The complete form of the constraints following from their conformal structure is extended so as to include constant mean curvature and other mean curvature foliations. This step is demonstrated using the momentum phase space approach. This approach yields equations of exactly the same form as the extended conformal thin sandwich approach. In solving the equations, it is never necessary actually to perform a tensor decomposition.

I. INTRODUCTION

The complete form of the constraints following from their conformal structure is extended so as to include constant mean curvature and other mean curvature foliations. This step is demonstrated using the momentum phase space approach. This approach yields equations of exactly the same form as the extended conformal thin sandwich approach. In solving the equations, it is never necessary actually to perform a tensor decomposition.

II. SOME GEOMETRY AND NOTATION

The spacetime metric $g_{\mu\nu}$ will be written in the “Cauchy-adapted” moving frame as

$$ds^2 = -N^2(dt)^2 + g_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

(1)

where the spatial scalar $N$ is the lapse function and $\beta^i$ is the (spatial) shift vector. In a natural (coordinate) basis, $\beta_i = g_{0i} (\beta_i = g_{ij}\beta^j$ and $g_{ij}, g^{kl}$ are taken as the 3x3 inverses of one another; they are riemannian). From this one can see that $\beta^i$ is a spatial vector and $\beta_i$ is a spatial one-form with respect to arbitrary spatial coordinate transformations provided these transformations are not time-dependent.

The spacetime cobasis is

$$\theta^0 = dt, \quad \theta^i = dx^i + \beta^i dt$$

(2)

and the dual vector basis is

$$e_0 \equiv \partial_0 = \partial/\partial t - \beta^i \partial/\partial x^i, \quad e_i \equiv \partial_i = \partial/\partial x^i.$$  

(3)

We see that $\partial_0$ is a Pfaffian derivative while $\partial_i$ and $\partial_i$ are natural derivatives. The basis vector $\partial_0$ can be generalized to the operator on spatial tensors

$$\hat{\partial}_0 = \partial_t - L_\beta$$

(4)

which, it should be noted, commutes with $\partial_i$ and propagates orthogonally to $t = \text{const.}$ slices. It is obvious that $\partial_0 = \partial/\partial x^t$ and $\partial_i = \partial/\partial t$ commute, because they are both natural derivatives. What is sometimes forgotten, but has been known for more than 50 years, is that (for example) the spatial Lie derivative $L_\beta$, for any vector field $\beta^i$, commutes with $\partial/\partial x^i$ when they act on tensors and more general objects such as spatial connections. This result holds in an even more general form. See, for example, Schouten’s *Ricci Calculus* [Sch], p. 105, Eq. (10.17). It is noteworthy that $\hat{\partial}_0$ acts orthogonally to $t = \text{constant}$ slices and that it is actually the only time derivative that ever occurs in the 3+1 formulation of general relativity based on an exact or locally exact (integrable) “time” basis one-form $\theta^0$, such as $dt$.

The connection coefficients in our “Cauchy-adapted” frame are given by

$$\omega^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} + g^{\alpha \delta} C^\gamma_{\delta \beta} + \frac{1}{2} C^\alpha_{\beta\gamma}$$

(5)

where $\Gamma$ denotes an ordinary Christoffel symbol, parentheses around indices denote the symmetric part (so $A_{(\beta\gamma)} = \frac{1}{2}(A_{\beta\gamma} + A_{\gamma\beta})$), and $C$ denotes the commutator

$$[e_\beta, e_\gamma] = C^\alpha_{\beta\gamma} e_\alpha.$$  

(6)
Our spacetime covariant derivative convention associated with \( g \) is
\[
D_\alpha V^\gamma = \partial_\alpha V^\gamma + \omega^\gamma_{\alpha\beta} V^\beta. \tag{7}
\]
The only non-vanishing \( C \)'s are
\[
C^i_{0j} = -C^i_{j0} = \partial_j \beta^i. \tag{8}
\]
For the spatial covariant derivative we write
\[
\nabla_i V^j = \partial_i V^j + \gamma^j_{ik} V^k = \partial_i V^j + \Gamma^j_{ik} V^k. \tag{9}
\]
Because the shift \( \beta^i \) is in the basis, the spacetime metric in the Cauchy basis that we use has no time - space components. A convenient consequence is that there is no ambiguity in writing \( g_{ij}, \ g^{ik}, \) or \( \Gamma^i_{jk}. \) For the spacetime metric determinant we will write \( \det g = -N^2g, \ g = \det g_{ij}; \) and \( g_{ij} \) is here and hereafter considered as a 3x3 symmetric tensor. Here, note that only \( \omega^i_{0j} \) and \( \omega^i_{j0} \) differ.

\[
\omega^i_{jk} = \Gamma^i_{jk}(g_{\mu\nu}) = \Gamma^i_{jk}(g_{mn}) = \gamma^i_{jk} \tag{10}
\]
\[
\omega^i_{aj} = -NK^i_{aj} + \partial_j \beta^a, \quad \omega^i_{j0} = -NK^i_{j0}, \quad \omega^0_{ij} = -N^{-1}K_{ij} \tag{11}
\]
\[
\omega^0_{00} = N\nabla^0 N, \quad \omega^0_{0i} = \omega^0_{0i} = \nabla_i \log N, \quad \omega^0_{00} = \partial_0 \log N \tag{12}
\]

The Riemann tensor satisfies the commutator
\[
(D_\alpha D_\beta - D_\beta D_\alpha)V^\gamma = (Riem)_{\alpha\beta}^{\gamma\delta} V^\delta \tag{13}
\]
where \( (Riem)_{\alpha\beta}^{\gamma\delta} \) would be denoted \( (Riem)^{\gamma\delta}_{\alpha\beta} \) in [MTW]. Again, we are using here the conventions of [CB-DeW].

