Inquiry for the $(\pi^+ - \pi^-)$ Bound state Conversion in two $\pi^0$ as being due to the Weinberg $\pi - \pi$-interaction Lagrangian.

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November 10, 2018
Abstract

In the work presented, the decay of the pionium, that is the \((\pi^+\pi^-)\) bound state, into two \(\pi^0\) is studied, the \(\pi\pi\)-interaction causing this transition being described by the underlying Weinberg lagrangian. The calculation with such a \(\pi\pi\)-lagrangian being carried out, the \(\pi\)-meson size \(r_0\) emerges to be allowed for and occurs in the final result. The bound \((\pi^+\pi^-)\)-system itself is presumed to be due to the instantaneous Coulomb interaction and is treated consistently nonrelativistic, the Bethe-Solpeter equation being utilized. Along calculating, the terms of the lowest order in fine structure constant \(\alpha\) and the terms \(\sim \ln(r_0)\) are retained. The obtained pionium lifetime \(\tau\) is thought to be compatible with the conceivable future experimental data. The dependence of the results on the effective lagrangian parameters is visualized. The investigation carried out persuades us that just the whole form of the genuine \(\pi\pi\)-interaction determines the pionium lifetime, but not simply the \(\pi\pi\) scattering lengths. The inquiry into pionium decaying promotes to specify the validity of various \(\pi\pi\)-interaction descriptions.

\textit{PACS:} 13.75.Lb; 10.11.Ef; 12.39.Fe; 10.11.St

\textit{Keywords:} \(\pi\pi\)-interaction, \(\pi^+\pi^-\) bound state decay into \(2\pi^0\).
1. Introduction. Agenda of $\pi\pi$-interaction.

At present, the stringent knowledge of $\pi\pi$-interaction is well understood to be of fundamental value in its own right as well as for the reliable treatment of the various phenomena, where pionic degrees of freedom prove to be substantial. Pion being the lightest and, properly speaking, simplest among strong-interacting particles, an inquiry into pion-pion interaction spread the way to visualization of the main features of hadrons interactions in general, in their immense complexity \[1\–5\]. At the same time, the pion-pion interaction is bound to be allowed for in describing the hot and dense hadronic systems abundant in pions which are known to be produced in heavy ions colliding \[3\] at high enough incident energy, the baryon number being rather negligible when compared with the number of genuine mesons. Even so, in treating the nuclear matter at large density and temperature, the phenomena non-linear in meson fields, that is the meson-meson interactions, are realized to play the crucial role, especially when the feasible phase transitions caused by the mesonic degrees of freedom softening are investigated \[7\]. Thus, to repose full confidence in adequacy of our perception of such systems behaviour, the pion-pion interaction must be properly accounted for, in particular along calculating the respective thermodynamic characteristics. Thereby, in all the cases, we must certainly conceive the pion-pion interaction to be provided by the well specified trustworthy lagrangian, but not in the least simply just by pion-pion scattering lengths.

Nowadays, in the lack of the pions interactions description strictly worked out from the first principles, we are in possession of the pion-pion interaction lagrangians \[1\–5\] which are thought to be as good as effective, obtained in frameworks of some plausible models, QCD-motivated at best. Consequently, there is to appeal to the experimental investigations the reliable information about the $\pi\pi$-interaction can be disentangled from. Then, confronting the results of processing experimental data and of theoretical calculations, we can check up the validity of a certain $\pi\pi$-interaction description and subsequently ameliorate the latter.

Up to now, the trustworthy cognizance concerning $\pi\pi$-interaction has been acquired, strictly speaking, solely from the analysis of the data obtained in the $\pi N \to \pi\pi N$ reaction which was studied at first time as far back as in 1965 \[8\] near threshold ($\varepsilon_\pi \sim 200 – 300 \text{ MeV}$) and afterwards for manifold incident pion energies, up to $\varepsilon_\pi \sim 1 – 2 \text{ GeV}$ as well (see, for instance, \[9\–11\]). The results of profound processing these experimental data carried out in the series of investigations \[9\–11\] make us visualize the effective lagrangians asserted in \[1\–3\] are thought to be expedient to describe the $\pi\pi$-interaction, at least for low and middle pion energy, $\varepsilon_\pi \sim m_\pi$. Unfortunately, the inescapable involvement of strong pion-nucleon interactions in such a process put a bound to an attainable reliability of the pure $\pi\pi$-interaction description because, on one hand, it is as good as impossible to get rid of the strong $\pi N$-interaction effect in the experimental measurements, and, on the other hand, one will scarcely maintain that a theoretical calculation can refine unambiguously the $\pi\pi$-interaction from $\pi N$ one in the reaction $\pi N \to \pi\pi N$ treatment. Thus, the further development of $\pi\pi$-interaction description by means of the profound effective lagrangian \[4\–7\], or my be along other approaches (see, for instance, \[12\]), calls on new experiment. For that matter, at first thought, the $K_{\ell 4}$-decay, $K \to \nu e \pi \pi$, \[13\] might appear of being fruitful to learn directly the pure $\pi\pi$-interaction occurring in the final state, but it has to realize the weak-decay vertex itself is not concisely known and needs to be approved in its own right \[14\]. Thus, as yet, the reaction $\pi N \to \pi\pi N$ was
and has been, as a matter of fact, the unique source of the data to check our perception about the $\pi\pi$-interaction.

