CAUCHY INTEGRALS FOR THE $p$-LAPLACE EQUATION
IN PLANAR LIPSCHITZ DOMAINS

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Abstract. We construct solutions to $p$-Laplace type equations in unbounded
Lipschitz domains in the plane with prescribed boundary data in appropriate
fractional Sobolev spaces. Our approach builds on a Cauchy integral representa-
tion formula for solutions.

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functional calculus.

1. Introduction

In [AAH], [AAMc], [AA], [AR], new representations and new methods for solving
boundary value problems for divergence form second order, real and complex, equa-
tions and systems were developed in domains Lipschitz diffeomorphic to the upper
half space $\mathbb{R}^{n+1}_+ := \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : t > 0\}$, $n \geq 1$. Focusing on the case of
equations, the authors consider equations

$$Lu(x, t) = \sum_{i,j=1}^{n+1} \partial_i (a_{i,j}(x, t) \partial_j u(x, t)) = 0, \quad \partial_{n+1} = \partial_t, \quad \partial_i = \partial_{x_i}, \quad (1.1)$$

with $A(x, t) = \{a_{i,j}(x, t)\}_{i,j=1}^{n+1} \in L_\infty(\mathbb{R}^{n+1}_+, C^{(n+1)^2})$, and with $A$ being strictly
accretive on a certain subspace $H$ of $L_2(\mathbb{R}^n, C^{(n+1)^2})$. The key idea/discovery in
these papers is that the equation in (1.1) becomes quite simple when expressing it in
terms of the conormal gradient $f = \nabla_A u = [\partial_{x} u, \nabla_x u]^*$, $*$ denotes the transpose,
$\partial_{x} u$ denotes the conormal derivative, instead of the potential $u$ itself. Indeed, $f$
solves a set of generalized Cauchy-Riemann equations expressed as a first order
system

$$\partial_t f + DBf = 0, \quad (1.2)$$

where $D$ is a self-adjoint first order differential operator with constant coefficients
and $B = B(x, t)$ is multiplication with a bounded matrix $B$, strictly accretive on $H$,
and pointwise determined by $A = A(x, t)$. The operator $DB$ is a bisectorial opera-
tor on $L_2(\mathbb{R}^n, C^{(n+1)^2})$ and if $A$, and hence $B$, is independent of the $t$-coordinate,
then it is proved that $DB$ satisfies certain square functions estimates which implies
that $DB$, when $B$ is independent of the $t$-coordinate, has an $L_2$-bounded holomor-
phic functional calculus. When $n = 1$ this non-trivial fact follows from [CAMM]
and for $n \geq 2$ it is a consequence of the technology developed in the context of the
resolutions of the Kato conjecture, see [AHLMcT], [AKMc]. Using the holomorphic
functional calculus for $DB$ one can then attempt to solve (1.2), when $B$ is independent
of the $t$-coordinate, by the semi-group formula $f = e^{-t|DB|}g$, with $g = g(x)$
in a suitable trace space and $f$ has non-tangential maximal and square function
estimates. The situation when $A$, and hence $B$, is dependent on the $t$-coordinate
can be addressed by perturbing the $t$-independent case and using a Picard iteration like argument, see [AA], [AR].

It is in general a very interested program to attempt to understand to what extent the approach outlined above can be used in the context of non-linear elliptic partial differential equations and in this paper we establish one such result in the non-linear setting of operators of $p$-Laplace type. Note that there has recently been significant progress concerning the boundary behaviour of non-negative solutions to the $p$-Laplace operator, in $\mathbb{R}^n$, $n \geq 1$, progress which gives at hand that many results previous established in the linear case of the Laplace operator, $p = 2$, see [CFMS], [D], [JK], remain valid also in the non-linear and potentially degenerate setting of the $p$-Laplace operator. Indeed, in [LN1], [LN2], [LN3], a number of results concerning the boundary behaviour of positive $p$-harmonic functions, $1 < p < \infty$, in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ were proved. In particular, the boundary Harnack inequality and the H"older continuity for ratios of positive $p$-harmonic functions, $1 < p < \infty$, vanishing on a portion of $\partial \Omega$ were established. Furthermore, the $p$-Martin boundary problem at $w \in \partial \Omega$ was resolved under the assumption that $\Omega$ is either convex, $C^1$-regular or a Lipschitz domain with small constant. Also, in [LN4] these questions were resolved for $p$-harmonic functions vanishing on a portion of certain Reifenberg flat and Ahlfors regular NTA-domains. The results and techniques developed in [LN1]-[LN4] concerning $p$-harmonic functions have also been used and further developed in [LN5], [LN6] in the context of free boundary regularity in general two-phase free boundary problems for the $p$-Laplace operator and in [LN7] in the context of regularity and free boundary regularity, below the continuous threshold, for the $p$-Laplace equation in Reifenberg flat and Ahlfors regular NTA-domains. These results are indications, and there are several others, that the many results valid in the linear case may still, with the right approach, be possible to prove also in the non-linear context of the $p$-Laplace operator. While we here restrict ourselves to the case $n = 1$, the planar case, for reasons to be discussed below, the ambition is to also understand the case $n \geq 2$ in future papers.

To outline our set-up, we let $\Omega \subset \mathbb{R}^2$ be an unbounded domain of the form $\Omega = \{(x,y) : x \in \mathbb{R}, y > \phi(x)\}$, where $\phi : \mathbb{R} \to \mathbb{R}$ denotes a Lipschitz function with constant $M$. Our main model equation is, given $1 < p < \infty$, the $p$-Laplace equation

\begin{equation}
\text{div} (|\nabla u|^{p-2} \nabla u) = 0.
\end{equation}

Given $1 < p < \infty$, we denote by $W^{1,p}(\Omega)$ the space of equivalence classes of functions $f \in L^p(\Omega)$ with distributional gradients $\nabla f = (\partial_x f, \partial_y f)$ which are in $L^p(\Omega)$ as well. Let $\|f\|_{1,p} = \|f\|_p + \|\nabla f\|_p$ be the norm in $W^{1,p}(\Omega)$ where $\|\cdot\|_p$ denotes the usual norm in $L^p(\Omega)$. Next, let $C_0^{\infty}(\Omega)$ be the set of infinitely differentiable functions with compact support in $\Omega$, and let $W_0^{1,p}(\Omega)$ be the closure of $C_0^{\infty}(\Omega)$ in the norm of $W^{1,p}(\Omega)$. We say that $u$ is a weak solution to (1.3) in $\Omega$ provided $u \in W^{1,p}(\Omega)$ and

\begin{equation}
\int_\Omega (|\nabla u|^{p-2} \nabla u, \nabla \theta) \, dx \, dy = 0
\end{equation}

whenever $\theta \in W_0^{1,p}(\Omega)$. In the special case $p = 2$ the equation in (1.3) reduces to the linear Laplace equation

\begin{equation}
\text{div}(\nabla u) = \partial_{xx} u + \partial_{yy} u = 0 \text{ in } \Omega.
\end{equation}

Let $\gamma = \{(x,\phi(x)) : x \in \mathbb{R}\} = \partial \Omega$ and consider, at a point $(x,y) \in \gamma$, the vector fields $(0,1)$, $(1,\phi'(x))$. Note that the vector field $(1,\phi'(x))$ is tangential to $\gamma$ at $(x,\phi(x))$. Based on these vector fields we introduce the first order differential
operators
\[
\begin{align*}
\partial_\perp & := (0, 1) \cdot (\partial_x, \partial_y) = \partial_y, \\
\partial_\parallel & := (1, \phi'(x)) \cdot (\partial_x, \partial_y) = \partial_x + \phi'(x)\partial_y = \partial_x + \phi'(x)\partial_\perp.
\end{align*}
\] (1.6)

Let, given $1 < p < \infty$, $u$ be a weak solution to (1.3) in $\Omega$. Then, using interior regularity results for the $p$-Laplace operator, see [DiH], [L], [T], $u$ is $C^{1,\gamma}$-regular locally, for some $\gamma \in (0, 1)$, and hence $\nabla u$ is well-defined pointwise. To proceed we first fix some notation. Here and below, we often identify $\mathbb{C}$ and $\mathbb{R}^2$, writing $a + ib = (a, b)^*$, where $*$ denotes transpose. Sometimes we also identity $a + ib$ with the multiplication operator $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. We parametrize $\Omega$ with
\[
y = t + \phi(x)
\]
so that $(x, y) \in \Omega$ corresponds to $(x, t) \in \mathbb{R}^2$. We sometimes write functions $f(x, t)$ as $f_t(x)$. Now, using this notation and the operators $\partial_\perp, \partial_\parallel$, introduced in (1.6), we in Section 2 prove that $u$ is a weak solution to (1.3) in $\Omega$ if and only
\[
f(x, t) = (f_1(x, t), f_2(x, t))^* = (\partial_x u(x, y), -\partial_y u(x, y))^*,
\]
that is $f(x, t) = \nabla u(x, t + \phi(x))$, is a solution to the first order system
\[
\partial_t f + B(f) Df = 0.
\] (1.7)
Here
\[
D := \begin{bmatrix} 0 & \partial_x \\ -\partial_x & 0 \end{bmatrix} = -i\partial_x
\]
and
\[
B(f) = \frac{1}{\Delta_p} \begin{bmatrix} B_{11}(f) & B_{12}(f) \\ B_{21}(f) & B_{22}(f) \end{bmatrix},
\] (1.8)
with
\[
\begin{align*}
B_{11}(f) &= (p - 2)f_2^2 + |f|^2, \\
B_{22}(f) &= (p - 2)f_1^2 + |f|^2, \\
B_{12}(f) &= ((p - 2)f_1^2 + |f|^2)\phi'(x), \\
B_{21}(f) &= -2(p - 2)f_1 f_2 - (\phi'(x))((p - 2)f_1^2 + |f|^2),
\end{align*}
\] (1.9)
and
\[
\Delta_p = ((p - 2)f_2^2 + |f|^2) \\
-2\phi'(x)(p - 2)f_1 f_2 + (\phi'(x))^2((p - 2)f_1^2 + |f|^2). 
\] (1.10)

