CURRENT OSCILLATIONS, INTERACTING HALL DISCS
AND BOUNDARY CFTs

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MIRAMARE–TRIESTE
December 1998

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Abstract

In this paper, we discuss the behavior of conformal field theories interacting at a single point. The edge states of the quantum Hall effect (QHE) system give rise to a particular representation of a chiral Kac-Moody current algebra. We show that in the case of QHE systems interacting at one point we obtain a “twisted” representation of the current algebra. The condition for stationarity of currents is the same as the classical Kirchoff’s law applied to the currents at the interaction point. We find that in the case of two discs touching at one point, since the currents are chiral, they are not stationary and one obtains current oscillations between the two discs. We determine the frequency of these oscillations in terms of an effective parameter characterizing the interaction. The chiral conformal field theories can be represented in terms of bosonic Lagrangians with a boundary interaction. We discuss how these one point interactions can be represented as boundary conditions on fields, and how the requirement of chirality leads to restrictions on the interactions described by these Lagrangians. By gauging these models we find that the theory is naturally coupled to a Chern-Simons gauge theory in 2 + 1 dimensions, and this coupling is completely determined by the requirement of anomaly cancellation.
1 Introduction

The motivation for studying interacting Hall systems is two-fold. Firstly there are interesting experiments that can be performed on these systems where a single point interaction comes about either from the experimental geometry of the sample or due to an applied external voltage that induces tunneling at one point between the edge currents of the quantum Hall system [1]. Secondly the effective field theories of these models involve interesting theoretical concepts like topological field theory, conformal field theory and the quantization of anomalous gauge theories. Studying these systems gives us a good testing ground for our understanding of these quantum field theoretic ideas. Related work on aspects of this subject involves the Chern-Simons description of the quantum Hall Effect (QHE) [2], this description is based on the fact that the low lying excitations of the QHE system are chiral edge excitations that can be described by a free field theory compactified at a radius that is related to the Hall conductivity. The fact that the effective field theory corresponding to the bulk incompressible behavior of the QHE system is a gauge theory requires one to understand various aspects of gauge theories on finite geometries [3, 4]. In particular the chiral scalar field theory when gauged (coupled to electromagnetism) is anomalous and the requirement of anomaly cancellation leads to a specific coupling of edge and bulk theories. Although much of this is by now standard, there are still some matters that we need to clarify in order to generalize to the case where we have “touching discs”, that is two quantum Hall systems with their respective discs, interacting through a single point on the boundary.

In this paper we begin with a brief discussion of the origin of edge currents in QHE systems in section 2. For simplicity, in this section, we study QHE on a strip of finite width, though later we change the geometry to that of a disc. (These different geometries differ in subtle details, but the essential physics remains unaltered.) In section 3, we discuss the relationship of edge states to current algebras, using the fact that the effective field theory of the low-lying excitations gives rise to a fermionic theory and this in turn gives us a particular representation of the Kac-Moody algebra on the Fock space of the fermionic theory. In section 4, we study how the current algebra is modified in the case of “touching discs” by the interaction at the junction. We also discuss the edge currents on the “touching discs”, and show how one obtains current oscillations between the discs. We then proceed to describe the quantization of the edge theory. In section 5, we generalize the situation to the case when there are several branches of edge excitations. In section 6, we show that the edge theories can be described in terms of a chiral bosonic Lagrangian with edge interactions. The edge interactions lead to a free chiral bosonic theory with non-trivial boundary conditions that we analyze in section 6. We also show in this section how the theory is highly constrained and the interaction parameters are not arbitrary. In section 7, we review the gauged chiral edge theory and its coupling to a bulk Chern-Simons theory with a view to generalizing the discussion to the case of touching discs. In section 8 we show that the interacting quantum Hall system can be described in terms of a twisted complex boson that enables one to diagonalize the boundary interaction, and that the requirement of anomaly cancellation determines the coupling of this complex boson to a bulk Chern-Simons theory.
2 Edge Currents on a Strip

The importance of edge states for the quantum Hall effect (QHE) was first recognized by [5, 6]. In this section, we will briefly review the edge states for a system of non-interacting electrons moving in a strong transverse magnetic field and their connection to current algebras. So, consider an electron confined to move in the cylindrical strip

\[-X/2 \leq x \leq X/2, \quad -Y/2 \leq y \leq Y/2\]  

(2.1)

with the lower and upper sides in the y direction identified. The strip is immersed in a strong magnetic field \( B \), directed along the positive z direction. The Hamiltonian for the electron is

\[ H = \frac{1}{2m}(\vec{p} + \frac{e}{c} \vec{A})^2 + V(x) \]  

(2.2)

where \( e \) is the absolute value of the charge of the electron, \( m \) its effective mass, \( \vec{A} \) the vector potential for the magnetic field and \( V(x) \) the potential in the \( x \)-direction confining the electrons to the strip. For simplicity, we assume for \( V(x) \) the form of an infinite-well potential localized at the edges. Moreover, it is convenient to choose the gauge for \( \vec{A} \) so that it is directed along the \( y \)-direction and depends only on \( x \):

\[ A_x = A_z = 0, \quad A_y = Bx. \]  

(2.3)

The appropriate boundary conditions for the wave function are:

\[ \psi(x, -Y/2) = \psi(x, Y/2), \]

\[ \psi(\pm X/2, y) = 0. \]  

(2.4)

The eigenfunctions then have the factorized form

\[ \psi_k(x, y) = \frac{1}{\sqrt{Y}} e^{-ik_y} \psi_{k,\nu}(x), \quad k = \frac{2n\pi}{Y}, \quad n = 0, \pm 1, \pm 2\ldots, \]  

(2.5)

where the meaning of \( \nu \) will be discussed below. The effective Hamiltonian for \( \psi_{k,\nu}(x) \) is

\[ H_{\text{eff}} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{\hbar}{2}(x - l^2k)^2 \]  

(2.6)

where \( \omega_c \) is the cyclotron frequency and \( l \) is the magnetic length:

\[ \omega_c = \frac{eB}{mc}, \quad l^2 = \frac{\hbar c}{eB}. \]  

(2.7)

In the conditions typical for the QHE,

\[ B \approx 10T, \quad \omega_c \approx 10^{12}s^{-1}, \quad l \approx 10^{-6}cm. \]  

(2.8)

\( H_{\text{eff}} \) is simply the Hamiltonian for a one-dimensional harmonic oscillator centered around \( X_k = l^2k \). Now, as long as \( X/2 - |X_k| \gg l \), the boundary conditions at \( x = \pm X/2 \) can be neglected and the eigenfunctions \( \psi_{k,\nu} \) correspond to the usual harmonic oscillator ones:

\[ \psi_{k,\nu}(x) = N_{\nu} e^{-\left(x-l^2k/2\right)^2/2l^2} H_{\nu}(x/l - lk). \]  

(2.9)
Here $H_{\nu}$ is the Hermite polynomial of order $\nu$ and $\nu$ is also its number of nodes between $-X/2$ and $X/2$. The corresponding energies are

$$E_{k,\nu} = E_\nu = \hbar \omega_c (\nu + \frac{1}{2}).$$

(2.10)

We see that the energy depends only on $\nu$ and not on $k$ (Landau levels).

That is not so when $X/2 - |X_k| \approx l$ or when $|X_k| > X/2$: the eigenfunctions $\psi_{k,\nu}$ can still be labeled by the number of nodes they have between $-X/2$ and $X/2$, but their energies now depend also on $k$. In particular, for a given $\nu$, the energies $E_{k,\nu}$ increase with $|X_k|$ without a limit and it can be shown that

$$\text{for } |X_k| = \pm \frac{X}{2}, \quad E_{k,\nu} = \hbar \omega_c (2\nu + \frac{3}{2})$$

(2.11)

while

$$\text{for } |X_k| - X/2 \gg l, \quad E_{k,\nu} \approx \left( \frac{l^2 k - X/2}{2m_0} \right)^2 \frac{e^2 B^2}{2m_0 \hbar^2}.$$  

(2.12)

We will refer to the set of states having the same quantum number $\nu$ as the $\nu$th Landau level. Let us now consider the current, $I_{k,\nu}$, carried in the $y$ direction by one of these states. The use of the Hellman-Feynman formula (see for example [7]) leads to the following expression for $I_{k,\nu}$:

$$I_{k,\nu} = \frac{1}{Y} \langle k, \nu | \frac{e}{m_c} (p_y + \frac{e}{c} A_y) | k, \nu \rangle$$

$$= \frac{e}{\hbar Y} \langle k, \nu | \frac{\partial H}{\partial k} | k, \nu \rangle = \frac{e}{Y \hbar} \frac{\partial E_{k,\nu}}{\partial k}. \quad (2.13)$$

From here we see that only the states localized near the edges carry a non-vanishing current. These currents are diamagnetic. We will refer to them as edge currents.