There are a number of possible definitions for the second fundamental tensor or extrinsic curvature tensor \( K_{ij}. \) This does not measure curvature in the sense of Gauss or Riemann, where curvature has dimensions of \((length)^{-2}\). The extrinsic curvature is a measure, at a point on a spatial slice, of the curvature of a spacetime geodesic curve relative to a spatial geodesic curve to which it is tangent at the point. The dimensions are therefore \((length)^{-1}\). (See, for example, the Appendix of [PE-Y] for a detailed discussion of extrinsic curvature tensors.) This is the same dimension as a connection symbol, and, in fact,
\[
K_{ij} = -N \omega^0_{ij} \tag{14}
\]
Also
\[
\hat{\partial}_0 g_{ij} = -2NK_{ij} = \partial_0 g_{ij} - (\nabla_i \beta_j + \nabla_j \beta_i) \tag{15}
\]
where \( \nabla_i \) is the spatial covariant derivative with connection \( \gamma^i_{jk} = \omega^i_{jk} = \Gamma^i_{jk}; \) and the final term is, apart from sign, \( \partial_0 \ nabla_i. \)

The Riemann tensor components are in accord with the Ricci identity \( [13] \)
\[
(Riem)_{\alpha\beta}^{\gamma\delta} = \partial_\alpha \omega^\gamma_{\beta\delta} - \partial_\beta \omega^\gamma_{\alpha\delta} + \omega^\gamma_{\alpha\epsilon} \omega^\epsilon_{\beta\delta} - \omega^\gamma_{\beta\epsilon} \omega^\epsilon_{\alpha\delta} - C^\gamma_{\alpha\beta} \omega^\epsilon_{\delta}. \tag{16}
\]
The spatial Riemann tensor is denoted \( R_{ik}^{\ k} \). These curvatures are related by the Gauss - Codazzi - Mainardi equations for codimension one (see, for example, \[Sel\])
\[
(Riem)_{ijkl} = R_{ijkl} + (K_{ik} K_{jl} - K_{il} K_{jk}) \tag{17}
\]
\[
(Riem)_{ijk0} = N(\nabla_j K_{ki} - \nabla_k K_{ji}) \tag{18}
\]
\[
(Riem)_{i0j0} = N(\partial_0 K_{ij} + NK_{ik} K_{kj} + \nabla_i \partial_j N) \tag{19}
\]
One can likewise form and decompose the Ricci tensor, which has the definition
\[
(Ric)_{\beta\delta} = (Riem)_{\alpha\beta}^{\alpha\delta} \tag{20}
\]
Then we can construct

\[(Ric)_{ij} = R_{ij} - N^{-1} \partial_i K_{ij} + K K_{ij} - 2 K_{ik} K^k_{\ j} - N^{-1} \nabla_i \partial_j N \quad (21)\]

\[(Ric)_{0j} = N (\partial_j K - \nabla_i K^i_{\ j}) \equiv N \nabla_j (\delta^i_{\ j} K - K^i_{\ j}) \quad (22)\]

\[(Ric)_{00} = N (\partial_0 K - N K_{ij} K^{ij} + \Delta N) \quad (23)\]

where $\Delta N$ denotes the spatial “rough” or scalar Laplacian acting on the lapse function: $\Delta N \equiv g^{ij} \nabla_i \nabla_j N \equiv \nabla^2 N$. The trace of $K_{ij}$ is $K$, the mean curvature.

It is important to know the spacetime scalar curvature, which we call $(C)$:

\[ (C) = g^{\alpha\beta} (Ric)_{\alpha\beta} = g^{\alpha\beta} R_{\lambda\alpha}^\lambda \beta \quad (24)\]

in the form

\[ N \sqrt{g}(C) = N \sqrt{g}(R + K_{ij} K^{ij} - K^2) - 2 \partial_i (\sqrt{g} K) + 2 \partial_i [\sqrt{g}(K^i - \nabla^i N)] \quad (25)\]

where $R$ is the spatial scalar curvature, because the spacetime scalar curvature density is the lagrangian density of the famous Hilbert action principle [Hil], explicitly modified in [Yo1] to conform to the ADM action principle [ADM], though different tensors are to be varied in the somewhat different perspectives in the different action principles. The scalar curvature itself will be needed later. It is found from (24), (21), (22), and (23) to be

\[ (C) = 2 N^{-1} \partial_0 K - 2 N^{-1} \Delta N + (R + K_{ij} K^{ij} - K^2) \quad (26)\]

