Nucleon Properties at Finite Volume: the $\epsilon'$-Regime

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Abstract

We study the properties of the nucleon in highly asymmetric volumes where the spatial dimensions are small but the time dimension is large in comparison to the inverse pion mass. To facilitate power-counting at the level of Feynman diagrams, we introduce $\epsilon'$-power-counting which is a special case of Leutwyler’s $\delta$-power-counting. Pion zero-modes enter the $\epsilon'$-counting perturbatively, in contrast to both the $\epsilon$- and $\delta$-power-countings, since $m_q\langle q\bar{q}\rangle V$ remains large. However, these modes are enhanced over those with non-zero momenta and enter at lower orders in the $\epsilon'$-expansion than they would in large volume chiral perturbation theory. We discuss an application of $\epsilon'$-counting by determining the nucleon mass, magnetic moment and axial matrix element at the first nontrivial order in the $\epsilon'$-expansion.

November 30, 2021
I. INTRODUCTION

Lattice QCD is the only known way of computing strong interaction observables rigorously from QCD. Even if (unquenched) lattice QCD simulations were to be performed at the physical values of the light-quark masses, \( m_q \), extrapolations would be required to obtain information about nature. As simulations are performed in lattice volumes of finite size, \( V = L_4 \times L^3 \) (where \( L_4 \) is the length of the time-dimension and \( L \) is the length of each spatial-dimension which are taken to be the same for simplicity), an extrapolation to the infinite-volume limit, \( L, L_4 \to \infty \), is required. In addition, as the lattice-spacing \( a \) is necessarily finite, an extrapolation to vanishing lattice spacing, \( a \to 0 \), is essential. Currently, an extrapolation in \( m_q \) from those used in the simulations down to those of nature is also required.

In order to perform these extrapolations, it is necessary to construct the effective field theory (EFT) appropriate for the lattice simulations. The dependence upon lattice spacing, lattice volume and quark masses computed in the EFT can be used to extract the coefficients of local operators that contribute to the observable of interest (at any given order in the EFT expansion). One can then use the EFT to compute the observable at the physical values of the quark masses, in the \( m_\pi L, m_\pi L_4 \gg 1 \) limit and in the \( a\Lambda \chi \ll 1 \) (where \( \Lambda \chi \) is the scale of chiral symmetry breaking) limit. Further, once appropriate counterterms have been determined, observables that are significantly more difficult to compute directly from lattice simulations can be computed with the EFT.

For simple one-body observables (we do not discuss the two-particle scattering amplitude or higher-body interactions on the lattice in this work) in large volumes, deviations from the infinite-volume limit are exponentially suppressed by factors of \( m_\pi L, m_\pi L_4 \gg 1 \) (the scale is set by \( m_\pi \) since the pions are the lightest hadrons) and can be computed with the \( p/\Lambda \chi \)-power-counting of infinite-volume chiral perturbation theory [1]. However, in small volumes with \( m_\pi L, m_\pi L_4 \sim \epsilon \ll 1 \) but with \( m_q \langle q\bar{q} \rangle L^3 L_4 \sim 1 \) (where \( \langle q\bar{q} \rangle \) is the infinite-volume quark condensate) one is in the \( \epsilon \)-regime, where the small expansion parameter is \( \epsilon \) and not \( p/\Lambda \chi \). Here, the contribution from the pion zero-modes are non-perturbative and must be resummed to all orders [2, 3, 4, 5, 6, 7, 8]. This regime has been explored both numerically and analytically in the light meson sector (e.g. Refs. [9, 10, 11, 12, 13, 14]), and efforts in the single nucleon [15], and heavy-meson [16] sectors are underway.

In this work we point out that for heavy-hadrons, such as nucleons, it is useful to explore the behavior of observables in highly asymmetric volumes where \( m_\pi L \ll 1 \) and \( m_\pi L_4 \gg 1 \). The underlying reason for this is simply that such objects are near their mass-shell when their kinetic energy and three-momentum are related via \( E \sim |k|^2 \). Such asymmetric volumes were first analyzed by Leutwyler [17] in the context of pion dynamics. Leutwyler introduced \( \delta \)-power-counting for which \( m_\pi L_4 \sim \delta^0 \) while \( m_\pi L \sim \delta^2 \) and \( \Lambda \chi L \sim 1/\delta \). This is similar to the \( \epsilon \)-regime as \( m_q \langle q\bar{q} \rangle L^3 L_4 \sim \delta^0 \) and the pion zero modes must be summed non-perturbatively. If \( L_4 \) becomes large, there is an additional small parameter, \( 1/L_4 \), and it is convenient to define a further power-counting describing this regime. We define a new expansion parameter \( \epsilon' \), and assign\(^1\)

\[
\Lambda \chi L \sim \frac{1}{\epsilon'}, \quad \frac{m_\pi^2}{\Lambda^2} \sim \frac{m_q}{\Lambda \chi} \sim \epsilon'^4, \quad \Lambda \chi L_4 \sim \frac{1}{\epsilon'^\alpha}, \quad (1)
\]

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\(^1\) There is some freedom in the counting of \( m_\pi \) and \( L_4 \); we simply require that \( m_q \langle q\bar{q} \rangle L^3 L_4 \gg 1 \).
in the power-counting, which provides \( m_\pi L \ll 1, m_\pi L_4 \gg 1 \) and \( \Lambda_\chi L, \Lambda_\chi L_4 \gg 1 \) for \( \alpha \gg 2 \).

