Real Time Correlators in Hot (2+1)d QCD

T.H. Hansson\footnote{Supported by the Swedish Natural Science Research Council.}, J. Wirstam
Institute of Theoretical Physics
University of Stockholm
Box 6730, S-113 85 Stockholm, Sweden

I. Zahed\footnote{Supported by the US Department of Energy under Grant No. DE-FG02-88ER40388.}
Department of Physics
SUNY at Stony Brook
Stony Brook, New York, 11794, USA

ABSTRACT

We use dimensional reduction techniques to relate real time finite $T$ correlation functions in (2+1) dimensional QCD to bound state parameters in a generalized ’t Hooft model with an infinite number of heavy quark and adjoint scalar fields. While static susceptibilities and correlation functions of the DeTar type can be calculated using only the light (static) gluonic modes, the dynamical correlators require the inclusion of the heavy modes. In particular we demonstrate that the leading $T$ perturbative result can be understood in terms of the bound states of the 2d model and that consistency requires bound state trajectories composed of both quarks and adjoint scalars. We also propose a non-perturbative expression for the dynamical DeTar correlators at small spatial momenta.
1. Introduction

About 10 years ago, DeTar [1] conjectured that the finite T excitation spectrum of QCD was much richer than previously expected - excitations with wavelengths longer than the magnetic scale $1/g^2 T$ are color singlets, at the electric scale $1/gT$ there are colored plasmons and at the hard scale $1/T$ colored quarks and gluons. He also proposed to test this conjecture by studying color singlet current-current correlators of the type

$$
G^{\alpha\alpha}(\vec{x}) = \frac{1}{\beta^2} \left\langle \int_0^\beta d\tau \bar{\psi}(\vec{x}, \tau) \Gamma^\alpha \psi(\vec{x}, \tau) \int_0^\beta d\tau' \bar{\psi}(\vec{0}, \tau') \Gamma^\alpha \psi(\vec{0}, \tau') \right\rangle,
$$

where $\langle \cdots \rangle$ is the average over the Gibbs ensemble. Lattice measurements of these “De-Tar correlators” in Euclidean space, relevant for studying screening effects in the finite temperature theory, suggest that the screening masses are about $2\pi T$ for the light vector and pseudovector excitations, and smaller for the scalar and pseudoscalar.

Some years ago, it was suggested [2, 3] that the lattice results on the screening lengths and the spatial extension of the static correlators at high temperature could be understood in terms of a dimensionally reduced theory which is known to confine in the high temperature limit. It was shown that the static correlation functions can be calculated from an effective theory involving nonrelativistic quarks with colored Coulomb interactions. The screening lengths were found to asymptote the free quark values, with spatially correlated quarks in the transverse directions at all temperatures. These ideas have been further developed in several papers, notably those of Huang and Lissia [4, 5], who have studied static properties at high temperature in QCD and in the Gross-Neveu model.

In spite of our increased understanding of the static aspects of high temperature QCD, DeTar’s main point, namely that the high temperature phase of QCD is confined in the sense that all low energy excitations are color singlets, is still neither proven nor disproven. The main difficulty is that although there are many sophisticated and efficient techniques for calculating Euclidean correlation functions, there are no good methods, neither analytical nor numerical, to obtain the real time, finite T correlators. In principle these can be obtained by analytically continuing the Euclidean functions, as is done.
routinely in perturbation theory, but on the lattice this is very difficult without having a rather detailed *a priori* knowledge about the analytic structure of the finite T spectral functions (for some alternative attempts in QCD see [6] and in simple model systems see [7]).

There are several lines of attack on this problem. The most direct is perturbation theory, where there are well developed methods for calculating real time correlators [8], but since QCD is known to be nonperturbative at large distances even at infinite $T$, this method is of limited value. Another way is to make some specific assumption about the spectral functions, as for instance pole dominance with $T$ dependent masses, and then use sum-rule techniques to check the consistency of the assumptions and estimate spectral parameters in terms of condensate values. This method is likely to be good at temperatures well below $T_c$ [9, 10, 11, 12], but has also been attempted at large $T$ [13]. The last approach is to attempt exact, or at least controlled, analytical calculations in simple models, with the hope that some of the features can be generalized to realistic cases. The present paper is devoted to a study of this kind using 2+1 dimensional QCD as a model theory.

The main idea of our calculation is to express the real time finite $T$ correlators in 2+1 d QCD in terms of spectral parameters (bound state masses, form factors) of a generalized 't Hooft model, which is obtained simply by rewriting the theory in 2d language. This is of course only advantageous if the heavy (*i.e.* nonstatic) modes can be totally, or to a large extent, ignored in the high $T$ limit. This is known to be the case for static observables, and we shall argue that for zero two-momentum, $\vec{q} = 0$, this is true also for certain dynamical correlation functions in the sense that only the heavy modes of the quark fields have to be retained. This claim is based on a comparison between the high temperature limit of perturbation theory in the 2+1 d model and the generalized 't Hooft model. This comparison also shows that for nonzero $\vec{q}$, the heavy scalars (originating from the non-static gluon modes) must be retained, which implies that the two-dimensional method is of limited value. However, in this case we can show that consistency requires
the existence of linear trajectories of bound states containing heavy adjoint scalars in addition to quarks and antiquarks. We shall briefly discuss this somewhat surprising result in relation to the recent work on QCD with adjoint scalars.

