TORSION OF RATIONAL ELLIPTIC CURVES
OVER CUBIC FIELDS

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Abstract. Let \( E \) be an elliptic curve defined over \( \mathbb{Q} \). We study the relationship between the torsion subgroup \( E(\mathbb{Q})_{\text{tors}} \) and the torsion subgroup \( E(K)_{\text{tors}} \), where \( K \) is a cubic number field. In particular, we study the number of cubic number fields \( K \) such that \( E(\mathbb{Q})_{\text{tors}} \neq E(K)_{\text{tors}} \).

1. Introduction

Let \( K \) be a number field. The Mordell-Weil Theorem states that the set of \( K \)-rational points of an elliptic curve \( E \) defined over \( K \) is a finitely generated abelian group. That is, \( E(K) \approx E(K)_{\text{tors}} \oplus \mathbb{Z}^r \), where \( E(K)_{\text{tors}} \) is the torsion subgroup and \( r \) is the rank. Moreover, it is well known that \( E(K)_{\text{tors}} \approx C_m \times C_n \) for two positive integers \( m, n \), where \( m \) divides \( n \) and where \( C_n \) is a cyclic group of order \( n \) from now on.

Let \( d \) be a positive integer. The set \( \Phi(d) \) of possible torsion structures of elliptic curves defined over number fields of degree \( d \) has been deeply studied by several authors. The case \( d = 1 \) was obtained by Mazur [15, 16]:

\[
\Phi(1) = \{C_n \mid n = 1, \ldots, 10, 12\} \cup \{C_2 \times C_{2m} \mid m = 1, \ldots, 4\}.
\]

The case \( d = 2 \) was completed by Kamienny [9] and Kenku and Momose [13]. There are not any other cases where \( \Phi(d) \) has been completely determined.

The second author [18] has extended this study to the set \( \Phi(\mathbb{Q})(d) \) of possible torsion structures over a number field of degree \( d \) of an elliptic curve defined over \( \mathbb{Q} \). He has obtained a complete description of \( \Phi(\mathbb{Q})(2) \) and \( \Phi(\mathbb{Q})(3) \). For convenience, we will write here only the latter set:

\[
\Phi(\mathbb{Q})(3) = \{C_n \mid n = 1, \ldots, 10, 12, 13, 14, 18, 21\} \cup \{C_2 \times C_{2m} \mid m = 1, \ldots, 7\}.
\]

Fix a possible torsion structure over \( \mathbb{Q} \), say \( G \in \Phi(1) \). Recently, in [5] the set \( \Phi(\mathbb{Q})(2, G) \) of possible torsion structures over a quadratic number field of an elliptic curve defined over \( \mathbb{Q} \) such that \( E(\mathbb{Q})_{\text{tors}} \approx G \in \Phi(1) \) was determined. The first goal of this paper is giving a complete description (see Theorem 2) of \( \Phi(\mathbb{Q})(3, G) \), as was done in [5, Theorem 2] for the case \( d = 2 \).

Moreover, in [6] the first and third author obtained, for \( d = 2 \) and for all \( G \in \Phi(1) \), the set

\[
\mathcal{H}_d(G) = \{S_1, \ldots, S_n\}
\]

where, for any \( i = 1, \ldots, n \), \( S_i = [H_1, \ldots, H_m] \) is a list, with \( H_i \in \Phi(\mathbb{Q})(d, G) \setminus \{G\} \), and there exists an elliptic curve \( E_i \) defined over \( \mathbb{Q} \) such that:

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• $E_i(\mathbb{Q})_{\text{tors}} = G$.
• There are number fields $K_1, \ldots, K_m$ (non–isomorphic pairwise) of degree $d$ with $E_i(K_j)_{\text{tors}} = H_j$, for all $j = 1, \ldots, m$.

Note that we are allowing the possibility of two (or more) of the $H_j$ being isomorphic. From these results, it follows [6, 19]:

**Corollary 1.** If $E$ is an elliptic curve defined over $\mathbb{Q}$, then there are at most four quadratic fields $K_i$, $i = 1, \ldots, 4$ (non–isomorphic pairwise), such that $E(K_i)_{\text{tors}} \neq E(\mathbb{Q})_{\text{tors}}$. That is,

$$\max_{G \in \Phi(1)} \left\{ \#S \mid S \in \mathcal{H}_Q(2, G) \right\} = 4.$$

Here, we obtain the equivalent description for the case $d = 3$. That is, we give a complete description of $\mathcal{H}_Q(3, G)$ for a given $G \in \Phi(1)$ (see Theorem 3). Precisely, the main results of this paper are the following:

**Theorem 2.** For $G \in \Phi(1)$, the set $\Phi_Q(3, G)$ is the following:

| $G$          | $\Phi_Q(3, G)$                          |
|--------------|----------------------------------------|
| $C_1$        | $\{C_1, C_2, C_3, C_4, C_6, C_7, C_{13}, C_2 \times C_2, C_2 \times C_{14}\}$ |
| $C_2$        | $\{C_2, C_6, C_{14}\}$                |
| $C_3$        | $\{C_3, C_6, C_7, C_9, C_{12}, C_{21}, C_2 \times C_6\}$ |
| $C_4$        | $\{C_4, C_{12}\}$                     |
| $C_5$        | $\{C_5, C_{10}\}$                     |
| $C_6$        | $\{C_6, C_{18}\}$                     |
| $C_7$        | $\{C_7, C_{14}\}$                     |
| $C_8$        | $\{C_8\}$                             |
| $C_9$        | $\{C_9, C_{18}\}$                     |
| $C_{10}$     | $\{C_{10}\}$                          |
| $C_{12}$     | $\{C_{12}\}$                          |
| $C_2 \times C_2$ | $\{C_2 \times C_2, C_2 \times C_6\}$ |
| $C_2 \times C_4$ | $\{C_2 \times C_4\}$                |
| $C_2 \times C_6$ | $\{C_2 \times C_6\}$                |
| $C_2 \times C_8$ | $\{C_2 \times C_8\}$                |

