The Completely Integrable Differential Systems are Essentially Linear Differential Systems

Jaume Llibre · Claudia Valls · Xiang Zhang

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Abstract Let $\dot{x} = f(x)$ be a $C^k$ autonomous differential system with $k \in \mathbb{N} \cup \{\infty, \omega\}$ defined in an open subset $\Omega$ of $\mathbb{R}^n$. Assume that the system $\dot{x} = f(x)$ is $C^r$ completely integrable, i.e., there exist $n - 1$ functionally independent first integrals of class $C^r$ with $2 \leq r \leq k$. As we shall see, we can assume without loss of generality that the divergence of the system $\dot{x} = f(x)$ is not zero in a full Lebesgue measure subset of $\Omega$. Then, any Jacobian multiplier is functionally independent of the $n - 1$ first integrals. Moreover, the system $\dot{x} = f(x)$ is $C^{r-1}$ orbitally equivalent to the linear differential system $\dot{y} = y$ in a full Lebesgue measure subset of $\Omega$. Additionally, for integrable polynomial differential systems, we characterize their type of Jacobian multipliers.

Keywords Differential systems · Completely integrability · Orbital equivalence · Normal form · Jacobian multiplier · Polynomial differential systems

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J. Llibre
Departament de Matemàtiques, Universitat Autònoma de Barcelona, Bellaterra, 08193 Barcelona, Catalonia, Spain
e-mail: jllibre@mat.uab.cat

C. Valls (✉)
Departamento de Matemática, Instituto Superior Técnico, Universidade Técnica de Lisboa, Av. Rovisco Pais, 1049-001 Lisbon, Portugal
e-mail: cvalls@math.ist.utl.pt

X. Zhang
Department of Mathematics, MOE–LSC, Shanghai Jiao Tong University, Shanghai 200240, People’s Republic of China
e-mail: xzhang@sjtu.edu.cn
1 Introduction and Statement of Results

Consider a $C^k$ autonomous differential system

$$\dot{x} = f(x), \quad x \in \Omega \subset \mathbb{R}^n,$$

where $k \in \mathbb{N} \cup \{\infty, \omega\}$, the dot denotes derivative with respect to the independent variable $t$, $\Omega$ is an open subset of $\mathbb{R}^n$, and $f(x) = (f_1(x), \ldots, f_n(x)) \in C^k(\Omega)$. Recall that $\mathbb{N}$ is the set of positive integers, and $C^\infty$ and $C^\omega$ are, respectively, the sets of infinitely smooth functions and analytic functions.

The main goal of this paper is to show that a completely integrable differential system defined in an open subset $\Omega$ of $\mathbb{R}^n$ (see below for a precise definition) is essentially linear in the sense that it is diffeomorphic to a linear differential system on a full Lebesgue measure subset of $\Omega$. Later on, we shall see that this result can be seen as a kind of global extension of the well-known flow box theorem.

A function $H(x)$ is a first integral of system (1) if it is continuous and defined in a full Lebesgue measure subset $\Omega_1$ of $\Omega$ (i.e., $\Omega \setminus \Omega_1$ has zero Lebesgue measure), and it is not locally constant on any positive Lebesgue measure subset of $\Omega_1$; moreover, $H(x)$ is constant along each orbit of system (1) in $\Omega_1$. System (1) is $C^r$ completely integrable, if it has $n - 1$ functionally independent $C^r$ first integrals in $\Omega$ with $1 \leq r \leq k$. See Llibre and Valls (2012) for an example of completely integrable differential systems.

Recall that $k$ smooth functions $H_1(x), \ldots, H_k(x)$ are functionally independent in $\Omega$ if their gradients $\nabla H_1, \ldots, \nabla H_k$ have rank $k$ in a full Lebesgue measure subset of $\Omega$. And that the $k$ smooth functions $H_1(x), \ldots, H_k(x)$ are functionally dependent in a subset $U$ of $\Omega$ if their gradients $\nabla H_1, \ldots, \nabla H_k$ have rank less than $k$ in each point of $U$. We note that from the definitions if the $k$ smooth functions $H_1(x), \ldots, H_k(x)$ are not functionally independent in $\Omega$, they must be functionally dependent in a positive Lebesgue measure subset of $\Omega$.

