Unimodular transformations of the supermanifolds and the calculation of the multi-loop amplitudes in the superstring theory

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Abstract

The modular transformations of the (1|1) complex supermanifolds in the like-Schottky modular parameterization are discussed. It is shown that these "supermodular" transformations depend on the spinor structure of the (1|1) complex supermanifold by terms proportional to the odd modular parameters. The above terms are calculated in the explicit form. The discussed terms are important for the study of the possible divergencies in the Ramond-Neveu-Schwarz superstring theory. In addition, they are necessary to calculate the dependence on the odd moduli of the fundamental domain in the modular space. The supermodular transformations of the multi-loop superstring partition functions calculated by the solution of the Ward identities are studied. In the present paper, it is shown that the above Ward identities are covariant under the supermodular transformations. Hence the considered partition functions necessarily possess the covariance under the supermodular transformations discussed. It is demonstrated in the explicit form the covariance of the above partition functions at zero odd moduli under those supermodular transformations in the Ramond sector, which turn a pair of even genus-1 spinor structures to a pair of the odd genus-1 spinor ones. The brief consideration of the cancellation of divergences is given.

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1 Introduction

In the Ramond-Neveu-Schwarz superstring theory \[1, 2\] the supermanifold formalism \[3\] occurs more appropriate for multi-loop calculations than a formalism of the Riemann surfaces. Indeed, in the superstring theory every \(2\pi\)-twist about \(A\)- or \(B\)-cycle on the Riemann surface is, generally, accompanied by a supersymmetrical transformation including, in addition, a boson-fermion mixing. The above mixing can be taken into account by an extension to the complex \((1|1)\) supermanifolds \[3\] of that complex \(z\)-plane where the genus-\(n\) Riemann surfaces are mapped. Every supermanifold is described by the supercoordinate \(t = (z|\theta)\) where \(z\) is the complex local coordinate and \(\theta\) is its Grassmann (odd) partner. Grassmann (odd) parameters of the discussed boson-fermion mixing are expressed in terms of complex Grassmann (odd) modular parameters, which are assigned to every complex \((1|1)\) supermanifold in addition to ordinary (even) complex Riemann moduli. In this case fermion strings are classified over ”superspin” structures instead the ordinary spin structures \[4\].

The superspin structures are defined for superfields on the discussed complex \((1|1)\) supermanifolds \[3\]. Being twisted about \((A, B)\)-cycles, the superfields are changed by mappings that present superconformal versions of fractional linear transformations. Generally, every considered mapping depends on \((3|2)\) parameters \[3\]. For odd parameters to be arbitrary, these mappings include, in addition, fermion-boson mixing above. It differs the superspin structures from the ordinary spin ones. Indeed, the ordinary spin structures \[4\] imply that boson fields are single-valued on Riemann surfaces. Only fermion fields being twisted about \((A, B)\)-cycles, may receive the factor of -1. For all odd parameters to be zero, every genus-\(n\) superspin structure \(L = (l_1, l_2)\) is reduced to the ordinary \((l_1, l_2)\) spin one. Here \(l_1\) and \(l_2\) are the theta function characteristics: \((l_1, l_2) = \bigcup_s (l_{1s}, l_{2s})\) where \(l_{is} \in (0, 1/2)\). The (super)spin structure is even, if \(4l_1l_2 = 4 \sum_{s=1}^n l_{1s}l_{2s}\) is even. It is odd, if \(4l_1l_2\) is odd.

One could to avoid the supermanifold formalism \[3\] using the prescription \[5, 6\] for the integration over the odd moduli. In this case, however, multi-loop amplitudes turn out to be depended on a choice of basis of the gravitino zero modes \[5, 7, 8\]. It means that the world-sheet supersymmetry is lost in the scheme discussed. Indeed, in the superstring theory both the vierbein and the gravitino field are the gauge fields. Owing to the gauge invariance the ”true” superstring amplitudes are independent of the choice of a gauge of the above gauge fields. Therefore, they have no dependence on the choice of basis of the gravitino zero modes. The discussed dependence on the choice of basis of the gravitino zero modes appears to be a serious difficulty in the scheme \[3\]. But the above difficulty is absent in the supermanifold formalism \[3\] that possesses the manifest world-sheet supersymmetry.

In the considered scheme \[3\] the multi-loop amplitudes are obtained \[3, 9, 10, 11, 12, 13\] by the summation over ”superspin” structure contributions integrated over both the even moduli and the odd ones and over the vertex supercoordinates, as well. Every superspin structure contribution presents the suitable partition function multiplied by the vacuum expectation of the vertex product. The above vacuum expectations are expressed in terms of superfield vacuum correlators. Different approaches to the calculation of the vacuum correlators and of the partition functions have been proposed \[3, 9, 10, 11, 12, 13, 14\]. In \[3\], the superfield vacuum correlators and the partition functions have been calculated in the explicit form for
all the even superspin structures. The integration over the modular parameters and over the vertex supercoordinates needs an additional investigation. Indeed, for every superspin structure contribution, both the integral over the even moduli and the integral over the \{z(\tau), \bar{z}(\tau)\} vertex local coordinates are divergent.\(^1\) The divergencies of the integrals over \{z(\tau), \bar{z}(\tau)\} arise from the region where all the vertices move to be closed together and from the region where all they move away from each other. The divergencies of the integrals over the modular parameters are due to a degeneration of the Riemann surfaces. Of the main difficulty for the investigation are the possible divergencies due to a degeneration of genus-n Riemann surfaces (n > 1) into a few ones of the lower genus. It is expected \[6, 9, 15, 16, 17\] that the above divergencies disappear after the summation over spinor structures to be performed, but this problem needs an additional study. In any case, the correct consideration of the divergency problem requires even if an implicit regularization procedure. The above regularization procedure must be chosen ensuring the supermodular group invariance of the superstring amplitudes. The supermodular group does be the superconformal extension of the modular group in the boson string theory. Generally, the supermodular transformations present the globally defined \(t \rightarrow \hat{t}(t, \{q_N\})\) holomorphic superconformal mappings \[3\] of the \(t = (z|\theta)\) supercoordinate, which are accompanied by the \(q_N \rightarrow \hat{q}_N(\{q_N\})\) holomorphic mappings of the complex moduli \(q_N\) and, generally, by the \(L \rightarrow \hat{L}\) change of the superspin structure, as well. To avoid the explicit regularization procedure, it seems attractive to write down the multi-loop superstring amplitude in the form of the integral over both \(\{q_N, \bar{q}_N\}\) and \(\{t^{(\tau)} , \bar{t}^{(\tau)}\}\) of the integrand covariant under the supermodular transformations. Being defined by the above integral, the considered superstring amplitude surely satisfies the restrictions due to the supermodular group, at least, if the above integrand has no non-integrable singularities. In this case the discussed construction solves the problem of the calculation of the superstring amplitudes. Simultaneously, the study of the divergency problem is reduced to the investigation of the singularities of the supermodular covariant integrand. Owing to the supermodular invariance, every superspin structure contribution possesses covariance under the supermodular transformations. So the desired integrand presents the sum over all the superspin structures contributions, every term being the partition function multiplied by the vacuum expectation of the vertex product. The discussed scheme is, however, complicated by the non-split in the sense of \[18\] of the supermanifolds. At least, the above non-split takes place, if the like-Schottky modular parameterization \[9, 10, 11, 12, 13, 14\] is used. In this case the modular group transformations \((t \rightarrow \hat{t}(t, \{q_N\}), q_N \rightarrow \hat{q}_N(\{q_N\}), L \rightarrow \hat{L})\) affect not only the bodies of the modular parameters, but on the soul components, as well. So the resulted modular parameters \(\hat{q}_N(\{q_N\})\) and the resulted supercoordinate \(t(t, \{q_N\})\) depend non-trivially on the odd modular parameters. Particular, among terms proportional to odd modular parameters, there are terms depending on the superspin structure \(L\). Because of the above \(L\) dependence of both \(t \rightarrow \hat{t}(t, \{q_N\})\) and \(q_N \rightarrow \hat{q}_N(\{q_N\})\), the discussed integrand is non-covariant under the supermodular group, if the \(q_N\) moduli are chosen to be the same for all the superspin structures. To build the supermodular covariant integrand, the calculation of the \(L\) dependence of both \(\hat{t}(t, \{q_N\})\) and \(\hat{q}_N(\{q_N\})\) is necessary. It seems that the

\(^1\) Throughout this paper the line over denotes complex conjugation.
knowledge of the above $L$ dependence is necessary also, if instead of the discussed scheme, one will attempt to construct an explicit regularization procedure for the integration of every superspin structure contribution.

In the present paper we calculate the explicit dependence on the odd modular parameters of both $\hat{t}(t, \{q_N\})$ and $\hat{q}_N(\{q_N\})$. Generally, the above dependence is obtained to be series in the odd modular parameters. We show that in both $\hat{t}$ and $\hat{q}_N$ among terms proportional to odd modular parameters, there are terms depending on the $L$ superspin structure. Furthermore, we propose method constructing the supermodular covariant integrand in the expression for the multi-loop amplitude. The above integrand presents the sum over all the superspin structures contributions, every term being calculated at its own moduli $\{q_{NL}\}$ and its own supercoordinates $\{t^{(r)}_L\}$, as well. These $(q_{NL}, t^{(r)}_L)$ variables are functions of the $(\{q_N\}, t^{(r)})$ variables of the integration: $q_{NL} = (q_{NL}(\{q_N\})$ and $t^{(r)}_L = t^{(r)}_L(t^{(r)}, \{q_N\})$. The above $(q_{NL}, t^{(r)}_L)$ functions are calculated from the condition that the same $(t \rightarrow \hat{t}, q_N \rightarrow \hat{q}_N)$ change of the $(t, q_N)$ variables corresponds to all the $(t_L \rightarrow \hat{t}_L)$ mappings associated with the particular supermodular transformation. To avoid misunderstands, it is necessary to note that the changes of $t$ under $2\pi$-twists about $(A_s, B_s)$-cycles remain depending on the $L$ superspin structure. Moreover, in this case the discussed changes of $t$ are not, generally, described by any simple expressions similar to the Schottky transformations. It is the fee for the supermodular transformations of $(t, q_N)$ to be independent of $L$. The desired supercovariant integrand turns out to be calculated uniquely by employing only a part of the supermodular transformations. So, to be sure in the self-consistency of the discussed scheme, one should verify that the above integrand is covariant under the whole supermodular group. This verification requires, however, an additional study of the supermodular transformations that is planned in another paper. Instead, in the present paper we discuss the changes under the supermodular transformations of the partition functions calculated in [14]. For the theory to be self-consistent, the multi-loop partition functions must be covariant under the supermodular group. We argue that the considered partition functions [14] possess the supermodular covariance required.

The discussed partition functions have been calculated [14] from equations [12, 13, 14] that are none other than Ward identities. These equations fully determine the partition functions up to constant factors only. The discussed equations are derived from the condition that the multi-loop superstring amplitudes are independent of a choice of both the vierbein and the gravitino field. So it seems natural to expect that the above equations are covariant under the supermodular group transformations. Such is indeed the case, and in the present paper we give the direct proof of the supermodular covariance of the equations discussed. Therefore, the partition functions [14] being the solution of these equations, with necessariness satisfy restrictions due to the supermodular group. Unfortunately, it is difficult to obtain a more direct evidence for the covariance of the discussed partition functions [14] under the whole supermodular group. Nevertheless, one can attempt to demonstrate the covariance of the above partition functions under some subgroups of the supermodular group. Particular, we demonstrate that at zero odd moduli the partition functions [14] are covariant under the supermodular transformations, which turn a pair of even genus-1
superspin structures in the Ramond sector to a pair of the odd genus-1 superspin ones.

Besides the application to the divergency problem, the dependence on the odd modular parameters of the supermodular transformations is necessary to calculate the dependence on the odd moduli of the region of the integration over the even moduli in the expressions for the multi-loop superstring amplitudes. Indeed, the moduli being defined modulo the supermodular group \([13]\), the even moduli are integrated over the fundamental domain that is determined by the condition that different varieties of moduli correspond to topologically non-equivalent supermanifolds. It is similar to the boson string theory where the region of the integration over moduli is determined by the modular invariance. Inasmuch as the supermanifolds are non-compact in the sense of \([19]\), the boundary of the discussed fundamental domain \(\Sigma\) depends on the odd moduli. When integrating over the odd moduli \(q_{od}\), the dependence of \(\Sigma\) on odd modular parameters must necessarily be taken into account. It is obvious that the discussed \(q_{od}\) dependence of \(\Sigma\) is just determined by the \(q_{od}\) dependence of the even moduli \(\hat{q}_{ev}(\{q_{ev}\})\) obtained by the \(q_{N} \to \hat{q}_{N}(\{q_{N}\})\) supermodular transformations of \(\{q_{N}\} = \{q_{ev}, q_{od}\}\).

The arising of the dependence on the superspin structure in both \(\hat{t}(t, \{q_{N}\})\) and \(\hat{q}_{N}(\{q_{N}\})\) when the odd moduli present, can be understood as it follows. For zero odd moduli, the supermodular transformations are reduced to the modular ones, which form the discrete group of globally defined holomorphic \(z \to \hat{z}^{(0)}(z, \{q_{ev}\})\) transformations accompanied by the \(q_{ev} \to \hat{q}_{ev}^{(0)}(\{q_{ev}\})\) change of the \(q_{ev}\) even moduli. In this case the modular transformations \(\omega^{(r)}(\{q_{ev}\}) \to \omega^{(r)}(\{\hat{q}_{ev}^{(0)}\})\) of the \(\omega^{(r)}(\{q_{ev}\})\) period matrices associated with Riemann surfaces determine in an implicit form all the new moduli \(\hat{q}_{ev}^{(0)}\) in terms of \(q_{ev}\) up to arbitrariness caused by possible fractionally linear transformations of Riemann surfaces. Since the \(\omega^{(r)}\) matrix does not depend on the superspin structure, both the \(\{\hat{q}_{ev}\}\) sets and the \(\hat{z}^{(0)}(z, \{q_{ev}\})\) local coordinate appear to be the same for all the spin structures, if the \(\{q_{ev}\}\) set is chosen to be the same for the spin structures considered. In the presence of the odd moduli, however, period matrices are assigned to \((1|1)\) supermanifolds rather than the Riemann surfaces \([3, 20]\). For the genus \(n \geq 2\) the above \(\omega(\{q_{N}\}; L)\) period matrices depend on the \(L\) superspin structure by terms proportional to odd moduli \([10, 21]\). These terms arise because in the considered scheme the fermions mix the bosons under \(2\pi\)-twists about \((A_{s}, B_{s})\)-cycles. So in the superstring theory, there are no reasons for \(\hat{q}_{N}(\{q_{N}\})\) and for \(\hat{t}(t, \{q_{N}\})\) to be independent of \(L\). Moreover, though the supermodular transformations of the above period matrices are described by the same relations as modular transformations of period matrices in the boson string theory, the above relations are insufficient to determine all the resulting moduli in terms of the "old" ones, if the odd moduli present. Only if the action of the supermodular group on the odd moduli to be determined, the discussed relations give in an implicit form the fundamental domain \(\Sigma\) in the modular space.

The calculation of the supermodular group action on the odd moduli is one of goals of the present paper. In general case the resulted \(\hat{q}_{od}(\{q_{N}\})\) odd moduli are calculated in terms of both the parameters of a suitable modular transformation at zero odd modular parameters and the \(q_{od}\) modular parameters, as well. The dependence of \(\hat{q}_{od}\) on \(q_{od}\) is obtained in the form of a series in \(q_{od}\). The \(\omega(\{q_{N}\}; L)\) period matrices in the Neveu-Schwarz sector have been calculated in \([3, 10]\). In the Ramond sector for the even superspin structures the
discussed period matrices have been calculated in [14]. For the odd superspin structures these \( \omega(\{q_N\}; L) \) period matrices can be calculated by factorization of the even superspin structure in two odd superspin structures when "handles" move away from each other. This calculation is planned to give in another place.

Like previous papers [9, 10, 11, 12, 13, 14], we use superconformal versions of the Schottky groups [22, 23]. Apparently, it is the only modular parameterization that allows to perform explicit calculations of the partition functions in the terms of the even and odd moduli. There are different ways to supersymmetrize ordinary spin structures, but supersymmetrizations do not all be suitable for the superstring theory. Especially, because the space of half-forms does not necessarily have a basis when there are odd moduli [24]. The super-Schottky groups appropriate for all superspin structures have been constructed in [21, 25, 26]. In the \( l_1 = 0 \) case the super-Schottky groups have been built before in [6, 9, 10]. The above \( l_1 = 0 \) case corresponds to the boson loop [9, 10]. The boson loops can be turned into another boson ones by the \((l_{1s} = 0, l_{2s} = 1/2) \rightarrow (l_{1s} = 0, l_{2s} = 0)\) supermodular transformations discussed already in [21, 25]. These transformations restrict the argument of every Schottky multiplier \( k_s \), for example, as \(|\arg k_s| \leq \pi\).

Additional restrictions on the fundamental domain in the modular space are due to the supermodular transformations that, for the given \( s \), interchange \( A_s \)-cycle and \( B_s \)-one. The above supermodular transformations change both the moduli \( q_N \) to \( \hat{q}_N \) and the \( t \) supercoordinate to be \( \hat{t}(t, \{q_N\}) \). We calculate both \( \hat{t}(t, \{q_N\}) \) and \( \hat{q}_N \) in terms of the considered transformation taken at zero odd moduli. The parameters of the above transformation can not be calculated in the explicit form. So we obtain the explicit dependence of \( \hat{t}(t, \{q_N\}) \) and of \( \hat{q}_N \) only on the odd moduli. We show that both \( \hat{t}(t, \{q_N\}) \) and \( \hat{q}_N \) depend on the \( L \) superspin structure. In the \( l_{1s} \neq 0 \) case the fermion fields are non-periodical about the \( A_s \)-cycle, superfields being branched on the complex \( z \)-plane where Riemann surfaces are mapped. Hence cuts arise on the complex \( z \)-plane above. Sets of these cuts can be drown in different ways, but the varieties of these cuts can be turn into each other by suitable supermodular transformations.

When the cuts on the \( z \)-plane present, there are the supermodular transformations due to going \( A_s \)-cycles over each other. Among these transformations, of the especial interest are those, which turn a pair of the even genus-1 structures into a pair of the odd genus-1 structures: \((l_{2s1} = 0, l_{2s2} = 0) \rightarrow (l_{2s1} = 1/2, l_{2s2} = 1/2)\), both \( l_{1s1} = l_{1s2} = 1/2 \) being unchanged. If odd moduli vanish, the moduli and the \((z|\theta)\) coordinates are unchanged under the discussed transformations. For non-zero odd moduli, both the moduli and the \((z|\theta)\) coordinates are changed. In this case we calculate in the explicit form both the resulted moduli and the resulted supercoordinates. The obtained results are used to demonstrate the covariance of the partition functions [14] at zero odd moduli under the considered supermodular transformations.

In the supersymmetrical formalism [3] the problem of the calculation of the partition functions and of the superfield vacuum correlators is concentrated, in mainly, on those superspin structures where at least one of the \( l_{1s} \) characteristics is unequal to zero. Indeed, for superspin structures where all the \( l_{1s} \) characteristics are equal to zero, the considered expressions can be derived [3] by a simple extension of the boson string results [27]. All the other
superspin structures can not be derived in this way. Generally, the procedure of "sewing" allows to consider the discussed superspin structures, but this scheme seems to be complicated, the results being obtained in the form that is rather difficult for an investigation. Main difficulties in the "sewing" scheme are due to the calculation of the Ramond zero mode contributions. The above shortcomings are absent in the scheme developed in \cite{12,13,14}. In the present paper we show in the explicit form that at zero odd moduli, the partition functions calculated in the considered scheme \cite{12,13,14} possess supermodular covariance under supermodular transformations turning two even genus-1 structures to a pair of the odd genus-1 ones.

