Effective Field Theory of Short–Range Forces

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Abstract

The method of effective field theories (EFTs) is developed for the scattering of two particles at wavelengths which are large compared to the range of their interaction. It is shown that the renormalized EFT is equivalent to the effective range expansion, to a Schrödinger equation with a pseudo-potential, and to an energy expansion of a generic boundary condition at the origin. The roles of regulators and potentials are also discussed. These ideas are exemplified in a toy model.

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1 Introduction

Separation of scales is essential in most problems of physics because it allows the selection of relevant degrees of freedom with a perhaps limited set of dominant interactions. Quantitative understanding then eventually requires a systematic treatment of the less relevant interactions, and methods to accomplish this are several. In nuclear physics this problem was faced early on when it was realized that the deuteron is large compared to the range of the nuclear force, and also in the analysis of slow-neutron scattering from bound protons, where the effects of the nuclear force had to be separated from those of the longer-range, electromagnetic interactions. This interest led to the treatment of interactions of short range by the equivalent techniques of boundary conditions [1, 2], pseudo–potentials [3, 2, 4], and the effective range expansion [5]. With time, these techniques came to be replaced by parametrizations of the nuclear force in terms of (single, sometimes double) meson exchange, which were perceived as more fundamental. Meanwhile, the method of effective field theories (EFTs) has been developed in particle physics. The idea is to start from the most general Lagrangian involving the relevant low-energy degrees of freedom and some chosen symmetries, short-range dynamics being described by local interactions with an arbitrary number of derivatives. Under the assumption that these local terms have natural size, a power counting argument shows that there exist an expansion in energy that allows the computation of low-energy observables in terms of parameters that summarize the effect of short-range interactions. This approach is now viewed as the paradigm for understanding the successes of the electroweak theory and of chiral Lagrangians. It is natural to ask what is the connection among these different methods.

In the last few years, the application of EFT to the nuclear force problem [6] has generated a lot of interest (see Ref. [7] for a review and further references). To a large extent, the potential usefulness of the approach lies in the fact that the general chiral Lagrangian allows a clear separation between pion effects constrained by the approximate chiral symmetry of QCD and shorter range effects that cannot at present be calculated from the QCD dynamics. Although a quantitative description of all nucleon-nucleon (NN) channels can be accomplished up to lab energies of about 100 MeV [8] and a qualitative understanding of otherwise mysterious nuclear regularities is possible [3, 11], several questions regarding the scope of such approach remain
unanswered, such as the role of cut-off vs. dimensional regularization, the usefulness of a dibaryon field, and the relation to the effective range expansion [11, 12, 13, 14, 15, 16, 17]. These issues do not typically arise in particle physics where most applications of EFT are to systems where the low-energy domain is perturbative—that is, involves a finite number of diagrams in the effective theory. Most of these issues are in fact more general than the \(NN\) problem itself, being independent of the complications generated by spin-dependent interactions, by several partial waves, or by (long-range) pion exchange. They arise from the necessity of summing up an infinite number of diagrams due to infrared quasi-divergences in any EFT that contains heavy stable particles.

Here I will therefore restrict myself to a system of two identical, stable spinless particles at energies much smaller than their mass, their internal excitation energy, and the range of their interaction. Generalization to spin and non-identical particles is straightforward, and is directly relevant to strong interactions in any nuclear reaction at sufficiently low energies. In the case of \(NN\) scattering, relevance is limited to center-of-mass momenta much smaller than the pion mass. Generalization to a potential with both long- and short-range components—appropriate for example to charged systems and to \(NN\) strong interactions at center-of-mass momenta comparable to the pion mass—will be presented in a separate publication.

I will show in detail how the infinite number of interactions in the EFT can be organized systematically by means of power counting, in any regularization scheme. The power counting itself depends on whether the underlying short-range dynamics is generic, or fine-tuned to display a shallow bound state, or fine-tuned to display a shallow amplitude zero. These three cases are treated in Sect. 3, after some generalities are presented in Sect. 2. We are going to see that the EFT method, when carried out to the end, is equivalent to the (generalized) effective range expansion. The Schrödinger equation that results from renormalization of the EFT problem will be derived in Sect. 4. I will show that it involves a delta-function pseudo-potential of the type previously considered in nuclear physics, or, equivalently, boundary conditions at the origin. I will discuss the relation to the traditional potential approach in Sect. 5. Necessary restrictions on the regularization scheme will be presented for the traditional approach to hold. A toy potential is played with in Sect. 6 to illustrate some of the consequences of power counting, before conclusion in Sect. 7.
Some of these results have been first presented briefly in Ref. [7], and applied to the three-nucleon problem in collaboration with P.F. Bedaque and H.-W. Hammer [18], with considerable success. In the case of a shallow bound state, the power counting has subsequently been re-discovered within specific regularization schemes in Refs. [19, 20], and enlarged to include low-momentum pions in Ref. [19], while the equivalence to the effective range expansion was recently confirmed using renormalization group methods [21]. Some of the implications of the power counting to the potential approach have been checked numerically for a toy underlying theory in Ref. [22].

2 Generalities

Consider the case of particles (described by a field $\psi$) with 3-momenta $Q$ much smaller than their mass $m$, the mass difference $\Delta$ to their first excited state, and the mass $\mu$ of the lightest particle that can be exchanged among them. The EFT appropriate to these small momenta contains only the field $\psi$ as a propagating, non-relativistic degree of freedom. The effect of all other states is to generate structure and interactions of range $\sim 1/\Delta$ or $\sim 1/\mu$, while relativistic corrections introduce further $\sim 1/m$ terms. Low-energy $S$-matrix elements can be obtained from an effective Lagrangian involving arbitrarily complicated operators of only $\psi$ and its derivatives. Naively, we expect these derivatives to be associated with factors of $1/m$, $1/\Delta$, or $1/\mu$ and, therefore, that the effective Lagrangian can be written as an expansion in $\partial/(m, \Delta, \mu)$. More generally, letting $M$ characterize the typical scale of all higher-energy effects, we seek an expansion of observables in powers of $Q/M$.

Because I am restricting myself to soft collisions, the relevant symmetry of the EFT is invariance under Lorentz transformations of small velocity, sometimes referred to as reparametrization invariance [23]. For simplicity, I only consider parity and time-reversal invariant theories. The particles $\psi$ are necessarily non-relativistic, and evolve only forward in time, in first approximation as static objects. Particle number is also conserved. By a convenient choice of fields, the effective Lagrangian can be written as

$$\mathcal{L} = \psi^\dagger \left( i\partial_0 + \frac{1}{2m} \vec{\nabla}^2 + \frac{1}{8m^3} \vec{\nabla}^4 + \ldots \right) \psi - \frac{1}{2} C_0 \psi^\dagger \psi \psi^\dagger \psi$$
\[ \frac{1}{8}(C_2 + C_2')\left[ \psi^\dagger (\vec{\nabla} - \vec{\nabla})\psi \cdot \psi^\dagger (\vec{\nabla} - \vec{\nabla})\psi - \psi^\dagger \psi \psi^\dagger (\vec{\nabla} - \vec{\nabla})^2\psi \right] \]
\[ + \frac{1}{4}(C_2 - C_2')\psi^\dagger \psi \vec{\nabla}^2(\psi^\dagger \psi) + \ldots, \]  

(1)

where the \( C_{2n} \)'s are parameters that depend on the details of the dynamics of range \( \sim 1/M \). I will restrict myself here to four space-time dimensions, although extension to three dimensions—which also has some phenomenological interest—is straightforward. In four space-time dimensions, \( C_{2n} \) has mass dimension \( -2(1 + n) \). The “…” include operators with more derivatives and/or more fields. The latter will not contribute to \( \psi \psi \) scattering.

Canonical quantization leads to familiar Feynman rules. For example, the \( \psi \) propagator at four-momentum \( p \) is given by

\[ S(p^0, \vec{p}) = \frac{i}{p^0 - \frac{\vec{p}^2}{2m} + \frac{\vec{p}^4}{8m^3} + \ldots + i\epsilon} \]  

(2)

and the four-\( \psi \) contact interaction by \(-iv(p, p')\), with

\[ v(p, p') = C_0 + C_2(\vec{p}^2 + \vec{p}'^2) + 2C_2' \vec{p} \cdot \vec{p}' + \ldots, \]  

(3)

\( \vec{p} \) (\( \vec{p}' \)) being the relative momentum of the incoming (outgoing) particles. If Fourier-transformed to coordinate space, these bare interactions consist of delta-functions and their derivatives.

