\[ \frac{1}{2} \]-Anyons in small atomic Bose-Einstein condensates

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We discuss a way of creating, manipulating and detecting anyons in rotating Bose-Einstein condensates consisting of a small number of atoms. By achieving a quasigstate in the atomic motional states we drive the system into a \[ \frac{1}{2} \]-Laughlin state for fractional quantum Hall bosons. Localized \[ \frac{1}{2} \]-quasiholes can be created by focusing lasers at the desired positions. We show how to manipulate these quasiholes in order to probe directly their \[ \frac{1}{2} \]-statistics.

The experimental achievement of Bose–Einstein condensation of weakly interacting atomic gases promises new possibilities to study the quantum properties of many–body systems. As compared to other systems, quantum degenerate atomic gases can be easily controlled and manipulated by electromagnetic fields, which makes them ideal candidates for the study of several intrinsically quantum phenomena. Until now, interest has focused mainly on using atomic condensates to study single particle quantum phenomena (as those occurring in atomic interferometry or atom optics), since for low temperatures atoms in condensates occupy essentially the same single particle state. In contrast, the possibility of observing entanglement, one of the most fascinating features of quantum mechanics, remains almost unexplored, requiring a way of making the gas to effectively behave in a strongly interacting manner. First steps in this direction are the recent proposals to entangle atomic beams and atomic spin squeezing in Bose-Einstein condensates (BEC), and the numerical prediction of new correlated phases appearing in rotating condensates.

In this Letter we show how a quantum degenerate Bose gas consisting of a small number of atoms can be used to study a quantum phenomenon highly collective in nature, namely the formation of quasiparticles exhibiting fractional statistics. This system offers the novel possibility of creating and manipulating anyons in a well controlled way that may allow for the experimental probing of their fractional statistics. The idea is to achieve a quasigstate of the atomic motional states by rotating the trap that confines the atoms, as it has successfully been done for the creation of vortices. Under appropriate conditions (like two dimensional confinement), the situation we describe can be understood in terms of the fractional quantum Hall effect for bosons. As in the theory of the fractional quantum Hall effect for electrons, elementary excitations exhibiting fractional statistics appear. We consider a situation where the atomic system is first prepared in a \[ \frac{1}{2} \]-Laughlin state, a highly correlated quantum liquid with nearly uniform density. We then show that piercing the system by offresonant laser light a single Laughlin quasihole localized at some chosen position \( z_0 \) can be created. This excitation involves a density profile in which exactly \( \frac{1}{2} \)-atom has been removed at \( z_0 \). In addition, lasers provide us with a tool for creating states with two quasiholes at the desired positions, and moreover, for creating superpositions of states having one and two quasiholes. Driving the superposition state along the proper path, as in a Ramsey-type interferometer, will allow to test the fractional statistical phase. Apart from interest of detecting such a phase directly, the ideas developed in the present work may pave the way for a physical implementation for quantum information processing based on anyons; as reported in [8], the most robust way of performing quantum computations seems to be based on excitations with fractional statistics since they have several fault–tolerant properties built–in. On the other hand, the conditions required to observe the effects predicted here are very demanding, and we expect that can be reached in the near future only with a small number of atoms.

We consider a set of \( N \) bosonic atoms confined in a potential which rotates in the \( x-y \) plane at a frequency \( \Omega \). We will assume that the confinement in the \( z \) direction is sufficiently strong so that we can ignore the excitations in that direction, and we will consider that the potential in two dimensions is isotropic and harmonic. The Hamiltonian describing this situation in a frame rotating with the trap is:

\[
H = \frac{1}{2} \sum_{i=1}^{N} \left( -\nabla_i^2 + r_i^2 - 2\Omega \frac{\omega}{\omega_L} L_{iz} \right) + \eta \sum_{i<j} \delta(r_i - r_j),
\]

with \( L_{iz} \) being the \( z \) component of the angular momentum of the \( i \)-th atom, and where we have used the trap energy, \( \hbar \omega_L \), as the unit of energy, and \( \ell = (\hbar/m_\omega)^{1/2} \) as the unit of length. The atoms are interacting via an effective contact potential, and the interacting coupling constant \( \eta \) is related to the \( s \)-wave scattering length, \( a \), and to the localization length in the \( z \) direction, \( \ell_z \), by \( \eta = \sqrt{2\pi a/\ell_z} \).

As pointed out recently in [8], in the limit \( \Omega = \omega \) the Hamiltonian is formally identical to the Hamiltonian of electrons in the quantum Hall effect with \( \omega \) playing the role of the magnetic field (with cyclotron frequency \( \omega_c = 2\omega \)), and the usual Coulomb interaction between electrons replaced by a contact interaction. Regardless

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that we are dealing with bosons instead of fermions the two problems are formally identical in this limit. We will now study how to create and manipulate anyons in our twin atomic system.

