Stochastic Navier-Stokes equation for a compressible fluid: two-loop approximation

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Abstract. A model of fully developed turbulence of a compressible fluid is briefly reviewed. It is assumed that fluid dynamics is governed by a stochastic version of Navier-Stokes equation. We show how corresponding field theoretic-model can be obtained and further analyzed by means of the perturbative renormalization group. Two fixed points of the RG equations are found. The perturbation theory is constructed within formal expansion scheme in parameter y, which describes scaling behavior of random force fluctuations. Actual calculations for fixed points’ coordinates are performed to two-loop order.

Keywords: stochastic Navier-Stokes equation, anomalous scaling, field-theoretic renormalization group, compressibility.

1 Introduction

Many natural phenomena are concerned with hydrodynamic flows. Ranging from microscopic up to macroscopic spatial scales fluids can exist in profoundly different states. Especially intrigued behavior is observed in case of turbulent flows. Such flows are ubiquitous in nature and are more common than generally believed [1,2]. Despite a substantial amount of effort that has been put into investigation of turbulence, the problem itself remains unsolved.

Most of studies are devoted to the case of incompressible fluid. However, particularly in an astrophysical context we have to deal with a compressible fluid rather than incompressible one [3]. In recent years there has also been an increased research activity of compressible turbulence in magnetohydrodynamic context [4,5,6,7,8]. In this work, our aim is to study compressible turbulence [9,10], partially motivated by previous studies [11,12,13]. In case of a compressible medium, we are in fact examining system in which sound modes are
generated. In fact, any compression leads to acoustic (sound) waves that are transmitted through the medium and serve as the prime source for dissipation. So the problem of the energy spectrum (and dissipation rate) of a compressible fluid is essentially one of stochastic acoustics.

The investigation of such behavior as anomalous scaling requires a lot of thorough analysis to be carried out. The phenomenon manifests itself in a singular power-like behavior of some statistical quantities (correlation functions, structure functions, etc.) in the inertial-convective range in the fully developed turbulence regime [12,15]. A quantitative parameter that describes intensity of turbulent motion is so-called Reynolds number Re that represents a ratio between inertial and dissipative forces. For high enough values of $\text{Re} \gg 1$ inertial interval is exhibited in which just transfer of kinetic energy from outer $L$ (input) to microscopic $l$ (dissipative) scales take place.

A very useful and computationally effective approach to the problems with many interacting degrees of freedom on different scales is the field-theoretic renormalization group (RG) approach which can be subsequently accompanied by the operator product expansion (OPE); see the monographs [16,17,18,19,20]. One of the greatest challenges is an investigation of the Navier-Stokes equation for a compressible fluid, and, in particular, a passive scalar field advection by this velocity ensemble. The first relevant discussion and analysis of passive advection emerged a few decades ago for the Kraichnan’s velocity ensemble [21,22,23,24]. Further studies developed its more realistic generalizations [25,26,27,28,29,30,31,32]. The RG+OPE technique was also applied to more complicated models, in particular, to the compressible case [11,33,34,35,36,37,38,39,40,41,42,43,44,45]. Our aim here is to improve existing (one-loop) results on compressible stochastic Navier-Stokes equation and determine relevant physical quantities to two-loop order. Note that in contrast to static phenomena transition from one-loop to two-loop approximation pose in stochastic dynamics much more demanding task.

The paper is a continuation of previous works [12,13,14] and it is organized as follows. In the introductory Sec. 2 we give a brief overview of the model and we reformulate stochastic equations into field-theoretical language. Sec. 3 is devoted to the renormalization group analysis. In Sec. 4 we present the fixed points’ structure, describe possible scaling regimes and calculate critical dimensions. The concluding Sec. 5 is devoted to a short discussion and future plans.