The sets $\Phi_Q(3, G)$ were first implies by the computations that can be found in the appendix. These computations also prove that all the listed groups actually are in $\Phi_Q(3, G)$.

**Theorem 3.** Let $E$ be an elliptic curve defined over $\mathbb{Q}$. Then:

(i) There is at most one cubic number field $K$, up to isomorphism, such that $E(K)_{\text{tors}} \simeq H \neq E(\mathbb{Q})_{\text{tors}}$, for a fixed $H \in \Phi_Q(3)$.

(ii) There are at most three cubic number fields $K_i$, $i = 1, 2, 3$ (non–isomorphic pairwise), such that $E(K_i)_{\text{tors}} \neq E(\mathbb{Q})_{\text{tors}}$.

Moreover, the elliptic curve $162b2$ is the unique rational elliptic curve where the torsion grows over three non–isomorphic cubic fields.
(iii) Let be $G \in \Phi(1)$ such that $\Phi(Q) \neq \{G\}$. Then the set $\mathcal{H}_Q(3, G)$ consists of the following elements (third row is $h = \#S$, for each $S \in \mathcal{H}_Q(3, G)$):

| $G$ | $\mathcal{H}_Q(3, G)$ | $h$ |
|-----|---------------------|-----|
| $C_2$ | $C_2$ | 1 |
| $C_4$ | $C_6$, $C_{14}$ | 1 |
| $C_6$ | $C_2 \times C_2$, $C_2 \times C_{14}$ | 2 |
| $C_2 \times C_2$ | $C_4$, $C_7$, $C_2 \times C_{13}$ | 3 |
| $C_2 \times C_3$, $C_2 \times C_7$ | $C_2 \times C_4$, $C_2 \times C_2$, $C_2 \times C_6$ | 1 |
| $C_2 \times C_2 \times C_2$ | $C_2 \times C_2 \times C_2$ | 1 |

The best result previously known [8, Lemma 3.3] stated that the torsion subgroup of a rational elliptic curve grows strictly in only finitely many cubic number fields.

**Notation:** Please mind that, in the sequel, for examples and precise curves we will use the Antwerp–Cremona tables and labels [1, 2]. We will write $G = H$ (respectively $G < H$ or $G \leq H$) for the fact that $G$ is isomorphic to $H$ (or to a subgroup of $H$) without further detail on the precise isomorphism.

## 2. Auxiliary results

We will fix once and for all some notations. We will use a short Weierstrass equation for an elliptic curve $E$,

$$E : Y^2 = X^3 + AX + B, \quad A, B \in \mathbb{Z},$$

with discriminant $\Delta$.

For such an elliptic curve $E$ and an integer $n$, let $E[n]$ be the subgroup of all points which order is a divisor of $n$ (over $\mathbb{Q}$), and let $E(K)[n]$ be the set of points in $E[n]$ with coordinates in $K$, for a number field $K$. Let us recall the following well-known result [21, Ch. III, 8.1.1]

**Proposition 4.** Let $E$ be an elliptic curve over a number field $K$. If $C_m \times C_m \leq E(K)$, then $K$ contains the cyclotomic field $\mathbb{Q}(\zeta_m)$ generated by the $m$-th roots of unity.

Let us fix the set–up, following [18]. Let $K/\mathbb{Q}$ be a cubic extension, and $L$ the normal closure of $K$ over $\mathbb{Q}$. Finally, let $M$ be the only subextension $\mathbb{Q} \subset M \subset L$ such that $[L : M] = 3$. Therefore, we have two possible situations:

- The extension $K/\mathbb{Q}$ is Galois. Then $\mathbb{Q} = M$ and $K = L$. \hfill
The extension $K/Q$ is not Galois. Then we have

```
\[
\begin{array}{c}
  \text{L} \\
  \downarrow 2 \\
  K \\
  \downarrow 3 \\
  M \\
  \downarrow 2 \\
  \text{Q}
\end{array}
\]
```

**Remark.** Let $\alpha \in \mathbb{Q}$. If there is some $\beta \in K$ with $\alpha = \beta^2$, then $\beta \in \mathbb{Q}$.

Now we will recall some results from [18] which will come in handy.

**Proposition 5.** Let $E$ be an elliptic curve defined over $\mathbb{Q}$, $K$, $L$ and $M$ as above, $G \in \Phi_\mathbb{Q}(1)$ and $H \in \Phi_\mathbb{Q}(3)$ such that $E(\mathbb{Q})_{\text{tors}} \simeq G$ and $E(K)_{\text{tors}} \simeq H$.

