On the geometric foundation of classical gauge gravitation theory

Gennadi Sardanashvily

Department of Theoretical Physics, Physics Faculty, Moscow State University, 117234 Moscow, Russia
E-mail: sard@grav.phys.msu.su

Abstract.
A number of recent works in E-print arXiv have addressed the foundation of gauge gravitation theory again. As is well known, differential geometry of fibre bundles provides the adequate mathematical formulation of classical field theory, including gauge theory on principal bundles. Gauge gravitation theory is formulated on the natural bundles over a world manifold whose structure group is reducible to the Lorentz group. It is the metric-affine gravitation theory where a metric (tetrad) gravitational field is a Higgs field.

1 Introduction

The present exposition of gauge gravitation theory follows Refs. [42, 63, 64].

By a world manifold throughout is meant a four-dimensional oriented smooth manifold coordinated by \((x^\lambda)\).

Let us first recall the notion of a gauge transformation [16, 42, 43]. In the physical literature, by a (general) gauge transformation is meant a bundle automorphisms \(\Phi\) of a fibre bundle \(Y \to X\) over a diffeomorphism \(f\) of its base \(X\). If \(f = \text{Id} X\), the \(\Phi\) is said to be a vertical gauge transformation. If \(P \to X\) is a principal bundle with a structure Lie group \(G\), a gauge transformation \(\Phi\) of \(P\) is an equivariant automorphism of \(P\), i.e., \(\Phi \circ R_g = R_g \circ \Phi\), \(g \in G\), where \(R_g\) denotes the canonical right action of \(G\) on \(P\) on the right. A diffeomorphism \(f\) of \(X\) need not give rise to an automorphism a fibre bundle \(Y \to X\), unless \(Y\) belongs to the category of natural bundles over \(X\). A natural bundle \(T \to X\), by definition, admits the canonical lift of any diffeomorphism \(f\) of its base \(X\) to a bundle automorphism \(\tilde{f}\) of \(T\) over \(f\) [34]. This lift \(\tilde{f}\) is called a general covariant transformation (or a holonomic automorphism). Natural bundles are exemplified by tensor bundles. They are associated with the principal bundle \(LX \to X\) of oriented linear frames in the tangent spaces to \(X\).

The familiar gauge theory of internal symmetries deals with only vertical gauge transformations, while generic gauge transformations in gravitation theory are general covariant transformations. Therefore, gauge gravitation theory cannot straightforwardly follow the scheme of gauge theory of internal symmetries. For instance, connections on a principal bundle \(P\) with the structure group of internal symmetries \(G\) are usually called the gauge potentials of the group \(G\). In gauge gravitation theory, this terminology leads to confusion.
Gauge gravitation theory deals with connections on the principle frame bundle \( LX \rightarrow X \) over a world manifold \( X \). Its structure group is \( GL_4 = GL^+(4, \mathbb{R}) \). However, gravitation theory is not the gauge theory of this group because gravitation Lagrangians need not be invariant under non-holonomic (e.g., vertical) \( GL_4 \)-gauge transformations.

In Utiyama’s seminal paper [72], only vertical Lorentz gauge transformations were considered. Therefore tetrad fields were introduced by hand. To overcome this inconsistency, T.Kibble, D.Sciana and others treated tetrad fields as gauge potentials of the translation group \( \mathbb{L}, 22, 29, 31, 38, 47, 65, 71 \). In the first variant of this approach \( \mathbb{L}, 31, 65 \), the canonical lift of vector fields on a world manifold \( X \) onto a tensor bundle (i.e., generators of one-dimensional groups of general covariant transformations) was falsely interpreted as infinitesimal gauge translations. Later, the horizontal lift of vector fields by a linear world connection was treated in the same manner \( \mathbb{L}, 22 \). Afterwards, the conventional gauge theory on affine fibre bundles was called into play. However, the translation part of an affine connection on a tangent bundle is a basic soldering form

\[
\sigma = \sigma^\alpha_\lambda(x) dx^\lambda \otimes \partial_\alpha
\]  

(1.1)

(see Section 2) and a tetrad field (see section 3) are proved to be different mathematical objects \( \mathbb{L}, 21, 55, 59 \). Nevertheless, one continues to utilize \( \sigma^\alpha_\lambda dx^\lambda \) as a non-holonomic coframe in the metric-affine gauge theory with non-holonomic \( GL(4, \mathbb{R}) \) gauge transformations \( \mathbb{L}, 23, 37 \). On another side, translation gauge potentials describe an elastic distortion in the gauge theory of dislocations in continuous media \( \mathbb{L}, 28, 40 \) and in the analogous gauge model of the fifth force \( \mathbb{L}, 56, 57, 59 \).

For the first time, the conception of a graviton as a Goldstone particle corresponding to breaking of Lorentz symmetries in a curved space-time was declared at the beginning of the 1960s by Heisenberg and D.Ivanenko. This idea was revived in the framework of constructing non-linear (induced) representations of the group \( GL(4, \mathbb{R}) \), containing the Lorentz group as a Cartan subgroup \( \mathbb{L}, 25, 50 \). However, the geometric aspect of gravity was not considered. The fact that, in the framework of the gauge theory on fibre bundles, a pseudo-Riemannian metric can be treated as a Higgs field responsible for spontaneous breaking of space-time symmetries has been pointed out by A.Trautman \( \mathbb{L}, 70 \) and by the author \( \mathbb{L}, 54 \). Afterwards, it has been shown that this symmetry breaking issues from the geometric equivalence principle and that it is also caused by the existence of Dirac fermion matter with exact Lorentz symmetries \( \mathbb{L}, 26, 27, 41, 58, 59, 61 \).

The geometric equivalence principle postulates that, with respect to some reference frames, Lorentz invariants can be defined everywhere on a world manifold \( \mathbb{L}, 27 \). This principle has the adequate fibre bundle formulation. It requires that the structure group \( GL_4 \) of natural bundles over a world manifold \( X \) is reducible to the Lorentz group \( SO(1, 3) \). Let \( LX \rightarrow X \) be the above mentioned principle frame bundle. By virtue of the well-known
theorem [32] (see Section 3), its structure group $GL_4$ is reducible to $SO(1, 3)$ if and only if there exists a global section of the quotient bundle

$$\Sigma_{PR} = LX / SO(1, 3),$$

(1.2)
called the metric bundle. Its global sections are pseudo-Riemannian metrics on $X$. Moreover, there is one-to-one correspondence between these global sections $g$ and the principal subbundles $L^gX$ of the frame bundle $LX$ with the structure group $SO(1, 3)$. These subbundles are called reduced Lorentz structures [16, 19, 33, 42, 74]. The metric bundle $\Sigma_{PR}$ (1.2) is usually identified with an open subbundle of the tensor bundle $\mathcal{Q}^2 TX$, and is coordinated by $(x^\lambda, \sigma^{\mu\nu})$ with respect to the holonomic frames $\{\partial_\lambda\}$ in $TX$.

In General Relativity, a pseudo-Riemannian metric describes a gravitational field. Then, following the general scheme of gauge theory with broken symmetries [16, 20, 42, 46, 60, 70] (see Section 3), one can treat a metric gravitational field as a Higgs field, associated to a reduced Lorentz structure. Principal connections on the principal frame bundle $LX \to X$ and associated connections on the natural bundles over $X$ (we agree to call them world connections on $X$) play the role of gauge fields. Thus, we come to metric-affine gravitation theory [23, 44, 45, 49]. Its configuration space is the bundle product $\Sigma_{PR} \times C_K$, where

$$C_K = J^1 LX / GL_4$$

(1.3)
is the fibre bundle, coordinated by $(x^\lambda, k_\lambda^\nu_{\alpha})$, whose sections are world connections

$$K = dx^\lambda \otimes (\partial_\lambda + K_\lambda^{\mu\nu} \hat{x}^\nu \hat{\partial}_\mu)$$

(1.4)
on $X$ (see Section 2). Given a pseudo-Riemannian metric, any world connection is expressed into the Christoffel symbols, torsion and non-metricity tensors.

Generic gauge transformations of metric-affine gravitation theory are general covariant transformations of natural. All gravitation Lagrangians, by construction, are invariant under these transformations, but need not be invariant under non-holonomic (e.g., vertical) $GL_4$-gauge transformations. At the same time, if a metric-affine Lagrangian factorizes through the curvature $R_{\lambda\mu}^{\alpha\beta}$ of a world connection, it is invariant under the (non-holonomic) dilatation gauge transformations

$$k_\mu^{\alpha\beta} \mapsto k_\mu^{\alpha\beta} + V_\mu \delta_\beta^{\alpha},$$

(1.5)
called the projective freedom. The problem is that this invariance implies the corresponding projective freedom of a matter Lagrangian which imposes rigorous constraints on matter sources. To avoid the projective freedom, one usually add non-curvature terms to metric-affine Lagrangians expressed into the irreducible parts of torsion and non-metricity tensors [49]. In this case, the Proca field [10] and the hypermomentum fluid [3, 49] can play the role of hypothetical matter sources of non-metricity.
To dispose of the projective freedom problem, one can consider only Lorentz connections. A Lorentz connection is a principal connection on the frame bundle $LX$ which is reducible to a principal connection on some Lorentz subbundle $L^gX$ of $LX$. This takes place iff the covariant derivative of the metric $g$ with respect to this connection vanishes. Conversely, any principal connection on a reduced Lorentz subbundle is extended to a principal connection on the frame bundle $LX$, i.e., it defines a world connection on $X$. The holonomy group of a Lorentz connection is $SO(1,3)$. Metric-affine gravitation theory restricted to Lorentz connections is gravitation theory with torsion \[66\]. The Levi–Civita connection of a pseudo-Riemannian metric $g$ is a torsion-free Lorentz connection on the reduced Lorentz bundle $L^gX$. Considering only Levi–Civita connections, one comes to General Relativity.

