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Abstract

In this paper we establish Springer correspondence for the symmetric pair $(\text{SL}(N), \text{SO}(N))$ using Fourier transform, parabolic induction functor, and a nearby cycle sheaf construction. As an application of our results we see that the cohomology of Hessenberg varieties can be expressed in terms of irreducible representations of Hecke algebras of symmetric groups at $q = -1$. Conversely, we see that the irreducible representations of Hecke algebras of symmetric groups at $q = -1$ arise in geometry.

1. Introduction

In this paper we consider the Springer correspondence in the case of symmetric spaces. We will concentrate on the split case of type $A$, i.e., the case of $\text{SL}(n, \mathbb{R})$. The case of $\text{SL}(n, \mathbb{H})$ was considered by Henderson in [Hen01], Grinberg in [Gri98] and Lusztig in [Lus11a], and the case of $\text{U}(p,q)$ was considered by Lusztig in [Lus11b] where he treats the general case of semisimple inner automorphisms. In both of these cases Springer theory closely resembles the classical situation. This turns out not to be so in the split case we consider here. In [CVX15a, CVX15b] we have computed Fourier transforms of IC sheaves supported on certain nilpotent orbits using resolutions of singularities of nilpotent orbit closures. In this paper we study the problem in general in the split case of type $A$ replacing the resolutions with a nearby cycle sheaf construction in [GVX18] based on earlier ideas of Grinberg [Gri01, Gri98].

Let us call an irreducible IC sheaf supported on a nilpotent orbit a nilpotent orbital complex. We show that the Fourier transform gives a bijection between nilpotent orbital complexes and certain representations of (extended) braid groups. We identify these representations of (extended) braid groups and construct them explicitly in terms of irreducible representations of Hecke algebras of symmetric groups at $q = -1$. This bijection can be viewed as Springer correspondence for the symmetric pair $(\text{SL}(N), \text{SO}(N))$. Let us note that the fact that representations of (affine) Hecke algebras at $q = -1$ arise in this situation was already observed by Grojnowski in his thesis [Gro92].

The proof of our main result, Theorem 4.1, makes use of a nearby cycle sheaf construction in [GVX18] and smallness property of maps associated to certain $\theta$-stable parabolic subgroups. The nearby cycle sheaves produce nilpotent orbital complexes whose Fourier transforms have

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full support. Those IC sheaves behave like ‘cuspidal sheaves’ in the sense that they do not
appear as direct summands of parabolic inductions. On the other hand, the smallness property
mentioned above implies a simple description of the images of parabolic induction functors
(Propositions 3.1, 3.2). Those results together with a counting lemma (Lemma 4.2) imply
Theorem 4.1. As corollaries, we obtain criteria for nilpotent orbital complexes to have full support
Fourier transforms (Corollaries 4.8, 4.9) and results on cohomology of Hessenberg varieties
(Theorem 5.2).

Our method appears to be applicable to general symmetric pairs and, more generally, polar
representations studied in [Gri98] and we hope to return to this in future work.

Let us mention that in [LY17], the authors show that one can obtain all nilpotent orbital
complexes using spiral induction functors introduced in [LY17] (in fact, they consider more
general setting of cyclically graded Lie algebras). Using their results and Theorem 4.1, we show
that all irreducible representations of Hecke algebras of symmetric groups at $q = -1$ appear in
the intersection cohomology of Hessenberg varieties, with coefficient in certain local systems (see
Theorem 6.3). This gives geometric constructions of irreducible representations of Hecke algebras
of symmetric groups at $q = -1$ and provides them with a Hodge structure.

The paper is organized as follows. In §2 we recall some facts about symmetric pairs and
introduce a class of representations of equivariant fundamental groups. Moreover, we recall the
definition of the parabolic induction functor. In §3 we study parabolic induction functors for
certain $\theta$-stable parabolic subgroups. In §4, we prove Theorem 4.1: the Fourier transform defines
a bijection between the set of nilpotent orbital complexes and the class of representations of
equivariant fundamental groups introduced in §2. In §§5 and 6, we discuss applications of our
results to cohomology of Hessenberg varieties and representations of Hecke algebras of symmetric
groups at $q = -1$. Finally, in §7, we propose a conjecture that gives a more precise description
of the bijection in Theorem 4.1.

2. Preliminaries

For convenience we work over $\mathbb{C}$. We adopt the usual convention of cohomological degrees for
perverse sheaves by having them be symmetric around 0. We also use the convention that all
functors are derived, so we write, for example, $\pi_*$ instead of $R\pi_*$. If $X$ is smooth we write $\mathbb{C}_X[-]$ for the constant sheaf placed in degree $-\dim X$ so that $\mathbb{C}_X[-]$ is perverse. If $U \subset X$ is a smooth
open dense subset of a variety $X$ and $\mathcal{L}$ is a local system on $U$, we write $\text{IC}(X, \mathcal{L})$ for the
IC-extension of $\mathcal{L}[-]$ to $X$; in particular, it is perverse. For simplicity of notation, when we have
a pair $(\mathcal{O}, \mathcal{E})$, where $\mathcal{O}$ is an orbit and $\mathcal{E}$ is a local system on $\mathcal{O}$, we also write $\text{IC}(\mathcal{O}, \mathcal{E})$ instead
of $\text{IC}(\mathcal{O}, \mathcal{E})$.

2.1 Notation

For $e \geq 2$, a partition $\lambda$ of a positive integer $k$ is called $e$-regular if the multiplicity of any part of
$\lambda$ is less than $e$. In particular, a partition is 2-regular if and only if it has distinct parts. Let us
denote by $\mathcal{P}(k)$ the set of all partitions of $k$ and by $\mathcal{P}_2(k)$ the set of all 2-regular partitions of $k$.

We denote by $\mathcal{H}_{k,-1}$ the Hecke algebra of the symmetric group $S_k$ with parameter $-1$. More
precisely, $\mathcal{H}_{k,-1}$ is the $\mathbb{C}$-algebra generated by $T_i, \ i = 1, \ldots, k-1$, with the following relations

\begin{equation}
T_i T_j = T_j T_i \quad \text{if} \ |i - j| \geq 2, \ i, j \in [1, k-1], \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad i \in [1, k-2], \\
T_i^2 = q + (q - 1) T_i, \quad \text{where} \ q = -1, \ i \in [1, k-1].
\end{equation}
It is shown in [DJ86] that the set of isomorphism classes of irreducible representations of \( \mathcal{H}_{k,-1} \) is parametrized by \( \mathcal{P}_2(k) \). For \( \mu \in \mathcal{P}_2(k) \), we write \( D_\mu \) for the irreducible representation of \( \mathcal{H}_{k,-1} \) corresponding to \( \mu \).

For a real number \( a \), we write \([a]\) for its integer part.

### 2.2 The split symmetric pair \((\text{SL}(N), \text{SO}(N))\)

Let \( G = \text{SL}(N) \) and \( \theta : G \to G \) an involution such that \( K = G^\theta = \text{SO}(N) \) and write \( g = \text{Lie}G \). We have \( g = g_0 \oplus g_1 \), where \( \theta|_{g_1} = (-1)^i \). The pair \((G, K)\) is a split symmetric pair, i.e., there exists a maximal torus \( A \) of \( G \) that is \( \theta \)-split, where \( \theta \)-split means that for all \( x \in A \), \( \theta(x) = x^{-1} \).

We also think of the pair \((G, K)\) concretely as \((\text{SL}(V), \text{SO}(V))\), where \( V \) is a vector space of dimension \( N \) equipped with a non-degenerate quadratic form \( Q \) such that \( \text{SO}(V) = \text{SO}(V, Q) \). We write the non-degenerate bilinear form associated to \( Q \) as \( \langle \, , \, \rangle \).

Let \( g^{\text{reg}} \) denote the set of regular elements in \( g \) and let \( g^{\text{rs}}_1 = g_1 \cap g^{\text{rs}} \). Similarly, let \( g^{\text{reg}}_1 \) denote the set of regular semisimple elements in \( g \) and let \( g^{\text{reg}}_1 = g_1 \cap g^{\text{reg}}_1 \).

Let \( \mathcal{N}_1 \) be the nilpotent cone of \( g \) and let \( \mathcal{N}_1 = \mathcal{N} \cap g \). When \( N \) is odd, the set of \( K \)-orbits in \( \mathcal{N}_1 \) is parametrized by \( \mathcal{P}(N) \). When \( N \) is even, the set of \( O(N) \)-orbits in \( \mathcal{N}_1 \) is parametrized by \( \mathcal{P}(N) \), moreover, each \( O(N) \)-orbit remains one \( K \)-orbit if \( \lambda \) has at least one odd part, and splits into two \( K \)-orbits otherwise. For \( \lambda \in \mathcal{P}(N) \), we write \( O_\lambda \) for the corresponding nilpotent \( K \)-orbit in \( \mathcal{N}_1 \) when \( \lambda \) has at least one odd part, and write \( O_\lambda^1 \) and \( O_\lambda^2 \) for the corresponding two nilpotent \( K \)-orbits in \( \mathcal{N}_1 \) when \( \lambda \) has only even parts. Suppose that \( \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s > 0) \).

For \( x \in O_\lambda \) (or \( O_\lambda^1, \omega = I, II \)), we have

\[
\dim Z_K(x) = \sum_{i=1}^{s} (i - 1) \lambda_i. \tag{2.2}
\]

Let \( a \) be a maximal abelian subspace of \( g_1 \), which is also a Cartan subspace of \( g \). We have the ‘little’ Weyl group

\[ W = N_K(a)/Z_K(a) = S_N. \]

#### 2.3 Equivariant fundamental group and its representations

As was discussed in [CVX15a], the equivariant fundamental group

\[ \pi^K_1(g^{\text{rs}}_1) \cong Z_K(a) \times B_N \cong (\mathbb{Z}/2\mathbb{Z})^{N-1} \times B_N, \]

where \( B_N \) is the braid group of \( N \) strands and it acts on

\[ Z_K(a) \cong \left\{ (i_1, \ldots, i_N) \in (\mathbb{Z}/2\mathbb{Z})^N \mid \sum_{k=1}^{N} i_k = 0 \right\} \cong (\mathbb{Z}/2\mathbb{Z})^{N-1} \]

via the natural map \( B_N \to S_N \). For simplicity we write

\[ \overline{B}_N = (\mathbb{Z}/2\mathbb{Z})^{N-1} \rtimes B_N \quad \text{and} \quad I_N = (\mathbb{Z}/2\mathbb{Z})^{N-1}. \]

It is easy to see that the action of \( B_N \) on \( I_N^\prime \) has \([N/2] + 1 \) orbits. We choose a set of representatives

\[ \chi_m \in I_N^\prime, \quad 0 \leq m \leq [N/2], \]

of the \( B_N \)-orbits as follows. Let \( \tau_i' \in (\mathbb{Z}/2\mathbb{Z})^N \) be the element with all entries 0 except the \( i \)th position. Then \( \{ \tau_i = \tau_i' + \tau_{i+1}', \quad i = 1, \ldots, N-1 \} \), is a set of generators for \( I_N \). For \( 0 \leq m \leq [N/2] \), we define a character \( \chi_m \) as follows:

\[ \chi_m(\tau_m) = -1 \quad \text{and} \quad \chi_m(\tau_i) = 1 \quad \text{for} \ i \neq m. \tag{2.3} \]

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For \( \chi \in I_N^\chi \), we set
\[
B_\chi = \text{Stab}_{B_N} \chi.
\]

Let \( s_i, i = 1, \ldots, N - 1 \), be the simple reflections in \( W = S_N \). It is easy to check that
\[
\text{Stab}_{S_N}(\chi_m) = \langle s_i, i \neq m \rangle \cong S_m \times S_{N-m} \quad \text{if } m \neq N/2,
\]
and
\[
\text{Stab}_{S_N}(\chi_m) \text{ contains } S_m \times S_m \text{ as an index-2 normal subgroup if } m = N/2.
\]

Let us define
\[
B_{m,N-m} = \text{the inverse image of } S_m \times S_{N-m} \cong \langle s_i, i \neq m \rangle \text{ under the map } B_N \to S_N.
\]

Then it follows from (2.4) that
\[
B_{\chi_m} = B_{m,N-m}, \quad \text{when } m \neq N/2,
\]
and \( B_{\chi_m} \) contains \( B_{m,N-m} \) as an index-2 normal subgroup when \( m = N/2 \).

Let \( \sigma_i, i = 1, \ldots, N - 1 \), be the standard generators of \( B_N \) which are lifts of the \( s_i \) under the map \( B_N \to S_N \). Then \( B_{m,N-m} \) is generated by \( \sigma_i, i \neq m \), and \( \sigma_m^2 \). We have a natural quotient map
\[
\mathbb{C}[B_{m,N-m}] \twoheadrightarrow \mathcal{H}_{m,-1} \otimes \mathcal{H}_{N-m,-1} \cong \mathbb{C}[B_{m,N-m}]/((\sigma_i - 1)^2, i \neq m, \sigma_m^2 - 1).
\]

Note that in the above formula the \( \sigma_i \) corresponds to \(-T_i\) of the Hecke algebra in (2.1). For us the \( \sigma_i \) are more natural generators since they arise from geometry as unipotent monodromy operators.

