Overconvergent global analytic geometry

Frédéric Paugam

October 30, 2014

“The rain has stopped, the clouds have drifted away,
    and the weather is clear again.”
    – Ryôkan, One robe, one bowl

Abstract

We define a notion of global analytic space with overconvergent structure sheaf. This gives an analog on a general base Banach ring of Große-Klönn’s overconvergent p-adic spaces and of Bambozzi’s generalized affinoid varieties over $\mathbb{R}$. This also gives an affinoid version of Berkovich’s and Poineau’s global analytic spaces. This affinoid approach allows the introduction of a notion of strict global analytic space, that has some relations with the ideas of Arakelov geometry, since the base extension along the identity morphism on $\mathbb{Z}$ (from the archimedean norm to the trivial norm) sends a strict global analytic space to a usual scheme over $\mathbb{Z}$, that we interpret here as a strict analytic space over $\mathbb{Z}$ equipped with its trivial norm. One may also interpret some particular analytification functors as mere base extensions. We use our categories to define overconvergent pro-étale motives and an overconvergent stable homotopy theory of global analytic spaces. These have natural Betti, de Rham and pro-étale realizations. We also define derived overconvergent global analytic spaces and their (derived) de Rham cohomology. Finally, we use Toen and Vezzosi’s derived geometric methods to define a natural (integral) Chern character on Waldhausen’s $K$-theory with values in cyclic homology. The compatibility of our constructions with Banach base extensions gives new perspectives both on global analytic spaces and on the various realizations of the corresponding motives.

Contents

1 Introduction 2

2 Seminormed algebraic objects 5
    2.1 $\mathbb{R}_+$-graded sets and seminorms ........................................ 5
    2.2 Seminormed modules ........................................................................ 8
    2.3 Seminormed polynomials and convergent power series ..................... 13
1 Introduction

Berkovich defined analytic spaces over a general Banach ring \((R, |\cdot|_R)\) in [Ber90], and Poineau studied them in a refined way in [Poi10] and [Poi13]. Their local models are given by coherent sheaves of ideals in the ring of analytic functions on an open subset of the analytic affine space \(A^n_{(R,|\cdot|_R)}\). This affine space is given by the space of multiplicative seminorms

\[
|\cdot| : R[X_1,\ldots,X_n] \to \mathbb{R}_+
\]

on the polynomial ring whose restriction to \(R\) is bounded by \(|\cdot|_R\), and equipped with its natural sheaf of analytic functions, defined as local uniform limits of rational functions without poles.
On a $p$-adic field, it is convenient to work with a refined notion of analytic space, that was also defined by Berkovich in [Ber93], using building blocks similar to those of Tate’s rigid analytic geometry [Tat71]: affinoid algebras. These are given by quotients of rings of power series $\mathbb{Q}_p\{\rho^{-1}T\}$ that converge (in a $p$-adic sense) on a given polydisc of arbitrary positive real radius $\rho$ (Tate’s theory reduces to $\rho = 1$, but this is not so well adapted to trivially valued fields, or to more general Banach rings, like the Banach ring $\mathbb{Z} = (\mathbb{Z}, |\cdot|_\infty)$ of integers with its archimedean absolute value).

The natural question that we answer in this paper is to give a setting for global analytic geometry that allows to change the base Banach ring along a bounded morphism $\varphi: (A, |\cdot|_A) \to (B, |\cdot|_B)$, and also to work with a notion of affinoid algebra over a general base Banach ring, in a way that is compatible to Berkovich’s original constructions when the base Banach ring is $\mathbb{Q}_p = (\mathbb{Q}_p, |\cdot|_p)$.

If we want to get back Berkovich’s analytic spaces over $\mathbb{Q}_p$, we have to work with uniform base Banach rings, given by Banach rings whose seminorm is power-multiplicative. Indeed, in this setting, we will have

$$\mathbb{Q}_p\{\rho^{-1}T\} \cong \mathbb{Z}\{\rho^{-1}T\} \otimes_{\mathbb{Z}} \mathbb{Q}_p,$$

where $\otimes$ denotes the coproduct of uniform Banach rings, and we do the base change along the bounded morphism $\mathbb{Z} = (\mathbb{Z}, |\cdot|_\infty) \to (\mathbb{Q}_p, |\cdot|_p) = \mathbb{Q}_p$.

It was already quite clear in Berkovich’s original work [Ber90] that, if the base Banach ring is $\mathbb{C} = (\mathbb{C}, |\cdot|_\infty)$, the naive definition of affinoid spaces over $\mathbb{C}$ would lead to important difficulties: the uniform ring of convergent power series on the unit disc identifies with the ring of continuous functions on the disc that are holomorphic in its interior. A simple quotient of such a ring will give the ring of all continuous functions on the unit circle, that is not something we would like to call an affinoid algebra over $\mathbb{C}$. As shown by Bambozzi in his thesis [Bam14], rings of overconvergent power series on closed polydiscs over an archimedean field are better behaved than usual rings of convergent power series, and allow the definition of a complete archimedean analog of Berkovich’s $p$-adic analytic spaces. They will also allow us, in the $p$-adic setting, to circumvent in a very natural way all the boundary-related difficulties that appear in Berkovich’s theory.

It is known since the work of Monsky and Washnitzer [MW68] that the de Rham cohomology of non-proper $p$-adic analytic spaces does not work well with convergent power series, because the Poincaré Lemma fails in this setting. Its correct formulation must be done in terms of rings of overconvergent power series (Große-Klönne subsequently developed the overconvergent analogs of rigid analytic varieties in [GK00], and studied their de Rham cohomology in [GK02] and [GK04]).

The ring $R\{\rho^{-1}T\}^1$ of overconvergent power series of radius $\rho$ is a formal filtered colimit of the uniform Banach rings $R\{\nu^{-1}T\}$ of convergent power series of radii $\nu > \rho$. The natural setting to study such formal filtered colimits is the setting of uniform ind-Banach rings. We will thus define overconvergent power series rings as particular kinds of uniform ind-Banach rings.
However, a uniform Banach ring is necessarily reduced. This prevents us from using nilpotent elements in uniform ind-Banach rings to formulate differential calculus algebraically, as one does in scheme theory and in complex analytic geometry.

We thus need to define a category of overconvergent analytic rings over a given Banach ring \((R, |·|_R)\) that contains the rings of overconvergent power series, but that also allows us to use nilpotent elements. The construction of this “completion” of the category of overconvergent power series rings is done in a way similar to the one used by Lawvere in synthetic differential geometry [Law79], by Dubuc and Zilber in synthetic analytic geometry [DZ94] and by Lurie in derived geometry in [Lur09a], using “functors of functions”.

The advantage of this categorical approach to analytic rings is that it generalizes directly to the derived setting, and allows a natural (i.e., functorial in the \(\infty\)-categorical sense) definition of (derived) de Rham cohomology for non-smooth spaces. Remark that there is also in [BK13] an approach to non-archimedean analytic geometry using a geometry relative to the symmetric monoidal category of Banach spaces.

Once a convenient category of overconvergent rings is defined, it is easy to use the “functor of point” approach to define a natural notion of overconvergent analytic space over a given Banach ring.

The great interest of this affinoid approach to global analytic geometry is that it allows the definition of strict global analytic spaces that is stable by base extension along a bounded morphism of Banach ring. For example, strict dagger analytic spaces over over \((\mathbb{Z}, |·|_{\infty})\) have a base extension to \((\mathbb{Z}, |·|_0)\) given by usual schemes. Such strict global analytic models for schemes over \(\mathbb{Z}\) may be considered as “Arakelov type” models. This will be discussed further in Section 5.

Using this new setting for global analytic geometry, we will define in Section 6 various cohomological invariants, as étale, analytic motivic cohomology, and global analytic \(K\)-theory. We will explain the relation of these invariants to the ones that were already developed before in the theory of schemes, that is given in our theory by a mere base extension along a bounded morphism of Banach rings, in the case of schemes that admit “Arakelov type” models in our sense.

Finally, motivated by applications to global comparison isomorphisms between étale cohomology and de Rham cohomology, we will define global analytic derived analytic spaces and their derived de Rham cohomology.

Acknowledgments:

During the preparation of this work, the author was supported by the university Pierre and Marie Curie and the ANR project “Espaces de Berkovich globaux”. I thank J. Ayoub, F. Bambozzi, O. Ben-Bassat, B. Bhatt, B. Conrad, D.-C. Cisinski, F. Deglise, B. Drew, A. Ducros, M. Flach, E. Große-Klönne, F. Ivorra, M. Karoubi, K. Kedlaya, T. Lemanissier, R. Liu, F. Loeser, B. Morin, J. Poineau, M. Porta, J. Riou, M. Robalo, M. Temkin, P. Schapira, J. Scholbach, P. Scholze and A. Vezzani for useful discussions. Special thanks are due to мудрец.
2 Seminormed algebraic objects

2.1 $\mathbb{R}_+$-graded sets and seminorms

Let $\mathbb{R}_+ := \mathbb{R}_{\geq 0}$ be the set of positive real numbers. We will now define various categories of $\mathbb{R}_+$-graded sets that give the natural setting for the theory of seminormed algebraic objects.

**Definition 1.** An $\mathbb{R}_+$-graded set is a pair $(X, | \cdot |_X)$ composed of a set $X$ and a map $| \cdot |_X : X \to \mathbb{R}_+$. A graded map (resp. contracting map, resp. bounded map) of $\mathbb{R}_+$-graded set is a map $f : X \to Y$ such that

$$|f(x)|_Y = |x|_X$$

(resp. $|f(x)|_Y \leq |x|_X$ for all $x \in X$,

resp. there exists $C > 0$ such that $|f(x)|_Y \leq C \cdot |x|_X$ for all $x \in X$).

The category of $\mathbb{R}_+$-graded sets with graded (resp. contracting, resp. bounded) maps is denoted $\mathbb{R}_+^{\text{Sets}}$ (resp. $\mathbb{R}_+^{\text{Sets}_{\leq 1}}$, resp. $\mathbb{R}_+^{\text{Sets}_{\leq}}$).

**Lemma 1.** The category $\mathbb{R}_+^{\text{Sets}}$ has arbitrary limits and colimits and has internal homomorphisms for the product monoidal structure. The category $\mathbb{R}_+^{\text{Sets}_{\leq 1}}$ has arbitrary colimits, and in particular, the degree zero set $\{0_0\}$ as terminal object. The category $\mathbb{R}_+^{\text{Sets}_{\leq}}$ has finite colimits and in particular $\{0_0\}$ as a terminal object. More generally, $\mathbb{R}_+^{\text{Sets}}$ has uniformly bounded colimits, meaning that if $X : I \to \mathbb{R}_+^{\text{Sets}}$ is a uniformly bounded diagram (i.e., a diagram such that there exists $C$ such that for all $\varphi : i \to j$ in $I$, $|X(\varphi)(x)|_{X_j} \leq C \cdot |x|_{X_i}$ for all $x \in X_i$) and $f : X \to Z$ is a uniformly bounded cocone (meaning that there exists $D > 0$ such that for every $i \in I$, $f(i) : X_i \to Z$ is $D$-bounded), then $\text{colim}_{i \in I} f_i : \text{colim}_{i \in I} X_i \to Z$ exists. The categories $\mathbb{R}_+^{\text{Sets}_{\leq 1}}$ and $\mathbb{R}_+^{\text{Sets}_{\leq}}$ have finite limits and internal homomorphisms for the product monoidal structure.

**Proof.** The category $\mathbb{R}_+^{\text{Sets}} = \text{Hom}_{\text{Cat}}(\mathbb{R}_+^{\text{disc}}, \text{Sets})$, being the functor category of $\text{Sets}$-valued functors on the discrete category with set of objects $\mathbb{R}_+$, has arbitrary limits and colimits. The set $\mathbb{R}_+$, graded by the identity, is its final object. The disjoint union $\bigsqcup_i X_i$ of the underlying sets of a family $(X_i, | \cdot |_i)$ of $\mathbb{R}_+$-graded sets, equipped with the grading $\bigsqcup | \cdot |_{X_i}$ is always a graded and a contracting coproduct. If the family is finite, it is a bounded coproduct. If we have an $\mathbb{N}$-indexed family $\{X_n\}$ of non-empty sets of degree $1$, we may define a family of bounded maps to $Y = \mathbb{R}_+$ by sending $X_n$ to $n$. There is no bounded map that extends this family to the coproduct set. From this counter-example, we see that the obstruction to having a colimit (i.e., a coproduct) for a cocone $f : X \to Z$, with $X : I \to \mathbb{R}_+^{\text{Sets}}$ a discrete diagram, disappears if the cocone is uniformly bounded. If $f, g : (X, | \cdot |_X) \to (Y, | \cdot |_Y)$ are two parallel bounded maps, and $\pi : Y \to Z = \text{coker}(f, g)$ is the coequalizer of the underlying sets, one may equip it with the grading (the infimum is that of a constant if both maps are graded)

$$|z| = \inf_{y \in \pi^{-1}(z)} |y|_Y.$$
This gives a coequalizer in the categories $\mathbb{R}^{Sets}_{+\leq}$ and $\mathbb{R}^{Sets}_{+\leq 1}$. This proves all the desired results about colimits. The product $\prod_{i=1}^{n} X_i$ of the underlying sets of a finite family $(X_i, |\cdot|_i)_{i=1,\ldots,n}$ of $\mathbb{R}_+$-graded sets, equipped with the grading

$$|(x_1, \ldots, x_n)| := \max_i |x_i|$$

is a product in the categories $\mathbb{R}^{Sets}_{+\leq 1}$ and $\mathbb{R}^{Sets}_{+\leq}$. Given a parallel pair $f, g : (X, |\cdot|_X) \to (Y, |\cdot|_Y)$ of bounded (resp. contracting) maps, the kernel of the pair $(f, g)$ of set maps equipped with the grading induced by that of $X$ is a kernel for the pair $(f, g)$ in the category $\mathbb{R}^{Sets}_{+\leq}$ (resp. $\mathbb{R}^{Sets}_{+\leq 1}$). Indeed, if $h : (Z, |\cdot|_Z) \to (X, |\cdot|_X)$ is a bounded (resp. contracting) map such that $f \circ h = g \circ h$, then $h$ factors set theoretically in a unique way through the kernel, and this factorization is bounded (resp. contracting). The internal homomorphisms between two objects $X$ and $Y$ of $\mathbb{R}^{Sets}_{+\leq}$ (resp. $\mathbb{R}^{Sets}_{+\leq 1}$) are given by the set $\text{Hom}(X, Y)$ of bounded (resp. contracting) maps with the grading given by

$$|f|_{\text{Hom}(X, Y)} := \inf \{ C > 0, |f(x)|_Y \leq C \cdot |x|_X \text{ for all } x \in X \}.$$

The set $\mathbb{R}_+$ has a natural family of commutative monoid structures indexed by $p \in [0, +\infty]$ given by $(+p, 0)$, where

$$r + p s := \sqrt[p]{r^p + s^p}$$

for $p < +\infty$ and

$$r + \infty s := \max(r, s).$$

Remark that we have $x + p s \leq r + p' s$ if $p' \leq p$. One also has the multiplicative monoid structure $(\times, 1)$, given by $r \times s := r.s$. We now define the associated symmetric monoidal structures on $\mathbb{R}_+$-graded sets.

**Definition 2.** The **multiplicative** (resp. $p$-additive, resp. maximum) tensor product of two $\mathbb{R}_+$-graded sets $(X, |\cdot|_X)$ and $(Y, |\cdot|_Y)$ is the product $X \times Y$ equipped with the grading

$$|(x, y)|_{X \otimes_p Y} := |x|_X \cdot |y|_Y$$

(resp. $|(x, y)|_{X \otimes_{p'} Y} := |x|_X + p |y|_Y$, resp. $|(x, y)|_{X \otimes_{\infty} Y} := \max(|x|_X, |y|_Y)$).

The three tensor products give monoidal structures on the categories $\mathbb{R}^{Sets}_{+\leq}$, $\mathbb{R}^{Sets}_{+\leq 1}$ and $\mathbb{R}^{Sets}_{+\leq}$. The unit object of the multiplicative monoidal structures is the one element set $\{1\}$. The unit object of the $p$-additive and maximum monoidal structure is the one element set $\{0\}$.

Remark that the maximal tensor product $\otimes_{\infty}$ is equal to the product $\times$ in the categories $\mathbb{R}^{Sets}_{+\leq}$ and $\mathbb{R}^{Sets}_{+\leq 1}$.
Definition 3. 1. A weakly seminormed additive monoid (resp. abelian group) is a commutative monoid (resp. an abelian group) object in the symmetric monoidal category \((\mathbb{R}_+^{\leq 1}, \otimes, \{0\}_0)\). More concretely, this is a monoid (resp. an abelian group) \((M, +, 0)\) equipped with a grading \(| \cdot |_M\) with \(|0|_M = 0\), and such that there exists \(C > 0\) with 
\[
|m + n| \leq C \cdot \max(|m|, |n|) \quad \text{(resp. } |m - n| \leq C \cdot \max(|m|, |n|))\).
\]

2. A seminormed monoid (resp. abelian group) is a commutative monoid (resp. an abelian group) object in the symmetric monoidal category \((\mathbb{R}_+^{\leq 1}, \otimes, \{0\}_0)\). More concretely, this is a monoid (resp. an abelian group) \((M, +, 0)\) equipped with a grading \(| \cdot |_M\) such that \(|0|_M = 0\) and 
\[
|m + n| \leq |m| + |n| \quad \text{(resp. } |m - n| \leq |m| + |n|)\).
\]

3. A weakly seminormed multiplicative monoid is a commutative monoid object in the symmetric monoidal category \((\mathbb{R}_+^{\leq 1}, \otimes, \{1\}_1)\). More concretely, this is a monoid \((M, \times, 1)\) equipped with a grading \(| \cdot |_M\) such that there exists \(C > 0\) with 
\[
|m \cdot n|_M \leq C \cdot |m|_M \cdot |n|_M.
\]

4. A seminormed multiplicative monoid is a commutative monoid object in the symmetric monoidal category \((\mathbb{R}_+^{\leq 1}, \otimes, \{1\}_1)\). More concretely, this is a monoid \((M, \times, 1)\) equipped with a grading \(| \cdot |_M\) such that \(|1|_M \leq 1\) and 
\[
|m \cdot n|_M \leq |m|_M \cdot |n|_M.
\]

If \((\mathcal{C}, \otimes)\) is one of the above symmetric monoidal categories, we will denote \(\text{MON}(\mathcal{C}, \otimes)\) the category of (commutative) monoids in \(\mathcal{C}\). There is a natural forgetful functor 
\[
\text{MON}(\mathcal{C}, \otimes) \to \mathcal{C}.
\]

Remark that if \((M, | \cdot |_M)\) is a weakly seminormed abelian group (resp. a weakly seminormed multiplicative monoid) then \((M, | \cdot |_M^t)\) is also a weakly seminormed abelian group (resp. multiplicative monoid) for every \(t > 0\).

Definition 4. A weakly seminormed ring (resp. seminormed ring) is a tuple \((R, | \cdot |_R, \times, +)\) composed of a weakly seminormed (resp. seminormed) abelian group \((R, | \cdot |_R, +)\) and a weakly seminormed (resp. seminormed) multiplicative monoid \((R, | \cdot |_R, \times)\) such that \((R, +, \times)\) is a ring in the classical sense. More concretely, an \(\mathbb{R}_+\)-graded set \((R, | \cdot |_R)\) equipped with a ring structure \((R, \times, +)\) such that \(|0|_R = 0\) is a weakly seminormed ring if there exists \(C > 0\) and \(D > 0\) such that 
\[
|a - b|_R \leq C \cdot \max(|a|_R, |b|_R) \quad \text{and} \quad |a \cdot b|_R \leq D \cdot |a|_R \cdot |b|_R.
\]

It is a seminormed ring if we can choose \(C = 2, D = 1\), we have \(|1|_R \leq 1\), and we further have 
\[
|a - b|_R \leq |a|_R + |b|_R.
\]
By definition, a weakly seminormed ring has an underlying $\mathbb{R}_+^{\text{sets}}$-graded set in $\mathbb{R}_+^{\leq \text{sets}}$, whereas a seminormed ring has an underlying $\mathbb{R}_+^{\text{sets}}$-graded set in $\mathbb{R}_+^{\leq 1}$.

A seminormed ring is called complete (or a Banach ring) if it is complete for the topology induced by its seminorm. We will denote $\text{SNRings}$ (resp. $\text{BanRings}$) the category of seminormed (resp. complete seminormed) rings.

### 2.2 Seminormed modules

**Definition 5.** Let $(R, |\cdot|_R)$ be a seminormed ring. A **seminormed module** over $(R, |\cdot|_R)$ is a module over $(R, |\cdot|_R)$ in $(\mathbb{R}_+^{\text{sets}}, \otimes, \otimes_m)$. More concretely, this is a seminormed abelian group $(M, +, |\cdot|_M)$ together with a multiplication map $\cdot : R \times M \to M$ that makes $M$ an $R$-module in the usual sense and such that

$$|a \cdot m|_M \leq |a|_R \cdot |m|_M.$$ 

We will denote $\text{SNMOD}(R, |\cdot|_R)$ the category of modules over $(R, |\cdot|_R)$.

Let $(R, |\cdot|_R, +, \times)$ be a seminormed ring, and let $X$ be an object in $\mathbb{R}_+^{\text{sets}}$. One may equip the set $R^{(X)} := \text{Hom}_{\text{sets}-f_{s}}(X, R)$ of all finitely supported maps from $X$ to $R$ with the $\ell^1$-grading given by

$$\left\| \sum_X a_x \{x\} \right\|_1 := \sum |a_x|_R \cdot |x|_X.$$ 

This gives a seminormed $R$-module structure $(R^{(X)}, \| \cdot \|_1)$ called the free seminormed module on $R$. If $f : X \to Y$ is a morphism in $\mathbb{R}_+^{\text{sets}}$, and

$$f_* : R^{(X)} \to R^{(Y)}$$

is the associated module map, given by

$$(f_* a)_y := \sum_{x \in X, f(x) = y} a_x,$$

then we have

$$\|f_* a\|_1 := \left\| \sum_{y \in Y} \left( \sum_{f(x) = y} a_x \right) \{y\} \right\|_1 = \sum_{y \in Y} \left( \sum_{f(x) = y} a_x \right) R \cdot |y|_Y \leq \sum_{y \in Y} \sum_{f(x) = y} (|a_x|_R \cdot |f(x)|_Y)$$

so that

$$\|f_* a\|_1 \leq \sum_{y \in Y} \sum_{f(x) = y} (|a_x|_R \cdot |x|_X) = \sum_{x \in X} |a_x|_R \cdot |x|_X = \|a\|_1.$$
which implies that $f_*$ is a morphism in $\mathbb{R}^{\text{SETS}}_{+\leq 1}$, so that $X \mapsto R^{(X)}$ gives a functor

$$R^{(\cdot)} : \mathbb{R}^{\text{SETS}}_{+\leq 1} \to \text{SNMOD}(R, | \cdot |_R).$$

The free seminormed module $R^{(X)}$ on an $\mathbb{R}_{+}$-graded set $X$ has the following universal property: if $f : X \to M$ is a contracting morphism from $X$ to a seminormed $R$-module, there exists a unique extension of $f$ to a morphism of seminormed $R$-modules

$$[f] : R^{(X)} \to M.$$

The extension is given by $[f](a) = \sum_x a_x \cdot f(x)$. It is a contracting map since

$$|f(a)|_M = |\sum_x a_x f(x)|_M \leq \sum_x |a_x|_R \cdot |f(x)|_M \leq \sum_x |a_x|_R \cdot |x|_X = \|a\|_1.$$  

Composition with the forgetful functor $\text{SNMOD}(R, | \cdot |_R) \to \mathbb{R}^{\text{SETS}}_{+\leq 1}$ thus gives an endofunctor

$$\Sigma^m_R : \mathbb{R}^{\text{SETS}}_{+\leq 1} \to \mathbb{R}^{\text{SETS}}_{+\leq 1}$$

which is monadic (using the usual composition of linear combinations). One may recover $R$ from $\Sigma^m_R$ by setting $R = \Sigma_R\{\{1\}\}$. This gives an embedding $R \mapsto \Sigma^m_R$ of seminormed rings into monads in $\mathbb{R}^{\text{SETS}}_{+\leq 1}$, similar to the embedding of usual rings in monads in $\text{SETS}$, used by Durov [Dur07] in his theory of generalized rings.

If $R$ is a Banach ring, a seminormed module over $R$ is called a Banach module if the underlying abelian group is complete for the topology induced by its group seminorm. We denote $\text{BANMOD}(R, | \cdot |_R)$ the category of Banach modules over $R$. There is a natural completion functor

$$\hat{\cdot} : \text{SNMOD}(R, | \cdot |_R) \to \text{BANMOD}(R, | \cdot |_R),$$

and the composition of the free seminormed module functor $R^{(\cdot)}$ with $\hat{\cdot}$ gives an endofunctor

$$\hat{\Sigma}^m_R : \mathbb{R}^{\text{SETS}}_{+\leq 1} \to \mathbb{R}^{\text{SETS}}_{+\leq 1}$$

that is monadic. One may still recover $R$ from $\hat{\Sigma}^m_R$ by setting $R = \hat{\Sigma}^m_R\{\{1\}\}$.

One may wonder what happens to these monadic embeddings if we work in the category $\mathbb{R}^{\text{SETS}}_{+\leq 1}$ of $\mathbb{R}_{+}$-graded sets with bounded maps, and with a weakly seminormed ring $(R, | \cdot |_R, +, \cdot, \times)$. We will denote $C$ the norm of the map $(x, y) \mapsto x + y$ and $D$ the norm of the multiplication map $(x, y) \mapsto x \cdot y$ on $R$. In this setting, the free $R$-module $R^{(X)}$ on an $\mathbb{R}_{+}$-graded set $X$, defined as the set of finitely supported maps $a : X \to R$, gives a functor

$$R^{(\cdot)} : \mathbb{R}^{\text{SETS}}_{+\leq 1} \to \text{MOD}(R)$$

with values in (non-graded) $R$-modules. One may define the $\ell^\infty$ grading on $R^{(X)}$ by setting

$$\max_{x \in X} |a_x|_R \cdot |x|_X.$$
The problem is that for $f : X \to Y$ a bounded map, the associated map

$$f_* : R^X \to R^Y$$

given by

$$f_* a_y := \sum_{x \in X, f(x) = y} a_x$$

is not anymore bounded in general. Similarly, the composition of linear combinations

$$\mu : R(R^X) \to R^X$$

is often not bounded. This shows that $X \mapsto R^X$ induces only a monad $\Sigma_R : R_+ \to R_+ Y$ in the category $R_+ Y$ of $R_+$-graded sets with unbounded (i.e., arbitrary) maps, not in the category $R_+ Y_\leq$. If $f : R \to S$ is a bounded morphism of weakly seminormed rings with norm $C_f$, and $X$ is an $R_+$-graded set, we get a natural map

$$[f] : R^X \to S^X$$

that is $R_+$-bounded for the $\ell^\infty$-gradings. Indeed, we have

$$\|f(a)\|_\infty = \left| \sum_x f(a_x) \{x\} \right|_\infty := \max_x (|f(a_x)|_S \cdot |x|_X) \leq C_f \cdot \max_x (|a_x|_R \cdot |x|_X) = C_f \cdot |a|_\infty.$$

We thus have a natural embedding $R \mapsto \Sigma^m_R$ of the category of weakly seminormed rings into the category of monads on $R_+ Y$ with morphisms given by monad morphisms $f : \Sigma \to \Sigma'$ such that for all $R_+$-graded set, $f_X : \Sigma(X) \to \Sigma'(X)$ is bounded.

