Quadratic cosine-Gauss beams – the new family of localized solutions of the paraxial wave equation

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Abstract. We propose a new class of localized solutions of the paraxial wave equation. They have a form of a product of a Gaussian term and an amplitude which contains only elementary coordinate functions. Solutions are obtained by summing of the quadratic Bessel–Gauss beams with odd indices. Due to the configuration of the obtained solutions, we named them quadratic cosine-Gauss beams.

1. Introduction

The purpose of this paper is to introduce a new family of localized solutions of the paraxial wave equation [1], otherwise referred to as the parabolic equation [2,3,4]. This equation has the form

\[ U_{xx} + U_{yy} + 2ikU_z = 0, \]  

(1)

where \( k \) is a real parameter called longitudinal wave number, \( z \) is longitudinal coordinate, and \( x \) and \( y \) are transverse ones. The solutions of equation (1) are interesting not only from the point of view of mathematics, but also have important physical applications. In particular, their solutions (more precisely, products of the form \( U = \exp[i(kz - \omega t)] \), where \( \omega \) is the circular frequency) describe, in the paraxial approximation, the propagation of wave beams localized along the \( z \) axis. Such narrow beams arise, for example, in lasers and optical fibers. In addition, equation (1) up to the notation coincides with the two-dimensional Schrödinger equation for a free particle [5].

Most of the localized analytical solutions of equation (1) known to the authors are represented as the product of two factors:

\[ U = GA. \]  

(2)

The first of them, called a fundamental mode, is itself a solution of equation (1) and has the form of a Gaussian exponential function of the transverse coordinates, the parameters of which depend on the longitudinal coordinate \( z \). In the simplest axisymmetric case, the fundamental mode has the form

\[ G(\rho, z) = Cq(z)\exp\left(\frac{ik\rho^2}{2q(z)}\right), \]  

(3)

where \( \rho^2 = x^2 + y^2 \), \( C \) is a constant, \( q(z) = z - iz_R \), the parameter \( z_R > 0 \) is called the Rayleigh length.
The second function $A(x,y,z)$, following [6], will be called the amplitude. In particular, for the Bessel–Gauss beams constructed in [7], the amplitude factor has the form

$$A_m(\rho, z, \varphi) = J_m \left( \frac{K \rho}{q(z)} \right) \exp \left( \frac{i k^2 x^2}{2 k q(z)} \right) \exp(i m \varphi).$$  \hspace{1cm} (4)

Here $\varphi$ is the polar angle in the $x, y$ plane, $J_m$ is a Bessel $m$-order function of the first kind [8] and $K$ is an arbitrary complex constant. Later, numerous generalizations of Bessel–Gauss beams were found, such as asymmetric [9], non-coaxial [10] and shifted [11, 12] Bessel–Gauss beams in which the amplitude is expressed in terms of Bessel functions. All these families belong to a class of Helmholtz–Gaussian beams [11, 13, 14], containing arbitrary solution of the Helmholtz equation in the plane. Note that the approach used in [9], where asymmetric Bessel–Gauss beams are constructed using the summation formula of a series containing Bessel functions, is the basis of this work.

2. Quadratic Bessel–Gauss and Helmholtz–Gauss beams

Now we will consider another class of localized solutions of equation (1) called Bessel–modulated Gaussian beams with quadratic radial dependence, or, briefly, quadratic Bessel–Gauss beams. These solutions were found in [15] and have the form

$$U_m(\rho, z, \varphi) = C \frac{w_0}{W(z)} \exp(\xi) J_{|m|/2}(\zeta) \exp(i m \varphi),$$  \hspace{1cm} (5)

where

$$\xi = -\frac{4(\mu^2+1)\rho^2}{\pi R} \frac{z_R}{W^2(z)}, \quad \zeta = \frac{\mu \rho^2}{W^2(z)}.$$  \hspace{1cm} (6)

the function $W(z)$ is determined by the equality

$$W(z) = w_0 \sqrt{1 - (\mu^2 + 1) \left( \frac{z}{z_R} \right)^2 + 2i \frac{z}{z_R}}.$$  \hspace{1cm} (7)

The parameters $w_0$ (the half-width of the beam in the waist) and $z_R$ (the Rayleigh length) are related to the longitudinal wave number $k$ by the equality $z_R = k w_0^2/2$, $C$ is the amplitude constant, and $\mu$ is an additional complex parameter.

Functions (5) with obvious external similarities with ordinary Bessel–Gauss beams have an important specificity. First of all, they do not belong to the class (2), since the Gaussian factor

$$\hat{G}(\rho, z) = C \frac{w_0}{W(z)} \exp(\xi)$$  \hspace{1cm} (8)

is not a fundamental mode and does not satisfy equation (1). The second factor

$$\hat{A}_m(\rho, z, \varphi) = J_{|m|/2}(\zeta) \exp(i m \varphi),$$  \hspace{1cm} (9)

which we will continue to call the amplitude, also differs from (4) due to the absence of an additional exponential factor, the index under the Bessel function, and the fact that the dependence of the argument of this function on $\rho$ is not linear but quadratic.

