Partial regularity of suitable weak solutions to the multi-dimensional generalized magnetohydrodynamics equations

Wei Ren
School of Mathematics and Systems Science
Beihang University, Beijing 100191, P. R. China
renwei4321@163.com

Yanqing Wang*
Department of Mathematics and Information Science
Zhengzhou University of Light Industry
Zhengzhou, Henan 450002, P. R. China
wangyanqing20056@gmail.com

Gang Wu
School of Mathematical Sciences
University of Chinese Academy of Sciences
Beijing 100049, P. R. China
wugangmaths@gmail.com

Received 19 May 2015
Revised 18 December 2015
Accepted 19 January 2016
Published 16 March 2016

In this paper, we are concerned with the partial regularity of the suitable weak solutions to the fractional MHD equations in $\mathbb{R}^n$ for $n = 2, 3$. In comparison with the work of the 3D fractional Navier–Stokes equations obtained by Tang and Yu in "Partial regularity of suitable weak solutions to the fractional Navier–Stokes equations, Comm. Math. Phys. 334 (2015) 1455–1482", our results include their endpoint case $\alpha = 3/4$ and the external force belongs to a more general parabolic Morrey space. Moreover, we prove some interior regularity criteria just via the scaled mixed norm of the velocity for the suitable weak solutions to the fractional MHD equations.

Keywords: Magnetohydrodynamics equations; suitable weak solutions; partial regularity.
Mathematics Subject Classification 2010: 76D03, 76D05, 35B33, 35Q35

*Corresponding author.
1. Introduction

We consider the following generalized incompressible magnetohydrodynamics (MHD) equations in $\mathbb{R}^n$ ($n = 2, 3$)

\[
\begin{cases}
  u_t + (-\Delta)^\alpha u + u \cdot \nabla u - h \cdot \nabla h + \nabla p = f, \\
  h_t + (-\Delta)^\beta h + u \cdot \nabla h - h \cdot \nabla u = 0, \\
  \text{div} u = \text{div} h = 0, \\
  (u, h)|_{t=0} = (u_0, h_0),
\end{cases}
\]

(1.1)

where $u, h$ describe the flow velocity field and the magnetic field, respectively, the scalar function $p = \pi + \frac{1}{2} h^2$ stands for the total pressure and the external force is denoted by $f$ with $\text{div} f = 0$. The fractional Laplacian $(-\Delta)^\alpha$ as the infinitesimal generator of a Lévy process is defined by $(-\Delta)^\alpha f(\xi) = |\xi|^{2\alpha} \hat{f}(\xi)$, where $\hat{f}(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(x) e^{-i\xi \cdot x} dx$. The initial data $(u_0, h_0)$ satisfies $\text{div} u_0 = \text{div} h_0 = 0$.

When $\alpha = \beta = 1$, the system (1.1) reduces to the classical MHD equations. MHD equations play an important role in electrically conducting fluids such as plasmas (see, e.g., [1]). There have been extensive studies on various topics concerning the MHD system and fractional MHD equations (see, e.g., [4, 6, 8, 10–13, 15, 19, 20, 29, 33–36] and references therein). MHD equations without the magnetic field degenerate to the Navier–Stokes equations. It is well-known that both the Navier–Stokes system and the Euler system in $\mathbb{R}^2$ are globally well-posed. Sermange and Teman [24] showed that the weak solutions of the 2D MHD system are regular. Very recently, the 2D generalized MHD system has been mathematically investigated in several works (see, e.g., [4, 15, 29, 36]). However, due to strong coupling between the magnetic field and the velocity field, to our knowledge, whether smooth solutions of the 2D MHD equations with fractional power dissipation $\alpha = \beta < 1$ breakdown in a finite time remains open. In [36], the global smooth solutions of 2D generalized MHD with $\alpha > 1$ and $\beta < 1$ were established by Wu. One goal of this paper is to prove partial regularity of solutions satisfying local energy inequality to the 2D fractional MHD system for $1/2 < \alpha = \beta < 1$.

The global weak solutions and the local strong solutions to the 3D MHD equations were constructed by Duvaut and Lions [8] and Sermange and Teman [24]. Regularity criteria of weak solutions to the 3D MHD equations only in terms of velocity field were proved in [6, 11]. Partial regularity of suitable weak solutions to the 3D MHD equations was investigated by He and Xin in [12] (see also [13]). The interior regularity criteria are shown for the suitable weak solutions via the velocity field with sufficiently small local scaled norm and the magnetic field with bounded local scaled norm in [12]. Very recently, Wang and Zhang [33] removed the magnetic field hypothesis for the regularity criteria for the suitable weak solutions to the 3D MHD equations. These results indicate that the velocity field plays a more dominant role than the magnetic field on the regularity of solutions to the magnetohydrodynamic equations, which is consistent with the numerical simulations in [10, 19].
Partial regularity of suitable weak solutions to the generalized MHD equations

The partial regularity of suitable weak solutions to the 3D MHD equations obtained in [12] is an analogue of the celebrated Caffarelli–Kohn–Nirenberg theorem to the 3D Navier–Stokes equations, namely, one-dimensional Hausdorff measure of the set of the possible space-time singular points of suitable weak solutions to the system is zero. The partial regularity of weak solutions obeying the local energy inequality to the 3D Navier–Stokes equations was originated from Scheffer [21–23]. The optimal Hausdorff dimension estimate of the possible space-time singular points set of suitable weak solutions to the 3D Navier–Stokes system was obtained by Caffarelli, Kohn and Nirenberg in [2]. Since then, there have been extensive studies on the partial regularity of solutions to the Navier–Stokes equations, MHD equations and the related models with fractional dissipation (see, e.g., [9, 12–14, 16, 18, 20, 25–28, 33, 31]). In particular, Katz and Pavlović [16] proved that the Hausdorff dimension of the singular set for classical solutions to the generalized Navier–Stokes equations with $1 < \alpha < 5/4$ at the first blow-up time is at most $5 - 4\alpha$, which was extended to the generalized MHD equations in [20]. It is shown in [14] that the $(5 - 4\alpha)/2\alpha$-dimensional Hausdorff measure of possible time singular points of weak solutions to the 3D fractional Navier–Stokes equations on the interval $(0, \infty)$ is zero if $5/6 \leq \alpha < 5/4$. Very recently, Tang and Yu [27] showed that the solutions of the 3D stationary fractional Navier–Stokes equations are regular away from a compact set whose $(5 - 6\alpha)$-Hausdorff measure is zero in the case $1/2 < \alpha < 5/6$.

Based on Caffarelli and Silvestre’s generalized extension for the fractional Laplacian operator, Tang and Yu [25] successfully established the partial regularity of suitable weak solutions to the fractional Navier–Stokes equations in the case $3/4 < \alpha < 1$ in [25], where the Hausdorff dimension of the potential space-time singular points set of suitable weak solutions is at most $5 - 4\alpha$. Since [25, Lemma 2.6] collapses when $\alpha = 3/4$, it seems that the limiting case $\alpha = 3/4$ cannot be covered in their work. One objective of this work is to address this borderline case. This is partially motivated by the previous investigation of partial regularity to the solutions of the 4D Navier–Stokes equations in [31]. The following observation plays an important role in our proof. Just as the 4D Navier–Stokes equations, from the interpolation inequality and Sobolev embedding theorem, we find that it holds $u \in L^{3/2}_{t,x}$ and $p \in L^{3/2}_{t,x}$ for the suitable weak solutions of the 3D generalized Navier–Stokes equations for $\alpha = 3/4$, which ensures that every term in the local energy inequality (2.13) makes sense. Meanwhile, this means a recurrence relation that the left-hand can control the right-hand in the local energy inequality. Specifically, we devote ourselves to treating the partial regularity of suitable weak solutions of the fractional magnetohydrodynamic equations (1.1).

Throughout this paper, $v^*$ denotes the extension of $v$ associated with the fractional Laplacian operator $(-\Delta)^\alpha$ in the sense of [3], for more details see Sec. 2. The norm of parabolic Morrey space $M_{2\alpha, \gamma}$ will be defined at the end of this section. In what follows, we consider the system (1.1) in the case $n/4 \leq \alpha = \beta < 1 (n = 2, 3)$ and $\alpha \neq 1/2$ unless otherwise stated. We are now ready to state the main theorems of this paper.
Theorem 1.1. Suppose that the triplet \((u, h, p)\) is a suitable weak solution to (1.1) and \(f \in L_{t,x}^q\) with \(q > \frac{n+2\alpha}{2\alpha}\) if \(1/2 < \alpha < 3/4\); \(f \in M_{2\alpha, \gamma}\) with \(\gamma > 0\) if \(3/4 \leq \alpha < 1\) in \(\mathbb{R}^n\) with \(n = 2, 3\). Then \(u\) and \(h\) can be bounded by 1 on \([-\frac{t}{s^{\alpha}}, 0] \times B(\frac{1}{4})\) provided the following condition holds,
\[
\sup_{t \in [-1,0]} \int_{B(1)} (|u|^2 + |h|^2) + \int_{Q^*_{t,1}} y^{1-2\alpha} (|\nabla^s u^*|^2 + |\nabla^s h^*|^2)
+ \left( \int_{Q^*_{t,1}} |p|^{3/2} \right)^{2/3} + \|f\| \leq \varepsilon_1,
\]
for an absolute constant \(\varepsilon_1 > 0\), where
\[
\|f\| = \begin{cases} 
\|f\|_{L^q(Q^*_{t,1})}, & \frac{1}{2} < \alpha < \frac{3}{4}, \\
\|f\|_{M_{2\alpha, \gamma}}, & \frac{3}{4} \leq \alpha < 1.
\end{cases}
\]

Remark 1.1. A slightly different version of Theorem 1.1 was obtained for the 3D Navier–Stokes equations by Vasseur in [30] and the 4D Navier–Stokes equations in [31] (see [13] for the MHD equations), where all the proofs rely on the De Giorigi iteration. Here, we mainly follow the pathway of [2, 25] to prove this theorem. In contrast with the work of [2, 25], we will use estimate on pressure \(p\) in \(L_{t,x}^{3/2}\) norm instead of \(L_t^3 L_x^1(p > 1)\) norm utilized there. Moreover, Theorem 1.1 without the magnetic field seems to be a new regularity criterion for the suitable weak solutions of the fractional Navier–Stokes equations in \(\mathbb{R}^n\), which is of independent interest.

Theorem 1.2. Assume that \((u, h, p)\) is a suitable weak solution to (1.1), then \((0, 0)\) is a regular point of \((u(x, t)\) and \(h(x, t)\) if the following condition holds,
\[
\limsup_{r \to 0^+} \frac{1}{r^{n+4\alpha}} \int_{Q^*_{t,r}} y^{1-2\alpha} (|\nabla^s u^*|^2 + |\nabla^s h^*|^2) \leq \varepsilon_2, \quad n = 2, 3,
\]
for an universal constant \(\varepsilon_2 > 0\).

Remark 1.2. In \(\mathbb{R}^n\), Vitali covering lemma utilized in [2] together with Theorem 1.2 implies that the \((5 - 4\alpha) (3/4 \leq \alpha < 1)\)-dimensional parabolic Hausdorff measure of the possible singular points set of \(u\) and \(h\) is zero for any suitable solution of (1.1), which extends the recent work of Tang and Yu [25] in the case \(3/4 < \alpha < 1\). It should be pointed out that, just as the 4D Navier–Stokes equations, it is not known whether the suitable weak solution to the system (1.1) exists. At least, we can obtain the partial regularity of smooth solutions of the 3D generalized Navier–Stokes equations with \(\alpha = 3/4\) at the first blow-up time.

Remark 1.3. It is worth noting that Tang and Yu [25] proved (1.3) without the magnetic field in \(\mathbb{R}^n\) under the condition that the force \(f\) lies in \(L_q^q\) with \(q' > \frac{3+6\alpha}{4\alpha+1}\). Notice that \(M_{2\alpha, \gamma} \supset L_q^q\) for \(q' > \frac{2\alpha+\gamma}{2\alpha}\) when \(0 < \gamma \leq 2\alpha\) and \(M_{2\alpha, \gamma} = \{0\}\) if \(\gamma > 2\alpha\). Hence, the assumption on the force \(f\) in [25] is relaxed. Furthermore, under the
definition of regular point, the hypothesis that $f \in L^q, q > \frac{n+2\alpha}{n}$ is optical in the sense of scaling. We mention that the Caffarelli–Kohn–Nirenberg theorem to the 3D Navier–Stokes equations with the external force belonging to parabolic Morrey space is due to the work of Ladyzenskaja and Seregin in [17], where the proof relies on a blow-up procedure and compactness argument.

Since the existence of the magnetic field in MHD equations (1.1), the proof of Theorem 1.2 is more involved than the generalized Navier–Stokes equations. Particularly, a difficulty arises when we deal with the case $\alpha = \beta \leq 3/4$. As mentioned above, [25, Lemma 2.6] seems to break down in this case. To build an effective iteration scheme via local energy inequality (2.13), our observation is that, under the hypothesis (1.3), the right-hand side of the local energy inequality (2.13) should be seen as the magnitude like $\|u\|_{L^3_{t,x}}^3 (\|h\|_{L^3_{t,x}}^2)$ rather than $\|u\|_{L^3_{t,x}}^3$ as usual. Based on this, we find that Lemma 2.1 established in the next section instead of Lemma 2.6 with $\alpha > 3/4$ in [25] works for $\alpha > 1/2$. However, this causes the estimate of term $\iint u \cdot \nabla \varphi p$ to be subtle. To this end, making full use of the interior estimate of harmonic function, we could establish the decay estimate of pressure $p - \bar{p}$ in $L^\infty_{t,x}$ norm. Meanwhile, the divergence-free algebraical structure of (1.1) plays a crucial role in dealing with the interaction terms between the magnetic field and the velocity field in the local energy inequality to avoid the appearance of terms similar to $\|u\|_{L^3_{t,x}}^3$. This enables us to achieve the proof of Theorem 1.2.

Partially motivated by the works [6, 11, 12, 33], we show that the velocity field plays a more important role than the magnetic field in the local regularity theory of the MHD equations (1.1). Precisely, we shall prove some interior regularity criteria which do not explicitly involve the magnetic field for the suitable weak solutions to the MHD equations (1.1). Without loss of generality, we assume that $f = 0$ in the following theorems.

**Theorem 1.3.** Assume that the triplet $(u, h, p)$ is a suitable weak solution to (1.1), then $(0, 0)$ is regular point of $(u, h)$ provided the following condition holds,

$$\limsup_{r \to 0^+} \frac{1}{r^{n+3-4\alpha}} \iint_{Q(r)} |u|^3 \, dx \, dt \leq \varepsilon_3, \quad n = 2, 3, \quad (1.4)$$

for an universal constant $\varepsilon_3 > 0$.

**Remark 1.4.** It is remarkable that the regularity criterion (1.4) is just in terms of the velocity field $u$ instead of the combination between the velocity field and the magnetic field. Moreover, even for the 3D fractional Navier–Stokes system, although the extension of $u$ appears in the right-hand side of its local energy inequality, this sufficient regularity condition (1.4) does not involve its extension $u^*$. The key issue to prove (1.4) is to resort the appropriate test function to circumvent the straightforward control of the terms involving the magnetic field $h$ and the extensions $u^*, h^*$ on the right-hand side of the local energy inequality (2.13).
Furthermore, the magnitude of the left-hand of the local energy inequality likes \( ||h||_{L_t^3L_x^2}^{2/3} \) helps us to handle the terms \( \int \int |u||h|^2 + |u||p - p_p| \). This allows us to complete the proof of Theorem 1.3. Due to the pressure in terms of the magnetic field and the velocity field, it seems difficult to extend the integral norms with different exponents in space and time in (1.4). Using the completely different iteration scheme involving the pressure and a slight variant of treating the terms \( \int \int |u||h|^2 + |u||p - p_p| \) in the proof of Theorem 1.3, we give the regularity conditions via the velocity with sufficiently small local scaled norm and the magnetic field with bounded local scaled norm. However, we would like to point out that we can remove the hypothesis of the magnetic field when \((n + 4)/8 < \alpha = \beta < 1\).

**Theorem 1.4.** Assume that the triplet \((u, h, p)\) is a suitable weak solution to (1.1), then \((0, 0)\) is regular point of \((u, h)\) provided one of the following conditions holds.

