Solving Homotopy Domain Equations

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Abstract
In order to get $\lambda$-models with a rich structure of $\infty$-groupoid, which we call “homotopy $\lambda$-models”, a general technique is described for solving domain equations on any cartesian closed $\infty$-category (c.c.i.) with enough points. Finally, the technique is applied in a particular c.c.i., where some examples of homotopy $\lambda$-models are given.

Keywords: Homotopy domain theory, Homotopy domain equation, Homotopy lambda model, Kan complex, Infinity groupoid,

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1. Introduction

The purpose of this work is to give a follow-up on the project of generalisation of Dana Scott’s Domain Theory [1] and [2], to a Homotopy Domain Theory, which began in [3], in the sense of providing methods to find $\lambda$-models that allow raising the interpretation of equality of $\lambda$-terms (e.g., $\beta$-equality, $\eta$-equality etc.) to a semantics of higher equalities.

In theoretical terms, one would be looking for a type of $\lambda$-models with $\infty$-groupoid properties, such as CW complexes, Kan complexes etc. For this task, the strategy is to search in the closed cartesian $\infty$-category (cci) $Kl(P)$ [3] (generalized version of the Kleisli bicategory of [4]), reflexive Kan complexes with relevant information, by means of the solution of domain equations proposed in this cci, which we call homotopy domain equations.

To guarantee the existence of Kan complexes with good information, we define the split Kan complexes, which in intuitive terms, are those that have
holes in all higher dimensions and also holes in the locality or class of any vertex. And we establish conditions to prove the existence of split Kan complexes which are reflexive, which we call homotopy $\lambda$-models; stronger than the homotopic $\lambda$-models initially defined in [5], and developed in [3].

It should be clarified that the literature around the Kan complexes is related to some computational theories, such as Homotopy Type Theory (HoTT) [6], so that to ensure the consistency of HoTT Voevodsky [7] (see [8] for higher inductive types) proved that HoTT has a model in the category of Kan complexes (see [6]).

To meet the goal of getting homotopy $\lambda$-models, in Section 2 we introduce the complete homotopy partial orders (c.h.p.o) as a direct generalization of the c.p.o’s, where the sets are replaced by Kan complexes and the order relations by weakly order relations, in Section 3 we propose a method for solving homotopy domain equations on any c.c.i., namely: first, one solves the contravariant functor problem in a similar way to classical Domain Theory, and later one uses a version from fixed point theorem to find some solutions of this equation. Finally, in Section 4 the methods of the previous section are applied in a particular c.c.i., and thus one ends up guaranteeing the existence of homotopy $\lambda$-models which, as previously mentioned, present an $\infty$-groupoid structure with relevant information in higher dimensions.

2. Complete Homotopy Partial Orders

In this chapter we introduce the complete homotopy partial orders (c.h.p.o) as a direct generalization of the c.p.o’s, where the sets are replaced by Kan complexes and the order relations $\leq$ by weakly order relations $\preceq$. The proofs of the propositions, lemmas and theorems are very similar to the classic case of the c.p.o’s [9].

Definition 2.1 (h.p.o). Let $\hat{K}$ be an $\infty$-category. The largest Kan complex $K \subseteq \hat{K}$ is a homotopy partial order (h.p.o), if for every $x, y \in K$ one has that
\( \hat{K}(x, y) \) is contractible or empty. Hence, the Kan complex \( K \) admits a relation of h.p.o \( \preceq \) defined for each \( x, y \in K \) as follows: \( x \preceq y \) if \( \hat{K}(x, y) \neq \emptyset \), hence the duple \((K, \preceq)\) is a h.p.o. (we denote only \( K \)). The \( \infty \)-category \( \hat{K} \) is also called a h.p.o.

**Definition 2.2 (c.h.p.o).** Let \( K \) be h.p.o.

1. A h.p.o \( X \subseteq K \) is directed if \( X \neq \emptyset \) and for each \( x, y \in X \), there exists \( z \in X \) such that \( x \preceq z \) and \( y \preceq z \).
2. \( K \) is a complete homotopy partial order (c.h.p.o) if
   a. There are initial objects, i.e., \( \bot \in K \) is an initial object if for each \( x \in K \), \( \bot \preceq x \).
   b. For each directed \( X \subseteq K \) the supremum (or colimit) \( \bigvee X \in K \) exists.

**Definition 2.3 (Continuity).** Let \( K \) and \( K' \) be c.h.p.o’s. A functor \( f : K \to K' \) is continuous if \( f(\bigvee X) \simeq \bigvee f(X) \), where \( f(X) \) is the essential image.

**Proposition 2.1.** Continuous functors on c.h.p.o’s are always monotonic.

*Proof.* Let \( f : K \to K' \) be a functor continuous between c.h.p.o’s and suppose the non-trivial case \( a \prec b \) in \( K \). Since the h.p.o \( \{a, b\} \subseteq K \) is directed, by continuity of \( f \) we have

\[
f(a) \preceq \bigvee \{f(a), f(b)\} \simeq f(\bigvee \{a, b\}) = f(b).
\]

\( \square \)

The cartesian product between c.h.p.o’s can be considered again as a c.h.p.o.

**Proposition 2.2.** Given the c.h.p.o’s \( K, K' \), let \( K \times K' \) the cartesian product partially ordered by

\[
(x, x') \preceq (y, y') \text{ if } x \preceq y \text{ and } x' \preceq y'.
\]

Then \( K \times K' \) is a c.h.p.o with for directed \( X \subseteq K \times K' \)

\[
\bigvee X = (\bigvee X_0, \bigvee X_1)
\]

where \( X_0 \) is the projection of \( X \) on \( K \), and \( X_1 \) is the projection of \( X \) on \( K' \).
Proof. \((⊥, ⊥')\) is a initial object of \(K \times K'\). On other hand, if \(X \subseteq K \times K'\) is directed, by definition of order on \(K \times K'\) are also directed, so the supremum \(∀ X \in K \times K'\) exists. □

Definition 2.4. Let \(K, K'\) be c.h.p.o's. Define the full subcategory \([K \to K'] \subseteq \text{Fun}(K, K')\) of the continuous functors. Since \(\hat{K}\) has enough points (is weakly contractible), we can define the order pointwise on \([K \to K']\) by:

\[f \preceq g \iff ∀ x \in K, f(x) \preceq' g(x).\]

Notation 2.1. Let \(K\) be a h.p.o and \(P\) a predicate. Denote by \(\langle x \in K \mid P(x)\rangle\) the h.p.o induced by the order of \(K\), whose objects are the \(x \in K\) which satisfy the property \(P\).

Lemma 2.1. Let \(\langle f_i \rangle_i \subseteq [K \to K']\) be a indexed directed of functors. Define

\[f(x) = \bigcurlyvee_i f_i(x).\]

Then \(f\) is well defined and continuous.

Proof. Since \(\langle f_i \rangle_i\) is directed, \(\langle f_i(x) \rangle_i\) is directed for each \(x \in K\), and since \(\hat{K}\) has enough points, the functor \(f\) exists. One other hand, for directed \(X \subseteq K\)

\[f(\bigcurlyvee X) = \bigcurlyvee_i f_i(\bigcurlyvee X) \simeq \bigcurlyvee_i f_i(X) \simeq \bigcurlyvee_i f_i(X) = \bigcurlyvee f(X).\]

□

Proposition 2.3. \([K \to K']\) is a c.h.p.o with supremum of a directed \(F \subseteq [K \to K']\) defined by

\[\bigcurlyvee F(x) = \bigcurlyvee (f(x) \mid f \in F).\]

Proof. The constant functor \(\lambda x.⊥'\) is a initial object of \([K \to K']\). By Lemma 2.1 the functor \(\lambda x.\bigcurlyvee (f(x) \mid f \in F)\) is continuous, which is the supremum of \(F\). □
Lemma 2.2. Let $f : K \times K' \to K''$. Then $f$ is continuous iff $f$ is continuous in its arguments separately, that is, iff $\lambda x.f(x, x'0)$ and $\lambda x'.f(x0, x')$ are continuous for all $x0, x'_0$.