In the light of the aforesaid agenda, the advent of the experiments dealing with the pure $\pi\pi$-interaction, without imposition of other strong (or weak) interactions, proves to be extremely desirable.

2. Up to now Pionium Treatment.

Long since, the inquiry into the properties of the $\pi^+\pi^-$ bound state, pionium, had been understood of being very instructive to study of the pure $\pi\pi$-interaction, free of effect of any other strong or weak interactions [15]. The feasible measurement of the pionium lifetime having been firstly considered in the early investigations [15], setting up the correspondent experiment has been profoundly elaborated in Refs. [16, 17], and the respective investigations are for now already under way [17], the results are liable to arrive in the nearest future.

Pionium typifies the bound hadrons systems their very coming into existence is due to electromagnetic interactions, but their decay is, as a matter of fact, caused by strong interactions. All the time ago, as far back as in 1954, the handy semiquantitative approach to treat such systems was set out [18], the strong interaction corrections to energy levels and wave functions of the $\pi$-atom, the $\pi^-P$ bound state, as well as the transition rate $\pi^-P \rightarrow \pi^0N$ being expressed through the free pion-nucleon scattering lengths $a_T^L$ and the $\pi$-atom wave function at the origin $\psi(0)$. Here, $T,L$ indices denote various isotopic and angular states. Subsequently following this method, the pionium lifetime, that is the $\pi^+\pi^- \rightarrow \pi^0\pi^0$ reaction rate, in the ground state was asserted in Refs. [15] to be the simple plain function

$$\tau^{-1} = \frac{16\pi}{9} \sqrt{\frac{2\Delta m}{m}} |a_0^0 - a_2^0|^2 \cdot |\psi(0)|^2$$  \hspace{1cm} (1)

of the $s$-wave $\pi\pi$-scattering lengths $a_0^0, a_2^0$, the pionium wave function at the origin $\psi(0)$, and the mass difference $\Delta m = m - m_0$, $m$ being charge pion mass. Thus, if the original approach of Ref. [18] had been strictly valid in the pionium case, all we need for the pionium lifetime precise calculation would have been the exact values of the quantities $|a_0^0 - a_2^0|, |\psi(0)|,$ and $\Delta m$. It is to take cognizance of the fact that only the difference of the scattering lengths would have come into picture, regardless the complete form of the genuine $\pi\pi$-interaction. This is due to the main original presumption of the approach of Ref. [18] that irrespective to the $\pi\pi$-interaction form the calculation of probability of the pionium decay into two $\pi^0$ is quite equivalent to the calculation of the annihilation probability of an free pair $\pi^+\pi^-$ with zero momenta into two $\pi^0$, $\pi^+\pi^- \rightarrow \pi^0\pi^0$, the initial state density being not the free particles state density, but the density of state of the particles in the bound state of pionium $|\psi(0)|^2$. Up to now, the authors of all the succeeding investigations [19–26] have been taking for granted that the pionium lifetime formula (1) asserted according to [18] in Ref. [15] holds true strictly, and all the efforts were devoted to acquire somehow the precise values of the quantities $a_T^L, \psi(0)$, the pure point-like Coulomb nonrelativistic $\psi(0)$ value and the free-particles scattering lengths $a_T^L$ values gained according to Refs. [1–4] being assumed as a starting point in all the calculations. Then, there was to calculate the corrections to that $\psi(0)$ value, especially due to strong interactions, and simultaneously the $a_T^L$-modifications on account of strong and electromagnetic interactions in the coupled $\pi^0\pi^0, \pi^+\pi^-$ channels.
In the several investigations [19–24] the various effective potentials were managed to describe this strong $\pi\pi$-interaction. The most profound calculations within such a potential approach have been carried out in Ref. [21] and especially in the work [22], where the aforesaid corrections have been thoroughly calculated in the framework of the model of the two-channels $\pi^0\pi^0, \pi^+\pi^-$ system, the effective range approximation being used to account for the strong pion-pion interaction. Thereby, once an effective radius is chosen (equal in both channels), the strong potentials in the channels are determined merely by the correspondent scattering lengths $a_{0T}^T$. In such a calculation, the electromagnetic corrections are due to the pion mass difference in different channels along with the Coulomb interaction imposition in the $\pi^+\pi^-$ channel. The coupled Schrödinger equations determining the pions wave functions in the coupled channels having been solved, the corrected, generalized scattering lengths as well as subsequently corrected $\psi(0)$ values are obtained which must be substituted in the original formula (1) for $\tau$ to acquire its eventual corrected value. The scrutinized corrections to $a_{LT}^T$ values (and to $\psi(0)$ as well) proved to amount not more than few per-cents, being substantially less than the uncertainties in the $a_{LT}^T$ predictions following from Ref. [4], as the authors of [22] have inferred.

