CONVEXITY PROPERTIES OF GRADIENT MAPS ASSOCIATED TO REAL REDUCTIVE REPRESENTATIONS

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Abstract. Let $G$ be a connected real reductive Lie group acting linearly on a finite dimensional vector space $V$ over $\mathbb{R}$. This action admits a Kempf-Ness function and so we have an associated gradient map. If $G$ is Abelian we explicitly compute the image of $G$ orbits under the gradient map, generalizing a result proved by Kac and Peterson [38]. A similar result is proved for the gradient map associated to the natural $G$ action on $\mathbb{P}(V)$. We also investigate the convex hull of the image of the gradient map restricted on the closure of $G$ orbits. Finally, we give a new proof of the Hilbert-Mumford criterion for real reductive Lie groups avoiding any algebraic result.

1. Introduction

Let $U$ be a compact connected Lie group and let $U^C$ be its complexification. Let $(Z, \omega)$ be a Kähler manifold on which $U^C$ acts holomorphically. Assume that $U$ acts in a Hamiltonian fashion with momentum map $\mu : Z \to \mathfrak{u}^*$. This means that $\omega$ is $U$-invariant, $\mu$ is $U$-equivariant and for any $\beta \in \mathfrak{u}$ we have $d\mu^\beta = i_{\beta_Z}\omega$, where $\mu^\beta(x) = \mu(x)(\beta)$ and $\beta_Z$ denotes the fundamental vector field on $Z$ induced by the action of $U$. It is well-known that the momentum map represents a fundamental tool in the study of the action of $U^C$ on $Z$. Of particular importance are convexity theorems [1, 27, 41], which depend on the fact that the functions $\mu^\beta$ are Morse-Bott with even indices. Assume that $U$ is a compact torus. If $Z$ is compact, then Atiyah proved a convexity Theorem along $U^C$ orbits [1]. Recently, Biliotti and Ghigi [16] proved a convexity Theorem along orbits in a very general setting using only so-called Kempf-Ness function. The original setting for Kempf-Ness function is the following: let $V$ be a unitary representation of $U$. For $x = [v] \in \mathbb{P}(V)$ and $g \in U^C$ we set $\Psi(x, g) = \log \|gv\|/\|v\|$ [10]. The behavior of the corresponding gradient map is encoded in the $\Psi$. In 1990 Richardson and Slodowoy [59] proved that the Kempf-Ness Theorem extends to the case of real reductive representations. This pioneering work has allowed to prove many results exploiting tools from geometric invariant theory. This is the perspective taken, amongst many others, in the papers [22, 45, 47]. Recently Deré and Lauret [23] use nice convexity properties of the moment map for the variety of nilpotent Lie algebras to investigate which nilpotent Lie algebras admit a Ricci negative solvable extension. This motivated us to investigate convexity properties of gradient maps associated to real reductive representations.

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We point out that there exist several non equivalent definitions of real reductive Lie group in the literature ([1, 29, 42, 66]). Since we are interested in real reductive representations, we restricted ourselves to linear groups, i.e., subgroups of GL(V), where V is a finite dimensional real vector space. By a Theorem of Mostow [53], if \( G \subset \text{GL}(n, \mathbb{R}) \) is closed under transpose then it is compatible with respect to the Cartan decomposition of GL(V). Hence we fix the following setup.

Let \( \rho : G \rightarrow \text{GL}(V) \) be a faithful representation on a finite dimensional real vector space. We identify \( G \) with \( \rho(G) \subset \text{GL}(V) \) and we assume that \( G \) is closed and it is closed under transpose. This means there exists a scalar product \( \langle \cdot, \cdot \rangle \) on \( V \) such that \( G = K \exp(p) \), where \( K = G \cap \text{O}(V) \) and \( p = g \cap \text{Sym}(V) \). Here we denote by \( \text{O}(V) \) the orthogonal group with respect to \( \langle \cdot, \cdot \rangle \), by \( \text{Sym}(V) \) the set of symmetric endomorphisms of \( V \) and finally with \( g \) the Lie algebra of \( G \). Then \( g = \mathfrak{k} \oplus \mathfrak{p} \) is the Cartan decomposition, that is \( [\mathfrak{t}, \mathfrak{t}] \subset \mathfrak{t}, [\mathfrak{t}, \mathfrak{p}] \subset \mathfrak{p} \) and \( [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{t} \).

Moreover, \( K \) is a maximal compact subgroup of \( G \), the map \( K \times \mathfrak{p} \mapsto G, (k, \xi) \mapsto k \exp(\xi) \), is a diffeomorphism, any two maximal Abelian subalgebras of \( \mathfrak{p} \) are conjugate by an element of \( K \) and the decomposition \( G = KTK \) holds, where \( T = \exp(\mathfrak{t}) \) is the connected Abelian subgroup corresponding to a maximal Abelian subalgebra \( \mathfrak{t} \) contained in \( \mathfrak{p} \) [36, 42]. In this setting, the function

\[
\Psi : G \times V \rightarrow \mathbb{R}, \quad (g, x) \mapsto \frac{1}{2}(\langle gx, gx \rangle - \langle x, x \rangle).
\]

is a Kempf-Ness function (see section 3) and the corresponding gradient map is given by

\[
\mathfrak{F}_p : V \rightarrow \mathfrak{p}^*, \quad \mathfrak{F}_p(x)(\xi) = \langle \xi x, x \rangle.
\]

If \( \mathfrak{a} \subset \mathfrak{p} \) is an Abelian subalgebra, then \( \Psi_{|_{A \times V}} \) is a Kempf-Ness function with respect to the \( A = \exp(\mathfrak{a}) \) action on \( V \) and the corresponding gradient map is given by \( \mathfrak{F}_a(x) = \mathfrak{F}_p(x)|_a \). The Kempf-Ness Theorem provides geometric criterion for the closedness of orbits of a representation of a real reductive Lie group and the existence of quotient [22, 32, 40, 53, 50, 62]. We point out that recently Böhm and Lafuente [19] proved the Kempf-Ness Theorem for linear actions of real reductive Lie groups avoiding any deep algebraic result. The basic tools are the notions of stable, semistable and polystable points. Biliotti, Ghigi, Raffero and Zedda, [13, 14, 15, 16], see also [55, 56, 64], identify an abstract setting to develop the geometrical invariant theory for actions of real reductive Lie groups and give numerical criteria for stability, semistability and polystability. These results have been applied for actions of real reductive groups on the set of probability measure of a compact Riemannian manifold. This problem is motivated by an application to upperbounds for the first eigenvalue of the Laplacian acting on functions [8, 9, 21, 37]. In this paper, using the properties of the Kempf-Ness functions, we explicitly compute the image of the gradient map corresponding to the \( A \) action on \( V \), respectively to the \( A \) action on \( \mathbb{P}(V) \), along \( A \) orbits (Theorems 16 and 33) generalizing a result due to Kac and Peterson in the complex setting [38], see also [7]. Roughly speaking, if \( A \) acts linearly on \( V \), respectively on \( \mathbb{P}(V) \), then \( A \cdot x \) is homeomorphic to a polyhedral \( C \), respectively to a polytope \( P \), and any \( A \) orbit contained in \( A \cdot x \) is diffeomorphic to the relative interior of a face of \( C \), respectively of a face of \( P \). As an application we obtain the Hilbert-Mumford criterion and the
algebraicity of the null cone for Abelian groups (Theorems 18 and 20). We also prove a convexity
Theorem of the gradient map, associated to $A$, restricted on the closure of $G$ orbits (Theorems
23 and 33). Applying results proved in [10, 11, 12], we completely describe the convex hull of the
image of the gradient map, with respect to $G$, restricted on the closure of $G$ orbits (Theorems
26, 34). Finally, using in a different context original ideas from [25], which are themselves of
some interest, we give a probably new proof of the Hilbert-Mumford criterion for real reductive
groups (Theorem 47) avoiding any algebraic result.

This paper is organized as follows.

In the second section we recall basic notions of convex geometry. In particular we recall
the definition of polyhedral, polytope, extremal points and exposed faces. In the third section
we recall the abstract setting on which we are able to develop a geometrical invariant theory
for actions of real reductive Lie groups. We also recall the notions of stable, semistable and
polystable points. In the fourth section we consider a closed subgroup $G$ of $GL(V)$, where $V$ is a
finite dimensional real vector space endowed by a scalar product $\langle \cdot, \cdot \rangle$, which is also closed under transpose. Given an Abelian subalgebra $a \subset p$, we explicitly compute the image of the gradient
map, with respect to $A = \exp(a)$, along $A$ orbits and restricted on the closure of $G$ orbits. As
an application we get the Hilbert-Mumford criterion for Abelian groups and the algebraicity
of the null cone. In the fifth section we completely describe the convex hull of the image of the
gradient map, with respect to $G$, restricted on the closure of $G$ orbits. We also discuss the
Hilbert-Mumford criterion for real reductive groups. In the sixth section, we investigate the $G$
action on $P(V)$. We prove convexity Theorems for the gradient map associated to $A$ and we
describe the convex hull of the image of the gradient map, with respect to $G$, restricted on the
closure of $G$ orbits. In the last section we give a proof of the Hilbert-Mumford criterion for real
reductive groups.

2. Convex geometry

It is useful to recall a few definitions and results regarding convex sets. A good references,
amongst many other, are [60, 61] (see also [10, 11, 12, 65]).

Let $V$ be a real vector space with a scalar product $\langle \cdot, \cdot \rangle$ and let $E \subset V$ be a convex subset.
The relative interior of $E$, denoted $\text{relint} \ E$, is the interior of $E$ in its affine hull. A face $F$ of $E$
is a convex subset $F \subset E$ with the following property: if $x, y \in E$ and $\text{relint}[x, y] \cap F \neq \emptyset$, then
$[x, y] \subset F$. The extreme points of $E$ are the points $x \in E$ such that $\{ \{x\}$ is a face. If $E$ is closed
and nonempty then the faces are closed [60, p. 62]. A face distinct from $E$ and $\emptyset$ will be called
a proper face. The support function of $E$ is the function $h_E : V \to \mathbb{R}, h_E(u) = \max_{x \in E} \langle x, u \rangle$. If
$u \neq 0$, the hyperplane $H(E, u) := \{ x \in E : \langle x, u \rangle = h_E(u) \}$ is called the supporting hyperplane
of $E$ for $u$. The set

$$F_u(E) := E \cap H(E, u)$$

is a face and it is called the exposed face of $E$ defined by $u$. In general not all faces of a convex
subset are exposed. A simple example is given by the convex hull of a closed disc and a point
outside the disc: the resulting convex set is the union of the disc and a triangle. The two vertices of the triangle that lie on the boundary of the disc are non-exposed 0-faces. Another example is given in [12] p.432.

A subset $E \subset V$ is called a **convex cone** if $E$ is convex, not empty and closed under multiplication by non negative real numbers. It is easy to check that $E$ only if $E$ is closed under addition and under multiplication by non negative real numbers. The cone generated by the vectors $f_1, \ldots, f_n \in V$ is the set $C(f_1, \ldots, f_n) := \{ \lambda_1 x_1 + \cdots + \lambda_n x_n : \lambda_1 \geq 0, \ldots, \lambda_n \geq 0 \}$. A cone arising in this way is called **finitely generated cone**. A **polytope** is the convex hull of a finite number of points of $V$. If $f_1, \ldots, f_n \in V$ then the set $P(f_1, \ldots, f_n) = \{ \alpha_1 f_1 + \cdots + \alpha_n f_n : \alpha_1, \ldots, \alpha_n \geq 0 \text{ and } \alpha_1 + \cdots + \alpha_n = 1 \}$ is the polytope generated by $f_1, \ldots, f_n$. The following result goes back by Farkas, Minkowsky and Weyl (see [61] p.87 for a proof).