Even if $\Sigma^m_R$ is not a monad in $R_+ \leq Y$, it has a complete $R_+$-filtration by a natural family of monads on $R_+ \leq Y$, defined in the following way: for $\rho \geq 0$, let $R^X_\leq \rho \subset R^X$ be the subset of finitely supported maps $a : X \to R$ such that for every partition $\bigcup_i Z_i = Z$ of a subset $Z$ of $X$, we have

$$\left| \sum_{z \in Z} a_z \right|_R \leq \rho \cdot \max_i \left( \left| \sum_{z \in Z_i} a_z \right|_R \right),$$

together with the grading induced by the $\ell^\infty$ grading on $R^X$. If $\nu \leq \rho$, we will have

$$R^X_\leq \nu \subset R^X_\leq \rho.$$

Moreover, the free module $R^X$ may be described as the union

$$R^X = \bigcup_{\rho \to \infty} R^X_\leq \rho.$$

If $t > 0$ and $\rho > 0$, we will have

$$(R, |\cdot| R^X_\leq \rho) = (R, |\cdot| R^X_\leq \rho).$$

We thus have a natural embedding $R \mapsto \Sigma^m_R$ of the category of weakly seminormed rings into the category of monads on $R_+ Y$ with morphisms given by monad morphisms $f : \Sigma \to \Sigma'$ such that for all $R_+$-graded set, $f_X : \Sigma(X) \to \Sigma'(X)$ is bounded.
Remark that by definition, the natural embedding $R_{\leq \rho}^{(X)} \hookrightarrow R^{(X)}$ is a bijection if $\rho \geq C$ and $X$ has a cardinal in $\{0, 1, 2\}$. Remark also that $R_{\leq \rho}^{(X)}$ is not an $R$-module in general. For example, if $(R, |\cdot|_R) = (\mathbb{Z}, |\cdot|_{\infty})$, $\rho = C = 2$, and $X = \{1, 2, 3\}$, we get that $\mathbb{Z}_{\leq 2}^{(X)} \subset \mathbb{Z}^3$ is the set of triples $(n_1, n_2, n_3)$ of integers such that

$$|n_1|_{\infty} + |n_2|_{\infty} + |n_3|_{\infty} \leq 2 \cdot \max_i |n_i|_{\infty},$$

and this is not stable by addition, because $(1, 1, 2)$ and $(0, 0, -1)$ are in it, but not $(1, 1, 1)$. We will see that in spite of these defects, the $\mathbb{R}^\text{Sets}$-monad structure on $X \mapsto R^{(X)}$ given by the composition of linear combinations extends to an $\mathbb{R}^\text{Sets}_{+\leq}$ monad structure on each $R_{\leq \rho}^{(X)}$. If $f : (X, |\cdot|_X) \to (Y, |\cdot|_Y)$ is a bounded map of norm $\|f\| = C_f$, then the map

$$f_* : R_{\leq \rho}^{(X)} \to R_{\leq \rho}^{(Y)}$$

given by

$$(f_* a)_y := \sum_{x \in X, f(x) = y} a_x$$

is well defined. Indeed, a partition $\coprod_i Z'_i$ of a subset $Z'$ of $Y$ gives a partition of the subset $Z = f^{-1}(Z')$ of $X$ by $Z_i = f^{-1}(Z'_i)$, so that

$$\left| \sum_{z' \in Z'} (f_* a)_{z'} \right|_R \leq \sum_{z \in Z} \sum_{f(z) = z'} a_z \leq \sum_{a \in Z} a_z \leq \rho \cdot \max_i \left( \sum_{z \in Z_i} a_z \right) = \rho \cdot \max_i \left( \left| \sum_{z' \in Z'_i} f_* a_{z'} \right|_R \right).$$

It is also bounded since

$$\|f_* a\|_{\infty} := \left| \sum_{y \in Y} \left( \sum_{f(x) = y} a_x \right) \{y\} \right|_{\infty} = \max_{y \in Y} \left( \left| \sum_{f(x) = y} a_x \right|_R \cdot |y|_Y \right) \leq \rho \cdot \max_{y \in Y} \max_{f(x) = y} (|a_x|_R \cdot |f(x)|_Y)$$

so that

$$\|f_* a\|_{\infty} \leq \rho \cdot |C_f| \cdot \max_{y \in Y} \max_{f(x) = y} (|a_x|_R \cdot |x|_X) = \rho \cdot C_f \cdot \max_{x \in X} (|a_x|_R \cdot |x|_X) = \rho \cdot C_f \cdot \|a\|_{\infty},$$

which implies that $f_*$ is a morphism in $\mathbb{R}^\text{Sets}_{+\leq}$ of norm smaller than $\rho \cdot C_f$. The endofunctor

$$R_{\leq \rho}^{(\cdot)} : \mathbb{R}^\text{Sets}_{+\leq} \to \mathbb{R}^\text{Sets}_{+\leq}$$

is also monadic since the composition map for linear combinations

$$\mu : R_{\leq \rho}^{(R_{\leq \rho}^{(X)})} \to R_{\leq \rho}^{(X)}$$
is bounded. Indeed, if \( a \in \mathbb{R}_{\leq \rho}^{(\mathbb{R}^{(1)})} \) is given by

\[
a = \sum_{b \in \mathbb{R}_{\leq \rho}^{(\mathbb{R}^{(1)})}} a_{\sum_{x} b_{x} \{x\}} \{\sum_{x} b_{x} \{x\}\},
\]

then we have

\[
|a|_{\infty} = \max_{b \in \mathbb{R}_{\leq \rho}^{(\mathbb{R}^{(1)})}} \left( |a_{\sum_{x} b_{x} \{x\}}|_{R} \cdot |\sum_{x} b_{x} \{x\}|_{\infty} \right)
\]

\[
= \max_{b \in \mathbb{R}_{\leq \rho}^{(\mathbb{R}^{(1)})}} \left( |a_{\sum_{x} b_{x} \{x\}}|_{R} \cdot \max_{x} (|b_{x}|_{R} \cdot |x|_{x}) \right)
\]

\[
= \max_{b \in \mathbb{R}_{\leq \rho}^{(\mathbb{R}^{(1)})}} \left( |a_{\sum_{x} b_{x} \{x\}}|_{R} \cdot \max_{x} (|b_{x}|_{R} \cdot |x|_{x}) \right)
\]

and also

\[
|\mu(a)|_{\infty} = \left| \sum_{x \in X} \left( \sum_{b \in \mathbb{R}_{\leq \rho}^{(\mathbb{R}^{(1)})} \quad b_{x} \neq 0 \quad \sum_{x} a_{\sum_{x} b_{x} \{x\}} \cdot b_{x} \right) \{x\} \right|_{\infty}
\]

\[
= \max_{x \in X} \left( \sum_{b \in \mathbb{R}_{\leq \rho}^{(\mathbb{R}^{(1)})} \quad b_{x} \neq 0 \quad \sum_{x} a_{\sum_{x} b_{x} \{x\}} \cdot b_{x} \right) \cdot |x|_{x}
\]

\[
\leq \rho \cdot D \cdot \max_{x \in X} \left( \max_{b \in \mathbb{R}_{\leq \rho}^{(\mathbb{R}^{(1)})} \quad b_{x} \neq 0 \quad |a_{\sum_{x} b_{x} \{x\}}|_{R} \cdot |b_{x}|_{R} \cdot |x|_{x} \right)
\]

\[
= \rho \cdot D \cdot \max_{b \in \mathbb{R}_{\leq \rho}^{(\mathbb{R}^{(1)})} \quad b_{x} \neq 0 \quad \sum_{x} a_{\sum_{x} b_{x} \{x\}} \cdot R \cdot |b_{x}|_{R} \cdot |x|_{x}
\]

so that

\[
|\mu(a)|_{\infty} \leq \rho \cdot D \cdot |a|_{\infty}.
\]

One may recover \((\mathbb{R}_{\leq \rho}^{(\mathbb{R}^{(1)})}, +, \times, |.|_{R})\) as \(R = \mathbb{R}_{\leq C}^{(11)}\) (where \(C\) denotes, as before, the norm of the addition map \(+\) on \(R\)). The addition and multiplication maps can be found back by composition of the monadic multiplication

\[
\mu : \mathbb{R}_{\leq C}^{(\mathbb{R}^{(1)})} \rightarrow \mathbb{R}_{\leq C}^{(\mathbb{R}^{(1)})}
\]

with the (bounded) embeddings

\[
[+] : \mathbb{R}^{2} \rightarrow \mathbb{R}_{\leq C}^{(\mathbb{R}^{(1)})}
\]

\[
(a, b) \mapsto 1 \cdot \{a\} + 1 \cdot \{b\}
\]
and

\[
[x] : \ \mathbb{R}^2 \to R_{\leq C}^{(R_{\leq C}(\{11\}))},
\]

\[
(a, b) \mapsto a \cdot \{b\}.
\]

Recall however that a bounded morphism \( f : R \to S \) of weakly seminormed rings only induces a morphism \( R^{(i)} \to S^{(i)} \) of monads in \( \mathbb{R}_{\leq 1}^{\text{sets}} \), and not any morphism \( R_{\leq \rho}^{(i)} \to S_{\leq \rho}^{(i)} \) in general. This makes the theory of weakly seminormed rings algebraically much more complicated than the theory of seminormed rings, and explains why we will mostly work with seminormed structures and the category \( \mathbb{R}_{\leq 1}^{\text{sets}} \) from now on.

**Remark 1.** We may think of

\[ R_{\rho}^{(i)} : X \mapsto R_{\leq \rho}^{(X)} \]

as a kind of ring of \( \rho \)-integers in \( R \), analogous to Durov’s archimedean ring of integers \( \mathbb{Z}_\infty \subset \mathbb{R} \) from [Dur07]. It is possible to define an ideal

\[ R_{\rho}^{(i)} : X \mapsto R_{\leq \rho}^{(X)} \]

in this monad, with quotient given denoted by \( \tilde{R}_{\rho} := R_{\rho}/R_{\rho}^{(i)} \). The \( \mathbb{R}_{\leq 1}^{\text{sets}} \)-graded monad

\[ \text{Gr}(R, | \cdot |_R) := \bigoplus_{r \in \mathbb{R}_{|R|}^{\leq 1}} \tilde{R}_{\rho} \]

then gives a natural archimedean analog of Temkin’s reduction from [Tem04], Section 3, that may be useful to describe the archimedean points of the dagger analytic topoi, if we start from a usual seminormed ring \( R \).

### 2.3 Seminormed polynomials and convergent power series

Let \((X, | \cdot |_X)\) be an object of \( \mathbb{R}_{\leq 1}^{\text{sets}} \) and \((\mathbb{N}, +, 0)\) be the additive monoid of non-negative integers. Let

\[ (X)^{\mathbb{N}} := \text{Hom}_{\text{sets-fs}}(X, \mathbb{N}) \]

be the monoid of monomials on \( X \), given by the set of finitely supported maps from \( X \) to \( \mathbb{N} \), seen as a multiplicative monoid whose generic element is of the form \( X^\alpha \) for \( \alpha : X \to \mathbb{N} \). The multiplicative grading on \((X)^{\mathbb{N}}\) is given by

\[ \|X^\alpha\|_m := \prod_{x \in X, \alpha(x) \neq 0} |x|^{\alpha(x)}, \]

with the convention that the empty product is equal to 1. This is a multiplicative monoid map \( \| \cdot \|_m : (X)^{\mathbb{N}} \to \mathbb{R}_+ \). Indeed, if \( \alpha, \beta : X \to \mathbb{N} \) are two finitely supported maps, then
we have
\[
\|X^\alpha \cdot X^\beta\|_m := \|X^{\alpha+\beta}\|_m := \prod_{x \in X, (\alpha+\beta)(x) \neq 0} |x|_{X}^{(\alpha+\beta)(x)}
\]
\[
= \left( \prod_{x \in X, \alpha(x) \neq 0, \beta(x) = 0} |x|_{X}^{\alpha(x)} \right) \cdot \left( \prod_{x \in X, \alpha(x) \neq 0, \beta(x) \neq 0} |x|_{X}^{(\alpha+\beta)(x)} \right) \cdot \left( \prod_{x \in X, \beta(x) \neq 0, \alpha(x) = 0} |x|_{X}^{\beta(x)} \right)
\]
and also
\[
\|X^\alpha\|_m \cdot \|X^\beta\|_m := \left( \prod_{x \in X, \alpha(x) \neq 0} |x|_{X}^{\alpha(x)} \right) \cdot \left( \prod_{x \in X, \beta(x) \neq 0} |x|_{X}^{\beta(x)} \right)
\]
\[
= \left( \prod_{x \in X, \alpha(x) \neq 0, \beta(x) = 0} |x|_{X}^{\alpha(x)} \right) \cdot \left( \prod_{x \in X, \alpha(x) \neq 0, \beta(x) \neq 0} |x|_{X}^{\alpha(x)} \right) \cdot \left( \prod_{x \in X, \beta(x) \neq 0, \alpha(x) = 0} |x|_{X}^{\beta(x)} \right) \cdot \left( \prod_{x \in X, \beta(x) \neq 0, \alpha(x) \neq 0} |x|_{X}^{\beta(x)} \right)
\]
so that
\[
\|X^\alpha \cdot X^\beta\|_m = \|X^\alpha\|_m \cdot \|X^\beta\|_m.
\]
Now suppose given a contracting map \(f : X \to Y\) and \(\alpha \in (X)^N\). As before, we define
\[
f_* \alpha(y) = \sum_{x \in f^{-1}(y)} \alpha(x).
\]
Suppose given a fixed \(y \in Y\) such that \(f_* \alpha(y) := \sum_{x \in f^{-1}(y)} \alpha(x) \neq 0\). Then
\[
|y|_{Y}^{f_* \alpha(y)} = |y|_{Y}^{\sum_{x \in f^{-1}(y)} \alpha(x)} = \prod_{x \in f^{-1}(y), \alpha(x) \neq 0} |f(x)|_{Y}^{\alpha(x)}.
\]
This implies that
\[
\|f_* \alpha\|_m := \prod_{y \in Y, f_* \alpha(y) \neq 0} |y|_{Y}^{f_* \alpha(y)} = \prod_{x \in f^{-1}(Y), \alpha(x) \neq 0} |f(x)|_{Y}^{\alpha(x)},
\]
and since \(f : X \to Y\) is contracting, we get
\[
\|f_* \alpha\|_m \leq \prod_{x \in f^{-1}(Y), \alpha(x) \neq 0} |x|_{X}^{\alpha(x)} = \prod_{x \in X, \alpha(x) \neq 0} |x|_{X}^{\alpha(x)} = \|\alpha\|_m.
\]
This shows that \(X \mapsto (X)^N\) gives a functor
\[
(\cdot)^N : \mathbb{R}^N_{+, \leq 1} \to \text{MON}(\mathbb{R}^N_{+, \leq 1}, \otimes_m).
\]
This free multiplicatively seminormed monoid has the following universal property: if \( f : X \to N \) is a contracting map from an \( \mathbb{R}_+ \)-graded set to a (commutative) multiplicatively seminormed monoid, there is a unique extension

\[ [f] : (X)^N \to N \]

of \( f \) to a morphism of multiplicatively seminormed monoids, given by

\[ [f](\alpha) := \prod_{x \in X, \alpha(x) \neq 0} f(x)^{\alpha(x)}. \]

Now we may combine the above two constructions: if \((X, |\cdot|_{X})\) is an \( \mathbb{R}_+ \)-graded set and \((R, |\cdot|_{R}, +, \times)\) is a seminormed ring, we may define the seminormed polynomial ring \( R[X] \) by setting \( R[X] := R^{(X)^N} \), together with the \( \ell^1 \)-seminorm \( |\cdot|_{m} \) associated to the multiplicative seminorm \( |\cdot|_{m} \) on \((X)^N\). By construction, this is a seminormed \( R \)-module together with a multiplication given by the usual polynomial multiplication

\[
\left( \sum a_{\alpha} X^{\alpha} \right) \cdot \left( \sum b_{\alpha} X^{\alpha} \right) := \sum_{\gamma} \left( \sum_{\alpha + \beta = \gamma} a_{\alpha} \cdot b_{\beta} \right) X^{\gamma}.
\]

We also have

\[
\left| \left( \sum a_{\alpha} X^{\alpha} \right) \cdot \left( \sum b_{\alpha} X^{\alpha} \right) \right| \leq \sum_{\gamma} \left( \sum_{\alpha + \beta = \gamma} |a_{\alpha}|_{R} \cdot |b_{\beta}|_{R} \right) |X^{\alpha}| \cdot |X^{\gamma}| = \left| \sum a_{\alpha} X^{\alpha} \right| \cdot \left| \sum b_{\alpha} X^{\alpha} \right|.
\]

Combined with our previous results, this shows that we have defined a functor

\[ R[\cdot]_{1} : \mathbb{R}_{+}^{\leq} \to \text{SNRings}. \]

The polynomial \( R \)-algebra has the following universal property: if \( X \) is an \( \mathbb{R}_+ \)-graded set, \( R \to S \) is a (contracting) morphism of seminormed rings, and \( f : X \to S \) is a contracting map, then \( f \) extends uniquely to a morphism of seminormed \( R \)-algebras

\[ [f] : R[X]_{1} \to S. \]

Composition with the forgetful functor \( \text{SNRings} \to \mathbb{R}_{+}^{\leq} \) gives an endofunctor

\[ \Sigma_{R}^{\cdot} : \mathbb{R}_{+}^{\leq} \to \mathbb{R}_{+}^{\leq} \]

which is monadic, since it is constructed as the composition of two monadic functors. One may recover \( R \) from \( \Sigma_{R}^{\cdot} \) by setting \( R = \Sigma_{R}^{\cdot}(\{\emptyset\}) \).

There is a natural completion functor

\[ \hat{-} : \text{SNRings} \to \text{BanRings} \]

that sends a seminormed ring to its completion. If \( R \) is a Banach ring, we will denote by

\[ R(\cdot) : \mathbb{R}_{+}^{\leq} \to \text{BanRings} \]

the composition of the completion functor with \( R[\cdot]_{1} \), that sends \( X \) to \( R(X) := \hat{R}[X] \). It still induces a monadic functor on \( \mathbb{R}_{+}^{\leq} \).
Definition 6. A weakly seminormed ring \((A, \cdot |\cdot|_A)\) is called uniform if its seminorm is power-multiplicative, meaning that 
\[ |a^n|_A = |a|^n \] for all \(a \in A\) and \(n \in \mathbb{N}\).

We denote \(\text{SNRINGS}_u\) (resp. \(\text{BANRINGS}_u\)) the category of uniform (resp. complete uniform) seminormed rings.

We will now show that, if we work with the category of uniform weakly seminormed rings, which are the basic building blocks of global analytic geometry, we may restrict to the category of uniform seminormed rings without loosing any information.

Lemma 2. The natural functor
\[ \text{SNRINGS}_u \to \text{SNRINGS}_{w,u} \]
from uniform seminormed rings to uniform weakly seminormed rings is fully faithful and every weakly seminormed ring is of the form \((R, |\cdot|_t^R)\) for some \(t > 0\) and \((R, |\cdot|_R)\) a seminormed ring.

Proof. Let \((R, |\cdot|_R, +, \times)\) be a uniform weakly seminormed ring. Then by [Art67], Theorem 3, there exists \(t > 0\) such that \(|\cdot|_t^R\) fulfills the triangle inequality
\[ |a + b|_t^R \leq |a|_t^R + |b|_t^R. \]
Moreover, if we have \(|a \cdot b|_R \leq C|a|_R \cdot |b|_R\), then taking \(n\)-th powers of the arguments, \(n\)-th roots of the terms in the inequality, and passing to the limit \(n \to \infty\), we get \(|a \cdot b|_R \leq |a|_R \cdot |b|_R\). This shows the last statement. Similarly, if \(f : R \to S\) is a bounded morphism of uniform weakly seminormed rings, the above “\(n\)-th roots of unity argument” shows that \(f\) is contracting. This shows that the inclusion functor is full. It is clearly faithful, which finishes the proof that it is an equivalence of categories. 

There is a natural “uniformization” functor
\[ U : \text{SNRINGS} \to \text{SNRINGS}_u \]
that sends \((R, |\cdot|_R)\) to \(R\) equipped with the seminorm given on \(a \in A\) by
\[ |a|_{R,u} := \lim_{n \to +\infty} \sqrt[n]{|a^n|}. \]

We will denote by
\[ R\{\cdot\} : \mathbb{R}^{\text{SETS}}_{+ \leq 1} \to \text{BANRINGS}_u \]
the composition of \(U\) with \(R\{\cdot\}\). This still induces a monadic functor on \(\mathbb{R}^{\text{SETS}}_{+ \leq 1}\).

If \(X = \rho = (\rho_1, \ldots, \rho_n) \in \mathbb{R}_+^n\), we denote \(\rho^{-1}T\) the set of variables \(\{\rho_1^{-1}T_1, \ldots, \rho_n^{-1}T_n\}\) equipped with the \(\mathbb{R}_+\)-grading given by \(|\rho_i^{-1}T_i| = \rho_i\). We will denote
\[ R\langle \rho^{-1}T \rangle := R\langle X \rangle \quad \text{and} \quad R\{\rho^{-1}T\} := R\{X\}. \]
Proposition 1. If \((R, | \cdot |_R)\) is a seminormed ring, the category \(\text{MOD}(R, | \cdot |_R)\) has arbitrary colimits, finite limits and internal homomorphisms. It also has a natural symmetric monoidal structure called the projective tensor product. Seminormed rings (resp. uniform seminormed rings) have finite colimits and finite limits.

Proof. The fact that seminormed modules have arbitrary colimits, finite limits and internal homomorphisms follows from the fact that the category \(\mathbb{R}^{\text{Sets}}_\leq 1\) has these properties. For example, if \((M, | \cdot |_M)\) is a seminormed abelian group and \(R \subset M\) be a submodule, the map \(M/R \to \mathbb{R}^+_\leq 1\) defined by

\[
|x| = \inf_{m \in \pi^{-1}(x)} |m|_M
\]

is a group seminorm on \(M/R\) called the residue seminorm. If \(R\) is a seminormed ring and \(I \subset R\) is an ideal, then the residue seminorm on \(R/I\) is a ring seminorm that makes \(A/I\) the quotient seminormed ring. Similarly, if \(f, g : (R, | \cdot |_R) \to (S, | \cdot |_S)\) are two parallel morphisms of seminormed rings, then the coequalizer \(\text{coker}(f, g)\) is given by the ring \(S/I\), where \(I\) is the ideal generated by elements of the form \(f(x) - g(x)\) for \(x \in R\), equipped with the residue seminorm. Let \((R, | \cdot |_R)\) be a seminormed ring and \((M, | \cdot |_M)\) and \((N, | \cdot |_N)\) be two seminormed modules over \((R, | \cdot |_R)\). The seminorm on \(M \otimes_R N\) given by the residue seminorm of the \(\ell^1\) module seminorm

\[
\| \cdot \|_1 : \quad R^{(M \times N)} \to \mathbb{R}^+_\leq 1 \quad a_{(m,n)}(m,n) \mapsto \sum |a_{(m,n)}| \cdot |m|_M \cdot |n|_N
\]

along the quotient map \(R^{(M \times N)} \to M \otimes_A N\) is called the projective tensor product seminorm and denoted \(\| \cdot \|_p : M \otimes_A N \to \mathbb{R}^+_\leq 1\). This gives a symmetric monoidal structure \(\otimes_{R,1}\) on \(\text{MOD}(R)\). The initial seminormed ring (resp. uniform seminormed ring) is given by the ring of integers \(\mathbb{Z}\) together with its archimedean seminorm \(\| \cdot \|_\infty\). Indeed, \(\mathbb{Z}\) is the initial ring, and the maximal ring seminorm on \(\mathbb{Z}\) must fulfill \(|1| = 1\) and \(|1+1| = |1| + |1| = 2\), so that all ring seminorms on \(\mathbb{Z}\) are bounded by this one. If \((R_i, | \cdot |_i)_{i \in I}\) is a finite family of seminormed rings, the projective tensor product seminorm on \(\otimes_{(\mathbb{Z}, | \cdot |_\infty)}(R_i, | \cdot |_i)\) is a ring seminorm that gives the coproduct of the family. We already showed that seminormed rings have coequalizers. The same is true for uniform seminormed rings using the uniformization functor \(U\). This shows the statement about finite colimits of seminormed rings. Now the non-empty finite limits of the underlying \(\mathbb{R}^+_\leq 1\)-graded sets in \(\mathbb{R}^{\text{Sets}}_\leq 1\) give limits in seminormed rings and uniform seminormed rings.

3 Dagger algebras

We will start this section by explaining the motivations underlying the introduction of the category of dagger algebras.

An important drawback of Banach rings like \(R\langle \rho^{-1}T \rangle\) or \(R\{\rho^{-1}T\}\) is that they are often non-noetherian (for example, if \(\rho = 1\) and \(R = \mathbb{C}\), one gets the ring of continuous
functions on the complex unit disc that are analytic on its interior). Moreover, Banach rings of convergent power series don’t have a nice differential calculus, at least over \( \mathbb{Q}_p \), since the Poincaré lemma fails for them. These two facts are our main motivations for the introduction of rings of overconvergent power series.

A power series \( f \in R[[T]] \) in one variable over a Banach ring \( R \) is overconvergent on a disc of radius \( \rho \) if it converges on a disc of radius \( \nu > \rho \). We would like to define the ring of overconvergent power series as the filtered colimit of the rings \( R\langle \nu^{-1}T \rangle \). However, this filtered colimit in the category of Banach rings is simply \( R\langle \rho^{-1}T \rangle \). So we need to work in a category that contains Banach rings and that allows us to keep track of the overconvergence properties. This will be the category of ind-Banach rings, that is already discussed (in the complex situation) in Bambozzi’s thesis [Bam14], Section 3.3.

Since we want to get back \( p \)-adic analytic spaces in the sense of Berkovich when we work over the Banach ring \( \mathbb{Q}_p = (\mathbb{Q}_p, |·|_p) \), we will not work with general Banach rings, but with uniform Banach rings. If we denote \( \otimes \) the coproduct in the category \( \text{BANRINGS}_u \) of uniform Banach rings, we get for example for \( \mathbb{Z} = (\mathbb{Z}, |·|_\infty) \), natural isomorphisms
\[
\mathbb{Q}_p\{\rho^{-1}T\} \cong \mathbb{Z}\{\rho^{-1}T\} \otimes_{\mathbb{Z}} \mathbb{Q}_p
\]
and
\[
\mathbb{C}\{T\} \cong \mathbb{Z}\{\rho^{-1}T\} \otimes_{\mathbb{Z}} \mathbb{C},
\]
where we use the two contracting morphisms \( \mathbb{Z} = (\mathbb{Z}, |·|_\infty) \to (\mathbb{C}, |·|_\mathbb{C}) = \mathbb{C} \) and \( \mathbb{Z} = (\mathbb{Z}, |·|_\infty) \to (\mathbb{Q}_p, |·|_p) = \mathbb{Q}_p \) to extend the scalars. The coproduct of uniform Banach rings thus exactly gives, by scalar extension, the rings of analytic functions on a disc that we would like to use both in the \( p \)-adic and in the complex case.

We will thus define the ring \( R\{\rho^{-1}T\}^\dagger \) of overconvergent power series on a given polydisc of radius \( \rho \) as the uniform ind-Banach ring given by the filtered colimit of the uniform Banach rings \( R\{\nu^{-1}T\} \) for \( \nu > \rho \).

An important drawback of the category of uniform Banach rings is that they are reduced, so that they exclude the use of nilpotent elements, that has proved (since Weil) to be so useful to do differential calculus in a geometric setting. This will lead us to the definition of the category of dagger algebras over a given Banach ring, that is inspired by the work of Dubuc-Zilber [DZ94] on analytic models for synthetic analytic geometry, of Lurie [Lur09a] on general geometries and of Porta on complex derived geometry [Por14].

A dagger rational domain in an overconvergent polydisc \( D^\dagger(0, \rho) \) (given by the Berkovich spectrum of the ind-Banach ring \( R\{\rho^{-1}T\}^\dagger \), to be defined in this section) is a subspace defined by finitely many inequalities of the form
\[
D(g, 1|r_1, f_1, \ldots, r_n, f_n) = \{ x \in D^\dagger(0, \rho), |f_i(x)| \leq r_i|g(x)| \},
\]
for \( (f_i, g) \) generating the underlying ring of \( R\{\rho^{-1}T\}^\dagger \) as an ideal and \( r_i > 0 \). One may define the ring of overconvergent functions on \( D \) as the quotient
\[
R\{\rho^{-1}T, r^{-1}X\}^\dagger/(gX_1 - f_1, \ldots, gX_n - f_n)
\]
in the category of uniform ind-Banach rings, and morphisms of dagger rational domains will be defined as morphisms of the corresponding uniform ind-Banach algebras over $R$.

A dagger algebra $A$ over a Banach ring $R$ will be a "functor of functions"

$$A : \text{Rat}_{(R, |\cdot|_R)}^\dagger \to \text{Sets}$$
on the category $\text{Rat}_{(R, |\cdot|_R)}^\dagger$ of dagger rational domains in finitely generated overconvergent power series rings $R\{\rho^{-1}T\}^\dagger$ for varying $\rho$ and $T$, that commutes with finite products and sends pullback diagrams of the form

$$
\begin{array}{ccc}
D_1 & \xrightarrow{\zeta} & D_2 \\
\downarrow & & \downarrow \\
D_3 & \xleftarrow{i} & D_4
\end{array}
$$

with $i$ an embedding of a sub-rational domain, to pullbacks. There is a natural fully faithful embedding

$$\text{RATALG}_{(R, |\cdot|_R)}^\dagger := \text{Rat}_{(R, |\cdot|_R)}^\dagger_{\text{op}} \to \text{ALG}_{(R, |\cdot|_R)}^\dagger$$
of the category of rational domain algebras to the category of dagger algebras.

### 3.1 Overconvergent power series

We will first use the indization of the category of uniform Banach rings to get a convenient category of overconvergent algebras over a general Banach ring. We will use [KS06], Chapter 6, as a general reference on indization of categories.

There is a natural Yoneda embedding

$$\text{BANRINGS}_{u} \to \text{BANRINGS}_{\gamma} = \text{Hom} (\text{BANRINGS}_{u}^{\text{op}}, \text{SETS})$$

$$(A, |\cdot|_A) \mapsto (B, |\cdot|_B) \mapsto \text{Hom}_{\text{BANRINGS}_{u}} ((B, |\cdot|_B), (A, |\cdot|_A)) .$$

**Definition 7.** The category ind-$\text{BANRINGS}_{u}$ of (uniform) ind-Banach rings is the full subcategory of $\text{BANRINGS}_{\gamma}$ whose objects are isomorphic to filtered colimits of uniform Banach rings.

Recall some basic facts about ind-objects in a given category $\mathcal{C}$. If $\alpha : I \to \mathcal{C}$ and $\beta : J \to \mathcal{C}$ are two filtrant systems in $\mathcal{C}$, we may compute the morphisms between the corresponding ind-objects by setting

$$\text{Hom}_{\text{ind-}\mathcal{C}} (\text{colim}_i \alpha(i), \text{colim}_j \beta(j)) := \lim_{i,j} \text{colim} \text{Hom}_{\mathcal{C}} (\alpha(i), \beta(j)) .$$

If $f : A \to B$ is a morphism between two ind-objects in $\mathcal{C}$, there exists a filtrant category $I$, two functors $\alpha : I \to \mathcal{C}$ and $\beta : I \to \mathcal{C}$ and a morphism of functors $\varphi : \alpha \to \beta$ such that $\text{colim}_I \varphi = f$ in ind-$\mathcal{C}$. More generally, if $f, g : A \to B$ is a pair of parallel morphisms between two ind-objects in $\mathcal{C}$, there exists a filtrant category $I$, two functors $\alpha : I \to \mathcal{C}$
and $\beta : I \to C$ and two morphisms of functors $\varphi, \psi : \alpha \to \beta$ such that $f = \text{colim}_I \varphi$ and $g = \text{colim}_I \psi$. This implies that the cokernel in ind-$C$ of $f$ and $g$ may be computed as the colimit of the objectwise cokernels

$$\text{coker}(f, g : A \to B) = \text{colim}_I (\text{coker}(\varphi(i), \psi(i) : \alpha(i) \to \beta(i))).$$

We have more generally the following result (see [KS06], Proposition 6.1.18):

**Proposition 2.** Let $C$ be a category that admits cokernels (resp. finite coproducts, resp. finite colimits). Then ind-$C$ admits cokernels (resp. small coproducts, resp. small colimits) and the natural embedding $C \to \text{ind}$-$C$ commutes with cokernels (resp. finite coproducts, resp. finite colimits).

Since BanRings$_u$ has finite colimits, ind-BanRings$_u$ has small colimits and the natural embedding

$$\text{BanRings}_u \to \text{ind}$\text{-BanRings}_u$$

sends finite colimits to finite colimits, so that the coproduct of a diagram of (uniform) Banach rings seen as a (uniform) ind-Banach ring is simply given by their uniform projective tensor product, and the cokernel of a pair of morphisms of (uniform) Banach rings is given by the quotient uniform Banach ring. More generally, the colimit of a finite diagram of uniform ind-Banach rings may be computed by using filtered colimits of uniform projective tensor products and uniform quotients of the component uniform Banach rings of the given diagram.

