Note on Many-Quark Model with $su(4)$ Algebraic Structure

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(Received April 24, 2009; Revised June 8, 2009)

In a many-quark model developed in our previous paper where two-body color pairing and particle-hole type interactions are active, the exact energy eigenstates are re-formed with physically clearer expressions than those derived in our previous paper. By using the re-formed energy eigenstates, two types of the eigenstates in which the pairing correlation and the quark triplet formation separately appear definitely, are unified and this model can be treated for both the strong color correlations and the quark triplet formations.

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§1. Introduction

One of the recent interests of quark and hadron physics may be the investigation of the various properties which many-quark system reveals. In our previous paper$^1$ which is referred to as (I), we have investigated the many-quark model where the two-body pairing interaction is active. This model is known as the Bonn model$^2$ and this model is known to lead to the formation of the quark triplet while only two-body interaction is contained.$^3$ As was pointed out in (I), the Bonn model has the $su(4)$ algebraic structure. Thus, from the viewpoint of the color $su(3)$-symmetry in quantum chromodynamics (QCD), this model can be extended to the many-quark model which does not have the $su(4)$ symmetry but has only the color $su(3)$-symmetry. In (I), we have formulated the above-mentioned idea in terms of the Schwinger boson representation for many-fermion system. As a result, we have given the many-quark model with the color pairing plus particle-hole type interactions under the color $su(3)$-symmetry, which we have called the modified Bonn model.

In (I), we gave the orthogonal set for the quantum states of this model in the boson realization for the many-quark system. Then, the exact eigenstates and eigenvalues for this modified Bonn model Hamiltonian were derived. As for the set of the eigenstates and the eigenenergies, there are two possibilities: One is classified as a form with the pairing correlation and the other is as a quark triplet formation.

In this paper, we reformulate the orthogonal set or the eigenstates for the modified Bonn model. Because the model is formulated in the boson space using the boson realization, the physical quantities peculiar to many-quark model are transcribed in boson representation. After that, the orthogonal set or the eigenstates are constructed in terms of the quark triplet, quark pair and single quark creation.
operators which are denoted as \( \hat{B}^* \), \( \hat{S}^i \) and \( \hat{q}^i \) respectively, by acting on the minimum weight state for the \( su(1, 1) \) algebra appearing in the boson realization. By using the re-formed orthogonal set, which we denote as \( \{ | lsr\omega \rangle \} \), of the quantum states, it is shown that the eigenstates with the pairing-type correlation and with the quark triplet formation can be described in a unified way. This is one of our main purposes of this paper. Further, it is indicated that two types of the minimum weight states for the \( su(1, 1) \)-algebra are necessary in order to construct the orthogonal set \( \{ | lsr\omega \rangle \} \). This fact may be an interesting feature for the model with the algebraic structure including the non-compact \( su(1, 1) \) sub-algebra.

As for the complete orthogonal set for the modified Bonn model, eight quantum numbers are needed. The complete orthogonal states are constructed from \( \{ | lsr\omega \rangle \} \), which degenerate with respect to the energy eigenvalues in the modified Bonn model. An idea to dissociate the degeneracy will be given and a possible method will be formulated by using the technique developed in the theory of the nuclear collective rotational motion.

This paper is organized as follows. In the next section, the outline of our modified Bonn model in the Schwinger boson representation is given by paying attention to the algebraic structure of this many-quark system. In \$3\), three kinds of the basic building blocks to construct the orthogonal set are presented together with their physical and mathematical properties. The re-formed construction of the orthogonal set, \( \{ | lsr\omega \rangle \} \), is given in \$4\) by using the three kinds of the building blocks, namely, the quark triplet, the color pair and the single quark creations. In \$5\), the relevance to the two forms developed in (I) is indicated and these two forms are described in the unified way by means of the re-formed orthogonal set \( \{ | lsr\omega \rangle \} \). The expressions in terms of the degeneracy \( \Omega \) and the quark numbers with color \( i \), \( N_i \), are given and the physical meaning of the state \( \{ | lsr\omega \rangle \} \) is also clarified in \$6\). The last section is devoted to the concluding remarks and the possible idea to dissociate the degeneracy with respect to the energy is mentioned in this last section. The proofs of various formulae and eigenvalue problem are given in Appendix A.

\section{Outline of the model in the Schwinger boson representation}

In this section, we will recapitulate the basic framework of many-quark model presented in (I). As was mentioned in \$1\), this model obeys the \( su(4) \)-algebra and is called the Bonn model. The Schwinger boson representation for the \( su(4) \)-algebra adopted in (I) is composed of eight kinds of bosons \((a, a^*, b, b^*, a^*_i, \hat{b}_i, \hat{b}^*_i ; i = 1, 2, 3)\), in which the symbol \( i \) denotes the color \( i \). With the use of the above bosons, the \( su(4) \)-generators are expressed in the form

\[
\begin{align*}
\hat{S}^i &= a^*_i \hat{b} - \hat{b}^* a_i , \\
\hat{S}_i &= \hat{b}^* a_i - \hat{b}_i a , \\
\hat{S}^*_i &= (a^*_i a_j - \hat{b}_j \hat{b}_i) + \delta_{ij}(\hat{a}^* a - \hat{b}^* b) .
\end{align*}
\]

(2.1)

Naturally, the Bonn model belongs to many-fermion models. We intend to describe this fermion system in the space spanned by the above boson operators. Therefore, it is indispensable to transcribe physical quantities peculiar to many-fermion model under investigation in boson space. Typical examples in the present case are \( \hat{S}^1 \),...
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\[ \hat{S}^2 \text{ and } \hat{S}^3, \] which correspond to the quark-pair creations in colors (2, 3), (3, 1) and (1, 2), respectively. The degeneracy of the single-particle level $i$, $2\Omega_i$, is a parameter in the fermion space and it is positive integer. In the boson space, we treat this parameter as an operator $2\tilde{\Omega}_i$, which is expressed in terms of the bosons. The quark number operator in the color $i$, $\tilde{N}_i$, is also the same. The operator $\tilde{N}_i$ should be transcribed in the boson space. We denote it as $\hat{N}_i$. In (I), we obtained $\hat{\Omega}$, $\tilde{N}_i$ and $\hat{N}$ ($= \sum_i \tilde{N}_i$) in the following form:

\[
\hat{\Omega} = n_0 + \frac{1}{2} \left[ (\hat{a}^* \hat{a} + \hat{b}^* \hat{b}) + \sum_i (\hat{a}^*_i \hat{a}_i + \hat{b}^*_i \hat{b}_i) \right], \tag{2.2}
\]

\[
\tilde{N}_i = n_0 + \hat{a}^* \hat{a} + \sum_j \hat{a}^*_j \hat{a}_j - (\hat{a}^*_i \hat{a}_i - \hat{b}^*_i \hat{b}_i), \tag{2.3}
\]

\[
\hat{N} = 3n_0 + 3\hat{a}^* \hat{a} + 2 \sum_i \hat{a}^*_i \hat{a}_i + \sum_i \hat{b}^*_i \hat{b}_i . \tag{2.4}
\]

Here, $n_0$ denotes a positive integer which corresponds to the seniority number in the $su(2)$-pairing model.

In the present boson space, there exists the $su(1, 1)$-algebra which does not exist in the original fermion system:

\[
\hat{T}_{\pm,0} = \hat{t}_{\pm,0} + \hat{\tau}_{\pm,0} . \tag{2.5}
\]

Here, $\hat{t}_{\pm,0}$ and $\hat{\tau}_{\pm,0}$ are defined as

\[
\hat{t}_+ = \hat{b}^* \hat{a}^* , \quad \hat{t}_- = \hat{a} \hat{b} , \quad \hat{t}_0 = \frac{1}{2} (\hat{b}^* \hat{b} + \hat{a}^* \hat{a}) + \frac{1}{2} , \tag{2.6}
\]

\[
\hat{\tau}_+ = \sum_i \hat{b}^*_i \hat{a}_i^* , \quad \hat{\tau}_- = \sum_i \hat{a}_i \hat{b}_i , \quad \hat{\tau}_0 = \frac{1}{2} \sum_i (\hat{b}^*_i \hat{b}_i + \hat{a}^*_i \hat{a}_i) + \frac{3}{2} . \tag{2.7}
\]

The sets $\{ \hat{t}_{\pm,0} \}$ and $\{ \hat{\tau}_{\pm,0} \}$ form two independent $su(1, 1)$-algebras. The most important property of $\{ \hat{T}_{\pm,0} \}$ is that $\{ \hat{T}_{\pm,0} \}$ commutes with $\{ \hat{S}_i, \hat{S}_1^i, \hat{S}_2^i, \hat{S}_3^i \}$.