In this paper, we denote by $\mathcal{X}$ the vector field associated to system (1). We will use $\partial_i$ to denote the partial derivative with respect to $x_i$ for $i = 1, \ldots, n$. By convention $C^{r-1} = C^r$ if $r = \infty$ or $\omega$; and $\text{div}\mathcal{X} = \text{div}f = \partial_1f_1 + \cdots + \partial_nf_n$ denotes the divergence of the vector field $\mathcal{X}$ or $f$.

A $C^1$ function $J$ is a Jacobian multiplier of system (1) if it is defined in a full Lebesgue measure subset $\Omega^* \subset \Omega$, and satisfies

$$\text{div}(Jf) \equiv 0, \quad \text{i.e.,} \quad \mathcal{X}(J) = -J\text{div}\mathcal{X}, \quad x \in \Omega^*.$$

If system (1) is two dimensional, a Jacobian multiplier is called an integrating factor.

We extend the usual ordering of $\mathbb{N}$ to the set $\mathbb{N} \cup \{\infty, \omega\}$ as follows: For all $k \in \mathbb{N}$, we have $k < \infty < \omega$. Two differential systems $\dot{x} = f(x)$ and $\dot{y} = g(y)$ are $C^r$ orbitally equivalent in an open subset $\Omega$ of $\mathbb{R}^n$ with $1 \leq r \leq \omega$ if there exists a $C^r$ invertible transformation $y = \Phi(x)$ defined on $\Omega$ such that $\Phi_* f \circ \Phi^{-1}(y) = q(y)g(y)$, where $\Phi_*$ is the tangent map of $\Phi$ and $q(y)$ is a nonvanishing scalar function defined on $\Omega$.

**Remark 1** Assume that $f(x)$ is not identically zero in any positive Lebesgue subset of $\Omega$, where $f(x) \in C^k(\Omega)$. If the divergence of the differential system $\text{d}x/\text{d}t = \dot{x} =
$f(x)$ is identically zero in a full Lebesgue measure subset of $\Omega$, doing the change of time $dt = h(x) \, ds$ being $h(x)$ a convenient positive function we obtain an orbitally equivalent differential system $dx/ds = h(x) \, f(x)$ whose divergence is not identically zero. Hence, in the rest of this paper, without loss of generality we assume that the divergence of the differential system $\dot{x} = f(x)$ is not zero in a full Lebesgue measure subset of $\Omega$.

The next is our first main result.

**Theorem 2** Let $\dot{x} = f(x)$ be the $C^k$ autonomous differential system (1) defined in $\Omega$ with $k \in (\mathbb{N}\setminus\{1\}) \cup \{\infty, \omega\}$, and $f(x) \neq 0$ in any open subset of $\Omega$. Assume that system (1) is $C^r$ completely integrable in $\Omega$ with $2 \leq r \leq k$ and that the Lebesgue measure of the set of its singularities is zero. Let $H_1(x), \ldots, H_{n-1}(x)$ be $n - 1$ functionally independent $C^r$ first integrals. Then, the following statements hold.

(a) System (1) has always a $C^{r-1}$ Jacobian multiplier defined in a full Lebesgue measure subset of $\Omega$.

(b) If $J(x)$ is a $C^{r-1}$ Jacobian multiplier of system (1), then $J(x)$ is functionally independent of $H_1(x), \ldots, H_{n-1}(x)$.

(c) There exists a full Lebesgue measure subset $\Omega_0 \subset \Omega$ in which system (1) is $C^{r-1}$ orbitally equivalent to the linear differential system

$$\dot{y} = y.$$  

We remark that statement (a) is not new and it can be obtained from Zhang (2014b, Theorem 1.1). We include its proof here for completeness. Statement (b) is new. Statement (c) generalizes and improves Theorem 1 of Giné and Llibre (2011) extending it from two-dimensional differential systems to any finite-dimensional differential systems.

