EXTINCTION AND COEXISTENCE OF SPECIES FOR A DIFFUSIVE INTRAGUILD PREDATION MODEL WITH B-D FUNCTIONAL RESPONSE

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Abstract. Extinction and coexistence of species are two fundamental issues in systems with IGP. In this paper, we constructed a mathematical model with IGP by introducing heterogeneous environment and B-D functional response between the predator and prey. First, some sufficient conditions for the extinction and permanence of the time-dependent system were obtained by using comparison principle and upper and lower solution method. Second, we got some necessary and sufficient conditions for the existence of coexistence states by means of the fixed point index theory. In addition, we discussed the uniqueness and stability of coexistence state under some conditions. Finally, we studied the effects of the parameters in system on the spatial distribution of species and obtained some interesting results about the extinction and coexistence of species by using numerical simulations.

1. Introduction. Intraguild predation, or IGP, is a widespread ecological phenomenon in natural communities, which occurs when two consumers share a common resource and also engage in a predator-prey interaction [35, 36, 4]. One of the pioneering work to rigorously model intraguild predation was by Holt and Polis in [18], where a basic ODE models were developed for a three species community. It was showed in [18] that IGP significantly influences the distribution, abundance and coexistence of many species.

Following the model in [18], Tanabe and Namba in [41] and Abrams and Fung in [31] mainly obtained some results about the existence of chaos by numerical simulations; Hsu, Ruan and Yang in [19] completely classify the parameter space of model in article [18] to obtain some results about extinction and uniform persistence. On the other hand, a growing number of biological and mathematical models including IGP have been proposed by incorporating some more realistic ecological factors, such as delay [39, 10], age or stage structure [45, 38, 21], adaptive foraging [32, 26], defense [20, 42, 44, 23], refuge [29] and additional species [27, 17]. The

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main goal in the studies above is to ascertain mechanisms for extinction and coexistence of species in systems with IGP. We remark that different functional responses between predator and prey have been assumed in models including IGP, such as linear functional response, Holling type II, Holling-Type III and ratio-dependent functional response respectively \cite{1, 43, 22, 15}. We point out that, to our knowledge, few model is constructed with assuming the B-D functional response between the predator and prey to investigate the IGP.

We remark that all the models aforementioned assumed a homogeneous environment with no migration dynamics. As pointed in \cite{18}, spatial heterogeneity may have a important effect on the coexistence and extinction of the species in the IGP communities. Amarasekare in \cite{2, 3} first model IGP in a heterogeneous environment by using an environment consisting of 3 distinct patches and investigated the effects of different dispersal strategies on the coexistence based on numerical simulations. On the other hand, \cite{37} model IGP in a heterogeneous environment with space to be a continuous variable resulting in a parabolic system of PDEs with Neumann boundary condition. They assumed that the IGprey employs a fitness based avoidance strategy and proved the existence of a global attractor for this system and derived conditions for the uniform persistence of the IGprey.

Motivated by the articles above, we construct a mathematical model with heterogeneous environment by introducing the diffusion and B-D functional response between the predator and prey and obtain the following IGP system:

\[
\begin{align*}
\frac{\partial u}{\partial t} - \Delta u &= u(r_1 - u) - \frac{a_1 uv}{1 + b_1 u + e_1 v} - \frac{a_2 uv}{1 + b_2 u + e_2 w} \\
\frac{\partial v}{\partial t} - \Delta v &= \frac{m_1 uv}{1 + b_1 u + e_1 v} - \frac{a_3 v w}{1 + b_3 v + e_3 w} - r_2 v \\
\frac{\partial w}{\partial t} - \Delta w &= w(r_3 - w) + \frac{m_2 u w}{1 + b_2 u + e_2 w} + \frac{m_3 v w}{1 + b_3 v + e_3 w} \\
k_1 \frac{\partial u}{\partial \nu} + u &= k_2 \frac{\partial v}{\partial \nu} + v = k_3 \frac{\partial w}{\partial \nu} + w = 0 \\
(u(0, x), v(0, x), w(0, x)) &= (u_0(x), v_0(x), w_0(x)) \geq (0, 0, 0)
\end{align*}
\]

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ ($N \geq 1$ is an integer) with a smooth boundary $\partial \Omega$ and $\nu$ is the outward unit rector on $\partial \Omega$. Here we assumed a Robin boundary conditions for the system which means that that the flux of individuals across the boundary point is proportional to the density. One can find more details about the biological meaning of the Robin boundary condition in page 31 in \cite{9}. The three functions $u$, $v$ and $w$ represent the densities of the resource, intermediate predator (IG prey) and top predator (IG predator) respectively. The terms $\frac{a_1 u}{1 + b_1 u + e_1 v}$ and $\frac{a_3 v}{1 + b_3 v + e_3 w}$ represent the B-D functional response, which turn to Holling-II functional response if $e_i = 0$ and linear functional response if both $b_i = 0$ and $e_i = 0$. Here $b_i/a_i$ represent the handing time of the predators and $e_i$ describes that mutual interference among individual of predators \cite{5, 14}. The positive constants $a_i (i = 1, 2, 3)$ are the consumption rate and $m_i (i = 1, 2, 3)$ represent the conversion rates of the prey to predator and $r_1$ is the growth rate of resource. We assume that $r_3$ may be positive or negative. If $r_3 > 0$, it represents the growth rate of the IGpredator, which implies that the IGpredator is a generalist with a exclusive resources; if $r_3 < 0$, the death rate of the IGpredator is $r_3 - w$, which the IGpredator has no exclusive resources. Similarly, we also assume that parameter
r_2 may be positive or negative and it represents the death rate of the IG prey if r_2 > 0 and the immigration rate if r_2 < 0. We point out that the model (1) turn to food chain system if a_2 = m_2 = 0 which was studied in [47]; while if a_3 = m_3 = 0, model (1) turn to a system with two predators competing for one prey, which was investigated in [46].

In our work here, one of the main purposes is to study the existence of positive stationary solutions of (1) by using fixed point index theory, which are the positive solutions of (2) are also called coexistence states. Assume that the function f(x) denotes the principle eigenvalue of the following eigenvalue problem

\[ -\Delta u = u(r_1 - u) - \frac{a_1 uv}{1 + b_1 u + e_1 v} - \frac{a_2 uw}{1 + b_2 u + e_2 w} \]

\[ -\Delta v = \frac{m_1 uw}{1 + b_1 u + e_1 v} - \frac{a_3 vw}{a_3 vw} - \frac{r_2 v}{1 + b_3 v + e_3 w} \]

\[ -\Delta w = w(r_3 - w) + \frac{v}{1 + b_2 u + e_2 w} + \frac{m_3 vw}{1 + b_2 u + e_2 w} \quad \text{in } \Omega, \]

\[ k_1 \frac{\partial u}{\partial v} + u = k_2 \frac{\partial v}{\partial v} + v = k_3 \frac{\partial w}{\partial v} + w = 0 \quad \text{on } \partial \Omega. \]

The positive solutions of (2) are also called coexistence states.

The rest of this paper is organized as follows. In Section 2, some sufficient conditions for the extinction and permanence of species in system (1) are obtained by using upper and lower solution method. In Section 3, the existence and nonexistence of coexistence states of model (2) are studied by using some degree theorems developed. In Section 4, we further discuss the uniqueness and stability of coexistence state under some conditions. In Section 5, some interesting results about the extinction and coexistence of species are obtained by using numerical simulations. Finally in Section 6, we give some discussions about the conclusions obtained above.

2. Extinction and permanence. In this section, we would like to investigate the asymptotic behavior of the time-dependent solutions of system (1). We first give some definitions and theorems which will be used in the following.

2.1. Some preliminaries.

**Theorem 2.1.** (see [47, 24].) For each h(x) \( \in C^k(\Omega)(0 < \alpha < 1) \) and \( k \geq 0 \), let \( \lambda_{1,k}(h(x)) \) denote the principle eigenvalue of the following eigenvalue problem

\[ -\Delta u + h(x)u = \lambda u \quad \text{in } \Omega, \]

\[ k \frac{\partial u}{\partial v} + u = 0 \quad \text{on } \partial \Omega, \]

and denote \( \lambda_{1,k}(0) \) by \( \lambda_{1,k} \) for simplicity. Then \( \lambda_{1,k}(h(x)) \) is strictly increasing in the sense that \( h_1(x) \leq h_2(x) \) and \( h_1(x) \neq h_2(x) \) implies that \( \lambda_{1,k}(h_1(x)) < \lambda_{1,k}(h_2(x)) \).

Consider the following partial differential equation with Robin boundary boundary

\[ -\Delta u = uf(x, u) \quad \text{in } \Omega, \]

\[ k \frac{\partial u}{\partial v} + u = 0 \quad \text{on } \partial \Omega. \]

Assume that the function \( f(x, u) : \overline{\Omega} \times [0, \infty) \to R \) satisfies the following hypotheses:

(H1) \( f(x, u) \) is \( C^\alpha \)-function in \( x \), where \( 0 < \alpha < 1 \);

(H2) \( f(x, u) \) is \( C^1 \)-function in \( u \) with \( f_u(x, u) < 0 \) for all \( (x, u) \in \overline{\Omega} \times [0, \infty) \);

(H3) \( f(x, u) \leq 0 \) on \( (x, u) \in \overline{\Omega} \times [C, \infty) \) for some positive constant \( C \).
Theorem 2.2. (see [40, 33].)

(i) The nonnegative solution \( u(x) \) of (3) satisfies \( u(x) \leq C \) for all \( x \in \overline{\Omega} \).
(ii) If \( \lambda_{1,k}(-f(x,0)) \geq 0 \), then (3) has no positive solutions. Moreover, the trivial solution \( u(x) = 0 \) is globally asymptotically stable.
(iii) If \( \lambda_{1,k}(-f(x,0)) < 0 \), then (3) has a unique positive solution which is globally asymptotically stable. In this case, the trivial solution \( u(x) = 0 \) is unstable.

In particular, consider the following partial differential equation

\[
\begin{cases}
-\Delta \varphi = \varphi(\rho(x) - \varphi) & \text{in } \Omega, \\
k \frac{\partial \varphi}{\partial \nu} + \varphi = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( \rho(x) \in C^\alpha(\Omega)(0 < \alpha < 1) \) is a positive function. It follows from Theorem 2.2 that there exists a unique positive solution of (4) if \( \lambda_{1,k}(\rho(x)) < 0 \). Let \( \Theta_k(\rho(x)) \) with \( \Theta_k(\rho(x)) \leq \max_{x \in \overline{\Omega}}(\rho(x)) \) be the unique positive solution of (4) in the following.

2.2. Extinction. In this subsection, we study the extinction of species in system (1). We can obtain the following sufficient conditions for the extinction of species in system (1).

Theorem 2.3. Let \( (u, v, w) \) be a positive solution of (1).

(i) If \( r_1 < \lambda_{1,k_1} \) \( r_2 > 0 \) and \( r_3 < \lambda_{1,k_3} \), then \( (u, v, w) \rightarrow (0, 0, 0) \) as \( t \rightarrow \infty \).
(ii) If \( r_1 > \lambda_{1,k_1}, -r_2 < \lambda_{1,k_2} \left( \frac{m_1 \Theta_k(r_1)}{1+m_1 \Theta_k(r_1)} \right) \) and \( r_3 < \lambda_{1,k_3} \left( \frac{m_2 \Theta_k(r_2)}{1+m_2 \Theta_k(r_2)} \right) \), then \( (u, v, w) \rightarrow (\Theta_k(r_1), 0, 0) \) as \( t \rightarrow \infty \).
(iii) If \( r_1 < \lambda_{1,k_1}, -r_2 < \lambda_{1,k_2} \) and \( r_3 > \lambda_{1,k_3} \), then \( (u, v, w) \rightarrow (0, 0, \Theta_k(r_3)) \) as \( t \rightarrow \infty \).

Proof. (i) First, it is easy to see that any time-dependent solution \( (u, v, w) \) of (1) satisfies

\[
\begin{cases}
\frac{\partial u}{\partial t} - \Delta u \leq u(r_1 - u) & \text{in } \Omega \times (0, \infty), \\
k_1 \frac{\partial u}{\partial \nu} + u = 0 & \text{on } \partial \Omega \times (0, \infty).
\end{cases}
\]

Since \( r_1 < \lambda_{1,k_1} \), it follows from (5), Theorem 2.2 and comparison principle that \( u \rightarrow 0 \) as \( t \rightarrow \infty \) uniformly. Then for sufficiently small positive constant \( \epsilon \), there exists a \( T_1(\epsilon) \) such that \( u(x, t) \leq \epsilon \) for all \( t > T_1(\epsilon) \). Take \( \epsilon < \min\left\{ \frac{r_2}{m_2}, \frac{\lambda_{1,k_3}-r_3}{m_2+m_3} \right\} \). So we have

\[
\begin{cases}
\frac{\partial v}{\partial t} - \Delta v \leq v(m_1 \epsilon - r_2) < 0 & \text{in } \Omega \times (T_1(\epsilon), \infty), \\
k_2 \frac{\partial v}{\partial \nu} + v = 0 & \text{on } \partial \Omega \times (T_1(\epsilon), \infty),
\end{cases}
\]

Since \( \epsilon < \frac{r_2}{m_1} \), we know that \( v \rightarrow 0 \) as \( t \rightarrow \infty \) uniformly by (6), Theorem 2.2 and comparison principle. Then, there exists a \( T_2(\epsilon) > T_1(\epsilon) \) such that \( v(x, t) \leq \epsilon \) for all \( t > T_2(\epsilon) \).

Then \( u(x, t) \leq \epsilon \) and \( v(x, t) \leq \epsilon \) for all \( t > T_2(\epsilon) \). So we have

\[
\begin{cases}
\frac{\partial w}{\partial t} - \Delta w \leq w(r_3 + (m_2 + m_3) \epsilon - w) & \text{in } \Omega \times (T_2(\epsilon), \infty), \\
k_3 \frac{\partial w}{\partial \nu} + w = 0 & \text{on } \partial \Omega \times (T_2(\epsilon), \infty).
\end{cases}
\]
Since \( r_3 < \lambda_{1,k_3} \), we know that \( w \to 0 \) as \( t \to \infty \) uniformly by (7), Theorem 2.2 and comparison principle again.

(ii) Since \( r_1 > \lambda_{1,k_1} \), it follows from (5), Theorem 2.2 and comparison principle that

\[
\limsup_{t \to \infty} u(x,t) \leq \Theta_{k_1}(r_1). \tag{8}
\]

Then there exists a \( T(\epsilon) \geq 0 \) such that \( u(x,t) \leq \Theta_{k_1}(r_1) + \epsilon \) for all \( t > T(\epsilon) \).