**Theorem 2.** A convex cone is finitely generated if and only is the intersection of finitely many closed linear half spaces.

The above theorem implies $C(f_1, \ldots, f_n)$ is closed. In the literature the intersection of finitely many closed linear half spaces is called polyhedral. Hence $C(f_1, \ldots, f_n)$ is a polyhedral. The concept of polytope and polyhedral are related and this statement is usually attribute to Minkowski (see [61] p. 89).

**Theorem 3.** A convex set is a polytope if and only if it is a bounded polyhedral.

If $f_1, \ldots, f_n \in V$, we denote by $C^o(f_1, \ldots, f_n) = \{ \lambda_1 x_1 + \cdots + \lambda_n x_n : \lambda_1 > 0, \ldots, \lambda_n > 0 \}$.

**Lemma 4.** $C^o(f_1, \ldots, f_n)$ is closed if and only if $0 \in C^o(f_1, \ldots, f_n)$.

**Proof.** If $0 \in C^o(f_1, \ldots, f_n)$, then there exist $\alpha_1, \ldots, \alpha_n > 0$ such that

$$0 = \alpha_1 f_1 + \cdots + \alpha_n f_n.$$

Let $v \in C(f_1, \ldots, f_n)$. Since $v = v + \alpha_1 f_1 + \cdots + \alpha_n f_n$ it follows $v \in C^o(f_1, \ldots, f_n)$ and so $C^o(f_1, \ldots, f_n) = C(f_1, \ldots, f_n)$ is closed. Vice-versa, assume $C^o(f_1, \ldots, f_n)$ is closed. Then $C^o(f_1, \ldots, f_n) = C(f_1, \ldots, f_n)$ and so $0 \in C^o(f_1, \ldots, f_n)$.

Now, we investigate the boundary structure of a polyhedral by means of convex geometry.

**Lemma 5.** Let $F$ be an exposed proper face of $C(f_1, \ldots, f_n)$. Then there exists $u \in V - \{0\}$ such that

$$F = \{ x \in C(f_1, \ldots, f_n) : \max_{y \in C(f_1, \ldots, f_n)} \langle y, u \rangle = \langle x, u \rangle = 0 \}.$$

Moreover, $F$ is itself a polyhedral.

**Proof.** Let $u \in V$ such that $F = \{ x \in C(f_1, \ldots, f_n) : \max_{y \in C(f_1, \ldots, f_n)} \langle y, u \rangle = \langle x, u \rangle = c \}$. Since $C(f_1, \ldots, f_n)$ is a cone, if $x \in C(f_1, \ldots, f_n)$, then $tx \in C(f_1, \ldots, f_n)$ for any positive $t$. Hence the maximum $c$ must be zero and so $F$ is a closed cone. We claim that there exists, $J = \{ j_1, \ldots, j_s \} \subset \{ 1, \ldots, n \}$ such that $\langle f_j, u \rangle = 0$ per $j = 1, \ldots, s$ and $\langle f_r, u \rangle < 0$ for any $r \in$...
\{1, \ldots, n\} - \{j_1, \ldots, j_s\}. Otherwise there exist \(a, \beta \in \{1, \ldots, n\}\) such that \(\langle f_a, u \rangle \langle f_\beta, u \rangle < 0\) and so \(\text{relint}[f_a, f_\beta] \cap F \neq \emptyset\) which is a contradiction. Now, it is easy to check that \(F = C(f_{i_1}, \ldots, f_{i_s})\) and so the result is proved. □

Any face of a polytope is exposed \([60]\). The following statement proves that any face of a polyhedral is exposed as well. A proof is given in \([61]\) p.101 (see also Proposition 1.22 p.6 in \([65]\)).

**Proposition 6.** Let \(F\) be an exposed face of \(C(f_1, \ldots, f_n)\) and \(F_1 \subset F\) be an exposed face of \(F\). Then \(F_1\) is an exposed face of \(C(f_1, \ldots, f_n)\). Hence any face of a polyhedral is exposed.

**Remark 7.** Let \(E\) be a closed convex set of \(V\). Let \(F \subset E\) be face. If \(F_1 \subset F\) is an exposed face of \(F\), it is not true in general that \(F_1\) is an exposed face of \(E\). This means that the above result holds for a polyhedral but it is not true in general.

### 3. Kempf-Ness functions

In this section we briefly recall the abstract setting introduced in \([14]\) (see also \([13, 15, 16]\)).

Let \(\mathcal{M}\) be a Hausdorff topological space and let \(G\) be a connected real reductive group which acts continuously on \(\mathcal{M}\). Observe that with these assumptions we can write \(G = K \exp(p)\), where \(K\) is a maximal compact subgroup of \(G\). Starting with these data we consider a function \(\Psi : \mathcal{M} \times G \rightarrow \mathbb{R}\), subject to four conditions.

\((P1)\) For any \(x \in \mathcal{M}\) the function \(\Psi(x, \cdot)\) is smooth on \(G\).

\((P2)\) The function \(\Psi(x, \cdot)\) is left–invariant with respect to \(K\), i.e.: \(\Psi(x, kg) = \Psi(x, g)\).

\((P3)\) For any \(x \in \mathcal{M}\), and any \(v \in p\) and \(t \in \mathbb{R}\):

\[
\frac{d^2}{dt^2} \Psi(x, \exp(tv)) \geq 0.
\]

Moreover:

\[
\frac{d^2}{dt^2} \bigg|_{t=0} \Psi(x, \exp(tv)) = 0
\]

if and only if \(\exp(\mathbb{R}v) \subset G_x\).

\((P4)\) For any \(x \in \mathcal{M}\), and any \(g, h \in G\):

\[
\Psi(x, g) + \Psi(gx, h) = \Psi(x, hg).
\]

This equation is called the cocycle condition.

For \(x \in \mathcal{M}\), we define \(\mathfrak{F}(x) \in p^*\) by requiring that:

\[
\mathfrak{F}(x)(\xi) := \frac{d}{dt} \bigg|_{t=0} \Psi(x, \exp(t\xi)).
\]

We call \(\mathfrak{F} : \mathcal{M} \rightarrow p^*\) the gradient map of \((\mathcal{M}, G, K, \Psi)\). As immediate consequence of the definition of \(\mathfrak{F}\) we have the following result.

**Proposition 8.** The map \(\mathfrak{F} : \mathcal{M} \rightarrow p^*\) is \(K\)-equivariant.
Proof. It is an easy application of the cocycle condition and the left-invariance with respect to $K$ of $\Psi(x, \cdot)$. Indeed,

$$
\mathcal{F}(kx)(\xi) = \left. \frac{d}{dt} \right|_{t=0} \Psi(kx, \exp(t\xi)) = \left. \frac{d}{dt} \right|_{t=0} \Psi(x, \exp(t\xi)k) = \left. \frac{d}{dt} \right|_{t=0} \Psi(x, k^{-1}\exp(t\xi)k) = \left. \frac{d}{dt} \right|_{t=0} \Psi(x, \exp(t\text{Ad}(k^{-1})(\xi))) = \text{Ad}^*(k)(\mathcal{F}(x))(\xi).
$$

□

The following definition summarizes the above discussion.

**Definition 9.** Let $G$ be a real reductive Lie group, $K$ a maximal compact subgroup of $G$ and $\mathcal{M}$ a topological space with a continuous $G$–action. A Kempf-Ness function for $(\mathcal{M}, G, K)$ is a function

$$
\Psi: \mathcal{M} \times G \to \mathbb{R},
$$

that satisfies conditions (P1)–(P4).

Let $(\mathcal{M}, G, K)$ be as above and let $\Psi$ be a Kempf-Ness function.

**Definition 10.** Let $x \in \mathcal{M}$. Then:

a) $x$ is polystable if $G \cdot x \cap \mathcal{F}^{-1}(0) \neq \emptyset$.

b) $x$ is stable if it is polystable and $g_x$ is conjugate to a subalgebra of $\mathfrak{k}$.

c) $x$ is semi–stable if $G \cdot x \cap \mathcal{F}^{-1}(0) \neq \emptyset$.

d) $x$ is unstable if it is not semi–stable.

**Remark 11.** The four conditions above are $G$-invariant in the sense that if a point $x$ satisfies one of them, then every point in the orbit of $x$ satisfy the same condition. This follows directly from the definition for polystability, semi–stability and unstability, while for stability it is enough to recall that $g_{gx} = \text{Ad}(g)(g_x)$.

The following result establishes a relation between the Kempf-Ness function and polystable points. A proof is given in [14, p.2190] (see also [13, 64, 56]).

**Proposition 12.** Let $x \in \mathcal{M}$. The following conditions are equivalent:

a) $g \in G$ is a critical point of $\Psi(x, \cdot)$;

b) $\mathcal{F}(gx) = 0$;

c) $g^{-1}K$ is a critical point of $\psi_x$.

**Proposition 13.** If $\mathcal{F}(x) = 0$, then the stabilizer of $x$, i.e., $G_x = \{g \in G : gx = x\}$ is compatible with respect to the Cartan decomposition of $G = K \exp(\mathfrak{p})$. Moreover, if $G = \exp(\mathfrak{p})$, with $\mathfrak{p}$ Abelian, then any stabilizer is compatible.
Proof. The first part of the statement is well-known. A proof is given in [14], see also [32]. The second part is also easy to check. For the sake of completeness, we give a proof.

Let \( x \in \mathcal{M} \) and let \( g \in G_x \). Then \( g = \exp(v) \) for some \( v \in \mathfrak{p} \). Let \( f(t) := \mathfrak{T}''(\exp(tv)x) = \mathfrak{T}(\exp(tv)x)(v) \). Then \( f(0) = f(1) = 0 \) and

\[
\frac{d}{dt} f(t) = \frac{d}{dt} \mathfrak{T}''(\exp(tv)x) = \frac{d^2}{dt^2} \Psi(x, \exp(tv)) \geq 0.
\]

Therefore \( \frac{d^2}{dt^2} \Psi(x, \exp(tv)) = 0 \) for \( 0 \leq t \leq 1 \). It follows from (P3) that \( \exp(tv)x = x \) for any \( t \in \mathbb{R} \) and thus \( \exp(t\xi) \in G_x \). \( \square \)

Given \( \xi \in \mathfrak{p} \) for any \( t \in \mathbb{R} \) we define \( \lambda(x, \xi, t) = \mathfrak{T}(\exp(t\xi)x)(\xi) \). Applying the cocycle condition we get

\[
\mathfrak{T}(\exp(tv)x)(\xi) = \frac{d}{dt} \Psi(x, \exp(t\xi))
\]

and so

\[
\frac{d}{dt} \mathfrak{T}(\exp(tv)x)(\xi) = \frac{d^2}{dt^2} \Psi(x, \exp(tv)) \geq 0.
\]

This means that

\[
\lambda(x, \xi, t) = \mathfrak{T}(x)(\xi) + \int_0^t \frac{d^2}{ds^2} \Psi(x, \exp(sv)) ds
\]

is a non decreasing function as a function of \( t \). Moreover,

\[
\Psi(x, \exp(t\xi)) = \int_0^t \lambda(x, \xi, \tau) d\tau,
\]

and so

\[
\lambda(x, \xi) := \lim_{t \to +\infty} \frac{d}{dt} \Psi(x, \exp(t\xi)) \in \mathbb{R} \cup \{\infty\}.
\]

The function \( \lambda(x, \cdot) \) is called maximal weight of \( x \) in the direction \( \xi \). For a reference see, amongst many others, [13, 14, 54, 56, 64]. We point out that the maximal weight is well defined for any convex function.

