Robustness Among Multiwinner Voting Rules

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Abstract. We investigate how robust are results of committee elections to small changes in the input preference orders, depending on the voting rules used. We find that for typical rules the effect of making a single swap of adjacent candidates in a single preference order is either that (1) at most one committee member can be replaced, or (2) it is possible that the whole committee can be replaced. We also show that the problem of computing the smallest number of swaps that lead to changing the election outcome is typically NP-hard, but there are natural FPT algorithms. Finally, for a number of rules we assess experimentally the average number of random swaps necessary to change the election result.

1 Introduction

We study how multiwinner voting rules (i.e., procedures used to select fixed-size committees of candidates) react to (small) changes in the input votes. We are interested both in the complexity of computing the smallest modification of the votes that affects the election outcome, and in the extent of the possible changes. We start by discussing our ideas informally in the following example.

Consider a research-funding agency that needs to choose which of the submitted project proposals to support. The agency asks a group of experts to evaluate the proposals and to rank them from the best to the worst one. Then, the agency uses some formal process—here modeled as a multiwinner voting rule—to aggregate these rankings and to select $k$ projects to be funded. Let us imagine that one of the experts realized that instead of ranking some proposal $A$ as better than $B$, he or she should have given the opposite opinion. What are the consequences of such a “mistake” of the expert? It may not affect the results at all, or it may cause only a minor change: Perhaps proposal $A$ would be dropped (to the benefit of $B$ or some other proposal) or $B$ would be selected (at the expense of $A$ or some other proposal). We show that while this indeed would be the case under a number of multiwinner voting rules (e.g., under the $k$-Borda rule; see Section 2 for definitions), there exist other rules (e.g., STV or the Chamberlin–Courant rule) for which such a single swap could lead to selecting a completely disjoint set of proposals. The agency would prefer to avoid situations where small changes in the experts’ opinions lead to (possibly large) changes in the outcomes; so the
agency would want to be able to compute the smallest number of swaps that would change the result. In cases where this number is too small, the agency might invite more experts to gain confidence in the results.

More formally, a multiwinner voting rule is a function that, given a set of rankings of the candidates and an integer \( k \), outputs a family of size-\( k \) subsets of the candidates (the winning committees). We consider the following three issues (for simplicity, below we assume to always have a unique winning committee):

1. We say that a multiwinner rule \( \mathcal{R} \) is \( \ell \)-robust if (1) swapping two adjacent candidates in a single vote can lead to replacing no more than \( \ell \) candidates in the winning committee, and (2) there are examples where exactly \( \ell \) candidates are indeed replaced; we refer to \( \ell \) as the robustness level of \( \mathcal{R} \).\(^5\) Notably, the robustness level is between 1 and \( k \), with 1-robust being the strongest form of robustness one could ask for. We ask for the robustness levels of several multiwinner rules.

2. We say that the robustness radius of an election \( E \) (for committee size \( k \)) under a multiwinner rule \( \mathcal{R} \) is the smallest number of swaps (of adjacent candidates) which are necessary to change the election outcome. We ask for the complexity of computing the robustness radius (referred to as the Robustness Radius problem) under a number of multiwinner rules (this problem is strongly related to the Margin of Victory \([20, 7, 26, 4]\) and Destructive Swap Bribery problems \([13, 25]\); in particular, it follows up on the study of robustness of single-winner rules of Shiryaev et al. \([25]\)).

3. We ask how many random swaps of adjacent candidates are necessary, on average, to move from a randomly generated election to one with a different outcome. Doing experiments, we assess the practical robustness of our rules.

There is quite a number of multiwinner rules. We consider only several of them, selected to represent a varied set of ideas from the literature (ranging from variants of scoring rules, through rules inspired by the Condorcet criterion, to the elimination-based STV rule). We find that all these rules are either 1-robust (a single swap can replace at most one committee member) or are \( k \)-robust (a single swap can replace the whole committee of size \( k \)).\(^6\) Somewhat surprisingly, this phenomenon is deeply connected to the complexity of winner determination. Specifically, we show (under mild assumptions) that if a rule has a constant robustness level, then it has a polynomial-time computable refinement (that is, it is possible to compute one of its outcomes in polynomial time). Since for many rules the problem of computing such a refinement is NP-hard, we get a quick way of finding out that such rules have non-constant robustness levels.