**III. EINSTEIN’S EQUATIONS**

Einstein used his insights about the principle of equivalence and his principle of general covariance (spacetime coordinate freedom plus a pseudo-riemannian metric not given a priori) in arriving at the final form of his field equations. As is now well known, his learning tensor analysis from Marcel Grossmann was an essential enabling step. The equations, using the Einstein tensor

\[ (Ein)_{\mu\nu} \equiv G_{\mu\nu} \equiv (Ric)_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (C), \quad (27)\]

are

\[ G_{\mu\nu} = \kappa T_{\mu\nu} \quad (28)\]

where $\kappa = 8 \pi G$, $c = 1$, $G = Newton’s$ constant, and the stress-energy-momentum tensor of fields other than gravity (the “source” tensor) $T_{\mu\nu}$ must satisfy, as Einstein reasoned, in analogy to the conservation laws of special relativity,

\[ \nabla_\mu T^{\mu\nu} = 0 \quad (29)\]

corresponding to

\[ \nabla_\mu G^{\mu\nu} \equiv 0 \quad (30)\]

which is an identity, the “third Bianchi identity” or the “(twice) contracted Bianchi identity.” For purposes of the discussion below, besides $c = 1$ we also take Dirac’s form of Planck’s constant to be one: $\hbar = 1$. This means $G$ has the dimension (length)$^2$; thus also does $\kappa$. (Mass is now inverse length.)

Here we will consider only the vacuum theory. This is non-trivial because the equations are non-linear (gravity acts as a source of itself) and because the global topology and (or) boundary conditions are not prescribed by the equations. We regard that the object of solving the equations is to find the metric. In the 3+1 form of the equations, which is very close to a hamiltonian framework, the object is to obtain $g_{ij}$ and $K_{ij}$, along with a workable specification of $\alpha = N g^{-1/2}$ and $\beta^i$ which, as we shall see shortly, are not determined by Einstein’s equations. For the vacuum case, one can use in four spacetime dimensions either of these two equations

\[ G_{\mu\nu} = 0 \quad (31)\]

or

\[ (Ric)_{\mu\nu} = 0. \quad (32)\]
For completeness, we note that the form of equation (32) with “sources” using the Ricci tensor is

$$(\text{Ric})_{\mu\nu} = \kappa(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}) \equiv \kappa\rho_{\mu\nu}$$

(33)

where $T = T^\alpha_\alpha$, the trace of the stress-energy tensor.

We now remark on a couple of points that are sometimes useful to be ar in mind. The curvature equations we have given purely geometric. They can be converted into physical gravity equations explicitly by assigning the physical dimensions length ($L$), mass ($M$), and time ($T$). We have already chosen $c = 1$, so $T = L$. We shall work in terms of $L$. Next we choose to make action dimensionless by setting $\hbar = 1$, which yields $M = L^{-1}$. Then $G$ (and $\kappa$) have dimension $L^2$. This is a handy viewpoint for the physicist and mathematician even if quantum effects are not considered. It enables us to display rather easily the nonlinear self-coupling that arises even in vacuum from the particular geometric nature of General Relativity.

We take the metric to be dimensionless while $t$ and $x^i$ have dimension $L$. (Think of natural locally Riemannian normal coordinates to make this view palatable.) The canonical form of the action based on $(2\kappa)^{-1}g^{1/2}N(C)$ is given by [Yo1] [ADM] [An-Yo]. It yields in place of $K_{ij}$ the closely related field canonical momentum [ADM]

$$\pi^{ij} = (2\kappa)^{-1}g^{1/2}(K_{grs} - K_{rs})(g^{ij}g^{kl})$$

(34)

Inverting this expression in three dimensions yields

$$K_{ij} = (2\kappa)\left[2^{-1/2}g^{1/2} - \frac{1}{2}g_{ij}\right]$$

(35)

where $\pi = \pi^k_k$ is the trace. We note that from (35) we can obtain for the mean curvature

$$K = (2\kappa)^{-1}2^{-1/2} = \frac{1}{2}(2\kappa)g^{1/2}\pi$$

(36)

$$g^{1/2}(2\kappa) = \pi$$

(37)

the well-known, and vital, integrand of the boundary term of the boundary term of the Hilbert action [Hil] that can be seen [Yo1] to convert it to the canonical action [ADM] [An-Yo].

Denoting the terms in the rectangular brackets in (35) by $\mu_{ij}$, which has dimension $L$, then

$$K_{ij} = (2\kappa)\mu_{ij}$$

(38)

Now we could rewrite the curvature equations with the gravitational interaction explicit. For example, the Gauss - Codazzi - Mainardi equations become

$$\text{(Riem)}_{ijkl} = R_{ijkl} + 4\kappa^2(\mu_{ik}\mu_{jl} - \mu_{il}\mu_{jk})$$

(39)

$$\text{(Riem)}_{0ijk} = 2\kappa N(\nabla_j\mu_{ki} - \nabla_k\mu_{ji})$$

(40)

$$\text{(Riem)}_{0ij0} = 2\kappa N\partial_0\mu_{ij} + 4N\kappa^2\mu_{ik}\mu_{j}^k + N\nabla_i\nabla_j N$$

(41)

