On Jordan type bounds for finite groups of diffeomorphisms of 3-manifolds and Euclidean spaces

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Abstract. By a classical result of Jordan, each finite subgroup $G$ of $\text{GL}_n(\mathbb{C})$ has an abelian subgroup whose index in $G$ is bounded by a constant depending only on $n$. We consider the problem if this remains true for finite subgroups $G$ of the diffeomorphism group of a smooth manifold, and show that it is true for all compact 3-manifolds as well as for Euclidean spaces $\mathbb{R}^n$, $n \leq 6$. The question remains open at present e.g. for odd-dimensional spheres $S^n$, $n \geq 5$ and for Euclidean spaces $\mathbb{R}^n$, $n \geq 7$.

1. Introduction

By a classical result of Jordan, each finite subgroup $G$ of $\text{GL}_n(\mathbb{C})$ has an abelian subgroup $A$ whose index in $G$ is bounded by a constant depending only on $n$ (see [C] for the optimal bound for each $n$). Recently there has been much interest in generalizations, replacing $\text{GL}_n(\mathbb{C})$ by more general geometrically interesting groups such as diffeomorphism groups of smooth manifolds ([MR1,2],[P1]), automorphism groups of algebraic varieties and the Cremona groups of birational self-maps of the affine $n$-dimensional space (cf. [P2], [Se, Theoreme 3.1]).

Following [P1,2], we say that a group $E$ is a Jordan group or has the Jordan property if there exits a constant such that every finite subgroup $G$ of $E$ has an abelian subgroup of index bounded by this constant. For a smooth manifold $M$, let $\text{Diff}(M)$ denote its diffeomorphism group. The present paper is motivated by the following general:

Question: For which (classes of) smooth manifolds $M$ is $\text{Diff}(M)$ a Jordan group?

Whereas this is in general not true for non-compact manifolds ([P1]), it has been conjectured that it is true for compact manifolds (see [MR1,2]); however, it should be true e.g. also for $\text{Diff}(\mathbb{R}^n)$.

Note that a Jordan group contains only finitely many finite non-abelian simple subgroups, up to isomorphism; in this regard, it has been shown in [GZ] that $\text{Diff}(S^n)$ contains only finitely many finite non-abelian simple subgroups, up to isomorphism (and, more generally, for any closed homology $n$-sphere, see also [Z1]). It has been
shown in [MR1] that $\text{Diff}(M)$ is a Jordan group if $M$ is a compact manifold without odd cohomology; in particular, $\text{Diff}(S^n)$ is a Jordan group for even dimensions $n$, but this remains open for odd dimensions $n \geq 5$.

On the basis of the geometrization of 3-manifolds and results of Kojima [K] and the author [Z2], in our first main result we consider the case of compact 3-manifolds:

**Theorem 1.** $\text{Diff}(M)$ is a Jordan group for compact 3-manifolds $M$.

In dimension three, this leaves open the question for non-compact 3-manifolds. Concerning dimension four, it is shown in [P2] that there are noncompact, simply connected, smooth 4-manifolds $M$ such that $\text{Diff}(M)$ is not a Jordan group. On the other hand, it is shown in [MR2] that, for compact smooth 4-manifolds $M$ with non-zero Euler characteristic, $\text{Diff}(M)$ is a Jordan group. The case of the 4-sphere $S^4$ is considered in [MeZ1,2] where it is shown that, up to 2-fold extension in the case of solvable groups, any finite group with an orientation-preserving smooth action on $S^4$ (or on any homology 4-sphere) is isomorphic to a subgroup of the orthogonal group $\text{SO}(5)$, presenting also a short list of such groups.

Next we consider Euclidean spaces $\mathbb{R}^n$. The following is proved in [GMZ]:

**Theorem 2.** ([GMZ]) Let $G$ be a finite subgroup of $\text{Diff}(\mathbb{R}^n)$ (or of $\text{Diff}(M)$, for any acyclic $n$-manifold $M$). Suppose that $n \leq 4$; then $G$ is isomorphic to a subgroup of the orthogonal group $\text{O}(n)$. In particular, the classical Jordan bound applies to $G$, so $\text{Diff}(\mathbb{R}^n)$ is a Jordan group for $n \leq 4$.

In [GMZ] the case of finite groups of diffeomorphisms of $\mathbb{R}^5$ is also considered; the classification in this case is not complete but the results imply easily that also $\text{Diff}(\mathbb{R}^5)$ is a Jordan group (more generally, the results in [GMZ] apply to arbitrary acyclic manifolds).

A main tool for the proof of our second main result is a recent group-theoretical result of Mundet i Riera and Turull [MT] (on the basis of the classification of the finite simple groups).

