Existence results for the Hadamard fractional differential equations and inclusions

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Abstract. In this paper, concerning non-local multi-point and integral boundary conditions, we investigate the existence and uniqueness of solutions for Hadamard fractional differential equations (HFDEs) and inclusions. In the case of equations, we use the Krasnoselskii fixed point theorem for the sum of two operators and Banach fixed point theorem, and Leray-Schauder’s alternative for multivalued maps and the fixed point theorem attributed to Covitz and Nadler for multivalued contractions for inclusions. Examples are given to clarify our main results. Finally, we discuss some variants of the given problem.

1. Introduction
Differential equations of fractional-order arise from a range of applications together within numerous fields of science and engineering. In particular, problems concerning analysis of the quality of such solutions for non-integer order differential equations have received the attention of the many authors, see [4, 5, 10, 11, 24, 25, 28, 30–38, 40, 41, 44, 46, 48], and the references therein. Most authors analyze extensively differential equations, including fractional derivatives Riemann-Liouville and Caputo. Nevertheless, Hadamard-type literature on fractional differential equations (FDEs) is not as enriched. The Hadamard-based fractional derivative introduced in 1892 [14] differs from the Riemann-Liouville and Caputo derivatives because the integration kernel within the Hadamard derivative description includes the absolute exponent’s logarithmic function. For the theoretical development on HFDEs, see [2,6,7,12,15,20,22,23,39,42,47] and the references cited therein. The monographs have commonly mentioned the hypothesis of fractional integrals, derivatives, and applications to FDEs. For more details, see [16–18,26,29] and also the references therein. Recently, some authors begun to study the Hadamard fractional boundary value problems (BVPs). The nonlinear HFDEs

\[ H^\alpha D^\alpha x(t) = f(t, x(t)), \quad 1 < t < e, \quad 1 < \alpha \leq 2, \]

\[ x(1) = 0, \quad A^H\mathcal{I}^\gamma x(\eta) + Bx(e) = c, \quad \gamma > 0, \quad 1 < \eta < e, \]
has been discussed in [1], where $H^\alpha$ is the Hadamard fractional derivative (HFD) of order $\alpha$, $f : [1, e] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, and $A, B, c$ are real constants. The existence and uniqueness results are proved by various fixed point theorems. In [3], Ahmad et al. studied BVPs for Hadamard fractional differential inclusions

$$H^\alpha x(t) \in F(t, x(t)), \quad 1 \leq t \leq e, \quad 1 < \alpha \leq 2,$$

$$x(1) = 0, \quad x(e) = H^\beta x(\eta), \quad 1 < \eta < e,$$

where $H^\alpha$ is the HFD, $F : [1, e] \times \mathbb{R} \to P(\mathbb{R})$ is a multivalued map, $P(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, $H^\beta$ is the Hadamard fractional integral (HFI) of order $\beta$. Yukunthorn et al. [49] discussed existence results for HFDEs on infinite domain:

$$H^\alpha u(t) + a(t)f(u(t)) = 0, \quad t \in (1, \infty), \quad 1 < \alpha \leq 2,$$

$$u(1) = 0, \quad H^\alpha u(\infty) = \sum_{i=1}^{m} \lambda_i H^\beta u(\eta),$$

where $H^\alpha$ denotes the HFD of order $\alpha$, $\eta \in (1, \infty)$, and $H^\beta$ is the HFI of order $\beta > 0$, $i = 1, 2, \ldots, m$, and $\lambda_i > 0$, $i = 1, 2, \ldots, m$, are arbitrary constants. The results are obtained through the applying of fixed-point theorems Leggett-Williams and Guo-Krasnoselskii. Qinghua et al. [27] investigated a Lyapunov-type inequality with the HFD. Similarly, with Hadamard integral and discrete boundary conditions, Wang et al. [43] studied the non-local Hadamard fractional BVP. Recently Muthaiah et al. [21] examined the existence of solutions for Hadamard’s non-local BVP.

In the present paper, we aim to study the existence and uniqueness of fractional equations and inclusions of solutions for the Hadamard type, supplemented by multi-point and integral boundary conditions. For Hadamard-type FDEs and inclusions, we consider the following boundary value problem:

$$H^\gamma y(\tau) = g(\tau, y(\tau)), \quad \tau \in \mathcal{J} := [1, T],$$

$$y(1) = y'(1) = 0, \quad H^\omega y(T) = \nu H^\omega y(\vartheta) + \varepsilon \sum_{j=1}^{k-2} v_j y(\varphi_j), \quad (1)$$

$$H^\gamma y(\tau) \in G(\tau, y(\tau)), \quad \tau \in \mathcal{J} := [1, T],$$

$$y(1) = y'(1) = 0, \quad H^\omega y(T) = \nu H^\omega y(\vartheta) + \varepsilon \sum_{j=1}^{k-2} v_j y(\varphi_j), \quad (2)$$

where $H^\gamma, H^\omega$ are the HFDs of order $2 < \gamma \leq 3$, $1 < \omega < 2$, $1 < \vartheta < T$, $H^\gamma$ is the HFD of order $1 < \vartheta < 2$, and $g : \mathcal{J} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, $G : \mathcal{J} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $Q(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$ and $\nu, \varepsilon$ are positive real constants.

We define space $\mathcal{P} = \mathcal{C}(\mathcal{J}, \mathbb{R})$ the Banach space of all continuous functions from $\mathcal{J} \to \mathbb{R}$ endowed with a topology of uniform convergence with the norm defined by $||y|| = \sup\{y(\tau), \tau \in \mathcal{J}\}$. The rest of the paper is structured as follows: Section 2 focuses on certain basic concepts of fractional calculus with the fundamental lemmas associated with this problem. The existence and uniqueness results can be accomplished using the fixed point theorems Krasnoselskii and Banach for equations, the Leray-Schauder alternative multivalued maps, and the multi-valued contractions in the case of inclusions in section 3. Examples are given in section 4 to verify the results. Two new problems are considered similar to (1) and (2), and Section 5 defines the strategy for resolving them.
2. Preliminaries
We begin with some basic definitions, properties, and lemmas recalled from [16,26].

**Definition 2.1.** The HFI of order $\varsigma \in \mathbb{R}^+$ of a function $g \in L^p[b,c]$, $0 \leq b \leq \tau \leq c \leq \infty$, is defined as

$$
(H^\varsigma g)(\tau) = \frac{1}{\Gamma(\varsigma)} \int_b^\tau \left( \log \frac{\sigma}{\tau} \right)^{\varsigma-1} g(\sigma) \frac{d\sigma}{\sigma}, \quad \varsigma > 0.
$$

**Definition 2.2.** Let $0 < b < c < \infty$, $\delta = \left( \frac{d}{d\tau} \right)$ and $AC^n_\delta[b,c] = \{ g : [b,c] \to \mathbb{R} : \delta^{n-1}g(\tau) \} \in AC^0[b,c]$]. The HFD of order $\varsigma > 0$ for a function $g \in AC^n_\delta[b,c]$ is defined as

$$
(H^\varsigma g)(\tau) = \frac{1}{\Gamma(n-\varsigma)} \left( \frac{d}{d\tau} \right)^n \int_1^\tau \left( \log \frac{\sigma}{\tau} \right)^{n-\varsigma} g(\sigma) \frac{d\sigma}{\sigma},
$$

where $n - 1 < \varsigma < n$, $n = [\varsigma] + 1$, $[\varsigma]$ denote the integer part of the real number $\varsigma$ and $\log(\cdot) = \log_{\varphi}(\cdot)$.