Let \( \epsilon \) be a sufficiently small positive constant which satisfies the following conditions:

\[
\begin{align*}
(a) & \quad \epsilon < \frac{\lambda_{1,k_2} \left( -\frac{m_1 \Theta_{k_1}(r_1)}{1+b_1 \Theta_{k_1}(r_1)} \right) + r_2}{m_1}, \\
(b) & \quad \epsilon < \frac{\lambda_{1,k_3} \left( -\frac{m_2 \Theta_{k_1}(r_1)}{1+b_2 \Theta_{k_1}(r_1)} \right) - r_3}{m_1 + m_2}, \\
(c) & \quad \epsilon < \frac{r_1 - \lambda_{1,k_1}}{a_1 + a_2}.
\end{align*}
\]

Therefore, we have

\[
\begin{cases}
\frac{\partial v}{\partial t} - \Delta v \leq v \left( \frac{m_1 (\Theta_{k_1}(r_1) + \epsilon)}{1 + b_1 \Theta_{k_1}(r_1) + b_1 \epsilon + e_1 v} - r_2 \right) & \text{in } (T(\epsilon), \infty) \times \Omega, \\
\frac{\partial v}{\partial r} + v = 0 & \text{on } (T(\epsilon), \infty) \times \partial \Omega.
\end{cases} \tag{9}
\]

Since \( \epsilon \) satisfies (a) and \( -r_2 < \lambda_{1,k_2} \left( -\frac{m_1 \Theta_{k_1}(r_1)}{1+b_1 \Theta_{k_1}(r_1)} \right) \), we conclude that \( v \to 0 \) uniformly as \( t \to \infty \) by (9), Theorem 2.2 and comparison principle. Then there exists a \( T'(\epsilon) > T(\epsilon) \) such that \( v(x,t) \leq \epsilon \) for all \( t > T'(\epsilon) \). So we can show that

\[
\begin{cases}
\frac{\partial w}{\partial t} - \Delta w \leq w \left( r_3 + \frac{m_2 (\Theta_{k_1}(r_1) + \epsilon)}{1 + b_2 \Theta_{k_1}(r_1) + b_2 \epsilon + e_2 w} + m_3 \epsilon - w \right) & \text{in } (T'(\epsilon), \infty) \times \Omega, \\
\frac{\partial w}{\partial r} + w = 0 & \text{on } (T'(\epsilon), \infty) \times \partial \Omega. \tag{10}
\end{cases}
\]

Since \( \epsilon \) satisfies (b) and \( r_3 < \lambda_{1,k_3} \left( -\frac{m_2 \Theta_{k_1}(r_1)}{1+b_2 \Theta_{k_1}(r_1)} \right) \), we conclude that \( w \to 0 \) uniformly as \( t \to \infty \) by (10), Theorem 2.2 and comparison principle.

Then, there exists a \( T''(\epsilon) > T'(\epsilon) \) such that \( v(x,t), w(x,t) < \epsilon \) for all \( t > T''(\epsilon) \). So we have

\[
\begin{cases}
\frac{\partial u}{\partial t} - \Delta u \geq u \left( r_1 - u - a_1 \epsilon - a_2 \epsilon \right) & \text{in } \Omega \times (T''(\epsilon), \infty), \\
k_1 \frac{\partial u}{\partial \nu} + u = 0 & \text{on } \partial \Omega \times (T''(\epsilon), \infty). \tag{11}
\end{cases}
\]

Since \( \epsilon \) satisfies (c), by using (11), Theorem 2.2 and comparison principle, we have

\[
\liminf_{t \to \infty} u(x,t) \geq \Theta_{k_1}(r_1 - a_1 \epsilon - a_2 \epsilon). \tag{12}
\]

So, by (8) and (12), we have
\[ \Theta_k_1 (r_1 - a_1 \epsilon - a_2) \leq \liminf_{t \to \infty} u(x, t) \leq \limsup_{t \to \infty} u(x, t) \leq \Theta_k_1 (r_1). \quad (13) \]

By the continuity for \( \epsilon \to 0 \), we conclude \( u(x, t) \to \Theta_k_1 (r_1) \) uniformly as \( t \to \infty \) from (13).

(iii) It follows from the third equation that

\[
\begin{cases}
\frac{\partial w}{\partial t} - \Delta w \geq w(r_3 - w) & \text{in } (0, \infty) \times \Omega, \\
k_3 \frac{\partial w}{\partial \nu} + w = 0 & \text{on } \partial \Omega \times (0, \infty). 
\end{cases}
\] (14)

Then by (14), Theorem 2.2 and comparison principle, we have

\[ \liminf_{t \to \infty} w(x, t) \geq \Theta_k_3 (r_3). \] (15)

Since \( r_1 < \lambda_{1,k_1}, -r_2 < \lambda_{1,k_2} \), we conclude that \( u \to 0 \) and \( v \to 0 \) as \( t \to \infty \) uniformly by Theorem 2.2 and comparison principle. Then for sufficiently small positive constant \( \epsilon \), there exists a \( T(\epsilon) \) such that \( u(x, t) \leq \epsilon \) and \( v(x, t) \leq \epsilon \) for all \( t > T(\epsilon) \). So we have

\[
\begin{cases}
\frac{\partial w}{\partial t} - \Delta w \leq w(r_3 + (m_2 + m_3)\epsilon - w) & \text{in } (T(\epsilon), \infty) \times \Omega, \\
k_3 \frac{\partial w}{\partial \nu} + w = 0 & \text{on } (T(\epsilon), \infty) \times \partial \Omega. 
\end{cases}
\] (16)

Take \( \epsilon < \frac{r_3 - \lambda_{1,k_1}}{m_2 + m_3} \). Since \( r_3 > \lambda_{1,k_1} \), we conclude from (16), Theorem 2.2 and comparison principle that

\[ \limsup_{t \to \infty} w(x, t) \leq \Theta_k_3 (r_3 + (m_1 + m_2)\epsilon). \] (17)

So, by (15) and (17), we have

\[ \Theta_k_3 (r_3) \leq \liminf_{t \to \infty} w(x, t) \leq \limsup_{t \to \infty} w(x, t) \leq \Theta_k_3 (r_3 + (m_1 + m_2)\epsilon). \] (18)

By the continuity for \( \epsilon \to 0 \), we conclude \( w(x, t) \to \Theta_k_3 (r_3) \) uniformly as \( t \to \infty \) from (18). This completes the proof.

2.3. Permanence and global attractor. In this subsection, we would like to investigate the permanence and global attractor of the time dependent system (1) by using upper and lower solution method. We first give the following definition about ordered upper and lower solution of (2).

**Definition 2.4.** A pair of functions \((\bar{u}, \bar{v}, \bar{w})\) and \((u, v, w)\) in \( C(\Omega) \cap C^2(\Omega) \) are called ordered upper and lower solution of \((\bar{u}, \bar{v}, \bar{w})\) if they satisfy the relation \( \bar{u} \geq u, \bar{v} \geq v, \bar{w} \geq w \) and the following inequalities:
\[- \Delta \pi \geq \bar{u}(r_1 - \bar{u}) - \frac{a_1 \bar{u} \pi}{1 + b_1 \bar{u} + c_1 \bar{u}} - \frac{a_2 \bar{u} \pi}{1 + b_2 \bar{u} + c_2 \bar{u}} \]
\[- \Delta u \leq \bar{u}(r_1 - \bar{u}) - \frac{a_1 u \pi}{1 + b_1 u + c_1 u} - \frac{a_2 u \pi}{1 + b_2 u + c_2 u} \]
\[- \Delta \pi \geq \frac{m_1 \pi \pi}{1 + b_1 \bar{u} + c_1 \bar{u}} - \frac{a_3 \pi \pi}{1 + b_3 \bar{u} + c_3 \bar{u}} - r_2 \pi \]
\[- \Delta u \leq \frac{m_1 u \pi}{1 + b_1 u + c_1 u} - \frac{a_3 u \pi}{1 + b_3 u + c_3 u} - r_2 u \]
\[- \Delta \pi \geq \bar{u}(r_3 - \bar{u}) + \frac{m_2 \pi \pi}{1 + b_2 \bar{u} + c_2 \bar{u}} + \frac{m_3 \pi \pi}{1 + b_3 \bar{u} + c_3 \bar{u}} \]
\[- \Delta u \leq \bar{u}(r_3 - \bar{u}) + \frac{m_2 u \pi}{1 + b_2 u + c_2 u} + \frac{m_3 u \pi}{1 + b_3 u + c_3 u} \quad \text{in } \Omega, \]
\[k_1 \frac{\partial \pi}{\partial \nu} + \bar{u} \geq 0 \quad k_1 \frac{\partial u}{\partial \nu} + u \]
\[k_2 \frac{\partial \pi}{\partial \nu} + \bar{u} \geq 0 \quad k_2 \frac{\partial u}{\partial \nu} + u \]
\[k_3 \frac{\partial \pi}{\partial \nu} + \bar{u} \geq 0 \quad k_3 \frac{\partial u}{\partial \nu} + u \quad \text{on } \partial \Omega. \]

Let \( v^\oplus \) be the unique positive solution of the following problem
\[
\left\{ \begin{array}{l}
- \Delta v = v \left( \frac{m_1 \Theta k_1 (r_1 - \frac{a_1}{c_1} - \frac{a_2}{c_2})}{1 + b_1 \Theta k_1 (r_1 - \frac{a_1}{c_1} - \frac{a_2}{c_2}) + c_1 u} - \frac{a_3}{c_3} - r_2 \right) \quad \text{in } \Omega, \\
k_2 \frac{\partial v}{\partial \nu} + v = 0 \quad \text{on } \partial \Omega,
\end{array} \right.
\]
and \( v^\ominus \) be the unique positive solution of the following problem
\[
\left\{ \begin{array}{l}
- \Delta v = v \left( \frac{m_1 \Theta k_1 (r_1)}{1 + b_1 \Theta k_1 (r_1) + c_1 u} - r_2 \right) \quad \text{in } \Omega, \\
k_2 \frac{\partial v}{\partial \nu} + v = 0 \quad \text{on } \partial \Omega,
\end{array} \right.
\]

Let \( w^\ominus \) be the unique positive solution of the following problem
\[
\left\{ \begin{array}{l}
- \Delta w = w \left( r_3 + \frac{m_2 \Theta k_1 (r_1 - \frac{a_1}{c_1} - \frac{a_2}{c_2})}{1 + b_2 \Theta k_1 (r_1 - \frac{a_1}{c_1} - \frac{a_2}{c_2}) + c_2 w} + \frac{m_3 v^\ominus}{1 + b_3 v^\ominus + c_3 w} \right) \quad \text{in } \Omega, \\
k_3 \frac{\partial w}{\partial \nu} + w = 0 \quad \text{on } \partial \Omega,
\end{array} \right.
\]
and \( w^\ominus \) be the unique positive solution of the following problem
\[
\left\{ \begin{array}{l}
- \Delta w = w \left( r_3 + \frac{m_2 \Theta k_1 (r_1)}{1 + b_2 \Theta k_1 (r_1) + c_2 w} + \frac{m_3 v^\ominus}{1 + b_3 v^\ominus + c_3 w} \right) \quad \text{in } \Omega, \\
k_3 \frac{\partial w}{\partial \nu} + w = 0 \quad \text{on } \partial \Omega.
\end{array} \right.
\]
In order to give the main results, we introduce the following assumptions:

\[
\begin{align*}
    r_1 &> \frac{a_1}{e_1} + \frac{a_2}{e_2}, \\
    -r_2 &> \lambda_{1,k_2} \left( - \frac{m_1 \Theta_{k_1} (r_1 - \frac{a_1}{e_1} - \frac{a_2}{e_2})}{1 + b_1 \Theta_{k_1} (r_1 - \frac{a_1}{e_1} - \frac{a_2}{e_2})} + \frac{a_3}{e_3}, \right) \\
    r_3 &> \lambda_{1,k_3} \left( - \frac{m_2 \Theta_{k_1} (r_1 - \frac{a_1}{e_1} - \frac{a_2}{e_2})}{1 + b_2 \Theta_{k_1} (r_1 - \frac{a_1}{e_1} - \frac{a_2}{e_2})} - \frac{m_3 v^\varphi}{1 + b_3 v^\varphi} \right). 
\end{align*}
\]

(19)

**Remark 2.5.** By Theorem 2.2, it is easy to see that the existence and uniqueness of \(v^\varphi, v^\delta, w^\psi, w^\phi\) follow from the assumptions (19).

The following theorem provides sufficient conditions for permanence of the time-dependent system (1).

**Theorem 2.6.** Assume that the conditions in (19) hold. Then, there exist a pair of functions \((u^+, v^+, w^+)^n\) and \((u^-, v^-, w^-)^n\) in \(C(\bar{\Omega}) \cap C^2(\Omega)\) such that

\[
\begin{align*}
    -\Delta u^+ &= u^+ (r_1 - u^+) - \frac{a_1 u^+ v^-}{1 + b_1 u^+ + e_1 v^-} - \frac{a_2 u^+ w^-}{1 + u^+ + e_2 w^-} \\
    -\Delta u^- &= u^- (r_1 - u^-) - \frac{a_1 u^- v^+}{1 + b_1 u^- + e_1 v^+} - \frac{a_2 u^- w^+}{1 + b_2 u^- + e_2 w^+} \\
    -\Delta v^+ &= \frac{m_1 u^+ v^+}{1 + b_1 u^+ + e_1 v^+} - \frac{a_3 v^+ w^+}{1 + b_3 v^+ + e_3 w^+} - r^+ v^+ \\
    -\Delta v^- &= \frac{m_1 u^- v^-}{1 + b_1 u^- + e_1 v^-} - \frac{a_3 v^- w^-}{1 + b_3 v^- + e_3 w^-} - r^- v^- \\
    -\Delta w^+ &= w^+ (r_3 - w^+) + \frac{m_2 u^+ w^+}{1 + b_2 u^+ + e_2 w^+} + \frac{m_3 v^+ w^+}{1 + b_3 v^+ + e_3 w^+} \\
    -\Delta w^- &= w^- (r_3 - w^-) + \frac{m_2 u^- w^-}{1 + b_2 u^- + e_2 w^-} + \frac{m_3 v^- w^-}{1 + b_3 v^- + e_3 w^-} \quad \text{in } \Omega.
\end{align*}
\]

and satisfy the following relations \(\Theta_{k_1} (r_1 - \frac{a_1}{e_1} - \frac{a_2}{e_2}) \leq u^- \leq u^+ \leq \Theta_{k_1} (r_1), v^\varphi \leq v^- \leq v^+ \leq v^\delta, w^\psi \leq w^- \leq w^+ \leq w^\phi\). Furthermore, \([u^-, u^+] \times [v^-, v^+] \times [w^-, w^+]\) is a positive global attractor of (1).

**Remark 2.7.** We point out such functions \((u^+, v^+, w^+)^n\) and \((u^-, v^-, w^-)^n\) are called quasimonotone of system (2) in [34].

**Proof.** It is easy to show that \((\Theta_{k_1} (r_1), v^\delta, w^\phi)^n\) and \((\Theta_{k_1} (r_1 - \frac{a_1}{e_1} - \frac{a_2}{e_2}), v^\varphi, w^\psi)^n\) are a pair of ordered upper and lower solution of (1) under assumption (19). From the definition 8.1 in page 425 in [33], we know that the reaction functions of the system (2) are quasimonotone. Then the existence of \((u^+, v^+, w^+)^n\) and \((u^-, v^-, w^-)^n\) can be proved easily by using the iteration scheme (10.3) or Theorem 10.1 in page 438 in [33].