**Theorem 3.** $\text{Diff}(\mathbb{R}^5)$ and $\text{Diff}(\mathbb{R}^6)$ are Jordan groups (and, more generally, $\text{Diff}(M)$ for any acyclic 5- or 6-manifold $M$).

We will present the proof of Theorem 3 for the new case $n = 6$; the same proof works also for $n = 5$ where it is, in fact, easier. As noted above, the proof for $n = 6$ uses the full classification of the finite simple groups; the proof for $n = 5$ instead requires "only" a smaller part of the classification of the finite simple groups (the Gorenstein-Harada classification of the finite simple groups of sectional 2-rank at most four), see [GMZ], [Z1].

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Two interesting cases where the Jordan property is not known at present are those of $\text{Diff}(S^5)$ and $\text{Diff}(\mathbb{R}^7)$. However, it seems likely that $\text{Diff}(S^n)$ and $\text{Diff}(\mathbb{R}^n)$ are Jordan groups for all values of $n$.

2. Proof of Theorem 1

It is easy to see that, if $\tilde{M}$ is a finite covering of $M$ such that $\text{Diff}(\tilde{M})$ is a Jordan group then also $\text{Diff}(M)$ is a Jordan group. So it is sufficient to consider the case of orientable manifolds, and also of orientation-preserving finite group actions $G$ (passing eventually to a subgroup of index two of $G$). Also, it is sufficient to consider the case of closed manifolds since, for a compact manifold $M$ with non-empty boundary, one can reduce to the closed case by taking the double of $M$ along the boundary, doubling also a given finite group action on $M$.

So let $M$ be a closed orientable 3-manifold and $G$ a finite group of orientation-preserving diffeomorphisms of $M$. If $\pi_1(M)$ is finite then, by the geometrization of 3-manifolds after Perelman, $M$ is a spherical 3-manifold and finitely covered by $S^3$; also, any finite group of diffeomorphisms of $M$ is conjugate to a linear (orthogonal) action. By the classical Jordan bound for linear groups, $\text{Diff}(S^3)$ is a Jordan group, and hence also $\text{Diff}(M)$ is a Jordan group.

Assume next that $M$ is irreducible and has infinite fundamental group; again by the geometrization of 3-manifolds, we can assume that $M$ is a geometric. Then, if $M$ does not admit a circle action, by [K, Theorem 4.1] there is a bound on the order of finite subgroups of $\text{Diff}(M)$ and we are done.

Suppose that $M$ has a circle action and infinite fundamental group. Then $M$ is a Seifert fiber space, and by the geometrization of finite group actions on Seifert fiber spaces ([MS]), we can assume that the action of the finite group $G$ of diffeomorphisms of $M$ is geometric, and in particular fiber-preserving and normalizing the $S^1$-action of $M$. Considering a suitable finite covering of $M$, we can moreover assume that $M$ has no exceptional fibers, and hence that the base space of the Seifert fibration (the quotient of the $S^1$-action) is a closed orientable surface $B$ without cone points. The finite group $G$ projects to a finite group $\bar{G}$ of diffeomorphisms of the base-surface $B$, and we can again assume that $\bar{G}$ is orientation-preserving.

If $B$ is a hyperbolic surface (of genus $g \geq 2$) then, by the formula of Riemann-Hurwitz, the order of the finite group $\bar{G}$ of diffeomorphisms of $B$ is bounded, and hence $G$ has a finite cyclic subgroup of bounded index (the intersection of $G$ with the $S^1$-action).

If $B$ is a torus $T^2$ then there are two cases. First, $M$ may be a 3-dimensional torus $T^3$; this acts by rotations on itself. Since the action of $G$ is geometric, the subgroup $G_0$ of $G$ acting trivially on the fundamental group is a subgroup of the $T^3$-action and hence abelian of rank at most three (see [Sc] for the geometries of 3-manifolds and their
isometry groups). The factor group $G/G_0$ acts faithfully on the fundamental group $\mathbb{Z}^3$ of the 3-torus and is isomorphic to a subgroup of $GL_3(\mathbb{Z})$. Since, by a well-known result of Minkowski, there is a bound on the finite subgroups of $GL_n(\mathbb{Z})$ for each $n$, $G$ has an abelian subgroup $G_0$ of bounded index.