In the next section we show how to reformulate 2+1 d QCD in a two dimensional language, recall some basic results about the ’t Hooft model and generalize certain formulas to the case of unequal mass bound states. Section 3 is devoted to a study of the $\vec{x}$ and $t$ dependent quark susceptibility which is calculated both in perturbation theory and from the 2d bound states. In section 4 we suggest a non-perturbative expression for non-static DeTar correlators at small $\vec{q}$, and we close in section 5 with a discussion of our results and some general comments.

2. Dimensional reduction

Our starting point is the Euclidean action for massless QCD$_3$, 

$$S_{3E} = \int d^3\hat{x} \mathcal{L}_{3E}(\hat{A}^\mu, \hat{\psi}, \hat{\bar{\psi}})$$

(2.1)

where

$$\mathcal{L}_{3E}(\hat{A}^\mu, \hat{\psi}, \hat{\bar{\psi}}) = \frac{1}{2} \text{Tr} \hat{F}_{\alpha\beta} \hat{F}^{\alpha\beta} + \hat{\bar{\psi}}(\gamma_\alpha \partial_\alpha + g_3 \gamma_\alpha \hat{A}_\alpha) \hat{\psi},$$

(2.2)

with the conventions $x^\alpha = (x^i, \tau)$, $\alpha, \beta = 1, 2, 3$ and the gamma matrices are defined by $\gamma_1, \gamma_2, \gamma_3 = (-\sigma^1, \sigma^2, \sigma^3)$ with $\sigma^i$ the usual Pauli matrices. $g_3$ is the (dimensionful) coupling constant in three dimensions. Next we simply rewrite the action in two-dimensional language, using dimensional reduction techniques. We differ however from e.g. ref. [2] in that we keep all the nonstatic modes as well. These heavy modes will then manifest themselves as massive fields in the 2d action. We start by making the following mode expansion of the quark and gluon fields

$$\hat{\psi}(\hat{x}^i, \tau) = \sum_{n=-\infty}^{\infty} \hat{\psi}_n(\hat{x}^i)e^{-i\omega_n^i \tau},$$

$$\hat{A}^\alpha(\hat{x}^i, \tau) = \sum_{m=-\infty}^{\infty} \hat{A}_m^\alpha(\hat{x}^i)e^{-i\omega_m^\alpha \tau},$$
where \( \omega_n^f = (2n + 1) \pi T \) and \( \omega_m^b = 2m\pi T \). Using the static gauge condition \( \partial_\tau \hat{A}^3(\hat{x}^i, \tau) = 0 \), which implies that the only remaining mode of the \( \hat{A}^3 \) field is a (perturbatively) massless Higgs \( \hat{A}^3(\hat{x}^i, \tau) = \phi(\hat{x}^i) \), we get the following final 2d form of the action,

\[
S_{3E} \rightarrow S_{2E}^{YM}[\hat{A}^i, \phi] - \beta \sum_n \int d^2 \hat{x} \hat{\psi}_n (\sigma^3 \omega_n^f + i\gamma_5 \not\partial_j - g_3 \gamma_j \hat{A}_j - g_3 \sigma^3 \phi) \hat{\psi}_n + \\
\beta \sum_{n,m} \int d^2 \hat{x} \hat{\psi}_n g_3 \gamma_j (\hat{A}_j^{(n-m)} \hat{\psi}_m + S_{2E}^{NS}[\hat{A}_j, \hat{A}_j^u]) ,
\]

(2.3)

where \( j = 1, 2 \) and \( \beta = T^{-1} \). \( S_{2E}^{YM}[A^i, \phi] \) is the action for a two-dimensional Euclidean YM-Higgs model, as discussed in detail in e.g. [2, 16], and \( S_{2E}^{NS}[\hat{A}_j, \hat{A}_j^u] \) is the action for the nonstatic gluonic modes including the coupling to the static ones.

As discussed in [2], static DeTar correlation functions can, up to \( O(1/T) \) corrections, be calculated from an effective dimensionally reduced theory, although this is not possible in 3+1 d QCD. The only effect of the heavy scalar fields is then to renormalize the coupling in the 2d model and to add new vertices. One must of course keep the heavy quark fields since there are no massless fermionic modes due to the anti-periodic boundary conditions. Even this model is too difficult to treat analytically, and we will make the simplifying assumption that the scalar field \( \phi \) can be neglected altogether. Arguments for why this does not change the basic properties of the model were given in [17], where it was argued that the net effect of the scalar field is to renormalize the string tension and to introduce an effective \( 1/R \) potential at large distances. A strong indirect evidence for the validity of this approximation was given in ref. [18], where it was shown that the leading high temperature behavior of the susceptibility is correctly reproduced by the contributions from 2d QCD bound states solely, neglecting any effects of the Higgs field.

When it comes to nonstatic correlation functions the situation is much more complicated. There is no \textit{a priori} reason for neglecting the heavy gluon modes, and as we shall see below in the case of the non-static susceptibility, in general they do contribute.

