MODULI OF QUANTUM RIEMANNIAN GEOMETRIES ON $\leq 4$ POINTS

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Abstract. We classify parallelizable noncommutative manifold structures on finite sets of small size in the general formalism of framed quantum manifolds and vielbeins introduced in [10]. The full moduli space is found for $\leq 3$ points, and a restricted moduli space for 4 points. Generalised Levi-Civita connections and their curvatures are found for a variety of models including models of a discrete torus. The topological part of the moduli space is found for $\leq 9$ points based on the known atlas of regular graphs.

1. Introduction

There has been a lot of interest over the years [1, 2, 3, 4, 5, 6, 7, 8] in the specific application of noncommutative geometry [9] to the commutative algebra of functions on a finite set $\Sigma$ (usually a finite group) in which the differential forms do not commute with functions. This provides a systematic way of handling geometry on finite lattices which, at the level of cohomology, electromagnetism and Yang-Mills theory has already proven interesting and computable. Notably, [8] contains the moduli of $U(1)$-Yang-Mills on the permutation group $S_3$ while [7] quantizes $U(1)$-Yang-Mills theory on the finite group $Z_2 \times Z_2$.

In this paper we want systematically to extend this theory to the gravitational case. Some first steps are in [10], to which the present paper is a sequel. It was shown there that finite groups have indeed a natural Riemannian geometry in a vielbein and frame-bundle formalism [11] which was worked out in detail for $S_3$ (it turns out to have Ricci essentially proportional to the metric, i.e. an ‘Einstein manifold’). Similarly, the alternating group $A_4$ was considered in [12] and has an essentially unique invariant metric with 4-bein and an associated spin connection with nonzero curvature but with Ricci=0, i.e. solves the vacuum Einstein equations. Hence the system of equations for a framed quantum Riemannian manifold is already known to have interesting nontrivial solutions. However, for quantum gravity (or classical but finite gravity) we need a better understanding of the moduli spaces of all metrics, connections etc. and this is what we study now on small sets. Once one has this, one can in principle begin to quantize this moduli space in a path integral approach, i.e. quantum gravity.

Section 2 starts with a brief account of the formalism for algebras which we then rapidly specialise to the case $\mathbb{C}(\Sigma)$, the algebra of functions in a finite set. That the theory is a specialisation of a functorial construction that is formulated for general algebras ensures that it is not ad-hoc (indeed, this same theory can be specialised to classical geometry and to q-deformed geometry for other choices of algebra[13]).

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Following [10], we find that for finite sets \( \Sigma \) the classification of ‘differential forms’ or exterior algebras of parallelizable type reduces to the classification of finite regular graphs with vertices \( \Sigma \) and a fixed number \( n \) arrows from every vertex. New results are Theorem 2.1 showing in detail that the calculus is then inner, and Theorem 2.2 for the construction of 2-forms. Both are needed in the paper. Further ingredients in the formalism are a choice of \( n \)-beins and a frame group \( G \) (in our case a finite group) acting on the vector space spanned by them. This gives the moduli of ‘quantum framed manifold’ structure on \( \Sigma \). After this, one may look for a compatible connection \( \nabla \), find the Riemann curvature and from this the Ricci tensors and Ricci scalar. In this way we set up the theory that we are going to explore for small numbers of points.

In Section 3.1 we analyze the case \( \Sigma = \{x, y\} \) of two points and frame groups \( S_2, S_3 \) acting on an einbein \( e_1 \) parametrised by a function \( \Theta \). We find that for each einbein there is a natural generalized Levi-Civita connection
\[
\nabla (f e_1) = df \otimes e_1 + 2f(\Theta)e_1 \otimes e_1
\]
for any function \( f \), where \( \langle \rangle \) is the average value over the two points. This has zero Riemannian curvature, which emerges as a typical feature on two points. In our spin connection approach we find also the moduli of spin connections; for \( S_2 \) framing we have a unique spin connection underlying \( \nabla \). For \( S_3 \) we find a larger moduli of spin connections, with gauge curvature, underlying the Riemannian geometry itself (all giving the same \( \nabla \)).

In Section 3.2 we similarly cover the case \( \Sigma = \{x, y, z\} \) of three points and frame groups \( S_2, S_3 \) acting on a zweibein. The zweibein moduli space is itself nontrivial as an algebraic variety but we show how put a generic point into a canonical form, and then study spin connections for a fixed zweibein. A general feature for three points emerges, namely that in all our models the Ricci scalar vanishes, but the Riemann and Ricci tensors themselves generically do not. For \( S_2 \) we have a linear constraint on the zweibein to admit a connection, after which there is a 1-parameter family of connections. For \( S_3 \) there is no constraint on the zweibein and a 8-dimensional moduli of connections.

The canonical form for the vielbeins obtained in our analysis of 2 and 3 points in Section 3 is one where (after linear transformations), one may restrict to vielbeins which have only a scalar \( \Theta_a \) associated to each edge. In Section 4 we proceed to restrict attention to this canonical form, now for four point sets. Physically, the modulus of the vielbein assigns a ‘length’ to each edge, while the natural connectivity for 4 points is that of \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) (interpreted as a discrete model of a torus), which we consider in Section 4.1 to 4.3; we consider various frame groups, among them an interesting choice (Section 4.2) is a frame group \( \mathbb{Z}_4 \) of ‘quarter rotations’ again as a discrete model of a torus; we find the most general connection, its Riemann and Ricci curvatures, etc. This model has the feature (Theorem 4.4) that a fully metric compatible spin connection is determined uniquely by the zweibein, but with the latter further constrained. By contrast, our weaker ‘skew metric compatible’ or cotorsion free condition admits further parameters \( a, b \) with the zweibein relatively unconstrained. We also see what happens if one takes too big a calculus on the frame group, namely additional unphysical modes emerge which do not, however, enter into the covariant derivative. This seems to us an important lesson for finite manifold-building by these methods. Section 4.4 completes the picture by covering the alternative connectivity of 4 points joined in a tetrahedron, which is more like
a sphere. Here with $\mathbb{Z}_3$ frame group of ‘one third’ rotations we find an unusual but interesting calculus, moduli etc without classical analogue.

Later, in Section 5 we make some first remarks on the quantum theory, including a look at the discrete torus model on $\mathbb{Z}_2 \times \mathbb{Z}_2$. Mainly, we find what we show on this model to be a reasonable unitarity or $^*$-structure on the system which is needed to reduce the functional integrals to real variables. We do not try to do the integrals themselves, which would be beyond the scope of our current analysis.

Finally, Section 6 return to a more qualitative account of all bidirectional framed geometries up to 9 points, deduced from the known atlas of graphs [14]. This covers the connectivity or topological aspect of the vielbein moduli space. At this level a vielbein amounts to a colouring of the graph into $n$-colours. For each such vielbein, there are further continuous degrees of freedom for matrices $e_a$ labelled according to the colouring $a$ (as seen in detail in Section 3). If we ignore these then we have a in principle a ‘combinatorial quantum gravity’ in which one sums over all such colourings.

Let us note that ‘geometry’ on finite sets in some form or other has a long pedigree. Common to all approaches is the basic data of ‘differentials’ as defined by directed edges between vertices (a ‘digraph’ or quiver). Such objects are used in representation theory for quivers formed on Dynkin diagrams. One also considers in that context some kind of ‘vector bundles’ with vector spaces over each vertex albeit of varying dimension. Similarly in physics as well as in simplicial cohomology one may ‘approximate’ a manifold by a finite triangulation and work on that. From the algebraic point of view one does not actually need bidirectional edges, e.g. every poset defines a connectivity graph and differential calculus with $x \to y$ if $x < y$ (albeit not a parallelisable one if it is finite). This would be relevant to modelling Lorentzian manifolds [15] with $x \to y$ modeling a time-like path from $x$ to $y$. Hence the deeper notions of vielbeins and Riemannian geometry that we develop on such data potentially has several applications.

2. Preliminaries: formalism of quantum Riemannian manifolds

Here we briefly recall the formalism of [10]. To tie in with the general theory we start with a brief recap over general algebras in Section 2.1. Then in Section 2.2 we specialise to the finite set case in more detail than outlined in [10]. We cover here only the parallelizable case where the frame bundle algebra has a trivial tensor product form. There is a still more general theory where the bundle is nontrivial, see [10], but this needs much more machinery and we do not cover it here. It would be needed for finite posets, for example.

2.1. Over general algebras. Let $M$ be a unital algebra. We equip $M$ with a differential structure in the sense $(\Omega^1(M), d)$, where $\Omega^1(M)$ is an $M - M$ bimodule, and $d : M \to \Omega^1(M)$. This is a notion common to all approaches to noncommutative geometry including [9]. We also need $\Omega^2(M)$ or (in principle) higher $\Omega^k(M)$ with $d^2 = 0$, for which we can take the maximal prolongation of $\Omega^1(M)$ or any of its quotients.

In this context we define a (left) vielbein of $V$-bein as a collection $\{e_a\}$ forming an $M$-basis of 1-forms $e_a \in \Omega^1(M)$, i.e. $\Omega^1(M) \cong M \otimes V$ where $V = \text{span}\{e_a\}$. One can also think equivalently of the $V$-bein as a map $e : V \to \Omega^1(M)$ as in [10] if we regard $V$ as a fixed abstract vector space. Given a vielbein we deduce operators
\( \rho^b_a, \partial^a : M \to M \) where
\[
e_a f = \sum_b \rho^{b \in} (f) e_b, \quad df = \sum_a (\partial^a f) e_a, \quad \forall f \in M
\]
as an expression of the bimodule and exterior derivative structure.

Next, we assume that we actually have an \( A \)-vielbein, i.e. we require \( V \) to be an \( A \)-comodule under a Hopf algebra \( A \). There is also a more general theory with \( A \) merely a coalgebra, i.e. this is not a critical assumption. We fix a left-covariant differential structure \( \Omega(A) \) on the fiber of the frame bundle. Like Lie groups, quantum groups are always parallelisable and hence \( \Omega(A) = A \otimes \Lambda^1 \) for some space of invariant 1-forms \( \Lambda^1 \). This is a quotient of the augmentation ideal \( A_+ \) of \( A \) (classically it means the functions that vanish at the group identity), i.e. \( \Lambda^1 = A_+ / Q_A \) for some left ideal \( Q_A \subseteq A_+ \). We call \( \Lambda^1 \) the 'quantum tangent space', where we suppose \( \Lambda^1 \) is finite dimensional and \( H \) a Hopf algebra dually paired with \( A \) (it plays the role classically of the enveloping algebra of the Lie algebra of the frame group). We will be interested only in the bicovariant case as in [10] where one knows from the Woronowicz theory [16] that \( \Lambda^1 \) is Ad-stable or that \( \Lambda^1 \) inherits Ad as a 'quantum Lie bracket'. When \( A \) is coquasitriangular one knows that \( \Lambda^1 \) is in fact a braided-Lie algebra [17]. However, neither assumption is critical for the geometry.

We let \( \{ f^i \} \) be a basis of \( \Lambda^1 \) and denote by \( \triangleright \) its left action inherited from the left action of \( H \) on \( V \) corresponding to the coaction of \( A \). It is only this action which is needed in the formulae below. In this basis a spin connection means a collection of 1-forms \( \{ A_i \} \). Its torsion tensor corresponds to
\[
d e_a + \sum_i A_i \wedge f^i \triangleright e_a
\]
and we are interested in torsion-free connections. We also (optionally) impose a regularity or 'differentiability' condition linking \( \Omega^2(M) \) and \( \Omega^1(A) \), namely
\[
\sum_{ij} A_i \wedge A_j \langle f^i f^j, q \rangle = 0 \quad \forall q \in Q_A.
\]
This ensures that the component 2-forms \( \{ F_i \} \) of the curvature of the spin connection, namely
\[
F_i = dA_i + \sum_{jk} c^{ijk} A_j \wedge A_k
\]
have a proper geometrical interpretation as a curvature 2-form with values in \( \Lambda^1 \). Here \( c^{ijk} = \langle e_i, f^j f^k \rangle \) are structure constants of the product of \( H \) projected to \( \Lambda^1 \) (where \( \{ e_i \} \) is a dual basis of \( \Lambda^1 \)).

For metrics we specialise to the case \( g = \sum_{a,b} \eta^{ab} e_a \otimes_M e_b \) where \( \eta \in V \otimes V \) is a nondegenerate \( H \)-invariant ‘local metric’. This is not the most general setup in [11] [10], where one can consider \( g \) an arbitrary (but nondegenerate) 2-form. In our case the cotorsion-free condition, which is the natural generalisation of Levi-Civita metric compatibility in [11] [10], is vanishing of
\[
d e_a + \sum_{i} S^{-1}(f^i) \triangleright e_a \wedge A_i
\]
where \( S \) denotes the antipode of \( H \).
Finally, we specialise to the case of $\Omega^2(M)$ constructed from an $H$-equivariant projection $\pi : V \otimes V \to V \otimes V$ according to the scheme indicated in [10]. From the above, we know that $\Omega^1(M) \otimes_M M \cong \omega \otimes \omega$ allowing us to define surjections

$$\Omega^1(M) \otimes \Omega^1(M) \to \Omega^2(M)$$

where we quotient out $M \otimes \ker \pi$. In fact we define $\Lambda$ as a quadratic algebra on $\omega$ with relations $\ker \pi$, and $\Omega(M) \cong \omega \otimes \Lambda$. Such a scheme imposes constraints on $\pi$. In this setting there is a canonical lift

$$(6) \quad \pi : \Omega^2(M) \to \Omega^1(M) \otimes \Omega^1(M), \quad \pi(e_a \wedge e_b) = \pi(e_a \otimes e_b).$$

Finally, we let $i(F_i) = \sum_{a,b} i(F_i)^{ab} e_a \otimes_M e_b$ define the components in the V-bein basis of the lifted $F_i$. Then

$$(7) \quad \text{Ricci} = \sum_{i,a,b} i(F_i)^{ab} e_b \otimes f^i \circ e_a.$$

The full Riemann curvature of the connection and the covariant derivative acting on 1-forms are

$$(8) \quad \text{Riemann}(\alpha) = \sum_{i,a} \alpha^a F_i \otimes f^i \circ e_a, \quad \nabla \alpha = \sum_a \partial \alpha^a \otimes e_a - \sum_{i,a} \alpha^a A_i \otimes f^i \circ e_a$$

where $\alpha = \sum_a \alpha^a e_a$. The derivation of these local formulae from a more abstract theory is in [10], in an equivalent comodule notation.

2.2. Over finite sets. We now specialise the above to the case $M = \mathcal{C}(\Sigma)$ where $\Sigma$ is a finite set and $H = \mathcal{C}(G)$ where $G$ is a finite group. In this case the possible $\Omega^1(\Sigma)$ are given by subsets

$$E \subset \Sigma \times \Sigma \text{ minus diagonal}$$

of ‘allowed directions’. This is already known from [1] and $E$ is the same as the structure of a quiver or digraph with vertex set $\Sigma$ and the notation $x \to y$ whenever $(x, y) \in E$. In the geometrical examples we typically expect $E$ symmetric or ‘bidiirectional’ i.e. for every edge $x \to y$ there is an edge $x \leftarrow y$, but we do not assume this in general. Explicitly,

$$\Omega^1(\Sigma) = \text{span}\{\delta_x \otimes \delta_y | x \to y\}, \quad df = \sum_{y \in F_x} (f(y) - f(x)) \delta_x \otimes \delta_y, \quad F_x = \{y | x \to y\}.$$

Note also that the $k$-fold product

$$\Omega^1(\Sigma) \otimes \cdots \otimes \Omega^1(\Sigma) = \text{span}\{\delta_x \otimes \delta_{x_1} \otimes \cdots \otimes \delta_{x_k} | x \to x_1 \to x_2 \to \cdots \to x_k\}$$

i.e. the linear span of the set of $k$-arcs. The bimodule structures are the pointwise ones for products from the extreme left and right.