We begin by writing the Hamiltonian in the following form:

\[ H = H_B + H_L + V. \]  

(2)

Here, \( H_B = \sum_{i=1}^{N} -\nabla_i^2/2 + r_i^2/2 - L^z_i \) is the quantum Hall single particle Hamiltonian, whose single particle energy levels are the Landau levels equally spaced by the cyclotron energy \( 2\hbar\omega \). The Hamiltonian \( H_L = (1 - \Omega/\omega) L^z \) is proportional to the \( z \) component of the total angular momentum, \( L^z = \sum_{i=1}^{N} L^z_i \), and \( V \) is the interaction term. From now on we consider the limit in which the energy scales characterizing Hamiltonians \( H_B \) and \( V \) are much larger than the one corresponding to \( H_L \). This means that both the trap energy and the typical interaction energy are large compared to the angular momentum term. In this limit the ground state and elementary excitations of the system will lie on the subspace of common zero energy eigenstates of \( H_B \) and \( V \). We now derive the spatial form of the many-body wave functions \( \Psi \) lying within this subspace. In order to be a zero energy eigenstate of \( H_B \), \( \Psi \) must lie within the subspace generated by tensor products of lowest Landau level single particle states.

\[ \Psi[z] = P(z_1, \ldots, z_N) \prod_k e^{-|z_k|^2/4}, \]  

(3)

where \( P[z] \) is a polynomial in each of the atomic coordinates \( z_k = x_k + i y_k \). Let’s assume that \( \Psi[z] \) is also an eigenstate of \( V \) with eigenvalue zero, and let’s choose any pair of particles \( i \) and \( j \). The dependence of \( \Psi[z] \) on \( z_i \) and \( z_j \) can be reexpressed in terms of the relative and center mass coordinates, \( z_{ij}, \bar{z}_{ij} \), so that we can expand \( P[z] = \sum_m F_m \), where \( F_m \) depends on \( \bar{z}_{ij} \) and on the positions of all the other particles. As we are dealing with bosons only even values of \( m \) appear in the sum. In order for \( \Psi[z] \) to be annihilated by the hard-core interaction \( V \), \( F_0 \) must be identically zero. It follows that \( z_{ij}^2 \) is a factor of \( P[z] \) so that

\[ P[z] = Q[z] \prod_{i<j} (z_i - z_j)^2. \]  

(4)

We then diagonalize Hamiltonian \( H_L \) within the truncated Hilbert space of wave functions of the form specified by (3) and (4). We note that when \( P[z] \) is a homogeneous polynomial in \( z \) the state (3) is an eigenstate of \( H_L \) with eigenvalue \( E(M) = (1 - \Omega/\omega)M \), where the total angular momentum \( M \) equals the homogeneous degree of \( P[z] \). It follows that the ground state of the system is the state with the lowest angular momentum, that is, the one with \( Q[z] = 1 \):

\[ \psi[z] = \prod_{i<j} (z_i - z_j)^2 \prod_k e^{-|z_k|^2/4}. \]  

(5)

This state is the bosonic variant of the Laughlin wave function for quantum Hall electrons.

We have confirmed the above arguments by diagonalizing the Hamiltonian exactly for \( N = 5 \) bosons. Fig. 1 shows the energy spectrum. There is a branch of states well separated in energy from the rest of the spectrum. These states are polynomial states of the form given by (3) and (4), and the ground state is the five atoms Laughlin state with angular momentum \( M_0 = N(N - 1)/2 = 20 \).

To create a single fractional quasiparticles in the Laughlin state we insert a laser localized (within an area \( \sim L^2 \)) at some position \( z_0 \). We require \( |z_0| \) to be within the size of the Laughlin state \( ( \sim 2\sqrt{N - 1}) \). The presence of the laser can be described by a localized repulsive potential, so that the new Hamiltonian of the system can be approximated by:

\[ H_0 = H_L + V_0 \sum_i \delta(z_i - z_0). \]  

(6)

We have studied the time evolution of the ground state as the intensity \( V_0 \) of the laser increases with time, under the assumption that the system remains always in the truncated Hilbert space of polynomial wave functions of the form (3). We note that the total angular momentum \( L \) does not commute anymore with the Hamiltonian, since the \( \delta \)-potential breaks the rotational symmetry. We have solved the dynamics exactly for the model system of \( N = 5 \) bosons. Fig. 2 shows the time evolution of the ground state. For low intensities of the laser the angular momentum term dominates and the system remains in the Laughlin state. But when the laser power becomes sufficiently large the system evolves to a Laughlin quasi-hole state with wave function

\[ \psi_{z_0}[z] = \prod_i (z_i - z_0) \psi[z], \]  