**Theorem 2.3** (Arzela-Ascoli Theorem (see [45])). A subset $F$ in $C(J,\mathbb{R})$ is relatively compact if and only if it is uniformly bounded and equicontinuous on $J$.

**Theorem 2.4** (Krasnoselskii Fixed Point Theorem (see [19])). Let $G$ be a Banach space, let $H$ be a bounded closed convex subset of $G$ and let $W$, $P$ be mappings of $H$ into $G$ such that $Wy + Pz \in H$ for every pair $y,z \in H$. If $W$ is a contraction and $P$ is completely continuous, then the equation $Wy + Pz = y$ has a solution on $H$.

**Theorem 2.5** (Banach Fixed Point Theorem (see [45])). Let $(G,d)$ be a complete metric space, and $P : H \to H$ a contraction mapping: $d(Py,Pz) \leq \kappa d(y,z)$, where $0 < \kappa < 1$, for each $y,z \in H$. Then, $\exists$ a unique fixed point $y$ of $P$ in $H$.

**Lemma 2.6.** If $W : Y \to Q_d(Z)$ is upper semi-continuous, then $G\nabla(W)$ is a closed subset of $Y \times Z$; i.e., for every sequence $\{y_n\}_{n \in \mathbb{N}} \subset Y$ and $\{z_n\}_{n \in \mathbb{N}} \subset Z$, if $y_n \to y_*$ and $z_n \to z_*$, then $y_n \to Wz_*$. Conversely, if $W$ is completely continuous and has a closed graph, then it is upper semi-continuous.

**Theorem 2.7** (Leray-Schauder for Kakutani Maps (see [13])). Let $P$ be a Banach space, $G$ a closed convex subset of $P$; $V$ an open subset of $G$ and $0 \in V$. Suppose that $T : \overline{V} \to Q_{cp,e}G$ is an upper semicontinuous compact map. Then either

(i) $T$ has a fixed point in $\overline{V}$; or

(ii) there is a $v \in \partial Y$ and $\mu \in (0,1)$ with $v \in \mu T(v)$.

**Theorem 2.8** (Covitz and Nadler for multivalued contraction (see [9])). Let $(Y,d)$ be a complete metric space. If $M : Y \to Q_d(Y)$ is a contraction, then $\text{Fix}M \neq 0$.

**Definition 2.9.** A function $y \in C^3(J,\mathbb{R})$ is called a solution of problem (2) if $\exists$ a function $\beta(\tau) \in L^1(J,\mathbb{R})$ with $\beta(\tau) \in G(\tau,y(\tau))$ such that

$$
D^\varsigma y(\tau) = \beta(\tau), \quad 2 < \varsigma \leq 3, \quad \forall \ \tau \in J \text{ and }
$$

$$
y(1) = y'(1) = 0, \quad H^\varsigma y(T) = \nu^H T^\varphi y(\varphi) + \varepsilon \sum_{j=1}^{k-2} \nu_j y(\varphi_j),
$$

where $1 < \varphi < \varphi_1 < \varphi_2 < \cdots < \varphi_{k-2} < T$. 
Lemma 2.10. For any \( \hat{y} \in \mathcal{C}(\mathcal{J}, \mathbb{R}) \), \( y \in \mathcal{C}^3(\mathcal{J}, \mathbb{R}) \), the function \( y \) is the solution of the problem
\[
^H\mathcal{D}^\gamma y(\tau) = \hat{y}(\tau), \quad \tau \in \mathcal{J},
\]
\[
y(1) = y'(1) = 0, \quad ^H\mathcal{D}^\nu y(T) = \nu^H\mathcal{I}^\theta y(\vartheta) + \varepsilon \sum_{j=1}^{k-2} v_j y(\varphi_j),
\]
if and only if
\[
y(\tau) = \frac{1}{\Gamma(\xi)} \int_1^\tau \left( \log \frac{\sigma}{\tau} \right)^{-1} \hat{g}(\sigma) \frac{d\sigma}{\sigma} + \frac{(\log \tau)^{-1}}{\Lambda} \left[ \nu \frac{1}{\Gamma(\xi + \theta)} \int_1^\theta \left( \log \frac{\vartheta}{\sigma} \right)^{-1} \hat{g}(\sigma) \frac{d\sigma}{\sigma} \right. \\
\left. + \varepsilon \sum_{j=1}^{k-2} \frac{1}{\Gamma(\xi)} \int_1^{\varphi_j} \left( \log \frac{\varphi_j}{\sigma} \right)^{-1} \hat{g}(\sigma, y(\sigma)) \frac{d\sigma}{\sigma} - \frac{1}{\Gamma(\xi - \omega)} \int_1^T \left( \log \frac{T}{\sigma} \right)^{-1} \hat{g}(\sigma, y(\sigma)) \frac{d\sigma}{\sigma} \right],
\]
where
\[
\Lambda = \frac{\Gamma(\xi)(\log T)^{-\omega-1}}{\Gamma(\xi - \omega)} - \frac{\Gamma(\xi)(\log \theta)^{e-1}}{\Gamma(\xi + \theta)} - \varepsilon \sum_{j=1}^{k-2} v_j (\log \varphi_j)^{-1}.
\]

3. Existence and Uniqueness Results
In view of Lemma 2.10, we define an operator \( \mathcal{T} : \mathcal{P} \to \mathcal{P} \) as
\[
\mathcal{T}(y)(\tau) = \frac{1}{\Gamma(\xi)} \int_1^\tau \left( \log \frac{\sigma}{\tau} \right)^{-1} \hat{g}(\sigma, y(\sigma)) \frac{d\sigma}{\sigma} + \frac{(\log \tau)^{-1}}{\Lambda} \left[ \nu \frac{1}{\Gamma(\xi + \theta)} \int_1^\theta \left( \log \frac{\vartheta}{\sigma} \right)^{-1} g(\sigma, y(\sigma)) \frac{d\sigma}{\sigma} \right. \\
\left. + \varepsilon \sum_{j=1}^{k-2} v_j \frac{1}{\Gamma(\xi)} \int_1^{\varphi_j} \left( \log \frac{\varphi_j}{\sigma} \right)^{-1} g(\sigma, y(\sigma)) \frac{d\sigma}{\sigma} - \frac{1}{\Gamma(\xi - \omega)} \int_1^T \left( \log \frac{T}{\sigma} \right)^{-1} g(\sigma, y(\sigma)) \frac{d\sigma}{\sigma} \right]
\]

We represent suitable for computing:
\[
\Delta = \frac{1}{\Lambda \Gamma(\xi + 1)} \left( \Lambda (\log T)^{e} + (\log T)^{e-1} \left( \varepsilon \sum_{j=1}^{k-2} v_j (\log \varphi_j)^{e} \right) \right)
\]
\[
+ \frac{(\log T)^{e-1}}{\Lambda} \left( \nu (\log \theta)^{e+1} + (\log T)^{e-\omega} \right).
\]

Let \( q : \mathcal{J} \times \mathbb{R} \to \mathbb{R} \) be a continuous function. To prove the existence and uniqueness results, we need the following assumptions.

\((E_1)\) \( |g(\tau, p_1) - g(\tau, p_2)| \leq S|p_1 - p_2|, \forall \tau \in \mathcal{J}, p_1, p_2 \in \mathbb{R}, S > 0.\)

\((E_2)\) \( |g(\tau, y(\tau))| \leq \xi(\tau) \) for \( (\tau, y) \in \mathcal{J} \times \mathbb{R} \), and \( \xi \in \mathcal{C}(\mathcal{J}, \mathbb{R}^+) \) with
\[
\|\xi\| = \max_{\tau \in \mathcal{J}} \xi(\tau).
\]

\((E_3)\) \( \mathcal{G} : \mathcal{J} \times \mathbb{R} \to \mathcal{Q}(\mathbb{R}) \) is Carathéodory and has nonempty compact and convex values.