Unlikely the effective potential approach of the Refs. [19–24], the investigation [25] utilized the Bethe-Solpeter equation to allow for the effect of strong interactions on the $\psi(0)$ value in the pionium lifetime (1) (via the pionium eigenstate energy shift $\Delta E$), the correction proving to be rather negligible.

The $\tau$ (1) value modification on account of pionium relativistic treatment, especially the allowance for retardation effect in the $\pi^+\pi^-$ electromagnetic interaction has been found $\sim 1\%$ in Ref. [26]. Thereby, the scattering lengths difference $a_{0}^0 - a_{2}^0$ was presumed of rendering the total strong interaction responsible of $\pi^+\pi^- \rightarrow \pi^0\pi^0$ transition, likewise in all the aforecited investigations [19–24], in spite of treating the retardation effect in the $\pi^+\pi^-$ system which implies the $\pi^+\pi^-$ relative velocity to be comparable with light velocity $c$.

Profound as are all the before discussed calculations of the quantities $a_{0T}^T, \psi(0)$, we ought to realize the expression (1) itself, insofar as originating from the very plausible, but semiquantitative approach [18], is, properly speaking, as good as semiquantitative in turn. But this did not mean to say that any results obtained accordingly to the method set out in Ref. [18] must be regarded as untenable and scarcely able to describe experimental data with high enough accuracy. There is to visualize the validity and accuracy of this very approach in each certain treated case is caused crucially by the form of the genuine strong interaction inducing the bound hadronic system decay. The very germ of the idea set forth in Ref. [18] makes us comprehend the approach of [18] itself will hold true with high precision, if the hadron-hadron interaction is as good as point-like and constant, especially momentum-independent which is thought to be well acceptable for $P\pi^-$-interaction in the $s$-state in [18], but not in the least for the $\pi\pi$-interactions asserted and used in Refs. [1, 2, 3, 12]. Consequently, pionium properties being studied, we must refrain from to pursue the way paved in Ref. [18] and reject, in turn, the handy expression (1) for pionium lifetime.

3. Interactions inducing pionium decay into two $\pi^0$. 

According to our lights, a general aim of the theoretical investigations of pionium lifetime is to visualize whether a certain form of the $\pi\pi$-interaction is eligible to provide
the experimental \(\tau\) value. In the work presented, we set out the calculation of \(\tau\), the \(\pi\pi\)-interaction being determined by the Weinberg lagrangian according to Refs. [1,2]. The probability of two-photons pionium annihilation, \(\pi^+\pi^- \rightarrow 2\gamma\), being practically negligible when compared with decay probability due to strong interaction, will not be discussed henceforth.

We treat pionium as the beforehand prepared \(\pi^+\pi^-\) bound state which is stable when strong pion fields interaction is turned off. The coupling of this state, pionium field, to the charge (complex) pion field is implemented via the virtual decay of the \(\pi^+\pi^-\) bound state \(|D_\lambda >\), pionium or di-meson, into free \(\pi^+\pi^-\) pair:

\[
\pi^+ + \pi^- \leftarrow |D_\lambda >
\] (2)

In our nowaday consistently nonrelativistic approach, we presume the formation of the initial \(\pi^+\pi^-\) bound state \(|D_\lambda >\) is caused by pure-instantaneous potential interaction \(U(y_1, y_2)\), where \(y_1, y_2\) are spatial coordinates of the \(\pi^+(y_1, t), \pi^-(y_2, t)\) mesons composing the pionium, the time coordinates coinciding. Subsequently, the vertex operator

\[
\hat{\mathcal{L}}_D = -[\pi^+(y_1, t)\pi^-(y_2, t) + \pi^-(y_1, t)\pi^+(y_2, t)] \hat{\mathcal{F}}(y_1, y_2, t),
\] (3)

\[
\hat{\mathcal{F}}(y_1, y_2, t) = \sum_\lambda [c_\lambda \mathcal{F}_\lambda(y_1, y_2, t) + c_\lambda^* \mathcal{F}_\lambda^*(y_1, y_2, t)]
\] (4)