If the electrons are non-interacting, the multi-particle states are obtained by placing each electron in a distinct Landau level (the spins are aligned along the $z$-direction due to $B$). In the ground state, all the levels up to the Fermi energy $E_F$, are filled. If $E_F$ lies between the first and second Landau bands, a typical value for the total current flowing in the proximity of the edge is

$$I_{\text{edge}} \approx 10^{-7} - 10^{-8} \text{ A.}$$

(2.14)

The question of the stability of these edge states in the presence of impurities in the sample was discussed in the paper of Halperin we referred to before. What happens is that, if the average distance between the impurities is large compared to the magnetic length, the electrons populating the edge states will simply go around them along equipotential lines, and there will be no backscattering from the states on one edge to the other.

### 3 Edge States and Current Algebras

The edge states we have discussed in the previous section form a set of states localized within a few magnetic lengths from the edge(s) of the sample. At zero temperature, only the states below the Fermi energy will be occupied.
Instead of a strip, we will study a disc of circumference \( L \), so that we have only one edge, and suppose, for the sake of simplicity, that \( E_F \) lies in the gap between the first and the second bulk Landau levels, so that the occupied edge states belong only to the first Landau level. If one is interested only in the low energy excitations of the system, it is then a good approximation to linearize the spectrum of the edge states around the Fermi energy. The energy spectrum that we obtain is the spectrum of a chiral, massless, complex Fermi field on the edge, propagating with a velocity, \( v_F \) equal to the group velocity at the Fermi energy. The chirality of the electrons is determined by the direction of the external magnetic field. Let the boundary of the disc \( \partial D \), of circumference \( L \), be coordinatized by its curvilinear coordinate \( x \) (with \( x \) and \( x + 2\pi \) identified), increasing in the counterclockwise direction. And assume that the direction of the external magnetic field is such that the electrons near the edge move in the clockwise direction. Such a theory is then described by the action

\[
S = i \int dx dt (\dot{\psi}_+^\dagger \psi_+ - v_F \psi_+^\dagger \partial_x \psi_+),
\]

\( 0 \leq x \leq L. \)

\( \psi_+ \) fulfills canonical anti-commutation relations:

\[
[\psi_+^\dagger(x, t), \psi_+(x', t)]_+ = \hbar \delta(x - x').
\]

The equation of motion implied by this action is

\[
\partial_- \psi_+ = (\partial_t - v_F \partial_x) \psi_+ = 0
\]

which simply means that \( \psi_+ \) is a left- moving field:

\[
\psi_+(x, t) = \psi_+(x + v_F t).
\]

Consider now the chiral current \( J_+(x, t) \), associated with the field \( \psi_+ \),

\[
J_+(x) =: \psi_+^\dagger(x) \psi_+(x):
\]

where the normal ordering is defined with respect to the annihilation and creation operators for \( \psi_+ \). Due to one loop contributions, the commutator of two such currents acquires an anomalous term (see for example [8]), known as the Schwinger term:

\[
[J_+(x, t), J_+(y, t)] = \frac{i\hbar^2}{2\pi} \delta'(x - y).
\]

The Fourier components

\[
K_n = \frac{1}{\hbar} \int dx J_+(x) e^{i2\pi nx/L}, \quad K_n^\dagger = K_{-n}, \quad n \in \mathbb{Z}
\]

of the current fulfill the commutation relations defining an abelian Kac-Moody (K-M) algebra,

\[
[K_n, K_m] = n \delta_{n+m,0}.
\]
As is well known, the charge sectors of the theory are completely described by the unitary, irreducible representations of this K-M algebra. Moreover, the Hamiltonian associated with the action can be expressed uniquely in terms of the current \( J_+ (x) \) as

\[
H = \frac{2\pi v_F}{\hbar} \int dx \, : J_+ (x) J_+ (x) :.
\]

The normal ordering \( : \times \times \times : \) here is defined with respect to the K-M generators:

\[
: K_n K_m : = K_m K_n \quad \text{if} \quad n > 0,
\]

\[
: K_n K_m : = K_n K_m \quad \text{if} \quad n < 0.
\]

4 Edge Currents on Two Touching Discs.

In the previous section, we have discussed the edge currents flowing on the rim of a disc in the presence of a strong transverse magnetic field. A question which arises naturally is how the behavior of these currents changes if two discs are put in contact at one point. We will not attempt here to analyze the detailed nature of the junction: the approach we will pursue in this section is to see if the behavior of the edge currents can be predicted on general grounds, starting from the current algebra discussed in the previous section. What one expects is that the edge currents will scatter at the junction, leading to a mixing of the edge states for the two discs. Moreover, the scattering pattern should be such that their chirality is preserved.

Consider two touching discs, \( D_1 \) and \( D_2 \), which we will assume, for simplicity, to have the same circumference \( L \). Let the boundaries of \( D_i \), \( i = 1, 2 \) be coordinatized by \( x_i : 0 \leq x_i \leq L \), measured starting from the junction and increasing for both the discs in, say, the anti-clockwise direction.

We now observe that, if a strong transverse magnetic field is turned on, there will be, as before, states localized near the edges of the two discs and that the corresponding edge currents will still fulfill a current algebra of the form Eq. (3.6). The problem, now, is that there is a singular point, the junction, and we have to define the behavior of the currents there. In order to do that, it is useful to observe that, fields are operator valued distributions in quantum field theory, so that their densities do not make sense as operators. Therefore, it is necessary to smear the fields with suitable test functions in order to get well defined self-adjoint operators. The test functions belong to a linear space and encode the boundary conditions on the fields. In the case of two touching discs, the test functions \( \Lambda \), for our current algebra on the edge are maps from the union of the boundaries of the two discs to real numbers. As we shall soon see, they need not be continuous at the junction.

So, define the smeared currents

\[
K(\Lambda) = \sum_i \int_{\partial D_i} dx_i \Lambda^i (x_i) J_+^i (x_i)
\]

(4.1)

where \( \Lambda^i (x_i) \) and \( J_+^i (x_i) \) denote the restrictions of \( \Lambda \) and \( J_+ \) to \( \partial D_i \). From Eq.(3.6) we get, for the commutator of two \( K \)'s,

\[
[K(\Lambda), \bar{K}(\bar{\Lambda})] = \frac{i\hbar^2}{2\pi} \sum_k \int_{\partial D_k} dx_k \Lambda^k \bar{\Lambda}^k.
\]

(4.2)
The boundary conditions on the smearing functions \( \Lambda \) at the junction must be such that the commutator Eq. (4.2) is antisymmetric with respect to the interchange of \( K(\Lambda) \) and \( K(\Lambda) \). Thus it must be that

\[
0 = [K(\Lambda), K(\Lambda)] + [K(\Lambda), K(\Lambda)] = \frac{i\hbar^2}{2\pi} \sum_k \int_{\partial D_k} dx_k (\Lambda^k d\Lambda^k + \Lambda^k d\Lambda^k) =
\]

\[
= \frac{i\hbar^2}{2\pi} \sum_k (\Lambda^k(L)\Lambda^k(L) - \Lambda^k(0)\Lambda^k(0)).
\] (4.3)

This implies that for all test functions \( \Lambda \),

\[
\Lambda^k(L) = \mathcal{O}_f^k \Lambda^i(0)
\] (4.4)

where \( \mathcal{O}_f^k \) is some fixed orthogonal matrix characterizing the junction.

Classically, the smearing functions can be identified with the possible current distributions (at a given time). Thus the boundary conditions Eq. (4.4) are nothing but the boundary conditions for the classical currents at the junction:

\[
J^k(L) = \mathcal{O}_f^k J^i(0).
\] (4.5)

We see in this manner that a scattering matrix for the currents, at the junction, arises naturally in this approach. In the next section we will study the physical implications of Eq. (4.5) on the behavior of the currents circulating around the edges of two touching discs. Here, we notice only that the classical currents fulfilling the boundary conditions Eq. (4.5) do not satisfy, in general, Kirchhoff’s law at the junction. This implies, classically, that at different times there will either be an accumulation or a depletion of electric charge at the junction. Consequently, the total charge distributed on the rims of the two discs \( Q \), will not be conserved in time. Actually this fact has, as we will see, an even stronger consequence in quantum theory: we will find that the corresponding charge operator, \( \hat{Q} \) cannot be defined at all (in a sense that will be explained later).