Let \( V \) be a finite dimensional real vector space and let \( f : V \to \mathbb{R} \) be a convex function. For any \( \xi \in V \), the function \( g(t) = f(t\xi) \) is convex and so

\[
\lambda_f(\xi) = \lim_{t \to +\infty} \frac{d}{dt} f(t\xi) \in \mathbb{R} \cup \{\infty\}
\]

is well-defined. We conclude this section proving a useful lemma.

**Lemma 14.** Let \( f : V \to \mathbb{R} \) be a convex function. Assume that for any \( \xi \in V - \{0\} \), we have \( \lambda_f(\xi) > 0 \). Then \( f \) is an exhaustion and so it has a critical point which is a global minimum.

**Proof.** We may assume that \( V \) is endowed by a scalar product \( \langle \cdot, \cdot \rangle \). Denote by \( S(V) \) the unit sphere with respect to \( \langle \cdot, \cdot \rangle \). Let \( \xi \in S(V) \). Since \( \lambda_f(\xi) > 0 \), keeping in mind that \( \frac{d^2}{dt^2} f(\exp(tv)) \geq 0 \), it follows there exist \( t(\xi) > 0 \) and \( C_0 > 0 \) such that

\[
\frac{d}{dt} f(\exp(t(\xi)\xi)) \geq C_0 > 0,
\]
for any $t \geq t(\xi)$. Hence there exists a neighborhood $U_\xi$ of $\xi$ in $S(V)$ such that $\frac{d}{dt} f(tv) > C \frac{v}{2}$ for any $t \geq t(\xi)$ and for any $v \in U_\xi$. By usual compactness argument, there exist two constants $C > 0$ and $t_o > 0$ such that $\frac{d}{dt} f(t\xi) \geq C$, for any $\xi \in S(V)$ and for any $t \geq t_o$. Therefore, for any $v \in V$ such that $\|v\| \geq t_o$, we get

$$f(v) = f(t_o \frac{v}{\|v\|}) + \int_{t_o}^{\|v\|} \frac{d}{dt} f\left( t \frac{v}{\|v\|} \right) dt$$

and so $f(v) \geq \min_{\|w\|=t_o} f(w)$. This means that $f$ is an exhaustion and so it has a critical point which is a global minimum. \qed

4. Convexity Theorems for Abelian groups

Let $G$ be a connected real reductive Lie group and let $\rho : G \rightarrow GL(V)$ be a faithful representation on a finite dimensional real vector space $V$. We identify $G$ with $\rho(G) \subset GL(V)$ and we assume that $G$ is closed and it is closed under transpose. This means that there exists a scalar product $\langle \cdot, \cdot \rangle$ on $V$ such that $G = K \exp(\mathfrak{p})$, where $K \subset O(V)$ and $\mathfrak{p} \subset \mathfrak{g} \cap \text{Sym}(V)$. In the sequel, we denote by $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$. We define

$$\Psi : V \times G \rightarrow \mathbb{R} \quad \Psi(x, g) = \frac{1}{2} (\|gx\|^2 - \|x\|^2).$$

Lemma 15. $\Psi : V \times G \rightarrow \mathbb{R}$ is a Kempf-Ness function and the corresponding gradient map $\mathcal{F}_\rho : V \rightarrow \mathfrak{p}^*$ is given by $\mathcal{F}(x)(\xi) = \langle \xi, x \rangle$.

Proof. $(P1)$ and $(P2)$ are easy to check. Let $\xi \in \mathfrak{p}$ and let $f(t) = \Psi(x, \exp(t\xi))$. Then

$$f'(t) = \langle \exp(t\xi)x, \exp(t\xi)x \rangle, \quad f''(t) = \langle \exp(t\xi)x, \exp(t\xi)x \rangle.$$ 

Hence $f''(t) \geq 0$ and $f''(0) = 0$ if and only if $\xi = 0$ and $\exp(R\xi) \subset G_x$. Now,

$$\Psi(x, hg) = \frac{1}{2} (\|hgx\|^2 - \|x\|^2)$$

$$= \frac{1}{2} (\|hgx\|^2 - \|gx\|^2) + \frac{1}{2} (\|gx\|^2 - \|x\|^2)$$

$$= \Psi(gx, h) + \Psi(x, g),$$

proving the cocycle condition. Finally

$$\mathcal{F}(x)(\xi) = \frac{1}{2} \left. \frac{d}{dt} \right|_{t=0} \langle \exp(t\xi)x, \exp(t\xi)x \rangle = \langle \xi, x \rangle,$$

concluding the proof. \qed

Let $A = \exp(\mathfrak{a})$, where $\mathfrak{a} \subset \mathfrak{p}$ is an Abelian subalgebra. It is easy to check that $\Psi_{|A \times V}$ is a Kempf-Ness function with respect to $A$ and $\mathcal{F}_\mathfrak{a} := (\mathcal{F}_\rho)_{|\mathfrak{a}}$ is the corresponding gradient map. Since $\mathfrak{a}$ is an Abelian subalgebra of symmetric endomorphisms, they are simultaneously diagonalizable. Hence there exists an orthonormal basis $\{v_1, \ldots, v_n\}$ of $V$ and functionals $\alpha_1, \ldots, \alpha_n \in \mathfrak{a}^*$ such that

$$\xi(v) = \sum_{i=1}^n \alpha_i(\xi) \langle v, v_i \rangle v_i.$$
This means that if \( v = x_1 v_1 + \cdots + x_n v_n \), then
\[
\exp(\xi)(v) = e^{\alpha_1(\xi)} x_1 v_1 + \cdots + e^{\alpha_n(\xi)} x_n v_n.
\]

In particular,
\[
\mathcal{F}_a(v) = \sum_{i=1}^{n} \| x_i \|^2 \alpha_i,
\]
and so the image of \( \mathcal{F}_a \) is contained in the polyhedral \( C(\alpha_1, \ldots, \alpha_n) \).

Let \( x = x_1 v_1 + \cdots + x_n v_n \). We define \( \text{supp}_x \{ i \in \{1, \ldots, n\} : x_i \neq 0 \} \). If \( I \subset \{1 \ldots, n\} \), we denote by \( C_I \) the polyhedral generated by \( \{ \alpha_i : i \in I \} \), i.e.,
\[
C_I = \left\{ \sum_{i \in I} s_i \alpha_i : i \in I \text{ and } s_i \geq 0 \right\}.
\]

We also denote by \( C_I^0 = \{ \sum_{i \in I} s_i \alpha_i : i \in I \text{ and } s_i > 0 \} \). In the theory of real reductive representations, a fundamental problem is to compute the image of the gradient map. The following theorem generalizes a result proved by Kac-Peterson \[38\], see also \[7\], to the real case.

**Theorem 16.** Let \( x \in V \) and let \( I = \text{supp}_x \). Then the map \( \mathcal{F}_a : A \cdot x \rightarrow a^* \) satisfies:

a) \( \mathcal{F}_a : A \cdot x \rightarrow C_I^0 \) is a diffeomorphism onto. Hence \( A/A_x \cong C_I^0 \);

b) \( \mathcal{F}_a : A \cdot x \rightarrow C_I \) is an homeomorphism and for any face \( \sigma \subset C_I \) there exists a unique \( A \)-orbit \( Y \) such that \( \mathcal{F}_a^{-1}(\sigma) = Y \). Therefore \( \mathcal{F}_a(A \cdot x) \) is a polyhedral.

**Proof.** By Proposition 2.1 in [16], the map \( \mathcal{F}_a : A \cdot x \rightarrow a^* \) is a diffeomorphism and its image is a convex open subset of \( \mu(x) + a_x^0 \), where \( a_x^0 = \{ \varphi \in a^* : \varphi|_{a_x} = 0 \} \). It is easy to check that \( \mathcal{F}_a(A \cdot x) \) is invariant under multiplication of non negative real numbers and so it is an open convex cone contained in \( C_I^0 \). Now, we shall prove that \( \mathcal{F}_a(A \cdot x) = C_I^0 \).

Let \( b \subset a \) such that \( a = a_x \oplus b \). Let \( c \in C_I^0 \). Then \( c = \sum_{i \in I} c_i \alpha_i \) with \( c_i > 0 \). We define
\[
f : b \rightarrow \mathbb{R} \quad f(\xi) = \Psi(x, \exp(\xi)) - c(\xi).
\]

The equation \( \mathcal{F}_a(\exp(\xi_0)x) = c \) means the existence of a critical point of \( f \). We prove that \( f \) is strictly convex and an exhaustion.

Fix \( v, w \in b \) with \( w \neq 0 \) and consider the curve \( \gamma(t) = v + tw \). Set \( u(t) = \mathcal{F}_a(\gamma(t)) \). It is easy to check that \( u''(t) = \frac{d^2}{dt^2} \mid_{t=0} \Psi(\exp(v)x, \exp(tw)) > 0 \) since \( w \notin a_x \). This proves \( f \) is a strictly convex function on \( b \).

Let \( \xi \in b - \{0\} \). Then
\[
\frac{d}{dt} f(t\xi) = \sum_{i \in I} (e^{t\alpha_i(\xi)} \| x_i \|^2 - c_i) \alpha_i(\xi).
\]

Since \( \xi \notin a_x \), it follows \( \alpha_i(\xi) \neq 0 \) for some \( i \in I \). If \( \alpha_i(\xi) > 0 \), then
\[
\lim_{t \rightarrow +\infty} (e^{t\alpha_i(\xi)} \| x_i \|^2 - c_i) \alpha_i(\xi) = +\infty.
\]
If $\alpha_i(\xi) < 0$, then $\lim_{t \to +\infty} e^{t\alpha_i(\xi)} \| x_i \|^2 = 0$ and so, keeping in mind that $c_i > 0$, we get

$$\lim_{t \to +\infty} (e^{t\alpha_i(\xi)} \| x_i \|^2 - c_i)\alpha_i(\xi) = -c_i\alpha_i(\xi) > 0.$$ 

Hence $\lambda_f(\xi) > 0$ for every $\xi \in b - \{0\}$. By Lemma 14 the function $f$ is an exhaustion and so it has a critical point concluding the proof of item (a).

Let $\sigma$ be a face of $C_I$. By Proposition 6 there exists $\xi \in a$ such that $\sigma = F_\xi(C_I)$. By Lemma 5 there exists $J \subset I$ such that $\alpha_i(\xi) = 0$ for $i \in J$ and $\alpha_i(\xi) < 0$ otherwise. In particular

$$\lim_{t \to +\infty} \exp(t\xi)x = \sum_{i \in J} x_i v_i = \theta$$

and $\mathfrak{F}(\theta) \in \sigma$. We prove

$$\mathfrak{F}_a^{-1}(\sigma) = \{ v \in A \cdot x : \max_{z \in A \cdot x} \mathfrak{F}_a(v)(\xi) = \mathfrak{F}_a(z)(\xi) = 0 \} = A \cdot \theta$$

Let $u \in \mathfrak{F}_a^{-1}(\sigma)$. Write $s(t) = \mathfrak{F}_a(\exp(t\xi)u)(\xi)$. The function $s$ has a maximum in $t = 0$ and so

$$s(0) = 2(\xi u, \xi u) = 0.$$ 

This implies $\mathfrak{F}_a^{-1}(\sigma) \subset Ker \xi$. Since $\xi$ commutes with $A$ it follows $Ker \xi$ is $A$-invariant. Moreover, the gradient flow of the function $\mathfrak{F}_a^\xi(x) := \mathfrak{F}_a(x)(\xi)$ is given by $\exp(t\xi)$. Therefore $\text{Crit} \mathfrak{F}_a^\xi = Ker \xi$ and so $\mathfrak{F}_a^{-1}(\sigma)$ is $A$-invariant as well. This implies $A \cdot \theta \subset \mathfrak{F}_a^{-1}(\sigma)$.

