ON K-STABILITY OF SOME DEL PEZZO SURFACES OF FANO INDEX 2

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ABSTRACT. For every integer $a \geq 2$, we relate the K-stability of hypersurfaces in the weighted projective space $\mathbb{P}(1,1,a,a)$ of degree $2a$ with the GIT stability of binary forms of degree $2a$. Moreover, we prove that such a hypersurface is K-polystable and not K-stable if it is quasi-smooth.

1. INTRODUCTION

It is an important problem in algebraic geometry and in differential geometry to decide if a given Fano variety $X$ admits a Kähler–Einstein (KE) metric. The Yau–Tian–Donaldson (YTD) Conjecture predicts that the existence of a KE metric on $X$ is equivalent to the K-polystability of $X$. Using Cheeger–Colding–Tian theory, the YTD Conjecture was first proved when $X$ is smooth [CDS15, Tia15, Ber16], when $X$ is $\mathbb{Q}$-Gorenstein smoothable [LWX19, SSY16], or when $X$ has dimension 2 [LTW21]. Later, a different method, namely the variational approach, was introduced in [BBJ21]. The analytic side of the variational approach was completed in [LTW21b, Li19] which shows that a $\mathbb{Q}$-Fano variety $X$, that is, a Fano variety with klt singularities, admits a KE metric if and only if $X$ is reduced uniformly K-stable, a concept introduced in [His16] as an equivariant version of uniform K-stability (see also [XZ20]). Recently, using purely algebro-geometric methods, the work [LXZ21] establishes the equivalence between K-polystability and reduced uniform K-stability. This work, combining with the variational approach, proves the YTD Conjecture for all $\mathbb{Q}$-Fano varieties.

K-stability of del Pezzo surfaces which are quasi-smooth hypersurfaces in weighted projective 3-spaces has been studied extensively. Johnson and Kollár [JK01] classified those which are anticanonically polarised (i.e. have Fano index 1) and decided the existence of a KE metric on many of these, by using Tian’s criterion which relates KE metrics to global log canonical thresholds (also called $\alpha$-invariants) [Tia87, Nad90, DK01, Che08, OS12, Fuj19]. This method was applied to most of these del Pezzo surfaces by Araujo [Ara02], Boyer–Galicki–Nakamaye [BGN03], and Cheltsov–Park–Shramov [CPS10]. One case was missing and was finally solved in [CPS21] by using delta invariants (see [FO18, BJ20]).

The (non-)existence of KE metrics on many del Pezzo surfaces which are quasi-smooth hypersurfaces in weighted projective 3-spaces with Fano index $\geq 2$ has been studied in [CPS10, CPS21, CS13, KW21].

In this paper, we study K-polystability of quasi-smooth degree $2a$ hypersurfaces in the weighted projective space $\mathbb{P}(1,1,a,a)$. When $a \in \{2,4\}$, such del Pezzo surfaces are $\mathbb{Q}$-Gorenstein smoothable, and their K-polystability was determined by Mabuchi–Mukai [MM93] and Odaka–Spotti–Sun [OSS16] (see Remark 5). To the authors’ knowledge it is not known if they are K-polystable for an integer $a = 3$.
or $a \geq 5$. In [KW21] Kim and Won conjecture that these surfaces are K-polystable and not K-stable.

Our main result relates the K-polystability (resp. K-semistability) of degree $2a$ hypersurfaces in $\mathbb{P}(1, 1, a, a)$ to GIT polystability (resp. GIT semistability) of degree $2a$ binary forms (see [MFK94, Chapter 4]).

**Theorem 1.** Let $a \geq 2$ be an integer and let $\mathbb{P}(1, 1, a, a)$ be the weighted projective space with coordinates $[x, y, z, w]$ with weights $\deg x = \deg y = 1$ and $\deg z = \deg w = a$. Let $X$ be a hypersurface of degree $2a$ in $\mathbb{P}(1, 1, a, a)$.