Proof. ($\Rightarrow$) Let $g = \lambda x.f(x, x'0)$. Then for the directed $X \subseteq K$

$$g(\bigvee X) = f(\bigvee X, x'0)$$

$$\simeq \bigvee f(X \times \{x0\}); \quad f \text{ is continuous and } X \times \{x0\} \text{ is directed}$$

$$= \bigvee g(X).$$

Similarly $\lambda x'.f(x0, x')$ is continuous.

($\Leftarrow$) Let $X \subseteq K \times K'$ be directed. So

$$f(\bigvee X) = f(\bigvee X_0, \bigvee X_1)$$

$$\simeq \bigvee f(X_0, \bigvee X_1); \quad f \text{ by hypothesis,}$$

$$= \bigvee f(X); \quad X \text{ is directed.}$$

Proposition 2.4 (Continuity of application). Define application

$$Ap : [K \to K'] \times K \to K'$$

by the functor $Ap(f, x) = f(x)$. The $Ap$ is continuous.

Proof. The functor $\lambda x.f(x) = f(x)$ is continuous by continuity of $f$. Let $h = f.f(x)$. Then for directed $F \subseteq [K \to K']$

$$h(\bigvee F) = (\bigvee F)(x)$$

$$= \bigvee \{f(x) | f \in F\} \quad \text{by Proposition 2.2}$$

$$= \bigvee \{h(f) | f \in F\}$$

$$= \bigvee h(F).$$

So $h$ is continuous, and by Lemma 2.2 the functor $Ap$ is continuous. \qed
**Proposition 2.5** (Continuity of abstraction). Let \( f \in [K \times K' \to K''] \). Define the functor \( \hat{f} (x) = \lambda y \in K'. f(x, y) \). Then

1. \( \hat{f} \) is continuous, i.e., \( \hat{f} \in [K \to [K' \to K'']] \);
2. \( \lambda f. \hat{f} : [K \times K' \to K''] \to [K \to [K' \to K'']] \) is continuous.

**Proof.** (1) Let \( X \subseteq K \) be directed. Then

\[
\hat{f}(\bigvee X) = \lambda y. f(\bigvee X, y) \\
\simeq \lambda y. \bigvee f(X, y) \\
\simeq \bigvee \lambda y. f(X, y); \text{ by Proposition 2.3 it takes } F = \lambda y.f(X, y),
\]

(2) Let \( L = \lambda f. \hat{f} \). Then for \( F \subseteq [K \times K' \to K''] \) directed

\[
L(\bigvee F) = \lambda x \lambda y. (\bigvee f)(x, y) \\
= \lambda x \lambda y. \bigvee_{f \in F} f(x, y) \\
\simeq \bigvee_{f \in F} \lambda x \lambda y. f(x, y) \\
= \bigvee L(F).
\]

\( \square \)

**Definition 2.5** (CHPO). Define the subcategory \( \text{CHPO} \subseteq \text{CAT}_\infty \) whose objects are the c.h.p.o’s and the morphisms are the continuous functors.

**Proposition 2.6.** \( \text{CHPO} \) is a cartesian closed \( \infty \)-cartesian.

**Proof.** One has that the product of c.h.p.o’s \( K \times K' \in \text{CHPO} \). The singleton c.h.p.o \( \Delta^0 \) is a terminal object. By 2.4 and 2.5 for each continuous functor \( f : K \times K' \to K'' \) there is an unique continuous functor \( \hat{f} : K \to [K' \to K''] \) such that

\[
\begin{array}{c}
K \times K' \searrow \downarrow f \swarrow K'' \\
\downarrow f \times id_{K'} \downarrow \downarrow \downarrow \downarrow Ap \\
[K' \to K''] \times K'
\end{array}
\]
commutes in $sSet$ (functors are morphisms of simplicial sets). Thus, the functor $K \times (-)$ has a right adjoint functor $[K \to (-)]$.

Another alternative:

**Remark 2.1.** Let $g \in [K \to [K' \to K'']]$. Define the functor $\bar{g}(x, y) = \lambda (x, y) \in K \times K'.g(x)(y)$.

Then

1. $\bar{g}$ is continuous, i.e., $\bar{g} \in [K \times K' \to K'']$;
2. $\lambda g.\bar{g} : [K \to [K' \to K'']] \to [K \times K' \to K'']$ is continuous.
3. $\lambda g.\bar{g}$ is an inverse of $\lambda f.\hat{f}$.

Hence, CHPO is cartesian closed.

Now, one shows that in CHPO, the projective limits exist.

**Definition 2.6 (The Kan complex $K_\infty$).** Let $\{K_i\}_{i \in \omega}$ be countable sequence of c.h.p.o’s and let $f_i \in [K_i \to K_{i+1}]$ for each $i \in \omega$.

1. The diagram $(K_i, f_i)$ is called a projective (or inverse) system of c.h.p.o’s.
2. The projective (or inverse) limit of the system $(K_i, f_i)$ (notation $\lim_{\leftarrow}(K_i, f_i)$) is the h.p.o $(K_\infty, \preceq_\infty)$, where $K_\infty$ is the full subcategory of $(\bigcup_i K_i)^\omega$ ($\omega$-times cartesian product) whose objects are the sequences $(x_i)_{i \in \omega}$ (or $x : \omega \to \bigcup_i K_i$) such that $x_i \in K_i$, $f(x_{i+1}) \simeq x_i$ and

   $$(x_i)_{i \in \omega} \preceq_\infty (y_i), \text{ if } \forall i, x_i \preceq_i y_i \text{ (in } K_i).$$

**Proposition 2.7.** Let $(K_i, f_i)$ be a projective system. Then $\lim_{\leftarrow}(K_i, f_i) = K_\infty$ is c.h.p.o with

$$\bigvee X = \lambda i. \bigvee (x(i) \mid x \in X),$$

for directed $X \subseteq \lim_{\leftarrow}(K_i, f_i)$.

**Proof.** If $X$ is directed, then $(x(i) \mid x \in X)$ is directed for each $i$. Let

$$y_i = \bigvee (x(i) \mid x \in X).$$
Then by continuity of $f_i$
\[
f_i(y_{i+1}) \simeq \bigvee f_i(\langle x(i+1) \mid x \in X \rangle)
= \bigvee \langle x(i) \mid x \in X \rangle
= y_i
\]
Thus, $(y_i)_i \in \varprojlim (K_i, f_i)$. Clearly it is the supremum of $X$. □

Therefore, the c.h.p.o $K_\infty$ satisfies the equation $X \simeq [X \to X]$ in the $\infty$-category $\text{CHPO}$.

**Definition 2.7** (Compact objects and Algebraic c.h.p.o’s).

1. $x \in K$ is compact if for every directed $X \subseteq K$ one has
\[
x \not\preceq \bigvee X \implies x \not\preceq x_0 \text{ for some } x_0 \in X.
\]

2. $K$ is a algebraic c.h.p.o if for all $x \in K$ the h.p.o $x \downarrow = \langle y \not\preceq x \mid y \text{ compact} \rangle$ is directed and $x \simeq \bigvee (x \downarrow)$.

**Proposition 2.8.** Let $K$ be algebraic and $f : K \to K$. Then $f$ is continuous iff $f(x) \simeq \bigvee (f(e) \mid e \not\preceq x \text{ and } e \text{ compact})$.