To ease notation, we here suppress the dependence of $B(f)$ on $p$ and on $\phi'(x)$. Note that if $\phi' \equiv 0$, that is $\Omega = \mathbb{R}^2$, then
\[
B(f) = \frac{1}{(p - 2)f_2^2 + |f|^2} \begin{bmatrix} (p - 2)f_2^2 + |f|^2 & 0 \\ -2(p - 2)f_1 f_2 & (p - 2)f_1^2 + |f|^2 \end{bmatrix},
\] (1.11)
and if $p = 2$, then $B(f) = B_0$ where
\[
B_0(x) := \frac{1}{1 + (\phi'(x))^2} \begin{bmatrix} 1 & \phi'(x) \\ -\phi'(x) & 1 \end{bmatrix} = \frac{1}{1 + i\phi'(x)}
\] (1.12)
In particular, when $p = 2$ and $\phi' \equiv 0$, then the system in (1.7) reduces to the classical Cauchy-Riemann equations.

Define the boundary Cauchy integral
\[
(S_0 h)_t(x) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{h(y)(1 + i\phi'(y))}{(y + i\phi(y)) - (x + i(t + \phi(x)))} dy,
\]
h : $\mathbb{R}$ $\rightarrow$ $\mathbb{C}$,
and the solid Cauchy integral
\[
(\tilde{S}h)_t(x) = \frac{1}{2\pi i} \int_{\mathbb{R}^+} \frac{h(y,s)(1 + i\phi'(y))}{(y + it + \phi(y)) - (x + it + \phi(x))} dyds, \quad h : \mathbb{R}^2_+ \to \mathbb{C}.
\]

Let, given \(0 < \sigma < 1\), \(\dot{H}^\sigma(\mathbb{R})\) denote the homogeneous fractional Sobolev space of order \(\sigma\). We prove the following two theorems. The first result gives a Cauchy integral representation for solutions to the \(p\)-Laplace equation.

**Theorem 1.1.** Let \(1 < p < \infty\), \(0 < \sigma < 1\), and \(0 \leq M < \infty\) be given. Assume that \(\phi : \mathbb{R} \to \mathbb{R}\) is a Lipschitz function with \(\|\phi'\|_\infty \leq M\) and assume that \(u\) is a weak solution to (1.3) in \(\Omega = \{(x,y) : x \in \mathbb{R}, y > \phi(x)\}\) satisfying
\[
(1.14) \quad \iint_{\Omega} |\nabla^2 u|^2 (y - \phi(x))^{1-2\sigma} \, dx \, dy < \infty.
\]

Let \(f(x,t) = (\partial_x u(x,y), -\partial_y u(x,y))^t\), \(y = t + \phi(x)\). Then there exists \(g \in \dot{H}^\sigma(\mathbb{R})\) such that the Cauchy integral representation
\[
(1.15) \quad f = S_0g + \tilde{S}(B_0 - B(f))Df,
\]
holds in \(\mathbb{R}^2\). Furthermore we have estimates
\[
(1.16) \quad \sup_{t>0} t^{-1} \int_t^{2t} \|f_s\|^2_{\dot{H}^\sigma} \, ds \leq c \iint_{\Omega} |\nabla^2 u|^2 (y - \phi(x))^{1-2\sigma} \, dx \, dy
\]
and limits
\[
(1.17) \quad \lim_{t \to 0^+} t^{-1} \int_t^{2t} \|f_s - f_0\|^2_{\dot{H}^\sigma} \, ds = 0,
\]
where the trace is
\[
f_0 = g - \int_0^\infty (S_0(B_0 - B(f_s))Df_s)_{-s} ds \in \dot{H}^\sigma(\mathbb{R}).
\]
Moreover, if \(0 < \sigma < 1/2\), we have the estimate
\[
\sup_{t>0} \|f_t\|^2_{\dot{H}^\sigma} \leq c \iint_{\Omega} |\nabla^2 u|^2 (y - \phi(x))^{1-2\sigma} \, dx \, dy
\]
and \(\lim_{t \to 0^+} \|f_t - f_0\|_{\dot{H}^\sigma} = 0\) and \(\lim_{t \to \infty} \|f_t\|_{\dot{H}^\sigma} = 0\).

Using the Cauchy integral representation in Theorem 1.1 we also prove solvability of the following boundary value problem for the \(p\)-Laplace equation.

**Theorem 1.2.** Let \(p, \sigma, M, \phi\) be as in Theorem 1.1. Then there exists \(\delta = \delta(\sigma, M), \\delta > 0\) such that if \(|p-2| < \delta\), then the following is true. Given any boundary data \(h \in \dot{H}^\sigma(\mathbb{R})\), there exists a weak solution \(u\) to (1.3) in \(\Omega = \{(x,y) : x \in \mathbb{R}, y > \phi(x)\}\) satisfying
\[
(1.18) \quad \iint_{\Omega} |\nabla^2 u|^2 (y - \phi(x))^{1-2\sigma} \, dx \, dy < \infty,
\]
and the boundary condition
\[
\partial_x u(x, \phi(x)) = h(x), \quad x \in \mathbb{R},
\]
where the trace of \(\nabla u\) is taken in the sense of Theorem 1.1. The same solvability result also holds true for the boundary condition \(\partial_y u(x, \phi(x)) = h(x)\).
1.1. Organization of the paper. In section 2 we first show how quasi-linear pdes in the plane can be reduced to a vector valued ode. In this section we also show that our system of odes is closely related to the theory of quasiconformal and quasiregular mappings in the plane. Section 3 is devoted to functional calculus and Cauchy type formulas in our setting and we here prove key quantitative estimates. Theorem 1.1 and Theorem 1.2 are proved in section 4. In section 5 we give a few concluding remarks discussing, in particular, generalizations of our main results to more general quasi-linear equations. We emphasize that our proofs of Theorem 1.1 and Theorem 1.2 rely heavily on the fact that we are working in the plane. For example, in the proof of Theorem 1.2 we construct the solutions by perturbing the linear case \((p=2)\) using a Picard type iteration scheme. To be able to pass to the limit we need to ensure that the zero sets \(\{(x,t) \in \mathbb{R}^2_+ : f^k(x,t) = 0\}\), \(\{f^k\}\) appearing in the construction, have measure zero. To conclude this we here make use of the connection to quasiregular mappings and the detailed results available concerning quasiregular mappings in the plane, see [AIM, IM, IM1].

2. Quasi-linear pdes in the plane

To stress generalities, we in this section consider quasilinear equations of the more general type

\[
\text{(2.1)} \quad \text{div } a(\nabla u) = \partial_x(a_1(\partial_x u, \partial_y u)) + \partial_y(a_2(\partial_x u, \partial_y u)) = 0,
\]

where \(a(z) = (a_1(z), a_2(z))\). Given \(p, 1 < p < \infty\), we assume that the vector field \(a : \mathbb{R}^2 \to \mathbb{R}^2\) is \(C^1\)-regular and satisfies the growth and ellipticity assumptions

\[
\begin{cases}
|a(z)| + |\nabla a(z)||z| \leq L|z|^{p-1} \\
|\nu|^{|p-2|} \leq (\nabla a(z) \xi, \xi)
\end{cases}
\]

whenever \(z, \xi \in \mathbb{R}^2\) and for some fixed parameters \(0 < \nu \leq L\). Here \(\nabla a(z)\) denote the Jacobian matrix of \(a\). We say that \(u\) is a weak solution to (2.1) in \(\Omega\) provided \(u \in W^{1,p}(\Omega)\) and

\[
\int \langle a(\nabla u), \nabla \theta \rangle \, dx = 0
\]

whenever \(\theta \in W_0^{1,p}(\Omega)\). If \(a(z) = \nu|z|^{p-2}z\) then a solution to (2.3) is referred to as a \(p\)-harmonic function and we emphasize this main example of equations (2.1), (2.2), given by the \(p\)-Laplace equation introduced in [LLO].

