Collaborative Deep Learning in Fixed Topology Networks

Zhanhong Jiang$^1$, Aditya Balu$^1$, Chinmay Hegde$^2$, and Soumik Sarkar$^1$

$^1$Department of Mechanical Engineering, Iowa State University, zhjiang, baditya, soumiks@iastate.edu
$^2$Department of Electrical and Computer Engineering, Iowa State University, chinmay@iastate.edu

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Abstract

There is significant recent interest to parallelize deep learning algorithms in order to handle the enormous growth in data and model sizes. While most advances focus on model parallelization and engaging multiple computing agents via using a central parameter server, aspect of data parallelization along with decentralized computation has not been explored sufficiently. In this context, this paper presents a new consensus-based distributed SGD (CDSGD) (and its momentum variant, CDMSGD) algorithm for collaborative deep learning over fixed topology networks that enables data parallelization as well as decentralized computation. Such a framework can be extremely useful for learning agents with access to only local/private data in a communication constrained environment. We analyze the convergence properties of the proposed algorithm with strongly convex and nonconvex objective functions with fixed and diminishing step sizes using concepts of Lyapunov function construction. We demonstrate the efficacy of our algorithms in comparison with the baseline centralized SGD and the recently proposed federated averaging algorithm (that also enables data parallelism) based on benchmark datasets such as MNIST, CIFAR-10 and CIFAR-100.

1 Introduction

In this paper, we address the scalability of optimization algorithms for deep learning in a distributed setting. Scaling up deep learning is becoming increasingly crucial for large-scale applications where the sizes of both the available data as well as the models are massive. Among various algorithmic advances, many recent attempts have been made to parallelize stochastic gradient descent (SGD) based learning schemes across multiple computing agents. An early approach called Downpour SGD, developed within Google’s disbeliefs software framework, primarily focuses on model parallelization (i.e., splitting the model across the agents). A different approach known as elastic averaging SGD (EASGD) attempts to improve perform multiple SGDs in parallel; this method uses a central parameter server that helps in assimilating parameter updates from the computing agents. However, none of the above approaches concretely address the issue of data parallelization, which is an important issue for several learning scenarios: for example, data parallelization enables privacy-preserving learning in scenarios such as distributed learning with a network of mobile and Internet-of-Things (IoT) devices. A recent scheme called Federated Averaging SGD attempts such a data parallelization in the context of deep learning with significant success; however, they still use a central parameter server.

In contrast, deep learning with decentralized computation can be achieved via gossip SGD algorithms, where agents communicate probabilistically without the aid of a parameter server. However, decentralized
Table 1: Comparisons between different optimization approaches

| Method        | $f$     | $\nabla f$ | Step Size | Con.Rate | D.P. | D.C. | C.C.T. |
|---------------|---------|-----------|-----------|----------|------|------|--------|
| SGD           | Str-con | Lip.      | Con.      | $\mathcal{O}(\gamma^k)$ | No   | No   | No     |
| Downpour SGD  | Nonconvex | Lip. | Con.&Ada.  | N/A      | Yes  | No   | No     |
| EASGD         | Str-con | Lip.      | Con.      | $\mathcal{O}(\gamma^k)$ | No   | No   | No     |
| Gossip SGD    | Str-con | Lip.&Bou. | Dim.      | $\mathcal{O}(\epsilon^k)$ | No   | Yes  | No     |
| FedAvg        | Nonconvex | Lip.    | Con.      | N/A      | Yes  | No   | No     |
| CDSGD [This paper] | Str-con | Lip.&Bou. | Dim. | $\mathcal{O}(\gamma^k)$ | Yes  | Yes  | Yes    |
|               | Nonconvex | Lip.&Bou. | Dim. | N/A | Yes  | No   | No     |

Con.Rate: convergence rate, Str-con: strongly convex, Lip.&Bou.: Lipschitz continuous and bounded, Con.: constant and Con.&Ada.: constant&adagrad, Dim.: diminishing. $\gamma \in (0,1)$ is a positive constant. $\epsilon \in (0.5,1]$ is a positive constant. D.P.: data parallelism. D.C.: decentralized computation. C.C.T.: constrained communication topology.

computation in the sense of gossip SGD is not feasible in many real life applications. For instance, consider a large (wide-area) sensor network [8, 9] or multi-agent robotic network that aims to learn a model of the environment in a collaborative manner [10, 11]. For such cases, it may be infeasible for arbitrary pairs of agents to communicate on-demand; typically, agents are only able to communicate with their respective neighbors in a communication network in a fixed (or evolving) topology.

**Contribution:** This paper introduces a new class of approaches for deep learning that enables both data parallelization and decentralized computation. Specifically, we propose consensus-based distributed SGD (CDSGD) and consensus-based distributed momentum SGD (CDMSGD) algorithms for collaborative deep learning that, for the first time, satisfies all three requirements: data parallelization, decentralized computation, and constrained communication over fixed topology networks. Moreover, while most existing studies solely rely on empirical evidence from simulations, we present rigorous convergence analysis for both (strongly) convex and non-convex objective functions, with both fixed and diminishing step sizes using a Lyapunov function construction approach. Our analysis reveals several advantages of our method: we match the best existing rates of convergence in the centralized setting, while simultaneously supporting data parallelism as well as constrained communication topologies; to our knowledge, this is the first approach that achieves all three desirable properties; see Table 1 for a detailed comparison.

Finally, we validate our algorithms’ performance on benchmark datasets, such as MNIST, CIFAR-10, and CIFAR-100. Apart from centralized SGD as a baseline, we also compare performance with that of Federated Averaging SGD as it also enables data parallelization. Empirical evidence (for a given number of agents and other hyperparametric conditions) suggests that while our method is slightly slower, we can achieve higher accuracy compared to the best available algorithm (Federated Averaging (FedAvg)).

**Related work:** Apart from the algorithms mentioned above, a few other related works exist, including a distributed system called Adam for large deep neural network (DNN) models [12] and a distributed methodology by Strom [13] for DNN training by controlling the rate of weight-update to reduce the amount of communication. Natural Gradient Stochastic Gradient Descent (NG-SGD) based on model averaging [14] and staleness-aware async-SGD [15] have also been developed for distributed deep learning. A method called CentralVR [16] was proposed for reducing the variance and conducting parallel execution with linear convergence rate. Moreover, a decentralized algorithm based on gossip protocol called the multi-step dual accelerated (MSDA) [17] was developed for solving deterministically smooth and strongly convex distributed optimization problems in networks with a provable optimal linear convergence rate. A new class of decentralized primal-dual methods [18] was also proposed recently in order to improve inter-node communication efficiency for distributed convex optimization problems. To minimize a finite sum of nonconvex functions over a network, the authors in [19] proposed a zeroth-order distributed algorithm (ZENITH) that was globally convergent with a sublinear rate. From the perspective of distributed optimization, the proposed algorithms
have similarities with the approaches of [20] [21]. However, we distinguish our work due to the collaborative learning aspect with data parallelization and extension to the stochastic setting and nonconvex objective functions. In [20] the authors only considered convex objective functions in a deterministic setting, while the authors in [21] presented results for non-convex optimization problems in a deterministic setting.

The rest of the paper is organized as follows. While section 2 formulates the distributed, unconstrained stochastic optimization problem, section 3 presents the CD SGD algorithm and the Lyapunov stochastic gradient required for analysis presented in section 4. Validation experiments and performance comparison results are described in section 5. The paper is summarized, concluded in section 6 along with future research directions. Detailed proofs of analytical results, extensions (e.g., effect of diminishing step size) and additional experiments are included in the supplementary section 7.

