A general procedure to construct criteria for identifying genuine multipartite continuous variable entanglement is presented. It relies on the proper definition of adequate global operators describing the multipartite system, the positive partial transpose criterion of separability, and quantum mechanical uncertainty relations. As a consequence, each criterion encountered consists of a single inequality that is nicely computable and experimentally feasible, and that when violated is sufficient condition for genuine multipartite entanglement. Additionally we show that the previous work of van Loock and Furusawa [Phys. Rev. A, 67, 052315 (2003)] is a special case of our result that includes strongest criteria to detect entanglement.
Genuine multipartite entanglement – entanglement between three or more quantum systems – is essential to harness the full power of quantum computing in either the circuit \[1\] or one-way models \[2\] as well as to the security in multi-party quantum encryption protocols \[3\]. Additionally, it provides increased precision in quantum metrology \[4\], furnishes the resource to solve the Byzantine agreement problem \[5\], and allows for multi-party quantum information protocols such as open destination quantum teleportation \[6\]. There is also evidence that it is responsible for efficient transport in biological systems \[7\], and is linked to fundamental aspects of phase transitions in spin chains \[8\].

In this context, the experimental identification of genuine multipartite entanglement is essential, since in order to reliably realize any multipartite entanglement-based task, it is necessary to confirm the presence of genuine multipartite entangled states. Although quantum state tomography can provide all of the available information about the system, it requires a number of measurements that increases exponentially with the number of subsystems. In this regard, entanglement witnesses composed of an abbreviated number of measurements are the most viable method for identifying genuine multipartite entanglement. This is true especially when the system is of high dimension, due to the dimension of the Hilbert space of each subsystem and/or the number of constituent subsystems.

In particular, for a continuous variable (CV) system composed of \(n\) subsystems or modes, quantum state tomography is not viable in general. This is rapidly becoming an important experimental concern, since possibly genuine multipartite CV entanglement has been produced for three degenerate \[9\] and non-degenerate modes \[10\], and more recently for a large number of temporal modes \[11, 12\]. Even for the specific case of two-modes, the difficulty of state tomography has also led to several witnesses involving second-order \[13–15\] and higher-order moments \[16–20\] of the canonical variables. Some of these criteria are based on the positive partial transpose (PPT) argument \[21, 22\] and uncertainty relations involving the variance \[13\] or entropy \[19, 20\] of marginal distributions. There are also some criteria to detect bipartite entanglement in the multipartite scenario \[18, 23\].

To identify genuine multipartite CV entanglement, distinct entanglement witnesses must be employed. A first step in this direction for CV systems was made by van Loock and Furusawa \[24\] using a variance criterion. However, Reid \[25\] has shown that the extension of these criteria to \(n\) modes does not account for all possible biseparable states \[26\], and has provided distinct solutions for three and four modes. In general, to prove genuine
multipartite entanglement, one must show that the state cannot be written as a convex sum of biseparable states. This possibility is also not excluded by the criteria developed in [27] formulated via a hierarchy of inequalities for minors in terms of moments of the given state.

In this work, we solve this problem in general by presenting a systematic method for construction of genuine multipartite entanglement criteria for $n$ CV modes. Our method consists of adequate definitions of global operators of the $n$-partite system, which can be employed in conjunction with a wide range of quantum mechanical uncertainty relations, including those based on entropy functions, producing unique inequalities that test genuine $n$-partite entanglement. In particular, we derive the unique family of a single pair of global $n$-mode position and momentum operators that test entanglement in all possible bipartitions simultaneously and also genuine $n$-partite entanglement. Our results reproduce the criteria of van Loock and Furusawa and Reid for the variance, showing that they are indeed based on PPT arguments, and extend it to a wide range of uncertainty relations.

For a $n$-mode state $\hat{\rho} \in \otimes_{i=1}^{n} \mathcal{H}_i$ we define a bipartition $\vec{l} | \vec{m}$ where the vectors of integer indexes $\vec{l} \equiv (l_1, \ldots, l_{n_A})$ and $\vec{m} \equiv (m_1, \ldots, m_{n_B})$ indicate the modes belonging to each part, and $l_i, m_i$ are integers in the set $\{1, \ldots, n\}$. Thus, we have Alice’s part with $n_A$ modes and the Bob’s part with $n_B = n - n_A$ modes, and for convenience we order $l_i < l_{i+1}$ and $m_i < m_{i+1}$.

A $n$-partite state is said to be (genuinely) $n$-partite entangled if it cannot be prepared by mixing states that are separable with respect to some bipartition, i.e. they do not belong to the family of “biseparable” states [26],

$$\hat{\rho}_{\text{bs}} \equiv \sum_{\{\vec{l}\}} p_{\{\vec{l}\}} \hat{\rho}_{\{\vec{l}\}} = \sum_{\{\vec{l}\}} p_{\{\vec{l}\}} \left( \sum_{j} \eta_j \hat{\rho}_{j}^{\{\vec{l}\}} \otimes \hat{\rho}_{j}^{\{\vec{m}\}} \right), \quad (1)$$

where the sum in $\{\vec{l}\}$ runs over the set $\{\vec{l}|\vec{m}\}$ of all the bipartition’s of the system (there are a total of $L$), and $\sum_{\{\vec{l}\}} p_{\{\vec{l}\}} = 1 = \sum_{j} \eta_j$. We label the different bipartition classes that can appear by the number of modes in each part, thus the two numbers $(n_{A}, n_{B})$ identify the bipartition class. For $n$ even there are $n/2$ different bipartition classes, i.e. $n_{A} = 1, \ldots, n/2$, and for $n$ odd there are $(n - 1)/2$, i.e. $n_{A} = 1, \ldots, (n - 1)/2$. For a fixed value $n_{A}$ there are $N_{n_{A}}$ bipartitions of the same class that correspond to different labels $\vec{l}|\vec{m}$. For $n_{A} \neq n/2$, $N_{n_{A}} = \binom{n}{n_{A}}$, and for $n_{A} = n/2$, $N_{n_{A}} = \frac{1}{2} \binom{n}{n_{A}}$. It is easy to see that for either $n$ odd or even, there are a total of $L = 2^{n-1} - 1$ different bipartitions.
For example, in the case of a three mode system \((n = 3)\) there is only one class, \(i.e.\) \((n_A = 1, n_B = 2)\), that contains \(N_{n_A} = 3\) bipartitions: \(\bar{l}|\bar{m} = 123, 213\) or \(312\). The total number of bipartitions in this case is \(L = N_{n_A=1} = 2^{3-1} - 1 = 3\). For \(n = 4\), we have two classes: i) \((n_A = 1, n_B = 3)\) that corresponds to the \(N_{n_B} = 4\) bipartitions, \(\bar{l}|\bar{m} = 1234, 2134, 3124, 4123\), and ii) \((n_A = 2, n_B = 2)\) that corresponds to the \(N_{n_B} = 3\) bipartitions, \(\bar{l}|\bar{m} = 1234, 1324, 1423, 2314, 2413, 3412\). The total number of bipartitions in this case is \(L = N_{n_A=1} + N_{n_A=2} = 4 + 3 = 2^{4-1} - 1 = 7\).

A genuine multipartite entanglement criterion needs to test entanglement in all the possible bipartitions that can be drawn in the system, and more generally refute state \((\Pi)\) as a possible description of the system. We will first concentrate on a particular bipartition \(\bar{l}|\bar{m}\), and then extend our results to consider state \((\Pi)\). Let us start defining the local canonical operators for \(n\) bosonic modes \(\hat{z} \equiv (\hat{a}, \hat{p})^T = (\hat{x}_1, ..., \hat{x}_n, \hat{p}_1, ..., \hat{p}_n)^T\) (\(T\) means transposition), with \(\hat{x}_j\) and \(\hat{p}_j\) the canonically conjugated observables with continuous spectra such that \([\hat{x}_j, \hat{p}_k] = i\delta_{jk}\), that we will generically call position and momentum respectively. Now, we define, for each of the \(N_{n_A}\) bipartitions \(\bar{l}|\bar{m}\) of the same type \((n_A, n_B)\), the auxiliary non-local operators:

\[
\hat{y}'_{\vec{\alpha}} \equiv (\hat{u}_{\vec{\alpha}}, \hat{v}_{\vec{\alpha}})^T \equiv (\hat{u}_{1,\vec{\alpha}}, ..., \hat{u}_{n,\vec{\alpha}}, \hat{v}_{1,\vec{\alpha}}, ..., \hat{v}_{n,\vec{\alpha}})^T = M_{\vec{\alpha}}\hat{z},
\]

(2)

with \(\vec{\alpha} = \vec{l}\) or \(\vec{\alpha} = \vec{m}\), \(M_{\vec{\alpha}} \equiv diag(M_{x,\vec{\alpha}}, M_{p,\vec{\alpha}})\) a \(2n \times 2n\) real matrix and \(M_{x,\vec{\alpha}}\) and \(M_{p,\vec{\alpha}}\) are non-singular real \(n \times n\) matrices. We impose that the auxiliary non-local operators satisfy the commutation relation \([\hat{u}_{j,\vec{\alpha}}, \hat{v}_{k,\vec{\alpha}}] = i\gamma_{\vec{\alpha}}\delta_{jk}\) with \(\gamma_{\vec{\alpha}}\) any real number. This means that \(M_{p,\vec{\alpha}}^TM_{x,\vec{\alpha}} = M_{x,\vec{\alpha}}M_{p,\vec{\alpha}}^T = \gamma_{\vec{\alpha}}1\), so

\[
M_{x,\vec{\alpha}} = \gamma_{\vec{\alpha}}(M_{p,\vec{\alpha}}^{-1})^T\ 
\]

(3)

Our systematic method to construct genuine multipartite entanglement criteria for \(n\) bosonic modes will use the PPT criterion of separability and any uncertainty relation that can be written in the form:

\[
F[\hat{\rho}, P_a(\xi), P_b(\xi)] - f(|\gamma|) \geq 0
\]