Then $X$ is K-semistable (resp. K-polystable) if and only if, after an automorphism of $\mathbb{P}(1, 1, a, a)$, the equation of $X$ is given by $z^2 + w^2 + g(x, y) = 0$ where $g \neq 0$ is GIT semistable (resp. GIT polystable) as a degree $2a$ binary form. Moreover, $X$ is not K-stable.

As a consequence we prove the K-polystability of quasi-smooth hypersurfaces in $\mathbb{P}(1, 1, a, a)$ of degree $2a$, hence partially confirming [KW21, Conjecture 1.3].

**Corollary 2.** Let $a \geq 2$ be an integer and let $X$ be a degree $2a$ quasi-smooth hypersurface in $\mathbb{P}(1, 1, a, a)$. Then $X$ is K-polystable and not K-stable. Moreover, $X$ admits a KE metric.

Recently the result of this corollary has been independently announced by Viswanathan using different methods.

It is possible to give a proof of K-polystability for a general hypersurface in $\mathbb{P}(1, 1, a, a)$ of degree $2a$, when $a$ is odd, by analysing the deformation theory of the toric surface appearing in Proposition 3 similarly to [KP21] and without using Theorem 1.

**Notation and conventions.** We always work over $\mathbb{C}$. A del Pezzo surface is a normal projective surface whose anticanonical divisor is $\mathbb{Q}$-Cartier and ample. Every toric variety we consider is normal. We do not even try to write down the definitions of K-(poly/semi)stability of Fano varieties and of log Fano pairs: we refer the reader to the excellent survey [Xu20], the paper [ADL19], and to the references therein.

**Acknowledgements.** The second author wishes to thank Anne-Sophie Kaloghiros for many fruitful conversations and Yuji Odaka for helpful e-mail exchanges; he is grateful also to Ivan Cheltsov and Jihun Park for useful remarks on an earlier draft of this manuscript and for sharing a preliminary version of [KW21]. The first author is partially supported by the NSF Grant DMS-2001317.

2. **Proofs**

In what follows $a$ is a fixed integer greater than 1. We consider the weighted projective space $\mathbb{P}(1, 1, a, a)$ with coordinates $[x, y, z, w]$ with weights $\deg x = \deg y = 1$ and $\deg z = \deg w = a$.

**Proposition 3.** If $Y$ is the hypersurface in $\mathbb{P}(1, 1, a, a)$ defined by the equation $zw - ax^ay^a = 0$, then $Y$ is a K-polystable toric del Pezzo surface.

**Proof.** We fix the lattice $N = \mathbb{Z}^2$ and its dual $M = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. Elements of $N$ will be columns and elements of $M$ will be rows.
Let $Q$ be the convex hull of the points 
\[(0,0), (0,1), (a^{-1}, 0), (-a^{-1}, 1)\]
in $M_\mathbb{R}$. Let $\Sigma$ be the inner normal fan of $Q$; thus $\Sigma$ is the complete normal fan in $N$ whose rays are generated by the vectors
\[
\begin{pmatrix} a \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -a \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}.
\]
We want to show that $Y$ is the toric variety associated to the fan $\Sigma$.

Provisionally, let $TV(\Sigma)$ denote the toric variety associated to $\Sigma$. Consider the cone $\tau$ in $M \oplus \mathbb{Z}$ spanned by $Q \times \{1\}$. Consider the finitely generated monoid $\tau \cap (M \oplus \mathbb{Z})$ and the semigroup algebra $\mathbb{C}[\tau \cap (M \oplus \mathbb{Z})]$, which is $\mathbb{N}$-graded via the projection $M \oplus \mathbb{Z} \to \mathbb{Z}$. Toric geometry says that $TV(\Sigma) = \text{Proj} \mathbb{C}[\tau \cap (M \oplus \mathbb{Z})]$. One can see that the minimal set of generators of the semigroup $\tau \cap (M \oplus \mathbb{Z})$ is made up of the vectors
\[(0,0,1), (0,1,1), (1,0,a), (-1,a,a);\]
these vectors satisfy a unique relation:
\[
a(0,0,1) + a(0,1,1) = (1,0,a) + (-1,a,a).
\]
Hence the $\mathbb{N}$-graded ring $\mathbb{C}[\tau \cap (M \oplus \mathbb{Z})]$ coincides with $\mathbb{C}[x,y,z,w]/(zw - x^ay^a)$, where $\text{deg } x = \text{deg } y = 1$ and $\text{deg } z = \text{deg } w = a$. Therefore $Y = TV(\Sigma)$.