Let us write
\[
\mathcal{H}_{m,-1} \otimes \mathcal{H}_{N-m,-1} = \mathcal{H}_{\chi_{m,-1}}.
\]

We consider a family of representations of \( \tilde{B}_N \) as follows. For \( 0 \leq m \leq \lfloor N/2 \rfloor \), we define
\[
L_{\chi_m} := \text{Ind}_{\mathbb{C}[B_{m,N-m}]}^{\mathbb{C}[B_N]} \mathcal{H}_{\chi_{m,-1}} \cong \mathbb{C}[B_N] \otimes_{\mathbb{C}[B_{m,N-m}]} \mathcal{H}_{\chi_{m,-1}},
\]
where in the tensor product \( \mathbb{C}[B_{m,N-m}] \) acts on \( \mathcal{H}_{\chi_{m,-1}} \) via the quotient map (2.6) and on \( \mathbb{C}[B_N] \) by right multiplication. The module \( L_{\chi_m} \) has a natural \( \tilde{B}_N \)-action defined as follows. We let \( B_N \) act on \( L_{\chi_m} \) by left multiplication and we let \( I_N \) act on \( L_{\chi_m} \) via \( a.(b \otimes v) = ((b, \chi_m)(a))(b \otimes v) \) for \( a \in I_N, b \in B_N \) and \( v \in \mathcal{H}_{\chi_{m,-1}} \). We view \( L_{\chi_m} \) as a representation of the equivariant fundamental group \( \tilde{B}_N \) in this manner.

We will next identify the composition factors of the modules \( L_{\chi_m} \). Let \( \mu^1 \in \mathcal{P}_2(m) \) and \( \mu^2 \in \mathcal{P}_2(N - m), m \in [0, \lfloor N/2 \rfloor] \). Proceeding just as in the definition of \( L_{\chi_m} \), one obtains the following representation of \( \tilde{B}_N \):
\[
V_{\mu^1, \mu^2} := \text{Ind}_{\mathbb{C}[B_{m,N-m}]}^{\mathbb{C}[B_N]} (D_{\mu^1} \otimes D_{\mu^2}) \cong \mathbb{C}[B_N] \otimes_{\mathbb{C}[B_{m,N-m}]} (D_{\mu^1} \otimes D_{\mu^2}).
\]

Using (2.5), one readily checks that \( V_{\mu^1, \mu^2} \) is an irreducible representation of \( \tilde{B}_N \) when \( m \neq N/2 \), or when \( m = N/2 \) and \( \mu^1 \neq \mu^2 \). When \( m = N/2 \) and \( \mu^1 = \mu^2 \), \( V_{\mu^1, \mu^2} \) breaks into the direct sum of two non-isomorphic irreducible representations of \( \tilde{B}_N \), which we denote by \( V^I_{\mu^1, \mu^2} \) and \( V^II_{\mu^1, \mu^2} \), i.e., we have
\[
V_{\mu, \mu} \cong V^I_{\mu, \mu} \oplus V^II_{\mu, \mu}.
\]

Moreover,
\[
\begin{align*}
\text{when } m &\neq N/2, \quad V_{\mu^1, \mu^2} \cong V^I_{\mu^1, \mu^2} \quad \text{if and only if } (\mu^1, \mu^2) = (\nu^1, \nu^2); \\
\text{when } m &= N/2, \quad V^I_{\mu^1, \mu^2} \cong V^II_{\mu^1, \mu^2} \quad \text{if and only if } \left(\mu^1, \mu^2\right) = \left(\nu^1, \nu^2\right);
\end{align*}
\]

either \( (\mu^1, \mu^2) = (\nu^1, \nu^2) \) or \( (\mu^1, \mu^2) = (\nu^2, \nu^1) \).

As the \( D_{\mu^1} \otimes D_{\mu^2} \) are the composition factors of \( \mathcal{H}_{\chi_{m,-1}} \) we conclude the following.
Consider the diagram be the natural projection map, where defined in [Hen01, Lus11b]. In the following we recall its definition. Let We will make use of the parabolic induction functor where p
2.1. Lemma Let 2.4 Parabolic induction functor and Fourier transform Let K
3. Maximal \( \theta \)-stable parabolic subgroups and parabolic induction
where $V_m^0 = \text{span}\{e_i, 1 \leq i \leq m\}$. We define $L_m$ to be the $\theta$-stable Levi subgroup of $P_m$ which consists of diagonal block matrices of sizes $m$, $N-2m$, $m$. When $N = 2n$, for $\omega = I, II$, we define $P_{m, \omega}$ to be the parabolic subgroup of $G$ that stabilizes the flag

$$0 \subset V_n^\omega \subset V_n^{\perp} \subset \mathbb{C}^{2n},$$

where $V_n^1 = \text{span}\{e_i, 1 \leq i \leq n\}$ and $V_n^{1\perp} = \text{span}\{e_i, 1 \leq i \leq n-1, e_{n+1}\}$. Let $L_{n, \omega}$ be a $\theta$-stable Levi subgroup of $P_{n, \omega}$. According to [BH00], every maximal $\theta$-stable parabolic subgroup of $G$ is $K$-conjugate to one of the above form.

Let $\mathfrak{p}_m = \text{Lie} P_m$, $\mathfrak{p}_m^1 = \mathfrak{p}_m \cap \mathfrak{g}_1$, and $(\mathfrak{n}_{P_m})_1 = \mathfrak{n}_{P_m} \cap \mathfrak{g}_1$, where $\mathfrak{n}_{P_m}$ is the nilpotent radical of $\mathfrak{p}_m$, etc.

**Proposition 3.1.** We have the following.

(i) The map

$$\pi_m^N : K \times P_m^\mathfrak{n} (\mathfrak{n}_{P_m})_1 \to N_1, \quad (k, x) \mapsto \text{Ad} k(x)$$

is a small map onto its image, generically one-to-one.

(ii) The map

$$\pi_m^{n, \omega} : K \times P_m^\mathfrak{n} \mathfrak{p}_1^m \to \mathfrak{g}_1, \quad (k, x) \mapsto \text{Ad} k(x).$$

is a small map onto its image, generically one-to-one.

The same holds for the two maps $\pi_n^{2n, \omega}$ and $\pi_n^{2n, \omega}$ defined using $P_{n, \omega}$, $\omega = I, II$.

We define

$$\mathfrak{g}_1^m = \text{Im} \pi_m^N, \quad 1 \leq m < N/2, \quad \mathfrak{g}_1^{n, \omega} = \text{Im} \pi_n^{2n, \omega}, \quad \omega = I, II. \quad (3.1)$$

For $m < N/2$, $\mathfrak{g}_1^m$ consists of elements in $\mathfrak{g}_1$ with eigenvalues $a_1, a_1, \ldots, a_m, a_m, a_j, j \in [2m+1, N]$, where $\sum_{k=1}^m 2a_k + \sum_{j=2m+1}^N a_j = 0$. Let

$$Y_m^r = \{ x \in \mathfrak{g}_1^{\text{reg}} \mid x \text{ has eigenvalues } a_1, a_1, \ldots, a_m, a_m, a_j, j \in [2m+1, N],$$

where $a_i \neq a_j$ for $i \neq j$.

One checks readily that $Y_m^r = \mathfrak{g}_1^m$.

Consider the case $m = N/2 = n$. For $\omega = I, II$, let

$$Y_n^{r, \omega} = \{ x \in \mathfrak{g}_1^{\text{reg}} \mid x \text{ has eigenvalues } a_1, a_1, \ldots, a_n, a_n, \text{ where } a_i \neq a_j \text{ for } i \neq j, \text{ and the nilpotent part of } x \text{ lies in the orbit } \mathcal{O}_n^{\omega} \},$$

where $\mathcal{O}_n^{\omega}$ is the nilpotent orbit given by the partition $2m$ and defined by the equation $\text{Im} \pi_n^{2n, \omega} = \mathcal{O}_n^{\omega}$. Then $Y_n^{r, \omega}$ is an open dense subset of $\mathfrak{g}_1^{n, \omega}$.

Let $(\mathfrak{p}_1)^m_r = \mathfrak{p}_1^m \cap Y_m^r$ and $(\mathfrak{i}_1)^{rs} = \mathfrak{i}_1^m \cap (\mathfrak{i}_1)^{rs}$.

**Proposition 3.2.** (1) Suppose that $1 \leq m < N/2$. There is a natural surjective map

$$\pi_1^K Y_m^r : \pi_1^K Y_m^r \to \pi_1^L (\mathfrak{i}_1^m)^{rs} \cong B_m \times \bar{B}_{N-2m} \quad (3.2)$$

such that for an $L_K^m$-equivariant local system $\mathcal{T}$ on $(\mathfrak{i}_1^m)^{rs}$ associated to a $\pi_1^L (\mathfrak{i}_1^m)^{rs}$-representation $E$, we have

$$\text{Ind}_{\pi_1^L (\mathfrak{i}_1^m)^{rs}}^{\pi_1^L (\mathfrak{i}_1^m)^{rs}} \text{IC}(\mathfrak{i}_1^m, \mathcal{T}) \cong \text{IC}(\mathfrak{g}_1^m, \mathcal{T}).$$
where \( T' \) is the \( K \)-equivariant local system on \( Y_\omega \) associated to the representation of \( \pi_1^K(Y_\omega) \) which is obtained from \( E \) by pull-back under the map (3.2).

(2) We have a natural surjective map

\[
\pi_1^K(Y_\omega) \to \pi_1^{L_1^\omega}, \quad \omega = I, II,
\]

(3.3)
such that for an \( L_1^\omega \)-equivariant local system \( T \) on \( (L_1^\omega) \) associated to a \( \pi_1^{L_1^\omega} \)-representation \( E \), we have

\[
\text{Ind}_{(L_1^\omega)}^{L_1^\omega} \text{IC}(T_1^\omega, T) \cong \text{IC}(E_1^\omega, T'),
\]

where \( T' \) is the \( K \)-equivariant local system on \( Y_\omega \) associated to the representation of \( \pi_1^K(Y_\omega) \) which is obtained from \( E \) by pull-back under the map (3.3).

### 3.1 Proof of Proposition 3.1

We begin with the proof of (1). Consider the following projection

\[
\tau_N^m : \{(x, 0 \subset V_m \subset V) | x \in g_1, xV_m = 0, xV_m^\perp \subset V \} \to N_1.
\]

When \( m \neq N/2 \), the map \( \tau_N^m \) can be identified with the map \( \tau_N^m \). When \( N = 2m \), the image of the map \( \tau_N^{2m} \) has two irreducible components, i.e., closures of the two orbits \( O_{2m}^I \) and \( O_{2m}^{II} \). The two maps \( \tau_N^{2m} \) and \( \tau_N^{2m} \) can be identified with the map \( \tau_N^{2m} \) restricted to \( (\tau_N^{2m})^{-1}(O_{2m}^I) \) and \( (\tau_N^{2m})^{-1}(O_{2m}^{II}) \) respectively. Thus it suffices to show that

the map \( \tau_N^m \) is small over its image and generically one-to-one.

(3.4)

When \( m \neq N/2 \), one can check that the image of \( \tau_N^m \) is as follows

\[
\text{Im} \tau_N^m = \bar{O}_{3m} \quad \text{if} \quad N \geq 3m, \quad \text{Im} \tau_N^m = \bar{O}_{3m} \quad \text{if} \quad N < 3m.
\]

Assume that \( N \geq 3m \) and \( x \in O_{3m} \). Then \( \tau_N^{2m}(x) \) consists of the flag \( 0 \subset V_m \subset V \), where \( V_m = \text{Im} x^2 \). Assume that \( N < 3m \) and \( x \in O_{3m} \). Then \( \tau_N^{2m}(x) \) consists of the flag \( 0 \subset V_m \subset V \), where \( V_m = \ker x \). This proves that \( \tau_N^m \) is generically one-to-one.