IV. THE 3+1-FORM OF EINSTEIN’S EQUATIONS

It is helpful to write out the ten vacuum equations using both (Ric) and (Ein):

$$\text{(Ric)}_{ij} = 0, \quad 2N(\text{Ric})^i_i = 0, \quad 2G^0_0 = 0$$

(42)

This form was noted by Lichnerowicz [Lich] in the case of zero shift as being revealing. First, recall the geometric identity [15]

$$\partial_0 g_{ij} = -2NK_{ij}$$

or

$$\partial_t g_{ij} = -2NK_{ij} + \mathcal{L}_\beta g_{ij} = -2NK_{ij} + (\nabla_i\partial_j + \nabla_j\partial_i)$$

(43)
From the first equation in (12), one can obtain

\[
\dot{\partial}_0 K_{ij} = -\nabla_i \partial_j N + N(R_{ij} - 2K_{il}K^l_j + KK_{ij}) \\
\equiv \partial_t K_{ij} - L_\beta K_{ij} \\
\equiv \partial_t K_{ij} - (\beta^i \nabla_i K_{ij} + K_{il}\nabla_j \beta^l + K_{ij}\nabla_i \beta^j).
\] (44)

The second and third equations in (12) contain no terms \(\partial_t K_{ij}\) (i.e., no “accelerations” \(\partial_0 g_{ij}\)) and are, therefore, constraints on the initial values of \(g_{ij}\) and \(K_{ij}\). As previously mentioned, in this “canonical”-like 3+1 form, there are no time derivatives of \(N = \alpha g^{1/2}\) or of \(\beta^i\). In a second-order formalism, \(\partial_t N\) and \(\partial_t \beta^i\) would appear, as we see from (13). To make second order wave operators on all components of the spacetime metric, the (original) harmonic coordinate conditions \((-g)^{-1/2}\partial_{\mu}((-g)^{1/2}g^{\mu \nu}) = 0\) in natural coordinates were introduced and shown, along with the constraints, to be conserved by the resulting “reduced” equations if the constraints were assumed to hold at the “initial” time \([CB1]\). But no powers of \(\dot{N}\) and \(\dot{\beta}^i\) appear in (13), the lagrangian of Hilbert’s action principle for the Einstein’s equations. We have thus an easy way of seeing that \(\dot{N}\) and \(\dot{\beta}^i\) are dynamically irrelevant. We find from the second and third equations of (12), respectively,

\[
2N R^0_i = C_i = 2\nabla_j(K^j_i - K_{ij}^j),
\]

\[
2C^0_i = C = K_{ij} K^{ij} - K^2 - R.
\] (45) (46)

These equations were derived in detail and displayed in \([Yo2]\), without a 3+1 splitting of the basis frames and coframes. An arbitrary spacetime basis was used there in order to remove what some people regarded as the “taint” of using particular coordinates. They are the standard 3+1 equations, wrongly called the standard ADM equations. In regard to evolving \(g_{ij}\) and \(K_{ij}\), I do not claim that (45) and (46) are preferred for any other reason other than their maximum simplicity and absolute correctness when written in explicitly canonical form, using the variable \(\pi^\mu\), defined below, in place of \(K_{ij}\). I say nothing here about the numerical properties of (45) and (46).

The canonical equations derived by Arnowitt, Deser, and Misner \([ADM]\) and by Dirac \([Dir1, Dir2]\), are not equivalent to (45) given above, even when written in the same formalism, that is, with \(g_{ij}\) and \(K_{ij}\). This is because their equation of motion is \(G_{ij} = 0\) rather than \(R_{ij} = 0\). Although \(G_{\mu \nu} = 0\) and \(R_{\mu \nu} = 0\) are equivalent, this is not true of spatial components. Instead, one has the key identity \([An-Yo]\)

\[
G_{ij} + g_{ij} G^0_\theta = (\text{Ric})_{ij} - g_{ij} g^{kl}(\text{Ric})_{kl},
\] (47)

or

\[
G_{ij} + \frac{1}{2} g_{ij} C = (\text{Ric})_{ij} - g_{ij} g^{kl}(\text{Ric})_{kl}.
\] (48)

Therefore, \(G_{ij} = 0\) is not the correct equation of motion unless the constraint \(C = 0\) also holds.

For the interested reader, I remark that this means the hamiltonian vector field of the ADM and Dirac canonical formalisms is not well-defined throughout the phase space. There is an easy cure in the ADM approach, which is based on a canonical action principle. When the metric \(g_{ij}\) is varied, one must hold fixed the “weighted” or “densitized” lapse function \(\alpha = g^{-1/2} N\), instead of just the scalar lapse \(N\). Thus, one carries out independent variations of \(g_{ij}\), \(\alpha\), \(\beta^i\), and \(\pi^{ij} = (2k)^{-1} g^{1/2}(kg^{ij} - K^{ij})\). \([Ash]\) \([An-Yo]\).

V. CONFORMAL TRANSFORMATIONS

A very useful technique for transforming the constraints \(C_i\) and \(C\) (15) and (16) into a well posed problem involving elliptic partial differential equations is to use conformal transformations of the essential spatial objects \(g_{ij}\), \(K_{ij}\), \(N\), and \(\beta^i\). Along the way we will again encounter the densitized lapse

\[
\alpha = g^{-1/2} N
\] (49)

which has, as one sees, weight (-1): \(N\) is a scalar with respect to spatial time-independent coordinate transformations (weight zero by definition).