**Proof.** ($\Rightarrow$) Let $f$ be continuous. Then
\[
f(x) = f(\bigvee \langle e \not\preceq x \mid e \text{ compact} \rangle)
\simeq \bigvee \langle f(e) \mid e \not\preceq x \text{ and } e \text{ compact} \rangle.
\]

($\Leftarrow$) First we check that $f$ is monotonic. If $x \not\preceq y$, then
\[
\langle e \not\preceq x \mid e \text{ compact} \rangle \subseteq \langle e \not\preceq y \mid e \text{ compact} \rangle,
\]
hence
\[
f(x) \simeq \bigvee \langle f(e) \mid e \not\preceq x \text{ and } e \text{ compact} \rangle
\simeq \bigvee \langle f(e) \mid e \not\preceq y \text{ and } e \text{ compact} \rangle
\simeq f(y).
\]
Now let $X \subseteq K$ directed. Then

$$f(\bigvee X) \simeq \bigvee \{f(e) \mid e \preceq \bigvee X \text{ and } e \text{ compact}\}$$

$$\simeq \bigvee \{f(x) \mid x \in X\}; \quad \text{by compactness}$$

$$\simeq f(\bigvee X); \quad \text{by monotonicity.}$$

Thus, $f(\bigvee X) \simeq \bigvee f(X)$. □

**Proposition 2.9.** Let $K, K'$ be c.h.p.o's.

1. $(x, y) \in K \times K'$ is compact iff $x$ and $y$ are compact.
2. $K$ and $K'$ are algebraic, then $K \times K'$ is algebraic.

**Proof.**

1. $(\Rightarrow)$ Let $X \subseteq K$ and $Y \subseteq K'$ be directed such that $x \preceq \bigvee X$ and $y \preceq \bigvee Y$. Hence

$$(x, y) \preceq (\bigvee X, \bigvee Y) = \bigvee (X \times Y).$$

By hypothesis, $(x, y) \preceq (x_0, y_0)$ for some $(x_0, y_0) \in X \times Y$. Thus, $x$ and $y$ are compact.

$(\Leftarrow)$ Let $X \subseteq K \times K'$ be directed such that

$$(x, y) \preceq \bigvee X = (\bigvee X_0, \bigvee X_1).$$

By hypothesis, there are $x_0 \in X_0$ and $y_0 \in X_1$ such that $x \preceq x_0$ and $y \preceq y_0$. Hence $(x_0, y_0)$ is compact.

2. Let $(x, y) \in K \times K'$. Then

$$\bigvee ((x, y) \downarrow) = \bigvee \{(e, d) \preceq (x, y) \mid (e, d) \text{ compact}\}$$

$$\quad = \bigvee \{(e, d) \preceq (x, y) \mid e \text{ and } d \text{ are compact}\}; \quad \text{by 1.}$$

$$\simeq (\bigvee (x \downarrow), \bigvee (y \downarrow))$$

$$\simeq (x, y); \quad \text{by hypothesis.}$$

□
**Definition 2.8 (Alg).** Define the subcategory \( \text{Alg} \subseteq \text{CHP O} \), whose object are the algebraic c.h.p.o’s and the morphism are continuous functors.

Note that \( \text{Acc} \), the \( \infty \)-category of accessible \( \infty \)-categories, is the generalization of \( \text{Alg} \), since all directed is filtered, all supremum is a filtered colimit, and for each \( X \in \text{Alg} \), \( X \) has all supremum and the full subcategory \( X_c \subseteq X \) of compacts objects generates \( X \) under supremum.

**3. Homotopy Domain Equation on an arbitrary cartesian closed \( \infty \)-category**

This section is a direct generalization of the traditional methods for solving domain equations in cartesian closed categories (see \([2]\) and \([1]\)), in the sense of obtaining solutions for certain types of equations, which we call Homotopy Domain Equations in any cartesian closed \( \infty \)-category.

**Definition 3.1.**

1. An \( \omega \)-diagram in an \( \infty \)-category \( K \) is a diagram with the following structure:

\[
K_0 \xrightarrow{f_0} K_1 \xrightarrow{f_1} K_2 \rightarrow \cdots \rightarrow K_n \xrightarrow{f_n} K_{n+1} \rightarrow \cdots
\]

(dually, one defines \( \omega^{\text{op}} \)-diagrams by just reversing the arrows).

2. An \( \infty \)-category \( K \) is \( \omega \)-complete (\( \omega \)-cocomplete) if has limits (colimits) for all \( \omega \)-diagrams.

3. A functor \( F : K \rightarrow K \) is \( \omega \)-continuous if it preserves (under equivalence) all colimits of \( \omega \)-diagrams.

**Theorem 3.1.** Let \( K \) be an \( \infty \)-category. Let \( F : K \rightarrow K \) be a \( \omega \)-continuous (covariant) functor and take a vertex \( K_0 \in K \) such that there is an edge \( \delta \in \text{K}(K_0, FK_0) \). Assume also that \( (K, \{ \delta_{i,\omega} \in \text{K}(F^i K_0, K) \}_{i \in \omega}) \) is a colimit for the \( \omega \)-diagram \( \{ (F^i K_0)_{i \in \omega}, \{ F^i \delta \}_{i \in \omega} \} \), where \( F^0 K_0 = K_0 \) and \( F^0 \delta = \delta \). Then \( K \simeq FK \).

**Proof.** We have that \( (FK, \{ F \delta_{i,\omega} \in \text{K}(F^{i+1} K_0, FK) \}_{i \in \omega}) \) is a colimit for

\[
\{ (F^{i+1} K_0)_{i \in \omega}, \{ F^{i+1} \delta \}_{i \in \omega} \}
\]
and \((K, \{\delta_{i+1,\omega} \in \mathcal{K}(F^{i+1}K_0, K)\}_{i \in \omega})\) is a cocone for the same diagram. Then, there is a unique edge (under homotopy; the space of choices is contractible) \(h : FK \to K\) such that \(h.F\delta_i = \delta_{i+1,\omega}\) for each \(i \in \omega\). We add to \((FK, \{F\delta_i \in \mathcal{K}(F^{i+1}K_0, FK)\}_{i \in \omega})\) the edge \(F\delta_{0,\omega}, \delta \in \mathcal{K}(K_0, FK)\). This gives a cocone for \((\{F^iK_0\}_{i \in \omega}, \{F^i\delta\}_{i \in \omega})\) and, since \((K, \{\delta_{i,\omega} \in \mathcal{K}(F^iK_0, K)\}_{i \in \omega})\) is its colimit, there is a unique edge (under homotopy) \(k : K \to FK\) such that \(k.\delta_i = \delta_{i+1,\omega}\) for each \(i \in \omega\). But, \(h.k.\delta_{i+1,\omega} = h.F\delta_i = \delta_{i+1,\omega}\) and \(h.k.\delta_{0,\omega} = \delta_{0,\omega}\) for each \(i \in \omega\), thus \(h.k\) is a mediating edge between the colimit \((K, \{\delta_{i,\omega} \in \mathcal{K}(F^iK_0, K)\}_{i \in \omega})\) and itself (besides \(I_K\)). Hence, by unicity (under homotopy) \(h.k \simeq I_K\). In the same way, we prove that \(k.h \simeq I_K\), and we conclude that \(F\) has a fixed point.