Let \( x \in O_{3m} \). We assume that \( 3^{m}2^{1N-3i-2j} \neq 3^{m}1^{N-3m} \) if \( N \geq 3m \), and \( 3^{i}2^{1N-3i-2j} \neq 3^{m}2^{m}2^{3m-N} \) if \( N < 3m \). It suffices to show that

\[
\dim(\tau_N^{m})^{-1}(x) < \text{codim}_{\text{Im} \tau_N^m} O_{3^{i}2^{1N-3i-2j}}/2.
\]

Let \( x_0 \in O_{3^{i}2^{1N-3i-2j}} \subset \text{Im} \tau_N^{m-i} \). (Note that \( \tau_N^{m-i} \) is defined since \( m - i \leq (N - 3i)/2 \).) One checks readily that (using (2.2) for the second equality)

\[
(\tau_N^{m})^{-1}(x) = (\tau_N^{m-i})^{-1}(x_0) \quad \text{and} \quad \text{codim}_{\text{Im} \tau_N^m} O_{3^{i}2^{1N-3i-2j}} = \text{codim}_{\text{Im} \tau_N^{m-i}} O_{2^{1N-3i-2j}}.
\]

Thus it suffices to show that

\[
\dim(\tau_N^{m-i})^{-1}(x_0) < \text{codim}_{\text{Im} \tau_N^{m-i}} O_{2^{1N-3i-2j}}/2.
\]

Let us write

\[
\Omega_N^{m-j} = (\tau_N^{m-i})^{-1}(\zeta_j) \quad \text{for} \quad \zeta_j \in O_{2^{1N-2j}} \subset \text{Im} \tau_N^N
\]

and

\[
a_N^{m,j} = \text{codim}_{\text{Im} \tau_N^N} O_{2^{1N-2j}} = m(2N - 3m) - j(N - j).
\]
To prove that the map $\tau_m^N$ is small, we are reduced to proving that

$$\dim \Omega_{m,j}^N < \frac{a_{m,j}}{2}. \quad (3.5)$$

To prove this we recall the partitioning of $\Omega_{m,j}^N$ into $\Omega_{m,j}^{N,k}$ given in [CVX15b, §2] as follows:

$$\Omega_{m,j}^{N,k} = \{(0 \subset V_m \subset V_m^\perp \subset V = \mathbb{C}^N) \mid \dim(V_m \cap \zeta_j V) = k\}.$$

We have

$$\Omega_{m,j}^{N,k} \neq \emptyset \Leftrightarrow \max\{m + j - N/2, j/2\} \leqslant k \leqslant \min\{j, m\}.$$

Recall that we have a surjective map $\Omega_{m,j}^{N,k} \to \text{OGr}(j - k, j) \times \text{OGr}(m - k, N - 2j)$ with fibers being affine spaces $\mathbb{A}^{(m-k)(j-k)}$. We have

$$\dim \Omega_{m,j}^{N,k} = -2k^2 + (-N + 3j + 2m + 1)k + mN - mj - \frac{j^2 + 3m^2 + j + m}{2}.$$

One checks that

if $j \geqslant N - 2m$, \ \ \ $\dim \Omega_{m,j}^{N,k}$ is maximal when $k = m + j - \left\lfloor \frac{N}{2} \right\rfloor$,

if $j < N - 2m$, \ \ \ $\dim \Omega_{m,j}^{N,k}$ is maximal when $k = \left\lfloor \frac{j + 1}{2} \right\rfloor$.

Thus a direct calculation shows that

$$\dim(\pi_m^N)^{-1}(\zeta_j) = \begin{cases} 
\frac{a_{m,j}}{2} + \frac{j + m - N}{2} & \text{if } j \geqslant N - 2m \text{ and } N \text{ even, or } j < N - 2m \text{ and } j \text{ odd,} \\
\frac{a_{m,j}}{2} - \frac{m}{2} & \text{if } j \geqslant N - 2m \text{ and } N \text{ odd, or } j < N - 2m \text{ and } j \text{ even.}
\end{cases}$$

This proves (3.5) (note that $m + j < N$). The proof of (3.4) is complete. This finishes the proof of the claim (1) in the proposition.

It then follows that we have

$$(\pi_m^N)_* \mathcal{C}[-] \cong \text{IC}(\mathcal{O}_\lambda, \mathbb{C}) \quad \text{(respectively } ((\pi_{N/2}^N)_\omega)_* \mathcal{C}[-] \cong \text{IC}(\mathcal{O}_\lambda^\omega, \mathbb{C}), \ \omega = I, II) \text{), \quad (3.6)}$$

where

$$\lambda = 3m1^{N-3m} \quad \text{if } N \geqslant 3m, \quad \lambda = 3^{N-2m}2^{3m-N} \quad \text{if } N < 3m.$$

Note that $K \times P_K^m p_1^m$ is the orthogonal complement of $K \times P_K^m (\mathfrak{n}_{P_m})_1$ in the trivial bundle $K \times g_1$ over $K/P_K^m$. By the functoriality of Fourier transform, we have that

$$\mathfrak{F}((\pi_m^N)_* \mathcal{C}[-]) \cong (\pi_m^N)_* \mathcal{C}[-]. \quad \quad (3.7)$$

Since Fourier transform sends simple perverse sheaves to simple perverse sheaves, we can conclude from (3.6) and (3.7) that

$$(\pi_m^N)_* \mathcal{C}[-] \cong \text{IC}(\text{Im} \ \hat{\pi}_m^N, \mathbb{C}).$$

This proves the claim (2) of the proposition. The argument for $(\pi_{2n}^n)_\omega$, $\omega = I, II$, is the same. The proof of the proposition is complete.
3.2 Proof of Proposition 3.2

Note that we have that
\[ L^m_K \cong GL(m) \times SO(N - 2m), \quad \text{and} \quad (l^m)_1 \cong \mathfrak{gl}(m) \oplus \mathfrak{s}(N - 2m)_1. \quad (3.8) \]

To ease notation, let us now write that \( L = L^m, \quad P = P^m, \) and \( \tilde{\pi} = \tilde{\pi}^N_m \) etc.

We first show the following.

The map \( \tilde{\pi} \) (respectively \( \tilde{\pi}^\omega_n \)), when restricted to \( \tilde{\pi}^{-1}(Y^r) \) (respectively \( \tilde{\pi}^{-1}(Y_n^r,\omega) \)), is one-to-one. \quad (3.9)

Each element in \( Y^r \) is \( K \)-conjugate to an element \( x_0 \in p_1 \) (see [KR71, Theorem 7]), where
\[
\begin{align*}
x_0 e_i &= a_i e_i, \quad x_0 e_{N+1-i} = e_i + a_i e_{N+1-i} \quad \text{for} \quad i \in [1, m], \\
x_0 e_j &= b_j e_j + c_j e_{N+1-j}, \quad x_0 e_{N+1-j} = c_j e_j + b_j e_{N+1-j} \quad \text{for} \quad j \in [m+1, [N/2]] \\
x_0 e_{(N+1)/2} &= b_{(N+1)/2} e_{(N+1)/2} \quad \text{if} \quad N \text{ is odd}
\end{align*}
\]

and the numbers \( a_i, i = 1, \ldots, m, b_j + c_j, b_j - c_j, j = m + 1, \ldots, [N/2], b_{(N+1)/2} \) are distinct.

Thus it suffices to show that \( \tilde{\pi}^{-1}(x_0) \) consists of one point. Note that \( \tilde{\pi}^{-1}(x_0) \) consists of \( x_0 \)-stable \( m \)-dimensional isotropic subspaces of \( V \). Assume that \( U_m \in \tilde{\pi}^{-1}(x_0) \). Then \( U_m \) is a direct sum of generalized eigenspaces of \( x_0 \). The generalized eigenspace of \( x_0 \) with eigenvalue \( a_i \) is \( \text{span}\{ e_i, e_{N+1-i} \}, \quad i = 1, \ldots, m, \) and the eigenspace of \( x_0 \) with eigenvalue \( b_j + c_j, b_j - c_j, b_{(N+1)/2} \) is \( \text{span}\{ e_j e_{N+1-j} \}, \quad \text{span}\{ e_j + e_{N+1-j} \}, \quad \text{span}\{ e_{(N+1)/2} \} \), respectively. Since \( U_m \) is isotropic, \( U_m \) has to equal \( \text{span}\{ e_i, i = 1, \ldots, m \} \). This proves (3.9) for \( \tilde{\pi}_m, \quad m < N/2 \). The proof for \( \tilde{\pi}^\omega_n \) is entirely similar and omitted.

Now we show the following.

The image of \( p^*_1 \) under the map \( \text{pr} : p_1 \rightarrow I_1 \) is \( I^*_1 \). \quad (3.11)

Let \( x \in p^*_1 \). By the above proof of (3.9) we can assume that \( \text{Ad}(k)x = x_0 \) for some \( k \in K \), where \( x_0 \) is as in (3.10). Thus \( (k, x) \in \tilde{\pi}^{-1}(x_0) \). It follows from (3.9) that \( (k, x) = (1, x_0) \in K \times P_K p_1 \). Hence \( k \in P_K \). Assume that \( k = lu \) where \( l \in L_K \) and \( u \in U_K = U \cap K \) (\( U \) is the unipotent radical of \( P \)). Then we have \( \text{pr}(x) = \text{pr}(\text{Ad}(u^{-1}l^{-1})x_0) = \text{pr}(\text{Ad}(l^{-1})x_0) = \text{Ad}(l^{-1}) \text{pr}(x_0) \). It is clear that \( \text{pr}(x_0) \in l^*_1 \). Thus (3.11) follows.

By (3.9) and (3.11), we have the following diagram, when restricting (2.10) to \( Y^r \),
\[
\begin{align*}
I^*_1 & \xleftarrow{\text{pr}} p^*_1 \xrightarrow{p_1} K \times p^*_1 \xrightarrow{p_2} K \times P_K p^*_1 \rightarrow K \times P_K (Y^r).
\end{align*}
\]

Using (3.9), we see that
\[
\pi^K_1(Y^r) \cong \pi^K_1 \times P_K (Y^r) \cong \pi^K_1 \times P_K (K \times P_K p^*_1) \cong \pi^K_1 \times P_K (K \times p^*_1) \cong \pi^P_K (p^*_1).
\]

Finally, the canonical map \( \pi^P_1(p^*_1) \rightarrow \pi^P_1(I^*_1) \cong \pi^L_1(I^*_1) \) is surjective. We see this as follows. First, the canonical map above can be identified with the canonical map \( \pi^P_1(p^*_1) \rightarrow \pi^P_1(\text{pr}^{-1}(I^*_1)) \). Now, because \( p^*_1 \) is an open subset in \( \text{pr}^{-1}(I^*_1) \), which is smooth, the map \( \pi_1(p^*_1) \rightarrow \pi_1(\text{pr}^{-1}(I^*_1)) \) is a surjection. To conclude that this property persists when we pass to the equivariant fundamental group it suffices to remark that the equivariant fundamental group is always a quotient of the ordinary fundamental group as long as the group is connected. We now conclude the argument making use of Proposition 3.1.
4. Fourier transform of nilpotent orbital complexes for \((\text{SL}(N), \text{SO}(N))\)

Consider the symmetric pair \((G, K) = (\text{SL}(N), \text{SO}(N))\). Let us write \(A_N\) for the set of all simple \(K\)-equivariant perverse sheaves on \(N_1\) (up to isomorphism), that is, the set of IC complexes \(\text{IC} (\mathcal{O}, \mathcal{E})\), where \(\mathcal{O}\) is a \(K\)-orbit in \(N_1\) and \(\mathcal{E}\) is an irreducible \(K\)-equivariant local system on \(\mathcal{O}\) (up to isomorphism). An IC complex in \(A_N\) is called a nilpotent orbital complex.

Let \(n = \lceil N/2\rceil\). We set

\[
\Sigma_N = \{(\nu; \mu^1, \mu^2) \mid 0 \leq m \leq n, \nu \in \mathcal{P}(m), 0 \leq k \leq n - m, \mu^1 \in \mathcal{P}_2(k), \mu^2 \in \mathcal{P}_2(N - 2m - k)\}.
\]

In the case when \(N\) is even, we identify the triple \((\nu; \mu^1, \mu^2)\) with \((\nu; \mu^2, \mu^1)\) if \(|\mu^1| = |\mu^2|\) and \(\mu^1 \neq \mu^2\), and the triples \((\nu; \mu, \mu)\) attain two labels I and II.

Given a triple \((\nu; \mu^1, \mu^2) \in \Sigma_N\) (respectively \((\nu; \mu, \mu)\) in \(\Sigma_N\), \(\omega = I, II\)), where \(|\nu| = m < N/2\), we define an irreducible \(K\)-equivariant local system \(\mathcal{T}(\nu; \mu^1, \mu^2)\) (respectively \(\mathcal{T}(\nu; \mu, \mu)\)) on \(Y_m^{\nu}\) (here we write \(Y_0^\nu = g_1^{s_\nu}\) as follows. We obtain a map

\[
\tau : \pi^K_1(Y_m^{\nu}) \to B_m \times \tilde{B}_{N-2m} \to S_m \times \tilde{B}_{N-2m}
\]

by composing the map in (3.2) with the natural map \(B_m \times \tilde{B}_{N-2m} \to S_m \times \tilde{B}_{N-2m}\). Note that the map \(\tau\) is surjective. Then \(\mathcal{T}(\nu; \mu^1, \mu^2)\) (respectively \(\mathcal{T}(\nu; \mu, \mu)\)) is the irreducible local system associated to the irreducible representation of \(\pi^K_1(Y_m^{\nu})\) given by pulling back the irreducible representation \(\rho_\nu \boxtimes V_{\mu^1, \mu^2}\) (respectively \(\rho_\nu \boxtimes V_{\mu, \mu}\)) via the map \(\tau\); here \(\rho_\nu \in S_{\mu^1, \mu^2}\) is the irreducible representation of \(S_{\mu^1, \mu^2}\) corresponding to \(\nu \in \mathcal{P}(m)\) and \(V_{\mu^1, \mu^2}\) (respectively \(V_{\mu, \mu}\)) is the irreducible representation of \(B_{N-2m}\) defined in (2.7) (respectively (2.9)).

