DISCRETE SUBGROUPS OF LOCALLY DEFINABLE GROUPS

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ABSTRACT. We work in the category of locally definable groups in an o-minimal expansion of a field. Eleftheriou and Peterzil conjectured that every definably generated abelian connected group $G$ in this category, is a cover of a definable group. We prove that this is the case under a natural convexity assumption inspired by the same authors, which in fact gives a necessary and sufficient condition. The proof is based on the study of the zero-dimensional compatible subgroups of $G$. We prove that the rank of such groups is bounded by the dimension of $G$. We also obtain the finiteness of the $n$-torsion subgroup under a divisibility assumption. Under a convexity hypothesis we show that the fundamental group of $G$ is finitely generated.

1. INTRODUCTION

We consider groups which are locally definable in an o-minimal expansion $M = (M, <, +, \cdot, \ldots)$ of a field. Most of our results hold more generally for an o-minimal expansion of a group, but the field assumption simplifies things. The study of definable groups has been a main theme of research in the model theory of o-minimal structures. More recently various authors considered the larger category of those groups which are locally definable, namely the domain and the group operation are given by countable unions of definable sets. One of the motivations for working in this larger category is that the universal cover of a definable group is locally definable [EE07]. Unlike the case of definable groups, in general one cannot expect a “tame” behaviour for all locally definable groups. For instance every countable group is obviously locally definable. However under some natural additional assumptions, such as connectedness (in the sense of [BO10]), the known examples of locally definable groups seem to exhibit a tame behavior. In particular it is natural to conjecture that a locally definable abelian connected group $G$ behaves in many respects like a finite product of copies of $(\mathbb{R}, +)$ and $\mathbb{R}/\mathbb{Z}$, and in particular it is divisible and has an $n$-torsion subgroup isomorphic to $(\mathbb{Z}/n\mathbb{Z})^s$ for some $s \leq \dim(G)$. This is indeed true in the definable case (with $s = \dim(G)$ if $G$ is definably compact [EO04]), and remains true for those abelian connected locally definable groups $G$ which are covers of definable groups (namely there is a locally definable surjective homomorphism from $G$ to a definable group $H$ of the same dimension). Let us observe that a cover of a locally definable group is always definably generated (see [EP12a]), namely it is generated as an abstract group by a definable subset. For an

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example of a locally definable group which is not definably generated one can take
a convex subgroup of \((M, <, +)\) of the form \(\bigcup_{i \in \mathbb{N}} (-a_i, a_i)\) where \(na_i < a_{i+1}\) for all
\(n \in \mathbb{N}\) [EP12a, Ele12]. Eleftheriou and Peterzil made the following conjecture:

**Conjecture 1.1.** [EP12a, EP12b] Let \(G\) be a definably generated abelian connected
group. Then \(G\) is a cover of a definable group.

A solution of the conjecture would reduce many questions about locally definable
abelian connected groups to similar questions about definable groups, even without
assuming that the group is definably generated. This depends on the fact that
every locally definable group \(G\) is a directed union of definably generated subgroups
(which can be taken to be connected if \(G\) is connected). For instance a positive
solution of Conjecture 1.1 would yield a positive solution to the following conjecture:

**Conjecture 1.2.** [Edm03, Edm05, EP12a] Every locally definable connected abelian
group is divisible.

In [EP12b] Eleftheriou and Peterzil introduced a notion of convexity which they
used to prove Conjecture 1.1 for subgroups of \((M^n, +)\). They also proved that
Conjecture 1.1 is equivalent to the following:

**Conjecture 1.3.** [EP12b, Conjecture B] Let \(G\) be a definably generated abelian
connected group.

1. There is a maximal \(k \in \mathbb{N}\) such that \(G\) contains a compatible subgroup \(\Gamma\)
   isomorphic to \(\mathbb{Z}^k\).
2. If \(G\) is not definable, \(k \geq 1\).

If Conjecture 1.3 holds and we take \(\Gamma\) as in the conjecture, then \(G/\Gamma\) is definable,
and \(G\) covers it.

We are now ready to discuss the results of this paper. In Section 4 we prove part
(1) of Conjecture 1.3. In Section 6 we prove part (2) under a convexity assumption
suggested by the work of Eleftheriou and Peterzil. As a result of this analysis we
show:

**Theorem** (see Theorem 6.6). A definably generated connected abelian group \(G\) is
a cover of a definable group if and only if for every definable set \(X \subset G\) there is a
definable set \(Y \subset G\) which contains the convex hull of \(X\) (in the sense of Definition
6.3).

We also consider locally definable connected abelian groups which are not nec-
essarily definably generated. Let \(G\) be a locally definable abelian connected group.
We show that if \(G\) is divisible then \(G\) has a finite \(n\)-torsion subgroup \(G[n]\) (Corollary
5.7). Note that in the definable case one can easily deduce the divisibility of \(G\) from
the finiteness of \(G[n]\), while here the implication goes the other way around. Since
our results may turn out to be useful to approach the divisibility conjecture we point
out that our proof yields the finiteness of \(G[n]\) under the weaker hypothesis that the
universal cover of \(G\) is torsion free (Theorem 5.6). Under an additional convexity
assumption we show that \(\pi_1(G)\) is finitely generated (Theorem 7.2) and that every
zero-dimensional compatible subgroup of \(G\) is finitely generated (Corollary 7.4)

The problem of the finiteness of the \(n\)-torsion subgroup (assuming divisibility)
was considered in the unpublished note [Edm03] by the second author, but the
proof of finiteness of \(G[n]\) contained therein had some gaps. One of the original
motivation for the writing up of this paper was to provide a correct proof.
In Section 8 we give an example of a locally definable H-space whose fundamental group is \(\mathbb{Q}\). This may explain some of the difficulties in showing that \(\pi_1(G)\) is finitely generated using only homotopy information, thus giving some indirect motivation for the introduction of the convexity assumption. In the same section we give an example of a locally definable connected group which has a generic definable subset, but does not cover a definable group. This shows that the results of [CP12a] do not extend to non-commutative groups.

The reader may gain some insight on the problems considered in this paper by first taking a look at Question 9.1 at the end of the paper.

Sections 2 and 3 contain some definitions and background results.