Let $z \in \mathfrak{F}_a^{-1}(\sigma)$. Since $z \in A \cdot x$, there exists $\{ a_n \}_{n \in \mathbb{N}} \in A$ such that $a_n x \mapsto z$. The flow $\exp(t\xi)$ commutes with $A$ and so for any $n \in \mathbb{N}$, we have

$$\lim_{t \to +\infty} \exp(t\xi)(a_n x) = a_n \theta \in Ker \xi.$$ 

This means $\lim_{t \to +\infty} \exp(t\xi)a_n x = P(a_n x) = a_n P(x) = a_n \theta$, where $P : V \to Ker \xi$ is the orthogonal projection on $Ker \xi$. Since $a_n x \mapsto z$ and $z \in Ker \xi$, it follows $P(a_n x) = a_n \theta \mapsto z$. This implies $z \in A \cdot \theta$ and so $\mathfrak{F}_a^{-1}(\theta) = A \cdot \theta$. Now, applying again item (a), we get $\mathfrak{F}_a : A \cdot \theta \to C_{I'}$, is a diffeomorphism, where $I' = \text{supp} \theta$. In particular $\theta \in \text{relint} \sigma$. Summing up we have proved that the map $\mathfrak{F}_a : A \cdot x \to C_I$ is an homeomorphism. \hfill \Box

**Corollary 17.** Let $x \in V$ and let $I = \text{supp} x$. Let $F$ be a face of $C_I$ and let $J \subset I$ be the subset of $I$ associated to $F$ as in the proof of Theorem 16. Let $v_F = \sum_{i \in J} x_i v_i$. Then

$$\mathfrak{F}_a : A \cdot v_F \to \text{relint} F,$$

is a diffeomorphism onto. Moreover, $A \cdot x$ is the disjoint union of orbits $A \cdot v_F$, as $F$ runs over the set of faces of the polyhedral $C_I$.

**Proof.** Let $v_F$ the element associated to $F$. In Theorem 16 we have proved that $\mathfrak{F}_a(v_F) \in \text{relint} F$, $\mathfrak{F}_a : A \cdot v_F \to \text{relint} F$ is a diffeomorphism and $\mathfrak{F}_a^{-1}(F) = A \cdot v_F$. Therefore $A \cdot x$ is the disjoint union of orbits $A \cdot v_F$, as $F$ runs over the set of faces of the polyhedral $C_I$. \hfill \Box

**Theorem 18** (Hilbert-Mumford criterion). Let $u \in A \cdot x$. Then there exist $\xi \in a$ and $a \in A$ such that

$$\lim_{t \to +\infty} \exp(t\xi)ax = u.$$
Proof. By Corollary \[17\] there exists a unique face \( F \) such that \( \mathcal{F}(u) \in \text{relint } F \). Therefore \( u = av_F \). Taking \( \xi \in a \) such that \( F = F_\xi(C_I) \) it follows
\[
\lim_{t \to +\infty} \exp(t\xi)ax = av_F = u.
\]

**Corollary 19.** \( A \cdot x \) is closed if and only if \( 0 \in C_I^\circ \). Therefore \( A \cdot x \) is closed if and only if \( A \cdot x \cap \mathcal{F}_a^{-1}(0) \neq \emptyset \).

Proof. By Theorem \[16\] \( A \cdot x \) is closed if and only if \( C_I^\circ \) is closed. By Lemma \[4\] we get \( A \cdot x \) is closed if and only if \( 0 \in C_I^\circ \) and so if and only if \( A \cdot x \cap \mathcal{F}_a^{-1}(0) \neq \emptyset \).

**Theorem 20.** The set \( \{ x \in V : 0 \in \text{relint } A \cdot x \} \) is a real algebraic subset of \( V \) and so it is closed.

Proof. By Theorem \[15\] \( 0 \in \text{relint } A \cdot x \) if and only if \( 0 \) is a face of \( C_I \), where \( I = \text{supp}_x \). Since there exist a finite numbers of \( C_I \) where \( I \subset \{1, \ldots, n\} \), it follows that there exist \( I_1, \ldots, I_k \subset \{1, \ldots, 1\} \) such that \( 0 \in \text{relint } A \cdot x \) if and only if \( \text{supp}_x = I_j \) for some \( j \in \{1, \ldots, k\} \). Since any face of \( C_I \) is exposed, there exist \( \xi_1, \ldots, \xi_k \in a \) such that \( 0 \in A \cdot x \) if and only if \( \exp(t\xi_s)x \to 0 \) for some \( s \in \{1, \ldots, k\} \). Now, \( \exp(t\xi_s)x \to 0 \) if and only if \( x = \sum_{i \in J} x_i v_i \) with \( \alpha_i(\xi_s) < 0 \) for any \( i \in J \).

Let \( Z_s = \{ i \in \{1, \ldots, n\} : \alpha_i(\xi_s) < 0 \} \), for \( s = 1, \ldots, k \). Define \( H_s = \{ x \in V : \langle x, v_k \rangle = 0 \} \) for \( k \in \{1, \ldots, n\} - Z_s \). It is easy to check \( \exp(t\xi_s)x \to 0 \) if and only if \( x \in H_s \). Therefore
\[
\{ x \in V : 0 \in \text{relint } A \cdot x \} = H_1 \cup \cdots \cup H_k,
\]
concluding the proof.

**Proposition 21.** The image of \( \mathcal{F}_a : V \to a^* \) is a polyhedral and the set \( \{ x \in V : \mathcal{F}_a(\text{relint } A \cdot x) = \mathcal{F}_a(V) \} \) is an open and dense subset of \( V \).

Proof. Let \( x = x_1v_1 + \cdots + x_nv_n \) such that \( x_i \neq 0 \) for any \( i = 1, \ldots, n \). Since \( \text{supp}_x = I = \{1, \ldots, n\} \), by Theorem \[10\] we get \( \mathcal{F}_a(\text{relint } A \cdot x) = C_I \). On the other hand, by definition of \( \mathcal{F}_a \), it follows \( \mathcal{F}_a(V) \subset C_I \) and so the image of \( \mathcal{F}_a \) is the polyhedral \( C_I \).

Let \( x \in V \) and let \( I = \text{supp}_x \). By Theorem \[10\] there exists a neighborhood \( U \) of \( x \) such that for any \( y \in U \) we have \( C_I^\circ \subset \mathcal{F}_a(\text{relint } A \cdot y) \) for any \( y \in U \). Therefore the set \( \{ x \in V : \mathcal{F}_a(\text{relint } A \cdot x) = \mathcal{F}_a(V) \} \) is open and it contains an open dense subset of \( V \). Therefore it is an open dense subset of \( V \).

Now we investigate convexity Theorems for \( A \)-invariant subsets. The proof of next result uses original ideas from [7].

**Proposition 22.** Let \( M \) be a closed real algebraic irreducible subset of \( V \). Assume that \( M \) is \( A \)-invariant. Then \( \mathcal{F}_a(M) = C_I \) where \( I = \text{supp}_v \) for some \( v \in M \) and so the image is a polyhedral.

Proof. Let \( v \in M \) and let \( I = \text{supp}_v \). Define \( U_v := \{ u \in M : C_I \subset C_{\text{supp}_v} \} \). The set \( \{ u \in V : C_I \subset C_{\text{supp}_v} \} \) is Zariski open and \( v \in U_v \). Therefore \( U_v \) is Zariski open. Now, keeping
in mind there exist finitely many subsets of \( \{1, \ldots, n\} \), there exist a finite numbers of open subset \( U_v \) and so
\[
M = U_{v_1} \cup \cdots \cup U_{v_k},
\]
for some \( v_1, \ldots, v_k \in M \). Since \( M \) irreducible, it follows that \( U_{v_1} \cap \cdots \cap U_{v_k} \neq \emptyset \), and so it is Zariski open. Therefore there exists \( x \in U_{v_1} \cap \cdots \cap U_{v_k} \) and so \( \tilde{\mathcal{S}}_a(M) = \tilde{\mathcal{S}}_a(A \cdot x) = C_I \) where \( I = \text{supp}_x \).

We conclude this section computing the image of \( G \) orbits under the gradient map.

**Theorem 23.** Let \( x \in V \). There exists \( v \in G \cdot x \) such that \( \tilde{\mathcal{S}}_a(G \cdot x) = \tilde{\mathcal{S}}_a(A \cdot v) \) and so the image is a polyhedral.

**Proof.** We give two proofs. In the first proof we assume that \( G \) is an algebraic real reductive group and \( \rho : G \rightarrow \text{GL}(V) \) is a rational representation. Let \( G^C \) be the complexification of \( G \) acting on \( V^C \). Let \( G_R = G^C \cap \text{GL}(V) \). Then \( G^C \cdot x \cap V \) is a closed real algebraic irreducible set and \( G^C \cdot x \cap V \) is a finite union of \( G_R \) orbits \([13, 22, 67]\). Moreover, any \( G^C_R \) orbit throughout an element \( v \in G^C \cdot x \cap V \) is open \([13, \text{Proposition 2.3}]\). Since \( (G_R)^o = G \cdot [4, 20] \) it follows that any \( G \) orbit throughout an element of \( G^C \cdot x \cap V \) is also open. Write \( M = G^C \cdot x \cap V \). By Proposition \([22]\) there exist \( v_1, \ldots, v_k \in M \) such that \( U_{v_1} \cap \cdots \cap U_{v_k} \neq \emptyset \) and \( M = U_{v_1} \cup \cdots \cup U_{v_k} \). Since \( U_{v_1} \cap \cdots \cap U_{v_k} \) is Zariski open, it follows \( G \cdot x \cap U_{v_1} \cap \cdots \cap U_{v_k} \neq \emptyset \), and so \( \tilde{\mathcal{S}}_a(G \cdot x) = \tilde{\mathcal{S}}_a(A \cdot z) \) for some \( z \in G \cdot x \).

The second proof works without any algebraic assumption. The main tool is a Theorem of Baire.

Let \( y = y_1v_1 + \cdots + y_nv_n \) with \( y_1 \cdots y_n \neq 0 \). If \( y \in G^C \cdot x \), then \( \tilde{\mathcal{S}}_a(A \cdot y) = \tilde{\mathcal{S}}_a(V) \) and so the result is proved. Otherwise,
\[
G^C \cdot x = G^C \cdot x \cap \{ v \in V : \langle v, v_1 \rangle = 0 \} \cup \cdots \cup G^C \cdot x \cap \{ v \in V : \langle v, v_n \rangle = 0 \}.
\]
By a Theorem of Baire, there exists \( k \in \{1, \ldots, n\} \) such that \( G^C \cdot x \cap \{ v \in V : \langle v, v_k \rangle = 0 \} \) has an interior point. Therefore, there exists \( y \in G \cdot x \) and a neighborhood \( V \) of the origin of the Lie algebra \( \mathfrak{g} \) of \( G \) such that \( \exp(\theta)y \in \{ v \in V : \langle v, v_1 \rangle = 0 \} \) for any \( \theta \in V \). Assume by contradiction that \( G \cdot y \) is not contained in \( V_k = \{ v \in V : \langle v, v_k \rangle = 0 \} \). Write \( A = \{ z \in G \cdot y : z \in V_k \} \).

