ON THE $\kappa$-SOLUTIONS OF THE RICCI FLOW ON NONCOMPACT 3-MANIFOLDS

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ABSTRACT. In this paper we prove that there is no $\kappa$-solution of Ricci flow on 3-dimensional noncompact manifold with strictly positive sectional curvature and blow up at some finite time $T$ satisfying \[ \int_0^T \sqrt{T-t} R(p_0, t) dt < \infty \] for some point $p_0$. This partially confirms a conjecture of Perelman.

1. Introduction

In [15] Perelman conjectured that there is no three-dimensional noncompact $\kappa$-solution with positive sectional curvature and blow-up at finite time. Recall the $\kappa$-solution is a complete non-flat ancient solution of Ricci flow that is $\kappa$-noncollapsed on all scales and has bounded nonnegative curvature at each time slice.

Our result partially confirms Perelman’s conjecture with an extra condition. The main result of this paper is following theorem.

Theorem 1.1. There is no $\kappa$-solution of Ricci flow on 3-dimensional noncompact manifold with strictly positive sectional curvature and blow up at some finite time $T > 0$ satisfying

\[ \int_0^T \sqrt{T-t} R(p_0, t) dt < \infty \] (1.1)

for some point $p_0$.

Remark 1.2. We remark that (1.1) holds if there exists some point $p_0$ such that $R(p_0, t) \leq \frac{c}{(T-t)^{\alpha}}$ with any $\alpha < \frac{3}{2}$.

Since the only orientable 2-dimensional $k$-solution for Ricci flow is $S^2$, by Hamilton’s strong maximum principle Theorem 1.1 implies the following classification theorem for the $\kappa$-solutions of 3-dimensional noncompact manifold satisfying the condition (1.1):

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Corollary 1.3. Let \((M^3, g(t))\) be the \(\kappa\)-solution to Ricci flow blow-up at finite time \(T\) on 3-dimensional noncompact manifold. If there exists a point \(p_0 \in M\) such that the condition (1.1) holds, \((M^3, g(t))\) must be \(S^2 \times \mathbb{R}\) or its quotient.

The main tool for the proof of Theorem 1.1 is the singular reduced entropy, which is obtained by taking the limit of the reduced length as the based point \((p_0, s_i)\) to the singular time. The singular reduced entropy was used by Naber [10] to show the asymptotic limit of Type I singularity of Ricci flow is shrinking soliton. By virtue of Hamilton’s harnack inequality and Perelman’s blow-down arguments, we can show that under the conditions of Theorem 1.1 the singular reduced volume share the same value as time goes to blow-up time \(T\) and goes to \(-\infty\). Then the singular reduced volume is constant independent of time and hence the Ricci flow is a gradient Ricci soliton which contradicts to the positive sectional curvature.

Finally, it is worth mentioning some related results. Hamilton [12] showed that the only orientable 2-dimensional \(k\)-solution is \(S^2\). Series works by Hamilton [12], Daskalopoulos, Hamilton, Sesum [5], Daskalopoulos and Sesum [4] classified the 2-dimensional ancient solutions with bounded curvature which is either \(S^2, \mathbb{R}^2\), cigar steady soliton, the King-Rosenau solution or their quotient. Ding [6] proved that the only simply connected noncompact 3-dimensional \(\kappa\)-solution that forms a forward singularity of Type I is the \(S^2 \times \mathbb{R}\). Zhang [18] and Hallgren [8] independently showed only simply connected noncompact 3-dimensional backward Type I \(\kappa\)-solution is the \(S^2 \times \mathbb{R}\). Ni [11] has proved that a closed Type I \(\kappa\)-solution with positive curvature operator of every dimension is a shrinking sphere or one of its quotients. For the mean curvature flow, Brendle and Choi [1] recently proved that every noncompact ancient solution of mean curvature flow in \(\mathbb{R}^3\) which is strictly convex and noncollapsed must be bowl soliton.

