We introduce proper display calculi for intuitionistic, bi-intuitionistic and classical linear logics with exponentials, which are sound, complete, conservative, and enjoy cut elimination and subformula property. Based on the same design, we introduce a variant of Lambek calculus with exponentials, aimed at capturing the controlled application of exchange and associativity. Properness (i.e., closure under uniform substitution of all parametric parts in rules) is the main technical novelty of the present proposal, allowing both for the smoothest proof of cut elimination and for the development of an overarching and modular treatment for a vast class of axiomatic extensions and expansions of intuitionistic, bi-intuitionistic, and classical linear logics with exponentials. Our proposal builds on an algebraic and order-theoretic analysis of linear logic and applies the guidelines of the multi-type methodology in the design of display calculi.

ACM Reference format:
Giuseppe Greco and Alessandra Palmigiano. 2023. Linear Logic Properly Displayed. ACM Trans. Comput. Logic 24, 2, Article 13 (January 2023), 56 pages.
https://doi.org/10.1145/3570919
perspective, this fact translates into the well-known stipulation that unrestricted (left- and right-) weakening and contraction rules cannot be included in Gentzen-style presentations of linear logic.

However, resources might exist that are available \textit{unlimitedly}; that is, one or more copies of these special resources have the same proof-power. To account for the co-existence of general and unlimited resources, the language of linear logic includes, along with the connectives $\otimes$ and $\multimap$ (sometimes referred to as the multiplicative conjunction and disjunction, respectively), also additive conjunction and disjunction, respectively, denoted $\&$ and $\oplus$, which are idempotent (i.e., $A \& A = A = A \oplus A$ for every formula $A$). Moreover, the language of linear logic includes the modal operators $!$ and $?$, called exponentials, which, respectively, govern the controlled application of left- and right-weakening and contraction rules for formulas under their scope, algebraically encoded in the following identities:

\[ !(A \& B) = !A \otimes !B \quad ?(A \oplus B) = ?A \multimap ?B. \]

The interplay between additive and multiplicative connectives, mediated by exponentials, is encoded in rules the parametric parts (or contexts) of which are not arbitrary, and hence closed under arbitrary substitution, but are restricted in some way. These restricted contexts create additional complications in the definition of reduction strategies for syntactic cut elimination.

In the present article, proof calculi for intuitionistic, bi-intuitionistic and classical linear logics are introduced in which all parameters in rules occur \textit{unrestricted}. This is possible, thanks to the introduction of a \textit{richer} language in which general and unlimited resources are assigned different \textit{types}, each of which interpreted by a different type of algebra (linear algebras for general resource-type terms, and (bi-)Heyting algebras or Boolean algebras for unlimited resource-type terms), and the interaction between these types is mediated by pairs of adjoint connectives, the composition of which captures Girard’s exponentials $!$ and $?$ as defined connectives. The proof-theoretic behavior of the adjoint connectives is that of standard normal modal operators. Moreover, the information capturing the essential properties of the exponentials can be expressed in the new language by means of identities of a syntactic shape called \textit{analytic inductive} (cf. Reference [63]), which guarantees that they can be equivalently encoded into analytic rules. The metatheory of these calculi is smooth and encompassed in a general theory (cf. References [18, 42, 63]), so one obtains soundness, completeness, conservativity, cut elimination, and subformula property as easy corollaries of general facts. Moreover, the same general theory guarantees that these meta-properties transfer to all analytic variants of the basic framework, such as non-commutative, affine, or relevant linear logic, and their analytic extensions and expansions. This makes it possible to embed in a modular way the proof theory of linear logic into a much wider family of logics, and to transfer insights and results.

The possibility of appealing to the general multi-type methodology in the specific case of linear logic is justified by an analysis of the algebraic semantics of linear logic, aimed at identifying the different types and their key interactions. The calculi introduced in the present article are designed according to the \textit{multi-type} methodology, introduced in References [40, 43] to provide DEL and PDL with analytic calculi based on motivations discussed in References [44, 58], and further developed in References [11, 42]. The multi-type methodology refines and generalizes \textit{proper display calculi} (cf. Reference [84, Section 4.1]) to extend the benefits of their meta-theory to logics that

\footnote{In Appendix \textit{G}, we will also consider the \textit{distributive} versions of these logics, i.e., those in which the additive conjunction and disjunction distribute over each other. Moreover, in the same Appendix, we will discuss an alternative approach for introducing proper display calculi for general linear logics, which is based on a modular "merge" of the calculi introduced in the present article with the proper display calculus for the basic propositional logic of general lattices introduced in Reference [64].}
are not properly displayable in their single-type presentation, according to the characterization results given in References [18, 63, 69]. Examples of such logics include very well known and widely used logical frameworks such as inquisitive logic [45], DEL [43], PDL [41], semi De Morgan logic and some of its extensions [60, 61], bilattice logic [62], non-normal and conditional logics [14], and logics of rough algebras [57, 59]. Multi-type calculi allow to capture large classes of axiomatic extensions of given logics [63]. Moreover, they provide a powerful and flexible environment for the design of new families of logics, such as those introduced in Reference [11]. A suitable multi-type calculus for first-order logic was introduced in Reference [6]. In this calculus, the side conditions of introduction rules for the quantifiers are encoded into analytic structural rules involving different types, paving the way for dealing with the many challenges arising when trying to interpret quantifiers in a non-classical setting. Also the long-standing problem of the identity of proofs can be usefully approached via the multi-type methodology: For instance, Reference [65] introduces a focused proof-theoretic environment in which different proof sections in a derivation are assigned different types.

Structure of the article. In Section 2, we highlight the ideas on which the approach of the present article is based as they have occurred in the literature in linear logic and in the proof theory of neighboring logics. Section 3 outlines the algebraic and order-theoretic environment that motivates our proposed treatment for the various linear logics that are also presented algebraically in this section. In Section 4, we introduce the syntactic counterpart of the multi-type semantic environment introduced in Section 3, in the form of a multi-type language, define translations between the single-type and the multi-type languages for linear logics, and discuss how equivalent analytic reformulations can be given of non-analytic axioms in linear logic. In Section 5, we introduce calculi for the various linear logics. In Section 6, we prove their soundness, completeness, conservativity, cut elimination, and subformula property. In Section 7, we discuss two directions that are opened up by the theory of proper display calculi now made available also to linear logics: In Section 7.1, we apply the same techniques developed in the previous sections to the problem, pertaining to the field of categorial grammar, of developing flexible enough formal frameworks to account for exceptions in rules for the generation of grammatical sentences in natural language. Specifically, we introduce exponentials that control exchange and associativity in the same way in which exponentials in linear logic control weakening and contraction. In Section 7.2, we discuss how the linear counterparts of intermediate logics can be captured and studied in the framework introduced in the present article. We discuss conclusions and further directions in Section 8. Because the present article draws from many areas, we have included several appendices for the reader’s convenience, each of which contains the definitions and facts from a given area that are used in the article. Specifically, Appendix A reports on the definition of proper multi-type calculi and their meta-theorem; Appendix B on the definition of analytic inductive inequalities; Appendix C on basic definitions and facts on canonical extensions. Appendix D collects the proof of the inversion lemmas that are needed for verifying the completeness. Various derivations are collected in Appendixes E, F, and G.

2 PRECURSORS OF THE MULTI-TYPE APPROACH TO LINEAR LOGIC
The multi-type methodology is aimed at developing analytic calculi with a uniform design and excellent properties for general classes of logics, but as we will argue in the present article, it is particularly natural for linear logic, given that precisely the proof-theoretic, algebraic, and category-theoretic methods developed for linear logic have been the growth-bed for many key insights and ideas the multi-type methodology builds on. In the present section, we review some
of these solutions with the specific aim of highlighting those aspects that anticipate the multi-type approach. Our presentation is certainly not an exhaustive survey of the literature, for which we refer to References [9, 74, 81].

Right from the first paper [49], Girard describes the connectives of linear logic as arising from the decomposition of classical connectives; this decomposition makes it possible to isolate, e.g., the constructive versus the non-constructive behavior of connectives, but also the linear versus the non-linear behavior, using structural rules to capture each type of behavior. In Reference [52], Girard further expands on a conceptual framework of reference for these ideas, and in particular describes the linear behavior as the behavior of general actions, which can be performed at a cost, and the non-linear behavior in terms of the behavior of situations, or actions that can be performed at no cost (or the cost of which is negligible). The purpose of the exponential connectives is then to bridge the linear and the non-linear behavior, thus making the language of linear logic expressive enough to support a Gödel-type translation of intuitionistic/classical logic into linear logic.

While exponentials are essential precisely for embedding intuitionistic and classical logic into linear logic, they pose significant complications when it comes to cut elimination and normalization, as is witnessed in the so-called Girard-Tait [50, 80] approach to cut elimination for Girard’s calculus, discussed also by Melliès [74]. Indeed, due to the fact that exponentials have non-standard introduction rules (the so-called promotion and dereliction rules), ad hoc transformation steps need to be devised to account for the permutation of cuts on !- or ?-formulas with restricted weakening and contraction and with promotion, giving rise to cut elimination proofs that, besides being lengthier and more cumbersome, are not structured in such a way that the same uniform reduction strategy applies.

Motivated both by these technical issues and by more general questions about ways of making different logics coexist and interact with one other, in Reference [51] Girard introduces a calculus for sequents $\Gamma; \Gamma' \vdash \Delta; \Delta'$ with different maintenance zones: the classical zones $\Gamma'$ and $\Delta'$, where all structural rules are applicable, and the linear zones $\Gamma$ and $\Delta$, where the application of structural rules is limited according to the stipulations of the linear behavior. The intended reading of the sequent above in terms of the linear logic language is then $\Gamma, !\Gamma' \vdash ?\Delta', \Delta$. A very similar environment is Andreoli’s calculus of dyadic sequents [1], introduced to address the issues of cut elimination and proof search. The strategy underlying these solutions aims at enriching the language of the calculus in a way that accounts at the structural level for a neat separation of the linear behavior from the non-linear behavior.

Although developed independently, Girard’s and Andreoli’s approach has strong similarities with Belnap’s general proof-theoretic paradigm of display logic [7]. What Belnap refers to as display logic is not in fact one logical system presented as a sequent calculus, but rather a methodology refining Gentzen sequent calculi through the systematic enrichment of the logical language with an additional layer of structural connectives besides the comma. This richer environment makes it possible to enforce a neat separation of roles between introduction rules (i.e., rules introducing principal formulas) and structural rules (i.e., those rules expressed purely in terms of structural variables and structural connectives). Indeed, while introduction rules are defined along very uniform and rigid lines and only capture the most basic information on each logical connective (its arity and the tonicity of each coordinate), structural rules encode most information on the behavior of each connective and on the interaction between different connectives. Thanks to this neat separation of roles, a suitable environment is created in which the most important technical contribution of Belnap’s paradigm can be stated and proved; namely, a general, smooth and robust proof strategy for cut elimination, hinging on the design principles we summarize as follows: (a) uniform shape
of introduction rules for all connectives, and (b) information on the specific behavior of each connective encoded purely at the structural level. These ideas are very much aligned with Girard’s considerations about the crucial role of structural rules (e.g., in telling the linear and non-linear behavior apart) in the genesis of linear logic, and succeed in creating an explicit mathematical environment in which they can be developed further.

The alignment of Belnap’s and Girard’s programs is also reflected in Belnap’s major motivation in introducing display calculi: creating a proof-theoretic environment capable of simultaneously accounting for “an indefinite number of logics all mixed together including boolean […], intuitionistic, relevance and (various) modal logics.” This motivation drives Belnap’s bookkeeping mechanism “permitting control in the presence of multiple logics” that is remarkable in its elegance and power and is based on the fact that (structural) connectives in all these logics are residuated. The engine of this bookkeeping mechanism is the property that gives display logic its name, namely, the display property, requiring that for any (derivable) sequent \( X \vdash Y \) and any substructure \( Z \) of \( X \) or \( Y \), the sequent \( X \vdash Y \) is provably equivalent to either a sequent of the form \( Z \vdash Y' \) or of the form \( X' \vdash Z \) (and exactly one of these cases occurs). Informally, a sequent calculus enjoys the display property if it is always possible to display any substructure \( Z \) of a given (derivable) sequent \( X \vdash Y \) either in precedent position \( Z \vdash Y' \) or in succedent position \( X' \vdash Z \) in a way that preserves logical equivalence.

However, when designing his display calculus for linear logic [8], Belnap derogates from design principle (a) and does not introduce structural counterparts for the exponentials. Without their structural counterparts, it is of course not possible to express the key properties of exponentials purely at the structural level, which also results in a violation of (b). Hence, in Belnap’s calculus, these properties are still encoded in the introduction rules for \(!\) and \(?\), which are hence non-standard. This latter fact is the key reason why the cut elimination theorem for Belnap’s display calculus for linear logic does not make as significant an improvement in smoothness over those for previous calculi as it could have made.

Belnap’s derogation has a technical reason, which stems from the fact that, when seen as operations on linear algebras (cf. Definition 3.1), the exponentials \(!\) and \(?\) are not residuated. Hence, if their structural counterparts were allowed in the language of Belnap’s display calculus for linear logic, then the resulting calculus would lose the display property, due to the fact that this property critically hinges on the presence of certain structural rules (the so-called display rules) for all structural connectives, which would be unsound in the case of exponentials, precisely due to their not being residuated. Hence, to preserve the display property for his calculus for linear logic (which is the ultimate drive for his cut elimination metatheorem), Belnap is forced to give up on the full enforcement of design principles (a) and (b).

As one of the reviewers observes, principle (a) becomes established in subsequent work of many authors adopting the display framework, while Belnap himself derogates from it in various ways, some of which are discussed in the present section. However, we believe it is fair to say that Belnap recognizes the importance of this principle both from a technical and a conceptual standpoint, as witnessed, e.g., by the following quote from Reference [7], Section 3.3: “It is a further striking feature of Display Logic that the same set of formula-connective postulates is used for every family; which is Display Logic’s own way of making sense out of everyone’s sense of family resemblance.”

Let \( P_i \), \( Q \) be posets, for \( 1 \leq i \leq n \). The maps \( f : P_1 \times \cdots \times P_n \rightarrow Q \) and \( g_i : P_1 \times \cdots \times P_{i-1} \times Q \times P_{i+1} \times \cdots \times P_n \rightarrow P_i \) are residuated (respectively, Galois-residuated) in their \( i \)th coordinates if \( f(x_1, \ldots, x_n) \leq y \) (respectively, \( y \leq f(x_1, \ldots, x_n) \)) iff \( x_i \leq g(x_1, \ldots, y, \ldots, x_n) \) for every \( x_i \in P_i \) and \( y \in Q \). In this case, \( f \) is the left residual of \( g_i \), and \( g_i \) is the right residual of \( f \). A map \( f \) as above is residuated if its (Galois-)residuates in each of its coordinates exist. Logical connectives are residuated if their algebraic interpretations are. A logic is residuated if its signature is closed under the residuals of each connective in each coordinate. For instance, tense modal logic [38], bi-intuitionistic logic [56], and the multiplicative fragment of the Lambek calculus [72] are residuated, while basic normal modal logic [13] and intuitionistic logic [82] are not.

Similar considerations apply for Belnap’s treatment of the additive connectives, which also lack their structural counterparts. We address this issue in the companion paper [64].
The differences in Belnap’s and Girard-Andreoli’s approaches reflect the tradeoff we are facing: Unlike Belnap, Girard-Andreoli are not bound to a general strategy for cut elimination (such as the one hinging on Belnap’s display property), which leaves them the freedom to include exponentials at the structural level, at the price of not being able to develop a cut elimination strategy via a meta-theorem. However, it is precisely the attempt to achieve cut elimination via a general meta-theorem that prevents Belnap from using the full power of design principles (a) and (b), and hence forces him to settle for a less than completely smooth cut elimination result. Both solutions make improvements, which leave open space for further improvement.

Our way out of this impasse takes its move from Girard’s initial idea about linear logic arising from the decomposition of classical and intuitionistic connectives and is based on viewing exponentials as compositions of adjoint maps. This is indeed possible and has been already pursued in the category-theoretic setting \[9, 10, 74\]. Benton \[9\], in particular, takes the environment of adjoint models (each consisting of a symmetric monoidal adjunction between a symmetric monoidal closed category and a Cartesian closed category) as the semantic justification for the introduction of the logical system LNL, in which the linear and non-linear behavior “exist on an equal footing.” Benton explores several sequent calculi for LNL, some of which fail to be cut-free. His final proposal is perhaps the closest precursor of the multi-type calculus that is introduced in the present article, in that it features both linear and nonlinear sequents and rules indicating how to move from one to the other. However, the proof of cut elimination does not appeal to a meta-theorem, and hence does not straightforwardly transfer to axiomatic extensions of linear logic such as classical, affine, or relevant linear logic \[33, 68, 74\].

In a nutshell, the approach pursued in the present article is based semantically on some well-known facts, which yield the definition of the algebraic counterparts of adjoint models. First, the algebraic interpretation of ! (respectively, ?) is an interior operator (respectively, a closure operator) on any linear algebra \( \mathbb{L} \). Hence, by general order theory, the operation ! (respectively, ?) can be identified with the composition of two adjoint maps: the map \( \iota : \mathbb{L} \rightarrow K_\pi \) (respectively, \( \gamma : \mathbb{L} \rightarrow K_\varepsilon \)) onto the range of ! (respectively, ?) and the natural order-embedding \( e_! : K_\pi \hookrightarrow \mathbb{L} \) (respectively, \( e_\varepsilon : K_\varepsilon \hookrightarrow \mathbb{L} \)). Second, the interaction between !, \( \otimes \), and & (respectively, between ?, \( \exists \), and \( \oplus \)) can be equivalently rephrased by saying that the poset \( K_\pi \) (respectively, \( K_\varepsilon \)) has a natural algebraic structure (cf. Proposition 3.6). Third, the algebraic structures of \( \mathbb{L} \) and \( K_\pi \) (respectively, \( \mathbb{L} \) and \( K_\varepsilon \)) are in fact compatible with the adjunction \( e_! \dashv \iota \) (respectively, \( \gamma \vdash e_\varepsilon \)) in a sense that is both mathematically precise and general (cf. proof of Proposition 3.6), and that provides the algebraic underpinning of the translation from classical and intuitionistic logic to linear logic. Interestingly, a very similar compatibility condition also accounts for the Gödel-Tarski translation from intuitionistic logic to the modal logic S4 (cf. algebraic analysis in Reference \[27\], Section 3.1).

The composite mathematical environment consisting of the algebras \( \mathbb{L} \), \( K_\pi \), and \( K_\varepsilon \) together with the adjunction situations between them naturally provides the interpretation for a logical language that is polychromatic in Melliès’ terminology \[74\], in the sense that admits terms of as many types as there are algebras in the environment. The mathematical properties of this environment provide the semantic justification for the design of the calculi introduced in Section 5, in which both display property and design principles (a) and (b) are satisfied. The change of perspective we pursue, along with Benton \[9\], Melliès \[74\], and Jacobs \[68\], is that this environment can also be taken as primary rather than derived. Accordingly, at least for proof-theoretic purposes, linear logic can be more naturally accommodated by such a composite environment than the standard linear algebras.

3 MULTI-TYPE SEMANTIC ENVIRONMENT FOR LINEAR LOGIC

In the present section, we introduce the algebraic environment that justifies semantically the multi-type approach to linear logic that we develop in Section 5. In the next subsection, we take
the algebras introduced in References [76, 81] as starting point. We enrich their definition, make it more modular, and expand on the properties of the images of the algebraic interpretation of exponentials that are briefly mentioned in Reference [81], leading to our notion of (right and left) "kernels." In Section 3.2, we expand on the algebraic significance of the interaction axioms between exponentials in terms of the existence of certain maps between kernels. In Section 3.3, we show that linear algebras with exponentials can be equivalently presented in terms of composite environments consisting of linear algebras without exponentials and other algebraic structures, connected via suitable adjoint maps. In Section 3.4, we report on results pertaining to the theory of canonical extensions applied to the composite environments introduced in the previous subsections. These results will be used in the development of the next sections.

3.1 Linear Algebras and their Kernels

Definition 3.1. \( L = (L, \& \oplus, \top, 0, \otimes, \neg, 1, \bot, \rightarrow) \) is an intuitionistic linear algebra (IL-algebra) if:

IL1. \( (L, \& \oplus, \top, 0, \otimes, \neg, 1, \bot, \rightarrow) \) is a bi-intuitionistic linear algebra (BiL-algebra) if:

B1. \( (L, \& \oplus, \top, 0, \otimes, \neg, 1, \bot, \rightarrow) \) is an IL-algebra;
B2. \( a \rightarrow b \leq c \) iff \( b \leq a \rightarrow c \) for all \( a, b, c \in L \);
B3. \( \neg \) preserves all finite meets, hence also the empty meet \( \top \), in each coordinate.

An IL-algebra is a classical linear algebra (CL-algebra) if

\[ C \cdot (a \rightarrow \bot) \rightarrow \bot = a \] for every \( a \in L \).

We will sometimes abbreviate \( a \rightarrow \bot \) as \( a^\bot \), and write \( C \) above as \( a^{\perp} = a \).

\[ \mathbb{L} = (L, \& \oplus, \top, 0, \otimes, \neg, 1, \bot, \rightarrow) \) is an intuitionistic linear algebra with storage (ILS-algebra) if:

IL5. \( \neg \) preserves all infinite meets, hence also the empty meet \( \top \), in each coordinate.

An ILS-algebra is a classical linear algebra with storage (CLS-algebra) if:

\[ \mathbb{L} = (L, \& \oplus, \top, 0, \otimes, \neg, 1, \bot, \rightarrow) \] is an intuitionistic linear algebra with storage and co-storage (ILSC-algebra) if:

\[ \mathbb{L} = (L, \& \oplus, \top, 0, \otimes, \neg, 1, \bot, \rightarrow) \] is a bi-intuitionistic linear algebra with storage and co-storage (BLSC-algebra) if:

\[ \mathbb{L} = (L, \& \oplus, \top, 0, \otimes, \neg, 1, \bot, \rightarrow) \] is a bi-intuitionistic linear algebra with storage and co-storage (BLSC-algebra) if:

BLSC1. \( (L, \& \oplus, \top, 0, \otimes, \neg, 1, \bot, \rightarrow) \) is an ILSC-algebra;
BLSC2. \( (L, \& \oplus, \top, 0, \otimes, \neg, 1, \bot, \rightarrow) \) is a BiL-algebra.
A BLSC-algebra \( \mathbb{L} \) is paired (BLP-algebra) if:

BLP1. \((L, \&, \oplus, \top, 0, \otimes, \neg, 1, \bot, \neg, !, ?)\) is an ILP-algebra;

BLP2. \(!a \to !b \leq ?(a \to b)\) for all \(a, b \in L\).

