Solving imperfect-information games via exponential counterfactual regret minimization

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Abstract—Two agents’ decision-making problems can be modeled as the game with two players, and a Nash equilibrium is the basic solution conception representing good play in games. Counterfactual regret minimization (CFR) is a popular method to solve Nash equilibrium strategy in two-player zero-sum games with imperfect information. The CFR and its variants have successfully addressed many problems in this field. However, the convergence of the CFR methods is not fast, since they solve the strategy by iterative computing. To some extent, this further affects the solution performance. In this paper, we propose a novel CFR based method, exponential counterfactual regret minimization, which also can be called as ECFR. Firstly, we present an exponential reduction technique for regret in the process of the iteration. Secondly, we prove that our method ECFR has a good theoretical guarantee of convergence. Finally, we show that, ECFR can converge faster compared with the prior state-of-the-art CFR based methods in the experiment.

Index Terms—Decision-making, Counterfactual regret minimization, Nash equilibrium, zero-sum games, imperfect information.

1 INTRODUCTION

GAME theory has always been regarded as a touchstone to verify the theory of computation and artificial intelligence, and it is also a very attractive research field of artificial intelligence. Generally, the game can be divided into perfect information game (PIG) and imperfect information game (IIG), according to whether the player can observe the game state completely. The PIG means that the player can observe the game state completely, such as Go and Chess. On the contrary, the IIG contains some private information, thus the player cannot completely observe the game state, such as poker. The IIG attracts more and more researchers’ attention, since the IIG is closer to the real life and more challenging compared with the PIG. In this paper, we mainly focus on the problem of decision-making in the field of the IIG.

In recent years, reinforcement learning has made great success in many fields, whether in the field of the PIG or the IIG [1], [2]. DeepMind team’s pioneering agent, which combines reinforcement learning [3] with deep learning, has achieved excellent performance in Atari games [1]. In addition, AlphaGo [2] and AlphaStar [4] based on deep reinforcement learning (DRL), as well as the agents they developed in recent years, have made great achievements [3]. Although the DRL based method has been successful in many fields, the strategy based on the DRL lacks enough theoretical guarantee, which limits the further application of the DRL methods [3]. Therefore, the strategy solving method based on the game theory has attracted researchers’ attention. Since its good theoretical guarantee, counterfactual regret minimization (CFR) becomes a classical method to solve the game strategy with imperfect information [3] in two-player zero-sum games.

The CFR has many variants in recent years, which also attracts the attention of more and more researchers [5]. Monte Carlo counterfactual regret minimization (MCCFR) is a sampling CFR-based method, which applies Monte Carlo technique to the vanilla CFR [6]. CFR+ is the main method of the Cepheus, the first computer program which solved the heads-up limit Texas Hold’em poker efficiently [7]. Double neural CFR [8] and Deep CFR [9] combine the deep neural network with the vanilla CFR and the linear CFR (LCFR) respectively. In addition, single deep CFR (SDCFR) [10] is a simplified variant of the Deep CFR, which only uses one neural network to approximate the value in the LCFR. Moreover, public chance sampling in CFR (PCCFR) [11], variance reduction in MCCFR (VR-MCCFR) [12] and discount CFR (DCFR) [13] are all variants of the vanilla CFR.

The CFR based methods have achieved a great success in the field of the IIG, especially in poker games. Libratus is the first agent based on the CFR method, which beat the professional human player in heads-up no-limit Texas Hold’em poker [14]. DeepStack defeated the professional poker player and its method is sound with theoretical proof [15]. Pluribus, the latest computer program based on the related CFR method, defeated the top poker player in six-player no-limit Texas Hold’em poker, which is recognized as the milestone in the field of artificial intelligence and game theory [16]. However, these successful applications are all aimed at specific areas using some specific settings to solve the strategy combined with the CFR, and do not improve the vanilla CFR. Due to the vanilla CFR is an iterative strategy solving method, with the increase of the number of iterations, the strategy obtained is more and more accurate. Therefore, it is worth studying how to get an acceptable strategy through less iterations. In other words, we need to explore how to accelerate the convergence of the CFR method.

Thus, this paper aims to the study of speeding up the convergence of the vanilla CFR. In this paper, we firstly...
present an exponential reduction technique that applies to the vanilla CFR, which we call this new variant exponential CFR (ECFR). Secondly, we give a theoretical proof of the ECFR, which shows that our method ECFR has a good theoretical guarantee the same as the vanilla CFR. Finally, four kinds of games (Kuhn poker, Leduc poker, Royal poker, Liar ‘s Dice) are used to test the ECFR. Kuhn poker and Leduc poker are classic test platforms in the field of the IIG. Extensive experimental results show that the ECFR converges faster in the four kinds of games compared with the state-of-the-art CFR based methods in recent years.