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2. Locally definable groups

We work in an o-minimal expansion \(M\) of a field. A set is definable if it is definable in \(M\) with parameters. There are several different definitions and variants of locally definable groups in the literature. In this paper a locally definable group is a countable union of definable sets (i.e. a locally definable set), equipped with a group operation whose restriction to each definable set is definable. Strictly speaking this definition works well only if \(M\) is \(\aleph_1\)-saturated, since for instance we would like to identify the universal cover of the 1-dimensional torus \(T\) with the locally definable group \(\text{Fin}(M) := \bigcup_{n \in \mathbb{N}} [-n, n] < (M,+),\), but if we take \(M = \mathbb{R}\) we run into the problem that \(M = \text{Fin}(M)\) while on the other hand we do not want to consider \(M\) as a definable subset of \(\text{Fin}(M)\) (since the inclusion does not hold passing to an elementary extension of \(M\)). So we assume, as in [BO10], that every locally definable set comes equipped (often implicitly) with a presentation as a countable union of definable sets \(\bigcup_{n \in \mathbb{N}} X_n\), and that a definable set \(Y\) counts as a subset of \(\bigcup_{n \in \mathbb{N}} X_n\) if it is included in \(\bigcup_{n \in J} X_n\) for some finite \(J \subset \mathbb{N}\). With this convention our results hold even if the base model \(M\) is not assumed to be \(\aleph_1\)-saturated, and in particular they hold over o-minimal expansions of the reals.

Every definable group has a unique group topology making it into a definable manifold over \(M\) [Pil88]. This result was generalized to locally definable groups in [BO10, Theorem 3.9] (see also [PS10, Proposition 2.2] and [Edm06, Theorem 2.3] for related results). So every locally definable group admits a group topology making it into a locally definable space with a countable atlas. We call the resulting group topology the \(t\)-topology. For the definition and a recent systematic treatment of locally definable spaces we refer to [BO10]. In [BO10] a well behaved subclass of the locally definable spaces is discussed: the paracompact ones. They admit the following characterization: a locally definable space \(X\) is paracompact if the closure of each definable subspace of \(X\) is definable [BO10, Fact 2.7].

Proposition 2.1. A locally definable group \(G\), with the \(t\)-topology, is always paracompact in the sense of [BO10].

Proof. Let \(X\) be a definable subset of \(G\). Let \(O\) be a definable open neighbourhood of the identity. Then the closure of \(X\) is contained in the definable set \(XO\) (the set of products \(xy\) with \(x \in X\) and \(y \in O\)), and therefore it is definable. \(\square\)
Thus we can make free use of the results of [BO10] on locally definable paracompact spaces when dealing with locally definable groups. Finally, let us recall the following definition: a subset $X$ of a locally definable group (or space) $G$ is compatible if its intersection with any definable subset of $G$ is definable. By [Edm05, Lemma 3.3 and Theorem 4.2] a normal subgroup $A \triangleleft B$ of a locally definable group $B$ is compatible if and only if it is the kernel of a locally definable surjective homomorphism $f: B \to C$ between locally definable groups. So the quotients $B/A$, with $A$ compatible in $B$, exist in the category of locally definable groups and they are unique up to locally definable isomorphisms. Let us also recall that a locally definable space is connected if it is not the union of two non-trivial clopen compatible subsets [BO10]. This is equivalent to the condition that every two points can be joined by a definable path.

3. Covers

We recall a few facts from the theory of covering spaces in the locally definable category, as developed in [Edm05, EE07] (see [EO04] for the definable case). Given two locally definable connected groups $U$ and $G$, a surjective locally definable homomorphism $f: U \to G$ is a covering homomorphism if and only if its kernel is zero-dimensional, or equivalently if $\dim(U) = \dim(G)$ [Edm05, Theorem 3.6]. Given such a covering $f: U \to G$, there is an induced injective homomorphism $f_*: \pi_1(U) \to \pi_1(G)$ on the o-minimal fundamental groups (see [Edm05, Proposition 4.6] or [BO10, Proposition 6.2]). By [Edm05] Prop. 3.4 and 3.12] we have:

$$\ker(f) \cong \pi_1(G)/f_*(\pi_1(U)).$$

In particular if $U$ is the universal cover of $G$ (as in [EE07]), then $\pi_1(U) = 0$ and $\ker(f) \cong \pi_1(G)$. So $\pi_1(G)$ is isomorphic to a zero-dimensional subgroup of the universal cover of $G$. Another way of saying the same thing is $\Gamma \cong \pi_1(U/\Lambda)$, provided $\pi_1(U) = 0$ and $\Lambda$ is a zero-dimensional compatible normal subgroup of $U$. More generally if $\Gamma$ is a zero-dimensional compatible normal subgroup of a connected locally definable group $G$, then $\Gamma$ is a quotient of $\pi_1(G/\Gamma)$ (consider the covering $G \to G/\Gamma$). So there is a strong connection between compatible zero-dimensional subgroups and fundamental groups.

4. Rank of the fundamental group

Homology and homotopy in the locally definable category has been studied by various authors. See [BO10] for some bibliography and recent results. Given a locally definable space $X$, we let $S_*(X)$ denote the o-minimal singular chain complex of $X$, and $H_*(X; R)$ the corresponding graded homology group with coefficients in $R$. Let us also recall that the rank of an abelian group $L$ is the cardinality of a maximal linearly independent (over $\mathbb{Z}$) subset of $L$; it coincides with the dimension of $L \otimes \mathbb{Q}$ as a $\mathbb{Q}$-vector space. So for $\mathbb{Q}$-vector spaces the rank coincides with the dimension.

**Theorem 4.1.** Let $G$ be a locally definable abelian connected group. Then its first homology group $H_1(G; \mathbb{Q})$ with rational coefficients has finite rank $\leq \dim(G)$.

For the proof we will make use of the theory of Hopf-algebras (see [Dol95]). The same tool has been used in [EO04] for the study of definable groups but working in cohomology rather than homology. We have chosen to work in homology since
the homological Kunneth formula $H_n(G \times G; \mathbb{Q}) \cong H_\ast(G; \mathbb{Q}) \otimes H_\ast(G; \mathbb{Q})$ does not require the assumption that $H_\ast(G; \mathbb{Q})$ is of finite type (unlike the cohomological version). Note that in the locally definable category the finite type assumption is not granted \textit{a priori}. Indeed it is easy to construct locally definable spaces which do not satisfy it. The Kunneth formula in homology is used in the proof of the following fact.

**Fact 4.2.** Let $R$ be a field and let $G$ be an locally definable abelian connected group. The group multiplication $\mu : G \times G \to G$ induces on $H_\ast(G; R)$ the structure of a skew commutative $R$-algebra, which is in fact a connected Hopf-algebra over $R$ in the sense of [Dol95]. If $R$ has characteristic zero, the Hopf-algebra is free.

