Fermionic model with a non-Hermitian Hamiltonian

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Abstract

This paper deals with the mathematical spectral analysis and physical interpretation of a fermionic system described by a non-Hermitian Hamiltonian possessing real eigenvalues. A statistical thermodynamical description of such a system is considered. Approximate expressions for the energy expectation value and the number operator expectation value, in terms of the absolute temperature $T$ and of the chemical potential $\mu$, are obtained, based on the Euler-Maclaurin formula.

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1 Introduction: fermionic creation and annihilation operators

The observables of a physical system are usually Hermitian operators, which describe measurable quantities. Elementary particles are either bosons or fermions. Fermions, contrarily to bosons, are described by bounded operators. The fermionic
operators act on an infinite-dimensional Hilbert space $\mathcal{H}$. More concretely, $\mathcal{H}$ is the direct sum of the spaces of \textit{completely antisymmetric tensors} of rank $k$, $k = 0, 1, 2, 3, \ldots$ over $\mathbb{C}^\infty$,

$$\mathcal{H} = \mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \ldots,$$

with the inner product $\langle \cdot, \cdot \rangle$ defined below. The Hilbert space of individual fermionic states is $\mathbb{C}^\infty$. Let $\{e_j\}$ be an orthonormal (o.n.) basis of $\mathbb{C}^\infty$. By $\{e_{j_1} \wedge \cdots \wedge e_{j_k} : j_1, \ldots, j_k = 1, 2, 3, \ldots\}$, we denote a basis of $\mathcal{A}_k$, constituted by the following tensors

$$e_{j_1} \wedge \cdots \wedge e_{j_k} = \frac{1}{\sqrt{k!}} \sum_{\sigma \in S_k} \text{sign}(\sigma) e_{j_{\sigma(1)}} \otimes \cdots \otimes e_{j_{\sigma(k)}},$$

where $S_k$ is the \textit{symmetric group of degree} $k$ and “sign” represents the $\pm$ sign of the permutation. Clearly

$$e_{j_1} \wedge \cdots \wedge e_{j_k} = \text{sign}(\sigma) e_{j_{\sigma(1)}} \wedge \cdots \wedge e_{j_{\sigma(k)}}.$$

The inner product in $\mathcal{A}_k$ is defined as

$$\langle \phi, \psi \rangle_k := \sum_{1 \leq j_1 < \cdots < j_k \leq \infty} \overline{\psi_{j_1, \ldots, j_k}} \phi_{j_1, \ldots, j_k},$$

where,

$$\phi = \sum_{1 \leq j_1 < \cdots < j_k \leq \infty} \phi_{j_1, \ldots, j_k} e_{j_1} \wedge \cdots \wedge e_{j_k}, \quad \psi = \sum_{1 \leq j_1 < \cdots < j_k \leq \infty} \psi_{j_1, \ldots, j_k} e_{j_1} \wedge \cdots \wedge e_{j_k}.$$

The inner product in $\mathcal{H}$ is defined as

$$\langle \phi, \psi \rangle_{\mathcal{H}} := \overline{\psi_{j_0}} \phi_{j_0} + \sum_{k=1}^{\infty} \sum_{1 \leq j_1 < \cdots < j_k \leq \infty} \overline{\psi_{j_1, \ldots, j_k}} \phi_{j_1, \ldots, j_k}, \quad \psi_{j_0}, \phi_{j_0} \in \mathbb{C},$$

where, for $\psi_{j_1, \ldots, j_k}, \phi_{j_1, \ldots, j_k} \in \mathbb{C},$

$$\phi = \phi_{j_0} + \sum_{k=1}^{\infty} \sum_{1 \leq j_1 < \cdots < j_k \leq \infty} \phi_{j_1, \ldots, j_k} e_{j_1} \wedge \cdots \wedge e_{j_k},$$

$$\psi = \psi_{j_0} + \sum_{k=1}^{\infty} \sum_{1 \leq j_1 < \cdots < j_k \leq \infty} \psi_{j_1, \ldots, j_k} e_{j_1} \wedge \cdots \wedge e_{j_k}.$$

We consider \textit{fermionic creation operators} $c^\dagger_j : \mathcal{A}_k \to \mathcal{A}_{k+1}$,

$$c^\dagger_j e_{j_1} \wedge \cdots \wedge e_{j_k} = e_j \wedge e_{j_1} \wedge \cdots \wedge e_{j_k},$$
and fermionic annihilation operators $c_j : A_k \to A_{k-1}$,

\[ c_j \psi = 0 \quad \text{if} \quad \psi \in A_0, \quad c_{j_1} e_{j_1} \wedge \cdots \wedge e_{j_k} = e_{j_2} \wedge \cdots \wedge e_{j_k}, \]

and $c_j e_{j_1} \wedge \cdots \wedge e_{j_k} = 0$ if $j \not\in \{j_1, \ldots, j_k\}$.

The operator $c_j^\dagger$ is the adjoint of $c_j$, 

\[ \langle c_j^\dagger \phi, \psi \rangle = \langle \phi, c_j \psi \rangle, \quad \phi, \psi \in \mathcal{H}. \]

The following anticommutation relations for the fermionic operators $c_j^\dagger, c_j$ hold:

\[ \{c_i^\dagger, c_j\} = c_i^\dagger c_j + c_j c_i^\dagger = \delta_{ij}, \quad \{c_i^\dagger, c_j^\dagger\} = \{c_i, c_j\} = 0, \quad i, j = 1, 2, 3, \ldots, n, \]

where $\delta_{ij}$ is the Kronecker symbol. Notice that

\[ [c_i^\dagger c_j, c_k] = -\delta_{jk} c_i^\dagger, \quad [c_i^\dagger c_j, c_k] = \delta_{ik} c_j. \]

The number operator in state $i$ is the Hermitian operator given by

\[ N_{op_i} = c_i^\dagger c_i, \]

and its eigenvalues are 0 and 1, being the number of fermions in that state. The total number operator is $N_{op} = N_{op_1} + N_{op_2} + N_{op_3} \ldots$.

In the last two decades, the quantum physics of systems described by non-Hermitian Hamiltonians has attracted the attention of researchers, from mathematicians to theoretical and applied physicists. Several classes of models have been investigated including bosonic systems, relevant in the so-called PT- and pseudo-Hermitian-quantum mechanics, see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 15] and references therein.

In the context of quantum systems with non-Hermitian Hamiltonians, pseudo fermionic operators appear, instead of fermionic operators, and the anticommutation relations are replaced by $\{a, b\} = 1$. In this case, $a^2 = b^2 = 0$, but $b$ is not assumed to be the adjoint of $a$.