In particular, the Banach ring $\mathbb{Z} = (\mathbb{Z}, \| \cdot \|_{\infty})$ is the initial uniform ind-Banach ring.

**Example 1.** Here is an interesting example of a non-trivial ind-Banach algebra. For every finite étale extension $K$ of $\mathbb{Q}$, we denote $O_K = (O_K, \| \cdot \|_{\infty})$ its ring of integers equipped with the norm

$$\|f\|_{\infty} := \max_{\sigma : K \to \mathbb{C}} |\sigma(f)|_{\infty}.$$  

Then the ring of integers $\bar{\mathbb{Z}}$ of $\bar{\mathbb{Q}}$ may be equipped with an ind-Banach ring structure $\bar{\mathbb{Z}}$ given by the fact that it is the union of all $O_K$ for $K$ finite étale over $\mathbb{Q}$. There is a natural action of $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on $\bar{\mathbb{Z}}$.

The forgetful functor $\text{Ring} : \text{BanRings}_u \to \text{Rings}$ giving the underlying ring of a Banach ring extends to an “underlying ring” functor

$$\text{Ring} : \text{ind}$\text{-BanRings}_u \to \text{Rings},$$

which is given by taking the filtered colimit of the underlying rings of a given diagram of uniform Banach rings.

**Definition 8.** The Berkovich spectrum $\mathcal{M}(A)$ of an ind-Banach ring $A$ is the set of equivalence classes of morphisms $\chi : A \to (K, \| \cdot \|)$ where $(K, \| \cdot \|)$ is a multiplicatively
normed Banach field. If $A$ is a Banach ring, we equip $\mathcal{M}(A)$ with the coarsest topology that makes all evaluation maps

$$|a(\cdot)| : \mathcal{M}(A) \to \mathbb{R}_+$$

for $a \in A$ continuous. If $A$ is an ind-Banach ring, we equip $\mathcal{M}(A)$ with the projective limit topology

$$\mathcal{M}(A) = \lim M(A_i)$$

for a description $A = \text{colim} A_i$ of $A$ as a colimit of uniform Banach rings. If $x$ is a point of $\mathcal{M}(A)$, the minimal Banach field $(\mathcal{K}(x), |\cdot|_x)$ in the corresponding equivalence class is called the residue field at $x$.

The Berkovich spectrum is clearly functorial.

We now define overconvergent power series over a given Banach ring.

**Definition 9.** Let $(X, |\cdot|_X)$ be an $\mathbb{R}_+$-graded set. A grading $|\cdot|_1$ on $X$ is called an over-grading if

$$|x|_X < |x|_1 \text{ for all } x \in X.$$  

If $(R, |\cdot|_R)$ is a (uniform) seminormed ring, an over-grading of $|\cdot|_R$ that is also a (uniform) seminorm will be called a (uniform) over-seminorm. Let $R$ be a uniform Banach ring and $X$ be an $\mathbb{R}_+$-graded set. The uniform ind-Banach ring of overconvergent power series on $R$ is defined as the (formal) filtered colimit, i.e., ind-uniform Banach ring given by

$$R\{X\}^\dagger := \text{colim}_{|\cdot| : X \to \mathbb{R}_+ \text{ over-seminorms}} R\{(X, |\cdot|_1)\}.$$  

In particular, if $\rho = (\rho_1, \ldots, \rho_n) \in \mathbb{R}_{>0}^n$ is a polyradius, the uniform ind-Banach ring $R\{\rho_1^{-1}T_1, \ldots, \rho_n^{-1}T_n\}^\dagger$ of overconvergent power series of radius $\rho$ is given by the colimit

$$R\{\rho_1^{-1}T_1, \ldots, \rho_n^{-1}T_n\}^\dagger := \text{colim}_{\nu > \rho} k\{\nu_1^{-1}T_1, \ldots, \nu_n^{-1}T_n\}.$$  

Recall that the Banach ring $R\{\rho^{-1}T\}$ has the following universal property: for every bounded morphism of uniform Banach rings $R \to R'$ and every element $f \in R'$ such that $|f|_{R'} \leq \rho$, there exists a unique commutative diagram:

$$\begin{array}{ccc}
\bigcirc & \longrightarrow & \bigcirc \\
\uparrow & & \uparrow \\
\bigcirc & & \bigcirc \\
\end{array}$$

Similarly, the ind-Banach ring $R\{\rho^{-1}T\}^\dagger$ has the following universal property: there is an isomorphism natural in $R' = \text{colim} R'_i$:

$$\text{Hom}_R(R\{\rho^{-1}T\}^\dagger, \text{colim} R'_i) \cong \text{Hom}(R, R') \times \lim_{\rho' > \rho} \text{colim}_{i} \{f \in R'_i, |f|_{R'_i} \leq \rho'\}.$$
There is a natural coproduct on the category of overconvergent power series algebras which is given by
\[ R\{\rho^{-1}T\}^\dagger \otimes_R R\{\nu^{-1}S\}^\dagger \cong R\{\rho^{-1}T,\nu^{-1}S\}^\dagger. \]
More generally, this formula also works for quotients by finitely generated ideals
\[ R\{\rho^{-1}T\}^\dagger / I \otimes_R R\{\nu^{-1}S\}^\dagger / J \cong R\{\rho^{-1}T,\nu^{-1}S\}^\dagger / (I, J) \]
because the indization functor commutes with finite colimits. The formation of overconvergent power series also commutes with extensions of the base Banach ring, meaning that for \( f : R \to S \) a bounded morphism, we have
\[ S\{X\}^\dagger \cong R\{X\}^\dagger \otimes_R S. \]
This allows us to define overconvergent power series over general uniform ind-Banach rings.

**Definition 10.** Let \( R \) be a uniform ind-Banach ring and \((X, | \cdot |_X)\) be an \( \mathbb{R}_+ \)-graded set. Let \( Z = (\mathbb{Z}, | \cdot |_{\infty}) \to R \) be the canonical morphism. The uniform ind-Banach ring of overconvergent power series on \( R \) with variables in \( X \) is defined by
\[ R\{X\}^\dagger := Z\{X\}^\dagger \otimes_Z R. \]

### 3.2 Rational domains and dagger algebras

In this subsection, we will denote \( R \) a fixed uniform ind-Banach ring. We want to describe rational domains in the overconvergent polydisc \( D^\dagger_{R}(0, \nu) := \mathcal{M}(R\{\nu^{-1}T\}^\dagger) \) defined by finitely many inequalities of the form
\[ D(f_0, \rho_0|\rho_1, f_1, \ldots, \rho_n, f_n) = \{ x \in D^\dagger_{R}(0, \nu), \ \rho_0|f_i(x)| \leq \rho_i|f_0(x)| \}, \]
for \((f_i)\) generating \( \text{Ring}(R\{\nu^{-1}T\})^\dagger \) as an ideal and \( \rho_i > 0 \). This will be done by defining directly the associated ind-uniform algebras.

**Definition 11.** Let \( A \) be a uniform ind-Banach ring and let \((f_0, \ldots, f_n) \in \text{Ring}(A)^{n+1} \) be a finite family of elements of the underlying ring of \( A \) that generate \( \text{Ring}(A) \) as an ideal, and \( \rho = (\rho_0, \ldots, \rho_n) \in \mathbb{R}_{>0}^{n+1} \).

1. The rational domain (resp. dagger rational domain) algebra \( A\{\rho_0, f|\rho_1, f_1, \ldots, \rho_n, f_n\} \) (resp. \( A\{\rho_0, f|\rho_1, f_1, \ldots, \rho_n, f_n\}^\dagger \)) is the uniform ind-Banach algebra over \( A \) given by the quotient
\[ A\{(\rho_1/\rho_0)^{-1}T_1, \ldots, (\rho_n/\rho_0)^{-1}T_n\}/(f_0T_1 - f_1, \ldots, f_0T_n - f_n) \]
(resp. \( A\{(\rho_1/\rho_0)^{-1}T_1, \ldots, (\rho_n/\rho_0)^{-1}T_n\}^\dagger/(f_0T_1 - f_1, \ldots, f_0T_n - f_n) \)).
2. The associated rational domain (resp. dagger rational domain) is the morphism
\[ D(\rho_0, f_0|\rho_1, f_1, \ldots, \rho_n, f_n) := \mathcal{M}(A(\rho_0, f_0|\rho_1, f_1, \ldots, \rho_n, f_n)) \to \mathcal{M}(A) \]
(resp. \[ D^\dagger(\rho_0, f_0|\rho_1, f_1, \ldots, \rho_n, f_n) := \mathcal{M}(A(\rho_0, f_0|\rho_1, f_1, \ldots, \rho_n, f_n)^\dagger) \to \mathcal{M}(A) \]).

3. The family of morphisms
\[ \left\{ D(f_i, \rho_i|f_0, \rho_0 \ldots, \hat{f}_i, \rho_i, \ldots, f_n, \rho_n) \to \mathcal{M}(A) \right\} \]
(resp. \[ \left\{ D^\dagger(f_i, \rho_i|f_0, \rho_0 \ldots, \hat{f}_i, \rho_i, \ldots, f_n, \rho_n) \to \mathcal{M}(A) \right\} \])
is called the standard covering associated to the family \[ \{(f_i, \rho_i)\}_{i=0,\ldots,n} \].

4. A dagger rational domain algebra (resp. rational domain covering) is called strict if the polyradius \( \rho = (\rho_0, \ldots, \rho_n) \) has all components equal to 1.

It is clear from the definition and properties of the coproduct of uniform ind-Banach rings that the pushout of a rational domain (resp. dagger rational domain) algebra along a morphism of uniform ind-Banach rings is again a rational domain (resp. dagger rational domain) algebra, and that a rational domain (resp. dagger rational domain) algebra on a rational domain (resp. dagger rational domain) algebra over \( R \) is a rational domain (resp. dagger rational domain) algebra over \( R \).

We will denote \( \text{RAT} \text{ALG}_R^{an} \) (resp. \( \text{RAT} \text{ALG}_R^\dagger \)) the category of ind-Banach algebras that are isomorphic to rational domain algebras over power series algebras \( R\{\nu^{-1}T\} \) (resp. dagger rational domain algebras over overconvergent power series algebras \( R\{\nu^{-1}T\}^\dagger \)) for various multiradii \( \nu \). We will denote \( \text{RAT} \text{ALG}_R^{an,s} \) (resp. \( \text{RAT} \text{ALG}_R^\dagger,s \)) the subcategory of strict rational domain algebras (resp. strict dagger rational domain algebras) over power series (resp. overconvergent power series) algebras of polyradii \( (1, \ldots, 1) \). To treat all these categories in a unified formalism, we introduce the following notation.

**Definition 12.** A type of analytic spaces is an element \( t \) of the set \( \{ an, \{ an, s \}, \dagger, \{ \dagger, s \} \} \). The category of rational domain \( t \)-algebras is denoted \( \text{RAT} \text{ALG}_R^t \).

**Example 2.** Let \( (R, | \cdot |) \) be a uniform Banach ring.

1. If \( R \) is an integral ring equipped with the trivial seminorm \( | \cdot |_0 \), the strictly convergent or overconvergent power series are given by polynomials, and we will see in Lemma 5 that strict rational domain algebras are given by localizations.

2. One may look at \( \mathbb{Z}_p \) and \( \mathbb{Q}_p \) as (non-strict) rational domain algebras over the base Banach ring \( R = \mathbb{Z}_0 := (\mathbb{Z}, | \cdot |_0) \), by using the formulas
\[ \mathbb{Z}_p = \mathcal{O}(\{ 2 \cdot |p| \leq |1| \}) := \mathbb{Z}_0\{2T\}/(T - p) \]
and
\[ \mathbb{Q}_p = \mathcal{O}(\{ 2 \cdot |p| \leq |1|, |1| \leq 3 \cdot |p| \}) := \mathbb{Z}_0\{2T, (1/3)S\}/(T - p, pS - 1) \).

This will play an essential role in the analytic derived de Rham cohomology approach to \( p \)-adic period rings, that we will discuss in Section 7.
3. Over $R = \mathbb{Q}_p = (\mathbb{Q}_p, |\cdot|_p)$, the above notions of rational domains give back the usual rational domains of Berkovich’s geometry [Ber90], and the strict rational domains of Tate’s rigid analytic geometry [Tat71]. Their overconvergent analogs were already used by Große-Klönne [GK00].

4. Over $R = \mathbb{C} = (\mathbb{C}, |\cdot|_\infty)$, we find back essentially the same strict overconvergent affinoid rational domains as those used by Bambozzi in his thesis [Bam14].

**Example 3.** Let $\mathbb{Z} := (\mathbb{Z}, |\cdot|_\infty)$ be the Banach ring of integers with its archimedean absolute value.

1. The strict convergent (resp. overconvergent) power series are given by polynomials, equipped with the sup norm (resp. over-seminorms of the sup norm) on the unit polydisc. Non strict rational domain algebras over them include ind-Banach rings like $\mathbb{Z}_p$ and $\mathbb{Q}_p$ (isomorphic to the ind-Banach rings $\mathbb{Z}_p$ and $\mathbb{Q}_p$ described in Example 2), but also the ind-Banach ring

$$\mathbb{R} = \mathcal{O}(\{2 \cdot |1| \leq |2|\}) := \mathbb{Z}[2T]/(2T - 1).$$

2. Strict rational domains over the polynomial ring $\mathbb{Z}[X]$ (equipped with its sup norm on the global unit disc $\mathcal{D}(0, 1)_\mathbb{Z}$), will be for example given by

$$\{|nX| \leq |1|\},$$

and more generally $$\{|qsX - ps| \leq |r|\},$$

with $(n, p, q, r, s) \in \mathbb{N}^6$ such that $qsX - ps$ and $r$ are relatively prime in $\mathbb{Z}[X]$. The archimedean fiber of these two examples give the disc $\mathcal{D}(0, 1/n)$ and (when $s$ and $q$ are non-zero) the discs $\mathcal{D}(p/q, r/s)$ with arbitrary rational center and arbitrary rational radius, intersected with the unit disc. Remark that this intersection may be empty.

3. The natural map $\mathbb{Z}\{X_0, 1/X_0\}^\dagger \to \mathbb{Z}\{X_1, 1/X_1\}^\dagger$ given by $X_0 \mapsto 1/X_1$ gives a well-defined isomorphism between the rational domains $\{1 \leq |X_0| \leq 1\}$ and $\{1 \leq |X_1| \leq 1\}$, with underlying algebras of functions the Zariski domain algebras $\mathbb{Z}[X_0, 1/X_0]$ and $\mathbb{Z}[X_1, 1/X_1]$ on the polynomial algebras $\mathbb{Z}[X_0]$ and $\mathbb{Z}[X_1]$. We will see in Proposition 11 that this allows us to put on the algebraic projective space $\text{Proj}(\mathbb{Z}[X_1, X_1])$ the structure of a strict dagger analytic space over the Banach ring $\mathbb{Z}$, giving a kind of additional “Arakelov structure” on it.

4. The natural “multiplicative” comultiplication $\Delta_m$ on $\mathbb{Z}[X_0, 1/X_0]$ naturally extends to a comultiplication $\Delta_m^\dagger$ on $\mathbb{Z}\{X_0, 1/X_0\}^\dagger$. We will call the corresponding analytic group over $\mathbb{Z}$ simply $U(1)$. Over the non-archimedean base $\mathbb{Z}_0 = (\mathbb{Z}, |\cdot|_0)$, one has $U(1) = \mathbb{G}_m$, but over $\mathbb{R}$, we get $U(1)_\mathbb{R}$ and over $\mathbb{Q}_p$, we also get $U(1)_{\mathbb{Q}_p}$. This may be quite disturbing to an algebraic geometer that the natural base extension (i.e., analytification) of the algebraic multiplicative group $\mathbb{G}_m$ to $K = \mathbb{R}$ or $\mathbb{Q}_p$ is not $\mathbb{G}_m^{an}$, but $U(1)_K$. 

24
Proposition 3. The category $\text{RatAlg}_{R}^t$ for $t \in \{an, \{an, s\}, \dagger, \{\dagger, s\}\}$ a type of analytic spaces, equipped with the admissible subcategories of rational domain embeddings and the Grothendieck topology generated by rational domain coverings is a pre-geometry in the sense of Lurie [Lur09a], Definition 3.1.1.

Proof. We already know that there are finite coproducts and pushout diagrams along rational domain embeddings. Let us show that if a triangle

\[
\begin{array}{ccc}
X & \xrightarrow{h} & Z \\
\downarrow{f} & & \downarrow{g} \\
Y & \xrightarrow{g} & \quad \\
\end{array}
\]

of rational domains is such that $g$ and $h$ are rational domain embeddings, then $f$ is also a rational domain embedding. Indeed, suppose we have a commutative diagram (multi-index notation)

\[
\begin{array}{ccc}
R\{\rho^{-1}T\}/(sT - r) & \xrightarrow{f} & R\{\nu^{-1}S\}/(uS - v) \\
\downarrow{g} & & \downarrow{h} \\
R & \xrightarrow{h} & R\{\nu^{-1}S\}/(uS - v) \\
\end{array}
\]

Then we get a natural morphism

\[
(R\{\rho^{-1}T\}/(sT - r))\{\nu^{-1}S\}/(uS - v) \twoheadrightarrow R\{\nu^{-1}S\}/(uS - v)
\]

that induces a natural morphism

\[
R\{\rho^{-1}T\}/(sT - r)\{\nu^{-1}S\}/(uS - v, T - f(T)) \twoheadrightarrow R\{\nu^{-1}S\}/(uS - v).
\]

This last map is an isomorphism, so that $f$ is indeed a rational domain embedding. This argument extends to the overconvergent setting. Let us now show that every retract of a rational domain embedding is a rational domain embedding. Suppose given a retraction diagram

\[
\begin{array}{ccc}
A & \xrightarrow{id} & A \\
\downarrow{id} & & \downarrow{id} \\
B & \xrightarrow{i} & R\{\rho^{-1}T\}/(gT - f) & \xrightarrow{r} & B \\
\end{array}
\]

By construction, there is a canonical retraction

\[
\begin{array}{ccc}
B & \xrightarrow{id} & A\{\rho^{-1}T\}/(r(g)T - r(f)) & \xrightarrow{r_1} & B \\
\end{array}
\]
where the map $i_1$ is given by the composition
\[
B \xrightarrow{i} R\{\rho^{-1}T\}/(gT - f) \xrightarrow{r} A\{\rho^{-1}T\}/(r(g)T - r(f))
\]
and the map $r_1$ is given by the universal property of the quotient. It remains to show that $i_1 \circ r_1 = \text{id}$ to get that $i_1$ and $r_1$ are inverse isomorphisms, which will finish the proof. ∎

For $t \in \{an, \{an, s\}, †, \{†, s\}\}$ a type of analytic spaces, the "underlying ring" functor
\[
\text{Ring} : \text{ind-BANRINGS}_u \to \text{RINGS}
\]
restricts naturally to an "underlying $R$-algebra" functor
\[
\text{Alg} : \text{RATALG}^t_R \to \text{ALG}_R.
\]
If $f : R \to S$ is a morphism of uniform ind-Banach algebras, there are natural base extension functor
\[
- \otimes_R S : \text{RATALG}^t_R \to \text{RATALG}^t_S.
\]
For $u \in \{an, †\}$, there is also a natural fully faithful embedding
\[
\text{RATALG}^{u,s}_R \to \text{RATALG}^{u}_R.
\]

Lemma 3. There are natural uniform completion functors
\[
\text{RATALG}^†_R \to \text{RATALG}^{an}_R
\]
and
\[
\text{RATALG}^{†,s}_R \to \text{RATALG}^{an,s}_R,
\]
and these functors commute with finite coproducts and pushouts along rational domain embeddings.

Proof. The uniform completion functors are given by restriction of the natural functor
\[
\text{ind-BANRINGS}_u \to \text{BANRINGS}_u
\]
sending a filtered diagram of uniform Banach rings to its colimit in the category of uniform Banach rings. The commutation with finite coproducts is clear. The pushout along a dagger rational embedding is given by a dagger rational embedding, whose associated analytic algebra is given by the pushout of the ind-Banach algebras associated to the given dagger algebras. ∎

We will now introduce the categories of analytic and dagger algebras, that will allow the use of nilpotent elements, which is not possible with uniform ind-Banach rings. We carefully inform the reader that the category \textit{SETS} used in this definition is a category of small enough sets.
Definition 13. For $t \in \{an, \{an, s\}, \dagger, \{\dagger, s\}\}$ a type of analytic spaces, a $t$-algebra will be a functor

$$A : (\text{RatAlg}_R^t)^{op} \to \text{SETS}$$

that sends finite coproducts of algebras to products, and pushout diagrams of the form

$$\begin{array}{ccc}
A_1 & \xrightarrow{i} & A_2 \\
\downarrow & & \downarrow \\
A_3 & \xrightarrow{i} & A_4
\end{array}$$

where $i : A_1 \to A_2$ corresponds to the embedding of a rational sub-domain, to pullbacks. We denote $\text{Alg}_R^t$ the category of $t$-algebras. Objects of $\text{Alg}_R^{an}$ (resp. $\text{Alg}_R^{an,s}$, resp. $\text{Alg}_R^\dagger$, resp. $\text{Alg}_R^{\dagger,s}$) will be called analytic (resp. strict analytic, resp. dagger, resp. strict dagger) algebras over $R$.

For $t \in \{an, \{an, s\}, \dagger, \{\dagger, s\}\}$ a type of analytic spaces, there is a natural Yoneda embedding

$$\text{RatAlg}_R^t \to \text{Alg}_R^t$$

that commutes by definition with finite coproducts and pushout diagrams of the form

$$\begin{array}{ccc}
A_1 & \xrightarrow{i} & A_2 \\
\downarrow & & \downarrow \\
A_3 & \xrightarrow{i} & A_4
\end{array}$$

where $i$ corresponds to an embedding of a rational sub-domain.

Lemma 4. For $u \in \{an, \dagger\}$, there is a natural functor

$$\text{Alg}_R^u \to \text{Alg}_R^{u,s}$$

from general $u$-algebras to strict $u$-algebras that makes the following diagram

$$\begin{array}{ccc}
\text{RatAlg}_R^{u,s} & \xrightarrow{} & \text{RatAlg}_R^u \\
\downarrow & & \downarrow \\
\text{Alg}_R^{u,s} & \xleftarrow{} & \text{Alg}_R^u
\end{array}$$

commutative.

Proof. This follows from the fact that the natural inclusion

$$\text{RatAlg}_R^{u,s} \to \text{RatAlg}_R^u$$

commutes with finite coproducts and pushouts along rational domain embeddings. \qed

27
**Proposition 4.** If \( f : R \to S \) is a morphism of uniform ind-Banach algebras and \( t \in \{ \text{an}, \{ \text{an}, s \}, \dagger, \{ \dagger, s \} \} \) is a type of analytic spaces, there is a natural base extension functor

\[- \otimes_R S : \text{Alg}_R^t \to \text{Alg}_S^t.\]

**Proof.** The base extension functor is given by the following construction: every \( A \) in \( \text{Alg}_R^t \) may be presented as a colimit of representable functors

\[A = \colim_i \text{Hom}_{\text{RatAlg}_R^t}(-, A_i) = \colim_i \text{Hom}_{\text{ind-BanRings}_{u/R}}(-, A_i),\]

with \( A_i \in \text{RatAlg}_R^t \). One then defines \( A \otimes_R S \) by

\[A \otimes_R S := \colim_i \text{Hom}_{\text{ind-BanRings}_{u/S}}(-, A_i \otimes_R S),\]

where the colimit is taken in the category \( \text{Alg}_S^t \).

By construction, the diagram

\[
\begin{array}{ccc}
\text{RatAlg}_R^t & \xrightarrow{- \otimes_R S} & \text{RatAlg}_S^t \\
\downarrow & & \downarrow \\
\text{Alg}_R^t & \xrightarrow{- \otimes_R S} & \text{Alg}_S^t
\end{array}
\]

commutes.

The following lemma will imply that strict analytic geometry on a trivially seminormed ring is essentially equivalent to scheme theory.

**Lemma 5.** If \( R \) is an integral ring with its trivial seminorm, then the categories of strict dagger and of strict analytic algebras are equivalent to the category of usual \( R \)-algebras, and rational domains are given by localizations.

**Proof.** This follows from the fact that rational domain algebras correspond to localization of polynomial algebras, and that a functor that sends coproducts to products and localization pushouts to pullbacks on them is equivalent to a functor on polynomial algebras that sends coproduct to products, which is well known (since Lawvere [Law04]) to be the same thing as an arbitrary algebra. So let us prove that rational domain algebras in polynomial rings over a trivially normed integral ring correspond to their localizations. First remark that the spectral seminorm on the unit polydisc for the polynomial algebra on \( R \) is given by the trivial seminorm. Indeed, it is the uniform seminorm associated to the canonical Banach norm, given by

\[|\sum a_\alpha X^\alpha| := \sum |a_\alpha|_0,\]

which gives

\[|\sum a_\alpha X^\alpha|_0 = \max (|a_\alpha|_0).\]
If \( A = R[X_1, \ldots, X_n] \) is a polynomial algebra and \( f_0, \ldots, f_n \) are elements in \( A \) such that there exist \( a_i \in A \) with \( 1 = \sum_{i \in I} a_i f_i \) then for every point in \( \mathcal{M}(A) = \mathcal{M}(A, | \cdot |_0) \), we have \( |a_i(x)| \leq |a_i|_0 \leq 1 \) and

\[
1 = |1| \leq \max_{i \in I} |f_i(x)| \leq \max_{i \in I} |f_i|_0 \leq 1
\]

so that

\[
\{ x, |f_i(x)| \leq |f_0(x)| \text{ for all } i > 0 \} = \{ x, 1 \leq |f_0(x)| \}.
\]

If we look at the corresponding rational domain algebra, this gives the quotient

\[
R[X_1, \ldots, X_n\{f_0, 1\}|1, 1] = R[X_1, \ldots, X_n, T]/(f_0 T - 1),
\]

equipped with the quotient seminorm of the trivial norm on the polynomial ring, that is also the trivial norm in this case, so that we get

\[
\{ x, 1 \leq |f_0(x)| \} = \{ x, f_0(x) \neq 0 \}.
\]

so that

\[
A\{f_0, 1|f_1, 1, \ldots, f_n, 1\} = A\{f_0, 1|1, 1\} = A[1/f_0].
\]

The same argument applies also to the overconvergent setting, since the Berkovich spectrum does not change.

For \( t \in \{\text{an}, \dagger\} \) a non-strict type of analytic spaces, the underlying \( R \)-algebra functor

\[
\text{Alg} : \text{RATALG}_R \to \text{ALG}_R
\]

extends naturally to a functor \( \text{Alg} : \text{ALG}_R^t \to \text{ALG}_R \) given by

\[
\text{Alg}(A) := \text{colim}_{\rho \to \infty} A(R(\rho^{-1} T)) \text{ for } t = \text{an}, \text{ and}
\]

\[
\text{Alg}(A) := \text{colim}_{\rho \to \infty} A(R(\rho^{-1} T)^\dagger) \text{ for } t = \dagger,
\]

with \( T \) a variable.

**Definition 14.** If \( A \) is an ind-Banach \( R \)-algebra and \( t \in \{\text{an}, \dagger\} \) is a type of analytic spaces, the \( t \)-completion of \( A \) is the \( t \)-algebra defined by setting its values on a rational domain \( t \)-algebra \( B \) in \( \text{RATALG}_R^t \) to be

\[
A^t(B) := \text{Hom}_{\text{ind-BANRINGS}_{u/R}}(B, A).
\]

By definition, the \( t \)-completion \( A^t \) of \( A \) naturally commutes with all pushout diagrams, and in particular with those needed to be a \( t \)-algebra over \( R \). This construction thus gives a \( t \)-completion functor

\[
(\cdot)^t : \text{ind-BANRINGS}_{u/R} \to \text{ALG}_R^t
\]

\[
A \mapsto A^t
\]

**Example 4.** We may apply the dagger completion to the ind-Banach ring \( \mathbb{Z} \) over \( (\mathbb{Z}, | \cdot |_\infty) \), to get a dagger algebra \( \mathbb{Z}^\dagger \) equipped with a natural \( \text{Gal}(\mathbb{Q}/\mathbb{Q}) \)-action.
We may apply the $t$-completion to morphisms $R \to (K, | \cdot |)$ to a multiplicatively normed Banach field $(K, | \cdot |)$, to define the Berkovich spectrum of a $t$-algebra.

**Definition 15.** Let $A$ be a $t$-algebra over $R$ for $t \in \{an, î\}$ a non-strict type of analytic spaces. The *Berkovich spectrum of $A$* is the set $\mathcal{M}(A)$ of equivalence classes of morphisms

$$\chi : A \to (K, | \cdot |)^t$$

for the family of morphisms $R \to (K, | \cdot |)$ from $R$ to a multiplicatively seminormed field, together with the coarsest topology that makes the maps

$$\mathcal{M}(A) \to \mathbb{R}_+$$
given by $x \mapsto |a(x)|$ for $a \in \text{Alg}(A)$ continuous.

The following proposition shows that using spectra of dagger algebras does not give more general Berkovich spectra. As we said before, the interest is to allow the introduction of nilpotent elements, which will be useful for the development of differential calculus.

**Proposition 5.** Let $A$ be a $t$-algebra over $R$ for $t \in \{an, î\}$ a non-strict type of analytic spaces. There exists a uniform ind-Banach ring $\text{ib}(A)$ that is initial with the property that there exists a morphism

$$A \to \text{ib}(A)^{î}.$$

The natural morphism

$$\mathcal{M}(\text{ib}(A)) \to \mathcal{M}(A)$$

is an isomorphism.

**Proof.** By abstract nonsense, the $t$-algebra $A$ is a canonical colimit of (representable) rational domain algebras $A_i$. Setting $\text{ib}(A)$ to be the colimit of these rational domain algebras $A_i$ in the category of uniform ind-Banach rings, we find that $\text{ib}(A)$ has the desired universal property. The map $\mathcal{M}(\text{ib}(A)) \to \mathcal{M}(A)$ is a bijection by the universal property of $\text{ib}(A)$ and it is clearly bicontinuous.

**Definition 16.** We will say that $A$ is a *uniform $t$-algebra* if the canonical morphism

$$A \to \text{ib}(A)^{î}$$

is an isomorphism.

