Compact spaces that do not map onto finite products

by

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Abstract. We provide examples of nonseparable compact spaces with the property that any continuous image which is homeomorphic to a finite product of spaces has a maximal prescribed number of nonseparable factors.

1. Introduction. The motivation of this work are several problems coming from [1], [2] and [9], dealing with the possibility of mapping a finite product of spaces onto another product space with more factors.

Let $B$ be the closed euclidean ball of a nonseparable Hilbert space, endowed with its weak topology. It was proven in [2] that $B^2$ is not homeomorphic to $B$, and the natural question arises whether $B^2$ is at least a continuous image of $B$. We shall prove that the answer is also negative. More generally:

Theorem 1. Let $n < m$ be natural numbers and suppose $f : B^n \to X_1 \times \cdots \times X_m$ is an onto continuous map. Then there exists $i \leq m$ such that $X_i$ is metrizable.

We shall also provide an alternative proof of the fact shown in [2] that if $B^n$ is homeomorphic to $L^m$ for some $L$ and some $m$, then $m$ divides $n$. Properties of this kind are also proven in [2] for some spaces of probability measures, but we shall show that the methods in our paper do not apply to such spaces.

The second problem deals with linearly ordered spaces. The following result was first obtained by Treybig [13], though there exists a shorter proof by Bula, Dębski and Kulpa [5]:

Theorem 2 (Treybig). Let $L$ be a linearly ordered compact space, and $X_0$ and $X_1$ two infinite compact spaces. If there is a continuous surjection $f : L \to X_0 \times X_1$, then both $X_0$ and $X_1$ are metrizable.

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Mardešić [9] tried to generalize this theorem to higher dimensions, proposing the following conjecture:

**Conjecture 3 (Mardešić).** Let $L_1, \ldots, L_n$ be linearly ordered compact spaces, let $X_0, \ldots, X_n$ be infinite compact spaces and let $f : \prod_{i=1}^n L_i \to \prod_{j=0}^n X_j$ be an onto continuous map. Then there exist $i,j \leq n$, $i \neq j$, such that $X_i$ and $X_j$ are metrizable.

He proved [9] that the conjecture holds under the assumption that all spaces $X_i$ are separable. Our methods enable us to obtain the following:

**Theorem 4.** Let $L_1, \ldots, L_n$ be linearly ordered compact spaces, let $X_0, \ldots, X_n$ be infinite compact spaces and let $f : \prod_{i=1}^n L_i \to \prod_{j=0}^n X_j$ be an onto continuous map. Then there exist $0 \leq i,j \leq n$, $i \neq j$, such that $X_i$ and $X_j$ are separable.

Notice that Mardešić’s partial answer to the conjecture and our own seem completely unrelated since his hypothesis is stronger than our conclusion. However, in the case when $n = 2$ both results can be combined to show that at least one factor must always be metrizable:

**Corollary 5.** Let $L_1$ and $L_2$ be linearly ordered compact spaces, let $X_0$, $X_1$ and $X_2$ be infinite compact spaces and let $f : L_1 \times L_2 \to X_0 \times X_1 \times X_2$ be an onto continuous map. Then there exists $i \in \{0, 1, 2\}$ such that $X_i$ is metrizable.

**Proof.** By Theorem 4, two factors of the last product are separable, say $X_0$ and $X_1$. Let $Y_3$ be an infinite quotient space of $X_3$ of countable weight. Then there is a quotient $L_1 \times L_2 \to X_0 \times X_1 \times Y_3$ where all factors are separable, so by Mardešić’s result [9] at least two of them are metrizable, so either $X_0$ or $X_1$ is metrizable.

The third problem deals with the spaces $\sigma_n(\Gamma) = \{x \in 2^\Gamma : |\text{supp}(x)| \leq n\}$. In our previous work [1] we studied the homeomorphic classification of finite and countable products of these spaces. In this paper, we shall determine when such a product is a continuous image of another one.

We want to express our gratitude to Stevo Todorcevic for calling our attention to the work of Mardešić.

2. Indecomposability properties (I). All along this paper, we work with compact Hausdorff topological spaces. When we talk about compact spaces, the Hausdorff $T_2$ separation axiom is implicitly assumed.

**Definition 6.** Let $X$ be a compact space. A *pseudoclopen* of $X$ is a pair $a = (a[0], a[1])$ such that $a[0]$ and $a[1]$ are open subsets of $X$ and $\overline{a[0]} \subset a[1]$. 

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Notice that every clopen set \( c \) can be identified with a pseudoclopen \((c, c)\). Conversely, if \( K \) is a totally disconnected compact space, then every pseudoclopen \( a \) is interpolated by a clopen set \( c \), meaning \( a[0] \subset c \subset a[1] \). The notion of pseudoclopen substitutes the notion of clopen sets in general (not totally disconnected) compact spaces.

**Definition 7.** An uncountable family \( F \) of sets will be called a Knaster-disjoint family if every uncountable subfamily \( G \subset F \) contains two disjoint elements. An uncountable family \( F \) of pseudoclopes of \( X \) will be called Knaster-disjoint if \( \{a[1]: a \in F\} \) is a Knaster-disjoint family of sets.

The terminology is motivated by Knaster’s well known chain condition: *Every uncountable family of nonempty opens sets contains an uncountable family in which any two elements have nonempty intersection.* Thus, for a completely regular space, the failure of Knaster’s condition is equivalent to the existence of an uncountable Knaster-disjoint family of strongly nonempty pseudoclopes (we call a pseudoclopen \( a \) strongly nonempty if \( a[0] \neq \emptyset \)).

**Definition 8.** Let \( X \) be a compact space, and \( n \) a natural number. We say that \( X \) has property \( I_n \) if for every collection of \( n + 1 \) Knaster-disjoint families \( F_0, \ldots, F_n \) of pseudoclopes, there exist uncountable subfamilies \( G_i \subset F_i \) for \( i = 0, \ldots, n \) such that for every choice \( a_i \in G_i \) for \( i = 0, \ldots, n \) we have \( a_0[0] \cap \cdots \cap a_n[0] = \emptyset \).

**Proposition 9.** Let \( X \) be a compact space with property \( I_n \), and \( f : X \to Y \) a continuous onto map. Then \( Y \) has property \( I_n \).

