Asymptotically Safe Starobinsky Inflation

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We revisit Starobinsky inflation in a quantum gravitational context, by means of the exact Renormalisation Group (RG). We calculate the non-perturbative beta functions for Newton’s ‘constant’ $G$ and the dimensionless $R^2$ coupling, and show that an attractive asymptotically free UV fixed point exists for the latter, while an asymptotically safe one exists for the former, and we provide the corresponding beta functions. The smallness of the $R^2$ coupling, required for agreement with inflationary observables, is naturally ensured by the presence of the asymptotically free UV fixed point. We discuss the corresponding RG dynamics, showing both how inflationary and classical observations define the renormalisation conditions for the couplings, and also how the UV regime is connected with lower energies along the RG flow.

I. INTRODUCTION

The Inflationary paradigm has led to a very successful framework within which we can explain the evolution of our Universe from a Hot Big Bang, in particular it has provided us with a mechanism to generate the primordial fluctuations observed in the Cosmic Microwave Background (CMB). There exist a plethora of models in the literature as can be seen through many of the excellent reviews on the subject [1–5]. Arguably the first model proposed, certainly the first that did not involve an evolving scalar field was due to Starobinsky [6], and it remains perfectly consistent with the most recent Planck data [7]. It is a particular example of modified gravity in which the action can be written as a general function $f(R)$ of the Ricci scalar $R$

$$S = \int d^4x \sqrt{-g} f(R)$$  \hspace{1cm} (1)

where in this case we have

$$S = \int d^4x \sqrt{-g} \left( \frac{1}{16\pi G} R + \frac{1}{b} R^2 \right)$$  \hspace{1cm} (2)

with the dimensionless coupling $b$ usually expressed as $b \equiv 6M^2/f m_p^2$, with $M$ a constant of mass dimension one, where $m_p \equiv G^{-1/2}$ is the Planck mass, $G$ is Newton’s constant which will become scale dependent and where $g$ is the determinant of the metric.

One of the key features of the Starobinsky action is that inflation is driven purely by the gravitational sector, i. e. without explicit introduction of new fields apart from the metric. Originally, the model was motivated by the one-loop corrections to the Einstein–Hilbert action resulting from vacuum quantum fluctuations in the matter sector at sufficiently high energies [6, 8, 9], which can be taken into account effectively by adding an $R^2$-term to the gravitational action. Recently, possibly motivated by its excellent fit to the CMB data (see for example [7]) there has been increased interest in the model in the context of supergravity [10–16] (see Ref. [17] for a more complete discussion on the subject), as well as in other quantum gravity contexts [18–20].

Starobinsky inflation is realised at a sufficiently high curvature regime in the early universe, where the $R^2$ term dominates the action, resulting in an unstable inflationary period with (quasi-) exponential expansion. As the curvature decreases with time, the Einstein–Hilbert term eventually comes to dominate at some lower curvature scale, and inflation ends with a graceful exit [6, 21–23].

The main goal of this work is to study the Starobinsky model beyond the semiclassical level by considering quantum fluctuations in the gravity sector. In particular, we shall calculate the (non-perturbative) Renormalisation Group (RG) flow for the two couplings in the action (2), namely Newton’s ‘constant’ $G$ and the $R^2$ coupling $b$. We will show that for this action, there exists a non-trivial UV fixed point under the RG, where $G$ is asymptotically safe and the $R^2$ coupling $b$ asymptotically free. The asymptotic freedom of the coupling $b$ turns out to be of great importance for a successful inflationary scenario in this context. We will further demonstrate that there naturally exist viable inflationary solutions along the RG flow from the UV to IR, in agreement with the Planck data. Notice that, in our analysis we will neglect a running cosmological constant in the action, as we are only interested in the original Starobinsky action ansatz. We will consider the inclusion of cosmological constant along with higher order curvature operators in a work to follow [24].

Let us first introduce the idea of Asymptotic Safety (AS) and briefly discuss the previous attempts in the literature to implement inflation in this context. AS, first suggested by Weinberg [25], proposes a UV complete theory for gravity by assuming that (metric) gravity is non-perturbatively renormalisable through the existence of a non-trivial (interacting) fixed point under the RG. The existence of such a fixed point, with a finite number of at-
tractive directions, which can be successfully connected to the low-energy regime, then yields a predictive theory for gravity. Essential to this is the non–perturbative nature of the AS scenario which makes no assumption about the smallness of the couplings in the UV. In the context of metric theories of gravity, a number of recent results in the literature have provided strong indicators for the existence of such a suitable UV fixed point [26–46], for reviews, see [47–50].

For a RG improved action and in the context of AS, inflation has been shown to work consistently for the case of a canonical scalar field coupled minimally to gravity, for particular choices of scalar field potentials [51–54]. Whereas scalar field inflation models can be made viable by adjusting potentials, this is much more of a challenge when inflation is derived from the gravitational sector alone because possible RG trajectories might never reach the required relative size of couplings. Such difficulties have been found in the approaches considered in the literature to date. In Refs. [55, 56], with and without higher-derivative terms in the action, it was found that viable inflationary solutions with a sufficient number of e-foldings are hard to realise. Furthermore, in Ref. [57] it was shown for a RG improved Einstein–Hilbert action that, although the inflationary period was able to generate sufficient number of e-foldings, the primordial fluctuations were too large to agree with observations. The nature of the latter result can be traced to the position of the UV fixed point for the running of Newton’s ‘constant’ $G$ and the cosmological constant $\Lambda$ which yields for the dimensionless combination

$$ (G \times \Lambda)_{\text{fixed point}} \sim O(10^{-3}). $$

However this combination also sets the scale of the gravitational wave power spectrum which in this case is clearly too large to agree with CMB observations where one would require the product to be more like $O(10^{-10})$. A similar result was also found in Ref. [58] at the 1–loop level.