By Lemma 3, we know that one may complete a rational domain dagger algebra to an analytic rational domain algebra. Since this completion is compatible with products and pullbacks along rational domain immersions, we get the following result.
Proposition 6. The uniform completion functors induce natural functors

$$\text{Alg}^\text{an}_R \to \text{Alg}^\dagger_R$$ and $$\text{Alg}^\text{an,s}_R \to \text{Alg}^\dagger,s_R,$$

and also natural functors

$$\text{Alg}^\dagger_R \to \text{Alg}^\text{an}_R$$ and $$\text{Alg}^\dagger,s_R \to \text{Alg}^\text{an,s}_R.$$

Proof. The natural functor given by the uniform completion

$$\text{RatAlg}^\dagger_R \to \text{RatAlg}^\text{an}_R$$

commutes with finite coproducts and pushouts along rational domain embeddings. This will induce a functor

$$\text{Alg}^\text{an}_R \to \text{Alg}^\dagger_R$$

that sends an analytic algebra $$A$$ to the dagger algebra

$$A^\dagger(R\{\nu^{-1}T\}\{f_0, \rho_0|f_i, \rho_i\}^\dagger) := A\left(R\{\nu^{-1}T\}\{\bar{f}_0, \rho_0|\bar{f}_i, \rho_i\}\right).$$

Conversely, given a dagger algebra $$A^\dagger$$, one may define the associated analytic algebra by writing down $$A^\dagger$$ as a colimit $$A^\dagger = \text{colim}_i A_i^\dagger$$ of rational domain dagger algebras (i.e., representable functors) and taking the same $$A$$ to be the colimit of the associated diagram of rational domain algebras $$A_i$$.

Remark 2. Let $$(R, |·|_R)$$ be a uniform Banach ring. Proposition 6 allows us to relate dagger algebras to usual analytic algebras, that are usually used on a non-archimedean Banach field, e.g., by Berkovich [Ber90] and Tate [Tat71]. The problem with the approach to analytic geometry using convergent power series and analytic algebras is that convergent power series on polydiscs are not well behaved in the archimedean setting (e.g., not Noetherian), and that it may lead to looking at the unit circle $$S^1$$ with its algebra of continuous complex valued functions as an analytic space (see [Ber90], Remark 1.5.5). As explained by Bambozzi in his thesis, the overconvergent setting solves this problem with $$S^1$$ (see [Bam14], Example 6.4.38). It is also well known that $$p$$-adic de Rham cohomology behaves better in the overconvergent setting [GK00].

3.3 Affinoid algebras

If $$\varphi : R \to S$$ is a morphism of uniform ind-Banach rings, it is in general not an easy task to compute the values of the functor

$$- \otimes_R S : \text{Alg}^\dagger_R \to \text{Alg}^\dagger_S.$$

We now introduce the definition of dagger affinoid algebras for which such a computation is easy. Categories of affinoid algebras will also give geometries in the sense of Lurie [Lur09a], Definition 1.2.5, that can be used to develop the corresponding notions of structured topoi. In this paper, we will use the functor of point approach to the definition of analytic spaces.
Definition 17. Let $t \in \{an, \{an, s\}, \dagger, \{\dagger, s\}\}$ be a type of analytic spaces. A $t$-algebra $A$ over $R$ that is the coequalizer

$$
\begin{array}{c}
C \\
\xrightarrow{f}
\end{array}
\xrightarrow{s} B \xrightarrow{g} A
$$

of two morphisms in $RAT\text{ALG}^t_R$ (i.e., that is finitely presented) is called a $t$-affinoid $R$-algebras. For $t \in \{an, \{an, s\}, \dagger, \{\dagger, s\}\}$, we denote $\text{Aff}^t_R$ the category of $t$-affinoid $R$-algebras. For $u \in \{an, \dagger\}$ a non-strict type of analytic space, an $u$-affinoid algebra $A$ over $R$ is called quasi-strictly $u$-affinoid if it is the coequalizer of two morphisms in $RAT\text{ALG}^{u,s}_R$. We denote $\text{Aff}^{u,qs}_R$ the category of quasi-strictly $u$-affinoid algebras.

By definition, there is a natural fully faithful functor

$$RAT\text{ALG}^t_R \to \text{Aff}^t_R,$$

and a natural fully faithful functor

$$\text{Aff}^t_R \to \text{Alg}^t_R.$$

The following proposition will allow us to test strict affinoi d algebras on non-strict rational domain algebras.

**Proposition 7.** For $u \in \{an, \dagger\}$, the forgetful functor

$$F : \text{Aff}^{u,qs}_R \to \text{Aff}^{u,s}_R$$

is an equivalence of categories.

**Proof.** Let us show that the forgetful functor

$$F : \text{Aff}^{u,qs}_R \to \text{Alg}^{u,s}_R$$

is a fully faithful embedding with essential image the finitely presented strict $u$-algebras. It is clear that the objects of the essential image are given by finitely presented strict $u$-algebras, so that we only have to prove the full faithfulness assessment. Let $A$ and $B$ be two quasi-strict $u$-affinoid algebras. Then we can write $A$ as a coequalizer

$$R_1 \xrightarrow{f} R_2 \xrightarrow{g} A$$

with $R_1$ and $R_2$ two strict $u$-rational domain algebras, so that

$$\text{Hom}_{\text{Aff}^{u,qs}_R}(A, B) := \text{Hom}_{\text{Alg}^{u}_R}(A, B) \cong \ker \left( \text{Hom}_{\text{Alg}^{u}_R}(R_2, B) \xrightarrow{f^*} \text{Hom}_{\text{Alg}^{u}_R}(R_1, B) \right)$$
which gives by definition of $B$ exactly

$$\text{Hom}_{\text{Aff}^{u,qs}_R}(A, B) \cong \ker \left( B(R_2) \xrightarrow{f^*} B(R_1) \right).$$

The right hand side of this isomorphism is also given by $\text{Hom}_{\text{Alg}^{u,s}_R}(F(A), F(B))$ where $F : \text{Aff}^{u,qs}_R \to \text{Alg}^{u,s}_R$ is the forgetful functor.

**Corollary 1.** For all $t \in \{\text{an}, \{\text{an}, s\}, \dagger, \{\dagger, s\}\}$, there is a natural forgetful functor

$$\text{Alg} : \text{Aff}^t_R \to \text{Alg}_R.$$

**Proof.** We already know that for $u \in \{\text{an}, \dagger\}$, the forgetful functor

$$\text{Alg} : \text{Alg}^u_R \to \text{Alg}_R$$

induces (by restriction) a natural forgetful functor

$$\text{Alg} : \text{Aff}^u_R \to \text{Alg}_R.$$

Using the natural equivalence

$$\text{Aff}^{u,qs}_R \xrightarrow{\sim} \text{Aff}^{u,s}_R,$$

and the embedding $\text{Aff}^{u,qs}_R \subset \text{Alg}^u_R$, we may define

$$\text{Alg} : \text{Aff}^{u,s}_R \to \text{Alg}_R$$

by composition. \qed

If $\varphi : R \to S$ is a morphism of uniform ind-Banach rings, the functor of extension of scalars on $t$-algebras restricts to a functor

$$- \otimes_R S : \text{Aff}^t_R \to \text{Aff}_S^t$$

which may be computed by

$$\text{coker}(f, g) \otimes_R S = \text{coker}(f \otimes_R S, g \otimes_R S).$$

If $A$ is a $t$-affinoid $R$-algebra, with presentation

$$A = \text{coker}(f, g : C \to B),$$

with $B$ and $C$ in $\text{RatAlg}_R^t$, then $\text{ib}(A)$ is the uniform ind-Banach algebra given by the same quotient

$$\text{ib}(A) = \text{coker}(f, g)$$

taken in the category of uniform ind-Banach algebras.
Proposition 8. For \( t \in \{ an, \{ an, s \}, \dagger, \{ \dagger, s \} \} \) a type of analytic spaces, the category \( G^t_R \) opposite to the category \( Aff^t_R \) of affinoid \( t \)-algebras is a geometry in the sense of Lurie [Lur09a, Definition 1.2.5] that gives the 0-truncated geometric envelope of the pregeometry \( \mathcal{T}^t_R := (\text{RAT} \text{ALG}^t_R)^\text{op} \) in the sense of loc. cit., Definition 3.4.9.

Proof. We will show that \( G^t_R \) is the 0-truncated geometric envelope of \( \mathcal{T}^t_R \). This follows from the construction of the geometric envelope, that uses the colimit completion process of [Lur09c], Proposition 5.3.6.2. The main observation is that \( \mathcal{T}^t_R \) have finite products and pullbacks along rational embeddings, and that one defines its geometric envelope by using objects of its presheaf category that are given by finite limits of its objects, and that commute with the already existing finite limits given by products and colimits along rational embeddings. One only has to add equalizers to the already given products to get all finite limits. \( \square \)

Proposition 9. For \( t \in \{ an, \{ an, s \}, \dagger, \{ \dagger, s \} \} \), there is a natural equivalence

\[
\text{ind-Aff}^t_R \xrightarrow{\sim} \text{Alg}^t_R.
\]

Proof. Since \( \text{Alg}^t_R \) has small filtered colimits, the inclusion functor

\[
\text{Aff}^t_R \to \text{Alg}^t_R
\]

commutes with finite colimits (by definition) and induces a functor

\[
\text{ind-Aff}^t_R \to \text{Alg}^t_R.
\]

This functor is an equivalence because every \( t \)-algebra (with values in the category of small sets) is a small colimit of rational domain \( t \)-algebras, and such a colimit can be constructed using a combination of finite colimits and small filtered colimits. \( \square \)

3.4 Perfectoid dagger algebras

We discuss shortly the perfectoid analogs (see [Sch12b] for an introduction to perfectoid spaces) of the constructions of the previous subsections, that may be useful for non-archimedean applications.

Definition 18. A perfectoid field is a non-archimedean Banach field \( K \) of residue characteristic \( p \) whose seminorm is non-discrete and whose Frobenius map is surjective on the residue field \( K^p/p \).

Definition 19. Let \( K \) be a perfectoid field with uniformizer \( \omega \in K^p \). Let \( \rho \in \mathbb{R}_{>0}^n \) be a multiradius. The non-strict dagger perfectoid \( K \)-algebra of power series of multiradius \( \rho \) is the ind-Banach algebra

\[
K\{((\rho^{-1} T)^{1/p^n})^{\dagger} := \left( \colim_{\nu \to \infty} K^p \{ \nu^{-1} T, \nu^{-1/p^n} S \}/(S^p - T) \right) [\omega^{-1}].
\]
Scholze [Sch12b] defines a tilt operation on convergent power series that sends $K\{T^{1/p^n}\}$ to $K^\triangleright\{T^{1/p^n}\}$. Diekert’s master thesis [Die12] describes the Tilt operation on overconvergent power series of radius 1. This can actually be extended to non-strict overconvergent power series, by using the formula
\[
K\{(\rho^{-1}T)^{1/p^n}\} \rightarrow K^\triangleright\{(\rho^{-1}T)^{1/p^n}\}.
\]

We may define perfectoid rational domain algebras and overconvergent perfectoid algebras in a way similar to the one used in the previous section, by replacing everywhere the rings of overconvergent power series by those of dagger perfectoid $K$-algebra of power series. This will give rise to categories $\text{RatAlg}^\triangleright\text{perf}_K$ and $\text{Alg}^\triangleright\text{perf}_K$ of dagger perfectoid rational algebras and dagger perfectoid algebras over $K$ that may be used to define a perfectoid version of dagger geometry, that gives back an overconvergent version of Scholze’s perfectoid spaces [Sch12b] in the strict setting. As pointed out by Scholze, this category only contains reduced rings, so that it is not adapted to the infinitesimal definition of differential notions. The tilt operation extends directly to rational domain algebras giving a natural tilting functor
\[
\text{RatAlg}^\triangleright\text{perf}_K \rightarrow \text{RatAlg}^\triangleright\text{perf}_K^\triangleright
\]
that extends, since it is compatible with coproducts and pushforwards along rational domain immersions, to a tilting functor
\[
\text{Alg}^\triangleright\text{perf}_K \rightarrow \text{Alg}^\triangleright\text{perf}_K^\triangleright.
\]
This will also give an equivalence between the corresponding étale and pro-étale topoi, to be define in 4.3.

Since a dagger rational domain algebra over a perfectoid algebra is still perfectoid (see [Sch12b], Theorem 6.3 for the strictly convergent situation), we may define dagger perfectoid rational domain algebras by using rational domain algebras in dagger perfectoid algebras of power series. This gives a natural fully faithful functor
\[
\text{RatAlg}^\triangleright\text{perf}_K \rightarrow \text{Alg}^\triangleright_K.
\]
This functor sends pushouts along rational domain immersions to pushouts, so that it extends to a functor
\[
\text{Alg}^\triangleright\text{perf}_K \rightarrow \text{Alg}^\triangleright_K.
\]
This allows us to do infinitesimal calculus on perfectoid dagger algebras by using infinitesimal extensions in the category of all dagger algebras.

**Remark 3.** One may use the above viewpoint to propose a global notion of perfectoid algebras. Let $\mathbf{Z}_0 := (\mathbb{Z}, \cdot, |_0)$ be the non-archimedean Banach ring of integers and $\mathbb{Q}$ its fraction field. One may easily define as above the global perfectoid overconvergent power series of multiradius $\rho$ by setting
\[
\mathbb{Q}\{\rho^{-1}T^{1/[p^n]}\} := \left(\colim_{\nu > \rho, n \to \infty} \mathbf{Z}_0\{\nu^{-1}T, \nu^{-1/n}S\}/(S^n - T)\right) \otimes_{\mathbf{Z}_0} \mathbb{Q}.
\]
The fields $K = (\mathbb{Q}(\mu^\infty), | \cdot |_0)$ or $K = (\overline{\mathbb{Q}}, | \cdot |_0)$ may play the role of the perfectoid base field. One may then easily define the category $\text{RAT}_{\text{alg}}^{\dagger, \text{perf}}_K$ and $\text{Alg}^{\dagger, \text{perf}}_K$ as before. Remark that there is no clear analog of the tilt operation in this situation, but Witt vector constructions with rings of integers in the spirit of those done by Davis and Kedlaya in [DK14] may be enough to define interesting period dagger rings and period sheaves. It is also possible to use $\hat{\mathbb{Z}}$-completed derived de Rham cohomology of $\overline{\mathbb{Z}}/\mathbb{Z}$ as proposed by Bhatt in [Bha12b], Remark 10.22.

4 Overconvergent geometry

In all this section, $R$ will denote a uniform ind-Banach ring. We will now define dagger analytic spaces over $R$, that are essentially given by pasting dagger algebras along their natural coverings by rational domains. This pasting operation can be done using $\text{Sets}$-valued sheaves on the category of dagger algebras. To ease comparisons with other kinds of analytic spaces, we will also define convergent analogs of dagger spaces.

4.1 Dagger analytic spaces

We first extend to affinoid $t$-algebras the notion of rational domain, and also define rational coverings. Let $t \in \{an, \{an, s\}, \dag, \{\dag, s\}\}$ be a type of analytic spaces.

Definition 20. Let $A$ be an affinoid $t$-algebra over $R$. Let $(f_0, \ldots, f_n) \in \text{Ring}(A)^{n+1}$ be a finite family of elements of $A$ that generate $\text{Ring}(A)$ as an ideal, and $\rho = (\rho_0, \ldots, \rho_n) \in \mathbb{R}_{>0}^{n+1}$.

1. The rational domain $t$-algebra $A\{\rho_0, f_0|\rho_1, f_1, \ldots, \rho_n, f_n\}^t$ is the affinoid $t$-algebra over $A$ given by the quotient
$$A\{(\rho_1/\rho_0)^{-1}T_1, \ldots, (\rho_n/\rho_0)^{-1}T_n\}^t/(f_0T_1 - f_1, \ldots, f_0T_n - f_n).$$

2. The associated rational domain is the morphism
$$D^t(\rho_0, f_0|\rho_1, f_1, \ldots, \rho_n, f_n) := \mathcal{M}(A\{\rho_0, f_0|\rho_1, f_1, \ldots, \rho_n, f_n\}^t) \longrightarrow \mathcal{M}(A).$$

3. The family
$$\left\{D^t(f_i, \rho_i|f_0, \rho_0 \ldots, \widehat{f_i}, \rho_i, \ldots, f_n, \rho_n) \longrightarrow \mathcal{M}(A) \right\}_{i}$$

is called the standard covering associated to the family $\{(f_i, \rho_i)\}_{i=0, \ldots, n}$.

4. A rational domain algebra (resp. rational domain) is called strict if $A$ is strict and the polyradius $\rho = (\rho_0, \ldots, \rho_n)$ has all components equal to 1.
By definition of the category $\text{Aff}^t_R$ of affinoid $t$-algebras, the natural functor

$$i : \text{RatAlg}^t_R \to \text{Aff}^t_R$$

sends rational domain algebras on a rational domain algebra $R$ to rational domain algebras over the associated affinoid $t$-algebra $R^t$, so that the notation of the above definition for rational domain algebras remains consistent with the previous one.

Let $(\text{Alg}^t_R, \tau_{\text{Rat}})$ be the category of $t$-algebras over $R$ equipped with the topology given by standard coverings by rational domains (naturally induced from the topology on $\text{Aff}^t_R$ and the equivalence $\text{Alg}^t_R \cong \text{ind-Aff}^t_R$, as in [Lur09a], Definition 2.4.3).

**Definition 21.** We now work in the category $\text{Shv}^{\widehat{\text{Sets}}}_{\text{Sets}}(\text{Alg}^t_R, \tau_{\text{Rat}})$ of sheaves on $t$-algebras over $R$ with values in a category $\widehat{\text{Sets}}$ of big sets. We denote $\mathbb{M}(A)$ the sheaf associated to the representable presheaf given by $A$.

1. A morphism $D \to X$ of sheaves is called a **rational domain** if its pullback along any morphism $\mathbb{M}(A) \to Y$ is a rational domain.

2. A finite family of rational domains $\{D_i \to X\}_{i \in I}$ is called a standard rational covering if its pullback along any morphism $x : \mathbb{M}(A) \to X$ is a standard rational covering.

3. A morphism $D \to X$ is called a **quasi-compact domain** if it is a union (colimit) of finitely many rational domains.

4. A morphism $U \to X$ is called a **pre-domain** if it is the colimit of an arbitrary family of rational domains.

5. A pre-domain $U \to X$ is called a **domain (or an admissible domain)** if for all points $x : \mathbb{M}(A) \to X$ that factorize through $U$, there exists a quasi-compact domain $D \to U$ with a factorization $x : \mathbb{M}(A) \to D$.

6. If $U \to X$ is a domain, a family of domains $\{U_i \to U\}$ is called an **admissible covering** if for all rational domain $D \to U$, the pullback of the family $\{U_i \to U\}$ to $D$ can be refined by a standard rational covering.

We then define a **$t$-analytic space** to be a sheaf $X$ in $\text{Sh}(\text{Alg}^t_R, \tau_{\text{Rat}})$ that can be written as the colimit

$$X = \colim_{\mathbb{M}(A) \to X} \mathbb{M}(A)$$

along the system of all representable domains $\mathbb{M}(A) \to X$. The category of $t$-analytic spaces is denoted $\text{AN}^t_R$. 

37
Remark 4. As explained by Temkin in [Tem10], it is difficult to describe a general open affine subscheme but one can easily characterize it by a universal property. For example, the Zariski open \( \mathbb{A}^2 - \{0\} \) of the affine plane \( \mathbb{A}^2 \) is not described by a localization of the polynomial ring, but as the union of the two subspaces given by taking out the axis, with coordinate rings \( \mathbb{Z}[x, y, 1/x] \) and \( \mathbb{Z}[x, y, 1/y] \). The analytic analog of general Zariski open subsets are given by (admissible) domains.

We may now define the notion of finitely presented analytic space, which is necessary to explain the relation of \( t \)-analytic spaces with complex analytic or \( p \)-adic analytic geometries.

**Definition 22.** A morphism \( f : X \to Y \) of \( t \)-analytic spaces is called:

1. affine and finitely presented if for every affinoid \( t \)-algebra point \( M(A) \to Y \), the pullback is an affinoid \( t \)-algebra point \( M(B) \to X \) such that \( B \) is an affinoid \( t \)-algebra over \( A \);

2. locally finitely presented if it is locally affine and finitely presented for the \( G \)-topology.

An analytic space is locally finitely presented if the morphism \( X \to M(R) \) is locally finitely presented.

For \( u \in \{ an, \dagger \} \), the natural functor \( \text{Aff}^{u,s}_R \to \text{Aff}^u_R \) is compatible with the rational domain topologies, so that it induces a functor between categories of sheaves, which gives a “destrictification” functor

\[
\text{AN}^{u,s}_R \to \text{AN}^u_R.
\]

**Remark 5.** It is likely that this functor is fully faithful at least in the non-archimedean case, following arguments similar to the ones used by Temkin [Tem04]. In the global case over \( \mathbb{Z} = (\mathbb{Z}, | \cdot |_{\infty}) \), Temkin’s graded approach to the description of the points of the \( G \)-topology can’t be directly adapted because of the lack of archimedean rings of integers (see however Remark 1 for a possible replacement), but one may try (following a suggestion of Temkin) to start from loc. cit., Proposition 2.5 to show that the fibers of the corresponding morphism of \( G \)-topoi are connected.

**Remark 6.** The destrification functor sometimes plays the role of the analytification functor: if we work on a trivially seminormed integral ring \( (R, | \cdot |_0) \), we have shown in Proposition 5 that strict dagger algebras are simply usual \( R \)-algebras, and rational domains are given by localizations. Then domains correspond to Zariski open subsets, and strict analytic spaces to usual schemes. The associated non-strict analytic spaces give their non-archimedean analytification over \( (R, | \cdot |_0) \). One may probably identify the points of the \( G \)-topology on the non-strict analytic spectrum of \( (R, | \cdot |_0) \) with the valuation spectrum of \( R \). This gives a relation between non-strict analytic geometry in the trivially valued case and Krull and Zariski’s valuative approach to algebraic geometry. This relation of course extend to Huber’s approach to non-archimedean geometry, since the Huber space
Remark 7. Let \( u \in \{\dagger, \text{an} \} \) be a non-strict type of analytic spaces. We must warn the reader here about the interpretation of algebraic geometry as strict analytic geometry over a trivially valued ring: looking at a scheme \( X \) over a ring \( R \) as a strict \( u \)-analytic space \( X_{u,s} \) over \( (R, | \cdot |_0) \) with \( u \in \{\dagger, \text{an} \} \), is not a completely harmless operation: for example, the base extension of the strict \( u \)-unit disc over \( \mathbb{Z}_0 := (\mathbb{Z}, | \cdot |_0) \), with function algebra \( (\mathbb{Z}[X], | \cdot |_0) \) along \( \mathbb{Z}_0 \to \mathbb{Q}_p := (\mathbb{Q}_p, | \cdot |_p) \), will give the \( p \)-adic unit disc. If we want to really get back the (non-strict) analytic affine line \( \mathbb{A}^1_{\mathbb{Q}_p} \), we need to start from the affine line \( \mathbb{A}^1_{\mathbb{Z}_0} \), that is a the non-strict analytic space given by the union

\[
\mathbb{A}^1_{\mathbb{Z}_0} = \colimit_{\rho > 0} D^{u,s}(0, \rho)
\]

of all discs with arbitrary radius and center 0. There are thus two very different ways of looking at a scheme over \( \mathbb{Z} \): as a non-strict analytic space, which gives its global analytification over \( \mathbb{Z}_0 \) and as a strict analytic space, which really gives the corresponding algebraic analytic space (that may be used in a GAGA theorem, for example). We will discuss the relation of this problem with Arakelov geometry in Section 5. In particular, we will see that in the case of projective varieties, the above two kinds of analytifications are identified.

We may also apply the “destrictification process” to the situation over the initial ind-Banach ring \( \mathbb{Z} := (\mathbb{Z}, | \cdot |_\infty) \): the base extension functor

\[
\mathbb{A}^1_{\mathbb{Z}_0} \to \mathbb{A}^1_{(\mathbb{Z}, | \cdot |_0)}
\]

sends a strict analytic space over \( \mathbb{Z} \) to the underlying scheme, and the functor

\[
\mathbb{A}^1_{\mathbb{Z}_0} \to \mathbb{A}^1_{\mathbb{Z}}
\]

sends it to the associated global dagger space.

**Proposition 10.** If \( \mathcal{C} = (\mathbb{C}, | \cdot |_\infty) \), the functor

\[
\mathrm{RAT}_{\mathbb{Z}_0} \to \mathrm{RAT}_{\mathbb{Z}}
\]

is an equivalence of pre-geometries, meaning that it induces an equivalence

\[
\mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}.
\]

**Proof.** This comes from the fact that the value set of \( | \cdot |_\infty \) is \( \mathbb{R}_+ \). Indeed, we have for \( \rho < 1 \), the isomorphism

\[
\mathbb{C}\{\rho^{-1}T\} = \mathbb{C}\{T\}\{S\}/(\rho S - T)
\]

and for \( \rho > 1 \), one may cover admissibly the disc of radius \( \rho \) by the strict rational domain \( \{|T| \leq 1\} \) (which is the spectrum of \( \mathbb{C}\{T\} \)) and the rational domain \( \{1 \leq |T| \leq \rho\} \) (which identifies with the strict rational domain \( \{\rho^{-1} \leq |S| \leq 1\} \) in the unit disc, with rational domain algebra \( \mathbb{C}\{S\}\{T\}/(TS - \rho^{-1}) \)). One shows in a similar way that a rational domain over \( \mathbb{C} \) may be admissibly covered by a strict rational covering. \( \square \)
The base extension functor corresponding to the bounded map $\mathbb{Z} \to \mathbb{C}$ thus gives a kind of “associated complex analytic space”

$$\mathcal{A}N_{\mathbb{Z}}^{t,s} \to \mathcal{A}N_{\mathbb{C}}^{t,s}.$$  

We will see in Proposition 11 that this operation works well in the case of projective varieties. However, in the $p$-adic case, we get two distinct functors on $\mathcal{A}N_{\mathbb{Z}}^{t,s}$ with values in strict (i.e., rigid analytic) and non-strict $p$-adic dagger spaces over $\mathbb{Q}_p$, because discs with radius not in $\sqrt[p]{\mathbb{Z}} \subset \mathbb{R}_+$ may not be covered admissibly by strict rational domain algebras.

**Definition 23.** Let $t \in \{an, \dagger\}$ be a non-strict type of analytic spaces and $X$ be a $t$-analytic space. The set of points $|X_{\text{top}}|$ of the underlying topological space $X_{\text{top}}$ is defined to be the set of equivalence classes of morphisms

$$\mathcal{M}((K, |\cdot|_K) \to X$$

for $R \to (K, |\cdot|_K)$ a morphism of uniform ind-Banach rings form $R$ to a complete multiplicatively seminormed field. A morphism of this form that is minimal in a given equivalence class $x \in X_{\text{top}}$ is denoted $K(x)$ and called the *residue field* at $x$. If $f : X \to Y$ is a morphism of $t$-analytic spaces, then there is a canonical map

$$|f_{\text{top}}| : |X_{\text{top}}| \to |Y_{\text{top}}|.$$

Let $X$ be a $t$-analytic space. A subset $U \subset |X_{\text{top}}|$ is called a *Berkovich open subset* if for every morphism $f : \mathcal{M}(B) \to X$, with induced set map

$$|f| : \mathcal{M}(B) \cong |\mathcal{M}(B)_{\text{top}}| \to X_{\text{top}},$$

the inverse image set $|f|^{-1}(U)$ is open in $\mathcal{M}(B)$. The *Berkovich topology* $\tau_B$ is defined to be the set of Berkovich open subset of $|X_{\text{top}}|$. We call $X_{\text{top}} := (|X_{\text{top}}|, \tau_B)$ the *underlying topological space* of $X$.

The underlying topological space is indeed a topological space (clear) and it is functorial in morphisms of $t$-analytic spaces. Indeed, if $f : X \to Y$ is a morphism of $t$-analytic spaces, $U \subset |X_{\text{top}}|$ is open, and $g : \mathcal{M}(B) \to X$ is a morphism, then $|g|^{-1}(|f|^{-1}(U)) = |f \circ g|^{-1}(U)$ is also open, so that $|f| : |X_{\text{top}}| \to |Y_{\text{top}}|$ is continuous.

We have thus showed that every $t$-analytic space gives rise to two natural topologies: the $G$-topology, and the Berkovich topology. There is natural condition to impose on them to get nicely behaved underlying topological spaces.

If $U = U_{\text{top}} \subset X_{\text{top}}$ is an open subset, we will still denote $U \subset X$ the sub-functor defined by the condition that every point $x : \mathcal{M}(A) \to X$ such that $x_{\text{top}} : \mathcal{M}(A) \to X_{\text{top}}$ factorizes through $U_{\text{top}}$ is in $U(A)$.

**Definition 24.** A $t$-analytic space $X$ is called a *$t$-Berkovich space* if it is locally finitely presented and every open subset $U \subset X_{\text{top}}$ is an admissible domain. We will denote $\text{Ber}^t_R$ the category of $t$-Berkovich spaces.
Remark that in a $t$-Berkovich space, every point $x \in |X_{\text{top}}|$ has a neighborhood given by a finite union $\bigcup_i \mathcal{M}(A_i)$ for $\mathcal{M}(A_i) \to X$ some rational domains.

The interest of imposing the above conditions is given by the following consequence of the admissibility of Berkovich open subsets: there is a natural continuous morphism of sites

$$\pi : (X, \tau_G) \to (X_{\text{top}}, \tau_B).$$

Remark that if $K = (K, |\cdot|_K) \to (L, |\cdot|_L) = L$ is a Banach field extension, and $X$ is a $t$-Berkovich space over $K$, then $X_L$ is a $t$-Berkovich space over $L$, but we may also take the base extension

$$\tilde{X}_L := X \times_{\mathcal{M}(K)} \mathcal{M}(L)$$

of $X$ inside the category of $t$-analytic spaces over $K$, which will be a $K$-dagger analytic space but not a $K$-dagger Berkovich space anymore, since it is not locally finitely presented in general (i.e., not locally modeled on $t$-affinoid algebras).