The conformal factor will be denoted by \(\varphi\). It will be assumed that \(\varphi > 0\) throughout. The conformal transformation is defined by its action on the metric

\[
\tilde{g}_{ij} = \varphi^4 g_{ij}.
\] (50)
It is called “conformal” because it preserves angles between vectors intersecting at a given point, whether one constructs the scalar product and vector magnitudes with $g_{ij}$ or $\bar{g}_{ij}$. The power “4” in (50) is convenient for three dimensions. For dimension $n \geq 3$, the “convenient power” is $4(n-2)^{-1}$ for the metric conformal deformation. The neatness of this choice comes out most clearly in the relation of the scalar curvatures $\bar{R} = R(\bar{g})$ and $R = R(g)$ below. From (50) and the fact that the spatial connection is simply the “Christoffel symbol of the second kind $\{^i_{jk}\}$ ” which we denote by $\Gamma^i_{jk}$,

$$\Gamma^i_{jk} \equiv \left\{^i_{jk}\right\} = \frac{1}{2} g^{il} (\partial_j g_{lk} + \partial_k g_{lj} - \partial_l g_{jk})$$

we find that

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \frac{1}{2} \varphi^{-1} (\delta^i_j \partial_k \varphi + \delta^i_k \partial_j \varphi - g^{il} g_{jk} \partial_l \varphi).$$

From $\bar{\Gamma}^i_{jk}$ we can find the relationship between $\bar{R}_{ij}$ and $R_{ij}$.

$$\bar{R}_{ij} = R_{ij} - 2 \varphi^{-1} \nabla_i \partial_j \varphi + 6 \varphi^{-2} (\partial_i \varphi)(\partial_j \varphi) - g_{ij} [2 \varphi^{-1} \Delta \varphi + 2 \varphi^{-2} (\nabla^k \varphi)(\partial_k \varphi)]$$

There is no need to derive the transformation of the Riemann tensor, for in three dimensions $R_{ijkl}$ can be expressed in terms of $g_{ij}$ and $R_{ij}$. One can see that this must be so, for both the Ricci and Riemann tensors have six algebraically independent components. The formula relating them has long been known. It is displayed for example in [Yo4]. This formula can be obtained from the identical vanishing of the Weyl conformal curvature tensor in three dimensions.

For $n \geq 3$, dimensions, For dimension $n \geq 3$, the “convenient power” is $4(n-2)^{-1}$ for the metric conformal deformation. The neatness of this choice comes out most clearly in the relation of the scalar curvatures $\bar{R} = R(\bar{g})$ and $R = R(g)$ below. From (50) and the fact that the spatial connection is simply the “Christoffel symbol of the second kind $\{^i_{jk}\}$ ” which we denote by $\Gamma^i_{jk}$,

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we find that

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \frac{1}{2} \varphi^{-1} (\delta^i_j \partial_k \varphi + \delta^i_k \partial_j \varphi - g^{il} g_{jk} \partial_l \varphi).$$

From $\bar{\Gamma}^i_{jk}$ we can find the relationship between $\bar{R}_{ij}$ and $R_{ij}$.

$$\bar{R}_{ij} = R_{ij} - 2 \varphi^{-1} \nabla_i \partial_j \varphi + 6 \varphi^{-2} (\partial_i \varphi)(\partial_j \varphi) - g_{ij} [2 \varphi^{-1} \Delta \varphi + 2 \varphi^{-2} (\nabla^k \varphi)(\partial_k \varphi)]$$

This formula can be obtained from the identical vanishing of the Weyl conformal curvature tensor in three dimensions. But riemannian three-spaces are $\textit{not}$ conformally flat, in general. The Weyl tensor in three dimensions is replaced by the Cotton tensor $C_{ijk}$, which is conformally invariant and vanishes if the three-space is conformally flat

$$C_{ijk} = \nabla_j L_{ik} - \nabla_k L_{ij}$$

where

$$L_{ik} = R_{ik} - \frac{1}{4} R g_{ik}.$$

Its dual $\bar{C}^{ijkl}$, found by using the inverse volume form $e^{mjk}$ on the skew pair $[jk]$ and by raising the index $i$ to $l$, in three dimensions, is a symmetric tensor $\ast C^{lm}$ with trace identically zero and covariant divergence identically zero. The Cotton tensor has third derivatives of the metric and is therefore not a curvature tensor, but rather is a differential curvature tensor with dimensions (length)$^{-3}$. Therefore, the dual Cotton tensor is

$$\ast C^{lm} = g^{li} e^{mjk} C_{ijk}.$$ (56)

Under conformal transformations, we have that $\bar{g}^{ij} = \varphi^{-4} g^{ij}$, $\bar{e}_{ij} = \varphi^6 e_{ijk}$ (the volume three-form), $\bar{C}_{ijk} = C_{ijk}$, and $\bar{e}^{ijk} = \varphi^{-6} e^{ijk}$. Therefore,

$$\ast \bar{C}^{ij} = \varphi^{-10} (\ast C^{ij}).$$ (57)

