Removal of $Z_3$-symmetry breaking from Fermionic Determinants

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We consider prescriptions that are free from the direct charge-screening effects by quark loops and enable us to clarify the confining nature of a vacuum. We test two candidates for an order parameter, a Polyakov loop ($P$) evaluated in zero-triality backgrounds and fermionic determinants ($D_{1,2}$) with non-zero triality. Especially, $D_{1,2}$ has very small fluctuations in comparison with a Polyakov loop in zero-triality sector, and seems to well reflect the characteristic of a vacuum. Such prescriptions could be still usable for the clarification of the confinement property of a vacuum.

I. INTRODUCTION

Clarification of the chiral phase transition and the confinement/deconfinement transition are the longstanding central issues in QCD. There have been numerous efforts to understand the phase structure of QCD at zero and finite temperature or density regions [1]. Chiral phase transition is detected by monitoring the order parameter, chiral condensate $\langle \bar{\psi} \psi \rangle$, whereas a Polyakov loop is widely used in order to know whether a system is in the confinement phase or not. The expectation value of a Polyakov loop is zero when a system is in the confinement phase, and it can be finite in the deconfinement phase. A Polyakov loop may be considered as an order parameter for the color confinement. These two “order parameters” unfortunately do not go together. (Interestingly, a Polyakov loop and Dirac eigenvalues, which also reflect the chiral phase transition, can be related [2, 3].) While chiral condensate $\langle \bar{\psi} \psi \rangle$ can be a good order parameter if we neglect the small current quark masses, a Polyakov loop can serve as an order parameter only in pure gauge systems without quarks (or with infinitely heavy quarks), where deconfinement transition is expressed by the spontaneous $Z_3$-symmetry breaking. The presence of dynamical quarks explicitly breaks the $Z_3$ symmetry and a Polyakov loop results in an approximate order parameter. The color fields induced by a Polyakov loop is readily screened by the dynamical quark loops that twist along the Euclidean time direction. Nevertheless, a Polyakov loop surely reflects the confinement nature and has been used in many studies.

It may be however desired to clarify the confinement nature in a clearer manner. For example, the chiral symmetry breaking surely has an impact on the properties of quarks. Current quarks would change their natures due to the spontaneous chiral symmetry breaking acquiring the large effective masses [4, 5], and therefore a Polyakov loop screened by “constituent” quarks could be affected by the chiral phase transition. The apparent coincidence of the critical temperatures for the chiral and confinement transitions is still under debate. Even if we evaluate a Polyakov loop correlator $\langle P(0)P(x) \rangle$, it is not so straightforward to clarify the color-confinement nature since we always have string-breaking effects. In high-density systems, the concept of color confinement could be obscure. Quarks may freely move from one hadron to another in sufficiently dense hadronic systems, even if the vacuum is still in “color confinement” phase. In fact, “quarkyonic” phase has been proposed recently [6, 7], where quarks are confined but their degrees of freedom can dominate the system. In any case, as long as we insist on a Polyakov loop, it is needed to single out the vacuum property independently of the direct charge-screening effects. Search for order parameters have a long-standing history, and several studies have been performed so far [8, 9, 10, 11]. We consider prescriptions that remove the dynamical quark loops which directly screen the color fields from a Polyakov loop.

II. FERMIONIC DETERMINANT

We introduce the basic ideas [8, 9, 10, 11] in this section to make this paper self-contained. We assume SU(3) lattice gauge theory with single quark-flavor in what follows. (Extensions to other cases are simple.) The QCD partition function is expressed as

$$Z = \int dU \det D e^{-S_{G[U]}},$$

(1)

with $D$ a (lattice) Dirac operator, in which non-zero chemical potential can be introduced, and $S_{G[U]}$ being the gauge action. In advance of the gauge fields’ integration, quark fields are integrated out, and all the quark dynamics is encoded in $D$. The quark loops that can directly screen the color fields from a Polyakov loop originate from such a fermionic determinant.