renders the virtual pionium state \(|D_\lambda >\) decay into a free \(\pi^+\pi^-\) pair. Here, \(\pi^\pm(y, t)\) are charge pion field operators, whereas \(\hat{\mathcal{F}}(y_1, y_2, t)\) stands for the pionium field, the quantities \(c_\lambda, c_\lambda^*\) being the pionium production and distraction operators in the state \(\lambda\). So far as the interaction \(U(y_1, y_2)\) is instantaneous, all the fields operators in (3,4) act at the same time point \(t\). In our calculation, the common relations are adopted

\[
\pi^+(x) = \frac{1}{\sqrt{2}}(\pi_1(x) + i\pi_2(x)), \quad \pi^-(x) = -(\pi^+(x))^*, \quad \pi^0(x) = \pi_3(x),
\]

\[
\pi^+(x) = \sum_p \frac{1}{\sqrt{2\varepsilon_p}}[a_p e^{-it\varepsilon_p + px} + b_p e^{it\varepsilon_p - px}],
\] (5)

the operator \(a_p\) destructing of \(\pi^+\)-meson and \(b_p^+\) producing \(\pi^-\)-meson. The vertex functions \(\mathcal{F}_\lambda(y_1, y_2, t)\) in (6) and the correspondent pionium eigenenergies \(E_\lambda\) in the states \(\lambda\) are well known (see, for instance, Refs. [27,28]) to be determined by the homogeneous Bethe-Solpeter equation

\[
\mathcal{F}_\lambda(y_1, y_2, t) = U(y_1, y_2) \cdot \int dt' \int d\mathbf{y}_1' \int d\mathbf{y}_2' D(y_1 - y_1')D(y_2 - y_2')\mathcal{F}_\lambda(y_1', y_2', t'),
\] (6)

where

\[
D(x) = \frac{1}{i(2\pi)^4} \int \frac{d^4k \cdot e^{ikx}}{k^2 - m^2 + i\delta}
\] (7)
is the usual pion propagator. In presumed non-relativistic approach, the vertex function $F_\lambda(y_1, y_2, t)$ proves to be reduced as follows (see, for instance, Refs. \[27, 28\] and also \[29\])

$$F_\lambda(y_1, y_2, t) = -i N \cdot U(y_1, y_2) \cdot \Phi_\lambda(y_1, y_2, t),$$  \hspace{1cm} (8)

where $\Phi_\lambda(y_1, y_2, t)$ is the non-relativistic $\pi^+\pi^-$ system wave function. The function $F_\lambda$ being determined by the homogeneous equation (0), the normalization factor $N$ emerges in (8) which calculation we defer for a while. The wave function $\Phi_\lambda(y_1, y_2, t)$ of such a nonrelativistic system is known (see, for instance, \[30\]) to be the product

$$\Phi_\lambda(y_1, y_2, t) = \psi_{nl}(z) \cdot \Psi_p(R) \cdot e^{iE_\lambda t}, \quad E_\lambda = 2m + \frac{P^2}{4m} + \varepsilon_{nl}, \quad \lambda = (nl, P)$$  \hspace{1cm} (9)

of the depending on the center of mass coordinate $R = (y_1 + y_2)/2$ wave function

$$\Psi_p(R) = \frac{1}{\sqrt{2E_\lambda}} e^{iR}$$  \hspace{1cm} (10)

of the free motion of the two-pion system as a whole with the total momentum $P$, and the intrinsic pionium wave function $\psi_{nl}(z)$ depending on the relative $\pi^+\pi^-$ coordinate $z = y_1 - y_2$. The functions $\psi_{nl}$ simultaneously with pionium energy levels $\varepsilon_{nl}$ are determined by the Schrödinger equation \[30\]

$$- \frac{1}{m} \nabla^2 \psi_{nl}(z) + U(z) \psi_{nl}(z) = \varepsilon_{nl} \psi_{nl}(z)$$  \hspace{1cm} (11)

with relevant boundary conditions at $z = 0, z \to \infty$. Here $m = 139.57 MeV$ is the $\pi^\pm$-meson mass \[31\]. We utilize the units $c = \hbar = 1$. For the pure-Coulomb point-like interaction

$$U(z) = -\frac{\alpha}{z}$$  \hspace{1cm} (12)

the ground state wave function $\psi_{10} \equiv \psi$, properly normalized, and energy $\varepsilon_{10} \equiv \varepsilon$ are known \[30\] to be

$$\psi(z) = \frac{1}{\sqrt{4\pi}} \cdot \sqrt{\frac{a^3}{2}} e^{-z^2/a}, \quad \varepsilon = -m\alpha^2/4;$$  \hspace{1cm} (13)

where $a = ma$ and $2/a$ is “Bhor radius”. Consequently, we denote $|D_{10}> \equiv |D>$ Henceforth, we consider this pionium ground state decay. The $\pi\pi$-interaction of the type \[\[3\] including dependence on pion momenta being put to use in our further calculations, the finite pion size $r_0$ emerges to come into the picture, which we allow for in due course replacing (12) by the electrostatic potential between two homogeneously charged spheres, $z$ being the distance between their centrepoints, that explicit expression, a bit long, is set out in Ref. \[32\]. Magnitude of the quantity $r_0$ itself have been estimated in some theoretical and experimental investigations \[33, 34\], accordingly which we have adopted $r_0 = 0.6 fm$ as realistic. If anything, it my be noted the calculations with generalized, but yet instantaneous potential accounting for the relativistic corrections up to $(1/c^2)$-order (the kind of Breit potential \[27, 35\]) would not provide the additional difficulties of principle.

