The Cartan domains are among the important subjects in many problems of representation
theory and mathematical physics [1, 4]. The methods of quantum groups theory [2] were used
in [7] to produce q-analogues of Cartan domains, in particular, q-analogues of balls in the
spaces of complex matrices.

The point of this work is to consider those quantum matrix balls and the associated
Hilbert spaces of 'functions'. As a main result, we present an explicit formula for the weighted
Bergman kernel.

It is implicit everywhere in the sequel that
\( q \in (0, 1), \quad m, n \in \mathbb{N}, \quad \text{and} \quad m \leq n. \)

We need q-analogues for \(*\)-algebras \( \text{Pol} (\text{Mat}_{m n}) \) of polynomials on the space \( \text{Mat}_{m n} \) of
complex matrices and a \(*\)-algebra \( D(\mathbb{U}) \) of smooth finite functions in the matrix ball \( \mathbb{U} = \{ A \in \text{Mat}_{m n} | \| A \| < 1 \} \). We start with forming a q-analogue for the algebra \( \text{Fun}(\mathbb{U}) = \text{Pol}(\text{Mat}_{m n}) + D(\mathbb{U}) \).

The \(*\)-algebra \( \text{Fun}(\mathbb{U})_q \) is given by its generators \( f_0, z_\alpha^a, a = 1, 2, \ldots, n, \alpha = 1, 2, \ldots, m, \) and the relations
\[
\begin{align*}
(z_\alpha^a z_\beta^b)_q &= q \cdot z_\beta^b z_\alpha^a, \quad a = b \quad \text{or} \quad a < b & \alpha < \beta \\
z_\alpha^a z_\beta^b &= z_\beta^b z_\alpha^a, \quad \alpha < \beta \quad \text{and} \quad a > b \\
z_\alpha^a z_\beta^b - z_\beta^b z_\alpha^a &= (q - q^{-1}) z_\beta^b z_\alpha^a, \quad \alpha < \beta \quad \text{and} \quad a < b
\end{align*}
\]
(1)

\[
\left( z_\alpha^a \right)^* = q^2 \sum_{a', b'=1}^n R(b, a, b', a') R(\beta, \alpha, \alpha', \alpha') \cdot z_\alpha^{a'} \cdot \left( z_\beta^{b'} \right)^* + (1 - q^2) \delta_{ab} \delta_{\alpha \beta},
\]
(2)

\[
(z_\alpha^a)^* f_0 = f_0 z_\alpha^a = 0,
\]
(3)

Here \( a, b = 1, 2, \ldots, n, \alpha, \beta = 1, 2, \ldots, m, \)

\[
R(i, j, i', j') = \begin{cases} 
q^{-1}, & i \neq j \quad \text{and} \quad i = i' \quad \text{and} \quad j = j' \\
1, & i = j = i' = j' \\
-(q^{-2} - 1), & i = j \quad \text{and} \quad i' = j' \quad \text{and} \quad j' > j \\
0, & \text{otherwise}
\end{cases}
\]

In this setting, the \(*\)-subalgebra \( \text{Pol}(\text{Mat}_{m n})_q \subset \text{Fun}(\mathbb{U})_q \) generated by \( z_\alpha^a, a = 1, 2, \ldots, n, \alpha = 1, 2, \ldots, m \) is a q-analogue of the \(*\)-algebra \( \text{Pol}(\text{Mat}_{m n}) \), and the bilateral ideal \( D(\mathbb{U})_q = \)}
Pol(Mat_{mn})_q f_0 Pol(Mat_{mn})_q is a q-analogue of the *-algebra $D(U)$. (The element $f_0$ works here as a delta-function, as one can see from (3)).

To motivate our subsequent constructions, observe that (see (3)) the *-algebra $D(U)_q$ is a $U_q \mathfrak{su}_{mn}$-module algebra [3]. Remind the explicit formula for invariant integral from [6].

Consider the representation $T$ of $\text{Fun}(U)_q$ in the space $\mathcal{H} = \text{Fun}(U)_q f_0 = \text{Pol}((\text{Mat}_{mn})_q f_0)$. There exists a unique positive scalar product in $\mathcal{H}$ such that $(f_0, f_0) = 1$, and

$$(T(f)\psi, \phi) = (\psi, T(f^*)\phi), \quad f \in \text{Fun}(U)_q, \quad \psi, \phi \in \mathcal{H}.$$ \hspace{1cm} (4)

One can prove that the *-algebra $\text{Pol}((\text{Mat}_{mn})_q)$ admits a unique up to unitary equivalence faithful irreducible *-representation by bounded operators in a Hilbert space. This *-representation can be produced via extending the operators $T(f), f \in \text{Pol}((\text{Mat}_{mn})_q)$, onto the completion of the pre-Hilbert space $\mathcal{H}$.

