Theory of ferrimagnetism in the Hubbard model on bipartite lattices with spectral symmetry

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Abstract

The Hubbard model is one of the most important models in condensed matter physics. In this paper, we developed a theory of ferrimagnetism in the Hubbard model on bipartite lattices with spectral symmetry. By taking three models as examples, we studied the ferrimagnetic orders that emerge from three typical fermionic systems—metal, semi-metal and (Chern) insulator. In particular, we found that there may exist various ferrimagnetic orders and explored the universal features.

Keywords: ferrimagnetism, Hubbard model, vacancy

(Some figures may appear in colour only in the online journal)

Introduction

The two-dimensional fermionic Hubbard model is one of the most interesting issues depicting the strongly correlated electronic systems including the Mott transition, quantum magnetism and possible superconducting order. It carries the essence of a correlated electronic system with intrinsic competition between on-site repulsive interaction and hopping processes of charge-carriers. Ever since the discovery of high $T_c$ cuprates, tremendous efforts have contributed to investigations of this model. However, there are still some ambiguities to be clarified.

The magnetic orders are important properties of the 2D Hubbard model, to which researchers have also paid much attention. At the half filling case, the ground state is known to be the long-range (LR) antiferromagnetic (AF) order. Nagaoka made a surprising discovery about the induced ferromagnetic (FM) by a single hole for the infinite coupling limit [1]. In the heavily doped region, a possible FM order may exist. The starting point on the issue of ferrimagnetism of the Hubbard model is a theorem by Lieb [2]. According to this theorem, the total spin $S$ of the ground state for the Hubbard model on a bipartite lattice with particle-hole (PH) symmetry and real hopping parameters is given by $S = |N_A - N_B|/2$ (we call it the Lieb spin moment) where $N_A$ and $N_B$ are the numbers of lattice sites on the A and B sublattices, respectively. Subsequently, according to [3], the ground state of the 2D sublattice-unbalanced Hubbard model on a bipartite lattice with the same conditions (PH symmetry and real hopping parameters) is really the ferrimagnetic (FR) order that possesses a finite total magnetic moment and the long range AF order and obeys $m(Q) = (\hat{s}_i \cdot \hat{s}_j) / N$ (we call it the Shen–Qiu–Tian (SQT) inequality) where $m(0) = \sum_j (\hat{s}_i \cdot \hat{s}_j) / N$ and $m(Q) = \sum_j g_i g_j e^{iQ \cdot \hat{r}_{ij}} / N$.

3 After a (partial) particle-hole transformation, the repulsive Hubbard model can be transformed into an attractive Hubbard model of which the ground state always has singlet superconducting (SC) pairing. This singlet SC just corresponds to the ground state with a total spin $|N_A - N_B|/2$ [2, 18, 19]. Then, the SQT inequality corresponds to the non-negative off-diagonal long-range SC pairing of the attractive Hubbard model in the earlier paper [18].
arbitrary small electron–electron interactions will drive the system into a long range ferrimagnetic order. We found that there may exist various FR orders. In particular, the total spin of all of these FR orders is found to be equal to that of the Lieb spin moment. However, the SQT inequality is valid for the model with only the nearest neighbour (NN) hoppings and can be violated by the next-nearest neighbour (NNN) hoppings.

The 2D Hubbard model with spectral symmetry

Our starting point is the following Hamiltonian

$$H = -\sum_{\langle i,j \rangle} t_{ij} (\hat{c}_i^\dagger \hat{c}_j + \text{h.c.}) + U \sum_i \hat{n}_i^e \hat{n}_i^e - \mu \sum_i \hat{c}_i^\dagger \hat{c}_i^\dagger \hat{c}_i \hat{c}_i,$$

(1)

where $\hat{c}_i = (\hat{c}_i^e, \hat{c}_i^h, \hat{c}_i^l, \hat{c}_i^r)^T$, with $\hat{c}_i^\dagger$ representing the fermion annihilation operator at site $i$. $\hat{n}_i = \hat{c}_i^\dagger \hat{c}_i$ denotes the spin index.

For this bipartite lattice, we have two sub-lattices, A and B. $t_{ij}$ is the hopping amplitude. $U$ is the strength of the repulsive interaction. $\mu$ is the chemical potential which is set to $\mu = 0$ for the half filling case. The lattice constant is $a$.