The remainder of our paper is organized as follows. In Sect.2 we introduce the concepts of the extensive form game and describe the Nash equilibrium and the CFR. Our method is described in Sect.3, which includes the ECFR and its convergence proof. In Sect.4, we evaluate the performance of the ECFR on four kinds of games. Finally, in Sect.5, we make a conclusion of the whole paper.

## 2 NOTATIONS AND PRELIMINARIES

In this section, the notations and definitions of the extensive-form game are introduced firstly. Secondly, the concept of Nash equilibrium is described. Finally, the overview of the CFR is described.

### 2.1 Extensive-form game

Normal-form game and extensive-form game are two classical models, which are used to model related problems of the game in the IIG. Generally, the extensive-form game is a widely used model to study the complex game with large scales. The game tree can be used to represent the extensive-form game in the field of the IIG, which is the same as that in the PIG (such as Go, Chess). Fig. 1 is a game tree of the game, Coin Toss.

In the game tree, each node represents a game state. A leaf node, which is known as the terminal node, indicates that the game has ended. Meanwhile, the corresponding payoff is returned after the game ends. In addition, the edge between two nodes represents the action or the decision taken by the player in the turn. Remarkably, the information set is a unique concept in the field of the IIG, which represents a game player cannot distinguish between two states in the same information set. As shown in the Fig. 1 the player \( P_1 \) can choose between actions Left and Right, with the action Left leading to obtain the payoff directly. If the action Right is selected by the player \( P_1 \), then the \( P_2 \) has the opportunity to guess how the coin landed. If the \( P_2 \) guesses correctly, the \( P_2 \) will receive a reward of -1 and the \( P_2 \) will receive a reward of 1 \[17\].

Generally, a finite extensive-form game with imperfect information has six components \( < N, H, P, f, I, u > : N \) is game player. \( H \) is a limited set of sequences, which is the possible historical actions. \( P \) is the player function. \( P(h) = c \) means that the chance determines the action \( a \) after the history \( h \). \( f_a \) is a function that associates with every history \( h \) for which \( P(h) = c \) is a probability measures \( f_a (c \mid h) \) on \( A(h) \) \( (f_a (a \mid h) \) is the probability that \( a \) occurs given \( h \)). \( I \) is the information set where the player cannot distinguish the state. \( u \) is an utility function for every termination state. Due to there are many symbols in the paper, especially in this part. Thus we give a brief list of variables for further reading in the Table 1.

### 2.2 Nash Equilibrium

Nash equilibrium is an important theory in game theory, which lays a theoretical foundation for many studies. Nash equilibrium is usually used to solve the strategy of the two-player extensive-form game in the field of the IIG, which can be also called non-cooperative game equilibrium (NE) \[18\]. To better understand Nash equilibrium, we first introduce the concept of the strategy in the following.

The strategy \( \sigma \) is a probability vector over actions in the extensive-form game, and \( \sigma_i \) is the strategy of the player \( i \). \( \sigma_i (I, a) \) represents the probability of the action \( a \) of the player \( i \) in the information set \( I \). \( \sigma_{-i} \) refers to all the strategies in \( \sigma \) except the player \( i \)’s strategy \( \sigma_i \). \( \pi^\sigma (h) \) can be the probability of the history \( h \) that occurs, only if the game player takes the legal action according to the strategy \( \sigma \). \( u_i (\sigma_i, \sigma_{-i}) \) is the expected payoff for the player \( i \) if all players play according to the strategy profile \( (\sigma_i, \sigma_{-i}) \).

A best response to \( \sigma_{-i} \) is a player \( i \)’s strategy \( BR(\sigma_{-i}) \) such that \( u_i (BR(\sigma_{-i}), \sigma_{-i}) = \max_{\sigma_i} u_i (\sigma_i, \sigma_{-i}) \). A Nash

| Variable | The meaning |
|----------|-------------|
| \( u \)  | the utility function, and \( u_i \) represents the utility of the player \( i \) |
| \( H \)  | a limited set of sequences, which is the possible historical actions |
| \( \sigma \) | the strategy, and \( \sigma_i \) the strategy of the player \( i \), \( \sigma_{-i} \) is the strategy of the player except the player \( i \) |
| \( I_i \) | the information set of the player \( i \) |
| \( \pi^\sigma (h) \) | the joint probability of reaching \( h \) if all players play according to \( \sigma \), \( \pi_i^\sigma (h) \) is the probability of reaching \( h \) if the player \( i \) play according to \( \sigma \) |
equilibrium $\sigma^*$ is a strategy profile where everyone plays a best response: $\forall i, u_i(\sigma^*_i, \sigma^*_-i) = \max_{\sigma'_i} u_i(\sigma'_i, \sigma^*_-)$. \[18\]

### 2.3 Counterfactual regret minimization

Counterfactual regret minimization (CFR) is a popular method to solve the Nash equilibrium strategy in two-player zero-sum games with imperfect information \[5\]. We provide an overview of the vanilla CFR in the following.

