Helical magnetogenesis with reheating phase from higher curvature coupling and baryogenesis

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We investigate the generation of helical magnetic fields and address the baryon asymmetry of the universe from an inflationary magnetogenesis scenario, in which the conformal and parity symmetries of the electromagnetic field are broken through its coupling to the Ricci scalar and to the Gauss-Bonnet invariant via the dual field tensor, so that the generated magnetic field can have a helical nature. Depending on the reheating mechanism, we consider two different cases - (1) instantaneous reheating scenario, in which case the reheating e-fold number is zero, and (2) Kamionkowski reheating scenario which is parametrized by a non-zero e-fold number, a reheating equation of state parameter and a given reheating temperature. It is demonstrated that for both the reheating mechanisms, the generated magnetic fields can be compatible with the observations for suitable range of the model parameter present in the non-minimal coupling of the electromagnetic field. Actually the present magnetogenesis model does not produce sufficient hierarchy between the electric and magnetic fields at the end of inflation, and thus the electric field is not able to sufficiently induce (or enhance) the magnetic field during the Kamionkowski reheating stage. This in turn makes both the reheating cases almost similar from the perspective of magnetic field’s evolution. Furthermore we find that the magnetic fields at the galactic scale with strength $\sim 10^{-13} G$ can lead to the resultant value of the ratio of the baryonic number density to the entropy density as large as $\sim 10^{-10}$, which is consistent with the observational data.

I. INTRODUCTION

Large scale magnetic field and baryon asymmetry of the universe are some of the important unresolved issues in the standard model of cosmology and particle physics. Despite of large attempts being made to understand those problems, it is still an active area of research. Although both the issues are seemingly disconnected, endeavour towards unified theoretical description would be extremely interesting. In order to proceed, one needs to go beyond standard model of cosmology and particle physics. Various astrophysical observations indicate that the universe is magnetized over a wide range of length scales. The magnetic fields have been detected in galaxies, galaxy clusters and even in intergalactic voids [1,8]. On the other hand, it is observationally known that the universe carries a net excess of baryons over antibaryons [7,11]. Such baryon asymmetry of the universe can be realized from the Sakharov’s three conditions [17]: (1) baryon number non-conservation, (2) Charge(C) and Charge parity(CP) violation, and (3) departure from thermal equilibrium.

Our current understanding on the origin of large scale magnetic fields are broadly classified into: (1) the astrophysical origin associated with dynamo mechanism [18,20] and (2) the primordial origin where the magnetic field is generated from the primordial quantum fluctuations during the inflationary epoch [21,44]. The inflationary magnetogenesis earned significant attention due to the fact that the inflation concomitantly solves the well known flatness and horizon problems and predicts almost scale invariant curvature perturbation which is in agreement with the Planck data [45,49] to a great precision. Due to this background curvature fluctuation the standard Maxwell’s action may produce magnetic field out of quantum vacuum, but its magnitude will naturally be very feeble and lie far below the observational constraints. The reason behind this feeble production of magnetic field is the conformal invariant nature of the standard electromagnetic theory and conformal flat nature of the background spacetime. Therefore, simplest way to enhance the strength of the magnetic field is to introduce the conformal non-invariant coupling in the electromagnetic sector. Several inflationary magnetogenesis models have been proposed so far, where the conformal non-invariant and non-minimal electromagnetic coupling has been introduced through the scalar field such as inflaton, axion and also through higher curvature terms [21,20,51,42,44,50,56]. Depending upon the basic physical requirements, those couplings can be classified into: parity invariant and parity violating magnetogenesis scenario. Interestingly, parity violating and non-conformally coupled electromagnetic field can generate helical magnetic field which has been shown to play crucial role in generating baryon asymmetry in the universe.

Apart from the inflationary magnetogenesis set-up, we should also mention a different proposal for magnetic field
generation, in particular, from the bouncing scenario \cite{57,59}. Similar to inflation, bounce scenario is also able to generate sufficient strength of magnetic field at present universe. However bounce models are plagued with some problems, like the instability of curvature perturbation near the bounce, BKL instability, the dark energy issue etc \cite{61,63}. Here it may be mentioned that these severe problems can be solved to some extent in some modified theories of gravity \cite{61,65}.

In regard to the relation between magnetogenesis and baryogenesis; it is well known that the presence of magnetic field leads to: (a) the dynamics of plasma to be out-of-thermal-equilibrium as in thermal equilibrium the photon distribution must be thermal which is incompatible with existence of long range magnetic field, (b) violation of time reversal (T) or in other words CP symmetry as well as C symmetry, and (c) being a vector quantity, the presence of magnetic field breaks the rotational invariance or SO(3) symmetry \cite{67}. Those are precisely the Sakharov’s conditions for the successful generation of baryogenesis. Therefore, once we assume the genesis of magnetic field in the early universe that can act as a potential source for the production of baryon asymmetry in the universe. Motivated by this, Giovannini and Shaposhnikov \cite{9,10} first investigated the generation of baryon asymmetry through the abelian anomaly in the form of $U(1)$ Chern-Simons term which assumes non-zero vacuum expectation value due to helical magnetic field. This in turn produces net baryon asymmetry through the non-conservation of baryon number current. Related proposals are investigated in \cite{65,71}. Hence, it is important to have non-zero helicity of the magnetic field which is varying with the cosmological evolution. Axion- electrodynamics has been widely studied in this regard. Non-trivial dynamics of axion field produces helical magnetic field which finally leads to the baryon asymmetry of the universe \cite{33}. Recently, the baryogenesis has been addressed from the production of helical magnetic fields, where the electromagnetic field gets coupled with the background Riemann tensor by the dual field tensor \cite{72}. Such coupling between the electromagnetic field and the Riemann tensor gives rise to helical nature of magnetic field, which in turn leads to a net baryonic number density as of the order $n_B/s_0 \sim 10^{-10}$, that is consistent with the cosmic microwave background (CMB) observations (where $n_B$ is the net baryonic number density and $s_0$ is the entropy density of the present universe). From a different perspective, the article \cite{73} discussed the baryogenesis from magnetic fields, however from MHD amplification.

In the present work, we propose a helical magnetogenesis scenario from inflationary set-up and address the baryon asymmetry of the universe (BAU), from higher curvature coupling of the electromagnetic (EM) field. In particular, the EM field couples with the background higher curvature term(s), namely with the Ricci scalar and with the Gauss-Bonnet invariant, via the dual field tensor. The cosmology in Gauss-Bonnet gravity theory from various perspectives have been discussed in \cite{74,79}. The presence of the non-minimal coupling in the EM action breaks the conformal and the parity symmetry, however preserves the $U(1)$ invariance, of the EM field, which in turn leads to the gauge field production from the primordial Bunch-Davies vacuum. Moreover the positive and negative helicity modes of the EM field get different amplitudes along their cosmological evolution, and thus give rise to helical nature of the magnetic field. The Chern-Simons number stored in the helical magnetic fields is directly connected to the net baryonic number density of the universe. In regard to the the background spacetime, it is considered to be the $\alpha$-attractor scalar-tensor theory which is known to provide viable inflationary scenario consistent with the Planck results. Owing to the fact that the non-minimal coupling of the EM field occurs via the dual field tensor, the kinetic term of the EM field remains canonical even in presence of such non-minimal coupling, and thus the strong coupling problem is naturally resolved in the present magnetogenesis model. In such scenario, we explore the cosmological evolution of the EM field and consequently determine the net baryon density of the universe. Most of the earlier magnetogenesis scenarios did not explore the effect of reheating phase until recently. Reheating can be important in this regard has recently been realized in \cite{33}, and further explored in \cite{29,31}. During the expansion, the universe enters into a reheating phase after inflation and depending on the reheating dynamics, we consider two different cases: (1) the instantaneous reheating case where the universe suddenly jumps to the radiation dominated epoch after the end of inflation, and thus the reheating era has zero e-fold number, and (2) the case where the reheating era possesses a non-zero e-fold number, in particular, we consider the Kamionkowski reheating mechanism that parametrizes the reheating dynamics by a non-zero e-fold number, a constant equation of state parameter and a given reheating temperature \cite{80} (see also \cite{81,60}). These two cases make certain differences in the evolution of the helicity power spectrum. Here we would like to mention that the baryogenesis from helical magnetic fields with curvature couplings have been proposed earlier, however in quite different contexts \cite{72}. It may be noted that in our present analysis, we include the higher curvature Gauss-Bonnet coupling in the set-up and also discuss the possible effects of the reheating phase in the production of helical magnetic field and consequently in the baryogenesis, which makes the present model essentially different from earlier ones.

The paper is organized as follows: in Sec. II, we describe the model; in Sec. III, we give the general expressions for the EM power spectra and the helicity spectrum in the present; Sec. IV is reserved for the solution of the vector potential, the electromagnetic energy density and the helicity spectrum during inflation. The corresponding calculations during the reheating phase are carried out in Sec. V and Sec. VI, which correspond to the cases of instantaneous reheating and a Kamionkowski like reheating model, respectively. The paper ends with some conclusions.
II. THE MODEL

The action is,

\[ S = S_{\text{grav}} + S_{\text{em}}^{(\text{can})} + S_{\text{CB}}, \]

where \( S_{\text{grav}} \), namely the gravitational action, determines the background evolution and a non-minimally coupled electromagnetic (EM) field propagates over the background spacetime, where \( S_{\text{em}}^{(\text{can})} \) and \( S_{\text{CB}} \) represent the canonical kinetic term and the non-minimal coupling function of the EM field respectively. The EM field couples with the higher curvature of the background spacetime, which actually provides the non-minimal coupling term in the action.

We focus on a specific inflationary model that permits slow roll inflation. In particular, we consider the so-called \( \alpha \)-attractor model, which unifies a large number of inflationary potentials \[89\]. If \( \Phi \) is the canonical scalar field driving inflation, the model is described by the following action,

\[ S_{\text{grav}} = \int d^4x \sqrt{-g} \left[ \frac{R}{2\kappa^2} - \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - V(\Phi) \right], \]

where \( \kappa^2 = 8\pi G \) (\( G \) is the Newton’s constant) and \( V(\Phi) \) denotes the scalar field potential having the form,

\[ V(\Phi) = \Lambda^4 \left[ 1 - \exp \left( -\kappa \Phi \sqrt{\frac{2}{3\alpha}} \right) \right]^{2n}. \]

Here the scale \( \Lambda \) can be determined using the constraints from the observations of the anisotropies in the CMB. It is worth pointing out here that, for \( \alpha = 1 \) and \( n = 1 \), the above potential reduces to the well known Starobinsky model. We should also mention that the potential in Eq.\[3\] contains a plateau at suitably large values of the field, which is favored by the CMB data. Finally, the \( \alpha \)-attractor inflationary model, i.e the \( S_{\text{grav}} \), is known to provide viable inflationary scenario, with the observable quantities like the spectral index, tensor-to-scalar ratio being compatible with the Planck 2018 constraints \[7\].

The canonical kinetic term of the EM field is given by,

\[ S_{\text{em}}^{(\text{can})} = \int d^4x \sqrt{-g} \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right], \]

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the EM field tensor and \( A_\mu \) is the corresponding field. In regard to the \( S_{\text{CB}} \), as mentioned earlier, the EM field non-minimally couples with the spacetime curvature of the background spacetime, in particular,

\[ S_{\text{CB}} = \int d^4x \sqrt{-g} f(R, G) \left[ -\lambda F_{\mu\nu} \tilde{F}^{\mu\nu} \right], \]

where \( \tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \) with \( \epsilon^{\mu\nu\alpha\beta} \) is the four dimensional Levi-Civita tensor and defined by \( \epsilon^{\mu\nu\alpha\beta} = -\frac{1}{2} \{ \mu\nu\alpha\beta \} \), the \( \{ \mu\nu\alpha\beta \} \) denotes the completely antisymmetric permutation having \( \{0123\} = 1 \). The EM coupling function \( f(R, G) \) is considered to depend on the background Ricci scalar and Gauss-Bonnet curvature respectively, particularly the form of \( f(R, G) \) is given by,

\[ f(R, G) = \kappa^2 q (R^q + G^{q/2}) \]

with \( q \) being a parameter of the model. Later we will show that \( q \sim O(1) \) is compatible with the large scale observations of present magnetic strength. It may be observed that the EM field gets the non-minimal coupling via the dual field tensor, in particular by \( F_{\mu\nu} \tilde{F}^{\mu\nu} \) controlled by the dimensionless quantity \( \lambda f(R, G) \), which acts as a parity violating agent in the EM field action and ensures the helical nature of the magnetic field. Here it deserves mentioning that the \( S_{\text{CB}} \) spoils the conformal invariance of the EM field, although preserves the U(1) invariance of the gauge field. The broken conformal symmetry is the key ingredient in the magnetogenesis scenario, otherwise the EM field energy density redshifts as \( 1/a^4 \) with the cosmological expansion of the universe and results to a very feeble magnetic strength at present epoch – not compatible with the CMB observation at all. During the early universe when the spacetime curvature is high, the conformal breaking coupling leads to a non-trivial contribution to the EM field equations, however at late times (in particular at the end of inflation) the \( S_{\text{CB}} \) will not contribute and consequently the EM field behaves as standard Maxwellian theory. Due to the presence of \( F_{\mu\nu} \tilde{F}^{\mu\nu} \) (\( \equiv \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \)) in Eq.\[5\], the conformal breaking coupling does not contain the term like \( (A_i')^2 \) (where the prime denotes the derivative with
respect to the conformal time) and thus the kinetic term of the EM field, i.e. $(A')^2$, comes only through the $S_{\text{em}}^{(\text{can})}$ given in Eq. (4). This in turn indicates that the EM kinetic term remains canonical in the present magnetogenesis model where the EM field gets coupled with the spacetime curvature through the corresponding dual field tensor. As a consequence, the model is free from the strong coupling problem. Moreover we will show later that the EM field energy density has negligible backreaction on the background inflationary FRW spacetime, which ensures the resolution of the backreaction issue in the present context.

