Nonlinear phenomena in mechanical system dynamics

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Abstract. The goal of the paper is to present selected, untypical, and intuitively unexpected phenomena from nonlinear mechanics. Particular attention is paid to the dynamics of self-, parametric and external excited systems. Interactions between these various vibration types lead mainly to quasi-periodic responses. However, in the selected domains of system parameters, the effect of frequency locking is observed. Furthermore, external harmonic force imposed on such a system produces a specific internal loop inside a resonance zone. As an example of nonlinear autoparametric systems, a structure (oscillator) with an attached pendulum is presented. The nonlinear terms introduced by pendulum motion cause instabilities in the resonance region. This instability transits the pendulum to rotation or chaotic motion. An application of nonlinear couplings for the reduction of unwanted vibrations is also studied. In order to reduce vibrations, the main structure is coupled to an electrical oscillator by a quadratic term. It has been shown that such a coupling leads to the amplitude saturation phenomenon which can then be used to design a nonlinear control strategy.

1. Introduction
In many dynamical systems nonlinear phenomena may completely alter intuitively expected behaviour and can drastically change their dynamical responses [1]. Mechanical vibrations can be caused by different reasons and, depending on the method of excitations, they are divided into three main types. We may distinguish between forced vibration, parametric vibrations and self-excited vibrations. The two former categories belong to mathematical models represented by nonautonomous equations. All the above systems are well known and deeply investigated in the literature, however mainly separately.

Paper [2] is one of the first that considered interactions between two different vibration types. There is only a preliminary conclusion offered on possible interactions between parametric and self-excitations. More extensive research is presented in [3], and advanced analysis is delivered in monograph [4] and papers [5, 6, 7, 8]. Coupled van der Pol – Mathieu oscillators are studied there for one and many degree of freedom systems. The influence of different nonlinear terms on the localisation of the frequency locking zones which follow the second kind of Hopf bifurcation, is studied using numerical and analytical methods in [9, 10].

The dynamics of parametrically and self-excited vibrations taking into account the influence of external force is presented in paper [11]. The authors conclude that five solutions may appear in the studied one–degree–of–freedom system. In fact this phenomenon was shown in former works such as [12, 13], where an additional external force influence, acting on a self- and parametrically excited system, was studied in detail for one and two degree of freedom models. It was shown that if the excitation frequency of the parametric and external terms was
tuned in the ratio 2:1, then essential changes in the main parametric resonance region occurred. For small values of excitation amplitude the phenomenon of an internal resonance loop was discovered. Comprehensive research considering interactions between parametric, external and self-excitations, as well as for regular, chaotic and hyperchaotic motion is published in [14]. The discussion of non-ideal systems is delivered in that monograph and in paper [15] as well.

Pendulum-like systems are commonly used in many practical applications, including special dynamical dampers [16, 1]. We may take advantage of the pendulum’s dynamics to suppress vibrations of high buildings, bridges or helicopter blades under flutter conditions [17]. But, a harmonically excited pendulum may also undergo complicated dynamics. Particular behaviour may occur if the pendulum and the main structure such as an oscillator are coupled by an inertial resonance condition. Such a system generates various motions, from simple periodic oscillation to complex chaos [18]. The presence of the coupling terms can lead to a certain type of instability which is referred to as autoparametric resonance. This kind of phenomenon takes place when the external resonance and the internal resonance meet, due to the coupling terms. Small parametric excitations, or a small change of initial conditions, then produce large response [1]. Similar dynamics governs the behaviour of coupled beam structures where the L-shaped beam is a good example [19]. The concept of the use of a combination of the magnetoreheological damper together with the nonlinear spring applied in the autoparametric system suspension is presented in [16, 1]. By activating magnetoreheological damping or varying nonlinear stiffness of the supporting spring, the unwanted dynamics can be eliminated or moved away. The solution with the MR damper gives reliable control possibilities and can help to react properly in critical situations.

Sometimes we may take advantage from nonlinear coupling. As is shown in [20] nonlinear coupling can be used for vibration suppression. Introducing quadratic terms on purpose, which couple the cantilever beam and a controller, leads to the saturation phenomenon. The system’s response depends on the controller damping and gains. The same control algorithm for the reduction of flexural and torsional vibrations of a plate is presented in [21]. To increase the effectiveness of the control the measurement of the frequency of excitation is added to the system. Analytical, numerical and experimental results for Nonlinear Saturation Control (NSC) and the linear position feedback algorithm (PPF) are studied in [22, 23, 24]. Those two algorithms are combined and then applied in experimental tests [22]. Advantages of hybrid, proportional linear and saturation nonlinear, controllers are presented in that paper. The influence of non-uniformities in the beam properties for a system with a saturation controller is presented in [25], where the additional effect of modal coupling for system response is studied. The effectiveness of the NSC method for vibration suppression of a light composite beam with Macro Fibre Composites (MFC) actuators is also confirmed in [26]. Advantages of MFC actuators in comparison with traditional PZT elements are stated in [27]. The proposed nonlinear controllers may work very effectively, under the condition that their nonlinear behaviour is known and well designed.