First, we defined the spectral symmetry. For the Hamiltonian in equation (1), with the spectral symmetry (S symmetry), the density of states (DOS) $\rho(E)$ is always symmetric via $E$ as $\rho(E) = \rho(-E)$. As a result, each energy level with positive energy $E$ must be paired with an energy level with negative energy $-E$. For clarity, we define an operator of the S symmetry as $S = P \cdot D$ where $P$ is the PH transformation operator $P = R \cdot K$ introduced in [8, 9] and $D$ is an operator that leads to $\hat{c}_i \rightarrow (1)\hat{c}_i^\dagger$, $K$ is the complex conjugate operator, and $D$ is a discrete transformation operator that commutes with the lattice translation operators $T_{s,x}$ as, $[D, T_s] = [D, T_x] = 0$.

For the generalised Hamiltonian of the 2D Hubbard model with spectral symmetry, we have $H = -S^H HS$. Thus, each energy level $|\psi\rangle$ with positive energy $E$ is paired with an energy level $S|\psi\rangle$ with negative energy $-E$. As a result, the PH symmetry is a special case of the S symmetry and for the case of $D = 1$, the S symmetry is reduced into the PH symmetry.

Vacancy-induced zero-modes

We next consider the case of a free Hamiltonian ($U = 0$) with a lattice-defect—the vacancy by adding a potential on given lattice site $R$. $H_R = -\sum_{\langle i,j \rangle} t_{ij} (\hat{c}_i^\dagger \hat{c}_j + \text{h.c.}) + V \hat{c}_i^\dagger \hat{c}_i^\dagger \hat{c}_i \hat{c}_i$. In the unitary limit, the lattice-defect becomes a missing lattice site that is just a vacancy on which we have an infinite on-site potential, i.e. $V \rightarrow \infty$. Note that the vacancy does not break the S symmetry, $H_R = -S^H H_R S$.

Because we consider the Hubbard model on a bipartite lattice with a single vacancy, the quantum levels of the system becomes an odd integer number. As a result, there must exist an unpaired electronic state when we remove a site to create a vacancy. Due to spectrum symmetry ($E \sim -E$), the corresponding unpaired electronic state $|\psi_0\rangle$ must have exactly zero energy. For the zero-mode (ZM) $|\psi_0\rangle$, we have $|\psi_0\rangle = S|\psi_0\rangle$. We denote the wave-function of the ZM $|\psi_0\rangle$ by $\psi_0(r-R)$ where $R$ is the position of the vacancy. When there exist $n$ vacancies on sublattice A, $n = |N_A - N_B|$ zero energy modes will necessarily appear. In addition, for the case with only NN hopping $t_{ij} = t_{ij}(1)(1)$ is the real or complex NN hopping parameter), these vacancy-induced (VI) zero-modes localise only on the sublattice B and are orthotropic to each other.

For some models with S symmetry, the DOS may be finite at the Fermi level. Except for the VI ZMs, there may exist additional zero energy states $|\psi\rangle \neq S|\psi\rangle$. However, the additional zero energy states are not protected by symmetry and are fragile against perturbations. On the contrary, because the VI ZMs are protected by S symmetry ($|\psi_0\rangle = S|\psi_0\rangle$), they are fixed precisely at the Fermi level, and the interaction term is highly relevant.

Effective spin model of vacancy-induced zero-modes

From the above discussion, we already know that there exists a ZM for each vacancy. In the small $U$ limit, $U \rightarrow 0$, the low energy physics is dominated by these ZMs and the low energy effective model is a FM-coupled spin model.

First, we consider the local moment of a VI ZM $|\psi_0\rangle$ on sublattice B, of which the vacancy lies on sublattice A. For each zero mode around a vacancy, the two energy levels are denoted by $|\psi_0\rangle$ and $|\psi_0\rangle$, respectively. We define the spin operator of the local spin moment as $\hat{S}_R = \hat{\psi}_0^\dagger (r_- - R) \sigma \hat{\psi}_0 (r_- - R) / 2$. For $|\psi_0\rangle$, we have $\hat{S}_R^+ |\psi_0\rangle = \frac{1}{2} |\psi_0\rangle$; For $|\psi_0\rangle$, we have $\hat{S}_R^- |\psi_0\rangle = -\frac{1}{2} |\psi_0\rangle$.