In such magnetogenesis set-up, we aim to determine the magnetic strength at the present epoch and consequently the baryon asymmetry of the universe from the helical nature of the magnetic field. Varying the action (4) with respect to $A_\mu$, we get the field equation for the EM field as,

$$\partial_\alpha \left[ \sqrt{-g} \left\{ g^{\mu\nu} \partial_\mu A_\nu + 8\lambda f(R, G) \epsilon^{\mu\nu\alpha\beta} \partial_\mu A_\nu \right\} \right] = 0 \ ,$$  

(7)

where it may be noticed that the conformal breaking coupling modifies the EM field equation compared to the standard Maxwell’s equation. A spatially flat FRW metric ansatz will fulfill our purpose in the present context, i.e

$$ds^2 = a^2(\eta) \left[ -d\eta^2 + d\vec{x}^2 \right] \ ,$$  

(8)

where $a(\eta)$ is the scale factor of the universe with $\eta$ being the conformal time and related to the cosmic time $(t)$ by $d\eta = \frac{dt}{a(t)}$. For this metric ansatz, the temporal component of the EM field Eq.(7) becomes,

$$\partial_\mu \left[ \partial_\mu A_0 - \partial_0 A_i \right] = 0 \ .$$  

(9)

Due to the antisymmetric nature of $\epsilon^{\mu\nu\alpha\beta}$, the temporal component Eq.(9) seems to free from the effect of the coupling function $f(R, G)$. Here we consider the Coulomb gauge condition, in particular $\partial_\mu A_\mu = 0$, due to which, Eq.(9) leads to the solution as $A_0 = 0$. Therefore in effect of $\partial_i A^i = 0$ and $A_0 = 0$, the spatial component of the EM field Eq.(7) takes the following form,

$$A_i''(\eta, \vec{x}) - \partial_i \partial_\mu A_\mu + 8\lambda' f(R, G) \epsilon_{ijk} \partial_j A_k = 0 \ ,$$  

(10)

where $\epsilon_{ijk} = [0ijk]$. As mentioned earlier, the $\alpha$-attractor model is able to drive successful inflationary scenario during the early universe for suitable values of $\alpha$ and $n$ respectively. Keeping this in mind, we consider a quasi-de-Sitter inflationary background spacetime in the present context, where the scale factor has the following form,

$$a(\eta) = \left( \frac{-\eta}{\eta_0} \right)^{\beta + 1} \quad \text{with} \quad \beta = -2 - \epsilon = -3 + \frac{H'}{H^2} .$$  

(11)

Here, and also in the subsequent calculation, a prime denotes $\frac{d}{d\eta}$, $H$ is the conformal Hubble parameter and defined by $H = a'/a$. Moreover $\epsilon$ is known as the slow roll parameter having the expression $\epsilon = -\frac{H'}{H^2} + 1$. Eq.(11) evidents that for $\epsilon = 0$, the scale factor becomes $a(\eta) \propto (\eta)^{-1}$ which results to the de-Sitter evolution of the universe. The scale factor immediately leads to the conformal Hubble parameter, Ricci scalar and the Gauss-Bonnet invariant as,

$$H = \frac{\beta + 1}{\eta}$$  

(12)

and

$$R = \frac{6}{a^2} (H'^2 + H^2) = \frac{6(\beta + 1)}{\eta_0^2} \left( \frac{-\eta}{\eta_0} \right)^{2\epsilon}$$

$$G = \frac{24}{a^4} H^2 H' = -\frac{24(\beta + 1)^3}{\eta_0^4} \left( \frac{-\eta}{\eta_0} \right)^{4\epsilon} ,$$  

(13)

respectively. The cosmic Hubble parameter (defined by $H = \dot{a}/a$, where an overdot represents $\frac{d}{dt}$) is related to the conformal Hubble parameter as $H = \frac{1}{a} \dot{H}$ and thus we can express $H$ as

$$H = \frac{1}{\eta_0} \exp (-\epsilon N) \ ,$$  

(14)

in terms of the e-folding number (symbolized by $N$). The e-folding number up-to the time $\eta$ is defined as $N = \int^0 aH \ d\eta$, where $N = 0$ refers the instance of the beginning of inflation which, in the present context, we consider to happen
When the CMB scale mode ($\sim 0.05 \text{Mpc}^{-1}$) crosses the Hubble horizon. If $N_t$ denotes the total e-fold number of inflation [89], then

$$N_t = \frac{3\alpha}{4n} \left[ \exp \left( \kappa \Phi_1 \frac{2}{3\alpha} \right) - \exp \left( \kappa \Phi_t \frac{2}{3\alpha} \right) - \kappa \frac{2}{3\alpha} (\Phi_t - \Phi_1) \right],$$

where $\Phi_1$ and $\Phi_t$ are given by,

$$\Phi_1 = \frac{1}{\kappa} \sqrt{\frac{3\alpha}{2}} \ln \left( 1 + \frac{4n + \sqrt{16n^2 + 24\alpha n (1 - n_s) (1 + n)}}{3\alpha (1 - n_s)} \right),$$

$$\Phi_t = \frac{1}{\kappa} \sqrt{\frac{3\alpha}{2}} \ln \left( 1 + \frac{2n}{\sqrt{3\alpha}} \right)$$

respectively, with $n_s$ being the spectral index of curvature perturbation. Moreover the Hubble parameter at the beginning of inflation (i.e. $H_0$), in terms of $\alpha$, $n$ and $n_s$, is given by [89],

$$H_0 = \frac{8n\pi\sqrt{A_s}}{\kappa \sqrt{6A \left( e^{\Phi_1 \sqrt{\frac{2\eta}{\kappa}} - 1} \right)}.$$

Here $A_s$ denotes the amplitude of the curvature perturbation. Recall that the latest Planck 2018 data [7] puts a constraint on the scalar spectral index as $n_s = [0.9649 \pm 0.0042]$. Throughout the paper, we will consider $\alpha = 1$, $n = 1$, $n_s = 0.9625$ and $A_s = 2.2 \times 10^{-9}$, for which one gets $N_t = 51.27$ and $H_0 = 1.6 \times 10^{13}\text{GeV}$.

Eq. (14) indicates that at the beginning of inflation, the cosmic Hubble parameter acquires $H_0 = 1/\eta_0$. Plugging the above expressions of $R$ and $G$ into Eq. (6) and a little bit of simplification yield the form of $f(R, G)$ as,

$$f(R, G) = \alpha = \frac{\kappa^{2n} \left[ 6\beta (\beta + 1) \right]^q + \left[ -24\beta (\beta + 1)^3 \right]^{q/2}}{\eta_0^{nq/2}} \left( \frac{-\eta}{\eta_0} \right)^{2eq}.$$

It may be observed from the above expression that the condition $\epsilon = 0$ leads to the function $f(R, G)$ as a constant, for which the EM field Eq. (10) becomes similar to the standard Maxwell’s equation. This is however expected because for constant $f(R, G)$, the conformal breaking term $\mathcal{L}_{CB} = \sqrt{-g} \lambda f(R, G) \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}$ present in the action becomes a total surface term and thus gives no contributions in the field equations. As a result, the EM action preserves the conformal symmetry and thus the EM energy density decays by $1/a^4$ with the cosmological expansion of the universe, which in turn results to a very feeble magnetic amplitude at the current epoch. Thus the present model where the EM field couples with the background spacetime curvature via the term like $\sim \sqrt{-g} \lambda f(R, G) \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}$, requires $\epsilon \neq 0$ in order to break the conformal invariance of the EM field. Hence in order to generate sufficient strength of magnetic field and consequently the baryon asymmetry of the present universe in such higher curvature helical magnetogenesis set-up, we consider the background inflationary scenario as a quasi-de-Sitter one, in which case $\epsilon \neq 0$ and $\epsilon < 1$, in the subsequent calculations.

III. ELECTRIC, MAGNETIC AND HELICITY POWER SPECTRA

In this section we aim to calculate the energy-momentum tensor of the EM field and subsequently we will determine the individual energy density of electric and magnetic field. In this regard, it may be mentioned that the electric and magnetic field are frame dependent, and for this purpose, here we consider the comoving observer, in which case the four velocity is given by $u^\mu = (a^{-1}(\eta), 0, 0, 0)$ in the $(\eta, \vec{x})$ coordinate system. The calculation of electric, magnetic energy density at time $\eta$ requires the energy-momentum tensor and the state (at $\eta$) of the EM field respectively. We will consider that the EM field starts from the Bunch-Davies vacuum state at distant past (i.e the “no-particle” state), which is indeed compatible with the equation of motion, as we will show in the later section. Moreover, since we work in the Heisenberg picture, the state of the field remains fixed over time; however due to the interaction between the EM field and the background FRW spacetime, the particle production occurs and thus the EM field does not remain in vacuum at later time. From action (1), we determine the energy-momentum tensor of the EM field as,

$$T_{\alpha\beta} = -\frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\alpha\beta}} \left[ \sqrt{-g} \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \lambda f(R, G) \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \right) \right]$$
where we use \( \frac{\delta \sqrt{-g}}{\delta g_{ab}} = -\frac{1}{2} \sqrt{-g} g_{ab} \) and \( [\mu \nu \rho \sigma] \) represents the antisymmetric permutation. It may be observed from Eq.\([10]\) that \( T_{ab} \) contains an interaction energy density between electric and magnetic fields, which is generated entirely due to the presence of the parity violating term \( \mathcal{L}_{CB} = -\sqrt{-g} \lambda f(R, G) \varepsilon^{\mu \nu \rho \sigma} F_{\mu \nu} F_{\rho \sigma} \) in the action. As mentioned earlier, the parity violating term \( \mathcal{L}_{CB} \) in the action leads to the helical nature of the magnetic field, in which case the helicity can be quantified by the quantity \( \rho_h = -A_i B^i \), where \( B^i \) specifies the magnetic field. At this stage, we proceed to quantize the EM field, i.e. \( A_i \) is promoted to a hermitian operator \( \hat{A}_i \) as,

\[
\hat{A}_i(\eta, \vec{x}) = \int \frac{dk}{(2\pi)^3} \sum_{r=+,-} \epsilon_r \left[ \hat{b}_r(\vec{k}) A_r(k, \eta) e^{i\vec{k} \cdot \vec{x}} + \hat{b}_r^+(\vec{k}) A_r^*(k, \eta) e^{-i\vec{k} \cdot \vec{x}} \right],
\]

where \( \vec{k} \) is the Fourier mode momentum or equivalently the EM wave vector, \( r \) is the polarization index, \( \epsilon_+ \) and \( \epsilon_- \) are two polarization vectors, and \( A_r(\vec{k}) \) is the EM mode function corresponds to the mode \( \vec{k} \). Due to the helical nature of the magnetic field, it will be useful if we work with the helicity basis set in the present context, in which case the parity violating term \( \rho_h = -A_i B^i \) is promoted to a hermitian operator \( \hat{A}_i \), which will be useful later in solving the EM mode function. Furthermore \( \hat{b}_r(\vec{k}) \) and \( \hat{b}_r^+(\vec{k}) \) are the creation and annihilation operators of \( \eta \to -\infty \) respectively, in particular they are defined over the vacuum state of distant past by the condition \( \hat{b}_r(\vec{k}) |0\rangle = 0 \forall \vec{k} \). Using the Fourier mode decomposition of the EM field along with the commutation relation \( [\hat{b}_r(\vec{k}), \hat{b}^+_s(\vec{k}')] = \delta_{rs} \delta(\vec{k} - \vec{k}') \), the expectation values (over the Bunch-Davies vacuum state) of electric and magnetic energy densities come as,

\[
\langle \rho(\vec{E}) \rangle = \sum_{r=+,-} \int \frac{dk}{2\pi} \frac{k^2}{a^4} |A_r'(k, \eta)|^2
\]

\[
\langle \rho(\vec{B}) \rangle = \sum_{r=+,-} \int \frac{dk}{2\pi} \frac{k^4}{a^4} |A_r(k, \eta)|^2
\]

respectively. Similarly, the helicity energy density in the Fourier basis is given by,

\[
\langle \rho_h \rangle = \int \frac{dk}{2\pi} \frac{k^3}{a^3} \left\{ |A_+(k, \eta)|^2 - |A_-(k, \eta)|^2 \right\}.
\]