The paper is organized as it follows. In Section 2 we give the description of the superspin structures in terms of super-Schottky group. We discuss also the fundamental domain in the modular space. Mainly, this Section presents a brief review of the results \cite{21,25,26} essential for understanding the following Sections. In Section 3 we consider the supermodular transformations, which, for the given \( s \), interchange \( A_s \)-cycle and \( B_s \)-one. In Section 4 we consider the supermodular transformations, which turn pair of the even genus-1 spinor structures with \( l_{1s_1} = l_{1s_2} = 1/2 \) into a pair of the odd genus-1 spinor ones. The supermodular covariance of the multi-loop partition functions is discussed in Section 5 and in Section 6. In Section 7 the supermodular covariant integrand in the expression for the multi-loop amplitudes is constructed. The integration region over moduli is defined. A brief discussion of the divergency problem is given.

\section{Superspin structures}

Generally, every superspin structure given on a genus-n complex (1|1) supermanifold is defined by the transformations \( (\Gamma_{a,s}(l_{1s}), \Gamma_{b,s}(l_{2s})) \) that are associated with rounds about \((A_s, B_s)\)-cycles, respectively \((s = 1, 2, ..., n)\). Like the previous Section, we map the supermanifolds by the supercoordinate \( t = (z|\theta) \) where \( z \) is a local complex coordinate and \( \theta \) is its odd partner. Following to \cite{3,4,11,12,13} we use for \( \Gamma_{b,s}(l_{2s}) \) the superconformal versions of Schottky transformations. For zero odd modular parameters these transformations \( \Gamma_{b,s}^{(o)}(l_{2s}) \) are defined as

\begin{equation}
\Gamma_{b,s}^{(o)}(l_{2s}) = \left\{ z \rightarrow g_s(z), \quad \theta \rightarrow -\frac{(-1)^{l_{2s}}\theta}{c_sz + d_s} \right\}
\end{equation}

where \( g_s(z) \) is the Schottky transformations:

\begin{equation}
g_s(z) = \frac{a_sz + b_s}{c_sz + d_s} \quad \text{with} \quad a_sd_s - b_sc_s = 1.
\end{equation}

Eq. \cite{4} takes into account \cite{3} that for \( l_{2s} = 0 \), the spinors are multiplied by -1. Furthermore, the \((a_s, b_s, c_s, d_s)\) parameters can be expressed \cite{23} in terms of two fixed points \( u_s \) and \( v_s \) on the complex \( z \)-plane together with the multiplier \( k_s \) by

\begin{equation}
a = \frac{u - kv}{\sqrt{k(u - v)}}, \quad d = \frac{ku - v}{\sqrt{k(u - v)}} \quad \text{and} \quad c = \frac{1 - k}{\sqrt{k(u - v)}}
\end{equation}
In this case the $\Gamma_{b,s}$ parameters, they being $\mu_{b,s}$ in the explicit form the discussed $\Gamma_{b,s}$ can also be written down as

$$\Gamma_{b,s}(l) = \{ z : |c_s z + d_s| = 1 \} \quad \text{and} \quad C_{u_s} = \{ z : |c_s z + a_s| = 1 \}. \quad (4)$$

It is useful to note that every $v_s$ point is situated inside of $C_{v_s}$ circle and every $u_s$ point is situated inside of $C_{u_s}$. The exterior of all the circles above is chosen to be the fundamental domain $\Omega$ on the complex $z$-plane. A path about $C_{v_s}$ circle ( or about $C_{u_s}$-circle ) corresponds to $2\pi$-twist about $A_s$-cycle. Under the above path the spinors are multiplied by $-1$ in the $l_{1s} = 1/2$ case, for $l_{1s} = 0$ they being unchanged $[3]$. Therefore, $2\pi$-twists about $A_s$-cycles are associated with the following $\Gamma^{(o)}_{a,s}(l_{1s})$ mappings:

$$\Gamma^{(o)}_{a,s}(l_{1s}) = \{ z \rightarrow z, \quad \theta \rightarrow (-1)^{2l_1}\theta \}. \quad (5)$$

To extend the discussed mappings $[1]$ and $[3]$ to arbitrary odd moduli it is necessary to find a relation between odd parameters in $\Gamma^{(o)}_{a,s}(l_{1s} = 1/2)$ and those in $\Gamma_{b,s}(l_{2s})$. Especially, because in the general case the space of half-forms does not have a basis when there are odd moduli $[24]$. To derive the desired relation, we employ $[14, 21, 25, 26]$ that for genus $n = 1$, there are no odd moduli. Indeed, the genus-1 amplitudes are obtained in terms of ordinary spin structures $[4]$. Hence for every particular $s$, all the odd parameters in both $\Gamma_{a,s}(l_{1s})$ and $\Gamma_{b,s}(l_{2s})$ can be reduced to zero by a suitable transformation $\tilde{\Gamma}_s$, which is the same for both the transformations above:

$$\Gamma_{a,s}(l_{1s}) = \tilde{\Gamma}_s^{-1}\Gamma^{(o)}_{a,s}(l_{1s})\tilde{\Gamma}_s, \quad \Gamma_{b,s}(l_{2s}) = \tilde{\Gamma}_s^{-1}\Gamma^{(o)}_{b,s}(l_{2s})\tilde{\Gamma}_s \quad (6)$$

where $\Gamma^{(o)}_{a,s}(l_{1s})$ is given by $[3]$, $\Gamma^{(o)}_{b,s}(l_{2s})$ is given by $[1]$ and $\tilde{\Gamma}_s$ depends, in addition, on two odd parameters, they being $(\mu_s, \nu_s)$. We choose $[14, 21, 25, 26]$ the $\tilde{\Gamma}_s$ mapping as

$$\tilde{\Gamma}_s : \quad z \rightarrow z_s + \theta\varepsilon_s(z_s); \quad \theta \rightarrow \theta_s(1 + \varepsilon_s\varepsilon'_s/2) + \varepsilon_s(z_s);$$

$$\varepsilon'_s = \delta \varepsilon_s(z), \quad \varepsilon_s(z) = [\mu_s(z - v_s) - \nu_s(z - u_s)(u_s - v_s)]^{-1}. \quad (7)$$

In this case the $\Gamma_{b,s}(l_{2s} = 1/2)$ mappings appear to be identical to those proposed in $[6, 9]$. In the explicit form the discussed $\Gamma_{b,s}(l_{2s} = 1/2)$ mappings are given in $[3, 9, 10, 13, 14, 21]$. They can also be written down as

$$\Gamma_{b,s}(l_{2s}) : \quad z \rightarrow g_s(z) + \frac{\theta \varepsilon_s(z, l_{2s})}{(c_sz + d_s)^2} - \frac{(-1)^{2l_2}\varepsilon_s(z)\varepsilon'_s(z)}{(c_sz + d_s)} + \frac{\theta\varepsilon_s(z)\varepsilon'_s(z)}{(c_sz + d_s)^2}, \quad (8)$$

where $g_s(z)$ is the Schottky transformation $[2]$. Both $\varepsilon_s(z)$ and $\varepsilon'_s(z)$ are defined in $[7]$. In addition, $\varepsilon_s(z, l_{2s})$ is defined by

$$\varepsilon_s(z, l_{2s}) = -(-1)^{2l_2}(c_sz + d_s)\varepsilon_s(g_s(z)) - \varepsilon_s(z). \quad (9)$$

Eq.$(8)$ shows that $\Gamma_{b,s}(l_{2s} = 0)$ are obtained from $\Gamma_{b,s}(l_{2s} = 1/2)$ by the $\sqrt{k_s} \rightarrow -\sqrt{k_s}$ replacement $[14, 21, 25, 26]$. Employing $[7]$, one can prove that transformations $[6]$ remain
to be fixed the supermanifold points \((u_s|\mu_s)\) and \((v_s|\nu_s)\), and that \(k_s\) is the multiplier of the \(\Gamma_{b,s}(l_{2s} = 0)\) transformation. Furthermore, it is obvious from (3) and (4) that \(\Gamma_{a,s}^2(l_{1s})\) is the identical transformation, as well as \(\Gamma_{a,s}(l_{1s} = 0)\): \(\Gamma_{a,s}^2(l_{1s}) = I\), \(\Gamma_{a,s}(l_{1s} = 0) = I\). Simultaneously, it is follows from (3)-(7) that \(\Gamma_{a,s}(l_{1s} = 1/2)\) is given by

\[
\Gamma_{a,s}(l_{1s} = 1/2) = \{z \rightarrow z - 2\varepsilon_s(z), \quad \theta \rightarrow -\theta(1 + 2\varepsilon_s\varepsilon'_s) + 2\varepsilon_s(z)\}. \tag{10}
\]

It is useful to note that the right side of (10) is equal to \(\Gamma_{b,s}(l_{2s} = 1/2)\) at \(\sqrt{\kappa_s} = -1\). Since \(\Gamma_{a,s}(l_{1s} = 1/2) \neq I\) and \(\Gamma_{a,s}^2(l_{1s} = 1/2) = I\), a square root cut on the considered \(z\)-plane is associated with every \(l_{1s} \neq 0\). One of its endcut points is placed inside the \(C_{u_s}\) circle and another endcut point is placed inside the \(C_{v_s}\) one. Explicit formulae for conformal tensors \([3, 23, 27]\) show that the above endcut points are situated at \(v_s\) and \(u_s\), respectively.

Superconformal \(p\)-tensors \(T_p(t)\) being considered, every \(\Gamma_{a,s}(l_{1s} = 1/2)\) transformation relates \(T_p(t)\) with its value \(T_p^{(s)}(t)\) obtained from \(T_p(t)\) by \(2\pi\)-twist about \(C_{u_s}\)-circle (3). So, \(T_p(t)\) is changed under the \(\Gamma_{a,s}(l_{1s}) = \{t \rightarrow t_a^s\}\) and \(\Gamma_{b,s} = \{t \rightarrow t_b^s\}\) mappings as

\[
T_p(t_a^s) = T_p^{(s)}(t)Q^p_{\Gamma_{a,s}}(t), \quad T_p(t_b^s) = T_p(t)Q^p_{\Gamma_{b,s}}(t). \tag{11}
\]

where \(Q_{\Gamma_{b,s}}(t)\) and \(Q_{\Gamma_{a,s}}(t)\) are the factors, which the spinor derivative \(D(t)\) receives under the \(\Gamma_{b,s}l_{2s}\) mapping, and, respectively, under the \(Q_{\Gamma_{a,s}}(t)\) one. The \(D(t)\) spinor derivative is defined as

\[
D(t) = \theta\partial_z + \partial_\theta. \tag{12}
\]

In (12) the \(\partial_\theta\) derivative is meant to be the "left" one. For an arbitrary superconformal mapping \(\Gamma = \{t \rightarrow t_\Gamma = (z_\Gamma(t)|\theta_\Gamma(t))\}\), the \(Q_\Gamma(t)\) factor is given by

\[
Q^{-1}_\Gamma(t) = D(t)\theta_\Gamma(t) \quad ; \quad D(t_\Gamma) = Q_\Gamma(t)D(t). \tag{13}
\]

It is obvious from (3) that all the even genus-1 superspin structures can be derived by supermodular transformations of a fixed structure because it can be done if the odd parameters are zero. Moreover, in this case the half-forms, as well as all vacuum superfield correlators associated with every superspin structure, can be derived by transformations (3) from those taken at zero odd parameters. Hence, at least for \(n = 1\), there is no the problem of constructing a basis of the half-forms, which has been observed in [24]. The discussed half-forms can be constructed [14] also for the higher genus supermanifolds. It is fairly clear because the above supermanifolds are all degenerated, in essential part, to the genus-1 supermanifolds when handles move far from each other. In addition, all even (odd) superspin structures can be derived by supermodular transformations of a fixed even (odd) structure that is the necessary condition of the supermodular invariance of the multi-loop amplitudes.

At it is was noted in the Introduction, the \(t\) supercoordinate is transformed under the supermodular group by holomorphic supersymmetrical transformations \(t \rightarrow \hat{t}(t; \{q_M\})\). Simultaneously, \(q_N \rightarrow \hat{q}_N(\{q_M\})\). Also, generally, the above transformations turn out the superspin structures into each other: \(L \rightarrow \hat{L}\). In the theory of Riemann surfaces the action of the modular group on the modular parameters can be given in an implicit form by
the relations between the $\omega^{(r)}(\{q_{\text{ev}}\})$ period matrices and those obtained by the action on $\omega^{(r)}(\{q_{\text{ev}}\})$ of the modular group. The above relations are as follows

$$\omega^{(r)}(\{q_{\text{ev}}\}) = [A\omega^{(r)}(\{\hat{q}_{\text{ev}}^{(0)}\}) + B][C\omega^{(r)}(\{\tilde{q}_{\text{ev}}^{(0)}\}) + D]^{-1} \quad (14)$$

where $A$, $B$, $C$ and $D$ are integral matrices (see also [13]). The $\omega^{(r)}$ period matrices in terms of the Schottky group parameters have been calculated in [23, 27]. The above matrices are given in the Appendix B of the present paper. In the genus $n > 3$ case a number of equations (14) being $n(n+1)/2$, is greater than a number $3n-3$ of the complex moduli, but only $3n-3$ among the equations are independent of each other. So eqs. (14) determine in an implicit form all the new $\hat{q}_{\text{ev}}^{(0)}$ Schottky parameters in terms of the old ones $\{q_{\text{ev}}\}$ up to arbitrariness due to possible fractionally linear transformations of Riemann surfaces. To determine in the similar way the action on the even super-Schottky group parameters of the supermodular group, one must add (14) by the calculation of a dependence of $\hat{q}_{\text{ev}}$ on odd modular parameters. As it explained in two following Sections, the above dependence is determined simultaneously with the calculation of the action of the supermodular group on the odd super-Schottky group parameters. One can also use, instead of (14), supermodular transformations of the period matrices assigned to complex (1|1) supermanifolds. Supermodular transformations $\omega(\{q_N\}; L) \to \omega(\{\tilde{q}_N\}; \tilde{L})$ of the above matrices have the same form (14) as in the theory of the Riemann surfaces:

$$\omega(\{q_N\}; L) = [A\omega(\{\hat{q}_N\}; \hat{L}) + B][C\omega(\{\tilde{q}_N\}; \tilde{L}) + D]^{-1}. \quad (15)$$

In the super-Schottky parameterization the above $\omega(\{q_N\}; L)$ matrices in the Neveu-Schwarz sector have been calculated in [3, 4]. In the Ramond sector the discussed period matrices for the even superspin structures have been calculated in [14]. The period matrices for the odd superspin structures can be calculated in the similar way. This calculation is planned in another paper. For all the even superspin structures the above period matrices is presented in Appendix B.

To determine even if in the implicit form the action of the supermodular group on the even super-Schottky group parameters, eqs. (15) must be added by expressions of the odd super-Schottky parameters $\tilde{q}_{\text{od}}$ in terms of $\{q_N\}$. The simplest supermodular transformations are those, which turn the $(l_{1s} = 0, l_{2s} = 1/2)$ characteristics to $(l_{1s} = 0, l_{2s} = 0)$ and conversely. The above transformations have already been discussed in [21, 23]. Under the considered transformations, arg $k_s$ is replaced by arg $k_s + 2\pi$, other modular parameters being unchanged, as well as the $t$ supercoordinate. Indeed, as it has been explained above, the $\Gamma_{h,s}(l_{2s} = 0)$ transformations are obtained from the $\Gamma_{h,s}(l_{2s} = 1/2)$ transformations (8) just by the arg $k_s \to \text{arg} k_s + 2\pi$ replacement. Hence the considered transformations restrict the argument of every $k_s$, for example, as $|\text{arg} k_s| \leq \pi$. Additional restrictions on the fundamental domain in the modular space arise from supermodular transformations discussed in Section 3 and Section 4. Unlike the above considered transformations, these transformations affect the super-Schottky group parameters. In addition, they appear depending on the superspin structure.
3 Interchanging $A_s$-cycles and $B_s$-cycles.

In this Section we consider those supermodular transformations, which, for a number of the handles, interchange $A$-cycle and $B$-one. These transformations are associated with the $t \to \hat{t}(t, \{q_N\})$ supersymmetrical mappings of the $t = (z, \theta)$ supercoordinate as follows

$$
\hat{z} = f(z) + f'(z)\theta \xi(z) \quad \text{and} \quad \hat{\theta} = \sqrt{f'(z)} \left[ \left(1 + \frac{1}{2} \xi(z)\xi'(z)\right) \theta + \xi(z) \right] \quad (16)
$$

where $f(z)$ (respectively, $\xi(z)$) is an ordinary (respectively, Grassmann) holomorphic function \cite{3,29}. Below it is implied that $z$ in (16) belongs to the fundamental domain \( \Omega \), which is the exterior of all the circles \( \{ \} \). We calculate the dependence of both $f(z)$ and $\xi(z)$ on the odd modular parameters for the supermodular transformations in question. It is obvious that the discussed transformations are determined up to the superconformal fractionally linear transformations. We fix the solution of (16) by the condition that the above transformations remain unchanged the \( \{N_\ast\} \) set of (3|2) of the super-Schottky parameters chosen to be no moduli, which is the same for all the genus-$n$ supermanifolds.

For the resulted superspin structure, $2\pi$ twists about $(A_s, B_s)$-cycles are associated with the $(\hat{\Gamma}_{a,s}(l_{1s}), \hat{\Gamma}_{b,s}(l_{2s}))$ transformations instead of $(\Gamma_{a,s}(l_{1s}), \Gamma_{b,s}(l_{2s}))$. The above transformations are defined by eqs.(3)-(10) in terms of the resulted $(k_s, \hat{u}_s, \hat{v}_s, \hat{\mu}_s, \hat{\nu}_s)$ Schottky parameters instead of the $(k_s, u_s, v_s, \mu_s, \nu_s)$ Schottky ones. In this case both $g_s(z)$, $\varepsilon(z)$ and $\epsilon(z, l_s)$ are replaced by $\hat{g}_s(\hat{z})$, $\hat{\varepsilon}(\hat{z})$ and $\hat{\epsilon}(\hat{z}, l_s)$, respectively. Above $\hat{g}_s(\hat{z})$, $\hat{\varepsilon}(\hat{z})$ and $\hat{\epsilon}(\hat{z}, l_s)$ are expressed in terms of the resulted moduli just the same as $g_s(z)$, $\varepsilon(z)$ and $\epsilon(z, l_s)$ are expressed in terms of $(k_s, u_s, v_s, \mu_s, \nu_s)$. As it was already explained in the Introduction, we calculate both $\hat{t}$ and the $\{\hat{k}_s, \hat{u}_s, \hat{v}_s, \hat{\mu}_s, \hat{\nu}_s\}$ in term of $\hat{z}(0)$ and of $\{\hat{k}_s(0), \hat{u}_s(0), \hat{v}_s(0)\}$, which are equal to above $\hat{z}$ and $\{\hat{k}, \hat{u}, \hat{v}\}$ taken at zero odd modular parameters.