Let $\sigma_t$ be the strategy on the iteration $t$. The instantaneous regret $r^t(I,a)$ on the iteration $t$ is $r^t(I,a) = \nu^t(I,a) - \nu^t(I)$ and the regret on the iteration $T$ is $R^T(I,a) = \sum_{t=1}^{T} r^t(I,a)$. Here, $\nu^t(I)$ is defined as $\nu^t(I) = \sum_{h \in T} \left( \pi_{\sigma_t}(h | I) v^t_q(h) \right)$, and $\nu^t(I,a) = \sum_{h \in T} \left( \pi_{\sigma_t}(h | I) v^t_q(h \cdot a) \right)$, where $\pi_{\sigma_t}(h | I) = \pi_{\sigma_t}(h) / \pi_{\sigma_t}(I)$. Generally, $R^T(I,a) = \max \{ R^T(I,a), 0 \}$. The CFR updates its strategy iteratively through the regret matching algorithm (RM) on each information set. In the RM, a player picks a distribution over actions in an information set in proportion to the positive regret on those actions. Formally, on the iteration $t + 1$, the player selects actions $a \in A(I)$ according to probabilities:

$$
\sigma^{t+1}(I,a) = \left\{ \begin{array}{ll}
\frac{R^T_{+}(I,a)}{\sum_{a' \in A(I)} R^T_{+}(I,a')} & \text{if } \sum_{a'} R^T_{+}(I,a') > 0 \\
0 & \text{otherwise}
\end{array} \right.
$$

### 3 Our Method

In this section, firstly, the exponential reduction technique is presented in Sect. 3.1. Secondly, the process of the ECFR is introduced in Sect. 3.2. Thirdly, the proof of the ECFR is discussed in Sect. 3.3. Finally, the difference to other CFR based methods will be discussed.

#### 3.1 Exponential reduction technique

In current years, there are many improved methods based on the vanilla CFR. Discount CFR (DCF) \[13\], Linear CFR (LCFR) \[9\] and dynamic thresholding for the CFR \[19\] are aiming at the study of speeding up the convergence of the vanilla CFR.

Among these methods, LCFR and DCFR are mainly to balance the weight of regret value generated in the early iteration and the later iteration. In \[19\], it introduces dynamic thresholding for the CFR, in which a threshold is set at every iteration such that any action with probability below the threshold is set to zero probability. In addition, the essence of the CFR is an iterative strategy solution method, and the strategy can become more and more accurate with the increase of iterations. Thus, intuitively, we need to balance the weight of regret value and the iteration. However, whether the vanilla CFR or several improved methods for balancing weights, the ultimate goal is to accelerate the iteration speed of the CFR, that is, to improve the convergence speed of the solution strategy. Based on this idea, we propose an exponential reduction technique, which can make the strategy converge faster.

We first analyze several improvement methods in recent years. As LCFR \[9\] and DCFR \[13\], their improvements to the vanilla CFR are mainly reflected in the weighting of the regret value. The LCFR weights the regret value, which is the same as the number of iterations. The DCFR also weights regret values, but this weight is respectively for positive and negative regret values, which are $(t/(t+1))^\alpha$ and $(t/(t+1))^\beta$ ($t$ is the number of the iteration). In other words, the weights of the regret values in these methods are different. However, they all give a certain weight to the regret value, so that the importance of the regret value does not change much with the number of iterations in the same situation.

Through the above analysis, we propose an exponential reduction technique. The core of this technique is exponential weight. Its formally description is as follows:

$$
f(x) = \left\{ \begin{array}{ll}
e^\alpha \cdot x & \text{if } x > 0 \\
e^\beta & \text{if } x \leq 0
\end{array} \right.
$$

where $\alpha$ is a parameter corresponding with the variable $x$, $\beta$ is a parameter with small value, and $f(x)$ is the output through the exponential reduction.

Besides, given the variable $x$, specifically in the CFR, there is a negative value in this variable $x$. The previous processing is to set the negative value to 0, no matter what the weight is, the final processed value is zero. However, for the exponential weight, when the negative value is set to 0, we give the variable a new minimum value $e^\alpha \cdot \beta$, which makes the final result not zero.