The paper is organized as follows. In Section 2 the spectral analysis of a non-Hermitian semi-infinite matrix $\hat{H}$ is performed. In Section 3 a metric matrix which renders $\hat{H}$ Hermitian is constructed. The obtained results are crucial in the remaining parts of the paper. In Section 4 a fermionic model characterized by a non-Hermitian Hamiltonian with real eigenvalues is introduced. In Section 5 the fermionic Hamiltonian is expressed in terms of dynamical pseudo-fermionic operators, using the results in Section 2. In Section 6 the so called Physical Hilbert space is introduced. In Section 7 statistical thermodynamics considerations are applied to the studied fermionic Hamiltonian. A numerical Example is also presented. In Section 8 some conclusions are drawn.
2 A non-Hermitian matrix with real eigenvalues

Let us consider the semi-infinite tridiagonal matrix, which has a central role in the paper,

\[
\hat{H} = \frac{1}{2\sqrt{2}} \begin{bmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\gamma & 0 & 0 & \ldots \\
\frac{1}{\sqrt{2}}\gamma & \frac{5}{\sqrt{2}} & -\sqrt{3} \cdot 4\gamma & 0 & \ldots \\
0 & \sqrt{3} \cdot 4\gamma & \frac{9}{\sqrt{2}} & -\sqrt{5} \cdot 6\gamma & \ldots \\
0 & 0 & \sqrt{5} \cdot 6\gamma & \frac{13}{\sqrt{2}} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}, \quad \gamma \in \mathbb{R}. \quad (3)
\]

We will be concerned with the following matrices

\[
\hat{S}_+ = \frac{1}{2\sqrt{2}} \begin{bmatrix}
0 & 0 & 0 & 0 & \ldots \\
\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \ldots \\
0 & \sqrt{3} \cdot 4\gamma & 0 & 0 & \ldots \\
0 & 0 & \sqrt{5} \cdot 6\gamma & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix},
\]

\[
\hat{S}_- = (\hat{S}_+)^T
\]

\[
\hat{S}_0 = \text{diag}(1/4, 5/4, 9/4, \ldots).
\]

The following commutation relations are easily seen to be satisfied

\[
[\hat{S}_-, \hat{S}_0] = \hat{S}_-, \quad [\hat{S}_0, \hat{S}_+] = \hat{S}_+, \quad [\hat{S}_-, \hat{S}_+] = \hat{S}_0.
\]

Obviously,

\[
\hat{H} = \hat{S}_0 + \gamma(\hat{S}_+ - \hat{S}_-).
\]

**Proposition 2.1** *The spectrum of* \(\hat{H}\) *is*

\[
\sigma(\hat{H}) = \sqrt{1 + 2\gamma^2} \{1/4, 5/4, 9/4, \ldots\},
\]

*and the corresponding eigenvectors of* \(\hat{H}\) *are given by*

\[
\hat{\psi}_n = \hat{S}_+^{n-1}\hat{\psi}_1, \quad n = 1, 2, \ldots
\]

*where* \(\hat{\psi}_1\) *is such that*

\[
\hat{S}_-\hat{\psi}_1 = 0.
\]

**Proof.** In order to obtain the eigenvalues and eigenvectors of \(\hat{H}\) the *equation of motion method* (EMM) \[11, 6\] is used,

\[
[\hat{H}, z\hat{S}_0 + x\hat{S}_+ + y\hat{S}_-] = \hat{S}_0(-\gamma x - \gamma y) + \hat{S}_+(-\gamma z + x) + \hat{S}_-(-\gamma z - y)
\]

\[
= \Lambda(z\hat{S}_0 + x\hat{S}_+ + y\hat{S}_-), \quad \Lambda, x, y, z \in \mathbb{R}.
\]

\[4\]
This method leads to the $3 \times 3$ matrix eigenproblem,

\[
\begin{bmatrix}
  0 & -\gamma & -\gamma \\
  -\gamma & 1 & 0 \\
  -\gamma & 0 & -1
\end{bmatrix}
\begin{bmatrix}
  z \\
  x \\
  y
\end{bmatrix} = \Lambda
\begin{bmatrix}
  z \\
  x \\
  y
\end{bmatrix}, \quad \Lambda \in \mathbb{R}.
\]

The eigenvalues are easily obtained,

\[\Lambda_0 = 0, \quad \Lambda_1 = -\sqrt{1 + 2\gamma^2}, \quad \Lambda_2 = \sqrt{1 + 2\gamma^2},\]

as well as the respective eigenvectors

\[u_0 = [(1, \gamma, -\gamma)]^T,\]
\[u_- = \left[1, -\frac{1 - \sqrt{1 + 2\gamma^2}}{2\gamma}, \frac{1 + \sqrt{1 + 2\gamma^2}}{2\gamma}\right]^T,\]
\[u_+ = \left[1, -\frac{1 + \sqrt{1 + 2\gamma^2}}{2\gamma}, \frac{1 - \sqrt{1 + 2\gamma^2}}{2\gamma}\right]^T.\]

From the normalized eigenvectors, the following matrices are constructed:

\[
\hat{T}_0 = \frac{1}{\sqrt{1 + 2\gamma^2}} (\hat{S}_0 + \gamma(\hat{S}_+ - \hat{S}_-)),
\]
\[
\hat{T}_- = \frac{\gamma}{\sqrt{1 + 2\gamma^2}} \hat{S}_0 - \frac{1 - \sqrt{1 + 2\gamma^2}}{2\sqrt{1 + 2\gamma^2}} \hat{S}_+ + \frac{1 + \sqrt{1 + 2\gamma^2}}{2\sqrt{1 + 2\gamma^2}} \hat{S}_-,
\]
\[
\hat{T}_+ = -\frac{\gamma}{\sqrt{1 + 2\gamma^2}} \hat{S}_0 + \frac{1 + \sqrt{1 + 2\gamma^2}}{2\sqrt{1 + 2\gamma^2}} \hat{S}_+ - \frac{1 - \sqrt{1 + 2\gamma^2}}{2\sqrt{1 + 2\gamma^2}} \hat{S}_-. \tag{5}
\]

These matrices obey the same commutation relations as the matrices $\hat{S}_0, \hat{S}_+, \hat{S}_-$:

\[
[\hat{T}_-, \hat{T}_0] = \hat{T}_-, \quad [\hat{T}_0, \hat{T}_+] = \hat{T}_+, \quad [\hat{T}_-, \hat{T}_+] = \hat{T}_0,
\]

and they characterize the $su(1,1)$ algebra.

In order to determine $\hat{\psi}_1$, we notice that

\[
\hat{T}_- = \frac{1}{4}
\begin{bmatrix}
  1 & \sqrt{1 \cdot 2} \eta^{-1} & 0 & 0 & \ldots \\
  \sqrt{1 \cdot 2} \eta & \frac{5}{5} \sqrt{3 \cdot 4} \eta^{-1} & \sqrt{3 \cdot 4} \eta^{-1} & 0 & \ldots \\
  0 & \sqrt{3 \cdot 4} \eta & \frac{9}{9} \sqrt{5 \cdot 6} \eta^{-1} & \sqrt{5 \cdot 6} \eta^{-1} & \ldots \\
  0 & 0 & \sqrt{5 \cdot 6} \eta & 13 & \ldots \\
  \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix},
\]

where

\[
\eta = \frac{\sqrt{1 + 2\gamma^2} - 1}{\sqrt{2} \gamma}.
\]
We easily find that
\[
\hat{\psi}_1 = \left[ 1, -\sqrt{\frac{1}{2}} \eta, \sqrt{\frac{1}{2} \cdot 4} \eta^2, -\sqrt{\frac{1}{2} \cdot 4 \cdot 6} \eta^3, \ldots \right]^T,
\]
satisfies
\[
\hat{H} \hat{\psi}_1 = \frac{1}{4} \hat{\psi}_1.
\]
Next, we notice the following: If \( \Lambda \) is an eigenvalue of \( \hat{H} \) with eigenvector \( \hat{\psi} \),
\[
\hat{H} \hat{\psi} = \Lambda \hat{\psi},
\]
then \( \Lambda + \sqrt{1 + 2 \gamma^2} \) is an eigenvalue of \( \hat{H} \) with eigenvector \( \hat{T}_+ \hat{\psi} \), that is,
\[
\hat{H} \hat{T}_+ \hat{\psi} = \left( \Lambda + \sqrt{1 + 2 \gamma^2} \right) \hat{T}_+ \hat{\psi}.
\]
Similarly, if \( \Lambda - \sqrt{1 + 2 \gamma^2} \) is an eigenvalue of \( \hat{H} \) with eigenvector \( \hat{T}_- \hat{\psi} \),
\[
\hat{H} \hat{T}_- \hat{\psi} = \left( \Lambda - \sqrt{1 + 2 \gamma^2} \right) \hat{T}_- \hat{\psi},
\]
provided \( \hat{T}_- \hat{\psi} \neq 0 \). Now, the claim easily follows. \( \blacksquare \)