**Proof.** Let \( F_0, \ldots, F_n \) be families of Knaster-disjoint pseudoclopes in \( Y \). For every \( i \leq n \), let \( F_i' = \{(f^{-1}(a[0]), f^{-1}(a[1])) : a \in F_i\} \). Then \( F_i' \) is a Knaster-disjoint family of pseudoclopes in \( X \). Because \( X \) has property \( I_n \), we can find uncountable subfamilies \( G_i \subset F_i \) such that whenever \( a_i \in G_i \), then \( f^{-1}(a_0[0]) \cap \cdots \cap f^{-1}(a_n[0]) = \emptyset \), which implies that \( a_0[0] \cap \cdots \cap a_n[0] = \emptyset \) because \( f \) is onto. \( \blacksquare \)

**Proposition 10.** Let \( X_0, \ldots, X_n \) be compact spaces such that \( X = X_0 \times \cdots \times X_n \) has property \( I_n \). Then there exists \( i \leq n \) such that \( X_i \) satisfies Knaster’s condition.

**Proof.** For a set \( s \subset X_i \), we write \( u_i(s) = \{(x_0, \ldots, x_n) \in X : x_i \in s\} \). Also, for a pseudoclopen \( a \) of \( X_i \), \( u_i(a) = (u_i(a[0]), u_i(a[1])) \) is a pseudoclopen of \( X \). Now, suppose no \( X_i \) satisfies Knaster’s condition. Then for every \( i \) we can find an uncountable Knaster-disjoint family \( F_i \) of strongly nonempty pseudoclopes in \( X_i \). Associated with them, we have Knaster-disjoint families \( F_i' = \{u_i(a) : a \in F_i\} \) of pseudoclopes in \( X \). These families contradict property \( I_n \), because \( a_0[0] \cap \cdots \cap a_n[0] \neq \emptyset \) whenever \( a_i \in F_i' \). \( \blacksquare \)
We now introduce some operations and relations. For two pseudoclopen sets $a$ and $b$, we write:

- $b \subset a$ if $b[0] \subset a[0]$ and $b[1] \subset a[1]$,
- $b \prec a$ if $a[0] \subset b[0] \subset b[1] \subset a[1]$ (we say that $b$ is finer than $a$),
- $a \cup b = (a[0] \cup b[0], a[1] \cup b[1])$ (this is a new pseudoclopen).

**Definition 11.** A family $\mathcal{B}$ of pseudoclopen sets of $X$ will be called a basis if for every pseudoclopen set $a$ of $X$ there exist finitely many pseudoclopen sets $b_1, \ldots, b_m$ from $\mathcal{B}$ such that $b_1 \cup \cdots \cup b_m \prec a$.

**Lemma 12.** If $\mathcal{B}$ is basis for the topology of a compact space $X$, then the family of all pseudoclopen sets $a$ such that $a[0] \in \mathcal{B}$ and $a[1] \in \mathcal{B}$ constitutes a basis of pseudoclopen sets.

**Lemma 13.** If $X$ is a totally disconnected compact space and $\mathcal{B}$ is a basis for the topology of $X$ consisting of clopen sets, then $\mathcal{B}' = \{(c, c) : c \in \mathcal{B}\}$ is a basis of pseudoclopen sets in $X$.

**Lemma 14.** In the definition of property $I_n$ we may assume that the families $\mathcal{F}_i$ are subfamilies of some basis of pseudoclopen sets for the space.

*Proof. Let $\mathcal{F}_0, \ldots, \mathcal{F}_n$ be arbitrary Knaster-disjoint families of pseudoclopen sets of $X$, and let $\mathcal{B}$ be a basis of pseudoclopen sets. Because $\mathcal{B}$ is a basis, we can suppose (by passing to finer pseudoclopen sets) that every $a \in \mathcal{F}_i$ is a finite union of elements of $\mathcal{B}$. Because we are looking for uncountable subfamilies, we can suppose without loss of generality that there exist natural numbers $m_0, \ldots, m_n$ such that every element $a \in \mathcal{F}_i$ is a union of exactly $m_i$ elements of $\mathcal{B}$, and we write it in the form $a = a_i^1 \cup \cdots \cup a_i^{m_i}$ where each $a_i^j$ is an element of $\mathcal{B}$. For every $i \leq n$ and every $j \leq m_i$, let $\mathcal{F}_i^j = \{a_i^j : a \in \mathcal{F}_i\}$. For every choice of numbers $j = (j_0, \ldots, j_n)$ with $j_i \leq m_i$, let $\mathcal{G}_i^j$ be the family of all $G \in \mathcal{F}_i$ such that whenever $a \in \mathcal{G}_i^j$, we have $a_0^{j_0}[0] \cap \cdots \cap a_n^{j_n}[0] = \emptyset$. There are only finitely many choices for the tuple $j = (j_0, \ldots, j_n)$, call them $j^{(1)}, \ldots, j^{(k)}$, so we can successively find the subfamilies $\mathcal{G}_i^{j(r)}$ satisfying $\mathcal{G}_i^{j(r)} \supseteq \mathcal{G}_i^{j(r+1)}$. The uncountable families $\mathcal{G}_i^{j(k)} \subset \mathcal{F}_i$ satisfy the desired conclusion.*

The following fact is well known; it is the key property behind the fact that, unlike the countable chain condition, Knaster’s condition is productive [12]:

**Lemma 15.** Let $\{a^0 \times a^1 : a \in \mathcal{F}\}$ be a Knaster-disjoint family of subsets of $X \times Y$ consisting of rectangles. Then one of the two families $\{a^0 : a \in \mathcal{F}\}$ or $\{a^1 : a \in \mathcal{F}\}$ contains an uncountable Knaster-disjoint subfamily.
PROPOSITION 16. Let $X$ and $Y$ be compact spaces with property $I_n$ and $I_m$ respectively. Then $X \times Y$ has property $I_{n+m}$.

Proof. Let $\mathcal{F}_0, \ldots, \mathcal{F}_{n+m}$ be uncountable Knaster-disjoint families of pseudoclopens of $X \times Y$. By Lemma 14, we can suppose that every $a \in \mathcal{F}_i$ is of the form $a = a^0 \times a^1 = (a^0[0] \times a^1[0], a^0[1] \times a^1[1])$ with $a^0$ and $a^1$ pseudoclopens of $X$ and $Y$. By Lemma 15, we can also suppose that for every $i \leq n + m$ there exists $j(i) \in \{0, 1\}$ such that the family $\{a^{j(i)} : a \in \mathcal{F}_i\}$ is Knaster-disjoint. By elementary cardinality reasons, either there exists $S \subseteq \{0, \ldots, n + m\}$ with $|S| = n + 1$ such that $(\forall i \in S)(j(i) = 0)$, or else there exists $T \subseteq \{0, \ldots, n + m\}$ with $|T| = m + 1$ such that $(\forall i \in T)(j(i) = 1)$. In the first case, we finish the proof by appealing to property $I_n$ of $X$, and in the second case by appealing to property $I_m$ of $Y$.