1. For any constant \(M > 0\), there exists a positive constant \(\varepsilon_4(M)\) such that
   \[
   \limsup_{r \to 0^+} r^{-\left(\frac{\alpha}{q} + \frac{2\alpha}{\ell} - (2\alpha - 1)\right)} \left( \int_{-r^{2\alpha}}^0 \left( \int_{B(r)} |u|^q \, dx \right) \, ds \right)^{\frac{q}{q}} \leq \varepsilon_4, \tag{1.5}
   \]
   where the pair \((q, \ell)\) satisfies
   \[
   2\alpha - 1 \leq \frac{n}{\ell} + \frac{2\alpha}{q} \leq 2\alpha, \quad 1 \leq q \leq \infty, \quad n = 2, 3. \tag{1.6}
   \]

2. Let \((n + 4)/8 < \alpha = \beta < 1\). There exists a positive constant \(\varepsilon_5\) such that
   \[
   \limsup_{r \to 0^+} r^{-\left(\frac{\alpha}{q} + \frac{2\alpha}{\ell} - (2\alpha - 1)\right)} \left( \int_{-r^{2\alpha}}^0 \left( \int_{B(r)} |h|^q \, dx \right) \, ds \right)^{\frac{q}{q}} \leq \varepsilon_5, \tag{1.7}
   \]
   where the pair \((\ell, q)\) satisfies
   \[
   2\alpha - 1 \leq \frac{n}{\ell} + \frac{2\alpha}{q} \leq 2\alpha, \quad \max\left\{ 1, \frac{n}{4\alpha - n} \right\} < \ell, \quad n = 2, 3. \tag{1.8}
   \]

**Remark 1.5.** As a straightforward consequence of (1.4) and (1.5), sufficient regular conditions for the suitable weak solutions to the 3D generalized Navier–Stokes equations are obtained. More precisely, let \((u, p)\) be the suitable weak solutions to the 3D fractional Navier–Stokes system for \(3/4 \leq \alpha < 1\), then \(u\) is regular on \(Q(r/2)\) provided that \(u\) lies in \(L^{\infty, \ell}(Q(r))\) with \(2\alpha/q + 3/\ell = 2\alpha - 1\) (\(\ell > 3/(2\alpha - 1)\)) or \(||u||_{L^{\infty, 3/(2\alpha - 1)}(Q(r))}\) is sufficiently small.

**Remark 1.6.** We emphasize that the magnetic field with bounded local scaled norm in (1.5) is only used for treating the term \(\int \int u \cdot \nabla \phi p\).
Remark 1.7. By a different method, we will derive new interior regularity criteria for the suitable weak solutions to the generalized Navier–Stokes system in the case $3/4 \leq \alpha < 1$ in [32].

The remainder of this paper is organized as follows. In the next section, we will begin with some facts related to Caffarelli and Silvestre’s generalized extension and collect some useful inequalities associated with this extension. Then we will present the definition of the suitable weak solutions and establish various dimensionless decay estimates. In Sec. 3, by means of induction argument, we complete the proof of Theorem 1.1. Finally, combining Theorem 1.1 proved in Sec. 3 with the preliminary lemmas in Sec. 2, we prove Theorem 1.2 as well as Theorems 1.3 and 1.4 in Sec. 4.

**Notations:** Throughout this paper, we denote

\[
B(x, \mu) := \{ y \in \mathbb{R}^n | |x - y| \leq \mu \}, \quad B(\mu) := B(0, \mu),
\]

\[
\bar{B}(\mu) := B(x_0, \mu), \quad B^*(x, \mu) := B(x, \mu) \times (0, \mu),
\]

\[
B^*(\mu) := B^*(0, \mu), \quad \bar{B}^*(\mu) := \bar{B}^*(x_0, \mu),
\]

\[
Q(x, t, \mu) := B(x, \mu) \times (t - \mu^{2\alpha}, t), \quad Q(\mu) := Q(0, 0, \mu),
\]

\[
\hat{Q}(\mu) := Q(x_0, t_0, \mu), \quad Q^*(x, t, \mu) := B^*(x, \mu) \times (t - \mu^{2\alpha}, t),
\]

\[
\hat{Q}^*(\mu) := Q^*(0, 0, \mu), \quad \hat{Q}^*(\mu) := Q^*(x_0, t_0, \mu),
\]

\[
r_k = 2^{-k}, \quad \bar{B}_k := \bar{B}(r_k), \quad B^*_k := B^*(r_k),
\]

\[
\hat{Q}_k := \hat{Q}(r_k), \quad \hat{Q}^*_k := \hat{Q}^*(r_k).
\]

The classical Sobolev norm $\| \cdot \|_H^s$ is defined as $\| f \|_{\mathcal{H}^s} = \int_{\mathbb{R}^n} (1 + |\xi|)^{2s} |\hat{f}(\xi)|^2 d\xi$, $s \in \mathbb{R}$. We denote by $\mathcal{H}^s$ homogenous Sobolev spaces with the norm $\| f \|^2_{\mathcal{H}^s} = \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi$. For $q \in [1, \infty]$, the notation $L^q(0, T; X)$ stands for the set of measurable functions on the interval $(0, T)$ with values in $X$ and $\| f(t, \cdot) \|_X$ belongs to $L^q(0, T)$. For simplicity, we write

\[
\| f \|_{L^q_t(Q(\mu))} := \sup_{(x, t) \in X} \| f \|_{L^q(-\mu^2, 0; L^q(B(\mu)))} \quad \text{and} \quad \| f \|_{L^q_t(Q(\mu))} := \sup_{(x, t) \in X} \| f \|_{L^q_t(Q(\mu))}.
\]

The parabolic Morrey space $M_{2\alpha, \gamma}$ ($0 < \gamma \leq 2\alpha$) is equipped with the norm

\[
\| f \|_{M_{2\alpha, \gamma}} = \sup_{(x, t) \in \mathbb{R}^n \times (-T, 0)} \left\{ \sup_{R > 0} \left( \frac{1}{T^{1 - 2\alpha}} \left( \int_{Q_{R}(x, t)} |f|^2 \right)^{\frac{1}{2}} \right) \right\}.
\]

Denote the average of $f$ on the set $\Omega$ by $\overline{f}_{\Omega}$. For convenience, $\overline{f}_x$ represents $\overline{f}_{B(x)}$ and $\overline{f}_{\bar{B}_k}$ is denoted by $\hat{f}_{\bar{B}_k}$. $K$ stands for the standard normalized fundamental solution of Laplace equation in $\mathbb{R}^n$ with $n \geq 2$. We denote by $\text{Div}$ the divergence operator in $\mathbb{R}^{n+1}$ and $\nabla^*$ the gradient operator in $\mathbb{R}^{n+1}$. $|\Omega|$ represents the Lebesgue measure of the set $\Omega$. We will use the summation convention on repeated indices. $C$ is an absolute constant which may be different from line to line unless otherwise stated in this paper.
2. Preliminaries

In this section, we first recall Caffarelli and Silvestre’s generalized extension for the fractional Laplacian operator \((-\Delta)^s\) with \(0 < s < 1\) in [3]. The fractional Laplacian can be interpreted as

\[
(-\Delta)^s u = -C_s \lim_{y \to 0} y^{1-2s} \partial_y u^*,
\]

where \(C_s\) is a constant depending only on \(s\) and \(u^*\) satisfies

\[
\begin{cases}
\text{Div} (y^{1-2s} \nabla^* u^*) = 0 & \text{in } \mathbb{R}^{n+1}, \\
u^*|_{y=0} = u.
\end{cases}
\]

Furthermore, the Poisson formula below is valid

\[
u^*(x, y) = C_{n,s} \int_{\mathbb{R}^n} \frac{y^{2s} u(\xi)}{(x - \xi^2 + y^2)^{\frac{n+2}{2}}} d\xi,
\]

where \(C_{n,s}\) is a constant depending only on \(n\) and \(s\). In addition, from [3, Sec. 3.2], an equivalent definition of the \(\dot{H}^s\) norm reads

\[
\|u\|_{\dot{H}^s}^2 = \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi = \int_{\mathbb{R}^{n+1}} y^{1-2s} |\nabla^* u^*|^2 dxdy.
\]

As a by-product of (2.2), for any \(v|_{y=0} = u\), it holds

\[
\int_{\mathbb{R}^{n+1}} y^{1-2s} |\nabla^* u^*|^2 dxdy \leq \int_{\mathbb{R}^{n+1}} y^{1-2s} |\nabla^* v|^2 dxdy.
\]

With (2.4) and (2.5) in hand, one can prove the following inequalities frequently used later:

\[
\|u - \overline{\mu}\|_{L^2_{|x|} (B(\mu/2))} \leq C \left( \int_{B^*(\mu)} y^{1-2s} |\nabla^* u^*|^2 \right)^{1/2},
\]

\[
\|u\|_{L^2_{|x|} (B(\mu/2))} \leq C \left( \int_{B^*(\mu)} y^{1-2s} |\nabla^* u^*|^2 \right)^{1/2} + C \mu^{-s} B \left( \int_{B(\mu)} |u|^2 \right)^{1/2},
\]

\[
\|v\|_{L^2_{\mu} (Q(\mu/2))} \leq C \left( \int_{Q^* (\mu)} y^{1-2s} |\nabla^* u^*|^2 \right)^{1/2} + C \left( \sup_{-\mu^2 \leq t < 0} \int_{B(\mu)} |u|^2 \right)^{1/2},
\]

\[
\int_{B^*(\mu)} y^{1-2s} |u^*|^2 \leq C \mu^{2-2s} \int_{B(\mu)} |u|^2 + C \mu^2 \int_{B^*(\mu)} y^{1-2s} |\nabla^* u^*|^2,
\]

\[1650018-8\]
Partial regularity of suitable weak solutions to the generalized MHD equations

\[
\int_{Q^*(\mu)} y^{-2s}|u^*|^2 \leq C \mu^{2-2s} \int_{Q^*(\mu)} |u|^2 + C \mu^2 \int_{Q^*(\mu)} y^{-2s} |\nabla^s u^*|^2, \tag{2.10}
\]

\[
\|u\|_{L^{\infty/\alpha} (B(2/3))} \leq C \left( \int_{B^*(1)} y^{-2s} |\nabla^s u^*|^2 \right)^{1/2} + C \left( \int_{B(1)} |u|^2 \right)^{1/2}, \tag{2.11}
\]

\[
\|u\|_{L^{\infty/\alpha} (Q(2/3))} \leq C \left( \int_{Q^*(1)} y^{-2s} |\nabla^s u^*|^2 \right)^{1/2} + C \left( \sup_{s \mu^2 \leq t < 0} \int_{B(1)} |u|^2 \right)^{1/2}. \tag{2.12}
\]

For the proof, we refer the reader to [25, Proposition 2.2, p. 1461 and (2.14) in p. 1463].

Now we present the definition of the suitable weak solution to the MHD equations (1.1).

**Definition 2.1.** A triplet \((u, h, p)\) is called a suitable weak solution to the generalized MHD equations (1.1) provided the following conditions are satisfied,

1. \(u, h \in L^\infty(0, T; L^2(\mathbb{R}^n)) \cap L^2(0, T; \dot{H}^\alpha(\mathbb{R}^n)), p \in L^{3/2}(0, T; L^{3/2}(\mathbb{R}^n))\).
2. \((u, h, p)\) solves (1.1) in \(\mathbb{R}^n \times (0, T)\) in the sense of distributions.
3. \((u, h, p)\) satisfies the following inequality

\[
\int_{\mathbb{R}^n} (|u|^2 + |h|^2) \varphi_1 (x, t) + 2C \int_{t-r_0}^{t} \int_{R^{n+1}_{t+1}} \varphi_2 (x, y, t) y^{1-2\alpha} (|\nabla^s u^*|^2 + |\nabla^s h^*|^2)
\]

\[
\leq C \int_{t-r_0}^{t} \int_{R^{n+1}_{t+1}} (|u^*|^2 + |h^*|^2) \text{Div} (y^{1-2\alpha} \nabla^s (\varphi_2))
\]

\[
+ \int_{t-r_0}^{t} \int_{\mathbb{R}^n} (|u|^2 + |h|^2) \left[ \partial_t \varphi_1 + C \lim_{y \to 0^+} y^{1-2\alpha} \partial_y (\varphi_2) \right]
\]

\[
+ \int_{t-r_0}^{t} \int_{\mathbb{R}^n} u \cdot \nabla \varphi_1 (|u|^2 + |h|^2 + 2p) - 2 \int_{t-r_0}^{t} \int_{\mathbb{R}^n} h \cdot \nabla \varphi_1 (u \cdot h)
\]

\[
+ 2 \int_{t-r_0}^{t} \int_{\mathbb{R}^n} u \cdot f \varphi_1, \tag{2.13}
\]

where \(\varphi_1 (x, t) \in C^\infty_0 (\mathbb{R}^n \times (0, T))\) and \(\lim_{y \to 0} \varphi_2 (x, y, t) = \varphi_1 (x, t)\).

A point \((x, t)\) is said to be a regular point of the suitable weak solutions to the system (1.1) if one has the boundedness of \((u, h)\) in some neighborhood of \((x, t)\). The remaining points are called singular point and denoted by \(S\).

Before we present the decay-type lemmas, according to the natural scaling property of system (1.1), we introduce the following dimensionless quantities:

\[
E_t (u, r) = \frac{1}{r^{n+2\alpha-(2\alpha-1)t}} \int_{Q(r)} |u|^t dx dt,
\]

where \(r = |x| = \sqrt{\sum_{i=1}^{n} x_i^2}\).
Note that, by the Hölder inequality, it is enough to prove Theorem 1.4 for the borderline case $n/\ell$ and $\rho$, Lemma 2.1.

With the help of the triangle inequality, the Hölder inequality and (2.6),

Proof. With the help of the triangle inequality, the Hölder inequality and (2.6), we see that

$$\int_{B(\rho)} |u|^3 \leq C \int_{B(\rho)} |u - \bar{u}_\rho|^3 + C \int_{B(\rho)} |\bar{u}_\rho|^3$$

$$\leq C \left( \int_{B(\frac{\rho}{2})} |u - \bar{u}_\rho|^2 \right) \frac{\mu^{n-2\alpha}}{\rho^{n-2\alpha}} \left( \int_{B(\frac{\rho}{2})} |u - \bar{u}_\rho|^\frac{2(n-2\alpha)}{n-2\alpha} \right)^\frac{2}{n}$$

$$+ C \frac{\mu^n}{\rho^2} \left( \int_{B(\rho)} |u|^2 \right)^{3/2}$$
Partial regularity of suitable weak solutions to the generalized MHD equations

\[ \leq C \left( \int_{B(\rho)} |u|^2 \right)^{\frac{6\alpha-n}{\alpha}} \left( \int_{B^*(\rho)} y^{1-2\alpha} |\nabla^* u|^2 \right)^{\frac{n}{\alpha}} \]

\[ + \frac{\mu^n C}{\rho^{\frac{n-2}{2}}} \left( \int_{B(\rho)} |u|^2 \right)^{\frac{3}{2}}. \tag{2.15} \]

Integrating in time on \((-\mu^{2\alpha}, 0)\) this inequality, we obtain

\[ \iint_{Q(\mu)} |u|^3 \leq C \left( \sup_{-\mu^{2\alpha} \leq t \leq 0} \int_{B(\rho)} |u|^2 \right)^{\frac{6\alpha-n}{\alpha}} \left( \int_{B^*(\rho)} y^{1-2\alpha} |\nabla^* u|^2 \right)^{\frac{n}{\alpha}} \]

\[ + C \frac{\mu^{n+2\alpha}}{\rho^{\frac{n-2}{2}}} \left( \sup_{-\mu^{2\alpha} \leq t \leq 0} \int_{B(\rho)} |u|^2 \right)^{\frac{3}{2}}, \]

which leads to

\[ E_3(u, \mu) \leq C \left( \frac{\rho}{\mu} \right)^{\frac{6\alpha-3-12\alpha}{n}} E^{\frac{6\alpha-n}{\alpha}}(\mu) E^{\frac{6\alpha-n}{\alpha}}(\rho) + C \left( \frac{\mu}{\rho} \right)^{\frac{6\alpha-3}{n}} E^{3/2}(\mu, \rho). \]

This achieves the proof of this lemma. \( \Box \)

To prove Theorems 1.2–1.4, we need different decay-type estimates involving the pressure.

