Local Gaussian operations can enhance continuous-variable entanglement distillation

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Entanglement distillation is a fundamental building block in long-distance quantum communication. Though known to be useless on their own for distilling Gaussian entangled states, local Gaussian operations may still help to improve non-Gaussian entanglement distillation schemes. Here we show that by applying local squeezing operations, both the performance and the efficiency of existing distillation protocols can be enhanced. We derive the optimal enhancement through local Gaussian unitaries, which can be obtained even in the most natural scenario when Gaussian mixed entangled states are shared after their distribution through a lossy-fiber communication channel.

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Entangled quantum states can be seen as one of the essential resources in quantum information processing (QIP) [1]. However, entanglement is fragile and easily degraded by uncontrollable environment-induced noise, for instance, during its distribution over longer distances. In order to circumvent this problem, various entanglement distillation protocols have been proposed capable of improving shared entanglement by local operations and classical communication [2]. In the realm of continuous-variable QIP, although Gaussian entangled states and Gaussian operations [3, 4] are widely used for teleportation, dense coding, and other protocols [5], distilling Gaussian entanglement in the potentially most efficient way through only Gaussian operations has been shown to be impossible [6–8].

In the mean time, efforts have been made to incorporate the necessary non-Gaussian element into optical entanglement distillation schemes. For example, Opatrný et al. introduced the photon subtraction (PS) strategy employing one of the most readily available non-Gaussian operations to distill stronger entanglement [9]. This type of distillation was recently demonstrated experimentally [10]. In other, complementary experiments, non-Gaussian entangled states were initially prepared and subsequently, since no longer restricted by the above no-go results, distilled through Gaussian operations [11, 12]. All these results and developments seem to imply that local Gaussian operations are useless for the distillation of Gaussian entangled states (see, for example, Ref. [13]; however, note [14]).

In the present work, we shall demonstrate that Gaussian operations, though insufficient for distilling Gaussian entanglement on their own, can still be used to enhance existing non-Gaussian distillation schemes. More specifically, we present examples of a distillation protocol that makes use of local squeezing operations in addition to Opatrný et al.’s non-Gaussian PS strategy. We then show that after distillation, both the success probability and the entanglement (in terms of logarithmic negativity) are improved in a regime when an initial Gaussian two-mode squeezed vacuum state (TMSS) is weakly squeezed. Moreover, this result even holds when the TMSS is first subject to an amplitude damping channel, as it would be the case, for instance, during an optical-fiber-based entanglement distribution. In fact, we are able to show that the local Gaussian pre-processing prior to PS-based distillation (but after the channel transmission) may become even more important for a lossy channel.

Our result is of both conceptual and practical significance. From a more fundamental point of view, it shows that in order to optimize quantum information tasks as important as entanglement distillation, discrete-variable and continuous-variable techniques should go hand in hand [15]. Since the PS technique [16] as well as the implementation of online squeezing [17] is becoming state of the art, our protocol is of practical relevance too.

Preliminaries—For representing a continuous-variable system consisting of \(N\) bosonic modes, we shall employ the phase-space description, where every mode \(k\) can be conveniently expressed by field quadrature position

![Diagram of entanglement distillation](image)