The invariant integral is of the form (see [6]):

$$\int_{U_q} f d\nu = \text{tr}(T(f)q^{-2T(\hat{\rho})}), \quad f \in D(U)_q,$$

with $\Gamma : \mathfrak{h} \rightarrow \text{End}(_{\mathcal{H}})$ being a subrepresentation of the natural representation of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{sl}_N$ in $\text{Fun}(U)_q$, and $\hat{\rho} \in \mathfrak{h}$ the element of this Cartan subalgebra determined by the half sum of positive roots $\rho$ under the standard pairing of $\mathfrak{h}$ and $\mathfrak{h}^*$. (To see that this integral is well defined, observe that the operators $T(f), f \in D(U)_q$, are finite dimensional, and $\mathcal{H}$ is decomposable into a sum of weight subspaces associated to non-negative weights.)

Our immediate intention is to produce q-analogues of weighted Bergman spaces. In the case $q = 1$ one has

$$\det(1 - zz^*) = 1 + \sum_{k=1}^{m} (-1)^k z^{\wedge k} z^{* \wedge k}, \quad (5)$$

with $z^{\wedge k}, z^{* \wedge k}$ being the "exterior powers" of the matrices $z, z^*$, that is, matrices formed by the minors of order $k$. The operators $(1 - q^2)^{-1/2}(z^a_0), (1 - q^2)^{-1/2}(z^a_0^*)$, $a = 1, 2, \ldots, n$, are respectively the q-analogues of creation and annihilation operators. The "creation operators" are placed in the right hand side of (5) to the left of the "annihilation operators". This allows one to produce a q-analogue of the polynomial $\det(1 - zz^*)$ in a standard way as follows.

Let $1 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_k \leq m$, $1 \leq a_1 < a_2 < \ldots < a_k \leq n$. Introduce q-analogues of minors for the matrix $z$:

$$z^{\wedge k}_{\{a_1, a_2, \ldots, a_k\}} = \sum_{s \in S_k} (-q)^{l(s)} z_{a_1}^{\alpha_s(1)} z_{a_2}^{\alpha_s(2)} \cdots z_{a_k}^{\alpha_s(k)},$$

with $l(s) = \text{card}\{\{i, j\} \mid i < j \quad \& \quad s(i) > s(j)\}$ being the length of the permutation $s$.

The q-analogue $y \in \text{Pol}((\text{Mat}_{mn})_q)$ for the polynomial $\det(1 - zz^*)$ is defined by

$$y = 1 + \sum_{k=1}^{m} (-1)^k \sum_{\{J^e\} \text{ card}(J^e) = k} \sum_{\{J^m\} \text{ card}(J^m) = k} z^{\wedge k}_{J^e} z^{* \wedge k}_{J^m} \cdot \left(z^{\wedge k}_{J^e} z^{* \wedge k}_{J^m}\right)^*.$$

2
Let $\lambda > m + n - 1$. Now (5) allows one to define the integral with weight $y^\lambda$ as follows:
\[
\int f \, d\nu_\lambda \overset{\text{def}}{=} C(\lambda) \text{tr} (T(f) T(y)^\lambda q^{-2\Gamma(\rho)}), \quad f \in D(\mathbb{U})_q,
\]
with $C(\lambda) = \prod_{j=0}^{n-1} \prod_{k=0}^{m-1} (1 - q^{2(\lambda+1-N)} q^{2(j+k)})$ being a normalizing multiple that provides $\int_{\mathbb{U}} 1 \, d\nu_\lambda = 1$.

The Hilbert space $L^2(d\nu_\lambda)_q$ is defined as a completion of the space $D(\mathbb{U})_q$ of finite functions with respect to the norm $\|f\|_\lambda = \left( \int_{\mathbb{U}} f^* f \, d\nu_\lambda \right)^{1/2}$. The closure $L^2_\alpha(d\nu_\lambda)_q$ in $L^2(d\nu_\lambda)_q$ of the unital subalgebra $\mathbb{C}[\text{Mat}_{mn}]_q \subset \text{Pol}(\text{Mat}_{mn})_q$ generated by $z^\alpha_a$, $a = 1, 2, \ldots, n$, $\alpha = 1, 2, \ldots, m$, will be called a weighted Bergman space.

Note that the relations (1) and the algebra $\mathbb{C}$ are considered in many works on quantum groups (see [2]). The *-algebra $\text{Pol}(\text{Mat}_{mn})_q$ determined by the relations (1) and (2), is a $q$-analogue of the Weyl algebra. This becomes plausible after one performs the 'change of variables' as follows:
\[
z^\alpha_a \mapsto (1 - q^2)^{-1/2} z^\alpha_a; \quad (z^\alpha_a)^* \mapsto (1 - q^2)^{-1/2} (z^\alpha_a)^*.
\]

Consider the orthogonal projection $P_\lambda$ in $L^2(d\nu_\lambda)_q$ onto the weighted Bergman space $L^2_\alpha(d\nu_\lambda)_q$. It is possible to show that $P_\lambda$ could be written as an integral operator (see [3])
\[
P_\lambda f = \int_{\mathbb{U}} K_\lambda(z, \zeta^*) f(\zeta) \, d\nu_\lambda(\zeta), \quad f \in D(\mathbb{U})_q.
\]

Our intention is to introduce the algebra $\mathbb{C}[\text{Mat}_{mn} \times \overline{\text{Mat}_{mn}}]_q$ of kernels of integral operators and to determine an explicit form of the weighted Bergman kernel $K_\lambda \in \mathbb{C}[\text{Mat}_{mn} \times \overline{\text{Mat}_{mn}}]_q$ involved in (5).