We mentioned in (I) that as a sub-algebra, the present $su(4)$-algebra contains the $su(3)$-algebra, which includes the $su(2)$-algebra as a sub-algebra. The generators are listed as

\[
\text{the } su(3)\text{-algebra } \{ \hat{S}_1^2, \hat{S}_2^1, \hat{S}_1^3, \hat{S}_2^3, \hat{S}_3^2, \hat{S}_3^3, \hat{Q}_0, \hat{R}_0 \} , \tag{2.8}
\]

\[
\text{the } su(2)\text{-algebra } \{ \hat{R}_+, \hat{R}_-, \hat{R}_0 \} . \tag{2.9}
\]

Here, $\hat{Q}_0$ and $\hat{R}_0$ are functions of $(\hat{S}_1^1, \hat{S}_2^2, \hat{S}_3^3)$ and there are several possibilities. Depending on $\hat{R}_0$, the operators $\hat{R}_\pm$ are determined. In (I), we adopted the following form:

\[
\hat{Q}_0 = \hat{S}_1^1 - \frac{1}{2} (\hat{S}_2^2 + \hat{S}_3^3) , \quad \hat{R}_0 = \frac{1}{2} (\hat{S}_2^2 - \hat{S}_3^3) . \tag{2.10}
\]

Then, $\hat{R}_\pm$ can be chosen as

\[
\hat{R}_+ = \hat{S}_2^3 , \quad \hat{R}_- = \hat{S}_3^2 . \tag{2.11}
\]
The Casimir operators denoted $\hat{P}^2$, $\hat{Q}^2$ and $\hat{R}^2$ for the $su(4)$-, the $su(3)$- and the $su(2)$-algebra are expressed, respectively, as

$$
\hat{P}^2 = 2(\hat{S}^1\hat{S}_1 + \hat{S}^2\hat{S}_2 + \hat{S}^3\hat{S}_3) + 2(\hat{S}^2\hat{S}_1 + \hat{S}^3\hat{S}_1 + \hat{R}_+\hat{R}_-) \\
+ \frac{3}{4}\hat{P}_0(\hat{P}_0 - 4) + \frac{2}{3}\hat{Q}_0(\hat{Q}_0 - 3) + 2\hat{R}_0(\hat{R}_0 - 1) , 
(2.12)
$$

$$
\hat{Q}^2 = 2(\hat{S}^1\hat{S}_2 + \hat{S}^2\hat{S}_3 + \hat{R}_+\hat{R}_-) + \frac{2}{3}\hat{Q}_0(\hat{Q}_0 - 3) + 2\hat{R}_0(\hat{R}_0 - 1) , 
(2.13)
$$

$$
\hat{R}^2 = \hat{R}_+\hat{R}_- + \hat{R}_0(\hat{R}_0 - 1) . 
(2.14)
$$

Here, $\hat{P}_0$ is defined as

$$
\hat{P}_0 = \frac{1}{3}(\hat{S}^1_1 + \hat{S}^2_2 + \hat{S}^3_3) . 
(2.15)
$$

In (I), we investigated the following Hamiltonian:

$$
\hat{H} = -(\hat{S}^1\hat{S}_1 + \hat{S}^2\hat{S}_2 + \hat{S}^3\hat{S}_3) , 
(2.16)
$$

$$
\hat{H}_m = \hat{H} + \chi \hat{Q}^2 . 
(\chi : \text{a real parameter}) 
(2.17)
$$

The Hamiltonian $\hat{H}$ is given in the original Bonn model and a possible modification, $\chi \hat{Q}^2$ is added in (I). The most fundamental relation in this model is as follows:

$$
[\hat{S}^j_i, \hat{H}] = 0 , \quad [\hat{S}^j_i, \hat{H}_m] = 0 . \quad (i, j = 1, 2, 3) 
(2.18)
$$

Here, the relation $[\hat{S}^j_i, \hat{Q}^2] = 0$, should be noted. With the use of the relations (2.12) and (2.13), $\hat{H}$ can be expressed as

$$
\hat{H} = -\frac{1}{2}\left(\hat{P}^2 - \hat{Q}^2 - \frac{3}{4}\hat{P}_0(\hat{P}_0 - 4)\right) . 
(2.19)
$$

In this paper, we will investigate $\hat{H}$ and $\hat{H}_m$ under an idea which is different from that in (I).

§3. Three kinds of basic building blocks and their properties

First, we note that in the present boson space, there exists one more $su(4)$-algebra, the generators of which are expressed as

$$
\hat{q}^i = \hat{b}^*_i\hat{b} - \hat{a}^*_i\hat{a} , \quad \hat{q} = \hat{b}^*_i\hat{b}_i - \hat{a}^*_i\hat{a}_i \\
\hat{q}_i^j = (\hat{b}^*_i\hat{b}_j - \hat{a}^*_i\hat{a}_j) - \delta_{ij}(\hat{b}^*_0\hat{b}_0 - \hat{a}^*_0\hat{a}_0) . 
(3.1)
$$

Any generator $\{\hat{q}^i, \hat{q}_i, \hat{q}_i^j\}$ commutes with any of $\{\hat{T}_{\pm0}\}$, but some generators do not commute with $\{\hat{S}^i, \hat{S}_i, \hat{S}_i^j\}$. With the use of $\hat{S}^i$ and $\hat{q}^i$, we define the following operators:

$$
\hat{B}^* = \sum_i \hat{S}^i \hat{q}^i , \quad \hat{B} = \sum_i \hat{q}_i \hat{S}_i . 
(3.2)
$$
The operators \( \hat{B}^*, \hat{B} \) also commute with any of \( \{ \hat{T}_{\pm,0} \} \). Main aim of this section is to investigate various properties of the operators \( \hat{S}^i \), \( \hat{q}^i \) and \( \hat{B}^* \). Through this task, we can conclude that these operators can be regarded as building blocks for constructing a possible form of the orthogonal set for the boson space.

We notice, first, the following relation

\[
\left[ \hat{S}^i, \hat{S}^j \right] = \left[ \hat{S}^i, \hat{q}^j \right] = \left[ \hat{q}^i, \hat{q}^j \right] = 0 \quad \text{(any } i \text{ and } j) \tag{3.3}
\]

Then, we have

\[
\left[ \hat{S}^i, \hat{B}^* \right] = \left[ \hat{q}^i, \hat{B}^* \right] = 0 \tag{3.4}
\]

From the above relations, we learn that the ordering of \( \hat{S}^i \), \( \hat{q}^i \) and \( \hat{B}^* \) for their products is arbitrary. Next, we show the properties of \( \hat{S}^i \), \( \hat{q}^i \) and \( \hat{B}^* \) related to the \( su(3) \)- and the \( su(2) \)-generators shown in the relations (2.8) and (2.9), respectively. We know the relation

\[
\left[ \hat{S}^j_i, \hat{S}^k_i \right] = \delta_{ij} \hat{S}^k_i + \delta_{jk} \hat{S}^i_i, \\
\left[ \hat{S}^j_i, \hat{q}^k_i \right] = \delta_{ij} \hat{q}^k_i - \delta_{ik} \hat{q}^j_i. \tag{3.5}
\]

The relations (3.2) and (3.5) give us

\[
\left[ \hat{S}^j_i, \hat{B}^* \right] = 0. \tag{3.6}
\]

Then, we have

\[
\left[ \hat{Q}_0, \hat{B}^* \right] = 0, \quad \left[ \hat{R}_{\pm,0}, \hat{B}^* \right] = 0. \tag{3.7}
\]

The relations (3.6) and (3.7) tell us that \( \hat{B}^* \) is a color-neutral operator, and naturally, \( su(2) \)-scalar. The operators \( \hat{q}^1 \) and \( \hat{S}^3 \) do not possess such properties. However, we notice \( \hat{q}^1 \) and \( \hat{S}^3 \). In the case \( \hat{q}^1 \), we have

\[
\left[ \hat{S}^2_i, \hat{q}^1_i \right] = \left[ \hat{S}^3_i, \hat{q}^1_i \right] = 0, \quad \left[ \hat{Q}_0, \hat{q}^1_i \right] = -\hat{q}^1_i. \tag{3.8}
\]