(i) If $G$ has a non-trivial $2$-Sylow subgroup, $G$ and $H$ have the same $2$-Sylow subgroup [18, Lemma 8].

(ii) If $C_4 \not\leq G$, then $C_8 \not\leq H$ and, if $C_4 \leq H$, then $M = \mathbb{Q}(i)$ and $\Delta \in (-1)(\mathbb{Q}^*)^2$ [4, 18, Corollary 12].

(iii) $E(K)[5] = E(\mathbb{Q})[5]$ [18, Lemma 21].

(iv) If $H = C_{21}$, then $E$ is the elliptic curve $162b1$ and $K = \mathbb{Q}(\zeta_9)^+$ [18, Theorem 2].

(v) If $G = C_7$ then $H \neq C_2 \times C_{14}$ [18, Proof Prop. 29].

(vi) If $E(M)$ has no points of order $3$, neither does $E(L)$ [18, Lemma 13].

Also some results on isogenies will be needed:

**Proposition 6.** Let $E$ be an elliptic curve defined over $\mathbb{Q}$, $K$ and $L$ as above.

(i) Assume $E$ has a rational n-isogeny. Then either $1 \leq n \leq 19$, or $n \in \{21, 25, 27, 37, 43, 67, 163\}$ [16, 10, 11, 12].

(ii) Assume $n$ is odd and not divisible by $3$. If $E(K)$ has a point of order $n$, then $E$ has a rational isogeny of degree $n$ [18, Lemma 18].

(iii) If $F$ is a number field and $E$ has two independent isogenies over $F$ with degrees $n$ and $m$, $E$ is isogeneous (over $F$) to an elliptic curve with an $mn$-isogeny [18, Lemma 7].

(iv) If $K = L$, $n$ is an odd integer and $E(K)$ has a point of order $n$, then $E$ has a rational n-isogeny [18, Lemma 19].

(v) Let $F$ be a quadratic number field, $n$ an odd integer and $E/Q$ an elliptic curve such that $C_n \leq E(F)$. Then $E$ has a rational n-isogeny [18, Lemma 5].

(vi) Assume $E(K)$ has a point of order $9$. Then either $E/Q$ has a 9-isogeny or it has two independent 3-isogenies [18, Proposition 14].

**Lemma 7.** Let $p$ be prime, $f$ a $p$-isogeny on $E/Q$, and let $\ker(f)$ be generated by $P$. Then the field of definition $\mathbb{Q}(P)$ of $P$ (and all of its multiples) is a cyclic (Galois) extension of $\mathbb{Q}$ of order dividing $p - 1$. 
Proof. First note that the fact that $F = \mathbb{Q}(P)$ is Galois over $\mathbb{Q}$ follows immediately from the Galois-invariance of $\langle P \rangle$. Let $\chi$ be the character of the isogeny,

$$\chi : \text{Gal}(F/\mathbb{Q}) \longrightarrow \text{Aut}(\langle P \rangle),$$

which, to each element of Gal$(F/\mathbb{Q})$, adjoins its action on $\langle P \rangle$. It is easy to check that this is a homomorphism.

Suppose that $\chi$ is not an injection. Then there exists an element $\sigma$, not the identity, such that $\chi(\sigma) = \text{id}$, so $\langle \sigma \rangle$ acts trivially on $P$. Denoting $F_0 = F^\sigma$ (the fixed field of $\langle \sigma \rangle$), every automorphism of Gal$(F/F_0)$ fixes $P$, and hence $P$ is $F_0$-rational, which is in contradiction with the minimality of $F$.

Since Gal$(F/\mathbb{Q})$ is isomorphic to a subgroup of Aut$(\langle P \rangle)$, which is isomorphic to $C_{p-1}$, we are finished. \hfill \Box

Lemma 8. If $E(K)$ has a point of order 3 over a cubic field $K$, then $E$ has a 3-isogeny over $\mathbb{Q}$. \hfill \Box

Proof. $E(L)$ has a point of order 3, so $E(M)$ has a point of order 3 from Proposition 5 (vi). And by Proposition 6 (v), $E$ has a 3-isogeny over $\mathbb{Q}$. \hfill \Box

Lemma 9. If $E(K)$ has a point of order 9, then $E(\mathbb{Q})$ has a point of order 3. \hfill \Box

Proof. By Proposition 3 (vi) $E/\mathbb{Q}$ has either an isogeny of degree 9 or 2 isogenies of degree 3.

First suppose it has 2 isogenies of degree 3 and no 3-torsion. Then it follows that $\mathbb{Q}(E[3])$ is a biquadratic field and the intersection of $\mathbb{Q}(E[3])$ and $K$ must be trivial (that is, $\mathbb{Q}$), which contradicts the fact that $E(K)$ has non-trivial 3-torsion. Hence $E(\mathbb{Q})$ has a 3-torsion point.

