Monotone expansion

Jean Bourgain* Amir Yehudayoff†

Abstract

This work, following the outline set in [B2], presents an explicit construction of a family of monotone expanders. The family is essentially defined by the Möbius action of $SL_2(\mathbb{R})$ on the real line. For the proof, we show a product-growth theorem for $SL_2(\mathbb{R})$.

0 Introduction

Expanders are sparse graphs with “strong connectivity” properties. Such graphs are extremely useful in various applications (see the survey [HLW]). Most sparse graphs are expanders, but for applications explicit constructions are needed. Indeed, explicit constructions of expanders graphs are known, e.g. [LPS, RVW]. Here we describe an explicit construction of monotone expanders (for more on such expanders see [DW]).

The construction of monotone expander we present first builds a “continuous” expander, which in turn can be discretized to the required size. A continuous expander is a finite family of maps $\Psi$ for which there exists a constant $c_0 > 0$ so that the following holds. Every $\psi \in \Psi$ is a smooth map from the interval $[0,1]$ to itself, and for all measurable $A \subset [0,1]$ with $|A| \leq 1/2$,

$$|\Psi(A)| \geq (1 + c_0)|A|,$$

where $\Psi(A) = \bigcup_{\psi \in \Psi} \psi(A)$. We say that $\Psi$ is a continuous monotone expander if in addition every $\psi \in \Psi$ is monotone, i.e., $\psi(x) > \psi(y)$ for $x > y$.

Theorem 0. There exists an explicit continuous monotone expander.

The word explicit in the theorem can be interpreted as follows. The family $\Psi$ can be (uniformly) described by a constant number of bits, and given a rational $x \in [0,1]$ that can be described by $b$ bits, $\psi(x)$ is rational and can be computed in time polynomial in $b$, for all $\psi \in \Psi$.

*Institute for Advanced Study, Princeton NJ, bourgain@math.ias.edu.
†Technion–IIT, Haifa, Israel, amir.yehudayoff@gmail.com.
The family \( \Psi \) also satisfies
\[
\| \psi - \text{id} \|_\infty, \| \psi' - 1 \|_\infty \leq c,
\]
for every \( \psi \in \Psi \), for a small constant \( c > 0 \), where \( \text{id} \) is the identity map.

The proof of the theorem follows the outline described in \([B2]\), which in turn uses ideas from recent works on growth and expansion in matrix groups. Most relevant is the work of Bourgain and Gamburd \([BG1]\) showing expansion in \( \text{SU}(2) \). Also related, is the work of Bourgain and Gamburd \([BG2]\) proving expansion in \( \text{SL}_2(\mathbb{F}_p) \), and the work of Helfgott \([H]\) showing growth in \( \text{SL}_2(\mathbb{F}_p) \).

The theorem describes the existence of a continuous monotone expander. By partitioning \([0, 1]\) to \( n \) equal-length intervals, \( \Psi \) naturally defines a discrete bi-partite monotone expander on \( 2n \) vertices. Namely, a bi-partite graph \( G \) with two color classes \( L, R \) of size \( n \) each so that (i) for every \( A \subset L \) of size \( |A| \leq n/2 \), the size of \( B = \{ b \in R : (a, b) \in E(G) \text{ for some } a \in A \} \) is at least \( (1 + c)|A| \), \( c > 0 \) a constant independent of \( n \), and (ii) the edges \( E(G) \) can be partitioned to finitely many sets \( E_1, \ldots, E_k \), \( k \) independent of \( n \), so that in each \( E_i \) edges do not “cross” each other (viz., \( E_i \) defines a partial monotone map). Since \( \Psi \) is explicit, the graph \( G \) is explicit as well. (If \( \Psi \) was continuous but not monotone, the same reduction would yield a family of discrete bi-partite expanders.)

No other proof of existence of discrete monotone expanders is known, not even using the probabilistic method. A partial explanation to that is the following. Natural probability distributions on partial monotone functions give, w.h.p., functions that are “close” to affine. Klawe, however, showed in \([K]\) that if one tries to construct expanders using affine transformations, then the minimal number of generators required is super-constant (in the number of vertices), and so no construction “that is close to affine” can work. Two more related comments: (i) The construction in this text uses “generators” that are defined as the ratio of two affine transformations. (ii) Dvir and Wigderson \([DW]\) showed that any proof of existence of a family of monotone expanders yields an explicit construction of monotone expanders.

Implicit in the work of Dvir and Shpilka \([DS]\) it is shown that an explicit discrete monotone expander easily yields an explicit dimension expander. Specifically, the existence of a constant number of \( n \times n \) zero-one matrices \( M_1, \ldots, M_k \) so that for every field \( F \) and for every subspace \( V \) of \( F^n \) of dimension \( D \leq n/2 \), the dimension of the span of \( M_1(V) \cup \ldots \cup M_k(V) \) is at least \( (1 + c)D \). The work of Lubotzky and Zelmanov \([LZ]\) shows that over the real numbers any explicit (perhaps non-monotone) expander yields an explicit dimension expander.

Here is an outline of the proof. To present the main ideas, we ignore many of the problematic issues.

**Defining maps.** Every matrix \( g \in \text{SL}_2(\mathbb{R}) \) acts on \( \mathbb{R} \) in a monotone way via the Möbius action. The maps \( \Psi \) will be defined by the actions of a set of matrices \( \mathcal{G} \subset \text{SL}_2(\mathbb{R}) \). This ensures that the maps in \( \Psi \) are monotone. Choose \( \mathcal{G} \) as a family of matrices that freely generate a group.
(with some extra properties, see Lemma 1.1 for exact statement). To find \(\mathcal{G}\), use the strong Tits alternative of Breuillard [Br], which roughly states that in a ball of constant radius in \(\text{SL}_2(\mathbb{R})\) there are elements that freely generate a group.

**Proving expansion.** As in many expanders constructions, the expansion follows by proving that the operator \(T\) defined by \(\Psi\) has a (restricted) spectral gap. As in recent works, the spectral gap is established as follows. Let \(\nu\) be the probability distribution defined by \(\Psi\). Then, the \(\ell\)-fold convolution of \(\nu\) with itself, \(\nu^{(\ell)}\), is flat, even for \(\ell\) relatively small. This statement implies the rapid mixing of the random walk defined by \(\nu\), and hence implies expansion. The proof consists of three steps.

(i) **Small \(\ell\).** To show that \(\nu^{(\ell)}\) is “somewhat” flat for small \(\ell\), use the fact that the group generated by \(\mathcal{G}\) is free, and Kesten’s estimates for the behavior of random walks on free groups. Roughly, as \(\mathcal{G}\) freely generates a group, the convolution “grows along a tree” and hence flat. Here we also need to use a “diophantine” property of \(\mathcal{G}\), i.e., that elements of \(\mathcal{G}\) have constant rational entries.

(ii) **Intermediate-size \(\ell\).** This is the main part of the argument. We prove a product-growth theorem for \(\text{SL}_2(\mathbb{R})\): if \(S\) is a subset of \(\text{SL}_2(\mathbb{R})\) with certain properties, then the size of \(S(3) = \{s_1s_2s_3 : s_i \in S\}\) is much larger than the size of \(S\). (The outline of the proof of the product theorem appears in Section 5.) Such a product theorem implies that \(\nu^{(2\ell)}\) is much flatter than \(\nu^{(\ell)}\), unless it is already pretty flat.

(iii) **Large \(\ell\).** By steps (i) and (ii), we can conclude that \(\nu^{(\ell)}\) is pretty flat, even for \(\ell\) relatively small. It remains to show that \(\nu^{(C\ell)}\) is very flat, for \(C > 0\) a constant. In previous works, this last step follows Sarnak and Xue’s multiplicities argument. As \(\text{SL}_2(\mathbb{R})\) is not compact, such an argument can not be applied here. Instead, use the subgroup structure of \(\text{SL}_2(\mathbb{R})\), or in other words the two-transitivity of the Möbius action. To do so, also use knowledge of the Fourier spectrum of the set \(A\). We are able to obtain knowledge on the spectrum of \(A\) by adding to \(\Psi\) the translate map. The translate map implies that, w.l.o.g., we can assume that the spectrum of \(A\) does not have low frequencies.

# 1 A monotone expander

In essence, the maps defining the monotone expander are induced by the action of \(\text{SL}_2(\mathbb{R})\) on \(\mathbb{R}\). To find the relevant elements of \(\text{SL}_2(\mathbb{R})\), use the following lemma. The proof of the lemma is given in Section 2.

**Lemma 1.1.** There is a constant \(C > 0\) so that the following holds. For \(\varepsilon > 0\) small, there is a positive integer \(Q\) and a subset \(\mathcal{G}\) of \(\text{SL}_2(\mathbb{R})\) so that

1. \((1/\varepsilon)^{1/C} < Q < (1/\varepsilon)^C\),
2. $Q < |G|^C$, 

3. elements of $G$ freely generate group,

4. elements of $G$ have entries of the form $\mathbb{Z}/Q$, and

5. every $g \in G$ admits 
\[ \|g - 1\|_2 = (g_{1,1} - 1)^2 + (g_{1,2})^2 + (g_{2,1})^2 + (g_{2,2} - 1)^2 \leq \varepsilon. \]

The lemma summarizes all the properties $G$ should satisfy in order to yield a monotone expander. When applying the lemma, $\varepsilon$ is a small universal constant. An important (and useful) property of the lemma is that both $|G|$ and $Q$ are polynomially comparable to $1/\varepsilon$. Without this property, the lemma immediately follows from the strong Tits alternative [Br]. Property 4 yields the non-commutative diophantine property of $G$, roughly, that for every $w \neq w'$ that are words of length $k$ in the element of $G$, the distance between $w$ and $w'$ is at least $(1/Q)^k$. This property is defined and used in [BG1]. Property 5 is crucial for handling the non-compactness of $SL_2(\mathbb{R})$.

Consider the Möbius action: Given $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL_2(\mathbb{R})$, denote by $\overline{g}$ the map defined by
\[ \overline{g}(x) = \frac{ax + b}{cx + d}. \]
For all $g$ in $SL_2(\mathbb{R})$, the derivative of the map $\overline{g}$ is
\[ \overline{g}'(x) = \frac{1}{(cx + d)^2}. \]
So $\overline{g}$ is monotone in any interval not containing $-d/c$.

**Construction.** Let $\Psi$ be the family of monotone smooth maps $\psi$ from sub-intervals of $[0, 1]$ to $[0, 1]$ defined as follows.

Let $\varepsilon > 0$ be a small universal constant (to be determined). Let $G$ be the family of matrices given by Lemma 1.1. Define 
\[ \Psi_G = \{ \overline{g} : g \in G \cup G^{-1} \}. \]
Here we restrict $\overline{g}$ to output values in $[0, 1]$, i.e., if $\psi \in \Psi_G$ is defined by $g$, then $\psi$ is a map from the interval $\overline{g}^{-1}([0, 1]) \cap [0, 1]$ to $[0, 1]$.

Let $K = K(\varepsilon)$ be a large integer (to be determined). Define the map $\psi_+ : [0, 1 - 1/K] \to [1/K, 1]$ by $\psi_+(x) = x + 1/K$, and the map $\psi_- : [1/K, 1] \to [0, 1 - 1/K]$ by $\psi_-(x) = x - 1/K$.

Finally, 
\[ \Psi = \Psi_G \cup \{ \psi_+, \psi_-, \text{id} \}, \]
where $\text{id}$ is the identity map.
Theorem 1.2. There is a constant \( c_0 > 0 \) so that the following holds. Let \( A \) be a measurable subset of \([0, 1]\) with \(|A| \leq 1/2\). Then, \(|\Psi(A)| \geq (1 + c_0)|A|\).

Theorem 1.2 implies Theorem 0 and follows from the following “restricted spectral gap” theorem. (To see that \( \Psi \) is explicit, add to \( \Psi \) all maps from “the large ball” in the proof of Lemma 1.1.) The Möbius action induces a unitary representation of \( \text{SL}_2(\mathbb{R}) \) on \( L^2(\mathbb{R}) \) defined by

\[
T_g^{-1} f(x) = \sqrt{g}(x)f(g(x)).
\]

For a positive integer \( K \), denote by \( F_K \) the family of maps \( f \in L^2(\mathbb{R}) \) with \( \text{supp}(f) \subset [0, 1] \) and \( \|f\|_2 = 1 \) so that for all \( k \in \{1, 2, \ldots, K\} \),

\[
\int_{I(k)} f(x)dx = 0,
\]

where

\[
I(k) = [(k - 1)/K, k/K].
\]

Theorem 1.3. Let \( \varepsilon > 0 \) be a small enough constant. Let \( \mathcal{G} \) be the set given by Lemma 1.1. If \( K = K(\varepsilon) \) is a large enough positive integer, then for all \( f \in F_K \),

\[
\left< \sum_g \nu(g)T_g f, f \right> < 1/2,
\]

with the probability measure

\[
\nu = (2|\mathcal{G}|)^{-1} \sum_{g \in \mathcal{G}} 1_g + 1_{g^{-1}},
\]

where \( 1_g \) is the delta function at \( g \).