Assume now that \(N = 2n\). Given a triple \((\nu; \emptyset, \emptyset)\) in \(\Sigma_N\), \(\omega = I, II\), we define the irreducible \(K\)-equivariant local system \(\mathcal{T}(\nu; \emptyset, \emptyset)\) on \((Y_n^{\nu})^\omega\) as the local system associated to the representation of \(\pi^K_1((Y_n^{\nu})^\omega)\) obtained by pulling back the representation \(\rho_\nu \in S_{\nu}^\omega\) corresponding to \(\nu \in \mathcal{P}(n)\) under that map

\[
\pi^K_1((Y_n^{\nu})^\omega) \to B_n \to S_n.
\]

Now we are ready to formulate our main result.

**Theorem 4.1.** The Fourier transform \(\mathfrak{F} : \text{Perv}_K(g_1) \to \text{Perv}_K(g_1)\) induces a bijection

\[
\mathfrak{F} : A_N \simeq \{\text{IC}(g_1^m, \mathcal{T}(\nu; \mu^1, \mu^2)) \mid (\nu; \mu^1, \mu^2) \in \Sigma_N, \mu^1 \neq \mu^2, |\nu| = m < N/2\}
\]

\[
\cup \{\text{IC}(g_1^m, \mathcal{T}(\nu; \mu, \mu)) \mid (\nu; \mu, \mu) \in \Sigma_N, \omega = I, II, |\nu| = m < N/2\} \quad \text{(if } N \text{ is even)},
\]

\[
\cup \{\text{IC}(g_1^{n, \omega}, \mathcal{T}(\nu; \emptyset, \emptyset)) \mid (\nu; \emptyset, \emptyset) \in \Sigma_N, \omega = I, II, |\nu| = n = N/2\} \quad \text{(if } N \text{ is even)},
\]

where \(g_1^m = g_1\), \(g_1^m\) and \(g_1^{n, \omega}\) are defined in (3.1).

**4.1 Proof of Theorem 4.1**

Let \(p(k)\) denote the number of partitions of \(k\) and let \(q(k)\) denote the number of 2-regular partitions of \(k\). We write \(p(0) = q(0) = 1\). Let us define

\[
d(k) = \sum_{s=0}^{k} q(s) q(2k + 1 - s), \quad (4.1)
\]

\[
e(k) = \sum_{s=0}^{k-1} q(s) q(2k - s) + \frac{q(k)^2 + 3q(k)}{2}. \quad (4.2)
\]
Lemma 4.2. We have
\[ |A_{2n+1}| = \sum_{k=0}^{n} p(n-k)d(k) = |\Sigma_{2n+1}|, \quad (4.3) \]
\[ |A_{2n}| = \sum_{k=0}^{n} p(n-k)e(k) = |\Sigma_{2n}|. \quad (4.4) \]

Proof. Note that
\[ \sum_{k \geq 0} p(k)x^k = \prod_{s \geq 1} \frac{1}{1-x^s} \quad \text{and} \quad \sum_{k \geq 0} q(k)x^k = \prod_{s \geq 1} (1+x^s). \quad (4.5) \]

Let \( p(l,k) \) denote the number of partitions of \( l \) into (not necessarily distinct) parts of exactly \( k \) different sizes. We have (see for example [Wil83])
\[ \sum_{l,k \geq 0} p(l,k)x^ly^k = \prod_{s \geq 1} \left( 1 + \frac{yx^s}{1-x^s} \right). \quad (4.6) \]

Assume first that \( N = 2n+1 \). Note that if \( \lambda \) is a partition of \( N \) with parts of \( k \) different sizes, then the component group \( A_K(x) \) of the centralizer \( Z_K(x) \) for \( x \in \mathcal{O}_\lambda \) is \((\mathbb{Z}/2\mathbb{Z})^{k-1}\). Thus there are \( 2^{k-1} \) irreducible \( K \)-equivariant local systems on \( \mathcal{O}_\lambda \) (up to isomorphism). Hence using (4.6), we see that
\[ |A_{2n+1}| = \sum_{k \geq 0} p(2n+1,k)2^{k-1} = \text{Coefficient of } x^{2n+1} \text{ in } \frac{1}{2} \prod_{s \geq 1} \left( 1 + x^s \right). \]

Using (4.5), we see that
\[ \prod_{s \geq 1} \left( 1 + x^s \right) = \left( \sum_{k \geq 0} p(k)x^k \right)^2. \quad (4.7) \]

It then follows that \( |A_{2n+1}| \) is the desired number. The fact that \( |\Sigma_{2n+1}| \) equals the same number is clear from the definition. Thus (4.3) holds.

Assume now that \( N = 2n \). Suppose that \( \lambda \) is a partition of \( N \) with parts of exactly \( k \) different sizes. If \( \lambda \) has at least one odd part, then there are \( 2^{k-1} \) irreducible \( K \)-equivariant local systems on \( \mathcal{O}_\lambda \) (up to isomorphism). If \( \lambda \) has only even parts, then there are \( 2^k \) irreducible \( K \)-equivariant local systems on each \( \mathcal{O}_\lambda^\omega \) (up to isomorphism), \( \omega = 1, II \).

Thus we have that
\[ |A_{2n}| = \sum_{k \geq 0} p(2n,k)2^{k-1} + \sum_{k \geq 0} p(n,k)3 \cdot 2^{k-1} \]
\[ = \text{Coefficient of } x^{2n} \text{ in } \frac{1}{2} \prod_{s \geq 1} \left( 1 + x^s \right) + \text{Coefficient of } x^n \text{ in } \frac{3}{2} \prod_{k \geq 1} \left( 1 + x^s \right) \]
\[ = \frac{1}{2} \left( \sum_{k=0}^{n} p(n-k) \left( 2 \sum_{s=0}^{k-1} q(s)q(2k-s) + q(k)^2 \right) \right) + \frac{3}{2} \sum_{k=0}^{n} p(n-k)q(k) = \sum_{k=0}^{n} p(n-k)e(k). \]

Here we have used (4.7) and the following equation
\[ \prod_{s \geq 1} \left( 1 + x^s \right) = \left( \sum_{k \geq 0} p(k)x^k \right)^2. \]
Again the fact that \( |\Sigma_{2n}| \) equals the desired number is clear from the definition. \( \square \)
Note that the IC sheaves appearing on the right-hand side of the Fourier transform map $\hat{\chi}$ in Theorem 4.1 are pairwise non-isomorphic. Thus, in view of Lemma 4.2, Theorem 4.1 follows from the following.

**Proposition 4.3.** Let $(\nu; \mu^1, \mu^2) \in \Sigma_N$ (respectively $(\nu; \mu, \mu^\omega) \in \Sigma_N$, $\omega = I, II$) and write $m = |\nu|$. The Fourier transform of $IC(g_1^{m_\nu}, T(\nu; \mu^1, \mu^2))$ (respectively $IC(g_1^{m_\nu}, T(\nu; \mu, \mu^\omega))$, $IC(g_1^{m_\nu}, T(\nu; \emptyset, \emptyset^\omega))$ is supported on a $K$-orbit in $N_1$.

**Proof.** Let $n = [N/2]$. We begin the proof by showing the following for $(\emptyset; \mu^1, \mu^2) \in \Sigma_N$ (respectively $(\emptyset; \mu, \mu^\omega) \in \Sigma_N$, $\omega = I, II$).

The Fourier transform of $IC(g_1, T(\emptyset; \mu^1, \mu^2))$ (respectively $IC(g_1, T(\emptyset; \mu, \mu^\omega))$)

is supported on a $K$-orbit in $N_1$.

Recall that $T(\emptyset; \mu^1, \mu^2)$ (respectively $T(\emptyset; \mu, \mu^\omega)$) is the irreducible $K$-equivariant local system on $\mathfrak{g}_1^{m_\nu}$ corresponding to $V_{\mu^1, \mu^2}$ (respectively $V_{\mu, \mu^\omega}$).

We will now appeal to the nearby cycle construction in [GVX18]. Let us recall the characters $\chi_m \in I_N^1$, $0 \leq m \leq n$ of (2.3). In [GVX18] we apply a nearby cycle construction to local systems associated to the $\chi_m$ and obtain a $K$-equivariant perverse sheaf $\mathcal{P}_{\chi_m}$ on the nilpotent cone $N_1$. More precisely, for each character $\chi_m$, let us write $W_{\chi_m} = Stab_W(\chi_m)$. Consider the following base-change diagram of the adjoint quotient map.

$$
\begin{array}{ccc}
g_{1,\chi_m} & \longrightarrow & \mathfrak{g}_1 \\
\downarrow f_{\chi_m} & & \downarrow f \\
\mathfrak{a}/W_{\chi_m} & \longrightarrow & \mathfrak{a}/W
\end{array}
$$

Let us write $g_{1,\chi_m}^{rs}$ for the base change of the regular semisimple locus $g_{1}^{rs}$. Denote by $F_{\chi_m}$ the $K$-equivariant local system on $g_{1,\chi_m}^{rs}$ corresponding to the representation of $\pi_1^K(g_{1,\chi_m}^{rs}) = I_N \rtimes B_{\chi_m}$, where $I_N$ acts via the character $\chi_m$ and $B_{\chi_m}$ acts trivially (recall that $B_{\chi_m} = Stab_B(\chi_m)$). We form the nearby cycle sheaf $\mathcal{P}_{\chi_m} = \psi_{F_{\chi_m}} F_{\chi_m}$, appropriately shifted, so that $\mathcal{P}_{\chi_m} \in \text{Perv}_K(N_1)$.

Applying [GVX18, Theorem 3.2], we obtain that

$$\hat{\chi}(\mathcal{P}_{\chi_m}) = IC(g_1, M_{\chi_m}),$$

where $M_{\chi_m}$ is the $K$-equivariant local system on $g_{1,\chi_m}^{rs}$ corresponding to the $\tilde{B}_\chi$-representation

$$M_{\chi_m} = \mathbb{C}[\tilde{B}_N] \otimes_{\mathbb{C}[\tilde{B}_0]} (\mathbb{C}_{\chi_m} \otimes \mathcal{H}_{W_{\chi_m}^0}).$$

Let us explain the notation in the above formula in our setting. Let $W_{\chi_m}^0$ be the Coxeter subgroup of $W$ generated by $s_\alpha$ with $\chi_m(\tilde{\alpha}(-1)) = 1$, $\alpha \in \Phi(g, \mathfrak{a})$, where $\Phi(g, \mathfrak{a})$ is the root system of $\mathfrak{g}$ with respect to $\mathfrak{a}$. Note that $\tilde{\alpha}(-1) \in I_N$. Then $\mathcal{H}_{W_{\chi_m}^0}$ is the Hecke algebra associated to the Coxeter group $W_{\chi_m}^0$ with parameter $\tilde{\alpha}(-1)$. Let $B_0_{\chi_m} \subset B_N$ be the inverse image of $W_{\chi_m}^0 \subset W$ under the natural map $B_N \rightarrow W = S_N$. Then $\mathcal{H}_{W_{\chi_m}^0} = I_N \rtimes B_{\chi_m}^0$. In our setting, one readily checks that $W_{\chi_m}^0 = \langle s_i, i \neq m \rangle$, $\mathcal{H}_{W_{\chi_m}^0} = \mathcal{H}_{\chi_m,-1}$ and $B_{\chi_m}^0 = B_{m,N-m}$ (here we use the notation in § 2.3).

The action of $B_{\chi_m}^0$ on $(\mathbb{C}_{\chi_m} \otimes \mathcal{H}_{W_{\chi_m}^0})$ is given by $I_N$ acting via the character $\chi_m$ and $B_{\chi_m}^0$ acting via the quotient map $\mathbb{C}[B_{\chi_m}^0] \rightarrow \mathcal{H}_{W_{\chi_m}^0}$. Thus we conclude

$$\hat{\chi}(\mathcal{P}_{\chi_m}) \cong IC(g_1, \mathcal{L}_{\chi_m}),$$

4.10
where $\mathcal{L}_{\chi_m}$ is the $K$-equivariant local system on $\mathfrak{g}_1$ corresponding to the representations $L_{\chi_m}$ of $\pi_1^L(\mathfrak{g}_1) = B_N$ defined in (2.8).

By Lemma 2.1 the IC sheaves $\text{IC}(g_1, T(\theta; \mu^1, \mu^2))$ and $\text{IC}(g_1, T(\theta; \mu, \mu^\omega))$ are composition factors of the IC($g_1, \mathcal{L}_{\chi_m}$). Hence (4.8) follows from (4.10).

