SHORT NOTE ON AN OPEN PROBLEM

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Abstract. In this work, we investigate a problem posed by Feng Qi and Bai-Ni Guo in their paper Complete monotonicities of functions involving the gamma and digamma functions.  
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1. Introduction and statement of the main results
A function $f$ is said to be completely monotonic on an interval $I$ if $f$ has derivatives of all orders on $I$ which alternate successively in sign, that is
$$(−1)^n f^{(n)}(x) \geq 0,$$
for $x \in I$ and $n \geq 0$. If the inequality above is strict for all $x \in I$ and for all $n \geq 0$, then $f$ is said to be strictly completely monotonic. For a good monograph about this subject see for instance, [2] and [3]. In [1], the author study a class of monotonic function involving the gamma and digamma functions. He shows that the function $f_{\alpha}(x) = (x + 1)^\alpha / (\Gamma(x + 1))^{1/x}$ is strictly completely monotonic on $(-1, +\infty)$ provided that $\alpha \leq 1/(1 + \tau_0) < 1$, where $\tau_0$ is the maximum of the function defined on $\mathbb{N} \times (0, +\infty)$ by
$$\tau(n, t) = \frac{1}{n} \left( t - (t + n + 1) \left( \frac{1}{1 + t} \right)^{n+1} \right).$$
For $n = 2, 3$, the author compute numerically the maximum, his finds $\max_{t>0} \tau(2, t) \simeq 0.264076$, and $\max_{t>0} \tau(3, t) \simeq 0.271807$. At my knowledge, no analytic method is given to compute $\tau_0$.

He posed the problem: find the maximum value
$$\alpha = \max_{(n,t) \in \mathbb{N} \times (0, +\infty)} \tau(n, t).$$
Using mathematica computations the author conjectured in [1] that $\alpha > 0.298$.

In this work, we answered the question, and we show that $\alpha$ is given by
$$\alpha = \frac{\ell}{1 + \ell + \ell^2},$$
where \( \ell \) is the unique solution of following equation
\[
\ell^3 - \frac{1}{\ell^2} - \frac{1}{\ell} - 1 = 0.
\]
Numerically \( \ell \approx 0.5576367386 \ldots \), and \( \alpha = 0.2984256075 \ldots \)

First of all, we show for every \( n \geq 1 \), there is a unique sequence \( t_n \in (0,n) \)
which is increasing and such that \( \max_{t \geq 0} \tau(n, t) = \tau(n, t_n) := \alpha_n \), and in second time,
we prove that the sequence \( \alpha_n \) converges towards growth to \( \alpha \). Which allows us
to deduce that
\[
\max_{(n, t) \in \mathbb{N} \times (0, +\infty)} \tau(n, t) = \alpha.
\]

2. Proof of main results

Let’s define on \([1, +\infty) \times [0, +\infty)\) the function
\[
\tau(x, t) = \frac{1}{x}(t - (t + x + 1)(\frac{t}{1 + t})^{x+1}).
\]

**Proposition 2.1.** For every \( x \geq 1 \), the function \( t \mapsto \tau(x, t) \) attains its maximum in only one point \( t(x) \in (0, x) \). The value of the maximum \( \alpha(x) := \tau(x, t(x)) \) is given by
\[
\alpha(x) = \frac{(1 + x)t(x)}{x^2 + (1 + t(x))^2 + x(2 + t(x))}.
\]