The “restricted spectral gap” theorem is proved in Section 3.

Proof of Theorem 1.2. We first reduce the general case to the “restricted spectral gap” case. Let \( \sigma > 0 \) be a small universal constant, to be determined. If there is \( k \in \{1, \ldots, K - 1\} \) so that

\[
\left| |A \cap I(k + 1)| - |A \cap I(k)| \right| \geq \sigma |A|,
\]

then, using the maps \( \psi_+, \psi_- \) and \( \text{id} \),

\[
|\Psi(A)| \geq (1 + \sigma)|A|.
\]

It thus remains to consider the case that \( \left| |A \cap I(k + 1)| - |A \cap I(k)| \right| < \sigma |A| \) for all \( k \). Thus, for all \( k \),

\[
|K|A \cap I(k)| - |A| < \sigma K^2 |A|.
\]

(1.2)
Assume towards a contradiction that the theorem does not hold.

Since $\|g - 1\|_2 \leq \varepsilon$, for all $x \in [0, 1]$,

$$\frac{1}{(1 + 2\varepsilon)^2} < \overline{g}'(x) < \frac{1}{(1 - 2\varepsilon)^2}.$$ 

Thus, for every $x \in [0, 1]$,

$$0 \leq \overline{g}(x) - x < 10\varepsilon.$$ 

We need to ensure that even after applying the maps in $\Psi_G$ we remain in $[0, 1]$. To this end, let

$$A' = A \cap [k'/K, 1 - k'/K]$$

with $k'$ the smallest integer so that $k' \geq 10\varepsilon K$. By (1.2),

$$0.99|A| \leq |A'| \leq |A|,$$

as long as $\sigma, \varepsilon$ are small.

Denote

$$f = 1_{A'} - |A'|.$$

For all $g \in G \cup G^{-1}$,

$$\langle T_g f, f \rangle \geq \frac{1}{1 - 7\varepsilon} \int (1_{A'}(\overline{g}(x)) - |A'|)(1_{A'}(x) - |A'|)dx \geq 0.9|A'|(1 - |A'|) \geq 0.8 \|F\|_2,$$

as long as $\sigma, \varepsilon, c_0$ are small.

Project $A'$ on $\mathcal{F}_K$. Define $F$ as follows: for all $x \in [0, 1]$, if $x \in I(k)$, then

$$F(x) = 1_{A'}(x) - K|A' \cap I(k)|.$$

Hence, $F/\|F\|_2 \in \mathcal{F}_K$. In addition, for $\sigma$ small, using (1.2),

$$\|f - F\|_2^2 = \sum_{k = k'}^{K - k'} \int_{I(k)} (|A' - K|A' \cap I(k)|)^2 dx \leq 2\sigma^2 K^4 |A'|^2 \leq 0.01 \|F\|_2^2.$$

Therefore,

$$0.8 \|F\|_2^2 \leq \left\langle \sum_g \nu(g) T_g (f - F + F), f - F + F \right\rangle \leq 0.1 \|F\|_2^2 + \left\langle \sum_g \nu(g) T_g F, F \right\rangle,$$

which contradicts Theorem 1.3.
2 Finding set of generators

Notation. For convenience, we use the following notation throughout the text. For a constant \( c \in \mathbb{R} \), we denote by \( c + \) a constant slightly larger than \( c \), and by \( c - \) a constant slightly smaller than \( c \). Typically, the meaning of “slightly” depends on other parameters that are clear from the context. We also use the following asymptotic notation. Write \( a \sim b \) if \( a \leq Cb \) with \( C \) a universal constant. Write \( a \gtrsim b \) if \( b \leq a \), and \( a \sim b \) if \( a \lesssim b \leq a \).

For \( \delta > 0 \), denote by \( B_\delta(x) \) the ball of radius \( \delta \) around \( x \) and by \( \Gamma_\delta(A) \) the \( \delta \)-neighborhood of the set \( A \). We consider the \( L^2 \)-metric on \( SL_2(\mathbb{R}) \).

Proof of Lemma [7.7]. Breuillard [Br] proved a strong Tits alternative: there is a constant \( r \in \mathbb{Z} \) so that if \( S \) is a finite symmetric subset of \( SL_2(\mathbb{R}) \), which generates a non-amenable subgroup, then \( S(r) = \{ s_1 s_2 \cdots s_r : s_i \in S \} \) contains two elements, which freely generate a group.

Let
\[
\begin{align*}
    h_1 &= \begin{pmatrix} 1 & 1/q \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad h_2 = \begin{pmatrix} 1 & 0 \\ 1/q & 1 \end{pmatrix}.
\end{align*}
\]

Observe
\[
\begin{align*}
    h_1^q &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad h_2^q = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.
\end{align*}
\]

Hence, \( h_1, h_2 \) generate a non-amenable group. Apply the strong Tits alternative on the set \( S = \{ h_1, h_2, h_1^{-1}, h_2^{-1} \} \). There are thus \( g_1, g_2 \in S(r) \) that freely generate a group.

It remains to convert \( g_1, g_2 \) to many elements that are close to identity and freely generate a group. Let \( \ell \sim \log(1/\epsilon) \) so that the following holds. Consider
\[
W = \{ w^2 : w = s_1 \cdots s_\ell, \ s_1 = g_1, \ s_\ell = g_2, \ s_i \in \{ g_1, g_2, g_1^{-1}, g_2^{-1} \}, \ s_{i+1} \neq s_i^{-1} \}.
\]

Say that a word \( \sigma_1 \sigma_2 \cdots \sigma_k \) in an alphabet \( \Sigma \cup \Sigma^{-1} \) is \( \langle \Sigma \rangle \)-reduced if \( \sigma_{i+1} \neq \sigma_i^{-1} \) for all \( i \in \{1, \ldots, k-1\} \). The size of \( W \) is order \( 3^\ell \) and \( W \) consists of words of \( \langle g_1, g_2 \rangle \)-reduced-length exactly \( 2\ell \).

Claim 2.1. The elements of \( W \) freely generate a group.

Proof. Let \( w_1 \neq w_2^{-1} \) in \( W \cup W^{-1} \). Write
\[
    w_1 = (g_{a_1} s_1 g_{b_1})^2 \quad \text{and} \quad w_2 = (g_{a_2} s_2 g_{b_2})^2
\]
with \( s_1, s_2 \) reduced words in \( \langle g_1, g_2 \rangle \), and \( g_{a_1}, g_{b_1}, g_{a_2}, g_{b_2} \) in \( \{ g_1, g_2, g_1^{-1}, g_2^{-1} \} \). If either \( w_1, w_2 \in W \) or \( w_1, w_2 \in W^{-1} \), then \( g_{a_2} \neq g_{b_1}^{-1} \) and so
\[
    w_1 w_2 = g_{a_1} s_1 g_{b_1} g_{a_1} s_1 g_{b_1} g_{a_2} s_2 g_{b_2} g_{a_2} s_2 g_{b_2}.
\]
in \( \langle g_1, g_2 \rangle \)-reduced form. If either \( w_1 \in W, w_2 \in W^{-1} \) or \( w_1 \in W^{-1}, w_2 \in W \), then, since \( s_1 \neq s_2^{-1} \) and the reduced-length of both \( s_1, s_2 \) is \( \ell - 2 \),
\[
w_1 w_2 = g_a s_1 g_b g_1 s' g_2 g_2 s_2 g_2
\]
in \( \langle g_1, g_2 \rangle \)-reduced form, with \( s' \) non-trivial.

Any non-trivial \( \langle W \rangle \)-reduced word is not the identity of \( \langle g_1, g_2 \rangle \): For \( w = g_a szsg_b \in W \cup W^{-1} \), where \( z \) is a product of two elements of \( \{ g_1, g_2, g_1^{-1}, g_2^{-1} \} \), call \( z \) the center of \( w \). The above implies that if \( w_1 \neq w_2^{-1} \) then the centers of both \( w_1, w_2 \) are not reduced in the \( \langle g_1, g_2 \rangle \)-reduced form of \( w_1 w_2 \).

Hence, if \( w = w_1 w_2 \cdots w_k \) is a non-trivial \( \langle W \rangle \)-reduced word, then even in its \( \langle g_1, g_2 \rangle \)-reduced form \( w \) is not the identity (as all centers are not reduced).

Observe that for every \( w \in W \),
\[
\|w\|_2 \|w^{-1}\|_2 \leq (1 + 1/q)^{2r\ell} := N.
\]

Cover the ball \( B_N(1) \) in \( SL_2(\mathbb{R}) \) with balls of radius \( \varepsilon/N \). There exists \( w_0 \in W \) so that
\[
|B_{\varepsilon/N}(w_0) \cap W| \gtrsim |W| (\varepsilon/N^2)^3 \gtrsim \varepsilon^{3\ell} (1 + 1/q)^{-12r\ell}.
\]

Define
\[
\mathcal{G} = (w_0^{-1} B_{\varepsilon/N}(w_0) \cap W) \setminus \{1\}.
\]

Choose \( q \) as a universal constant so that \((1 + 1/q)^{12r} < 1.01\). Hence,
\[
|\mathcal{G}| = |W| - 1 \gtrsim 2^\ell.
\]

In addition, for \( g \in \mathcal{G} \),
\[
\|1 - g\|_2 \leq N \|w_0 - w_0 g\|_2 \leq \varepsilon,
\]
and the entries of \( g \) are of the form \( \mathbb{Z}/Q \) with \( Q = q^{4r\ell} \) and \( \log Q \sim \log(1/\varepsilon) \). Finally, as \( \mathcal{G} \) is of the form \( w_0^{-1} W \setminus \{1\} \) with \( W \) freely generating a group, the elements of \( \mathcal{G} \) freely generate a group as well.

\section{Restricted spectral gap via flattening}

To prove the “restricted spectral gap” property, we prove the following theorem that roughly states that after enough iterations \( \nu \) becomes very flat. Denote by \( P_\delta \) the approximate identity on \( SL_2(\mathbb{R}) \), namely, the density of the uniform distribution on the ball of radius \( \delta \) around 1 in \( SL_2(\mathbb{R}) \),
\[
P_\delta = \frac{1_{B_\delta(1)}}{|B_\delta(1)|}.
\]
For two distributions $\mu, \mu'$ on $\text{SL}_2(\mathbb{R})$ denote by $\mu * \mu'$ the convolution of $\mu$ and $\mu'$. Denote by $\mu^{(\ell)}$ the $\ell$-fold convolution of $\mu$ with itself.

**Theorem 3.1.** Let $\gamma > 0$. Assume that $\varepsilon > 0$, the parameter from \ref{lemma1.1} in Lemma \ref{lemma1.1} and $\delta > 0$ are small enough as a function of $\gamma$. If

$$\ell > C_1 \frac{\log(1/\delta)}{\log(1/\varepsilon)}$$

with $C_1 = C_1(\gamma) > 0$, then

$$\left\| \mu^{(\ell)} * P_\delta \right\|_\infty < \delta^{-\gamma}.$$

The proof of the theorem is given in Section \ref{section4} (When applying the theorem, $\gamma$ is a universal constant.)

**Proof of Theorem \ref{theorem1.3}.** Let $f \in \mathcal{F}_K$. Assume that \eqref{eqn1.1} does not hold, i.e.,

$$\left( \sum_g \nu(g) T_g f, f \right) \geq 1/2. \tag{3.1}$$

We start by finding a level set of the Fourier transform that “violates \eqref{eqn1.1} as well.” The Littlewood-Paley decomposition of $f$ is

$$f = \sum_{k=0}^\infty \Delta_k f,$$

where for every $k$ and for every $\lambda \in \text{supp} \hat{\Delta_k f}$,

$$|\lambda| \sim 2^k.$$

We are interested in the Hecke operator

$$T = \sum_g \nu(g) T_g.$$

As $f \in \mathcal{F}_K$, we can consider the part of $f$ with high frequencies.

**Claim 3.2.** For $k_0 \geq 0$, define

$$f_0 = \sum_{k \geq k_0} \Delta_k f.$$

If $K$ is large enough, depending on $k_0$, then

$$\langle T f_0, f_0 \rangle > 1/4.$$
Isolate one frequency-level of $f_0$, using the following claim.