PROPOSITION 17. Let $X$ be a compact space and $d$, $n$ and $q$ natural numbers. Suppose that $X^d$ has property $I_n$ and $(q + 1)d > n$. Then $X$ has property $I_q$.

Proof. Let $\mathcal{F}_0, \ldots, \mathcal{F}_q$ be Knaster-disjoint families of pseudoclopens of $X$. For every $i \in \{1, \ldots, d\}$ let $p_i : X^d \to X$ be the projection on the $i$th coordinate. Let

$$\mathcal{F}_j^i = \{ (p_i^{-1}a[0], p_i^{-1}a[1]) : a \in \mathcal{F}_j \}.$$ 

These are $(q + 1)d$ Knaster-disjoint families of pseudoclopens of $X^d$. Since $(q + 1)d > n$, and $X^d$ has property $I_n$ it follows that there are uncountable subfamilies $\mathcal{G}_j^i \subset \mathcal{F}_j^i$ such that

$$\bigcap_{i=1}^d \bigcap_{j=0}^q c_j^i[0] = \emptyset$$

whenever $c_j^i \in \mathcal{G}_j^i$.

We claim that there exists $i \in \{1, \ldots, d\}$ such that $b_0^i[0] \cap \cdots \cap b_q^i[0] = \emptyset$ whenever $b_0 \in \mathcal{G}_0^i, \ldots, b_q \in \mathcal{G}_q^i$. This claim concludes the proof because then the families $\mathcal{G}_j = \{ a \in \mathcal{F}_j : (p_i^{-1}a[0], p_i^{-1}a[1]) \in \mathcal{G}_j^i \}$ are the uncountable subfamilies of $\mathcal{F}_j$ we are looking for. The claim is proved by contradiction. If it were false we could find for every $i$ elements $b_0^i \in \mathcal{G}_0^i, \ldots, b_q^i \in \mathcal{G}_q^i$ such that $b_0^i[0] \cap \cdots \cap b_q^i[0] \neq \emptyset$. Since a clopen from a family $\mathcal{G}_j^i$ depends only on the coordinate $i$, this implies that $\bigcap_{i=1}^d \bigcap_{j=0}^q b_j^i[0] \neq \emptyset$, which is a contradiction.

COROLLARY 18. Let $K$ be a compact space with property $I_n$ but not $I_{n-1}$. If $K$ is homeomorphic to $X^d$ for some space $X$, then $d$ divides $n$.

Proof. If $d$ does not divide $n$, then there exists an integer $q$ such that $q < n/d < q + 1$. The previous proposition shows that $X$ has property $I_q$, so by Proposition 16, $X^d \approx K$ has $I_{qd}$, and therefore $I_{n-1}$ because $qd < n$, so $qd \leq n - 1$. ■
3. The euclidean ball. Let $\Gamma$ be an uncountable set, and let
\[ B = B(\Gamma) = \left\{ x \in [-1, 1]^\Gamma : \sum_{\gamma \in \Gamma} |x_\gamma| \leq 1 \right\}. \]
We consider this set as a compact space endowed with the pointwise topology.
This $B(\Gamma)$ is actually homeomorphic to the ball of the Banach space $\ell_p(\Gamma)$
in the weak topology for $1 < p < \infty$ and also to the dual ball of $c_0(\Gamma)$ in the weak* topology.

**Theorem 19.** The space $B$ has property $I_1$.

We observe that this result implies Theorem 1. Let $n < m$ and let $f : B^n \to X_1 \times \cdots \times X_n$ be a continuous surjection. By Propositions 9 and 16, $X_1 \times \cdots \times X_n$ has property $I_n$, and by Proposition 10 there exists $i \leq m$ such that $X_i$ satisfies Knaster’s condition, therefore also the countable chain condition (every disjoint family of open sets is countable). But $B$ is an Eberlein compact, so its continuous image $X_i$ is also Eberlein compact [4] and a result of Rosenthal [10] establishes that the countable chain condition implies countable weight for an Eberlein compact.

It is easy to see that $B_+(\Gamma) = B(\Gamma) \cap [0, 1]^\Gamma$ maps continuously onto $B(\Gamma)$; an onto continuous map $B_+(\Gamma \times 2) \to B(\Gamma)$ is given by $x \mapsto (x_{(\gamma, 0)} - x_{(\gamma, 1)})_{\gamma \in \Gamma}$. For notational simplicity, we write $A_n = 1/2^{n+1}$. The following totally disconnected compact space maps onto $B_+(\Gamma)$:
\[ L = \left\{ x \in \{0, 1\}^{\Gamma \times \omega} : \sum_{(\gamma, n) \in \Gamma \times \omega} A_n x_{\gamma,n} \leq 1 \right\}. \]
A surjection $g : L \to B_+(\Gamma)$ is given by $g(x)_{\gamma} = \sum_{n<\omega} A_n x_{\gamma,n}$. By Proposition 9, Theorem 19 follows from

**Theorem 20.** The space $L$ has property $I_1$.

**Proof.** We consider the basis $\mathcal{B}$ of clopen subsets of $L$ consisting of the sets of the form
\[ a_{U,V} = \left\{ x \in L : \forall (\gamma, n) \in U \ x_{\gamma,n} = 1, \ \forall (\gamma, n) \in V \ x_{\gamma,n} = 0 \right\}, \]
where $U$ and $V$ are two disjoint finite subsets of $\Gamma \times \omega$. It will be convenient to use the following notations: given $a = a_{U,V} \in \mathcal{B}$, we write $U(a) = U$ and $V(a) = V$. Also, for a finite set $U \subseteq \Gamma \times \omega$, we define
\[ \sigma(U) = \sum_{(\gamma, n) \in U} A_n = \sum_{n<\omega} A_n |U \cap \Gamma \times \{n\}|. \]
Notice the fundamental property that if $U \cap U' = \emptyset$ then $\sigma(U \cup U') = \sigma(U) + \sigma(U')$. We need to know when two elements of $\mathcal{B}$ are disjoint:

**(DC)** Let $a, b \in \mathcal{B}$. Then $a \cap b = \emptyset$ if and only if one of the following three conditions holds: either $U(a) \cap V(b) \neq \emptyset$ or $V(a) \cap U(b) \neq \emptyset$ or $\sigma(U(a) \cup U(b)) > 1$. 
Let $\mathcal{F}_0$ and $\mathcal{F}_1$ be two Knaster-disjoint families of clopen sets from $\mathcal{B}$. By Lemmas 13 and 14, it is enough to check property $I_1$ on such two families of clopen sets.