\((E_4)\) \exists a continuous non-decreasing function \( \psi : [0, \infty) \to [0, \infty] \) and a function \( q \in \mathcal{C}(\mathcal{J}, \mathbb{R}^+) \) such that
\[
\|\mathcal{G}(\tau, y)\| = \sup \{ |z| : z \in \mathcal{G}(\tau, y) \} \leq q(\tau) \psi(\|y\|) \text{ for each } (\tau, y) \in \mathcal{J} \times \mathbb{R}.
\]
Proof. Defining $\mathcal{M}_h(\tau, y) : \mathcal{J} \to \mathcal{Q}_{cp}(\mathbb{R})$ is such that $\mathcal{G}(\cdot, y) : \mathcal{J} \to \mathcal{Q}_{cp}(\mathbb{R})$ is measurable for each $y \in \mathbb{R}$.

Theorem 3.1. Suppose $(\mathcal{E}_1)$ and $(\mathcal{E}_2)$ holds. In addition, if

$$\frac{\nu(\log \theta)^{c+e}}{\Gamma(\beta + 1)} + \varepsilon \sum_{j=1}^{k-2} \left( \log \varphi_j \right)^{\sigma} \frac{\rho_j}{\Gamma(\beta + 1)} + \frac{(\log T)^{\sigma - \omega}}{\Gamma(\beta - \omega)} \right) < 1. \tag{8}$$

At least one solution to the problem $(1)$ exists on $\mathcal{J}$.

Proof. Defining $\mathcal{B}_\theta = \{ y \in \mathcal{P} : \| y \| \leq \theta \}$, where $\theta \geq \| \xi \| \Delta$. To prove Theorem 2.4’s hypothesis, we split operator $\mathcal{J}$ given by $(6)$ as $\mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2$ on $\mathcal{B}_\theta$, where

$$\begin{align*}
(\mathcal{J}_1 y)(\tau) &= \frac{1}{\Gamma(\beta)} \int_1^\tau \left( \log \frac{\tau}{\sigma} \right)^{\sigma - 1} g(\sigma, y(\tau)) \frac{d\sigma}{\sigma}, \\
(\mathcal{J}_2 y)(\tau) &= \frac{(\log T)^{\beta - 1}}{\Gamma(\beta + 1)} \left( \log \frac{\theta}{\sigma} \right)^{c+e-1} g(\sigma, y(\tau)) \frac{d\sigma}{\sigma} \\
&\quad + \varepsilon \sum_{j=1}^{k-2} \frac{1}{\Gamma(\beta)} \int_1^{\varphi_j} \left( \log \frac{\varphi_j}{\sigma} \right)^{\sigma - 1} g(\sigma, y(\tau)) \frac{d\sigma}{\sigma} \\
&\quad - \frac{1}{\Gamma(\beta - \omega)} \int_1^\tau \left( \log \frac{T}{\sigma} \right)^{\sigma - \omega - 1} g(\sigma, y(\tau)) \frac{d\sigma}{\sigma}.
\end{align*}$$

For $\hat{y}_1, \hat{y}_2 \in \mathcal{B}_\theta$,

$$\begin{align*}
\| (\mathcal{J}_1 \hat{y}_1)(\tau) + (\mathcal{J}_2 \hat{y}_2)(\tau) \| &\leq \sup_{\tau \in \mathcal{J}} \left\{ \frac{1}{\Gamma(\beta)} \int_1^\tau \left( \log \frac{\tau}{\sigma} \right)^{\sigma - 1} \left| g(\sigma, y(\tau)) \right| \frac{d\sigma}{\sigma} \\
&\quad + \frac{(\log T)^{\beta - 1}}{\Gamma(\beta + 1)} \left( \log \frac{\theta}{\sigma} \right)^{c+e-1} \left| g(\sigma, y(\tau)) \right| \frac{d\sigma}{\sigma} \\
&\quad + \varepsilon \sum_{j=1}^{k-2} \frac{1}{\Gamma(\beta)} \int_1^{\varphi_j} \left( \log \frac{\varphi_j}{\sigma} \right)^{\sigma - 1} \left| g(\sigma, y(\tau)) \right| \frac{d\sigma}{\sigma} \\
&\quad + \frac{1}{\Gamma(\beta - \omega)} \int_1^\tau \left( \log \frac{T}{\sigma} \right)^{\sigma - \omega - 1} \left| g(\sigma, y(\tau)) \right| \frac{d\sigma}{\sigma} \right\} \\
&\leq \| \xi \| \left\{ \frac{(\log T)^{\beta}}{\Gamma(\beta + 1)} + \frac{(\log T)^{\beta - 1}}{\Gamma(\beta + 1)} \left( \log \frac{\theta}{\sigma} \right)^{c+e} \frac{\rho_j}{\Gamma(\beta + 1)} + \varepsilon \sum_{j=1}^{k-2} \frac{1}{\Gamma(\beta + 1)} + \frac{(\log T)^{\sigma - \omega}}{\Gamma(\beta - \omega)} \right\} \\
&\leq \| \xi \| \Delta \leq \theta,
\end{align*}$$
which implies that $\mathcal{T}_1 \tilde{y}_1 + \mathcal{T}_2 \tilde{y}_2 \in \mathcal{B}_\theta$. Now, we proving $\mathcal{T}_2$ is a contractive. Let $p_1, p_2 \in \mathbb{R}$, $\tau \in \mathcal{J}$. Then, together with (8) with assumption $(E_1)$, we get

$$
\|\mathcal{T}_2 p_1 - \mathcal{T}_2 p_2\| \leq \frac{8(\log T)^{\zeta-1}}{\Lambda} \left( \frac{\nu(\log \theta)^{\chi+\varphi}}{\Gamma(\zeta+\varphi+1)} + \varepsilon \sum_{j=1}^{k-2} u_j (\log \varphi_j)^{\zeta} \right) + \frac{(\log T)^{\zeta-\omega}}{\Gamma(\zeta-\omega+1)} \|p_1 - p_2\|.
$$

According to statement $(E_1)$, operator $\mathcal{T}_2$ is a contraction. First, we'll demonstrate $\mathcal{T}_1$’s compact and continuous. $g$-continuity implies operator $\mathcal{T}_1$ is continuous. $\mathcal{T}_1$ is also uniformly bounded as $\mathcal{B}_\theta$,

$$
\|\mathcal{T}_1 y\| \leq \|\xi\|(\log T)^{\chi}.
$$

Moreover, with $\sup_{(\tau,y) \in \mathcal{J} \times \mathcal{B}_\theta} |g(\tau, y)| = \hat{g} < \infty$ and $\tau_1 < \tau_2$, $\tau_1, \tau_2 \in \mathcal{J}$, we have

$$
|\mathcal{T}_1 y(\tau_2) - \mathcal{T}_1 y(\tau_1)| = \left| \frac{1}{\Gamma(\zeta)} \int_1^{\tau_2} \left( \log \frac{\tau_2}{\sigma} \right)^{\chi-1} g(\sigma, y(\sigma)) \frac{d\sigma}{\sigma} \right|
$$

$$
- \left| \frac{1}{\Gamma(\zeta)} \int_1^{\tau_1} \left( \log \frac{\tau_1}{\sigma} \right)^{\chi-1} g(\sigma, y(\sigma)) \frac{d\sigma}{\sigma} \right|
$$

$$
\leq \frac{\hat{g}}{\Gamma(\zeta)} \int_1^{\tau_1} \left[ \left( \log \frac{\tau_2}{\sigma} \right)^{\chi-1} - \left( \log \frac{\tau_1}{\sigma} \right)^{\chi-1} \right] \frac{d\sigma}{\sigma} + \int_1^{\tau_1} \left( \log \frac{\tau_2}{\sigma} \right)^{\chi-1} \frac{d\sigma}{\sigma}.
$$

Clearly, the RHS of (9) tends to zero independent of $y$ as $\tau_2 - \tau_1 \to 0$. Thus, $\mathcal{T}_1$ is relatively compact on $\mathcal{B}_\theta$. Therefore, by Theorem 2.3, $\mathcal{T}_1$ is compact on $\mathcal{B}_\theta$. Thus, all premises of Theorem 2.4 are fulfilled. Therefore, at least one solution exists for the problem (1) on $\mathcal{J}$.