Without loss of generality, we can assume that under the transformations considered, $(l_{1s}, l_{2s})$-characteristics are changed only for $s = 1, 2...p$, they being unchanged for $p < s \leq n$. Hence in the $s \leq p$ case every $2\pi$-twist about $A_s$-cycle is turned to $2\pi$-twist about $B_s$-cycle and conversely. And every $2\pi$-twist about $A_s$-cycle ( $B_s$-cycle ) is turned to itself in the $s > p$ case. Hence the set of the equations arises as follows

$$
\hat{\Gamma}_{a,s}(l_{1s})(\hat{t}) = \hat{t} \left( \Gamma_{b,s}(l_{2s})(t) \right) \quad \text{and} \quad \hat{\Gamma}_{b,s}(l_{2s})(\hat{t}) = \hat{t} \left( \Gamma_{a,s}(l_{1s})(t) \right) \quad \text{for} \quad s \leq p
$$

$$
\hat{\Gamma}_{b,s}(l_{2s})(\hat{t}) = \hat{t} \left( \Gamma_{b,s}(l_{2s})(t) \right) \quad \text{and} \quad \hat{\Gamma}_{a,s}(l_{1s})(\hat{t}) = \hat{t} \left( \Gamma_{a,s}(l_{1s})(t) \right) \quad \text{for} \quad s > p. \quad (17)
$$

The $\hat{t}(s)(t)$ value in (17) obtained by $2\pi$-twist of $\hat{t}(t)$ about $C_{v_s}$-circle \( \{ \} \) on the complex $z$-plane. We write down (17) in term of $f_0(z)$ and $\hat{g}_s(0)(f)$ presenting $f(z)$ and, respectively, $\hat{g}_s(f)$ calculated at zero modular parameters. These $f_0(z)$ and $\hat{g}_s(0)(f)$ obey eqs.(17) taken at zero odd parameters. The above equations can be written down as follows

$$
f_0(z) - f_0(g_s) = \hat{g}_s(0)(f_0) - f_0^{(s)}(z) = 0 \quad \text{for} \quad s \leq p,
$$

$$
\hat{g}_s(0)(f_0) - f_0(g_s) = f_0 - f_0^{(s)} = 0 \quad \text{for} \quad s > p \quad (18)
$$

where $g_s \equiv g_s(z)$ and $f_0 \equiv f_0(z)$. Furthermore, $f_0^{(s)}(z)$ is derived from $f_0(z)$ by $2\pi$-twist about $A_s$-cycle. Eqs.(18) determine both $f_0(z)$ and $\hat{g}_s(0)(f)$ up to fractionally linear transformations.
We fix the choice of both $f_0(z)$ and $\hat{g}_s^{(0)}(f)$ by the condition that the above \{$N_0$\} set of the Schottky parameters is unchanged. In this case both $f(z)$ and $\hat{g}_s(f)$ in question differ from $f_0(z)$ and $\hat{g}_s^{(0)}(f)$ only by terms proportional to odd modular parameters. It will be convenient to define $y(z)$ and $h_s(f)$ functions as follows

$$ y(z) = \frac{f(z) - f_0(z)}{f_0'(z)} \quad \text{and} \quad h_s(f) = \hat{g}_s(f) - \hat{g}_s^{(0)}(f). \quad (19) $$

Every of eqs.\([17]\) presents the set of two equations, every equation being the first order polynomial in $\theta$. So, there are four equations associated with every $2\pi$-twist, but one can verify that two of these equations follow from two other ones. The full set of the independent relations determining $\hat{g}_s(f)$ and $\hat{g}_s^{(0)}(f)$ are given in Appendix A. Eqs.\([20]\) must be complected by the equations, which follow from the relations (17) for $\hat{g}_s^{(0)}(f)$. The explicit formulae for the $\rho^{(pq)}$ functions ( with $p = a, b$ and $q = a, b$ ) are given in Appendix A. Eqs.\([20]\) must be complected by the equations, which follow from the relations (17) for $\hat{g}_s^{(0)}(f)$. To derive these equations, we calculate $\hat{g}_s^{(0)}(f) = \partial_f \hat{g}_s(f)$ using for this purpose the linear in $\theta$ terms in the relations discussed. We substitute this $\hat{g}_s^{(0)}(f)$ to the relations determining $\hat{\theta}$ at $\theta = 0$. In this case the desired equations turn out to be as follows

$$ y(z) - y(g_s(c_s z + d_s)^2 = - \frac{h_s(f)}{f_0'(g_s) g_s'(z)} + \rho_s^{(bb)}(z) \quad \text{for} \quad s > p, $$

$$ y(z) - y(g_s(c_s z + d_s)^2 = \rho_s^{(ab)}(z) \quad \text{for} \quad s \leq p, $$

$$ y(z) - y_s(z) = - \frac{h_s(f)}{f_0'(z)} + \rho_s^{(sa)}(z) \quad \text{for} \quad s \leq p, $$

$$ y(z) - y_s(z) = \rho_s^{(aa)}(z) \quad \text{for} \quad s > p \quad (20) $$

where $f \equiv f(z)$ and $h_s(f)$ is defined by (19). Every $y_s(z)$ in (20) is obtained by $2\pi$-twist of $y(z)$ about $A_s$-cycle. The explicit formulae for the $\rho^{(pq)}$ functions ( with $p = a, b$ and $q = a, b$ ) are given in Appendix A. Eqs.\([20]\) must be complected by the equations, which follow from the relations (17) for $\hat{\theta}$. To derive these equations, we calculate $\hat{g}_s^{(0)}(f) = \partial_f \hat{g}_s(f)$ using for this purpose the linear in $\theta$ terms in the relations discussed. We substitute this $\hat{g}_s^{(0)}(f)$ to the relations determining $\hat{\theta}$ at $\theta = 0$. In this case the desired equations turn out to be as follows

$$ \xi(z) + (-1)^{2l_{2s}} \xi(g_s(c_s z + d_s) = \epsilon_s(z, l_{2s}) + \frac{1 - (-1)^{2l_{2s}} \xi_s(f)}{\sqrt{f'(z)}} + \eta_s^{(ab)}(z) \quad \text{for} \quad s \leq p, $$

$$ \xi(z) + (-1)^{2l_{2s}} \xi(g_s(c_s z + d_s) = \epsilon_s(z, l_{2s}) - \frac{\hat{\epsilon}_s(f, l_{1s})}{\sqrt{f'(z)}} + \eta_s^{(bb)}(z) \quad \text{for} \quad s > p, $$

$$ \xi(z) - (-1)^{2l_{1s}} \xi_s(z) = \left[ 1 - (-1)^{2l_{1s}} \right] \xi_s(z) + \frac{\hat{\epsilon}_s(f, l_{1s})}{\sqrt{f'(z)}} + \eta_s^{(ba)}(z) \quad \text{for} \quad s \leq p, $$

$$ \xi(z) - (-1)^{2l_{1s}} \xi_s(z) = \left[ 1 - (-1)^{2l_{1s}} \right] \xi_s(z) - \frac{\hat{\epsilon}_s(f)}{\sqrt{f'(z)}} + \eta_s^{(aa)}(z) \quad \text{for} \quad s > p \quad (21) $$

where $f \equiv f(z)$ and $g_s \equiv g_s(z)$. The $\eta_s^{(pq)}$ functions with $p = a, b$ and $q = a, b$ are given in Appendix A. Every $\xi_s(z)$ value in (21) is obtained by $2\pi$-twist of $\xi(z)$ about $C_{v_s}$-circle (4) on the complex $z$-plane. The first pair of the equations in (20) and in (21) is derived from those of eqs.\([17]\), which associated with $2\pi$-twists about $B_s$-cycles on $z$-plane. And the
The changes of the considered Green functions under the \( z \to g_s(z') \) transformations on \( z' \)-plane. We require also that the above Green functions have no singularities in the \( \Omega \) fundamental domain on both \( z \)-plane and \( z' \)-one, except only at \( z = z' \). Particular, being a non-singular conformal 2-tensor, \( G^{(b)}_{gh}(z, z') \) decreases not slower than \((z')^{-4}\) at \( z' \to \infty \). And \( G^{(f)}_{gh}(z, z') \) being a non-singular conformal 3/2-tensor, decreases not slower than \((z')^{-3}\) at \( z' \to \infty \). It is useful to note that under \( z \to g_s(z) \) transformations the above Green functions necessarily have the depending on \( z \) periods. So they are not to be conformal tensors under the above transformations. The discussed Green functions can be obtained from ghost Green functions considered in [12, 13, 14]. These \( G_{gh}(t, t') \) Green functions appear in the special ghost scheme [12, 13] that allows to calculate, among of other things, both the moduli volume form and zero mode contributions to the partition functions. Both \( G^{(b)}_{gh}(z, z') \) and \( G^{(f)}_{gh}(z, z') \) are calculated from the \( G_{gh}(t, t') \) defined in [14] by taking all modular parameters to be zero. In this case, \( G_{gh}(t, t') \) can be written down in terms of \( G^{(b)}_{gh}(z, z') \) and \( G^{(f)}_{gh}(z, z') \) as

\[
G_{gh}(t, t') = G^{(b)}_{gh}(z, z') \theta' + \theta G^{(f)}_{gh}(z, z').
\]

We normalize both \( G^{(b)}_{gh}(z, z') \) and \( G^{(f)}_{gh}(z, z') \) as follows

\[
G^{(b)}_{gh}(z, z') \to -(z - z')^{-1} \quad \text{and} \quad G^{(f)}_{gh}(z, z') \to (z - z')^{-1} \quad \text{at} \quad z \to z'.
\]

Then the changes of the considered Green functions under the \( z \to g_s(z) \) transformation are given by [14]

\[
G^{(f)}_{gh}(g_s(z), z') = (-1)^{2d_s-1}(c_s z + d_s)^{-1} \left( G^{(f)}_{gh}(z, z') + \sum_{F_s} \theta \bar{P}_{F_s}(z) \tilde{\chi}^{(0)}_{F_s}(z') \right)
\]

\[
G^{(b)}_{gh}(g_s(z), z') = (c_s z + d_s)^{-2} \left( G^{(b)}_{gh}(z, z') + \sum_{R_s} \bar{P}_{R_s}(z) \tilde{\chi}^{(0)}_{R_s}(z') \right),
\]

where \( \tilde{\chi}^{(0)}_{R_s}(z') \) are the conformal 2-tensor zero modes ( in number of \( 3n - 3 \) ), and \( \tilde{\chi}^{(0)}_{F_s}(z') \) are the conformal 3/2-tensor zero modes ( in number of \( 2n - 2 \) ).\(^2\) We use for both \( R_s \) and \( F_s \) the same notation as for the Schottky parameters: \( R_s = (k_s, u_s, v_s) \) and \( F_s = (\mu_s, v_s) \). The summation in (23) is performed over only those \( R_s = (k_s, u_s, v_s) \) and \( F_s = (\mu_s, v_s) \), which do not belong to the \( \{N_0\} \) set chosen to be the same for all the genus-\( n \) surfaces. Furthermore,

\(^2\) In terms of the \( \tilde{\chi}^{(0)}_{N_s}(t') \) zero modes defined in [14], the above \( \tilde{\chi}^{(0)}_{R_s}(z') \) and \( \tilde{\chi}^{(0)}_{F_s}(z') \) are expressed as \( \theta' \tilde{\chi}^{(0)}_{R_s}(z') = \tilde{\chi}_{R_s}(t') \) and \( \tilde{\chi}^{(0)}_{F_s}(z') = \tilde{\chi}_{F_s}(t') \), \( \tilde{\chi}_{N_s}(t') \) being taken at zero odd modular parameters.
$P_{F,a}(z)$ and $P_{R,a}(z)$ in [23] present polynomials of degree-1 and, respectively, of degree-2. In the explicit form the above polynomials are given by [12, 13, 14, 28]

$$
P_{k_s} = (c_s + d_s)^2 \frac{\partial g_s(z)}{\partial k_s}, \quad P_{u_s} = (c_s + d_s)^2 \frac{\partial g_s(z)}{\partial u_s}, \quad P_{v_s} = (c_s + d_s)^2 \frac{\partial g_s(z)}{\partial v_s},$$

where $P_{k_s} = P_{k_s}(z)$, $P_{u_s} = P_{u_s}(z)$, $P_{v_s} = P_{v_s}(z)$, and $\epsilon_s(z, l_{2s})$ is defined by (3). It is obvious that the $G^{(f)}_{gh}$ Green functions depend on the spin structure. In the $l_{1s} \neq 0$ case they change also under $2\pi$-twist about $C_{v_s}$ cycle as follows

$$
G^{(f)}_{gh}(z, z') = -\frac{1 - (-1)^{2l_{1s}}}{2} \left[ G^{(f)(s)}_{gh}(z, z') + \sum_{F_s} P^{(a)}_{F_s}(z) \tilde{\chi}_{F_s}(z') \right]
$$

(25)

where $G^{(f)(s)}_{gh}(z, z')$ denotes the $G^{(f)}_{gh}$ Green function $2\pi$-twisted about $A_s$-cycle. Like (23), the summation in (23) is performed over only those $F_s = (\mu_s, \nu_s)$, which chosen to be moduli, and $\tilde{\chi}_{F_s}(z')$ are the same as in (23). Furthermore, $P^{(a)}_{F_s}(z)$ are degree-1 polynomials in $z$. The above polynomials are equal to $P_{F,a}(z)$ polynomials in eq. (24) taken at $(-1)^{2l_{2s}} \sqrt{k_s} = -1$. In the explicit form

$$
P^{(a)}_{\mu_s}(z) = -\frac{4(z - v_s)}{u_s - v_s} \quad \text{and} \quad P^{(a)}_{\nu_s}(z) = \frac{4(z - u_s)}{u_s - v_s}.
$$

(26)

It is proved in [14] that eqs.(23) and eqs.(25) are self-consistent. In [14] the discussed Green functions were considered only for even spin structures, but they can be extended without any difficulties to odd spin ones of genus $n > 1$.

To derive the set of integral equations for both $\xi(z)$ and $y(z)$ in question we start with the following relations

$$
\xi(z) = -\int_{C(z)} G^{(f)}_{gh}(z, z') \xi(z') \frac{dz'}{2\pi i} \quad \text{and} \quad y(z) = \int_{C(z)} G^{(b)}_{gh}(z, z') y(z') \frac{dz'}{2\pi i}
$$

(27)

where infinitesimal contour $C(z)$ gets around $z$-point in the positive direction. Then we deform this contour to those, which surround both $C_{v_s}$ and $C_{u_s}$ circles [4] together with the $C_s$ cuts that, generally, present because both $y(z)$ and $\xi(z)$ are branched. Every integral along $C_{u_s}$ is reduced to the integral along the $C_{v_s}$ contour by the $z' \to g_s(z')$ transformation. As the result, in the integrand, either $[y(z') - y(g_s(z'))(s_sz' + d_s)^2]$ or $[\xi(z') - \xi(g_s(z'))(c_sz' + d_s)^2]$
appears. We replace every above value by its value given by eqs. (20) and (21). Eqs. (20)-(21) are used also to calculate both \[\xi(z') - \xi^{(a)}(z')\] and \[y(z') - y^{(a)}(z')\] in every integral along the \(C_s\) cut. The desirable equations turn out to be

\[
\xi(z) = \sum_{s=1}^{p} \int_{C_s} G_{gh}^{(f)}(z, z') \left[ 1 - (-1)^{2n_s} \varepsilon_s(z') + \frac{\dot{\varepsilon}_s(f, l_{1s})}{\sqrt{f'(z')}} - \eta^{(ba)}_s(z') \right] \frac{dz'}{2\pi i} \\
+ \sum_{s=p+1}^{n} \int_{C_s} G_{gh}^{(f)}(z, z') \left[ \varepsilon_s(z') \frac{1}{\sqrt{f'(z')}} \right] \frac{dz'}{2\pi i} \\
+ \sum_{s=1}^{p} \int_{C_{v_s}} G_{gh}^{(f)}(z, z') \left[ \varepsilon_s(z') \frac{1}{\sqrt{f'(z')}} + \eta^{(ab)}_s(z') \right] \frac{dz'}{2\pi i} \\
+ \sum_{s=p+1}^{n} \int_{C_{v_s}} G_{gh}^{(f)}(z, z') \left[ \varepsilon_s(z') \frac{1}{\sqrt{f'(z')}} \right] \frac{dz'}{2\pi i}
\]

(28)

together with the following ones

\[
y(z) = \sum_{s=1}^{p} \int_{C_s} G_{gh}^{(b)}(z, z') \left[ \rho^{(ba)}_s(z', l_{1s}) - \frac{h_s(f)}{f_0(s)'(z')} \right] \frac{dz'}{2\pi i} \\
+ \sum_{s=p+1}^{n} \int_{C_s} G_{gh}^{(b)}(z, z') \rho^{(aa)}_s(z') \frac{dz'}{2\pi i} \\
+ \sum_{s=1}^{p} \int_{C_{v_s}} G_{gh}^{(b)}(z, z') \left[ \frac{h_s(f)}{f_0(g_s)g_s'(z')} - \rho^{(bb)}_s(z) \right] \frac{dz'}{2\pi i} \\
+ \sum_{s=p+1}^{n} \int_{C_{v_s}} G_{gh}^{(b)}(z, z') \rho^{(ab)}_s(z') \frac{dz'}{2\pi i}.
\]

(29)

In (28) and (29) the definition are the same as in (21) and (21). Both \(\eta^{(ar)}(z')\) and \(\rho^{(ar)}(z')\) are defined in Appendix A. Every \(C_s\) path in (28) and (24) goes along the upper edge of the cut from the \(z_s'^{(−)}\) point to the \(z_s'^{(+)}\) point where \(z_s'^{(−)} = g_s(z_s'^{(−)})\) is an arbitrary point on the \(C_{v_s}\) circle. Every \(C_{v_s}\) circle in (28) and (24) is rounded in the positive direction starting from the \(z_s'^{(−)}\) point above.

Generally, the right sides of (28)-(29) has a singularity at \(z = z_s'^{(−)}\) and at \(z = z_s'^{(+)}\), as well. At the same time, both \(y(z)\) and \(\xi(z)\) are assumed to have singularity neither at the boundary of the fundamental domain \(\Omega\) nor inside \(\Omega\). So, generally, the left sides of (29)-(21) being calculated from (28)-(29), differ from those given on the right side of above eqs. (20)-(21). The additional contributions to the left sides of eqs.(24)-(21) are caused by the proportional to zero mode terms on the right side of (24). Therefore, eqs. (28)-(29) are equivalent to (20)-(21) only if the discussed contributions are equal to zero. Hence eqs. (28) and (29) must be added by the following relations

\[
0 = \sum_{s=1}^{p} \int_{C_s} \tilde{X}_F_r(z') \left[ 1 - (-1)^{2n_s} \varepsilon_s(z') + \frac{\dot{\varepsilon}_s(f, l_{1s})}{\sqrt{f'(z')}} - \eta^{(ba)}_s(z') \right] \frac{dz'}{2\pi i} \\
+ \sum_{s=p+1}^{n} \int_{C_s} \tilde{X}_F_r(z') \left[ 1 - (-1)^{2n_s} \right] \left( \varepsilon_s(z') - \frac{\dot{\varepsilon}_s(f)}{\sqrt{f'(z')}} \right) \frac{dz'}{2\pi i}
\]

(28)
\[ + \sum_{s=1}^{p} \int_{\tilde{C}_{v_s}} \tilde{\chi}_{F_s}(z') \left[ \epsilon_s(z', l_{2s}) + \frac{1 - (-1)^{2l_{2s}}}{\sqrt{f'(z')}} + \eta^{(ab)}_s(z') \right] \frac{dz'}{2\pi i} \]

\[ + \sum_{s=p+1}^{n} \int_{\tilde{C}_{v_s}} \tilde{\chi}_{F_s}(z') \left[ \epsilon_s(z', l_{2s}) - \frac{\hat{\epsilon}_s(f, l_{2s})}{\sqrt{f'(z')}} + \eta^{(bb)}_s(z') \right] \frac{dz'}{2\pi i} \quad (30) \]

where \( r = 1, \ldots n \), and to the following ones

\[ 0 = \sum_{s=1}^{p} \int_{\tilde{C}_s} \tilde{\chi}_{R_s}(z') \left[ \frac{h_s(f)}{f_0^{(s)'}(z')} - \rho^{(bb)}_s(z', l_{1s}) \right] \frac{dz'}{2\pi i} + \sum_{s=p+1}^{n} \int_{\tilde{C}_s} \tilde{\chi}_{R_s}(z') \rho^{(aa)}_s(z') \frac{dz'}{2\pi i} \]

\[ - \sum_{s=p+1}^{n} \int_{\tilde{C}_{v_s}} \tilde{\chi}_{R_s}(z') \left[ \frac{h_s(f)}{f_0^{(s)'}(g_s')} - \rho^{(bb)}_s(z) \right] \frac{dz'}{2\pi i} + \sum_{s=1}^{p} \int_{\tilde{C}_{v_s}} \tilde{\chi}_{R_s}(z') \rho^{(bb)}_s(z') \frac{dz'}{2\pi i} \quad (31) \]

Eqs. (28)-(31) determine both \( \xi(z) \) and \( g(z) \), as well as both \( (\hat{\mu}_s, \hat{v}_s) \) and the \( (\delta k_s, \delta u_s, \delta v_s) \) differences defined to be

\[ \delta k_s = \hat{k}_s - k_s^{(0)}, \quad \delta u_s = \hat{u}_s - u_s^{(0)} \quad \text{and} \quad \delta v_s = \hat{v}_s - v_s^{(0)}. \quad (32) \]

The \( (k_s^{(0)}, u_s^{(0)}, v_s^{(0)}) \) Schottky parameters in (32) are associated with the \( g_s^{(0)}(f) \). A various choice of the shape of the \( \tilde{C}_s \) lines is associated with various supermodular transformations.