The main purpose of this paper is to point out nonlinear interactions which, in consequence, can lead to very complex dynamics, including chaotic motion, even for models having a very simple structure. Advantages and disadvantages of the nonlinear couplings will be demonstrated for selected examples.

2. A loop phenomenon in the dynamics of a self and parametrically exited system driven by an external force

Interactions between self and parametric vibrations can lead to very complex behaviour, even for very simple models [12, 13]. Near the parametric resonance regions the frequency locking phenomenon takes place (this is sometimes called frequency synchronisation). Outside these regions the response is quasi-periodic, represented by quasi-periodic limit cycles on Poincaré
The lumped mass dynamics of the considered system is described by the following differential equations of motion which take the matrix form

$$\mathbf{m} \ddot{\mathbf{x}} + \mathbf{f}_d (\mathbf{x}, \dot{\mathbf{x}}) + \mathbf{f}_p (\mathbf{x}, \omega_p, t) + \mathbf{f}_x (\mathbf{x}) = \mathbf{f}_e (\omega_e, t),$$

(1)

where the subscripts used in the notation denote, $d$ – self (damping), $p$ – parametric and $e$ – external excitation terms and $\mathbf{x}$ is a general coordinates vector.

Let us consider, as an example, a two degree of freedom system as presented in Fig.1. In this example we assume that the parametric resonance frequency is $2\omega$ while the external excitation frequency is $\omega$. This case corresponds to practical engineering applications [29]. The differential equations of motion of the model in Fig. 1 have the following form

$$m_1 \ddot{X}_1 + f_{d1} \left( X_1, \dot{X}_1 \right) + \delta_1 X_1 + \gamma_1 X_1^3 + (\delta_{12} - \mu \cos 2\omega t) (X_1 - X_2) = q \cos \omega t$$

$$m_2 \ddot{X}_2 + f_{d2} \left( X_2, \dot{X}_2 \right) + \delta_2 X_2 + \gamma_2 X_2^3 - (\delta_{12} - \mu \cos 2\omega t) (X_1 - X_2) = 0,$$

(2)

where, $f_{d1}(X_1, \dot{X}_1)$, $f_{d2}(X_2, \dot{X}_2)$ are nonlinear self-excitation damping functions. They can be modelled by van der Pol or Rayleigh type terms. For the Rayleigh model considered for further analysis, these functions are defined as $f_{d1} = -\alpha_1 X_1 + \beta_1 X_1^3$, $f_{d2} = -\alpha_2 X_2 + \beta_2 X_2^3$.

![Figure 1. Model of coupled self, parametrically and externally excited oscillators with 2 DOF.](image)

![Figure 2. Resonance curves versus external excitation amplitude $q$.](image)

At first, to check the possible dynamics and interactions between self and parametric vibrations driven by an external harmonic force, we consider a one degree of freedom system. This case corresponds to a single oscillator (No. I in Fig. 1) described by Eq. (2). The resonance curves obtained for $m_1 = 1$, $\alpha_1 = 0.01$, $\beta_1 = 0.05$, $\delta_1 = 1$, $\gamma_1 = 0.1$, $\delta_{12} = 1$, $\mu = 0.2$ are presented in Fig.2. We see that the resonance curves have an unexpected shape. For a small amplitude of external force (parameter $q$) the internal loop occurs inside the resonance curve. This loop may exist only below a certain threshold of excitation amplitude $q$. It means that even for a single degree of freedom system five solutions, which represent steady states, exist. However, stability analysis shows that only two of them are stable (solid lines of the resonance curve). For large values of $q$ (above the threshold) the loop disappears and the curve has a classical form. The loop occurrence phenomenon is studied in detail in [12].