**Proof.** Given two locally definable spaces $X$ and $Y$ there is a natural chain homotopy equivalence

$$S_\ast(X \times Y) \cong S_\ast(X) \otimes S_\ast(Y)$$

This has been verified in the definable case in [EO04 Proposition 3.2] and the proof in the locally definable case is identical. The Kunneth formula

$$H_\ast(X \times Y; R) \cong H_\ast(X; R) \otimes H_\ast(Y; R)$$

then follows (see [Dol95 Chapter VI Theorem 9.13]). Identifying $H_\ast(G \times G; \mathbb{Q})$ with $H_\ast(G; \mathbb{Q}) \otimes H_\ast(G; \mathbb{Q})$, the group multiplication $\mu : G \times G \to G$ induces a multiplication $\mu_\ast : H_\ast(G; \mathbb{Q}) \otimes H_\ast(G; \mathbb{Q}) \to H_\ast(G; \mathbb{Q})$ in homology making $H_\ast(G; \mathbb{Q})$ into a Pontryagin’s ring (see [Dol95 Chapter VII Section 3]). The diagonal map $\Delta : G \to G \times G$, defined by $\Delta(x) = (x, x)$, induces a natural co-multiplication (or diagonal) $D : H_\ast(G; \mathbb{Q}) \to H_\ast(G; \mathbb{Q}) \otimes H_\ast(G; \mathbb{Q})$ making the Pontryagin’s ring $H_\ast(G; \mathbb{Q})$ into a connected Hopf-algebra (see [Dol95 Chapter VII 10.10]). Over a field of characteristic zero, any connected Hopf-algebra is free by the Hopf-Leray theorem (see [Dol95 Chapter VII Proposition 10.16]).

**Proof of Theorem 4.1.** Let $d = \dim(G)$ and for a contradiction take $d + 1$ elements in $H_1(G; \mathbb{Q})$ independent over $\mathbb{Q}$. Since the Hopf $\mathbb{Q}$-algebra $H_\ast(G; \mathbb{Q})$ is free, their Pontryagin product is a non-zero element of $H_{d+1}(G, \mathbb{Q})$. And this is absurd since $H_n(G, \mathbb{Q})$ vanishes if $n > d$. In fact, the homology of a locally definable space is the direct limit of the homologies of its definable subsets by [BO10 Theorem 3.1], and a definable set of dimension $d$ has trivial $H_n$ for $n > d$ (see for instance [EO04 Lemma 3.1]).

**Corollary 4.3.** Let $G$ be a locally definable abelian connected group. Then $\pi_1(G)$ has finite rank $\leq \dim(G)$.

**Proof.** Since $G$ is a locally definable group, $\pi_1(G)$ is abelian. Hence, by the locally definable version of the Hurewicz theorem [BO10 Theorem 6.15], $\pi_1(G)$ is isomorphic to $H_1(G; \mathbb{Z})$. It follows that $\pi_1(G) \otimes \mathbb{Q}$ is isomorphic to $H_1(G; \mathbb{Q})$, so it has finite rank $\leq d$. Hence the same holds for $\pi_1(G)$.

The following corollary proves part (1) of Conjecture B in [EP12].

**Corollary 4.4.** Let $G$ be a locally definable abelian connected group. Let $\Gamma < G$ be a zero-dimensional compatible subgroup of $G$. Then $\text{rank}(\Gamma) \leq \dim(G)$.

**Proof.** Let $\pi : U \to G$ be the universal cover of $G$ and let $\Lambda = \pi^{-1}(\Gamma) < U$. Then $\Lambda$ is a compatible zero-dimensional subgroup of $U$. By Corollary 4.3 we know that $\pi_1(U/\Lambda)$ has finite rank $\leq \dim(U) = \dim(G)$. Since $U$ is simply connected,
5. Divisibility

Let $G$ be a locally definable abelian connected group. To understand the $n$-torsion subgroup $G[n]$ of $G$, a natural approach would to consider the homomorphism $n : G \to G$, $x \mapsto nx$. This has been proved to be effective in the definable case [EO04]. However to prove that this map is surjective, hence a covering, we would need the divisibility of $G$, which in the locally definable case is still conjectural. So we follow a different approach, using the covering map $G \to G/G[n]$ (this is the particular case of Theorem 5.4 with $\Gamma = G[n]$). This will allow us to prove the finiteness of $G[n]$ without assuming that $G$ is divisible, but under the weaker hypothesis that the universal cover of $G$ is torsion free. We begin by showing that this “weaker hypothesis” is indeed implied by the divisibility of $G$.

**Proposition 5.1.** Let $\pi : G \to H$ be a locally definable covering homomorphism between locally definable connected groups. If $H$ is divisible, then $G$ is divisible.

**Proof.** Without loss of generality we can assume $M$ sufficiently saturated. We must show that $nG = G$ for all $n \in \mathbb{N}$. Since $H$ is divisible we know that $\pi(nG) = nH = H$. It follows that $(nG)(\ker \pi) = G$. So $nG$ has bounded index in $G$ (see Definition 6.1). A locally definable subgroup of bounded index is always a compatible subgroup [EP12a, Fact 2.3]. Therefore $nG$ is a compatible subgroup of $G$. But since $nG$ is open and $G$ is connected, it must coincide with $G$. □

**Proposition 5.2.** The universal cover of a locally definable abelian connected divisible group is divisible and torsion free.

**Proof.** Let $G$ be our locally definable abelian connected divisible group, and let $U$ be its universal cover. Using Proposition 5.1 we have the divisibility of $U$, so for every $n \in \mathbb{N}^+$ multiplication by $n$ is a covering homomorphism $n : U \to U$. Its induced homomorphism on the fundamental group is again given by multiplication by $n$, so by the results on coverings in Section 3 we have $U[n] \cong \pi_1(U)/n\pi_1(U)$. Since $\pi_1(U) = 0$, we get $U[n] = 0$. So $U$ is torsion free. □

Given an abelian group $\Gamma$ and a prime $p$, the $p$-rank of $\Gamma$ is the dimension of its $p$-torsion subgroup $\Gamma[p]$ as a vector space over $\mathbb{Z}/p\mathbb{Z}$. Note that $\Gamma \otimes \mathbb{Z}/p\mathbb{Z}$ coincides with its $p$-torsion subgroup, so the $p$-rank of $\Gamma \otimes \mathbb{Z}/p\mathbb{Z}$ is its dimension over $\mathbb{Z}/p\mathbb{Z}$.

**Remark 5.3.**

1. If $\Gamma$ is torsion free, then the $p$-rank of $\Gamma \otimes \mathbb{Z}/p\mathbb{Z}$ is equal to the rank of $\Gamma$.
2. If $\Lambda$ is a quotient of $\Gamma$, the $p$-rank of $\Lambda \otimes \mathbb{Z}/p\mathbb{Z}$ is less or equal than the $p$-rank of $\Gamma \otimes \mathbb{Z}/p\mathbb{Z}$.