As explained in [10] a vielbein in this setting is possible iff $E$ fibers over $\Sigma$ i.e. $F_x$ have cardinality $n$ (say), independent of $x$. In this case an $n$-bein is the specification of invertible $n \times n$ matrices $e_{-x,-}$, for each $x \in \Sigma$. Here $e_{axy}$ has indices $a \equiv 1, \cdots, n$ and $y \in F_x$. We write the inverses as $e_a^{-1}y$ with

$$\sum_{y \in F_x} e_a^{-1}y e_{bxy} = \delta_{a,b}, \quad \sum_a e_a^{-1}y e_{axy} = \delta_y'$$
In this case the operators $\Omega^1$ are

\begin{equation}
\tag{10}
p_a^b(f)(x) = \sum_{y \in F_x} e_b^{-1}xy f(y)e_{axy}, \quad (\partial^a f)(x) = \sum_{y \in F_x} (f(y) - f(x)) e_a^{-1}xy.
\end{equation}

A calculus on an algebra $M$ is inner if there is a 1-form $\theta$ with $df = [\theta, f]$ for $f \in M$.

**Theorem 2.1.** cf[10] A finite set calculus equipped with a vielbein is inner,

$$\theta = \sum_a \Theta_a e_a, \quad \Theta_a(x) = \sum_y e_a^{-1}xy.$$  

Moreover, the maximal prolongation exterior algebra $\Omega(\Sigma)$ has likewise $d = [\theta, \cdot]$ (graded anticommutator) and is generated by $\mathbb{C}(\Sigma)$ and the quadratic algebra on the $\{e_a\}$ with relations

$$\sum_{y \in F_{x,z}} \sum_{a,b} e_a^{-1}xy e_b^{-1}yz e_a \wedge e_b = 0, \quad \forall (x, z) \notin E \cup \text{diag}; \quad F_{x,z} = \{y\mid x \to y \to z\}.$$  

**Proof.** We define $\theta$ as stated. Then the explicit formulae (10) allow one to verify that $df = [\theta, f]$ for any function $f$, as required. The maximal prolongation of the $\Omega^1$ is defined as the tensor algebra over $M = \mathbb{C}(\Sigma)$ modulo the relations in degree 2 imposed by extending $d$ as a superderivation with $d^2 = 0$. More precisely, we lift any 1-form to the universal differential calculus over $\mathbb{C}(\Sigma)$, apply the universal exterior derivative there, and then project down to $\Omega^2$. That this should be well-defined defines the minimal relations in degree 2 (which are the only ones imposed in the maximal prolongation). In our case as basis of the kernel of the projection to $\Omega^1$ is given by $\delta_x d \delta_z = 0$ whenever $(x, z) \notin E \cup \text{diag}$, so we require for each such $(x, z)$ the relation

$$d \delta_x \wedge d \delta_z = 0.$$  

We compute

$$d \delta_x = \sum_a \sum_{y \in F_x} (\delta_x(y) - \delta_x) e_a^{-1}y e_a = \sum_a (e_a^{-1}x - \delta_x \Theta_a(x)) e_a,$$  

where $\cdot$ denotes a functional dependence on points in $\Sigma$ and we adopt the convention that $e_a^{-1}wx = 0 = e_{aux}$ if $x \notin F_w$. Note also that

$$\sum_a e_a^{-1}x \rho_a^c(f) = \sum_{y \in F} \sum_a e_a^{-1}x e_c^{-1}y f(y) e_a = f(x) e_c^{-1}x$$  

by (10) and (9). The latter also implies that $\sum_a \Theta_a(x) e_{axy} = 1$ if $y \in F_x$. Hence

$$d \delta_x \wedge d \delta_z = \sum_{a,b,c} e_a^{-1}x \rho_a^c (e_b^{-1}z) e_c \wedge e_b - \sum_{a,b,c} e_a^{-1}x \Theta_b(z) \rho_a^c (\delta_z) e_c \wedge e_b$$  

$$- \sum_{a,b,c} \delta_x \Theta_a(x) \rho_a^c (e_b^{-1}z) e_c \wedge e_b + \sum_{a,b,c} \delta_x \Theta_a(x) \Theta_b(z) \rho_a^c (\delta_z) e_c \wedge e_b$$  

$$= \sum_{b,c} e_c^{-1}x e_b^{-1}xz e_c \wedge e_b - \sum_{b,c} \delta_z(x) \Theta_b(z) e_c^{-1}z e_c \wedge e_b$$  

$$- \delta_x \sum_{b,c} \sum_{y \in F_x} e_c^{-1}xy e_b^{-1}yz e_c \wedge e_b + \delta_x \sum_{a,b,c} \Theta_a(x) \Theta_b(z) e_{axy} e_c^{-1}xz e_c \wedge e_b.$$
The first and last term vanish for \((x, z) \notin E\) and the second term for \(x \neq z\). Hence in this case we obtain precisely the relation stated from the remaining third term. This completes the proof of the result mentioned in [10].

It is then a computation to write

\[
e_a = \sum_{(x,y) \in E} e_{axy} \delta_x \delta_y = \sum_{y \in F} e_a \cdot y \delta_y
\]

and obtain \(de_a = \theta e_a + e_a \theta\). Note that the compatibility of \(d\) with the relations (11) for all \(f\) more or less requires this relation since applying \(d\) to (11) gives \((de_a - \{ \theta, e_a \}) f = \rho_a^b(f)(de_b - \{ \theta, e_b \})\) after using (11) and that the calculus is inner.

We note in passing that that by similar computations the maximal prolongation has

\[
\theta \wedge \theta = \sum_{a,b} \Theta_a e_a \Theta_b e_b = \sum_{a,b,c} \Theta_a \sum_{y \in F} e_c^{-1} - y \Theta_b(y) e_a - y e_c \wedge e_b
\]

\[
= \sum_{b,c} \sum_{y \in F} e_c^{-1} - y \Theta_b(y) e_c \wedge e_b = \sum_{a,b} \sum_{y \in F} \sum_{\cdots} e_a^{-1} - y e_b e_{b} e_a \wedge e_b
\]

which (in view of the relations for \(\Omega^2(\Sigma)\)) has contributions only from \(z = \cdot\) and \(\cdot \to z\). This is not necessarily zero, i.e. \(\theta\) is not necessarily closed (rather, \(d\theta = 2\theta \wedge \theta\) so that \(d\theta = -2\theta\) is always a zero curvature \(U(1)\) connection).

We also require for a \(G\)-covariant vielbein that \(V = \text{span}\{e_a\}\) is a \(G\) module. The above constructions are all \(G\)-covariant under these local transformations of \(V\). To define more general exterior algebras \(\Omega(\Sigma)\) we let \(\pi : V \otimes V \to V \otimes V\) be a \(G\)-equivariant projection operator, with components defined by \(\pi_x (e_a \otimes e_b) = \sum_{c,d} \pi_{ab} cd e_c \otimes e_d\). We define operators

\[
(11) \quad \pi_{x,z} : \mathbb{C}F_{x,z} \to \mathbb{C}F_{x,z}, \quad \pi_{x,z}^{y'z} = \sum_{a,b,c,d} \pi_{ab} cd e_a^{-1} xy e_b^{-1} y z e_{cx y'} e_{dy'}
\]

on the space spanned by 2-arcs with fixed endpoints \(x, z\).

**Theorem 2.2.** \(\pi\) defines an exterior algebra with \(d^2 = 0\) as a quotient of the tensor algebra on \(V\) by the quadratic relations

\[
\ker \pi = 0
\]

iff

\[
(i) \quad \sum_{a,b,c,d} \pi_{ab} cd e_a^{-1} xy e_b^{-1} y z e_{cx y'} e_{dy'} = 0, \quad \forall z \neq z', y \in F_{x,z}, y' \in F_{x,z'}
\]

\[
(ii) \quad \sum_{y \in F_{x,z}} \pi_{x,z}^{y'z} = 0, \quad \forall (x, z) \notin E \cup \text{diag}, \quad y' \in F_{x,z}.
\]

**Proof.** We identify \(\Omega^1(\Sigma) \otimes_M \Omega^1(\Sigma)\) with \(\mathbb{C}(\Sigma) \otimes V \otimes V\) via the vielbein so that \(\pi\) induces left-module projection operators on this. These are therefore given by projection matrices \(\pi_x\) on the space spanned by the 2-arcs from \(x\), for each \(x\). Their components are

\[
\pi_{x,y'z'} = \sum_{a,b,c,d} \pi_{ab} cd e_a^{-1} xy e_b^{-1} y z e_{cx y'} e_{dy'}
\]

We require that these are also right module maps, which is the condition (i) stated. It means that \(\pi_{x,y'z'} = \pi_{x,z}^{y'z} \delta^{y'z}\) for a family of projections \(\pi_{x,z}\) for each fixed \(x, z\). These are the operators (11). As explained in [10] there is then a condition on the family of projectors to ensure that the quotient \(\Omega^1(\Sigma) \otimes_M \Omega^1(\Sigma) \to \Omega^2(\Sigma)\) factors through the maximal prolongation, namely the condition (ii). This is necessary
and sufficient for the relations in $\Omega^2(\Sigma)$ defined by $\ker \pi$ to be compatible with the extension of $d$ to 2-forms via the graded Leibniz rule.

The maximal prolongation in Theorem 2.1 can be viewed as given by a generalisation of this construction in which the projection $\pi$ is allowed to vary from point to point, i.e. a field of projections $\pi_x$. The more specific construction in Theorem 2.2 is necessarily a quotient of it by further relations.

Finally, we fix an $\text{Ad}$-stable subset $C \subset G$ with $e \notin C$ (e here the group identity), e.g. a nontrivial conjugacy class. These describe the bicovariant calculi $\Omega^1(G)$ in the Woronowicz theory\cite{16}. The space of invariant forms $\Lambda^1$ in $\Omega^1(G)$ has basis $\{e| i \in C\}$. The dual basis of $\Lambda^{1^*}$ is $\{f^i\}$ with $f^i = i - e$. The torsion and cotorsion equations then have the same form \cite{21} and \cite{5}, with $Si = i^{-1}$ the group algebra antipode. The regularity condition now reads

\begin{equation}
\sum_{ij=q} A_i \wedge A_j = 0, \quad \forall q \notin C \cup \{e\}.
\end{equation}

This is empty if we chose the universal calculus on $G$ (where $C = G - \{e\}$), but in general it is a quadratic constraint. The curvature form is then

\begin{equation}
F_i = dA_i + \sum_{jk=i} A_j \wedge A_k - \{A_i, \sum_j A_j\}.
\end{equation}

The formulae for the Ricci and Riemann tensors and $\nabla$ have the same form \cite{8}.

3. Moduli of geometries on two or three points

In this section we describe the moduli space of possible vielbeins and metrics on 2 or 3 points, and moduli of spin connections and their curvature for some points in the moduli of vielbeins with respect to frame group $S_2$ or $S_3$.

More precisely, the moduli of possible vielbeins is in the first place labelled by two natural numbers $m = |\Sigma|$ and $n$ a fixed number of arcs from each point. For each $m, n$, the combinatorial part of the moduli space consists of determining all possible quiver structures with no self-arcs, i.e. all $E \subseteq \Sigma \times \Sigma - \text{diag}$ with $F_x$ of cardinality $n$ at each $x \in \Sigma$. We interpret it as finding all possible parallelizable $\Omega^1(\Sigma)$ with $n$-dimensional cotangent space. Note that $\tilde{E}$ where we flip the entries of $E$ defines another calculus $\Omega^1(\Sigma)$ and in the asymmetric case one could (although we do not do it here) demand this to also be parallelizable, with an associated number $\tilde{n}$.

There is a corresponding moduli of geometries built on this arrow-reversed calculus.

For $m = 2$ or $\Sigma = \{x,y\}$ there is only one possibility, namely $n = 1$ and the quiver

\[ x \leftrightarrow y \]

up to relabellings. This is the universal calculus on $\Sigma$ where $E$ is as large as possible.

For $m = 3$ or $\Sigma = \{x,y,z\}$ there are two cases for $n = 1$, namely

\[ x \leftrightarrow y \leftrightarrow z \]

\[ \quad x \leftrightarrow y \leftrightarrow z \]

up to relabellings. These are asymmetric. For $n = 2$ there is only one possibility, the universal calculus on $\Sigma$ again, which is always symmetric. It is given by

\[ x \leftrightarrow y \leftrightarrow z \].
Next, for our projection matrix $\pi$ to define $\Omega^2(\Sigma)$ we make the ‘naive’ choice
\begin{equation}
\pi = \frac{1}{2} (\text{id} - \tau)
\end{equation}
where $\tau$ is the usual ‘flip’ operator on the tensor product, i.e. we assume the basic 1-forms anticommute. This seems to give reasonable results for $n = 2$ and a small number of points (in general it would be too restrictive). For $n = 1$ we choose $\pi = 1$ (the choice $\pi = 0$ is also allowed but not very interesting). More generally, we should determine all possible equivariant projection matrices $\pi : V \otimes V \to V \otimes V$ for choice of frame group $G$ and a representation $V$ of dimension $n$. The representation theory of $G$ then dictates the possible equivariant projection matrices $\pi$:
\end{equation}

This is the representation theoretic part of the moduli space. In our case, we take symmetric groups $S_2, S_3$ appropriate to our small number of points. For $n = 1$, $V$ has to be trivial (we denote this by $\mathbb{C}$) or the sign representation given by $(-1)^{l(g)}$ where $l$ is the length function. For $n = 2$ we have $V = \mathbb{C} \oplus \text{sign}$, $V = \text{sign} \oplus \text{sign}$ or, in the case of $S_3$ also its 2-dimensional representation. In all three cases $V \otimes V = \mathbb{C} \oplus \text{sign} \oplus V$ and the ‘naive’ $\pi$ projects out all but the sign representation here (cf in classical geometry the top form transforms by the determinant under a linear transformation). The invariant local metric $\eta$ up to a normalisation is also classified by representation theory and we take it as the generator of the natural trivial representation in the decomposition of $V \otimes V$.

Fixing all the above quasi-combinatorial data, we have a moduli space
\begin{equation}
\text{Vielbeins}_m, n, E, \pi = \{e_{a,x,y}\}/GL_n
\end{equation}
consisting of $m \times n \times n$ invertible matrices subject to the constraints in Theorem 2.2. We divide by an overall $GL_n$ acting on the left and corresponding to a change of basis of $V$. We arrive at a certain algebraic variety which we shall describe first.

Finally, for a fixed vielbein and the above data, we look at the moduli of spin connections for $\eta$. This last part requires us to fix a differential structure on $G$. For $S_2$ the only choice is the universal calculus $\Omega^1(S_2)$. For $S_3$ there is the universal calculus and the calculus corresponding to the 2-cycles conjugacy class. The remaining conjugacy class does not give a reasonable geometry of $S_3$ (it is not connected) and does not appear to give interesting results, so we omit it. In all cases we assume that the action of $G$ on $V$ is not trivial when restricted to the braided-Lie algebra generators $f_i$. Otherwise, they would act as zero, the Riemann curvature would be automatically zero and $\nabla$ would be just given by $d$ for any spin connection. So we omit this uninteresting case in our analysis.

