On generating series of classes of equivariant Hilbert schemes of fat points

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Abstract

In previous papers the authors gave formulae for generating series of classes (in the Grothendieck ring $K_0(V_C)$ of complex quasi-projective varieties) of Hilbert schemes of zero-dimensional subschemes on smooth varieties and on orbifolds in terms of certain local data and the, so called, power structure over the ring $K_0(V_C)$. Here we give an analogue of these formulae for equivariant (with respect to an action of a finite group on a smooth variety) Hilbert schemes of zero-dimensional subschemes and compute some local generating series for an action of the cyclic group on a smooth surface.

1 Introduction

For a complex $d$-dimensional quasi-projective variety $X^d$, let $\text{Hilb}_X^k$ be the Hilbert scheme of zero-dimensional subschemes (sets of “fat points”) of length $k$ of $X$. For a locally closed subvariety $Y \subset X$, let us denote by $\text{Hilb}_{X,Y}^k$ the Hilbert scheme of zero-dimensional subschemes of length $k$ of the variety $X$ supported at points of the variety $Y$, and for a point $x \in X$, $\text{Hilb}_{X,x}^k := \text{Hilb}_{X,\{x\}}^k$.

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Let $K_0(V_C)$ be the Grothendieck ring of complex quasi-projective varieties. This is the abelian group generated by the classes $[X]$ of all complex quasi-projective varieties $X$ modulo the relations:

1) if varieties $X$ and $Y$ are isomorphic, then $[X] = [Y];$

2) if $Y$ is a Zariski closed subvariety of $X$, then $[X] = [Y] + [X \setminus Y].$

The multiplication in $K_0(V_C)$ is defined by the Cartesian product of varieties: $[X_1] \cdot [X_2] = [X_1 \times X_2].$ The class $[\text{Spec } \mathbb{C}] \in K_0(V_C)$ of the complex affine line is denoted by $\mathbb{L}.$ For a quasi-projective variety $X,$ the class $[X],$ being an additive invariant of the variety, can be considered as a generalized Euler characteristic $\chi_g(X)$ of the variety $X.$ The additivity of $\chi_g(\bullet)$ (property 2) above) permits to use it as a measure for the notion of the integral with respect to the generalized Euler characteristic. If $\psi : X \to A$ is a constructible function on a variety $X$ with values in an abelian group $A,$ then the integral of $\psi$ with respect to the generalized Euler characteristic is defined as

$$\int_X \psi d\chi_g := \sum_{a \in A} \chi_g(\psi^{-1}\{a\}) \cdot a \in K_0(V_C) \otimes \mathbb{Z} A. \quad (1)$$

Let

$$H_X(T) := 1 + \sum_{k=1}^{\infty} [\text{Hilb}^k_X] T^k \in 1 + T \cdot K_0(V_C)[[T]] \quad \text{and}$$

$$H_{X,Y}(T) := 1 + \sum_{k=1}^{\infty} [\text{Hilb}^k_{X,Y}] T^k \in 1 + T \cdot K_0(V_C)[[T]]$$

be the generating series of classes of Hilbert schemes.

In [6], there was defined a notion of a power structure over a ring and there was described a natural power structure over the Grothendieck ring $K_0(V_C)$ of complex quasi-projective varieties. This means that for a series $A(T) = 1 + a_1 T + a_2 T^2 + \ldots \in 1 + T \cdot K_0(V_C)[[T]]$ and for an element $m \in K_0(V_C)$ one defines a series $(A(T))^m \in 1 + T \cdot K_0(V_C)[[T]]$ so that all the usual properties of the exponential function hold. For the natural power structure over the ring $K_0(V_C)$ and for $a_i = [A_i], m = [M],$ where $A_i$ and $M$ are quasi-projective varieties, the series $(A(T))^m$ has the following geometric description. The coefficient at $T^k$ in the series

$$(1 + [A_1]T + [A_2]T^2 + \ldots)^M$$
is represented by the configuration space of pairs \((K, \varphi)\) consisting of a finite subset \(K\) of the variety \(M\) and a map \(\varphi\) from \(K\) to the disjoint union \(\bigsqcup_{i=1}^{\infty} A_i\) of the varieties \(A_i\), such that \(\sum_{x \in K} I(\varphi(x)) = k\), where \(I : \bigsqcup_{i=1}^{\infty} A_i \to \mathbb{Z}\) is the tautological function sending the component \(A_i\) of the disjoint union to \(i\). This power structure is connected with the \(\lambda\)-structure on the ring \(K_0(\mathcal{V}_C)\) defined by the Kapranov zeta function \([10]\):