(4)

where \(F\) is a functional, \(\hat{u}\) and \(\hat{v}\) are a pair of operators such that \([\hat{u}, \hat{v}] = i\gamma \hat{1}\), and \(P_a(\xi)\) and \(P_b(\xi)\) are the marginal probability distributions of these operators in the state \(\hat{\rho}\). Examples of the generic uncertainty relation of the type in Eq.(4) are the entropic uncertainty relation \([28]\) where \(F_E[\hat{\rho}, P_a, P_b] \equiv h[P_a] + h[P_b]\) and \(f_E(|\gamma|) = \ln(\pi e|\gamma|)\), the variance product criteria.
where $F_H[\hat{\rho}, P_\hat{u}, P_\hat{v}] \equiv \Delta u \Delta \hat{v}$ and $f_H(|\gamma|) = 1/2|\gamma|$ and the variance sum criteria \[14\] where $F_{Lin}[\hat{\rho}, P_\hat{u}, P_\hat{v}] \equiv \Delta^2 u + \Delta^2 \hat{v}$ and $F_{Lin}(|\gamma|) = |\gamma|$. Here, $h[P] \equiv - \int dx P(x) \ln(P(x))$ is the Shannon entropy of the marginal probability distribution $P(x)$. All these inequalities can be condensed in the single inequality \[29\]:

$$\ln [\pi e (\Delta^2 \hat{u} + \Delta^2 \hat{v})] \geq \ln (2\pi e \Delta \hat{u} \Delta \hat{v}) \geq h[P_\hat{u}] + h[P_\hat{v}] \geq \ln (\pi e |\gamma|). \tag{5}$$

The partial transposition with respect to the modes of each part we generically denote $T_{\tilde{\alpha}}$. The partial transposition $T_{\tilde{\alpha}}$ over any separable state $\hat{\rho}_{\{\tilde{l}\}}$ in the bipartition $\tilde{l}|\tilde{m}$ result in another possible physical realizable state $\hat{\rho}^T_{\tilde{\alpha}}{\{\tilde{l}\}}$. The PPT criterion of separability \[13, 21, 22\] establishes that, for an arbitrary $n$-mode state $\hat{\rho} \in \bigotimes_{i=1}^{n} \mathcal{H}_i$, if $\hat{\rho}^T_{\tilde{\alpha}}$ is a non-physical state then the original state is entangled in the considered bipartition. The application of the PPT criterion in a given bipartition $\tilde{l}|\tilde{m}$ involves either the partial transposition $T_{\tilde{l}}$ or $T_{\tilde{m}}$. Therefore, an entanglement witness on the bipartition $\tilde{l}|\tilde{m}$ based on a given uncertainty relation and the PPT criterion of separability is given by the inequality:

$$F[\hat{\rho}^T_{\tilde{\alpha}}, P^T_{\hat{u}k, \tilde{\alpha}}, P^T_{\hat{v}k, \tilde{\alpha}}] - f(|\gamma_{\tilde{\alpha}}|) \geq 0, \tag{6}$$

that must be violated in order to detect entanglement. Here $P^T_{\hat{u}k, \tilde{\alpha}}(\xi)$ and $P^T_{\hat{v}k, \tilde{\alpha}}(\xi)$ are marginal probability distributions of the Wigner function $W_{T_{\tilde{\alpha}}}$ associated with the operator $\hat{\rho}^T_{\tilde{\alpha}}$.

Violation of the inequality in Eq.(6) constitutes a useful entanglement criterion in the bipartition once we provide a prescription of how to measure $F[\hat{\rho}^T_{\tilde{\alpha}}, P^T_{\hat{u}k, \tilde{\alpha}}, P^T_{\hat{v}k, \tilde{\alpha}}]$ in the original state $\hat{\rho}$ given that $\hat{\rho}^T_{\tilde{\alpha}}$ could be a non-physical state when the original state is entangled. In order to do this we are going to find a pair of new non-local operators $\hat{\mu}_{k, \tilde{\alpha}}$ and $\hat{\nu}_{k, \tilde{\alpha}}$ such that

$$P^T_{\hat{u}k, \tilde{\alpha}}(\xi) = P_{\hat{\mu}_{k, \tilde{\alpha}}}(\xi), \quad P^T_{\hat{v}k, \tilde{\alpha}}(\xi) = P_{\hat{\nu}_{k, \tilde{\alpha}}}(\xi) \tag{7}$$

and therefore

$$F[\hat{\rho}^T_{\tilde{\alpha}}, P^T_{\hat{u}k, \tilde{\alpha}}, P^T_{\hat{v}k, \tilde{\alpha}}] = F[\hat{\rho}, P_{\hat{\mu}_{k, \tilde{\alpha}}}, P_{\hat{\nu}_{k, \tilde{\alpha}}}] \geq f(|\gamma_{\tilde{\alpha}}|). \tag{8}$$

From now on we refer to the non-local operators that appear in the functional $F$ with the partial transpose operator $\hat{\rho}^T_{\tilde{\alpha}}$ as the “old” ones and the non-local operators that appear in the functional $F$ with the original state $\hat{\rho}$ as the “new” ones. So, first we remember that
where $W$ is the Wigner function of the original state $\hat{\rho}$, and we define the diagonal matrices $\Lambda_{\vec{\alpha}}$ with ones in the location of modes that are not transposed and minus ones in the location of modes that are transposed. Now, we proceed as follows:

\[
P_{y_{k,\vec{\alpha}}}(\xi) \equiv \int \frac{d\mu d\nu}{|\gamma_\alpha|^n} W_{T_{\vec{\alpha},y}} \left( M_{x,\vec{\alpha}}^{-1} u, M_{p,\vec{\alpha}}^{-1} v \right) \delta(y_{k,\vec{\alpha}} - \xi) =
\]

\[
= \int \frac{d\mu d\nu}{|\gamma_\alpha|^n} W \left( M_{x,\vec{\alpha}}^{-1} M_{\mu,\vec{\alpha}}^{-1} \mu, M_{p,\vec{\alpha}}^{-1} M_{\nu,\vec{\alpha}}^{-1} \nu \right) \delta \left( \sum_{i=1}^{n} (M_{y,\vec{\alpha}}^{-1})_{ki} y_{i,\vec{\alpha}} - \xi \right) = P_{y_{k,\vec{\alpha}}}(\xi)
\]

where $y' = u$ or $y' = v$ and $y = \mu$ or $y = \nu$, we used Eq. (9) and we made the change of variables $\mu = M_{\mu,\vec{\alpha}} u$ and $\nu = M_{\nu,\vec{\alpha}} v$ with the Jacobian equal to one. In order for $P_{y_{k,\vec{\alpha}}}(\xi)$ to be the marginal probability distribution of the “new” non-local operator $\hat{y}_{k,\vec{\alpha}}$ in the original state $\hat{\rho}$, the following conditions must be fulfilled: $M_{x,\vec{\alpha}}^{-1} M_{\mu,\vec{\alpha}}^{-1} = M_{x,\vec{\alpha}}^{-1}$ and $\Lambda_{\vec{\alpha}} M_{p,\vec{\alpha}}^{-1} M_{\nu,\vec{\alpha}}^{-1} = M_{p,\vec{\alpha}}$. This is equivalent to:

\[
M_{\mu,\vec{\alpha}} = 1,
\]

\[
M_{\nu,\vec{\alpha}} = M_{p,\vec{\alpha}} \Lambda_{\vec{\alpha}} M_{p,\vec{\alpha}}^{-1} = M_{p,\vec{\alpha}} \Lambda_{\vec{\alpha}} M_{x,\vec{\alpha}}^{-1} \gamma_{\vec{\alpha}}^{-1},
\]

with $\det(M_{\nu,\vec{\alpha}}) = \det(\Lambda_{\vec{\alpha}}) = (-1)^s$ where $s = n_A$ for $\vec{\alpha} = \vec{l}$ and $s = n_B$ when $\vec{\alpha} = \vec{m}$. Note that Eq. (11b) shows that the matrix $M_{\nu,\vec{\alpha}}$ is an involutory matrix, i.e., $M_{\nu,\vec{\alpha}}^2 = 1$ with signature $s$ (the signature is the number of elements equal to $-1$ in $\Lambda_{\vec{\alpha}}$ [30]). Therefore, the involuntary property allows us to recognise in Eq. (10) the “new” non-local operators as:

\[
\hat{\mu}_{k,\vec{\alpha}} = \hat{u}_{k,\vec{\alpha}} = \sum_{j=1}^{n} (M_{x,\vec{\alpha}})_{k,j} \hat{x}_j,
\]

\[
\hat{\nu}_{k,\vec{\alpha}} = \sum_{j=1}^{n} (M_{\nu,\vec{\alpha}})_{k,j} \hat{v}_j = \sum_{j=1}^{n} (M_{p,\vec{\alpha}} \Lambda_{\vec{\alpha}})_{k,j} \hat{p}_j;
\]

which correspond to the $k$th components of the linear transformations $\hat{\mu}_{\vec{\alpha}} = M_{\mu,\vec{\alpha}} \hat{u}_{\vec{\alpha}}$ and $\hat{\nu}_{\vec{\alpha}} = M_{\nu,\vec{\alpha}} \hat{v}_{\vec{\alpha}}$ ($M_{\mu,\vec{\alpha}} = 1$), with $[\hat{\mu}_{k,\vec{\alpha}}, \hat{\nu}_{k,\vec{\alpha}}] = i\gamma_{\vec{\alpha}} (M_{\nu,\vec{\alpha}})_{kk} = i(M_{p,\vec{\alpha}} \Lambda_{\vec{\alpha}} M_{x,\vec{\alpha}}^T)_{kk} = i\delta$. Thus, to test entanglement in a given bipartition $\vec{l}|\vec{m}$ through the criteria given in Eq. (8), we only need to specify a matrix $M_{p,\vec{\alpha}}$, and consequently the matrix $M_{x,\vec{\alpha}} = \gamma_{\vec{\alpha}} (M_{p,\vec{\alpha}}^{-1})^T$, where the only important part of these matrices are their $k$th rows. Indeed, without loss of generality we choose $k = 1$ so in the supplementary material [31] we give the general structure of a matrix
$M_{p,\vec{\alpha}}$ with the property that the first rows of $M_{p,\vec{\alpha}}$ and $M_{x,\vec{\alpha}}$ are the coefficient of the arbitrary "old" non-local operators that appear in Eq.(8). This gives $\hat{\nu} = \hat{\nu}_{1,\vec{\alpha}} \hat{\mu} = \hat{\mu}_{1,\vec{\alpha}}$ respectively, where:

$$\hat{\nu} = \sum_{j=1}^{n} \bar{g}_j \hat{p}_j \quad \text{and} \quad \hat{\mu} = \sum_{j=1}^{n} h_j \hat{x}_j,$$

such that $[\hat{\nu}, \hat{\mu}] = i\gamma_{\vec{\alpha}}$ with

$$\gamma_{\vec{\alpha}} = \sum_{j=1}^{n} h_j \bar{g}_j. \quad (14)$$

Here $\bar{g}_j = -g_j$ if $j$ is a component of the vector $\vec{\alpha}$ or $\bar{g}_j = g_j$ otherwise. This structure guarantees that the corresponding first row of the matrix $M_{p,\vec{\alpha}}$ is composed by the coefficients of an arbitrary "new" non-local operator $\hat{\nu} = \hat{\nu}_{1,\vec{\alpha}}$, so the "new" operators are:

$$\hat{\nu} = \sum_{j=1}^{n} g_j \hat{p}_j \quad \text{and} \quad \hat{\mu} = \sum_{j=1}^{n} h_j \hat{x}_j,$$

such that $[\hat{\mu}, \hat{\nu}] = i\sum_{j=1}^{n} h_j g_j = i\delta$.

