Characteristics of Algebraic Thinking of Junior High School Students

D I Rahmawati
Graduate School of Indonesia University of Eduation, Bandung, Indonesia
dwiinayah28@student.upi.edu

Abstract. Algebraic thinking is a process of thinking that involves not only changing an object that is uncertain or changing into a mathematical symbol but it involves some activity such as representing or language mathematics, problem-solving, reasoning, arithmetic generalization, modelling and function. The aim of this study was to describe the characteristics of algebraic thinking of junior high school students who have high, medium, and low mathematics skills. This study used descriptive qualitative method. The subject of this study was junior high school students who have high, medium, and low mathematics skills based on the final exam results of the previous semester. Data collection methods used in this study was essay test and interview. The result show that characteristics of students who have high skill has achieved algebraic thinking characteristics as arithmetic generalizations, problem-solving, and quantitative reasoning, while the characteristics of students who have medium skill has achieved algebraic thinking characteristics as arithmetics generalizations, modelling, representation, and the characteristics of students who have low skill has achieved algebraic thinking characteristics as modelling, language of mathematics, and arithmetic generalization.

1. Introduction
Mathematics is one of the most important science needed in solving the problems of daily life. In addition, mathematics is also closely related to other fields of science, such as physics, biology, economics, and so on. For that reason, mathematics is a science that must be taught and applied in learning of schools; so that students are trained and able understand the concepts of mathematics. Algebra is very important to be mastered by students, because either implicitly or explicitly algebra is used in the activities of daily life [1]. Therefore, the students students should be capable of engaging in algebraic thinking.

Algebraic thinking itself is one of the most important processes of mathematical thinking. Algebraic thinking is using mathematical symbols and tools to analyze different situations by: (1) extracting information from situations, (2) representing information mathematically in words, diagrams, tables, graphs, and equations; and (3) interpreting and applying mathematical findings, such as solving unknowns, testing allegations, and identifying functional relationships in similar situations and related new situations [2]. If the students are capable of engaging in algebraic thinking well, it will help and support students in their cognitive process to learn algebra [3]. Moreover, algebraic thinking has become a catch-all phrase for the mathematics teaching and learning that will be accomplished in algebra [4]. Furthermore, Kriegler describes the components of algebraic thinking, they are 1) the development of mathematical thinking tools, and 2) the fundamental idea of algebra. The mathematical thinking tool consists of three characteristics: a) problem solving skills consists of using
problem solving strategies and exploring multiple approaches/multiple solutions, b) representation skills consists of displaying relationships visually, symbolically, numerically, verbally; translating among different representations; interpreting information within representations; and c) quantitative reasoning skills consists of analyzing problems to extract and quantify essential features and inductive and deductive reasoning [4]. The fundamental algebraic ideas is algebra as: a) generalized arithmetics consists of conceptually based computational strategies; ratio and proportion; estimation, b) algebra as the language of mathematics consists of meaning of variables and variable expressions; meaning of solutions; understanding and using properties of the number system; reading, writing, manipulating numbers and symbols using algebraic conventions; using equivalent symbolic representations to manipulate formulas, expressions, equations, inequalities; and c) algebra as a tool for functions and mathematical modeling consists of seeking, expressing, generalizing patterns and rules in real-world contexts; representing mathematical ideas using equations, tables, graphs, or words; tasking with input/output patterns; developing coordinate graphing skills [4].

Meanwhile, Lins mentions there are three characteristics of algebraic thinking, which are as follows. To think algebraically is:

1. To think arithmetically, which means modelling in numbers;
2. To think internally, which means reference only to the operations and equality relation, in other words solutions in the boundaries of the semantic field of numbers and arithmetical operations;
3. To think analytically, which means what is unknown has to be treated as known. [5]

However, when looking at the current facts, many students have difficulty in learning algebra, even becoming one of the feared mathematics topics by students at school. One of these difficulties is students are less able to interpret and represent a problem situation into mathematical model such as mathematical equations (algebra) or into other forms of representation. The most frequently observed difficulty is the skill to translate between problem situations and mathematical situations [6]. Besides, this skill is very important in learning algebra. This difficulty can be caused by the process of students’ thinking transition from arithmetical thinking to algebraic thinking. The transition process of arithmetic thinking to algebraic thinking is the most difficult steps in the life of mathematics students [7]. Students who are familiar with numerical answer calculations, they suddenly have to switch to the abstract symbolic language. Therefore, it is needed more attention to know the thinking process of students in depth, especially the process of algebraic thinking for the thinking transition process goes well, so that researcher wanted to know the process or the characteristic of algebraic thinking of Junior High School student in solving mathematics problems. Also, in this study, researcher describe the errors made by students in solving mathematics problems.