Now let $(\nu; \mu^1, \mu^2) \in \Sigma_N$ with $|\nu| = m > 0$. Let
\[ K(\rho_\nu \boxtimes V_{\mu^1, \mu^2}) \]
denote the irreducible $L_K$-equivariant local system on $P_1$ associated to the irreducible representation of $\pi_1^{L_K}(P_1)$ obtained as a pullback of $\rho_\nu \boxtimes V_{\mu^1, \mu^2}$ via the map $\pi_1^{L_K}(P_1) \cong B_m \times \tilde{B}_{N-2m} \to S_m \times \tilde{B}_{N-2m}$.

By Proposition 3.2, we have that
\[ \text{IC}(g_1^m, T(\nu; \mu^1, \mu^2)) = \text{Ind}_{P_1 \cap P_1}^{g_1} \text{IC}(1, K(\rho_\nu \boxtimes V_{\mu^1, \mu^2})). \] (4.11)

Since Fourier transform commutes with induction (see (2.11)), it suffices to show that the Fourier transform of $\text{IC}(1, K(\rho_\nu \boxtimes V_{\mu^1, \mu^2}))$ is supported on an $L_K$-nilpotent orbit in $P_1$. This follows from the classical Springer correspondence for $\mathfrak{g}(m)$ and (4.8) applied to the symmetric pair $(\text{SL}(N-2m), \text{SO}(N-2m))$ (see (3.8)).

The proof for $\text{IC}(g_1^m, T(\nu; \mu, \mu^\omega))$, $\text{IC}(g_1^m, T(\nu; \theta, \theta^\omega))$ proceeds in the same manner; in the latter case one uses the corresponding $\theta$-stable Levi and parabolic subgroups. We omit the details. \hfill \Box

4.2 More on induction
Let $(\nu; \mu^1, \mu^2) \in \Sigma_N$. Assume that $|\nu| = m > 0$. Let $L^m \subset P^m$ be as in § 3. Recall that $L_K^m \cong \text{GL}(m) \times \text{SO}(N-2m)$ and $P_1^m \cong \mathfrak{g}(m) \oplus \mathfrak{s}(N-2m)$.

A nilpotent $L_K^m$-orbit in $P_1^m$ is given by a nilpotent orbit in $\mathfrak{g}(m)$ and a nilpotent $\text{SO}(N-2m)$-orbit in $\mathfrak{s}(N-2m)$. Thus the nilpotent $L_K^m$-orbits in $P_1^m$ are parametrized by $P(m) \times P(N-2m)$, with extra labels I and II for partitions in $P(N-2m)$ with all parts even. For $\alpha \in P(m)$ and $\beta \in P(N-2m)$, we denote by $O_{\alpha, \beta}$ (or $O_{\alpha, \beta}^\omega$) the nilpotent $L_K^m$-orbit in $P_1^m$ given by the nilpotent orbit $O_\alpha$ in $\mathfrak{g}(m)$ and the nilpotent $\text{SO}(N-2m)$-orbit $O_\beta$ (or $O_\beta^\omega$) in $\mathfrak{s}(N-2m)$.

In the following we will omit the labels I and II with the understanding that everything should have corresponding labels, for example, $O_{\lambda}^\omega = \text{Ind}_{P_1^m \cap P_1^m}^{g_1} O_{\alpha, \beta}$ etc.

**Proposition 4.4.** Let $\alpha \in P(m)$ and $\beta \in P(N-2m)$. Let $O_\lambda = \text{Ind}_{P_1^m \cap P_1^m}^{g_1} O_{\alpha, \beta}$, i.e., $\lambda_i = \beta_i + 2\alpha_i$. Assume that $u \in O_{\alpha, \beta}$ and $v \in O_\lambda \cap (u + (n_{P^m}))$. We have a natural surjective map
\[ \psi : A_{L_K^m}(v) \twoheadrightarrow A_{L_K^m}(u). \]
Moreover, let $C \boxtimes E$ be an $L_K^m$-equivariant irreducible local system on $O_{\alpha, \beta}$ and let $\tilde{E}$ be the $K$-equivariant local system on $O_{\lambda}$ obtained from $C \boxtimes E$ via the map $\psi$ above. Then $IC(O_{\lambda}, \tilde{E})$ is a direct summand of $\text{Ind}_{P_1^m \cap P_1^m}^{g_1} IC(O_{\nu, \mu}, C \boxtimes E)$.

**Corollary 4.5.** If moreover $(O_{\mu, \nu}) \in \mathcal{A}_{N-2m}$ is a pair such that $\mathfrak{F}(IC(O_{\nu, \mu}))$ has full support, then we have
\[ \text{Ind}_{P_1^m \cap P_1^m}^{g_1} IC(O_{\nu, \mu}, C \boxtimes E) \cong IC(O_{\lambda}, \tilde{E}). \]

As before let us now write $L = L^m$ and $P = P^m$ etc. We begin the proof of the above proposition with the following lemma.
Lemma 4.6. The map
\[ \gamma : K \times P_{\mathcal{K}} \to \tilde{\mathcal{O}}_{\alpha,\beta} + (n_{\mathcal{P}})_1 \to \tilde{\mathcal{O}}_{\lambda} \]
is generically one-to-one.

Proof. Let \( x_0 \in \mathcal{O}_{\lambda} \). We can and will assume that \( x_0 \in \mathcal{O}_{\alpha,\beta} + (n_{\mathcal{P}})_1 \). We show that \( \gamma^{-1}(x_0) \) is a point. Assume that \( \gamma(k, x) = x_0 \), i.e., \( \text{Ad} k(x) = x_0 \). Then \( x \in \mathcal{O}_{\alpha,\beta} + (n_{\mathcal{P}})_1 \). Let \( \tilde{\mathcal{O}}_{\lambda} \) (respectively \( \tilde{\mathcal{O}}_{\alpha,\beta} \)) be the (unique) \( G \)-orbit (respectively \( L \)-orbit) in \( g \) (respectively \( l \)) that contains \( \mathcal{O}_{\lambda} \) (respectively \( \mathcal{O}_{\alpha,\beta} \)). We have that
\[ \tilde{\mathcal{O}}_{\lambda} = \text{Ind}^{\mathfrak{g}}_{\mathfrak{l}} \tilde{\mathcal{O}}_{\alpha,\beta} \]
in the notation of Lusztig and Spaltenstein [LS79]. By [LS79, Theorem 1.3], we have \( Z^0_G(x_0) \subset P \).
In fact, we have that \( Z_G(x_0) \subset P \). This can be seen by enlarging the group \( G \) to \( \text{GL}(N) \) and using the fact that \( Z_{\text{GL}(N)}(x_0) \) is connected. Thus \( Z_K(x_0) \subset P_{\mathcal{K}} \). Furthermore, \( \tilde{\mathcal{O}}_{\lambda} \cap \tilde{\mathcal{O}}_{\alpha,\beta} + (n_{\mathcal{P}}) \) is a single orbit under \( P \). Thus there exists \( p \in P \) such that \( \text{Ad} p(x) = x_0 \). It follows that \( k^{-1} p \in Z_G(x_0) \), \( k \in P \cap K = P_K \). Now we have that \( (k, x) = (1, \text{Ad} k(x)) = (1, x_0) \). \( \square \)

Proof of Proposition 4.4. Note that the proof of the above lemma shows that \( Z_G(v) = Z_P(v) \).

We have \( Z_P(v) \subset Z_L(u) U_P \). Thus \( Z_K(v) = Z_{P_{\mathcal{K}}}(v) \subset Z_{L_{\mathcal{K}}}(u)(U_P \cap K) \). It follows that we have a natural projection map
\[ Z_K(v)/Z^0_{\mathcal{K}}(v) = Z_{P_{\mathcal{K}}}(v)/Z^0_{P_{\mathcal{K}}}(v) \to Z_{L_{\mathcal{K}}}(u)/Z^0_{L_{\mathcal{K}}}(u). \]
We show that this gives us the desired map \( \psi \). Following [LS79], we have that \( Z_{L_{\mathcal{K}}}(u)(U_P \cap K) \) has a dense orbit, i.e., the orbit of \( v \), in the irreducible variety \( u + (n_{\mathcal{P}})_1 \). Thus \( Z_{P_{\mathcal{K}}}(v) = Z_K(v) \) meets all the irreducible components of \( Z_{L_{\mathcal{K}}}(u)(U_P \cap K) \), which implies that \( \psi \) is surjective.

It is easy to see that
\[ \text{supp} (\text{Ind}^{\mathfrak{g}}_{\mathfrak{l}} \mathcal{IC}(\mathcal{O}_{\alpha,\beta}, \mathbb{C} \boxtimes \mathcal{E})) = \tilde{\mathcal{O}}_{\lambda}. \quad (4.12) \]
The proposition follows from the definition of parabolic induction functor and Lemma 4.6. \( \square \)

Remark 4.7. The proof of Lemma 4.6 and the existence and surjectivity of the map \( \psi \) in Proposition 4.4 works for any \( \theta \)-stable Levi contained in a \( \theta \)-stable parabolic subgroup.

Proof of Corollary 4.5. Note that the assumption implies that \( \mathfrak{F}(\mathcal{IC}(\mathcal{O}_{\alpha,\beta}, \mathbb{C} \boxtimes \mathcal{E})) \) has full support, i.e., \( \mathcal{IC}(\mathcal{O}_{\alpha,\beta}, \mathbb{C} \boxtimes \mathcal{E}) = \mathcal{IC}((1, \mathcal{G})) \) for some irreducible \( L_{\mathcal{K}} \)-equivariant local system \( \mathcal{G} \) on \( f \). We have that
\[ \mathfrak{F}(\text{Ind}^{\mathfrak{g}}_{\mathfrak{l}} \mathcal{IC}(\mathcal{O}_{\alpha,\beta}, \mathbb{C} \boxtimes \mathcal{E})) = \text{Ind}^{\mathfrak{g}}_{\mathfrak{l}} \mathfrak{F}(\mathcal{IC}(\mathcal{O}_{\alpha,\beta}, \mathbb{C} \boxtimes \mathcal{E})) = \text{Ind}^{\mathfrak{g}}_{\mathfrak{l}} \mathcal{IC}(1, \mathcal{G}). \]
It suffices to show that \( \text{Ind}^{\mathfrak{g}}_{\mathfrak{l}} \mathcal{IC}(1, \mathcal{G}) \) is irreducible. This follows from the definition of the induction functor and Proposition 3.1. \( \square \)

Corollary 4.8. The Fourier transform of a nilpotent orbital complex \( \mathcal{IC}(\mathcal{O}, \mathcal{E}) \in \mathcal{A}_{\mathcal{K}} \) has full support, i.e., \( \text{supp} \mathfrak{F}(\mathcal{IC}(\mathcal{O}, \mathcal{E})) = g_1 \), if and only if it is not of the form \( \text{Ind}^{\mathfrak{g}}_{\mathfrak{l}} \mathcal{IC}(\mathcal{O}', \mathcal{E}') \) where \( \text{supp} \mathfrak{F}(\mathcal{IC}(\mathcal{O}', \mathcal{E}')) = l_1 \), and \( L \subset P \) is a pair chosen as in \( \S 3 \).

Proof. The only if part follows from the facts that Fourier transform commutes with parabolic induction and that \( \text{supp} \text{Ind}^{\mathfrak{g}}_{\mathfrak{l}} \mathcal{A} \subset g_1 \). The if part follows from (4.11), Corollary 4.5 and Theorem 4.1. \( \square \)
Corollary 4.9. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots) \in \mathcal{P}(N)$.

(1) If $\lambda_i - \lambda_{i+1} \geq 3$ for some $i$, then supp $\widetilde{\mathbb{F}}(\text{IC}(\mathcal{O}_\lambda, \mathcal{E})) \neq \mathcal{g}_1$ for any $K$-equivariant local system $\mathcal{E}$ on $\mathcal{O}_\lambda$. The same holds for $\mathcal{O}_\lambda^\omega$ if $\lambda$ has only even parts.

(2) Suppose that $\lambda_i - \lambda_{i+1} \leq 2$ for all $i$. Let $f_\lambda$ be the number of different sizes of parts of $\lambda$, and $g_\lambda$ the number of $i$ such that $\lambda_i - \lambda_{i+1} = 2$.

(a) If at least one part of $\lambda$ is odd, then there are $2^{f_\lambda - 1 - g_\lambda}$ irreducible $K$-equivariant local systems $\mathcal{E}$ on $\mathcal{O}_\lambda$ such that supp $\widetilde{\mathbb{F}}(\text{IC}(\mathcal{O}_\lambda, \mathcal{E})) = \mathcal{g}_1$.

(b) If all parts of $\lambda$ are even, then there is exactly one irreducible $K$-equivariant local system $\mathcal{E}^\omega$ on each orbit $\mathcal{O}_\lambda^\omega$, $\omega = I, II$, such that supp $\widetilde{\mathbb{F}}(\text{IC}(\mathcal{O}_\lambda^\omega, \mathcal{E}^\omega)) = \mathcal{g}_1$.