To apply the 2d QCD technology, we follow the approach in [2] and rotate to a fictitious 2d Minkowski space by:

\[
t = x^0 = -i\hat{x}^2 \quad \hat{\partial}_2 = -i\partial_0 \quad A_0 = i\hat{A}_2 \quad \gamma^0 = \hat{\gamma}^2 \quad \gamma^1 = i\hat{\gamma}^1 \quad \{\gamma^\mu, \gamma^\nu\} = 2g^\mu\nu
\]

(2.4)
with $\mu, \nu = 0, 1$ and the Minkowski metric $(+-)$. We also perform a chiral rotation on the quark fields and a rescaling (in order to get the mass term on the conventional form and to give the quark fields the right canonical dimension [3])

$$\hat{\psi}_n = \sqrt{T} e^{i\frac{\pi}{4}\sigma^3} \psi_n$$
$$\hat{\bar{\psi}}_n = i\sqrt{T} \overline{\psi}_n e^{i\frac{\pi}{4}\sigma^3}.$$  

With this, we get the following final form for the action we shall use in the static sector,

$$\tilde{S}_{3E} = -i S_{YM}^2 [A^\mu] - i \sum_n \int d^2 x \overline{\psi}_n (i\gamma^\mu \partial_\mu - g\gamma^\mu A_\mu - M_n) \psi_n.$$  

In (2.6) the gluon field has been rescaled as well and $g = \sqrt{T} g_3$ is the 2-dimensional coupling constant. The model described by (2.6) is QCD in two dimensions with an infinite number of fermion fields with masses $M_n = (2n + 1)\pi T$, which can be solved in the large $N_c$ limit, as originally shown by ’t Hooft [19] and subsequently elaborated by many others. We will show that the methods developed in e.g. [20] and [21] can be adapted to the present problem, and that the difficulties associated with the infinite number of quark fields can be handled.

The bilinear quark currents in the original 3d Euclidean theory takes the following form, for the vector current $\hat{V}^a$,

$$\hat{V}^3 = -T \sum_{m,n} e^{-i(\omega_n - \omega_m)\tau} \overline{\psi}_m \psi_n$$
$$\hat{V}^j = \xi T \sum_{m,n} e^{-i(\omega_n - \omega_m)\tau} \overline{\psi}_m \gamma^j \psi_n$$

with $j = 1, 2$ and $\xi = i^{(j-1)}$.

We now state some key results for the ’t Hooft model that we will use in the following. For a review and references to original papers, see e.g. [22]. Due to the linear Coulomb potential $V(r) = \sigma r$, with the string tension $\sigma = g^2 N_c / \pi$, there are no free quark states, but only color neutral hadrons, and at energies much larger than the quark masses, the bound-state spectrum is approximately given by

$$M_k^2 = \mu_k^2 \sigma = \pi^2 \sigma k \quad k \gg 1.$$  

(2.9)
In real QCD, the Greens functions for the vector and axial vector currents are of special importance, since they couple to external electro-magnetic and weak probes. In a model problem one is of course free to consider any arbitrary Greens function, but we shall concentrate on the vector current two point function, i.e. the color singlet polarization tensor, \( \Pi_{\mu\nu}(\omega, \vec{q}) \), that at \( T = 0 \) determines the (would be) \( e^+e^- \) production rate, at \( T \neq 0 \) and \( \omega = 0 \) gives the screening of external electric and magnetic fields, and for \( \omega \geq |\vec{q}| \) contains information about the finite \( T \) quasi particle spectrum. At \( T = 0 \) this function has been extensively studied, and, in the case of one quark flavor and to leading order in \( 1/N_c \), the following exact expression has been derived \[19, 21, 22\]

\[
\Pi_{\mu\nu}(\omega, \vec{q}) = \left( g^2 g_{\mu\nu} - q_\mu q_\nu \right) \left( \frac{g^2 N_c}{\pi} \right) \sum_k \frac{(g_k^{(V)})^2}{q^2 - M^2_k + i\epsilon}, \tag{2.10}
\]

where

\[
g_k^{(V)} = \int_0^1 dx \phi_n(x), \tag{2.11}
\]

and \( \phi_n(x) \) is the meson wave function and \( M^2_k \) its invariant mass. Note that the mesons have zero-width to leading order in \( 1/N_c \).

A similar analysis for the scalar current two point function in the leading \( N_c \) approximation yields

\[
\Pi_S(\mu^2) = -\left( \frac{g^2 N_c}{\pi} \right) \sum_n \frac{(g_n^{(S)})^2}{\mu^2 - \mu^2_n + i\epsilon}, \tag{2.12}
\]

where \( \mu, \mu_n \) is the four momentum of the current and the bound state mass, in units of \( g^2 N_c/\pi \), with the coupling of the the bound state to the respective source given by

\[
g_n^{(S)} = \frac{1}{2} \int_0^1 dx \left( \frac{\sqrt{\gamma_a}}{x} - \frac{\sqrt{\gamma_b}}{1-x} \right) \phi_n(x). \tag{2.13}
\]

\( \gamma_a, \gamma_b \) are the quark masses, also in units of \( g^2 N_c/\pi \). However, due to parity,

\[
m_a \int_0^1 dx \frac{1}{x} \phi_n(x) = (-1)^{n+1} m_b \int_0^1 dx \frac{1}{1-x} \phi_n(x), \tag{2.14}
\]

so the sum over the bound states in eq. \ref{2.12} is effectively restricted to even \( n \). One can also show that, asymptotically,

\[
g_n^{(S)} \to \frac{\pi}{\sqrt{2}}, \tag{2.15}
\]

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Similar relations can easily be derived for the pseudoscalar current.