I will concentrate here on the two-particle system at energy \( E = k^2/m - k^4/4m^3 + \ldots \) in the center-of-mass frame. Conservation of particle number reduces the \( T \)-matrix to a sum of bubble diagrams. These bubbles will give contributions containing

\[ \int \frac{dl^0}{2\pi} S \left( \frac{E}{2} + l^0, \vec{p} + \vec{l} \right) S \left( \frac{E}{2} - l^0, -\left( \vec{p} + \vec{l} \right) \right). \]  

(4)

Upon integration over the zeroth component of the loop momentum, the two particles evolve formally according to the familiar non-relativistic Schrödinger propagator

\[ G_0(l; k) = -\frac{m}{l^2 - k^2 - i\epsilon} \]  

(5)

in momentum space, or

\[ G_0(r; k) = -imk \frac{e^{ikr}}{4\pi r} \]  

(6)
in coordinate space. Relativistic corrections can be accounted for by a two-legged vertex

\[ u(l; k) = -\frac{l^4 - k^4}{8m^3} + \ldots \] (7)

Loops are then associated with non-zero integrals of the form

\[ I_{2n}(k) = \int \frac{d^3 l}{(2\pi)^3} l^{2n}G_0(l; k) = -m \int \frac{d^3 l}{(2\pi)^3} \frac{l^{2n}}{l^2 - k^2 - i\epsilon}, \] (8)

with \( n \geq 0 \) an integer that depends on the number of derivatives at the vertices. Such integrals are ultraviolet divergent, which is the field-theoretical translation of the well-known fact that our bare interactions are too singular for propagation with the Schrödinger propagator. The problem therefore requires regularization and renormalization. This introduces a mass scale \( \Lambda \), which might be a cut-off in momentum space or a mass scale generated by extending the dimension of space in Eq. (8) to a sufficiently small \( D - 1 \) such that the integral converges —I will refer to this as a cut-off scale, although I am not assuming a particular regularization scheme. In general, power-like dependence on \( \Lambda \) appears and can be encoded in

\[ L_n \equiv \int dl \ l^{n-1} \equiv \theta_n \Lambda^n, \] (9)

where \( \theta_n \) is a number that depends on the regularization scheme chosen. In dimensional regularization with minimal subtraction, for example, \( \theta_n = 0 \), but in general \( \theta_n \) can be non-vanishing with either sign. The loop integrals are then

\[ I_{2n}(k) = -\frac{m}{2\pi^2} \left[ \sum_{i=0}^{n} k^{2i}L_{2(n-i)+1} + \frac{\pi}{2} k^{2n+1} + \frac{k^{2(n+1)}}{\Lambda} R(k^2/\Lambda^2) \right], \] (10)

where \( R(x) \) is a regularization-scheme-dependent function that approaches a finite limit as \( x \to 0 \).

The severe \( \Lambda \)-dependence comes from the region of high momenta that cannot be described correctly by the EFT; it can be removed by lumping these terms together with the unknown bare parameters into “renormalized” coefficients \( C_{2n}^{(R)} \). Only the latter are observable. The regulator-independent, non-analytic piece in Eq. (10), on the other hand, is characteristic of loops: it cannot be mocked up by re-shuffling contact interactions. From Eq. (10) we see that we can estimate its effects by associating a factor \( mQ/4\pi \) to each loop and a factor \( Q \) to each derivative at the vertices.
3 The amplitude

The goal is to find a power counting scheme that justifies an order by order truncation of the effective theory to a finite number of interactions. The observable results of the EFT should be independent of the choice of $\Lambda$, but only to the accuracy implied by this power counting scheme.

All $\psi\psi$ observables can be obtained from the on-shell $T$-matrix $T_{os}(k, \hat{p}' \cdot \hat{p})$. (Off-shell behavior cannot be separated from three-body force effects, and plays no role in the EFT treatment of the two-body system.) Scattering information is encoded in the phase shifts. In the case of two non-relativistic particles of equal mass the relation between the $l$-wave amplitude $\left(T_{os}(k)\right)_l$ and the phase shift $\delta_l(k)$ is

\[
\frac{4\pi}{mk} \left(1 + \frac{k^2}{2m^2} + O\left(\frac{k^4}{m^4}\right)\right)^{-1} (\cot \delta_l - i)^{-1} = \int_{-1}^{1} dx P_l(x) T_{os}(k, x). \tag{11}
\]

Here $1 + k^2/2m^2 + O(k^4/m^4)$ stems from the small relativistic corrections in the inverse of the density of states. At sufficiently small energy it is customary to Taylor-expand

\[
k^{2l+1} \cot \delta_l = -\frac{1}{a_l} + \frac{r_l}{2} k^2 - P_l r_l^3 k^4 + \ldots, \tag{12}
\]

or alternatively,

\[
k^{-(2l+1)} \tan \delta_l = -a_l \left(1 + \frac{r_l a_l}{2} k^2 - \left(P_l - \frac{a_l}{4 r_l}\right) a_l r_l^3 k^4 + \ldots\right). \tag{13}
\]

This is known as the effective range expansion, $a_l$ being the $l$-wave scattering length, $r_l$ the $l$-wave effective range, $P_l$ the $l$-wave shape parameter, etc. A real (virtual) bound state can arise as a pole of the amplitude \([11]\) at imaginary momentum $k = i\kappa$, $\kappa \geq 0$ ($\leq 0$). If this pole is sufficiently shallow, Eq. (12) gives $\kappa$ in terms of the effective range parameters.

The $T$-matrix is given by the diagrams in Fig. [11]; it is simply a sum of bubble graphs, whose vertices are the four-$\psi$ contact terms that appear in the Lagrangian \([\mathbb{I}1]\). For the first few terms one finds

\[
T_{os}(k, \hat{p}' \cdot \hat{p}) = -C_0[1 + C_0 I_0 + (C_0 I_0)^2 + (C_0 I_0)^3 + 2C_2 I_2 + \ldots]
\]
\[-2C_2k^2[1 + C_0I_0 + \ldots] - C_0^2[-\frac{L_3}{8\pi^2m} + \frac{k^2}{2m^2I_0}]\]
\[-2C'_2k^2\hat{p}' \cdot \hat{p}[1 + \ldots] + \ldots\]
\[\begin{equation}
- C_0^{(R)} \left\{ 1 - \left( \frac{mC_0^{(R)}}{4\pi} \right)^2 \left( \frac{mC_0^{(R)}}{4\pi} \right)^2 \left( \frac{mC_0^{(R)}}{4\pi} \right)^3 \right. \\
+ 2C_2^{(R)}k^2 \left[ 1 - 2 \left( \frac{mC_0^{(R)}}{4\pi} \right)^2 \right] + 2C'_2^{(R)}k^2\hat{p}' \cdot \hat{p} \\
+ \left( \frac{mC_0^{(R)}}{4\pi} \right)^2 \frac{k^2}{2m^2} + \ldots \right\}
\end{equation}\]

Here dependence on the cut-off was eliminated by defining renormalized parameters $C_0^{(R)}$, $C_2^{(R)}$, and $C'_2^{(R)}$ from the bare parameters $C_0(\Lambda)$, $C_2(\Lambda)$, and $C'_2(\Lambda)$:

\[C_0^{(R)} \equiv C_0 \left[ 1 - \frac{mC_0L_1}{2\pi^2} + \left( \frac{mC_0L_1}{2\pi^2} \right)^2 - \left( \frac{mC_0L_1}{2\pi^2} \right)^3 \right. \]
\[\left. - mC_0L_3 \left( \frac{2C_2}{C_0} + \frac{1}{4m^2} \right) + \ldots \right], \quad (15)\]

or

\[\frac{1}{C_0^{(R)}} = \frac{1}{C_0} + \frac{m}{2\pi^2} \left[ L_1 + \left( \frac{2C_2}{C_0} + \frac{1}{4m^2} \right) L_3 \right] + \ldots, \quad (16)\]

plus

\[\frac{C_2^{(R)}}{(C_0^{(R)})^2} \equiv \frac{C_2}{C_0^2} - \frac{m}{4\pi^2} \left( \frac{R(0)}{\Lambda} + \frac{L_1}{2m^2} \right) + \ldots, \quad (17)\]

and

\[C'_2^{(R)} \equiv C'_2 + \ldots \quad (18)\]

Note that I chose to absorb a finite piece $-m(C_0^{(R)})^2 R(0)/4\pi^2\Lambda$ in $C_2^{(R)}$. Other terms coming from $R(k^2/\Lambda^2)$ are $\propto k^4$ or higher powers, and cannot be separated from higher-order contact interactions that I have not written down explicitly.

It is important to realize that Eq. (14) consists of two different expansions: a loop expansion and an expansion in the number of insertions of derivatives at the vertices or particle lines. The derivative expansion depends on the

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The determination of the remaining unrenormalized coupling via the two-loop expansion is discussed in Sec. 4. Here we say only that $C_0(\Lambda)$ and $C_2(\Lambda)$ are renormalized in the way discussed in Sec. 3, while $C'_2(\Lambda)$ is kept bare. The connection between the bare and renormalized couplings is provided by the formulas in Eqs. (15) and (18).
Figure 1: First few terms of the two-particle amplitude $T$ in a natural EFT. Two solid lines represent a Schrödinger propagator; a circle on a line represents a $Q^4$ relativistic correction; a heavy dot stands for a $Q^0$ contact interaction, and a dot within a circle for a $Q^2$ contact interaction.

relative size of the coefficients $C^{(R)}_{2n}$, for example $C^{(R)}_2 Q^2 / C^{(R)}_0$. The loop expansion is governed by $mC^{(R)}_0 / 4\pi$. The sizes of terms in both expansions thus depend on the absolute size of the $C^{(R)}_{2n}$’s. Three different cases will be considered in turn.

### 3.1 Natural EFT

Let us consider first the simplest type of underlying theory: one that is natural. This is a theory with a single mass scale $M$ and no fine-tuning. We expect all parameters to scale with $M$ according to their mass dimension. It is convenient, however, to factor in a $4\pi/m$: I write $C^{(\ell)}_{2n} = 4\pi \gamma^{(\ell)}_{2n} / m M^{2n+1}$ with $\gamma^{(\ell)}_{2n}$ dimensionless parameters of $O(1)$.

In this case, the derivative expansion is in $C^{(R)}_2 Q^2 / C^{(R)}_0 \sim (Q/M)^2$, while the loop expansion is in $mC^{(R)}_0 / 4\pi \sim Q/M$. The $T$-matrix in Eq. (14) is
thus a simple expansion in powers of $Q/M$.