#### 3.2 Exponential Counterfactual Regret Minimization

We propose a novel CFR-based variant, exponential Counterfactual Regret Minimization, which can be also called ECFR, in this section. Our method ECFR is based on the vanilla CFR, which redistributes the weight of instant regret value through the exponential reduction technique introduced before.

As depicted before, the parameter $\alpha$ in the exponential reduction technique is corresponding with the variable $x$. Specifically in the ECFR, we firstly redefine a loss function, which needs to be closely related to the instant regret value in each iteration (we regard the instant regret value on each iteration as the variable). We call it $L_1$, loss, which is more suitable for the CFR. It can be defined as follows:

$$
L_1 = r^t(I,a) - EV_I
$$

where $r^t(I,a)$ has the same definition in the vanilla CFR that is the immediate counterfactual regret. $EV_I$ is the average counterfactual regret value on each iteration, $EV_I = \frac{1}{|I|} \sum_{a \in A_I} r(I,a)$, $A_I$ is the legal actions on the information set $I$.

The vanilla CFR decomposes the total regret value into the regret value above each information set. Our approach still uses this feature for every action on each information set at every iteration. Different from the vanilla CFR, our method ECFR uses a certain weight for the calculation of the immediate regret value. ECFR balances the importance of immediate regret values by introducing $L_1$ loss and weighting in exponential form, with the iteration increasing.

The regret for all action $a \in A(I)$ on each information set $I$, $R^T_{+,ECFR}(I,a)$ can be depicted as follows:

$$
R^T_{+,ECFR}(I,a) = \left\{ \begin{array}{ll}
\sum_{t=1}^{T} e^{t-1} F^t(I,a) & \text{if } r^t(I,a) > 0 \\
\frac{1}{\alpha} \sum_{t=1}^{T} e^{t-1} \beta & \text{if } r^t(I,a) \leq 0
\end{array} \right.
$$

\[4\]
where $\beta$ is a parameter that will be set in the following.

Then, the strategy of next iteration strategy for $T + 1$ iterations, which can be computed with regret matching algorithm (RM) is depicted as follows:

$$
\sigma_i^{T+1}(I, a) = \frac{e^{t_i}R_i^{T,ECFR}(I, a)}{\sum_{a \in A(I)} e^{t_i}R_i^{T,ECFR}(I, a')} \tag{5}
$$

In the vanilla CFR, the regret $R^T \leq \sum_{t \leq T} R^T(I)$ if the player $i$ plays according to the CFR on each iteration. Thus, as $T \to \infty, R^T_T \to 0$ [5]. If both players' average regret satisfies $\frac{R^T}{T} \leq \epsilon$, then their average strategies $\bar{\sigma}_1^T, \bar{\sigma}_2^T$ will be a 2-$\epsilon$-Nash equilibrium in the two-player zero-sum game [20], where the average strategy $\bar{\sigma}_i^T(I) = \frac{\sum_{t=1}^{T} \pi_{i}^T(I, \sigma^T_{i}(I))}{\sum_{t=1}^{T} \pi_{i}^T(I)}$.

For regret matching in the CFR [5], it proves that if $\sum_{t=1}^{\infty} w_t = \infty$ then the weighted average regret, which is defined as $R_{w,T}^i = \max_{a \in A} \frac{\sum_{t=1}^{T} (w_t \sigma_t^i(a))}{\sum_{t=1}^{T} w_t}$ is bounded by:

$$
R_{w,T}^i = \frac{\Delta A \sqrt{T} \sqrt{\sum_{t=1}^{T} w_t^2}}{\sum_{t=1}^{T} w_t} \tag{6}
$$

The work of [21] has shown that the weighted average strategy $\bar{\sigma}_{w,T}^i(I) = \frac{\sum_{t=1}^{T} (w_t \sigma_t^i(I))}{\sum_{t=1}^{T} w_t}$ is a 2-$\epsilon$-Nash equilibrium if the weighted average regret is $\epsilon$ in two-player zero-sum games. The same conclusion can be applied to our method ECFR, the average strategy of players computed with ECFR will eventually converge to Nash equilibrium, and the average strategy of the ECFR is as follows:

$$
\bar{\sigma}_i^T(I, a) = \frac{\sum_{t=1}^{T} e^{t_i} \pi_{i}^T(I, \sigma^T_{i}(I))}{\sum_{t=1}^{T} e^{t_i} \pi_{i}^T(I)} \tag{7}
$$

The detailed proof of the CFR will be introduced in the next section. The ECFR algorithm is depicted in the algorithm 1.