(7)

where \( \psi[z] \) is the Laughlin wave function. Note that the state \( \psi_{z_0} \) is a superposition of homogeneous polynomial states with angular momenta running from \( M_0 \) to

FIG. 1: Eigenvalues of the Hamiltonian for \( N = 5 \) bosons as a function of the total angular momentum \( M \). For illustrating purposes we chose \( (1 - \Omega/\omega)/\eta = 0.001 \). Energy is measured in units of \( \eta \hbar \omega \).
ψ, and in the quasihole state $\Psi_{z_1}$. The crucial point is that at a certain point of the evolution the system reaches a superposition state $\Psi \sim (\psi_{z_0} + \psi_{z_0,1})$. This is precisely the superposition we need to test the statistical angle. We then stop the evolution of the system at this point, and remove instantaneously the laser located at $z_1$. In this way the state of the system remains unchanged but we go back to Hamiltonian $H_z$. Note that the state $\Psi$ is no longer an eigenstate of this Hamiltonian. We are now in a position to detect the statistical phase.

2) Second step: Statistical phase accumulation. We adiabatically move the remaining laser along a path enclosing position $z_1$, so that the time dependent Hamiltonian is: $H(t) = H_L + V_0 \sum_i \delta(z_i - z(t))$, with $z(t) = |z_0 - z_1 e^{i\beta t}$. As follows from the adiabatic theorem the evolved state at the end of the process will be:

$$\Psi' = e^{-iE_1T + \varphi_1} \psi_{z_0} + e^{-iE_2T + \varphi_2} \psi_{z_0,1},$$

(8)

where besides the dynamical phase each state picks up a Berry phase. The difference between the Berry phases $\varphi = \varphi_1 - \varphi_2$ of the two states reflects the extra phase that the quasihole at $z_0$ picks up due to the presence of the other quasihole at $z_1$, thus it is the statistical phase.

We have simulated this experiment with our 5 bosons model system. In order to have a more precise detection of the Berry phase we have considered a regime in which the dynamical relative phase 1 is much smaller than the relative Berry phase, so that $(E_1 - E_2)/\beta \sim 5(1 - \omega/\Omega) \ll 1$.

We have performed the simulation for different closed paths and different positions of the two quasiholes. The relative phase emerging from the numerical calculation is $\varphi = 1.031\pi$, so that $\Psi' \sim \psi_{z_0} - \psi_{z_0,1}$. Since the closed loop we have performed is equivalent to two consecutive interchanges, this result states that, as it should be, $\frac{1}{2}$–quasiholes pick up a phase $\varphi = -\pi$ when we interchange them. Thus, they do not behave as bosons nor fermions, but as anyons with statistics $\frac{1}{2}$.

3) Third step: Detecting the statistical phase. Suppose that we come back to the point at which we stopped the evolution of the system. This means that we instantaneously restore the laser we had removed at position $z_1$ and continue increasing its intensity. If we had not made the step 2 of the experiment, the system would be in the state $\Psi$ and would evolve to the two-quasihole state $\psi_{z_0,1}$. However, after performing step 2 the state $\Psi$ has changed into $\Psi'$ and the system will now evolve in a different way. Fig. 2 shows how the minus sign is reflected in the density profile of the final state. Instead of getting a two hole state we go back to the state with only one quasihole at $z_0$.

We discuss now the set of conditions that a system of $N$ atoms must fulfill to perform an experiment as the one we have described above. First of all, we have made a two-dimension approximation, so that we need $\ell \ll \ell_z$.

We have considered also a contact interaction between

$M_0 + N$. In this quasihole state all the particles are expelled from the position where the laser is located, so that the potential energy is minimized. This gain in potential energy compensates the cost of confining energy due to the spreading of the gas.

It is remarkable that for intermediate laser intensities the ground state is approximately equal to a superposition of the Laughlin state and the quasihole state: $\Psi \sim \alpha \psi + \alpha_0 \psi_{z_0}$, with no other states participating significantly in the evolution of the system. To understand this behavior let us consider a state quasi-orthogonal to this subspace, that is a state having a quasihole localized at some other position $z_i \neq z_0$. To reach this state the system has to pay confining energy (because of the system spreading around $z_i$), but there is no gain in potential energy since the hole has been created at the wrong position. It follows that any state out of the subspace generated by $\psi$ and $\psi_0$ will be always much higher in energy (no matter the intensity of the laser), and will not take part in the evolution of the system.