In particular, if $\lambda_i - \lambda_{i+1} \leq 1$ for all $i$, then supp $\widetilde{\mathbb{F}}(\text{IC}(\mathcal{O}_\lambda, \mathcal{E})) = \mathcal{g}_1$ for any $K$-equivariant local system $\mathcal{E}$ on $\mathcal{O}_\lambda$.

Proof. (1) Assume that $\lambda_i - \lambda_{i+1} \geq 3$. Let $m = i_o$, $\alpha = 1^{i_o}$, $\beta = (\lambda_1 - 2, \ldots, \lambda_{i_0} - 2, \lambda_{i_0+1}, \ldots)$. Then $\mathcal{O}_\lambda = \text{Ind}_{\mathcal{P}_1^{\omega} \subset \mathcal{P}_1^{\omega}}^\mathcal{O}_{\alpha, \beta}$. Let $u \in \mathcal{O}_{\alpha, \beta}$ and $v \in \mathcal{O}_\lambda \cap (u + (n_{pm})_1)$. Note that $A_K(v) \cong A_{L_K}(u)$. It then follows from Proposition 4.4 that for each irreducible $K$-equivariant local system $\mathcal{E}$ on $\mathcal{O}_\lambda$, $\text{IC}(\mathcal{O}_\lambda, \mathcal{E})$ is a direct summand of $\text{Ind}_{\mathcal{P}_1^{\omega} \subset \mathcal{P}_1^{\omega}}^\mathcal{O}_{\alpha, \beta} \text{IC}(\mathcal{O}_\alpha^\omega, \mathcal{E}_\alpha^\omega)$ for some irreducible $L_K$-equivariant local system $\mathcal{E}_\alpha^\omega$ on $\mathcal{O}_\alpha^\omega$. As before, this shows that $\widetilde{\mathbb{F}}(\text{IC}(\mathcal{O}_\lambda, \mathcal{E}))$ has smaller support.

In the case when $\lambda$ has only even parts, we let $\mathcal{O}_\lambda^\omega = \text{Ind}_{\mathcal{P}_1^{\omega} \subset \mathcal{P}_1^{\omega}}^\mathcal{O}_{\alpha, \beta}$, if $m < N/2$, and we let $\mathcal{O}_\alpha^\omega = \text{Ind}_{\mathcal{P}_1^{\omega} \subset \mathcal{P}_1^{\omega}}^\mathcal{O}_{\alpha, \beta}$, if $m = N/2 = n$, where $\omega = I, II$. The proof for $\mathcal{O}_\lambda^\omega$ then proceeds in the same way.

(2) We argue by induction on $g_\lambda$. If $g_\lambda = 0$, then (2) follows from (4.12) and Corollary 4.8. Assume by induction hypothesis that (2) holds for all $\mu$ with $g_\mu < g_\lambda$.

Assume first that $\lambda$ has at least one odd part. Suppose that $i_1, \ldots, i_k$ are such that $\lambda_{i_l} - \lambda_{i_l+1} = 2$, where $k = g_\lambda$.

Let $a = (a_1 \geq a_2 \geq \cdots \geq a_k \geq 0)$ be a partition such that $a \neq \emptyset$, $a_k \leq 1$, and $a_l \leq a_{l+1} - 1$. Note that the number of such partitions is $2^k - 1$. Consider a partition $\mu(a)$ such that $\mu_l = \lambda_l - 2a_j$ for $l \in [2j - 1, 2j]$.

Consider the case when at least one part of $\lambda$ is odd (respectively when all parts of $\lambda$ are even).

We conclude from the discussion above that

$$m_\lambda \leq 2^{f_\lambda - g_\lambda - 1}$$

if $\lambda$ has at least one odd part, respectively

$$m_\lambda^\omega \leq 1$$

if all parts of $\lambda$ are even.

(4.13)
Theorem 4.1 implies that the number of pairs $IC(O, E) \in A_N$ such that $\text{supp}(IC(O, E)) = g_1$ is $d(n)$ (see (4.1)), when $N = 2n + 1$, and $e(n)$ (see (4.2)), when $N = 2n$. In view of (4.13) and claim (1) of the corollary, it suffices to show that

$$\sum_{\lambda \in P_t(2n+1)} 2f_\lambda - g_\lambda - 1 = d(n), \quad \sum_{\lambda \in P_t(2n), \lambda_i - \lambda_{i+1} \leq 2, \text{ not all parts of } \lambda \text{ even}} 2f_\lambda - g_\lambda - 1 + 2q(n) = e(n).$$

This can be seen as follows. Note that when $N$ is even, the number of orbits of the form $O(\omega, \lambda)$, where all parts of $\lambda$ are even and $\lambda_i - \lambda_{i+1} \leq 2$, is $2q(n)$. We know that $d(n) = \text{Coefficient of } x^{2n+1} \text{ in } \frac{1}{2} \prod_{s \geq 1} (1 + x^s)^2$, $e(n) = \frac{3}{2} q(n) + \text{Coefficient of } x^{2n} \text{ in } \frac{1}{2} \prod_{s \geq 1} (1 + x^s)^2$.

A partition $\lambda$ satisfies that $\lambda_i - \lambda_{i+1} \leq 2$ if and only if each part of the transpose partition $\lambda'$ has multiplicity at most 2. We have $f_\lambda = f_{\lambda'}$ and $g_\lambda$ equals the number of parts in $\lambda'$ with multiplicity 2. It is easy to see that each $\lambda'$ whose parts have multiplicity at most 2 appears in $\prod_{s \geq 1} (1 + x^s)^2$ exactly $2f_\lambda - g_\lambda$ times. Hence (4.14) follows.

**Remark 4.10.** In [CVX15a, Conjecture 1.2], we conjectured that one can obtain all nilpotent orbital complexes by induction from those of smaller groups whose Fourier transforms have full support. This conjecture follows from Corollary 4.8.

5. Cohomology of Hessenberg varieties

Hessenberg varieties, defined generally in [GKM06], arise naturally in our setting (for details, see [CVX15b]). In particular, they arise as fibers of maps $\pi$ and $\tilde{\pi}$ in the following diagram

$$\begin{array}{ccc}
K/P_K \times g_1 & \xrightarrow{\pi} & K \times P_K E \\
\downarrow & & \downarrow \tilde{\pi} \\
\mathcal{N}_1 & \rightarrow & g_1
\end{array}$$

where $P_K = P \cap K$ for a $\theta$-stable parabolic subgroup $P$ of $G$, $E$ is a $P_K$-stable subspace of $g_1$ consisting of nilpotent elements, and $E^\perp$ is the orthogonal complement of $E$ in $g_1$ via a $K$-invariant non-degenerate form on $g_1$. The generic fibers of maps $\tilde{\pi}$ are Hessenberg varieties.

In this section we discuss an application of our result to cohomology of Hessenberg varieties. Let us fix $s \in g_1^\theta$ and consider the corresponding Hessenberg variety

$$\text{Hess} := \tilde{\pi}^{-1}(s) = \{gP_K \in K/P_K \mid g^{-1}sg \in E^\perp\}.$$ 

The centralizer $Z_K(s)$ acts naturally on Hess and it induces an action of the component group $\pi_0(Z_K(s)) \cong I_N$ on the cohomology groups $H^*(\text{Hess}, \mathbb{C})$. Let

$$H^*(\text{Hess}, \mathbb{C}) = \bigoplus_{\chi \in I_N^\vee} H^*(\text{Hess}, \mathbb{C})_{\chi}$$

be the eigenspace decomposition with respect to the action of $I_N$. 

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Definition 5.1. The stable part $H^\ast(Hess, \mathbb{C})_{st}$ of $H^\ast(Hess, \mathbb{C})$ is the direct summand $H^\ast(Hess, \mathbb{C})_{\chi_{triv}}$ where $\chi_{triv} \in I_N^\vee$ is the trivial character.

For simplicity we now assume $\bar{\pi}$ is onto. In this case $\bar{\pi}$ is smooth over $g_1^{\text{rs}}$ (e.g. see [CVX15b, Lemma 2.1]) and the equivariant fundamental group $\pi_1^g(g_1^{\text{rs}}, s) \cong I_N \times B_N$ acts on $H^\ast(Hess, \mathbb{C})$ by the monodromy action. Recall that for $\chi \in I_N^\vee$, $B_\chi$ stands for the stabilizer of $\chi$ in $B_N$. Clearly, each summand $H^\ast(Hess, \mathbb{C})_\chi$ is stable under the action of $B_\chi$. Let $\chi_m \in I_N^\vee$, $B_\chi$, and $B_{m,N-m}$ be as in §2.3. Assume that $\chi$ is in the $B_N$-orbit of $\chi_m$. Then for any $b \in B_N$ with $b_\chi = \chi_m$ we have an isomorphism $t_b : B_\chi \cong B_{\chi_m}, u \mapsto bub^{-1}$. Note that $\chi_{triv} = \chi_0$ and $B_{\chi_m} = B_{m,N-m}$ except when $N$ is even and $m = N/2$. In that case, $B_{m,N-m}$ is an index-2 subgroup of $B_{\chi_m}$.

Recall the algebra $\mathcal{H}_{\chi_m,-1} = \mathcal{H}_{m,-1} \otimes \mathcal{H}_{N-m,-1}$ and their representations $D_{\mu^1} \otimes D_{\mu^2}$ introduced in §2.3. Each $\mathcal{H}_{\chi_m,-1}$ is a quotient of the group algebra $\mathbb{C}[B_{m,N-m}]$ and $\mathcal{H}_{\chi_0,-1} = \mathcal{H}_{\chi_{triv},-1} = \mathcal{H}_{N,-1}$ is the Hecke algebra of $S_N$ at $q = -1$.

Theorem 5.2. (i) Let $\chi_m \in I_N^\vee$ be the representatives of $B_N$-orbits in §2.3. To every $\chi \in I_N^\vee$ in the orbit of $\chi_m$ and an element $b \in B_N$ satisfying $b(\chi) = \chi_m$, the monodromy action of $b$ on $H^\ast(Hess, \mathbb{C})$ induces an isomorphism $H^\ast(Hess, \mathbb{C})_\chi \cong H^\ast(Hess, \mathbb{C})_{\chi_m}$ compatible with the actions of $B_\chi \cong B_{\chi_m}$ on both sides.

(ii) The action of $\mathbb{C}[B_{m,N-m}]$ on $H^\ast(Hess, \mathbb{C})_{\chi_m}$ factors through the algebra $\mathcal{H}_{\chi_m,-1}$ and the resulting representation is a direct sum of $D_{\mu^1} \otimes D_{\mu^2}$, $\mu^1 \in P_2(m)$, $\mu^2 \in P_2(N-m)$. In particular, the stable part $H^\ast(Hess, \mathbb{C})_{st}$ is generated by irreducible representations of the Hecke algebra of $S_N$ at $q = -1$.

Proof. Part (1) is clear. To prove part (2) we proceed as follows. By the decomposition theorem $\pi_\ast\mathcal{C}$ is a direct sum of shifts of nilpotent orbital complexes. Since $\mathcal{F}(\pi_\ast\mathcal{C}) \cong \mathcal{F}_\ast\mathcal{C}$ (up to shift), Theorem 4.1 implies that a generic stalk of $\mathcal{F}(\pi_\ast\mathcal{C})$, which is isomorphic to $H^\ast(Hess, \mathbb{C})$, is a direct sum of the local systems $V_{\mu^1,\mu^2} = \text{Ind}^{\mathbb{C}[B_N]}_{\mathbb{C}[B_{m,N-m}]} D_{\mu^1} \otimes D_{\mu^2}$ introduced in (2.8). Since $I_N$ acts on $V_{\mu^1,\mu^2}$ by the formula $a.(b \otimes v) = ((b,\chi_m)(a))(b \otimes v)$ for $a \in I_N$, $b \in B_N$ and $v \in D_{\mu_1} \otimes D_{\mu_2}$, we have $(V_{\mu^1,\mu^2})_\chi \cong D_{\mu_1} \otimes D_{\mu_2}$. The theorem follows.

Example 5.3. Let $C$ be the hyper-elliptic curve with affine equation $y^2 = \prod_{j=1}^N (x - a_j)$ (here $a_i \neq a_j$ for $i \neq j$). Assume $N = 2n + 2$ is even. Then according to [CVX17, §2.3] the Jacobian $\text{Jac}(C)$ is an example of a Hessenberg variety and the monodromy action of $\pi_\ast(g_1^{\text{rs}}, s)$ factors through $B_N$, that is, $H^\ast(\text{Jac}(C), \mathbb{C}) \cong H^\ast(\text{Jac}(C), \mathbb{C})_{st}$. Let $\mu_k = (N - k, k) \in P_2(N)$ and $D_{\mu_k}$ be the corresponding representation of $\mathcal{H}_{N,-1}$. Using [A’Ca79], one can check that the induced action of the group algebra $\mathbb{C}[B_N]$ on $H^\ast(\text{Jac}(C), \mathbb{C})$ factors through $\mathcal{H}_{N,-1}$ and for $i \leq n$ the resulting representation of $\mathcal{H}_{N,-1}$ is isomorphic to

$$H^i(\text{Jac}(C), \mathbb{C}) \cong \bigoplus_{j=0}^{[i/2]} D_{\mu_{i-2j}}$$

with the primitive part $H^i(\text{Jac}(C), \mathbb{C})_{\text{prim}} \cong D_{\mu_i}$.