**Theorem 3.2.** Suppose $(E_1)$ hold. Furthermore, $8 \Delta < 1$ is assumed, where $\Delta$ is described in (7). Then, $\exists$ a unique solution for (1) on $\mathcal{J}$.

**Proof.** Let us define $\sup_{\tau \in \mathcal{J}} |g(\tau, 0)| = \Omega < \infty$. Selecting $\theta \geq \frac{\Omega \Delta}{1 - 8 \Delta}$, we demonstrate that $\mathcal{T} \mathcal{B}_\theta \subset \mathcal{B}_\theta$, where $\mathcal{B}_\theta = \{ y \in \mathcal{P} : \|y\| \leq \theta \}$. For $y \in \mathcal{B}_\theta$, we have

$$
|\mathcal{T}_1 y(\tau)| \leq (8 \theta + \Omega) \sup_{\tau \in \mathcal{J}} \left\{ \frac{1}{\Gamma(\zeta)} \int_1^{\varphi_1} \left( \log \frac{\varphi_1}{\sigma} \right)^{\chi-1} \frac{d\sigma}{\sigma} \right. 
$$

$$
+ \frac{(\log \tau)^{\chi-1}}{\Lambda} \left[ \frac{\nu}{\Gamma(\zeta+\varphi)} \int_1^{\varphi} \left( \log \frac{\varphi}{\sigma} \right)^{\chi+\varphi-1} \frac{d\sigma}{\sigma} \right. 
$$

$$
+ \varepsilon \sum_{j=1}^{k-2} u_j \frac{1}{\Gamma(\zeta)} \int_1^{\varphi_j} \left( \log \frac{\varphi_j}{\sigma} \right)^{\chi-1} \frac{d\sigma}{\sigma}
$$

$$
+ \frac{1}{\Gamma(\zeta-\omega)} \int_1^{T} \left( \log \frac{T}{\sigma} \right)^{\chi-\omega-1} \frac{d\sigma}{\sigma} \right) \left( \log \frac{\tau}{\sigma} \right)^{\chi-1} \frac{d\sigma}{\sigma} \right\} \leq (8 \theta + \Omega) \Delta.
$$

(10)
Therefore, \(|\langle \mathcal{J}y \rangle \| \leq \theta \) follows from (10). Now, for \( y, \hat{y} \in \mathcal{P} \), we get

\[
|\mathcal{J}y(\tau) - \mathcal{J}\hat{y}(\tau)| \leq S\|y - \hat{y}\| \left\{ \frac{1}{1 - \lambda} \int_{1}^{\tau} \left( \log \frac{T}{\sigma} \right)^{\lambda - 1} \frac{d\sigma}{\sigma} + \frac{\log \tau}{\Lambda} \left[ \frac{1}{\Gamma(\zeta + \theta)} \int_{1}^{\theta} \left( \log \frac{\phi(y)}{\sigma} \right)^{\lambda - 1} \frac{d\sigma}{\sigma} \\
+ \varepsilon \sum_{j=1}^{k-2} \frac{1}{\Gamma(\zeta)} \int_{1}^{\varphi_j} \left( \log \frac{\varphi_j}{\sigma} \right)^{\lambda - 1} \frac{d\sigma}{\sigma} \\
- \frac{1}{\Gamma(\zeta - \omega)} \int_{1}^{T} \left( \log \frac{T}{\sigma} \right)^{\lambda - 1} \frac{d\sigma}{\sigma} \right] \right\} = S\|y - \hat{y}\|.
\]

Thus,

\[
\|\mathcal{J}y - \mathcal{J}\hat{y}\| \leq S\|y - \hat{y}\|.
\]

Since \( S\|y - \hat{y}\| < 1 \) by assumption, \( \mathcal{J} \) is a contraction. Theorem 2.5 follows that equation (1) has a unique solution on \( \mathcal{J} \).

**Theorem 3.3.** Suppose \((E_3)-(E_5)\) holds. Then problem (2) contains at least one solution on \( \mathcal{J} \).

**Proof.** Define an operator \( \Upsilon_G : \mathcal{P} \rightarrow \mathcal{Q}(\mathcal{P}) \) by

\[
\Upsilon_G(y) = \begin{cases}
    f \in \mathcal{P} : \\
    f(\tau) = \left\{ \frac{1}{\Gamma(\zeta)} \int_{1}^{\tau} \left( \log \frac{T}{\sigma} \right)^{\lambda - 1} g(\sigma, y(\sigma)) \frac{d\sigma}{\sigma} \\
    + \frac{\log \tau}{\Lambda} \left[ \frac{1}{\Gamma(\zeta + \theta)} \int_{1}^{\theta} \left( \log \frac{\phi(y)}{\sigma} \right)^{\lambda - 1} g(\sigma, y(\sigma)) \frac{d\sigma}{\sigma} \\
    + \varepsilon \sum_{j=1}^{k-2} \frac{1}{\Gamma(\zeta)} \int_{1}^{\varphi_j} \left( \log \frac{\varphi_j}{\sigma} \right)^{\lambda - 1} g(\sigma, y(\sigma)) \frac{d\sigma}{\sigma} \\
    - \frac{1}{\Gamma(\zeta - \omega)} \int_{1}^{T} \left( \log \frac{T}{\sigma} \right)^{\lambda - 1} g(\sigma, y(\sigma)) \frac{d\sigma}{\sigma} \right] \right\} 
\end{cases}
\]

for \( \beta \in \mathcal{U}_{\Upsilon_G,y} \). We will demonstrate that \( \Upsilon_G \) satisfies the assumptions of the Theorem 2.7. First, we present that \( \Upsilon_G \) is convex for each \( y \in \mathcal{P} \). This is evident as \( \mathcal{U}_{\Upsilon_G,y} \) is convex. Next, we show \( \Upsilon_G \) maps in \( \mathcal{P} \) bounded sets into bounded sets. For a positive number \( \theta \), let \( \mathcal{B}_{\theta} = \{ y \in \mathcal{P} : \|y\| \leq \theta \} \) be a bounded ball in \( \mathcal{P} \). Then, for each \( f \in \Upsilon_G(y) \), \( y \in \mathcal{B}_{\theta} \) \( \exists \beta \in \mathcal{U}_{\Upsilon_G,y} \) such that