Next, we consider the case of two vacancies on sublattice A. Because the ZMs induced by two different vacancies at $R$ and $R'$ are orthotropic to each other, when considering the on-site interaction there exists the effective FM spin coupling, expressed as $-J_{\text{eff}}(R, R') \hat{S}_R \cdot \hat{S}_{R'}$, where $J_{\text{eff}}(R, R') > 0$ is the effective FM spin coupling constant that comes from the RKKY interaction mediated by the itinerant electron $J_{\text{RKKY}}(R, R)$ and direct Heisenberg exchange coupling between local spin moments [10] $J_{\text{H}}(R, R)$ or $J_{\text{eff}}(R, R') = J_{\text{RKKY}}(R, R) + J_{\text{H}}(R, R')$.

For the RKKY interaction, the coupling strength is given by $J_{\text{RKKY}}(R, R') = -\sum_{k} J_{\text{R}}(q) \psi_0^\dagger (q) \psi_0(q) g_{\text{R}}(k-q)$ where $J_{\text{R}}(q)$ is expressed as $J_{\text{R}}(q) = U \sum_{i \in B} |\psi_0^\dagger (i)|^2 e^{iqR}$. $\chi(q) = \chi_{\text{AA}}(q) \chi_{\text{AB}}(q)$ is the renormalised spin susceptibility from a random phase approximation calculation, and $\chi(q) = \frac{\chi_0(q)}{1 - U \chi_0(q)}$ where $\chi_0(q)$ is defined by $\chi_0(q) = -\frac{1}{N} \sum_{k} G_{\text{R}}(k) G_{\text{R}}(k - q)$.

For the direct Heisenberg exchange coupling, the coupling constant between two local spin moments on $R$ and $R'$ is given by $J_{\text{H}}(R, R') = -U \sum_{q} |\psi_0^\dagger (R, R')|^2 |\psi_0^\dagger (R, R')|^2 \chi_0(q) \cdot \chi_0(q)$ and $\psi_0(q) \cdot \psi_0(q)$ are wave functions of the two zero modes of the vacancies at $R$ and $R'$.

Finally, we derive an effective spin model induced by a super-lattice of vacancies on sublattice A, of which the Hamiltonian is
\[ H_{\text{eff}} = -\sum_{\mathbf{R}, \mathbf{R}'} J_{\text{eff}}(\mathbf{R}, \mathbf{R}') \hat{S}_\mathbf{R} \cdot \hat{S}_{\mathbf{R}'} > 0. \] (2)

In the small \( U \) limit, the effective FM spin coupling constant is proportional to \( U \). Thus, an arbitrarily small repulsive interaction will drive the spin moments of the VI ZMs into an FM ordered state and the resulting ground state of the original Hamiltonian turns into an LR FR order.

**Ferrimagnetism**

Let us show the universal features of the FR order in the Hubbard model in the small \( U \) limit.

For a system with \( L_x \times L_y \) vacancy-lattice (\( L_x \) along the \( x \) direction and \( L_y \) along the \( y \) direction), in a unit cell (UC) there are \( L_x \times L_y / a^2 - 1 \) lattice sites. Here, we consider all vacancies on the same sublattice and \( L_x \) and \( L_y \) are all even numbers. In general, we have \( L_x \times L_y / a^2 - 1 \) magnetic order parameters \( M_i = \langle \hat{s}^z_i \rangle = (\hat{c}^\dagger_i \hat{c}_{i+1} - \hat{c}^\dagger_{i+1} \hat{c}_i) / 2 \) to denote the local magnetisations on the lattice sites in a UC. To simply characterise the FR order, we introduce two order parameters: the total FM moment in a unit cell \( \mathcal{M} = \sum_{\text{unit-cell}} \langle \hat{c}^\dagger_i \hat{c}_{i+1} - \hat{c}^\dagger_{i+1} \hat{c}_i \rangle \) and the total AF moment in a unit cell \( \mathcal{N} = \sum_{\text{unit-cell}} (\hat{c}^\dagger_i \hat{c}_{i+1} + \hat{c}^\dagger_{i+1} \hat{c}_i) \).

When \( \langle \hat{S}_\mathbf{R}^z \rangle = 1 / 2 \), there exists an FM order of the local spin moments induced by VI ZMs. The FR order is a direct consequence of the FM order of the spin moments induced by vacancies. The local magnetisations are given by \( M_i = \langle \hat{s}^z_i \rangle \) and the total AF moment in a UC is \( \mathcal{N} = 2 \sum_{\text{unit-cell}} (-1)^i M_i \).