2 Model

Let us start with a discussion of a model for compressible velocity fluctuations. The dynamics of a compressible fluid is governed by the stochastic Navier-Stokes equation [9] taken in the form

$$\rho \nabla_t v_i = \nu_0 [\delta_{ik} \partial^2 - \partial_i \partial_k] v_k + \mu_0 \partial_i \partial_k v_k - \partial_i p + f_i^v,$$

where the operator $\nabla_t$ stands for an expression $\nabla_t = \partial_t + v_k \partial_k$, also known as a Lagrangian (or convective) derivative. Further, $\rho = \rho(t, x)$ is a fluid
density field, \( v_i = v_i(t, x) \) is the velocity field, \( \partial_i = \partial/\partial x_i \) is the \( i \)th component of spatial gradient, \( \partial^2 = \partial_i \partial_i \) is the Laplace operator, \( p = p(t, x) \) is the pressure field, and \( f_i^v \) is the external force, which is specified later. In what follows we employ a condensed notation in which we write \( x = (t, x) \), where a spatial vector variable \( x \) equals \( (x_1, x_2, \ldots, x_d) \) with \( d \) being a dimensionality of space. Although it is possible to consider \( d \) as an additional free parameter [14], in this work spatial dimension \( d \) implicitly takes most physically relevant value 3. Two parameters \( \nu_0 \) and \( \mu_0 \) in Eq. (1) are two viscosity coefficients [9]. Summations over repeated vector indices (Einstein summation convention) are always implied in this work.

Let us make two important remarks regarding the physical interpretation of Eq. (1). First, this equation should be regarded as a dynamic equation only for a fluctuating part of the total velocity field. In other words, it is assumed that the mean (regular) part of the velocity field has already been subtracted [1,2]. Second, the random force \( f_i^v \) mimics not only an input of energy, but to some extent it is responsible for neglected interactions between fluctuating part of the velocity field and the mean part [16,19]. In reality, the latter interactions are always present and their mutual interplay generates turbulence [2].

Let us note that stochastic theory of turbulence is similar to a fluctuation theory for critical phenomena [16,46]. The main difference is lack of Hamilton-like operator for turbulence. Nevertheless, it is still possible to take advantage of well-established theoretical tools borrowed from quantum field theory and employ them on turbulence [16,18].

To complete the theoretical set-up of the model, Eq. (1) has to be augmented by additional two relations. They are a continuity equation and a certain thermodynamic relation [9]. The former one can be written in the form

\[
\partial_t \rho + \partial_i (\rho v_i) = 0 \quad (2)
\]

and the latter we choose as follows

\[
\delta p = \epsilon_0^2 \delta \rho, \quad (3)
\]

where \( \delta \rho \) and \( \delta \rho \) describe deviations from the equilibrium values of pressure field and density field, respectively.

Viscous terms in Eq. (1) characterize dissipative processes in the system and in a turbulent state it is expected their relevance at small length scales. Without a continuous input of energy, turbulent processes would eventually die out because of dissipation and the flow would eventually become regular. There are various possibilities for modeling of energy input [19]. For translationally invariant theories it is convenient to specify properties of the random force \( f_i \) in time-momentum representation

\[
\langle f_i(x) f_j(x') \rangle = \frac{\delta(t - t')}{(2\pi)^d} \int_{k > m} d^d k \ D_{ij}^v(k) e^{i k \cdot (x - x')}, \quad (4)
\]

where the delta function in time variable ensures Galilean invariance of the model [19]. The integral in Eq. (4) is infrared (IR) regularized with a parameter \( m \sim L_v^{-1} \), where \( L_v \) denotes outer scale, i.e., scale of the biggest turbulent
eddie. More details can be found in the literature [19,47]. The kernel function $D_{ij}^{v}(k)$ is now assumed in the following form

$$D_{ij}^{v}(k) = g_0^3 k^{4-d-y} \left\{ P_{ij}(k) + \alpha Q_{ij}(k) \right\},$$

(5)

where $g_0$ is a coupling constant, $k = |k|$ is the wave number, $y$ is a suitable scaling exponent, and $\alpha$ is a free dimensionless parameter. Parameter $\alpha$ basically measures intensity with which energy flows into a system via longitudinal modes.

Further, the projection operators $P_{ij}$ and $Q_{ij}$ in the momentum space read

$$P_{ij}(k) = \delta_{ij} - \frac{k_i k_j}{k^2}, \quad Q_{ij} = \frac{k_i k_j}{k^2},$$

(6)

and correspond to the transversal and longitudinal projector, respectively.