The Robustness Radius problem tends to be NP-hard (sometimes even for a single swap) and, thus, we seek fixed-parameter tractability (FPT) results. For example, we find several FPT algorithms parametrized by the number of voters (useful, e.g., for scenarios with few experts, such as our introductory example). See Table 1 for an overview on our theoretical results.

\(^5\) Indeed, the formal definition is more complex due to taking care of ties.

\(^6\) We also construct somewhat artificial rules with robustness levels between 1 and \( k \).
Table 1. Summary of our results. Together with each rule, we provide the complexity of its winner determination. The parameters \(m\), \(n\), and \(B\) mean, respectively, the number of candidates, voters, and the robustness radius; NP-hard\((B)\) means NP-hard even for constant \(B\). (♣) For STV there is a polynomial-time algorithm for computing a single winning committee but deciding if a given committee wins is NP-hard [10].

| Voting Rule       | Robustness Level | Complexity of Robustness Radius |
|-------------------|------------------|--------------------------------|
| SNTV, Bloc, \(k\)-Borda (P) | 1                | P                              |
| \(k\)-Copeland (P)          | 1                | NP-hard, FPT\((m)\)            |
| NED (NP-hard [1])          | \(k\)            | NP-hard, FPT\((m)\)            |
| STV (P) (♣)               | \(k\)            | NP-hard\((B)\), FPT\((m)\), FPT\((n)\) |
| \(\beta\)-CC (NP-hard [23, 19, 3]) | \(k\)            | NP-hard\((B)\), FPT\((m)\), FPT\((n)\) |

We complement our work with an experimental evaluation of how robust are our rules with respect to random swaps. On the average, to change the outcome of an election, one needs to make the most swaps under the \(k\)-Borda rule. All the omitted proofs are present in the long version of our paper [5].

2 Preliminaries

Elections. An election \(E = (C, V)\) consists of a set of candidates \(C = \{c_1, \ldots, c_m\}\) and of a collection of voters \(V = (v_1, \ldots, v_n)\). We consider the ordinal election model, where each voter \(v\) is associated with a preference order \(\succ_v\), that is, with a ranking of the candidates from the most to the least desirable one (according to this voter). A multiwinner voting rule \(R\) is a function that, given an election \(E = (C, V)\) and a committee size \(k\), outputs a set \(R(E, k)\) of size-\(k\) subsets of \(C\), referred to as the winning committees (each of these committees ties for victory).

(Committee) Scoring Rules. Given a voter \(v\) and a candidate \(c\), by \(\text{pos}_v(c)\) we denote the position of \(c\) in \(v\)'s preference order (the top-ranked candidate has position 1 and the following candidate has position 2, and so on). A scoring function for \(m\) candidates is a function \(\gamma_m : [m] \to \mathbb{R}\) that associates each candidate-position with a score. Examples of scoring functions include (1) the Borda scoring functions, \(\beta_m(i) = m - i\); and (2) the \(t\)-Approval scoring functions, \(\alpha_t(i)\) defined so that \(\alpha_t(i) = 1\) if \(i \leq t\) and \(\alpha_t(i) = 0\) otherwise (\(\alpha_1\) is typically referred to as the Plurality scoring function). For a scoring function \(\gamma_m\), the \(\gamma_m\)-score of a candidate \(c\) in an \(m\)-candidate election \(E = (C, V)\) is defined as \(\gamma_m\)-score\(_E(c) = \sum_{v \in V} \gamma_m(\text{pos}_v(c))\).

For a given election \(E\) and a committee size \(k\), the SNTV score of a size-\(k\) committee \(S\) is defined as the sum of the Plurality scores of its members. SNTV outputs the committee(s) with the highest score (i.e., the rule outputs the committees that consist of \(k\) candidates with the highest plurality scores; there may be more than one such committee due to ties). Bloc and \(k\)-Borda rules are
defined analogously, but using $k$-Approval and Borda scoring functions, respectively. The Chamberlin–Courant rule [8] (abbreviated as $\beta$-CC) also outputs the committees with the highest score, but computes these scores in a different way: The score of committee $S$ in a vote $v$ is the Borda score of the highest-ranked member of $S$ (the score of a committee is the sum of the scores from all voters).

SNTV, Bloc, $k$-Borda, and $\beta$-CC are examples of committee scoring rules [12, 14]. However, while the first three rules are polynomial-time computable, winner determination for $\beta$-CC is well-known to be NP-hard [23, 19, 3].