We show the relationship between the order parameters \( \mathcal{M} \) and \( \mathcal{N} \) and \( m(0) \), \( m(\mathcal{Q}) \) in the FR order. In FR order, the local moments of VI ZMs have a long range FM order, \( \langle \hat{s}_\mathbf{R}^z \rangle = 1 / 2 \). Due to \( \langle \hat{c}^\dagger_i \hat{c}_j \hat{c}^\dagger_j \hat{c}^\dagger_i \rangle = (M_i)^2 \), we have \( m(0) = \mathcal{M} (L_x \times L_y / a^2 - 1) \) and \( m(\mathcal{Q}) = \mathcal{N} (L_x \times L_y / a^2 - 1) \). As a result, for the FR order in the Hubbard model with VI ZMs, we can check the SQT inequality \( m(0) \leq m(\mathcal{Q}) \) by calculating \( \mathcal{M} \) and \( \mathcal{N} \).

In the small \( U \) limit, \( U \to 0 \), the low energy physics is dominated by these ZMs. The ground state exhibits a long-range FM order denoted by \( \langle \hat{S}_\mathbf{R}^z \rangle = 1 / 2 \) or \( \mathcal{M} = 1 \). As a result, the total spin moment of the ground state must be the total number of spin moments of the VI ZMs \( \sum_i \mathcal{M} \), which is just the Lieb spin moment \( \mathcal{S} = |N_b - N_g|/2 \).

The Hubbard model on a square lattice

In condensed matter physics, the Hubbard model on a square lattice is the simplest model and always a good starting point to learn the correlated electron system on a lattice. For the Hubbard model on a square lattice, the hopping parameters are \( t_{ij} = t \).

For free electrons, \( U = 0 \), after diagonalisation of the Hamiltonian in momentum space, the energy spectra are obtained as \( E_\mathbf{k} = -2\cos(k_x) + \cos(k_y) - \mu \). The DOS has a maximum value at \( E = 0 \). At half filling, \( \mu = 0 \), the ground state is a metal. In this case, the Fermi surface has the perfect nesting condition. Because the Hubbard model on bipartite lattices at half-filling is unstable against AF instability, the ground state becomes an insulator with AF order for the case of finite \( U \). Such AF order is described by the following mean field ansatz \( \langle \hat{c}^\dagger_i \hat{c}_{i+\sigma} \rangle = \frac{1}{2} (1 + (-1)\sigma N_0) \), where \( N_0 \) is the staggered magnetisation. For the cases of spin up and spin down, we have \( \sigma = +1 \) and \( \sigma = -1 \), respectively. Then, in the mean field theory, by minimising the ground state energy in the reduced Brillouin zone (BZ), we could solve the staggered magnetisation. Due to the perfect nesting, adding arbitrary onsite repulsive interaction (or \( U \neq 0 \)), the ground state turns into an AF insulator.

For this model, the operator of the \( S \) symmetry is \( S = \mathcal{P} \). The \( S \) symmetry protected VI ZM is an extended state, of which the wave-function|\( \psi_0 \rangle \) can naturally be \( \psi_0|_{\mathbf{R} = \mathbf{0}} = 0, \psi_0|_{\mathbf{R} = \mathbf{1}} = \frac{1}{N_0} \).

See the particle density distribution of this ZM on a \( 60a \times 60a \) lattice shown in figure 1(a).

Then, we use the mean field approach to study the Hubbard model on a square lattice with vacancy-lattice. The lattice constant of the vacancy-lattice is set to be \( d = 6a \). We choose \( 6a \times 6a \) sites to be a UC (\( 6a \) along the \( x \) direction and \( 6a \) along the \( y \) direction). To determine the ground state with the lowest energy, we need to solve \( 6 \times 6 - 1 = 35 \) order parameters that denote the local magnetisations \( M_i = \langle \hat{c}^\dagger_i \hat{c}_{i+1} - \hat{c}^\dagger_{i+1} \hat{c}_i \rangle / 2 \) on 35 lattice sites in a UC.