\[
f(\tau) = \frac{1}{\Gamma(\zeta)} \int_{1}^{\tau} \left( \log \frac{T}{\sigma} \right)^{\lambda - 1} g(\sigma, y(\sigma)) \frac{d\sigma}{\sigma} \\
+ \frac{\log \tau}{\Lambda} \left[ \frac{1}{\Gamma(\zeta + \theta)} \int_{1}^{\theta} \left( \log \frac{\phi(y)}{\sigma} \right)^{\lambda - 1} g(\sigma, y(\sigma)) \frac{d\sigma}{\sigma} \\
+ \varepsilon \sum_{j=1}^{k-2} \frac{1}{\Gamma(\zeta)} \int_{1}^{\varphi_j} \left( \log \frac{\varphi_j}{\sigma} \right)^{\lambda - 1} g(\sigma, y(\sigma)) \frac{d\sigma}{\sigma} \\
- \frac{1}{\Gamma(\zeta - \omega)} \int_{1}^{T} \left( \log \frac{T}{\sigma} \right)^{\lambda - 1} g(\sigma, y(\sigma)) \frac{d\sigma}{\sigma} \right].
\]
Then, for $\tau \in J$, we have
\[
|f(\tau)| = \frac{1}{\Gamma(\varphi)} \int_{1}^{\tau} \left( \log \frac{\tau}{\sigma} \right)^{-1} |\beta(\sigma)| \frac{d\sigma}{\sigma} + \frac{1}{\Gamma(\varphi)} \int_{1}^{\tau} \left( \log \frac{\varphi}{\sigma} \right)^{\varsigma-1} |\beta(\sigma)| \frac{d\sigma}{\sigma}
\]
\[
+ \epsilon \sum_{j=1}^{k-2} v_j \int_{1}^{\varphi_j} \left( \log \frac{\varphi_j}{\sigma} \right)^{-1} |\beta(\sigma)| \frac{d\sigma}{\sigma} + \frac{1}{\Gamma(\varphi)} \int_{1}^{\tau} \left( \log \frac{T}{\sigma} \right)^{\varsigma-1} |\beta(\sigma)| \frac{d\sigma}{\sigma}
\]
\[
\leq \psi(\|y\|) \|q\| \left\{ \frac{1}{\Gamma(\varphi)} \int_{1}^{\tau} \left( \log \frac{\tau}{\sigma} \right)^{-1} d\sigma + \frac{1}{\Gamma(\varphi)} \int_{1}^{\tau} \left( \log \frac{T}{\sigma} \right)^{\varsigma-1} d\sigma \right\}
\]
\[
\leq \psi(\|y\|) \|q\| \Delta.
\]
Consequently,
\[
\|f\| \leq \psi(\|y\|) \|q\| \Delta.
\]
We show that maps bounded sets into equicontinuous sets of $P$. Let $\tau_1, \tau_2 \in J$ with $\tau_1 < \tau_2$ and $y \in B_\delta$. For each $f \in Y_\varphi(y)$, we obtain
\[
|f(\tau_2) - f(\tau_1)| \leq \frac{\psi(\|y\|) \|q\|}{\Gamma(\varphi)} \left| \int_{1}^{\tau_2} \left[ \left( \log \frac{\tau_2}{\sigma} \right)^{\varsigma-1} - \left( \log \frac{\tau_1}{\sigma} \right)^{\varsigma-1} \right] d\sigma \right|
\]
\[
+ \int_{\tau_1}^{\tau_2} \left( \log \frac{\tau_2}{\sigma} \right)^{-1} d\sigma + \frac{\psi(\|y\|) \|q\| (\log \frac{\tau_2}{\sigma})^{\varsigma-1} - (\log \frac{\tau_1}{\sigma})^{\varsigma-1}}{\Lambda}
\]
\[
\times \left[ \frac{1}{\Gamma(\varphi)} \int_{1}^{\tau_2} \left( \log \frac{\varphi}{\sigma} \right)^{\varsigma-1} d\sigma \right] + \epsilon \sum_{j=1}^{k-2} v_j \int_{1}^{\varphi_j} \left( \log \frac{\varphi_j}{\sigma} \right)^{-1} d\sigma
\]
\[
+ \frac{1}{\Gamma(\varphi)} \int_{1}^{\tau} \left( \log \frac{T}{\sigma} \right)^{\varsigma-1} d\sigma \right\}.
\]
Obviously the RHS of the above inequality tends to zero, independently of $y \in B_\delta$ as $\tau_2 - \tau_1 \to 0$. As $Y_\varphi$ satisfies the assumptions $(E_3 - (\mathcal{E}_5))$, hence it follows by the Theorem 2.3 that $Y_\varphi : P \to \mathcal{Q}(P)$ is completely continuous. $Y_\varphi$ is shown to be upper semicontinuous if we determine that it has a closed graph, since $Y_\varphi$ is shown to be completely continuous already. So we’re going to prove $Y_\varphi$ has a closed graph. Let $y_n \to y_*$, $f_n \in Y_\varphi(y_n)$ and $f_n \to f_*$. Then we have to prove that $Y_\varphi(y_*)$. Associated with $f_n \in Y_\varphi(y_n)$, $\exists \beta_n \in Y_\varphi(y_n)$, such that for each $\tau \in J$,
\[
f_n(\tau) = \frac{1}{\Gamma(\varphi)} \int_{1}^{\tau} \left( \log \frac{\tau}{\sigma} \right)^{\varsigma-1} \beta_n(\sigma) \frac{d\sigma}{\sigma} + \frac{1}{\Gamma(\varphi)} \int_{1}^{\tau} \left( \log \frac{\varphi}{\sigma} \right)^{\varsigma-1} \beta_n(\sigma) \frac{d\sigma}{\sigma}
\]
Observe that
\[
\int_{\gamma} \left( \log \frac{\varphi_j}{\sigma} \right)^{-1} \beta_n(\sigma) \d\sigma - \frac{1}{\Gamma(\zeta - \omega)} \int_{1}^{T} \left( \log \frac{T}{\sigma} \right)^{-\omega-1} \beta_n(\sigma) \d\sigma
\]

Thus it suffices to demonstrate that \( \exists \beta_* \in \mathcal{U}_{G;y} \), such that for each \( \tau \in J \),
\[
f_* = \frac{1}{\Gamma(\zeta)} \int_{1}^{\tau} \left( \log \frac{T}{\sigma} \right)^{-\beta}(\sigma) \d\sigma + \frac{(\log \tau)^{\zeta-1}}{\Lambda} \int_{1}^{\theta} \left( \log \frac{\vartheta}{\sigma} \right)^{\zeta+e-1} \beta_*(\sigma) \d\sigma
\]
\[
+ \varepsilon \sum_{j=1}^{k-2} \frac{1}{\Gamma(\zeta)} \int_{1}^{\varphi_j} \left( \log \frac{\varphi_j}{\sigma} \right)^{-1} \beta_* (\sigma) \d\sigma - \frac{1}{\Gamma(\zeta - \omega)} \int_{1}^{T} \left( \log \frac{T}{\sigma} \right)^{-\omega-1} \beta_*(\sigma) \d\sigma
\]

Let us consider the linear operator \( \mathcal{K} : \mathcal{L}^1(J, \mathbb{R}) \to \mathcal{P} \) given by
\[
f \mapsto \mathcal{K}(\beta)(\tau) = \frac{1}{\Gamma(\zeta)} \int_{1}^{\tau} \left( \log \frac{T}{\sigma} \right)^{-\beta}(\sigma) \d\sigma + \frac{(\log \tau)^{\zeta-1}}{\Lambda} \int_{1}^{\theta} \left( \log \frac{\vartheta}{\sigma} \right)^{\zeta+e-1} \beta_*(\sigma) \d\sigma
\]
\[
+ \varepsilon \sum_{j=1}^{k-2} \frac{1}{\Gamma(\zeta)} \int_{1}^{\varphi_j} \left( \log \frac{\varphi_j}{\sigma} \right)^{-1} \beta_* (\sigma) \d\sigma - \frac{1}{\Gamma(\zeta - \omega)} \int_{1}^{T} \left( \log \frac{T}{\sigma} \right)^{-\omega-1} \beta_*(\sigma) \d\sigma
\]

Observe that
\[
\|f_n(\tau) - f_*(\tau)\| = \left\| \frac{1}{\Gamma(\zeta)} \int_{1}^{\tau} \left( \log \frac{T}{\sigma} \right)^{-\beta}(\sigma) \d\sigma - \frac{1}{\Gamma(\zeta)} \int_{1}^{\tau} \left( \log \frac{T}{\sigma} \right)^{-\beta}(\sigma) \d\sigma \right\| \to 0,
\]
as \( n \to \infty \).