This fermionic determinant can be expanded in terms of two types of quark loops (See Fig. 1): ordinary quark loops and wraparound quark loops. Ordinary loops exist entirely in a system and can be smoothly shrunk. Wraparound loops twist along the imaginary-time direction, and they are the very loops that can directly screen a Polyakov loop. Such wraparound loops are also responsible for the finiteness of quark density at finite chemical potential, and are important ingredients.

A fermionic determinant can be generally divided into
three terms:
\[ D = \det D = \sum_{i=0,1,2} D_i \]
with \( D_i \) \((i = 0, 1, 2)\) the terms which contain \((3k + i)\) wraparound quark loops, respectively. Here \( k \) is an integer and can be negative, and a wraparound anti-quark loop is counted as \(-1\). One connected quark loop which twists \( m \)-times along the Euclidean time direction is counted as \( m \) quark loops. The total number of quark loops is defined as a net number: In case a contribution contains as many wraparound quark loops as wraparound anti-quark loops, the number is defined as zero. We concentrate only on wraparound loops, since ordinary quark loops are irrelevant in the following argument. Next, we consider the uniform \( Z_3 \) transformation; \( U_{t}(t = 0) \rightarrow z U_{t}(t = 0) \) with \( z \equiv \exp(2/3 \pi i) \). Here \( U_{t}(t = 0) \) are the link variables on a lattice. By such a transformation, a Polyakov loop \( P \) is rotated as \( P \rightarrow zP \). This \( Z_3 \) transformation also affects dynamical (wraparound) quark loops, and \( D_i \) are transformed as \( D_i \rightarrow z^i D_i \). Using this property, we can single out each \( D_i \) by means of such \( Z_3 \) link-variable transformations. Denoting the fermionic determinant after \( n \)-times such \( Z_3 \) transformations as \( D(n) \), \( D_i \) can be obtained as
\[ D_i = \frac{1}{3} \sum_{n=0,1,2} D(n) \times z^{-in}. \] (3)

(See earlier works \[8, 9, 10, 11\].) With vanishing quark chemical potential, \( D_0 = D_0 \) and \( D_1 = D_2 \) hold due to the charge conjugation symmetry. As we have mentioned above, \( D_i \) contains \((3k + i)\) wraparound quark loops, and hence what screens the Polyakov loop’s color fields are the quarks encoded in \( D_2 \). In fact, the complex phase associated with \( D_2 \) rotates a Polyakov loop into the real sector. \((D_0 \) and \( D_1 \)) do not.) The important note is that quarks in \( D_0 \) and \( D_1 \) cannot screen a fundamental charge completely. Then, if we evaluate only \( D_0 \), the response of a Polyakov loop would be the same as that obtained in a quenched system. \( D_0 \) is in fact \( Z_3 \)-transformation invariant and does not cause the explicit \( Z_3 \)-symmetry breaking. We note here that this removal of the explicit \( Z_3 \)-symmetry breaking is essentially different from the quark-loop quenching. Evaluating \( D_i \) corresponds to computing the partition function \( Z_i \) in the \( i \)-triality sector \( (T = i) \),
\[ Z_i = \sum_{3k+i} \langle 3k+i | e^{-\beta H} | 3k+i \rangle. \] (4)

Here, \( |3k+i\rangle \) denote all the possible states that contain \((3k + i)\) net quarks \((k \) is an arbitrary integer\). The vacuum still contains dynamical quark loops, and if the quark chemical potential is finite, there will exist \( 3k \) net quarks \((k \) net “baryons”) in the system. String-breaking phenomena caused by dynamical quarks will be also reproduced. It was also shown in Ref. \[11\] that such a projection does not change thermal properties. We expect that it would be possible to unambiguously clarify the confining property of a vacuum independently of the direct charge-screening effect by dynamical quarks. (If we employ an anti Polyakov loop, the roles of \( D_1 \) and \( D_2 \) switch positions with each other.)