**Proof.** 1) Deriving the function \( \tau(x, t) \) with respect to \( t \), gives
\[
\partial_t \tau(x, t) = \frac{1}{x}(1 - (\frac{t}{1 + t})^{x+1}) \frac{1}{x} - (t + x + 1)(x + 1)(\frac{t}{1 + t})^{x+1},
\]
\[
\partial_t \tau(x, t) = \frac{1}{x}(1 - (\frac{t}{1 + t})^{x+1}) - (t + x + 1)(x + 1)\frac{1}{t(1 + t)}(\frac{t}{1 + t})^{x+1}.
\]
The second derivative is given as follow
\[
\partial^2_t \tau(x, t) = (1 + x)t^{-1+x}(1 + t)^{-3-x}(-1 - x + t).
\]
Hence, \( \partial^2_t \tau(x, t) = 0 \) on \((0, +\infty)\) if and only if \( t = x + 1 \). Moreover, \( \partial_t \tau(x, 0) = \frac{1}{x} \),
and
\[
\partial_t \tau(x, x) = \frac{1}{x}(1 - 3(\frac{x}{1 + x})^{x+1}) - \frac{1}{x}(\frac{x}{1 + x})^{x+1}).
\]
For \( x \geq 1 \), consider the function \( h(x) = 1 - (3 + \frac{1}{x})(\frac{x}{1 + x})^{x+1} \). We have by successive differentiation
\[
h'(x) = -x^2(1 + x)^{-1-x}h_1(x),
\]
where \( h_1(x) = 3 + (1 + 3x)\log(1 + \frac{1}{x}) \),
\[
h'_1(x) = \frac{1}{x} + \frac{2}{1 + x} + 3\log(x) - 3\log(1 + x),
\]
and
\[ h_1''(x) = \frac{x-1}{x^2(1+x)^2} \geq 0. \]

Since, \( \lim_{x \to +\infty} h_1'(x) = 0 \), hence \( h_1'(x) \leq 0 \). Moreover, \( \lim_{x \to +\infty} h_1(x) = 0 \), and hence \( h_1(x) \geq 0 \). One deduces that \( h \) is non increasing on \([1, +\infty]\), and \( h(1) = 0 \). Thus, for every \( x > 1 \)
\[ \partial_t \tau(x, x) < 0. \]
Which implies that the derivative \( t \mapsto \partial_t \tau(x, t) \) decreases strictly on \((0, x+1)\). By the fact that the second derivative across the \( x \)-axis only one time on \((0, +\infty)\), then there is a unique \( t(x) \in (0, x+1) \) such that \( \partial_t \tau(x, t(x)) = 0 \), and
\[ t(x) \in (0, x), \]
in view of \( \partial_t \tau(x, x) < 0 \).
Remark that in the interval \((0, x)\), \( \partial^2_t \tau(x, t) < 0 \), hence the critical point \( t(x) \) is a maximum for the function \( t \mapsto \tau(x, t) \).

2) Recall that
\[ \alpha(x) = \tau(x, t(x)). \]
Furthermore,
\[ 1 - \left( \frac{t(x)}{1 + t(x)} \right)^x + 1 - (t(x) + x + 1)(x + 1) \frac{1}{t(x)(1 + t(x))} \left( \frac{t(x)}{1 + t(x)} \right)^x + 1 = 0. \]
and
\[ \alpha(x) = \frac{1}{x}(t(x) - (t(x) + x + 1)(\frac{t(x)}{1 + t(x)}))^{x+1}. \]

By substituting equation (1) in (2), we get
\[ \alpha(x) = \frac{(1 + x)t(x)}{x^2 + (1 + t(x))^2 + x(2 + t(x))}. \]

Lemma 2.2. The function \( x \mapsto t(x) \) is well defined, defines a \( C^1 \)-diffeomorphism and satisfies
\[ \frac{(x+1)^2}{2x+3} \leq t(x) < x. \]

Proof. 1) \( x \mapsto t(x) \) defines a function by uniqueness proved in Proposition 2.1

2) Let \( a > 0 \), we saw that \( \partial_t \tau(a, t(a)) = 0 \), and \( \partial^2_t \tau(a, t(a)) < 0 \). Applying implicit theorem, then there is a neighborhood \( V_a \) of \( a \) and neighborhood \( W_{t(a)} \) of \( t(a) \) and \( C^1 \)-diffeomorphism \( \varphi : V_a \to W_{t(a)} \) such that for every \( x \in V_a \), and \( y \in W_{t(a)} \)
\[ \partial_t \tau(x, y) = 0 \iff y = \varphi(x). \]
By uniqueness \( \varphi(x) = t(x) \).
3) The right inequality \( t(x) < x \) has been proved. To show the left one, it is enough to prove that
\[
\partial_t(x, \frac{(x+1)^2}{2x+3}) \geq 0,
\]
in view of the decay of \( t \mapsto \partial_t \tau(x,t) \) throughout the interval \((0, t(x))\), and the definition of \( t(x) \), and uniqueness the result follows.