Next, we prove that \([u^-, u^+] \times [v^-, v^+] \times [w^-, w^+]\) is a positive global attractor of (1). Note that \((u^-, v^-, w^-)^n\) is positive in \(\Omega\) by maximum principle, it suffice to prove that \([u^-, u^+] \times [v^-, v^+] \times [w^-, w^+]\) is a global attractor.
Let $\epsilon$ be sufficiently small such that

\[
(A) \quad \epsilon < \frac{-\lambda_{1,k_2} \left( - \frac{m_1 \Theta_{k_1} (r_1 - \frac{a_1}{e_1} - \frac{a_2}{e_2})}{1 + b_1 \Theta_{k_1} (r_1 - \frac{a_1}{e_1} - \frac{a_2}{e_2})} - \frac{a_3}{e_3} - r_2 \right)}{m_1} < \frac{-\lambda_{1,k_2} \left( - \frac{m_1 \Theta_{k_1} (r_1)}{1 + b_1 \Theta_{k_1} (r_1)} - r_2 \right)}{m_1}.
\]

\[
(B) \quad \epsilon < \frac{-\lambda_{1,k_3} \left( - \frac{m_2 \Theta_{k_1} (r_1 - \frac{a_1}{e_1} - \frac{a_2}{e_2})}{1 + b_2 \Theta_{k_1} (r_1 - \frac{a_1}{e_1} - \frac{a_2}{e_2})} - \frac{m_3 v}{1 + b_3 v} \right) + r_3}{m_2 + m_3} < \frac{-\lambda_{1,k_3} \left( - \frac{m_2 \Theta_{k_1} (r_1)}{1 + b_2 \Theta_{k_1} (r_1)} - \frac{m_3 v}{1 + b_3 v} \right) + r_3}{m_2 + m_3}.
\]

It can be seen that the assumption (19) implies the positivity of the middle terms in (A) and (B). The 2nd inequalities in (A) and (B) follow from the monotonicity of the expression. So we may choose $\epsilon$ small enough such that the inequalities in (A) and (B) hold.

First, we can obtain from the first equation of (1) that

\[
\begin{cases}
\frac{\partial u}{\partial t} - \Delta u \leq u(r_1 - u) & \text{in } (0, \infty) \times \Omega, \\
k_1 \frac{\partial u}{\partial \nu} + u = 0 & \text{on } (0, \infty) \times \partial \Omega,
\end{cases}
\]

and

\[
\begin{cases}
\frac{\partial u}{\partial t} - \Delta u \geq u \left( r_1 - u - \frac{a_1}{e_1} - \frac{a_2}{e_2} \right) & \text{in } \Omega \times (0, \infty), \\
k_1 \frac{\partial u}{\partial \nu} + u = 0 & \text{on } \partial \Omega \times (0, \infty).
\end{cases}
\]

Then by using Theorem 2.2, comparison principle and assumption (19), we can get from (20) and (21) that $\limsup_{t \to \infty} u(x, t) \leq \Theta_{k_1} (r_1)$ and $\liminf_{t \to \infty} u(x, t) \geq \Theta_{k_1} (r_1 - \frac{a_1}{e_1} - \frac{a_2}{e_2})$ respectively.

So there exists a $T(\epsilon) \geq 0$ such that for all $t > T(\epsilon)$, we have

\[
u(x, t) \leq \Theta_{k_1} (r_1) + \epsilon
\]

and

\[
u(x, t) \geq \Theta_{k_1} \left( r_1 - \frac{a_1}{e_1} - \frac{a_2}{e_2} \right) - \epsilon.
\]

Then, it follows from (22), (23) and the second equation of (1) that

\[
\begin{cases}
\frac{\partial v}{\partial t} - \Delta v \leq \frac{m_1 (\Theta_{k_1} (r_1) + \epsilon)}{1 + b_1 \Theta_{k_1} (r_1) + \epsilon + e_1 v} - r_2 \\
\leq \frac{m_1 \Theta_{k_1} (r_1)}{1 + b_1 \Theta_{k_1} (r_1) + e_1 v} + m_1 \epsilon - r_2 & \text{in } \Omega \times (T(\epsilon), \infty), \\
k_2 \frac{\partial v}{\partial \nu} + v = 0 & \text{on } \partial \Omega \times (T(\epsilon), \infty),
\end{cases}
\]

(24)
and
\[ \begin{align*}
\frac{\partial v}{\partial t} - \Delta v & \geq v \left( \frac{m_1(\Theta_{k_1}(r_1 - \frac{a_1}{e_1} - \frac{a_2}{e_2}) - \epsilon)}{1 + b_1 \Theta_{k_1}(r_1 - \frac{a_1}{e_1} - \frac{a_2}{e_2}) - \epsilon + e_1 v} - \frac{a_3}{e_3} - r_2 \right) \\
& \geq v \left( \frac{m_1 \Theta_{k_1}(r_1 - \frac{a_1}{e_1} - \frac{a_2}{e_2}) - \epsilon + e_1 v}{1 + b_1 \Theta_{k_1}(r_1 - \frac{a_1}{e_1} - \frac{a_2}{e_2}) + e_1 v} - \frac{a_3}{e_3} - r_2 \right) \quad \text{in } \Omega \times (T(\epsilon), \infty), \\
k_2 \frac{\partial v}{\partial \nu} + v & = 0 \quad \text{on } \partial \Omega \times (T(\epsilon), \infty).
\end{align*} \]

(25)

Since \( \epsilon \) satisfy (A), by using Theorem 2.2, comparison principle and assumption (19), we can get from (24) and (25) that \( \limsup_{t \to \infty} v(x, t) \leq v^\omega \) and \( \liminf_{t \to \infty} v(x, t) \geq v^\gamma \) respectively. Thus there exists a \( T'(\epsilon) > T(\epsilon) \) such that for all \( t > T'(\epsilon) \), we have
\[ v(x, t) \leq v^\omega + \epsilon, \]
and
\[ v(x, t) \geq v^\gamma - \epsilon. \]

Then for \( (x, t) \in \Omega \times (T'(\epsilon), \infty) \), we can get from (26), (27) and the third equation of (1) that
\[ \frac{\partial w}{\partial t} - \Delta w \leq w \left( r_2 + \frac{m_2(\Theta_{k_1}(r_1) + \epsilon)}{1 + b_2(\Theta_{k_1}(r_1) + \epsilon) + e_3 w} + \frac{m_3(v^\omega + \epsilon)}{1 + b_3(v^\omega + \epsilon) + e_3 w} - w \right) \]
\[ \leq w \left( r_2 + \frac{m_2 \Theta_{k_1}(r_1)}{1 + b_2 \Theta_{k_1}(r_1) + e_3 w} + \frac{m_3 v^\omega}{1 + b_3 v^\omega + e_3 w} + (m_2 + m_3) \epsilon - w \right), \]
and
\[ \frac{\partial w}{\partial t} - \Delta w \geq w \left( r_3 + \frac{m_2(\Theta_{k_1}(r_1 - \frac{a_1}{e_1} - \frac{a_2}{e_2}) - \epsilon)}{1 + b_2(\Theta_{k_1}(r_1 - \frac{a_1}{e_1} - \frac{a_2}{e_2}) - \epsilon) + e_2 w} + \frac{m_3(v^\gamma - \epsilon)}{1 + b_3(v^\gamma - \epsilon) + e_3 w} \right. \]
\[ \geq w \left( r_3 + \frac{m_2 \Theta_{k_1}(r_1 - \frac{a_1}{e_1} - \frac{a_2}{e_2})}{1 + b_2 \Theta_{k_1}(r_1 - \frac{a_1}{e_1} - \frac{a_2}{e_2}) + e_2 w} + \frac{m_3 v^\gamma}{1 + b_3 v^\gamma + e_3 w} \right. \]
\[ \left. \left. - (m_2 + m_3) \epsilon - w \right). \right) \]
\[ (28) \]

(29)

Since \( \epsilon \) satisfies (B), we obtain \( \limsup_{t \to \infty} w(x, t) \leq w^\omega \) and \( \liminf_{t \to \infty} w(x, t) \geq w^\gamma \) by (28), (29), assumption (19). So there exists a \( T''(\epsilon) > T'(\epsilon) \) such that for all \( t > T''(\epsilon) \), we have
\[ w(x, t) \leq w^\omega + \epsilon, \]
and
\[ w(x, t) \geq w^\gamma - \epsilon. \]

Finally, by (22), (23), (26), (27), (30) and (31), we conclude that for any non-trivial initial condition \((u_0(x), v_0(x), w_0(x))\), the time-dependent solution \((u, v, w)\) of (1) satisfies
\[ (u, v, w) \in [\Theta_{k_1}(r_1 - \frac{a_1}{e_1} - \frac{a_2}{e_2}) - \epsilon, \Theta_{k_1}(r_1) + \epsilon] \times [v^\gamma - \epsilon, v^\omega + \epsilon] \times [w^\gamma - \epsilon, w^\omega + \epsilon] \]
for all \( t > T''(\epsilon) \). Then, by Corollary 2.1 and Theorem 2.1 in [34], we complete the proof. \( \square \)
The next theorem gives sufficient conditions for a global attractor in the case that exactly one species is dying out. This can be proved similarly as in the above theorem. So we omit the proof.

**Theorem 2.8.** (i) If $r_1 > \frac{a_1}{c_1} + \frac{a_2}{c_2}$, $-r_2 > \lambda_{1,k_2}\left(-\frac{m_1\Theta_{m_1}(r_1)}{1+b_1\Theta_{m_1}(r_1)} - \frac{m_2}{1+b_2\Theta_{m_2}(r_1)}\right) + \frac{a_2}{c_2}$ and $r_3 \leq \lambda_{1,k_3}\left(-\frac{m_2\Theta_{m_1}(r_1)}{1+b_2\Theta_{m_2}(r_1)} - \frac{m_3K_2}{1+b_2K_2}\right)$, then there exists a pair of quasisolution $(u^+, v^+)$ and $(u^-, v^-)$ of the following system

\[
\begin{align*}
-\Delta u &= u(r_1 - u) - \frac{a_1uv}{1+b_1u + e_1v} \\
-\Delta v &= \frac{m_1uv}{1+b_1u + e_1v} - r_2v \\
k_1\frac{\partial u}{\partial \nu} + u &= k_2\frac{\partial v}{\partial \nu} + v = 0
\end{align*}
\]

in $\Omega$, on $\partial\Omega$ with $u^+ \geq u^-$ and $v^+ \geq v^-$. Moreover, $[u^-, u^+] \times [v^-, v^+] \times \{0\}$ is a global attractor of (1).

(ii) If $r_1 > \frac{a_1}{c_1} + \frac{a_2}{c_2}$, $-r_2 \leq \lambda_{1,k_2}\left(-\frac{m_1\Theta_{m_1}(r_1)}{1+b_1\Theta_{m_1}(r_1)}\right)$ and $r_3 > \lambda_{1,k_3}\left(-\frac{m_2\Theta_{m_1}(r_1)}{1+b_2\Theta_{m_2}(r_1)} - \frac{m_3}{1+b_2K_2}\right)$, then there exists a pair of quasisolution $(u^+, w^+)$ and $(u^-, w^-)$ of the following system

\[
\begin{align*}
-\Delta u &= u(r_1 - u) - \frac{a_1uv}{1+b_1u + e_1v} \\
-\Delta w &= w(r_3 - w) + \frac{m_3uv}{1+b_2u + e_2v} \\
k_1\frac{\partial u}{\partial \nu} + u &= k_3\frac{\partial w}{\partial \nu} + w = 0
\end{align*}
\]

in $\Omega$, on $\partial\Omega$ with $u^+ \geq u^-$ and $w^+ \geq w^-$. Moreover, $[u^-, u^+] \times \{0\} \times [w^-, w^+]$ is a global attractor of (1).

3. Coexistence states. In this section, we shall investigate the existence of coexistence states for (2) by using the fixed point index theory. We first give some theorems to calculate the indexes at the trivial and semi-trivial states of (2).

3.1. Some preliminaries.

**Theorem 3.1.** (see [16, 28].) Assume $h(x) \in C^\alpha(\Omega)(0 < \alpha < 1)$ and $M$ is a sufficiently large number such that $M > h(x)$ for all $x \in \Omega$. Define a positive and compact operator $L := (\Delta + M)^{-1}(M - h(x)) : C_k^1(\Omega) \to C_k^1(\Omega) = \{u \in C^1(\Omega) : k\frac{\partial u}{\partial \nu} + u = 0 \text{ on } \partial\Omega\}$ for $k \geq 1$. Denote the spectral radius of $L$ by $\gamma_k(L)$.

(i) $\lambda_{1,k}(h) > 0$ if and only if $\gamma_k(L) < 1$.

(ii) $\lambda_{1,k}(h) < 0$ if and only if $\gamma_k(L) > 1$.

(iii) $\lambda_{1,k}(h) = 0$ if and only if $\gamma_k(L) = 1$.

From Theorem 3.1, we can see that it is crucial to know the sign of the eigenvalue $\lambda_{1,k}(h)$ to determine the spectral radius of $L$. The following theorem is established to determine the sign of the principle eigenvalue $\lambda_{1,k}(h)$.

**Theorem 3.2.** (see [7, 8].) Let $h(x) \in L^\infty(\Omega)$ and $\varphi \geq 0$, $\varphi \not= 0$ in $\Omega$ with $k\frac{\partial \varphi}{\partial \nu} + \varphi = 0$ on $\partial\Omega$ for $k \geq 0$.

(i) If $0 \not= -\Delta \varphi + h(x)\varphi \leq 0$, then $\lambda_1(h(x)) < 0$. 
(ii) If $0 \neq -\Delta \varphi + h(x)\varphi \geq 0$, then $\lambda_1(h(x)) > 0$.

(iii) If $-\Delta \varphi + h(x)\varphi = 0$, then $\lambda_1(h(x)) = 0$.

Now, we state the fixed point index theory which plays an crucial role in getting the sufficient conditions for the existence of positive solution of model (2).

Let $E$ be a real Banach space and $\mathbb{W} \subset E$ be the natural positive cone of $E$. For $y \in \mathbb{W}$, define $\mathbb{W}_y = \{x \in E : y + \eta \in \mathbb{W} \text{ for some } \eta > 0\}$ and $S_y = \{x \in \mathbb{W}_y : -x \in \mathbb{W}_y\}$. Then $\mathbb{W}_y$ is a wedge containing $\mathbb{W}$, $y$, $-y$, while $S_y$ is a closed subset of $E$ containing $y$. Let $T$ be a compact linear operator on $E$ which satisfies $T(\mathbb{W}_y) \subset \mathbb{W}_y$. We say that $T$ has property $\alpha$ on $\mathbb{W}_y$ if there is a $t \in (0, 1)$ and an $\omega \in \mathbb{W}_y \setminus S_y$ such that $(I - tT)\omega \in S_y$. Let $\mathcal{A} : \mathbb{W} \to \mathbb{W}$ is a compact operator with a fixed point $y \in \mathbb{W}$ and $\mathcal{A}$ is a Fréchet differentiable at $y$. Let $L = \mathcal{A}'(y)$ be the Fréchet derivative of $\mathcal{A}$ at $y$. Then $L$ maps $\mathbb{W}_y$ into itself. We denote by $\deg_{\mathbb{W}}(I - \mathcal{A}, \mathbb{W})$ the degree of $I - \mathcal{A}$ in $\mathbb{W}$ relative to $\mathbb{W}$, $\text{index}_{\mathbb{W}}(\mathcal{A}, y)$ the fixed point index of $\mathcal{A}$ at $y$ relative to $\mathbb{W}$ and $\text{deg}_{\mathbb{W}}(I - \mathcal{A}, S) = \sum_{y \in S} \text{index}_{\mathbb{W}}(\mathcal{A}, y)$ where $S$ only contains discrete points. Then the following theorem can be obtained.