In this work, we will show an important new result, namely that the original Starobinsky inflationary scenario can work naturally in the context of AS through the existence of an asymptotically safe UV fixed point for $G$, and an asymptotically free one for the $R^2$ coupling, $b$. In section II we present the non–perturbative beta functions, the associated fixed point structure and RG dynamics, while in section III we discuss analytic solutions and the implementation of renormalisation conditions for the gravitational couplings. In section IV we proceed to study how the slow–roll inflationary solutions obtained in this context are in agreement with the Planck observations. Our results are summarised and discussed in section V, and appendices are included where we present explicit expressions and equations.

II. THE BETA FUNCTIONS

The goal of this section is the presentation of the fundamental ingredient for our subsequent analysis, the non–perturbative beta functions under the RG for the action (2). The first step towards this derivation is the evaluation of an Exact Renormalisation Group equation (ERG) for the coarse grained effective action [59] $\Gamma_k[g_{\mu\nu}]$,

$$ k \partial_b \Gamma_k[g_{\mu\nu}] = \frac{1}{2} \text{Tr} \left[ \left( \Gamma_k^{(2)} + R_k \right)^{-1} \partial_b R_k \right], \quad (4) $$

with $\Gamma_k^{(2)}$ denoting the inverse, full propagator, and $\partial_b \equiv \partial / \partial k$, where the infrared (IR) cut-off scale $k$ sets the coarse graining or RG scale. In particular, momentum modes below $k$ are suppressed, while those above $k$ are not, and therefore integrated out. The regulator $R_k$ ensures IR regularisation as well as finiteness of the trace under very generic requirements [49, 50, 60, 61].

Evaluation of the ERG yields the flow equation for the effective action, which describes the RG dynamics of $\Gamma_k[g]$ as a function of the scale $k$. In general, the complexity of (4) requires us to assume a particular form for $\Gamma_k[g]$ to solve it. The flow equation for an effective action assuming the general $f(R)$ form in (1) has been calculated in Refs. [33–35], for different cut–off schemes on a Euclidean spherical background, and we refer the reader to these references for its explicit form and details of the calculation. The result is

$$ 384\pi^2 \left( \partial_\mu \tilde{f} + 4 \tilde{f} - 2 \tilde{R} \tilde{f} \right) = \frac{d\Gamma_k}{d\mu} \left[ \tilde{f}, \tilde{f}_R, \tilde{f}_{RR}, \partial_\mu \tilde{f} \right], \quad (5) $$

with $\mu$ being the “RG-time” $\mu \equiv \ln k / k_0$ for some reference scale $k_0$ (which will eventually become $m_\mu$), $\tilde{R} \equiv R / k^2$ is the dimensionless Ricci scalar, and we have defined $\tilde{f} \equiv f(R, \mu) / k^4$, $\tilde{f}_R \equiv \partial f(R, \mu) / \partial R$, $\tilde{f}_{RR} \equiv \partial^2 f(R, \mu) / \partial R^2$. The right hand side of the flow equation is a non–linear equation in $\tilde{R}$, the couplings and their first derivatives with respect to the cut–off scale $k$. In view of (5), and for an effective action associated with a Wick-rotated Starobinsky action (2), the two beta functions are

$$ k \frac{d}{dk} \tilde{G}(k) = \beta_G(\tilde{G}, b), \quad k \frac{d}{dk} b(k) = \beta_b(\tilde{G}, b), \quad (6) $$

where we have introduced the dimensionless Newton’s coupling $\tilde{G}(k) \equiv k^2 G(k)$. The beta functions are given explicitly in appendix A. It is important to notice that, although their derivation assumes a Euclidean signature, their validity is expected to carry over to Lorentzian signatures as well. Concrete evidence supporting this expectation has been found in Ref. [41]. We further notice that, while at the classical level, the action (1) can be re-expressed in a dynamically equivalent fashion in the so–called Einstein frame, where the Ricci scalar is mini-
nally coupled to a canonical scalar field [62–65], there is no reason to believe that in principle the two representations are equivalent at the quantum level. Evidence for the non-equivalence of the two frames, and in the context of the exact RG, has been recently found in Ref. [66].

At this point we remind the reader that if we were only considering the action (2) with just the $R^2$ operator, and no linear term, then by itself, it is marginal under standard perturbation theory. The corresponding beta function at leading order in $b$ (derived by sending $\tilde{G} \to 0$ in (2)) is then given by

$$k \frac{d}{dk} b = -\frac{1117}{8640\pi^2} b^2,$$

and is similar in spirit to the QCD one, entailing asymptotic freedom of $b(k)$. The beta function (7) exhibits one fixed point, $b = 0$, with an associated eigenvalue equal to zero. We will show below that the asymptotic freedom of the coupling $b$ persists after the inclusion of the linear curvature term, as described by action (2). Furthermore, we will see that a crucial difference with the beta function (7) will be the corresponding eigenvalue, which in that case will equal minus two - evidence that we are seeing a pure non-perturbative effect in action here.

The fixed points of the system of beta functions (6) can be found by setting the corresponding right hand sides to zero. Using the cut-off scheme of Ref. [34] we find that each fixed point gives rise to its own RG dynamical analysis; around the UV fixed point with $b<0$, a result known from perturbation theory.

At the same time, the negative eigenvalues (12)–(13) indicate that both UVFP1 and UVFP2 are fully UV-attractive. As a consequence, there cannot be any RG trajectory connecting UVFP1 and UVFP2 and their respective basins of attraction are disjoint.

In particular, the non-trivial eigenvalue of the coupling $b$ in (13) is not what one would infer from a simple dimensional analysis: around the UV fixed point with $b = 0$, the coupling $b$ acquires an anomalous dimension equal to minus two, which is a pure non-perturbative effect. We further discuss the origin of this property in appendix A.