We can complete the renormalization procedure by relating the parameters of the EFT to observables. Up to higher-order terms, Eq. (14) is equivalent to

$$T_{os}(k, \hat{p}' \cdot \hat{p}) = (T^{(0)}_{os}(k))_0 - 2C_2^{(R)}k^2\hat{p}' \cdot \hat{p} + O((4\pi/mM)(Q/M)^4),$$

(19)

where

$$(T^{(0)}_{os}(k))_0 = -\left[\frac{1}{C_0^{(R)}} - 2\frac{C_2^{(R)}}{C_0^{(R)}}k^2 + \frac{imk}{4\pi} \left(1 + \frac{k^2}{2m^2}\right)^{-1} \left(1 + O((Q/M)^4)\right)\right].$$

(20)

This is in the form of an effective range expansion in each partial wave. (Note in particular that the (non-analytic) relativistic corrections come out in agreement with Eq. (11), as they should.) In the $S$-wave we obtain a scattering length

$$a_0 = \frac{mC_0^{(R)}}{4\pi},$$

(21)

and an effective range

$$r_0 = \frac{16\pi}{mC_0^{(R)}} \left(\frac{C_2^{(R)}}{C_0^{(R)}} + \frac{1}{4m^2}\right).$$

(22)

In the $P$-wave, we find a scattering volume

$$a_1 = \frac{mC_2^{(R)}}{6\pi},$$

(23)

Higher moments in Eq. (12) can likewise be obtained from higher-order terms in the $Q/M$ expansion of the amplitude. For example, the shape parameter is

$$P_0r_0^3 = \frac{16\pi}{mC_0^{(R)}} \left(\frac{C_2^{(R)}}{C_0^{(R)}}\right)^2 - \frac{C_4^{(R)}}{C_0^{(R)}} + \frac{C_2^{(R)}}{4m^2C_0^{(R)}} + \frac{3}{32m^4},$$

(24)

where $C_4$ represents a certain combination of $Q^4$ contact interactions. We see that an $l$-wave effective range parameter of mass dimension $\delta$ is $O(1/(2l + 1)M^\delta)$. This scaling of effective range parameters is indeed what
one gets in simple potential models, like a square well of range $R \sim 1/M$ and depth $V_0 \sim M$, as we will see in Sect. 6. It is a simple manifestation of the existence of a single mass scale $M$. The expansion parameter $Q/M$ can then be written alternatively as $Qa_0 \sim Qr_0 \sim \ldots$ Note, moreover, that unitarity effects in an $l$-wave are further suppressed by a factor of $2l + 1$, being $\sim Q^{2l+1}a_l \sim Q^{2l+1}/(2l + 1)M^{2l+1}$.

The assumption of naturalness thus implies a perturbative amplitude. Since terms in the $Q/M$ expansion of the amplitude are in correspondence to terms in the derivative expansion of the effective Lagrangian, the accuracy of description of low-energy data can be improved systematically by considering higher-order terms in the Lagrangian (1). Not surprisingly, the EFT here is quite boring: it can only describe scattering; bound states, if they exist, have typical momenta $\kappa \sim M$, and are outside the region of validity of the expansion (14).

3.2 An unnatural EFT

In nuclear or molecular physics, we are also interested in situations where there are shallow bound states. A (real or virtual) bound state appears at threshold when a parameter $\alpha$ of the underlying theory takes a critical value $\alpha_c$. A shallow bound state means that $\alpha$ is close to $\alpha_c$. That is, the underlying theory is so fine-tuned that it has two distinct scales: the obvious $M$ and another $\Re = |\alpha/\alpha_c - 1|M \ll M$ induced by the fine-tuning. I will consider the case of an $S$-wave shallow bound state by taking $C^{(R)}_{2n} = 4\pi\gamma_{2n}/m\Re(M\Re)^n$ and $C'_{2n}^{(R)} = 4\pi\gamma'_{2n}/mM^{2n+1}$ with $\gamma_{2n}^{(i)}$ dimensionless parameters of $O(1)$. This course recovers the previous, natural scenario when $\alpha$ is tuned out of $\alpha_c$, and $\Re$ becomes comparable to $M$.

Now one can see that the terms in $C^{(R)}_0$ in Eq. (14) are the dominant ones at each order in $k$. For example, the two terms contributing to the coefficient of $k^2$, one from the loop expansion, the other from the derivative expansion, are order $1/\Re^2$ and $1/\Re M$, respectively. The loop and derivative expansions contained in Eq. (14) are still in the same combination of parameters as before. Now, however, $mQC^{(R)}_0/4\pi \sim Q/\Re$, while $C^{(R)}_2/Q^2/C^{(R)}_0 \sim Q^2/\Re M$.

For $Q \ll \Re$, the theory is still perturbative, now in $Q/\Re$. The situation is similar to a natural theory, except that perturbation theory breaks down at the smaller scale $\Re$, much before the scale $M$ associated with short-range
As \( Q \) becomes comparable with \( \aleph \), the most important terms come from the loop expansion. The great advantage of short-range interactions is that we can perform a summation of these terms analytically. The main contribution to the bubble comes from the Schrödinger propagator \( G_0 \) with the \( Q^0 \) term in the vertex \( v(p,p') \). The bubbles summing to a geometric series, one gets as the leading order amplitude in Fig. 2

\[
T_{os}^{(0)}(k) = -\left( \frac{1}{C_0} - I_0(k) \right)^{-1},
\]

which is of \( O(4\pi/m(\aleph + Q)) \).

Higher-order terms in the derivative expansion can now be accounted for perturbatively —see Fig. 3. The first order correction comes from one insertion of the \( C_2 \) term in the vertex. We find

\[
T_{os}^{(1)}(k) = -2\frac{C_2}{C_0^2} \left( k^2 - \frac{m}{2\pi^2} L_3 \right) (T_{os}^{(0)}(k))^2,
\]

which is of \( O(Q^2/M(\aleph + Q)) \) relative to \( T_{os}^{(0)}(k) \) in Eq. (23). A similar procedure can be followed in higher orders. For example, the next corrections come in the \( S \)-wave. There are two corrections which are nominally of \( O(Q^4/M^2(\aleph + Q)) \) relative to leading: (i) one insertion of a combination —denoted by \( C_4 \)— of \( Q^4 \) terms in the Lagrangian (1); (ii) two insertions of \( C_2 \). There is also one insertion of the relativistic correction in the propagator, which is of \( O(Q^3/m^2(\aleph + Q)) \). The sum of these three corrections I call \( T_{os}^{(2)}(k) \). The first \( P \)-wave contribution, one insertion of \( C'_2 \), starts at \( O(Q^2(\aleph + Q)/M^3) \):

\[
T_{os}^{(3p)}(k, \hat{p}' \cdot \hat{p}) = -2C'_2k^2\hat{p}' \cdot \hat{p}.
\]

The amplitude is

\[
T_{os}(k, \hat{p}' \cdot \hat{p}) = T_{os}^{(0)}(k) + T_{os}^{(1)}(k) + T_{os}^{(2)}(k) + \ldots + T_{os}^{(3p)}(k, \hat{p}' \cdot \hat{p}) + \ldots
\]

\[
= (T_{os}^{(0)}(k))_0 - 2C'_2k^2\hat{p}' \cdot \hat{p} + O((4\pi/m\aleph)(\aleph/M)^4); \tag{28}
\]

the \( S \)-wave component can be rewritten, up to higher order terms, as

\[
(T_{os}(k))_0 = [(T_{os}^{(0)}(k))^{-1} + T_{os}^{(1)}(k)(T_{os}^{(0)}(k))^{-2} + T_{os}^{(2)}(k)(T_{os}^{(0)}(k))^{-2} + \ldots]^{-1}
\]
As before, cut-off dependence of integrals and bare parameters was lumped together in renormalized parameters. Here for simplicity I took $Q \sim \aleph$ in displaying higher orders. This amplitude is nominally correct including $O((4\pi/m\aleph)(\aleph/M)^2)$ only. I will argue soon that it is actually correct includ-
By re-summing the largest terms, we have produced a new expansion where the leading amplitude is of $O(4\pi/m(\mathcal{R} + Q))$ and corrections go as $Q^2/M(\mathcal{R} + Q)$ or similar combinations. For $Q \ll \mathcal{R}$, we regain the perturbative expansion. The difference is that for $Q \sim \mathcal{R}$, although all the terms in the loop expansion are of the same order $(O(4\pi/m\mathcal{R}))$, corrections from derivatives go in relative powers of $\mathcal{R}/M \ll 1$ and do not have to be included all at once: they can be accounted for systematically, at each order in $\mathcal{R}/M$. As we consider larger $Q$, $Q \gtrsim \mathcal{R}$, the leading terms become $O(4\pi/mQ)$ and corrections get relatively more important, growing in powers of $Q/M$. The new expansion fails only at momenta $\sim M$, as desired.

What has been achieved that is new compared to the natural theory is that the low-energy $S$-wave bound state can arise as a pole of the amplitude within the range of validity of the EFT, at $k = i\kappa$,

$$\kappa = \frac{4\pi}{mC_0^{(R)}} \left( 1 + 2\frac{C_2^{(R)}}{C_0^{(R)}}\kappa^2 + O(\kappa^3) \right)$$

$$= \frac{4\pi}{mC_0^{(R)}} \left( 1 + 2\frac{C_2^{(R)}}{C_0^{(R)}} \left( \frac{4\pi}{mC_0^{(R)}} \right)^2 + O((\mathcal{R}/M)^2) \right),$$

that is, $\kappa \sim \mathcal{R}$. $C_0^{(R)} > 0$ ($< 0$) implies $\kappa > 0$ ($< 0$) and represents a real (virtual) bound state. (It can be checked easily that the residue of $i$ times the $S$-matrix at this pole is indeed positive (negative) if $\kappa > 0$ ($< 0$).) The binding energy is $B = \kappa^2/m + O(\mathcal{R}^4/M^3)$, which to this accuracy is, of course, just the usual effective range relation among $B$, $a_0$ and $r_0$. It is clear that the bound state can be treated in a systematic expansion in $\mathcal{R}/M$.