**Definition 25.** Let $u \in \{\text{an}, \dagger\}$ be the projection of $t$ along the natural map

$$\{\text{an}, \{\text{an}, s\}, \dagger, \{\dagger, s\}\} \to \{\text{an}, \dagger\}.$$

The $u$-affine line over $R$ is the (non-strict) $u$-space over $R$ defined by the colimit of discs in one variable over $R$:

$$\mathbb{A}^1_{R,u} := \text{colim}_\rho D^u_{R}(0, \rho).$$

The structural sheaf $\mathcal{O}^t$ of $t$-algebras on a $t$-analytic space $(X, \tau_G)$ is given on a domain $D \to X$ by

$$\mathcal{O}^t_{X,G}(D) := \text{Hom}_{\mathbb{A}^1_{R,u}}(D, \mathbb{A}^1_{R,u}).$$

If $X$ is a $t$-Berkovich space, the structural sheaf on $X_{\text{top}}$ is given by

$$\mathcal{O}^t_{X_{\text{top}}}(U) = \pi_* \mathcal{O}^t_{X,G} := \text{colim}_{D \subset U} (\mathcal{O}^t_{X,G}(D)),$$

where the colimit is taken in the category $\text{Alg}^{t}_R$ and over all domains (or even rational domains) contained in $U$.

Remark that one needs to pass to the non-strict category to define the affine line, even if we work in the strict case, because when we work with strict analytic spaces on a trivially valued integral ring $(R, |\cdot|_0)$, i.e., with schemes, there are no elements in $R$ of arbitrary big seminorm, so that one would need to give another definition for the affine line in this case (for example, as the unit disc, that is the usual algebraic affine line in this strict trivially valued situation).

**Example 5.** We now explain how the notion of weak formal scheme from Meredith [Mer72] is naturally related to dagger analytic spaces over a $p$-adic ring of integers. Let $K$ be a complete non trivially valued non-archimedean field, with ring of integers $O_K$ and residue field $k$. Let $\pi \in O_K$ be a uniformizer. A weak formal scheme is a strict dagger
analytic space over \((O_K, |·|_K)\). The generic fiber of a weak formal scheme \(X\) is the extension of scalars of \(X\) along the bounded morphism \((O_K, |·|_K) \to (K, |·|_K)\). Its special fiber is the scheme given by the extension of scalars of \(X\) along the bounded morphism \((O_K, |·|_K) \to (k, |·|_0)\). The case of formal schemes can be treated similarly using strict analytic spaces over \((O_K, |·|_K)\). It is easy to find weak formal schemes that gives dagger models over \(O_K\) of finitely presented affine schemes over \(k\). Indeed, a natural dagger model for the affine space \(A^n_k\) is of course given by the unit polydisc \(D^n,\dagger(0,1)\) over \(O_K\), and given finitely many equations and inequations \(f_i = 0\) and \(g_j \neq 0\) in \(k[X_1, \ldots, X_n]\), one may extend them to \(D^n,\dagger(0,1)\) by taking pre-images in \(O_K[X_1, \ldots, X_n]\) of the corresponding polynomials, and using the strict domain \(D\) defined by \(|f_i| \leq |\pi|\) and \(|g_j| \geq 1\), for example. A similar argument applies to the projective space: the strict projective dagger space over \(O_K\) may be obtained by pasting various polydiscs (instead of various affine spaces). There is also a general theorem due to Arabia [Ara01]: every smooth scheme \(\overline{X}\) over \(k\) can be extended to a smooth weak formal scheme \(X\) over \(O_K\). In the general case, one can always extend a quasi-projective scheme over \(k\) to a weak formal scheme over \(O_K\), that is not smooth anymore in general. One may however see it as a derived dagger space over \(O_K\) in the sense of Section 7, and the (Hodge-completed) de Rham cohomology of its generic fiber, that is a derived dagger space over \(K\), will give us a nice cohomology theory. It is also quite sure that the (Hodge completed) derived de Rham cohomology of the dagger model \(\overline{X}\) over \(O_K\) is an interesting invariant, because it also contains an important \(p\)-torsion information, similar to the one used in \(p\)-adic cohomology theories.

**Remark 8.** Let \(X\) be an analytic space over a Banach ring \(R = (R, |·|_R)\) in the sense of Berkovich (see [Ber90] and [Poi13]). To every dagger rational domain algebra \(A\) over \(R\) we associate the corresponding germ of global analytic subspace \(D_A\) in the corresponding analytic affine space \(A^n_{(R, |·|_R)}\). This correspondence is actually an equivalence, since we can find back that algebra using global sections on the germ. The presheaf \(X^\dagger\) on rational domain dagger algebras defined by setting \(X^\dagger(A)\) to be given by the set of morphisms of germs (pro-global analytic spaces) \(D_A \to X\) is a sheaf on the category of rational domain algebras for the standard rational coverings, and this extends naturally to a sheaf on the category \(\text{Alg}^\dagger_{R}\) by writing a dagger algebra as a colimit of rational dagger algebras. This gives a natural functor

\[
(-)^\dagger : \text{Ber}^\text{glob}_R \to \text{AN}^\dagger_R
\]

from global analytic spaces over \(R\) in the sense of Berkovich [Ber90] to dagger analytic spaces over \(R\). This will allow us to define the de Rham cohomology of a global analytic space, and to relate dagger analytic motives over \(\mathbb{C}\) to Ayoub’s analytic motives.

**Remark 9.** The functor that sends a dagger algebra to the associated analytic algebra, being compatible with rational domains and standard rational coverings, induces a natural functor between the corresponding category of sheaves, which in turn induces a natural functor

\[
\text{AN}^\dagger_R \to \text{AN}^\text{an}_R
\]
from dagger analytic spaces to the category of convergent analytic spaces. This reduces naturally to a functor

\[ \text{Ber}_R^+ \to \text{Ber}_R^{an}. \]

If \((R, | \cdot |_R) = (K, | \cdot |_K)\) is a non-archimedean valued field, this functor is close to being an equivalence, as already shown in a similar (strict) situation by Große-Klönne in [GK00]. The category \(\text{Ber}_K^{an}\) is very close to the category of Berkovich’s non-archimedean analytic spaces from [Ber93]. If we denote \(\text{Ber}_K\) Berkovich’s original category, there is a natural functor

\[ \text{Ber}_K \to \text{Ber}_K^{an} \]

that sends \(X\) to the sheaf \(\text{Hom}_{\text{Ber}_K}(-, X)\). We defined our notion of dagger Berkovich spaces so that this functor is likely to be an equivalence of categories, with the aim of making the comparison with Berkovich’s theory easier. Remark that, as shown by Bambozzi in his thesis [Bam14], the theory of dagger spaces seems better adapted to the archimedean situation.

### 4.2 Tate’s acyclicity theorem

We now follow closely the approach of Tate [Tat71] (see also [BGR84], Chap. 7), by adapting its archimedean version given by Bambozzi in [Bam14], Chapter 4. The arguments are almost the same, but we write them for the reader’s convenience. In this section, we denote \((R, | \cdot |_R)\) a given base Banach ring, and \(t = \{an, \{an, s\}, \dagger, \{\dagger, s\}\}\) a type of analytic spaces. Let \(X = \mathbb{M}(A)\) be an affinoid \(t\)-analytic space over \(R\). Recall that we have defined the presheaf \(\mathcal{O}_X\) on the rational domain topology by setting

\[ \mathcal{O}_X(U) := \text{Hom}_\text{AN}_R(U, \mathbb{A}_R^{1, u}), \]

where \(u \in \{an, \dagger\}\) is the non-strict type analytic space associated to the type \(t\). One may compute this explicitly on rational domain algebras, and this gives simply the functor

\[ B \mapsto \text{ALG}(B) \]

that sends a \(t\)-rational domain algebra \(B = A\{f_0, \rho_0|f_1, \rho_1, \ldots, f_n, \rho_n\}\) to the underlying algebra. We will now check that this presheaf is actually a sheaf.

**Lemma 6.** Let \(X = \mathbb{M}(A)\) be an affinoid \(t\)-analytic space over \(R\) and \(X = \cup_i U_i\) be an affinoid \(t\)-covering. Then

\[ \mathcal{O}_X(X) \to \prod_i \mathcal{O}_X(U_i) \]

is injective.

**Proof.** This comes from the definition of \(\mathcal{O}_X\) as morphisms with values in the (non-strict) space given by affine line, and that analytic functions (overconvergent or not) are determined by their germs at every point of the Berkovich space. \(\square\)
Lemma 7. Let $A$ be a $t$-affinoid $R$-algebra. For any $f \in \text{ALG}(A)$, there exists a point $x_0 \in \mathcal{M}(A)$ such that

$$|f(x_0)| = \sup_{x \in \mathcal{M}(A)} |f(x)|.$$  

Proof. The dagger analytic situation restricts to the strict situation by using the fact that the Berkovich spectrum of a dagger affinoid algebra may be identified with the Berkovich spectrum of the associated analytic affinoid algebra. One may then use the fact that we work with uniform Banach rings, so that the given norm is equal to the spectral norm

$$\|f\|_{\infty} := \sup_{x \in \mathcal{M}(A)} |f(x)|.$$  

Since $x \mapsto |f(x)|$ is continuous for the Berkovich topology (by definition) on the compact topological space $\mathcal{M}(A)$, there exists $x_0 \in \mathcal{M}(A)$ such that $\|f\|_{\infty} = |f(x_0)|$. $\square$

Lemma 8. Let $X = \mathcal{M}(A)$ with $A$ $t$-affinoid $R$-algebra and $f_1, \ldots, f_n \in \text{ALG}(A)$, $\rho_1, \ldots, \rho_n \in \mathbb{R}_{>0}$, then the function

$$\alpha(x) := \max_{i=1, \ldots, n} \rho_i^{-1} |f_i(x)|$$

assume it’s minimum in $X$.

Proof. If the $f_i$’s have a common zero, there is nothing to show. Otherwise, they generate the unit ideal in $A$, and one may consider the rational covering of $X$ given by

$$X_i = D(f_i, \rho_i|f_1, \rho_1, \ldots, \hat{f_i}, \rho_i, \ldots, f_n, \rho_n) = \{x \in X, \alpha(x) = \rho_i^{-1} |f_i(x)|\}.$$  

By Lemma 7, $\rho_i^{-1} |f_i(x)|$ assume its minimum on $X_i$ (because $\rho_i |f_i^{-1}(x)|$ has a maximum), and so $\alpha$ has assumes its minimum in $X$, which is the least of the minimum of the $\rho_i^{-1} |f_i(x)|$ over $X_i$. $\square$

Definition 26. Let $A$ be a $t$-affinoid algebra, $f_1, \ldots, f_n \in \text{ALG}(A)$, $\rho_1, \ldots, \rho_n \in \mathbb{R}_{>0}$ and $X = \mathcal{M}(A)$. If $t = \{an, \dot{t}\}$, we further suppose that $\rho_1 = \cdots = \rho_n = 1$. Then each

$$\mathcal{U}_i = \{D^t(1, 1|f_i, \rho_i), D^t(1, 1|f_i^{-1}, \rho_i^{-1})\}$$

is a $t$-rational covering of $X$. We denote by $\mathcal{U}_1 \times \cdots \times \mathcal{U}_n$ the covering consisting of all intersections of the form $U_1 \cap \cdots \cap U_n$ where $U_i \in \mathcal{U}_i$ and call this the $t$-Laurent covering of $X$ generated by $\{(f_i, \rho_i)\}_{i=1, \ldots, n}$.

More explicitly, the elements of the Laurent covering generated by $\{(f_i, \rho_i)\}$ are rational domains of the form

$$D(1, 1|f_1^{\mu_1}, \rho_1^{\mu_1}, \ldots, f_n^{\mu_n}, \rho_n^{\mu_n})$$

with $\mu_i = \pm 1$.  

44
**Lemma 9.** Suppose that \( t = \{an, \dagger\} \) is a non-strict type of analytic spaces. Let \( \mathcal{U} \) be a \( t \)-rational covering of \( X = \mathbb{M}(A) \). There exists a \( t \)-Laurent covering \( \mathcal{V} \) of \( X \) such that for any \( V \in \mathcal{V} \), the covering \( \mathcal{U}_V \) is a \( t \)-rational covering of \( V \), which is generated by units in \( \mathcal{O}_X(V) \).

**Proof.** Let \( f_1, \ldots, f_n \in \text{ALG}(A) \), \( \rho_1, \ldots, \rho_n \in \mathbb{R}_{>0} \) be the datum of definition of the \( t \)-rational covering \( \mathcal{U} \). We may choose a constant \( c \in \mathbb{R}_{>0} \) such that

\[
    c^{-1} \leq \inf_{x \in X} (\max_{1 \leq i \leq n} \rho_i^{-1}|f_i(x)|),
\]

which is well defined by Lemma 8. Let \( \mathcal{V} \) be the \( t \)-Laurent covering generated by \( \{(f_i, c^{-1}\rho_i)\}_{i=1,\ldots,n} \). Consider the set

\[
    V = D(1,1|f_1^\mu_1, (c^{-1}\rho_1)^\mu_1, \ldots, f_n^\mu_n, (c^{-1}\rho_n)^\mu_n) \in \mathcal{V},
\]

where \( \mu_i = \pm 1 \). We can assume that there exists an \( s \in \{0, \ldots, n\} \) such that \( \alpha_1 = \cdots = \alpha_s = 1 \), and \( \alpha_{s+1} = \cdots = \alpha_n = -1 \). Then

\[
    D(f_i, \rho_i|f_1, \rho_1, \ldots, f_i, \rho_i, \ldots, f_n, \rho_n) \cap V = \emptyset
\]

for \( i = 1, \ldots, s \), since

\[
    \max_{1 \leq i \leq n} \rho_i^{-1}|f_i(x)| \leq c^{-1} \leq \max_{i=1,\ldots,n} \rho_i^{-1}|f_i(x)|
\]

for all \( x \in V \). In particular, for all \( x \in V \), we have

\[
    \max_{i=1,\ldots,n} \rho_i^{-1}|f_i(x)| = \max_{i=s+1,\ldots,n} \rho_i^{-1}|f_i(x)|,
\]

hence \( \mathcal{U}_V \) is the rational covering of \( V \) gee rated by \( \{(f_i|V, \rho_i)\} \) for \( s + 1 \leq i \leq n \), which are units in \( V \) if \( t = an \) since their spectral norm is positive. They are also units in \( V \) if \( t = \dagger \).

\[\square\]

**Lemma 10.** Let \( \mathcal{U} \) be a \( t \)-rational covering of \( X \) which is generated by units \( f_1, \ldots, f_n \in \mathcal{O}_X(X) \), then there exists a \( t \)-Laurent covering \( \mathcal{V} \) of \( X \) which is a refinement of \( \mathcal{U} \).

**Proof.** Let \( \mathcal{V} \) be the Laurent covering generated by the family \( \{(f_i, \rho_i f^{-1}_j, \rho_j \rho_i)^{-1}\} \), with \( 1 \leq i < j \leq n \). Consider \( V \in \mathcal{V} \). Defining \( I = \{(i,j) \in \mathbb{N}^2, 1 \leq i < j \leq n\} \), we can find a partition \( I = I_1 \sqcup \cdots \sqcup I_2 \) such that

\[
    V = \cap_{(i,j) \in I_1} (D(1,1|f_i f_j^{-1}, \rho_i \rho_j^{-1})) \cap \cap_{(i,j) \in I_2} (D(1,1|f_j f_i^{-1}, \rho_j \rho_i^{-1})).
\]

One defines a partial order on \( \{1, \ldots, n\} \) requiring that if \( (i,j) \in I_1 \), then \( i \sim < j \) or if \( (i,j) \in I_2 \) then \( j \sim < i \). For each \( i, j \in \{1, \ldots, n\} \) with \( i \neq j \), then \( i \sim < j \) or \( j \sim < i \). Consider a maximal chain (which always exists) \( i_1 \sim \cdots \sim i_r \) of elements of \( \{1, \ldots, n\} \). Since \( i_r \) is maximal, we have that for any \( i \in \{1, \ldots, n\} \), \( i \sim < i_r \), which implies \( \rho_i |f_j(x)| \leq \rho_j |f_j(x)| \) for all \( x \in V \), i.e.,

\[
    V \subset D(f_i, \rho_i|f_1, \rho_1, \ldots, f_{i_r}, \rho_{i_r}, \ldots, f_n, \rho_n).
\]

\[\square\]
Theorem 1 (Tate’s acyclicity theorem). Suppose either that the base Banach ring is non-archimedean, or that \( t \in \{\hat{t}, \{\hat{t}, s\}\} \) is an overconvergent type of analytic spaces. The presheaf \( \mathcal{O}_X \) is acyclic for the \( G \)-topology on \( \mathbb{M}(A) \).

Proof. Since the \( G \)-topology is generated by rational coverings, we may reduce to them. Using Lemma 9 and 10, we can refine a given rational covering by a (non-strict) Laurent covering, and then by induction to the case of the covering \( \mathcal{U} = \{D(1, 1|f, \rho), D(1, 1|f^{-1}, \rho^{-1})\} \). The proof now follows closely the Tate’s one, adapted by Bambozzi to the archimedean setting in [Bam14], Theorem 4.0.18: one has to check that the sequence

\[
0 \to A \to A\{\rho^{-1}X\}/(X-f) \times A\{\rho Y\}/(Yf-1) \to A\{\rho^{-1}X, \rho Y\}/(X-f, Xf-1) \to 0
\]

is exact, which is done by Tate’s tricks of the trade, as explained in loc. cit. \( \square \)

4.3 The étale, Nisnevich and pro-étale topology

Let \( R \) be a uniform ind-Banach ring and \( t \in \{an, \{an, s\}, \hat{t}, \{\hat{t}, s\}\} \).

We will now define general notions of differential calculus in a way that is similar to the method used in scheme theory. We carefully inform the reader that these notions will be interesting mostly for dagger analytic spaces, but the general definitions will be useful for comparison purposes.

Definition 27. Let \( f : A \to B \) be a morphism of \( t \)-algebras over \( R \).

1. The morphism \( f \) is called \( \text{formally étale} \) (resp. \( \text{formally unramified}, \text{resp. formally smooth} \)) if for every commutative square

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
C & \xrightarrow{g} & C/I
\end{array}
\]

of \( t \)-algebras, with \( I \) a nilpotent ideal, the dotted arrow exists and is unique (resp. is unique if it exists, resp. exists).

2. It is called \( \text{flat} \) (resp. \( \text{finite} \)) if the underlying morphism \( \text{Alg}(f) : \text{Alg}(A) \to \text{Alg}(B) \) is flat (resp. finite).

3. A morphism \( f : X \to Y \) of \( t \)-analytic spaces is called \( \text{quasi-étale} \) (resp. \( \text{quasi-unramified}, \text{resp. quasi-smooth} \)) if it is locally finitely presented and it is locally formally étale (resp. formally unramified, resp. formally smooth) for the \( G \)-topology.

4. A quasi-étale morphism \( f : X \to Y \) of non-strict Berkovich spaces is said to \( \text{have the Nisnevich property} \) if every point \( y \in Y_{\text{top}} \) has a domain neighborhood \( U = \mathbb{M}(A) \).
such that $f$ has a section on $U$, meaning that there is a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
U & \longrightarrow & Y
\end{array}
$$

**Definition 28.** Let $X$ be a $t$-analytic space over $R$. A family $\mathcal{R} = \{Y_i \to X\}_{i \in I}$ of étale morphisms is called an étale covering if, for all morphism $\mathcal{M}(A) \to X$ with $A$ a $t$-algebra over $R$, the family $\mathcal{R} \times_X U := \{Y_i \times_X \mathcal{M}(A) \to \mathcal{M}(A)\}_{i \in I}$ can be refined by a finite family $\{\mathcal{M}(B_j) \to \mathcal{M}(A)\}_{j \in J}$ of étale morphisms. If we further suppose that $X$ and $Y$ are Berkovich spaces, we will say that it is a Nisnevich covering if the morphism $\coprod_{j \in J} \mathcal{M}(B_j) \to \mathcal{M}(A)$ has the Nisnevich property.

We now define the pro-étale topology following closely Scholze's approach from [Sch13], Section 3. This topology gives a better take at completed étale cohomology. Let $X$ be a $t$-analytic space and $X_{\text{fet}}$ be the category of finite étale morphisms $f: Y \to X$.

**Definition 29.** The pro-finite étale site is defined to be the category pro-$X_{\text{et}}$ of pro-objects of the category $X_{\text{fet}}$. A morphism $U \to V$ of objects of pro-$X_{\text{et}}$ is called étale (resp. finite étale) if it is induced by an étale (resp. finite étale) morphism $U_0 \to V_0$ of objects in $X_{\text{et}}$ by base extension along a morphism $V \to V_0$. A morphism $U \to V$ of objects of pro-$X_{\text{et}}$ is called pro-étale if it can be written as a cofiltered inverse limit $U = \lim U_i$ of objects $U_i \to V$ étale over $V$, such that $U_i \to U_j$ is finite étale and surjective for large $i > j$. Such a presentation $U = \lim U_i \to V$ is called a pro-étale presentation. The pro-étale site $X_{\text{proet}}$ has as underlying category the full subcategory of pro-$X_{\text{et}}$ of objects that are pro-étale over $X$. Finally, a covering in $X_{\text{proet}}$ is given by a family of pro-étale morphisms $\{f_i: U_i \to U\}$ such that $\coprod f_i: \coprod U_i \to U$ is an épimorphism.

There is a natural projection

$$
\eta: X_{\text{proet}} \to X_{\text{et}}.
$$

We will denote $\mathbb{Z}_\ell$ the limit of the constant pro-étale sheaves $\mathbb{Z}/\ell^n\mathbb{Z}$, and $\hat{\mathbb{Z}}$ the limit of the constant pro-étale sheaves $\mathbb{Z}/n\mathbb{Z}$. We will also denote $\mathbb{Q}_\ell := \mathbb{Z}_\ell \otimes \mathbb{Q}$ and $\mathbb{A}_f := \hat{\mathbb{Z}} \otimes \mathbb{Q}$.

**Definition 30.** Let $X$ be a dagger analytic space. The $\ell$-adic (resp. complete integral) étale cohomology of $X$ is defined by

$$
H^*_\text{et}(X, \mathbb{Z}_\ell) := H^*(X_{\text{proet}}, \mathbb{Z}_\ell)
$$

(resp. $H^*_\text{et}(X, \hat{\mathbb{Z}}) := H^*(X_{\text{proet}}, \hat{\mathbb{Z}})$).

**Example 6.** Let $X$ be a proper non-strict analytic space over $\mathbb{Z}_0 = (\mathbb{Z}, | \cdot |_0)$ and denote $\mathbb{Q} = (\mathbb{Q}, | \cdot |_0)$. It may be interesting to try to extend Scholze's result on $p$-adic Hodge theory in [Sch13] to give a comparison theorem between the pro-étale cohomology of $X_\mathbb{Q}$.
with coefficients in \( \mathring{\mathbb{Z}} \) and the derived de Rham cohomology of \( X/\mathbb{Z}_0 \). Such an extension has already been discussed in the semistable algebraic setting (which is a strict analytic situation over \( \mathbb{Z}_0 \)) by Bhatt in [Bha12b], Remark 10.22, but the use of analytic methods may allow for a treatment of the question based only on Faltings’ almost mathematical methods (like in Scholze’s approach), avoiding the semistable reduction hypothesis. This will be further discussed in Subsection 7.1.

A notion of overconvergent sub-analytic subset will be necessary for the study of direct images of analytically constructible sheaves in the pro-étale topology. We refer to Martin’s paper [Mar12] for the notion of overconvergent sub-analytic subsets in strict \( p \)-adic geometry. We give here an adaptation of his definition to our general setting.

**Definition 31.** Let \( R \) be a uniform ind-Banach algebra and \( X_{\text{top}} \) be the underlying topological space of a dagger (resp. a strict dagger) space \( X \) over \( R \). A **semi-analytic subset** of \( X \) is a subspace in the boolean algebra generated by its rational (resp. strictly rational) subsets (by finite union, finite intersections and complements). A **sub-analytic subset** in \( X \) is a subset given by the projection of a semi-analytic subset in \( X \times_R D^{\dagger}_n(0,\rho) \) (resp. in \( X \times_R D^{\dagger}_n(0,1) \)) for some \( n \geq 0 \) along the natural projection map to \( X \).

It is not clear that subanalytic subsets defined as above form a boolean algebra, as it is the case in the strict \( p \)-adic setting thanks to the result of Martin, loc. cit., Proposition 1.39, but this may be an interesting question to ask to model theorists.

**5 Dagger analytic geometry and Archimedean compactifications**

Let \( \mathbb{Z} := (\mathbb{Z}, | \cdot |_\infty) \) be the global analytic basis and \( \mathbb{Z}_0 := (\mathbb{Z}, | \cdot |_0) \) be its non-archimedean counterpart. It is quite clear that any scheme locally of finite type over \( \mathbb{Z} \) may be equipped with a structure of non-strict dagger space over \( \mathbb{Z} \). Indeed, closed affine subschemes of \( \mathbb{A}^n_{\mathbb{Z}} \) may be described as closed analytic subspaces of the overconvergent analytic affine space \( \mathbb{A}^{\dagger,n}_{\mathbb{Z}} \), and since algebraic maps are overconvergent, they can be used to define dagger spaces over \( \mathbb{Z} \) from schemes locally of finite type over \( \mathbb{Z} \). We thus get a (not very natural) “base restriction functor”

\[
\mathbb{A}^{\dagger,n}_{\mathbb{Z}_0} \cong \text{Sch}_{\mathbb{Z}} \longrightarrow \mathbb{A}^{\dagger,n}_{\mathbb{Z}}
\]

from the category of schemes over \( \mathbb{Z} \) (i.e., strict dagger spaces over \( \mathbb{Z}_0 \)) to the category of dagger spaces over \( \mathbb{Z} \). All this shows that the category of non-strict global analytic spaces is a natural recipient both for algebraic geometry and analytic geometry over various bases like \( \mathbb{R} \), \( \mathbb{Q}_p \) or \( \mathbb{Z}_p \).

**Definition 32.** The above defined functor

\[
\mathbb{A}^{\dagger} : \text{Sch}_{\mathbb{Z}} \longrightarrow \mathbb{A}^{\dagger}_{\mathbb{Z}}
\]

will be called the non-strict (dagger) analytification functor.
We now ask the natural question of describing which schemes over $\mathbb{Z}$ may be seen as strict dagger spaces over $\mathbb{Z}$, i.e., are isomorphic to schemes in the image of the functor

$$\mathbb{A}^\dagger_{\mathbb{Z}}^{+,s} \longrightarrow \mathbb{A}^\dagger_{\mathbb{Z}}.$$  

This will lead us to the idea that an extension of a scheme structure over $\mathbb{Z}$ to a strict dagger analytic space structure over $\mathbb{Z}$ may be naturally thought of as a kind of Archimedean compactification, in the sense that is usually meant in Arakelov geometry.

**Definition 33.** Let $X$ be a scheme over $\mathbb{Z}$. An Archimedean compactification of $X$ is a strict dagger analytic space $\mathbb{A}^\dagger_{\mathbb{Z}}^{+,s}(X)$ over $\mathbb{Z}$ together with an isomorphism

$$\mathbb{A}^\dagger_{\mathbb{Z}}^{+,s}(X) \sim \longrightarrow \mathbb{A}^\dagger_{\mathbb{Z}}(X)$$

of non-strict analytic spaces over $\mathbb{Z}$.

**Example 7.** The unit disc $D^\dagger(0,1)_{\mathbb{Z}}$ is a strict dagger space with the affine line as base extension to $\mathbb{Z}_0$, but it is not an Archimedean compactification of the algebraic affine line $\mathbb{A}^\dagger_{\mathbb{Z}}$ because there is no isomorphism

$$D^\dagger(0,1)_{\mathbb{Z}} \sim \longrightarrow \mathbb{A}^\dagger_{\mathbb{Z}}.$$  

Indeed, such an isomorphism would identify the associated topological spaces, but one of them is compact and the other is non-compact.

### 5.1 Archimedean compactifications of projective schemes

To illustrate the notion of Archimedean compactification, we will describe it for the projective space $\mathbb{P}^1_\mathbb{Z}$. Recall from Example 3 that the natural isomorphism

$$\mathbb{Z}[X_0, 1/X_0] \longrightarrow \mathbb{Z}[X_1, 1/X_1]$$

given by $X_0 \mapsto 1/X_0$ is the underlying ring map of an isomorphism

$$\mathbb{Z}\{X_0, 1/X_0\}^\dagger \longrightarrow \mathbb{Z}\{X_1, 1/X_1\}^\dagger$$

of overconvergent rational domain algebras over $\mathbb{Z}$. One may paste the overconvergent global analytic discs

$$\mathbb{D}^\dagger_\mathbb{Z} := \mathcal{M}(\mathbb{Z}\{X_0\}^\dagger) \text{ and } \mathbb{D}^\dagger_{\mathbb{Z}} := \mathcal{M}(\mathbb{Z}\{X_1\}^\dagger)$$

(with functions given by the polynomial algebras over $\mathbb{Z}$ with their sup norms on all discs containing the global unit disc) to get a global analytic version $\mathbb{P}^1_{\mathbb{Z}}$ of the projective line, that will be the Archimedean compactification that we were looking for. Adding the polydisc seminorm structures on the polynomial rings gives an important additional information that may be thought as some kind of “Arakelov compactification”. Remark
that one may see the (overconvergent analytic) complex projective space $\mathbb{P}^1_C$ either as the pasting of two copies of $A^1_C$ along $G^\dagger_m,C$ (which gives also an algebraic model for it over $C$), or as the pasting of two (overconvergent) discs $D^1_C$ along $D^1_C - \{0\}$. Using the disc viewpoint “breaks the $G^\dagger_m$-symmetry” of the algebraic situation. It is quite striking that such a “breaking of the $G^\dagger_m$” symmetry can be also done in the global analytic setting.

Lemma 11. If $X$ is a scheme over $Z$ that admits an archimedean compactification

$$\text{An}^{1,s}(X) \xrightarrow{\sim} \text{An}^\dagger(X),$$

then any closed subscheme $Z$ of $X$ also has an archimedean compactification and the inclusion $Z \to X$ may be extended to $\text{An}^{1,s}(Z) \subset \text{An}^{1,s}(X)$.

Proof. Let $Z \subset X$ be a closed subscheme, and let $\text{An}^{1,s}(X) = \cup_i M(A_i)$ be a covering of the archimedean compactification of $X$ by strict representable domains over $Z$. Then one may write the equations of $Z$ in these charts and they are compatible by construction with the pasting maps, so that $Z$ also has an archimedean compactification. □

Not every affine variety over $Z$, written in explicit coordinates, can be easily seen as an affine overconvergent analytic variety over $Z$, because the solutions of an equation in the affine line are not always contained in the unit disc. However, the case of projective varieties is different, as we will see from the following proposition.

