A NOTE ON THE GROWTH OF REAL FUNCTIONS IN SETS OF
POSITIVE DENSITY

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Abstract. Motivated by earlier results from the complex function theory, the growth
of non-decreasing and unbounded real-valued functions is studied in sets of positive
linear/logarithmic density. The results improve several existing results and they are of
interest from the real analysis and the complex analysis points of view.

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density, meromorphic functions, order of growth.

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1. Background

We study the growth of non-decreasing and unbounded functions $T : [t_0, \infty) \to (0, \infty)$
in sets of positive linear/logarithmic density (defined below). The growth is categorized
in terms of the order and lower order of $T$, which are given, respectively, by
\[
\rho = \rho(T(r)) = \limsup_{r \to \infty} \frac{\log T(r)}{\log r} \quad \text{and} \quad \mu = \mu(T(r)) = \liminf_{r \to \infty} \frac{\log T(r)}{\log r}.
\]
Clearly $\mu(T(r)) \leq \rho(T(r))$ holds in general. The following quick example shows that a
strict inequality is also possible.

Example 1. Let $0 < \mu < \rho < 1$, and let $T : (2, \infty) \to (0, \infty)$ be a function whose graph
$y = T(x)$ is a polygonal path as follows: Start from the point $(2, 2^\rho)$ on $y = x^\rho$ and move
horizontally to the right until you hit the curve $y = x^\mu$. Then move diagonally parallel
to the line $y = x$ and to the right until you hit the curve $y = x^\rho$. Proceed repeatedly
until you get the graph as in the figure below. Then $T$ is continuous, non-decreasing, and
$x^\mu \leq T(x) \leq x^\rho$ for all $x \geq 2$. It is also clear that $T$ has order $\rho$ and lower order $\mu$.

The motivation for this note comes from the complex function theory, although the
results that follow are stated for real-valued functions $T(r)$. If so preferred, $T(r)$ can
be replaced with the Nevanlinna characteristic $T(r, f)$ or with the integrated counting
function $N(r, c, f)$ of the $c$-points of a meromorphic $f$, where $c \in \hat{\mathbb{C}}$. 

[Graph of the polygonal path example]
The order \( \rho(f) \) and the lower order \( \mu(f) \) of \( f \) are defined by \( \rho(f) = \rho(T(r, f)) \) and \( \mu(f) = \mu(T(r, f)) \). For entire functions, \( T(r, f) \) can be replaced with the logarithmic maximum modulus \( \log M(r, f) \). The quantities measuring the growth of \( N(r, c, f) \) are called exponents of convergence of \( c \)-points of \( f \), and they are defined by \( \lambda(c, f) = \rho(N(r, c, f)) \) and \( \sigma(c, f) = \mu(N(r, c, f)) \). It is known that the functions \( T(r, f) \), \( \log M(r, f) \) and \( N(r, c, f) \) are convex functions in \( \log r \), but this property is not needed in this note.

It is known that, for any fixed \( \mu \) and \( \rho \) satisfying \( 0 \leq \mu \leq \rho \leq \infty \) there exists a meromorphic function \( f \) of order \( \rho(f) = \rho \) and of lower order \( \mu(f) = \mu \) [4, p. 238]. If \( \mu < \rho \), it is natural to ask what are the sizes of sets of \( r \)-values, where the growth of \( T(r, f) \) is near maximal or near minimal. To this end, we recall Theorem A from [12, Corollary 3.7], where the size of such sets \( D \subset [1, \infty) \) is measured in terms of upper and lower (linear) densities given by

\[
\overline{\text{dens}}(D) = \limsup_{r \to \infty} \frac{\int_{[1,r]} dt}{r} \quad \text{and} \quad \underline{\text{dens}}(D) = \liminf_{r \to \infty} \frac{\int_{[1,r]} dt}{r}.
\]

**Theorem A.** Let \( f \) be a meromorphic function such that \( 0 \leq \mu(f) < \rho(f) \leq \infty \), and let \( \mu(f) < a \leq b < \rho(f) \). Then the sets

\[
E = \{ r \geq 1 : T(r, f) \leq r^a \} \quad \text{and} \quad F = \{ r \geq 1 : T(r, f) > r^b \}
\]

are of upper density one and of lower density zero.