### Algorithm 1 The ECFR algorithm

**Input:** The game $G$, the strategy $\sigma^i$, the regret $v_i(\sigma, I)$ and $R_i^T(I, a)$, $\beta$

**Output:** The strategy $\sigma_i^{T+1}$ of next iteration, the average strategy $\bar{\sigma}_i^T(I, a)$.

1. for iteration $t = 1 \rightarrow T$
2. **Initialize** the strategy $\sigma^i$, regret $v_i(\sigma, I)$ and $R_i^T(I, a)$
3. for Player $i$ = 1, 2 do
4. $v_i(\sigma, I) = \sum_{h \in H} \pi_{i}(h) \sum_{h' \in H} \pi_{i}(h') u_i(h')$
5. $r_i(I, a) = v_i(\sigma_{i-1}^T(I), a) - v_i(\sigma^T_{i}(I))$
6. $R_{i,ECFR}(I, a) = \left\{ \begin{array}{ll}
R_i^T(I, a) & r_i > 0 \\
\sum_{t=1}^{T} e^{t_i} \beta_i & r_i \leq 0
\end{array} \right.$
7. $\sigma_i^{T+1}(I, a) = \sum_{a \in A(I)} e^{t_i} \pi_{i}^T(I, \sigma^T_{i}(I))$
8. $\bar{\sigma}_i^T(I, a) = \frac{\sum_{a \in A(I)} e^{t_i} \pi_{i}^T(I, \sigma^T_{i}(I))}{\sum_{a \in A(I)} e^{t_i} \pi_{i}^T(I)}$
9. end for
10. end for
11. return $\sigma_i^{T+1}(I, a), \bar{\sigma}_i^T(I, a)$

### 3.3 Theoretical guarantee of convergence for the ECFR

A brief but sufficient theoretical proof of convergence for the ECFR is given in this section. As described in the previous section, our method ECFR also use the regret matching algorithm as the core algorithm. We give the bigger argument to weight the regret. Thus, we will give a proof that there is a bound for the sequence of weights, when the average strategy is calculated. Meanwhile, the bound on the convergence can be never lower compared with the vanilla CFR.

**Theorem 1** Assume that the number of the iteration $T$ of the ECFR, which is played in a two-player zero-sum game. Then the weighted average strategy profile is a $\Delta |I| \sqrt{A} \sqrt{T}$-Nash equilibrium.

The proof is provided in the following, which combines the the proof for the vanilla CFR [5], CFR+ [7, 22] and the discount CFR [13].

**Proof.** The lowest amount of the instant regret on any iteration is $-\Delta$. Consider the weighted sequence of iterations $\sigma^1, \ldots, \sigma^T$, where $\sigma^T$ is identical to $\sigma^I$, but the weight is $w_{a,I} = \prod_{t=1}^{T-1} e^{t_i} = e^{\frac{(T-1)(T+1)}{2}}$ rather than $w_{a,I} = \prod_{t=1}^{T-1} e^{T_i}$. $R^T(I, a)$ is the regret of the action $a$ on the information set $I$ at the iteration $t$, for this new sequence.

In addition, for the regret matching [23], which proves that if $\sum_{t=1}^{\infty} w_t = \infty$, then the weighted average regret that defined as $R_{w,T}^i = \max_{a \in A} \frac{\sum_{t=1}^{T} (w_t \sigma_t^i(a))}{\sum_{t=1}^{T} w_t}$ can be bounded by the Eq.6 depicted in the section 3.2.

We can find that $R^T(I, a) \leq \frac{\Delta \sqrt{A} \sqrt{T}}{\sqrt{A}}$ for the player $i$ action $a$ on the information set $I$, from the Lemma 3. We can use the Lemma 1, which uses the weight $w_{a,I}$ for the iteration $t$ with $B = \frac{\Delta \sqrt{A} \sqrt{T}}{\sqrt{A}}$ and $C = 0$. It means that $R^T(I, a) \leq w_T(B - C) \leq \frac{\Delta \sqrt{A} \sqrt{T}}{\sqrt{A}}$ from the Lemma 1. Furthermore, we get the weighted average regret is at most $\frac{\Delta |I| \sqrt{A} \sqrt{T}}{\sqrt{A}}$ from the Lemma 3. Since for the information set $|I_1| + |I_2| = |I|$, thus the weighted average strategies form a $\frac{\Delta |I| \sqrt{A} \sqrt{T}}{\sqrt{A}}$-Nash equilibrium. (With the iteration $T$ increasing, $\frac{\Delta |I| \sqrt{A} \sqrt{T}}{\sqrt{A}}$ is infinitely close to zero).