**Lemma 2.2.** For \( 0 < \mu \leq \frac{1}{2} \rho \), there exists an absolute constant \( C \) independent of \( \mu \) and \( \rho \) such that

\[ P_{n+2\alpha}(\mu) \leq C \left( \frac{\rho}{\mu} \right)^{\frac{2\alpha+0+\alpha+2-4\alpha}{n}} E_{\alpha, \ell}(\mu) E_{\alpha, \ell}(\rho) + C \left( \frac{\mu}{\rho} \right)^{\frac{6\alpha-3-12\alpha}{n}} P_{n+2\alpha}(\rho), \tag{2.16} \]

\[ P_{\ell', \ell}(\mu) \leq C \left( \frac{\rho}{\mu} \right)^{n+2-4\alpha} E_{\frac{\alpha}{2}}(\mu) E_{\frac{\alpha-1}{2}}(\rho) + C \left( \frac{\mu}{\rho} \right)^{\frac{6\alpha-n}{n}} P_{\ell', \ell}(\rho), \tag{2.17} \]

\[ P_{3/2}(\mu) \leq C \left( \frac{\rho}{\mu} \right)^{\frac{6\alpha-3-12\alpha}{n}} E_{\frac{6\alpha-n}{n}}(\mu) E_{\frac{6\alpha-n}{n}}(\rho) + C \left( \frac{\mu}{\rho} \right)^{\frac{6\alpha-3}{n}} P_{3/2}(\rho), \tag{2.18} \]

\[ P_{3/2}(\mu) \leq C \left( \frac{\rho}{\mu} \right)^{\frac{6\alpha-3-12\alpha}{n}} E(\mu)^{\frac{1}{2}} E_{\frac{1}{2}}(\mu, \ell) + C \left( \frac{\mu}{\rho} \right)^{\frac{6\alpha-3}{n}} P_{3/2}(\rho), \tag{2.19} \]

where the pair \((q', \ell')\) is the conjugate index of \((q, \ell)\) in (1.6).
Proof. We consider the usual cut-off function $\phi \in C_0^\infty(B(\rho/2))$ such that $\phi \equiv 1$ on $B(\frac{\rho}{2})$ with $0 \leq \phi \leq 1$ and $|\nabla \phi| \leq C\rho^{-1}$, $|\nabla^2 \phi| \leq C\rho^{-2}$.

Due to the divergence free condition on $u$ and $h$, we may write

$$\partial_t \partial_i (p \phi) = -\phi \partial_i \partial_j [U_{i,j} - H_{i,j}] + 2\partial_j \phi \partial_i p + p \partial_i \partial_t \phi,$$

where $U_{i,j} = (u_j - \overline{u}_j \rho/2)(u_i - \overline{u}_i \rho/2)$ and $H_{i,j} = (h_j - \overline{h}_j \rho/2)(h_i - \overline{h}_i \rho/2)$. This yields that, for any $x \in B(\frac{\rho}{2})$,

$$p(x) = K \{ -\phi \partial_i \partial_j [U_{i,j} - H_{i,j}] + 2\partial_i \phi \partial_j p + p \partial_i \partial_j \phi \}
= -\partial_i \partial_j K \{ \phi[U_{i,j} - H_{i,j}] + 2\partial_i K \{ \partial_j \phi[U_{i,j} - H_{i,j}] \}
- K \{ \partial_i \partial_j \phi[U_{i,j} - H_{i,j}] + 2\partial_i K \{ \partial_j \phi p - K \{ \partial_i \partial_j \phi p \}
=: P_1(x) + P_2(x) + P_3(x), \quad (2.20)$$

where $K$ represents the standard normalized fundamental solution of Laplace equation. Since $\phi(x) = 1$ when $x \in B(\rho/4)$, we know that

$$\Delta(P_2(x) + P_3(x)) = 0.$$

By the interior estimate of harmonic function and the Hölder inequality, we see that, for every $x_0 \in B(\rho/8)$,

$$|\nabla(P_2 + P_3)(x_0)| \leq \frac{C}{\rho^{n+1}} \|(P_2 + P_3)\|_{L^1(B(x_0, \rho/8))}
\leq \frac{C}{\rho^{n+1}} \|(P_2 + P_3)\|_{L^1(B(\rho/4))}
\leq \frac{C}{\rho^{n+1}} \rho^{n(1-\frac{q}{n})} \|(P_2 + P_3)\|_{L^q(B(\rho/4))},$$

which in turn implies

$$\|\nabla(P_2 + P_3)\|_{L^\infty(B(\rho/8))}^q \leq C\rho^{-(n+q)} \|(P_2 + P_3)\|_{L^q(B(\rho/4))}^q.$$

The latest inequality above together with the mean value theorem leads to, for any $\mu \leq \frac{1}{8} \rho$,

$$\|(P_2 + P_3) - (P_2 + P_3)_\mu\|_{L^\infty(B(\mu))}^q \leq C\mu^n \|(P_2 + P_3) - (P_2 + P_3)_\mu\|_{L^\infty(B(\mu))}^q
\leq C\mu^{n+q} \|\nabla(P_2 + P_3)\|_{L^\infty(B(\rho/8))}^q
\leq C \left(\frac{\mu}{\rho}\right)^{n+q} \|\nabla(P_2 + P_3)\|_{L^\infty(B(\rho/4))}^q.$$

Integrating in time on $(-\mu^{2\alpha}, 0)$ for $q = \frac{n+2\alpha}{\alpha}$, we infer that

$$\|(P_2 + P_3) - (P_2 + P_3)_\mu\|_{L^{n+\frac{2\alpha}{\alpha}}(B(\mu))} \leq C \left(\frac{\mu}{\rho}\right)^{n+\frac{2\alpha}{\alpha}} \|\nabla(P_2 + P_3)\|_{L^{n+\frac{2\alpha}{\alpha}}(B(\rho/4))}.$$
Notice that \((P_2 + P_3) - (P_2 + P_3)_{\rho/4}\) is also a harmonic function on \(B(\rho/4)\), then the following estimate is valid
\[
\| (P_2 + P_3) - (P_2 + P_3)_{\rho/4} \| \leq C \left( \frac{\mu}{\rho} \right)^{\frac{n+4a}{n}} \| (P_2 + P_3) - (P_2 + P_3)_{\rho/4} \|^{\frac{n+4a}{n}} (B(\rho))
\]
In the light of the triangle inequality, we have
\[
\| (P_2 + P_3) - (P_2 + P_3)_{\rho/4} \| \leq \| p - \tilde{p}_{\rho/4} \| + \| P_1 - \tilde{P}_{\rho/4} \| + \| P_1 \| + \| P_1 \|^{\frac{n+4a}{n}} (B(\rho/4))
\]
which in turns yields
\[
\| (P_2 + P_3) - (P_2 + P_3)_{\rho/4} \| \leq C \left( \frac{\mu}{\rho} \right)^{\frac{n+4a}{n}} \| p - \tilde{p}_{\rho/4} \| + \| P_1 \| + \| P_1 \|^{\frac{n+4a}{n}} (B(\rho/4))
\]
Utilizing the Hölder inequality and (2.6), we see that
\[
\int_{B(\rho/2)} |u - \tilde{u}_{\rho/2}|^\frac{2(n+2a)}{n} \leq C \left( \int_{B(\rho/2)} |u - \tilde{u}_{\rho/2}|^\frac{2(n+2a)}{n} \right)^{\frac{n}{2(n+2a)}} \left( \int_{B(\rho/2)} |u - \tilde{u}_{\rho/2}|^\frac{2(n+2a)}{n} \right)^{\frac{n}{2(n+2a)}}
\]
\[
\leq C \left( \int_{B(\rho/2)} |u|^2 \right)^\frac{n}{2(n+2a)} \left( \int_{B(\rho/2)} |u - \tilde{u}_{\rho/2}|^\frac{2(n+2a)}{n} \right)^{\frac{n}{2(n+2a)}}
\]
\[
\leq C \left( \int_{B(\rho)} |u|^2 \right)^\frac{n}{2(n+2a)} \left( \int_{B(\rho)} y^{1-2a} |\nabla \ast u|^2 \right).
\]
According to the classical Calderón–Zygmund theorem and (2.22), we get that
\[
\int_{B(\rho/4)} |P_1(x)|^{\frac{n+4a}{n}} \leq C \int_{B(\rho/2)} |u - \tilde{u}_{\rho/2}|^{\frac{2(n+2a)}{n}} + |h - \tilde{h}_{\rho/2}|^{\frac{2(n+2a)}{n}}
\]
\[
\leq C \left( \int_{B(\rho)} |u|^2 \right)^\frac{2n}{n+4a} \left( \int_{B(\rho)} y^{1-2a} |\nabla \ast u|^2 \right)
\]
\[
+ C \left( \int_{B(\rho)} |h|^2 \right)^\frac{2n}{n+4a} \left( \int_{B(\rho)} y^{1-2a} |\nabla \ast h|^2 \right).
\]
which obviously implies that, for any \( \mu \leq \frac{\rho}{q} \),
\[
\int_{B(\mu)} |P_1(x)| \frac{\alpha + 2 \alpha_0}{\alpha} \leq C \left( \int_{B(\rho)} |u|^2 \right)^{\frac{\alpha}{\alpha}} \left( \int_{B^*(\rho)} y^{1-2\alpha} |\nabla^* u^*|^2 \right)^{\frac{\alpha}{\alpha}} + C \left( \int_{B(\rho)} |h|^2 \right)^{\frac{\alpha}{\alpha}} \left( \int_{B^*(\rho)} y^{1-2\alpha} |\nabla^* h^*|^2 \right)^{\frac{\alpha}{\alpha}},
\]
\tag{2.24}
\]

Let \( \tau = (n + 2 - 4\alpha)(1 + \frac{2\alpha_0}{\alpha}) \), then, it follows from (2.21)–(2.24) that
\[
\frac{1}{\mu^\tau} \int_{\rho \mu} |p - \overline{\rho}_\mu| \frac{\alpha + 2 \alpha_0}{\alpha} \leq C \left( \frac{\rho}{\mu} \right)^\tau \left( \frac{1}{\rho^{n+2-4\alpha}} \sup_{-\rho^{2\alpha} \leq t < 0} \int_{B(\rho)} |u|^2 \right)^{\frac{\alpha}{\alpha}} \times \left( \frac{1}{\rho^{n+2-4\alpha}} \int_{\rho \mu} y^{1-2\alpha} |\nabla^* u^*|^2 \right)^{\frac{\alpha}{\alpha}} + C \left( \frac{\rho}{\mu} \right)^\tau \left( \frac{1}{\rho^{n+2-4\alpha}} \sup_{-\rho^{2\alpha} \leq t < 0} \int_{B(\rho)} |h|^2 \right)^{\frac{\alpha}{\alpha}} \times \left( \frac{1}{\rho^{n+2-4\alpha}} \int_{\rho \mu} y^{1-2\alpha} |\nabla^* h^*|^2 \right)^{\frac{\alpha}{\alpha}} + C \left( \frac{\mu}{\rho} \right)^{\frac{\alpha_0^2 + 2(\alpha - 1) - n}{\alpha}} \frac{1}{\rho^\tau} \int_{\rho \mu} |p - \overline{\rho}_\mu| \frac{\alpha + 2 \alpha_0}{\alpha} ,
\]
which turns out that
\[
P_{\frac{n+2\alpha}{\alpha}} (\mu) \leq C \left( \frac{\rho}{\mu} \right)^{\frac{2\alpha_0^2 + 2(\alpha_0 - 1) - n}{\alpha}} E_\mu (\rho) E_\mu (\rho) + C \left( \frac{\mu}{\rho} \right)^{\frac{\alpha_0^2 + 2(\alpha - 1) - n}{\alpha}} P_{\frac{n+2\alpha}{\alpha}} (\rho).
\]
\tag{2.26}
\]

In view of (2.6), we know that
\[
\begin{align*}
||u - \overline{\rho}_{\rho/2}||_{L^{2q'}(Q(\rho/2))} & \leq C ||u - \overline{\rho}_{\rho/2}||_{L^\infty(Q(\rho/2))} ||u - \overline{\rho}_{\rho/2}||_{L^2(Q(\rho/2))} \frac{1}{\alpha} \frac{1}{\alpha'} \frac{1}{L^2(\rho/2)} \frac{1}{L^2(\rho/2)} \frac{1}{L^2(\rho/2)} \frac{1}{L^2(\rho/2)} \frac{1}{L^2(\rho/2)} \\
& \leq C ||u||_{L^\infty(Q(\rho))} ||y^{1/2-\alpha} \nabla^* u^*||_{L^2(\rho)} \frac{1}{\alpha} \frac{1}{\alpha'} \frac{1}{L^2(\rho)} \frac{1}{L^2(\rho)} \frac{1}{L^2(\rho)} \frac{1}{L^2(\rho)} \frac{1}{L^2(\rho)} \\
& \leq C ||u||_{L^\infty(Q(\rho))} ||y^{1/2-\alpha} \nabla^* u^*||_{L^2(\rho)} , \tag{2.27}
\end{align*}
\]
where \( \alpha/q' + n/(2\ell') = n/2 \). A slight modification about the proof of (2.25) together with (2.27) gives (2.17).
that the following estimate is valid
\[
\|u - \overline{u}_{\rho/2}\|_{L^{3}(Q(\rho/2))}^{3} \leq C\|u - \overline{u}_{\rho/2}\|_{L^{6}(Q(\rho/2))}^{2} \|u - \overline{u}_{\rho/2}\|_{L^{2\nu'}(Q(\rho/2))}^{2}
\]
\[
\leq C\|u\|_{L^{6}(Q(\rho))}\|u\|_{L^{\infty}(Q(\rho))}^{\frac{2}{3}}\|y^{1/2-\alpha}\nabla^{+}u\|_{L^{2}(Q(\rho))}^{2},
\]
(2.28)
where \(\alpha/q' + n/(2\ell') = n/2\). Along the exact same line as the proof of (2.25), we obtain (2.19).

It remains to show (2.18). Indeed, in the light of the Hölder inequality and (2.6), we get
\[
\int_{B(\rho/2)}|u - \overline{u}_{\rho/2}|^{3} \leq C\left(\int_{B(\rho/2)}|u - \overline{u}_{\rho/2}|^{2}\right)^{\frac{3n-n}{n}}\left(\int_{B(\rho)}|y|^{3-2\alpha}\|\nabla^{+}u\|^{2}\right)^{\frac{n}{n}}.
\]
Integrating this inequality in time, we deduce that
\[
\int_{-\mu^{2\alpha}}^{0} \int_{B(\rho)}|u - \overline{u}_{\rho/2}|^{3} \leq C\left(\sup_{-\mu^{2\alpha} \leq t \leq 0} \int_{B(\rho)}|u|^{2}\right)^{\frac{3n-n}{n}}\left(\int_{Q^{\ast}(\rho)}|y|^{3-2\alpha}\|\nabla^{+}u\|^{2}\right)^{\frac{n}{n}}.
\]
\[
\leq C\mu^{\frac{3n-n}{n}}\left(\sup_{-\mu^{2\alpha} \leq t \leq 0} \int_{B(\rho)}|u|^{2}\right)^{\frac{3n-n}{n}}\left(\int_{Q^{\ast}(\rho)}|y|^{3-2\alpha}\|\nabla^{+}u\|^{2}\right)^{\frac{n}{n}}.
\]
With this inequality in hand, the rest proof of (2.18) is parallel to the one of (2.26). Thus, the proof of this lemma is completed.

A slight variant of the proof of \([2, \text{Lemma 3.2, p. 786}]\) yields the following lemma.