Given a reduced Lorentz subbundle $L^gX$ and Lorentz connections on it, one can consider an associated vector bundle with the structure Lorentz group whose sections are treated as matter fields. The only realistic example of these matter fields is a Dirac spinor field. Its symmetry group $L_s = SL(2, \mathbb{C})$ (see Section 5) is the two-fold universal covering of the proper (connected) Lorentz group $L = SO^0(1,3)$. Therefore, the existence of Dirac spinor fields on a world manifold implies the contraction of the structure group $GL_4$ of the principal frame bundle $LX$ to the proper Lorentz group $L$. In accordance with the above mentioned theorem, this contraction takes place iff there exists a global section $h$ of the quotient bundle

$$\Sigma = LX/L,$$  \hspace{1cm} (1.6)

called the tetrad bundle. It is the two-fold covering of the metric bundle $\Sigma_{PR}$ \[\Sigma_{PR}\] Its global sections are called tetrad fields. A tetrad field $h$ is represented by local sections $h_\lambda^a(x)$ of the corresponding $L$-principal subbundle $L^hX$ of the frame bundle $LX$ (see Section 4). They are called tetrad functions. Obviously, every tetrad field $h: X \to \Sigma$ defines a unique pseudo-Riemannian metric $g: X \to \Sigma_{PR}$ such that the familiar relation

$$g_{\mu\nu} = h_\mu^a h_\nu^b \eta^{ab},$$  \hspace{1cm} (1.7)

where $\eta$ is the Minkowski metric, holds. Therefore, one can think of $h$ as being a tetrad gravitational field.

It should be emphasized that tetrad functions of a tetrad field $h$ are defined up to vertical Lorentz gauge transformations of the principal bundle $L^hX$. Therefore, tetrad gravitation theory possesses additional vertical Lorentz gauge transformations. However, these transformations are not similar to vertical gauge transformations in gauge theory of internal symmetries because they do not transform a tetrad field, but change only its representation by local tetrad functions.

To include into consideration Dirac spinor fields, one should consider spinor bundles over a world manifolds. A Dirac spin structure on a world manifold $X$ is defined as a pair $(P^h, z_s)$ of an $L_s$-principal bundle $P^h \to X$ and a bundle morphism

$$z_s : P^h \to LX$$  \hspace{1cm} (1.8)

from $P^h$ to the frame bundle $LX$ [2, 3, 30]. Every morphism (1.8) factorizes through a morphism

$$z_h : P^h \to L^h X$$

(1.9)
of $P^h$ to some $L$-principal subbundle $L^h X$ of the frame bundle $LX$. It follows that the necessary condition for existence of a Dirac spin structure on a world manifold $X$ is that $X$ admits a Lorentz structure and, consequently, a gravitational field exists.

Dirac spinor fields in the presence of a tetrad field $h$ are described by sections of the $P^h$-associated spinor bundle $S^h \to X$. In order to construct the Dirac operator on this bundle, one should define: (i) the representation

$$dx^\lambda \mapsto \gamma_h(dx^\lambda) = h^\lambda_a(x)\gamma^a$$

(1.10)
of coframes $\{dx^\lambda\}$ by the Dirac $\gamma$-matrices on elements of $S^h$ and (ii) the covariant derivative of sections of $S^h$ with respect to a world connection on $X$ [16, 42, 53, 54]. The result is the following (see Section 5).

(i) One can show that, for different tetrad fields $h$ and $h'$, the representations $\gamma_h$ and $\gamma_{h'}$ (1.10) are not equivalent. This fact exhibits the physical nature of gravity as a Higgs field [16, 42, 59, 64].

(ii) Given a tetrad field $h$, there is one-to-one correspondence between Lorentz connections on the $L$-principal bundle $L^h X$ and the principal connections, called spin connections, on the $h$-associated principal spinor bundle. For instance, every Levi–Civita connection yields the Fock–Ivanenko spin connection. Furthermore, one can show that, though a world connection need not be a Lorentz connection, any world connection yields a Lorentz connection on each $L$-principal subbundle $L^h X$ and, consequently, a spin connection [1, 16, 42, 52, 53]. It follows that gauge gravitation theory in the presence of Dirac spinor fields comes to the metric-affine theory, too.

This theory admits vertical spinor gauge transformations of Dirac spinor fields and vertical Lorentz gauge transformations of tetrad functions which lead to the equivalent representations (1.10). The problem lies in extension of general covariant transformations to spinor fields.

The group $GL_4$ has the universal two-fold covering group $\tilde{GL}_4$ such that the diagram

$$\begin{array}{ccc}
\tilde{GL}_4 & \longrightarrow & GL_4 \\
\downarrow \quad & & \downarrow \\
L_s & \overset{z_L}{\longrightarrow} & L
\end{array}$$

(1.11)

commutes. Let us consider the corresponding two-fold covering bundle $\tilde{LX} \to X$ of the frame bundle $LX$ [3, 36, 51, 68]. The group $\tilde{GL}_4$ admits the spinor representation, but it is infinite-dimensional. As a consequence, the $\tilde{LX}$-associated spinor bundle over $X$ describes
infinite-dimensional ”world” spinor fields, but not the Dirac ones [23]. At the same time, since the fibre bundle
\[ \tilde{LX} \to \Sigma \] (1.12)
is an \( L_s \)-principal bundle over the tetrad bundle \( \Sigma = \tilde{LX}/L_s \), the commutative diagram
\[ \begin{array}{ccc}
\tilde{LX} & \xrightarrow{\tilde{z}} & LX \\
\downarrow & & \downarrow \\
\Sigma & & \\
\end{array} \] (1.13)
provides a Dirac spin structure on the tetrad bundle \( \Sigma \). This spin structure is unique and possesses the following important property [17, 42, 64].

Let us consider the spinor bundle \( S \to \Sigma \), associated with the \( L_s \)-principal bundle (1.12). We have the composite bundle
\[ S \to \Sigma \to X. \] (1.14)

Given a tetrad field \( h \), the restriction \( h^*S \) of the spinor bundle \( S \to \Sigma \) to \( h(X) \subset \Sigma \) is isomorphic to the \( h \)-associated spinor bundle \( S^h \to X \) (see Section 3), i.e., its sections are Dirac spinor fields in the presence of a tetrad gravitational field \( h \). Conversely, every Dirac spin bundle \( S^h \) over a world manifold can be obtained in this way. The key point is that \( S \to X \) is not a spinor bundle, and admits general covariant transformations. The corresponding canonical lift onto \( S \) of a vector field on \( X \) can be constructed [16, 42, 64]. The goal is the energy-momentum conservation law in metric-affine gravitation theory in the presence of fermion fields. One can show that the corresponding energy-momentum current reduces to the generalized Komar superpotential [16, 42, 63].

The work is organized as follows. Section 2 is devoted to geometry of linear and affine frame bundles and world connections. Section 3 is concerned with the reduced structures and geometry of spontaneous symmetry breaking. Section 4 addresses the reduced Lorentz structures. Section 5 is devoted to spin structures on a world manifold. For the convenience of the reader, the relevant material on geometry of fibre bundles, jet manifolds and connections is summarized in Sections 6 – 7 (see [16, 42] for the detailed exposition).

2 Frame bundles and world connections

Let \( X \) be a world manifold. Its coordinate atlas \( \Psi_X \) and the corresponding holonomic bundle atlas \( \Psi \) [6, 8] of the tangent bundle \( TX \) hold fixed.

Let \( \pi_{LX} : LX \to X \) be the \( GL_4 \)-principal bundle of oriented linear frames in the tangent spaces to a world manifold \( X \). Its sections are called frame fields. Given holonomic frames \( \{ \partial_\mu \} \) in the tangent bundle \( TX \), every element \( \{ H_a \} \) of the frame bundle \( LX \) takes the
form $H_a = H_a^\mu \partial_\mu$, where $H_a^\mu$ is a matrix element of the natural representation of the group $GL_4$ in $\mathbb{R}^4$. These matrix elements constitute the bundle coordinates

$$(x^\lambda, H_a^\mu), \quad H_a^\mu = \frac{\partial x^\mu}{\partial x^\lambda} H_a^\lambda,$$

on $LX$. In these coordinates, the canonical action (7.38) of the structure group $GL_4$ on $LX$ reads

$$R_g : H_a^\mu \mapsto H_b^\mu g^b_a, \quad g \in GL_4.$$  

The frame bundle $LX$ is equipped with the canonical $\mathbb{R}^4$-valued 1-form

$$\theta_{LX} = H_a^\mu dx^\mu \otimes t_a, \quad (2.1)$$

where $\{t_a\}$ is a fixed basis for $\mathbb{R}^4$ and $H_a^\mu$ is the inverse matrix of $H_a^\mu$.

The frame bundle $LX \rightarrow X$ belongs to the category of natural bundles. Every diffeomorphism $f$ of $X$ gives rise to the automorphism

$$\tilde{f} : (x^\lambda, H_a^\lambda) \mapsto (f^\lambda(x), \partial_\mu f^\lambda H_a^\mu) \quad (2.2)$$

of $LX$. The lift (2.2) implies the canonical lift $\tilde{\tau}$ of every vector field $\tau$ on $X$ onto the principal bundle $LX$. It is defined by the relation

$$L_{\tilde{\tau}} \theta_{LX} = 0,$$

where $\tilde{\tau}$ (6.13) is the canonical lift of $\tau$ onto the tangent bundle $TX$.