Set
\[
\phi(x) = \partial_t \left( x, \frac{(x+1)^2}{2x+3} \right).
\]
By some algebra we get
\[
\phi(x) = \left( x+1 \right)^{2x+2} \frac{(x+2)^{2x+4}}{x(x+2)^{2x+4}} \left( x+1 \right)^{2x+2} - 7x^2 - 21x - 16.
\]

Let
\[
\Phi(x) = 2 \log(x+2) + 2(x+1) \log \left( \frac{x+2}{x+1} \right) - \log(7x^2 + 21x + 16).
\]
Differentiate gives
\[
\Phi'(x) = 2 \log \left( \frac{x+2}{x+1} \right) - \frac{21 + 14x}{16 + 21x + 7x^2},
\]
and
\[
\Phi''(x) = \frac{-35x^2 - 105x - 78}{(1 + x)(2 + x)(16 + 7x(3 + x))^2} < 0.
\]
Since \( \lim_{x \to +\infty} \Phi'(x) = 0 \), hence \( \Phi'(x) > 0 \) for every \( x \geq 1 \). Thus,
\[
\Phi(x) \geq \Phi(1) > \Phi(0) = 4 \log 2 - \log 16 = 0.
\]
Which gives,
\[
\phi(x) > 0,
\]
and the result follows.

**Proposition 2.3.** The function \( x \mapsto \alpha(x) \) is of class \( C^1 \) non decreasing on \([1, +\infty[, \) with derivative
\[
\alpha'(x) = \partial_t \tau(x, t(x)),
\]
and satisfies for every \( x \geq 1 \) the inequality
\[
0 \leq \alpha(x) < \frac{x}{3x + 1}.
\]

**Proof.** 1) Recall that
\[
\alpha(x) = \tau(x, t(x)).
\]
\( \alpha(x) \) is \( C^1 \) as composed of the \( C^1 \) functions \( x \mapsto (x, t(x)) \) and \((x, t) \mapsto f(x, t)\).

By differentiation we get
\[
\alpha'(x) = \partial_1 \tau(x, t(x)) + t'(x) \partial_2 \tau(x, t(x)) = \partial_1 \tau(x, t(x)),
\]
where we used $\partial_2 \tau(x, t(x)) = 0$. Deriving the expression of the function $\tau(x, t)$ with respect to $x$, gives

$$
\partial_1 \tau(x, t(x)) = \frac{t(x) \left( -1 - t(x) + \left( \frac{t(x)}{1 + t(x)} \right)^x \left( 1 + t(x) + x(1 + t(x) + x) \log(1 + \frac{1}{t(x)}) \right) \right)}{(1 + t(x))^2} = (1 + t(x)) \left( -1 + \frac{(1 + t(x))(1 + t(x) + x(1 + t(x) + x) \log(1 + \frac{1}{t(x)}))}{t(x)(1 + t(x)) + (t(x) + x + 1)(x + 1)} \right),
$$

where we used equation (1).

For $0 < u \leq x$, and $x \geq 1$, let’s define

$$
k(u) = -1 + \frac{(1 + u)(1 + u + x(1 + u + x) \log(1 + \frac{1}{u}))}{u(1 + u) + (u + x + 1)(x + 1)}. \quad (4)
$$

First of all

$$
\alpha'(x) = \partial_1 \tau(x, t(x)) = (1 + t(x))k(t(x)).
$$

So, it is enough to show that $k(t(x)) \geq 0$.

Differentiate yields

$$
k'(u) = \frac{x \left( u^2 - (1 + x)^3 - ux(3 + 2x) + ux(1 + x)(2 + 2u + x) \log \left( 1 + \frac{1}{u} \right) \right)}{u(1 + u) + (u + x + 1)(x + 1)},
$$

Using the inequality $\log(1 + 1/u) \leq 1/u$, we get

$$
k'(u) \leq \frac{(1 + u)(u - 1 - x)}{u(1 + u)^2 + u(2 + x)^2} < 0.
$$

Thus $k(u)$ decreases, since $t(x) < x$, hence for every $x \geq 1$

$$
k(t(x)) \geq k(x) = -1 + \frac{(1 + x)(1 + x + x(1 + x + x) \log(1 + \frac{1}{x}))}{x(1 + x) + (x + x + 1)(x + 1)},
$$