Remark 5.4. It would be nice to have an explicit decomposition of $H^\ast(Hess, \mathbb{C})_{\chi_m}$ into irreducible representations of $\mathcal{H}_{\chi_m,-1}$. For this one needs finer information for the bijection in Theorem 4.1 (see §7). In [CVX15a, CVX17], we establish an explicit bijection for certain nilpotent orbital complexes and we work out an explicit decomposition for the cohomology of the Hessenberg varieties that are isomorphic to Fano varieties of $k$-planes in smooth complete intersections of two quadrics in projective space.
6. Representations of $\mathcal{H}_{N,-1}$

In this section we show that all irreducible representations of the Hecke algebra $\mathcal{H}_{N,-1}$ come from geometry. Indeed they all appear in intersection cohomology of a Hessenberg variety with coefficient in a local system. In particular, this shows that all irreducible representations of $\mathcal{H}_{N,-1}$ carry a Hodge structure. In particular, the irreducible representations of $\mathcal{H}_{N,-1}$ can be viewed as variations of Hodge structure.

Let $\mathcal{O}$ be a nilpotent $K$-orbit on $\mathfrak{g}_1$ and $\mathcal{L}$ an irreducible $K$-equivariant local system on $\mathcal{O}$. We call $(\mathcal{O}, \mathcal{L})$ a nilpotent pair. Following [LY17], we associate to each nilpotent pair $(\mathcal{O}, \mathcal{L})$ two families of Hessenberg varieties $\text{Hess}_{\mathcal{L},\pm 1} \to \mathfrak{g}_1$ together with local systems $\hat{\mathcal{L}}_{\pm 1}$ on open subsets $\text{Hess}_{\mathcal{L},\pm 1} \subset \text{Hess}_{\mathcal{L},\pm 1}$.

Let $x \in \mathfrak{g}_1$ be a nilpotent element in $\mathcal{O}$. Choose a normal sl$_2$-triple $\{x, h, y\}$ and let

$$\mathfrak{g}(i) = \{v \in \mathfrak{g} \mid [h, v] = iv\}, \quad \mathfrak{g}_0(i) = \mathfrak{g}(i) \cap \mathfrak{g}_0, \quad \text{and} \quad \mathfrak{g}_1(i) = \mathfrak{g}(i) \cap \mathfrak{g}_1.$$ 

For any $N \in \mathbb{Z}$ we write $N \in \{0, 1\}$ for its image in $\mathbb{Z}/2\mathbb{Z}$. Define

$$\mathfrak{p}_N^x = \bigoplus_{k \geq 2N} \mathfrak{g}_N(k), \quad \mathfrak{l}_N^x = \mathfrak{g}_N(2N), \quad \text{and} \quad \mathfrak{l}^x = \bigoplus_{N \in \mathbb{Z}} \mathfrak{l}_N^x.$$ 

One can check that $\mathfrak{l}^x \subset \mathfrak{g}$ is a graded Lie subalgebra of $\mathfrak{g}$ and $x \in \mathfrak{l}_1^x = \mathfrak{g}_1(2)$. Let $L_0^x \subset K$ be the reductive subgroup with Lie algebra $\mathfrak{l}_0^x = \mathfrak{g}_0(0)$. By [LY17, 2.9(c)], the restriction

$$\mathcal{L}_1' := \mathcal{L} |_{\mathfrak{l}_1^x}$$

is an irreducible $L_0^x$-equivariant local system on the unique open $L_0^x$-orbit $\tilde{\mathfrak{l}}_1^x$ on $\mathfrak{l}_1^x$.

According to [Lus95], there exists a graded parabolic subalgebra $\mathfrak{q} = \bigoplus_{N \in \mathbb{Z}} \mathfrak{q}_N$ of $\mathfrak{l}^x$, a Levi subalgebra $\mathfrak{m} = \bigoplus_{N \in \mathbb{Z}} \mathfrak{m}_N$ of $\mathfrak{q}_0$, and a cuspidal local system $\mathcal{L}_1$ on the open $M_0$-orbit $\tilde{\mathfrak{m}}_1$ of $\mathfrak{m}_1$ (here $M_0$ is the reductive subgroup of $L_0^x$ with Lie algebra $\mathfrak{m}_0$) such that some shift of the IC-complex $\text{IC}(\mathfrak{l}_1^x, \mathcal{L}_1')$ is a direct summand of $\text{Ind}_{\mathfrak{m}_1 \subset \mathfrak{q}_1} \text{IC}(\mathfrak{m}_1, \mathcal{L}_1)$.

In addition, we have

$$\mathfrak{g}(\text{IC}(\mathfrak{m}_1, \mathcal{L}_1)) \cong \text{IC}(\mathfrak{m}_{-1}, \mathcal{L}_{-1}),$$

where $\mathcal{L}_{-1}$ is a cuspidal local system on the unique open orbit $\tilde{\mathfrak{m}}_{-1} \subset \mathfrak{m}_{-1}$.

Define $\hat{\mathfrak{q}}_N$ to be the pre-image of $\mathfrak{q}_N$ under the projection map $\mathfrak{p}_N^x \to \mathfrak{l}_N^x$. Let $Q_K \subset K$ be the parabolic subgroup with Lie algebra $\hat{\mathfrak{q}}_0$. Denote by $\hat{\mathfrak{q}}_{\pm 1}$ the preimage of $\mathfrak{m}_{\pm 1}$ under the projection map $\hat{\mathfrak{q}}_{\pm 1} \to \mathfrak{q}_{\pm 1} \to \mathfrak{m}_{\pm 1}$. The group $Q_K$ acts naturally on $\hat{\mathfrak{q}}_{\pm 1}$ and $\hat{\mathfrak{q}}_{\pm 1}$ and we define

$$\text{Hess}_{\mathcal{L},\pm 1} := K \times Q_K \hat{\mathfrak{q}}_{\pm 1}, \quad \text{Hess}_{\mathcal{L},\pm 1} := K \times Q_K \hat{\mathfrak{q}}_{\pm 1}.$$ 

Let

$$\pi_{\mathcal{L},\pm 1} : \text{Hess}_{\mathcal{L},\pm 1} \to \mathfrak{g}_1, (x, v) \to xvx^{-1}$$

and let $\hat{\pi}_{\mathcal{L},\pm 1}$ be its restriction to $\text{Hess}_{\mathcal{L},\pm 1}$. For any $s \in \mathfrak{g}_1$, we denote by $\text{Hess}_{\mathcal{L},\pm 1, s}$ and $\hat{\text{Hess}}_{\mathcal{L},\pm 1, s}$ the fiber of $\pi_{\mathcal{L},\pm 1}$ and $\hat{\pi}_{\mathcal{L},\pm 1}$ over $s$, respectively.
Springer correspondence for the split symmetric pair in type $A$

There are natural maps

$$h_{\mathcal{L} \pm 1} : \text{Hess}_{\mathcal{L} \pm 1} \to [m_{\pm 1}/M_0], \quad \hat{h}_{\mathcal{L} \pm 1} : \text{Hess}_{\mathcal{L} \pm 1} \to [\hat{m}_{\pm 1}/M_0]$$

sending $(k, v) \in \text{Hess}_{\mathcal{L} \pm 1} = K \times K \hat{q}_{\pm 1}$ to $\tilde{v}$, the image of $v \in \hat{q}_{\pm 1}$ under the map $\hat{q}_{\pm 1} \to m_{\pm 1} \to [m_{\pm 1}/M_0]$. We define the following local system

$$\hat{\mathcal{L}}_{\pm 1} := (\hat{h}_{\mathcal{L} \pm 1})^* \mathcal{L}_{\pm 1}$$
on $\text{Hess}_{\mathcal{L} \pm 1}$. Here we view the $M_0$-local systems $\mathcal{L}_{\pm 1}$ as sheaves on $[\hat{m}_{\pm 1}/M_0]$.

**Example 6.1.** Consider the nilpotent pair $(\mathcal{O}, \mathcal{L} = \mathcal{L}_{\text{triv}})$ where $\mathcal{L}_{\text{triv}}$ is the trivial local system on $\mathcal{O}$. Using [Lus95, Proposition 7.3] one can check that in this case $q = \bigoplus_{N \in \mathbb{Z}} q_N$ is a Borel subalgebra of $V$ and $m = \bigoplus_{N \in \mathbb{Z}} m_N$ is a Cartan subalgebra. Moreover the grading on $m$ is concentrated in degree zero, i.e., $m = m_0$, and the cuspidal local system $\hat{\mathcal{L}}_{\pm 1}$ is the skyscraper sheaf supported on $m_{\pm 1} = \{0\}$. It follows that in this case $\text{Hess}_{\mathcal{L}_{\text{triv}}, 1} = 0$ and $\hat{\mathcal{L}}_{\pm 1}$ is the constant local system.

In [LY17, §7], the authors prove the following:

$$(\pi_{\mathcal{L}, -1})_* \text{IC} (\text{Hess}_{\mathcal{L}, -1}, \hat{\mathcal{L}}_{-1})$$
is the Fourier transform of $(\pi_{\mathcal{L}, 1})_* \text{IC} (\text{Hess}_{\mathcal{L}, 1}, \hat{\mathcal{L}}_1)$. \quad (6.1)

Some shift of $\text{IC}(\mathcal{O}, \mathcal{L})$ (respectively the Fourier transform of $\text{IC}(\mathcal{O}, \mathcal{L})$) appears in

$$(\pi_{\mathcal{L}, -1})_* \text{IC} (\text{Hess}_{\mathcal{L}, 1}, \hat{\mathcal{L}}_1)$$
(respectively $(\pi_{\mathcal{L}, 1})_* \text{IC} (\text{Hess}_{\mathcal{L}, -1}, \hat{\mathcal{L}}_{-1})$) as a direct summand. \quad (6.2)

Assume from now on that $\pi_{\mathcal{L}, -1} : \text{Hess}_{\mathcal{L}, -1} \to \mathfrak{g}_1$ is surjective. Then the sheaf $(\pi_{\mathcal{L}, -1})_* \text{IC} (\text{Hess}_{\mathcal{L}, -1}, \hat{\mathcal{L}}_{-1})$ is smooth over $\mathfrak{g}_1^{\text{rs}}$. One sees this as follows. According to the first statement of (6.1) the characteristic variety of $(\pi_{\mathcal{L}, -1})_* \text{IC} (\text{Hess}_{\mathcal{L}, -1}, \hat{\mathcal{L}}_{-1})$ coincides with that of $(\pi_{\mathcal{L}, 1})_* \text{IC} (\text{Hess}_{\mathcal{L}, 1}, \hat{\mathcal{L}}_1)$ as they are Fourier transforms of each other. But $(\pi_{\mathcal{L}, 1})_* \text{IC} (\text{Hess}_{\mathcal{L}, 1}, \hat{\mathcal{L}}_1)$ is $K$-equivariant and supported on the nilpotent cone. A straightforward calculation then shows the smoothness of $(\pi_{\mathcal{L}, -1})_* \text{IC} (\text{Hess}_{\mathcal{L}, -1}, \hat{\mathcal{L}}_{-1})$ on $\mathfrak{g}_1^{\text{rs}}$. Thus, by the decomposition theorem, we conclude that

$$(\pi_{\mathcal{L}, -1})_* \text{IC} (\text{Hess}_{\mathcal{L}, -1}, \hat{\mathcal{L}}_{-1})|_{\mathfrak{g}_1^{\text{rs}}}$$
is a direct sum of shifts of irreducible local systems. \quad (6.3)

In addition, the $\text{IC}(\text{Hess}_{\mathcal{L}, -1}, \hat{\mathcal{L}}_{-1})$ and hence $(\pi_{\mathcal{L}, -1})_* \text{IC} (\text{Hess}_{\mathcal{L}, -1}, \hat{\mathcal{L}}_{-1})$ has a canonical structure as a Hodge module and thus the direct summands are IC-extensions of irreducible variations of pure Hodge structure, see, [Sai88].

We fix a generic $s \in \mathfrak{g}_1^{\text{rs}}$ and then

$$H^* ((\pi_{\mathcal{L}, -1})_* \text{IC} (\text{Hess}_{\mathcal{L}, -1}, \hat{\mathcal{L}}_{-1}))_s = \text{IH}_* (\text{Hess}_{\mathcal{L}, -1, s}, \hat{\mathcal{L}}_{-1}).$$

Thus we obtain an action of the fundamental group $\pi^K_1 (\mathfrak{g}_1^{\text{rs}}, s)$ on $\text{IH}_* (\text{Hess}_{\mathcal{L}, -1, s}, \hat{\mathcal{L}}_{-1})$ and by the discussion above this action breaks into a direct sum of irreducible representations which are also variations of Hodge structure.