The vectors in (1) are the vertices of a polytope $P$ in $N$. This implies that $Y$ is a del Pezzo surface, i.e. $-K_Y$ is $\mathbb{Q}$-Cartier and ample.

Let $P^o$ be the polar of $P$; thus $P^o$ is the convex hull of $(0, \pm 1)$ and $\pm (\frac{2}{a}, -1)$ in $M_\mathbb{R}$. The polygon $P^o$ is the moment polytope of the toric boundary of $Y$, which is an anticanonical divisor. Since $P$ is centrally symmetric, also $P^o$ is centrally symmetric, thus the barycentre of $P^o$ is the origin. By [Ber16] $Y$ is K-polystable.

\[\square\]

**Remark 4.** (1) Another way to show K-polystability of $Y$ is by realising $Y \cong (\mathbb{P}^1 \times \mathbb{P}^1)/(\mathbb{Z}/a\mathbb{Z})$, where the $\mathbb{Z}/a\mathbb{Z}$-action on $\mathbb{P}^1 \times \mathbb{P}^1$ is given by
\[
\zeta \cdot ([u_0, u_1], [v_0, v_1]) := ([\zeta u_0, u_1], [\zeta^{-1}v_0, v_1]) \quad \text{with } \zeta = e^{\frac{2\pi i}{a}}.
\]
Since the above action is free away from finitely many points, and it preserves the product of Fubini-Study metrics on $\mathbb{P}^1 \times \mathbb{P}^1$, we know that $Y$ admits a KE metric and hence is K-polystable by [Ber16].

(2) A degree $2a$ hypersurface in $\mathbb{P}(1,1,a,a)$ is defined by an equation
\[
q(z, w) + f(x, y)z + h(x, y)w + g(x, y) = 0
\]
where $q$ is a quadratic form, $f$ and $h$ are forms of degree $a$, and $g$ is a form of degree $2a$. With an automorphism of $\mathbb{P}(1,1,a,a)$ which is induced by a linear change of the coordinates $z, w$, we can diagonalise the quadratic form $q$, so that the term $zw$ disappears. Furthermore, if $q$ has full rank, with an automorphism of $\mathbb{P}(1,1,a,a)$ induced by $z \mapsto z + \frac{q}{2}$ and $w \mapsto w + \frac{q}{2}$, the equation becomes
\[
z^2 + w^2 + g(x, y) = 0.
\]
Proof of Theorem 1. We start from the “if” part. Suppose \( X \subset \mathbb{P}(1, 1, a, a) \) is defined by the equation \( z^2 + w^2 + g(x, y) = 0 \) with \( g \neq 0 \). Then the “if” part states that \( X \) is K-semistable (resp. K-polystable) if \( g \) is GIT semistable (resp. GIT polystable).

By forgetting the \( w \)-coordinate, we obtain a double cover \( \pi : X \to \mathbb{P}(1, 1, a) \) with branch locus \( D = (z^2 + g(x, y) = 0) \). Thus by [LZ20, Zhu21] we know that \( X \) is K-semistable (resp. K-polystable) if and only if \( (\mathbb{P}(1, 1, a), \frac{1}{2}D) \) is K-semistable (resp. K-polystable).