For nonstatic correlation functions the situation is more complicated. First, we must extend the above treatment to include bound states of quarks with different masses. To do this we have generalized the formulas (2.10) and (2.12) to the case of unequal masses, i.e. to couplings of the form \( \bar{\psi}_n \gamma_\mu \psi_m \) where the field \( \psi_n \) has mass \( M_n \). The resulting expression for the polarization tensor is identical to (2.10), but with the wave functions being solutions to the unequal mass 't Hooft equation; the same holds for the scalar correlator (2.12). Although the result is simple and intuitive, the derivation again involves several nontrivial cancellations. There is however a more subtle issue that regards the nonstatic gluon modes. As seen from (2.3), they manifest themselves in the 2d theory as heavy vector fields in the adjoint representation. Since there is no physical light gluon in two dimensions, we expect these heavy vector particles to be confined, with a string tension \( \sigma_{Ad} = (C_{Ad}/C_F)\sigma \), where \( C_{Ad} \) and \( C_F \) are the Casimir operators in the adjoint and fundamental representations respectively. For a general discussion of 2d Yang-Mills theory with adjoint matter fields, see e.g. [14, 15, 17]. In the large \( N_c \) limit we expect asymptotically straight trajectories of stable bound states of the type \( \hat{\psi}_n A_{m_1} \ldots A_{m_k} \hat{\psi}_n \), and there is no reason to expect that they should not contribute to correlation functions. In fact we would expect a general formula of the form

\[
G(\omega, \vec{q}) = - \left( \frac{N_c}{\pi} \right) \sum_n \frac{1}{q^2 - (M_{n}^B)^2} \left( g_{n}^{(B)} \right)^2 , \tag{2.16}
\]

where \( M_n^B \) and \( g_n^{(B)} \) are masses and couplings for these trajectories. An important point, to be discussed below, is that Lorentz invariance requires the vector particles to couple either in pairs or via derivative couplings.

3. Static and dynamical quark susceptibilities

In the large \( N_c \) limit, all contributions to color singlet correlation functions will be of the form given in (2.10) and (2.12). Using the results from the 't Hooft model, we can calculate the contribution from the \( \bar{q}q \) trajectory to the finite temperature correlator of
vector currents,

\[ G^{\mu\nu}(\Omega_N, \vec{q}) = \int_0^\beta d\tau \int d^2r \, e^{-i(\tau \Omega_N + \vec{q} \cdot \vec{r})} \langle V^\mu(\tau, \vec{r}) V'^\nu(0, \vec{0}) \rangle , \quad (3.1) \]

where the expectation value is with respect to the action \( \hat{S}_{\text{3E}} \). We have set \( \Omega_N = 2\pi NT \) and used \( \vec{r} = (x, y) \). Using (2.6) and (2.7) we can relate \( G^{33} \) to the 2d scalar correlator, and by combining this with (2.12) we get

\[ G^{33}_{\eta_0}(\Omega_N, \vec{q}) = - \left( \frac{g^2 N_c}{\pi} \right)^2 \sum_{n=-\infty}^{\infty} \sum_k \frac{(g_{k}^{n+N,n})^2}{q^2 - M_k^2(n + N, n) + i\epsilon} , \quad (3.2) \]

where \( M_k^2(m, n) \) is the mass and \( g_{k}^{m,n} \), given by (2.13), the form factor of the \( k \):th pole on the trajectory describing mesons consisting of quarks with masses \( (2n + 1)\pi T \) and \( (2m + 1)\pi T \). Similar expressions can be given for correlators of other currents. Note that since we have rotated to a 2d Minkowski space, \( q^2 = q_0^2 - q_1^2 \). After having evaluated (3.2) using 2d QCD methods, we can obtain the corresponding real time finite \( T \) correlation function by the analytic continuation \( i\Omega_N = \omega + i\epsilon \).

As shown in appendix A, perturbation theory to one loop gives the following leading behavior at large \( T \) and \( |\vec{q}| = q \gg \omega \),

\[ G^{33}(\omega, \vec{q}) = - \frac{g_3^2 N_c}{\pi} \left( T \ln 2 - \frac{q\pi}{16} \right) + O(q^2/T) \quad \text{for} \ T \gg q \gg \omega . \quad (3.3) \]

This expression is also leading in \( N_c \), so we can hope to reproduce it using the \'t Hooft model, and we shall now demonstrate that it can be calculated from (3.2), \textit{i.e.} by neglecting both the contributions from the bound states containing heavy scalars, and the effect of the light Higgs field. The derivation is a straightforward extension of the calculation in [13], which we now briefly recall. Since each term in the sum (3.2) is \( \sim O(1) \) in the temperature, a result \( \sim T \) must emanate from the infinite range of the summation. Thus we can use the asymptotic expressions (2.9) and (2.15) for bound state masses and scalar form factors, and furthermore replace the discrete sum with an integral of a continuous bound state spectrum,

\[ \sum_k \rightarrow \frac{1}{\pi^2 \sigma} \int_{(m_1+m_2)^2} dM^2 , \quad (3.4) \]
with $m_1$ and $m_2$ the two different quark masses. The resulting expression for the relevant correlation function (after the rotation $q^2 \to -\vec{q}^2$) reads,

$$G_{qq}^{33}(\omega, \vec{q}) = \frac{g_3^2 N_c T}{2\pi} \sum_n \int_0^{\Lambda^2} \frac{dM^2}{q^2 + M^2 + (m_1 + m_2)^2} , \quad (3.5)$$

where we introduced an ultraviolet cutoff $\Lambda$. The static limit corresponds to $\Omega_N = 0$, where $m_1^2 = m_2^2 = \omega_n^2$. Performing the $M^2$-integration and rewriting the sum over $\omega_n$ as a contour integral we get for the $T$-dependent piece,

$$G_{qq}^{33}(\omega, \vec{q}) = -\frac{g_3^2 N_c}{\pi} \int_{-\infty}^{\infty} dz \frac{dz}{2\pi i} n_F(\beta z) \ln \left( \frac{\Lambda^2}{q^2 - 4z^2} \right) , \quad (3.6)$$

where $n_F$ is the Fermi distribution function. The logarithmic function has a cut along the real axis; in the positive real half-plane the cut starts at $q/2$. Following \[18\] we can deform and close the contour to get

$$G_{qq}^{33}(\omega = 0, \vec{q}) = -\frac{g_3^2 N_c}{\pi} \left( T \ln 2 - \frac{q}{4} \right) \quad (3.7)$$

and comparing this with (3.3) we see that the leading terms agree, but the subleading ones deviate, although they have the same $k, T$ dependence and the same sign.