Now suppose $E/\mathbb{Q}$ has a 9-isogeny $f$, such that ker$(f) = \langle P \rangle$, and such that $P$ is $K$-rational. Then the isogeny character

$$\chi : \text{Gal}(K/\mathbb{Q}) \longrightarrow \text{Aut}(\langle P \rangle)$$

sends the generator $\sigma$ of Gal$(K/\mathbb{Q})$ into an element of order 3 in Aut$(\langle P \rangle)$, i.e. into $[4]$ or $[7]$. Both of these act trivially on $\langle 3P \rangle$, implying that $E(\mathbb{Q})$ has non-trivial 3-torsion. \hfill \Box

Remark. Now and then we will consider the case where we have $K_1$ and $K_2$ two different cubic number fields. Let us write as usual $K_1K_2$ for the compositum field of both extensions. Then one of these two situations hold:

- $|K_1K_2 : \mathbb{Q}| = 9.$
- $|K_1K_2 : \mathbb{Q}| = 6.$ In this case, $K_1$ and $K_2$ are isomorphic and $K_1K_2$ is the Galois closure of both fields over $\mathbb{Q}$.

3. Proof of Theorem 2

Note that from Proposition 3 (i), if $G = C_{2n}$, for some $n \neq 0$, then $C_2 \times C_2 \not\subset H$. Also from Proposition 4 (i) and the description of $\Phi_3(3)$, we can solve the non-cyclic cases from Theorem 2 easily, as we know that

$$\Phi_3(3, C_2 \times C_{2n}) \leq \begin{cases} \{C_2 \times C_2, C_2 \times C_6, C_2 \times C_{14}\} & \text{if } n = 1, \\ \{C_2 \times C_{2n}\} & \text{if } n \neq 1. \end{cases}$$

The only case that will not happen and we cannot discard already is $G = C_2 \times C_2$, $H = C_2 \times C_{14}$. But this case cannot happen as, from Proposition 3 (ii) and (iii),
that would imply $E$ has a 28–isogeny, contradicting Proposition 6 (i). This finishes the non–cyclic case.

Let us move therefore to the cyclic case. The groups $H$ from $\Phi_\mathbb{Q}(3)$ that do not appear in some $\Phi_\mathbb{Q}(3, G)$, with a $G < H$ and $G$ cyclic can be ruled out from $\Phi_\mathbb{Q}(3, G)$ most of the times using the previous results. In the table below we indicate:

- With (i) - (vi), which part of Proposition 5 is used,
- With (9), the case is ruled out from Lemma 9
- With $-$, the case is ruled out because $G \not\subset H$,
- With $\checkmark$, the case is possible (and in fact, it occurs).

The table (row $= H$, column $= G$) deals with the case $G$ cyclic.

|   | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ | $C_6$ | $C_7$ | $C_8$ | $C_9$ | $C_{10}$ | $C_{12}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $C_1$ | $\checkmark$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ |
| $C_2$ | $\checkmark$ | $\checkmark$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ |
| $C_3$ | $\checkmark$ | $-$ | $\checkmark$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ |
| $C_4$ | $\checkmark$ | (i) | $-$ | $\checkmark$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ |
| $C_5$ | (iii) | $-$ | $-$ | $-$ | $\checkmark$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ |
| $C_6$ | $\checkmark$ | $\checkmark$ | $\checkmark$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ |
| $C_7$ | $\checkmark$ | $-$ | $-$ | $-$ | $-$ | $\checkmark$ | $-$ | $-$ | $-$ | $-$ | $-$ |
| $C_8$ | (ii) | (i) | $-$ | (i) | $-$ | $-$ | $\checkmark$ | $-$ | $-$ | $-$ | $-$ |
| $C_9$ | (9) | $-$ | $\checkmark$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $\checkmark$ | $-$ |
| $C_{10}$ | (iii) | (iii) | $-$ | $\checkmark$ | $-$ | $-$ | $-$ | $\checkmark$ | $-$ | $-$ | $-$ |
| $C_{12}$ | (?) | (i) | $\checkmark$ | $-$ | (i) | $-$ | $-$ | $-$ | $-$ | $\checkmark$ | $-$ |
| $C_{13}$ | $\checkmark$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ |
| $C_{14}$ | (?) | $\checkmark$ | $-$ | $-$ | $-$ | $-$ | $\checkmark$ | $-$ | $-$ | $-$ | $-$ |
| $C_{18}$ | (9) | (9) | (?) | $-$ | $-$ | $\checkmark$ | $-$ | $\checkmark$ | $-$ | $-$ | $-$ |
| $C_{21}$ | (iv) | $-$ | $\checkmark$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ |
| $C_2 \times C_2$ | $\checkmark$ | (i) | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ |
| $C_2 \times C_4$ | (?) | (i) | $-$ | (i) | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ |
| $C_2 \times C_6$ | (?) | (i) | $-$ | (i) | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ |
| $C_2 \times C_8$ | (ii) | (i) | $-$ | (i) | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ | $-$ |
| $C_2 \times C_{14}$ | $\checkmark$ | (i) | $-$ | $-$ | $-$ | (v) | $-$ | $-$ | $-$ | $-$ | $-$ |

Let us now discard the remaining cases.

The case $G = C_1$, $H = C_{12}$. In this case, from Proposition 5 (ii,vi), we already know that $M = \mathbb{Q}(i)$ and $E(M)[3] \neq \{O\}$. Again as above, having points of order 3 in both $M$ and $K$ implies that these are independent points and hence $E[3](L) \simeq C_3 \times C_3$, from which it follows that $M = \mathbb{Q}(\zeta_3)$, which is a contradiction.