Further variations of the above problem is to compare the growth of \( T(r, f) \) and \( T(r, g) \) in the case \( \rho(f) < \rho(g) \), or the growth of \( N(r, c, f) \) and \( T(r, f) \) in the case \( \lambda(c, f) < \rho(f) \). Recall that \( \lambda(c, f) \leq \rho(f) \) holds in general for every \( f \) and for every \( c \in \hat{\mathbb{C}} \).

Due to the logarithms appearing in the definitions of orders, it seems more natural to study these growth questions in terms of logarithmic densities. Recall that the upper and lower logarithmic densities of a set \( D \subset [1, \infty) \) are given by

\[
\overline{\text{logdens}}(D) = \limsup_{r \to \infty} \frac{\int_{[1,r]} \frac{dt}{\log r}}{\log r} \quad \text{and} \quad \underline{\text{logdens}}(D) = \limsup_{r \to \infty} \frac{\int_{[1,r]} \frac{dt}{\log r}}{\log r}.
\]

The connection between linear and logarithmic densities is apparent from the inequalities

\[
0 \leq \underline{\text{dens}}(D) \leq \underline{\text{logdens}}(D) \leq \overline{\text{logdens}}(D) \leq \overline{\text{dens}}(D) \leq 1, \tag{1.1}
\]

which can be found in [13, p. 121].

The main results and their consequences are stated and discussed in Section 2, while the corresponding situation on bounded intervals is briefly discussed in Section 3. Lemmas for the proofs are given in Section 4, while the proofs of the main results are postponed to Sections 5 and 6. The growth results in this paper yield slight improvements of several earlier results in the complex function theory [2, 6, 9, 10, 11, 12, 8, 14, 15].
2. Functions on unbounded intervals

We begin with the following general result.

**Theorem 2.1.** Let $T_1, T_2 : [1, \infty) \to (0, \infty)$ be non-decreasing and unbounded functions such that $\xi(T_1) < \xi(T_2)$, where $\xi$ stands for either the order $\rho$ or the lower order $\mu$, the same order on both sides of the inequality. Let $\psi : [1, \infty) \to (0, \infty)$ be any non-decreasing function such that $\log \psi(r) = o(\log r)$ as $r \to \infty$. Then the set

$$G = \{r \geq 1 : \psi(r)T_1(r) < T_2(r)\}$$

satisfies

$$\log \text{dens}(G) \geq 1 - \frac{\xi(T_1)}{\xi(T_2)}.$$

A special case of Theorem 2.1 is implicitly proved in [10, p. 347] in the case $\rho(T_2) < \infty$. Our proof is based on a different method, which applies more generally for different orders, different $\psi$’s as well as in the case $\rho(T_2) = \infty$.

A possible choice for $\psi$ in Theorem 2.1 is $\psi(r) = (\log r)^{\beta}$, where $\beta > 0$. If we replace $T_1(r)$ and $T_2(r)$ by $T(r, f)$ and $T(r, g)$, respectively, where $f$ and $g$ are meromorphic functions, and if $\psi$ is unbounded, then Theorem 2.1 states in particular, that

$$T(r, f) = o(T(r, g)), \quad r \in G. \quad (2.1)$$

Thus $f$ is a small function of $g$ relative to the set $G$. Recall that in the complex function theory, a meromorphic function $f$ is said to be a small function of another meromorphic function $g$, if $T(r, f) = o(T(r, g))$ for all $r$ outside of a set of finite linear measure (or sometimes outside of a set of finite logarithmic measure). Small functions appear frequently in the theories of complex differential and functional equations, which in turn typically rely on growth estimates for logarithmic derivatives and for logarithmic differences. The former estimates are usually valid outside of exceptional sets of finite linear/logarithmic measure, while the exceptional sets in the latter estimates may go up to upper logarithmic density $< \varepsilon$. Hence, in most cases, the definition of small functions could be relaxed to (2.1), where the set $G$ is required to have positive logarithmic upper density.

**Corollary 2.2.** Let $T : [1, \infty) \to (0, \infty)$ be a non-decreasing unbounded function of order $\rho = \rho(T)$ and lower order $\mu = \mu(T)$.