The flow box theorem states the existence of $n - 1$ functionally independent first integrals in a neighborhood of a regular point of the differential system $\dot{x} = f(x)$ by making it diffeomorphic to the differential system $(\dot{y}_1, \ldots, \dot{y}_n) = (1, 0, \ldots, 0)$. Theorem 2 under the assumptions of the existence of $n - 1$ functionally independent first integrals for the $C^k$ differential system $\dot{x} = f(x)$ defined in an open subset $\Omega$ of $\mathbb{R}^n$ shows that the system is diffeomorphic to the linear differential system $(\dot{y}_1, \ldots, \dot{y}_n) = (y_1, \ldots, y_n)$ in an open and dense subset of $\Omega$.

In the next proposition, we characterize the zero Lebesgue measure subset mentioned in statement (c) of Theorem 2.

**Proposition 3** Under the assumptions of Theorem 2, we can assume without loss of generality that

$$D(x) = \det \begin{pmatrix} \partial_1 H_1(x) & \cdots & \partial_{n-1} H_1(x) \\ \vdots & \ddots & \vdots \\ \partial_1 H_{n-1}(x) & \cdots & \partial_{n-1} H_{n-1}(x) \end{pmatrix} \neq 0.$$
Then, the zero Lebesgue measure subset $\Omega \setminus \Omega_0$ of statement (c) of Theorem 2 can be chosen as

$$\{x \in \Omega \mid D(x) f_n(x) \text{div}(f(x)) = 0\}.$$ 

Theorem 2 shows the existence of Jacobian multipliers for completely integrable differential systems. We now characterize the class of the Jacobian multipliers of these integrable differential systems.

Let $\mathbb{C}[x]$ be the ring of all polynomials in the variables $(x_1, \ldots, x_n) = x$ with coefficients in $\mathbb{C}$. A function $H(x)$ is of Darboux type if it is of the form

$$f_1^{k_1}(x) \cdots f_r^{k_r}(x) \exp \left( \frac{g(x)}{h(x)} \right),$$

where $f_i, g, h \in \mathbb{C}[x]$ with $g$ and $h$ coprime, and $k_i \in \mathbb{C}$ for $i = 1, \ldots, r$. Recall that the notion of Darboux function was considered by Darboux (1878a, b) in 1878 for studying the existence of first integrals through invariant algebraic curves (or surfaces or hypersurfaces) of polynomial differential systems. A Darboux first integral of the system $\dot{x} = f(x)$ is a first integral of Darboux type. A polynomial differential system $\dot{x} = f(x)$ in $\mathbb{R}^n$ or $\mathbb{C}^n$ is Darboux integrable if it has $n - 1$ functionally independent Darboux first integrals.

Let $\mathbb{C}(x)$ be the field of all rational functions in the variables $x$ with coefficients in $\mathbb{C}$. A function is Liouvillian if it belongs to the Liouvillian field extension of $\mathbb{C}(x)$, for more details on the Liouvillian field extension, see, for instance, Singer (1992). A polynomial differential system $\dot{x} = f(x)$ in $\mathbb{R}^n$ or $\mathbb{C}^n$ is Liouvillian integrable if it has $n - 1$ functionally independent Liouvillian first integrals.

Our next result provides the class of functions which belong to the Jacobian multipliers of an integrable polynomial differential system.

**Theorem 4** Assume that system (1) with $f = (f_1, \ldots, f_n)$ is an $n$-dimensional polynomial differential system with $f_1, \ldots, f_n$ relatively prime. Then, the following statements hold.

(a) If system (1) is Liouvillian integrable, then it has a Darboux Jacobian multiplier.
(b) If system (1) is Darboux integrable, then it has a rational Jacobian multiplier.
(c) If system (1) is polynomial integrable, then it has a polynomial Jacobian multiplier.