**Proposition 2.2** The eigenvectors of \( \hat{H}^T \) are given by
\[
\hat{\psi}_n = (\hat{T}_-^T)^{n-1} \hat{\psi}_1, \ n = 1, 2, \ldots
\]
where \( \hat{\psi}_1 \) is such that
\[
(\hat{T}_+^T) \hat{\psi}_1 = 0.
\]

**Proof.** We easily find that
\[
\hat{\psi}_0 = \left[ 1, \sqrt{\frac{1}{2}} \eta, \sqrt{\frac{1}{2} \cdot 4} \eta^2, \sqrt{\frac{1}{2} \cdot 4 \cdot 6} \eta^3, \ldots \right]^T,
\]
satisfies
\[
\hat{H} \hat{\psi}_0 = \frac{1}{4} \hat{\psi}_0.
\]
It is now clear that the claim holds. \( \blacksquare \)

Some observations are in order.

1. For a convenient normalization, the eigensystems \( \{\hat{\psi}_n\}, \{\tilde{\psi}_n\} \) become biorthogonal
\[
\langle \tilde{\psi}_m, \hat{\psi}_n \rangle = \delta_{mn} \langle \hat{\psi}_m, \tilde{\psi}_n \rangle.
\]

2. The matrix \( \hat{T}_+ \) is a raising matrix, and \( \hat{T}_- \) is a lowering matrix. However, \( \hat{T}_+ \) is not the adjoint of \( \hat{T}_- \), \( \hat{T}_- \neq (\hat{T}_+)^\dagger \) and \( \hat{T}_0 \) is not Hermitian, \( \hat{T}_0 \neq \hat{T}_0^\dagger \). Due to these facts, we say that the matrices \( \hat{T}_0, \hat{T}_+, \hat{T}_- \) generate a pseudo-su(1,1) algebra.
3 Metric matrix

We may easily construct a positive definite matrix $\hat{D}$ and a Hermitian matrix $\hat{H}_0$ such that

$$\hat{H} = \hat{D}^{-1}\hat{H}_0\hat{D}, \quad \hat{H}^\dagger = \hat{D}\hat{H}_0\hat{D}^{-1},$$

so that

$$\hat{H} = \hat{D}^{-2}\hat{H}_0^\dagger\hat{D}^2.$$ 

For $\hat{S}_0$, $\hat{S}_+$, $\hat{S}_-$ in (4), using the commutation relations

$$[\hat{S}_0, (\hat{S}_+ + \hat{S}_-)] = \hat{S}_+ - \hat{S}_-, \quad [(\hat{S}_+ - \hat{S}_-), (\hat{S}_+ + \hat{S}_-)] = -2\hat{S}_0,$$

by some calculations, we get

$$e^{-\alpha(\hat{S}_+ + \hat{S}_-)}\hat{S}_0e^{\alpha(\hat{S}_+ + \hat{S}_-)} = \cos\sqrt{2\alpha} \hat{S}_0 + \frac{\hat{S}_+ - \hat{S}_-}{\sqrt{2}} \sin \sqrt{2\alpha}, \quad (6)$$

$$e^{-\alpha(\hat{S}_+ + \hat{S}_-)}(\hat{S}_+ - \hat{S}_-)e^{\alpha(\hat{S}_+ + \hat{S}_-)} = \cos \sqrt{2\alpha} (\hat{S}_+ - \hat{S}_-) - \hat{S}_0\sqrt{2} \sin \sqrt{2\alpha},$$

$$e^{-\alpha(\hat{S}_+ + \hat{S}_-)}(\hat{S}_+ + \hat{S}_-)e^{\alpha(\hat{S}_+ + \hat{S}_-)} = \hat{S}_+ + \hat{S}_-. \quad (7)$$

It follows that

$$e^{-\alpha(\hat{S}_+ + \hat{S}_-)}\hat{S}_+e^{\alpha(\hat{S}_+ + \hat{S}_-)} = \frac{\cos \sqrt{2\alpha}}{2} (\hat{S}_+ - \hat{S}_-) + \frac{1}{2}(\hat{S}_+ + \hat{S}_-) - \hat{S}_0 \frac{\sqrt{2} \sin \sqrt{2\alpha}}{2},$$

$$e^{-\alpha(\hat{S}_+ + \hat{S}_-)}\hat{S}_-e^{\alpha(\hat{S}_+ + \hat{S}_-)} = \frac{\cos \sqrt{2\alpha}}{2} (\hat{S}_- - \hat{S}_+) + \frac{1}{2}(\hat{S}_+ + \hat{S}_-) + \hat{S}_0 \frac{\sqrt{2} \sin \sqrt{2\alpha}}{2}. \quad (7)$$

We have shown the following result.

**Proposition 3.1** Let

$$\gamma = \frac{\tan \sqrt{2\alpha}}{\sqrt{2}}, \quad \frac{1}{\sqrt{1 + 2\gamma^2}} = \cos \sqrt{2\alpha},$$

in (6) and (7). Then, for $\hat{T}_0$, $\hat{T}_+$ and $\hat{T}_-$ in (3),

$$e^{-\alpha(\hat{S}_+ + \hat{S}_-)}\hat{S}_0e^{\alpha(\hat{S}_+ + \hat{S}_-)} = \frac{1}{\sqrt{1 + 2\gamma^2}}(\hat{S}_0 + \gamma(\hat{S}_+ - \hat{S}_-)) = \hat{T}_0,$$

$$e^{-\alpha(\hat{S}_+ + \hat{S}_-)}\hat{S}_+e^{\alpha(\hat{S}_+ + \hat{S}_-)} =$$

$$= \frac{1}{\sqrt{1 + 2\gamma^2}} \left( \frac{1}{2}(\hat{S}_+ - \hat{S}_-) + \frac{1}{2}\sqrt{1 + 2\gamma^2}(\hat{S}_+ + \hat{S}_-) - \gamma \hat{S}_0 \right) = \hat{T}_+,$$

$$e^{-\alpha(\hat{S}_+ + \hat{S}_-)}\hat{S}_-e^{\alpha(\hat{S}_+ + \hat{S}_-)} =$$

$$= \frac{1}{\sqrt{1 + 2\gamma^2}} \left( \frac{1}{2}(\hat{S}_- - \hat{S}_+) + \frac{1}{2}\sqrt{1 + 2\gamma^2}(\hat{S}_+ + \hat{S}_-) + \gamma \hat{S}_0 \right) = \hat{T}_-. \quad 7$$
Next we observe that, if \( \tilde{e}_j \) is an eigenvector of \( \tilde{S}_0 \) associated with the eigenvalue \( (j - 3/4) \), then the eigenvectors \( \tilde{\psi}_j \) and \( \check{\psi}_j \) of \( \tilde{T}_0 \) and \( \tilde{T}_0^\dagger \) associated with the same eigenvalue satisfy
\[
\tilde{\psi}_j = e^{-\alpha(\tilde{S}_+ + \tilde{S}_-)}\tilde{e}_j, \quad \check{\psi}_j = e^{\alpha(\tilde{S}_+ + \tilde{S}_-)}\tilde{e}_j.
\]
Indeed,
\[
\tilde{T}_0 e^{-\alpha(\tilde{S}_+ + \tilde{S}_-)}\tilde{e}_j = e^{-\alpha(\tilde{S}_+ + \tilde{S}_-)}\tilde{S}_0 \tilde{e}_j = \left( j - \frac{3}{4} \right) e^{-\alpha(\tilde{S}_+ + \tilde{S}_-)}\tilde{e}_j,
\]
\[
\tilde{T}_0^\dagger e^{\alpha(\tilde{S}_+ + \tilde{S}_-)}\tilde{e}_j = e^{\alpha(\tilde{S}_+ + \tilde{S}_-)}\tilde{S}_0 \tilde{e}_j = \left( j - \frac{3}{4} \right) e^{\alpha(\tilde{S}_+ + \tilde{S}_-)}\tilde{e}_j.
\]