The existence of the second fixed point UVFP2 in (10) is crucial for the resulting inflationary dynamics as it provides us with a mechanism for naturally producing small inflationary fluctuations at the perturbative level. What is more, this type of behaviour is able to overcome the previously found problems in the context of AS inflation discussed in the introduction, i.e. combining a sufficient number of e-folds with the requirement of obtaining the correct amplitude for the metric fluctuations.

We are now in a position to discuss the relevant dynamics of the system (6). The non-linearity of the beta functions allows only for a numerical solution, however close to the fixed point we are most interested in, i.e. the asymptotically free one UVFP2 (10)) we obtain an analytic solution by expanding the beta functions around $b = 0$, while keeping $G$ general. Doing so, the beta functions organise schematically as

$$\beta_G(\tilde{G}, b) \approx 3\left( \frac{17\tilde{G} - 24\pi)\tilde{G}}{5\tilde{G} - 36\pi} + \sum_{n=1}^{\infty} \sum_{m=0}^{2n} C_{(nm)} b^n \tilde{G}^{m-2n+2} \right) (5\tilde{G} - 36\pi)^{n+1},$$

(14)

$$\beta_b(\tilde{G}, b) \approx 4\tilde{G} b \tilde{G}^{m-2n+1} \left( \frac{2n-2}{2n} \sum_{n=2}^{\infty} \sum_{m=0}^{2n-2} D_{(nm)} b^n \tilde{G}^{m-2n+3} \right).$$

(15)

Here $C_{(nm)}$ and $D_{(nm)}$ are constants which can easily be obtained from equations (A1)–(A2). For the family of RG trajectories we will be interested in ($b \ll 1$), the second and higher order terms in $b$ in (14)–(15) can be

\[1\]

To confirm the stability of the UV fixed points under change of cut-off scheme, we checked that using the cut-off scheme of Ref. [35] instead, the fixed point values show differences only in the fourth decimal place.
neglected. Towards lower cut-off energies, trajectories emerging from UVFP2 will approach the GFP with $\tilde{G}$ dropping to smaller values, as one can also see from Fig. 1.

As we search for analytic solutions to (6) we focus on the regime $b \ll \tilde{G} \lesssim 1$, as this allows us to further expand (14)–(15) in powers of $\tilde{G}$. As we shall see, the solutions will allow us to capture very accurately the actual RG flow of the system. To leading order in $\tilde{G}$ we then obtain,

$$k \frac{d}{dk} \tilde{G} \simeq 2\tilde{G} - \frac{41}{36\pi} \tilde{G}^2 + \mathcal{O}(\tilde{G}^3, b), \quad (16)$$

$$k \frac{d}{dk} b \simeq -\frac{41}{36\pi} \tilde{G} b + \mathcal{O}(\tilde{G}^2, b^2). \quad (17)$$

The negative sign in (17) is crucial for the asymptotic freedom of the coupling $b$. We will discuss the solutions of the above system in the next section.

Before we close this section, let us remark on two interesting properties of the model (2) in view of the asymptotic freedom of the coupling $b$. It is well known that models of the $f(R)$ type propagate an extra, massive scalar degree of freedom, (often called a “scalaron”) compared to General Relativity [67, 68]. For the general $f(R)$ action (1) the effective mass of the scalaron is

$$m^2_{\text{eff}} = \frac{f_R - R f_{RR}}{3f_{RR}}. \quad (18)$$

In terms of the Starobinsky action (2) this corresponds to

$$m^2_{\text{eff}} = \frac{m_p^2}{96\pi} b, \quad (19)$$

which shows that the value of $m_{\text{eff}}$ compared to $m_p$ at some cut-off scale $k$ is set by $b \equiv b(k)$. The point to make here is that, in the limit $k \to \infty$, for the asymptotically free coupling $b$, the scalaron mass vanishes ($m_{\text{eff}} \to 0$), reflecting the absence of any length scales in that limit, and the restoration of scale-invariance.

What is more, as we explain later in section (IV), when the curvature squared term dominates the action (2) the universe expands (quasi-) exponentially. The second fixed point UVFP2 (10) ensures that for a sufficiently high value of the cut-off scale $k$, the coupling $b$ will be sufficiently small for the $R^2$ term to dominate the action and the universe to inflate.

### III. SOLUTIONS AND RENORMALISATION CONDITIONS

The approximate system of beta functions (16)–(17) for $\tilde{G}(k) \equiv k^2 \tilde{G}(k)$ and $b(k)$ can be solved analytically, after first solving (16) by separation of variables, and then substituting the solution into (17), to find

$$\tilde{G}(k) \simeq \frac{\tilde{G}_0}{1 + \frac{41}{72\pi} (k/k_0)^2},$$

$$b(k) \simeq \frac{b_0}{1 + \frac{41}{72\pi} (k/k_0)^2}, \quad (20)$$

with $\tilde{G}_0$ and $b_0$ being constants of integration, and $k_0$ a constant non-vanishing reference scale, which we will choose to be the Planck mass as it is measured today, i.e. $k_0 = m_p$. This is a convenient choice that will allow us to measure everything in units of the Planck mass.

The values of all physical observables are then defined with respect to the chosen scale. It is easy to see that as $k \to \infty$, $b \to 0$ and $\tilde{G} \to 72\pi/41$, while as $k \to 0$, $b \to b_0$, and $\tilde{G} \to 0$. This behaviour is in very good agreement with the full numerical solution of equations (A1)-(A2) (presented in Fig. 1) in the vicinity of the GFP.