FIG. 1. Distillation of Gaussian continuous-variable entanglement through PS and local squeezing, \(S_A(r'_A)\) and \(S_B(r'_B)\). Here, \(|\psi\rangle_{AB}\) is a pure TMSS. Beam splitters with transmission coefficient \(T\) and on-off detectors \(C\) and \(D\) are used to realize the corresponding non-Gaussian PS operations.
and momentum operators, $\hat{x}_k = (\hat{a}_k + \hat{a}_k^\dagger)/\sqrt{2}, \hat{p}_k = (\hat{a}_k - \hat{a}_k^\dagger)/(i\sqrt{2})$, with $\hat{a}_k, \hat{a}_k^\dagger$ being the mode annihilation and creators operators. Throughout, we use hats to denote operators in Hilbert space. The canonical commutation relations for an $N$-mode system can be conveniently written as $[\hat{X}_m, \hat{X}_n] = i\Omega_{mn}, 1 \leq m, n \leq 2N$, with $\hat{X} \equiv (\hat{x}_1, \hat{p}_1, \cdots, \hat{x}_N, \hat{p}_N)$ and $\Omega = \sum_{k=1}^{N} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Furthermore, the density matrix of the above $N$-mode system $\rho$ in infinite-dimensional Hilbert space is represented by the characteristic function $\chi(\xi) = \text{Tr}\{\rho \exp[i\hat{X}^\dagger \xi]\}, \xi \in \mathbb{R}^{2N}$, or equivalently by its Fourier transform, i.e., the Wigner function $W(x) = \int_{\mathbb{R}^{2N}} \exp(-i\xi^\dagger \hat{X}) \chi(\xi)$. In particular, Gaussian states are those states whose characteristic or Wigner functions are Gaussian: $\chi(\xi) = \exp\left[-\frac{1}{2}(\xi^\dagger TV\xi + i\bar{x}T\xi)\right]$, and $W(x, \bar{x}, V) = \exp\left[-\frac{1}{2}(x^\dagger TV^\dagger-x\bar{x})\right]$. Such Gaussian states are fully characterized by displacements $\bar{x} = \text{Tr}(\hat{X}\rho)$ and a covariance matrix (CM) $V$, with entries $V_{lm} = \frac{1}{2}(\langle \hat{X}_l \hat{X}_m \rangle + \langle \hat{X}_m \hat{X}_l \rangle) - \langle \hat{X}_l \rangle \langle \hat{X}_m \rangle$. For example, for the pure TMSS, we have $\bar{x} = 0$ and the CM is given by

$$V_{AB} = \frac{1}{2} \begin{pmatrix} c & s & c & s \\ s & c & -s & c \\ c & s & c & s \\ -s & c & c & s \end{pmatrix},$$

with $c = \cosh 2r, s = \sinh 2r$. Unitary state evolutions in Hilbert space generated by Hamiltonians quadratic in the canonical operators correspond to symplectic transformations, $S \in Sp(2N, \mathbb{R})$, in phase space. In what follows, we will frequently use the symplectic single-mode squeezing, $S(r) = \text{diag}[e^r, e^{-r}]$, and two-mode beam-splitting operations. The latter is performed via the 4 × 4 matrix $S_{BS}^{kl}$ with non-vanishing elements $(S_{BS}^{kl})_{14} = \sqrt{T}, (S_{BS}^{kl})_{31} = \sqrt{1-T}, (S_{BS}^{kl})_{12} = (S_{BS}^{kl})_{34} = (S_{BS}^{kl})_{24} = \sqrt{1-T}$. Here, $T$ represents the transmission coefficient and $kl$ the two input modes of the beam splitter.

**Pure-state distillation**— with the notations above, we can now proceed to derive the state evolution in our modified entanglement distillation protocol, as shown in Fig. 1. As a start, let us first assume that the initial state is the pure TMSS, $|\psi\rangle_{AB} = \sum_n \sqrt{\lambda^n} |n\rangle_A |n\rangle_B, \lambda = \tanh r$, with $|n\rangle$ denoting the Fock basis and $r$ the squeezing parameter. To perform entanglement distillation, we propose to use two local squeezing operations $S_A(r'_A)$ and $S_B(r'_B)$ (with squeezing parameters $r'_A$ and $r'_B$) before the PS operation. For the latter, we employ a beam splitter (with transmission coefficient $T$) and conventional on-off detectors [18]. Throughout this paper, we refer to a successful distillation event when both detectors register nonzero photon counts.

Now assume modes $C$ and $D$ are initially prepared in a vacuum state. Thus, the CM of the four-mode state $ABCD$ (before the photon detections) becomes

$$V_{ABCD} = S_{tot} \left(V_{AB} \oplus \frac{1}{2} I_{CD}\right) S_{tot}^T,$$

$$S_{tot} = \left[S_{BSAC}^{AC} \oplus S_{BSB}^{BD}\right] \left[S_A(r') \oplus S_B(r')\right],$$

under the additional assumption that optimal distillation will occur for an initial symmetric state through symmetric local PS and squeezing operations, $r'_A = r'_B = r'$.