We observe that \( e^{-\alpha(\tilde{S}_+ + \tilde{S}_-)} \) goes from \( \text{span}\{\tilde{e}_j\} \) to \( \text{span}\{\tilde{\psi}_j\} \) and from \( \text{span}\{\check{\psi}_j\} \) to \( \text{span}\{\check{\psi}_j\} \), while \( e^{\alpha(\tilde{S}_+ + \tilde{S}_-)} \) goes from \( \text{span}\{\check{\psi}_j\} \) to \( \text{span}\{\tilde{e}_j\} \) and from \( \text{span}\{\check{\psi}_j\} \) to \( \text{span}\{\check{\psi}_j\} \), according with
\[
e^{-\alpha(\tilde{S}_+ + \tilde{S}_-)}(\text{span}\{\tilde{e}_j\}) = \text{span}\{\tilde{\psi}_j\}, \quad e^{-\alpha(\tilde{S}_+ + \tilde{S}_-)}(\text{span}\{\check{\psi}_j\}) = \text{span}\{\check{\psi}_j\},
\]
and
\[
e^{\alpha(\tilde{S}_+ + \tilde{S}_-)}(\text{span}\{\tilde{\psi}_j\}) = \text{span}\{\tilde{e}_j\}, \quad e^{\alpha(\tilde{S}_+ + \tilde{S}_-)}(\text{span}\{\check{\psi}_j\}) = \text{span}\{\check{\psi}_j\}.
\]

It follows that,
\[
\tilde{\psi}_j = e^{-2\alpha(\tilde{S}_+ + \tilde{S}_-)}\check{\psi}_j
\]
and so,
\[
e^{2\alpha(\tilde{S}_+ + \tilde{S}_-)}(\text{span}\{\tilde{\psi}_j\}) = \text{span}\{\check{\psi}_j\}, \quad e^{-2\alpha(\tilde{S}_+ + \tilde{S}_-)}(\text{span}\{\check{\psi}_j\}) = \text{span}\{\tilde{\psi}_j\}.
\]

As consequence,
\[
e^{2\alpha(\tilde{S}_+ + \tilde{S}_-)}\tilde{H} e^{-2\alpha(\tilde{S}_+ + \tilde{S}_-)} = \tilde{H}^\dagger,
\]
and we find the metric matrix
\[
\tilde{D}^2 = e^{2\alpha(\tilde{S}_+ + \tilde{S}_-)},
\]
which induces the inner product
\[
\langle \cdot, \cdot \rangle_{\tilde{D}^2} := \langle \tilde{D}^2 \cdot, \cdot \rangle.
\]

The following result is now easily obtained.

**Proposition 3.2** With respect to the inner product \( \langle \cdot, \cdot \rangle_{\tilde{D}^2} \), the infinite matrix \( \tilde{H}_0 \) is Hermitian. Moreover, \( \tilde{T}_0 \) is also Hermitian and \( \tilde{T}^+ \) is the adjoint of \( \tilde{T}_- \).
Proof. The first assertion is proved noting that

$$\langle \hat{H} \hat{\psi}, \hat{\psi} \rangle_{\hat{D}^2} = \langle \hat{D}^2 \hat{H} \hat{\psi}, \hat{\psi} \rangle = \langle \hat{\psi}, \hat{H} \hat{D}^2 \hat{\psi} \rangle = \langle \hat{D}^2 \hat{\psi}, \hat{H} \hat{\psi} \rangle = \langle \hat{\psi}, \hat{H} \hat{\psi} \rangle_{\hat{D}^2}. $$

The next assertion is similarly shown. The last assertion clearly follows, observing that

$$\hat{D}^{-1} \hat{S} \hat{D} = \hat{T}_+, \quad \hat{D}^{-1} \hat{S} \hat{D}^2 = \hat{T}_-, $$

implies

$$\hat{D}^2 \hat{T}_+ = \hat{T}_+^\dagger \hat{D}^2. $$

4 A fermionic model

We are concerned with the following non-Hermitian Hamiltonian

$$H = \frac{1}{4} c_1^\dagger c_1 + \frac{5}{4} c_2^\dagger c_2 + \frac{9}{4} c_3^\dagger c_3 + \ldots $$

$$+ \gamma \left( \sqrt{\frac{1 \cdot 2}{8}} (c_2^\dagger c_1 - c_1^\dagger c_2) + \sqrt{\frac{3 \cdot 4}{8}} (c_3^\dagger c_2 - c_2^\dagger c_3) + \sqrt{\frac{5 \cdot 6}{8}} (c_4^\dagger c_3 - c_3^\dagger c_4) + \ldots \right), \quad \gamma \in \mathbb{R},$$

where $c_j^\dagger$ and its adjoint $c_j$ are fermionic operators.

In terms of the Hermitian operators

$$S_0 = \frac{1}{4} c_1^\dagger c_1 + \frac{5}{4} c_2^\dagger c_2 + \frac{9}{4} c_3^\dagger c_3 + \ldots $$

$$S_- = \sqrt{\frac{1 \cdot 2}{8}} c_1^\dagger c_2 + \sqrt{\frac{3 \cdot 4}{8}} c_2^\dagger c_3 + \sqrt{\frac{5 \cdot 6}{8}} c_3^\dagger c_4 + \ldots $$

$$S_+ = \sqrt{\frac{1 \cdot 2}{8}} c_2^\dagger c_1 + \sqrt{\frac{3 \cdot 4}{8}} c_3^\dagger c_2 + \sqrt{\frac{5 \cdot 6}{8}} c_4^\dagger c_3 + \ldots ,$$

the Hamiltonian $H$ is expressed as

$$H = S_0 + \gamma (S_+ - S_-).$$

The following commutation relations are easily seen to hold

$$[S_-, S_0] = S_- , \quad [S_0, S_+] = S_+ , \quad [S_-, S_+] = S_0.$$
4.1 Raising and lowering operators

Since $H, S_0, S_+, S_-$ commute with $N_{op}$, the eigenspaces of $N_{op}$, namely $A_0, A_1, A_2, \ldots$ are invariant spaces of $H, S_0, S_+, S_-$. We also notice that $H, H^\dagger$ and $N_{op}$ have the same vacuum $\phi_0 \in A_0$:

$$H\phi_0 = H^\dagger\phi_0 = N_{op}\phi_0 = 0.$$  

In order to obtain the eigenvalues and eigenvectors of $H$ the equation of motion method (EMM) is also used,

$$[H, zS_0 + xS_+ + yS_-] = S_0(-\gamma x - \gamma y) + S_+(-\gamma z + x) + S_-(-\gamma z - y)
= \Lambda(zS_0 + xS_+ + yS_-), \quad \Lambda, x, y, z \in \mathbb{R}.$$  