According to Lemma 2.6, \( \mathcal{K} \circ \mathcal{U}_{G;y} \) is a closed graph operator. We have \( f_n(\tau) \in \mathcal{K}(\mathcal{U}_{G;y_n}) \). So, because of \( y_n \to y_* \), we have
\[
f_*(\tau) = \frac{1}{\Gamma(\zeta)} \int_{1}^{\tau} \left( \log \frac{T}{\sigma} \right)^{-\beta}(\sigma) \d\sigma
\]
\[
\frac{(\log \tau)^{\zeta-1}}{\Lambda} \int_{1}^{\theta} \left( \log \frac{\vartheta}{\sigma} \right)^{\zeta+e-1} \beta_*(\sigma) \d\sigma
\]
\[
+ \varepsilon \sum_{j=1}^{k-2} \frac{1}{\Gamma(\zeta)} \int_{1}^{\varphi_j} \left( \log \frac{\varphi_j}{\sigma} \right)^{-1} \beta_* (\sigma) \d\sigma
\]
\[
\frac{1}{\Gamma(\zeta - \omega)} \int_{1}^{T} \left( \log \frac{T}{\sigma} \right)^{-\omega-1} \beta_*(\sigma) \d\sigma
\]
f for some \( \beta_* \in \mathcal{U}_{G;y_*} \). There is an open set \( \mathcal{Z} \subseteq \mathcal{P} \) with \( \gamma \notin \mathcal{U}_{G}(\mu) \) for every \( \mu \in (0,1) \) and all \( y \in \partial \mathcal{Z} \). Then there’s \( \beta \in \mathcal{L}^1(J, \mathbb{R}) \) with \( \beta \in \mathcal{U}_{G;y} \) so we have
\[
y(\tau) = \frac{1}{\Gamma(\zeta)} \int_{1}^{\tau} \left( \log \frac{T}{\sigma} \right)^{-\beta}(\sigma) \d\sigma + \frac{(\log \tau)^{\zeta-1}}{\Lambda} \int_{1}^{\theta} \left( \log \frac{\vartheta}{\sigma} \right)^{\zeta+e-1} \beta(\sigma) \d\sigma
\]
\[ (+e \sum_{j=1}^{k-2} v_j \frac{1}{\Gamma(\zeta)} \int_1^{\varphi_j} (\log \frac{\varphi_j}{\sigma})^{\zeta-1} \beta(\sigma) \frac{d\sigma}{\sigma} - \frac{1}{\Gamma(\zeta - \omega)} \int_1^T (\log \frac{T}{\sigma})^{\zeta-1} \beta(\sigma) \frac{d\sigma}{\sigma}) \right) \right), \quad \tau \in \mathcal{J}. \]

As in the second step, one can have \( \|y\| \leq \psi(\|y\|)\|q\| \Delta \), which implies that \( \frac{\|y\|}{\psi(\|y\|)\|q\|} \leq 1 \). In view of (\( \mathcal{E}_3 \)), \( \exists \mathcal{V} \) such that \( \|y\| \neq \mathcal{V} \). Let us set \( \mathcal{Z} = \{y \in \mathcal{P} : \|y\| < \mathcal{V}\} \). It should be noted that the \( \mathcal{Y}_G : \mathcal{Z} \to \mathcal{Q}(\mathcal{P}) \) operator is upper semicontinuous and completely continuous. There is no \( y \in \partial \mathcal{Z} \) from \( \mathcal{Z} \)'s choice to \( y \in \mu \mathcal{Y}_G(y) \) for some \( \mu \in (0, 1) \). Hence we deduce from the Theorem 2.7 that the \( \mathcal{Y}_G \) has a fixed \( y \in \mathcal{Z} \) point, which is a solution to this problem (2).

**Theorem 3.4.** Suppose \( (\mathcal{E}_6) \) and \( (\mathcal{E}_7) \) holds. Then problem (2) contains at least one solution for \( \mathcal{J} \) if \( \gamma = \|\kappa\| \Delta < 1 \), where the \( \Delta \) is defined by (7).

**Proof.** Remember that for each \( y \in \mathcal{P} \), the set \( \mathcal{U}_{G,y} \) is nonempty by assumption (\( \mathcal{E}_6 \)), so \( \mathcal{G} \) has a measurable selection (see Theorem III.6 [8]). We show that the \( \mathcal{Y}_G \), defined at the start of the Theorem 3.3 proof, satisfies the Theorem 2.8 assumptions. To demonstrate that \( \mathcal{Y}_G(y) \in \mathcal{Q}_G(\mathcal{P}) \) for each \( y \in \mathcal{P} \), let \( \{v_n\}_{n \geq 0} \in \mathcal{Y}_G(y) \) be such that \( v_n \to v \in \mathcal{P} \) as \( n \to \infty \). Then \( v \in \mathcal{P} \) and \( \exists \beta_n \in \mathcal{U}_{G,y_n} \) such that, for each \( \tau \in \mathcal{J} \),

\[
v_n(\tau) = \frac{1}{\Gamma(\zeta)} \int_1^\tau \left( \left( \log \frac{\tau}{\sigma} \right)^{\zeta-1} \beta_n(\sigma) \frac{d\sigma}{\sigma} + \frac{1}{\Gamma(\zeta + \varrho)} \int_1^\varrho \left( \left( \log \frac{\varrho}{\sigma} \right)^{\zeta-1} \beta_n(\sigma) \frac{d\sigma}{\sigma} \right) \right) \right) \right) \right) + \varepsilon \sum_{j=1}^{k-2} v_j \frac{1}{\Gamma(\zeta)} \int_1^{\varphi_j} \left( \left( \log \frac{\varphi_j}{\sigma} \right)^{\zeta-1} \beta_n(\sigma) \frac{d\sigma}{\sigma} + \frac{1}{\Gamma(\zeta - \omega)} \int_1^\omega \left( \left( \log \frac{\omega}{\sigma} \right)^{\zeta-1} \beta_n(\sigma) \frac{d\sigma}{\sigma} \right) \right) \right) \right).

Since \( \mathcal{G} \) has compact values, we move a sub-sequence to get \( \beta_n \) converging to \( \beta \) in \( \mathcal{L}^1(\mathcal{J}, \mathbb{R}) \), we have

\[
\beta_n(\tau) \to \beta(\tau) = \frac{1}{\Gamma(\zeta)} \int_1^\tau \left( \left( \log \frac{\tau}{\sigma} \right)^{\zeta-1} \beta(\sigma) \frac{d\sigma}{\sigma} + \frac{1}{\Gamma(\zeta + \varrho)} \int_1^\varrho \left( \left( \log \frac{\varrho}{\sigma} \right)^{\zeta-1} \beta(\sigma) \frac{d\sigma}{\sigma} \right) \right) \right) \right) + \varepsilon \sum_{j=1}^{k-2} v_j \frac{1}{\Gamma(\zeta)} \int_1^{\varphi_j} \left( \left( \log \frac{\varphi_j}{\sigma} \right)^{\zeta-1} \beta(\sigma) \frac{d\sigma}{\sigma} + \frac{1}{\Gamma(\zeta - \omega)} \int_1^\omega \left( \left( \log \frac{\omega}{\sigma} \right)^{\zeta-1} \beta(\sigma) \frac{d\sigma}{\sigma} \right) \right) \right) \right).