Now, Eq. (29) has the same structure as Eq. (21), the difference resting on the relative orders of the various terms. There is no change in the formulas for $l(> 0)$-wave parameters —such as Eq. (23) for the $P$-wave scattering volume—and they scale with $M$ as in the natural scenario. The $S$-wave is trickier. The scattering length $a_0$ is given by the same Eq. (24) as before, but it now scales with $\mathcal{R}$ rather than $M$, $a_0 = O(\mathcal{R}^{-1})$. The formula for the effective range $r_0$ is still Eq. (22), but the two terms come at different orders. The main contribution originates in the contact interactions and scales with $M$, so that $r_0 = O(M^{-1})$. The relativistic correction, on the other hand, is smaller by $\sim M\mathcal{R}/m^2 \sim \mathcal{R}/M$; this is what justifies the neglect of relativistic
corrections in many situations and the usefulness of a non-relativistic framework in shallow-bound-state problems. An underlying theory consisting of a non-relativistic potential can thus serve as a test for the scalings of $a_0$ and $r_0$. Indeed, it has been explicitly shown [25] that if the underlying theory consists of a non-relativistic potential of strength $\alpha$ and range $1/M$, then near critical binding effective range parameters behave precisely in way derived above. A square-well example will be shown in Sect. 6.

As for the shape parameter $P_0$ of Eq. (24), to the order we have worked so far we recover only the interaction pieces:

$$P_0 = \left(\frac{\gamma_2}{\gamma_0}\right)^2 - \frac{\gamma_4}{\gamma_0} + O(\frac{1}{M^3}).$$

The problem is, such an $M/\pi$ behavior is seen neither in phenomenological analyses nor in models. This implies that $\gamma_4$ is $O(1)$ alright, but must be precisely $\gamma_4 = \frac{\gamma_2^2}{\gamma_0} + O(\frac{1}{M})$ so that $P_0$ be $O(1/16)$. In other words, although each interaction term in Eq. (24) comes at relative $O(\frac{1}{M^2})$ in the amplitude, they are correlated and their sum is only of relative $O(\frac{1}{M^3})$.

We conclude that the shape parameter appears at the same order as the first $P$-wave contribution for $Q \gtrsim \pi$. Eqs. (29) and (30) are therefore good up to order $O(\frac{1}{M^3})$, as advertised. For $Q \sim \pi$, this means that the first three orders of the expansion of the amplitude $-O(\frac{4\pi}{M})$, $O(\frac{4\pi}{mM})$, and $O(\frac{4\pi}{mM^2})$, depicted in Fig. 2 — are pure $S$-wave and given solely by $C_0^{(R)}$ and $C_2^{(R)}$, which can be determined from $a_0$ and $r_0$. We can write

$$T_{os}(k) = -\left[\frac{1}{C_0^{(R)}} - 2\frac{C_2^{(R)}}{(C_0^{(R)})^2}k^2 + \frac{imk}{4\pi} \left(1 + \frac{k^2}{2m^2}\right)\right]^{-1} \left(1 + O(\frac{\pi^3}{M^3})\right).$$

(31)

It is easy to verify that, in order to avoid $M/\pi$ enhancements in higher effective range parameters, there must be such correlations in higher order coefficients as well: $\gamma_{2n} = \frac{\gamma_2^n}{\gamma_0^n} + O(\frac{1}{M})$, or $C_{2n}^{(R)} = C_0^{(R)}(C_2^{(R)})^n + O(1)$, or $C_{2n}^{(R)} = C_0^{(R)}(C_2^{(R)})^n + O(1)$. Although not necessary, it is possible to reorganize the EFT in order to account explicitly for the dominant correlations. Because they form a geometric series in $C_2^{(R)}/C_0^{(R)}$, we can sum the dominant correlation terms as we have done above with the $C_0^{(R)}$ contributions themselves. This procedure generates a four-$\psi$ interaction of the type $C_0^{(R)}(1 - 2C_2^{(R)}k^2/C_0^{(R)})$; Eq. (31) follows immediately.

Now, this new re-summation resembles the exchange of an $s$-channel par-
ticle, $\sigma(g^{(R)})^2/(E - \Delta^{(R)})$ at center-of-mass energy $E = k^2/m$, provided (i) the sign $\sigma$ of its kinetic term and its coupling $g^{(R)}$ to $\psi$ satisfy

$$\sigma(g^{(R)})^2 = - \frac{(C_0^{(R)})^2}{2mC_2^{(R)}},$$

that is $(g^{(R)})^2/2\pi = O(M/m^2)$; and (ii) its mass be $2m$ plus

$$\Delta^{(R)} = \frac{C_0^{(R)}}{2mC_2^{(R)}},$$

that is, $\Delta^{(R)} = O(M\mathcal{R}/2m)$. This justifies the suggestion by Kaplan [12] that one can use a new EFT which involves, besides particles, also a dibaryon-baryon field with the quantum numbers of the shallow (real or virtual) bound state. In the case under consideration, we are talking a scalar field, which I denote $T$. The leading terms in the most general Lagrangian consistent with the same symmetries as before are

$$\mathcal{L} = \psi^\dagger \left( i\partial_0 + \frac{1}{2m} \vec{\nabla}^2 + \frac{1}{8m^3} \vec{\nabla}^4 + \ldots \right) \psi$$

$$+ \sigma T^\dagger \left( i\partial_0 + \frac{1}{4m} \vec{\nabla}^2 - \Delta + \ldots \right) T - \frac{g}{\sqrt{2}} (T^\dagger \psi \psi + \text{h.c.}) + \ldots$$

(34)

Note the sign of the kinetic term: the bare $T$ is a ghost (normal) field if $C_2^{(R)} > 0 \ (< 0)$, but this is not a problem because this field does not correspond to an asymptotic state. “...” stand for terms with more derivatives, which are suppressed by powers of $\mathcal{R}/M$. Effects of non-derivative four-$\psi$ contact term can be absorbed in $g^2/\Delta$ and terms with more derivatives.

The coupling to two-particle states dresses the dibaryon propagator. The dressed propagator contains bubbles as in Fig. 3 plus insertions of relativistic corrections. This amounts to a self-energy contribution proportional to the bubble integral. As we know, this integral is ultraviolet divergent and requires renormalization of the parameters of the Lagrangian (34). Relativistic corrections can be accounted for as in the EFT without the dibaryon. The dibaryon propagator in its center-of-mass ($p^0 = k^2/m - k^4/4m^3 + \ldots$) has the form

$$S(p^0, \vec{0}) = \sigma \frac{i}{p^0 - \Delta - \sigma g^2 I_0(\sqrt{mp})}$$
Figure 3: The dressed dibaryon propagator: a bar is the bare dibaryon propagator, while the bubbles represent two-particle propagation.

\[ = \sigma \frac{(g^{(R)})^2}{g^2} \frac{i}{k^2/m - \Delta^{(R)} + \frac{\sigma m (g^{(R)})^2}{4\pi} i k (1 + \frac{k^2}{2m^2}) + \ldots + i\epsilon}, \quad (35) \]

where

\[ \frac{\Delta^{(R)}}{(g^{(R)})^2} = \Delta - \frac{m}{2\pi^2} \left( L_1 + \frac{L_3}{4m^2} \right) + \ldots \quad (36) \]

and

\[ \frac{1}{(g^{(R)})^2} = \frac{1}{g^2} + \frac{m^2}{2\pi^2} \left( \frac{R(0)}{\Lambda} + \frac{L_1}{2m^2} \right) + \ldots \quad (37) \]

In leading orders, the amplitude in this EFT is —see Fig. 4—

\[ T_{\alpha\sigma}(0) = ig^2 S(p^0, \vec{0}) \]

\[ = - \sigma \frac{\Delta^{(R)}}{(g^{(R)})^2} + \frac{\sigma}{m (g^{(R)})^2} k^2 + \frac{i m k}{4\pi} \left( 1 + \frac{k^2}{2m^2} \right)^{-1} (1 + \ldots). \quad (38) \]

The result is identical to Eq. (31), as it should. This is a more direct way to re-derive the effective range expansion, but proceeding this way mixes different orders in the \( \mathcal{N}/M \) expansion. The non-relativistic dibaryon includes the first two orders in \( \mathcal{N}/M \) correctly; the addition of relativistic corrections regains the first three orders. In either case a number of higher-order terms is included as well, but this is irrelevant for the expansion is in a small parameter. Corrections stemming from higher-derivative operators can be

Figure 4: The two-particle amplitude \( T \) in the EFT with a dibaryon field.
accounted for perturbatively, analogously to what we did in Eq. (29). They will contribute to the $\psi\psi$ amplitude starting at relative $O((\mathcal{N}/M)^3)$.

The same real or virtual bound state as before exists near zero energy. The field $T$ corresponds to a light (ghost or normal) state of mass $\Delta^{(R)} \sim \mathcal{N}$ relative to $2m$. It serves to introduce through its bare propagator two bare poles at the $k = \pm \sqrt{m\Delta}$. (In dimensional regularization with minimal subtraction, for example, these bare poles are on the real (imaginary) axis for $a_0 r_0 > 0$ ($< 0$), at the geometric mean between the two scales $\mathcal{N}$ and $M$.) This is an undesired pole structure, but fortunately it changes character as the amplitude gets dressed. Upon dressing, the poles move to the imaginary axis. One ends up at the position of the shallow bound state,

$$\kappa_s = \kappa = -\sigma \frac{4\pi \Delta^{(R)}}{m(g^{(R)})^2} \left( 1 + \frac{1}{m\Delta^{(R)}} \kappa^2 + O(\kappa^3) \right)$$

(39)

that is, $\kappa_s = O(\mathcal{N})$; $\kappa_s > 0$ ($< 0$) for $a_0 > 0$ ($< 0$). The other pole ends up at

$$\kappa_d = -\sigma \frac{m^2(g^{(R)})^2}{4\pi} \left( 1 + \frac{\sigma}{m\Delta^{(R)}} \frac{4\pi \Delta^{(R)}}{m(g^{(R)})^2} \right)^2 + O(\mathcal{N}/M)^2) \right)$$

(40)

that is, $\kappa_d = O(M)$; $\kappa_d > 0$ ($< 0$) for $r_0 > 0$ ($< 0$).