This method leads to the $3 \times 3$ matrix eigenproblem,

$$\begin{bmatrix} 0 & -\gamma & -\gamma \\ -\gamma & 1 & 0 \\ -\gamma & 0 & -1 \end{bmatrix} \begin{bmatrix} z \\ x \\ y \end{bmatrix} = \Lambda \begin{bmatrix} z \\ x \\ y \end{bmatrix}, \quad \Lambda \in \mathbb{R},$$

whose eigenvalues are readily obtained,

$$\Lambda_0 = 0, \quad \Lambda_1 = -\sqrt{1 + 2\gamma^2}, \quad \Lambda_2 = \sqrt{1 + 2\gamma^2},$$

as well as the respective eigenvectors,

$$u_0 = [(1, \gamma, -\gamma)]^T,$$

$$u_- = \begin{bmatrix} 1, -\frac{1 - \sqrt{1 + 2\gamma^2}}{2\gamma}, \frac{1 + \sqrt{1 + 2\gamma^2}}{2\gamma} \end{bmatrix}^T,$$

$$u_+ = \begin{bmatrix} 1, -\frac{1 + \sqrt{1 + 2\gamma^2}}{2\gamma}, \frac{1 - \sqrt{1 + 2\gamma^2}}{2\gamma} \end{bmatrix}^T.$$  

From the normalized eigenvectors the following operators are constructed:

$$T_0 = \frac{1}{\sqrt{1 + 2\gamma^2}}(S_0 + \gamma(S_+ - S_-)),$$

$$T_- = \frac{\gamma}{\sqrt{1 + 2\gamma^2}}S_0 - \frac{1 - \sqrt{1 + 2\gamma^2}}{2\sqrt{1 + 2\gamma^2}}S_+ + \frac{1 + \sqrt{1 + 2\gamma^2}}{2\sqrt{1 + 2\gamma^2}}S_-,$$

$$T_+ = -\frac{\gamma}{\sqrt{1 + 2\gamma^2}}S_0 + \frac{1 + \sqrt{1 + 2\gamma^2}}{2\sqrt{1 + 2\gamma^2}}S_+ - \frac{1 - \sqrt{1 + 2\gamma^2}}{2\sqrt{1 + 2\gamma^2}}S_-.$$  

(8)

These operators obey the same commutation relations as the operators $S_0, S_+, S_-$ which characterize the $su(1,1)$ algebra,

$$[T_-, T_0] = T_-, \quad [T_0, T_+] = T_+, \quad [T_-, T_+] = T_0.$$  

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We say that $T_+$ is a *raising operator*, because, if $\lambda$ is an eigenvalue of $H$ with eigenvector $\psi$, i.e.,

$$H\psi = \lambda \psi,$$

then $(\lambda + \sqrt{1 + 2\gamma^2})$ is an eigenvalue of $H$ with eigenvector $T_+\psi$, that is,

$$HT_+\psi = \left(\lambda + \sqrt{1 + 2\gamma^2}\right)T_+\psi.$$ 

Similarly, $T_-$ is a *lowering operator*, because $(\lambda - \sqrt{1 + 2\gamma^2})$ is an eigenvalue of $H$ with eigenvector $T_-\psi$, that is,

$$HT_-\psi = \left(\lambda - \sqrt{1 + 2\gamma^2}\right)T_-\psi,$$

provided $T_-\psi \neq 0$. However, $T_+$ is not the adjoint of $T_-$, $T_+ \neq (T_+)^\dagger$ and $T_0$ is not Hermitian, $T_0 \neq T_0^\dagger$. Due to these facts, we say that the operators $T_0, T_+, T_-$ generate a *pseudo-su(1,1)* algebra.

We have shown the following.

**Proposition 4.1** The eigenvalues of $H$ associated with eigenvectors in $\mathcal{A}_1$ are $\sqrt{1 + 2\gamma^2}$ $(1/4, 5/4, 9/4, \ldots)$. The eigenvectors of $H$ in $\mathcal{A}_1$ are

$$\psi_n = T_+^{n-1}\psi_1, \quad n = 2, 3, 4, \ldots,$$

where $\psi_1 \in \mathcal{A}_1$ is such that

$$T_-\psi_1 = 0.$$

**Proof.** The result follows, observing that the eigenvalues of $T_0$ associated with eigenvectors in $\mathcal{A}_1$ are $1/4, 5/4, 9/4, \ldots.$ $\blacksquare$

**Proposition 4.2** The eigenvectors of $H^\dagger$ in $\mathcal{A}_1$ are

$$\tilde{\psi}_n = (T_-)^n\tilde{\psi}_1,$$

where $\tilde{\psi}_1 \in \mathcal{A}_1$ satisfies

$$(T_+)^\dagger\tilde{\psi}_1 = 0.$$

The eigenvector systems $\{\psi_n\}, \{\tilde{\psi}_n\}$ may be made biorthonormal so that

$$\langle \tilde{\psi}_m, \psi_n \rangle = \delta_{mn}\langle \tilde{\psi}_n, \psi_n \rangle.$$

We observe that the restriction of the operator $H$ to $\mathcal{A}_1$ is identified with the matrix $\hat{H}$ acting on $\mathbb{C}$. 

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5 Dynamical fermionic operators

The dynamical fermionic operators are linear combinations $x_1 c_1^\dagger + x_2 c_2^\dagger + x_3 c_3^\dagger + \ldots$, $x_1, x_2, x_3, \ldots \in \mathbb{R}$, such that

\[
\begin{align*}
H, (x_1 c_1^\dagger + x_2 c_2^\dagger + x_3 c_3^\dagger + \ldots) &= \left(\frac{1}{4} x_1 - \sqrt{\frac{1}{8} \gamma x_2} \right) c_1^\dagger \\
+ \left(\sqrt{\frac{1}{2} \gamma x_1} + \frac{5}{4} x_2 - \sqrt{\frac{3}{4} \gamma x_3} \right) c_2^\dagger + \left(\sqrt{\frac{3}{4} \gamma x_2} + \frac{9}{8} x_3 - \sqrt{\frac{5}{6} \gamma x_4} \right) c_3^\dagger + \ldots
= \lambda(x_1 c_1^\dagger + x_2 c_2^\dagger + x_3 c_3^\dagger + \ldots), \quad \lambda \in \mathbb{R}.
\end{align*}
\]

The EMM leads to the eigenproblem,

\[
\begin{bmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & \ldots \\
\frac{1}{\sqrt{2}} & 0 & \frac{9}{\sqrt{2}} & 0 & \ldots \\
0 & \frac{9}{\sqrt{2}} & 0 & \frac{13}{\sqrt{2}} & \ldots \\
0 & 0 & \frac{13}{\sqrt{2}} & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\vdots
\end{bmatrix}
= \lambda
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\vdots
\end{bmatrix},
\]

involving the matrix $\hat{H}$ in [3], Section 2.