The case $G = C_1$, $H = C_{14}$. In this case $E$ must have a rational 7–isogeny, from Proposition 6 (ii). Then, from Lemma 7 we know that $K$ is a cyclic cubic Galois extension, hence $K = L$. Under these circumstances, $E(K)[2]$ cannot be $C_2$, as $K$ is either the splitting field of $X^3 + AX + B$ (in which case $E(K)[2] = C_2 \times C_2$) or is irreducible over $K$, in which case there are no points of order 2 in $E(K)$.

The case $G = C_1$, $H = C_2 \times C_4$. Assume our curve is given in Weierstrass short form

$$Y^2 = X^3 + AX + B.$$
If \( G \) is cyclic and \( H \) is not, \( K \) must be the splitting field of \( X^3 + AX + B \). So in this case \( \mathbb{Q} = M \), and \( K = L \), but this contradicts Proposition\(^5\)(ii).

**The case \( G = C_1 \), \( H = C_2 \times C_6 \).** As in the previous case, \( \mathbb{Q} = M \), and \( K = L \). But there are points of order 3 in \( E(L) \), so \( E(M)[3] \neq \{O\} \), but this contradicts \( G = C_1 \), as \( \mathbb{Q} = M \).

**The case \( G = C_3 \), \( H = C_{18} \).** As we gain exactly one 2–torsion point in the passing from \( \mathbb{Q} \) to \( K \), we already know that \( K \) is not Galois and, in fact, \( L \) must be the splitting field of \( X^3 + AX + B \). Then, from Lemma\(^7\) and Proposition\(^6\)(vi) we have that \( E(\mathbb{Q}) \) must have 2 isogenies of degree 3.

Now we look at how \( \text{Gal}(L/\mathbb{Q}) \) acts on \( E[9] \). The \( L \)-rational points have to be sent to \( L \)-rational points. So if \( P \) is an \( L \)-rational point of order 9, the generators of \( \text{Gal}(L/\mathbb{Q}) \) cannot both send \( P \) to a multiple of \( P \), because this would imply that \( \langle P \rangle \) is \( \text{Gal}(L/\mathbb{Q}) \)-invariant (and hence \( \text{Gal}(\mathbb{Q}/\mathbb{Q}) \)-invariant), which would imply a 9–isogeny over \( \mathbb{Q} \). So this means that \( E[9](L) \) is strictly larger than \( C_9 \). The only possibility is that \( E[9](L) = C_3 \times C_9 \) and this implies \( M = \mathbb{Q}(\sqrt{-3}) \) because of Proposition\(^4\).

As \( L \) is the splitting field of \( X^3 + AX + B \), this really implies \( E(L)_{\text{tors}} \leq C_6 \times C_{18} \). Moreover, as the quadratic subextension of \( L \) is \( \mathbb{Q}(\sqrt{-3}) \), \( L \) is a pure cubic field and our curve is a Mordell curve \( Y^2 = X^3 + n \), for some \( n \in \mathbb{Z} \). But the only elliptic curve with \( j \)-invariant 0 defined over \( \mathbb{Q} \) which has full 3–torsion over \( \mathbb{Q}(\sqrt{-3}) \) is \( 27a1 \) (and also its \(-3 \) twist), and by simply computing that this curve has \( L \)-torsion \( C_6 \times C_6 \), we are finished.

### 4. Proof of Theorem\(^3\)

**Proof of (i).** Let \( E \) be an elliptic curve defined over \( \mathbb{Q} \) such that \( E(\mathbb{Q})_{\text{tors}} \cong G \in \Phi(1) \) and \( H \in \Phi_3(3) \). Let us prove that there is at most one cubic number field \( K \) such that \( E(K)_{\text{tors}} \cong H \neq G \).

First, let be \( H = G \times C_m \) such that \( \gcd(|G|, m) = 1 \). Suppose that there exist two cubic fields \( K_1 \) and \( K_2 \) such that \( E(K_i)_{\text{tors}} \cong H \), \( i = 1, 2 \). Then \( C_m \times C_m \leq E(L)_{\text{tors}} \), where \( L \) is the degree 9 number field obtained by composition of \( K_1 \) and \( K_2 \). Therefore, \( \mathbb{Q} \times \mathbb{Q} \subset L \), which implies that \( \varphi(m) \) divides 9. This eliminates the following possibilities:

- \( G = C_1 \) and \( H \in \{C_3, C_4, C_6, C_7, C_{13}\} \);
- \( G = C_2 \) and \( H \in \{C_6, C_{14}\} \);
- \( G = C_3 \) and \( H \in \{C_{12}, C_{21}\} \);
- \( G = C_4 \) and \( H = C_{12} \);
- \( G = C_2 \times C_2 \) and \( H = C_2 \times C_6 \);

On the other hand, if the order of \( G \) is odd then there is at most one \( H \) of even order with \( G < H \). The cubic field is the one defined by the 2–division polynomial of the elliptic curve. This argument therefore crosses out the cases:

- \( G = C_1 \) and \( H \in \{C_2, C_2 \times C_2, C_2 \times C_{14}\} \);
- \( G = C_3 \) and \( H \in \{C_6, C_2 \times C_6\} \);
- \( G = C_5 \) and \( H = C_{10} \);
- \( G = C_7 \) and \( H = C_{14} \);
- \( G = C_9 \) and \( H = C_{18} \);
The remaining cases to be dealt with are $G = C_3$ with $H = C_9$ and $G = C_9$ with $H = C_{18}$. These are essentially the same since $C_6 = C_3 \times C_3$ and $C_{18} = C_2 \times C_9$.