In contrast to their behavior in the \( \epsilon \)-regime, in the \( \epsilon' \)-regime, the pion zero-modes (modes with zero spatial momentum) are perturbative as \( m_q \langle q\overline{q} \rangle L^3 \sim \epsilon'(1 - \alpha) \) is large. As for the \( m_\pi L, m_\pi L_4 \gg 1 \) regime, the vacuum structure in the \( \epsilon' \)-regime is perturbatively close to that at infinite volume. Correspondingly, the \( \epsilon' \)-regime is that part of the \( \delta \)-regime where \( m_q \langle q\overline{q} \rangle L^3 \) is large.

Here we shall take all three spatial dimensions to be of order \( \epsilon' \) but the same modified power-counting will emerge even if only one of spatial directions is small. In what follows we will work in the \( \alpha = \infty \) limit in which the time-direction is infinite and will assume the lattice spacing vanishes. In principle deviations from these limits can be included perturbatively.

Finite volume effects on the properties of heavy hadrons computed in lattice QCD have recently been investigated for \( L_4 = \infty \) using the standard power-counting of chiral perturbation theory valid for \( m_\pi L \gg 1 \) (see Refs. \[18, 19, 20, 21, 22, 23\]). The \( \epsilon' \)-counting introduced here allows for consistent calculations with \( m_\pi L \ll 1 \), and in what follows we calculate the nucleon mass, magnetic moment and axial matrix element in this regime.

### II. NUCLEONS AT FINITE VOLUME

At leading order in the \( p \)-expansion, the Lagrange density describing the low-energy dynamics of the nucleons, \( \Delta \)'s and pions (pseudo-Goldstone bosons) that is consistent with the spontaneously broken \( SU(2)_L \otimes SU(2)_R \) approximate chiral symmetry of QCD is \[24, 25\]

\[
\mathcal{L} = \overline{N} iv \cdot D N - \mathcal{T}_\mu iv \cdot D T^\mu + \Delta \mathcal{T}_\mu T^\mu + \frac{f^2}{8} \text{Tr} \left[ \partial_\mu \Sigma^{\dagger} \partial^\mu \Sigma \right] + \lambda \frac{f^2}{4} \text{Tr} \left[ m_q \Sigma^{\dagger} \right. + \text{h.c.} \left. \right] + 2g_A \overline{N} S^\mu A_\mu N + g_{\Delta N} \left[ T^{a\nu,\mu} A^d_{\mu, \nu} N_b \epsilon_{cd} + \text{h.c.} \right] + 2g_{\Delta\Delta} \overline{T}_\nu S^\mu A_\mu T^\nu ,
\]

(2)

where \( D \) is the chiral covariant derivative, \( N \) is the nucleon field operator, and \( T^\mu \) is the Rarita-Schwinger field containing the quartet of spin-\( \frac{3}{2} \) \( \Delta \)-resonances as defined in heavy-baryon \( \chi \)PT \[24, 25\]. The mass difference between the \( \Delta \)-resonances and the nucleon is \( \Delta \) (taken to be \( \sim m_\pi \)), and \( g_A \sim 1.26, g_{\Delta N} \) and \( g_{\Delta\Delta} \) are the (infinite volume, chiral limit) axial couplings between the baryons and pions. \( S_\mu \) is the covariant spin vector \[24, 25\], and \( v_\nu \) is the heavy-baryon four-velocity, with \( v^2 = 1 \). Pions appear in eq. (2) through \( \Sigma \) and \( A_\mu \) which are defined to be

\[
\Sigma = \exp \left( \frac{2iM}{f} \right) = \xi^2, \quad A^\mu = \frac{i}{2} \left( \xi \delta^\mu \xi^\dagger - \xi^\dagger \delta^\mu \xi \right), \quad M = \left( \begin{array}{c} \pi^0/\sqrt{2} \\ \pi^- \\ -\pi^0/\sqrt{2} \end{array} \right) ,
\]

(3)

and \( f \sim 132 \text{ MeV} \) is the pion decay constant. With this leading order Lagrange density one can determine the leading chiral loop corrections to observables in the single nucleon sector.

\[2\] In the naive constituent quark model, \( |g_{\Delta N}|/g_A = 6/5 \) and \( |g_{\Delta\Delta}|/g_A = 9/5 \).
A. The Nucleon Mass