2. PRELIMINARIES

Perelman introduced in [15] the reduced entropy (i.e. reduced distance and reduced volume), which becomes one of powerful tools for studying Ricci flow. The reduced entropy enjoys very nice analytic and geometric properties, including in particular the monotonicity of the reduced volume. These properties can be used, as demonstrated by Perelman, to show the limit of the suitable rescaled Ricci flows is a gradient shrinking soliton.

We recall some basic formulas and properties about reduced entropy in [15]. Let \(g(t)\) solves the Ricci flow

\[
\frac{\partial g}{\partial t} = -2\text{Rc}.
\]
on $M \times (-\infty, T)$. Denote $h(\tau) = g(t)$ with $\tau(t) = T - t$. The $l$-length is defined

$$l_{p,s}(q, \tau) = \inf_{\gamma} \left\{ \frac{1}{2\sqrt{\tau - s}} \int_s^\tau \sqrt{\eta - s} (R_{h(\eta)}(\gamma(\eta))) + |\gamma'(\eta)|^2_{h(\eta)} d\eta \right\}, \quad (2.2)$$

where the infimum is taken over all curves $\gamma : [s, \tau] \to M$ with $\gamma(s) = p$ and $\gamma(\tau) = q$. The reduced volume is defined as

$$\mathcal{V}_{p,s}(\tau) = \int_M (4\pi(\tau - s))^{-\frac{n}{2}} e^{-l_{p,s}(x,\tau)} d\text{vol}_{h(\tau)}(x), \quad (2.3)$$

**Theorem 2.1** (Perelman). Let $l_{p,s}(q, \tau)$ be the reduced length defined in (2.2) and $L_{p,s}(q, \tau) = 2 \sqrt{\tau - s} l_{p,s}(q, \tau)$. Then $l_{p,s}(q, \tau)$ satisfies the following properties:

$$2 \frac{\partial l_{p,s}}{\partial \tau} + |\nabla l_{p,s}|^2 - R + \frac{l_{p,s}}{\tau - s} = 0, \quad (2.4)$$

$$|\nabla l_{p,s}|^2 = \frac{l_{p,s}}{\tau - s} - R - \frac{K}{(\tau - s)^2}, \quad (2.5)$$

$$\left(\frac{\partial}{\partial \tau} + \Delta\right) L_{p,s} \leq 2n, \quad (2.6)$$

$$\frac{\partial l_{p,s}}{\partial \tau} - \Delta l_{p,s} + |\nabla l_{p,s}|^2 - R + \frac{l_{p,s} - n}{2(\tau - s)} \geq 0, \quad (2.7)$$

$$2\Delta l_{p,s} - |\nabla l_{p,s}|^2 + R + \frac{l_{p,s} - n}{\tau - s} \leq 0. \quad (2.8)$$

For any $s$ and $\tau > s$, there exists $q^* \in M$ such that $l_{p,s}(q^*, \tau) \leq \frac{2}{\tau}$. The reduced volume defined in (2.3) is monotone non-increasing along the backward Ricci flow $(M, h(\tau))$. Moreover, if the Ricci flow has nonnegative curvature operator,

$$|\nabla l_{p,s}|^2 + R \leq c \frac{l_{p,s}}{\tau - s} \quad (2.9)$$

$$l_{p,s}(x, \tau) \geq -l_{p,s}(y, \tau) - 1 + c \frac{d_{h(\tau)}^2(x, y)}{\tau - s} \quad (2.10)$$

$$l_{p,s}^{\frac{1}{2}}(x, \tau) \leq l_{p,s}^{\frac{1}{2}}(y, \tau) + c \frac{d_{h(\tau)}(x, y)}{\sqrt{\tau - s}} \quad (2.11)$$

**Proof.** One can find the details of the proofs of Theorem 2.1 for the version $s = 0$ in [15] (also see in [14] and [3]). Just shifting the time as $\tilde{h}(\tau - s) = h(\tau)$, one can get Theorem 2.1. □