Consequently the power spectra of electric and magnetic fields (defined by the energy density within unit logarithmic interval of \( k \)) become,

\[
\mathcal{P}(\vec{E}) = \frac{\partial \langle \rho(\vec{E}) \rangle}{\partial \ln k} = \sum_{r=+,-} \frac{k^2}{2\pi^2} \frac{k^2}{a^4} |A_r'(k, \eta)|^2
\]

\[
\mathcal{P}(\vec{B}) = \frac{\partial \langle \rho(\vec{B}) \rangle}{\partial \ln k} = \sum_{r=+,-} \frac{k^4}{2\pi^2} \frac{k^4}{a^4} |A_r(k, \eta)|^2.
\]

Furthermore, the helicity power spectrum is given by,

\[
\mathcal{P}_h = \frac{\partial \langle \rho_h \rangle}{\partial \ln k} = \frac{k^3}{2\pi^2} \frac{k^3}{a^3} \left\{ |A_+(k, \eta)|^2 - |A_-(k, \eta)|^2 \right\}
\]

Here it may be observed that the scale dependence of \( \mathcal{P}(\vec{E}), \mathcal{P}(\vec{B}) \) and \( \mathcal{P}_h \) are different. In particular, the electric power spectrum gets the scale dependency from the \( k^3 \) factor as well as from the time derivative of the EM mode function, while the magnetic power spectrum has the scale dependency due to the \( k^5 \) factor and also due to the mode function itself. On the other hand, the scale dependency of the helicity spectrum comes through the \( k^3 \) factor and from the difference between the two EM mode functions. Therefore it is very unlikely, that the electric, magnetic and helicity power spectra have similar dependency on \( k \). In order to extract the scale dependence of such power spectra, we need to solve the EM mode functions from Eq.\([10]\), which is the subject of the next section.
IV. SOLUTION OF THE MODE FUNCTION AND SCALE DEPENDENCE OF THE POWER SPECTRA DURING INFLATION

In this section we aim to determine the solution of EM mode functions and the corresponding power spectra during the inflationary era when the scale factor behaves as of Eq.\([11]\). For this purpose, we need Eq.\([10]\) which, due to the Fourier decomposition of \(A_\pm(\eta,\bar{x})\), takes the following form:

\[
A_\pm''(k,\eta) + (k^2 \pm 8\lambda k f'(R, G)) A_\pm(k,\eta) = 0 ,
\]

where \(f'(R, G) = \frac{df}{dG}\). The form of \(f(R, G)\) of Eq.\([6]\) immediately leads to

\[
A_\pm''(k,\eta) + \left( k^2 \pm k \left( \frac{\zeta^2}{\eta^2} \right) \left( -\frac{\eta_0}{\eta} \right)^{2\alpha} \right) A_\pm(k,\eta) = 0 ,
\]

where \(\zeta^2\) and \(\alpha\) have the following forms,

\[
\zeta^2 = \left( 16\epsilon q \lambda \eta_0 \right) \left( \frac{\kappa}{\eta_0} \right)^2 \left\{ \left[ 6\beta(\beta + 1) \right]^q + \left[ -24(\beta + 1)^3 \right]^{q/2} \right\} ,
\]

\[
\alpha = -\frac{1}{2} - \epsilon q
\]

Here it may be mentioned that \(\zeta^2\) is proportional with \(\epsilon\), \(q\) and thus for \(\epsilon = 0\) or \(q = 0\), both the EM mode functions seem to satisfy the standard Maxwell’s equation in vacuum, as expected. Thereby the non-zero values of \(\epsilon\) (the slow roll parameter) and \(q\) modify the EM field equations non-trivially by Eq.\([26]\), compared to the standard Maxwell’s equation.

Eq.\([26]\) may not be solved in a closed form. To obtain the solution, we individually consider the sub-horizon (region I) and super-horizon (region II) limits respectively. In Region I (sub-horizon limit), the wavelength of the mode is smaller than the Hubble radius, i.e. \(H \ll k\), and thus one can neglect the term containing \(\zeta^2\) in Eq.\([26]\). In Region II (super-horizon scales), the mode lies outside the Hubble radius, i.e. \(k \ll H\), and thus we can neglect \(k^2\) in Eq.\([26]\).

While evaluating the mode-functions is trivial in Region I, it is highly non-trivial in Region II.

In region I, Eq.\([26]\) can be expressed as,

\[
A_\pm''(k,\eta) + k^2 A_\pm(k,\eta) = 0 .
\]

The Bunch-Davies vacuum state is considered to be the initial states of the EM mode functions, due to which, the solution of Eq.\([28]\) is

\[
A_\pm(k,\eta) = \frac{1}{\sqrt{2k}} e^{-ik\eta} .
\]

In region II (super-horizon scale), Eq.\([26]\) turns out to be,

\[
A_\pm''(k,\eta) \mp k \left( \frac{\zeta^2}{\eta^2} \right) \left( -\frac{\eta_0}{\eta} \right)^{2\alpha} A_\pm(k,\eta) = 0 ,
\]

on solving which, we get \(A_\pm(k,\eta)\) as (see the Appendix-A in Sec.\([VIII]\)),

\[
A_+(k,\eta) = \left( -\frac{\eta}{\eta_0} \right)^{1/2} \left\{ C_1 J_{\frac{\zeta}{\alpha}} \left( -\frac{\zeta}{\alpha} \left( -\frac{\eta_0}{\eta} \right)^{\alpha} \right) + C_2 Y_{\frac{\zeta}{\alpha}} \left( -\frac{\zeta}{\alpha} \left( -\frac{\eta_0}{\eta} \right)^{\alpha} \right) \right\} ,
\]

\[
A_-(k,\eta) = \left( -\frac{\eta}{\eta_0} \right)^{1/2} \left\{ C_3 J_{\frac{\zeta}{\alpha}} \left( \frac{\zeta}{\alpha} \left( -\frac{\eta_0}{\eta} \right)^{\alpha} \right) + C_4 Y_{\frac{\zeta}{\alpha}} \left( \frac{\zeta}{\alpha} \left( -\frac{\eta_0}{\eta} \right)^{\alpha} \right) \right\} ,
\]

where \(J_{\frac{\zeta}{\alpha}}(z)\) and \(Y_{\frac{\zeta}{\alpha}}(z)\) represent the Bessel functions of first and second kind respectively, and \(C_i\) \((i = 1, 2, 3, 4)\) are the integration constants. The integration constants can be determined by matching \(A_\pm(k,\eta)\) and \(A_\perp(k,\eta)\) at the transition time between the regions I and II, i.e when the mode \(k\) crosses the horizon. If the horizon crossing instant of \(k\)-th mode is symbolized by \(\eta_0\), then we can write,

\[
|k\eta_0| = 1 + \epsilon .
\]
The matching conditions of \( A_\pm(k, \eta) \) and \( A'_\pm(k, \eta) \) at \( \eta_* \) lead to the explicit expressions of \( C_i \) \((i = 1, 2, 3, 4)\) which we present in the Appendix-A in Sec. \( \text{VIII} \). The above expressions of \( A_\pm(k, \eta) \) immediately yields the electric and magnetic power spectra in the superhorizon regime as (see the Appendix-A in Sec. \( \text{VIII} \) for detailed calculations),

\[
\mathcal{P}(\vec{E}) = \left( \frac{k}{2 \pi^2} \right)^4 \left( \frac{H_0}{k} \right)^2 \left( \frac{k}{a} \right)^4 \left( \frac{\zeta \sqrt{k}}{2 \alpha} \right)^{2 \alpha} \frac{\Gamma \left( \frac{1}{2 \alpha} \right)}{\Gamma \left( 1 + \frac{\alpha}{2 \alpha} \right)} \left\{ |C_2|^2 + |C_4|^2 \right\}
\]

and

\[
\mathcal{P}(\vec{B}) = \left( \frac{k}{2 \pi^2} \right)^4 \left( \frac{k}{a} \right)^4 \left( \frac{\zeta \sqrt{k}}{2 \alpha} \right)^{2 \alpha} \frac{\Gamma \left( \frac{1}{2 \alpha} \right)}{\Gamma \left( 1 + \frac{\alpha}{2 \alpha} \right)} \left\{ |C_1 - C_2 \cot \left( -\frac{\pi}{2 \alpha} \right)|^2 + |C_3 - C_4 \cot \left( -\frac{\pi}{2 \alpha} \right)|^2 \right\}
\]

respectively. Eqs.\( (33) \) and \( (34) \) clearly demonstrate that the electric and magnetic power spectra are not scale invariant during the superhorizon inflationary era, however they have different scale dependence. Furthermore both the \( \mathcal{P}(\vec{E}) \) and \( \mathcal{P}(\vec{B}) \) tends to zero as \( |k \eta| \to 0 \). Such characteristics of the power spectra hints to the resolution of the backreaction problem in the present magnetogenesis model.

However in order to ensure that the backreaction of the EM field on the background spacetime stays small during the inflation, we first consider the electric field and magnetic field energy density at \( \eta = \eta_c \), which are given by,

\[
\rho(\vec{E}, \eta_c) = \int_{k_i}^{k_f} \mathcal{P}(\vec{E}) \, d\ln k \quad \text{and} \quad \rho(\vec{B}, \eta_c) = \int_{k_i}^{k_f} \mathcal{P}(\vec{B}) \, d\ln k .
\]

The integration limits \( k_i \) and \( k_f \) are the mode momentum which cross the horizon at the beginning of inflation and at \( \eta = \eta_c \) respectively. The beginning of inflation is considered to be the instance when the CMB scale mode crosses the horizon, and thus \( k_i = k_{\text{CMB}} = 0.05 \text{Mpc}^{-1} \). Moreover we have \( |\eta_c| = k_c^{-1} \) where \( \eta_c \) is any intermediate time instance during the inflation and thus \( k_c \) is greater than the CMB scale momentum. The quantity \( N_c \) represents the inflationary e-folding number up-to \( \eta = \eta_c \) measured from the beginning of inflation, i.e \( N_c = \ln \left( \frac{a}{a_0} \right) \) with \( a_c = a(\eta_c) \) and \( a_0 = a(\eta) \). In order to resolve the backreaction issue, we need to examine whether the EM energy density is lower than the background energy density which is given by \( \rho_{bg} = 3H_0^2M_{Pl}^2 \) where \( M_{Pl} \) is the reduced Planck mass. However due to the dependence of \( C_i = C_i(k) \) \((i = 1, 2, 3, 4)\) (see the Sec. \( \text{VIII} \)), the integration in Eq.\( (35) \) may not be performed in a closed form, and thus we numerically integrate Eq.\( (35) \) to obtain the electric and magnetic energy density at \( \eta_c \). For this purpose, we consider \( H_0 = 1.6 \times 10^{13} \text{GeV}, \; N_f = 51 \) and \( \epsilon = 0.1 \) (see the discussion after Eq.\( (17) \)). Considering these set of values, we perform the numerical integration of Eq.\( (35) \), and give a plot of the quantity

\[
R = \frac{\left( \rho(\vec{E}, \eta_c) + \rho(\vec{B}, \eta_c) \right)}{\rho_{bg}} ,
\]

with respect to the parameter \( q \) for different values of \( k_c \), see Fig.\( 1 \). In particular, we consider \( k_c = 10^{-2} \text{GeV} \) and \( 10^{-25} \text{GeV} \) in the left and right plot of Fig.\( 1 \) respectively. Here it deserves mentioning that the mode \( k_c = 10^{-2} \text{GeV} \) crosses the horizon near the end of inflation, i.e \( N_c \approx N_f \); while the mode \( k_c = 10^{-25} \text{GeV} \) crosses the horizon near \( N_c = 35 \) measured from the beginning of inflation.

The Fig.\( 1 \) clearly demonstrates that \( R \leq 10^{-4} \) for \( q \leq 3.3 \). This argues that for \( q \leq 3.3 \), the EM field has negligible backreaction on the background inflationary spacetime — leading to the resolution of the backreaction problem in the present context.