1. Nucleon Mass in the \( p \)-expansion

The nucleon mass calculated at the first nontrivial order in the \( p \)-expansion in the isospin limit is known to be \( [24, 25, 26] \)

\[
M_N(\infty) = M_0 - 2m(\alpha_M + 2\sigma_M) - \frac{1}{8\pi f^2} \left[ \frac{3}{2} g_A^2 m_\pi^3 + \frac{4g_{\Delta N}^2}{3\pi} F_\pi \right], \tag{4}
\]

where \( M_0 \) is the chiral limit nucleon mass and the constants \( \alpha_M \) and \( \sigma_M \) are coefficients in the Lagrange density

\[
\mathcal{L}_M = 2\alpha_M \overline{N} \mathcal{M}_+ N + 2\sigma_M \overline{NN} \text{Tr} [\mathcal{M}_+] , \tag{5}
\]

where \( \mathcal{M}_+ = \frac{1}{2} (\xi^\dagger m_q \xi + m_q \xi^\dagger) \). The function \( F_\pi \) is given by

\[
F_\pi = (m_\pi^2 - \Delta^2) \left( \sqrt{\Delta^2 - m_\pi^2} \log \left( \frac{\Delta - \sqrt{\Delta^2 - m_\pi^2 + i\epsilon}}{\Delta + \sqrt{\Delta^2 - m_\pi^2 + i\epsilon}} \right) - \Delta \log \left( \frac{m_\pi^2}{\mu^2} \right) \right)
\]

\[-\frac{1}{2} \Delta m_\pi^2 \log \left( \frac{m_\pi^2}{\mu^2} \right) \] . \tag{6}

The three terms in eq. (4) scale as \( p^0 \), \( p^2 \) and \( p^3 \) in the \( p \)-expansion, respectively, and the loop contribution entering at order \( p^3 \) results from the diagrams shown in Fig. 1.

The modifications to the nucleon mass in a volume with all dimensions large compared to the inverse pion mass have been recently computed by Beane \( [18] \). It is found that the leading finite-volume corrections in the \( m_\pi L, m_\pi L_4 \gg 1 \) limit also arise from the one-loop diagrams that give the order \( p^3 \) contributions to the nucleon mass in the infinite-volume limit \( [18] \),

\[
M_N(L) = M_N(\infty) - \frac{1}{12\pi^2} \left[ \frac{9g_A^2}{2f^2} \mathcal{K}(0) + \frac{4g_{\Delta N}^2}{f^2} \mathcal{K}(\Delta) \right], \tag{7}
\]

where the function \( \mathcal{K}(\Delta) \) is

\[
\mathcal{K}(\Delta) = \sum_{n \neq 0} \int_{m_\pi}^{\infty} d\beta \frac{\beta^3}{\sqrt{\beta^2 + \Delta^2 - m_\pi^2}} \left[ \frac{1}{\beta |\mathbf{n}|} K_1(\beta L |\mathbf{n}|) - K_0(\beta L |\mathbf{n}|) \right], \tag{8}
\]
where the \( K_\nu(z) \) are modified Bessel functions of order \( \nu \). For \( \Delta = 0 \) this expression reduces to \[ K_0 = -\frac{\pi m_\pi^2}{2} \sum_{n \neq 0} \frac{1}{m_\pi L|n|} e^{-m_\pi L|n|} . \] (9)

2. Nucleon Mass in the \( \epsilon' \)-expansion

The order at which operators contribute in the \( \epsilon' \)-expansion is different from the order at which they contribute in the \( p \)-expansion. The local operators given in eq. (5) contribute at order \( p^2 \) in the \( p \)-expansion, but contribute at order \( \epsilon'^4 \) in the \( \epsilon' \)-expansion. Power-counting the loop contributions is a little more complicated than in the \( p \)-regime, and is similar in some respects to the counting in the \( \epsilon \)-regime. Since \(|p| \sim 1/L \sim \epsilon' \) and \( m_\pi \sim \epsilon'^2 \), the spatial momentum zero-mode and non-zero modes must be treated separately as their counting is different,

\[
\frac{i}{k_0^2 - |k|^2 - m_\pi^2 + i\eta} \sim \begin{cases} L^2 \sim \epsilon'^{-2} & \text{non-zero mode} \\ m_\pi^2 \sim \epsilon'^{-4} & \text{zero mode (}|k| = 0) \end{cases} .
\] (10)

In calculating loop diagrams one picks out the pole at \( k_0 = \sqrt{|k|^2 + m_\pi^2 + i\eta} \), so \( k_0 \sim |k| \sim \epsilon' \) if \( |k| \neq 0 \) and \( k_0 \sim m_\pi \sim \epsilon'^2 \) otherwise.