\[
\zeta_M(T) = 1 + [S^1 M] \cdot T + [S^2 M] \cdot t^2 + [S^3 M] \cdot T^3 + \ldots
\]

where \(S^k M\) is the \(k\)-th-symmetric power of the variety \(M\): \(\zeta_M(T) = (1 - T)^{-[M]}\).

There are two natural homomorphisms from the Grothendieck ring \(K_0(\mathcal{V}_C)\) to the ring \(\mathbb{Z}\) of integers and to the ring \(\mathbb{Z}[u, v]\) of polynomials in two variables: the Euler characteristic (alternating sum of ranks of cohomology groups with compact support) \(\chi : K_0(V) \to \mathbb{Z}\) and the Hodge-Deligne polynomial \(e : K_0(V) \to \mathbb{Z}[u, v]\). These homomorphisms respect the power structures over the corresponding rings (see e.g. \([7]\)).

In \([7]\), it was shown that, for a smooth quasi-projective variety \(X\) of dimension \(d\), the following equation holds:

\[
\mathbb{H}_X(T) = \left( \mathbb{H}_{d,0}(T) \right)^{[X]} \tag{2}
\]

where \(A^d_C\) is the complex affine space of dimension \(d\). For \(d = 2\), i.e. for surfaces, in other terms this equation was proved in the Grothendieck ring of motives by L. Göttsche \([5]\). In this case one has

\[
\mathbb{H}_{d,0}(T) = \prod_{i=1}^{\infty} \frac{1}{1 - T^{i-1} L_i}. \tag{3}
\]

For an arbitrary dimension \(d\), the reduction of the equation (2) for the Hodge-Deligne polynomial was proved by J. Cheah in \([1]\).

A generalization of the equation (2) for orbifolds was given in \([8]\). In this case the function \(\mathbb{H}_{X,x}(T)\) is a constructible function on \(X\) with values in the abelian group \(1 + T \cdot K_0(\mathcal{V}_C)[[T]]\) (with respect to multiplication) and one has

\[
\mathbb{H}_X(T) = \int_X \mathbb{H}_{X,x}(T)^{d_{\chi g}}. \tag{4}
\]
(Here $d\chi_g$ is put into the exponent since the group operation in $1 + T \cdot K_0(\mathcal{V}_C)([T])$ is the multiplication. In this case the sum in RHS of the equation (1) is substituted by the product.) The equation (1) reduces the computation of the generating series $H_X(T)$ to the computation of the local data $H_{X,x}(T)$.

Here we shall give an analogue of equation (1) for equivariant (with respect to an action of a finite group $G$ on a smooth variety) Hilbert schemes of zero dimensional subschemes and compute some local generating series for an action of a cyclic group on a smooth surface.

## 2 Equivariant Hilbert scheme of fat points

Let $G$ be a finite group (of order $|G|$) acting on a smooth complex $d$-dimensional quasi-projective complex variety $X^d$. The group $G$ also acts on the Hilbert schemes $\text{Hilb}_X^k$ of zero-dimensional subschemes on $X$. One can say that there are (at least) three natural notions of equivariant Hilbert schemes of zero-dimensional subschemes on $X$ (see e.g. [3], [11], [12]).

First, one can define the equivariant Hilbert scheme $^{(1)}\text{Hilb}_X^{G,k}$ as the $G$-invariant part of the action of the group $G$ on $\text{Hilb}_X^k$.

Second, as the equivariant Hilbert scheme $^{(2)}\text{Hilb}_X^{G,k}$ one can take the (unique) component (or union of components if the variety $X$ is reducible) of $^{(1)}\text{Hilb}_X^{G,k}$ which maps birationally on the $k/|G|$-th symmetric power of $X$.