Every world connection $K$ (1.4) on a world manifold $X$ is associated with a principal connection on $LX$. Consequently, there is one-to-one correspondence between the world connections and the sections of the quotient fibre bundle $C_K$ (1.3), called the bundle of world connections. With respect to the holonomic frames in $TX$, the fibre bundle $C_K$ is provided with the bundle coordinates $(x^\lambda, k_{\lambda}^\nu \alpha)$ such that, for any section $K$ of $C_K \rightarrow X$,

$$k_{\lambda}^\nu \alpha \circ K = K_{\lambda}^\nu \alpha$$

are the coefficients of the world connection $K$ (1.4). Though the bundle of world connections $C_K$ (1.3) is not an $LX$-associated bundle, it is a natural bundle and admits the canonical lift

$$\tilde{\tau} = \tau^\mu \partial_\mu + [\partial_\nu \tau^\alpha k_{\mu}^\nu \beta - \partial_\beta \tau^\nu k_{\mu}^\alpha \nu - \partial_\mu \tau^\nu k_{\nu}^\alpha \beta + \partial_\mu \tau^\alpha] \frac{\partial}{\partial k_{\mu}^\alpha \beta} \quad (2.3)$$

of any vector field $\tau$ on $X$ [14, 42].
The torsion of a world connection is the vertical-valued 2-form $T$ (7.23) on $TX$. Due to the canonical vertical splitting (6.22), the torsion (7.23) is represented by the tangent-valued 2-form (7.15) on $X$ and the soldering form
\[ T = T_\mu^\nu \lambda \hat{x}^\lambda d\mu \otimes \hat{\partial}_\nu. \] (2.4)

Every world connection $K$ yields the horizontal lift
\[ K\tau = \tau^\lambda(\partial_\lambda + K_\lambda^\beta \hat{x}^\beta \hat{\partial}_\lambda) \] (2.5)
of a vector field $\tau$ on $X$ onto $TX$. One can think of this lift as being the generator of a local 1-parameter group of non-holonomic automorphisms of this bundle. Note that, in the pioneer gauge gravitation models, the canonical lift (6.15) and the horizontal lift (2.5) were treated as generators of the gauge group of translations [22, 27] (see [39] for some other lifts onto $TX$ of vector fields on $X$).

Being a vector bundle, the tangent bundle $TX$ over a world manifold has a natural structure of an affine bundle. Therefore, one can consider affine connections on $TX$, called affine world connections. Let us study them as principal connections.

Let $Y \to X$ be an affine bundle with a $k$-dimensional typical fibre $V$. It is associated with a principal bundle $AY$ of affine frames in $Y$ whose structure group is the general affine group $GA(k, \mathbb{R})$. Then any affine connection on $Y \to X$ can be seen as an associated with a principal connection on $AY \to X$. These connections are represented by global sections of the affine bundle $J^1P/GA(k, \mathbb{R}) \to X$.

Every affine connection $\Gamma$ (7.24) on $Y \to X$ defines a linear connection $\Gamma$ (7.25) on the underlying vector bundle $Y \to X$. This connection $\Gamma$ is associated with a linear principal connection on the principal bundle $LY$ of linear frames in $Y$ whose structure group is the general linear group $GL(k, \mathbb{R})$. We have the exact sequence of groups
\[ 0 \to T_k \to GA(k, \mathbb{R}) \to GL(k, \mathbb{R}) \to 1, \] (2.6)
where $T_k$ is the group of translations in $\mathbb{R}^k$. It is readily observed that there is the corresponding principal bundle morphism $AY \to LY$ over $X$, and the principal connection $\mathcal{C}$ on $LY$ is the image of the principal connection $\Gamma$ on $AY \to X$ under this morphism in accordance with Theorem 7.4.

The exact sequence (2.6) admits a splitting $GL(k, \mathbb{R}) \hookrightarrow GA(k, \mathbb{R})$, but one usually loses sight of the fact that this splitting is not canonical (see, e.g., [32]). It depends on the morphism
\[ V \ni v \mapsto v - v_0 \in \mathcal{V}, \]
i.e., on the choice of an origin $v_0$ of the affine space $V$. Given $v_0$, the image of the corresponding monomorphism $GL(k, \mathbb{R}) \hookrightarrow GA(k, \mathbb{R})$ is the stabilizer $G(v_0) \subset GA(k, \mathbb{R})$ of $v_0$. Different subgroups $G(v_0)$ and $G(v'_0)$ are related to each other as follows:
\[ G(v'_0) = T(v'_0 - v_0)G(v_0)T^{-1}(v'_0 - v_0), \]
where \( T(v'_0 - v_0) \) is the translation along the vector \((v'_0 - v_0) \in V\).

Note that the well-known morphism of a \( k \)-dimensional affine space \( V \) onto a hypersurface \( \mathbb{y}^{k+1} = 1 \) in \( \mathbb{R}^{k+1} \) and the corresponding representation of elements of \( GA(k, \mathbb{R}) \) by particular \((k + 1) \times (k + 1)\)-matrices also fail to be canonical. They depend on a point \( v_0 \in V \) sent to vector \((0, \ldots, 0, 1) \in \mathbb{R}^{k+1}\).

If \( Y \to X \) is a vector bundle, it is provided with the canonical structure of an affine bundle whose origin is the canonical zero section \( \hat{0} \). In this case, we have the canonical splitting of the exact sequence \((2.6)\) such that \( GL(k, \mathbb{R}) \) is a subgroup of \( GA(k, \mathbb{R}) \) and \( GA(k, \mathbb{R}) \) is the semidirect product of \( GL(k, \mathbb{R}) \) and the group \( T_k \) of translations in \( \mathbb{R}^k \). Given a \( GA(k, \mathbb{R})\)-principal bundle \( AY \to X \), its affine structure group \( GA(k, \mathbb{R}) \) is always reducible to the linear subgroup since the quotient \( GA(k, \mathbb{R})/GL(k, \mathbb{R}) \) is a vector space \( \mathbb{R}^k \) provided with the natural affine structure. The corresponding quotient bundle is isomorphic to the vector bundle \( Y \to X \). There is the canonical injection of the linear frame bundle \( LY \hookrightarrow AY \) onto the reduced \( GL(k, \mathbb{R})\)-principal subbundle of \( AY \) which corresponds to the zero section \( \hat{0} \) of \( Y \to X \). In this case, every principal connection on the linear frame bundle \( LY \to Y \) gives rise to a principal connection on the affine frame bundle in accordance with Theorem \((3.7)\). It follows that any affine connection \( \Gamma \) on a vector bundle \( Y \to X \) defines a linear connection \( \Gamma^L \) on \( Y \to X \) and that every linear connection on \( Y \to X \) can be seen as an affine one. Hence, any affine connection \( \Gamma \) on the vector bundle \( Y \to X \) falls into the sum of the associated linear connection \( \Gamma^L \) and a basic soldering form \( \sigma \) on \( Y \to X \). Due to the vertical splitting \((6.11)\), this soldering form is represented by a global section of the tensor product \( T^*X \otimes Y \).

Let now \( Y \to X \) be the tangent bundle \( TX \to X \) considered as an affine bundle. Then the relationship between affine and linear world connections on \( TX \) is the repetition of that we have said above. In particular, any affine world connection

\[
K = dx^\lambda \otimes (K_\lambda^\alpha(x) \partial_\mu + \sigma_\lambda(x)) \partial_\alpha
\]  

on \( TX \to X \) is represented by the sum of the associated linear world connection

\[
\kappa = K_\lambda^\alpha(x) \partial_\mu dx^\lambda \otimes \partial_\alpha
\]

on \( TX \) and a basic soldering form \( \sigma \) \((1.1)\) on \( TX \). For instance, if \( \sigma = \theta_X \), we have the Cartan connection \((7.27)\).

As was mentioned above, the tensor field \( \sigma \) \((1.1)\) was falsely identified with a tetrad field in some gauge gravitation models.

### 3 Geometry of spontaneous symmetry breaking

Spontaneous symmetry breaking is a quantum phenomenon. In classical field theory, spontaneous symmetry breaking is modelled by classical Higgs fields. In gauge theory on a principal bundle \( P \to X \), a symmetry breaking is said to occur when the structure group \( G \) of
$P$ is reducible to a closed subgroup $H \subset G$ of exact symmetries \[10, 27, 30, 42, 46, 60, 70\]. This structure group reduction takes place iff a global section $h$ of the quotient bundle $P/H \to X$ exists (see Theorem 3.2 below). In gauge theory, such a global section $h$ is treated as a Higgs field. From the mathematical viewpoint, one speaks on the Klein–Chern geometry and the reduced $G$-structure \[74\].

Let $\pi_{P_X} : P \to X$ be a $G$-principal bundle (see Section 7) and $H$ a closed subgroup of $G$. It is also a Lie group. We assume that $\dim H > 0$. We have the composite fibre bundle
\[ P \to P/H \to X \] (3.1)
(see Section 7), where
\[ P_\Sigma = P \xrightarrow{\pi_{P_\Sigma}} P/H \] (3.2)
is a principal bundle with the structure group $H$ and
\[ \Sigma = P/H \xrightarrow{\pi_{\Sigma X}} X \] (3.3)
is a $P$-associated fibre bundle with the typical fibre $G/H$ on which the structure group $G$ acts naturally on the left. Note that the canonical surjection $G \to G/H$ is an $H$-principal bundle.

One says that the structure group $G$ of a principal bundle $P$ is reducible to a Lie subgroup $H$ if there exists a $H$-principal subbundle $P^h$ of $P$ with the structure group $H$. This subbundle is called a reduced $G \downarrow H$-structure \[10, 19, 33, 42, 74\].

Two reduced $G \downarrow H$-structures $P^h$ and $P^{h'}$ of a $G$-principal bundle $P$ are said to be isomorphic if there is an automorphism $\Phi$ of $P$ which provides an isomorphism of $P^h$ and $P^{h'}$. If $\Phi$ is a vertical automorphism of $P$, reduced structures $P^h$ and $P^{h'}$ are called equivalent. Note that, in \[19, 33\] (see also \[1\]), only reduced structures of the frame bundle $LX$ are considered, and a class of isomorphisms of such reduced structures is restricted to holonomic automorphisms of $LX$.