The relation (3.9) tells us that \( \hat{q}^1 \) is \( su(2) \)-scalar, but, not color-neutral. The relation (3.8) will play an important role for constructing the orthogonal set, which will be shown in the next section. The operator \( \hat{S}^3 \) obeys the relations

\[
\left[ \hat{S}^1_2, \hat{S}^3 \right] = \left[ \hat{S}^1_3, \hat{S}^3 \right] = 0, \quad \left[ \hat{Q}_0, \hat{S}^3 \right] = -\frac{1}{2} \hat{S}^3, \tag{3.10}
\]

\[
\left[ \hat{R}_{-}, \hat{S}^3 \right] = 0, \quad \left[ \hat{R}_{0}, \hat{S}^3 \right] = -\frac{1}{2} \hat{S}^3. \tag{3.11}
\]

The above indicates that \( \hat{S}^3 \) is not color-neutral and also, not \( su(2) \)-scalar, but the relations (3.10) and (3.11) also play an important role for constructing the orthogonal set. The commutation relation of \( \hat{B}^*, \hat{q}^1 \) and \( \hat{S}^3 \) for \( \hat{P}_0 \) is given as

\[
\left[ \hat{P}_0, \hat{B}^* \right] = 2\hat{B}^*, \quad \left[ \hat{P}_0, \hat{q}^1 \right] = \frac{2}{3} \hat{q}^1, \quad \left[ \hat{P}_0, \hat{S}^3 \right] = \frac{4}{3} \hat{S}^3. \tag{3.12}
\]

The above are the mathematical properties of \( \hat{S}^i \), \( \hat{q}^i \) and \( \hat{B}^* \).
Next, we investigate physical properties in relation to $\hat{\Omega}$ and $\hat{N}$ presented in the relations (2.2) and (2.4), respectively. It is very easy to show the following relations:

\[
[ \hat{\Omega} , \hat{B}^* ] = [ \hat{\Omega} , \hat{S}^3 ] = [ \hat{\Omega} , \hat{q}^1 ] = 0 , \tag{3.13}
\]

\[
[ \hat{N} , \hat{B}^* ] = 3\hat{B}^* , \quad [ \hat{N} , \hat{S}^3 ] = 2\hat{S}^3 , \quad [ \hat{N} , \hat{q}^1 ] = \hat{q}^1 . \tag{3.14}
\]

The relation (3.13) shows that the degeneracy of the single-particle level does not depend on $\hat{B}^*$, $\hat{S}^3$ and $\hat{q}^1$. The relation (3.14) tells us that the operators $\hat{B}^*$, $\hat{S}^3$ and $\hat{q}^1$ change the quark number 3, 2 and 1, respectively. Therefore, it may be permitted for $\hat{B}^*$, $\hat{S}^3$ and $\hat{q}^1$ to play a role of the quark-triplet, the quark-pair and single-quark creation, respectively. Combining with the mathematical properties, we have the following image: The quark-triplet is color-neutral and $su(2)$-scalar. The quark-pair is colored, but $su(2)$-scalar. As a final remark of this section, we will mention the role of the boson operator $\hat{b}^*$ itself. The operator $\hat{b}^*$ obeys

\[
[ \hat{\Omega} , \hat{b}^* ] = \frac{1}{2} \hat{b}^* , \quad [ \hat{N} , \hat{b}^* ] = 0 . \tag{3.15}
\]

The relation (3.15) tells us that the degeneracy $\Omega$ is determined by $\hat{b}^*$, but, it does not change the quark number.

§4. Construction of the orthogonal set

On the basis of the above argument in §3, we search the orthogonal set. We introduce the following state:

\[
\|lsr\omega\rangle = (\hat{S}^3)^{2l}(\hat{q}^1)^{2s}(\hat{B}^*)^r(\hat{b}^*)^{2\omega}|0\rangle . \tag{4.1}
\]

In this paper, we omit numerical factor such as normalization constant for any state except some cases. Here, we construct the orthogonal set in terms of the Schwinger boson realization for the $su(4)$ algebra. If the connection to the states in the original fermion space is necessary, the additional quantum numbers except for the quantum numbers related to the $su(4)$ algebra are needed. We do not concern this point which is beyond a scope in this paper. The state (4.1) is an eigenstate for $\hat{P}_0$, $\hat{Q}_0$, $\hat{R}_0$ and $\hat{T}_0$:

\[
\hat{P}_0\|lsr\omega\rangle = -2 \left( \omega - 2 \left( r + \frac{1}{3}(2l + s) \right) \right) \|lsr\omega\rangle , \tag{4.2}
\]
\[
\hat{Q}_0\|lsr\omega\rangle = -(l + 2s)\|lsr\omega\rangle , \quad \hat{R}_0\|lsr\omega\rangle = -l\|lsr\omega\rangle , \tag{4.3}
\]
\[
\hat{T}_0\|lsr\omega\rangle = (\omega + 2)\|lsr\omega\rangle . \tag{4.4}
\]

The above indicates that the set $\{|\|lsr\omega\rangle\}$ forms an orthogonal set. However, our present system is composed of the eight kinds of bosons, and then, the complete set is specified by eight quantum numbers. In order to obtain the complete orthogonal set, we pay attention to the relations

\[
\hat{S}^1_2\|lsr\omega\rangle = \hat{S}^1_3\|lsr\omega\rangle = \hat{R}_-\|lsr\omega\rangle = 0 , \tag{4.5}
\]
\[
\hat{T}_-\|lsr\omega\rangle = 0 . \tag{4.6}
\]
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The relations (4.5) and (4.6) tell us that the state $|\ell \sigma \omega \rangle$ is the minimum weight state for the $su(3)$- and the $su(1,1)$-algebra. Then, a possible complete orthogonal set is given in the form

$$|k\kappa\kappa_0 \ell \sigma \omega \rangle = (\hat{T}_+)^{\omega_0 - \omega}(\hat{Q}_+)(k\kappa\kappa_0)|\ell \sigma \omega \rangle.$$  

(4.7)

Here, $\hat{Q}_+ (k\kappa\kappa_0)$ is a certain function of $\hat{S}_1^2$, $\hat{S}_1^3$ and $\hat{R}_+$ and the concrete expression is given in the relation (I.A.8). Instead of the form (4.7), we can adopt the following form:

$$|k\kappa\kappa_0 \ell \sigma \omega \rangle = (\hat{T}_+)^{\omega_0 - \omega}(\hat{R}_+)(k\kappa\kappa_0)|\ell \sigma \omega \rangle.$$  

(4.8)

The operator $\hat{S}_1^4$ is defined as

$$\hat{S}_1^4 = \hat{S}_1^3 \hat{R}_0 - \frac{1}{2} \hat{S}_1^2 \hat{R}_+ .$$  

(4.9)

In the Appendix, we will give the reason why the expression (4.8) is permitted. The state $|k\kappa\kappa_0 \ell \sigma \omega \rangle$ is an eigenstate of $\hat{P}^2$, $\hat{R}_0$, $\hat{Q}^2$, $\hat{Q}_0$, $\hat{R}^2$, $\hat{R}_0$, $\hat{T}^2$ and $\hat{T}_0$, the eigenvalues of which are summarized as follows:

$$\hat{P}^2 : (2(r + s) - \omega)^2 + 2\omega(\omega + 3) , \quad \hat{R}_0 : 2 \left( r + \frac{1}{3}(2l + s) \right) - \omega ,$$

$$\hat{Q}^2 : \frac{2}{3}(l + 2s)(l + 2s + 3) + 2l(l + 1) , \quad \hat{Q}_0 : 3k - (l + 2s) ,$$

$$\hat{R}^2 : \kappa(\kappa + 1) , \quad \hat{R}_0 : \kappa_0 ,$$

$$\hat{T}^2 : (\omega + 2)(\omega + 1) , \quad \hat{T}_0 : \omega + 2 .$$  

(4.10a)

(4.10b)

Straightforward calculation gives us the above eigenvalues except the case $\hat{P}^2$. In the Appendix, we will show the derivation of the eigenvalue of $\hat{P}^2$. The quantum numbers obey the conditions

$$l + s + r \leq \omega ,$$

$$k \leq s ,$$

$$|l - k| \leq \kappa \leq l + k ,$$

$$-\kappa \leq \kappa_0 \leq \kappa , \quad \omega \leq \omega_0 .$$  

(4.11a)

(4.11b)

(4.11c)

(4.11d)

The conditions (4.11a), (4.11b) and (4.11c) are discussed in the Appendix. The condition (4.11d) comes from the rules of the $su(2)$- and the $su(1,1)$-algebra. The conditions (4.11a) and (4.11b) will play a central role in §5 and 7, respectively.