In our present calculation, the $\pi\pi$-interaction inducing the $\pi^+\pi^- \to 2\pi^0$ transition is specified by well known Weinberg lagrangian

$$\hat{L}_{\pi\pi}(x) = -\frac{1}{(2\pi)^2} [\partial_i \pi(x) \partial^i \pi(x) - \beta \bar{m}^2(\pi(x))^2] \pi^2(x)$$  \hspace{1cm} (14)
elaborated and scrutinized in Refs. [1–3]. Here \( f_\pi = 92.4 \text{MeV} \) [31]. Dependence of the results of calculations on the parameters \( \beta, \bar{m} \) in the chiral symmetry violating term will be discussed in the last Section.

Let us recall the validity of the lagrangian (14) have been inferred from processing the experimental data on the \( N\pi \to N\pi\pi \) reaction, see Refs. [8–11], at least for not very high pions energies.

The difference of the masses of a charge pion, \( m = 139.57 \text{MeV} \), and neutral one, \( m_0 = 134.98 \text{MeV} \), \( \Delta m = m - m_0 = 4.59 \text{MeV} \) being greater than pionium binding energy \( \varepsilon \), the initial \( \pi^+\pi^- \) bound state \( |D> \) transition into the final two \( \pi^0 \) state turns out to be possible via processes presenting by (3, 14). All the effective interactions between pion (charge and neutral) and pionium fields are described by the total interaction lagrangian

\[
\hat{L}_{\text{tot}} = \hat{L}_D + \hat{L}_{\pi\pi},
\]

which determines eventually pionium lifetime \( \tau \).

4. Pionium decay amplitude.

The matrix element

\[
S^{0\pi\pi}_D = \langle \pi^0\pi^0 | \hat{S} | D >
\]

of the \( \hat{S} \)-matrix dictated by the lagrangian (15) determines the initial pionium state \( |D> \) decay into two final \( \pi^0 \). To the first order in \( \hat{L}_{\pi\pi} \) (14), the \( S \)-matrix element (16) takes the form (see, for instance, [27, 28])

\[
S^{1\pi\pi}_D = -\int dR \int dz \int dt \int d^4 x < \pi^0\pi^0 | \hat{T} | \hat{L}_D (R, z, t) \cdot \hat{L}_{\pi\pi} (x) | D > =
\]

\[
= \frac{iN8}{(2f_\pi)^2 2\sqrt{2E_\lambda \varepsilon_1 \varepsilon_2}} \int dR \int dz \int dt \int d^4 x U(z) \psi_\lambda (z) \{ 2\beta \bar{m}^2 - (\varepsilon_1 \varepsilon_2 - p_1 p_2) + \partial \nu \partial \mu \} \times
\]

\[
\times D(R + z/2 - x, t - x_0) \cdot D(R - z/2 - x, t - x_0) \cdot e^{-iE_\lambda + iP_{\text{PR}}} \cdot e^{ix_0(\varepsilon_1 + \varepsilon_2) - i\text{x}(p_1 + p_2)},
\]

where \( \hat{T} \) is usual time-ordering operator and \( \varepsilon_{1,2}, p_{1,2} \) denote final \( \pi^0 \) energies and momenta. Certainly, when necessary, the high \( \hat{L}_{\pi\pi} \)-order contributions in (16) could be allowed for in usual way. These terms, if calculated, would render, in particular, the effect of strong \( \pi\pi \)-interaction on the pionium state. In course of our to-day calculation,
we restricted ourselves by accounting for the first $L_{\pi\pi}$-order. If anything, it may be recall the analysis of $N\pi \rightarrow N\pi\pi$ reaction was carried out in Refs. [3-11], as a matter of fact, in the same first order in $L_{\pi\pi}$ approximation. For the ground state pionium decay at rest, the relations hold

\[ \mathbf{P} = 0, \quad E_n \mathbf{P} = E = 2m + \varepsilon, \quad \varepsilon_1 = \varepsilon_2 = \varepsilon_0 = E/2, \]

\[ \mathbf{P}_1 = -\mathbf{P}_2, \quad |\mathbf{P}_1| = |\mathbf{P}_0| = p_0 = \sqrt{(E/2)^2 - m_0^2}, \]