**Theorem 3.3.** (see [47, 46, 24] ) Assume that $I - L$ is invertible on $\mathbb{W}_y$.

(i) If $L$ has property $\alpha$ on $\mathbb{W}_y$, then $\text{index}_{\mathbb{W}}(\mathcal{A}, y) = 0$.

(ii) If $L$ does not have property $\alpha$ on $\mathbb{W}_y$, then $\text{index}_{\mathbb{W}}(\mathcal{A}, y) = (-1)^\sigma$, where $\sigma$ is the sum of algebraic multiplicities of the eigenvalues of $L$ which are greater than 1.

Finally, we introduce the following theorem about degree calculations, which was introduced by E. N. Dancer and Y. H. Du in [13] and we state here for convenience.

Assume that $E_1$ and $E_2$ are ordered Banach spaces with positive cones $\mathbb{W}_1$ and $\mathbb{W}_2$, respectively. Let $E = E_1 \oplus E_2$ and $\mathbb{W} = \mathbb{W}_1 \oplus \mathbb{W}_2$. Then $E$ is an ordered Banach space with positive cone $\mathbb{W}$. Let $D$ be an open set in $\mathbb{W}$ containing 0 and $\mathcal{R}_i := D \to \mathbb{W}_i$ be compact operators, $i = 1, 2$. Assume $(u, v)$ is a general element in $\mathbb{W}$ with $u \in \mathbb{W}_1$ and $v \in \mathbb{W}_2$. Define $\mathcal{R} := D \to \mathbb{W}$ by $\mathcal{R}(u, v) = (\mathcal{R}_1(u, v), \mathcal{R}_2(u, v))$ and $\mathbb{W}_2(\delta) = \{v \in \mathbb{W}_2 : \|v\|_{E_2} < \delta\}$. Then the following theorem can be obtained.

**Theorem 3.4.** Suppose $U \subset \mathbb{W}_1 \cap D$ is relatively open and bounded, and $\mathcal{R}_1(u, 0) \neq u$ for $u \in \partial U$, $\mathcal{R}_2(u, 0) \equiv 0$ for $u \in U$. Suppose $\mathcal{R}_2 := D \to \mathbb{W}_2$ extends to a continuously differentiable mapping of a neighborhood of $D$ into $E_2$, $\mathbb{W}_2 - \mathbb{W}_2$ is dense in $E_2$ and $S = \{u \in U : u = \mathcal{R}_1(u, 0)\}$.

(i) If for any $u \in S$, the spectral radius $\gamma(\mathcal{R}_2(u, 0)) \|v\|_{E_2} > 1$ and 1 is not an eigenvalue of $\mathcal{R}_2(u, 0) \|v\|_{E_2}$ corresponding to a positive eigenvector, then $\text{deg}_{\mathbb{W}}(I - \mathcal{R}, U \times \mathbb{W}_2(\delta), 0) = 0$ for $\delta > 0$ small.

(ii) If for any $u \in S$, the spectral radius $\gamma(\mathcal{R}_2(u, 0)) \|v\|_{E_2} < 1$, then $\text{deg}_{\mathbb{W}}(I - \mathcal{R}, U \times \mathbb{W}_2(\delta), 0) = \text{deg}_{\mathbb{W}}(I - \mathcal{R}_1, |\mathbb{W}_1|, U, 0)$ for $\delta > 0$ small.

### 3.2. Nonexistence of coexistence states.

We start from the following lemma which give the priori bounds for the coexistence states of (2).

**Lemma 3.5.** If $m_1 r_1 > r_2 (1 + b_1 r_1)$, then any coexistence state $(u, v, w)$ of (2) has an a priori bounds:

\[ u(x) \leq K_1, \quad v(x) \leq K_2, \quad w(x) \leq K_3, \]

where

\[ K_1 = r_1, \quad K_2 = \frac{m_1 r_1 - r_2 (1 + b_1 r_1)}{r_2 e_1}, \quad K_3 = r_3 + \frac{m_2 K_1}{1 + b_2 K_1} + \frac{m_3 K_2}{1 + b_3 K_2}. \]
Proof. From the first equation of (2), we can obtain
\[
\begin{cases}
- \Delta u \leq u(r_1 - u) & \text{in } \Omega, \\
k \frac{\partial u}{\partial \nu} + u = 0 & \text{on } \partial \Omega.
\end{cases}
\] (32)

Then by (32), the maximum principle and Theorem 2.2, we know
\[u(x) \leq \Theta_k(r_1) \leq r_1.\]

From the second equation of (2), we have
\[
\begin{cases}
- \Delta v \leq v \left( \frac{m_1 r_1}{1 + b_1 r_1 + e_1 v} - r_2 \right) & \text{in } \Omega, \\
k \frac{\partial v}{\partial \nu} + v = 0 & \text{on } \partial \Omega.
\end{cases}
\] (33)

Hence by (33), we have
\[v(x) \leq \frac{m_1 r_1 - r_2 (1 + r_1)}{r_2 e_1}.\]

Then from the third equation of (2), we have
\[
\begin{cases}
- \Delta w \leq w \left( r_3 - w + \frac{m_2 K_1}{1 + b_2 K_1} + \frac{m_3 K_2}{1 + b_3 K_2} \right) & \text{in } \Omega, \\
k \frac{\partial w}{\partial \nu} + w = 0 & \text{on } \partial \Omega.
\end{cases}
\] (34)

By using (34), the maximum principle and Theorem 2.2, we have
\[w(x) \leq r_3 + \frac{m_2 K_1}{1 + b_2 K_1} + \frac{m_3 K_2}{1 + b_3 K_2}.\]

The proof is completed. \(\square\)

Remark 3.6. From the proof of Lemma 3.5, we can see that \(m_1 r_1 > r_2 (1 + r_1)\) is the necessary condition for problem (2) have coexistence states. So, throughout this subsection, we assume that: \((H) m_1 r_1 > r_2 (1 + r_1)\).

Now we can obtain the following necessary conditions for the existence of coexistence states for system (2).

Theorem 3.7. \(i)\) If \(r_1 \leq \lambda_{1,k_1}\), then there is no positive solution of problem (2).

\(ii)\) If \(r_3 < \lambda_{1,k_3}\), then \(r_1 > \lambda_{1,k_1}\), \(r_2 < \frac{m_1 r_1}{1 + b_1 r_1} - \lambda_{1,k_2}\) and \(r_3 + \frac{m_2 r_1}{1 + b_2 r_1} + \frac{m_3 K_2}{1 + b_3 K_2} > \lambda_{1,k_3}\) are necessary conditions for the existence of positive solutions to (2).

\(iii)\) If \(r_3 > \lambda_{1,k_3}\), then \(r_1 > \lambda_{1,k_1}\left( \frac{a_2 \Theta_{k_1}(r_3)}{1 + e_2 \Theta_{k_3}(r_3)} \right)\) and \(-r_2 > \lambda_{1,k_2}\left( -\frac{m_1 \Theta_{k_1}(r_1)}{1 + b_1 \Theta_{k_1}(r_1)} + \frac{a_3 \Theta_{k_3}(r_3)}{1 + e_3 \Theta_{k_3}(r_3)} \right)\) are necessary conditions for the existence of positive solutions to (2).

Proof. \(i)\) Assume \((u, v, w)\) is a positive solution of (2). Then it follows from the first equation and the comparison principle of eigenvalues that
\[r_1 = \lambda_{1,k_1}(u + \frac{a_1 v}{1 + b_1 u + e_1 v} + \frac{a_2 w}{1 + b_2 u + e_2 w}) > \lambda_{1,k_1},\]
which is a contradiction.
(ii) Let \((u, v, w)\) be a positive solution of (2). Then \(r_1 > \lambda_{1,k_1}\) from (i) and \(u(x) \leq \Theta_{k_1}(r_1) \leq r_1, \ v \leq K_2, \ w \leq K_3\) from Lemma 3.5. We can get from the second equation and the comparison principle of eigenvalues that
\[
-r_2 = \lambda_{1,k_2}\left(-\frac{m_1u}{1+b_1u+e_1v} + \frac{a_3w}{1+b_3v+e_3w}\right) > \lambda_{1,k_2}\left(-\frac{m_1\Theta_{k_1}(r_1)}{1+b_1\Theta_{k_1}(r_1)}\right) > \lambda_{1,k_2}\left(-\frac{m_1r_1}{1+b_1r_1}\right),
\]
which implies that \(r_2 < \frac{m_1r_1}{1+b_1r_1} - \lambda_{1,k_2}\). Similarly, it follows from the third equation that
\[
r_3 = \lambda_{1,k_3}\left(w - \frac{m_2u}{1+b_2u+e_2v} - \frac{m_3v}{1+b_3v+e_3w}\right) > \lambda_{1,k_3}\left(-\frac{m_2r_1}{1+b_2r_1} - \frac{m_3K_2}{1+b_3K_2}\right),
\]
which implies that \(r_3 < \frac{m_2r_1}{1+b_2r_1} + \frac{m_3K_2}{1+b_3K_2} > \lambda_{1,k_3}\).

(iii) Let \((u,v,w)\) be a positive solution of (2). Then \(r_1 > \lambda_{1,k_1}\) from (i) and \(u(x) \leq \Theta_{k_1}(r_1) \leq r_1, \ v \leq \Theta_{k_2}(r_2) \leq r_2, \ w \leq \Theta_{k_3}(r_3) \leq r_3\). We can get from the first equation and the comparison principle of eigenvalues that
\[
r_1 = \lambda_{1,k_1}\left(u + \frac{a_1v}{1+b_1u+e_1v} + \frac{a_2w}{1+b_2u+e_2w}\right) > \lambda_{1,k_1}\left(\frac{a_2\Theta_{k_2}(r_2)}{1+e_2\Theta_{k_2}(r_2)}\right),
\]
and from the second equation that
\[
-r_2 = \lambda_{1,k_2}\left(-\frac{m_1u}{1+b_1u+e_1v} + \frac{a_3w}{1+b_3v+e_3w}\right) > \lambda_{1,k_2}\left(-\frac{m_1\Theta_{k_1}(r_1)}{1+b_1\Theta_{k_1}(r_1)} + \frac{a_3\Theta_{k_2}(r_2)}{1+e_3\Theta_{k_2}(r_2)}\right).
\]
The proof is completed.

3.3. Existence of coexistence states. In the rest of this section, we shall give sufficient conditions for (2) has coexistence states by using the fixed point index theory. Now, we introduce the following notations:

\[
\begin{align*}
E &= C_{k_1}^1(\Omega) \oplus C_{k_2}^1(\Omega) \oplus C_{k_3}^1(\Omega), \\
N_i &= \{\phi \in C_{k_i}^1(\Omega) : \phi \geq 0 \text{ in } \Omega\}, \quad i = 1, 2, 3, \\
W &= N_1 \oplus N_2 \oplus N_3, \\
D &= \{(u,v,w) \in W : u \leq (K_1 + 1), v \leq (K_2 + 1), w \leq (K_3 + 1)\},
\end{align*}
\]

where \(C_{k_i}^1(\Omega) = \{\phi \in C^1(\Omega) : k_i \frac{\partial \phi}{\partial n} + \phi = 0 \text{ on } \partial \Omega, i = 1, 2, 3\}\) and \(K_1, K_2, K_3\) are defined in Lemma 3.5.

From Lemma 3.5, we can see that the coexistence state of (2) must be in \(D\). Take \(q\) sufficiently large such that \(u(r_1 - u) - \frac{a_1 uv}{1+b_1u+e_1v} - \frac{a_2 uv}{1+b_2u+e_2w} + qw, \frac{a_2 uv}{1+b_2u+e_2w} - \frac{a_3 uv}{1+b_3u+e_3w} - r_2v + qw\) and \(r_3w - w^2 + \frac{a_3 uv}{1+b_3u+e_3w} + qw, \frac{a_2 uv}{1+b_2u+e_2w} - \frac{a_3 uv}{1+b_3u+e_3w} + qw\) are respectively monotone increasing with respect to \(u, v\) and \(w\) for all \((u,v,w) \in [0, K_1] \times [0, K_2] \times [0, K_3]\).
Define a positive and compact operator \( \Re : \mathbb{E} \rightarrow \mathbb{E} \) by
\[
\Re(u,v,w) = (-\Delta + q)^{-1} \begin{pmatrix}
    w(r_1 - u) - \frac{a_1 uv}{1 + b_1 u + e_1 v} - \frac{a_2 uv}{1 + b_2 u + e_2 v} + qu \\
    w(r_3 - w) + \frac{m_2 uv}{1 + b_2 u + e_2 v} + \frac{m_3 uv}{1 + b_3 u + e_3 v} + qw
\end{pmatrix}.
\]

**Remark 3.8.** Observe that (2) is equivalent to \((u,v,w) = \Re(u,v,w)\), and then it is sufficient to prove that \(\Re\) has a coexistence state.

From the remark above, we can see that it is necessary to calculate the degree of \(I - \Re\) in \(D\) relative to \(\mathbb{W}\) and the fixed point index of \(\Re\) at \((0,0,0)\) relative to \(\mathbb{W}\). The following lemma gives the corresponding results about \(\text{deg}_W(I - \Re, \mathbb{D})\) and \(\text{index}_W(\Re, (0,0,0))\).

**Lemma 3.9.**
(i) \(\text{deg}_W(I - \Re, \mathbb{D}) = 1\).
(ii) If \(r_1 > \lambda_1, r_2 \neq \lambda_1, k_2, r_3 \neq \lambda_1, k_3\), then \(\text{index}_W(\Re, (0,0,0)) = 0\).