\begin{definition}
An \(\infty\)-category \(\mathcal{K}\) is a \((0, \infty)\)-category if
\begin{enumerate}
\item every Kan complex \(\mathcal{K}(A, B)\) is a c.h.p.o with a least element \(0_{A,B}\) under homotopy,
\item composition of morphisms is a continuous operation with respect to the homotopy order,
\item for every \(f \in \mathcal{K}(A, B)\), \(0_{B,C}.f \simeq 0_{A,C}\).
\end{enumerate}
\end{definition}

\begin{definition}[h-projection par]
Let \(\mathcal{K}\) be a \((0, \infty)\)-category, and let \(f^+ : A \to B\) and \(f^- : b \to a\) be two morphisms in \(\mathcal{K}\). Then \((f^+, f^-)\) is a homotopy projection (or h-projection) pair (from \(A\) to \(B\)) if \(f^+.f^- \simeq I_A\) and \(f^- .f^+ \preceq I_B\). If \((f^+, f^-)\) is a h-projection pair, \(f^+ \in \mathcal{K}^{HE}(A, B)\) is an h-embedding and \(f^- \in \mathcal{K}^{HP}(A, B)\) is an h-projection. Where \(\mathcal{K}^{HE}\) is the subcategory of \(\mathcal{K}\) with the same objects and the h-embeddings as morphisms, and \(\mathcal{K}^{HP}\) is the subcategory of \(\mathcal{K}\) with the same objects and the h-projections as morphisms.
\end{definition}

\begin{definition}[h-projections pair (0, \infty)-category]
Let \(\mathcal{K}\) be a \((0, \infty)\)-category. The \((0, \infty)\)-category \(\mathcal{K}^{HPrj}\) is the \(\infty\)-category embedding in \(\mathcal{K}^{HE}\) with the same objects of \(\mathcal{K}\) and h-projection pairs \((f^+, f^-)\) as morphisms.
\end{definition}

\begin{remark}
Every h-embedding \(i\) has unique (under homotopy) associated h-projection \(j = i^R\) (and, conversely, every h-projection \(j\) has a unique (under
homotopy) associated h-embedding $i = j^L$, $\mathcal{K}^{HPri}$ is equivalent to a subcategory $\mathcal{K}^{HE}$ of $\mathcal{K}$ that has h-embeddings as morphisms (as well to a subcategory $\mathcal{K}^{HP}$ of $\mathcal{K}$ which has h-projections as morphisms).

**Definition 3.5.** Given a $(0,\infty)$-category $\mathcal{K}$, and a contravariant functor in the first component $F : \mathcal{K}^{op} \times \mathcal{K} \to \mathcal{K}$, the functor covariant $F^{+−} : \mathcal{K}^{HPri} \times \mathcal{K}^{HPri} \to \mathcal{K}^{HPri}$ is defined by

$$F^{+−}(A, B) = F(A, B),$$

$$F^{+−}((f^{+}, f^{−}), (g^{+}, g^{−})) = (F(f^{−}, g^{+}), F(f^{+}, g^{−})),
$$

where $A, B$ are vertices and $(f^{+}, f^{−}), (g^{+}, g^{−})$ are n-simplexes pairs in $\mathcal{K}^{HPri}$.

Given the $ω$-chain $\{\{K_i\}_{i\in ω}, \{f_i\}_{i\in ω}\}$ in a h-projective $(0,\infty)$-category. Let $(K, \{γ_i\}_{i\in ω})$ be a limit for $\{\{K_i\}_{i\in ω}, \{f_i\}_{i\in ω}\}$ in $\mathcal{K}$. Note that $δ_i.γ_i ≃ δ_{i+1}.f_i^{+}.f_i^{−}.γ_{i+1} ≤ δ_{i+1}.γ_{i+1}$, for each $i \in ω$. Then, $\{δ_i.γ_i\}$ is an $ω$-chain and its colimit is $Θ = \bigcurlyvee_{i\in ω}\{δ_i.γ_i\}$. Now for each $j \in ω$ one has

$$γ_j.Θ_j = γ_j.\bigcurlyvee_{i\in ω}\{δ_i.γ_i\}$$
$$≃ γ_j.\bigcurlyvee_{i \geq j}\{δ_i.γ_i\}$$
$$≃ \bigcurlyvee_{i \geq j}\{γ_j.δ_i.γ_i\}$$
$$≃ \bigcurlyvee_{i \geq j}\{f_{i,j}.γ_i\}$$
$$≃ γ_j.
$$

Thus, $Θ$ is a mediating edge between the limit $(K, \{γ_i\}_{i\in ω})$ for $ω^{op}$-diagram $\{\{K_i\}_{i\in ω}, \{f_i\}_{i\in ω}\}$ and itself (besides $I_K$). So, by unicity (under equivalence) $Θ ≃ I_K$. This result guarantees the proof of the following theorems of this section. All these proofs are similar to case of the 0-categories (except for uniqueness proofs, which are under homotopy) and for that the reader is referred to [2].
Theorem 3.2. Let \( K \) be a \((0, \infty)\)-category. Let \( \{\{K_i\}_{i \in \omega}, \{f_i\}_{i \in \omega}\} \) be an \( \omega \)-diagram in \( K^{HPr} \). If \( \{K_i, \gamma_i\}_{i \in \omega} \) is a limit for \( \{\{K_i\}_{i \in \omega}, \{f_i\}_{i \in \omega}\} \) in \( K \), then \( \{K, \{\delta_i, \gamma_i\}_{i \in \omega}\} \) is a colimit for \( \{\{K_i\}_{i \in \omega}, \{f_i\}_{i \in \omega}\} \) in \( K^{HPr} \) (that is, every \( \gamma_i \) is a right member of a projection pair).

Proof. Fix \( K_j \). For each \( i \) define \( f_{j,i} : K_j \to K_i \) by:

\[
  f_{j,i} = \begin{cases} 
    f_i - f_{i+1} \cdots f_{j-1} & \text{if } i < j, \\
    I_{K_j} & \text{if } i = j, \\
    f_i^+ \cdots f_{j+1}^+ & \text{if } i > j.
  \end{cases}
\]

\( \{K_i, \{f_{j,i}\}_{i \in \omega}\} \) is a cone for \( \{\{K_i\}_{i \in \omega}, \{f_i\}_{i \in \omega}\} \), since \( f_i - f_{j,i+1} \simeq f_{j,i} \). Thus there is a unique morphism (under homotopy) \( \delta_j : K_j \to K \) such that \( \gamma_i \cdot \delta_j \simeq f_{j,i} \) for each \( i \in \omega \). If \( i = j \), \( \gamma_j \cdot \delta_j \simeq I_{K_j} \).

Since \( \Theta = \bigvee_{i \in \omega} \{\delta_i, \gamma_i\} \simeq I_K \), \( \delta_i, \gamma_i \not\simeq I_K \) for each \( i \in \omega \). Thus \( \{\delta_i, \gamma_i\} \) is a h-projection pair for each \( i \in \omega \).

One still has to check that \( (f_{j+1}^+, f_{j-1}^-) \cdot (\delta_{j+1}, \gamma_{j+1}) \simeq (\delta_j, \gamma_j) \). We have that \( f_j^- \cdot \gamma_{j+1} \simeq \gamma_j \) by the definition of cone in \( K \). In order of to prove that \( \delta_{j+1} \cdot f_j^+ \simeq \delta_j \), note that \( \delta_i \cdot (\delta_{j+1} \cdot f_j^+) \simeq f_{j+1,i} \cdot f_j^+ \simeq f_{j,j} \simeq I_{K_j} \simeq \gamma_i \cdot \delta_i \), by unicity (under homotopy) of \( \delta_j : K_j \to K \), \( \delta_{j+1} \cdot f_j^+ \simeq \delta_j \). Thus, \( \{K, \{\delta_i, \gamma_i\}_{i \in \omega}\} \) is a cone in \( K^{HPr} \).