The following theorem by Perelman can be found the proof in [15] (also see in [14] and [3]).
Theorem 2.2 (Perelman). If there exists \( l \in C^{0,1}(M \times (0, +\infty)) \cap W^{1,2}(M \times (0, +\infty)) \) with for \( s = 0 \) \((2.4), (2.5), (2.9)\) hold almost everywhere, \((2.10), (2.11)\) hold and \((2.6), (2.7)\) hold for distribution sense, moreover \( V(\tau) \triangleq \int_M (4\pi\tau)^{-\frac{1}{2}} e^{-l(x, \tau)} dvol_{h(\tau)}(x) \) is constant under the backward Ricci flow \((M, h(\tau))\), then \((M, h(\tau))\) is gradient shrinking soliton and \( l \) is the smooth soliton function satisfying

\[
R_{h} + \text{Hess} l = \frac{h}{2\tau},
\]

and

\[
2\Delta l - |\nabla l|^2 + R_{h} + \frac{l - n}{\tau} = 0.
\]

3. The proof of Theorem 1.1

Before presenting the proof of Theorem 1.1, we need the following \( \epsilon \)-regularity theorem for \( \kappa \)-solutions of Ricci flow.

Lemma 3.1. Let \((M, g(t))\) be the \( \kappa \)-solution to the Ricci flow on the 3-dimensional manifold for \( t \in (-\infty, T]\). Denote \( h(\tau) = g(t) \) with \( \tau(t) = T - t \) and \( V_{p, s} \) be the reduced volume defined in \((2.3)\). Then there exists an universal constant \( \epsilon_0 \) such that if \( V_{p, s}(\tau_0) \geq 1 - \epsilon_0 \), we have \( r_{Rm}(p, s) \geq \epsilon_0(\tau_0 - s) \) where \( r_{Rm}(p, s) = \sup\{r > 0 : \sup_{B_{g(t)}(x, r)(s - r^2, s)} |Rm| \leq r^2\} \).

Proof. By shifting time and rescaling, we may assume \( s = 0 \) and \( \tau_0 = 1 \). We argue by contradiction. In this case we have that there exists a sequence of \( \kappa \)-solutions \((M_i, g_i(t), p_i)\) such that

\[
V_{p_i, 0}(1) \geq 1 - \frac{1}{i}
\]

and

\[
r_{Rm}(p_i, 0) \leq \frac{1}{i}.
\]

Define \( r_i = r_{Rm}(p_i, 0) \). Consider the rescaled flows \((M_i, \tilde{g}_i(t), p_i)\) with \( \tilde{g}_i(t) = r_i^{-2} g_i(t^2) \). Clearly, \( \tilde{r}_{Rm}(p_i, 0) = 1 \). By Perelman’s result, we have \( R_{\tilde{g}_i}(q, 0) \leq C(r) \) if \( d_{\tilde{g}_i(0)}(q, p_0) \leq r \). Then by Hamilton’s harnack inequality, we get \( R_{\tilde{g}_i}(q, t) \leq C(r) \) if \( d_{\tilde{g}_i(0)}(q, p_0) \leq r \). This allows us to pass to a subsequence to derive a pointed limit

\[
(M_i, \tilde{g}_i(t), p_i) \underset{C^\infty}{\rightarrow} (M_\infty, \tilde{g}_\infty(t), p_\infty),
\]

with

\[
\tilde{r}_{Rm}(p_\infty, 0) = 1.
\]
Take the constant path $\gamma$ in $p_i$, we have $\tilde{l}_{p_i,0}(p_i, \tau) \leq \tilde{l}(\gamma) \leq \frac{3}{4}$ for $0 \leq \tau \leq 1$, where $\tilde{l}_{p_i,0}$ is the reduced length with respect to $\tilde{g}_i$. Then by (2.10) and (2.11)

$$-rac{7}{4} + c \frac{d^2_{h(\tau)}(q, p_i)}{\tau} \leq \tilde{l}_{p_i,0}(q, \tau) \leq \left(\frac{\sqrt{3}}{2} + c \frac{d_{h(\tau)}(q, p_i)}{\sqrt{\tau}}\right)^2.\quad (3.3)$$