CLSC-algebras reflect the logical signature of (classical) linear logic as originally defined by Girard, although no link between the exponentials is assumed. ILP-algebras are a slightly modified version of Ono’s modal FL-algebras [76, Definition 6.1]. The notion of BLSC-algebra is a symmetrization of that of ILSC-algebra, where the linear co-implication \( \to \) (a.k.a. subtraction, or exclusion) has been added to the signature as the left residual of \( \neg \), as was done in References [55, 79, 81]. The proof-theoretic framework introduced in Section 5 will account for each of these environments modularly and conservatively.

Condition IL4 implies that \( \otimes \) preserves all existing joins (hence, all finite joins and the empty join 0 in particular) in each coordinate, and \( \to \) preserves all existing meets in its second coordinate and reverses all existing joins in its first coordinate. Hence, the following De Morgan law holds in IL-algebras:

\[
(a \oplus b)^\perp = (a)^\perp \& (b)^\perp.
\] (1)

The main difference between IL-algebras and CL-algebras is captured by the next Lemma.

**Lemma 3.2.** The following De Morgan law also holds in CL-algebras:

\[
(a \& b)^\perp = (a)^\perp \oplus (b)^\perp.
\] (2)

**Proof.** Since \((\cdot)^\perp\) is antitone by definition, \((a \& b)^\perp\) is a common upper bound of \(a^\perp\) and \(b^\perp\).

Condition C implies that \((\cdot)^\perp : L \to L\) is surjective. Hence, to prove the converse inequality, it is enough to show that if \(a^\perp \leq c^\perp\) and \(b^\perp \leq c^\perp\), then \((a \& b)^\perp \leq c^\perp\). The assumptions imply that \(c = c^\perp = a\), and likewise \(c \leq b\). Hence, \(c \leq a \& b\), which implies that \((a \& b)^\perp \leq c^\perp\) as required. \(\square\)

By conditions S2 and S3, the operation \(! : L \to L\) is an interior operator on \(L\) seen as a poset. Dually, by SC2 and SC3, the operation \(? : L \to L\) is a closure operator on \(L\) seen as a poset. By general order-theoretic facts (cf. Reference [31, Chapter 7]) this means that

\[
! = e_i \circ \iota \quad \text{and} \quad ? = e_\gamma \circ \gamma,
\]

where \(\iota : L \to \text{Range}(!)\) and \(\gamma : L \to \text{Range(?)}\), defined by \(\iota(a) = ![a]\) and \(\gamma(a) = ?a\) for every \(a \in L\), are adjoints of the natural embeddings \(e_i : \text{Range}(!) \hookrightarrow L\) and \(e_\gamma : \text{Range(?)} \hookrightarrow L\) as follows:

\[
e_i \circ \iota \quad \text{and} \quad \gamma \circ e_\gamma,
\]

i.e., for every \(a \in L\), \(o \in \text{Range}(!)\), and \(c \in \text{Range(?)}\),

\[
e_i(o) \leq a \quad \text{iff} \quad o \leq \iota(a) \quad \text{and} \quad \gamma(a) \leq c \quad \text{iff} \quad a \leq e_\gamma(c).
\]

In what follows, we let \(K_!\) and \(K_?\) be the subposets of \(L\) identified by \(\text{Range}(!) = \text{Range}(\iota)\) and \(\text{Range(?)} = \text{Range}(\gamma)\), respectively. Sometimes, we will refer to elements in \(K_!\) as “open” and elements in \(K_?\) as “closed.”

**Lemma 3.3.** For every ILSC-algebra \(\mathbb{L}\), every \(\alpha \in K_!\) and \(\xi \in K_?\),

\[
i(\varepsilon_i(\alpha)) = \alpha \quad \text{and} \quad \gamma(\varepsilon_i(\xi)) = \xi.
\] (3)

**Proof.** We only prove the first identity, the proof of the remaining one being dual. By adjunction, \(\alpha \leq i(\varepsilon_i(\alpha))\) iff \(\varepsilon_i(\alpha) \leq \varepsilon_i(\alpha)\), which always holds. As to the converse inequality \(i(\varepsilon_i(\alpha)) \leq \alpha\), since \(\varepsilon_i\) is an order-embedding, it is enough to show that \(\varepsilon_i(i(\varepsilon_i(\alpha))) \leq \varepsilon_i(\alpha)\), which by adjunction is equivalent to \(i(\varepsilon_i(\alpha)) \leq i(\varepsilon_i(\alpha))\), which always holds. \(\square\)
In what follows,

1. \( \vee \) and \( \bigvee \) denote joins in \( K_1 \);
2. \( \wedge \) and \( \bigwedge \) denote meets in \( K_2 \);
3. \( \oplus \) and \( \bigoplus \) denote joins in \( L \);
4. \& and \( \&_L \) denote meets in \( L \).

The following fact shows that \( L \)-joins of open elements are open, and \( L \)-meets of closed elements are closed.

**Fact 3.4.** For all (finite) set \( I \),

1. if \( \bigvee_{i \in I} i(a_i) \) exists, then \( \bigoplus_{i \in I} !a_i = !\left( \bigoplus_{i \in I} !a_i \right) \);
2. if \( \bigwedge_{i \in I} \gamma(a_i) \) exists, then \( \&_{i \in I} ?a_i = ?(\&_{i \in I} ?a_i) \).

**Proof.** We only prove the first item, the proof of the second one being dual. The following chain of identities holds:

\[
!\left( \bigoplus_{i \in I} !a_i \right) = !\left( \bigoplus_{i \in I} e_\iota(i(a_i)) \right) = !e_\iota \circ i = e_\iota \text{ preserves existing joins}.
\]

Hence, the subposets \( K_1 \) and \( K_2 \) are, respectively, a \( \oplus \)-subsemilattice and a \( \& \)-subsemilattice of \((L, \oplus, \&, 0, \top)\): indeed,

\[
e_\iota \left( \bigvee_{i \in I} i(a_i) \right) = \bigoplus_{i \in I} e_\iota(i(a_i)) = \bigoplus_{i \in I} !a_i \quad \text{and} \quad e_\iota \left( \bigwedge_{i \in I} \gamma(a_i) \right) = \&_{i \in I} e_\iota(\gamma(a_i)) = \&_{i \in I} ?a_i.
\]

**Definition 3.5.** For any ILS-algebra \( \mathbb{L} = (L, \& \oplus, \&, 0, \top) \), let the **left-kernel** of \( \mathbb{L} \) be the algebra \( \mathbb{K}_1 = (K_1, \wedge_1, \vee_1, t_1, f_1, \to) \) defined as follows:

- **LK1.** \( K_1 := \text{Range}(!) = \text{Range}(\iota) \), where \( \iota : L \to K_1 \) is defined by letting \( i(a) = !a \) for any \( a \in L \);
- **LK2.** \( \alpha \vee_1 \beta := i(e_\iota(\alpha) \oplus e_\iota(\beta)) \) for all \( \alpha, \beta \in K_1 \);
- **LK3.** \( \alpha \wedge_1 \beta := i(e_\iota(\alpha) \& e_\iota(\beta)) \) for all \( \alpha, \beta \in K_1 \);
- **LK4.** \( t_1 := i(\top) \);
- **LK5.** \( f_1 := i(0) \);
- **LK6.** \( \alpha \to \beta := i(e_\iota(\alpha) \to e_\iota(\beta)) \).

We also let

- **LK7.** \( \neg_1 \alpha := - \text{to} \to f_1 := i(e_\iota(\alpha) \to e_\iota(f_1)) = i(e_\iota(\alpha) \to 0) = i(e_\iota(\alpha)^{\bot}) \).

For any ILSC-algebra \( \mathbb{L} = (L, \& \oplus, \&, 0, \top, \to, \leftrightarrow, \boxdot, !, ?) \), let the **right-kernel** of \( \mathbb{L} \) be the algebra \( \mathbb{K}_2 = (K_2, \vee_2, \wedge_2, t_2, f_2, \to) \) defined as follows:

- **RK1.** \( K_2 := \text{Range}(?) = \text{Range}(\gamma) \), where \( \gamma : L \to K_2 \) is defined by letting \( \gamma(a) = ?a \) for any \( a \in L \);
- **RK2.** \( \xi \vee_2 \chi := \gamma(e_{\nu_2}(\xi) \& e_{\nu_2}(\chi)) \);
- **RK3.** \( \xi \wedge_2 \chi := \gamma(e_{\nu_2}(\xi) \oplus e_{\nu_2}(\chi)) \);
- **RK4.** \( f_2 := \gamma(0) \);
- **RK5.** \( t_2 := \gamma(\top) \).

For any BLSC-algebra \( \mathbb{L} = (L, \& \oplus, \&, 0, \top, \to, \leftrightarrow, \boxdot, !, ?) \), let the **right-kernel** of \( \mathbb{L} \) be the algebra \( \mathbb{K}_3 = (K_3, \wedge_3, \vee_3, t_3, f_3, \to) \) such that \( \mathbb{K}_3 = (K_3, \wedge_3, \vee_3, t_3, f_3, \to) \) is defined as above, and moreover:

- **RK6.** \( \xi \to \chi := \gamma(e_{\nu_2}(\xi) \to e_{\nu_2}(\chi)) \).

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It is interesting to notice the fit between the definition of the algebraic structure of the kernels given above and the translation by which intuitionistic formulas embed into linear formulas given in Reference [49, Chapter 5]. The following proposition develops and expands on an observation made by Troelstra (cf. Reference [81, Exercise after Lemma 8.17]).

**Proposition 3.6.** For any \( L \),

1. if \( L \) is an ILS-algebra, then \( K_1 \) is a Heyting algebra;
2. if \( L \) is an CLS-algebra, then \( K_1 \) is a Boolean algebra;
3. if \( L \) is an ILSC-algebra, then \( K_1 \) and \( K_2 \) are a Heyting algebra and a distributive lattice, respectively;
4. if \( L \) is an BLSC-algebra, then \( K_1 \) and \( K_2 \) are a Heyting algebra and a co-Heyting algebra, respectively.

**Proof.** We only prove the first and second item, the proofs of the remaining items being dual to the proof of item 1. Let us show that for all \( a, b \in L \), the greatest lower bound of \( \iota(a) \) and \( \iota(b) \) exists and coincides with \( \iota(a \& b) \). From \( b \preceq \top \) and S2, we get \( !b \leq !\top \). Hence, by S4 and the monotonicity of \( \otimes \) in both coordinates,

\[
\iota(a \& b) = (a \& b) = a \otimes b \leq a \otimes !\top = !a \otimes 1 = !a = \iota(a),
\]

which implies that \( \iota(a \& b) \preceq \iota(a) \), since \( \iota : K_1 \hookrightarrow L \) is an order-embedding. Likewise, one shows that \( \iota(a \& b) \preceq \iota(b) \), which finishes the proof that \( \iota(a \& b) \) is a common lower bound for \( \iota(a) \) and \( \iota(b) \).

To finish the proof of the claim, one needs to show that, if \( c \in L \) and \( \iota(c) \preceq \iota(a) \) and \( \iota(c) \preceq \iota(b) \), then \( \iota(c) \preceq \iota(a \& b) \). Indeed, the assumptions imply that \( !c = \iota(c) \preceq \iota(a) = a \preceq a \), and likewise, \( !c \preceq a \& b \), which implies, by S2 and S3, that \( \iota(c) = !c = !(a \& b) = \iota(a \& b) \). This implies that \( \iota(c) \preceq \iota(a \& b) \), as required.

Let us now show that \( \land \) preserves coordinatewise all existing joins in \( K_1 \). That is, if \( \bigvee_{i \in I} \iota(a_i) \) exists, then for every \( b \in L \),

\[
\bigvee_{i \in I} \iota(a_i) \land \iota(b) = \bigvee_{i \in I} (\iota(a_i) \land \iota(b)).
\]

The following chain of identities holds:

\[
\begin{align*}
(\bigvee_{i \in I} \iota(a_i)) \land \iota(b) & = \iota\left( \bigoplus_{i \in I} !a_i \right) \land \iota(b) \quad \text{(Definition 3.5 (LK2))} \\
& = \iota\left( \bigoplus_{i \in I} !a_i \& b \right) \quad \text{(Definition 3.5 (LK3))}
\end{align*}
\]

\[
\begin{align*}
\bigvee_{i \in I} (\iota(a_i) \land \iota(b)) & = \bigvee_{i \in I} \iota(a_i \& b) \quad \text{(Definition 3.5 (LK3))} \\
& = \iota\left( \bigoplus_{i \in I} !a_i \& b \right) \quad \text{(Definition 3.5 (LK2))} \\
& = \iota\left( \bigoplus_{i \in I} !a_i \otimes !b \right) \quad \text{(S4)} \\
& = \iota\left( \bigoplus_{i \in I} !a_i \right) \otimes !b \quad \text{(\( \otimes \) preserves existing joins)}.
\end{align*}
\]

Hence, to finish the proof of the claim it is enough to show that

\[
\iota\left( \bigoplus_{i \in I} !a_i \right) \& b = \iota\left( \bigoplus_{i \in I} !a_i \otimes !b \right).
\]

Since \( \iota : K_1 \hookrightarrow L \) is injective and \( ! = \iota \circ \iota \), it is enough to show that

\[
!\left( \bigoplus_{i \in I} !a_i \right) \& b = !\left( \bigoplus_{i \in I} !a_i \otimes !b \right).
\]
Indeed, by S3, S4, and Fact 3.4,

\[ !\left( \bigoplus_{i \in I} !a_i \right) \land b = !\left( \bigoplus_{i \in I} !a_i \right) \land !b = !\left( \bigoplus_{i \in I} !a_i \right) \otimes !b, \]

as required. By S2 and S4, \( i(a) \leq i(\top) = 1 \) for every \( a \in L \), which motivates Definition 3.5 (LK4), and moreover, S3 implies that \( i(0) \leq 0 \), which motivates Definition 3.5 (KL5). Let us show that for all \( a, b, c \in L \),

\[ i(a) \land i(b) \leq i(c) \iff i(b) \leq i(a) \rightarrow i(c). \]

This concludes the proof of item 1. As to item 2, let us assume that \( a^{\perp\perp} = a \) for any \( a \in L \). Then, by Definition 3.5 (LK7),

\[ \neg\neg i(a) = i(a^{\perp\perp}) = i(a), \]

which is enough to establish that the Heyting algebra \( \mathbb{K} \) is a Boolean algebra.

Summing up, the axiomatization of the exponentials as interior and closure operators, respectively, generates a composite algebraic environment of linear logic, in which linear algebras come together with one or two other algebras, namely, the kernels, which, depending on the original linear signature, can be endowed with a structure of Heyting algebras, distributive lattices, or co-Heyting algebras. Moreover, the adjunctions relating the linear algebra and its kernel(s) also guarantee the natural embedding maps to enjoy the following additional properties:

**Proposition 3.7.** For every ILS(C)-algebra \( L \), all \( \alpha, \beta \in \mathbb{K}_\alpha \), and all \( \xi, \chi \in \mathbb{K}_\beta \),

\[ e_1(\alpha) \otimes e_1(\beta) = e_1(\alpha \land_1 \beta) \quad \text{and} \quad e_2(\xi) \supset e_2(\chi) = e_2(\xi \lor_2 \chi) \quad e_1(t_1) = 1 \quad \text{and} \quad e_2(t_2) = \bot. \]

**Proof.** We only prove the identities involving \( e_1 \), the proof of the remaining ones being dual. Since \( i : L \rightarrow \mathbb{K}_\alpha \) is surjective and order-preserving, proving the required identity is equivalent to showing that, for all \( a, b \in L \),

\[ e_1(i(a)) \otimes e_1(i(b)) = e_1(i(a) \land_1 i(b)) \quad \text{and} \quad e_1(i(\top)) = 1. \]

Since \( i \) preserves meets, \( e_1(i(a) \land_1 i(b)) = e_1(i(a & b)) = !(a & b) \). Hence, the displayed identities above are equivalent to

\[ !a \otimes !b = !(a & b) \quad \text{and} \quad \top = 1, \]

which are true by S4.
3.2 The Interaction of Exponentials in Paired Linear Algebras

Proposition 3.8. (1) The following are equivalent in any ILSC-algebra L:
(a) for all a, b, c ∈ L, if !a ⊗ b ≤ ?c, then !(a ?) ⊗ b ≤ ?c;
(b) for all a, b, c ∈ L, if b ≤ !a → ?c, then ?b ≤ !a → ?c;
(c) ?!(a → ?b) = !(a → ?b) for all a, b ∈ L.

(2) If L is also a BLSC-algebra, then the following are equivalent:
(a) for all a, b, c ∈ L, if !a ≤ b ▷ ?c, then !a ≤ b ▷ ?c;
(b) for all a, b, c ∈ L, if c → !a ≤ b, then ?c → !a ≤ b;
(c) !(b → !a) = ?b → !a for all a, b ∈ L.

Proof. We only prove item 1, the proof of item 2 being order-dual. Clearly, (a) and (b) are equivalent by IL4. Let us assume (c), and let a, b, c ∈ L such that b ≤ !a → ?c. Then by SC2,

?b ≤ ?!(a → ?c) = !(a → ?c),

which proves (b). Let us assume (b) and let a, b ∈ L. Then !a → ?b ≤ !a → ?b implies that !(a → ?b) ≤ !a → ?b, which, together with !a → ?b ≤ ?!(a → ?b), which holds by SC3, proves (c).

The following proposition develops and expands on Reference [76, Lemma 6.1(6)]

Proposition 3.9. (1) For any ILP-algebra L, any (hence, all) of the conditions (a)–(c) in Proposition 3.8.1 holds.

(2) For any BLP-algebra L, any (hence, all) of the conditions (a)–(c) in Proposition 3.8.1 and 2 hold.

Proof. We only prove item 1, the proof of (the second part of) item 2 being order-dual. Let a, b, c ∈ L. By IL2 and IL4, it is enough to show that if !a ≤ b → ?c, then !a ≤ ?b → ?c. Indeed, by the assumption, S3, S2, P, and SC3,

!a = !(b → ?c) ≤ ?b → ?c = ?b → ?c,

as required.

Proposition 3.9 and the two items (c) of Proposition 3.8 imply that in any ILP-algebra (respectively, BLP-algebra) L, a binary map (respectively, binary maps)

→ : K ⊗ K → K \quad \text{and} \quad \multimap : K × K → K

can be defined such that, for any α ∈ K, and ξ ∈ K, the element α \multimap ξ ∈ K (respectively, x \multimap ξ ∈ K) is the unique solution to the following equation(s):

e_1(α) \multimap e_2(ξ) = e_1(α → ξ) \quad \text{and} \quad e_2(ξ) \multimap e_1(α) = e_1(ξ → α).

(4)

Proposition 3.10. (1) For any ILP-algebra L,

(a) the map → preserves finite meets in its second coordinate and reverses finite joins in its first coordinate.

(b) γ(a) ≤ i(a → b) → γ(b) for all a, b ∈ L.

(2) For any BLP-algebra L,

(a) the map → preserves finite joins in its second coordinate and reverses finite meets in its first coordinate.

(b) γ(a → b) → i(b) ≤ i(a) for all a, b ∈ L.

Proof. We only prove item 1, the proof of item 2 being order-dual. As to (a), to show that α → (ξ ∧ ξ) = (α → ξ) ∧ (α → ξ), it is enough to show that

e_1(α → ξ) ∧ e_2(ξ) = e_1((α → ξ) ∧ (α → ξ)).

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Indeed,
\[
e_\top((\alpha \multimap \xi) \land \top((\alpha \multimap \chi))) = e_\top(\alpha \multimap \xi) \land e_\top(\alpha \multimap \chi) \tag{e_\top preserves existing meets (4)}
\]
\[
= (e_\top(\alpha \rightarrow \xi)) \land (e_\top(\alpha \rightarrow \chi)) \\
= e_\top(\alpha \rightarrow (e_\top(\xi) \land e_\top(\chi))) \tag{\rightarrow preserves meets in 2nd coord. (4)}
\]
To show that \( \multimap t_1 = t_1 \), it is enough to show that
\[
e_\top(\alpha \multimap t_1) = e_\top(t_1) = \top.
\]
Indeed,
\[
e_\top(\alpha \multimap t_1) = e_\top(\alpha \rightarrow e_\top(t_1)) \tag{e_\top preserves existing meets (4)}
\]
\[
= e_\top(\alpha \rightarrow \top) \tag{\rightarrow preserves meets in 2nd coord.}
\]
The verification that \( \multimap \) reverses finite joins in its first coordinate is similar and makes use of \( e_\top \) preserving existing joins and \( \rightarrow \) reversing joins in its first coordinate. As to (b), let \( a, b \in \mathbb{L} \). Since \( \mathbb{L} \) is an ILP-algebra, axiom P holds (cf. Definition 3.1):
\[
!(a \multimap b) \leq ?a \rightarrow ?b \tag{P (IL2 and IL4)}
\]
\[
\text{iff } ?a \leq !(a \multimap b) \rightarrow ?b \tag{IL2 and IL4}
\]
\[
\text{iff } e_\top(y(a)) \leq e_\top(i(a \rightarrow b)) \rightarrow e_\top(y(b)) \tag{! = e_\top \circ i \text{ and } ? = e_\top \circ y (4)}
\]
\[
\text{iff } e_\top(y(a)) \leq e_\top(i(a \rightarrow b) \multimap y(b)) \tag{e_\top order-embedding}
\]
\[
\text{iff } y(a) \leq i(a \multimap b) \multimap y(b). \tag{e_\top order-embedding}
\]
\[
\text{□}
\]

### 3.3 Reverse-engineering Exponentials

In the previous subsections, we have seen that certain mathematical structures (namely, the kernels and their adjunction situations, and the maps \( \multimap \) and \( \multimap \rightarrow \)) arise from the axiomatization of linear algebras with exponentials. In the present subsection, we take these mathematical structures as primary (we call them heterogeneous algebras, adopting the terminology of Reference [12]), and capture linear algebras with exponentials in terms of these. The results of the present subsection establish the equivalence between the multi-type and the standard (single-type) algebraic semantics for linear logic, and hence provide the mathematical justification for an alternative viewpoint that takes the multi-type semantics as primary.

Heterogeneous algebras for the various linear logics can be understood as the algebraic counterparts of Benton’s LNL-models [9, Definition 9], and the identities and inequalities defining them can be understood as the multi-type counterparts of the identities defining the algebraic behavior of exponentials. The latter understanding will be developed further in Section 4.