Due to its functional form with respect to momentum dependence, function (5) corresponds to a non-local term in ensuing field theoretic action. However, physical and plausible mathematical considerations [16] justify this choice. One of the reasons is a straightforward modeling of a steady input of energy into the system from outer scales. In what follows we attack the problem with the RG approach. The value of the scaling exponent $y$ in Eq. (5) describes a deviation from a logarithmic behavior (that is obtained for $y = 0$). In the stochastic theory of turbulence the main interest is in the limit behavior $y \to 4$ that yields an ideal pumping from infinite spatial scales [19].

Let us make a brief remark about possible generalization of the model. Although, we present our results with a general spatial dimension $d$, we have always implicitly in mind its most realistic value $d = 3$. However, it would be possible to generalize the model [14] and consider $d$ as additional small parameter, similar to the well-known $\phi^4$–theory in critical statics [18,20]. Usually the spatial dimension $d$ plays a passive role and is considered only as an independent parameter. However, Honkonen and Nalimov [48] showed that in the vicinity of space dimension $d = 2$ additional divergences appear in the model of the incompressible Navier-Stokes ensemble and these divergences have to be properly taken into account. Their procedure also naturally leads into improved perturbation expansion [49,50]. As can be seen from the RG discussion in the next section a similar situation occurs for the model in the vicinity of space dimension $d = 4$. In this case an additional divergence appears in the 1-irreducible Green function $\langle v'v' \rangle_{1-ir}$. Utilizing this feature one can employ a double expansion scheme, in which the formal expansion parameters are $y$, and $\varepsilon = 4 - d$, i.e., a deviation from the space dimension $d = 4$ [32,48].

Our main theoretical tool is the renormalization group theory. Its proper application requires a proof of a renormalizability of the model, i.e., a proof that only a finite number of divergent structures exists in a diagrammatic expansion [17,18]. As was shown in [51], this requirement can be accomplished by the following procedure: first the stochastic equation (1) is divided by density field $\rho$, then fluctuations in viscous terms are neglected, and finally. Using the expressions (2) and (3) the problem is formulated into a system of two coupled
differential equations
\[ \nabla_t v_i = \nu_0 [\delta_{ik} \partial^2 - \partial_i \partial_k] v_k + \mu_0 \partial_i \partial_k v_k - \partial_i \phi + f_i, \]  
\[ \nabla_t \phi = -c_0^2 \partial_i v_i, \]  
where a new field \( \phi = \phi(x) \) has been introduced for convenience. It is related to the density fluctuations via the relation \( \phi = c_0^2 \ln(\rho/\bar{\rho}) \) \[11,40\]. A parameter \( c_0 \) denotes the adiabatic speed of sound, \( \bar{\rho} \) is the mean value of density field \( \rho \), and \( f_i = f_i(x) \) is the external force normalized per unit mass.

According to the general theorem \[16,18\], the stochastic problem given by Eqs. (7), and (8), is tantamount to the field theoretic model with a doubled set of fields \( \Phi = \{v_i, v'_i, \phi, \phi'\} \) and given De Dominicis-Janssen action functional. The latter can be written in a compact form as a sum of two terms
\[ S_{\text{total}}[\Phi] = S_{\text{vel}}[\Phi] + S_{\text{den}}[\Phi], \] where the first term describes a velocity part
\[ S_{\text{vel}}[\Phi] = \frac{v'_i D_{ij} v'_j}{2} + v'_i \left[ -\nabla_t v_i + \nu_0 (\delta_{ij} \partial^2 - \partial_i \partial_j) v_j + \mu_0 \nu_0 \partial_i \partial_j v_j - \partial_i \phi \right], \] and the second term is given by the expression
\[ S_{\text{den}}[\Phi] = \phi' [-\nabla_t \phi + v_0 \nu_0 \partial^2 \phi - c_0^2 (\partial_i v_i)]. \] Here, \( D_{ij} \) is the correlation function \[5\]. Note that we have introduced a new dimensionless parameter \( u_0 = \mu_0/\nu_0 > 0 \) and a new term \( v_0 \nu_0 \phi' \partial^2 \phi \) with another positive dimensionless parameter \( v_0 \), which is needed to ensure multiplicative renormalizability \[16,18\].