**Proof.** (i) It is easy to see that \(\Re\) has no fixed point on \(\partial \mathbb{D}\). So, the \(\text{deg}_W(I - \Re, \mathbb{D})\) is well defined. For \(\mu \in [0,1]\), define a positive and compact operator \(\Re_\mu : \mathbb{E} \rightarrow \mathbb{E}\) by
\[
\Re_\mu(u,v,w) = (-\Delta + q)^{-1} \begin{pmatrix}
    \mu vr_1 - u - \frac{a_1 uv}{1 + b_1 u + e_1 v} - \frac{a_2 uv}{1 + b_2 u + e_2 v} + qu \\
    \mu vr_3 - w + \frac{m_2 uv}{1 + b_2 u + e_2 v} + \frac{m_3 uv}{1 + b_3 u + e_3 v} + qw
\end{pmatrix}.
\]
Then \(\Re_1 = \Re\) and a fixed point of \(\Re_\mu\) is a solution of the following problem
\[
\begin{cases}
    -\Delta u = \mu v (r_1 - u - \frac{a_1 v}{1 + b_1 u + e_1 v} - \frac{a_2 v}{1 + b_2 u + e_2 v}) \\
    -\Delta v = \mu v (\frac{m_1 u}{1 + b_1 u + e_1 v} - \frac{a_3 w}{1 + b_3 u + e_3 w} - r_2) \\
    -\Delta w = \mu v (r_3 - w + \frac{m_2 u}{1 + b_2 u + e_2 v} + \frac{m_3 u}{1 + b_3 u + e_3 v}) \\
    k_1 \frac{\partial u}{\partial \nu} + u = k_2 \frac{\partial v}{\partial \nu} + v = k_3 \frac{\partial w}{\partial \nu} + w = 0
\end{cases}
\text{in } \Omega,
\]
\[
\begin{cases}
    \frac{\partial u}{\partial \nu} + u = \frac{\partial v}{\partial \nu} + v = \frac{\partial w}{\partial \nu} + w = 0
\end{cases}
\text{on } \partial \Omega.
\]

As the proof of Lemma 3.5, we can show that the fixed point \((u,v,w)\) of \(\Re_\mu\) also satisfy \(u \leq K_1, v \leq K_2\) and \(w \leq K_3\) for each \(\mu \in [0,1]\). Thus, \(\Re_\mu\) has no fixed point on \(\partial \mathbb{D}\) and \(\text{deg}_W(I - \Re_\mu, \mathbb{D})\) is well defined. Since \(\text{deg}_W(I - \Re_\mu, \mathbb{D})\) is independent of \(\mu\), we have \(\text{deg}_W(I - \Re, \mathbb{D}) = \text{deg}_W(I - \Re_1, \mathbb{D}) = \text{deg}_W(I - \Re_0, \mathbb{D})\).

Observe that (35) has only the trivial solution \((0,0,0)\) when \(\mu = 0\). Set
\[
\mathbb{L} = \Re_0(0,0,0) = (-\Delta + q)^{-1} \begin{pmatrix}
    q & 0 & 0 \\
    0 & q & 0 \\
    0 & 0 & q
\end{pmatrix}.
\]

Assume that \(\mathbb{L}(\xi_1, \xi_2, \xi_3) = (\xi_1, \xi_2, \xi_3)\) for some \((\xi_1, \xi_2, \xi_3) \in \mathbb{W}_0(0,0,0) = \mathbb{N} \times \mathbb{N} \times \mathbb{N}\).

It is easy to show that \((\xi_1, \xi_2, \xi_3) = (0,0,0)\) by maximum principle. So, \(I - \mathbb{L}\) is invertible on \(\mathbb{W}_0(0,0,0)\). Since \(\lambda_1, k_3 > 0\), we have \(\gamma_k(\mathbb{L}) < 1\) for \(k = 1, 2, 3\) by Theorem 3.1. This implies that \(\mathbb{L}\) does not have property \(\alpha\). So, by Theorem 3.3, we have \(\text{deg}_W(I - \Re, \mathbb{D}) = \text{deg}_W(I - \Re_0, \mathbb{D}) = \text{index}_W(\Re_0, (0,0,0)) = 1\).

(ii) Note that \(\Re(0,0,0) = (0,0,0)\). Let \(\mathbb{L} = \Re(0,0,0)\) and then
\[
\mathbb{L} = (-\Delta + q)^{-1} \begin{pmatrix}
    r_1 + q & 0 & 0 \\
    0 & -r_3 + q & 0 \\
    0 & 0 & r_3 + q
\end{pmatrix}.
\]
Assume that \( L(\xi_1, \xi_2, \xi_3) = (\xi_1, \xi_2, \xi_3) \in \mathbb{W}_{(0,0,0)} = N \times N \times N \).

Then

\[
\begin{cases}
- \Delta \xi_1 = r_1 \xi_1 \\
- \Delta \xi_2 = -r_2 \xi_2 \\
- \Delta \xi_3 = r_3 \xi_3 \\
k_1 \frac{\partial \xi_1}{\partial \nu} + \xi_1 = k_2 \frac{\partial \xi_2}{\partial \nu} + \xi_2 = k_3 \frac{\partial \xi_3}{\partial \nu} + \xi_3 = 0 \quad \text{on } \partial \Omega.
\end{cases}
\]  

Since \( r_1 > \lambda_{1,k_1} \), we can show \( \xi_1 \equiv 0 \). If not, we have \( \lambda_{1,k_1} = r_1 \) from the first equation of (36), which is a contradiction. Similarly, since \( r_2 \neq -\lambda_{1,k_2} \) and \( r_3 \neq -\lambda_{1,k_2} \), we can get \( \xi_2 \equiv 0 \) and \( \xi_3 \equiv 0 \). So, \( (\xi_1, \xi_2, \xi_3) \equiv (0, 0, 0) \) and \( I - L \) is invertible on \( \mathbb{W}_{(0,0,0)} \).

Note that \( r_1 > \lambda_{1,k_1} \) implies \( \gamma_0 = \gamma_{k_1} \left( (\Delta + q)^{-1}(r_1 + q) \right) > 1 \) by Theorem 3.1, where \( \gamma_0 \) is the principle eigenvalue of the operator \( (\Delta + q)^{-1}(r_1 + q) \) with a corresponding eigenfunction \( \phi(x) > 0 \), we have \( (\phi(x), 0, 0) \in \mathbb{W}_{(0,0,0)} \setminus S_{(0,0,0)} \) since \( S_{(0,0,0)} = (0, 0, 0) \). Then \( (I - \gamma_0^{-1}L)(\phi(x), 0, 0) = (0, 0, 0) \in S_{(0,0,0)} \), which shows that \( L \) has property \( \alpha \). Therefore, \( \text{index}_{\mathbb{W}}(\mathcal{R}, (0,0,0)) = 0 \) by Theorem 3.3. The proof is completed.

The next lemma gives the index at the semitrivial solution \( (\Theta_{k_1}(r_1), 0, 0) \) of (2).

**Lemma 3.10.** Assume that \( r_1 > \lambda_{1,k_1} \), \( r_2 \neq \lambda_{1,k_2} \) and \( r_3 \neq \lambda_{1,k_3} \).

(i) \( \text{index}_{\mathbb{W}}(\mathcal{R}, (\Theta_{k_1}(r_1), 0, 0)) = 0 \) if \( r_2 > \lambda_{1,k_2} \left( -\frac{m_1\Theta_{k_1}(r_1)}{1+b_1\Theta_{k_1}(r_1)} \right) \) or \( r_3 > \lambda_{1,k_3} \left( -\frac{m_2\Theta_{k_1}(r_1)}{1+b_2\Theta_{k_1}(r_1)} \right) \).

(ii) \( \text{index}_{\mathbb{W}}(\mathcal{R}, (\Theta_{k_1}(r_1), 0, 0)) = 1 \) if \( r_2 < \lambda_{1,k_2} \left( -\frac{m_1\Theta_{k_1}(r_1)}{1+b_1\Theta_{k_1}(r_1)} \right) \) and \( r_3 < \lambda_{1,k_3} \left( -\frac{m_2\Theta_{k_1}(r_1)}{1+b_2\Theta_{k_1}(r_1)} \right) \).

**Proof.** (i) Note that \( \mathcal{R}(\Theta_{k_1}(r_1), 0, 0) = (\Theta_{k_1}(r_1), 0, 0) \). Let \( L = \mathcal{R}'(\Theta_{k_1}(r_1), 0, 0) \) and then

\[
L = (-\Delta + q)^{-1} \begin{pmatrix}
  r_1 - 2\Theta_{k_1}(r_1) + q & -\frac{a_1\Theta_{k_1}(r_1)}{1+b_1\Theta_{k_1}(r_1)} & -\frac{a_2\Theta_{k_1}(r_1)}{1+b_2\Theta_{k_1}(r_1)} \\
  0 & \frac{m_1\Theta_{k_1}(r_1)}{1+b_1\Theta_{k_1}(r_1)} - r_2 + q & 0 \\
  0 & 0 & r_3 + \frac{m_2\Theta_{k_1}(r_1)}{1+b_2\Theta_{k_1}(r_1)} + q
\end{pmatrix}.
\]

If \( L(\xi_1, \xi_2, \xi_3) = (\xi_1, \xi_2, \xi_3) \in \mathbb{W}_{(\Theta_{k_1}(r_1), 0, 0)} = C^1(\overline{\Omega}) \times N \times N \), then

\[
\begin{cases}
- \Delta \xi_1 + 2(\Theta_{k_1}(r_1) - r_1)\xi_1 = -\frac{a_1\Theta_{k_1}(r_1)}{1+b_1\Theta_{k_1}(r_1)}\xi_2 - \frac{a_2\Theta_{k_1}(r_1)}{1+b_2\Theta_{k_1}(r_1)}\xi_3 \\
- \Delta \xi_2 + \left( r_2 - \frac{m_1\Theta_{k_1}(r_1)}{1+b_1\Theta_{k_1}(r_1)} \right)\xi_2 = 0 \\
- \Delta \xi_3 + \left( r_3 - \frac{m_2\Theta_{k_1}(r_1)}{1+b_2\Theta_{k_1}(r_1)} \right)\xi_3 = 0 \\
k_1 \frac{\partial \xi_1}{\partial \nu} + \xi_1 = k_2 \frac{\partial \xi_2}{\partial \nu} + \xi_2 = k_3 \frac{\partial \xi_3}{\partial \nu} + \xi_3 = 0 \quad \text{in } \Omega, \\
k_1 \frac{\partial \xi_1}{\partial \nu} + \xi_1 = k_2 \frac{\partial \xi_2}{\partial \nu} + \xi_2 = k_3 \frac{\partial \xi_3}{\partial \nu} + \xi_3 = 0 \quad \text{on } \partial \Omega.
\end{cases}
\]  

Take account of \( \xi_2 \in N \). If \( \xi_2 \neq 0 \), we can see from the second equation of (37) that

\[-r_2 = \lambda_{1,k_2} \left( -\frac{m_1\Theta_{k_1}(r_1)}{1+b_1\Theta_{k_1}(r_1)} \right).\]

This contradicts \( -r_2 \neq \lambda_{1,k_2} \left( -\frac{m_1\Theta_{k_1}(r_1)}{1+b_1\Theta_{k_1}(r_1)} \right) \). So, \( \xi_2 \equiv 0 \).
Similarly, since \(r_3 \neq \lambda_1, k_3 \left( - \frac{m_2 \Theta_k (r_1)}{1 + b_1 \Theta_k (r_1)} \right) \), we can prove that \( \xi_3 \equiv 0 \). Then from the first equation of (37), we can get that

\[
\begin{aligned}
- \Delta \xi_1 + (2 \Theta_k (r_1) - r_1) \xi_1 &= 0 \quad \text{in } \Omega, \\
\frac{k_1 \partial \xi_1}{\partial v} + \xi_1 &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\] (38)

If \( \xi_1 \neq 0 \), then \( \lambda_{1,k_1} (2 \Theta_k (r_1) - r_1) = 0 \). On the other hand, \( \lambda_{1,k_1} (2 \Theta_k (r_1) - r_1) > \lambda_{1,k_1} (\Theta_k (r_1) - r_1) = 0 \) by Theorem 2.1, which is a contradiction. Therefore, \((\xi_1, \xi_2, \xi_3) \equiv (0, 0, 0) \) and \( I = \mathbb{L} \) is invertible on \( \mathbb{W}(\phi_{k_1}, 0) \).

We claim that \( \mathbb{L} \) has property \( \alpha \) on \( \mathbb{W}(\phi_{k_1}, 0) \). We set

\[
A = (-\Delta + q)^{-1} \left( \frac{m_1 \Theta_k (r_1)}{1 + b_1 \Theta_k (r_1)} - r_2 + q \right).
\]

Since \(-r_2 > \lambda_1, k_2 \left( - \frac{m_1 \Theta_k (r_1)}{1 + b_1 \Theta_k (r_1)} \right) \), it can be seen that \( \gamma_{r_2} (A) > 1 \) is an eigenvalue of \( A \) with a corresponding eigenfunction \( \phi_{r_2} \). Therefore, \((\xi_1, \xi_2, \xi_3) \equiv (0, 0, 0) \) on \( \mathbb{W}(\phi_{k_1}) \). Then we can obtain that

\[
(I - \gamma_{r_2}^{-1} \mathbb{L}) \begin{pmatrix} \phi_{r_2} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \gamma_{r_2}^{-1} (-\Delta + q)^{-1} \begin{pmatrix} \frac{m_1 \Theta_k (r_1)}{1 + b_1 \Theta_k (r_1)} \phi_{r_2} \\ 0 \end{pmatrix} \in \mathbb{W}(\phi_{k_1}).
\] (39)

This establishes our claim. Hence, \( \text{index}_\mathbb{W}(\mathbb{L}, (\phi_{k_1}, 0, 0)) = 0 \) by Theorem 3.3.

(ii) First, we show that \( \mathbb{L} \) has no property \( \alpha \) on \( \mathbb{W}(\phi_{k_1}, 0) \). On the contrary, if \( \mathbb{L} \) has property \( \alpha \) in \( \mathbb{W}(\phi_{k_1}, 0) \), then there exists \( \eta \in (0, 1) \) and \((\varphi_1, \varphi_2, \varphi_3) \in \mathbb{W}(\phi_{k_1}, 0) \) such that \((I - \eta \mathbb{L})(\varphi_1, \varphi_2, \varphi_3) \in \mathbb{W}(\phi_{k_1}, 0) \). Therefore, \(( -\Delta + q)^{-1} \left( \frac{m_1 \Theta_k (r_1)}{1 + b_1 \Theta_k (r_1)} - r_2 + q \right) \varphi_2 = \frac{1}{\eta} \varphi_2 \), which implies that \( \frac{1}{\eta} > 1 \) is an eigenvalue of the operator \(( -\Delta + q)^{-1} \left( \frac{m_1 \Theta_k (r_1)}{1 + b_1 \Theta_k (r_1)} - r_2 + q \right) \).

On the other hand, since \(-r_2 < \lambda_1, k_2 \left( - \frac{m_1 \Theta_k (r_1)}{1 + b_1 \Theta_k (r_1)} \right) \), we know the principle eigenvalue \( \gamma_{k_2} \left( -\Delta + q \right)^{-1} \left( \frac{m_1 \Theta_k (r_1)}{1 + b_1 \Theta_k (r_1)} - r_2 + q \right) < 1 \). This contradiction shows that \( \mathbb{L} \) does not have property \( \alpha \) on \( \mathbb{W}(\phi_{k_1}, 0) \). So by Theorem 3.3, we have

\[
\text{index}_\mathbb{W}(\mathbb{L}, (\phi_{k_1}, 0, 0)) = (-1)^\nu,
\]

where \( \sigma \) is the sum of the multiplicities of all eigenvalues of \( \mathbb{L} \) which are greater than 1.

