Stress Analysis of Orthotropic Wedge Loaded on the Apex

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Abstract. The complex variable function including a material parameter is analyzed for the solving the mechanic problem. Typical stress boundary condition for a wedge is considered to orthotropic materials. By constructing new stress function, the mechanic analysis of the wedge subjected to a concentrated force is carried out. The stress boundary problem and the governing equation are studied. The formulae of the stress fields in rectangular and polar coordinates are derived for the composite wedge.

1. Introduction
The elasticity analysis of the plane stress problem is of great importance to the usual engineering application. The distribution of any stress depends on the forces acting on the complete closed boundary. The complex variable theory provides a very powerful tool for the solution of many boundary value problems in the elastic body. Such theory was originally found by some researchers for solving general boundary problems in isotropic materials. Furthermore, the complex variable technique has also been expanded to use for anisotropic materials [1~3]. Complex variable methods prove to be very useful for the solution of many full-space and half-space problems. Fiber-reinforced polymer matrix materials are the most typical composites, which are also as anisotropic materials at the macroscopic level [4, 5]. The orthotropic plate may have been the base of composites in common engineering use. Typical examples include concentrated force and moment systems applied to the boundary of the plate. The feasible method to solve stress-field problems in anisotropic composites is to use the analytic function theory, and the results have been reported [6, 7]. The purpose of this paper is to focus attention on a new solution of the boundary-value problem for the orthotropic materials.

2. Typical Plane Problem and Basic Equations
Let us now consider a long wedge subjected to a concentrated force $P$ at its apex angle as shown in Figure 1. The force $P$ is in the x-y plane and this is a simple symmetrical loading case. The thickness of the wedge in the direction perpendicular to the x-y plane is taken as unity, so that $P$ is the load per unit thickness. The distribution of the load along the thickness of the plate is uniform. The conditions along the boundary faces of the wedge are free ($\theta = \pm \alpha$) and can be satisfied by taking the values to be zero for the stress components, $\sigma_\theta$, $\tau_{r\theta}$. Thus, it is derived that: $\sigma_\theta = 0$, $\tau_{r\theta} = 0$.

The plane stress problems of composite materials are common and very important for the application. It is the key point to solve stress-field problems in orthotropic materials. Suppose the principal elastic directions of the plate coincide with the coordinate directions (x, y), and let the directions 1, 2 parallel to the axes x, y, respectively. By using the linear elastic strain-stress relations, the linear constitutive equations for the orthotropic materials are given as follows:
\[ \varepsilon_x = \frac{\sigma_x}{E_1} - \frac{\nu_{12} \sigma_y}{E_1}, \quad \varepsilon_y = \frac{\sigma_y}{E_2} - \frac{\nu_{12} \sigma_x}{E_1}, \quad \gamma_{xy} = \frac{\tau_{xy}}{G_{12}} \]  

(1)

\[ \alpha \]

**Figure 1.** Wedge and loading condition

It is well known that the compatibility condition of strains must satisfy as follows:

\[ \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \]  

(2)

In the case of plane stress state, the equilibrium equations are as (body forces are absent):

\[ \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \]  

(3)

Usually, the method of solving the equations is by introducing a new function \( U \) of \( x \) and \( y \), called the stress function. Through taking any real function \( U \), it is easily checked that the equilibrium equations are satisfied by putting the following expressions for the stress components:

\[ \sigma_x = \frac{\partial^2 U}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 U}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 U}{\partial x \partial y} \]  

(4)

By means of above stress and strain relations, the governing equation of the compatibility condition can be expressed by the stress function \( U(x, y) \), which is

\[ \frac{\partial^4 U}{\partial y^4} + 2\left( \frac{E_1}{2G_{12}} - \nu_{12} \right) \frac{\partial^4 U}{\partial x^2 \partial y^2} + \frac{E_2}{E_1} \frac{\partial^4 U}{\partial x^4} = 0 \]  

(5)

Thus, the solution of a plane problem can be reduce to finding a solution of equation (5) that must also satisfy the boundary conditions.

3. Complex Variable and Stress Function

In order to solve the boundary loading problem about an orthotropic plate, the complex function shall be introduced for the convenience of stress investigation.