In our scheme, we propose to use on-off photon detectors, as commonly employed in quantum optics experiments. Such a detector is represented by two measurement outcomes: ‘off’ when no photons are detected and ‘on’ when one or more photons are detected. Through a successful distillation event, modes $C$ and $D$ are projected onto non-vacuum components and the state of modes $A$ and $B$ is reduced to $\rho = \text{Tr}_{CD} \left[\rho_{ABCD} I_{AB} \otimes \Pi_{C}^{(on)} \otimes \Pi_{D}^{(on)}\right] / \text{P succ},$ with $\Pi_{C}^{(on)} = I_2 - |0\rangle \langle 0| = \sum_{n=1}^{\infty} |n\rangle \langle n|$. Throughout, we use $I_m$ to represent an $m$-dimensional identity matrix. In order to obtain analytical results, we shall again employ the phase-space formalism. In fact, although the single-mode operator $\Pi_{C}^{(on)}$ leads to a non-Gaussian Wigner function, $W(x) = \frac{1}{T} - \frac{1}{2} \exp[-x^T \Sigma^{-1} x]$ [19], by expressing every single operator through a Wigner function and carrying out the corresponding integrals, we find that the Wigner function of the distilled state is a linear combination of four Gaussian functions:

$$W_\rho(x) \cdot \text{P succ} = \sum_{j=1}^{4} P_j W(x, 0, V_j),$$

with $P_1 = 1, P_2 = -|\det (V_C + I_2/2)|^{-1/2}, P_3 = -|\det (V_D + I_2/2)|^{-1/2}, P_4 = |\det (\Gamma_{CD} + I_4/2)|^{-1/2}, V_1 = \Gamma_{AB}, V_2 = \Gamma_{AB} - \sigma_1 (V_C + I_2/2)^{-1} \sigma_1^\dagger, V_3 = \Gamma_{AB} - \sigma_2 (V_D + I_2/2)^{-1} \sigma_2^\dagger, V_4 = \Gamma_{AB} - \sigma (\Gamma_{CD} + I_4/2)^{-1} \sigma^\dagger,$ and $\Gamma_{AB}, \Gamma_{CD}, CD, V_C, V_D$ defined by partitioning $V_{ABCD}$ as

$$V_{ABCD} = \begin{pmatrix} \Gamma_{AB} & \sigma & \sigma^T \\ \sigma^T & \Gamma_{CD} \end{pmatrix}, \quad \Gamma_{CD} = \begin{pmatrix} V_C & \varsigma \\ \varsigma^T & V_D \end{pmatrix}.$$
Theorem 1: let $V$ be the CM of a two-mode zero-displacement Gaussian state and $\rho_{AB}$ be the corresponding density matrix in the Fock basis, then the normalized logarithmic negativity of the output state follows the linear rule

$$\mathcal{L} = V^2 \rho_{AB} V^2 \mathcal{L}$$

with

$$M = \sigma_x \otimes I_2 + \sigma_z \otimes I_2 (L_2^+ \Lambda^{-1} L_2^+) \sigma_z \otimes I_2,$$

$$\Lambda = V + \frac{1}{2} L_2^{-1} (\sigma_x \otimes I_2) L_2^+,$$

$$L_2 = \begin{pmatrix} -i \sqrt{2} & -\sqrt{2} \\ i \sqrt{2} & -\sqrt{2} \\ -i \sqrt{2} & \sqrt{2} \\ i \sqrt{2} & \sqrt{2} \end{pmatrix},$$

where $\sigma_x, \sigma_z$ are the usual $2 \times 2$ Pauli matrices.

For our non-Gaussian state whose Wigner function is a linear combination of Gaussian functions, $W(x) = \sum_j P_j W(x, 0, V_j)$, it can be easily proved that the density matrix follows the linear rule $\rho = \sum_j P_j \rho(V_j)$, with $\rho(V_j)$ determined by Theorem 1.

Using the above methods, we now present a numerical evaluation of the performance of our modified distillation protocol. Fig. 2 shows the logarithmic negativity of the distilled state for $r = 0.025, r' \in [0.01, 0.20]$ [25]. The logarithmic negativity attains a maximal value at an intermediate point $r'_{\text{opt}} = 0.1565$, whereas the success probability increases monotonically with local squeezing $r'$. At the optimal point $r'_{\text{opt}}$, with success probability $3.5029 \times 10^{-6}$ and $E_N = 1.5278$, we obtain a significant improvement over Opatrný's original PS strategy (with $1.5645 \times 10^{-6}$ and $E_N = 0.1352$ [26, 27]). In other words, the local filter operations corresponding to the beam splitters and on-off detectors (i.e., the local map for say mode $A$, $\mathcal{E} : \rho_{AB} \rightarrow \sum_{i=1}^{\infty} \bar{E}_i \rho_{AB} \bar{E}_i^\dagger / \text{Tr} \left( \sum_{i} \bar{E}_i^\dagger \bar{E}_i \rho_{AB} \right)$, with $\bar{E}_i = \sum_{n=1}^{\infty} (-1)^i \sqrt{\binom{n}{i} T^{n-i}/2} (1-T)^{i/2} |n-i\rangle \langle n|$, become more efficient when replaced by the set of operators $\lbrace \bar{E}_i S_A(r'_A) \rbrace_{i=1,2,...}$ At the same time, local squeezing increases each mode's average photon number and thus enhances the probability that photons are detected.