For the one-loop diagrams contributing to the nucleon mass, the momentum zero-mode does not contribute due to the derivative couplings in eq. (2) and therefore diagram (a) in Fig. 1 becomes

\[
\text{loop} \sim \frac{1}{L^3} \sum_k \int \frac{dk_0}{2\pi} \frac{k_0}{k_0^2 - |k|^2} \frac{1}{k_0} |k|^2 \sim \epsilon'^3 \epsilon' \epsilon'^{-2} \epsilon'^{-1} \epsilon'^2 \sim \epsilon'^3 .
\] (11)

Since the \( k_0 \) integral gives \( k_0 \sim |k| \sim \epsilon' \), and \( \Delta \sim m_\pi \sim \epsilon'^2 \),

\[
\frac{1}{k_0 - \Delta} = \frac{1}{k_0} + \frac{\Delta}{k_0^2} + O(\epsilon') ,
\] (12)

and the leading contribution from loops with \( \Delta \)-intermediate states can be found by setting \( \Delta = 0 \). After some elementary manipulations, it is easy to show that in the \( \epsilon' \)-regime, the function \( \mathcal{K}(\Delta) \) in eq. (8) becomes \[ \mathcal{K}(\Delta) = -\frac{2\pi^2}{L^3} \left[ 1 + \frac{\Delta L}{2\pi} c_1 + \frac{(m^2 - \Delta^2) L^2}{4\pi^2} c_1 \right] + O(\epsilon'^6) , \] (13)

\[ \text{In particular we have used the relation}
\]

\[
\sum_{n \neq 0} \frac{1}{|n|} e^{-z|n|} = \sum_{n \neq 0} \frac{1}{|n|} e^{-|n|} + \frac{4\pi}{z^2} - 4\pi - 1 - \sum_{p \neq 0} \frac{4\pi(z^2 - 1)}{(z^2 + 4\pi^2|p|^2)(1 + 4\pi^2|p|^2)} + z ,
\]

and define divergent sums through dimension regularization, e.g.

\[
\sum_{n \neq 0} \frac{1}{|n|} = \lim_{z \to 0} \left[ \sum_{n \neq 0} \frac{1}{|n|} e^{-z|n|} \right] .
\] 5
where \( c_1 = -2.8372974 \) is the same geometric number that appears in the \( 1/L \) expansion of the lowest two-particle continuum energy-level in a finite volume [27].

From this expression, we see that the one-loop diagrams that produce the order \( p^3 \) contribution in eq. (14) contribute at order \( \epsilon'^3 \) in the \( \epsilon' \)-expansion, and provides the leading volume dependence in this regime. Combining this result with eq. (7), we arrive at

\[
M_N(L) = M_0 + \frac{1}{f^2 L^3} \left[ \frac{3}{4} g_A^2 + \frac{2}{3} g_{\Delta N}^2 \right] + O(\epsilon'^4) .
\]

A detailed study of the volume dependence of the nucleon mass in the \( \epsilon' \)-regime (including higher order corrections) would provide a clean method of determining this particular combination of axial couplings. This may provide the simplest way of determining the transition axial coupling, \( g_{N\Delta} \), as it cannot be reliably determined from analysis of a three-point function at the physical values of the quark masses.

B. The Nucleon Magnetic Moment

1. Nucleon Magnetic Moment in the \( p \)-expansion

The magnetic moment of the nucleon has been computed in two-flavor and three-flavor infinite-volume chiral perturbation theory [28, 29, 30, 31], and in the isospin-limit of the two-flavor case is known to be

\[
\hat{\mu}(\infty) = \mu_0 + \mu_1 \tau^3 - \frac{M_N}{4\pi f^2} \left[ g_A^2 m_\pi + \frac{2}{9} g_{\Delta N}^2 F_\pi \right] \tau^3 ,
\]

where

\[
\pi F_\pi = \sqrt{\Delta^2 - m_\pi^2} \log \left( \frac{\Delta - \sqrt{\Delta^2 - m_\pi^2 + i\epsilon}}{\Delta + \sqrt{\Delta^2 - m_\pi^2 + i\epsilon}} \right) - \Delta \log \left( \frac{m_\pi^2}{\mu^2} \right) .
\]

In the limit \( \Delta \to 0 \), \( F_\pi = m_\pi \). The quantities \( \mu_{0,1} \) are the coefficients of the dimension-five operators in the Lagrange density

\[
\mathcal{L} = \frac{e}{4M_N} F_{\mu\nu} \left[ \mu_0 \overline{N} \sigma^{\mu\nu} N + \mu_1 \overline{N} \sigma^{\mu\nu} \tau^3 \xi N \right] ,
\]

where \( F_{\mu\nu} \) is the electromagnetic field-strength tensor and \( \tau^a_{\xi^+} = \frac{1}{2} (\xi^{\dagger} \tau^a \xi + \xi \tau^a \xi^{\dagger}) \). The leading contribution to the magnetic moment comes from these dimension-five operators and is of order \( p^0 \), while the contributions from the one-loop diagrams shown in Fig. 2 [the third term in eq. (13)] are of order \( p \).