Since \( G \) is analytic and the \( G \) action on \( V \) is analytic, \( G \cdot y \) is analytic. By the above discussion the interior of \( A \), that we denote by \( A^o \), is not empty. Let \( z \in \partial A^o \). Let \( \varphi : U \rightarrow U' \) be a chart of \( z \). Then \( z \in V_k \) and so \( \varphi^{-1}(U' \cap V_k) \) contains an open subset. Since \( \varphi \) is analytic it follows \( \varphi(U) \subset V_k \). A contradiction. Therefore \( G^C \cdot x = G^C \cdot y \subset V_k \). In particular, there exists a \( G \)-invariant subspace \( Z \subset V_k \) containing \( G \cdot x \) such that \( V = Z \oplus Z^\perp \) is a \( G \)-stable splitting. This follows by the Cartan decomposition \( G = K \exp(p) \) where \( K \subset O(V) \) and \( p \subset \text{Sym}(V) \). Moreover, if \( x \in Z \), then
\[
\tilde{\mathcal{S}}_a(G \cdot x) = (\tilde{\mathcal{S}}_a)\vert_Z (G \cdot x).
\]
After a finite number of steps, there exists a \( G \) invariant subspace \( W \) such that \( G \cdot x \subset W \) and \( G \cdot x \) is not contained in any subspace of \( W \), i.e., it is full. Hence, there exists \( y \in G \cdot x \) such
that $\mathcal{F}_a(G \cdot x) = \mathcal{F}_a(A \cdot y) = \mathcal{F}_a(W)$. Moreover, since $\{z \in W : \mathcal{F}_a(A \cdot z) = \mathcal{F}_a(W)\}$ is open and dense, we may choose $y \in G \cdot x$ such that $\mathcal{F}_a(G \cdot x) = \mathcal{F}_a(A \cdot y) = \mathcal{F}_a(W)$. □

5. Convexity Theorems for real reductive representations

Let $(V, \langle \cdot, \cdot \rangle)$ be a real finite dimensional vector space and let $G \subset GL(V)$ be a connected closed real reductive group such that $G = K \exp(p)$, where $K \subset O(V)$ and $p \subset g \cap \text{Sym}(V)$. Let $\mathcal{F}_p : V \rightarrow p^*$ be the corresponding gradient map. Using an $\text{Ad}(K)$-invariant scalar product on $p$ we can identify $p$ with $p^*$ and so we may think the gradient map as a $p$-valued map. If $a \subset p$ is an Abelian subalgebra, then $\mathcal{F}_a = \pi_a \circ \mathcal{F}_p$ is the gradient map associated to $A = \exp(a)$, where $\pi_a : p \rightarrow a$ is the orthogonal projection. In this section we study the convex hull of the image of the closure of $G$ orbits under the gradient map associated to $G$.

**Theorem 24.** Let $x \in V$ and let $E = \text{Conv}(\mathcal{F}_p(G \cdot x))$. Then $E$ is a closed convex set and any face of $E$ is exposed.

**Proof.** By Theorem 23 $C = \mathcal{F}_a(G \cdot x)$ is a polyhedral. Since $\mathcal{F}_a = \pi_a \circ \mathcal{F}_p$ and $\mathcal{F}_a(G \cdot x)$ is convex, it follows $\pi_a(E) = \pi_a(\mathcal{F}_p(G \cdot x)) = \mathcal{F}_a(G \cdot x)$. By Lemma 7 in [26] we have $E = KC$. In particular $E$ is closed, due to the fact that $K$ is compact, and $C$ is closed. By Theorem 3 any face of $E$ is exposed and so, by Lemma 1.4 and Proposition 1.4 in [12], any face of $E$ is exposed. □

**Remark 25.** In [12] the authors study $K$-invariant compact convex sets. However, Lemma 1.4 and Proposition 1.4 holds without the compactness assumption.

Similarly, one may prove the following more general result.

**Theorem 26.** Let $M$ be an $G$-invariant closed algebraic irreducible subset of $V$. Let $a \subset p$ a maximal Abelian subalgebra. Then $\text{Conv}(\mathcal{F}_p(M)) = KC$, where $C$ is a polyhedral. Therefore $\text{Conv}(\mathcal{F}_p(M))$ is closed and any face of $\text{Conv}(\mathcal{F}_p(M))$ is exposed.

Given $\beta \in p$, we define the following subgroups:

$$G^\beta = \{g \in G : \text{Ad}(g)(\beta) = \beta\}$$

$$G^\beta_- := \{g \in G : \lim_{t \rightarrow +\infty} \exp(t\beta)g\exp(-t\beta)\text{exists}\}$$

$$R^\beta_- := \{g \in G : \lim_{t \rightarrow +\infty} \exp(t\beta)g\exp(-t\beta) = e\}$$

The next lemma is well-known. A proof is given in [11].

**Lemma 27.** If $g \in G^\beta_-$ then $\lim_{t \rightarrow +\infty} \exp(t\beta)g\exp(-t\beta) \in G^\beta$. Moreover, $G^\beta_-$ is a parabolic subgroup of $G$ with Lie algebra $g^\beta_- = g^\beta \oplus v^\beta_-$ and $G = G^\beta_- K$. Every parabolic subgroup of $G$ equals $G^\beta_-$ for some $\beta \in p$. $R^\beta_-$ is the unipotent radical of $G^\beta_-$ and $G^\beta$ is a Levi factor. Finally, $G = KG^\beta_-$. 

We establish a connection between the Hilbert-Mumford criterion and the convex hull of the image of the gradient map restricted on the closure of a $G$ orbit.
Proposition 28. Let $x \in V$. If $0 \in \mathfrak{p}$ belongs to a face of $\text{Conv}(\mathfrak{g}_p(G \cdot x))$ then there exists $\xi \in \mathfrak{p}$ such that $\lim_{t \to +\infty} \exp(t\xi)x = 0$.

Proof. Assume $0 \in \text{Conv}(\mathfrak{g}_p(G \cdot x))$ belongs to a face. By Theorems 26 and 23 there exists $z \in G \cdot x$ such that $\text{Conv}(\mathfrak{g}_p(G \cdot x)) = K\mathfrak{g}_a(A \cdot z)$. By Lemma 1.4 [12], $0$ belongs to a face of $A \cdot z$. Applying the Hilbert Mumford criterion for Abelian groups, there exists $\nu \in \mathfrak{p}$ such that

$$\lim_{t \to +\infty} \exp(t\nu)z = 0.$$ 

and so $0 \in A \cdot \mathfrak{g}$. Now, $z = gx$ and $g = h k$ where $h \in G^\nu$ and $k \in K$. Therefore, keeping in mind that the limit $\lim_{t \to +\infty} \exp(t\nu)h \exp(-t\nu)$ exists, we have

$$0 = \lim_{t \to +\infty} \exp(t\nu)h k x = \lim_{t \to +\infty} (\exp(t\nu)h \exp(-t\nu)) \exp(t\nu)k x = \lim_{t \to +\infty} \exp(t\nu)k x$$

Since $\exp(t\nu)k x = k \exp(t\text{Ad}(k^{-1})(\nu)x)$ it follows

$$\lim_{t \to +\infty} \exp(\text{Ad}(k^{-1})(\nu))x = 0$$

and so the result is proved. \hfill \square

A conjecture might be that the vice-versa holds as well. This means that $0 \in A \cdot \mathfrak{g}$ if and only if $0$ belongs to a face of $\text{Conv}(\mathfrak{g}_p(G \cdot x))$. Roughly speaking the Hilbert-Mumford criterion for real reductive groups follows from the boundary structure, by means of the convex geometry, of the convex hull of the image of the gradient map restricted on the closure of $G$ orbits. We briefly recall a proof of the Hilbert-Mumford criterion [59, 19, 32]. The main point is the existence of $k \in K$ such that $0 \in A \cdot k x$ and so $0 \in A' \cdot x$ where $A' = k^{-1}A k$. Now, if $\mathfrak{g}_{a'}(G \cdot x) = \mathfrak{g}_{a'}(A' \cdot x)$, where $\mathfrak{a}'$ is the Lie algebra of $A'$, then Theorem 23 implies $\text{Conv}(\mathfrak{g}_p(G \cdot x)) = K\mathfrak{g}_{a'}(A' \cdot x)$ and so, by Lemma 1.4 [12], $0$ is a face of $\text{Conv}(\mathfrak{g}_p(G \cdot x))$. However $\mathfrak{g}_{a'}(G \cdot x)$ does not coincide in general with $\mathfrak{g}_{a'}(A' \cdot x)$. Indeed, $0$ does not belong in general to a face of $\text{Conv}(\mathfrak{g}_p(G \cdot x))$.

Example 29. Let $\text{SL}(2, \mathbb{R})$ acting on $\mathbb{R}^2$ in a natural way. It has 2 orbit: the origin and $\mathbb{R}^2 \setminus \{0\}$. Let $\xi$ be any non zero symmetric matrices with trace 0. If we consider a orthonormal basis $\{e_1, e_2\}$ of eigenvectors of $\xi$, up to rescaling, we may assume $\xi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The gradient map with respect to $A = \exp(\mathbb{R}\xi)$ is given by $\mathfrak{g}_{a}(x, y) = x^2 - y^2$. Then $\mathfrak{g}_{a}(\mathbb{R}^2) = \mathbb{R}$ and so $\mathfrak{g}_{p}(\mathbb{R}^2) = \mathfrak{p}$.

Now, we come back to the proof of the Hilbert-Mumford criterion. We have proved that $0 \in A' \cdot x$. Applying Hilbert Mumford criterion for Abelian groups, there exists $\xi \in \mathfrak{a}'$, where $\mathfrak{a}'$ is the Lie algebra of $A'$, such that $\exp(t\xi)x \to 0$. We claim that the orbit $G^\xi \cdot x$ goes to 0 under the flow $\exp(t\xi)$ when $t \to +\infty$. Indeed,

$$\lim_{t \to +\infty} \exp(t\xi)g x = \lim_{t \to +\infty} (\exp(t\xi)g \exp(-t\xi)) \exp(t\xi)x = 0.$$ 

Therefore, the function $x \mapsto \mathfrak{g}_p(x)(\xi)$ restricted on $G^\xi \cdot x$ has a maximum achieved in 0. Now, if we denote by $E = \text{Conv}(\mathfrak{g}_p(G^\xi \cdot x))$, the above discussion proves that $0$ belongs to an exposed
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face of $E$. Roughly speaking, it seems that we can provide some appropriate connection between
the Hilbert-Mumford criterion and the convex hull of the image of the gradient map restricted
on the closure of orbits of parabolic subgroups. We leave this problem for future investigation.