**Definition 3.11.** (1) A heterogeneous ILS-algebra (respectively, heterogeneous CLS-algebra) is a structure \( (\mathbb{L}, \mathbb{A}, e, i) \) such that \( \mathbb{L} \) is an IL-algebra (respectively, CL-algebra), \( \mathbb{A} \) is a Heyting algebra, \( e_\top : \mathbb{A} \multimap \mathbb{L} \) and \( i : \mathbb{L} \rightarrow \mathbb{A} \) such that \( e_\top i + i \), and \( i(e_\top(\alpha)) = \alpha \) for every \( \alpha \in \mathbb{A} \), and moreover, for all \( \alpha, \beta \in \mathbb{A} \),
\[
e_\top(\alpha) \otimes e_\top(\beta) = e_\top(\alpha \land \beta) \quad \text{and} \quad e_\top(t) = 1.
\]
A heterogeneous ILS-algebra (respectively, CLS-algebra) is perfect if both \( \mathbb{L} \) and \( \mathbb{A} \) are perfect (cf. Definition C.5).
(2) A heterogeneous ILSC-algebra (respectively, heterogeneous BLSC-algebra) is a structure \((L, A, B, e, t, e, \gamma)\) such that \(L\) is an IL-algebra (respectively, BiL-algebra), \((L, A, e, t)\) is a heterogeneous ILS-algebra, \(B\) is a distributive lattice (respectively, co-Heyting algebra), \(e_\gamma : B \hookrightarrow L\) and \(\gamma : L \twoheadrightarrow B\) such that \(\gamma \circ e_\gamma\), and \(\gamma(e_\gamma(\xi)) = \xi\) for every \(\xi \in B\), and moreover, for all \(\xi, \chi \in B\),
\[
e_\gamma(\xi) \triangledown e_\gamma(\chi) = e_\gamma(\xi \lor \chi) \quad \text{and} \quad e_\gamma(f) = \bot.
\]

A heterogeneous ILSC-algebra (respectively, BLSC-algebra) is perfect if \(L, A\) and \(B\) are perfect (cf. Definition C.5).

(3) A heterogeneous ILP-algebra is a structure \((L, A, B, e, t, e, \gamma, \rightarrow)\) such that \((L, A, B, e, t, e_\gamma, \gamma)\) is a heterogeneous ILSC-algebra, and
\[
\rightarrow : A \times B \rightarrow B
\]
is such that, for any \(\alpha \in A\), \(\xi \in B\), and \(a, b \in L\),
\[
e_\gamma(a) \rightharpoonup e_\gamma(\xi) = e_\gamma(a \rightharpoonup \xi) \quad \text{and} \quad a(b) \leq a \rightarrow b \rightarrow a(b).
\]

A heterogeneous ILP-algebra is perfect if \(L, A\), and \(B\) are perfect (cf. Definition C.5) and \(\rightarrow\) is completely meet-preserving in its second coordinate and completely join-reversing in its first coordinate.

(4) A heterogeneous BLP-algebra is a structure \((L, A, B, e, t, e, \gamma, \rightarrow, \rightarrow)\) such that \((L, A, B, e, t, e_\gamma, \gamma)\) is a heterogeneous ILP-algebra, and
\[
\rightarrow : B \times A \rightarrow A
\]
is such that, for any \(\alpha \in A\), \(\xi \in B\), and \(a, b \in L\),
\[
e_\gamma(\xi) \rightharpoonup e_\gamma(a) = e_\gamma(\xi \rightharpoonup a) \quad \text{and} \quad a(b) \leq a \rightarrow b \rightarrow a(b).
\]

A heterogeneous BLP-algebra is perfect if \(L, A\), and \(B\) are perfect (cf. Definition C.5), \(\rightarrow\) is completely meet-preserving in its second coordinate and completely join-reversing in its first coordinate, and \(\rightarrow\) is completely join-preserving in its second coordinate and completely meet-reversing in its first coordinate.

In the light of the definitions above, the results of Sections 3.1 and 3.2 can be summarized by the following:

**Proposition 3.12.** For any algebra \(L\),

1. If \(L\) is an ILS-algebra (respectively, CLS-algebra), then \((L, K_1, e, t)\) is a heterogeneous ILS-algebra (respectively, heterogeneous CLS-algebra), where \(K_1\) is as in Definition 3.5.
2. If \(L\) is an ILSC-algebra (respectively, BLSC-algebra), then \((L, K_2, e, t, e, \gamma)\) is a heterogeneous ILSC-algebra (respectively, heterogeneous BLSC-algebra), where \(K_2\) and \(K_\gamma\) are as in Definition 3.5.
3. If \(L\) is an ILP-algebra, then \((L, K_3, K_\gamma, e, t, e, \gamma, \rightarrow)\) is a heterogeneous ILP-algebra, where \(K_3\) and \(K_\gamma\) are as in Definition 3.5, and \(\rightarrow\) as indicated right after the proof of Proposition 3.9.
4. If \(L\) is an BLP-algebra, then \((L, K_4, K_\gamma, e, t, e, \gamma, \rightarrow, \rightarrow)\) is a heterogeneous BLP-algebra, where \(K_4\) and \(K_\gamma\) are as in Definition 3.5, and \(\rightarrow\) and \(\rightarrow\) as indicated right after the proof of Proposition 3.9.

Together with the proposition above, the following proposition shows that heterogeneous algebras are an equivalent presentation of linear algebras with exponentials:
PROPOSITION 3.13. For any algebra $\mathbb{L}$,

(1) If $(\mathbb{L}, \wedge, \vee, !)$ is a heterogeneous ILS-algebra (respectively, heterogeneous CLS-algebra), then $\mathbb{L}$ can be endowed with the structure of ILS-algebra (respectively, CLS-algebra) defining $! : \mathbb{L} \rightarrow \mathbb{L}$ by $a := e!(a(a))$ for every $a \in \mathbb{L}$. Moreover, $\mathbb{A} \simeq \mathbb{K}_\mathbb{A}$.

(2) If $(\mathbb{L}, \wedge, \vee, !)$ is a heterogeneous ILS-algebra (respectively, heterogeneous BLSC-algebra), then $\mathbb{L}$ can be endowed with the structure of ILS-algebra (respectively, BLSC-algebra) by defining $!$ as in the item above, and $?: \mathbb{L} \rightarrow \mathbb{L}$ by $a := e?((\gamma(a)))$ for every $a \in \mathbb{L}$. Moreover, $\mathbb{K}_\mathbb{B} \simeq \mathbb{K}_\mathbb{A}$.

(3) If $(\mathbb{L}, \wedge, \vee, !, \otimes, \tau)$ is a heterogeneous ILSC-algebra, then $\mathbb{L}$ can be endowed with the structure of ILSC-algebra by defining $!$ and $?$ as in the items above.

(4) If $(\mathbb{L}, \wedge, \vee, !, \otimes, \tau)$ is a heterogeneous BLP-algebra, then $\mathbb{L}$ can be endowed with the structure of BLP-algebra by defining $!$ and $?$ as in the items above.

PROOF. Let us prove item 1. By assumption, $\mathbb{L}$ is an IL-algebra (respectively, CL-algebra), which verifies S1. The assumption that $e_t \mapsto t$ implies that both $e_t$ and $t$ are monotone, and hence so is their composition $!$, which verifies S2. Also from $e_t \mapsto t$ and $t(a) \leq t(a)$, it immediately follows that $!a = e_t((a)) \leq a$ for every $a \in \mathbb{L}$. To finish the proof of S3, we need to show that $!a = !a$ for every $a \in \mathbb{L}$. By definition of $!$, this is equivalent to showing that $e_t((a)) \leq e_t((a))$. By the monotonicity of $e_t$, it is enough to show that $a \leq e_t((a))$, which by adjunction is equivalent to $e_t((a)) \leq e_t((a))$, which is always true. This finishes the proof of S3. As to S4, let us observe preliminarily that, since $t$ is a right-adjoint, it preserves existing meets, and hence $t(a_\otimes b) = t(a) \land t(b)$ for all $a, b \in \mathbb{L}$. By definition, showing that $! (a_\otimes b) = ! a \otimes ! b$ and $! T = 1$ is equivalent to showing that

$$e_t((a_\otimes b)) = e_t((a)) \otimes e_t((b)) \quad \text{and} \quad e_t(T) = 1,$$

which, thanks to the preliminary observation, can be equivalently rewritten as follows:

$$e_t((a) \land (b)) = e_t((a)) \otimes e_t((b)) \quad \text{and} \quad e_t(T) = 1,$$

which are true, by the assumptions on $e_t$. This completes the proof that $(\mathbb{L}, !)$ is an ILS-algebra. As to the second part of the statement, let us show preliminarily that the following identities hold:

- $\text{LK}_2^\mathbb{A}$. $\alpha \lor \beta = e_t((\alpha) \lor (\beta))$ for all $\alpha, \beta \in \mathbb{A}$;
- $\text{LK}_3^\mathbb{A}$. $\alpha \land \beta = e_t((\alpha) \land (\beta))$ for all $\alpha, \beta \in \mathbb{A}$;
- $\text{LK}_4^\mathbb{A}$. $t = e_t(T)$;
- $\text{LK}_5^\mathbb{A}$. $f = e_t(0)$;
- $\text{LK}_6^\mathbb{A}$. $\alpha \rightarrow \beta = e_t((\alpha) \rightarrow (\beta))$ for all $\alpha, \beta \in \mathbb{A}$.

Let us observe that, since $e_t$ is a left-adjoint, it preserves existing joins, and hence $e_t(f) = 0$ and $e_t(\alpha \lor \beta) = e_t(\alpha) \lor e_t(\beta)$. Together with $t \circ e_t = Id$, these identities imply that

$$f_! = e_t(f) = 0 \quad \text{and} \quad \alpha \lor \beta = e_t((\alpha) \lor (\beta)) = e_t((\alpha) \lor e_t(\beta)),$$

which proves $\text{LK}_2^\mathbb{A}$ and $\text{LK}_5^\mathbb{A}$. As to $\text{LK}_3^\mathbb{A}$ and $\text{LK}_4^\mathbb{A}$, since $t$ preserves existing meets, and $t \circ e_t = Id$, $e_t((\alpha) \land (\beta)) = e_t((\alpha)) \land e_t((\beta)) = \alpha \land \beta$ and $t = e_t(T)$, as required. As to $\text{LK}_6^\mathbb{A}$, let $\alpha, \beta \in \mathbb{A}$. Since $\mathbb{A}$ is a Heyting algebra, the inequality $\alpha \land (\alpha \rightarrow \beta) \leq \beta$ holds. Also:

\[
\begin{align*}
\alpha \land (\alpha \rightarrow \beta) \leq \beta & \quad \text{iff} \quad e_t((\alpha) \land (\alpha \rightarrow \beta)) \leq e_t(\beta) \\
& \quad \text{(e_\otimes order-embedding)} \\
\alpha \land (\alpha \rightarrow \beta) \leq e_t(\alpha) \otimes e_t(\alpha \rightarrow \beta) \leq e_t(\beta) & \quad \text{(assumption on e_\otimes)} \\
& \quad \\
\alpha \land (\alpha \rightarrow \beta) \leq e_t(\alpha) \rightarrow (\beta) & \quad \text{(II4)}
\end{align*}
\]
Since $i$ is monotone and $i \circ e_1 = Id_A$, this implies that:

$$\alpha \rightarrow \beta = i(e_1(\alpha \rightarrow \beta)) \leq i(e_1(\alpha) \rightarrow e_1(\beta)).$$

Conversely, for all $\alpha, \beta \in A$,

$$i(e_1(\alpha) \rightarrow e_1(\beta)) \leq \alpha \rightarrow \beta$$

iff

$$i(e_1(\alpha) \rightarrow e_1(\beta)) \leq \alpha \rightarrow i(e_1(\beta)) \quad (i \circ e_1 = Id_A)$$

iff

$$\alpha \land i(e_1(\alpha) \rightarrow e_1(\beta)) \leq i(e_1(\beta)) \quad (\text{residuation in } A)$$

iff

$$e_1(\alpha \land i(e_1(\alpha) \rightarrow e_1(\beta))) \leq e_1(\beta) \quad (e_1 + i)$$

iff

$$e_1(i(e_1(\alpha) \rightarrow e_1(\beta))) \leq e_1(\beta) \quad (\text{assumption on } e_1)$$

iff

$$i(e_1(\alpha) \rightarrow e_1(\beta))) \leq i(e_1(\alpha) \rightarrow e_1(\beta)) \quad (\text{IL4})$$

iff

$$i(e_1(\alpha) \rightarrow e_1(\beta)) \leq i(e_1(\alpha) \rightarrow e_1(\beta)), \quad (e_1 + i)$$

and the last inequality is clearly true. This finishes the proof of $LK_6_A$. To show that $A$ and $K_i$ are isomorphic as Heyting algebras, notice that the domain of $K_i$ is defined as $K_i := \text{Range}(!) = \text{Range}(e_i \circ i)$. Since by assumption $i$ is surjective, $K_i = \text{Range}(e_i)$, and, since $e_i$ is an order-embedding, $K_i$, regarded as a sub-poset of $L$, is order-isomorphic (hence, lattice-isomorphic) to the domain of $A$ with its lattice order. Let $e_i' : K_i \rightarrow L$ and $i' : L \rightarrow K_i$ denote the adjoint maps arising from $!$. Let $e : A \rightarrow K_i$ denote the order-isomorphism between $A$ and $K_i$. Thus, $e_1 = e_i' \circ e$ and $i' = e \circ i$. To finish the proof of item 1, we need to show that for all $\alpha, \beta \in A$,

$$e(\alpha \rightarrow_A \beta) = e(\alpha) \rightarrow_{K_i} e(\beta).$$

The proof of item 2 is similar to that of item 1 and is omitted. As to item 3, to finish the proof it is enough to show that, for all $a, b \in L$,

$$!(a \rightarrow b) \leq ?a \rightarrow ?b.$$  

iff

$$?a \leq !(a \rightarrow b) \rightarrow ?b \quad (\text{IL2 and IL4})$$

iff

$$e_1(y(a)) \leq e_1(i(a \rightarrow b)) \rightarrow e_1(y(b)) \quad (! := e \circ i \text{ and } ? := e_1 \circ y)$$

iff

$$e_1(y(a)) \leq e_1(i(a \rightarrow b) \rightarrow y(b)) \quad (\text{first assumption on } \rightarrow)$$

iff

$$y(a) \leq i(a \rightarrow b) \rightarrow y(b), \quad (e_1 \text{ order-embedding})$$

and the last inequality is true by assumption. The proof of item 4 is order-dual to one of item 3 and is omitted. □

### 3.4 Canonical Extensions of Linear Algebras and their Kernels

In the previous subsection, we showed that heterogeneous algebras for the various linear logics (cf. Definition 3.11) are equivalent presentations of linear algebras with exponentials, and hence can serve as equivalent semantic structures for each linear logic, which can also be taken as the primary semantics. This change in perspective is particularly advantageous when it comes to defining the canonical extension of a linear algebra with exponential(s) in a way that uniformly applies general criteria. Indeed, the canonical extension of a normal (distributive) lattice expansion $A = (L, \mathcal{F}, \mathcal{G})$ (cf. Definition C.3) is defined in a uniform way for any signature as the normal (distributive) lattice expansion $\hat{A} := (L^\delta, \mathcal{F}^\sigma, \mathcal{G}^\pi)$, where $L^\delta$ is the canonical extension of $L$ (cf. Definition C.1), and $\mathcal{F}^\sigma := \{f^\sigma \mid f \in \mathcal{F}\}$ and $\mathcal{G}^\pi := \{g^\pi \mid g \in \mathcal{G}\}$ (cf. Definition C.2).
However, since the exponentials are not normal when regarded as operations on linear algebras, when taking them as primary, we do not have general guidelines in choosing whether to take the $\sigma$- or the $\pi$-extension of each (cf. Definition C.2), given that the $\sigma$-extensions and $\pi$-extensions of exponentials do not coincide in general, and different settings or purposes might provide different reasons to choose one extension over the other. So, we would need to motivate our choice on the basis of considerations that might not easily be portable to other settings (as done in Reference [30]).

In contrast, when defining exponentials as compositions of pairs of adjoint maps, thanks to the fact that adjoint maps are normal (in the sense of Definition C.3) in a lattice-based environment, the general criterion for defining the canonical extensions of normal lattice expansions can be straightforwardly exported to the heterogeneous algebras of Definition 3.11. Following the general guidelines, we take the $\sigma$-extensions of left adjoint maps and the $\pi$-extensions of the right adjoint maps (which are themselves adjoints, by the general theory) and then define the canonical extension of exponentials as the composition of these. Since the adjoint maps are normal and unary, they have the extra benefit of being smooth (that is, their $\sigma$- and $\pi$-extensions coincide; cf. Section C), but this is not essential. The essential aspect is that this definition is not taken on a case-by-case basis, but rather instantiates a general criterion.

As a practical benefit of this defining strategy, we are now in a position to obtain two key properties of the identities and inequalities defining the heterogeneous algebras of Definition 3.11 as instances of general results in the theory of unified correspondence [22–26, 32, 77] for (multi-type) normal (distributive) lattice expansions. Specifically, these inequalities are all of a certain syntactic shape called analytic inductive (cf. Definition B.3). By unified correspondence theory, (analytic) inductive inequalities (1) are canonical (cf. Theorem B.4, Reference [24, Theorem 7.1]), and (2) can be equivalently encoded into analytic rules of a proper display calculus (cf. References [15, 63, Proposition 59]). Property (1) guarantees that the validity of all the identities and inequalities defining the heterogeneous algebras of the lower rows of the diagrams in the statement of Proposition 3.14 below transfers to the heterogeneous algebras in the upper rows of the same diagrams, and hence, the algebraic completeness of each original logical system transfers to the corresponding proper subclass of perfect heterogeneous algebras. Moreover, the heterogeneous algebras in the upper rows are such that all the various maps involved (including $\rightarrow$, $\pi$, and $\sigma$) are residuated in each coordinate, which implies that the display postulates relative to these connectives are sound. Property (2) guarantees that the identities and inequalities defining the heterogeneous algebras of Definition 3.11 can be equivalently encoded into (multi-type) analytic rules, which will form part of the calculi introduced in Section 5.

In what follows, we let $L^\delta$, $A^\delta$, and $B^\delta$ denote the canonical extensions of $L$, $A$, and $B$, respectively.

**Proposition 3.14.** For any algebra $L$,

(1) If $(L, A, e, t)$ is a heterogeneous ILS-algebra (respectively, heterogeneous CLS-algebra), then $(L^\delta, A^\delta, e^\sigma, t^\pi)$ is a perfect heterogeneous ILS-algebra (respectively, CLS-algebra).
(2) If \((L, A, B, e_1, e_7, \gamma)\) is a heterogeneous ILSC-algebra (respectively, heterogeneous BLSC-algebra), then \((L^\delta, A^\delta, B^\delta, e^\sigma, i^\pi, e^\gamma, y^\sigma)\) is a perfect heterogeneous ILSC-algebra (respectively, BLSC-algebra).

(3) If \((L, A, B, e_1, i, e_7, y, \rightarrow\) ) is a heterogeneous ILP-algebra, then
\[
(L^\delta, A^\delta, B^\delta, e^\sigma, i^\pi, e^\gamma, y^\sigma, \rightarrow^\pi)
\]
is a perfect heterogeneous ILP-algebra. Hence,
\[
\rightarrow^\pi : A^\delta \times B^\delta \rightarrow B^\delta
\]
has residuals \(\delta : A^\delta \times B^\delta \rightarrow B^\delta\) and \(\leftarrow : B^\delta \times A^\delta \rightarrow A^\delta\) in each coordinate.

(4) If \((L, A, B, e_1, i, e_7, y, \rightarrow, \rightarrow)\) is a heterogeneous BLP-algebra, then
\[
(L^\delta, A^\delta, B^\delta, e^\sigma, i^\pi, e^\gamma, y^\sigma, \rightarrow^\pi, \rightarrow^\sigma)
\]
is a perfect heterogeneous ILP-algebra. Hence, not only \(\rightarrow^\pi\) has residuals as in the item above, but also
\[
\rightarrow^\sigma : B^\delta \times A^\delta \rightarrow A^\delta\ 
\]
has residuals \(\upsilon : B^\delta \times A^\delta \rightarrow A^\delta\) and \(\leftarrow : A^\delta \times A^\delta \rightarrow B^\delta\) in each coordinate.

**Proof.** As to item 1, it is a basic fact (cf. Section C) that \(e_1 + i\) implies that \(e_1^\sigma + i^\pi\). This in turn implies that \(i^\pi(e_1^\sigma(u)) \leq v\) for every \(v \in A^\delta\). The converse inequality \(u \leq i^\pi(e_1^\sigma(v))\) also holds, since the original inequality \(\alpha \leq i(e(\alpha))\) is valid in \(A\) and is analytic inductive (cf. Definition B.3), and hence canonical (cf. Theorem B.4). The identity \(e_1^\sigma(t_1) = 1\) clearly holds, since \(A\) and \(L\) are subalgebras of \(A^\delta\) and \(L^\delta\), respectively, \(e_1^\sigma\) coincides with \(e_1\) on \(A\) and \(e_1(t_1) = 1\). Finally, the two inequalities \(e_1(\alpha) \otimes e_1(\beta) \leq e_1(\alpha \land \beta)\) and \(e_1(\alpha \land \beta) \leq e_1(\alpha) \otimes e_1(\beta)\) are also analytic inductive, hence canonical, which completes the proof that \((L^\delta, A^\delta, e^\sigma, i^\pi)\) is a heterogeneous ILS-algebra (respectively, CLS-algebra), which is also perfect, since \(L^\delta\) and \(A^\delta\) are perfect (cf. Section C). The remaining items are proved in a similar way, observing that all the inequalities mentioned in these statements are analytic inductive (cf. Definition B.3), hence canonical. The existence of the residuals of \(\rightarrow\) and \(\rightarrow\), as well as the claim that the heterogeneous algebras are perfect, can be argued as follows: by a proof analogous to the proof of the first (respectively, second) item (b) of Proposition 3.10, one shows that \(\rightarrow\) (respectively, \(\rightarrow\)) is finitely meet-preserving (respectively, join-preserving) in its second coordinate and finitely join-reversing (respectively, meet-reversing) in its first coordinate. As discussed in Section C, this implies that \(\rightarrow^\pi\) (respectively, \(\rightarrow^\sigma\)) is completely meet-preserving (respectively, join-preserving) in its second coordinate and completely join-reversing (respectively, meet-reversing) in its first coordinate. Since these maps are defined between complete lattices, this is sufficient to infer the existence of the required residuals. \(\square\)
The following is an immediate consequence of Propositions 3.12, 3.13, and 3.14.