Solving (28)-(31), it is useful to use the following relations [14]

\[ \int_{C_{v_s}} G_{gh}^{(f)}(z, z') P_{F_s}(z') \frac{dz'}{2\pi i} - [1 - (-1)^{2l_{1s}}] \int_{\tilde{C}_s} G_{gh}^{(f)}(z, z') P_{F_s}^{(a)}(z') \frac{dz'}{4\pi i} = 0, \]

\[ \int_{C_{v_s}} G_{gh}^{(b)}(z, z') P_{R_s}(z') \frac{dz'}{2\pi i} = 0 \quad (33) \]

where \( P_{R_s}(z'), P_{F_s}(z') \) and \( P_{F_s}^{(a)}(z') \) are defined by (24) and (25). To prove (33) one can start with integrating a suitable Green function product taken along the infinitesimal contour around \( z \)-point. Deforming this contour to that, which surrounds both \( C_{v_s} \) and \( C_{u_s} \) circles (4) together with the \( \tilde{C}_s \) cut (if it exists), one uses (23) and (25). In more details the proof of the relations similar to (33) is discussed in [14]. Particular, owing to (33), the sum of the integrals of \( \varepsilon_s \) and \( \epsilon_s \) disappears in eq.(28). Furthermore, using (22), (23) and (33), one derives the following relations [14]

\[ \int_{C_{v_s}} \tilde{\chi}_{F_s}(z') P_{F_s}(z') \frac{dz'}{2\pi i} - [1 - (-1)^{2l_{1s}}] \int_{\tilde{C}_s} \tilde{\chi}_{F_s}(z') P_{F_s}^{(a)}(z') \frac{dz'}{4\pi i} = -\delta_{F_s', F_s}, \]

\[ \int_{C_{v_s}} \tilde{\chi}_{R_s}(z') P_{R_s}(z') \frac{dz'}{2\pi i} = \delta_{R_s', R_s}. \quad (34) \]
where $\tilde{\chi}_{R_s}(z')$ and $\tilde{\chi}_{F_s}(z')$ are the same as in (30) and in (31), $\delta_{NN'}$ being the Kronecker symbol. Particular, using (31), one can perform explicitly the integrals of $\varepsilon_s$ and $\varepsilon_s$ in (33). In this case (30) are written down as

$$
\hat{\mu}_r = \sum_{s=1}^{n} \left[ \hat{X}_{\mu;\mu_s}^{-1}(L)\mu_s + \hat{X}_{\nu;\nu_s}^{-1}(L)\nu_s \right] + \sum_{s=1}^{n} \hat{X}_{\nu;F_s}^{-1}(L)\eta_{F_s},
$$

$$
\hat{\nu}_r = \sum_{s=1}^{n} \left[ \hat{X}_{\nu;\mu_s}^{-1}(L)\mu_s + \hat{X}_{\nu;\nu_s}^{-1}(L)\nu_s \right] + \sum_{s=1}^{n} \hat{X}_{\nu;F_s}^{-1}(L)\eta_{F_s}
$$

(35)

where $\eta_{F_s}$ is defined as

$$
\eta_{F_s} = 2 \sum_{s=1}^{p} \int_{C_{v_s}} \tilde{\chi}_{F_s}(z)\eta^{(ab)}_s(z) \frac{dz}{2\pi i} + 2 \sum_{s=p+1}^{n} \int_{C_{v_s}} \tilde{\chi}_{F_s}(z)\eta^{(bb)}_s(z) \frac{dz}{2\pi i} - 2 \sum_{s=1}^{p} \int_{C_{v_s}} \tilde{\chi}_{F_s}(z)\eta^{(ba)}_s(z) \frac{dz}{2\pi i} - 2 \sum_{s=p+1}^{n} \int_{C_{v_s}} \tilde{\chi}_{F_s}(z)\eta^{(aa)}_s(z) \frac{dz}{2\pi i}.
$$

(36)

The $\hat{X}_{F_s;F_s}(L)$ matrix in (33) is given by

$$
\hat{X}_{F_s;\mu_s}(L) = 2 \frac{1 - (-1)^{2l_s}}{\hat{u}_s - \hat{v}_s} \int_{C_{v_s}} \frac{dz}{2\pi i} \frac{\tilde{\chi}_{F_s}(z)[f(z) - \hat{v}_s]}{f'(z)}
$$

$$
-2 \frac{(-1)^{2l_s}}{\hat{u}_s - \hat{v}_s} \int_{C_{v_s}} \frac{dz}{2\pi i} \frac{\tilde{\chi}_{F_s}(z)[f(z) - \hat{v}_s]}{\sqrt{k_s}f'(z)}
$$

for $s \leq p$;

$$
\hat{X}_{F_s;\nu_s}(L) = 2 \frac{(-1)^{2l_s}}{\hat{u}_s - \hat{v}_s} \int_{C_{v_s}} \frac{dz}{2\pi i} \frac{\tilde{\chi}_{F_s}(z)[f(z) - \hat{v}_s]}{\sqrt{k_s}f'(z)}
$$

$$
-2 \frac{1 - (-1)^{2l_s}}{\hat{u}_s - \hat{v}_s} \int_{C_{v_s}} \frac{dz}{2\pi i} \frac{\tilde{\chi}_{F_s}(z)[f(z) - \hat{v}_s]}{f'(z)}
$$

$$
+2 \frac{(-1)^{2l_s}}{\hat{u}_s - \hat{v}_s} \int_{C_{v_s}} \frac{dz}{2\pi i} \frac{\tilde{\chi}_{F_s}(z)[f(z) - \hat{v}_s]}{\sqrt{k_s}f'(z)}
$$

for $s \geq p$;

$$
\hat{X}_{F_s;F_s}(L) = -2 \frac{(-1)^{2l_s}}{\hat{u}_s - \hat{v}_s} \int_{C_{v_s}} \frac{dz}{2\pi i} \frac{\tilde{\chi}_{F_s}(z)[f(z) - \hat{v}_s]}{\sqrt{k_s}f'(z)}
$$

$$
+2 \frac{1 - (-1)^{2l_s}}{\hat{u}_s - \hat{v}_s} \int_{C_{v_s}} \frac{dz}{2\pi i} \frac{\tilde{\chi}_{F_s}(z)[f(z) - \hat{v}_s]}{f'(z)}
$$

(37)

Since the terms on the right sides of (28), (29), (31) and (33) are proportional to Grassmann parameters, the above equations can be solved by an iteration procedure. In this case the
solution of the considered equations is obtained to be series in \{\mu_s, \nu_s\}. The first step of the discussed procedure is to replace \( f(z') \) in (28) and in (33) by \( f_0(z') \) (assumed to be known), the terms with \( \eta^{(qr)} \) \( q = a, b \) and \( r = a, b \) being neglected. In this case (28) determines the linear in \{\mu_s, \nu_s\} terms in \( \xi(z) \). In the same approximation eqs. (33) determine the action of the supermodular group on the odd super-Schottky parameters. It is obvious that both \( \xi(z) \) and \( y(z) \) have no singularities inside the fundamental domain \( \Omega \) and at its boundary, as well (the proof is omitted here). Simultaneously, all the resulted values in question depend on the \( L \) superspin structure, if the \{\mu_s, \nu_s, v_s, \mu_v, \nu_v\} set is chosen to be the same for all the superspin structures. The above \( L \) dependence arises because both \( G_{gth}^{(f)}(z, z') \) in (28) and \( \tilde{\chi}_F(z') \) in (31) depend on \( L \). One can see also that all the even superspin structures \( S_{ev} \) without odd genus-1 structures can be obtained by a suitable supermodular transformation of the only superspin structure. In the next Section we consider the supermodular transformations turning the above \( S_{ev} \) structures to those containing an even number of the odd genus-1 spinor structures.

4 Transformation of two even genus-1 structures to a pair of the odd genus-1 ones

In this Section we consider the supermodular transformations of \( S_{ev} \rightarrow S_2 \), which turn a pair of the genus-1 structures to a pair of the odd genus-1 ones, say \( l_{1s_1} = l_{1s_2} = 1/2, l_{2s_1} = l_{2s_2} = 0 \rightarrow l_{1s_1} = l_{1s_2} = l_{2s_1} = l_{2s_2} = 1/2 \). Without loss of generality we assume that \( s_1 = 1 \) and \( s_2 = 2 \). Under the discussed transformation, the \( \omega_{12}^{(r)} \) element of the \( \omega^{(r)} \) period matrix turns to \( \omega_{12}^{(r)} \pm 1 \), the other \( \omega^{(r)} \) matrix elements being unchanged. It is worth while to note that the expression (31) of \( \omega_{12}^{(r)} \) contains, among of other things, the following term

\[
\omega_{12}^{(r)} = \frac{1}{2\pi i} \ln \frac{(u_1 - u_2)(v_1 - v_2)}{(u_1 - v_2)(v_1 - u_2)} + ...
\]

The desirable \( \omega_{12}^{(r)} \rightarrow \omega_{12}^{(r)} \pm 1 \) replacement is achieved by the addition of 2\( \pi \) to the argument of one of the differences inside the round brackets in (33). So the discussed transformation of \( \omega \) appears to be the result of suitable rounds of the \( (u_1, u_2, v_1, v_2) \) fixed points over each other. As example, we consider the clock-wise going of the \( u_2 \) point about the \( u_1 \) one. On
z-plane the discussed round corresponds to the going of $C_{u_2}$ circle (4) about $C_{u_1}$ circle, as it is shown in Fig.1. In this case we start with the $S_{u_1}$ superspin structure with $l_{11} = l_{12} = 1/2$, $l_{21} = l_{22} = 0$. The initial position of the circles and the cuts is shown in Fig.1(a). The $(v_1)(u_1)$ cut is situated between $v_1$ and $u_1$ fixed points and the $(v_2)(u_2)$ cut is situated between $v_2$ and $u_2$. After the discussed round to be performed, the circles are returned again to the same position, but the cuts are deformed, as it is shown in Fig.1(b). A shape of the cuts is unessential because, as it has been explained in Section 2, it can be changed by a supermodular transformation. Hence we can close the cuts together, as it is shown in Fig.1(c). In Fig.1(c) the resulted cuts are represented by the bold lines. As it is follows from Fig.1(b), the bold line about $C_{u_1}$ circle is formed by the $(v_2)(u_2)$ cut sandwiched by $(v_2)(u_2)$ cut. The bold line about $C_{u_2}$ circle is formed by the $(v_2)(u_2)$ cut. And the bold line drawing from $C_{u_1}$ circle to $C_{u_2}$ one, is formed by both $(v_1)(u_1)$ and $(v_2)(u_2)$ cuts sandwiched every other. The cuts surrounding both the circles in Fig.1(c) are removed by the re-definition of superholomorphic functions at ( and inside ) the considered circles to be the analytical continuation of the above superholomorphic functions from the fundamental domain $\Omega$. As we noted already in Section 2, the $\Omega$ domain presents the exterior of all the $C_{u_1}$ circles and of the $C_{u_2}$ ones. Under the above re-definition, both the $(\Gamma^{(ch)}_{b,s}(l_{2s} = 0)$ transformations for both $s = 1$ and $s = 2$ are also changed to be $\Gamma^{(ch)}_{b,s}$ as follows

$$\Gamma^{(ch)}_{b,s} = \delta_{s1} \Gamma_{a,1} \Gamma_{a,2} \Gamma_{b,1} (l_2 = 0) + \delta_{s2} \Gamma_{a,1} \Gamma_{b,2} (l_2 = 0) \quad \text{for} \quad s = 1, 2 \quad (40)$$

In eq.(40), the $\Gamma_{a,s} = \Gamma_{a,s}(l_{1s} = 1/2)$ mappings are defined by (10) and $\delta_{sr}$ is the Kronecker symbol. As it follows from (41) and (42), in the case of the odd modular parameters to be zero ($\mu_1 = \nu_1 = \mu_2 = \nu_2 = 0$), the transformation (40) just corresponds to the transformation of two considered even genus-$1$ spinor structures into the odd genus-$1$ spinor structures: $l_{21} = l_{22} = 0 \rightarrow l_{21} = l_{22} = 1/2$, $l_{11} = l_{12} = 1/2$ being unchanged. For arbitrary modular parameters, every $T_p(t)$ superconformal $p$-tensor is changed in going across the $(u_1)(u_2)$ line in Fig.1(c) by the $\Gamma_{2121}$ transformation turned out to be

$$\Gamma_{2121} = \Gamma_{a,2} \Gamma_{a,1} \Gamma_{a,2} \Gamma_{a,1} \quad (41)$$

where $\Gamma_{a,s}$ are the same as in (40). Eq.(41) follows directly from (41) and Fig.1(b). Hence in this case the cut arises to be between $C_{u_1}$ and $C_{u_2}$. Nevertheless, we show that this cut can be removed by a suitable superholomorphic mapping of the $t = (z|\bar{\theta})$ supercoordinate. As the result, we again obtain two odd genus-$1$ structures. We write down the desired mapping $t \rightarrow \tilde{t} = (\tilde{z}|\tilde{\bar{\theta}})$ of the $t$ supercoordinate as follows

$$z = \tilde{f}(\tilde{z}) + \tilde{f}'(\tilde{z}) \tilde{\bar{\theta}} \tilde{\xi}(\tilde{z}) \quad \text{and} \quad \theta = \sqrt{\tilde{\bar{\theta}}(\tilde{z})} \left[ \left( 1 + \frac{1}{2} \tilde{f}(\tilde{z}) \tilde{\xi}(\tilde{z}) \right) \tilde{\bar{\theta}} + \tilde{\xi}(\tilde{z}) \right] \quad (42)$$

where both $\tilde{f}(\tilde{z})$ and $\tilde{\xi}(\tilde{z})$ are proportional to the odd modular parameters. In this case the $\{\tilde{u}_s, \tilde{v}_s, \tilde{k}_s\}$ resulted Schottky parameters differ from the old Schottky ones only by terms proportional to the odd modular parameters. Furthermore, $A_s \rightarrow \tilde{A}_s$ and $B_s \rightarrow \tilde{B}_s$ under the above mapping (42). Like the previous Section, we calculate that solution of (42), which
does not change the \( \{N_0\} \) set of \((3|2)\) Schottky parameters chosen to be no moduli. Both \( \tilde{f}(\tilde{z}) \) and \( \xi(\tilde{z}) \) are calculated from the condition that superconformal tensors have no discontinuity going across the \( (\tilde{u}_1)(\tilde{u}_2) \) line on \( \tilde{z} \)-plane. In this case

\[
\tilde{\xi}^{(t)}(\tilde{z}) - \tilde{\xi}^{(r)}(\tilde{z}) = 4[\varepsilon_2(\tilde{f}) - \varepsilon_1(\tilde{f})] - 12\varepsilon_1(\tilde{f})\varepsilon_2(\tilde{f}) + 4\varepsilon_1(\tilde{f})\varepsilon_2(\tilde{f})\varepsilon_1(\tilde{f})
\]
and
\[
\tilde{f}^{(t)}(\tilde{z}) - \tilde{f}^{(r)}(\tilde{z}) = -8\varepsilon_1(\tilde{f})\varepsilon_2(\tilde{f})
\]

where \( \tilde{f} \equiv \tilde{f}(\tilde{z}) \) and the \((l)\) symbol (and, respectively, the \((r)\) symbol) at the right top shows that the value being marked by the above symbol, is calculated at the left (and, respectively, the right) edge of the \( (\tilde{u}_1)(\tilde{u}_2) \) cut. Eqns.\((43)\) follow directly from \((11)\). Above eqns.\((43)\) must be completed by the relations

\[
\Gamma_{b,s}(t) = t \left( \tilde{\Gamma}_{b,s}(l_{2s})(\hat{t}) \right) \quad \text{and} \quad \Gamma_{a,s}(l_{1s})(\hat{t}) = t^{(s)}(\tilde{\Gamma}_{a,s}(l_{1s})(\hat{t})) \quad \text{for} \quad s = 1, 2;
\]

\[
\Gamma_{b,s}(l_{2s})(t) = t \left( \tilde{\Gamma}_{b,s}(l_{2s})(\hat{t}) \right) \quad \text{and} \quad \Gamma_{a,s}(l_{1s})(t) = t^{(s)}(\tilde{\Gamma}_{a,s}(l_{1s})(\hat{t})) \quad \text{for} \quad s > 2.
\]

where \( (\tilde{\Gamma}_{a,s}(l_{1s}), \tilde{\Gamma}_{b,s}(l_{2s})) \) transformations are associated with \( 2\pi \)-twists about \( (A_s, B_s) \)-cycles on \( \hat{t} \)-supermanifold. Furthermore, \( l_{1s} = l_{2s} = 1/2 \) for \( s = 1, 2 \) and \( \Gamma_{b,s}^{(ch)} \) are given by \((14)\). Eqs.\((43)\) determine the discontinuities of both \( \tilde{f}(\tilde{z}) \) and \( \xi(\tilde{z}) \) under \( 2\pi \)-twists about \( (A_s, B_s) \)-cycles. Eqns.\((43)\)-\((44)\) can be solved by the same method, which was developed in the previous Section for the solution of eqns.\((17)\). In this case both \( \tilde{f}(\tilde{z}) \) and \( \xi(\tilde{z}) \) are found to be quite similar to \( f(z) \) and \( \xi(z) \) in Section 3 except only the additional term due to eqns.\((43)\). To avoid unwieldy expressions we give in the explicit form only the terms linear in odd modular parameters. In this approximation, \( \tilde{f}(\tilde{z}) = \tilde{z} \). In addition, there is no difference between \( \xi(\tilde{z}) \) and \( \xi(z) \). In the considered approximation, the desired \( \xi(z) \) turns out to be

\[
\xi(z) = -\int_{\tilde{C}_{12}} G_{gh}^{(f)}(z, z')\gamma(z')\frac{dz'}{2\pi i} + \int_{C_{u_2}} G_{gh}^{(f)}(z, z')\gamma(z')\frac{dz'}{4\pi i} - \int_{C_{u_2}} G_{gh}^{(f)}(z, z')\gamma(z')\frac{dz'}{4\pi i}
\]

In \((15)\) the \( \tilde{C}_{12} \) path goes along the \( u_1u_2 \) cut in Fig.\(1(c) \) from the \( z_1^{(+)} \) point to the \( z_2^{(+)} \) one. Every \( z_s^{(+)} \) point \((s = 1, 2) \) is the intersect of the \( \tilde{C}_{12} \) path with the \( C_{u_s} \) circle. Every \( C_{u_s} \)-circle \((s = 1, 2) \) in \((13)\) is rounded in the positive direction starting from the \( z_s^{(-)} \) point defined above. The \( \gamma(z) \) local parameter in \((13)\) is given by

\[
\gamma \equiv \gamma(z) = 4\varepsilon_2(z) - 4\varepsilon_1(z)
\]

where \( \varepsilon_s(z) \) are defined by \((9)\) for \( s = 1, 2 \). In deriving \((43)\) we transform the integrals along the \( C_{v_{1s}} \) circles to the integrals along the \( C_{u_s} \) circles by the \( z \rightarrow (d_sz - b_s)(-c_sz + a_s)^{-1} \) replacements. In addition, \((43)\), eqns.\((13)\) have been taken into account. Using \((1)\) and \((23)\) one can express \( \gamma(z) \) in terms of \( P_{F_s}^{(a)} \) polynomials as follows

\[
\gamma(z) = \mu_2 P_{\mu_2}^{(a)}(z) + \nu_2 P_{\nu_2}^{(a)}(z) - \mu_1 P_{\mu_1}^{(a)}(z) - \nu_1 P_{\nu_1}^{(a)}(z)
\]

One can verify that \( \tilde{\xi}(z) \) being given by \((13)\), satisfies eq.\((43)\) taken in the considered linear approximation in odd modular parameters. In addition, \((13)\) has no singularities at \( z = z_1^{(+)} \).
and at $z = \zeta_{2}^{(+)}$. Since the dependence on Grassmann parameters of the even moduli begins with quadratic terms, only the $(\mu_{s}, \nu_{s})$ Schottky parameters are changed in the considered linear approximation in odd modular parameters, the transformed parameters being $(\tilde{\mu}_{s}, \tilde{\nu}_{s})$. The $(\tilde{\mu}_{s}, \tilde{\nu}_{s})$ in question are calculated from the relations quite similar to (30). Eqs. (34) being taken into account, the desired $(\tilde{\mu}_{s}, \tilde{\nu}_{s})$ are found to be

$$
\tilde{\mu}_{s} = \mu_{s} + \sum_{j=1}^{2} X_{\mu_{s} \mu_{j}} \mu_{j} + X_{\mu_{s} \nu_{j}} \nu_{j} \quad \text{and} \quad \tilde{\nu}_{s} = \nu_{s} + \sum_{j=1}^{2} X_{\nu_{s} \mu_{j}} \mu_{j} + X_{\nu_{s} \nu_{j}} \nu_{j}
$$

(48)

where $X_{\mu_{s} \mu_{j}}$, $X_{\mu_{s} \nu_{j}}$, $X_{\nu_{s} \mu_{j}}$ and $X_{\nu_{s} \nu_{j}}$ are the non-zero matrix elements $X_{F_{s}F'_{j}}$ of the $X$ matrix, which is defined by

$$
(-1)^{j} X_{F_{s}F'_{j}} = \int_{C_{u_{1}}} \tilde{\chi}_{F_{s}}(z') P^{(a)}_{F'_{j}}(z') \frac{dz'}{2\pi i} - 2 \int_{C_{v_{2}}} \tilde{\chi}_{F_{s}}(z') P^{(a)}_{F'_{j}}(z') \frac{dz'}{2\pi i} - \int_{C_{u_{2}}} \tilde{\chi}_{F_{s}}(z') P^{(a)}_{F'_{j}}(z') \frac{dz'}{2\pi i}
$$

for $F'_{j} = \mu_{j}, \nu_{j}$ with $j = 1, 2$; $F_{s}F'_{j} = 0$ for $F'_{j} = \mu_{j}, \nu_{j}$ with $j > 2$. (49)

where $X_{F_{s}F'_{j}}$ matrix elements are labeled by $F_{s} = (\mu_{s}, \nu_{s})$ and $F'_{j} = (\mu_{j}, \nu_{j})$ where $1 \leq s \leq n$, $1 \leq j \leq n$, $n$ being the genus. In deriving (43) eq. (41) is used. It is worth-while to note that $X_{F_{s}F'_{j}} = 0$, if $F_{s} \in \{N_{0}\}$. In the next section we employ eqs. (49) to discuss the supermodular covariance of the multi-loop partition functions.