Let us assume for the moment that \( g \) is an arbitrary degree 2a binary form. Denote by \( D_0 := (z^2 = 0) \) as a divisor on \( \mathbb{P}(1, 1, a) \). It is clear that \( \mathbb{P}(1, 1, a) \) is the projective cone over \( \mathbb{P}^1 \) with polarization \( \mathcal{O}_{\mathbb{P}^1}(a) \), and \( \frac{1}{2}D_0 \) is the section at infinity. Since \( \mathbb{P}^1 \) is Kähler–Einstein, [LL19, Proposition 3.3] shows that \( (\mathbb{P}(1, 1, a), (1 - \frac{a}{2})\frac{1}{2}D_0) \) admits a conical KE metric, where \( r \in \mathbb{Q}_{>0} \) satisfies \( \mathcal{O}_{\mathbb{P}^1}(a) \sim_\mathbb{Q} -r^{-1}K_{\mathbb{P}^1} \), i.e. \( r = \frac{2}{a} \). By computation, \( (1 - \frac{a}{2})\frac{1}{2} = \frac{a-1}{2a} \). Thus \( (\mathbb{P}(1, 1, a), \frac{a-1}{2a}D_0) \) admits a conical KE metric and hence is K-polystable. It is clear that under the \( \mathbb{G}_m \)-action \( \sigma \) on \( \mathbb{P}(1, 1, a) \) given by \( \sigma(t) \cdot [x, y, z] = [x, ty, tz] \), the log Fano pair \( (\mathbb{P}(1, 1, a), \frac{a-1}{2a}D) \) specially degenerates to \( (\mathbb{P}(1, 1, a), \frac{a-1}{2a}D_0) \) as \( t \to 0 \). Thus by openness of K-semistability [BLX19, Xu20b] we know that \( (\mathbb{P}(1, 1, a), \frac{a-1}{2a}D) \) is K-semistable.

Next, we assume that \( g \neq 0 \) is GIT semistable. By GIT of binary forms, we know that each linear factor in \( g(x, y) \) has multiplicity at most \( a \). In other words, the curve \( D \) has only \( A_{k-1} \)-singularities (i.e. locally analytically given by \( x^2 + y^k = 0 \)) where \( k \leq a \). Thus we have that \( \text{lct}(\mathbb{P}(1, 1, a); D) \geq \frac{1}{2} + \frac{1}{a} = \frac{a+2}{2a} \). This implies that \( (\mathbb{P}(1, 1, a), \frac{a+2}{2a}D) \) is a log canonical log Calabi–Yau pair. Thus interpolation for K-stability [ADL19, Proposition 2.13] implies that \( (\mathbb{P}(1, 1, a), \frac{1}{2}D) \) is K-semistable.

Next, we assume that \( g \neq 0 \) is GIT polystable. There are two cases: \( g \) is strictly GIT polystable (i.e. GIT semistable but not GIT stable), or \( g \) is GIT stable. In the first case, under a suitable coordinate we may write \( g(x, y) = x^a y^a \). Thus the double cover \( X \) is toric, and as shown in Proposition 3 \( X \) is K-polystable. In the second case, we know that each linear factor in \( g(x, y) \) has multiplicity at most \( a-1 \). Thus the curve \( D \) has only \( A_{k-1} \)-singularities where \( k \leq a-1 \). Thus we have that \( \text{lct}(\mathbb{P}(1, 1, a); D) \geq \frac{1}{2} + \frac{1}{a-1} > \frac{a+2}{2a} \), which implies that \( (\mathbb{P}(1, 1, a), \frac{a+2}{2a}D) \) is a klt log Calabi–Yau pair. Thus interpolation for K-stability [ADL19, Proposition 2.13] implies that \( (\mathbb{P}(1, 1, a), \frac{1}{2}D) \) is K-stable. This finishes the proof of the “if” part.