We now turn to the non-static case. In appendix A we derive the following perturbative result,

$$G_{qq}^{33}(\omega, \vec{q}) = g_3^2 N_c T \frac{1}{2\pi} \ln 2 \left( \frac{q^2}{\omega^2} \right) + O(Tq^4/\omega^4) \quad \text{for } T \gg \omega \gg q . \quad (3.8)$$

By comparing (3.8) with (3.3) we see that the two limits $q = 0, \omega \to 0$ and $\omega = 0, q \to 0$ do not commute, just as in 3+1 dimensions.

The limit (3.8) is a bit harder to reproduce with the 2-dimensional technique than the static one, since $m_1 \neq m_2$ which implies a more complicated analytic structure in the $z$-plane. Performing the $M^2$-integration we get

$$G_{qq}^{33}(\omega, \vec{q}) = -\frac{g_3^2 N_c}{\pi} \int_C dz \frac{dz}{4\pi i} \tanh \left( \frac{\beta z}{2} \right) \ln \left[ q^2 + \left( \sqrt{-z^2 + \sqrt{-(z - \omega)^2}} \right)^2 \right] , \quad (3.9)$$

where we analytically continued back to real time by $i\Omega_N = \omega + i\epsilon$, and where the integration contour $C$ and the square root cuts in the $z$-plane are shown in fig. 2(a) in
appendix B. After some exercise in complex analysis, which is outlined in appendix B, we get

\[
G^{33}(\omega, \vec{q}) = \frac{g_\pi^2 N_c}{\pi} \int_0^\infty \frac{dx}{2\pi i} n_F(\beta x) \left[ \ln \left( q^2 + \Omega_N^2 - 4x^2 - 4ix\Omega_N \right) - \ln \left( q^2 + \Omega_N^2 - 4x^2 + 4ix\Omega_N \right) \right].
\]  

(3.10)

Taking \( \vec{q} = \vec{0} \) in (3.10) the logarithms cancel for \( \Omega_N = 0 \), so we conclude that in the high-temperature limit,

\[
G^{33}_{\vec{q}}(\omega, \vec{0}) \sim \omega + O(1/\omega^2) \quad \text{for} \quad \omega \ll T.
\]  

(3.11)

Note that the non-analytic properties of the perturbative results are also manifest in (3.9), since if we take the limits in reverse order we get the logarithmic cut leading to (3.7), instead of (3.11). We have thus reproduced the nonleading behavior for the perturbative result (3.8), in the limit \( \vec{q} = \vec{0} \), but from the expression (3.10) it is also clear that for \( \vec{q} \neq \vec{0} \) the leading term will not pick up any linear \( T \)-dependent piece, but only \( O(q^2/\omega) \).

At the end of the previous section, we pointed out that there is no compelling reason to neglect the contribution from trajectories containing heavy vector particles. We now want to argue that the leading term in (3.8) is a likely sign of such a trajectory. The simplest possible contribution would come from bound states of the type \( \hat{\psi}_{m_1} A_n \hat{\psi}_{m_2} \) as illustrated in fig. 1. Since \( A_n \) carries a vector index the coupling must be derivative, \( i.e. \) of the form \( g_{\vec{q}Aq}q^\mu \), where the coupling constant \( g_{\vec{q}Aq} \) has dimension 1/mass. Without solving the resulting 3-body problem we can not calculate this coupling, but we can nevertheless predict that the contribution to the correlation function from the trajectory with \( m_1 = m_2 \) and \( m_N = \Omega_N = 2\pi NT \) will be

\[
G_{\vec{q}Aq}^{33}(\omega, \vec{q}) = \frac{g_{\vec{q}Aq}^2}{(g^{(S)})^2} G_{\vec{q}}^{33}(0, \vec{q}) = -g_{\vec{q}Aq}^2 q^2 \frac{2N_cT}{\pi^3} \ln 2.
\]  

(3.12)

Although we have not determined the coupling, the presence of the factor \( T \ln 2 \) in this formula is very suggestive when comparing to the perturbative result (3.8). It is very natural to expect it to have the same origin, \( i.e. \) from an infinite summation over states on a linear trajectory. For (3.12) to be consistent with (3.8) we must have \( g_{\vec{q}Aq} \sim 1/m_n \).
i.e. the scale is set by the mass of the scalar particle. We have not been able to prove this.

![Diagram](image)

**Fig. 1.** *Contribution to the bound states containing a heavy vector particle.*

To summarize this section, we have found a rather pleasing picture in terms of the interplay (high temperature duality) between perturbation theory and the dimensionally reduced theory at high temperature. In the static sector only the lightest gluon modes need to be retained in order to correctly reproduce the screening length, whereas the non-static sector for consistency requires bound states containing heavy vector particles.