(i) If $\mu < \infty$, then for any $a, b \in (\mu, \infty)$, the sets

$$H = \{r \geq 1 : T(r) \leq r^a\} \quad \text{and} \quad I = \{r \geq 1 : T(r) > r^b\}$$

satisfy

$$\log \text{dens}(H) \geq 1 - \frac{\mu}{a} \quad \text{and} \quad \log \text{dens}(I) \leq \frac{\mu}{b}. \quad (2.2)$$
(ii) If $\rho > 0$, then for any $a, b \in (0, \rho)$, the sets $H$ and $I$ satisfy
\[
\logdens(H) \leq \frac{a}{\rho} \quad \text{and} \quad \logdens(I) \geq 1 - \frac{b}{\rho}.
\] (2.3)

Proof. To prove the first inequality in (2.2), set $\psi(r) \equiv 1$, $T_1(r) = T(r)$ and $T_2(r) = r^a$. It is clear that $\mu(T_1) < \mu(T_2)$. The set $H$ can be re-written as
\[
H = \{r \geq 1 : \psi(r)T_1(r) < T_2(r)\}.
\]
Then from Theorem 2.1, we have
\[
\logdens(H) \geq 1 - \frac{\mu(T_1)}{\mu(T_2)} = 1 - \frac{\mu}{a}.
\]

To prove the second inequality in (2.2), we notice from the previous reasoning that the set $I^c$, which is the complement of $I$ in $[1, \infty)$, satisfies
\[
\logdens(I^c) \geq 1 - \frac{\mu}{b}.
\]
Recall that
\[
\logdens(D^c) = 1 - \logdens(D)
\] (2.4)
is true for every set $D \subset [1, \infty)$ [13, p. 121]. Then the second inequality in (2.2) follows.

The inequalities in (2.3) can be proved similarly. \qed

From Corollary 2.2, we see that the sets $E$ and $F$ in Theorem A satisfy
\[
\logdens(E) \geq 1 - \frac{\mu(f)}{a}, \quad \logdens(E) \leq \frac{a}{\rho(f)},
\]
\[
\logdens(F) \geq 1 - \frac{b}{\rho(f)}, \quad \logdens(F) \leq \frac{\mu(f)}{b}.
\]
Thus, Corollary 2.2 is an improvement of Theorem A. Moreover, the second inequality in (2.3) improves [9, Lemma 2.2], [11, Lemma 3] and [15, Lemma 2.7].

We give some immediate consequences of Corollary 2.2.

Corollary 2.3. Let $T : [1, \infty) \to (0, \infty)$ be a non-decreasing function of order $\rho \in (0, \infty)$, and let $\varepsilon > 0$. Then the set
\[
J_1 = \{r \geq 1 : r^{\rho - \varepsilon} \leq T(r) \leq r^{\rho + \varepsilon}\}
\]
measures
\[
\logdens(J_1) \geq \frac{\varepsilon}{\rho}.
\]
Proof. By the definition of the order, the inequality $T(r) \leq r^{\rho + \varepsilon}$ holds for all $r$ large enough. Hence the assertion follows from the second inequality in (2.3). \qed

Corollary 2.3 improves [6, Corollary 3.3], which claims that the set $J_1$ has infinite logarithmic measure. Similarly, if the order is replaced with the lower order in Corollary 2.3, then we get, by using the first inequality in (2.2), the following result.
Corollary 2.4. Let \( T : [1, \infty) \to (0, \infty) \) be a non-decreasing function of lower order \( \mu \in (0, \infty) \), and let \( \varepsilon > 0 \). Then the set
\[
J_2 = \{ r \geq 1 : r^{\mu - \varepsilon} \leq T(r) \leq r^{\mu + \varepsilon} \}
\]
satisfies
\[
\logdens(J_2) \geq \frac{\varepsilon}{\mu + \varepsilon}.
\]
Replacing \( T(r) \) with \( T(r, f) \) for a meromorphic function \( f \), we see that Corollary 2.4 improves \([15, \text{Lemma 2.2}]\), which claims that the set \( J_2 \) has infinite logarithmic measure.

A weaker version of the next result is applied in the theory of complex differential equations in \([12]\).