Let us express the eigenvector $\hat{\psi}_n$ of $\hat{H}$ as

\[
\hat{\psi}_n = \begin{bmatrix} x_1^{(n)} & x_2^{(n)} & x_3^{(n)} & \ldots \end{bmatrix}^T,
\]

and the eigenvector $\check{\psi}_n$ of $\hat{H}^T$ as

\[
\check{\psi}_n = \begin{bmatrix} y_1^{(n)} & y_2^{(n)} & y_3^{(n)} & \ldots \end{bmatrix}^T.
\]

Pseudo-fermionic operators may now be constructed

\[
d_i^\dagger := x_1^{(i)} c_1^\dagger + x_2^{(i)} c_2^\dagger + x_3^{(i)} c_3^\dagger + \ldots,
\]

\[
d_i := y_1^{(i)} c_1 + y_2^{(i)} c_2 + y_3^{(i)} c_3 + \ldots.
\]

The following anticommutation relations hold

\[
\{d_i^\dagger, d_j\} = d_i^\dagger d_j + d_j d_i^\dagger = \delta_{ij}, \quad \{d_i^\dagger, d_j^\dagger\} = \{d_i, d_j\} = 0, \quad i, j = 1, 2, 3, \ldots, n.
\]

These operators are called pseudo-fermionic because $d_i^\dagger \neq d_i^\dagger$.

The proof of the next result is independent from the proofs of Propositions 4.1 and 4.2.
Theorem 5.1 In terms of the pseudo-fermionic operators, the Hamiltonian \( H \) may be expressed as
\[
H = \sqrt{1 + 2\gamma^2} \left( \frac{1}{4} d_1^\dagger d_1 + \frac{5}{4} d_2^\dagger d_2 + \frac{9}{4} d_3^\dagger d_3 + \ldots \right).
\]

Further,
\[
\sigma(H) = \left\{ \sqrt{1 + 2\gamma^2} \sum_{k=1}^{\infty} \frac{4k-3}{4} n_k, \ n_k \in \{0, 1\}, \ k = 1, 2, 3, \ldots \right\},
\]
and the associated eigenvectors are expressed as
\[
\Psi_{n_1, n_2, n_3, \ldots} = \left( (d_1^\dagger)^{n_1} (d_2^\dagger)^{n_2} (d_3^\dagger)^{n_3} \ldots \right) \psi, \ \psi \in \mathcal{A}_0, \ n_k \in \{0, 1\}, \ k = 1, 2, 3, \ldots.
\]

Proof. Denote by \( \lambda_n \) the common eigenvalue of \( \hat{H} \) and \( \hat{H}^T \) associated, respectively with the eigenvectors \( \hat{\psi}_n \) and \( \check{\psi}_n \),
\[
\hat{H} \hat{\psi}_n = \lambda_n \hat{\psi}_n, \quad \hat{H} \check{\psi}_n = \lambda_n \check{\psi}_n.
\]

Let \( \hat{U} \) and \( \check{U} \) be the matrices
\[
\hat{U} = [\hat{\psi}_1, \hat{\psi}_2, \hat{\psi}_3, \ldots], \quad \check{U} = [\check{\psi}_1, \check{\psi}_2, \check{\psi}_3, \ldots],
\]
whose columns are the eigenvectors \( \hat{\psi}_n \) and \( \check{\psi}_n \) of \( \hat{H} \) and \( \hat{H}^\dagger \) and let
\[
\hat{H}_{\text{diag}} = \text{diag}(\lambda_1, \lambda_1, \lambda_3, \ldots),
\]
(cf. Propositions 2.1 and 2.2). Then, we may write
\[
\hat{H} \hat{U} = \hat{U} \hat{H}_{\text{diag}}, \quad \hat{H} \check{U} = \check{U} \hat{H}_{\text{diag}}.
\]
Moreover from the biorthonormality of the eigenvectors we get,
\[
\hat{U} \check{U}^T = \check{U}^T \hat{U} = I.
\]
Notice that
\[
H = [c_1^\dagger, c_2^\dagger, c_3^\dagger, \ldots] \hat{H} [c_1, c_2, c_3, \ldots]^T.
\]
Indeed,
\[
[d_1^\dagger, d_2^\dagger, d_3^\dagger, \ldots] \hat{H}_{\text{diag}} [d_1, d_2, c_3, \ldots]^T
= [d_1^\dagger, d_2^\dagger, d_3^\dagger, \ldots] \check{U}^T \hat{H}_{\text{diag}} \hat{U}^T [d_1, d_2, c_3, \ldots]^T
= [c_1^\dagger, c_2^\dagger, c_3^\dagger, \ldots] (\hat{U} \check{U}^T) \hat{H} (\hat{U} \check{U}^T) [c_1, c_2, c_3, \ldots]^T
= [c_1^\dagger, c_2^\dagger, c_3^\dagger, \ldots] \hat{H} [c_1, c_2, c_3, \ldots]^T = H.
\]
It is now easy to show that
\[
H \Psi_{n_1, n_2, \ldots} = (n_1 \lambda_1 + n_2 \lambda_2 + \ldots) \Psi_{n_1, n_2, \ldots}.
\]
Theorem 5.2 In terms of the pseudo-fermion operators, $T_0, T_+, T_-$, defined in (8) may be expressed as

$$T_0 = \frac{1}{4} x_1^1 + \frac{5}{4} x_2^1 + \frac{9}{4} x_3^1 + \ldots,$$

$$T_- = \sqrt{\frac{1\cdot 2}{8}} x_1^1 + \sqrt{\frac{3\cdot 4}{8}} x_2^1 + \sqrt{\frac{5\cdot 6}{8}} x_3^1 + \ldots,$$

$$T_+ = \sqrt{\frac{1\cdot 2}{8}} x_2^1 + \sqrt{\frac{4\cdot 4}{8}} x_3^1 + \sqrt{\frac{5\cdot 6}{8}} x_4^1 + \ldots.$$

Proof. Analogous to the previous proof.

6 Physical Hilbert space

Let

$$\mathcal{S} = \text{span}\left\{ \psi_i, \psi_i \wedge \psi_j, \psi_i \wedge \psi_j \wedge \psi_k, \ldots : i < j < k < \ldots ; i, j, k, \ldots = 1, 2, 3, \ldots \right\},$$

$$\tilde{\mathcal{S}} = \text{span}\left\{ \tilde{\psi}_i, \tilde{\psi}_i \wedge \tilde{\psi}_j, \tilde{\psi}_i \wedge \tilde{\psi}_j \wedge \tilde{\psi}_k, \ldots : i < j < k < \ldots ; i, j, k, \ldots = 1, 2, 3, \ldots \right\}.$$

We find

$$\langle \psi_i, \tilde{\psi}_{i'} \rangle = 0, \langle \psi_i \wedge \psi_j, \tilde{\psi}_{i'} \wedge \tilde{\psi}_{j'} \rangle = 0, \langle \psi_i \wedge \psi_j \wedge \psi_k, \tilde{\psi}_{i'} \wedge \tilde{\psi}_{j'} \wedge \tilde{\psi}_{k'} \rangle = 0, \ldots,$$

$$i \neq i', (i, j) \neq (i', j'), (i, j, k) \neq (i', j', k'), \ldots , i, j, k, \ldots = 1, 2, 3, \ldots.$$

Let us define $D : \mathcal{S} \rightarrow \tilde{\mathcal{S}}$ such that

$$D \psi_i := \tilde{\psi}_i, D(\psi_i \wedge \psi_j) := \tilde{\psi}_i \wedge \tilde{\psi}_j, D(\psi_i \wedge \psi_j \wedge \psi_k) := \tilde{\psi}_i \wedge \tilde{\psi}_j \wedge \tilde{\psi}_k = 0, \ldots,$$

$$i < j < k < \ldots, i, j, k, \ldots = 1, 2, 3, \ldots.$$

For $\Phi, \Psi \in \mathcal{S}$, we define the inner product

$$\langle \Phi, \Psi \rangle_D = \langle D \Phi, \Psi \rangle.$$

Following Mostafazadeh, we say that the physical Hilbert space [13] [14] is the set $\mathcal{S}$ endowed with the inner product $\langle \cdot, \cdot \rangle_D$. It may easily be seen that, for $0 \neq \Phi \in \mathcal{S}$, we have $\langle \Phi, \Phi \rangle_D > 0$.