There is an advantage to consider commuting "new" non-local operators (i.e. $\delta = 0$) to test entanglement with the inequality in Eq.(8) since they typically have entangled states as mutual eigenstates, increasing the ability to detect entanglement. For arbitrary commuting operators of the sort given in Eq. (15), we show in Theorem.1 of Ref. [31] that all separable states in the bipartition $\vec{l}|\vec{m}$ must satisfy inequality (8) with $\gamma_{\vec{\alpha}} = \pm \gamma_{\vec{l}}$ (see Eq.(14)), where the plus sign stands when $\vec{\alpha} = \vec{l}$ and the minus sign when $\vec{\alpha} = \vec{m}$. This is our first result, which is an entanglement criteria for the bipartition in question.

We now extend this result to derive a genuine entanglement criteria based on inequalities like the one in Eq.(8) but now excluding the possibility of a violation by biseparable states $\hat{\rho}_{bs}$ given in (11). To do so we need the result of Theorem 3 in [31]: if $F[\hat{\rho}, P_\mu(\xi), P_\nu(\xi)] \geq G[\hat{\rho}, P_\mu(\xi)] + \tilde{G}[\hat{\rho}, P_\nu(\xi)]$ with $G$ and $\tilde{G}$ two arbitrarily concave functionals with respect to the probabilities distributions $P_\mu$ and $P_\nu$ [32] and $f$ a strictly increasing function, then for every pair of commuting operators of the type given in Eqs.(15), biseparable states $\hat{\rho} = \hat{\rho}_{bs}$ satisfy the inequality:

$$F[\hat{\rho}, P_\mu(\xi), P_\nu(\xi)] \geq f(\gamma_{\min}) \geq 0,$$

where $\gamma_{\min} = \min_{\vec{l}} \{ |\gamma_{\vec{l}}| \} \geq 0$, and $\{\vec{l}\}$ runs over all the $L$ bipartitions of the system. This means we must consider all possible different location of minus sign in $\bar{g}_j$ (see Eq.(14)) for fixed values of the coefficients $h_j$ and $g_j$ with $j = 1, \ldots, n$. So, Eq.(16) is a genuine
entanglement witness once we provide pairs of commuting "new" non-local operators $\hat{\mu}$ and $\hat{\nu}$ for which $\gamma_{\min} \neq 0$. It is important to note that $\gamma_{\min}$ depends only upon the set of operators chosen, and not the particular convex combination of bi-separable states in Eq. (10).

Thus, the problem of identifying criteria for genuine multipartite entanglement has been reduced to finding a suitable pair of non-local operators $\hat{\mu}$ and $\hat{\nu}$ for which $\gamma_{(l)} \neq 0$ for all the $L$ bipartition’s of the system, this means that this pair of non-local operators tests bipartite entanglement through Eq. (8). Thus, in order to test entanglement in all the $L$ bipartitions with this specific pair of operators $\hat{\mu}$ and $\hat{\nu}$, we only need to find $\gamma_{\min}$ between all the $\gamma_{(l)}$ (all different from zero) and use Eq. (16) that can not be violated by biseparable states.

In what follows we develop a systematic way to find these type of commuting operators. Let’s start observing that $\Lambda_{\bar{\alpha}} = \pm P_{\beta l}^T \Lambda_{\beta} P_{\beta l}$, where the minus sign applies when $\bar{\alpha} = \bar{m}$ and the plus sign when $\bar{\alpha} = \bar{l}$, $\beta \equiv (1, \ldots, n_{A})$. Here $P_{\beta l} = \prod_{i=1}^{n_{A}} P_{i l}$, with $P_{i l}$ defined as the permutation matrix between the mode $i$ and the mode $l_i$ (i.e., the matrix obtained from swapping the rows $i$ and $l_i$ of the identity matrix 1) and $P_{i l} = 1$. Note, that $P_{\beta l}^T P_{\beta l} = 1$ because $P_{i l}^2 = 1$. We call $\Lambda_{\beta}$ and $M_{p,\bar{\alpha}=\beta}$ the “seed” matrices associated with the bipartition class $(n_A, n_B)$. With these matrices we can test entanglement in what we call the “seed” bipartition $\bar{l}|\bar{m}$ with $\bar{l} = \bar{\beta}$. We will use the “new” non-local operators $\hat{\mu}_{1,\beta} = \sum_{j=1}^{n} (M_{x,\beta})_{i j} \hat{x}_j$ and $\hat{\nu}_{1,\beta} = \sum_{j=1}^{n} (M_{p,\beta} \Lambda_{\beta})_{i j} \hat{p}_j$ in Eq. (10), while the “old” ones are $\hat{u}_{1,\beta} = \hat{\mu}_{1,\beta}$ and $\hat{v}_{1,\beta} = \sum_{j=1}^{n} (M_{p,\beta} \Lambda_{\beta})_{i j} \hat{p}_j$ such that $[\hat{u}_{1,\beta}, \hat{v}_{1,\beta}] = i \gamma_{\beta}$, and therefore $M_{x,\beta} = \gamma_{\beta} (M_{p,\beta}^{-1})^T$. In order to test entanglement in the rest of the bipartitions of the same class $(n_A, n_B)$ we can use the matrices

$$M_{p,\bar{\alpha}} = P_{\beta l}^T M_{p,\beta} P_{\beta l}$$

$$M_{x,\bar{\alpha}} = \gamma_{\bar{\alpha}} P_{\beta l}^T (M_{p,\beta}^{-1})^T P_{\beta l}$$

so that according to Eq. (12), and choosing $k = l_1$, the “new” non-local operators are: $\hat{\mu}_{l_1,\bar{\alpha}} = \sum_{j=1}^{n} (M_{x,\beta} P_{\beta l_1}^T)_{i j} \hat{x}_j$ and $\hat{\nu}_{l_1,\bar{\alpha}} = \pm \sum_{j=1}^{n} (M_{p,\beta} \Lambda_{\beta} P_{\beta l_1}^T)_{i j} \hat{p}_j$. The important thing to realize here is that the “old” non-local operators in Eq. (10), $\hat{u}_{l_1,\bar{\alpha}} = \hat{\mu}_{l_1,\bar{\alpha}}$ and $\hat{v}_{l_1,\bar{\alpha}} = \sum_{l=1}^{n} (M_{p,\beta} P_{\beta l_1}^T)_{i j} \hat{x}_l$, are different from $\hat{u}_{l_1,\bar{\beta}} = \hat{\mu}_{l_1,\bar{\beta}}$ and $\hat{v}_{l_1,\bar{\beta}}$, but with the same commutator $[\hat{u}_{l_1,\bar{\alpha}}, \hat{v}_{l_1,\bar{\alpha}}] = i \gamma_{\alpha} = i \gamma_{\beta}$. So, using the matrices in Eqs. (17) the lower bound in Eq. (8) is equal for all bipartitions of the same class $(n_A, n_B)$. Thus, we can use this result to obtain a single pair of “new” non-local operators to be used in the genuine multipartite entanglement criterion in Eq. (16).
For bipartitions such that the vector $\vec{l}$ does not contain the mode $n$, we choose the seed matrix $(\hat{M}_{p,\beta})_{ij} = (-g, \ldots, -g, g, \ldots, g')$ where the minus sign stands in the first $n_A$ positions, $g = -\gamma/2h$ and $g' = \gamma(n-1)/2h'$. For bipartitions such that the vector $\vec{l}$ contains the mode $n$ we choose the seed matrix $(\hat{M}_{x,\beta})_{ij} = (-g, \ldots, -g', \ldots, -g, g, \ldots, g)$ where again the minus sign stands in the first $n_A$ positions and $g'$ is located in the position $i$ ($1 \leq i \leq n_A$) such that $l_i = n$ with $l_i$ a component of $\vec{l}$. Therefore, we have $(\hat{M}_{x,\beta})_{ij} = (h, h, \ldots, h, h')$ and $(\hat{M}_{x,\beta})_{ij} = (h, \ldots, h', h, \ldots, h)$ where the location of $h'$ is in the position $i$ ($1 \leq i \leq n_A$) such that $l_i = n$. For this choice of the seed matrices we have $[\hat{u}_{t_1,\alpha}, \hat{v}_{t_1,\alpha}] = [\hat{u}_{1,\beta}, \hat{v}_{1,\beta}] = in_A\gamma$ if the mode $n$ is not in $\vec{l}$ and $[\hat{u}_{t_1,\beta}, \hat{v}_{t_1,\beta}] = [\hat{u}_{1,\beta}, \hat{v}_{1,\beta}] = -i(n - n_A)\gamma$ if the mode $n$ is in $\vec{l}$. Then, because $(\hat{M}_{p,\beta}^{T} - \beta^{T} \hat{\beta})_{ij} = (\hat{M}_{p,\beta}^{T} - \beta^{T} \hat{\beta})_{ij} = (g, \ldots, g, g')$ and $(\hat{M}_{x,\beta}^{T} - \beta^{T} \hat{\beta})_{ij} = (\hat{M}_{x,\beta}^{T} - \beta^{T} \hat{\beta})_{ij} = (h, \ldots, h, h')$, the commuting operators:

$$\hat{\mu} = h\hat{x}_1 + h\hat{x}_2 + \ldots + h\hat{x}_{n-1} + h'\hat{x}_n,$$

$$\hat{\nu} = \frac{\gamma}{2} \left( -\frac{\hat{p}_1}{h} - \ldots - \frac{\hat{p}_{n-1}}{h} + \frac{(n-1)\hat{p}_n}{h} \right),$$

are the ones that must be used in our genuine entanglement criterion in Eq.(16) with $\gamma_{\text{min}} = \min_{\{\gamma\}} \{|\gamma|\} = \min_{n_A} \{n_A\gamma, (n - n_A)\gamma\}$. The minimization is over the values $n_A = 1, \ldots, n/2$ for $n$ even, and $n_A = 1, \ldots, (n - 1)/2$ for $n$ odd. This is our third main result. If we rename the mode 1 as $n$ and set $h' = 1$ and $h = -1/\sqrt{n-1}$, then $\gamma = 2/(n-1)$ and we recover the non-local operators that appear in Eq.(30) of [24]. Here the lower bound must be given by $\gamma$ according to our Eq.(16) with $F = F_{\text{Lin}}$ and $f_{\text{Lin}}(\gamma) = |\gamma|$ [34].

It is also possible to have a genuine entanglement criterion that, contrary to the one in Eq.(16), involves several pairs of commuting operators. This can be done due to the result in Theorem 4 of the Supplementary Material [31]. We define a set of pairs $\{\mu_m, \nu_m\}$ of commuting operators, i.e. $[\mu_m, \nu_m] = 0$ with $m = 1, \ldots, M$ ($M \leq L$) of the form

$$\mu_m = \sum_{j=1}^{n} h_{mj}\hat{x}_j$$

and

$$\nu_m = \sum_{j=1}^{n} g_{mj}\hat{p}_j$$

where $h_{mj}, g_{mj}$ are real numbers not all equal to zero), and $\gamma_{m,\vec{l}} \equiv \sum_{j=1}^{n} \hat{g}_{mj}h_{mj}$ where for all values of $m = 1, \ldots, M \leq L$, $\hat{g}_{mj} = -g_{mj}$ if $j$ is one component of the vector $\vec{l}$ or $\hat{g}_{mj} = g_{mj}$ otherwise. Let's denote the complementary subset where the values of $\gamma_{m,\vec{l}}$ are not zero as $\{\vec{l}\}_m$ (for every fixed value of $m$) so we call $\gamma_{m,\text{min}} = \min_{\{\vec{l}\}_m} \{\gamma_{m,\vec{l}}\} > 0$ and $\gamma_{m,\text{min}} = \min_m \{\gamma_{m,\text{min}}\}$. If we consider uncertainty relations like the one considered in Eq.(16) and if all the sets $\{\{\vec{l}\}_m, m = 1, \ldots, M\}$ cover all possible bipartitions of the system (with possible repetitions), then for all biseparable states $\hat{\rho} = \hat{\rho}_{bs}$,
it is always true that:

\[
\sum_{m=1}^{M} F[\hat{\rho}, P_{\mu_m}(\xi), P_{\nu_m}(\xi)] \geq \Theta \ f(\gamma_{\text{min}}) \geq f(\gamma_{\text{min}}),
\]  

(19)

where \( \Theta = \sum_{m=1}^{M} \sum_{\{\vec{l}\}_m} p_{\{\vec{l}\}_m} \geq 1 \). Therefore, for a general state \( \hat{\rho} \), violation of Eq. (19) constitutes a genuine entanglement criterion, where the first lower bound must be considered if all the sets \( \{\{\vec{l}\}_m, m = 1, \ldots, M\} \) contain all the bipartitions of the system an integer number of times \( \Theta \), otherwise the second one must be considered.

Consider for example in the case of four mode states the set of non-local operators considered in Eq. (39) of Ref. [24]: \( \{\hat{\mu}_1, \hat{\nu}_1, \hat{\mu}_2, \hat{\nu}_2, \hat{\mu}_3, \hat{\nu}_3, \hat{\mu}_4, \hat{\nu}_4, \hat{\mu}_5, \hat{\nu}_5, \hat{\mu}_6, \hat{\nu}_6\} = \{\hat{x}_1 - \hat{x}_2, \hat{p}_1 + \hat{p}_2 + g_1\hat{p}_3 + g_14\hat{p}_4, \hat{x}_2 - \hat{x}_3, g_2\hat{p}_1 + \hat{p}_2 + \hat{p}_3 + g_24\hat{p}_4, \hat{x}_1 - \hat{x}_3, \hat{p}_1 + g_32\hat{p}_2 + \hat{p}_3 + g_34\hat{p}_4, \hat{x}_3 - \hat{x}_4, g_41\hat{p}_1 + g_42\hat{p}_2 + \hat{p}_3 + \hat{p}_4, \hat{x}_2 - \hat{x}_4, g_5\hat{p}_1 + \hat{p}_2 + g_53\hat{p}_3 + \hat{p}_4, \hat{x}_1 - \hat{x}_4, \hat{p}_1 + g_62\hat{p}_2 + g_63\hat{p}_3 + \hat{p}_4\}. For these operators we have:

\[
\{\vec{l}\}_1 = \{1|234, 2|134, 13|24, 14|23\},
\]

(20a)

\[
\{\vec{l}\}_2 = \{2|134, 3|124, 12|34, 13|24\},
\]

(20b)

\[
\{\vec{l}\}_3 = \{1|234, 3|124, 12|34, 14|23\},
\]

(20c)

\[
\{\vec{l}\}_4 = \{3|124, 4|123, 13|24, 14|23\},
\]

(20d)

\[
\{\vec{l}\}_5 = \{2|134, 4|123, 12|34, 14|23\},
\]

(20e)

\[
\{\vec{l}\}_6 = \{1|234, 4|123, 12|34, 13|24\},
\]

(20f)

therefore \( \Theta = 3(p_{1|234} + p_{2|134} + p_{3|124} + p_{4|123}) + 3(p_{12|34} + p_{13|24} + p_{14|23}) = 3 \) and \( \gamma_{\text{min}} = 2 \) (see also [23]). It is important to note that each of the sub-equations in Eq. (39) of Ref. [24] are special cases of the inequality in Eq. (15) (with \( F = F_{L,m} \) and \( f(|\gamma_{m,i}|) = |\gamma_{m,i}| \), \( m = 1, \ldots, M = 6 \leq L = 7 \)) whose violations test entanglement in the bipartitions specified in Eqs. (20), however we only have a genuine entanglement criterion if we use Eq. (19).

To summarize, in this letter we have presented a general framework which provides criteria for genuine multipartite continuous variable entanglement. These criteria expressed in term of uncertainty relations for global canonical operators describing the multipartite system are experimentally feasible, do not demanding quantum state tomography, and are easily computable.
SUPPLEMENTARY MATERIAL

General structure of the matrices $M_{p,\vec{\alpha}}$.

From the set of operators of $n$-bosonic modes $\hat{z} \equiv (\hat{x}, \hat{p})^T = (\hat{x}_1, ..., \hat{x}_n, \hat{p}_1, ..., \hat{p}_n)^T$ ($T$ means transposition), with $\hat{x}_j$ and $\hat{p}_j$ the canonically conjugate pair of observables with continuous spectra, such that $[\hat{x}_j, \hat{p}_k] = i\delta_{jk}$, we define a set of $k = 1, ..., n$ auxiliary pairs of non-local operators that we name the "old" ones: $\hat{y}'_{\vec{\alpha}} \equiv (\hat{u}_{\vec{\alpha}}, \hat{v}_{\vec{\alpha}})^T \equiv (\hat{u}_{1,\vec{\alpha}}, ..., \hat{u}_{n,\vec{\alpha}}, \hat{v}_{1,\vec{\alpha}}, ..., \hat{v}_{n,\vec{\alpha}})^T = M_{\vec{\alpha}} \hat{z}$. Here, with $M_{\vec{\alpha}} \equiv \text{diag}(M_{x,\vec{\alpha}}, M_{p,\vec{\alpha}})$ a $2n \times 2n$ real matrix and $M_{x,\vec{\alpha}}$ and $M_{p,\vec{\alpha}}$ are non-singular real $n \times n$ matrices such that the pairs of operators,

$$\hat{u}_{k,\vec{\alpha}} = \sum_{j=1}^{n} (M_{x,\vec{\alpha}})_{k,j} \hat{x}_j$$

$$\hat{v}_{k,\vec{\alpha}} = \sum_{j=1}^{n} (M_{p,\vec{\alpha}})_{k,j} \hat{p}_j,$$

satisfy the commutation relation $[\hat{u}_{j,\vec{\alpha}}, \hat{v}_{k,\vec{\alpha}}] = i\gamma_{\vec{\alpha}} \delta_{jk}$ with $\gamma_{\vec{\alpha}}$ a real number. This implies the matrix relation:

$$M_{x,\vec{\alpha}} = \gamma_{\vec{\alpha}} (M_{p,\vec{\alpha}}^{-1})^T.$$

Here, the label is $\vec{\alpha} = \vec{l} \equiv (l_1, ..., l_n)$ or $\vec{\alpha} = \vec{m} \equiv (m_1, ..., m_n)$, with $l_i, m_i$ natural numbers in the set $\{1, ..., n\}$ (for convenience we choose $l_i < l_{i+1}$ and $m_i < m_{i+1}$). We can also define another set of $k = 1, ..., n$ auxiliary pairs of non-local operators, that we name the "new" ones, $\hat{y}_{\vec{\alpha}} \equiv (\hat{\mu}_{\vec{\alpha}}, \hat{\nu}_{\vec{\alpha}})^T \equiv (\hat{\mu}_{1,\vec{\alpha}}, ..., \hat{\mu}_{n,\vec{\alpha}}, \hat{\nu}_{1,\vec{\alpha}}, ..., \hat{\nu}_{n,\vec{\alpha}})^T = \text{diag}(M_{x,\vec{\alpha}}, M_{p,\vec{\alpha}} \Lambda_{\vec{\alpha}}) \hat{z}$ in the form:

$$\hat{\mu}_{k,\vec{\alpha}} = \hat{u}_{k,\vec{\alpha}} = \sum_{j=1}^{n} (M_{x,\vec{\alpha}})_{k,j} \hat{x}_j$$

$$\hat{\nu}_{k,\vec{\alpha}} = \sum_{j=1}^{n} (M_{p,\vec{\alpha}})_{k,j} \hat{p}_j,$$

The matrix $\Lambda_{\vec{\alpha}}$ is a diagonal matrix with minus one in the positions labeled by $\vec{\alpha}$ and one in the rest.