Corollary 2.5. Let \( f \) and \( g \) be meromorphic functions such that \( 0 \leq \mu(f) < \mu(g) \leq \infty \), and let \( \mu(f) < a \leq b < \mu(g) \). Then the set
\[
K = \{ r \geq 1 : T(r, f) \leq r^a, T(r, g) \geq r^b \}
\]
satisfies
\[
\logdens(K) \geq 1 - \frac{\mu(f)}{a}.
\]

Proof. By the definition of limit inferior, the inequality \( T(r, g) \geq r^b \) holds for all \( r \) large enough. Hence the assertion follows from the first inequality in \((2.2)\). \( \square \)

Corollary 2.5 is an improvement of \([12, \text{Lemma 3.4}]\), which is originally stated for entire functions, but it holds for meromorphic functions also. The conclusion of \([12, \text{Lemma 3.4}]\) is that the set \( K \) has positive upper logarithmic density.

Similarly, from the second inequality in \((2.3)\), if the lower orders in Corollary 2.5 are replaced with orders, then \( \logdens(K) \geq 1 - \frac{b}{\rho(g)} \).

Next, we give a result about comparing the growth of two functions in the case they have the same order.

Theorem 2.6. Let \( T_1, T_2 : [1, \infty) \to (0, \infty) \) be non-decreasing functions both having order \( \rho \in (0, \infty) \). Suppose that \( \tau(T_1) < \tau(T_2) \), where
\[
\tau(T) = \limsup_{r \to \infty} r^{-\rho}T(r)
\]
is the type of \( T \) with respect to its order \( \rho \). Let \( C \in (1, \tau(T_2)/\tau(T_1)) \). Then the set
\[
L = \{ r \geq 1 : CT_1(r) < T_2(r) \}
\]
satisfies
\[
\overline{\text{dens}}(L) \geq 1 - C^{1/\rho} \left( \frac{\tau(T_1)}{\tau(T_2)} \right)^{1/\rho}.
\]

As a consequence of Theorem 2.6, we give the following improvement of \([6, \text{Corollary 3.4}]\), which claims that the set \( M_1 \) defined below has infinite linear measure.
Corollary 2.7. Let $T : [1, \infty) \to (0, \infty)$ be a non-decreasing function of order $\rho \in (0, \infty)$ and of type $\tau \in (0, \infty)$, and let $\varepsilon > 0$. Then the set

$$M_1 = \{ r \geq 1 : (\tau - \varepsilon)r^\rho \leq T(r) \leq (\tau + \varepsilon)r^\rho \}$$

satisfies $\overline{\text{dens}}(M_1) \geq 1 - \left( \frac{\tau - \varepsilon}{\tau} \right)^{1/\rho}$.

Proof. From the definition of type, the inequality $T(r) \leq (\tau + \varepsilon)r^\rho$ is valid for all $r$ large enough. Hence the sets $M_1$ and $M'_1 = \{ r \geq 1 : (\tau - \varepsilon)r^\rho \leq T(r) \}$ have the same upper linear density. Take any $C > 1$. Set $T_1(r) = \frac{C - \varepsilon}{C}\tau r^\rho$ and $T_2(r) = T(r)$. Clearly $\tau(T_1) < \tau(T_2)$ and $C \in (1, \tau(T_2)/\tau(T_1))$. Then $M'_1 = \{ r \geq 1 : CT_1(r) \leq T_2(r) \}$. From Theorem 2.6,

$$\overline{\text{dens}}(M_1) \geq 1 - C^{1/\rho} \left( \frac{\tau - \varepsilon}{C\tau} \right)^{1/\rho} = 1 - \left( \frac{\tau - \varepsilon}{\tau} \right)^{1/\rho},$$

which completes the proof. $\square$

Let $f$ be an entire function of order $\rho \in (0, \infty)$ and of type $\tau \in (0, \infty)$ defined with respect to $\log M(r, f)$. Let $\varepsilon > 0$. Then [14, Lemma 8] claims that the set

$$M_2 = \{ r \geq 1 : (\tau - \varepsilon)r^\rho \leq \log M(r, f) \leq (\tau + \varepsilon)r^\rho \}$$

has infinite logarithmic measure. It is stated without proof in [7, p. 97] that a set of finite logarithmic measure has zero upper linear density. Hence we see that Corollary 2.7 is an improvement of [14, Lemma 8]. For the convenience of the reader, we give an elementary proof for the statement in [7, p. 97].

Lemma 2.8. If a set $D \subset [1, \infty)$ satisfies $\int_D \frac{dt}{t} < \infty$, then $\overline{\text{dens}}(D) = 0$.