Assume we have $\langle P \rangle \cong C_3$, $\langle Q \rangle \cong C_9$, where $P$ and $Q$ are defined over two non-isomorphic cubic fields. Therefore $P$ is not a multiple of $Q$ and $Q$ is not a multiple of $P$ and $C_3 \times C_3 \leq \langle P, Q \rangle$. This is impossible, since both $P$ and $Q$ would be defined over a field of degree 9, which cannot contain $\mathbb{Q}(\zeta_3)$.

This proves the first statement of Theorem 3.

**Proof of (ii) and (iii).** First note that if

$$E : Y^2 = f(X)$$

is an elliptic curve defined over $\mathbb{Q}$ such that $E(\mathbb{Q})_{\text{tors}} \cong G$ has odd order, then $f(X)$ is an irreducible cubic polynomial. Now, denote by $K$ the cubic field defined by $f(X)$, then $H = E(K)_{\text{tors}}$ satisfies that $G \neq H$ and $H$ is of even order. Moreover, $H$ is the unique group of even order such that $H \in S$, for any $S \in H_3(3, G)$ because $f(X)$ is the 2-division polynomial of $E$.

Now, for any $G \in \Phi(1)$ let us construct the elements $S \in H_3(3, G)$ in ascending order of $\#S$. In Table 1 (see Appendix) we show examples for all the possible cases of $S$ (after taking into account the preliminary remark) for any $G \in \Phi(1)$. Now, by (i) we know that there are not repeated elements in any $S \in H_3(3, G)$. Then the possible cases with $\#S > 1$ come from $G = C_1, C_2, C_3$:

$G = C_1$

We have examples in Table 1 for any $S \in H_3(3, C_1)$ with $\#S = 2$ except for the cases:

$[C_1, C_{13}], [C_3, C_6], [C_6, C_1], [C_6, C_{13}], [C_2 \times C_2, C_{13}], [C_2 \times C_{14}, C_3], [C_2 \times C_{14}, C_7], [C_2 \times C_{14}, C_{13}]$.

- As for $[C_3, C_{13}]$, if such a curve existed then it would have to have discriminant $-Y^2$ (as it gains 4-torsion - see Proposition 3 (ii)) for some rational $Y$. On the other hand, the curve must have a 13-isogeny over $\mathbb{Q}$, which implies its discriminant is of the form [18, Lemma 27]

$$\Delta = \Box \cdot t(t^2 + 6t + 13)$$

where $\Box$ is a rational square. Therefore such a curve would give a rational non-trivial (meaning $Y \neq 0$) solution of the equation

$$Y^2 = X^3 - 6X^2 + 13X,$$

but one easily checks that there are none.

- Let us look at the pair $[C_6, C_7]$. The existence of $C_6$ implies a 3-isogeny over $\mathbb{Q}$ and the existence of $C_7$ implies a rational 7-isogeny, hence $E$ has a 21-isogeny. Therefore $E$ is a twist of an elliptic curve in the 162b isogeny class. It can be seen that only one elliptic curve in each of the 4 family of twists gains 7-torsion in a cubic extension. Thus there are in fact 4 curves that we need to check, all in all. For each of the 4 curves we can
check whether the curve gains any 3-torsion in the fields where it gains 2-torsion, and discard all the cases.

- The case $[C_6, C_{13}]$ can be ruled out as, from Proposition 6 (iii) and Lemma 8, it would imply the existence of a curve with a rational 39-isogeny, contradicting Proposition 3 (i).
- The case $[C_2 \times C_2, C_{13}]$ is very similar to the first one, the only difference being that, gaining full 2-torsion over a cubic field, the discriminant must be a square. Anyway, the corresponding equation

$$Y^2 = X^3 + 6X^2 + 13X,$$

still has no solutions with $Y \neq 0$.
- Let us look at the case $[C_2 \times C_{14}, C_3]$. A curve featuring these torsion extensions would have a 21-isogeny from Proposition 6 (ii, iv) and Lemma 8 and also would gain full 2-torsion over a cubic field, so as in the previous case its discriminant must be a square. But the elliptic curves with a 21-isogeny have discriminant $-2 \Box$, where $\Box$ is a rational square [1, pp. 78–80]. Hence this case is not possible.
- We can remove the case $[C_2 \times C_{14}, C_7]$, similarly as the second case. In this case we would have two cubic extensions $K_1$ and $K_2$ which must verify $[K_1 K_2 : \mathbb{Q}] = 9$, as $X^3 + AX + B$ splits completely in one of them and remains irreducible in the other. As $\mathbb{Q}((\zeta)) \subset K_1 K_2$ using Proposition 3 above, we reach a contradiction.
- The last case, that of $[C_2 \times C_{14}, C_{13}]$, is also removable as it would similarly imply the existence of a rational elliptic curve with a 91-isogeny.

Now, we need to prove that the only $S \in \mathcal{H}_3(3, C_1)$ with $\#S = 3$ is $[C_2, C_3, C_7]$. For this purpose we have to remove the cases:

$[C_2, C_3, C_{13}], [C_2, C_7, C_{13}], [C_3, C_4, C_7], [C_2 \times C_2, C_3, C_7].$

- The first case can be ruled out as $[C_6, C_{13}]$ above, for it implies the existence of a rational curve with a 39-isogeny.
- The second case, as $[C_2 \times C_{14}, C_{13}]$ above, would imply the existence of a rational elliptic curve with a 91-isogeny. Hence it cannot happen.
- The third case is eliminated by noting that the discriminant of such a curve should be $-Y^2$ (for it gains 4-torsion) and $-2 \cdot \Box$, where $\Box$ is a rational square (for it has a 21-isogeny).
- The last case is similar to the case $[C_2 \times C_{14}, C_3]$ above.