4. **DeTar correlators at small $\vec{q}$.**

We argued above that the formula (3.2) can be used to calculate the non-static DeTar correlator in the limit $\vec{q} = \vec{0}$. However, to be able to make the final analytical continuation $\Omega_N = -i\omega$ we need expressions for both bound state masses and form factors that are analytic in the masses, *i.e.* in $\Omega_N$. Clearly some approximation is needed, and we have tried to use the semiclassical methods developed in [23]. Even this turns out to be too difficult, since the WKB expressions for the wave functions in general are very complicated, and do not allow us to derive explicit analytic expressions for the form factors. However, the nonrelativistic system of two particles with masses $m_1$ and $m_2$ moving in a linear potential $V(r) = \sigma r$ is relatively easy to handle. The bound states of this system are given by,

$$M_k^2(m_1, m_2) = M^2 + 2\lambda_k M \frac{\sigma^{2/3}}{(2m)^{1/3}} + O(m_1^{-4/3})$$

for $\sigma \ll m_1, m_2$, with $M = m_1 + m_2$, $m = m_1 m_2 / M$ and $\lambda_n$ the $n^{th}$ root of the Airy function $Ai'(x)$. We can also calculate the nonrelativistic form factors which are proportional
to the wave function at the origin,

$$(g_k^{n_1,n_2})^2 \sim |\psi_k(0)|^2 \sim (m\sigma)^{1/3}$$  \hspace{1cm} (4.2)

for even parity states. For odd parity states, the wave function vanishes at the origin.

Although in principle straightforward, we shall not elaborate on the evaluation of the normalization in (4.2). We are interested in the result after the analytic continuation $\Omega_N = -i\omega$, when the masses become,

$$m_1 = |2n + 1|\pi T$$  \hspace{1cm} (4.3)
$$m_2 = |i\omega + (2n + 1)\pi T|$$

where $|x| = (x^2)^{1/2}$ which implies that $\text{Re}(m_1 + m_2) = |2n + 1|2\pi T$. In particular, we get for the lowest masses,

$$M_k = 2\pi T \pm i\omega + \frac{\lambda_k\sigma^{2/3}}{(2\pi T)^{1/3}} \left( 1 \pm \frac{i\pi T\omega}{(\pi T)^2 + \omega^2} \right)$$  \hspace{1cm} (4.4)

where the upper and lower sign refers to $n = 0$ and $n = -1$ respectively.

The question now is whether we can use this information to calculate the correlator (3.1), at least in some specific kinematic region. This is not a simple problem. First notice that the correlation function cannot be calculated directly for fixed $\omega$ since the sum over the radial states does not converge fast enough for the nonrelativistic approximation to be useful. Also, the analytic structure in $\omega$ obtained from (3.10) and (3.12) is too complicated to allow us to evaluate the real time correlator by a fourier transformation. The way to proceed is instead to consider the following Fourier transform,

$$\tilde{G}^{\mu\nu}(\Omega_N, x) = i \int \frac{dq}{2\pi} e^{iqx} G^{\mu\nu}(\Omega_N, 0, q)$$  \hspace{1cm} (4.5)

and calculate the contribution from the $q\bar{q}$-trajectory only, to get

$$\tilde{G}^{11}_{q\bar{q}}(\Omega_N, x) = -\frac{g^2 N}{2\pi} \sum_{n=-\infty}^{\infty} \sum_k (g_k^{n+N,n})^2 M_k(n + N, n)e^{-M_k(n+N,n)x} .$$  \hspace{1cm} (4.6)

In the limit $xT \gg 1$ and $g^2 \ll T$, the expression (4.6) is saturated with a few nonrelativistic bound states. Note that in the static limit $\omega = 0$ we retain the static DeTar correlators
derived in \([2]\), and the main result of this section is that the \(x\)–dependence of the dynamical correlators are obtained by a straightforward extension of the earlier results. Also note that large \(x\) corresponds to small \(q\). Following the arguments of the previous sections, we expect trajectories containing heavy bosons to give only small contributions.

We shall now make the conditions that allow us to saturate (4.6) with only a few bound states more precise. As already stated, \(M_k\) depends linearly on \(|n|T\), so for \(xT \gg 1\) equation (4.6) will be dominated by the two lowest masses, corresponding to (4.4). To suppress higher radial excitations we must require

\[
x \gg \left( \frac{T}{\sigma^2} \right)^{\frac{1}{3}},
\]

(4.7)

which combined with the weak coupling condition \(T \gg g \sim \sqrt{\sigma}\), implies

\[
xT \gg \left( \frac{T^2}{\sigma} \right)^{\frac{2}{3}} \gg \gg 1,
\]

(4.8)

which means that we are in a region where the correlation function is very small.

5. Discussion and Conclusions

We have shown how to relate real time finite \(T\) correlation functions in 3d QCD to quantities that can, at least in a certain limit, be calculated analytically in a generalized \('t Hooft\) model with an infinite number of heavy quarks and scalars. Several assumptions and approximations went into our calculation and we want to comment upon their validity.

First, we neglected the Higgs field. This is a questionable step, and might well be improved upon. The effect of the Higgs field can be estimated in two limits. For the stringlike excitations far up on the “Regge” trajectories we can use the arguments of \([17]\) to conclude that the only effect of the Higgs field would be to renormalize the string tension and add a non-leading \(1/R\) term to the potential. This will clearly not change the leading behavior of the spectrum which was the only thing needed to retain the perturbative result for the susceptibility. For the nonrelativistic bound states we can estimate the effect on the masses from Higgs exchange to lowest order in perturbation
theory, using an effective four fermion potential of the kind discussed in [24]. However, as long as there are no numerical simulations to compare with it is hardly worth elaborating on corrections of this type.

Secondly we want to emphasize a point already made in section 3. Starting from the high temperature limit, we showed that consistency between the perturbative high temperature theory (i.e. 3d QCD) and the generalized 't Hooft model requires that the spectrum of the latter contains trajectories of bound states with heavy vector constituents, and that their decay strength decreases as $1/\omega$. These effects are dominant for time-like separations, whereas e.g. the screening length can be calculated by only taking into account the $\overline{q}q$-trajectories.