2 Formulation

We consider the standard (unconstrained) empirical risk minimization problem typically used in machine learning problems (such as deep learning):

$$\min f(x) := \frac{1}{n} \sum_{i=1}^{n} f^i(x),$$

where $x \in \mathbb{R}^d$ denotes the parameter of interest and $f : \mathbb{R}^d \to \mathbb{R}$ is a given loss function, and $f^i$ is the function value corresponding to a data point $i$. In this paper, we are interested in learning problems where the computational agents exhibit data parallelism, i.e., they only have access to their own respective training datasets. However, we assume that the agents can communicate over a static undirected graph $G = (V, E)$, where $V$ is a vertex set (with nodes corresponding to agents) and $E$ is an edge set. With $N$ agents, we have $V = \{1, 2, ..., N\}$ and $E \subseteq V \times V$. If $(j, l) \in E$, then Agent $j$ can communicate with Agent $l$. The neighborhood of agent $j \in V$ is defined as: $Nb(j) \triangleq \{l \in V : (j, l) \in E \text{ or } j = l\}$. Throughout this paper we assume that the graph $G$ is connected. Let $\mathcal{D}_j$, $j = 1, \ldots, n$ denote the subset of the training data (comprising $n_j$ samples) corresponding to the $j$th agents such that $\sum_{j=1}^{N} n_j = n$. With this setup, we have the following simplification of Eq. (1)

$$\min f(x) := \frac{1}{n} \sum_{j=1}^{N} \sum_{i \in \mathcal{D}_j} f^i(x) = \frac{N}{n} \sum_{j=1}^{N} \sum_{i \in \mathcal{D}_j} f^i_j(x),$$

where, $f_j(x) = \frac{1}{N} f(x)$ is the objective function specific to Agent $j$. This formulation enables us to state the optimization problem in a distributed manner, where $f(x) = \sum_{j=1}^{N} f_j(x)$. Furthermore, the problem (1) can be reformulated as

$$\min_{\mathbf{x}} \frac{N}{n} \mathbf{1}^T \mathbf{F}(\mathbf{x}) := \frac{N}{n} \sum_{j=1}^{N} \sum_{i \in \mathcal{D}_j} f^i_j(x^j)$$

s.t. $x^j = x^l \ \forall (j, l) \in \mathcal{E},$

where $\mathbf{x} := (x^1, x^2, \ldots, x^N)^T \in \mathbb{R}^{N \times d}$ and $\mathbf{F}(\mathbf{x})$ can be written as

$$\mathbf{F}(\mathbf{x}) = \left[ \sum_{i \in \mathcal{D}_1} f^i_1(x^1), \sum_{i \in \mathcal{D}_2} f^i_2(x^2), \ldots, \sum_{i \in \mathcal{D}_N} f^i_N(x^N) \right]^T$$

Note that with $d > 1$, the parameter set $\mathbf{x}$ as well as the gradient $\nabla \mathbf{F}(\mathbf{x})$ correspond to matrix variables. However, for simplicity in presenting our analysis, we set $d = 1$ in this paper, which corresponds to the case where $\mathbf{x}$ and $\nabla \mathbf{F}(\mathbf{x})$ are vectors.

1Note that in our formulation, we are assuming that every agent has the same local objective function while in general distributed optimization problems they can be different.
We now introduce several key definitions and assumptions that characterize the objective functions and the interaction matrix.

**Definition 1.** A function $f : \mathbb{R}^d \to \mathbb{R}$ is $H$-strongly convex, if for all $x, y \in \mathbb{R}^d$, we have $f(y) \geq f(x) + \nabla f(x)^T (y - x) + \frac{H}{2} \|y - x\|^2$.

**Definition 2.** A function $f : \mathbb{R}^d \to \mathbb{R}$ is $\gamma$-smooth if for all $x, y \in \mathbb{R}^d$, we have $f(y) \leq f(x) + \nabla f(x)^T (y - x) + \frac{\gamma}{2} \|y - x\|^2$.

As a consequence of Definition 2, we can conclude that $\nabla f$ is Lipschitz continuous, i.e., $\|\nabla f(y) - \nabla f(x)\| \leq \gamma \|y - x\|$ [22].

**Definition 3.** A function $c$ is said to be coercive if it satisfies: $c(x) \to \infty$ when $\|x\| \to \infty$.

**Assumption 1.** The objective functions $f_j : \mathbb{R}^d \to \mathbb{R}$ are assumed to satisfy the following conditions: a) Each $f_j$ is $\gamma_j$-smooth; b) each $f_j$ is proper (not everywhere infinite) and coercive; and c) each $f_j$ is $L_j$-Lipschitz continuous, i.e., $|f_j(y) - f_j(x)| < L_j \|y - x\| \forall x, y \in \mathbb{R}^d$.

As a consequence of Assumption 1, we can conclude that $\sum_{j=1}^{N} f_j(x^j)$ possesses Lipschitz continuous gradient with parameter $\gamma_m := \max_j \gamma_j$. Similarly, each $f_j$ is strongly convex with $H_j$ such that $\sum_{j=1}^{N} f_j(x^j)$ is strongly convex with $H_m = \min_j H_j$.

Regarding the communication network, we use $\Pi$ to denote the agent interaction matrix, where the element $\pi_{jl}$ signifies the link weight between agents $j$ and $l$.

**Assumption 2.** a) If $(j, l) \notin \mathcal{E}$, then $\pi_{jl} = 0$; b) $\mathbf{1}^T \Pi = \mathbf{1}^T$, $\Pi \mathbf{1} = \mathbf{1}$; c) $\ker(I - \Pi) = \text{span}\{\mathbf{1}\}$; and d) $I \succeq \Pi \succ 0$.

The main outcome of Assumption 2 is that the probability transition matrix is doubly stochastic and that we have $\lambda_1(\Pi) = 1 > \lambda_2(\Pi) \geq \cdots \geq \lambda_N(\Pi) > 0$, where $\lambda_z(\Pi)$ denotes the $z$-th largest eigenvalue of $\Pi$.

### 3 Proposed Algorithm

#### 3.1 Consensus Distributed SGD

For solving stochastic optimization problems, SGD and its variants have been commonly used to centralized and distributed problem formulations. Therefore, the following algorithm is proposed based on SGD and the concept of consensus to solve the problem laid out in Eq. 2.

\[ x_{k+1}^j = \sum_{l \in \mathcal{N}(j)} \pi_{jl} x_k^l - \alpha g_j(x_k^j) \tag{5} \]

where $\mathcal{N}(j)$ indicates the neighborhood of agent $j$, $\alpha$ is the step size, $g_j(x_k^j)$ is stochastic gradient of $f_j$ at $x_k^j$. While the pseudo-code of CDSGD is shown below in Algorithm 1, momentum versions of CDSGD based on Polyak momentum [23] and Nesterov momentum [24] are also presented in the supplementary section. Note, mini-batch implementations of these algorithms are straightforward, hence, are not discussed here in detail.

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**Algorithm 1: CDSGD**

| **Input** | $m, \alpha, N$ |
|-----------|----------------|
| **Initialize:** $x_0^j$ ($j = 1, 2, \ldots, N$) |
| **Distribute** the training dataset to $N$ agents. |
| **for each agent do** |
| for $k = 0 : m$ do |
| Randomly shuffle the corresponding data subset $\mathcal{D}_j$ |
| $w_{k+1}^j = \sum_{l \in \mathcal{N}(j)} \pi_{jl} x_k^l$ |
| $x_{k+1}^j = w_{k+1}^j - \alpha g_j(x_k^j)$ |
| end |
| end |

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3.2 Tools for convergence analysis

We now analyze the convergence properties of the iterates \{x_k^i\} generated by Algorithm 1. The following section summarizes some key intermediate concepts required to establish our main results.