Proof. Let $\chi_D(r)$ be the characteristic function of $D$. The function $\mu(r) = \int_1^r \frac{\chi_D(t)}{t} dt$ is increasing and bounded on $[1, \infty)$. By partial integration and L’Hospital’s rule,

$$\frac{1}{r} \int_1^r \chi_D(t) dt = \frac{1}{r} \int_1^r \mu'(t) dt = \mu(r) - \frac{1}{r} \int_1^r \mu(t) dt \to 0, \quad r \to \infty.$$

Thus $\overline{\text{dens}}(D) = 0$. $\square$

In [7, Lemma 4], it is shown that for a meromorphic function $f$ of order $\rho$, and for constants $C_1 > 1$ and $C_2 > 1$, the set

$$N_1 = \{ r : T(C_1 r, f) \geq C_2 T(r, f) \}$$

satisfies

$$\overline{\text{logdens}}(N_1) \leq \frac{\rho \log C_1}{\log C_2}. \quad (2.6)$$

If either $\rho = 0$ or $C_2^{1/\rho} \geq C_1$, then the inequality (2.6) is meaningful, and it gives information about size of the set $N_1$.

In the opposite case when $\rho > 0$ and $C_2^{1/\rho} < C_1$, the quantity $\frac{\rho \log C_1}{\log C_2}$ is larger than 1, and hence we conclude nothing from (2.6). In this case, the set $N_1$ is expected to be large,
and its size can be estimated by means of Theorem 2.6 with an additional assumption on the type of \( f \). In fact, we get the following corollary for a general function \( T(r) \).

**Corollary 2.9.** Let \( T : [1, \infty) \to (0, \infty) \) be a non-decreasing function of order \( \rho \in (0, \infty) \) and of type \( \tau \in (0, \infty) \). Let \( C_1 > 1 \) and \( C_2 > 1 \) be such that \( C_2^{1/\rho} < C_1 \). Then the set

\[
N_2 = \{ r : T(C_1 r) \geq C_2 T(r) \}
\]

satisfies

\[
\overline{\text{dens}}(N_2) \geq 1 - \frac{C_2^{1/\rho}}{C_1}.
\]

**Proof.** Take \( C = C_2 \), \( T_1(r) = T(r) \) and \( T_2(r) = T(C_1 r) \). It is not difficult to see that \( T_1 \) and \( T_2 \) have the same order \( \rho \), whereas \( \tau(T_1) = \tau \) and \( \tau(T_2) = C_1^\rho \tau \). Clearly, \( \tau(T_1) < \tau(T_2) \) and \( 1 < C = C_2 < C_1 \). Then, from Theorem 2.6, we obtain

\[
\overline{\text{dens}}(N_2) \geq 1 - C_2^{1/\rho} \left( \frac{\tau(T_1)}{\tau(T_2)} \right)^{1/\rho} = 1 - \frac{C_2^{1/\rho}}{C_1},
\]

which completes the proof. \( \square \)

Replacing \( T(r) \) in Corollary 2.9 with \( T(r, f) \), where \( f \) is a meromorphic function of order \( \rho \in (0, \infty) \) and of type \( \tau \in (0, \infty) \), we find that the set \( N_1 \) in (2.5) is large in the sense that \( \overline{\text{dens}}(N_1) \geq 1 - C_2^{1/\rho}/C_1 \).

### 3. Functions on bounded intervals

We now turn to bounded intervals, which we normalize to be the unit interval. Let \( T : [t_0, 1) \to (0, \infty) \) be a non-decreasing and unbounded function. The order and lower order of \( T \) are given, respectively, by

\[
\rho = \limsup_{r \to 1^-} \frac{\log T(r)}{-\log(1 - r)} \quad \text{and} \quad \mu = \liminf_{r \to 1^-} \frac{\log T(r)}{-\log(1 - r)}.
\]

If \( T \) has order \( \rho \in (0, \infty) \), then

\[
\tau(T) = \limsup_{r \to 1^-} (1 - r)^\rho T(r)
\]

is the type of \( T \) with respect to its order \( \rho \).