6. REAL REDUCTIVE REPRESENTATIONS ON PROJECTIVE SPACES

Let $V$ be a real finite dimensional vector space endowed by a scalar product $\langle \cdot, \cdot \rangle$. Let
$G \subset \text{GL}(V)$ a closed reductive subgroup satisfying $G = K \exp(p)$, where $K = G \cap \text{O}(V)$ is
a maximal compact subgroup of $G$ and $p = g \cap \text{Sym}(V)$. We denote by $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$. The $G$
action on $V$ induces, in a natural way, an action on the projective space $P(V)$ given by

$$G \times P(V) \longrightarrow P(V) \quad (g, [v]) \mapsto [gv].$$

Lemma 30. The function

$$\tilde{\Psi} : G \times P(V) \longrightarrow \mathbb{R} \quad (g, [x]) \mapsto \log \left( \frac{\| gx \|}{\| x \|} \right),$$

is a Kempf-Ness function and the corresponding gradient map $\tilde{\delta}_p : P(V) \longrightarrow p^*$ is given by

$$\tilde{\delta}_p([x])(\xi) = \frac{\langle \xi x, x \rangle}{\| x \|^2}.$$

Proof. $P(1)$ and $P(2)$ are easy to check. Let $\xi \in p$ and let $f(t) = \tilde{\Psi}(x, \exp(t\xi))$. Then

$$f'(t) = \frac{\langle \exp(t\xi) x, \exp(t\xi)x \rangle}{\langle \exp(t\xi)x, \exp(t\xi)x \rangle}$$

$$f''(t) = 2 \frac{\langle \exp(t\xi)x, \exp(t\xi)x \rangle^2 (\langle \exp(t\xi)x, \exp(t\xi)x \rangle^2 - \langle \exp(t\xi)x, \exp(t\xi)x \rangle^2)}{\langle \exp(t\xi)x, \exp(t\xi)x \rangle^2}.$$ 

By the Cauchy-Schwartz’s inequality we have $f''(t) \geq 0$. Moreover, $f''(0) = 0$ if and only if $\xi x$
and $x$ are linearly dependent and so if and only if $\exp(R\xi) \subset G[x]$. Now,

$$\Psi(x, hg) = \log \left( \frac{\| hgx \|}{\| x \|} \right)$$

$$= \log \left( \frac{\| hgx \|}{\| gx \|} \right) + \log \left( \frac{\| gx \|}{\| x \|} \right)$$

$$= \tilde{\Psi}(gx, h) + \tilde{\Psi}(x, g),$$

proving the cocycle condition. Finally

$$\tilde{\delta}_p(x)(\xi) = \frac{d}{dt} \bigg|_{t=0} \log \left( \frac{\| \exp(t\xi)x, \exp(t\xi)x \|}{\| x \|} \right)$$

$$= \frac{\langle \xi x, x \rangle}{\| x \|^2}.$$

concluding the proof. \hfill \Box

Let $a \subset p$ be an Abelian subalgebra. It is easy to check that $\tilde{\delta}_a = \pi_a \circ \tilde{\delta}_p$ is the gradient map
associated to $A$. We denote by $P(V)^A = \{ y \in P(V) : A \cdot y = y \}$, the fixed points set.
Theorem 31. Let $x \in \mathbb{P}(V)$. The map $\tilde{\mathcal{A}} : \overline{A \cdot x} \to \mathfrak{a}^*$ satisfies the following properties:

a) \( \tilde{\mathcal{A}}(A \cdot x) \) is diffeomorphic to an open convex subset of $\tilde{\mathcal{A}}(x) + a_1^\perp$, its closure coincides with $\tilde{\mathcal{A}}(A \cdot x)$, it is a polytope $P$ and it is the convex hull of $\tilde{\mathcal{A}}(\mathbb{P}(V)^A \cap A \cdot x)$.

b) $\tilde{\mathcal{A}} : \overline{A \cdot x} \to P$ is an homomorphism. Moreover, any face of $P$ is homeomorphic to a closure of a unique $A$ orbit contained in $A \cdot x$.

Proof. The first statement is well-known. A proof using the Kempf-Ness function is given in [16]. Item (b) is known in the complex setting [1].

Let \( \sigma \subset P \) be a face of $P$. It is an exposed face, so there exists $\xi \in \mathfrak{a}$ such that

\[
\sigma = \{ z \in P : \langle z, \xi \rangle = \max_{y \in \tilde{\mathcal{A}}(A \cdot x)} \langle y, \xi \rangle \}.
\]

Since for any $y \in \mathbb{P}(V)$, the function $t \mapsto \tilde{\mathcal{A}}(\exp(t\xi)y)(\xi)$ is monotone, it follows that $\tilde{\mathcal{A}}^{-1}(\sigma) \subset \mathbb{P}(V)^\Gamma$, where $\Gamma = \exp(\mathbb{R} \xi)$ and so $\tilde{\mathcal{A}}^{-1}(\sigma)$ is $A$-invariant. In the sequel we denote by $\tilde{\mathcal{A}}^\xi$ the function $x \mapsto \tilde{\mathcal{A}}(x)(\xi)$ which is non-degenerate in the sense of Morse-Bott [1, 27, 32, 33], see also [33] p. 589 for a proof. Applying the Linearization Theorem [32, 33], for any $x \in \mathbb{P}(V)$ the limit $\lim_{t \to +\infty} \exp(t\xi)x$ exists and the set of all points of $\mathbb{P}(V)$ for which $\lim_{t \to +\infty} \exp(t\xi) \cdot$ lies on a given critical manifold $C$ forms a submanifold and they are called unstable manifolds. We denote by $\varphi_\infty$ the limit map. Let $\xi \in \mathfrak{p}$ and let $\lambda_1 > \cdots > \lambda_k$ be its eigenvalues. We denote by $V_1, \ldots, V_k$ the corresponding eigenspaces. In view of the orthogonal decompositions $V = V_1 \oplus \cdots \oplus V_k$, the critical points of $\tilde{\mathcal{A}}^\xi$ are given by $\mathbb{P}(V_1) \cup \cdots \cup \mathbb{P}(V_k)$ and the corresponding unstable manifolds are given by:

\[
W^\xi_1 = \mathbb{P}^n(\mathbb{R}) - \mathbb{P}(V_2 \oplus \cdots \oplus V_k),
\]

\[
W^\xi_2 = \mathbb{P}(V_2 \oplus \cdots \oplus V_k) - \mathbb{P}(V_3 \oplus \cdots \oplus V_k),
\]

\[
\vdots
\]

\[
W^\xi_{k-1} = \mathbb{P}(V_{k-1} \oplus V_k) - \mathbb{P}(V_k).
\]

\[
W^\xi_k = \mathbb{P}(V_k).
\]

(see [14]). Assume that $x \in W^\xi_j$ for some $1 \leq j \leq k$. Then $x_\infty = \lim_{t \to +\infty} \exp(t\xi)x \in \mathbb{P}(V_j)$ and $A \cdot x \subset \mathbb{P}((V_j \oplus \cdots \oplus V_k)$. Since $\tilde{\mathcal{A}}^\xi$ restricted to $\mathbb{P}(V_j \oplus \cdots \oplus V_k)$ has $\mathbb{P}(V_j)$ as a unique maximum it follows $x_\infty \in \tilde{\mathcal{A}}^{-1}(\sigma)$. Therefore $\overline{A \cdot x_\infty} \subset \tilde{\mathcal{A}}^{-1}(\sigma)$. Let $a_n$ be a sequence of elements of $A$ such that $a_n \cdot x \to \theta \in \tilde{\mathcal{A}}^{-1}(\sigma)$. Since

\[
\varphi_\infty : W^\xi_k \longrightarrow \mathbb{P}(V_j), \quad y \mapsto \lim_{t \to +\infty} \exp(t\xi)y
\]

is smooth, it follows

\[
\theta = \lim_{n \to \infty} \varphi_\infty(a_n \cdot x) = \lim_{n \to \infty} a_n \cdot x_\infty,
\]

and so it lies in $\overline{A \cdot x_\infty}$. Therefore

\[
\tilde{\mathcal{A}}^{-1}(\sigma) = \overline{A \cdot x_\infty}.
\]

Now, again applying item (1), the map $\tilde{\mathcal{A}} : A \cdot x_\infty \to \text{relint} \sigma$ is a diffeomorphism, proving item (b). \qed
The following result is a direct consequence of the above Theorem.

**Corollary 32.** Let \( x \in \mathbb{P}(V) \). Then

a) \( \overline{A \cdot x} - A \cdot x \) is a finite union of \( A \) orbits.

b) \( u \in \overline{A \cdot x} \) if and only if there exists \( \xi \in \mathfrak{a} \) and \( a \in A \) such that \( \lim_{t \to +\infty} \exp(t\xi)ax = u \).

Note that the second item provides a Hilbert-Mumford criterion for Abelian groups acting on \( \mathbb{P}(V) \). Now we completely describe the image of the gradient map along \( A \) orbits. In the sequel we denote by \( \pi : V - \{0\} \to \mathbb{P}(V) \) the natural projection. Since \( A \) is Abelian, there exists an orthonormal basis \( \{v_1, \ldots, v_n\} \) of \( V \) such that diagonalize simultaneously any element of \( A \). Therefore

\[
\tilde{\mathfrak{a}}([x_1v_1 + \cdots + v_ne_n])(\xi) = \frac{x_1^2\alpha_1(\xi) + \cdots + x_n^2\alpha(\xi)}{x_1^2 + \cdots + x_n^2}.
\]

If \( I \subset \{1, \ldots, n\} \), we denote by \( P_I = \text{Conv}(\alpha_k : k \in I) \) the polytope generated by \( \alpha_k \), as \( k \) runs in \( I \).

**Theorem 33.** Let \( x \in \mathbb{P}(V) \) and let \( \tilde{x} \in V \) such that \( \pi(\tilde{x}) = x \). Let \( I = \text{supp}_x \). Then

a) \( \tilde{\mathfrak{a}}(\overline{A \cdot x}) = P_I \), where \( P_I = \text{Conv}(\alpha_i : i \in I) \).

b) \( A \cdot x \) is closed if and only if \( x \in \mathbb{P}(V)^A \).

**Proof.** By definition of \( \tilde{\mathfrak{a}} \) it follows \( \tilde{\mathfrak{a}}(\overline{A \cdot x}) \subseteq P_I \). It is well known that \( P_I \) is the convex hull of its extremal points \([40]\). Since \( P_I \) is generated by \( \alpha_k \), as \( k \) runs on \( I \), the extremal points of \( P_I \) are contained in \( \{\alpha_j : j \in I\} \). Assume that \( \alpha_s \) is an extremal point of \( P_I \) for some \( s \in I \). Let \( J = \{r \in I \text{ such that } \alpha_r = \alpha_j\} \). Since any face of \( P_I \) is exposed, \( \{\alpha_s\} \) is an exposed face and so there exists \( \xi \in \mathfrak{a} \) and \( c > 0 \) such that \( \langle \xi, \alpha_i \rangle < c \) for \( i \notin J \) and \( \langle \xi, \alpha_s \rangle = c \) for \( s \in J \). Therefore

\[
\tilde{\mathfrak{a}}(\exp(t\xi)x) = \sum_{i \in I} \frac{x_i^2e^{2t\langle \alpha_i, \xi \rangle}}{x_i^2e^{2\langle \alpha_i, \xi \rangle}} + \sum_{m \in I\setminus J} \frac{x_m^2e^{2t\langle \alpha_m, \xi \rangle}}{x_m^2e^{2\langle \alpha_m, \xi \rangle}} - c.
\]

Taking the limit \( t \to +\infty \), it follows \( \alpha_s \in \tilde{\mathfrak{a}}(\overline{A \cdot x}) \). This holds for any extremal point of \( P_I \) and so \( \tilde{\mathfrak{a}}(\overline{A \cdot x}) = P_I \) proving the first item.

Since \( \tilde{\mathfrak{a}}(\overline{A \cdot x}) = \tilde{\mathfrak{a}}(\overline{A \cdot x}) \), \( A \cdot x \) is closed if and only if \( \tilde{\mathfrak{a}}(\overline{A \cdot x}) \) is closed. Since the image of any \( A \)-orbit is the relative interior of a polytope, it follows that \( A \cdot x \) is closed if and only if \( A \cdot x = x \).

We now compute the image of the closure of a \( G \) orbit under the gradient map.