**Corollary 3.15.** For any algebra \( \mathbb{L} \),

1. If \((\mathbb{L}, !)\) is an ILS-algebra (respectively, CLS-algebra), then \( \mathbb{L}^\delta \) can be endowed with the structure of ILS-algebra (respectively, CLS-algebra) by defining \( !^\delta : \mathbb{L} \to \mathbb{L}^\delta \) by \( !^\delta := e_\pi^\delta \circ !^\circ \). Moreover, \( \mathbb{K}_\mathbb{L}^\delta \equiv \mathbb{K}_\mathbb{L}^\circ \).

2. If \((\mathbb{L}, ?!)\) is an ILSC-algebra (respectively, BLSC-algebra), then \( \mathbb{L}^\delta \) can be endowed with the structure of ILSC-algebra (respectively, BLSC-algebra) by defining \( ?^\delta : \mathbb{L} \to \mathbb{L}^\delta \) by \( ?^\delta := e_\pi^\delta \circ ?^\circ \). Moreover, \( \mathbb{K}_\mathbb{L}^\delta \equiv \mathbb{K}_\mathbb{L}^\circ \).

3. If \((\mathbb{L}, !, ?)\) is an ILP-algebra (respectively, BLP-algebra), then \( \mathbb{L}^\delta \) can be endowed with the structure of ILP-algebra (respectively, BLP-algebra) by defining \( !^\delta \) and \( ?^\delta \) as in the items above.

### 4 Multi-Type Hilbert-Style Presentation of Linear Logic

In Section 3.3, the heterogeneous algebras associated with the various linear logics have been introduced (cf. Definition 3.11) and shown to be equivalent presentations of linear algebras with exponentials. These constructions motivate from a semantic perspective the syntactic shift we take in the present section, from the original single-type language to a multi-type language. Indeed, the heterogeneous algebras of Definition 3.11 provide a natural interpretation for the following multi-type language \( \mathcal{L}_{MT} \), defined by simultaneous induction from a given set AtProp of atomic propositions (the elements of which are denoted by letters \( p, q \)):\(^5\)

- Kernel \( \vdash \alpha := (A) \mid t_1 \mid f_1 \mid A \mid A \land \mid A \mid A \to \mid A \mid \alpha \to \alpha \).
- \( ? \)-Kernel \( \models \xi := (A) \mid t_2 \mid f_2 \mid \xi \mid \xi \land \mid \xi \mid \xi \to \mid \xi \mid \alpha \to \xi \).

Linear \( \models A := p \mid e_1(\alpha) \mid e_1(\xi) \mid 1 \mid \bot \mid A \mid A \otimes A \mid A \triangleright A \mid A \to A \mid A \mid \top \mid 0 \mid A \land A \mid A \lor A \).

The interpretation of \( \mathcal{L}_{MT} \)-terms into heterogeneous algebras of compatible signature is defined as the straightforward generalization of the interpretation of propositional languages in algebras of compatible signature.

The toggle between linear algebras with exponentials and heterogeneous algebras (cf. Sections 3.1, 3.2, and 3.3) is reflected syntactically by the following translation \( (\cdot)^\tau : \mathcal{L} \to \mathcal{L}_{MT} \) between the original language(s) \( \mathcal{L} \) of linear logic(s) and (their corresponding multi-type languages) \( \mathcal{L}_{MT} \):

\[
\begin{align*}
p^\tau & = p \\
\tau^\tau & = \top \\
1^\tau & = 1 \\
(A \& B)^\tau & = A^\tau \& B^\tau \\
(A \otimes B)^\tau & = A^\tau \otimes B^\tau \\
(A \triangleright B)^\tau & = A^\tau \triangleright B^\tau
\end{align*}
\]

\( ^5 \)There are clear similarities between \( \mathcal{L}_{MT} \) and the language of Linear Non-Linear logic LNL \( [9] \), given that they both aim at capturing the interplay between the linear and the non-linear behavior. However, there are also differences: For instance, in \( \mathcal{L}_{MT} \), only the Linear type has atomic propositions, whereas in LNL each type has its own atomic propositions. This difference reflects a difference in the aims of Reference \( [9] \) and of the present article: While Reference \( [9] \) aims at studying the environment of adjunction models in their own right, the present article aims at studying Girard’s linear logic and its variants through the lenses of the multi-type environment, and hence focuses on the specific multi-type language adequate for this task. As we will discuss in the following section, we will present a slightly different version of this language, which accounts for the residuals of \( \to \) \& and \( \to \) \& in each coordinate, and the residuals of \( \lor \) \& and \( \lor \) \&. Finally, in the next section, we will use a different notation for the heterogeneous unary connectives, which is aimed at emphasizing their standard proof-theoretic behavior rather than their intended algebraic interpretation.

\( ^6 \)We specify the language corresponding to BLP-algebras, which is the richest signature. The multi-type languages corresponding to the other linear algebras are defined analogously, suitably omitting the defining clauses that are not applicable.
\[(A \rightarrow B)^t = A^t \rightarrow B^t \quad (A \leftrightarrow B)^t = A^t \leftrightarrow B^t \quad (!A)^t = e_{\Gamma}(A^t) \quad (?A)^t = e_{\gamma}(A^t).\]

Not only does the translation \((\cdot)^t : L \rightarrow L_{MT}\) elicit the switch from the single-type language to the multi-type language, but it is also compatible with the underlying toggle between linear algebras with exponentials and their associated heterogeneous algebras. Indeed, for every \(L\)-algebra \(L\), let \(L^*\) denote its associated heterogeneous algebra (cf. Proposition 3.12). The following proposition is proved by a routine induction on \(L\)-formulas, using the deduction-detachment theorem of linear logic:

**Proposition 4.1.** For all \(L\)-formulas \(A\) and \(B\) and every \(L\)-algebra \(L\),

\[L \models A \leq B \iff L^* \models A^t \leq B^t.\]

The main technical difference between the single-type and the multi-type settings is that, while \(!\) and \(?\) are not normal (i.e., their algebraic interpretations are not finitely join-preserving or meet-preserving), all connectives in \(L_{MT}\) are normal, which allows to apply the standard proof-theoretic treatment for normal connectives to them (e.g., to associate each connective to its structural counterpart, have sound display rules), according to the general definitions and results of multi-type algebraic proof theory. In particular, the general definition of analytic inductive inequalities can be instantiated to inequalities in the \(L_{MT}\)-signature (cf. Definition B.3). Hence, we are now in a position to translate the identities and inequalities for the interpretations of the exponentials in linear algebras into \(L_{MT}\) using \((\cdot)^t\) and verify whether the resulting translations are analytic inductive.

\[
\begin{align*}
!(A \& B) &= !A \otimes !B \mapsto e_{\iota}(A \& B) \leq e_{\iota}A \otimes e_{\iota}B \quad (i) \\
&= e_{\iota}A \otimes e_{\iota}B \leq e_{\iota}(A \& B) \quad (ii) \\
!\top &= 1 \mapsto e_{\iota}\top \leq 1 \quad (iii) \\
&= 1 \leq e_{\iota}\top \quad (iv) \\
?(A \oplus B) &= ?A \odot ?B \mapsto e_{\gamma}(A \oplus B) \leq e_{\gamma}A \odot e_{\gamma}B \quad (v) \\
&= e_{\gamma}A \odot e_{\gamma}B \leq e_{\gamma}(A \oplus B) \quad (vi) \\
?0 &= \perp \mapsto e_{\gamma}\gamma0 \leq \perp \quad (vii) \\
&= \perp \leq e_{\gamma}\gamma0 \quad (viii) \\
!(A \rightarrow B) &\leq ?A \rightarrow ?B \mapsto e_{\iota}(A \rightarrow B) \leq e_{\iota}A \rightarrow e_{\gamma}B \quad (ix) \\
&= e_{\iota}A \rightarrow e_{\iota}B \leq e_{\iota}(A \rightarrow B) \quad (x)
\end{align*}
\]

It is easy to see that \((iii)\), \((iv)\), \((vii)\), and \((viii)\) are the only analytic inductive inequalities of the list above. Indeed, recall (cf. Definition B.3) that \(e_{\iota}, \gamma, \otimes, \&\), and \(\rightarrow\) (respectively, \(e_{\gamma}, \iota, \odot, \odot\), and \(\rightarrow\)) are \(F\)-connectives (respectively, \(G\)-connectives), since their interpretations preserve finite joins (respectively, meets) in each positive coordinate and reverse finite meets (respectively, joins) in each negative coordinate.\(^7\) Then \((i)\) and \((ii)\) violate analyticity because of \(\iota\) occurring in the scope of \(e_{\iota}\) in the right-hand side, \((v)\) and \((vi)\) because of \(\gamma\) occurring in the scope of \(e_{\gamma}\) in the left-hand side, and \((ix)\) and \((x)\) because of the subterms \(e_{\gamma}A\) and \(e_{\iota}A\), respectively.

In the light of the general result characterizing analytic inductive inequalities as exactly those equivalently captured by analytic rules of proper display calculi (cf. Reference [63, Propositions 59 and 61]), the failure of the inequalities above to be analytic inductive gives a clear identification of the main hurdle towards the definition of a proper display calculus for linear logic.

\(^7\)Recall that, for the sake of the present article, we have confined ourselves to distributive linear logic, but the failure of analyticity transfers of course also to the non-distributive setting.
However, the order-theoretic analysis developed in Section 3 also provides a pathway to a solution:

**Proposition 4.2.** Each (in)equality in the left column of the following table is semantically equivalent on heterogeneous algebras of the appropriate signature to the corresponding (in)equality in the right column:

| Left Column | Right Column |
|-------------|--------------|
| $e_\iota(A \& B) = e_\iota A \otimes e_\iota B$ | $e_\iota (\alpha \land_1 \beta) = e_\iota \alpha \otimes e_\iota \beta$ |
| $e_\iota T = 1$ | $e_\iota 1 = 1$ |
| $e_\gamma(A \oplus B) = e_\gamma A \ominus e_\gamma B$ | $e_\gamma (\xi \lor_? \chi) = e_\gamma \xi \ominus e_\gamma \chi$ |
| $e_\gamma 0 = \bot$ | $e_\gamma \bot = \bot$ |
| $e_\iota(A \rightarrow B) \leq e_\iota \gamma A \rightarrow e_\iota \gamma B$ | $\gamma A \leq \iota (A \rightarrow B) \rightarrow_\gamma \gamma B$ |
| $e_\iota A \rightarrow e_\iota B \leq e_\iota \gamma (A \rightarrow B)$ | $\gamma (A \rightarrow B) \rightarrow_\gamma \iota B \leq \iota A$. |

**Proof.** The proof of the equivalence of the identities in the rows from the first to the fourth immediately follows from the fact that the map $\iota$ (respectively, $\gamma$) is surjective and preserves finite meets (respectively, joins). The equivalences of the last two rows are shown in the proof of Proposition 3.13. □

The identities and inequalities in the right column of the statement of Proposition 4.2 can be then taken as an alternative multi-type Hilbert-style presentation of the (!, ?)-fragment of) linear logic. Finally, it is easy to verify that these identities and inequalities are all analytic inductive (cf. Definition B.3), and hence can be equivalently encoded into analytic rules of a proper (multi-type) display calculus. In the next section, we introduce the calculi resulting from this procedure.

## 5 PROPER DISPLAY CALCULI FOR LINEAR LOGICS

In the present section, we introduce display calculi for the various linear logics captured by the algebras of Definition 3.1. As is typical of similar existing calculi, the language manipulated by each of these calculi is built up from structural and operational (a.k.a. logical) term constructors. In the tables of Section 5.1, each structural symbol in the upper rows corresponds to one or two logical symbols in the lower rows. The idea, which will be made precise in Section 6.1, is that the interpretation of each structural connective coincides with that of the corresponding logical connective on the left-hand (respectively, right-hand) side (if it exists) when occurring in precedent (respectively, succedent) position.

The language $L_{MT}$ introduced in the previous section and the language introduced in the following subsection are clearly related. However, there are differences. Besides the fact that the language below has an extra layer of structural connectives, the main difference is that the pure kernel-type connectives are represented only at the structural level (as are the heterogeneous binary connectives and their residuals). This choice is in line with the main aim of the present article, which revolves around the original system of linear logic defined by Girard, and its intuitionistic and bi-intuitionistic variants. Accordingly, we include at the operational level only the connectives that are directly involved in capturing original linear formulas. Nonetheless, calculi for the logics of the various heterogeneous algebras would be easily obtainable as variants of the calculi introduced below, just by adding their corresponding standard introduction rules. A complete and cut-free calculus for MELL can be obtained by removing the logical rules introducing additive connectives from the calculus D.LL introduced in Section 5.2.

---

8In the synoptic tables of the next subsection, the operational symbols that are represented only at the structural level will appear between round brackets.
5.1 Language

In the present subsection, we introduce the language of the display calculi for the various linear logics (we will use D.LL to refer to them collectively). Below, we introduce the richest signature, i.e., the one intended to capture the linear logic of BLP-algebras. This signature includes the types Linear, !-Kernel, and ?-Kernel, sometimes abbreviated as L, K!, and K?, respectively.

\[
\begin{align*}
A &::= p | 1 | \bot | A \otimes A | A \oslash A | A \to A | T | 0 | A & A | A \oplus A | \Diamond_! \alpha | \Box_? \xi \\
X &::= A | \Phi | X | X \geq X | I | \circ \Gamma | \circ \Pi \\
\alpha &::= \square_! A \\
\Gamma &::= \alpha | \bullet X | \lozenge_! \Gamma | \lozenge \Gamma | \Pi \triangleright \Gamma | \Pi \triangleright \Pi | \Pi \triangleright \Pi | \Pi
\end{align*}
\]

\[
\begin{align*}
\xi &::= \Diamond_? A \\
\Pi &::= \xi | \bullet_? X | \lozenge_? \Pi | \Pi \triangleright \Pi | \Pi \triangleright \Pi | \Pi \triangleright \Pi | \Pi \triangleright \Pi | \Pi \triangleright \Pi
\end{align*}
\]

Our notational conventions assign different variables to different types, and hence allow us to drop the subscripts of the pure Kernel connectives and of the unary multi-type connectives, given that the parsing of expressions such as \( \Pi > \Pi \) and \( \Diamond_! \alpha \) is unambiguous.

- **Structural and operational pure K-type connectives:**

|                  | \( \Diamond_! \) | \( \Diamond_? \) | \( \lozenge_! \) | \( \lozenge_? \) | \( \triangleright_! \) | \( \triangleright_? \) |
|------------------|------------------|------------------|------------------|------------------|------------------|------------------|
| \( t_! \)        | \( t_? \)        | \( \land_! \)    | \( \land_? \)    | \( \to_! \)      | \( \to_? \)      | \( \to_? \)      |
| \( f_! \)        | \( f_? \)        | \( \lor_! \)     | \( \lor_? \)     | \( \lor_? \)     | \( \lor_? \)     | \( \lor_? \)     |

- **Structural and operational pure L-type connectives:**

|                  | Multiplicative connectives | Additive connectives |
|------------------|-----------------------------|----------------------|
|                  | \( \Phi \)                  | \( I \)               |
| \( I \)          | \( \bot \)                  | \( \otimes \)         |
| \( \otimes \)    | \( \rightarrow \)           | \( \circ \)           |
| \( \rightarrow \) | \( \circ \)                 | \( \ominus \)         |
| \( \ominus \)    | \( \ominus \)               | \( \ominus \)         |

- **Structural and operational unary multi-type connectives:**

|                  | \( \Diamond_! \) | \( \Diamond_? \) | \( \lozenge_! \) | \( \lozenge_? \) | \( \triangleright_! \) | \( \triangleright_? \) |
|------------------|------------------|------------------|------------------|------------------|------------------|------------------|
| \( L \rightarrow K! \) | \( L \rightarrow K? \) | \( K! \rightarrow L \) | \( K? \rightarrow L \) |
| \( \bullet_! \)   | \( \bullet_? \)   | \( \circ_! \)     | \( \circ_? \)     |
| \( \square_! \)   | \( \square_? \)   | \( \Diamond_! \)  | \( \Diamond_? \)  |

The connectives \( \square_! \), \( \lozenge_? \), \( \Diamond_! \), and \( \Box_? \) are interpreted in heterogeneous algebras of appropriate signature as the maps \( t, \gamma, e_! \), and \( e_? \), respectively. Exponentials in the language of D.LL are defined as follows:

\[
\begin{align*}
! A &::= \Diamond_! \bullet_! A \\
? A &::= \Box_? \lozenge_? A
\end{align*}
\]

In what follows, we will omit the subscripts of the unary modalities.
5.2 Rules

In what follows, structures of type Linear are denoted by the variables \(X, Y, Z,\) and \(W\); structures of type !-Kernel are denoted by the variables \(\Gamma, \Delta, \Theta,\) and \(\Lambda;\) structures of type ?-Kernel are denoted by the variables \(\Pi, \Sigma, \Psi,\) and \(\Omega.\) With these stipulations, in the present subsection, we omit the subscripts of pure Kernel-type structural connectives and unary multi-type structural connectives.

**Basic intuitionistic linear environment.**

- Identity and cut rules

\[
\frac{p \vdash p}{Id_L} \quad \frac{X \vdash A \quad A \vdash Y}{X \vdash Y \quad Cut_L}
\]

- Pure Linear-type display rules

\[
\frac{\tau}{\Gamma \vdash \alpha \quad \alpha \vdash \Delta}{\Gamma \vdash \alpha \quad \alpha \vdash \Delta} \quad \frac{\Pi \vdash \xi \quad \xi \vdash \Sigma}{\Pi \vdash \xi \quad \xi \vdash \Sigma}
\]

- Pure Kernel-type display rules

\[
\frac{\Gamma \vdash \Delta \quad \Delta \vdash \Theta}{\Gamma \vdash \Delta \quad \Delta \vdash \Theta} \quad \frac{\Pi \vdash \Psi \quad \Psi \vdash \Sigma}{\Pi \vdash \Psi \quad \Psi \vdash \Sigma}
\]

- Multi-type display rules

\[
\frac{\Gamma \vdash \bullet X}{\tau \vdash \bullet X} \quad \frac{\Gamma \vdash \bullet X}{\tau \vdash \bullet X}
\]

- Pure Linear-type structural rules

**additive**

\[
\frac{Y \vdash I \quad X \vdash Y}{X \vdash Y} \quad \frac{Y \vdash I \quad X \vdash Y}{X \vdash Y}
\]

**multiplicative**

\[
\frac{X \vdash Y}{\Phi, X \vdash Y} \quad \frac{X \vdash Y}{\Phi, X \vdash Y}
\]

\[
\frac{X, Y \vdash Z}{Y, X \vdash Z} \quad \frac{X, Y \vdash Z}{Y, X \vdash Z}
\]

\[
\frac{X, (Y, Z) \vdash W}{(X, Y), Z \vdash W} \quad \frac{X, (Y, Z) \vdash W}{(X, Y), Z \vdash W}
\]

\[
\frac{X \vdash (Y \gg Z, W)}{X \vdash (Y \gg Z, W)} \quad \frac{X \vdash (Y \gg Z, W)}{X \vdash (Y \gg Z, W)}
\]
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- Pure Kernel-type structural rules

**!-Kernel**

\[\begin{align*}
\text{Θ}_1 & \vdash \Delta \\
\quad & \quad \vdash \Delta \\
\text{Ω}_1 & \vdash \Delta; \quad \vdash \Delta
\end{align*}\]

**?-Kernel**

\[\begin{align*}
\text{Θ}_? & \vdash \Pi; \quad \vdash \Pi \\
\quad & \quad \vdash \Pi; \quad \vdash \Pi
\end{align*}\]

\[\begin{align*}
E & \vdash \Delta; \quad \Lambda \quad \vdash \Delta; \\
\quad & \quad \vdash \Delta; \quad \Lambda
\end{align*}\]

**Multi-type structural rules**

\[\begin{align*}
\text{coreg / reg}_L & \vdash \circ \Gamma, \circ \Delta \vdash X \\
\quad & \vdash (\circ \Gamma; \Delta) \vdash X
\end{align*}\]

\[\begin{align*}
\text{reg / coreg}_r & \vdash X \vdash \circ \Pi, \circ \Sigma \\
\quad & \vdash (X \vdash \circ \Pi; \circ \Sigma)
\end{align*}\]

\[\begin{align*}
\text{conec / nec}_L & \vdash \Phi \vdash X \\
\quad & \vdash X \vdash \circ \Phi
\end{align*}\]

- Pure Linear-type operational rules

**additive**

\[\begin{align*}
& 0 \\
& 1 \\
& \top \\
& X \vdash I \quad \vdash 0 \quad 0 \\
& X \vdash 0 \\
& \top \vdash X \\
& X \vdash \top \\
& X \vdash X \\
& X \vdash \top \\
& X \vdash \top \\
& X \vdash \top \\
& A_{1 \in \{1, 2\}} \vdash X \\
& A_1 \land A_2 \vdash X \\
& A \oplus B \vdash X
\end{align*}\]

**multiplicative**

\[\begin{align*}
& \otimes \\
& \Box \\
& \Box \\
& \top \\
& \top \\
& \top \\
& \top \\
& \top \\
& \top \\
& A \vdash X \quad B \vdash Y \\
& A \otimes B \vdash X \quad Y \vdash B
\end{align*}\]

- Operational rules for multi-type unary operators

\[\begin{align*}
K_1 \to L \\
L \to K_1
\end{align*}\]

\[\begin{align*}
& \circ \alpha \vdash X \\
& \Box \alpha \vdash X \\
& \circ \Gamma \vdash \alpha \\
& \Box \Gamma \vdash \alpha
\end{align*}\]

\[\begin{align*}
& A \vdash X \\
& A \vdash \alpha \\
& \Box A \vdash X \\
& \Box A \vdash \alpha
\end{align*}\]

\[\begin{align*}
& X \vdash \circ \xi \\
& X \vdash \circ \xi \\
& \Box \xi \vdash \circ \Pi \\
& \Box \xi \vdash \circ \Pi
\end{align*}\]
Co-intuitionistic and bi-intuitionistic variants. The calculus for the bi-intuitionistic (respectively, co-intuitionistic) variant of linear logic with exponentials is defined by adding (respectively, replacing the introduction rules for $\rightarrow$ with) the following introduction rules in the calculus given above:

$$
\begin{align*}
\Gamma \vdash A \rightarrow B &\vdash Z \\
A &\vdash B \rightarrow Z \\
A \vdash X &\vdash Y \vdash B \\
X &\vdash Y \vdash A \rightarrow B \\
\end{align*}
$$

Paired variants. Paired variants of each calculus given above (i.e., intuitionistic, co-intuitionistic, bi-intuitionistic) are defined by adding one, the other, or both rows of display postulates below (depending on whether one, the other, or both binary maps $\rightarrow$ and $\leftrightarrow$ are part of the definition of the heterogeneous algebras associated with the given linear logic), and, accordingly, one, the other, or both pairs of FS/co-FS rules, corresponding to the defining properties of the maps $\rightarrow$ and $\leftrightarrow$, respectively.