Statement (a) with $n = 2$ is due to Singer (1992), and Christopher (1999) provided a different proof, see also Christopher and Li (2007) and Pereira (2002). Statement (a) with $n > 2$ was proved recently by Zhang (2014a). We include this statement here for completeness. Statement (b) was proved in Chavarriga et al. (2003) for $n = 2$. The proof of statement (c) with $n = 2$ follows from Chavarriga et al. (2003) and Ferragut et al. (2007).

This paper is organized as follows. In the next section, we prove our results. In Sect. 3, we present an application of Theorem 2.
2 Proof of the Main Results

For proving Theorem 2, we need the following result, which is due to Olver, see Olver (1993, Theorem 2.16), and it reveals the essential property of functional dependence.

**Theorem 5** Assume that $M \subset \mathbb{R}^n$ is a $C^\infty$ manifold, and $g_1, \ldots, g_k$ are real $C^1$ functions on $M$. Then, $g_1, \ldots, g_k$ are functionally dependent on $M$ if and only if for all $x \in M$, there exists a neighborhood $U$ of $x$ and a $C^1$ real function $F(z_1, \ldots, z_k)$ in $k$ variables such that

$$F(g_1(x), \ldots, g_k(x)) \equiv 0, \quad x \in U.$$  

We must mention that the idea of the proof of statement (a) of Theorem 2 partially comes from Zhang (2014a, b), and the proof of statement (c) partially comes from Giné and Llibre (2011).

**Proof of Theorem 2** Since $H_1(x), \ldots, H_{n-1}(x)$ are $C^r$ first integrals of system (1) in $\Omega$, by definition we have

$$\partial_1 H_1 f_1 + \cdots + \partial_{n-1} H_1 f_{n-1} + \partial_n H_1 f_n = 0,$$

$$\vdots$$

$$\partial_1 H_{n-1} f_1 + \cdots + \partial_{n-1} H_{n-1} f_{n-1} + \partial_n H_{n-1} f_n = 0. \quad (3)$$

Since $H_1, \ldots, H_{n-1}$ are functionally independent in $\Omega$, we can assume without loss of generality that

$$D(\mathcal{H}) := \det (\partial_1 \mathcal{H}, \ldots, \partial_{n-1} \mathcal{H}) \neq 0, \quad x \in \Omega_0 \subset \Omega,$$

where $\Omega_0$ is a full Lebesgue measure subset of $\Omega$, and

$$\mathcal{H} := (H_1, \ldots, H_{n-1})^T,$$

$$\partial_i \mathcal{H} := (\partial_i H_1, \ldots, \partial_i H_{n-1})^T, \quad i = 1, \ldots, n.$$  

where $T$ denotes the transpose of a matrix.

For $i = 1, \ldots, n-1$, set

$$D_i(\mathcal{H}) := \det (\partial_1 \mathcal{H}, \ldots, \partial_{i-1} \mathcal{H}, \partial_i \mathcal{H}, \partial_{i+1} \mathcal{H}, \ldots, \partial_{n-1} \mathcal{H}).$$

Using Cramer’s rule to solve (3) with respect to $f_1, \ldots, f_{n-1}$, we get

$$f_i(x) = -\frac{D_i(\mathcal{H})}{D(\mathcal{H})} f_n(x), \quad i = 1, \ldots, n-1. \quad (4)$$

It follows that

$$D(\mathcal{H})(f_1(x), \ldots, f_{n-1}(x)) = -(D_1(\mathcal{H}), \ldots, D_{n-1}(\mathcal{H})) f_n(x). \quad (5)$$
Set
\[ Q(x) = \frac{D(H)}{f_n(x)}. \]  
(6)