### 4.1 Stationary Currents on the Edges of Two Touching Discs

Before quantizing the current algebra Eq. (4.2), in this section we wish to present simple considerations on the behavior of the edge currents on two touching discs. The average current in a quantum edge-eigenstate should correspond with a classical, stationary current distribution. We will study the stationary current distributions on the edges of two touching discs, fulfilling the boundary conditions Eq. (4.5).

In the stationary regime there cannot be an accumulation of charge at the vertex, and so besides the boundary condition Eq. (4.5), we must have

\[
\sum_{i=1,2} J^i_+(L) = \sum_{i=1,2} J^i_+(0),
\] (4.6)

which is nothing but the statement of Kirchhoff’s law at the junction. The above equation, along with Eq. (4.5), implies that either \( J^i(L) = J^i(0) \) for \( i = 1, 2 \) or, \( J^1(L) = J^2(0) \) and \( J^2(L) = J^1(0) \).
4.2 Oscillating Edge Currents on Two Touching Discs

As we discussed in the previous section the Kirchoff law condition Eq. (4.6) is equivalent to demanding stationarity of currents. However there are physical situations where we cannot require the currents to be stationary in this sense. In particular, when we have a chiral edge theory as in the quantum Hall effect, we cannot have stationary edge excitations, as the conditions for stationarity are incompatible with chirality. In fact, for the case of a single quantum Hall disc, there is an edge current that propagates around the Hall disc with a frequency \( \nu \) determined by the Fermi velocity \( v_F \) where

\[
\nu = \frac{v_F}{2\pi L}.
\]  

(4.7)

In this section we will see that in the absence of a stationarity condition, we obtain transport of current from one disc to another and this leads to current oscillations between the discs, with a frequency characterized by the parameter \( \alpha \) that is used to parameterize the interaction between the currents on the two discs.

For two discs with equal circumference \( L \), the frequency \( \nu \) of the oscillations is given by

\[
\nu = \frac{\alpha v_F}{2\pi L}.
\]  

(4.8)

These oscillations can be understood if one considers the propagation of a classical impulse on the boundary. There are now two cases, depending on the sign of \( \det O \). Let us consider now the case when \( \det O = 1 \) and hence,

\[
O = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}.
\]  

(4.9)

Suppose that, at time 0, there is a small wave packet on the left disc, propagating in the clockwise direction. When the impulse reaches the junction, it splits according to the boundary conditions Eq. (4.5) into a transmitted impulse, propagating clockwise on the edge of the left disc, and another transmitted impulse propagating in the clockwise direction on the edge of the right disc. There is no reflected impulse on the left disc as the currents are chiral. The transmitted impulse is equal to \( \cos \alpha \) times the initial one, and the scattered one to \( -\sin \alpha \) the initial one. When these two impulses reach again the junction, they again split according to Eq. (4.5) and then recombine and it is easy to check that the resulting impulse on the left disc is equal to \( \cos 2\alpha \) times the initial pulse, while that on the right disc is equal to \( -\sin 2\alpha \) times the initial one. After \( n \) scatterings, the impulse on the left disc is equal to \( \cos n\alpha \) and that on the right disc equal to \( -\sin n\alpha \), times the starting impulse. As discussed above the oscillations arise basically because the relation Eq. (4.5) also implies that after the \( n \)-th passage through the junction

\[
J^i(L) = O^{(n)}(\alpha)J^i(0).
\]  

(4.10)

For simplicity we can take the case \( O^{(n)}(\alpha) = 1 \). This means that the current oscillates between the two discs with a frequency given by Eq. (4.8). This also explains why there are no stationary current distributions: they arise from taking time averages of these oscillating currents, which of course vanish. A similar analysis, for the case \( \det O = -1 \), that is

\[
O = \begin{pmatrix} -\cos \alpha & \sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix},
\]  

(4.11)
gives a different result: the initial impulse on the first disc re-composes after two scatterings since \( O^2 = 1 \). The average current on each disc does not vanish and this explains the possibility of non-vanishing stationary currents in this case.

4.3 Quantization of the Edge Currents on Two Touching Discs

In this section, we will quantize the edge currents on the boundaries of two touching discs in the case \( \det O = 1 \). An interesting aspect is that the quantization cannot be done, in this case, directly in terms of a Fermi field \( \psi_+ \) on the boundary, as was possible on a circle. The reason is that there are no boundary conditions for the Fermi field which imply the boundary conditions (4.5) for the currents. A simple way to understand this is that the boundary conditions on \( \psi \) which make the Dirac Hamiltonian self-adjoint always imply Kirchoff’s law at the junction and we have seen before that our currents will not satisfy it in general.

Thus, we have to quantize the currents directly. In order to do so, we introduce a basis of smearing functions fulfilling the boundary conditions (4.4). A convenient choice is

\[
\Lambda_n \equiv \begin{pmatrix} \Lambda^1_n \\ \Lambda^2_n \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} e^{i(\alpha + 2\pi n) \frac{\pi}{\hbar}},
\]

\[
\Lambda_n^\dagger \equiv \begin{pmatrix} \Lambda^{1\dagger}_n \\ \Lambda^{2\dagger}_n \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-i(\alpha + 2\pi n) \frac{\pi}{\hbar}},
\]

\[n \in \mathbb{Z}, \quad 0 < \alpha < 2\pi.\] (4.12)

We next introduce the corresponding smeared charges:

\[
K_n \equiv \frac{1}{\hbar} K(\Lambda_n),
\]

\[
K_n^\dagger \equiv \frac{1}{\hbar} K(\Lambda_n^\dagger).
\] (4.13)

They fulfill the algebra

\[
[K_n, K_m] = [K_n^\dagger, K_m^\dagger] = 0.
\]

\[
[K_n, K_m^\dagger] = (\alpha + 2\pi n) \delta_{n,m}.
\] (4.14)

What we have got (modulo a normalization) is an infinite set of annihilation and creation operators. The vacuum, \(|0\rangle\), is defined by the conditions

\[
K_n |0\rangle = 0, \quad n \geq 0,
\]

\[
K_n^\dagger |0\rangle = 0, \quad n < 0.
\] (4.15)

As the Hamiltonian we assume the same expression, in terms of the currents that we had on a disc:

\[
H = \frac{2\pi v_F}{\hbar} \sum_k \int_{\partial D_k} dx_k \varepsilon^k J_+^k(x_k) J_+^k(x_k) \varepsilon^k.
\] (4.16)
When expanded in terms of the generators $K_n$, this Hamiltonian coincides with that of an infinite set of harmonic oscillators:

$$H = \frac{2\pi v_F^2}{L} \sum_n :K_n^\dagger K_n:$$

(4.17)

Let us look now at the total charge $\hat{Q}$ of the discs:

$$\hat{Q} \equiv \sum_k \int_{\partial D_k} dx_k J^k_+(x_k) = \hbar e^{\pi/4} (e^{-i\alpha} - 1) \sum_n \frac{1}{\alpha + 2\pi n} K_n + c.c.$$ (4.18)

It can be checked that, when acting on the vacuum, $\hat{Q}$ creates a state of infinite norm.

5 The Multichannel Case

In this section we wish to examine the edge currents on two touching discs when either the filling factor $n_1$ of the disc $D_1$ or that, $n_2$, of disc $D_2$ or both are larger than 1. Because of the junction, there can now be mixing between the several branches of edge currents circulating at the boundaries of the two discs. We will consider, for simplicity, the case of equal circumference $L$.