Combining (2.9) and (3.3), we obtain that passing by a subsequence $\tilde{l}_{p_i,0} \rightarrow \tilde{l}_{p_{\infty},0}$ for $0 \leq \tau \leq 1$, where $\tilde{l}_{p_{\infty},0}$ is the reduced length with respect to $\tilde{g}_{\infty}(t)$. Hence $\tilde{V}_{p_{\infty},0}(\tau) \rightarrow \tilde{V}_{p_{\infty},0}(\tau)$ for $0 \leq \tau \leq 1$. It follows from (3.1) that $\tilde{V}_{p_{\infty},0}(1) = 1$. This implies $(M_{\infty}, \tilde{h}_{\infty}(t))$ is the gradient solution with singular type at $\tau = 0$. But then the only way for the curvature to stay bounded as $\tau \rightarrow 0$ is if $(M_{\infty}, \tilde{g}_{\infty}(t))$ is flat for all time. This contradicts (3.2) and thus proves the lemma.

□

Now we give the proof of Theorem 1.1.

**Proof of Theorem 1.1.**

We argue by contradiction. If there exists $k$-solution of Ricci flow on 3-dimensional noncompact manifold with strictly positive sectional curvature and blow up at some finite time $T$ satisfying the condition (1.1) for some point $p_0$. Denote $h(\tau) = g(t)$ with $\tau(t) = T - t$. We divide the proof of Theorem 1.1 into the following four steps.

**Step 1:** By taking $s_i \rightarrow 0^-$, we show $l_{p_{si}}$ subconverges to a limit $l_{p_{0,0}}$.

Take the constant path $\gamma$ in $p_0$, the condition (1.1) implies

$$l_{p_{0,si}}(p_0, \tau) \leq l(\gamma) \leq \frac{\int_{T-\tau}^{T} \sqrt{T - tR(p_0, t)} dt}{2 \sqrt{T-s_i}} \leq C(\tau)\quad (3.4)$$

as $s_i \rightarrow 0$. Combining this with (2.4), we obtain that there exists $l_{p_{0,0}} \in C^{0,1}(M \times (0, +\infty))$

$$l_{p_{0,si}} \rightarrow l_{p_{0,0}},$$

in $C^{0,\alpha}_{loc}(M \times (0, +\infty))$ and weakly in $W^{1,2}_{loc}(M \times (0, +\infty))$ with (2.4), (2.5), (2.9), (2.10), (2.11) hold and (2.6), (2.7) hold for distribution sense for $s = 0$. Hence $\tilde{V}_{p_{0,0}}(\tau) = \frac{4\pi(\gamma)^2}{\tau} e^{-\int_{p_0}^{p_0}(x, \tau) d\nu_{h(\tau)}(x)}$ is monotone non-increasing under the backward Ricci flow $(M, h(\tau))$.

By Theorem 2.1, for any fixed $\tau$ there exists $q^\tau_i \in M$ such that $l_{p,si} (q^\tau_i, \tau) \leq \frac{\#}{2}$. Moreover, it follows from (2.10) that

$$cd_{h(\tau)}(p_0, q^\tau_i) \leq (\tau - s_i)(l_{p_{0,si}}(p_0, \tau) + l_{p_{0,si}}(q^\tau_i, \tau) + 1) \leq C(\tau)$$

as $s_i \rightarrow 0$. Then for any fixed $\tau$, we have $q^\tau_i \rightarrow q^\tau$ and

$$l_{p_{0,0}}(q^\tau, \tau) \leq \frac{n}{2}.\quad (3.5)$$
Step 2: Rescale the backward Ricci flow as $h_i(\tau) = (\tau_i)^{-1} h(\tau, \tau_i)$. Taking $\tau_i \to 0^-$ and $\tau_i \to +\infty$, we show that $(M, h_i(\tau), q_{\tau_i})$ both subconverge to the gradient shrinking solitons which is $\kappa$–noncollapsed on all scales.