A $G$-invariant zero-dimensional subscheme of length $k$ (i.e. a closed point of $^{(1)}\text{Hilb}_X^{G,k}$) has a decomposition into parts supported at different $G$-orbits. For each of these parts one has the natural representation of the group $G$ on the fibre of the tautological bundle on the Hilbert scheme. The third version $^{(3)}\text{Hilb}_X^{G,k}$ of the equivariant Hilbert scheme consists of those points of $^{(1)}\text{Hilb}_X^{G,k}$ for which all these representations corresponding to parts supported at different $G$-orbits are multiples of the regular one.

In the last two cases $k$ must be a multiple of the order $|G|$ of the group $G$. One always has

$^{(1)}\text{Hilb}_X^{G,k} \supset ^{(3)}\text{Hilb}_X^{G,k} \supset ^{(2)}\text{Hilb}_X^{G,k}$

and all these equivariant Hilbert schemes are quasi-projective varieties. They are smooth if $\text{Hilb}_X^k$ is smooth. In particular this holds if $X$ is a smooth surface ($d = 2$). In many cases, in particular for surfaces, the last two notions coincide. This is not true in general (see e.g. [3]). The equivariant
Hilbert scheme \( \text{Hilb}_{C^2}^{G,k} \) is always larger than the other two. In particular \( \text{Hilb}_{C^2}^{G,1} \neq \emptyset \). It seems that the last two notions are more interesting from geometrical point of view. In particular, for a finite group \( G \subset SL(2, \mathbb{C}) \) acting on \( C^2 \) in the natural way, \( \text{Hilb}_{C^2}^{G,\lvert G \rvert} \) (or \( \text{Hilb}_{C^2}^{G,\lvert G \rvert,0} \)) is a crepant resolution of the factor space \( C^2/G \). However, the first one could be interesting as well. In particular, it seems that formulae for the generating series of classes of \( \text{Hilb}_{C^2}^{G,k} \) (or of \( \text{Hilb}_{C^2}^{G,k} \)) are somewhat simpler in this case.

Let

\[
\text{Hilb}_{X}^{G,k}(T) := 1 + \sum_{k=1}^{\infty} [\text{Hilb}_{X}^{G,k}] T^k
\]

and, for a locally closed \( G \)-invariant subvariety \( Y \subset X \),

\[
\text{Hilb}_{X,Y}^{G,k}(T) := 1 + \sum_{k=1}^{\infty} [\text{Hilb}_{X,Y}^{G,k}] T^k
\]

be the generating series of classes of the equivariant Hilbert schemes.

3 Generating series of classes of equivariant Hilbert scheme of fat points through local data

Statements of this section have the same form for all three notions of the equivariant Hilbert scheme of zero-dimensional subschemes discussed above. Since we do not have to specify the notion we shall use the notation \( \text{Hilb}_{X}^{G,k}, \text{Hilb}_{X,x}^{G,k}, \ldots \) without indicating the number corresponding to the version.

Let \( G \) be a finite group (of order \( |G| \)) acting on a smooth \( d \)-dimensional quasi-projective complex variety \( X^d \), let \( Y := X^d/G \) be the corresponding factor space and let \( p : X \to Y \) be the projection map. For a point \( x \in X \), let \( G_x = \{ g \in G : g \cdot x = x \} \subset G \) be the isotropy subgroup of the point \( x \). Let \( y \) be a point of \( Y = X^d/G \), i.e. a \( G \)-orbit in \( X \). For all points \( x \) in this \( G \)-orbit the isotropy subgroups \( G_x \) are conjugate to each other. One has the natural representation \( \alpha_x \) of the isotropy group \( G_x \) on the tangent space \( T_x X \cong \mathbb{C}^d \) at the point \( x \).

For a conjugate class \( h \) of subgroups of \( G \), for a representative \( H \subset G \) of this class, and for a representation \( \alpha \) of the group \( H \) on \( \mathbb{C}^d \), let \( Y_{h,\alpha} \) be
the set of points $y \in Y$ such that the isotropy subgroup of each point in the corresponding orbit belongs to $h$ and (for those of them whose isotropy group coincides with $H$) the representation of the isotropy subgroup coincides with $\alpha$. For a representation $\alpha$ of a subgroup $H \subset G$ on $\mathbb{C}^d$, one considers the equivariant Hilbert scheme $(\bullet)\text{Hilb}^{H,k}_{\mathbb{C}^d,0}$ and the corresponding generating series

$$(\bullet)\text{Hilb}^H_{\mathbb{C}^d,\alpha}(T) := 1 + \sum_{k=1}^{\infty} [(\bullet)\text{Hilb}^{H,k}_{\mathbb{C}^d,0}] T^k.$$ 