By the familiar $\Delta$-system lemma (cf. [6, Theorem 9.18]), we can assume that the families $\{U(a) : a \in \mathcal{F}_0\}$, $\{U(a) : a \in \mathcal{F}_1\}$, $\{V(a) : a \in \mathcal{F}_0\}$ and $\{V(a) : a \in \mathcal{F}_1\}$ are $\Delta$-systems with roots $R_0$, $R_1$, $S_0$ and $S_1$ respectively. We write $U(a) = R_i \cup U_i(a)$ and $V(a) = S_i \cup V_i(a)$ for $a \in \mathcal{F}_i$, separating the root and the disjoint part of the $\Delta$-system in such a way that $\{U_i(a) : a \in \mathcal{F}_i\}$ and $\{V_i(a) : a \in \mathcal{F}_i\}$ are disjoint families of finite sets for $i = 0, 1$. By a further refinement we can also suppose that the whole family

\begin{equation}
(U_0(a), U_1(b), V_0(a), V_1(b) : a \in \mathcal{F}_0, b \in \mathcal{F}_1) \cup \{R_0 \cup R_1 \cup S_0 \cup S_1\}
\end{equation}

is a disjoint family.

The number $\sigma(U)$ is always a rational number, so we can also suppose that $\sigma(U_0(a)) = q_0$ and $\sigma(U_1(b)) = q_1$ are rational numbers independent of $a \in \mathcal{F}_0$ and $b \in \mathcal{F}_1$.

**Claim.** $\sigma(R_i) + 2q_i > 1$ for $i = 0, 1$.

**Proof of the claim.** Since $\mathcal{F}_i$ is a Knaster-disjoint family, we can pick two different elements $a, b \in \mathcal{F}_i$ such that $a \cap b = \emptyset$. Thus, one of the three alternatives of the disjointness criterion (DC) must hold. But the first two alternatives are impossible. For example, the disjointness of the family $(\star)$ above implies that $U(a) \cap V(b) = R_i \cap S_i \subset U(a) \cap V(a) = \emptyset$. Therefore, the third alternative holds:

\[ 1 < \sigma(U(a) \cup U(b)) = \sigma(R_i) + \sigma(U_i(a)) + \sigma(U_i(b)) = \sigma(R_i) + 2q_i. \]

We finish the proof by showing that, after all these refinements, $a \cap b = \emptyset$ whenever $a \in \mathcal{F}_0$ and $b \in \mathcal{F}_1$. Using again the disjointness criterion (DC) we prove that $\sigma(U(a) \cup U(b)) > 1$. We have

\[ \sigma(U(a) \cup U(b)) = \sigma(R_0 \cup R_1) + \sigma(U_0(a)) + \sigma(U_1(b)) = \sigma(R_0 \cup R_1) + q_0 + q_1. \]

If say $q_i = \min(q_0, q_1)$, then by the Claim,

\[ \sigma(U(a) \cup U(b)) = \sigma(R_0 \cup R_1) + q_0 + q_1 \geq \sigma(R_i) + 2q_i > 1. \]

**Corollary 21** (Avilés, Kalenda). Let $X$ be a compact space and $m, n$ natural numbers such that $B^n$ is homeomorphic to $X^m$. Then $m$ divides $n$.

**Proof.** Apply the preceding theorem and Corollary 18. □

**4. Remarks about spaces $P(K)$.** Given a compact space $K$, we denote by $P(K)$ the space of Radon probability measures on $K$ endowed with the weak* topology. Results analogous to Corollary 21 are proven in [2] for certain spaces of probability measures (like $P(\sigma_n(\Gamma))$ and $P(\sigma_1(\Gamma)^n))$, so it
is a natural question whether such spaces have property $I_1$. We show in this 
section that property $I_n$ on $P(K)$ imposes very restrictive conditions on $K$.

**Proposition 22.** Let $X$ be a compact space which contains $n$ open subsets 
whose closures are pairwise disjoint and fail the countable chain condition. Then $P(X)$ maps continuously onto $B^n$, and in particular $P(X)$ does not have property $I_{n-1}$.

**Proof.** Let $V_1, \ldots, V_n$ be open subsets of $X$ with $V_i \cap V_j = \emptyset$ for $i \neq j$, and for every $i$, let $\mathcal{U}_i$ be an uncountable disjoint family of nonempty open subsets of $V_i$. For every $u \in \bigcup_{i=1}^n \mathcal{U}_i$ let $h_u : X \to [0,1]$ be a continuous function such that $h_u(X \setminus u) = 0$ and $\max\{h_u(x) : x \in u\} = 1$. For every $i \leq n$ let also $g_i : X \to [0,1]$ be a continuous function such that $g_i(x) = 0$ if $x \in V_i$ and $g_i(x) = 1$ if $x \in V_j$ for some $j \neq i$. Let

$$
\chi_n(t) = \begin{cases} 
1 & \text{if } t \geq 1 - 1/n, \\
1/(1-t) & \text{if } t < 1 - 1/n. 
\end{cases}
$$

For a Radon measure $\mu$ on $X$ and a continuous function $\phi : X \to \mathbb{R}$, we put $\mu(\phi) = \int \phi d\mu$. We define $f : P(X) \to [0,1]^{\mathcal{U}_1} \times \cdots \times [0,1]^{\mathcal{U}_n}$ as follows:

$$
f(\mu)_u = \chi_n(\mu(g_i)) \cdot \mu(h_u), \quad u \in \mathcal{U}_i.
$$

For a set $\Gamma$, recall that $B_+(\Gamma) = \{x \in [0,1]^\Gamma : \sum_{\gamma \in \Gamma} x_\gamma \leq 1\}$. Then $B(\Gamma)$ is a continuous image of $B_+(\Gamma)$, and we shall show that $f(P(X)) = B_+(\mathcal{U}_1) \times \cdots \times B_+(\mathcal{U}_n)$.

First, we check that $f(P(X)) \subset B_+(\mathcal{U}_1) \times \cdots \times B_+(\mathcal{U}_n)$. For fixed $\mu$ in $P(X)$ and $i \leq n$,

$$
\sum_{u \in \mathcal{U}_i} f(\mu)_u = \chi_n(\mu(g_i)) \sum_{u \in \mathcal{U}_i} \mu(h_u) \leq \frac{1}{1 - \mu(g_i)} \sum_{u \in \mathcal{U}_i} \mu(h_u) \leq 1
$$

because $g_i$ has disjoint support from all $h_u$’s, so $\mu(g_i) + \sum_{u \in \mathcal{U}_i} \mu(h_u) \leq 1$.

