Uniform convergence of proliferating particles to the FKPP equation

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Abstract

In this paper we consider a system of Brownian particles with proliferation whose rate depends on the empirical measure. The dependence is more local than a mean field one and has been called moderate interaction by Oelschläger [17, 18]. We prove that the empirical process converges, uniformly in the space variable, to the solution of the Fisher-Kolmogorov-Petrowskii-Piskunov equation. We use a semigroup approach which is new in the framework of these systems and is inspired by some literature on stochastic partial differential equations.

Keywords: Macroscopic limit; particle system with proliferation; FKPP equation; stochastic PDEs; semigroup approach.

MSC Subject Classification: Primary: 60K35, 35K57; Secondary: 60F17, 35K58, 92C17.

1 Introduction

Consider the so called Fisher-Kolmogorov-Petrowskii-Piskunov (FKPP) equation - with all constants equal to 1, which is always possible by suitable rescalings

$$\frac{\partial u}{\partial t} = \Delta u + u(1 - u), \quad u|_{t=0} = u_0. \tag{1}$$

This is a paradigm of equations arising in biology and other fields. For instance, in the mathematical description of cancer growth, although being too simplified to capture several features of true tumors, it may serve to explore mathematical features of diffusion and proliferation. In such applications, it describes a density of cancer cells which diffuse and
proliferate with proliferation modulated by the density itself, such that, starting with an initial density $0 \leq u_0 \leq 1$, the growth due to proliferation cannot exceed the threshold 1. Having in mind this example, it is natural to expect that this equation is the macroscopic limit of a system of microscopic particles, like cancer cells, which are subject to proliferation. To be biologically realistic, we have to require that the proliferation rate is not uniform among particles but depends on the concentration of particles: wherever particles are more concentrated, there is less space and more competition for nutrients, which slows down proliferation. We prove a result of convergence of such kind of proliferation particle systems - as described in detail in section 1.2 below - to the FKPP equation. A key point of the microscopic model that should be known in advance, to understand this introduction, is that the proliferation rate of particle “a” (see below the meaning of this index) is given by the random time-dependent rate

$$\lambda_{a,N}^t = \left(1 - (\theta_N \ast S_N^t) \left(X_{a,N}^t\right)\right)^+, \tag{2}$$

where $N$ is the number of initial particles, $X_{a,N}^t$ is the position of particle “$a$”, $S_N^t$ is the empirical measure, $\theta_N$ is a family of smooth mollifiers - hence $\theta_N \ast S_N^t$ is a smoothed version of the empirical density. Formula (2) quantifies the fact that proliferation is slower when the empirical measure is more concentrated, and stops above a threshold. Since there is no reason why the mollified empirical measure $\theta_N \ast S_N^t$ is smaller than one, we have to cut with the positive part, in (2). Hence, initially the limit PDE will have the proliferation term $u(1 - u)^+$, which is meaningful also for $u > 1$, but by a uniqueness result, the term reduces to $u(1 - u)$ when $0 \leq u_0 \leq 1$.

The final result is natural and expected but there is a technical difficulty which, in our opinion, is not sufficiently clarified in the literature. The proof of convergence of the particle system to the PDE relies on the tightness of the empirical measure and a passage to the limit in the identity satisfied by the empirical measure. This identity includes the nonlinear term

$$\left\langle (1 - \theta_N \ast S_N^t)^+ S_N^t, \phi \right\rangle$$

where $\phi$ is a smooth test function. Since $S_N^t$ converges only weakly, it is required that $\theta_N \ast S_N^t$ converges uniformly, in the space variable, in order to pass to the limit. Maybe in special cases one can perform special tricks but the question of uniform convergence is a natural one in this problem and it is also of independent interest, hence we investigate when it holds true.

Following the proposal of K. Oelschl"ager [17], [18], we assume

$$\theta_N(x) = N^{\beta} \theta \left(\frac{N^\beta}{d} x\right). \tag{3}$$

Recall that the case $\beta = 0$ is the mean field one (long range interaction), the case $\beta = 1$ corresponds to local (like nearest neighbor) interactions, while the case $0 < \beta < 1$
corresponds to an intermediate regime, called "moderate" by [17]. Our main result is that uniform convergence of \( \theta_N \ast S_t^N \) to \( u \) holds under the condition \( \beta < \frac{1}{2} \).

In addition to our main result, Theorem 1, see also Appendix 7 where we show that this condition arises with other proofs of uniform convergence. We believe this condition is strict for the uniform convergence. A second motivation for the analysis of uniform convergence, besides the problem of passage to the limit in the nonlinear term outlined above, is the question whether a "front" of microscopic particles which moves due to proliferation approximates the traveling waves of FKPP equation. Results in this direction seem to be related to uniform convergence of mollified empirical measure but they require also several other ingredients and go beyond the scope of the present paper, hence they are not discussed here.

1.1 Comparison with related problems and results

First, let us clarify that the problem treated here is more correct and difficult than a two-step approach which does not clarify the true relation between the particle system and the PDE, although it gives a plausible indication of the link. The two-step approach freezes first the parameter in the mollifier, namely it treats particles proliferating with rate

\[
\lambda_t^{a,N_0,N} = \left( 1 - \left( \theta_{N_0} \ast S_t^{N_0} \right) \left( X_t^{a,N_0,N} \right) \right)^+.
\]

and proves that \( S_t^{N_0,N} \) weakly converges as \( N \to \infty \), to the solution \( u_{N_0} \) of the following equation with non-local proliferation

\[
\frac{\partial u_{N_0}}{\partial t} = \Delta u_{N_0} + u_{N_0} (1 - \theta_{N_0} \ast u_{N_0})^+.
\]  

The second step consists in proving that \( u_{N_0} \) converges to the solution \( u \) of the FKPP equation. The link between the particle system \( X_t^{a,N_0,N} \) and the solution \( u \) of the FKPP equation is only conjectured by this approach. In principle the conjecture could be even wrong. Take a system of particle interactions with short range couplings, where the two-steps approach leads to the porous media equation with the non-linearity \( \Delta u^2 \) (see [20]). But a direct link between the particle system and the limit PDE (the so called hydrodynamic limit problem) leads to a non-linearity of the form \( \Delta f(u) \) where \( f(u) \) is not necessarily \( u^2 \) (see [27], [25]). For a proof of the mean field result of convergence of \( S_t^{N_0,N} \) to \( u_{N_0} \) as \( N \to \infty \), see for instance [6], [10]. The issue of uniform convergence of \( \theta_N \ast S_t^N \) to \( u \) does not arise and weak convergence of the measures \( S_t^{N_0,N} \) is sufficient.

Going back to the problem with the rates [2], K. Oelschl¨ ager papers [17], [18] have been our main source of inspiration. Our attempt in the present work is to clarify a result
of convergence in the case of diffusion and proliferation under assumptions comparable to those of [17], [18] but possibly with some additional degree of generality and with a new proof.

We have extended the assumption \( \beta < \frac{d}{(d+1)(d+2)} \) and removed the restriction \( V = W \ast W \) of [18] and, hopefully, we have given a modern proof which clarifies certain issues of the tightness and the convergence problem. Concerning extensions of the range of \( \beta \), maybe there are other directions, as remarked in [18], page 575; our specific extension is however motivated not only by the generality but also by the property of uniform convergence (not proved in [18]), which seems relevant in itself.

Other interesting works related to the problem of particle approximation of FKPP equation are [14], [13], [15], [16], [23] and [1], [3] from the more applied literature. For the FKPP limit of discrete lattice systems, even the more difficult question of the hydrodynamic limit has been solved, see [8] with completely local interaction, but the analogous problem for diffusions is more difficult and has not been done.

To solve the problem of uniform convergence, we propose a new approach, by semigroup theory. Traces of this approach can be found in [15] and [6], but have been used for other purposes. In the work [10] it is remarked that uniform convergence can be obtained as a by-product of energy inequalities and Sobolev convergence, under the assumption \( \beta < \frac{d}{d+2} \), but only in dimension \( d = 1 \), where the condition is more restrictive than \( \beta < 1/2 \).

The approach extends to other models, in particular with interactions. With the same technique, under appropriate assumptions on the convolution kernels \( \theta_N \) below, we may recover a result, under different assumptions, of [17], where the macroscopic PDE has the form

\[
\frac{\partial u}{\partial t} = \Delta u - \text{div} (uF(u)) + u (1 - u), \quad u|_{t=0} = u_0
\]

and \( F \) is a local nonlinear function, not a non-local operator as in mean field theories.

Let us insist on the fact that our proliferation rate is natural from the viewpoint of Biology. It is very different from the constant rate used in the probabilistic formulae used by McKean and others to represent solutions of the FKPP equations; these formula have several reasons of interest but do not have a biological meaning - constant proliferation rate would lead to exponential blow-up of the number of particles. Constant rates do not pose the difficulties described above in taking the limit in the nonlinear term. Approximation by finite systems of these representation formula therefore pose different problems. For this and other directions, different from our one, see [12], [21] and references therein.

1.2 The microscopic model

We consider a particle system on filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, P)\) with \( N \in \mathbb{N} \) initial particles. We label particles by \( a \in \Lambda^N \), where

\[
\Lambda^N = \{(k, i_1, ..., i_n) : i_1, ..., i_n \in \{1, 2\}, k = 1, ..., N, n \in \mathbb{N}_0\}
\]
is the set of all particles. For a non-initial particle \( a = (k, i_1, \ldots, i_n) \) we denote its parent particle by \((a, -) = (k, i_1, \ldots, i_{n-1})\). Each particle has a lifetime, which is the random time interval \( I^{a,N} = [T_0^{a,N}, T_1^{a,N}] \subset [0, \infty) \), where \( T_0^{a,N}, T_1^{a,N} \) are \( \mathcal{F}_t \)-stopping times. We have \( T_0^{a,N} = 0 \) for initial particles \( a = (k), k = 1, \ldots, N \) and \( T_0^{a,N} = T_1^{(a,-),N} \) for other particles. The time \( T_1^{a,N} \) at which a particle dies and splits into two (we call this a proliferation event) is described more precisely below.

Particles are born at the position their parent died, i.e. \( X^{a,N}_{T_0^{a,N}} = X^{(a,-),N}_{T_1^{(a,-),N}} \) with the convention \( X^{a,N}_{T_1^{a,N}} := \lim_{t \uparrow T_1^{a,N}} X^{a,N}_t \). During its lifetime the position of \( a \in \Lambda^N \), \( X^{a,N}_t \in \mathbb{R}^d \), is given by

\[
    dX^{a,N}_t = \sqrt{2} dB^a_t
\]

where \( B^a_t \) are independent Brownian motions in \( \mathbb{R}^d \).