**Claim 3.3.** There is $k \geq k_0$ so that

$$
\|T\Delta_k f_0\|_2 \geq c_1 \|\Delta_k f_0\|_2
$$

with $c_1 > 0$ a universal constant.

**Proof.** Bound

$$
\|Tf_0\|^2 \leq \sum_{k,k'} |\langle T\Delta_k f_0, T\Delta_{k'} f_0 \rangle| = \sum_{|k-k'| \leq C} |\langle T\Delta_k f_0, T\Delta_{k'} f_0 \rangle| + \sum_{|k-k'| > C} |\langle T\Delta_k f_0, T\Delta_{k'} f_0 \rangle|
$$

with $C > 0$ a universal constant to be determined. Bound each of the two terms in the sum separately. Firstly,

$$
\sum_{|k-k'| \leq C} |\langle T\Delta_k f_0, T\Delta_{k'} f_0 \rangle| \leq \sum_{|k-k'| \leq C} \|T\Delta_k f_0\|_2 \|T\Delta_{k'} f_0\|_2 \lesssim C \sum_k \|T\Delta_k f_0\|^2_2
$$

Secondly, consider $k > k' + C$. Recall that (the absolute value of) the spectrum of $\Delta_k f_0$ is of order $2^k$. Similarly, the spectrum of $\Delta_{k'} f_0$ is of order $2^{k'}$, which, since $T_g$ for $g \in (G \cup G^{-1})(G \cup G^{-1})$ is a smooth $L^\infty$-perturbation of identity, implies that the norm of the derivative of $T_g \Delta_{k'} f_0$ is at most order $2^{k'}$. Hence,

$$
|\langle T\Delta_k f_0, T\Delta_{k'} f_0 \rangle| \lesssim 2^{-k} \|\Delta_k f_0\|_2 \|T\Delta_{k'} f_0\|_2 \lesssim 2^{-C} \|f_0\|^2_2.
$$

Thus,

$$
\sum_{k > k' + C} |\langle T\Delta_k f_0, T\Delta_{k'} f_0 \rangle| \lesssim \sum_{k > k' + C} 2^{k-k} \|\Delta_k f_0\|_2 \|\Delta_{k'} f_0\|_2 \lesssim 2^{-C} \|f_0\|^2_2,
$$

and so, for appropriate $C$,

$$
\sum_{|k-k'| > C} |\langle T\Delta_k f_0, T\Delta_{k'} f_0 \rangle| < 1/20.
$$

Concluding, using Claim 3.2,

$$
\sum_{k \geq k_0} \|\Delta_k f_0\|^2 \lesssim \|f_0\|^2_2 \lesssim 1/16 - 1/20 < \|Tf_0\|^2_2 - 1/20 \lesssim C \|T\Delta_k f_0\|^2_2.
$$

Set

$$
F = \frac{\Delta_k f_0}{\|\Delta_k f_0\|_2}
$$

10
with \( k \) from Claim \ref{claim3.3}. Thus, \((TF, TF) \geq c_1^2\) and so \(\|T^2F\|_2 \geq c_1^2\). Iterating, for all \(\ell > 0\) a power of two,
\[
\left\| T^\ell F \right\|_2 \geq c_1^\ell. \tag{3.2}
\]

To prove the theorem, argue that the norm of \( T^\ell F \) is actually small, thus obtaining the required contradiction: Let \( \gamma > 0 \) be a small universal constant (to be determined). Let \( \ell \) be the smallest power of two so that
\[
\ell > C_1(\gamma)k/\log(1/\varepsilon)
\]
and by Theorem \ref{thm3.1}
\[
\left\| \nu(\ell) * P_\delta \right\|_\infty < \delta^{-\gamma},
\]
with \( \varepsilon > 0 \) a small enough universal constant to be determined, and
\[
\delta = 4^{-k}.
\]

As \( \delta \) is small and the spectrum of \( F \) is controlled, the following claim holds.

**Claim 3.4.**
\[
\left\| \int_{\text{SL}_2(\mathbb{R})} (T_gF)((\nu(\ell) * P_\delta)(g)) \right\|_2 \gtrsim c_1^\ell.
\]

**Proof.** If \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) satisfies \( \|g - 1\|_2 \leq \eta \leq 1/20 \), then for all \( x \in \mathbb{R} \) so that \( |x| \leq 2 \),
\[
|x - gx| = \left| \frac{cx^2 + dx - ax - b}{cx + d} \right| \lesssim \eta.
\]
In addition, if \( h \in B_\delta(g) \) for \( g \in \text{supp}(\nu(\ell)) \), then
\[
\|h^{-1}g - 1\|_2 \leq \delta(1 + \varepsilon)^\ell.
\]
Recall, \( 2^k\delta(1 + \varepsilon)^\ell \) is much smaller than \( c_1^\ell \). Hence, since the norm of the derivative of \( F \) is at most order \( 2^k \),
\[
\|T_gF - T_hF\|_2 = \|F - T_{h^{-1}g}F\|_2 \lesssim 2^k\delta(1 + \varepsilon)^\ell.
\]
So,
\[
\left\| T^\ell F - \int_{\text{SL}_2(\mathbb{R})} (T_hF)((\nu(\ell) * P_\delta)(h)) \right\|_2 \lesssim 2^k(1 + \varepsilon)^\ell \delta \leq c_1^\ell/2.
\]
The claim follows by \ref{eq:3.2}. \( \square \)
The claim above contradicts the following proposition, as shown below. In short, the proposition follows by the flatness lemma and the subgroup structure of $\text{SL}_2(\mathbb{R})$.

**Proposition 3.5.** There exists universal constants $\sigma_0, C > 0$ so that

$$
\left\| \int_{\text{SL}_2(\mathbb{R})} (T_g F)((\nu(\ell) * P_\delta)(g)) dg \right\|_2 \lesssim \delta^{-\gamma} (1 + \varepsilon)^C \ell^{-2 - \sigma_0 \ell}.
$$

**Proof.** Bound, using Theorem 3.1 and unitarity of $T_h$, since the support of $\nu(\ell) * P_\delta$ is contained in $B_{2(1+\varepsilon)\ell}(1)$,

$$
\left\| \int (T_g F)((\nu(\ell) * P_\delta)(g)) dg \right\|_2^2 = \int \int \langle T_g F, T_h F \rangle ((\nu(\ell) * P_\delta)(g))(\nu(\ell) * P_\delta)(h)) dgdh
$$

$$
\lesssim \delta^{-2\gamma} (1 + \varepsilon)^{3\ell} \int_{B_{2(1+\varepsilon)\ell}(1)} |\langle T_g F, F \rangle| dg. \tag{3.3}
$$

Approximate $B_{2(1+\varepsilon)\ell}(1)$ by a smooth function: let $\kappa : \text{SL}_2(\mathbb{R}) \to \mathbb{R}_{\geq 0}$ be a smooth function so that $\|\kappa\|_\infty = 1$, and so that $\kappa(g) = 1$ if $\|g - 1\|_2 \leq 4(1 + \varepsilon)^{2\ell}$ and $\kappa(g) = 0$ if $\|g\|_2 > 8(1 + \varepsilon)^{2\ell}$. Using Cauchy-Schwarz inequality,

$$
|\langle T_g F, F \rangle\rangle_2 \lesssim \delta^{-2\gamma} (1 + \varepsilon)^{5\ell} \left( \int \langle T_g F, F \rangle^2 \kappa(g) dg \right)^{1/2}. \tag{3.4}
$$

Write

$$
\int |\langle T_g F, F \rangle|^2 \kappa(g) dg \leq \int \int |F(x)||F(y)| \left| \int T_g F(x)T_g F(y)\kappa(g) dg \right| dx dy.
$$

Separate to two cases, according to the distance between $x$ and $y$. Choose $\eta > 0$ small, to be determined. In both cases, use the following (convenient) parameterization of $\text{SL}_2(\mathbb{R})$:

$$
g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} u \cos \theta & v \cos \phi \\ u \sin \theta & v \sin \phi \end{pmatrix}
$$

with

$$
uv \sin(\phi - \theta) = 1.
$$

On the chart $a \neq 0$, we have

$$
dg = \frac{dadbdc}{|a|} = \frac{dud\theta d\phi}{|u| \sin^2(\theta - \phi)}.
$$

**Case one.** The first case is when $x, y$ are close: Bound

$$
\int \int_{|x - y| < \eta} |F(x)||F(y)| \int |T_g F(x)||T_g F(y)|\kappa(g) dg dx dy. \tag{3.5}
$$
Write $F = F_1 + F_\infty$ with
\[ \|F_1\|_1 \leq 2^{-\sigma k} \quad \text{and} \quad \|F_\infty\|_\infty \leq 2^{\sigma k} \]
for a universal constant $\sigma > 0$ to be determined. Equation (3.5) can be bounded from above by a sum of several terms (with different combinations of $F_1, F_\infty$ replacing $F$). Consider, e.g., substituting $F_1$ instead of the leftmost $F$ in (3.5),
\[
\int \int_{|x-y|<\eta} |F_1(x)||F(y)| \int |T_g F(x)||T_g F(y)||\kappa(g)dgdx dy \leq \int |F_1(x)| \int |T_g F(x)||\kappa(g)dgdx.
\]
(3.6)

Fix $x$, and denote
\[
M = (x + 1)^{-1/2} \left( \begin{array}{cc} 1 & -x \\ 1 & 1 \end{array} \right) \in \text{SL}_2(\mathbb{R}),
\]
so that $M(x) = 0$. (The matrix $M$ shows two-transitivity of the Möbius action: $M$ maps $x$ to zero and $-1$ to infinity. Note that $x, -1$ are far.) Change variables and use parametrization given above,
\[
\int |T_g F(x)||\kappa(g)dg = \int |T_{M^{-1}g^{-1}} F(x)||\kappa(M^{-1}g^{-1})|dg
\]
\[
\lesssim \int \int \int |F(\cot \phi)||\kappa(M^{-1}g^{-1})|\frac{1}{|\sin \phi||\sin(\theta - \phi)|}dud\theta d\phi.
\]
(3.7)

If $\kappa(M^{-1}g^{-1}) \neq 0$, then $\|g\|_2 \lesssim (1 + \varepsilon)^{2\ell}$, and so in the integral above $|\sin(\theta - \phi)| \gtrsim (1 + \varepsilon)^{-4\ell}$. Change variables again,
\[
|\text{(3.7)}| \lesssim (1 + \varepsilon)^{4\ell} \int \int \int |F(\xi)||\kappa(M^{-1}g^{-1})|\frac{1}{|\xi + 1|^{1/2}}dud\theta d\xi \lesssim (1 + \varepsilon)^{6\ell}.
\]

Hence,
\[
|\text{(3.6)}| \lesssim (1 + \varepsilon)^{6\ell} \|F_1\|_1 \leq (1 + \varepsilon)^{6\ell}2^{-\sigma k}.
\]

The same bound holds also if we replace each of the other three $F$’s by $F_1$ in (3.5). It thus remains to trivially bound
\[
\int \int_{|x-y|<\eta} |F_\infty(x)||F_\infty(y)| \int |T_g F_\infty(x)||T_g F_\infty(y)||\kappa(g)dgdx dy \lesssim \eta(1 + \varepsilon)^{6\ell}2^{4\sigma k},
\]
and conclude
\[
|\text{(3.5)}| \lesssim (1 + \varepsilon)^{6\ell} \left( \eta 2^{4\sigma k} + 2^{-\sigma k} \right).
\]
(3.8)
Case two. Next, understand what happens for far $x$ and $y$. The argument in this case is more elaborate and uses knowledge of the spectrum of $F$. Start by

$$\int \int_{|x-y| \geq \eta} |F(x)||F(y)| \left| \int T_g F(x)T_g F(y) \kappa(g) dg \right| dxdy \leq \left( \int \int_{|x-y| \geq \eta} \left| \int_{\text{SL}_2(\mathbb{R})} T_g F(x)T_g F(y) \kappa(g) dg \right|^2 dxdy \right)^{1/2}. \quad (3.9)$$

In this case, argue for fixed $x$ and $y$ in $[0, 1]$ so that $x \geq y + \eta$. Denote $M = (x - y)^{-1/2} \begin{pmatrix} 1 & -x \\ 1 & -y \end{pmatrix} \in \text{SL}_2(\mathbb{R})$, so that $M(x) = 0$ and $M(y) = \infty$. Change variables,

$$\left| \int T_g F(x)T_g F(y) \kappa(g) dg \right| = \left| \int T_{M^{-1}g^{-1}} F(x)T_{M^{-1}g^{-1}} F(y) \kappa(M^{-1}g^{-1}) dg \right| = (x - y)^{-1} \left| \int \frac{F(\cot \phi)F(\cot \theta)}{|\sin \phi \cdot \sin \theta|} \kappa(M^{-1}g^{-1}) \frac{dud\theta d\phi}{|u||\sin(\theta - \phi)|} \right|.$$  