One proves next that \( \{K, \{\delta_i, \gamma_i\}_{i \in \omega}\} \) is a colimit. Let \( \{K', \{\delta'_i, \gamma'_i\}_{i \in \omega}\} \) be another cocone for \( \{\{K_i\}_{i \in \omega}, \{f_i\}_{i \in \omega}\} \). That is, for each \( i \in \omega \):

\[
  \delta'_i \cdot \gamma_i \simeq \delta'_{i+1} \cdot f_{i+1}^- \cdot \gamma_{i+1} \lesssim \delta'_{i+1} \cdot \gamma_{i+1},
\]

\[
  \delta_i \cdot \gamma'_i \simeq \delta_{i+1} \cdot f_{i+1}^+ \cdot \gamma'_{i+1} \lesssim \delta_{i+1} \cdot \gamma'_{i+1}.
\]

Define thus:

\[
  h = \bigvee_{i \in \omega} \{\delta'_i, \gamma_i\} : K \to K',
\]

\[
  k = \bigvee_{i \in \omega} \{\delta_i, \gamma'_i\} : K' \to K.
\]
Observe that \((h, k)\) is a h-projection pair, since:

\[
\begin{align*}
k.h &= \bigvee_{i \in \omega} \{\delta_i, \gamma_i\} \cdot \bigvee_{i \in \omega} \{\delta_i', \gamma_i\} \\
&\simeq \bigvee_{i \in \omega} \{\delta_i, (\delta_i', \gamma_i')\} \\
&\simeq \bigvee_{i \in \omega} \{\delta_i, \gamma_i\} \\
&\simeq \Theta = I_K
\end{align*}
\]

and

\[
\begin{align*}
h.k &= \bigvee_{i \in \omega} \{\delta_i, \gamma_i\} \cdot \bigvee_{i \in \omega} \{\delta_i', \gamma_i'\} \\
&\simeq \bigvee_{i \in \omega} \{\delta_i', (\gamma_i, \delta_i')\} \\
&\simeq \bigvee_{i \in \omega} \{\delta_i', \gamma_i'\} \\
&\simeq I_K.
\end{align*}
\]

Moreover, \((h, k)\) is a mediating morphism between \((K, \{(\delta_i, \gamma_i)\}_{i \in \omega})\) and \((K', \{(\delta_i', \gamma_i')\}_{i \in \omega})\), since for each \(i \in \omega\):

\[
(h, k). (\delta_j, \gamma_j) = (h, \delta_j, \gamma_j, k) \\
\simeq (\bigvee_{i \geq j} \{\delta_i, \gamma_i\}, \bigvee_{i \geq j} \{\gamma_j, \delta_i, \gamma_i'\}) \\
\simeq (\bigvee_{i \geq j} \{\delta_i', \gamma_i\}, \bigvee_{i \geq j} \{\gamma_j, \delta_i, \gamma_i'\}) \\
\simeq (\bigvee_{i \geq j} \{\delta_i', \gamma_i\}, \bigvee_{i \geq j} \{\gamma_j, \delta_i, \gamma_i'\}) \\
\simeq (\delta_j', \gamma_j').
\]

Thus, for each \(j \in \omega\), \(\mathcal{K}_{K_j/}^{HPr}(\delta_j, \gamma_j, (\delta_j', \gamma_j')) \neq \emptyset\), that is \(\mathcal{K}_{K_j/}^{HP}(\delta_j', \gamma_j)\) and \(\mathcal{K}_{K_j/}^{HE}(\delta_j, \delta_j')\) spaces that are not empty.

Since \(\mathcal{K}_{K_j/}^{HP}(\gamma_j', \gamma_j) \subseteq \mathcal{K}_{K_j/}^{HP}(\gamma_j', \gamma_j)\) and by hypothesis \(\gamma_j\) is an object final in \(\mathcal{K}_{K_j/}^{HP}\) for all \(j \in \omega\), then \(\mathcal{K}_{K_j/}^{HP}(\gamma_j', \gamma_j)\) and \(\mathcal{K}_{K_j/}^{HE}(\delta_j, \delta_j')\) are contractible for each
j ∈ ω. Thus, the Kan complex $K^{HP_{rj}}(\{(\delta_j, \gamma_j), (\delta'_j, \gamma'_j)\})$ is contractible for each $j ∈ ω$, that is, $(h,k)$ is unique (under homotopy) in the mediating morphism between $(K,\{(\delta_i, \gamma_i)\}_{i ∈ ω})$ and $(K',\{(\delta'_i, \gamma'_i)\}_{i ∈ ω})$.

Therefore, the following Corollary is an immediate consequence of the proof from theorem before.

**Corollary 3.1.** The cocone $(K,\{(\delta_i, \gamma_i)\}_{i ∈ ω})$ for the $ω$-chain $(\{K_i\}_{i ∈ ω}, \{(f^+_i, f^-_i)\}_{i ∈ ω})$ in $K^{HP_{rj}}$ is universal (a cocone colimit) iff $Θ = \bigcurlyvee_{i ∈ ω} δ_i.γ_i ≃ I_K$.

**Definition 3.6** (Locally monotonic). Let $K$ be a $(0, ∞)$-category. A functor $F : K^{op} × K → K$ is locally $h$-monotonic if it is monotonic on the Kan complexes of 1-simplexes, i.e., for $f, f' ∈ K^{op}(A, B)$ and $g, g' ∈ K(C, D)$ one has

$$f \preceq f', g \preceq g' \implies F(f,g) \preceq F(f',g').$$

**Proposition 3.1.** If $F : K^{op} × K → K$ is locally $h$-monotonic and $(f^+, f^-)$, $(g^+, g^-)$ are $h$-projection pairs, then $F^{+−}((f^+, f^-), (g^+, g^-))$ is also an $h$-projection pair.

**Proof.** By definition $F^{+−}((f^+, f^-), (g^+, g^-)) = (F(f^−, g^+), F(f^+, g^−))$. Then

$$F(f^+, g^−).F(f^−, g^+) \simeq F((f^+, g^−).(f^−, g^+)) = F(f^−.f^+, g^−.g^+) \simeq F(id, id) \simeq id$$

and

$$F(f^−, g^+).F(f^+, g^−) \simeq F((f^−, g^+).(f^+, g^−)) = F(f^+.,f^−, g^+.g^−) \simeq F(id, id) \simeq id.$$
Definition 3.7 (Locally continuous). Let $\mathcal{K}$ be a $(0,\infty)$-category. A $F : \mathcal{K}^{\text{op}} \times \mathcal{K} \to \mathcal{K}$ is locally continuous if it is $\omega$-continuous on the Kan complexes of 1-simplices. That is, for every directed diagram $\{f_i\}_{i \in \omega}$ in $\mathcal{K}^{\text{op}}(A, B)$, and every directed diagram $\{g_i\}_{i \in \omega}$ in $\mathcal{K}(C, D)$, one has

$$F(\bigvee_{i \in \omega}\{f_i\}, \bigvee_{i \in \omega}\{g_i\}) \simeq \bigvee_{i \in \omega} F(f_i, g_i).$$

Remark 3.2. If $F$ is locally continuous, then it is also locally monotonic.

Theorem 3.3. Let $\mathcal{K}$ be a $(0,\infty)$-category. Let also $F : \mathcal{K}^{\text{op}} \times \mathcal{K} \to \mathcal{K}$ be a locally continuous functor. Then the functor $F^{+\omega} : \mathcal{K}^{HPrj} \times \mathcal{K}^{HPrj} \to \mathcal{K}^{HPrj}$ is $\omega$-continuous.

Proof. Let $((A_i), \{(f_i^+, f_i^-)\}_{i \in \omega})$ and $((B_i), \{(f_i^+, f_i^-)\}_{i \in \omega})$ be two $\omega$-chains in $\mathcal{K}^{HPrj}$ and let $(A, \{\rho_i^+, \rho_i^-\}_{i \in \omega})$ and $(B, \{\sigma_i^+, \sigma_i^-\}_{i \in \omega})$ be the respective limits. We have that

$$\Theta = \bigvee_{i \in \omega}\{\rho_i^+, \rho_i^-\} \simeq I_A$$
$$\Psi = \bigvee_{i \in \omega}\{\sigma_i^+, \sigma_i^-\} \simeq I_B.$$

One must shows that

$$(F^{+\omega}(A, B), \{F^{+\omega}(\rho_i^+, \rho_i^-), F^{+\omega}(\sigma_i^+, \sigma_i^-)\}_{i \in \omega}) = (F(A, B), \{(F(\rho_i^+, \sigma_i^+), F(\rho_i^+, \sigma_i^-))\}_{i \in \omega})$$

is a colimit for the $\omega$-chain $((F^{+\omega}(A_i), B_i)), \{(f_i^+, f_i^-)\}_{i \in \omega}, \{F^{+\omega}((f_i^+, f_i^-), (g_i^+, g_i^-))\}_{i \in \omega})$.