Introduce the notation
\[
k_i = \sum_{J' \subset \{1, 2, \ldots, m\}} \sum_{\text{card}(J') = i} \sum_{\text{card}(J'') = j} z^{\wedge i} J' \otimes \left( z^{\wedge i} J'' \right)^*.
\]

Let $\mathbb{C}[\text{Mat}_{mn}]_q \subset \text{Pol}(\text{Mat}_{mn})_q$ be the unital subalgebra generated by $(z^\alpha_a)^*$, $a = 1, 2, \ldots, n$, $\alpha = 1, 2, \ldots, m$, and $\mathbb{C}[\text{Mat}_{mn}]_q^{\text{op}}$ the algebra which differs from $\mathbb{C}[\text{Mat}_{mn}]_q$ by a replacement of its multiplication law to the opposite one (this replacement is motivated in [3]). The tensor product algebra $\mathbb{C}[\text{Mat}_{mn}]_q^{\text{op}} \otimes \mathbb{C}[\text{Mat}_{mn}]_q$ will be called an algebra of polynomial kernels. It is possible to show that in this algebra $k_i k_j = k_j k_i$ for all $i, j = 1, 2, \ldots, m$.

We follow [3, 4] in equipping $\text{Pol}(\text{Mat}_{mn})_q$ with a $\mathbb{Z}$-grading: $\deg(z^\alpha_a) = 1$, $\deg((z^\alpha_a)^*) = -1$, $a = 1, 2, \ldots, n$, $\alpha = 1, 2, \ldots, m$. In this context one has:
\[
\mathbb{C}[\text{Mat}_{mn}]_q^{\text{op}} = \bigoplus_{i=0}^{\infty} \mathbb{C}[\text{Mat}_{mn}]_q^{\text{op}, i}, \quad \mathbb{C}[\text{Mat}_{mn}]_q = \bigoplus_{j=0}^{\infty} \mathbb{C}[\text{Mat}_{mn}]_q, -j
\]
The kernel algebra \( \mathbb{C}[\text{Mat}_{mn}]_{q} \) will stand for a completion of \( \mathbb{C}[\text{Mat}_{mn}]_{q}^{\text{op}} \otimes \mathbb{C}[\text{Mat}_{mn}]_{q,-j} \) in the topology associated to the grading in (8). The kernel algebra is constituted by formal series

\[
\psi = \sum_{i,j=0}^{\infty} \psi_{ij}, \quad \psi_{ij} \in \mathbb{C}[\text{Mat}_{mn}]_{q,i} \otimes \mathbb{C}[\text{Mat}_{mn}]_{q,-j}.
\]

Our main result is the following formula for the weighted Bergman kernel:

\[
K_{\lambda} = \prod_{j=0}^{\infty} \left( 1 + \sum_{i=1}^{m} (-q^{2(\lambda+j)})^{i} k_{i} \right) \cdot \prod_{j=0}^{\infty} \left( 1 + \sum_{i=1}^{m} (-q^{2j})^{i} k_{i} \right)^{-1}
\]

with \( k_{i} \) being the polynomial kernels (8). (The right hand side of (9) determines an element of \( \mathbb{C}[\text{Mat}_{mn} \times \text{Mat}_{mn}]_{q} \) since \( k_{i} \in \mathbb{C}[\text{Mat}_{mn}]_{q,i} \otimes \mathbb{C}[\text{Mat}_{mn}]_{q,-i} \) for all \( i = 1, 2, \ldots, m \).

In the special case \( m = n = 1 \) we get a well known result (3):

\[
K_{\lambda} = \prod_{j=0}^{\infty} \left( 1 - q^{2(\lambda+j)} z \otimes \zeta^{*} \right) \cdot \left( \prod_{j=0}^{\infty} (1 - q^{2j} z \otimes \zeta^{*}) \right)^{-1} = \prod_{i=0}^{\infty} \frac{(1 - q^{2\lambda}) (1 - q^{2(\lambda+1)}) \ldots (1 - q^{2(\lambda+i)})}{(1 - q^2) (1 - q^4) \ldots (1 - q^{2i})} z^{i} \otimes \zeta^{*i}.
\]

Now passage to a limit as \( q \to 1 \) and replacement of \( \otimes \) by a dot yields \( K_{\lambda} \to (1 - z \zeta^{*})^{-\lambda} \).

A q-analogue of an ordinary Bergman kernel for the matrix ball (see 3) is derivable from (9) by a substitution \( \lambda = m + n \):

\[
K = \prod_{j=0}^{m+n-1} \left( 1 + \sum_{i=1}^{m} (-q^{2j})^{i} k_{i} \right)^{-1} \to_{q \to 1} (\det(1 - z \cdot \zeta^{*}))^{-(m+n)}.
\]

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