If $M$ fibers over $T^2$ but is not a 3-torus then it belongs to the nilpotent geometry Nil given by the Heisenberg group (see again [Sc]). Now the subgroup $G_0$ of $G$ acting trivially on the fundamental group, up to inner automorphisms, is a cyclic subgroup of the $S^1$ action on $M$, and $G/G_0$ injects into the outer automorphism group $Out(\pi_1 M)$ of the fundamental group. The fundamental group of $M$ has a presentation

$$\pi_1 M = \langle a, b, t \mid [a, b] = t^k, \ [a, t] = [b, t] = 1 \rangle,$$

with $k \neq 0$. Now an easy calculation shows that the subgroup of the outer automorphism group of $\pi_1 M$ inducing the identity of the factor group $\pi_1 M \sim \mathbb{Z}$ is finite. Since the orders of finite subgroups of $GL_2(\mathbb{Z})$ are also bounded, $G$ has a finite cyclic subgroup $G_0$ of bounded index.

Finally, if the base-surface is the 2-sphere then either $M$ has finite fundamental group and is a spherical manifold, or homeomorphic to $S^2 \times S^1$ (and hence non-irreducible). We note that $S^2 \times S^1$ belongs to the $(S^2 \times \mathbb{R})$-geometry, one of Thurston’s eight 3-dimensional geometries; this is the easiest of the eight geometries and can be easily handled, see [Sc] for the isometry group of this geometry.

Summarizing, we have shown that for any closed irreducible 3-manifold $M$ (and also for $S^2 \times S^1$), $Diff(M)$ is a Jordan group.

Suppose that $M$ is non-irreducible but not $S^2 \times S^1$. If $M$ has a summand other than lens spaces and $S^2 \times S^1$ then, by [K, Theorem 4.2], the orders of finite diffeomorphism groups of $M$ are again bounded and we are done.

Suppose next that $M$ is a connected sum $\sharp_g(S^2 \times S^1)$ of $g$ copies of $S^2 \times S^1$, with $g > 1$. By [Z2], $G$ has a finite cyclic normal subgroup (the subgroup acting trivially on the fundamental group, up to inner automorphisms) such that the order of the factor group is bounded by a polynomial which is quadratic in $g$, so we are done also in this case. Finally, if $M$ is a connected sum of lens spaces, including $S^2 \times S^1$, then $M$ has a finite covering by a 3-manifold of type $\tilde{M} = \sharp_g(S^2 \times S^1)$ as before. Now $Diff(\tilde{M})$ is a Jordan group and hence also $Diff(M)$.

We have considered all possibilities for $M$ and completed the proof of Theorem 1.
3. Proof of Theorem 3

We prove the theorem for $n = 6$; for $n = 5$ the theorem follows from [GMZ, Theorem 3], and also a shorter version of the following proof for $n = 6$ applies.

We want to show that Diff($\mathbb{R}^6$) is a Jordan group, i.e. that there is a constant such that every finite subgroup $G$ of Diff($\mathbb{R}^6$) has an abelian subgroup of index bounded by this constant. By the main result of [MT], if this is true for all finite subgroups $G$ of Diff($\mathbb{R}^6$) which are a semidirect product $G = P \rtimes Q$, for a finite normal $p$-group $P$ and a finite $q$-group $Q$, with distinct primes $p$ and $q$, then it is true for all finite subgroups $G$ of Diff($\mathbb{R}^6$) (this uses the classification of the finite simple groups). So we have to consider only groups of type $G = P \rtimes Q$: given such a group, we have to find an abelian subgroup $A$ of $G$ whose index is bounded by a constant not depending on the specific group.

Let $G = P \rtimes Q$ be as before; we can assume that the action of $G$ is orientation-preserving. By general Smith fixed point theory, a finite $q$-group acting on $\mathbb{R}^n$ (or on any acyclic $n$-manifold) has non-empty fixed point set (see [B], [GMZ, section 2]). So $Q$ has a global fixed point and is isomorphic to a subgroup of the orthogonal group SO(6) (considering the induced linear action on the tangent space of a global fixed point). Hence, by the classical Jordan bound for linear groups, we may assume that $Q$ is an abelian $q$-group.

Let $F$ denote the fixed point set of $P$; since $P$ is normal, $F$ is invariant under the action of $Q$ and, since the action is orientation-preserving, $F$ is a submanifold of dimension at most four (i.e., of codimension at least two).

Suppose first that $F$ has dimension four. Then $P$ acts as a group of rotations around its fixed point set $F$ and hence is a cyclic group (isomorphic to a subgroup of SO(2)). By conjugation, every element of $G$ acts as $\pm$-identity on $P$ (conjugates a minimal rotation to a minimal rotation). Let $G_0$ be the subgroup of index one or two of $G$ acting trivially on $P$, and let $Q_0$ be its image in $Q$. Then $G_0 \cong P \times Q_0$ is an abelian subgroup of index at most two in $G$, so we are done in this case.