First, we construct an appropriate Lyapunov function that will enable us to establish convergence. Observe that the update law in Alg. 1 can be expressed as:

\[
x_{k+1} = \Pi x_k - \alpha g(x_k),
\]

where

\[
g(x_k) = [g_1(x_k^1)g_2(x_k^2) \ldots g_N(x_k^N)]^T
\]

Denoting \(w_k = \Pi x_k\), the update law can be re-written as \(x_{k+1} = w_k - \alpha g(x_k)\). Moreover, \(x_{k+1} = x_k - \alpha (g(x_k) + \alpha^{-1}(I - \Pi)x_k)\). Rearranging the last equality yields the following relation:

\[
x_{k+1} = x_k - \alpha (g(x_k) + \alpha^{-1}(I - \Pi)x_k) = x_k - \alpha (g(x_k) + \alpha^{-1}(I - \Pi)x_k)
\]

where the last term in Eq. 7 is the Stochastic Lyapunov Gradient. From Eq. 7 we observe that the “effective” gradient step is given by \(g(x_k) + \alpha^{-1}(I - \Pi)x_k\). Rewriting \(\nabla J^i(x_k) = g(x_k) + \alpha^{-1}(I - \Pi)x_k\), the updates of CDSGD can be expressed as:

\[
x_{k+1} = x_k - \alpha \nabla J^i(x_k).
\]

The above expression naturally motivates the following Lyapunov function candidate:

\[
V(x, \alpha) := \frac{N \cdot 1^T F(x) + 1}{2\alpha} \|x\|^2_{I - \Pi}
\]

where \(\|\cdot\|_{I - \Pi}\) denotes the norm with respect to the PSD matrix \(I - \Pi\). Since \(\sum_{j=1}^N f_j(x^j)\) has a \(\gamma_m\)-Lipschitz continuous gradient, \(\nabla V(x)\) also has a Lipschitz continuous gradient with parameter:

\[
\hat{\gamma} := \gamma_m + \alpha^{-1}\lambda_{\max}(I - \Pi) = \gamma_m + \alpha^{-1}(1 - \lambda_N(\Pi)).
\]

Similarly, as \(\sum_{j=1}^N f_j(x^j)\) is \(H_m\)-strongly convex, then \(V(x)\) is strongly convex with parameter:

\[
\hat{H} := H_m + (2\alpha)^{-1}\lambda_{\min}(I - \Pi) = H_m + (2\alpha)^{-1}(1 - \lambda_2(\Pi)).
\]

Based on Definition 1 \(V\) has a unique minimizer, denoted by \(x^*\) with \(V^* = V(x^*)\). Correspondingly, using strong convexity of \(V\), we can obtain the relation:

\[
2\hat{H}(V(x) - V^*) \leq \|\nabla V(x)\|^2 \text{ for all } x \in \mathbb{R}^N.
\]

From strong convexity and the Lipschitz continuous property of \(\nabla f_j\), the constants \(H_m\) and \(\gamma_m\) further satisfy \(H_m \leq \gamma_m\) and hence, \(\hat{H} \leq \hat{\gamma}\).

Next, we introduce two key lemmas that will help establish our main theoretical guarantees. Due to space limitations, all proofs are deferred to the supplementary material in Section 7.

**Lemma 1.** Under Assumptions 1 and 2 the iterates of CDSGD satisfy \(\forall k \in \mathbb{N}\):

\[
E[V(x_{k+1})] - V(x_k) \leq -\alpha \nabla V(x_k)^T E[\nabla J^i(x_k)] + \frac{\hat{\gamma}}{2} \alpha^2 E[\|\nabla J^i(x_k)\|^2]
\]

At a high level, since \(E[\nabla J^i(x_k)]\) is the unbiased estimate of \(\nabla V(x_k)\), using the updates \(\nabla J^i(x_k)\) will lead to sufficient decrease in the Lyapunov function. However, unbiasedness is not enough, and we also need to control higher order moments of \(\nabla J^i(x_k)\) to ensure convergence. Specifically, we consider the variance of \(\nabla J^i(x_k)\):

\[
\text{Var}[\nabla J^i(x_k)] := E[\|\nabla J^i(x_k)\|^2] - \|E[\nabla J^i(x_k)]\|^2
\]

To bound the variance of \(\nabla J^i(x_k)\), we use a standard assumption presented in [25] in the context of (centralized) deep learning. Such an assumption aims at providing an upper bound for the “gradient noise” caused by the randomness in the minibatch selection at each iteration.
Algorithm 1 satisfies \( \forall s \) and discussed in Lemma 4 in the supplementary section 7. Assumption 3.

\[
\alpha a) \text{There exist scalars } \zeta_2 \geq \zeta_1 > 0 \text{ such that } \nabla V(x_k)^T \mathbb{E}[\nabla J^i(x_k)] \geq \zeta_1 \| \nabla V(x_k) \|^2 \text{ and } \mathbb{E}[\nabla J^i(x_k)] \leq \zeta_2 \| \nabla V(x_k) \| \text{ for all } k \in \mathbb{N}; \\
b) \text{There exist scalars } Q \geq 0 \text{ and } Q_V \geq 0 \text{ such that } \\
\text{Var}[\nabla J^i(x_k)] \leq Q + Q_V \| \nabla V(x_k) \|^2 \text{ for all } k \in \mathbb{N}.
\]

Remark 1. While Assumption 3(a) guarantees the sufficient descent of \( V \) in the direction of \(-\nabla J^i(x_k)\), Assumption 3(b) states that the variance of \( \nabla J^i(x_k) \) is bounded above by the second moment of \( \nabla V(x_k) \).

The constant \( Q \) can be considered to represent the second moment of the “gradient noise” in \( \nabla J^i(x_k) \).

Therefore, the second moment of \( \nabla J^i(x_k) \) can be bounded above as \( \mathbb{E}[\| \nabla J^i(x_k) \|^2] \leq Q + Q_m \| \nabla V(x_k) \|^2 \), where \( Q_m := Q_V + \zeta_2^2 \geq \zeta_2^2 > 0 \).

Lemma 2. Under Assumptions 3 and 4 the iterates of CDSGD satisfy \( \forall k \in \mathbb{N} \):

\[
\mathbb{E}[V(x_{k+1})] - V(x_k) \leq - (\zeta_1 - \frac{\hat{\gamma}}{2} \alpha Q_m) \alpha \| \nabla V(x_k) \|^2 + \frac{\hat{\gamma}}{2} \alpha^2 Q.
\]  

(13)

In Lemma 2 the first term is strictly negative if the step size satisfies the following necessary condition:

\[
0 < \alpha \leq \frac{2 \zeta_1}{\gamma Q_m}
\]  

(14)

However, in latter analysis, when such a condition is substituted into the convergence analysis, it may produce a larger upper bound. For obtaining a tight upper bound, we impose a sufficient condition for the rest of analysis as follows:

\[
0 < \alpha \leq \frac{\zeta_1}{\gamma Q_m}
\]  

(15)

As \( \hat{\gamma} \) is a function of \( \alpha \), the above inequality can be rewritten as \( 0 < \alpha \leq \frac{\zeta_1 - (1 - \lambda_2(\Pi)) \gamma Q_m}{\gamma Q_m} \).

4 Main Results

We now present our main theoretical results establishing the convergence of CDSGD. First, we show that for most generic loss functions (whether convex or not), CDSGD achieves consensus across different agents in the graph, provided the step size (which is fixed across iterations) does not exceed a natural upper bound.

Proposition 1. (Consensus with fixed step size) Under Assumptions 1 and 2, the iterates of CDSGD (Algorithm 1) satisfy \( \forall k \in \mathbb{N} \):

\[
\mathbb{E}[\| x_k^l - s_k \|] \leq \frac{\alpha L}{1 - \lambda_2(\Pi)}
\]  

(16)

where \( \alpha \) satisfies \( 0 < \alpha \leq \frac{\zeta_1 - (1 - \lambda_2(\Pi)) \gamma Q_m}{\gamma Q_m} \) and \( L \) is an upper bound of \( \mathbb{E}[\| g(x_k) \|], \forall k \in \mathbb{N} \) (defined properly and discussed in Lemma 4 in the supplementary section 1) and \( s_k = \frac{1}{N} \sum_{j=1}^{N} x_k^j \) represents the average parameter estimate.

The proof of this proposition can be adapted from [26, Lemma 1].

Next, we show that for strongly convex loss functions, CDSGD converges linearly to a neighborhood of the global optimum.

