Lattice HQET Calculation of the Isgur-Wise Function

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1. The Tadpole-Improved Simulation

The Isgur-Wise function is the form factor of a heavy-light meson in which the heavy quark is taken to be much heavier than the energy scale, $m_Q \gg \Lambda_{QCD}$. This calculation adds perturbative corrections to the simulation results of Draper & McNeile [1]. The Isgur-Wise function is calculated using the action first suggested by Mandula & Ogilvie [2]:

$$iS = \sum_x \left\{ v_0 \left[ \psi^\dagger(x) \psi(x) - \psi^\dagger(x) \frac{U_t(x)}{u_0} \psi(x + \hat{t}) \right] + \sum_{j=1}^{3} \frac{-i v_j}{2} \left[ \psi^\dagger(x) \frac{U_j(x)}{u_0} \psi(x + \hat{j}) \right. \right.$$

$$\left. \left. - \psi^\dagger(x) \frac{U_j(x - \hat{j})}{u_0} \psi(x - \hat{j}) \right] \right\}$$

This leads to the evolution equation:

$$G(x + \hat{t}) = \frac{U_t(x)}{u_0} \left\{ G(x) \right. \right.$$

$$\left. \left. - \sum_{j=1}^{3} \frac{i v_j}{2} \left[ \frac{U_j(x)}{u_0} G(x + \hat{j}) - \frac{U_j(x - \hat{j})}{u_0} G(x - \hat{j}) \right] \right\}$$

2. The Isgur-Wise Function

The Isgur-Wise function was extracted from the lattice simulation as a ratio of three-point functions which was suggested by Mandula & Ogilvie [2]: $|\xi_{\text{lat}}(v \cdot v')|^2$ is the large $\Delta t$ limit of

$$\frac{4 v_0 v_0'}{(v_0 + v_0')^2} \frac{C_{v'v}(\Delta t) C_{v'v}(\Delta t)}{C_{vv}(\Delta t) C_{vv}(\Delta t)}$$

where $\Delta t$ is the time separation between the current operator and each $B$-meson interpolating field.

Draper & McNeile have presented [1] the non-tadpole-improved unrenormalized slope of the lattice Isgur-Wise function to demonstrate the efficiency of the computational techniques.

3. Tadpole Improvement for HQET

Tadpole improvement grew from the observation that lattice links, $U$, have mean field value, $u_0 \neq 1$. Therefore, it is better to use an action written as a function of $U/u_0$. In the Wilson action, each link has a coefficient $\kappa$; $u_0$ can be paired with $\kappa$ to easily tadpole improve $a \text{ posteriori}$ any previous non-tadpole-improved calculation.
In the HQET, there is no common coefficient (analogous to $\kappa$) for both $U_i$ and $U_j$. Correspondence between tadpole-improved and non-tadpole-improved HQET actions with $v^2 = 1$ cannot be made via a simple rescaling of parameters, as is done for the Wilson action with $\kappa$.

Fortunately, the evolution equation can be written (as noticed by Mandula & Ogilvie) such that the $u_0$ is grouped with $\tilde{v}_j$. Thus, tadpole-improved Monte-Carlo data can be constructed from the non-tadpole-improved data by replacing $v^\text{nt} \rightarrow v^\text{tad}$ and by including two overall multiplicative factors:

$$ G^\text{tad}(t; \tilde{v}_j, v^\text{nt}_0) = u_0^{-t} G^\text{nt}(t; \tilde{v}_j, v^\text{nt}_0) \quad (1) $$

In addition to the multiplicative factors $u_0^{-t}$ and $v^\text{nt}_0/v^\text{tad}_0$, the tadpole-improvement of the simulation requires adjusting the velocity according to $\tilde{v}_j = u_0 \tilde{v}_j^\text{nt}$, subject to $(v^\text{tad})^2 = 1$ and $(v^\text{nt})^2 = 1$. Thus,

$$ v^\text{tad}_0 = v^\text{nt}_0 [1 + (1 - u_0^2) (v^\text{nt}_j)^2]^{-\frac{1}{2}} $$

$$ v^\text{tad}_j = u_0 v^\text{nt}_j [1 + (1 - u_0^2) (v^\text{nt}_j)^2]^{-\frac{1}{2}} $$

4. Tadpole-Improved Renormalization

By comparing the unrenormalized propagator

$$ \left[ \frac{e^{ik_k}}{v^\text{nt}_0} - 1 + \frac{v^\text{nt}_b}{u_0} \sin(k_z) + M^b_0 - \Sigma(k, v) \right]^{-1} $$

to the renormalized propagator

$$ iH(k, v) = Z_Q \left[ e^{i(k_k)} + v^\text{nt}_j \sin(k_z) + M^j \right]^{-1} $$

at $O(k^2)$ and using $(v^\text{nt})^2 = (v^b)^2 = 1$, the perturbative renormalizations can be obtained. Aglietti has done this for a different non-tadpole-improved action, for the special case $\tilde{v} = v_z \tilde{z}$.