Beane [18] computed the leading finite-volume corrections to the nucleon magnetic moment in the \( p \)-expansion for \( m_\pi L, m_\pi L_4 \gg 1 \), finding

\[
\hat{\mu}(L) = \hat{\mu}(\infty) + \frac{M_N}{6\pi^2 f^2} \left[ g_A^2 \mathcal{V}(0) + \frac{2}{9} g_{\Delta N}^2 \mathcal{V}(\Delta) \right] \tau^3 ,
\]
FIG. 2: One-loop graphs that contribute to the nucleon magnetic moment. The solid, thick-solid and dashed lines denote a nucleon, $\Delta$-resonance, and a meson, respectively. The solid-squares denote an axial coupling given in eq. (2) and the solid-circles denote a leading-order electromagnetic interaction.

FIG. 3: One-loop operator insertion that contributes to the nucleon magnetic moment at next-to-leading order in the $\epsilon'$-regime. At large volume, this contribution is suppressed, contributing at order $p^2$. The crossed-circle indicates the leading two-pion correction to the isovector electromagnetic current from eq. (17).

where the function $\mathcal{Y}(\Delta)$ is given by

$$
\mathcal{Y}(\Delta) = \sum_{n \neq 0} \int_{m_\pi}^{\infty} d\beta \frac{\beta}{\sqrt{\beta^2 + \Delta^2 - m_\pi^2}} \left[ 3K_0(\beta L|n|) - \beta L|n|K_1(\beta L|n|) \right],
$$

and in the $\Delta = 0$ limit one has

$$
\mathcal{Y}(0) = -\frac{\pi m_\pi}{2} \sum_{n \neq 0} \left[ 1 - \frac{2}{m_\pi L|n|} \right] e^{-m_\pi L|n|}.
$$

\textsuperscript{4} The function $\mathcal{Y}(\Delta)$ is related to the function $K(\Delta)$ defined in eq. (8) by

$$
\mathcal{Y}(\Delta) = -2 \frac{\partial K(\Delta)}{\partial m_\pi^2}.
$$
2. **Nucleon Magnetic Moment in the $\epsilon'$-expansion**

The finite-volume contributions to the magnetic moment in the $\epsilon'$-regime behave quite differently from those of the nucleon mass. The dimension-5 operators in eq. (17) again give the leading contribution, entering at order $\epsilon'^0$. Part of the leading finite-volume correction, order $\epsilon'$, comes from the one-loop diagrams of Fig. 2 and it is straightforward to calculate their contribution to the nucleon magnetic moment in the $\epsilon'$-regime. However, at the same order in the $\epsilon'$-expansion, the two-pion contribution to the isovector magnetic moment operator generates the tadpole loop diagram shown in Fig. 3. This diagram contributes at sub-leading order ($p^2$) in the $p$-expansion, but in the $\epsilon'$-regime, the modified counting of zero modes enhances this to order $\epsilon'$ if the pion has zero three-momentum. The infinite volume integral corresponding to this diagram (defined in dimensional regularization) is

$$ R_\pi = \mu^{4-n} \int \frac{d^n q}{(2\pi)^n} \frac{1}{q^2 - m_\pi^2 + i\eta} . $$

In the $\epsilon'$-regime, this becomes

$$ \frac{1}{L^3} \sum_\mathbf{q} \int dq_0 \frac{1}{q_0^2 - |\mathbf{q}|^2 - m_\pi^2 + i\eta} , $$

and again, $q_0 \sim |\mathbf{q}| \sim \epsilon'$ generically, but for $|\mathbf{q}| = 0$, $q_0 \sim m_\pi \sim \epsilon'^2$. Thus the contribution of this diagram is order $\epsilon'$ for zero-modes, and $\epsilon'^2$ otherwise.

Taking these various contributions into account, we see that the nucleon magnetic moment in the $\epsilon'$-regime is

$$ \hat{\mu}(L) = \mu_0 + \mu_1 \tau^3 \left[ 1 - \frac{1}{m_\pi f^2 L^3} \right] + \frac{M_N c_1}{6\pi f^2 L} \left[ g_A^2 + \frac{2}{9} g_{\Delta N}^2 \right] \tau^3 + \mathcal{O}(\epsilon'^2) , $$

and the leading volume dependence of the isovector magnetic moment is $\sim 1/L, 1/(m_\pi L^3)$ occurring at order $\epsilon'$. For the isoscalar magnetic moment, the first correction occurs at order $\epsilon'^2$.

The appearance of the mass of the pion in the denominator of one of the leading corrections might be cause for worry. However, one is not able to take the $m_\pi \to 0$ limit of this result and remain in the $\epsilon'$-regime. For small enough pion mass, $m_\pi L$ is no longer of order $\epsilon'$ and the perturbative series must be rearranged to yield sensible results. One would move into the $\delta$-regime, where the pion zero-modes would need to be resummed.