2. Research Methods
This study used descriptive qualitative method. The subjects of this study are eighth grade students in one of public schools Purbalingga city. They are students who have studied algebra-related materials. They consist of the 32 students, selected 6 students to be interviewed with two students of high mathematical skill, two students of medium mathematical skill, and two students of low mathematical skill based on the final exam results of the previous semester.

Researchers are the key instruments that collect the data by themselves through documentation, behavioral observation, or interviews with research subjects [8]. Therefore, researchers have a very important role in this research. The supporting instruments used in this study are essay and interview guides. The researcher initially gave essay in the form of five items essay to the students. Then, students’ task characterization was conducted. Based on this characterization, the researcher conducted interview to the students that representing their group to reveal the necessary things related to the algebraic thinking process that occurred in the students. The following is the essay used by researchers.
1. Mr. Dhani has a rectangular shaped garden behind his house. The circumference of the garden is 42 m. The difference between the length and the width of the garden is 10 m. Determine the length and the width!

2. Mr. Budi wants to make a cube-shaped cage with a frame made of iron. The length of the planned cage side is 40 cm. Mr. Budi has a metal material along 30 m. Then Mr. Budi wants to add a roof for the chicken coop pyramid-shaped.
   a. Illustrate the shape of the chicken coop above so it is easy to understand
   b. Write a mathematical model to calculate the number of cages that can be made by Mr. Budi and solve it by giving the reasons.

3. Dad has a strand with a length of 40 cm to make a frame toy beam-shaped.
   a. Describe several ways to determine the volume of the toy.
   b. Then solve it include what concept is you used?
   c. Choose the best answer from assignment a by giving the reasons

4. Note the following picture. Laila and Dahlia play the metal beams and scales such as the picture below.

   a. Laila argues that the weight of metal beam is more than 6 kg. Do you think Laila's opinion is correct? Give your reasons.
   b. When Dahlia came, she said that the weight of 3 pieces of metal beam is equal to 12 kg. Do you think Dahlia's opinion is correct? Give your reasons.
   c. How do you determine the weight of single metal beam?
   d. If the weight of five metal beams is less than x and weighs of seven metal blocks more than y, then write down the set of all possible weights of metal beam?

5. Given a painting hung in a row. The hanging paintings next to each other divide the two nails as shown below.

   a. How many nails are needed if there are 4 paintings hung in a row?
   b. How many nails are needed if there are 10 and 20 paintings that are hung in a row?
   c. Suppose the number of paintings is y, then how many nails are needed?
   d. Write a linear equation to determine the number of nails by specifying p is the number of nails and x is the number of paintings. From the equations you have made, apply to complete how many paintings to hang if the number of nails is 100.

Problem adapted from Margayanti (2015) [9]
e. Can you create a new linear equation to represent the number of spikes \((t)\) for any number of paintings \((p)\).

*Problem adapted from Lian & Idris (2006)* [10]

Furthermore, data analysis was done, using several techniques, they are: data reduction, data presentation, and the conclusion [11]. The obtained data are described based on the actual circumstances to obtain a natural description of the students’ algebraic thinking characteristics.

3. Result and Discussion

This study describes algebraic thinking characteristics of junior high school students. To select the subject of research, the researcher gave an essay to all students of a class in the participated school consisting of 32 students. The result of the students' task on the essay was reviewed by the researcher to determine the algebraic thinking characteristics of each student. The selected 6 subjects were 2 students with high mathematics skills (S1 and S2), 2 students with medium mathematics skills (S3 and S4), and 2 students with low mathematics skills (S5 and S6). The selection of this subject is based on the final exam result in the previous semester. From each item in problem above, identified student’s algebraic thinking characteristics based on algebraic thinking characteristics according to Kriegler.