### III. NUMERICAL TESTS

#### A. Polyakov loop

In this subsection, we individually and explicitly investigate the distributions of a Polyakov loop evaluated with \( D_i \) \((i = 0, 1, 2)\). We generate quenched gauge configurations with the plaquette action at \( \beta = 5.7 \) on \( 4^4 \) lattice, and perform reweighting using \( D_i \), as well as the full determinant \( D \) generated with the Wilson fermion at \( \kappa = 0.1600 \).

Fig. 2(upper) and Fig. 2(lower) are the Polyakov-loop distributions in the complex plain evaluated with \( D \) and \( D_0 \), respectively, at \( \beta = 5.7 \), where the broken \( Z_3 \) symmetry is observed. As expected, the strength in the real sector is much enhanced by the effect of \( D \), which can be seen in Fig. 2(upper). On the other hand, Fig. 2(lower) shows the similar strength to the quenched case, which was expected in the previous section. Especially one can find a three-peak structure in the complex plain.

We show the same plots obtained at \( \beta = 5.0 \) in Fig. 3, where no \( Z_3 \) breaking can be found. The strength in the real sector is again (but weakly) enhanced by the effect of \( D \) as can be seen in Fig. 3(upper), whereas Fig. 3(lower) shows the similar distributions to the quenched system. This prescription, where we measure a Polyakov loop in zero-triality backgrounds (evaluating \( D_0 \)), is free from the direct screening effects or the string-breaking effects by quark loops, and seems to enable us to clarify the confining nature of a vacuum.

For further clarification, we present the distributions reweighted with \( D_1, 2 \) in Fig. 4. One can observe that only \( D_2 \) can rotate a Polyakov loop into the real sector, which implies that directly-screening quarks are coming from \( D_2 \). The Polyakov loops evaluated with \( D_1 \) are rather uniformly distributed around the origin. Then, \( D_1 \) or \( D_0 + D_1 \) may be also employed for this prescription. The nonvanishing Polyakov loops are sometimes related to the explicit \( Z_3 \)-symmetry breaking caused by fermions. Including \( D_1 \) actually breaks the \( Z_3 \) symmetry explicitly, but it gives a similar Polyakov loop distribution to the \( Z_3 \)-symmetric quenched case, which is natural if we take into account that quarks in \( D_1 \) cannot screen a Polyakov loop.
loop completely and hence the situation is physically similar to the quenched system.

So far, the response of the Polyakov loop evaluated with $D_0$ has been simply similar to the quenched case, and the difference is unclear. In order to show the difference, we perform a test with non-vanishing quark chemical potentials. We hereafter focus on the phase angles of $D_0$ and $D$ at finite chemical potential. (Averaging the fermionic determinants at finite chemical potential was suggested and performed some years ago in the context of the reduction of phase fluctuations [12, 13, 14].)

Fig. 5 shows the phase angles of $D_0$ and $D$ obtained at $\beta = 5.7$, $\kappa = 0.1640$, and $0 \leq \mu \leq 0.5$, with one gauge configuration. The phase angle of the full determinant $D$ grows as we increase the chemical potential $\mu$, which is caused by the asymmetry between quarks and anti-quarks. However, as can be found in Fig. 5, the phase angle of $D_0$ remains zero in the small-$\mu$ region, and rapidly grows above some value of $\mu$. The reason for this behavior would be that the number of wraparound quarks in $D_0$ is $3k$. When $\mu$ is small enough, it is not likely for the system to accommodate three net quarks, and $D_0$ is still symmetric between quarks and anti-quarks, which implies $k \sim 0$. The phase angle at small-$\mu$ region is brought about by $D_1$ and $D_2$. At large $\mu$, the charge conjugation symmetry in $D_0$ is largely broken, and the phase angle can be finite, which implies $k > 0$.