We may expect that for many degrees of freedom system the loop should also appear inside the resonance curve. In order to verify this the set of coupled nonlinear equations (2) is transformed to normal coordinates $Y_j$, $(j = 1, 2)$, by a transformation $\mathbf{X} = \mathbf{uY}$, where $\mathbf{u}$ is a modal matrix. Taking into account that around resonances mainly one $Y_j$ normal coordinate in involved, the problem is solved analytically, assuming weakly nonlinear coupling [28]. This
approach cannot be accepted if the system is strongly nonlinear or various types of nonlinear terms are involved in its dynamics. Therefore, as an alternative, the nonlinear normal modes (NNMs) formulation is proposed. The modes are determined on the basis of free vibrations of a nonlinear system. Details of that formulation are presented in [29]. The NNMs include geometric nonlinear terms therefore these terms vanish after the coordinate transformation. However, self- and parametric excitations still exist. After the NNM projection the system is uncoupled into two single nonlinear oscillators

\[
M_j \ddot{Y}_j + M_j \omega_{0j}^2 Y_j - C_{\alpha j} \dot{Y}_j + C_{\beta j} \dot{Y}_j^3 - C_{\mu j} Y_j \cos 2\omega t = C_{qj} \cos \omega t ,
\]

where, \( M_j \) is modal mass and \( \omega_{0j} \) natural frequency \((j = 1, 2)\). Because the NNMs are amplitude dependent, the modal mass, natural frequency and modal matrix are also amplitude dependent. Equation (3) is solved by the multiple time scales method together with the constraint equation for NMNs. The resonance curves obtained for \( j = 2 \) and \( M_2 = 1.089, \omega_{02} = 1.16752, C_{\alpha2} = 0.01044, C_{\beta2} = 0.0501, C_{\mu2} = 0.292988 \), are presented in Figs.3 and 4. We see that for the two degree of freedom system the loop exists as well. However, the loop found by the LNM transformation (Fig.3) is located on the left part of the resonance curve. Contrary to this the NNM transformation gives a much smaller loop, located on the right side of the curve. The solution obtained by NNMs is much closer to real system dynamics and is in a good agreement with direct numerical simulations.

3. Regular and chaotic dynamics of an autoparametric system with magnetorheological damping

As has been mentioned in the introduction systems with pendulums are very sensitive and often, due to their strong nonlinearity, may behave unpredictably. The autoparametric pendulum-like system presented in Fig.5 is composed of the main structure i.e. a nonlinear oscillator and the pendulum. The oscillator’s suspension is considered as a classical linear suspension with viscous damping or a nonlinear suspension with a nonlinear spring and a magnetorheological (MR) damper. The nonlinear spring is introduced by a structural modification while the MR damper behaviour can be modified on-line from viscous effects, if the system is not activated, to a mixed mode system with viscous and dry friction components, when the damper is activated.

The differential equations of coupled motion of the oscillator and the pendulum have the form

\[
\ddot{X} + F_d (X, \dot{X}) + F_s (X) + \mu \lambda (\ddot{\varphi} \sin \varphi + \dot{\varphi}^2 \cos \varphi) = q \cos \omega t
\]

\[
\ddot{\varphi} + \alpha_2 \dot{\varphi} + \lambda (\ddot{X} + 1) \sin \varphi = 0.
\]

Figure 3. Resonance curve around the second natural frequency; linear normal mode (LNM) projection.

Figure 4. Resonance curve around the second natural frequency; nonlinear normal mode (NNM) projection.
Functions $F_d$ and $F_s$ represent the nonlinear damping and stiffness and they are defined respectively as, $F_d = \alpha_1 \dot{X} + \alpha_3 \tanh (\epsilon \dot{X})$, $F_s = X + \gamma X^3$.

![Figure 5. Model of an autoparametric system with a pendulum and a nonlinear semi-active suspension.](image)

A sample resonance curve of the pendulum obtained for data as follows: $\alpha_1 = 0.261354$, $\alpha_2 = 0.1$, $\alpha_3 = 0$, $\mu = 17.2279$, $\lambda = 0.127213$, $q = 2.45094$ and $\gamma = 0$ is presented in Fig.6(a).

![Figure 6. Resonance curve of the pendulum with marked unstable regions (dashed line) (a), chaotic attractor for $\omega = 1.1$, $\alpha_3 = 0$ (b), and bifurcation diagram of the chaotic attractor versus the MR damping parameter $\alpha_3$ (c); $\alpha_1 = 0.2613$, $\alpha_2 = 0.1$.](image)

Analysis of Eqs.4 exhibits instability zones in the pendulum motion. In Fig.6(a) we see an instability near the middle of the resonance curve. In this region the pendulum may go to chaotic motion or to regular rotations [16, 18, 14]. A strange chaotic attractor obtained for frequency $\omega = 1.1$ is presented in Fig.6(b). The attractor has been determined for parameter $\alpha_3 = 0$, which means that the MR damper is not activated. The bifurcation diagram in Fig.6(c) shows the influence of MR damping. We see that an increase of MR damping effectively suppresses rotations or oscillations of the pendulum. By applying a semi-active MR device we may control the dynamics of the pendulum and the structure treated as a whole.