Note that the propagator $S$ of Eq. (33) is not simply a normalized particle propagator. The sign of the residue of $-iS$ at each of the poles depends on the signs of $a_0$, $r_0$, and $(g^{(R)})^2/g^2$. The latter depends on the regularization scheme, but this dependence of course disappears from the sign of the residue of $i$ times the $S$-matrix. One finds that the latter is positive (negative) at $k = i\kappa_s$ for $a_0 > 0$ ($< 0$), and negative (positive) at $k = i\kappa_d$ for $r_0 > 0$ ($< 0$). So the shallow pole corresponds to a regular (composite) particle, while the deep pole corresponds to a ghost. In any case, the deep bound state is outside the domain of the EFT, while the physics of the shallow bound state is correctly described by the EFT.

The field $T$ can be used to describe the shallow bound state in processes other than $\psi\psi$ scattering, if one keeps in mind that the residue of $-iS$ at
the pole is not 1, but \((g(R)^2/g^2)(\sigma/(1 + \sigma m^2 g^2/8\pi \sqrt{mB_s}))\). Despite the resemblance, this is not an EFT with an “elementary” field for the bound state. The dibaryon field shows structure at \(Q \gtrsim \aleph\), which arises from \(\psi\) loops. At \(Q \ll \aleph\), the EFT (with or without dibaryon) can be matched onto a lower-energy EFT containing a “heavy” field for the bound state which does not couple to \(\psi\psi\) states. This can be seen by expanding \(S\) for \(p^0 = -B_s + \delta p^0\), in which case

\[
S(p^0, \vec{p}) \rightarrow \frac{iZ}{\delta p^0 - \frac{\vec{p}^2}{2(2m - B_s)} + i\epsilon}
\]

with a wave function normalization

\[
Z = -2\sigma \frac{(g(R)^2)}{g^2} \frac{B_s}{B_d}
\]

up to terms suppressed by \(Q/\aleph\).

### 3.3 Another unnatural EFT

Without fine-tuning, the scattering length comes out of natural size. In the previous subsection we saw how, adjusting the potential to produce a shallow bound state, the scattering length results big. Let us consider now the remaining case: an unnaturally small scattering length. This happens when a parameter \(\alpha\) of the underlying theory is very close to the value \(\alpha'_c\) that generates a zero of the amplitude right at threshold. Here again, we are dealing with two scales: the natural scale \(M\) and a small scale \(\aleph' = |\alpha/\alpha'_c - 1|M \ll M\); it is convenient to introduce an alternative small scale \(\Omega = \sqrt{|\alpha/\alpha'_c - 1|M} \ll M\). I will consider the case of an \(S\)-wave shallow zero by taking \(C_0^{(R)} = 4\pi \gamma_0 \aleph'/mM^2 = 4\pi \gamma_0 \Omega^2/mM^3\), and \(C_{2n}^{(R)} = 4\pi \gamma_{2n}^{(R)}/mM^{2n+1}\) for \(n \geq 1\), with \(\gamma_{2n}^{(R)}\) dimensionless parameters of \(O(1)\). This of course recovers the natural scenario when \(\alpha\) is tuned out of \(\alpha'_c\), and \(\Omega\) becomes comparable to \(M\).

In this case, the loop expansion in Eq. (14) is in \(mQC_0^{(R)}/4\pi \sim \aleph' M^2 = (Q/\Omega)(\Omega/M)^3\). The derivative expansion in other channels behaves as in the natural case, but in the \(S\)-channel it is (again) less trivial than in the natural scenario: \(C_{2n+1}^{(R)} Q^2/C_{2n}^{(R)} \sim Q^2/M^2\) for \(n \geq 1\), yet \(C_2^{(R)} Q^2/C_0^{(R)} \sim Q^2/\aleph'M = (Q/\Omega)^2\).
For $Q \ll \Omega$ both expansions are still perturbative: the leading order is
one $C_0$ contact interaction, with corrections at $(Q/\Omega)(\Omega/M)^3$ from two $C_0$’s,
at $(Q/\Omega)^2$ from $C_2$ and $C'_2$, and so on. This perturbative expansion is similar
to the natural case.

When $Q \sim \Omega$, however, the $C_2$ and $C'_2$ terms become comparable to the
$C_0$ term. The leading order, shown in Fig. 5, is then simply

$$T_{os}^{(0)}(k, \hat{p}' \cdot \hat{p}) = - \left( C_0 + 2C_2 k^2 + 2C'_2 k^2 \hat{p}' \cdot \hat{p} \right), \quad (43)$$

which is of $O((4\pi \Omega^2/\pi M^3)(1 + (Q/\Omega)^2))$.

Further derivative terms come in powers of $\pi' / M = (\Omega / M)^2$, so that the
first corrections come from four-derivative contact terms: $C_4$, and other $P$-
and $D$-wave terms. Waves higher than $S$ again behave as in the natural case,
so they are not very interesting. I write

$$T_{os}(k, \hat{p}' \cdot \hat{p}) = T_{os}^{(0)}(k, \hat{p}' \cdot \hat{p}) + \ldots$$

$$= (T_{os}(k))_0 - 2C_2 k^2 \hat{p}' \cdot \hat{p} + O((4\pi \Omega^2/\pi M^3)(\Omega/M)^2), \quad (44)$$

and focus on the $S$-wave component $(T_{os}(k))_0$.

The first $S$-wave correction is just

$$T_{os}^{(2)}(k) = -4C_4 k^4, \quad (45)$$

which is of $O((Q/\Omega)^2(\Omega/M)^3)$ relative to $T_{os}^{(0)}(k)$ in Eq. (43). The second
corrections come from the one-loop graphs with $C_0$ and $C_2$ vertices,

$$T_{os}^{(3)}(k) = - \left[ (C_0 + C_2 k^2)I_0(k) + 2C_2 (C_0 + C_2 k^2)I_2(k) + C_2^2 I_4(k) \right], \quad (46)$$

which are of $O((Q/\Omega)(\Omega/M)^3(1 + (Q/\Omega)^2 + (Q/\Omega)^4)$ relative to $T_{os}^{(0)}(k)$ in
Eq. (43). And so on. See Fig. 5.

The $S$-wave amplitude is

$$(T_{os}(k))_0 = T_{os}^{(0)}(k) + T_{os}^{(2)}(k) + T_{os}^{(3)}(k) + \ldots$$

$$= - \left[ C_0^{(R)} + 2C_2^{(R)} k^2 + 4C_4^{(R)} k^4 - \frac{imk}{4\pi} (C_0^{(R)} + 2C_2^{(R)} k^2)^2 + \ldots \right]$$

$$= - \left[ C_0^{(R)} + 2C_2^{(R)} k^2 \left( 1 - \frac{4C_4^{(R)} k^4}{C_0^{(R)} + 2C_2^{(R)} k^2} \right) + \frac{imk}{4\pi} \right]^{-1}$$
Figure 5: First three orders of the two-particle amplitude $T$ in an EFT with a shallow zero. Two solid lines represent a Schrödinger propagator; a heavy dot stands for a $Q^0$ contact interaction, a dot within a circle for a $Q^2$ contact interaction, and a dot within two circles for a $Q^4$ contact interaction.

\[ T^{(0)} = \quad + \quad T^{(2)} = \]

\[ T^{(1)} = \]

\[
\begin{align*}
T^{(0)} &= \left(1 + O((\Omega/M)^4)\right) \\
&= -\left[\frac{1}{C_0^{(R)} + 2C_2^{(R)}k^2 + 4C_4^{(R)}k^4} + \frac{imk}{4\pi}\right]^{-1} \left(1 + O((\Omega/M)^4)\right).
\end{align*}
\]

(47)

As before, cut-off dependence of integrals and parameters was lumped together in renormalized parameters. For simplicity in writing the higher-order terms, I here took $Q \sim \Omega$.

One can see that Eq. (47) is equivalent to a power expansion of $k^{-1}\tan\delta$, and comparison with Eq. (13) shows that the effective range parameters have the same expressions as in Eqs. (21)–(24). There is no change in the scaling of $l (> 0)$-wave parameters with $M$, but the $S$-wave is once again modified. The scattering length $a_0$ now scales with $M^3/\Omega^2$ rather than $M$, $a_0 = O(\Omega^2/M^3)$, the effective range $r_0$ scales with $\Omega^4/M^3$ rather than $M$, $r_0 = O(M^3/\Omega^4)$, the shape parameter $P_{0r_0^3}$ with $\Omega^6/M^3$ rather than $M^3$, $P_0 = O(\Omega^6/M^6)$, and so on. That this is the correct scaling of the effective
range parameters will be shown in the case of the square-well potential in Sect. 6.

Alternatively, we can rewrite Eq. (47) as

\[
(T_{os}(k))_0 = \frac{4\pi/m}{R - 1 + Pk^2 - A + \ldots - ik}
\]

where each of the parameters \(R, P, A, \ldots\) has an expansion in powers of \((\Omega/M)^2\):

\[
R = \frac{4\pi}{mC_0^{(R)}} \left( 1 + \frac{C_0^{(R)}C_4^{(R)}}{(C_2^{(R)})^2} + \ldots \right),
\]

\[
P = \frac{2C_2^{(R)}}{C_0^{(R)}} \left( 1 - \frac{C_0^{(R)}C_4^{(R)}}{(C_2^{(R)})^2} + \ldots \right),
\]

\[
A = -\frac{4\pi}{mC_0^{(R)}} \frac{C_0^{(R)}C_4^{(R)}}{(C_2^{(R)})^2} (1 + \ldots),
\]

\[
\ldots
\]

They are \(R \sim M^3/\Omega^2 \gg M, P \sim 1/\Omega^2 \gg 1/M^2, A \sim M, \ldots\) Clearly, the effect of this kind of fine-tuning is to generate a shallow pole in \(k\cot\delta\), at \(k = O(\Omega)\). This not only forces the amplitude to be small right at threshold — which translates into small \(a_0\) — but also imparts huge energy dependence to the amplitude near threshold — which translates into large \(r_0\). The usual effective range expansion is confined to very small momenta \(Q \ll \Omega\). Eq. (49) generalizes this expansion to momenta which include the position of the pole.