The relation (2.18) teaches us that the energy eigenvalue is determined by $|\ell \sigma \omega \rangle$. Therefore, we pay special attention to this state. The state $|\ell \sigma \omega \rangle$ is constructed by operating the building blocks $\hat{S}_3^3$, $\hat{q}_1^1$ and $\hat{B}^*$ for $2l$-, $2s$- and $2r$-times, respectively, on the state $(\hat{b}^*)^{2\omega}|0\rangle$. Of course, the ordering of the operation is arbitrary and each building block plays its own role which was discussed in §3. However, $(\hat{S}_3, \hat{S}_3^3)$,
(\hat{q}_1, \hat{q}_1^\dagger) and \(\hat{B}, \hat{B}^*\) do not behave independently of one another. For example, we have

\[
\begin{align*}
[ [ \hat{q}_1, \hat{S}^3 ] , \hat{S}^3 ] &= 0 , \quad \hat{q}_1 (\hat{b}^*)^{2\omega} |0\rangle &= [ [ \hat{q}_1, \hat{S}^3 ] (\hat{b}^*)^{2\omega} |0\rangle = 0 , \quad (4.12) \\
[ [ \hat{S}_3 , \hat{q}_1 ] , \hat{q}_1^\dagger ] &= 0 , \quad \hat{S}_3 (\hat{b}^*)^{2\omega} |0\rangle &= [ [ \hat{S}_3 , \hat{q}_1^\dagger ] (\hat{b}^*)^{2\omega} |0\rangle = 0 . \quad (4.13)
\end{align*}
\]

The relations (4.12) and (4.13) give us

\[
\begin{align*}
\hat{q}_1 (\hat{S}^3)^{2l} (\hat{b}^*)^{2\omega} |0\rangle &= 0 , \\
\hat{S}_3 (\hat{q}_1^\dagger)^{2s} (\hat{b}^*)^{2\omega} |0\rangle &= 0 .
\end{align*}
\] (4.14)

Therefore, if restricting \(\{(\hat{S}^3)^{2l}(\hat{q}_1^\dagger)^{2s}(\hat{b}^*)^{2\omega}|0\rangle\}, (\hat{S}_3, \hat{q}_1^\dagger)\) and \((\hat{q}_1, \hat{q}_1^\dagger)\) behave independently of each other. But, if \((\hat{B}, \hat{B}^*)\) is included, the situation becomes complicated. For example, we have the relation

\[
\begin{align*}
\hat{q}_1 (\hat{B}^*)^{2r} (\hat{b}^*)^{2\omega} |0\rangle &= \hat{S}_3 (\hat{B}^*)^{2r-1} (\hat{b}^*)^{2\omega} |0\rangle , \\
\hat{S}_3 (\hat{B}^*)^{2r} (\hat{b}^*)^{2\omega} |0\rangle &= q^3 (\hat{B}^*)^{2r-1} (\hat{b}^*)^{2\omega} |0\rangle .
\end{align*}
\] (4.15)

The above tells us that our building blocks are not elementary but composite. In relation to the above consideration, the work presented by Pittel et al. may be interesting.

Finally, we present the energy eigenvalue based on the use of the expression (2.19). The state \(|lsr\omega\rangle\) is the eigenstate of \(\hat{P}^2\) and \(\hat{Q}^2\), the eigenvalues of which are the same as those shown in the relation (4.10a). Further, \(|lsr\omega\rangle\) is also the eigenstate of \(\hat{P}_0\), the eigenvalue of which is shown in the relation (4.2). Under the above consideration, the form (2.19) gives us the following energy eigenvalue:

\[
\begin{align*}
E_{lsr\omega}^{(m)} &= E_{lsr\omega} + \chi F_{ls} , \quad (4.16a) \\
E_{lsr\omega} &= -[2l(2(\omega - s - 2r - l) + 1) + 2r(2(\omega - r) + 3)] , \quad (4.16b) \\
F_{ls} &= 2l(l + 1) + \frac{2}{3}(l + 2s)(l + 2s + 3) . \quad (4.16c)
\end{align*}
\]

Concerning the energy eigenvalue (4.16a), we must note the following: the operators \((\hat{T}_+, \hat{Q}_+(lk\xi\eta\rho\zeta))\) and \((\hat{T}_+, \hat{R}_+, \hat{S}_1^2, \hat{S}_1^4)\) appearing in the forms (4.7) and (4.8), respectively, commute with \(\hat{H}\) and \(\hat{H}_m\) in the relations (2.16) and (2.17), respectively. Therefore, the state \(|lk\xi\eta\rho\zeta\rangle\) gives us the same energy eigenvalue as that given by \(|lsr\omega\rangle\). This means that these states are degenerate. Of course, there exist exceptions. Subject on this degeneracy will be taken up again in §§6 and 7.

§5. Relevance to the two forms presented in (I)

As was mentioned in §1, in (I), we have investigated the present system under the two forms. In this section, we will examine the relevance of the state (4.1) to the states (I.3-25) and (I.4-24). First, we discuss the case of the state (I.4-24), which is given as

\[
|\lambda \rho \sigma_0 \sigma_1 = (\hat{S}^3)^{2\lambda} (\hat{S}_1^4)^{2\rho} (\hat{b}_1)^{2(\sigma_1 - \sigma_0)} (\hat{b}^*)^{2\sigma_0} |0\rangle .
\] (5.1)

Here, we omit the interpretation of the quantum numbers. The operator \(\hat{S}^4\) is defined as

\[
\hat{S}^4 = \hat{S}^1 \hat{Q}_0 + \hat{S}^2 \hat{S}_1^2 + \hat{S}^3 \hat{S}_1^3 .
\] (5.2)
In (I), \( \hat{S}^4 \) played a central role for the description of the present system under the name of the pairing correlation. Our aim is to show that the state (5.1) can be re-formed to the state (4.1).

First, we note the following relation:

\[
[ \hat{q}^1 , \hat{S}^4 ] = \hat{B}^* , \quad [ \hat{B}^* , \hat{S}^4 ] = 0 , \quad [ \hat{B}^* , \hat{q}^1 ] = 0 . \tag{5.3}
\]

Further, we notice the formula

\[
(\hat{S}^4)^n(\hat{q}^1)^n = (-)^n(\hat{B}^* - \hat{q}^1 \hat{S}^4)(2\hat{B}^* - \hat{q}^1 \hat{S}^4) \cdots (n\hat{B}^* - \hat{q}^1 \hat{S}^4) . \tag{5.4}
\]

Proof of the formula (5.4) can be performed by the mathematical induction with the relation (5.3) and the relation shown as

\[
(m\hat{B}^* - \hat{q}^1 \hat{S}^4)\hat{q}^1 = \hat{q}^1((m + 1)\hat{B}^* - \hat{q}^1 \hat{S}^4) . \tag{5.5}
\]

With the use of the formula (5.4), for \( m \geq 0 \), we have

\[
(\hat{S}^4)^n(\hat{q}^1)^{n+m}(\hat{b}^* p)|0\rangle = (\hat{S}^4)^n(\hat{q}^1)^n \cdot (\hat{q}^1)^m(\hat{b}^* p)|0\rangle
\]

\[
= (-)^n \frac{(n+m)!}{m!}(\hat{B}^*)^n \cdot (\hat{q}^1)^m(\hat{b}^* p)|0\rangle . \tag{5.6}
\]

Here, we used

\[
\hat{S}^4(\hat{b}^* p)|0\rangle = 0 , \tag{5.7}
\]

\[
(k\hat{B}^* - \hat{q}^1 \hat{S}^4)(\hat{q}^1)^m(\hat{b}^* p)|0\rangle = (k + m)\hat{B}^*(\hat{q}^1)^m(\hat{b}^* p)|0\rangle . \tag{5.8}
\]

Further, for \( m \geq 1 \), we have

\[
(\hat{S}^4)^{m+n}(\hat{q}^1)^n(\hat{b}^* p)|0\rangle = (\hat{S}^4)^m \cdot (\hat{S}^4)^n(\hat{q}^1)^n(\hat{b}^* p)|0\rangle
\]

\[
= (\hat{S}^4)^m \cdot (-)^n(\hat{B}^*)^n(\hat{b}^* p)|0\rangle = (-)^n(\hat{B}^*)^n(\hat{S}^4)^m(\hat{b}^* p)|0\rangle = 0 . \tag{5.9}
\]

With the use of the operator \( \hat{q}^1 \), the state (5.1) can be expressed as

\[
|\lambda \rho \sigma_0 \sigma_1\rangle = \frac{(2\sigma_0)!}{(2\sigma_1)!} (\hat{S}^3)^{2\lambda}(\hat{q}^1)^{2\rho}(\hat{b}^* \sigma_1)|0\rangle . \tag{5.10}
\]