5 Supermodular covariance of the superstring partition functions in the particular case

It is commonly to write n-loop superstring amplitudes $A_{n}$ as follows

$$
A_{n} = \prod_{N} dq_{N} d\overline{q}_{N} \prod_{r} dt^{(r)} d\overline{t}^{(r)} \sum_{L, L'} \hat{Z}_{L, L'}(\{q_{N}, \overline{q}_{N}\}) < \prod_{r} V(t^{(r)}, \overline{t}^{(r)}) >_{L, L'}
$$

(50)

where $\hat{Z}_{L, L'}$ are the measures (partition functions) and the $< ... >_{L, L'}$ symbol denotes the vacuum expectations calculated for the $(L, L')$ superspin structure. The index $L$ $(L')$ labels superspin structures of right (left) fields. The integration in (50) is performed over both $(3n - 3)2n - 2)$ complex moduli $q_{N}$ and over their complex conjugated $\overline{q}_{N}$ and, in addition, over the $(z^{(r)}, \overline{z}^{(r)})$ vertex local coordinates and over their odd partners $(\theta^{(r)}, \overline{\theta}^{(r)})$, as well. As it was already noted in Section 1, in fact eq. (50) needs the regularization. In this Section we employ eq. (50) only to clean the definition of the $\hat{Z}_{L, L'}$ partition functions. The holomorphic structure $[10, 21]$ of the above partition functions is determined by the following equation

$$
\hat{Z}_{L, L'}(\{q_{N}, \overline{q}_{N}\}) = [\det 2\pi i \omega(\{q_{N_{s}}\}, L') - \omega(\{q_{N_{s}}\}, L)]^{-5} Z_{L}(\{q_{N_{s}}\}) \overline{Z}_{L'}(\{q_{N_{s}}\})
$$

(51)

where $Z_{L}(\{q_{N_{s}}\})$ is a holomorphic function of the $q_{N_{s}}$ moduli, and $\omega(\{q_{N_{s}}\}, L)$ is the period matrix associated with the supermanifold under consideration. In terms of super-Schottky
group parameters \(\{k_s, u_s, v_s, \mu_s, \nu_s\}\) both \(Z_L(\{q_N\})\) and \(\omega(\{q_N\}, L)\) for the Neveu-Schwarz sector have been obtained in \(\textbf{[9, 10, 12, 13]}\). For all the even superspin structures in the Ramond sector they have been calculated in \(\textbf{[13]}\). In this case the result is given in the form of series over Grassmann modular parameters. For the sake of completeness we present these results in Appendix B.

It is necessary for the considered theory to be self-consistent, that the above \(\hat{Z}_{L,L'}\) partition functions do be covariant under \(q_N \rightarrow \hat{q}_N(\{q_N\})\) supermodular group transformations of the \(q_N\) modular parameters as follows

\[
\hat{Z}_{L,L'}(\{q_N, \bar{q}_N\}) = \hat{Z}_{L,L'}(\{q_N, \bar{q}_N\}) |Jac(\partial q_N/\partial \hat{q}_{N'})|^2
\]

where \(Jac(\partial q_N/\partial \hat{q}_{N'})\) is the Jacobian of the considered supermodular transformation and \(\hat{L} (\hat{L}')\) is the resulted superspin structure of right (left) fields. We give a direct evidence that, for zero odd moduli, the partition function calculated in \(\textbf{[14]}\) satisfy eq.(52) under the supermodular transformations turning a pair of the genus-1 structures to a pair of the odd genus-1 ones. In the next Section we argue that the considered partition functions are covariant under the whole supermodular group.

At zero odd moduli the \(\omega(\{q_N\}, L)\) period matrix in \(\textbf{(51)}\) is reduced to that associated with the Riemann surface, and, therefore, it is independent of the \(L\) spin structure. In this case, as it follows from \(\textbf{(52)}\) and from \(\textbf{(53)}\), the \(Z_{S_{ev}}(\{q_N\})\) holomorphic partition function associated to the \(S_{ev}\) spin structure is changed under the \(S_{ev} \rightarrow S_2\) supermodular transformation discussed in Section 4, as follows

\[
Z_{S_{ev}}(\{q_N\}) = \frac{Z_{S_2}(\{\hat{q}_N\})}{\det(I + X)} \tag{53}
\]

where \(Z_{S_2}\) is the partition function associated with the \(S_2\) spin structure. The above \((S_{ev}, S_2)\) spin structures were defined in the end of Section 3 and in the beginning of the previous Section. The \(X\) matrix in \(\textbf{(53)}\) is defined by \(\textbf{(49)}\). We show that the partition functions obtained in \(\textbf{[14]}\) satisfy the conditions \(\textbf{(53)}\). One can see from Appendix B that discussed \(Z_L\) partition functions at zero odd modular parameters can be written down as follows

\[
Z_L(\{k_s, u_s, v_s\}) = \frac{\hat{Z}_L(\{k_s, u_s, v_s\}, \{\sigma_p\})}{\sqrt{\det M(\{\sigma_p\}) \det M(-\{\sigma_p\})}} \tag{54}
\]

where \(\hat{Z}_L(\{k_s, u_s, v_s\})\) is invariant under the discussed \(S_{ev} \rightarrow S_2\) supermodular transformations and \(M(\{\sigma_p\})\) is the matrix defined below. Both \(M(\{\sigma_p\})\) and \(\hat{Z}_L(\{k_s, u_s, v_s\}, \{\sigma_p\})\) depend on the choice of the \(\{\sigma_p\}\) set where \(\sigma_p = \pm 1\), and \(p\) labels those genus-1 spin structures, which are associated with \(l_1p = 1/2\). Nevertheless, the right side of \(\textbf{(54)}\) turns out to be independent of the choice of the above \(\{\sigma_p\}\) set \(\textbf{[14]}\). To define the \(\hat{M}(\{\sigma_p\})\) matrix in \(\textbf{(54)}\) we consider \(\textbf{[14]}\) for every spin structure the Green functions \(G_{(\sigma)}(z, z')\) and the Green function \(G_f(z, z')\) as it follows just below.

The \(G_{(\sigma)}(z, z')\) functions are defined \(\textbf{[14]}\) by

\[
G_{(\sigma)}(z, z') = \sum_{\Gamma} \exp i[\Omega_T(\{\sigma_p\}) + \sum_s 2l_s \sigma_s (J_{(\sigma)s}(z) - J_{(\sigma)s}(z'))]/[z - g_T(z')][z' - d_T^3] \tag{55}
\]
where $J_{(o)}$ are the functions having the periods to be $2\pi i \omega_{sr}^{(r)}$, and $\omega_{sr}^{(r)}$ is the period matrix at zero odd moduli. The summation in (53) is performed over all the group products $\Gamma = \{z \to g_{r}(z)\}$ of the basic group elements $\Gamma_{s} = \{z \to g_{s}(z)\}$ including $\Gamma = I$. Furthermore, $\Omega_{\Gamma}(\{\sigma_{s}\})$ in (53) is defined as

$$\Omega_{\Gamma}(\{\sigma_{s}\}) = -\sum_{s,r} 2l_{1s} \sigma_{s} \omega_{sr}^{(c)} n_{r}(\Gamma) + \sum_{r} (2l_{2r} - 1) n_{r}(\Gamma)$$

(56)

where $n_{r}(\Gamma)$ is the number of times that the $\Gamma_{r}$ generators are present in $\Gamma$ (for its inverse $n_{r}(\Gamma)$ is defined to be negative ) and $\sigma_{s} = \pm 1$. So, $G_{(o)}$ depends on a choice of the $\{\sigma_{s}\}$ set. It is follows from (55) that the changes of $G_{(o)}$ under the $z \to g_{s}(z)$ mappings are as follows

$$G_{(o)}(g_{s}(z), z') = (-1)^{2l_{2s}}(c_{r}z + d_{s})^{-1} \left(G_{(o)}(z, z') + \sum_{F_{r} = \mu_{r}, \nu_{r}} \tilde{Y}_{\sigma,F_{r}}(z) \Phi_{\sigma,F_{r}}^{(0)}(z') \right)$$

(57)

where $\Phi_{\sigma,F_{r}}^{(0)}(z')$ are 3/2-tensors, $F_{r} = (\mu_{r}, \nu_{r})$ and $\tilde{Y}_{\sigma,F_{r}}(z)$ is given by

$$\tilde{Y}_{\sigma,F_{r}}(z) = \exp[\pi i \sum_{s} 2l_{1s} \sigma_{s} J_{(o)s}(z)] P_{F_{r}}(z)$$

(58)

where $P_{F_{r}}(z)$ is given by (24). In addition, we define the $G_{f}(z, z')$ Green function, which is changed under $2\pi$-twists about $A_{s}$-cycles and about $B_{s}$-ones as follows

$$G_{f}(g_{s}(z), z') = (-1)^{2l_{2s}}(c_{s}z + d_{s})^{-1} \left(G_{f}(z, z') + \sum_{F_{s} = \mu_{s}, \nu_{s}} P_{F_{s}}(z) \chi_{F_{s}}(z') \right)$$

$$G_{f}(z, z') = -\frac{1 - (-1)^{2l_{1s}}}{2} \left(G_{f}^{(*)}(z, z') + \sum_{F_{s} = \mu_{s}, \nu_{s}} P_{F_{s}}^{(*)}(z) \chi_{F_{s}}(z') \right)$$

(59)

where, unlike (23) and (24), the summation is performed over all $F_{s} = (\mu_{s}, \nu_{s})$ including the $\{N_{0}\}$ set, too. Furthermore, $G_{f}^{(*)}(z, z')$ in (53) is the $G_{f}$ Green function $2\pi$-twisted about $A_{s}$-cycle. In addition, $P_{F_{s}}(z)$ and $P_{F_{s}}^{(*)}(z)$ in (53) are degree-1 polynomials defined by (24) and (24), $\chi_{F_{s}}(z')$ being conformal 3/2-tensors.4 Above $\chi_{F_{s}}(z')$ have no singularities in the fundamental domain on $z'$-plane, except only at $z' \to \infty$. It is worth-while to note that the $G_{gh}^{(f)}(z, z')$ Green function discussed in the previous Sections can be expressed in terms of the $G_{f}(z, z')$ function as

$$G_{gh}^{(f)}(z, z') = G_{f}(z, z') - \sum_{F \in \{N_{0}\}} \tilde{P}_{F}(z) \chi_{F}(z')$$

(60)

where $\tilde{P}_{F}(z)$ are degree-1 polynomials in $z$. The $F$ indices in (60) are associated with those odd Schottky parameters, which chosen to be the same for all genus-$n$ supermanifolds. The $\tilde{P}_{F}(z)$ polynomials in (60) are determined from the condition for $G_{gh}^{(f)}(z, z')$ to decrease at

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4 In terms of the $\chi_{N_{0}}(t')$ superconformal 3/2-tensors defined in (14), every above $\chi_{F_{s}}(z')$ is equal to $\chi_{F_{s}}(t')$ taken at zero odd modular parameters.
expressed in the terms of $\Phi(0)$ where the $\tilde{G}$ spin structure there is the only Green function $[14]$ satisfying eqs. (59), the right side of eqs. (59), the right side of Running this contour away and using (23) and (25), we obtain (61). As soon as for every $z$ the same as in (28) and (29). Furthermore, the $\tilde{G}$ of the spin structure.

It can be also proved $[14]$ that the $\tilde{G}$ none other than the $\phi$ and (25), we obtain (61). As soon as for every $z$ the same as in (28), (29) and (61). The $\Phi$ under 2

The desired $\tilde{M}(\{\sigma_p\})$ matrix in (64) is worth-while to note that the above $G_f(z, z')$ functions can be expressed in terms $G_{(\sigma)}(z, z')$ as

$$G_f(z, z') = G_{(\sigma)}(z, z') - \sum_{s=1}^{n} \sum_{F_z=\mu_s, \nu_s} \int G_{(\sigma)}(z, z'') \frac{dz''}{2\pi i} P_{F_z}(z'') \chi_{F_z}(z')$$

+ $\sum_{s=1}^{n} \sum_{F_z=\mu_s, \nu_s} \left[ 1 - (-1)^{2l_z} \right] \int G_{(\sigma)}(z, z'') \frac{dz''}{2\pi i} P_{F_z}(z'') \chi_{F_z}(z')$ (61)

where $\chi_{F_z}(z')$ are 3/2-tensors defined by (59). Both the $C_{v_s}$ contours and the $\tilde{C}_s$ ones are the same as in (28) and (29). Furthermore, the $P_{F_z}(z'')$ polynomials are defined by (24) and the $P_{F_z}^{(a)}(z'')$ ones are given by (26). To derive (61) we represent $G_f(z, z')$ to be the integral over $z''$ performed along the infinitesimal contour around $z'$, the integrand being $G_{(\sigma)}(z, z'')$. Running this contour away and using (23) and (25), we obtain (61). As soon as for every spin structure there is the only Green function $[14]$ satisfying eqs. (54), the right side of (61) is, in fact, independent of $\{\sigma_s\}$. Furthermore, the $\chi_{F_z}(z')$ conformal 3/2-tensors are expressed in the terms of $\Phi_{\sigma,F}^{(0)}$ in (57) as $[14]$

$$\Phi_{\sigma,F}^{(0)} = \sum_{F_z=\mu_s, \nu_s} \tilde{M}_{F_z,F_z}^{s} \{\sigma_q\} \chi_{F_z}$$

(62)

where the $\tilde{M}_{F_z,F_z}^{s} \{\sigma_q\}$ elements of the $\tilde{M}(\sigma)$ matrix are given by

$$\tilde{M}_{F_z,F_z}^{s} \{\sigma_q\} = \int_{C_{v_s}} \Phi_{\sigma,F_z}^{(0)}(z) \frac{dz}{2\pi i} P_{F_z}(z) - \frac{1 - (-1)^{2l_z}}{2} \int_{\tilde{C}_z} \Phi_{\sigma,F_z}^{(0)}(z) \frac{dz}{2\pi i} P_{F_z}^{(a)}(z).$$

The desired $\tilde{M}(\{\sigma_p\})$ matrix in (64) is just the same as that defined by (63). Eqs. (62) and (63) are derived from the condition that, being calculated from (61), the changes of $G_f(z, z')$ under 2π-twists about $(A_s, B_s)$-cycles are given by (59). Both the $C_{v_s}$ contours and the $\tilde{C}_z$ ones are the same as in (28), (23) and (21). The $\Phi_{\sigma,F_z}^{(0)}(z)$ conformal 3/2-tensors are calculated explicitly from eqs. (54) and (57). It is useful to note that going certain of $A_s$-cycles about each other turns both $G_{(\sigma)}(z, z')$ and $\Phi_{\sigma,F_z}^{(0)}(z)$ associated with the $S_{ev}$ superspin structures to the superspin structures containing pairs of the odd genus-1 superspin ones. Particular, it is follows from this fact and from (64) that under the discussed $S_{ev} \rightarrow S_2$ transformation, $Z_{S_{ev}}(\{q_{N_s}\})$ holomorphic partition function turns into $Z_{S_{2}}(\{q_{N_s}\})$ with exception only the factor due to both the change of the contours of the integration in the $\tilde{M}(\{\sigma_p\})$ matrix (63) and to the modification of the $(P_{F_1}, P_{F_2})$ polynomials (24) in (63). Indeed, before the discussed going of $C_{u_2}$ about $C_{u_1}$ circle being performed, the $(C_1, \tilde{C}_2)$ contours in (23) present none other than the $z_{s}^{-}(z_{s}^{+})$ lines in Fig.1(a) where $z_{s}^{+} = g_{s}(z_{s}^{-})$ and $s = 1, 2$. The $z_{s}^{-}$ point is the intersect of the $C_{v_s}$ circle with the $(v_s)(u_s)$ line. After the going of $C_{u_2}$
about $C_{u_1}$ circle to be performed, the $(z_s^{(-)}) (z_s^{(+)})$ lines are deformed to be as in Fig.1(b). So the integral along the additional paths arises. In the $s = 2$ case the above integral can be reduced to the one taken along the $C_{u_1}$ circle together with the integral along the $(z_1^{(+)})(z_2^{(+)})$ line in Fig.1(c). In the $s = 1$ case the integral along the $C_{u_1}$ circle is added. To express the discussed $M(\{\sigma_p\})$ matrix in terms of that assigned to the $S_2$ spin structure, one must also to take into account that the $(P_{F_1}, P_{F_2})$ polynomials \([24]\) associated with the $S_2$ spin structure are calculated from those associated with the $S_{ev}$ spin structures by suitable $\sqrt{k_s} \rightarrow -\sqrt{k_s}$ replacements. Furthermore, the integral along every $C_{u_2}$ circle can be turned to the integral along the $C_{u_1}$ circle by the $z \rightarrow (d_sz - b_s)(-c_sz + a_s)^{-1}$ replacement. In this case one obtains the expression of $Z_{S_{ev}}(\{q_N\})$ in terms of $Z_{S_2}(\{\tilde{q}_N\})$ as follows

$$Z_{S_{ev}}(\{q_N\}) = \frac{Z_{S_2}(\{\tilde{q}_N\})}{\det(I + \tilde{X})} \quad (64)$$

where the $\tilde{X}_{F_i,F_j}$ elements of the $\tilde{X}$ matrix are given by

$$(-1)^j \tilde{X}_{F_i,F_j} = \int_{C_{u_1}} \chi_{F_i}(z) P_{F_j}^{(a)}(z') \frac{dz'}{2\pi i} - 2 \int_{C_{u_2}} \chi_{F_i}(z) P_{F_j}^{(a)}(z') \frac{dz'}{2\pi i} - \int_{C_{u_2}} \chi_{F_i}(z') P_{F_j}^{(a)}(z) \frac{dz'}{2\pi i}$$

for $F_j' = \mu_j, \nu_j$ with $j = 1, 2$;

$$\tilde{X}_{F_i,F_j} = 0 \quad \text{for} \quad F_j' = \mu_j, \nu_j \quad \text{with} \quad j > 2. \quad (65)$$

In deriving (64) and (65) we employ eqs.(62) and (63). One can see from (63) and (19) that the $\tilde{X}$ matrix differs from the $X$ matrix by the $\chi_{F_i}(z') \rightarrow \chi_{F_i}(z')$ replacement. Nevertheless, we prove that $\det(I + \tilde{X}) = \det(I + X)$ and, therefore, eq. (64) is the same as (53).