With momentum shift, $p \rightarrow p' = (p_3 + i \ln(u_0), \vec{p})$, with $\frac{1}{u_0} \exp(i k_3) = \exp(i(k_k + i \ln(u_0)))$ and with $X_{\mu} = \frac{\delta}{\delta p_\mu} \Sigma(p)|_{p_0 = 0}$, the tadpole-improved perturbative renormalizations are found to be

$$ \delta M = -\Sigma(0) - v_0 \ln(u_0) $$

$$ \delta Z_Q = Z_Q - 1 = -i v_0 X_4 - u_0 \sum_{j=1}^3 v_j X_j $$

$$ \frac{\delta v_0}{v_0} = -i v_0 v_j X_4 - (1 + v_j^2) X_i - v_j \sum_{j \neq i} v_j X_j $$

$$ u_0 \text{ is the perturbative expansion and } $$

$$ v^\text{r, tad}_j = v^\text{b, tad}_j Z^\text{tad}_v \quad, \quad v^\text{r, tad}_0 = v^\text{b, tad}_0 Z^\text{tad}_v $$

$$ Z^\text{tad}_v \equiv 1 + \frac{\delta v_0}{v_0} \quad, \quad Z^\text{tad}_v \equiv 1 + \frac{\delta v_0}{v_0} $$

If one fits to $\exp\{-t\}$ rather than $\exp\{-t + 1\}$, the tadpole-improved wave-function renormalization is reduced to $\delta Z^\text{Q} = \delta Z^\text{Q} + (\delta M^\text{tad} + v_0 \ln(u_0))/v_0$. Thus the $\ln(u_0)$ term cancels explicitly and, as in the static case, tadpole-improvement has no effect on $\delta Z^\text{Q}$, to order $\alpha$.

5. Perturbative Renormalizations

We will present our computations of the renormalization factors elsewhere, but include this comment: Although the tadpole-improved functions include factors of $u_0|_{\text{pt}}$, these effects are higher order in $\alpha$ and are dropped. Only the velocity renormalization is explicitly affected by the perturbative expansion of $u_0$:

$$ Z^\text{tad}_{v_j} = \left( 1 + \frac{\delta v_0}{v_0} \right) - \frac{g^2 C_F}{16\pi^2} (\pi^2) $$

The perturbative renormalizations favor a scale of $g^* a = 1.9(1)$ for $\alpha$, which yields $\alpha \approx 0.19(1)$.

6. Velocity Renormalization

Mandula & Ogilvie consider the perturbative velocity renormalization expanded in orders of $\tilde{v}$. Our numbers for the velocity renormalization agree with theirs.

Another option is to consider, as did both Mandula & Ogilvie and Hashimoto & Matsufuru, the non-perturbative renormalization of the velocity. From Hashimoto’s & Matsufuru’s graph, we estimate their $Z^\text{nt}_{v_\text{np}} \approx 1.05(5)$. From Mandula & Ogilvie’s result, we notice that $Z^\text{nt}_{v_\text{np}} = 4$.
\( u_0 \times 1.01(1) \). This is very close to the effect of tadpole-improving, and implies \( Z_\xi^\text{tad} = 1.01(1) \).

We therefore use \( Z_\xi^\text{tad} \approx 1 \) as the non-perturbative velocity renormalization in our calculation to renormalize the slope of the Isgur-Wise function.

7. Renormalization of \( \xi'(v \cdot v') \)

We claim that we can convert our Monte-Carlo data into tadpole-improved results and can calculate a renormalized tadpole-improved slope for the Isgur-Wise function.

We use the notation \( Z_\xi' = 1 + \delta Z_\xi' \) for the renormalization of the Isgur-Wise function, with \( \delta Z_\xi' \):

\[
\frac{g^2}{12\pi^2} \left[ 2(1 - v \cdot v') r(v \cdot v') \ln(\mu a)^2 - f'(v, v') \right]
\]

with \( r(v \cdot v') \) defined in [9] and primes on \( Z \) and \( f \) to indicate the “reduced value.”

For simplicity, we use the local current, which is not conserved on the lattice; \( Z_\xi'(1) \neq 1 \). However, the construction in [2] guarantees that the extracted renormalized Isgur-Wise function is properly normalized, \( \xi_{\text{ren}}(1) = 1 \).

8. Conclusions

After renormalization of our tadpole-improved results, we obtain \( \xi_{\text{ren}}(1) = -0.64(13) \) for the slope of the Isgur-Wise function in continuum HQET in the \( \overline{\text{MS}} \) scheme at a scale of 4.0 GeV. Without renormalization, the slope is \( \xi_{\text{unren}}(1) = -0.56(13) \). Without tadpole-improvement, the slope is \( \xi_{\text{unren}}(1) = -0.43(10) \).

We found that the tadpole-improved action (and therefore the tadpole-improved data) cannot be obtained from the non-tadpole-improved action (or data) by a simple rescaling of any parameter. However, the form of the evolution equation allows the construction of the tadpole-improved Monte-Carlo data from the non-tadpole-improved data as described in [2]. After tadpole improvement, non-perturbative corrections to the velocity are negligible. Furthermore, tadpole improvement greatly reduces the perturbative corrections to the slope of the Isgur-Wise function.

### Table 1

| \( \Delta t \) | Not Tadpole Improved | Tadpole Improved |
|----------------|----------------------|-----------------|
| \( \kappa_c \) | \( \xi_{\text{ren}}(1) \) | \( \xi_{\text{ren}}(1) \) |
| 2 | 0.38^{+1}_{-1} | 0.38^{+1}_{-1} |
| 3 | 0.42^{+2}_{-2} | 0.41^{+2}_{-2} |
| 4 | 0.50^{+3}_{-3} | 0.48^{+3}_{-3} |

The negative of the slope at the normalization point, \( \xi'(1) \), from both the unrenormalized and the renormalized ratio of three-point functions. This ratio gives the (un)-renormalized Isgur-Wise function \( \xi(v \cdot v') \) at asymptotically-large times \( \Delta t \).

## References

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