As \(Q\) increases, \(Q \gtrsim \Omega\), the \(C_2\) term becomes dominant, and the most important corrections come from repeated insertions of \(C_2\), generating powers of \(Q/M\). The EFT expansion becomes progressively worse until it breaks down at \(M\).

**3.4 Moral**

In summary, I have shown in the subsections above that, in the case of short-range interactions, the EFT approach, when applied consistently, is
completely equivalent to the (generalized) effective range expansion. The details of how this works depend on the power counting, which in turn depends on whether we are considering a straightforward natural underlying theory, an unnatural theory with a shallow pole, or an unnatural theory with a shallow zero. In any case, if the EFT parameters are known from the underlying, more fundamental theory, then the effective range parameters can be predicted; otherwise, the EFT parameters can be determined by fitting $\psi\psi$ scattering data with the effective range expansion. We will consider a toy example in Sect. 6. Before that, we will explore the connection with the Schrödinger equation.

4 Pseudo-potential and boundary conditions

The system we have been considering —that of two heavy, stable particles with short-range interactions— is traditionally dealt with in the framework of non-relativistic quantum mechanics, where a solution is attained by solving the Schrödinger equation with a potential believed to describe the short-range dynamics —a model. If no approximations are used in the solution, this procedure is equivalent to obtaining the full scattering amplitude by iteration of the potential to all orders. The potential is naturally defined as the sum of diagrams which do not contain $\psi$ poles. For short-range forces, the bare potential coincides —apart from a phase— with the vertex (3). An expansion of the bare potential is essentially an expansion of the Lagrangian itself. I have shown that the leading part of the two-particle amplitude can be obtained from the leading term(s) of the bare potential (3) and from the Schrödinger propagator (5), followed by renormalization. Renormalization is the price paid for the model-independence of the EFT approach.

One might wonder what (if any) Schrödinger equation is generated by the renormalization procedure. This is particularly relevant because in Ref. [13] it was argued —following Wigner [26]— that $r_0$ cannot be positive if we take a sequence of ever shorter-range, (possibly non-local) hermitian potentials in a coordinate space Schrödinger equation. This procedure can be thought of as a particular regularization of the Schrödinger equation with a delta-function potential, followed by removal of the regulator. This suggests that the Schrödinger equation for the “renormalized potential” (if it exists) violates the assumptions made in Ref. [13] about the nature of the potential.
I am going to show now that the “renormalized” Schrödinger equation — i.e. the equation in terms of the observable, renormalized parameters — contains the pseudo-potential discovered long ago in Refs. [1, 2, 4]. This, in turn, is equivalent to a normal Schrödinger equation with unusual boundary conditions at the origin [3, 2].

I have shown in Sect. 3 that what is to be taken as the leading part of the potential and whether this leading piece has to be iterated to all orders depend on fine-tuning (or lack thereof) in the underlying theory. For simplicity in this section I will (i) leave implicit which terms should be iterated to all orders, and which ones can be treated perturbatively; and (ii) neglect relativistic corrections, which we have seen are less important than the $Q^2$ four-$\psi$ interaction in both fine-tuning scenarios. I will also concentrate on the S-wave, inclusion of other waves following analogous arguments.

The bare Schrödinger equation can be obtained by acting with the operator $E - H_0$ on the (scattering) wave-function $|\Psi>$, which in coordinate space is, asymptotically,

$$\Psi(\vec{r}) = \exp(i\vec{p} \cdot \vec{r}) + T_{os}(k)G_0(r; k)$$

with $|\vec{p}| = k$ and $G_0(r; k)$ given by Eq. (6). Not surprisingly we find

$$<\vec{r}'|(E - H_0)|\Psi> = <\vec{r}'|v_{os}|\Psi>,$$

where

$$v_{os}(k) = C_0 + 2C_2k^2 + \ldots$$

in momentum space.

As we saw, the rhs of Eq. (51) does not make sense as it stands. The effects of renormalization on the on-shell amplitude are two-fold. One is to replace $v(p, p')$ by

$$v^{(R)}(k) = C_0^{(R)} + 2C_2^{(R)}k^2 + \ldots$$

The other is to substitute the (free) Schrödinger propagator $G_0(p; k)$ by

$$G_0^{(R)}(p; k) = \frac{-m}{p^2 - k^2 - i\epsilon} D(k^2/p^2),$$

where $D(k^2/p^2)$ is such that

$$\int \frac{d^3l}{(2\pi)^3} G_0^{(R)}(l; k) = \frac{m}{2\pi^2} \int_{0}^{\infty} dl \frac{l^2 D(k^2/l^2)}{l^2 - k^2 - i\epsilon} = -\frac{imk}{4\pi}. (55)$$
Eq. (55) does not define $D(k^2/p^2)$ uniquely. The simplest solution is

$$D(k^2/p^2) = \frac{k^2}{p^2},$$

(56)

and the corresponding coordinate-space propagator is

$$G_0^{(R)}(r; k) = G_0(r; k) - G_0(r; 0).$$

(57)

Another solution is

$$D(k^2/p^2) = \frac{2k^2}{p^2 - k^2 - i\epsilon},$$

(58)

giving

$$G_0^{(R)}(r; k) = \frac{\partial}{\partial r}(rG_0(r; k)).$$

(59)

An infinite number of other solutions exist, but their important common feature is that they soften the $r \to 0$ behavior of the propagator: as it is obvious from the defining Eq. (55),

$$G_0^{(R)}(r; k) = -\frac{imk}{4\pi} (1 + O(kr)).$$

(60)

Renormalization therefore defines the rhs of Eq. (51) as

$$\langle \vec{r}|v_{os}|\Psi > \equiv \langle \vec{r}|v^{(R)}|\Psi_R >,$$

(61)

where

$$\Psi_R(\vec{r}) = \exp(i\vec{p} \cdot \vec{r}) + T_{os}(k)G_0^{(R)}(r; k)$$

(62)

satisfies

$$\delta(\vec{r})\Psi_R(\vec{r}) = \delta(\vec{r})\frac{\partial}{\partial r}(r\Psi(\vec{r})).$$

(63)

due to Eq. (60). We can now define an “effective renormalized potential” $\hat{v}^{(R)}(k)$ acting on the original wave-function,

$$\langle \vec{r}|v^{(R)}|\Psi_R > \equiv \langle \vec{r}|\hat{v}^{(R)}|\Psi >,$$

(64)

finding

$$\hat{v}^{(R)}(\vec{r}) = (C_0^{(R)} + 2C_2^{(R)}k^2 + \ldots)\delta(\vec{r})\frac{\partial}{\partial r}.$$

(65)
This is the “potential” which encodes all the effects of renormalization: used with free Schrödinger propagation it produces by construction the correct, renormalized amplitude. That is, the renormalized Schrödinger equation is

$$-\frac{1}{m}(\nabla^2 + k^2)\Psi(\vec{r}) = -(C_0^{(R)} + 2C_2^{(R)}k^2 + \ldots)\delta(\vec{r})\frac{\partial}{\partial r}(r\Psi(\vec{r})).$$  \hspace{1cm} (66)

It is easy to invert the preceding reasoning and check that this Schrödinger equation indeed leads to the on-shell \(T\)-matrix obtained previously.

The above derivation can be sketched in diagrammatic terms —see Fig. 3. The steps leading to Eq. (66) can be repeated starting with the bound state wave-function

$$\Psi(\vec{r}) = -\frac{N}{m}G_0(r; i\kappa).$$  \hspace{1cm} (67)

\((N\) is a normalization factor.)

Recalling the connection (21), (22), etc., between EFT and effective range parameters, we see that this is nothing but a generalization of the well-known pseudo-potential of Fermi and Breit [3, 2, 4]. Therefore, renormalization of an EFT with only short-range interactions leads to a Schrödinger equation with a pseudo-potential determined by renormalized parameters, rather than a Schrödinger equation with a local, smeared delta-function-type potential determined by bare parameters.

Further insight into the effect of renormalization can be gained by looking at the solution of Eq. (66). Recall that an ordinary potential does not generate an \(1/r\) behavior of the wave-function close to the origin because action of the Laplacian on such singularity produces a delta function: \(\nabla^2(1/r) = -4\pi\delta(\vec{r})\). A local delta-function potential —as our bare potential — in turn is too singular to admit a solution of this type because then the potential side of the Schrödinger equation becomes \(v\Psi \sim \delta(\vec{r})/r\), more singular than the kinetic side. (This is the quantum-mechanical manifestation of our original divergence problem.) The effect of the \(\frac{\partial}{\partial r}r\) operator in the pseudo-potential is to soften this singularity at the origin, replacing \(\delta(\vec{r})/r\) by \(\delta(\vec{r})\). Indeed, we can find the solution of Eq. (66) by expanding \(\Psi(\vec{r}) = \sum_{n=-\infty}^{+\infty} A_n r^n\) at small \(r\). The result is completely determined by \(A_{-1}\) as

$$\Psi(\vec{r}) = A_{-1}\left[\frac{1}{r} - \frac{4\pi}{mC_0^{(R)}} \left(1 - 2\frac{C_2^{(R)}}{(C_0^{(R)})^2}k^2 + \ldots\right) + O(r)\right].$$  \hspace{1cm} (68)
Now, the same solution follows from the free Schrödinger equation,

\[ -\frac{1}{m}(\nabla^2 + k^2)\Psi(\vec{r}) = 0, \]  

with a peculiar boundary condition at the origin,

\[
\left. \frac{\partial}{\partial r} \ln(r\Psi(\vec{r})) \right|_{r=0} = -\frac{4\pi}{mC_0^{(R)}} \left( 1 - 2 \frac{C_2^{(R)}}{(C_0^{(R)})^2} k^2 + \ldots \right).
\]

Again, using Eqs. (21), (22), etc., we recognize the first term in Eq. (70) as the boundary condition of Breit [2]. The renormalized EFT of only short-range interactions is equivalent to an energy expansion of the most general condition on the logarithmic derivative of the wave-function at the origin.