**Theorem 5.4.** Let $G$ be a locally definable abelian connected group whose universal cover $U$ is torsion free. Let $\Gamma$ be a zero-dimensional compatible subgroup of $G$. Then the $p$-rank of $\Gamma \otimes \mathbb{Z}/p\mathbb{Z}$ is less or equal than $\dim(G)$.

**Proof.** By the theory of covers in Section 3 the group $\Gamma$ is a quotient of $\pi_1(G/\Gamma)$, which has rank $\leq \dim(G)$ by Corollary 4.3. Again by the theory of covers, $\pi_1(G/\Gamma)$ is isomorphic to a compatible subgroup of the universal cover of $G/\Gamma$, which coincides with the universal cover $U$ of $G$. Therefore, by our assumption on $U$, $\pi_1(G/\Gamma)$
is torsion free. It then follows, by Remark 5.3(1), that the rank of \( \pi_1(G/\Gamma) \) is equal to the \( p \)-rank of \( \pi_1(G/\Gamma) \otimes \mathbb{Z}/p\mathbb{Z} \), so the latter is also \( \leq \dim(G) \). Finally since \( \Gamma \) is a quotient of \( \pi_1(G/\Gamma) \), by Remark 5.3(2) we conclude that the \( p \)-rank of \( \Gamma \otimes \mathbb{Z}/p\mathbb{Z} \) is \( \leq \dim(G) \).

**Fact 5.5.** [EP12a, Proposition 3.1] Let \( G \) be a locally definable abelian connected group. Then for each positive integer \( n \), \( G[n] \) is a compatible zero-dimensional subgroup of \( G \).

So we can take \( \Gamma = G[p] \) in Theorem 5.4 and noting that \( G[p] \otimes \mathbb{Z}/p\mathbb{Z} \cong G[p] \) we obtain that \( G[p] \) has \( p \)-rank \( \leq \dim(G) \). In other words we have point (1) in the following:

**Corollary 5.6.** Let \( G \) be a locally definable abelian connected group of dimension \( d \) whose universal cover is torsion free. Then:

1. For each prime \( p \), the \( p \)-rank of \( G[p] \) is \( \leq d \).
2. For each prime \( p \), the torsion subgroup \( G[p^k] \) is isomorphic to \( (\mathbb{Z}/p^k\mathbb{Z})^s \) for some \( s \leq d \) possibly depending on \( p \) but not on \( k \).
3. The \( n \)-torsion subgroup \( G[n] \) of \( G \) is finite, and it has at most \( n^d \) elements.

**Proof.** We obtain (1) by Theorem 5.4. The rest follows by general arguments in abstract group theory using only the fact that \( G \) is a divisible abelian group. The details are as follows.

(2): Consider \( G[p^k] \). Let \( x_1, \ldots, x_s \in G[p] \) be a basis of \( G[p] \) as a vector space over \( \mathbb{Z}/p\mathbb{Z} \). Since \( G \) is divisible, there are \( y_1, \ldots, y_s \in G \) such that \( p^{k-1}y_i = x_i \) for \( i = 1, \ldots, s \). It is then easy to verify, by induction on \( k \), that \( y_1, \ldots, y_s \) have order \( p^k \), and \( G[p^k] \) is a free \( \mathbb{Z}/p^k\mathbb{Z} \)-module generated by \( y_1, \ldots, y_s \).

(3): It suffices to note that if \( n = p_1^{\alpha_1} \cdots p_l^{\alpha_l} \) is the prime decomposition of \( n \), then

\[
G[n] \cong G[p_1^{\alpha_1}] \oplus \cdots \oplus G[p_l^{\alpha_l}]
\]

using only the fact that \( G \) is an abelian group.

By Proposition 5.2 and Corollary 5.6 we now obtain:

**Corollary 5.7.** Let \( G \) be a locally definable connected divisible abelian group of dimension \( d \). Then the conclusion of Theorem 5.6 holds. In particular \( G[n] \) is finite with at most \( n^d \) elements.

Note that, besides the bound on the cardinality, Corollary 5.7 does not give us much information on the structure of the \( n \)-torsion subgroup. For instance it does not answer the question whether \( G[6] \) can be isomorphic to \( (\mathbb{Z}/3\mathbb{Z})^2 \times \mathbb{Z}/2\mathbb{Z} \) (it is easy to find abstract abelian divisible groups with this 6-torsion subgroup). Later however we will show, under an additional convexity assumption, that \( G[n] \cong (\mathbb{Z}/n\mathbb{Z})^r \) for some \( s \leq \dim(G) \) not depending on \( n \).

6. Convexity

In this section we prove the Conjectures of Eleftheriou and Peterzil mentioned in the introduction under a convexity assumption. Let us first recall the following definition.

**Definition 6.1.** Let \( G \) be a locally definable group in a \( \kappa \)-saturated o-minimal structure \( M \) for some sufficiently big cardinal \( \kappa \).
(1) A subset $X \subset G$ is type-definable if it is the intersection $< \kappa$ definable sets.

(2) A type-definable subgroup $H < G$ has bounded index if there are no new cosets of $H$ in $G$ in elementary extensions of $M$. Equivalently $[G : H] < \kappa$.

(3) If $H \trianglelefteq G$ has bounded index, we say that $X \subset G/H$ is open in the logic topology if its preimage in $G$ is the union of $< \kappa$ definable sets.

(4) If there is a smallest type-definable subgroup of $G$ of bounded index we call it $G^{00}$ and say that $G^{00}$ exists.

For $G$ definable, $G^{00}$ exists and $G/G^{00}$ is a real Lie group [BOPP05]. For $G$ locally definable we have:

**Theorem 6.2.** [EP12a] Let $G$ be an abelian, connected, definably generated group of dimension $d$. Then:

1. The subgroup $G^{00}$ exists if and only if $G$ covers a definable group;
2. If $G^{00}$ exists, then $G^{00}$ is divisible and $G/G^{00}$ is a Lie group isomorphic to $\mathbb{R}^k \times (\mathbb{R}/\mathbb{Z})^r$ for some $k, r$ with $k + r \leq d$.

Although we will not need it explicitly, let us also recall that, under the same hypothesis, in [EP12a] it is also shown that $G^{00}$ exists if and only if $G$ has a generic definable subset, where a subset $X$ of $G$ is generic if for every definable $Y \subset G$ finitely many translates of $X$ cover $Y$.