- Display postulates for multi-type binary operators

  $\Gamma \uparrow \Pi \vdash \Sigma$

  $\Pi \vdash \Gamma \downarrow \Sigma$

  $\Gamma \Pi \vdash \Sigma$

  $\Gamma \vdash \Sigma \downarrow \Pi$

- Structural rules corresponding to the pairing axioms

  $\circ \Pi \rightarrow \circ X$

  $\circ (\Pi \leftrightarrow \Gamma) \vdash X$

  $X \vdash \circ (\Gamma \rightarrow \Pi)$

  $\circ \Pi \rightarrow \circ X$

  $\circ (\Pi \leftrightarrow \Gamma) \vdash X$

  $X \vdash \circ (\Gamma \rightarrow \Pi)$

Classical linear variants. The propositional linear base of each calculus introduced above turns classical by adding the following rules:

$$
\begin{align*}
\text{coGi}_m &\vdash (X \rightarrow Y) \cdot Z \rightarrow W \\
&\vdash X \rightarrow (Y, Z) \rightarrow W \\
&\vdash X \rightarrow Y \rightarrow (Z, W) \\
&\vdash X \rightarrow (Y \rightarrow Z) \rightarrow W \\
\text{coGi}_m &\vdash X \rightarrow Y \rightarrow (Z, W) \\
\text{coGi}_m &\vdash X \rightarrow (Y \rightarrow Z) \rightarrow W
\end{align*}
$$

adding which, not only the sequents $(A \rightarrow \bot) \rightarrow \bot \rightarrow A$ become derivable, but also $(A \rightarrow 1) \rightarrow 1 \rightarrow 1 \rightarrow A$.

Relevant and affine variants. Relevant and affine variants of each calculus given above are defined by adding one or the other row of structural rules below:

$$
\begin{align*}
\text{C}_m &\vdash X, X \rightarrow Y \rightarrow X \rightarrow Y \\
&\vdash X \rightarrow Y, Y \rightarrow X \rightarrow Y \\
\text{W}_m &\vdash X \rightarrow Y \rightarrow X \rightarrow Y \\
&\vdash X \rightarrow Y, Z \rightarrow X \rightarrow Y, Z
\end{align*}
$$

Adding both rows would erase the distinction between the multiplicative and the additive behavior.

5.3 Linear Negations as Primitive Connectives

In the present article, we have taken the intuitionistic linear setting as basic and, as is usual in this setting, linear negation (and dual linear negation) are defined connectives. Namely, linear negation $A^{\bot}$ is defined as $A \rightarrow \bot$ and dual linear negation $A^!$ as $A \rightarrow 1$ (cf. Reference [55]). However, one can alternatively stipulate that negation(s) are primitive. In this case, the following pure Linear-type structural and operational connectives need to be added to the language of D.LL:

$$
\begin{align*}
\text{C}_m &\vdash X, X \rightarrow Y \rightarrow X \rightarrow Y \\
&\vdash X \rightarrow Y, Y \rightarrow X \rightarrow Y \\
\text{W}_m &\vdash X \rightarrow Y \rightarrow X \rightarrow Y \\
&\vdash X \rightarrow Y, Z \rightarrow X \rightarrow Y, Z
\end{align*}
$$
In the present subsection, we discuss this alternative.

\textit{(Bi-)intuitionistic linear negations.}

- Display postulates for linear negations
  
  \[
  \text{Gal}_m \quad \frac{\star X \vdash Y}{\star Y \vdash X} \quad \frac{X \vdash Y}{Y \vdash Y} \quad \text{Gal}_m
  \]

- Operational rules for linear negations
  
  \[
  \frac{X \vdash A}{A^\perp \vdash \neg A} \quad \frac{X \vdash \neg A}{X \vdash A^\perp} \quad \frac{A \vdash X}{\neg A \vdash X^\perp} \quad \frac{A^\perp \vdash X}{A \vdash X^\perp}.
  \]

In the calculus extended with the rules above, and the following rules

\[
\text{coimp - left neg} \quad \frac{X \gg \Phi \vdash Y}{\star X \vdash \Phi} \quad \frac{X \vdash Y \gg \Phi}{X \vdash \Phi}\]

the sequents \(A \to \perp \vdash A^\perp\) and \(A \to 1 \vdash A^1\) are then derivable as follows:

- \[
  \frac{A \vdash A}{A \to \perp \vdash \Phi} \quad \frac{A \vdash \Phi}{A \to \perp \vdash \Phi} \quad \frac{A \vdash \Phi}{A \vdash \Phi} \quad \frac{A \vdash \Phi}{A \vdash \Phi} \quad \frac{A \vdash \Phi}{A \vdash \Phi}.
  \]

\textit{Paired variants.} In paired linear logics, either one or both heterogeneous negations \(\alpha \gg t_1 \in K_1\) and \(\xi \gg t_2 \in K_2\) can be defined as \(\alpha \gg t_1\) and \(\xi \gg t_2\), respectively, (and, at the structural level, also the “symmetric” negations \(t_1 \gg \alpha \in K_1\) and \(t_2 \gg \xi \in K_2\) defined as \(t_1 \gg \alpha\) and \(t_2 \gg \xi\), respectively). When these negations are taken as primitive, the following heterogeneous structural and operational connectives need to be added to the language of D.LL:

- \[
  K_1 \to K_2 \quad K_2 \to K_1 \quad K_1 \to K_2 \quad K_2 \to K_1
  \]
- \[
  \Gamma \ominus \Sigma \vdash \Gamma \quad \Sigma \vdash \Omega \]
- \[
  \Gamma \vdash \Gamma \ominus \Sigma \quad \Sigma \vdash \Omega \]
- \[
  \Gamma \vdash \Gamma \ominus \Sigma \quad \Sigma \vdash \Omega \]

- Display postulates for heterogeneous negations

\[
\text{Gal}_? \quad \frac{\Omega \Sigma \vdash \Gamma}{\Gamma \vdash \Omega \Sigma} \quad \frac{\Sigma \vdash \Omega \Gamma}{\Gamma \vdash \Omega \Sigma} \quad \text{Gal}_?.
\]
As usual, we will drop the subscripts, since the reading is unambiguous.

- Operational rules for heterogeneous negations

\[
\begin{align*}
\Gamma \vdash \alpha & \quad \Gamma \vdash \circ \alpha \\
\circ \alpha & \vdash \circ \Gamma & \circ \alpha & \vdash \circ \Gamma
\end{align*}
\]

In the calculus extended with the rules above, and the following rules

\[
\text{dual het neg} \quad \frac{\Pi \vdash \circ \Gamma}{\circ \Pi \vdash \Gamma} \quad \frac{\Pi \vdash \circ \circ \Gamma}{\Pi \vdash \circ \Gamma}
\]

the sequents \( \alpha \rightarrow f \) and \( \xi \rightarrow t \) become derivable (we omit the corresponding derivations). In this language, it becomes possible to formulate the following alternative versions of the FS/coFS rules (which are equivalent to them if the linear implications are defined connectives):

\[
P_{coP_{FL}} \quad \frac{\circ \Pi \land \circ \Gamma \vdash X}{\circ (\circ \Pi ; \Gamma) \vdash X} \quad \frac{\circ \Pi \land \circ \Gamma \vdash X}{\circ \circ \Pi \land \circ \Gamma \vdash X}
\]

**Interdefinable exponentials.** A connection that is at least as strong as the one captured by the P/coP rules is encoded in the following rules:

\[
\text{swap-in / -out} \quad \frac{\circ \circ \Pi \land \circ \Gamma \vdash X}{\circ \circ \Pi \land \circ \Gamma \vdash X} \quad \frac{\circ \circ \Pi \land \circ \Gamma \vdash X}{\circ \circ \Pi \land \circ \Gamma \vdash X}
\]

Indeed, in the presence of the rules above, P and coP are derivable as follows:

\[
\begin{align*}
\ast \circ \Pi \land \circ \Gamma \vdash X & \quad \circ \circ \Pi \land \circ \Gamma \vdash X \\
\circ \circ \Pi \land \circ \Gamma \vdash X & \quad \circ \circ \Pi \land \circ \Gamma \vdash X
\end{align*}
\]

Using swap-out and swap-in one can prove that the following sequents are derivable:

\[
\begin{align*}
A & \vdash A \\
\ast A & \vdash A & \ast A & \vdash \circ A & \ast A & \vdash \circ A
\end{align*}
\]

\[
\begin{align*}
\ast \circ \circ A & \vdash A & \ast \circ \circ A & \vdash A \\
\ast \circ \circ A & \vdash A & \ast \circ \circ A & \vdash A
\end{align*}
\]

\[
\begin{align*}
\ast \circ \circ A & \vdash !A & \ast \circ \circ A & \vdash !A \\
\ast \circ \circ A & \vdash !A & \ast \circ \circ A & \vdash !A
\end{align*}
\]

\[
\begin{align*}
\ast A & \vdash \circ !A & \ast A & \vdash \circ !A \\
\ast A & \vdash \circ !A & \ast A & \vdash \circ !A
\end{align*}
\]

\[
\begin{align*}
?A & \vdash \circ \circ A & \ast \circ \circ A & \vdash \circ \circ !A & \ast \circ \circ A & \vdash \circ \circ !A
\end{align*}
\]
Classical linear negations. When negations are primitive, the classical linear propositional base can be captured by adding the following structural rules:

\[
\begin{align*}
\text{pseudo contr} & \quad \frac{X + Y}{*Y + *X} \\
\text{left neg} & \quad \frac{X + Y, Z}{*Y, X + Z} \quad \frac{X, Y + Z}{Y + *X, Z} \quad \text{right neg}
\end{align*}
\]

The rules above are the counterparts of the classical Grishin rules \(\text{Gri}_m\) and \(\text{coGri}_m\) in the language in which negation is primitive and the two linear implications are defined. In the calculus extended accordingly, the sequents \(A \to B \vdash A^\bot \land B\) and \(A \to B \vdash A^1 \otimes B\) are then derivable as follows:

\[
\begin{align*}
A + A & \quad B + B \\
A, A \to B + B & \\
A \to B + *A, B & \\
A \to B + *B, *A & \\
B \gg A & \to B + *A \\
B \gg A & \to B + A^1 \\
A & \to B + *B, A^1 \\
A & \to B + A^1, *B & \\
A & \to B + A^1 \land B \\
A \to B + A^1 \gg B
\end{align*}
\]

Moreover, left and right negation are interderivable:

\[
\begin{align*}
A + A & \quad B + B \\
A \gg B + A & \to B & \\
B + A, A \to B & & \\
*A + A^1 & \quad B + B & \\
*A, B + A^1 \otimes B & \\
B + A, A^1 \otimes B & \\
A \gg B + A^1 \otimes B & \\
A \to B + A^1 \otimes B & \\
A^1 \otimes B + A & \to B
\end{align*}
\]

Augmenting the paired setting with the following rules:

\[
\begin{align*}
\text{pseudo contr} & \quad \frac{\Gamma \vdash A}{\top \Delta \vdash \top \Gamma} \quad \frac{\Pi \vdash \Sigma}{\top \Sigma \vdash \top \Pi} \quad \text{pseudo contr} \\
\text{swap-in / -out} & \quad \frac{\otimes \bullet X \vdash \Sigma}{\otimes \bullet X \vdash \Sigma} \quad \frac{\bullet X \vdash \Sigma}{\bullet X \vdash \Sigma} \quad \text{swap-in / -out}
\end{align*}
\]
the following sequents become derivable as well:

\[
\begin{align*}
\text{swap-in} & : & A \vdash A & \\
& & \bullet A \vdash \bullet A & \\
& & \circ A \vdash \circ A & \\
& & \circ A \vdash \circ A & \\
\text{def} & : & \Box A \vdash \circ A & \\
& & \circ A \vdash \circ A & \\
& & \circ A \vdash \circ A & \\
& & \circ A \vdash \circ A & \\
& & \circ A \vdash \circ A & \\
\end{align*}
\]

6 PROPERTIES

6.1 Soundness

In the present subsection, we discuss the verification of the soundness of the rules of D.LL w.r.t. the semantics of perfect heterogeneous algebras (cf. Definition 3.11). This verification specializes and applies the steps of the argument for the soundness of the proper display calculi for (D)LE logics discussed in Reference [63, Section 4.2.1]. Namely, the first step consists in interpreting structural symbols as logical symbols according to their (precedent or succedent) position, as indicated in the synoptic tables of Section 5.1. This makes it possible to interpret sequents as inequalities, and rules as quasi-inequalities in the language of the corresponding (perfect) heterogeneous algebras that provide the intended semantics of the given multi-type logic. The next step consists in showing that each quasi-inequality obtained in the first step is semantically equivalent to some analytic inductive (cf. Appendix B) inequality that is valid in the above-mentioned class of heterogeneous algebras; the proof of these semantic equivalences can be obtained using the algorithm ALBA, as discussed in Reference [63, Section 3]. The final step consists in verifying that, for those analytic inductive inequalities mentioned in the second step in which multi-type connectives occur that are involved in the translation of exponentials in the original language, these analytic inductive axioms are in turn semantically equivalent to the translations of the original linear logic axioms. The approach taken in the present article is precisely about showing that, while simply translating the axioms of linear logic into the multi-type language does not yield axioms of a syntactic shape (i.e., the analytic inductive shape, cf. Appendix B) that makes them amenable to being captured by a proper display calculus, the crucial step of Proposition 4.2 essentially says that the information contained in these axioms can be equivalently expressed by analytic inductive axioms.

For any sequent \( x \vdash y \), we define the signed generation trees \( +x \) and \( -y \) by labelling the root of the generation tree of \( x \) (respectively, \( y \)) with the sign \( + \) (respectively, \( - \)) and then propagating the sign to all nodes according to the polarity of the coordinate of the connective assigned to each node. Positive (respectively, negative) coordinates propagate the same (respectively, opposite) sign to the corresponding child node. Then, a substructure \( z \) in \( x \vdash y \) is in precedent (respectively, succedent) position if the sign of its root node as a subtree of \( +x \) or \( -y \) is \( + \) (respectively, \( - \)).
Let us illustrate how these steps are carried out on some heterogeneous rules of D.LL (the procedure works analogously on the remaining ones). Below, we illustrate the first step, in which the rules on the left-hand side are interpreted as the quasi-inequalities on the right-hand side:

\[
\begin{align*}
\text{coFS}_I & : \frac{\sigma \Pi \Rightarrow \sigma \Gamma + \mathcal{X}}{\sigma (\Pi \gg \Gamma) + \mathcal{X}} \quad \implies \quad \forall \xi \forall \alpha \forall c [\square \xi \to \diamond \alpha \leq c \Rightarrow \diamond (\xi \gg \alpha) \leq c]
\end{align*}
\]

\[
\begin{align*}
\text{FS}_I & : \frac{X + \sigma \Pi \Rightarrow \sigma \Pi \gg \Pi}{X + \sigma (\Pi \gg \Pi)} \quad \implies \quad \forall \xi \forall \alpha \forall c [c \leq \diamond \alpha \to \square \xi \Rightarrow c \leq \square (\alpha \rightarrow \xi)]
\end{align*}
\]

\[
\begin{align*}
\text{reg}_I & : \frac{X + \sigma \Pi, \sigma \Sigma}{X + \sigma (\Pi ; \Sigma)} \quad \implies \quad \forall c \forall \xi \forall \chi [c \leq \square \xi \land \square \chi \Rightarrow c \leq \square (\xi \lor \chi)]
\end{align*}
\]

\[
\begin{align*}
\text{core}_I & : \frac{\sigma \Gamma, \sigma \Delta + \mathcal{X}}{\sigma (\Gamma ; \Delta) + \mathcal{X}} \quad \implies \quad \forall \alpha \forall \beta \forall c [\diamond \alpha \land \diamond \beta \leq c \Rightarrow (\alpha \land \beta) \leq c]
\end{align*}
\]

Let us show that the quasi-inequalities above are valid in the appropriate classes of perfect heterogeneous algebras (second and third steps). As to the second step, let us show, via ALBA reductions, that the quasi-inequalities above are semantically equivalent to analytic inductive inequalities in the multi-type language of (appropriate classes of) heterogeneous algebras:

\[
\forall a \forall b [\bullet (a \rightarrow b) \rightarrow \bullet b \leq \bullet a]
\]

\[
\begin{align*}
\text{iff} & \quad \forall \xi \forall \alpha \forall \forall c [\bullet (a \rightarrow b) \leq \xi \land \alpha \leq \bullet b \land \alpha \leq c] \Rightarrow \xi \rightarrow \alpha \leq \bullet c
\end{align*}
\]

\[
\begin{align*}
\text{iff} & \quad \forall \xi \forall \alpha \forall \forall c [\bullet (c \rightarrow b) \leq \xi \land \alpha \leq \bullet b] \Rightarrow \xi \rightarrow \alpha \leq \bullet c
\end{align*}
\]

\[
\begin{align*}
\text{iff} & \quad \forall \xi \forall \alpha \forall \forall c [\bullet (c \rightarrow \diamond \alpha) \leq \xi \Rightarrow \xi \rightarrow \alpha \leq \bullet c]
\end{align*}
\]

\[
\begin{align*}
\text{iff} & \quad \forall \xi \forall \alpha \forall \forall c \rightarrow \diamond \alpha \leq \square \xi \Rightarrow \diamond (\xi \rightarrow \alpha) \leq c.
\end{align*}
\]

\[
\begin{align*}
\forall a \forall b [\bullet a \leq \bullet (a \rightarrow \longrightarrow \bullet b)]
\end{align*}
\]

\[
\begin{align*}
\text{iff} & \quad \forall \xi \forall \alpha \forall \forall c [\bullet (c \rightarrow b) \leq \xi \land \alpha \leq \bullet c \land \alpha \leq c] \Rightarrow \xi \rightarrow \alpha \leq \bullet c
\end{align*}
\]

\[
\begin{align*}
\text{iff} & \quad \forall \xi \forall \alpha \forall \forall c [\bullet (c \rightarrow \diamond \alpha) \leq \xi \Rightarrow \xi \rightarrow \alpha \leq \bullet c]
\end{align*}
\]

\[
\begin{align*}
\text{iff} & \quad \forall \xi \forall \alpha \forall \forall c \rightarrow \diamond \alpha \leq \square \xi \Rightarrow \diamond (\alpha \rightarrow \xi) \leq c.
\end{align*}
\]

\[
\begin{align*}
\forall \xi \forall \chi [\square \xi \land \square \chi \leq \square (\xi \lor \chi)]
\end{align*}
\]

\[
\begin{align*}
\text{iff} & \quad \forall \alpha \forall \forall c [\bullet (a \rightarrow \longrightarrow \bullet c) \leq \square \xi \land \square \chi \Rightarrow a \leq \square (\xi \lor \chi)].
\end{align*}
\]

\[
\begin{align*}
\forall c \forall \xi \forall \forall c \leq \diamond c \rightarrow \bullet \alpha \rightarrow \square \xi \Rightarrow c \leq \bullet (a \rightarrow \xi).
\end{align*}
\]

\[
\begin{align*}
\forall \xi \forall \forall c \leq \diamond a \rightarrow \bullet b \rightarrow \bullet c \leq \bullet (\alpha \rightarrow \xi),
\end{align*}
\]

It can readily be checked that the ALBA rewriting rules applied in the computations above (adjunction rules and Ackermann rules) are sound on perfect heterogeneous algebras. As discussed in Reference [63], the soundness of these rewriting rules only depends on the order-theoretic properties of the interpretation of the logical connectives and their adjoints and residuals. The fact that some of these maps are not internal operations but have different domains and codomains does not make any substantial difference.

The third step immediately follows from the fact that the axioms in the first lines of each of the computations above appear in the right column of the statement of Proposition 4.2 and are valid by construction on heterogeneous algebras of suitable similarity type (this readily follows from the results in Sections 3.1, 3.2 and 3.3), and hence in particular on perfect heterogeneous algebras.
6.2 Completeness

Completeness of proper display calculi follows straightforwardly from their general theory. Specifically, in Reference [15] it was shown that any analytic inductive axiom in arbitrary (single-type) logical signature can be derived in the basic proper display calculus augmented with the analytic structural rules generated from it with the methodology introduced in Reference [63], which is the same methodology used in the present article to generate the structural analytic rules. The completeness result in Reference [15] straightforwardly extends to the multi-type setting, readily yielding the completeness of D.LL.

However, in the present subsection, we take a more concrete route to this result and show that the translations of the axioms and rules of Girard’s calculus for linear logic (cf. Reference [49]) are derivable in D.LL. Since Girard’s calculus is complete w.r.t. the appropriate class of perfect linear algebras, and hence w.r.t. their associated perfect heterogeneous algebras, this is enough to show the completeness of the version of D.LL corresponding to Girard’s calculus.10

The derivations of axioms and rules not involving exponentials are standard and we omit them. In the remainder of the present subsection, we focus on the rules involving exponentials, namely,

- left (respectively, right) dereliction and right (respectively, left) promotion rules:

\[
\begin{align*}
X, A \vdash Y & \quad X, !A \vdash Y \\
X, !A \vdash Y & \quad X, A \vdash Y \\
!X, A, ?Y & \quad !X, A, ?Y \\
!X, A \vdash ?Y & \quad !X, A \vdash ?Y
\end{align*}
\]

- left (respectively, right) weakening and left (respectively, right) contraction rules:

\[
\begin{align*}
X \vdash Y & \quad X \vdash Y \\
X, !A \vdash Y & \quad X, !A \vdash Y \\
X, !A \vdash Y & \quad X, !A \vdash Y \\
X \vdash ?A, ?A, Y & \quad X \vdash ?A, ?A, Y
\end{align*}
\]

where, abusing notation, !X and ?Y denote structures that are built from formulas of the form !A and ?B, respectively, using only the structural counterpart of □ and □.

Translating these rules in the language of D.LL, we obtain:

\[
\begin{align*}
X, A \vdash Y & \quad X, !A \vdash Y \\
X \vdash □A, Y & \quad X \vdash □A, Y \\
□X, □A \vdash □Y & \quad □X, □A \vdash □Y
\end{align*}
\]

where, abusing notation, □X and □Y denote structures that are built from formulas of the form □A and □B, respectively, using only the structural counterpart of □ and □.