Which gives

$$
k(t(x)) \geq -1 + \frac{1 + x + x(1 + 2x) \log(1 + \frac{1}{x})}{3x + 1}. \quad (5)
$$

Set

$$
\Theta(x) = 1 + x + x(1 + 2x) \log(1 + \frac{1}{x}) - 3x - 1,
$$

Differentiate, by straightforward computation it yields

$$
\Theta'(x) = -4 + \frac{1}{1 + x} + (1 + 4x) \log \left( \frac{1 + \frac{1}{x}}{x(1 + x)^2} \right),
$$

$$
\Theta''(x) = \frac{-2x(3 + 2x)}{x(1 + x)^2} + 4 \log \left( \frac{1 + \frac{1}{x}}{x} \right).
$$
\[ \Theta''(x) = \frac{1-x}{x^2(1+x)^3}. \]

For \( x \geq 1 \), \( \Theta''(x) < 0 \), and \( \Theta''(x) \geq \lim_{x \to +\infty} \Theta''(x) = 0 \). Hence \( \Theta'(x) \leq \lim_{x \to +\infty} \Theta'(x) = 0 \). Then \( \Theta \) decreases, moreover, \( \Theta(x) = 1/(6x) + o(1/x) \), then \( \Theta(x) \geq 0 \). By equation (5), one deduces the positivity of \( k(t(x)) \), namely for every \( x \geq 1 \), \( k(t(x)) \geq 0 \). and,

\[ \alpha'(x) = \partial_t \tau(x,t(x)) \geq 0. \]

2) We saw by Proposition 2.1 \( \alpha(x) = \psi(t(x)) \), where

\[ \psi(u) = \frac{(x+1)u}{x^2 + (u+1)^2 + x(u+2)}. \]

Deriving with respect to \( u \), it yields

\[ \psi'(u) = \frac{(1+n)^2 - u^2}{(n^2 + (1+u)^2 + n(2+u))^2} \geq 0, \]

for every \( u \geq x+1 \).

Since, by Lemma 2.2, \( t(x) < x \), and \( \psi(x) = x/(3x+1) \). One deduces that, for every \( x \geq 1 \)

\[ 0 \leq \alpha(x) \leq \frac{x}{3x+1}. \]

**Proposition 2.4.**  (i) The sequence \( t_n \) increases, and the sequence \( \frac{t_n}{n} \) converges to \( \ell \), the unique solution of the equation

\[ e^{\frac{t}{\ell}} - \frac{1}{\ell^2} - \frac{1}{\ell} - 1 = 0. \]

Numerical computation gives \( \ell \approx 0.5577 \).

(ii) The sequence \( \alpha_n \) is bounded. Moreover, \( \alpha_n \) converges to \( \alpha \), where

\[ \alpha = \frac{\ell}{1 + \ell + \ell^2}. \]

Numerically \( \alpha \approx 0.298438 \).

(iii) \( \max_{(n,t) \in \mathbb{N} \times (0,\infty)} \tau(n,t) = \alpha. \)

**Proof.** i)a) Let \( t_{n+1} \) denotes the unique zero of the function \( t \mapsto \partial_t \tau(n+1,t). \). Straightforward computation gives

\[ \partial_t \tau(n,t_{n+1}) = \frac{1}{n} \left( 1 - \left( 1 + \frac{(t_{n+1}+n+1)(n+1)}{t_{n+1}(1+n+1)} \right) \left( \frac{t_{n+1}}{1+t_{n+1}} \right)^{n+1} \right). \]
Furthermore, by equation \( \partial_t \tau(n + 1, t_{n+1}) = 0 \), we get
\[
\left( \frac{t_{n+1}}{1 + t_{n+1}} \right)^{n+2} = \frac{t_{n+1}(1 + t_{n+1})}{t_{n+1}(1 + t_{n+1}) + (t_{n+1} + n + 2)(n + 2)}.
\]
Which implies that
\[
\partial_t \tau(n, t_{n+1}) = \frac{1}{n} \left( 1 - \frac{t_{n+1}(t_{n+1} + 1) + (t_{n+1} + n + 1)(n + 1) t_{n+1} + 1}{t_{n+1}(t_{n+1} + 1) + (t_{n+1} + n + 2)(n + 2) - t_{n+1}} \right).
\]
In other words,
\[
\partial_t \tau(n, t_{n+1}) = - \frac{(1 + n)(1 + n - t_{n+1})}{nt_{n+1} ((2 + n)^2 + (3 + n)t_{n+1} + t_{n+1}^2)} < 0,
\]
in view of \( t_{n+1} \leq n + 1 \).