The following table of algebraic thinking characteristics experienced by students in solving mathematics problems.

| Algebraic thinking characteristics | Item | Total | Percentage characteristics |
|-----------------------------------|------|-------|---------------------------|
| Modelling                         | 15   | 19    | 10 10 17 71              | 44,38%                    |
| Mathematics language              | 15   | 15    | 10 10 14 64              | 40%                      |
| Representation                    | 13   | 14    | 15 5 13 60              | 37,5%                    |
| Arithmetics generalization        | 25   | 24    | 25 10 18 102            | 63,75%                   |
| Problem solving                   | 13   | 10    | 10 3 3 39              | 24,38%                   |
| Quantitative reasoning            | 13   | 10    | 5 3 3 34              | 21,25%                   |

From the results above, it shows that among the six algebraic thinking characteristics, modeling and arithmetic generalizations is most widely used by students in algebraic thinking to solve mathematics problems above. This can be seen from the percentage of algebraic thinking characteristics as modeling and as arithmetic generalizations, i.e. 44.3% and 63.75% respectively. Then, the analysis result of students’ algebraic thinking characteristics for each problem is described as follows.

3.1. Description of S1’s of Algebraic Thinking Characteristics

The result of the S1’s task on the provided essay indicates that S1 can answer correctly towards the number 1 to number 4. In questions of number 5, S1 made a mistake in solving the problem, especially on the point (e); he was asked to write a new linear equation or new formula based on the situation in question number 5. Overall, in solving the fifth problem, it appears that S1 engages the algebraic thinking as problem-solving skills because he can answer correctly the five questions, as representation skill because S1 can recognize the relationship visually (images), symbols, numerically and verbally. Then, S1 also engages algebraic thinking as arithmetic generalization, it was seen when he can write and can manipulate numbers and symbols using algebraic rules, using symbolic representations for manipulation of formulas, expressions, equations, and inequalities. S1 also engages algebraic thinking as the tool for mathematical functions and modeling, it was seen when he can...
imagine a problem situation into a mathematical model in number 2, as well as S1 engaging algebraic thinking as mathematical language because he can explain the used variable means in each question. Lastly, S1 also engages algebraic thinking as quantitative reasoning well because he used the reasoning and revealed the reason well in solving the problem, especially on number 4. It appears in interview. When the researcher confirmed all the S1’s task and looked for the S1’s algebraic thinking characteristic, and the results give the appropriate results.

3.2. Description of S2’s of Algebraic Thinking Characteristics
The results of the S2’s task on the provided essay indicates that S2 can answer correctly towards the numbers 1, 2, 3. In item 4, S2 could not finish well. Similarly in question 5, S2 made a mistake in solving the problem, especially in points (c), (d), and (e); he was asked to write a linear equation of the formed pattern from the question at the previous point and making a new linear equation or new formula based on the situation in questions of number 5. Overall, in completing these five questions, it appears that S2 engages algebraic thinking as problem-solving, as representation, as arithmetic generalization, as a tool for function and mathematical modeling. However, S2 is less able to use algebraic thinking as quantitative reasoning well. It was seen in question 4 that he can not solve the problem properly because S2 was unable to give a clear reason in answering the question. In conducted interviews with S2 also shows that that he has a low skill in quantitative reasoning. Here is a few transcript of interview with the S2 about quantitative reasoning.

R : Why did not you give a reason for question 4a?
S2 : I feel confused to write my opinion in a word.
R : Oh I see. Then, how can you get the answer for question 4a?
S2 : So like this ma’am, I see that the first scales of the weight of one beam is less than 5 kg. Then the second scales, the weight of his two beams more than 7 kg. Means the weight of one beam is more than 3.5 kg. Then on the first scale weighs one beam is less than 5 kg, means the weight of one beam between 3.5 kg and 5 kg. So the answer is wrong if the weight of one beam is more than 6 kg.
R : Alright, you are correct. Next time, you should write down your reason as you said before on the answer sheet so it is not only answering correct or incorrect.
S2 : Okay, Ma’am.