The limited results of this paper may be compared with the very detailed ones previously obtained in the Schwinger model [25] and the large $N$ limit of the 2d Gross-Neveu model [4], where the analysis is simpler. The Schwinger model is exactly solvable in terms of an elementary scalar field, and the 2d Gross-Neveu model is tractable in the $1/N$ approximation. In contrast, 3d QCD studied here is rather similar to real QCD, and the difficulties are accordingly increased - in fact we find it encouraging that nonperturbative statements about the asymptotics of the Detar correlator, admittedly in some specific kinematical domain, can at all be made using analytical methods.

Finally we want to comment upon to which extent the results of this 3d model can be generalized to real, (3+1)d QCD at finite temperature. Starting from a high T phase, the correlators in 4d QCD show a similar non-analytical behavior as encountered in 3d; in the static case we then have $G_{\overline{q}q}^{44} \sim T^2$, and in the nonstatic, $G_{\overline{q}q}^{44} \sim T^2(q^2/\omega^2)$. Since it is also known that QCD in 3d is confining, it is from this point of view not unlikely that the dynamical correlators in the generalized 3d theory have to be accompanied by a summation over trajectories of bound states consisting of heavy vector fields as well, with a derivative coupling and a coupling constant $g_{\overline{A}q} \sim 1/\omega$, just as in the 2d case. However, since 3d QCD is not solvable even in the large $N$ limit, this conjecture is not easy to verify. For the dynamical DeTar correlators, we expect the asymptotic $x$-behavior to be
generic, *i.e.* that the sum over the bound states is saturated by only a few nonrelativistic states and that the leading $\omega$ dependence of the correlation functions comes from the $\omega$ dependence of the screening masses just like in (4.4) and (4.6). Again there will be corrections from the mass-dependence of the binding energy.

For *non-asymptotic* $x$, nothing can be said for sure, since many bound states will contribute, and the nonrelativistic approximation can no longer be trusted. One should note, however, that to the extent that the constituent masses enter the total mass additively, the leading $\omega$ dependence in (4.6) could still be correct, since the factor $e^{i\omega x}$ is common to all the terms in the sum. If valid, this argument, which is the same in both three and four dimensions, predicts a very simple $\omega$ dependence for the DeTar correlators, and might even be related to dynamical quark-gluon plasma parameters. Again we stress that these comments on the non-asymptotic behaviour are highly speculative.

**Appendix A. The polarization tensor at high $T$**

In this appendix we give the derivation of the asymptotic results, equations (3.3) and (3.8). The polarization tensor $G(\Omega_N, \vec{q}) = (N_c g_0^2)^{-1}G^{33}(\Omega_N, \vec{q})$ is given by

$$G(\Omega_N, \vec{q}) = T \sum_n \int \frac{d^2 p}{(2\pi)^2} \text{Tr} \left( \sigma_3 \frac{1}{p} \frac{1}{\vec{p} + \vec{q}} \right)$$

$$= T \sum_n \int \frac{d^2 p}{(2\pi)^2} \left( \frac{p_0^2 + \vec{p}^2 + p_0 q_0 + \vec{p} \cdot \vec{q}}{p^2 (p + q)^2} \right),$$

where $p_0 = (2n + 1)\pi Ti$ and $q_0 = 2N\pi Ti$. Evaluating the frequency summation with the help of contour integration, omitting the vacuum piece and performing the analytic continuation $i\Omega_N = \omega + i\epsilon$, we find for the imaginary part

$$\text{Im}G(\omega, \vec{q}) = -\theta(q - \omega) \int_{q+}^{\infty} \frac{dp}{2\pi} \left[ \frac{4p^2 - 4p\omega + \omega^2 - q^2}{\sqrt{4p^2 q^2 - (\omega^2 - q^2 - 2p\omega)^2}} \right] \left( n_F(\beta(p - \omega)) - n_F(\beta p) \right)$$

$$-\theta(\omega - q) \int_{\omega-}^{\omega+} \frac{dp}{4\pi} \left[ \frac{4p^2 - 4p\omega + \omega^2 - q^2}{\sqrt{4p^2 q^2 - (\omega^2 - q^2 - 2p\omega)^2}} \right] \left( n_F(\beta p) + n_F(\beta(\omega - p)) \right),$$

where $\theta$ is the step function. The integration limits are $q_+ = (q + \omega)/2$, $\omega_\pm = (\omega \pm q)/2$ and
\( q = |\vec{q}|, \ p = |\vec{p}|. \) From this expression it is clear that the imaginary part is proportional to \( \omega \) when \( q \gg \omega \) and to \( q \) for \( \omega \gg q \).