The physical numerical range of $A$ is defined as

$$W_{\text{phys}}(A) = \{ \langle \Phi A, \Phi \rangle_D : \langle \Phi, \Phi \rangle_D = 1, \Phi \in \mathcal{H} \}.$$

and is useful in the next section.
7 Statistical thermodynamics of non-Hermitian Hamiltonians with real eigenvalues

The main objective of this section is to present the description, according to statistical thermodynamics, of a system characterized by a non-Hermitian Hamiltonian possessing real eigenvalues. Conserved quantities are operators that have real eigenvalues and commute with $H$. Since $H$ is not Hermitian, the conserved quantities may not be Hermitian, but they have the same eigenvectors as $H$, and are Hermitian with respect to the norm induced by $\langle \cdot, \cdot \rangle_D$.

In statistical thermodynamics, pure states are given by vectors and mixed states are described by density matrices, i.e., positive semidefinite Hermitian matrices with trace 1. Observable quantities are represented by Hermitian matrices. For a system with a Hermitian Hamiltonian $H$ and fermionic number operator

$$N_{op} = \sum_{i=1}^{\infty} c_i^\dagger c_i,$$

the density matrix of the equilibrium thermal state is

$$\rho_{eq} = \frac{e^{-\beta H - \zeta N_{op}}}{\text{Tr} e^{-\beta H - \zeta N_{op}}},$$

where $\beta$ is the inverse of the absolute temperature $T$ and $\zeta$ is related to the so called chemical potential $\mu$ according to $\zeta = -\beta \mu$.

If $H$ is not Hermitian, also $\rho_{eq}$ is not Hermitian, but it has real eigenvalues. The density matrix encapsulates the statistical properties of the system.

The partition function $Z$ is

$$Z = \text{Tr} \exp(-\beta H - \zeta N_{op}).$$

According to statistical thermodynamics, the equilibrium properties of the system may be derived from the logarithm of the partition function, while in classical thermodynamics, the equilibrium properties of a system may be derived from its thermodynamical potential

$$F = E - \mu N - TS,$$

where $E$ is the internal energy, $\mu$ is the chemical potential, $N$ is the number of particles, understood as the amount of some chemical compound, and $S$ the classical entropy. In statistical thermodynamics, $E$ becomes the expectation value of $H$, while $N$ is identified with $\langle N_{op} \rangle$, the expectation value of $N_{op}$. Obviously, $F$ is identified with $-\log Z/\beta$, since the roles played by both quantities are parallel,

$$F = -\frac{1}{\beta} \log Z.$$
This identification provides the statistical definition of entropy. We note that
\[ E = \text{Tr}(\rho_{eq} H) = -\frac{\partial \log Z}{\partial \beta} \]
and
\[ N = \text{Tr}(\rho_{eq} N_{op}) = -\frac{\partial \log Z}{\partial \zeta} \]
are, respectively, the expected values of \( H \) and of \( N_{op} \) at statistical equilibrium, that is, the equilibrium expectation values of the respective physical measurements.

By the following computation
\[
-\text{Tr}(\rho_{eq} \log \rho_{eq}) \\
= \text{Tr} \left( \frac{e^{-\beta H - \zeta N_{op}}}{\text{Tr}(e^{-\beta H - \zeta N_{op}})} \times (\beta H + \zeta N_{op} + \log Z) \right) \\
= \beta E + \zeta \langle N \rangle + \log Z \\
= \beta (E - \mu \langle N \rangle + \frac{1}{\beta} \log Z) \\
= \beta (E - \mu \langle N \rangle + (TS - E + \mu \langle N \rangle)) = S,
\]
we get
\[ S = -\text{Tr}(\rho_{eq} \log \rho_{eq}). \]
Recall that the von Neumann entropy is given by \(-\text{Tr}(\rho \log \rho)\) for an arbitrary \( \rho \), \( \rho \neq \rho_{eq} \).

We observe that, if \( \lambda_k \) are the eigenvalues of the matrix \( \hat{H} \), then the eigenvalues of \( \hat{H} - \mu I \) are \( \lambda_k - \mu \). Thus, the number operator \( N_{op} = \sum_{k=1}^{\infty} c_k^\dagger c_k \) may be replaced, in the expression of \( Z \), by the pseudo-fermionic number operator
\[
N_{op} = \sum_{k=1}^{\infty} d_k^\dagger d_k. \tag{10}
\]
This is in consonance with the corresponding expression for \( H \),
\[
H = \sum_{k=1}^{\infty} \lambda_k d_k^\dagger d_k. \tag{11}
\]
In the definition of the partition function, (10) and (11) shall be used. This ensures that \( Z \) is real and positive even though \( \lambda_k d_k^\dagger d_k \) and \( d_k^\dagger d_k \) are non-Hermitian.

Next, we obtain approximations for \( E \) and \( N \), which are valid if the temperature is sufficiently high.
Figure 1: We have considered $\gamma = 3/5$. Dashed lines represent $\langle H \rangle$ vs $\langle N_{op} \rangle$ for variable $\mu$ with fixed values of $\beta$. Full lines, represent $\langle H \rangle$ vs $\langle N_{op} \rangle$ for variable $\beta$ with fixed values of $\mu$. The crossing points define pairs $(\langle E \rangle, \langle N_{op} \rangle)$ corresponding to pairs $(\beta, \mu)$. The horizontal scale should be divided by 100. The thick line represents the boundary of $W_{phys}(H + iN_{op})$, the physical numerical range of $H + iN_{op}$.

### 7.1 A numerical example

The eigenvalues of $\beta H + \zeta N_{op}$ are

$$(\beta \sqrt{1 + 2\gamma^2} (k - 3/4) + \zeta)n_k, \ k = 1, 2, 3, \ldots, n_k = 0, 1.$$ 

Thus

$$Z = \sum_{n_1, n_2, n_3, \ldots \in \{0, 1\}} \exp \left( -\sum_k (\beta \sqrt{1 + 2\gamma^2} (k - 3/4) + \zeta)n_k \right),$$

so that,

$$\log Z = \sum_{k=1}^{\infty} \log \left( 1 + \exp(-\beta \sqrt{1 + 2\gamma^2} k - \zeta') \right),$$

where $\zeta' = \zeta - \frac{3}{4} \sqrt{1 + 2\gamma^2} \beta$. 

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The Euler-Maclaurin (E-M) formula is an important tool in numerical analysis. It estimates a sum \( \sum_{k=0}^{n} g(k) \) through the integral \( \int_{0}^{n} g(t)dt \) with an error term involving Bernoulli numbers and polynomials [16].

Lemma 7.1 Let \( g(t) \) be a real function of class \( C^2 \). Then,

\[
\sum_{k=0}^{n-1} g(k) = \int_{0}^{n} g(t)dt - \frac{1}{2}(g(n) - g(0)) + \frac{1}{12}(g'(n) - g'(0)) - \frac{1}{2} \int_{0}^{n} B_2(\{t\}) g''(t)dt,
\]

where \( k \) is a non negative integer, \( B_2(x) = x^2 - x + 1/6 \) is the second Bernoulli polynomial and \( \{t\} \) denotes the fractional part of \( t \).