By reading \( n = 2\rho, \ n + m = 2(\sigma_1 - \sigma_0) \) and \( p = 2\sigma_1 \) in the formula (5.6), the state (5.10) can be expressed in the form

\[
|\lambda \rho \sigma_0 \sigma_1\rangle = (\hat{S}^3)^{2\lambda}(\hat{B}^*)^{2\rho}(\hat{q}^1)^{2(\sigma_1 - \sigma_0) - 2\rho}(\hat{b}^* \sigma_1)|0\rangle
\]

\[
= (\hat{S}^3)^{2\lambda}(\hat{q}^1)^{2(\sigma_1 - \sigma_0 - \rho)}(\hat{B}^*)^{2\rho}(\hat{b}^* 2\sigma_1)|0\rangle
\]

\[
= |\lambda, \sigma_1 - \sigma_0 - \rho, \rho, \sigma_1\rangle . \tag{5.11}
\]

Here, we used \([\hat{B}^*, \hat{q}^1] = 0\) and omitted the numerical factor such as factorial. We can see that the quantum numbers \( l, s, r \) and \( \omega \) in the state (4.1) are nothing but

\[
l = \lambda , \quad s = \sigma_1 - \sigma_0 - \rho , \quad r = \rho , \quad \omega = \sigma_1 . \tag{5.12}
\]
Then, the energy eigenvalue (4.16) becomes the same expression as that shown in the form (I.4.45) with (I.4.43b).

Next, we will investigate the relevance to the state (I.3.25), which is copied in the form

\[
\| \lambda \tau T \rangle = \left( \hat{O}_+ (t \tau) \right)^{T-(t+\tau)} | \lambda \rangle \otimes | t \rangle ,
\]

\[
\| \lambda \tau \rangle = ( \hat{a}_3^* )^{2 \lambda} ( \hat{b}_1^* )^{2 \tau - 2 \lambda} | 0 \rangle ,
\]

\[
| t \rangle = ( \hat{b}^* )^{2t - 1} | 0 \rangle .
\]

Here, \( \hat{O}_+ (t \tau) \) is defined in the relation (I.3.22). The states \( \| \lambda \tau T \rangle \), \( \| \lambda \tau \rangle \) and \( | t \rangle \) obey

\[
\hat{T}_- \| \lambda \tau T \rangle = 0 , \quad \hat{T}_0 \| \lambda \tau T \rangle = T \| \lambda \tau T \rangle ,
\]

\[
\hat{\tau}_- \| \lambda \tau \rangle = 0 , \quad \hat{\tau}_0 \| \lambda \tau \rangle = \tau \| \lambda \tau \rangle ,
\]

\[
\hat{\imath}_- | t \rangle = 0 , \quad \hat{\imath}_0 | t \rangle = t | t \rangle .
\]

\[
T = t + \tau , \quad t + \tau + 1 , \quad t + \tau + 2 , \ldots
\]

The states \( \| \lambda \tau T \rangle \), \( \| \lambda \tau \rangle \) and \( | t \rangle \) are the minimum weight states of the \( su(1,1) \)-algebra, \( \{ T_{\pm,0} (= \hat{\tau}_{\pm,0} + \hat{\imath}_{\pm,0}) \} \), \( \{ \hat{\tau}_{\pm,0} \} \) and \( \{ \hat{\imath}_{\pm,0} \} \) specified by \( T_0 = T \), \( \tau_0 = \tau \) and \( t_0 = t \), respectively. The \(( T - (t + \tau))\)-times operation of \( \hat{O}_+ (t \tau) \) on the state \( \| \lambda \tau \rangle \otimes | t \rangle \) gives us the state \( \| \lambda \tau T \rangle \).

On the other hand, the state (4.1) can be rewritten as

\[
| lsr \omega \rangle = ( \hat{B}^* )^{2r} ( \hat{S}^3 )^{2l} ( \hat{q}^1 )^{2s} ( \hat{b}^* )^{2\omega} | 0 \rangle
\]

\[
= ( \hat{B}^* )^{2r} ( \hat{a}_3^* )^{2l} ( \hat{b}_1^* )^{2s} ( \hat{b}^* )^{2(\omega - (s + l))} | 0 \rangle .
\]

The state \( | lsr \omega \rangle \) satisfies

\[
\hat{T}_- | lsr \omega \rangle = 0 , \quad \hat{T}_0 | lsr \omega \rangle = T | lsr \omega \rangle , \quad T = \omega + 2 .
\]

Further, \( | lsr \omega \rangle \) can be rewritten in the form

\[
| lsr \omega \rangle = ( \hat{B}^* )^{2r} ( \hat{b}^* )^{4r} | l \tau \rangle \otimes | t \rangle .
\]

Here, \( \tau \) and \( t \) are defined through

\[
2\tau - 3 = 2(s + l) , \quad 2t - 1 = 2(\omega - (l + s) - 2r) .
\]

The relations (5.20) and (5.22) give us

\[
T = t + \tau + 2r , \quad i.e., \quad 2r = T - (t + \tau) .
\]

Since \( | lsr \omega \rangle \) is the minimum weight state of the \( su(1,1) \)-algebra \( \{ \hat{T}_{\pm,0} \} \), we have the following expression:

\[
| lsr \omega \rangle = \left( \hat{O}_+ (t \tau) \right)^{T-(t+\tau)} | l \tau \rangle \otimes | t \rangle .
\]

The state (5.24) is the state (5.13) itself. It may be interesting to see that the operation of \( ( \hat{B}^* )^{2r} ( \hat{b}^* )^{4r} \) on the state \( | l \tau \rangle \otimes | t \rangle \) is equivalent to that of \( ( \hat{O}_+ (t \tau) )^{T-(t+\tau)} \).
From the above argument, we have the correspondence
\[ l = \lambda, \quad s = \frac{1}{2}(2\tau - 3 - 2\lambda), \quad r = \frac{1}{2}(T - t - \tau), \quad \omega = T - 2. \] (5.25)

Of course, the energy eigenvalue (4.16) under the relation (5.25) is identical to the form (I-3.29). However, we must notice the re-formation from the form (5.19) to the form (5.21). Since \( 2t - 1 \geq 0 \), the relation (5.22) gives us
\[ r \leq \frac{1}{2}(\omega - (l + s)). \] (5.26)

This means that the form (5.21) is valid under the condition (5.26). However, our present system obeys the condition (4.11a), which is rewritten as \( r \leq \omega - (l + s) \). Therefore, inevitably, we must investigate the case \( (\omega - (l + s))/2 \leq r \leq \omega - (l + s) \).

In connection with the state \( |l\tau r\omega\rangle \), we introduce the following state in the present boson space:
\[ |l\tau r\omega\rangle = (\hat{B})^2(\hat{S}_1)^2s(\hat{\mathcal{B}})^2(\omega-(l+s)-r)(\hat{a}^*)^2\omega|0\rangle. \] (5.27)

The exponent \( 2(\omega -(l + s) - r) \) in the state (5.27) should be positive or zero, and then, we have
\[ r \leq \omega - (l + s). \] (5.28)

The relation (5.28) is nothing but the condition (4.11a). Straightforward calculation gives us the following relations:
\[ \hat{P}_0|l\tau r\omega\rangle = -2\left[ \omega - 2\left( r + \frac{1}{3}(2l + s) \right) \right]|l\tau r\omega\rangle, \] (5.29)
\[ \hat{Q}_0|l\tau r\omega\rangle = -(l + 2s)|l\tau r\omega\rangle, \quad \hat{R}_0|l\tau r\omega\rangle = -l|l\tau r\omega\rangle, \] (5.30)
\[ \hat{T}_0|l\tau r\omega\rangle = (\omega + 2)|l\tau r\omega\rangle, \] (5.31)
\[ \hat{S}_2^1|l\tau r\omega\rangle = \hat{S}_2^2|l\tau r\omega\rangle = \hat{R}_-|l\tau r\omega\rangle = 0, \] (5.32)
\[ \hat{T}_-|l\tau r\omega\rangle = 0. \] (5.33)

If we compare the relations (5.29)–(5.33) with the relations (4.2)–(4.6), the state \( |l\tau r\omega\rangle \) should be identical with the state \( |l\tau r\omega\rangle \) except the normalization constant:
\[ |l\tau r\omega\rangle = |l\tau r\omega\rangle. \] (5.34)

Of course, we can define \( |k\kappa\kappa_0lr\tau r\omega\omega_0\rangle \) in the same form as \( |k\kappa\kappa_0lr\tau r\omega\omega_0\rangle \). The state \( |l\tau r\omega\rangle \) is the re-formed expression of \( |l\tau r\omega\rangle \).