Thus, \( v \in \mathcal{Y}_G(y) \). We demonstrate that \( \exists \gamma < 1 \) such that

\[
\mathcal{M}_k(\mathcal{Y}_G(y), \mathcal{Y}_G(\mathcal{V})) \leq \gamma \|y - \mathcal{V}\|, \quad \text{for each } y, \mathcal{V} \in \mathcal{P}.
\]

Let \( y, \mathcal{V} \in \mathcal{P} \) and \( f_1 \in \mathcal{Y}_G(y) \). Then \( \exists \beta_1(\tau) \in \mathcal{G}(\tau, y(\tau)) \) such that, for each \( \tau \in \mathcal{J} \),

\[
f_1(\tau) = \frac{1}{\Gamma(\zeta)} \int_1^\tau \left( \left( \log \frac{\tau}{\sigma} \right)^{\zeta-1} \beta_1(\sigma) \frac{d\sigma}{\sigma} + \frac{1}{\Gamma(\zeta + \varrho)} \int_1^\varrho \left( \left( \log \frac{\varrho}{\sigma} \right)^{\zeta-1} \beta_1(\sigma) \frac{d\sigma}{\sigma} \right) \right) \right) \right) + \varepsilon \sum_{j=1}^{k-2} v_j \frac{1}{\Gamma(\zeta)} \int_1^{\varphi_j} \left( \left( \log \frac{\varphi_j}{\sigma} \right)^{\zeta-1} \beta_1(\sigma) \frac{d\sigma}{\sigma} + \frac{1}{\Gamma(\zeta - \omega)} \int_1^\omega \left( \left( \log \frac{\omega}{\sigma} \right)^{\zeta-1} \beta_1(\sigma) \frac{d\sigma}{\sigma} \right) \right) \right) \right).

By the assumption (\( \mathcal{E}_7 \)), we have

\[
\mathcal{M}_k(\mathcal{G}(\tau, y), \mathcal{G}(\tau, \mathcal{V})) \leq h(\|y(\tau) - \mathcal{V}(\tau)\|).
\]
So, \( \exists z \in G(\tau, \overline{y}(\tau)) \) such that
\[
|\beta_1(\tau - z)| \leq h(\tau)\|y(\tau) - \overline{y}(\tau)\|, \quad \tau \in \mathcal{J}.
\]

Define \( \mathcal{H} : \mathcal{J} \to \mathcal{Q}(\mathbb{R}) \) by
\[
\mathcal{H}(\tau) = \{ z \in \mathbb{R} : |\beta_1(\tau - z)| \leq h(\tau)\|y(\tau) - \overline{y}(\tau)\| \}.
\]

Because multivalued operator \( \mathcal{H}(\tau) \cap G(\tau, \overline{y}(\tau)) \) is measurable, a function \( \beta_2(\tau) \) exists, which is a measurable selection for \( \mathcal{H} \). So \( \beta_2(\tau) \in G(\tau, \overline{y}(\tau)) \) and \( |\beta_1(\tau) - \beta_2(\tau)| \leq h(\tau)\|y(\tau) - \overline{y}(\tau)\| \) for every \( \tau \in \mathcal{J} \). Defining
\[
f_2(\tau) = \frac{1}{\Gamma(\varsigma)} \int_{1}^{\tau} \left( \log \frac{\tau}{\sigma} \right)^{\varsigma-1} \beta_2(\sigma) \frac{d\sigma}{\sigma} + \frac{(\log \tau)^{\varsigma-1}}{\Lambda} \left[ \nu \frac{1}{\Gamma(\varsigma + \varrho)} \int_{1}^{\varrho} \left( \log \frac{\varrho}{\sigma} \right)^{\varsigma+\varrho-1} \beta_2(\sigma) \frac{d\sigma}{\sigma} \right] + \varepsilon \sum_{j=1}^{k-2} v_j \frac{1}{\Gamma(\varsigma)} \int_{1}^{\varphi_j} \left( \log \frac{\varphi_j}{\sigma} \right)^{\varsigma-1} \beta_2(\sigma) \frac{d\sigma}{\sigma} - \frac{1}{\Gamma(\varsigma - \omega)} \int_{1}^{T} \left( \log \frac{T}{\sigma} \right)^{\varsigma-\omega-1} \beta_2(\sigma) \frac{d\sigma}{\sigma}, \quad \text{for each} \quad \tau \in \mathcal{J}.
\]

Thus,
\[
|f_1(\tau) - f_2(\tau)| = \frac{1}{\Gamma(\varsigma)} \int_{1}^{\tau} \left( \log \frac{\tau}{\sigma} \right)^{\varsigma-1} |\beta_1(\sigma) - \beta_2(\sigma)| \frac{d\sigma}{\sigma} + \frac{(\log \tau)^{\varsigma-1}}{\Lambda} \left[ \nu \frac{1}{\Gamma(\varsigma + \varrho)} \int_{1}^{\varrho} \left( \log \frac{\varrho}{\sigma} \right)^{\varsigma+\varrho-1} |\beta_1(\sigma) - \beta_2(\sigma)| \frac{d\sigma}{\sigma} \right] + \varepsilon \sum_{j=1}^{k-2} v_j \frac{1}{\Gamma(\varsigma)} \int_{1}^{\varphi_j} \left( \log \frac{\varphi_j}{\sigma} \right)^{\varsigma-1} |\beta_1(\sigma) - \beta_2(\sigma)| \frac{d\sigma}{\sigma} + \frac{1}{\Gamma(\varsigma - \omega)} \int_{1}^{T} \left( \log \frac{T}{\sigma} \right)^{\varsigma-\omega-1} |\beta_1(\sigma) - \beta_2(\sigma)| \frac{d\sigma}{\sigma} \right] \leq \|\kappa\| \left\{ \frac{1}{\Gamma(\varsigma)} \int_{1}^{\tau} \left( \log \frac{\tau}{\sigma} \right)^{\varsigma-1} \frac{d\sigma}{\sigma} + \frac{(\log \tau)^{\varsigma-1}}{\Lambda} \left[ \nu \frac{1}{\Gamma(\varsigma + \varrho)} \int_{1}^{\varrho} \left( \log \frac{\varrho}{\sigma} \right)^{\varsigma+\varrho-1} \frac{d\sigma}{\sigma} \right] + \varepsilon \sum_{j=1}^{k-2} v_j \frac{1}{\Gamma(\varsigma)} \int_{1}^{\varphi_j} \left( \log \frac{\varphi_j}{\sigma} \right)^{\varsigma-1} \frac{d\sigma}{\sigma} + \frac{1}{\Gamma(\varsigma - \omega)} \int_{1}^{T} \left( \log \frac{T}{\sigma} \right)^{\varsigma-\omega-1} \frac{d\sigma}{\sigma} \right\} \|y - \overline{y}\| \leq (\|\kappa\|\Delta)\|y - \overline{y}\|.
\]

Hence,
\[
\|f_1 - f_2\| \leq (\|\kappa\|\Delta)\|y - \overline{y}\|.
\]

Therefore, we get
\[
\mathcal{M}_h(\Upsilon_G(y), \Upsilon_G(\overline{y})) \leq (\|\kappa\|\Delta)\|y - \overline{y}\|,
\]
if we interchange the functions of \( y \) and \( \overline{y} \). As \( \Upsilon_G \) is a contraction, Theorem 2.8 follows that \( \Upsilon_G \) has a fixed point \( y \), which is a solution to the problem (2). \( \square \)
4. Examples
Example 4.1. Consider the following problem

\begin{equation}
\mathcal{H}D_{\omega}^\zeta \, y(\tau) = \frac{1}{25} \cdot (1 + \tau^2) y(\tau), \quad \tau \in [1, 2],
\end{equation}

with the boundary conditions

\begin{equation}
y(1) = y'(1) = 0, \quad \mathcal{H}D_{\omega}^\zeta \, y(T) = \nu \mathcal{H}L_{\mu}^\varphi \, y(\varphi) + \varepsilon \sum_{j=1}^{k-2} \nu_j y(\varphi_j).
\end{equation}

Here, \( \zeta = \frac{47}{20}, \ \omega = \frac{27}{20}, \ \varphi = \frac{8}{5}, \ \nu = \frac{1}{40}, \ \varphi = \frac{5}{4}, \ T = 2, \ \nu_1 = \frac{7}{20}, \ \nu_2 = \frac{11}{20}, \ \nu_3 = \frac{3}{4}, \ \varphi_1 = \frac{27}{20}, \ \varphi_2 = \frac{3}{2}, \ \varphi_3 = \frac{7}{4} \).