Looking with greater detail at the case $[C_2, C_3, C_7]$ we find that if a curve gains torsion in such a way in three non-isomorphic cubic fields, it must have a 21-isogeny and in fact (as in the $[C_6, C_7]$ case) it can only be a very precise curve a family of twists in the 162b isogeny class. There are only 4 such curves and 162b2 is the only one that grows strictly in three cubic extensions.

$G = C_2$

The only case to discard here is $[C_6, C_{14}]$. If such a curve (say $E$) existed, it would follow that $E$ would have a 3-isogeny and 7-isogeny and hence a 21-isogeny. $E$ would also have to contain $C_2$, since the odd isogeny cannot kill this torsion. But there do not exist elliptic curves with 21-isogenies and non-trivial 2-torsion over $\mathbb{Q}$ [1, pp. 78–80].
We have examples in Table 1 for any $S \in \mathcal{H}(3, C_3)$ with $\#S = 2$ except for the cases:

- $[C_9, C_{12}], [C_{12}, C_{21}], [C_2 \times C_6, C_9], [C_2 \times C_6, C_{21}]

- $[C_2 \times C_6, C_9], [C_2 \times C_6, C_{21}].$

From Proposition 6 (vi) our curve has either a 9–isogeny or two independent 3–isogenies and $\mathbb{Q}(E[3]) = \mathbb{Q}(\zeta_3)$. Moreover from Proposition 5 (iii) $\Delta \in (-1) \cdot (\mathbb{Q}^*)^2$.

Assume that $E$ has two independent 3–isogenies and $\mathbb{Q}(E[3]) = \mathbb{Q}(\zeta_3)$. From [20, p. 147] we get

$$\Delta = -216 \frac{b^3(h^6 - 6b^2h^2 + 12b^3)}{h^6}, \quad b, h \in \mathbb{Q}.$$ 

As $\Delta = -y^2$ for some $y \in \mathbb{Q}$, the existence of $E$ implies there are $b, h, y \in \mathbb{Q}$ with

$$\left(\frac{y}{bh}\right)^2 = 6 \left(\frac{b}{h^2}\right)^2 \left[1 - 6 \left(\frac{b}{h^2}\right)^2 - 12 \left(\frac{b}{h^2}\right)^3\right],$$

that is a rational point on the curve

$$Y^2 = 6X \left(1 - 6X^2 - 12X^3\right),$$

but its Mordell–Weil group is trivial, and the trivial point do not yield an elliptic curve $E$.

So we are bound to assume $E$ has a 9–isogeny. From [7, Appendix], it follows that $E$ is a twist of $u^2 = v^3 + av + b$, where

$$a = -3x(x^3 - 24), \quad b = 2(x^6 - 36x^6 + 216),$$

for some $x \in \mathbb{Q}$. Then the discriminat of this curve is

$$2^{12}3^6(c^3 - 27)u^{12},$$

where the twelfth power may appear because of twisting. As this should be in $(-1) \cdot (\mathbb{Q}^*)^2$, it should give a point on

$$Y^2 = X^3 - 27.$$

The points in this curve can be easily computed (we have done so with Magma [9]; there is only the point at infinity and a point of order 2 that discriminant 0, so we are done.

- Second and fourth cases are not possible, as the only curve whose torsion grows to $C_{21}$ is $162b1$, and this curve fits neither of these cases (see Table 1).
- $[C_2 \times C_6, C_9]$. This case parallels the first one. The only formal change is that, as we gain full 2–torsion in a cubic extension, $\Delta \in (\mathbb{Q}^*)^2$. Hence, the same arguments lead us to state that such a curve must yield either a point on

$$Y^2 = -6X \left(1 - 6X^2 - 12X^3\right),$$

if it has two independent rational 3–isogenies, or a point on

$$Y^2 = X^3 + 27.$$

\footnote{Note there is a misprint in the original article, $h^4$ in the numerator should be replaced by $h^6$.}
should it have a rational 9–isogeny. As both cases can be checked to be impossible, we are finished.

Finally, we see that there are no $S \in \mathcal{H}_{\mathbb{Q}}(3, \mathcal{C}_3)$ with $\#S = 3$. Such $S$ should have two groups of odd order. These must be $\mathcal{C}_9$ and $\mathcal{C}_{21}$. But again the unique elliptic curve over $\mathbb{Q}$ with $\mathcal{C}_{21}$ over a cubic field is $162b1$ and for this curve, this is not the case (see Table I).

**APPENDIX: Computations**

Let $G \in \Phi(1)$, $S = [H_1, ..., H_m] \in \mathcal{H}_{\mathbb{Q}}(3, G)$, $E$ an elliptic curve defined over $\mathbb{Q}$ such that $E(\mathbb{Q})_{\text{tors}} = G$ and let $K_1, ..., K_m$ cubic fields, such that $E(K_i)_{\text{tors}} = H_i$ for $i = 1, ..., m$.