The case $\pi = 0$ is trivial and we deal with it here. In this case $\Omega^1(\Sigma)$ is the top degree so that there is no constraint on the $\{e_{axy}\}$ other than being invertible. I.e.
\begin{equation}
\text{Vielbeins}_{m, n, E, 0} = (GL_n)^{m-1}.
\end{equation}
Similarly the torsion, cotorsion and regularity conditions are empty and any collection of 1-forms $\{A_i\}$ are trivially a spin connection, with zero curvature.

3.1. Two points. For $\Sigma = \{x, y\}$ the only choice is the universal calculus as explained above, which has $n = 1$, i.e. we look for a 1-bein $e_1$. We write
\begin{equation}
e_1 = \alpha \delta_x \otimes \delta_y + \beta \delta_y \otimes \delta_x; \quad e_1xy = \alpha, \quad e_{1yx} = \beta, \quad \alpha, \beta \neq 0.
\end{equation}
The partial derivatives and commutation relations are
\begin{equation}
e_1 f = f e_1, \quad \partial^1 f = (\bar{f} - f) \Theta; \quad \Theta(x) = \alpha^{-1}, \quad \Theta(y) = \beta^{-1};
\end{equation}
\[ \tilde{f}(x) = f(y), \quad \tilde{f}(y) = f(x). \]

The generating 1-form and exterior derivative are

\[ \theta = \Theta e_1, \quad df = (\tilde{f} - f)\theta \]

For \( \Omega^2(\Sigma) \) we have only one nontrivial possibility, namely \( \pi = 1 \), which gives the maximal prolongation with no relations in the exterior algebra (it is the universal exterior algebra on \( \Sigma \)). The conditions in Theorem 2.2 are empty as the assumptions are never satisfied. Here \( \Lambda = \mathbb{C}[e_1] \) and each \( \Omega^k(\Sigma) \) is 1-dimensional. The exterior derivatives are defined by the graded-Leibniz rule and

\[ de_1 = (\Theta + \tilde{\Theta})e_1^2. \]

**Proposition 3.1.** For any \( \Theta \) the dimensions of \( H^i \) are \( 1 : 0 : 1 \). Here

\[ H^0 = \mathbb{C}.1, H^2 = \mathbb{C}.\delta_\varepsilon e_1^2 \]

**Proof.** First of all we show explicitly that, in accordance with theorem 2.2, \( d^2 = 0 \).

\[ d(df) = d(\Theta(\tilde{f} - f)) = \Theta \tilde{\Theta}(f - \tilde{f}) + \Theta \tilde{\Theta}(\tilde{f} - f) = 0 \]

The functions \( f \) such that \( df = 0 \) are the constant ones so the nullspace of \( d \) acting on \( \mathbb{C}[\Sigma] \) is 1-dimensional (and therefore, \( p_0 = 1 \)). Since the dimension of \( \mathbb{C}[\Sigma] \otimes \Omega^1(\Sigma) = 2 \) this means that the image of \( d \) in \( \mathbb{C}[\Sigma] \otimes \Omega^1(\Sigma) \) is 1-dimensional. If \( \omega = f e_1 \) is a one-form, the \( d\omega = 0 \) if and only if \( f = \Theta(\delta_x - \delta_y) \) which implies that the nullspace of \( d \) contained in \( \mathbb{C}[\Sigma] \otimes \Omega^1(\Sigma) \) is one dimensional.

Then, \( p_1 = 1 - 1 = 0 \). In turn, the image of \( d \) in \( \mathbb{C} \otimes \Omega^2(\Sigma) \) is one-dimensional, and so \( p_2 = 2 - 1 = 1 \) (every two form is in the kernel of \( d \)). \( H^2 \) is spanned over \( \mathbb{C} \) by \( \delta_\varepsilon e_1^2 \) (or \( \delta_\varepsilon e_1^2 \)). \( \diamond \)

Since we are working modulo an overall change of basis including normalisation, only \( \alpha^{-1}\beta \) is significant, so

\[ \text{Vielbeins}_{2,1,\text{univ},1} = \mathbb{C}^* \]

Next we look at spin connections. For group \( G \) we assume a symmetric group acting in the only nontrivial possibility, the sign representation on \( e_1 \). Thus \( f^i \triangleright e_1 = 0 \) if the permutation \( i \) is even and \( f^i \triangleright e_1 = -2e_1 \) if \( i \) is odd. For \( S_2 \) we have only the universal calculus, hence only one \( f^i \) where \( i = (12) \). We write \( A = ae_1 \) for a function \( a \). Then the torsion-free condition becomes

\[ \Theta + \tilde{\Theta} - 2a = 0 \]

which is also the cotorsion-free condition, while the regularity condition is empty. Hence for each 1-bein there is a unique spin connection

\[ a = \frac{\alpha + \beta}{2\alpha\beta} = \frac{\Theta + \tilde{\Theta}}{2} \]

which is a constant function. Its curvature is

\[ F = dA - 2A^2 = 0 \]

which means that the Riemann tensor is also zero. The covariant derivative is

\[ \nabla(f e_1) = df \otimes e_1 + f(\Theta + \tilde{\Theta})e_1 \otimes e_1. \]

For \( S_3 \) we with its 3-dimensional (2-cycles) calculus we have \( f^i \triangleright e_1 = -2e_1 \) and writing the three components functions \( a_1, a_2, a_3 \) of the spin connections in directions
(12), (23), (13), the torsion and cotorsion conditions for any fixed 1-bein become
\[ \Theta + \bar{\Theta} - 2(a_1 + a_2 + a_3) = 0 \]
while the regularity (which does not depend on the representation) is
\[ a_1\bar{a}_2 + a_2\bar{a}_3 + a_3\bar{a}_1 = 0. \]
There are different classes of solutions including a 2-dimensional part of the moduli space of spin connections for a generic 1-bein. The curvature is
\[ F_i = (a_i - \bar{a}_i)(\bar{\Theta} - \Theta)e_1 \wedge e_1 \]
and is typically nonzero if the factor \((\bar{\Theta} - \Theta)(x) = \frac{\alpha - \beta}{\alpha}\) is nonzero. On the other hand, we find (16) again, with zero Riemann curvature.

For \(S_3\) with its 5-dimensional (universal) calculus, a spin connection consists of components \(b_1, b_2\) in the 3-cycles directions which are unconstrained, and \(a_1, a_2, a_3\) in the 2-cycles directions, with the single linear equation
\[ \Theta + \bar{\Theta} - 2(a_1 + a_2 + a_3) = 0 \]
for vanishing of torsion and cotorsion. The regularity condition is empty. So here the moduli space of connections is linear for each vielbein. There is typically curvature at the frame bundle level but again the Riemann curvature vanishes since \(\nabla\) is still given by (14).

We conclude for 2 points that increasing the frame braided Lie algebra allows more spin connections but these do not enter into the Riemannian geometry itself. Instead, we find a unique generalised Levi-Civita type covariant derivative (16) for each einbein, and it has zero Riemannian curvature.

3.2. Three points. For \(\Sigma = \{x, y, z\}\) there are two fibrations for \(n = 1\) and one for \(n = 2\) as explained above.

For \(n = 1\) a vielbein means three invertible numbers \(\{e_x\}, \{e_y\}, \{e_z\}\). However, both types of fibrations for \(n = 1\) imply \(\pi = 0\) as the only solution. This is forced by the conditions in Theorem 2.2 as follows. For the triangular fibration the 2-arcs are
\[ x \rightarrow y \rightarrow z, \ y \rightarrow z \rightarrow x, \ z \rightarrow x \rightarrow y \]
but then condition (ii) requires \(\pi_{x,z}y = 0\), which implies \(\pi = 0\). For the case of the other fibration the 2-arcs are
\[ z \rightarrow x \rightarrow y, \ x \rightarrow y \rightarrow x, \ y \rightarrow x \rightarrow y. \]
In this case condition (ii) requires \(\pi_{z,y}^x = 0\) and hence \(\pi = 0\). Hence for \(n = 1\) only the trivial case \(\pi = 0\) already covered in general above is allowed.

For \(n = 2\) we have only one fibration, which is the universal \(\Omega^1(\Sigma)\). Then a vielbein means in the first place three invertible matrices
\[ e_x = X, \ e_y = Y, \ e_z = Z. \]
Because of the cyclic nature of the graph, we label the columns of \(X\) at \(y, z\), of \(Y\) as \(z, x\) and of \(Z\) as \(x, y\). There are two types of 2-arcs, namely
\[ x \rightarrow y \rightarrow z, \ x \rightarrow z \rightarrow y, \ y \rightarrow x \rightarrow z, \ y \rightarrow z \rightarrow x \]
or
\[ x \rightarrow y \rightarrow x, \ x \rightarrow z \rightarrow x, \ y \rightarrow x \rightarrow y, \ y \rightarrow z \rightarrow y, \ z \rightarrow x \rightarrow y, \ z \rightarrow y \rightarrow x. \]
Finally, we take the ‘naive’ form \(\Omega^1(\Sigma)\) for \(\pi\). The condition (ii) in Theorem 2.2 is empty because \(\Omega^1(\Sigma)\) is universal. Condition (i) gives equations of the form

\[
0 = \pi_{xyz} = \frac{1}{2} (e_1^{-1} e_{xy} e_2^{-1} e_{yz} - e_2^{-1} e_{xy} e_1^{-1} e_{yz}) (e_{1xy} e_{2y'z'} - e_{2xy} e_{1y'z'})
\]

for \(z' \neq z\) and \(x \to y' \to z'\). Similarly for other 2-arcs in place of \(x \to y \to z\). The allowed cases are vanishing of \(\pi_{x'yz}, \pi_{xy'z}, \pi_{xyz}, \pi_{x'yx}, \pi_{xzy}, \pi_{xyz}, \pi_{y'zx}, \pi_{xyz}, \pi_{xyz}\) and the cyclic rotations of \((xyz)\). Finally, keeping in mind the factorisation in the formula for \(\pi\), we define

\[
f(x, y, z) = X_{1y} Y_{2z} - X_{2y} Y_{1z} = X_{11} Y_{21} - X_{21} Y_{11}
\]

\[
f(x, y, x) = X_{1y} Y_{2x} - X_{2y} Y_{1x} = X_{11} Y_{22} - X_{21} Y_{12}
\]

etc. Here the first two entries of \(f\) determine the matrices used, while the second two entries of \(f\) label the indices on the matrices. Similarly, we define \(\bar{f}(x, y, z)\) etc. in the same way but with \(X^{-t}, Y^{-t}, Z^{-t}\) the inverse-transposed matrices. With these notations we see that

\[
\text{vielbeins}_{3, 2, \text{univ}, \text{flip}}
\]

is the variety consisting of 3 invertible matrices \(X, Y, Z\) subject to the relations

\[
0 = \bar{f}(x, y, z) f(x, z, y), \quad 0 = \bar{f}(x, y, z) f(x, y, x), \quad 0 = \bar{f}(x, y, x) f(x, z, y)
\]

\[
0 = \bar{f}(x, z, x) f(x, y, z), \quad 0 = \bar{f}(x, z, x) f(x, z, y), \quad 0 = f(x, z, y) f(x, z, y)
\]

\[
0 = f(x, y, z) f(x, y, x), \quad 0 = f(x, y, x) \bar{f}(x, z, y)
\]

and their cyclic rotations of \((xyz)\), and modulo an overall \(GL_2\).

In principle this could have several cases depending on which factor vanishes in each case. One special case is

\[
f(x, y, z) = 0, \quad f(x, z, y) = 0, \quad \bar{f}(x, y, z) = 0, \quad \bar{f}(x, z, y) = 0
\]

and its cyclic rotations. These equations reduce to

\[
X_{22} Y_{11} = X_{12} Y_{22}, \quad Y_{22} X_{12} = X_{11} Y_{21}, \quad X_{12} Z_{12} = Z_{11} X_{21}, \quad X_{22} Z_{22} = X_{12} Z_{22}.
\]

Up to an overall \(GL_2\), this component of the moduli space of vielbeins has the general solution

\[
X = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}, \quad Y = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix}, \quad Z = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix}
\]

modulo a remaining \(C^* \times C^*\) (e.g. up to \(GL_n\) one can assume \(\alpha_1 = \alpha_2 = 1\). We have

\[
\Theta_a(x) = \alpha_a^{-1}, \quad \Theta_a(y) = \beta_a^{-1}, \quad \Theta_a(z) = \gamma_a^{-1}
\]

and

\[
e_a f = R_a(f) e_a, \quad \partial^a f = (R_a(f) - f) \Theta_a
\]

where we identify \(\Sigma\) with \(Z_3\) and use its addition law according to the conventions above to define \(R_a(f) = f \left( (1 + a) \right)\). For the exterior algebra, by our choice of \(\pi\), the exterior algebra relations are \(e_1 \wedge e_2 = -e_2 \wedge e_1\) and \(e_1^2 = e_2^2 = 0\). We are finally ready to look at compatible spin connections, which we do for groups \(S_2\) and then \(S_3\), with their natural nontrivial representations and calculi. Note that since we
have already chosen a diagonal form of the vielbein moduli space, different actions are not all equivalent.

**Proposition 3.2.** For any values of $\Theta_1, \Theta_2$ the dimensions $p_i$ of $H^i$ are $1 : 2 : 1$. Here

$$H^0 = \mathbb{C}.1, H^1 = \mathbb{C}. < -\Theta_1 \delta_y e_1 + \Theta_2 \delta_z e_2, -\Theta_1 \delta_x e_1 + \Theta_2 \delta_y e_2 >, H^2 = \mathbb{C}. e_1 \wedge e_2$$

**Proof.** Since in this case $R_2 = R_1^{-1}$ we have

$$d(df) = (\Theta_1 R_1 \Theta_2 - \Theta_2 R_2 \Theta_1)(R_1 R_2 f - f) = 0, \forall f \in \mathbb{C}[\Sigma]$$

and $d$ is cohomological. By observing that $dc = 0$ for any constant $c$, we have $p_0 = 1$ and the dimension of the image of $d$ in $\mathbb{C}[\Sigma] \otimes \Omega^1(\Sigma)$ (which is itself of dimension 6) is 2. If $\omega = f e_1 + g e_2$ is a one form, then $d\omega = 0$ if and only if $(-\Theta_2 \delta^2 f + f \tilde{\partial}^i \Theta_2 + \Theta_1 \tilde{\partial}^i g - g \tilde{\partial}^2 \Theta_1) = 0$. This equation admits a 4 dimensional space of solutions, therefore $p_1 = 4 - 2 = 2$. Moreover the image of $d$ in the two-forms is of dimension 2, and given that $d$ sends every two-form to zero, one obtains $p_2 = 3 - 2 = 1$. $H^1$ is spanned by $< -\Theta_1 \delta_y e_1 + \Theta_2 \delta_z e_2, -\Theta_1 \delta_x e_1 + \Theta_2 \delta_y e_2 >$, and $H^2 = \mathbb{C}. e_1 \wedge e_2$.