In [16], a class of quadratic Helmholtz–Gauss beams, that are localized solutions of equation (1), was constructed. The solutions have the form of the product of the function (8) and an amplitude, expressed through an arbitrary solution of the Helmholtz equation on an auxiliary two-sheet surface. This class, in addition to (5), contains families of asymmetric and shifted quadratic Bessel–Gauss beams similar to those considered in [9] and [11, 12], as well as simple astigmatic and Arnaud-Kogelnik [17] general astigmatic Gaussian beams. This work is devoted to the construction of another family which belongs to the same class.
3. Cosine-Gauss beams

In the case when the index \( m \) takes odd values, i.e. \( m=2n-1 \), expression (5) contains Bessel functions of half-integer order. For such functions the identity [8]

\[
\left( \frac{2}{\pi \zeta} \right)^{1/2} \cos \sqrt{\zeta^2 - 2 \zeta t} = \sum_{n=0}^{\infty} \frac{t^n}{n!} J_{n-\frac{1}{2}}(\zeta) \tag{10}
\]

is carried out.

Expression (10) tends to infinity as \( \zeta \to 0 \). If we discard the first term in the series (10)

\[
J_{-\frac{1}{2}}(\zeta) = \left( \frac{2}{\pi \zeta} \right)^{1/2} \cos \zeta
\]

and start the summation with \( m=1 \), then as a result we get a function which is bounded in a neighborhood of zero

\[
\sum_{n=1}^{\infty} \frac{t^n}{n!} J_{n-\frac{1}{2}}(\zeta) = \left( \frac{2}{\pi \zeta} \right)^{1/2} \left( \cos \sqrt{\zeta^2 - 2 \zeta t} - \cos \zeta \right) \tag{12}
\]

As \( \zeta \to 0 \), this function tends to zero as \( (2\zeta/n)^1/2 \).

We use formula (12) to find the sum of functions (5) with odd indices:

\[
\sum_{n=1}^{\infty} \frac{t^n}{n!} U_{\pm (2n-1)} = C \frac{w_0}{W(z)} \left( \frac{2}{\pi \zeta} \right)^{1/2} \left( \cos \sqrt{\zeta^2 - 2 t \zeta e^{\pm i \varphi}} - \cos \zeta \right) e^{\xi \pm i \varphi}. \tag{13}
\]

Here \( t \) is an arbitrary complex value. Series (13) converges uniformly with the derivatives, and its sum, like each of the terms, satisfies equation (1). This fact is easy to verify by direct substitution. Thus, we have obtained a new family of analytical solutions of the equation (1). These solutions, in addition to the parameters included in (5), contain an extra parameter \( t \). The solutions are odd with respect to the simultaneous replacement of the signs of the transverse coordinates \( x \) and \( y \), vanish on the beam axis and have a vortex with a topological charge of \( \pm 1 \) on this axis. Optical vortices are also formed on spatial curves, on which the cosine difference vanishes, it corresponds to the condition

\[
\frac{\mu \rho^2}{W^2(z)} (2k \pi - t e^{\pm i \varphi}) = 2k^2 \pi^2.
\]

Another regularization method is to consider the difference between functions of the form (10) for various values of \( t \). Such differences will again be bounded as \( \zeta \to 0 \), since the singular terms corresponding to \( n=0 \) are canceled:

\[
\sum_{n=1}^{\infty} \frac{t^n-t^n}{n!} J_{n-\frac{1}{2}}(\zeta) = \left( \frac{2}{\pi \zeta} \right)^{1/2} \left( \cos \sqrt{\zeta^2 - 2 \zeta t} - \cos \sqrt{\zeta^2 - 2 \zeta t'} \right). \tag{14}
\]

Formula (12) is a special case of (14) with \( t'=0 \). The solution of the parabolic equation (1) corresponding to the formula (14) has the form

\[
\sum_{n=1}^{\infty} \frac{t^n-t^n}{n!} U_{\pm (2n-1)} = C \frac{w_0}{W(z)} \left( \frac{2}{\pi \zeta} \right)^{1/2} \left( \cos \sqrt{\zeta^2 - 2 t \zeta e^{\pm i \varphi}} - \cos \sqrt{\zeta^2 - 2 t' \zeta e^{\pm i \varphi}} \right) e^{\xi \pm i \varphi}. \tag{15}
\]