Next, we treat the “only if” part. In fact, this follows from moduli comparison arguments as in [ADL19]. Let \( \mathcal{A} := H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2a)) \) be the affine space parametrizing degree 2a binary forms. Let \( \mathcal{A}^{ss} \subset \mathcal{A} \setminus \{0\} \) be the open subset of GIT semistable binary forms. Consider the universal family of weighted hypersurfaces \( \mathcal{X} \to \mathcal{A}^{ss} \) where \( \mathcal{X} \subset \mathbb{P}(1, 1, a) \times \mathcal{A}^{ss} \) has fibre \( (z^2 + w^2 + g(x, y) = 0) \) over each \( g \in \mathcal{A}^{ss} \). By the “if” part we know that each fibre of \( \mathcal{X} \to \mathcal{A}^{ss} \) is K-semistable. Consider the \( (\mathbb{G}_m \times \text{SL}_2) \)-action \( \lambda \) on \( \mathcal{A} \) given by \( \lambda(t, A) \cdot g(x, y) = t^2g(A^{-1}(x, y)) \). It is clear that \( \mathcal{A}^{ss} \) is a \( (\mathbb{G}_m \times \text{SL}_2) \)-invariant open subset. Then there is a \( (\mathbb{G}_m \times \text{SL}_2) \)-action \( \hat{\lambda} \) on \( \mathcal{X} \) as a lifting of \( \lambda \) given by

\[ \hat{\lambda}(t, A) \cdot ([x, y, z, w], g) := ([A(x, y), tz, tw], \lambda(t, A) \cdot g). \]

Denote by \( \mathcal{M}^{\text{GIT}} := [\mathcal{A}^{ss}/(\mathbb{G}_m \times \text{SL}_2)] \) and \( M^{\text{GIT}} := \mathcal{P} \sslash \text{SL}_2 \) where \( \mathcal{P} := \mathbb{P}(\mathcal{A}) \). It is clear that \( M^{\text{GIT}} \) is the good moduli space of \( \mathcal{M}^{\text{GIT}} \). Taking quotient of
the family \( \mathcal{X} \to A^m \) by \( \tilde{\lambda} \), we obtain a \( \mathbb{Q} \)-Gorenstein flat family of \( K \)-semistable \( \mathbb{Q} \)-Fano varieties over \( M^{\text{GIT}} \), where fibres over closed points are precisely \( K \)-polystable fibres.

From a series of important recent works [Jia20, LWX21, CP21, BX19, ABHLX20, Xu20b, BLX19, XZ20, XZ21, BHLX21, LXZ21], we know that there exists an Artin stack of finite type \( M_{K^{ss}}^{2,8/a} \) parametrizing \( K \)-semistable (possibly singular) del Pezzo surfaces of degree 8/a. Moreover, \( M_{K^{ss}}^{2,8/a} \) admits a projective good moduli space \( M_{K^{ps}}^{2,8/a} \) parametrizing \( K \)-polystable ones. Let \( M^K \) be the Zariski closure (with reduced structure) of the locally closed substack in \( M_{K^{ss}}^{2,8/a} \) parametrizing \( K \)-semistable degree 2a weighted hypersurfaces \( X \subset \mathbb{P}(1,1,a,a) \). Let \( M^K \) be the good moduli space of \( M^K \) as a closed algebraic subspace of \( M_{K^{ps}}^{2,8/a} \). Then the above construction and the “if” part produces a morphism \( \Phi : M^{\text{GIT}} \to M^K \) which descends to a morphism \( \phi : M^{\text{GIT}} \to M^K \). Since a general weighted hypersurface \( X \) has the form \( z^2 + w^2 + g(x,y) = 0 \) in a suitable coordinate where \( g \neq 0 \) has no multiple linear factors, we know that \( \Phi \) is dominant. The “if” part shows that \( \Phi \) sends closed points to closed points. Since \( M^{\text{GIT}} \) is projective, we know that \( \phi \) is proper and dominant, which implies that \( \phi \) is surjective. Moreover, since \( \text{SL}_2 \) has no non-trivial characters, we have injections