For $S_2$ with its universal calculus, we choose the natural action on $i = (12)$ on $V = \text{span}\{e_1, e_2\}$ that flips the basis vectors (hence by orientation reversal of the frame). The invariant metric here is

$$\eta = e_1 \otimes e_1 + e_2 \otimes e_2$$

and the action of the braided-Lie algebra generator of $S_2$ is $f^i \triangleright e_1 = e_2 - e_1$ and $f^i \triangleright e_2 = e_1 - e_2$. Let us denote by $\tilde{\partial}^i \equiv R_i - \text{id}$ the usual finite difference on the group $\mathbb{Z}_3$, and $\langle \rangle$ denotes the average value over the three points.

**Proposition 3.3.** For 3 points, 2-dimensional cotangent space and $S_2$ frame group, existence of a torsion free cotorsion free connection requires the zweibein to obey

$$\Theta_1 + R_1 \Theta_2 = (\Theta_1 + \Theta_2).$$

In this case there is a 1-parameter family of connections of the form

$$A = (\Theta_1 - \lambda) e_1 + (-R_2 \Theta_1 + \lambda) e_2$$

for an arbitrary constant $\lambda$. The covariant derivative is

$$\nabla e_1 = -\nabla e_2 = ((\Theta_1 - \lambda) e_1 + (-R_2 \Theta_1 + \lambda) e_2) \otimes (e_1 - e_2)$$

Its Riemannian and Ricci curvatures are

$$\text{Riemann}(e_1) = -\text{Riemann}(e_2) = \rho e_1 \wedge e_2 \otimes (e_2 - e_1), \text{ Ricci} = \frac{\rho}{2} (e_1 + e_2) \otimes (e_1 - e_2).$$

where

$$\rho = (2\lambda - (\Theta_1 + \Theta_2)) \tilde{\partial}^2 \Theta_1.$$

The Ricci scalar vanishes identically.

**Proof.** Writing a spin connection $A = ae_1 + be_2$, the torsion and cotorsion equations reduce to

$$\tilde{\partial}^i \Theta_2 = \tilde{\partial}^i \Theta_1 = -(a + b) = R_2(a) + R_1(b)$$

and there is no regularity condition since the calculus on $S_2$ is universal. The third of these equations has solution $b = -R_2(a)$ since $(\text{id} + R_1)$ is invertible on $\mathbb{Z}_3$. The
full system for a vielbein and spin connection then reduces to invertibility of the \( \Theta_i \) values, the stated constraint on the zweibein, and

\[
a = \Theta_1 - \lambda, \quad b = -R_2 \Theta_1 + \lambda
\]

for an arbitrary constant \( \lambda \). A straightforward computation then gives the curvature as

\[
F = dA - A^2 = \rho e_1 \wedge e_2
\]

for \( \rho \) as stated, which Riemann curvature as the action of \( F \). One may also compute this directly from the covariant derivative stated. Finally, for the antisymmetrization projector that we use, the lifting map \( i \) is

\[
i(e_1 \wedge e_2) = \frac{1}{2} (e_1 \otimes e_2 - e_2 \otimes e_1).
\]

Using this to lift the 2-form values of the Riemann tensor and contracting as in (17) we obtain the Ricci tensor as stated. Its further contraction by the inverse metric is then zero. ⊗

We see among other things that \( \Theta_2 \) is determined up to a constant from \( \Theta_1 \), i.e. not every zweibein is allowed. On the other hand, for a generic allowed zweibein we have zero full curvature for a unique spin connection in the family, given by \( \lambda = \frac{1}{2}(\Theta_1 + \Theta_2) \). Otherwise the curvatures are nonzero.

For \( S_3 \) with its standard 2-dimensional irreducible representation and 2-cycles calculus, we have now \( i = (12), (23), (13) \) (as \( i \) ranges 1,2,3) with the above flip action of (12) extended to a permutation of \( e_1, e_2, e_3 \equiv -e_1 - e_2 \). The invariant metric is

\[
\eta = e_1 \otimes e_1 + e_2 \otimes e_2 + \frac{1}{2}(e_1 \otimes e_2 + e_2 \otimes e_1)
\]

and the action of \( S_3 \) on the vielbein is

\[
\begin{align*}
f^1 &\mapsto e_1 = e_2 - e_1, & f^2 &\mapsto e_1 = 0, & f^3 &\mapsto e_1 = -2e_1 - e_2 \\
f^1 &\mapsto e_2 = e_1 - e_2, & f^2 &\mapsto e_2 = -e_1 - 2e_2, & f^3 &\mapsto e_2 = 0.
\end{align*}
\]

**Proposition 3.4.** For 3 points, 2-dimensional cotangent space and \( S_3 \) frame group, the zweibein is unconstrained and the torsion free cotorsion free connections are of the form

\[
A_i = a_i e_1 + b_i e_2; \quad a_1 = a, \quad b_1 = b
\]

\[
a_2 = \frac{1}{2}(\Theta_1 - a), \quad b_2 = R_2 \Theta_1 + b, \quad a_3 = R_1 \Theta_2 + a, \quad b_3 = \frac{1}{2}(\Theta_2 - b)
\]

for arbitrary functions \( a, b \) and constants \( \lambda, \mu \). Here \( \Theta_1 \equiv \Theta_1 - \lambda \) and \( \Theta_2 \equiv \Theta_2 - \mu \) are notations. For a regular connection we would need in addition:

\[
a R_1 b_2 - b R_2 a_2 + a_3 R_1 b - b_3 R_2 a + a_2 R_1 b_3 - b_2 R_2 a_3 = 0
\]

\[
a R_1 b - b_2 R_2 a + a R_1 b_3 - b R_2 a_3 + a_3 R_1 b_2 - b_3 R_2 a = 0.
\]

The covariant derivative for the connection is

\[
\nabla e_1 = (3a + 2R_1 \Theta_2) e_1 \otimes e_1 + R_1 \Theta_2 e_1 \otimes e_2 + \Theta_2 e_2 \otimes e_1 + \frac{1}{2}(\Theta_2 - 3b)e_2 \otimes e_2
\]

\[
\nabla e_2 = \frac{1}{2}(\Theta_1 - 3a) e_1 \otimes e_1 + \Theta_2 e_1 \otimes e_2 + R_2 \Theta_1 e_2 \otimes e_1 + (3b + 2R_2 \Theta_1) e_2 \otimes e_2
\]
Proof. The torsion equations for a spin connection with components \( a_i, b_i \) are
\[
\bar{\partial}^i \Theta_2 + a_1 + b_1 + 2b_3 - a_3 = 0, \quad -\bar{\partial}^2 \Theta_1 - a_1 - b_1 + b_2 - 2a_2 = 0
\]
and the cotorsion equations
\[
\bar{\partial}^i \Theta_2 - (R_2 a_1 + R_1 b_1 - R_2 a_3 + 2R_1 b_3) = 0
\]
\[
-\bar{\partial}^2 \Theta_1 + R_1 b_1 + R_2 a_1 - R_1 b_2 + 2R_2 a_2 = 0.
\]
By combining these equations and using similar methods as in the previous \( S_2 \) examples, one finds that their general solution is of the form:
\[
\Theta_1 = 2a_2 + a_1 + \lambda, \quad R_1(\Theta_2) = a_3 - a_1 + \mu
\]
\[
b_1 + 2b_3 = R_2(a_3 - a_1), \quad 2b_3 + b_2 = R_2(2a_2 + a_3)
\]
for some constants \( \lambda, \mu \). This means that for a fixed vielbein and constants \( \mu, \lambda \) the equations for a connection are solved as stated. One then writes out the covariant derivative and the optional regularity condition. \( \ast \)

We can see here (and also in our previous examples) why full metric compatibility \( \nabla \eta = 0 \) is too strong in finite noncommutative geometry (which is why we need our weaker cotorsion-free condition):

**Proposition 3.5.** The covariant derivatives above do not fully preserve the metric unless \( a = b = 0 \) and \( \Theta_1 = \lambda, \Theta_2 = \mu \) are constant.

Proof. We compute
\[
\nabla \eta = \left( \frac{a}{2} + 4R_1 \Theta_2 + \frac{R_1}{2} \right)e_1 \otimes e_1 \otimes e_1 + \left( 2R_1 \Theta_2 + \Theta_1 \right)e_1 \otimes e_1 \otimes e_2
\]
\[
+ \left( 2R_1 \Theta_2 + \Theta_1 \right)e_2 \otimes e_2 \otimes e_1 + \left( 2R_1 \Theta_2 + R_2 \Theta_1 \right)e_2 \otimes e_1 \otimes e_1
\]
\[
+ \left( 2R_2 \Theta_1 \right)e_1 \otimes e_1 \otimes e_2 + \left( 2R_2 \Theta_1 \right)e_2 \otimes e_1 \otimes e_2 + \left( \frac{a}{2} - 4R_2 \Theta_1 + \Theta_1 \right)e_2 \otimes e_2 \otimes e_2
\]
For this to be zero forces \( a = b = \Theta_1 = \Theta_2 = 0 \) which translates as stated since \( \lambda, \mu \) are arbitrary. \( \ast \)

One may proceed to compute the curvatures etc. for a general solution. Here we present the results for the special case where the zweibein is constant with \( \Theta_1 = \lambda, \Theta_2 = \mu \) say, but the \( a, b \) are arbitrary, i.e. the flat background but not flat spin connection case.

**Proposition 3.6.** For constant zweibein but \( a, b \) arbitrary, the Riemann and Ricci curvatures take the form
\[
\text{Riemann}(e_1) = (3\partial^2 a - \partial^1 b)e_1 \otimes e_2 \otimes e_1 - 2\partial^1 be_1 \otimes e_2 \otimes e_2
\]
\[
\text{Riemann}(e_2) = -\partial^2 ae_1 \otimes e_2 \otimes e_1 + 3\partial^1 be_1 \otimes e_2 \otimes e_2
\]
\[
\text{Ricci} = -\frac{3}{4} \partial^2 ae_1 \otimes e_1 + \frac{3}{2} \partial^1 be_1 \otimes e_2 - \frac{1}{2} (3\partial^2 a - \partial^1 b)e_2 \otimes e_1 + \partial^1 be_2 \otimes e_2.
\]
The Ricci scalar vanishes identically. The regularity condition is
\[
aR_1(b) = 0.
\]
Proof. We compute the gauge curvature of the spin connection as

\[ F_1 = (\mu \bar{\partial}^2 a + \lambda \bar{\partial}^1 b)e_1 \wedge e_2 \]
\[ F_2 = -\left(\frac{1}{2} \mu \bar{\partial}^2 a - \lambda \bar{\partial}^1 b\right)e_1 \wedge e_2 \]
\[ F_3 = -\left(-\mu \bar{\partial}^2 a + \lambda \bar{\partial}^1 b\right)e_1 \wedge e_2. \]

Its action on the zweibein then determines the Riemann curvature as stated, using (8). We recall that \( \partial_i = \Theta_i \bar{\partial}^i \) is the geometrical partial derivative defined by \( d \) and we revert to this. We use the same lifting map as in Proposition 3.1 and (7) to find

\[ \text{Ricci} = \frac{1}{2} \begin{pmatrix} F_2 - F_1 & F_1 + 2F_2 \\ -F_1 - 2F_3 & F_1 - F_3 \end{pmatrix} \]

in the \( e_i \) basis. This gives the result stated. Finally, note that the inverse of the matrix in \( \eta \) is

\[ \eta^{-1} = \frac{4}{3} \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix} \]

in the dual basis and it is this which we use to contract against the Ricci tensor to obtain the Ricci scalar. Independently of the details of \( F_1 \), we have this as \( \frac{2}{3}(F_2 - F_1 + F_1 - F_3 - \frac{1}{2}(F_1 + 2F_2 - F_1 - 2F_3)) = 0 \), i.e. vanishes identically. \( \diamond \)

The general case may be worked out in the same way: the formulae for the \( F_i \) are rather more complicated functions of the \( a, b, \lambda, \mu, \Theta_i \), but the other steps follow the same pattern. In particular, the Ricci tensor has the same asymmetric form and the Ricci scalar vanishes in general. We see that with 3 points, the conditions with frame group \( S_2 \) are a little strong and constrain the zweibein, while with \( S_3 \) there is an abundance of spin connections compatible with any zweibein, namely \( a, b \) arbitrary (and two further parameters which one might fix for example by \( \lambda = \langle \Theta_1 \rangle \) and \( \mu = \langle \Theta_2 \rangle \)) and that in all cases with three points, the Ricci scalar vanishes. Note that we have not covered it here, but one has a similar picture for \( S_3 \) with its universal calculus; then there are five 1-forms \( A_i \) for the spin connection with linear equations for the torsion and cotorsion that prescribe the derivatives of \( \Theta_1, \Theta_2 \) in terms of the fields, and an empty equation for regularity.

4. Geometries on 4 points

For four points we will not be fully general as above but restrict to the more interesting class of models featuring already in our analysis for 2,3 points. First of all, we shall focus on the case of all arrows bidirectional, i.e. a symmetric subset \( E \) to define the calculus. This means for four points that we have (a) the square connectivity which is a 2-dimensional calculus or (b) the universal or 3-dimensional calculus. We look mainly at the former since it has a clear geometrical interpretation as the connectivity of a torus, namely in Sections 4.1-4.3. Indeed, this is the natural calculus for the group \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) viewed as a discrete model of a torus. Section 4.4 covers the alternative of the universal calculus on the basis which has the connectivity of a tetrahedron or discrete model of a sphere.

Next, rather than the full analysis, we shall restrict attention to the diagonal vielbeins given by scalars attached to the edges as we deduced up to equivalence for the 3 points case above. These scalars are our remaining continuous degrees of freedom and allow our square to ‘pulse’ by stretching or contracting edges. Such
a restricted class is interesting for any fixed combinatorics. On the other hand, we will have more choices for the frame group and its calculus, still giving several models.

4.1. **Discrete torus as base space.** Thus, in this section, and the next two, we write the vertices as $\Sigma = \{(0,0), (1,0), (0,1), (1,1)\}$, using an additive group notation.

Over each point we have a fiber

$$F_{(0,0)} = \{(1,0), (0,1)\}, \quad F_{(1,0)} = \{(0,0), (1,1)\}, \quad F_{(0,1)} = \{(0,0), (1,1)\}, \quad F_{(1,1)} = \{(1,0), (0,1)\}$$

of order 2. We fix the connectivity by identifying these fibers by vielbeins of the diagonal form

$$e_1^x, y = \Theta_1^{x} \delta_{y-x}, (1,0), \quad e_2^x, y = \Theta_2^{x} \delta_{y-x}, (0,1)$$

for any two points $x, y \in \Sigma$. This is the natural vielbein on $\mathbb{Z}_2 \times \mathbb{Z}_2$ with additional continuous nowhere-zero functional parameters $\Theta_i$. We have the picture

```
(0,0) -------- (0,1)
          |          |
          |          |
          |          |
(1,1) -------- (1,0)
```

From each point in the lattice it is possible to move in two directions, which correspond to the vectors $e_1$ and $e_2$. Here $e_1$ translates adding the element $(1,0)$ of $\mathbb{Z}_2 \times \mathbb{Z}_2$, and $e_2$ corresponds to moving by adding $(0,1)$. We define the translation operators acting on the functions $f$ as

$$(R_1 f)(x) = f(x + (1,0)), \quad (R_2 f)(x) = f(x + (0,1))$$

and obtain the partial derivatives

$$\partial^1 f(x) = (f(x + (1,0)) - f(x)) \Theta_1, \quad \partial^2 f(x) = (f(x + (0,1)) - f(x)) \Theta_2$$

and commutation relations (which define the right multiplication on $\Omega^1(M)$, left multiplication being the obvious one) as:

$$e_1 f = R_1(f) e_1, \quad e_2 f = R_2(f) e_2$$

for all functions $f$. This has the same form as we found up to $GL_2$ for three points in the preceding section and completes our description of $\Omega^1$ and its $\mathbb{C}[\mathbb{Z}_2 \times \mathbb{Z}_2]$-basis $\{e_1, e_2\}$.