Analogues of linear and logarithmic densities for sets \( D \subset (0, 1) \) are introduced in [3] as follows. The upper and lower final densities of \( D \) are

\[
\overline{\mathcal{F}} - \text{dens}(D) = \limsup_{r \to 1^-} \frac{\int_{D \cap (r, 1)} dt}{1 - r} \quad \text{and} \quad \overline{\mathcal{F}} - \text{dens}(D) = \liminf_{r \to 1^-} \frac{\int_{D \cap (r, 1)} dt}{1 - r},
\]

while the upper and lower final \( \mathcal{L} \)-densities of \( D \) are

\[
\overline{\mathcal{L}} - \text{dens}(D) = \limsup_{r \to 1^-} \frac{\int_{D \cap (r, 1)} \frac{dt}{1-t}}{-\log(1 - r)} \quad \text{and} \quad \overline{\mathcal{L}} - \text{dens}(D) = \liminf_{r \to 1^-} \frac{\int_{D \cap (r, 1)} \frac{dt}{1-t}}{-\log(1 - r)}.
\]
The term "final" reflects the fact that integration is conducted over points that are larger than \( r \) as opposed to integrating over values that are at most \( r \). An analogue for (1.1) is proved in [3] as follows:

\[
0 \leq \overline{F} - \text{dens}(D) \leq \underline{L} - \text{dens}(D) \leq \overline{L} - \text{dens}(D) \leq \overline{F} - \text{dens}(D) \leq 1.
\]

Results analogous to those discussed in the previous section can now be obtained. Similarly as in Lemma 2.8, finite logarithmic measure implies zero final density.

**Lemma 3.1.** If a set \( D \subset (0, 1) \) satisfies \( \int_D \frac{dt}{1-t} < \infty \), then \( \overline{F} - \text{dens}(D) = 0 \).

**Proof.** The function \( \nu(r) = \int_r^1 \frac{\chi(t)}{1-t} \, dt \) is decreasing and tends to zero as \( r \to 1^- \). Clearly \( \nu'(r) = -\frac{\chi(t)}{1-t} \), so that by partial integration and L’Hospital’s rule,

\[
\frac{1}{1-r} \int_r^1 \chi_D(t) \, dt = - \frac{1}{1-r} \int_r^1 (1-t)\nu'(t) \, dt = \nu(r) - \frac{1}{1-r} \int_r^1 \nu(t) \, dt \to 0,
\]

as \( r \to 1^- \). Thus \( \overline{F} - \text{dens}(D) = 0 \). \( \square \)

An analogue of Theorem 2.1 is given as follows.

**Theorem 3.2.** Let \( T_1, T_2 : [1/2, 1) \to (0, \infty) \) be non-decreasing and unbounded functions such that \( \xi(T_1) < \xi(T_2) \), where \( \xi \) stands for either the order \( \rho \) or the lower order \( \mu \), the same order on both sides of the inequality. Let \( \psi : [1/2, 1) \to (0, \infty) \) be any non-decreasing function such that \( \log \psi(r) = o(- \log(1 - r)) \) as \( r \to 1^- \). Then the set

\[
G_0 = \{ r \in [1/2, 1) : \psi(r)T_1(r) < T_2(r) \}
\]

satisfies

\[
\overline{L} - \text{dens}(G_0) \geq 1 - \frac{\xi(T_1)}{\xi(T_2)}.
\]

Analogues of the sets \( H \) and \( I \) in Corollary 2.2 are 

\[
H_0 = \{ r \in [1/2, 1) : T(r) \leq (1 - r)^{-a} \} \quad \text{and} \quad I_0 = \{ r \in [1/2, 1) : T(r) > (1 - r)^{-b} \},
\]

and an analogue of Corollary 2.2 can be deduced from Theorem 3.2. Meanwhile, an analogue of Theorem 2.6 is given as follows.

**Theorem 3.3.** Let \( T_1, T_2 : [1/2, 1) \to (0, \infty) \) be non-decreasing functions both having order \( \rho \in (0, \infty) \). Suppose that \( \tau(T_1) < \tau(T_2) \), and let \( C \in (1, \tau(T_2)/\tau(T_1)) \). Then the set

\[
L_0 = \{ r \in [1/2, 1) : CT_1(r) < T_2(r) \}
\]

satisfies

\[
\overline{F} - \text{dens}(L_0) \geq 1 - C^{1/\rho} \left( \frac{\tau(T_1)}{\tau(T_2)} \right)^{1/\rho}.
\]

If applications to the complex function theory are of interest, then functions meromorphic in the unit disc \( \mathbb{D} \) should be considered. We omit the details of these discussions.
4. Lemmas for avoiding exceptional sets

Our first auxiliary result for exceptional sets originates from [1, 5], where the exceptional set $D$ has finite linear measure or finite logarithmic measure, respectively.