Let \( \Lambda^N_t \) denote the set of all particles alive at time \( t \). We define the empirical measure as

\[
    S^N_t = \frac{1}{N} \sum_{a \in \Lambda^N_t} \delta_{X^{a,N}_t}.
\]

Take a family of standard Poisson processes \( (\mathcal{N}^{0,a})_{a \in \Lambda^N} \) which is independent of the Brownian motion and the initial condition \( X^{(k),N}_0, k = 1, \ldots, N \). The branching time \( T_1^{a,N} \) of particle \( a \in \Lambda^N \) is the first (and only) jump time of \( \mathcal{N}^{a,N}_t := \mathcal{N}^{0,a}_{\Lambda^N_t} \), where \( \Lambda^N_t = \int_0^t 1_{s \in I^{a,N}} \lambda^{a,N}_s \, ds \) and the random rate \( \lambda^{a,N}_t \) is given by

\[
    \lambda^{a,N}_t = \left( 1 - (\theta_N * S^N_t) \left( X^{a,N}_t \right) \right)^+
\]

where

\[
    \theta_N(x) = \epsilon_N^{-d} \theta \left( \epsilon_N^{-1} x \right)
\]

is a family of mollifiers with \( \epsilon_N = N^{-\frac{d}{2}} \), namely we assume (3).

### 1.3 Assumptions and main result

Throughout this paper we assume that

\[
    \beta \in \left( 0, \frac{1}{2} \right)
\]

and that \( \theta : \mathbb{R}^d \to \mathbb{R} \) is a probability density of class

\[
    \theta \in W^{\alpha_0,2} \left( \mathbb{R}^d \right) \text{ for some } \alpha_0 \in \left( \frac{d}{2}, \frac{d(1 - \beta)}{2\beta} \right)
\]
(notice that, for $\beta > 0$, the inequality $\frac{d}{2} < \frac{d(1-\beta)}{2\beta}$ is equivalent to $\beta < \frac{1}{2}$). The weaker assumption $\beta = 1$ corresponds to nearest-neighbor (or contact) interaction and it is just the natural scaling to avoid that the kernel is more concentrated than the typical space around a single particle, when the particles are uniformly distributed. The case $\beta = 0$ corresponds to mean field interaction. The explanation for condition (7) is given at the beginning of Section 4. At the biological level it means that the modulation of proliferation by the local density of cells is not completely local, but has a certain range of action, which is less than long range as a mean field model.

Let us introduce the mollified empirical measure (the theoretical analog of the numerical method of kernel smoothing) $h^N_t$ defined as

$$h^N_t(x) = (\theta^{N*}_N * S^N_t)(x).$$

Concerning the initial condition, assume that $u_0 \in L^1(\mathbb{R}^d)$, $0 \leq u_0(x) \leq 1$, $u_0$ is uniformly continuous and $S^N_0$ converges weakly to $u_0(x) dx$, as $N \to \infty$, in probability. Moreover, assume that for some $\rho_0 \geq \alpha_0 - 1$

$$\sup_N E \left[ \int_{\mathbb{R}^d} \left| (I - A)_{\rho_0/2} h^N_0(x) \right|^2 dx \right] < \infty. \quad (9)$$

When the initial positions $X^i_0$, $i = 1, ..., N$, are independent identically distributed with common probability density $u_0 \in W^{\rho_0, 2}(\mathbb{R}^d)$, with $\alpha_0 - 1 \leq \rho_0 \leq \alpha_0$, this condition is satisfied, see Proposition 14 below. Finally, the definition of weak solution of the PDE (1) is given below in Section 6.3.

**Theorem 1** Assume that $S^N_0$ converges weakly to $u_0(x) dx$, as $N \to \infty$, in probability, where $u_0$ satisfies the assumptions above. Further, assume (7), (8) and (9). Then, for every $\alpha \in (d/2, \alpha_0)$, the process $h^N_t$ converges in probability in the

- weak star topology of $L^\infty(0, T; L^2(\mathbb{R}^d))$,
- weak topology of $L^2(0, T; W^{\alpha, 2}(\mathbb{R}^d))$
- strong topology of $L^2(0, T; W^{\alpha, 2}_{\text{loc}}(\mathbb{R}^d))$

as $N \to \infty$, to the unique weak solution of the PDE (1).

Note that the topology of convergences of $h^N_t$ includes the convergence in $L^2(0, T; C(D))$ for every regular bounded domain $D \subset \mathbb{R}^d$. The notion of weak solution is given by Definition 7.
2 Preparation

2.1 Analytic Semigroup and Sobolev Spaces

For every \( \alpha \in \mathbb{R} \), the Sobolev spaces \( W^{\alpha,2}(\mathbb{R}^d) \) are well defined, see [24] for the material recalled here. For positive \( \alpha \) the restriction of \( f \in W^{\alpha,2}(\mathbb{R}^d) \) to a ball \( B(0,R) \) is in \( W^{\alpha,2}(B(0,R)) \). The family of operators, for \( t \geq 0 \),

\[
(e^{tA}) = \int_{\mathbb{R}^d} \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x-y|^2}{4t}} f(y) \, dy
\]
defines an analytic semigroup in each space \( W^{\alpha,2}(\mathbb{R}^d) \). With little abuse of notation, we write \( e^{tA} \) for each value of \( \alpha \). The infinitesimal generator, say in \( L^2(\mathbb{R}^d) \), is the operator \( A : D(A) \subset L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d) \) defined as \( Af = \Delta f \). Fractional powers \( (I - A)^{\beta} \) are well defined for every \( \beta \in \mathbb{R} \) and \( \| (I - A)^{\alpha/2} f \|_{L^2(\mathbb{R}^d)} \) is equivalent to the norm in \( W^{\alpha,2}(\mathbb{R}^d) \).

Recall also that (see [19]), for every \( \beta > 0 \), and given \( T > 0 \), there is a constant \( C_{\beta,T} \) such that

\[
\left\| (I - A)^{\beta} e^{tA} \right\|_{L^2 \rightarrow L^2} \leq \frac{C_{\beta,T}}{t^\beta}
\]
for \( t \in (0, T] \).

2.2 Equation for the empirical measure and its mild formulation

Starting from this section, we drop the suffix \( N \) in \( X^a,N \), \( I^a,N \), \( T^a,N \), \( \lambda^a,N \), \( N^a,N \) to simplify notations. Let \( \delta \) denote a point outside \( \mathbb{R}^d \), the so called grave state, where we assume the processes \( X^a \) live when \( t \notin I^a \). Hence, whenever a particle proliferates and therefore dies, it stays forever in the grave state \( \delta \). In the sequel, the test functions \( \phi \) are assumed to be defined over \( \mathbb{R}^d \setminus \{ \delta \} \) and be such that \( \phi(\delta) = 0 \). Using Itô formula over random time intervals, one can show that \( \phi(X^a_t) \), with \( \phi \in C^2(\mathbb{R}^d) \), satisfies

\[
\phi(X^a_t) = \phi(X^a_{T^a_0}) 1_{t \geq T^a_0} - \phi(X^a_{T^a_1}) 1_{t > T^a_1} + \sqrt{2} \int_0^t 1_{s \in I^a} \nabla \phi(X^a_s) \, dB^a_s + \int_0^t 1_{s \in I^a} \Delta \phi(X^a_s) \, ds.
\]

With a few computations, one can see that the empirical measure \( S^N_t \) satisfies

\[
d \langle S^N_t, \phi \rangle = \langle S^N_t, \Delta \phi \rangle + \left( (1 - h^N_t)^+ \right) S^N_t, \phi \rangle dt + dM^1_t,\phi,N + dM^2_t,\phi,N
\]
for every \( \phi \in C^2(\mathbb{R}^d) \) and where

\[
M^1_t,\phi,N := \frac{\sqrt{2}}{N} \sum_{a \in A^N} \int_0^t 1_{s \in I^a} \nabla \phi(X^a_s) \cdot dB^a_s,
\]

\[
M^2_t,\phi,N := \frac{1}{N} \sum_{a \in A^N} \phi(X^a_{T^a_0}) 1_{t \geq T^a_0} - \frac{1}{N} \sum_{a \in A^N} \int_0^t \phi(X^a_s) \lambda^a_s \, ds.
\]
We deduce that \( h_t^N(x) \) satisfies
\[
d h_t^N(x) = \Delta h_t^N(x) \, dt + \left( \theta_N \ast \left( (1 - h_t^N)^+ S_t^N \right) \right)(x) \, dt + dM_{1,N}^t(x) + dM_{2,N}^t(x),
\]
where
\[
M_{1,N}^t(x) := -\frac{\sqrt{2}}{N} \sum_{a \in \Lambda^N} \int_0^t 1_{s \in I^a} \nabla \theta_N(x - X_s^a) \cdot dB_s^a,
\]
\[
M_{2,N}^t(x) := \frac{1}{N} \sum_{a \in \Lambda^N} \theta_N(x - X_{T_1}^a) 1_{t \geq T_1} - \frac{1}{N} \int_0^t \sum_{a \in \Lambda^N} \theta_N(x - X_s^a) \lambda_s^a \, ds
\]
\[
= \frac{1}{N} \sum_{a \in \Lambda^N} \int_0^t \theta_N(x - X_s^a) \, d(\mathcal{N}_s^a - \Lambda_s^a).
\]

Following a standard procedure, used for instance by [7], we may rewrite this equation in mild form:
\[
h_t^N = e^{tA} h_0^N + \int_0^t e^{(t-s)A} \left( \theta_N \ast \left( (1 - h_s^N)^+ S_s^N \right) \right) ds + \int_0^t e^{(t-s)A} dM_{1,N}^s + \int_0^t e^{(t-s)A} dM_{2,N}^s.
\]

(11)

This opens the possibility of a semigroup approach, which is a main novelty of this paper.

2.3 Total mass and useful inequalities

The total relative mass
\[
[S_t^N] := S_t^N(R^d) = \langle S_t^N, 1 \rangle = \frac{\text{Card}(A_t^N)}{N}
\]
plays a central role. Since, in our model, the number of particles may only increase, we have

\[
[S_t^N] \leq [S_T^N] \quad \text{for all } t \in [0, T].
\]

(12)

The quantity \([S_T^N]\) is, moreover, exponentially integrable, uniformly in \(N\), see Lemma 13 below. We also repeatedly use the identity
\[
\int_{\mathbb{R}^d} h_t^N(x) \, dx = [S_t^N],
\]
which follows from Fubini theorem. Another simple rule of calculus we often use is
\[
| \left( \theta_N \ast (f S_t^N) \right)(x) | \leq \| f \|_{\infty} h_t^N(x)
\]
for every bounded measurable \( f : \mathbb{R}^d \rightarrow \mathbb{R} \). Moreover, since \( h_s^N \geq 0 \), we have
\[
(1 - h_s^N(x))^+ \in [0, 1].
\]

(15)
Finally, we often use the inequality

$$\frac{1}{N} \int_{\mathbb{R}^d} |\theta_N(x)|^2 \, dx \leq C, \quad (16)$$

which holds with a suitable constant $C > 0$. Indeed, it holds

$$\frac{1}{N} \int_{\mathbb{R}^d} |\theta_N(x)|^2 \, dx = \frac{\epsilon_N^{-d}}{N} \int_{\mathbb{R}^d} \epsilon_N^{-d} |\theta(\epsilon_N^{-1} x)|^2 \, dx = \frac{\epsilon_N^{-d}}{N} \int_{\mathbb{R}^d} |\theta(x)|^2 \, dx.$$

Inequality (16) follows from the assumptions $\theta \in L^2(\mathbb{R}^d)$ and $\sup_N \epsilon_N^{-d}/N < \infty$.