**Theorem 1** The following equations holds:

$$(\bullet)\text{Hilb}^G_X(T) = \prod_{h,\alpha} \left((\bullet)\text{Hilb}^H_{\mathbb{C}^d,\alpha}(T^{[G]/[H]}) \right)^{[Y_{h,\alpha}]} = \int_{X/G} (\bullet)\text{Hilb}^G_{X,x}(T^{[G]/[G_x]}) d\chi.$$  

**Proof.** The proof essentially follows the lines of the proof of Theorem 1 from [8]. If $Z$ is a $G$-invariant Zariski closed subset of the variety $X$, one has:

$$(\bullet)\text{Hilb}^G_X(T) = (\bullet)\text{Hilb}^G_{X,Z}(T) \cdot (\bullet)\text{Hilb}^G_{X,X\setminus Z}(T).$$  

This implies that one has to prove that $(\bullet)\text{Hilb}^G_{X,p^{-1}(Y_{h,\alpha})}(T) = \left((\bullet)\text{Hilb}^H_{\mathbb{C}^d,\alpha}(T^{[G]/[H]}) \right)^{[Y_{h,\alpha}]}$. Moreover, by the same reason, it is sufficient to prove that $(\bullet)\text{Hilb}^G_{X,p^{-1}(Y_i)}(T) = \left((\bullet)\text{Hilb}^H_{\mathbb{C}^d,\alpha}(T^{[G]/[H]}) \right)^{[Y_i]}$ for elements $Y_i$ of a Zariski open covering of $Y_{h,\alpha}$. Without loss of generality one may assume that $X$ (and therefore $p^{-1}(Y_{h,\alpha})$) lies in an affine space $\mathbb{A}^N_\mathbb{C}$ (an affine chart of the ambient projective space).

Let us fix linear equations corresponding to the representation $\alpha$ of the group $H$:

$$g^*z_i = \sum_{j=1}^d \alpha_{i,j}(g)z_j,$$  

where $(\alpha_{i,j}(g)) = \alpha(g)$. For a point $x \in p^{-1}(Y_{h,\alpha})$ with the isotropy subgroup $G_x$ coinciding with $H$ (not only conjugate to it), let $u_1, u_2, \ldots, u_d$ be a regular system of parameters on the manifold $X$ at the point $x$. For example, one may suppose that $x$ is the origin in $\mathbb{A}^N_\mathbb{C}$ and $u_1, u_2, \ldots, u_d$ are standard coordinates in $\mathbb{A}^N_\mathbb{C}$ such that the projection of the tangent space $T_xX$ to the corresponding $d$-dimensional coordinate subspace is non-degenerate. In this case $u_1 - u_1(x')$, $u_2 - u_2(x')$, $u_d - u_d(x')$ is a regular system of parameters at each point $x'$ from a Zariski open neighbourhood of the point $x$ in $X$. 

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Moreover, let us suppose that the parameters $u_1, u_2, \ldots, u_d$ are chosen in such a way that the representation of the subgroup $H$ in the tangent space $T_x X$ is given by the standard equations (7). Define functions $\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_d$ on $X$ by the equations

$$\tilde{u}_i = \frac{1}{|H|} \sum_{g \in H} \sum_{j=1}^{d} \alpha_{i,j}(g^{-1}) g^* u_j.$$  

One has

$$g^* \tilde{u}_i = \sum_{j=1}^{d} \alpha_{i,j}(g) \tilde{u}_j.$$  

At the point $x$ one has $d \tilde{u}_i = du_i$. Therefore $\tilde{u}_1 - \tilde{u}_1(x'), \tilde{u}_2 - \tilde{u}_2(x'), \ldots, \tilde{u}_d - \tilde{u}_d(x')$ is a regular system of parameters at each point from a Zariski open neighbourhood $Z_x$ of the point $x$ in the corresponding irreducible component of $p^{-1}(Y_{h,\alpha})$. The choice of the regular set of parameters identifies $H$-invariant zero-dimensional subschemes on $X$ supported at a point $x'$ from $Z_x$ and thus $G$-invariant zero-dimensional subschemes on $X$ supported at $GZ_x$ with $H$-invariant zero-dimensional subschemes on the space $\mathbb{A}_C^d$ (with the action defined by the equations (7)) supported at the origin.