4. Nonlinear control
We may take some advantages from strongly nonlinear couplings which may be applied to get special effects in a control strategy. Let us consider a light composite cantilever beam with active elements. Embedded Macro Fiber Composite (MFC) actuators are used for the real implementation of the control strategy. A sketch of a real system used for the validation of the theoretical results is presented in Fig.7.

The model of the beam (the plant) is assumed to be nonlinear, including geometrical and inertia nonlinear terms. The beam is forced kinematically by vertical motion of the clamped end. The basic structure is then coupled with an additional oscillator realised by the electrical part. In order to suppress vibrations, the beam is coupled with the controller by means of quadratic
terms. The mathematical model of this nonlinear control problem, taking into account the first mode response, has the form

\[ \ddot{v} + 2\mu \omega_s \dot{v} + \omega_s^2 v + \beta v^3 + \delta \left( \dot{v}^2 + v^2 \ddot{v} \right) = f \sin(\Omega t) + \gamma u^2 \]

\[ \ddot{u} + 2\zeta \omega_c \dot{u} + \omega_c^2 u = \alpha uv, \]

(5)

where \( f = y_0 \xi \Omega^2 \) and the excitation frequency \( \Omega \) is close to \( \omega_s \). Due to the quadratic coupling the effect of amplitude saturation can be achieved. But this requires tuning of the natural frequency of the controller to the natural frequency of the beam in the ratio \( \omega_c = \frac{1}{2} \omega_s \). The physical parameters of the beam taken for numerical and experimental tests have values, Young’s modulus: \( E = 25.5 \times 10^9 \) Pa, density: \( \rho = 2100 \) kg/m\(^3\), length: \( L = 236 \) mm, width: \( b = 12.8 \) mm. The dimensionless parameters calculated for various thickness \( h \) and tip mass \( M \) values are presented in Table 1. They allow a study of the dynamics of the beam with hard (\( \beta > 0 \)) and soft (\( \beta < 0 \)) elasticity terms. However, the global characteristic depends also on the nonlinear inertia terms (parameter \( \delta \)), which may overturn the slope of the resonance characteristic. The other parameters for all cases take these values: \( \mu = 0.01, \zeta = 0.001, \alpha = 2, \gamma = 0.01 \). The amplitude of excitation \( y_0 \) is varied.

| Physical parameters \( h \) (mm) | Dimensionless parameters \( M \) (gram) | \( \omega_s^2 \) | \( \beta \) | \( \delta \) | \( \xi \) |
|---|---|---|---|---|---|
| 1.0 | 0.5 | 9.3825 | 14.4108 | 3.2746 | 0.89663 |
| 2.1 | 0.5 | 10.7430 | 25.2913 | 3.8922 | 0.83866 |
| 2.1 | 5.0 | 4.8883 | -7.1401 | 1.0827 | 1.24173 |
| 2.1 | 15 | 2.2012 | -7.5974 | -0.32638 | 1.85326 |

As we may observe in Fig. 8, for a well tuned structure and properly selected parameters around a relative frequency \( \Omega / \omega_s \approx 1 \), the response of the system is very well suppressed. Moreover, the system’s response is well suppressed even if the excitation amplitude is increased. The diagrams
frequency-amplitude and frequency-gain $\alpha$ (Fig. 9) present regions of low amplitude vibration (dark-blue colour). The red colour represents vibrations of high amplitude. This vibration suppression is caused by nonlinear coupling, leading to the saturation phenomenon [28, 30].

5. Conclusions and remarks
A few selected phenomena of nonlinear dynamics are presented in this paper. It has been shown that the interaction between self, parametric and external vibrations leads to the occurrence of an internal loop inside the resonance curve. Then, five solutions, even for a simple one-degree-of-freedom system are found. Nonlinear couplings in the autoparametric pendulum-like structures result in instabilities in the resonance regions. Then the pendulum may go to full rotations or chaotic motion. It has been shown that the application of a semi-active MR damper may successfully reduce the pendulum’s vibration and allow control of the system behaviour. The quadratic nonlinearity, coupling the light composite beam and the MFC active element, may effectively suppress the structure vibrations. The saturation phenomenon maintains the system response at the design level even if the amplitude of excitation increases.

6. References
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Acknowledgments
The research leading to these results has received funding from the European Union Seventh Framework Programme (FP7/2007-2013), FP7-REGPOT-2009-1, under grant agreement No: 245479. The support by the Polish Ministry of Science and Higher Education-grant no 1471-1/7.PR UE/2010/7-is also acknowledged.