In regard to the helicity of the magnetic field, Eq.\( (24) \) leads to the helicity power spectrum during the inflation as,

\[
\mathcal{P}_h = \left( \frac{k}{2 \pi^2} \right)^3 \left( \frac{k}{a} \right)^3 \left( \frac{\zeta \sqrt{k}}{2 \alpha} \right)^{1/(2 \alpha)} \frac{\Gamma \left( \frac{1}{2 \alpha} \right)}{\Gamma \left( 1 + \frac{\alpha}{2 \alpha} \right)} \left\{ |C_1 - C_2 \cot \left( -\frac{\pi}{2 \alpha} \right)|^2 - |C_3 - C_4 \cot \left( -\frac{\pi}{2 \alpha} \right)|^2 \right\} .
\]

Eq.\( (36) \) reveals that the comoving helicity remains conserved with the expansion of the universe during the inflationary super-Hubble regime. Moreover it is evident from Eq.\( (31) \) that \( A_\pm(k, \eta) \) contains a complex argument within the Bessel function, which in turn makes the amplitude of the positive helicity mode much larger than that of the negative helicity mode, i.e \( |A_+(k, \eta)| \gg |A_-(k, \eta)| \). Thus the helicity power spectrum of the EM field turns out to be positive.
\(N_c = N_f = 51\)

\(N_c = 35\)

**V. PRESENT MAGNETIC STRENGTH AND MAGNETIC HELICITY FOR THE INSTANTANEOUS REHEATING CASE**

In this section, we aim to calculate the magnetic strength and consequently the magnetic helicity at current universe. For this purpose, we need to know about the conductivity of the universe during and after the inflation. During the inflation, the universe remains a poor electrical conductor where the conductivity is low, however in the post-inflationary era, the universe becomes a good conductor. The large conductivity during the post-inflation lies in the consideration that the universe experiences an instantaneous reheating phase just after the inflation, in which case, the e-fold number of the reheating epoch is zero and thus the universe makes a sudden jump from the inflation to a radiation dominated epoch where the Hubble parameter evolves as \(H \propto t^{-1}\) with \(t\) being the cosmic time (here we would like to mention that at a later section, we will relax the assumption of “instantaneous reheating” and will consider a reheating epoch having non-zero e-fold number after inflation). The high conductivity in the post-inflation phase leads to an electric current of the form \(J^i = \sigma E^i\) (where \(\sigma\) is the electrical conductivity and \(E^i\) is the electric field) which in turn acts as a source term in the EM field equation. Due to the presence of such source term and owing to the large \(\sigma\), the solution of the EM field after inflation behaves as \(A_{\pm}(k, \eta) = C_{\pm} + D_{\pm} e^{-\sigma t}\) (with \(C_{\pm}\) and \(D_{\pm}\) being the integration constants), where the term \(e^{-\sigma t}\) becomes soon negligible and thus \(A_{\pm}(k, \eta)\) becomes constant with time. As a result, the electric field becomes vanishingly small and the magnetic field remains the dominant piece in the electromagnetic energy density. Moreover we mentioned earlier that at the late time, the conformal breaking coupling \(f(R, G)\) does not contribute and thus the EM action restores the conformal symmetry of the EM field. In particular, the function \(f(R, G)\) is considered to be zero from the end of inflation, which is also connected from the continuity point of view, as Eq.(18) leads to \(f(R, G)\) tends to zero as \(|k\eta| \to 0\), i.e. at the end of inflation. From the conformal symmetry of the EM field, we may argue that the EM energy density decays as \(1/a^4\) with the expansion of the universe, or equivalently the magnetic energy density goes by \(1/a^4\), as the electric field practically vanishes in the post-inflationary epoch. As a result, the magnetic energy density at the end of inflation is related to that of at the present epoch by the following relation,

\[
\mathcal{P}(\vec{B})\big|_0 = \left(\frac{a_f}{a_0}\right)^4 \mathcal{P}(\vec{B})\big|_f ,
\]

where the suffix ‘f’ and ‘0’ denote the end instance of inflation and the current time of universe respectively. Moreover the helicity power spectrum evolves as \(1/a^3\) in the post-inflationary era, and thus \(\mathcal{P}_h\) at the end of inflation is connected to that of at the present epoch by following:

\[
\mathcal{P}_h\big|_0 = \left(\frac{a_f}{a_0}\right)^3 \mathcal{P}_h\big|_f .
\]
Eqs. (34) and (37) lead to the magnetic field’s current amplitude as,

$$B_0 = \left(\frac{k}{\pi^2}\right)^{1/2}\left(\frac{\sqrt{\pi}}{\pi}\right)^{1/(2\alpha)} \left| C_1 - C_2 \cot\left(-\frac{\pi}{2\alpha}\right) \right| \left(\frac{a_f}{a_0}\right)^2 \left(\frac{k}{a_f}\right)^2 ,$$

(39)

with, recall that $C_i$ ($i = 1, 2$) are given in the Sec. VIII and $k = 0.05\,\text{Mpc}^{-1}$ lies around the large scale CMB mode. Here we consider the contribution from the positive helicity mode only, due to the reason that $|A_+(k, \eta)| \gg |A_-(k, \eta)|$ during the inflationary era. If this generated magnetic fields on the scale $k = 0.05\,\text{Mpc}^{-1}$ at the present time play the role of seed magnetic fields of galactic magnetic fields, then by assuming the magnetic flux conservation, $B r^2 = \text{constant}$ [16], where $r$ is a scale, and using the scale ratio $kr_{\text{gal}}/(2\pi) \sim 10^{-5}$, where $r_{\text{gal}}$ is the scale of galaxies, we can estimate the strength of the magnetic fields at the galactic scale as,

$$B_{0}^{(\text{gal})} = \left(\frac{2\pi}{kr_{\text{gal}}}\right)^2 \left(\frac{k}{\pi^2}\right)^{1/2} \left(\frac{\sqrt{\pi}}{\pi}\right)^{1/(2\alpha)} \left| C_1 - C_2 \cot\left(-\frac{\pi}{2\alpha}\right) \right| \left(\frac{a_f}{a_0}\right)^2 \left(\frac{k}{a_f}\right)^2 .$$

(40)

Here $a_f \approx (-H_f \eta_f)^{-1}$. Therefore in order to estimate $B_{0}^{(\text{gal})}$ from Eq. (40), we need to know $a_f/a_0$ and $|k\eta_f|$. In regard to $a_f/a_0$, we use the entropy conservation in the post-inflation era, i.e $g T^3 a^3 = \text{constant}$, where $g$ symbolizes the effective relativistic degrees of freedom and $T$ is the corresponding temperature – this finally results to $a_f/a_0 = 10^3 (H_f/10^{-5}M_{\text{Pl}})^{1/2}$, where $H_f$ is the Hubble parameter at the end of inflation and can be determined from Eq. (14) as $H_f = H_0 \exp(-\epsilon N_f)$, with $N_f$ being the total e-fold of the inflationary phase. The set: $H_0 = 1.6 \times 10^{15}\,\text{GeV}$, $\epsilon = 0.1$ and $N_f = 51$ immediately leads to $H_f = 9 \times 10^{10}\,\text{GeV}$, Moreover, for the purpose of determining $|k\eta_f|$, the relation $|k\eta_f| = 1/kf$ stands to be useful where $k_f$ is the momentum of the mode which crosses the Hubble horizon at the end instance of inflation, and $k$ lies around the CMB scale. As a consequence, $|k\eta_f|$ is given by $|k\eta_f| = \exp(-N_f)$ where $k = 0.05\,\text{Mpc}^{-1} = 3.2 \times 10^{-40}\,\text{GeV}$. Using such expressions and considering a sample value of $q = 2.18$, we get,

$$\left|\frac{\sqrt{\pi}}{\pi}\right|^{1/(2\alpha)} \left| C_1 - C_2 \cot\left(-\frac{\pi}{2\alpha}\right) \right| \sim 10^{54} .$$

Consequently, for $q = 2.18$, the $B_{0}^{(\text{gal})}$ is estimated as,

$$B_{0}^{(\text{gal})} \sim 10^{-16}\,\text{Gauss} ,$$

where the conversion 1G = $1.95 \times 10^{-20}\,\text{GeV}^2$ is used. For a better understanding, we give a plot of $B_{0}^{(\text{gal})}$ vs. $q$ from Eq. (40) in Fig. 2.

In order to confront the model with observations, we use the CMB results, according to which, the present magnetic field around the CMB scale lies within $10^{-22}\,\text{G} \lesssim B_0^{(\text{gal})} \lesssim 10^{-10}\,\text{G}$. Thereby the Fig. 2 demonstrates that the theoretically predicted $B_0^{(\text{gal})}$ gets consistent with the observational constraints for the parametric regime: $2.1 \leq q \leq 2.26$. Here we need to recall that $q$ should lie within $q < 3.3$ to have a negligible backreaction of the EM field on the background spacetime (see Fig. 4). Thus we may argue that the magnetogenesis model, where the EM field gets non-minimally coupled by the term $\mathcal{L}_{CB} = \sqrt{-g} \lambda(R\cdot \mathcal{G}) \epsilon^{\rho\sigma\alpha\beta} F_{\rho\mu} F_{\sigma\beta}$, seems to generate sufficient magnetic strength at present epoch of the universe and concomitantly resolves the backreaction issue.

In regard to the generation of BAU from helical magnetic fields, the net baryonic number density ($n_B$) is defined by [10, 16],

$$n_B = -\frac{n_f}{2} \Delta n_{\text{CS}} \quad , \quad \Delta n_{\text{CS}} = -\frac{g^2}{4\pi^2} \int \tilde{E} \tilde{B} \, dt .$$

(41)

Here, $n_f$ is the number of fermionic generations (throughout this paper we use $n_f = 3$), $\Delta n_{\text{CS}}$ is the Chern-Simons number density and $g^2/(4\pi) = \alpha_{\text{EM}}$ is the fine structure constant. The electric and magnetic fields are defined by [16],

$$E_i = \frac{1}{a} \dot{A}_i \quad \text{and} \quad B_i = \frac{1}{a^2} \epsilon_{ijk} \partial_j A_k .$$

(42)
FIG. 2: \( B_0^{\text{gal}} \) vs \( q \) from Eq.(39). We take \( \lambda = 1 \).

Owing to the mode decomposition of \( A_i(\vec{x}, \eta) \), \( n_B \) can be determined as,

\[
n_B = \left( \frac{n_f}{2} \right) \left( \frac{g'^2}{4\pi^2} \right) \left( \frac{k}{2\pi} \right)^3 \left( \frac{k_0}{a_0} \right)^3 \left( \frac{\sqrt{\frac{T}{2\pi}}}{\Gamma(1 + \frac{1}{2\alpha})} \right)^2 \left| C_1 - C_2 \cot \left( -\frac{\pi}{2\alpha} \right) \right|^2.
\]

Here \( k_0 = a_0 H_0 \), i.e. \( k_0^{-1} \) is the present horizon scale. Due to the Eq.(40), the above expression of \( n_B \) can be equivalently written as [16],

\[
n_B = \left( \frac{n_f}{2} \right) \left( \frac{g'^2}{4\pi^2} \right) \left( \frac{a_0}{k_0} \right) \left( \frac{kr_{\text{gal}}}{2\pi} \right)^4 \left( B_0^{\text{gal}} \right)^2.
\] (43)

In regard to estimate the net baryon density, we will use Eq.(43) where \( n_B \) has been expressed in terms of the \( B_0^{\text{gal}} \). However it is more convenient to get the ratio of net baryon density to the entropy density of the present universe, i.e \( n_B/s_0 \), which is dimensionless and has observational constraints as \( n_B/s_0 = (6.09 \pm 0.06) \times 10^{-10} \) from the CMB data. The current entropy density \( (s_0) \) is given by,

\[
s_0 = 2.97 \times 10^3 \left( \frac{T_0}{2.75[K]} \right)^3 \, \text{cm}^{-3},
\] (44)

with \( T_0 = 2.73K \) is the present temperature of the cosmic microwave background (CMB) radiation. Fig. 3 depicts that the model predicts sufficient magnetic strength for a suitable regime of the reheating EoS parameter. Using this information along with the parameter values that we considered earlier (i.e \( H_0 = 1.6 \times 10^{13}\text{GeV} \), \( \epsilon = 0.1 \), \( N_f = 51 \) and \( H_f = 9 \times 10^{10}\text{GeV} \)), we estimate \( n_B/s_0 \) for different values of \( B_0^{\text{gal}} \), with \( \frac{k_0}{a_0} = 0.05\text{Mpc}^{-1} \approx 3.2 \times 10^{-40}\text{GeV} \). These are given in Table I. From Table I, we see that the magnetic fields of current strength \( B_0^{\text{gal}} \sim 10^{-13}\text{G} \) leads to the resultant value of \( n_B/s_0 \) as of the order \( \sim 10^{-10} \).

Here we would like to mention that the above arguments are based on instantaneous reheating where, as mentioned earlier, the reheating epoch has zero e-fold and thus the universe gets a large conductivity after the end of inflation. However it would be more physical if the universe undergoes through a reheating phase having a non-zero e-fold number. Motivated by this, in the next section, we will relax the assumption of “instantaneous reheating” and will consider a prolonged reheating stage in-between the end of inflation and the beginning of radiation dominated epoch.