There are the following two theorems \[32\].

**Theorem 3.1.** A structure group $G$ of a principal bundle $P$ is reducible to its closed subgroup $H$ iff $P$ has an atlas $\Psi_P$ with $H$-valued transition functions. \[\square\]

Given a reduced subbundle $P^h$ of $P$, such an atlas $\Psi_P$ is defined by a family of local sections \(\{z_\alpha\}\) which take their values into $P^h$ (see Section 7).

**Theorem 3.2.** There is one-to-one correspondence $P^h = \pi_{P_\Sigma}^{-1}(h(X))$ between the reduced $H$-principal subbundles $P^h$ of $P$ and the global sections $h$ of the quotient fibre bundle $P/H \to X$ (3.3). \[\square\]
Given such a section \( h \), let us consider the restriction \( h^*P_\Sigma \) (7.31) of the \( H \)-principal bundle \( P_\Sigma \) (3.2) to \( h(X) \subset \Sigma \). This is a \( H \)-principal bundle over \( X \), which is equivalent to the reduced subbundle \( P^h \) of \( P \).

In general, there are topological obstructions to the reduction of a structure group of a principal bundle to its subgroup. In accordance with Theorem 6.1, the structure group \( G \) of a principal bundle \( P \) is always reducible to its closed subgroup \( H \) if the quotient \( G/H \) is homeomorphic to an Euclidean space \( \mathbb{R}^k \).

**Theorem 3.3.** [67]. A structure group \( G \) of a principal bundle is always reducible to its maximal compact subgroup \( H \) since the quotient space \( G/H \) is homeomorphic to an Euclidean space. \( \Box \)

Two \( H \)-principal subbundles \( P^h \) and \( P^{h'} \) of a \( G \)-principal bundle \( P \) need not be isomorphic.

**Proposition 3.4.** [42]. (i) Every vertical automorphism \( \Phi \) of the principal bundle \( P \to X \) sends an \( H \)-principal subbundle \( P^h \) onto an equivalent \( H \)-principal subbundle \( P^{h'} \).

(ii) Conversely, let two reduced subbundles \( P^h \) and \( P^{h'} \) of a principal fibre bundle \( P \) be isomorphic to each other, and \( \Phi : P^h \to P^{h'} \) be an isomorphism. Then \( \Phi \) is extended to a vertical automorphism of \( P \). \( \Box \)

**Proposition 3.5.** [67]. If the quotient \( G/H \) is homeomorphic to an Euclidean space \( \mathbb{R}^k \), all \( H \)-principal subbundles of a \( G \)-principal bundle \( P \) are equivalent to each other. \( \Box \)

Given a reduced subbundle \( P^h \) of a principal bundle \( P \), let

\[
Y^h = (P^h \times V)/H
\]

be the associated fibre bundle with a typical fibre \( V \) (see Section 7). Let \( P^{h'} \) be another reduced subbundle of \( P \) which is isomorphic to \( P^h \), and

\[
Y^{h'} = (P^{h'} \times V)/H
\]

The fibre bundles \( Y^h \) and \( Y^{h'} \) are isomorphic, but not canonically isomorphic in general.

**Proposition 3.6.** [42]. Let \( P^h \) be an \( H \)-principal subbundle of a \( G \)-principal bundle \( P \). Let \( Y^h \) be the \( P^h \)-associated bundle (3.4) with a typical fibre \( V \). If \( V \) carries a representation of the whole group \( G \), the fibre bundle \( Y^h \) is canonically isomorphic to the \( P \)-associated fibre bundle

\[
Y = (P \times V)/G.
\]
In accordance with Theorem 3.2, the set of reduced $H$-principal subbundles $P^h$ of $P$ is in bijective correspondence with the set of Higgs fields $h$. Given such a subbundle $P^h$, let $Y^h$ be the associated vector bundle with a typical fibre $V$ which admits a representation of the group $H$ of exact symmetries, but not of the whole symmetry group $G$. Its sections $s_h$ describe matter fields in the presence of the Higgs fields $h$ and some principal connection $A_h$ on $P^h$. In general, the fibre bundle $Y^h$ is not associated with other $H$-principal subbundles $P^{h'}$ of $P$. It follows that, in this case, $V$-valued matter fields can be represented only by pairs with Higgs fields. The goal is to describe the totality of these pairs $(s_h, h)$ for all Higgs fields $h$.

For this purpose, let us consider the composite fibre bundle (3.1) and the composite fibre bundle

$$Y \xrightarrow{\pi_Y} \Sigma \xrightarrow{\pi_\Sigma} X$$

(3.5)

where $Y \to \Sigma$ is a vector bundle

$$Y = (P \times V)/H$$

associated with the corresponding $H$-principal bundle $P_\Sigma$. Given a global section $h$ of the fibre bundle $\Sigma \to X$ and the $P^h$-associated fibre bundle (3.4), there is the canonical injection

$$i_h : Y^h = (P^h \times V)/H \hookrightarrow Y$$

over $X$ whose image is the restriction

$$h^*Y = (h^*P \times V)/H$$

of the fibre bundle $Y \to \Sigma$ to $h(X) \subset \Sigma$, i.e.,

$$i_h(Y^h) \cong \pi_{\Sigma}^{-1}(h(X))$$

(3.6)

(see Proposition (7.1)). Then, by virtue of Proposition (7.2) every global section $s_h$ of the fibre bundle $Y^h$ corresponds to the global section $i_h \circ s_h$ of the composite fibre bundle (3.5). Conversely, every global section $s$ of the composite fibre bundle (3.5) which projects onto a section $h = \pi_Y \circ s$ of the fibre bundle $\Sigma \to X$ takes its values into the subbundle $i_h(Y^h) \subset Y$ in accordance with the relation (3.6). Hence, there is one-to-one correspondence between the sections of the fibre bundle $Y^h$ (3.4) and the sections of the composite fibre bundle (3.5) which cover $h$.

Thus, it is the composite fibre bundle (3.3) whose sections describe the above-mentioned totality of pairs $(s_h, h)$ of matter fields and Higgs fields in gauge theory with broken symmetries [16, 42, 60].
Turn now to the properties of connections compatible with a reduced structure \[32\].

**Theorem 3.7.** Since principal connections are equivariant, every principal connection \(A_h\) on a reduced \(H\)-principal subbundle \(P^h\) of a \(G\)-principal bundle \(P\) gives rise to a principal connection on \(P\). \(\Box\)

**Theorem 3.8.** A principal connection \(A\) on a \(G\)-principal bundle \(P\) is reducible to a principal connection on a reduced \(H\)-principal subbundle \(P^h\) of \(P\) iff the corresponding global section \(h\) of the \(P\)-associated fibre bundle \(P/H \to X\) is an integral section of the associated principal connection \(A\) on \(P/H \to X\). \(\Box\)

**Theorem 3.9.** Given the composite fibre bundle \((3.1)\), let \(A_{\Sigma}\) be a principal connection on the \(H\)-principal bundle \(P \to P/H\). Then, for any reduced \(H\)-principal subbundle \(P^h\) of \(P\), the pull-back connection \(i^*_h A_{\Sigma}\) \((7.34)\) is a principal connection on \(P^h\). \(\Box\)

This assertion is a corollary of Theorem \(7.3\).

As a consequence of Theorem \(3.3\), there is the following peculiarity of the dynamics of field systems with symmetry breaking. Let the composite fibre bundle \(Y\) \((3.3)\) be provided with coordinates \((x^\lambda, \sigma^m, y^i)\), where \((x^\lambda, \sigma^m)\) are bundle coordinates on the fibre bundle \(\Sigma \to X\). Let

\[
A_{\Sigma} = dx^\lambda \otimes (\partial_{\lambda} + A^i_{\lambda} \partial_i) + d\sigma^m \otimes (\partial_m + A^i_m \partial_i) \tag{3.7}
\]

be a principal connection on the vector bundle \(Y \to \Sigma\). This connection defines the splitting \((7.36)\) of the vertical tangent bundle \(VY\) and leads to the vertical covariant differential \((7.37)\) which reads

\[
\widetilde{D} = dx^\lambda \otimes (y_i^\lambda - A^i_{\lambda} - A^i_m \sigma^m_{\lambda}) \partial_i. \tag{3.8}
\]

The operator \((3.8)\) possesses the following property (see Section 7). Given a global section \(h\) of \(\Sigma \to X\), its restriction

\[
\widetilde{D}_h = \widetilde{D} \circ J^1_i h: J^1 Y^h \to T^* X \otimes VY^h, \tag{3.9}
\]

\[
\widetilde{D}_h = dx^\lambda \otimes (y_i^\lambda - A^i_{\lambda} - A^i_m \partial_{\lambda} h^m) \partial_i,
\]

to \(Y^h\) is the familiar covariant differential relative to the pull-back principal connection \(A_h\) \((7.34)\) on the fibre bundle \(Y^h \to X\). Thus, a Lagrangian on the jet manifold \(J^1 Y\) of a composite fibre bundle usually factorizes through the vertical differential \(\widetilde{D}_A\) \([16, 12]\).
4 Lorentz structures

Let us apply the above scheme of symmetry breaking in gauge theory to gravitation theory.

As was mentioned above, the geometric formulation of the equivalence principle requires that the structure group $GL_4$ of the frame bundle $LX$ over a world manifold $X$ is reducible to the Lorentz group $SO(1, 3)$, while the condition of existence of fermion fields implies that $GL_4$ is reducible to the proper Lorentz group $L$. The latter is homeomorphic to $RP^3 \times \mathbb{R}^3$, where $RP^3$ is a real 3-dimensional projective space. Unless otherwise stated, by a Lorentz structure we will mean a reduced $L$-principal subbundle $L^hX$ of $LX$ called the Lorentz subbundle.