\[
\text{Pic}(M^{\text{GIT}}) = \text{Pic}(P \sslash \text{SL}_2) \hookrightarrow \text{Pic}_{\text{SL}_2}(P^{\text{ss}}) \hookrightarrow \text{Pic}(P^{\text{ss}})
\]

by [KKV89, Proposition 4.2 and Section 2.1]. It is clear that \( P \setminus P^{\text{ss}} \) has codimension at least 2 in \( P \). Thus we have \( \text{Pic}(P^{\text{ss}}) \cong \text{Pic}(P) \cong \mathbb{Z} \). In particular, the GIT quotient \( M^{\text{GIT}} \) has Picard rank 1. It is clear that \( M^K \) is not a single point. Thus \( \phi : M^{\text{GIT}} \to M^K \) is a finite surjective morphism by Zariski’s main theorem.

Next, we show that \( K \)-poly/semistability implies GIT poly/semistability. Since \( \phi \) is surjective, a \( K \)-polystable hypersurface \( X \subset \mathbb{P}(1,1,a,a) \) satisfies that \( [X] = \phi([g]) \in M^K \) for some GIT polystable binary form \( g \in A \setminus \{0\} \). Thus \( X \) has the form \( z^2 + w^2 + g(x,y) = 0 \) with \( g \neq 0 \) being GIT polystable. If \( X \subset \mathbb{P}(1,1,a,a) \) is \( K \)-semistable, then it specially degenerates to a \( K \)-polystable point \( [X_0] \in M^K \) by [LWX21]. Clearly \( X_0 \) has the form \( z^2 + w^2 + g_0(x,y) = 0 \) with \( g_0 \neq 0 \) being GIT polystable. Since the rank of quadratic forms cannot jump up under degeneration, the quadratic terms in \( (z,w) \) of the equation of \( X \) has rank 2, which implies that \( X = (z^2 + w^2 + g(x,y) = 0) \) for some \( g \). By [Fuj19b, Corollary 1.7], we know that \( (\mathbb{P}(1,1,a),\frac{1}{2}D) \) is \( K \)-semistable where \( D = (z^2 + g(x,y) = 0) \). Since \( X \) carries a \( \mathbb{Z}/2\mathbb{Z} \)-action given by \( w \mapsto -w \), we may assume that the special degeneration from \( X \) to \( X_0 \) is \( \mathbb{Z}/2\mathbb{Z} \)-equivariant by [LZ20, Zhu21]. In particular, this shows that \( (\mathbb{P}(1,1,a),\frac{1}{2}D) \) specially degenerates to \( (\mathbb{P}(1,1,a),\frac{1}{2}D_0) \) where \( D_0 = (z^2 + g_0(x,y) = 0) \). By the lower semi-continuity of \( \text{lct} \) (see e.g. [DK01]), we know that \( \text{lct}(\mathbb{P}(1,1,a);D) \geq \text{lct}(\mathbb{P}(1,1,a);D_0) \geq \frac{a^2}{2} \) where the latter inequality was proven in the “if” part due to the fact that \( g_0 \) is GIT polystable. Thus this shows that \( g \neq 0 \), and each linear factor in \( g(x,y) \) has multiplicity at most \( a \). Thus we obtain the GIT semistability of \( g \). The proof of the “only if” part is finished.

Finally, we show that any hypersurface \( X \subset \mathbb{P}(1,1,a,a) \) of degree 2a is not \( K \)-stable. If \( X \) were \( K \)-stable, then it would have equation \( z^2 + w^2 + g(x,y) = 0 \), or equivalently the equation \( zw + g(x,y) = 0 \). It is clear that \( t \cdot (z,w) = (tz,t^{-1}w) \) defines an effective action of \( \mathbb{G}_m \) on \( X \). Thus \( X \) is not \( K \)-stable by definition. \( \square \)
Proof of Corollary 2. It is clear that $X$ is quasi-smooth if and only if, up to an automorphism of $\mathbb{P}(1, 1, a, a)$, $X$ has the equation $z^2 + w^2 + g(x, y) = 0$ where $g$ has no multiple linear factors. Thus by Theorem 1 we conclude that $X$ is K-polystable and not K-stable. The existence of KE metrics on $X$ follows from [LTW21].