One might worry that the validity of solutions (20) breaks down as soon as the two couplings become of comparable magnitude, $\tilde{G} \sim b \ll 1$, i.e. when the assumption that $b \ll \tilde{G} \lesssim 1$ is violated. However, even in that case, the solutions (20) remain quite accurate, as the full, numerical solution shows in Fig. 1. Another way to understand this is by looking at the approximate beta functions around the GFP. Notice that according to (20), $\tilde{G}(k)$ changes faster than $b(k)$ due to the extra $k^2$ dependence. At the point where $\tilde{G} \ll b \lesssim 1$, the beta function $\beta_b$, to leading order, will be given by (7), while the one for $\tilde{G}(k)$ has the same form as (14), but with the $\tilde{G}^2$-coefficient now being equal to $75/24\pi$. The corresponding solutions read as,

$$\tilde{G}(k) \simeq \frac{\tilde{G}_0}{1 + \frac{75}{24\pi} (k/k_0)^2},$$

$$b(k) \simeq \frac{b_0}{1 + \left(\frac{1117}{8640\pi^3}\right) b_0 \ln(k/k_0)}, \quad (22)$$

with $b_0 \ll 1$.

At the scale where the denominator of (22) becomes zero, the analytical solution breaks down, and one has to resort to a numerical solution of the full system of beta functions (see also Fig. 1.) Notice that the pole in (22) is of similar nature to that of the QCD running coupling at 1–loop.

The integration constants $\tilde{G}_0$, $b_0$ in the solutions (20) have to be fixed by applying appropriate renormalisation conditions at a particular scale $k = k_0$ (recall we will take $k_0 = m_p$). However, an important point is that the measurements available to us for each of these couplings correspond to different scales (energies). On the one hand, Newton’s ‘constant’ $G$ has been measured from micrometer to solar distance scales (low energies), while as we shall do in section IV, the $R^2$ coupling $b$ should be determined from CMB observations (high energies). To
overcome this obstacle, our strategy will be to match the measured value of each coupling at the corresponding energy scale, \( k \equiv k_{\text{measured}} \), through the analytic solutions (20), and then use (20) to extrapolate their values for example to the Planck scale \( k \sim m_p \) as this is the reference scale we have chosen for \( k_0 \).

In the classical regime, corresponding to \( k \ll k_0 = m_p \), we know that \( G = \tilde{G}(k)k^{-2} = 1/m_p^2 \), hence it follows that we need to choose our initial constant as \( \tilde{G}_0 \simeq 1 \). Using the solution (20) to extrapolate up to the Planck scale, it then follows that

\[
\tilde{G}(k = m_p) \simeq 0.85.
\]  

Relation (23) will provide our renormalisation condition for \( \tilde{G} \) at the Planck scale. Notice that, as the cut–off energy \( k \) drops much below the Planck scale \( (k/m_p \ll 1) \), the couplings’ evolution enters the classical regime, where \( \tilde{G}(k) = k^2G(k) \simeq k^2m_p^{-2} \) and \( b \simeq b_0 \). The appropriate value for \( b_0 \) which will define the renormalisation condition for the coupling \( b = b(k) \) will be determined in section IV, using the recently published Planck data.

IV. INFLATONARY SOLUTIONS

The solutions for the running couplings (20) are functions of the RG scale \( k \), and when inserted into the action (2) they result in a continuous family of actions parametrized by the value of the RG scale \( k \). Now, in cosmology, variables and physical quantities generally depend on the space-time coordinates \( x^i \). One is naturally led to think that in an expanding universe the process of integrating out degrees of freedom occurs as a function of time and space, which in turn demands promoting the IR cut–off \( k \) to a (monotonic) space-time dependent function, \( k = k(t, x) \). Such a procedure has been implemented in a number of ways \([51, 55–58, 69–80]\), with one particular example being associating \( k^2 \sim H^2 \), since the Hubble parameter \( H \) naturally provides an IR cut–off. Our approach here will be to RG-improve the \( k \)-dependent action (2) in a covariant way \([57, 77, 78]\).

We will do this by relating the cut–off scale \( k \) with the Ricci scalar through

\[
k^2 = \alpha R,
\]

with \( \alpha > 0 \) constant, a choice which is very similar in spirit to the RG improvement of the effective potential in scalar field theory as applied in Ref. [81]. The effect of the identification (24) will be to implement the running of the gravitational couplings under the RG as non–linear curvature corrections in the original action ansatz (2).

The dynamics of the resulting action, which is shown below in (26), can then be treated with classical methods as we shall also see.

The value of the constant \( \alpha \equiv k^2/R \) has a particular physical interpretation: its magnitude describes the importance of radiative corrections in the classical equations. In particular, the case of \( \alpha \ll 1 \) implies that higher order curvature corrections in the action (2) are important, but at the same time radiative corrections are small; in the opposite case where \( \alpha \gg 1 \), higher order curvature corrections are negligible, but radiative ones not necessarily \([55]\). As also argued in Ref. \([55]\), the optimal case corresponds to \( \alpha \sim 1 \), which is the case when radiative corrections start to become important and higher order corrections in the action become negligible. From now on, we will set \( \alpha = 1 \), which is the case when the results of Ref. \([55]\) apply. Notice though that one could also proceed keeping \( \alpha \) undetermined, and constrain its value from observations.

Notice that with this choice, and during slow-roll inflation \( |H/\pi|^2 \ll 1 \) from (24) one has

\[
k^2 = R = 6H^2(2 + H/\pi^2) \simeq 12H^2.
\]

In other words, the Hubble parameter naturally sets the RG cut-off scale, or equivalently the cosmological horizon sets the typical scale of correlations between quantum degrees of freedom.

Let us now look at the solutions we found for \( \tilde{G}(k) \) and \( b(k) \) given in (20). We conveniently chose \( \tilde{G}_0 = 1 \), so that according to the renormalisation condition (23) there are sizable running coupling corrections at the Planck scale. Inserting the cut–off identification (24) with \( \alpha = 1 \), as described above, into the solutions (20) for \( b(k) \) and \( \tilde{G}(k) \), and in turn plugging the latter into the action (2), the RG-improved action reads

\[
S_{\text{RG-improved}} = \int d^4x \sqrt{-g} \left( \frac{m_p^2}{16\pi} R + \left( \frac{1}{b_0} + \frac{41}{1152\pi^2} \right) R^2 + \frac{1}{b_0} \frac{41}{72\pi} R^3 \right).
\]

We will assume that \( b_0 \ll 1152\pi^2/41 \approx 277 \) which allows us to neglect the second term proportional to \( R^2 \), a choice that will be justified in the next section.