For this purpose we note that the $\chi_{F_i}(z)$ conformal zero modes in (19) are expressed in terms of the $\chi_{F_i}(z)$ conformal 3/2-tensors as follows

$$\tilde{\chi}_{F_i}(z) = \chi_{F_i}(z) - \sum_{F_i' \in \{N_0\}} A_{F_i,F_i'} \chi_{F_i'}(z) \quad (66)$$

where $F_p = (\mu_p, \nu_p)$ and the $\{N_0\}$ set of the indices is the same as in (19). The above indices are associated with those Schottky parameters, which chosen to be the same for all genus-$n$ supermanifolds. Furthermore, $A_{F_i,F_i'}$ elements of the $A$ matrix are defined by

$$A_{\mu_s,\mu_r} = \frac{u_s - v_r}{u_r - v_r}, \quad A_{\mu_s,\nu_r} = \frac{u_r - u_s}{u_r - v_r}, \quad A_{\nu_s,\mu_r} = \frac{v_s - u_r}{u_r - v_r}, \quad A_{\nu_s,\nu_r} = \frac{u_r - v_s}{u_r - v_r}, \quad \text{if} \quad (\mu_r, \nu_r) \in \{N_0\}; \quad \text{otherwise} \quad A_{F_i,F_i'} = 0. \quad (67)$$

In (67), like throughout above, $(u_p, v_p)$ are the fixed points of the $z \rightarrow g_p(z)$ Schottky transformation (3). To derive (66) and (67), we calculate the $\tilde{P}_{F_s}$ polynomials in (19) from the condition that in the sum on the right side of (23), there are no terms proportional to $P_{F_s}$ with $F_s \in \{N_0\}$. Employing, in addition, eqs.(64), one obtains both (66) and (67). Hence the $X$ matrix can be written down as

$$X = \tilde{X} - A\tilde{M}^{-1} \tilde{X} \quad (68)$$
where the $A$ matrix is defined by (67). Eq.(68) follows directly from (49), (62), (65) and (66). Furthermore, it is obvious from (68) that

$$\det(I + X) = \det(I + \tilde{X}) \det[1 - A\tilde{M}^{-1}\tilde{X}(I + \tilde{X})^{-1}]$$

(69)

that is the same as follows

$$\det(I + X) = \det(I + \tilde{X}) \det[1 - (I + \tilde{X})^{-1}\tilde{X}A\tilde{M}^{-1}]$$

(70)

In addition, one can verify by the direct calculation that

$$\tilde{X}A = 0$$

(71)

and, therefore, the desirable relation

$$\det(I + \tilde{X}) = \det(I + X)$$

(72)

takes place. So the partition functions (54) obey eq.(53), and, therefore, they are covariant under the particular modular transformations considered.

6 Supermodular covariance of the superstring partition functions in the general case

As it has been noted in the Introduction, partition functions (54) have been calculated [14] from equations that are none other than Ward identities. The above equations are obtained from the condition that the discussed amplitudes are independent of a choice of vierbein and the gravitino field. Hence it is natural to expect that these equations do be supermodular covariant. Below we give the direct proof that the considered equations really possess the covariance under the supermodular group discussed. Since the above equations fully determine the partition functions (up to constant factors only), the partition functions (54) necessarily satisfy restrictions due to the whole supermodular group. The desired equations have the following form [12, 13, 14]

$$\sum_N \tilde{\chi}_N(t; L) \frac{\partial}{\partial q_N} \ln \tilde{Z}_{L,L'}(\{q_N, \bar{q}_n\}) = \langle T_{gh} + T_m > - \sum_N \frac{\partial}{\partial q_N} \tilde{\chi}_N(t; L)$$

(73)

together with the equations to be complex conjugated to (73). The derivatives with respect to odd moduli in (73) are implied to be the "right" ones. The $\tilde{\chi}_N(t; L)$ superconformal 3/2-tensor zero modes will be defined below. Furthermore, $T_{gh}$ and $T_m$ are the stress tensors of the ghost and string superfields, respectively. In the explicit form

$$T_m = 10(\partial X)DX/2 \quad \text{and} \quad T_{gh} = -(\partial \hat{F})\hat{B} - \partial(\hat{F} \hat{B}) + D[(D\hat{F})\hat{B}]|/2$$

(74)

\footnote{These superconformal zero modes are denoted in [14] as $\tilde{\chi}_N(t)$.}
where $D$ denotes the spinor derivative $\partial_{\gamma}$ and $X$ is the scalar superfield, the space-time dimension being 10. In addition, $\hat{B}$ is 3/2-tensor ghost superfield and $\hat{F}$ is the vector ghost one. In (74) the explicit dependence on the supercoordinate $t = (z|\theta)$ is omitted. The above $T_m$ is calculated in term of the $G_{(m)}(t, t'; L)$ vacuum correlator defined as follows

$$G_{(m)}(t, t'; L) = -D(t')\partial_{\bar{\gamma}} < X(t, \bar{t})X(t', \bar{t}'>)$$

(75)

Furthermore, $T_{gh}$ is calculated in terms of the ghost superfield vacuum correlator $G_{gh}(t, t'; L)$ where

$$G_{gh}(t, t'; L) = <\hat{F}(t, \bar{t})\hat{B}(t', \bar{t})>.$$  

(76)

It is quite essential that, unlike the well known ghost scheme [24, 23], the vacuum correlator (74) has depending on $t$ periods under rounds about $(A_s, B_s)$-cycles. In the explicit form (the explicit dependence on $L$ is omitted)

$$G_{gh}(t^a_s, t') = Q_{1_{a,s}}^{-2}(t) \left( G_{gh}^{(s)}(t, t') + \sum_N Y_{a,N}^{(s)}(t)\tilde{\chi}_N(t') \right),$$

$$G_{gh}(t^b_s, t') = Q_{1_{b,s}}^{-2}(t) \left( G_{gh}(t, t') + \sum_N Y_{b,N}^{(s)}(t)\tilde{\chi}_N(t') \right)$$

(77)

where 3/2-zero modes $\tilde{\chi}_N$ are the same as in (73). Both the $t^a_s = \Gamma_{a,s}(l_{1s})(t)$ transformations and the $t^b_s = \Gamma_{b,s}(l_{2s})(t)$ ones (3) are defined in Section 2. Furthermore, $G_{gh}^{(s)}(t, t')$ is obtained from $G_{gh}(t, t')$ by 2\pi-twist about $C_{(1_s)}$-circle (3). At least, $Y_{a,N}^{(s)}$ and $Y_{b,N}^{(s)}$ are polynomials of degree 2 in $(z, \theta)$. The sum over $N$ in (77) includes only those $Y_{p,N}^{(s)}(t)$, which associated with the Schottky parameters that are moduli. We use for the $\{N_r \}$ indices the same notation $(k_r, u_r, v_r, \mu_r, \nu_r)$ as for the Schottky parameters. In this notation, particularly, $q_{k_r} = k_r$, $q_{u_r} = u_r$ and so one. In this case the above polynomials are given as follows [13, 14]

$$Y_{p,N_p}(t) = Y_{p,N_p}(s)\delta_{rs} \quad \text{where} \quad p = a, b \quad \text{and} \quad Y_{p,N_p}(t) = Q_{1_{p,s}}^2 \left[ \frac{\partial_{q_{N_p}}}{\partial_{q_{N_p}}} + \frac{\partial_{q_{N_p}}}{\partial_{q_{N_p}}} \right].$$

(78)

Eqs.(77) being taken into account, the condition for $G_{gh}(t, t')$ to be 3/2-supertensor on $t'$-supermanifold, fully determines [12, 13, 14] both $G_{gh}(t, t')$ and 3/2-supconformal zero modes $\tilde{\chi}_N(t')$. At the odd modular parameters to be zero, eqs. (77) reduce to (23) and (25). Unlike the ghost correlators considered in [3, 24], $G_{gh}(t, t')$ has no unphysical poles [12, 13, 14]. In the calculation of $T_m + T_{gh}$, the singularity at $z \rightarrow z'$ in both $G_{(m)}(t, t')$ and $G_{gh}(t, t')$ is removed by the usual prescription [24].

Eqs. (73) resemble the equations discussed in [3, 24]. But, unlike those in [3, 24], they take into account, in addition, the factors due to both ghost zero modes and the moduli volume form. The above terms are taken into account in (73) owing to the using of the $G_{gh}(t, t')$ ghost superfield vacuum correlator satisfying eqs.(77) and owing to the presence of the proportional to $\partial_{q_N} \tilde{\chi}_N(t)$ terms on the right side of (73). The difference between eqs.(73) and those in [3, 24] is especially for those superspin structures where at least one of the $l_{1s}$ characteristics is unequal to zero. Indeed, in this case the equations [3, 24] have no
solutions at all. To the contrary, \( \tilde{G} \) allows to obtain explicit formulae for the partition functions associated with the superspin structures discussed.

One can see that both \( T_{gh} \) and \( T_m \) have the usual form \( \tilde{G} \) in the terms of the ghost or string superfields, but in the considered scheme, \( T_m + T_{gh} \) is not superconformal 3/2-form under the \( (\Gamma_{a,s}(l_{1s}), \Gamma_{b,s}(l_{2s})) \) mappings because the \( G_{gh}(t, t') \) ghost superfield vacuum correlators have the periods under the mappings above. Nevertheless, the right side of eq.\( (73) \), as well as the left side, appears to be superconformal 3/2-form under the considered mappings. Moreover, we prove that eqs.\( (77) \) are covariant under supermodular transformations.

As it was already noted in this paper, the discussed transformations, generally, present globally defined \( t \to \hat{t}(t, \{ q_N \}) = (\hat{z}|\hat{\theta}) \) mappings that accompanied by both the \( L \to \hat{L} \) change of the superspin structure and the change \( q_N \to \hat{q}_N(\{ q_M \}) \) of the moduli. Under the considered transformations, \( G_{gh}(t, t'; L) \to G_{gh}(\hat{t}, \hat{t}'; \hat{L}) \) and \( G_{(m)}(t, t'; L) \to G_{(m)}(\hat{t}, \hat{t}'; \hat{L}) \). Since \( G_{(m)}(t, t'; L) \) is defined by \( (75) \) to be the tensor under globally defined superconformal transformations, the desired \( G_{(m)}(\hat{t}, \hat{t}'; \hat{L}) \) is given by

\[
G_{(m)}(t, t') = \hat{Q}(\hat{t})G_{(m)}(\hat{t}, \hat{t}')\hat{Q}(t) \tag{79}
\]

To calculate \( G_{(m)}(\hat{t}, \hat{t}'; \hat{L}) \), it is useful to note that a number of rounds about \( (A, B) \) cycles on \( \hat{t} \) supermanifold corresponds to every 2\( \pi \)-twist about either \( A \)-cycle or \( B \)-cycle on the \( t \) one. Therefore, to every \( t \to \Gamma_{b,s}(l_{2s})(t) \) mapping and to every \( t \to \Gamma_{a,s}(l_{1s})(t) \) mapping, the appropriate mappings on \( \hat{t} \)-supermanifold can be assigned. For the condition that under the above mappings, \( G_{(m)}(\hat{t}, \hat{t}'; \hat{L}) \) is changed in the accordance with eqs.\( (77) \) written in terms of the variables assigned to \( \hat{t} \)-supermanifold, the desired supermodular transformation of \( G_{gh}(t, t'; L) \) turns out to be

\[
G_{gh}(t, t'; L) = \hat{Q}^{-2}(\hat{t}) \left( G_{gh}(\hat{t}, \hat{t}'; \hat{L}) + \sum_N \left[ \frac{\partial z(\hat{t})}{\partial \hat{q}_N} + \theta(\hat{t}) \frac{\partial \theta(\hat{t})}{\partial \hat{q}_N} \right] \hat{\chi}_N(\hat{t}'; \hat{L}) \right) \hat{Q}^3(\hat{t}'). \tag{80}
\]

In \( (80) \) the \( \hat{Q}(t) \) factor is defined to be

\[
\hat{Q}^{-1}(\hat{t}) = D(\hat{t})\theta(\hat{t}') \tag{81}
\]

and \( D(\hat{t}) \) is the spinor derivative \( (12) \) with respect to \( \hat{t} = (\hat{z}|\hat{\theta}) \). Furthermore, the \( \hat{\chi}_N(t, L) \) superconformal 3/2-zero modes in \( (77) \) are written down in terms of \( \chi_N(\hat{t}'; \hat{L}) \) in \( (80) \) as follows

\[
\hat{\chi}_N(t, L) = \sum_{N'} \frac{\partial q_N}{\partial \hat{q}_{N'}} \hat{\chi}_{N'}(\hat{t}; \hat{L})\hat{Q}^3(\hat{t}) \tag{82}
\]

In addition to eqs.\( (80), (81) \) and \( (73) \), one must take into account eq.\( (12) \), which describes the supermodular transformation of the partition functions. In this case one can verify by the direct calculation that eqs.\( (73) \) appear to be covariant under the transformation discussed.

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7 Supermodular invariance of the multi-loop superstring amplitudes

In the self-consistent theory the multi-loop superstring amplitudes $A_n$ must satisfy the restrictions due to the supermodular group. Naively, eq. (50) satisfies the above restrictions because every even (odd) superspin structure contribution in $A_n$ can be derived by supermodular transformations of the contribution due to a fixed even (odd) structure. In fact, however, the supermodular invariance of eq. (50) must be ensured by a suitable regularization procedure because the integration of every superspin contribution in (50) is divergent. It is worth-while to note that the above regularization procedure is necessary even if in the whole integrand the singularities are cancelled after the summation over the superspin structures to be performed. Indeed, the $(q_N \rightarrow \hat{q}_N, t^{(r)} \rightarrow \hat{t}^{(r)})$ change, being associated with the particular supermodular transformation, depends on the superspin structure in terms proportional to odd modular parameters, as it has been shown in the previous Sections. As the result, this case the supermodular invariance of (50) could be the result of an appropriate integration of every superspin structure contribution. Being divergent, this integration needs the regularization procedure. To avoid the above regularization procedure, it seems attractive to re-write down the right side of (50) to be the integral of the supermodular covariant function. For this purpose we assign to every superspin structure contribution in (50) the suitable regularization procedure because the integration of every superspin contribution in (50) is divergent. It is however, the supermodular invariance of eq. (50) must be ensured by a suitable regularization procedure because the integration of every superspin contribution in (50) is divergent. It is implied that the consideration given in the previous Sections is referred to $t_L(t, \{q_N\})$ and $q_{LN}(\{q_N\})$. Particular, eqs. (15), (28)-(31), (35), (45) and (48) determine the action of the supermodular group on $t_L$ and $q_{LN}$. The $(t, \{q_N\})$ dependence of $t_L$ and of $q_{LN}$ is calculated from the condition that the $(t \rightarrow \hat{t}, q_N \rightarrow \hat{q}_N)$ change under every supermodular transformation associated with the given integral matrices in (14) and (15) is the same for all the superspin structures. In this case the integrand in (83) appears to be non-covariant under the discussed transformations. In this case the supermodular invariance of (50) could be the result of an appropriate integration of every superspin structure contribution. Being divergent, this integration needs the regularization procedure. To avoid the above regularization procedure, it seems attractive to re-write down the right side of (50) to be the integral of the supermodular covariant function.