### 3 Main estimates on martingale terms

Let $\alpha \in (d/2, \alpha_0)$, as in the statement of Theorem 11.

**Lemma 2** There exists a constant $C' > 0$ such that for all $N \in \mathbb{N}$, $t \in [0, T]$, small $h > 0$

$$\left\| \int_0^t (I - A)^{\alpha/2} e^{(t+h-s)A} dM_s^{1,N} \right\|_{L^2(\Omega \times \mathbb{R}^d)} \leq C'.$$

**Proof.**

$$\left\| \int_0^t (I - A)^{\alpha/2} e^{(t+h-s)A} dM_s^{1,N} \right\|_{L^2(\Omega \times \mathbb{R}^d)}^2$$

$$= \frac{2}{N^2} \int_{\mathbb{R}^d} E \left[ \sum_{a \in \Lambda^N} \int_0^t \left( (I - A)^{\alpha/2} e^{(t+h-s)A} 1_{s \in I^a} \nabla \theta_N (\cdot - X_s^a) \right)(x) \cdot dB_s^a \right]^2 \, dx$$

$$= \frac{2}{N^2} \int_{\mathbb{R}^d} E \left[ \sum_{a \in \Lambda^N} \int_0^t \left( (I - A)^{\alpha/2} e^{(t+h-s)A} 1_{s \in I^a} \nabla \theta_N (\cdot - X_s^a) \right)(x) \right]^2 \, ds \, dx$$

$$= \frac{2}{N^2} E \left[ \sum_{a \in \Lambda^N} \int_0^t 1_{s \in I^a} \left( \int_{\mathbb{R}^d} \left( (I - A)^{\alpha/2} e^{(t+h-s)A} \nabla \theta_N (\cdot - X_s^a) \right)(x) \right)^2 \, dx \right] \, ds.$$

We have

$$\left( (I - A)^{\alpha/2} e^{(t+h-s)A} \nabla \theta_N (\cdot - X_s^a) \right)(x) = \left( (I - A)^{\alpha/2} e^{(t+h-s)A} \nabla \theta_N \right)(x - X_s^a).$$

Then, by change of variable,

$$\int_{\mathbb{R}^d} \left( (I - A)^{\alpha/2} e^{(t+h-s)A} \nabla \theta_N (\cdot - X_s^a) \right)(x)^2 \, dx$$

$$= \int_{\mathbb{R}^d} \left( (I - A)^{\alpha/2} e^{(t+h-s)A} \nabla \theta_N \right)(x)^2 \, dx.$$
Therefore, since \( \frac{1}{N} \sum_{a \in A^N} 1_{s \in I^a} = [S^N_s] \leq [S^N_T] \),
\[
\left\| \int_0^t (I - A)^{\alpha/2} e^{(t+h-s)A} dM^1_{s,N} \right\|_{L^2(\Omega \times R^d)}^2
= \frac{2}{N} E \left[ \int_0^t \left( \frac{1}{N} \sum_{a \in A^N} 1_{s \in I^a} \right) \left( \int_{\Omega \times R^d} \left\| (I - A)^{\alpha/2} e^{(t+h-s)A} \nabla \theta_N \right\| dx \right) ds \right]
\leq \frac{2}{N} E \left( [S^N_T] \right) \int_0^t \left\| (I - A)^{\alpha/2} e^{(t+h-s)A} \nabla \theta_N \right\|_{L^2}^2 ds.
\]

From assumption [S] and the condition \( \alpha \in (d/2, \alpha_0) \), we have \( \frac{2}{\alpha} (2\alpha + \varepsilon + d) \leq 1 \) and at the same time \( \alpha + \frac{\varepsilon}{2} \leq \alpha_0 \). Denoting by \( C > 0 \) any constant independent of \( N \) and recalling that \( \varepsilon_N = N^{-\frac{\alpha}{2}} \), we have
\[
\leq \frac{C}{N} \int_0^t \left\| (I - A)^{1-\varepsilon/2} e^{(t-s)A} \right\|_{L^2 \rightarrow L^2}^2 \left\| \nabla (I - A)^{-1/2} e^{hA} \theta_N \right\|_{L^2}^2 ds
\leq \frac{C}{N} \| \theta_N \|_{W^{\alpha + \varepsilon/2, 2}}^2 \int_0^t \frac{1}{(t-s)^{1-\varepsilon/2}} ds \leq C \frac{e^{-2\alpha - \varepsilon - d}}{N} \leq C
\]
where we have used Lemma [15] below. 

**Lemma 3** There exists a constant \( C > 0 \) such that for all \( N \in \mathbb{N}, t \in [0, T] \), small \( h > 0 \)
\[
\left\| \int_0^t (I - A)^{\alpha/2} e^{(t+h-s)A} dM^2_{s,N} \right\|_{L^2(\Omega \times R^d)} \leq C.
\]

**Proof.** Since
\[
M^2_{t,2,N} = \frac{1}{N} \sum_{a \in A^N} \int_0^t \theta_N (x - X^a_{s-}) d (\Lambda_{s-}^a - \Lambda_{s-})
\]
we have
\[
\left\| \int_0^t (I - A)^{\alpha/2} e^{(t+h-s)A} dM^2_{s,2,N} \right\|_{L^2(\Omega \times R^d)}^2
= \frac{1}{N^2} \int_{\Omega \times R^d} E \left[ \sum_{a \in A^N} \int_0^t \left( (I - A)^{\alpha/2} e^{(t+h-s)A} dM^2_{s,2,N} \right) \nabla \theta_N (x - X^a_{s-}) d (\Lambda_{s-}^a - \Lambda_{s-}) \right] dx.
\]
Write \( g_{t,s,h}^a (X^a_{s-}) \) for \( (I - A)^{\alpha/2} e^{(t+h-s)A} dM^2_{s,2,N} \nabla \theta_N (x - X^a_{s-}) \). Since the jumps of \( \Lambda_{s-}^a \) and \( \Lambda_{s-}^{a'} \), for \( a \neq a' \), never occur at the same time, we have
\[
E \left[ \left( \int_0^t g_{t,s,h}^a (X^a_{s-}) d (\Lambda_{s-}^a - \Lambda_{s-}^{a'}) \right) \left( \int_0^t g_{t,s,h}^{a'} (X^{a'}_{s-}) d (\Lambda_{s-}^{a'} - \Lambda_{s-}^{a',N}) \right) \right] = 0.
\]
Hence the last expression is equal to

\[
= \frac{1}{N^2} \sum_{a \in \Lambda^N} \int_{\mathbb{R}^d} E \left[ \left| \int_0^t g_{t,s,h}^{a,N} (X_s^a) \cdot d (N_s^a - \Lambda_s^a) \right|^2 \right] dx.
\]

It is known that

\[
E \left[ \left| \int_0^t g_{t,s,h}^{a,N} (X_s^a) \cdot d (N_s^a - \Lambda_s^a) \right|^2 \right] = E \left[ \left| \int_0^t g_{t,s,h}^{a,N} (X_s^a) \right|^2 d\Lambda_s^a \right].
\]

Therefore, the last expression simplifies to

\[
= \frac{1}{N^2} \sum_{a \in \Lambda^N} \int_{\mathbb{R}^d} E \left[ \left| \int_0^t g_{t,s,h}^{a,N} (X_s^a) \right|^2 \lambda_s^a ds \right] dx
\]

\[
= \frac{1}{N^2} \sum_{a \in \Lambda^N} E \left[ \int_0^t \left( \int_{\mathbb{R}^d} \left| (I - A)^{\alpha/2} e^{(t+h-s)A} 1_{s \in I^a} \nabla \theta_N (\cdot - X_s^a) \right| (x) \right|^2 dx \right] \lambda_s^a ds .
\]

As in the previous proof, and taking into account the boundedness of \( \lambda_s^a \) (by definition),

\[
= \frac{1}{N^2} \sum_{a \in \Lambda^N} E \left[ \int_0^t \left( \int_{\mathbb{R}^d} \left| (I - A)^{\alpha/2} e^{(t+h-s)A} 1_{s \in I^a} \nabla \theta_N \right| (x) \right|^2 dx \right] \lambda_s^a ds
\]

\[
\leq \frac{1}{N} E \left[ \int_0^t \left( \frac{1}{N} \sum_{a \in \Lambda^N} 1_{s \in I^a} \right) \left\| (I - A)^{\alpha/2} e^{(t+h-s)A} \nabla \theta_N \right\|_{L^2}^2 ds \right]
\]

\[
\leq \frac{1}{N} E \left( \left[ S_T^N \right] \right) \int_0^t \left\| (I - A)^{\alpha/2} e^{(t+h-s)A} \nabla \theta_N \right\|_{L^2}^2 ds .
\]

This is the same expression as in the previous proof, which is bounded by a constant, uniformly in \( N \). ■

4 Main estimate on \( h_t^N \)

As described above, we need an estimate on \( h_t^N \) in a Hölder norm (in space) which we gain by Sobolev embedding theorem. Since we work in an \( L^2 \)-setting (computations not reported here in the \( L^p \) setting do not help since they re-introduce difficulties from other sides), we have

\[
W^{\alpha,2} \left( \mathbb{R}^d \right) \subset C_0^\varepsilon \left( \mathbb{R}^d \right) \quad \text{if } (\alpha - \varepsilon) 2 \geq d.
\]

This is the reason for the restriction on \( \alpha \), namely \( 2\alpha > d \). Recall that \( \alpha_0 \) and \( \rho_0 \) were introduced in (8) and (9) respectively.
Lemma 4 Assume $\alpha \in (d/2, \alpha_0)$. Then there exist constants $C, C' > 0$ such that for all $N \in \mathbb{N}, t \in (0, T]$

$$\|h_t^N\|_{L^2(\Omega; W^{\alpha, 2}(\mathbb{R}^d))} \leq C \mathcal{E} \left[ \| (I - A)^{\frac{\alpha}{2}} h_t^N \|_{L^2(\mathbb{R}^d)}^2 \right]^{1/2} \leq C' \left( 1 + \frac{1}{t^{(\alpha - \rho_0)/2}} \right).$$