Conversely, we now prove that any $x \in B_+(\mathcal{U}_1) \times \cdots \times B_+(\mathcal{U}_n)$ belongs to $f(P(X))$. For every $i \leq n$ let $\xi_i \in V_i \setminus \bigcup\{u : u \in \mathcal{U}_i\}$, and for every $u \in \mathcal{U}_i$ let $\zeta_u \in u$ be such that $h_u(\zeta_u) = 1$. We define $\mu \in P(X)$ to be a discrete probability measure on $X$ such that $\mu\{\zeta_u\} = x_u/n$ for every $u$, and $\mu\{\xi_n\} = (1 - \sum_{u \in \mathcal{U}_n} x_u)/n$. We have $\mu(h_u) = x_u/n$ for $u \in \mathcal{U}_n$, $\mu(g_i) = 1 - 1/n$, and $f(\mu) = x$. ■

Recall that $\sigma_1(\Gamma)$ is the one-point compactification of a discrete set $\Gamma$. The space $P(\sigma_1(\Gamma))$ is homeomorphic to $B_+(\Gamma)$, a continuous image of $B(\Gamma)$, so it has property $I_1$ by Theorem 19. Also, if $K_n$ is a discrete union of $n$ disjoint copies of $\sigma_1(\Gamma)$, then $P(K_n)$ has property $I_n$, because it is a continuous image of $P(\sigma_1(\Gamma))^n \times B_+(\{1, \ldots, n\})$. We may ask if there is some sufficient condition on $L$ so that $P(L)$ has property $I_n$. 


5. Indecomposability properties (II). A family $\mathcal{F}$ of pseudoclopens will be called disjoint if $a[1] \cap b[1] = \emptyset$ for any $a, b \in \mathcal{F}, a \neq b$.

DEFINITION 23. Let $X$ be a compact space, and $n$ a natural number. We say that $X$ has property $I^*_n$ if for every collection of $n$ Knaster-disjoint families $\mathcal{F}_1, \ldots, \mathcal{F}_n$ of pseudoclopens, and every infinite disjoint family $\mathcal{F}_0$ of pseudoclopens, there exist uncountable subfamilies $\mathcal{G}_i \subset \mathcal{F}_i$ for $i = 1, \ldots, n$, and an infinite subfamily $\mathcal{G}_0 \subset \mathcal{F}_0$ such that for every choice $a_i \in \mathcal{G}_i$ for $i = 0, \ldots, n$ we have $a_0[0] \cap \cdots \cap a_n[0] = \emptyset$.

PROPOSITION 24. Let $X$ be a compact space with property $I^*_n$, and $f : X \to Y$ a continuous onto map. Then $Y$ has property $I^*_n$.

Proof. Analogous to Proposition 9.

PROPOSITION 25. Let $X_0, \ldots, X_n$ be infinite compact spaces such that $X = X_0 \times \cdots \times X_n$ has property $I^*_n$. Then there exist $i, j \leq n, i \neq j$, such that $X_i$ and $X_j$ satisfy Knaster’s condition.

Proof. This is equivalent to saying that for every $i \leq n$ there exists $j \neq i$ such that $X_j$ satisfies Knaster’s condition. We prove this statement for $i = 0$. By contradiction, if this were false, then we could find an uncountable Knaster-disjoint family $\mathcal{F}_j$ of strongly nonempty pseudoclopens in $X_j$, for every $j = 1, \ldots, n$. Let also $\mathcal{F}_0$ be an infinite disjoint family of pseudoclopens of $X_0$. Just as in Proposition 10, these families can be lifted to families of pseudoclopens of $X$ that violate property $I^*_n$.

DEFINITION 26. A family $\mathcal{B}_0$ of pseudoclopens of $X$ will be called a strong basis if for every pseudoclopen $a$ of $X$ there exists $b \in \mathcal{B}_0$ such that $b \prec a$.

Notice that if $X$ is a totally disconnected compact space, then the family $\mathcal{B}_0$ of pseudoclopen sets of the form $(c, c)$, $c$ clopen, constitutes a strong basis.

LEMMA 27. Let $\mathcal{B}$ and $\mathcal{B}_0$ be a basis and a strong basis of pseudoclopens of $X$ respectively. Then the following condition is sufficient for $X$ to have property $I^*_n$: For every collection of $n$ Knaster-disjoint families $\mathcal{F}_1, \ldots, \mathcal{F}_n$ of pseudoclopens from $\mathcal{B}$ and every infinite disjoint family $\mathcal{F}_0$ of pseudoclopens from $\mathcal{B}_0$, there exist uncountable subfamilies $\mathcal{G}_i \subset \mathcal{F}_i$ for $i = 1, \ldots, n$ and an infinite subfamily $\mathcal{G}_0 \subset \mathcal{F}_0$ such that for every choice $a_i \in \mathcal{G}_i$ for $i = 0, \ldots, n$ we have $a_0[0] \cap \cdots \cap a_n[0] = \emptyset$.

Proof. Analogous to Lemma 14. Just note that we need $\mathcal{B}_0$ to be a strong basis and not just a basis, because when dealing with infinite instead of uncountable families, it is not possible to fix the length of finite unions by passing to a further subfamily.
6. Linearly ordered spaces

**Lemma 28.** Let \( F \) be a Knaster-disjoint family of sets. Then there exist at most countably many elements \( a \in F \) such that \( \{ b \in F : a \cap b = \emptyset \} \) is countable.

**Proof.** Suppose by contradiction that there are uncountably many such elements. Then it is possible to construct by induction a transfinite sequence \( \{ a_\alpha : \alpha < \omega_1 \} \subset F \) of such elements such that \( a_\alpha \cap a_\beta \neq \emptyset \) for all \( \alpha < \beta < \omega_1 \). This contradicts the assumption that \( F \) is Knaster-disjoint. \( \blacksquare \)