Now, in the same idea as that for the form (5.19), we can rewrite the state (5.27):
\[ |l\tau r\omega\rangle = (\hat{B})^2(\omega-(l+s)-r)(\hat{a}^*)^4(\omega-(l+s)-r)|l\tau \otimes |t\rangle). \] (5.35)

Here, \( |l\tau \rangle \) and \( \tau \) are given in the relations (5.14) and (5.22), respectively, and \( |t\rangle \) and \( t \) are defined as
\[ |t\rangle = (\hat{a}^*)^{2t-1}|0\rangle, \quad 2t - 1 = 2(2r + (l + s) - \omega). \] (5.36)
Of course, as is shown in the relations (5.31) and (5.33), $|lsrw\rangle$ is the minimum state for $\{\tilde{T}_{\pm,0}\}$ with $T = \omega + 2$. In the same way as the relation (5.23), we have

$$T = t + \tau + 2(\omega - (l + s) - r) , \quad \text{i.e.,} \quad 2(\omega - (l + s) - r) = T - (t + \tau) . \quad (5.37)$$

Therefore, $|lsrw\rangle$ can be expressed in the form

$$|lsrw\rangle = (\hat{O}_+(t\tau))^{T-(t+\tau)}|l\rangle \otimes |t\rangle . \quad (5.38)$$

The conditions $2t - 1 \geq 0$ for the relation (5.36) and (5.28) give us the following inequality:

$$\frac{1}{2}(\omega - (l + s)) \leq r \leq \omega - (l + s) . \quad (5.39)$$

Thus, we arrived at our aim.

From the above argument, we have the correspondence

$$l = \lambda , \quad s = \frac{1}{2}(2\tau - 3 - 2\lambda) , \quad r = \frac{1}{2}(T - (1-t) - \tau) , \quad \omega = T - 2 . \quad (5.40)$$

The correspondence (5.40) should be compared with that shown in the relation (5.25). As was demonstrated in the relations (I.3.43)–(I.3.47), the quantum number $t$ is replaced with $(1-t)$ in the quantum number $r$. Of course, the energy eigenvalue (4.16) under the relation (5.40) becomes the form (I.3.48).

§6. Expressions in terms of the degeneracy $\Omega$ and the quark numbers $N_1$, $N_2$ and $N_3$

As were shown in the relations (4.2)–(4.4), the state $|lsrw\rangle$ is an eigenstate of $\hat{P}_0$, $\hat{Q}_0$, $\hat{R}_0$ and $\hat{T}_0$. But, the eigenvalues of these operators are not directly connected to the present many-quark model. In order to make the connection, we take up the operators $\hat{O}$, $\hat{N}_1$, $\hat{N}_2$ and $\hat{N}_3$ presented in the relations (2.2) and (2.3). For obtaining their eigenvalues, the following formulae are useful:

$$[\hat{O}, \hat{S}^3] = [\hat{O}, \hat{q}^1] = [\hat{O}, \hat{B}^*] = 0 ,$$

$$[\hat{O}, \hat{b}^*] = \frac{1}{2}\hat{b}^* , \quad \hat{O}|0\rangle = n_0|0\rangle , \quad (6.1)$$

$$[\hat{N}_i, \hat{S}^3] = (1 - \delta_{i3})\hat{S}^3 , \quad [\hat{N}_i, \hat{q}^1] = \delta_{i1}\hat{q}^1 , \quad [\hat{N}_i, \hat{B}^*] = \hat{B}^* , \quad (6.2)$$

Here, some of the above relations are copied from the relations (3.13) and (3.15). The state $|lsrw\rangle$ is the eigenstate of $\hat{O}$, $\hat{N}_1$, $\hat{N}_2$ and $\hat{N}_3$ and their eigenvalues $\Omega$, $N_1$, $N_2$ and $N_3$ are given in the form

$$\Omega = n_0 + \omega \quad , \quad (6.3)$$

$$N_1 = n_0 + 2l + 2s + 2r \quad , \quad N_2 = n_0 + 2l + 2r \quad , \quad N_3 = n_0 + 2r \quad . \quad (6.4)$$

Conversely, we have

$$\omega = \Omega - n_0 \quad , \quad (6.5)$$

$$2s = N_1 - N_2 \quad , \quad 2l = N_2 - N_3 \quad , \quad 2r = N_3 - n_0 . \quad (6.6)$$
Then, $|lsr\omega\rangle$ can be rewritten as

$$
|lsr\omega\rangle = (\hat{q}^1)^{N_1-2N_2}(\hat{S}^3)^{N_2-N_3}(\hat{B}^*)(N_3-n_0)(\Omega-n_0)|0\rangle .
$$

(6.7)

In the expression (6.7), the order of $\hat{q}^1$, $\hat{S}^3$ and $\hat{B}^*$ is changed. The eigenvalues of $\hat{H}$ and $\hat{Q}^2$ are expressed in terms of the new quantum numbers by substituting the expressions (6.5) and (6.6) to the relations (4.16b) and (4.16c).

Since $2s$, $2l$, $2r$, $2\omega \geq 0$, we have the inequalities

$$
N_1 \geq N_2 \geq N_3 \geq n_0 ,
$$

(6.8)

$$
\Omega \geq n_0 .
$$

(6.9)

Figure 1 shows the behaviors of $N_1$, $N_2$, $N_3$ and $n_0$. The quantities $(N_1 - N_2)$, $(N_2 - N_3)$, $(N_3 - n_0)$ and $n_0$ denote the single-quark number in color 1, the quark-pair number in the pair (1,2), the quark-triplet number and the quark-triplet in $\Delta$-excitation in the sense of the idea proposed by Petry et al. in the state $|lsr\omega\rangle$.

By varying $N_1$, $N_2$, $N_3$ and $n_0$, we obtain various phases. For example, in the case $N_1 = N_2 = N_3$ and $n_0 = 0$, the system consists of only the triplets. As the other extreme case, there exists the case $N_2 = N_3 = n_0 = 0$. In this case, the system consists only of the single-quarks. If $N_1 = N_2$ and $N_3 = n_0 = 0$, the system is composed only of the quark-pairs. Last two cases are in colored states. Of course, the first case is in color-neutral state. Anyhow, it may be interesting to investigate these various phases systematically.

Next, we investigate the eigenvalues of $\hat{\Omega}$, $\hat{N}_1$, $\hat{N}_2$ and $\hat{N}_3$ calculated by the state (4.7). The part $(\hat{T}_+)^{\omega_0-\omega}$ does not give any effect because the sub-space with $\omega_0 = \omega$, i.e., $T_0 = T$ corresponds to the original fermion space. Judging from the role of the $su(1,1)$-algebra in the present formalism, it may be natural. Therefore, hereafter, we regard $\omega_0$ as $\omega$. The eigenvalue of $\hat{\Omega}$ does not change from $\Omega$ calculated in the state $|lsr\omega\rangle$. Therefore, the problem is reduced to investigating the effect of $\hat{Q}_+(lk\kappa\kappa_0)$ on $N_1$, $N_2$ and $N_3$. Straightforward calculation gives us

$$
[ \hat{N}_1 , \hat{Q}_+(lk\kappa\kappa_0) ] = -2k\hat{Q}_+(lk\kappa\kappa_0) ,
$$

(6.10)

Fig. 1. The classification of the inequalities (6.8) and (6.9).
\[ [\hat{N}_2, \hat{Q}_+(lk\kappa_0)] = (k - (l + \kappa_0))\hat{Q}_+(lk\kappa_0), \]
\[ [\hat{N}_3, \hat{Q}_+(lk\kappa_0)] = (k + (l + \kappa_0))\hat{Q}_+(lk\kappa_0). \]

Therefore, the eigenvalues of \( \hat{N}_1, \hat{N}_2 \) and \( \hat{N}_3 \) for the state \( \hat{Q}_+(lk\kappa_0)|lsr\omega\rangle \), which are denoted as \( N_1^0, N_2^0 \) and \( N_3^0 \), respectively, are as follows:

\[ N_1^0 = N_1 - 2k, \]
\[ N_2^0 = N_2 + k - (l + \kappa_0), \]
\[ N_3^0 = N_3 + k + (l + \kappa_0). \]

Of course, we have \( \sum_i N_i^0 = \sum_i N_i = N \). Energetically, the state \( \hat{Q}_+(lk\kappa_0)|lsr\omega\rangle \) is degenerate to \( |lsr\omega\rangle \), but, the quark number distribution fluctuates around the distribution \( (N_1, N_2, N_3) \) which determine the energy eigenvalue.