Proposition 11. If $X$ is a projective variety over $Z$, then it has a natural Archimedean compactification

$$\text{An}^{1,s}(X) \xrightarrow{\sim} \text{An}^\dagger(X).$$

Proof. The case of the projective space $\mathbb{P}_Z^n$ is obtained by generalizing directly the above example: the natural morphism

$$\mathbb{Z}[t_0, \ldots, t_n, 1/t_i] \to \mathbb{Z}[t_0, \ldots, t_n, 1/t_j],$$

$$t_k \mapsto t_k/t_j \text{ si } k \neq i$$

$$t_i \mapsto 1/t_j$$

is bounded enough, so that it induces a morphism of overconvergent rational domain algebras

$$\mathbb{Z}\{t_0, \ldots, t_n, 1/t_i\}^\dagger \to \mathbb{Z}\{t_0, \ldots, t_n, 1/t_j\}^\dagger$$

over the base Banach ring $\mathbb{Z} = (\mathbb{Z}, |.|_\infty)$ (these are just localizations of polynomial algebras, but equipped with a family of norms induced by the over-seminorms of the sup norm on the global unit polydisc). If $Z \subset \mathbb{P}_Z^n$ is a closed sub-scheme, then we may apply Lemma 11 to get an Archimedean compactification of $Z$. □

Remark 10. It is clear from what we explained in Example 7 that general schemes usually don’t have Arithmetic compactifications. Indeed, the affine line $A^1_Z$ is not representable in strict dagger spaces.

50
Remark 11. Another approach to finding strict dagger models over \( \mathbb{Z} \) of projective schemes over \( \mathbb{Z} \) may be given by the isomorphism

\[
\mathbb{P}^1_{\mathbb{Z}} \cong \mathbb{A}^2_{\mathbb{Z}} - \{(0,0)\}/\mathbb{G}_{m,\mathbb{Z}}.
\]

One may define a strict dagger model of \( \mathbb{P}^1_{\mathbb{Z}} \) on \( \mathbb{Z} \) by using the quotient analytic space \( \mathbb{D}^{2,\dagger}(0,1)_{\mathbb{Z}} - \{(0,0)\}/U(1)_{\mathbb{Z}} \), where \( U(1)_{\mathbb{Z}} := \mathbb{M}(\mathbb{Z}\{X,1/X\})^{\dagger} \). The analytic space

\[
\mathbb{D}^{2,\dagger}(0,1)_{\mathbb{Z}} - \{(0,0)\}
\]

may be defined by pasting \( \mathbb{D}^{\dagger}(0,1)_{\mathbb{Z}} \times U(1)_{\mathbb{Z}} \) and \( U(1)_{\mathbb{Z}} \times \mathbb{D}^{\dagger}(0,1)_{\mathbb{Z}} \) along their common rational domain \( U(1)_{\mathbb{Z}} \times U(1)_{\mathbb{Z}} \). This gives another way of presenting the strict dagger projective space, as the quotient analytic sheaf (i.e., set-valued sheaf with values in sets on the rational domain topology on strict dagger algebras)

\[
\mathbb{P}^{1,\dagger}_{\mathbb{Z}} := \mathbb{D}^{2,\dagger}(0,1)_{\mathbb{Z}} - \{(0,0)\}/U(1)_{\mathbb{Z}}.
\]

**Proposition 12.** The following diagram of functors

\[
\begin{array}{ccc}
\text{ProjAn}^{\dagger,s}_{\mathbb{Z}} & \xrightarrow{\sim} & \text{ProjAn}^{\dagger,s}_{\mathbb{q}} \cong \text{Proj}_{\mathbb{Z}} \\
\downarrow & & \downarrow \\
\text{An}^{\dagger,s}_{\mathbb{Z}} & \xrightarrow{\sim} & \text{An}^{\dagger}_{\mathbb{Z}}
\end{array}
\]

is \( (2) \)-commutative, with vertical arrows fully faithful and the upper horizontal arrow an equivalence.

**Proof.** The vertical arrows are fully faithful by definition. The fact that the upper horizontal arrow is essentially surjective follows from proposition 11. The fact that it is faithful is clear. The fact that it is full is less clear. If \( f : X \to Y \) is a morphism of projective varieties, then its graph \( \Gamma_f \), defined as the pullback

\[
\begin{array}{ccc}
\Gamma_f & \xrightarrow{f \times \text{id}} & X \times Y \\
\downarrow & & \downarrow \\
Y & \xrightarrow{\Delta_Y} & Y \times Y
\end{array}
\]

is projective. By proposition 11, this graph has an Arithmetic compactification. Since \( f : X \to Y \) may be written as the pullback of the projection \( \Gamma_f \to Y \) along the identity map, it also has an Archimedean compactification, so that the upper horizontal arrow of the diagram in the statement of the proposition is an equivalence. \( \square \)
5.2 Logarithmic Archimedean compactifications of quasi-projective schemes

We would like to have a way to associate to a quasi-projective variety over $\mathbb{Z}$ some kind of strict global dagger space over $\mathbb{Z}$ with generic fiber the given variety, that will give an Arakelov model of the given variety. For example, if we start from $X = \mathbb{A}^1_\mathbb{Z}$, this wish can't be fulfilled stricto sensu. A way to overcome this problem with non-projective schemes, at least at the cohomological level, was paved by Deligne in [Del71], and then formalized geometrically by Fontaine-Illusie and Hyodo-Kato, by the use of logarithmic analytic spaces. In the above example, one replaces the affine line over $\mathbb{Z}$ by the logarithmic analytic space over $\mathbb{Z}$ given by the projective line over $\mathbb{Z}$, together with the sheaf of monoids $\mathcal{M} := j_* \mathcal{O}^*_\mathbb{A}^1 \cap \mathcal{O}_{\mathbb{P}^1} \subset \mathcal{O}_{\mathbb{P}^1}$, where $j : \mathbb{A}^1 \rightarrow \mathbb{P}^1$ is the natural embedding. The definition of this monoid of course uses some non-strict dagger geometry, since even $\mathbb{A}^1$ and the structural sheaf can’t be defined in the strict setting in general, but the analytic space in play (here $\mathbb{P}^1$) is really a strict analytic space. So strict logarithmic geometry gives an intermediary setting between strict analytic geometry and non-strict analytic geometry.

**Definition 34.** Let $t \in \{an, \{an, s\}, \dagger, \{\dagger, s\}\}$ be a type of analytic spaces. Let $X$ be a $t$-analytic space over an ind-Banach ring $R$. A pre-logarithmic structure on $X$ is given by a sheaf of monoids $\mathcal{M}$ on $X_{et}$, together with a morphism of multiplicative monoids $\alpha : \mathcal{M} \rightarrow \mathcal{O}_X$.

The pre-log structure is called a log structure if $\alpha$ induces an isomorphism

$$\alpha^{-1}(\mathcal{O}_X^*) \xrightarrow{\sim} \mathcal{O}_X^*.$$

It is easy to generalize the notion of Archimedean compactification to the logarithmic setting.

**Definition 35.** Let $(X, \mathcal{M})$ be a log scheme over $\mathbb{Z}$, and $\text{AN}^\dagger(X, \mathcal{M})$ be the associated non-strict logarithmic dagger space over $\mathbb{Z}$. An **Archimedean compactification** of $(X, \mathcal{M})$ is a strict logarithmic dagger space $\text{AN}^{\dagger,s}(X, \mathcal{M})$ over $\mathbb{Z}$ together with an isomorphism

$$\text{AN}^{\dagger,s}(X, \mathcal{M}) \xrightarrow{\sim} \text{AN}^\dagger(X, \mathcal{M})$$

of non-strict dagger logarithmic spaces over $\mathbb{Z}$.

We now may now extend Proposition 11 to the case of semistably compactifiable schemes.

**Proposition 13.** Let $X$ be a smooth scheme over $\mathbb{Z}$ that admits a projective compactification $\bar{X}$ over $\mathbb{Z}$ such that $D := \bar{X} \setminus D$ is a divisor with normal crossings. Then the associated log-scheme $(\bar{X}, \mathcal{M}_D)$, where $\mathcal{M}_D := j_* \mathcal{O}_X^* \cap \mathcal{O}_X$, and where $j : X \rightarrow \bar{X}$ is the natural embedding, has a canonical Archimedean compactification $\text{AN}^{\dagger,s}(\bar{X}, \mathcal{M})$. 

52
Proof. The closed inclusion $D \to \bar{X}$ is a morphism of projective schemes over $\mathbb{Z}$ that has an Archimedean compactification $\text{An}^{t,s}(D) \to \text{An}^{t,s}(\bar{X})$ by Proposition 12. The associated logarithmic strict dagger space $\text{An}^{t,s}(X, \mathcal{M}_D)$ will do the job.

Remark 12. We can still say something in the non-semistable case, using de Jong’s resolution of singularities, as explained by Beilinson in [Bei11]. Let $X$ be a smooth quasi-projective scheme over $\overline{\mathbb{Q}}$. De Jong’s theorem implies that a basis for the $h$-topology on $X$ is given by arithmetic semistable pairs $(U, \bar{U})/\mathbb{Z}$ (a smooth compactification of a smooth variety with boundary a normal crossing divisor). To each such pair, one may associate a logarithmic scheme over $\overline{\mathbb{Z}}$ that has an Archimedean compactification over $\overline{\mathbb{Z}}$. So we may say that in some sense, every scheme over $\overline{\mathbb{Q}}$ may be $h$-locally logarithmically Arithmetically compactified. This construction may give a natural setting to explain geometrically Arakelov-motivic cohomology [HS10] in a way that avoids the direct use of Deligne cohomology. This point will be further discussed in Remark 17. This may also give a natural setting to define global period rings by derived periods à la Beilinson-Bhatt. This point will be further discussed in Subsection 7.1.

5.3 A dagger arithmetic Riemann-Roch problem

It is quite tempting to generalize Rion’s homotopy theoretic approach to the Riemann-Roch theorem from [Rio10] by looking at it as written in the setting of homotopy theory of strict analytic spaces over the base $\mathbb{Z}_0 = (\mathbb{Z}, |·|_0)$ (in a sense to be explained in Section 6), and trying to extend it to strict analytic spaces over $\mathbb{Z} = (\mathbb{Z}, |·|_\infty)$. We will call the question of this extension the dagger arithmetic Riemann-Roch problem.

One may define the dagger general linear group as the sheaf on dagger algebras over $\mathbb{Z}$ given by

$$\text{GL}_n : A \mapsto \text{GL}_n(\text{Alg}(A)).$$

Since $A^1 : A \mapsto \text{Alg}(A)$ is not representable in the category of strict dagger spaces over $\mathbb{Z}_0$, it is quite reasonable to imagine that the same applies to the general linear group for $n > 1$. It is quite easy, however, to define the strict global dagger analog of the classifying space $\text{BGL}$ used by Rion: in $A^1$-homotopy theory, this space is described as the infinite Grassmannian $\text{Gr}_{\infty}$ given by the colimit of the systems $(\text{Gr}_{d,n})_{(d,n) \in \mathbb{N}^2}$, where the transition morphisms are of the form $\text{Gr}_{d,r} \to \text{Gr}_{1+d,r}$ and $\text{Gr}_{d,r} \to \text{Gr}_{d,r+1}$. We thus only have to show that these varieties and maps have a dagger Archimedean compactification, i.e., a strict model over $\mathbb{Z}$, which is quite probable since they are projective, so that we can apply Proposition 11.

Another approach to this problem, that is followed by Karoubi and Villamayor in [KV73], and more recently by Tamme [Tam11], is to replace the group $\text{GL}_n$ by the simplicial group $\text{GL}^\bullet_n$ given by

$$\text{GL}^\bullet_n : A \mapsto \text{GL}_n(\text{Alg}(A[\Delta^\bullet]^1)).$$
where the simplicial ring $A\{\Delta^\bullet\}^\dagger$ is defined by

$$A\{\Delta^n\}^\dagger := A\{T_0, \ldots, T_n\}^\dagger / (\sum T_i - 1).$$

The classifying space may then be defined as the total $\infty$-stack associated to the functor

$$\text{BGL} : A \mapsto \mathbb{Z} \times B_\ast \text{GL}(A\{\Delta^\bullet\}^\dagger)$$

with values in bisimplicial sets. One then defines the (overconvergent) Karoubi-Villamayor $K$-theory of $A$ as

$$\text{KV}_i(A) = \pi_i(\text{BGL}).$$

Following [Tam11], 2.4, this gives back algebraic $K$-theory for $i \geq 1$ in the case of a trivially normed integral ring $R$ that is supposed to be regular.

Once given the correct Archimedean dagger compactification of $\text{BGL}$, and using the notion of rational motivic cohomology proposed in Section 6, one may ask the following question: if $f : X \to S$ is a projective smooth morphism of strict dagger analytic spaces over $\mathbb{Z}$, does the following diagram

$$
\begin{array}{ccc}
\mathbb{R}f_\ast \text{BGL}_\mathbb{Q} & \xrightarrow{\mathbb{R}f_\ast (\text{ch}. \text{Td}(T_f))} & \prod_{i \in \mathbb{Z}} \mathbb{R}f_\ast \text{H}_\mathbb{Q}(i)[2i] \\
\downarrow f_\ast & & \downarrow f_\ast \\
\text{BGL}_\mathbb{Q} & \xrightarrow{\text{ch}} & \prod_{i \in \mathbb{Z}} \text{H}_\mathbb{Q}(i)[2i]
\end{array}
$$

commute in the strict rational stable homotopy theory $\text{SH}^{\dagger, s}(S)$? The same question may apply in the quasi-projective case by replacing $f_\ast$ by the proper direct image $f_!$ and motivic cohomology $\text{H}$ by its version with proper support $\text{H}_c$.

**Remark 13.** As a corollary of this homotopy theoretic Riemann-Roch theorem, one would get a Riemann-Roch theorem relating the direct image of higher Artin-Verdier $K$-theory classes to the direct image of their Chern classes in higher Artin-Verdier motivic cohomology (to be defined as motivic cohomology of strict analytic spaces over $\mathbb{Z}$ in the sense of Section 6). This would give a kind of higher arithmetic Riemann-Roch for Artin-Verdier motivic étale cohomology, that is quite different in nature from the Riemann-Roch statements proved on Arakelov-motivic cohomology in [Sch12a], since the Hodge filtration is not included in our approach.

**Remark 14.** To get a global analytic interpretation of Arakelov-motivic cohomology, one really needs to combine the global analytic information given by strict global motivic cohomology with the differential information given by Hodge-filtered de Rham cohomology. This problem may be approached by trying to globalize the period isomorphism of $p$-adic Hodge theory (see Subsection 7.1).

**Remark 15.** A global analytic interpretation of Arakelov-motivic cohomology may also be attained by taking inspiration in the work of Karoubi on multiplicative $K$-theory [Kar86].
As explained to the author by Gregory Ginot, this viewpoint has the great advantage on
the usual geometric approach to avoid the introduction of denominators of the form $\frac{1}{n!}$ in
the definition of the Chern character map

$$\text{ch} : K_*(X) \longrightarrow HC_*(X).$$

In any case (i.e., even in the Karoubi approach), to get the right $p$-torsion information,
one needs to work with a version of (maybe Hodge-completed) derived Hodge-filtered de
Rham cohomology relative to the global analytic base $\mathbb{Z}$, to be defined in Subsection 7.3.
A global analytic version of the Chern character will be discussed in Section 8.

6 Global analytic motives

The aim of this section is to give a formalism for analytic motives à la Morel-Voevodsky
[MV99], that gives back usual motives in the strict case over a trivially valued integral ring,
and that also gives back good categories of rigid and complex analytic motives, similar to
those defined by Ayoub in [Ayo10] and [Ayo11]. We will define étale, Nisnevich and pro-
étale motives, with a preference to étale motives, since they seem to have better properties
than Nisnevich motives with respect to the integral Hodge and Tate conjectures (see
[RS14]), and they also allow a direct definition of the étale realization functor with finite
effects (remark however that over a characteristic $p$-basis, they are with coefficients
in $\mathbb{Z}[1/p]$, so that they don’t give a good information on the $p$-part of the cohomology).
We will also use the pro-étale topology that gives a better take at the completed étale
realization functor.

We refer to Ayoub’s thesis [Ayo07a] and [Ayo07b] for a systematic treatment of the
homotopy theory of schemes and to Cisinski and Deglise for a refined treatment of the
theory of motives with rational coefficients [CD09]. We will use the language of $\infty$-
categories (for which we refer to Lurie’s books [Lur09c] and [Lur09b]) to get a shorter
presentation, but the language of model categories and symmetric spectra in presheaves,
developed by Ayoub in [Ayo07b] has the advantage of allowing more explicit computations.
We will give a presentation of our theory that is a neat combination of the viewpoint used
by Roballo in his thesis [Rob12] and of Ayoub in his works on motives and analytic
motives.

6.1 Stable homotopy theory of sheaves

The analog in global analytic geometry of the affine line used in algebraic homotopy
theory (and of the unit interval used in classical homotopy theory) will be the unit disc.
It indeed gives back the algebraic affine line in the strict situation over a trivially valued
integral ring. The stable homotopy theory of analytic spaces will be constructed by using
$\infty$-sheaves (aka $\infty$-stacks) on the étale (resp. Nisnevich, resp. pro-étale) site of analytic
spaces with values in a stable presentable symmetric monoidal $\infty$-category $(\mathcal{M}, \otimes)$, that
will be the stable ∞-category (Sp, ∧) of spectra, the stable ∞-category (Sp(\text{Mod}_s(A)), ⊗) of simplicial module spectra on a sheaf \(A\) of rings (that will often be \(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}\) of \(\mathbb{Q}\)), or (in the characteristic zero situation) the stable ∞-category (\text{Mod}_{dg}(A), ⊗) of differential graded modules over \(A\). The main difference between the simplicial and the differential graded setting is that commutative differential graded algebras give correct strictifications of homotopy commutative algebras only over \(\mathbb{Q}\). If we work with modules, we will get categories of motives, and if we work with spectra, we will get stable homotopy categories.

We refer to Robalo [Rob12] (see also [Rob14]) for a short introduction to the ∞-categorical tools used in this subsection, and to Lurie’s book [Lur09b] for a complete reference on homotopical algebraic tools. We start by recalling from Robalo’s [Rob12] important facts about the stabilization of symmetric monoidal ∞-categories.

**Theorem 2.** Let \((\mathcal{C}, ⊗)\) be a presentable symmetric monoidal ∞-category and \(T \in (\mathcal{C}, ⊗)\) be an object. There exists a natural monoidal functor \((\mathcal{C}, ⊗) \to (\mathcal{C}[T^{⊗^{-1}}], ⊗)\) from \(\mathcal{C}\) to a presentable symmetric monoidal ∞-category such that for every symmetric monoidal category \(\mathcal{D}\), the natural morphism

\[
\text{Map}((\mathcal{C}[T^{⊗^{-1}}], ⊗), (\mathcal{D}, ⊗)) \longrightarrow \text{Map}_{T^{⊗^{-1}}}(\mathcal{C}, (\mathcal{D}, ⊗)),
\]

from symmetric monoidal functors to symmetric monoidal functors that make \(T\) invertible, is an equivalence. If the object \(T\) is further symmetric, meaning that there is a natural 2-equivalence in \(\mathcal{C}\) between the cyclic permutation \(σ_{(123)}\) on \(T ⊗ T ⊗ T\) and the identity map, given by a 2-morphism:

then the underlying ∞-category of \(\mathcal{C}[T^{⊗^{-1}}]\) is identified with the stabilization

\[
\text{Stab}_T(\mathcal{C}) := \text{colim}(... T^{⊗^{-1}} \to \mathcal{C} T^{⊗^{-1}} \to \mathcal{C} T^{⊗^{-1}} \to \cdots).
\]

We denote \((\mathcal{S}, ×)\) the symmetric monoidal ∞-category of spaces, obtained as the ∞-localization of the monoidal category \((\text{SSets}, ×)\) of simplicial sets by weak equivalences. The symmetric monoidal ∞-category of pointed spaces with the wedge product is denoted \((\mathcal{S}_*, ∧)\). The symmetric monoidal ∞-category of spectra is obtained by

\[
(\text{Sp}, ∧) := ((\mathcal{S}_*, ∧)[(S^1)^{⊗^{-1}}], ⊗).
\]

Since \(S^1\) is symmetric in \(\mathcal{S}_*\), the underlying ∞-category of \(\text{Sp}\) may be described as the stabilization of \(\mathcal{S}_*\) with respect to the wedge product by \(S^1\).

**Definition 36.** An object \(X\) of a stable ∞-category \(\mathcal{M}\) is called *homotopically compact* if for all \(n\), the functor \(\text{Hom}_h(\mathcal{M}, X, -[n])\) commutes to small filtered colimits.

We essentially give here an ∞-categorical analog of Ayoub’s notion of coefficient category from [Ayo07b], Definition 4.4.23.
Definition 37. Let \((\mathcal{M}, \otimes)\) be a symmetric monoidal \(\infty\)-category. We say that \((\mathcal{M}, \otimes)\) is a category of coefficients if

1. \(\mathcal{M}\) is stable and presentable,

2. there exists a set \(\mathcal{E}\) of homotopically compact objects of \(\mathcal{M}\) that generate the triangulated category with infinite sums \(h(\mathcal{M})\).

By definition, a symmetric monoidal model category \((\mathcal{M}, \otimes)\) that is a category of coefficients in the sense of Ayoub loc. cit. will give a symmetric monoidal \(\infty\)-category \((\mathcal{M}, \otimes)\) that is a coefficient category in the above sense. The model category setting gives a better take at explicit computations, but we chose the \(\infty\)-category setting because it sometimes allows easier universal constructions.

Proposition 14. Let \((X, \tau)\) be a small \(\infty\)-site and \((\mathcal{M}, \otimes)\) be an \(\infty\)-category of coefficients. Then the categories

\[ \text{PreShv}(X, \tau, \mathcal{M}) \quad \text{and} \quad \text{Shv}(X, \tau, \mathcal{M}) \]

of presheaves and sheaves on \((X, \tau)\) with values in \(\mathcal{M}\) are stable presentable symmetric monoidal \(\infty\)-categories.

Proof. See [Rob12] for a closely related result. This follows from the fact (explained to us by Brad Drew) that one may write

\[ \text{PreShv}(X, \tau, \mathcal{M}) = \text{PreShv}(X, \tau, \text{Sp}) \otimes_{\text{Sp}} \mathcal{M} \]

and similarly for sheaves. The fact that \(\text{PreShv}(X, \tau, \text{Sp})\) is stable presentable and symmetric monoidal is already known, because it may be obtained by stabilizing presheaves with values in \(\text{SSets}\), that are presentable. \(\square\)

We refer to Robalo [Rob12], Section 5 for the following.

Definition 38. Let \((X, \tau)\) be a small \(\infty\)-site, \((\mathcal{M}, \otimes)\) be a coefficient \(\infty\)-category, and \(I \in \text{Shv}(X, \tau, \mathcal{M})\) be an object.

1. The associated unstable homotopy category is the \(\infty\)-localization

\[ \text{H}(X, \tau, I, \mathcal{M}) = L_I(\text{Shv}(X, \tau, \mathcal{M})) \]

of the \(\infty\)-category of sheaves with respect to the class of morphisms \(X \times I \to X\).

2. The pointed unstable homotopy category is the associated pointed symmetric monoidal \(\infty\)-category \(\text{H}(X, \tau, I, \mathcal{M})_+\).
3. If $T$ is a symmetric object in $H(X, \tau, I, \mathcal{M})_*$, we define the associated stable homotopy category as the universal presentable symmetric monoidal $\infty$-category in which $T$ becomes $\otimes$-invertible:

$$\text{SH}(X, \tau, I, T, \mathcal{M}) := H(X, \tau, I, \mathcal{M})_*[T^\otimes -1].$$

The underlying $\infty$-category of $\text{SH}(X, \tau, I, \mathcal{M})$ is equivalent to the $T$-stabilization of $H(X, I, \mathcal{M})_*$, which is given by the $\infty$-categorical colimit of the sequence

$$\cdots \rightarrow H(X, I, \mathcal{M})_* \rightarrow H(X, I, \mathcal{M})_* \rightarrow \cdots$$

### 6.2 Analytic motives and spectra

Let $t \in \{an, \{an, s\}, \dag, \{\dag, s\}\}$ be a type of analytic spaces. We now apply the constructions of the previous section to the category of smooth $t$-analytic spaces with its étale, Nisnevich and pro-étale topologies. We follow quite closely the approach of Ayoub in the complex [Ayo10] and $p$-adic analytic [Ayo11] situations.

Let $R$ be an ind-Banach ring and $X$ be a $t$-analytic space over $R$. The category $\text{ANSM}_X^t$ of smooth $t$-spaces over $X$ is small. It will be equipped with a topology $\tau$ that is either the étale topology $\tau_{et}$, the Nisnevich topology $\tau_{Nis}$ or the pro-étale topology $\tau_{proet}$. We fix an $\infty$-category $(\mathcal{M}, \otimes)$ of coefficients. We will denote $T$ the object in $\text{SH}(\text{ANSM}_X^t, \tau, \mathcal{M})$ given by

$$T = \text{cof}(\mathbb{G}_m, X \otimes 1 \rightarrow \mathbb{A}_X^1 \otimes 1).$$

The proof of Ayoub that $T$ is symmetric in the algebraic setting in [Ayo07b], Lemme 4.5.65, being based on elementary integer valued matrix computations, extends directly to the strict and non-strict overconvergent setting.

**Definition 39.** The $\tau$-stable homotopy category $\text{SH}_t^\mathcal{M}(X, \tau)$ with coefficients in $\mathcal{M}$ is defined by

$$\text{SH}_t^\mathcal{M}(X, \tau) := \text{SH}(\text{ANSM}_X^t, \tau, D_X^t(0, 1) \otimes 1, T, \mathcal{M}).$$

If $(\mathcal{M}, \otimes) = (\text{Sp}, \wedge)$ is the symmetric monoidal $\infty$-category of spectra, we will denote $\text{SH}_t^\mathcal{M}(X, \tau)$ simply by $\text{SH}_t(X, \tau)$. If $\Lambda$ is a commutative ring and $(\mathcal{M}, \otimes) = (\text{Mod}_{dg}(\Lambda), \otimes)$, the $\infty$-category

$$\text{DA}_t^\Lambda(X, \Lambda) = \text{SH}_t^\mathcal{M}(X, \tau)$$

is called the category of $\tau$-motivic sheaves with coefficients in $\Lambda$. More generally, if $\Lambda$ is a sheaf of rings for the given topology $\tau$, we will still denote

$$\text{DA}_t^\Lambda(X, \Lambda) := \text{DA}_t^\Lambda(X, \mathbb{Z}) \otimes_{\text{SH}(X, \tau, \text{Mod}_{dg}(\mathbb{Z}_X))} \text{SH}(X, \tau, \text{Mod}_{dg}(\Lambda))$$

the associated category of $\tau$-motivic sheaves with coefficients in $\Lambda$. 
The notation of the above definition are consistent, because if \( \Lambda_X \) is a constant sheaf of rings with values \( \Lambda \), we will have a canonical equivalence

\[
\text{SH}_{\text{Mod}}^{t}(\Lambda)(X, \tau) \cong \text{SH}_{\text{Mod}}^{t}(\Lambda)(X, \tau) \otimes_{\text{SH}(X, \tau, \text{Mod}_{\text{dg}}(\Lambda_X))} \text{SH}(X, \tau, \text{Mod}_{\text{dg}}(\Lambda_X)).
\]

It is natural, following what we said in Remark 5, to ask if the natural functor

\[
\text{SH}_{3R}^{t, s}(X, \tau) \to \text{SH}_{3R}^{t}(X, \tau)
\]

is fully faithful. This question seems to have a positive answer over \( C \) (where it is even an equivalence), and may also have a positive answer on a non-archimedean field \( K \), if one can adapt the work of Temkin [Tem04]. If we work over \( (\mathbb{Z}, | \cdot |_\infty) \), this adaptation does not seem to be so easy, but the question remains interesting: this would relate usual algebraic motives to global analytic motives, which are still quite rigid objects.

**Remark 16.** It is quite natural to try to extend Ayoub’s formulation of the six operation formalism from [Ayo07b] (partially extended to the \( \infty \)-categorical setting by Robalo [Rob14]) to our more general setting. Ayoub’s papers [Ayo11] and [Ayo10] show us that there is no essential obstructions to this possibility. We will use this idea in some of our discussions.

**Example 8.** Let \( X \) be a scheme, seen as a strict analytic space over \( \mathbb{Z}_0 := (\mathbb{Z}, | \cdot |_0) \). Then the stable homotopy categories \( \text{SH}_{3R}(X, \tau) \) for \( \tau = \tau_{et} \) and \( \tau = \tau_{Nis} \) give back the usual étale and Nisnevich stable homotopy categories. This will be useful to get various strict non-archimedean analytifications over \( \mathbb{Q}_p \) and \( \mathbb{Z}_p \) for (say) projective schemes by a mere base change. One must not forget however, that the base extension of \( \mathbb{A}^1_{\mathbb{Z}} \), seen as the strict analytic space over \( \mathbb{Z}_0 \) given by the unit disc, only give the unit disc on \( \mathbb{Q}_p \) and \( \mathbb{Z}_p \), and not the affine line.

**Example 9.** If the base Banach ring \( \mathbb{Q}_p = (\mathbb{Q}_p, | \cdot |_p) \) is seen as a strict analytic algebra over itself, the category \( \text{SH}^{t, s}_{3R}(\mathbb{Q}_p, \tau_{Nis}) \) gives back Ayoub’s category \( \text{RigSH}_{3R}(\mathbb{Q}_p) \) of rigid analytic motives over \( \mathbb{Q}_p \) with coefficients in \( \mathfrak{M} \). We will use the overconvergent analog, because it carries a natural de Rham realization functor. We may also work with smooth perfectoid spaces over the completion of \( \mathbb{Q}_p[p^{1/p^n}] \), seen as analytic spaces over this field. Using Nisnevich coverings, we find a perfectoid version of Ayoub’s rigid analytic motives (see Vezzani’s article [Vez14] for a description of the tilt operation in the setting of rigid analytic motives). If we work over the Banach ring \( \mathbb{Z}_p = (\mathbb{Z}_p, | \cdot |_p) \) and \( X \) is a strict dagger analytic space over \( \mathbb{Z}_p \), we find a homotopy category \( \text{SH}_{3R}^{t, s}(X, \tau_{Nis}) \) that is an overconvergent (sometimes called “weakly convergent” in the litterature) analog of Ayoub’s homotopy category \( \text{RigSH}_{3R}(Y) \) over a \( \mathbb{Z}_p \)-rigid scheme \( Y \) (see [Ayo11]). It may (or may not) be possible to represent syntomic cohomology in this new category following closely the approach of Deglise and Mazzari [DM12].