and the Eq. (17) reduces as follows

\[ S_{\pi^0\pi^0D}^1 = i(2\pi)^4 \cdot T_{\pi^0\pi^0D} \cdot \delta(\mathbf{p}_1 + \mathbf{p}_2) \delta(\varepsilon_2 + \varepsilon_1 - E), \]

\[ T_{\pi^0\pi^0D} = \frac{1}{(2\pi)^4 (2\pi f_\pi)^2 E \sqrt{2E}} \int dz \cdot U(z) \cdot \psi(z) \times \]

\[ \times \int d^4q \frac{-2\beta \bar{m}^2 + m_0^2 - E^2/2 + q_0^2 - q_0 E}{[q_0^2 - q^2 - m^2 + i0] \cdot [(E - q_0)^2 - q^2 - m^2 + i0]} \cdot e^{-izq}. \]

Let us take cognizance of the quantities $q^2, q_0^2$ emergence in the nominator in (19) which is due to the term

\[ (\partial_\mu \pi \cdot \partial^\mu \pi)(\pi)^2 \]

in the $\pi\pi$-interaction (14), this fact substantially affected the integrand behaviour in (19), especially at extremely large $q$ values. Integrating over $dq_0$ and over directions of the vectors $q$ and $z$ having been carried out, the Eq. (19) reduces to

\[ T_{\pi^0\pi^0D} = \frac{-i8N}{\pi(2\pi f_\pi)^2 E \sqrt{2E}} \int_0^\infty \frac{q dq}{\omega(q)} \cdot [1 - \frac{b}{q^2 + c^2}] \cdot \int_0^\infty dz \cdot U(z) \cdot z \cdot \psi(z) \cdot \sin(qz), \]

where the notations are introduced:

\[ \omega(q) = \sqrt{q^2 + m^2}, \quad c^2 = m^2 - (E/2)^2, \quad b = (-2\beta \bar{m}^2 + m_0^2 + m^2 - E^2)/2 \]

It is not difficult to realize the behaviour of the integrand in (20) at large momenta, $q \rightarrow \infty$, and subsequently the convergence of the integral in (20) itself are governed by the behaviour of the quantity $zU(z)\psi(z)\sin(qz)$ when $z$ value tends to zero, $z \rightarrow 0$. There is to calculate the contributions arising from two terms in brackets in integrand (20): from “unit”, 1, and from $b/(q^2 + c^2)$. We take up firstly integrating the term with “unit” and then set out the integral with quantity $b/(q^2 + c^2)$.

Not hard thing is to become convinced the integral in (20) with “unit” in brackets would diverge logarithmically, if the pure point-like Coulomb values (12), (13) were adopted for quantities $U(z), \psi(z)$ in (20). This divergency emerges because a pion size is neglected. To remove this puzzling, but spurious contradiction we allow for the finite pion size $r_0$, $r_0 a \ll r_0 m \ll 1$ in course of calculating this integral, $U(z)$ being the electrostatic potential between two homogeneously charged spheres of the radius $r_0$ [32], as discussed already after Eq. (13). Then, integrating over $dq$ having been performed, the integral in (20) originating due to the “unit” in brackets transforms to (see Ref. [33])

\[-\int_0^\infty dz \cdot zU(z)\psi(z) \frac{d}{dz} K_0(mz) = -zU(z)\psi(z)K_0(mz)|_0 + \]

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\[
\int_0^{2r_0} dz K_0(mz) \psi(z) \frac{d}{dz} [z U(z)] + \alpha \int_0^\infty dz K_0(mz) \frac{d}{dz} \psi(z) + \int_0^{2r_0} dz K_0(mz) z U(z) \frac{d}{dz} \psi(z),
\]

(21)