The generalization of the current algebra to this case is:

$$\{ K(\Lambda), K(\Lambda') \} = \sum_{l=1}^{n_1} k^1_l \int_{\partial D_1} \Lambda^1_l d\tilde{\Lambda}^1 + \sum_{j=1}^{n_2} k^2_j \int_{\partial D_2} \Lambda^2_j d\tilde{\Lambda}^2;$$ (5.1)

where $k^i_j$ are real numbers, reminiscent of the coefficients appearing in Chern-Simons theory. The smearing functions $\Lambda$ and $\tilde{\Lambda}$ can now be regarded as $N \equiv (n_1 + n_2)$-dimensional column vectors:

$$\Lambda \equiv \begin{pmatrix} \Lambda_1^1 \\ \vdots \\ \Lambda_{n_1}^1 \\ \Lambda_1^2 \\ \vdots \\ \Lambda_{n_2}^2 \end{pmatrix}.$$ (5.2)

If we now define the $N \times N$ diagonal matrix $\mathcal{K}$,

$$\mathcal{K} \equiv diag(k_1^1, ..., k_{n_1}^1, k_1^2, ..., k_{n_2}^2),$$ (5.3)

we see that the antisymmetry of the Poisson bracket implies that the smearing functions fulfill the boundary conditions

$$\Lambda(L) = A\Lambda(0)$$ (5.4)

where $A$ is some (fixed) $N \times N$ real matrix, which leaves $\mathcal{K}$ invariant:

$$A^\dagger \mathcal{K} A = \mathcal{K}$$ (5.5)
As in Eq. (4.5), we again identify the boundary conditions Eq. (5.4) with the boundary conditions for the edge currents at the junction:

\[ J(L) = AJ(0) \]  (5.6)

where

\[
I = \begin{pmatrix}
I_1^1 \\
\vdots \\
I_1^{\alpha_1} \\
\vdots \\
I_2^{\alpha_2}
\end{pmatrix} .
\]  (5.7)

The simplest case is when the matrix \( A \) can be diagonalized. This happens, for example, if we assume that \( \mathcal{K} \) is proportional to the identity matrix, because, then, \( A \) is an orthogonal matrix. This is the case we wish to consider now. As is well known, if one regards an orthogonal matrix \( A \) as a linear transformation acting on a real \( N \) dimensional vector space \( V \), it is always possible to decompose \( V \) into invariant two-dimensional and one-dimensional orthogonal subspaces, so that \( A \) acts as the identity on the one-dimensional ones and as a rotation on the two-dimensional ones:

\[
V = \bigoplus_p V_p^{(2)} \bigoplus_q V_q^{(1)}, \quad \text{dim } V_p^{(2)} = 2, \quad \text{dim } V_q^{(1)} = 1,
\]

\[
AV_p^{(2)} = R(\alpha_p)V_p^{(2)}, \quad AV_q^{(1)} = V_q^{(1)}. \]  (5.8)

Notice that it follows from this that, if \( \text{dim } A \) is odd, \( q \geq 1 \). The behavior of the currents is then clear: for each subspace \( V_q^{(1)} \), there is a set of stationary currents (\( I \) is along the direction of the corresponding eigenvector of \( A \)) while for each \( V_p^{(2)} \), there is a set of oscillating currents, the frequency \( \nu_p \) of the oscillations being given by Eq. (4.8) with \( a_p \) in the place of \( a \).

6 Compact Scalar Field Theory with Boundaries

The first issue we discuss in this section is regarding the proper quantization of a compact chiral scalar field theory on manifolds with boundaries. Some of the issues that arise here are due to this latter circumstance: we can now add boundary interactions to the theory, interactions that will be required to preserve conformal invariance and chirality. The boundary equations of motion then lead to non-trivial boundary conditions on the fields. Thus apart from the standard Neumann boundary conditions, we also have situations where there are Dirichlet boundary conditions [9] or more general mixed boundary interactions.

The Lagrangian for a single scalar field is given by

\[
L = \frac{R^2}{2} \int d\sigma \; dt \; \partial_{\mu}X \partial^{\mu}X + \int_{\text{boundary}} dt \; (\text{boundary terms}).
\]  (6.1)

We have to deal with two cases here: the spatial geometry parameterized by "\( \sigma \)" can be a circle or a line segment. They correspond to closed and open string theories respectively.
(Note: The closed string theory corresponds to the situation where the Hall effect sample has a disc geometry, while the open string theory is probably relevant if the geometry is a rectangular bar/strip with length \( l \).) If we are dealing with a closed string, then the boundary term corresponds to a given but arbitrary marked point on the circle. The only constraints on the boundary terms are that they should not spoil the conformal invariance of theory and should also maintain the chirality. The possible candidates for these terms for the disc geometry are of the form

\[
\int dt \left[ Ae^{i\alpha X(0,t)} + Be^{-i\beta X(2\pi,t)} + C[X(0,t)\partial_t X(2\pi,t) - X(2\pi,t)\partial_t X(0,t)] \right]
\]

where \( \alpha \) and \( \beta \) are fixed by requiring conformal invariance i.e. the operators have conformal dimension one and they do not acquire anomalous dimensions in perturbation theory. The last term describes an interaction between the fields at one point, physically this allows the field \( X \) to have discontinuities at this point. The exponential terms are very interesting. In a fermionized version they correspond to bilinears in fermions and can describe tunneling of fermions.

In order to deal with compact chiral scalars we need to impose the condition \( X(\sigma + 2\pi, t) = X(\sigma, t) + 2\pi RN, N \in \mathbb{Z} \) (which allows for winding modes), and the chirality condition \( \partial_z X = 0 \), where \( z = \sigma + it \). The chirality constraint is actually second class which means that it does not commute with itself for different values of \( \sigma \). But one can quantize the system using the Dirac or some other prescription.

To make contact with the edge currents of the quantum Hall effect we will have to gauge this theory. The required analysis is done in [10, 11, 12], where it is shown that one has to deal with an anomalous chiral theory living on the edge and the anomaly has to be cancelled by an appropriate Chern-Simons term in the bulk. The reason that this chirality condition has to be dealt with delicately is because some choices of boundary conditions will violate chirality and so there is a tension between allowed boundary conditions and constraints. Similarly the gauge invariance of the boundary conditions has to be checked explicitly too.

### 6.1 Boundary Conditions

Consider two scalar fields living on a circle of perimeter \( L \), and interacting only at the point \( \theta = 0 \) (which is identified with \( L \)). We can write the action for this system as

\[
S = \frac{R_i^2}{2} \int \partial_\mu X_1 \partial^\mu X_1 + \frac{R_i^2}{2} \int \partial_\mu X_2 \partial^\mu X_2 + B_{i,\alpha,\beta,j} \int X^j_i \partial_\tau X^j_\beta dt,
\]

where \( \alpha, \beta \) are 0 or \( 2\pi \) and \( X^i_\alpha = X^i(\alpha) \). The last term \( B_{i,\alpha,\beta,j} \) represents a boundary interaction. The equations of motion are

\[
\Box X_1 = 0, \quad (6.4)
\]
\[
\Box X_2 = 0. \quad (6.5)
\]

The boundary conditions can be compactly written as

\[
R_i^2 \delta_{ij} (\sigma^3)_{\alpha\beta} \partial_{\sigma} X^j_\beta + K_{i,\alpha,\beta,\delta} \partial_{t} X^j_\beta = 0, \quad i, \alpha \text{ fixed} \quad (6.6)
\]
where \( K_{i;\sigma;j;\beta} = B_{i;\sigma;j;\beta} - B_{j;\beta;i;\sigma} \) and \( \sigma^3 \) is the Pauli matrix. If the fields \( X_i \) are chiral then we have

\[
\begin{align*}
\partial_\sigma X^1 + \epsilon_1 \partial_\tau X^1 &= 0, \\
\partial_\sigma X^2 + \epsilon_2 \partial_\tau X^2 &= 0,
\end{align*}
\]

(6.7) (6.8)

where \( \epsilon_1, \epsilon_2 = \pm 1 \).

Using the chirality conditions, we can eliminate \( \partial_\tau X \) in favor of \( \partial_\sigma X \), and substitute in the boundary conditions. The resulting equations can be compactly written as

\[
\begin{bmatrix}
R_1 & -\epsilon_1 R_1 & -\epsilon_2 R_2 & -\epsilon_3 R_2 \\
R_1 & -\epsilon_1 R_1 & -\epsilon_2 R_2 & -\epsilon_3 R_2 \\
\epsilon_2 R_1 & \epsilon_2 R_1 & R_2 & -\epsilon_2 R_2 \\
\epsilon_3 R_1 & \epsilon_3 R_1 & \epsilon_3 R_2 & -R_2 \\
\end{bmatrix}
\begin{bmatrix}
\partial_\sigma X^1_{2\pi} \\
\partial_\sigma X^1_0 \\
\partial_\sigma X^2_{2\pi} \\
\partial_\sigma X^2_0 \\
\end{bmatrix}
\equiv MY = 0,
\]

(6.9)

where \( K_{1,2;\pi;1,0} = aR_1^2, K_{1,2;\pi;2;2\pi} = bR_1 R_2, K_{1,0;2,2\pi} = cR_1 R_2, K_{1,0;2,2\pi} = dR_1 R_2, K_{1,0;2,2\pi} = eR_1 R_2, K_{1,2;\pi;2,2\pi} = fR_2^2 \).