Using the fact that \( t \mapsto \partial_t \tau(n, t) \) decreases on \((0, n+1)\), \( t_n, t_{n+1} \in (0, n+1) \) and \( \partial_t \tau(n, t_n) = 0 \), we get \( t_n < t_{n+1} \).

b) First of all, by Lemma 2.2, \( (n + 1)^2/n(2n + 3) \leq t_n/n \leq 1 \) and the left hand side is bounded, then \( t_n/n \) is bounded too. Let \( t_{n_k}/n_k \) be a some convergent subsequence, and \( \ell = \lim_{k \to +\infty} t_{n_k}/n_k \in [1/2; 1] \).

Using equation (2) with \( x = n_k \), one deduces that, as \( k \to +\infty \)
\[
e^{-1/\ell(-1 - \ell + (-1 + e^\ell)\ell^2)} = 0,
\]
or if we set \( a = \frac{1}{\ell} \), with \( a \geq 1 \),
\[
e^a = a^2 + a + 1.
\] (6)

Let \( \eta(a) = e^a - a^2 - a - 1 \), then \( \eta'(a) = e^a - 2a - 1 \), and \( \eta''(a) = e^a - 2 > 0 \) for \( a \geq 1 \). Hence, \( \eta' \) increases on \((1, +\infty)\). Since, \( \eta'(1) = e - 3 < 0 \), and \( \eta'(2) = e^2 - 5 > 0 \). Hence there is a unique \( a_0 \in ]1, 2[ \), \( \eta'(a_0) = 0 \), \( a_0 \approx 1.26 \). Remark that \( \eta(1) = e - 3 < 0 \), and \( \eta(a_0) = 2a_0 + 1 - a_0^2 - a_0 - 1 = a_0(1 - a_0) < 0 \). Moreover on \((1, a_0]\) the function \( \eta \) decreases and is strictly negative, and increases on \((a_0, +\infty)\) with \( \lim_{a \to +\infty} \eta(a) = +\infty \). Thus equation (3) admit a unique solution \( x_0 \) in \((a_0, +\infty)\). Which implies that \( \ell \) is the unique limit of a subsequence and the sequence \( t_n/n \) converges to \( \ell \). Numerically \( x_0 \approx 1.793 \), and \( \ell = 1/x_0 \approx 0.5577 \).

b) Form Proposition 2.1 with \( x = n \), \( \alpha_n = \alpha(n) \), and \( t_n = t(n) \),
\[
\alpha_n = \frac{(1 + n)t_n}{n^2 + (1 + t_n)^2 + n(2 + t_n)}.
\]

As \( n \) goes to \( +\infty \), and the fact that \( t_n/n \to \ell \), one gets
\[
\alpha := \lim_{n \to +\infty} \alpha_n = \frac{\ell}{1 + \ell + \ell^2}.
\]
Numerically \( \alpha \approx 0.298438 \).
iii) We saw by Proposition 2.3 that the sequence $\alpha_n$ increases and converges to $\alpha = \sup_n \alpha_n$. First of all, for every $n \geq 1$,

$$\tau(n, t) \leq \max_{t > 0} \tau(n, t) = \alpha_n \leq \alpha.$$ 

Hence, $\max_{(n, t) \in \mathbb{N} \times (0, +\infty)} \tau(n, t)$ is well defined and

$$\max_{(n, t) \in \mathbb{N} \times (0, +\infty)} \tau(n, t) \leq \alpha.$$ 

Moreover,

$$\alpha_n = \tau(n, t_n) \leq \max_{(n, t) \in \mathbb{N} \times (0, +\infty)} \tau(n, t).$$

One deduces that

$$\max_{(n, t) \in \mathbb{N} \times (0, +\infty)} \tau(n, t) = \sup_{n \geq 1} \alpha_n = \alpha.$$ 

Figure 1: Plot of the function $f(a) = e^a - a^2 - a - 1$, and $x_0 = 1.793$.

Figure 2: Plot of the function $\tau(n, t)$, $n = 1, 2, 3$
References

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