The component group $\pi_0 (Z_K (s)) \cong I_N$ acts on $\text{IH}_* (\text{Hess}_{\mathcal{L}, -1, s}, \hat{\mathcal{L}}_{-1})$ and we write

$$\text{IH}_* (\text{Hess}_{\mathcal{L}, -1, s}, \hat{\mathcal{L}}_{-1}) = \bigoplus_{\chi \in I_N^s} \text{IH}_* (\text{Hess}_{\mathcal{L}, -1, s}, \hat{\mathcal{L}}_{-1}) \chi$$

for the corresponding eigenspace decomposition.
**Definition 6.2.** The stable part $\text{IH}^*(\text{Hess}_{\mu,-1}, \hat{L}_{-1})_{\text{st}}$ of $\text{IH}^*(\text{Hess}_{\mu,-1}, \hat{L}_{-1})$ is the direct summand $\text{IH}^*(\text{Hess}_{\mu,-1}, \hat{L}_{-1})_{\chi_{\text{triv}}}$ where $\chi_{\text{triv}} \in I^*_N$ is the trivial character.

Observe that $\text{IH}^*(\text{Hess}_{\mu,-1}, \hat{L}_{-1})_{\text{st}}$ is stable under the monodromy action of $\pi^K_1(g_1^{\text{rs}}, s)$. Moreover, the action factors through the braid group $B_N$ via the quotient map $\pi^K_1(g_1^{\text{rs}}, s) \to B_N$.

For every irreducible representation $D_{\mu}$ of $\mathcal{H}_{N,-1}$, let $V_{\mu}$ be the local system on $g_1^{\text{rs}}$ associated to $D_{\mu}$. By Theorem 4.1, there exists a unique nilpotent pair $(O_\mu, \mathcal{L}_\mu)$ such that $\hat{\mathcal{F}}(\text{IC}(O_\mu, \mathcal{L}_\mu)) \cong \text{IC}(g_1, V_{\mu})$.

**Theorem 6.3.** Let $D_{\mu}$ be an irreducible representation of $\mathcal{H}_{N,-1}$ and let $(O_\mu, \mathcal{L}_\mu)$ be the associated nilpotent pair as above. We have the following.

(i) The map $\pi_{\mathcal{L}_\mu,-1}$ is onto, the action of the braid group $B_N$ on $\text{IH}^*(\text{Hess}_{\mathcal{L}_\mu,-1}, \hat{L}_{\mathcal{L}_\mu,-1})_{\text{st}}$ factors through the Hecke algebra $\mathcal{H}_{N,-1}$ and $\text{IH}^*(\text{Hess}_{\mathcal{L}_\mu,-1}, \hat{L}_{\mathcal{L}_\mu,-1})_{\text{st}}$ is a direct sum of irreducible representations of $\mathcal{H}_{N,-1}$.

(ii) $D_{\mu}$ appears in $\text{IH}^*(\text{Hess}_{\mathcal{L}_\mu,-1}, \hat{L}_{\mathcal{L}_\mu,-1})_{\text{st}}$ with non-zero multiplicity.

**Proof.** Since for every irreducible representation $W$ of $\text{IH}^*(\text{Hess}_{\mathcal{L}_\mu,-1}, \hat{L}_{\mathcal{L}_\mu,-1})_{\text{st}}$ the corresponding Fourier transform $\hat{\mathcal{F}}(\text{IC}(g_1, W))$ is supported on the nilpotent cone (here $W$ is the local system on $g_1^{\text{rs}}$ associated to $W$), the same argument as in the proof of Theorem 5.2 implies part (1). Part (2) follows from (6.1), (6.2), and (6.4). □

**7. Conjecture on more precise matching**

In Theorem 4.1 we show that the Fourier transform establishes a bijection between two sets of intersection cohomology sheaves. In this section we formulate a conjecture which refines the bijection in Theorem 4.1. We also relate the conjecture to our earlier conjectures in [CVX15b].

Our conjecture is not strong enough to produce an exact matching. The exact description of the bijection is crucial for applications, for example, computing cohomologies of Hessenberg varieties as explained in §5.

We begin with associating to each nilpotent orbit $O_\lambda$ (respectively $O_\mu^\prime$, $\omega = 1, \Pi$) a subset $\Sigma_\lambda \subset \Sigma_N$ (respectively $\Sigma_\mu^\prime \subset \Sigma_N$), if $\lambda \in \mathcal{P}(N)$ has at least one odd part (respectively has only even parts). Let $\lambda$ be a partition of $N$ and let $\lambda'$ be the transpose partition of $\lambda$. Suppose that

$$\lambda' = (\lambda'_1)^{2m_1} \cdots (\lambda'_{l+1})^{2m_{l+1}-1} \cdots (\lambda'_k)^{2m_k-1}, \quad (7.1)$$

where $m_i \geq 1$, $i = 1, \ldots, k$. Here and in what follows we write the parts in a partition in the order which is most convenient for us. In particular, in (7.1) we place the parts with even multiplicity before the parts with odd multiplicity.

Let $\delta_i \in \{0, 1\}$ for $i \in [1, l]$ and let

$$\nu(\delta_1, \ldots, \delta_l) = (\lambda'_1)^{m_1-\delta_1} \cdots (\lambda'_{l+1})^{m_{l+1}-\delta_l} (\lambda'_{l+1})^{m_{l+1}-1} \cdots (\lambda'_k)^{m_k-1},$$

$$\mu(\delta_1, \ldots, \delta_l) = (\lambda'_1)^{2\delta_1} \cdots (\lambda'_{l+1})^{2\delta_l} (\lambda'_{l+1}) \cdots (\lambda'_k).$$

Note that $2|\nu(\delta_1, \ldots, \delta_l)| + |\mu(\delta_1, \ldots, \delta_l)| = N$. Let

$$J \subset J_0 := \{l + 1, \ldots, k\}$$

such that $\sum_{j \in J} \lambda'_j < \sum_{j \in J_0 - J} \lambda'_j$. 

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We define
\[ \mu^1(\delta_1, \ldots, \delta_l; J) = (\lambda_1^1)^{\delta_1} \cdots (\lambda_l^1)^{\delta_l}(\lambda_{l+1}^1) \cdots (\lambda_{l+k-l-s}^1), \quad J = \{j_1, \ldots, j_s\}. \]
\[ \mu^2(\delta_1, \ldots, \delta_l; J) = (\lambda_1^2)^{\delta_1} \cdots (\lambda_l^2)^{\delta_l}(\lambda_{l+1}^2) \cdots (\lambda_{l+k-l-s}^2), \quad J_0 - J = \{i_1, \ldots, i_{k-l-s}\}. \]

Note that \( \lambda_{l+1}^1 = 0 \) if and only if all parts of \( \lambda \) are even. In this case, \( J_0 = \emptyset = J \) and \( \mu^1(\delta_1, \ldots, \delta_l; J) = \mu^2(\delta_1, \ldots, \delta_l; J) \) and we write \( \mu(\delta_1, \ldots, \delta_l; J) = \mu^i(\delta_1, \ldots, \delta_l; J), \ i = 1, 2. \)

If \( \lambda \) has at least one odd part, then let
\[ \Sigma_\lambda := \left\{ (\nu(\delta_1, \ldots, \delta_l); \mu^1(\delta_1, \ldots, \delta_l; J), \mu^2(\delta_1, \ldots, \delta_l; J)) \mid \delta_i \in \{0, 1\}, i = 1, \ldots, l, \right. \]
\[ \left. J \subset \{l + 1, \ldots, k\}, \text{ such that } \sum_{j \in J} \lambda^1_j < \sum_{j \in J_0 - J} \lambda^2_j \right\}. \]

If all parts of \( \lambda \) are even (in which case \( \lambda_{l+1}^1 = 0 \)), then let
\[ \Sigma^\omega_\lambda = \left\{ (\nu(\delta_1, \ldots, \delta_l); \mu(\delta_1, \ldots, \delta_l); \mu(\delta_1, \ldots, \delta_l))^\omega \mid \delta_i \in \{0, 1\}, i = 1, \ldots, l, \right\}. \quad \omega = I, II. \]

We have \( |\Sigma_\lambda| = 2^{k-1} \) (respectively \( |\Sigma^\omega_\lambda| = 2^l \)), which equals the number of non-isomorphic irreducible \( K \)-equivariant local systems on \( O_\lambda \) (respectively \( O^\omega_\lambda \)).

**Conjecture 7.1.** Let \( \lambda \) be a partition of \( N. \)

(1) If \( \lambda \) has at least one odd part, then the Fourier transform \( \mathcal{F} \) induces the following bijection:
\[ \mathcal{F} : \{ \text{IC}(O_\lambda, E) \mid E \text{ irreducible } K\text{-equivariant local system on } O_\lambda(\text{up to isomorphism}) \} \]
\[ \xrightarrow{\sim} \{ \text{IC}(g_1^{[|\nu|]}, T(\nu; \mu_1, \mu_2)) \mid (\nu; \mu_1, \mu_2) \in \Sigma_\lambda \}. \]
Moreover,
\[ \mathcal{F}(\text{IC}(O_\lambda, C)) = \text{IC}(g_1^{[|\nu|]}, T(\nu_0; \mu_1^1, \mu_0^2)) \]
where \( (\nu_0; \mu_1^1, \mu_0^2) \in \Sigma_\lambda \) is the unique triple such that \( |\nu_0| = \max\{|\nu|, (\nu, \mu_1, \mu_2) \in \Sigma_\lambda\} \) and the parts of \( \mu_0^2 \) and the parts of \( \mu_0^1 \) have the opposite parity (in particular, all parts of \( \mu_0^1 \) have the same parity).

(2) If all parts of \( \lambda \) are even, then the Fourier transform induces the following bijection
\[ \mathcal{F} : \{ \text{IC}(O^\omega_\lambda, E) \mid \omega = I, II, E \text{ irreducible } K\text{-equivariant local system on } O^\omega_\lambda(\text{up to isom}) \} \]
\[ \xrightarrow{\sim} \{ \text{IC}(g_1^{[|\nu|]}, T(\nu; \mu, \mu)^\omega) \mid \omega = I, II, (\nu; \mu, \mu)^\omega \in \Sigma^\omega_\lambda, \mu \neq \emptyset \} \]
\[ \cup \{ \text{IC}(g_1^{[|\nu|]}, T(\nu; \emptyset, \emptyset)) \mid \omega = I, II, (\nu; \emptyset, \emptyset) \in \Sigma^\omega_\lambda \}. \]
Moreover,
\[ \mathcal{F}(\text{IC}(O^\omega_\lambda, C)) = \text{IC}(g_1^{[|\nu|]}, T(\nu_0; \emptyset, \emptyset)), \]
where \( |\nu_0| = n \) and \( (\nu_0; \emptyset, \emptyset) \in \Sigma_\lambda \).

Note that \( \mathcal{F}(\text{IC}(O_\lambda, E)) \) has full support if and only if \( \nu(\delta_1, \ldots, \delta_l) = \emptyset. \) Thus we see that the conjecture is compatible with Corollary 4.9. We also remark that special cases of the conjecture are verified by [CVX15a, Theorems 4.1 and 4.3].

Let us relate the conjecture above to our previous conjectures in [CVX15b]. In [CVX15b] we constructed local systems \( E_{n,i,j}^{2n+1} \) and \( E_{n,i,j}^{2n+1} \) on \( g_1^n. \) In terms of the parametrization introduced
in this paper, we have
\[ E_{i,j}^{2n+1} = T(\emptyset; (2i - j, j), (2n + 1 - 2i)) \]
\[ \tilde{E}_{i,j}^{2n+1} = T(\emptyset; (2i - 1 - j, j), (2n + 2 - 2i)). \]

Thus we see that Conjecture 7.1 applied to \( E_{i,j}^{2n+1} \) agrees with [CVX15b, Conjectures 6.1 and 6.3]. Applied to \( \tilde{E}_{i,j}^{2n+1} \), Conjecture 7.1 implies that the supports of \( \mathfrak{g}(IC(g_1, \tilde{E}_{i,j}^{2n+1})) \) are as follows:
\[
\begin{align*}
O_{3j/2i-2j-12n+3-4i+j} & \quad \text{if } 4i - j \leq 2n + 3, \\
O_{3j/2n+2-2i-j14i-j-2n-3} & \quad \text{if } 2i + j \leq 2n + 2 \text{ and } 4i - j \geq 2n + 3, \\
O_{3j/2n-2i+22i+j-2n-213i-2j-1} & \quad \text{if } 2i + j \geq 2n + 2.
\end{align*}
\]

Note that the above orbits are all of even dimension and each of the even-dimensional orbits appears twice there.

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