2.1. Reduction of the pde to a system of odes. Let \(\Omega \subset \mathbb{R}^2\) be an unbounded domain of the form \(\Omega = \{(x,y) : x \in \mathbb{R}, \ y > \phi(x)\}\) where \(\phi : \mathbb{R} \to \mathbb{R}\) denotes a Lipschitz function with constant \(M\). Recall the first order operators \(\partial_\perp, \partial_\parallel\), introduced in (1.4). Using \(\partial_\perp, \partial_\parallel\) we see, given a vector field \(v = (v_1, v_2)\), that

\[
\begin{align*}
\text{curl } v &= \partial_x v_2 - \partial_y v_1 = \partial_\parallel v_2 - \partial_\perp v_1 - \phi'(x) \partial_\perp v_1 \\
\text{div } v &= \partial_x v_1 + \partial_y v_2 = \partial_\parallel v_1 - \phi'(x) \partial_\perp v_1 + \partial_\perp v_2.
\end{align*}
\]

Let \(w(x,y) = (w_1(x,y), w_2(x,y)) = (\partial_x u(x,y), -\partial_y u(x,y))\), where \(u\) is weak solution to (2.1). Then using (2.3) and (2.4) we have that

\[
\begin{align*}
0 &= -\partial_\parallel w_2 - \partial_\perp w_1 + \phi'(x) \partial_\perp w_2, \\
0 &= \partial_\parallel a_1(\bar{w}) - \phi'(x) \partial_\perp a_1(\bar{w}) + \partial_\perp a_2(\bar{w}).
\end{align*}
\]

Simply writing \(a\) for \(a(\bar{w})\) and \(\phi'\) for \(\phi'(x)\) we see that the second relation (2.5) can be expressed as

\[
\begin{align*}
0 &= (\partial_1 a_1) \partial_\parallel w_1 - (\partial_2 a_1) \partial_\parallel w_2 - \phi'((\partial_1 a_1) \partial_\perp w_1 - (\partial_2 a_1) \partial_\perp w_2) \\
&\quad + (\partial_1 a_2) \partial_\perp w_1 - (\partial_2 a_2) \partial_\perp w_2.
\end{align*}
\]
We next want to solve for \((\partial_\perp w_1, \partial_\perp w_2)\) in the system

\[
0 = -\partial_\parallel w_2 - \partial_\perp w_1 + \phi' \partial_\perp w_2, \\
0 = (\partial_1 a_1) \partial_\parallel w_1 - (\partial_2 a_1) \partial_\parallel w_2 - \phi'(x)((\partial_1 a_1) \partial_\perp w_1 - (\partial_2 a_1) \partial_\perp w_2)
\]

(2.7)

Let

\[
A := \begin{bmatrix} (\partial_1 a_2) - \phi'(\partial_1 a_1) & - (\partial_2 a_2) + \phi'(\partial_2 a_1) \end{bmatrix}, \\
D := \begin{bmatrix} 0 & -\partial_\parallel \\ -\partial_\parallel & 0 \end{bmatrix}.
\]

Using this notation the system in (2.7) can be written

\[
A \begin{bmatrix} \partial_\perp w_1 \\ \partial_\perp w_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} D \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \partial_\parallel w_1 \\ -\partial_\parallel w_2 \end{bmatrix}.
\]

In the following we let

\[
\Delta := -((\partial_1 a_2) - \phi'(\partial_1 a_1))\phi' + ((\partial_2 a_2) - \phi'(\partial_2 a_1))
\]

(2.9)

Using this we have

\[
A^{-1} = \frac{1}{\Delta} \begin{bmatrix} (\partial_2 a_2) - \phi'(\partial_2 a_1) & \phi' \\ (\partial_1 a_2) - \phi'(\partial_1 a_1) & 1 \end{bmatrix},
\]

and

\[
A^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} \partial_2 a_2 \\ \partial_1 a_1 \end{bmatrix} - \frac{\partial_1 a_2}{\partial_1 a_1} (\partial_1 a_1) \phi'.
\]

Let, for \(w = (w_1, w_2)\) and \(\phi\) given,

\[
B^w,\phi(x, t) := \frac{1}{\Delta} \begin{bmatrix} (\partial_2 a_2)(\bar{w}) \\ \partial_1 a_1(\bar{w}) \phi'(x) \end{bmatrix}.
\]

Then (2.8) can be restated as

\[
A \begin{bmatrix} \partial_\perp w_1 \\ \partial_\perp w_2 \end{bmatrix} + B^w,\phi D \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0.
\]

We summarize our findings as follows.

**Lemma 2.1.** A function \(u\) is a weak solution to (2.1) in \(\Omega\) if and only if

\[
f(x, t) = (f_1(x, t), f_2(x, t)) := (\partial_\alpha u(x, t + \phi(x)), -\partial_\gamma u(x, t + \phi(x)))^*
\]

satisfies

\[
\partial_\alpha \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + B^{f,\phi} D \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = 0, \\
D := \begin{bmatrix} 0 & \partial_\gamma \\ -\partial_\gamma & 0 \end{bmatrix},
\]

in \(\mathbb{R}_+^2 := \{(x, t) \in \mathbb{R}^2 : t > 0\}\) where

\[
B^{f,\phi}(x, t) := \frac{1}{\Delta} \begin{bmatrix} \partial_2 a_2(\bar{f}) \\ \partial_1 a_1(\bar{f}) \phi'(x) \end{bmatrix},
\]

and

\[
\Delta := (\partial_2 a_2(\bar{f}) - \phi'(x)((\partial_1 a_2) \bar{f} + \partial_2 a_1(\bar{f})) + (\phi'(x))2(\partial_1 a_1(\bar{f})).
\]

Recall that a 2 × 2-dimensional matrix \(B\), defined in \(\mathbb{R}^2\) and potentially complex valued, is said to be accretive if

\[
\kappa := \text{essinf}_{(x, t) \in \mathbb{R}^2} \inf_{\xi \in \mathbb{C} \setminus \{0\}} \frac{\text{Re}(B(x, t) \xi, \xi)}{|\xi|^2} > 0.
\]
Lemma 2.2. Let $a: \mathbb{R}^2 \to \mathbb{R}^2$ be a $C^1$-regular vector field satisfying (2.2) for some fixed parameters $0 < \nu \leq L$. Let $B = B^i \phi$ be as in (2.12), (2.13). Then $B \in L_\infty(\mathbb{R}^2_+, \mathbb{C}^2)$ and $B$ is accretive in the sense of (2.14). Furthermore, the $L_\infty$-bound on $B$, and the parameter of accretivity $\kappa$, depend only on $p, M, \nu, \text{ and } L$.

Proof. First, using the ellipticity type condition in (2.2) we see that
\[
\Delta \geq \nu |f|^{p-2}(1 + (\phi'(x))^2).
\]
Hence, using also the upper bound in (2.2) we can conclude that
\[
|B| \leq \frac{c(L)}{\Delta} |f|^{p-2}(1 + \phi'(x)) \leq c(p, M, \nu, L).
\]
To estimate the parameter of accretivity, let $\xi \in \mathbb{C}^2 \setminus \{0\}$, $(x, t) \in \mathbb{R}^2_+$, and note that
\[
\text{Re}(B(x, t)\xi, \xi) = B_{11}|\xi_1|^2 + B_{22}|\xi_2|^2 + B_{12}\text{Re}(\xi_1\xi_2) + B_{21}\text{Re}(\xi_1\bar{\xi}_2)
\]
\[
= (\partial_1 a_1(\bar{f}))|\xi_1|^2 + (\partial_2 a_2(\bar{f}))|\xi_2|^2
\]
\[
+ ((\partial_1 a_2(\bar{f})) + (\partial_2 a_1(\bar{f})))\text{Re}(\xi_1\bar{\xi}_2)
\]
(2.15)
and the estimate now follows from (2.2). □