**Lemma 1.** Call a sequence $x_1, \ldots, x_T$ of bounded real values BC-plausible if $B > 0, C \leq 0, \sum_{i} x_i \geq C$ for all $i$, and $\sum_{i} x_i \leq B$. For any BC-plausible sequence and any sequence of non-decreasing weights $w_t \geq 0, \sum_{t=1}^{\infty} w_t x_t \leq w_T(B - C)$.

**Lemma 2.** Given a group of actions $A$ and any sequence of rewards $v_i$, such that $|v^i(a) - v^i(b)| \leq \Delta$ for all $t$ and all $a, b \in A$, after conducting a set of strategies decided by the regret matching, however applying the regret-like value $Q^i(a)$ instead of $R^T(I, a)$, $Q^T(I, a) \leq \Delta \sqrt{|A|T}$ for all $a \in A$.

**Proof.** This Lemma is closely resembling Lemma 1, which are both from [22], thus here we donot give the detailed proof of these two lemmas.

**Lemma 3.** Suppose the player $i$ conducts $T$ iterations of the ECFR, then the weighted regret for the player $i$ is at most $\Delta |I| \sqrt{A} \sqrt{T}$, and the weighted average regret for the player $i$ is at most $\frac{\Delta |I| \sqrt{A} \sqrt{T}}{\sqrt{A}}$.

**Proof.** The weight of the iteration $t < T$ is $w_t = \prod_{t=1}^{T} e^{t_i}$ and $w_T = 1$. Therefore, for all iteration $t$, $\sum_{t=1}^{T} w_t^2 \leq T e^{T^2}$. 
In addition, \( \sum_{t=1}^{T} w_{t} \geq T e^{\frac{T^{2}}{2}} \geq T e^{T^2} \).

Through the Eq.5 and the Lemma 2, we can find that \( Q_{i}^{w,T} \leq \Delta \sqrt{|A|/\sum_{t=1}^{T} w_{t}^2} \leq \Delta \sqrt{|A|} e^{2T^2} \). Due to \( R^{T}_{i} \leq \sum_{I \in I_{d}} R^{T}(I) \frac{T}{\Delta \sqrt{|A|} e^{2T^2}} \), we can find that \( Q_{i}^{w,T} \leq \Delta \sqrt{|I_{d}|/\sum_{t=1}^{T} w_{t}^2} \Delta \sqrt{|A|} e^{2T^2} \). Since \( R^{T}_{i} \leq Q_{i}^{w,T} \), thus \( R^{T}_{i} \leq Q_{i}^{w,T} \).

### 3.4 Difference from Other CFR based methods

In this section we mainly analyze the difference between our method ECFR and other three major CFR based methods, LCFR [9], DCFR [9] and dynamic thresholding for the CFR [19] (we call this method dynamic CFR for short). Because these methods all the study of speeding up the convergence of the vanilla CFR.

The LCFR and the DCFR both improve the CFR through a certain weight. The DCFR is completed by discounting the immediate regret value obtained in each iteration. For the positive immediate regret value, multiplied by the weight of \( \frac{1}{\sqrt{t}} \), and for the negative immediate regret value, multiplied by the weight of \( \frac{\beta^{t}}{\sqrt{t}} \). In addition, for the average strategy, the vanilla CFR directly uses the average strategy as the final strategy. In the DCFR, the average strategy is multiplied by \( \beta^{t} \) as the final strategy. After the above three forms of discount, the regret value of each iteration and the final strategy are realized by the weight reallocated. The LCFR uses the iteration corresponding to the immediate regret value. That is to say, the iteration is weighted to the regret value in every iterations with the number of iteration increasing. In the dynamic CFR, it speeds up the convergence by pruning parts of decision tree with dynamic thresholding.

Although the ECFR also reweights the regret value in each iteration, it is quite different from the DCFR and the LCFR both in the original idea and the implementation. First of all, our approach comes from an intuitive idea. No matter which method is improved based on the vanilla CFR, the ultimate goal is to further accelerate the convergence of the strategy by reweighting the regret value. In the iterative method, when the number of iterations increases and the strategy becomes better, the importance of the same level of the immediate regret is totally different. Secondly, in terms of implementation details, our method reallocates the weight through the exponential form of the redefined \( L_{1} \) loss, and the DCFR is realized by some discount on the number of iterations. Therefore, this is a totally different approach.