It is clearly a cone, by the property of functors. We show that it is universal by proving that $\bigvee_{i \in \omega}\{F(\rho_i^+, \sigma_i^+), F(\rho_i^-, \sigma_i^-)\} \simeq I_{F(A, B)}$; according to the Corollary 3.1. Computing we have

$$\bigvee_{i \in \omega}\{F(\rho_i^+, \sigma_i^+), F(\rho_i^-, \sigma_i^-)\} \simeq \bigvee_{i \in \omega}\{F(\rho_i^+, \rho_i^-), \sigma_i^+, \sigma_i^-\}$$
$$\simeq F(\bigvee_{i \in \omega}(\rho_i^+, \rho_i^-), \bigvee_{i \in \omega}(\sigma_i^+, \sigma_i^-))$$
$$\simeq F(\Theta, \Psi)$$
$$\simeq F(I_A, I_B) = I_{F(A, B)}.$$
Remark 3.3. Let $\mathcal{K}$ be a cartesian closed $(0, \infty)$-category, $\omega^{op}$-complete and with final object. Since the exponential functor $\Rightarrow: \mathcal{K}^{op} \times \mathcal{K} \to \mathcal{K}$ and the diagonal functor $\Delta: \mathcal{K} \to \mathcal{K} \times \mathcal{K}$ are locally continuous, by the Theorem 3.3, the associated functors

$$(\Rightarrow)^{+}: \mathcal{K}^{HPrj} \times \mathcal{K}^{HPrj} \to \mathcal{K}^{HPrj}, \quad (\Delta)^{+}: \mathcal{K}^{HPrj} \to \mathcal{K}^{HPrj} \times \mathcal{K}^{HPrj}$$

are $\omega$-continuous. But composition of $\omega$-continuous functors is still an $\omega$-continuous functor. Thus, the functor

$$F = (\Rightarrow)^{+} \cdot (\Delta)^{+}: \mathcal{K}^{HPrj} \to \mathcal{K}^{HPrj},$$

is $\omega$-continuous. By Theorem 3.3 the functor $F$ has a fixed point, that is, there is a vertex $K \in \mathcal{K}$ such that $K \simeq (K \Rightarrow K)$. The $\infty$-category of the fixed points of $F$ is denoted by $\text{Fix}(F)$.

4. Homotopy Domain Equation on $Kl(P)$

In this section we consider $Kl(P)$ of [3] be an $\infty$-category, in order to apply the homotopy domain theory of the previous section.

For the next proposition, let $\mathcal{P}_{\kappa}^L$ be the subcategory of $\text{CAT}_\infty$ whose objects are $\kappa$-compactly generated $\infty$-categories and whose morphisms are functors which preserve small colimits and $\kappa$-compact objects. Also, let $\text{CAT}_{\infty}^{Rex(\kappa)}$ denote the subcategory of $\text{CAT}_\infty$ whose objects are $\infty$-categories which admit $\kappa$-small colimits and whose morphisms are functors which preserve $\kappa$-small colimits. Finally, let $L^*: \text{Cat}_\infty \to \text{Cat}_{\infty}^{Rex(\kappa)}$ the functor which closes an $\infty$-category to an $\infty$-category which admit $\kappa$-small colimits, so $L^* A \simeq (PA)^{\kappa}$.

Proposition 4.1. The $\infty$-category $Kl(P)$ admits limits for $\omega^{op}$-diagrams in $\mathcal{P}_{\kappa}^L$. 

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Proof. Given an $\omega^{op}$-diagram of pro-functors in $Kl(P)$ such that this is associated to $\omega^{op}$-diagram $p$ of functors

$$PK_0 \xleftarrow{f_0} PK_1 \xleftarrow{f_1} PK_2 \xleftarrow{f_2} \cdots$$

in the $\infty$-category $Pr^L_\kappa$, which admit all the small limits. Then, there is a limit $(T, \{\gamma_k\}_{k\in\omega})$ in $Pr^L_\kappa$ for this $\omega^{op}$-diagram $p$. Since $T$ is presentable, so for any regular cardinal $\kappa$, the full subcategory of $\kappa$-compacts $T^\kappa$ is essentially small \[10\]. Thus $(T^\kappa, \{\gamma_k.i\}_{k\in\omega})$ is a cone for the $\omega^{op}$-diagram in $Kl(P)$, where $i$ is the inclusion functor $T \supseteq T^\kappa$.

Let $(T', \{\tau_k\}_{k\in\omega})$ be another cone for $\{(PK_k)_{k\in\omega}, \{f_k\}_{k\in\omega}\}$ in $Kl(P)$. Then, $(PT', \{\tau_k^\# \in Pr^L_\kappa(PT', PK_k)\}_{k\in\omega})$ is a cone from $\omega^{op}$-diagram $p$ in $Pr^L_\kappa$. Thus, there is a unique edge (under homotopy) $h : PT' \to T$ in $Pr^L_\kappa$ such that $\gamma_k.h \simeq \tau_k^\#$ for each $k \in \omega$. Applying the full faithful functor $(-)^\kappa : Pr^L_\kappa \to CAT^{\text{ReEx}(\kappa)}_\infty$ \[10\], one has that $h^\kappa \in Fun^\kappa(L^*T', T^\kappa) \simeq Fun(T', T^\kappa)$ is the unique edge (under homotopy) such that $(\gamma_k.i).h^\kappa \simeq \tau_k^\# : j$, for each $k \in \omega$ with $j : L^*T' \to PT'$ being a Yoneda embedding. Thus, there is a unique (under homotopy) $h' : T' \to T^\kappa$ such that $(\gamma_k.i).h' \simeq \tau_k$.

Since each $Fun(A, PB)$ has initial object, any $Fun(A, PB)(F, G)$ is contractible or empty. Thus, $K(P)(A, B) \subseteq Fun(A, PB)$ admits a homotopy partial order (h.p.o.). For the following theorem, denote by $0_{P,A,PB}$ in $Fun^L(PA, PB)$ as the constant functor in empty Kan complex $\emptyset$, that is

$$0_{P,A,PB} f := \lambda x \in B. \emptyset$$

**Theorem 4.1.** The $\infty$-category $Kl(P)$ is a $(0, \infty)$-category.

Proof. Let $PB$ be presentable, $Fun[A, PB]$ is presentable, thus $(Kl(P)(A, B), \preceq)$ is complete. On the other hand, let $F$ be an object in $Fun^L(PA, PB)$, then

$$0_{P,A,PB} f x = (\lambda x \in B. \emptyset)x = \emptyset \subseteq F f x,$$
for every object $f$ in $PA$ and $x$ in $B$. Thus, $0_{PA, PB}$ is the least element (under homotopy) in $Fun^L(PA, PB)$, i.e., $0_{A, B}$ is the least element in $KL(P)(A, B)$.