**Theorem 29.** Every linearly ordered compact space \( L \) has property \( I_1^* \).

**Proof.** Every compact linearly ordered space \( L \) is the continuous image of a compact linearly ordered totally disconnected space (one can consider the lexicographical product \( L \times \{0, 1\} \)). Therefore, we can suppose that \( L \) is totally disconnected. By Lemma 14, we have to show that for any Knaster-disjoint families \( F_0 \) and \( F_1 \) of clopen intervals of \( L \), there are further uncountable subfamilies for which all crossed intersections are empty. By Lemma 28, we can suppose that each element of \( F_i \) is disjoint from uncountably many elements of \( F_i \), \( i = 0, 1 \). Notice that Knaster-disjoint families are point-countable, that is, every element of \( L \) belongs to at most countably many intervals from \( F_i \). Suppose that some interval \( a \in F_0 \) intersects uncountably many intervals from \( F_1 \). Except those that contain one of the two extremes of \( a \), which are at most countably many, the others are actually contained in \( a \). In this case it is enough to take \( G_0 = \{ b \in F_0 : b \cap a = \emptyset \} \) and \( G_1 = \{ b \in F_1 : b \subset a \} \). The remaining case is that every element of \( F_0 \) intersects at most countably many elements of \( F_1 \) and vice versa. In this case we can produce by induction two \( \omega_1 \)-sequences \( (a_\alpha)_{\alpha < \omega_1} \subset F_0 \) and \( (b_\alpha)_{\alpha < \omega_1} \subset F_1 \) with \( a_\alpha \cap b_\beta = \emptyset \) for all \( \alpha, \beta < \omega_1 \). \( \blacksquare \)

**Theorem 30.** If \( L_1, \ldots, L_n \) are linearly ordered compact spaces, then \( K = L_1 \times \cdots \times L_n \) has property \( I_n^* \).

**Proof.** Again, we assume that the spaces \( L_j \) are totally disconnected. Using Lemma 27, let \( F_0 \) be a countably infinite disjoint family of clopens of \( K \) and \( F_1, \ldots, F_n \) be Knaster-disjoint families of clopen boxes of \( K \) (by a box we mean a product of clopen intervals),

\[
F_i = \{ a_\alpha[i] = a_\alpha^0[i] \times \cdots \times a_\alpha^n[i] : \alpha < \omega_1 \}, \quad i > 0.
\]

By Lemma 15 we can suppose that for every \( i \) there exists \( j(i) \) such that \( \{ a_\alpha[i] : \alpha < \omega_1 \} \) is Knaster-disjoint. We can actually assume that the map \( i \mapsto j(i) \) is a bijection, since if there existed \( i \neq i' \) with \( j(i) = j(i') = j \) we would be done by applying the fact that \( L_j \) has property \( I_1 \). After relabelling we suppose that each family \( H_i = \{ a_\alpha^i[i] : \alpha < \omega_1 \} \) is a Knaster-disjoint family of clopen intervals of \( L_i \). We consider two cases:
Case 1: There exists \( c \in \mathcal{F}_0 \) and a box \( b_1 \times \cdots \times b_n \subset c \) such that \( A_i = \{ \alpha < \omega_1 : a^\alpha_i[b] \subset b_i \} \) is uncountable for all \( i = 1, \ldots, n \). In this case, we can take \( \mathcal{G}_0 = \mathcal{F}_0 \setminus \{ c \} \) and \( \mathcal{F}_i = \{ a^\alpha[i] : \alpha \in A_i \} \).

Case 2: The previous case does not hold, so for every \( c \in \mathcal{F}_0 \) and every box \( b = b_1 \times \cdots \times b_n \subset c \) there exists \( i = i(b) \in \{1, \ldots, n\} \) such that the set \( A_i(b) = \{ \alpha < \omega_1 : a^\alpha_i[b] \subset b_i \} \) is countable. Actually, the set \( A_i'(b) = \{ \alpha < \omega_1 : a^\alpha_i[b] \cap b_i \neq \emptyset \} \) is also countable, because the family \( \mathcal{H}_i = \{ a^\alpha[i] : \alpha < \omega_1 \} \) is point-countable, so only countably many intervals from it can hit one of the two extremes of \( b_i \). Every clopen \( c \in \mathcal{F}_0 \) is a finite union of boxes, so there is a countable family \( \mathcal{B} \) of boxes contained in elements of \( \mathcal{F}_0 \) such that every element of \( \mathcal{F}_0 \) is a finite union of elements of \( \mathcal{B} \). The subfamilies \( \mathcal{G}_0 = \mathcal{F}_0 \) and \( \mathcal{G}_i = \{ a^\alpha[i] : \alpha < \omega_1, \alpha \not\in \bigcup_{b \in \mathcal{B}} A_i'(b) \} \) have the desired properties. ■

In order to complete the proof of Theorem 4, having Theorem 30, Proposition 24 and Proposition 25, it only remains to pass from Knaster’s condition to separability. It is a classical result of Knaster [8] that a linearly ordered space which satisfies Knaster’s condition is separable. We need just a little more:

**Proposition 31.** Let \( L_1, \ldots, L_n \) be linearly ordered compact spaces and \( f : L_1 \times \cdots \times L_n \to X \) an onto continuous map. If \( X \) satisfies Knaster’s condition, then \( X \) is separable.

**Proof.** Let \( Y \subset L_1 \times \cdots \times L_n \) be a closed subspace such that \( f : Y \to X \) is an irreducible onto map. Recall that a continuous function is irreducible if \( f(Y') \neq X \) whenever \( Y' \neq Y \) is a proper closed subset of \( Y \). A standard argument using Zorn’s lemma yields the existence of such a \( Y \). Knaster’s condition is preserved by irreducible preimages, so \( Y \) satisfies Knaster’s condition (to see this, associate to an uncountable family \( \mathcal{F} \) of nonempty open sets of \( Y \) the family \( \{ X \setminus f(X \setminus U) : U \in \mathcal{F} \} \) of nonempty open sets of \( X \)). Now let \( p_i : L_1 \times \cdots \times L_n \to L_i \) be the natural projection, and \( K_i = p_i(Y) \). Then \( p_i(Y) \) is a linearly ordered compact space with Knaster’s property, so by Knaster’s result [8] it is separable. On the other hand, \( Y \subset p_1(Y) \times \cdots \times p_n(Y) \), so \( f(p_1(Y) \times \cdots \times p_n(Y)) = X \), therefore \( X \) is also separable. ■

**7. Spaces of finite sets.** For a natural number \( n \) and an uncountable set \( \Gamma \), let \( \sigma_n(\Gamma) \) denote the family of subsets of \( \Gamma \) of cardinality less than or equal to \( n \). This is a closed subset of \( 2^\Gamma \), so we view \( \sigma_n(\Gamma) \) as a compact topological space. A basis for its topology consists of the sets of the form

\[
\Phi_A^B = \{ C \in \sigma_n(\Gamma) : A \subset C \subset \Gamma \setminus B \},
\]

where \( A \) and \( B \) are finite subsets of \( \Gamma \).
The topological classification of the spaces which are finite or countable products of spaces $\sigma_n(\Gamma)$ is studied in [1]. In the case of finite products, which we are now interested in, if $\sigma_1(\Gamma)^{e_1} \times \cdots \times \sigma_n(\Gamma)^{e_n}$ is homeomorphic to $\sigma_1(\Gamma)^{f_1} \times \cdots \times \sigma_n(\Gamma)^{f_n}$, where $n$ and each $e_i$, $f_i$ are natural numbers, then $e_i = f_i$ for every $i$.