Now, we want to give the general structure of the matrix $M_{p,\vec{\alpha}}$ in a way that the following conditions must be satisfied:

i) $(M_{p,\vec{\alpha}})_{1j} = \bar{g}_j$, 
ii) $(M_{x,\vec{\alpha}})_{1j} = h_j$, 

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iii) \[ \sum_{j=1}^{n} \bar{g}_j h_j = \gamma \vec{\alpha}, \]
iv) \[ \sum_{j=1}^{n} g_j h_j = \delta, \]
where \( \bar{g}_j = -g_j \) if \( j \) is one component of the vector \( \vec{\alpha} \) or \( \bar{g}_j = g_j \) otherwise (\( \vec{\alpha} = \vec{l} \) or \( \vec{\alpha} = \vec{m} \)), \( g_j \) and \( h_j \) are real numbers not all identical to zero. All the conditions mean that the first rows of the matrices \( M_{x,\vec{\alpha}} \) and \( M_{p,\vec{\alpha}} \) are the coefficient of the pair of "old' operators, i.e.,

\[
(M_{x,\vec{\alpha}})_{1j} = h_j \rightarrow \hat{u} = \hat{u}_{1,\vec{\alpha}} = \sum_{j=1}^{n} h_j \hat{x}_j \quad (26)
\]

\[
(M_{p,\vec{\alpha}})_{1j} = \bar{g}_j \rightarrow \hat{v} = \hat{v}_{1,\vec{\alpha}} = \sum_{j=1}^{n} \bar{g}_j \hat{p}_j \quad (27)
\]
such that

\[
[\hat{u}, \hat{v}] = i(M_{p,\vec{\alpha}}^T M_{x,\vec{\alpha}})_{11} = i \left( \sum_{j=1}^{n} \bar{g}_j h_j \right) = i\gamma \vec{\alpha}, \quad (28)
\]

and the first rows of the matrices \( M_{x,\vec{\alpha}} \) and \( M_{p,\vec{\alpha}} \) \( \vec{\alpha} \) are the coefficient of the pair of "new" operators, i.e.,

\[
(M_{x,\vec{\alpha}})_{1j} = h_j \rightarrow \hat{\mu} = \hat{\mu}_{1,\vec{\alpha}} = \hat{\mu}_{1,\vec{\alpha}} = \sum_{j=1}^{n} g_j \hat{x}_j \quad (29)
\]

\[
(M_{p,\vec{\alpha}})_{1j} = g_j \rightarrow \hat{\nu} = \hat{\nu}_{1,\vec{\alpha}} = \sum_{j=1}^{n} g_j \hat{p}_j \quad (30)
\]
such that

\[
[\hat{\mu}, \hat{\nu}] = i(M_{p,\vec{\alpha}}^T M_{x,\vec{\alpha}})_{11} = i \sum_{j=1}^{n} g_j h_j = i\delta. \quad (31)
\]

Because of the relation in Eq. (23) we only need to give the general structure of the matrix \( M_{p,\vec{\alpha}} \). The explicit form of this matrix satisfying conditions i)-iv) is:

\[
(M_{p,\vec{\alpha}})_{ij} = \begin{cases} 
\bar{g}_1 = \bar{g}_1(\gamma \vec{\alpha}, \delta, \bar{g}_2, \ldots, \bar{g}_n, h_1, \ldots, h_n) & \text{for } i = j = 1 \\
\bar{g}_j & \text{for } i = 1 \text{ and } 1 < j < n \\
\bar{g}_n = \bar{g}_n(\gamma \vec{\alpha}, \delta, \bar{g}_2, \ldots, \bar{g}_n, h_1, \ldots, h_n) & \text{for } i = 1 \text{ and } j = n \\
Q_{ij} & 1 < i \leq n \text{ and } 1 \leq j \leq n - 1 \\
-\frac{1}{h_n} \left( \sum_{l=1}^{n-1} h_l Q_{il} \right) & \text{for } 1 < i \leq n \text{ and } j = n
\end{cases} \quad (32)
\]

where \( \bar{g}_j = -g_j \) if \( j \) is one component of the vector \( \vec{\alpha} \) or \( \bar{g}_j = g_j \) otherwise (\( \vec{\alpha} = \vec{l} \) or \( \vec{\alpha} = \vec{m} \)), \( \bar{g}_1 \) and \( \bar{g}_n \) are solutions of the equations:

\[
\sum_{j=1}^{n} \bar{g}_j h_j = \gamma \vec{\alpha} \quad (33)
\]

\[
\sum_{j=1}^{n} g_j h_j = \delta \quad (34)
\]
and the matrix elements $Q_{ij}$ ($1 < i \leq n$ and $1 \leq j \leq n - 1$) are arbitrary. That the matrix in Eq. (32) verifies conditions $i) - iv)$, which can be easily corroborated by direct inspection.

In the case when $\vec{\alpha} = \vec{\beta} = (1, \ldots, n_A)$ the explicit form of $M_{p,\vec{\beta}}$ is:

$$
\begin{align*}
(M_{p,\vec{\beta}})_{ij} &= \begin{cases} 
-g_1 = \frac{1}{n_A} \left( \frac{d - \gamma_{\vec{\beta}}}{2} - \sum_{l=2}^{n_A} g_l h_l \right) & \text{for } i = j = 1 \\
g_j & \text{for } i = 1 \text{ and } 1 \leq j \leq n_A \\
g_j & \text{for } i = 1 \text{ and } n_A < j < n \\
Q_{ij} & \text{for } i = 1 \text{ and } j = n \\
g_n = \frac{1}{n_A} \left( \frac{d + \gamma_{\vec{\beta}}}{2} - \sum_{l=n_A+1}^{n-1} g_j h_l \right) & 1 < i \leq n \text{ and } 1 \leq j \leq n - 1 \\
-\frac{1}{n_A} (\sum_{l=1}^{n-1} h_l Q_{il}) & \text{for } 1 < i \leq n \text{ and } j = n
\end{cases}.
\end{align*}
$$

Theorems

**Theorem 1** Let

$$
\hat{\mu} = \sum_{j=1}^{n} h_j \hat{x}_j \quad \text{and} \quad \hat{\nu} = \sum_{j=1}^{n} g_j \hat{p}_j
$$

be two non-local operators such that

$$
[\hat{\mu}, \hat{\nu}] = i \sum_{j=1}^{n} h_j g_j = 0
$$

(where $h_j, g_j$ are real numbers not all equal to zero), and let $\hat{\rho}_{(\vec{l})} = \sum_{\vec{j}} \eta_{\vec{j}} \hat{\rho}_{\vec{j}}^{(\vec{l})} \otimes \hat{\rho}_{\vec{j}}^{(\vec{m})}$ be an arbitrary $n$-mode separable state on a bipartition $\vec{l}\vec{m}$, where the vectors of integer indexes, $\vec{l} \equiv (l_1, \ldots, l_{n_A})$, $\vec{m} \equiv (m_1, \ldots, m_{n_B})$, indicate the modes belonging to each part. (i.e. Alice’s part contains $n_A$ modes and Bob’s part $n_B = n - n_A$ modes, where for convenience we order $l_i < l_{i+1}$, $m_i < m_{i+1}$, and $l_i, m_i$ are integers in the set $\{1, \ldots, n\}$). For any uncertainty relation that can be written in the form:

$$
F[\hat{\rho}, P_{\hat{\nu}}(\xi), P_{\hat{\nu}}(\xi)] - f(|\gamma_{\vec{l}}|) \geq 0
$$

(38)

where $F$ is a functional, $\hat{\nu}'$ and $\hat{\nu}'$ are a pair of operators such that $[\hat{\mu}', \hat{\nu}'] = i \gamma_{\vec{l}} \hat{1}$, and $P_{\hat{\nu}}(\xi)$ and $P_{\hat{\nu}}(\xi)$ are the marginal density probability distributions corresponding to these operators in the state $\hat{\rho}$, then for the commuting operators in Eq. (36) above it is always true that:

$$
F[\hat{\rho}_{(\vec{l})}, P_{\hat{\nu}}(\xi), P_{\hat{\nu}}(\xi)] - f(|\gamma_{\vec{l}}|) \geq 0,
$$

(39)

where $\gamma_{\vec{l}} = \pm \sum_{j=1}^{n} g_j h_j \ (g_j = -g_j$ if $j$ is one component of the vector $\vec{l}$ or $\vec{g}_j = g_j$ otherwise).
Proof We are going to prove that:

\[ F[\hat{\rho}, P_\mu(\xi), P_\nu(\xi)] = F[\hat{\rho}^{T_{\vec{\alpha}}}, P_\mu(\xi), P_\nu(\xi)], \]  

(40)

where

\[ \hat{u} = \hat{u}_{1,\vec{\alpha}} = \hat{\mu} \equiv \hat{\mu}_{1,\vec{\alpha}} = \sum_{l=1}^{n} h_l \hat{x}_l, \]  

(41)

\[ \hat{v} = \hat{v}_{1,\vec{\alpha}} \equiv \pm \left( \sum_{l=1}^{n} \bar{g}_l \hat{p}_l \right), \]  

(42)

such that

\[ [\hat{u}, \hat{v}] = \pm i \left( \sum_{l=1}^{n} \bar{g}_l h_l \right) = \pm i \gamma_{\vec{\alpha}}, \]  

(43)

and \( T_{\vec{\alpha}} \) means partial transposition with respect to the modes \( \vec{\alpha} = \vec{l} \) (then the sign plus stands in Eq.(42)) or \( \vec{\alpha} = \vec{m} \) (then the sign minus stands in Eq.(42)). Note that because the state \( \hat{\rho} = \sum \eta_j \hat{\rho}_j^{\{l\}} \otimes \hat{\rho}_j^{\{m\}} \) is separable in the bipartition \( \vec{l}/\vec{m} \), the partial transpose state \( \hat{\rho}^{T_{\vec{\alpha}}} \) is a physical state so the operators \( \hat{u} \) and \( \hat{v} \) must satisfy the uncertainty relation:

\[ F_E[\hat{\rho}^{T_{\vec{\alpha}}}, P_\mu(\xi), P_\nu(\xi)] - f(|\gamma_{\vec{\alpha}}|) \geq 0. \]  

(44)

Q.E.D.