Before moving to the next section, we would like to explore the following two cases in regard to the non-minimal coupling of the EM field – (1) when the EM field couples with the background Ricci scalar alone, i.e \( f(R, G) = \kappa^2 R^0 \)
and, (2) when the EM field couples with the background Gauss-Bonnet scalar alone, i.e \( f(R, \mathcal{G}) = \kappa^{2q} \mathcal{G}^{q/2} \). In the first case, by using \( R = R(\eta) \) from Eq. (13), we get the non-minimal coupling function (in terms of conformal time) as,

\[
f(R, \mathcal{G}) = \kappa^{2q} \left\{ \frac{6\beta(\beta + 1)^3}{\eta_0^{2q}} \right\} \left( \frac{-\eta}{\eta_0} \right)^{2q},
\]

(45)

while for the second case, it comes as

\[
f(R, \mathcal{G}) = \kappa^{2q} \left\{ \frac{-24(\beta + 1)^{3q/2}}{\eta_0^{2q}} \right\} \left( \frac{-\eta}{\eta_0} \right)^{2q}.
\]

(46)

Owing to the above non-minimal coupling functions, the present magnetic field at the galactic scale comes with the same expression as of Eq. (40), with \( \zeta^2 \) replaced by

\[
\zeta^2 = (16\epsilon q \lambda \eta_0) \left( \frac{\kappa}{\eta_0} \right)^{2q} \left\{ [6\beta(\beta + 1)]^q \right\}
\]

for the first case, and

\[
\zeta^2 = (16\epsilon q \lambda \eta_0) \left( \frac{\kappa}{\eta_0} \right)^{2q} \left\{ [-24(\beta + 1)^{3q/2}] \right\}
\]

for the second case. Therefore we estimate the current magnetic strength in terms of the parameter \( q \) for the two aforementioned cases. These are depicted in the left and the right plot of Fig. 3, respectively.

The Fig. 3 clearly demonstrates that for both the cases, whether the EM field couples with the Ricci scalar alone or with the GB scalar alone, the present magnetic strength comes within the observational constraints for a suitable range of \( q \). However it seems that the viable range of \( q \) changes for the two cases – in particular, \( q \) should lie within \( 2.1 \leq q \leq 3.0 \) for the case when \( f(R, \mathcal{G}) = \kappa^{2q} R^q \), while for \( f(R, \mathcal{G}) = \kappa^{2q} \mathcal{G}^{q/2} \) we get \( 2.9 \leq q \leq 3.2 \) in order to make the corresponding magnetogenesis model compatible with the observation. Thus as a whole, the viable range of \( q \) for different cases are given by,

\[
\begin{align*}
2.1 \leq q & \leq 2.26 \text{ for } f(R, \mathcal{G}) = \kappa^{2q} (R^q + \mathcal{G}^{q/2}), \\
2.1 \leq q & \leq 2.30 \text{ for } f(R, \mathcal{G}) = \kappa^{2q} R^q, \\
2.9 \leq q & \leq 3.2 \text{ for } f(R, \mathcal{G}) = \kappa^{2q} \mathcal{G}^{q/2}.
\end{align*}
\]

It may be observed that the viable range of \( q \) does not change considerably for the above cases. This is expected from the fact that \( R^q \) and \( \mathcal{G}^{q/2} \) have same order of magnitude during the inflationary era, in particular, \( R^q \sim \mathcal{G}^{q/2} \sim H_0^{2q} \) during the inflation (where \( H_0 \) is the inflationary Hubble parameter). Therefore one important point to be argued that in the realm of higher curvature magnetogenesis scenario, any higher curvature coupling of the EM field, that has same order of magnitude with \( R^q \) (or \( \mathcal{G}^{q/2} \)) and couples with \( F_{\mu\nu} F^{\mu\nu} \), can predict sufficient strength of magnetic field for suitable range of the corresponding model parameter.
VI. PRESENT MAGNETIC STRENGTH AND MAGNETIC HELICITY FOR A KAMIONKOWSKI LIKE REHEATING MODEL WITH NON-ZERO E-FOLDING NUMBER

Instead of instantaneous reheating (that we have considered in Sec. [V]), it would be more physical if the universe undergoes through a reheating phase which has a non-zero e-fold number. In the case of instantaneous reheating, the electrical conductivity immediately turns on after the end of inflation and thus the electric field practically goes to zero. However if the reheating phase is considered to have a non-zero e-fold number, then there is no reason to consider a large conductivity immediately after the inflation. Indeed, the conductivity remains non-zero and small, due to which, the existence of electric field may induce the magnetic field during the stage between the end of inflation and the end of reheating [34]. As a result, the magnetic field’s strength and consequently the helicity spectrum at present universe may become larger compared to those in the instantaneous reheating case. In such situation, it is important to calculate the magnetic field’s strength and the helicity spectrum at current time by considering the reheating phase with a non-zero e-fold number. This is our aim in the present section.

In regard to the reheating dynamics, we consider the conventional reheating mechanism proposed by Kamionkowski et al. [80], where the main idea is to parametrize the reheating dynamics by – a non-zero e-fold number, a constant equation of state (EoS) parameter and a given reheating temperature. Clearly, the Hubble parameter during the reheating era is connected to that of at the end of inflation by the constant EoS parameter (symbolized by \( \omega_{\text{eff}} \)), in particular, \( H \propto a^{-\frac{2}{3}(1+\omega_{\text{eff}})} \) in the reheating stage. Moreover the inflaton energy density is supposed to instantaneously convert to radiation energy density after the end of reheating, which indicates the beginning of radiation dominated epoch. The characterized quantities of the reheating phase, i.e the e-fold number \( (N_{\text{re}}) \) and the temperature \( (T_{\text{re}}) \) can be expressed in terms of \( \omega_{\text{eff}} \) and of some inflationary parameters by following [80, 81],

\[
N_{\text{re}} = \frac{4}{1 - 3\omega_{\text{eff}}} \left[ -\frac{1}{4} \ln \left( \frac{45}{\pi^2 g_{\text{r,c}}} \right) - \frac{1}{3} \ln \left( \frac{11g_{s,\text{re}}}{43} \right) - \ln \left( \frac{k}{a_0 T_0} \right) - \ln \left( \frac{(3H_0^2 M_{\text{Pl}}^2)^{1/4}}{H_0} \right) - N_f \right] ,
\]

\[
T_{\text{re}} = H_0 \left( \frac{43}{11g_{s,\text{re}}} \right)^{1/2} \left( \frac{a_0 T_0}{k} \right) \exp \left[ -(N_f + N_{\text{re}}) \right] ,
\]

where \( T_0 = 2.73 \text{K} \) is the present temperature of the universe, \( \frac{k}{a_0} \) is the pivot scale \( \approx 0.05 \text{Mpc}^{-1} \) in case of large scale modes and \( a_0 \) is the present cosmological scale factor. Here, for simplicity, we have taken both the values of the degrees of freedom (d.o.f) as \( g_{s,\text{re}} = g_{\text{r,c}} \approx 100 \), where \( g_{s,\text{re}} \) is the d.o.f for entropy at reheating and \( g_{\text{r,c}} \) is the effective number of relativistic species upon thermalization. With this reheating model in hand, we now solve the electromagnetic mode function and consequently determine the power spectra during the reheating epoch, in the next subsection.
A. Solution of the mode function and power spectra during the reheating epoch

We mentioned earlier that the function \( f(R, \mathcal{G}) \) is considered to be zero after the end of inflation. As a consequence, the production of the gauge field from the quantum vacuum stops, which is accounted by the fact that the Bogoliubov coefficients in the post-inflation become explicitly time independent and get equal to that of at the end of inflation. Therefore, in the reheating epoch, the EM field follows the standard Maxwell’s equation, in particular,

\[
A'^{(re)}_\pm (k, \eta) + k^2 A^{(re)}_\pm (k, \eta) = 0 ,
\]

where \( A^{(re)}_\pm (k, \eta) \) is the EM mode function during the reheating phase. Eq. (49) points that the electrical conductivity of the universe during the reheating era is considered to be negligible, in particular to be zero. Solving Eq. (49), one gets,

\[
A^{(re)}(k, \eta) = \frac{1}{\sqrt{2k}} \left[ c_\pm e^{-ik(\eta-\eta_f)} + d_\pm e^{ik(\eta-\eta_f)} \right] ,
\]

where \( c_\pm \) and \( d_\pm \) are integration constants, that can be determined from the continuity conditions of the EM field at the junction between the end of inflation and the beginning of reheating, i.e.,

\[
A^{(re)}_\pm (k, \eta_f) = A_\pm (k, \eta_f) \quad \text{and} \quad A'^{(re)}_\pm (k, \eta_f) = A'_\pm (k, \eta_f),
\]

with \( A_\pm (k, \eta_f) \) are the EM mode functions at the end of inflation and have the forms as in Eq. (51), see the Appendix-A in Sec. VIII. Such continuity relations immediately lead to the integration constants in terms of \( A_\pm (k, \eta_f) \) and \( A'_\pm (k, \eta_f) \) as,

\[
c_\pm = \sqrt{\frac{k}{2}} A_\pm (k, \eta_f) + \frac{i}{\sqrt{2k}} A'_\pm (k, \eta_f) \quad , \quad d_\pm = \sqrt{\frac{k}{2}} A_\pm (k, \eta_f) - \frac{i}{\sqrt{2k}} A'_\pm (k, \eta_f) .
\]

One of the important quantities in magnetogenesis scenario are the Bogoliubov coefficients which actually account the production of gauge particles from the quantum vacuum. In particular, due to the interaction of the EM field with the background FRW spacetime, the Hamiltonian of the EM field becomes explicit time dependent, which in turn makes the electromagnetic vacuum state explicit time dependent. Thus if the EM field starts from the vacuum state at distant past (namely the Bunch-Davies state which we consider in the present context), then it will not remain in vacuum at a later time. This corresponds to the production of gauge particles from the Bunch-Davies state and the Bogoliubov coefficients provide the number of produced particles at a certain time. More precisely, the Bogoliubov coefficients at time \( \eta \) relate the mode functions correspond to the vacuum state of distant past with the set of mode functions that lead to the instantaneous vacuum of the field at the \( \eta \). Based on this, the Bogoliubov coefficients during the reheating epoch, in the present context, are determined as,

\[
\alpha_\pm (k, \eta) = \sqrt{\frac{k}{2}} A^{(re)}_\pm (k, \eta) + \frac{i}{\sqrt{2k}} A'^{(re)}_\pm (k, \eta), \quad \beta_\pm (k, \eta) = \sqrt{\frac{k}{2}} A^{(re)}_\pm (k, \eta) - \frac{i}{\sqrt{2k}} A'^{(re)}_\pm (k, \eta) .
\]

Eqs. (50) and (53) immediately result \( \alpha_\pm \) and \( \beta_\pm \) as,

\[
\alpha_\pm (k, \eta) = c_\pm e^{-ik(\eta-\eta_f)} , \quad \beta_\pm (k, \eta) = d_\pm e^{ik(\eta-\eta_f)},
\]

which relate the Bogoliubov coefficients with the integration constants that appear in the solution of \( A^{(re)}_\pm (k, \eta) \). The absolute value of \( \beta_\pm \) (i.e \( |\beta_\pm| = |\beta_\pm (k, \eta_f)| = d_\pm \)) during the reheating era seems to be explicit time independent and equal to that of at the end of inflation, i.e \( |\beta_\pm (k, \eta)| = |\beta_\pm (k, \eta_f)| = d_\pm \) where \( \eta \) is any time during reheating, from Eq. (54). This is a direct consequence of the fact that the coupling function \( f(R, \mathcal{G}) \) is considered to be zero in the reheating stage, due to which the EM action restores the conformal symmetry and thus the EM field equations in the background FRW spacetime become similar to that of in the Minkowski spacetime. As a result, the gauge production ceases to occur after the inflation. Moreover Eq. (54) leads to \( c_\pm = \alpha_\pm (k, \eta_f) \) and \( d_\pm = \beta_\pm (k, \eta_f) \), and thus from Eq. (50),

\[
A^{(re)}_\pm (k, \eta) = \frac{1}{\sqrt{2k}} \left[ \alpha_\pm (k, \eta_f) e^{-ik(\eta-\eta_f)} + \beta_\pm (k, \eta_f) e^{ik(\eta-\eta_f)} \right] .
\]
Plugging the above solution of $A^{(re)}_\pm(k,\eta)$ into Eq. (24) along with the relation $|\alpha_\pm|^2 - |\beta_\pm|^2 = 1$ yield the magnetic and electric power spectra during the reheating phase as,

$$\mathcal{P}(\vec{B}) = \frac{1}{2\pi^2} \sum_{r=+,-} \frac{k^5}{a^4} |A^{(re)}_r(k,\eta)|^2$$

$$= \frac{1}{2\pi^2} \sum_{r=+,-} \left( \frac{k^4}{a^4} \right) \left[ |\alpha_r(k,\eta_f)|^2 + |\beta_r(k,\eta_f)|^2 + 2|\alpha_r(k,\eta_f)| \beta_r(k,\eta_f)| \cos \left\{ \theta^{(r)}_1 - \theta^{(r)}_2 - 2k(\eta - \eta_f) \right\} \right]$$

and

$$\mathcal{P}(\vec{E}) = \frac{1}{2\pi^2} \sum_{r=1,2} \frac{k^2}{a^4} |A^{(re)}_r(k,\eta)|^2$$

$$= \frac{1}{2\pi^2} \sum_{r=+,-} \left( \frac{k^4}{a^4} \right) \left[ |\alpha_r(k,\eta_f)|^2 + |\beta_r(k,\eta_f)|^2 - 2|\alpha_r(k,\eta_f)| \beta_r(k,\eta_f)| \cos \left\{ \theta^{(r)}_1 - \theta^{(r)}_2 - 2k(\eta - \eta_f) \right\} \right]$$

respectively, where $\theta^{(r)}_1 = \text{Arg} \left( \alpha_r(k,\eta_f) \right)$ and $\theta^{(r)}_2 = \text{Arg} \left( \beta_r(k,\eta_f) \right)$. Consequently, the total electromagnetic power spectrum turns out to be,

$$\mathcal{P}_{em} = \mathcal{P}(\vec{B}) + \mathcal{P}(\vec{E}) = \frac{1}{\pi^2} \sum_{r=+,-} \left( \frac{k^4}{a^4} \right) \left[ |\alpha_r(k,\eta_f)|^2 + |\beta_r(k,\eta_f)|^2 \right].$$