The real part of the polarization tensor is

\[
-\text{Re}G(\omega, \vec{q}) = \frac{T}{\pi} \ln 2 - \theta(\omega - q) \int_0^\infty \frac{dp}{2\pi} n_F(\beta p) \left( \frac{4p^2 + 4p\omega + \omega^2 - q^2}{\sqrt{(\omega^2 - q^2 + 2p\omega)^2 - 4p^2q^2}} \right)
\]

\[
+\theta(\omega - q) \int_{k_+}^\infty \frac{dp}{2\pi} n_F(\beta p) \left( \frac{4p^2 - 4p\omega + \omega^2 - q^2}{\sqrt{(\omega^2 - q^2 - 2p\omega)^2 - 4p^2q^2}} \right)
\]

\[
-\theta(\omega - q) \int_0^{q_+} \frac{dp}{2\pi} n_F(\beta p) \left( \frac{4p^2 - 4p\omega + \omega^2 - q^2}{\sqrt{(\omega^2 - q^2 + 2p\omega)^2 - 4p^2q^2}} \right)
\]

\[
+\theta(q - \omega) \int_{k_-}^\infty \frac{dp}{2\pi} n_F(\beta p) \left( \frac{4p^2 + 4p\omega + \omega^2 - q^2}{\sqrt{(\omega^2 - q^2 - 2p\omega)^2 - 4p^2q^2}} \right)
\]

\[
+\theta(q - \omega) \int_0^{q_-} \frac{dp}{2\pi} n_F(\beta p) \left( \frac{4p^2 - 4p\omega + \omega^2 - q^2}{\sqrt{(\omega^2 - q^2 + 2p\omega)^2 - 4p^2q^2}} \right).
\]  

(A.3)

Expanding in powers of \( q/\omega \), we find

\[
\text{Re}G(\omega, \vec{q}) \simeq \frac{T}{2\pi} \left( \frac{q^2}{\omega^2} \right) \ln 2
\]  

(A.4)

in the limit \( T \gg \omega \gg q \).

In the other limit, \( T \gg q \gg \omega \), the expansion of (A.3) gives

\[
\text{Re}G(\omega, \vec{q}) = -\frac{T}{\pi} \ln 2 + \frac{q}{16}.
\]  

(A.5)

Equation (A.3) gives the screening mass

\[
m_{el}^2 = -g_3^2 G(\omega, \vec{q} = \vec{0}) = g_3^2 \frac{T}{\pi} \ln 2,
\]

(A.6)

a result that also follows from the pressure of the free quark-antiquark gas, with a non-vanishing chemical potential.

**Appendix B. The polarization tensor from 2d, \( \vec{q} = \vec{0} \)**

Here we show that the polarization tensor calculated from the 2d theory has no leading
$T$-behavior for $\vec{q}^2 = 0$. Starting from the expression (3.4) and performing the $M^2$-integration, we can rewrite the Matsubara summation as

$$G(\Omega_N, \vec{q}) = -\frac{1}{2\pi} \int_C \frac{dz}{4\pi i} \tanh \left( \frac{\beta z}{2} \right) \ln \left[ q^2 + \left( \sqrt{-z^2 + \sqrt{-(z - i\Omega_N)^2}} \right)^2 \right], \quad (B.1)$$

where the contour $C$ is as shown in fig. 2(a). We have two different cuts for the square roots, shown as wiggly lines (for definiteness we have chosen $\Omega_N > 0$); the cut for $z = i\Omega_N$ is always at an even multiple of $i\pi T$ and hence it will not interfere with the poles of $\tanh(\beta z/2)$. These cuts correspond to having the cut for $\sqrt{f(z)}$ along the negative real axis in the complex $f$-plane.

Fig. 2. The integration contour for $G^{33}$, before (a) and after (b) deformation.

We can now deform the contour and arrive at the contour shown in fig. 2(b), where we have neglected $T$-independent pieces; this follows from the identity

$$\frac{1}{2} \tan \left( \frac{\beta z}{2} \right) = \pm \frac{1}{2} \frac{1}{e^{\pm \beta z} + 1}. \quad (B.2)$$

The cut at $z = 0$ then gives

$$-\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dx}{4\pi i} \tanh \left( \frac{\beta x}{2} \right) \left[ \ln \left( \vec{q}^2 + \Omega_N^2 \right) - \ln \left( -4x^2 + 4ix\Omega_N + \Omega_N^2 + \vec{q}^2 \right) \right] \quad (B.3)$$

and by using (B.2), equation (B.3) becomes for the temperature dependent piece

$$\frac{1}{2\pi} \int_0^{\infty} \frac{dx}{2\pi i} n_F(\beta x) \left[ \ln \left( -4x^2 - 4ix\Omega_N + \Omega_N^2 + \vec{q}^2 \right) - \ln \left( -4x^2 + 4ix\Omega_N + \Omega_N^2 + \vec{q}^2 \right) \right]. \quad (B.4)$$
In the same way, we find for the second cut at \( z = i\Omega_N \),

\[
\frac{1}{2\pi} \int_0^\infty \frac{dx}{2\pi i} n_F(\beta x) \left[ \ln \left( -4x^2 - 4ix\Omega_N + \Omega_N^2 + q^2 \right) - \ln \left( -4x^2 + 4ix\Omega_N + \Omega_N^2 + (-q)^2 \right) \right] 
\]

by using

\[
\tanh \left( \frac{\beta x}{2} \right) = \tanh \left( \frac{\beta (x + i\Omega_N)}{2} \right) 
\]

and so equation (B.1) becomes

\[
G(\Omega_N, \vec{q}) = \frac{1}{\pi} \int_0^\infty \frac{dx}{2\pi i} n_F(\beta x) \left[ \ln \left( \Omega_N^2 + \vec{q}^2 - 4x^2 - 4ix\Omega_N \right) - \ln \left( \Omega_N^2 + \vec{q}^2 - 4x^2 + 4ix\Omega_N \right) \right] .
\]

Taking \( \vec{q} = 0 \) we see that by putting \( \Omega_N = 0 \) the integral vanishes. Since it also has to vanish for \( \beta \to \infty \) we conclude that, after analytic continuation,

\[
G^{33}(\omega, \vec{q} = 0) \sim \omega \exp \left( -\frac{\omega}{T} \right) ,
\]

which is a subleading term and goes like \( \sim \omega \) for high temperatures.

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