In order to prove the equality in Eq.(40), let us define, as in Section I, a set of auxiliary \( k = 1, \ldots, n \) pairs of non-local operators (the type we named “new” in the Section):

\[ \hat{\mu}_k,\vec{\alpha} = \sum_{j=1}^{n} (M_{x,\vec{\alpha}})_{k,j} \hat{x}_j, \]  

(45)

\[ \hat{v}_k,\vec{\alpha} = \sum_{j=1}^{n} (M_{p,\vec{\alpha}})_{k,j} \hat{p}_j, \]  

(46)

and a set of auxiliary \( k = 1, \ldots, n \) pairs of non-local operators (the type we named “old” in the Section):

\[ \hat{u}_k,\vec{\alpha} = \sum_{j=1}^{n} (M_{x,\vec{\alpha}})_{k,j} \hat{x}_j \]  

(47)

\[ \hat{v}_k,\vec{\alpha} = \sum_{j=1}^{n} (M_{p,\vec{\alpha}})_{k,j} \hat{p}_j, \]  

(48)
such that \([\hat{u}_{j,\vec{\alpha}}, \hat{v}_{k,\vec{\alpha}}] = i\gamma_\vec{\alpha}\delta_{jk}\), i.e. \(M_{x,\vec{\alpha}} = \gamma_\vec{\alpha}(M_{p,\vec{\alpha}})^T\). We identify the operators \(\hat{\mu}\) and \(\hat{\nu}\), that commute, with the first pair of the “new” operators, i.e.

\[
\hat{\mu} \equiv \sum_{j=1}^{n} h_j \hat{x}_j = \hat{\mu}_{1,\vec{\alpha}}
\]

(49)

\[
\hat{\nu} \equiv \sum_{j=1}^{n} g_j \hat{p}_j = \hat{\nu}_{1,\vec{\alpha}}.
\]

(50)

Therefore

\[
[\hat{\mu}, \hat{\nu}] = [\hat{\mu}_{1,\vec{\alpha}}, \hat{\nu}_{1,\vec{\alpha}}] = i \delta_{11} \gamma_\vec{\alpha} \delta_{jk} = i\gamma_\vec{\alpha} = 0
\]

(51)

with \(\vec{\alpha} = \vec{l}\) or \(\vec{\alpha} = \vec{m}\). Thus, we choose the matrix \(M_{p,\vec{\alpha}}\) to have the structure given in Eq.(32) so it verifies Eq.(51) and also:

\[
\hat{u} = \hat{u}_{1,\vec{\alpha}} = \hat{\mu} \equiv \hat{\mu}_{1,\vec{\alpha}} = \sum_{j=1}^{n} (M_{x,\vec{\alpha}})_{1,j} \hat{x}_j = \sum_{j=1}^{n} h_j \hat{x}_j
\]

(52)

\[
\hat{\nu} = \hat{\nu}_{1,\vec{\alpha}} = \sum_{j=1}^{n} (M_{p,\vec{\alpha}})_{1,j} \hat{p}_j = \sum_{j=1}^{n} g_j \hat{p}_j
\]

(53)

with

\[
[\hat{u}, \hat{\nu}] = i (M_{p,\vec{\alpha}}M_{x,\vec{\alpha}})^T_{11} = i \left( \sum_{l=1}^{n} g_l h_l \right) = i\gamma_\vec{\alpha}.
\]

(54)

Now, we proceed as follows: for the probability distribution function \(P_{y_{1,\vec{\alpha}}}(\xi) (\hat{y} = \hat{\mu}, \hat{\nu})\) in the state \(\hat{\rho} = \sum_j \eta_j \rho_j^{(\vec{l})} \otimes \rho_j^{(\vec{m})}\), we have

\[
P_{y_{1,\vec{\alpha}}}(\xi) \equiv \int \frac{d\vec{\mu} d\vec{\nu}}{|\gamma_\vec{\alpha}|^n} W(M_{x,\vec{\alpha}}^{-1} \mu, M_{p,\vec{\alpha}}^{-1} \nu) \delta(y_{1,\vec{\alpha}} - \xi),
\]

(55)

where \(W\) is the Wigner function of the state \(\hat{\rho}\) and \(\vec{\mu} \equiv (\mu_1, \ldots, \mu_n), \vec{\nu} \equiv (\nu_1, \ldots, \nu_n)\). So,

\[
\begin{align*}
P_{y_{1,\vec{\alpha}}}(\xi) & \equiv \int \frac{d\vec{\mu} d\vec{\nu}}{|\gamma_\vec{\alpha}|^n} W(M_{x,\vec{\alpha}}^{-1} \mu, M_{p,\vec{\alpha}}^{-1} \nu) \delta(y_{1,\vec{\alpha}} - \xi) = \\
& = \int \frac{d\vec{\mu} d\vec{\nu}}{|\gamma_\vec{\alpha}|^n} W_T(\mu, \nu) \times \\
& \times \delta \left( \sum_{i=1}^{n} (M_{x,\vec{\alpha}}^{-1})_{1,i} y_{i,\vec{\alpha}} - \xi \right) = \\
& = \int \frac{d\vec{\mu} d\vec{\nu}}{|\gamma_\vec{\alpha}|^n} W_T(\mu, \nu) \times \\
& \times \delta \left( \sum_{i=1}^{n} (M_{x,\vec{\alpha}}^{-1})_{1,i} y_{i,\vec{\alpha}} - \xi \right) = P_{y_{1,\vec{\alpha}}}(\xi),
\end{align*}
\]

(56)
where \( \hat{y}' = \hat{u} \) or \( \hat{y}' = \hat{v} \) and we use

\[
W_{T_\alpha}(x, p) = W(x, \alpha p),
\]

(58)

and in Eq.(56) we made the change of variables: \( \mu = M_{\mu, \alpha}^{-1} \mu' \) and \( \nu = M_{\nu, \alpha}^{-1} \nu' \) with Jacobian equal to one because \( M_{\mu, \alpha} = 1 \) and \( M_{\nu, \alpha} = \alpha M_{\nu, \alpha}^{-1} \). Then \( \det(M_{\nu, \alpha}) = \det(\bar{\alpha}) = (-1)^s \) (\( s = n_A \) for \( \bar{\alpha} = \bar{l} \) and \( s = n_B \) when \( \bar{\alpha} = \bar{m} \)). Thus,

\[
P_{\hat{\mu}_1, \alpha}(\xi) = P_{\hat{\mu}_1, \alpha}(\xi) \quad \text{with} \quad \hat{\mu}_1, \alpha = \hat{\mu}_1, \alpha \quad \text{given in Eq.(52),}
\]

(59)

and

\[
P_{\hat{\nu}_1, \alpha}(\xi) = P_{\hat{\nu}_1, \alpha}(\xi),
\]

(60)

with

\[
\hat{\nu}_1, \alpha = \sum_{j=1}^n (M_{\nu, \alpha})_1 j \hat{\nu}_j, \alpha = \sum_{j=1}^n (M_{\nu, \alpha} M_{\nu, \alpha}^{-1} \Lambda^{-1})_1 j \hat{\nu}_j = \sum_{j=1}^n (M_{\nu, \alpha})_1 j \hat{\nu}_j \quad \text{given in Eq.(53).}
\]

(61)

Finally, we note that because \( \bar{m} = -\bar{l} \) we necessarily need that \( M_{\mu, \bar{m}} = M_{\mu, \bar{l}} \) in order to have \( \hat{\nu} = \hat{\nu}_{1, \bar{m}} \) in Eq.(46). Thus,

\[
P_{\hat{\mu}_1, \alpha}(\xi) = P_{\hat{\mu}_1, \alpha}(\xi),
\]

\[
P_{\hat{\nu}_1, \alpha}(\xi) = P_{\hat{\nu}_1, \alpha}(\xi),
\]

with \( \hat{\nu}_1, \alpha = \sum_{j=1}^n (M_{\nu, \alpha})_1 j \hat{\nu}_j, \alpha = \sum_{j=1}^n (M_{\nu, \alpha} M_{\nu, \alpha}^{-1} \Lambda^{-1})_1 j \hat{\nu}_j \quad \text{given in Eq.(53).}
\]

(61)

Theorem 2 Consider any two non-local operators \( \hat{\mu} = \sum_{l=1}^n h_l \hat{x}_l \) and \( \hat{\nu} = \sum_{l=1}^n g_l \hat{p}_l \) and any functional \( F[\hat{\rho}, P_\mu(\xi), P_\nu(\xi)] \) that verifies:

\[
F[\hat{\rho}, P_\mu(\xi), P_\nu(\xi)] \geq G[\hat{\rho}, P_\mu(\xi)] + \bar{G}[\hat{\rho}, P_\nu(\xi)]
\]

(63)

with \( G \) and \( \bar{G} \) two arbitrarily concave functionals with respect to the probabilities distributions \( P_\mu \) and \( P_\nu \). When these probability distributions correspond to the measurement of the observables \( \hat{\mu} \) and \( \hat{\nu} \) in any “biseparable” state of a \( n \)-mode bosonic system i.e.:

\[
\hat{\rho}_{bs} \equiv \sum_{\{\bar{l}\}} P_{\{\bar{l}\}} \hat{\rho}_{\{\bar{l}\}} = \sum_{\{\bar{l}\}} P_{\{\bar{l}\}} \left( \sum_j \eta_j \hat{\rho}_j^{\{\bar{l}\}} \otimes \hat{\rho}_j^{\{\bar{m}\}} \right),
\]

(64)

(where the vectors of integer indexes, \( \bar{l} \equiv (l_1, \ldots, l_{n_A}) \), \( \bar{m} \equiv (m_1, \ldots, m_{n_B}) \), indicate the modes belonging to each bipartition \( \bar{l}|\bar{m} \), i.e. the Alice’s part with \( n_A \) modes and the Bob’s
part with \( n_B = n - n_A \) modes, for convenience we order \( l_i < l_{i+1}, m_i < m_{i+1}, \) and \( l_i, m_i \) are integers in the set \( \{1, \ldots, n\} \), we have:

\[
F[\hat{\rho}_{bs}, P_\mu(\xi), P_\nu(\xi)] \geq \sum_{\{i\}} p_{\{i\}} \left( G[\hat{\rho}_{i1}, P_{\hat{\mu}, \hat{\nu}_{i1}}(\xi)] + \tilde{G}[P_{\hat{\nu}_{i1}}(\xi)] \right),
\]

(65)

where \( P_{\hat{\mu}, \hat{\nu}_{i1}} \) and \( P_{\hat{\nu}_{i1}} \) are the probability distributions of measure the observables \( \hat{\mu} \) and \( \hat{\nu} \) in the separables states \( \hat{\rho}_{i1} \equiv \sum_j \eta_j \hat{\rho}_j^{i1} \otimes \hat{\rho}_j^{m1} \).