**Theorem 34.** Let \( x \in \mathbb{P}(V) \) and let \( \tilde{x} \in V \) such that \( \pi(\tilde{x}) = x \). There exists \( v \in G \cdot \tilde{x} \) such that \( \tilde{\mathfrak{a}}(\overline{G \cdot x}) = \tilde{\mathfrak{a}}(\overline{A \cdot x}) = P_I \), where \( I = \text{supp}_x \), and so it is a polytope. Moreover, the set \( \text{Conv}(\tilde{\mathfrak{a}}(\overline{G \cdot x})) = K P_I \) and so any face of \( \text{Conv}(\tilde{\mathfrak{a}}(G \cdot x)) \) is exposed.
Proof. Let \( \tilde{x} \in V \) such that \( \pi(\tilde{x}) = x \). By Theorem 23 it follows that \( \tilde{\mathcal{F}}_a(G \cdot \tilde{x}) = \tilde{\mathcal{F}}_a(A \cdot \tilde{v}) = C_I \), where \( I = \text{supp}_G \). By the above corollary \( \tilde{\mathcal{F}}_a(A \cdot \tilde{v}) = P_I \), where \( v = \pi(\tilde{v}) \). Since for any \( \hat{y} \in G \cdot \tilde{x} \), we get \( \tilde{\mathcal{F}}_a(A \cdot \hat{y}) \subset C_I \), it follows \( \tilde{\mathcal{F}}_a(A \cdot \hat{v}) \subset P_I \), for any \( y \in G \cdot x \) and so the first part is proved.

Set \( E = \text{Conv}(\tilde{\mathcal{F}}_p(G \cdot x)) \). By Lemma 7 in [26] we get \( \pi_0(E) = \tilde{\mathcal{F}}_a(A \cdot \pi(v)) \) and so \( E = K \tilde{\mathcal{F}}_a(A \cdot \tilde{v}) = CP_I \). By Theorem 0.3 in [12], any face of \( E \) is exposed. \( \square \)

7. Hilbert-Mumford criterion for reductive groups

In this section we prove the Hilbert-Mumford criterion for real reductive groups. We use in a different context original ideas from [25].

Let \( G \subset \text{GL}(V) \) a closed reductive subgroup satisfying \( G = K \exp(p) \), where \( K = G \cap O(V) \) is a maximal compact subgroup of \( G \) and \( p = g \cap \text{Sym}(V) \). The function \( \tilde{\Psi} : G \times \mathbb{P}(V) \to \mathbb{R} \), \( (g, [x]) \mapsto \log \left( \frac{\|gx\|}{\|x\|} \right) \), is a Kempf-Ness function and the corresponding gradient map \( \tilde{\mathcal{F}}_p : \mathbb{P}(V) \to p^* \) is given by \( \tilde{\mathcal{F}}_p([x])(\xi) = \langle \xi x, x \rangle \frac{x^T x}{\|x\|^2} \).

We may fix Ad\( (K) \)-invariant scalar product \( \langle \cdot, \cdot \rangle_g \) on \( g \) such that \( g = k \oplus p \) is an orthogonal splitting. Hence we may think the gradient map as a \( p \)-valued map, \( \tilde{\mathcal{F}}_p : \mathbb{P}(V) \to p \).

We define \( f : \mathbb{P}(V) \to \mathbb{R}, \quad [x] \mapsto \frac{1}{2} \langle \tilde{\mathcal{F}}_p([x]), \tilde{\mathcal{F}}_p([x]) \rangle_g \).

In the sequel if \( \xi \in g \), we denote by \( \xi^\#([x]) := \frac{d}{dt}|_{t=0} \exp(t\xi)[x] \) the vector field induced by the \( G \) action.

Lemma 35. The function \( f \) is analytic and its gradient is given by \( \nabla f(x) = \tilde{\mathcal{F}}_p^\#(x) \).

Proof. We identify \( V \) with \( \mathbb{R}^n \). Assume that \( G = \text{SL}(n) \). Since

\[
\tilde{\mathcal{F}}_{\text{Sym}(n)}([x]) = \frac{x x^T}{\|x\|^2} - \frac{1}{2n} I_d,
\]

\( \tilde{\mathcal{F}}_{\text{Sym}(n)} \) is a polynomial and so it is analytic. If \( G \subset \text{SL}(V) \), then \( \tilde{\mathcal{F}}_p = \pi_p \circ \tilde{\mathcal{F}}_{\text{Sym}(n)} \), where \( \pi_p \) is the orthogonal projection, and so it analytic as well. The second part of the statement is easy to check. \( \square \)

The negative gradient flow of \( f \) throughout \( y \in \mathbb{P}(V) \) is then solution of the differential equation

(36) \[
\begin{cases}
\dot{x}(t) = -\tilde{\mathcal{F}}_p^\#(x(t)) \\
x(0) = y
\end{cases}
\]

Since \( \mathbb{P}(V) \) is compact, the solution is defined in all \( \mathbb{R} \).
Lemma 37. Let $g : \mathbb{R} \to G$ be the unique solution of the differential equation
\[
\begin{aligned}
g^{-1}(t) \dot{g}(t) &= \tilde{\delta}_p^\#(x(t)) \\
g(0) &= e
\end{aligned}
\]
Then $x(t) = g^{-1}(t)y$.

Proof. The solution $g$ is defined in all $\mathbb{R}$ (see [43] p. 69). Since $g^{-1} = -g^{-1}\dot{g} g^{-1}$, it follows
\[
\dot{x}(t) = -g^{-1} \dot{g} g^{-1}y = -\tilde{\delta}_p^\#(x(t))
\]
and so the result is proved. \hfill \Box

The following Theorem arises from Lojasiewicz gradient inequality, see [17, 48]. A proof is given in [25, Theorem 3.3 p.14].

Theorem 38. In the above assumption, the limit $\lim_{t \to +\infty} x(t) = x_\infty$ exists. Moreover, there exists positive constants $\alpha, c, \beta$, $\frac{1}{2} < \gamma < 1$ and $T > 0$ such that for any $t > T$ we have
\[
d(x(t), x_\infty) \leq \int_t^\infty \| \dot{x}(s) \| \, ds \\
\leq \frac{\alpha}{1-\gamma} (f(x(t)) - f(x_\infty))^{1-\gamma} \\
\leq \frac{c}{(t-T)} \beta.
\]

The following Theorem is a consequence of the Stratification Theorem in [33] and results proved in [28, 46].

Theorem 39. Let $y \in \mathbb{P}(V)$ and let $x : \mathbb{R} \to \mathbb{P}(V)$ be the solution of 36. Let $x_\infty = \lim_{t \to +\infty} x(t)$. Then
\[
\| \tilde{\delta}_p(x_\infty) \| = \inf_{z \in G \cdot y} \| \tilde{\delta}_p(z) \|.
\]
Moreover, if $z \in G \cdot y$ satisfies $\| \tilde{\delta}_p(z) \| = \inf_{z \in G \cdot y} \| \tilde{\delta}_p(z) \|$, then $z \in K \cdot x_\infty$.

Proof. Let $C_p$ be the set of critical points of $f$. Set $B_p = \tilde{\delta}_p(C_p)$. Since $f$ is $K$-invariant, the sets $C_p$ and $B_p$ are $K$-invariant. By the Stratification Theorem [33], there exists $\beta \in B_p$ such that $\tilde{\delta}_p(G \cdot y) \cap B_p = K \beta$ and $\inf_{z \in G \cdot y} \| \tilde{\delta}_p(z) \| = \| \beta \|$. Since $x_\infty$ is a critical point of $f$ it follows $\inf_{z \in G \cdot y} \| \tilde{\delta}_p(z) \| = \| \tilde{\delta}_p(x_\infty) \|$. By Theorem 5.1 in [28], $G \cdot y$ collapses to a single $K$-orbit under the negative gradient flow of $f$. Let $C$ be the connected component of the critical set of $f$ corresponding to $\beta$ and let $S_\beta$ the corresponding stratum. In [46], in the complex setting, the author proves that the map
\[
\varphi_\infty : S_\beta \to C \quad \varphi_\infty(x) = x_\infty,
\]
is a continuous retraction. This result follows by the Lojasiewicz gradient inequality (see Lemma 2.3, p.124). Therefore the same holds in our situation.

Let $g_n y \to r$ be such that
\[
\inf_{z \in G \cdot y} \| \tilde{\delta}_p(z) \| = \| \tilde{\delta}_p(r) \|.
\]
Then
\[ \lim_{n \to +\infty} \varphi_\infty(g_n y) = \varphi_\infty(r) = r; \]
and so \( r \) belongs to \( K \cdot x_\infty \). \( \Box \)

We define on \( G \) the left-Riemannian metric which agree with \( \langle \cdot, \cdot \rangle_g \) on the tangent space \( T_x G \). This metric is \( K \)-invariant with respect the right \( K \)-action. Let \( \Phi_x : G \to \mathbb{R} \), defined as
\[ \Phi_x(g) = \log \left( \frac{\| g^{-1} x \|}{\| x \|} \right). \]

**Lemma 40.** The differential of \( \Phi_x \) is given by \( (d\Phi_x)_g(v) = -\langle \tilde{\delta}_p(g^{-1} x), dL_{g^{-1}}(v) \rangle \). Therefore \( \nabla \Phi_x(g) = v_x(g) \) where \( v_x(g) = -dL_g(\tilde{\delta}_p(g^{-1} x)) \).

**Proof.** Let \( g \in G \) and let \( X \in \mathfrak{g} \). Then
\[ (d\Phi_x)_g(dL_g(X)) = \frac{-\langle X(g^{-1} x), g^{-1} x \rangle}{\| g^{-1} x \|^2}. \]
If \( X \in \mathfrak{p} \) then \( (d\Psi_x)_g(dL_g(X)) = -\langle \tilde{\delta}_p(g^{-1} x), X \rangle_\mathfrak{g} \). If \( X \in \mathfrak{k} \), then
\[ 0 = (d\Phi_x)_g(dL_g(X)) = \langle \tilde{\delta}_p(g^{-1} x), X \rangle_\mathfrak{g}. \]
This means
\[ (d\Phi_x)_g(dL_g(X)) = -\langle \tilde{\delta}_p(g^{-1} x), X \rangle_\mathfrak{g} = -\langle dL_g(\tilde{\delta}_p(g^{-1} x)), L_g(X) \rangle, \]
concluding the proof. \( \Box \)

Define \( \Theta_x : G \to G(x) \) as follows
\[ \Theta_x(g) = g^{-1} x. \]

**Lemma 41.** The map \( \Theta_x \) intertwines the gradient of \( \Phi_x \) and \( \nabla f \).

**Proof.** Let \( \xi \in \mathfrak{p} \). Since
\[ \Theta_x(g \exp(t\xi)) = \exp(-t\xi)g^{-1} x, \]
we get
\[ (d\Theta_x)_g(dL_g(\xi)) = -\xi^\#(g^{-1} x), \]
and so the result is proved. \( \Box \)

Since \( \Phi_x \) is \( K \)-invariant, it descends to a smooth map \( \Phi_x : G/K \to \mathbb{R} \), that we also denote by \( \Phi_x \). A proof of the next Lemma is given in \[25\] (Lemma A.3 p.140).

**Lemma 42.** Let \( F : G/K \to \mathbb{R} \) be a smooth function that is convex along geodesics. Let \( c_0, c_1 : \mathbb{R} \to M \) be the negative gradient flow of \( F \) and let \( \rho(t) = d_M(c_0(t), c_1(t)) \). Then \( \rho(t) \) is nonincreasing function.