**Example 10.** Let \( X \) be a complex analytic space. This is also a global analytic space over \( C \) in the sense of Berkovich (see [Poi13]), which has, by Remark 8, a naturally associated dagger analytic space \( X^\dagger \) over \( C \). There is a natural functor

\[
\text{SH}_{3R}^{Ayoub}(X, \tau_{usu}) \to \text{SH}_{3R}^{t}(X^\dagger, \tau_{et})
\]
from the complex analytic stable homotopy category over $X$ with the usual topology and with coefficients in $\mathcal{M}$ (in the sense of Ayoub [Ayo10]) to the stable homotopy category over $X^\dagger$ with coefficients in $\mathcal{M}$. It is likely a fully faithful functor. It may even be an equivalence. In any case, the same methods as those of Ayoub in loc. cit. should allow to prove that a convenient version of $\text{SH}^s_{\mathcal{M}}(X^\dagger, \tau_{et})$ is equivalent to the $\infty$-category $\text{SH}((X(C)/\sigma), \tau_{usu}, \mathcal{M})$ where $\sigma$ is complex conjugation. Indeed, the analytic etale $\infty$-topos of $X$ is identified with the quotient topos $[X(C)/\sigma]$. Example 11. Suppose that a given scheme over $\mathbb{Z}$ may be seen as the extension of a strict dagger analytic space over $\mathbb{Z}_c := (\mathbb{Z}, | \cdot |_\infty)$. The associated stable homotopy categories $\text{SH}^s_{\mathcal{M}}(X, \tau_{Nis})$ give a category of (strict) motives over $\mathbb{Z}$ that has a natural analytification by base change to $\mathbb{C} = (\mathbb{C}, | \cdot |_\infty)$ that is very close to usual homotopy theory of $\mathcal{M}$-valued sheaves (by Example 10), and also natural non-archimedean analytifications over $\mathbb{Q}_p$ and $\mathbb{Z}_p$ that are close to Ayoub’s rigid analytic motives. It is likely that the Artin-Verdier étale cohomology theory can be represented as the étale cohomology spectrum in the stable homotopy category $\text{SH}^s_{\mathcal{M}}(X^\dagger, \tau_{et})$ with the $\infty$-category $\text{SH}((X(C)/\sigma), \tau_{usu}, \mathcal{M})$ where $\sigma$ is complex conjugation. Indeed, the analytic etale $\infty$-topos of $X$ is identified with the quotient topos $[X(C)/\sigma]$. 

Remark 17. It is quite tempting to define (an étale Artin-Verdier version of) Arakelov motivic cohomology (defined by Holmstrom and Scholbach in [HS10]) with coefficients in $\mathbb{R}$ by using the various Grothendieck operations that may be available on the motivic categories. By definition, real Beilinson-Deligne cohomology is representable by a spectrum $H_{BD}$ in

$$
\text{SH}^s_{\mathcal{M}}(\mathbb{Z}_0, \mathbb{Z}) = \text{SH}^s_{\mathcal{M}}(\mathbb{Z}, \mathbb{Z}),
$$

where $\mathbb{Z}_0 = (\mathbb{Z}, | \cdot |_0)$. The same is true for the étale motivic cohomology spectrum $H_{mot,et,\mathbb{Z}}$. Remark that one can’t hope to represent Beilinson-Deligne cohomology in the
non-strict analytic category because its construction is based on smooth compactifications with boundary given by a normal crossing divisor, which are not available in general in the analytic setting. The Beilinson-Deligne component $H_{BD}$ should however be more naturally explained by a nice spectrum in the category of strict global analytic motives $\mathrm{SH}^*_{\text{et}}(\mathbb{Z}, \mathbb{Z})$ with $\mathbb{Z} = (\mathbb{Z}, | \cdot |_\infty)$. Indeed, it is related to the Betti realization which is naturally available only over the global base $\mathbb{Z}$. We may work with the global Artin-Verdier analog $H^\text{av}_Z$ over $\mathbb{Z}$ of $H_{\text{mot,et},Z}$. There is a natural morphism

$$H_{\text{mot},\mathbb{R}} \xrightarrow{\text{id} \wedge H_{BD}} H_{\text{mot},\mathbb{R}} \wedge H_{BD}$$

and Holmstrom and Scholbach define the Arakelov-motivic cohomology spectrum $\hat{H}$ as the homotopy fiber of this morphism (recall the motivic étale cohomology with coefficients in $\mathbb{Q}$ identifies with usual motivic cohomology). One may try to extend, the interpretation of special values of $L$-functions as proposed by Scholbach in his thesis [Sch10] (with probably some additional truncational cares) to the study of special values up to a factor in $\mathbb{Z}^\times$: the determinant of the pairing

$$H^*_{\text{mot,et}}(X, \mathbb{Z}) \times \hat{H}^*(X, \mathbb{R}) \to \mathbb{R}$$

between real motivic Arakelov cohomology and integral motivic homology with values in $\mathbb{R}$ (in the case $X/\mathbb{Z}$ smooth projective) may give the special value up to a factor in $\{\pm 1\}$ (an argument against this idea, explained to the author by Baptiste Morin, is that the use of étale motives destroys the $p$-torsion information in characteristic $p$). This comes from the fact, explained to the author by Jakob Scholbach, that there are natural isomorphisms:

$$\det(\hat{H}^*(X, \mathbb{R})) = \det(H^*(X, \mathbb{R})) \otimes_{\mathbb{R}} \det(H^*_B(X, \mathbb{R}))$$

$$= \det(H^*_{\text{mot,et}}(X, \mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{R}$$

$$\otimes_{\mathbb{R}} \det(H^*_B(X, \mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{R}$$

$$\otimes_{\mathbb{R}} \det^{-1}(H^*_dR,\text{fil}(X/\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{R}$$

where $H^*_B(X, \mathbb{Z})$ is the usual Betti cohomology. Working with a smooth log-scheme over $\mathbb{Z}$ would treat the semistable case. Remark that the integral structure on the Deligne cohomology determinant is —not— given by integral Deligne cohomology (that is a locally compact group; this may have relations to Morin’s global arithmetic cohomology, however), but by a combination of Betti cohomology with filtered absolute de Rham cohomology. To give a global analytic interpretation of Scholbach’s constructions, here is how we proceed: We interpret the factor

$$\det(H^*_{\text{mot,et}}(X, \mathbb{Z})) \otimes_{\mathbb{Z}} \det(H^*_B(X, \mathbb{Z}))$$

in his determinant as the determinant of the Artin-Verdier étale motivic cohomology $H^*_{\text{et,av}}(X, \mathbb{Z})$ (which contains both integral étale motivic information and Betti information; maybe a $p$-part information should be added following Milne and Ramachandran
[MR13]; the Weil-étale motivic cohomology would even be better) and the factor
\[
\det^{-1}(H^*_{dR,fil}(X/\mathbb{Z}))
\]
as the determinant of filtered de Rham cohomology over \((\mathbb{Z}, | \cdot |_0)\). We conjecture that the base extension
\[
(\mathbb{Z}, | \cdot |_\infty) \to (\mathbb{Z}, | \cdot |_0)
\]
of an algebraic motive does not change its filtered de Rham cohomology, so that we may interpret \(H^*_{dR,fil}(X/\mathbb{Z})\) as the de Rham cohomology of \(X\) over \((\mathbb{Z}, | \cdot |_\infty)\). We refer to subsection 5.2 for a discussion of the problem of finding a model over \((\mathbb{Z}, | \cdot |_\infty)\) of a scheme over \(\mathbb{Z}\), seen as a strict dagger space over \((\mathbb{Z}, | \cdot |_0)\). This allows us to seek for the definition of a regulator from Artin-Verdier motivic cohomology to filtered de Rham cohomology over \((\mathbb{Z}, | \cdot |_\infty)\), given by a filtered de Rham realization functor over this global analytic base. The fiber of this map of spectra gives back the integral structure on Arakelov motivic cohomology, and the pairing between motivic homology over \(U = \{|2| \leq |1|\} \subset X = \mathcal{M}(\mathbb{Z}, | \cdot |_\infty)\) and Arakelov motivic cohomology may be defined in a natural way.

### 6.3 Realizations

Let \(t \in \{an, \{an, s\}, \dagger, \{\dagger, s\}\}\) be a type of analytic spaces. Let \(\Lambda\) be a sheaf of rings for the pro-étale topology on \(\text{AnSm}^t_X\). As before, we denote
\[
\eta : \text{AnSm}^\dagger_{X,\text{proet}} \to \text{AnSm}^\dagger_{X,\text{et}}.
\]

**Definition 40.** The étale realization of integral motives with coefficients in \(\Lambda\) is given by the composition
\[
DA^\dagger_{et}(X, \mathbb{Z}) \xrightarrow{\eta^*} DA^\dagger_{\text{proet}}(X, \mathbb{Z}) \to DA^\dagger_{\text{proet}}(X, \Lambda).
\]

In some particular torsion cases like for example \(\Lambda = \mathbb{Z}/n\mathbb{Z}\), it is possible to show that \(DA_{et}(X, \Lambda)\) is equivalent to the \(\infty\)-category of sheaves of \(\Lambda\)-modules \(\text{Shv}(X, \tau_{et}, \text{Mod}_{dg}(\Lambda))\). This is the approach used by Ayoub in [Ayo14] to define the étale realization. This may extend nicely to the pro-étale situation with coefficients in pro-étale sheaves like \(\mathbb{Z}_\ell\) or \(\hat{\mathbb{Z}}\).

Recall that any scheme over \(\mathbb{Z}\) may be seen as a non-strict dagger space over \((\mathbb{Z}, | \cdot |_\infty)\). We will now define a Betti realization for these objects.

**Definition 41.** Let \(X\) be a non-strict dagger analytic space over \(\mathbb{Z} = (\mathbb{Z}, | \cdot |_\infty)\). The Betti realization with coefficients in a coefficient category \(\mathcal{M}\) is given by the base extension
\[
\text{SH}^\dagger_{\mathcal{M}}(X, \tau) \to \text{SH}^\dagger_{\mathcal{M}}(X_{\mathbb{R}}, \tau).
\]
The fact that the above definition is reasonable follows from what we said in Remark 10: Ayoub’s methods allow us to show that there is a natural equivalence
\[ \text{Shv}(X^{\text{ber}}_R, \tau_{\text{usu}}, \mathcal{M}) \xrightarrow{\sim} \text{SH}^\dagger_{2\mathfrak{m}}(X_R, \tau), \]
where \( X^{\text{ber}}_R \) is the Berkovich space associated to \( X_R \).

Remark 18. If \( X \) is a strict dagger analytic space over \( \mathbb{Z} \), then the diagram
\[
\begin{array}{ccc}
\text{SH}^\dagger_{2\mathfrak{m}}(X, \tau) & \rightarrow & \text{SH}^\dagger_{2\mathfrak{m}}(X, \tau) \\
\downarrow & & \downarrow \\
\text{SH}^\dagger_{2\mathfrak{m}}(X_R, \tau) & \rightarrow & \text{SH}^\dagger_{2\mathfrak{m}}(X_R, \tau)
\end{array}
\]
is commutative and the down horizontal arrow is an equivalence (this last fact follows from Proposition 10). This means that we may see the Betti realization of a strict dagger motive over \( X \) as a strict dagger motive over \( X_R \).

The main interest of the theory of overconvergent analytic spaces is that they have a nice de Rham cohomology theory, as was already showed by Große-Klönne in [GK02].

We will now define the de Rham realization of dagger motives over a given base by using the associated sheaves on the de Rham space.

Let \( R \) be a base ind-Banach ring and \( X \) be a dagger analytic space over \( R \).

**Definition 42.** The de Rham space of \( X \) is the presheaf on \( \text{Alg}^\dagger_R \) given by
\[ X(A) := X(A/I) \]
where \( I \subset \text{Alg}(A) \) is the nilradical. A sheaf of quasi-coherent modules on \( X_{dR} \) is called a cristal on \( X \).

The de Rham space is functorial in presheaves, so that it is in particular functorial in morphisms \( f : Y \rightarrow X \) of analytic spaces.

Let \( X \) be a dagger analytic space that is flat over \( \mathcal{M}(\mathbb{Z}, | \cdot |_{\infty}) \). The category of de Rham coefficients on \( X \) is the category \( \text{Shv}(X_{dR}, \text{Mod}_{dR}(\mathcal{O}_{X_{dR}} \otimes \mathbb{Z} \mathbb{Q})) \).

**Theorem 3.** There are natural realization functors
\[ \text{SH}^\dagger(X, \tau) \rightarrow \text{Shv}(X_{dR}, \text{Mod}_{dR}(\mathcal{O}_{X_{dR}} \otimes \mathbb{Z} \mathbb{Q})) \]
and for \( \Lambda \subset \mathbb{Q} \),
\[ \text{DA}^\dagger(X, \Lambda) \rightarrow \text{Shv}(X_{dR}, \text{Mod}_{dR}(\mathcal{O}_{X_{dR}} \otimes \mathbb{Z} \mathbb{Q})). \]

**Proof.** The realization functor will extend the natural relative de Rham cohomology functor
\[ \text{ANSm}^\dagger_{X, \tau} \rightarrow \text{Shv}(X_{dR}, \text{Mod}_{dR}(\mathcal{O}_{X_{dR}} \otimes \mathbb{Z} \mathbb{Q})) \]
\[ [Y \rightarrow X] \mapsto (f_*\mathcal{O}_{Y_{dR}}) \otimes \mathbb{Q} \]
We have to show that if \( Y \to X \) is a smooth morphism, then there is a natural homotopy equivalence
\[
(f_* \mathcal{O}_{Y \times D^1(0,1)_X}^{dR}) \otimes \mathbb{Q} \cong (f_* \mathcal{O}_Y^{dR}) \otimes \mathbb{Q}.
\]
Using the compatibility of the Kunneth formula with direct image, this will follow from the computation of the de Rham cohomology of the disc (which works only in the overconvergent setting), that may be done over the initial base \((\mathbb{Z}, | \cdot |_\infty)\). Using the fact that we work with \(\mathbb{Q}\)-coefficients, we get the Poincaré Lemma for overconvergent power series on the disc. We also need to show that \((f_* \mathcal{O}_Y^{dR}) \otimes \mathbb{Q}\) is \(\otimes\)-invertible in \(\text{Shv}(X_{dR}, \text{MOD}_{dg}(\mathcal{O}_{X_{dR}} \otimes_{\mathbb{Z}} \mathbb{Q}))\). This follows from the formula
\[
f_* \mathcal{O}_\mathbb{T}^{dR} \cong \text{cof}(f_* \mathcal{O}_{\text{gm}, X}^{dR} \to f_* \mathcal{O}_{\mathbb{A}_X}^{dR}),
\]
that gives that \(f_* \mathcal{O}_\mathbb{T}^{dR}\) is locally free on \(X\) of rank 1, and thus dualizable over \(\mathcal{O}_X\). The dual will give the tensor inverse. \(\square\)

7 Derived dagger analytic geometry

We have defined dagger analytic spaces, by following the usual method of synthetic geometry, explained in the introduction of the book [Pau14b]: we started by improving the category of rational domain dagger algebras by adding natural solution spaces for ideals. This was done using a “functor of function” viewpoint. We then used “functors of points” to define spaces. These ideas are close to the ones used by Lawvere [Law79] in synthetic differential geometry and Dubuc and Taubin [DT83] in synthetic analytic geometry. Our main motivation for using this “synthetic” approach, as opposed to the usual approach to classical analytic geometry using locally ringed spaces, is that it generalizes directly to the derived setting. We are very much inspired by Lurie’s approach to derived analytic geometry from [Lur09a] and [Lur11] and by the related work in progress of Mauro Porta on complex analytic derived geometry [Por14]. We refer to this last work for a complete and neat description of the complex analytic derived theory, including a good theory of modules. We will now extend the above categories of analytic spaces to \(\infty\)-categories of derived analytic spaces. This can be done by using homotopical functors of functions on categories \(\text{RAT}_R\) of rational \(t\)-algebras, that will give derived analytic algebras, and homotopical functors of points on them.

Before diving into the abstract theory, we will give some motivations for its development.

7.1 Motivation: global periods

One of our main motivations for developing overconvergent global derived analytic geometry comes from the work of Beilinson and Bhatt (see [Bei11] and [Bha12b]) on \(p\)-adic Hodge theory: they define a ring of \(p\)-adic periods by the formula
\[
A_{cris} := \text{DR}(\mathbb{Z}_p/\mathbb{Z}_p) \hat{\otimes} \mathbb{Z}_p.
\]
where DR denotes algebraic de Rham cohomology and the completed tensor product means the homotopy colimit
\[
A_{\text{cris}} := \underset{n}{\text{hocolim}} \text{DR}(\bar{\mathbb{Z}}_p/\mathbb{Z}_p) \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}.
\]

Using Hodge-completed derived de Rham cohomology instead of derived de Rham cohomology, one gets
\[
A_{dR} := \hat{\text{DR}}(\bar{\mathbb{Z}}_p/\mathbb{Z}_p) \hat{\otimes} \mathbb{Z}_p
\]
and also
\[
B_{dR}^+ := \hat{\text{DR}}(\bar{\mathbb{Z}}_p/\mathbb{Z}_p) \hat{\otimes} \mathbb{Q}_p.
\]

Seeking for a geometric interpretation of these derived completed tensor product, we may interpret
\[
A_{\text{cris}} \simeq \text{DR}^\text{an}(\bar{\mathbb{Z}}_p \hat{\otimes}_{\mathbb{Z}} \mathbb{Z}_p/\mathbb{Z}_p),
\]
where \(\mathbb{Z}_p\) denotes here the strict derived analytic ring over \(\mathbb{Z}_0 := (\mathbb{Z}, | \cdot |_0)\), given by
\[
A_{\text{cris}} \simeq \text{DR}^\text{an}(\bar{\mathbb{Z}}_p \hat{\otimes}_{\mathbb{Z}_0} \mathbb{Z}_p/\mathbb{Z}_p),
\]
where \(\mathbb{Z}_p\) denotes the Banach ring \((\mathbb{Z}_p, | \cdot |_p)\) and \(\bar{\mathbb{Z}}_p \hat{\otimes}_{\mathbb{Z}_0} \mathbb{Z}_p\) denotes the derived analytic ring over \(\mathbb{Z}_p\) obtained by extension of scalars of the non-strict analytic algebra \(\bar{\mathbb{Z}}_p\) over \((\mathbb{Z}, | \cdot |_0)\) along the bounded morphism \(\mathbb{Z}_0 = (\mathbb{Z}, | \cdot |_0) \to (\mathbb{Z}_p, | \cdot |_p) = \mathbb{Z}_p\) of Banach rings. Similarly, one would have
\[
B_{dR}^+ \simeq \hat{\text{DR}}^\text{an}(\bar{\mathbb{Z}}_p \hat{\otimes}_{\mathbb{Z}_0} \mathbb{Q}_p/\mathbb{Q}_p).
\]

This interpretation may help to sheafify the construction in the spirit of Scholze’s work [Sch13] and to globalize it in the spirit of Bhatt’s paper loc. cit., Remark 11.10: one gets global analogs
\[
A_{\text{ccris}} \simeq \text{DR}^\text{an}(\hat{\mathbb{Z}} \hat{\otimes}_{\mathbb{Z}} \hat{\mathbb{Z}}/\hat{\mathbb{Z}}),
\]
and
\[
B_{dR}^+ \simeq \hat{\text{DR}}^\text{an}(\hat{\mathbb{Z}} \hat{\otimes}_{\mathbb{A}_f} \hat{\mathbb{A}}_f/\hat{\mathbb{A}}_f),
\]
given by the extension of scalar of derived de Rham cohomology of \(\hat{\mathbb{Z}}/\mathbb{Z}\) to the ring \(\mathbb{A}_f\) of finite adeles, seen as an analytic ring over \((\mathbb{Z}, | \cdot |_0)\). In our global analytic viewpoint, one may even define naturally, using the base \(\mathbb{Z} = (\mathbb{Z}, | \cdot |\infty)\), a new period ring
\[
B_{g, dR}^+ := \hat{\text{DR}}^\text{an}(\hat{\mathbb{Z}} \hat{\otimes}_{\mathbb{Z}} \hat{\mathbb{A}}/\hat{\mathbb{A}}).
\]
that also takes care of the archimedean component

\[ B_{\infty, dR}^+ := \mathcal{D}RK^\infty (\hat{\mathbb{Z}} \otimes \mathbb{R}/\mathbb{R}). \]

In the archimedean situation of Hodge theory, one usually only uses the Galois group of \( \mathbb{C} \) over \( \mathbb{R} \), but knowing that a variety is defined over \( \mathbb{Z} \) may be an important information to be used in archimedean Hodge theory. One may even study the groupoid derived stack

\[ \mathbb{R} \text{Hom}_{\mathcal{A}_1}(D_* \mathbb{A}_\mathbb{Z}^1, \mathbb{A}_\mathbb{Z}^2) \Rightarrow \mathbb{A}_\mathbb{Z}^1 \]

(where \( D_* \) is the cosimplicial scheme that is given in degree \( n \) by the union of the \( n+1 \) coordinate axis in \( \mathbb{A}^{n+1} \) and the face and degeneracies on \( [n \mapsto \mathbb{A}^{n+1}] \) are given by addition of coordinates and insertion of zeroes) that encodes (when derived pullbacked to \( \mathbb{A} \) and completed along the unit section) the derived Hodge filtration on \( B_{g, dR}^+ \). Its pullback at the archimedean place gives back the archimedean Hodge filtration of \( \hat{\mathbb{Z}} \), and its pullback on \( \hat{\mathbb{Z}} \) gives back the Hodge filtration on the global period ring.

Now, if we want to adapt Beilinson’s strategy from [Bei11] in this global case, we can proceed in the following way: define the sheaf \( B_{dR}^+ \) of filtered dg-algebras on the \( h \)-topology on the category \( \text{VAR}_\mathbb{Q} \) of quasi-projective varieties by sheafifying for the \( h \)-topology the presheaf that sends a semistable pair \((U, \bar{U})\) over \( \bar{\mathbb{Z}} \) (with \( \bar{U} \) projective over \( \bar{\mathbb{Z}} \)) to the Hodge completed analytic derived de Rham cohomology

\[ B_{dR}^+(U, \bar{U}) := \mathcal{D}RK^\infty ((\bar{U}, \mathcal{M}_U) \otimes_{\mathbb{Z}} \mathbb{A}/\mathbb{A}) \]

of the corresponding strict dagger logarithmic space over \( \mathbb{Z} \) (defined in Subsection 5.2), extended to \( \mathbb{A} \). One then defines the global Arithmetic de Rham complex of \( X \) as

\[ \mathbb{R} \Gamma^+_{dR}(X) := \mathbb{R} \Gamma(X_h, B_{dR}^+). \]

There is a natural diagram

\[ H^*_{dR}(X) \xrightarrow{\alpha} H^*(\mathbb{R} \Gamma^+_{dR}(X)) \xleftarrow{\beta} H^*_{\text{proet}}(X, \mathbb{A}) \otimes_{\mathbb{A}} B_{g, dR}^+ \]

Now the global analog of the Poincaré Lemma would be that the natural morphism of sheaves on the \( h \)-topology

\[ B_{g, dR}^+ \xrightarrow{\sim} B_{dR}^+ \]

is a filtered quasi-isomorphism. This would imply that \( \beta \) is an isomorphism, and the base extension to \( B_{g, dR}^+ \) of the corresponding morphism

\[ H^*_{dR}(X) \rightarrow H^*_{\text{proet}}(X, \mathbb{A}) \otimes_{\mathbb{A}} B_{g, dR}^+ \]

would then be the global period isomorphism.

We may also try to follow Scholze’s approach from [Sch13] to propose a strategy to prove a global version of his \( p \)-adic comparison theorem. Let \( X/\mathbb{Z} \) be a flat scheme over
an open subset of $\text{Spec}(\mathbb{Z})$, and whose generic fiber $X_{\mathbb{Q}}/\mathbb{Q}$ is proper and smooth. One may define a period sheaf $\mathbb{B}^+_{dR}$ on the pro-étale site of $X_{\mathbb{Q}}$ by

$$\mathbb{B}^+_{dR}(U) := \widehat{\text{DR}}^\text{an}(U_{\mathbb{A}_f}/\mathbb{A}_f).$$

This induces a period sheaf $\mathbb{B}^+_{dR}$ on the pro-étale site of $X_{\mathbb{Q}}$ by

$$\mathbb{B}^+_{dR}(U) := \widehat{\text{DR}}^\text{an}(U_{\mathbb{A}_f}/\mathbb{A}_f).$$

One should have an isomorphism

$$H^*_\text{proet}(X_{\mathbb{Q}}, \mathbb{A}_f) \otimes_{\mathbb{A}_f} B^+_{dR} \sim H^*_\text{proet}(X_{\mathbb{Q}}, \mathbb{B}^+_dR),$$

and an isomorphism

$$H^*_dR(X_{\mathbb{Q}}/\mathbb{Q}) \otimes_{\mathbb{Q}} B_{dR} \sim H^*_\text{proet}(X_{\mathbb{Q}}, \mathbb{B}^+_dR) \otimes_{B^+_{dR}} B_{dR}$$

given by a Poincaré Lemma similar to the one proved by Scholze in the local setting, that identifies the global sheaf of constants $\mathbb{B}^+_{dR}$ to the horizontal sections of the natural connection on $\mathcal{O}\mathbb{B}^+_{dR}$. All this would imply that there exists a natural isomorphism

$$H^*_\text{proet}(X_{\mathbb{Q}}, \mathbb{A}_f) \otimes_{\mathbb{A}_f} B_{dR} \sim H^*_dR(X_{\mathbb{Q}}, \mathbb{Q}) \otimes_{\mathbb{Q}} B_{dR}$$

compatible with the filtration and the Galois action on both sides. Adding the archimedean information to the above reasoning is a quite tempting task, if one uses the global period sheaf

$$\mathbb{B}^+_{g,ddR}(U) := \widehat{\text{DR}}^\text{an}(U_{\mathbb{A}}/\mathbb{A})$$

and the global period ring

$$B^+_{g,ddR} := \widehat{\text{DR}}^\text{an}(\mathbb{Z} \otimes_{\mathbb{Q}} \mathbb{A}/\mathbb{A}).$$

For this naive idea to work, one needs to think of $\mathbb{Q}$ not as a Banach ring but as a non-strict analytic ring $\mathbb{Q}$ over $\mathbb{Z} := (\mathbb{Z}, |\cdot|_\infty)$ given by germs of functions around the trivial norm $|\cdot|_0 \in \mathcal{M}(\mathbb{Z})$. One then defines $\mathbb{Q} := \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}$. This way, it is meaningful to study the pro-étale cohomology of $X_{\mathbb{Q}}$ with coefficients in the full ring of adèles, and one may still have a comparison isomorphism

$$H^*_\text{proet}(X_{\mathbb{Q}}, \mathbb{A}) \otimes_{\mathbb{A}} B_{g,ddR} \sim H^*_dR(X_{\mathbb{Q}}/\mathbb{Q}) \otimes_{B_{g,ddR}}.$$

**Remark 19.** Since Betti cohomology may be nicely computed as analytic motivic cohomology, and the classical comparison isomorphism is between Betti cohomology and de Rham cohomology, one may try to generalize this isomorphism in the motivic direction, by trying to relate (étale) motivic cohomology to a kind of global derived analog of Deligne cohomology (i.e., a motivically graded version of global multiplicative $K$-theory, that combines global analytic $K$-theory with the Hodge filtration). This question will be studied later.
7.2 Derived dagger algebras

Let $R$ be a uniform ind-Banach ring. We denote $S$ the $\infty$-category of spaces, obtained by the $\infty$-localization of the category $SSets$ of simplicial sets by weak equivalences, and as before, $Sp = S_r(S^1)_{\otimes}^{-1}$ the monoidal $\infty$-category of spectra, given by its universal tensor stabilisation with respect to the smash $S^1$ action. Let $t \in \{an, \{an, s\}, \dagger, \{\dagger, s\}\}$ be a type of analytic space.

**Definition 43.** A bounded (resp. unbounded) derived $t$-algebra over $R$ is a functor

$$A : (\text{RatAlg}_R^t)^{\text{op}} \to S$$

(resp. $A : (\text{RatAlg}_R^t)^{\text{op}} \to Sp$)

that commutes to finite products and sends pullbacks along rational domain immersions to pullbacks. A bounded (resp. unbounded) derived $t$-algebra over $R$ is called an affinoid $t$-algebra if it is finitely presented (i.e., the finite colimit of a diagram of rational domain $t$-algebras). We will denote $\text{DALG}_R^t$ (resp. $\text{DALG}_{u,R}^t$, resp. $\text{DAff}_R^t$, resp. $\text{DAff}_{u,R}^t$) the $\infty$-category of bounded derived $t$-algebras (resp. derived $t$-algebras, resp. bounded derived affinoid $t$-algebras, resp. derived affinoid $t$-algebras).

**Proposition 15.** The $\infty$-category opposite to $\text{DAff}_R^t$, equipped with the Grothendieck topology generated by standard rational domain coverings is a geometry in the sense of Lurie [Lur09a], Definition 1.2.5. One has

$$\text{DALG}_R^t = \text{ind-DAff}_{u,R}^t$$

and $\text{DALG}_{u,R}^t = \text{ind-DAff}_{u,R}^t$.

The geometry given by $\text{DAff}_R^t$ is the geometric envelope of the pre-geometry given by $\text{RatAlg}_R^t$.

**Proof.** The construction of $\text{DAff}_R^t$ from $\text{RatAlg}_R^t$ shows that it is its geometric envelope, as described by Lurie in [Lur09a], Lemma 3.4.3 (see also [Lur09c], 5.3.6.2). The statement about ind-objects follows from the fact that the category of small derived $t$-algebras is generated under the combination of finite colimits and filtered colimits by rational domain algebras.

**Definition 44.** A derived $t$-analytic scheme is an $\infty$-stack $X \in \text{SHV}(\text{DALG}_R^t, \tau_{\text{Rat}}, S)$ that is locally isomorphic to a representable stack $M(A) := \text{Map}_{\text{DALG}_R^t}(A, \_)$ for the usual topology. An étale (resp. pro-étale) Artin derived $t$-analytic stack is an $\infty$-stack $X \in \text{SHV}(\text{DALG}_R^t, \tau_{\text{et}}, S)$ (resp $X \in \text{SHV}(\text{DALG}_R^t, \tau_{\text{pro-et}}, S)$) that can be locally written for the étale (resp. pro-étale) topology as a smooth quotient of a representable $\infty$-groupoid $M(A_\_)$ associated to a cosimplicial derived analytic algebra.