In this section, we will determine when a finite product of spaces $\sigma_n(\Gamma)$ can be mapped continuously onto another such product. An obvious sufficient condition for the existence of a continuous onto map between such finite products is the following:

**Lemma 32.** Let $(n_1, \ldots, n_r)$ and $(m_1, \ldots, m_s)$ be two finite sets of natural numbers. Suppose that there exist sets $S_i \subset \{1, \ldots, r\}$ for $i = 1, \ldots, s$ which are pairwise disjoint and such that $m_i \leq \sum_{j \in S_i} n_j$ for every $i \in \{1, \ldots, s\}$. Then the space $\sigma_{n_1}(\Gamma) \times \cdots \times \sigma_{n_r}(\Gamma)$ maps continuously onto $\sigma_{m_1}(\Gamma) \times \cdots \times \sigma_{m_s}(\Gamma)$.

**Proof.** First, $\sigma_n(\Gamma)$ maps continuously onto $\sigma_m(\Gamma)$ if $m \leq n$. Namely, fix $\gamma_0 \in \Gamma$ and then define $f : \sigma_n(\Gamma) \to \sigma_{n-1}(\Gamma \setminus \{\gamma_0\})$ by $f(x) = x \setminus \{\gamma_0\}$ if $\gamma_0 \in x$, and $f(x) = \emptyset$ if $\gamma_0 \notin x$. Secondly, for a finite set $S$ of natural numbers whose sum is $\Sigma(S)$, we have the union map

$$u : \prod_{n \in S} \sigma_n(\Gamma) \to \sigma_{\Sigma(S)}(\Gamma), \quad u(x) = \bigcup_{n \in S} x_n.$$

These two remarks show that $\prod_{n \in S} \sigma_n(\Gamma)$ maps onto $\sigma_m(\Gamma)$ whenever $m \leq \Sigma(S)$. The proof of the lemma is obtained by applying this fact to $S = S_i$ for every $i$, and considering product maps. \(\blacksquare\)

In Theorem 34 below we shall prove that the sufficient condition of the previous lemma is actually necessary. Indeed, we shall obtain stronger indecomposability properties. An $m$-point family of sets is a family $F$ such that every subfamily of cardinality $m + 1$ has empty intersection.

**Definition 33.** Let $m_* = (m_1, \ldots, m_s)$ be a finite sequence of natural numbers. We say that a compact space has property $I[m_*]$ if for any uncountable families $F_1, \ldots, F_s$ of clopen sets such that $F_i$ is an $m_i$-point family for every $i$, there exist uncountable subfamilies $G_i \subset F_i$ such that $\bigcup_{i=1}^s G_i$ is a $((\sum_{i=1}^s m_i) - 1)$-point family.

**Theorem 34.** Let $(n_1, \ldots, n_r)$ and $(m_1, \ldots, m_s)$ be two finite sets of natural numbers. The following are equivalent:

1. $K = \sigma_{n_1}(\Gamma) \times \cdots \times \sigma_{n_r}(\Gamma)$ does not have property $(m_1, \ldots, m_s)$.
2. $\sigma_{n_1}(\Gamma) \times \cdots \times \sigma_{n_r}(\Gamma)$ maps continuously onto $\sigma_{m_1}(\Gamma) \times \cdots \times \sigma_{m_s}(\Gamma)$.
3. There exist disjoint sets $S_i \subset \{1, \ldots, r\}$ for $i = 1, \ldots, s$ such that $m_i \leq \sum_{j \in S_i} n_j$ for every $i \in \{1, \ldots, s\}$. 


Proof. (3)⇒(2) is Lemma 32, (2)⇒(1) is clear: Let \( g : \sigma_{m_1}(\Gamma) \times \cdots \times \sigma_{m_s}(\Gamma) \rightarrow \sigma_m(\Gamma) \) be onto, and consider the families 
\[ F_i = \{ \{ x : \gamma \in g(x) \} : \gamma \in \Gamma \}, \quad i = 1, \ldots, s. \]
These families witness the failure of \( I[m_s] \). It remains to prove (1)⇒(3). As \( K \) does not have property \((m_1, \ldots, m_s)\), there exist families \( F_1, \ldots, F_s \) in \( K \) such that \( F_i \) is an \( m_i \)-point family, but for any uncountable subfamilies \( G_i \subset F_i \), the union \( \bigcup_{i=1}^{s} G_i \) is not a \((\sum_{i=1}^{s} m_i - 1)\)-point family.

For a fixed \( i \), we can suppose that each clopen \( x \in F_i \) is of the form
\[
(*) \quad x = \bigcup_{p_1=1}^{k(i)} \Phi_{a(x,p_1)}^{b(x,p_1)} \times \cdots \times \Phi_{a(x,p_r)}^{b(x,p_r)},
\]
where \( a(x,p,j) \) and \( b(x,p,j) \) are finite subsets of \( \Gamma \), and \( \Phi_{a(x,p,j)}^{b(x,p,j)} \) is a basic clopen subset of \( \sigma_{n_j}(\Gamma) \). Moreover, we can suppose that for fixed \( i, p, j \), \( \{ a(x,p,j) : x \in F_i \} \) and \( \{ b(x,p,j) : x \in F_i \} \) form \( \Delta \)-systems of constant cardinality with roots \( A(i,p,j) \) and \( B(i,p,j) \). We write \( a(x,p,j) = A(i,p,j) \cup \alpha(x,p,j) \) and \( b(x,p,j) = B(i,p,j) \cup \beta(x,p,j) \) separating the root and the disjoint part of the \( \Delta \)-system. We also write \( |\alpha|(i,p,j) = |\alpha(x,p,j)| \), \( x \in F_i \). By passing to further uncountable subfamilies we can also assume that
\[
(**) \quad (\alpha(x,p,j) \cup \beta(x,p,j)) \cap (a(x',p',j') \cup b(x',p',j')) = \emptyset
\]
whenever \( x \neq x' \).