Next, we shall prove that \( \sigma = 0 \). Suppose \( \mu > 1 \) is an eigenvalue of \( \mathbb{L} \) with corresponding eigenfunction \((\xi_1, \xi_2, \xi_3) \), then we have
that we can obtain the following results.

\[
\begin{align*}
\Delta & \xi_1 + q\xi_1 = \frac{1}{\mu} \left( (r_1 - 2\Theta_{k_1}(r_1)) + q \right) \xi_1 - \frac{a_1 \Theta_{k_1}(r_1)}{1 + b_1 \Theta_{k_1}(r_1)} \xi_2 - \frac{a_2 \Theta_{k_1}(r_1)}{1 + b_2 \Theta_{k_1}(r_1)} \xi_3 \\
\Delta & \xi_2 + q\xi_2 = \frac{1}{\mu} \left( \frac{m_1 \Theta_{k_1}(r_1)}{1 + b_1 \Theta_{k_1}(r_1)} - r_2 + q \right) \xi_2 \\
\Delta & \xi_3 + q\xi_3 = \frac{1}{\mu} \left( \frac{m_2 \Theta_{k_1}(r_1)}{1 + b_2 \Theta_{k_1}(r_1)} + r_3 + q \right) \xi_3 \quad \text{in } \Omega, \\
k_1 \frac{\partial \xi_1}{\partial \nu} + \xi_1 = k_2 \frac{\partial \xi_2}{\partial \nu} + \xi_2 = k_3 \frac{\partial \xi_3}{\partial \nu} + \xi_3 = 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

(40)

If \( \xi_2 \neq 0 \), we can get from the second equation of (40) that:

\[
0 = \lambda_{1,k_2} \left( q(1 - \frac{1}{\mu}) - \frac{1}{\mu} \left( \frac{m_1 \Theta_{k_1}(r_1)}{1 + b_1 \Theta_{k_1}(r_1)} - r_2 \right) \right) > \lambda_{1,k_2} \left( - \frac{m_1 \Theta_{k_1}(r_1)}{1 + b_1 \Theta_{k_1}(r_1)} + r_2. \right)
\]

This contradicts \(-r_2 < \lambda_{1,k_2} \left( - \frac{m_1 \Theta_{k_1}(r_1)}{1 + b_1 \Theta_{k_1}(r_1)} \right) \). So, \( \xi_2 \equiv 0 \). Similarly, since \( r_3 < \lambda_{1,k_3} \left( - \frac{m_1 \Theta_{k_3}(r_1)}{1 + b_2 \Theta_{k_3}(r_1)} \right) \), we can prove that \( \xi_3 \equiv 0 \). Then from the first equation of (40), we can get that

\[
0 = \lambda_{1,k_1} \left( q(1 - \frac{1}{\mu}) - \frac{1}{\mu} \left( r_1 - 2\Theta_{k_1}(r_1) \right) \right) \geq \lambda_{1,k_1} \left( 2\Theta_{k_1}(r_1) - r_1 \right) > \lambda_{1,k_1} \left( \Theta_{k_1}(r_1) - r_1 \right) = 0
\]

This contradiction shows that \( \xi_1 \equiv 0 \). So, \( \xi_1, \xi_2, \xi_3 \equiv (0, 0, 0) \), which implies that \( L \) has no eigenvalue being greater than 1. Consequently, \( \sigma = 0 \) and then \( \text{index}_W(\mathcal{R}, \Theta_{k_3}(r_1), 0, 0) = 1 \) by Theorem 3.3. The proof is completed.

Similarly, if \( r_3 > \lambda_{1,k_3} \), (2) exists another semi-trivial solutions \((0, 0, 0, \Theta_{k_3}(r_3))\) and we can obtain the following results.

**Lemma 3.11.** Assume that \( r_3 > \lambda_{1,k_3} \), \( r_1 \neq \lambda_{1,k_1} \left( \frac{a_2 \Theta_{k_3}(r_3)}{1 + b_2 \Theta_{k_3}(r_3)} \right) \) and \( r_2 \neq \lambda_{1,k_2} \left( - \frac{a_3 \Theta_{k_3}(r_3)}{1 + b_3 \Theta_{k_3}(r_3)} \right) \).

(i) \( \text{index}_W(\mathcal{R}, (0, 0, \Theta_{k_3}(r_3))) = 0 \) if \( r_1 > \lambda_{1,k_1} \left( \frac{a_2 \Theta_{k_3}(r_3)}{1 + b_2 \Theta_{k_3}(r_3)} \right) \) or \( r_2 > \lambda_{1,k_2} \left( - \frac{a_3 \Theta_{k_3}(r_3)}{1 + b_3 \Theta_{k_3}(r_3)} \right) \).

(ii) \( \text{index}_W(\mathcal{R}, (0, 0, \Theta_{k_3}(r_3))) = 1 \) if \( r_1 < \lambda_{1,k_1} \left( \frac{a_2 \Theta_{k_3}(r_3)}{1 + b_2 \Theta_{k_3}(r_3)} \right) \) and \( r_2 < \lambda_{1,k_2} \left( - \frac{a_3 \Theta_{k_3}(r_3)}{1 + b_3 \Theta_{k_3}(r_3)} \right) \).

**Proof.** (i) Note that \( \mathcal{R}(0, 0, \Theta_{k_3}(r_3)) = (0, 0, \Theta_{k_3}(r_3)) \). Let \( L = \mathcal{R}'(0, 0, \Theta_{k_3}(r_2)) \) and then

\[
L = (-\Delta + q)^{-1} \begin{pmatrix}
1 & 0 & 0 \\
0 & a_2 \Theta_{k_3}(r_3) & -a_3 \Theta_{k_3}(r_3) \\
m_2 \Theta_{k_3}(r_3) & e_2 \Theta_{k_3}(r_3) - r_2 + q & 0
\end{pmatrix}.
\]

The following proof is similar to that of Lemma 3.9, so we omit it. 

To study the other semi-trivial solutions of (2), consider the following three possible subsystems:

\[
\begin{align*}
-\Delta u &= u(r_1 - u) - \frac{a_1 uv}{1 + b_1 u + e_1 v} \\
-\Delta v &= \frac{m_1 uv}{1 + b_1 u + e_1 v} - r_2 v \quad \text{in } \Omega, \\
k_1 \frac{\partial u}{\partial \nu} + u = k_2 \frac{\partial v}{\partial \nu} + v = 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

(41)
Then (42) has a positive solution \( u, v, w \) in \( \Omega \), and a semi-trivial solution \((v, w) = (0, \Theta_k (r_1))\). About the positive solutions of systems (41) and (42), we can obtain some results from [6, 30] and [16] respectively and simple comparison arguments for elliptic problems. We point out that the corresponding main results are still valid under Robin boundary conditions even if the results in the above references were obtained under Dirichlet boundaries.

**Theorem 3.12.** (41) has a positive solution \((u^*, v^*)\) with \( u^* \leq \Theta_k (r_1) \) and \( v^* \leq v^\ast \) if and only if \( r_1 > \lambda_{1,k_1} \) and \( -r_2 > \lambda_{1,k_2} \) \(- m_1 \Theta_{k_1} (r_1) \), where \( v^\ast \) is the unique positive solution of the following problem

\[
\begin{cases}
-\Delta v = v \left( \frac{m_1 \Theta_{k_1} (r_1)}{1 + b_1 \Theta_{k_1} (r_1) + e_1 v} - r_2 \right) & \text{in } \Omega, \\
k_2 \frac{\partial v}{\partial n} + v = 0 & \text{on } \partial \Omega.
\end{cases}
\]

In addition, if \( r_1 - \frac{a_1}{e_1} > \lambda_{1,k_1} \) and \( -r_2 > \lambda_{1,k_2} \left( - \frac{m_1 \Theta_{k_1} (r_1 - \frac{a_1}{e_1})}{1 + b_1 \Theta_{k_1} (r_1 - \frac{a_1}{e_1})} \right) \), then the positive solution \((u^*, v^*)\) satisfies \( u^* \geq \Theta_k (r_1 - \frac{a_1}{e_1}) \) and \( v^* \geq v^\ast \), where \( v^\ast \) is the unique positive solution of the following problem

\[
\begin{cases}
-\Delta v = v \left( \frac{m_1 \Theta_{k_1} (r_1 - \frac{a_1}{e_1})}{1 + b_1 \Theta_{k_1} (r_1 - \frac{a_1}{e_1}) + e_1 v} - r_2 \right) & \text{in } \Omega, \\
k_2 \frac{\partial v}{\partial n} + v = 0 & \text{on } \partial \Omega.
\end{cases}
\]

**Theorem 3.13.** Assume one of the following conditions hold:

(i) \( r_1 > \lambda_{1,k_1} \) and \( \lambda_{1,k_1} \left( - m_2 \Theta_{k_1} (r_1) \right) < r_2 < \lambda_{1,k_3} \);

(ii) \( r_1 > \lambda_{1,k_1} \left( \frac{a_2 \Theta_{k_1} (r_2)}{1 + c_2 \Theta_{k_3} (r_2)} \right) \) and \( r_2 > \lambda_{1,k_3} \).

Then (42) has a positive solution \((u^\ast, w^\ast, v^\ast)\) with \( u^\ast \leq \Theta_k (r_1) \) and \( w^\ast \leq \Theta_k (r_2 + \frac{a_2}{e_2}) \).

Furthermore, if \( r_1 - \frac{a_2}{e_2} > \lambda_{1,k_1} \) and \( r_3 > \lambda_{1,k_3} \left( - m_2 \Theta_{k_1} (r_1 - \frac{a_2}{e_2}) \right) \), then the positive solution \((u^\ast, v^\ast)\) satisfies \( u^\ast \geq \Theta_k (r_1 - \frac{a_2}{e_2}) \) and \( w^\ast \geq w^\ast \), where \( w^\ast \) is the unique positive solution of the following problem

\[
\begin{cases}
-\Delta w = w \left( r_3 - w + \frac{m_2 \Theta_{k_1} (r_1 - \frac{a_2}{e_2})}{1 + b_2 \Theta_{k_1} (r_1 - \frac{a_2}{e_2}) + e_2 w} \right) & \text{in } \Omega, \\
k_3 \frac{\partial w}{\partial n} + w = 0 & \text{on } \partial \Omega.
\end{cases}
\]
Let $S_1 = \{(u^*, v^*, 0)\}$ where $(u^*, v^*)$ is the positive solution of (41) and $S_2 = \{(u^0, 0, w^0)\}$ where $(u^0, w^0)$ is the positive solution of (42). Then by using Theorem 3.4, we can get the following results.

**Lemma 3.14.** Assume that $r_1 > \lambda_{1,k_1}$ and $-r_2 > \lambda_{1,k_2}$.

(i) If $r_3 > \lambda_{1,k_3}(m_{2u^*} - m_{3v^*})$ for any $(u^*, v^*, 0) \in S_1$, then $\deg (I - \mathbb{R}, S_1) = 0$.

(ii) If $r_3 < \lambda_{1,k_3}(m_{2u^*} - m_{3v^*})$ for any $(u^*, v^*, 0) \in S_1$, then $\deg (I - \mathbb{R}, S_1) = 1$.

**Proof.** Recall the definitions of $E$, $W$ and $D$ in section 3, we define $E_1 = C^1_k(\overline{\Omega}) \bigoplus C^1_k(\overline{\Omega})$, $E_2 = C^1_k(\overline{\Omega})$ and denote $W_1 = N_1 \bigoplus N_2$, $W_2 = N_1$. Define

$$\Re_1(u, v, w) = (-\Delta + q)^{-1} \left( u(r_1 - u - \frac{a_1 v}{1+b_1 u+c_1 v} - \frac{a_2 w}{1+b_2 u+c_2 w}) + qu \right),$$

$$\Re_2(u, v, w) = (-\Delta + q)^{-1} \left( w(r_3 - w + \frac{m_2 u^*}{1+b_2 u} + \frac{m_3 v^*}{1+b_3 v}) + qw \right).$$

Then $\mathbb{R} = (\Re_1, \Re_2)$. We choose a neighborhood $U \subset W_1 \cap D$ of $S_1 \cap W_1$ such that $(\Theta_{k_1}(r_1), 0) \notin U$. Now, $\Re_1(u, v, 0) = (u, v)$ with $(u, v) \in U$ if and only if $(u, v, 0) \in S_1$. Then we are in the framework to use Theorem 3.4.

For any $(u^*, v^*, 0) \in S_1$, we have

$$\Re_2'(u^*, v^*, 0)|_{W_2} w = (-\Delta + q)^{-1} \left( q + r_3 + \frac{m_2 u^*}{1+b_2 u} + \frac{m_3 v^*}{1+b_3 v} \right) w.$$}

Notice that $q + r_3 + \frac{m_2 u^*}{1+b_2 u} + \frac{m_3 v^*}{1+b_3 v} > 0$ in $\Omega$ for any $(u^*, v^*, 0) \in S_1$ by our choice of $q$. So, by using maximum principle, we can see that $\Re_2'(u^*, v^*, 0)|_{W_2}$ is $u_0$-positive in the sense of [25] with $u_0 = (-\Delta)^{-1}1$. Hence $\gamma(\Re_2'(u^*, v^*, 0)|_{W_2}) > 0$ and is the only eigenvalue corresponding to a positive eigenvector.

From the definition of $\Re_2'(u^*, v^*, 0)|_{W_2}$, we can easily show that $\gamma(\Re_2'(u^*, v^*, 0)|_{W_2}) > 1$ if and only if $r_2 > \lambda_{1,k_3}(m_{2u^*} - m_{3v^*})$ and $\gamma(\Re_2'(u^*, v^*, 0)|_{W_2}) < 1$ if and only if $r_2 < \lambda_{1,k_3}(m_{2u^*} - m_{3v^*})$. So by Theorem 3.4, we have

$$\deg (I - \mathbb{R}, U \times W_2(\epsilon), 0) = \begin{cases} 0 & \text{if } r_3 > \lambda_{1,k_3}(m_{2u^*} - m_{3v^*}), \\ \deg (I - \mathbb{R}_1, U, 0) & \text{if } r_3 < \lambda_{1,k_3}(m_{2u^*} - m_{3v^*}). \end{cases}$$

(47)

On the other hand, following the results in [11] and [12], we have

$$\deg (I - \mathbb{R}_1, U, 0) = \begin{cases} 1 & \text{if } r_1 > \lambda_{1,k_1} \text{ and } -r_2 > \lambda_{1,k_2}, \\ -1 & \text{if } r_1 < \lambda_{1,k_1} \text{ and } -r_2 < \lambda_{1,k_2}, \\ 0 & \text{if } (r_1 - \lambda_{1,k_1})(-r_2 - \lambda_{1,k_2}) < 0, \end{cases}$$

(48)

Therefore, from (47) and (48) and the conditions of Lemma 3.14, we can get

$$\deg (I - \mathbb{R}, U \times W_2(\epsilon), 0) = \begin{cases} 0 & \text{if } r_3 > \lambda_{1,k_3}(m_{2u^*} - m_{3v^*}), \\ 1 & \text{if } r_3 < \lambda_{1,k_3}(m_{2u^*} - m_{3v^*}). \end{cases}$$

(49)
Since the degree discussed above does not depend on the particular choice of $U$ and $\epsilon$ and $S_1 \neq \emptyset$ implies $r_1 > \lambda_{1,k_1}$ and $-r_2 > \lambda_{1,k_2}$, by using (49), we complete the proof.

Similarly, we can obtain the following lemma by using the similar methods as above.