This way, a $G$-invariant zero-dimensional subscheme on $X$ supported at points of the subvariety $GZ_x$ is defined by a finite subset $K \subset Y_x = p(Z_x) \equiv Z_x$ to each point of which there corresponds an $H$-invariant zero-dimensional subscheme on $\mathbb{A}_C^d$ supported at the origin. The length of the subscheme is equal to $|G|/|H|$ times the sum of lengths of the corresponding subschemes of $\mathbb{A}_C^d$. As it follows from the geometric description of the power structure over the Grothendieck ring of quasi-projective varieties, the coefficient at $T^n$ in the right hand side of the equation (7) is represented just by the configuration space of such objects. This proves the statement. □

4 Generating series of classes of equivariant Hilbert schemes for two-dimensional representations of a cyclic group.

Let the cyclic group $\mathbb{Z}_M$ act on the plane $\mathbb{C}^2$ by $\sigma \ast (x, y) = (\sigma x, \sigma^N y)$ where $\sigma = \exp \left( \frac{2\pi i}{M} \right)$ is the generator of $\mathbb{Z}_M$. For $N = -1$ (or rather $N \equiv -1 \mod M$) the factor space $\mathbb{C}^2/\mathbb{Z}_M$ has the $A_{M-1}$ singularity.
We shall denote \( \mathbb{H}_{Z_M}^{M} (T) \), \( \mathbb{H}_{Z_M}^{M} (T, 0) \), . . . for this action by \( \mathbb{H}_{Z_M}^{M, N} (T) \), \( \mathbb{H}_{Z_M}^{M, N} (T, 0) \), . . .

**Theorem 2** The following equation holds:

\[
\begin{align*}
(1) \mathbb{H}_{C^2,0}^{M, -1} (T) &= \prod_{i=1}^{\infty} \left( \frac{(1 - T^{M_i})^M}{1 - T^i} \cdot \frac{1}{(1 - \ell^i T^{M_i})^{M-1} \cdot (1 - \ell^{i-1} T^{M_i})} \right), \\
(2) \mathbb{H}_{C^2,0}^{M, -1} (T) &= \prod_{i=1}^{\infty} \frac{1}{(1 - \ell^i T^{M_i})^{M-1} \cdot (1 - \ell^{i-1} T^{M_i})}. 
\end{align*}
\]

**Proof.** It appears to be somewhat simpler to describe the computation of \( \mathbb{H}_{C^2,0}^{M, -1} (T) \) where \( C \) is an invariant line of the \( Z_M \)-action on the plane \( C^2 \) and then to apply Theorem 1 to get \( \mathbb{H}_{C^2,0}^{M, -1} (T) \). To compute \( \mathbb{H}_{C^2,0}^{M, N} (T) \), \( \mathbb{H}_{C^2,0}^{M, N} (T) \), or \( \mathbb{H}_{C^2,0}^{M, N} (T) \), one uses the method of G. Ellingsrud and S.A. Strømme [4] based on a result of A. Bialynicki-Birula. For that, one considers the natural action of the complex 2-torus \( C^* \times C^* \) on the projective plane \( \mathbb{CP}^2 \) and the corresponding action on the Hilbert schemes of zero-dimensional subschemes on it. This action has a finite number of fixed points. For a subgroup \( G \subset C^* \times C^* \) (say, for a cyclic one) the torus acts on the corresponding equivariant Hilbert schemes as well. This action (with a fixed generic subgroup of \( C^* \times C^* \) isomorphic to \( C^* \)) defines cell decompositions of the Hilbert schemes of zero-dimensional subschemes and of the equivariant one(s). Cells (locally closed subvarieties isomorphic to complex affine spaces) correspond to fixed points of the action on the Hilbert schemes. The dimension of a cell is equal to the dimension of the subspace of the tangent space corresponding to representations of \( C^* \) with positive characters. Fixed points of the natural action of the torus \( C^* \times C^* \) on \( \text{Hilb}_{C^2}^k \) and also on \( \text{Hilb}_{C^2,0}^{M, N; k} \) are the monomial ideals in \( C[[x, y]] \) of length (codimension) \( k \). Monomial ideals of length \( k \) correspond to partitions of \( k \), i.e. to Young diagrams of size \( k \). Fixed points of the \( (C^* \times C^*) \)-action on \( \text{Hilb}_{C^2}^{M, N; k} \) are those monomial ideals of length \( k \), for which the corresponding Young diagram has the same numbers of boxes with the quasi-homogeneous weights 0, 1, . . . , \( M - 1 \). (The box corresponding to the monomial \( x^k y^\ell \) has the quasi-homogeneous weight equal to \( k + N \ell \mod M \). In the last case the Hilbert scheme (and the set of fixed points) is empty if \( k \) is not divisible by \( M \).)