Table I shows an example of every possible situation, where

- the first column is $G$,
- the second column is $S \in \mathcal{H}_{\mathbb{Q}}(3, G)$,
- the third column is $\#S$,
- the fourth column is the label of the elliptic curve $E$ with minimal conductor satisfying the conditions above,
- the fifth column displays the coefficients of corresponding defining cubic polynomial to the respective $H$’s in $S$.

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Table 1. $h = \#S$ for $S \in \mathcal{H}_Q(3,G)$

| $G$ | $\mathcal{H}_Q(3,G)$ | $h$ | label | cubics |
|-----|------------------------|-----|-------|--------|
| $\mathcal{C}_2$ | 11a2 | 1 | $[-12419196912, -10135152, 0, 1]$ |
| $\mathcal{C}_4$ | 338b2 | 1 | $[872683713, 799551, -513, 1]$ |
| $\mathcal{C}_6$ | 108a2 | 1 | $[-80, -24, -24, 1]$ |
| $\mathcal{C}_2 \times \mathcal{C}_2$ | 196a1 | 1 | $[-5832, -2916, 18, 1]$ |
| $\mathcal{C}_2 \times \mathcal{C}_{14}$ | 1922c1 | 1 | $[191319746769, -8017245, -216621, 1]$ |
| $\mathcal{C}_2, \mathcal{C}_3$ | 19a2 | 2 | $[432, -864, 0, 1]$, $[577, 1155, 2307, 1]$ |
| $\mathcal{C}_2, \mathcal{C}_7$ | 294a1 | 2 | $[89009298, -29835, 324, 1]$, $[2000376, -142884, -126, 1]$ |
| $\mathcal{C}_2, \mathcal{C}_{13}$ | 147b1 | 2 | $[1928016, -8208, 648, 1]$, $[2000376, -142884, -126, 1]$ |
| $\mathcal{C}_3, \mathcal{C}_1$ | 162d2 | 1 | $[-5200640, -19968, -600, 1]$, $[-2010032, 28944, 90, 1]$ |
| $\mathcal{C}_3, \mathcal{C}_2 \times \mathcal{C}_2$ | 196b2 | 1 | $[-4076477, -8565, -6927, 1]$, $[-16003008, -571536, 252, 1]$ |
| $\mathcal{C}_4, \mathcal{C}_7$ | 338b1 | 3 | $[100472373, 1906011, -153, 1]$, $[64064520, -492804, -1170, 1]$ |
| $\mathcal{C}_7, \mathcal{C}_2 \times \mathcal{C}_2$ | 3969a1 | 3 | $[30005640, -142884, -1890, 1]$, $[-6578496, -46656, 1269, 1]$ |
| $\mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_7$ | 162b2 | 3 | $[190144, 32880, -15, 1]$, $[-1324783, 70851, 51, 1]$, $[1417176, -20244, -486, 1]$ |
| $\mathcal{C}_2$ | $\mathcal{C}_6$ | 1 | 14a3 | $[-5581197, -3861, -3231, 1]$ |
| $\mathcal{C}_{14}$ | 49a3 | 1 | $[26004888, -142884, -1638, 1]$ |
| $\mathcal{C}_3$ | $\mathcal{C}_6$ | 1 | 19a1 | $[857881, 18003, -69, 1]$ |
| $\mathcal{C}_{12}$ | 162d1 | 1 | $[-95707, -933, -777, 1]$ |
| $\mathcal{C}_2 \times \mathcal{C}_6$ | 196b1 | 1 | $[2000376, -142884, -126, 1]$ |
| $\mathcal{C}_6, \mathcal{C}_9$ | 19a3 | 2 | $[-432, 864, 0, 1]$, $[40824, -2916, -126, 1]$ |
| $\mathcal{C}_6, \mathcal{C}_{21}$ | 162b1 | 2 | $[8882, -267, 132, 1]$, $[14984, -564, -570, 1]$ |
| $\mathcal{C}_4$ | $\mathcal{C}_{12}$ | 1 | 90c1 | $[-11243584, -11472, -2892, 1]$ |
| $\mathcal{C}_5$ | $\mathcal{C}_{10}$ | 1 | 11a1 | $[-74308, -384, -336, 1]$ |
| $\mathcal{C}_6$ | $\mathcal{C}_{18}$ | 1 | 14a4 | $[5832, -2916, -18, 1]$ |
| $\mathcal{C}_7$ | $\mathcal{C}_{14}$ | 1 | 26b1 | $[44396, -960, 87, 1]$ |
| $\mathcal{C}_8$ | 0 |
| $\mathcal{C}_9$ | $\mathcal{C}_{18}$ | 1 | 54b3 | $[-12331008, -13824, -72, 1]$ |
| $\mathcal{C}_{10}$ | 0 |
| $\mathcal{C}_{12}$ | 0 |
| $\mathcal{C}_2 \times \mathcal{C}_2$ | $\mathcal{C}_2 \times \mathcal{C}_6$ | 1 | 30a6 | $[-3621888, -8640, -1476, 1]$ |
| $\mathcal{C}_2 \times \mathcal{C}_4$ | 0 |
| $\mathcal{C}_2 \times \mathcal{C}_6$ | 0 |
| $\mathcal{C}_2 \times \mathcal{C}_8$ | 0 |