The action (26) is the result of the RG-improvement of the original \( R^2 \) action (2), based on the solutions (20). We see that its key effect is to generate an effective \( R^3 \) term in addition to the linear and quadratic one. However, for the inflationary relevant energy scales, where
the energy (or curvature) of the universe decreases compared to $m_p$, $k^2 \sim R \lesssim m_p^2$, the cubic term will remain negligible as long as
\begin{equation}
\frac{41}{12\pi^2} \frac{R}{m_p^2} \ll 1,
\end{equation}
a condition that is always satisfied in the curvature regime we are interested in, i.e. $R/m_p^2 \ll 1$. In fact the condition (27) is simply equivalent to saying that in the RG solutions (20), the second term in the denominators is much less than unity, i.e. $(41G_\parallel/12\pi^2)(k/m_\chi)^2 \ll 1$.

It follows, given the above requirements, the RG improved action (26) now simply reduces to the standard Starobinsky form
\begin{equation}
S \simeq \int d^4x \sqrt{-g} \left( \frac{m_p^2}{16\pi} R + \frac{1}{b_0} R^2 \right).
\end{equation}
The remaining input required for the action (28) is to determine the initial value of $b_0$, and this will be done by considering the inflationary observables at the corresponding high energy scales. In particular, the required value of $b_0$ for successful inflation will define its renormalisation condition in the UV, and select the particular family of RG trajectories. As we will see, the (very small) observationally required value is in perfect agreement with RG solutions connecting the UV with the IR regime in a viable way (see also Fig.1.)

To determine the key inflationary observables for the action (28) and compare them with the Planck data, we follow a similar analysis to Refs. [6, 8, 68, 82–84].

Let us first derive the background equation of motion for the action (28), or initially for the more general action (1). Varying with respect to the metric field, and evaluating on the Friedman–Robertson–Walker (FRW) metric,
\begin{equation}
ds^2 = -dt^2 + a(t)^2 dx^2
\end{equation}
we get for the $00$-component of the Einstein-equations,
\begin{equation}
\frac{f_R}{H^2}(1 - \epsilon) + 6 f_{RR} \left( 4 \epsilon + \frac{\dot{H}}{H} - 2 \dot{\epsilon}^2 \right) - \frac{1}{6} \frac{f}{H^4} = 0,
\end{equation}
with
\begin{equation}
\epsilon \equiv -\dot{H}/H^2
\end{equation}
the slow-roll parameter, $f_R \equiv df/dR$, $f_{RR} \equiv d^2f/dR^2$, $R = 6H^2(2 - \epsilon)$ and $\dot{\epsilon} \equiv d\dot{\epsilon}/dt$.

For $\epsilon = 0$ and $\dot{\epsilon} = 0$, the solution of equation (30), if it exists, yields an exact de Sitter expansion where $H = H_0 = \text{const.}$, and it is easy to check that an action $f(R) \propto R^2$ possesses such a solution. In the context of the action (28), inflation starts off at some sufficiently high curvature scale, where the $R^2$ term dominates, then as the curvature decreases over time, eventually the linear term becomes significant, and inflation ends.

To describe the inflationary dynamics of (28) we will be interested in slow-roll solutions of (30), (the appropriate $f(R)$ being given by (28)), where $\epsilon^2, \dot{\epsilon}/H \ll 1$, $R \simeq 12H^2, \dot{R} \simeq -24H^3 \epsilon$ and $R^2 \simeq 144H^4$. Under these assumptions, and neglecting terms of order $(\epsilon^2, \dot{\epsilon}/H)$, equation (30) can then be integrated to give
\begin{equation}
H(t) \simeq H_0 - \frac{1}{576\pi} b_0 m_p^2 (t - t_0),
\end{equation}
with $H_0 \equiv H(t_0) = \text{constant}$, and $t_0$ the time when inflation begins.

The number of $e$-foldings $N$ between some scale $H(t)$ and the end of inflation, is given by
\begin{equation}
N \equiv \int_t^{t_{\text{end}}} H(t) dt = - \int_H^{H_{\text{end}}} \frac{d\log H}{H} \epsilon \simeq \frac{288\pi}{b_0 m_p^2} (H^2 - H_{\text{end}}^2),
\end{equation}
with the slow-roll parameter $\epsilon$ calculated from the solution (32) as
\begin{equation}
\epsilon \equiv \epsilon(t) = \frac{1}{36} \frac{m_p^2/16\pi}{b_0} \frac{1}{H(t)^2},
\end{equation}
and where $H_{\text{end}}$ is the value of $H$ when inflation ends which corresponds to when $\epsilon = 1$ in (34). Notice that (34) implies that the universe inflates as long as the $R^2$ term dominates the action (28).

Small, inhomogeneous fluctuations of the metric field during inflation contribute to the temperature inhomogeneities observed in the CMB. For the case of the action (28) the cosmologically relevant propagating degrees of freedom around the FRW background (29) are the two polarisation modes of the transverse-traceless (spin-two) field and a scalar (spin-zero) degree of freedom respectively (see e.g. [82] for an explicit analysis).