Next we fix the projector $\pi$ as the naive antisymmetrizer $\{e_1\}$ on the $e_1$ basis, again as we took in Section 3.2 for three points. We check that for four points it obeys the condition needed in Theorem 2.2 to define the 2-forms $\Omega^2$. Thus, labelling the points as $x, y, z, t$ where we start at $(0,0)$ and go around the square clockwise,
we have the possible 2 arcs of two different forms:

\[
\begin{align*}
    &x \rightarrow y \rightarrow z \quad y \rightarrow z \rightarrow t \\
    &x \rightarrow t \rightarrow z \quad y \rightarrow x \rightarrow t \\
    &z \rightarrow t \rightarrow x \quad t \rightarrow x \rightarrow y \\
    &z \rightarrow y \rightarrow x \quad t \rightarrow z \rightarrow y
\end{align*}
\]

or

\[
\begin{align*}
    &x \rightarrow y \rightarrow x \quad y \rightarrow z \rightarrow y \\
    &z \rightarrow y \rightarrow z \quad x \rightarrow t \rightarrow x \\
    &y \rightarrow x \rightarrow y \quad z \rightarrow t \rightarrow z \\
    &t \rightarrow x \rightarrow t \quad t \rightarrow z \rightarrow t
\end{align*}
\]

to check out condition (i) in Theorem 2.2 we will consider the two-arcs leaving from \( x \), the starting point being irrelevant in the reasoning which follows. This implies that we have to verify the vanishing of \( \pi_{xyz}^{yz} \), \( \pi_{xyz}^{xe} \), \( \pi_{xyz}^{yz} \), \( \pi_{xyz}^{yx} \). Now, in general

\[
\pi_{xyz}^{yz} = (e_1^{−1}xy_e z^{−1}yz − e_2^{−1}xy_e 1^{−1}yz)(e_1xy_e 2^{−1}yz − e_2^{−1}xy_e 1^{−1}yz)
\]

replacing in this expression the actual form of the arc, we establish that

\[
\pi_{xyz}^{yz}, \pi_{xyz}^{xe}, \pi_{xyz}^{yz}, \pi_{xyz}^{yx}
\]

are zero. (We obtain a similar result swapping upper and lower indices). The second constraint of Theorem 2.2, is not trivially satisfied in this case. The conditions \( \pi_{x,z}y_t + \pi_{x,z}t_y = 0 \) and \( \pi_{x,z}y_t + \pi_{x,z}t_y = 0 \) both give

\[
(18) \quad \Theta_1 R_1 \Theta_2 = \Theta_2 R_2 \Theta_1, \quad \text{i.e.,} \quad \partial_1 \theta_2 − \partial_2 \theta_1 = 0.
\]

Finally, we will take as a metric the element \( \eta = e_1 \otimes e_1 + e_2 \otimes e_2 \) and we will take the lifting (17) which is the natural choice for the antisymmetrizer projector. The exterior differentials of the base elements are:

\[
\begin{align*}
    &d e_1 = \bar{\partial} \Theta_2 e_1 \wedge e_2, \quad d e_2 = − \bar{\partial}^2 \Theta_1 e_1 \wedge e_2
\end{align*}
\]

where we recall that \( \bar{\partial}^a = \Theta_a^{-1} \partial^a = R_a − \text{id} \) are the usual group finite differences.

**Proposition 4.1.** For generic values of \( \Theta_1, \Theta_2 \) the dimensions \( p_i \) of \( H^i \) are \( 1 : 2 : 1 \).

Here

\[
H^0 = \mathbb{C}.1, \quad H^1 = < (\Theta_2(y) \delta_z + \Theta_2(z) \delta_y) e_1, (\Theta_2(z) \delta_x + \Theta_2(y) \delta_t) e_2 >, \quad H^2 = \mathbb{C}. e_1 \wedge e_2
\]

**Proof.**

\[
d(df) = (\Theta_1 R_1 \Theta_2 − \Theta_2 R_2 \Theta_1)(R_1 R_2 f − f) = 0, \forall f \in \mathbb{C}[\Sigma]
\]

which is zero due to the constraint we imposed on the \( \Theta \)s. In the usual way, \( p_0 = 1 \), and the dimension of the image of \( d \) in \( \mathbb{C}[\Sigma] \otimes \Omega^1(\Sigma) \) (itself of dimension 8) is 3. A one form \( fe_1 + ge_2 \) is in the nullspace of \( d \) if and only if it satisfies

\[
(−\Theta_2 \bar{\partial}^2 f + f \bar{\partial} \Theta_2 + \Theta_2 \bar{\partial}^1 g − g \bar{\partial} \Theta_1) = 0
\]

it’s easy to find that the solution space of this equation has, for generic values of \( \Theta_1 \) satisfying (18), dimension 5 (and therefore, \( p_1 = 5 − 3 = 2 \)). This, in turn, implies the image of \( d \) inside \( \mathbb{C}[\Sigma] \otimes \Omega^2(\Sigma) \) is 3 dimensional. Then, \( p_2 = 4 − 3 = 1 \). For generic values of \( \Theta, e_1 \wedge e_2 \) is not in the image of \( d \), and gives a representative for \( H^2 \). \( \diamond \)

For the remaining aspects of the geometry we fix the frame group and its calculus. Then we can solve for the connections, curvature etc. Even with all of the above choices, we have several models.
4.2. Torus model with $\mathbb{Z}_4 \subset SO(2)$ frame group. Here we think of the additive group $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ as a discrete model of $SO(2)$, i.e. 90-degree notations. This is in keeping with our discrete model of a torus. We have on $\mathbb{Z}_4$ either the 3-dimensional universal calculus or the natural 2-dimensional calculus given by a square.

4.2.1. 3D calculus on $\mathbb{Z}_4$. We choose the three-dimensional (universal) calculus defined on $\mathbb{Z}_4$ by $\{1, 2, 3\}$. This corresponds to $f^1, f^2, f^3$ acting as the corresponding rotation of the vielbein vectors $e_1$ and $e_2$, minus the identity, that is

$$f^1 \triangleright e_1 = e_2 - e_1, \quad f^1 \triangleright e_2 = -e_1 - e_2$$
$$f^2 \triangleright e_1 = -2e_1, \quad f^2 \triangleright e_2 = -2e_2$$
$$f^3 \triangleright e_1 = -e_1 - e_2, \quad f^3 \triangleright e_2 = e_1 - e_2.$$

**Proposition 4.2.** The moduli space of torsion free cotorsion free connections on the quantum Riemannian manifold above is given by:

$$A_1 = \alpha e_1 + \beta e_2$$
$$A_2 = \frac{1}{2}(-\alpha + \beta - \gamma - \delta - \overrightarrow{\Theta}_1)e_1 + \frac{1}{2}(-\alpha - \beta + \gamma - \delta - \overrightarrow{\Theta}_2)e_2$$
$$A_3 = \gamma e_1 + \delta e_2$$

for four functions $\alpha, \beta, \gamma, \delta$ subject to the linear constraint

$$(R_1 + R_2)a = 0, \quad (R_1 + R_2)b = 0$$

where $a = \gamma - \alpha$ and $b = \beta - \delta$. The corresponding covariant derivative is

$$\nabla e_1 = (b - \overrightarrow{\Theta}_1)e_1 \otimes e_1 + ae_1 \otimes e_2 + (a - \overrightarrow{\Theta}_2)e_2 \otimes e_1 - be_2 \otimes e_2$$
$$\nabla e_2 = -ae_1 \otimes e_1 + (b - \overrightarrow{\Theta}_1)e_1 \otimes e_2 + be_2 \otimes e_1 + (a - \overrightarrow{\Theta}_2)e_2 \otimes e_2$$

**Proof.** We want the connection to be torsion free, i.e., it has to satisfy the following two linear equations:

$$A_1^1 + A_1^2 + 2A_2^1 - A_3^1 + A_3^2 = -\overrightarrow{\Theta}_2$$
$$-A_1^2 + A_2^1 - 2A_1^1 - A_3^1 - A_3^2 = \overrightarrow{\Theta}_1$$

from which we obtain the general solution above (without constraints on $\alpha, \beta, \gamma, \delta$). In conformity to what we have done so far, we also demand that the cotorsion of the connection be zero. Notice that, differently from the previous cases investigated in this paper, the elements of the fibre group $\mathbb{Z}_4$ are not of order 2, which implies that the action of $S$ on the $f^i$s is not trivial; we have infact $S^{-1}(f^1) = f^3, S^{-1}(f^2) = f^2, S^{-1}(f^3) = f^1$, and the zero-cotorsion condition can be put down (following [?]) as

$$-R_1 A_1^2 + R_2 A_1^1 - 2R_1 A_2^1 - R_2 A_1^1 - R_1 A_3^2 = -\overrightarrow{\Theta}_2$$
$$R_1 A_1^2 + R_2 A_1^1 + 2R_2 A_1^2 - R_1 A_3^1 + R_2 A_3^2 = \overrightarrow{\Theta}_1$$

The requirement on the cotorsion translates into the constraint $(R_1 + R_2)a = (R_1 + R_2)b = 0$, where $a, b$ are defined in the statement of the proposition. The regularity condition is empty, because the calculus on $\mathbb{Z}_4$ is the universal one. We then compute the covariant derivative using $\nabla$.

Note that the covariant derivative depends only on $a, b$ so these parametrize the ‘physical’ or effective moduli space, which is therefore 4 dimensional: two functions on $\mathbb{Z}_2 \times \mathbb{Z}_2$ modulo the linear constraint. One may check that the torsion is indeed zero, which is to say $\nabla \wedge e_1 = de_1$ and $\nabla \wedge e_2 = de_2$. The cotorsion condition means
that $\nabla$ respects $\eta$ in a skew sense, as one may also directly verify evaluating the cotorsion of the metric (when the torsion is null)

$$\Gamma\eta = (\nabla \wedge \text{id} - \text{id} \wedge \nabla)(e_1 \otimes e_1 + e_2 \otimes e_2) = 0$$

where

$$(\nabla \wedge \text{id})\eta = \overline{\partial}^1 \Theta_2 e_1 \wedge e_2 \otimes e_1 - \overline{\partial}^2 \Theta_1 e_1 \wedge e_2 \otimes e_2$$

and

$$(\text{id} \wedge \nabla)\eta = -(R_1 + R_2)a + \overline{\partial}^1 \Theta_2 e_1 \wedge e_2 \otimes e_1 - ((R_1 + R_2)b + \overline{\partial}^2 \Theta_1)e_1 \wedge e_2 \otimes e_2$$

(later on we will consider $\nabla\eta = 0$ in the usual full sense).

**Proposition 4.3.** The Ricci scalar for the covariant derivative above is:

$$R = \overline{\partial}^1 b + \overline{\partial}^2 a + \overline{\partial}^3 R_1 b + \overline{\partial}^1 \Theta_2 R_2 a - 2bR_1 b - 2aR_2 a$$

**Proof.** From the action of the $f^i$ we can then compute the Riemann curvature using the general theory in Section 2, finding now

$$\begin{align*}
\text{Riemann}(e_1) &= -(F_1 - 2F_2 - F_3) \otimes e_1 + (F_1 + F_3) \otimes e_2 \\
\text{Riemann}(e_2) &= -(F_1 + F_3) \otimes e_1 + (F_1 - 2F_2 - F_3) \otimes e_2
\end{align*}$$

(Where we are denoting, for brevity, the coefficient functions of $e_1 \wedge e_2$ in the curvature components by $F_i$). In the same way the Ricci tensor is:

$$\text{Ricci} = \frac{1}{2}((F_1 - F_3) e_1 \otimes e_1 + (F_1 + 2F_2 + F_3) e_1 \otimes e_2 - (F_1 + 2F_2 + F_3) e_2 \otimes e_1 + (F_1 - F_3) e_2 \otimes e_2),$$

where we identify the 2-forms $F_i$ with their scaler coefficients as multiples of the top form $e_1 \wedge e_2$. and, taking the trace in a standard way, the Ricci scalar is $R = F_1 - F_3$

(this and the other component $F_1 + 2F_2 + F_3$ occur also in the Riemann tensor so we see that the Ricci tensor vanishes if and only if the entire Riemann tensor does). We can compute $F_1, F_2, F_3$ by means of:

$$\begin{align*}
F_1 &= dA_1 + A_2 \wedge A_3 + A_3 \wedge A_2 - 2A_1 \wedge A_1 - A_2 \wedge A_2 - A_3 \wedge A_3 \\
F_2 &= dA_2 + A_1 \wedge A_3 + A_3 \wedge A_1 - A_1 \wedge A_2 - A_2 \wedge A_1 - 2A_2 \wedge A_2 - A_3 \wedge A_3 \\
F_3 &= dA_3 + A_2 \wedge A_1 + A_1 \wedge A_2 - A_3 \wedge A_3 - A_2 \wedge A_3 - A_3 \wedge A_2 - 2A_3 \wedge A_3
\end{align*}$$

as inferred from (18) where

$$\begin{align*}
dA_1 &= (-\Theta_2 \overline{\partial}^1 \alpha + a \overline{\partial}^1 \Theta_2 + \Theta_1 \overline{\partial}^1 \beta - \beta \overline{\partial}^2 \Theta_1)e_1 \wedge e_2 \\
dA_2 &= \frac{1}{2}(\Theta_1 \overline{\partial}^1 (-\alpha - \beta + \gamma - \delta) + \Theta_2 \overline{\partial}^2 (\alpha - \beta + \gamma + \delta) \\
&\quad + \overline{\partial}^1 \Theta_2 (-\alpha + \beta - \gamma - \delta + 2\Theta_1) + \overline{\partial}^2 \Theta_1 (\alpha + \beta - \gamma + \delta - 2\Theta_2))e_1 \wedge e_2 \\
dA_3 &= (-\Theta_2 \overline{\partial}^1 \gamma + \gamma \overline{\partial}^1 \Theta_2 + \Theta_1 \overline{\partial}^1 \delta - \delta \overline{\partial}^2 \Theta_1)e_1 \wedge e_2
\end{align*}$$