From [4], it follows that the dimension of the cell in \( \text{Hilb}_{C^2,0}^{M, -1; k} \) corresponding to the fixed point described by a Young diagram of the partition
\{b_0 \geq b_1 \geq \ldots \geq b_{r-1} > 0\} of the integer \(k\) is equal to the number of monomials in the (equivariant) expression

\[
T^+ = \sum_{1 \leq i \leq j \leq r} b_{j-1} \sum_{s=b_j}^{b_{j-1}-1} \lambda^{i-j-1} \mu^{b_{j-1}-s-1},
\]

for the “positive part” of the tangent space to the Hilbert scheme \(\text{Hilb}_{C^2, C}^{M, N;k}\) with the weights \(N(i-j-1)+(b_{i-1}-s-1) \equiv 0 \mod M\). For \(N = -1\) these weights are just the lengths of hooks of the corresponding Young diagram. For a fixed \(M\), Young diagrams of size \(k\) are in one-to-one correspondence with the sets consisting of the so-called \(M\)-core (of size \(k'\)) of the diagram (empty for diagrams with equal numbers of boxes with different weights) and its “star \(M\)-diagram”, i.e. a collection of “usual” diagrams of sizes, say, \(k_0, k_1, \ldots, k_{M-1}\), such that \(k' + M(k_0 + k_1 + \ldots + k_{M-1}) = k\): see, e.g., [9]. The dimension of the corresponding cell (i.e. the number of hooks with length divisible by \(M\)) is equal to \(k_0 + k_1 + \ldots + k_{M-1}\). For \(\text{Hilb}_{C^2, C}^{M,-1;k}\) (\(k' = 0\)) this means that the cells correspond to partition of sizes \(k_0, k_1, \ldots, k_{M-1}\) for representations of \(k/M\) as an ordered sum \(k_0 + k_1 + \ldots + k_{M-1}\). The dimension of the corresponding cell is equal to \(k_0 + k_1 + \ldots + k_{M-1}\). Therefore

\[
(2) \mathbb{H}_{C^2, C}^{M,-1}(T) = \prod_{i=1}^{\infty} \frac{1}{(1-T^i)^M}. 
\]

The generating series for the numbers of \(M\)-cores of different sizes is

\[
\prod_{i=1}^{\infty} \frac{(1-T^i)^M}{1-T^i}
\]

(see, e.g., [2]). Therefore

\[
(1) \mathbb{H}_{C^2, C}^{M,-1}(T) = \prod_{i=1}^{\infty} \frac{(1-T^i)^M}{1-T^i} \cdot \prod_{i=1}^{\infty} \frac{1}{(1-T^i)^M}. 
\]

From Theorem \([1]\) equation (2) and the equation \((1-\mathbb{L}^i T)^{-L} = (1-\mathbb{L}^{i+1} T)^{-1}\) it follows that

\[
(\ast) \mathbb{H}_{C^2,0}^{M,-1}(T) = (\ast) \mathbb{H}_{C^2, C}^{M,-1}(T) \cdot \left( \prod_{i=1}^{\infty} \frac{1}{1-\mathbb{L}^i T^M i} \right)^{1-L} = (\ast) \mathbb{H}_{C^2, C}^{M,-1}(T) \cdot \prod_{i=1}^{\infty} \frac{1-\mathbb{L}^i T^M i}{1-\mathbb{L}^{i-1} T^M i}. 
\]
Therefore

\[
\begin{align*}
(1) \mathbb{H}^{M,-1}_{C^2,0}(T) &= \prod_{i=1}^{\infty} \frac{(1 - t^{M_i})^M}{(1 - T^i)(1 - L_i T^{M_i})^{M-1}(1 - L_i^{-1} T^{M_i})} \cdot \prod_{i=1}^{\infty} \frac{1}{(1 - L_i T^{M_i})^{M-1}(1 - L_i^{-1} T^{M_i})}, \\
(2) \mathbb{H}^{M,-1}_{C^2,0}(T) &= \prod_{i=1}^{\infty} \frac{1}{(1 - L_i T^{M_i})^{M-1}(1 - L_i^{-1} T^{M_i})}.
\end{align*}
\]

\[\Box\]