Note that there is the topological obstruction to the existence of a reduced Lorentz structure on a world manifold $X$. All non-compact manifolds and compact manifolds whose Euler characteristic equals zero admit a reduced $SO(1, 3)$-structure and, as a consequence, a pseudo-Riemannian metric [11]. A reduced $L$-structure exists if $X$ is additionally time-orientable. In gravitational models, certain conditions of causality should also be satisfied (see [21]). A compact space-time does not possess this property. At the same time, a non-compact world manifold $X$ has a spin structure if it is parallelizable (i.e., the tangent bundle $TX \to X$ is trivial) [14, 73].

Let us assume that a Lorentz structure on a world manifold exists. Then one can show that different Lorentz subbundles $L^hX$ and $L^{h'}X$ of the frame bundle $LX$ are equivalent as $L$-principal bundles [24]. It means that there exists a vertical automorphism of the frame bundle $LX$ which sends isomorphically $L^hX$ onto $L^{h'}X$ (see Proposition 3.4).

By virtue of Theorem 3.2 there is one-to-one correspondence between the $L$-principal subbundles $L^hX$ of the frame bundle $LX$ and the global sections $h$ of the tetrad bundle $\Sigma$ (1.6). This is an $LX$-associated fibre bundle with the typical fibre $GL_4/L$, homeomorphic to $S^3 \times \mathbb{R}^7$. Its global sections are tetrad fields. Note that the typical fibre of the metric bundle $\Sigma_{PR}$ (1.2) is homeomorphic to $RP^3 \times \mathbb{R}^7$.

Every tetrad field $h$ defines an associated Lorentz atlas $\Psi^h = \{(U_\zeta, z^h_\zeta)\}$ of the frame bundle $LX$ such that the corresponding local sections $z^h_\zeta$ of $LX$ take their values into the Lorentz subbundle $L^hX$. They are called tetrad functions, and are given by the coordinate expression

$$h^\mu_a = H^\mu_a \circ z^h_\zeta.$$  \hspace{1cm} (4.1)

In accordance with Theorem 3.1 the transition functions of the Lorentz atlastes of the frame bundle $LX$ and associated bundles are $L$-valued.

Given a Lorentz atlas $\Psi^h$, the pull-back $z^{h*}_\zeta \theta_{LX}$ of the canonical form $\theta_{LX}$ (2.1) by a tetrad function $z^h_\zeta$ is called a (local) tetrad form. It reads

$$h^a \otimes t^a = z^{h*}_\zeta \theta_{LX} = h^a_\lambda dx^\lambda \otimes t^a,$$  \hspace{1cm} (4.2)
where $h^a_\lambda$ are elements of the inverse matrix to $h^\mu_a$ (4.1). The tetrad form (4.2) determines the tetrad coframes
\[
h^a = h^a_\mu(x) dx^\mu, \quad x \in U_\zeta,
\] (4.3)
in the cotangent bundle $T^*X$. These coframes are associated with the Lorentz atlas $\Psi^h$. In particular, the relation (1.7) between the tetrad functions and the metric functions of the corresponding pseudo-Riemannian metric $g : X \to \Sigma_{\text{PR}}$ takes the form
\[
g = \eta_{ab} h^a \otimes h^b.
\]
It follows that this pseudo-Riemannian metric $g$ has the Minkowski metric functions with respect to any Lorentz atlas $\Psi^h$. It exemplifies a Lorentz invariant mentioned in the geometric equivalence principle.

Let $M = \mathbb{R}^4$ be the dual of $\mathbb{R}^4$ provided with the Minkowski metric $\eta$. Given a tetrad field $h$, let us consider the $L^hX$-associated fibre bundle of Minkowski spaces
\[
M^hX = (L^hX \times M)/L.
\] (4.4)
By virtue of Proposition 3.6, it is isomorphic to the cotangent bundle
\[
T^*X = (LX \times \mathbb{R}^4)/GL_4 = (L^hX \times M)/L = M^hX.
\] (4.5)
Given the isomorphism (4.5), we say that the cotangent bundle $T^*X$ is provided with a Minkowski structure. Note that different Minkowski structures $M^hX$ and $M^{h'}X$ on $T^*X$ are not equivalent.

As was mentioned above, a principal connection on a Lorentz subbundle $L^h$ of the frame bundle $LX$ is called the Lorentz connection. It reads
\[
A_h = dx^\lambda \otimes (\partial_\lambda + \frac{1}{2} A^{ab}_\lambda \varepsilon_{ab})
\] (4.6)
where $\varepsilon_{ab} = -\varepsilon_{ba}$ are generators of the Lorentz group. Recall that the Lorentz group $L$ acts on $\mathbb{R}^4$ by the generators
\[
\varepsilon_{ab}^c d = \eta_{ad} \delta_b^c - \eta_{bd} \delta_a^c.
\] (4.7)
By virtue of Theorem 3.7, every Lorentz connection (4.6) is extended to a principal connection on the frame bundle $LX$ and, thereby, it defines a world connection $K$ whose coefficients are
\[
K_\lambda^\mu \nu = h^b_\nu \partial_\lambda h^\mu_b + \eta_{ka} h^a_b h^b_{\nu} A^{ab}_\lambda.
\] (4.8)
This world connection is also called the Lorentz connection. Its holonomy group is a subgroup of the Lorentz group $L$. Conversely, let $K$ be a world connection with the holonomy
group $L$. By virtue of the well-known theorem \[32\], it defines a Lorentz subbundle of the frame bundle $LX$, and is a Lorentz connection on this subbundle.

Though a world connection is not a Lorentz connection in general, any world connection $K$ defines a Lorentz connection $K_h$ on each $L$-principal subbundle $L^hX$ of the frame bundle as follows.

It is readily observed that the Lie algebra of the general linear group $GL_4$ is the direct sum

$$\mathfrak{g}(GL_4) = \mathfrak{g}(L) \oplus \mathfrak{m}$$

of the Lie algebra $\mathfrak{g}(L)$ of the Lorentz group and a subspace $\mathfrak{m}$ such that $ad(l)(\mathfrak{m}) \subset \mathfrak{m}$ for all $l \in L$. Let $\overline{K}$ be the connection form of a principal connection $K$ on $LX$. Then, by virtue of the well-known theorem \[32\], the pull-back on $L^hX$ of the $\mathfrak{g}(L)$-valued component $K_L$ of $K$ is the connection form of a principal connection $K_h$ on the Lorentz subbundle $L^hX$. To obtain the connection parameters of $K_h$, let us consider the local connection 1-form of the connection $K$ with respect to a Lorentz atlas $\Psi^h$ of $LX$ given by the tetrad forms $h^a$. This reads

$$z^h_\xi \overline{K} = -K^b_{\lambda a}dx^\lambda \otimes \varepsilon^a_b, \quad K^b_{\lambda a} = -h^b_{\mu \lambda} \partial_\lambda h^\mu_a + K^\mu_{\lambda \nu} h^b_{\mu \nu} h^a_{\lambda},$$

where $\{\varepsilon^a_b\}$ is the basis for the right Lie algebra of the group $GL_4$. Then, the Lorentz part of this form is the local connection 1-form of the connection $K_h$ on $L^hX$. We have

$$z^h_\xi \overline{K}_L = -\frac{1}{2} A_{\lambda}^{ab} dx^\lambda \otimes \varepsilon_{ab}, \quad A_{\lambda}^{ab} = \frac{1}{2}(\eta^{k b} h^a_{\mu} - \eta^{k a} h^b_{\mu})(\partial_\lambda h^\mu_k - h^\nu_k K^\nu_{\lambda \nu}). \quad (4.9)$$

If $K$ is a Lorentz connection extended from $L^hX$, then obviously $K_h = K$.

5 Dirac spin structures

We describe Dirac spinors in terms of Clifford algebras (see, e.g., \[3, 8, 36, 48, 53\]).

Let $M$ be the Minkowski space equipped with the Minkowski metric $\eta$, and $\{e^a\}$ be a fixed basis for $M$. By $\mathbb{C}_{1,3}$ is denoted the complex Minkowski metric $\eta$, and $\{e^a\}$ be a fixed basis for $M$. By $\mathbb{C}_{1,3}$ is denoted the complex Clifford algebra generated by elements of $M$. This is the complexified quotient of the tensor algebra $\otimes M$ of $M$ by the two-sided ideal generated by elements

$$e \otimes e' + e' \otimes e - 2\eta(e, e') \in \otimes M, \quad e, e' \in M.$$ 

The complex Clifford algebra $\mathbb{C}_{1,3}$ is isomorphic to the real Clifford algebra $\mathbb{R}_{2,3}$, whose generating space is $\mathbb{R}^5$ equipped with the metric

$$\text{diag}(1, -1, -1, -1, 1).$$
Its subalgebra generated by elements of $M \subset \mathbb{R}^5$ is the real Clifford algebra $\mathbb{R}_{1,3}$.

A spinor space $V$ is defined as a minimal left ideal of $\mathbb{C}_{1,3}$ on which this algebra acts on the left. We have the representation

$$\gamma : M \otimes V \to V, \quad \gamma(e^a) = \gamma^a,$$  \tag{5.1}

of elements of the Minkowski space $M \subset \mathbb{C}_{1,3}$ by the Dirac $\gamma$-matrices on $V$. Different ideals $V$ lead to equivalent representations (5.1).