Theorem 1. (Convergence of CDSGD with fixed step size, strongly convex case) Under Assumptions 1, 2, and 3 the iterates of CDSGD satisfy the following inequality \( \forall k \in \mathbb{N} \):

\[
\mathbb{E}[V(x_k) - V^*] \leq (1 - \alpha \hat{H} \zeta_1)^{k-1} (V(x_1) - V^*) + \frac{\alpha^2 \hat{\gamma} Q}{2} \sum_{l=0}^{k-1} (1 - \alpha \hat{H} \zeta_1)^l
\]

\[
= (1 - (\alpha H_m + 1 - \lambda_2(\Pi)) \zeta_1)^{k-1} (V(x_1) - V^*)
\]

\[
+ \frac{\alpha^2 \gamma_m + \alpha(1 - \lambda_2(\Pi)) \zeta_1}{2} \sum_{l=0}^{k-1} (1 - (\alpha H_m + 1 - \lambda_2(\Pi)) \zeta_1)^l
\]  

(17)

when the step size satisfies \( 0 < \alpha \leq \frac{\zeta_1 - (1 - \lambda_2(\Pi)) \gamma Q_m}{\gamma Q_m} \).
A detailed proof is presented in the supplementary section 7. We observe from Theorem 1 that the sequence of Lyapunov function values \( \{V(x_k)\} \) converges linearly to a neighborhood of the optimal value, i.e., \( \lim_{k \to \infty} E[V(x_k)] \leq \frac{\alpha \gamma_m + 1 - \lambda_N(\Pi)Q}{2H_{\gamma_1}} \). We also observe that the term on the right hand side decreases with the spectral gap of the agent interaction matrix \( \Pi, \) i.e., \( 1 - \lambda_2(\Pi), \) which suggests an interesting relation between convergence and topology of the graph. Moreover, we observe that the upper bound is proportional to the step size parameter \( \alpha, \) and smaller step sizes lead to smaller radii of convergence. (However, choosing a very small step-size may negatively affect the convergence rate of the algorithm). Finally, if the gradient in this context is not stochastic (i.e., the parameter \( Q = 0 \)), then linear convergence to the optimal value is achieved, which matches known rates of convergence with (centralized) gradient descent under strong convexity and smoothness assumptions.

**Remark 2.** Since \( E[\frac{N}{n} 1_T^T F(x_k)] \leq E[V(x_k)] \) and \( \frac{N}{n} 1^T F(x^*) = V^*, \) the sequence of objective function values are themselves upper bounded as follows: \( E[\frac{N}{n} 1^T F(x_k) - \frac{N}{n} 1^T F(x^*)] \leq E[V(x_k) - V^*] \). Therefore, using Theorem 1 we can establish analogous convergence rates in terms of the true objective function values \( \{\frac{N}{n} 1^T F(x_k)\} \) as well.

The above convergence result for CDSGD is limited to the case when the objective functions are strongly convex. However, most practical deep learning systems (such as convolutional neural network learning) involve optimizing over highly non-convex objective functions, which are much harder to analyze. Nevertheless, we show that even under such situations, CDSGD exhibits a (weaker) notion of convergence.

**Theorem 2.** (Convergence of CDSGD with fixed step size, nonconvex case) Under Assumptions 2 and 3, the iterates of CDSGD satisfy \( \forall m \in \mathbb{N} \):

\[
E[\sum_{k=1}^{m} \|\nabla V(x_k)\|^2] \leq \frac{\tilde{\gamma} m \alpha Q}{\zeta_1} + \frac{2(V(x_1) - V_{inf})}{\zeta_1 \alpha} = \frac{(\gamma m \alpha + 1 - \lambda_N(\Pi))mQ}{\zeta_1} + \frac{2(V(x_1) - V_{inf})}{\zeta_1 \alpha}.
\]

(18)

where the step size satisfies \( 0 < \alpha \leq \frac{\zeta_1 - (1 - \lambda_N(\Pi))Q}{\gamma_m Q_m \zeta_1}. \)

**Remark 3.** Theorem 2 states that when in the absence of "gradient noise" (i.e., when \( Q = 0 \)), the quantity \( E[\sum_{k=1}^{m} \|\nabla V(x_k)\|^2] \) remains finite. Therefore, necessarily \( \{\|\nabla V(x_k)\|\} \to 0 \) and the estimates approach a stationary point. On the other hand, if the gradient calculations are stochastic, then a similar claim cannot be made. However, for this case we have the upper bound \( \lim_{m \to \infty} E[\frac{1}{m} \sum_{k=1}^{m} \|\nabla V(x_k)\|^2] \leq \frac{(\gamma m \alpha + 1 - \lambda_N(\Pi))Q}{\zeta_1 \alpha}. \)

This tells us that while we cannot guarantee convergence in terms of sequence of objective function values, we can still assert that the average of the second moment of gradients is strictly bounded from above even for the case of nonconvex objective functions.

Moreover, the upper bound cannot be solely controlled via the step-size parameter \( \alpha \) (which is different from what is implied in the strongly convex case by Theorem 1). In general, the upper bound becomes tighter as \( \lambda_N(\Pi) \) increases; however, an increase in \( \lambda_N(\Pi) \) may result in a commensurate increase in \( \lambda_2(\Pi), \) leading to worse connectivity in the graph and adversely affecting consensus among agents. Again, our upper bounds are reflective of interesting tradeoffs between consensus and convergence in the gradients, and their dependence on graph topology.

The above results are for fixed step size \( \alpha \), and we can prove complementary results for CDSGD even for the (more prevalent) case of diminishing step size \( \alpha_k \). These are presented in the supplementary material due to space constraints.
5 Experimental Results

This section presents the experimental results using the benchmark image recognition dataset, CIFAR-10. We use a deep convolutional neural network (CNN) model (with 2 convolutional layers with 32 filters each followed by a max pooling layer, then 2 more convolutional layers with 64 filters each followed by another max pooling layer and a dense layer with 512 units, ReLU activation is used in convolutional layers) to validate the proposed algorithm. We use a fully connected topology with 5 agents and uniform agent interaction matrix except mentioned otherwise. A mini-batch size of 128 and a fixed step size of 0.01 are used in these experiments. The experiments are performed using Keras and TensorFlow [27, 28] and the codes will be made publicly available soon. While we included the training and validation accuracy plots for the different case studies here, the corresponding training loss plots, results with other benchmark datasets such as MNIST and CIFAR-100 and decaying as well as different fixed step sizes are presented in the supplementary section.

5.1 Performance comparison with benchmark methods

We begin with comparing the accuracy of CDSGD with that of the centralized SGD algorithm as shown in Fig. 1(a). While the CDSGD convergence rate is significantly slower compared to SGD as expected, it is observed that CDSGD can eventually achieve high accuracy, comparable with centralized SGD. However, another interesting observation is that the generalization gap (the difference between training and validation accuracy as defined in [29]) for the proposed CDSGD algorithm is significantly smaller than that of SGD which is an useful property. We also compare both CDSGD and CDMSGD with the Federated averaging SGD (FedAvg) algorithm which also performs data parallelization (see Fig. 1(b)). For the sake of comparison, we use same number of agents and choose $E = 1$ and $C = 1$ as the hyperparameters in the FedAvg algorithm as it is close to a fully connected topology scenario as considered in the CDSGD and CDMSGD experiments. As CDSGD is significantly slow, we mainly compare the CDMSGD with FedAvg which have similar convergence rates (CDMSGD being slightly slower). The main observation is that CDMSGD performs better than FedAvg at the steady state and can achieve centralized SGD level performance. It is important to note that FedAvg does not perform decentralized computation. Essentially it runs a brute force parameter averaging on a central parameter server at every epoch (i.e., consensus at every epoch) and then broadcasts the updated parameters to the agents. Hence, it tends to be slightly faster than CDMSGD which uses a truly decentralized computation over a network.