Because each \( F_i \) is an \( m_i \)-point family but the family \( \bigcup F_i \) is not a \((\sum_{i=1}^{s} m_i - 1)\)-point family, there must exist \( x_1^i, \ldots, x_{m_i}^i \in F_i \) such that 
\[ \bigcap_{i=1}^{r} \bigcap_{q=1}^{m_i} \Phi_{a(x_i^q,p_i^q,1)}^{b(x_i^q,p_i^q,1)} \times \cdots \times \Phi_{a(x_i^q,p_i^q,r)}^{b(x_i^q,p_i^q,r)} \neq \emptyset. \]

We define the sets \( S_i \) appearing in statement (3) of the theorem in the following way:
\[ S_i = \left\{ j \in \{1, \ldots, r\} : \exists q : |\alpha|(i,p_i^q,j) + \bigcup_{q=1}^{m_i} a(x_{i}^q,p_i^q,j) > n_j \right\}. \]

**Claim 1.** \( S_i \cap S_{i'} = \emptyset. \)

**Proof.** Suppose \( j \in S_i \cap S_{i'} \). Consider witnesses \( \overline{q} \) and \( \overline{q}' \) for \( j \in S_i \) and \( j \in S_{i'} \) respectively. Assume that \( |\alpha|(i,p_i^q,j) \leq |\alpha|(i',p_i'^{q'},j) \). We know that
\[
\left( \bigcap_{q=1}^{m_i} \Phi_{a(x_i^q,p_i^q,j)}^{b(x_i^q,p_i^q,j)} \right) \cap \left( \bigcap_{q=1}^{m_i'} \Phi_{a(x_i'^{q'},p_i'^{q'},j)}^{b(x_i'^{q'},p_i'^{q'},j)} \right) \neq \emptyset.
\]
In particular,
\[ |a(x_1^q, p_i^q, j) + \bigcup_{q=1}^{m_i} a(x_i^q, p_i^q, j) | \leq n_j, \]
but
\[ n_j < |\alpha|(i, p_i^q, j) + |\alpha|(i', p_i^{q'}, j) \leq n_j. \]

Thus there exists \( j \) such that
\[ \Phi a(x_i^q, p_i^q, j) \cap \bigcap_{q=1}^{m_i} \Phi a(x_i^q, p_i^q, j) = \emptyset. \]

Since we know that \( \bigcap_{q=1}^{m_i} \Phi a(x_i^q, p_i^q, j) \neq \emptyset \), there are only three possibilities that such an intersection of basic clopen sets of the form \( \Phi a(x_i^q, p_i^q, j) \) is empty in \( \sigma_{n_j}(\Gamma) \): either

1. \(|a(x_i^q, p_i^q, j) \cup \bigcup_{q=1}^{m_i} a(x_i^q, p_i^q, j) | > n_j, \) or
2. \(a(x_i^q, p_i^q, j) \cap \bigcup_{q=1}^{m_i} b(x_i^q, p_i^q, j) \neq \emptyset, \) or
3. \(b(x_i^q, p_i^q, j) \cap \bigcup_{q=1}^{m_i} a(x_i^q, p_i^q, j) \neq \emptyset. \)

The first option leads immediately to the desired conclusion. The second and third options cannot occur. The reason is that we assumed the \( \Delta \)-systems to satisfy the disjointness property (**) Thus, for instance if (2) happened we would have \( A(i, p_i^q, j) \cap B(i, p_i^q, j) \neq \emptyset \) for some \( \hat{q} \), which implies that
\[ \Phi a(x_i^q, p_i^q, j) \cap \bigcap_{q=1}^{m_i} \Phi a(x_i^q, p_i^q, j) = \emptyset, \] which is a contradiction.

CLAIM 3. \( \sum_{j \in S_i} n_j \geq m_i. \)

Proof. Consider the function \( J : \{1, \ldots, m_i \} \to \{1, \ldots, r \} \) which associates to every \( \hat{q} \) an element \( j = J(q) \) as in Claim 2. It is enough to notice
that $|J^{-1}(j)| \leq n_j$ for every $j$. Namely,
\[
\bigcap_{q \in J^{-1}(j)} \Phi_{b_q(x^{q_i}, p^{q_i}_j)}^{a(x^{q_i}, p^{q_i}_j)} \neq \emptyset,
\]
so $|\bigcup_{q \in J^{-1}(j)} a(x^{q_i}, p^{q_i}_j)| \leq n_j$. But on the other hand, $|\alpha(i, p^{q_i}_j,j) > 0$ for every $q \in J^{-1}(j)$, so $|J^{-1}(j)| \leq n_j$. ■

8. Remarks

REMARK 35. Theorem 1 is in a sense best possible, because $B^n$ maps continuously onto $B^n \times [0, 1]^{\omega}$. This is a consequence of a result of Kalenda [7] that $B_+(\Gamma) \approx P(\sigma_1(\Gamma))$ is homeomorphic to $B_+(\Gamma) \times [0, 1]$ together with the facts that $B_+$ and $B$ are continuous images of each other, and $[0, 1]^{\omega}$ is a continuous image of $[0, 1]$.

REMARK 36. Mardešić’s conjecture is also best possible. For example, any nonmetrizable nonscattered compact space maps continuously onto $[0, 1]^{\omega}$. Hence we can easily get continuous maps $L_1 \times \cdots \times L_n \to L_1 \times \cdots \cdots \times L_{n-1} \times [0, 1]^{\omega}$.

REMARK 37. The reader might wonder why we deal with Knaster-disjoint families instead of simply disjoint families. The reason is that Knaster-disjointness behaves better with respect to products. For instance, if we define similar properties to $I_n$ or $I_n^*$ with disjoint families instead of Knaster-disjoint, then the proofs of Proposition 16 and Theorem 30 do not work any more, unless we assume that certain colorings of the uncountable have uncountable monochromatic sets: In the first case this can be overcome if the compact spaces $X$ and $Y$ have property (Q) of Bell [3], and in the second case one may need to assume that Suslin lines do not exist.

REMARK 38. Given an uncountable regular cardinal $\aleph$, we can define indecomposability properties $I_n(\aleph)$, $I_n^*(\aleph)$ or $I[m^*](\aleph)$ in a similar way but substituting “for any uncountable families... there exist uncountable subfamilies” by “for any families of cardinality $\aleph$... there exist subfamilies of cardinality $\aleph$”. All the results in this note can be rewritten in this more general way, and in particular the assertions of Theorems 19, 29, 30 and 34 hold for these properties relative to $\aleph$.

REMARK 39. After Rudin’s result [11] that every monotonically normal compact space is the continuous image of a linearly ordered space, in Theorem 4 the assumption that the spaces $L_i$ are linearly ordered can be substituted by the assumption that the spaces $L_i$ are monotonically normal.
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