B. Distributions of Fermionic determinants

We next take a look at the distributions of fermionic determinants themselves in quenched QCD in this subsection. Fig. 6 shows the distributions of $D_1$ in the complex plain computed at $\beta = 5.7$ (upper) and $\beta = 5.0$ (lower). These distributions are qualitatively similar to the Polyakov-loop distributions evaluated with $D_0$. Such behaviors can be understood intuitively: $D_1$ contains $(3k + 1)$ wraparound quark loops, and hence corresponds to the free energy of $(3k + 1)$ light quarks. The physical situation is similar to the $3k$-quark vacuum (obtained by evaluating $D_0$) with one Polyakov loop, where $(3k + 1)$ net charges exist. $D_{1,2}$ can be another candidate for an order parameter.

If we directly generate unquenched gauge configurations adopting $D_0$ instead of $D$ via some adequate algo-
FIG. 4: The distributions of the Polyakov loops evaluated with $D_1$(upper) and $D_2$(lower) at $\beta = 5.0$.

FIG. 5: The phase angles of $D_0$ and $D$ obtained as a function of chemical potential $\mu$ are plotted. They were evaluated with one gauge configuration.

FIG. 6: The distributions of $D_1$ computed at $\beta = 5.7$(upper) and $\beta = 5.0$(lower) are plotted in the complex plain. These distributions are qualitatively similar to the Polyakov-loop distributions evaluated with $D_0$.

in Ref. [8],

$$\frac{\int DUD_n e^{-S_g[U]} }{ \int DU D e^{-S_g[U]} }.$$  \hspace{1cm} (6)

The latter can be obtained by measuring

$$\frac{\langle D_n/D_0 \rangle^{T=0} }{ \langle D/D_0 \rangle^{T=0} } = \frac{\int DUD_n e^{-S_g[U]} }{ \int DU D e^{-S_g[U]} }.$$  \hspace{1cm} (7)

in zero-triality sector. $D_{1,2}/D_0$ as well as a Polyakov loop evaluated in zero-triality sector also seems usable for the clarification of the color confinement.

If we allow non-zero triality sectors in unquenched gauge-updation processes (ordinary updations), the order parameter should be $D_{1,2}/D$, which also turns out to be the original form proposed in Ref. [8] (Eq.(6)). This implementation, however, would not be suitable for such purposes, simply because ergodicity is worse. Though the configurations that have smaller statistical weights $e^{-S_g}$ will produce larger $D_{1,2}/D$ and $Z_3$ symmetry in $\langle D_{1,2}/D \rangle$ could be finally recovered, such configurations would less appear in actual calculations.

We note here that the inevitable failure of such treatments in the thermodynamic limit was discussed in Ref. [9]. Even so, if we directly measure $\langle D_n \rangle/\langle D \rangle$ in quenched backgrounds (like our analyses in this paper and the original form in Ref. [8]), the ergodicity itself is not lost [8] and this simple prescription could be valid, though it is time-consuming.
C. Spontaneous $Z_3$-symmetry breaking

The deconfinement phase transition, if it exists, would be detected still as the spontaneous $Z_3$-symmetry breaking, as in the quenched case. Even when the transition is crossover, the argument is not invalidated. We have to introduce a small explicit $Z_3$-symmetry breaking effect and take the thermodynamic limit before removing the explicit breaking, in finite volume systems. Practically, it will be workable for the detection of transition to evaluate (Ref5)\textsuperscript{#}. Here $Ω$ is a Polyakov loop $P$ or the fermionic determinants $D_{1,2}$. Especially, the fluctuation of $D_{1,2}$ is much suppressed than that of a Polyakov loop in zero-triality sector, which can be seen in Fig. 6(upper): The values of $D_{1,2}$ are very sharply distributed on the $Z_3$-axes, and we can find three “lines” in the complex plain. In any case, the use of $D_{1,2}$ seems much more advantageous than a Polyakov loop, since it gives us more clearer signals.