5.2 Effect of network size and topology

In this section, we investigate the effects of network size and topology on the performance of the proposed algorithms. Figure 2(a) shows the change in training performance as the number of agents grow from 2 to 8 and to 16. Although with increase in number of agents, the convergence rate slows down, all networks are able to achieve similar accuracy levels. Finally, we investigate the impact of network sparsity (as quantified...
Figure 2: Average training (solid lines) and validation (dash lines) accuracy along with accuracy variance over agents for CDMSGD algorithm with (a) varying network size and (b) varying network topology by the second largest eigenvalue) on the learning performance. The primary observation is convergence of average accuracy value happens faster for sparser networks (higher second largest eigenvalue). This is similar to the trend observed for FedAvg algorithm while reducing the Client fraction ($C$) which makes the (stochastic) agent interaction matrix sparser. However, from the plot of the variance of accuracy values over agents (a smooth version using moving average filter), it can be observed that the level of consensus is more stable for denser networks compared to that for sparser networks. This is also expected as discussed in Proposition 1. Note, with the availability of a central parameter server (as in federated averaging), sparser topology may be useful for a faster convergence, however, consensus (hence, topology density) is critical for a collaborative learning paradigm with decentralized computation.

6 Conclusion and Future Work
This paper addresses the collaborative deep learning (and many other machine learning) problem in a completely distributed manner (i.e., with data parallelism and decentralized computation) over networks with fixed topology. We establish a consensus based distributed SGD framework and proposed associated learning algorithms that can prove to be extremely useful in practice. Using a Lyapunov function construction approach, we show that the proposed CDSGD algorithm can achieve linear convergence rate with sufficiently small fixed step size and sublinear convergence rate with diminishing step size (see supplementary section 7 for details) for strongly convex and Lipschitz differentiable objective functions. Moreover, decaying gradients can be observed for the nonconvex objective functions using CDSGD. Relevant experimental results using benchmark datasets show that CDSGD can achieve centralized SGD level accuracy with sufficient training epochs while maintaining a significantly low generalization error. The momentum variant of the proposed algorithm, CDMSGD can outperform recently proposed FedAvg algorithm which also uses data parallelism but does not perform a decentralized computation, i.e., uses a central parameter server. The effects of network size and topology are also explored experimentally which conforms to the analytical understandings. While current and future research is focusing on extensive testing and validation of the proposed framework especially for large networks, a few technical research directions include: (i) collaborative learning with extreme non-IID data; (ii) collaborative learning over directed time-varying graphs; and (iii) understanding the dependencies between learning rate and consensus.
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7 Supplementary Materials for “Collaborative Deep Learning in Fixed Topology Networks"

7.1 Additional analytical results and proofs

We begin with proofs of the lemmas and theorems that are presented in the main body of the paper without proof. The statements of the lemmas and theorems are presented again for completeness.

**Lemma 1** Let Assumptions 1 and 2 hold. The iterates of CDSGD (Algorithm 1) satisfy the following inequality \( \forall k \in \mathbb{N} \):

\[
E[V(x_{k+1})] - V(x_k) \leq -\alpha_k \nabla V(x_k)^T \mathbb{E}[\nabla J^i(x_k)] + \frac{\hat{\gamma}}{2} \alpha_k^2 \mathbb{E}[\|\nabla J^i(x_k)\|^2]
\]  

\( (19) \)

**Proof.** By Assumption 1, the iterates generated by CDSGD satisfy

\[
V(x_{k+1}) - V(x_k) \leq \nabla V(x_k)^T (x_{k+1} - x_k) + \frac{1}{2} \hat{\gamma} \|x_{k+1} - x_k\|^2
\]  

\( (20) \)

Taking expectations on both sides, we can obtain

\[
E[V(x_{k+1})] - V(x_k) \leq E[-\alpha \nabla V(x_k)^T \nabla J^i(x_k) + \frac{1}{2} \hat{\gamma} \alpha^2 \|\nabla J^i(x_k)\|^2]
\]  

\( (21) \)

While \( V(x_k) \) is deterministic, \( V(x_{k+1}) \) is stochastic due to the random sampling aspect. Therefore, we have

\[
E[V(x_{k+1})] - V(x_k) \leq -\alpha \nabla V(x_k)^T \mathbb{E}[\nabla J^i(x_k)] + \frac{1}{2} \hat{\gamma} \alpha^2 \mathbb{E}[\|\nabla J^i(x_k)\|^2]
\]  

\( (22) \)

which completes the proof.

**Lemma 2** Let Assumptions 1, 2, and 3 hold. The iterates of CDSGD (Algorithm 1) satisfy the following inequality \( \forall k \in \mathbb{N} \):

\[
E[V(x_{k+1})] - V(x_k) \leq -(\zeta_1 - \hat{\gamma} \alpha Q_m) \alpha \|\nabla V(x_k)\|^2 + \frac{\hat{\gamma}}{2} \alpha^2 Q
\]  

\( (23) \)

**Proof.** Recalling Lemma 1 and using Assumption 3 and Remark 1, we have

\[
E[V(x_{k+1})] - V(x_k) \leq -(\zeta_1 - \hat{\gamma} \alpha Q_m) \alpha \|\nabla V(x_k)\|^2 + \frac{\hat{\gamma}}{2} \alpha^2 \mathbb{E}[\|\nabla J^i(x_k)\|^2]
\]  

\( (24) \)

which completes the proof.

**Lemma 3.** \( V \) has a lower bound denoted by \( V_{\inf} \) over an open set which contains the iterates \( \{x_k\} \) generated by CDSGD (Algorithm 1).

Lemma 3 can be obtained as each \( f_j \) is proper and coercive. Such a lemma is able to help characterize the nonconvex case in which the global optimum may not be achieved.
Lemma 4. Let Assumption 7 holds. There exists some constant $0 < L < \infty$ such that $\mathbb{E}[\|g(x_k)\|] \leq L$.

The proof of Lemma 4 directly follows from the Assumption 1(c) and $L = \max_j L_j$.

**Theorem** Convergence of CDSGD with fixed step size, strongly convex case: Let Assumptions 1, 2 and 3 hold. The iterates of CDSGD (Algorithm 1) satisfy the following inequality for all $k \in \mathbb{N}$, when the step size satisfies

$$0 < \alpha \leq \frac{\zeta_1 - (1 - \lambda N(\Pi))Q_m}{\gamma_m Q_m}$$

$$\mathbb{E}[V(x_k) - V^*] \leq (1 - \alpha \hat{H}\zeta_1)k - (1 - \alpha \hat{H}\zeta_1)j (V(x_1) - V^*) + \frac{\alpha^2 \hat{\gamma} Q}{2} \sum_{l=0}^{k-1} (1 - \alpha \hat{H}\zeta_1)^l$$

(25)

Proof. Recalling Lemma 2 and using Eq. 10 yield that

$$\mathbb{E}[V(x_{k+1})] - V(x_k) \leq -(\zeta_1 - \frac{\hat{\gamma}}{2} \alpha Q_m)\alpha \|\nabla V(x_k)\|^2 + \frac{\hat{\gamma}}{2} \alpha^2 Q$$

$$\leq -\frac{1}{2} \zeta_1 \|\nabla V(x_k)\|^2 + \frac{\alpha^2 \hat{\gamma} Q}{2}$$

(26)

The second inequality follows from the relation: $\alpha \leq \frac{\zeta_1 - (1 - \lambda N(\Pi))Q_m}{\gamma_m Q_m}$. The third inequality follows from the strong convexity. The expectation taken in the above inequalities is only related to $x_{k+1}$. Hence, recursively taking the expectation and subtracting $V^*$ from both sides requires the following inequality to hold

$$\mathbb{E}[V(x_{k+1})] - V^* \leq (1 - \alpha \hat{H}\zeta_1)\mathbb{E}[V(x_k) - V^*] + \frac{\alpha^2 \hat{\gamma} Q}{2}$$

(27)

As $0 < \alpha \hat{H}\zeta_1 \leq \frac{\hat{\gamma} \alpha^2 Q}{\zeta_1} \leq \frac{\hat{\gamma} \alpha^2 Q}{\zeta_1} \leq 1$, the conclusion follows by applying Eq. 27 recursively through iteration $k \in \mathbb{N}$. 