- Derivations of left- and right-dereliction:

\[
\begin{align*}
X, A \vdash Y & \quad X \vdash A, Y \\
A \vdash X \Rightarrow Y & \quad Y \Rightarrow X \vdash A \\
□A \vdash □(X \Rightarrow Y) & \quad □Y \Rightarrow X \vdash □A \\
□A \vdash □X \Rightarrow Y & \quad □Y \Rightarrow □X \vdash □A
\end{align*}
\]

\[
\begin{align*}
X, □A \vdash Y & \quad X \vdash □A, Y \\
X \vdash □A, Y & \quad X \vdash □A, Y
\end{align*}
\]

10In Section F, we derive the axioms of the Hilbert-style presentations of intuitionistic linear logic. In a similar fashion, it is possible to transfer the completeness of the Hilbert-style presentation of each variant of linear logic w.r.t. its associated class of perfect heterogeneous algebras to the associated proper display calculus. These results are special instances of the general syntactic completeness result proved in Reference [15]. For the sake of self-containment and for providing concrete examples of how the calculus works, we include some derivations here.
• Derivations of left- and right-weakening:

\[
\begin{array}{c}
W_m \\
\frac{X \vdash Y}{\diamond A \vdash X \gg Y} \\
\frac{\diamond A \vdash X \gg Y}{X, \diamond A \vdash Y} \\
\end{array}
\quad
\begin{array}{c}
W_m \\
\frac{X \vdash Y}{X, \diamond A \vdash Y} \\
\frac{X \vdash Y, \circ A}{X \vdash Y, \circ A} \\
\end{array}
\]

For the purpose of showing that the promotion and contraction rules are derivable, it is enough to show that the following rules are derivable in D.LL:

\[
\begin{array}{c}
X, (\circ \bullet A, \circ \bullet A) \vdash Y \\
X \vdash (\circ \bullet A, \circ \bullet A), Y \\
\circ \Gamma \vdash A, \circ \Pi \\
\circ \Gamma \vdash \diamond \bullet A, \circ \Pi \\
\circ \Gamma, \circ \bullet A \vdash \circ \Pi \\
\end{array}
\quad
\begin{array}{c}
X \vdash Y, \circ \bullet A \\
Y \vdash X + \circ \bullet A \\
Y \gg X + \circ \bullet A \\
X \vdash Y, \quad \Box A \\
X \vdash \Box A, Y \\
\end{array}
\]

Indeed, as discussed in Section 6.3 Conservativity, left-introduction rules for \(\mathcal{F}\)-connectives (such as diamonds) and right-introduction rules for \(\mathcal{G}\)-connectives (such as boxes) are invertible. Hence, if, e.g., \(\diamond \bullet X + A, \Box \bullet Y\) is derivable, using the inversion lemmas, associativity, exchange, regrouping, and display, one can show that \(\circ \Gamma \vdash A, \circ \Pi\) is derivable for some \(\Gamma\) and \(\Pi\). Also, if \(\circ \Gamma \vdash \diamond \bullet A, \circ \Pi\) is derivable, then using associativity, exchange, coregrouping, display, and box- and diamond-introduction rules, one can show that \(\diamond \bullet X \vdash \diamond \bullet A, \Box \bullet Y\) is derivable. In what follows, we show that the rules displayed above are derivable, which completes the proof that the promotion rules are derivable.

• Derivations of left- and right-promotion:

\[
\begin{array}{c}
\diamond \bullet X, A \vdash \Box \bullet Y \\
\circ \Gamma, A \vdash \circ \Pi \\
\hline
\end{array}
\quad
\begin{array}{c}
\diamond \bullet X, A \vdash \Box \bullet Y \\
\circ \Gamma \vdash A, \circ \Pi \\
\hline
\end{array}
\]

\[
\begin{array}{c}
A \vdash \circ \Gamma \gg \circ \Pi \\
A \vdash (\circ \Gamma \gg \circ \Pi) \quad \text{coFS} \\
\hline
\end{array}
\quad
\begin{array}{c}
\circ \Pi \gg \circ \Gamma \vdash A \\
\circ \Pi \vdash \circ \Gamma \gg \circ \Pi \\
\hline
\end{array}
\]

\[
\begin{array}{c}
\circ \Gamma \gg \circ \Pi \vdash \circ \Gamma \\
\circ \Gamma \vdash \circ \Pi \gg \circ \Pi \quad \text{coFS} \\
\hline
\end{array}
\quad
\begin{array}{c}
\circ \Gamma \vdash \circ \Pi \gg \circ \Pi \\
\circ \Gamma \vdash \circ \Pi \gg \circ \Pi \\
\hline
\end{array}
\]

\[
\begin{array}{c}
\circ \Pi \gg \circ \Gamma \vdash \circ \Pi \gg \circ \Gamma \\
\circ \Pi \vdash \circ \Pi \gg \circ \Gamma \\
\hline
\end{array}
\quad
\begin{array}{c}
\circ \Gamma \vdash \circ \Pi \gg \circ \Gamma \\
\circ \Gamma \vdash \circ \Pi \gg \circ \Gamma \\
\hline
\end{array}
\]

\[
\begin{array}{c}
\circ \Gamma \gg \circ \Pi \vdash \circ \Gamma \gg \circ \Pi \\
\circ \Gamma \gg \circ \Pi \vdash \circ \Gamma \gg \circ \Pi \\
\hline
\end{array}
\quad
\begin{array}{c}
\circ \Gamma \gg \circ \Pi \vdash \circ \Gamma \gg \circ \Pi \\
\circ \Gamma \gg \circ \Pi \vdash \circ \Gamma \gg \circ \Pi \\
\hline
\end{array}
\]

\[
\begin{array}{c}
\circ \Gamma \gg \circ \Pi \vdash \circ \Gamma \gg \circ \Pi \\
\circ \Gamma \gg \circ \Pi \vdash \circ \Gamma \gg \circ \Pi \\
\hline
\end{array}
\quad
\begin{array}{c}
\circ \Gamma \gg \circ \Pi \vdash \circ \Gamma \gg \circ \Pi \\
\circ \Gamma \gg \circ \Pi \vdash \circ \Gamma \gg \circ \Pi \\
\hline
\end{array}
\]

6.3 Conservativity

To argue that (each) calculus introduced in Section 5 adequately captures its associated linear logic, we follow the standard proof strategy discussed in References [58, 63]. Let LL denote a Hilbert-style presentation of (one of the variants of) linear logic (viz. those given in References [76, 81]); let \(\vdash_{LL}\)}
denote the syntactic consequence relation arising from LL, and let \( \models_{HA_{LL}} \) denote the semantic consequence relation arising from (perfect) heterogeneous LL-algebras. We need to show that, for all formulas \( A \) and \( B \) of the original language of linear logic, if \( A' \vdash B' \) is a D.LL-derivable sequent, then \( A \vdash_{LL} B \). This claim can be proved using the following facts: (a) the rules of D.LL are sound w.r.t. perfect heterogeneous LL-algebras (cf. Section 6.1), (b) LL is strongly complete w.r.t. perfect LL-algebras, and (c) perfect LL-algebras are equivalently presented as perfect heterogeneous LL-algebras (cf. Section 3.3), so the semantic consequence relations arising from each type of structures preserve and reflect the translation (cf. Proposition 4.1). Then, let \( A, B \) be formulas of the original LL-language. If \( A' \vdash B' \) is a D.LL-derivable sequent, then, by (a), \( A' \models_{HA_{LL}} B' \). By (c), this implies that \( A \models_{LL} B \), where \( \models_{LL} \) denotes the semantic consequence relation arising from (perfect) LL-algebras. By (b), this implies that \( A \vdash_{LL} B \), as required.

### 6.4 Cut Elimination and Subformula Property

In the present section, we outline the proof of cut elimination and subformula property for the calculi introduced in Section 5. As discussed earlier on, these calculi have been designed so the cut elimination and subformula property do not need to be proved via the original argument by Gentzen, but can rather be inferred from Theorem A.2, a Belnap-style meta-theorem that is the version restricted to proper display calculi of the Belnap-style metatheorem proved in Reference [42]. The meta-theorem to which we will appeal for the calculi of Section 5 was proved in Reference [42], and in Appendix A, we report on a restricted version of it (cf. Theorem A.2) that specifically applies to proper multi-type display calculi (cf. Definition A.1).

By Theorem A.2, it is enough to verify that the calculi of Section 5 meet the conditions listed in Definition A.1. All conditions except \( C_8 \) are readily satisfied by inspecting the rules. In what follows, we verify \( C_3 \) (see Appendix A). This requires to check that reduction steps are available for every application of the cut rule in which both cut-formulas are principal, which either remove the original cut altogether or replace it by one or more cuts on formulas of strictly lower complexity.

**Atomic propositions:**

\[
\frac{p \vdash p}{p \vdash p} \quad \frac{p \vdash p}{p \vdash p} \quad \Rightarrow \quad \frac{p \vdash p}{p \vdash p}.
\]

**Constants:**

\[
\frac{\vdash \pi_1}{\Phi \vdash X} \quad \frac{\Phi \vdash \pi_1 \vdash X}{\Phi \vdash X} \quad \Rightarrow \quad \frac{\Phi \vdash X}{\Phi \vdash X}.
\]

The cases for \( \bot, \top, 0 \) are standard and similar to the one above.

**Binary connectives monotone in each coordinate:**

\[
\frac{\vdash \pi_2}{Y \vdash B} \quad \frac{A, B \vdash Z}{\overline{A \Rightarrow Z}} \quad \frac{Y \vdash A}{B \vdash A \Rightarrow Z}.
\]

\[
\frac{\vdash \pi_3}{A \vdash B} \quad \frac{A, B \vdash Z}{\overline{A \Rightarrow Z}} \quad \frac{A \vdash Y}{A \Rightarrow Z}.
\]

\[
\frac{\vdash \pi_1}{X \vdash A} \quad \frac{Y \vdash B}{X \vdash A \otimes B} \quad \frac{A, B \vdash Z}{A \otimes B \vdash Z} \quad \frac{X \vdash Y}{X \vdash Y} \quad \frac{X \vdash Z}{X \vdash Y}.
\]

\[
\frac{\vdash \pi_3}{X, Y \vdash Z} \quad \frac{\overline{Y, X \vdash Z}}{X, Y \vdash Z}.
\]
The cases for $A \otimes B$, $A \& B$, $A \multimap B$, $A \oplus B$ are standard and similar to the one above.

**Binary connectives with some antitone coordinate:**

\[
\begin{array}{c}
\pi_1 : X \vdash A \multimap B \\
\pi_2 : A, X \vdash B \\
\pi_3 : Y \vdash A \multimap B \\
\pi_3 : Y, X \vdash B \\
\end{array}
\]

\[
\begin{array}{c}
\vdash \pi_1 : X \vdash A \multimap B \\
\vdash \pi_2 : A, X \vdash B \\
\vdash \pi_3 : Y \vdash A \multimap B \\
\vdash \pi_3 : Y, X \vdash B \\
\end{array}
\]

\[
\begin{array}{c}
\pi_1 : X \vdash A \multimap B \\
\pi_2 : A, X \vdash B \\
\pi_3 : Y \vdash A \multimap B \\
\pi_3 : Y, X \vdash B \\
\end{array}
\]

\[
\begin{array}{c}
\vdash \pi_1 : X \vdash A \multimap B \\
\vdash \pi_2 : A, X \vdash B \\
\vdash \pi_3 : Y \vdash A \multimap B \\
\vdash \pi_3 : Y, X \vdash B \\
\end{array}
\]

The case for $A \multimap B$ is standard and similar to the one above.

**Unary multi-type connectives:**

\[
\begin{array}{c}
\pi_1 : \Gamma \vdash \alpha \\
\pi_2 : \alpha \vdash X \\
\pi_3 : \alpha \vdash X \\
\pi_4 : \alpha \vdash X \\
\pi_5 : \alpha \vdash X \\
\end{array}
\]

\[
\begin{array}{c}
\vdash \pi_1 : \Gamma \vdash \alpha \\
\vdash \pi_2 : \alpha \vdash X \\
\vdash \pi_3 : \alpha \vdash X \\
\vdash \pi_3 : \alpha \vdash X \\
\vdash \pi_3 : \alpha \vdash X \\
\end{array}
\]

\[
\begin{array}{c}
\pi_1 : \Gamma \vdash \alpha \\
\pi_2 : \alpha \vdash X \\
\pi_3 : \alpha \vdash X \\
\pi_3 : \alpha \vdash X \\
\pi_3 : \alpha \vdash X \\
\end{array}
\]

\[
\begin{array}{c}
\vdash \pi_1 : \Gamma \vdash \alpha \\
\vdash \pi_2 : \alpha \vdash X \\
\vdash \pi_3 : \alpha \vdash X \\
\vdash \pi_3 : \alpha \vdash X \\
\vdash \pi_3 : \alpha \vdash X \\
\end{array}
\]

The cases for $\Box \xi$ and $\diamondsuit A$ are standard and similar to the ones above.

7 APPLICATIONS

7.1 Structural Control

We have argued that, because modularity is built in proper display calculi, embedding the proof theory of linear logic in the framework of proper display calculi helps to make the connections between linear logic and other neighboring logics more systematic. Specifically, by adding the appropriate analytic rules to the version of D.LL corresponding to each linear logic (intuitionistic, bi-intuitionistic, and classical) considered in this article, we can capture, e.g., the affine and relevant counterparts of each linear logic, while preserving all properties of the basic systems. Moreover, fragments and expansions of linear logic can be captured in the same way, thus creating a framework that accounts for substructural logics. Finally, each of these calculi can be further embedded in richer calculi, for instance to obtain proper display calculi for dynamic epistemic logics on substructural propositional bases.

Establishing these connections is very useful for transferring techniques, insights, and results from one logical setting to another. In the present section, we give one example of this transfer of techniques from linear logic to categorial grammar in linguistics.

In categorial grammar, the proof-theoretic framework of Lambek calculus and some of its extensions are used for generating grammatically well-formed sentences in natural language by
means of logical derivations. One crucial problem in this area is accounting for the fact that grammar rules often admit exceptions, understood as rules that yield grammatically non-well-formed constructions if applied unrestrictedly, but grammatically well-formed sentences if applied in a controlled way. In Reference [70], Kurtonina and Moortgat propose a proof-theoretic framework that accounts for the controlled application of associativity, commutativity, and their combination (among others). Their proposal is conceptually akin to the exponentials in linear logic. Indeed, the basic language of their proposal is an expansion of the basic Lambek calculus with two modal operators adjacent to one another, inspired to the modal operators into which ! decomposes. In fact, the requirements on the modal operators of Reference [70] would perfectly match the multi-type modal operators introduced in the present article, were it not for the fact that they are captured algebraically as operations internal to an FL-algebra, rather than having different algebras as domain and codomain. That is, Kurtonina and Moortgat adopt a single-type environment. In what follows, we recast (a fragment of) Kurtonina and Moortgat’s framework for structural control in a multi-type setting.

We consider three different multi-type environments, each of which includes two types: a General type (corresponding to Lambek calculus), and a Special type, corresponding to associative, commutative, and associative+commutative Lambek calculus, respectively. The three environments have the same language, specified as follows:

\[
\text{General} \ni A ::= p \mid \Diamond \alpha \mid A \otimes A \mid A \rightarrow A \mid A \leftarrow A
\]

\[
\text{Special} \ni \alpha ::= \Box A \mid \alpha \otimes \alpha \mid \alpha \rightarrow \alpha \mid \alpha \leftarrow \alpha.
\]

The language above is interpreted into algebraic structures \((L, A, \Box, \Diamond)\) such that:

- FL1. \(L = (L, \leq, \otimes, \rightarrow, \leftarrow)\) is a partially ordered algebra;
- FL2. \(a \otimes b \leq c\) iff \(a \leq c \leftarrow b \) iff \(b \leq a \rightarrow c\) for all \(a, b, c \in L\);
- FL3. \(A = (A, \leq, \otimes, \rightarrow, \leftarrow)\) is a partially ordered algebra;
- FL4. \(\alpha \circ \beta \leq \gamma\) iff \(\alpha \leq \gamma \rightarrow \beta\) iff \(\beta \leq \alpha \leftarrow \gamma\) for all \(\alpha, \beta, \gamma \in A\);
- FL5. \(\Box : L \rightarrow A\) and \(\Diamond : A \rightarrow L\) are such that \(\Diamond \Diamond A\) and \(\Box \Box = \text{Id}_L\);
- FL6. \(\Diamond \alpha \otimes \Box \beta = \Diamond (\alpha \circ \beta)\) for all \(\alpha, \beta \in A\).

Structures \((L, A, \Box, \Diamond)\) satisfying FL1-FL6 will be referred to as heterogeneous FL-algebras. Any such structure is associative if in addition

- FL7. \(\alpha \circ (\beta \circ \gamma) = (\alpha \circ \beta) \circ \gamma\) for all \(\alpha, \beta, \gamma \in A\),

and is commutative if

- FL8. \(\alpha \circ \beta = \beta \circ \alpha\) for all \(\alpha, \beta \in A\).

With an argument similar to the one given in Proposition 3.13, one shows that any heterogeneous FL-algebra gives rise to an algebra \(\mathbb{L} = (L, \leq, \otimes, \rightarrow, \leftarrow, !)\) such that \(\mathbb{L} = (L, \leq, \otimes, \rightarrow, \leftarrow)\) satisfies FL1 and FL2, and \(! : L \rightarrow L\) defined as \(! a : = \Diamond \Box a\) satisfies S2 and S3. With the help of ! (or, equivalently, of \(\Diamond \Box\)), the controlled commutativity and associativity in \(\mathbb{L}\) can be expressed as follows:

\[!A \otimes (!B \otimes !C) = (!A \otimes !B) \otimes !C \quad \text{and} \quad !A \otimes !B = !B \otimes !A,\]

which corresponds to the full internal labelling discussed in Reference [70, Section 3.2]. A basic multi-type display calculus D.FL can be straightforwardly introduced along the lines of D.LL in the following language:
The rules of D.FL include identity axioms for atomic formulas (of General type), cut rules for both types, display postulates for the pure-type connectives modelled on conditions FL2 and FL4, display postulates for the multi-type connectives modelled on FL5, standard introduction rules for all connectives,\(^{11}\) and the following regrouping/co-regrouping rule, which captures FL6:

\[
\frac{\circ \Theta, \circ \Delta \vdash X}{\circ (\Gamma; \Delta) \vdash X}
\]

The associative, commutative, and associative + commutative extensions of the basic calculus D.FL are, respectively, defined by adding one, the other, or both of the following rules, which hold unrestricted in the appropriate Special type:

\[
\begin{align*}
A_s \quad & \frac{\Gamma; (\Delta; \Theta) \vdash \Lambda}{(\Gamma; \Delta); \Theta \vdash \Lambda} \\
E_s \quad & \frac{\Gamma; \Delta \vdash X}{\Delta; \Gamma \vdash X}
\end{align*}
\]

Then, the appropriate extension of D.FL derives the restricted associativity and commutativity holding in the General type using the interaction between regrouping and co-regrouping, display rules, and the unrestricted associativity and commutativity holding in the Special type:

### 7.2 Intermediate Linear Logics

*Gödel-Dummett logic.* Among non-classical logics, the family of intermediate logics (i.e., the logics that are stronger than intuitionistic logic but weaker than classical logic) is one of the earliest introduced and most thoroughly studied. A very well-known member of this family is the **Gödel-Dummett logic**, obtained by extending intuitionistic logic with the following prelinearity

\(^{11}\)i.e., the left-introduction rules for \(\otimes\), \(\odot\), and \(\circ\) are invertible and right-introduction rules of the remaining connectives are invertible.
This logic was introduced by Gödel in Reference [54] as the limit of an infinite chain of progressively weaker logics between classical and intuitionistic logic. The existence of this chain was used by Gödel to refute the finite-valuedness\(^\text{12}\) of intuitionistic logic. Dummett [34] later axiomatized this logic, which was further studied in connection with, e.g., the provability logic of Heyting’s arithmetics [83], relevance logic [37], and many-valued logics [66]. A rich mathematical theory has been developed over the years for Gödel-Dummett logic: Its algebraic semantics is given by the variety of Gödel algebras, which is locally finite, and its subdirectly irreducible elements are chains. This variety is generated by any infinite Heyting chain, and its proper subvarieties are precisely the varieties generated by finite chains [67]. Finite Gödel algebras are dually equivalent to finite forests, occurring ubiquitously as data structures; hence, the importance of this logic in theoretical computer science.

**Proof calculi for intermediate logics.** It is impossible to capture the prelinearity axiom by analytic structural rules in any Gentzen calculus (cf. Reference [16, Section 5]). Hence, no Gentzen calculus can possibly exist for Gödel-Dummett logic. This situation is almost universal for intermediate logics and has provided the motivation for introducing more powerful calculi than Gentzen calculi. For instance, axiom (1) can be captured by analytic structural rules in hypersequent calculi [4] and display calculi [17], among others. Moreover, display calculi capture all the intermediate logics defined by analytic inductive axioms [63, Definition 55]. The structural display rule capturing axiom (1) is:

\[
\begin{array}{c}
X \vdash Z \quad W \vdash Y \\
I \vdash (X > Y); (W > Z)
\end{array}
\]

Gödel-Dummett linear logic and other intermediate linear logics. Classical linear logic and intuitionistic linear logic have been studied very intensively over the years and have given rise to a wide range of variants. The development of systematic methods to capture and study these variants has become the focus of much recent work in this field [74]. Also in the context of these results, it is very natural to conjecture that systematically exploring the logical space between ILL and CLL would yield a theory of intermediate linear logics of comparable or even greater richness than that of intermediate logics.

This pursuit would greatly increase the flexibility and scope of the overall linear logic framework in capturing the interaction between actions and situations (cf. Reference [52, Section 1.1.1.]), as “intermediate” settings would be accounted for, which cannot be captured by ILL nor by CLL.

As starting examples, consider the intermediate linear logic obtained by adding the following prelinearity axiom:

\[(A \rightsquigarrow B) \vee (B \rightsquigarrow A)\]  \hspace{1cm} (6)

or its controlled variant:

\[(!A \rightsquigarrow !B) \vee (!B \rightsquigarrow !A).\]  \hspace{1cm} (7)

**Proof theory of intermediate linear logics.** The same situation that was described for intermediate logics reappears in this setting. For instance, Axiom (2) above:

– cannot be captured by analytic structural rules of any Gentzen calculus for intuitionistic linear logic;

\(^\text{12}\) In the sense explained by Troelstra in Reference [39, p. 222].
can be captured by the following analytic structural rule of Belnap’s display calculus for linear logic:

\[ \frac{X \vdash Z \quad W \vdash Y}{I \vdash (X > Y); (W > Z)} \]

Axiom (3) above:

– cannot be captured by analytic structural rules of any Gentzen calculus for intuitionistic linear logic;
– cannot be captured by analytic structural rules of Belnap’s display calculus for linear logic;
– can be captured by the following multi-type analytic structural rule:

\[ \frac{\circ \Gamma \vdash Z \quad \circ \Delta \vdash Y}{I \vdash (\circ \Gamma > Y); (\circ \Delta > Z)} \]

where \( \Gamma \) and \( \Delta \) are of type Kernel, \( Y \) and \( Z \) are of type Linear, > is the structural connective for linear implication, and \( \circ \) is the structural connective for the multi-type diamond operator from Kernel to Linear.