**Mixed-state distillation**— in the above distillation scheme, the local squeezers could be as well seen as part of the initial state preparation. One could then argue that so far we have simply found the optimal Gaussian pure state for PS-based entanglement concentration which turns out to be different from the TMSS. It is therefore intriguing to examine whether the TMSS would also benefit from local Gaussian pre-processing after its transmission through an imperfect channel such as a lossy fiber. This would lead to a clear distinction between local operations before and after the entanglement distribution, corresponding to alternate state preparations or alternate state distillations, respectively.

For this purpose, we use beam splitters and auxiliary vacuum modes to model the optical amplitude-damping channel [27]. Our method introduced above still applies, except for replacing $c \rightarrow c' = 1 - \eta + \eta \cosh 2r, s \rightarrow s' = \eta \sinh 2r$ in Eq. (1), with $\eta$ being the transmission efficiency of the lossy channel. We find that local

FIG. 2. Distillation of a pure TMSS ($r = 0.025$) using local squeezers ($r'$) and PS method ($T = 0.95$). (a) logarithmic negativity of the output state; the blue circle indicates the optimal $r'$ which maximizes the logarithmic negativity ($r'_{\text{opt}} = 0.1565, E_N = 1.5278$) (b) success probability of distillation, i.e., the probability that both detectors obtain ‘on’ results.

FIG. 3. Mixed-state distillation with $\eta = 0.5$. Comparing the distillation performances between Opatrný’s PS strategy (blue, dashed line) and our local-squeezing enhanced PS strategy (red, dot dashed line), for $r \in [0.005, 0.4], T = 0.95, r' = r = \arctanh(\lambda)$. The black (solid) line indicates the logarithmic negativity of the mixed state before distillation.
squeezing still helps to improve entanglement distillation of mixed states. In Fig. 3, the logarithmic negativity and success probability of distillation are shown. For simplicity, we consider the case when each mode of the TMSS is transmitted through a 3dB ($\eta = 0.5$) amplitude-damping channel. In the low-squeezing regime, e.g. for $r \in [0.005, 0.4]$, a significant improvement is obtained.

Quantum teleportation – the improvement in our modified scheme can be experimentally verified in an operational fashion through quantum teleportation [9, 26]. Let us consider standard unit-gain teleportation in which the entangled state after distillation is pre-shared and an unknown coherent state $|\alpha_0\rangle$ is to be teleported. The teleportation fidelity is $F_{\text{tel}} = \langle \alpha_0 | \rho_{\text{out}}(\alpha_0) | \alpha_0 \rangle$, with $\rho_{\text{out}}(\alpha_0)$ being the ensemble average of all output states conditioned upon different Bell measurement results. Due to its linearity, the teleportation fidelity using the entangled state in Eq. (4) can be conveniently written as $F = \frac{1}{4} \sum_{j=1}^{4} P_j F_{\text{tel}}(V_j)/P_{\text{succ}}$, with (for any $\alpha_0$) $F_{\text{tel}}(V) = \det(R \alpha R + \gamma^T R + R \gamma + \beta + I_2)^{-1/2}$, $R = \text{diag}(-1, 1)$, where $\alpha, \beta, \gamma$ are defined by $V \equiv \left( \begin{array}{c c} \alpha & \gamma \\ \gamma^T & \beta \end{array} \right)$.