2.2. Quasi-regular mappings in the plane. Consider a function $f: \Omega \to \Omega'$ where $\Omega, \Omega' \subset \mathbb{C}$. Let $z = x + iy \in \mathbb{C}$ and assume that $f$ has a derivative $\nabla f$ at $z$. We let $\partial_z f = (\partial_x f - i\partial_y f)/2, \partial_{\bar{z}} f = (\partial_x f + i\partial_y f)/2$ and we write the derivative as
\[
\nabla f(z)h = \partial_z f(z)h + \partial_{\bar{z}} f(z)\bar{h}, \quad h \in \mathbb{C} = \mathbb{R}^2.
\]
Note that
\[
|\nabla f(z)|^2 = |\partial_z f(z)|^2 + |\partial_{\bar{z}} f(z)|^2
\]
and that the Jacobian equals
\[
J(z, f) = |\partial_z f(z)|^2 - |\partial_{\bar{z}} f(z)|^2.
\]
Recall that if the mapping $f$ satisfies $f \in W^{1,2}_{\text{loc}}(\Omega)$, $f$ is orientation preserving so that $J(z, f) \geq 0$ a.e., and if the following holds
\[
|\nabla f(z)|^2 \leq K J(z, f) \text{ for almost every } z \in \Omega,
\]
then $f$ is called $K$-quasiregular. The smallest number $K$ for which (2.16) holds is called the dilation of $f$ and we denote this number by $K(f)$. If, in addition, $f$ is a homeomorphism, then $f$ is called $K$-quasiconformal. Note that (2.16) can also be expressed as
\[
|\partial_z f(z)|^2 + |\partial_{\bar{z}} f(z)|^2 \leq K (|\partial_z f(z)|^2 - |\partial_{\bar{z}} f(z)|^2)
\]
(2.17)
or equivalently
\[
|\partial_z f(z)|^2 \leq \frac{K-1}{K+1} |\partial_{\bar{z}} f(z)|^2 \text{ for almost every } z \in \Omega.
\]
(2.18)
In particular, $f: \Omega \to \Omega'$ is $K$-quasiregular if and only if $f \in W^{1,2}_{\text{loc}}(\Omega)$, $f$ is orientation preserving and
\[
\partial_z f(z) = \mu(z) \partial_{\bar{z}} f(z) \text{ for almost every } z \in \Omega
\]
where $\mu$, called the Beltrami coefficient of $f$, is a bounded measurable function satisfying
\[
\|\mu\|_\infty \leq \sqrt{\frac{K-1}{K+1}} < 1.
\]
(2.19)
Note that the differential equation in (2.19) is called the Beltrami equation and it is this equation that provides the link from the geometric theory of quasiconformal...
mappings to complex analysis and to elliptic partial differential equations. For accounts of these connections we refer to [AIM], [IM], and [IM1]. The following lemma connects the notion of quasiregular mappings to the setup used in this paper.

**Lemma 2.3.** Let \( f = (f_1, f_2) \in W^{1,2}_{\text{loc}}(\Omega) \). If \( \partial_t f + \text{BD} f = 0 \) for some bounded and accretive \( B \) then \( f \) is quasiregular. Furthermore, if \( f \) is quasiregular and if we define the complex linear multiplier \( B := -\partial_t f / \text{BD} f \), then \( B \) is bounded and accretive and \( \partial_t f + \text{BD} f = 0 \).

**Proof.** Assume that \( \partial_t f + \text{BD} f = 0 \) for some bounded and accretive \( B \). Simply note that
\[
|\nabla f|^2 = |\partial_x f_1|^2 + |\partial_x f_2|^2 + |\partial_y f_1|^2 + |\partial_y f_2|^2 = |\partial_t f|^2 + |\text{BD} f|^2,
\]
(2.21) \( J(f) = |\partial_x f_1| |\partial_y f_2| - |\partial_x f_2| |\partial_y f_1| = |\partial_t f| \cdot |\text{BD} f| \), and hence \( |\nabla f|^2 \approx |\text{BD} f|^2 \lesssim (\text{BD} f, \text{DF}) = -(\partial_t f, \text{DF}) \), so \( f \) is quasiregular. To prove the other direction, assume that \( f \) is quasiregular and let \( B := -\partial_t f / \text{BD} f \) by complex division. Then \( \partial_t f + \text{BD} f = 0 \) and \( B \) is bounded since \( |\partial_t f|^2 + |\text{BD} f|^2 \lesssim |\partial_t f| \cdot |\text{BD} f| \). Moreover
\[
1 \approx \frac{J(f)}{|\nabla f|^2} = \frac{|\partial_t f| \cdot |\text{BD} f|}{|\partial_t f|^2 + |\text{BD} f|^2} \approx \frac{(\text{BD} f, \text{DF})}{|\text{DF}|^2} = \text{Re}(B),
\]
so \( B \) is accretive. This completes the proof of the lemma. \( \square \)

We next note the following existence and uniqueness result for the Beltrami equation in (2.19), assuming that \( \mu \) has compact support, as well as the Stoilow factorization of quasiregular mappings with subsequent corollary. Besides the more modern references given below for these results, we also refer the reader to the very readable lecture notes [Ahl, Chapter V].

**Theorem 2.1.** Let \( \mu \) bounded measurable function on \( C \) with compact support and assume that
\[
\|\mu\|_\infty \leq k \text{ for some } k < 1.
\]
Then there exists a unique \( f \in W^{1,2}_{\text{loc}}(\Omega) \) such that
\[
\partial_z f(z) = \mu(z) \partial_z f(z) \text{ for almost every } z \in C,
\]
(2.24) \( f(z) = z + O(z^{-1}) \text{ as } z \to \infty. \)

Moreover, there exists \( p(k) \) such that \( f \in W^{1,p}_{\text{loc}}(\Omega) \) for all \( p \), \( 2 \leq p < p(k) \).

**Proof.** This is Theorem 5.1.2 in [AIM]. \( \square \)

**Theorem 2.2.** Let \( f : \Omega \to \Omega' \) be a homeomorphic solution to the Beltrami equation in (2.19) with \( |\mu(z)| \leq k < 1 \) almost everywhere on \( \Omega \), and assume that \( f \in W^{1,1}_{\text{loc}}(\Omega) \). Suppose that \( g \in W^{1,2}_{\text{loc}}(\Omega) \) is any other solutions to (2.19). Then there exists a holomorphic function \( \Phi : \Omega' \to \mathbb{C} \) such that
\[
g(z) = \Phi(f(z)), \quad z \in \Omega.
\]
Conversely, if \( \Phi \) is holomorphic in \( \Omega' \), then the composition \( \Phi \circ f \) is a \( W^{1,2}_{\text{loc}} \)-solution to (2.19) in \( \Omega \).

**Proof.** This is Theorem 5.5.1 in [AIM]. \( \square \)

**Corollary 2.1.** Let \( f \) be a quasiregular mapping defined on a domain \( \Omega \subset \mathbb{C} \). Then
1. \( f \) is open and discrete.
2. \( f \) is locally H"older continuous with exponent \( \alpha = 1/K \), \( K = K(f) \).
3. \( f \) is differentiable with non-vanishing Jacobian almost everywhere.
Remark 2.1.  

Recall that a mapping $f : \Omega \rightarrow \mathbb{R}^2$ is discrete if $f^{-1}(y)$ is a discrete set for all $y \in \mathbb{R}^2$, and $f$ is open if it takes open sets onto open sets. That $f^{-1}(y)$ is a discrete set means that it is made up by isolated points. We also note the following lemma concerning the convergence of $K$-quasiregular mappings.

**Lemma 2.4.** Let $f_j : \Omega \rightarrow \mathbb{R}^2$, $j = 1, \ldots$, be a sequence of $K$-quasiregular mappings converging locally uniformly to a mapping $f$. Then $f$ is quasiregular and

$$K(f) = \limsup_{j \to \infty} K(f_j).$$

**Proof.** See, for example Theorem 8.6 in [R] and the discussion above Theorem 2.4 in the same reference.

Beltrami equations can be reduced to real elliptic divergence form equations and the following lemma can be verified by a straightforward calculation.

**Lemma 2.5.** Let $f = (f_1, f_2) \in W^{1,2}_{\text{loc}}(\Omega)$ satisfy (2.19) for some $\mu \in L_{\infty}(\Omega, \mathbb{C})$ satisfying (2.20). Define a $2 \times 2$-matrix $A = A_\mu = \{a_{ij}\}$ as follows. Given $\mu = \mu_1 + i\mu_2$, $\mu = \mu(z) \in \mathbb{R}$, $z \in \mathbb{C}$, we let

$$a_{11}(z) = \frac{1 - 2\mu_1(z) + |\mu(z)|^2}{1 - |\mu(z)|^2}, \quad a_{22}(z) = \frac{1 + 2\mu_1(z) + |\mu(z)|^2}{1 - |\mu(z)|^2},$$

(2.25)  $$a_{12}(z) = a_{21}(z) = -\frac{2\mu_2(z)}{1 - |\mu(z)|^2}.$$

Then $A$ is bounded, symmetric and satisfies $\det A(z) = 1$ for a.e. $z \in \Omega$. Furthermore, $f_1$ and $f_2$ are weak solutions to the equation

$$\text{div}(A \nabla \cdot) = 0 \text{ in } \Omega.$$

**Remark 2.1.** Consider the matrix $A$ in the statement of Lemma 2.5. Using that $\det A = 1$ one easily see that the eigenvalues of $A(z)$ are

$$\lambda_{\pm}(z) := \frac{1 + |\mu(z)|^2}{1 - |\mu(z)|^2} \pm \sqrt{\frac{1 + |\mu(z)|^2}{1 - |\mu(z)|^2} - 1}.$$

Since $|\mu(z)| < 1$ we immediately see that $\lambda_{\pm}(z)$ are greater or equal to 1 and that

$$\sup_{z \in \Omega} \lambda_-(z) \leq \sup_{z \in \Omega} \lambda_+(z) \leq \frac{1 + \beta^2}{1 - \beta^2} + \sqrt{\frac{1 + \beta^2}{1 - \beta^2} - 1}$$

(2.26)

if $|\mu(z)| < \beta$ for all $z \in \Omega$. In particular, if this is the case then $A$ is uniformly elliptic. Naturally an upper bound can also be derived by simply using the explicit expression of the coefficients $\{a_{ij}\}$.

The following lemma is essentially (2) of Corollary 2.1 but we include it, and a short discussion of its proof based on pde-techniques, to stress the connection between the Beltrami equation and quasi-linear pdes.