**Condorcet-Inspired Rules.** A candidate $c$ is a Condorcet winner (resp. a weak Condorcet winner) if for each candidate $d$, more than (at least) half of the voters prefer $c$ to $d$. In the multiwinner case, a committee is Gehrlein strongly-stable (resp. weakly-stable) if every committee member is preferred to every non-member by more than (at least) half of the voters [15], and a multiwinner rule is Gehrlein strongly-stable (resp. weakly-stable) if it outputs exactly the Gehrlein strongly-stable (weakly-stable) committees whenever they exist. For example, let the NED score of a committee $S$ be the number of pairs $(c,d)$ such that (i) $c$ is a candidate in $S$, (ii) $d$ is a candidate outside of $S$, and (iii) at least half of the voters prefer $c$ to $d$. Then, the NED rule [9], defined to output the committees with the highest NED score, is Gehrlein weakly-stable. In contrast, the $k$-Copeland$^\alpha$ rule is Gehrlein strongly-stable but not weakly-stable (the Copeland$^\alpha$ score of a candidate $c$, where $\alpha \in [0,1]$, is the number of candidates $d$ such that a majority of the voters prefer $c$ to $d$, plus $\alpha$ times the number of candidates $e$ such that exactly half of the voters prefer $c$ to $e$; winning $k$-Copeland$^\alpha$ committees consist of $k$ candidates with the highest scores). Detailed studies of Gehrlein stability mostly focused on the weak variant of the notion [2, 17]. Very recent findings, as well as results from this paper, suggest that the strong variant is more appealing [1, 24]: for example, all Gehrlein weakly-stable rules are NP-hard to compute [1], whereas there are strongly-stable rules (such as $k$-Copeland$^\alpha$) that are polynomial-time computable.

**STV.** For an election with $m$ candidates, the STV rule executes up to $m$ rounds as follows. In a single round, it checks whether there is a candidate $c$ who is ranked first by at least $q = \left\lfloor \frac{n}{k+1} \right\rfloor + 1$ voters and, if so, then it (i) includes $c$ into the winning committee, (ii) removes exactly $q$ voters that rank $c$ first from the election, and (iii) removes $c$ from the remaining preference orders. If such a candidate does not exist, then a candidate $d$ that is ranked first by the fewest voters is removed. Note that this description does not specify which $q$ voters to remove or which candidate to remove if there is more than one that is ranked first by the fewest voters. We adopt the parallel-universes tie-breaking model and we say that a committee wins under STV if there is any way of breaking such internal ties that leads to him or her being elected [10].

We can compute some STV winning committee by breaking the internal ties in some arbitrary way, but it is NP-hard to decide if a given committee wins [10].

**Parametrized Complexity.** We assume familiarity with basic notions of parametrized complexity, such as parametrized problems, FPT-algorithms, and $W[1]$-hardness. For details, we refer to the textbook of Cygan et al. [11].
3 Robustness Levels of Multiwinner Rules

In this section we identify the robustness levels of our multiwinner rules. We start by defining this notion formally; note that the definition below has to take into account that a voting rule can output several tied committees.

**Definition 1.** The robustness level of a multiwinner rule \( R \) for elections with \( m \) candidates and committee size \( k \) is the smallest value \( \ell \) such that for each election \( E = (C,V) \) with \( |C| = m \), each election \( E' \) obtained from \( E \) by making a single swap of adjacent candidates in a single vote, and each committee \( W \in R(E,k) \), there exists a committee \( W' \in R(E',k) \) such that \( |W \cap W'| \geq k - \ell \).

All rules that we consider belong to one of two extremes: Either they are 1-robust (i.e., they are very robust) or they are \( k \)-robust (i.e., they are possibly very non-robust). We start with a large class of rules that are 1-robust.

**Proposition 1.** Let \( R \) be a voting rule that assigns points to candidates and selects those with the highest scores. If a single swap in an election affects the scores of at most two candidates (decreases the score of one and increases the score of the other), then the robustness level of \( R \) is equal to one.

The proof uses the observation that, after a single swap, either the candidate whose score increases can push out a single (lowest-scoring) member of the winning committee \( W \), or a member of \( W \) who loses score can be replaced by the highest-scoring candidate outside \( W \). This suffices to deal with four of our rules.

**Corollary 1.** SNTV, Bloc, \( k \)-Borda, and \( k \)-Copeland\(^\alpha \) (for each \( \alpha \)) are 1-robust.

In contrast, Gehrlein weakly-stable rules are \( k \)-robust.