Since \( Y \neq 0 \) in (6.9), we require that \( \det M = 0 \) in order to obtain non-trivial solutions i.e.

\[
[(af + cd - be)^2 + (b^2 + c^2 - d^2)\epsilon_1 \epsilon_2 - a^2 - f^2 + 1] = 0.
\]

(6.10)

This seems to be a general case, but in order to study the oscillation scenario, we make the simplifying assumption that \( b = c = 0 \) and also make the assumption that \( R_1 = R_2 = R \).

The motivation for the choice \( b = c = 0 \) is that this is the simplest boundary condition that gives rise to the kind of current oscillations discussed in section 4.

Let us now study the case \( \epsilon_1 = \epsilon_2 = 1, b = c = 0 \) in greater detail. The boundary conditions in this case are

\[
\begin{align*}
\partial_\sigma X^1_{2\pi} &= a\partial_\sigma X^1_0 + c\partial_\sigma X^2_0, \\
\partial_\sigma X^2_{2\pi} &= -d\partial_\sigma X^1_0 + f\partial_\sigma X^2_0.
\end{align*}
\]

(6.11) (6.12)

Comparing these equations with Eq.(4.5) we find that \( c = d = \sin \alpha \) and \( a = f = \cos \alpha \). It is trivial to check that Eq.(6.10) is satisfied.

Now if we define the complex field \( Z = X^1 + iX^2 \) and its complex conjugate \( \bar{Z} = X^1 - iX^2 \), then we find that Eq.(6.11) can be rewritten as

\[
\begin{align*}
\partial_\sigma Z(L) &= e^{i\alpha} \partial_\sigma Z(0), \\
\partial_\sigma \bar{Z}(L) &= e^{-i\alpha} \partial_\sigma \bar{Z}(0).
\end{align*}
\]

(6.13) (6.14)

The solution for this boundary condition is that the field \( Z \) (upto an additive constant) is quasi-periodic and satisfies \( Z(L) = e^{i\alpha} Z(0) \). The mode expansions of \( Z(\sigma) \) will involve quasi-periodic functions of the form \( e^{i(n + \frac{2\pi}{L})\sigma} \).
7 Chern-Simons Theory on a Disc

In this section we will discuss some of the results that have already been studied in the literature, [13, 14, 15], the main reason being to set up the stage for generalizing to the case of interacting Hall systems. In addition we think that there are still some interesting issues to be understood here. One such issue has to do with a proper mode analysis. Careful mode analysis is indispensable already in classical canonical formalism to resolve subtle mathematical problems; the latter arise from the fact that there are fields supported both in the bulk and at the edge. The bulk and edge field theories are coupled and the nature of the coupling is dictated by anomaly cancellation or gauge invariance. One thing that the mode analysis should allow one to see in detail is how the Gauss law goes from being a second class constraint (as in the case of the chiral Schwinger model) to a first class constraint. This is another way of saying that gauge invariance is restored by anomaly cancellation due to the extra bulk contribution.

Let us rework our favorite example to illustrate our method of canonical quantization. This is the usual Chern-Simons theory on a disc, along with a boundary interaction term:

\[ S = -\frac{k}{4\pi} \int_{D \times \mathbb{R}} A dA + \frac{k}{4\pi} \epsilon \int_{\partial D \times \mathbb{R}} d\phi A. \]  

(7.1)

This is invariant under the gauge transformation

\[ A \rightarrow A + d\lambda, \]

(7.2)

\[ \phi \rightarrow \phi - \epsilon \lambda. \]  

(7.3)

The action does not contain the kinetic energy term for the \( \phi \) field: we will add it later. As it stands, this action describes a topological field theory.

If we abstract the Lagrangian from the action, we notice that in terms of the variables \( z, \bar{z} \), \( A_0(z, \bar{z}) \) is a zero-form on the spatial slice while the other components of \( A \) can be written in terms of a one-form \( A \). It is this one-form that we will expand in terms of an appropriate basis set. The bulk Lagrangian is

\[ L_{\text{bulk}} = \frac{k}{4\pi} \int A \dot{A} - \frac{k}{2\pi} \epsilon \int A dA_0. \]  

(7.4)

The wedge product is implicit in the above expression.

The edge Lagrangian is

\[ L_{\text{edge}} = \frac{k}{4\pi} \int_{\partial D} A_0 A_\theta + \frac{k}{4\pi} \epsilon \int_{\partial D} \{ \dot{\phi} A_\theta - \partial_\theta \phi A_0 \}. \]  

(7.5)

The strategy for finding the modes for \( A \) will be the same as the one adopted in [15]. There are 4 kinds of 1-forms. The first two are \( h_n(z) \) and \( \bar{h}_n(\bar{z}) \) where

\[ h_n(z) = \frac{dz^n}{\sqrt{2\pi n R^n}}, \quad \bar{h}_n(\bar{z}) = \frac{d\bar{z}^n}{\sqrt{2\pi n R^n}}, \quad n \geq 1. \]

(7.6)

Here \( R \) is the radius of the disc. The other two modes are \( \psi_{nM} \) and \( \star \psi_{nM} \), where \( \star \psi_{nM} = N_{nM} dF_{nM} \). The \( \star \) is the Hodge dual defined as in [15]. Also, \( F_{nM} = e^{in\sigma} G_{nM} \) where \( G_{nM} \) satisfies the Bessel equation

\[ \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + (\omega^2 - \frac{n^2}{r^2}) \right) G_{nM}(r) = 0, \]  

(7.7)
and the boundary condition \( G_{nM}(R) = 0 \), and \( N_{nM} \) are chosen so that \((\psi_{nM}, \psi_{n'M'}) = \delta_{n'n} \delta_{M'M} \). Thus

\[
G_{nM} = J_n(w_{nM}) \quad n \in \mathbb{Z} \quad \text{and} \quad M \in \mathbb{Z}_+
\]

(7.8)

and \( \omega_{nM} \) are the solutions of \( J_n(\omega_{nM} R) = 0 \).

The modes described above form a complete orthonormal set:

\[
\begin{align*}
(h_n, h_m) &= (\tilde{h}_n, \tilde{h}_m) = \delta_{nm}, \\
(\psi_{nM}, \psi_{n'M'}) &= (\phi_{nM}, \phi_{n'M'}) = \delta_{n'n} \delta_{M'M}.
\end{align*}
\]

(7.9)

and

\[
(h_n, \tilde{h}_m) = (h_n, \phi_{n'M'}) = (h_n, *\psi_{n'M'}) = (\psi_{nM}, *\psi_{n'M'}) = 0 \text{ etc.}
\]

The inner product \((\alpha, \beta)\) of one-forms \( \alpha = \alpha_i dx^i \) and \( \beta = \beta_i dx^i \) is defined here as

\[
(\alpha, \beta) = \int \alpha_i \beta_i d^2x.
\]

(7.10)

Therefore we have,

\[
A = \sum_{nM} \left( a_{nM}(t) \psi_{nM} + b_{nM}(t) \psi_{n'M} \right) + \sum_{n=1}^{\infty} \hat{a}_n(t) h_n + \sum_{n=1}^{\infty} \hat{b}_n(t) \tilde{h}_n,
\]

\[
dA_0 = \sum_{nM} b_{0nM}(t) \psi_{nM} + \sum_{n=1}^{\infty} \alpha_{0n}(t) h_n + \sum_{n=1}^{\infty} \hat{b}_n(t) \tilde{h}_n
\]

(7.11)

where we have used the notation \( a_{nM}^\dagger = a_{-nM} \), \( b_{nM}^\dagger = b_{-nM} \), and \( b_{0nM}^\dagger = b_{0-nM} \) to make the expressions more concise.