**Lemma 2.3.** For any \(\mu \leq 1/2\rho\), a constant \(C\) independent of \(\mu\) and \(\rho\) exists such that the following estimate is valid
\[
\frac{1}{\mu^{n+3-4\alpha}} \int_{\bar{Q}(\rho)} |u||p - \overline{p}_{B(\rho)}|
\]
\[
\leq C\left(\frac{1}{\mu^{n+3-4\alpha}} \int_{\bar{Q}(\rho)} |u|^{3}\right)^{1/3}\left(\frac{1}{\mu^{n+3-4\alpha}} \int_{\bar{Q}(\rho)} |u|^{3} + |h|^{3}\right)^{2/3}
\]
\[
+ C\mu^{\alpha-1} \left(\frac{1}{\mu^{n+3-4\alpha}} \int_{\bar{Q}(\rho)} |u|^{3}\right)^{1/3} \sup_{-\mu^{2\alpha} \leq t \leq 0} \int_{2\mu < |y-x| < \rho} \frac{|u|^{2} + |h|^{2}}{|y-x|^{n+1}} dy.
\]
By a straightforward computation, for any $P$ we still utilize the notations $\mu$ on $B$ and $\rho\phi$.

\[
\nabla \phi = \frac{1}{\mu^{n+3-4\alpha}} \int_{Q(\mu)} |u|^3 \right)^{1/3} + \left( \frac{1}{\rho^{n+3-4\alpha}} \int_{Q(\rho)} |u|^3 + |h|^3 \right)^{2/3}.
\]

**Proof.** Replace the cut-off function $\phi$ in (2.20) with $\psi$ chosen such that $\psi(y) = 1$ on $B(x_0, 3/4\rho)$, $\psi(y) = 0$ on $B^c(x_0, \rho)$ and $\rho^3 |\nabla^k \psi| \leq C (k = 1, 2)$. Just as (2.20), we still utilize the notations $P_1$, $P_2$ and $P_3$ below. First, we may write

\[
P_1 = -\partial_i \partial_j K * (\psi[U_{i,j} - H_{i,j}])
\]

\[
= -\int_{|y - x_0| < 2\mu} \partial_i_j[K(|x - y|)]\psi[U_{i,j} - H_{i,j}]
\]

\[
+ \int_{|y - x_0| \geq 2\mu} \partial_i_j[K(|x - y|)]\psi(U_{i,j} - H_{i,j})
\]

\[
=: P_{11} + P_{12}.
\]

Due to the classical Calderón–Zygmund theorem, for any $\mu \leq 1/2\rho$, we see that

\[
\|P_{11}\|_{L^{3/2}(\hat{B}(\mu))} \leq C(\|u\|^2_{L^3(\hat{B}(2\mu))} + \|h\|^2_{L^3(\hat{B}(2\mu))}),
\]

which together with the Hölder inequality implies that

\[
\int_{\hat{B}(\mu)} |n||P_{11} - P_{11}| \leq \int_{\hat{B}(\mu)} |n||P_{11}| + \int_{\hat{B}(\mu)} |n||P_{11}|
\]

\[
\leq \|u\|_{L^3(\hat{B}(\mu))}(\|P_{11}\|_{L^{3/2}(\hat{B}(\mu))} + C\|P_{11}\|_{L^{3/2}(\hat{B}(\mu))})
\]

\[
\leq C\|u\|_{L^3(\hat{B}(\mu))}(\|u\|^2_{L^3(\hat{B}(2\mu))} + \|h\|^2_{L^3(\hat{B}(2\mu))}).
\]

By a straightforward computation, for any $|x - x_0| \leq \mu$, we have

\[
|\nabla P_{12}(x)| \leq C \int_{2\mu < |y - x_0| < \rho} \frac{|u|^2 + |h|^2}{|y - x_0|^{n+1}} dy,
\]

\[
|\nabla P_2(x)| \leq C \rho^{-(n+1)} \int_{\hat{B}(\rho)} |u|^2 + |h|^2
\]

\[
\leq C \rho^{-(2n+3)/3} \left( \int_{\hat{B}(\rho)} |u|^3 + |h|^3 \right)^{2/3},
\]

\[
|\nabla P_3(x)| \leq C \rho^{-(n+1)} \int_{\hat{B}(\rho)} |p| \leq C \rho^{-(2n+3)/3} \left( \int_{\hat{B}(\rho)} |p|^3/2 \right)^{2/3}.
\]
Partial regularity of suitable weak solutions to the generalized MHD equations

Using the Hölder inequality and the mean value theorem, we observe that
\[
\int_{B(\mu)} |u| |P_{12} - \overline{P_{12}(\mu)}| \leq C \mu^{2n/3} \left( \int_{B(\mu)} |u|^3 \right)^{1/3} \sup_{x \in B(\mu)} |P_{12} - \overline{P_{12}(\mu)}| \\
\leq C \mu^{(2n+3)/3} \left( \int_{B(\mu)} |u|^3 \right)^{1/3} \sup_{x \in B(\mu)} |\nabla P_{12}| \\
\leq C \mu^{(2n+3)/3} \left( \int_{B(\mu)} |u|^3 \right)^{1/3} \int_{2 \mu < |y - x_0| < \rho} \frac{|u|^2 + |h|^2}{|y - x_0|^n} dy.
\]

By similar arguments, we can get
\[
\int_{B(\mu)} |u| |P_2 - \overline{P_2(\mu)}| \leq C \left( \frac{\mu}{\rho} \right)^{(2n+3)/3} \left( \int_{B(\mu)} |u|^3 \right)^{1/3} \left( \int_{B(\mu)} |u|^3 + |h|^3 \right)^{2/3},
\]
\[
\int_{B(\mu)} |u| |P_3 - \overline{P_3(\mu)}| \leq C \left( \frac{\mu}{\rho} \right)^{(2n+3)/3} \left( \int_{B(\mu)} |u|^3 \right)^{1/3} \left( \int_{B(\mu)} |p|^{3/2} \right)^{2/3}.
\]

Since \( p = P_{11} + P_{12} + P_2 + P_3 \), putting together with the above estimates and integrating in time, we obtain the desired estimate. \( \square \)

3. Induction Arguments

Based on the induction arguments developed in [2,25,26], this section contains the proof of Theorem 1.1. Before proving Theorem 1.1, we present a key proposition, which can be seen as the bridge between the previous step and the next step for the given statement in the induction arguments. Moreover, as a by-product of this proposition, we obtain a corollary, which help us to circumvent the straightforward control of the terms involving the magnetic field \( h \) and the extensions \( u^* \), \( h^* \) in the local energy inequality (2.13) to conclude the proof of Theorems 1.3 and 1.4.

**Proposition 3.1.** There is a constant \( C \) such that the following result holds. For any given \( (x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}^- \) and \( k_0 \in \mathbb{N} \), we have for any \( k > k_0 \),
\[
\sup_{-r_k^2 \leq t-t_0 \leq 0} \int_{B_{k_0}} \left( |u|^2 + |h|^2 \right) + r_k^{-n} \int_{Q_k} y^{1-2\alpha} (|\nabla^* u^*|^2 + |\nabla^* h^*|^2) \\
\leq C \sup_{-r_{k_0}^2 \leq t-t_0 \leq 0} \int_{B_{k_0}} \left( |u|^2 + |h|^2 \right) + C r_{k_0}^{-n} \int_{Q_{k_0}} y^{1-2\alpha} (|\nabla^* u^*|^2 + |\nabla^* h^*|^2)
\]
\[
+ C \sum_{l=k_0}^k r_l^{2n-1} \int_{Q_l} \left( |u|^3 + |h|^3 + |u||p - \bar{p}_l| \right) + C \sum_{l=k_0}^k r_l^{\gamma} \left( \int_{Q_l} |u|^3 \right)^{1/3} \|f\|,
\]

(3.1)
Proof. Without loss of generality, we suppose that \((x_0, t_0) = (0, 0)\). Let \(\Gamma(x, r_k^{2\alpha} - t)\) be the fundamental solution of the backward fractional heat equation, that is,

\[
\Gamma_t - (-\Delta)^\alpha \Gamma = \Gamma_t + C_\alpha \lim_{y \to 0^+} y^{1-2\alpha} \partial_y \Gamma^* = 0,
\]

where \(\Gamma^* = \Gamma^*(x, y, r_k^{2\alpha} - t)\) is the extension of \(\Gamma(x, r_k^{2\alpha} - t)\) in the sense of (2.2). Then, by means of the Poisson formula (2.3), we have

\[
\Gamma^* = \int_{\mathbb{R}^n} \frac{y^{2\alpha} \Gamma(\xi, r_k^{2\alpha} - t)}{|x - \xi|^2 + y^2} d\xi.
\]

To proceed further, we list some properties of the test function \(\Gamma(r_k^{2\alpha} - t, x)\) (whose deduction can be found in [7]):

\[
\int_{\mathbb{R}^n} \Gamma(x, r_k^{2\alpha} - t) dx = 1,
\]

\[
\frac{C^{-1}(r_k^{2\alpha} - t)}{(r_k^{2\alpha} - t)^{n+2\alpha} + |x|^{n+2\alpha}) \leq \Gamma(x, r_k^{2\alpha} - t) \leq \frac{C(r_k^{2\alpha} - t)}{(r_k^{2\alpha} - t)^{n+2\alpha} + |x|^{n+2\alpha}},
\]

\[
|\nabla \Gamma(x, r_k^{2\alpha} - t)| \leq \frac{C(r_k^{2\alpha} - t)^{1-\frac{2\alpha}{n}}}{(r_k^{2\alpha} - t)^{n+2\alpha} + |x|^{n+2\alpha}},
\]

\[
|\Gamma(x_1, r_k^{2\alpha} - t) - \Gamma(x_2, r_k^{2\alpha} - t)| \leq \frac{C|x_1 - x_2|^{\delta} (r_k^{2\alpha} - t)^{1-\frac{2\alpha}{n}}}{(r_k^{2\alpha} - t)^{n+2\alpha} + (|x_1| \wedge |x_2|)^{n+2\alpha}},
\]

where \(\delta \in (0, 2\alpha \wedge 1)\).

Consider the smooth cut-off functions below

\[
\phi_1(x, t) = \begin{cases} 
1, & (x, t) \in Q(r_{k_0+1}), \\
0, & (x, t) \in Q^c \left( \frac{3}{2} r_{k_0+1} \right).
\end{cases}
\]

and

\[
\phi_2(y) = \begin{cases} 
1, & 0 \leq y \leq r_{k_0+1}, \\
0, & y > \frac{3}{2} r_{k_0+1};
\end{cases}
\]

satisfying

\[
0 \leq \phi_1, \quad \phi_2 \leq 1, \quad r_{k_0}^{2\alpha} |\partial_t \phi_1(x, t)| + r_{k_0} |\partial_x \phi_1(x, t)| \leq C \quad \text{and} \quad r_{k_0}^l |\partial_y \phi_2(y)| \leq C.
\]
Notice that $\lim_{y \to 0^-} \Gamma^* \varphi_2(y) = \Gamma$, therefore, setting $\varphi_1 = \phi_1 \Gamma$ and $\varphi_2 = \phi_1 \phi_2 \Gamma^*$ in the local energy inequality (2.13), we see that

$$\int_{\mathbb{R}^n} (|u|^2 + |h|^2) \phi_1(x, t) \Gamma + 2C \int_{-r_k^2}^t \int_{\mathbb{R}^{n+1}} \phi_1 \phi_2 \Gamma^* y^{1-2\alpha} (|\nabla^* u|^2 + |\nabla^* h|^2)$$

$$\leq C \int_{-r_k^2}^t \int_{\mathbb{R}^{n+1}} (|u|^2 + |h|^2) \text{Div} (y^{1-2\alpha} \nabla^* (\phi_1 \phi_2 \Gamma^*))$$

$$+ \int_{-r_k^2}^t \int_{\mathbb{R}^n} (|u|^2 + |h|^2) \left[ (\phi_1 \Gamma)_t + C \lim_{y \to 0^+} y^{1-2\alpha} \partial_y (\phi_1 \phi_2 \Gamma^*) \right]$$

$$+ \int_{-r_k^2}^t \int_{\mathbb{R}^n} u \cdot \nabla (\phi_1 \Gamma) (|u|^2 + |h|^2 + 2p) - 2 \int_{-r_k^2}^t \int_{\mathbb{R}^n} h \cdot \nabla (\phi_1 \Gamma) (u \cdot h)$$

$$+ 2 \int_{-r_k^2}^t \int_{\mathbb{R}^n} u \cdot f (\phi_1 \Gamma).$$

(3.8)

First, we present the low bound estimates of the terms on the left-hand side of this inequality. Indeed, with the help of (3.5), we find

$$\Gamma(x, r_k^{2\alpha} - t) \geq \frac{C^{-1} (r_k^{2\alpha} - t)}{((r_k^{2\alpha} - t) + |x|)^{n+2\alpha}} \geq Cr_k^{-n}, \quad -r_k^{2\alpha} \leq t \leq 0, \quad x \in \bar{B}_k,$$

which means that

$$\int_{\bar{B}_k} (|u|^2 + |h|^2) \phi_1 \Gamma \geq C \int_{\bar{B}_k} (|u|^2 + |h|^2).$$

(3.9)

For each $y \in [0, r_k]$, $x \in \bar{B}_k$, the triangle inequality allows us to get $|\xi| \leq 2r_k$ under the hypothesis $|x - \xi| < |y|$, then, arguing in the same manner as (3.9), we know that $\Gamma (\xi, r_k^{2\alpha} - t) \geq Cr_k^{-n}$ if $|\xi| \leq 2r_k$. According to the Poisson formula (3.3), we see that

$$\Gamma^* \geq \int_{|x - \xi| < |y|} \gamma_{2\alpha} \gamma (\xi, r_k^{2\alpha} - t) \frac{d \xi}{|x - \xi|^2 + y^2} \geq C r_k^{-n},$$

which in turns implies

$$\int_{-r_k^{2\alpha}}^t \int_{\mathbb{R}^{n+1}} \phi_1 \phi_2 \Gamma^* y^{1-2\alpha} (|\nabla^* u|^2 + |\nabla^* h|^2)$$

$$\geq r_k^{-n} \int_{Q_k^r} y^{1-2\alpha} (|\nabla^* u|^2 + |\nabla^* h|^2).$$

Secondly, we turn our attentions to the right-hand side of (3.8). Since $\Gamma^*$ is the extension of $\Gamma$ in the sense of (2.2), namely, $\text{Div} (y^{1-2\alpha} \nabla^* \Gamma^*) = 0$, we write

$$\text{Div} (y^{1-2\alpha} \nabla^* (\phi_1 \phi_2 \Gamma^*))$$

$$= (1 - 2\alpha) y^{-2\alpha} \phi_1 \phi_2 \phi_2 + y^{1-2\alpha} \Gamma^* \text{Div} (\nabla^* (\phi_1 \phi_2)) + 2y^{1-2\alpha} \nabla^* \Gamma^* \nabla^* (\phi_1 \phi_2)$$

1650018-18
\[\begin{align*}
\Delta \phi_1 & \leq \left|\frac{y^{2\alpha} \Gamma(\xi, r_k^{2\alpha} - t)}{|x - \xi|^2 + y^2} \frac{2^{2\alpha}}{2} \right| d\xi \\
& \leq \sup_{r_k^2 \leq t \leq 0} \left| \frac{\Gamma(\xi, r_k^{2\alpha} - t)}{|x - \xi|^2 + y^2} \right| \leq C r_k^{-n-2}.
\end{align*}\]

Thanks to the support property of \(\partial_y \phi_2\), it is easy to estimate \(I_1, I_{21}, I_{31}\). Using the Poisson formula (3.3) and (3.4), we arrive at

\[\partial_y \phi_2^{*} \leq Cy^{-n-1} \int_{\mathbb{R}^n} \Gamma(\xi, r_k^{2\alpha} - t) d\xi \leq Cy^{-n-1},\]

which together with (2.10) leads to

\[\int_{-r_k^{2\alpha}}^{r_k^{2\alpha}} \int_{R_n^+} |u|^2 (I_1 + I_{21}) \leq Cr_k^{-(n+2)} \int_{Q_k^*} y^{1-2\alpha} |u|^2 \leq C \sup_{-r_k^2 \leq t \leq 0} \int_{B_k} |u|^2 + Cr_k^{-n} \int_{Q_k^*} y^{1-2\alpha} |\nabla \phi^*|^2.\]

Likewise,

\[\int_{-r_k^{2\alpha}}^{r_k^{2\alpha}} \int_{R_n^+} |h|^2 (I_1 + I_{21}) \leq C \sup_{-r_k^2 \leq t \leq 0} \int_{B_k} |h|^2 + Cr_k^{-n} \int_{Q_k^*} y^{1-2\alpha} |\nabla h|^2.\]

It follows from the Poisson formula (3.3) that

\[\partial_y \Gamma^*(x, y, r_k^{2\alpha} - t) = \int_{\mathbb{R}^n} \frac{2\alpha |x - \xi|^2 y^{2\alpha - 1} - ny^{2\alpha + 1}}{|x - \xi|^2 + y^2} \Gamma(\xi, r_k^{2\alpha} - t) d\xi \leq Cy^{-n-1},\]

which implies

\[\int_{-r_k^{2\alpha}}^{r_k^{2\alpha}} \int_{R_n^+} (|u|^2 + |h|^2) I_{31} \leq C \sup_{-r_k^2 \leq t \leq 0} \int_{B_k} (|u|^2 + |h|^2) + r_k^{-n} \int_{Q_k^*} y^{1-2\alpha} (|\nabla u|^2 + |\nabla h|^2),\]

where we have used (2.10) again.