From the mean field calculation, we find that for the weak coupling limit, \( U \to 0 \), the ground state exhibits a uniform FR order, \( M_{\mathbf{R} = \mathbf{0}} = 0, M_{\mathbf{R} = \mathbf{1}} = \frac{1}{35} \). The FR ordered state is illustrated in figure 1(b). From figure 1(c), one can see that the total FM moment in a UC is indeed a constant, \( \mathcal{M} \equiv 1 \), which is consistent with the prediction of the Lieb spin moment \( |N_b - N_g|/2 \). However, the total AF moment in a UC is also \( \mathcal{N} = 1 \). The
FM-AF-balance character \((\mathcal{M} = \mathcal{N} \text{ or } m(0) = m(Q))\) comes from the fact that the VI ZMs are only distributed on one sublattice. With an increase in the interaction strength, the average magnetisations on the sites of sublattice B become finite values with an opposite polarisation to those on the sites of sublattice A. With an increase in the interaction strength, we have an AF-dominated FR order with \(\mathcal{M} < \mathcal{N}\) (or \(m(0) < m(Q)\)). In this case, the SQT inequality \(m(0) \leq m(Q)\) is satisfied. In the large \(U\) limit, we have a saturated value, \(\mathcal{N} \rightarrow 35\) but \(\mathcal{M} \equiv 1\).

Figure 1(d) shows the DOS of the FR order, of which there exists an energy gap. Near the gap, the DOS is enhanced due to the VI ZMs.

### The staggered-flux Hubbard model on a square lattice

Recently, people realised the photon-assisted tunneling on an optical lattice and then generated a large effective (staggered) magnetic flux on an optical lattice [11–13]. When two-component fermions with repulsive interaction are put into such an optical lattice, one can obtain an effective staggered-flux Hubbard model. Such progress in the relevant research may provide a new research platform for the study of ferrimagnetism. With such a platform, it is easy to change the potential barrier by varying the laser intensities to tune the Hamiltonian parameters, including the hopping strength \((t\text{-term})\), the staggered flux \((\phi)\) and the particle interaction \((U\text{-term})\). For this reason, we take the Hubbard model on a square lattice with staggered-flux (SF) as the second example, where \(t_{i,x\pm\delta_x} = e^{i\phi_x} t\) for \(i \in A\), \(t_{i,y\pm\delta_y} = t\) for \(i \in A\) and \(t_{i,x\pm\delta_y} = t_{i,y\pm\delta_x} = t\) for \(i \in B\). See the illustration in figure 2(a).

In particular, for this model, the operator of the \(S\) symmetry is \(S_F = \mathcal{P} \cdot T\) where \(T\) is the time-reversal transformation operator that commutes with lattice translation operators \([T, T_i] = [T, T_j] = 0\).}

For free electrons, \(U = 0\), after diagonalisation of the Hamiltonian in momentum space, the energy spectra are obtained as

\[E_k = -\frac{1}{2} \cos(c_1 k_x + c_2 k_y) = -\frac{1}{2} \cos\left(2c_1 k_x + 2c_2 k_y\right)\]

For the case of \(\pi\)-flux, the energy spectrum turns into

\[E_k = \pm 2t \sqrt{\cos^2 k_x + \cos^2 k_y}\]

and the system becomes a semi-metal and also has the perfect nesting condition. For the band, there exists a hole pocket for the band, there exists an electron pocket. The DOS \(\rho(E)\) near the Fermi surface is finite but reduced with increasing \(\phi\). See the illustration in figure 2(b).

For the \(\pi\)-flux case, the DOS near Fermi surface is zero and the critical point between the semi-metal and AF insulator is approximately \(U_c = 3.11t\) [14, 15]. However, for the \(\phi \neq 0, \pi\) case, due to the perfect nesting condition, the arbitrarily small interaction term leads to an AF order and then the BZ is reduced into a quarter one.

For the case of \(\phi \neq 0\), the VI ZMs is a quasi-localised state. See the particle density distribution of this ZM for the case of \(\phi = \pi\) in figure 3(a) and that for the case of \(\phi = \pi/2\) in figure 4(a), respectively. In the continuum limit, for the case of \(\phi = \pi\), the wave function of VI ZM distributes on the sublattice \(B\) that has a simple form of

\[\psi_0(x, y) \approx e^{\phi_{x+y}} + e^{-\phi_{x+y}}\]

The amplitude of this bound state decays with the distance to the vacancy as \(1/r\). However, for the case of \(\phi = \pi/2\), the VI ZMs are anisotropic due to the rotation-symmetry breaking of the original Hamiltonian.