The physics behind these Schrödinger formulations is clear if we look at
the resulting radial wave-functions $r \psi(r) = u(r)$. (i) For $k^2 = -\kappa^2 < 0$,

$$u(r) = c_{(-)} e^{-\kappa r}, \quad (71)$$

with $c_{(-)}$ a normalization factor and $\kappa$ given by Eq. (30). (ii) For $k^2 \geq 0$,  

$$u(r) = c_{(+)} \left( \sin kr - \frac{a_0 k}{1 - \frac{a_0 r_0 k^2}{2}} \cos kr \right), \quad (72)$$

with $c_{(+)}$ a normalization factor and $a_0$ and $r_0$ given in Eqs. (21) and (22). By changing the unspecified dependence of $c_{(+)}$ on $k$ and $\kappa_{s,d}$ we do not change the underlying pole structure of the $\psi\psi$ amplitude, but we do change the Jost functions. This is a reflection of the fact that the EFT cannot distinguish among short-range potentials which produce the same asymptotic behavior. Indeed, it is not difficult to see that with appropriate choices of $c_{(+)}$ we can cover all situations considered in Ref. \[27\]. The EFT shrinks to the origin the complicated wiggles that the “underlying” wave-function might have at small distances, thus replacing them with a smooth behavior that matches onto the tail of the underlying wave-function. The effective wave-function is all tail for the tail is all one can see from far away.

5 Potential, Regularization issues

I have so far avoided using specific regularization schemes in order to emphasize the features of the renormalized theory, which is all that determines observables anyway. The power countings derived in Sect. 3 are valid independent of the details of the regularization scheme adopted. The conclusion that the EFT approach is completely equivalent to the effective range expansion, the method of pseudo-potentials, and to analytic boundary conditions had previously been hidden by the use of specific regulators. The issue of regularization has also clouded the relation to the traditional approach of using a potential in the Schrödinger equation. Here a few words are spent on these related issues: the roles of a potential and of regularization schemes. (Some of these remarks have been presented in Ref. \[7\], and been shown since to be supported by explicit numerical investigation \[22\].)

In the case of a natural theory, it is clear that iteration of a potential is superfluous. Within its region of validity, the EFT is perturbative: nothing needs to be iterated to all orders, and any bound states, if they exist,
exist beyond the range of the EFT. Iteration of the potential includes only part of the higher-order terms, and this incomplete set cannot \textit{a priori} be taken seriously. Indeed, it has been shown that the iteration of momentum-independent contact interactions does not reproduce the non-analytical behavior of a toy natural underlying theory \cite{14}. On the other hand, since the error induced by iteration is small, it does not affect observables much and does no harm—as long as one remembers that higher orders have not been accounted for correctly.

The more interesting cases are the ones with fine-tuning in the underlying theory. For definiteness I focus on the case of a shallow bound state, which has attracted a lot of attention lately in the context of nuclear forces.

The appropriate formula for the amplitude, correct up to and including $O((\mathcal{R}/M)^2)$, is Eq. (31), which I obtained using perturbation theory for the corrections to the $Q^0$ interaction in the bare potential (3). The same result can be obtained more easily by first summing the bubbles with the full $v(p', p)$ and then expanding in powers of the energy. For example, keeping track only of the $Q^2$ terms in the $S$-wave,

\begin{equation}
(T_{os}(k))_0 = - \left( \frac{1}{C_0 + 2C_2k^2} - I(k) \right)^{-1} , \tag{73}
\end{equation}

with

\begin{equation}
I(k) = \left( 1 + \frac{k^2}{2m^2} \right) I_0(k) - \left( \frac{C_2}{C_0} + \frac{1}{4m^2} \right) \frac{mL_3}{2\pi^2} . \tag{74}
\end{equation}

This corresponds to the standard procedure of iterating the full potential, but it is now obvious that to remove the infinities we have to expand

\begin{equation}
\frac{1}{C_0 + 2C_2k^2} = \frac{1}{C_0} - \frac{2C_2}{C_0}k^2 + \ldots , \tag{75}
\end{equation}

and consistently neglect $Q^4$ terms. We are then back to Eq. (31). Terms of higher order in the $Q/M$ expansion can only be made regulator-independent by renormalizing higher order parameters in the Lagrangian. Indeed, as we have seen before, $O(Q^4)$ terms in $v(p, p')$ will contribute $k^4$ terms to Eq. (73).

Regulator-dependence is allowed insofar as it is small, that is, of the same magnitude as smaller terms that have been neglected. In practice, in the perturbative treatment of corrections some cut-off dependence will remain from $R(k^2/\Lambda^2)$, for it contains arbitrarily high powers of $Q/\Lambda$. Now, what
we have just done in Eq. (73) was to re-sum some sub-leading terms, then to throw them away in Eq. (75). The first step is carried out by iterating the potential, but the last step requires extra work. Omitting it introduces further cut-off dependence that creeps in through bare parameters rather than $R$; the size of the error induced by such laziness depends on the relative size of the bare parameters.

For example, to $O((\mathcal{N}/M)^2)$, by solving the the Schrödinger equation with a finite cut-off we induce a shape parameter

$$P_0r_0^3 = \frac{16\pi C_2^2}{mC_0^3} + \frac{2}{\pi\Lambda^3} \left( R'(0) + \frac{\Lambda^2}{2m^2} R(0) \right) + \frac{r_0}{8m^2}. \quad (76)$$

We see that apart from a regularization-independent relativistic correction, cut-off dependence enters through both $R$ and the bare parameter ratio $C_2^2/C_0^3$.

The induced error in the amplitude coming from $R$ is $O(\mathcal{N}/\Lambda)$ and will be small as long as $\Lambda \gtrsim M$. It can in particular be completely removed by taking $\Lambda \to \infty$. It is natural to ask whether there are regulator schemes in which we can forfeit the last step in Eq. (75) and still manage to remove the errors coming from bare parameters, too. In more complicated situations dealt with in the literature —where there exists a long-range potential which can only be solved numerically— both of these spurious effects can be present and impossible to remove explicitly. It is desirable to find schemes that at least minimize these spurious effects; yet, we must be prepared to discover that not all schemes are flexible enough to achieve this.

To any given order, the equations relating bare and renormalized parameters will be truncated, and beyond leading order, be highly non-linear —see Eqs. (16), (17), for example. Inverting them, we can find the bare parameters in terms of the renormalized parameters (known from data) and the cut-off: $C_{2n} = C_{2n}(C_{2n}^{(R)}; \Lambda)$. This of course means that the bare parameters are not observables, depending on the scheme chosen. Given $C_{2n}^{(R)}$’s, whether one finds real solutions $C_{2n}$ for any $\Lambda$ depends on the actual values of the $\theta_n$’s in Eq. (9). In particular, to $O((\mathcal{N}/M)^2)$ the important term for large $\Lambda$ is $C_2^{(R)}\theta_3\Lambda^3$; the sign of $C_2^{(R)}$ being that of $r_0$, it is the sign of $\theta_3$ which is of concern. It was pointed out in Ref. [13] that the most obvious cut-off schemes (such as a sharp momentum-space cut-off and a coordinate-space regularization by square wells) imply $\theta_3 > 0$. Imposing real bare parameters, it is
found that \( r_0 \leq 2/\Lambda \) for large \( \Lambda \), or in other words, that the limit \( \Lambda \to \infty \) can only be taken if \( r_0 \leq 0 \). This is a reformulation in field-theoretical language of Wigner’s theorem [26].

While such a constraint is unusual, it is doubtful that it is of much relevance to EFTs applied consistently to a certain order. First, this is obviously a regularization-scheme-dependent issue. Dimensional regularization with minimal subtraction has the feature that \( \theta_n = 0 \), and it is easy to construct (smooth) cut-off schemes such that \( \theta_3 \leq 0 \) [28]. In the worst case scenario, the above constraint is a result of attempting to remove the cut-off in an inadequate regularization scheme. But actually this constraint only arises from a constraint imposed on unobservable bare parameters. No matter what process one is considering in the low-momentum regime, the bare parameters never appear, except in the right combinations with the cut-off to produce renormalized parameters: an imaginary bare parameter is of no relevance. Finally, even if one insists on the unfortunate choice of regulator and the reality constraint on bare parameters, a positive \( r_0 \) can still be achieved by keeping the cut-off finite and of the order of the mass scale of the underlying theory, \( \Lambda \sim M \).

Regardless, keeping a cut-off \( \Lambda \sim M \) has one useful consequence. In this case, the large — containing inverse powers of \( \mathcal{R} \) — renormalized parameters \( C_{2n}(R) \) can be generated by natural-size bare coefficients \( C_{2n} = O(4\pi/mM^{2n+1}) \). For example, \( C_0 = -(2\pi^2/\theta_1\Lambda)(1 + \pi\mathcal{R}/2\gamma_0\theta_1\Lambda + \ldots) \) is of natural size even though \( C_0(R) = 4\pi\gamma_0/m\mathcal{R} \) is large. Similarly, \( C_2 = (\pi\gamma_2/2\theta_1^2\gamma_0m\Lambda^2M)(1 + \pi\mathcal{R}/\gamma_0M\Lambda + \ldots) \). What a cut-off does is to give a scale for the bare parameters to be fine-tuned against. This way the ratio of bare parameters can be made of natural size as well. The difference between a truncation of the rhs of Eq. (75) and the lhs is then no bigger than other neglected small effects such as higher-order interactions in the Lagrangian: for example, the shape parameter \( \mathcal{F} \) induced by the cut-off is in this case \( O(1/M^3) \). A power expansion of the bare potential followed by its iteration to all orders is under these circumstances no worse than a perturbation treatment of the corrections to leading order.