By Lurie [Lur09a], Theorem 2.4.1, one may also interpret derived $t$-analytic schemes as some particular kinds of $\text{RatAlg}_R^t$-structured $\infty$-topoi.
7.3 The dagger cotangent complex and derived de Rham cohomology

Let \( t \in \{ \dagger, \{ \dagger, s \} \} \) be an overconvergent type of analytic spaces.

One uses the tangent \( \infty \)-category approach (stabilization of the overcategory) of [Lur07] and [Lur09b], 7.3, to define quasi-coherent modules on derived analytic algebras and derived analytic spaces. This also gives a definition of the cotangent complex and of the de Rham space \( X_{dR} \) associated to a derived dagger analytic space.

**Definition 45.** If \( A \in DAlg^t_R \), we denote \( DAlg^t_A \) the pointed \( \infty \)-category with finite colimits whose objects are morphisms \( A \to B \). The \( \infty \)-category \( \text{Mod}(A) \) of modules over \( A \) is defined as the tangent \( \infty \)-category of \( DAlg^t_R \) at \( A \), given by the stabilization

\[
\text{Mod}(A) = T_A DAlg^t := \text{Stab}(DAlg^t_A).
\]

**Definition 46.** The right adjoint \( L \) to the natural forgetful functor \( \text{Mod}(A) \to DAlg^t_A \) is called the dagger cotangent complex, and we denote \( L(B) \) by \( L_{B/A} \).

**Proposition 16.** The \( \infty \)-category \( \text{Mod}(A) \) is equipped with a natural symmetric monoidal structure \( \otimes \) with unit object \( \underline{1}_A \) that makes it a symmetric monoidal \( \infty \)-category \( (\text{Mod}(A), \otimes, \underline{1}_A) \).

**Proof.** The tensor product of two modules \( M \) and \( N \) with corresponding spectral \( A \)-algebras \( B_M \) and \( B_N \) is defined as the cofiber

\[
B_M \otimes N := \text{cof}(B_M \oplus B_N \to B_M \otimes_A B_N).
\]

If \( A \) is a \( t \)-rational \( R \)-algebra, we will denote \( \underline{1}_A \) the module over \( A \) given by the \( t \)-affinoid \( R \)-algebra

\[
\underline{1}_A := A\{\text{ALG}(A)\}^t/(\{(a, a \in \text{ALG}(A))\}^2, (a([b]+[c])-[a(b+c)], a \in \text{ALG}(A), b \in \text{ALG}(A))),
\]

where the set \( \text{ALG}(A) \) is equipped with the grading given by the uniform norm on \( \mathcal{M}(A) \), which is well defined since \( \mathcal{M}(A) \) is compact. If \( A \) is a derived \( t \)-analytic \( R \)-algebra, we may write it as a colimit

\[
A = \colim_i A_i
\]

of \( t \)-rational \( R \)-algebras \( A_i \), and we define \( \underline{1}_A \) as the colimit of the corresponding modules \( \underline{1}_{A_i} \). The above binary tensor product operation extends naturally to a symmetric monoidal \( \infty \)-category structure with unit object \( \underline{1}_A \) (model of the Lawvere theory of commutative monoids in the \( \infty \)-category \( \infty \text{-CAT}^{pr} \) of \( \infty \)-categories).

**Definition 47.** Let \( A \) be a derived \( t \)-analytic algebra. The symmetric monoidal \( \infty \)-category \( (\text{PERF}(A), \otimes, \underline{1}_A) \) of perfect complexes over \( A \) is the symmetric monoidal stable sub-\( \infty \)-category of \( \text{Mod}(A) \) generated by \( \underline{1}_A \).
Remark 20. There is a nice theory that gives a notion of "prime spectrum" for a stable symmetric monoidal ∞-category, due to Hopkins, Neeman and Thomason. In the strict trivially normed case, i.e., with usual rings, this gives back the usual Zariski spectrum. One may actually see a stable symmetric monoidal ∞-category as a symmetric rig-∞-category, i.e., an ∞-category with two monoidal structures, together with a distributivity relation between them, by using the fact that it has direct sums. It is of course possible to define the notion of prime ideal in a rig-∞-category. To be more precise, consider the category with finite product (Lawvere algebraic theory) given by the category \((\text{Sets}^\text{op}, \times)\) of affine spaces, opposite the category of finitely generated polynomial rings over \(\mathbb{Z}\). A model of this theory with values in the category \(\text{Sets}\) of (small) sets is nothing but a general ring \(A\), with underlying ring \(A(\mathbb{A}^1)\). We will define a rig-∞-category as a model of this theory with values in the ∞-category with finite products \((\infty\text{-Cat}^\text{op}, \times)\).

Remark 21. The problem with the above "analytic ring" approach to the definition of an analytic chromatic spectrum is that it is not easy to define a convenient analog of...
the stable symmetric monoidal ∞-category of spectra in this context. One may try to replace the use of analytical rig categories over a given Banach ring \((R, |\cdot|)\) by the use of the rig-category \((\mathbb{R}_{+}^{\text{Sets}}, \otimes_{m}, \otimes_{1})\) of \(\mathbb{R}_{+}\)-graded sets with contracting maps, and by using simplicial \(\mathbb{R}_{+}\)-graded sets

\[
X : \Delta^{\text{op}} \to \mathbb{R}_{+}^{\text{Sets}}
\]

together with their monoidal structures \(\otimes_{m}\) and \(\otimes_{1}\). One may put on the standard simplicial sets \(\Delta^{n} = \text{Hom}(\cdot, [n])\) the grading given by \(1\). This induces a natural grading on \(X_{0} = \text{Hom}(\Delta^{0}, X)\) for every \(\mathbb{R}_{+}\)-graded simplicial set \(X\), and thus a quotient grading on \(\pi_{0}(X)\). Since \(\pi_{0}\) commutes with products in the ungraded situation, it should also commute with the two given monoidal structures \(\otimes_{m}\) and \(\otimes_{1}\) on \(\mathbb{R}_{+}\)-graded sets, if we use the quotient grading on \(\pi_{0}\). One defines fibrant objects by their lifting properties, and then the associated weak equivalences, as in the non-graded situation. One may define the graded analog of \(S^{1}\) by using the grading \(1\) on all standard simplices. Then there is certainly a way to invert the additive tensor product \(\otimes_{1}\) by \(S^{1}\) in the rig-∞-category \((\mathbb{R}_{+}^{\text{Sets}}^{\leq 1}, \otimes_{m}, \otimes_{1})\) to get a rig-∞-category \((\mathbb{R}_{+}^{\text{Sp}}^{\leq 1}, \wedge_{m}, \wedge_{1})\), following a method similar to the one used by Robalo in [Rob14], that may give a nice \(\mathbb{R}_{+}\)-graded analog of the stable symmetric monoidal ∞-category \((\text{Sp}, \otimes)\) of spectra. One may then define the \(\mathbb{R}_{+}\)-graded sphere spectrum \(S^{+}\) to be given by the (symmetric) suspension rig-spectrum of the point \(\Delta^{0}\) graded by \(1\), for the monoidal structure \(\otimes_{1}\). This should give a rig-object for the bimonoidal structure on \(\mathbb{R}_{+}\)-graded spectra such that \(\pi_{0}(S^{+}) = (\mathbb{Z}, |\cdot|_{\infty})\).

Remark 22. Combining the above \(\mathbb{R}_{+}\)-graded line of ideas with the notion of analytic spectrum of rig categories developed by the author in [Pau14a], one may ask if it is possible to define an analytic spectrum for a general rig-∞-category. So one needs to define a reasonable homotopical enhancement of the notion of seminorm on a rig category. Recall that a seminorm on a rig category \((\mathcal{C}, \oplus, \otimes)\) is the datum for each objects \(M\) and \(N\) in \(\mathcal{C}\) of an \(\mathbb{R}_{+}\)-grading

\[
|\cdot|_{M,N} : \text{Hom}_{\mathcal{C}}(M, N) \to \mathbb{R}_{+}
\]

on the set of morphisms between \(M\) and \(N\) such that

\[
|f \oplus g| = \max(|f|, |g|) \\
|f \circ g| \leq |f| \cdot |g| \\
|f \otimes g| \leq |f| \cdot |g| \\
|0| = 0 \\
|\text{id}_{M}| = 1 \quad \text{if } 0_{M} \neq \text{id}_{M}
\]

and that it is called multiplicative if \(|f \otimes g| = |f| \cdot |g|\). One may thus encode such a seminormed rig category as a (particular kind of) rig category that is rig-enriched over the category of \(\mathbb{R}_{+}\)-graded sets with its two monoidal structures \(\otimes_{m}\) and \(\otimes_{\infty}\) given by the multiplication and the maximum operations on \(\mathbb{R}_{+}\). To get a homotopical version of this, we need to use a rig-∞-category that is weakly enriched in the category \((\mathbb{R}_{+}^{\text{Sp}}^{\leq 1}, \otimes_{m}, \otimes_{\infty})\) of \(\mathbb{R}_{+}\)-graded spectra for the supremum seminorm. One may also consider a version with
\( \otimes_1 \) instead of \( \otimes_\infty \). In this way, homotopical seminorms are given by particular kinds of enrichments, that one may try to parametrize.

One must be careful in extending the definition of the cotangent complex of a morphism of dagger algebras to the geometric situation of a morphism of derived dagger analytic spaces \( f : X \to Y \). Actually, even for the spectrum \( f : X = \mathcal{M}(A) \to \mathcal{M}(R) = Y \) of an \( R \)-affinoid \( t \)-analytic algebra, one needs to define the category of quasi-coherent modules on \( X \) in a local way for the \( G \)-topology, as was pointed out to the author, on a \( p \)-adic example due to Gabber by Brian Conrad (see [Con06], Remark 2.1.5 and Example 2.1.6).

**Definition 48.** Let \( X \) be a \( t \)-analytic space over \( R \). The category \( \text{QCoh}(X) \) of quasi-coherent modules over \( X \) is defined as the (opposite) tangent category

\[
\text{QCoh}(X)^{op} := T_X(\text{An}^t)^{op}
\]

to the category of analytic spaces at \( X \), defined as the category of abelian co-group objects in the category of morphisms \( f : Y \to X \). Similarly, if \( X \) is a derived \( t \)-analytic space over \( R \), the derived category \( \text{DQCoh}(X) \) of quasi-coherent modules over \( X \) is defined as the (opposite) \( \infty \)-tangent category

\[
\text{DQCoh}(X)^{op} := T_X(\text{DAn}^t)^{op}
\]

to the category of derived analytic spaces at \( X \), defined as the stabilization of the category of morphisms \( f : Y \to X \). The cotangent complex functor \( L \) is given by the adjoint of the forgetful functor

\[
\text{DQCoh}(X) \to \text{DAn}^t.
\]

We denote the module \( L(Y) \) by \( \mathbb{L}_{Y/X} \).

We may still define the symmetric monoidal \( \infty \)-category of perfect complexes in this geometric situation.

**Proposition 17.** The \( \infty \)-category \( \text{DQCoh}(X) \) is equipped with a natural symmetric monoidal structure \( \otimes \) with unit object denoted \( \mathbb{1}_X \). The symmetric monoidal stable \( \infty \)-category generated by \( \mathbb{1}_X \) is denoted \( (\text{Perf}(X), \otimes, \mathbb{1}_X) \). Every object of \( \text{Perf}(X) \) is strongly dualizable for the monoidal structure.

**Proof.** The same constructions work as in the situation of analytic algebras. The fact that every perfect complex is strongly dualizable follows from the fact that \( \mathbb{1}_X \) is the unit object for the stable monoidal structure. \( \square \)

**Example 12.** If \((R, | \cdot |_0)\) is a trivially seminormed integral ring and if we work with strict analytic spaces, then we get back the usual cotangent complex of algebraic geometry defined originally by Illusie in his thesis [Ill71], and the usual notion of perfect complexes on an \( R \)-scheme.

One may also define derived de Rham cohomology of dagger analytic spaces.
Definition 49. Let $f : X \to Y$ be a morphism of derived analytic spaces and $\mathbb{L}_{X/Y}$ be the corresponding cotangent complex. The derived de Rham complex (resp. completed derived de Rham complex) is defined by

$$
\Omega^*_{X/Y} := \left| \Lambda^* \mathbb{L}_{X/Y} \right|
$$

(resp. $\hat{\Omega}^*_{X/Y} := \left| \Lambda^* \mathbb{L}_{X/Y} \right|$),

where the exterior product is meant as the derived exterior product in the derived category of $\mathcal{O}_X$-modules and the sign $|\cdot|$ (resp. $|\cdot|_*$) means the totalization (resp. the product totalization) of the bicomplex. The derived de Rham (resp. Hodge completed derived de Rham) cohomology of $X$ over $Y$ is given by

$$
\text{DR}^*(X/Y) := \mathbb{R}\Gamma(X, \Omega^*_{X/Y})
$$

(resp. $\hat{\text{DR}}^*(X/Y) := \mathbb{R}\Gamma(X, \hat{\Omega}^*_{X/Y})$).

The derived and completed derived de Rham complexes are equipped with natural commutative algebra structures ($E_\infty$-algebras).

Remark 23. In characteristic 0, it is explained by Bhatt in [Bha12b], Remark 2.6, that derived de Rham cohomology is trivial in the algebraic setting, and that one really needs to pass to the Hodge completed setting to get back usual de Rham cohomology in the case of schemes (this is another important result due to Bhatt [Bha12a]). However, the non-completed version plays a central role in the definition of refined period rings, such as $A_{\text{cris}}$, so that one really needs to use it.

8 Chern characters and global regulators

We will now use the formalism of Toen and Vezzosi [TV09] for the cyclic Chern character, and adapt it to higher $K$-theory by getting inspiration from Blanc’s thesis [Bla13]. The fact that the Chern character in cyclic homology is integral (explained to the author by Gregory Ginot) is our main motivation for working on its adaptation to our global analytic setting, with the aim of defining various local and global “regulator type” maps.

8.1 Analytic Waldhausen $K$-theory and cyclic homology

For the following constructions to work with integral coefficients, we need a notion of analytic $K$-theory that is not necessarily homotopy invariant, so that neither the usual Karoubi-Villamayor approach [KV73], neither the dagger $\mathbb{D}^1$-homotopical approach described in Section 6 will give us what we need. We will thus use the Waldhausen approach explained on the nlab contributive website (following [TV02]; see also [Lur09b], Remark 11.4).

First recall that from a stable $\infty$-category, one may define the associated Waldhausen $K$-theory by the following definition.
Definition 50. Let $\mathcal{C}$ be an $\infty$-category. The core of $\mathcal{C}$ is the maximal sub-$\infty$-groupoid $\text{Core}(\mathcal{C})$ of $\mathcal{C}$. The $n$-gap of $\mathcal{C}$ is the full sub-$\infty$-category $\text{Gap}(\mathcal{C}^{\Delta^n})$ of $\text{Func}(\text{ARR}(\Delta^n), \mathcal{C})$ on those objects $F$ for which

- the diagonal $F(n,n)$ is inhabited by zero objects, for all $n$;
- all diagrams of the form

$$
\begin{array}{ccc}
F(i,j) & \rightarrow & F(i,k) \\
\downarrow & & \downarrow \\
F(j,j) & \rightarrow & F(j,k)
\end{array}
$$

is an $\infty$-pushout.

The (connective) $K$-theory spectrum of an $\infty$-category $\mathcal{C}$ with pushouts is defined by

$$K(\mathcal{C}) := \text{colim Core}(\text{Gap}(\mathcal{C}^{\Delta^n})).$$

The universal completion of the functor

$$K : \infty\text{CAT}^{st} \rightarrow \text{Sp}$$

from stable $\infty$-categories to spectra that sends homotopy cofibers of stable $\infty$-category to homotopy cofibers of spectra is the corresponding unconnective $K$-functor

$$K^{uc} : \infty\text{CAT}^{st} \rightarrow \text{Sp}.$$

If we have a geometric derived analytic stack $X$, we will be interested by the associated monoidal $\infty$-category $(\text{Perf}(X), \otimes)$ of perfect complexes, and the corresponding $K$-theory spectrum

$$K(X) := K(\text{Perf}(X)).$$

Let $\infty\text{Mon}$ be the $\infty$-category of $\infty$-monoids, given by models of the Lawvere theory $(\mathcal{T}_{\text{Mon}}, \times)$ of commutative monoids (category with finite products opposite to that of free finitely generated commutative monoids) with values in the $\infty$-category $\infty\text{Grpd}$ of $\infty$-groupoids (i.e., simplicial sets up to weak equivalences). Let $t \in \{ an, \{ an, s \}, \dagger, \{ \dagger, s \} \}$ be a type of analytic spaces.

Definition 51. The Hochshild homology pre-stack

$$\text{HH}_{pr} : \text{DAN}^t_R \rightarrow \infty\text{Mon}$$

is defined on the $\infty$-category of derived $t$-analytic spaces over $R$ by sending an analytic space $X$ to the $\infty$-monoid $\text{End}_{\text{Perf}(X^{S^1})}(\mathbb{1})$ of endomorphisms of the unit object in the $\infty$-symmetric monoidal category of perfect complexes on the derived loop space $X^{S^1}$ of $X$. The associated stack is called the Hochshild homology stack and denoted

$$\text{HH} : \text{DAN}^t_R \rightarrow \infty\text{Mon}.$$
The Hochshild homology pre-stack

$$\text{HH}_{pr} : \text{DAN}_R^t \to \infty \text{MON}$$

has actually a natural lifting

$$\widetilde{\text{HH}}_{pr} : \text{DAN}_R^t \to \mathbb{R} \text{Hom}(BS^1, \infty \text{MON}) =: S^1 – \infty \text{MON}$$

to the $\infty$-category of $S^1$-equivariant $\infty$-monoids. The associated stack also gives

$$\widetilde{\text{HH}} : \text{DAN}_R^t \to S^1 – \infty \text{MON}.$$  

**Definition 52.** The functors obtained by composing the above lifting of the Hochshild homology functors with the functor of homotopy fixed point

$$(\cdot)^{hS^1} := \lim_{BS^1} : S^1 – \infty \text{MON} \to \infty \text{MON},$$

defines two $\infty$-functors

$$\text{HC}^{neg}_{pr} : \text{DAN}_R^t \to \infty \text{MON}$$

$$\text{HC}^{neg} : \text{DAN}_R^t \to \infty \text{MON}$$

called respectively the *negative cyclic pre-stack and negative cyclic stack*. Similarly, composing with the $\infty$-functor of homotopy coinvariants

$$(\cdot)_{hS^1} := \text{colim}_{BS^1} : S^1 – \infty \text{MON} \to \infty \text{MON},$$

defines two new $\infty$-functors

$$\text{HC}^{pr} : \text{DAN}_R^t \to \infty \text{MON}$$

$$\text{HC} : \text{DAN}_R^t \to \infty \text{MON}$$

called the *cyclic homology pre-stack and the cyclic homology stack*.

There is actually an equivalence of $\infty$-categories

$$\infty \text{MON} \sim \text{Sp}^{\text{con}}$$

between the $\infty$-category of $\infty$-monoids and the $\infty$-category $\text{Sp}^{\text{con}}$ of connective spectra, that we may use to define the cyclic homology spectra of an analytic space,

**Definition 53.** Let $X$ be a $t$-analytic space. The cyclic homology spectra of $X$ are defined as the spectra $\text{HC}^{neg}(X)$ and $\text{HC}(X)$ associated to the $\infty$-monoids $\text{HC}^{neg}(X)$ and $\text{HC}(X)$.

**Definition 54.** There is a natural morphism from homotopical $S^1$-coinvariants to homotopical $S^1$-invariants that induces a natural morphism

$$\text{HC} \to \text{HC}^{neg}.$$
8.2 The cyclic Chern character

We want to adapt Toen-Vezzosi’s construction of the Chern character from [TV09] to get morphisms of spectra

\[ \text{ch} : \mathbb{K}(X) \to \mathbb{H} \text{C}_{\text{neg}}(X) \]

and

\[ \text{ch} : \mathbb{K}^{\text{nc}}(X) \to \mathbb{H} \text{T}(X) \]

that induce both the usual Chern character (for the negative one) if we work over a regular Banach ring of characteristic 0, and an interesting new Chern character from non-connective $K$-theory if we work in a highly singular, global analytic or characteristic $p$ situation. This may be easily extended to logarithmic analytic spaces if one uses the tangent category definition for their category of perfect complexes, as we did for classical analytic spaces.

To show that Toen and Vezzosi’s work extends to a Chern character

\[ \text{ch} : \mathbb{K}(X) \to \mathbb{H} \text{C}_{\text{neg}}(X) \]

on the full connective $K$-theory spectrum, we need to show that the trace morphism may be extended to every $\text{Gap}(\mathcal{C}^{\Delta^n})$ in a compatible way with the simplicial maps. This should follow from the derived additivity of traces alluded to in the introduction of loc. cit., Subsection 2.4: one has to show that the cyclic trace $\text{Tr}^{\text{st}}$ restricted to a stable symmetric monoidal $\infty$-category is compatible with exact triangles.

Then, the universal property of the non-connective $K$-theory spectrum and the fact that homotopy cofibers of stable $\infty$-categories are sent to homotopy cofibers of their associated Tate cyclic homology spectra should do the rest of the job of defining

\[ \text{ch} : \mathbb{K}^{\text{nc}}(X) \to \mathbb{H} \text{T}(X). \]

We leave the details of these constructions to a later publication.

References

[Ara01] Alberto Arabia. Relèvements des algèbres lisses et de leurs morphismes. Comment. Math. Helv., 76(4):607–639, 2001.

[Art67] Emil Artin. Algebraic numbers and algebraic functions. Gordon and Breach Science Publishers, New York, 1967.

[Ayo07a] Joseph Ayoub. Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique. I. Astérisque, 314:x+466 pp. (2008), 2007.
Joseph Ayoub. Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique. II. *Astérisque*, 315:vi+364 pp. (2008), 2007.

Joseph Ayoub. Note sur les opérations de Grothendieck et la réalisation de Betti. *J. Inst. Math. Jussieu*, 9(2):225–263, 2010.

Joseph Ayoub. Motifs des variétés analytiques rigides. *preprint*, 2011.

Joseph Ayoub. La réalisation étale et les opérations de grothendieck. *Preprint*, 2014.

F. Bambozzi. On a generalization of affinoid varieties. *ArXiv e-prints*, January 2014.

A. Beilinson. p-adic periods and derived de Rham cohomology. *ArXiv e-prints*, February 2011.

Vladimir G. Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields*, volume 33 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1990.

Vladimir G. Berkovich. Étale cohomology for non-Archimedean analytic spaces. *Inst. Hautes Études Sci. Publ. Math.*, 78:5–161 (1994), 1993.

S. Bosch, U. Güntzer, and R. Remmert. *Non-Archimedean analysis*, volume 261 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1984. A systematic approach to rigid analytic geometry.

B. Bhatt. Completions and derived de Rham cohomology. *ArXiv e-prints*, July 2012.

B. Bhatt. p-adic derived de Rham cohomology. *ArXiv e-prints*, April 2012.

Oren Ben-Bassat and Kobi Kremnitzer. Non-Archimedean analytic geometry as relative algebraic geometry. *ArXiv e-prints*, December 2013.

A. Blanc. Invariants topologiques des Espaces non commutatifs. *ArXiv e-prints*, July 2013.

D.-C. Cisinski and F. Déglise. Triangulated categories of mixed motives. *ArXiv e-prints*, December 2009.

Brian Conrad. Relative ampleness in rigid geometry. *Ann. Inst. Fourier (Grenoble)*, 56(4):1049–1126, 2006.
Pierre Deligne. Théorie de Hodge. II. *Inst. Hautes Études Sci. Publ. Math.*, 40:5–57, 1971.

Nikolai Diekert. *Der Tilt von überkonvergenten Potenzreihen*. Diploma. Freiburg Universität, 2012.

C. Davis and K. S. Kedlaya. Almost purity for overconvergent Witt vectors. *ArXiv e-prints*, March 2014.

F. Déglise and N. Mazzari. The rigid syntomic ring spectrum. *ArXiv e-prints*, November 2012.

Eduardo Dubuc and Gabriel Taubin. Analytic rings. *Cahiers Topologie Géom. Différentielle*, 24(3):225–265, 1983.

Nikolai Durov. *New Approach to Arakelov Geometry*. arXiv.org:0704.2030, 2007.

Eduardo J. Dubuc and Jorge G. Zilber. On analytic models of synthetic differential geometry. *Cahiers Topologie Géom. Différentielle Catég.*, 35(1):49–73, 1994.

Elmar Grosse-Klönne. Rigid analytic spaces with overconvergent structure sheaf. *J. Reine Angew. Math.*, 519:73–95, 2000.

Elmar Grosse-Klönne. Finiteness of de Rham cohomology in rigid analysis. *Duke Math. J.*, 113(1):57–91, 2002.

Elmar Große-Klönne. De Rham cohomology of rigid spaces. *Math. Z.*, 247(2):223–240, 2004.

A. Holmstrom and J. Scholbach. Arakelov motivic cohomology I. *ArXiv e-prints*, December 2010.

Luc Illusie. *Complexe cotangent et déformations. I*. Lecture Notes in Mathematics, Vol. 239. Springer-Verlag, Berlin, 1971.

Max Karoubi. *K*-théorie multiplicative. *C. R. Acad. Sci. Paris Sér. I Math.*, 302(8):321–324, 1986.

Masaki Kashiwara and Pierre Schapira. *Categories and sheaves*, volume 332 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006.

Max Karoubi and Orlando Villamayor. *K*-théorie algébrique et *K*-théorie topologique. II. *Math. Scand.*, 32:57–86, 1973.
[Law79] F. William Lawvere. Categorical dynamics. In Topos theoretic methods in geometry, volume 30 of Various Publ. Ser., pages 1–28. Aarhus Univ., Aarhus, 1979.

[Law04] F. William Lawvere. Functorial semantics of algebraic theories and some algebraic problems in the context of functorial semantics of algebraic theories. Repr. Theory Appl. Categ., 5:1–121, 2004. Reprinted from Proc. Nat. Acad. Sci. U.S.A. 50 (1963), 869–872 [MR0158921] and Reports of the Midwest Category Seminar. II, 41–61, Springer, Berlin, 1968 [MR0231882].

[Lur07] J. Lurie. Derived Algebraic Geometry IV: Deformation Theory. ArXiv e-prints, September 2007.

[Lur09a] Jacob Lurie. Derived Algebraic Geometry V: Structured Spaces. Preprint, 2009.

[Lur09b] Jacob Lurie. Higher Algebra. Preprint, 2009.

[Lur09c] Jacob Lurie. Higher topos theory, volume 170 of Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009.

[Lur11] Jacob Lurie. Derived Algebraic Geometry IX: Closed Immersions. Preprint, 2011.

[Mar12] F. Martin. Overconvergent subanalytic subsets in the framework of Berkovich spaces. ArXiv e-prints, November 2012.

[Mer72] David Meredith. Weak formal schemes. Nagoya Math. J., 45:1–38, 1972.

[MR13] J. Milne and N. Ramachandran. Motivic complexes and special values of zeta functions. ArXiv e-prints, November 2013.

[MV99] Fabien Morel and Vladimir Voevodsky. $\mathbb{A}^1$-homotopy theory of schemes. Inst. Hautes Études Sci. Publ. Math., 90:45–143 (2001), 1999.

[MW68] P. Monsky and G. Washnitzer. Formal cohomology. I. Ann. of Math. (2), 88:181–217, 1968.

[Pau14a] Frédéric Paugam. Analytic spectrum of rig categories. Theory and Applications of Categories, 29, 2014.

[Pau14b] Frédéric Paugam. Towards the mathematics of quantum field theory, volume 59 of Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer Verlag, 2014.

[Poi10] Jérôme Poineau. La droite de Berkovich sur $\mathbb{Z}$. Astérisque, 334:viii+xii+284, 2010.
[Poi13] Jérôme Poineau. Espaces de Berkovich sur $\mathbb{Z}$ : étude locale. *Invent. Math.*, 194(3):535–590, 2013.

[Por14] Mauro Porta. *Géométrie analytique dérivée*. Thèse. université de Paris 7, 2014.

[Rio10] Joël Riou. Algebraic $K$-theory, $\mathbb{A}^1$-homotopy and Riemann-Roch theorems. *J. Topol.*, 3(2):229–264, 2010.

[Rob12] M. Robalo. Noncommutative Motives I: A Universal Characterization of the Motivic Stable Homotopy Theory of Schemes. *ArXiv e-prints*, June 2012.

[Rob14] Marco Robalo. *Théorie homotopique motivique des espaces non commutatifs*. Université de Montpellier 2s, Montpellier, 2014. Thèse, université de Montpellier 2.

[RS14] Andreas Rosenschon and Vasudevan Srinivas. *Etale motivic cohomology and algebraic cycles*. Preprint, 2014.

[Sch10] J. Scholbach. Special L-values of geometric motives. *ArXiv e-prints*, March 2010.

[Sch12a] J. Scholbach. Arakelov motivic cohomology II. *ArXiv e-prints*, May 2012.

[Sch12b] Peter Scholze. Perfectoid spaces. *Publ. Math. Inst. Hautes Études Sci.*, 116:245–313, 2012.

[Sch13] Peter Scholze. $p$-adic Hodge theory for rigid-analytic varieties. *Forum Math. Pi*, 1:e1, 77, 2013.

[Tam11] G. Tamme. Karoubi’s relative Chern character, the rigid syntomic regulator, and the Bloch-Kato exponential map. *ArXiv e-prints*, November 2011.

[Tat71] John Tate. Rigid analytic spaces. *Invent. Math.*, 12:257–289, 1971.

[Tem04] M. Temkin. On local properties of non-archimedean spaces ii. *Isr. J. of Math.*, 140(1):1–27, 2004.

[Tem10] M. Temkin. Introduction to Berkovich analytic spaces. *ArXiv e-prints*, October 2010.

[TV02] B. Toen and G. Vezzosi. A remark on K-theory and S-categories. *ArXiv Mathematics e-prints*, October 2002.

[TV09] B. Toen and G. Vezzosi. Caractères de Chern, traces équivariantes et géométrie algébrique d’après Wintenberger. *ArXiv e-prints*, March 2009.

[Vez14] A. Vezzani. A motivic version of the theorem of Fontaine and Wintenberger. *ArXiv e-prints*, May 2014.