The other ingredient that we need is the following notion of convexity.

**Definition 6.3.** Let $C$ be a subset of an abelian group $G$. We say that $C$ is convex if for every $a, b \in C$ and $m, n \in \mathbb{N}$, not both null, $C$ contains every solution $x \in G$ of the equation $(m + n)x = ma + nb$. Note that if $G$ is divisible, there will be at least one solution. Given a subset $X \subset G$, the convex hull $\text{ch}(X)$ of $X$ is the smallest convex set containing $X$. It can be equivalently defined as the set of all $x \in G$ which satisfy an equation of the form $(n_1 + \ldots + n_k)x = n_1a_1 + \ldots + n_ka_k$ for some $a_1, \ldots, a_k \in X$ and non-negative integers $n_1, \ldots, n_k$ not all zero (since the points $a_i$ are not assumed to be distinct, without loss of generality we can take $n_i = 1$ for all $i$). Note that the convex hull of a definable set need not be definable. We say that a locally definable abelian group has definably bounded convex hulls if for all definable $X \subset G$ there is a definable $Y \subset G$ containing $\text{ch}(X)$.

**Notation 6.4.** Let $X$ be a subset of an abelian group $G$. We write $nX$ for $\{nx \mid x \in X\}$ and $\Sigma_nX$ for $\{x_1 + \ldots + x_n \mid x_1, \ldots, x_n \in X\}$, where $+$ is the group operation.

**Remark 6.5.** If $G$ is divisible and torsion free, then $X$ is convex if and only if $nX = \Sigma_nX$ for every $n \in \mathbb{N}^+$. In this case $G$ is a $\mathbb{Q}$-vector space and convexity has the usual meaning (the $\mathbb{Q}$-segment between two points in $X$ is contained in $X$). If $G$ is only assumed to be divisible, then $X$ is convex if and only if we have both $nX = \Sigma_nX$ for every $n \in \mathbb{N}^+$ and $g + X = X$ for every torsion element $g \in G$.

Example: the convex hull of the zero element of $G$ is the torsion subgroup of $G$.

We are now ready to state the main result of this section.

**Theorem 6.6.** Given a definably generated abelian connected group $G$, the following are equivalent:

1. For every definable set $X \subset G$, there is a definable set $Y \subset G$ such that $\Sigma_nX \subset nY$ for all $n \in \mathbb{N}$.  

(2) The group $G$ is a cover of a definable group.
(3) The group $G$ is divisible and has definably bounded convex hulls.

Moreover these properties are stable under elementary extensions (i.e. preserved upwards and downwards).

**Remark 6.7.** Condition (3) is equivalent to the conjunction of (1) and the condition that the torsion subgroup of $G$ is contained in a definable set.

**Proof.** Let $X \subset G$ be definable and let $C$ be the convex hull of $X$. Then $nC = \Sigma_n C$ for every $n$. Assuming (3) there is a definable set $Y$ containing $C$. So $nY \supseteq nC = \Sigma_n C \supseteq \Sigma_n X$ and we have (1). Moreover the convex hull of the identity of the group is the torsion subgroup of $G$, so if (3) holds we have both (1) and the fact that the torsion subgroup is contained in a definable set. The other direction will not be needed and is left to the reader. □

In the proof of the Theorem we will make use of the following remark, which justifies the technical condition (1).

**Remark 6.8.** Condition (1) in Theorem 6.6 is inherited by the quotients of $G$, namely if $G$ satisfies (1) and $H < G$ is compatible, also $G/H$ satisfies (1).

**Remark 6.9** (The question of saturation). None of the three properties in Theorem 6.6 is *prima facie* stable under elementary extensions. Nevertheless we claim that, under the hypothesis of the theorem, all of them will turn out to be. In fact, assume that $G$ is $M$-locally definable and $M < M'$ with $M'$ saturated enough. Clearly, property (2) passes from $G(M)$ to $G(M')$. Both properties (1) and (3) pass form $G(M)$ to $G(M')$ observing that, since $G$ is locally definable set over $M$, each $M'$-definable subsets of $G(M')$ is contained in some $M$-definable subset. By Lemma 6.10 below, property (2) goes from $G(M')$ to $G(M)$. Also, property (3) implies property (1) by Remark 6.7. So, finally, to establish the theorem in full generality, suffices to prove that (1) implies (2) assuming saturation and to prove that (2) implies (3) without requiring saturation.

**Lemma 6.10.** Let $M'$ be an elementary extension of $M$. Let $G$ be an $M$-locally definable connected abelian group. If $G(M')$ covers an $M'$-definable group, then $G(M)$ covers some $M$-definable group.

**Proof.** In [EP12a] Eleftheriou and Peterzil proved that $G$ covers of a definable group if and only if $G$ has a compatible subgroup $\Gamma$ such that $G/\Gamma$ is definable and $\Gamma$ is isomorphic to $\mathbb{Z}^k$ for some $k$. One direction is clear: if $\Gamma$ exists, $G$ covers the definable group $G/\Gamma$. The proof of the other direction in [EP12a] however requires to work in a saturated structure $M$, since one makes use of $G/G^{00}$ in order to find the appropriate $\Gamma < G$. It turns out however that a modification of the proof in [EP12a] gives the desired result. So assume that $G(M')$ covers an $M'$-definable group. We can assume $M'$ sufficiently saturated. As in [EP12a], working in $M'$, we have that $G^{00}$ exists, and $G/G^{00}$ is an abelian Lie group isomorphic to $\mathbb{R}^k \times (\mathbb{R}/\mathbb{Z})^l$ for some $k,l \in \mathbb{N}$. So we can write $G/G^{00}$ as the direct sum $L + K$ of two subgroups with $L \cong \mathbb{R}^k$ and $L \cong (\mathbb{R}/\mathbb{Z})^l$. Note that $K$ is uniquely determined (it is the closure of the torsion subgroup), but $L$ is not. Now fix a subgroup $\Gamma$ of $G/G^{00}$ isomorphic to $\mathbb{Z}^k$, say $\mathbb{Z} z_1 + \ldots + \mathbb{Z} z_k < G/G^{00}$, with $z_1, \ldots, z_k \in L$ (so $\Gamma < L$). Choose $u_1, \ldots, u_k \in G(M')$ such that $\pi(u_i) = z_i$, where $\pi : G \to G/G^{00}$ is the projection, and let $\Gamma = \mathbb{Z} u_1 + \ldots + \mathbb{Z} u_k < G(M')$. Eleftheriou and Peterzil show
that $G(M')/\Gamma$ is $M'$-definable. If we could choose $u_1, \ldots, u_k$ to be $M$-rational points, the same argument would show that $G(M)/\Gamma$ is $M$-definable and we would be done. However it may happen that there are no $M$-rational points $u_i$ in $G$ mapping to the given points $z_i \in G/G^{00}$. To overcome the impasse, we pick a small open neighbourhood $V_i \subset G/G^{00}$ of each $z_i$. The inverse image $U_i \subset G$ of $V_i$ is a union of subsets of $G$ definable over the ground model $M$. Hence $U_i$ contains a non-empty $M$-definable set $D_i$, and we pick each $u_i$ among the $M$-rational points of the corresponding $D_i$. Let $z'_i = \pi(u_i) \in V_i$. It is easy to see that, if the neighbourhoods $V_i$ were chosen small enough, there is an automorphism of $G/G^{00}$ mapping each $z'_i$ to the corresponding $z_i$. In fact, considering the universal cover $R^{k+l}$ of $R^k \times (R/Z)^l$ this amount to show that if we have a $k$-tuple of points in $R^{k+l}$ whose $R$-linear span is a $k$-dimensional subspace transversal to the subspace $0 \times R^l$, then the same remain true after a small perturbation of the points. To complete the proof that $G(M)/\Gamma$ is definable we must find a definable set containing representatives for all the cosets of $\Gamma$ in $G(M)$. For this we can reason as in [EP12, Lemma 3.3], going to a saturated model $M'$ if needed, but then observing that every $M'$-definable set is contained in an $M$-definable set. \hfill $\square$