**Lemma 2.6.** Let $f = (f_1, f_2)$ be as in the statement of Lemma 2.5 and assume that $|\mu(z)| \leq \beta < 1$ on $\Omega$. Then there exist $c = c(\beta)$, $1 \leq c < \infty$, and $\sigma = \sigma(\beta) \in (0, 1)$, such that if $B(z, 2R) \subset \Omega$ then

$$\sup_{z_1, z_2 \in B(z, r)} |f(z_2) - f(z_1)| \leq c(r/R)^{\sigma} \left(R^{-2} \int_{B(z, 2R)} |f|^2 \, dz\right)^{1/2}.$$
that we have a topological splitting and

\[ \lambda \text{Re } \psi \]

denotes the open bisector, where

Note here, in particular, that

In particular, as a consequence of \[CMcM\] we note that both

\[ DB \]

Let

Proof. Let \( f = (f_1, f_2) \) be as in the statement of Lemma \[2.5\] and assume that \( |\mu(z)| \leq \beta < 1 \) on \( \Omega \). Consider a ball \( B(z, R) \) such that \( B(z, 2R) \subset \Omega \). Then, using Lemma \[2.5\] and Moser iteration we have that

\[
\sup_{z_1, z_2 \in B(z, R)} |f(z_2) - f(z_1)| \leq c(R |R|^{\sigma} \left( R^{-2} \int_{B(z, 2R)} |f|^2 dz \right)^{1/2}
\]

where \( c, \sigma \in (0, 1) \) are independent of \( f, r \) and \( R \). In fact, \( c \) and \( \sigma \) only depend on the operator through the ellipticity and the bounded on the coefficients and if \( |\mu(z)| \leq \beta < 1 \) on \( \Omega \), see Lemma \[2.4\] then \( c \) and \( \sigma \) will depend on \( \mu \) through \( \beta \). \( \square \)

3. Functional calculus and Cauchy operators

Recall the definition of \( D \) introduced below \[1.7\] and the matrix \( B_0 \) defined in \[1.3\]. Given \( p, f, \phi \), we in the following write \( B(f) = B_0^p \phi \) for this generic \( t \) dependent matrix. In line with \[AA\], \[AR\] we approach the system in \[1.7\] using a functional calculus build on the \( t \)-independent matrix \( B_0 \). As references for functional calculus we refer to \[ADMC\], \[AZ\], \[AMCN\], \[DN\], \[H\], \[Mc\], \[Mc1\], \[McM\]. Let \( L_2(\mathbb{R}) = L_2(\mathbb{R}, \mathbb{C}) \) and let, for \(-1 \leq \alpha \leq 1\),

\[
L_2(\mathbb{R}^2, t^\alpha) := \left\{ f : \mathbb{R}^2 \to \mathbb{C}^2 : \int_{\mathbb{R}^2} |f(x, t)|^2 t^\alpha dx < \infty \right\}.
\]

Then, both as an operator in \( L_2(\mathbb{R}) \) and in \( L_2(\mathbb{R}^2, t^\alpha) \), acting in the \( x \)-variable for each fixed \( t > 0 \), \( DB_0 \) and \( B_0 D \) define closed and densely defined operators with spectrum contained in a bisector \( S_\omega = S_{\omega^+} \cup (-S_{\omega^+}) \) where

\[
S_{\omega^+} := \{ \lambda \in \mathbb{C} : |\arg \lambda| \leq \omega \} \cup \{0\}, \quad \omega < \pi/2.
\]

In particular, as a consequence of \[CMcM\] we note that both \( DB_0 \) and \( B_0 D \) have bounded holomorphic functional calculi which supply estimates of operators \( \psi(DB_0) \) and \( \psi(B_0 D) \) formed by applying holomorphic functions \( \psi : S_{\mu}^o \to \mathbb{C}, \omega < \mu \), to the operators \( DB_0 \) and \( B_0 D \) respectively. Here \( S_{\mu}^o = S_{\mu^+}^o \cup (-S_{\mu^+}^o) \) denotes the open bisector, where

\[
S_{\mu^+}^o := \{ \lambda \in \mathbb{C} : |\arg \lambda| < \mu \} \setminus \{0\},
\]

Note here, in particular, that \( \psi \) need not be analytic across 0 or at \( \infty \).

Applying the functional calculi with the scalar holomorphic functions \( \lambda \to |\lambda| := \pm \lambda \), if \( \pm \text{Re } \lambda > 0 \), \( \lambda \to e^{-t |\lambda|} \), \( \lambda \to \chi_{\pm}(\lambda) := 1 \) if \( \pm \text{Re } \lambda > 0 \) and 0 elsewhere, here \( \text{Re } \lambda \) is the real part of \( \lambda \), we get operators

\[
\Lambda_0 := [DB_0], \quad e^{-t |DB_0|}, \quad t > 0, \quad E_{\mu}^0 := \chi_{\pm}(DB_0),
\]

\[
\tilde{\Lambda}_0 := [B_0 D], \quad e^{-t |B_0 D|}, \quad t > 0, \quad \tilde{E}_{\mu}^0 := \chi_{\pm}(B_0 D),
\]

acting as operators in \( L_2(\mathbb{R}) \). We note that the operators \( \tilde{E}_{0}^+ \) are projections and that we have a topological splitting

\[
L_2(\mathbb{R}) = \tilde{E}_{0}^+ L_2(\mathbb{R}) \oplus \tilde{E}_{0}^- L_2(\mathbb{R}).
\]

Using this notation we define the operators

\[
(S_0 h)_t(x) = \begin{cases}
(e^{-t \Lambda_0} \tilde{E}_{0}^+ h)(x), & t > 0, \\
(e^{-t \Lambda_0} \tilde{E}_{0}^- h)(x), & t < 0,
\end{cases}
\]

and

\[
(\tilde{S} h)_t(x) := \int_0^t e^{-(t-s) \Lambda_0} \tilde{E}_{0}^+ B_0 h_s(x) ds + \int_t^\infty e^{-(s-t) \Lambda_0} \tilde{E}_{0}^- B_0 h_s(x) ds, \quad t > 0.
\]
As we will see in Lemma 3.1 below, these operators coincide with the boundary and solid Cauchy integrals also mentioned in the introduction.

We intend to derive a representation formula for solutions to the equation

\( (3.11) \)
\[ \partial_t f_t + B(f_t)D f_t = 0, \quad f_t \in L_2(\mathbb{R}). \]

To do this we first write

\( (3.7) \)
\[ \mathcal{E}_t = \mathcal{E}_t(x) := (\mathcal{E}(f))_t(x) := I - (B_0(x))^{-1}B(f(x,t)). \]

Applying the projections \( \mathcal{E}_t^+ \) we see that

\( (3.8) \)
\[ \partial_t f_t^+ + \lambda_0 f_t^+ = \mathcal{E}_t^+ B_0 \mathcal{E}_t D f_t, \]
\[ \partial_t f_t^- - \lambda_0 f_t^- = -\mathcal{E}_t^- B_0 \mathcal{E}_t D f_t. \]

where now \( f_t^+ = \mathcal{E}_t^+ f_t \). Formally assuming that

\( \lim_{t \to 0^+} f_t = f_0, \lim_{t \to \infty} f_t = 0, \)
we can integrate the equations in (3.9) to conclude that

\( (3.9) \)
\[ f_t^+ - e^{-t\lambda_0} f_0^+ = \int_0^t e^{-(t-s)\lambda_0} \mathcal{E}_t^+ B_0 \mathcal{E}_s D f_s ds, \]
\[ 0 - f_t^- = -\int_t^\infty e^{-(s-t)\lambda_0} \mathcal{E}_t^- B_0 \mathcal{E}_s D f_s ds. \]

Subtracting the equations in (3.9), we see that

\( (3.10) \)
\[ f_t = (S_0 f_0)_t + \tilde{S}\mathcal{E}_t D f_t, \]
and we have derived a representation formula for (3.6). To continue we note that

\( (3.11) \)
\[ D f_t = D e^{-t\lambda_0} \mathcal{E}_t^+ f_0 + D \tilde{S}\mathcal{E}_t D f_t. \]

However, using that relations \( D \psi(B_0 D) = \psi(DB_0)D \) through the holomorphic functional calculus we first see that \( D e^{-t\lambda_0} \mathcal{E}_t^+ f_0 = e^{-t\lambda_0} \mathcal{E}_t^+ f_0 \). Furthermore, by the same argument we have that

\( (3.12) \)
\[ D e^{-(t-s)\lambda_0} \mathcal{E}_t^+ B_0 = \lambda_0 e^{-(t-s)\lambda_0} \mathcal{E}_s^+ B_0, \]
\[ D e^{-(s-t)\lambda_0} \mathcal{E}_t^- B_0 = \lambda_0 e^{-(s-t)\lambda_0} \mathcal{E}_s^- B_0. \]

Define

\( (3.13) \)
\[ (Sh)_t(x) := D(\tilde{S}h)(x) = \int_0^t \lambda_0 e^{-(t-s)\lambda_0} \mathcal{E}_s^+ h_s(x) ds \]
\[ + \int_t^\infty \lambda_0 e^{-(s-t)\lambda_0} \mathcal{E}_s^- h_s(x) ds, \]
for \( h \in L_2(\mathbb{R}) \) so that

\( (3.14) \)
\[ D f_t = e^{-t\lambda_0} \mathcal{E}_t^+ f_0 + S\mathcal{E}_t D f_t. \]

For a rigorous definition of the singular integral operator \( S \), see [AA]. We here note the following lemma concerning the operators \( S_0, \tilde{S} \) and \( S \) and their relation to complex analysis.