and

$$\begin{align*}
A_1 \wedge A_1 &= \alpha R_1 \beta - \beta R_2 \alpha \\
A_1 \wedge A_2 &= \frac{a}{2}(-R_1 \alpha - R_1 \beta + R_1 \gamma - R_1 \delta + \overline{\partial}^1 \Theta_2) + \frac{b}{2}(R_2 \alpha - R_2 \beta + R_2 \gamma + R_2 \delta - \overline{\partial}^2 \Theta_1)
\end{align*}$$
The Ricci scalar is given by
\[ R = -\partial^2 a + \partial^1 b + \alpha(-R_1 \beta - R_1 \delta - R_2 \alpha + R_2 \gamma + \frac{\partial^1 \Theta_2}{2}) \]
\[ + \beta(R_2 \alpha + R_2 \gamma + R_2 \beta - R_2 \delta - \frac{\partial^2 \Theta_1}{2}) + \gamma(-R_1 \delta - R_1 \beta - R_1 \alpha + R_1 \gamma + \frac{\partial^1 \Theta_2}{2}) \]
\[ + \delta(R_1 \beta - R_1 \delta + R_2 \gamma + R_2 \alpha - \frac{\partial^1 \Theta_1}{2} + R_1(\beta - \delta) - \frac{\partial^1 \Theta_2}{2} R_2(\alpha - \gamma) \]
\[ F_1 = -\partial^2 \alpha + \partial^1 \beta + \alpha(-R_1 \beta - R_1 \delta - R_2 \alpha + R_2 \gamma + \frac{\partial^1 \Theta_2}{2}) \]
\[ + \beta(R_2 \alpha + R_2 \gamma + R_2 \beta - R_2 \delta - \frac{\partial^2 \Theta_1}{2}) + \gamma(-R_1 \delta - R_1 \beta - R_1 \alpha + R_1 \gamma + \frac{\partial^1 \Theta_2}{2}) \]
\[ + \delta(R_1 \beta - R_1 \delta + R_2 \gamma + R_2 \alpha - \frac{\partial^1 \Theta_1}{2} + \partial^2 \partial \Theta_2 + \partial^2 \Theta_1 - R_2 \gamma) \]
\[ F_2 = \frac{1}{2} \partial^1(-\alpha - \beta + \gamma - \delta) + \frac{1}{2} \partial^2(\alpha - \beta + \gamma + \delta) \]
\[ + \frac{\partial^1}{2}(\partial^1 \Theta_2 - R_2(\beta - \delta) - \alpha + \beta - \gamma - \delta) + \frac{\partial^2}{2}(\partial^1 \Theta_1 - R_1(\alpha - \gamma) + \alpha + \beta - \gamma + \delta) \]
\[ + \frac{\alpha}{2}(3R_1 \beta - R_2(\beta - \delta) - \partial^1 \Theta_1 + R_1 \delta) + \frac{\beta}{2}(-3R_2 \alpha + \partial^2 \Theta_2 - R_2 \gamma) \]
\[ + \frac{\gamma}{2}(3R_1 \delta + R_2(\beta - \delta) + \partial^2 \Theta_1 + R_1 \beta) + \frac{\delta}{2}(-3R_2 \gamma - R_1 \alpha + \partial^1 \Theta_2 - R_2 \alpha) \]
\[ F_3 = -\partial^2 \gamma + \partial^1 \delta + \frac{\partial^1}{2}R_2(\alpha - \gamma) - \frac{\partial^2}{2}R_1(\beta - \delta) \]
\[ + \alpha(\partial R_1 \delta - R_1 \alpha + R_1 \gamma - R_1 \beta + \frac{\partial^1 \Theta_2}{2}) \]
\[ + \beta(R_2 \alpha + R_2 \gamma + R_1 \beta - R_1 \delta - \frac{\partial^1 \Theta_2}{2}) + \gamma(-R_1 \delta + R_1 \alpha - R_1 \gamma + \frac{\partial^1 \Theta_2}{2} - R_1 \beta) \]
\[ + \delta(R_2 \gamma + R_2 \alpha + R_2 \beta - R_2 \delta - \frac{\partial^2 \Theta_2}{2}) \]
from which we compute the Ricci curvature etc. as above, and write in terms of \( a, b \).

It is useful to observe that it’s not mandatory to compute the curvature two-form in order to get hold of the Riemann tensor. One could also use the formula
\[ \text{Riemann}(e_1) = ((\text{id} \wedge \nabla) - (d \otimes \text{id})) \circ \nabla(e_1) \]
and similarly for \( e_2 \), which provides a useful check. Either way, the Riemann tensor turns out to have the form
\[ \text{Riemann}(e_1) = \rho e_1 \wedge e_2 \otimes e_1 + R e_1 \wedge e_2 \otimes e_2 \]
\[ \text{Riemann}(e_2) = -R e_1 \wedge e_2 \otimes e_1 + \rho e_1 \wedge e_2 \otimes e_2 \]
with
\[ \rho = \partial^2 b - \partial^1 a + 2bR_1 a - 2aR_2 b - \partial^1 \Theta_1 R_1 a + \partial^2 \Theta_2 R_2 b \]
and \( R \) the Ricci scaler computed above. We see in particular that \( a = b = 0 \) is a natural point in the effective moduli space where the Ricci tensor (and the entire curvature) is zero.

Next we consider full metric compatibility as opposed to the weaker cotorsion condition.

**Theorem 4.4.** The metric \( \eta \) satisfies the equation \( \nabla \eta = 0 \) if and only if
\[ a = \partial^1 \Theta_2, \quad b = \partial^2 \Theta_1, \quad \partial^1(\partial^1 \Theta_2) = 0, \quad \partial^2(\partial^2 \Theta_1) = 0 \]
The Ricci scalar is given by
\[ R = -(\partial^2 \Theta_2)^2 - (\partial^1 \Theta_1)^2 \]
The torsion free, cotorsion free connections is given by:

\[ \nabla(e_1 \otimes e_1 + e_2 \otimes e_2) = 2((b - \overline{\nabla}^j \Theta_1)e_1 \otimes e_1 \otimes e_1 + (a - \overline{\nabla}^i \Theta_2)e_2 \otimes e_1 \otimes e_1 \\
+ (b - \overline{\nabla}^j \Theta_1)e_1 \otimes e_2 \otimes e_2 + (a - \overline{\nabla}^i \Theta_2)e_2 \otimes e_2 \otimes e_2) = 0 \]

the solution to the above equation is \( a = \overline{\nabla}^j \Theta_2 \), \( b = \overline{\nabla}^i \Theta_1 \). The kernel constraint on \( a, b \) then requires the constraint on the vielbein. 

**Proof.** We only need to state explicitly the equality \( \nabla \eta = 0 \), as in:

\[ \nabla(e_1 \otimes e_1 + e_2 \otimes e_2) = 2((b - \overline{\nabla}^j \Theta_1)e_1 \otimes e_1 \otimes e_1 + (a - \overline{\nabla}^i \Theta_2)e_2 \otimes e_1 \otimes e_1 \\
+ (b - \overline{\nabla}^j \Theta_1)e_1 \otimes e_2 \otimes e_2 + (a - \overline{\nabla}^i \Theta_2)e_2 \otimes e_2 \otimes e_2) = 0 \]

We see that not every vielbein admits a strictly metric compatible condition – in general we need our weaker cotorsion-free condition. However, when it does so, the covariant derivative is uniquely determined as in classical Riemannian geometry.

### 4.2.2. 2D calculus on \( \mathbb{Z}_4 \)

We also consider the 2D calculus on \( \mathbb{Z}_4 \), defined by \( \{1, 3\} \) with \( f^1 \) and \( f^3 \), acting as before. Our interesting result is that the geometric content is the same as the universal calculus above except that some redundant modes in the universal case are not present, but replaced by a quadratic regularity condition.

**Proposition 4.5.** With the above specification for the action, the moduli space of torsion free, cotorsion free connections is given by:

\[ A_1 = (-\alpha - \overline{\nabla}^2 \Theta_1)e_1 + (\beta - \overline{\nabla}^3 \Theta_2)e_2 \\
A_3 = (\beta - \overline{\nabla}^3 \Theta_1)e_1 + (\alpha - \overline{\nabla}^2 \Theta_2)e_2 \]

with the conditions

\[ (R_1 + R_2)a = 0, (R_1 + R_2)b = 0 \]

where \( a = \alpha + \beta \) and \( b = \beta - \alpha \). In terms of \( a, b \) the covariant derivative \( \nabla \) is as before, in Proposition 4.1, and the regularity condition reads

\[ \overline{\nabla}^2 \Theta_1 \overline{\nabla}^3 a - \overline{\nabla}^3 \Theta_2 \overline{\nabla}^2 b = 0 \]

**Proof.** Here the parameters \( \alpha, \beta \) are not the same as in the previous section (but related to them). We solve the zero torsion condition

\[ A_1 + A_2 - A_3 = \overline{\nabla}^1 \Theta_2 \\
A_2 - A_3 + A_3 = \overline{\nabla}^3 \Theta_1 \]

which gives the solution above in terms of \( \alpha, \beta \) or the combinations \( a, b \), but free of any constraint on the \( a, b \). Next we require the connection to have zero cotorsion:

\[ -R_1 A_1^2 - R_1 A_2^3 - R_2 A_4^1 + R_2 A_4^1 = -\overline{\nabla}^1 \Theta_2 \\
-R_1 A_1^2 + R_1 A_3^3 + R_2 A_4^1 + R_2 A_4^1 = \overline{\nabla}^3 \Theta_1 \]

and obtain the constraint \( (R_1 + R_2)a = (R_1 + R_2)b = 0 \). We then compute the covariant derivative using the action of \( f^1, f^3 \). The regularity condition in this case is given by:

\[ A_1 \wedge A_1 + A_3 \wedge A_3 = 0 \]

\( \diamond \)

**Corollary 4.6.** The Riemann and Ricci tensors corresponding to the connection above have the form (in terms of \( a \) and \( b \)) as in Proposition 4.2.
Proof. This follows since the Riemann and Ricci tensors are determined by $\nabla$ which has the same form. It is also instructive (but a different computation) to compute them directly; as usual from the definition of the curvature

$$F_1 = dA_1 - 2A_1 \wedge A_1 - A_3 \wedge A_1 - A_1 \wedge A_3$$
$$F_3 = dA_3 - A_1 \wedge A_3 - A_3 \wedge A_1 - 2A_3 \wedge A_3$$

we compute the expression for the Riemann tensor:

$$\text{Riemann}(e_1) = (-F_1 - F_3) \otimes e_1 + (F_1 - F_3) \otimes e_2$$
$$\text{Riemann}(e_2) = (-F_1 + F_3) \otimes e_1 + (-F_1 - F_3) \otimes e_2$$

inserting the actual form of $F_1$ and $F_3$ and the regularity condition. The Ricci tensor is

$$\text{Ricci} = \frac{1}{2}((-F_1 - F_3)e_1 \otimes e_1 + (F_1 + F_3)e_1 \otimes e_2 + (-F_1 - F_3)e_2 \otimes e_1 + (F_1 - F_3)e_2 \otimes e_2)$$

If we want Ricci flatness, we must force $F_1 = F_3 = 0$. Note that if the Ricci flat is null, so is the Riemann tensor.

The condition for the metric compatibility is the same as in Proposition 4.4

**Proposition 4.7.** The metric $\eta$ satisfies the equation $\nabla \eta = 0$, if and only if

$$a = \overline{\partial}^1 \Theta_2, \quad b = \overline{\partial}^2 \Theta_1, \quad \partial^1 (\overline{\partial}^2 \Theta_2) = 0, \quad \partial^1 (\overline{\partial}^1 \Theta_1) = 0.$$

The regularity condition holds and the Riemann and Ricci tensors are as in Theorem 4.4

Proof. We impose the condition $\nabla \eta = 0$, which has the same shape as in the previous case. We then check that the regularity condition in Proposition 4.1 indeed holds for these $a, b$. \quad \diamond

We conclude that moving to the 2D calculus on $\mathbb{Z}_4$ gives essentially the same Riemannian geometry as using the 3D calculus but without some of the superfluous modes that we found there. Instead, these are replaced by a regularity condition. This gives us some insight into the 'correct' choice of calculus for the frame group and what happens if one chooses one that is too big.

4.3. Torus model with translations $\mathbb{Z}_2 \times \mathbb{Z}_2$ as frame group. We take now the frame group to be $\mathbb{Z}_2 \times \mathbb{Z}_2$ acting by ‘translation’ on our base space which we recall is also the group $\mathbb{Z}_2 \times \mathbb{Z}_2$. We write the frame group elements as $\bar{0}0, \bar{0}1, \bar{1}0, \bar{1}1$, say. As before, we have two choices for the calculus on the frame group.

4.3.1. 3D calculus on $\mathbb{Z}_2 \times \mathbb{Z}_2$. This is the universal calculus defined by $\{\bar{1}0, \bar{0}1, \bar{1}1\}$

The corresponding $f$s act by

$$f^{\bar{1}0} \circ e_1 = -2e_1, \quad f^{\bar{0}1} \circ e_2 = 0$$
$$f^{\bar{1}0} \circ e_1 = 0, \quad f^{\bar{0}1} \circ e_2 = -2e_2$$
$$f^{\bar{1}1} \circ e_1 = -2e_1, \quad f^{\bar{1}1} \circ e_2 = -2e_2$$
Proposition 4.8. The moduli space of torsion free, cotorsion free connections is given by:

\[
A_{10} = \alpha e_1 - (\delta + \frac{\bar{\theta}^1 e_2}{2})e_2 \\
A_{01} = -\gamma + \frac{\bar{\theta}^1 e_2}{2}e_1 + \beta e_2 \\
A_{11} = \gamma e_1 + \delta e_2
\]

We will use the ansatz \(a = \alpha + \gamma, b = \beta + \delta\). The covariant derivative corresponding to this connection is:

\[
\nabla e_1 = 2ae_1 \otimes e_1 - \bar{\Theta}^1 \Theta_2 e_2 \otimes e_1, \quad \nabla e_2 = -\bar{\Theta}^2 \Theta_1 e_1 \otimes e_2 + 2be_2 \otimes e_2
\]

Proof. We solve the torsion condition

\[
2A_{10}^2 + 2A_{11}^2 = -\bar{\Theta}^1 \Theta_2, \quad -2A_{01}^1 - 2A_{11}^1 = \bar{\Theta}^2 \Theta_1
\]

and the zero cotorsion condition

\[
-2R_1 A_{10}^2 - 2R_1 A_{11}^2 = -\bar{\Theta}^1 \Theta_2, \quad 2R_2 A_{01}^1 + 2R_2 A_{11}^1 = \bar{\Theta}^2 \Theta_1
\]

then we work out the covariant derivative using (8). \(\diamondsuit\)

There is no regularity condition for the universal calculus on the frame group (because there is no element different from the identity which lies outside the subset defining the calculus).