It may be observed from the above expressions that the total EM power spectrum redshifts by $\frac{1}{\sqrt{a}}$ with the expansion of the universe, however the individual magnetic and electric spectra do not evolve as $\frac{1}{\sqrt{a}}$ due to the presence of the time dependent factor $k(\eta - \eta_f)$ within the cosine argument of Eqs. (56) and (57) respectively. Thus the comoving EM energy density seems to be conserved in the reheating era, which, once again, is a consequence of the fact that the EM action restores the conformal symmetry in the post-inflationary epoch. Furthermore Eqs. (56) and (24) lead to the helicity power spectrum in-between the end of inflation and the end of reheating as,

$$\mathcal{P}_h(k,\eta) = \left( \frac{k^4}{\pi^2a^3} \right) \left\{ |\beta_+|^2 - |\beta_-|^2 + \sqrt{1 + |\beta_+|^2} |\beta_-| \cos \left\{ \theta^{(+)}_1 - \theta^{(+)2} - 2k(\eta - \eta_f) \right\} \right. \right.$$

$$- \sqrt{1 + |\beta_-|^2} |\beta_+| \cos \left\{ \theta^{(-)}_1 - \theta^{(-)2} - 2k(\eta - \eta_f) \right\} \right\}.$$ 

(59)

Here we would like to mention that both the EM mode functions follow the same equation of motion (see Eq. (10)) in the reheating era, however during inflation, the conformal coupling $f(R,G)$ differs the amplitude of $A_+(k,\eta)$ compared to that of $A_-(k,\eta)$, and as a result, $A_+(k,\eta_f) \neq A_-(k,\eta_f)$ and $A'_+(k,\eta_f) \neq A'_-(k,\eta_f)$, i.e positive and negative helicity modes obey different initial conditions at the beginning of reheating (recall, $\eta_f$ represents the end instance of inflation or equivalently the start of reheating). Thus as a whole, $A_+(k,\eta)$ have same equations of motion but different initial conditions during the reheating stage, in effect of which, $A_+(k,\eta)$ and $A_-(k,\eta)$ get different and makes a non-zero helicity during the same, as given in Eq. (59).

Following, we determine the explicit forms of Bogoliubov coefficients and the factor $k(\eta - \eta_f)$ present in the expressions of various kinds of power spectra.

- **Determination of $\alpha_\pm(k,\eta_f)$ and $\beta_\pm(k,\eta_f)$:** In view of Eqs. (52) and (54), we can express $\alpha_r(k,\eta_f)$ and $\beta_r(k,\eta_f)$ as,

$$\alpha_r(k,\eta_f) = \sqrt{\frac{k}{2}} \left[ A_r(k,\eta_f) + i \frac{dA_r}{dk}(\eta_f) \right]_{\eta_f},$$

$$\beta_r(k,\eta_f) = \sqrt{\frac{k}{2}} \left[ A_r(k,\eta_f) - i \frac{dA_r}{dk}(\eta_f) \right]_{\eta_f},$$

(60)

with recall, the index $r$ specifies the two polarization modes of the EM field. Using the super-Hubble solution of $A_\pm(k,\eta)$, we determine the Bogoliubov coefficients as follows:

$$\alpha_+(k,\eta_f) = \sqrt{\frac{k}{2}} \left[ \left( \frac{C_1 - C_2 \cot \left( \frac{\pi \eta_f}{2\alpha} \right)}{\Gamma \left( 1 + \frac{1}{2\alpha} \right) } \right)^{1/(2\alpha)} - \frac{iH_0}{k} \left( \frac{C_2 \Gamma \left( \frac{1}{2\alpha} \right)}{\pi} \right) \left( -i \frac{\sqrt{k}}{2\alpha} \right)^{-1/(2\alpha)} \right].$$
\[ \beta_+(k, \eta_f) = \sqrt{\frac{k}{2}} \left[ \left( \frac{C_1 - C_2 \cot \left( \frac{\pi}{2} \right)}{\Gamma (1 + \frac{2}{3})} \right) \left( -\frac{\zeta \sqrt{k}}{2a^3} \right)^{1/(2\alpha)} - iH_0 \left( \frac{C_4 \Gamma (\frac{1}{2})}{\pi} \right) \left( -\frac{\zeta \sqrt{k}}{2a^3} \right)^{-1/(2\alpha)} \right], \]

\[ \alpha_-(k, \eta_f) = \sqrt{\frac{k}{2}} \left[ \left( \frac{C_1 - C_2 \cot \left( \frac{\pi}{2} \right)}{\Gamma (1 + \frac{2}{3})} \right) \left( \frac{\zeta \sqrt{k}}{2a^3} \right)^{1/(2\alpha)} + iH_0 \left( \frac{C_4 \Gamma (\frac{1}{2})}{\pi} \right) \left( \frac{\zeta \sqrt{k}}{2a^3} \right)^{-1/(2\alpha)} \right], \]

\[ \beta_-(k, \eta_f) = \sqrt{\frac{k}{2}} \left[ \left( \frac{C_1 - C_4 \cot \left( \frac{\pi}{3} \right)}{\Gamma (1 + \frac{2}{3})} \right) \left( \frac{\zeta \sqrt{k}}{2a^3} \right)^{1/(2\alpha)} - iH_0 \left( \frac{C_4 \Gamma (\frac{1}{2})}{\pi} \right) \left( \frac{\zeta \sqrt{k}}{2a^3} \right)^{-1/(2\alpha)} \right]. \] (61)

The \( C_i \) (\( i = 1, 2, 3, 4 \)) are given in the Appendix-A. Since \( C_1 \) and \( C_2 \) contain complex arguments within the Bessel function, one can show from Eq.(61) that \( |\beta_+(\eta_f)| \) is much larger than unity for \( q \sim O(1) \), in particular \( |\beta_+(\eta_f)| \sim 10^{37} \) for \( q = 2.18 \) (later we will show that \( q = 2.18 \) leads to sufficient magnetic strength at current epoch). On the other hand, \( \beta_- \) (or \( \alpha_- \)), the term which is proportional to \( \alpha \), gets suppressed compared to the \( \beta_- \) (or \( \alpha_- \)), i.e \( |\beta_-| \ll |\beta_+| \).

**Determination of “\( \eta - \eta_f \)” during reheating:** The factor \( k(\eta - \eta_f) \) in Eqs.(56) and (59) leads to the non-conventional dynamics of magnetic and helicity energy density. The constant equation of state during reheating dynamics results to this special term as,

\[ k(\eta - \eta_f) = \left. \frac{2k}{3\omega_{eff} + 1} \right| \frac{1}{aH} - \frac{1}{af H_f}. \] (62)

where we use \( \eta - \eta_f = \int_{a_f}^{a_c} \frac{d\eta}{\omega(\eta)} \) and the quantities with suffix 'I' represent the respective quantities at the end of inflation.

With the above expressions of \( \alpha_+(k, \eta_f) \), \( \beta_+(k, \eta_f) \) and \( k(\eta - \eta_f) \): the magnetic, electric and helicity power spectra during the reheating era (from Eqs.(56), (57) and (59)) turn out to be,

\[ P(\vec{B}) = \frac{1}{\pi^2} \left( \frac{k^3}{a^3} \right) |\beta_+(\eta_f)|^2 \left\{ \operatorname{Arg}[\alpha_+(\eta_f)\beta_+^*(\eta_f)] - \frac{4k}{3\omega_{eff} + 1} \left( \frac{1}{aH} - \frac{1}{af H_f} \right) \right\}^2, \] (63)

\[ P(\vec{E}) = \frac{1}{\pi^2} \left( \frac{k^3}{a^3} \right) |\beta_+(\eta_f)|^2, \] (64)

and

\[ P_h = \frac{1}{\pi^2} \left( \frac{k^3}{a^3} \right) |\beta_+(\eta_f)|^2 \left\{ \operatorname{Arg}[\alpha_+(\eta_f)\beta_+^*(\eta_f)] - \frac{4k}{3\omega_{eff} + 1} \left( \frac{1}{aH} - \frac{1}{af H_f} \right) \right\}^2, \] (65)

respectively, where we neglect the contribution coming from the negative helicity modes. Therefore the magnetic power spectrum during the reheating seems to be controlled by two terms: (1) the conventional one that redshifts by \( 1/a^4 \) with the expansion of the universe and (2) the term which is proportional to \( (a^2 H)^{-2} \) emerged from \( \frac{(k/a)^2}{\omega_{eff}} \) present in the expression of Eq.(63). In regard to the helicity power spectrum \( P_h \) during reheating dynamics, Eq.(65) reveals that, similar to the magnetic spectrum, \( P_h \) is also controlled by two terms: (1) the term that evolves by \( 1/a^3 \) and (2) the other one which is proportional to \( 1/ (a^2 H^2) \). However due to the constant equation of state, the Hubble parameter during the reheating behaves (in terms of scale factor) as \( H \propto a^{-\frac{2}{3}}(1+\omega_{eff}) \) and thus the term \( 1/ (a^2 H^2) \) is further proportional to \( a^{-2+3\omega_{eff}} \). The presence of such special terms in the magnetic as well as in the helicity power spectra provide the main difference in the magnetogenesis scenario between the instantaneous reheating case and the case where the reheating phase has a non-zero e-fold number. For the magnetic power spectrum \( P(\vec{B}) \): in the instantaneous reheating case, it evolves by \( \frac{1}{\beta} \) from the very end of inflation to the present epoch; however in the case of a non-zero e-fold reheating phase, \( P(\vec{B}) \) is controlled by Eq.(63) in-between the end of inflation to the end of reheating and then goes as \( 1/a^3 \) from the end of reheating to the present epoch. On the other hand, for the helicity power spectrum: in the instantaneous reheating case, the comoving helicity remains conserved (i.e \( P_h \propto 1/a^3 \)) after inflation and until today; however, when the reheating phase is considered to have a non-zero e-fold number, \( P_h \) follows Eq.(65) from the end of inflation to the end of reheating and then \( P_h \propto 1/a^4 \) until the present epoch. Finally we would like to mention that unlike to the magnetic or helicity power spectra, the electric one decays by \( 1/a^4 \) during the reheating era, as evident from Eq.(64).
B. Current magnetic strength and baryon asymmetry of the universe: Constraints on $\omega_{\text{eff}}$