The next lemma contains the main idea of our argument.

**Lemma 6.11.** (Assume $M$ saturated.) Let $G$ be a locally definable abelian group generated by a definable set $X$ such that $0 \in X$ and $X = -X$. Assume that for some definable subset $Y$ of $G$ and for all $n \in \mathbb{N}^+$ we have $nY \nsubseteq \Sigma_nX$. Then there is an infinite cyclic compatible subgroup of $G$.

**Proof.** By our assumptions there is an element $a_n \in Y$ with $na_n \notin \Sigma_nX$. Let $p_n(x) \in S(Y)$ be the type of $a_n$ over all relevant parameters (needed to locally define $G$ and to define $X$ and $Y$). Since $S(Y)$ is compact, there is some $q(x) \in S(Y)$ which is an accumulation point of $\{p_n : n \in \mathbb{N}^+\}$. Let $b \models q$. We will show that $Zb$ is an infinite compatible subgroup of $G$. To this aim note that for each $n$ there is a formula $\phi_n(x)$ in $p_n(x)$ which says “$x \in Y$ and $nx \notin \Sigma_nX$”. (This can be said in a first-order way since $X$ and $Y$ are definable and $x \mapsto nx$ is locally definable.) It is not clear whether $\phi_n(x)$ should belong to $q(x)$. However we have the following:

**Claim:** For each $m \in \mathbb{N}^+$ there is a formula $\phi^2_m(x)$ in $q(x)$ expressing “$x \in Y$ and $2mx \notin \Sigma_mX$”.

 Granted the claim, we have $2mb \notin \Sigma_mX$ for all $m$, and this clearly implies that $Zb$ is an infinite compatible subgroup of $G$. (Indeed, given a definable set $D \subset G$, if we choose $m_0$ such that $D + D \subset \Sigma_{m_0}X$, we have $mb \notin D$ for all $m \geq m_0$.)

To prove the claim it suffices to show that $\phi_n(x)$ implies $\phi^2_m(x)$ for all big enough $n$. Choose $k \in \mathbb{N}^+$ so big that $Y \subset \Sigma_kX$ and take $n > 4km$. Suppose for a contradiction that $x$ satisfies $\phi_n(x)$ but not $\phi^2_m(x)$, namely $x$ is an element of $Y$ such that $nx \notin \Sigma_nX$ and $2mx \in \Sigma_mX$. Let $n = 2mq + r$ with $r < 2m$, and observe that $mq + kr < n$. Then

$$nx = 2mqx + rx \in \Sigma_{mq}X + \Sigma_{kr}X \subset \Sigma_nX$$

which is the desired contradiction. \hfill $\square$

**Corollary 6.12.** (Assume $M$ saturated.) Let $G$ be a locally definable abelian group generated by a definable set. If $G$ is divisible and has property $[\Sigma]$ of Theorem 6.6, then either $G$ is definable or $G$ has an infinite cyclic compatible subgroup.
Proof. Let $X$ be a definable set generating $G$. Without loss of generality $0 \in X$ and $X = -X$. Assume that $G$ is not definable, i.e. $n < m$ implies $\Sigma_m X \not\subseteq \Sigma_n X$. Let $Y'$ be $2Y$ where $Y$ is a definable set witnessing property (1) for $X$. We claim that for our sets $X$ and $Y'$ the hypothesis of Lemma 6.11 holds. In fact $nY' = 2nY \supset \Sigma_n X \not\subseteq \Sigma_n X$. □

We are now ready to complete the proof of the Theorem.

Proof of Theorem 6.6. We have already remarked that (3) implies (1).

Now assume (1) with the aim of proving (2). By Remark 6.9 we may assume $M$ saturated. We need to prove that $G$ covers a definable group. Let $\Gamma$ be a torsion free discrete compatible subgroup of $G$ of maximal rank, which exists by Corollary 4.4. Consider $G/\Gamma$. Observe that $G/\Gamma$ inherits property (1) from $G$ by Remark 6.7. We claim that $G/\Gamma$ is definable. If not, then by Corollary 6.12 it has an infinite cyclic compatible subgroup $\Lambda$ and, as in the proof of [EP12b, Theorem 2.5], the inverse image of $\Lambda$ in $G$ has rank greater than rank($\Gamma$). This contradiction establishes (2).

Now assume (2) with the aim of proving (3). We must prove this without assuming that the ground model $M$ is saturated. Consider a saturated $M' \succ M$. By [EP12a, Theorem 3.9] $G^{00}$ exists. By [EP12a, Proposition 3.5] $G^{00}$ is divisible and $G(M')/G^{00}(M')$, endowed with the logic topology, is an abelian connected real Lie group. Let $X$ be an $M$-definable subset of $G$. The image of $X(M')$ in $G(M')/G^{00}(M')$ is compact. Since $G/G^{00} \cong \mathbb{R}^k \times \mathbb{R}/\mathbb{Z}^l$ for some $k,l \in \mathbb{N}$, clearly there is a compact convex subset $C'$ of $G(M')/G^{00}(M')$ containing the said image. Since $G^{00}$ is divisible, the preimage in $G(M')$ of a convex subset of $G/G^{00}$ is easily seen to be convex. So in particular the preimage $C \subset G(M')$ of $C'$ is convex. Also, being the preimage of a compact set, $C$ must be contained in a definable set (see [EP12a]), and since $G$ is locally definable over $M$, $C$ is actually contained in a $M$-definable set. □

Corollary 6.13. Let $G$ be a locally definable abelian connected group. If $G$ covers a definable group, then also any quotient of $G$ by a compatible subgroup covers a definable group.