Letting $s_i \to 0$ in (2.10) and (2.11), we obtain

$$l_{p_{0},0}(x,\tau) \geq -l_{p_{0},0}(y,\tau) - 1 + c \frac{d^2 h_i(\tau)(x, y)}{\tau}, \quad (3.6)$$

$$l^p_{p_{0},0}(x,\tau) \leq l^p_{p_{0},0}(y,\tau) + c \frac{d h_i(\tau)(x, y)}{\sqrt{\tau}}. \quad (3.7)$$

By the scaling property for $l_{p_{0},0}$, we obtain

$$l^i_{p_{0},0}(x,\tau) \geq -l^i_{p_{0},0}(y,\tau) - 1 + c \frac{d^2 h_i(\tau)(x, y)}{\tau}, \quad (3.8)$$

$$(l^i_{p_{0},0})^2(x,\tau) \leq (l^i_{p_{0},0})^2(y,\tau) + c \frac{d h_i(\tau)(x, y)}{\sqrt{\tau}}. \quad (3.9)$$

where $l^i_{p_{0},0}$ is the reduced length with respect to $h_i(\tau)$. Since $l_{p_{0},0}(q_{\tau_i}, \tau_i) \leq \frac{n}{2}$, $l^i_{p_{0},0}(q_{\tau_i},1) \leq \frac{n}{2}$. It follows from (2.9) and (2.4) that $l^i_{p_{0},0}(q_{\tau_i}, \tau) \leq C(\tau)$. Then

$$-C(\tau) - 1 + c \frac{d^2 h_i(\tau)(x, q_{\tau_i})}{\tau} \leq l^i_{p_{0},0}(x,\tau) \leq (C(\tau)^\frac{1}{2} + c \frac{d h_i(\tau)(x, q_{\tau_i})}{\sqrt{\tau}})^2, \quad (3.10)$$

$$|\nabla l^i_{p_{0},0}|^2(x,\tau) + R_{h_i(\tau)}(x,\tau) \leq \frac{c}{\tau} (C(\tau)^\frac{1}{2} + c \frac{d h_i(\tau)(x, q_{\tau_i})}{\sqrt{\tau}})^2. \quad (3.11)$$

Hence $(M, h_i(\tau), q_{\tau_i})$ both subconverge to the $\kappa$–noncollapsed on all scales Ricci flows $(M^\pm_\infty, h^\pm_\infty(\tau), q^\pm_\infty)$ as $\tau_i \to 0^-$ and $\tau_i \to +\infty$. Moreover, there exist $l^\pm \in C^0(M^\pm_\infty \times (0, +\infty))$ such that $l^i_{p_{0},0} \circ \phi_i \to l^\pm$ in $L^2(M^\pm_\infty \times (0, +\infty))$ and weakly in $W^{1,2}(M^\pm_\infty \times (0, +\infty))$ as $\tau_i \to 0^-$ and $\tau_i \to +\infty$ with (2.4), (2.5), (2.9), (2.10), (2.11) hold and (2.6), (2.7) hold for distribution sense for $s = 0$. By (3.10), scaling and monotone property of reduced volume, we have

$$\mathcal{V}^\pm_+(\tau) = \lim_{\tau_i \to +\infty} \mathcal{V}^\tau_{p_{0},0}(\tau_i) = \lim_{\tau_i \to +\infty} \mathcal{V}^\tau_{p_{0},0}(\tau_i,\tau) \equiv c^+, \quad (3.12)$$

and

$$\mathcal{V}^\pm_-(\tau) = \lim_{\tau_i \to 0^-} \mathcal{V}^\tau_{p_{0},0}(\tau_i) = \lim_{\tau_i \to 0^-} \mathcal{V}^\tau_{p_{0},0}(\tau_i,\tau) \equiv c^-. \quad (3.13)$$