3.3. Description of S3’s of Algebraic Thinking Characteristics
The result of the S3’s task on the provided essay indicates that S3 can solve the problem of numbers 1, 2, 3 and 5, while question number 4 did not answered at all. Overall, it appears that S3 engages algebraic thinking as arithmetic generalization, as a tool for mathematical functions and modeling, as representation and mathematics languages well. However, the S3 can not use the problem-solving and algebraic thinking as quantitative reasoning well. It can be seen from the task of number 4 that he could not answer the question; which shows that he has a low skill in quantitative reasoning. Also, the found error in numbers 2 and 3 indicates that he has a low skill in problem-solving. In conducted interviews with S3 also shows that he does not have the algebraic thinking characteristics in problem-solving and quantitative reasoning. The following is a few transcript of the interview with S3.

R : Please tell me what the method you use to finish the question 2b?
S3 : The length of the iron divides the length of the cage frame
R : Is it only the way you write?
S3 : I thinks yes, Ma’am
R : Then, why did not you answer the questions 4?
S3 : I feel confused, Ma’am... I can not understand the question
3.4. Description of S4’s of Algebraic Thinking Characteristics
The result of the S4’s task on the provided essay indicates that S4 can solve problems 1, 2, 3, 4, and 5. However, in the process it is not complete as in the number 4 where S4 only does the 4a question. Similarly with S3, overall, it appears that S4 engage algebraic thinking as: arithmetic generalization, a tool for mathematical functions and modeling, representation and mathematics language well. However, the S4 can not use the problem-solving skills and algebraic thinking as quantitative reasoning well. It can be seen from the task of number 4 that he could not answer the question; which shows that he has a low skill in quantitative reasoning. Additionally, the found error in numbers 2 and 5 shows that he has a low skill in problem-solving. In conducted interviews with S4 also shows that he does not have the algebraic thinking characteristics in problem-solving and quantitative reasoning.

3.5. Description of S5’s of Algebraic Thinking Characteristics
The result of the S5’s task on the provided essay indicates that S5 only able to solve the problems of number 1 and 2 correctly. For numbers 3, 4, and 5, he did not finish the questions or did not give a solution, he only wrote down what the known and what the asked in questions. This certainly shows that S5 engages algebraic thinking as mathematical modeling and mathematical language because S5 can represent the problem situation of number 1 and 2 into the mathematical model, that is equations/expressions mathematics. Also, the given solution in question of number 1 and 2 were correct, it shows that S5 is capable of engaging in algebraic thinking as arithmetic generalization because he can use the algebraic rules and problem-solving skills well so that numbers 1 and 2 can be answered correctly. Meanwhile, the problems of numbers 3, 4, and 5 were unsolved at the time indicates that he has a low skills in problem-solving and arithmetic generalization. Likewise, his quantitative reasoning is still low because it appeared that S5 can not engage algebraic thinking as a reasoning; as seen on the unanswered questions of number 3 and 4. Here is a few transcript of the interview with S5.
R : Why did not you answer the questions of number 3, 4, and 5?
S5 : I feel difficult to understand those questions, especially number 5.
R : But you wrote down the known questions and the asked questions on number 3, 4, and 5. It means you can understand the questions.
S5 : But I feel confused to solve it.

3.6. Description of S6’s of Algebraic Thinking Characteristics
The result of the S6’s task on the provided written test indicates that S6 can answer correctly towards the numbers 1 and 2. The questions of number 3 was done by S6 but it is not complete, the number 4 was not answered at all, and the number 5 was only tasked on the point (a). It shows the same thing as the S5 did that S6 can only use algebraic thinking skills as modeling and mathematical language. Other algebraic thinking characteristics such as problem-solving, arithmetic generalizations, and quantitative reasoning are still quite low with many questions relating to these characteristics and not being tasked out.