**Lemma 3.15.** Assume that one of the following conditions hold:

(i) $r_1 > \lambda_{1,k_1}$ and $\lambda_{1,k_3} \left( - \frac{m_2 \Theta_{k_3}(r_1)}{1+b_1 \Theta_{k_1}(r_1)} \right) < r_3 < \lambda_{1,k_3}$;

(ii) $r_1 > \lambda_{1,k_1} \left( \frac{a_2 \Theta_{k_1}(r_1)}{1+c_2 \Theta_{k_3}(r_3)} \right)$ and $r_3 > \lambda_{1,k_3}$.

Then we have

(i) If $-r_2 > \lambda_{1,k_2} \left( - \frac{m_1 u^t}{1+b_1 u^t} + \frac{a_3 w^t}{1+c_3 w^t} \right)$ for any $(u^t, 0, w^t) \in S_2$, then $\text{deg}_W(I-R, S_2) = 0$.

(ii) If $-r_2 < \lambda_{1,k_2} \left( - \frac{m_1 u^t}{1+b_1 u^t} + \frac{a_3 w^t}{1+c_3 w^t} \right)$ for any $(u^t, 0, w^t) \in S_2$, then $\text{deg}_W(I-R, S_2) = 1$.

Based on above analysis, we can give the following results about the existence of coexistence states of (2).

**Theorem 3.16.** If one of the following conditions hold, then (2) has at least one coexistence state.

\[
\begin{align*}
(i) & \\ r_1 > \lambda_{1,k_1}, \\
-r_2 > \lambda_{1,k_2} \left( - \frac{m_1 u^t}{1+b_1 u^t} + \frac{a_3 w^t}{1+c_3 w^t} \right), \\
& \max \left\{ \lambda_{1,k_3} \left( - \frac{m_2 \Theta_{k_3}(r_1)}{1+b_2 \Theta_{k_1}(r_1)} \right), \lambda_{1,k_3} \left( - \frac{m_2 \Theta_{k_3}(r_3)}{1+b_3 \Theta_{k_3}(r_3)} \right) \right\} < r_3 < \lambda_{1,k_3}; \\
& (50) \\
(ii) & \\
& \begin{cases}
 r_1 > \lambda_{1,k_1} \left( \frac{a_2 \Theta_{k_3}(r_3)}{1+c_2 \Theta_{k_3}(r_3)} \right), \\
 -r_2 > \lambda_{1,k_2} \left( - \frac{m_1 u^t}{1+b_1 u^t} + \frac{a_3 w^t}{1+c_3 w^t} \right), \\
 r_3 > \lambda_{1,k_3}; \\
(51)
\end{cases} \\
(iii) & \\
& \begin{cases}
 r_1 > \lambda_{1,k_1}, \\
 \lambda_{1,k_2} \left( - \frac{m_1 \Theta_{k_3}(r_1)}{1+b_1 \Theta_{k_1}(r_1)} \right) < -r_2 < \lambda_{1,k_2} \left( - \frac{m_1 u^t}{1+b_1 u^t} + \frac{a_3 w^t}{1+c_3 w^t} \right), \\
 \lambda_{1,k_3} \left( - \frac{m_2 \Theta_{k_3}(r_1)}{1+b_2 \Theta_{k_3}(r_1)} \right) < r_3 < \lambda_{1,k_3} \left( - \frac{m_2 u^t}{1+b_2 u^t} - \frac{m_3 w^t}{1+b_3 w^t} \right). \\
(52)
\end{cases}
\]

**Proof.** Since $r_1 > \lambda_{1,k_1}$, we can obtain $\text{deg}_W(I-R, D) = 1$ and $\text{ind}_{W}(R, (0, 0, 0)) = 0$ from Lemma 3.9. Thus, it suffices to show that

$$
\text{ind}_{W}(R, (\Theta_{k_1}(r_1), 0, 0)) + \text{ind}_{W}(R, (0, 0, \Theta_{k_3}(r_3))) + \text{deg}_W(I-R, S_1) + \text{deg}_W(I-R, S_2) \neq 1.
$$

(i) Since $r_3 < \lambda_{1,k_3}$, we know $(0, 0, \Theta_{k_3}(r_3))$ does not exist. Note that $u^* \leq \Theta_{k_1}(r_1)$ and $w^* \leq \Theta_{k_1}(r_1)$ hold. We have

$$
-r_2 > \lambda_{1,k_2} \left( - \frac{m_1 u^t}{1+b_1 u^t} + \frac{a_3 w^t}{1+c_3 w^t} \right) > \lambda_{1,k_2} \left( - \frac{m_1 \Theta_{k_3}(r_1)}{1+b_1 \Theta_{k_1}(r_1)} \right),
$$

which implies that $\text{ind}_{W}(R, (\Theta_{k_3}(r_1), 0, 0)) = 0$ from Lemma 3.10. Since $-r_2 > \lambda_{1,k_2} \left( - \frac{m_1 u^t}{1+b_1 u^t} + \frac{a_3 w^t}{1+c_3 w^t} \right)$, we have $\text{deg}_W(I-R, S_2) = 0$ from Lemma 3.15. Since
\[ r_3 > \lambda_{1,k_3} \left( \frac{m_2 u^*}{1+b_2 u^*} - \frac{m_3 v^*}{1+b_3 v^*} \right), \] we have \( \deg_W(I - \mathcal{R}, S_1) = 0 \) from Lemma 3.14. So we can obtain
\[
\text{index}_W(\mathcal{R}, (\Theta_{k_1}(r_1), 0, 0)) + \deg_W(I - \mathcal{R}, S_1) + \deg_W(I - \mathcal{R}, S_2) = 0 \neq 1.
\]

(ii) Note that \( r_3 > \lambda_{1,k_3} > \lambda_{1,k_1} \left( \frac{m_1 \Theta_{k_1}(r_1)}{1+b_1 \Theta_{k_1}(r_1)} \right) \). We know from Lemma 3.11 that \( \text{index}_W(\mathcal{R}, (0, 0, \Theta_{k_3}(r_3))) = 0 \). The remainder of the proof is the same as that of (i).

(iii) Since \( r_1 > \lambda_{1,k_1} \) and \( -r_2 > \lambda_{1,k_2} \left( \frac{m_4 \Theta_{k_2}(r_1)}{1+b_4 \Theta_{k_2}(r_1)} \right) \), we know from Lemma 3.10 that \( \text{index}_W(\mathcal{R}, (\Theta_{k_1}(r_1), 0, 0)) = 0 \). Since \( -r_2 < \lambda_{1,k_2} \left( \frac{m_1 u^*}{1+b_1 u^*} + \frac{a_3 w^*}{1+e_3 w^*} \right) \), \( r_3 > \lambda_{1,k_3} \left( \frac{m_2 \Theta_{k_1}(r_1)}{1+b_2 \Theta_{k_1}(r_1)} - \frac{m_3 v^*}{1+b_3 v^*} \right) \) we have \( \deg_W(I - \mathcal{R}, S_2) = 1 \) from Lemma 3.15. Since \( r_3 < \lambda_{1,k_1} \left( \frac{m_2 u^*}{1+b_2 u^*} - \frac{m_3 v^*}{1+b_3 v^*} \right) \), we know \( (0, 0, \Theta_{k_3}(r_3)) \) does not exist, and \( \deg_W(I - \mathcal{R}, S_1) = 1 \) from Lemma 3.14. So we can obtain
\[
\text{index}_W(\mathcal{R}, (\Theta_{k_1}(r_1), 0, 0)) + \deg_W(I - \mathcal{R}, S_1) + \deg_W(I - \mathcal{R}, S_2) = 2 \neq 1.
\]
The proof is completed. \( \square \)

4. Stability and uniqueness of coexistence states. In this section, we investigate the stability and uniqueness of coexistence states of (2). We first introduce some notations to give our main results.

Let \( u^\dagger = \Theta_{k_1}(r_1) \) be the unique positive solution of the following system
\[
\begin{cases}
-\Delta u = u(r_1 - u) & \text{in } \Omega, \\
k_1 \frac{\partial u}{\partial \nu} + u = 0 & \text{on } \partial \Omega
\end{cases}
\] (53)
when \( r_1 > \lambda_{1,k_1} \).

Denote \( v^\dagger \) be the unique positive solution of the following system
\[
\begin{cases}
-\Delta v = v \left( \frac{m_1 \Theta_{k_1}(r_1)}{1+b_1 \Theta_{k_1}(r_1)} - r_2 \right) & \text{in } \Omega, \\
k_2 \frac{\partial v}{\partial \nu} + v = 0 & \text{on } \partial \Omega
\end{cases}
\] (54)
if \( -r_2 > \lambda_{1,k_2} \left( \frac{m_1 \Theta_{k_1}(r_1)}{1+b_1 \Theta_{k_1}(r_1)} \right) \).

Let \( w^\dagger \) denote the unique positive solution of the following system
\[
\begin{cases}
-\Delta w = w \left( r_3 - w + \frac{m_2 u^\dagger}{1+b_2 u^\dagger + e_2 w} + \frac{m_3 v^\dagger}{1+b_3 v^\dagger + e_3 w} \right) & \text{in } \Omega, \\
k_3 \frac{\partial w}{\partial \nu} + w = 0 & \text{on } \partial \Omega
\end{cases}
\] (55)
if \( r_3 > \lambda_{1,k_3} \left( \frac{m_2 u^\dagger}{1+b_2 u^\dagger} - \frac{m_3 v^\dagger}{1+b_3 v^\dagger} \right) \).

We remark that the existence and uniqueness of positive solution of (53), (54) and (55) follows from Theorem 4. We can also note that the condition (ii) in Theorem 3.16 implies the existence and uniqueness of positive solution of (53), (54) and (55).

The following theorem shows the uniqueness, non-degeneracy and linear stability of the coexistence states for (2) under some assumptions.
Theorem 4.1. Assume that the condition (ii) in Theorem 3.16 hold.

(i) The coexistence state of (2) converge to \((u^i, v^i, w^i)\) as \(a_i \to 0\) for \(i = 1, 2, 3\).

(ii) There exists a positive constant \(\tilde{a} = \tilde{a}(a_1, a_2, a_3)\) such that for \(a_1, a_2, a_3 < \tilde{a}\), (2) has exactly one coexistence state which is non-degenerate and linearly stable.

(iii) The coexistence states of (2) converge to \((\Theta_k_i(r_1), 0, \Theta_k_i(r_3))\) as \(e_i \to \infty\) for \(i = 1, 2, 3\).

(iv) There exists a positive constant \(\tilde{e} = \tilde{e}(e_1, e_2, e_3)\) such that for \(e_1, e_2, e_3 > \tilde{e}\), (2) has exactly one coexistence state which is non-degenerate and linearly stable.

Proof. (i) It is easy to see that the compact operator \(\mathcal{R}(u, v, w)\) defined in Section 3 converges to the operator

\[
\tilde{\mathcal{R}}(u, v, w) = (-\Delta + q)^{-1} \begin{pmatrix}
    u(r_1 - u) + qu \\
    v\left( \frac{m_1 u}{1+b_1 u + e_1 v} - r_2 \right) + qu \\
    w(r_3 - w + \frac{m_2 u}{1+b_2 u + e_2 w} + \frac{m_3 v}{1+b_3 v + e_3 w}) + qw
\end{pmatrix}
\]

as \(a_i \to 0\) for \(i = 1, 2, 3\). So the fixed points of (2) converge to the fixed points of \(\tilde{\mathcal{R}}(u, v, w)\). It can be shown that \((u^i, v^i, w^i)\) is the only fixed point of \(\tilde{\mathcal{R}}(u, v, w)\), the conclusion follows.

(ii) We first show that the coexistence state is non-degenerate and linearly stable. In view of [40, Theorem 11.20], it is sufficient to show that the corresponding linearized eigenvalue problem of (2) has no eigenvalue \(\lambda\) with \(\Re(\lambda) \leq 0\). We argue by contradiction. Suppose that (2) has a coexistence state \((u_i, v_i, w_i)\) which is either degenerate or linearly unstable for sequences \(\{a_{1,i}\}, \{a_{2,i}\}\) and \(\{a_{3,i}\}\) with \(a_{1,i}, a_{2,i}, a_{3,i} \to 0\) as \(i \to \infty\). So there exist \(\lambda_i\) with \(\Re(\lambda_i) \leq 0\) and \((\xi_i, \zeta_i, \eta_i) \neq (0, 0, 0)\) such that

\[
-\Delta \xi_i - \left( r_1 - 2u_i - \frac{a_1 v_i(1+e_1 v_i)}{(1+b_1 u_i + e_1 v_i)^2} - \frac{a_2 w_i(1+e_2 w_i)}{(1+b_2 u_i + e_2 w_i)^2} \right) \xi_i + \frac{a_1 u_i(1+b_1 u_i)}{(1+b_1 u_i + e_1 v_i)^2} \xi_j + \frac{a_2 u_i(1+b_2 u_i)}{(1+b_2 u_i + e_2 w_i)^2} \eta_i = \lambda_i \xi_i \\
-\Delta \zeta_i - \left( \frac{m_1 u_i(1+b_1 u_i)}{(1+b_1 u_i + e_1 v_i)^2} - a_3 w_i(1+e_3 w_i) \right) \zeta_i + \frac{a_3 v_i(1+b_3 v_i)}{(1+b_3 v_i + e_3 w_i)^2} \eta_i = \lambda_i \zeta_i \\
-\Delta \eta_i - \left( \frac{m_2 u_i(1+e_2 w_i)}{(1+b_2 u_i + e_2 w_i)^2} \right) \eta_i - \left( \frac{m_3 w_i(1+b_3 w_i)}{(1+b_3 w_i + e_3 w_i)^2} \right) \eta_i - \left( \frac{m_3 v_i(1+b_3 v_i)}{(1+b_3 v_i + e_3 w_i)^2} \right) \eta_i = \lambda_i \eta_i \text{ in } \Omega, \\
k_1 \frac{\partial \xi_i}{\partial n} + \zeta_i = k_2 \frac{\partial \zeta_i}{\partial n} + \eta_i = k_3 \frac{\partial \eta_i}{\partial n} + \eta_i = 0 \text{ on } \partial \Omega.
\]