The properties of $\ast C^{ij}$ hold for an entire conformal equivalence class, that is, for all sets of conformally related riemannian metrics as in (50) for all $0 < \varphi < \infty$. The divergence of $\ast C^{ij}$ is $\textit{identically}$ zero, so it need not be surprising that

$$\bar{\nabla}_j (\ast \bar{C}^{ij}) = \varphi^{-10} \nabla_j (\ast C^{ij})$$ (58)

using the barred and unbarred connections to form the covariant divergence operators $\bar{\nabla}_i$ and $\nabla_j$. We see that (57) is the natural conformal transformation law for symmetric, traceless type $(1,0)$ tensors in three dimensions, whose divergence may or may not vanish. Note that in obtaining (58), the scaling (57) was used. But $\textit{no}$ properties of $\ast C^{ij}$ were employed except the symmetry type of the tensor representation: symmetric with zero trace.

We now pass to the famous formula for the conformal transformation of the scalar curvature $R$. It was introduced in connection with with an early treatment of the constraints in [Lich]. From (53) it follows that

$$\bar{R} = \varphi^{-4} R - 8 \varphi^{-5} \Delta \varphi.$$ (59)

So far every transformation has followed from the defining relation $\bar{g}_{ij} = \varphi^4 g_{ij}$. A glance at both the constraints (44) and (49) shows that we must deal with $K_{ij}$. The method here can be deduced by writing $K_{ij}$ as

$$K^{ij} = A^{ij} + \frac{1}{3} K g^{ij}$$ (60)
where \( A^{ij} \) is the traceless part of \( K^{ij} \). We treat \( A^{ij} \) and \( K \) differently because \( A^{ij} \) and \( Kg^{ij} \) can be regarded as different irreducible types of symmetric two-index tensors. They also have different conformal transformations. Lichnerowicz took \( K = 0 \) \cite{Lich}. But this is too restrictive, and even then the simplified “momentum constraint” \cite{Yo3} was not solved. Mme. Choquet-Bruhat first argued that one has to solve the momentum constraint with a second-order operator on a vector potential \cite{CB2}. A very useful result was given in \cite{Yo3} and it was used for many years to solve the momentum constraint. Its imperfections were noted first by O’Murchadha \cite{O'M}, Isenberg \cite{Is}, and the author. The inference was that the early method was only an ansatz. A better method yet, with no ambiguities, was displayed in \cite{Yo2} and \cite{Pf-Yo}. It is given below.

Suppose an overbar denotes a solution of the constraints and the corresponding object without an overbar denotes a “trial function.” The strategy is to “deform” the trial objects conformally into barred quantities, that is, into solutions. Every object we deal with has, in effect, a “conformal dimension,” which is not given by its physical dimension or by its tensorial character, that is, how it transforms under a change of the basis or of the natural coordinates.

### VI. AN ELLIPTIC SYSTEM

We write (45) and (46) in the barred variables as

\[
\bar{\nabla}_j \bar{A}^{ij} - \frac{2}{3} \bar{g}^{ij} \partial_j \bar{K} = 0 \tag{61}
\]

\[
\bar{A}^{ij} \bar{A}^{ij} - \frac{2}{3} \bar{K} - \bar{R} = 0 \tag{62}
\]

Conformal transformations for objects that are purely concomitants of \( \bar{g}^{ij} \) (or \( g^{ij} \)) are derived as above in a straightforward manner. But the extrinsic curvature variables have to be handled with a modicum of care. The transformations obtained by extending \( \bar{g}_{ij} = \varphi^4 g_{ij} \) to all of the spacetime metric variables is not appropriate because the view that spacetime structure is primary is not helpful in a situation, as here, where there is as yet no spacetime.

We begin with \( \bar{K} \). We hold it fixed because its inverse in the simpler cosmological models is the “Hubble time,” without a knowledge of which the epoch is not known. Data astronomers obtain from different directions in the sky, or at different “depths” back in time are basically correlated and they fix \( \bar{K} \) implicitly. Therefore, I long ago adopted the rule \cite{Yo1} of fixing the “mean curvature” \( \bar{K} \)

\[
\bar{K} = K \tag{63}
\]

under conformal transformations. Thus it is specified \textit{a priori}.

What to do about the the symmetric tracefree tensor \( A^{ij} \)? The prior discussion of *\( C^{ij} \) indicates the transformation

\[
\bar{A}^{ij} = \varphi^{-10} A^{ij} \tag{64}
\]