**Proof** The Wigner function \( W(\mathbf{x}, \mathbf{p}) \) of the biseparable state \( \hat{\rho}_{bs} \) (in Eq.(64)) is:

\[
W(\mathbf{x}, \mathbf{p}) = \sum_{\{i\}} p_{\{i\}} W_{\hat{\rho}_{i1}}(\mathbf{x}, \mathbf{p}),
\]

so for the marginal distribution we have:

\[
P_{\hat{\mu}}(\xi) = \sum_{\{i\}} p_{\{i\}} P_{\hat{\mu}, \hat{\nu}_{i1}}(\xi) \quad \text{and} \quad P_{\hat{\nu}}(\xi) = \sum_{\{i\}} p_{\{i\}} P_{\hat{\nu}, \hat{\rho}_{i1}}(\xi).
\]

(67)

Then we use the concave property of \( G \) and \( \tilde{G} \) to arrive to the conclusion of the theorem.

Now we can use Theorem.1 and Theorem.?? to prove the following theorem:

**Theorem .3** Let be \( \hat{\mu} = \sum_{j=1}^n h_j \hat{x}_j \) and \( \hat{\nu} = \sum_{j=1}^n g_j \hat{p}_j \) two non-local operators such that \([\hat{\mu}, \hat{\nu}] = i \sum_{j=1}^n h_j g_j = 0 \) (where \( h_j, g_j \) are real numbers not all equal to zero), and let \( \hat{\rho}_{bs} \) be the biseparable state described in Eq.(64). For any uncertainty relation that can be written in the form:

\[
F[\hat{\rho}, P_{\hat{\mu}'}(\xi), P_{\hat{\nu}'}(\xi)] \geq G[\hat{\rho}, P_{\hat{\mu}'}(\xi)] + \tilde{G}[\hat{\rho}, P_{\hat{\nu}'}(\xi)] \geq f(|\gamma|)
\]

(68)

where \( \hat{\mu}' \) and \( \hat{\nu}' \) are a pair of operators such that \([\hat{\mu}', \hat{\nu}'] = i \gamma' \hat{1}, \) \( F, G \) and \( \tilde{G} \) are functionals with \( G \) and \( \tilde{G} \) concave with respect to the probabilities distributions, \( P_{\hat{\mu}'} \) and \( P_{\hat{\nu}'} \) respectively, and a strictly increasing function \( f \), then it is always true that:

\[
F[\hat{\rho}_{bs}, P_{\hat{\mu}}(\xi), P_{\hat{\nu}}(\xi)] \geq f(\gamma_{\min})
\]

(69)

where

\[
\gamma_{\min} = \min_{\{i\}} (|\gamma_i|) \geq 0,
\]

(70)

and \( \gamma_i = \pm \sum_{j=1}^n \tilde{g}_j h_j \) (\( \tilde{g}_j = -g_j \) if \( j \) is one component of the vector \( \tilde{l} \) or \( \tilde{g}_j = g_j \) otherwise). Note that if the probabilities \( p_{\{i\}} \) are all zero except one we recover Theorem.1.
Proof From Theorem?? and Theorem[1] we have:

$$F[\hat{\rho}_{bs}, P_\mu(\xi), P_\nu(\xi)] \geq \sum_{\{\tilde{l}\}} p_{\tilde{l}} \left( G[\hat{\rho}_{\tilde{l}}], P_{\hat{\mu}_{\tilde{l}}(\xi)}(\xi) \right) + \tilde{G}[P_{\hat{\nu}_{\tilde{l}}(\xi)}(\xi)] \geq \sum_{\{\tilde{l}\}} p_{\tilde{l}} f(|\gamma_{\tilde{l}}|) \geq \left( \sum_{\{\tilde{l}\}} p_{\tilde{l}} \right) f(|\gamma_{\text{min}}|) \geq f(|\gamma_{\text{min}}|),$$

(71)

where we use the fact that $f$ is a strictly increasing function and $\sum_{\{\tilde{l}\}} p_{\tilde{l}} = 1$.

Theorem .4 Consider a set of pairs $\{\hat{\mu}_m, \hat{\nu}_m\}$ of commuting non-local operators, i.e. $[\hat{\mu}_m, \hat{\nu}_m] = 0$ with $m = 1, \ldots, M$ ($M \leq L$ the total number of bipartitions of a n-mode continuous variable system) of the form $\hat{\mu}_m = \sum_{j=1}^{n} h_{mj} \hat{x}_j$ and $\hat{\nu}_m = \sum_{j=1}^{n} g_{mj} \hat{\rho}_j$ (where $h_{mj}, g_{mj}$ are real numbers not all equal to zero). Here, as in the theorems above, we labels the bipartition $\tilde{l}|\tilde{m}$ by the vectors of integer indexes, $\tilde{l} \equiv (l_1, \ldots, l_{n_A})$, $\tilde{m} \equiv (m_1, \ldots, m_{n_B})$, that indicate the modes belonging to each part, i.e. Alice’s part with $n_A$ modes and Bob’s part with $n_B = n - n_A$ modes (for convenience we order $l_i < l_{i+1}$, $m_i < m_{i+1}$, and $l_i, m_i$ are integers in the set $\{1, \ldots, n\}$). We also define: $\gamma_{m,\tilde{l}} = \sum_{j=1}^{n} \tilde{g}_{mj} h_{mj}$ ($\tilde{g}_{mj} = -g_{mj}$ if $j$ is one component of the vector $\tilde{l}$ or $\tilde{g}_{mj} = g_{mj}$ otherwise for all values of $m = 1, \ldots, M \leq L$). We consider that for every fixed value of $m$ the values of $\gamma_{m,\tilde{l}}$ are zero for a subset of the bipartition $\{\tilde{l}\}$. Thus, we denote the complementary subset where the values of $\gamma_{m,\tilde{l}}$ are not zero as $\{\tilde{l}\}_m$, so for every fixed value of $m$ we define $\gamma_{m,\text{min}} = \min_{\{\tilde{l}\}_m} \{|\gamma_{m,\tilde{l}}|\} > 0$ and $\tilde{\gamma}_{\text{min}} = \min_m \gamma_{m,\text{min}}$. If we consider uncertainty relations of the type used in Theorem[3] and if the set $\{\{\tilde{l}\}_m, m = 1, \ldots, M\}$ covers all possible bipartitions of the system (with possible repetitions), then it is always true that for all biseparable states $\hat{\rho}_{bs} \equiv \sum_{\{\tilde{l}\}} p_{\tilde{l}} \hat{\rho}_{\tilde{l}} = \sum_{\{\tilde{l}\}} p_{\tilde{l}} \left( \sum_j \eta_j \hat{\rho}_j \otimes \hat{\rho}_j^{m} \right)$:

$$\sum_{m=1}^{M} F[\hat{\rho}_{bs}, P_{\hat{\mu}_m}(\xi), P_{\hat{\nu}_m}(\xi)] \geq \Theta f(\tilde{\gamma}_{\text{min}}) \geq f(\tilde{\gamma}_{\text{min}}),$$

(73)

where $\Theta = \sum_{m=1}^{M} \sum_{\{\tilde{l}\}_m} p_{\tilde{l}}$. Note that $\Theta$ is an integer if all the sets $\{\{\tilde{l}\}_m, m = 1, \ldots, M\}$ cover all the bipartition’s of the system with $\Theta$ number of repetitions.
Proof According to Theorem?? and Theorem.1 we have:
\[ \sum_{m=1}^{M} F[\hat{\rho}_{bs}, P_{\hat{\mu}_m}(\xi), P_{\hat{\nu}_m}(\xi)] \geq \sum_{m=1}^{M} \sum_{\{\tilde{l}\}} p_{\{\tilde{l}\}_m} \left( G[\hat{\rho}_{\{\tilde{l}\}_m}, P_{\hat{\mu}_m,\hat{\nu}_m}(\xi)] + \tilde{G}[P_{\hat{\mu}_m,\hat{\nu}_m}(\xi)] \right) \geq (74) \]
\[ \geq \sum_{m=1}^{M} \sum_{\{\tilde{l}\}_m} p_{\{\tilde{l}\}_m} f(\gamma_{m,\{\tilde{l}\}_m}) \geq \sum_{m=1}^{M} \left( \sum_{\{\tilde{l}\}_m} p_{\{\tilde{l}\}_m} \right) f(\gamma_{m,\min}) \geq \Theta f(\tilde{\gamma}_{\min}) \geq f(\tilde{\gamma}_{\min}) (75) \]
where we use that \( f \) is strictly crescent and \( \Theta = \sum_{m=1}^{M} \sum_{\{\tilde{l}\}_m} p_{\{\tilde{l}\}_m} \geq 1 \).