It is clear that \( Q \) is defined in a full Lebesgue measure subset \( \Omega_Q \subset \Omega_0 \subset \Omega \) and is a \( C^{r-1} \) function in \( \Omega_Q \). Moreover, we get from (5) that
\[ D_i(H)(x) = -Q(x) f_i(x), \quad i = 1, \ldots, n - 1. \]  
(7)

We claim that \( Q(x) \) is a Jacobian multiplier of system (1) in \( \Omega_0 \). We now prove this claim. It follows from (6) and (7) that
\[ \sum_{i=1}^{n-1} \partial_i (Q f_i) + \partial_n (Q f_n) = - \sum_{i=1}^{n-1} \partial_i D_i(H) + \partial_n D(H). \]  
(8)

Next, we only need to prove that the right-hand side of (8) is identically zero. Using the derivative of a determinant, we get easily that
\[
\partial_n D(H) = \sum_{i=1}^{n-1} \det(\partial_1 H, \ldots, \partial_{i-1} H, \partial_n, \partial_i H, \partial_{i+1} H, \ldots, \partial_{n-1} H).
\]
\[
\partial_i D_i(H) = \det(\partial_1 H, \ldots, \partial_{i-1} H, \partial_i, \partial_{n} H, \partial_{i+1} H, \ldots, \partial_{n-1} H)
+ \sum_{j=1, j \neq i}^{n-1} \det(\partial_1 H, \ldots, \partial_{j-1} H, \partial_j, \partial_{n} H, \partial_{j+1} H, \ldots, \partial_{i-1} H, \partial_{i+1} H, \ldots, \partial_{n-1} H).
\]
Hence, we have
\[
\partial_n D(H) - \sum_{i=1}^{n-1} \partial_i D_i(H)
= - \sum_{i=1}^{n-1} \sum_{j=1, j \neq i}^{n-1} \det(\partial_1 H, \ldots, \partial_{j-1} H, \partial_j, \partial_{n} H, \partial_{j+1} H, \ldots, \partial_{i-1} H, \partial_{i+1} H, \ldots, \partial_{n-1} H)
= 0,
\]
because in the last equality we have used the fact that
\[
\det(\partial_1 H, \ldots, \partial_{j-1} H, \partial_j, \partial_{n} H, \partial_{j+1} H, \ldots, \partial_{i-1} H, \partial_{i+1} H, \ldots, \partial_{n-1} H)
+ \det(\partial_1 H, \ldots, \partial_{j-1} H, \partial_n, \partial_{j+1} H, \ldots, \partial_{i-1} H, \partial_j, \partial_{i+1} H, \ldots, \partial_{n-1} H)
= 0.
\]
Now it follows from (8) that
\[ \sum_{i=1}^{n-1} \partial_i(Qf_i) + \partial_n(Qf_n) \equiv 0. \]

That is \( Q(x) \) is a Jacobian multiplier, and consequently, statement (a) follows.

For proving (b), we note that \( \mathcal{X}(J) = -J \text{div}\mathcal{X} \) and \( \text{div}\mathcal{X} \neq 0 \) in a full Lebesgue measure subset of \( \Omega \), so \( \mathcal{X}(J) \neq 0 \) in a full Lebesgue measure subset of \( \Omega \). By contrary, if \( J \) is not functionally independent of \( H_1, \ldots, H_{n-1} \) in \( \Omega \), then they are functionally dependent in a positive Lebesgue measure subset of \( \Omega \). Since \( J, H_1, \ldots, H_{n-1} \) are smooth functions in a full Lebesgue measure subset of \( \Omega \), there exists an open subset \( \Omega_0 \) of \( \Omega \), in which \( J, H_1, \ldots, H_{n-1} \) are functionally dependent. By Theorem 5 for any \( x_0 \in \Omega_0 \), there exist an open subset \( \Omega_1 \subset \Omega_0 \) with \( x_0 \in \Omega_1 \) and a smooth function \( g(z_1, \ldots, z_n) \) such that \( g(J(x), H_1(x), \ldots, H_{n-1}(x)) \equiv 0 \) in \( \Omega_1 \) and \( \frac{\partial g}{\partial z_1} \neq 0 \). Then, by the implicit function theorem, we have \( J(x) = h(H_1(x), \ldots, H_{n-1}(x)) \) in \( \Omega_2 \) (an open subset of \( \Omega_1 \)) where \( x_0 \in \Omega_2 \) and \( h \) is a smooth function. Since \( H_1, \ldots, H_{n-1} \) are first integrals of the vector field \( \mathcal{X} \), it follows that
\[ \mathcal{X}(J) = \mathcal{X}(h(H_1, \ldots, H_{n-1})) \equiv 0, \quad x \in \Omega_2. \]