The presence of a reheating phase with non-zero e-fold number results to the existence of an electric field in the post-inflationary evolution of the universe till the time when the universe becomes purely conductive and this generally occurs at the end of reheating. Such electric field in turn induces the magnetic field according to the Faraday’s law of induction \cite{34}, due to which, the magnetic field energy density in the reheating epoch redshifts slower than that of in the case of instantaneous reheating. Therefore it is important to investigate whether the presence of a reheating phase having non-zero e-fold yields a sufficient amount of magnetic strength and consequently the helicity at the present time. The Hubble parameter during the reheating phase evolves as $H \propto a^{-\frac{3}{2}(1+\omega_{\text{eff}})}$, and thus the Hubble parameter at the end of reheating is connected to that of at the end of inflation as,

$$H_{re} = H_f \left(\frac{a_{re}}{a_f}\right)^{-\frac{3}{2}(1+\omega_{\text{eff}})},$$  \hspace{1cm} (66)

where the suffix ‘re’ denotes the end instance of reheating, and $\ln \left(\frac{a_{re}}{a_f}\right) = N_{re}$ represents the e-fold number of the reheating era, the $N_{re}$ in terms of the reheating parameters is given in Eq. (47). The above expression of $H_{re}$ immediately leads to the magnetic and helicity power spectra (from Eqs. (63) and (64)) at the end of reheating as,

$$\mathcal{P}(\vec{B})|_{re} = \frac{1}{\pi^2} \left(\frac{k^4}{a_{re}^4}\right) |\beta_+(\eta_f)|^2 \left\{ \text{Arg} \left[ \alpha_+(\eta_f)\beta_+^*(\eta_f) \right] - \pi - \frac{4}{3\omega_{\text{eff}}+1} \left( \frac{k}{a_f H_f} \right) \text{exp} \left[ \left( \frac{3\omega_{\text{eff}}+1}{2} \right) N_{re} \right] \right\}^2,$$ \hspace{1cm} (67)

and

$$\mathcal{P}_h|_{re} = \frac{1}{\pi^2} \left(\frac{k^3}{a_{re}^3}\right) |\beta_+(\eta_f)|^2 \left\{ \text{Arg} \left[ \alpha_+(\eta_f)\beta_+^*(\eta_f) \right] - \pi - \frac{4}{3\omega_{\text{eff}}+1} \left( \frac{k}{a_f H_f} \right) \text{exp} \left[ \left( \frac{3\omega_{\text{eff}}+1}{2} \right) N_{re} \right] \right\}^2,$$ \hspace{1cm} (68)

respectively, where we use $H_f/H_{re} = \exp [3(1 + \omega_{\text{eff}})N_{re}/2]$ from Eq. (66). After the end of reheating, the universe becomes a good conductor, which in turn makes the electric field zero. Moreover due to the conformal invariance, the comoving EM energy density as well as the comoving helicity get conserved during the post-reheating era. Hence, the present magnetic and the present helicity power spectra is connected to that of at the end of reheating by following:

$$\mathcal{P}(\vec{B})|_0 = \left(\frac{a_{re}}{a_0}\right)^4 \mathcal{P}(\vec{B})|_{re}$$ \hspace{1cm} and \hspace{1cm} $$\mathcal{P}_h|_0 = \left(\frac{a_{re}}{a_0}\right)^3 \mathcal{P}_h|_{re},$$ \hspace{1cm} (69)

where the suffix ‘0’ denotes the present time of the universe. Now using Eq. (67), we get the magnetic field’s current amplitude as,

$$B_0 = \frac{\sqrt{2}}{\pi} \left(\frac{k}{a_0}\right)^2 |\beta_+(\eta_f)| \left\{ \text{Arg} \left[ \alpha_+(\eta_f)\beta_+^*(\eta_f) \right] - \pi - \frac{4}{3\omega_{\text{eff}}+1} \left( \frac{k}{a_f H_f} \right) \text{exp} \left( \frac{1+3\omega_{\text{eff}}}{2} N_{re} \right) \right\},$$ \hspace{1cm} (70)

Furthermore, $\mathcal{P}(\vec{B})|_0 = \frac{1}{2}B_0^2$, with $B_0$ being the present amplitude of the magnetic field on the scale $k = 0.05\text{Mpc}^{-1}$.

Considering this generated magnetic fields at the present time play the role of seed magnetic fields of galactic magnetic fields and by assuming the magnetic flux conservation \cite{10}, we can estimate the strength of the galactic magnetic fields as,

$$B_{0^{(gal)}} = \frac{\sqrt{2}}{\pi} \left(\frac{2\pi}{k\gamma_{gal}}\right)^2 \left(\frac{k}{a_0}\right)^2 |\beta_+(\eta_f)| \left\{ \text{Arg} \left[ \alpha_+(\eta_f)\beta_+^*(\eta_f) \right] - \pi - \frac{4}{3\omega_{\text{eff}}+1} \left( \frac{k}{a_f H_f} \right) \text{exp} \left( \frac{1+3\omega_{\text{eff}}}{2} N_{re} \right) \right\}.$$ \hspace{1cm} (71)

Finally, from Eq. (68), we obtain the helicity power at present epoch as,

$$\mathcal{P}_h|_0 = \frac{1}{\pi^2} \left(\frac{k^3}{a_0^3}\right) |\beta_+(\eta_f)|^2 \left\{ \text{Arg} \left[ \alpha_+(\eta_f)\beta_+^*(\eta_f) \right] - \pi - \frac{4}{3\omega_{\text{eff}}+1} \left( \frac{k}{a_f H_f} \right) \text{exp} \left( \frac{1+3\omega_{\text{eff}}}{2} N_{re} \right) \right\}^2.$$ \hspace{1cm} (72)

Here we need to recall that $\mathcal{P}_h|_0$ yields the net baryon density in present universe ($n_B$). Therefore it may be observed that both the $B_{0^{(gal)}}$ and $n_B$ depend on the reheating dynamics (through $N_{re}$ and $\omega_{\text{eff}}$) as well as on the background
inflationary dynamics (through \( H_f \) and the absolute value of the Bogoliubov coefficients i.e. \( |\beta_+(k,\eta_f)|^2 \)). As a result, the magnetic field’s current strength and the net baryon density of the present universe get a one-to-one correspondence with the reheating equation of state parameter.

Thus as a whole, Eqs. (71) and (72) are the key equations to determine the magnetic field’s current strength and the net baryon density of the universe. Having obtained such expressions, we now confront the model with the observations which put a constraint on the present magnetic strength as \( 10^{-22}G \lesssim B_0^{(gal)} \lesssim 10^{-10}G \). In regard to this, we take \( H_0 = 1.6 \times 10^{13}\text{GeV}, \epsilon = 0.1 \) and \( N_f = 51 \), which immediately leads to \( H_f = 9 \times 10^{10}\text{GeV} \). Moreover the parameter \( q \) is constrained by \( q \leq 3.3 \) in order to resolve the backreaction issue. With such considerations and by using Eq. (71), we plot the current magnetic strength (\( B_0^{(gal)} \)) vs \( q \) for two different reheating temperatures (\( T_{re} \)), see Fig. 4. In particular, we consider \( T_{re} = 6 \times 10^4\text{GeV} \) and \( T_{re} = 2 \times 10^4\text{GeV} \), which correspond to \( \omega_{\text{eff}} = 0.16 \) and 0.14 respectively. For detailed numerical estimation of \( B_0^{(gal)} \), see the Appendix-B in Sec. IX.

![FIG. 4: \( B_0^{(gal)} \) vs \( q \) where we take \( \lambda = 1 \). The left and right plot correspond to the sets \( [T_{re},\omega_{\text{eff}}] = [600\text{GeV},0.16] \) and \( [T_{re},\omega_{\text{eff}}] = [2 \times 10^4\text{GeV},0.14] \) respectively.](image)

The Fig. 4 clearly demonstrates that \( B_0^{(gal)} \) lies within the observational constraints if the model parameter \( q \) satisfies: \( 2.1 \leq q \leq 2.25 \) for both the values of \( T_{re} \) considered.

By comparing Fig. 2 and Fig. 4, it is clear that the viable range of \( q \) remains almost same in both the instantaneous and Kamionkowski reheating scenario, or we may argue that the generation of helical magnetic fields in the present context is not much affected by the reheating era. This is due to the fact that the present magnetogenesis model does not produce sufficient hierarchy between the electric and magnetic fields at the end of inflation, and thus the electric field is not able to sufficiently induce (or enhance) the magnetic field during the Kamionkowski reheating stage – which in turn makes both the reheating cases almost similar from the perspective of magnetic field’s evolution. The demonstration goes as follows: actually the information of the reheating era in the magnetic power spectrum comes through the term \((-k\eta_f)\varepsilon^{\left(\frac{2\omega_{\text{eff}}+1}{2}\right)}N_{re}^{-\frac{1}{2}}\) present in Eq. (71). However due to the reason that the electric and magnetic fields do not have much hierarchy at the end of inflation (particularly \( \mathcal{P}(\vec{E})/\mathcal{P}(\vec{B}) \approx 10^4 \), where the suffix ‘f’ represents the end instant of inflation), the term \((-k\eta_f)\varepsilon^{\left(\frac{2\omega_{\text{eff}}+1}{2}\right)}N_{re}^{-\frac{1}{2}}\) gets heavily suppressed compared to the other term sitting within the parenthesis of Eq. (71), i.e. \( \arg [\alpha_+(\eta_f)\beta_+^*(\eta_f)] = \pi \), namely

\[
\left( \frac{k}{a_f H_f} \right) \exp \left[ \left( \frac{1+3\omega_{\text{eff}}}{2} \right) N_{re} \right] \ll \left| \arg [\alpha_+(\eta_f)\beta_+^*(\eta_f)] - \pi \right|
\]

(73)

(see the estimation in the Appendix-B). As a result, the \( B_0^{(gal)} \) of Eq. (71) becomes almost independent of \( \omega_{\text{eff}} \). In particular owing to Eq. (73), the scaling of the magnetic spectrum during the Kamionkowski reheating stage is approximately given by \( \sim a^{-\pi} \) – which is independent of \( \omega_{\text{eff}} \) and same as the magnetic field scales after the end of inflation in the instantaneous reheating case. The above arguments reveal that why the generation of helical magnetic
fields in the present context is not much affected by the presence of the Kamionkowski reheating stage characterized by $\omega_{\text{eff}}$. Here we would like to mention that this is unlike to the higher curvature magnetogenesis scenario proposed in [29] where the spacetime curvature couples with $F_{\mu\nu}F^{\mu\nu}$ (i.e. the magnetic field is not helical in nature) and consequently, the electric field at the end of inflation becomes much stronger that that of the magnetic field, in particular, the ratio of electric and magnetic field at the end of inflation comes as $\mathcal{P}(E)/\mathcal{P}(B) \sim e^{2N_f} \gg 1$ (with $N_f$ being the inflationary e-folding number). Due to such strong hierarchy between the electric and the magnetic fields, the relative phase between $\alpha_+(\eta_f)$ and $\beta_+(\eta_f)$ becomes equal to $\pi$ from Eq. (66), and consequently the term $(-k\eta_f)e^{\left(\frac{2\omega_{\text{eff}}+1}{3}\right)N_f}$ dominates over $[\text{Arg} \left(\alpha_+(\eta_f)\beta_+^*(\eta_f)\right)] - \pi$. In effect, the magnetic spectrum during the Kamionkowski reheating stage scales as $(a^3H)^{-2}e^{\left(3\omega_{\text{eff}}+1\right)N_f}$ which indeed depends on the reheating equation of state parameter. Thus in the magnetogenesis scenario [29], the Kamionkowski reheating stage shows considerable effects on the magnetic field's evolution, in particular, the magnetic field seems to be enhanced in the Kamionkowski reheating case compared to that of in the instantaneous reheating case. The above arguments clearly demonstrate that the hierarchy between electric and magnetic fields at the end of inflation plays a significant role whether the Kamionkowski reheating phase affects the generation of magnetic field in a magnetogenesis scenario or not. In the present context of higher curvature helical magnetogenesis scenario, the electric and the magnetic fields do not get a considerable hierarchy at the end of inflation, and consequently, the generation of the helical magnetic field is not much affected by the reheating physics.

In regard to the generation of BAU from helical magnetic fields, the net baryonic number density ($n_B$) can be determined as,

$$n_B = \frac{n_f}{2} \left(\frac{g^{*}_f}{4\pi^2}\right) \left(\frac{a^3_0}{a^3_f}\right) \left|\beta_+(\eta_f)\right|^2 \left\{\text{Arg} \left[\alpha_+(\eta_f)\beta_+^*(\eta_f)\right] - \pi - \frac{4}{3\omega_{\text{eff}}+1} \left(\frac{k_0}{a_f\eta_f}\right) \exp \left[\left(1+3\omega_{\text{eff}}\right)N_f\right]\right\}^2,$$

(74)

where, recall that $k_0 = a_0H_0$, i.e $k_0^{-1}$ is the present horizon scale. Owing to Eq. (71), the above expression of $n_B$ can be equivalently written as,

$$n_B = \frac{n_f}{2} \left(\frac{g^{*}_f}{4\pi^2}\right) \left(\frac{a_0}{k_0}\right) \left(\frac{kr_{\text{gal}}}{2\pi}\right)^4 \left(B_0^{(\text{gal})}\right)^2.$$

(75)

In regard to estimate the net baryon density, we will use Eq. (75) where $n_B$ has been expressed in terms of the $B_0^{(\text{gal})}$. As mentioned earlier, we are interested to determine $n_B/s_0$ where $s_0$ is given earlier in Eq. (44). With the set of values that we considered earlier (i.e $H_0 = 1.6 \times 10^{13}\text{GeV}$, $\epsilon = 0.1$, and $N_f = 51$), we estimate $n_B/s_0$ for different values of $B_0$, with $T_{re} = 6 \times 10^{12}\text{GeV}$ and $k_0^{\text{max}} = 3.2 \times 10^{-4}\text{GeV}$. Such estimation of $n_B/s_0$ in Kamionkowski reheating scenario resembles with that of presented in the Table [11] which clearly demonstrates that the magnetic fields of current strength $B_0^{(\text{gal})} \sim 10^{-15}\text{G}$ leads to the resultant value of $n_B/s_0$ as of the order $\sim 10^{-10}$.