**Proposition 2.** The robustness level of each Gehrlein weakly-stable rule is \( k \).

**Proof.** Consider the following election, described through its majority graph (in a majority graph, each candidate is a vertex and there is a directed arc from candidate \( u \) to candidate \( v \) if more than half of the voters prefer \( u \) to \( v \); the classic McGarvey’s theorem says that each majority graph can be implemented with polynomially many votes [22]). We form an election with candidate set \( C = A \cup B \cup \{c\} \), where \( A = \{a_1, \ldots, a_k\} \) and \( B = \{b_1, \ldots, b_k\} \), and with the following majority graph: The candidates in \( A \) form one cycle, the candidates in \( B \) form another cycle, and there are no other arcs (i.e., for all other pairs of candidates \( (x,y) \) the same number of voters prefers \( x \) to \( y \) as the other way round). We further assume that there is a vote, call it \( v \), where \( c \) is ranked directly below \( a_1 \) (McGarvey’s theorem easily accommodates this need).

In the constructed election, there are two Gehrlein weakly-stable committees, \( A \) and \( B \). To see this, note that if a Gehrlein weakly-stable contains some \( a_i \), then it must also contain all other members of \( A \) (otherwise there would be a candidate outside of the committee that is preferred by a majority of the voters to a committee member). An analogous argument holds for \( B \).
If we push \( c \) ahead of \( a_1 \) in vote \( v \), then a majority of the voters prefers \( c \) to \( a_1 \). Thus, \( A \) is no longer Gehrlein weakly-stable and \( B \) becomes the unique winning committee. Since (1) \( A \) and \( B \) are disjoint, (2) \( A \) is among the winning committees prior to the swap, and (3) \( B \) is the unique winning committee after the swap; we have that every Gehrlein weakly-stable rule is \( k \)-robust.

To conclude this section we show that the robustness levels of \( \beta \)-CC and STV are \( k \). Such negative results seem unavoidable among rules that—like \( \beta \)-CC and STV—provide diversity or proportionality (we discuss this further in Section 4).

**Proposition 3.** Both \( \beta \)-CC and STV are \( k \)-robust.

One may wonder whether there exist any voting rules with robustness level between 1 and \( k \). Although we could not identify any classical rules with this property, we found natural hybrid multi-stage rules which satisfy it. For example, the rule which first elects half of the committee as \( k \)-Borda does and then the other half as \( \beta \)-CC does has robustness level of roughly \( k/2 \).

**Proposition 4.** For each \( \ell \), there is an \( \ell \)-robust rule.

### 4 Computing Refinements of Robust Rules

It turns out that the dichotomy between 1-robust and \( k \)-robust rules is strongly connected to the one between polynomial-time computable rules and those that are NP-hard. To make this claim formal, we need the following definition.

**Definition 2.** A multiwinner rule \( R \) is scoring-efficient if the following holds:

1. For each three positive integers \( n, m, \) and \( k \) (\( k \leq m \)) there is a polynomial-time computable election \( E \) with \( n \) voters and \( m \) candidates, such that at least one member of \( R(E,k) \) can be computed in polynomial time.
2. There is a polynomial-time computable function \( f_R \) that for each election \( E \), committee size \( k \), and committee \( S \), associates score \( f_R(E,k,S) \) with \( S \), so that \( R(E,k) \) consists exactly of the committees with the highest \( f_R \)-score.

The first condition from Definition 2 is satisfied, e.g., by weakly unanimous rules.

**Definition 3 (Elkind et al. [12]).** A rule \( R \) is weakly unanimous if for each election \( E = (C,V) \) and each committee size \( k \), if each voter ranks the same set \( W \) of \( k \) candidates on top (possibly in different order), then \( W \in R(E,k) \).

All voting rules which we consider in this paper are weakly unanimous (indeed, voting rules which are not weakly unanimous are somewhat “suspicious”). Further, all our rules, except STV, satisfy the second condition from Definition 2. For example, while winner determination for \( \beta \)-CC is indeed NP-hard, computing the score of a given committee can be done in polynomial time. We are ready to state and prove the main theorem of this section.
Theorem 1. Let $\mathcal{R}$ be a 1-robust scoring-efficient multiwinner rule. Then there is a polynomial-time computable rule $\mathcal{R}'$ such that for each election $E$ and committee size $k$ we have $\mathcal{R}'(E,k) \subseteq \mathcal{R}(E,k)$.