Remark 5. For $a = 2$, the del Pezzo surface $X$ admits an embedding into $\mathbb{P}^4$ as a complete intersection of two hyperquadrics. This is induced by the linear system $| - K_X|$ which is very ample.

For $a = 4$, $X$ (as a double cover of $\mathbb{P}(1, 1, 4)$) appeared in [OSS16] where it lies in the exceptional divisor of Kirwan blow-up of the GIT moduli space. Hence $X$ admits a $\mathbb{Q}$-Gorenstein smoothing to degree 2 smooth del Pezzo surfaces.

Therefore, in both cases ($a = 2$ or $a = 4$) our K-moduli space $M^K_2$, introduced in the proof of Theorem 1, form a divisor in the K-moduli spaces of $\mathbb{Q}$-Gorenstein smoothable del Pezzo surfaces of degree $\frac{a}{2}$ studied in [MM93, OSS16]. We will see in Proposition 6 what happens for $a = 3$ or $a \geq 5$.

Proposition 6. If $a = 3$ or $a \geq 5$, then the locus of K-polystable degree $2a$ hypersurfaces in $\mathbb{P}(1, 1, a, a)$ is a connected component of $\mathcal{M}^{\text{Kps}}_{2, 8/a}$.

Proof. We denote by $\Gamma$ the connected component of $\mathcal{M}^{\text{Kps}}_{2, 8/a}$ containing K-polystable degree $2a$ hypersurfaces in $\mathbb{P}(1, 1, a, a)$. In the proof of Theorem 1 we showed that the locus of K-polystable degree $2a$ hypersurfaces in $\mathbb{P}(1, 1, a, a)$ is closed in $\Gamma$; this locus is denoted by $M^K$. We need to prove that $M^K$ coincides with $\Gamma$. We will achieve this by a dimension count. Using the notation of the proof of Theorem 1, there is a finite surjective morphism $\phi : M^{\text{GIT}} \to M^K$. Thus we have

$$\dim M^K = \dim M^{\text{GIT}} = \dim \mathcal{P} - \dim \text{SL}_2 = 2a - 3.$$ 

Let us now compute the dimension of $\Gamma$ by analysing the deformation theory of the K-polystable toric del Pezzo surface $Y$ introduced in Proposition 3. Note that a similar study was discussed in [MGS21].

Let $\mathcal{R}_Y^0$ denote the sheaf of derivations on $Y$, i.e. the dual of $\Omega_Y^1$. Let $\mathcal{R}_Y^{G, 1}$ denote the sheaf of 1st order $\mathbb{Q}$-Gorenstein deformations of $Y$. The singular locus of $Y$, which consists of 4 points, contains the set-theoretic support of $\mathcal{R}_Y^{G, 1}$.

Since $Y$ is a toric Fano, we have $H^1(\mathcal{R}_Y^0) = H^2(\mathcal{R}_Y^0) = 0$ by [Pet19, §4.3]. Via a standard argument about the local-to-global spectral sequence for Ext, we deduce that the tangent space of the $\mathbb{Q}$-Gorenstein deformation functor of $Y$ is $H^0(\mathcal{R}_Y^{G, 1})$. The $\mathbb{Q}$-Gorenstein deformation functor of $Y$ is unobstructed because $Y$ is a del Pezzo surface with cyclic quotient singularities [ACC+16, Lemma 6]. Therefore the germ at the origin of the vector space $H^0(\mathcal{R}_Y^{G, 1})$ is the base of the miniversal (Kuranishi) $\mathbb{Q}$-Gorenstein deformation of $Y$.