In the dynamic CFR, the exponential weight is used in computing the next iteration strategy when using Hedge to minimize regret. However, our approach ECFR is quite different from the dynamic CFR in six aspects. Firstly, our approach uses exponential decay to calculate cumulative regret and the next iteration strategy, while the dynamic CFR only uses exponential weight in computing the next iteration strategy. Secondly, our approach gives a small value to the negative immediate regret. Because we think that the actions corresponding to the negative regret in the early stage are also valuable. While the dynamic only deals with the positive regret. Thirdly, the exponential weight is only used in the dynamic CFR when using the Hedge as the algorithm to minimize the regret. While our approach uses regret matching algorithm as regret minimization method for the strategy iteration. Fourthly, the dynamic CFR does not traverse all nodes in the game tree, but prunes the nodes below the threshold to accelerate the convergence. While our approach traverses all nodes in the game tree, and accelerates convergence by redistributing the weight of regret. Fifthly, the parameter setting for exponential weight of the two methods are different. \( e^{\frac{T}{\sqrt{T}}} \) is set in the dynamic CFR, while \( e^{\frac{T^{2}}{\sqrt{T}}} \) is set in the iteration \( t \) up to the iteration \( t \), while in our approach the \( L_{1} \) is set \( (L_{1} = \frac{\Delta \sqrt{|I_{d}|/\sum_{t=1}^{T} w_{t}^2} \Delta \sqrt{|A|} e^{2T^2}}{\sqrt{T}}) \). Finally, from the theoretical analysis, the regret bound of the two methods are different. The regret bound of the dynamic CFR is \( R^{T}(I) \leq C \sqrt{\Delta(I) \sqrt{2 \ln |A(I)|}} \sqrt{T} \) \((C \geq 1 \text{ on every iteration } t)\), while the regret bound in our method is \( R^{T}(I) \leq \frac{\Delta \sqrt{|A|} e^{2T^2}}{\sqrt{T}} \).

### 4 Experiment

#### 4.1 Experiment Setup

Due to the game of the poker has all elements of the IIG, it has been used to test the performance of CFR based methods by researchers in recent years. Therefore, we test and compare the ECFR with other CFR based methods on three different poker games. The three kinds of poker are Kuhn, Leduc and Royal, which are all simple versions of Texas hold’em poker. Compared with Texas hold’em poker, although the number of game states and actions is smaller, they still have all the characteristics of the IIG. We choose them as the test platform, mainly because they are small enough to evaluate the performance of the algorithm.

It should be pointed out that the three kinds of poker we used in the experiment are about two-player games. Among these three kinds of poker, Kuhn poker is the simplest. Kuhn poker has three cards in total, only one round, each player has one hand, and no public cards. Leduc poker has six cards in two rounds. Each player has a private hand in the first round and a public card in the second round. There are eight cards in the Royal poker game, and there will be three rounds. In the first round, each player gets a private card, and a public card is issued in the second round and the third round respectively. The table Tab.2 gives some descriptions of three types of poker.

In addition, in order to further verify the effectiveness of our method, we also test it on the game of Liar’s Dice. The Liar’s Dice is different from poker game, but it still belongs to the field of the IIG. Five dice are used for each player with dice cups used for concealment. In the each round, each player can roll a “hand” of dice under their cup and look at their hand while keeping it concealed from other players. The first player begins bidding, which announces any face value and the number of dice that the player believes are in the field of the IIG. Five dices are used for each player in this round, and a public card in the second round. There are eight cards in the Royal poker game, and there will be three rounds. In the first round, each player gets a private card, and a public card is issued in the second round and the third round respectively. The table Tab.2 gives some descriptions of three types of poker.
### 4.2 Experimental Results

A natural and standard evaluation metric is measuring the exploitability of the strategy solved by the algorithm in recent years. The strategy is better when the strategy’s exploitability is lower. When the exploitability is zero, it means that this strategy cannot be beat by any strategy.

Two groups experiments are conducted in the paper. The first experiment is to verify the effectiveness of the method on four different games. The second group experiments is the ablation study, which mainly analyzes the parameter setting.

#### 4.2.1 Compared with state-of-the-art methods

We conduct the first group experiments with three state-of-the-art methods of recent years, which are CFR [5], LCFR [9], and DCFR [13] respectively. It is found that when the number of iterations 10000, the reduction of exploitability has been very small. Thus, we only carried out 10000 iterations, which is enough to show the effectiveness of each method. The experimental results are shown in the Fig. 2. Note that figures in the left column are the overall curve of the experimental results and figures in the right column are a further detailed display of the figures in the left column.

As shown in the Fig. 2, four methods are totally used in the experiment. The CFR [5], LCFR [9], and DCFR [13] are used to compare with our approach ECFR. The CFR [5] is the first method to solve the strategy through regret matching in the IIG. The LCFR and the DCFR are both CFR based methods, which discount regrets from iterations in various ways. Among them, we choose the parameters $\alpha = 1.5$, $\beta = 0$, and $\gamma = 2$ in the DCFR, which is an optimal parameter given in the [13].