Change variables,

$$\int \frac{F(\cot \phi)F(\cot \theta)}{|\sin \phi \cdot \sin \theta|} \kappa(M^{-1}g^{-1}) \frac{dud\theta d\phi}{|u||\sin(\theta - \phi)|} = \int \int F(\xi)F(\zeta)E(\xi, \zeta)d\xi d\zeta,$$

with

$$E(\xi, \zeta) = \frac{\sqrt{(1 + \xi^2)(1 + \zeta^2)}}{|\sin(\cot^{-1} \zeta - \cot^{-1} \xi)|} \int \kappa(M^{-1}g^{-1}) \frac{du}{|u|}. $$

Continue by using that Fourier basis diagonalize $\nabla$. Start by bounding the norms of $E$ and $\nabla E$. First, if $\kappa(M^{-1}g^{-1}) \neq 0$, then

$$\|g\|_2 \lesssim (1 + \varepsilon)^{2\ell} \eta^{-1/2}.$$

Hence, in the definition of $E$ we can assume

$$(1 + \varepsilon)^{-2\ell} \eta^{1/2} \lesssim |u| \lesssim (1 + \varepsilon)^{2\ell} \eta^{-1/2},$$

and

$$\frac{1}{|\sin(\cot^{-1} \zeta - \cot^{-1} \xi)|} \gtrsim (1 + \varepsilon)^{-4\ell} \eta.$$ 

Therefore, there is a universal constant $C > 0$ so that

$$\|E\|_\infty, \|\nabla E\|_\infty \lesssim (1 + \varepsilon)^{C\ell} \eta^{-C}.$$
Since the support of the Fourier transform of $F$ is of absolute value at least order $2^k$, bound
\[ \left| \int \int F(z)F(w)E(z,w)dzdw \right| \lesssim 2^{-k}(1 + \varepsilon)^{C\ell}\eta^{-C}. \]
Thus,
\[ |3.9| \lesssim 2^{-k}(1 + \varepsilon)^{C\ell}\eta^{-C-1}. \] (3.10)

**Concluding.** By (3.8) and (3.10),
\[ \sqrt{|3.3|} \lesssim \delta^{-\gamma}(1 + \varepsilon)^{C\ell} \left( \eta^{2\sigma_k} + 2^{-\sigma_k} + 2^{-k}\eta^{-C} \right)^{1/2} \leq \delta^{-\gamma}(1 + \varepsilon)^{C\ell}2^{-\sigma_k/4} \]
for appropriate choice of $\eta$, and with $\sigma > 0$ a universal constant.

We can finally conclude, using Claim 3.3 and Proposition 3.5
\[ c_1^\ell \lesssim |3.4| \lesssim \delta^{-\gamma}(1 + \varepsilon)^{C\ell}2^{-\sigma_k}, \] (3.11)
which is a contradiction for $\gamma = \sigma_0/4$, $k_0$ large and $\varepsilon$ small.

**4 Flatness via a product theorem**

Theorem 3.1 follows from the following flattening lemma, which roughly states that if
\[ \mu = \nu^{(\ell_0)} * P_\delta \]
is a little flat then $\mu * \mu$ is much flatter (unless $\mu$ is already very flat). The proof of the lemma is given in Section 4.1.

**Lemma 4.1.** Let $0 < \gamma < 3/2$. With the notation above, assume that
\[ \delta^{-\gamma} < \|\mu\|_2 < \delta^{-3/2 + \gamma} \]
and
\[ \ell_0 > C_2 \frac{\log(1/\delta)}{\log(1/\varepsilon)} \]
with $C_2 = C_2(\gamma) > 0$. Also assume that $\varepsilon > 0$, the parameter from 5 in Lemma 1.1 and $\delta > 0$ are small enough as a function of $\gamma$. Then, there exists $\sigma = \sigma(\gamma) > 0$ so that
\[ \|\mu * \mu\|_2 < \delta^\sigma \|\mu\|_2. \]
We apply the flattening lemma iteratively. To start iterating, we need to show that \( \mu \) is “a little flat” to begin with.

**Proposition 4.2.** If

\[ \ell_0 \geq \log_Q(1/\delta) \]

with \( Q \) from Lemma 1.1, then

\[ \| \mu \|_2 \leq \delta^{-3/2+\gamma} \]

with \( \gamma > 0 \) a universal constant.

This follows from Kesten’s bound, the following proposition about random walks on free groups.

**Proposition 4.3.** Assume \( H \) is a finite set freely generating a group. Denote

\[ \pi = (2|H|)^{-1} \sum_{h \in H} 1_h + 1_{h^{-1}}. \]

Denote by \( p^{(t)}(x,x) \) the probability of being at \( x \) after \( t \) steps in a random walk according to \( \pi \) started at \( x \). Then,

\[ \lim_{t \to \infty} \sup_{t}(p^{(t)}(x,x))^{1/t} = \frac{\sqrt{2k} - 1}{k}. \]

Denote by \( W_k(G) \) the set of words of length at most \( k \) in \( G \cup G^{-1} \).

**Proof of Proposition 4.2.** Let \( k \) be the maximal integer so that

\[ 1/Q^k \geq \delta^{1/2}. \]

For every \( y \in \text{supp}(\nu^{(k)}) \),

\[ \|y\|_2 \leq (1 + \varepsilon)^k \leq \delta^\varepsilon, \]

for \( \varepsilon \) small. By Lemma 1.1, the entries of elements in \( W_k(G) \) are in \( \mathbb{Z}/Q^k \). So, for all \( y \neq y' \) in \( W_k(G) \),

\[ \|y - y'\|_2 \geq \delta^{1/2}, \]

which implies

\[ (yB_\delta(1)) \cap (y'B_\delta(1)) = \emptyset, \]

for \( \varepsilon \) small. Hence,

\[ \left\| \sum_y \nu^{(k)}(y) P_\delta(y^{-1}) \right\|_2 \leq \left( \sum_y (\nu^{(k)}(y))^2 \| P_\delta(y^{-1}) \|_2^2 \right)^{1/2} \leq \left\| \nu^{(k)} \right\|_\infty^{1/2} \| P_\delta \|_2. \]

Finally, by Propositions 4.3 and Lemma 1.1 since convolution does not increase norms,

\[ \|\mu\|_2 \lesssim \left( \frac{2|G| - 1}{|G|^2} \right)^{k/4} \delta^{-3/2} < \delta^{-3/2+\gamma}. \]

\[ \square \]
Proof of Theorem 3.1. By Proposition 4.2 and Lemmas 4.1 and 1.1,

\[ \|\mu^{(k)}\|_2 = \| (\nu^{(\ell_0)} * P_\delta)^{(k)} \|_2 \leq \delta^{-\gamma/4} \]  

(4.1)

with \( k = k(\gamma) > 1 \) and

\[ \ell_0 \leq C_3 \frac{\log(1/\delta)}{\log(1/\varepsilon)} \]

with \( C_3 > 0 \) a constant. For every \( g \),

\[ \|\mu^{(2k)}(g)\| = \left\| \int_h \mu^{(k)}(h)\mu^{(k)}(h^{-1}g)dh \right\| \leq \|\mu^{(k)}\|_2 \leq \delta^{-\gamma/2} \]

Lemma 2.5 in [BG1] states

\[ cP_\delta \leq P_\delta * P_\delta \leq \frac{1}{c}P_\delta \]

with \( c > 0 \) a constant. Hence,

\[ \left\| \nu^{(\ell)} * P_\delta \right\|_\infty \leq C_4 (1 + \varepsilon)^{C_4\ell_0} \left\|\mu^{(2k)}\right\|_\infty \leq C_4 (1 + \varepsilon)^{C_4\ell_0} \delta^{-\gamma/2} \leq \delta^{-\gamma} \]

with \( C_4 = C_4(\gamma) > 0 \) and \( \ell \leq C_4\ell_0 \), for \( \varepsilon, \delta \) small.

4.1 A product theorem

The flattening lemma follows from the following product theorem. (The proof of the product theorem is deferred to Section 5.) We need to use metric entropy: for a subset \( S \) of a metric space denote by \( \mathcal{N}_\delta(S) \) the least number of balls of radius \( \delta \) needed to cover \( S \).

Theorem 4.4. For all \( \sigma_1, \tau > 0 \), there is \( \varepsilon_5 > 0 \) so that the following holds. Let \( \delta > 0 \) be small enough. Let \( A \subset \text{SL}_2(\mathbb{R}) \cap B_\alpha(1) \), \( \alpha > 0 \) a small universal constant, be so that

1. \( A = A^{-1} \),

2. \( \mathcal{N}_\delta(A) = \delta^{-3+\sigma_0} \),

\[ \sigma_1 \leq \sigma_0 \leq 3 - \sigma_1, \]

3. for every \( \delta < \rho < \delta^{\varepsilon_5} \), there is a finite set \( X \subset A \) so that \( |X| \geq \rho^{-\tau} \) and for every \( x \neq x' \) in \( X \) we have \( \|x - x'\|_2 \geq \rho \), and

4. w.r.t. every complex basis change diagonalizing some matrix in \( \text{SL}_2(\mathbb{R}) \cap B_1(1) \), there is \( g \in A_4 \) so that \( |g_{1,2}g_{2,1}| \geq \delta^{\varepsilon_5} \).
Then,
\[ \mathcal{N}_3(\text{AAA}) > \delta^{-65} \mathcal{N}_3(A). \]

The condition that \( A \) is contained in a small ball is not necessary, but simplifies the statement and the proof. The condition \( A = A^{-1} \) is, of course, not necessary as well, but simplifies notation. Condition \( 4 \) above implies that \( A \) is far from strict subgroups.

**Proof of Lemma 4.1.** We prove the lemma for 
\[ \ell_0 \sim C_2(\gamma) \frac{\log(1/\delta)}{\log(1/\varepsilon)}. \]

The proof for larger \( \ell_0 \) follows, as convolution does not increase the norm.

Assume towards a contradiction that 
\[ \| \mu * \mu \|_2 > \delta^{0+} \| \mu \|_2. \]

To prove the theorem, we shall find a set \( A \) that violates the product theorem. The set \( A \) will be one of the level sets of \( \mu \) in the following decomposition. Decompose \( \mu \) as
\[ \mu \sim \sum_j 2^j \chi_j, \]
where the sum is over \( O(\log(1/\delta)) \) values of \( j \) (recall that \( \mu \) is point-wise bounded by \( O(1/\delta^3) \) and we can ignore points with too small \( \mu \)-measure), and where \( \chi_j \) is the characteristic function of a set \( A_j \subset \text{SL}_2(\mathbb{R}) \) so that
\[ A_j = A_j^{-1}. \]

Choose \( j_1 < j_2 \) so that
\[ 2^{j_1 + j_2} \| \chi_{j_1} * \chi_{j_2} \|_2 \gtrsim \| \mu * \mu \|_2 / \log^2(1/\delta) \geq \delta^{0+} \| \mu \|_2. \]

Using Young’s inequality, bound
\[ 2^{j_1 + j_2} \| \chi_{j_1} \|_2 \| \chi_{j_2} \|_1 \geq \delta^{0+} \| \mu \|_2 \geq \delta^{0+} 2^{j_2} \| \chi_{j_2} \|_2. \]

So, since \( 2^{j_2} |A_{j_2}| \leq 1, \)
\[ 2^{j_1/2} |A_{j_1}|^{1/2} \geq 2^{j_1-j_2/2} |A_{j_1}|^{1/2} \geq 2^{j_1} |A_{j_1}|^{1/2} |A_{j_2}|^{1/2} \geq \delta^{0+}. \]

Similarly,
\[ 2^{j_1/2-j_2/2} \geq 2^{j_1/2} |A_{j_2}|^{1/2} \geq \delta^{0+}, \]
which implies
\[2^{j_1} < 2^{j_2} \leq \delta^{0-2^{j_1}}.\]

Since \(2^{j_2}|A_{j_2}| \leq 1\), using Young’s inequality and (4.2), we thus have
\[
\delta^{0+2^{-2j_2}}|A_{j_1}| \leq \langle \chi_{j_1} \ast \chi_{j_2}, \chi_{j_1} \ast \chi_{j_2} \rangle \leq \|\chi_{j_2}\|_2 \|\chi_{j_1} \ast \chi_{j_2}\|_2 \\
\leq \|\chi_{j_2}\|_2 \|\chi_{j_1}\|_1 \|\chi_{j_1} \ast \chi_{j_2}\|_2 \leq 2^{-3j_2/2} \|\chi_{j_1} \ast \chi_{j_1}\|_2.
\]