Now suppose that the fixed point set $F$ of $P$ has dimension three (and also codimension three). This implies that $p = 2$ since, if $p$ is odd, by an inductive argument on the $p$-group $P$, its fixed point set $F$ has even codimension. Considering the action of $P$ on a 3-ball transverse to $F$ in some point, $P$ is a subgroup of the orthogonal group SO(3) and hence isomorphic to a cyclic or dihedral 2-group.

If $P \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is isomorphic to the Klein 4-group then the subgroup $G_0$ of $G$ acting by conjugation trivially on $P$ has index at most three in $G$ (since $Q$ is a $q$-group of odd order) and is abelian, so we are done. If $P$ is a cyclic 2-group then its automorphism group is also a 2-group; since $Q$ has odd order, $G$ acts by conjugation trivially on $P$, so $G$ is abelian and we are done. If $P$ is a dihedral 2-group of order at least eight then it
has a cyclic characteristic subgroup $P_0$ of index two, so $G$ has a subgroup $G_0 = P_0 \times Q$ of index two; by the previous case, $G_0$ is abelian and we are done.

Suppose next that $F$ has dimension two. By Smith fixed point theory, $F$ is an acyclic manifold mod $p$ (i.e., for homology with coefficients in $\mathbb{Z}_p$). Since $F$ has dimension two, it is in fact acyclic also for integer coefficients (see [GMZ, proof of Lemma 3]). Then the finite $q$-group $Q$ has a fixed point in $F$, and hence $G$ has a global fixed point. Now $G$ is isomorphic to a subgroup of $\text{SO}(6)$, so we are done by the classical Jordan bound.

The cases that $F$ has dimension one or zero are similar.

This completes the proof of Theorem 3.

Remark. Considering the next case of $\mathbb{R}^7$, if the fixed point set $F$ of $P$ has codimension two or three, or if it has dimension at most two, the proof works exactly as before. The case we cannot handle at present is when $F$ has dimension three (and codimension four). In this case $P$ is isomorphic to a subgroup of $\text{SO}(4)$, e.g. isomorphic to $\mathbb{Z}_p$ or $\mathbb{Z}_p \times \mathbb{Z}_p$, and we don’t know how to bound the index of the subgroup of $G$ (or $Q$) acting trivially on $P$ (independent of the prime $p$).

References

[B] G. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, New York 1972

[C] M.J. Collins, *On Jordan’s theorem for complex linear groups*, J. Group Theory 10, 411-423 (2007)

[GMZ] A. Guazzi, M. Mecchia, B. Zimmermann, *On finite groups acting on acyclic low-dimensional manifolds*, Fund. Math. 215, 203-217 (2011)

[GZ] A. Guazzi, B. Zimmermann, *On finite simple groups acting on homology spheres*, Monatsh. Math. 169, 371-381 (2013)

[K] S. Kojima, *Bounding finite groups acting on 3-manifolds*, Math. Proc. Camb. Phil. Soc. 96, 269-281 (1984)

[MeZ1] M. Mecchia, B. Zimmermann, *On finite simple and nonsolvable groups acting on homology 4-spheres*, Top. Appl. 153, 2933-2942 (2006)

[MeZ2] M. Mecchia, B. Zimmermann, *On finite groups acting on homology 4-spheres and finite subgroups of SO(5)*, Top. Appl. 158, 741-747 (2011)

[MS] W.H. Meeks, P. Scott, *Finite group actions on 3-manifolds*, Invent. math. 86, 287-346 (1986)

[MR1] I. Mundet i Riera, *Finite groups acting on manifolds without odd cohomology*, arXiv: 1310.6565

[MR2] I. Mundet i Riera, *Finite group actions on 4-manifolds with nonzero Euler characteristic*, arXiv:1312.3149
[MT] I. Mundet i Riera, A. Turull, *Boosting an analogue of Jordan’s theorem for finite groups*, arXiv:1310.6518

[P1] V.L. Popov, *Finite subgroups of diffeomorphism groups*, arXiv:1310.6548

[P2] V.L. Popov, *Jordan groups and automorphism groups of algebraic varieties*, arXiv:13007.5522

[Sc] P. Scott, *The geometries of 3-manifolds*, Bull. London Math. Soc. 15, 401-487 (1983)

[Se] J.-P. Serre, *Le groupe de Cremona et ses sous-groupes finis*, Sem. Bourbaki 1000, 75-100 (2008)

[Z1] B. Zimmermann, *On finite groups acting on spheres and finite subgroups of orthogonal groups*, Sib. Electron. Math. Rep. 9, 1 - 12 (2012) (http://semr.math.nsc.ru)

[Z2] B. Zimmermann, *On finite groups acting on a connected sum of 3-manifolds $S^2 \times S^1$*, arXiv:1202.5427 (to appear in Fund. Math. 2014)