As a closing note in this section, we must mention a small comment. The state (4.1) gives us the relation (6.8). Then, we have the following question: How do we treat the cases which do not obey the relation (4.1), for example, \( N_2 \geq N_1 \geq N_3 \)? If the colors \( i = 1 \) and \( i = 2 \) read the colors \( i = 2 \) and \( i = 1 \), respectively, we can treat this case in a way similar to the case (6.8). The other cases are also the same. Therefore, it may be enough to treat only the case (6.8).

\section*{7. Concluding remarks}

In this paper, we have reformulated the Bonn model and its modification developed in (I). We could show the energy eigenstates in a unified form and understood their structures in relation to the quark numbers in colors \( i = 1, 2 \) and 3. In (I), we showed various inequalities for the physical quantities used in (I). Of course, these relations are available in the present form. In this Bonn model and its modification, both the quark triplet and quark pairing states are treated in an equal footing. Of course, non-color singlet states are contained in these models. We will investigate the minimum energy states in these models, in which the color neutral states are realized in a certain condition. However, the study of the non-color-singlet states themselves will be useful for the regime of color superconductivity. This is an open problem.

As a final remark, we will discuss a second modification of the original Bonn model. The Hamiltonian may be defined as

\[ \hat{H}_{m'} = \hat{H} + \chi \hat{Q}^{2'}, \]
\[ \hat{Q}^{2'} = 2(\hat{S}_1^2\hat{S}_2^1 + \hat{S}_1^3\hat{S}_3^1 + \hat{R}_+\hat{R}_-) \]
\[ = \hat{Q}^2 - \left[ \frac{2}{3}\hat{Q}_0(\hat{Q}_0 - 3) + 2\hat{R}_0(\hat{R}_0 - 1) \right], \]

where \( \hat{Q}^2 \) is the Casimir operator of the \( su(3) \) algebra and is defined in Eq. (2.13). Then, we have

\[ [\hat{S}_i^j, \hat{H}_{m'}] \neq 0. \quad (i \neq j) \]
Judging from the Hamiltonian (7.1), we can diagonalize $\hat{H}_{m'}$ in the framework of the orthogonal set (4.7) or (4.8) with (4.1). Of course, we are interested in the case $\omega_0 = \omega$: $|kK\theta\omega_0\rangle = |kK\theta\omega\rangle$. We can calculate the energy eigenvalue of $\hat{Q}^{2'}$ defined in the relation (7.2) by using the relation (4.10a). Then, the energy eigenvalue is expressed in the form

$$E_{lsr\omega}^{(m')} = E_{lsr\omega} + \chi F_{ls}^{kK\theta\omega} \ ,$$

(7.4a)

$$F_{ls}^{kK\theta\omega} = 2l(l + 1) + 2k[(2l + 4s + 3) - 3k] - 2\kappa_0(\kappa_0 - 1) .$$

(7.4b)

Here, $E_{lsr\omega}$ is given in the form (4.10a). The term $F_{ls}^{kK\theta\omega}$ can be rewritten as

$$F_{ls}^{kK\theta\omega} = 4k(2s - k + 1) + 2(k + l - \kappa)(k + l + \kappa + 1) + 2(\kappa + \kappa_0)(\kappa - \kappa_0 + 1) .$$

(7.5)

The conditions (4.11b)–(4.11d) lead us to $F_{ls}^{kK\theta\omega} \geq 0$, which is, of course, consistent with the positive-definiteness of $\hat{Q}^{2'}$ shown in the form (7.2). The case ($k = 0$, $\kappa = l$, $\kappa_0 = -l$) gives us $F_{ls}^{0l-l} = 0$.

In the case of the Bonn model and its modification, the states $|kK\theta\omega_0\rangle$ are energetically degenerate to the state $|lsr\omega\rangle$ ($= |0l - llsr\omega\rangle$) and the energy eigenvalues are given in the form (4.16). However, in the case the second modification, the degeneracy disappears. Typically, such a sector forms the rotational band and the state with $I = K$ is called band head. In our present case, the state $|lsr\omega\rangle$ may be called the band head and on this band head, the band structure is formed with the energy $F_{ls}^{kK\theta\omega}$.

$$\hat{H} = E(\hat{l}^2_z) + \frac{1}{2G}(\hat{l}^2_x + \hat{l}^2_y) = E(\hat{l}^2_z) + \frac{1}{2G} \hat{l}^2 - \frac{1}{2G} \hat{l}^2_z ,$$

(7.6)

where $(\hat{l}_x, \hat{l}_y, \hat{l}_z)$ denotes angular momentum operator and $\hat{l}^2$ is the Casimir operator of the $su(2)$ algebra formed by this angular momentum operator. Here, $(x, y, z)$ shows the axes in the body-fixed frame. This operator obeys the relations $[\hat{l}_y, \hat{l}_z] = -i\hat{l}_x (x, y, z: \text{cyclic})$. The quantity $G$ denotes the moment of inertia. The eigenvalue of $\hat{H}$ is given in the form

$$E_{IK} = E(K^2) + \frac{1}{2G}(I(I + 1) - K^2) , \quad I = K, K + 1, K + 2, \cdots . \quad (K \neq 0)$$

(7.7)

If there does not exist the rotational term, the energy eigenvalue is given by $E(K^2)$ and all the states with $I = K$, $K + 1$, $K + 2$, $\cdots$ are degenerate. But, if the rotational term is switched on, the degenerate energies are split and the rotational states are constructed on the state with $I = K$. Usually, such a sector forms the rotational band and the state with $I = K$ is called band head. In our present case, the state $|lsr\omega\rangle$ may be called the band head and on this band head, the band structure is formed with the energy $F_{ls}^{kK\theta\omega}$.
As was shown in the above, the form discussed in this section may be permitted as a modified Bonn model. In the case of the nuclear rotational motion of the axially symmetric deformed nuclei, the ground state has no three-dimensional rotational symmetry which is described by the $su(2)$ algebra. However, the rotational symmetry around $z$ axis is rest and this symmetry is described by the $u(1)$ algebra. Actually, the Hamiltonian (7.6) has still $u(1)$ symmetry, but no $su(2)$ symmetry. The same situation is realized in the modified Bonn model with the second modification represented by Eqs. (7.1) and (7.2). As is pointed out in §3, for the building blocks of the energy eigenstates, the operators $\hat{S}^3$ and $\hat{q}^1$ are not color $su(3)$ neutral. Therefore, the energy eigenstates are not the color $su(3)$ singlet in general. Further, the operator $\hat{S}^3$ is not $su(2)$ scalar, while $\hat{B}^*$ and $\hat{q}^1$ are still $su(2)$-scalars. Thus, the eigenstates has no longer the $su(2)$ symmetry as a sub-algebra of the color $su(3)$ algebra. However, the eigenstates are still possessed of a certain symmetry. Actually, the operators $\hat{Q}_0$ and $\hat{R}_0$ appearing in Eqs. (7.1) and (7.2) correspond to the two generators with only diagonal elements in the $su(3)$ algebra, namely form the Cartan-Weyl sub-algebra. Thus, the second modification in Eqs. (7.1) and (7.2) has only $u(1) \times u(1)$ symmetry while the eigenstates has no $su(3)$ and $su(2)$ symmetry. The above treatment may parallel the treatment of the nuclear rotational motion with axially symmetric deformed nuclei developed in the theory of nuclear collective motion.

Acknowledgements

One of the authors (Y.T.) is partially supported by the Grant-in-Aid of the Scientific Research No.18540278 from the Ministry of Education, Culture, Sports, Science and Technology in Japan.