The above result has a number of interesting features. The first is that pion mass dependent and pion mass independent volume corrections enter at the same order. For a given volume, the numerical value of $m_\pi L^2 \sim \mathcal{O}(1)$ can enhance or suppress different parts of the correction. A further, somewhat curious property of this result is the strong dependence upon the geometry of the spatial dimensions as is evidenced by the appearance of $c_1$ in the leading correction. In a study of the two-nucleon system in the presence of background electroweak fields [32], it was observed that there are particular spatial geometries for which the coefficient $c_1$ vanishes (e.g. in a volume with sides in the ratio 1 : 1 : 3,72448) and others where it is much larger than its symmetric volume value (see also Ref. [33]). Thus, by changing the shape of the spatial volume, the interplay of the different order $\epsilon'$ contributions can also be modified. In lattice calculations, such freedom should enable one to eliminate or enhance the leading corrections.
C. The Axial Current Matrix Element in the Nucleon

1. The Axial Current Matrix Element in the Nucleon in the $p$-expansion

The nucleon matrix element of the axial current has been extensively studied in the $p$-expansion, and is known to be \[ 19 \]

\[
\Gamma_{NN}(\infty) = g_A - i \frac{4}{3f^2} \left[ 4g_A^3 J_\pi(0) + 4 \left( g_{\Delta N}^2 g_A + \frac{25}{81} g_{\Delta N}^2 g_{\Delta \Delta} \right) J_\pi(\Delta) \right. \]

\[
+ \frac{3}{2} g_A R_\pi - \frac{32}{9} g_{\Delta N}^2 g_A N_\pi(\Delta) \left. \right] + \text{counterterms,} \tag{24}
\]

where the loop contributions, defined with dimensional regularization, are

\[
J_\pi(\Delta) = \mu^{4-n} \int \frac{d^4q}{(2\pi)^4} \frac{(S \cdot q)^2}{v \cdot q - (v \cdot q - (v \cdot q + i\eta) q^2 - m_\pi^2 + i\eta)} \tag{25}
\]

\[
N_\pi(\Delta) = \mu^{4-n} \int \frac{d^4q}{(2\pi)^4} \frac{(S \cdot q)^2}{v \cdot q - (v \cdot q + i\eta) q^2 - m_\pi^2 + i\eta},
\]

and $R_\pi$ is defined in eq. (21). These result from the diagrams shown in Fig. 4 and to be clear, we emphasize that $g_A, g_{\Delta N}$ and $g_{\Delta \Delta}$ are the infinite volume, chiral limit axial couplings.

Recently, the finite volume corrections in the $m_\pi L \gg 1$ limit of the $p$-expansion have been computed \[ 19 \], and are

\[
\Gamma_{NN}(L) = \Gamma_{NN}(\infty) + \frac{m_\pi^2}{3\pi^2 f^2} \left[ g_A^3 F_1 + \left( g_{\Delta N}^2 g_A + \frac{25}{81} g_{\Delta N}^2 g_{\Delta \Delta} \right) F_2 + g_A F_3 + g_{\Delta N}^2 g_A F_4 \right], \tag{26}\]

where

\[
F_1 = \sum_{n \neq 0} \left[ K_0(m_\pi L|n|) - \frac{K_1(m_\pi L|n|)}{m_\pi L|n|} \right],
\]

\[
F_2 = -\sum_{n \neq 0} \left[ \frac{K_1(m_\pi L|n|)}{m_\pi L|n|} + \frac{\Delta^2 - m_\pi^2}{m_\pi^2} K_0(m_\pi L|n|) \right. \]

\[
- \frac{\Delta}{m_\pi^2} \int_{m_\pi}^{\infty} d\beta \frac{2\beta K_0(\beta L|n|) + (\Delta^2 - m_\pi^2) L|n| K_1(\beta L|n|)}{\sqrt{\beta^2 + \Delta^2 - m_\pi^2}} \left. \right], \tag{27}\]

\[
F_3 = -\frac{3}{2} \sum_{n \neq 0} \frac{K_1(m_\pi L|n|)}{m_\pi L|n|},
\]

\[
F_4 = \frac{8}{9} \sum_{n \neq 0} \left[ \frac{K_1(m_\pi L|n|)}{m_\pi L|n|} - \frac{\pi e^{-m_\pi L|n|}}{2\Delta L|n|} - \frac{\Delta^2 - m_\pi^2}{m_\pi^2 \Delta} \int_{m_\pi}^{\infty} d\beta \frac{\beta K_0(\beta L|n|)}{\sqrt{\beta^2 + \Delta^2 - m_\pi^2}} \right].
\]

These finite-volume corrections enter at order $p^2$ in the $p$-expansion, and arise from the one-loop diagrams shown in Fig. 4.
FIG. 4: One-loop graphs that contribute to the matrix elements of the axial current in the nucleon. A solid, thick-solid and dashed line denote a nucleon, a $\Delta$-resonance, and a pion, respectively. The solid-squares denote an axial coupling given in eq. (2), while the crossed circle denotes an insertion of the axial-vector current operator. Diagrams (a) to (e) are vertex corrections, while diagrams (f) and (g) give rise to wave-function renormalization.