Proposition 4.9. The Riemann and Ricci tensors corresponding to the above connection are:

\[
\text{Riemann}(e_1) = -2(-\partial^2 a + \bar{\Theta}^1 \Theta_2 (\Omega - R_2 a) + \frac{\bar{\Theta}^1 e_2 \Theta_2}{2})e_1 \wedge e_2 \otimes e_1 \\
\text{Riemann}(e_2) = -2(\bar{\Theta}^2 \Theta_1 (R_1 b - \Theta_2) + \partial^1 b - \frac{\bar{\Theta}^2 e_1 \Theta_2}{2})e_1 \wedge e_2 \otimes e_2
\]

\[
\text{Ricci} = -(-\partial^2 a + \bar{\Theta}^1 \Theta_2 (\Omega - R_2 a) + \frac{\bar{\Theta}^1 e_2 \Theta_2}{2})e_1 \otimes e_2 \\
-\bar{\Theta}^2 \Theta_1 (R_1 b - \Theta_2) + \partial^1 b - \frac{\bar{\Theta}^2 e_1 \Theta_2}{2})e_2 \otimes e_1
\]

Proof. We have:

\[
F_{10} = dA_{10} + A_{01} \wedge A_{11} + A_{11} \wedge A_{01} - 2A_{10} \wedge A_{10} - A_{10} \wedge A_{01} \\
- A_{01} \wedge A_{10} - A_{10} \wedge A_{11} - A_{11} \wedge A_{01} \\
F_{01} = dA_{01} + A_{10} \wedge A_{11} + A_{11} \wedge A_{01} - A_{10} \wedge A_{01} - A_{01} \wedge A_{10} \\
- A_{01} \wedge A_{10} - A_{10} \wedge A_{11} - A_{11} \wedge A_{01} \\
F_{11} = dA_{11} + A_{10} \wedge A_{01} + A_{01} \wedge A_{10} - A_{10} \wedge A_{10} - A_{10} \wedge A_{01} \\
- A_{11} \wedge A_{01} - A_{01} \wedge A_{11}
\]

and

\[
\text{Riemann}(e_1) = -2(F_{10} + F_{11}) \otimes e_1, \quad \text{Riemann}(e_2) = -2(F_{01} + F_{11}) \otimes e_2
\]

The same result is obtained by \(\text{Riemann}(e_a) = ((\text{id} \wedge \nabla) - (d \otimes \text{id})) \circ \nabla(e_a)\). \(\diamondsuit\)

Proposition 4.10. The condition \(\nabla \eta = 0\) is satisfied if and only if \(\alpha = -\gamma\) and \(\beta = -\delta\) and \(\bar{\Theta}^2 \Theta_2 = \bar{\Theta}^1 \Theta_1 = 0\). In this case, both the Riemann and the Ricci tensor are zero.
Proof. The first part of the proposition is easily proved by computing
\[
\nabla(e_1 \otimes e_1 + e_2 \otimes e_2) = 4ae_1 \otimes e_1 - 2\Theta_1 e_2 \otimes e_1 \otimes e_1
- 2\Theta_2 e_1 \otimes e_2 - 4be_2 \otimes e_2 \otimes e_2 = 0
\]
which means \(a = b = \Theta_1 = \Theta_2 = 0\) have; this implies that the Riemann and
the Ricci tensor are both zero.
\(\square\)

4.3.2. 2D calculus on \(\mathbb{Z}_2 \times \mathbb{Z}_2\). The calculus on the fibre will be defined now by
\{10, 01\}

Proposition 4.11. the moduli space of torsion free, cotorsion free connections is
given by:
\[
A_{10} = \alpha e_1 - \partial_1 \Theta_2 e_2 \\
A_{01} = -\partial_2 \Theta_1 e_1 + \beta e_2
\]
We set \(a = \alpha, b = \beta\) (as in the case before but with \(\gamma = \delta = 0\)), then the covariant
derivative has the same form as in Proposition 4.8. The regularity condition is
\[
a\partial_1 b - b\partial_2 a = 0.
\]
Proof. We solve the torsion equations
\[
2A^2_{10} = -\Theta^1 \Theta_2, \quad -2A^1_{01} = \Theta^2 \Theta_1
\]
and the cotorsion equations
\[
-2R_1 A^2_{10} = -\Theta^1 \Theta_2, \quad 2R_2 A^1_{01} = \Theta^2 \Theta_1
\]
The regularity condition is, in this case, \(A_{10} \wedge A_{01} + A_{01} \wedge A_{10} = 0\), which comes
out as \(aR_1 b - bR_2 a = 0\), which can be written as stated.
\(\square\)

Corollary 4.12. The Riemann and Ricci tensors are (as functions of \(a, b\)) as in
Proposition 4.9
Proof. This follows from \(\nabla\) but can also be computed directly as useful check; the curvature two form corresponding to the regular connection above, is given by:
\[
F_{10} = (-\Theta_2 \partial^2 a + \Theta^2 \Theta_2 (\Theta_1 - R_2 a) + \frac{\Theta^2 \Theta_2}{2} e_1 \wedge e_2
F_{01} = (\Theta^2 \Theta_1 (R_1 b - \Theta_2) + \Theta_1 \partial^1 b - \frac{\Theta^2 \Theta_2}{2} e_1 \wedge e_2
\]
computed from the expression for \(F\) (regularity condition applied)
\[
F_{10} = dA_{10} - 2A_{10} \wedge A_{10} \\
F_{01} = dA_{01} - 2A_{01} \wedge A_{01}
\]
the Ricci tensor is \(F_{01} e_1 \otimes e_2 - F_{10} e_2 \otimes e_1\), Riemann is given by \(\text{Riemann}(e_1) = -2F_{10} \otimes e_1, \text{Riemann}(e_2) = -2F_{01} \otimes e_2\).
\(\square\)
Finally, the only connection fulfilling the condition
\[
\nabla \eta = \nabla (e_1 \otimes e_1 + e_2 \otimes e_2) = -4ae_1 \otimes e_1 + 2be_2 \otimes e_2 + \mathcal{J} \Theta_2(e_1 \otimes e_2 \otimes e_1 + e_1 \otimes e_1 \otimes e_2) - 2\mathcal{J}^2 \Theta_1 e_1 \otimes e_2 \otimes e_2 = 0
\]
is, in this case, the null connection.

We see again the same phenomenon as in Section 4.2; working with the 'correct' 2D calculus rather than the universal 3D eliminates redundant fields that do not enter into the Riemannian geometry, trading them for an optional regularity condition.

4.4. Discrete sphere base with $\mathbb{Z}_3 \subset SO(2)$ frame group. As the main alternative to the above models, we look at the case of the universal calculus on the 4 points of our base space, which has the connectivity of a tetrahedron or discrete model of a sphere:

Our results are rather unusual, probably due to the small number of points in the model. As a projector we are led to $\pi$ defined by
\[
\pi(e_a \otimes e_b) = i(e_a \wedge e_b) = \begin{cases} 
0 & a \neq b \\
\frac{1}{3}(e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3) & a = b
\end{cases}
\]
This means that $\Omega^2$ has the relations
\[
e_1^2 = e_2^2 = e_3^2 \equiv \text{Top}, \quad e_a \wedge e_b = 0, \quad \forall a \neq b.
\]
This projector obeys the compatibility condition (i) of Theorem 2.2 as follows. We are required to list all the two-arcs contained in the graph. Naming the vertices as $x, y, z, t$ (starting from $(0,0)$ and going clockwise) the possible two arcs from $x$ are:
\[
x \to y \to x, x \to y \to z, x \to y \to t \\
x \to z \to x, x \to z \to y, x \to z \to t \\
x \to t \to x, x \to t \to y, x \to t \to z
\]
Now we have to make sure all the possible expression of the form
\[
\pi_{xyz}^{yz}, \pi_{xyz}^{yt}, \pi_{xyz}^{zy}, \pi_{xyz}^{zt}, \ldots
\]
(60 of them in total) vanish, which happens to be the case. Note that we just considered the two arcs departing from \(x\), since the choice of "start point" is immaterial here due to the symmetry of the graph. The second condition of Theorem 2.2 is empty in this case, because the calculus on \(\Sigma\) is the universal one. \(\Omega^2(\Sigma)\) The action of the external derivative on the vielbein elements \(e_i\) is computed from 1 and is 
\[
de_1 = (\Theta_1 + R_1 \Theta_1)\text{Top} = \tilde{\Theta}_1 \text{Top}, \quad de_2 = (\Theta_2 + R_2 \Theta_2)\text{Top} = \tilde{\Theta}_2 \text{Top} \\
d e_3 = (\Theta_3 + R_3 \Theta_3)\text{Top} = \tilde{\Theta}_3 \text{Top}
\]
Next, we take \(Z_3 = \{0, 1, 2\}\) as a frame group, with calculus defined by \(\{1, 2\}\). \(f^1, f^2\) will acting on \(e_1, e_2, e_3\) as \(e_1 \rightarrow e_3 \rightarrow e_2 \rightarrow e_1\) (notice that the definition of the projector is invariant under this action), or:
\[
f^1 \triangleright e_1 = e_3 - e_1, \quad f^2 \triangleright e_1 = e_2 - e_1 \\
f^1 \triangleright e_2 = e_1 - e_2, \quad f^2 \triangleright e_2 = e_3 - e_2 \\
f^1 \triangleright e_3 = e_2 - e_3, \quad f^2 \triangleright e_3 = e_1 - e_3
\]
(it’s an anticlockwise rotation in the picture below, which is the tetrahedron from the viewpoint of the vertex \((0, 0, 0)\))

![Diagram](image)

**Proposition 4.13.** The moduli space of torsion free connections is 4-dimensional, given by 6 parameters \(\alpha_1, \ldots, \alpha_3, \beta_1, \ldots, \beta_3\) with two independent equations given by 
\[
\tilde{\Theta}_1 + \alpha_3 - \alpha_1 - \beta_1 + \beta_2 = 0
\]
and cyclic permutations. The additional conditions for zero-cotorsion are the two independent equations given by 
\[
\bar{\Theta}^1(\alpha_1 + \beta_1) - \bar{\Theta}^2 \alpha_2 - \bar{\Theta}^3 \beta_3 - \alpha_2 + \alpha_3 + \beta_2 - \beta_3 = 0
\]
and cyclic permutations.

**Proof.** Firstly, we write down the zero torsion condition, but with a notation of the form 
\[
A^1_1 = \alpha_1, \quad A^2_1 = \alpha_2, \quad A^3_1 = \alpha_3, \quad A^1_2 = \beta_1, \quad A^2_2 = \beta_2, \quad A^3_2 = \beta_3
\]
to underline a symmetry of the theory with respect to cyclical permutations in the upper indexes of the \(A^i_j\) (as usual, the lower index refers to the frame group directions). The vanishing of the cotorsion corresponds to 
\[
\tilde{\Theta}_1 + R_2 \alpha_2 - R_1 \alpha_1 + R_3 \beta_3 - R_1 \beta_1 = 0
\]
and cyclic permutations. Combining the torsion and cotorsion equations we obtain the equations as stated. \(\diamondsuit\)
The covariant derivative shows the same rotational symmetry. Infact, given
\[ \nabla e_1 = (\omega + \tilde{\omega}) \otimes e_1 - \omega \otimes e_2 - \tilde{\omega} \otimes e_3 \]
\[ \omega = \sum_a \alpha_a e_a, \quad \tilde{\omega} = \sum_a \beta_a e_a \]
\( \nabla e_2 \) and \( \nabla e_3 \) can be found by cyclical rotations of \( \nabla e_1 \).

**Proposition 4.14.** The Riemann tensor corresponding to the connection above is

Riemann\((e_1) = -(\rho + \tilde{\rho})\) Top \( \otimes e_1 + \tilde{\rho}\) Top \( \otimes e_2 + \rho\) Top \( \otimes e_3 \)

(Riemann\((e_2)\), Riemann\((e_3)\) can be found by cyclical rotation) where

\[ \rho = \partial^i a_1 + a_1 \tilde{\Theta}_1 + \beta_1 R_1 (\beta_1 - a_1) - a_1 R_1 (2a_1 + \beta_1) + \text{cycl.} \]

\[ \tilde{\rho} = \partial^i \tilde{\beta}_1 + \tilde{\beta}_1 \Theta_1 + a_1 R_1 (a_1 - \beta_1) - \beta_1 R_1 (2\beta_1 + a_1) + \text{cycl.} \]

and the Ricci scalar, \( R = -(\rho + \tilde{\rho}) \)

**Proof.** Riemann tensor is obtained in the usual way from the curvature components \( F_i \); the expression for Ricci then comes out as

\[ \text{Ricci} \begin{aligned} \frac{1}{2} & \left[ (F_1 + F_2) e_1 \otimes e_1 + F_2 e_1 \otimes e_2 + F_1 e_1 \otimes e_3 + F_1 e_2 \otimes e_1 - (F_1 + F_2) e_2 \otimes e_2 \\ & + F_2 e_2 \otimes e_3 + F_2 e_3 \otimes e_1 + F_1 e_3 \otimes e_2 - (F_1 + F_2) e_3 \otimes e_3 \right] \end{aligned} \]

(from which the Ricci scalar \( R = -(F_1 + F_2) \)). The curvature two-form is:

\[ F_1 = dA_1 + A_2 \wedge A_2 - 2A_1 \wedge A_1 - A_2 \wedge A_1 - A_1 \wedge A_2, \quad F_2 = dA_2 + A_1 \wedge A_1 - A_1 \wedge A_2 - A_2 \wedge A_1 - 2A_2 \wedge A_2 \]

and similarly for the other components.

We know that the moduli space of connections is 2-parameter, which we see here is reflected in the two physical curvature parameters \( \rho, \tilde{\rho} \). This model is obviously far from classical, but we see that it has several reasonable features including a cyclic symmetry and a degree 2 top form, i.e. a nonclassical 'surface'.

5. Remarks on the Quantum Theory

So far we have solved only for the classical geometry which could form the basis for classical equations of motion for gravity and matter in a classical background. For quantum theory at least in a path integral approach one must integrate over all such moduli spaces with respect to an action weighting. Here quantum gravity, in particular, diverges badly. The advantage of working only on a finite number of points as we have done above is that now such functional integrals become finite dimensional integrals, which may still diverge but which are surely much more tractable. Such integrals for gauge theory on \( S_3 \) are discussed in [14] and carried to fruition for Yang-Mills on \( Z_2 \times Z_2 \) in [17], where the theory was found to be divergent but renormalisable. Here we make some first remarks about how to extend this in principle to the gravitational case. The new ingredient not yet covered is the correct 'unitarity' or reality conditions on the spin connection, which we now propose.

Thus, until now we could have worked above over a generic field, but now we must really we work over \( \mathbb{C} \) and specify reality or 'unitarity' conditions which should be expected for a physical interpretation. This cuts down our moduli still further and also reduces us to integration over real variables in our finite setting. To do this, we note that \( C(\Sigma) \) is a *-algebra with * given by pointwise complex conjugation. We extend this to inner calculi with the assumption \( \theta^* = \theta \) so that * anticommutes with \( d \) (other conventions are also possible). For models based on groups and conjugacy
classes with elements of order 2 this is naturally implemented by \( e^*_a = e_a \) (more generally, \( e_{a^{-1}} \), as in [8, 9, 7]. For the models based on \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) connectivity in Section 4 we take \( e^*_a = e_a \). We likewise, and more importantly, we take

\[ \theta^* = \theta \]

which ensures that \( d = [\theta, \cdot] \) behaves as usual for a * structure in the differential graded algebra (so \( d \) graded-commutes with *). In terms of field components this translates to

\[ \overline{\Theta_a(x)} = R_a \Theta_a(x). \]

This is consistent with our model in Section 4, for example, where the condition [18] on \( \Theta \) in Section 4.1 for a 2-form projector is invariant under *.

Next, we consider the spin connection components. For a unitary action for the braided-Lie algebra generators \( f^i \) we would take \( A^*_i = A_i \). What is a unitary action is motivated from Hopf algebra theory where the action of a Hopf *-algebra one would require \( (f^i \triangleright e_a)^* = S^{-1}(f^i) \triangleright e^*_{a} \), where in our case \( S f^i = f^{i-1} \) is inversion in the frame group algebra. The *-structure on the braided-Lie algebra generators which is not so clear, but if we assumed that \( f^{i*} = f^{i-1} \) as for elements in a group algebra, these two inverses cancel and we would be led to require \( (f^i \triangleright e_a)^* = f^i \triangleright e_a \). This indeed holds for the actions in the present paper, particularly those in Section 4, since these are obtained from permutations. Next, if the generators are unitary in this sense, we want the frame group connection to be 'antihermitian' so we propose here

\[ A^*_i = A_{i^{-1}}. \]

for the component 1-forms. This has the reasonable consequence that applying * to the torsion equations gives the cotorsion equations, i.e. these are related by complex conjugation in the unitary version of the theory. This is desirable as it suggests that imposing the unitarity condition on the moduli space of torsion and cotorsion free connections is not so likely to give no solutions. This too is borne out when we look closely at the moduli of connections on our \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) in Proposition 4.2 or 4.5.