In Fig. 4(a), we consider a pure TMSS state ($\eta = 1, r = 0.025$) being distilled and then used for teleporting an unknown coherent state. Fig. 4(b) shows the optimal local squeezing $r'_{\text{optF}}$ as a function of $r$. Clearly, $r'_{\text{optF}}$ begins to drop towards zero for a stronger pure TMSS state ($r \sim 0.13$), representing a threshold beyond which our modified scheme ceases to improve Opatrný’s PS strategy ($r' = 0$). In Fig. 4(c), the fidelity at $r' = r'_{\text{optF}}$ is compared with Opatrný’s PS strategy, while Fig. 4(d) shows the fidelity for using a distilled 3dB-damped TMSS state. In this case, a larger threshold value, $r \sim 0.26$, occurs. Thus, we find that for an amplitude-damped resource state, the local Gaussian pre-processing prior to distillation becomes even more significant.

Optimal Gaussian unitaries – the most general symplectic transformation applicable to our local filters can be written as a sequence of phase rotation, local squeezing, and another phase rotation [28], $U(r, \theta, \phi) = \left( \begin{array}{cc} \cos \theta & \sin \theta e^{-r} \\ -\sin \theta & \cos \theta \end{array} \right) \left( \begin{array}{cc} e^r & 0 \\ 0 & e^{-r} \end{array} \right) \left( \begin{array}{cc} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{array} \right)$. For a fixed squeezing $r = \text{arctanh}(\lambda)$ in the initial pure TMSS of Eq. (1), we applied two local unitary operations $U(r, \theta_A, \phi_A)$ and $U(r, \theta_B, \phi_B)$ to modes $A$ and $B$, respectively. We computed the logarithmic negativity and success probability for 3000 randomly chosen $\theta_A, \phi_A, \theta_B, \phi_B$, and we found that both figures of merit attain maximal values for $\theta_A = \phi_A = \theta_B = \phi_B = 0$.

Summary – we demonstrated that local Gaussian operations can be useful to enhance the distillation capabilities of non-Gaussian operations applied upon Gaussian entangled states, even when the initial states are Gaussian mixed states such as those emerging from an imperfect channel transmission for realistic entanglement distribution in quantum communication. In our protocol, for the distribution of one entangled-state copy, we considered a photon loss channel, and for one-copy distillation, we used the optimal local unitary squeezers in addition to photon subtraction. It remains an open question whether local Gaussian non-unitary maps could further improve non-Gaussian entanglement distillation schemes.

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it should be noted that the conclusion in Ref. [13] is mainly drawn from optimizing the purity of the distilled entangled state. Furthermore, what is considered there is an asymptotic and iterative, multi-copy entanglement distillation scheme where almost infinitely many copies of initial entangled states are involved. This is obviously different from our single-copy distillation protocol. Another difference is that in Ref. [13], all figures of merit are calculated for the zero-one-photon subspaces only, whereas in our calculations, there is no such restriction.

see, for example, A. Furusawa and P. van Loock, Quantum teleportation and entanglement: a hybrid approach to optical quantum information processing, Wiley VCH 2011.

Beyond this, we present a stronger theorem for which the density matrix obtained is automatically normalized.

in general, Theorem 1 applies to arbitrary CMs $V_j$, including arbitrary two-mode squeezing $r$. Here we choose initial squeezing $r = 0.025$, although we could as well use larger squeezing $r$ and, with the same method, derive the optimal local squeezing $r_{opt}$. However, for stronger squeezing $r$, $r_{opt}$ will be so large that we have to keep many (e.g. $\geq 15$) photons in each mode to make sure that every CM $V_j$ ($j = 1, 2, 3, 4$) is transformed to a density matrix within a precision of $10^{-6}$. To double-check our computer program, we also transformed the density matrices back to the CMs and always kept enough photons in each mode to make sure that the elements of every newly obtained CM matrix $V_j'$ are only negligibly different ($<10^{-6}$) from those of the original CM $V_j$. For example, the calculation of the density matrix and logarithmic negativity for $r = 0.025$, $r' = 0.025$ takes only 12 minutes and a 60 MB memory, while only 10 photons are kept in each mode, with our MATLAB program and Intel Core i5 2.27 GHz CPU. However, for larger $r$, say $r = 0.04$, $r' = 0.10$, we have to consider up to 12 photons corresponding to a computation time of 2.5 hours with 100 MB memory. Further, if we keep 14 photons, the numbers become 15 hours and 600 MB memory, etc.

note that there is an error in Eq. (6) of Ref. [23], and it should correctly read $t_j t'_j - t_j \mu_j + t'_j \mu'_j$ in that equation. Beyond this, we present a stronger theorem for which the

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