Proof. The first inequality follows from the fact that the two norms are equivalent. From the mild formulation (11) we have

$$\|\cdot\|_{W^{\alpha, 2}(\mathbb{R}^d)} \text{ and } \| (I - A)^{\frac{\alpha}{2}} \|_{L^2(\mathbb{R}^d)}$$

are equivalent. From the mild formulation (11) we have

$$\left\| (I - A)^{\alpha/2} e^{hA} h_t^N \right\|_{L^2(\Omega \times \mathbb{R}^d)} \leq (I - A)^{\alpha/2} e^{(t+h)A} h_0^N \right\|_{L^2(\Omega \times \mathbb{R}^d)}$$

$$+ \int_0^t \left\| (I - A)^{\alpha/2} e^{(t+h-s)A} \left( \theta_N \ast \left( (1 - h_s^N) + S_s^N \right) \right) \right\|_{L^2(\Omega \times \mathbb{R}^d)} ds$$

$$+ \int_0^t (I - A)^{\alpha/2} e^{(t+h-s)A} dM_s^{1,N} \right\|_{L^2(\Omega \times \mathbb{R}^d)} + \int_0^t (I - A)^{\alpha/2} e^{(t+h-s)A} dM_s^{2,N} \right\|_{L^2(\Omega \times \mathbb{R}^d)}.$$

The last two terms are bounded by a constant, by Lemmata 2 and 3. For the first term, where $C > 0$ is a constant that may change from instance to instance, we have

$$\left\| (I - A)^{\alpha/2} e^{(t+h)A} h_0^N \right\|_{L^2(\Omega \times \mathbb{R}^d)} \leq \left\| (I - A)^{\alpha-\rho_0/2} e^{(t+h)A} \right\|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} \| (I - A)^{\rho_0/2} h_0^N \|_{L^2(\Omega \times \mathbb{R}^d)}$$

$$\leq \frac{C}{(t + h)^{(\alpha-\rho_0)/2}}$$

where we have used assumption (9). About the second one,

$$\int_0^t \left\| (I - A)^{\alpha/2} e^{(t+h-s)A} \left( \theta_N \ast \left( (1 - h_s^N) + S_s^N \right) \right) \right\|_{L^2(\Omega \times \mathbb{R}^d)} ds$$

$$\leq \int_0^t \left\| e^{(t-s)A} \right\|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} \left\| (I - A)^{\alpha/2} e^{hA} \left( \theta_N \ast \left( (1 - h_s^N) + S_s^N \right) \right) \right\|_{L^2(\Omega \times \mathbb{R}^d)} ds.$$

Since the operator $f \mapsto (I - A)^{\alpha/2} e^{hA} f$ is positive on $L^2(\mathbb{R}^d)$, see Lemma 16, it holds $(I - A)^{\alpha/2} e^{hA} f \leq (I - A)^{\alpha/2} e^{hA} g$ if $f \leq g$. Because of

$$0 \leq \left( \theta_N \ast \left( (1 - h_s^N) + S_s^N \right) \right) (x) \leq h_s^N (x),$$

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we deduce
\[ \theta_N \ast \left( (1 - h^N_s)^+ S^N_s \right) \leq (I - A)^{\alpha/2} e^{hA} h^N_s. \]

Hence,
\[
\int_0^t \left\| (I - A)^{\alpha/2} e^{(t+h-s)A} \left( \theta_N \ast \left( (1 - h^N_s)^+ S^N_s \right) \right) \right\|_{L^2(\Omega \times \mathbb{R}^d)} ds \\
\leq C \int_0^t \left\| (I - A)^{\alpha/2} e^{hA} h^N_s \right\|_{L^2(\Omega \times \mathbb{R}^d)} ds.
\]

Until now we have proved
\[
\left\| (I - A)^{\alpha/2} e^{hA} h^N_t \right\|_{L^2(\Omega \times \mathbb{R}^d)} \leq \frac{C}{(t + h)^{\frac{(\alpha - \rho_0)/2}{2}}} + C \int_0^t \left\| (I - A)^{\alpha/2} e^{hA} h^N_s \right\|_{L^2(\Omega \times \mathbb{R}^d)} ds + C.
\]

By Gronwall’s lemma we deduce
\[
\left\| (I - A)^{\alpha/2} e^{hA} h^N_t \right\|_{L^2(\Omega \times \mathbb{R}^d)} \leq \frac{C}{(t + h)^{\frac{(\alpha - \rho_0)/2}{2}}} + C.
\]

We may now take the limit as \( h \to 0 \). The proof is complete. ■

**Remark 5** The result is true also for \( \alpha = 0 \):
\[
\sup_{t \in [0,T]} E \left[ \left\| h^N_t \right\|_{L^2(\mathbb{R}^d)}^2 \right] \leq C. \tag{17}
\]

### 5 Other estimates on \( h^N_t \)

In order to show tightness of the family of the functions \( \{ h^N \}_N \), in addition to the previous bound which shows a regularity in space, we also need a regularity in time. See the compactness criteria below.

**Lemma 6** Given any \( \gamma \in (0, 1/2) \), it holds
\[
\lim_{R \to \infty} \sup_{N \in \mathbb{N}} P \left( \int_0^T \int_0^T \frac{\left\| h^N_t - h^N_s \right\|_{W^{-2,2}}^2}{|t-s|^{1+2\gamma}} ds dt > R \right) = 0.
\]

**Proof. Step 1.** We need to estimate \( \| h^N_t - h^N_s \|_{W^{-2,2}} \) in such a way that it cancels with the singularity in the denominator at \( t = s \). Notice that \( L^2 \subset W^{-2,2} \) with continuous embedding, namely there exists a constant \( C > 0 \) such that \( \| f \|_{W^{-2,2}} \leq C \| f \|_{L^2} \) for all
of them. Accordingly, we split the estimate of $\mathcal{M}_{t}^{1,N} - \mathcal{M}_{s}^{1,N}$ and now using (14)

\[2\] $\mathcal{M}_{t}^{2,N} - \mathcal{M}_{s}^{2,N}\] in three more elementary estimates, that now we handle separately; the final result will be a consequence of them.

The number $C_{\gamma} = \int_{0}^{T} \int_{0}^{T} \frac{1}{t-s} ds dt$ is finite, hence the first addend is bounded by (renaming the constant $C$)

\[4\]

and both these terms are, uniformly in $N$, small for large $R$, due to Lemma 13 and estimate (17).

Step 2. Concerning the martingale terms, we now prove that

$E \left\| \mathcal{M}_{t}^{1,N} - \mathcal{M}_{s}^{1,N} \right\|_{L^{2}}^{2} \leq C |t-s|$

and

$E \left\| \mathcal{M}_{t}^{2,N} - \mathcal{M}_{s}^{2,N} \right\|_{L^{2}}^{2} \leq C |t-s|$
for some constant $C > 0$. By Chebyshev’s inequality it follows that

$$\lim_{R \to \infty} \sup_{N \in \mathbb{N}} P \left( \int_0^T \int_0^T \frac{\|M_{t,N}^{1,N} - M_{s,N}^{1,N}\|^2_{W^{-1,2}}}{|t-s|^{1+2\gamma}} \, ds \, dt > R \right) = 0,$$

and the proof will be complete. For notational convenience, we abbreviate, for $= 1, 2$,

$$M^{i,N}_t(x) = \frac{1}{N} \sum_{a \in A^N} M^{i,a}_t(x).$$

Note, that for every $x \in \mathbb{R}^d$ the processes $M^{1,N}(x)$ and $M^{2,N}(x)$ are martingales. It follows, with computations similar to those of Lemma 2, for $t \geq s$

$$E \left\| M^{1,N}_t - M^{1,N}_s \right\|^2_{W^{-1,2}} = \int_{\mathbb{R}^d} \frac{1}{N^2} \sum_{a \in A^N} E \left[ \int_s^t 1_{r \in I^a} (I - \Delta)^{-\frac{1}{2}} \nabla \theta_N (x - X^a_r)^2 \, dr \right] \, dx
\leq \frac{1}{N} \left\| (I - \Delta)^{-\frac{1}{2}} \nabla \theta_N \right\|^2_{L^2} E \int_s^t \frac{1}{N} \sum_{a \in A^N} 1_{r \in I^a} \, dr
\leq \frac{1}{N} \| \theta_N \|^2_{L^2} (t - s) \leq C (t - s).$$

Similarly, for the second martingale, in analogy with Lemma 3

$$E \left\| M^{2,N}_t - M^{2,N}_s \right\|^2_{L^2} = \int_{\mathbb{R}^d} \frac{1}{N^2} \sum_{a \in A^N} E \left[ \int_s^t 1_{r \in I^a} \theta_N (x - X^a_r)^2 \, dr \right] \, dx
\leq C_F \frac{1}{N} \| \theta_N \|^2_{L^2} E \int_s^t \frac{1}{N} \sum_{a \in A^N} 1_{r \in I^a} \, dr \leq C (t - s).$$

6 Passage to the limit

6.1 Criterion of compactness

A version of Aubin-Lions lemma, see [11], [9], [2], states that when $E_0 \subset E \subset E_1$ are three Banach spaces with continuous dense embedding, $E_0$, $E_1$ reflexive, with $E_0$ compactly
embedded into $E$, given $p, q \in (1, \infty)$ and $\gamma \in (0, 1)$, the space $L^q(0, T; E_0) \cap W^{\gamma, p}(0, T; E_1)$ is compactly embedded into $L^q(0, T; E)$.

Given the number $\alpha_0$ in assumption \([3]\), we take any pair $\alpha' < \alpha$ in the interval $(d/2, \alpha_0)$. We use Aubin-Lions lemma with $E = W^{\alpha', 2}(D)$, $E_0 = W^{\alpha, 2}(D)$, $0 < \gamma < \frac{1}{2}$ and $E_1 = W^{-2, 2}(\mathbb{R}^d)$, where $D$ is any regular bounded domain. The lemma states that $L^2(0, T; W^{\alpha, 2}(D)) \cap W^{\gamma, 2}(0, T; W^{-2, 2}(\mathbb{R}^d))$ is compactly embedded into $L^2 \left(0, T; W^{\alpha', 2}(D)\right)$.

Notice that for $\gamma p > 1$, the space $W^{\gamma, p}(0, T; E_1)$ is embedded into $C \left([0, T]; E_1\right)$, so it is not suitable for our purposes since we have to deal with discontinuous processes. However, for $\gamma p < 1$ the space $W^{\gamma, p}(0, T; E_1)$ includes piecewise constant functions, as one can easily check. Therefore it is a good space for càdlàg processes.