In addition, we find that

\begin{equation*}
|g(\tau, y(\tau))| = \frac{1}{25} \cdot (1 + \tau^2)|y| \quad \text{as} \quad |g(\tau, p_1(\tau)) - g(\tau, p_2(\tau))| \leq \frac{1}{25} \|p_1 - p_2\|.
\end{equation*}

With these details, we find \( \Lambda \cong 0.628930676188709, \ \Delta \cong 0.7673808510410263. \ Theorem 3.1's\ presumptions were satisfied. Therefore, the problem (11)-(12) has at least one solution on \([1, 2]\).

Example 4.2. Consider the following problem

\begin{equation}
\mathcal{H}D_{\omega}^\zeta \, y(\tau) = \frac{1}{1 + \tau} + \frac{|y(\tau)|}{1 + |y(\tau)|} \cdot \frac{1}{(\tau + 5)^2}, \quad \tau \in [1, 2],
\end{equation}

with the boundary conditions (12). In addition, we find that

\begin{equation*}
|g(\tau, y(\tau))| = \frac{1}{1 + \tau} + \frac{|y|}{1 + |y|} \cdot \frac{1}{(\tau + 5)^2} \quad \text{as} \quad |g(\tau, p_1(\tau)) - g(\tau, p_2(\tau))| \leq \frac{1}{36} \|p_1 - p_2\|.
\end{equation*}

The above details show that \( \Lambda = 1.2458726240293911 \) and \( \Delta = 0.26351630694877304. \ Thus the Theorem 3.2 presumptions are fulfilled. Accordingly, the problem (13) with (12) has a unique solution on \([1, 2]\) by Theorem 3.2.

5. Discussion
We discussed the existence and uniqueness of solutions for HFDEs and inclusions supplemented by non-local discrete and Hadamard integral conditions through Krasnoselskii fixed-point theorem and Banach fixed-point theorem for equations, and Leray-Schauder’s alternative for multivalued maps and fixed-point theorem due to Covitz for inclusions. When we have fixed the parameters involved in the problem \((\nu, \varepsilon) (1)\), our results correspond to certain specific problems.

Suppose that taking \( \nu = 0 \) in the results provided, we are given the problems (1) with the form:

\begin{equation}
\mathcal{H}D^\zeta y(T) = \varepsilon \sum_{j=1}^{k-2} \nu_j y(\varphi_j), \quad \text{while the results are} \quad \mathcal{H}D^\zeta y(T) = \nu \mathcal{H}L^\varphi y(\varphi), \quad \text{followed by} \ \varepsilon = 0.
\end{equation}

Besides, the approach employed in the previous section can solve some problems close to the problem (1). Next, by modifying the condition, we considered two new problems:

\begin{equation}
\mathcal{H}D^\omega y(T) = \nu \mathcal{H}L^\varphi y(\varphi) + \varepsilon \sum_{j=1}^{k-2} \nu_j y(\varphi_j),
\end{equation}

(14)
in problem (1) with

\[ H^\omega \mathcal{D}^\nu y(T) = \nu \sum_{i=1}^{m} H^\omega_i y(\vartheta_i), \]  
\[ (15) \]

\[ H^\omega \mathcal{D}^\epsilon y(T) = \epsilon \sum_{j=1}^{k-2} \nu_j H^\rho_j y(\varphi_j). \]  
\[ (16) \]

Regarding problem (1) with (15) instead of (14), we get operator \( T_1 : \mathcal{P} \rightarrow \mathcal{P} \) defined by

Concerning the problem (1) with (15) instead of (14), we obtain the operator \( T_1 : \mathcal{P} \rightarrow \mathcal{P} \) defined by

\[ (T_1 y)(\tau) = \frac{1}{\Gamma(\varsigma)} \int_1^T \left( \log \frac{\tau}{\sigma} \right)^{\varsigma-1} g(\sigma, y(\sigma)) \frac{d\sigma}{\sigma} \]

\[ + \frac{(\log \tau)^{\varsigma-1}}{\Lambda} \left[ \nu \sum_{i=1}^{m} \frac{1}{\Gamma(\varsigma + \vartheta_i)} \int_1^{\vartheta_i} \left( \log \frac{\vartheta_i}{\sigma} \right)^{\varsigma+\vartheta_i-1} g(\sigma, y(\sigma)) \frac{d\sigma}{\sigma} \right. \]

\[ - \frac{1}{\Gamma(\varsigma - \omega)} \int_1^T \left( \log \frac{T}{\sigma} \right)^{\varsigma-\omega-1} g(\sigma, y(\sigma)) \frac{d\sigma}{\sigma} \],

where

\[ \Lambda = \frac{\Gamma(\varsigma)(\log T)^{\varsigma-\omega-1}}{\Gamma(\varsigma - \omega)} - \nu \sum_{i=1}^{m} \frac{\Gamma(\varsigma)(\log \vartheta_i)^{\varsigma+\vartheta_i-1}}{\Gamma(\varsigma + \vartheta_i)}. \]

Similarly, the problem (1) related to operator \( T_2 : \mathcal{P} \rightarrow \mathcal{P} \) with conditions (16) rather than (14) is

\[ (T_2 y)(\tau) = \frac{1}{\Gamma(\varsigma)} \int_1^T \left( \log \frac{\tau}{\sigma} \right)^{\varsigma-1} g(\sigma, y(\sigma)) \frac{d\sigma}{\sigma} \]

\[ + \frac{(\log \tau)^{\varsigma-1}}{\Lambda} \left[ \epsilon \sum_{j=1}^{k-2} \frac{1}{\Gamma(\varsigma - \rho_j)} \int_1^{\varphi_j} \left( \log \frac{\varphi_j}{\sigma} \right)^{-\rho-1} g(\sigma, y(\sigma)) \frac{d\sigma}{\sigma} \right. \]

\[ - \frac{1}{\Gamma(\varsigma - \omega)} \int_1^T \left( \log \frac{T}{\sigma} \right)^{\varsigma-\omega-1} g(\sigma, y(\sigma)) \frac{d\sigma}{\sigma} \],

where

\[ \Lambda = \frac{\Gamma(\varsigma)(\log T)^{\varsigma-\omega-1}}{\Gamma(\varsigma - \omega)} - \epsilon \sum_{j=1}^{k-2} \frac{\Gamma(\varsigma)(\log \varphi_j)^{-\rho-1}}{\Gamma(\varsigma - \rho)}. \]

The existence and uniqueness of solutions for the new problems can be defined by \( T_1 \) and \( T_2 \) operators similar to those obtained for (1) and (2).

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