3.1. Complex Variable and Coordinates

The basic complex variable and its conjugate, \( z = x + iy \), \( \bar{z} = x - iy \) are normally appeared in general books (\( i = \sqrt{-1} \)). For the convenience of next investigation, another complex variable \( w \) and its conjugate \( \bar{w} \) are also introduced as follows:
\[ w = X + iY = x + ihy, \quad \overline{w} = X - iY = x - ihy \]

And so: \( X = x, \quad Y = hy \)

Where, the constant \( h \) is real. The derivative relations must be as follows:

\[
\begin{align*}
\frac{\partial w}{\partial X} &= \frac{\partial w}{\partial x} = \frac{\partial \overline{w}}{\partial x} = \frac{\partial \overline{w}}{\partial X} = 1, \\
\frac{\partial w}{\partial Y} &= \frac{1}{h} \frac{\partial w}{\partial y} = i, \\
\frac{\partial \overline{w}}{\partial Y} &= \frac{1}{h} \frac{\partial \overline{w}}{\partial y} = -i
\end{align*}
\]  \quad (6)

For the plane problem, rectangular and polar coordinates are shown in Figure 2. In terms of the polar coordinates (in \( oxy \)-plane and \( OXY \)-plane), the complex variables can be written as:

\[
\begin{cases}
z = x + iy = r \cos \theta + ir \sin \theta = r \cdot e^{i\theta} \\
w = r \cos \theta + iyr \sin \theta = R \cos \Theta + iR \sin \Theta = R \cdot e^{i\Theta}
\end{cases}
\]  \quad (7)

Thus, there are following relations:

\[
\begin{align*}
z\overline{z} &= r^2, \\
w\overline{w} &= R^2, \\
R \cos \Theta &= r \cos \theta, \\
R \sin \Theta &= hr \sin \theta, \\
\tan \Theta &= h \tan \theta
\end{align*}
\]

\[
\begin{align*}
w + \overline{w} &= 2x, \\
w - \overline{w} &= 2ihy, \\
w^2 + \overline{w}^2 &= 2(x^2 - h^2y^2), \\
w^2 - \overline{w}^2 &= 4ihxy
\end{align*}
\]

When \( h = 1 \), then \( \Theta = \theta, \quad R = r \). The \( w \)-plane is also reduced to \( z \)-plane.

**Figure 2.** Scheme of the coordinates in the wedge plane

### 3.2. Stress Expression by Complex Function

The complex function \( \ln w \) shall be considered firstly to use for solving the boundary loading problem as shown in Figure 1. In terms of the boundary condition and experimental knowledge, the real stress function \( U \) can be determined by the form:

\[ U = C(w - \overline{w})(\ln w - \ln \overline{w}) \]  \quad (8)

Where, \( C \) is arbitrary constant. The derivatives of \( U \) with respect to \( x \) or \( y \) are given by:

\[
\begin{align*}
\frac{\partial U}{\partial x} &= C(w - \overline{w})(\frac{1}{w} - \frac{1}{\overline{w}}) = 2Cihy(\frac{1}{w} - \frac{1}{\overline{w}}) \\
\frac{\partial U}{\partial y} &= 2Cih(\ln w - \ln \overline{w}) - 2Ch^2y(\frac{1}{w} + \frac{1}{\overline{w}})
\end{align*}
\]  \quad (9)

And then,
On the basis of above equations, the governing equation (5) becomes:

\[
[h^2 - (\frac{E_1}{2G_{12}} - \nu_{12})](\frac{1}{w^3} + \frac{1}{w^7}) + \frac{3iy}{4h} [2(\frac{E_1}{2G_{12}} - \nu_{12})h^2 - \frac{E_1}{E_2} - h^4](\frac{1}{w^4} - \frac{1}{w^4}) = 0
\]  

(12)

Thus the solution must be followed by the characteristic equations, and also reduced to:

\[
h^2 - (\frac{E_1}{2G_{12}} - \nu_{12}) = 0, \quad h^4 - 2(\frac{E_1}{2G_{12}} - \nu_{12})h^2 + \frac{E_1}{E_2} = 0
\]  

(13)

The solution of the characteristic equation is given as:

\[
h^2 = \frac{E_1}{2G_{12}} - \nu_{12}, \quad h^4 = \frac{E_1}{E_2}
\]  

(14)

So the positive real root can be obtained, that is:

\[
h = \sqrt[4]{\frac{E_1}{2G_{12}} - \nu_{12}} = \sqrt[4]{\frac{E_1}{E_2}}
\]  

(15)

Substituting the derivative equation (10) into the equation (4), the stresses can be expressed as:

\[
\begin{align*}
\sigma_x &= -4Ch^2(\frac{1}{w} - \frac{1}{w}) + 2Ch^3iy(\frac{1}{w^3} - \frac{1}{w^3}) = -8Ch^2 \frac{x^3}{(wW)^2} \\
\sigma_y &= -2Cihy(\frac{1}{w^7} - \frac{1}{w^7}) = -8Ch^2 \frac{xy^2}{(wW)^2} \\
\tau_{xy} &= -2Cih(\frac{1}{w} - \frac{1}{w}) - 2Ch^2y(\frac{1}{w^2} + \frac{1}{w^2}) = -8Ch^2 \frac{x^2y}{(wW)^2}
\end{align*}
\]  