where \( K_0(z) \) is the Mackdonald’s cylindrical function (see [36]). The first term in the righthand side (21) vanishes due to \( \psi(z) \approx e^{-az/2} \to 0 \) when \( z \to \infty \), and it disappears at \( z = 0 \) owing to \( zU(z) = 0 \) at \( z = 0 \) because, in turn, the potential \( U(z) \) for finite size charged particles has got at \( z = 0 \) a finite value \( U(0) \), in particular, for the afore adopted potential of homogeneously charged spheres \( U(0) = -6a/(5r_0) \). Further, in our treatment, we are on the point to carry out all the calculations in the lowest \( \alpha \)-order. All the expression (20) (as well as (21)) is proportional to \( \alpha \sqrt{\alpha^2} \) due to the \( \alpha \)-dependence of the functions \( U, \psi \). Calculating the integrals in (20), (21), we retain only the terms which besides this \( \alpha \)-dependence are inversely proportional to \( \alpha, \sim 1/\alpha \), and \( \alpha \)-independent. Even so, we retain in the asymptotic expansion in \( r_0 \) only the terms \( \sim \ln(r_0) \), but drop out the terms \( \sim r_0^2 n \geq 1 \). Consequently, the second and third integrals in righthand side in (21) are realized to be neglected. Indeed, at \( z \geq 2r_0 \) in the second integral, the function \( \psi(z) \) behaves likely (13), \( \sim e^{-za'/2} \), the quantity \( a' \) being of the same order in \( \alpha \) as \( a, a' \approx a = ma \) (see, for instance, Refs. [32, 37]). Then, we have got \( d\psi(z)/dz \sim am\psi(z)/2 \), and, subsequently, this integral gets the additional factor of \( \alpha \) and must be dropped out. Then, in the third integral in righthand side (21), for \( z \leq 2r_0 \ll 2/a \), the function \( \psi(z) \) varying smoothly (see, for instance, Refs. [32, 37]), \( \psi(z) \sim \psi(0)(1 + za'') \), where \( a'' \) is of the same order as \( a \), the derivative \( d\psi(z)/dz \approx \psi(z)/a \), so as the whole integral comes out \( \sim ar_0a\psi(0) \) and is to be omitted as well. Thus, eventually, there is to calculate the first integral in the righthand side of (21). Its upper limit turned out to be \( 2r_0 \) because \( \frac{d}{dz}[U(z)z] = 0 \) at \( z \geq 2r_0 \), \( U(z) \) being point-like Coulomb potential \( -\alpha/z \) when \( z \geq 2r_0 \). For this \( z \) values, the relations \( r_0a \ll 1, r_0m \ll 1 \) being valid, the replacements hold true

\[
\psi(z) = \psi(0), \quad K_0(mz) = -\ln(mz/2) - C
\]

(22)

with the accuracy up to order \( \sim r_0a, \sim r_0m \). Here \( C \approx 0.577 \) is Euler constant (see [30]). Then, the approximation (22) being put to use, the first integral in righthand side in (21) is straightforward calculated, and the whole expression (21) results in

\[
\psi(0)[\alpha(\ln(mr_0) + C) + \bar{U}], \quad \bar{U} = \int_0^{2r_0} dz U(z)
\]

(23)

The quantity \( \bar{U} \) is calculated accordingly to Ref. [32] which gives \( \bar{U} \approx -\alpha \cdot (3/2) \).

While treating the integral with the term \( b/(c^2 + q^2) \) within brackets in (21), the presence of an additional \( q^2 \) in the denominator provides this integral convergence even without the finite pion size \( r_0 \) being taken into account. This does mean to say that asymptotic expanding this integral in \( r_0 \) begins with a term \( \sim r_0 \) which is beyond our to-day accuracy, as has been presumed abov. Then, the Eqs. (12, 13) being adopted, this integral in the lowest \( \alpha \)-order reduces as follows

\[
\sqrt{\frac{a^3}{8\pi}} ba \int_0^\infty dq \cdot q^2 \frac{d}{dq} \frac{(q^2 + (a/2)^2)(q^2 + c^2)}{(q^2 + (a/2)^2)(q^2 + c^2)} \omega(q) = \sqrt{\frac{a^3}{8\pi}} b \frac{\pi}{m} \left[ \frac{1}{2} - \alpha \right]
\]

(24)
After all, with allowance for results \((23, 24)\), the transition amplitude \((20)\) takes the form

\[
\mathcal{T}_{\pi^0\pi^0D} = -\frac{i8\mathcal{N}}{\pi(2f_\pi)^2E\sqrt{2}E} \cdot \sqrt{\frac{a^3}{8\pi}} \cdot \left[ \frac{b}{m^2}\frac{(\pi^2 - \alpha)}{2} + \alpha(ln(mr_0) + C) + \bar{U} \right] \quad (25)
\]

The normalization factor \(\mathcal{N}\) residing in the Eqs. \((4, 17 - 20, 25)\) is to determined by equating the energy \(E\) of the state \(|D\rangle\) of a pionium at rest and the expectation value in the \(|D\rangle\)-state of the operator of the \(\hat{T}^{00}\)-component of energy-momentum tenser of a charge (complex) pion field:

\[
\mathcal{T}^{00}(\xi_0, \xi) = -\frac{[\partial\pi^-(\xi)\cdot \partial\pi^+(\xi)}{\partial\xi_0} + \frac{\partial\pi^-(\xi)}{\partial\xi}\cdot \frac{\partial\pi^+(\xi)}{\partial\xi} + m^2\pi^-(\xi)\pi^+(\xi)], \quad (27)
\]

where the \(\hat{S}_D\)-matrix is dictated by the lagrangian \((4)\), so as

\[
E = \mathcal{N}^2 \sum_p [a^+_pa_p + b^+_pb_p] \quad (29)
\]

which apparently shows up no divergency by integrating over \(dq\) which is due to the integrand steep enough decrease at \(q \to \infty\) on the account, in turn, of the high power of \(q\) in the denominator of \((29)\). Evaluating \((29)\), we are to retain only the terms of the lowest \(\alpha\)-order: the \(\alpha\)-independent terms and terms \(\sim \alpha\) (if they would have appeared), omitting the terms \(\sim \alpha^n, n > 1\). Then, Eq. \((29)\) reduces to

\[
E = \frac{\mathcal{N}^2}{Em}, \quad \mathcal{N}^2 = 4m^3 \quad (30)
\]