Assume that \(||\xi_i||_L^2 + ||\zeta_i||_L^2 + ||\eta_i||_L^2 = 1\). Then we can get from the system (56) that

\[
\lambda_i = \int_\Omega |\nabla \xi_i|^2 dx - \left( r_1 - 2u_i - \frac{a_1 v_i(1+e_1 v_i)}{(1+b_1 u_i + e_1 v_i)^2} - \frac{a_2 w_i(1+e_2 w_i)}{(1+b_2 u_i + e_2 w_i)^2} \right) ||\xi_i||^2 dx + \int_\Omega \frac{a_1 u_i(1+b_1 u_i)}{(1+b_1 u_i + e_1 v_i)^2} \xi_i dx + \int_\Omega \frac{a_2 u_i(1+b_2 u_i)}{(1+b_2 u_i + e_2 w_i)^2} \eta_i dx + \tau_1 \int_{\partial \Omega} |\nabla \xi_i|^2 dx
\]
where $\overline{\xi}$, $\overline{\zeta}$ and $\overline{\eta}$ are the respective complex conjugates of $\xi$, $\zeta$ and $\eta$. Moreover, $\tau_i$ is defined by $\frac{1}{\tau_i}$ for $k_i > 0$ and $0$ for $k_i = 0$. It is easy to show that $\lambda_i \to \lambda$ and then $\Re \lambda \leq 0$. By $L^p$ estimate, we have $\|\xi\|_{W^{2,2}}, \|\zeta\|_{W^{2,2}}$ and $\|\eta\|_{W^{2,2}}$ are bounded and we may assume that $\xi_i \to \xi$, $\zeta_i \to \zeta$ and $\eta_i \to \eta$ in $H^1$ strongly. Then by taking the limit in (56), we can obtain

\[
\begin{align*}
- \Delta \xi - (r_1 - 2u^1)\xi &= \lambda \xi \\
- \Delta \zeta - \frac{m_1 u_1 (1 + e_1 u^1)}{(1 + b_1 u^1 + e_1 v^1)^2} \xi - \left( \frac{m_1 u_1 (1 + b_1 u^1)}{(1 + b_1 u^1 + e_1 v^1)^2} - r_2 \right) \zeta &= \lambda \zeta \\
- \Delta \eta - \frac{m_2 u_1 (1 + e_2 v^1)}{(1 + b_2 u^1 + e_2 v^1)^2} \xi - \left( \frac{m_2 u_1 (1 + e_2 v^1)}{(1 + b_2 u^1 + e_2 v^1)^2} - \frac{m_2 u_1 (1 + b_2 u^1)}{(1 + b_2 u^1 + e_2 v^1)^2} \right) \eta &= \lambda \eta \\
&\text{in } \Omega,
\end{align*}
\]

\[
\begin{align*}
&k_1 \frac{\partial \xi}{\partial \nu} + \xi = k_2 \frac{\partial \zeta}{\partial \nu} + \zeta = k_3 \frac{\partial \eta}{\partial \nu} + \eta = 0 &\text{on } \partial \Omega.
\end{align*}
\]

(57)

Obviously $\lambda \in \mathbb{R}$. If $\xi \neq 0$, then $\lambda = \lambda_{1,k_1} (2u^1 - r_1) = \lambda_{1,k_1} (\Theta_{k_1} (r_1) - r_1) > \lambda_{1,k_1} (\Theta_{k_1} (r_1) - r_1) = 0$. However, $\Re \lambda \leq 0$, which is a contradiction. Hence we have $\xi = 0$. Then it follows from the second equation of (57) that

\[
\begin{align*}
- \Delta \zeta - \left( \frac{m_1 u_1 (1 + b_1 u^1)}{(1 + b_1 u^1 + e_1 v^1)^2} - r_2 \right) \zeta &= \lambda \zeta \\
- \Delta \eta - \left( r_3 - 2u^1 + \frac{m_2 u_1 (1 + e_2 v^1)}{(1 + b_2 u^1 + e_2 v^1)^2} + \frac{m_2 u_1 (1 + b_2 u^1)}{(1 + b_2 u^1 + e_2 v^1)^2} \right) \eta &= \lambda \eta \\
&\text{in } \Omega,
\end{align*}
\]

\[
\begin{align*}
&k_2 \frac{\partial \zeta}{\partial \nu} + \zeta = k_3 \frac{\partial \eta}{\partial \nu} + \eta = 0 &\text{on } \partial \Omega.
\end{align*}
\]

If $\zeta \neq 0$, then $\lambda = \lambda_{1,k_2} r_2 - \frac{m_1 u_1 (1 + b_1 u^1)}{(1 + b_1 u^1 + e_1 v^1)^2} = 0$, which is a contradiction with $\Re \lambda \leq 0$. Hence we have $\zeta = 0$. Then it follows from the third equation of (57) that

\[
\begin{align*}
- \Delta \eta - \left( r_3 - 2u^1 + \frac{m_2 u_1 (1 + b_2 u^1)}{(1 + b_2 u^1 + e_2 v^1)^2} + \frac{m_3 u_1 (1 + b_3 u^1)}{(1 + b_3 u^1 + e_3 v^1)^2} \right) \eta &= \lambda \eta &\text{in } \Omega,
\end{align*}
\]

\[
\begin{align*}
&k_3 \frac{\partial \eta}{\partial \nu} + \eta = 0 &\text{on } \partial \Omega.
\end{align*}
\]
If $\eta \neq 0$, we have
\[
\lambda = \lambda_{1, k_3} \left( 2w^\dagger - r_3 - \frac{m_2u^\dagger (1 + b_2u^\dagger)}{(1 + b_2u^\dagger + e_2w^\dagger)^2} - \frac{m_3u^\dagger (1 + b_3u^\dagger)}{(1 + b_3u^\dagger + e_3w^\dagger)^2} \right)
\]
which is a contradiction. Hence we have $\xi = \zeta = \eta = 0$. This contradiction indicates that the coexistence state of (2) is non-degenerate and linearly stable.

Next, we prove the uniqueness of coexistence state of (2). First the compactness implies that $\mathcal{R}$ has at most finitely many positive fixed points in the region $\mathcal{N}$ defined in Section 4.3 and denote them by $(u_i, v_i, w_i)$ for $i = 1, \ldots, k$. Thus it follows from Theorem 3.3 that $\text{index}(\mathcal{R}(u_i, v_i, w_i)) = (-1)^i = 1$ for $i = 1, \ldots, k$. Then by using the additivity property of the degree, we have
\[
k = \sum_{i=1}^{k} \text{index}_W(\mathcal{R}, (u_i, v_i, w_i)) = \text{deg}_W(I - \mathcal{R}, \mathcal{D}) - \text{index}_W(\mathcal{R}, (0, 0, 0))
\]
as $e_i \to \infty$ for $i = 1, 2, 3$. Hence the fixed points of (2) converge to the fixed points of $\mathcal{R}(u, v, w)$. It can be shown that $(\Theta_{k_1}(r_1), 0, \Theta_{k_3}(r_3))$ is the only fixed point of $\mathcal{R}(u, v, w)$ as in [24] and the conclusion follows.

(iv) As in the proof of (ii), we can derive a contradiction to prove that the coexistence state is non-degenerate and linearly stable and get the uniqueness by using the additivity property of the degree. So we omit the proof.

5. Numerical simulation. In this section, some numerical results on spatial patterns to the system (1) are provided to verify and complement the analytic results in section 4. For simplicity, the simulations are performed on a one dimensional spatial grid with length of $L$ under Dirichlet boundary condition.

\[
\begin{align*}
\begin{cases}
    u_t = u_{xx} + u(r_1 - u) - \frac{a_1uw}{1 + b_1u + e_1v} - \frac{a_2uw}{1 + b_2u + e_2w} \\
    v_t = v_{xx} + \frac{m_1uw}{1 + b_1u + e_1v} - \frac{a_3uw}{1 + b_2u + e_2w} - r_2v \\
    w_t = w_{xx} + w(r_3 - w) + \frac{m_2uw}{1 + b_2u + e_2w} + \frac{m_3uw}{1 + b_3u + e_3w} \quad (x, t) \in (0, L) \times \mathbb{R}^+,
\end{cases}
\end{align*}
\]

The initial conditions consist of random perturbations about the homogeneous state of $u_0 = 1, v_0 = 1, w_0 = 0.5$. Here we point out that the characteristics of
stationary state remain unchanged even the initial conditions are displace by the other forms such as $u_0 = |\sin x|, v_0 = |\cos x|, w_0 = 0.5$. The system (1) will be solved with the approach of discretizing the diffusion operator in space, applying forward Euler integration to the finite-difference equations with time steps $\Delta t = 0.002$. We run the simulations until they reach a stationary state or until they show a behavior that does not seem to change its characteristics anymore.

5.1. Effect of growth rate. Here we first investigate the effects of $r_1$, the growth rate of resource, on distribution of species. To proceed, we vary the values of $r_1$ with the other parameters fixed. Fig. 1 illustrates the spatial distribution of resource, IGprey and IGpredator in a one dimension region with different levels of growth rates $r_1$ for resource. One can see that, the population IGprey is in the state of extinction when the growth rate of resource is low (Fig. 1(a)). With the increase of growth rate (bigger $r_1$), the population IGprey can coexist with resource and IGpredator (Fig. 1(b)). One can also note that larger growth rate of resource leads to larger mass of IGprey and IGpredator (Fig. 1(c)). Fig. 2 illustrates the spatial distribution of resource, IGprey and IGpredator in a one dimension region with different levels of $r_2$. Here we recall that the parameter $r_2$ describes the death rate of IGprey if $r_2 > 0$ and the immigration rate of IGprey if $r_2 < 0$. One can see that, the population IGprey is driven to extinction when the death rate $r_2$ is high (Fig. 2(a)). With the decrease of death rate of IGprey (smaller $r_2$), the
population IGprey can coexist with resource and IGpredator (Fig. 2(b)). When there is immigration of IGprey, that is \( r_2 < 0 \), one can note that larger mass of IGprey can coexist with resource and IGpredator (Fig. 2(c)). Fig. 3 illustrates the spatial distribution of resource, IGprey and IGpredator in a one dimension region with different levels of \( r_3 \). Here we recall that the parameter \( r_3 \) describes exclusive resource of IGpredator if \( r_3 > 0 \) and the death rate of IGpredator if \( r_3 < 0 \). When there is no exclusive resource of IGpredator, that is \( r_3 < 0 \) and describe death rate, one can see that the population IGpredator is driven to extinction when the death rate is high (Fig. 3(a)). With the decrease of death rate of IGprey (\( r_3 < 0 \) with smaller absolute value), the population IGpredator can coexist with source and IGpredator (Fig. 3(b)). When there is exclusive resources for IGpredator, that is \( r_3 > 0 \), one can note that the IGprey is driven to extinction by the growth of IGpredator (Fig. 3(c)).

One can conclude from Fig. 1 that, under some circumstances, the increase of the growth rate \( r_1 \) of resource mediate the coexistence of species in system, while it follows from Fig. 2 that the decrease of death rate \( r_2 \) of IGprey mediate the coexistence. The effects of the parameter \( r_3 \) can be complex and it leads to the extinction of IGpredator if it is small enough, leads to the extinction of IGprey if it is large enough and to coexistence of species when it lies in the mediate space.

5.2. Effect of interference of predators. Here we first investigate the effects of \( e_2 \), which describes that mutual interference among individual of IGpredators [29,30], on distribution of species. To proceed, we vary the values of \( e_2 \) with the other parameters fixed. Fig. 4 illustrates the spatial distribution of source, IGprey and IGpredator in a one dimension region with different interference levels \( e_2 \) among IGpredators. One can see that, the population IGprey is in the state of extinction when interference level of predators is very low (Fig. 4(a)). With the increase of interference levels among IGpredators (bigger \( e_2 \)), the population IGprey can coexist with resource and IGpredator (Fig. 4(b)). One can also note that stronger interference among predators leads to larger mass of IGprey and smaller mass of IGpredator (Fig. 4(c)). Similarly, Fig. 5 illustrates the spatial distribution of resource, IGprey and IGpredator in a one dimension region with different interference levels \( e_3 \) among IGpredators. Fig. 6 illustrates the spatial distribution of source, IGprey and IGpredator in a one dimension region with different interference levels among IGprey. One can see that, the population IGprey can coexist with resource and IGpredator when interference level among IGprey is very low (Fig. 6(a)). With the
increase of interference level among IGprey (bigger $e_1$), the mass of population IGprey decrease (Fig.6(b)) and turn to extinction at last (Fig.6(c)). Fig.7 illustrates the spatial distribution of source, IGprey and IGpredator in a one dimension region with different interference levels $e_2$ and $e_3$ among predators. One can see that, the population IGpredator can coexist with resource and IGprey when interference levels among IGpredators is very low (Fig.7(a)). With the increase of interference levels of IGpredators (bigger $e_2$ and $e_3$), the mass of population IGpredator decreases (Fig.7(b)) and turn to extinction at last (Fig.7(c)).
One can conclude from Fig.4 and Fig.5 that different interference levels $e_2$ and $e_3$ among predators can lead to coexistence of species under some circumstances; while from Fig.6 and Fig.7, we see that they can also lead to extinction of species under some other circumstances.

5.3. Effect of intraguild predation. Here we investigate the effects of $a_2$ and $m_2$, which describes Intraguild predation in a food chain model, on distribution of species. To proceed, we vary the values of $a_2$ and $m_2$ with the other parameters fixed. Fig.8 illustrates the spatial distribution of resource, IGprey and IGpredator in a one dimension region with different intraguild predation strengths $a_2$.

![Figure 7. Extinction of IGpredator driven by $e_2$ and $e_3$ with fixed parameter values $a_1 = 5, a_3 = 3, a_3 = 2, m_1 = 2.5, m_2 = 1.5, m_3 = 1, b_1 = b_2 = b_3 = 0.2, r_1 = 2, r_2 = 0.3, r_3 = -0.5, e_1 = 1.$](image)

![Figure 8. Coexistence induced by IGP ($a_2$ and $m_2$) with fixed parameter values $a_1 = 5, a_3 = 0.5, m_1 = 2.5, m_3 = 0.25, b_1 = b_2 = b_3 = 0.2, r_1 = 2, r_2 = 0.1, r_3 = -0.15, e_1 = e_2 = e_3 = 10.$](image)
population IGprey coexist with resource and IGpredator (Fig.9(a)). With the introduction of IGpredation (bigger $a_2$), the mass of the population IGprey decrease (Fig.9(b)) and turn to the state of extinction at last (Fig.9(c)).

From Fig.8 and Fig.9 above, we can see that the introduction of IGpredation can lead to coexistence or extinction of species under some circumstances.

6. Conclusions and discussions. In recent years, the study of the dynamic relationship between predator and prey with IGP has long been one of the most important themes in population dynamics because of its universal existence in nature. Extinction and coexistence of species are two important issues in systems with IGP. Most studies about predator and prey models with IGP were assumed in a homogeneous environment in space and different ODE models were set up. In this paper, we constructed a mathematical model with IGP in the presence of diffusion and B-D functional response between the predator and prey. Here we assumed that the IGpredator may be a generalist with a exclusive resources or just a specialist. We obtained some sufficient conditions for the extinction and permanence of the time-dependent system. We also obtained some necessary and sufficient conditions for the existence of coexistence states and further discussed the uniqueness and stability of coexistence state under some special conditions. The existence of a non-constant time-independent positive solution, also called stationary pattern, always indicates the dynamical richness of the system. We found that the dynamic relationship between predator and prey with IGP may be more complex than that of the models without IGP such as food chain or two predators one prey models.

Generally, it is not enough to just know some conditions on extinction and existence of coexistence states and we hope to know how the parameters in system affect the distribution of species. Obviously, it is difficult to obtain these results by just using the analytic methods. Thus by using numerical simulations, we studied the effects of the parameters in model on the spatial distribution of species and obtain some interesting results about the extinction and coexistence of species. For example, we found that different interference levels among predators can both lead to coexistence of species under some circumstances and to extinction of species under some other circumstances. We also found that the introduction of IGpredation in a food chain model and the growth rate (or death rate) of IGpredator have similar effects on the extinction and coexistence of species. These results verify and complement the analytic results in section 4.
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