Proof. By the equivalence between condition (1) and (2) in Theorem 6.6 and the fact that (1) is preserved under quotients. □

As already observed in Remark 6.5 if the group is divisible and torsion free then we can regard it as a $\mathbb{Q}$-vector space and the notion of convex hull assumes the usual meaning. In this context we consider the following conjecture:

Conjecture 6.14. Every locally definable abelian connected group which is divisible and torsion free has definably bounded convex hull.

By [EP12b] the above conjecture holds for subgroups of $(M^n,+)$. 

Corollary 6.15. Under Conjecture 6.14 every definably generated connected abelian divisible group $G$ is a cover of a definable group.

Proof. By Proposition 5.2 the universal cover $U$ of $G$ is divisible and torsion free. Assuming the conjecture, $U$ covers a definable group (by Theorem 6.6). Now apply Corollary 6.13 □
7. Further Consequences of Convexity

In this section we will show that the convexity hypothesis considered in Section 6 can be used to study locally definable connected abelian groups even without assuming that the group is definably generated. In particular, we will prove that under suitable hypothesis the fundamental group \( \pi_1(G) \) is finitely generated. We recall that every definable set \( X \) has a finitely generated fundamental group [BO02], but this result does not extend to locally definable sets. So we will need to make essential use of the group structure of \( G \).

Remark 7.1. Let \( \Gamma \) be a subgroup of \( \mathbb{Q}^s \) for some \( s \in \mathbb{N} \). Let \( v_1, \ldots, v_s \) be \( \mathbb{Q} \)-independent elements of \( \Gamma \), and assume that \( \Gamma \cap \text{ch}(v_1, \ldots, v_s) \) is finite. Then \( \Gamma \) is isomorphic to \( \mathbb{Z}^s \).

Proof. The hypothesis implies that the group identity is an isolated point of \( \Gamma \) in the topology inherited by \( \mathbb{Q}^d \) as a topological subgroup of \( \mathbb{R}^d \). Now it suffices to recall that the only discrete subgroups of \( \mathbb{R}^d \) are of form \( \mathbb{Z}^s \). □

Theorem 7.2. Let \( G \) be a locally definable abelian connected group. Assume that the universal cover \( U \) of \( G \) is divisible and has definably bounded convex hulls. Then \( \pi_1(G) \cong \mathbb{Z}^s \) for some \( s \leq \dim(G) \).

Proof. By Proposition 5.2 \( U \) is divisible and torsion free, so it is a vector space over \( \mathbb{Q} \). By the theory of covers \( \pi_1(G) \) is isomorphic to a zero-dimensional compatible subgroup \( \Gamma \) of \( U \). So in particular \( \pi_1(G) \) is torsion free. Moreover it has rank \( \leq \dim(G) \) by Theorem 4.3. It follows that \( \Gamma \) is isomorphic to a subgroup of \( \mathbb{Q}^s \) where \( s \) is the rank of \( \pi_1(G) \). Let \( v_1, \ldots, v_s \in \Gamma \) be \( \mathbb{Q} \)-linearly independent elements. By the hypothesis \( \text{ch}(v_1, \ldots, v_s) \) is contained in a definable subset \( D \) of \( U \). Since \( \Gamma \) is compatible and zero-dimensional, \( \Gamma \cap D \) is finite. So a fortiori \( \Gamma \cap \text{ch}(v_1, \ldots, v_s) \) is finite. Hence, by Remark 7.1, \( \Gamma \) is isomorphic to \( \mathbb{Z}^s \). □

Corollary 7.3. Under the same assumptions \( G[n] \cong (\mathbb{Z}/n\mathbb{Z})^s \).

Proof. If \( G \) is locally definable connected abelian and divisible, then the multiplication by \( n \) is a covering homomorphism \( n: G \to G \), and from the theory of covers \( G[n] \cong \pi_1(G)/n\pi_1(G) \) [Edm05, Theorem 3.15]. □

Corollary 7.4. Let \( G \) be a locally definable, abelian, connected group. Suppose that the universal cover of \( G \) is divisible and has definably bounded convex hulls. Then every zero dimensional compatible subgroup \( \Gamma \) of \( G \) is finitely generated, with at most \( \dim(G) \) generators.

Proof. From the theory of covers it follows that \( \Gamma \) is a quotient of \( \pi_1(G/\Gamma) \), which is isomorphic to \( \mathbb{Z}^s \) for some \( s \leq \dim(G) \) by Theorem 7.2. □

8. Examples

It was conjectures in [EP12a] that every definably generated abelian connected group is a cover of a definable group. We show that this cannot be generalized to non-abelian groups.

Example 8.1. There is a non-abelian locally definable group \( G \) such that \( G \) is generated by a definable generic set, but \( G \) does not cover a definable group.
Proof. Let \( H \) be any definably connected centerless group of positive dimension definable in an \( \omega \)-saturated real closed field \( M \). We will show that no locally definable connected proper subgroup \( G \) of \( H \) of maximal dimension can cover a definable group; then reasoning as in [HPP08, Proposition 7.8] we will construct such a subgroup which is generated by a definable generic set. For the first part, let \( G \) be a connected locally definable proper subgroup of \( H \) with \( \dim(G) = \dim(H) \). Suppose that \( G \) covers a definable group \( L \). Take any non-trivial element \( x \) of \( G \) in the fibre over the identity of \( L \). Clearly \( x \) must belong to the center of \( G \), hence the centralizer \( C_H(x) \) of \( x \) in \( H \) has maximal dimension. By the connectedness of \( H \) we get \( C_H(x) = H \), hence \( x \in Z(H) \), which is impossible since \( H \) has trivial center. Now we construct a locally definable subgroup \( G \) as before. \( H \) has a \( C^1 \) group-manifold structure. Taking a local chart of \( H \) around the identity \( e \), we can assume \( e \in U \subset H \) for some open definable subset \( U \) of \( M^{\dim(G)} \) such that on \( U \) the group topology coincides with the subset topology. Without loss of generality \( e \) is \( 0 \in U \). Take a definable open neighbourhood \( V \) of \( e \) such that the group operation restricted to \( V \times V \) is a differentiable function taking values in \( U \). Hence, for all \( x \) and \( y \) in \( V \), we have \( x \cdot y = x + y + f(x, y) \) with \( f(x, y) \in o(|x| + |y|) \) for \( (x, y) \rightarrow (e, e) \). By saturation, take an \( \epsilon > 0 \) such that \( |f(x, y)| < |x| + |y| \) for all \( x \) and \( y \) with \( |x| + |y| < \epsilon \) (where \( a < b \) means \( na < b \) for all \( n \in \mathbb{N} \).) Basically, for elements smaller that \( \epsilon \), the group structure of \( H \) and the group structure of \( M^n \) are infinitesimally close, this enables us to construct our example in a straightforward way. Pick a positive \( \delta < \epsilon \) and let \( X = \{ x \mid |x| < \delta \} \). We claim that the group \( G \) generated by \( X \) is \( \bigcup_{i \in \mathbb{N}} X_i \), where \( X_i = \{ x \mid |x| < i\delta \} \). In fact, clearly \( X_i \cdot X_j \subset X_{i+j+1} \), and for any \( x \) in \( X \) and any \( n \) in \( \mathbb{N} \), we have \( |(nx) \cdot x^{-n}| < |x| \), hence \( (nx) \cdot x^{-n} \in X \) and \( X_n \subset X^{n+1} \). To prove that \( X \) is generic in \( G \) observe that there is a countable set \( Y = \{ y_i \}_{i \in \mathbb{N}} \) such that \( G = Y + \frac{1}{2} X \), it is easy to see that \( G = Y \cdot X \). \( \square \)