By definition, the Clifford group $G_{1,3}$ consists of the invertible elements $l_s$ of the real Clifford algebra $\mathbb{R}_{1,3}$ such that the inner automorphisms defined by these elements preserve the Minkowski space $M \subset \mathbb{R}_{1,3}$, that is,

$$l_s l_s^{-1} = l(e), \quad e \in M,$$  \tag{5.2}

where $l$ is a Lorentz transformation of $M$. Hence, we have an epimorphism of the Clifford group $G_{1,3}$ onto the Lorentz group $O(1, 3)$. Since the action (5.2) of the Clifford group on the Minkowski space $M$ is not effective, one usually consider its pin and spin subgroups. The subgroup $Pin(1, 3)$ of $G_{1,3}$ is generated by elements $e \in M$ such that $\eta(e, e) = \pm 1$. The even part of $Pin(1, 3)$ is the spin group $Spin(1, 3)$. Its component of the unity is the above mentioned group two-fold universal covering group

$$z_L : L_s \to L = L_s/\mathbb{Z}_2, \quad \mathbb{Z}_2 = \{1, -1\},$$  \tag{5.3}

of the proper Lorentz group $L$.

The Clifford group $G_{1,3}$ acts on the spinor space $V$ by left multiplications

$$G_{1,3} \ni l_s : v \mapsto l_s v, \quad v \in V.$$  

This action preserves the representation (5.1), i.e.,

$$\gamma(l M \otimes l_s V) = l_s \gamma(M \otimes V).$$

The spin group $L_s$ acts on the spinor space $V$ by means of the generators

$$L_{ab} = \frac{1}{4} [\gamma_a, \gamma_b].$$  \tag{5.4}

Let $P^h \to X$ be a principal bundle with the structure group $L_s$ provided with the bundle morphisms $z_s$ (1.8) and $z_h$ (1.9). Dirac spinor fields in the presence of a tetrad field $h$ are described by sections of the $P^h$-associated spinor bundle

$$S^h = (P^h \times V)/L_s \to X,$$  \tag{5.5}

whose typical fibre $V$ carriers the spinor representation (5.4) of the spin group $L_s$. To define the Dirac operator on this fields, the spinor bundle $S^h$ (5.5) must be represented as
a subbundle of the bundle of Clifford algebras, i.e., as a spinor structure on the cotangent bundle $T^*X$.

Every fibre bundle of Minkowski spaces $M^hX$ over a world manifold $X$ is extended to the fibre bundle of Clifford algebras $C^hX$ with the fibres generated by the fibres of $M^hX$. This fibre bundle $C^hX$ has the structure group $\text{Aut}(\mathbb{C}_{1,3})$ of inner automorphisms of the Clifford algebra $\mathbb{C}_{1,3}$. In general, $C^hX$ does not contain a spinor subbundle because a spinor subspace $V$ is not stable under inner automorphisms of $\mathbb{C}_{1,3}$. As was shown [5], a spinor subbundle of $C^hX$ exists if the transition functions of $C^hX$ can be lifted from $\text{Aut}(\mathbb{C}_{1,3})$ to the Clifford group $G_{1,3}$. This agrees with the standard condition of existence of a spin structure on a world manifold $X$. Such a spinor subbundle is the bundle $S^h$ (5.5) associated with the universal two-fold covering (1.9) of the Lorentz bundle $L^hX$. We will call $P^h$ (and $S^h$) the $h$-associated Dirac spin structure on a world manifold.

Note that all spin structures on a manifold $X$ which are related to the two-fold universal covering groups possess the following two properties [20]. Let $P \to X$ be a principal bundle whose structure group $G$ has the fundamental group $\pi_1(G) = \mathbb{Z}_2$. Let $\tilde{G}$ be the universal covering group of $G$.

- The topological obstruction to the existence of a $\tilde{G}$-principal bundle $\tilde{P} \to X$ covering the bundle $P \to X$ is given by the Čech cohomology group $H^2(X; \mathbb{Z}_2)$ of $X$ with coefficients in $\mathbb{Z}_2$. Roughly speaking, the principal bundle $P$ defines an element of $H^2(X; \mathbb{Z}_2)$ which must be zero so that $P \to X$ can give rise to $\tilde{P} \to X$.

- Non-equivalent lifts of $P \to X$ to $\tilde{G}$-principal bundles are classified by elements of the Čech cohomology group $H^1(X; \mathbb{Z}_2)$.

In particular, the well-known topological obstruction to the existence of a Dirac spin structure is the second Stiefel–Whitney class $w_2(X) \in H^2(X; \mathbb{Z}_2)$ of $X$ [36]. In the case of 4-dimensional non-compact manifolds, all Riemannian and pseudo-Riemannian spin structures are equivalent [2, 14].

There exists the bundle morphism

$$\gamma_h : T^*X \otimes S^h = (P^h \times (M \otimes V))/L_s \to (P^h \times \gamma(M \otimes V))/L_s = S^h,$$

where by $\gamma$ is meant the left action (5.1) of $M \subset \mathbb{C}_{1,3}$ on $V \subset \mathbb{C}_{1,3}$. One can think of (5.6) as being the representation of covectors to $X^4$ by the Dirac $\gamma$-matrices on elements of the spinor bundle $S^h$. Relative to an atlas $\{z_\zeta\}$ of $L^hX$, the representation (5.6) reads

$$y^A(\gamma_h(h^a(x) \otimes v)) = \gamma^a_{A\ B}y^B(v), \quad v \in S^h_x,$$

where $y^A$ are the corresponding bundle coordinates of $S^h$, and $h^a$ are the tetrad coframes (1.3). For brevity, we will write

$$\tilde{h}^a = \gamma_h(h^a) = \gamma^a, \quad \tilde{d}x^\lambda = \gamma_h(dx^\lambda) = h^\lambda(x)\gamma^a.$$
Let $A_h$ be a principal connection on $S^h$ and let
\[ D : J^1 S^h \to T^* X \otimes S^h, \quad D = (y^A - A^{ab}_\lambda L_{ab}^A y^B) dx^\lambda \otimes \partial_A, \]
be the corresponding covariant differential (7.7), where
\[ VS^h = S^h \times_X S^h. \]
The first order differential Dirac operator is defined on $S^h$ by the composition
\[ \Delta_h = \gamma_h \circ D : J^1 S^h \to T^* X \otimes S^h \to S^h, \]
(5.7)
\[ y^A \circ \Delta_h = h^A_{\lambda B}(y^B - \frac{1}{2} A^{ab}_\lambda L_{ab}^A y^B). \]
Note that there is one-to-one correspondence between the principal connections, called spin connections, on the $h$-associated principal spinor bundle $P^h$ and the Lorentz connections on the $L$-principal bundle $L^hX$. Indeed, it follows from Theorem 7.4 that every principal connection
\[ A_h = dx^\lambda \otimes (\partial_\lambda + \frac{1}{2} A^{ab}_\lambda \varepsilon_{ab} ) \]
(5.8)
on $P^h$ defines a principal connection on $L^hX$ which is given by the same expression (5.8). Conversely, the pull-back $z_h^* \gamma_h$ on $P^h$ of the connection form $\gamma_h$ of a Lorentz connection $A_h$ on $L^hX$ is equivariant under the action of group $L_s$ on $P^h$ and, consequently, it is a connection form of a spin connection on $P^h$.
In particular, the Levi–Civita connection of a pseudo-Riemannian metric $g$ gives rise to the spin connection with the components
\[ A^{ab}_\lambda = \eta^{kb} h^a_\mu (\partial_\lambda h^\mu_k - h^\nu_k \{ \lambda \}_{\mu}). \]
(5.9)
We consider the general case of a spin connection generated on $P^h$ by a world connection $K$. The Lorentz connection $K_h$ induced by $K$ on $L^hX$ takes the form (1.6) with components (1.9). It defines the spin connection
\[ K_h = dx^\lambda \otimes [\partial_\lambda + \frac{1}{4} (\eta^{kb} h^a_\mu - \eta^{ka} h^b_\mu) (\partial_\lambda h^\mu_k - h^\nu_k K^\mu_{\lambda \nu}) L_{ab}^A y^B \partial_A ] \]
(5.10)
on $S^h$, where $L_{ab}$ are the generators (5.4) [12, 52, 63]. Substituting this spin connection in the Dirac operator (5.7), we obtain the dynamics of Dirac spinor fields in the presence of an arbitrary world connection on a world manifold, not only of the Lorentz type.
Motivated by the connection (5.10), one can obtain the lift
\[ \bar{\tau} = \tau^\lambda \partial_\lambda + \frac{1}{4} (\eta^{kb} h^a_\mu - \eta^{ka} h^b_\mu) (\tau^\lambda \partial_\lambda h^\mu_k - h^\nu_k \partial_\lambda \tau^\mu) L_{ab}^A y^B \partial_A \]
(5.11)
of vector fields $\tau$ on $X$ onto the spinor bundle $S^h$ [10, 12, 62]. This lift can be brought into the form
\[
\tilde{\tau} = \tau\{\} - \frac{1}{4}(\eta^{kb}h^a_{\mu} - \eta^{ka}h^b_{\mu})h_k^\nu \nabla_\nu \tau^\mu L_{ab} A_B y^B \partial_A,
\]
where $\tau\{\}$ is the horizontal lift of $\tau$ by means of the spin Levi–Civita connection for the tetrad field $h$, and $\nabla_\nu \tau^\mu$ are the covariant derivatives of $\tau$ relative to the Levi–Civita connection [12, 18, 35].

Let us further assume that a world manifold $X$ is non-compact, and let it be parallelizable in order to admit a spin structure. In this case, all Dirac spin structures are equivalent, i.e., the principal spinor bundles $P^h$ and $P^{h'}$ are isomorphic [2, 14]. Nevertheless, the associated structures of the bundles of Minkowski spaces $M^hX$ and $M^{h'}X$ [14] on the cotangent bundle $T^*X$ are not equivalent, and so are the representations $\gamma_h$ and $\gamma_{h'}$ [5] [14, 59]. It follows that every Dirac fermion field must be described in a pair $(s_h, h)$ with a certain tetrad field $h$, and Dirac fermion fields in the presence of different tetrad fields fail to be given by sections of the same spinor bundle. The goal is to construct a bundle over $X$ whose sections exhaust the whole totality of fermion-gravitation pairs [10, 12, 64]. Following the general scheme of describing symmetry breaking in Section 3, we will use the fact that the frame bundle $LX$ is the principal bundle $LX \to \Sigma$ over the tetrad bundle $\Sigma$ (1.6) with the structure Lorentz group $L$.