It is clear that

\[\begin{align*}
\Delta \phi_1 & \leq \left|\frac{y^{2\alpha} \Gamma(\xi, r_k^{2\alpha} - t)}{|x - \xi|^2 + y^2} \frac{2^{2\alpha}}{2} \right| d\xi \\
& \leq \sup_{r_k^2 \leq t \leq 0} \left| \frac{\Gamma(\xi, r_k^{2\alpha} - t)}{|x - \xi|^2 + y^2} \right| \leq C r_k^{-n-2}.
\end{align*}\]

Thanks to (3.5), we have

\[\int_{|\xi| > 2^{-2} r_k} \frac{y^{2\alpha} \Gamma(\xi, r_k^{2\alpha} - t)}{|x - \xi|^2 + y^2} \frac{2^{2\alpha}}{2} d\xi \leq C r_k^{-n} \int_{\mathbb{R}^n} \frac{y^{2\alpha} \Gamma(\xi, r_k^{2\alpha} - t)}{|x - \xi|^2 + y^2} \frac{2^{2\alpha}}{2} d\xi \leq C r_k^{-n}.\]
Moreover, some straightforward computations give

\[
\int_{-r_k^0}^{r_k^0} \int_{B_{km}} (|u|^2 + |\dot{u}|^2) I_{22} \leq C \sup_{-r_k^0 \leq t \leq 0} \int_{B_{r_k^0}} (|u|^2 + |\dot{u}|^2)
\]

+ C \int_{Q_{r_k^0}} y^{1-2\alpha} (|\nabla^* u|^2 + |\nabla^* h|^2).

Consequently, by (2.10) again, we find

\[
\int_{-r_k^0}^{r_k^0} \int_{B_{km}} (|u|^2 + |\dot{u}|^2) I_{22} \leq C \sup_{-r_k^0 \leq t \leq 0} \int_{B_{r_k^0}} (|u|^2 + |\dot{u}|^2)
\]

+ C \int_{Q_{r_k^0}} y^{1-2\alpha} (|\nabla^* u|^2 + |\nabla^* h|^2).

Moreover, some straightforward computations give

\[
\nabla \Gamma^*(x, y, r_k^0 - t) = -(n + 2\alpha) \int_{\mathbb{R}^n} \frac{y^{2\alpha}(x - \xi) \Gamma(\xi, r_k^{2\alpha} - t)}{(|x - \xi|^2 + y^2)^{\frac{n+1}{2} + 1}} d\xi
\]

\[
= -(n + 2\alpha) \int_{\mathbb{R}^n} \frac{\int_{\mathbb{R}^n} y^{2\alpha}(x - \xi) \Gamma(\xi, r_k^{2\alpha} - t) - \Gamma(x, r_k^{2\alpha} - t)]}{(|x - \xi|^2 + y^2)^{\frac{n+1}{2} + 1}} d\xi.
\]

This leads to

\[
I_{32} \leq C \phi_2 y \int_{\mathbb{R}^n} \frac{\nabla \phi_1 \Gamma(\xi, r_k^{2\alpha} - t) - \Gamma(x, r_k^{2\alpha} - t)]}{(|x - \xi|^2 + y^2)^{\frac{n+1}{2} + 1}} d\xi.
\]

By the Hölder inequality, for any \( y \leq r_k^0 \), we easily verify that

\[
|u^*(x, y)|^2 = \left| u(x) + \int_0^y \partial_z u^* dz \right|^2
\]

\[
\leq C |u(x)|^2 + Cy^{2\alpha} \int_0^{r_k^0} z^{1-2\alpha} |\nabla^* u|^2 dz,
\]

which in turns implies, abusing notation slightly,

\[
\int_{-r_k^0}^{r_k^0} \int_{\mathbb{R}^n} |u^*|^2 I_{32}
\]

\[
\leq C \int_{Q_{r_k^0}} I_{32} \left( |u(x)|^2 + y^{2\alpha} \int_0^{r_k^0} z^{1-2\alpha} |\nabla^* u|^2 dz \right)
\]

\[
\leq C \int_{Q_{r_k^0}} \left( |u(x)|^2 + y^{2\alpha} \int_0^{r_k^0} z^{1-2\alpha} |\nabla^* u|^2 dz \right)
\]

\[
\times \int_{\mathbb{R}^n} \frac{\nabla \phi_1 \Gamma(\xi, r_k^{2\alpha} - t) - \Gamma(x, r_k^{2\alpha} - t)]}{(|x - \xi|^2 + y^2)^{\frac{n+1}{2} + 1}} d\xi
\]

\[
\leq C \int_{Q_{r_k^0}} |u(x)|^2 \left( |\nabla \phi_1 \Gamma(\xi, r_k^{2\alpha} - t) - \Gamma(x, r_k^{2\alpha} - t)]}{(|x - \xi|^2 + y^2)^{\frac{n+1}{2} + 1}} d\xi
\]

\[
+ C \int_{Q_{r_k^0}} y^{1-2\alpha} \int_{\mathbb{R}^n} \frac{\nabla \phi_1 \Gamma(\xi, r_k^{2\alpha} - t) - \Gamma(x, r_k^{2\alpha} - t)]}{(|x - \xi|^2 + y^2)^{\frac{n+1}{2} + 1}} d\xi,
\]

(3.11)
where we have used the fact

\[
\int_{0}^{r_{k_0}} dy^2 \left( \frac{1}{|x - \xi|^n + 2} \right) \leq C.
\]

For any \(x\) belonging to the support of \(\nabla \phi_1\), it is valid that \(|x - \xi| \geq 1/r_{k_0}\) for either \(|\xi| \leq 1/4r_{k_0}\) or \(|\xi| \geq 2r_{k_0}\). Then, we deduce that

\[
\left( \int_{|\xi| \leq 1/4r_{k_0}} + \int_{|\xi| \geq 2r_{k_0}} \right) \frac{|\Gamma(\xi, r_{k_0}^2 - t) - \Gamma(x, r_{k_0}^2 - t)|}{|x - \xi|^{n+2\alpha}} d\xi
\]

\[
\leq C r_{k_0}^{-(n+2\alpha)} \int_{|\xi| \leq 1/4r_{k_0}} \Gamma(\xi, r_{k_0}^2 - t) d\xi + C r_{k_0}^{-(n+2\alpha)}
\]

\[
+ C r_{k_0}^{-(n+2\alpha)} \int_{|\xi| \geq 2r_{k_0}} \Gamma(\xi, r_{k_0}^2 - t) d\xi
\]

\[
+ C r_{k_0}^{-n} \int_{|x - \xi| \geq 1/4r_{k_0}} \frac{1}{|x - \xi|^{n+2\alpha}} d\xi
\]

\[
\leq C r_{k_0}^{-(n+2\alpha)},
\]

where we have used (3.4) and (3.5) with \(3r_{k_0}/4 \geq |x| \geq r_{k_0}/2\). In light of (3.7), for \(\delta \in (2\alpha - 1, 1)\), we see that

\[
\int_{1/4r_{k_0} \leq |\xi| \leq 2r_{k_0}} \frac{|\Gamma(\xi, r_{k_0}^2 - t) - \Gamma(x, r_{k_0}^2 - t)|}{|x - \xi|^{n+2\alpha}} d\xi
\]

\[
\leq C \int_{1/4r_{k_0} \leq |\xi| \leq 2r_{k_0}} \frac{(r_{k_0}^2 - t)^{1-\frac{n}{2}} |(r_{k_0}^2 - t) \frac{1}{2} + (|x| \wedge |\xi|))^{n-2\alpha}}{|x - \xi|^{n+2\alpha - \delta}} d\xi
\]

\[
\leq C r_{k_0}^{-n - \delta} \int_{|x - \xi| < 4r_{k_0}} \frac{1}{|x - \xi|^{n+2\alpha - \delta}} d\xi
\]

\[
\leq C r_{k_0}^{-(n+2\alpha)}.
\]

Substituting the above estimates into (3.11), we arrive at

\[
\int_{-r_{k_0}^2}^{t} \int_{B_{r_{k_0}^2}^+} |u^*|^2 I_{32} \leq C \sup_{-r_{k_0}^2 \leq t \leq 0} \int_{B_{r_{k_0}^2}} |u|^2 + r_{k_0}^{-n} \iint_{Q_{r_{k_0}^2}} y^{1-2\alpha} \nabla^* u^*|^2.
\]

Likewise, we have

\[
\int_{-r_{k_0}^2}^{t} \int_{B_{r_{k_0}^2}^+} |h^*|^2 I_{32} \leq C \sup_{-r_{k_0}^2 \leq t \leq 0} \int_{B_{r_{k_0}^2}} |h|^2 + r_{k_0}^{-n} \iint_{Q_{r_{k_0}^2}} y^{1-2\alpha} \nabla^* h^*|^2.
\]

We deduce from (3.2) that

\[
(\phi_1 \Gamma)_t + \lim_{y \to 0^+} y^{1-2\alpha} \partial_y (\phi_1 \phi_2 \Gamma^*) = \partial_t \phi_1 \Gamma.
\]
Partial regularity of suitable weak solutions to the generalized MHD equations

As the support of $\partial_y \phi_1$ is included in $\tilde{Q}(\frac{3r_k}{2k})/\tilde{Q}(\frac{r_k}{2})$, from (3.5), we get

$$\int_{-r_k^{2a}}^{t} \int_{\mathbb{R}^n} (|u|^2 + |h|^2) \left[ (\phi_1 \Gamma)_t + \lim_{y \to 0^+} y^{1-2a} \partial_y (\phi_1 \phi_2 \Gamma^*) \right]$$

$$\leq C \sup_{-r_k^{2a} \leq t \leq 0} \int_{B_k} (|u|^2 + |h|^2).$$

It follows from (3.5) and (3.6) that

$$\Gamma \leq C r_{l+1}^{-n}, \quad \nabla \Gamma \leq C r_{l+1}^{-(n+1)} \text{ on } Q_{l+1},$$

$$\Gamma \leq C r_{k}^{-n}, \quad \nabla \Gamma \leq C r_{k}^{-(n+1)} \text{ on } Q_k,$$

which yields

$$\int_{Q_{l+1}} u \cdot \nabla (\phi_1 \Gamma) (|u|^2 + |h|^2)$$

$$\leq \sum_{l=k_0}^{k-1} \int_{Q_l/Q_{l+1}} (|u|^3 + |h|^3) |\nabla (\phi_1 \Gamma)| + \int_{Q_k} (|u|^3 + |h|^3) |\nabla (\phi_1 \Gamma)|$$

$$\leq C \sum_{l=k_0}^{k} r_l^{2a-1} \int_{Q_l} (|u|^3 + |h|^3).$$

Exactly as in the above derivation, we derive from the Young inequality that

$$\int_{-r_k^{2a}}^{t} \int_{\mathbb{R}^n} h \cdot \nabla (\phi_1 \Gamma) (u \cdot h) \leq C \sum_{l=k_0}^{k} r_l^{2a-1} \int_{Q_l} (|u|^3 + |h|^3).$$

For any $1/2 < \alpha < 3/4$, using the Hölder inequality twice, we get

$$\int_{-r_k^{2a}}^{t} \int_{\mathbb{R}^n} f \phi_1 \Gamma u \leq C \sum_{l=k_0}^{k} r_l^{2a-1} \left( \int_{Q_l} |u|^3 \right)^{1/3} \left( \int_{Q_l} |f|^{3/2} \right)^{2/3}$$

$$\leq C \sum_{l=k_0}^{k} r_l^{2a-\frac{3-\alpha}{4}} \left( \int_{Q_l} |u|^3 \right)^{1/3} \left( \int_{Q_l} |f|^{9} \right)^{1/9}.$$
W. Ren, Y. Wang & G. Wu

of (3.5) and (3.6) again, we see that \(|\nabla(\chi_k \phi_1 \Gamma)| \leq C r_k^{-(n+1)}\). Thus, thanks to the divergence free condition, we have

\[
\int_{Q_k} u \cdot \nabla(\phi_1 \Gamma) p = \sum_{l=k_0}^{k-1} \int_{Q_l} u \cdot \nabla((\chi_l - \chi_{l+1}) \phi_1 \Gamma)(p - \bar{p}_l) + \int_{Q_k} u \cdot \nabla(\chi_k \phi_1 \Gamma)(p - \bar{p}_k) 
\leq C \sum_{l=k_0}^{k} r_l^{2\alpha - 1} \int_{Q_l} |u| |p - \bar{p}_l|.
\]

Finally, these collected estimates lead to (3.1).\)[122x646]

A slight variant of the above proof provides the following corollary, which allows us to complete the proof of Theorems 1.3 and 1.4.

**Corollary 3.1.** For any \(\mu \leq 1/8\rho\), there exists a constant \(C\) such that the following result is valid, under the condition \(f = 0\),

\[
E(\mu) + E_*(\mu) \leq C \left( \frac{\mu}{\rho} \right)^{4\alpha - 2} \left( E \left( \frac{\rho}{4} \right) + E_* \left( \frac{\rho}{4} \right) \right)
+ C \left( \frac{\rho}{\mu} \right)^{n+3-4\alpha} \rho^{-(n+3-4\alpha)} \int_{Q(\rho/4)} |u|^3
+ |u||h|^2 + |u||p - p_{\rho/4}|. \tag{3.12}
\]

Now, we are in a position to prove Theorem 1.1.