Hence,
\[
\|\chi_{j_1} \ast \chi_{j_1}\|_2^2 \geq \delta^{0+2^{-j_2}}|A_{j_1}|^2 \geq \delta^{0+2^{-j_1}}|A_{j_1}|^2 \geq \delta^0 + |A_{j_1}|^3. \tag{4.5}
\]

Use a version of Balog-Szemeredi-Gowers theorem proved in [T]. Denote
\[\mathcal{K} = B_r(1) \quad \text{with} \quad r = \delta^{-C_3(\gamma)\varepsilon} = \delta^{0-},\]
a compact subset of \(\text{SL}_2(\mathbb{R})\), with \(C_3(\gamma) \sim C_2(\gamma)\) to be determined. Specifically, if \(\varepsilon\) is small enough, then
\[A_{j_1} \subset \mathcal{K}.
\]

The \textit{multiplicative energy} of \(A_{j_1}\) is \(\|\chi_{j_1} \ast \chi_{j_1}\|_2^2\). Equation (4.5) implies that \(A_{j_1}\) has high energy. Theorem 5.4 (or, more precisely, its proof) in [T] implies that, for the appropriate \(C_3(\gamma)\), there exists \(H \subset \mathcal{K}\) which is an approximate group, namely,

\[H = H^{-1}\]

and there exists a finite set \(Y \subset \mathcal{K}\) of size
\[|Y| \leq \delta^{0-}\] (4.6)
satisfying
\[HH \subset YH\] (4.7)
so that
\[\delta^{0+}|A_{j_1}| \leq |H| \leq \delta^0 - |A_{j_1}|.\] (4.8)

In addition, there is \(y \in \mathcal{K}\) such that
\[|A_1| \geq \delta^{0+}|A_{j_1}|,\] (4.9)
where
\[A_1 = A_{j_1} \cap yH.\]
Finally, define
\[ A = ((A_1^{-1}A_1) \cup (A_1 A_1^{-1})) \cap B_\alpha(1), \]
for \( \alpha > 0 \) as in Theorem 4.4. Hence,
\[ |A| \geq \delta^{0+} |A_1| \geq \delta^{0+} |A_{j_1}|. \]  

(4.10)

We now prove that \( A \) violates the product theorem. We first show that it violates the conclusion of the product theorem and then show that it satisfies the assumptions of the product theorem.

Using (4.3) and Young’s inequality,
\[ 2^{j_1+j_2} |A_{j_2}|^{1/2} |A_{j_1}| = 2^{j_1+j_2} \| \chi_{j_2} \|_2 \| \chi_{j_1} \|_1 \geq \delta^{0+} \| \mu \|_2 \geq \delta^{0+} 2^{j_2} |A_{j_2}|^{1/2}. \]

Hence, using (4.9),
\[ \mu(yH) \geq \mu(A_1) \geq \delta^{0+} 2^{j_1} |A_{j_1}| \geq \delta^{0+}. \]  

(4.11)

On the other hand,
\[ \mu(yH) \lesssim \delta^{-3} \max_{z \in \supp(\mu(\ell_0))} |yH \cap B_{\delta_1^-(z)}|. \]

So, there is \( z_0 \in K \) so that
\[ |H \cap S| \geq \delta^{3+}, \]
with
\[ S = B_{\delta_1^-(z_0)}. \]

Let \( Z \) be a maximal set of points in \( H \) so that for all \( z \neq z' \) in \( Z \),
\[ zS \cap z'S = \emptyset. \]

Bound,
\[ \delta^{0-} |H| \geq |HH| \geq |Z||H \cap S| \geq \delta^{3+} \mathcal{N}_\delta(H). \]

Hence,
\[ \mathcal{N}_\delta(H) \leq \delta^{-3-} |H|. \]  

(4.12)

Finally,
\[ \mathcal{N}_\delta(AAA) \lesssim \mathcal{N}_\delta(H(6)) \leq \delta^{-3-} |H| \leq \delta^{-3-} |A| \leq \delta^{0-} \mathcal{N}_\delta(A). \]

So, indeed, the conclusion of the product theorem does not hold. It remains to prove that \( A \) satisfies the assumptions of the product theorem.

First,
\[ A = A^{-1}. \]
The second thing we show is that $A$ is not too small or too large. Equation (4.3) implies

$$\delta^{0+} \left\| \mu \right\|_2 \leq 2^{j_1+j_2} \left\| \chi_{j_1} \ast \chi_{j_2} \right\|_2 \leq 2^{j_1} \left\| \chi_{j_1} \right\|_2 2^{j_2} \left\| \chi_{j_2} \right\|_1 \leq 2^{j_1} |A_{j_1}|^{1/2},$$

which implies

$$\delta^{-\gamma^{+}} \leq 2^{j_1} |A_{j_1}|^{1/2} \lesssim \left\| \mu \right\|_2 \leq \delta^{-3/2+\gamma^{+}}.$$ 

Thus,

$$\delta^{-2\gamma^{+}} |A_{j_1}| \leq (2^{j_1} |A_{j_1}|)^2 \leq 1$$

and, using (4.4),

$$\delta^{0+} \leq (2^{j_1} |A_{j_1}|)^2 \lesssim \delta^{-3+2\gamma^{+}} |A_{j_1}|.$$ 

Therefore,

$$\delta^{3-2\gamma^{+}} \leq |A_{j_1}| \leq \delta^{2\gamma^{+}},$$ 

which implies, using (4.3),

$$\delta^{3-2\gamma^{+}} \leq |H| \leq \delta^{2\gamma^{+}}.$$ 

Therefore, using (4.10) and (4.9), (4.7), (4.12),

$$\delta^{-2\gamma^{+}} \leq \delta^{-3+|A_{j_1}|} \leq \delta^{-3+|A|} \leq N_{\delta}(A) \leq \delta^{-3-|H|} \leq \delta^{-3+2\gamma^+},$$

or

$$N_{\delta}(A) = \delta^{-3+\sigma_0},$$

with $\sigma_1 < \sigma_0 < 3 - \sigma_1$ and $\sigma_1 = 2\gamma^+ - 1$.

Thirdly, we prove that $A$ is well-distributed: Let $\varepsilon_5 = \varepsilon_5(\sigma_1, \tau) > 0$ be as given by Theorem 4.4 for $\tau > 0$ a universal constant to be determined, and let $\delta < \rho < \delta^{\varepsilon_5}$. We prove that there is a finite set $X \subset A$ so that $|X| \geq \rho^{-\tau}$ and for every $x \neq x'$ in $X$ we have $\left\| x - x' \right\|_2 \geq \rho$. Equation (4.11) says $\mu(A_1) \geq \delta^{0+}$. Write $\nu^{(\ell)} = \nu^{(\ell_0)} \ast \nu^{(|\ell|)}$, for $\ell < \ell_0$ the largest integer so that

$$Q^{-\ell} > \rho.$$ 

There thus exists $z_1 \in K$ so that

$$\nu^{(1)}(A_1 z_1) \geq \delta^{0+}.$$ 

By Lemma 1.1, for every $x \neq x'$ in $\text{supp}(\nu^{(\ell)}) \subseteq \mathcal{W}_\ell(\mathcal{G})$,

$$\left\| x - x' \right\|_2 \geq Q^{-\ell} > \rho.$$ 

By Proposition 4.3

$$\nu^{(\ell)}(A_1 z_1) \leq |\mathcal{W}_\ell(\mathcal{G}) \cap A_1 z_1| \left( \frac{2|\mathcal{G}| - 1}{|\mathcal{G}|^2} \right)^{\ell/2}.$$ 

Thus, using Lemma 1.1 again,

$$N_{\rho}(A) \geq \delta^{0+} N_{\rho}(A_1 z_1) \geq \delta^{0+} |\mathcal{W}_\ell(\mathcal{G}) \cap A_1 z_1| \geq \delta^{0+} \left( \frac{|\mathcal{G}|^2}{2|G| - 1} \right)^{\ell/2} \geq \rho^{-\tau},$$

or

$$N_{\rho}(A) = \delta^{-3+\sigma_0},$$

with $\sigma_1 < \sigma_0 < 3 - \sigma_1$ and $\sigma_1 = 2\gamma^+ - 1$. 

for $\tau \sim 1$.

It remains to show that $A$ contains matrices with certain properties. That is, w.r.t. every basis in a bounded domain, there is $g \in A(4)$ so that $|g_{1,2}g_{2,1}| \geq \delta_{\varepsilon_5}$. Fix a basis diagonalizing some matrix in $SL_2(\mathbb{R}) \cap B_1(1)$. Choose $\ell_1$ large, to be determined. By Proposition 8 from [BG2], since the elements of $G$ freely generate a group, if $S \subset W_{\ell_1}(G)$ is so that for all $g_1, g_2, g_3, g_4 \in S$, the bi-commutator $[[g_1, g_2], [g_3, g_4]]$ is 1, then $|S| \leq \ell_1^6$. As above, there is $z_2 \in K$ so that

$$|W_{\ell_1}(G) \cap A_{1}z_2| \geq \delta_0\left(\frac{|G|^2}{2|G| - 1}\right)^{\ell_1/2}.$$ 

The set $A_{1}z_2$ is contained in a ball of radius $r' = \delta_0^{-1}$ around 1. Cover the ball of radius $r'$ around 1 by balls of radius $\beta = \alpha/(r' + 1) \geq \delta_0^{-1}$. There thus exists $z_3 \in W_{\ell_1}(G) \cap A_{1}z_2$ so that

$$|W_{\ell_1}(G) \cap A_{1}z_2 \cap B_{\beta}(z_3)| \geq \delta_0\left(\frac{|G|^2}{2|G| - 1}\right)^{\ell_1/2} > \ell_1^6$$

(the last inequality is the first property $\ell_1$ should satisfy). Hence, there are $g_1, g_2, g_3, g_4 \in (W_{\ell_1}(G) \cap A_{1}z_2 \cap B_{\beta}(z_3))z_3^{-1} \subset A_1A_1^{-1}$ with non-trivial bi-commutator. For every $g' \in \{g_1, g_2, g_3, g_4\}$,

$$\|g' - 1\|_2 \leq \|g'z_3 - z_3\|_2 (r' + 1) \leq \beta(r' + 1) = \alpha,$$

which implies

$$g' \in A.$$ 

If $g' \in \{g_1, g_2, g_3, g_4\}$ is so that $|(g')(1,2)(g')_{2,1}| \neq 0$, then

$$|(g')_{1,2}(g')_{2,1}| \geq Q^{-20\ell_1} \geq \delta_{\varepsilon_5}$$

(this is the second property $\ell_1$ should satisfy). In this case, we are done. Otherwise, recall that if four $2 \times 2$ matrices are either all upper triangular or all lower triangular, then they have a trivial bi-commutator. So, w.l.o.g. $g_1$ is lower triangular and $g_2$ is upper triangular, which implies that $g_1g_2$ has the required property.

\[ \square \]

5 A product theorem

In this section we prove the product theorem, Theorem 4.3. The proof consists of several parts given in the following sub-sections. (The outline of the proof follows [BG1], but the proof in our case is more elaborate.) The theorem is finally proved in Section 5.5. We start this section with a brief outline of the proof of the product theorem. We note that not only field properties
are used but also metric properties, the argument is a multi-scale one. Here are the steps of the proof (ignoring many technicalities).

We wish to prove that a set $A$ with certain properties becomes larger when multiplied by itself.

(i) Assume toward a contradiction that $A_{(3)}$ is not larger than $A$.

(ii) Assuming (i), find a set $V$ of commuting matrices which is not too small and is close to $A_{(2)}$. To do so, use a version of the Balog-Szemerédi-Gowers theorem.

(iii) If $V$ is concentrated in a small ball, then $AV$ will “move $V$ around” and hence $AV$ will be much bigger than $A$. This is a contradiction, as $AV$ is close to $A_{(3)}$.

(iv) Otherwise, $V$ is not concentrated on any ball, which means that it is well-distributed. In this case, use the discretized ring conjecture, which roughly states that a well-distributed set in $\mathbb{R}$ becomes larger under sums and products. To move from $\text{SL}_2(\mathbb{R})$ to $\mathbb{R}$, use matrix-trace, which translates matrix-product to sums and products in the field.