Proof. We will show a polynomial-time algorithm that, given an election $E$ and committee size $k$, finds a committee $W \in \mathcal{R}(E,k)$; we let $\mathcal{R}'(E,k)$ output $\{W\}$.

Let $E = (C,V)$ be our input election and let $k$ be the size of the desired committee. Let $E' = (C,V')$ be an election with $|V'| = |V|$, whose existence is guaranteed by the first condition of Definition 2, and let $S'$ be a size-$k$ $\mathcal{R}$-winning committee for this election, also guaranteed by Definition 2.

Let $E_0, E_1, \ldots, E_t$ be a sequence of elections such that $E_0 = E'$, $E_t = E$, and for each integer $i \in [t]$, we obtain $E_i$ from $E_{i-1}$ by (i) finding a voter $v$ and two candidates $c$ and $d$ such that in $E_{i-1}$ voter $v$ ranks $c$ right ahead of $d$, but in $E$ voter $v$ ranks $d$ ahead of $c$ (although not necessarily right ahead of $c$), and (ii) swapping $c$ and $d$ in $v$’s preference order. We note that at most $|C||V|^2$ swaps suffice to transform $E'$ into $E$ (i.e., $t \leq |C||V|^2$).

For each $i \in \{0,1,\ldots,t\}$, we find a committee $S_i \in \mathcal{R}(E_i,k)$. We start with $S_0 = S'$ (which satisfies our condition) and for each $i \in [t]$, we obtain $S_i$ from $S_{i-1}$ as follows: Since $\mathcal{R}$ is 1-robust, we know that at least one committee $S''$ from the set $\{S'' \mid |S_{i-1} \cap S''| \geq k-1\}$ is winning in $E_i$. We try each committee $S''$ from this set (there are polynomially many such committees) and compute the $f_\mathcal{R}$-score of each of them (recall Condition 2 of Definition 2). The committee with the highest $f_\mathcal{R}$-score must be winning in $E_i$ and we set $S_i$ to be this committee (by Definition 2, computing the $f_\mathcal{R}$-scores is a polynomial-time task).

Finally, we output $S_t$. By our arguments, we have that $S_t \in \mathcal{R}(E,k)$. \qed

Theorem 1 generalizes to the case of $r$-robust rules for constant $r$: Our algorithm simply has to try more (but still polynomially many) committees $S''$.

Let us note how Theorem 1 relates to single-winner rules (that can be seen as multiwinner rules for $k = 1$). All such rules are 1-robust, but for those with NP-hard winner determination, even computing the candidates’ scores is NP-hard (see, e.g., the survey of Caragiannis et al. [6]), so Theorem 1 does not apply.

5 Complexity of Computing the Robustness Radius

In the ROBUSTNESS Radius problem we ask if it is possible to change the election result by performing a given number of swaps of adjacent candidates. Intuitively, the more swaps are necessary, the more robust a particular election is.

Definition 4. Let $\mathcal{R}$ be a multiwinner rule. In the $\mathcal{R}$ ROBUSTNESS Radius problem we are given an election $E = (C,V)$, a committee size $k$, and an integer $B$. We ask if it is possible to obtain an election $E'$ by making at most $B$ swaps of adjacent candidates within the rankings in $E$ so that $\mathcal{R}(E',k) \neq \mathcal{R}(E,k)$.

This problem is strongly connected to some other problems studied in the literature. Specifically, in the DESTRUCTIVE SWAP BRIBERY problem [13, 25, 16]...
(DSB for short) we ask if it is possible to preclude a particular candidate from winning by making a given number of swaps. DSB was already used to study robustness of single-winner election rules by Shiryaev et al. [25]. We decided to give our problem a different name, and not to refer to it as a multiwinner variant of DSB, because we feel that in the latter the goal should be to preclude a given candidate from being a member of any of the winning committees, instead of changing the outcome in any arbitrary way. In this sense, our problem is very similar to the MARGIN OF VICTORY problem [20, 7, 26, 4], which has the same goal, but instead of counting single swaps, counts how many votes are changed.

We find that ROBUSTNESS RADIUS tends to be computationally challenging. Indeed, we find polynomial-time algorithms only for the following simple rules.