The corresponding power spectra for an $f(R)$ type action have been calculated in Refs. [68, 82, 85, 86], and are presented in (B5) and (B6) for convenience. From (B5) and using the slow-roll approximations (B4), we obtain for the amplitude of the scalar fluctuations evaluated at horizon crossing $k_F = aH$, i.e. when the particular scale $k_F$ leaves the cosmological horizon $^2$,
\begin{equation}
P_S \simeq \frac{1}{48\pi^2} \frac{H^2}{f_R} \frac{1}{e^{2}} \simeq \frac{N^2}{288\pi b_0}.
\end{equation}
In the last approximate equality we used (34) to relate $\epsilon$ with $N$ at the time of horizon crossing of the mode $k_F$, by approximating $\epsilon \simeq 1/2N$, always in the slow-roll regime. Furthermore, we approximated $f_R = m_p^2/(16\pi) + (1/b_0)(2 - \epsilon)H^2 \simeq 2/b_0H^2$, since as we will see shortly $b_0 \ll 1$.

In a similar fashion, from relation (B6) for the tensor

$^2 k_F$ here stands for the Fourier wavenumber to make the distinction with the RG scale $k$. 
amplitude during slow-roll we find that

$$P_T \simeq \frac{1}{\pi^2} \frac{H^2}{f_R} \simeq \frac{1}{24\pi^2} b_0.$$  \hspace{1cm} (36)$$

The tensor to scalar ratio is easily found to be

$$r \equiv \frac{P_T}{P_S} \simeq 48\epsilon^2 \simeq \frac{12}{N^2},$$  \hspace{1cm} (37)$$

where we again have used (33) and (34) in the last approximate equality to relate $\epsilon$ with $N$. Notice that the tensor-to-scalar ratio is suppressed by a factor of $\epsilon^2$.

The spectral indices are defined by equations (B5) and (B6) in the appendix, and we will assume that they are independent of $k$. In other words we assume there is effectively no running of the spectral index. Under the slow-roll conditions described in (B4) they are approximated as (B7)

$$n_S - 1 \simeq -4\epsilon, \quad n_T \simeq 0.$$  \hspace{1cm} (38)$$

We need to make connection with observations which we do by using the recent Planck results combined with the WMAP large scale polarisation likelihood (Planck + WP) [7]. At the pivot scale $k_F = 0.002\text{Mpc}^{-1}$, for the tensor to scalar ratio they require that $r_{0.002} < 0.12$. Similarly, the bound for the spectral index and scalar amplitude reads as $n_S = (0.9603 \pm 0.0073)$, and $\ln(10^{10} P_S) = 3.089^{+0.024}_{-0.027}$ respectively.

The typical value of the minimum number of e-foldings needed to solve the flatness and horizon problem is between 50 – 60 [7]. Here, we shall use the value $N = 55$, which from (37) implies that $r \simeq 0.004$, certainly within the Planck bound. At the same time, from (37) we further find that $\epsilon \simeq 0.009$, which from (38) implies that $n_S \simeq 0.964$ also within the observational bound.

We can now use the Planck upper bound on $\epsilon$ and the approximate relation (34), to first determine an upper bound for $b_0$. This bound can then be used to put an upper bound on the Hubble parameter and, through (25), on the cut-off scale $k$ as well.

For $N = 55$ in relation (35), the Planck observation of $P_S$ provides us with the important result, the required value of the coupling $b_0$,

$$b_0 \simeq 2.063 \times 10^{-9}.$$  \hspace{1cm} (39)$$

Using (39) in (34) we get an upper bound for $H$ which can then be used to put an upper bound on the cut-off scale $k$ through (25). This way, we find

$$\frac{H}{m_p} \lesssim 1.126 \times 10^{-5} \leftrightarrow \frac{k}{m_p} \lesssim 3.89 \times 10^{-5},$$  \hspace{1cm} (40)$$

for $N = 55$ e-foldings before the end of inflation.

To find the renormalisation condition of $b(k)$ at the Planck scale we use (39) in the analytic solution (20) to obtain,

$$b(k = m_p) = 1.757 \times 10^{-9}.$$  \hspace{1cm} (41)$$

This important relation supplements the similar one for $\tilde{G}(k)$, given in (23), and together they select out the RG trajectory which matches with CMB observations.

It has to be stressed that a crucial fact regarding the viability of inflation in this context is the (almost) constancy of the coupling $b$ for energies below $m_p$. This can also be seen from the beta function (17); in particular, for $b \sim 10^{-9}$ and $\tilde{G} < 1$, (17) tells us that

$$\left| \frac{dB}{dk} \right| \simeq 10^{-9} \tilde{G} \ll 1,$$  \hspace{1cm} (42)$$

obviously in agreement with the upper and lower limits of the Planck data. We emphasise that the required value for $b$, relation (39), can be achieved for a wide range of cut-off scales, thanks to its asymptotic freedom property and its tiny variation for $k/m_p \ll 1$ along the RG flow.

What is more, the negligible variation of $b(k)$ for $k/m_p \ll 1$ implies that the renormalisation condition (41) will also provide a prediction for the value of the coupling $b$ at classical scales. Notice that purely from classical considerations, a rather weak low energy bound for $b$ can be found as follows: given that the average matter density at a distance $r = 10^{17.39}\text{Mpc}$ from the centre of the sun is $\rho_m \approx 10^{-24}\text{g/cm^3}$, one finds that $R = 8\pi G \rho_m \approx 10^{-117} m_p^2$, which in turn implies that, if $b > 10^{-115}$, the effect of the $R^2$ term is negligible, and General Relativity recovered. Therefore, the requirement of viable inflation in this context, provides with a much stronger bound on the coupling $b$ at classical scales.

We conclude this section, by noting that the asymptotic freedom of the curvature squared coupling, $b$, ensures the smallness of its value in the UV, while the asymptotic safety of Newton’s ‘constant’ $G$ ensures the absence of UV infinities on the theory space. What is more, the approximate constancy of the coupling $b$ during inflation further guarantees that the primordial fluctuations remain sufficiently small, allowing for a viable period of inflation driven by the $R^2$ term.