In the particular case, if we take \( t'=t \) in the sums (14) and (15), all terms with even values of \( n \) reduce and terms with odd \( n=2k-1 \) \((k \geq 1)\) double:

\[
2 \sum_{k=1}^{\infty} \frac{t^{2k-1}}{(2k-1)!} U_{\pm (4k-3)} = C \frac{w_0}{W(z)} \left( \frac{2}{\pi \zeta} \right)^{1/2}
\]
\[
\cdot \left( \cos \sqrt{\xi^2 - 2t\xi e^{\pm 2i\phi}} - \cos \sqrt{\xi^2 + 2t\xi e^{\pm 2i\phi}} \right) e^{\xi \mp i\phi}.
\]

(16)

If we take the sum of terms only with even numbers \(n=2k\) other than zero, then a combination of not two, but three cosines will correspond to it:

\[
2 \sum_{k=1}^{\infty} \frac{t^{2k}}{(2k)!} U_{\pm(4k-1)} = C \frac{w_0}{W(z)} \left( \frac{2}{\pi \xi} \right)^{1/2} \cdot \left( \cos \sqrt{\xi^2 - 2t\xi e^{\pm 2i\phi}} + \cos \sqrt{\xi^2 + 2t\xi e^{\pm 2i\phi}} - 2 \cos \xi \right) e^{\xi \mp i\phi}.
\]

(17)

Solution (17) has an optical vortex with a topological charge of \(\pm 3\) on the optical axis.

A further generalization of the obtained solutions are functions containing linear combinations of an arbitrary number of terms

\[
\frac{w_0}{W(z)} \left( \frac{2}{\pi \xi} \right)^{1/2} \sum_{j=1}^{N} C_j \cos \sqrt{\xi^2 - 2t_j \xi e^{\pm 2i\phi}} e^{\xi \mp i\phi},
\]

(18)

where

\[
\sum_{j=1}^{N} C_j = 0,
\]

and \(t\) are arbitrary complex constants. In such sums, the singular terms reduce again, and expressions (18) give regular solutions of equation (1). In particular, values of the coefficients \(C_j\) can be selected as

\[
C_j = C \exp \left( \frac{2\pi i}{N} jM \right),
\]

where \(0<M<N\). If we simultaneously choose

\[
t_j = \exp \left( \frac{2\pi i}{N} j \right) t,
\]

where \(t\) is some complex constant, then we obtain a generalization of a solution of the form (16). The expansion of the solution in powers of \(t\) contains only terms with numbers \(n\) for which the expression \(M+n\) is a multiple of \(N\), i.e. \(n=kN-M\) with some positive integer \(k\). Indeed, the factor for \((t^n/n!)U_{\pm(2n-1)}\) in such a sum is

\[
\sum_{j=1}^{N} \exp \left( \frac{2\pi i}{N} jM \right) \exp \left( \frac{2\pi i}{N} jn \right) = \begin{cases} 0, & M+n \neq kN \\ N, & M+n = kN \end{cases}
\]

and therefore

\[
C \frac{w_0}{W(z)} \left( \frac{2}{\pi \xi} \right)^{1/2} \sum_{j=1}^{N} \exp \left( \frac{2\pi i}{N} jM \right) \cos \sqrt{\xi^2 - 2 \exp \left( \frac{2\pi i}{N} j \right) t \xi e^{\pm 2i\phi} e^{\xi \mp i\phi}}
\]

\[
= N \sum_{k=1}^{\infty} \frac{t^{kN-M}}{(kN-M)!} U_{\pm(2kN-2M-1)}
\]

(19)

Such beams have a vortex on the optical axis with a topological charge equal to \(\pm(2N-2M-1)\). In addition, vortices with a topological charge of \(\pm 1\) are formed on the curves, where the linear combination of cosines vanishes.

Solution (19) is a generalization of (16) and goes into it for \(N=2, M=1\). An analog of function (17) is
\[ C \frac{w_0}{W(z)} \left( \frac{2}{\pi \xi} \right)^{1/2} \left( -N \cos \xi + \sum_{j=1}^{N} \cos \sqrt{\frac{\xi^2 - 2 \exp \left( \frac{2\pi i}{N} j \right) t \xi e^{\pm 2i\varphi}} \right) e^{\xi \mp i \varphi} \]

\[ = N \sum_{k=1}^{\infty} \frac{t^k}{(kN)!} U_{\pm(2kN-1)}. \tag{20} \]

Solution (20) has a vortex on the optical axis with a topological charge equal to \( \pm(2N-1) \).

4. Conclusion
We have constructed a new family of localized solutions of the paraxial wave equation (1), which are naturally to call quadratic cosine-Gauss beams. These solutions belong to the class of quadratic Helmholtz–Gaussian beams considered in [16] since they can be represented as superposition of quadratic Bessel–Gauss beams.

In the future we intend to continue work on the study of the properties of the obtained solutions.

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