**Theorem 2.** ROBUSTNESS RADIUS is computable in polynomial time for SNTV, Bloc, and $k$-Borda.

The rules in Theorem 2 are all 1-robust, but not all 1-robust rules have efficient ROBUSTNESS RADIUS algorithms. In particular, a simple modification of a proof of Kaczmarczyk and Faliszewski [16, Theorem 3] shows that for $k$-Copelandα rules (which are 1-robust) we obtain NP-hardness. We also obtain a general NP-hardness for all Gehrlein weakly-stable rules.

**Corollary 2.** $k$-Copeland ROBUSTNESS RADIUS is NP-hard.

**Theorem 3.** ROBUSTNESS RADIUS is NP-hard for Gehrlein weakly-stable rules.

Without much surprise, we find that ROBUSTNESS RADIUS is also NP-hard for STV and for $\beta$-CC. For these rules, however, the hardness results are, in fact, significantly stronger. In both cases it is NP-hard to decide if the election outcome changes after a single swap, and for STV the result holds even for committees of size one ($\beta$-CC with committees of size one is simply the single-winner Borda rule, for which the problem is polynomial-time solvable [25]).

**Theorem 4.** ROBUSTNESS RADIUS is NP-hard both for STV and for $\beta$-CC, even if we can perform only a single swap; for STV this holds even for committees of size 1. For $\beta$-CC, the problem is $W[1]$-hard with respect to the committee size.

For the case of $\beta$-CC, the proof of Theorem 4 gives much more than stated in the theorem. Indeed, our construction shows that the problem remains NP-hard even if we are given a current winning committee as part of the input. Further, the same construction gives the following corollary (whose first part is sometimes taken for granted in the literature, but has not been shown formally yet).

**Corollary 3.** The problem of deciding if a given candidate belongs to some $\beta$-CC winning committee (for a given election and committee size) is both NP-hard and coNP-hard.

We conclude this section by providing FPT algorithms for ROBUSTNESS RADIUS. An FPT algorithm for a given parameter (e.g., the number of candidates
or the number of voters) must have running time of the form $f(k)|I|^{O(1)}$, where $k$ is the value of the parameter and $|I|$ is the length of the encoding of the input instance. Using the standard approach of formulating integer linear programs and invoking the algorithm of Lenstra [18], we find that Robustness Radius is in FPT when parametrized by the number of candidates.

**Proposition 5.** Robustness Radius for $k$-Copeland, NED, STV, and $\beta$-CC is in FPT (parametrized by the number of candidates).

For STV and $\beta$-CC we also get algorithms parametrized by the number of voters. For the case of STV, we assume that the value of $k$ is such that we never need to “delete non-existent voters” and we refer to committee sizes, $k$, where such deleting is not necessary as normal. For example, $k$ is not normal if $k > n$.

Another example is to take $n = 12$ and $k = 5$: We need to delete $q = \left\lceil \frac{12}{5+1} \right\rceil + 1 = 3$ voters for each committee member, which requires deleting 15 out of 12 voters.

**Theorem 5.** STV Robustness Radius is in FPT when parametrized by the number $n$ of voters (for normal committee sizes).

**Proof.** Let $E = (C, V)$ be an input election and $k$ be the size of the desired committee. For each candidate $c$, we define “rank of $c$” as $\text{rank}(c) = \min_{v \in V}(\text{pos}_v(c))$.

First, we prove that a candidate with a rank higher than $n$ cannot be a member of a winning committee. Let us assume towards a contradiction that there exists a candidate $c$ with $\text{rank}(c) > n$ who is a member of some winning committee $W$. When STV adds some candidate to the committee (this happens when the number of voters who rank such a candidate first exceeds the quota $\left\lceil \frac{n}{k+1} \right\rceil + 1$), it removes this candidate and at least one voter from the election. Thus, before $c$ were included in $W$, STV must have removed some candidate $c'$ from the election without adding it to $W$ (since $c$ had to be ranked first by some voter to be included in the committee; for $c$ to be ranked first, STV had to delete at least $n$ candidates, so by the assumption that the committee size is normal, not all of them could have been included in the committee). Since STV always removes a candidate with the lowest Plurality score, and at the moment when $c'$ was removed the Plurality score of $c$ was equal to zero, the Plurality score of $c'$ also must have been zero. Thus, removing $c'$ from the election did not affect the top preferences of the voters, and STV, right after removing $c'$, removed another candidate with zero Plurality score. By repeating this argument sufficiently many times, we conclude that $c$ must have been eventually eliminated, and so could not have been added to $W$. This gives a contradiction and proves our claim.