**Lemma 3.1.** Let \( S_0, \tilde{S} \) and \( S \) be defined as above. Then \( S_0 \) is the boundary Cauchy integral

\[ (S_0 h)_t(x) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{h(y)(1 + i\gamma(y)) dy}{y + i\gamma(y) - (x + i\gamma(x))}. \]
Lemma 3.2. In particular, we have the following estimates.

\[ (\mathcal{S}h)_t(x) = \frac{1}{2\pi i} \int_{\mathbb{R}^2_+} \frac{h(y,s)(1 + i\phi'(y))}{(y + i(s + \phi(y)) - (x + i(t + \phi(x)))^2} dyds, \]

and \( S = D\mathcal{S} \) is the Beurling transform

\[ (Sh)_t(x) = -\frac{1}{2\pi} p.v. \int_{\mathbb{R}^2_+} \frac{h(y,s)(1 + i\phi'(y))}{((y + i(s + \phi(y)) - (x + i(t + \phi(x)))^2} dyds. \]

Proof. The result for the boundary Cauchy singular integral is well known, see for example [McQ]. Integration and derivation gives the result for \( \mathcal{S} \) and \( S \). \( \square \)

To further understand the appropriate function spaces we consider the free evolution \( e^{-t\tilde{\Lambda}_0} \tilde{E}_0^+ f \) and we note that

\[
\int_{\mathbb{R}^2_+} |D e^{-t\tilde{\Lambda}_0} \tilde{E}_0^+ f|^2 t^{1-2\sigma} dx dt \\
\approx \int_{\mathbb{R}^2_+} |B_0 D e^{-tB_0}\tilde{E}_0^+ f|^2 t^{1-2\sigma} dx dt \\
= \int_{\mathbb{R}^+} \left( \int_{\mathbb{R}} |t^{1-\sigma} B_0 D e^{-t\tilde{\Lambda}_0} \tilde{E}_0^+ f|^2 dx \right) \frac{dt}{t}.
\]

Note that \( \tilde{\Lambda}_0 = \text{sgn}(B_0 D)B_0 D \) and since \( \text{sgn}(B_0 D) \) is invertible we can conclude that

\[
\int_{\mathbb{R}^2_+} |D e^{-t\tilde{\Lambda}_0} \tilde{E}_0^+ f|^2 t^{1-2\sigma} dx dt \\
\approx \int_{\mathbb{R}^+} \left( \int_{\mathbb{R}} |(t\tilde{\Lambda}_0)^{1-\sigma} e^{-t\tilde{\Lambda}_0} \tilde{E}_0^+ f|^2 dx \right) \frac{dt}{t}.
\]

Using that \( \psi(tB_0) := (t\tilde{\Lambda}_0)^{1-\sigma} e^{-t\tilde{\Lambda}_0} \tilde{E}_0^+ f \) satisfy square function estimates when \( \sigma < 1 \), it follows from \( \text{3.16} \) that

\[
\int_{\mathbb{R}^2_+} |D e^{-t\tilde{\Lambda}_0} \tilde{E}_0^+ f|^2 t^{1-2\sigma} dx dt \approx \int_{\mathbb{R}} |\tilde{E}_0^+ f|^2 dx.
\]

However, \( \mathcal{D}(\tilde{\Lambda}_0) = \mathcal{D}(B_0 D) \approx \mathcal{D}(D) \approx \mathcal{D}(\nabla) = \dot{H}^1(\mathbb{R}) \) and \( \mathcal{D}(\tilde{\Lambda}_0^0) = L_2(\mathbb{R}) \). Hence, as in [Kr], by interpolation we see that

\[
\mathcal{D}(\tilde{\Lambda}_0^\sigma) = \dot{H}^{\sigma}(\mathbb{R}).
\]

In particular, we have the following estimates.

Lemma 3.2. Let \( 0 \leq \sigma \leq 1 \). For all \( f \in \mathcal{D}(\tilde{\Lambda}_0^\sigma) \) we have

\[
\|\tilde{\Lambda}_0^\sigma f\|_2 \approx \|f\|_{\dot{H}^{\sigma}(\mathbb{R})}.
\]

Lemma 3.3. The following estimates holds.

(1) For \( \sigma \in [0,1] \),

\[
\sup_{t>0} \|e^{-t\tilde{\Lambda}_0} \tilde{E}_0^+ f\|_{\dot{H}^{\sigma}(\mathbb{R})} \leq c \|f\|_{\dot{H}^{\sigma}(\mathbb{R})}.
\]

(2) For \( \sigma \in [0,1] \),

\[
\int_{\mathbb{R}^2_+} |D e^{-t\tilde{\Lambda}_0} \tilde{E}_0^+ f|^2 t^{1-2\sigma} dx dt \leq c \|f\|_{\dot{H}^{\sigma}(\mathbb{R})}^2.
\]
Remark 3.1. Below we refer to [R] for many results concerning solving linear boundary value problems for first order systems of the form
\[ \partial_t f + B_0 Df = 0, \]
with boundary trace \( f|_\mathcal{R} \in \dot{H}^\sigma(\mathbb{R}). \) Applying the isomorphism \( D : \dot{H}^\sigma(\mathbb{R}) \to \dot{H}^\sigma(\mathbb{R}), \) pointwise in \( t, \) to this equation, yields the system
\[ \partial_t f + DB_0 f = 0, \]
for \( \tilde{f} := Df \in \dot{H}^\sigma(\mathbb{R}). \) Thus the results stated in [R] for \( \tilde{f}, \) transfer directly to the setting in this paper for \( f. \)

To establish estimates for the operators \( S \) and \( \tilde{S} \) we first note that for a multiplier \( \mathcal{E} \) we have
\[ (3.19) \sup_{\|f\|_{L_2(R^2_+,t^{1-2\sigma})}=1} \|\mathcal{E} f\|_{L_2(R^2_+,t^{1-2\sigma})} = \|\mathcal{E}\|_{L_\infty(R^2_+)}. \]
We next prove the following lemma.

Lemma 3.4. The following estimates holds.

1. For \( \sigma \in [0,1/2), \)
\[ \sup_{t>0} \|\mathcal{E}_t Df_t\|_{L_\infty(R^2_+)}^2 \leq c \|\mathcal{E}\|_{L_\infty(R^2_+)}^2 \int R^2_+ |Df(x,t)|^2 t^{1-2\sigma} dx dt. \]

2. For \( \sigma \in [1/2, 1), \)
\[ \sup_{t>0} \frac{1}{t} \int R^2_+ \|\tilde{\mathcal{E}}_t Df_t\|_{L_\infty(R^2_+)}^2 ds \leq c \|\mathcal{E}\|_{L_\infty(R^2_+)}^2 \int R^2_+ |Df(x,t)|^2 t^{1-2\sigma} dx dt. \]

3. For \( \sigma \in (0,1), \)
\[ \int R^2_+ |Sf_t|^2 t^{1-2\sigma} dx dt \leq c \int R^2_+ |f(x,t)|^2 t^{1-2\sigma} dx dt. \]

Proof. The estimates (1) and (2) follow from [R] Thm 1.3. For the reader’s convenience we outline the proof here. Let \( h_t := \mathcal{E}_t Df_t \) and consider
\[ (3.20) \tilde{S} h_t = \int_0^t e^{-(s-t)\tilde{\Lambda}_0} \tilde{E}_0^+ B_0 h_s ds + \int_t^\infty e^{-(s-t)\tilde{\Lambda}_0} \tilde{E}_0^- B_0 h_s ds. \]
Then
\[ \tilde{\Lambda}^\sigma_0 \tilde{S} h_t = \int_0^t \tilde{\Lambda}^{1-\tilde{\sigma}}_0 e^{-(s-t)\tilde{\Lambda}_0} \tilde{E}_0^+ B_0 h_s ds + \int_t^\infty \tilde{\Lambda}^{1-\tilde{\sigma}}_0 e^{-(s-t)\tilde{\Lambda}_0} \tilde{E}_0^- B_0 h_s ds \]
\[ (3.21) \]
where \( \tilde{\sigma} := 1 - \sigma. \) Using Lemma 3.2 we see that we want to estimate \( \|\tilde{\Lambda}^\sigma_0 \tilde{S} h_t\|_2. \) To do this we, following [R], write
\[ (3.22) \tilde{\Lambda}^\sigma_0 \tilde{S} h_t = I_1 + I_2 + I_3 + I_4 \]
where
\[ I_1 = \int_{t/2}^{2t} \tilde{\Lambda}^{1-\tilde{\sigma}}_0 e^{-(s-t)\tilde{\Lambda}_0} \tilde{E}_0^+ \text{sgn}(s-t) B_0 h_s ds, \]
\[ I_2 = \int_0^{t/2} \tilde{\Lambda}^{1-\tilde{\sigma}}_0 e^{-(s-t)\tilde{\Lambda}_0} (I - e^{-2s\tilde{\Lambda}_0}) \tilde{E}_0^+ B_0 h_s ds, \]
\[ I_3 = \int_{2t}^{\infty} \tilde{\Lambda}^{1-\tilde{\sigma}}_0 e^{-(s-t)\tilde{\Lambda}_0} (I - e^{-2s\tilde{\Lambda}_0}) \tilde{E}_0^- B_0 h_s ds, \]
\[ I_4 = \int_{t/2}^{2t} \tilde{\Lambda}^{1-\tilde{\sigma}}_0 e^{-(s-t)\tilde{\Lambda}_0} \tilde{E}_0^- B_0 h_s ds. \]
Obviously a similar estimate holds for (3.26) when positive and as \( -\sigma \) when negative. Consider first the case when \( \sigma \in [0, 1/2), \) i.e., \( \tilde{\sigma} \in (1/2, 1]. \) We then immediately see that