In fact, the size of $V$ obtained is roughly $|A|^{1/3}$. To get back to the “correct” order of magnitude, we use that $A$ is far from strict subgroups in that it contains a matrix $g$ so that $g_1, g_2, g_1^{-1}$ is far from zero (w.r.t. any basis change). In rough terms, this property of $A$ is used to show that the size of $VgVgV$ is $|V|^3 \sim |A|$.

\section*{5.1 Finding commuting matrices}

In this sub-section we show that, under some non-degeneracy conditions, a set of matrices induces a not-too-small set of commuting matrices. To prove this, we also show that a set of matrices induces a not-too-small trace-set. We start by stating the results. The proofs follow.

The trace of a matrix $g$ is $\text{Tr}g = g_{1,1} + g_{2,2}$. Every $g$ in $\text{SL}_2(\mathbb{C})$ with $|\text{Tr}g| \neq 2$ can be diagonalized. (Elements $g$ in $\text{SL}_2(\mathbb{R})$ with $|\text{Tr}g| < 2$ have complex eigenvalues, so we must consider $\text{SL}_2(\mathbb{C})$.) Define $\text{Diag}$ to be the set of diagonal matrices $\nu$ in $\text{SL}_2(\mathbb{C})$ so that $\text{Tr} \nu \in \mathbb{R}$.

The following lemma shows that, at least in one “direction,” the trace-set of a set is not too small.

\textbf{Lemma 5.1.} Think of $\text{SL}_2(\mathbb{R})$ as a subset of $\mathbb{R}^4$, and let $g_0, g_1, g_2, g_3 \in \text{SL}_2(\mathbb{R}) \cap B_{1/2}(1)$ be so that

$$|\det(g_0, g_1, g_2, g_3)| \geq \delta_{0+},$$

and let $A \subset \text{SL}_2(\mathbb{R}) \cap B_{1/2}(1)$. Then, there is $I \subset \{0, 1, 2, 3\}$ of size $|I| = 3$ so that

$$\prod_{i \in I} \mathcal{N}_\delta(\text{Tr}g_i^{-1}A) \geq \delta_{0+} \mathcal{N}_\delta(A).$$
The following lemma allows to find a commuting set of matrices via trace.

**Lemma 5.2.** Let \( A \subset \text{SL}_2(\mathbb{C}) \cap B_\alpha(1) \), \( \alpha > 0 \) a small constant, be so that \( \text{dist}(A, \pm 1) \geq \delta^{0+} \). Then, there exists a set \( V \subset \text{SL}_2(\mathbb{C}) \) of commuting matrices so that

\[
\mathcal{N}_\delta(V) \geq \delta^{0+} \frac{\mathcal{N}_\delta(\text{Tr}A) \mathcal{N}_\delta(A)}{\mathcal{N}_\delta(A^2A^{-1})},
\]

and every \( v \in V \) satisfies \( \text{dist}(v, A^{-1}A) \leq \delta^1 \).

We shall also need the following corollary of the two lemmas.

**Corollary 5.3.** Let \( A \subset \text{SL}_2(\mathbb{R}) \cap B_\alpha(1) \), \( \alpha > 0 \) a small constant. Let \( g_1, g_2, g_3 \in \text{SL}_2(\mathbb{R}) \cap B_\alpha(1) \) be so that \( |\det(1, g_1, g_2, g_3)| \geq \delta^{0+} \). Then, there is a set of commuting matrices \( V \subset \text{SL}_2(\mathbb{C}) \) so that there is \( g_0 \in \{1, g_1, g_2, g_3\} \) so that

\[
\mathcal{N}_\delta(V) \geq \delta^{0+} \frac{\mathcal{N}_\delta(A)^{1/3}}{\mathcal{N}_\delta(Ag_0^{-1}AA^{-1})},
\]

and every \( v \in V \) satisfies \( \text{dist}(v, A^{-1}A) \leq \delta^1 \).

**Proof of Lemma 5.2**. For \( i \in \{0, 1, 2, 3\} \), denote

\[
ge_i' = \begin{pmatrix} d_i & -c_i \\ -b_i & a_i \end{pmatrix},
\]

where

\[
ge_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}.
\]

By (5.1),

\[
|\det(g_0', g_1', g_2', g_3')| = |\det(g_0, g_1, g_2, g_3)| \geq \delta^{0+}.
\]

Hence, let \( A' \subset A \) be contained in a ball of radius \( \delta^{0+} \) so that

\[
\mathcal{N}_\delta(A') \leq \delta^{0-} \mathcal{N}_\delta(A'),
\]

and so that there is a set \( I \subset \{0, 1, 2, 3\} \) of size \( |I| = 3 \) so that

\[
\mathcal{N}_\delta(A') \leq \delta^{0-} \mathcal{N}_\delta(PA'),
\]

where \( P \) is the projection to the sub-space \( \text{span}\{g_i' : i \in I\} \). (The map \( g \mapsto Pg \) restricted to a small ball is a diffeomorphism with bounded distortion.) For every \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) in \( \text{SL}_2(\mathbb{R}) \),

\[
\text{Tr}g_i^{-1} g = d_i a - b_i c - c_i b + a_i d = \langle g, g_i' \rangle,
\]

24
with the standard inner product over \( \mathbb{R}^4 \). Thus,
\[
\mathcal{N}_\delta(PA') \leq \delta^0 \prod_{i \in I} \mathcal{N}_\delta(\text{Tr}g_i^{-1}A') \leq \delta^0 \prod_{i \in I} \mathcal{N}_\delta(\text{Tr}g_i^{-1}A).
\]

\( \square \)

**Proof of Lemma 5.2.** Choose \( T \subset \text{Tr}A \) so that
\[
|T| \sim \mathcal{N}_\delta(\text{Tr}A),
\]
and so that for all \( t \neq t' \) in \( T \),
\[
|t - t'|, |t - 2|, |t + 2| > 2\delta.
\]
(If \( \mathcal{N}_\delta(\text{Tr}A) \) is small, the lemma trivially holds.) Since trace is continuous,
\[
\sum_{t \in T} \mathcal{N}_\delta \left( \{ g \in A^2A^{-1} : |\text{Tr}g - t| < \delta/4 \} \right) \lesssim \mathcal{N}_\delta(A^2A^{-1}).
\]
There thus exists \( t_0 \in T \) so that the set
\[
A_0 = \{ g \in A^2A^{-1} : |\text{Tr}g - t_0| < \delta/4 \}
\]
satisfies
\[
\mathcal{N}_\delta(A_0) \lesssim \frac{\mathcal{N}_\delta(A^2A^{-1})}{|T|}.
\]
Choose \( g_0 \in A \) so that \( \text{Tr}g_0 = t_0 \).
Choose \( A_1 \subset A_0 \) so that
\[
|A_1| = \mathcal{N}_\delta(A_0)
\]
and
\[
A_0 \subset \bigcup_{g \in A_1} B_\delta(g). \tag{5.3}
\]
For \( g \in A_1 \), define (with a slight abuse of notation)
\[
A_g = \{ x \in A : xg_0x^{-1} \in B_\delta(g) \}.
\]
Since for every \( x \) we have \( \text{Tr}xg_0x^{-1} = \text{Tr}g_0 = t_0 \), for every \( x \in A \) we have \( xg_0x^{-1} \in A_0 \). Equation \( \tag{5.3} \) thus implies
\[
A = \bigcup_{g \in A_1} A_g.
\]
Hence, there is \( g_1 \in A_1 \) so that
\[
N_\delta(A_{g_1}) \geq \frac{N_\delta(A)}{|A_1|} = \frac{N_\delta(A)}{N_\delta(A_0)} \geq \frac{N_\delta(A)}{N_\delta(A^2A^{-1})|T|}.
\] (5.4)

Fix \( x_1 \in A_{g_1} \). By definition, for every \( x \in A_{g_1} \),
\[
\|xg_1x^{-1} - x_1g_1x_1^{-1}\| \leq 2\delta.
\]
Since \( A \) is bounded,
\[
\|yg_1 - g_1y\| \lesssim \delta,
\]
where
\[
y = x_1^{-1}x \in x_1^{-1}A_{g_1}.
\]
Since \( g_1 \in A \) is far from \( \pm 1 \), conclude that diagonalizing \( g_1 \) makes \( x_1^{-1}A \) close to diagonal: Since \( |\text{Tr}g_1| \neq 2 \), there exists a matrix \( u \) so that \( v_1 = ug_1u^{-1} \) is diagonal. By assumption on \( A \),
\[
\text{dist}(v_1, \pm 1) \sim \text{dist}(g_1, \pm 1) \geq \delta^0+.
\]
So,
\[
|(v_1)_{1,1} - (v_1)_{2,2}| \geq \delta^0+.
\]
In addition,
\[
\|uyu^{-1}v_1 - v_1uyu^{-1}\| \lesssim \delta.
\]
Hence,
\[
|(uyu^{-1})_{1,2}|, |(uyu^{-1})_{2,1}| \lesssim \delta^1-.
\]
Since \( |\det(uyu^{-1})| = 1 \), there is thus a diagonal \( v \in \text{SL}_2(\mathbb{C}) \) so that
\[
\|uyu^{-1} - v\| \lesssim \delta^1-.
\]
We can thus conclude that \( x_1^{-1}A_{g_1} \subset A^{-1}A \) is in a \((\delta^1-)\)-neighborhood of a set \( V \subset \text{SL}_2(\mathbb{C}) \) of commuting matrices. In particular,
\[
N_\delta(V) \geq \delta^0+ N_\delta(A_{g_1}).
\]
Equations (5.4) and (5.5) imply the claimed lower bound on \( N_\delta(V) \).

\[\square\]

Proof of Corollary 5.3. Since \( |\det(1, g_1, g_2, g_3)| \geq \delta^0+ \), the pairwise distances between \( \pm 1, \pm g_1, \pm g_2, \pm g_3 \) are at least \( \delta^0+ \). Thus, there exists a subset \( A' \) of \( A \) so that
\[
N_\delta(A') \geq \delta^0+ N_\delta(A)
\]
and
\[ \text{dist}(A', \{ \pm 1, \pm g_1, \pm g_2, \pm g_3 \}) \geq \delta^{0+}. \]

By Lemma 5.1 there exists \( g_0 \in \{ 1, g_1, g_2, g_3 \} \) so that
\[ N_\delta(Tr g_0^{-1} A') \geq \delta^{0+} N_\delta(A')^{1/3}. \]

Now, apply Lemma 5.2 on the set \( g_0^{-1} A' \) to complete the proof.

5.2 Trace expansion via discretized ring conjecture

The following lemma is the main result of this section. The lemma roughly tells us that if a set \( V \) of commuting matrices is well-distributed then adding a non-commuting element to \( V \) makes its trace-set grow under products.

**Lemma 5.4.** For every \( 0 < \sigma < 2 \) and \( 0 < \kappa < 1 \), there is \( \varepsilon_4 > 0 \) so that the following holds. Let \( V \subset \text{SL}_2(\mathbb{C}) \cap B_\alpha(1), \alpha > 0 \) a small constant, be so that \( V = V^{-1} \), so that \( \text{dist}(v, \text{Diag}) \leq \delta^{1-} \) for all \( v \) in \( V \), so that
\[ N_\delta(V) = \delta^{-\sigma}, \]
and so that for all \( \delta < \rho < \delta^{\varepsilon_4} \),
\[ \max_a N_\delta(V \cap B_\rho(a)) < \rho^\kappa \delta^{-\sigma}. \quad (5.5) \]

Let \( g = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{C}) \cap B_\alpha(1) \) be so that \( \text{Tr} g \in \mathbb{R} \) and \( |bc| \geq \delta^{\varepsilon_4} \). Then,
\[ N_\delta(\text{Tr} W g W g) \geq \delta^{-\sigma-\varepsilon_4}, \]
where \( W = V_{(8)} \).

The starting point here is the discretized ring conjecture. This conjecture was first proved in \[B1\] and later strengthened in \[BG1\], see Proposition 3.2 in \[BG1\].

**Lemma 5.5.** For all \( 0 < \sigma, \kappa < 1 \), there is \( \varepsilon_2 > 0 \) so that for all \( \delta > 0 \) small, the following holds. Let \( A \subset [-1, 1] \) be a union of \( \delta \)-intervals so that
\[ |A| = \delta^{1-\sigma} \]
and for all \( \delta < \rho < \delta^{\varepsilon_2} \),
\[ \max_a |A \cap B_\rho(a)| < \rho^\kappa |A|. \]

Then,
\[ |A + A| + |AA| > \delta^{1-\sigma-\varepsilon_2}. \]
The discretized ring conjecture was used in [BG1] to prove “scalar amplification,” i.e., the following proposition.