**Theorem 43.** Let \( \Phi_x : G/K \to \mathbb{R} \). Then
a) \( \Phi_x \) is a Morse-Bott function and it is convex along geodesics;
b) The critical set, possibly empty, is the submanifold
\[ \{gK \in G/K : \tilde{\Phi}_p(g^{-1}x) = 0 \} . \]

c) If \( c : I \to M \) is a negative gradient flow of \( \Phi_x \), then
\[ \lim_{t \to +\infty} \Phi_x(c(t)) = \inf_{x \in G/K} \Phi_x \]

Proof. By Lemma 6 in [14], see also Lemma 1.19 p. 1115, \( \Phi_x \) is convex along geodesics. By Proposition 9 p.2191 it follows that a critical points of \( \Phi_x \) are the elements \( \pi(g) \in G/K \) such that \( \tilde{\Phi}_p(g^{-1}x) = 0 \). If \( gK = \pi(g) \) is a critical point, then
\[
\text{Hess}(\Phi_x)(dL_g(\xi), dL_g(\xi)) = \left. \frac{d^2}{dt^2} \right|_{t=0} \Phi_x(g \exp(tv)) = \left. \frac{d^2}{dt^2} \right|_{t=0} \log \left( \| \exp(-t\xi)g^{-1}x \| \right) \geq 0,
\]
and it is 0 if and only if \( \exp(t\xi) \subset G_{g^{-1}x} \). Let \( g \in \text{Crit} \Phi_x \). By Hadamard-Cartan Theorem, the map
\[ p \to G/K, \quad \xi \mapsto g \exp(\xi), \]
is a diffeomorphism. Therefore \( g \exp(\xi) \in \text{Crit} \Phi_x \) if and only if \( \tilde{\Phi}_p(\exp(-\xi)g^{-1}x) = 0 \). Since \( \tilde{\Phi}_p(g^{-1}x) = 0 \), by Proposition [13] it follows that \( \exp(t\xi) \subset G_{g^{-1}x} \). This implies
\[ \text{Crit} \Phi_x = \{ \pi(g \exp(t\xi)) : \exp(t\xi) \in G_{g^{-1}x} \} \]
and so it is submanifold and the Kernel of the Hessian. Therefore \( \Phi_x \) is a Morse-Bott function.

The gradient of \( \Psi_x : G \to \mathbb{R} \) is given by \( \nabla \Psi_x(g) = d\pi_g(v_g(g)) \). Hence the negative gradient flow of \( \Psi_x : G/K \to \mathbb{R} \) satisfies the differential equation \( \frac{dv}{dt} = -\nabla \Psi_x(v) \). By Lemma 40 the map \( \Theta_x \) intertwines the gradient of \( \Psi_x \) with \( \nabla f \). Therefore the negative gradient flow of \( \Psi_x : G \to \mathbb{R} \) satisfies the differential equation \( \frac{dv}{dt} = -\nabla \Psi_x(v) \). Vice-versa if \( g : \mathbb{R} \to G \) satisfies the equation 36 then one may prove that \( \pi \circ g \) is the negative gradient flow of \( \Psi_x : G/K \to \mathbb{R} \).

Let \( c_1, c_2 : \mathbb{R} \to M \) be negative gradient flow of \( \Phi_x \). Then there exist \( g_0, g_1 : \mathbb{R} \to G \) solution of 36 such that \( c_0 = \pi \circ g_0 \) and \( c_1 = \pi \circ g_1 \). Since \( G = K \exp(p) \), there exist \( \xi : \mathbb{R} \to \mathbb{R} \) and \( k : \mathbb{R} \to K \) such that \( g_1(t) = g_0(t) \exp(\xi(t))k(t) \). Write
\[ H : \mathbb{R} \times \mathbb{R} \to G/K, \quad H(t, s) = \pi(g_0(t) \exp(s\xi(t))). \]
In the sequel we denote by \( H_t(s) = H(t, s) \). The curve \( s \mapsto H_t(s) \) is the unique geodetic joining \( c_0(t) \) and \( c_1(t) \). By Lemma 42 the function
\[ \rho(t) = d_{G/K}(c_0(t), c_1(t)) = \| \xi(t) \|, \]
is nonincreasing. Assume \( \text{Crit} \Phi_x \) is not empty. Hence we may assume \( g_0(0) \in \text{Crit} \Phi_x \) and so the curve \( c_0 \) is constant. Since \( \rho \) is nonincreasing, the image \( c_1 \) is contained in a compact subset. This implies, keeping in mind that \( \Phi_x \) is Morse-Bott, the limit \( \lim_{t \to +\infty} c_1(t) \in \text{Crit} \Phi_x \) and so item (c) holds. In particular, every negative gradient flow converges to a critical point and so \( \Phi_x \) has a global minimum. Now, assume that \( \Phi_x \) does not have any critical point. Assume by contradiction
\[ \lim_{t \to +\infty} \Phi(c_0(t)) = a > \inf_{G/K} \Phi_x. \]
Hence $\Phi(c_0(t))$ is bounded from below. We may choose $c_0$ such way $\Phi_x(c_1(0)) < a$. By Lemma 42, $\rho$ is nonincreasing and so there exists $C > 0$ such that $\rho(t) = \| \xi(t) \| \leq C$. Hence

$$\frac{d}{ds} \Phi_x(H_t(s)) = (d\Phi_x)_{c_0(t)}(H_t'(0))$$

$$= -\langle \tilde{\xi}_p(g_0(t)^{-1}x), \xi(t) \rangle$$

$$\geq ||\tilde{\xi}_p(g_0(t)^{-1}x)|| \cdot ||\xi(t)||$$

$$\geq C \cdot ||\tilde{\xi}_p(g_0(t)^{-1}x)||.$$  

Since for $t$ fixed, $H_t(s)$ is a geodesic, it follows that the function $s \mapsto \Phi_x(H_t(s))$ is convex and so its derivative $\frac{d}{ds} \Phi_x(H_t(s))$ increases. Hence

$$\Phi_x(c_1(t)) = \Phi_x(H_t(1))$$

$$= \Phi_x(H_t(0)) + \int_0^1 \frac{d}{ds} \Phi_x(H_t(s))ds$$

$$\geq \Phi_x(c_0(t)) - C \cdot ||\tilde{\xi}_p(g_0(t)^{-1}x)||.$$  

Now, the function $\Phi \circ c_1$ is bounded and $\frac{d}{dt} \Phi_x(c_0(t)) = - ||\tilde{\xi}_p(g_0(t)^{-1}x)||$. Therefore, there exists a sequence $t_i \to +\infty$ such that $\frac{d}{dt} \Phi_x(c_0(t)) = - ||\tilde{\xi}_p(g_0(t)^{-1}x)||$ goes to 0. This implies

$$\lim_{t_i \to +\infty} \Phi_x(c_1(t_i)) \geq \Phi_x(c_0(t_i)) \geq a,$$

a contradiction since $(\Phi_x(c_0(t))) < a$ and so $\lim_{t_i \to +\infty} \Phi_x(c_1(t_i)) < a.$  

Now, we are able to prove the Hilbert-Mumford criterion for real reductive groups. We start recalling a well-known numerical criterion, a proof is given in [14], and some technical lemmata.

**Theorem 44.** Let $x \in \mathbb{P}(V)$. Then $x$ is semistable if and only if for any $\xi \in \mathfrak{p}$, $\lambda(x, \xi) \geq 0$;

**Lemma 45.** Let $x \in \mathbb{P}(V)$ and let $\tilde{x} \in V$ such that $\pi(\tilde{x}) = x$. Let $\alpha_1 > \cdots > \alpha_k$ be the eigenvalues of $\xi$. We denote by $V_i$ the corresponding eigenspaces. Write $\tilde{x} = v_1 + \cdots + v_k$. Then $\lambda(x, \xi) = \alpha_j$, where $j = \min\{1, \ldots, k\}$ such that $v_j \neq 0$. Moreover, $\lim_{t \to +\infty} \exp(t\xi)\tilde{x} = 0$ if and only if $\alpha_j < 0$.

**Proof.**

$$\lambda(x, \xi) = \lim_{t \to +\infty} \frac{d}{dt} \tilde{\Psi}(x, \exp(t\xi))$$

$$= \lim_{t \to +\infty} \frac{\alpha_j e^{2\alpha_j t} \cdot ||v_j||^2 + \cdots + \alpha_k e^{2\alpha_k t} \cdot ||v_k||^2}{e^{2\alpha_1 t} ||v_j||^2 + \cdots + e^{2\alpha_k t} ||v_k||^2}$$

$$= \lim_{t \to +\infty} \frac{\alpha_j \cdot ||v_j||^2 + \alpha_{j+1} e^{2(\alpha_{j+1} - \alpha_j)t} \cdot ||v_{j+1}||^2 + \cdots + \alpha_k e^{2(\alpha_k - \alpha_j)t} \cdot ||v_k||^2}{||v_j||^2 + e^{2(\alpha_{j+1} - \alpha_j)t} ||v_{j+1}||^2 + \cdots + e^{2(\alpha_k - \alpha_j)t} ||v_k||^2}$$

$$= \alpha_j.$$
Since
\[ \| \exp(t\xi) \tilde{x} \|^2 = \sum_{i=1}^{k} e^{2\alpha_i} \| v_i \|^2, \]
it follows that \( \exp(t\xi) \tilde{x} \to 0 \) if and only if \( \alpha_j < 0 \). Otherwise the limit does not exist or \( \exp(t\xi) \in G \cdot \tilde{x} \).

**Lemma 46.** Let \( x \in \mathbb{P}(V) \) be a semistable point. Let \( \tilde{x} \in V \) be such that \( \pi(\tilde{x}) = x \). Then \( 0 \notin G \cdot \tilde{x} \).

**Proof.** Let \( s : \mathbb{R} \to \mathbb{P}(V) \) and let \( g : \mathbb{R} \to G \) so that \( s(t) = g(t)^{-1}x \) is the solution of (36). By Theorem 33 the limit \( \lim_{t \to +\infty} s(t) = x_\infty \) exists and by Theorem 39 it satisfies
\[ \| \tilde{\Phi}_p(x_\infty) \| = \inf_{y \in \overline{G \cdot x}} \| \tilde{\Phi}_p(y) \|. \]
Since \( x \) is semistable it follows that \( \| \tilde{\Phi}_p(x_\infty) \| = 0 \). By the Lojasiewicz gradient inequality for \( f \), there exist positive constant \( \alpha, \beta \) and \( \frac{1}{2} < \gamma < 1 \) such that
\[ \| \tilde{\Phi}_p(x) \| \leq 2f(x) \leq 2f(x)^\gamma \leq C \| \text{grad} f \| = C \| \tilde{\Phi}_p^2(x) \|. \]
Therefore
\[ \| \tilde{\Phi}_p(s(t)) \| = \| -\frac{d}{dt} \Phi_{[x]} \circ c(t) \| \leq C \| \dot{s}(t) \|, \]
where \( c(t) = \pi \circ g(t) \) is the negative gradient flow of \( \Phi_x : G/K \to \mathbb{R} \). By Theorem 33 the function \( \| \dot{s}(t) \| \) is integrable over \( \mathbb{R}^+ \), and so the limit
\[ \lim_{t \to +\infty} \Phi_{[x]} \circ c(t) = a \in \mathbb{R}. \]
By Theorem 43 the function \( \Phi_x \) is bounded below. Since \( \Phi_{[x]}(g) = \left( \| g^{-1} \tilde{x} \| \right) \), we get \( 0 \notin G \cdot \tilde{x} \) concluding the proof.

**Theorem 47** (Hilbert-Mumford criterion for reductive groups). Let \( y \in V \) and assume that \( 0 \in \overline{G \cdot y} \). Then there exists \( \xi \in p \) such that \( \lim_{t \to +\infty} \exp(t\xi)y = 0 \).

**Proof.** Let \( y \in V \) such that \( 0 \in \overline{G \cdot y} \). By Lemma 46 the point \( \pi(y) \) is not semistable in \( \mathbb{P}(V) \). By Theorem 44 there exists \( \xi \in p \) such that \( \lambda_x(\pi(y), \xi) < 0 \). By Lemma 45 we get \( \lim_{t \to +\infty} \exp(t\xi)y = 0 \) concluding the proof.

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