Further, we employ a condensed notation, in which integrals over the spatial variable \( x \) and the time variable \( t \), as well as summation over repeated indices, are not explicitly written, for instance
\[ \phi' \partial_t \phi = \int dt \int d^d x \phi'(t, x) \partial_t \phi(t, x), \]
\[ v'_i D_{ik} v'_k = \sum_{ik} \int dt \int d^d x \int d^d x' v_i(t, x) D_{ik}(x - x') v_k(t, x'). \] (12)

In a functional formulation various stochastic quantities (correlation and structure functions) are calculated as path integrals with weight functional
\[ \exp(S_{\text{total}}[\Phi]). \]

The main benefits of such approach are transparency in a perturbation theory and potential use of powerful methods of the quantum field theory, such as Feynman diagrammatic technique and renormalization group procedure \[18,19,20\].
3 Renormalization group analysis

Ultraviolet renormalizability reveals itself in a presence of divergences in Feynman graphs, which are constructed according to simple laws \[16,20\] using a graphical notation from Fig. 1. From a practical point of view, an analysis of the 1-particle irreducible Green functions, later referred to as 1-irreducible Green functions following the notation in \[16\], is of utmost importance. In the case of translationally invariant models \[16,20\] two independent scales have to be introduced: the time scale \(T\) and the length scale \(L\). Thus the canonical dimension of any quantity \(F\) (a field or a parameter) is described by two numbers, the frequency dimension \(d_\omega^F\) and the momentum dimension \(d_k^F\), defined such that following normalization holds

\[
\begin{align*}
  d_k^F k &= -d_k^F x = 1, \\
  d_\omega^F \omega &= d_\omega^F t = 1, \\
  d_k^F \omega &= d_k^F t = 0,
\end{align*}
\]

(13)

and the given quantity then scales as

\[
[F] \sim [T]^{-d_\omega^F}[L]^{-d_k^F}.
\]

(14)

The remaining dimensions can be found from the requirement that each term of the action functional \([9]\) be dimensionless, with respect to both the momentum and the frequency dimensions separately.

Based on \(d_k^F\) and \(d_\omega^F\), the total canonical dimension \(d_F = d_k^F + 2d_\omega^F\) can be introduced, which in the renormalization theory of dynamic models plays the same role as the conventional (momentum) dimension does in static problems \[16\]. Setting \(\omega \sim k^2\) ensures that all the viscosity and diffusion coefficients in the model are dimensionless. Another option is to set the speed of sound \(c_0\) dimensionless and consequently obtain that \(\omega \sim k\), i.e., \(d_F = d_k^F + d_\omega^F\). This variant would mean that we are interested in the asymptotic behavior of the Green functions as \(\omega \sim k \to 0\), in other words, in modes in turbulent medium. Even though this problem is very interesting itself, it is not yet accessible for the RG treatment, so we do not discuss it here. The choice \(\omega \sim k^2 \to 0\) is the same as in the models of incompressible fluid, where it is the only possibility because the speed of sound is infinite. A similar alternative in dispersion laws exists, for example, within the so-called model H of equilibrium dynamical critical behavior, see \[16,20\].
The canonical dimensions for the model (9) are listed in Tab. 1. It then directly follows that the model is logarithmic (the coupling constant $g \sim [L]^{-y}$ becomes dimensionless) at $y = 0$. In this work we use the minimal subtraction (MS) scheme for the calculation of renormalization constants. In this scheme the UV divergences in the Green functions manifest themselves as pole in $y$.

Table 1. Canonical dimensions of the fields and parameters entering velocity part of the total action (9).

| $F$ | $v'_i$ | $v_i$ | $\phi'$ | $\phi$ | $m$, $\mu$, $A$ | $\rho_0$, $\nu$, $\nu_0$, $c$, $g_1$, $u_0$, $v_0$, $u$, $v$, $g$, $\alpha$ |
|-----|--------|-------|---------|--------|----------------|---------------------------------------------|
| $d_F$ | $d + 1$ | $-1$ | $d + 2$ | $-2$ | $1$ | $-2$ | $-1$ | $y$ | $0$ |
| $d_F^2$ | $-1$ | $1$ | $-2$ | $2$ | $0$ | $1$ | $1$ | $0$ | $0$ |
| $d_F^2$ | $d - 1$ | $1$ | $d - 2$ | $2$ | $1$ | $0$ | $1$ | $y$ | $0$ |

The total canonical dimension of any 1-irreducible Green function $\Gamma$ is given by the relation

$$\delta \Gamma = d + 2 - \sum_{\Phi} N_\Phi d_\Phi,$$

(15)

where $N_\Phi$ is the number of the given type of field entering the function $\Gamma$, $d_\Phi$ is the corresponding total canonical dimension of field $\Phi$, and the summation runs over all types of the fields $\Phi$ in function $\Gamma$ [16,18,20].