Appendix A

Proofs of Various Formulae

A.1. Proof of the form (4.8) with the conditions (4.11c) and (4.11d)

First, we note the following relations:

\[ \hat{R}_- |lsr\omega\rangle = 0 , \quad \hat{R}_0 |lsr\omega\rangle = -l |lsr\omega\rangle , \]  
\[ \hat{S}^1_2 |lsr\omega\rangle = 0 , \quad \hat{S}^1_4 |lsr\omega\rangle = 0 . \]  

Here, $\hat{S}^1_4$ denotes Hermite conjugate of $\hat{S}^1_4$. Further, we have

\[ [ \hat{R}_- , \hat{S}^2_1 ] = 0 , \quad [ \hat{R}_- , \hat{S}^4_1 ] = \hat{S}^3_1 \hat{R}_-, \]  
\[ [ \hat{R}_0 , \hat{S}^2_1 ] = -\frac{1}{2} \hat{S}^2_1 , \quad [ \hat{R}_0 , \hat{S}^4_1 ] = \frac{1}{2} \hat{S}^4_1 . \]

The relations (A.3) and (A.4) give us

\[ [ \hat{R}_- , (\hat{S}^2_1)^n ] = 0 , \quad [ \hat{R}_- , (\hat{S}^4_1)^n ] = \phi_n \cdot \hat{R}_- , \]  
\[ [ \hat{R}_0 , (\hat{S}^2_1)^n ] = -\frac{n}{2} (\hat{S}^2_1)^n , \quad [ \hat{R}_0 , (\hat{S}^4_1)^n ] = \frac{n}{2} (\hat{S}^4_1)^n . \]
The operator $\hat{\Phi}_n$ denotes a certain function of $\hat{S}_1^3$ and $\hat{S}_1^4$, but, in the present argument, the explicit form is not necessary.

With the use of the above relations, we obtain the relation
\[ \hat{R}_- \|kkl rs\rangle = 0 , \quad \hat{R}_0 \|kkl rs\rangle = -\kappa \|kkl rs\rangle, \] (A-7)
\[ \kappa \geq 0 . \] (A-8)

Here, $\|kkl rs\rangle$ is defined as
\[ \|kkl rs\rangle = (\hat{S}_1^2)^{k-l+\kappa}(\hat{S}_1^4)^{k+l-\kappa} \|lsr\rangle . \] (A-9)

The state $\|kkl rs\rangle$ is nothing but the minimum weight state of $(\hat{R}_\pm, 0)$, in which $-\kappa$ denotes the eigenvalue of $\hat{R}_0$. This state is also the minimum weight state of $(\hat{T}_\pm, 0)$, in which $\omega$ denotes the eigenvalue of $\hat{T}_0$. Then, with the use of the raising operators $\hat{R}_+$ and $\hat{T}_+$, we have the form (4.8) with the condition (4.11d).

We, further, note the relation
\[ \hat{R}_- (\hat{S}_1^4)^{k+l-\kappa} \|lsr\rangle = 0, \]
\[ \hat{R}_0 (\hat{S}_1^4)^{k+l-\kappa} \|lsr\rangle = -\frac{1}{2}(l-k+\kappa)(\hat{S}_1^4)^{k+l-\kappa} \|lsr\rangle . \] (A-10)

We can see that $(\hat{S}_1^4)^{k+l-\kappa} \|lsr\rangle$ is also the minimum weight state of $(\hat{R}_\pm, 0)$ with the eigenvalue of $\hat{R}_0$ being $-(l-k+\kappa)/2$. Therefore, in the same way as the condition (A-8), we have
\[ l-k+\kappa \geq 0 . \] (A-11)

The exponents $(k-l+\kappa)$ and $(k+l-\kappa)$ in the state (A-9) should be positive or zero:
\[ k-l+\kappa \geq 0 , \quad k+l-\kappa \geq 0 . \] (A-12)

The inequalities (A-11) and (A-12) lead us to the condition (4.11c). As is clear from the relation (4.10a), $k$ is related to the eigenvalue of $\hat{Q}_0$. Since $[\hat{S}_1^2, \hat{S}_1^4] = 0$, we have
\[ \|kkl rs\rangle = (\hat{S}_1^2)^{k+l+\kappa}(\hat{S}_1^2)^{k-l+\kappa} \|lsr\rangle . \] (A-13)

The case $(\hat{S}_1^2)^{k+l+\kappa} \|lsr\rangle$ is also treated in the same form as the case $(\hat{S}_1^4)^{k+l-\kappa} \|lsr\rangle$. In this case, the eigenvalue of $\hat{R}_0$ is given as $-(k+l+\kappa)/2$, which is automatically negative. Therefore, any condition is not obtained. The idea presented in this Appendix is very similar to that developed by the present authors.

A.2. Derivation of the eigenvalue of $\hat{P}^2$ shown in the relation (4.10)

First, we notice that straightforward calculation gives us the following expression:
\[ \|k\kappa_0 lsr\omega_0\rangle = (\hat{T}_+)^{\omega_0-\omega}(\hat{R}_+)^{\kappa+l}\kappa_0(\hat{S}_1^2)^{k-l+\kappa}(\hat{S}_1^4)^{k+l-\kappa} \times (\hat{S}_1^3)^2 \hat{P}_+(sr\omega) \|sr\omega\rangle . \] (A-14)
Here, \( \hat{P}_+(sr\omega) \) and \( \|sr\omega\rangle \) are defined as
\[
\hat{P}_+(sr\omega) = \left( \frac{2\omega}{2(s+r)} \right) \sum_{(n)} (2r)!(2s+n_1)!
\times (\hat{S}^1)^{n_1}(\hat{S}^2)^{n_2}(\hat{S}^3)^{n_3},
\]
\[
\|sr\omega\rangle = (\hat{a}^1)^{(s+r)}(\hat{b}^*)^2\omega|0\rangle.
\]
(A-15) \hspace{1cm} (A-16)

The symbol \( \sum_{(n)'} \) denotes the sum restricted \( n_1 + n_2 + n_3 = 2r \). If we note that the \( su(1,1) \)- and the \( su(4) \)-generators commute with \( \hat{P}_+^2 \) and the state \( \|sr\omega\rangle \) is the minimum weight state for the \( su(4) \)-algebra, the eigenvalue of \( \hat{P}_+^2 \) for \( \|kkk0sr\omega\omega0\rangle \) is equal to that for \( \|sr\omega\rangle \) and we have
\[
\hat{P}_+^2\|sr\omega\rangle = [(2(s+r) - \omega)^2 + 2\omega(\omega + 3)]\|sr\omega\rangle.
\]
(A-17)

The above result is nothing but the result shown in the relation (4.10).

A.3. Derivation of the inequality (4.11a)

The state \( \|lsr\omega\rangle \) can be rewritten as
\[
\|lsr\omega\rangle = (\hat{B}^*)^{2r}(\hat{a}_3^*)^{2l}(\hat{b}_1^*)^{2s}(\hat{b}^*)^{2(\omega - (l+s))}|0\rangle.
\]
(A-18)

The operator \( \hat{B}^* \) can be expressed explicitly as follows:
\[
\hat{B}^* = \hat{r}_+ \hat{b}^2 - 2\hat{r}_0 \hat{a}^* \hat{b} + \hat{r}_- \hat{a}^{*2}.
\]
(A-19)

Noticing \( \hat{r}_-(\hat{a}_3^*)^{2l}(\hat{b}_1^*)^{2s}(\hat{b}^*)^{2(\omega - (l+s))}|0\rangle = 0 \), the expression (A-18) can be rewritten in the form
\[
\|lsr\omega\rangle = \left[ \hat{c}_0 \cdot (\hat{b})^{4r} + \hat{c}_1 \cdot \hat{a}^* (\hat{b})^{4r-1} + \cdots + \hat{c}_{2r} \cdot (\hat{a}^*)^{2r} (\hat{b})^{2r} \right]
\times (\hat{a}_3^*)^{2l}(\hat{b}_1^*)^{2s}(\hat{b}^*)^{2(\omega - (l+s))}|0\rangle.
\]
(A-20)

Here, \( \hat{c}_k \) \((k = 0, 1, \cdots, 2r)\) denotes a certain function of \( \hat{r}_+ \) and \( \hat{r}_0 \). In the present argument, the concrete expression is not necessary to show. Then, the relation (A-20) shows us that the state \( \|lsr\omega\rangle \) does not vanish if there exists the condition
\[
2(\omega - (l+s)) - 2r \geq 0 \quad \text{ i.e., } \quad l + s + r \leq \omega.
\]
(A-21)

The relation (A-21) is nothing but the inequality (4.11a). We can see that under the condition (A-21), the exponent \( 2(\omega - (l+s)) \) in the relation (A-18) is positive or zero.

A.4. Derivation of the inequality (4.11b)

In (I), the following inequality was presented in the relation (I.3-17):
\[
2(l + k) \leq 2\tau - 3.
\]
(A-22)

The notations are changed from the original. The proof was given in Appendix B of (I). On the other hand, in the relation (5.22), \( 2\tau - 3 \) is shown in the form
\[
2\tau - 3 = 2(s + l).
\]
(A-23)
Then, we have

\[ 2(l + k) \leq 2(s + l), \text{ i.e., } k \leq s. \tag{A.24} \]

The relation (A.24) is nothing but the inequality (4.11b).

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