**Theorem** Convergence of CDSGD with fixed step size, nonconvex case: Let Assumptions 1, 2 and 3 hold. The iterates of CDSGD (Algorithm 1) satisfy the following inequality for all $m \in \mathbb{N}$, when the step size satisfies

$$0 < \alpha \leq \frac{\zeta_1 - (1 - \lambda N(\Pi))Q_m}{\gamma_m Q_m}$$

$$\mathbb{E}\left[\sum_{k=1}^{m} \|\nabla V(x_k)\|^2\right] \leq \frac{\hat{\gamma} maQ}{\zeta_1} + \frac{2(V(x_1) - V_{init})}{\zeta_1 \alpha}$$

(28)

Proof. Recalling Lemma 2 and also taking the expectation lead to the following relation,

$$\mathbb{E}[V(x_{k+1})] - \mathbb{E}[V(x_k)] \leq -(\zeta_1 - \frac{\hat{\gamma} \alpha Q_m}{2})\alpha \mathbb{E}[\|\nabla V(x_k)\|^2] + \frac{\hat{\gamma} \alpha^2 Q}{2}$$

(29)
As the step size satisfies that $\alpha \leq \frac{\zeta_1}{\gamma Q_m}$, it results in

$$E[V(x_{k+1})] - E[V(x_k)] \leq -\frac{\zeta_1}{2} \alpha \|\nabla V(x_k)\|^2 + \frac{\alpha^2 \gamma Q}{2}$$  \hspace{1cm} (30)

Applying the above inequality from 1 to $m$ and summing them up can give the following relation

$$V_{\inf} - V(x_1) \leq E[V(x_{k+1})] - V(x_1) \leq -\frac{\zeta_1 \alpha}{2} \sum_{k=1}^{m} E[\|\nabla V(x_k)\|^2] + \frac{m \alpha^2 \gamma Q}{2}$$  \hspace{1cm} (31)

The last inequality follows from the Lemma 3. Rearrangement of the above inequality and substituting $\hat{\gamma} = \gamma_m + \alpha^{-1}(1 - \lambda_N(\Pi))$ into it yield the desired result.

### 7.2 Proof with Diminishing Step Size

From results presented in section 4, it can be concluded that when the step size is fixed, the function value can only converge near the optimal value. However, in many deep learning models, noisy gradient is quite common due to the random data sampling. Hence, such a situation requires the step size to be adaptive and then with noise, the function value sequence is able to converge to the optimal value. Let $\{\alpha_k\}$ be defined as a diminishing step size sequence that satisfies the following properties:

$$\alpha_k > 0, \sum_{k=0}^{\infty} \alpha_k = \infty, \sum_{k=0}^{\infty} \alpha_k^2 < \infty$$

The implication of the above properties is that $\lim_{k \to \infty} \alpha_k = 0$. The next proposition states that when the step size is diminishing, consensus can be achieved asymptotically, i.e., $\lim_{k \to \infty} E[\|x_k - s_k\|] = 0$.

**Proposition 2.** (Consensus with diminishing step size) Let Assumptions 1 and 2 hold. The iterates of CD SGD (Algorithm 1) satisfy the following inequality $\forall k \in \mathbb{N}$ when $\alpha_k$ is diminishing,

$$\lim_{k \to \infty} E[\|x_k - s_k\|] = 0$$  \hspace{1cm} (32)

The proof is adapted from the Lemma 1 in [26], Lemmas 5 and 6 in [30].

Recalling the algorithm CD SGD

$$x_{k+1} = w_k - \alpha_k g(x_k) = x_k - \alpha_k (g(x_k) + \frac{1}{\alpha_k}(x_k - w_k))$$

We define $\nabla \hat{J}^i(x_k) = \nabla J^i(x_k, \alpha_k) = g(x_k) + \frac{1}{\alpha_k}(x_k - w_k)$, and the following Lyapunov function

$$\hat{V}(x) = V(x, \alpha_k) := \frac{N}{n} 1^T F(x) + \frac{1}{2\alpha_k} \|x\|_{\Pi}^2$$  \hspace{1cm} (33)

The general Lyapunov function is a function of the diminishing step size $\alpha_k$. However, the step size is independent of the variable $x$ such that it only affects the magnitude of $\|x\|_{\Pi}$ along with iterations. Note, from Proposition 2, we have that each agent eventually reaches the consensus with diminishing step size. Hence, the term $\frac{1}{\alpha_k} \|x\|_{\Pi}$ should not increase with increase in $k$ as the step size $\alpha_k \to 0$ for $k \to \infty$.

To show that CD SGD with diminishing step size enables convergence to the optimal value, the necessary lemmas and assumptions are directly used from the previous part of the paper with modified constants.

We next show that the Lyapunov function and stochastic Lyapunov gradient with the diminishing step size are bounded. More formally, we aim to show that $\|\nabla \hat{J}^i(x_k)\|$ is bounded above for all $k \in \mathbb{N}$. We have, $\|\nabla \hat{J}^i(x_k)\| \leq \|g(x_k)\| + \frac{1}{\alpha_k} \|(I - \Pi)x_k\|$ and $g(x_k)$ is bounded. Therefore, we have to show that $\frac{1}{\alpha_k} \|(I - \Pi)x_k\|$ is bounded for all $k \in \mathbb{N}$. 


Lemma 5. Let Assumptions 1 and 2 hold. The iterates of CDSGD (Algorithm 1) satisfy the following inequality \( \forall k \in \mathbb{N} \), when the step size is diminishing and satisfies that
\[
0 < \alpha_0 \leq \frac{\hat{\zeta}_1 - (1 - \lambda N(\Pi))\hat{Q}_m}{\gamma_m Q_m},
\]
\[
\frac{1}{\alpha_k} \mathbb{E}[\| (I - \Pi) x_k \|] < \infty \tag{34}
\]
and
\[
\lim_{k \to \infty} \mathbb{E}[\| (I - \Pi) x_k \|] = 0. \tag{35}
\]

\( \hat{\zeta}_1, \hat{Q}_m \) correspond to \( \hat{V} \).

The proof of Lemma 5 requires another auxiliary technical lemma as follows.

Lemma 6. Let Assumptions 1 and 2 hold. The iterates of CDSGD (Algorithm 1) satisfy the following inequality \( \forall k \in \mathbb{N} \), when the step size is diminishing and satisfies that
\[
0 < \alpha_0 \leq \frac{\hat{\zeta}_1 - (1 - \lambda N(\Pi))\hat{Q}_m}{\gamma_m Q_m},
\]
\[
\sum_{k=2}^{\infty} \alpha_k \mathbb{E}[\| (I - \Pi) x_k \|] < \infty. \tag{36}
\]

Proof. Recalling the CDSGD algorithm,
\[
x_{k+1} = \Pi x_k - \alpha_k g(x_k) \tag{37}
\]

Applying the above equality from 1 to \( k - 1 \) yields that
\[
x_k = \Pi^{k-1} x_1 - \sum_{l=1}^{k-1} \alpha_l \Pi^{k-1-l} g(x_l) \tag{38}
\]

Setting \( x_1 = 0 \) results in that \( x_k = -\sum_{l=1}^{k-1} \alpha_l \Pi^{k-1-l} g(x_l) \). With this setup, we have
\[
\sum_{k=2}^{\infty} \alpha_k \| (I - \Pi) x_k \| \leq \sum_{k=2}^{\infty} \alpha_k \| I - \Pi \| \| x_k \|
\]
\[
\leq \sum_{k=2}^{\infty} \alpha_k \| \sum_{l=1}^{k-1} \alpha_l \Pi^{k-1-l} g(x_l) \|
\]
\[
\leq \sum_{k=2}^{\infty} \alpha_k \sum_{l=1}^{k-1} \alpha_l \| \Pi^{k-1-l} g(x_l) \|
\]
\[
\leq \sum_{k=2}^{\infty} \alpha_k \sum_{l=1}^{k-1} \alpha_l \| \Pi^{k-1-l} \| \| g(x_l) \|
\]
\[
\leq \sum_{k=2}^{\infty} \alpha_k \sum_{l=1}^{k-1} \alpha_l \| \Pi^{k-1-l} \| \| g(x_l) \|
\]
With the step size being nonincreasing, taking expectation on both side leads to
\[
\mathbb{E}[\sum_{k=2}^{\infty} \alpha_k \| (I - \Pi) x_k \|] \leq \sum_{k=2}^{\infty} \sum_{l=1}^{k-1} \alpha_l^2 \| \Pi^{k-1-l} \| \mathbb{E}[\| g(x_l) \|] \tag{40}
\]