But symmetric tensors "\( \bar{T}^{ij} \)" have, in a curved space, three irreducible types that are formally \( L^2 \)-orthogonal. One is the trace \((\bar{g}^{ij} \bar{T}^i_k)\), another is like *\( C^{ij} \), that is, a part with vanishing covariant divergence. Finally, a symmetric tracefree tensor can be constructed from a vector

\[
(\bar{L}X)^{ij} = (\nabla^i X^j + \nabla^j X^i - \frac{2}{3} \bar{g}^{ij} \nabla_l X^l), \tag{65}
\]

the “conformal Killing form” of \( X^i \). (I have not found other constructions that are sufficiently well-behaved under conformal transformations to be useful in this problem.) This expression \cite{Yo1} vanishes iff \( X^i \) is a conformal killing vector of \( \bar{g}_{ij} \). Then, \( X^i \) would be a conformal killing vector of every metric conformal to \( \bar{g}_{ij} \). Therefore,

\[
\bar{X}^i = X^i, \quad \bar{g}_{ij} = \varphi^4 g_{ij} \tag{66}
\]

and

\[
(\bar{L}X)^{ij} = \varphi^{-4} (LX)^{ij}, \tag{67}
\]

which misses obeying our “rule” \cite{Yo1} or \cite{Yo2}. For a long time, the mismatched powers required a work-around to obtain an ansatz for solving the constraints \cite{Yo4}, but I arrived at a simple solution fairly recently (2001) \cite{Yo5, Pf-Yo}. The vectorial part \cite{Yo3} needs a weight factor and a corresponding change in the measure of orthogonality. The solution seems to me simple, beautiful, and absolutely correct in the present context.
Recall our statement that the densitized lapse $\alpha$ is the preferred undetermined multiplier (rather than the lapse $N$) in the action principle leading to 3+1 (or canonical) equations of motion. See An-Yd where this is made perfectly clear. This is not to say anything about the “best” form of the $\partial_t K_{ij}$ (or $\partial_t \pi^i$) equation of motion for calculational purposes. In fact, the system for $\partial_t g_{ij}$ and $\partial_t K_{ij}$ is only “weakly hyperbolic.” But I do say that only this form gives a Hamiltonian vector field well-defined in the entire momentum phase space.

To proceed, we note that one does not scale undetermined multipliers. Therefore
\[ \tilde{\beta}^i = \beta^i, \quad \tilde{\alpha} = \alpha. \] (68)

But because $\tilde{\alpha} = \tilde{g}^{-1/2} \tilde{N}$ and $\tilde{g}^{1/2} = \varphi^6 g^{1/2}$, then $\tilde{N} = \tilde{g}^{1/2} \tilde{\alpha} = g^{1/2} \alpha$ implies
\[ \tilde{N} = \varphi^6 N. \] (69)

I have known that the transformation (69) was useful since 1971. But, thinking that $\tilde{g}$ was correct all along. It made its first appearance in the conformal thin sandwich problem Yo2. Then I learned about the densitized lapse and saw its role in the action principle. It then dawned on me that (69) was correct all along. It made its first appearance in the conformal thin sandwich problem Yo2.

The lapse becomes, thus, a dynamical variable Ash, An-Yo, Pf-Yo. A look at (69) and at the relation between $\partial_t \tilde{g}_{ij}$ and $\tilde{K}_{ij}$ gives us the scalar weight factor $(-2N)^{-1}$ in the decomposition of $\tilde{A}^{ij}$
\[ \tilde{A}^{ij} = A^{ij}_{(\delta)} + (-2N)^{-1}(LX)^{ij}. \] (70)

The subscript $(\delta)$ indicates that the covariant divergence of $A^{ij}_{(\delta)}$ vanishes. Note that (70) does not mean that the extrinsic curvature is sensitive to $N$. It is not. What it does mean is that the identification of the divergence-free and trace-free part of the extrinsic curvature is, in part, dependent on $N$. Also note that the two parts of $\tilde{A}^{ij}$ are formally $L^2$-orthogonal both before and after a conformal transformation, with the geometrical spacetime measure
\[ \sqrt{-g} = Ng^{1/2} \] (71)

instead of the spatial measure $g^{1/2}$. Therefore, we have
\[
\int \tilde{A}^{ij}_{(\delta)}(-2\tilde{N})^{-1}(LX)^{kl}[\tilde{g}_{ik}\tilde{g}_{jl}](N\tilde{g}^{1/2})d^3x = \int A^{ij}_{(\delta)}(-2N)^{-1}(LX)^{kl}[g_{ik}g_{jl}](Ng^{1/2})d^3x.
\] (72)

Upon integration by parts, with suitable boundary conditions, or no boundary, each of the integrals (72) vanishes.

We construct $A^{ij}_{(\delta)}$ or $\tilde{A}^{ij}_{(\delta)}$ by extracting from a freely given symmetric tracefree tensor $\tilde{F}^{ij} = \varphi^{-10} F^{ij}$ its transverse-tracefree part, which will be our $A^{ij}_{(\delta)}$
\[ F^{ij} = A^{ij}_{(\delta)} + (-2N)^{-1}(LY)^{ij} \] (73)

with
\[ \nabla_j [(-2N)^{-1}(LY)^{ij}] = \nabla_j F^{ij} \] (74)

The momentum constraints
\[ \nabla_j \tilde{A}^{ij} - \frac{2}{3} \tilde{g}^{ij} \partial_j K = 0 \] (75)

become, with $Z^i = X^i - Y^i$,
\[ \nabla_j [(-2N)^{-1}(LZ)^{ij}] = \nabla_j F^{ij} - \frac{2}{3} \varphi^6 \tilde{g}^{ij} \partial_j K. \] (76)

The solution for $Z^i$, with given $N$, will give the parts of $\tilde{K}_{ij}$,
\[ \tilde{A}^{ij} = \varphi^{-10}[F^{ij} + (-2N)^{-1}(LZ)^{ij}], \] (77)

\[ \frac{1}{3} \tilde{K} \tilde{g}^{ij} = \frac{1}{3} \varphi^{-4} Kg^{ij}. \] (78)
However, \([Pf1]\) contains \(\varphi\) and is coupled to the “hamiltonian constraint” \([E2]\) unless the “constant mean curvature” (CMC) condition \(K = \text{constant}\) (in space, \(\partial_t K = 0\)) can be employed, as introduced in \([Yo1]\). This includes maximal slicing \(K = 0\) \([Lich]\). (Lichnerowicz did not propose the CMC condition as claimed in \([Tip-Mars]\).)