Consider the torus $T_N = N \otimes \mathbb{Z} \mathbb{G}_m$ acting on the toric variety $Y$. There is an action of $T_N$ on the vector space $H^0(\mathcal{R}_Y^{G, 1})$, hence $H^0(\mathcal{R}_Y^{G, 1})$ splits into the direct sum of irreducible representations (characters) of the torus $T_N$.

We observe that the singularities of $Y$ are:

- 2 points of type $\frac{a}{2}(1, -1) = A_{a-1}$, which correspond to the cones in $\Sigma$ spanned by

$$\pm \begin{pmatrix} a \\ 1 \end{pmatrix}, \pm \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$
2 points of type $\frac{1}{a}(1,1)$, which correspond to the cones in $\Sigma$ spanned by
\[\pm \left( \frac{a}{1} \right), \mp \left( \frac{0}{1} \right).\]
Since $a = 3$ or $a \geq 5$, the surface singularity $\frac{1}{a}(1,1)$ is $\mathbb{Q}$-Gorenstein rigid, so it does not contribute to $H^0(\mathcal{Z}_Y^{\mathbb{Q}G})$. One can see that the $T_N$-representation $H^0(\mathcal{Z}_Y^{\mathbb{Q}G})$ is the direct sum of the 1-dimensional representation of $T_N$ associated to the characters
\[(2) \quad (0, \pm 2), (0, \pm 3), \ldots, (0, \pm a) \in M.\]
In particular dim $H^0(\mathcal{Z}_Y^{\mathbb{Q}G}) = 2a - 2$, so the base of the miniversal $\mathbb{Q}$-Gorenstein deformation of $Y$ is a smooth germ of dimension $2a - 2$.

Since the weights in (2) are contained in a rank 1 sublattice of $M$, there exists a 1-dimensional subtorus of $T_N$ which acts trivially on $H^0(\mathcal{Z}_Y^{\mathbb{Q}G})$. More precisely one can prove that the affine quotient $H^0(\mathcal{Z}_Y^{\mathbb{Q}G})/T_N$ has dimension $2a - 3$.

Since every facet of the polytope $P^a$ has no interior lattice points, by [KP21, Proposition 2.6] the automorphism group of $Y$ is $T_N \times \text{Aut}(P)$, where $\text{Aut}(P) \subseteq \text{GL}(N)$ is the finite group consisting of the lattice automorphisms which keep the polytope $P$ invariant. Since the difference between $T_N$ and $\text{Aut}(Y)$ is just a finite group, we deduce that the affine quotient the affine quotient $H^0(\mathcal{Z}_Y^{\mathbb{Q}G})/\text{Aut}(Y)$ has dimension $2a - 3$. By the local structure of the K-moduli space [ABHLX20, AHR20] we know that the completion of the local ring of $\Gamma$ at $[Y]$ coincides with the completion at the origin of $H^0(\mathcal{Z}_Y^{\mathbb{Q}G})/\text{Aut}(Y)$. This proves that $\Gamma$ has dimension $2a - 3$ at $[Y]$. Since dim $M^K = 2a - 3$, we know that $M^K$ is an irreducible component of $\Gamma$.

Moreover, since all K-polystable del Pezzo surfaces in $M^K$ have cyclic quotient singularities by Theorem 1, they have unobstructed $\mathbb{Q}$-Gorenstein deformations by [ACC+16, Lemma 6]. Thus the stack $\mathcal{M}^{\text{Kss}}_{2+\frac{2}{a}}$ is smooth in an open neighbourhood of $\mathcal{M}^{M^K}$. In particular, this implies that $\Gamma$ is normal in an open neighbourhood of $M^K$. Since $M^K$ is an irreducible component of $\Gamma$, we have $M^K = \Gamma$. □

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