The same reasoning also shows that the number of committees winning according to STV is bounded by a function of $n$: In each step either one of at most $n$ voters is removed, or all candidates who are not ranked first by any voter are removed from the election (which leaves at most $n$ candidates in the election).

Second, we observe that the robustness radius for our election is at most $n^2$. Indeed, we can take any winning candidate, and with at most $n^2$ swaps we can push him or her to have rank at least $n + 1$. Such a candidate will no longer be a member of $W$ and, so, the outcome of the election will change.
Third, we observe that in order to change the outcome of an election, we should only swap such candidates that at least one of them has rank at most \( n^2 + n + 1 \). Indeed, consider a candidate \( c \) with rank \( c > n^2 + n + 1 \). After \( n^2 \) swaps, the rank of this candidate would still be above \( n \), so he or she still would not belong to any winning committee. Thus, a swap of any two candidates with ranks higher than \( n^2 + n + 1 \) does not belong to any of the sequences of at most \( n^2 \) swaps that change the election result (the exact positions of these two candidates would have no influence on the STV outcome).

As a result, it suffices to focus on the candidates with ranks at most \( n^2 + n + 1 \). There are at most \( n(n^2 + n + 1) \) such candidates. Consequently, there are at most \( (2n^3 + 2n^2 + 2n)n^2 \) possible \( n^2 \)-long sequences of swaps which we need to check in order to find the optimal one. This completes the proof. \( \Box \)

The algorithm for the case of \( \beta \)-CC is more involved. Briefly put, it relies on finding either the unique winning committee or two committees tied for victory. In the former case, it combines brute-force search with dynamic programming, and in the latter either a single swap or a clever greedy algorithm suffice.

**Theorem 6.** \( \beta \)-CC Robustness Radius is in FPT for the number of voters.

### 6 Beyond the Worst Case: An Experimental Evaluation

In this section we present results of experiments, in which we measure how many randomly-selected swaps are necessary to change election results.\(^7\)

We perform a series of experiments using four distributions of rankings obtained from PrefLib [21], a library that contains both real-life preference data and tools for generating synthetic elections. We use three artificial distributions: (i) Impartial Culture (IC), (ii) Mallows Model with parameter \( \phi \) between 0 and 1 drawn uniformly, and (iii) the mixture of two Mallows Models with separate parameters \( \phi \) chosen identically to the previous model. (Intuitively, in the Mallows model there is a single most probable preference order \( r_0 \), and the more swaps are necessary to modify a given order \( r \) to become \( r_0 \), the less probable it is to draw \( r \); in the IC model, drawing each preference order is equally likely.) Additionally, we use one dataset describing real-life preferences over sushi sets (we treat this dataset as a distribution selecting votes from this dataset uniformly at random). For each of these four distributions, and for every investigated rule (for \( k \)-Copeland we took \( \alpha = 0.5 \)), we perform 200 simulations. In each simulation we draw an election containing 10 candidates and 50 voters from the given distribution. Then we repeatedly draw a pair of adjacent candidates uniformly at random and perform a swap, until the outcome of the election changes or 1000 swaps are done (taking 1000 for the following computations). The average number of swaps required to change the outcome of an election for different

\(^7\) We found STV to be computationally too intensive for our experiments, so we used a simplified variant where all internal ties are broken lexicographically. We omit NED for similar reasons (but we expect the results to be similar as for \( k \)-Copeland).
rules and for different datasets is depicted in Figure 1. We present the results for committee size $k = 3$; simulations for $k = 5$ led to analogous conclusions.

Among our rules, $k$-Borda is the most robust one ($k$-Copeland holds second place), whereas rules that achieve either diversity ($\beta$-CC and, to some extent, SNTV) or proportionality (STV) are more vulnerable to small changes in the input. This is aligned with what we have seen in the theoretical part of the paper (with a minor exception of SNTV). As expected, the robustness radius decreases with the increase of randomness in the voters’ preferences. Indeed, one needs to make more swaps to change the outcome of elections which are highly biased towards the resulting winners. Thus we conclude that preferences in the sushi datasets are not highly centered, as the radiuses of these elections are small.

7 Conclusions

We formalized the notion of robustness of multiwinner rules and studied the complexity of assessing the robustness/confidence of collective multiwinner decisions. Our theoretical and experimental analysis indicates that $k$-Borda is the most robust among our rules, and that proportional rules, such as STV and the Chamberlin–Courant rule, are on the other end of the spectrum.

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