Then, we discuss the staggered-flux Hubbard model with a 6a \(\times\) 6a vacancy-lattice. Now the ground state is a cluster FR order for weak coupling cases. The word ‘cluster’ means that the magnetic order parameters \(M_i\) is larger near the vacancy but smaller far away from it. See the illustration figures 3(b) and 4(b). This ‘cluster’ behaviour is obviously a physical consequence of the quasi-localised zero-modes induced by the vacancies. In particular, from figure 4(b), for the case of \(\phi = \pi/2\), an anisotropic FR order appears
which is and and and and. See 0) and). In addition,

\[ H = \frac{\Delta}{2} (\mathbf{S}_1 \cdot \mathbf{S}_2 + \mathbf{S}_3 \cdot \mathbf{S}_4) + \sum_{\langle ij \rangle} \left( J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j + U \mathbf{n}_i \cdot \mathbf{n}_j \right) \]

is the Hamiltonian of the SF Hubbard model on a square lattice. When

\[ t_2 \ll t_1 \] (or \[ t_2 \approx t_1 \]), the ground state \[ E_{\text{gs}} \] becomes a defect-diluted AF order. The system then has no nontrivial topological properties and a phase transition occurs. The energy gap closes and opens again.

\[ \Delta = E_{\text{gs}} \] turns into an AF-dominated FR order with \[ \phi = \pi, \theta = 0.1t; \] and \[ H \) is also a cluster FR. In the small limit, the average magnetisation \[ M \] tends to zero.

\[ M \approx \frac{N}{2} \] and \( M \approx 1 \) \( U \to \infty \). In addition, from the DOS of the system, there exist the VI ZMs that dominate the low energy physics. All these features indicate an AF-dominated FR order from the FM order of the spin moments of the VI ZMs.

**The spinful Haldane model on a square lattice**

An interesting issue is the FR order in a correlated topological insulator with NNN hoppings. In this paper, our analysis focuses on the interacting spinful Haldane model on square lattice, of which the Hamiltonian is given by

\[ H_0 = \sum_{\langle ij \rangle} \left( t_{ij} \mathbf{c}_i \mathbf{c}^\dagger_j + \text{H.c.} \right) \]

The spinful Haldane model on a square lattice, of which the Hamiltonian is given by

\[ H_0 = \sum_{\langle ij \rangle} \left( t_{ij} \mathbf{S}_i \cdot \mathbf{S}_j + \text{H.c.} \right) \]

is also a cluster FR. In the small \( U \) limit, the average magnetisation on the sites of sublattice A becomes finite as

\[ \frac{M_{\text{subA}}}{|M_{\text{subA}}|} = \frac{M_{\text{subB}}}{|M_{\text{subB}}|} \neq 0. \]

In this case, the magnetic order in sublattice A has the same polarised direction to that of sublattice B but its value is much smaller than that of sublattice B. The total spin \( S \) of the ground state still obeys the prediction of the Lieb spin moment as

\[ |N_A - N_B|/2. \]

However, we have \( M > N \), i.e. \( m(0) > m(Q) \). The ground state is an FM-dominated FR order and the SQT inequality is violated. We conclude that the violation of the SQT inequality for this case is due to the NNN hoppings. When we increase the interaction strength, we have \( m(0) = m(Q) \) (or \( N = 1 \)) at \( U_0 = t \). When we further increase the interaction strength, the ground state turns into an AF-dominated FR order with \( m(0) < m(Q) \). See figure 5(c). For larger interaction strength, a topological quantum phase transition occurs. The energy gap closes and opens again. The system then has no nontrivial topological properties and becomes a defect-diluted AF order.
Conclusion

In this paper, we developed a universal formula of the ferromagnetism in the Hubbard model beyond Lieb’s theorem by taking into account the S symmetry with larger universality than the traditional PH symmetry. By taking three models as examples, we studied the defect-induced FR orders that emerge from three typical two-component fermionic systems—metal, semi-metal and (Chern) insulator. We found that there may exist various FR orders. The total spin of all of these FR orders is equal to that of the Lieb spin moment ($\frac{1}{2}N_a - \frac{1}{2}N_b$). From the common feature of these FR orders, we hypothesise that it is the S symmetry protected VI ZMs ($\mathcal{H}_n = \mathcal{S}\mathcal{H}_n\mathcal{S}^{-1}$) that dominate the low energy physics of the system in the small $U$ limit. In addition, we found that the SQT inequality ($m(0) \leq m(\mathcal{Q})$) is valid for the model with only the NN hoppings and can be violated by the NNN hoppings. In addition, the formula can be straightforwardly applied to the Hubbard model on other bipartite lattices with S symmetry.

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