Superficial UV divergences whose removal requires counterterms can be present only in those functions $\Gamma$ for which the formal index of divergence $\delta \Gamma$ is a non-negative integer. A dimensional analysis should be augmented by the several additional considerations. They are all explicitly stated in the previous works [11,14]. Therefore, we do not repeat them here and continue with a simple conclusion that model with the action (9) is renormalizable.

From a straightforward inspection of RG theory it is clear that for determination of scaling regimes only two Green functions have to be considered. The reason is that we study theory with three charges, $g$, $u$ and $v$. Once their fixed values are found, we would be able to study scaling regimes and their stabilities. Thus only graphs that are needed to be calculated are two-point Green functions $\langle vv \rangle_{1PI}$ and $\langle pp \rangle_{1PI}$. In a one-loop approximation [11,14,40] the calculation is simple as there are only two Feynman diagrams at this level.

For two-loop approximation, following graphs have to be computed for the velocity part

(16)
On the other hand, for the pressure part additional eight diagrams are needed

![Diagrams](18)

The remaining diagrams are needed only for determination of anomalous dimension of fields, which is left for future study.

In contrast to the incompressible case [49] compressible model (9) proved to be much more demanding from technical point of view. This is caused by three reasons. First, in compressible case there are six physical quantities \((\mu_0, v_0, \alpha, e_0)\) instead of just two \((v_0\) and charge \(g_0\)) for incompressible fluid. Second, propagators now contain both transversal and longitudinal parts and last, interaction vertices are not proportional to the momentum of prime field, what implies that the degree of UV divergence could not be lowered.

In evaluation of UV divergent parts of Feynman diagrams we have applied approach suggested in [49]. Using symbolic software [52] we were able to simplify some calculations and to determine divergent parts at least in numerical sense. Because the details of calculation are rather straightforward and proceed in a standard fashion [16,17,18,20], we refrain from mentioning them here.

### 4 Scaling regimes

The relation between the initial and renormalized action functionals \(S(\varphi, e_0) = S^R(Z_\varphi, \varphi, e, \mu)\) (where \(e_0\) is the complete set of bare parameters and \(e\) is the set of their renormalized counterparts) yields the fundamental RG differential equation:

\[
\left\{ \mathcal{D}_{RG} + N_\varphi \gamma_\varphi + N_{\varphi'} \gamma_{\varphi'} \right\} G^R(e, \mu, \ldots) = 0,
\]

where \(G = \langle \varphi \cdots \varphi \rangle\) is a correlation function of the fields \(\varphi\); \(N_\varphi\) and \(N_{\varphi'}\) are the counts of normalization-requiring fields \(\varphi\) and \(\varphi'\), respectively, which are the inputs to \(G\); the ellipsis in expression [19] stands for the other arguments of \(G\) (spatial and time variables, etc.). \(\mathcal{D}_{RG}\) is the operation \(\mathcal{D}_{\mu}\) expressed in the renormalized variables and \(\mathcal{D}_{\mu}\) is the differential operation \(\mu \partial_\mu\) for fixed \(e_0\).