Theorem .5 The differential entropy \( G[P(\xi)] \equiv -\int_S d\xi P(\xi) \ln(P(\xi)) \) \((S = (-\infty, \infty) \) is a strictly concave functional of \( P(\xi) \), i.e. if:
\[ P(\xi) = \sum_{m=1}^{M} p_m P_m(\xi), \]
then
\[ G[P(\xi)] \geq \sum_{m=1}^{M} p_m G[P_m(\xi)]. \]

Proof First we observe that
\[ P(\xi) = \sum_{m=1}^{M} p_m P_m(\xi) \geq P_m(\xi) \]
for all values of \( \xi \in S \) and \( 1 \leq m \leq M \). So, because the function \(-\ln(x)\) is strictly crescent we also have:
\[ -\ln(P(\xi)) > -\ln(P_m(\xi)), \]
for all values of \( \xi \in S \) and \( 1 \leq m \leq M \). Therefore, we immediately have:
\[ G[P(\xi)] = -\int_S d\xi \left( \sum_{m=1}^{M} p_m P_m(\xi) \right) \ln(P(\xi)) \geq \sum_{m=1}^{M} p_m \left( -\int_S d\xi P_m(\xi) \ln(P_m(\xi)) \right) = \sum_{m=1}^{M} p_m G[P_m(\xi)]. \]

Example of how to obtain the pair of “new” operators in Eq.(18) of the manuscript for the more unbalance bipartition class.

The more unbalance bipartition class is \((n_A = 1, n_B = n - 1)\) whose “seed” bipartition corresponds to \( \vec{\beta} = (1) \) and the rest to \( \vec{l} = (l_1) \) \((2 \leq l_1 \leq n) \). For the bipartition with \( \vec{\beta} = (1) \)
and for those with \( \vec{l} = (l_1) \) \((2 \leq l_1 \leq n - 1)\) we use the seed matrix such that \((\tilde{M}_{p,(1)})_{1j} = (-g, g, \ldots, g, g') = (\gamma/2)(1/h, -1/h, \ldots, -1/h, (n - 1)/h')\), and for the bipartition \( \vec{l} = (n) \) the seed matrix such that \((\tilde{M}'_{p,(1)})_{1j} = (-g', g, \ldots, g, g)\). Therefore, we have \((\tilde{M}_{x,(1)})_{1j} = (h, h, \ldots, h, h')\) and \((\tilde{M}'_{x,(1)})_{1j} = (h', h, \ldots, h)\). So for all bipartitions \( \vec{l} = (l_1) \) \((1 \leq l_1 \leq n - 1)\) we have different “old” position-type and momentum-type non-local operators \( \hat{u}_{1,\vec{a}}, \hat{v}_{1,\vec{a}} \) but all such that \([\hat{u}_{1,\vec{a}}, \hat{v}_{1,\vec{a}}] = [\hat{u}_{1,(1)}, \hat{v}_{1,(1)}] = i\gamma \) with \( \hat{u}_{1,(1)} = \hat{\mu} = h\hat{x}_1 + h\hat{x}_2 + \ldots + h'\hat{x}_n \) and \( \hat{v}_{1,(1)} = \hat{\nu} = -g\hat{p}_1 + g\hat{p}_2 + \ldots + g\hat{p}_{n-1} + g'\hat{p}_n \), and for the bipartition \( \vec{l} = (n) \) the “old” operators are \( \hat{u}_{n,\vec{a}} = \hat{u}' = h'\hat{x}_1 + h\hat{x}_2 + \ldots + h\hat{x}_n \) and \( \hat{v}_{n,\vec{a}} = \hat{v}' = -g'\hat{p}_1 + g'\hat{p}_2 + \ldots + g\hat{p}_{n-1} + g\hat{p}_n \), such that \([\hat{u}', \hat{v}'] = i\gamma(n) = -i(n - 2)\gamma \) (here remember that: \( \vec{\alpha} = \vec{l} = (l_1) \) or \( \vec{\alpha} = \vec{m} = (m_1, \ldots, m_n) \)).

Also, because for all \(2 \leq l_1 \leq n - 1\),

\[
(\tilde{M}_{p,(1)}(\begin{array}{c} 1 \end{array}, (1)(n)))_{1j} = (\tilde{M}'_{p,(1)}(\begin{array}{c} 1 \end{array}, (1)(l_1)))_{1j}, \tag{81a}
\]

\[
(\tilde{M}_{x,(1)}(\begin{array}{c} 1 \end{array}, (1)(n)))_{1j} = (\tilde{M}'_{x,(1)}(\begin{array}{c} 1 \end{array}, (1)(l_1)))_{1j} = (\tilde{M}_{x,(1)})_{1j}. \tag{81b}
\]

we have a single pair of “new” position-type and momentum-type non-local operators \( \hat{\mu} = \hat{\mu}_{n,\vec{a}} = \hat{\mu}_{l_1,\vec{a}} \) and \( \hat{\nu} = \hat{v}_{n,\vec{a}} = \hat{v}_{l_1,\vec{a}} \) respectively \((1 \leq l_1 \leq n - 1)\), with:

\[
\hat{\mu} = h\hat{x}_1 + h\hat{x}_2 + \ldots + h\hat{x}_{n-1} + h'\hat{x}_n, \tag{82a}
\]

\[
\hat{\nu} = \frac{\gamma}{2} \left( -\frac{\hat{p}_1}{h} - \frac{\hat{p}_2}{h} - \ldots - \frac{\hat{p}_{n-1}}{h} + \frac{(n - 1)\hat{p}_n}{h'} \right), \tag{82b}
\]

such that \([\hat{\mu}, \hat{\nu}] = 0\).

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[1] R. Jozsa and N. Linde, Proc. R. Soc. Lond. A 459, 2011 (2003).
[2] R. Raussendorf and H. J. Briegel, Phys. Lett. Lett. 86, 5188 (2001).
[3] S. Schauer, M. Huber, and B. C. Hiesmayr, Phys. Rev. A 82, 062311 (2010).
[4] V. Giovannetti, S. Lloyd, and L. Maccone, Science 306, 1330 (2004).
[5] R. Neigovzen, C. Rodó, G. Adesso, and A. Sanpera, Phys. Rev. A 77, 062307 (2008).
[6] Z. Zhao, Y.-A. Chen, A.-N. Zhang, T. Yang, H. J. Briegel, and J.-W. Pan, Nature 430, 54 (2004).
[7] M. Sarovar, A. Ishizaki, G. R. Fleming, and K. B. Whaley, Nature Physics 6, 462 (2010).
[8] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, England, 1999).

[9] T. Aoki, N. Takei, H. Yonezawa, K. Wakui, T. Hiraoka, A. Furusawa, and P. van Loock, Phys. Rev. Lett. **91**, 080404 (2003).

[10] A. S. Coelho, F. A. S. Barbosa, K. N. Cassemeiro, A. S. Villar, M. Martinelli, and P. Nussenzveig, Science **326**, 823 (2009).

[11] S. Yokoyama, R. Ukai, S. C. Armstrong, C. Sornphiphatphong, T. Kaji, S. Suzuki, J.-i. Yoshikawa, H. Yonezawa, N. C. Menicucci, and A. Furusawa, Nature Photonics **7**, 982 (2013).

[12] M. Chen, N. C. Menicucci, and O. Pfister, Physical Review Letters **112**, 120505 (2014).

[13] R. Simon, Phys. Rev. Lett. **84**, 2726 (2000).

[14] L.-M. Duan, G. Giedke, J. I. Cirac, and P. Zoller, Phys. Rev. Lett. **84**, 2722 (2000).

[15] S. Mancini, V. Giovannetti, D. Vitali, and P. Tombesi, Physical Review Letters **88**, 120401 (2002).

[16] E. Shchukin and W. Vogel, Phys. Rev. Lett. **95**, 230502 (2005).

[17] G. S. Agarwal and A. Biswas, New Journal of Physics pp. 1–7 (2005).

[18] M. Hillery and M. S. Zubairy, Phys. Rev. Lett. **96**, 050503 (2006).

[19] S. P. Walborn, B. G. Taketani, A. Salles, F. Toscano, and R. L. de Matos Filho, Phys. Rev. Lett. **103**, 160505 (2009).

[20] A. Saboia, F. Toscano, and S. P. Walborn, Phys. Rev. A **83**, 032307 (2011).

[21] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).

[22] P. H. M. Horodecki and R. Horodecki, Phys. Lett. A **223**, 1 (1996).

[23] A. Miranowicz, M. Piani, P. Horodecki, and R. Horodecki, Physical Review A **80**, 052303 (2009).

[24] P. van Loock and A. Furusawa, Phys. Rev. A **67**, 052315 (2003).

[25] M. D. Reid, arXiv.org>quant-ph>arXiv:1310.2690 (2013), URL http://arxiv.org/abs/1310.2690

[26] J.-D. Bancal, N. Gisin, Y.-C. Liang, and S. Pironio, Phys. Rev. Lett. **106**, 250404 (2011), URL http://link.aps.org/doi/10.1103/PhysRevLett.106.250404

[27] E. Shchukin and W. Vogel, Physical Review A **74**, 030302 (2006).

[28] I. Bialynicki-Birula and J. Mycielski, Commun. Math. Phys. **44**, 129 (1975).

[29] Y. Huang, Physical Review A **83**, 052124 (2011).

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In the Supplementary Material we show, for example, that the Shannon entropy $h[P]$ is indeed concave.

For example, for $n = 3$ we have only one bipartition class, i.e., $(n_A = 1, n_B = 2)$, thus we only have one seed matrix $\Lambda_1 = \text{diag}(-1, 1, 1)$. Then we can write, for $n = 3$, all the matrices in Eq. (9) as: $\Lambda_{\vec{\alpha}} = \pm P_{l_1} \Lambda_1 P_{l_1}$ with $l_1 = 1, 2, 3$.

The factor of 2 of our lower bound $\gamma = 2/(n - 1)$ do not appear in [24] because they define the canonical commutation relation as $[\hat{x}_j, \hat{p}_k] = i\delta_{jk}/2$ and not $[\hat{x}_j, \hat{p}_k] = i\delta_{jk}$ as we do. Thus, their lower bound is $\gamma' = \gamma/2$. 