**Proof of Theorem 1.1.** In what follows, let \((x_0, t_0) \in Q(1/8)\) and \(r_k = 2^{-k}\). According to the Lebesgue differentiation theorem, it suffices to show

\[
\int_{Q_k} |u|^3 + |h|^3 + \min\{1, r_k^{4\alpha - 3}\} \int_{Q_k} |u||p - \bar{p}_k| \leq \varepsilon_1^{2/3}, \quad k \geq 3. \tag{3.13}
\]

First, we show that (3.13) is valid for \(k = 3\). In fact, by means of interpolation inequality, the Young inequality and (2.12), we infer that

\[
\left( \int_{Q(2/3)} |u|^{2\alpha + \frac{4\alpha}{n}} \right)^{\frac{n}{2\alpha + 4\alpha}} \leq C \left( \sup_{-(\frac{2}{3})^{2\alpha} \leq t < 0} \int_{B(1)} |u|^{2\alpha} \right)^{\frac{n}{4\alpha + 2\alpha}} \left( \int_{-(\frac{2}{3})^{2\alpha}}^{0} \int_{B(2/3)} |u|^{\frac{2\alpha}{n - 2\alpha}} \right)^{\frac{n - 2\alpha}{n + 2\alpha}} \int_{-(\frac{2}{3})^{2\alpha}}^{0} \int_{B(2/3)} |u|^{\frac{2\alpha}{n - 2\alpha}} \right)^{\frac{n - 2\alpha}{n + 2\alpha}}.
\]

1650018-24
Partial regularity of suitable weak solutions to the generalized MHD equations

\[ \leq C\left( \sup_{-1 \leq t < 0} \int_{B(2/3)} |u|^2 \right)^{1/2} + C\left( \int_{-\left(\frac{4}{3}\right)^{2\alpha}}^0 \left( \int_{B(2/3)} |u|^{\frac{2\alpha}{n}} \right)^{\frac{n-2\alpha}{n}} \right)^{1/2} \]

\[ \leq C\left( \sup_{-1 \leq t < 0} \int_{B(1)} |u|^2 \right)^{1/2} + C\left( \int_{Q(1)} y^{1-2\alpha} |\nabla^* u|^2 \right)^{1/2}. \quad (3.14) \]

It turns out that

\[ \iint_{Q_3} |u|^3 + |h|^3 \leq C\varepsilon_1^{3/2}. \]

Applying Lemma 2.3 with \( \mu = 1/8 \) and \( \rho = 1/2 \), we observe that

\[ \iint_{Q_3} |u| |p - \bar{p}_3| \]

\[ \leq C \left( \iint_{Q_3} |u|^3 \right)^{1/3} \left( \iint_{Q_2} |u|^3 + |h|^3 \right)^{2/3} + C \left( \iint_{Q_3} |u|^3 \right)^{1/3} \sup_{-\left(\frac{4}{3}\right)^{2\alpha} \leq t - t_0 < 0} \int_{1/4 |y - x_0| < 1/2} \frac{|u|^2 + |h|^2}{|y - x_0|^{n+1}} \, dy \]

\[ + C \left( \iint_{Q_3} |u|^3 \right)^{1/3} \left( \iint_{Q(1/2)} |u|^3 + |h|^3 \right)^{2/3} + C \left( \iint_{Q_3} |u|^3 \right)^{1/3} \left( \iint_{Q(1/2)} |p|^{3/2} \right)^{2/3} \]

\[ \leq C \left( \iint_{Q(2/3)} |u|^3 \right)^{1/3} \left( \iint_{Q(2/3)} |u|^3 + |h|^3 \right)^{2/3} + C \left( \iint_{Q(2/3)} |u|^3 \right)^{1/3} \sup_{-\left(2/3\right)^{2\alpha} \leq t - t_0 < 0} \int_{|y| \leq 1} (|u|^2 + |h|^2) \, dy \]

\[ + C \left( \iint_{Q(2/3)} |u|^3 \right)^{1/3} \left( \iint_{Q(2/3)} |u|^3 + |h|^3 \right)^{2/3} + C \left( \iint_{Q(2/3)} |u|^3 \right)^{1/3} \left( \iint_{Q(2/3)} |p|^{3/2} \right)^{2/3}, \]

which means

\[ \iint_{Q_3} (|u|^3 + |h|^3) + \min\{1, r_3^{\frac{2\alpha}{n}}\} \iint_{Q_3} |u||p - \bar{p}_3| \leq \varepsilon_1^{2/3}. \]

1650018-25
This proves (3.13) in the case $k = 3$. Now, we assume that, for any $3 \leq l \leq k$,
\[
\int_{Q_l} |u|^3 + |h|^3 + \min\{1, r_l^{4\alpha-3}\} \int_{Q_l} |u|p - \tilde{p}x | \leq \varepsilon_1^{2/3}.
\]
For any $3 \leq i \leq k$, by Proposition 3.1, (1.2) and the above induction hypothesis, we find that
\[
\sup_{-r_i^{2\alpha-1} \leq t \leq 0} \int_{B_i} (|u|^2 + |h|^2) + r_i^{-n} \int_{Q_i} y^{1-2\alpha}(|\nabla^* u|^2 + |\nabla^* h|^2)
\leq C \sup_{-r_i^{2\alpha-1} \leq t \leq 0} \int_{B_i} (|u|^2 + |h|^2) + r_i^{-n} \int_{Q_i} y^{1-2\alpha}(|\nabla^* u|^2 + |\nabla^* h|^2)
+ C \sum_{l=3}^{i} r_i^{2\alpha-1} \int_{Q_i} (|u|^2 + |h|^2 + |u|p - \tilde{p}x) + C \sum_{l=3}^{i} r_i^{\gamma'} \left( \int_{Q_l} |u|^3 \right)^{1/3} \|f\|
\leq C \varepsilon_1 + C \sum_{l=3}^{i} \max\{r_l^{2\alpha-1}, r_l^{2\alpha}\} \varepsilon_1^{2/3} + C \sum_{l=3}^{i} r_i^{\gamma'} \varepsilon_1^{11/9}
\leq C \varepsilon_1^{2/3}. \quad (3.15)
\]
Making use of the Hölder inequality and (2.7), we obtain
\[
\int_{B_{k+1}} |u|^3 dx \leq C \left( \int_{B_{k+1}} |u|^2 dx \right)^{\frac{6\alpha-n}{4\alpha}} \left( \int_{B_{k+1}} |u|^{\frac{6\alpha-n}{2\alpha}} dx \right)^{\frac{(4\alpha-2\alpha)}{4\alpha}}
\leq C \left( \int_{B_k} |u|^2 \right)^{\frac{6\alpha-n}{4\alpha}} \left[ \left( \int_{B_k} y^{1-2\alpha}(|\nabla^* u|^2) \right)^{1/2} + r_k^{-\alpha} \left( \int_{B_k} |u|^2 \right)^{1/2} \right]^\frac{1}{\frac{4\alpha-2\alpha}{4\alpha}}
\]
which yields
\[
\int_{Q_{k+1}} |u|^3 \leq C \left( \sup_{-r_k^{2\alpha} \leq t \leq 0} \int_{B_k} |u|^2 \right)^{\frac{6\alpha-n}{4\alpha}} r_k^{2\alpha(1-\frac{n}{4\alpha})} \left( \int_{Q_k} y^{1-2\alpha}(|\nabla^* u|^2) \right)^{\frac{1}{\frac{4\alpha-2\alpha}{4\alpha}}}
+ r_k^{-\alpha+2\alpha} \left( \sup_{-r_k^{2\alpha} \leq t \leq 0} \int_{B_k} |u|^2 \right)^{3/2}.
\]
This inequality, combined with (3.15), implies that
\[
\frac{1}{r_{k+1}^{n+2\alpha}} \int_{Q_{k+1}} |u|^3
\leq C \left( \frac{1}{r_k^{2\alpha}} \sup_{-r_k^{2\alpha} \leq t \leq 0} \int_{B_k} |u|^2 \right)^{3/2} + C \left( \frac{1}{r_k^{2\alpha}} \sup_{-r_k^{2\alpha} \leq t \leq 0} \int_{B_k} |u|^2 \right)^{\frac{6\alpha-n}{4\alpha}}
\times \left( r_k^{-n} \int_{Q_k} y^{1-2\alpha}(|\nabla^* u|^2) \right)^{\frac{1}{\frac{4\alpha-2\alpha}{4\alpha}}} \leq C \varepsilon_1. \quad (3.16)
\]
Arguing in the same manner as above, we see that
\[
\frac{1}{r_{k+1}^{n+2\alpha}} \int_{\tilde{Q}_{k+1}} |h|^3 \leq C\varepsilon_1.
\] (3.17)

Employing Lemma 2.3 with \(\mu = r_{k+1}\) and \(\rho = 1/2\), we arrive at
\[
\frac{1}{r_{k+1}^{n+2\alpha}} \int_{\tilde{Q}_{k+1}} |u||p - \tilde{p}_{k+1}|
\leq C \left( \frac{1}{r_{k+1}^{n+3-4\alpha}} \int_{\tilde{Q}_{k+1}} |u|^3 \right)^{1/3} \left( \frac{1}{r_{k+1}^{n+3-4\alpha}} \int_{\tilde{Q}_{k}} |u|^3 + |h|^3 \right)^{2/3}
+ Cr_{k+1}^{2\alpha-1} \left( \frac{1}{r_{k+1}^{n+3-4\alpha}} \int_{\tilde{Q}_{k+1}} |u|^3 \right)^{1/3} \left( \int_{\tilde{Q}_{k(1/2)}} |u|^3 + |h|^3 \right)^{2/3}
\times \sup_{-r_{k+1}^2 \leq t_0 < 0} \int_{|y - x_0| < \frac{1}{4}} |u|^2 + |h|^2 dy
+ C(r_{k+1})^{\frac{n+1}{2} - \alpha} \left( \frac{1}{r_{k+1}^{n+3-4\alpha}} \int_{\tilde{Q}_{k+1}} |u|^3 \right)^{1/3} \left( \int_{\tilde{Q}_{(1/2)}} |u|^3 + |h|^3 \right)^{2/3}
+ C(r_{k+1})^{\frac{n+1}{2} - \alpha} \left( \frac{1}{r_{k+1}^{n+3-4\alpha}} \int_{\tilde{Q}_{k+1}} |u|^3 \right)^{1/3} \left( \int_{\tilde{Q}_{(1/2)^{2/3}}} |p|^3 \right)^{2/3}.
\]

A simple computation together with (3.15) yields
\[
\int_{|y - x_0| < \frac{1}{4}} |u|^2 + |h|^2 dy = \sum_{l=1}^{k-1} \int_{r_{l+1}^2 < |y - x_0| < r_l} \frac{|u|^2 + |h|^2}{|y - x_0|^{n+1}} dy
\leq C \sum_{l=1}^{k} r_l^{-1} \varepsilon_1^{2/3},
\]
which implies
\[
\frac{1}{r_{k+1}^{n+2\alpha}} \int_{\tilde{Q}_{k+1}} |u||p - \tilde{p}_{k+1}|
\leq C \left( \frac{1}{r_{k+1}^{n+2\alpha}} \int_{\tilde{Q}_{k+1}} |u|^3 \right)^{1/3} \left( \frac{1}{r_{k+1}^{n+2\alpha}} \int_{\tilde{Q}_{k}} |u|^3 + |h|^3 \right)^{2/3}
+ Cr_{k+1}^{2\alpha-1} \left( \frac{1}{r_{k+1}^{n+2\alpha}} \int_{\tilde{Q}_{k+1}} |u|^3 \right)^{1/3} \sum_{l=1}^{k} r_l^{-1}
+ C \varepsilon_1^{4/3} \left( \frac{1}{r_{k+1}^{n+2\alpha}} \int_{\tilde{Q}_{k+1}} |u|^3 \right)^{1/3} \left( \int_{\tilde{Q}_{(2/3)^{2/3}}} |u|^3 + |h|^3 \right)^{2/3}.
\]

1650018-27
Choosing and where we have used (3.16), (3.17), (3.14) and (1.2).

Collecting the above bounds, we eventually conclude that

$$\sup_{Q_{k+1}} (|u|^3 + |\bar{h}|^3) + \min\{1, r_{k+1}^{-\frac{4n-3}{2}}, \max\{1, r_{k+1}^{\frac{2}{3}}\} \} \sup_{Q_{k+1}} |\bar{p} - \bar{p}_{k+1}| \leq \varepsilon_1^{2/3}. $$

This completes the proof of this theorem.

4. Proofs of Theorems 1.2–1.4

This section is devoted to proving Theorems 1.2–1.4. More precisely, based on Theorem 1.1, we will exploit appropriate iteration scheme via local energy inequality (2.13) with respect to different smallness conditions in these theorems. Our fundamental tools are the decay-type lemmas established in Sec. 2 and Corollary 3.1 in Sec. 3. Eventually, an iteration argument helps us to finish the proof.

Proof of Theorem 1.2. Consider the smooth cut-off functions below

$$\phi_1(x, t) = \begin{cases} 1, & (x, t) \in Q(\mu), \\ 0, & (x, t) \in Q^c(2\mu); \end{cases}$$

and

$$\phi_2(y) = \begin{cases} 1, & 0 \leq y \leq \mu, \\ 0, & y > 2\mu; \end{cases}$$

satisfying

$$0 \leq \phi_1, \phi_2 \leq 1, \mu^{2\alpha} |\partial_x \phi_1(x, t)| + \mu^\alpha |\partial^\alpha_x \phi_1(x, t)| \leq C \quad \text{and} \quad \mu^\alpha |\partial_y \phi_2(y)| \leq C.$$

Choosing $\varphi_1 = \phi_1$ and $\varphi_2 = \phi_1 \phi_2$ in the local energy inequality (2.13) and utilizing the incompressible condition, we deduce from (2.10) that

$$\sup_{-\mu^{2\alpha} \leq t < 0} \int_{B(\mu)} |u|^2 + |h|^2 + 2C \int_{Q^*(\mu)} y^{1-2\alpha} (|\nabla^* u|^2 + |\nabla^* h|^2)$$

$$\leq C\mu^{-2\alpha} \int_{Q(2\mu)} (|u|^2 + |h|^2) + C \int_{Q^*(2\mu)} y^{1-2\alpha} (|\nabla^* u|^2 + |\nabla^* h|^2)$$

$$+ C\mu^{-1} \int_{Q(2\mu)} (|u||u|^2 - |\bar{u}_\mu|^2) + 2|u||p - \bar{p}|)$$

$$+ C\mu^{-1} \int_{Q(2\mu)} |h||u \cdot h - \bar{u}_{\mu} \cdot \bar{h}_{\mu}| + 2 \int_{Q(2\mu)} u f. \quad (4.1)$$
According to the Hölder inequality, (2.6) and (2.7), for any $\mu \leq \rho/4$, we know that

$$
\int_{B(2\mu)} |u||u|^2 - |\bar{u}_\rho|^2 \\
\leq C \mu^{(2n-1)/2} \left( \int_{B(2\mu)} |u|^{2n/(n-2\alpha)} \right)^{\alpha/(2\alpha)} \left( \int_{B(2\mu)} |u + \bar{u}_\rho|^2 \right)^{1/2} \\
\times \left( \int_{B(2\mu)} |u - \bar{u}_\rho|^{2n/(n-2\alpha)} \right)^{\alpha/(2\alpha)} \\
\leq C \mu^{(4n-1)/2} \left( \int_{B(\rho/2)} |u|^{2n/(n-2\alpha)} \right)^{\alpha/(2\alpha)} \\
\times \left( \int_{B(\rho/2)} |u + \bar{u}_\rho|^2 \right)^{1/2} \left( \int_{B(\rho/2)} |u - \bar{u}_\rho|^{2n/(n-2\alpha)} \right)^{\alpha/(2\alpha)} \\
\leq C \mu^{(4n-1)/2} \left( \int_{B^{*}(\rho)} y^{1-2\alpha}|\nabla^* u^*|^2 \right)^{1/2} + \rho^{-\alpha} \left( \int_{B(\rho)} |u|^2 \right)^{1/2} \\
\times \left( \int_{B(\rho)} |u|^2 \right)^{1/2} \left( \int_{B^{*}(\rho)} y^{1-2\alpha}|\nabla^* u^*|^2 \right)^{1/2} \\
\leq C \mu^{(4n-1)/2} \left( \int_{B^{*}(\rho)} y^{1-2\alpha}|\nabla^* u^*|^2 \right)^{1/2} \left( \int_{B(\rho)} |u|^2 \right)^{1/2} \\
+ C \rho^{-\alpha} \mu^{(4n-1)/2} \left( \int_{B(\rho)} |u|^2 \right)^{1/2} \left( \int_{B^{*}(\rho)} y^{1-2\alpha}|\nabla^* u^*|^2 \right)^{1/2}.
$$

From the triangle inequality, arguing as above, we infer that

$$
\int_{B(2\mu)} |h|u \cdot h - \bar{h}_\rho \cdot \bar{h}_\rho| \\
\leq \int_{B(2\mu)} |hu||h - \bar{h}_\rho| + |h||u - \bar{u}_\rho|\bar{h}_\rho| \\
\leq C \mu^{(4n-1)/2} \left( \int_{B(\rho/2)} |h|^{2n/(n-2\alpha)} \right)^{\alpha/(2\alpha)} \left( \int_{B(\rho/2)} |u|^2 \right)^{1/2} \\
\times \left( \int_{B(\rho/2)} |h - \bar{u}_\rho|^{2n/(n-2\alpha)} \right)^{\alpha/(2\alpha)} \left( \int_{B(\rho/2)} |u|^2 \right)^{1/2}.
$$
For any \(1\) proceed further, we claim that there is a constant \(\mu_1\) such that for any \(\mu \leq \mu_1\),

\[
W. Ren, Y. Wang & G. Wu
\]

\[
+ C\mu^{(4\alpha - n)} \left( \int_{B(\rho/2)} |h|^{2n/(n-2\alpha)} \right) \left( \int_{B(\rho/2)} |\bar{u}|^{2n/(n-2\alpha)} \right)^{1/2} \times \left( \int_{B(\rho/2)} |u - \bar{u}|^{2n/(n-2\alpha)} \right)^{\alpha - 2}/\alpha
\]

\[
\leq C\mu^{(4\alpha - n)/2} \left( \frac{\int_{B^+(\rho)} y^{1-2\alpha}(|\nabla^* u|^2 + |\nabla^* h|^2)}{\left( \int_{B(\rho)} (|u|^2 + |h|^2) \right)^{1/2}} \right) + C\rho^{-\alpha} \mu^{(4\alpha - n)/2} \left( \int_{B(\rho)} |u|^2 + |h|^2 \right) \left( \int_{B^+(\rho)} y^{1-2\alpha}(|\nabla^* u|^2 + |\nabla^* h|^2) \right)^{1/2}.
\]