Without loss of generality, one can take $t = t_{L_0}$ and $q_N = q_{L_0N}$ for the $L_0$ superspin structure. We choose $L_0$ to be the superspin structure $S(0)$ where $l_{1s} = l_{2s} = 0$ for every $s$. Without loss of generality, one can take $t = t_{L_0}$ and $q_N = q_{L_0N}$ for the $L_0$ superspin structure. We choose $L_0$ to be the superspin structure $S(0)$ where $l_{1s} = l_{2s} = 0$ for every $s$. Without loss of generality, one can take $t = t_{L_0}$ and $q_N = q_{L_0N}$ for the $L_0$ superspin structure. We choose $L_0$ to be the superspin structure $S(0)$ where $l_{1s} = l_{2s} = 0$ for every $s$. Without loss of generality, one can take $t = t_{L_0}$ and $q_N = q_{L_0N}$ for the $L_0$ superspin structure. We choose $L_0$ to be the superspin structure $S(0)$ where $l_{1s} = l_{2s} = 0$ for every $s$. Without loss of generality, one can take $t = t_{L_0}$ and $q_N = q_{L_0N}$ for the $L_0$ superspin structure. We choose $L_0$ to be the superspin structure $S(0)$ where $l_{1s} = l_{2s} = 0$ for every $s$. Without loss of generality, one can take $t = t_{L_0}$ and $q_N = q_{L_0N}$ for the $L_0$ superspin structure. We choose $L_0$ to be the superspin structure $S(0)$ where $l_{1s} = l_{2s} = 0$ for every $s$. Without loss of generality, one can take $t = t_{L_0}$ and $q_N = q_{L_0N}$ for the $L_0$ superspin structure. We choose $L_0$ to be the superspin structure $S(0)$ where $l_{1s} = l_{2s} = 0$ for every $s$. Without loss of generality, one can take $t = t_{L_0}$ and $q_N = q_{L_0N}$ for the $L_0$ superspin structure. We choose $L_0$ to be the superspin structure $S(0)$ where $l_{1s} = l_{2s} = 0$ for every $s$. Without loss of generality, one can take $t = t_{L_0}$ and $q_N = q_{L_0N}$ for the $L_0$ superspin structure. We choose $L_0$ to be the superspin structure $S(0)$ where $l_{1s} = l_{2s} = 0$ for every $s$. Without loss of generality, one can take $t = t_{L_0}$ and $q_N = q_{L_0N}$ for the $L_0$ superspin structure. We choose $L_0$ to be the superspin structure $S(0)$ where $l_{1s} = l_{2s} = 0$ for every $s$. Without loss of generality, one can take $t = t_{L_0}$ and $q_N = q_{L_0N}$ for the $L_0$ superspin structure. We choose $L_0$ to be the superspin structure $S(0)$ where $l_{1s} = l_{2s} = 0$ for every $s$.
It is convenient because the supermodular transformations discussed in Sections 3 and 4 map
the \( S(0) \) superspin structure onto itself. In this case action of the supermodular group on \( t \)
and \( q_N \) is determined by (35), (28)-(31), (32) and (15), all they being taken for \( L = S(0) \).
Particular, to calculate the quadratic in \( \{\mu_s, \nu_s\} \) terms in \( \hat{q}_{ev} \), one can substitute in (15)
eqs. (33) with \( \eta^{(r)} = 0 \) and with the \( X \) matrix taken at zero odd super-Schottky parameters.
After the above substitution to be performed in (15), eqs. (15) determine in the discussed
approximation the integration region in (33). Under the above \( L_0 = S(0) \) choice, the \( (t = t_L, q_N = q_{LN}) \) relations take place for all
the Neveu-Schwarz superspin structures \( S_1 \) (in this case, \( l_{1s} = 0 \) for every \( s \)). Indeed, as
it was discussed in Section 2, all these superspin structures can be derived from \( S(0) \) by
the \( \sqrt{k_s} \rightarrow -\sqrt{k_s} \) replacements. Furthermore, for the superspin structures with non-zero
\( l_{1s} \)-characteristics, the \( (t_L, q_{LN}) \) variables differ from \( (t, q_N) \) only by terms proportional to
the odd modular parameters. Employing the results obtained in the previous Sections, one
can calculate \( t_L(t, \{q_N\}) \) and \( q_{LN}(\{q_N\}) \) assigned to the above superspin structures. For
superspin structures \( S_{ev} \) without the odd genus-1 superspin ones we use the supermodular
transformations discussed in Section 3. In this case the desired relations for the calculation
of \( t_L(t, \{q_N\}) \) and \( q_{LN}(\{q_N\}) \) are given by
\[
\hat{t}_L(t_L, \{q_{LN}\}) = t_L(\hat{t}_0, \{\hat{q}_{N0}\}).
\] (85)
On the left side of (33), \( t_L \equiv t_L(t, \{q_N\}) \) and \( q_{LN} \equiv q_{LN}(\{q_N\}) \). Furthermore, \( \hat{t}_0 \equiv \hat{t}_0(t, \{q_N\}) \)
and \( \hat{q}_{N0} \equiv \hat{q}_{N0}(q_N) \). In (33) the \( (t \rightarrow \hat{t}_0, q_N \rightarrow q_{N0}) \) supermodular transformations are
calculated for the \( S(0) \) superspin structure defined above. Eqs. (33) follow directly from the
condition that the right side of (33) is covariant under the supermodular transformations
considered. In the \( L = S(0) \) case eqs. (33) degenerate to be the identity. To solve (33) for
the \( L \neq S(0) \), one can take \( \hat{L} = S_1 \). In this case \( t_{\hat{L}}(t) = t \). Hence eqs. (33) determine both
\( t_L(t, \{q_N\}) \) and \( q_{LN}(\{q_N\}) \) for all the \( S_{ev} \) superspin structures discussed. Particular, in the
linear approximation in \( \{\mu_r, \nu_r\} \), the desired \( \{\mu_{Lr}, \nu_{Lr}\} \) for \( L = S_{ev} \) are given by
\[
\mu_{Lr} = \sum_{s=1}^{n} \left[ \left( \hat{X}(L)\hat{X}^{-1}(L_0) \right)_{\mu_r \mu_s} \mu_s + \left( \hat{X}(L)\hat{X}^{-1}(L_0) \right)_{\nu_r \nu_s} \nu_s \right],
\]
\[
\nu_{Lr} = \sum_{s=1}^{n} \left[ \left( \hat{X}(L)\hat{X}^{-1}(L_0) \right)_{\nu_r \mu_s} \mu_s + \left( \hat{X}(L)\hat{X}^{-1}(L_0) \right)_{\nu_r \nu_s} \nu_s \right].
\] (86)
where both \( \hat{X}(L) \) and \( \hat{X}(L_0) \) are taken at zero odd super-Schottky parameters and \( L_0 = S(0) \).
Eqs. (33) follows from (33) and (33). Furthermore, if one substitute (33) into (15), one can
calculate from the obtained equations the region of the integration over \( \{q_{ev}\} \) in the quadratic
approximation in \( \{\mu|s, \nu_s\} \). The above integration region turns out to be calculated in terms of
the modular group parameters at zero odd moduli and in terms of the odd super-Schottky
parameters, as well. To calculate the \( t_{L}(t, \{q_N\}) \) functions for the even superspin structures
containing the odd genus-1 superspin ones, one must consider the transformations discussed
in Section 4. In this case (33) is replaced as
\[
\hat{t}_{L}(t_L, \{q_{LN}\}) = t_{L}^{(-)}(t, \{q_N\}).
\] (87)
where \( t_L \equiv t_L(t, \{q_N\}) \), \( q_{LN} \equiv q_{LN}(\{q_N\}) \) and \( t_L^{(-)}(t, \{q_N\}) \) is calculated from \( t_L(t, \{q_N\}) \) by the going of suitable \( A_n \)-cycles about each other. Starting with \( \hat{L} \) to be among the \( S_{ev} \) superspin structures, one calculate from (57) both \( t_L(t, \{q_N\}) \) and \( q_{LN}(\{q_N\}) \) for the superspin structures containing a number of pairs of the odd genus-1 superspin ones. This calculation is quite similar to that performed for the \( S_{ev} \) superspin structures.

It is obvious that (83) should be reduced to (50), if the integration of every superspin contribution were to be finite. The above integrations being divergent, we define the non-integrable singularities in (84) means the finiteness of the theory discussed. Particularly, the absence of potential divergences in (84) is free from non-integrable singularities. The study of potential singularities of (84) and, therefore, potential divergences in the theory. So one can hope that being supermodular covariant, the space-time supersymmetry prohibits the tadpoles appearing to be the only source of divergences in the theory. In turn, the space-time supersymmetry prohibits the tadpoles appearing to be the only source of divergences in string theories arise from the degenerativeness of Riemann surfaces 31. In fact the discussed singularity in (84) is compensated by a smallness of the integration volume associated with the configurations considered. Moreover, the domain where bodies of the \( k_{La}(\{q_N\}) \) Schottky multipliers are near to unity (up to the phase) is equivalent modulo of modular group to the domain where bodies of these Schottky multipliers are small 13. So the above domain can be excluded from the integration region. Furthermore, one can see from (51) and (24) that vanishing the \( k_{La}(\{q_N\}) \) body appears when \( k_s \to 0 \). In this case \( k_{La}(\{q_N\}) \to k_s[1 + o_s(L)] \) where \( o_s(L) \) is proportional to odd modular parameters. The highest \( k_s^{-3/2} \) singularity appears in the case when \( (l_{1s} = 0, l_{2s} = 1/2) \) or \( (l_{1s} = l_{2s} = 0) \). As it is usual 3, in every sum of two superspin structures distinguished only in the discussed genus-1 superspin ones, the above \( k_s^{-3/2} \) singularity is reduced to \( k_s^{-1} \) because \( o_s(L) \) is the same for both the superspin structures considered. In addition, the non-holomorphic factor in (51) gives the factor \((\ln |k|)^{-5}\) in (84). As the result, the integration over small \( k_s \) does not lead to divergency of (83). Moreover, one finds to be finite the integral over the region where the fixed points associated with a particular basic Schottky transformation go away from each other. Indeed, in this case the radius of the circles (4) associated with the considered
Schottky transformation is taken to be finite. Otherwise the above circles intersect the ones associated with other basic Schottky transformations. The finiteness of the above radius at \( |u_s - v_s| \to \infty \) requires \( k_s \) to be small as \( |k_s| \equiv |u_s - v_s|^{-1} \) that provides the finiteness of the integral discussed.

Because of the \((u_s - v_s - \mu_s v_s)^{-1}\) factors in (95), a potential singularity in (84) is also expected if, for a particular handle, \( |u_s - v_s| \to 0 \). The above singularity could lead to divergencies of (50). In the considered \(|u_s - v_s| \to 0\) limit the genus-\(n\) Riemann surface is degenerated in two separate Riemann, one of genus 1 and the other genus \((n - 1)\). If a number of the vertices in (84) present on both the above surfaces, the discussed singularity origins the threshold singularities of (50) at suitable magnitudes of the external 10-momenta.

But in the configuration where all the vertices appear to be whether on the genus-1 surface or on the one of genus \((n - 1)\), the considered singularity should cause a divergency of (50) independent of 10-momenta above. One can see from (84) and (95) that in the discussed region the integrand of (83) has the following form

\[
\begin{align*}
\left| \frac{1}{u_s - v_s} + \frac{\mu_s v_s + \hat{o}_s(L)}{(u_s - v_s)^2} \right|^2 \left[ I_1^{(0)} I_{n-1} + I_1 I_{n-1}^{(0)} + (u_s - v_s)B + (\bar{u}_s - \bar{v}_s)\bar{B} \right] 
\end{align*}
\]

where \(I_m\) is the integrand for the genus-\(m\) amplitude (with \(m = n - 1\) or \(m = 1\)) and \(I_m^{(0)}\) is the same for the genus-\(m\) vacuum one. For \(m = 1\), the discussed integrand is obtained by the factorization of (84) when the particular handle moves away from the others. Furthermore, \(\hat{o}_s(L)\) appears in (88) because of the difference between \((u_s, v_s)\) and the fixed points of the Schottky transformations. The line over denotes the complex conjugation and \(B\) describes the terms proportional to \((u_s - v_s)\). One can see, for the discussed singularity to be absent in (88), the necessary condition is \(I_m^{(0)} = 0\). The above \(I_m^{(0)} = 0\) could be the consequence of the space-time supersymmetry, which causes the vanishing of the vacuum amplitude [15]. Using the measure [14] presented in Appendix B, one can show without essential difficulties that \(I_1^{(0)} = 0\), but the verification of the discussed statement for \(m \geq 1\) needs an additional investigation. Furthermore, being necessary, the above \(I_m^{(0)} = 0\) condition is insufficient to remove the singularity in question because the second order pole presents in (88) due to the expansion in the series over the odd super-Schottky parameters. So for the discussed singularity to be absent, the \(B = 0\) condition must be added. One can see a reason for this \(B = 0\) condition to be because the second order pole in (88) is reduced to the first order one [17, 30] by a choice of the appropriate variables [10]. It is not evidently, however, whether the above choice is consistent with the supermodular invariance. So an additional study of the discussed subject seems to be necessary. The kindred divergencies appear in (83) when the Riemann surface is degenerated in two separate Riemann surfaces, one of genus \(n_1\) and the other of genus \((n - n_1)\) with \(n_1 > n - n_1 > 1\). The integral over the vertex local coordinates is potentially divergent, too. Indeed, when all the vertices move to be closed together, the vacuum expectations of the vertex product in (84) are ceased to be independent of \(q_N\). The discussed vacuum expectations begin to be covariant under the superconformal extension of the \(SL_2\) group that originates the divergency of the integral over the vertex coordinates. In this case the singularity in the integrand is appear to be similar to (88). To
Furthermore, both $\hat{\varphi}$ and $\No$ are necessary. For the similar reasons, divergency might arise from the region where all the vertices move away from each other.

It is necessary to note that we uniquely calculate the supercovariant integrand \((\ref{eq:24})\) taking into account only a part of the supermodular transformations. So, to be sure in the self-consistency of the discussed scheme, one should verify that the above integrand is covariant under the whole supermodular group. This verification requires, however, a more detailed study of the discussed supermodular transformations that is not finished at present. We plan to discuss this problem in another paper.

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**A**

To give the explicit definitions of the $\rho_s^{(pq)}$ functions in \((\ref{eq:24})\) and the $\eta_s^{(pq)}$ functions in \((\ref{eq:21})\) for $p = a, b$ and $q = a, b$ we present the above functions as

$$
\rho_s^{(bb)}(z) = \frac{c_s^{(0)} f'(z) y^2(z)}{c_s^{(0)} f + d_s^{(0)}} + \frac{\hat{\varphi}_s^{(b)}(z, l_{2s})}{\hat{f}_0(g_s) g_s'(z)} + \rho_s^{(b)}(z), \quad \rho_s^{(ab)}(z) = \frac{\hat{\varphi}_s^{(a)}(z, l_{2s})}{\hat{f}_0(g_s) g_s'(z)} + \rho_s^{(b)}(z),
$$

$$
\rho_s^{(ba)}(z) = \frac{c_s^{(0)} f'(z) y^2(z)}{c_s^{(0)} f + d_s^{(0)}} + \frac{\hat{\varphi}_s^{(b)}(z, l_{1s})}{\hat{f}_0(z, l_{1s})} + \rho_s^{(a)}(z), \quad \rho_s^{(aa)}(z) = \frac{\hat{\varphi}_s^{(a)}(z, l_{1s})}{\hat{f}_0(z, l_{1s})} + \rho_s^{(a)}(z),
$$

$$
\eta_s^{(bb)}(z) = \hat{\eta}_s^{(b)}(z, l_{2s}) + \eta_s^{(b)}(z), \quad \eta_s^{(ab)}(z) = \hat{\eta}_s^{(a)}(z, l_{2s}) + \eta_s^{(b)}(z), \quad \eta_s^{(ba)}(z) = \hat{\eta}_s^{(b)}(z, l_{1s}) + \eta_s^{(b)}(z), \quad \eta_s^{(aa)}(z) = \hat{\eta}_s^{(a)}(z, l_{1s}) + \eta_s^{(a)}(z)
$$

(89)

where both $\rho_s^{(p)}(z)$ and $\eta_s^{(p)}(z)$ with $p = a, b$ are defined as follows

$$
\eta_s^{(b)}(z) = \epsilon_s(z, l_{2s}) \xi^2(g_s) + \xi'_s(g_s)(z - g_s) + (1) \frac{\epsilon_s(z, l_{2s}) - \epsilon_s(z) + (1) \xi_s(g_s) c_s z + d_s}{f_0(g_s) g_s'(z) [c_s z + d_s]}
$$

$$
\eta_s^{(a)}(z) = \frac{\epsilon_s(z, l_{2s}) - \epsilon_s(z) + (1) \xi_s(g_s) c_s z + d_s}{f_0(g_s) g_s'(z) [c_s z + d_s]}
$$

$$
\eta_s^{(b)}(z) = \frac{\epsilon_s(z, l_{2s}) - \epsilon_s(z) + (1) \xi_s(g_s) c_s z + d_s}{f_0(g_s) g_s'(z) [c_s z + d_s]}
$$

(90)

Furthermore, both $\hat{\varphi}_s^{(p)}(z, l_s)$ and $\hat{\eta}_s^{(p)}(z, l_s)$ with $p = a, b$ in \((\ref{eq:89})\) are given by

$$
\hat{\varphi}_s^{(b)}(z, l_s) = (-1) \frac{\epsilon_s(z) c_s f + d_s}{[\epsilon_s(z)]^2}, \quad \hat{\varphi}_s^{(a)}(z, l_s) = (-1) \frac{\epsilon_s(z) c_s f + d_s}{[\epsilon_s(z)]^2}.
$$

(91)
\[ \dot{\rho}_s^{(a)}(z, l_s) = [1 - (-1)^{2l_s}] \xi(z) \dot{\xi}_s(f) \sqrt{f'(z)}, \]
\[ \dot{\eta}_s^{(a)}(z, l_s) = [1 - (-1)^{2l_s}] \left[ \xi(z) \dot{\xi}_s \dot{\xi}'_s - \frac{\xi(z) \xi'(z) \dot{\xi}_s(f)}{2 \sqrt{f'(z)}} \right], \]
\[ \dot{\eta}_s^{(b)}(z, l_s) = \frac{\xi(z) \xi'(z) \dot{\xi}_s(f, l_s)}{2 \sqrt{f'(z)}} - \xi(z) \dot{\xi}_s(f, l_s) \dot{\xi}'_s(f) \]  

where \( f = f(z) \).

### B Measure in terms of super-Schottky parameters

For the Neveu-Schwarz sector matrix elements \( \omega_{ps}(\{q_{N_s}\}, L) \) of the period matrix in (51) are given by [9, 10]

\[ 2\pi i \omega_{ps}(\{q_{N_s}\}, L) = \sum_k'' \ln \left[ u_s - g_T(u_p, \mu_p) - \mu_s \theta_T(\mu_p, u_p)[v_s - g_T(v_p, \nu_p) - \nu_s \theta_T(v_p, \nu_p)] \right] u_s - g_T(v_p, \nu_p) - \mu_s \theta_T(\mu_p, u_p), \]
\[ + \delta_{ps} \ln k_s. \]  

In (92) the summation is performed over all super-Schottky group transformations \( \Gamma = \{ z \to g_T(z, \theta), \theta \to \theta_T(\theta, z) \} \) except those that have the leftmost to be a power of \( \Gamma_s \), the rightmost being a power of \( \Gamma_p \). Besides, \( \Gamma \neq I \), if \( s = p \). The \( Z_L(\{q_{N_s}\}) \) factor in (51) for the Neveu-Schwarz sector has been found to be [11, 12, 13]

\[ Z_L(\{q_{N_s}\}) = H(\{q_{N_s}\}) \left( \prod_{s=1}^{n} \frac{(1 - (-1)^{2l_s} - 1)[1 - (-1)^{2l_s} \sqrt{k_s}]}{k_s^{3/2}} \right) \left[ 1 - (-1)^{\Sigma_k \sqrt{k}} \right]^{-2} \]
\[ \times \prod_{m=1}^{\infty} \frac{1 - (-1)^{\Sigma_k k^{m-1/2}}}{(1 - k^m)^8} \]  

where the product over \( (k) \) is taken over all the multipliers of super-Schottky group (3), which are not powers of other the ones and

\[ \Sigma_k = \sum_r (2l_{2r} - 1) n_r(\Gamma). \]  

In (94), \( n_r(\Gamma) \) is the number of times that the \( \Gamma_r \) generators are present in \( \Gamma \) (for its inverse \( n_r(\Gamma) \) is defined to be negative). At last, \( H(\{q_{N_s}\}) \) in (58) is defined by

\[ H(\{q_{N_s}\}) = g^{2n(u_1 - u_2)(v_1 - v_2)} \left( 1 - \frac{\mu_1 \mu_2}{2(u_1 - u_2)} - \frac{\nu_1 \mu_2}{2(v_1 - v_2)} \right) \left( \prod_{s=1}^{n} (u_s - v_s - \mu_s \nu_s) \right)^{-1} \]  

where \( g \) is the coupling constant. It is assumed that \( u_1, v_1, u_2, \mu_1 \) and \( \nu_1 \) are fixed to be the same for all the genus-\( n \) supermanifolds and, therefore, they are not the moduli.
The Ramond sector to be considered, we present $Z_L(\{q_{N_s}\})$ in (51) for every even super-spin structure as follows [14]

$$Z_L(\{q_{N_s}\}) = Z_{0(m)}(\{k_s, u_s, v_s\}, L) Z_{0(gh)}(\{k_s, u_s, v_s\}, L) \times H(\{q_{N_s}\}) \Upsilon_m^{(n)}(\{q_{N_s}\}, L) \Upsilon_{gh}^{(n)}(\{q_{N_s}\}, L)$$

(96)

where $H(\{q_{N_s}\})$ is given by (95) and the subscript "gh" ("m") labels the ghost (respectively, string superfield) contributions. Both $Z_{0(m)}(\{k_s, u_s, v_s\}, L)$ and $Z_{0(gh)}(\{k_s, u_s, v_s\}, L)$ are calculated at zero odd Schottky parameters. The $\Theta$ in the denominator associates with that spin structure where, as in (97), the product over $(\{q_{N_s}\}, \{q_{N_s}\}, \{q_{N_s}\}, \{q_{N_s}\})$ are not powers of the other ones. The period matrix $\Gamma(\{q_{N_s}\})$ in (51) for every even super-spin structure as follows [14] (67), which are not powers of the other ones. The $\tilde{Z}_0(\{k_s, l_{1s}, l_{2s}\})$ factors in (98) are defined by (98). Furthermore,

$$\Lambda(k, \{\sigma_p\}) = \exp \Omega_{\Gamma(k)}(\{\sigma_p\})$$

(99)

where $\Omega_{\Gamma(k)}(\{\sigma_p\})$ is given by (53) for the group products of the basic Schottky transformations having the multiplier to be $k$. The $\tilde{Z}_0(\{k_s, l_{1s}, l_{2s}\})$ factors in (98) are defined by

$$\tilde{Z}_0(\{k_s, l_{1s}, l_{2s}\}) = \frac{(-1)^{2l_{1s}+2l_{2s}-1} [1 - (-1)^{2l_{2s}} \sqrt{k_{2s}}^2]}{4^{2l_{1s}+2l_{2s}}}$$

(100)

Eq. (100) is slightly different from eq.(147) in [14] because we use in (98) the $\tilde{M}(\{\sigma_p\})$ matrix instead of $M(0)(\{\sigma_p\})$ employed in [14]. It is useful to remind that in (97) and in (98), the $k$ multipliers are calculated at zero odd Schottky parameters. Both $\Upsilon_m^{(n)}$ and $\Upsilon_{gh}^{(n)}$ in (96) depending on the above odd parameters have the following form

$$\ln \Upsilon_{gh}^{(n)}(\{q_{N_s}\}, L) = trace \ln \left[ I + \tilde{\Delta}_{gh}(\{\sigma_p\}) \right] - \ln det \tilde{U}(\{\sigma_p\}) + \ln sdet[U(\{\sigma_p\})U_0^{-1}(\{\sigma_p\})],$$

(101)
\[
\ln \Upsilon^{(n)}(\{q_{N_s}\}, L) = -5 \text{trace} \ln \left[ 1 + \tilde{\Delta}_m \right] \tag{102}
\]

where \(\tilde{\Delta}_m\) and \(\tilde{\Delta}_{gh}\) are integral operators and both \(U(\{\sigma_p\})\) and \(\tilde{U}(\{\sigma_p\})\) to be matrices, all they being defined below. Furthermore, \(U_0(\{\sigma_p\}) = \tilde{U}(\{\sigma_p\})\) at zero odd Schottky parameters. Both \(\tilde{\Delta}_m, \tilde{\Delta}_{gh}\) and \(U(\{\sigma_p\}) - I\) are proportional to the odd Schottky parameters. So \((101)\) and \((102)\) can be calculated to be series over odd Schottky parameters. The superdeterminant in \((101)\) is defined as

\[
s\det U = \frac{\det U_{(bb)}}{\det U_{(ff)}} \det[I - U_{(bb)}^{-1} U_{(bf)} U_{(ff)}^{-1} U_{(fb)}] \tag{103}
\]

where \(U_{(bb)}, U_{(bf)}, U_{(fb)}\) and \(U_{(ff)}\) are submatrices forming the above \(U\) matrix. The \(b\) index labels boson components and the \(f\) index labels the fermion ones.