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As we discussed earlier, we consider that there exists a constant $L$ that bounds $\mathbb{E}[\|g(x_k)\|]$ from above for $k \in \mathbb{N}$. Thus, the following relation can be obtained

$$
\mathbb{E}\left[\sum_{k=2}^{\infty} \alpha_k \|(I - \Pi)x_k\|\right] \leq \sum_{k=2}^{\infty} \sum_{l=1}^{k-1} \alpha_l^2 \lambda_2(\Pi)^{k-1-l} \mathbb{E}[\|g(x_l)\|] \\
\leq L \sum_{k=2}^{\infty} \sum_{l=1}^{k-1} \alpha_l^2 \lambda_2(\Pi)^{k-1-l}
$$

(41)

As $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$ and $\lambda_2(\Pi) < 1$ then by Lemma 5 in [40], the desired result follows. $\square$

**Proof of Lemma [3]** We first define that $h_k = \frac{\|(I - \Pi)x_k\|}{\alpha_k}$. Hence, the result of Lemma [6] can be rewritten as $\sum_{k=2}^{\infty} \alpha_k^2 \mathbb{E}[h_k] < \infty$. By defining $h_m = \sup\{\mathbb{E}[h_k]\}$, we have $h_m \sum_{k=2}^{\infty} \alpha_k^2 < \infty$, which implies that $h_m < \infty$ as $\sum_{k=1}^{\infty} \alpha_k^2 < \infty$. Hence, it is immediately seen that $\mathbb{E}[h_k] < \infty$. As $k \to \infty$, $\alpha_k \to 0$ and $\frac{1}{\alpha_k} \mathbb{E}[\|g(x_k)\|] < \infty$, then $\lim_{k \to \infty} \mathbb{E}[\|g(x_k)\|] = 0$, which completes the proof. $\square$

The implication of Lemma [3] is two folds: One can observe that $\hat{V}(x_k)$ and $\nabla \hat{J}(x_k)$ are finite even with diminishing step size such that based on Definition [2] there exists a finite positive constant $\gamma'$ to allow the smoothness of $\hat{V}(x_k)$ for all $k \in \mathbb{N}$ to hold true; another observation is that Assumption [3] still can be used in the main results. It can also be concluded that $\hat{V}$ is strongly convex with some constant $0 \leq H' < \infty$, where $H'$ corresponds to $\hat{V}$.

**Theorem 3. (Convergence of CDSGD with diminishing step size, strongly convex case)** Let Assumptions [4] and [3] hold. The iterates of CDSGD (Algorithm [7]) satisfy the following inequality for $k \in \mathbb{N}$, when the step size is diminishing and satisfies that

$$0 < \alpha_0 \leq \frac{\hat{c}_1 - (1 - \lambda_N(\Pi))\hat{Q}_m}{\gamma_m Q_m},$$

$$\mathbb{E}[\hat{V}(x_k) - \hat{V}^*] \leq \beta^{k-2}(\hat{V}(x_1) - \hat{V}^*) + \frac{\gamma' \hat{Q} \alpha_2^2}{2} \sum_{p=1}^{k-2} \beta^{k-p-2} \alpha_p^2$$

$$+ \frac{\gamma' \hat{Q} \alpha_2^2}{2}$$

(42)

where $\sup\{1 - \alpha_k H' \hat{c}_1\} \leq \beta < 1$, and $\gamma', \hat{Q}$ correspond to $\hat{V}$.

**Proof.** As $\alpha_1 \leq \frac{\hat{c}_1}{\gamma\hat{Q}_m}$, then it can be obtained that $\alpha_k \gamma' \hat{Q}_m \leq \alpha_1 \gamma' \hat{Q}_m \leq \hat{c}_1$ for all $k \in \mathbb{N}$. Recalling Lemma [2] and Eq. [10] subtracting $\hat{V}^*$ from both sides, and taking the expectation yield the following relation

$$\mathbb{E}[\hat{V}(x_{k+1}) - \hat{V}^*] \leq (1 - \alpha_k H' \hat{c}_1)\mathbb{E}[\hat{V}(x_k) - \hat{V}^*] + \frac{\gamma' \hat{Q} \alpha_2^2}{2}$$

(43)

Applying the above inequality recursively can give the following relation

$$\mathbb{E}[\hat{V}(x_{k+1}) - \hat{V}^*] \leq (1 - \alpha_k H' \hat{c}_1)(1 - \alpha_{k-1} H' \hat{c}_1)\mathbb{E}[\hat{V}(x_{k-1}) - \hat{V}^*]$$

$$+ (1 - \alpha_{k-1} H' \hat{c}_1) \frac{\gamma' \hat{Q} \alpha_2^2}{2} + \gamma' \hat{Q} \alpha_2^2$$

(44)

By induction, the following can be obtained

$$\mathbb{E}[\hat{V}(x_{k+1}) - \hat{V}^*] \leq$$

$$\prod_{q=1}^{k} (1 - \alpha_q H' \hat{c}_1)(\hat{V}(x_1) - \hat{V}^*) + \frac{\gamma' \hat{Q} \alpha_2^2}{2} \sum_{p=1}^{k-1} \prod_{r=p+1}^{k} (1 - \alpha_r H' \hat{c}_1)\alpha_p^2$$

$$+ \gamma' \hat{Q} \alpha_2^2$$

(45)
As $H' \leq \gamma'$, it can be derived that $0 < \alpha_k H' \hat{\zeta}_1 \leq 1$ for all $k \in \mathbb{N}$. Therefore, $1 - \alpha_k H' \hat{\zeta}_1 \in [0, 1)$ such that we can define a positive constant $\beta$ satisfies that $\sup \{1 - \alpha_k H' \hat{\zeta}_1\} \leq \beta < 1$. Hence, combining the last inequalities together, we have

$$
\begin{align*}
\mathbb{E} [\hat{V}(x_{k+1}) - \hat{V}^*] &\leq \beta^{k-1} (\hat{V}(x_1) - \hat{V}^*) + \frac{\gamma' \hat{Q} \alpha_k^2}{2} \\
&+ \frac{\gamma' \hat{Q} \alpha_k^2}{2}
\end{align*}
$$

(46)

which completes the proof by replacing $k + 1$ with $k$.

Remark 4. From Theorem $\text{3}$, we can conclude that the function value sequence $\{\hat{V}(x_k)\}$ asymptotically converges to the optimal value. (This holds regardless of whether the “gradient noise" parameter $\hat{Q}$ is zero or not.) In fact, we can establish the rate of convergence as follows: the first term on the right hand side decreases exponentially if $\beta < 1$, and the last term decreases as quickly as $\alpha_k^2$. For the middle term, we can use Lemma 3.1 of $\text{[31]}$ that establishes bounds on the convolution of two scalar sequences. If we choose $t > 0$ such that $\alpha_k = \frac{\epsilon}{k^{t+1}}$, where $\epsilon \in (0, 1]$, then the necessary growth conditions on $\alpha_k$ are satisfied; substituting this into Theorem $\text{3}$ yields the stated convergence rate of $O(1/k^t)$. In practice, $\alpha_k$ can be made adaptive to $\frac{\Theta}{k^t + \epsilon}$ for any constant $\Theta > 0$.

Similarly, we also present the convergence results for the nonconvex objective functions.