The mode expansion for the field \( \phi \) living on the boundary of the disc parameterized by the angular variable \( 0 \leq \theta \leq 2\pi \) is

\[
\phi = \frac{1}{\sqrt{2\pi}} \sum_n \left( \frac{\phi_n}{\sqrt{n}} e^{in\theta} + \frac{\phi_n^*}{\sqrt{n}} e^{-in\theta} \right) + p\theta + q, \quad n \in \mathbb{Z}_+
\]

(7.12)

where \( p, q \) correspond to the zero modes. We will also need the mode expansions for the fields at the boundary of the disc. They are

\[
A|_{\partial D} = a_0 d\theta + \sum_{n \geq 1} \left( \tilde{a}_n + i \sqrt{\frac{n}{2\pi}} \tilde{\alpha}_n \right) e^{in\theta} d\theta
\]

\[
+ \sum_{n \geq 1} \left( \tilde{a}_{-n} - i \sqrt{\frac{n}{2\pi}} \tilde{\alpha}_n^* \right) e^{-in\theta} d\theta
\]

(7.13)

\[
A_0|_{\partial D} = \alpha_0^{(0)} + \sum_{n \geq 1} \left( \frac{\alpha_{0n}}{\sqrt{2\pi n}} e^{in\theta} + \frac{\alpha_{0n}^*}{\sqrt{2\pi n}} e^{-in\theta} \right),
\]

(7.14)

where \( \tilde{a}_n = \sum_M a_{nM} N_{nM} \partial_r J_n(\omega_{nM} R)|_{r=R} \), and \( \alpha_0^{(0)} \) is a constant.

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The Lagrangian can be easily written in terms of these modes.

\[ L = \sum_{nM} \frac{Ke}{2} \left[ -a_{nM}^+ \dot{b}_{nM} + b_{nM}^+ \dot{a}_{nM} + i \left( \alpha_n^+ \alpha_n - \alpha_n^\dagger \alpha_n \right) \right] \]

\[ + \frac{Ke}{2} [2a_{nM}^+ b_{0nM} - 2i(\alpha_n^+ \alpha_{0n} - \alpha_n \alpha_{0n}^\dagger)] \]

\[ - \frac{Ke}{2} [\dot{a}_{0n}^+(a_n + i\alpha_n) + a_{0n}(a_n^+ - i\alpha_n^+) + \alpha_0 a_0] \]

\[ + \frac{K}{2} [\dot{\phi}_n(a_n^+ - i\alpha_n^+) + \phi_n^+(a_n + i\alpha_n) + a_0 \dot{q}] \]

\[ - \frac{K}{2} [i\alpha_{0n} \dot{\phi}_n - i\alpha_{0n} \phi_n^+ + a_0 \ddot{q}] \] (7.15)

where \( K = \frac{k}{2\pi} \) and \( a_n = \sqrt{\frac{2\pi}{n}} \tilde{a}_n, \alpha_n = \sqrt{\frac{2\pi}{n}} \tilde{\alpha}_n \).

The conjugate momenta are:

\[ \Pi(a_{nM}) = \frac{Ke}{2} b_{nM}^+ \]

\[ \Pi(b_{nM}) = -\frac{Ke}{2} a_{nM}^+ \]

\[ \Pi(\alpha_{0n}) = 0 = \Pi(\alpha_{0n}^+) \]

\[ \Pi(b_{0nM}) = 0 \]

\[ \Pi(\alpha_n) = \frac{K\alpha}{2} \alpha_n^+ \]

\[ \Pi(\dot{\alpha}_n) = -\frac{K\alpha}{2} \alpha_n \]

\[ \Pi(\dot{\phi}_n) = \frac{K}{2} (a_n^+ - i\alpha_n^+) \]

\[ \Pi(\phi_n^+) = \frac{K}{2} (a_n + i\alpha_n). \] (7.16)

All the above are primary constraints. Doing the constraint analysis we obtain further secondary constraints. They are

\[ a_{nM} \sim 0 \]

\[ \epsilon \alpha_n - \phi_n \sim 0 \]

\[ \epsilon \alpha_n^+ - \phi_n^+ \sim 0. \] (7.17)

The Poisson bracket matrix of the constraints has a zero determinant indicating the first class nature of some constraints. The set of constraints is

\[ \chi_1 \equiv \Pi(\alpha_n) - \frac{iKe}{2} \alpha_n^+ = 0, \quad \chi_2 \equiv \Pi(\alpha_n^+) + \frac{iKe}{2} \alpha_n = 0, \]

\[ \chi_3 \equiv \epsilon \alpha_n - \phi_n = 0, \quad \chi_4 \equiv \epsilon \alpha_n^+ - \phi_n^+ = 0, \]

\[ \chi_5 \equiv \epsilon \Pi(\phi_n) + \frac{iK}{2} \phi_n^+ = 0, \quad \chi_6 \equiv \epsilon \Pi(\phi_n^+) - \frac{iK}{2} \phi_n = 0. \] (7.18)

In writing this set of equations we have used \( \epsilon \alpha_n - \phi_n = 0 \) in the constraint for the momenta of \( \phi \) which amounts to taking some linear combinations of constraints and calling that as a new constraint.

We use the Poisson brackets

\[ \{\alpha_n, \Pi(\alpha_n)\} = \{\phi_n, \Pi(\phi_n)\} = 1. \] (7.19)
The Poisson bracket matrix is
\[
\{[\chi_i, \chi_j]\} = \begin{pmatrix}
0 & -iKe & -e & 0 & 0 & 0 \\
-iKe & 0 & 0 & -e & 0 & 0 \\
e & 0 & 0 & 0 & -e & 0 \\
0 & e & 0 & 0 & 0 & -e \\
0 & 0 & e & 0 & 0 & iKe \\
0 & 0 & 0 & e & -iKe & 0
\end{pmatrix}
\] (7.20)

This matrix has zero determinant. The eigenvectors for zero eigenvalue are (in the same basis of constraints that the matrix is written in (7.20))
\[
(0, 1, -iK, 0, 0, 1), \\
(1, 0, 0, iK, 1, 0).
\] (7.21)

These zero eigenvectors correspond to linear combinations of first class constraints that are the generators of gauge transformations on the edge of the disc. Doing the constraint analysis, we find that the Lagrange multipliers corresponding to the second class constraints get fixed, and the Hamiltonian can be written as a sum of constraints. Since this theory is completely topological, the Hamiltonian is weakly zero as expected.

The second class constraints force us to construct Dirac brackets for quantizing the theory. The Dirac brackets for the gauge invariant edge observables are
\[
\{\alpha_n + \frac{1}{e}\phi_n, \alpha_m^\dagger + \frac{1}{e}\phi_m^\dagger\}^{DB} = -\frac{4\pi i}{k}\delta_{nm}.
\] (7.22)

8 Dynamics for the Edge Field

We can now include the kinetic energy term for the scalar field at the edge. This is just an extra term in the Lagrangian of the form
\[
\frac{k}{8\pi \epsilon^2} \int D\mu \phi D^{\mu}\phi.
\] (8.1)

The analysis now, though identical in spirit to the previous one, differs in detail.

The new Lagrangian \(L'\) can be easily written in terms of these modes,
\[
L' = \sum_{nM} \frac{K\epsilon}{2} [-a_{nM}^\dagger b_{nM} + b_{nM}^\dagger a_{nM} + i(\alpha_n^\dagger \alpha_n - \alpha_n \alpha_n^\dagger)]
\]
\[
+ \frac{K\epsilon}{2} [2a_{nM}^\dagger b_{0nM} - 2i(\alpha_n^\dagger \alpha_{0n} - \alpha_n \alpha_{0n}^\dagger)]
\]
\[
- \frac{K\epsilon}{2} [\alpha_0^\dagger (a_n + i\alpha_n) + \alpha_{0n} (a_n^\dagger - i\alpha_n^\dagger) + \alpha_0 \alpha_0]
\]
\[
+ \frac{K}{2} [\phi_n (a_n^\dagger - i\alpha_n^\dagger) + \phi_n^\dagger (a_n + i\alpha_n) + \alpha_0 \phi]
\]
\[
- \frac{K}{2} [i\alpha_{0n} \phi_n - i\alpha_{0n} \phi_n^\dagger + \alpha_0 \phi]
\]
\[
+ \frac{K}{2\epsilon n} (\phi_n^\dagger + e\alpha_n^\dagger) (\phi_n + e\alpha_n)
\]
\[
+ \frac{Kn}{2\epsilon} (ea_n^\dagger - ie\alpha_n^\dagger)(ea_n - i\phi_n - ie\alpha_n)
\] (8.2)
where \( K = \frac{i}{2\pi e} \) and \( a_n = \sqrt{\frac{2\pi}{n} a_n}, \alpha_n = \sqrt{\frac{2\pi}{n} \tilde{a}_n}. \)