**Lemma 4.1 ([12]).** Suppose that $g, h : [1, \infty) \to (0, \infty)$ are non-decreasing functions such that

$$g(r) \leq h(r), \quad r \in [1, \infty) \setminus D,$$

holds. If $\overline{\text{dens}}(D) < 1$, then for any $\alpha > 1/(1 - \overline{\text{dens}}(D))$, there exists an $r_0 > 0$ such that $g(r) \leq h(\alpha r)$ for all $r > r_0$.

The following lemma is a refinement of Lemma 4.1 in the sense of (1.1). We give the elementary proof for the convenience of the reader.

**Lemma 4.2.** Suppose that $g, h : [1, \infty) \to (0, \infty)$ are non-decreasing such that (4.1) holds, where $\text{logdens}(D) < 1$. Then, for any $\alpha > 1/(1 - \text{logdens}(D))$, there exists an $r_0 > 1$ such that $g(r) \leq h(r^\alpha)$ for all $r > r_0$.

**Proof.** Let $D^c = [1, \infty) \setminus D$ denote the complement of $D$, and let $\alpha > 1/(1 - \text{logdens}(D))$. We claim that there exists an $r_0 > 1$ such that, for every $r \geq r_0$, the interval $[r, r^\alpha]$ meets the set $D^c$. Suppose on the contrary to this claim that there exists a sequence $\{r_n\} \subset (1, \infty)$ such that $r_n \to \infty$ as $n \to \infty$, and $[r_n, r_n^\alpha] \subset D$ for every $n \in \mathbb{N}$. Define

$$I = \bigcup_{n=1}^\infty [r_n, r_n^\alpha].$$

Then $I \subset D$, but

$$\frac{\text{logdens}(I)}{\alpha - 1} \leq \frac{\int_{I} \frac{dt}{\log r}}{\alpha \log r} \leq \lim_{n \to \infty} \frac{\int_{[r_n, r_n^\alpha]} \frac{dt}{\log r}}{\alpha \log r_n} = \frac{\alpha - 1}{\alpha} > \text{logdens}(D),$$

which is a contradiction. Finally, let $r \geq r_0$, and take $t \in [r, r^\alpha] \cap D^c$. Then

$$g(r) \leq g(t) \leq h(t) \leq h(r^\alpha)$$

by the monotonicity of $g(r)$ and $h(r)$. This completes the proof. \qed

We note that a unit interval analogue of Lemma 4.1 already exists, see [8, Lemma 2]. Lemma 4.3 below is a unit interval analogue of Lemma 4.2. The proof is an obvious modification of the proof of Lemma 4.2, where $[r, r^\alpha]$ is to be replaced with $[r, 1 - (1 - r)^\alpha]$.

**Lemma 4.3.** Suppose that $g, h : [1/2, 1) \to (0, \infty)$ are non-decreasing functions such that (4.1) holds, where $\overline{\text{L-dens}}(D) < 1$. Then, for any $\alpha > 1/(1 - \overline{\text{L-dens}}(D))$, there exists an $r_0 > 1/2$ such that $g(r) \leq h(1 - (1 - r)^\alpha)$ for all $r > r_0$.

It turns out that Lemmas 4.1 and 4.2 yield an immediate improvement and an extension of the following Doeringer’s lemma, which is used in the theory of differential polynomials.
Lemma 4.4 ([2]). Suppose that \( g(r) \) and \( h(r) \) are real-valued, non-decreasing and non-negative functions on \([1, \infty)\) such that \( g(r) \leq Ch(r) \) for some \( C > 0 \) and for all \( r \in [1, \infty) \setminus D \), where \( D \subset [1, \infty) \) has finite linear measure. Then
\[
\rho(g(r)) \leq \rho(h(r)) \quad \text{and} \quad \mu(g(r)) \leq \mu(h(r)).
\]

Indeed, the size of the set \( D \) in Lemma 4.4 can be relaxed to \( \text{dens}(D) < 1 \), and the conclusions remain the same by Lemma 4.1. If the size of \( D \) is relaxed even further to \( \beta = \log\text{dens}(D) < 1 \), then \( \rho(g(r)) \leq \alpha \rho(h(r)) \) and \( \mu(g(r)) \leq \alpha \mu(h(r)) \) for any \( \alpha > 1/(1 - \beta) \) by Lemma 4.2, and hence
\[
\rho(g(r)) \leq \frac{\rho(h(r))}{1 - \beta} \quad \text{and} \quad \mu(g(r)) \leq \frac{\mu(h(r))}{1 - \beta}.
\]