The research program. The multi-type methodology opens up the possibility of charting and investigating systematically, both algebraically and proof-theoretically, an infinite class of intermediate linear logics, each of which presented by a proper display calculus endowed by design with its package of basic properties (soundness, completeness, conservativity, cut elimination, and subformula property). In particular, the general theory associates each calculus with its appropriate class of heterogeneous algebras. For instance:

– in the heterogeneous algebras of the intermediate linear logic defined by Axiom (2), the algebra \( L \) satisfies prelinearity.
– in the heterogeneous algebras of the intermediate linear logic defined by Axiom (3), the “kernel” (Heyting) algebra \( A \) is a Gödel algebra.

8 CONCLUSIONS

Results. In the present article, we have introduced proper display calculi for several variants of classical, intuitionistic, and bi-intuitionistic linear logic, and proved soundness, syntactic completeness, conservativity, cut elimination, and subformula property for each. These results are key instances of results in the wider research program of multi-type algebraic proof theory, which, generalizing References [15, 63], integrates algebraic canonicity and correspondence techniques [19–21, 23–25, 28, 73, 77, 78] into structural proof theory and is aimed at endowing proof theoretically challenging logics with calculi with excellent properties. This theory guarantees, among other things, that in each setting, these properties modularly transfer to calculi corresponding to fragments of the original languages, and to their analytic axiomatic extensions and expansions. In particular, relative to the present framework, we have given a multi-type reformulation of the mathematically akin but independently motivated formal framework of Reference [70], aimed at extending the use of exponential-type connectives, decomposed into pairs of adjoint modal operators, for the controlled application of structural rules. Specifically, we have outlined a multi-type framework in which the applications of commutativity and associativity are controlled in the same way in which applications of weakening and contraction are controlled in linear logic.

Alternative symmetrization of intuitionistic linear logic. Closely connected to the modularity of (proper) display calculi is their symmetry. Indeed, in the original Gentzen calculi, the difference between, e.g., the classical and the intuitionistic behavior is captured by a restriction on the shape of sequents, which entails that precedent and succedent parts are managed differently, while display calculi sequents have the same unrestricted shape in every setting, with no difference in the
management of precedent and succedent parts. This is why, in the environment introduced in the present article, it has been natural to consider linear subtraction $\cdot$ along with linear implication. This connective makes it possible to realize a symmetrization of intuitionistic linear logic alternative to the one realized by switching to classical linear logic, and rather analogous to the one effected by switching from intuitionistic to bi-intuitionistic logic. As discussed in Section C, linear subtraction can always be interpreted in perfect intuitionistic linear algebras as the left residual of $\otimes$, and moreover, intuitionistic linear logic is complete w.r.t. perfect intuitionistic linear algebras. These facts imply that bi-intuitionistic linear logic, defined as the logic of bi-intuitionistic linear algebras (cf. Definition 3.1), conservatively extends intuitionistic linear logic. When it comes to the treatment of exponentials in the bi-intuitionistic setting, we have considered both Ono’s interaction axiom (cf. P1 in Definition 3.1) and its symmetric version expressed in terms of $\Rightarrow$ (cf. BLP2 in Definition 3.1). In Section 3.2, we have showed that, while Ono’s interaction axiom corresponds to the left-promotion rule (of ?), its symmetric version corresponds to the right-promotion rule (of !).

Bi-intuitionistic linear logic provides an environment in which all the original rules involving exponentials are derivable (i.e., restricted weakening and contraction, promotion and dereliction rules). We conjecture that ! and ? are not necessarily interdefinable in the bi-intuitionistic linear setting as in classical linear logic. These features are interesting and deserving further investigation. In particular, the residuation between $\Rightarrow$ and $\otimes$ can perhaps provide a handle towards an improvement in the understanding of the computational meaning of both connectives.

**Further directions.** Multi-type calculi form an environment in which it has been possible to settle the question concerning the analiticity of linear logic. However, and perhaps even more interestingly, this environment also helps to clearly formulate a broad range of questions at various levels of generality, spanning from the one concerning the alternative symmetrization of linear logic discussed above, to the concrete implementation of Girard’s research program on “the unity of logic” [51]. Another such question concerns the systematic exploration of the different versions of the analytic rules that encode the pairing axioms P1 and BLP2 (cf. Definition 3.1) and make it possible to derive the non-analytic promotion/demotion rules. Indeed, these different versions are not equivalent in every setting, which opens up the possibility of making finer-grained distinctions. We conjecture that these relations can be expressed also in fragments of the languages considered in the present article, such as the purely positive setting, along the lines of Dunn’s positive modal logic [35].

**Implicit complexity.** In implicit computational complexity, one is interested in characterizing complexity classes by restricting the features of programming languages or logical languages, rather than by referring to explicit bounds on resources or particular machine models. Within the proofs-as-program paradigm, various variants of linear logic [2, 5, 29, 53, 71] consider alternative exponential connectives inducing a complexity bound on the cut elimination procedure and, therefore, characterizing complexity classes. Another interesting line of research that can be developed on the basis of the present results would be to investigate how various versions of exponential connectives can be represented and captured via adequate representation theorems in a multi-type environment.
APPENDICES

A PROOF OF CUT ELIMINATION

As discussed in Section 6.4, cut elimination and subformula property for the calculi defined in Section 5 immediately follow from the Belnap-style metatheorem for quasi-proper multi-type calculi of [42]. Hence, it is enough to verify that the calculi defined in Section 5 verify the hypotheses of Reference [42, Theorem 4.1].

In fact, the calculi defined in Section 5 satisfy stronger properties than those required by Reference [42, Theorem 4.1]. Hence, below, we provide the corresponding restriction of the definition of quasi-proper multi-type calculus given in Reference [42], which applies specifically to the calculi of Section 5. The resulting definition, given below, is the exact counterpart in the multi-type setting of the definition of proper display calculi introduced in Reference [84] and generalized in Reference [63].

A sequent \( \Gamma \vdash \Delta \) is type-uniform if \( \Gamma \) and \( \Delta \) are of the same type.

Definition A.1. Proper multi-type display calculi are those satisfying the following list of conditions:

\( C_1 \): Preservation of operational terms. Each operational term occurring in a premise of an inference rule \( \inf \) is a subterm of some operational term in the conclusion of \( \inf \).

\( C_2 \): Shape-alikeness and type-alikeness of parameters. Congruent parameters\(^{13} \) are occurrences of the same structure, and are of the same type.

\( C_3 \): Non-proliferation of parameters. Each parameter in an inference rule \( \inf \) is congruent to at most one constituent in the conclusion of \( \inf \).

\( C_4 \): Position-alikeness of parameters. Congruent parameters are either all in precedent position or all in succedent position (cf. Footnote 9).

\( C_5 \): Display of principal constituents. If an operational term \( a \) is principal in the conclusion sequent \( s \) of a derivation \( \pi \), then \( a \) is in display.

\( C_6 \): Closure under substitution for succedent parts within each type. Each rule is closed under simultaneous substitution of arbitrary structures for congruent operational terms occurring in succedent position, within each type.

\( C_7 \): Closure under substitution for precedent parts within each type. Each rule is closed under simultaneous substitution of arbitrary structures for congruent operational terms occurring in precedent position, within each type.

\( C_8 \): Eliminability of matching principal constituents. This condition requests a standard Gentzen-style checking, which is now limited to the case in which both cut formulas are principal, i.e., each of them has been introduced with the last rule application of each corresponding

\(^{13}\)The congruence relation between non-active-parts in rule-applications is understood as derived from the specification of each rule; that is, we assume that each schematic rule of the system comes with an explicit specification of which elements are congruent to which (and then the congruence relation is defined as the reflexive and transitive closure of the resulting relation). Our convention throughout the article is that congruent parameters are denoted by the same structural variables. For instance, in the rule

\[
\frac{X; Y \vdash Z}{\overline{Y}; \overline{X} \vdash Z}
\]

the structures \( X \), \( Y \), and \( Z \) are parametric and the occurrences of \( X \) (respectively, \( Y \), \( Z \)) in the premise and the conclusion are congruent.
subduction. In this case, analogously to the proof Gentzen-style, condition $C_8$ requires being able to transform the given deduction into a deduction with the same conclusion in which either the cut is eliminated altogether or is transformed in one or more applications of the cut rule, involving proper subterms of the original operational cut-term.

$C_9$: Type-uniformity of derivable sequents. Each derivable sequent is type-uniform.

$C_{10}$: Preservation of type-uniformity of cut rules. All cut rules preserve type-uniformity.

Since proper multi-type display calculi are quasi-proper, the following theorem is an immediate consequence of Reference [42, Theorem 4.1]:

**Theorem A.2.** Every proper multi-type display calculus enjoys cut elimination and subformula property.

**Proposition A.3.** The calculus $mD.LL$ is a proper multi-type display calculus.

**Proof.** Condition $C_8$ has been discussed in Section 6.4. Condition $C_1$ immediately follows from the fact that the structural meta-variables occurring in the premises of each rule also occur in its conclusion, and moreover, the logical meta-variables occurring in the premises of each logical rule occur as immediate subformulas of the principal formula.

Conditions $C_2$, $C_3$, and $C_4$ are ingrained in the definition of the rules, thanks to the notational convention that we use the same metavariable in the premises and in the conclusion of a rule if and only if these parameters are congruent. One can verify by inspection on each rule that each occurrence of a structural metavariable in a premise is congruent to at most one structural metavariable in the conclusion, and that the (precedent or succedent) position of all structural metavariables in the premises is preserved in the conclusion.

Condition $C_5$ is also readily verified by inspection on the logical rules. In fact, not only the principal formula, but also the auxiliary formulas are in display.

Conditions $C_6$ and $C_7$ are also immediately verified, thanks to the notational conventions that we use to introduce the rules, in terms of congruent metavariables that can be instantiated by arbitrary structures within each type.

Condition $C_9$ readily follows from the fact that axioms are type-uniform, and rules preserve type-uniformity.

Condition $C_{10}$ readily follows from the fact that axioms are type-uniform, and rules preserve type-uniformity.

□

**B ANALYTIC INDUCTIVE INEQUALITIES**

In the present section, we specialize the definition of analytic inductive inequalities to the multi-type language $L_{MT}$, in the types Linear, !-Kernel, and ?-Kernel (respectively abbreviated as $L$, $K_！$, and $K_?$), defined in Section 4 and reported below for the reader’s convenience:

$K_！ \ni \alpha ::= t(A) \mid f_！ \mid f_？ \mid f \mid \alpha \lor_！ \alpha \mid \alpha \land_！ \alpha \mid \alpha \rightarrow_！ \alpha \mid \xi \rightarrow_！ \alpha$

$K_？ \ni \xi ::= t_？ \mid f_？ \mid f \mid \xi \lor_？ \xi \mid \xi \land_？ \xi \mid \xi \rightarrow_？ \xi \mid \alpha \rightarrow_？ \xi$

$L \ni A ::= p \mid e(\alpha) \mid e(\xi) \mid 1 \mid \bot \mid A^± \mid A \otimes A \mid A \Rightarrow A \mid A \rightarrow A \mid A \rightarrow_！ A \mid \top \mid 0 \mid A \& A \mid A \oplus A$.

We will make use of the following auxiliary definition: an order-type over $n \in \mathbb{N}$ is an $n$-tuple $\epsilon \in \{1, \partial\}^n$. For every order type $\epsilon$, we denote its opposite order type by $\epsilon_\partial$, that is, $\epsilon_\partial(i) = 1$ iff $\epsilon(i) = \partial$ for every $1 \leq i \leq n$. The connectives of the language above are grouped together into the families $\mathcal{F} := \mathcal{F}_K \cup \mathcal{F}_K' \cup \mathcal{F}_L \cup \mathcal{F}_{MT}$ and $\mathcal{G} := \mathcal{G}_K \cup \mathcal{G}_K' \cup \mathcal{G}_L \cup \mathcal{G}_{MT}$ defined as follows:

---

14The definition given in the present Appendix is applicable to the setting of distributive linear logic only.
Table 1. Skeleton and PIA Nodes

| Skeleton | PIA |
|----------|-----|
| \( \Lambda \)-adjoints | SRA |
| + \( \lor \), \( \land \), \( \otimes \) & \( \land \), \( \lor \), \( \otimes \) \& \( \gamma \) | + \( t \), \( t \), \( \top \), \( \land \), \( \lor \), \( \otimes \) \& \( e \) |
| \( \land \), \( \lor \) & \( \land \), \( \lor \), \( \otimes \) | - \( f \), \( f \), \( 0 \), \( \lor \), \( \land \), \( \otimes \) \& \( e \) |

\( \mathcal{F}_K := \{ t \}, \land \} \)

\( \mathcal{F}_K := \{ t, \land, \lor \} \)

\( \mathcal{F}_L := \{ 1, \otimes, \land \} \)

\( \mathcal{F}_M := \{ e, \land \} \)

\( \mathcal{G}_K := \{ f, \lor \} \)

\( \mathcal{G}_K := \{ f, \land \} \)

\( \mathcal{G}_L := \{ 1, \otimes, \land \} \)

\( \mathcal{G}_M := \{ e, \land \} \)

For any \( f \in \mathcal{F} \) (respectively, \( g \in \mathcal{G} \)), we let \( n_f \in \mathbb{N} \) (respectively, \( n_g \in \mathbb{N} \)) denote the arity of \( f \) (respectively, \( g \)), and the order-type \( \epsilon_f \) (respectively, \( \epsilon_g \)) on \( n_f \) (respectively, \( n_g \)) indicate whether the \( i \)th coordinate of \( f \) (respectively, \( g \)) is positive (\( \epsilon_f(i) = 1 \), \( \epsilon_g(i) = 1 \)) or negative (\( \epsilon_f(i) = 0 \), \( \epsilon_g(i) = 0 \)). The order-theoretic motivation for this partition is that the algebraic interpretations of \( \mathcal{F} \)-connectives (respectively, \( \mathcal{G} \)-connectives), preserve finite joins (respectively, meets) in each positive coordinate and reverse finite meets (respectively, joins) in each negative coordinate.

For any term \( s(p_1, \ldots p_n) \), any order type \( \epsilon \) over \( n \), and any \( 1 \leq i \leq n \), an \( \epsilon \)-critical node in a signed generation tree of \( s \) is a leaf node \( +p_i \) with \( \epsilon(i) = 1 \) or \( -p_i \) with \( \epsilon(i) = 0 \). An \( \epsilon \)-critical branch in the tree is a branch ending in an \( \epsilon \)-critical node. For any term \( s(p_1, \ldots p_n) \) and any order type \( \epsilon \) over \( n \), we say that \( +s \) (respectively, \( -s \)) agrees with \( \epsilon \), and write \( \epsilon(+s) \) (respectively, \( \epsilon(-s) \)), if every leaf in the signed generation tree of \( +s \) (respectively, \( -s \)) is \( \epsilon \)-critical. We will also write \( +s' < *s \) (respectively, \( -s' < *s \)) to indicate that the subterm \( s' \) inherits the positive (respectively, negative) sign from the signed generation tree \( *s \). Finally, we will write \( \epsilon(s') < *s \) (respectively, \( \epsilon^0(s') < *s \)) to indicate that the signed subtree \( s' \), with the sign inherited from \( *s \), agrees with \( \epsilon \) (respectively, with \( \epsilon^0 \)).

**Definition B.1 (Signed Generation Tree).** The positive (respectively, negative) generation tree of any \( \mathcal{L}_{	ext{MT}} \)-term \( s \) is defined by labelling the root node of the generation tree of \( s \) with the sign + (respectively, −), and then propagating the labelling on each remaining node as follows:

For any node labelled with \( \ell \in \mathcal{F} \cup \mathcal{G} \) of arity \( n_\ell \geq 1 \), and for any \( 1 \leq i \leq n_\ell \), assign the same (respectively, the opposite) sign to its \( i \)th child node if \( \epsilon_\ell(i) = 1 \) (respectively, \( \epsilon_\ell(i) = 0 \)).

Nodes in signed generation trees are positive (respectively, negative) if are signed + (respectively, −).

**Definition B.2 (Good Branch).** Nodes in signed generation trees will be called \( \Lambda \)-adjoints, syntactically left residual (SLR), syntactically right residual (SRR), and syntactically right adjoint (SRA), according to the specification given in Table 1. A branch in a signed generation tree \( *s \), with \( * \in \{+,−\} \), is called a good branch if it is the concatenation of two paths \( P_1 \) and \( P_2 \), one of which may possibly be of length 0, such that \( P_1 \) is a path from the leaf consisting (apart from variable nodes) only of PIA-nodes, and \( P_2 \) consists (apart from variable nodes) only of Skeleton-nodes.

---

For explanations of our choice of terminologies here, we refer to Reference [78, Remark 3.24].
Definition B.3 (Analytic Inductive Inequalities). For any order type $\epsilon$ and any irreflexive and transitive relation $\prec_\Omega$ on $p_1, \ldots, p_n$, the signed generation tree $\ast s (\ast \in \{\ast -, +\})$ of a term $s(p_1, \ldots p_n)$ is analytic ($\Omega, \epsilon$)-inductive if

1. every branch of $\ast s$ is good (cf. Definition B.2);
2. for all $1 \leq i \leq n$, every $m$-ary SRR-node occurring in any $\epsilon$-critical branch with leaf $p_i$ is of the form $\circ \ast (s_1, \ldots, s_{j-1}, \beta, s_{j+1} \ldots, s_m)$, where for any $h \in \{1, \ldots, m\} \setminus j$:
   a. $\epsilon^h(s_h) \prec_\Omega s$ (cf. discussion before Definition B.2), and
   b. $p_k \prec_\Omega p_i$ for every $p_k$ occurring in $s_h$ and for every $1 \leq k \leq n$.

We will refer to $\prec_\Omega$ as the dependency order on the variables. An inequality $s \leq t$ is analytic ($\Omega, \epsilon$)-inductive if the signed generation trees $\ast s$ and $-t$ are analytic ($\Omega, \epsilon$)-inductive. An inequality $s \leq t$ is analytic inductive if it is analytic ($\Omega, \epsilon$)-inductive for some $\Omega$ and $\epsilon$.

In each setting in which they are defined, analytic inductive inequalities are a subclass of inductive inequalities (cf. Reference [63]). In their turn, inductive inequalities are canonical (that is, preserved under canonical extensions, as defined in each setting). Hence, the following is an immediate consequence of general results on the canonicity of inductive inequalities:

Theorem B.4. Analytic inductive $L_{MT}$-inequalities are canonical.

C BACKGROUND ON CANONICAL EXTENSIONS

In the present section, we report on basic notions and facts of canonical extensions of bounded lattices that are used in the present article. Our presentation is based on Reference [24]. The proofs of many basic properties can be found in References [36, 46].

Definition C.1. The canonical extension of a bounded lattice $L$ is a complete lattice $L^\delta$ containing $L$ as a sublattice, such that:

1. (denseness) every element of $L^\delta$ can be expressed both as a join of meets and as a meet of joins of elements from $L$;
2. (compactness) for all $S, T \subseteq L$, if $\bigwedge S \leq \bigvee T$ in $L^\delta$, then $\bigwedge F \leq \bigvee G$ for some finite sets $F \subseteq S$ and $G \subseteq T$.

The canonical extension $L^\delta$ of a (distributive) lattice $L$ is a perfect (distributive) lattice, i.e., a complete (and completely distributive) lattice that is completely join-generated by its completely
Moreover, canonical extensions are unique up to isomorphisms fixing the original algebra.

An element \( k \in L^\delta \) (respectively, \( o \in L^\delta \)) is closed (respectively, open) if it is the meet (respectively, join) of some subset of \( L \). Let \( K(L^\delta) \) (respectively, \( O(L^\delta) \)) be the set of closed (respectively, open) elements of \( L^\delta \). It is easy to see that the denseness condition in Definition C.1 implies that \( f^\omega(L^\delta) \subseteq K(L^\delta) \) and \( M^\omega(L^\delta) \subseteq O(L^\delta) \).

**Definition C.2.** For every unary, order-preserving map \( f : L \to M \) between bounded lattices, the \( \sigma \)-extension of \( f \) is defined first by declaring, for every \( k \in K(L^\delta) \),

\[
\sigma^\omega(k) := \bigwedge \{ f(a) \mid a \in L \text{ and } k \leq a \},
\]

and then, for every \( u \in L^\delta \),

\[
\sigma^\omega(u) := \bigvee \{ \sigma^\omega(k) \mid k \in K(L^\delta) \text{ and } k \leq u \}.
\]

The \( \pi \)-extension of \( f \) is defined first by declaring, for every \( o \in O(L^\delta) \),

\[
\pi^\omega(o) := \bigvee \{ f(a) \mid a \in L \text{ and } a \leq o \},
\]

and then, for every \( u \in L^\delta \),

\[
\pi^\omega(u) := \bigwedge \{ \pi^\omega(o) \mid o \in O(L^\delta) \text{ and } u \leq o \}.
\]

It is easy to see that the \( \sigma \)- and \( \pi \)-extensions of monotone maps are monotone. Moreover, the \( \sigma \)-extension of a map that preserves finite joins (respectively, reverses finite meets) will preserve arbitrary joins (respectively, reverse arbitrary meets). Because canonical extensions are complete lattices, this implies (cf. Reference [31, Proposition 7.34]) that the \( \sigma \)-extension of any such map is a left (Galois) adjoint, that is, its right (respectively, Galois) adjoint exists.

Dually, the \( \pi \)-extension of a map that preserves finite meets (respectively, reverses finite joins) will preserve arbitrary meets (respectively, reverse arbitrary meets), and hence the \( \pi \)-extension of any such map is a right (Galois) adjoint, that is, its left (Galois) adjoint exists.

Finally, if \( f : L \to M \) and \( g : M \to L \) are such that \( f \circ g \), then \( f^\omega \circ g^\omega \).