With the next example we show that the fundamental group of a locally definable \( H \)-space may not be finitely generated. (The reader may consult [Dol95] for the definition of \( H \)-space, and figure out by himself the obvious adaptation of the definition to the locally definable category.) This may explain the difficulties in proving, without convexity assumptions, that the fundamental group of a locally definable connected abelian group is finitely generated.

Example 8.2. There is a locally definable set \( C \) with \( \pi_1(C) \cong \mathbb{Q} \) and such that \( C \) can be endowed with a locally definable \( H \)-space structure.

Proof. Our \( H \)-space is going to be “definably compact” in the sense that all definable paths in it have a limit. Moreover we believe that the construction can be modified to yield a locally definable manifold as well, however the details are very tedious to verify. Notice that the fundamental group of a compact topological manifold cannot be \( \mathbb{Q} \). This is the plan of the example: working in a real closed field \( M \), first we describe a definably compact locally definable space with \( \mathbb{Q} \) as fundamental group, then we show how to endow it with an \( H \)-space structure and how to embed it in \( M^3 \), finally with suggest how to turn it into a locally definable manifold. Let \( \{ G_i \}_{i=1,2,...} \) be infinitely many copies of \( SO(2) \), it is useful to consider them as distinct groups. Consider the maps \( m_i : G_i \rightarrow G_{i+1} \) where \( m_i(x) = ix \). Let \( C_i \) be the mapping cylinder of \( m_i \). By a slight abuse of notation we can consider \( G_i \) and \( G_{i+1} \) as subspaces of \( C_i \). Then the definable fundamental group of \( C = \bigcup_i C_i \)
is \( \mathbb{Q} \). In fact, let \( S_n \) denote the initial segment \( \bigcup_{i \leq n} C_i \) of the union. Clearly \( S_n \) retracts onto \( G_{n+1} \), hence \( \pi_1(S_n) = \mathbb{Z} \). Also \( \pi_1(C) = \lim_{n \to \infty} \pi_1(S_n) \) where the maps are those induced by the inclusion. Now, by the retraction, the inclusion of \( S_n \) in \( S_{n+1} \) induces on the fundamental groups the same map as the inclusion of \( G_{n+1} \) in \( C_{n+1} \), which is the multiplication by \( n + 1 \). So \( \pi_1(C) \) is the group generated by the generators \( \alpha_i \) of \( \pi_1(G_i) \) with relations \( i \alpha_i = \alpha_{i+1} \), and this is \( \mathbb{Q} \). The H-space structure is given by the following map. Let \( x = (x, t) \in C_i \setminus G_{i+1} \cong G_i \times [0, 1] \) and \( y = (y, s) \in C_j \setminus G_{j+1} \cong G_j \times [0, 1] \) be two elements of \( C \). If \( i = j \) then we let \( \mu(x, y) = (x, t \cdot y, \max(t, s)) \in C_i \setminus G_{i+1} \). If \( i < j \) we let \( \mu(x, y) = (\rho_j(x) \cdot y, s) \in C_j \setminus G_{j+1} \) where \( \rho_j \) is the retraction from \( C_1 \cup \cdots \cup C_{j-1} \) to \( G_j \). The case \( j < i \) is symmetric. For the embedding in \( M^5 \), we first embed \( SO(2) \times [0, 1] \) in \( M^5 \) in the standard way. Subsequently we map \( G_i \) to \( SO(2) \times \{(0, 0, 0)\} \subset M^5 \). The mapping cylinder \( C_i \) maps to
\[ \{(1-t) + ta^n, t(1-t) a, i + t \mid a \in SO(2), t \in [0, 1]\} \subset M^2 \times M^2 \times M = M^5 \]
\( C \) is the union of the mapping cylinders. Finally, our H-space is not a definable manifold. We suggest that taking a suitably small tubular neighbourhood of it (in \( M^5 \)) should yield an homotopy equivalent locally definable manifold. \( \square \)

9. Questions

The following questions remain open:

**Question 9.1.** Assume \( G \) is a locally definable abelian connected group.

(1) Is \( G \) divisible? \[ \text{Edm03, Edm05, EPI2a} \]

(2) Suppose \( G \) is definably generated. Is \( G \) a cover of a definable group? \[ \text{EP12a} \]

(3) Is the torsion subgroup of \( G \) contained in a definable set?

(4) Is the universal cover of \( G \) torsion free? (Yes if \( G \) is divisible.)

(5) Suppose \( G \) is divisible and torsion free. Is \( G \) definably simply connected?

(6) Can we have \( G[6] \cong (\mathbb{Z}/3\mathbb{Z})^2 \times \mathbb{Z}/2\mathbb{Z} \)? (No if \( G \) is divisible.)

(7) Can we have \( (\mathbb{Z}/2\mathbb{Z})^{\dim(G)+1} \subset G \)? (No if \( G \) is divisible.)

(8) Can we have \( \pi_1(G) \cong \mathbb{Q} \)?

Questions (3), (5), (8) can be settled under a convexity assumption (on \( G \) or on the universal cover of \( G \)). So the problem is to answer them without additional hypothesis.

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