The following conjugate momenta \( \Pi(\chi) \) of \( \chi (= a_{nM} \text{ etc}) \) are all constrained:

\[
\begin{align*}
\Pi(a_{nM}) &= \frac{Ke}{2} b^\dagger_{nM}, \\
\Pi(b_{nM}) &= -\frac{Ke}{2} a^\dagger_{nM}, \\
\Pi(\alpha_{On}) &= 0 = \Pi(\alpha^\dagger_{On}), \\
\Pi(b_{0nM}) &= 0 \\
\Pi(\alpha_n) &= \frac{Ke}{2} \alpha^\dagger_n, \\
\Pi(\alpha^\dagger_n) &= -\frac{Ke}{2} \alpha_n.
\end{align*}
\] (8.3)

We also have the canonical momenta for the field \( \phi \):

\[
\begin{align*}
\Pi(\phi_n) &= K/2(a_n^\dagger - ia_n^\dagger) + \frac{K}{2en} (\phi_n^\dagger + e\alpha_{On}^\dagger), \\
\Pi(\phi_n^\dagger) &= K/2(a_n + ia_n^\dagger) + \frac{K}{2en} (\phi_n^\dagger + e\alpha_{On}).
\end{align*}
\] (8.4)

The Hamiltonian can be found:

\[
H = \frac{2en}{k} \Pi_n(\phi_n^\dagger)\Pi_n(\phi_n) + ien[\Pi_n(\phi_n^\dagger)\alpha_n^\dagger - \Pi_n(\phi_n)\alpha_n] \\
+ \frac{Ke}{2} \alpha^\dagger_n \alpha_n - iKe\alpha_{On}^\dagger, e\Pi_n + \frac{iK}{2} \phi_n^\dagger + e\Pi_n^\dagger - iKe\phi_n^\dagger + \text{ (constraints)}
\] (8.5)

There are two first class constraints that generate gauge transformations namely \( \Pi(\alpha_{On}) \), \( \Pi(\alpha_{On}) \). There are no first class generators corresponding to the zero eigenvectors that we found in Eq (7.21) when there was no kinetic energy term for the chiral boson at the edge. There are four second class constraints \( \Pi(\alpha_n^\dagger), \Pi(\alpha_n) - iKe\alpha_{On}, e\Pi_n + \frac{iK}{2} \phi_n^\dagger, e\Pi_n^\dagger - iKe\phi_n \). These can be eliminated using Dirac brackets and imposing the second class constraints strongly.

The non-vanishing Dirac brackets are:

\[
\begin{align*}
\{\alpha_n, \alpha_m^\dagger\}^{DB} &= \frac{-2\pi i}{k} \delta_{nm}, \\
\{\phi_n, \Pi(\phi_m)\}^{DB} &= \frac{1}{2} \delta_{nm}, \\
\{\phi_n, (\phi_m^\dagger)\}^{DB} &= \frac{2\pi ie^2}{k} \delta_{nm}, \\
\{\Pi(\phi_n), \Pi^\dagger(\phi_m)\}^{DB} &= -\frac{iek}{8\pi e^2} \delta_{nm}.
\end{align*}
\] (8.6)

We can now write the Hamiltonian as

\[
H = \frac{nK}{4e} [(p_n + iea_n^\dagger)^\dagger(p_n + iea_n^\dagger) \\
+ (\phi_n + e\alpha_n)^\dagger(\phi_n + e\alpha_n)] + h.c.
\] (8.7)

where \( \frac{2e\Pi_n}{K} = p_n \).
In this section, we have just discussed the quantization of a Chern-Simons theory on the disc coupled to a scalar field on the edge. It can be extended to the case of the touching discs where we will have two Chern-Simons fields. Due to the edge interaction we find it convenient to work in terms of a complex combination of these real fields. We then find (as discussed in section six) that due to the interaction at the contact point the modes are twisted. The quantization of the twisted chiral complex scalar field coupled to a complex Chern-Simons field can be done using an analysis similar to the one in this section. We will summarize the results in the next section.

9 Touching Discs

Instead of two real fields \(X^1(\sigma, t)\) and \(X^2(\sigma, t)\) on the boundary, we can work with a single complex field \(Z = X^1 + iX^2\). In terms of this \(Z\), the boundary conditions can be written as

\[
\partial_\sigma Z_{2\pi} = e^{i2\pi \alpha} \partial_\sigma Z_0, \tag{9.1}
\]

\[
Z_{2\pi} = e^{i2\pi \alpha} Z_0. \tag{9.2}
\]

and complex conjugates of the above equations. Thus \(Z\) is “quasi-periodic”.

With \(A = A^1 + iA^2\), we write the action as

\[
S = -k \int_{D \times \mathbb{R}} (\tilde{A}dA + Ad\tilde{A}) + \frac{k}{\epsilon} \int_{\partial D \times \mathbb{R}} (d\tilde{Z}A + dZA), \tag{9.3}
\]

where \(\tilde{A}\) is the complex conjugate of \(A\).

The bulk Lagrangian is

\[
L_{\text{bulk}} = k \int (\tilde{A}A + A\tilde{A}) - k \int (\tilde{A}dA_0 + Ad\tilde{A}_0) + k \int (d\tilde{A}_0A + dA_0\tilde{A}). \tag{9.4}
\]

Similarly, the edge Lagrangian is

\[
L_{\text{edge}} = -k \int_{\partial D} (A_0A + A\tilde{A}) + \frac{k}{\epsilon} \int_{\partial D \times \mathbb{R}} [(\tilde{Z}A_{\sigma} + \tilde{Z}\tilde{A}_{\sigma}) - (\partial_\sigma ZA_0 + \partial_\sigma Z\tilde{A}_0)]. \tag{9.5}
\]

We need a convenient mode expansion for \(A, dA_0, Z\) and their complex conjugates. Also, \(A, dA_0\) and \(Z\) must be of the same quasi-periodicity for the action to be well-defined.

Here, any 1-form in the bulk can be written as a linear combination of the following 4 types of one-forms:

\[
h_n(z) = \frac{dz^{n+\alpha}}{\sqrt{2\pi(n+\alpha)R^{n+\alpha}}}, n \geq 0, \tag{9.6}
\]

\[
g(z) = \frac{d}{\sqrt{2\pi(n-\alpha)R^{n-\alpha}}}, n \geq 1. \tag{9.7}
\]

The other two modes are \(\psi_{nM}\) and \(\psi_{nM}^*\), where \(\psi_{nM} = N_{nM}dF_{nM}\). Here \(F_{nM} = e^{i(n+\alpha)\sigma}G_{nM}\), and \(G_{nM}\) satisfies the Bessel equation

\[
\left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + \left(\omega^2 - \frac{(n+\alpha)^2}{r^2}\right)\right]G_{nM}(r) = 0, \tag{9.8}
\]
while \( N_{nM} \) are so chosen that \((\ast \psi_{nM}, \ast \psi_{nM}) = 1\). Thus
\[
G_{nM} = J_{n+\alpha} (w_{n+\alpha, MR}), n \geq 0,
\]
\[
= J_{-(n+\alpha)} (w_{-(n+\alpha), MR}), n \leq -1
\]
and \( G_{nM} (R) = 0 \).