Firstly, we analyze the convergence of our approach. In the Fig. 2 (especially the figure on the left), we can find that the red solid line (stands for our approach ECFR) all ends up close to zero in four test games. In addition, from the overall trend of the curve, we can also find that with the increase of the number of iterations, the ECFR shows a similar trend with the other three methods. That is, the curve present a downward trend. Moreover, the convergence of the ECFR has been proved theoretically before in the Sec.3.3. Therefore, our approach has a good convergence, which has been verified from both experimental and theoretical perspectives.

#### 4.2.2 Ablation study

We conduct the second group experiments to study the different parameter settings. For the setting of $\beta$ in the ECFR in Eq.4, several different settings are tested. Specifically, we have made four different settings for $\beta$: $\beta = 1, \pm 0.1, \pm 0.01$, $\pm 0.001, \pm 0.0001, \pm 0.00001, \pm 0.000001; r, r^2, r^3; r, r^2, r^3$. $r$ is the instant regret for each action and $t$ is the number of iteration. The results are shown in the Fig. 3a. We start with a rough selection of four different sets of values and then make a further fine selection. For the further fine selection, the $\beta$ is set to: $-0.008, -0.009, -0.0001, -0.00001, -0.00012, -r^2, -2r^2, -3r^2$, and the results are shown in the Fig. 3b. Considering that we only choose the optimal parameter settings here, two games (Kuhn and Leduc) are used to test the setting. The iteration is set to 1000.

As shown in the Fig. 3a we can find that $\beta = -0.0001$, $\beta = -r^2$, $\beta = -0.00001$, and $\beta = r^3$ have a good performance in the Kuhn poker game. In the Leduc poker game, $\beta = -0.0001$ and $\beta = -r^2$ have a good performance, but $\beta = -0.00001$ and $\beta = r^3$ perform worse than $\beta = -0.0001$ and $\beta = -r^2$.

Table 2: The details of three kinds of poker

| Poker  | Total cards | Public cards | Private cards | Round | Ante | Betsize |
|--------|-------------|--------------|---------------|-------|------|---------|
| Kuhn   | 3           | 0            | 1             | 1     | 1    | 1       |
| Leduc  | 6           | 1            | 1             | 2     | 1    | 2; 4    |
| Royal  | 8           | 2            | 1             | 3     | 1    | 2; 4; 4 |

Fig. 2 shows the experimental results on the Leduc poker. We can find that our approach is not always better than other methods. It can be found that although in the iteration $t = 73 \sim 76,118 \sim 122$, the exploitability of the ECFR (the red curve) is slightly slower than that of the DCFR and the LCFR. But in addition to these limited number of iterations, our method always shows superiority over all other methods.

Fig. 2c shows the experimental results on the Royal poker. It can be seen clearly from the figure that the performance of our method performs better than that of other comparison methods. The red curve is always at the bottom compared with other curves.

Fig. 2a shows the experimental results on the Kuhn poker. It can be found that our approach is not always better than other methods. It can be found that although in the iteration $t = 73 \sim 76,118 \sim 122$, the exploitability of the ECFR (the red curve) is slightly slower than that of the DCFR and the LCFR. But in addition to these limited number of iterations, our method always shows superiority over all other methods.
On the whole, $\beta = -0.0001$ and $\beta = -r^2$ are chosen for further optimization. In order to further select the appropriate $\beta$, we set them more finely. For $\beta = -0.0001$, $-0.008, -0.009, -0.00011, -0.00012$ are added for compar-

Fig. 2. Compared with the prior state-of-the-art methods. (The Y-axis represents the exploitability, and the X-axis represents the number of iterations. The smaller of the exploitability, the better. The subfigures in the left column are the overall curve of the experimental results. The subfigures in the right column are a further detailed display of the subfigures in the left column, in order to better display the experimental results.)
In the Kuhn poker game, we can find that $\beta = -r^2$ performs the best compared with other settings in the Fig. 3b. In the Leduc poker game, although $\beta = -0.0001, 0.0001, 0.00012$ perform better in the first 400 iterations, the performance of $\beta = -r^2$ gradually exceeds others after 400 iterations. Therefore, $\beta = -r^2$ is selected as the final setting in the comparison experiments.

5 Conclusion

In this paper we proposed a new variant of the vanilla CFR, ECFR, which is used to solve the approximate Nash equilibria strategy of extensive-form games in the imperfect information game. We introduce the exponential reduction technique for regret in the process of iteration. Moreover, we give the theoretical guarantee of our method. Extensive experiments are conducted on four kinds of games. The experimental results show that our method not only has a good convergence, but also converges faster compared with state-of-the-art methods.
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