The polylogarithm is the function defined by the power series

\[
Li_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}, \quad s \in \mathbb{C}.
\]

The case \( s = 2 \) will be used in the next proposition.

Proposition 7.1 Let \( E \) and \( \langle N_{op} \rangle \) be the expectation values of \( H \) and \( N_{op} \), respectively. Then,

\[
E = \frac{1}{\beta^2 \sqrt{1 + 2\gamma^2}} Li_2(-e^{\zeta'}) - \frac{\sqrt{1 + 2\gamma^2}}{12(1 + e^{-\zeta'})} + \ldots - \frac{3}{4} \frac{\beta^2}{\sqrt{1 + 2\gamma^2}} (N_{op}),
\]

\[
\langle N_{op} \rangle = \frac{1}{\beta \sqrt{1 + 2\gamma^2}} \log(1 + e^{-\zeta'}) + \frac{1}{2(1 + e^{-\zeta'})} - \frac{\beta}{12(1 + e^{-\zeta'})^2} + \ldots.
\]

Proof. We observe that

\[
E = -\frac{\partial}{\partial \beta} \log Z, \quad \langle N_{op} \rangle = \frac{\partial}{\partial \zeta} \log Z.
\]

By the Euler-Maclaurin formula and using Mathematica,

\[
\log Z = \int_{0}^{\infty} dx \log \left(1 + e^{-\beta \sqrt{1 + 2\gamma^2} x - \zeta'}\right) + \frac{1}{2} \log(1 + e^{-\zeta'}) + \frac{\beta \sqrt{1 + 2\gamma^2} e^{-\zeta'}}{12(1 + e^{-\zeta'})} + \ldots
\]

\[
= -\frac{1}{\beta \sqrt{1 + 2\gamma^2}} Li_2(-e^{\zeta'}) + \frac{1}{2} \log(1 + e^{-\zeta'}) + \frac{\beta \sqrt{1 + 2\gamma^2} e^{-\zeta'}}{12(1 + e^{-\zeta'})} + \ldots.
\]

Thus, the result follows. ■
In Fig. 1, dashed lines represent \( \langle H \rangle vs \langle N_{op} \rangle \) for \( \gamma = 3/5 \) and variable \( \mu \) with \( \beta = 0.001, 0.01, 0.02, 0.03, 0.04, 0.08, 0.2 \), from top to bottom. Except for \( \beta = 0.001 \), we have taken \(-15\sqrt{1+2\gamma^2} < \mu < 15\sqrt{1+2\gamma^2}\). Full lines, represent \( \langle H \rangle vs \langle N_{op} \rangle \) for variable \( \beta \) with \( \mu = \sqrt{1+2\gamma^2} \) \((-14.75, -9.75, -4.75, \ldots, 10.25, 15.25)\), from left to right. The crossing points define pairs \( (\langle E \rangle, \langle N_{op} \rangle) \) corresponding to pairs \( (\beta, \mu) \). The horizontal scale should be divided by 100. The thick line represents the boundary of \( W_{phys}(H + iN_{op}) \), the physical numerical range of \( H + iN_{op} \) (the vertical line parallel to the \( y \) axis and the lower parabolic arc). The curve for \( \beta = 0.001 \), with \(-6000 < \mu < -4500\), was included to illustrate that points near the boundary of \( W_{phys}(H + iN_{op}) \), are described for very low \( \beta \).

We remark that in the present case the function \( g(x) \) is of class \( C^\infty \), so that the Euler-Maclaurin formula leads to an expansion in powers of \( \beta \sqrt{1+2\gamma^2} \) for \( \log Z \) which may be carried out indefinitely. However, in Proposition 7.1 only the first three terms of this expansion have been considered. The results are very good for points of \( W_{phys}(H + iN_{op}) \) which are not close to its boundary. For closer points, the full expansion may be needed.

8 Conclusions

We have investigated the spectrum of a non-Hermitian semi-infinite matrix \( \hat{H} \), and we have explicitly constructed a metric matrix which renders \( \hat{H} \) Hermitian. A fermionic model characterized by a non-Hermitian Hamiltonian with real eigenvalues has been investigated. Dynamical pseudo-fermionic operators have been constructed in terms of which the fermionic Hamiltonian acquires diagonal form. A physical Hilbert space, allowing for the probabilistic interpretation of the model according to quantum mechanics, has been introduced. Approximate expressions for the energy expectation value and the number operator expectation value, in terms of the absolute temperature \( T \) and of the chemical potential \( \mu \), are obtained, based on the Euler-Maclaurin formula. Statistical thermodynamics considerations, in which the physical Hilbert space plays an important role, are applied to the studied fermionic Hamiltonian.

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References

[1] F. Bagarello, *Construction of pseudo-bosons systems*, J. Math. Phys. 51 (2010) 023531.

[2] F. Bagarello, *More mathematics on pseudo-bosons*, J. Math. Phys. 51 (2013) 063512.

[3] F. Bagarello, J.-P. Gazeau, F.H. Szafraniec, M. Znojil, *Non-Selfadjoint Operators in Quantum Physics: Mathematical Aspects*, Wiley, 2015.

[4] F. Bagarello, *A concise review on pseudo-bosons, pseudo-fermions and their relatives*, Theoretical and Mathematical Physics, 193 (2027) 16801693.

[5] N. Bebiano, J. da Providência, J. P. da Providência, *Mathematical Aspects of Quantum Systems with a Pseudo-Hermitian Hamiltonian*, Brazilian Journal of Physics, 46 (2016) 152-156.

[6] N. Bebiano and J. da Providência, *The EMM and the Spectral Analysis of a Non Self-adjoint Hamiltonian on an Infinite Dimensional Hilbert Space*, Non-Hermitian Hamiltonians in Quantum Physics. Springer Proceedings in Physics, 184 (2016) 157-166.

[7] N. Bebiano, J. da Providência, J. P. da Providência, *Fermionic chain model with a non-Hermitian Hamiltonian*, Letters in mathematical science, accepted.

[8] N. Bebiano, J. da Providência, J. P. da Providência, *Towards non-Hermitian quantum statistical thermodynamics*, arXiv:1907.13221 [quant-ph].

[9] C.M. Bender and S. Boettcher, *Real Spectra in Non-Hermitian Hamiltonians Having PT Symmetry*, Phys. Rev. Lett., 80 (1998) 5243-5246.

[10] C.M. Bender, D.C. Brody and H.F. Jones, *Complex Extension of Quantum Mechanics*, Phys. Rev. Lett, 89 (2002) 27041.

[11] D. J. Rowe, *Equations-of-Motion Method and the Extended Shell Model*, Rev. Mod. Phys., 40 (1968) 153.

[12] L. D. Landau and E. M. Lifshitz, *Statistical Physics*, Pergamon Pres, 1969.

[13] A. Mostafazadeh, *Pseudo-Hermitian Quantum Mechanics with Unbounded Metric Operators*, arXiv:1203.6241 [math-ph], Phil. Trans. R. Soc. A 371 (2013) 20120050.

[14] A. Mostafazadeh, *Exact PT-symmetry is equivalent to Hermiticity*, J. Phys. A: Math. Gen. 36 (2003) 7081. Complex Extension of Quantum Mechanics, J. Math. Phys. 46 (2005) 102108;
[15] J. da Providência, N. Bebiano and JP. da Providência, *Non Hermitian operators with real spectra in Quantum Mechanics*, Brazilian Journal of Physics, 41 (2011) 78-85.

[16] M.Z. Spivey, *The Euler-Maclaurin formula and sums of powers*, Mathematics magazine, 79 (2006) 61-65.