The definitions above apply also to \( \epsilon \)-ary operations that are \( \epsilon \)-monotone for some order type \( \epsilon \) over \( n \) (cf. Section B). Indeed, let us first observe that taking order-duals interchanges closed and open elements: \( K((L^\delta)^\delta) = O(L^\delta) \) and \( O((L^\delta)^\delta) = K(L^\delta) \); similarly, \( K((L^n)^\delta) = K(L^\delta)^n \) and \( O((L^n)^\delta) = O(L^\delta)^n \). Hence, \( K((L^\delta)^\epsilon) = \prod_i K(L^\delta)^{\epsilon(i)} \) and \( O((L^\delta)^\epsilon) = \prod_i O(L^\delta)^{\epsilon(i)} \) for every lattice \( L \) and every order-type \( \epsilon \) over any \( n \in \mathbb{N} \), where

\[
K(L^\delta)^{\epsilon(i)} := \begin{cases} K(L^\delta) & \text{if } \epsilon(i) = 1 \\ O(L^\delta) & \text{if } \epsilon(i) = 0 \end{cases} \quad O(L^\delta)^{\epsilon(i)} := \begin{cases} O(L^\delta) & \text{if } \epsilon(i) = 1 \\ K(L^\delta) & \text{if } \epsilon(i) = 0 \end{cases}.
\]

From these observations it immediately follows that taking the canonical extension of a lattice commutes with taking order-duals and products, namely: \( (L^\delta)^\delta = (L^\delta)^\delta \) and \( (L^\delta \times L^\delta)^\delta = L^\delta \times L^\delta \). Hence, \( (L^\delta)^\epsilon \) can be identified with \( (L^\delta)^\delta \), \( (L^n)^\delta \), \( (L^\delta)^n \), and \( (L^\epsilon)^\delta \) with \( (L^\delta)^\epsilon \) for any order type \( \epsilon \) over \( n \), where \( L^\epsilon := \prod^n_{i=1} L^\delta(i) \). These identifications make it possible to obtain the

---

16 An element \( j \in L \) is completely join-irreducible if \( \neq \bot \) and for every \( S \subseteq L \), if \( j \leq \bigvee S \), then \( j \in S \). We let \( f^\omega(L) \) denote the set of the completely join-irreducible elements of \( L \). Completely meet-irreducible elements are defined order-dually, and their collection is denoted by \( M^\omega(L) \).

17 That is, are monotone (respectively, antitone) in each coordinate \( i \) such that \( \epsilon(i) = 1 \) (respectively, \( \epsilon(i) = 0 \)).
definition of $\sigma$- and $\pi$-extensions of $\varepsilon$-monotone operations of any arity $n$ and order-type $\varepsilon$ over $n$ by instantiating the corresponding definitions given above for monotone and unary functions. The $\sigma$- and $\pi$-extensions of the lattice operations coincide with the lattice operations of the canonical extension.

**Definition C.3.** For any lattice $L$, an operation $h$ on $L$ of arity $n_h$ is normal if it is order-preserving or order-reversing in each coordinate, and moreover one of the following conditions holds: (a) $h$ preserves finite (hence, possibly empty) joins in each coordinate in which it is order-preserving and reverses finite meets in each coordinate in which it is order-reversing; (b) $h$ preserves finite (hence, possibly empty) meets in each coordinate in which it is order-preserving and reverses finite joins in each coordinate in which it is order-reversing.

A normal (distributive) lattice expansion is an algebra $A = (L, \mathcal{F}, \mathcal{G})$ such that $L$ is a bounded (distributive) lattice and $\mathcal{F}$ and $\mathcal{G}$ are finite (possibly empty) and disjoint sets of operations on $L$ such that every $f \in \mathcal{F}$ is normal and verifies condition (a), and every $g \in \mathcal{G}$ is normal and verifies condition (b).

Intuitionistic and bi-intuitionistic linear algebras (without exponentials), Heyting, co-Heyting, and bi-Heyting algebras are all examples of normal (distributive) lattice expansions: for intuitionistic linear algebras, take $\mathcal{F} := \{\otimes, 1\}$ and $\mathcal{G} := \{\neg, \mathcal{N}, \bot\}$; for bi-intuitionistic linear algebras, take $\mathcal{F} := \{\bullet, \otimes, 1\}$ and $\mathcal{G} := \{\neg, \mathcal{N}, \bot\}$; for Heyting algebras, take $\mathcal{F} := \emptyset$ and $\mathcal{G} := \{\rightarrow\}$; for co-Heyting algebras, take $\mathcal{F} := \{\rightarrow\}$ and $\mathcal{G} := \emptyset$; for bi-Heyting algebras, take $\mathcal{F} := \{\rightarrow\}$ and $\mathcal{G} := \{\rightarrow\}$.

**Definition C.4.** The canonical extension of any normal (distributive) lattice expansion $A = (L, \mathcal{F}, \mathcal{G})$ is the normal (distributive) lattice expansion $A^\delta := (L^\delta, \mathcal{F}^\sigma, \mathcal{G}^\pi)$, where $L^\delta$ is the canonical extension of $L$ (cf. Definition C.1), and $\mathcal{F}^\sigma := \{f^\sigma \mid f \in \mathcal{F}\}$ and $\mathcal{G}^\pi := \{g^\pi \mid g \in \mathcal{G}\}$.

The definition above applies in a uniform way to any signature and is motivated by the fact that, as remarked above, it preserves existing residuations/Galois connections among operations in the original signature. Another noticeable, if more technical, feature of this definition is its being independent on whether the original maps are smooth (i.e., their $\sigma$- and $\pi$-extensions coincide). While all unary normal operations are smooth, normal operations of arity higher than 1 might not be smooth in general.

It follows straightforwardly from the facts above that the classes of linear algebras without exponentials, Heyting, co-Heyting, and bi-Heyting algebras are closed under taking canonical extensions. It also follows that the canonical extension of a normal LE $A$ is a perfect normal LE:

**Definition C.5.** A normal LE $A = (L, \mathcal{F}, \mathcal{G})$ is perfect if $L$ is a perfect lattice (cf. discussion above), and moreover the following infinitary distribution laws are satisfied for each $f \in \mathcal{F}$, $g \in \mathcal{G}$, $1 \leq i \leq n_f$ and $1 \leq j \leq n_g$: for every $S \subseteq L$,

\[
\begin{align*}
    f(x_1, \ldots, \vee S, \ldots, x_{n_f}) &= \vee \{f(x_1, \ldots, x_i, \ldots, x_{n_f}) \mid x_i \in S\} & \text{if } \varepsilon_f(i) = 1 \\
    f(x_1, \ldots, \wedge S, \ldots, x_{n_f}) &= \wedge \{f(x_1, \ldots, x_i, \ldots, x_{n_f}) \mid x_i \in S\} & \text{if } \varepsilon_f(i) = \partial \\
    g(x_1, \ldots, \vee S, \ldots, x_{n_g}) &= \vee \{g(x_1, \ldots, x_i, \ldots, x_{n_g}) \mid x_i \in S\} & \text{if } \varepsilon_g(i) = 1 \\
    g(x_1, \ldots, \wedge S, \ldots, x_{n_g}) &= \wedge \{g(x_1, \ldots, x_i, \ldots, x_{n_g}) \mid x_i \in S\} & \text{if } \varepsilon_g(i) = \partial.
\end{align*}
\]

Before finishing the present subsection, let us spell out the definitions of the extended operations.

Denoting by $\leq^\varepsilon$ the product order on $(L^\delta)^\varepsilon$, we have for every $f \in \mathcal{F}$, $g \in \mathcal{G}$, $\overline{u} \in (L^\delta)^{n_f}$ and $\overline{v} \in (L^\delta)^{n_g}$,
\[ f^\sigma (\bar{k}) := \land \{ f (\bar{a}) \mid \bar{a} \in (L^\delta)^{\epsilon_f} \text{ and } \bar{k} \leq^\delta \bar{a} \} \quad g^\sigma (\bar{a}) := \lor \{ g (\bar{a}) \mid \bar{a} \in (L^\delta)^{\epsilon_g} \text{ and } \bar{a} \leq^\delta \bar{a} \} \]

Two facts are worth being highlighted, since they follow a pattern that is key to the conservativity argument given in Section 6.3. First, the algebraic completeness of linear logic (without exponentials), intuitionistic, co-intuitionistic, and bi-intuitionistic logic, and the canonical embedding of the algebras corresponding to these logics into their respective canonical extensions immediately give completeness of each of these logics w.r.t. the corresponding class of perfect normal LEs, which is condition (a) in the general conservativity argument of which the one given in Section 6.3 is an instance. Second, the existence of the adjoints and residuals (in each coordinate) of the extensions of the original operations provides semantic interpretation to all structural connectives (including to those the operational counterparts of which do not belong to the original signature). For instance, the canonical extension of any Heyting algebra (respectively, co-Heyting algebra) is naturally endowed with a bi-Heyting algebra structure, since the finite distributivity of joins over meets (respectively, meets over joins) implies complete distributivity holds in the canonical extension, which guarantees the existence of the left residual \( \rightarrow \) of \( \lor \) (respectively, the right residual \( \rightarrow \) of \( \land \)) in the canonical extension. Likewise, the finite distributivity of \( \land \) over \( \lor \) in any intuitionistic linear algebra without exponentials guarantees the existence of the left residual \( \rightarrow \) of \( \land \) in the canonical extension, which is then naturally endowed with a structure of bi-intuitionistic linear algebra without exponentials. The existence of all adjoints and residuals makes it possible to interpret the structural rules of the display calculus intended to capture each logic, and verify their soundness, which is requirement (b) in the general conservativity argument (cf. Section 6.3).

## D INVERSION LEMMAS

In the present section, we prove the general inversion lemmas holding in any proper multi-type display calculus (cf. Definition A.1) Recall that rule \( R \) is invertible if every premise sequent of \( R \) may be derived from the conclusion sequent of \( R \). In what follows, we fix an arbitrary multi-type signature \( (F, G) \) (generalizing the presentations of Appendices B and C; see Reference [63] for an extended discussion). The language for (the corresponding fragment of) the associated calculus is

\[
\begin{array}{ll}
H & K \\
\hline
f & g
\end{array}
\]

In what follows, we omit reference to types and use variables \( x, y, z \) to denote structural terms of arbitrary type and variables \( a, b, c \) to denote operational terms of arbitrary type. All sequents are understood to be type-uniform (cf. Section A). In any proper multi-type display calculus, the introduction rules for any \( f \in F \) and \( g \in G \) have the following shape (cf. Reference [63]):

\[
\begin{align*}
\frac{H(a_1, \ldots, a_{n_f}) \vdash x}{f (a_1, \ldots, a_{n_f}) \vdash x} & \quad \frac{x \vdash a_i \quad a_j \vdash x_j \mid 1 \leq i, j \leq n_f, \epsilon_f (i) = 1 \text{ and } \epsilon_f (j) = \delta}{H(x_1, \ldots, x_{n_f}) \vdash f (a_1, \ldots, a_n)} & f_R \\
\frac{x_i \vdash a_i \quad a_j \vdash x_j \mid 1 \leq i, j \leq n_g, \epsilon_g (i) = 1 \text{ and } \epsilon_g (j) = \delta}{g (a_1, \ldots, a_n) \vdash K(x_1, \ldots, x_{n_g})} & \frac{x \vdash K(a_1, \ldots, a_{n_g})}{x \vdash g (a_1, \ldots, a_{n_g})} & g_R
\end{align*}
\]

In particular, if \( n_f = 0 = n_g \), then the rules \( f_R \) and \( g_L \) above reduce to the axioms (0-ary rules) \( H \vdash f \) and \( g \vdash K \). Using these rules (and the standard introduction rules for lattice connectives), the following lemma can be proved by a routine induction on terms:

ACM Transactions on Computational Logic, Vol. 24, No. 2, Article 13. Publication date: January 2023.
**Lemma D.1.** In any proper multi-type display calculus, all sequents $a ⊢ a$ are derivable.

**Lemma D.2.** In any proper multi-type display calculus, the left-introduction rule of any $f \in F$ and the right-introduction rule of any $g \in G$ are invertible.

**Proof.** Using Lemma D.1, the following derivation proves the claim for $f$:

```
\[
\frac{(a_i + a_j \mid 1 \leq i, j \leq n_f, \epsilon_f(i) = 1 \text{ and } \epsilon_f(j) = \partial)}{H(a_1, \ldots, a_{n_f}) \vdash f(a_1, \ldots, a_n)} \quad f(a_1, \ldots, a_{n_f}) \vdash x \quad \text{Cut}
\]
```

The proof of the remaining part of the statement is analogous. □

Using Lemma D.2, and suitably making use of the display property of proper multi-type display calculi, one can prove the following:

**Corollary D.3.** In any proper multi-type display calculus,

1. if $(x \vdash y)[f(a_1, \ldots, a_{n_f})]^{pre}$ is derivable, then so is $(x \vdash y)[H(a_1, \ldots, a_{n_f})]^{pre}$;
2. if $(x \vdash y)[g(a_1, \ldots, a_{n_g})]^{succ}$ is derivable, then so is $(x \vdash y)[K(a_1, \ldots, a_{n_g})]^{succ}$.

### E DERIVING HILBERT-STYLE AXIOMS AND RULES FOR EXPONENTIALS

The relevant axioms and rule capturing the behavior of $!$ were considered in References [3, 81]. In Reference [76] also the algebraic inequalities capturing the behavior of $? were considered. Below, we reproduce the axioms and rule for $!$ and the axioms for $?$ corresponding to the algebraic inequalities:

**Axioms**

| A1. $B \rightarrow (!A \rightarrow B)$ |
| A2. $(!A \rightarrow (!A \rightarrow B)) \rightarrow (!A \rightarrow !B)$ |
| A3. $!(A \rightarrow B) \rightarrow (!A \rightarrow !B)$ |
| A4. $!A \rightarrow A$ |
| A5. $!A \rightarrow !!A$ |
| Rule $!R \vdash A \Rightarrow \vdash !A$ |

**A6.** $(A \rightarrow B) \vdash ?A \rightarrow ?B$  
**A7.** $A \rightarrow ?A$  
**A8.** $??A \rightarrow ?A$  
**A9.** $? \perp \rightarrow ? \perp, ? \perp \rightarrow \perp \perp$  
**A10.** $\perp \rightarrow ?A$

The rule $!R$ is derivable in D.LL as follows:

```
\[
\frac{\Phi \vdash A}{\square \vdash A} \quad \frac{\square \vdash \bullet A}{\square \vdash \square A} \quad \frac{\Phi \vdash \square A}{\Phi \vdash \bullet \square A} \quad \frac{\Phi \vdash \bullet A}{\Phi \vdash \square \bullet A}
\]
```

All the previous axioms are derivable in D.LL as follows:

**A1.** $B \rightarrow (!A \rightarrow B)$  
**A2.** $(!A \rightarrow (!A \rightarrow B)) \rightarrow (!A \rightarrow B)$

18The notation $(x \vdash y)[f(a_1, \ldots, a_{n_f})]^{pre}$ (respectively, $(x \vdash y)[g(a_1, \ldots, a_{n_g})]^{succ}$) indicates that $f(a_1, \ldots, a_{n_f})$ (respectively, $g(a_1, \ldots, a_{n_g})$) occurs as a substructure of $x \vdash y$ in precedent (respectively, succeed) position.
A3. !(A → B) → (!A → !B)

A4. !A → A and A7. A → ?A
A5. \( !A \rightarrow !!A \) and \( ?A \rightarrow ?A \)

A6. \((A \rightarrow B) \vdash ?A \rightarrow ?B \)

A9. \( \bot \vdash \bot \) and \( A \vdash ?A \)

F \hspace{1em} **COMPARING DERIVATIONS IN GIRARD'S CALCULUS AND IN D.LL**

Earlier on, we have discussed that the multi-type approach makes it possible to design calculi particularly suitable as tools of analysis. In particular, the calculus D.LL was introduced in the present
article with the aim of understanding the interaction between the additive and the multiplicative
connectives mediated by the exponentials, as is encoded by the following derivable sequents:

\[
\begin{align*}
\top & \vdash 1 \\
A \otimes !B & \vdash !(A \land B) \\
\neg \top & \vdash \bot \\
\neg A \land \neg B & \vdash \neg (A \lor B)
\end{align*}
\]

In what follows, we compare the derivations of these sequents in Girard’s calculus and in D.LL.
Notice that the introduction via controlled weakening (emphasized by the label \( W \)) in the derivation of \( \top \vdash 1 \) in Girard’s calculus corresponds to the application of the unrestricted pure \( K \)-type weakening rule \( W \) in the D.LL derivation. Moreover, the right promotion with empty context in the derivation of \( 1 \vdash \top \) in Girard’s calculus is encoded in a sequence of standard display and introduction rules, elicited by the rules nec and conec.

Below, notice that the derivation of \( !A \otimes !B \vdash !(A \land B) \) in Girard’s calculus makes use of a controlled weakening (emphasized by the label \( W \)). Moreover, the right-introduction rule for & internalizes a left-contraction (we emphasize the second observation by labelling the right introduction of & with \( C \)). The derivation of \( !(A \land B) \vdash !A \otimes !B \) is somewhat dual to the derivation previously discussed, in that it makes use of a controlled contraction (emphasized by the label \( C \)), and the left-introduction rule for & internalizes a left-weakening (we emphasize the second observation labelling the left-introduction of & with \( W \)).
The controlled weakening and contraction rules $W$ and $C$ applied above correspond, in the D.LL derivations of the same sequents below, to the unrestricted pure $K_i$-type weakening rule $W_i$ and $C_i$; moreover, the internalized weakening and contraction in the additive introductions of $\&$ are explicitly performed via the unrestricted pure L-type rules $W_a$ and $C_a$.

derivations in D.LL

G  AN ALTERNATIVE APPROACH TO PROPER DISPLAY CALCULI FOR GENERAL LINEAR LOGIC

In the present article, when defining the logical rules for the additive connectives $\&$ and $\oplus$ of linear logic, as well as their corresponding counterparts in the Linear type of the multi-type language, we have followed Belnap’s and Girard’s presentations and have not associated these connectives with structural proxies. However, thanks to the modularity of proper multi-type calculi, a different approach can be taken; in particular, following the treatment of basic lattice logic introduced in Reference [64], properly displayable calculi for linear logic in suitable multi-type settings can be introduced in which each primitive connective has a structural counterpart. Linear logic formulas can be encoded in the language of such calculi via certain (positional) translations. In what follows, we do not expand on their precise definition\(^\text{19}\):

\[
\begin{align*}
!A & \otimes !B \vdash !(A \& B) & \rightsquigarrow & \Diamond \Box A \otimes \Box B \vdash \Diamond \Box (\Diamond \Diamond \Box A \lor \Diamond \Diamond \Box B) \\
!(A \& B) & \vdash !A \otimes !B & \rightsquigarrow & \Diamond \Box (\lor \Diamond \Box A \land \lor \Diamond \Box B) \vdash \Diamond \Box A \otimes \Box B.
\end{align*}
\]

\(^{19}\)The new heterogeneous connectives $\Box$, $\Diamond \Box$, and $\Diamond \Diamond \Box$ bridge the Linear type with two new types, algebraically interpreted as distributive lattices. We refer to Reference [64] for more details. The only properties that are relevant to the present section are the usual ones of adjoint normal modalities.
Below, we derive the sequents \( !A \otimes !B \vdash ! (A \odot B) \) in a version of D.LL that follows this alternative approach. Also in this setting, the implicit applications of weakening and contraction internalized by the additive introductions of \& are now explicitly performed via the \textit{unrestricted} weakening and contraction rules \( W_r \) and \( C_r \), respectively, which pertain to the new types.

![Diagram](image_url)

**G.1 Distributive Linear Logic Properly Displayed**

Distributive linear logic is a version of linear logic where the lattice reduct is distributive \([47, 48, 75]\). We expand the structural language of the Linear-type with the structural connectives \(-\) and \(\succ\) as follows:

\[
X ::= A \mid \Phi \mid X, X \mid X \succ X \mid I \mid X \times X \mid X \times X \mid o_1 \Gamma \mid o_2 \Pi.
\]

The structural connective \(\succ\) is supposed to be the proxy associated to \& and \(\circ\), while the structural connective \(\succ\) is the proxy of additive implication and co-implication (these logical connectives appear within brackets in the following table to signify that we do not add this logical operators in the signature of the language):

- Structural and operational pure additive Linear-type connectives:

| Additive connectives | 1 | - | \(\succ\) |
|----------------------|---|---|------|
| \(\top\) | 0 | \& | \(\ominus\) | \(\ominus\) |
The following rules are added to the previous calculus D.LL:

- Pure additive Linear-type display rules

\[
\begin{align*}
\text{res}_a & \quad \frac{X \cdot Y \vdash Z}{Y \cdot X \succ Z} \quad & \frac{X \vdash Y \cdot Z}{Y \succ X \vdash Z} \quad \text{res}_a \\
\end{align*}
\]

- Pure additive Linear-type structural rules

\[
\begin{align*}
\text{I} & \quad \frac{X \vdash Y}{X \cdot I \vdash Y} \quad & \frac{X \vdash Y}{X \cdot Y \vdash I} \\
\text{E}_a & \quad \frac{X \cdot Y \vdash Z}{Y \cdot X \vdash Z} \quad & \frac{X \cdot Y \vdash Z}{Z \cdot X \vdash Y} \quad \text{E}_a \\
\text{A}_a & \quad \frac{X \cdot (Y \cdot Z) \vdash W}{(X : Y) \cdot Z \vdash W} \quad & \frac{X \cdot (Y \cdot Z) \vdash W}{X \cdot (Z \cdot W) \vdash Y} \quad \text{A}_a \\
\text{W}_a & \quad \frac{X \vdash Y}{X \cdot Z \vdash Y} \quad & \frac{X \vdash Y}{X \cdot Y \cdot Z} \quad \text{W}_a \\
\text{C}_a & \quad \frac{X \cdot X \vdash Y}{X \vdash X} \quad & \frac{Y \vdash X \cdot X}{Y \vdash X} \quad \text{C}_a \\
\text{Gri}_a & \quad \frac{(X \succ Y) \cdot Z \vdash W}{X \succ (Y \cdot Z) \vdash W} \quad & \frac{X \succ (Y \cdot Z) \vdash W}{X \succ Y \cdot (Z \cdot W)} \quad \text{Gri}_a \\
\end{align*}
\]

The operational rules introducing \( \top \) and 0 are as in D.LL, while the introduction rules for \& and \( \oplus \) are now as follows:

- Pure additive Linear-type operational rules

\[
\begin{align*}
\& & \quad \frac{A \cdot B \vdash X}{A \& B \vdash X} \quad \frac{X \vdash A}{X \cdot Y \vdash A \& B} \quad \& \\
\oplus & \quad \frac{A \vdash X}{A \oplus B \vdash X \cdot Y} \quad \frac{B \vdash Y}{X \vdash A \cdot B} \quad \oplus \\
\end{align*}
\]

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Received 29 December 2021; revised 9 June 2022; accepted 26 September 2022