**Proposition 5.6.** For all $0 < \sigma, \kappa < 1$, there is $\varepsilon_3 > 0$ so that the following holds. Let $S \subset \mathbb{C}$ be a subset of the complex unit circle, so that $S$ is a union of $\delta$-arcs, $\delta > 0$ small enough, so that $S = S^{-1}$, so that

$$|S| = \delta^{1-\sigma}$$

(size is measured in the unit circle), and so that for all $\delta < \rho < \delta^{\varepsilon_3}$,

$$\max_a |S \cap B_\rho(a)| < \rho^\kappa |S|. \quad (5.6)$$

If $\gamma, \lambda \in \mathbb{R}$ are so that $\gamma > 0, |\lambda| \geq \delta^{\varepsilon_3}$, then the set

$$D = \{xy + \gamma/(xy) + \lambda(x/y + y/x) : x, y \in S(4)\}$$

satisfies

$$\mathcal{N}_\delta(D) \geq \delta^{-\varepsilon_3 - \sigma}.$$  

We also need and prove the following variant of scalar amplification.

**Proposition 5.7.** For all $0 < \sigma, \kappa < 1$, there is $\varepsilon_3 > 0$ so that the following holds. Let $S \subset [1/2, 2]$ be a union of $\delta$-intervals, $\delta > 0$ small enough, so that $S = S^{-1}$, so that

$$|S| = \delta^{1-\sigma},$$

and so that for all $\delta < \rho < \delta^{\varepsilon_3}$,

$$\max_a |S \cap B_\rho(a)| < \rho^\kappa |S|. \quad (5.7)$$

If $\gamma, \lambda \in \mathbb{R}$ are so that $\gamma > 0, |\lambda| \geq \delta^{\varepsilon_3}$, then the set

$$D = \{xy + \gamma/(xy) + \lambda(x/y + y/x) : x, y \in S(4)\}$$

satisfies

$$\mathcal{N}_\delta(D) \geq \delta^{-\varepsilon_3 - \sigma}.$$  

Lemma 5.4 follows from scalar amplification.

**Proof of Lemma 5.4.** Let $V_0 \subset \text{Diag}$ be so that $\text{dist}(v, V_0) \leq \delta_0 = \delta^{1-}$ for all $v$ in $V$ and $\text{dist}(v_0, V) \leq \delta_0$ for all $v_0$ in $V_0$. Specifically, for all $\delta_0 < \rho < \delta_0^{2\varepsilon_4}$,

$$\max_a \mathcal{N}_{\delta_0}(V_0 \cap B_\rho(a)) \leq \delta^{0-} \max_a \mathcal{N}_{\delta_0}(V \cap B_\rho(a)) \leq \delta^{0-} \rho^\kappa \delta^{-\sigma}. \quad (5.8)$$

28
Observe
\[
\text{Tr} \left( \begin{array}{c} x \\ 1/x \end{array} \right) g \left( \begin{array}{c} y \\ 1/y \end{array} \right) g = a^2 xy + d^2/(xy) + bc(x/y + y/x). \tag{5.9}
\]
Write
\[
V_0 = \left\{ \left( \begin{array}{c} x \\ 1/x \end{array} \right) : x \in T \right\}.
\]
The set \( T \) is contained in the real numbers union the complex unit circle. Denote by \( T_1 = T \cap \mathbb{R} \), and \( T_2 = T \setminus T_1 \). First, assume
\[
\mathcal{N}_{\delta_0}(T_1) \sim \mathcal{N}_{\delta_0}(V_0). \tag{5.10}
\]
Define \( S_1 \) to be a \( \delta_0 \)-neighborhood of \( T_1 \). Thus,
\[
|S_1| = \delta_0^{1-\sigma_1}
\]
with \( \sigma_1 \geq \sigma/2 \). Equation (5.8) implies that \( S_1 \) satisfies (5.7) with \( \kappa_1 = \kappa/2 \). As in Propositions 5.7 denote
\[
D_1 = a^2 \{ xy + \gamma/(xy) + \lambda(x/y + y/x) : x, y \in (S_1)_{(4)} \}.
\]
with \( \gamma = (d/a)^2 \) and \( \lambda = bc/a^2 \). Observe, \( ad - bc = 1 \) and \( a + d \in \mathbb{R} \) imply \( d/a \in \mathbb{R} \) and \( bc/a^2 \in \mathbb{R} \). In addition, \( |\lambda| \geq \delta_0^{0+} \). The proposition thus implies
\[
\mathcal{N}_{\delta_0}(D_1) \geq \delta_0^{-\varepsilon_3 - 1}|S_1| \geq \delta^{-\varepsilon_3 - \sigma_1}.
\]
Using (5.9), conclude
\[
\mathcal{N}_{\delta}(\text{TrWgWg}) \geq \delta^{-\sigma - \varepsilon_3^+}.
\]
When (5.10) does not hold, consider \( T_2 \) and use Proposition 5.6 instead of Proposition 5.7. \( \Box \)

**Proof of Proposition 5.7.** Assume towards a contradiction that the proposition does not hold. W.l.o.g., for every \( s \) in \( S \),
\[
\text{dist}(s, \{ \gamma^{1/4}, 1 \}) \geq \delta^{0+}. \tag{5.11}
\]
We first find a set \( A \) so that \( A + A \) is not much larger than \( A \). If \( s, s' \in S \), then \( x = s'/s \in S_{(2)} \) and \( y = ss' \in S_{(2)} \) satisfy \( xy = s'^2 \) and \( y/x = s^2 \). By assumption, we can thus conclude
\[
\left\| \left\{ (s'^2 + \gamma/s^2) + \lambda(s^2 + 1/s^2) : s', s \in S_{(2)} \right\} \right\| \lesssim \delta^{-\varepsilon_3}|S|.
\]
Denote
\[
A = \{ \lambda(s^2 + 1/s^2) : s \in S_{(2)} \}
\]
29
and

\[ A' = \{ s'^2 + \gamma/s'^2 : s' \in S(2) \}. \]

Since \(|\lambda| \geq \delta^{0+}\),

\[ |A| \geq \delta^{0+}|S|. \]

By (5.11), the derivative of the map \( s' \mapsto s'^2 + \gamma/s'^2 \) is bounded away from zero in the relevant range. Thus,

\[ |A'| \geq \delta^{0+}|S|. \]

Ruzsa's inequality in measure version for open sets \( A, A' \subset \mathbb{R} \) states

\[ |A + A| \leq |A + A'|^2/|A'|. \]

(see, e.g., Lemma 3.2 in [T]). Therefore,

\[ |A + A| \leq \delta^{0-}|S|. \] (5.12)

We now find a set that does not significantly increase its size under sums and products. Define

\[ A_1 = \{ s^2 + 1/s^2 : s \in S \}. \]

By (5.11),

\[ |A_1| \geq \delta^{0+}|S|. \]

Hence, by (5.12), since \(|\lambda| \geq \delta^{0+}\),

\[ |A_1 + A_1| \leq \delta^{0-}|A + A| \leq \delta^{0-}|A_1|. \]

Observe

\[ (s_1^2 + 1/s_1^2)(s_2^2 + 1/s_2^2) = ((s_1s_2)^2 + 1/(s_1s_2)^2) + ((s_1/s_2)^2 + 1/(s_1/s_2)^2). \]

Hence, using (5.12), since \(|\lambda| \geq \delta^{0+}\),

\[ |A_1A_1| \leq \delta^{0-}|A + A| \leq \delta^{0-}|A_1|. \]

So,

\[ |A_1 + A_1| + |A_1A_1| \leq \delta^{0-}|A_1|. \]

If \( \varepsilon_3 > 0 \) is small enough, we can set \( 0 < \sigma' < 1 \) so that

\[ |A_1| = \delta^{1-\sigma'}. \]

Choose \( \kappa' = \kappa/2 \). Set \( \varepsilon_2 = \varepsilon_2(\sigma', \kappa') > 0 \) as in Lemma 5.5. If \( \varepsilon_3 > 0 \) is small enough, then for every \( \delta < \rho < \delta^{0+} \),

\[ \max_a |A_1 \cap B_\rho(a)| \leq \delta^{0-} \max_a |S \cap B_\rho(a)| < \delta^{0-}\rho^{\kappa'}|S| \leq \delta^{0-}\rho^{\kappa'}|A_1| \leq \rho^{\kappa'}|A_1|. \]

This contradicts Lemma 5.5. □
5.3 Expansion using a non-commuting element

We shall use the following variant of a lemma from [BG1], see [H] as well. Roughly, the lemma states that adding a non-commuting element to a commuting set of matrices makes it grow under products.

**Lemma 5.8.** Let $V \subset \text{SL}_2(\mathbb{C}) \cap B_{\alpha}(1)$, $\alpha$ a small constant, be so that $\text{dist}(v, \text{Diag}) \leq \delta^{1-}$ for all $v$ in $V$. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Diag} \cap B_{\alpha}(1)$ be so that $|bc| \geq \delta^{0+}$. Then,

$$N_{\delta}(V g V g V) \geq \delta^{0+} N_{\delta}(V)^3.$$

**Proof.** Assume

$$N_{\delta}(V) > \delta^{0-}$$

(otherwise, the lemma trivially holds). There are several cases to consider.

1. Denote by $\text{Diag}_{\mathbb{R}}$ the set of matrices in $\text{Diag}$ with entries in $\mathbb{R}$. Consider the case that there is a subset of $\text{Diag}_{\mathbb{R}}$ with comparable metric entropy to that of $V$: Assume that there is $Z \subset \mathbb{R}$ so that $|Z| \geq \delta^{0+} N_{\delta}(V)$, so that for all $z \in Z$,

$$\text{dist} \left( \begin{pmatrix} z \\ 1/z \end{pmatrix}, V \right) \leq \delta^{1-},$$

and so that for all $z \neq z'$ in $Z$,

$$|z - z'| > \delta.$$

W.l.o.g., assume that $z \geq \sqrt{d/a}$ (the proof in the other case is similar). Furthermore, by (5.13), we can assume w.l.o.g. that

$$z - \sqrt{d/a}, |z - 1| \geq \delta^{0+}.$$

For $z = (z_1, z_2, z_3)$ in $Z^3$, denote

$$M_z = \begin{pmatrix} z_1 & 1/z_1 \\ 1/z_2 & z_2 \end{pmatrix} g \begin{pmatrix} z_2 & 1/z_2 \\ 1/z_3 & z_3 \end{pmatrix}.$$

To prove the lemma, we will show that for all $z \neq z'$ in $Z^3$,

$$\|M_z - M_{z'}\| \geq \delta^{1+}.$$

Observe

$$M_z = \begin{pmatrix} z_1 z_3 (a^2 z_2 + bc/z_2) & (z_1/z_3) b (a z_2 + d/z_2) \\ (z_3/z_1) c (a z_2 + d/z_2) & 1/z_1 z_3 (bc z_2 + d^2/z_2) \end{pmatrix}.$$
Consider the following two cases.

1.1. The first case is when $z_2 > z'_2$. We have two sub-cases to consider.

1.1.1. The first sub-case is $|z_1/z_3 - z'_1/z'_3| \geq \delta^{1+}$. Bound

$$|(M_z)_{1,2} / (M_z)_{1,2} - (M_{z'})_{1,2} / (M_{z'})_{1,2}| = |b/c| \cdot |(z_1/z_3)^2 - (z'_1/z'_3)^2| \geq \delta^{1+}.$$  

Thus,

$$\delta^{1+} \leq |(M_z)_{1,2} (M_z')_{1,2} - (M_{z'})_{1,2} (M_z)_{1,2}|$$

$$= |((M_z)_{1,2} - (M_{z'})_{1,2}) (M_{z'})_{1,2} + (M_{z'})_{1,2} ((M_{z'})_{1,2} - (M_z)_{1,2})|.$$  

So,

$$\|M_z - M_{z'}\| \geq \delta^{1+}.$$  

1.1.2. The second sub-case is $|z_1/z_3 - z'_1/z'_3| < \delta^{1+}$. Bound

$$|(M_z)_{1,2} - (M_{z'})_{1,2}| = |b/a| \cdot |(z_1/z_3) (z_2 + (d/a)/z_2) - (z'_1/z'_3) (z'_2 + (d/a)/z'_2)|$$

$$\geq |b/a| \cdot |z_2 + (d/a)/z_2 - z'_2 + (d/a)/z'_2| - \delta^{1+}.$$  

The map $z_2 \mapsto z_2 + (d/a)/z_2$ has derivative at least $\delta^{0+}$ for $z_2 \geq \sqrt{d/a} + \delta^{0+}$. So,

$$|(M_z)_{1,2} - (M_{z'})_{1,2}| \geq \delta^{1+}.$$  

1.2. The second case is $z_2 = z'_2$ and $(z_1, z_3) \neq (z'_1, z'_3)$. Assume w.l.o.g. $z_1 \neq z'_1$ (the argument in the other case is similar). Since the entries of $g \left( \begin{pmatrix} z_2 \\ 1/z_2 \end{pmatrix} \right) g$ are bounded away from 0 and $V$ is close to 1,

$$\|M_z - M_{z'}\| \geq \delta^{0+} \| (z_1 z_3 - z'_1 z'_3, z_1 z'_3 - z'_1 z_3) \|.$$  

Since $\|(z_3, z'_3)\| \geq 1$ and $\det \left( \begin{pmatrix} z_1 & -z'_1 \\ -z'_1 & z_1 \end{pmatrix} \right) \geq \delta$,

$$\|(z_1 z_3 - z'_1 z'_3, z_1 z'_3 - z'_1 z_3)\| \geq \delta.$$  

2. Otherwise, there is a subset of $\text{Diag} \setminus \text{Diag}_{R}$ with comparable metric entropy to that of $V$: There is a subset of the complex unit circle $Z$ so that $|Z| \geq \delta^{0+} N_0(V)$, so that for all $z \in Z$,

$$\text{dist} \left( \left( \begin{pmatrix} z \\ 1/z \end{pmatrix}, V \right) \leq \delta^{1-},$$  

and so that for all $z \neq z'$ in $Z$,

$$|z - z'| > \delta.$$
Assume w.l.o.g. that \( \text{dist}(Z, 1) \geq \delta^{0+} \). Also assume w.l.o.g. that every element of \( Z \) has positive imaginary part (the other case is similar).