Therefore we have,
\[
A = \sum_{nM} \left[ a_{nM} (t) \psi_{nM} + e_{nM} (t) * \psi_{nM} \right]
\]
\[
+ \sum_{n=0}^{\infty} \alpha_n (t) h_n + \sum_{n=1}^{\infty} \beta_n (t) \bar{g}_n,
\]
\[
dA_0 = \sum_{nM} e_{0nM} (t) * \psi_{nM} + \sum_{n=0}^{\infty} \alpha_{0n} (t) h_n + \sum_{n=1}^{\infty} \beta_{0n} (t) \bar{g}_n
\]
and
\[
Z = \frac{\sum_{n=-\infty}^{+\infty} z_n}{\sqrt{2\pi |n + \alpha|}} e^{i(n+\alpha)\sigma}.
\]

We also need \( A|_{\beta D} \) and \( A_0|_{\beta D} \). We find them to be
\[
A|_{\beta D} = \sum_{nM} a_{nM} f_{nM} e^{i(n+\alpha)\sigma} d\sigma + \sum_{n=0}^{\infty} \alpha_i \sqrt{\frac{n + \alpha}{2\pi}} e^{i(n+\alpha)\sigma} d\sigma
\]
\[
+ \sum_{n=1}^{\infty} \sqrt{\frac{n - \alpha}{2\pi}} e^{-i(n-\alpha)\sigma} d\sigma
\]
\[
A_0|_{\beta D} = \sum_{n=0}^{\infty} i\alpha_{0n} \sqrt{\frac{n + \alpha}{2\pi}} e^{i(n+\alpha)\sigma} d\sigma
\]
\[
\sigma + \sum_{n=1}^{\infty} i\beta_{0n} \sqrt{\frac{n - \alpha}{2\pi}} e^{-i(n-\alpha)\sigma} d\sigma
\]
where
\[
f_{nM} = N_{nM} w_{n+\alpha, MR} \mathcal{F}_{n+\alpha} (w_{n+\alpha, MR}), n \geq 0,
\]
\[
= N_{nM} w_{-(n+\alpha), MR} \mathcal{F}_{-(n+\alpha)} (w_{-(n+\alpha), MR}), n \leq -1.
\]

For future convenience, we define
\[
a_n \equiv \sum_{M} a_{nM} f_{nM}
\]

The full Lagrangian \( L = L_{bulk} + L_{edge} \) can be written in terms of the modes as
\[
L = k \sum_{nM} (a_{nM}^{\ast} \dot{a}_{nM} - a_{nM} \dot{a}_{nM}^{\ast}) + k \sum (i\alpha_n \dot{\alpha}_n - i\beta_n \dot{\beta}_n)
\]
\[+ 2k \sum a_{nM}^* \dot{a}_{0nM} + \sum (i\alpha_{0n} \dot{\alpha}_{0n} + i\beta_{0n} \dot{\beta}_{0n})
\]
\[+ k \sum \frac{\dot{\alpha}_{0n} a_n}{\sqrt{2\pi (n + \alpha)}} + \frac{\dot{\beta}_{0n} a_{-n}}{\sqrt{2\pi (n - \alpha)}} + \frac{\dot{\beta}_{0n} a_{-n}}{\sqrt{2\pi |n + \alpha|}}
\]
\[= k \sum \frac{i(\dot{z}_n \alpha_n - \dot{\beta}_n \beta_n) - k \sum i(z_n \alpha_{0n} - z_{-n} \beta_{0n}) + c.c.}{\sqrt{2\pi}}.
\]
We can now write the momenta conjugate to the various fields $q_n$, using $\Pi(q_n) \equiv \frac{\partial L}{\partial \dot{q}_n}$:

\[
\begin{align*}
\Pi(a_{nM}) &= \frac{k}{\epsilon} a_{nM}^{(\ast)}, \\
\Pi(a_{nM}^{(\ast)}) &= -\frac{k}{\epsilon} a_{nM}, \\
\Pi(\alpha_n) &= i\frac{k}{\epsilon} \alpha_n, \\
\Pi(\beta_n) &= i\frac{k}{\epsilon} \beta_n, \\
\Pi(\alpha_{0n}) &= 0, \\
\Pi(\beta_{0n}) &= 0, \\
\Pi(z_n) &= \frac{k}{\epsilon} a_n \frac{1}{\sqrt{2\pi |n + \alpha|}} - \frac{i}{\epsilon} \alpha_n, n \geq 0, \\
&= \frac{k}{\epsilon} a_n \frac{1}{\sqrt{2\pi |n + \alpha|}} - \frac{i}{\epsilon} \beta_{-n}, n \leq -1, \\
\Pi(q_n) &= \tilde{\Pi}(q_n).
\end{align*}
\]

The Hamiltonian $H$ can be calculated from the Lagrangian to be

\[
H = -2k \sum_{n,M} \tilde{a}_{nM} a_{nM}^{(\ast)} + k \sum \left( \frac{a_n}{2\pi(n + \alpha)} - i\alpha_n + \frac{i}{\epsilon} z_n \alpha_{0n} + \frac{a_{-n}}{2\pi(n - \alpha)} + i\beta_n + \frac{i}{\epsilon} z_{-n} \beta_{0n} + \text{c.c.} \right).
\]

The Poisson brackets follow their form $\{q_n, \Pi(q_m)\} = \delta_{nm}$ for the various fields. These relations in turn imply that

\[
\begin{align*}
\{a_{nM}, a_{n'M}^{(\ast)}\} &= \frac{1}{k} \delta_{nn'} \delta_{MM'}, \\
\{\alpha_n, \alpha_m\} &= -\frac{i}{k} \delta_{nm}
\end{align*}
\]

and so on.

Since $\Pi(\alpha_{0nM}^{(\ast)}) = 0$, we insist that $\tilde{\Pi}(\alpha_{0nM}^{(\ast)}) = \{\Pi(\alpha_{0nM}^{(\ast)}), H\}$ annihilates any physical state $|\phi\rangle$. This, and other such relations imply

\[
\begin{align*}
a_{nM}|\phi\rangle &= 0, \\
\tilde{a}_{nM}|\phi\rangle &= 0, \\
(\alpha_n - \frac{1}{\epsilon} z_n)|\phi\rangle &= 0, n \geq 0, \\
(\beta_n + \frac{1}{\epsilon} z_n)|\phi\rangle &= 0, n \geq 0, \\
(\beta_{-n} - \frac{1}{\epsilon} z_{-n})|\phi\rangle &= 0, n \leq -1, \\
(\beta_{-n} + \frac{1}{\epsilon} z_{-n})|\phi\rangle &= 0, n \leq -1.
\end{align*}
\]
The last four constraints do not commute amongst each other. So quantizing this system of constraints requires us to define the Dirac bracket. Again we can calculate the Dirac bracket of various operators with each other. The interesting ones are

\[
\{q_n, q_m\} = -\frac{4i}{k} \delta_{nm}, \tag{9.42}
\]

\[
\{z_m, \bar{z}_n\} = -\frac{4ie^2}{k} \delta_{mn} \tag{9.43}
\]

where

\[
q_n = \alpha_n, \ n \geq 0, \tag{9.44}
\]

\[
= \beta_{-n}, \ n \leq -1 \tag{9.45}
\]

This is just the affine \(U(1)\) Kac-Moody algebra.

## 10 Conclusions

In this paper we have studied quantum Hall systems that are coupled through an interaction at one point. We find that the interaction can lead to current oscillations between the edge currents of the Hall systems. We have derived the period of these oscillations in terms of a parameter characterizing the interaction. It would be interesting to investigate experimentally the possible values of these parameters. The fact that these oscillations arise in situations where there is a violation of the Kirchoff’s law applied to the currents at the interaction junction at any given point of time and the fact that the currents exhibit a characteristic periodicity seem to indicate that there may be a natural way to associate a capacitance to the junction point so that this allows one to understand the violation of Kirchoff’s law in conjunction with the origin of oscillations. Thus the kind of boundary conditions that we find may be an effective description of the mesoscopic quantum capacitance found in [16], the capacitance arises here from the microscopic details of the contact established at the junction.

In this paper we have discussed the modification of abelian current algebras and the fact that the interactions lead to a twisted current algebra. It would be interesting to study the corresponding modifications in the case of non-abelian current algebras as it is known that in the non-abelian case there are restrictions on the values of the allowed twists and this leads to constraints on parameters controlling the boundary interactions.

We have assumed the same filling fraction \(k\) for both the discs. We have to work out the analysis similar to the above for different filling fractions. More importantly, there is a much more general set of boundary conditions, with many parameters, that preserve chirality, as shown in the first section. We have put all except one of the parameters to zero in our detailed work. We have to try to understand how to exploit the more general boundary conditions and see if there are interesting solutions that are also physically relevant.

The connection of our work to string theory and D-branes can be found in [17, 18, 19] where the relations between open strings moving in external fields and boundary conditions of the conformal field theories are established. The connections to twisted current algebras are also noticed there. The fact that the boundary conditions are changed by the boundary
interactions discussed here indicates that there may be common features of our approach with
the one involving the “twist field” [20]. In the conformal field theory context and in the context
of dissipative quantum mechanics, similar boundary interactions were studied in [21, 22]. The
main difference between those discussions and what we have in this paper is the fact that we
are studying a theory of chiral bosons.

11 Acknowledgments

The work of APB, TRG, VJ and SV was supported in part by Department of Energy, U. S.
A. under contract number DE-FG02-ER40231. VJ would like to acknowledge The Institute of
Mathematical Sciences, Chennai for hospitality during his visit to Chennai where part of this
work was completed.

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