\[
||I_1||_2 \lesssim \int \frac{2t}{|t-s|^{1-\tilde{\sigma}}} ds \\
\lesssim \left( \int_0^{2t} \left( \frac{1}{|t-s|^{1-2\tilde{\sigma}}} \right)^{1/2} ds \right) \|\hat{h}\|_{L_2(\mathbb{R}^2_x, t^{1-2\tilde{\sigma}})}
\]

(3.24)

To estimate \( ||I_2||_2 \) we note that

\[
||\hat{\Lambda}_0^{1-\tilde{\sigma}} e^{-(t-s)\hat{\lambda}_0} (I - e^{-2s\hat{\lambda}_0})|| =
\]

(3.25)

\[
||s(t-s)^{2-\tilde{\sigma}}((t-s)(1-2\tilde{\sigma}))) e^{-(t-s)\hat{\lambda}_0} (I - e^{-2s\hat{\lambda}_0})/s\| \lesssim s/t^{2-\tilde{\sigma}}.
\]

Using this we see that

\[
||I_2||_2 \lesssim \int_0^{t/2} s/t^{2-\tilde{\sigma}} \|\hat{h}_s\|_2 ds \lesssim \|\hat{h}\|_{L_2(\mathbb{R}^2_x, t^{1-2\tilde{\sigma}})}.
\]

(3.26)

Obviously a similar estimate holds for \( ||I_3||_2 \). Finally, to estimate \( ||I_4||_2 \) we consider \( \phi \in L_2(\mathbb{R}), \phi||_2 = 1, \) and note that

\[
||I_4, \phi|| \lesssim \int_{s=0}^{\infty} \|s\hat{\Lambda}_0^{1-\tilde{\sigma}} e^{-s\hat{\lambda}_0} \phi||_2 \|\hat{h}_s\|_2 ds \lesssim \|\hat{h}\|_{L_2(\mathbb{R}^2_x, t^{1-2\tilde{\sigma}})}.
\]

(3.27)

These estimates complete the proof of (1). The proof of (2), in this case \( \sigma \in [1/2, 1], \) i.e., \( \tilde{\sigma} \in (0, 1/2), \) follow similar except that in this case we have to be slightly more careful when estimating

\[
\frac{1}{t} \int_t^{2t} ||I_1(s)||_2 ds
\]

(3.28)

where

\[
I_1(s) = \int_{s/2}^{2s} \hat{\Lambda}_0^{1-\tilde{\sigma}} e^{-(s-\tau)\hat{\lambda}_0} E_0^{s\text{sgn}(s-\tau)} B_0 h_\tau d\tau.
\]

(3.29)

Indeed, in this case we have

\[
\frac{1}{t} \int_t^{2t} ||I_1(s)||_2 ds \lesssim \frac{1}{t} \int_t^{2t} \left( \int_{s/2}^{2s} \frac{\|\hat{h}_\tau\|_2}{|s-\tau|^{1-\tilde{\sigma}}} d\tau \right) ds \\
\lesssim \frac{1}{t} \int_t^{2t} \left( \int_{s/2}^{2s} \frac{1}{|s-\tau|^{1-\tilde{\sigma}}} \left( \int_{s/2}^{2s} \frac{\|\hat{h}_\tau\|_2^2}{|s-\tau|^{1-\tilde{\sigma}}} d\tau \right) ds \\
\lesssim \frac{1}{t} \int_t^{2t} s^{\tilde{\sigma}} \left( \int_{s/2}^{2s} \frac{\|\hat{h}_\tau\|_2^2}{|s-\tau|^{1-\tilde{\sigma}}} d\tau \right) ds \\
\lesssim \frac{1}{t} \int_{t/2}^{4t} \left( \int_{\tau/2}^{2\tau} \frac{s^{\tilde{\sigma}}}{|s-\tau|^{1-\tilde{\sigma}}} ds \right) \|\hat{h}_s\|_2^2 ds \lesssim \|\hat{h}\|_{L_2(\mathbb{R}^2_x, t^{1-2\tilde{\sigma}})}.
\]

(3.30)

To complete the proof of the lemma it only remains to prove the statement in (3). This follows from [R] Thm. 2.3, which completes the proof of Lemma 5.3. □
4. Proof of the Main Results

In the following we let $B_0$ be as in $\text{[1,13]}$ and we recall the operators introduced in $\text{[2,3]}$ and acting in $L_2(\mathbb{R})$. Recall that $B_0D$ is an injective bisectorial operator and that

\begin{equation}
L_2(\mathbb{R}) = \tilde{E}_0^+ L_2(\mathbb{R}) \oplus \tilde{E}_0^- L_2(\mathbb{R}).
\end{equation}

Lemma 4.1. Let, for a given $1 < p < \infty$ and Lipschitz function $\phi$ with Lipschitz constant $M$, $B(f) = B_p^f \phi$ as in $\text{[1,8]}$. Then there exists $C(M, p) < \infty$ such that $|B(f)| \leq C(M, p)$ and

\[ \operatorname{Re}(B(f)v, v) \geq |v|^2 / C(M, p), \quad \text{for all } v \in \mathbb{C}^2, \]

uniformly for all $f \neq 0$. Moreover, for a fixed Lipschitz function $\phi$, we have

\[ \limsup_{p \to 2} \left( \sup_{f \neq 0} \frac{|B(f) - B_0|}{|p - 2|} \right) < \infty. \]

Proof. The proof is straightforward and we omit it. $\square$

4.1. Proof of Theorem 1.1. Let $p$, $1 < p < \infty$, be given and let $\sigma \in (0, 1)$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function with constant at most $M$. Assume that $u$ is $p$-harmonic in $\Omega = \{(x, y) : x \in \mathbb{R}, \ y > \phi(x)\}$ and that

\begin{equation}
\iint_{\Omega} |
abla^2 u|^2 (y - \phi(x))^{1-2\sigma} \, dx \, dy < \infty.
\end{equation}

Let $(x, y) \to (x, t)$, $t = y - \phi(x)$, $f(x, t) = (\partial_x u(x, y), -\partial_y u(x, y))^\ast$. Then,

\begin{equation}
\partial_t f + B_0 Df = B_0 \Sigma Df, \quad \Sigma := I - B_0^{-1} B_p^f \phi.
\end{equation}

We note that $\Sigma$ is a multiplier defined almost everywhere by Corollary 2.1 which is bounded by Lemma 4.1 with bound independent of $f$. Using (3.10) we see that

\begin{equation}
f_t = e^{-t B_0^+} \tilde{E}_0^+ g^+ + \tilde{S} \Sigma Df_t.
\end{equation}

Formally, we expect

\[ \|g^+\|^2_{H^\sigma(\mathbb{R})} \lesssim \iint_{\Omega} |
abla^2 u|^2 (y - \phi(x))^{1-2\sigma} \, dx \, dy \]

from (4.1). This is indeed the case, and can be proved rigorously as in $\text{[15]}$, which we refer to for detailed proofs of the stated bounds and limits. See Lemma 3.3 and Lemma 3.4 for some of the main details.