This is in contradiction to the fact that \( \mathcal{X}(J) \neq 0 \) in a full Lebesgue measure subset of \( \Omega \). This contradiction shows that \( J, H_1, \ldots, H_{n-1} \) are functionally independent in a full Lebesgue measure subset of \( \Omega \), and so statement (b) follows.

By statement (a), system (1) has a \( C^{r-1} \) Jacobian multiplier \( J(x) \). Since \( \text{div}\mathcal{X} \neq 0 \) in a full Lebesgue measure subset of \( \Omega \), it follows from (b) that the functions \( J, H_1, \ldots, H_{n-1} \) are functionally independent. So there exists a full Lebesgue measure subset \( \Omega_0 \subset \Omega \) such that \( \nabla J, \nabla H_1, \ldots, \nabla H_{n-1} \) have rank \( n \) at all points of \( \Omega_0 \).

Since \( \text{div}\mathcal{X} \neq 0 \) in a full Lebesgue measure subset of \( \Omega \), there exists a full Lebesgue measure subset \( \Omega_0 \subset \Omega \) such that \( \nabla J, \nabla H_1, \ldots, \nabla H_{n-1} \) have rank \( n \) at all points of \( \Omega_0 \). Taking the invertible change of variables
\[ y_n = J(x), \quad y_i = J(x)H_i, \quad i = 1, \ldots, n-1, \quad x \in \Omega_0, \]
we have
\[ \dot{y}_n = \dot{J} = -J \text{div} f = -y_n \text{div} f, \]
\[ \dot{y}_i = JH_i + J \dot{H}_i = \dot{J}H_i = -J \text{div} f H_i = -y_i \text{div} f. \]

This proves that system (1) is \( C^{r-1} \) orbitally equivalent to the linear system (2). Hence, statement (c) follows. This completes the proof of the theorem. \( \Box \)

**Proof of Proposition 3** We note that \( D(x) = D(H)(x) \). The latter was defined in the proof of Theorem 2. By assumption, it follows that \( D(x) \) is a \( C^1 \) function and does not
vanish on a full Lebesgue measure subset of \( \Omega \). From (4) of the proof of Theorem 2, we have that \( f_n(x) \) does not vanish on a full Lebesgue measure subset of \( \Omega \). Otherwise, all \( f_i \)s are almost zero, and so system (1) has a positive Lebesgue measure subset of singularities, a contradiction.

The proof of Theorem 2 shows that the vector field \( \mathcal{X} \) has the Jacobian multiplier \( Q = D(x)/f_n(x) \). Again from the proof of Theorem 2 and working with the vector field \( \mathcal{X} \), we get that

\[
\begin{vmatrix} 
\frac{\partial Q}{\partial Q H_1} & \cdots & \frac{\partial Q}{\partial Q H_{n-1}} \\
\vdots & \ddots & \vdots \\
\frac{\partial Q}{\partial Q H_{n-1}} & \cdots & \frac{\partial Q}{\partial Q H_{n-1}} 
\end{vmatrix} = (-1)^{n+1} Q^n \mathcal{X}(Q) \\
\quad = (-1)^{n+1} \left( \frac{D(x)}{f_n(x)} \right)^{n+1} \text{div}\mathcal{X}.
\]