(4.2)

As a consequence, we know that

\[
\frac{1}{\mu^{n+3-4\alpha}} \int_{Q(\rho)} |u||u|^2 - |\bar{u}|^2| + |h||u - \bar{u}| + p\mu \cdot R\rho
\]

\[
\leq C \left( \frac{\rho}{\mu} \right)^{2(n+2-4\alpha)} \left( \frac{1}{\rho^{n+2-4\alpha}} \sup_{-\rho^{2\alpha} \leq t < 0} \int_{B(\rho)} (|u|^2 + |h|^2) \right)^{1/2} \times \left( \frac{1}{\rho^{n+2-4\alpha}} \sup_{-\rho^{2\alpha} \leq t < 0} \int_{B(\rho)} (|u|^2 + |h|^2) \right)
\]

\[
+ C \left( \frac{\rho}{\mu} \right)^{2(n+2-4\alpha)} \left( \frac{1}{\rho^{n+2-4\alpha}} \sup_{-\rho^{2\alpha} \leq t < 0} \int_{B(\rho)} (|u|^2 + |h|^2) \right)^{1/2} \times \left( \frac{1}{\rho^{n+2-4\alpha}} \int_{Q^+(\rho)} y^{1-2\alpha}(|\nabla^* u|^2 + |\nabla^* h|^2) \right)^{1/2}
\]

\[
\leq C \left( \frac{\rho}{\mu} \right)^{2(n+2-4\alpha)} \left( E(\rho)^{1/2} E^*_\rho + E(\rho)E^*_\rho^{1/2}(\rho) \right).
\]

(4.3)

For any \(1/2 < \alpha < 3/4\), we find that

\[
F_q(\mu) \leq C\mu^{2\alpha} \left( \int_{R^n} |f(x, t)|^q dxdt \right)^{1/2}, \quad q > \frac{2\alpha + n}{2\alpha},
\]

(4.4)

which means that \(F_q(\mu)\) tends to 0 as \(\mu \to 0\). For \(\alpha \geq 3/4\), we see that

\[
\mu^{2\alpha - 1 + \gamma} ||f||_{2\alpha, \gamma} \to 0 \quad \text{as} \quad \mu \to 0.
\]

Thus, no matter in which case, the smallness of external force has been shown. To proceed further, we claim that there is a constant \(\mu_1\) such that for any \(\mu \leq \mu_1\),

\[
1650018-30
\]
Partial regularity of suitable weak solutions to the generalized MHD equations

\[ F_{3/2}(\mu) \leq \varepsilon_2. \] Indeed, by the Hölder inequality, for any \( \alpha \geq 3/4 \), it holds

\[
F_{3/2}(\mu) = \mu^{3/2(4\alpha-1)-(2\alpha+n)} \int_{Q(\mu)} |f(x,t)|^{3/2} \, dx \, dt
\]

\[
\leq C \mu^{3 \alpha - \frac{3}{2} + 3 \alpha - \frac{\alpha}{2}} \left( \frac{1}{\mu^{2(\gamma-2\alpha)}} \int_{Q(\mu)} |f(x,t)|^{2\alpha} \, dx \, dt \right)^{\frac{3}{4\alpha}}
\]

\[
\leq C \mu^{3 \alpha - \frac{3}{2} + 3 \alpha - \frac{\alpha}{2}} \|f\|^{\frac{3}{2}}_{L_{2\alpha, \gamma}}.
\]

Therefore, we see that \( F_{3/2}(\mu) \to 0 \) as \( \mu \to 0 \) when \( \alpha \geq 3/4 \). By similar arguments, the same statement is also valid if \( 1/2 < \alpha < 3/4 \). Therefore, the assertion follows.

Next, we will prove the smallness of the other terms in (1.2) under the condition (1.3). Notice that (1.3) implies that there exists \( \mu_0 \in (0, \mu_1) \) such that

\[ E_\ast(\mu) \leq \varepsilon_2, \quad \text{for any } \mu \leq \mu_0. \] (4.5)

Plugging (4.3), (2.16) in Lemma 2.2 into (4.1) and using the Cauchy-Schwarz inequality, we have

\[ E(\mu) + E_\ast(\mu) \leq CE_3^{2/3}(2\mu) + CE_\ast(2\mu) \]

\[ + C \left( \frac{\rho}{\mu} \right)^{\frac{3}{2}(n+2-4\alpha)} (E(\rho)^{1/2} E_\ast(\rho) + E(\rho) E_\ast^{1/2}(\rho)) \]

\[ + CE_3^{1/3}(u, 2\mu) F_{3/2}^{2/3}(2\mu) + CE_\ast^{1/3}(u, 2\mu) F_{3/2}^{2/3}(2\mu) \]

\[ \leq CE_3^{2/3}(2\mu) + CE_\ast(2\mu) + C \left( \frac{\rho}{\mu} \right)^{\frac{3}{2}(n+2-4\alpha)} \]

\[ \times (E(\rho)^{1/2} E_\ast(\rho) + E(\rho) E_\ast^{1/2}(\rho)) + P_{\frac{3n+2}{n+2}}(2\mu) + F_{3/2}^{4/3}(2\mu) \]

\[ \leq C \left( \frac{\rho}{\mu} \right)^{2n+4-8\alpha} E^{\frac{n}{4\alpha-n}}(\rho) E_\ast^{\frac{n}{4\alpha-n}} + C \left( \frac{\mu}{\rho} \right)^{4\alpha-2} E(\rho) \]

\[ + C \left( \frac{\rho}{\mu} \right)^{n+2-4\alpha} E_\ast(\rho) + C \left( \frac{\rho}{\mu} \right)^{\frac{3}{2}(n+2-4\alpha)} \]

\[ \times (E(\rho)^{1/2} E_\ast(\rho) + E(\rho) E_\ast^{1/2}(\rho)) \]

\[ + C \left( \frac{\rho}{\mu} \right)^{2n+4-8\alpha} E^{\frac{4\alpha}{n+2\alpha}}(\rho) E_\ast^{\frac{4\alpha}{n+2\alpha}}(\rho) \]

\[ + C \left( \frac{\mu}{\rho} \right)^{\frac{28n+4+2\alpha(n-1)-n}{n+2\alpha}} P_{\frac{3n+2}{n+2\alpha}}(\rho) + C \left( \frac{\mu}{\rho} \right)^{\frac{3}{2}n+2-4\alpha} F_{3/2}^{4/3}(\rho), \] (4.6)
It follows from (4.6) and (2.16) in Lemma 2.2 that

$$G(\mu) = E(\mu) + E_*(\mu) + \frac{n}{\mu^{2n-4\alpha}}(\mu).$$

It follows from (4.6) and (2.16) in Lemma 2.2 that

$$G(\mu) \geq C\left(\frac{\rho}{\mu}\right)^{2+n-4\alpha} E^{\frac{8n-2n}{2n-4\alpha}}(\mu) E_{*}^{\frac{8n}{2n-4\alpha}}(\mu) + C\left(\frac{\mu}{\rho}\right)^{4\alpha-2} E(\mu)$$

$$+ \left[\left(\frac{\rho}{\mu}\right)^{\frac{3n+2-4\alpha}{2n-4\alpha}} \left[\left(\frac{\rho}{\mu}\right)^{n+2-4\alpha} E_*(\mu) + \left(\frac{\rho}{\mu}\right)^{2n} F^{4/3}(\rho)\right] + \left(\frac{\rho}{\mu}\right)^{2n+4-8\alpha} E^{\frac{8n-2n}{2n-4\alpha}}(\mu) E_{*}^{\frac{8n}{2n-4\alpha}}(\mu) + C\left(\frac{\mu}{\rho}\right)^{2n+4-8\alpha} F^{4/3}(\rho)\right].$$

Notice that $\frac{4n}{n+2\alpha} < 1$ and $8\alpha^2 + 2\alpha(n-1) - n > 0$ for $n/4 \leq \alpha < n/2$. Thus

$$G(\mu) \leq C\left(\frac{\rho}{\mu}\right)^{2(n+2-4\alpha)} G^{\tau_1}(\mu) E_{*}^{\tau_3}(\mu) + \left(\frac{\mu}{\rho}\right)^{\tau_3} G(\mu) + \left(\frac{\rho}{\mu}\right)^{2n} F^{4/3}(\rho),$$

where $0 \leq \tau_1 < 1, \tau_2, \tau_3 > 0$. Using the Young inequality, we infer that

$$G(\mu) \leq C\left(\frac{\rho}{\mu}\right)^{6} G(\mu) E_{*}^{\tau_4}(\mu) + \left(\frac{\mu}{\rho}\right)^{\tau_4} G(\mu) + \left(\frac{\rho}{\mu}\right)^{6} \left[E_{*}^{\tau_4}(\mu) + F^{4/3}(\rho)\right]$$

$$\leq C_1 \lambda^{6} \varepsilon_2^{\tau_4} G(\mu) + C_2 \lambda^{\tau_3} G(\mu) + C_3 \lambda^{6} \varepsilon_2^{\tau_3},$$

where $\alpha > 0, \tau_4, \tau_3 > 0, \lambda = \frac{\mu}{\rho} \leq \frac{1}{2}$ and $\rho \leq \mu_0$.

Choosing $\lambda, \varepsilon_2$ such that $\frac{1}{2} = 2C_2 \lambda^{\tau_3} < 1$ and

$$\varepsilon_2 = \min\left\{\frac{1}{\lambda^{1/3}} \left(\frac{1 - \lambda}{2C_3}\right)^{\frac{1}{2(1+4\alpha)}}, \frac{1}{\lambda^{1/3}} \left(\frac{1 - \lambda}{2C_3}\right)^{\frac{1}{2(1+4\alpha)}}\right\},$$

we obtain

$$G(\lambda \rho) \leq qG(\rho) + C_3 \lambda^{-6} \varepsilon_2^{\tau_3}. \tag{4.7}$$

Iterating (4.7), we deduce that

$$G(\lambda^k \rho) \leq q^k G(\rho) + \frac{1}{2} \lambda^{2n+4-8\alpha} \varepsilon_1.$$

From the definition of $G(\mu)$, there exists a positive number $K_0$ such that

$$q^{K_0} G(\mu_0) \leq \frac{4}{\mu_0^{2n-4\alpha}} \left\{\|u\|_{L^{2,2}} \left\|\frac{u}{L^{2,2}}, \|p\|_{L^{2,2}}\right\| \right\}_{\mu_0^{2n+4-8\alpha}} \left(qK_0 \leq \frac{1}{2} \lambda^{2n+4-8\alpha} \varepsilon_1.\right.$$
Let $\mu_2 := \lambda^{K_0} \mu_0$, then, for all $0 < \mu \leq \mu_2$, there is a constant $k \geq K_0$ such that $\lambda^{k+1} \mu_0 \leq \mu \leq \lambda^k \mu_0$. Thus, we have

$$E(\mu) + E_*(\mu) + P_{3/2}^{2/3}(\mu)$$

$$\leq \frac{1}{\lambda^{2n+4-8\alpha}} G(\lambda^k \mu_0)$$

$$\leq \frac{1}{\lambda^{2n+4-8\alpha}} \left( \hat{q}^{k-K_0} \hat{q}^{K_0} G(\mu_0) + \frac{1}{2} \lambda^{2n+4-8\alpha} \varepsilon_1 \right) \leq \varepsilon_1. \quad (4.8)$$

By a scaling argument together with Theorem 1.1, we end the proof of Theorem 1.2.

**Proof of Theorem 1.3.** Thanks to (2.14) in Lemma 2.1 and (2.18) in Lemma 2.2, we get

$$E_3^{2/3} \left( \frac{\mu}{4} \right) \leq C(E(p) + E_*(p)) + CE(p),$$

$$P_{3/2}^{2/3} \left( \frac{\mu}{4} \right) \leq C(E(p) + E_*(p)) + CP_{3/2}^{2/3}(p).$$

By the Hölder inequality, (3.12) in Corollary 3.1 and the inequalities above, we see that

$$E(\mu) + E_*(\mu) \leq C \left( \frac{\mu}{\rho} \right)^{4\alpha - 2} \left( E \left( \frac{\mu}{4} \right) + E_* \left( \frac{\mu}{4} \right) \right) + \left( \frac{\rho}{\mu} \right)^{n+3-4\alpha} E_3 \left( u, \frac{\rho}{4} \right)$$

$$+ \left( \frac{\rho}{\mu} \right)^{n+3-4\alpha} E_3^{1/3} \left( u, \frac{\rho}{4} \right) E_3^{2/3} \left( h, \frac{\rho}{4} \right)$$

$$+ \left( \frac{\rho}{\mu} \right)^{n+3-4\alpha} E_3^{1/3} \left( u, \frac{\rho}{4} \right) P_{3/2}^{2/3} \left( \frac{\rho}{4} \right)$$

$$\leq C \left( \frac{\mu}{\rho} \right)^{4\alpha - 2} \left( E(p) + E_*(p) \right) + \left( \frac{\rho}{\mu} \right)^{n+3-4\alpha} E_3(u, \rho)$$

$$+ \left( \frac{\rho}{\mu} \right)^{n+3-4\alpha} E_3^{1/3}(u, \rho)(E(p) + E_*(p))$$

$$+ \left( \frac{\rho}{\mu} \right)^{n+3-4\alpha} E_3^{1/3}(u, \rho)(E(p) + E_*(p) + P_{3/2}^{2/3}(p)). \quad (4.9)$$
With the help of (2.18) again and Young’s inequality, we infer that
\[ P_{3/2}^{2/3}(\mu) \leq \left( \frac{\mu}{\rho} \right)^{\frac{6+3n-12\alpha}{2}} E_2^{\frac{n-n}{6\alpha}}(\rho) E_2^{\frac{2n}{\alpha}}(\rho) + C \left( \frac{\mu}{\rho} \right)^{\frac{8n-3}{2}} P_{3/2}^{2/3}(\rho) \]
\[ \leq \left( \frac{\mu}{\rho} \right)^{2+n-4\alpha} (E_2(\rho) + E_*(\rho)) + C \left( \frac{\mu}{\rho} \right)^{\frac{8n-3}{2}} P_{3/2}^{2/3}(\rho). \] 
(4.10)

Now, set
\[ G_1(\mu) = E(\mu) + E_*(\mu) + \varepsilon_3^{1/4} P_{3/2}^{2/3}(\mu). \]

It follows from (4.9) and (4.10) that
\[ G_1(\mu) \leq C \left( \frac{\mu}{\rho} \right)^{4\alpha - 2} G_1(\mu) + \frac{\mu}{\rho}^{n+3-4\alpha} E_3(u, \rho) + \frac{\mu}{\rho}^{n+3-4\alpha} E_3^{1/3}(u, \rho) G_1(\mu) \]
\[ + \left( \frac{\mu}{\rho} \right)^{n+3-4\alpha} E_3^{1/3}(u, \rho) G_1(\mu) + \varepsilon_3^{1/4} G_1(\mu) \]
\[ + \left( \frac{\mu}{\rho} \right)^{2+n-4\alpha} \varepsilon_3^{1/4} G_1(\mu) + C \left( \frac{\mu}{\rho} \right)^{\frac{8n-3}{2}} G_1(\mu) \]
\[ \leq C \left( \frac{\mu}{\rho} \right)^{n+3-4\alpha} \varepsilon_3^{1/12} G_1(\mu) + C \left( \frac{\mu}{\rho} \right)^{\tau_6} G_1(\mu) + \left( \frac{\mu}{\rho} \right)^{n+3-4\alpha} \varepsilon_3 \]
\[ \leq C \lambda^{4\alpha-n-3} \varepsilon_3^{1/12} G_1(\mu) + C \lambda^{10} G_1(\mu) + \lambda^{4\alpha-n-3} \varepsilon_3, \]
where \( \tau_6 > 0, \lambda = \frac{\mu}{\rho} \leq \frac{1}{32} \). An iteration argument completely analogous to that adopted in the proof of Theorem 1.2 yields the smallness of \( E(\mu) + E_*(\mu) \). This together with Theorem 1.2 completes the proof of Theorem 1.3. \[ \square \]