V. CONCLUSIONS AND DISCUSSION

In this work, we have revisited Starobinsky inflation, by calculating the non-perturbative beta functions for the (Euclidian) vacuum gravitational action (2), in the context of the exact Renormalisation Group (RG). We have presented the full, non-perturbative beta functions for the gravitational couplings (equations (A1)-(A2)), while approximate relations in the UV were derived in (14)-(15) and (20). The dynamics and predictions of the inflationary regime emerging from the non-trivial UV fixed point found were studied, showing that the asymptotic freedom of the curvature squared coupling guarantees a successful
inflationary period for a wide range of scales along the RG flow.

The main results of our analysis can be summarised as follows:

- We showed that under the RG an attractive, non-trivial UV fixed point exists, UVFP2 relation (10), where Newton’s coupling $G$ is asymptotically safe, and the $R^2$ coupling asymptotically free. We obtained indications that the fixed point is stable under the cut–off scheme adopted, and independent of the gauge choice due to its origin in the gauge-invariant transverse-traceless part of the flow equation, as discussed in section A of the appendix. Furthermore, the fixed point is UV-attractive and is connected with the IR regime along the RG flow, as the cut-off energy $k$ decreases, also shown in Fig. 1. A remarkable property of the new UV fixed point found is the non-trivial anomalous dimension (= −2) for the dimensionless $R^2$ coupling in its vicinity, which is not expected using a naive dimensional analysis. Apart from the non–trivial asymptotically safe/free fixed point, we also showed that the theory space of the action (2) exhibits a trivial, Gaussian fixed point (8) along with a second, non-trivial UV one (9).

- The asymptotic freedom of the $R^2$ coupling, $b(k \to \infty) = 0$, ensures that the universe enters into a de Sitter-like expansion at some sufficiently high cut–off scale, as the $R^2$ term comes to dominate the action ($1/b(k) \gg 1$). As the cut–off scale $k$ decreases along the RG flow, the curvature also decreases until eventually the Einstein–Hilbert term in the action comes to dominate and inflation ends.

- The UV fixed point UVFP2 further ensures that the primordial fluctuations of the metric during inflation remain sufficiently small, as one can see from equations (35)-(36). Provided that at sufficiently high energy the dimensionless couplings $\tilde{G}$ and $b$ start close to their fixed point value, given in (10), and in particular provided that $b(k_0) = b_0 \ll 1$, the RG evolution is stable, and connects smoothly the UV with the IR regime. At lower energies, with $k/m_p \ll 1$ we have $\tilde{G} \equiv k^2 G(k) \ll 1$ and $b \approx b_0 = \text{const.}$ Under these conditions, inflation can occur for a wide range of cut–off (or curvature) scales from the GUT scale down to the electroweak scale. The CMB data provides us with the appropriate renormalisation condition for the coupling $b$ at the energy scale where inflation occurs along the RG flow, relation (41), selecting out a particular RG trajectory among the infinitely many.

In the future, we plan to extend the analysis of this fascinating area [24], presenting a more detailed exposition of the RG dynamics of the action (2), from the UV to IR, together with a determination of further properties of the non-perturbative beta functions, including an investigation into how the RG, inflationary dynamics and predictions are modified by the inclusion of higher order curvature operators in the action.

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Appendix A: Explicit beta functions and the UVFP2

In this section, we present the explicit form of the beta functions (6) for the couplings $\tilde{G}$ and $b$ and comment on the nature of the non-trivial anomalous dimension of $b$ in the UV. We will leave a more explicit discussion on its nature to future work [24].

With the “optimised” cut–off of Ref. [87] and in the Landau type gauge of [34], the full, non-perturbative beta functions for the dimensionless couplings $\tilde{G} \equiv G/k^2$ and $b$ take the form

$$k \frac{d}{dk}\tilde{G} = \frac{A_0(1-B_2)+A_2B_0}{1-A_1-B_2+A_1B_2-A_2B_1}, \quad (A1)$$

$$k \frac{d}{dk} b = \frac{B_0(1-A_1)+A_0B_1}{1-A_1-B_2+A_1B_2-A_2B_1}, \quad (A2)$$

where the coefficients $A_i$ and $B_i$ are rational functions of $\tilde{G}$ and $b$, and are given below, in (A5)–(A10).

The fixed points of the flow result from the roots of the numerators as long as they are not roots of the denominator, too. In particular, equation (A1) requires for

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.pdf}
\caption{Solution of the full system of beta functions for the couplings $\tilde{G}(k)$ (black) and $b(k) \times 10^8$ (blue) for the initial conditions: $k_{UV} = 10^4 m_p$, $k_{IR} = 10^{-5} m_p$, $\tilde{G}(k = m_p) = 0.85$, $b(k = m_p) = 10^{-9}$. The couplings start in the UV close to UVFP2, with $\tilde{G}_{IR} \approx 4.44$, $b_{IR} = 0$, and evolve towards the IR as the cut–off energy $k$ decreases. We expect inflation to occur for energies smaller than the Planck mass, $k/m_p \ll 1$. Notice that in that regime, $b \approx \text{const.}$ and $\tilde{G} \approx (\text{const.}) k^2$.}
\end{figure}
a fixed point
\[ A_0 = -\frac{A_2 B_0}{1 - B_2}. \quad (A3) \]

Inserting the above relation into the beta function \((A2)\) gives
\[ k \frac{d}{dk} b = \frac{B_0}{1 - B_2} = -2b + \mathcal{O}(b^2) \quad (A4) \]
from which one sees that one of the eigenvalues of the stability matrix at the fixed point \(b = 0\) is \(-2\).