2. Let $\{p_i\}_{i \in \omega}$ be a non-decreasing chain of morphisms in $Fun^L(PA, PB)$. By (1), the colimit $\bigvee_{i \in \omega} p_i$ exists.

i. First let’s prove that for each object $z$ in $PA$, its colimit is given by

$$\left( \bigvee_{i \in \omega} p_i \right) z \simeq \bigvee_{i \in \omega} p_i z.$$

Since $p_i \preceq \bigvee_{i \in \omega} p_i$, given a vertex $z$ of $PA$, by Definition 2.1 $p_i z \preceq \bigvee_{i \in \omega} p_i z$. On the other hand, $\bigvee_{i \in \omega} p_i z$ is the supremum (under equivalence) of $\{p_i z\}$, hence

$$p_i z \preceq \bigvee_{i \in \omega} p_i z \preceq (\bigvee_{i \in \omega} p_i) z.$$

Let $q_z := \bigvee_{i \in \omega} p_i z$, by Definition 2.1

$$p_i \preceq q \preceq \bigvee_{i \in \omega} p_i,$$

but $\bigvee_{i \in \omega} p_i$ is the supremum (under equivalence) of $\{p_i\}_{i \in \omega}$, thus $\bigvee_{i \in \omega} p_i \simeq q$, by Definition 2.1

$$\left( \bigvee_{i \in \omega} p_i \right) z \simeq \bigvee_{i \in \omega} p_i z.$$

ii. Now let’s prove that the composition is continuous on the right. Take a functor $F$ in $Fun_\kappa(PA', PA)$ and a vertex $z$ of $PA'$, then $Fz$ is a vertex of $PA$, by (i) we have

$$((\bigvee_{i \in \omega} p_i) F) z \simeq (\bigvee_{i \in \omega} p_i) (F z)$$

$$\simeq \bigvee_{i \in \omega} p_i (F z)$$

$$\simeq \bigvee_{i \in \omega} (p_i F) z$$

$$\simeq (\bigvee_{i \in \omega} p_i F) z,$$
since $Kl(P)$ does have enough points, it follows $Fun^L(PA, PB)$,

$$(\bigvee_{i \in \omega} p_i).F \simeq \bigvee_{i \in \omega} p_i.F.$$ 

iii. Finally let’s prove that the composition is continuous on the right.

Let $G$ be a functor in $Fun^L(PB, PC)$ and $z$ an object in $PA$. By (i) and the continuity of $G$, we have

$$(G. \bigvee_{i \in \omega} p_i)z \simeq G((\bigvee_{i \in \omega} p_i)z)$$

$\simeq G(\bigvee_{i \in \omega} p_i z)$$

$\simeq \bigvee_{i \in \omega} G(p_i z)$$

$\simeq \bigvee_{i \in \omega} (G.p_i)z$$

$\simeq (\bigvee_{i \in \omega} G.p_i)z,$

since $Kl(P)$ does have enough points, it follows

$$G. \bigvee_{i \in \omega} p_i \simeq \bigvee_{i \in \omega} G.p_i.$$ 

3. Let $F$ be an object in $Fun^L(PA, PB)$ and $f$ in $PA$, hence

$$(0_{PB, PC}.F)f = 0_{PB, PC}(Ff) = \lambda x \in C.0 = 0_{PA, PC}f,$$

that is, $0_{PB, PC}.F = 0_{PA, PC}$.

\[
\sqrt{\ }
\]

**Proposition 4.2.** For any small $\infty$-category $A$, there is an $h$-projection from $A$ to $A \Rightarrow A$ in $Kl(P)$.

**Proof.** We have that there is a diagonal functor

$$\delta : PA \to [PA, PA]^L \simeq P(A^{op} \times A) = P(A \Rightarrow A),$$

where $[PA, PA]^L$ is the $\infty$-category of the functors which preserve small colimits or left adjoints. Since $\delta$ preserves all small colimits, by the Adjoint Functor
Theorem, δ has a right adjoint γ. One the other hand, the diagonal functor δ is an h-embedding, then the unit is an equivalence, i.e., γ.δ ≃ I_{P A} and the counit is an h.p.o., that is, δ.γ ≼ I_{P(A⇒A)} in the c.h.p.o. Kl(P)(A ⇒ A, A ⇒ A). Thus, (δ, γ) is a projection pair of A to A ⇒ A in Kl(P), which we call the diagonal projection.

Proposition 4.3. There is a reflexive non-contractible object in Kl(P).

Proof. First the trivial Kan complex ∆^0 is a fixed point from endofunctor FX = (X ⇒ X) on Kl(P), since

\[ P\Delta^0 \simeq P(\Delta^0 \times \Delta^0) = P(\Delta^0 \Rightarrow \Delta^0). \]

that is, \( \Delta^0 \simeq (\Delta^0 \Rightarrow \Delta^0) = F\Delta^0 \) in Kl(P). Let’s suppose that all the Kan complexes in Fix(F) are equivalent to \( \Delta^0 \). Since Kl(P) contains all the small ∞-categories, then there is a small non-contractible Kan complex \( K_0 \), i.e.,

\[ \Delta^0 \prec K_0 \overset{(\delta_0, \gamma_0)}{\rightarrow} F K_0 \]

in \( (Kl(P))^{HPrj} \) for all \( n \in \omega \), where \( \delta_0 \) is the diagonal functor, which has its equivalent functor in \( \mathcal{P}_\kappa^R = (\mathcal{P}_\kappa^L)^{op} \). Since \( \{F^iK_0\}_{i \in \omega}, \{F^i(\gamma_0)\}_{i \in \omega} \) is an \( \omega^{op} \)-diagram in \( \mathcal{P}_\kappa^L \), by Proposition 4.1 and Theorem 3.2 there is a colimit \( (K, \{(\delta_{i, \omega}, \gamma_{\omega, i})\}_{i \in \omega}) \) in \( (Kl(P))^{HPrj} \) for the \( \omega \)-diagram \( \{F^iK_0\}_{i \in \omega}, \{F^i(\delta_0, \gamma_0)\}_{i \in \omega} \).

Thus,

\[ \Delta^0 \prec K_0 \overset{\delta_{0, \omega}}{\rightarrow} K \in Fix(F) \]

which is a contradiction. □

The fact that a Kan complex X is not contractible does not imply that every vertex \( x \in X \) contains information, nor that it contains holes in all the higher dimensions. It motivates the following definition.

Definition 4.1 (Split Kan complex). A small Kan complex X is split if

1. \( \pi_0(X) \) is infinite.
2. for each \( n \geq 1 \), there is a vertex \( x \in X \) such that \( \pi_n(X, x) \not\simeq * \).
3. for each vertex $x$ of some $k$-simplex in $X$, with $k \geq 2$, there is $n \geq 1$ such $\pi_n(X,x) \not\simeq \ast$.

**Example 4.1.** For each $n \geq 0$, let the Kan complex $B^n \cong \partial \Delta^n_\ast$ (isomorphic). Where $\Delta^n_\ast$ have the same vertices and faces of $\Delta^n$ but invertible 1-simplexes. And take the Kan complex $B^\omega$, with $\omega$ different vertices $B^0_\omega, B^1_\omega, B^2_\omega, \ldots$, such that there is a map $f : \partial \Delta^n \to B^\omega$, with $f(i) = B^n_i$ for each $0 \leq i \leq n$, which set $f \partial \Delta \cong B^n$. Define the split Kan complex $B_0$ as the disjoint union

$$B_0 = \coprod_{n \leq \omega} B^n.$$  

Note that $B^n$ is “similar” to sphere $S^{n-1}$. Furthermore, $\pi_{n-1}(B^n) \not\simeq \ast$ for all $n \geq 2$, and there is $k \geq n$ such that $\pi_k(B^n) \not\simeq \ast$ for each $n \geq 3$ [11].