4.2. Proof of Theorem 1.2. Let $0 < \sigma < 1$ and a Lipschitz function $\phi$, with Lipschitz constant $M$, be given and assume that $|p - 2| < \delta$, where $\delta > 0$ is to be chosen. We recursively define functions $f^k = (f^k_1, f^k_2) : \mathbb{R}^2_+ \to \mathbb{R}^2$, solving the boundary value problem

\[
\begin{cases}
\partial_t f^{k+1} + B(f^k) Df^{k+1} = 0, & \text{in } \mathbb{R}^2_+,

(f^{k+1})_1 = h, & \text{on } \mathbb{R},
\end{cases}
\]

for $k \geq 1$. Indeed, for $k = 0$, we define $f^1$ to be the solution to this boundary value problem, with $B(f^0) := B_0$. This boundary value problem for $f^1$, which corresponds to Cauchy–Riemann equations in the Lipschitz domain $\Omega$, is well known to be well posed, and we have a unique solution with bounds

\[ \|Df^1\|_{L_2(\mathbb{R}^2, t^{1-2\sigma})} \lesssim \|h\|_{H^\sigma(\mathbb{R})}. \]

This follows by interpolation from the end point cases $\sigma \in \{0, 1\}$, where it is proved by Rellich type estimates. For $k \geq 1$, we note that $\|B(f^k) - B_0\|_{L_\infty(\mathbb{R}^2_+)}$ is small depending on $\delta$. By known stability results for boundary value problems, which
follows from the square function estimates for $B_0D$, see [R] Prop. 4.2, we may and do choose $\delta$ small enough so that we have unique solutions $f_k$ with bounds

$$
\|Df_k\|_{L^2(\mathbb{R}_+^2, t^{1-2\sigma})} \lesssim \|h\|_{H^\sigma(\mathbb{R})}.
$$

We obtain a sequence of uniformly quasi regular functions $\{f_k\}$, which are uniformly bounded

$$
\|Df_k\|_{L^2(\mathbb{R}_+^2, t^{1-2\sigma})} \leq C, \quad \|f_k\|_{C^\omega(K)} \leq C(K),
$$

for any compact subset $K \subset \mathbb{R}_+^2$. The latter uniform H"older bounds follow from Sobolev embedding theorem and interior estimates of Lemma 2.6.

To continue, consider now the equation

$$
\partial_t f_k + B(f_k)Df_k = 0
$$

as $k \to \infty$. We can and do pick a subsequence $\{f_{k_j}\}_{j=1}^\infty$ so that $Df_{k_j}$ converges weakly in $L^2(\mathbb{R}_+^2, t^{1-2\sigma})$, and $f_{k_j}$ converges locally uniformly on compact subsets, by Banach–Alaoglu’s and Arzelà–Ascoli’s theorems. Then the limit function $f$ is quasiregular by Lemma 2.4 and the union $Z \subset \mathbb{R}_+^2$ of the zeros of $f$ and all $f_{k_j}$, $j = 1, 2, \ldots$, has measure zero by (1) of Corollary 2.1. Thus $B(f_{k_j})$ are well defined and converges pointwise to $B(f)$ on $\mathbb{R}_+^2 \setminus Z$. Writing

$$
\int_{\mathbb{R}_+^2} (\varphi, B(f_{k_j})Df_{k_j})dxdt
$$

and using Lebesgue dominated convergence theorem on the second term, we can conclude that $B(f_{k_j})Df_{k_j}$ converges weakly to $B(f)Df$. Therefore $f$ solves

$$
\partial_t f + B(f)Df = 0.
$$

It now remains to show that the boundary condition $f_1|_\mathbb{R} = h$ holds for the trace $f|_\mathbb{R}$, which exists by Theorem 1.1. To this end, we note that a slight generalization of Theorem 1.1 yields the representation formula

$$
f_{k_j} = S_0(g_{k_j} + \tilde{S}(0) - B(f_{k_j}))Df_{k_j},
$$

which gives a representation of $f_{k_j}$ using the boundary function $g_{k_j} \in \tilde{E}_0^+ \dot{H}^\sigma(\mathbb{R})$, and the trace of $f_{k_j}$ can be expressed as

$$
f_{k_j} |_{\mathbb{R}} = g_{k_j} + \int_0^\infty (S_0(0) - B(f_{k_j}))Df_{k_j} - ds.
$$

The functions $\{g_{k_j}\}$, $g_{k_j} \in \tilde{E}_0^+ \dot{H}^\sigma(\mathbb{R})$, are uniformly bounded and

$$
DS_0(g_{k_j}) = f_{k_j} + \int_0^\infty (S_0(0) - B(f_{k_j}))Df_{k_j} - ds.
$$

converges weakly in $L^2(\mathbb{R}_+^2, t^{1-2\sigma})$. Using this the weak convergence of $g_{k_j}$ and of $f_{k_j} |_{\mathbb{R}} = g_{k_j} + \int_0^\infty (S_0(0) - B(f_{k_j}))Df_{k_j} - ds$ in $\dot{H}^\sigma(\mathbb{R})$ follows. We conclude that $f_1|_\mathbb{R} = h$, which completes the proof.

5. Concluding remarks

We here briefly discuss generalizations of Theorem 1.1 and Theorem 2.2 to the more general quasi-linear pdes in the plane considered in section 2. Indeed, consider (2.1) assuming (2.2) and recall Lemma 2.1. The lemma states that $u$ is a weak solution to (2.1) in $\Omega$ if and only if

$$
f(x, t) = (f_1(x, t), f_2(x, t)) := (\partial_x u(x, t + \phi(x)), -\partial_y u(x, t + \phi(x)))^*.
$$
Lemma 2.2 we have

\[ \partial_t \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + B_p^{a,f,\phi} D \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = 0, \]

in \( \mathbb{R}^2_+ := \{(x,t) \in \mathbb{R}^2 : t > 0\} \) where we now stress the dependence of \( B_p^{f,\phi} \) on the symbol \( a \) by writing \( B_p^{a,f,\phi} \). Furthermore, \( B_p^{a,f,\phi} \) is given in (2.12), (2.13) and by Lemma 2.2 we have \( B_p^{a,f,\phi} \in L_\infty(\mathbb{R}^2_+; \mathbb{C}^2) \) and \( B \) is accretive in the sense of (2.14). Furthermore, the \( L_\infty \)-bound on \( B_p^{a,f,\phi} \), and the parameter of accretivity \( \kappa \), depend only on \( p, M, \nu, \) and \( L \). In the following we let \( B_0 \) be as in (1.13) and we recall the arguments in section 4.

Theorem 5.2. Then, in the following we say that \( B_p^{a,f,\phi} \) is within \( \epsilon \) of \( B_0 \) if

\[ \sup_{f \neq 0} \frac{|B_p^{a,f,\phi} - B_0|}{|p - 2|} < \epsilon. \]

Note that when the underlying operator is the \( p \)-Laplace operator, i.e., \( a(\eta) = |\eta|^{p-2}\eta \), and \( B(f) = B_p^{a,f,\phi} \) as in (1.3), then Lemma 4.1 states that

\[ \limsup_{p \to 2} \sup_{f \neq 0} \frac{|B_p^{a,f,\phi} - B_0|}{|p - 2|} < \infty \]

and hence in this case \( B_p^{a,f,\phi} \) is within \( \epsilon \) of \( B_0 \), for any \( \epsilon \in (0, 1) \), as long as \( |p - 2| \) is small enough. The following two theorems, generalizing Theorem 1.1 and Theorem 1.2 to the more general quasi-linear pdes in the plane, can be proved by repeating the arguments in section 4.

**Theorem 5.1.** Let \( 1 < p < \infty, 0 < \sigma < 1, \) and \( 0 \leq M < \infty \) be given. Assume that \( \phi : \mathbb{R} \to \mathbb{R} \) is a Lipschitz function with \( \|\phi\|_\infty \leq M \) and assume that \( u \) is a weak solution to (2.1), assuming \( (2.2) \), in \( \Omega = \{(x,y) : x \in \mathbb{R}, y > \phi(x)\} \) satisfying

\[ \int_{\Omega} |\nabla^2 u|^2 (y - \phi(x))^{1-2\sigma} \, dx \, dy < \infty. \]

Let \( f(x,t) = \left( \partial_x u(x,y), -\partial_y u(x,y) \right)^*, \) \( y = t + \phi(x) \). Then there exists \( g \in \dot{H}^{\sigma}(\mathbb{R}) \) such that the Cauchy integral representation

\[ f = S_0 g + \tilde{S}((B_0 - B_p^{a,f,\phi})Df), \]

holds in \( \mathbb{R}^2_+ \). Furthermore, (1.10), (1.17), as well as the remaining statements of Theorem 1.3 remain valid also in this setting.

**Theorem 5.2.** Let \( p, \sigma, M, \phi \) be as in Theorem 5.1. Then there exists \( \epsilon_0 = \epsilon_0(p,\sigma,M) \), \( \epsilon_0 \in (0,1) \), such that the following is true. Let \( \epsilon \in (0,\epsilon_0) \) and assume that \( B_p^{a,f,\phi} \) is within \( \epsilon \) of \( B_0 \). Then, given any boundary data \( h \in \dot{H}^{\sigma}(\mathbb{R}) \), there exists a weak solution \( u \) to (2.1) in \( \Omega = \{(x,y) : x \in \mathbb{R}, y > \phi(x)\} \) satisfying

\[ \int_{\Omega} |\nabla^2 u|^2 (y - \phi(x))^{1-2\sigma} \, dx \, dy < \infty, \]

and the boundary condition

\[ \partial_x u(x, \phi(x)) = h(x), \quad x \in \mathbb{R}, \]

where the trace of \( \nabla u \) is taken in the sense of Theorem 5.1. The same solvability result also holds true for the boundary condition \( \partial_y u(x, \phi(x)) = h(x) \).
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