2.1. When \( z_2 \neq z'_2 \), bound

\[
\|z_2 - z'_2\| = |(M_z)_{1,2} - (M_{z'})_{1,2}| = |ba||z_2 + (d/a)/z_2 - |z'_2 + (d/a)/z'_2||.
\]

If we denote, \( z_2 = e^{i\theta_2} \) and \( z'_2 = e^{i\theta'_2} \), then

\[
\|z_2 + (d/a)/z_2\|^2 - |z'_2 + (d/a)/z'_2|^2 = 2(d/a)|\cos(2\theta_2) - \cos(2\theta'_2)| \geq \delta^{0+}|z_2 - z'_2| > \delta^{1+}.
\]

Hence,

\[
\|M_z - M_{z'}\| \geq \delta^{1+}.
\]

2.2. When \( z_2 = z'_2 \), the argument is similar to the one in case 1.2. above.

5.4 Finding “independent directions”

Roughly, we now show that two non-commuting matrices induce four “independent directions.”

Claim 5.9. Let \( g_1 \in SL_2(\mathbb{C}) \cap B_1(1) \) be so that \( \text{dist}(g_1, \pm 1) \geq \delta^{0+} \) and \( \text{Tr}g_1 \neq 2 \). Let \( g_2 \in SL_2(\mathbb{C}) \) be so that w.r.t. the basis that makes \( g_1 \) diagonal \( |(g_2)_{1,2}(g_2)_{2,1}| \geq \delta^{0+} \). Then,

\[
|\text{det}(1, g_1, g_2, g_1g_2)| \geq \delta^{0+}.
\]

Proof. Choose a basis so that \( g_1 \) is diagonal (this is a linear transformation on the \( g_i \)’s with bounded away from zero determinant). Denote \( \lambda = (g_1)_{1,1} \). In the new basis,

\[
|\text{det}(1, g_1, g_2, g_1g_2)| = \left| \begin{array}{ccc}
1 & \lambda & (g_1g_2)_{1,1} \\
(g_2)_{1,2} & (g_1g_2)_{1,2} \\
(g_2)_{2,1} & (g_1g_2)_{2,1} \\
1 & 1/\lambda & (g_1g_2)_{2,2}
\end{array} \right| = |(\lambda - 1/\lambda)((g_1g_2)_{1,2}(g_2)_{2,1} - (g_1g_2)_{2,1}(g_2)_{1,2})|.
\]

By choice,

\[
|\lambda - 1/\lambda| \geq \delta^{0+}.
\]

and

\[
|(g_2)_{1,2}(g_2)_{2,1}| \geq \delta^{0+}.
\]

Hence,

\[
|(g_1g_2)_{1,2}(g_2)_{2,1} - (g_1g_2)_{2,1}(g_2)_{1,2})| = |(\lambda - 1/\lambda)(g_2)_{1,2}(g_2)_{2,1}| \geq \delta^{0+}.
\]

\[\blacksquare\]
5.5 Proof of product theorem

Proof of Theorem 4.4

Assume towards a contradiction that

\[ N_\delta(\aaa) \leq \delta^{0-}N_\delta(A). \]

By [1], for every finite \( k \),

\[ N_\delta(A(k)) \leq \delta^{0-}N_\delta(A) \quad (5.14) \]

as well.

The first step is to find a large, commuting set of matrices. By assumption on \( A \) and using Claim 5.9, choose \( g_1, g_2, g_3 \in A(8) \) with \( |\det(1, g_1, g_2, g_3)| \geq \delta^{0+} \). Equation (5.14) and Corollary 5.3 imply that there is a set of commuting matrices \( V \subset \text{SL}_2(\mathbb{C}) \) so that

\[ N_\delta(V) \geq \delta^{0+}N_\delta(A)^{1/3} = \delta^{-1+\sigma/3+} \quad (5.15) \]

and so that

\[ V \subset \Gamma_{\delta^3}(A(2)). \]

Assume (by perhaps allowing \( V \subset \Gamma_{\delta^3}(A(4)) \)) that \( V = V^{-1} \) and

\[ V \subset B_{\delta^{3\varepsilon_5}}(1). \quad (5.16) \]

Proceed according to two cases.

The first case is when \( V \) is well-spread, i.e., the conditions for using the discretized ring conjecture are held. Define

\[ \sigma = 1 - \sigma_0/3 \quad \text{and} \quad \kappa = \tau/6 \]

so that \( N_\delta(V) = \delta^{-\sigma} \). Assume that for all \( \delta < \rho < \delta^{\varepsilon_4} \) with \( \varepsilon_4 = \varepsilon_4(\sigma, \kappa) \) from Lemma 5.4

\[ \max_a N_\delta(V \cap B_\rho(a)) < \rho^\sigma \delta^{-\sigma}. \]

By assumption on \( A \), there is \( g_0 \in A(4) \) so that (w.r.t. the basis that makes \( V \) diagonal) the distance between \( g_0 \) and 1 is at most a small constant, and \( |(g_0)_{1,2}(g_0)_{2,1}| \geq \delta^{\varepsilon_5} \). Even after the basis change \( \text{Tr}g_0 \in \mathbb{R} \). Thus, Lemma 5.4 implies

\[ N_\delta(\text{Tr}W_0) \geq \delta^{-\sigma-\varepsilon_4}, \]

where

\[ W_0 = Wg_0Wg_0W \]

and

\[ W = V(8). \]
(Here and below $C > 0$ will be a large universal constant, that may change its value.) By choice,
\[ \text{dist}(g_0^2, \pm 1) \gtrsim \delta^{\varepsilon_5}. \]
Thus, using (5.16),
\[ \text{dist}(W_0, \pm 1) \gtrsim \delta^{2\varepsilon_5}. \]
We can hence apply Lemma 5.2 with $W_0$ to obtain a set
\[ W_1 \subset \Gamma_{\delta_1} - (W_0^{-1}W_0) \]
of commuting matrices so that
\[ N_\delta(W_1) \geq \delta^{0+}N_\delta(V_0)V_0V_0^* \geq \delta^{0+}\frac{\delta^{-\sigma-\varepsilon_4}N_\delta(V)}{N_\delta(W_0^2W_0^{-1})}. \]
By (5.13) and Lemma 5.8 we thus have
\[ N_\delta(W_1) \geq \delta^{0+}\frac{\delta^{-\sigma-\varepsilon_4}N_\delta(V)^3}{N_\delta(A)}. \]
So, by (5.15),
\[ N_\delta(W_1) \geq \delta^{0+}\frac{\delta^{3\varepsilon_4/2}}{N_\delta(A)}. \]
Again, we can find $g_1 \in A_{(4)}$ so that (w.r.t. the basis that makes $W_1$ diagonal) $\text{dist}(g_1, 1)$ is at most a small constant, $\text{Tr}g_1 \in \mathbb{R}$, and $|(g_1)_{1,2}(g_1)_{2,1}| \geq \delta^{0+}$. So, we can apply Lemma 5.8 again and get
\[ N_\delta(A) \geq \delta^{0+}N_\delta(W_1g_1W_1W_1) \geq \delta^{0+}\frac{\delta^{-3\sigma-\varepsilon_4/2}}{N_\delta(A)} \geq \delta^{-3+\varepsilon_4/2}N_\delta(A). \]
This contradicts (5.14), and the proof is complete in this case.

The proof in the second case, when $V$ is not well-spread, is simpler. Indeed, we have
\[ N_\delta(V_0) \geq \rho^\kappa\delta^{-\sigma} \]
with
\[ V_0 = V \cap B_\rho(a) \]
(reusing notation). So, by Lemma 5.8
\[ N_\delta(V_1) \geq \delta^{0+}N_\delta(V_0)^3 \geq \rho^{3\kappa}\delta^{-3\sigma+}, \]
where
\[ V_1 = V_0g_0V_0g_0V_0 \subset \Gamma_{\delta_1} - (A_{(C)}) \]
with $g_0$ from above. By assumption on $A$, there is a finite $X \subset A$ so that
\[ |X| \geq \rho^{-\tau} \]
35
and for all \( x \neq x' \) in \( X \),
\[
\|x - x'\| \geq C\rho.
\]
Denote
\[
V_2 = \bigcup_{x \in X} xV_1.
\]
Therefore,
\[
\mathcal{N}_\delta(V_2) \geq |X|\mathcal{N}_\delta(V_1) \geq \rho^{-\tau} \rho^3 \delta^{-3\sigma +} \geq \rho^{-\tau/2} \delta^{-3+\sigma_0 +} \geq \delta^{-3+\sigma_0-\varepsilon_4 \tau/3} = \delta^{0-N_\delta(A)}.
\]
Since \( V_2 \subset \Gamma_{\delta_1}(A(C)) \), we obtained a contradiction to (5.14), and the proof is complete. \( \square \)

References

[B1] J. Bourgain. On the Erdos-Volkmann and Katz-Tao ring conjectures. Geom. Funct. Anal. 13, pages 334–365, 2003.

[B2] J. Bourgain. Expanders and dimensional expansion. Comptes Rendus Mathematique 347 (7), pages 357–362, 2009.

[BG1] J. Bourgain and A. Gamburd. On the spectral gap for finitely-generated subgroups of \( \text{SU}(2) \). Inventiones Mathematicae 171 (1), pages 83–121, 2007.

[BG2] J. Bourgain and A. Gamburd. Uniform expansion bounds for Cayley graphs of \( \text{SL}_2(\mathbb{F}_p) \). Annals of Mathematics 167 (2), pages 625–642, 2008.

[Br] E. Breuillard. A strong Tits alternative. Preprint, 2008.

[DS] Z. Dvir and A. Shpilka. Towards dimension expanders over finite fields. In Proceedings of the IEEE 23rd Annual Conference on Computational Complexity, pages 304–310, 2008.

[DW] Z. Dvir and A. Wigderson. Monotone expanders: constructions and applications. Theory of Computing, 6 (1), pages 291–308, 2010.

[H] H. A. Helfgott. Growth and generation in \( \text{SL}_2(\mathbb{Z}/p\mathbb{Z}) \). Annals of Mathematics 167 (2), pages 601–623, 2008.

[HLW] S. Hoory, N. Linial, and A. Wigderson. Expander graphs and their applications. Bull. Amer. Math. Soc. 43, pages 439-561, 2006.

[K] M. M. Klawe. Limitations on explicit constructions of expanding graphs. SIAM J. Comput. 13 (1), pages 156–166, 1984.

[LPS] A. Lubotzky, R. Phillips, and P. Sarnak. Ramanujan graphs. Combinatorica 8 (3), pages 261–277, 1988.
[LZ] A. Lubotzky and Y. Zelmanov. Dimension expanders. Journal of Algebra 319 (2), pages 730–738, 2008.

[RVW] Omer Reingold, Salil Vadhan, and Avi Wigderson. Entropy waves, the zig-zag graph product, and new constant-degree expanders. Annals of Mathematics, 155(1):157187, 2002.

[T] T. Tao. Product set estimates for non-commutative groups. Combinatorica 28 (5), pages 547–594, 2008.