Now, consider the space

$$ Y_0 := L^\infty \left(0, T; L^2 \left(\mathbb{R}^d\right)\right) \cap L^2 \left(0, T; W^{\alpha, 2} \left(\mathbb{R}^d\right)\right) \cap W^{\gamma, 2} \left(0, T; W^{-2, 2} \left(\mathbb{R}^d\right)\right). $$

Using the Fréchet topology on $L^2 \left(0, T; W^{\alpha, 2}_{\text{loc}} \left(\mathbb{R}^d\right)\right)$ defined as

$$ d(f, g) = \sum_{n=1}^\infty 2^{-n} \left(1 \wedge \int_0^T \| (f - g)(t, \cdot) \|_{W^{\alpha, 2}(B(0, n))}^p \, dt \right) $$

one has that $L^2 \left(0, T; W^{\alpha, 2}_{\text{loc}} \left(\mathbb{R}^d\right)\right) \cap W^{\gamma, 2}(0, T; W^{-2, 2}(\mathbb{R}^d))$ is compactly embedded into $L^2 \left(0, T; W^{\alpha, 2}_{\text{loc}} \left(\mathbb{R}^d\right)\right)$ (the proof is elementary, using the fact that if a set is compact in $L^2 \left(0, T; W^{\alpha, 2}_{\text{loc}} \left(\mathbb{R}^d\right)\right)$ for every $n$ then it is compact in $L^2 \left(0, T; W^{\alpha, 2}_{\text{loc}} \left(\mathbb{R}^d\right)\right)$ with this topology; see a similar result in \([3]\)). Denoting by $L^\infty_{\text{ws}} \left(0, T; L^2 \left(\mathbb{R}^d\right)\right)$ and $L^2_{\text{ws}} \left(0, T; W^{\alpha, 2}(\mathbb{R}^d)\right)$ the spaces $L^\infty \left(0, T; L^2 \left(\mathbb{R}^d\right)\right)$ and $L^2 \left(0, T; W^{\alpha, 2}(\mathbb{R}^d)\right)$ endowed respectively with the weak star and weak topology, we have that $Y_0$ is compactly embedded into

$$ Y := L^\infty_{\text{ws}} \left(0, T; L^2 \left(\mathbb{R}^d\right)\right) \cap L^2_{\text{ws}} \left(0, T; W^{\alpha, 2} \left(\mathbb{R}^d\right)\right) \cap L^2 \left(0, T; W^{\alpha, 2}_{\text{loc}} \left(\mathbb{R}^d\right)\right). \quad (18) $$

Notice that

$$ L^2 \left(0, T; W^{\alpha, 2}_{\text{loc}} \left(\mathbb{R}^d\right)\right) \subset L^2 \left(0, T; C(D)\right) $$

for every regular bounded domain $D \subset \mathbb{R}^d$.

Denote by $\{Q^N\}_{N \in \mathbb{N}}$ the laws of $\{h^N\}_{N \in \mathbb{N}}$ on $Y_0$. From the ”boundedness in probability” of the family $\{Q^N\}_{N \in \mathbb{N}}$ on $Y_0$, stated by Lemma \([3]\) (notice that square integrability in time of $\| h^N_t \|_{L^2(\Omega; W^{\alpha, 2}(\mathbb{R}^d))}$ comes from the assumption $\alpha_0 - \rho_0 \leq 1$ which implies $\alpha - \rho_0 < 1$) and Lemma \([3]\) it follows that the family $\{Q^N\}_{N \in \mathbb{N}}$ is tight in $Y$, hence relatively compact, by Prohorov theorem. From every subsequence of $\{Q^N\}_{N \in \mathbb{N}}$ it is possible to extract a further subsequence which converges to a probability measure $Q$ on $Y$. We shall prove that every such limit measure $Q$ is a Dirac measure $Q = \delta_u$ concentrated to the
same element \( u \in Y \), hence the whole sequence \( \{Q^N\}_{N \in \mathbb{N}} \) converges to \( \delta_u \); and also the processes \( \{h^N\}_{N \in \mathbb{N}} \) converge in probability to \( u \).

Finally, since \( \alpha' < \alpha \) are arbitrary in the interval \((d/2, \alpha_0)\), in Theorem 1 we have stated the weak convergence in \( L^2 (0, T; W^{\alpha,2} (\mathbb{R}^d)) \) and the strong convergence in \( L^2 (0, T; W^{\alpha,2}_{loc} (\mathbb{R}^d)) \) with the same symbol \( \alpha \in (d/2, \alpha_0) \).

### 6.2 Convergence

Let us consider also the auxiliary equation

\[
\frac{\partial u}{\partial t} = \Delta u + u (1 - u)^+ , \quad u|_{t=0} = u_0. 
\] (19)

**Definition 7** Given \( u_0 : \mathbb{R}^d \to \mathbb{R} \) measurable, with \( 0 \leq u_0(x) \leq 1 \) (resp. \( u_0(x) \geq 0 \)), we call a measurable function \( u : [0,T] \times \mathbb{R}^d \to \mathbb{R} \) a weak solution of equation (1) (resp. of equation (19)), if

\[
\langle u_t, \phi \rangle = \langle u_0, \phi \rangle + \int_0^t \langle u_r, \Delta \phi \rangle dr + \int_0^t \langle (1 - u_r) u_r, \phi \rangle dr 
\] (resp. with the term \((1 - u_r)^+\) in place of \((1 - u_r)\)) for all \( \phi \in C_\infty_c (\mathbb{R}^d) \) and a.e. \( t \in [0,T] \).

**Remark 8** If \( u : [0,T] \times \mathbb{R}^d \to \mathbb{R} \) is a measurable function, with \( 0 \leq u_t(x) \leq 1 \) (resp. \( u_t(x) \geq 0 \)), such that

\[
0 = \int_0^T \int_{\mathbb{R}^d} \left( \frac{\partial \phi_t}{\partial t} + \Delta \phi_t + (1 - u_t) \phi_t \right) u_t dx dt + \langle u_0, \phi_0 \rangle 
\] (resp. with the term \((1 - u_r)^+\) in place of \((1 - u_r)\)) for all time-dependent test functions \( \phi_t \) of class \( C_\infty_c ([0,T] \times \mathbb{R}^d) \), then one can prove (by taking test functions \( \eta^t \phi_t(x) \)) that, for every time-independent test function \( \phi \in C_\infty_c (\mathbb{R}^d) \) we have that identity (20) (resp. with the term \((1 - u_r)^+\) in place of \((1 - u_r)\)) holds.

**Lemma 9** Under the assumptions of Theorem 1 \( Q \) is supported on the set of weak solutions of equation (19).

**Proof.** Step 1. We apply remark 8. For each \( \phi \in C_\infty_c ([0,T] \times \mathbb{R}^d) \), we introduce two functionals

\[
u \mapsto \Psi_\phi (u) := \int_0^T \int_{\mathbb{R}^d} \left( \frac{\partial \phi_t}{\partial t} + \Delta \phi_t + (1 - u_t)^+ \phi_t \right) u_t dx dt + \langle u_0, \phi \rangle
\]
\[
u \mapsto \Psi_\phi^+ (u) := \int_0^T \int_{\mathbb{R}^d} u_t \phi_t \, dx \, dt.
\]

They are continuous on \( Y \): since \( \phi \) is bounded measurable and compact support and we have at most the quadratic term \( u_t^2 \) under the integral signs, the topology of \( L^2 (0, T; L^2_{\text{loc}} (\mathbb{R}^d)) \), weaker than the topology of \( Y \), is sufficient to prove continuity. Denote by \( Q^N \) the law of \( h_i^N \) and assume a subsequence \( Q^{N_k} \) weakly converges, in the topology of the space \( Y \) defined by (18), to a probability measure \( Q \). By Portmanteau theorem, for every \( \epsilon > 0 \),

\[
Q (u : |\Psi_\phi (u)| > \epsilon) \leq \lim \inf_{k \to \infty} Q^{N_k} (u : |\Psi_\phi (u)| > \epsilon) = \lim \inf_{k \to \infty} P \left( \left| \Psi_\phi (h_i^N) \right| > \epsilon \right).
\]

To show \( Q (u : |\Psi_\phi (u)| > \epsilon) = 0 \) we prove in Step 2 below that this \( \lim \inf \) is zero. Since this holds for every \( \epsilon > 0 \), we deduce \( Q (u : \Psi_\phi (u) = 0) = 1 \). By a classical argument of density of a countable set of test functions, we deduce

\[
Q \left( \Psi_\phi (u) = 0 \text{ for all } \phi \in C_c^\infty ([0, T] \times \mathbb{R}^d) \right) = 1.
\]

Similarly, if \( \phi_t \geq 0, \phi \in C_c^\infty ([0, T] \times \mathbb{R}^d) \), we apply the same argument to \( \Psi_\phi^+ \) and get

\[
Q \left( u : \int_0^T \int_{\mathbb{R}^d} u_t \phi_t \, dx \, dt < 0 \right) \leq \lim \inf_{k \to \infty} P \left( \int_0^T \int_{\mathbb{R}^d} h_i^N \phi_t \, dx \, dt < 0 \right) = 0
\]

hence \( Q \) is supported on functions \( u \) such that \( u_t (x) \geq 0 \) for a.e. \( (t, x) \in [0, T] \times \mathbb{R}^d \). These two properties prove that \( Q \) is supported on the set of weak solutions of equation (19).

**Step 2.** It remains to prove that \( \lim \inf P \left( \left| \Psi_\phi \left( h_i^{N_k} \right) \right| > \epsilon \right) = 0 \). Let us write \( N \) instead of \( N_k \) for simplicity of notation. We have

\[
\Psi_\phi (h_i^N) = \int_0^T \int_{\mathbb{R}^d} \left( \frac{\partial \phi_t}{\partial t} + \Delta \phi_t + (1 - h_i^N)^+ \phi_t \right) h_i^N \, dx \, dt + \langle u_0, \phi_0 \rangle.
\]

By Itô formula, for every \( \phi_t \in C_c^\infty ([0, T] \times \mathbb{R}^d) \), one has

\[
0 = \int_0^T \int_{\mathbb{R}^d} \left( \frac{\partial \phi_t}{\partial t} + \Delta \phi_t \right) h_i^N \, dx \, dt + \int_0^T \int_{\mathbb{R}^d} \theta_N * (1 - h_i^N)^+ S_i^N \phi_t \, dx \, dt
\]

\[
+ \langle h_0^N, \phi_0 \rangle + \int_0^T \int_{\mathbb{R}^d} \phi_t dM_{t, 1} \, dx + \int_0^T \int_{\mathbb{R}^d} \phi_t dM_{t, 2} \, dx.
\]

Hence,

\[
\Psi_\phi (h_i^N) = \int_0^T \int_{\mathbb{R}^d} \left[ (1 - h_i^N)^+ h_i^N - \theta_N * (1 - h_i^N)^+ S_i^N \right] \phi_t \, dx \, dt
\]

\[
- \int_{\mathbb{R}^d} \int_0^T \phi_t dM_{t, 1} \, dx - \int_{\mathbb{R}^d} \int_0^T \phi_t dM_{t, 2} \, dx
\]

\[
- \langle h_0^N, \phi_0 \rangle + \langle u_0, \phi_0 \rangle.
\]

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In order to prove \( \lim_{N \to \infty} P \left( |\Psi_\phi(h_N^N)| > \varepsilon \right) = 0 \), it is sufficient to prove the same result for each one of the previous terms. Lemma 10 deals with the first term and the two martingale terms can be treated by Chebyshev’s inequality and Lemma 11 below. The terms

\[ -\langle h_0^N, \phi_0 \rangle + \langle u_0, \phi_0 \rangle = -\langle S_0^N, \theta_N \ast \phi \rangle + \langle u_0, \phi_0 \rangle \]

converge to zero in probability by the assumption that \( S_0^N \) converges weakly to \( u_0(x) \, dx \), as \( N \to \infty \), in probability.

\[ \square \]

**Lemma 10** It holds

\[
\int_0^T \int_{\mathbb{R}^d} \left[ (1 - h_t^N)^+ h_t^N - \theta_N \ast \left( (1 - h_t^N)^+ S_t^N \right) \right] \phi_t \, dx \, dt \to 0 \quad \text{as} \quad N \to \infty
\]

in probability.