Let us consider the above mentioned two-fold covering bundle $\tilde{LX}$ of the frame bundle $LX$ and the Dirac spin structure (1.13) on the tetrad bundle $\Sigma$. Owing to the commutative diagram (1.11), we have the commutative diagram
\[
\begin{array}{ccc}
\tilde{LX} & \xrightarrow{\tilde{\tau}} & LX \\
\downarrow & & \downarrow \\
P^h & \xrightarrow{\tau_h} & L^h X
\end{array}
\] (5.12)
for any tetrad field $h$ [13, 12, 64]. This means that, given a tetrad field $h$, the restriction $h^*\tilde{LX}$ of the $L_s$-principal bundle (1.12) to $h(X) \subset \Sigma$ is isomorphic to the $L_s$-principal subbundle $P^h$ of the fibre bundle $LX \to X$ which is the Dirac spin structure associated with the Lorentz structure $L^h X$.

Let us consider the spinor bundle
\[
S = (LX \times V)/L_s \to \Sigma,
\] (5.13)
associated with the $L_s$-principal bundle (1.12), and the corresponding composite spinor bundle (1.14). Given a tetrad field $h$, there is the above mentioned canonical isomorphism
\[
i_h : S^h = (P^h \times V)/L_s \to (h^*\tilde{LX} \times V)/L_s
\]
of the \( h \)-associated spinor bundle \( S^h \) \((5.3)\) onto the restriction \( h^* S \) of the spinor bundle \( S \to \Sigma \) to \( h(X) \subset \Sigma \) (see Proposition \(7.1\)). Hence, every global section \( s_h \) of the spinor bundle \( S^h \) corresponds to the global section \( i_h \circ s_h \) of the composite spinor bundle \((1.14)\). Conversely, every global section \( s \) of the composite spinor bundle \((1.14)\), which projects onto a tetrad field \( h \), takes its values into the subbundle \( i_h(S^h) \subset S \) (see Proposition \(7.2\)).

Let the frame bundle \( LX \to X \) be provided with a holonomic atlas \( \{ U_\zeta, T\phi_\zeta \} \), and let the principal bundles \( \tilde{L}X \to \Sigma \) and \( LX \to \Sigma \) have the associated atlases \( \{ U_\zeta, z_\zeta^i \} \) and \( \{ U_\zeta, z_\zeta = \tilde{z} \circ z_\zeta^i \} \), respectively. With these atlases, the composite spinor bundle \((1.14)\) is equipped with the bundle coordinates \((x^\lambda, \sigma^a_\lambda, y^A)\), where \((x^\lambda, \sigma^a_\lambda)\) are coordinates of the tetrad bundle \( \Sigma \) such that \( \sigma^a_{\lambda} \) are the matrix components of the group element \((T\phi_\zeta \circ z_\zeta)(\sigma), \sigma \in U_\zeta, \pi_{\Sigma X}(\sigma) \in U_\zeta\). For any tetrad field \( h \), we have \((\sigma^a_\lambda \circ h)(x) = h^a_\lambda(x)\) where \( h^a_\lambda(x) = H^a_\lambda \circ z_\zeta \circ h \) are the tetrad functions \((4.1)\) with respect to the Lorentz atlas \( \{ z_\zeta \circ h \} \) of \( L^h X \).

The spinor bundle \( S \to \Sigma \) is a subbundle of the bundle of Clifford algebras which is generated by the bundle of Minkowski spaces

\[
E_M = (LX \times M)/L \to \Sigma \tag{5.14}
\]

associated with the \( L \)-principal bundle \( LX \to \Sigma \). Since the fibre bundles \( LX \to X \) and \( GL_4 \to GL_4/L \) are trivial, so is the fibre bundle \((5.14)\). Hence, it is isomorphic to the product \( \Sigma \times T^*X \). Then there exists the representation

\[
\gamma_\Sigma : T^*X \otimes S = (\tilde{L}X \times (M \otimes V))/L_\alpha \to (\tilde{L}X \times \gamma(M \otimes V))/L_\alpha = S, \tag{5.15}
\]

given by the coordinate expression

\[
dx^\lambda = \gamma_\Sigma(dx^\lambda) = \sigma^a_\lambda \gamma^a.
\]

Restricted to \( h(X) \subset \Sigma \), this representation recovers the morphism \( \gamma_h \) \((5.1)\).

Using the representation \( \gamma_\Sigma \) \((5.15)\), one can construct the total Dirac operator on the composite spinor bundle \( S \) as follows. Since the bundles \( \tilde{L}X \to \Sigma \) and \( \Sigma \to X \) are trivial, let us consider a principal connection \( A \) \((5.7)\) on the \( L_\alpha \)-principal bundle \( \tilde{L}X \to \Sigma \) given by the local connection form

\[
A = (A^{ab}_\lambda dx^\lambda + A^{kab}_\mu d\sigma^\mu_k) \otimes L_{ab}, \tag{5.16}
\]

\[
A^{ab}_\lambda = -\frac{1}{2}(\eta^{kb}\sigma^a_\mu - \eta^{ka}\sigma^b_\mu)\sigma^\mu_k K^\lambda_{\mu},
\]

\[
A^{kab}_\mu = \frac{1}{2}(\eta^{kb}\sigma^a_\mu - \eta^{ka}\sigma^b_\mu), \tag{5.17}
\]

where \( K \) is a world connection on \( X \). This connection defines the associated spin connection

\[
A_\Sigma = dx^\lambda \otimes (\partial_\lambda + \frac{1}{2}A^{ab}_\lambda L_{ab}A^B dy^B \partial_A) + d\sigma^\mu_k \otimes (\sigma^k_\mu + \frac{1}{2}A^{kab}_\mu L_{ab}A^B dy^B \partial_A) \tag{5.18}
\]
on the spinor bundle $S \to \Sigma$. The choice of the connection (5.10) is motivated by the fact that, given a tetrad field $h$, the restriction of the spin connection (5.18) to $S^h$ is exactly the spin connection (5.10).

The connection (5.18) yields the first order differential operator $\tilde{D}$ (7.37) on the composite spinor bundle $S \to X$ which reads

$$\tilde{D} : J^1S \to T^*X \otimes S,$$

$$\tilde{D} = dx^\lambda \otimes [y^A_\lambda - \frac{1}{2}(A^a_\lambda + A_{\mu k}^a \sigma^\mu_{\lambda k})L^A_{ab} B^B y^B] \partial_A =$$

$$dx^\lambda \otimes [y^A_\lambda - \frac{1}{4}(\eta^{kb}_a \sigma^a_\mu - \eta^{ka}_a \sigma^a_\mu)(\sigma^\mu_{\lambda k} - \sigma^\mu_{k \lambda \nu} L^A_{ab} B^B)] \partial_A.$$ (5.19)

The corresponding restriction $\tilde{D}_h$ (3.9) of the operator $\tilde{D}$ (5.19) to $J^1S^h \subset J^1S$ recovers the familiar covariant differential on the $h$-associated spinor bundle $S^h \to X^4$ relative to the spin connection (5.11).

Combining (5.15) and (5.19) gives the first order differential operator

$$\Delta = \gamma_\Sigma \circ \tilde{D} : J^1S \to T^*X \otimes S \to S,$$

$$y^B \circ \Delta = \sigma^a_\lambda \gamma_a^B A[y^A_\lambda - \frac{1}{4}(\eta^{kb}_a \sigma^a_\mu - \eta^{ka}_a \sigma^a_\mu)(\sigma^\mu_{\lambda k} - \sigma^\mu_{k \lambda \nu} L^A_{ab} B^B)],$$ (5.20)

on the composite spinor bundle $S \to X$. One can think of $\Delta$ as being the total Dirac operator on $S$ since, for every tetrad field $h$, the restriction of $\Delta$ to $J^1S^h \subset J^1S$ is exactly the Dirac operator $\Delta_h$ (5.4) on the spinor bundle $S^h$ in the presence of the background tetrad field $h$ and the spin connection (5.11).