We concentrate on the second of these as the more physical model with modes \( a, b \).

**Proposition 5.1.** The reality condition in the moduli of torsion free and cotorsion free connections on the discrete torus in Proposition 4.5 is \( \vec{a} = R_2 a, \vec{b} = R_1 b \). The regularity condition is invariant under conjugation and the Ricci scalar in Proposition 4.5 is real up to a ‘total divergence’ given by \( \partial^1, \partial^2 \).

**Proof.** From the above, we deduce from Proposition 4.5 and the reality condition on the \( \theta \), we find \( \vec{a} = -R_1 \beta, \vec{b} = R_2 \alpha \) which translates as stated given that the functions \( a, b \) reverse sign under \( R_1 R_2 \). The latter also means that \( R_1 (\partial^2 a) = \overline{\partial}^2 (-R_2 a) = \overline{\partial}^2 a \), and similarly \( \overline{\partial}^1 b \) is \( R_2 \)-invariant. Since \( (\partial^2 \theta_1)^* = R_1 (\partial^2 \theta_1) \), and similarly with \( R_2 \) for \( \overline{\partial}^1 \theta_2 \), we see that the regularity condition is invariant under *.

We then compute

\[
R = R_1 (\partial^1 b + \partial^2 \theta_1 R_1 b) + R_2 (\partial^1 a + \overline{\partial}^1 \theta_2 R_2 a) - 2bR_1 b - 2aR_2 a
\]

\[
= R + \overline{\partial}^1 (\partial^1 b + \partial^2 \theta_1 R_1 b) + \overline{\partial}^2 (\partial^2 a + \partial^1 \theta_2 R_2 a).
\]

where \( (\partial^1 b)^* = (R_1 \theta_1) \partial^1 R_1 b = R_1 (\partial^1 b) \), and similarly for \( \partial^2 a \). \( \Box \)

The reduced moduli space with full metric compatibility in Proposition 4.7 is also consistent with this *-structure, i.e. our reality condition holds for \( a, b \) given
by \( \theta \) as stated there. Moreover, the stated condition on the \( \theta \) required for this reduces simply to \( a, b \) real.

After that, for quantum gravity one should presumably take as action
\[
S = \sum_{x \in \Sigma} R(x)
\]
using the Ricci scalar curvature; we are not in a position to deduce field equations by a variational principle, so this is an assumption of one way to make sense of the quantum theory. To see how this works we again look at our discrete torus model on 4 points. We already know from Section 3 that for 2 or 3 points the Ricci scalar vanishes in all our models, so this model would be the first with nontrivial Ricci scalar. From the above Proposition 5.1 we see that the action \( S \) is real. Moreover, our fields \( a, b \) etc are functions on the four points but so highly constrained as to be fully determined each by a single complex number, which we denote \( A, B \). Here
\[
A = a(0, 0) = -a(0, 1) = -a(1, 0), \quad \bar{A} = a(0, 1) = -a(1, 0)
\]
\[
B = b(0, 0) = -b(1, 1), \quad \bar{B} = b(1, 0) = -b(0, 1).
\]
Note that the Ricci scalar splits up into two terms
\[
R = R_B + R_A, \quad R_B = \partial^1 b + \partial^2 \theta_1 R_1 b - 2b R_1 b = R_2 \theta_1 \bar{b} - \theta_1 b - 2\bar{b}b
\]
and the similar expression for \( R_A \) with 1, 2 interchanged. Writing
\[
\Theta = \theta_1(0, 0), \quad \bar{\Theta} = \theta_1(1, 0), \quad \tilde{\Theta} = \theta_1(0, 1), \quad \bar{\tilde{\Theta}} = \theta_1(1, 1)
\]
we find
\[
S = S_B + S_A: \quad S_B = -8BB + 2B(\Theta - \bar{\Theta}) + 2\bar{B}(\bar{\Theta} - \Theta).
\]
where we compute \( R_B \) at the four points in terms of our new variables and add up. Similarly for the \( A \) field and \( \theta_2 \). If we restrict to the full metric compatibility in Theorem 4.4 then the action is just \( S_B = -4B^2 \) and the dynamical variables are \( \Theta, \bar{\Theta} \) constrained such that \( B = \Theta - \bar{\Theta} \) is real. Again similarly for the \( A \) system.

Finally, we make a polar decomposition of the fields as
\[
B = \lambda e^{i\phi}, \quad \Theta = \mu e^{i\psi}, \quad \bar{\Theta} = \bar{\mu} e^{i\tilde{\psi}}
\]
in terms of real positive \( \lambda, \mu, \bar{\mu} \) and angles \( \phi, \psi, \tilde{\psi} \). In terms of these, we find
\[
S_B = -8\lambda^2 + 4\lambda\bar{\mu} \cos(\phi - \tilde{\psi}) - 4\lambda\mu \cos(\phi + \psi)
\]
with similar results for the \( A \) system. Then ‘quantum gravity’ is reduced to integrals over these real variables. There remains the constraint \( \int_{\Sigma} \) as well as the optional regularity condition to be imposed on the moduli in Proposition 4.5. These both cross-couple the \( A \) and \( B \) systems making even this simplest model nontrivial.

It is not our scope to consider the quantum theory in detail here, particularly since the geometries in this paper are low dimensional, where one does not expect very dynamical quantum gravity; for a compact surface in two dimensions the integral of the classical Ricci scalar is a constant by the Gauss-Bonnet theorem. For a classical torus this should be zero, so we see that the discrete torus model already exhibits non-standard behaviour, the meaning of which remains to be understood. It also remains to identify physical observables to be computed by such functional integral methods. However, our low-dimensional example does indicate the possibility of reasonable unitarity constraints and illustrate how a quantum gravity theory might proceed in principle.
6. Combinatorics of geometries up to nine points

For higher numbers of points we do not attempt a detailed classification but rather we overview the range of possibilities with a view to picking out the most interesting ones.

In the first place, we now limit ourselves to the more interesting case of symmetric (‘bidirectional’) differential calculi. These are just graphs with no self-edges and no more than one edge between vertices. For a fibration with fiber size \(n\), these are the so-called \(n\)-regular graphs. There is no classification theory for \(n\)-regular graphs (or \(n\)-regular simplicial approximation of a manifold gives one) but small ones are listed in [14]. From there we see that there is a reasonable number for \(m \leq 8\) after which the number grows rapidly. We deal only with connected graphs.

Note also for any \(m\) that here are none for \(n = 1\) (except \(m = 2\)). For \(n = 2\) there is just the \(m\)-gon for all \(m\). This is the differential calculus on \(\mathbb{Z}_n\) with \(C = \{-1, 1\}\). For \(n = m - 1\) there is exactly the universal calculus or totally connected graph. We observe that the \(m\)-gon and universal calculi are members of a ‘circulant graph’ family \(\mathbb{Z}_m^{(1,p,q,\ldots)}\) where \(p,q,\ldots\) are distinct integers modulo \(m\). They correspond to the calculus on \(\mathbb{Z}_m\) with \(C = \{\pm 1, \pm p, \pm q, \ldots\}\) where we only have \(p\) if \(2p = 0 \mod m\), etc. The direct product of circulants with \(C_1, C_2\) means with \(C = (C_1,0) \cup (0,C_2)\) (as for the product of any groups equipped with differential structures, see [10]). An example of a circulant is in Figure 1. Note also the ‘handshaking lemma’ in graph theory that \(nm\) has to be even. Then we have the following list of connected graphs which is complete up to \(m = 8\):

For \(m = 2\) we have only the universal calculus at \(n = 1\).

For \(m = 3\) we have only the universal calculus which equals the 3-gon calculus at \(n = 2\).

For \(m = 4\) we have only the 4-gon at \(n = 2\), which can also be viewed as \(\mathbb{Z}_2^{(1)} \times \mathbb{Z}_2^{(1)}\) (i.e. with the direct product calculus where \(C = \{(0,1), (1,0)\}\), and the universal calculus at \(n = 3\).

For \(m = 5\) we have only the 5-gon at \(n = 2\) and the universal at \(n = 4\).

For \(m = 6\) we have only the 6-gon at \(n = 2\) and two choices at \(n = 3\). These are the circulant \(\mathbb{Z}_6^{(1,3)}\), which is also the graph for the \(S_3\) calculus with its 2-cycles conjugacy class, and the circulant \(\mathbb{Z}_2^{(1)} \times \mathbb{Z}_3^{(1)}\). At \(n = 4\) we have only the circulant \(\mathbb{Z}_6^{(1,2)}\), which is a triangulation of the sphere and is also the graph for \(S_3\) with a left-covariant calculus. See Fig 1 (a),(b). At \(n = 5\) we just have the universal one. Note that the 3-cycles calculus on \(S_3\) is not connected so does not appear in this list.

For \(m = 7\) we have only the 7-gon at \(n = 2\), none at \(n = 3,5\) and two choices at \(n = 4\). One is the circulant \(\mathbb{Z}_7^{(1,2)}\) and the other is shown in Fig 1 (c). At \(n = 6\) we just have the universal one.

For \(m = 8\) we have the 8-gon at \(n = 2\) and five at \(n = 3\). One of these is the cube, which is \(\mathbb{Z}_2^{(1)} \times \mathbb{Z}_2^{(1)} \times \mathbb{Z}_2^{(1)}\). It can also be viewed as \(\mathbb{Z}_2^{(1)} \times \mathbb{Z}_4^{(1)}\). Another is the circulant \(\mathbb{Z}_8^{(1,4)}\). See Fig. 1(b). The remaining three are as in Fig. 1(c). At \(n = 4\) there are six, namely the circulants \(\mathbb{Z}_8^{(1,2)}\) and \(\mathbb{Z}_8^{(1,3)}\), and \(\mathbb{Z}_2^{(1)} \times \mathbb{Z}_4^{(1,2)}\) and the remaining three in Fig. 1(c). At \(n = 5\) there are three, namely the circulants \(\mathbb{Z}_8^{(1,2,4)}\) and \(\mathbb{Z}_8^{(1,3,4)}\) and the remaining one in Fig 1(c). At \(n = 6\) we have only the circulant \(\mathbb{Z}_8^{(1,2,3)}\). At \(n = 7\) we just have the universal calculus.
For $m = 9$ there is the 9-gon at $n = 2$, none at $n = 3$ and already sixteen at $n = 4$, of which three are groups, namely the circulants $\mathbb{Z}_9^{(1,3)}$, $\mathbb{Z}_9^{(1,4)}$ and a simplicial torus torus (see Fig. 1(b)), which is $\mathbb{Z}_9^{(1)} \times \mathbb{Z}_3^{(1)}$. There are none at $n = 5$ and three at $n = 6$ of which two are circulants on $\mathbb{Z}_9$, and so forth. Fig. 1(d) also shows an important $m = 10$ graph with $n = 3$ which is a $\mathbb{Z}_2$ quotient of the dodecahedron and can be thought of as a discrete $\mathbb{R}P^2$.

At the qualitative or ‘topological’ level of this section, we can immediately present one discrete moduli space of combinatorial solutions for vielbeins. Namely, for any $E$ that fibers over $\Sigma$ with $|F_x| = n$, an $n$-bein is provided by any choice of bijections $s_x : \{1, \cdots, n\} \to F_x$ by

\begin{equation}
e_{ax} = \delta_{s_x(a),y} = e_a^{-1} e_{xy} , \quad e_a = \sum_x \delta_x d \delta_{s_x(a)}
\end{equation}

giving

\begin{equation}
e_a f = f(s(a)) e_a , \quad (\partial a f)(x) = f(s_x(a)) - f(x).
\end{equation}

Here $s_x(a)$ is a function on $\Sigma$ (with $\cdot$ denoting the functional dependence). Pictorially, we label all 1-arcs arbitrarily by $\{1, \cdots, n\}$ and $s_x(a)$ is the endpoint of the arc labelled $a$ from $x$. The element $\theta$ and the relations of the maximal prolongation are

\begin{equation}
\theta = \sum_a e_a , \quad \sum_{x \to y \to z} e_a \wedge e_b = 0 , \quad \forall x \neq z , \ x \nrightarrow z.
\end{equation}
The corresponding projectors are

$$\pi(e_a \otimes e_b) = e_a \otimes e_b - \sum_x \delta_x \sum_{x \rightarrow y \rightarrow z} e_c \otimes e_d; \text{ where } x \rightarrow y \rightarrow z$$

and have a functional dependence. Since the wedge product is given by setting to zero the elements of the tensor product which are in the kernel of this projector, we have the lift \( i : \Omega^2 \rightarrow \Omega^1 \otimes \Omega^1 \) given by the same formula. These formulae are for general left-parallelizable calculi. In our bidirectional case each arc really means two arrows since we can move along it in either direction. In this case the combinatorial data \( \{ s_x(a) \} \) for this class of vielbeins is a bicolouring of the graph, with two colours \( a \in \{1, \cdots, n\} \) for each arc, namely one for each arrow. Moreover, we can follow the coloured arrows from vertex to vertex and in this way the doubled-up graph (in which each arc is a pair of arrows going in opposite directions) is decomposed into coloured loops. The loops of each colour need not be connected.

For the framed geometry one must also choose a frame group \( G \) acting on the vector space \( V \) spanned by the vielbeins, a calculus on the group given by an Ad-stable subset, and projectors \( \pi \). For the combinatorial solutions above it is natural to take \( G = S_n \) acting by permuting the colours, i.e. \( g e_a = e_{g(a)} \) for a permutation \( g \). We can then take (for example) the universal differential calculus on \( S_n \) where \( i \in S_n - \{e\} \) so that there is no regularity condition to solve when we use the braided-Lie algebra with basis \( \{ f_i \} \). Then the torsion and cotorsion equations for \( A_i \) are linear and hence determined by linear algebra. More generally, our choice of frame group and associated structures have to be chosen according to what geometry we want to model. I.e. for each choice of regular graph for the ‘topology’ of the finite set, we have further choices for the actual geometry we want to model. We have already seen how this goes for a small number of points; there are progressively more choices as the number of points increases.

Also, for the quantum theory on should sum over all topological configurations, i.e. graphs and colourings, and then integrate over all moduli spaces for each colouring (eg of the restricted variety as we have done in Section 5), weighted with some action such as the Einstein-Hilbert one. In this way one arrives in principle at a quantum gravity theory in which differential structures (which goes into the graph) are summed over as well as an additional variable. Let us note here a remarkable duality: the sum over all coloured graphs, which is the combinatorial part of our theory, is in the spirit of a Feynman diagram, i.e. in some sense the discrete quantum gravity theory is somewhat like a scalar theory in usual flat space (with \( \phi^n \) interaction if we look at \( n \)-regular graphs). If one wanted to take this further, one should sum over the number of points \( m \), i.e. take all finite sets with \( n \)-regular graphs or the \( n \)-dimensionality of the non-commutative manifold structures fixed.

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