2. The Axial Current Matrix Element in the Nucleon in the $\epsilon'$-expansion

The finite-volume contributions to the axial matrix element in the $\epsilon'$-regime behave differently again from those of both the nucleon mass and the magnetic moment. The one-loop diagrams with nucleon or $\Delta$ intermediate states, diagrams (a), (b), (c), (d), (f) and (g) of Fig. 4 contribute at order $\epsilon'^2$, as the momentum zero-modes do not contribute due to the derivative coupling to the baryons. That is,

$$\text{loop} \sim \frac{1}{L^3} \sum_q \int dq_0 \frac{1}{q_0^2 - |q|^2} \left[ \frac{1}{q_0^2} \right]^2 |q|^2 \sim \epsilon'^3 \epsilon' \epsilon'^{-2} \epsilon'^{-2} \epsilon'^2 \sim \epsilon'^2,$$

(28)

where $q_0 \sim |q| \sim \epsilon'$.

Pion zero-modes do contribute to the one-loop diagram (diagram (e) of Fig. 4) resulting from the two pion term in the expansion of the axial-current insertion (there is no derivative to eliminate such a contribution). As for the magnetic moment, this zero-mode contribution is order $\epsilon'$, while the non-zero-mode contributions remain at order $\epsilon'^2$. Thus, one can determine the leading volume dependence of the axial-current matrix element directly from
FIG. 5: A two-loop contribution to $g_A$ that contributes at order $\epsilon'^2$. This diagram enters at order $p^4$ in the $p$-expansion.

the zero-mode contribution to the function $F_3$ in eq. (27), and finds that

$$\Gamma_{NN}(L) = g_A \left[ 1 - \frac{1}{m_\pi f^2 L^3} \right] + O(\epsilon'^2),$$

(29)

in the $\epsilon'$-regime. Again, one cannot take the pion mass to zero at fixed $L$ in this expression as one rapidly leaves the realm of applicability of the $\epsilon'$-expansion.

It is tempting to simply take finite-volume loop corrections computed in the $p$-regime (e.g. eq. (26)) and look at the $m_\pi L \lesssim 1$ limit in an attempt to recover the result in the $\epsilon'$-regime. However, both the magnetic moment and the axial-current matrix element demonstrate why this will give incorrect results. As discussed above, at order $\epsilon'^2$ there will be contributions from the various one-loop diagrams in Fig. 4 and from the nonzero modes of the two-pion operator tadpole. In addition, there will be contributions from the two-loop diagram involving momentum zero-mode pions that arise from the four-pion piece of the axial-current, Fig. 5. Such contributions only enter at order $p^4$ in the $p$-expansion and are not present in eq. (26).

III. DISCUSSION AND CONCLUSIONS

We have defined a power-counting ($\epsilon'$-power-counting) to describe observables calculated in lattice QCD on highly asymmetric lattices which are long in the time-dimension and short in the spatial-dimensions, compared to the inverse pion mass. In such volumes, the relative size of contributions from counterterms and loop diagrams is modified from that of both the $p$-power-counting and the $\epsilon$-power-counting. Loop diagrams involving momentum zero-modes of the pion field are promoted over those involving the non-zero modes but still remain perturbative (as opposed to the more general $\delta$-regime of Leutwyler), while the quark-mass dependent counterterms are demoted.

One issue that is somewhat uncertain is the range of applicability of the $\epsilon'$-power-counting. Looking more closely at the expansion at the level of Feynman diagrams, it is clear that in the $L_4 \rightarrow \infty$ limit it is an expansion in factors of

$$\epsilon' \sim \frac{m_\pi L}{2\pi}, \frac{\Delta L}{2\pi}, \frac{2\pi}{\Lambda_\chi L},$$

(30)

and the factors of $2\pi$ are important. Of course, these same factors of $2\pi$ arise in the $\epsilon$-expansion. Therefore, the $\epsilon'$-power-counting applies in the region where $m_\pi L \ll 2\pi$ and
$\Lambda \chi L \gg 2\pi$. For the physical values of the $m_\pi$ and $\Lambda_\chi$, a lattice dimension of $L = 2.5 - 4$ fm is in the $\epsilon'$-regime ($\epsilon' < \frac{1}{2}$ in this range). Whilst for $m_\pi = 300$ MeV, boxes $L \sim 2.5$ fm are just inside the $\epsilon'$-regime.