**Proof.** We split the inner integral into

\[
\left| \left\langle \theta_N \ast (S_t^N (1 - h_t^N)^+) - h_t^N (1 - h_t^N)^+, \phi_t \right\rangle \right| \\
\leq \left| \left\langle \theta_N \ast (S_t^N (1 - h_t^N)^+) - S_t^N (1 - h_t^N)^+, \phi_t \right\rangle \right| \\
+ \left| \left\langle S_t^N (1 - h_t^N)^+ - S_t^N (1 - h_t^N)^+, \phi_t \right\rangle \right| \\
+ \left| \left\langle S_t^N (1 - h_t^N)^+ - h_t^N (1 - h_t^N)^+, \phi_t \right\rangle \right| \\
+ \left| \left\langle h_t^N (1 - h_t)^+ - h_t^N (1 - h_t)^+, \phi_t \right\rangle \right| \\
= I_t^N + II_t^N + III_t^N + IV_t^N,
\]

where \( h \) denotes the almost sure limit of \( (h_N^N)_{N \in \mathbb{N}} \) given by Skorokhod’s representation theorem. To prove Lemma 10 it is sufficient to show that each term on the right-hand side integrated in time converges in probability to zero. In order to prove that, it is sufficient to show that the expectation converge to zero for every \( t \in [0, T] \), because

\[
P \left( \int_0^T I_t^N \, dt > \varepsilon \right) \leq \frac{1}{\varepsilon} E \int_0^T I_t^N \, dt = \frac{1}{\varepsilon} \int_0^T EI_t^N \, dt \to 0.
\]

Note, there is a compact set \( K \), such that \( K \supset \cup_{t \in [0, T]} \text{supp}(\phi_t) \). For ease of notation we omit the subscript \( t \) in the following.

First,

\[ I_t^N = \left| \left\langle \theta_N \ast (S_t^N (1 - h_t^N)^+) - S_t^N (1 - h_t^N)^+, \varphi \right\rangle \right| = \left| \left\langle S_t^N (1 - h_t^N)^+, \theta_N \ast \varphi - \varphi \right\rangle \right| \leq \left[ S_t^N \right] \left\| \theta_N \ast \varphi - \varphi \right\|_\infty . \]
Hence,

\[ E1^N \leq E \left[ \bar{S}_t^N \right] \| \theta_N \ast \varphi - \varphi \|_\infty \leq \| \theta_N \ast \varphi \|_\infty \sup_{N \in \mathbb{N}} E \left[ S_T^N \right] \to 0 \]

Second, observe that

\[ II^N = \left\| S_t^N (1 - h_t^N) - S_t^N (1 - h_t) \right\| \leq |\varphi|_\infty \sup_{x \in K} \left| (1 - h_t^N(x)) - (1 - h_t(x)) \right| \]

and by Sobolev embedding and Lemma 4 we have

\[ \sup_{x \in K} |h_t^N(x) - h_t(x)| \to 0. \]

It follows

\[ EII^N \leq \| \varphi \|_\infty E \left( \left[ S_T^N \sup_{x \in K} |h_t^N - h_t(x)| \right] \right) \]

\[ \leq \| \varphi \|_\infty E \left( \left[ S_T^N \right]^2 \right) E \left( \left( \sup_{x \in K} |h_t^N(x) - h_t(x)| \right)^2 \right) \]

\[ \leq \| \varphi \|_\infty \sup_{N \in \mathbb{N}} \left( \left[ S_T^N \right]^2 \right) E \left( \left( \sup_{x \in K} |h_t^N(x) - h_t(x)| \right)^2 \right) \to 0. \]

The third term converges to zero pointwise due to the weak convergence of \( S^N \) and \( h^N \).

Finally, the last term also converges pointwise. From Section 6.1 we have

\[ \int_K |h_t^N(x) - h_t(x)|^2 \, dx \to 0. \]

Therefore,

\[ \left| \left\langle h_t^N (1 - h_t^N)^+ - h_t^N (1 - h_t)^+ , \varphi \right\rangle \right| \]

\[ \leq \left( \int_K |h_t^N|^2 \, dx \right)^{\frac{1}{2}} \left( \int_K \left| (1 - h_t^N)^+ - (1 - h_t)^+ \right|^2 \, dx \right)^{\frac{1}{2}} \]

\[ \leq \left( \int_K |h_t^N|^2 \, dx \right)^{\frac{1}{2}} \left( \int_K |h_t^N - h_t|^2 \, dx \right)^{\frac{1}{2}} \to 0. \]

\[ \to (f_K |h_t|^2 \, dx)^{\frac{1}{2}} < \infty \]

In the next lemma we denote by \( C \) any constant depending only on \( T \), \( \| \theta \|_{L^2} \), \( \sup_{x \in \mathbb{R}^d / N} \), \( \| \varphi \|_\infty \), \( \| \nabla \varphi \|_\infty \), \( E \left[ [S_T^N] \right] \).
Lemma 11 For $i = 1, 2$

\[
E \left[ \left| \int_{\mathbb{R}^d} \int_0^T \phi_t(x) dM^i_t(x) dx \right|^2 \right] \leq CN^{\beta-1}.
\]

Proof. Set

\[
g^N_t(y) := - \int_{\mathbb{R}^d} \phi_t(x) \nabla \theta_N (x - y) \, dx = \int_{\mathbb{R}^d} \nabla \phi_t(x) \theta_N (x - y) \, dx.
\]

For the first martingale term we have

\[
E \left[ \left| \int_{\mathbb{R}^d} \int_0^T \phi_t(x) dM^i_t(x) dx \right|^2 \right] = \frac{2}{N^2} \sum_{a \in A^N} E \left[ \int_0^T 1_{t \in I^a} \left| g^N_t(X^a_t) \right|^2 dt \right]
\leq \frac{2}{N} \left\| \theta_N \right\|_{L^2}^2 \left\| \nabla \phi \right\|_{L^\infty} E \int_0^T [S^N_t] dt.
\]

The assertion for $i = 1$ follows from Lemma 13 and

\[
\frac{1}{N} \left\| \theta_N \right\|_{L^2}^2 \leq C \frac{\epsilon N^d}{N} = CN^{\beta-1}.
\]

Set

\[
\tilde{g}^N_t(y) := - \int_{\mathbb{R}^d} \phi_t(x) \theta_N (x - y) \, dx,
\]

then for the second martingale term we have

\[
E \left[ \left| \int_{\mathbb{R}^d} \int_0^T \phi_t(x) dM^2_t(x) dx \right|^2 \right] = \frac{1}{N^2} \sum_{a \in A^N} E \left[ \int_0^T 1_{t \in I^a} \left| \tilde{g}^N_t(X^a_t) \right|^2 \lambda^a_t dt \right]
\leq \frac{1}{N} \left\| \theta_N \right\|_{L^2}^2 \left\| \phi \right\|_{L^\infty} E \int_0^T [S^N_t] dt
\]

and we conclude by the same argument. This completes the proof. ■

6.3 Auxiliary results

Theorem 12 There is at most one weak solution of equation (19). The unique solution has the additional property $u_t(x) \leq 1$, hence it is also the unique solution of (1).

Proof. Let $u^1, u^2$ be two weak solutions of the equation (19) with the same initial condition $u_0$. Let $\{\rho_\epsilon(x)\}_\epsilon$ be a family of standard symmetric mollifiers. For any $\epsilon > 0$ and $x \in \mathbb{R}^d$
we can use $\rho_{\varepsilon}(x - \cdot)$ as test function in the equation \cite{20}. Set $u^i_{\varepsilon}(t, x) = u^i(t, x) *_{x} \rho_{\varepsilon}(x)$ for $i = 1, 2$. Then we have

$$u^i_{\varepsilon}(t, x) = (u_0 * \rho_{\varepsilon})(x) + \int_0^t \Delta u^i_{\varepsilon}(s, x) \, ds + \int_0^t (\rho_{\varepsilon} * (1 - u^i)^+ u^i)(s, x) \, ds.$$  

Writing this identity in mild form we obtain (we write $u^i(t)$ for the function $u^i(s, \cdot)$ and $S(t)$ for $e^{tA}$)

$$u^i_{\varepsilon}(t) = S(t)(u_0 * \rho_{\varepsilon}) + \int_0^t S(t-s) \left( \rho_{\varepsilon} * ((1 - u^i)^+ u^i)(s) \right) \, ds.$$  

Write $g(u)$ for the function $u \to (1-u)^+ u$ from $[0, \infty)$ into $[0, \infty)$. The function $U = u^1 - u^2$ satisfies

$$\rho_{\varepsilon} * U(t) = \int_0^t S(t-s) \left( \rho_{\varepsilon} * \left[ g(u^1(s)) - g(u^2(s)) \right] \right) \, ds.$$  

Taking the limit as $\varepsilon \to 0$ we have

$$U(t) = \int_0^t S(t-s) \left[ g(u^1(s)) - g(u^2(s)) \right] \, ds.$$  

Hence,

$$\|U(t)\|_{\infty} \leq \int_0^t \|g(u^1(s)) - g(u^2(s))\|_{\infty} \, ds.$$  

Notice that the function $g$ is globally Lipschitz, with Lipschitz constant 1 (compute the derivative). It follows

$$\|U(t)\|_{\infty} \leq \int_0^t \|U(s)\|_{\infty} \, ds.$$  

By Gronwall’s lemma we conclude $U = 0$.

It is a classical result that equation \cite{11} has a unique weak solution, with the property $u \in [0, 1]$, being $u_0$ bounded, uniformly continuous and of class $L^2$ (see \cite{22}, Chapter 14, Section A). Hence, this solution is also a solution of equation \cite{19} and coincides with the unique weak solution of that equation. \hfill $\blacksquare$

**Lemma 13** There exists a $\gamma > 0$ such that $\sup_N E\left[e^{\gamma S^N_t}\right] < \infty$.

This lemma follows from the boundedness of the rates $\lambda^a N$. Indeed, this boundedness implies that the process $t \mapsto \left[ S^N_t \right]$ is stochastically dominated by $\frac{1}{N} Y_N[\left[ S_0^N \right]](\cdot)$, where $Y_k$ is a Yule process with birth rate 1 and $Y_k(0) = k$, see also \cite{10}.