We actually can invert the previous reasoning and use the freedom introduced by the cut-off to save some work. Given a regularization procedure, we can further fine-tune \( \Lambda \) to its “optimal” value that improves agreement with data. For example, if we are working to \( O((\mathcal{R}/M)^2) \), we can fine-tune
A so that Eq. (76) fits the experimental value for $P_0$; such a procedure was considered in Ref. [13] in the case of the $1S_0$ $NN$ phase shift. If we are dealing with a single channel, this is a consistent procedure: it simply means that $O(Q^4)$ contributions have been included, but they do not need to be written explicitly because the corresponding bare parameter is zero. In general, however, only part of the higher order effects can be accounted for in this manner.

The above discussion holds for any scheme where a cut-off mass scale appears explicitly. Dimensional regularization with minimal subtraction, first applied to this problem in Ref. [11], is special because it defines the integral (10) with $L_n = 0$ and $R = 0$. It suffers from no errors generated by $R$, but it forces the bare parameters to coincide with the renormalized ones and thus be unnaturally large. Because no cut-off dependence is visible, if one works to $Q^2$ terms it is not immediately apparent that $1/(C_0 + 2C_2k^2)$ has necessarily to be expanded in $k^2$. It was one of the results of Ref. [11] that a fitting of the $1S_0$ $NN$ phase shift based on the effective range expansion worked much better than one based on the form $1/(C_0 + 2C_2k^2)$, and a suggestion was made that the most useful expansion is that of $T^{-1}$ rather than $v$. One can understand the failure of the $1/(C_0 + 2C_2k^2)$ fit by noticing that the induced spurious shape parameter (76) is here $P_0r_0^3 = a_0r_0^2/4 - 3r_0/8m^2 + 1/4a_0m^4$, which is $O(1/M^2)$. Since there is no regulator to blame the fine-tuning on, the fine-tuning contaminates all other induced terms as well; they are all automatically large if $a_0$ is large. In dimensional regularization with minimal subtraction one cannot substitute the perturbative treatment of corrections by the iteration of the whole potential without reducing the range of applicability of the theory from $M$ to the ratio of parameters in the power expansion of the contact interactions, which is $\sqrt{RM}$.

The perturbative treatment of corrections can still be carried out with dimensional regularization. This is true of minimal subtraction or other subtraction schemes, for example the convenient one invented in Ref. [19]. (The power counting of Sect. 3.2 [7] was re-discovered independently using this new subtraction in Ref. [19].) As originally found in Ref. [12], the contamination problem of minimal subtraction can be avoided by using the dibaryon field, which in view of our regularization-independent arguments of Sect. 3 is not surprising.
The point of using an EFT is to describe low-energy physics with minimal assumptions about higher-energy behavior. The results of this paper hold regardless of the details of the physics that generates the short-range interactions. They will arise from exchange of (possibly many) particles of sufficiently high mass, perhaps in a theory without a small coupling constant. It might happen that this physics can be equally well described by some potential of range $R$ much smaller than the wavelength of the interacting asymptotic particles. Therefore we expect that the EFT results should be valid in the particular case of simple quantum-mechanical potentials.

Here I will illustrate some of the results of Sect. 3 about the scaling of effective range parameters with a simple attractive square-well potential of range $R$ and depth $V_0$

$$V(r) = -V_0 \theta(R - r). \quad (77)$$

Defining a dimensionless variable

$$\alpha \equiv \sqrt{mV_0 R} \quad (78)$$

it is easy to find the $S$-wave amplitude

$$(T(k))_0 = -i \left[ e^{-2ikR} \frac{\sqrt{\alpha^2 + (kR)^2} \cot \sqrt{\alpha^2 + (kR)^2} + ikR}{\sqrt{\alpha^2 + (kR)^2} \cot \sqrt{\alpha^2 + (kR)^2} - ikR} - 1 \right]. \quad (79)$$

It then follows that

$$a_0 = R \left( 1 - \frac{\tan \alpha}{\alpha} \right) \quad (80)$$

and

$$r_0 = R \left( 1 - \frac{R}{a_0 \alpha^2} - \frac{R^2}{3a_0^2} \right); \quad (81)$$

expressions for the shape and other parameters can also be worked out but are more complicated. The amplitude Eq. (79) has an interesting pole structure as a function of $\alpha$ [24]. In particular, it can be shown explicitly that all poles with $|k| < 1/R$ are indeed on the imaginary axis. These poles cross $k = 0$ when $\alpha$ takes one of the values $\alpha_c = (2n + 1)\pi/2$. The zeros of Eq. (79), on the other hand, cross $k = 0$ when $\alpha$ takes one of the values $\alpha'_c = \tan \alpha'_c$. 

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For generic values of $\alpha = O(1)$, there is only one mass scale $M = 1/R$ in the problem: $\tan \alpha/\alpha \sim 1$ and thus $a_0 = O(1/M)$, $r_0 = O(1/M)$, and $\kappa = O(M)$. As expected, in this generic case the effective range parameters, bound state momenta, and positions of the zeros are all given by the natural mass scale $M$.

As $\alpha - \alpha_c$ becomes small,

$$a_0 = \frac{R}{\alpha_c(\alpha - \alpha_c)} \left[ 1 - \frac{(1 - \alpha_c^2)}{\alpha_c}(\alpha - \alpha_c) + O((\alpha - \alpha_c)^2) \right],$$  \hspace{1cm} (82)

$$r_0 = R \left[ 1 - \frac{1}{\alpha_c}(\alpha - \alpha_c) + O((\alpha - \alpha_c)^2) \right],$$  \hspace{1cm} (83)

and the pole is at $k = \iota \kappa$,

$$\kappa = \frac{\alpha_c(\alpha - \alpha_c)}{R} \left[ 1 + \frac{(1 - \alpha_c^2/2)}{\alpha_c}(\alpha - \alpha_c) + O((\alpha - \alpha_c)^2) \right].$$  \hspace{1cm} (84)

If we identify $\aleph = |\alpha/\alpha_c - 1|/R$, we see that indeed the scaling of these parameters is as anticipated in Sect. 3, namely $a_0 = O(1/\aleph)$, $r_0 = O(1/M)$, and $\kappa = O(\aleph)$.

Likewise, as $\alpha - \alpha'_c$ becomes small,

$$a_0 = -R\alpha'_c(\alpha - \alpha'_c) \left[ 1 + \frac{(1 + \alpha_c^2/\alpha'_c^3)}{\alpha_c^3}(\alpha - \alpha'_c) + O((\alpha - \alpha'_c)^2) \right]$$  \hspace{1cm} (85)

and

$$r_0 = -\frac{R}{3\alpha_c^2(\alpha - \alpha'_c)^2} \left[ 1 - \left( 3 + \frac{2(1 + \alpha_c^2 + \alpha'_c^4)}{\alpha_c^3} \right)(\alpha - \alpha'_c) + O((\alpha - \alpha'_c)^2) \right].$$  \hspace{1cm} (86)

Again, identifying $\Omega = \sqrt{|\alpha/\alpha'_c - 1|/R}$, we see that indeed $a_0 = O(\Omega^2/M^3)$ and $r_0 = O(M^3/\Omega^4)$, as advertised in Sect. 3.

7 Discussion and Conclusion

I have developed here power countings for EFTs with natural and unnatural short-range interactions that are manifestly independent of the regularization
scheme. I have shown that the modern EFT method applied to the problem of short-range forces is, after renormalization, completely equivalent to the ancient effective range expansion, and to the methods of pseudo-potentials and boundary conditions. Some of the possible pitfalls in the implementation of EFT method were also discussed. Why, then, bother with the EFT method?

EFTs are nowadays thought to be the rationale for the successes of field-theory based particle physics. We have therefore gained insight into the ancient techniques by seeing how they arise from renormalization in the modern conceptual framework without extra dynamical assumptions. But it can be argued that this conceptual gain is compensated by a practical loss, in that one can more simply stick to the ancient techniques with the same observable results.

The gain is indeed marginal for the problem at hand, but there are advantages to a field-theoretical framework when other processes in the same energy scale are considered, and we want to treat them all consistently, free of off-shell ambiguities. Power counting applies to these processes as well, although it has to be enlarged to accommodate new operators associated with other particles. For example, recently the three-body system has been attacked using the power counting of Sect. 3.2. In the case of nucleon-deuteron scattering, successful model-independent predictions can be achieved very easily [18].

Yet the gain is potentially much more significant in more general cases where there are other, lighter degrees of freedom that generate longer-range forces. One should recall the reason why particle physics found its EFT paradigm: symmetries play a dominant role in physics because they restrict the form of S-matrix elements (and consequently also determine what the relevant low-energy degrees of freedom are), and a Lagrangian framework is by far the easiest way to incorporate symmetries. The problem considered in this paper had very weak symmetry constraints —essentially only on particle number and invariance under small Lorentz boosts. In more interesting problems, the role of symmetry is bound to be more important, and the EFT method more convenient than the ancient techniques. We only need to return to the $NN$ system to offer an example: at energies comparable to the pion mass the pion has to be retained as a degree of freedom, and (approximate) chiral symmetry is the most important constraint on pion interactions. At sufficiently low energies the inclusion of the pion is not mandatory, but can
be carried out with minor adaptations to the power counting presented above [13]. The EFT method of chiral Lagrangians is the only known, systematic way to treat this spontaneously broken symmetry, and it has been successful in dealing with low-energy phenomenology from pion-pion interactions to few-nucleon forces [30, 7].

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