Gathering the transformations for \(\bar{R}\) and \(\bar{K}^{ij}\) enables us to write the hamiltonian constraint as the general relativity version of the Laplace-Poisson equation

\[
\Delta \varphi - \frac{1}{8} [R \varphi + (F_{ij} + (-2N)^{-1}(LZ)_{ij})^2 \varphi^{-7} - \frac{2}{3} K^2 \varphi^5] = 0.
\]

Suppose, for example, we choose \(N = 1\). Then

\[
\bar{N} = \varphi^6 = \tilde{g}^{1/2}(g^{-1/2}).
\]

We are certainly entitled to have chosen \(g_{ij}\) such that \(g^{1/2} = 1\), without loss of generality. Thus we recover Teitelboim’s gauge for the lapse equation

\[
\bar{N} = \tilde{g}^{1/2}
\]

in his noted paper on the canonical path integral in general relativity \([Teit]\).

A bit more generally, if we choose

\[
\bar{\partial}_t \tilde{g}^{1/2} = 0,
\]

we see that \(\bar{N}\) automatically satisfies the time gauge equation used by Choquet-Bruhat and Ruggeri \([CB-Ru]\).

The constraints have the same form as they do in the thin sandwich formulation: see \([Yo2]\). Therefore, the space of solutions has the properties obtained in \([CB-Isen-Yo]\).

If we write out from the formula for \(\bar{\partial}_t \tilde{g}_{ij}\) its tracefree part \(\bar{u}_{ij}\), or the velocity of the conformal metric, we obtain

\[
\bar{u}_{ij} = -2\bar{N} \bar{F}_{ij} + [\bar{L}(\bar{Z} + \bar{\beta})]_{ij}
\]

with \((\bar{Z} + \bar{\beta})_{ij} = \bar{g}_{ij}(Z^i + \beta^i)\). This has the form of the solution in the conformal thin sandwich problem. The choice of shift \(\beta^i = -Z^i\) is possible here and renders a simple final form.

No splitting of tensors need be carried out. The quantities \(\tilde{\alpha}\) and \(\bar{\beta}^i\) can be freely specified. \(\bar{K}_{ij}\) determines the geometry not of \(t + \delta t\), but of a slice a fixed orthogonal proper time from \(t\). One adjusts \(\bar{N}\) and \(\bar{\beta}^i\) to obtain the metric of \(t + \delta t\).

### VII. Extension of the Initial Value Conditions

In either the conformal thin sandwich approach or the extrinsic curvature approach, a very useful extension can be made \([Pf-Yo]\). We can answer the question: how is the “trial” lapse \(\bar{N}\) to be chosen such that the physical lapse \(\tilde{N}\) produces a desirable foliation? (At the same time, this \(\tilde{N}\) will produce \(\bar{A}_{(\delta)}^{ij}\).)

This question has at least one useful answer. One has specified \(K\). We then construct a mean curvature slicing by specifying \(\partial_t K\). The physical equation is in vacuum

\[
\partial_t K - \beta^i \partial_i K = \bar{N}(\bar{R} + K^2) - \bar{\Delta} \bar{N}
\]

which has a simple extension to matter-filled spacetimes. Using the physical hamiltonian constraint in \([S4]\) gives

\[
\bar{\Delta} \bar{N} - \bar{N} (\bar{A}_{ij} \bar{A}^{ij} + \frac{1}{3} K^2) = -\partial_t K
\]

The laplacian has the conformal transformation

\[
\bar{\Delta} \bar{N} = \varphi^{-4} \Delta \bar{N} + 2 \varphi^{-4} (\nabla_i \bar{N})(\nabla^i \log \varphi).
\]

Combining the previous conformal transformations with \([S5]\) and \([S6]\) yields the equation for \(\Delta \bar{N}\) given in \([Pf-Yo]\) or the equivalent form \([Pf1]\)

\[
\Delta (N \varphi^7) - (N \varphi^7) \left[\frac{1}{8} R + \frac{5}{12} K^2 \varphi^4 + \frac{7}{8} A_{ij} A^{ij} \varphi^{-8}\right] = - (\partial_t K - \beta^i \partial_i K).
\]

We have now five coupled conformally covariant initial data conditions. While \([S7]\) looks formidable, and no complete mathematical results are presently available, the system of \(free\) equations has yielded a unique solution by numerical methods in many and all cases \([Pf2]\). There is a theorem waiting to be found.
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