3.7. Errors Made by Students in Solving The Problem
By knowing this student's thinking process, the teacher can (1) track the location and type of mistakes made by the students, and (2) know the errors of thinking that occur and fix the students’ knowledge netof students [12]. Overall, all students are actually can understand, write, manipulate numbers and use algebraic rules well. However, there are still some errors made by students. According to Subanji and Mulyoto, there are several types of errors that students make in doing mathematical problems include: 1) Errors of language interpretation. In this type of error students often make the mistake of declaring daily language in the language of mathematics. Because of so many symbols, graphics and tables that make students make mistakes in interpreting symbols, graphs and tables into the language of mathematics, 2) Errors of technical. In this type of error students often make calculations or computing errors in working on problems; 3) Errors of concept. In this type of error, students often make mistakes in determining or applying formulas to answer a problem. Student misuses the use of a
theorem or formula that is inconsistent with the condition of the applicable formula or does not write the theorem, 4) Error in using the data, i.e. the student does not use data that should be used in answering the questions. Students also make mistakes in entering data into variables and adding unnecessary data in answering a problem, and 5) Error in drawing conclusions, i.e. errors in making inferences without the correct supporters are also often done by students. The student's errors in concluding statements that are inconsistent with logical reasoning [13].

In this study, some of the errors found on students' essay result, they are 1) errors of concept, 2) errors of language interpretation, 3) errors of technical, 4) error in using the data, and 5) error in drawing conclusion. The following data relates to errors made by students presented in the table below.

| Types of Error                        | Item 1 | Item 2 | Item 3 | Item 4 | Item 5 | Percentage Total |
|---------------------------------------|--------|--------|--------|--------|--------|-----------------|
| Errors of concept                     | 14     | 15     | 16     | 23     | 27     | 59.58%          |
| Errors of language interpretation     | 14     | 14     | 10     | 11     | 15     | 40%             |
| Errors of technical                   | 13     | 10     | 24     | 16     | 20     | 51.88%          |
| Error in using the data               | 19     | 17     | 27     | 10     | 25     | 61.25%          |
| Error in drawing conclusion           | 3      | 5      | 9      | 10     | 6      | 20.63%          |
| Total                                 | 63     | 61     | 86     | 70     | 93     |                 |

Based on the table above, indicates that errors in using the data is the most error done by students with percentage 61, 25%. In addition, the question of number 5 is a matter of where the student is doing a lot of errors. It is perhaps because students have not understood well about number 5, especially on questions of 5d and 5e. Students have difficulty in solving it.

Errors of concept is carried out by students is using the concept or formula in solving the problem. It was seen on the S3's student answers, when he was incorrect in using the formula to answer the question of number 2. S3 used the formula to find the answer by dividing the length of the iron and the length of the cage frame, whereas the truth is dividing the length of the iron by the length of the whole strand, which is then sought in advance the overall length of the strand. Another example of misconceptions done by students as below. The student is incorrect in using the concept or formula to determine the length of a whole strand on question of number 3. The length of the whole strand should be 4 (length + width + height).

![Figure 1. One of the errors of concept of the student's answer.](image)

Then, the error of language interpretation is the error in declaring the daily life language into mathematical symbols or into mathematical language. Here is one of the following student responses.
From the answers above, it shows that the students misinterpreted towards the meaning of the problem and got incorrect in stating the problem situation into the mathematical model. Furthermore, they had the technical errors in calculating and errors in manipulating algebraic forms. It was seen when the student was calculating to solve the problem. Here is an example of the technical errors exercised by one of the students below. The student is wrong in doing the calculation, that is $4 \times 40 = 80$, it should be $160$.

**Figure 3.** One of the technical error of the student’s answer.

4. Conclusion

Based on the results of discussion above, it can be concluded that students either with high, medium, or low mathematics skills, all of them have algebraic thinking characteristics as modeling, mathematical language, arithmetic generalization. However, not all students can have algebraic thinking characteristics as problem solving and quantitative reasoning, such as students with low and medium mathematics skills. Students with medium and lower mathematical skills more accentuate the characteristics of algebraic thinking as modeling and arithmetic generalization in solving problem, but have not quite achieved algebraic thinking as problem solving and quantitative reasoning. Students only who have high mathematics skill were capable of algebraic thinking as problem solving and quantitative reasoning. In addition, there are errors made by students in solving the problem, including errors of concept, errors of technical, and errors of language interpretation, error in using the data, and error in drawing conclusion.

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