The following Proposition gives an easy sufficient condition for assumption \cite{9} on the initial condition.

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Proposition 14 Assume that $X_i^0, i = 1, \ldots, N,$ are independent identically distributed r.v with common probability density $u_0 \in W^{\rho_0, 2} (\mathbb{R}^d),$ that assumption (3) holds and that $\rho_0 \leq \alpha_0$. Then

$$\sup_{N \in \mathbb{N}} E \left\| \theta_N * S^N_0 \right\|_{W^{\rho, 2} (\mathbb{R}^d)}^2 < \infty.$$ 

Proof. Step 1. To clarify the proof below, for pedagogical reasons we first treat the case $\rho_0 = 0$. By the i.i.d. property

$$E \int_{\mathbb{R}^d} \left| (\theta_N * S^N_0) (x) \right|^2 dx = \frac{1}{N^2} \int_{\mathbb{R}^d} E \left[ \left( \sum_{i=1}^N \theta_N (x - X_i^0) \right)^2 \right] dx$$

$$= \frac{1}{N} \int_{\mathbb{R}^d} E \left[ \left| \theta_N (x - X_1^0) \right|^2 \right] dx + \frac{N (N - 1)}{N^2} \int_{\mathbb{R}^d} E \left[ \theta_N (x - X_1^0) \right]^2 dx.$$ 

For the last term notice that

$$E \left[ \theta_N (x - X_1^0) \right] = (\theta_N * u_0) (x)$$

hence,

$$\frac{N (N - 1)}{N^2} \int_{\mathbb{R}^d} E \left[ \theta_N (x - X_1^0) \right]^2 dx \leq \left\| \theta_N * u_0 \right\|_{L^2}^2 \leq C,$$

because $\theta_N * u_0 \rightarrow u_0$ in $L^2 (\mathbb{R}^d)$. About the first term, we have

$$\frac{1}{N} \int_{\mathbb{R}^d} E \left[ \left| \theta_N (x - X_1^0) \right|^2 \right] dx = \frac{1}{N} E \left[ \int_{\mathbb{R}^d} \left| \theta_N (x - X_1^0) \right|^2 dx \right] = \frac{1}{N} E \left[ \int_{\mathbb{R}^d} \left| \theta_N (x) \right|^2 dx \right] \leq C$$

by (16). Hence $E \int_{\mathbb{R}^d} \left| (\theta_N * S^N_0) (x) \right|^2 dx \leq C$.

If $\rho_0$ is an integer, the proof can be easily modified. Let us treat the general case in the next step.

Step 2. Similarly to a property already used in the proof of Lemma 2 one has the following translation invariance property:

$$\left( (I - A)^{\rho_0/2} \theta_N (\cdot - X_i^0) \right) (x) = \left( (I - A)^{\rho_0/2} \theta_N \right) (x - X_i^0).$$
Therefore

\[
E \int_{\mathbb{R}^d} \left| \left( (I - A)^{\rho_0/2} \theta_N * S_0^N \right)(x) \right|^2 \, dx = \frac{1}{N^2} \int_{\mathbb{R}^d} E \left[ \left( \sum_{i=1}^{N} \left( (I - A)^{\rho_0/2} \theta_N (\cdot - X_i^0) \right)(x) \right)^2 \right] \, dx \\
= \frac{1}{N^2} \int_{\mathbb{R}^d} E \left[ \left( \sum_{i=1}^{N} \left( (I - A)^{\rho_0/2} \theta_N \right)(x - X_i^0) \right)^2 \right] \, dx \\
= \frac{1}{N} \int_{\mathbb{R}^d} E \left[ \left( (I - A)^{\rho_0/2} \theta_N \right)(x - X_1^0) \right]^2 \, dx \\
+ \frac{N(N - 1)}{N^2} \int_{\mathbb{R}^d} E \left[ \left( (I - A)^{\rho_0/2} \theta_N \right)(x - X_1^0) \right]^2 \, dx.
\]

For the last term we have (using the fact that the operator \((I - A)^{\rho_0/2}\) is self-adjoint in \(L^2(\mathbb{R}^d)\))

\[
E \left[ \left( (I - A)^{\rho_0/2} \theta_N \right)(x - X_1^0) \right] = \int \left( (I - A)^{\rho_0/2} \theta_N \right)(x - y) u_0(y) \, dy \\
= \langle (I - A)^{\rho_0/2} \theta_N, u_0(x - \cdot) \rangle \\
= \langle \theta_N, (I - A)^{\rho_0/2} u_0(x - \cdot) \rangle \\
= \int \theta_N(z) \left( (I - A)^{\rho_0/2} u_0(x - \cdot) \right)(z) \, dz \\
= \int \theta_N(z) \left( (I - A)^{\rho_0/2} u_0 \right)(x - z) \, dz \\
= \left( \theta_N * (I - A)^{\rho_0/2} u_0 \right)(x)
\]

where we have used again a translation invariance property. Hence,

\[
\frac{N(N - 1)}{N^2} \int_{\mathbb{R}^d} E \left[ \left( (I - A)^{\rho_0/2} \theta_N \right)(x - X_1^0) \right]^2 \, dx \\
\leq \left\| \theta_N * (I - A)^{\rho_0/2} u_0 \right\|_{L^2}^2 \leq C \left\| (I - A)^{\rho_0/2} u_0 \right\|_{L^2}^2 \leq C
\]
because the convolutions with $\theta_N$ are equibounded in $L^2(\mathbb{R}^d)$. For the first term, we have
\[
\frac{1}{N} \int_{\mathbb{R}^d} E \left[ \left| \left( (I - A)^{\rho_0/2} \theta_N \right) (x - X_0^1) \right|^2 \right] dx = \frac{1}{N} E \left[ \int_{\mathbb{R}^d} \left| \left( (I - A)^{\rho_0/2} \theta_N \right) (x - X_0^1) \right|^2 dx \right]
\]

\[
= \frac{1}{N} E \left[ \int_{\mathbb{R}^d} \left| \left( (I - A)^{\rho_0/2} \theta_N \right) (x) \right|^2 dx \right]
\]

\[
\leq \frac{1}{N} C \epsilon_N^{-2\rho_0} \epsilon_N^{-d} \| \theta \|_{W^{\alpha,2}(\mathbb{R}^d)}^2 = \frac{1}{N} C N^{\frac{d}{2}} (2\rho_0 + d)
\]

\[
\leq \frac{1}{N} C N^{\frac{d}{2}} (2\alpha_0 + d) \leq C
\]

where we have used Lemma 15 below, $\epsilon_N = N^{\frac{d}{2}}$, $\rho_0 \leq \alpha_0$ and the condition $\alpha_0 \leq \frac{d(1-\beta)}{2\beta}$ imposed in assumption (S). Hence $E \int_{\mathbb{R}^d} \left| \left( (I - A)^{\rho_0/2} \left( \theta_N * S_0^N \right) \right) (x) \right|^2 dx \leq C$.

**Lemma 15** For every $\alpha \geq 0$ and $\theta \in W^{\alpha,2}(\mathbb{R}^d)$ there exists a constant $C \geq 0$ such that

\[
\| \theta_N \|_{W^{\alpha,2}(\mathbb{R}^d)} \leq C \epsilon_N^{\alpha} \epsilon_N^{-d/2} \| \theta \|_{W^{\alpha,2}(\mathbb{R}^d)}.
\]

**Proof.** First, we compute the Fourier transform of $\theta_N$, i.e. for all $\lambda \in \mathbb{R}^d$

\[
\hat{\theta}_N (\lambda) = \int_{\mathbb{R}^d} e^{i\lambda \cdot x} \theta_N (x) dx = \epsilon_N^{-d} \int_{\mathbb{R}^d} e^{i\lambda \cdot x} \left( \epsilon_N^{-1} x \right) dx
\]

\[
= \int_{\mathbb{R}^d} e^{i\epsilon_N \lambda \cdot y} \theta (y) dy = \hat{\theta} (\epsilon_N \lambda).
\]

Second, note that the norms

\[
f \mapsto \| f \|_{W^{\alpha,2}(\mathbb{R}^d)}^2 \quad \text{and} \quad f \mapsto \int_{\mathbb{R}^d} \left( 1 + |\lambda|^2 \right)^{\alpha} \left| \hat{f} (\lambda) \right|^2 d\lambda
\]

are equivalent. Therefore, there is a constant $C \geq 0$, which may change from instance to instance, such that

\[
\| \theta_N \|_{W^{\alpha,2}(\mathbb{R}^d)}^2 \leq C \int_{\mathbb{R}^d} \left( 1 + |\lambda|^2 \right)^{\alpha} \left| \hat{\theta}_N (\lambda) \right|^2 d\lambda = C \int_{\mathbb{R}^d} \left( 1 + |\lambda|^2 \right)^{\alpha} \left| \hat{\theta} (\epsilon_N \lambda) \right|^2 d\lambda
\]

\[
= C \epsilon_N^{-d} \int_{\mathbb{R}^d} \left( 1 + |\epsilon_N^{-1} \eta|^2 \right)^{\alpha} \left| \hat{\theta} (\eta) \right|^2 d\eta
\]

\[
= C \epsilon_N^{-d} \epsilon_N^{-2\alpha} \int_{\mathbb{R}^d} \left( \epsilon_N^2 + |\eta|^2 \right)^{\alpha} \left| \hat{\theta} (\eta) \right|^2 d\eta
\]

\[
\leq C \epsilon_N^{-d} \epsilon_N^{-2\alpha} \int_{\mathbb{R}^d} \left( 1 + |\eta|^2 \right)^{\alpha} \left| \hat{\theta} (\eta) \right|^2 d\eta
\]

\[
\leq C \epsilon_N^{-d} \epsilon_N^{-2\alpha} \| \theta \|_{W^{\alpha,2}(\mathbb{R}^d)}^2.
\]

"
Lemma 16 The linear bounded operator $f \mapsto (I - A)^{\epsilon/2} e^{tA} f$ on $L^2(\mathbb{R}^d)$ is positive, i.e. $f \geq 0$ implies $(I - A)^{\epsilon/2} e^{tA} f \geq 0$.