**Proof of Theorem 1.4.** (1) Recall that the pair \((q', \ell')\) is the conjugate index of \((q, \ell)\) in (1.6). By means of the Hölder inequality, (2.8) and the Young inequality, we arrive at
\[
\|u\|_{L^{2\ell'}\|L_{2q}(Q(\rho/2))} \leq C \|u\|_{L^{\ell'}\|L_{2q}(Q(\rho/2))} \|v\|_{L^{q}\|L_{2q}(Q(\rho/2))} \leq C \|u\|_{L^{\ell'}\|L_{2q}(Q(\rho/2))} \left(\|u^{1/2 - \alpha} \nabla u\|_{L^2(Q(\rho/2))} + \|u\|_{L^{\infty}(Q(\rho/2))} \right)^{1 - \frac{1}{2}},
\]
(4.11)

where \(\alpha/q + n/2\ell = n/2\). In view of the Hölder inequality and (4.11), we deduce that
\[
\rho^{-(n+3-4\alpha)} \int_{(\rho/2)^{2n}} \int_{B(\rho/2)} |u| |h|^2 \leq C \rho^{-(n+3-4\alpha)} \|u\|_{L^{q\ell'}(Q(\rho/2))} \|h\|_{L^{2q\ell'}(Q(\rho/2))} \leq C \left[ E(\rho) + E_*(\rho) \right] E_{q,\ell}(u, \rho).
\]
(4.12)
A similar procedure yields

$$\rho^{-(n+3-4\alpha)} \int_{-\infty}^{0} \int_{B(\rho/2)} |u|^3 \leq C[E(\rho) + E_*(\rho)]E_{q,\ell}(u, \rho),$$

(4.13)

which together with (4.12) and (3.12) in Corollary 3.1 implies that

$$E(\mu) + E_*(\mu) \leq C \left( \frac{\mu}{\rho} \right)^{4\alpha - 2} \left( E \left( \frac{\mu}{4} \right) + E_* \left( \frac{\mu}{4} \right) \right)$$

$$+ C \left( \frac{\rho}{\mu} \right)^{n+3-4\alpha} \left[ E(\rho) + E_*(\rho) \right] E_{q,\ell}(u, \rho)$$

$$+ C \left( \frac{\rho}{\mu} \right)^{n+3-4\alpha} E_3^{1/3}(u, \rho)P_{3/2}^2(\rho)$$

$$\leq C \left( \frac{\mu}{\rho} \right)^{4\alpha - 2} (E(\rho) + E_*(\rho))$$

$$+ \left( \frac{\rho}{\mu} \right)^{n+3-4\alpha} \left[ E(\rho) + E_*(\rho) \right] E_{q,\ell}(u, \rho)$$

$$+ \left( \frac{\rho}{\mu} \right)^{n+3-4\alpha} E_3^{1/3}(u, \rho) [E(\rho) + E_*(\rho)]^{1/3} P_{3/2}^2(\rho).$$

(4.14)

It follows from (2.19) and condition (1.5) that

$$\varepsilon_1^{1/4} P_{3/2}(\mu) \leq C \varepsilon_1^{1/4} \left( \frac{\rho}{\mu} \right)^{n+3-4\alpha} E(\rho)^{\frac{1}{2}} E_* (\rho)^{1-\frac{1}{4}} E_{q,\ell}(\rho)$$

$$+ C \left( \frac{\mu}{\rho} \right)^{4\alpha - 2} \varepsilon_1^{1/4} P_{3/2}(\rho)$$

$$\leq C(M) \varepsilon_1^{1/4} \left( \frac{\rho}{\mu} \right)^{n+3-4\alpha} E(\rho)^{\frac{1}{2}} E_* (\rho)^{1-\frac{1}{4}}$$

$$+ C \left( \frac{\mu}{\rho} \right)^{4\alpha - 2} \varepsilon_1^{1/4} P_{3/2}(\rho).$$

(4.15)

Set $G_2(\mu) = E(\mu) + E_*(\mu) + \varepsilon_1^{1/4} P_{3/2}(\mu)$ and $\lambda = \frac{\mu}{\rho}$, then, (4.15) together with (4.14) implies that

$$G_2(\lambda \rho) \leq C \lambda^{4\alpha - 2} (E(\rho) + E_*(\rho)) + \lambda^{4\alpha - n+3} [E(\rho) + E_*(\rho)] E_{q,\ell}(u, \rho)$$

$$+ \lambda^{4\alpha - n+3} E_{q,\ell}^{1/3}(u, \rho) \varepsilon_4^{1/4} + C \left( \frac{\mu}{\rho} \right)^{4\alpha - 2} \varepsilon_4^{1/4} P_{3/2}(\rho)$$

$$+ C \varepsilon_4^{1/4} \left( \frac{\rho}{\mu} \right)^{n+3-4\alpha} E(\rho)^{\frac{1}{2}} E_* (\rho)^{1-\frac{1}{4}} + C \left( \frac{\mu}{\rho} \right)^{4\alpha - 2} \varepsilon_4^{1/4} P_{3/2}(\rho).$$
It follows from (2.17) that
\[
G_{\tau} estimates imply that
\]
\[
\text{Theorem 1.4.}
\]
\[
\text{ment used in the proof Theorem 1.2, one can complete the proof of second part of}
\]
\[
W. Ren, Y. Wang & G. Wu
\]
\[
(2) By a slight modified proof of (4.14) and (2.17), we arrive at
\]
\[
E(\mu) + E_*(\mu) \leq C \left( \frac{\mu^7}{\rho} \right)^{\alpha - 2} (E(\rho) + E_*(\rho)) + \left( \frac{\rho}{\mu} \right)^{\alpha - 4} \left( E(\rho) + E_*(\rho) \right)
\]
\[
\times [E(\rho) + E_*(\rho)] E_{q, \ell} (u, \rho) + \left( \frac{\rho}{\mu} \right)^{\alpha - 4} \left( E(\rho) + E_*(\rho) \right) E_{q, \ell} (u, \rho)
\]
\[
\leq C \left( \frac{\mu^7}{\rho} \right)^{\alpha - 2} \left( \frac{\rho}{\mu} \right)^{\alpha - 4} \left( E(\rho) + E_*(\rho) \right)
\]
\[
+ \left( \frac{\rho}{\mu} \right)^{\alpha - 4} \left( \frac{\rho}{\mu} \right)^{\alpha - 4} \left( E(\rho) + E_*(\rho) \right) E_{q, \ell} (u, \rho)
\]
\[
+ \left( \frac{\rho}{\mu} \right)^{\alpha - 4} \left( \frac{\rho}{\mu} \right)^{\alpha - 4} \left( E(\rho) + E_*(\rho) \right) E_{q, \ell} (u, \rho)
\]
\[
\times \left[ \left( \frac{\rho}{\mu} \right)^{n + 2 - 4\alpha} E_7 (\rho) E_* \left( \frac{\rho}{\mu} \right) + C \left( \frac{\rho}{\mu} \right)^{\alpha - \frac{\alpha - 1}{\alpha - 4}} \right] \left( \frac{\rho}{\mu} \right)^{n + 2 - 4\alpha} \left( E(\rho) + E_*(\rho) \right) E_{q, \ell} (u, \rho).
\]

It follows from (2.17) that
\[
P_{q, \ell} (\mu) \leq \left( \frac{\rho}{\mu} \right)^{n + 2 - 4\alpha} E_7 (\rho) E_* \left( \frac{\rho}{\mu} \right) + C \left( \frac{\rho}{\mu} \right)^{\alpha - \frac{\alpha - 1}{\alpha - 4}} \left( \frac{\rho}{\mu} \right)^{n + 2 - 4\alpha} \left( E(\rho) + E_*(\rho) \right) E_{q, \ell} (u, \rho).
\]

We denote \( G_4 (\mu) = E(\mu) + E_*(\mu) + \varepsilon_5^{1/4} P_{q, \ell} (\mu) \) and \( \lambda = \frac{\mu}{\rho} = \frac{\rho}{\mu} \), then the above estimates imply that
\[
G_4 (\mu) \leq \lambda^{5\alpha - n - 4} G_4 (\rho) + \lambda^{5\alpha - 2n - 6} G_4 (\rho) E_{q, \ell} (u, \rho)
\]
\[
+ \lambda^{5\alpha - 2n - 6} E_{q, \ell} (u, \rho) G_4 (\rho) + \lambda^{5\alpha - n - 5 - \frac{\alpha}{\alpha - 4}} E_5^{1/4} G_4 (\rho) E_{q, \ell} (u, \rho)
\]
\[
+ \lambda^{5\alpha - n - 2} E_5^{1/4} G_4 (\rho) + \lambda^{5\alpha - \frac{\alpha}{\alpha - 4}} G_4 (\rho)
\]
\[
\leq C \lambda^{\alpha} G_4 (\rho) + \lambda^{\alpha - 1} G_4 (\rho),
\]

where \( \tau_9, \tau_{10}, \tau_{11} > 0 \). Based on this, along the same line of the iteration argument used in the proof Theorem 1.2, one can complete the proof of second part of Theorem 1.4.
Partial regularity of suitable weak solutions to the generalized MHD equations

Acknowledgments

The authors thank the anonymous referee and the associated editor for the invaluable comments and suggestions which helped to improve the paper greatly. The third author is supported in part by the National Natural Science Foundation of China under Grant No. 11101405.

Note added to the proof: After this paper was submitted for publication, Dr. Yukang Chen informed us that the partial regularity of the suitable weak solutions to the fractional Navier–Stokes equations with a zero force in the case $\alpha = 3/4$ was independently obtained in the preprint [5] with Prof. Zhe Lian and Dr. Changhua Wei. We would like to point out that our proof is different from that in [5], especially the estimates of pressure.

References

[1] D. Biskamp, Nonlinear Magnetohydrodynamics (Cambridge University Press, Cambridge, 1993).
[2] L. Caffarelli, R. Kohn and L. Nirenberg, Partial regularity of suitable weak solutions of Navier–Stokes equation, *Comm. Pure. Appl. Math.* 35 (1982) 771–831.
[3] L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, *Comm. Partial Differential Equations* 32 (2007) 1245–1260.
[4] C. Cao, J. Wu and B. Yuan, The 2D incompressible magnetohydrodynamics equations with only magnetic diffusion, *SIAM J. Math. Anal.* 46 (2014) 588–602.
[5] Y. Chen, Z. Lei and C. Wei, Partial regularity of solutions to the fractional Navier–Stokes Equations, preprint (2015).
[6] Q. Chen, C. Miao and Z. Zhang, On the regularity criterion of weak solution for the 3D viscous magento-hydrodynamics equations, *Comm. Math. Phys.* 284 (2008) 919–930.
[7] Z. Chen and X. Zhang, Heat kernels and analyticity of non-symmetric jump diffusion semigroups, to appear in *Probab. Theory Relat. Fields* DOI:10.1007/s00440-01s-0651-y.
[8] G. Duvaut and J. Lions, Inéquations en thermoélasticité et magnétohydrodynamique, *Arch. Ration. Mech. Anal.* 46 (1972) 241–279.
[9] S. Gustafson, K. Kang and T. Tsai, Interior regularity criteria for suitable weak solutions of the Navier–Stokes equations, *Comm. Math. Phys.* 273 (2007) 161–176.
[10] A. Hasegawa, Self-organization processes in continuous media, *Adv. Phys.* 34 (1985) 1–42.
[11] C. He and Z. Xin, On the regularity of weak solutions to the magnetohydrodynamic equations, *J. Differential Equations* 213 (2005) 235–254.
[12] ______: Partial regularity of suitable weak solutions to the incompressible magnetohydrodynamic equations, *J. Funct. Anal.* 227 (2005) 113–152.
[13] Q. Jiu and Y. Wang, Remarks on partial regularity for suitable weak solutions of the incompressible magnetohydrodynamic equations, *J. Math. Anal. Appl.* 409 (2014) 1052–1065.
[14] ______: On possible time singular points and eventual regularity of weak solutions to the fractional Navier–Stokes equations, *Dyn. Partial Differential Equations* 11 (2014) 321–343.
[15] Q. Jiu and J. Zhao, Global regularity of 2D generalized MHD equations with magnetic diffusion, *Z. Angew. Math. Phys.* 66 (2015) 677–687.
[16] N. Katz and N. Pavlović, A cheap Caffarelli–Kohn–Nirenberg inequality for the Navier–Stokes equation with hyper-dissipation, *Geom. Funct. Anal.* 12 (2002) 355–379.

[17] O. Ladyzenskaja and G. Seregin, On partial regularity of suitable weak solutions to the three-dimensional Navier–Stokes equations, *J. Math. Fluid Mech.* 1 (1999) 356–387.

[18] F. Lin, A new proof of the Caffarelli–Kohn–Nirenberg Theorem, *Comm. Pure Appl. Math.* 51 (1998) 241–257.

[19] H. Politano, A. Pouquet and P. L. Sulem, Current and vorticity dynamics in three-dimensional magnetohydrodynamics turbulence, *Phys. Plasmas.* 2 (1995) 2931–2939.

[20] W. Ren and G. Wu, Partial regularity for the 3D Magneto-hydrodynamics system with hyper-dissipation, *Acta Math. Sin. (Engl. Ser.)* 31 (2015) 1097–1112.

[21] V. Scheffer, Partial regularity of solutions to the Navier–Stokes equations, *Pacific J. Math.* 66 (1976) 535–552.

[22] Hausdorff measure and the Navier–Stokes equations, *Comm. Math. Phys.* 55 (1977) 97–112.

[23] The Navier–Stokes equations in space dimension four, *Comm. Math. Phys.* 61 (1978) 41–68.

[24] M. Sermange and R. Teman, Some mathematical questions related to the MHD equations, *Comm. Pure Appl. Math.* 36 (1983) 635–664.

[25] L. Tang and Y. Yu, Partial regularity of suitable weak solutions to the fractional Navier–Stokes equations, *Comm. Math. Phys.* 334 (2015) 1455–1482.

[26] Erratum to: Partial regularity of suitable weak solutions to the fractional Navier–Stokes equations, *Comm. Math. Phys.* 335 (2015) 1057–1063.

[27] Partial Hölder regularity for steady fractional Navier–Stokes equation, to appear in *Calc. Var. Partial Differential Equations*, DOI:10.1007/s00526-016-0967-x.

[28] G. Tian and Z. Xin, Gradient estimation on Navier–Stokes equations, *Comm. Anal. Geom.* 7 (1999) 221–257.

[29] C. Tran, X. Yu and Z. Zhai, On global regularity of 2D generalized magnetohydrodynamics equations, *J. Differential Equations* 254 (2013) 4194–4216.

[30] A. Vasseur, A new proof of partial regularity of solutions to Navier Stokes equations, *NoDEA Nonlinear Differential Equations Appl.* 14 (2007) 753–785.

[31] Y. Wang and G. Wu, A unified proof on the partial regularity for suitable weak solutions of non-stationary and stationary Navier–Stokes equations, *J. Differential Equations* 256 (2014) 1224–1249.

[32] Local regularity criteria of the 3D Navier–Stokes and related equations, submitted.

[33] W. Wang and Z. Zhang, On the interior regularity criteria for suitable weak solutions of the magnetohydrodynamics equations, *SIAM J. Math. Anal.* 45 (2013) 2666–2677.

[34] J. Wu, Generalized MHD equations, *J. Differential Equations* 195 (2003) 284–312.

[35] Regularity criteria for the generalized MHD equations, *Comm. Partial Differential Equations* 33 (2008) 285–306.

[36] Global regularity for a class of generalized magnetohydrodynamic equations, *J. Math. Fluid Mech.* 13 (2011) 295–305.