(16)

4. Solution of the Boundary Problem
In order to solve the stress boundary problem of the wedge, we can consider to taking the relations between stresses in the two coordinate systems. It is common knowledge that the transformation of stresses can be expressed by the following relations:
Substituting the stresses of the expression (16) into the above equations, and to simplify them, then the stress components \( \sigma_r, \sigma_\theta, \tau_{r\theta} \) in the polar coordinate system can be obtained:

\[
\begin{align*}
\sigma_r &= \sigma_x \cos^2 \theta + \sigma_y \sin^2 \theta + 2\tau_{xy} \sin \theta \cos \theta \\
\sigma_\theta &= \sigma_x \sin^2 \theta + \sigma_y \cos^2 \theta - 2\tau_{xy} \sin \theta \cos \theta \\
\tau_{r\theta} &= (\sigma_y - \sigma_x) \sin \theta \cos \theta + \tau_{xy} (\cos^2 \theta - \sin^2 \theta)
\end{align*}
\] (17)

By using the relations between two coordinate systems, \( x = r \cos \theta, \ y = r \sin \theta \), the above stresses can be simplified as follows:

\[
\begin{align*}
\sigma_r &= -8Ch^2 \frac{x}{(ww)^2} (x^2 \cos^2 \theta + y^2 \sin^2 \theta + 2xy \sin \theta \cos \theta) \\
\sigma_\theta &= -8Ch^2 \frac{x}{(ww)^2} (x^2 \sin^2 \theta + y^2 \cos^2 \theta - 2xy \sin \theta \cos \theta) \\
\tau_{r\theta} &= -8Ch^2 \frac{x}{(ww)^2} [(y^2 - x^2) \sin \theta \cos \theta + xy (\cos^2 \theta - \sin^2 \theta)]
\end{align*}
\] (18)

It is evident that above stress fields can satisfy the boundary conditions of the wedge in Figure 1. Next, the constant \( C \) ought to be determined. In terms of the loading condition and the coordinate systems as shown in Figure 2, the force must be in equilibrium. The main equilibrant equation is given as

\[
P + \int_{-\tan \alpha}^{\tan \alpha} \sigma_x \, dy = P - \int_{-\tan \alpha}^{\tan \alpha} \frac{8Ch^2 x^3}{(x^2 + h^2 y^2)^2} \, dy = 0
\]

By solving the integration (note that: \( h > 0 \)), then the result can be obtained as

\[
P = 8Ch^2 \left[ \frac{\tan \alpha}{1 + h^2 \tan^2 \alpha} + \frac{\tan \alpha}{h} \arctan(h \tan \alpha) \right] = 8Ch(\beta + \sin \beta \cos \beta)
\]

Where, \( \beta = \arctan(h \tan \alpha) \), \( \tan \beta = h \tan \alpha \). Thus, the constant \( C \) is determined as follows:

\[
C = \frac{P}{8h(\beta + \sin \beta \cos \beta)}
\] (20)

The radius stress component, \( \sigma_r \) in the polar coordinate plane can be obtained, that is:

\[
\sigma_r = -\frac{Ph \cos \theta}{r(\beta + \sin \beta \cos \beta)(\cos^2 \theta + h^2 \sin^2 \theta)^2}
\] (21)

And again, by substituting the constant \( C \) into the stress equation (16), the stresses in the rectangular coordinate system can be also determined. There must be as follows:
\[
\begin{align*}
\sigma_x &= -\frac{Phx^3}{(\beta + \sin \beta \cos \beta)(w\bar{w})^2} \\
\sigma_y &= -\frac{Phxy^2}{(\beta + \sin \beta \cos \beta)(w\bar{w})^2} \\
\tau_{xy} &= -\frac{Phx^2y}{(\beta + \sin \beta \cos \beta)(w\bar{w})^2}
\end{align*}
\] (22)

Where, \( w\bar{w} = x^2 + h^2y^2 \). Obviously, the stress components in the polar coordinate system are relatively simple, as the normal stress \( \sigma_y \) and the shearing stress \( \tau_{xy} \) are zero. The radial normal stress \( \sigma_r \) is only stress component. Nevertheless, the stress components in the rectangular coordinate system are complex relatively.

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