To present \(\tilde{\Delta}_{gh}\) in \((101)\), we define genus-1 Green functions \(S^{(1)}_{\sigma,s}(t, t')\) as

\[
S^{(1)}_{\sigma,s}(t, t') = Q_{\Gamma}(t_s)^{-2} \left[ G^{(1)}_{b}(z_s, z'_s) \theta_s + \theta_s G^{(1)}_{(\sigma)}(z_s, z'_s) - \varepsilon'_s \Sigma_{\sigma}(z'_s) \right] Q_{\Gamma}(t'_s)^3 \tag{104}
\]

where \(t_s = (z_s, \theta_s)\) is defined by \((7)\), the \(Q_{\Gamma}\) factor is defined by \((13)\) and \(G^{(1)}_{(\sigma)}\) is \(G_{(\sigma)}\) defined by \((55)\) for genus \(n = 1\). Furthermore, the boson contribution \(G^{(1)}_{b}\) in \((104)\) is \(G_{b}\) taken at genus \(n = 1\). For an arbitrary genus-\(n\), \(G_{b}\) is defined to be

\[
G_{b}(z, z') = -\sum_{\Gamma} \left[ \frac{1}{\left| z - g_{\Gamma}(z') \right|} \frac{1}{c_{\Gamma} z' + d_{\Gamma}} \right]^{14} \tag{105}
\]

where the summation is performed over all the group product of basic Schottky group elements \((2)\). The last term in \((104)\) is defined to be limit of \(z G^{(1)}_{(\sigma)}(z_s, z'_s)\) at \(z \to \infty\). Owing to this term, \(S^{(1)}_{\sigma,s}(t, t')\) decreases at \(z \to 0\) or at \(z' \to 0\). In \((101)\), the \(\Delta_{gh}(\{\sigma_r\})\) integral operator is formed by the \(\{\tilde{\Delta}_{gh}(p)(\{\sigma_r\})\}\) set of the \(\tilde{\Delta}_{gh}(p)(\{\sigma_r\})\) integral operators, the kernels being \(\tilde{\Delta}_{gh}(\{\sigma_r\})(t, t')dt'\). We define the kernel together with the differential \(dt' = dz'd\theta' / 2\pi i\) to have deal with the objects obeying bose statistics. Every the \(\tilde{\Delta}_{gh}(\{\sigma_r\})\) integral operator being applied to a function of \(t'\), performs integrating over \(t'\) along the \(C_p\) contour. The above \(C_p\)-contour gets around in the positive direction both \(C_{v^r}\) and \(C_{u^r}\) circles \((4)\) together with the \(\tilde{C}_p\) cut, if this cut presents (i.e. \(l_{1p} \neq 0\)). The \(\tilde{C}_p\) cuts are defined next to eq.\((27)\).

In the explicit form

\[
\tilde{\Delta}_{gh}(\{\sigma_r\})(t, t') = \int_{\tilde{C}_p} G^{(1)}(t, t_1; \{\sigma_q\}) \frac{dz_1d\theta_1}{2\pi i} \delta S^{(1)}_{\sigma}(t_1, t') \tag{106}
\]

\(^6\text{Eq.}(102)\) corresponds to eq.(134) of \((14)\). In \((102)\) we retrieved an factor \(-5\) and symbol "trace" missed mistakenly in front of the right side of above eq.(134). In addition, in \((14)\) a number of other inaccuracies sliced in formulas for the factors considered. In discussed eq.(134) of \((14)\) the expression inside the square brackets should read \(1 + \Delta_m\). In (137) of \((14)\), \(\delta S^{(1)}_{\sigma}\) should be dropped. In (138) of \((14)\), \(Y^{(1)}_{N_s}(t')\) should read \(Y^{(1)}_{N_s}(t')\) and \(C_s\) should read \(C_s^{(b)}\).
where it was explained above, \( C_p \)-contour gets around in the positive direction both circles (4) together with the \( \tilde{C}_p \) cut. The Green function \( G^0(t, t'; \{ \sigma_p \}) \) is defined as

\[
G^0(t, t'; \{ \sigma_p \}) = G_b(z, z') \theta' + \theta G_{(s)}(z, z')
\]  

(107)

where \( G_b(z, z') \) is defined by \( (105) \) and \( G_{(s)}(z, z') \) is given by \( (53) \).

To present the expression for \( \hat{U}_{N_r N_r}(\{ \sigma_p \}) \) in \( (101) \) we define 3/2-tensors \( \Psi_{\sigma, N_r}^{(0)}(z) \) by

\[
\Phi_{\sigma, N_r}^{(0)} = \sum_{N_r = \mu_r, \nu_r} \hat{M}_{N_s, N_r}(\{ \sigma_p \}) \Psi_{\sigma, N_r}^{(0)} \quad \text{where} \quad \hat{M}_{N_s, N_r}(\{ \sigma_p \}) = \int_{C_{\nu r}} \Phi_{\sigma, N_r}^{(0)}(z) \frac{dz}{2\pi i} \tilde{Y}_{\sigma, N_r}^{(1)}(z).
\]  

(108)

where \( \tilde{Y}_{\sigma, N_r}^{(1)}(z) \) is equal to \( \tilde{Y}_{\sigma, N_r}(z) \) defined by eq. \( (58) \) at the genus \( n = 1 \). In this case the \( U'_{N_r N_r}(\{ \sigma_p \}) \) elements of the \( \hat{U}(\{ \sigma_p \}) \) matrix are given by

\[
\hat{U}_{N_r N_r}(\{ \sigma_p \}) = I - \sum_p \int_{C_{\nu r}} \Phi_{\sigma, N_r}^{(0)}(z) dt \int_{C_{\nu r}} \delta S_{\sigma, b}^{(1)}(t, t_1) dt_1 
\times \sum_h \int_{C_{\nu s}} \Delta^{(h)}(t_1, t_2) dt_2 \int_{C_{\nu s}} \theta_2 G_{(s)}(z_2, z') dz' \tilde{Y}_{\sigma, s}^{(1)}(z').
\]  

(109)

where \( dt = dz d\theta/2\pi i \), \( G_{(s)}(z, z') \) is defined by \( (53) \), \( \tilde{Y}_{\sigma, s}^{(1)}(t') \) is defined by \( (58) \) at the genus \( n = 1 \) and \( \delta S_{\sigma, b}^{(1)} \) is referred to those terms in \( (104) \), which are proportional to the odd Schottky parameters. Furthermore, \( \Delta^{(h)}(t_1, t_2) dt_2 \) present the kernels of the \( \Delta^{(h)} \) integral operators. The \( \{ \Delta^{(h)} \} \) set of these operators forms the \( \Delta \) operator that can be given as

\[
\Delta = [I + \Delta_{gh}(\{ \sigma_p \})]^{-1}
\]  

(110)

where the \( \Delta_{gh}(\{ \sigma_p \}) \) operator is the same as in \( (101) \). Eqs. \( (101) \) and \( (109) \) correspond to eqs. (137) and (138) of [14] with \( \hat{U} \) to be \( U' \) of [14]. But in \( (101) \) and \( (109) \) both \( \tilde{Y}_{\sigma, b}^{(n)}(\{ q_{N_s}, L \}) \) and \( \hat{U} \) is given in terms of \( \Delta_{gh} \) instead of \( \Delta_{gh} \) defined in [14]. This leads to more compact, formulas, especially for the \( U(\{ \sigma_p \}) \) matrix presented below. The proof of \( (101) \) and \( (109) \) is achieved by an expansion in powers of \( \Delta_{gh} \) of (137) and of (138) in [14]. In this case sum of integrations over \( t' \) of every particular \( \delta S_{\sigma, b}^{(1)}(t, t') \) along the \( C_r \) contours \( (r \neq p) \) is reduced to the integral along the \( C_p \) contour. As the result, eqs. \( (101) \) and \( (109) \) arise.

To present the \( U(\{ \sigma_p \}) \) matrix in \( (101) \) we define 3/2-supertensors \( \Psi_{\sigma, N_r}^{(1)}(z) \) on the genus-1 supermanifolds by

\[
S_{\sigma, s}^{(1)}(t_b, t'') = Q_{\Gamma_{k_s}}^{-2}(t) \left( S_{\sigma, s}^{(1)}(t, t') + \sum_{N_s} \tilde{Y}_{\sigma, N_s}(t) \Psi_{\sigma, N_s}(t') \right)
\]  

(111)

where \( N_s = (k_s, u_s, v_s, \mu_s, \nu_s) \) and the \( t \to t'_b \) transformation is defined in [11]. For \( N_s = (k_s, u_s, v_s) \), the \( \tilde{Y}_{b, N_s}^{(1)}(t) \) polynomials in \( (111) \) are equal to \( P_{R_s}(z_s) Q_{\Gamma_{k_s}}^{-2}(t_s) \) where \( P_{R_s} \) are defined by (24). For \( N_s = (\mu_s, \nu_s) \), the above \( \tilde{Y}_{\sigma, N_s}(t) \) polynomials are equal to \( \tilde{Y}_{p, N_s}(t_s) Q_{\Gamma_{k_s}}^{-2}(t_s) \),
where \( D \) is the spinor derivative (12). The periods of \( R_L(t,t') \) are \( J_s(t;L) \) and the periods of \( J_s(t;L) \) form the \( 2\pi i\omega(\{q_N\},L) \) matrix, \( \omega(\{q_N\},L) \) being the period matrix in (51). It can be shown (14) that \( K_L(t,t') \) obeys the integral equation with the kernel to be none other that the kernel of the desired integral operator \( \tilde{\Delta}_m \) in (102).

To give in the explicit form the kernel of \( \Delta_m \) one can note that for zero odd modular parameters, \( R_L(t,t') \) is reduced to \( R_{(0)}(t,t';L) \) as

\[
R_{(0)}(t,t';L) = R_b(z,z') - \theta \theta' R_f(z,z';L)
\]
where $R_b(z, z')$ is the boson Green function and $R_f(z, z'; L)$ is the fermion Green one. The $R_b(z, z')$ Green function is given by \[24\]

\[ R_b(z, z') = \sum_r \ln \left( \frac{[z - g_t(z')] [-c_t z^{(o)} + a_t]}{[-c_t z + a_t] [z^{(o)} - g_t(z^{(l)})]} \right) \] (118)

$z^{(o)}$ and $z^{(l)}$ being arbitrary constants. The fermion Green function $R_f(z, z'; L)$ in (117) can be given as

\[ R_f(z, z'; L) = \exp \left\{ \frac{1}{2} [R_b(z, z) + R_b(z', z')] - R_b(z, z') \right\} \frac{\Theta[l_1, l_2](J[\omega^{(r)}])}{\Theta[l_1, l_2](0[\omega^{(r)}])} \] (119)

where Green function $R_b(z, z)$ for $z' = z$ is defined to be the limit of $R_b(z, z') - \ln(z - z')$ at $z \to z'$. Furthermore, $\Theta$ is the theta function and the symbol $J$ denotes the set of functions $(J_{(0)s}(z) - J_{(0)s}(z')) / 2\pi i, J_{(0)s}(z)$ being periods of $R_b(z, z')$. We define also for arbitrary odd moduli the genus-1 Green functions $R^{(1)}_{(f,s)}(t, t')$ as

\[ R^{(1)}_{(f,s)}(t, t') = R^{(1)}_{(0)s}(t, t') + \varepsilon'_s \varepsilon'_s \Xi_s(\infty, z_s') - \theta_s \varepsilon'_s \Xi_s(z_s, \infty) \quad \text{for} \quad s = 1, 2, \ldots n \] (120)

where both $t_s = (z_s \mid \theta_s)$, $t'_s = (z'_s \mid \theta'_s)$ and $\varepsilon'_s$ are defined by (1), and $R^{(1)}_{(0)s}(t, t')$ is $R^{(1)}_{(f,s)}(t, t')$ at zero odd moduli. Furthermore, $\Xi_s(z, z')$ is

\[ \Xi_s(z, z') = (z - z') R^{(1)}_{(f,s)}(z, z'; l_{1s}, l_{2s}) \] (121)

Two the last terms in (121) provide decreasing $K^{(1)}_{(s)}(t, t')$ at $z \to \infty$ or $z' \to \infty$ where $K^{(1)}_{(s)}(t, t')$ is defined by eq.(116) for $R = R^{(1)}_{(s)}$. Being twisted under $(A_s, B_s)$-circles, $R^{(1)}_{(f,s)}(t, t')$ is changed by $(\Gamma_{a,s}, \Gamma_{b,s})$-mappings (3). To calculate $R^{(1)}_{(f,s)}(t, t')$ in (120) for even genus-1 spin structures we use (118) and (119) at $n = 1$. The genus-1 spin structure being odd, we defined $R^{(1)}_{(f,s)}(z, z')$ as

\[ R^{(1)}_{(f,s)}(z, z') = \frac{\partial_s \left\{ \Theta[1/2, 1/2](J_{(1)}[\omega^{(1)}]) \right\} \left[ \partial'_s J^{(1)}_{(0)s}(z') \right]}{\Theta[1/2, 1/2](J_{(1)}[\omega^{(1)}]) \left[ \partial'_s J^{(1)}_{(0)s}(z') \right]} \] (122)

where $\Theta$ is the genus-1 theta function. Furthermore, $J_{(1)} = (J_{(0)s}(z) - J_{(0)s}(z')) / 2\pi i$ and $J_{(0)s}$ is the period of $R^{(1)}_{(b,s)}(z, z')$, the period of $J^{(1)}_{(0)s}$ being $2\pi i \omega^{(1)}_{(s)}$. In this case, for every $s$, the Green function $K^{(1)}_{(s)}(t, t') = D(t') R^{(1)}_{(f,s)}(t, t')$ is changed under $\Gamma_{b,s}$ transformation as

\[ K^{(1)}_{(s)}(t, t') = \left[ K^{(1)}_{(s)}(t, t') + \varphi_s(t) f_s(t') \right] Q_{\Gamma_{b,s}}(t') \]

\[ K^{(1)}_{(s)}(t, t') = K^{(1)}_{(s)}(t, t') + 2\pi i \eta^{(1)}_{(s)}(t') - \varphi_s(t) f_s(t') \] (123)

where $f_s(t') = D(t') \varphi_s(t')$. The above $\varphi_s(t')$ disappears, if $(l_{1s}, l_{2s})$-characteristics correspond to an even genus-1 spin structure. In this case the desired integral equation for $K_{(s)}(t, t')$ has the following form \[14\]

\[ K_{(s)}(t, t') = K_{(0)}(t, t'; L) - \sum_{r=1}^{n} \int_{C_r} K_{(0)}(t, t_1; L) dt_1 \delta K^{(1)}_{(r)}(t_1, t_2) dt_2 K_{(r)}(t_2, t') - \int_{C_r} K_{(0)}(t, t_1; L) \]

38
\[ \delta K \]

of \[14\] being used. It is also employed that sum of integrations over

The proof of (125) is achieved by an expansion in powers of both \( \Delta m \). As example, \( \delta K^{(1)}(t_1, t_2) = K^{(1)}(t_1, t_2) - K^{(1)}(t_1, t_2) \) where \( K^{(1)}(t_1, t_2) \) is the \( K^{(1)}(t_1, t_2) \) function taken at \( \mu_r = \nu_r = 0 \). The \( C_r \) contours are defined as in \[103\]. On the \( z_s \) complex plane \[7\] the \( C_{vs} \) contour is none other than the \( C_{vs} \) circle \[3\]. Only the odd genus-1 spin structures contribute in two last terms on the right side of \[124\]. The term \( K^{(0)}(t, t'; L) = D(t')R^{(0)}(t, t'; L) \) outside the integral on the right side of \[124\] is calculated in the terms of both \( R_b(z, z') \) and \( R_f(z, z'; L) \), as it has been explained above. Since the kernel of \[124\] is proportional to odd parameters, solution of \[124\] can be obtained by the iteration procedure, every posterior iteration being, at least, one more power in odd parameters than a previous one. Therefore, \( K^{(1)}(t, t') \) appears to be a series containing a finite number of terms. After \( K^{(1)}(t, t') \) being determined, the \( R_{L}(t, t') \) Green function is calculated without essential difficulties.

The kernel of integral operator \( \hat{\Delta}_m \) in \[102\] is just the kernel of \[124\]. The right side of \[102\] is calculated by an expansion in powers of \( \Delta m \). Eq. \[102\] is more convenient for the calculation than eq.(134) of \[14\] where \( Y^{(n)}_m(\{q_N\}, L) \) is given in terms of \( \Delta_m \) defined by \( (135) \) in \[14\]. To prove identity of \[102\] with \( (134) \) of \[14\] one can verify that

\[ \text{trace } \ln \left[ 1 + \hat{\Delta}_m \right] - \text{trace } \ln [1 + \Delta_m] = 0. \]  

(125)

The proof of \[125\] is achieved by an expansion in powers of both \( \Delta m \) and \( \Delta_m \), eqs. (50) and (51) of \[14\] being used. It is also employed that sum of integrations over \( t_1 \) of every particular \( \delta K^{(1)}(t, t_1) \) along the \( C_r \) contours \( (r \neq s) \) is reduced to the integral along \( C_s \)-contour.

The \( J_p \) periods of \( R_L(t, t') \) in \[127\] are calculated as \[14\]

\[ J_p(t; L) = \int_{C_p} K(t, t')J_p^{(1)}(t') \frac{d\theta dz'}{2\pi i} \]  

(126)

where \( J_p^{(1)}(t') \) is the period of the genus-1 Green function \( R^{(1)}_p(t, t') \). In \[127\] and \[126\] the integration contour \( C_p \) is defined as in \[106\]. The \( \omega_{rp} \) matrix elements of the period matrix \( \omega(\{q_N\}, L) \) in the measure \( (51) \) can be calculated as \[14\]

\[ 2\pi i \omega_{rp} = k_r \delta_{rp} + \int_{C_r} D(t)J_p(t; L)J^{(1)}_r(t) \frac{d\theta dz}{2\pi i} \]  

(127)

where \( k_r \) is the Schottky multiplier. The right side of \[127\] can be proved to be symmetrical in respect to interchanging \( r \) and \( p \).

For all the \( l_s \) theta characteristics to be zero ( that is the Neveu-Schwarz sector ) eq.(94) is reduced to \[93\] and \[127\] is reduced to \[92\].
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Figure 1: The going of $C_{u_2}$ circle round the $C_{u_1}$ one: the initial position (a), the final position (b), the cuts are reduced to be closed together (c).

Figure in G.S. Danilov’s paper "Unimodular transformations of the supermanifolds and the calculation of the multi-loop amplitudes in the superstring theory"