This shows that the transformation from system \( \dot{x} = h(x)f(x) \) to \( \dot{y} = y \) defined by

\[
y_1 = Q(x)H_1(x), \ldots, y_{n-1} = Q(x)H_{n-1}(x), y_n = Q(x),
\]

is a \( C^{r-1} \) diffeomorphism on \( \Omega_0 := \{ x \in \Omega : D(x)/f_n(x) \text{div}\mathcal{X} \neq 0 \} \). Obviously, \( \Omega \setminus \Omega_0 = \{ x \in \Omega : D(x)/f_n(x) \text{div}\mathcal{X} = 0 \} \) is a zero Lebesgue measure subset of \( \Omega \). This proves the proposition. \( \square \)

**Proof of Theorem 4** Recall that statement (a) was proved in Singer (1992) and Zhang (2014a). We now prove statements (b) and (c).

Let \( H_1(x), \ldots, H_{n-1}(x) \) be \( n-1 \) functionally independent Darboux first integrals of the polynomial differential system (1). Taking \( G_i(x) = \log H_i(x) \), \( i = 1, \ldots, n-1 \). Then, \( G_1(x), \ldots, G_{n-1}(x) \) are also functionally independent first integrals of system (1).

We assume without loss of generality that

\[
\mathcal{G}(x) := \det \begin{pmatrix}
\frac{\partial G_1(x)}{\partial G_1(x)} & \cdots & \frac{\partial G_{n-1}(x)}{\partial G_1(x)} \\
\vdots & \ddots & \vdots \\
\frac{\partial G_{n-1}(x)}{\partial G_{n-1}(x)} & \cdots & \frac{\partial G_{n-1}(x)}{\partial G_{n-1}(x)}
\end{pmatrix},
\]

is not zero in \( \mathbb{R}^n \) except perhaps a zero Lebesgue measure subset. Then, we get from the proof of Proposition 3 that \( f_n \neq 0 \) in a full Lebesgue measure subset of \( \Omega \). Since \( H_i(x) \) is a Darboux function, we assume that it is of the form

\[
H_i(x) = \frac{k_i}{h_i(x)} \exp \left( \frac{q_i(x)}{h_i(x)} \right),
\]

\( \square \) Springer
where $g_{ij}, q_i, h_i$ are polynomials, and $k_{ij} \in \mathbb{C}, j = 1, \ldots, r_i$. Computing the partial derivative of $G_i(x) = \log H_i(x)$ with respect to $x_s$ for $s = 1, \ldots, n$, we get
\[
\partial_s G_i(x) = \sum_{j=1}^{r_i} k_{ij} \frac{\partial_s g_{ij}}{g_{ij}} + \frac{h_i \partial_s q_i - q_i \partial_s h_i}{h_i^2}.
\]
This is a rational function. So $G(x)$ is also a rational function.

By Theorem 2 and its proof, it follows that system (1) has the rational Jacobian multiplier $Q(x) = G(x)/f_n(x)$, because $f_n$ is a polynomial. Hence, statement (b) follows.

Finally, we prove statement (c). Let $H_1, \ldots, H_{n-1}$ be $n-1$ functionally independent polynomial first integrals of the polynomial differential system (1). Here, we will use the notations defined in the proof of Theorem 2. We assume that $D(H) := \det(\partial_1 H, \ldots, \partial_{n-1} H) \neq 0$ in $\mathbb{R}^n$ except perhaps a zero Lebesgue measure subset. Recall that $H = (H_1, \ldots, H_{n-1})^T$ and $\partial_i H = (\partial_i H_1, \ldots, \partial_i H_{n-1})^T$, $i = 1, \ldots, n$. The proof of Theorem 2 shows that
\[
D(H)(f_1(x), \ldots, f_{n-1}(x)) = -(D_1(H), \ldots, D_{n-1}(H))f_n(x).
\]
Since $D(H), D_1(H), \ldots, D_{n-1}(H)$ are polynomials and $f_1(x), \ldots, f_n(x)$ are relatively prime polynomials, it verifies that $f_n(x)$ divides $D(H)(x)$. Hence, system (1) has the polynomial Jacobian multiplier
\[
Q(x) = D(H)(x)/f_n(x).
\]
This proves statement (c) and consequently the theorem.