Based on the creation of quasiholes and superposition states we describe now a possible experiment to reveal directly the statistics of anyons in a Ramsey-type interferometer:

1) First step: Preparation of the initial state. We focus a laser at position $z_0$ and increase its intensity until a single quasihole is created. Keeping constant the intensity of this laser we then adiabatically insert another laser at position $z_1$, far enough from $z_0$ ($|z_1 - z_0| \geq \ell$). The new Hamiltonian is: $H_{z_0} = H_L + V_0 \sum_i \delta(z_i - z_0) + V_1 \sum_i \delta(z_i - z_1)$. Following an analogous pattern to the one shown in Fig. 1 for a single quasihole, the system now evolves from the one-quasihole state to a two-quasihole state of the form: $\psi_{z_0,1}[z] = \prod_i (z_i - z_0)(z_i - z_1) \psi[z]$. The crucial point is that at a certain point of the evolution the system reaches a superposition state $\Psi \sim (\psi_{z_0} + \psi_{z_0,1})$. This is precisely the superposition we need to test the statistical angle. We then stop the evolution of the system at this point, and remove instantaneously the laser located at $z_1$. In this way the state of the system remains unchanged but we go back to Hamiltonian $H_z$. Note that the state $\Psi$ is no longer an eigenstate of this Hamiltonian. We are now in a position to detect the statistical phase.

FIG. 2: Coefficients of the ground state for the system of $N = 5$ particles as a function of time, in the Laughlin state $\psi$ (filled line), and in the quasihole state $\psi_{z_0}$ (dashed line). The laser is localized at $z_0 = 2.5\ell$ (the size of the droplet is $\sim 4.2\ell$), and its intensity increases with time as $V_0 = 0.1t \ (\text{time in units of } \omega^{-1})$. The three plots at the bottom show the density profile of the ground state at three steps of the time evolution (the size of the plane is $6\ell \times 6\ell$).
Step 1: Adiabatically inserting a laser at position $z_1$ we drive the system from $\psi_{z_0}$ (plot a) into $\Psi = \psi_{z_0} + \psi_{z_0,z_1}$ (plot b). Step 2: We instantaneously remove the laser at $z_1$ and drive the laser at $z_0$ along a closed path enclosing $z_1$. The resulting state is $\Psi' \sim \psi_{z_0} - \psi_{z_0,z_1}$ (plot c). Step 3: We restore the laser at $z_1$ and continue increasing its intensity. The final state (plot d) is the single quasihole state $\psi_{z_0}$. In contrast, if the laser at $z_0$ makes two complete loops around $z_1$ (step 2'). In this case the resulting state is again $\Psi \sim \psi_{z_0} + \psi_{z_0,z_1}$ (plot c'), which evolves to the state with two holes (plot d').

FIG. 3: Numerical simulation of the Ramsey–like experiment for $N = 5$ bosons. 

We have created a state with two Laughlin quasihole states. This implies a total angular momentum $M = M_0 + 2N$ and thus an angular momentum energy per particle $e_L \lesssim 2(1 - \Omega/\omega)N$. Remember that we have projected onto the subspace of common zero eigenstates of $H_B$ and $V$. In order for this projection to be valid we then need $e_L$ to be much smaller than both the trap energy and the typical interaction energy, so that the conditions $2(1 - \Omega/\omega)N \ll 1$, $\eta/4\pi$ are required [1].

For creating the $1/2$-quasihole we had to focus the lasers within a distance $\sim \ell$. For a localization length $\ell \sim 1\mu m$ this implies an upper limit for the trap frequency of $\sim 1 kHz$. The preparation of the superposition state used to test the statistical angle requires to adiabatically increase the laser intensity. In order to estimate how slowly the laser needs to be inserted for a system of $N$ particles we have used a two state approximation. Confining ourselves to a basis formed by states $\psi_{z_0}$ and $\psi_{z_0,z_1}$, we find that the width of the avoided crossing is $\Delta \sim N(1 - \Omega/\omega)(|z_1|/\sqrt{N})^N$. This is normally a small quantity, quickly decreasing when the hole is made closer to the center of the trap. The time scale $\Delta^{-1}$ gives us the estimation on how slowly the laser is to be set on/off in order to reach the necessary adiabaticity requirements. Finally, the most restrictive condition is the temperature. In order to freeze out the excitations we need $kT/\omega \ll (1 - \Omega/\omega)$, which together with the above conditions implies $kT/\omega \ll 1/N, \eta/N$ [12].

In conclusion, this letter provides a way of creating anyons in rotating Bose Einstein condensates, and proposes an experiment in which the fractional statistics can be tested. While the requirements are very demanding, we expect that creation and (effectively) ground state cooling of small ensembles of bosonic atoms to be within experimental reach in coming years. Development of these experimental techniques promises the controlled engineering of strongly correlated, entangled states of atoms, with novel applications in quantum information.

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