**Corollary.** Let the cyclic group \(Z_M\) act on a smooth surface \(S\) in such a way, that the factor space \(S/Z_M\) has only \(A_{M-1}\) singularities (i.e. at each of \(d\) fixed points \(P_1, \ldots, P_d\) one has the representation corresponding to \(N = -1\)). Then

\[
\begin{align*}
(1) \mathbb{H}^{M,-1}_{S}(T) &= \left( \prod_{i=1}^{\infty} \frac{(1 - t^{M_i})^M}{(1 - T^i)(1 - L_i T^{M_i})^{M-1}(1 - L_i^{-1} T^{M_i})} \right)^d \left( \prod_{i=1}^{\infty} \frac{1}{(1 - L_i^{-1} T^{M_i})} \right)^{|\{S\} \setminus \{P_i\}|/Z_M}, \\
(2) \mathbb{H}^{M,-1}_{S}(T) &= \left( \prod_{i=1}^{\infty} \frac{1}{(1 - L_i T^{M_i})^{M-1}(1 - L_i^{-1} T^{M_i})} \right)^d \left( \prod_{i=1}^{\infty} \frac{1}{(1 - L_i^{-1} T^{M_i})} \right)^{|\{S\} \setminus \{P_i\}|/Z_M}.
\end{align*}
\]

**Example.** Let the group \(Z_3\) act on the projective plane \(\mathbb{C}P^2\) by \(\sigma \ast(x_0 : x_1 : x_2) = (x_0 : \sigma x_1 : \sigma^2 x_2)\). Then

\[
(2) \mathbb{H}^{Z_3}_{\mathbb{C}P^2}(T) = 1 + (1 + 7L + L^2)T^3 + (1 + 8L + 36L^2 + 8L^3 + L^4)T^6 + (1 + 8L + 44L^2 + 149L^3 + 44L^4 + 8L^5 + L^6)T^9 + (1 + 8L + 45L^2 + 192L^3 + 543L^4 + 192L^5 + 45L^6 + 8L^7 + L^8)T^{12} + \ldots
\]

**Remarks.**

1. One can easily see that if \(N_1 N_2 \equiv 1 \text{ mod } M\), one has \((\ast) \mathbb{H}^{M,N}_{C^2,0}(T) = (\ast) \mathbb{H}^{M,N}_{C^2,0}(T)\).

2. It seems that, for \(N \neq -1\), one has somewhat better (less complicated) formulae for the series \((1) \mathbb{H}^{M,N}_{C^2,0}(T)\) than for the series \((2) \mathbb{H}^{M,N}_{C^2,0}(T)\) (at least in the form similar to \(\text{(3)}\) and \(\text{(8)}\)). To show that, it is convenient to write down the logarithms \(\text{Log}(\ast) \mathbb{H}^{M,N}_{C^2,0}(T)\) of the generating series \((\ast) \mathbb{H}^{M,N}_{C^2,0}(T)\) in the sense of \(\text{(8)}\): if \(A(T) = \prod_{i,j} (1 - L_i T^i)^{-k_{ij}}\), with \(k_{ij} \in \mathbb{Z}\), then by definition
Log $A(T) = \sum_{i,j} k_{ij}L^j T^i$. In particular, the equation (5) means that

$$\log (2) H_{C_2,0}^{M-1}(T) = \sum_{i=1}^{\infty} ((M - 1)L^i + L^{i-1}) T^{Mi}.$$ 