5. Proof of Theorem 2.1

Suppose first that \( \xi(T_1) < \xi(T_2) < \infty \). Suppose on the contrary to the assertion that \( \log\text{dens}(G) < 1 - \frac{\xi(T_1)}{\xi(T_2)} \). Then we may choose \( \varepsilon > 0 \) small enough such that
\[
\log\text{dens}(G) < 1 - \frac{\xi(T_1)}{\xi(T_2)} + \varepsilon \quad \text{and} \quad \frac{\xi(T_2) - \varepsilon}{\xi(T_1) + \varepsilon} > 1.
\]
Set \( \alpha = \frac{\xi(T_2) - \varepsilon}{\xi(T_1) + \varepsilon} \). Then, clearly \( \alpha > 1/(1 - \log\text{dens}(G)) \) and
\[
T_2(r) \leq T_1(r) \psi(r), \quad r \in [1, \infty) \setminus G. \tag{5.1}
\]
By applying Lemma 4.2 to avoid the set \( G \), we conclude that \( T_2(r) \leq T_1(r^\alpha) \psi(r^\alpha) \), for every \( r \) large enough. This yields \( \xi(T_2) \leq \alpha \xi(T_1) < \xi(T_2) - \varepsilon \), which is a contradiction. Hence \( \log\text{dens}(G) \geq 1 - \frac{\xi(T_1)}{\xi(T_2)} \).

Suppose next that \( \xi(T_1) < \xi(T_2) = \infty \). If \( \log\text{dens}(G) < 1 \), we may choose \( \alpha > 1/(1 - \log\text{dens}(G)) \). Then from (5.1) and Lemma 4.2, we see that \( \xi(T_2) \leq \alpha \xi(T_1) < \infty \), which is a contradiction. Hence \( \log\text{dens}(G) = 1 \).

6. Proof of Theorem 2.6

Suppose first that \( 0 < \tau(T_1) < \tau(T_2) < \infty \). Suppose on the contrary to the assertion that \( \text{dens}(L) < 1 - C^{1/\rho} \left( \frac{\tau(T_1)}{\tau(T_2)} \right)^{1/\rho} \). We may choose \( \varepsilon > 0 \) small enough such that
\[
\text{dens}(L) < 1 - C^{1/\rho} \left( \frac{\tau(T_1) + \varepsilon}{\tau(T_2) - \varepsilon} \right)^{1/\rho} \quad \text{and} \quad \frac{\tau(T_2) - \varepsilon}{C(\tau(T_1) + \varepsilon)} > 1.
\]
Choose \( \alpha = \left( \frac{\tau(T_2) - \varepsilon}{C(\tau(T_1) + \varepsilon)} \right)^{1/\rho} \). Since \( \alpha > 1/(1 - \text{dens}(L)) \) and
\[
T_2(r) \leq CT_1(r), \quad r \in [1, \infty) \setminus L, \tag{6.1}
\]
we conclude by Lemma 4.1 that
\[
T_2(r) \leq CT_1(\alpha r) \leq C(\tau(T_1) + \varepsilon)(\alpha r)^\rho = (\tau(T_2) - \varepsilon)r^\rho, \quad r \geq r_1 > r_0.
\]
This gives \( \tau(T_2) \leq \tau(T_2) - \varepsilon \), which is a contradiction. Hence \( \text{dens}(L) \geq 1 - C^{1/\rho} \left( \frac{\tau(T_1)}{\tau(T_2)} \right)^{1/\rho} \).
Suppose next that $\tau(T_1) < \tau(T_2) = \infty$. If $\text{dens}(L) < 1$, we take $\alpha > 1/(1 - \text{dens}(L))$. Then for $\varepsilon > 0$, it follows from (6.1) and Lemma 4.1 that

$$T_2(r) \leq CT_1(\alpha r) \leq C(\tau(T_1) + \varepsilon)(\alpha r)^\rho, \quad r \geq r_1,$$

which implies $\tau(T_2) < \infty$. This is a contradiction, and hence $\text{dens}(L) = 1$.

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