The reason for the tiny deviations from the $Z_3$ axes in Fig. 6(upper) is the discontinuous behavior in $D_{1,2}$ in the deconfined phase. To see this, we compute $D_1$ on the gauge configurations where Polyakov-loop’s absolute values are less than 0.4. Such Polyakov-loop’s distributions are round-shaped both at $β=5.7$ (deconfined) and 5.0(confined). On the other hand, $D_1$ at $β=5.7$ (Fig. 7(upper)) still reproduces the three-peak structure, which implies $D_1$ at small $|P|$ also reflects the phase. To have a closer look, we classify configurations into three categories, sector 1 ($-π/3 \leq \arg P \leq π/3$), sector 2 ($π/3 \leq \arg P \leq π$), sector 3 ($-π \leq \arg P \leq -π/3$), in terms of the phase angle (arg $P$) of a Polyakov loop. The scatter plot of $D_1$ in each sector can be found in Fig. 7. Though, at $β=5.7$, the distribution of a Polyakov loop ($|P|<0.4$) is round-shaped, that of $D_1$ is split onto $Z_3$ axes, which are drawn as three solid lines in the figure. This splitting indicates that $D_{1,2}$ “as a function of $P$” quickly changes its value at the boundaries between sectors in the deconfinement phase whereas they seem to be rather smooth functions in the confinement phase. $D_{1,2}$ would be discontinuous at the boundaries in the thermodynamic limit, which is considered as the consequence of the spontaneous $Z_3$-symmetry breaking. Even if the $Z_3$ symmetry in the $Z_3$-symmetrized canonical formulation is always spontaneously broken in the presence of dynamical quarks and the transition is crossover, it will be still reflected in $D_{1,2}$.

IV. SUMMARY

We have considered prescriptions that are free from the direct charge-screening effects by quark loops and enable us to clarify the confining nature of a vacuum. We have tested two candidates for an order parameter, a Polyakov loop ($P$) evaluated in zero-triality sector and fermionic determinants ($D_{1,2}$) with non-zero triality. We have also individually investigated the distribution of a Polyakov loop in each triality sector. Especially, $D_{1,2}$ has much smaller fluctuations in comparison with a Polyakov loop in zero-triality sector, and seems to well reflect the characteristic of a vacuum. Such prescription could single out the confinement nature of a vacuum properly and independently of the direct screening effects or the string-breaking effects.

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[1] O. Philipsen, Eur. Phys. J. ST 152 (2007) 29 [arXiv:0708.1293 [hep-lat]].
[2] C. Gattringer, Phys. Rev. Lett. 97 (2006) 032003
[3] F. Bruckmann, C. Gattringer and C. Hagen, Phys. Lett. B 647 (2007) 56 [arXiv:hep-lat/0612020].
[4] Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122 (1961) 345.
[5] Y. Nambu and G. Jona-Lasinio, Phys. Rev. 124 (1961) 246.
[6] L. McLerran and R. D. Pisarski, Nucl. Phys. A 796 (2007) 83 [arXiv:0706.2191 [hep-ph]].
[7] K. Miura and A. Ohnishi, arXiv:0806.3357 [nucl-th].
[8] C. E. Detar and L. D. McLerran, Phys. Lett. B 119, 171 (1982).
[9] K. Fukushima, Annals Phys. 304 (2003) 72 [arXiv:hep-ph/0204302].
[10] M. Faber, O. Borisenko and G. Zinovev, Nucl. Phys. B 444 (1995) 563 [arXiv:hep-ph/9504264].
[11] S. Kratochvila and P. de Forcrand, Phys. Rev. D 73 (2006) 114512 [arXiv:hep-lat/0602005].
[12] P. de Forcrand and V. Laliena, Phys. Rev. D 61 (2000) 034502 [arXiv:hep-lat/9907004].
[13] G. Aarts, O. Kaczmarek, F. Karsch and I. O. Stamatescu, Nucl. Phys. Proc. Suppl. 106 (2002) 456 [arXiv:hep-lat/0110145].
[14] Y. Sasai, A. Nakamura and T. Takaishi, Nucl. Phys. Proc. Suppl. 129 (2004) 539 [arXiv:hep-lat/0310046].
[15] A. Alexandru, M. Faber, I. Horvath and K. F. Liu, Phys. Rev. D 72 (2005) 114513 [arXiv:hep-lat/0507020].