Then Theorem [2.2] implies $(M^\pm_\infty, h^\pm_\infty(\tau), q^\pm_\infty)$ are gradient shrinking solitons and $l^\pm$ are soliton functions satisfying

$$Rc_{h^\pm_\infty} + Hess l^\pm = \frac{h^\pm_\infty}{2\tau}. \quad (3.14)$$
Step 3: Next we show that the limits \((M_\infty^\pm, h_\infty^\pm(\tau), q_\infty^\pm)\) are non-flat. If the limit shrinking soliton \((M_\infty^\pm, h_\infty^\pm(\tau), q_\infty^\pm)\) is flat, \((M_\infty^\pm, h_\infty^\pm(\tau))\) is isometric to Euclidean space. This implies \(\lim_{\tau \to \infty} V(\tau) = 1\) by (3.13). Hence there exists \(\tau_0 > 0\) such that \(V(p_0, 0)(\tau_0) \geq 1 - \frac{\epsilon_0}{2}\). By Lebesgue theorem, we have \(V(p_0, 0)(\tau) \to V(p_0, 0)(\tau_0)\) for any \(\tau\) as \(s_i \to 0\). Then there exists \(N > 0\) such that \(V(p_0, 0)(\tau_i) \geq 1 - \epsilon_0\) when \(i \geq N\). By Theorem 3.1, \(r_{Rm}(p_0, s_i) \geq \epsilon_0|\tau_0 - s_i|\) for \(i \geq N\). Hence \(\limsup_{s_i \to 0} r_{Rm}(p_0, s_i) \leq (\epsilon_0|\tau_0|)^{-2}\). This contradicts that \(p_0\) is a blow-up point. Then we have proved that \((M_\infty^\pm, h_\infty^\pm(\tau))\) is non-flat and \(\lim_{\tau \to \infty} V(p_0, 0)(\tau) < 1\). Since \(V(p_0, 0)(\tau)\) is monotone non-increasing in \(\tau\), we have \(\lim_{\tau \to \infty} V(p_0, 0)(\tau) < 1\). Then by (3.12), similar arguments show that \((M_\infty^\pm, h_\infty^\pm(\tau))\) is non-flat.

Step 4: Finally, we show that \((M, g(t))\) is shrinking soliton on \(S^2 \times \mathbb{R}\) which contradicts that \((M, g(t))\) has the strictly positive sectional curvature.

Note that only the non-flat \(\kappa\)-noncollapsed noncompact 3-dimensional shrinking soliton are \(S^2 \times \mathbb{R}, R\mathbb{P}^2 \times \mathbb{R}\) and \((S^2 \times \mathbb{R})/\mathbb{Z}_2\) (see [15]). Since \((M, g(t))\) has positive sectional curvature, it is diffeomorphic to \(\mathbb{R}^3\), and \(S^2 \times \mathbb{R}\) is the only one that can arise as the limit of a sequence of Ricci flows that are diffeomorphic to \(\mathbb{R}^3\). It follows that both \((M_\infty^\pm, h_\infty^\pm(\tau), q_\infty^\pm)\) are \(S^2 \times \mathbb{R}\).

Denoting by \(x\) the coordinates on \(S^2\) and by \(y\) the coordinates on \(\mathbb{R}\), the both potentials also can be splitted as \(I^\pm(x, y, \tau) = I^\pm_{S^2}(x, \tau) + I^\pm_{\mathbb{R}}(y, \tau)\) with \(I^\pm_{S^2}(x, \tau)\) and \(I^\pm_{\mathbb{R}}(y, \tau)\) both satisfying the soliton equation (3.14) on \(S^2\) and \(\mathbb{R}\) respectively. By (3.14) and the compactness for \(S^2\), we have both \(I^\pm_{S^2}(x, 1)\) are constants. It follows from the soliton equation on \(\mathbb{R}\) that we have \(I^\pm_{\mathbb{R}}(y, 1) = \frac{1}{2}y^2 + c^\pm_1 y + c^\pm_2\). Since \(I^\pm_{\mathbb{R}}(y, 1)\) satisfies (3.15), we conclude that \(c^\pm_2 = (c^\pm_1)^2 + 1\) and hence \(I^\pm(x, y, 1) = \frac{1}{2}y^2 + 2c^\pm_1y + c^\pm_2\). Then \(V^\infty(\tau) = \lim_{\tau \to \infty} V^\infty(\tau) < 1\) and hence \(V^\infty(\tau) \equiv constant\). It follows from Theorem 2.2 that \((M, g(t))\) is shrinking soliton on \(S^2 \times \mathbb{R}\) which contradicts that \((M, g(t))\) has the strictly positive sectional curvature. 

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