If anything, for verification’s sake, the \(\mathcal{N}\) value can be obtained by equating the expectation value of the particles number operator related to zeroth component of charged pion field current

\[
\hat{\mathcal{N}} = \sum_p [a^+_pa_p + b^+_pb_p] \quad (32)
\]

(see Eqs. \((3)\)) in the pionium state \(|D\rangle\) and the pions number \(N = 2\), that is from the equation

\[
< D|\hat{T}(\hat{N}\hat{S}_D)||D > = 2.
\]
All the calculations having been carried out in due course, we arrive at the same \( \mathcal{N} \) value (30). Let us take cognizance of the fact that the righthand side in Eqs. (27 - 29) proves having got no terms \( \sim \alpha \), its expansion in \( \alpha \) starting with a term \( \sim \alpha^2 \). Evidently, it must be just so because the quantity \( E = 2m - ma^2/4 \) in the lefthand side does not include terms \( \sim \alpha \).

5. Pionium life-time calculation results and concluding remarks.

Thus, we have at our disposal the expression (25) for the transition amplitude with \( \mathcal{N} \) defined by Eq. (30). Then, we acquire in the usual way (see, for instance, Ref. [27]) the total probability \( W \) of a pionium conversion into two \( \pi^0 \), that is the inverse pionium lifetime \( \tau \)

\[
W = \frac{1}{\tau} = \frac{a^3 p_0 m^3 \tilde{b}^2}{(2f_\pi)^4 2\pi^2 E^2 [1 - 4\alpha/\pi (1 - \frac{\tilde{U}/\alpha + \ln(mr_0) + C}{\tilde{b}})]},
\]

(31)

where \( \tilde{b} = [-2\beta -(\tilde{m}/m)^2 - (m^0/m)^2 - 3]/2 \) and all the other quantities have been set forth above.

Let us now inquire into how the \( \tau \) value (31) depends on the values of \( \tilde{m}, \beta \) which reside in the chiral symmetry violating term in lagrangian (14). Let firstly \( \tilde{m} = m_0 \), then we gain for the \( \beta \) values \( \beta = 1/2, \beta = 1/3, \beta = 1/4 \) asserted in Refs. [1–3]

\[
\tau_{m_0,1/2} = 4.95 \times 10^{-15} \text{sec}, \quad \tau_{m_0,1/3} = 6.18 \times 10^{-15} \text{sec}, \quad \tau_{m_0,1/4} = 6.90 \times 10^{-15} \text{sec}
\]

Thus, the dependence of \( \tau \) on \( \beta \) is thought to be sizeable, the deviations of these \( \tau \) values from each other amounting \( \approx 15\% \). On the other hand, if we adopt \( \tilde{m} = m \) instead \( \tilde{m} = m_0 \), we shall have got

\[
\tau_{m,1/2} = 4.71 \times 10^{-15} \text{sec}
\]

which deviates from \( \tau_{m_0,1/2} \) by about 5%. Let us also note the second term within brackets in (31) amounts \( \approx 2\% \) to the whole \( W \) value.

Our result proves do not contradict the up to now estimation \( \tau = 2.9^{+\infty}_{-2.1} \times 10^{-15} \text{sec} \) set out in Ref. [17]. It might be instructive to recall the results of \( \tau \) calculation obtained in the previous investigations, surveyed in Section 2, appear to be some smaller as compare to our one. For instance, the value \( \tau = 2.72 \times 10^{-15} \text{sec} \) has been asserted in Ref. [21] and \( \tau = 3.2 \times 10^{-15} \text{sec} \) in Ref. [24].

The investigation carried out makes us realize the pionium life-time (as well as its other properties) does depend crucially on the form of the genuine \( \pi\pi \)-interaction, but not simply just on the free pions scattering lengths only. Thus, the pionium decay as being due to the most plausible concise Weinberg lagrangian (14) having been studied, the investigations pursuing other present-day trustworthy \( \pi\pi \)-interaction descriptions are very desirable and instructive. If the consistent \( \tau \) calculation in framework of a certain method of \( \pi\pi \)-interaction description (see, for instance Refs. [1, 2, 12]) is carried out and, subsequently, its result is confronted to the experimental \( \tau \) value, the validity of this method will come to light. We are on the point of inquiring into the various \( \pi\pi \)-interaction representations in the course of pionium lifetime studying.
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