Thereby similar to the instantaneous reheating case, the current magnetic strength and the net baryonic number density in the Kamionkowski reheating scenario are found to be compatible with the observational constraints, provided the parameter $q$ lies within $2.1 \leq q \leq 2.5$.

VII. CONCLUSION

We provide a viable helical magnetogenesis scenario from inflation and explain the baryon asymmetry of the universe. The electromagnetic (EM) field couples with the background higher curvature term(s), in particular with the Ricci scalar and with the Gauss-Bonnet (GB) invariant, via the dual field tensor. Such non-minimal coupling of the EM field breaks the conformal and the parity symmetries, however preserves the $U(1)$ invariance, of the electromagnetic action. As a result, the positive and negative helicity modes get different amplitudes along their cosmological evolution, and thus the magnetic field becomes helical in nature, which is further connected to the net baryonic number density of the universe. The background spacetime is governed by the well studied $\alpha$-attractor scalar-tensor theory which is known to lead to a viable inflationary scenario consistent with the recent Planck data. The model has some good features: first – during the early universe when the curvature is large, the conformal breaking coupling remains significant and results to a non-trivial contribution to the EM field equations, however during the late times, the curvature becomes low, and as a consequence, the conformal breaking term does not actually contribute particularly from the end of inflation. Second – due to the fact that the non-minimal coupling of the EM field occurs via the dual field tensor, the kinetic term of the EM field remains canonical even in presence of such non-minimal coupling. This leads to the resolution of strong coupling problem in the present model. As a third strong point of our model – the EM field
is found to have a negligible backreaction on the background spacetime and consequently the backreaction issue is resolved in the model.

In such a scenario, we explore the cosmological evolution of electric and magnetic fields, starting from the inflationary era to the present epoch. Consequently we address the helicity power spectrum, that arises due to the difference in the amplitudes between the positive and negative helicity modes of the EM field. During the evolution, the universe enters to a reheating era after the end of inflation and depending on the reheating dynamics, we consider two different cases: (1) the instantaneous reheating case where the universe instantaneously jumps to the radiation dominated epoch as the inflation ends, i.e the e-fold number of the instantaneous reheating era is zero. (2) As a second case, the reheating phase is considered to have a non-zero e-fold number, in particular, we consider the conventional reheating mechanism proposed by Kamionkowski et al. [80], where the reheating phase is parametrized by a constant equation of state parameter ($\omega_{\text{eff}}$) and a reheating temperature ($T_{\text{re}}$).

In the case of instantaneous reheating, as the universe suddenly jumps from inflation to radiation era, the conductivity of the universe becomes huge in the post-inflationary stage and thus the electric field dies out quickly. Moreover, the EM action restores the conformal symmetry after inflation and consequently the EM energy density evolves as $a^{-4}$ with the expansion of the universe, or equivalently the magnetic energy density redshifts as $a^{-6}$ as the electric field practically vanishes. Consequently the helicity power spectrum goes by $a^{-3}$, i.e the comoving helicity remains conserved with cosmic time in the post-inflation phase. As a result, the theoretical predictions for the helicity power spectrum — it evolves by Eq.(63) during the reheating stage, and then, as $1/a^3$ from the end of reheating to the present time. It turns out that similar to the instantaneous reheating case, the current magnetic strength at present time can lead to the resultant value of the ratio of the baryonic number density to the entropy density, i.e $n_{B}/s_{0}$, as large as $10^{-10}$, which is consistent with the observational data of BAU.

Instead of the instantaneous reheating, it would be more physical if the universe undergoes through a reheating phase which has a non-zero e-fold number. As mentioned, we consider a Kamionkowski like reheating mechanism where, due to the constant EoS parameter symbolized by $\omega_{\text{eff}}$, the Hubble parameter during the reheating era is given by $H \propto a^{-\frac{\alpha}{2}(1+\omega_{\text{eff}})}$. After the end of reheating, the universe enters to a radiation era and the conductivity gets a huge value, which in turn makes the electric field practically zero. Specifically, in regard to the magnetic power spectrum – it evolves by Eq.(65) during the reheating stage, and after that, as $1/a^3$ till the present epoch. On other hand, the helicity power spectrum follows Eq.(65) in the reheating era, and then, as $1/a^3$ from the end of reheating to the present time. Furthermore we have found that the magnetic fields with strength $B_{0}^{(\text{gal})} \sim 10^{-13}G$ at present time can lead to the resultant value of the ratio of the baryonic number density to the entropy density, i.e $n_{B}/s_{0}$, as large as $10^{-10}$, which is consistent with the observational data of BAU.

VIII. APPENDIX-A: SOLUTIONS AND POWER SPECTRUM OF THE EM FIELD DURING INFLATION

To solve Eq.(30), we introduce a dimensionless variable,

$$\tau = \left(\frac{-\eta}{\eta_{0}}\right)^{\alpha}.$$  \hspace{1cm} (76)

In terms of $\tau$, Eq.(30) turns out to be,

$$\alpha^{2} \frac{d^{2}A_{\pm}}{d\tau^{2}} + \left(\frac{\alpha(\alpha + 1)}{\tau}\right) \frac{dA_{\pm}}{d\tau} + k\zeta^{2} A_{\pm}(k, \tau) = 0.$$  \hspace{1cm} (77)

The above equation is a Bessel differential equation and has a complete solution as,

$$A_{\pm}(k, \tau) = \tau^{-1/(2\alpha)} \left\{ C_{1} J_{\pm}^{\left(-i\frac{\zeta\sqrt{k}}{\alpha}\tau\right)} + C_{2} Y_{\pm}^{\left(-i\frac{\zeta\sqrt{k}}{\alpha}\tau\right)} \right\}.$$
\[ A_- (k, \eta) = \tau^{-1/(2\alpha)} \left\{ C_3 \frac{J_{\pm}}{\mp} \left( \frac{\zeta \sqrt{k}}{\alpha \tau} \right) + C_4 Y_{\pm} \left( \frac{\zeta \sqrt{k}}{\alpha \tau} \right) \right\}, \] (78)

which, in terms of \( \eta \), resembles with the form given in Eq.(31). By matching \( A_\pm (k, \eta) \) and \( A'_\pm (k, \eta) \) at \( \eta = \eta_* \) (recall, \( \eta_* \) is the horizon crossing instant of \( k \)-th mode), we get \( C_i \) \((i = 1, 2)\) as follows:

\[
C_1 = \frac{-1}{2\alpha \sqrt{-2k\eta_*/\eta_0}} \left[ i\pi e^{-i\eta_*} \left\{ -\zeta \sqrt{k} \tau_* \left( J_{+1} \mp \frac{i}{\eta_*} \right) + k\eta_* Y_{+} \left( \frac{\zeta \sqrt{k}}{\alpha \tau_*} \right) \right\} \right],
\]

\[
C_2 = \frac{-1}{4\alpha \sqrt{-2k\eta_*/\eta_0}} \left[ i\pi e^{-i\eta_*} \left\{ -\zeta \sqrt{k} \tau_* \left( J_{-1} \pm \frac{i}{\eta_*} \right) - J_{+1} \pm \frac{i}{\eta_*} \right\} \right].
\] (79)

Similarly the matching conditions of \( A_\pm (k, \eta) \) and \( A'_\pm (k, \eta) \) lead to \( C_3, C_4 \) as,

\[
C_3 = \frac{-1}{2\alpha \sqrt{-2k\eta_*/\eta_0}} \left[ -\pi e^{-i\eta_*} \left\{ -\zeta \sqrt{k} \tau_* \left( J_{+1} \mp \frac{i}{\eta_*} \right) \left( \frac{\zeta \sqrt{k}}{\alpha \tau_*} \right) \right\} \right],
\]

\[
C_4 = \frac{-1}{2\alpha \sqrt{-2k\eta_*/\eta_0}} \left[ \pi e^{-i\eta_*} \left\{ -\zeta \sqrt{k} \tau_* \left( J_{-1} \pm \frac{i}{\eta_*} \right) \left( \frac{\zeta \sqrt{k}}{\alpha \tau_*} \right) \right\} \right].
\] (80)

In the above expressions, \( \tau_* = (\eta_0/\eta_*)^\alpha \) and \( \eta_0 = (1 + \epsilon) \), as shown in Eq.(32). From Eq.(31), \( A_\pm (k, \eta) \) in the superhorizon limit \((i.e. |k\eta| \ll 1)\) come as,

\[
\lim_{|k\eta| \ll 1} A_+ (k, \eta) = \left( \frac{C_1 - C_2 \cot \left( \frac{\pi}{2\alpha} \right)}{\Gamma \left( 1 + \frac{1}{2\alpha} \right)} \right)^{1/(2\alpha)},
\]

\[
\lim_{|k\eta| \ll 1} A_- (k, \eta) = \left( \frac{C_3 - C_4 \cot \left( \frac{\pi}{2\alpha} \right)}{\Gamma \left( 1 + \frac{1}{2\alpha} \right)} \right)^{1/(2\alpha)}.
\] (81)

Moreover \( A'_\pm (k, \eta) \) (which are necessary for electric power spectrum) in the superhorizon regime are obtained as,

\[
\lim_{|k\eta| \ll 1} \frac{dA_+}{d(k\eta)} = H_0 \left( \frac{C_2 \Gamma \left( \frac{\pi}{2\alpha} \right)}{\pi} \right)^{1/(2\alpha)} \left( \frac{-i\zeta \sqrt{k}}{2\alpha} \right)^{-1/(2\alpha)},
\]

\[
\lim_{|k\eta| \ll 1} \frac{dA_-}{d(-k\eta)} = H_0 \left( \frac{C_4 \Gamma \left( \frac{\pi}{2\alpha} \right)}{\pi} \right)^{1/(2\alpha)} \left( \frac{\zeta \sqrt{k}}{2\alpha} \right)^{-1/(2\alpha)}.
\] (82)

Using such expressions, one arrives the electric and magnetic power spectra during inflation, as obtained in Eq.(33) and Eq.(34) respectively.

**IX. APPENDIX-B: NUMERICAL ESTIMATION OF MAGNETIC STRENGTH**

From Eq.(71), the magnetic field strength at present epoch (in presence of Kamionkowski reheating phase) is given by,

\[
E_{0}^{(gal)} = \sqrt{\frac{2}{\pi}} \left( \frac{2\pi}{k_{\tau}^{gal}} \right)^2 \left( \frac{k}{a_0} \right)^2 |\beta_+ (\eta_f)| \left\{ \text{Arg} [\alpha_+ (\eta_f) \beta_+^* (\eta_f)] - \pi - \frac{4}{3\omega_{\text{eff}} + 1} \left( -\frac{k}{a_f H_f} \right) \exp \left[ \frac{1 + 3\omega_{\text{eff}}}{2} N_{\text{re}} \right] \right\},
\] (83)

where \( k/(a_f H_f) = -k_{\eta_f} \). Moreover \( N_{\text{re}} \) represents the e-fold number of the reheating phase, and given in Eq.(47) as,

\[
N_{\text{re}} = \frac{4}{(1 - 3\omega_{\text{eff}})} \left[ -\frac{1}{4} \ln \left( \frac{45}{\pi^2 g_{\text{eff}}} \right) - \frac{1}{3} \ln \left( \frac{11g_{\text{re}}}{43} \right) - \ln \left( \frac{k}{a_0 T_0} \right) - \ln \left( \frac{3H_f^2 M_{\text{Pl}}^2}{H_0} \right)^{1/4} - N_f \right].
\] (84)
Now the estimations for different quantities are,

where we use the conversion $1 \text{K} = 8.6 \times 10^{-14} \text{GeV}$. Such considered values immediately lead to $N_{\text{re}} = 33$ and $H_{\text{re}} = 6 \times 10^{-13} \text{GeV}$ respectively. Consequently, we estimate the following quantities:

\[
\begin{align*}
|\beta_+(\eta_f)| &= 1.8 \times 10^{37}, \\
|\text{Arg} \left[ \alpha_+(\eta_f)\beta_+^*(\eta_f) \right] - \pi &= 3 \times 10^{-3}, \\
\left( \frac{4k}{a_f H_f} \right) \exp \left[ \left( 1 + \frac{3 \omega_{\text{eff}}}{2} \right) N_{\text{re}} \right] &= 1.3 \times 10^{-9}.
\end{align*}
\] (85)

Plugging all such values in Eq. (83), one gets,

\[
B_{0}^{(\text{gal})} \sim 10^{10} \times 10^{-80} \times 10^{37} \times 10^{-3} \times \left( \frac{1}{1.95 \times 10^{-20}} \right) \text{Gauss},
\]

\[
\sim 10^{-16} \text{Gauss}.
\] (86)

Similarly, one can estimate $B_{0}^{(\text{gal})}$ for other values of $q$, as we described in Fig. [4].

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