For the present model it takes the form

\[
\mathcal{D}_{RG} = \mathcal{D}_\mu + \beta_\varphi \partial_\varphi + \beta_\alpha \partial_\alpha + \beta_e \partial_e - \gamma_\varphi \mathcal{D}_\nu - \gamma_e \mathcal{D}_c.
\]

Here, we have denoted \(\mathcal{D}_x \equiv x \partial_x\) for any variable \(x\). The anomalous dimension \(\gamma_F\) of some quantity \(F\) (a field or a parameter) is defined as

\[
\gamma_F = Z^{-1}_F \hat{\mathcal{D}}_\mu Z_F = \hat{\mathcal{D}}_\mu \ln Z_F,
\]
and the \( \beta \) functions for the four dimensionless coupling constants \( g, u \) and \( v \), which we now redefine according to the following rule

\[
g \equiv g_1, \quad u \equiv g_2, \quad v \equiv g_3. \tag{22}
\]

for convenience. \( \beta \) functions express the flows of parameters under the RG transformation, and are defined through relation \( \beta_i = D_\mu g_i \). This yields

\[
\begin{align*}
\beta_1 &= g_1( -y - \gamma_1 ), \quad \beta_2 = -g_2 \gamma_2, \quad \beta_3 = -3 \gamma_3, \tag{23} \\
\gamma_1 &\equiv \gamma_g, \quad \gamma_2 \equiv \gamma_u, \quad \gamma_3 \equiv \gamma_v. \tag{24}
\end{align*}
\]

Based on the analysis of the RG equation (19) it follows that the large scale behavior with respect to spatial and time scales is governed by the IR attractive ("stable") fixed points \( g^* = \{ g_1^*, g_2^*, g_3^* \} \), whose coordinates are found from the conditions [16,17,18]:

\[
\beta_1(g^*) = \beta_2(g^*) = \beta_3(g^*) = 0. \tag{25}
\]

Let us consider a set of invariant couplings \( \overline{g}_i = \overline{g}_i(s, \{ g_i \}) \) with the initial data \( \overline{g}_i|_{s=1} = g_i \). Here, \( s = k/\mu \) and IR asymptotic behavior (i.e., behavior at large distances) corresponds to the limit \( s \to 0 \). An evolution of invariant couplings is described by the set of flow equations

\[
D_s \overline{g}_i = \beta_i(\overline{g}_j), \tag{26}
\]

whose solution as \( s \to 0 \) behaves approximately like

\[
\overline{g}_i(s, g^*) \cong g_i^* + \text{const} \times s^{\omega_i}, \tag{27}
\]

where \( \{ \omega_i \} \) is the set of eigenvalues of the matrix

\[
\Omega_{ij} = \partial \beta_i / \partial g_j|_{g^*}. \tag{28}
\]

The existence of IR attractive solutions of the RG equations leads to the existence of the scaling behavior of Green functions. From (27) it follows that the type of the fixed point is determined by the matrix (28): for the IR attractive fixed points the matrix \( \Omega \) has to be positive definite.

Altogether two IR attractive fixed points are found, which defines possible scaling regimes of the system. The fixed point FPI (the trivial or Gaussian point) is stable if \( y < 0 \). This regime is characterized by irrelevance of all his charges, i.e.,

\[
g_1^* = g_2^* = g_3^* = 0. \tag{29}
\]

On the other hand, the fixed point FPII is fully nontrivial, i.e. all his coordinates attain nonzero value. We have found the following numerical expressions for them

\[
\begin{align*}
g_1^* &= 2y + \frac{-2.00625\alpha^2 - 4.8847\alpha + 4.4206}{5\alpha + 12} y^2, \tag{30} \\
g_2^* &= 1 + \frac{0.125797\alpha^2 - 0.83854\alpha - 0.188233}{5\alpha + 12} y, \tag{31}
\end{align*}
\]
\[ g_3^* = 1 + \frac{0.217295\alpha^3 + 1.7247\alpha^2 - 1.27116\alpha - 6.9228}{(\alpha + 6)(5\alpha + 12)}y. \]  

(32)

To one-loop order we have thus obtained same results as has been claimed previously [11,40]. The initial analysis reveals that FPII is nontrivial for \( y > 0 \) and not very large values of \( \alpha \).

5 Conclusion

In the present paper the compressible fluid governed by the Navier-Stokes velocity ensemble has been examined. The fluid was assumed to be compressible and the space dimension was fixed to \( d = 3 \). The problem has been investigated by means of renormalization group and expansion scheme in \( y \) was constructed.

There are two nontrivial IR stable fixed points in this model and, therefore, the critical behavior in the inertial range demonstrates two different regimes depending on the the scaling exponent \( y \). Coordinates of nontrivial fixed points have been obtained for the first time to two-loop precision. This can be considered as a first step to full two-loop analysis of the model.

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