**Example 4.2.** For each $n \geq 2$, let $D^n$ be a Kan complex, such that its $k$-th face set the isomorphism

$$d_kD^n \cong \begin{cases} \Delta^{n-1} & \text{if } k = 0, \\ \partial \Delta^{n-1} & \text{if } 1 \leq k \leq n, \end{cases}$$

Now take the Kan complex $D^\omega$, with $\omega$ different vertices $D^0_\omega, D^1_\omega, D^2_\omega, \ldots$, such that there is a map $f : D^n \to D^\omega$, with $f(D^n_i) = D^n_i$ for each $0 \leq i \leq n$, which set $fD^n \cong D^n$. Define the split Kan complex $D_0$ as the disjoint union

$$D_0 = \coprod_{n \leq \omega} D^n.$$  

Note that each Kan complex $D^n$, with $n \geq 2$, is “similar” to the sphere $S^{n-1}$ with $n$ holes. Besides that its higher groups have the same properties from Example 4.1 we also have the additional property $\pi_{n-2}(D^n) \not\simeq \ast$ for all $n \geq 3$.

**Example 4.3.** For each $n \geq 2$, let $E^n$ be a Kan complex, such that its $k$-th face set the isomorphism

$$d_kE^n \cong \begin{cases} \partial^{n-2} \Delta^{n-1} & \text{if } 0 \leq k \leq 2, \\ \partial^{n-k} \Delta^{n-1} & \text{if } 3 \leq k \leq n. \end{cases}$$
Now take the Kan complex $E^\omega$, with $\omega$ different vertices $E_0^\omega, E_1^\omega, E_2^\omega, \ldots$, such that there is a map $f : E^n \to E^\omega$, with $fE_i^n = E_i^\omega$ for each $0 \leq i \leq n$, which set $fE^n \cong E^n$. Define the split Kan complex $E_0$ as the disjoint union

$$E_0 = \coprod_{n \leq \omega} E^n.$$ 

Note that $E_0$ has more information than the split Kan complexes $B_0$ and $D_0$ from the previous examples, in the sense that for all $n \geq 2$, it satisfies the property $\pi_k(E^n) \not\cong \ast$ for each $1 \leq k \leq n - 1$.

**Proposition 4.4.** There is a split Kan complex in the $\infty$-category of fixed points $\text{Fix}(F)$.

**Proof.** Let $K_0$ be a Kan complex split and $(K, \{ (\delta_i, \delta_\omega, i) \}_{i \in \omega})$ the limit from $\omega$-diagram $\{ (F^iK_0)_{i \in \omega}, \{ F^i(\delta_0, \gamma_0) \}_{i \in \omega} \}$ in $(KL(P))^{HPrJ}$, with $(\delta_0, \gamma_0)$ the first projection from $K_0$ to $K_1 := FK_0$. Let $z = (x, y)$ be a vertex of some $k$-simplex in $K_{i+1}$, with $k \geq 2$. By induction on $i$, there is $n_1, n_2 \geq 1$, such that $\pi_{n_1}(K_i, x), \pi_{n_2}(K_i, x) \not\cong \ast$. For any $n \in \{ n_1, n_2 \}$, one has

$$\pi_n(K_{i+1}, z) = \pi_n(K_i \Rightarrow K_i, z)$$
$$= \pi_n(K_i \times K_i, (x, y))$$
$$\cong \pi_n(K_i, x) \times \pi_n(K_i, y)$$
$$\not\cong \ast$$

Given any vertex $y$ of some $k$-simplex in $K$, with $k \geq 2$. There is $i \geq 0$ and $x \in K_i$ such that $\delta_i, x \simeq y$. Since there is $\pi_n(K_i, x) \not\cong \ast$, then

$$\pi_n(K, y) \cong \pi_n(K, \delta_i, x) \not\cong \ast.$$

\[\square\]

**Definition 4.2** (Split Homotopy $\lambda$-Model). Let $\mathcal{K} \hookrightarrow \text{Cat}_\infty$ be a cartesian closed $\infty$-category with enough points. A Kan complex $K \in \mathcal{K}$ is a split homotopy $\lambda$-model if $K$ is a reflexive split Kan complex.
Note that every split homotopy λ-model is a homotopic λ-model as defined in 
[5] and [3], which only captures information up to dimension 2 (for equivalences 
(2-paths) of 1-paths between points). While the split homotopy λ-models have 
no dimensional limit to capture relevant information, and therefore, those can 
generate a richer higher λ-calculus theory.

**Example 4.4.** Given the split Kan complexes $B_0$, $D_0$ and $E_0$ of the Examples 
4.1, 4.2 and 4.3 respectively. Starting from the diagonal projection as the initial 
projection pair, these initial objects will generate the respective split Kan com-
plexes $B$, $D$ and $E$ in $\text{Fix}(F)$. One can see that the split homotopy λ-model $E$ has more information than the homotopic λ-models $B$ and $D$.

5. Conclusions and further work

Some methods were established for solving homotopy domain equations, 
which further contributes to the project of a generalization of the Domain The-
ory to a Homotopy Domain Theory (HoDT). Using those methods of solving 
equations, it was possible to obtain some specific homotopy models in a carte-
sian closed $\infty$-category, which could help to define a general higher λ-calculus 
theory. Besides, we prove the existence of an extension of the set $D_\infty$ to a Kan 
complex $K_\infty$, which models a type-free version of HoTT, which we call HoTFT 
(Homotopy Type-Free Theory) [12], which could have the advantage of rescuing 
the $\beta\eta$-conversions as relations of intentional equality and not as relations of 
judgmental equality as occurs in HoTT. For future work, it would be interest-
ing to see what is the relationship between the theory of a homotopy λ-model 
and a version of HoTT based on computational paths ([13] and [14]).

References

[1] S. Abramsky, A. Jung, Domain theory, In S. Abramsky, D. M. Gabbay, 
and T. S. E. Maibaum, editors, Handbook of Logic in Computer Science, 
volume 3, pages 1–168, Clarendon Press, 1994.
[2] A. Asperti, G. Longo, Categories, Types and Structures: An Introduction to Category Theory for the working computer scientist, Foundations of Computing Series, M.I.T Press, 1991.

[3] D. Martínez-Rivillas, R. de Queiroz, Towards a homotopy domain theory, arXiv:2007.15082.
URL https://arxiv.org/abs/2007.15082

[4] M. Hyland, Some reasons for generalizing domain theory, Mathematical Structures in Computer Science 20 (2010) 239–265.

[5] D. Martínez-Rivillas, R. de Queiroz, The ∞-groupoid generated by an arbitrary topological λ-model, Logic Journal of the IGPL 30 (3) (2022) 465–488 https://doi.org/10.1093/jigpal/jzab015, (also arXiv:1906.05729).

[6] T. U. F. Program, Homotopy Type Theory: Univalent Foundations of Mathematics, Princeton, NJ: Institute for Advanced Study, 2013.

[7] C. Kapulkin, P. Lumsdaine, The simplicial model of univalent foundations (after voevodsky), arXiv:1211.2851.
URL https://arxiv.org/abs/1211.2851

[8] P. Lumsdaine, M. Shulman, Semantics of higher inductive types, Mathematical Proceedings of the Cambridge Philosophical Society 169 (2020) 159–208.

[9] H. Barendregt, The Lambda Calculus, its Syntax and Semantics, North-Holland Co., Amsterdam, 1984.

[10] J. Lurie, Higher Topos Theory, Princeton University Press, Princeton and Oxford, 2009.

[11] A. Hatcher, Algebraic Topology, Cambridge University Press, New York, NY, 2001.
[12] D. Martínez-Rivillas, R. de Queiroz, The theory of an arbitrary higher $\lambda$-model, arXiv:2111.07092.
URL https://arxiv.org/abs/2111.07092

[13] R. de Queiroz, A. de Oliveira, A. Ramos, Propositional equality, identity types, and direct computational paths, South American Journal of Logic 2 (2) (2016) 245–296.

[14] A. Ramos, R. de Queiroz, A. de Oliveira, On the identity type as the type of computational paths, Logic Journal of the IGPL 25 (4) (2017) 562–584.