The origin of this particular eigenvalue can be understood by studying the structure of the flow equation in the \(f(R)\) ansatz. In particular, it turns out that this eigenvalue is directly related to those terms in the right hand side of the flow equation \((5)\) which are projected on the \(R^2\) term and which have a prefactor of \(1/b\). Starting from the explicit expression for the RG flow of \(f(R)\)-theories, as it is presented in Ref. [34], it can be seen that for the case of our action ansatz \((2)\), after expanding in curvature \(\tilde{R}\) (see definitions after eq. \((5)\)) the only suitable term occurs in the tensor part, but not the scalar part obtained after the standard SO(5) (York) decomposition of the metric fluctuation \([88]\). A main difference between the scalar and tensor contribution in the flow equation is the fact that the scalar part has, due to higher-derivative terms not affecting the tensor part, an additional overall factor of \(b\) after expansion in \(\tilde{R}\). Therefore, the lowest order contribution in \(b\) to the beta function \(\beta_b\) comes entirely from the tensor part. In particular, it originates from a term proportional to \(k \partial_b \tilde{f}_R - 2 \tilde{R} f_{RR} \tilde{R}\) giving this way the required contributions to \(B_2\) and \(B_0\) respectively. The second term results from the quadratic mass dimension of the Ricci scalar and is responsible for the eigenvalue \(-2\). It is remarkable that this eigenvalue results entirely from the gauge-invariant tensor part and not the gauge-dependent scalar part. We plan to present a detailed explanation of this point in [24].

1. Beta function coefficients

The explicit form of the coefficients in eqs. \((A1)-(A2)\), is given by
\[ A_0 = \frac{\tilde{G} \left( b^3 (144\pi - 301\tilde{G}) + 3456\pi b^2 (17\tilde{G} - 8\pi)\tilde{G} + 9216\pi^2 b (144\pi - 323\tilde{G})\tilde{G}^2 + 17694720\pi^3 \tilde{G}^4 \right)}{72\pi b (b - 96\pi \tilde{G})^2} \quad (A5) \]
\[ A_1 = \frac{4\tilde{G} \left( b^3 - 225\pi b^2 \tilde{G} + 15840\pi^2 b \tilde{G}^2 - 276480\pi^3 \tilde{G}^3 \right)}{9\pi b (b - 96\pi \tilde{G})^2} \quad (A6) \]
\[ A_2 = \frac{16\tilde{G} \left( b^2 - 200\pi b \tilde{G} + 7680\pi^2 \tilde{G}^2 \right)}{b^2 (b - 96\pi \tilde{G})^2} \quad (A7) \]
\[ B_0 = \frac{-491b^5 - 157088\pi b^4 \tilde{G} + 18275328\pi^2 b^3 \tilde{G}^2 - 916586496\pi^3 b^2 \tilde{G}^3 + 17694720000\pi^4 b \tilde{G}^4 - 135895449600\pi^5 \tilde{G}^5}{2880\pi^2 (b - 96\pi \tilde{G})^3} \quad (A8) \]
\[ B_1 = \frac{-89b^5 + 31818\pi b^4 \tilde{G} - 4328064\pi^2 b^3 \tilde{G}^2 + 276203520\pi^3 b^2 \tilde{G}^3 - 8493465600\pi^4 b \tilde{G}^4 + 101921587200\pi^5 \tilde{G}^5}{4320\pi^2 \tilde{G} (b - 96\pi \tilde{G})^3} \quad (A9) \]
\[ B_2 = \frac{\tilde{G} \left( 73b^4 - 222912\pi b^3 \tilde{G} + 24247926\pi^2 b^2 \tilde{G}^2 - 1150156800\pi^3 b \tilde{G}^3 + 16986931200\pi^4 \tilde{G}^4 \right)}{720\pi b (b - 96\pi \tilde{G})^3} \quad (A10) \]

Appendix B: Slow–roll parameters and spectra

The slow–roll parameter \(\epsilon \equiv -\dot{H}/H^2\) was introduced in \((31)\), and here we introduce the second–order slow–roll parameters as \([89]\)
\[ \epsilon_2 \equiv \frac{\dot{R}}{2H f_R} = \frac{f_{RR}}{2H f_R} \tilde{R}, \quad (B1) \]
\[ \epsilon_3 \equiv \frac{\dot{R}}{H f_R} = \frac{f_{RR}}{f_R} \frac{\dot{R}}{H} + \frac{\dot{R}}{H \tilde{R}}. \quad (B2) \]

During slow-roll we have
\[ R = 6H^2 (2 - \epsilon), \quad \tilde{R} \simeq -24H^2 \epsilon, \quad \dot{R} \simeq 24H^4 \epsilon (3\epsilon - i/n), \quad (B3) \]
\[ i/n \epsilon \simeq 2\epsilon \]
and for the action \((28)\) one has \(f_{RRR} = 0\) and \(\dot{H}/H^2 \simeq 0\). We can can then approximate
\[ \epsilon_2 \simeq -\epsilon, \quad \epsilon_3 = \frac{\dot{R}}{H \tilde{R}} \simeq -\epsilon. \quad (B4) \]

The tree-level scalar and tensor power spectrum re-
spectively can be found to be \[68, 82, 85, 86\] (we follow the definitions in \[68\])

\[P_S = \frac{(1 + c_2)^2}{12 f_R} \left[1 - e^{\frac{r}{c_2}} \Gamma(\frac{2}{3}) \frac{H^2}{2 \pi} \left(\frac{1}{2} |k_F \eta|\right)^{n_S - 1}\right], \]  
(B5)

\[P_T \simeq \frac{H^2}{s^2 f_R} \left[1 - e^{\frac{r}{c_2}} \Gamma(\frac{2}{3}) \frac{H^2}{2 \pi} \left(\frac{1}{2} |k_F \eta|\right)^{n_T}\right], \]  
(B6)

with \(\eta \equiv \int a^{-1}dt\) the conformal time, and the corresponding spectral indices defined as \(n_S - 1 \simeq -4c - 2c_2 - 2\lambda_s\), and \(n_T \simeq -2c - 2c_2\) respectively. Notice that the Fourier wave number is denoted by \(k_F\) to distinguish it from the RG scale \(k\). Using the slow-roll approximations of (B4) we have that

\[n_S - 1 \simeq -4c, \quad n_T \simeq 0. \]  
(B7)

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