Theorem 4. (Convergence of CDSGD with diminishing step size, nonconvex case) Let Assumptions $\text{[1]}$ $\text{[2]}$ and $\text{[3]}$ hold. The iterates of CDSGD (Algorithm $\text{7}$) satisfy the following inequality $\forall m \in \mathbb{N}$, when the step size is diminishing and satisfies that

$$
0 < \alpha_0 \leq \hat{\zeta}_1 - (1 - \lambda_N(\Pi)) \hat{Q}_m,
$$

(47)

$$
\begin{align*}
\mathbb{E} \left[ \sum_{k=1}^{m} \alpha_k \| \nabla \hat{V}(x_k) \|^2 \right] &\leq \frac{2(\hat{V}(x_1) - \hat{V}_{inf})}{\hat{\zeta}_1} + \frac{\gamma' \hat{Q} \alpha_k^2}{2} \\
&+ \frac{\gamma' \hat{Q} \alpha_k^2}{2}
\end{align*}
$$

Proof. Assume that $\alpha_k \gamma' \hat{Q}_m \leq \hat{\zeta}_1$ for all $k \in \mathbb{N}$. Based on Eq. $\text{[29]}$ we consider the diminishing step size and Lyapunov function, then the following relation can be obtained

$$
\mathbb{E} [\hat{V}(x_{k+1})] - \mathbb{E} [\hat{V}(x_k)] \leq - (\hat{\zeta}_1 - \frac{\gamma' \alpha_k \hat{Q}_m}{2}) \alpha_k \mathbb{E} [\| \nabla \hat{V}(x_k) \|^2] + \frac{\gamma' \alpha_k \hat{Q}}{2}
$$

(48)

Combining the condition for the step size yields the following inequality

$$
\mathbb{E} [\hat{V}(x_{k+1})] - \mathbb{E} [\hat{V}(x_k)] \leq - \frac{\hat{\zeta}_1 \alpha_k}{2} \mathbb{E} [\| \nabla \hat{V}(x_k) \|^2] + \frac{\gamma' \alpha_k \hat{Q}}{2}
$$

(49)

Applying the last inequality from 1 to $m$ and summing them up,

$$
\hat{V}_{inf} - \mathbb{E} [\hat{V}(x_1)] \leq \mathbb{E} [\hat{V}(x_{k+1})] - \mathbb{E} [\hat{V}(x_1)] \leq
$$

$$
- \frac{\hat{\zeta}_1}{2} \sum_{k=1}^{m} \alpha_k \mathbb{E} [\| \nabla \hat{V}(x_k) \|^2] + \frac{\gamma' \hat{Q} \alpha_k^2}{2} \sum_{k=1}^{m} \alpha_k^2
$$

(50)

Dividing by $\hat{\zeta}_1/2$ and rearranging the terms lead to the desired results. 

Remark 5. Compared to Theorem $\text{2}$, Theorem $\text{4}$ has shown the decaying of gradient $\| \nabla \hat{V}(x_k) \|$ even with noise when the step size is diminishing in the nonconvex case. This is because when $k \to \infty$, the right hand side of Eq. $\text{[47]}$ remains finite such that $\| \nabla \hat{V}(x_k) \|^2$ approaches 0.
7.3 Additional pseudo-codes of the algorithms

Momentum methods have been regarded as effective methods to speed up the convergence in numerous optimization problems. While the Nesterov Momentum method has been extended widely to generate variants with provable global convergence properties, the global convergence analysis of Polyak Momentum methods is still quite challenging and an active research topic. Pseudo-codes of CDSGD combined with Polyak momentum and Nesterov momentum methods are presented below.

**Algorithm 2: CDSGD with Polyak Momentum**

```
Input : m, α, N, μ (momentum term)
1 Initialize: x₀, v₀
2 Distribute the training data set to N agents
3 For each agent:
4   for k = 0 : m do
5     Randomly shuffle the corresponding data subset;
6     w_jk+1 = \sum_{l \in Nh(j)} \pi_j \xi^l_k
7     v_jk+1 = μv_jk - α_k g_j(x_jk)
8     x_jk+1 = w_jk+1 + v_jk+1
9 end
```

**Algorithm 3: CDSGD with Nesterov Momentum**

```
Input : m, α, N, μ
1 Initialize: x₀, v₀
2 Distribute the training data set to N agents
3 For each agent:
4   for k = 0 : m do
5     Randomly shuffle the corresponding data subset
6     w_jk+1 = \sum_{l \in Nh(j)} \pi_j \xi^l_k
7     v_jk+1 = μv_jk - α_k g_j(x_jk + μv_jk)
8     x_jk+1 = w_jk+1 + v_jk+1
9 end
```

7.4 Additional Experimental Results

We begin with a discussion on the training loss profiles for the CIFAR-10 results presented in the main body of the paper.

7.4.1 Comparison of the loss for benchmark methods

Figure 3 (a) shows the loss (in log scale) with respect to the number of epochs for SGD and CDSGD algorithms. The solid curve means training and the dash curve indicates validation. From the loss results, it can be observed that SGD has the sublinear convergence rate for training and dominates among the two methods during the training process. While for the validation, SGD performs poorly after around 70 epochs. However, CDSGD shows linear convergence rate (in log scale as discussed in the analysis) for both training and validation. Though, it takes a lot of more time compared to SGD for convergence, it eventually performs better than SGD in the validation data and the gap between the training and validation loss (i.e., the generalization gap [29]) is very less compared to that in SGD.
7.4.2 Results on CIFAR-100 dataset

For the experiments on the CIFAR-100 dataset, we use a CNN similar to that used for the CIFAR-10 dataset. While the results of CIFAR-100 also converges fast for SGD, CDMSGD and Federated Averaging SGD (FedAvg) algorithms (CDMSGD being the slowest) as shown in Figure 4, it can be seen that eventually, the loss converges better than the FedAvg algorithm. Similar to the observation made for the CIFAR-10 dataset, we observe that CDMSGD achieves significantly higher validation accuracy compared to FedAvg while approaching similar accuracy level as that of (centralized) SGD. It can also be seen that as expected CDSGD’s convergence is very slow compared to the others.

7.4.3 Results on MNIST dataset

For the experiments on the MNIST dataset, the model used for training is a Deep Neural Network with 20 Fully Connected layers consisting of 50 ReLU units each and the output layer with 10 units having softmax activation. The model was trained using the categorical cross-entropy loss. Figure 4(c & d) shows the loss and accuracy obtained over the number of epochs. In this case, while the accuracy levels are significantly higher as expected for the MNIST dataset, the trends remain consistent with the results obtained for the other benchmark datasets of CIFAR-10 and CIFAR-100. Note, the generalization gap between the training and validation data for all the methods are very less (least for CDMSGD).

7.4.4 Effect of the decaying step size

Based on the analysis presented in section 7.2 it is evident that decaying step size has a significant effect on the accuracy as well as convergence. A performance comparison of SGD, Momentum SGD (MSGD) and CDMSGD with a decaying stepsize is performed using the MNIST dataset. It can be seen that the performance of the CDMSGD with decaying step size becomes slightly better than SGD with decaying step size while (centralized) MSGD has the best performance. Although CDMSGD sometimes suffers from large fluctuations, it demonstrates the least generalization gap among all the algorithms.

7.4.5 Effect of step size

The analysis presented in this paper shows that choice of step size is critical in terms of convergence as well as accuracy. To explore this aspect experimentally, we compare the performance of CDMSGD for three
Figure 4: Average training (solid lines) and validation (dash lines) (a) loss and (b) accuracy for SGD, CDSGD, CDMSGD and Federated averaging method for the CIFAR-100 dataset (c) loss and (d) accuracy for SGD, CDSGD, CDMSGD and Federated averaging method for the MNIST dataset
Figure 5: Average training (solid lines) and validation (dash lines) (a) loss and (b) accuracy for SGD, MSGD and CDMSGD method for the MNIST dataset for decaying step size. (c) loss and (d) accuracy for CDMSGD for the MNIST data with different learning rates.
different fixed step sizes using MNIST data. The results are presented in (c) & (d), where the (fixed) step size was varied from $0.1(1E - 1)$ to $0.01(1E - 2)$ and then to $0.001(1E - 3)$. While the fastest convergence of the algorithm is observed with step size 0.1, the level of consensus (indicated by the variance among the agents) is quite unstable. On the other hand, with very low step size 0.001, the level of consensus is quite stable (moving average of variance remains 0). However, the convergence is extremely slow. This observation conforms to the theoretical analysis described in the paper as well as justifies the choice of step size 0.01 in the experiments presented above.