3 Two Applications of Theorem 2

Consider the differential system
\[
\dot{x} = -y - z, \quad \dot{y} = x, \quad \dot{z} = xz,
\]
which is the only completely integrable case of the Rössler differential system constructed by Rössler (1987) in 1976. O.E. Rössler inspired by the geometry of three-dimensional flows, introduced several systems in the 1970s as prototypes of the simplest autonomous differential equations having chaos, the simplicity is in the sense of minimal dimension, minimal number of parameters, and minimal nonlinearities. In MathSciNet appear at this moment more than 197 articles about the Rössler's systems. This unique integrable Rössler differential system was first proved in Zhang (2004). Recently, system (9) was studied from the Poisson dynamics point of view, see Tudoran and Gîrban (2012).

We can check that system (9) has the two functionally independent first integrals
\[
H_1(x, y, z) = \frac{1}{2}(x^2 + y^2) + z, \quad H_2(x, y, z) = e^{-y}z,
\]
and the Jacobian multiplier $J = e^{-y}$. We can check that the transformation of variables

$$u = J(x, y, z), \quad v = J(x, y, z)H_1(x, y, z), \quad w = J(x, y, z)H_2(x, y, z),$$

is invertible in the region $\Omega_0 := \{(x, y, z) \in \mathbb{R}^3 | x \neq 0\}$, because the Jacobian determinant of this transformation is $xe^{-4y}$. By Theorem 2, system (9) is transformed to

$$\dot{u} = u, \quad \dot{v} = v, \quad \dot{w} = w,$$

via the change of variables (10) in $\Omega_0$. We note that the divergence of system (9) is $x$.

The Lotka–Volterra systems are classical differential systems introduced independently by Lotka and Volterra in the 1920s to model the interaction among species, see Lotka (1920), Volterra (1931), see also Kolmogorov (1936). A particular class of the three-dimensional Lotka–Volterra systems is the so-called May–Leonard models May and Leonard (1975). A completely integrable May and Leonard differential system is

$$\begin{align*}
\dot{x} &= x(1 - x + y + z), \\
\dot{y} &= y(1 + x - y + z), \\
\dot{z} &= z(1 + x + y - z).
\end{align*}$$

See Blé et al. (2013) for the study of the integrability of this system.

We can check that system (12) has the two functionally independent first integrals

$$H_1(x, y, z) = \frac{y(x - z)}{x(y - z)},$$

$$H_2(x, y, z) = \frac{2x^2(y - z)^2}{(x - y)y^{5/2}(x - z)^{3/2}} \left(2y(y - x) + \sqrt{\frac{(x - y)^2y^2}{x - z}} F \left(\arcsin \left(\frac{\sqrt{y}}{\sqrt{x}}\right) \left| \frac{x}{y(x - z)}\right.\right) - \sqrt{\frac{(x - y)y^2}{z}} F \left(\arcsin \left(\sqrt{1 - \frac{y}{x}}\right) \left| \frac{x}{(x - y)z}\right.\right)\right),$$

where $F(\Phi|m)$ is the elliptic integral of the first kind. The Jacobian multiplier is

$$J = \frac{1}{y^{3/2}(x - z)^{3/2}}.$$

We can check that the transformation of variables (10) is invertible in the region $\Omega_0 := \{(x, y, z) \in \mathbb{R}^3 | xyz(x - y)(x - z)(y - z) \neq 0\}$, because the Jacobian determinant of this transformation is $6/(y^6(x - z)^6)$ and the two first integrals $H_1$ and $H_2$ must be well defined. By Theorem 2, system (12) is transformed to system (11) via the change of variables (10) in $\Omega_0$. We note that the divergence of system (12) is 0.
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