As in the $\epsilon$-regime, with such narrow windows of applicability and with a marginally small expansion parameter, it will be important to extend our analysis to higher orders. Since zero-modes and non-zero modes are treated on different footings, this rapidly becomes complicated. A tractable way to formulate the problem is to define separate fields for the different modes and construct the low energy effective theory directly in terms of these degrees of freedom. Such an approach would be similar in spirit to that of soft-collinear effective theory (e.g., [35]), but we do not pursue it here.

Throughout this work, we have taken $\alpha = \infty$ so that the time direction is infinite and we have effectively considered a (Minkowski-space) Hamiltonian formulation of the problem. This was a convenient simplification and not strictly necessary for our analysis; all that is required is that $\alpha \gg 2$. Provided that this is the case, corrections to the $L_A \to \infty$ limit will appear perturbatively in the $\epsilon'$-expansion. Concomitant with the above constraints on $L$ is that $m_\pi L^4 \gg 2\pi$, so the time direction of $\epsilon'$ lattices is large, $L_A/L \sim \epsilon' (1 - \alpha)$.

It is obvious from the above discussion that most (if not all) current lattice simulations do not fall into the $\epsilon'$-regime. However, since large finite volume effects have been observed in lattice calculations of various hadronic quantities including $g_A$ [36, 37, 38], it is interesting to insert some numbers. Taking $m_\pi = 300$ MeV and $L = 2.5$ fm (which would require $L_A \gtrsim 6$ fm to be in the $\epsilon'$-regime), the leading $\epsilon'$ correction to $g_A$ is found to be 10% from eq. (29). That is, a measurement of the axial matrix element of this volume will give a result 10% lower than the infinite volume value at this pion mass. For the same parameters, the modification of the nucleon mass is $\sim 2\%$. If such large corrections are present in lattice calculations of $g_A$, they would go a long way towards explaining the observed discrepancy.

Here we have worked in two-flavour QCD, but it is also interesting to consider the $\epsilon'$-regime in quenched QCD. It has long been known that the singlet propagator of quenched theories leads to strongly enhanced finite size effects and the same is true in the $\epsilon'$-expansion. Loops involving the zero-mode of a $\eta'$ meson will contribute at order $\epsilon'^0$ and must be treated non-perturbatively by integrating over certain directions in the coset space of the graded group [39]. These modes are precociously in the $\epsilon$-regime.

It is interesting to consider two nucleons in the $\epsilon'$-regime. A fundamental issue in nuclear physics is to understand the quark mass dependence of nuclear properties and processes. Recent progress [40, 41, 42, 43] in this area has highlighted the fact that quark mass dependent four nucleon operators in nuclear effective field theories are extremely difficult to isolate experimentally and and in all likeliness will only be determined from lattice QCD. For large volumes where $r \ll L$ (where $r \sim m_\pi^{-1}$ is the range of the nuclear force), Lüscher's analysis [27] of two particle energy levels is applicable and one can determine the two-nucleon elastic scattering parameters [44]. However, to determine the contributions of the quark mass dependent operators, lattice calculations must be performed over a range of quark masses. Additionally, quark mass dependent contributions are only suppressed by one or two orders in the $p$-counting that is valid in this regime and make an important contribution to the scattering parameters. In contrast, in the regime where $\epsilon'$-counting is valid, the fine-tuning of the two-nucleon sector that leads to the unnaturally large scattering lengths and the anomalously weak binding of the deuteron found in nature will not persist. Therefore in this regime the power-counting of operators in the effective field theory will be according to their engineering dimensions. Thus the leading mass dependent operators will be relatively
suppressed by $\epsilon'$ and analysis of two-nucleon energy levels will cleanly determine the quark mass independent contributions. A potentially serious impediment to extracting fundamental information about the two-nucleon sector from this regime is that the ultraviolet cutoff of the two-nucleon effective theory with dynamical pions is $\Lambda_{NN} \sim 300$ MeV which makes it unlikely that the required hierarchy of length scales can be established.

Finally, the utility of the $\epsilon'$-regime is not restricted to heavy objects such as nucleons. The very large time-dimension allows for $q_0 \sim |q|^2$ kinematics to have heavy objects near their mass-shell, but does not preclude having $q_0 \sim |q|$, as is required to have light-particles near mass-shell. Therefore, in the analysis of, for instance, the matrix element of the twist-2 isovector operators in the pion, there is a contribution from the diagram with four-pions emerging from the operator insertion which contributes at the one-loop level. The momentum zero-mode contribution from such a diagram will be enhanced in the $\epsilon'$-regime over the non-zero-mode contributions, and it is clear that the leading volume dependence will be of the same form as that of the axial-current matrix element in the nucleon, $\sim 1/(m_\pi f^2 L^3)$.

**Acknowledgments**

We would like to thank Silas Beane, Paulo Bedaque, Jiunn-Wei Chen, Harald Griesshammer, David Lin, Gautam Rupak and Steve Sharpe for useful discussions. This work is supported in part by the U.S. Dept. of Energy under Grant No. DE-FG03-97ER4014.

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