Proof. Let $f$ be a non-negative function of class $L^2(\mathbb{R}^d)$. In order to prove that the function $g := (I - A)^{\epsilon/2} e^{tA} f$ is non-negative, it is sufficient to prove that its Fourier transform $\hat{g}$ is non-negative definite, namely $\text{Re} \sum_{i,j=1}^{n} \hat{g}(\lambda_i - \lambda_j) \xi_i \overline{\xi_j} \geq 0$ for every $n \in \mathbb{N}$, $\lambda_i \in \mathbb{R}^d$ and $\xi_i \in \mathbb{C}$, $i = 1, \ldots, n$. We have

$$\hat{g}(\lambda) = \left(1 + |\lambda|^2 \right)^{\epsilon/2} e^{-t|\lambda|^2} \hat{f}(\lambda)$$

and thus we have to prove that, given $n \in \mathbb{N}$, $\lambda_i \in \mathbb{R}^d$ and $\xi_i \in \mathbb{C}$, $i = 1, \ldots, n$, one has

$$\text{Re} \sum_{i,j=1}^{n} \left(1 + |\lambda_i - \lambda_j|^2 \right)^{\epsilon/2} e^{-t|\lambda_i - \lambda_j|^2} \hat{f}(\lambda_i - \lambda_j) \xi_i \overline{\xi_j} \geq 0$$

namely

$$\sum_{i=1}^{n} \text{Re} \hat{f}(0) \xi_i \overline{\xi_j} + \sum_{i<j} \left(1 + |\lambda_i - \lambda_j|^2 \right)^{\epsilon/2} e^{-t|\lambda_i - \lambda_j|^2} \left(\text{Re} \hat{f}(\lambda_i - \lambda_j) \xi_i \overline{\xi_j} + \text{Re} \hat{f}(\lambda_j - \lambda_i) \xi_j \overline{\xi_i} \right) \geq 0.$$

Corresponding to any couple $(i, j) \in \{1, \ldots, n\}^2$, let $\overline{\xi_1}, \ldots, \overline{\xi_n} \in \mathbb{C}$ be such that $\overline{\xi_i} = \xi_i$, $\overline{\xi_j} = \xi_j$, and $\overline{\xi_k} = 0$ for $k \notin \{i, j\}$. We know that $\sum_{i=1}^{n} \text{Re} \hat{f}(\lambda_i - \lambda_j) \xi_i \overline{\xi_j} \geq 0$, hence, if $i = j$

$$\text{Re} \hat{f}(0) \xi_i \overline{\xi_i} \geq 0$$

while for $i \neq j$

$$\text{Re} \hat{f}(\lambda_i - \lambda_j) \xi_i \overline{\xi_j} + \text{Re} \hat{f}(\lambda_j - \lambda_i) \xi_j \overline{\xi_i} \geq 0.$$

Using these two facts above we get the result. ■

7 Appendix

Since we deduced the threshold $\beta < 1/2$ from an approach based on Sobolev’s embedding theorem in the spaces $W^{\alpha, 2}$, it is natural to ask what happens if we use $W^{\alpha,p}$-topologies (which allow one to use much smaller $\alpha$, taking advantage of large $p$) or even Hölder topologies. We have done partial computations in these directions and the threshold $\beta < 1/2$ is the same in all approaches we have outlined. Let us show here just a partial computation in Hölder norms.

The restriction (in all approaches) seems to come from the estimate of the Brownian martingale. Recall it is given by
\[ \tilde{M}_{t}^{1,N}(x) := \frac{1}{N} \sum_{\alpha \in \Lambda^{N}} \int_{0}^{t} \left( e^{(t-s)A} \nabla \theta_{N} \right) (x - X_{s}^{\alpha}) 1_{s \in I_{\alpha}} \cdot dB_{s}^{\alpha}. \]

In order to investigate its Hölder properties, let us invoke Kolmogorov regularity criterion. Hence, we estimate, by the Burkholder-Davis-Gundy inequality,

\[
E \left[ \left| \tilde{M}_{t}^{1,N}(x) - \tilde{M}_{t}^{1,N}(x') \right|^{p} \right] \leq \frac{C}{N^{p}} \left[ \sum_{\alpha \in \Lambda^{N}} \int_{0}^{t} \left( \left| e^{(t-s)A} \nabla \theta_{N} \right| (x - X_{s}^{\alpha}) - \left| e^{(t-s)A} \nabla \theta_{N} \right| (x' - X_{s}^{\alpha}) \right)^{2} 1_{s \in I_{\alpha}} ds \right]^{p/2} 
\]

\[
= \frac{C}{N^{p}} \left[ \left| \int_{0}^{t} \sum_{\alpha \in \Lambda^{N}} \left[ e^{(t-s)A} \nabla \theta_{N} \right]_{\alpha}^{2} \left| x - x' \right|^{2\alpha} ds \right|^{p/2} \right] 
\]

\[
= \frac{C}{N^{p}} \left[ \left| \int_{0}^{t} N \left[ S_{N}^{2} \right] \left[ e^{(t-s)A} \nabla \theta_{N} \right]_{\alpha}^{2} ds \right|^{p/2} \right] |x - x'|^{\alpha p} 
\]

\[
\leq CE \left[ \left[ S_{T}^{N} \right]^{p/2} \right] \left( \frac{1}{N} \int_{0}^{t} \left[ e^{(t-s)A} \nabla \theta_{N} \right]_{\alpha}^{2} ds \right)^{p/2} |x - x'|^{\alpha p} 
\]

To apply Kolmogorov criterion we need \( \alpha p > d \). If we choose \( \alpha > 0 \) such that

\[
\frac{1}{N} \int_{0}^{t} \left[ e^{(t-s)A} \nabla \theta_{N} \right]_{\alpha}^{2} ds \leq C,
\]

then we can take \( p \) so large that \( \alpha p > d \). Hence, we may choose \( \alpha \) as small as we want. We denote the uniform norm in the space of continuous functions by \( \| \cdot \|_{0} \). A rough computation (we do not give details) gives us

\[
\left[ e^{(t-s)A} \theta_{N} \right]_{\alpha}^{2} \leq \left( \frac{C}{(t-s)^{\frac{1}{2} + \frac{\alpha}{2} - \alpha}} \right)^{2} \left\| (1 - A)^{-\frac{1}{2} + \alpha} \nabla \theta_{N} \right\|_{0}^{2}
\]

hence

\[
\frac{1}{N} \int_{0}^{t} \left[ e^{(t-s)A} \nabla \theta_{N} \right]_{\alpha}^{2} ds \leq C \frac{1}{N} \left\| (1 - A)^{-\frac{1}{2} + \alpha} \nabla \theta_{N} \right\|_{0}^{2} \leq C \frac{1}{N} \left\| (1 - A)^{\alpha} \theta_{N} \right\|_{0}^{2} \leq C \frac{N^{2\beta + 2\alpha \beta / d}}{N^{2\beta}}.
\]

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Since $\alpha$ can be taken arbitrarily small, the condition is $\beta < 1/2$.

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**References**

[1] R. E. Baker, M. J. Simpson, Correcting mean-field approximations for birth-death-movement processes, *Phys. Rev. E* 82 (2010), 041905.

[2] L. Banas, Z. Brzezniak, M. Neklyudov, A. Prohl, *Stochastic Ferromagnetism: Analysis and Numerics*, De Gruyter Studies in Mathematics 58, 2013.

[3] M. Bodnar, J. J. L. Velazquez, Derivation of macroscopic equations for individual cell-based models: A formal approach, *Math. Meth. Appl. Sci.* 28 (2005), 1757–1779.

[4] Z. Brzezniak, M. Ondrejat, E. Motyl, Invariant measure for the stochastic Navier-Stokes equations in unbounded 2D domains, [arXiv:1502.02637](https://arxiv.org/abs/1502.02637).

[5] Z. Brzezniak, E. Motyl, Existence of a martingale solution of the stochastic Navier-Stokes equations in unbounded 2D and 3D-domains, *J. Diff. Eq.* 254 (2013), no. 4, 1627-1685.

[6] N. Champagnat, S. Méléard, Invasion and adaptive evolution for individual-based spatially structured populations, *J. Math. Biol.* 55 (2007), n. 2, 147-188.

[7] G. Da Prato, J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge Univ. Press, Cambridge 1992.

[8] A. De Masi, P. A. Ferrari, J. L. Lebowitz, Reaction-diffusion equations for interacting particle systems, *J. Stat. Phys.* 44 (1986), n.3, pp 589-644.

[9] F. Flandoli, D. Gatarek, Martingale and stationary solutions for stochastic Navier-Stokes equations, *Probab. Theory Relat. Fields* 102 (1995), 367-391.

[10] F. Flandoli, Matti Leimbach, Mean field limit with proliferation, preprint.

[11] J. L. Lions, *Quelques Methodes de Resolution des Problemes aux Limites non Lineaires*, Dunod, Paris, 1969.

[12] McKean, H. P., Application of Brownian motion to the equation of Kolmogorov-Petrovskii-Piskunov, *Comm. Pure Appl. Math.*, 28(3): 323–331, 1975.
[13] S. Meleard, S. Roelly-Coppoletta, A propagation of chaos result for a system of particles with moderate interaction, *Stochastic Processes Appl.* **26** (1987), 317-332.

[14] S. Meleard, V. Bansaye, Some stochastic models for structured populations: scaling limits and long time behavior, [arXiv:1506.04165v1](https://arxiv.org/abs/1506.04165v1).

[15] M. Metivier, Quelques problèmes liés aux systèmes infinis de particules et leurs limites, *Séminaire de Probabilités* (Strasbourg) 20 (1986), 426-446.

[16] G. Nappo, E. Orlandi, Limit laws for a coagulation model of interacting random particles, *Annales de l'I.H.P.* section B, 24 (1988), no. 3, 319-344.

[17] K. Oelschläger, A law of large numbers for moderately interacting diffusion processes, *Zeitschrift fur Wahrsch. Verwandte Gebiete* **69** (1985), 279-322.

[18] K. Oelschläger, On the Derivation of Reaction-Diffusion Equations as Limit Dynamics of Systems of Moderately Interacting Stochastic Processes, *Probab. Th. Rel. Fields* **82** (1989), 565-586.

[19] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, 1983.

[20] R. Philipowski, Interacting diffusions approximating the porous medium equation and propagation of chaos, Stoch. Proc. Appl. **117** (2007), 526–538.

[21] Régnier, H. and Talay, D., Convergence rate of the Sherman and Peskin branching stochastic particle method, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 460(2014), 199–220, 2004.

[22] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, Springer, Berlin, 1994.

[23] A. Stevens, The derivation of chemotaxis equations as limit dynamics of moderately interacting stochastic many-particle systems, *SIAM J. Appl. Math.* **61** (2000), no. 1, pp. 183–212.

[24] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, Amsterdam, 1978.

[25] K. Uchiyama, Pressure in classical Statistical Mechanics and interacting Brownian particles in multi-dimensions, *Ann. H. Poincaré* 1 (2000) 1159–1202.

[26] J. M. A. M. van Neerven, M. C. Veraar, L. Weis, Stochastic integration in UMD Banach spaces, *The Annals of Probab.* **35** (2007), No. 4, 1438–1478.

[27] S.R.S. Varadhan, Scaling limit for interacting diffusions, *Comm. Math. Phys.* **135** (1991), 313–353.