Computations made by the use of Maple gave:

$$\log (1) H_{C_2,0}^{3,1}(T) = T + LT^2 + T^3 + LT^4 + L^2 T^5 + LT^6 + L^2 T^7 + L^3 T^8 + L^2 T^9 + L^3 T^{10} + L^4 T^{11} + L^2 T^{12} + L^4 T^{13} + L^5 T^{14} + L^4 T^{15} + L^5 T^{16} + L^6 T^{17} + L^5 T^{18} + L^6 T^{19} + L^7 T^{20} + L^6 T^{21} + \ldots$$

$$\log (2) H_{C_2,0}^{3,1}(T) = (1 + L)L T^3 + (2L + 2L^2 + L^3)T^6 + (2L^2 + 2L^3 + L^4)T^9$$
$$+ (L^2 + L^3 - L^6)T^{12} + (-L^3 - L^5 - L^6 - L^7)T^{15} + (2L^5 + L^7)T^{18}$$
$$+ (2L^4 + 3L^5 + 7L^6 + 6L^7 + 6L^8 + 3L^9 + 2L^{10})T^{21} + \ldots$$

One can make the following

Conjecture:

$$(1) H_{C_2,0}^{3,1}(T) = \prod_{i=1}^{\infty} \frac{1}{(1 - L^{i-1}T^{3(i-2)})(1 - L^iT^{3i-1})(1 - L^{i-1}T^{3i})}.$$ 

A conjectural equation for $\log (2) H_{C_2,0}^{3,1}(T)$ is not clear.

Some other examples:

$$\log (1) H_{C_2,0}^{4,1}(T) = T + LT^2 + T^3 + LT^4 + T^5 + (-1 + L + L^2)T^6 + T^7 + (-1 + L + L^2)T^8$$
$$+ T^9 + (-1 + L^2 + L^3)T^{10} + T^{11} + (-1 + L^2 + L^3)T^{12}$$
$$+ T^{13} + (-1 + L^3 + L^4)T^{14} + T^{15} + (-1 + L^3 + L^4)T^{16}$$
$$+ T^{17} + (-1 + L^4 + L^5)T^{18} + T^{19} + (-1 + L^4 + L^5)T^{20} + \ldots$$

$$\log (2) H_{C_2,0}^{4,1}(T) = (1 + L)L T^4 + (2L + 2L^2 + L^3)T^8 + (L + 4L^2 + 5L^3 + 3L^4 + L^5)T^{12}$$
$$+ (4L^3 + 5L^4 + 3L^5)T^{16} + (-L^2 - 3L^3 - 2L^4 - L^5 - 3L^6 - 3L^7$$
$$- L^8)T^{20} + \ldots$$

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\[ \text{Log}^{(1)} \mathbb{H}_{\mathbb{C}^2,0}^{5,2}(T) = T + T^2 + LT^3 + LT^4 + T^5 + LT^6 + LT^7 + L^2T^8 + L^2T^9 + LT^{10} + L^2T^{11} + L^2T^{12} + L^3T^{13} + L^3T^{14} + L^2T^{15} + L^3T^{16} + L^3T^{17} + L^4T^{18} + L^4T^{19} + L^3T^{20} + L^4T^{21} + L^4T^{22} + L^5T^{23} + L^5T^{24} + L^4T^{25} + \ldots \]

\[ \text{Log}^{(2)} \mathbb{H}_{\mathbb{C}^2,0}^{5,2}(T) = (1 + 2L)T^5 + (3L^2 + 5L^3 + 2L^4)T^{10} + (3L^2 + 5L^3 + 2L^4)T^{15} + (-3L^2 - 2L^3 - 4L^4 - 3L^5 - 3L^6)T^{20} + (-3L^3 - 6L^4 - 9L^5 - 7L^6 - 3L^7)T^{25} + \ldots \]

3. Though a conjectural formula for \( \mathbb{H}_{\mathbb{C}^2,0}^{M,1}(T) \) (or for \( \text{Log}^{(2)} \mathbb{H}_{\mathbb{C}^2,0}^{M,1}(T) \)) is not clear even for small \( M > 2 \), computations show that one could have the following stabilization. Let \( \text{Log}^{(2)} \mathbb{H}_{\mathbb{C}^2,0}^{M,1}(T) = \sum_{i=1}^{\infty} p_{i}^{M,1}(L) \cdot T^{M_i} \), where \( p_{i}^{M,1}(L) \) are polynomials in \( L \). The computations predict that \( p_{i}^{M,1}(L) = p_{i}^{M',1}(L) \) for \( M'' > M' > i \).

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