Thus, we come to the gauge model of metric-affine gravity and Dirac fermion fields. The total configuration space of this model is the jet manifold $J^1Y$ of the bundle product $Y = C_K \times S$, where $C_K$ is the bundle of world connections (1.3). This configuration space is coordinated by $(x^\mu, \sigma^\mu_\lambda, k^\mu_\alpha_\beta, y^A)$. Let $J^1S$ denote the first order jet manifold of the fibre bundle $Y \to \Sigma$. This fibre bundle can be endowed with the spin connection

$$A^\lambda_Y : Y \longrightarrow J^1S \to J^1Y \to J^1S,$$

$$A^\lambda_Y = dx^\lambda \otimes (\partial_\lambda + \overline{A}^a_\lambda L^A_{ab} B^B \partial_A) + d\sigma^\mu_\lambda \otimes (\partial^k_\mu + A^a_{\mu k} L^A_{ab} B^B \partial_A),$$ (5.21)

where $A^a_{\mu k}$ is given by the expression (5.14) and

$$\overline{A}^a_\lambda = -\frac{1}{2}(\eta^{kb}_a \sigma^a_\mu - \eta^{ka}_a \sigma^a_\mu) \sigma^\nu_{k \lambda \mu}.$$
Using the connection (5.21), we obtain the first order differential operator
\[ \tilde{D}_Y : J^1 Y \to T^* X \otimes S, \]
\[ \tilde{D}_Y = dx^\lambda \otimes [y^A_\lambda - \frac{1}{4}(\eta^{kb}\sigma^a_\mu - \eta^{ka}\sigma^b_\mu)(\sigma^\mu_{\lambda k} - \sigma^\nu_{\lambda k}k_{\nu}^\mu)L_{ab} A B y^B] \partial_A, \] (5.22)
and the total Dirac operator
\[ \Delta_Y = \gamma_\Sigma \circ \tilde{D} : J^1 Y \to T^* X \otimes S \to S, \]
\[ y^B \circ \Delta_Y = \sigma^\lambda_{\alpha B} A [y^A_\lambda - \frac{1}{4}(\eta^{kb}\sigma^a_\mu - \eta^{ka}\sigma^b_\mu)(\sigma^\mu_{\lambda k} - \sigma^\nu_{\lambda k}k_{\nu}^\mu)L_{ab} A B y^B], \] (5.23)
on the fibre bundle \( Y \to X \). Given a world connection \( K : X \to C_K \), the restrictions of the spin connection \( A_Y \) (5.21), the operator \( \tilde{D}_Y \) (5.22) and the Dirac operator \( \Delta_Y \) (5.23) to \( K^* Y \) are exactly the spin connection (5.18) and the operators (5.19) and (5.20), respectively.

Finally, since the spin structure (1.13) is unique, the \( GL_4 \)-principal bundle \( \tilde{L}X \to X^4 \) as well as the frame bundle \( LX \) admits the canonical lift of any diffeomorphism \( f \) of the base \( X \) \[9, 24, 42\]. This lift yields the general covariant transformation of the associated spinor bundle \( S \to \Sigma \) over the general covariant transformations of the tetrad bundle \( \Sigma \). The corresponding canonical lift onto \( S \) of a vector field on \( X \) can be constructed as a generalization of the lift (5.11) (see \[16, 64\] for detail).

6 Appendix. Fibre bundles and jet manifolds

All maps throughout are smooth and manifolds are real, finite-dimensional, Hausdorff, second-countable (hence, paracompact) and connected.

Fibre bundles

A manifold \( Y \) is called a fibred manifold over a base \( X \) if there is a surjective submersion
\[ \pi : Y \to X, \] (6.1)
i.e., the tangent map \( T\pi : TY \to TX \) is a surjection. A fibred manifold \( Y \to X \) is provided with an atlas of fibred coordinates \((x^\lambda, y^i)\), where \( x^\lambda \) are coordinates on the base \( X \), whose transition functions \( x^\lambda \to x'^\lambda \) are independent of the coordinates \( y^i \).

A fibred manifold \( Y \to X \) is called a fibre bundle if it is locally trivial. This means that the base \( X \) admits an open covering \( \{U_\xi\} \) so that \( Y \) is locally equivalent to the splittings
\[ \psi_\xi : \pi^{-1}(U_\xi) \to U_\xi \times V \] (6.2)
together with the transition functions
\[ \psi_\xi(y) = (\rho_\xi \circ \psi_\xi)(y), \quad y \in \pi^{-1}(U_\xi \cap U_\zeta). \] (6.3)

The manifold \( V \) is one for all local splittings (6.2). It is called the typical fibre of \( Y \). Local trivializations \( (U_\xi, \psi_\xi) \) make up a bundle atlas \( \Psi = \{(U_\xi, \psi_\xi)\} \) of the fibre bundle \( Y \). Given an atlas \( \Psi \), a fibre bundle \( Y \) is provided with the associated bundle coordinates \((x^\lambda, y^i)\) where
\[ y^i(y) = (y^i \circ \text{pr}_2 \circ \psi_\xi)(y), \quad y \in Y, \]
are coordinates on the typical fibre \( V \). Note that a fibre bundle \( Y \to X \) is uniquely defined by a bundle atlas \( \Psi \). Two bundle atlases are said to be equivalent if a union of these atlases is also a bundle atlas.

A fibre bundle \( Y \to X \) is called trivial if \( Y \) is diffeomorphic to the product \( X \times V \). Any fibre bundle over a contractible base is trivial.

By a (global) section of a fibre bundle (6.1) is meant a manifold morphism \( s : X \to Y \) such that \( \pi \circ s = \text{Id}X \). A section \( s \) is an imbedding, i.e., \( s(X) \subset Y \) is both a (closed) submanifold and a topological subspace of \( Y \). Similarly, a section \( s \) of a fibre bundle \( Y \to X \) over a submanifold \( N \subset X \) is a morphism \( s : N \to Y \) such that \( \pi \circ s : N \hookrightarrow X \) is a natural injection. A section of a fibre bundle over an open subset \( U \subset X \) is simply called a local section. A fibre bundle, by definition, admits a local section around each point of its base.

**Theorem 6.1.** Any fibre bundle \( Y \to X \) whose typical fibre is diffeomorphic to \( \mathbb{R}^m \) has a global section. Its section over a closed subset of \( N \subset X \) can always be extended to a global section. \( \Box \)

A bundle morphism of two bundles \( \pi : Y \to X \) and \( \pi' : Y' \to X' \) is a pair of maps \( \Phi : Y \to Y' \) and \( f : X \to X' \) such that the diagram
\[
\begin{array}{ccc}
Y & \xrightarrow{\Phi} & Y' \\
\downarrow{\pi} & & \downarrow{\pi'} \\
X & \xrightarrow{f} & X'
\end{array}
\] (6.4)
is commutative, i.e., \( \Phi \) sends fibres to fibres. In brief, one says that (6.4) is a bundle morphism \( \Phi \) over \( f \). If \( f = \text{Id}X \), then \( \Phi \) is called a bundle morphism over \( X \).

If a bundle morphism \( \Phi \) (6.4) is a diffeomorphism, it is called an isomorphism of fibre bundles. Two fibre bundles over the same base \( X \) are said to be equivalent if there exists their isomorphism over \( X \). A bundle morphism \( \Phi \) (6.4) over \( X \) (or its image \( \Phi(Y) \)) is called a subbundle of the fibre bundle \( Y' \to X \) if \( \Phi(Y) \) is a submanifold of \( Y' \).
Given a fibre bundle \( \pi : Y \to X \) and a morphism \( f : X' \to X \), the pull-back of \( Y \) by \( f \) is the manifold

\[
f^*Y = \{(x', y) \in X' \times Y : \pi(y) = f(x')\}
\]

(6.5)
together with the natural projection \((x', y) \mapsto x'\). It is a fibre bundle over \( X' \) such that the fibre of \( f^*Y \) over a point \( x' \in X' \) is that of \( Y \) over the point \( f(x') \in X \).

In particular, if \( i_{X'} : X' \subset X \) is a submanifold, then the pull-back \( i^*_X Y = Y |_{X'} \) is called the restriction of a fibre bundle \( Y \) to the submanifold \( X' \subset X \).

Let \( \pi : Y \to X \) and \( \pi' : Y' \to X \) be fibre bundles over the same base \( X \). Their fibred product \( Y \times_{X} Y' \) over \( X \) is defined as the pull-back

\[
Y \times_{X} Y' = \pi^*Y' \quad \text{or} \quad Y \times_{X} Y' = \pi'^*Y.
\]

**Vector and affine bundles**

A vector bundle is a fibre bundle \( Y \to X \) such that its typical fibre \( V \) and all fibres \( Y_x = \pi^{-1}(x), x \in X \), are real finite-dimensional vector spaces, and there is a linear bundle atlas \( \Psi = \{(U_\xi, \psi_\xi)\} \) of \( Y \to X \) whose trivialization morphisms \( \psi_\xi(x) : Y_x \to V, x \in U_\xi \) are linear isomorphisms. The associated bundle coordinates \((y^i)\) have linear transition functions. By a morphism of vector bundles is meant a bundle morphism whose restriction to each fibre of \( Y \) is a linear map.

There are the following standard constructions of new vector bundles from old ones.

- **Let** \( Y \to X \) be a vector bundle with a typical fibre \( V \). By \( Y^* \to X \) is meant the dual vector bundle with the typical fibre \( V^* \) dual of \( V \). The interior product of \( Y \) and \( Y^* \) is defined as a bundle morphism

\[
\iota : Y \otimes_{X} Y^* \longrightarrow X \times \mathbb{R}.
\]

- **Let** \( Y \to X \) and \( Y' \to X \) be vector bundles with typical fibres \( V \) and \( V' \), respectively. Their Whitney sum \( Y \oplus_{X} Y' \) is a vector bundle over \( X \) with the typical fibre \( V \oplus V' \).

- **Let** \( Y \to X \) and \( Y' \to X \) be vector bundles with typical fibres \( V \) and \( V' \), respectively. Their tensor product \( Y \otimes_{X} Y' \) is a vector bundle over \( X \) with the typical fibre \( V \otimes V' \). Similarly, the exterior product of vector bundles \( Y \wedge_{X} Y \) is defined.
By virtue of Theorem 6.1, vector bundles have global sections. Moreover, there exists the canonical zero-valued global section $\hat{0}$ of vector bundles.

Let us consider an exact sequence of vector bundles over the same base $X$

$$0 \to Y' \overset{i}{\to} Y \overset{j}{\to} Y'' \to 0,$$

where $Y' \to Y$ is an injection and $Y \to Y''$ is a surjection of vector bundles such that $\text{Im} \ i = \text{Ker} \ j$. This is equivalent to the fact that $Y'' = Y/Y'$ is the quotient bundle. One says that the exact sequence (6.6) of vector bundles admits a splitting if there exists a linear bundle monomorphism $\Gamma : Y'' \to Y$ over $X$ such that $j \circ \Gamma = \text{Id} \ Y''$. Then

$$Y = i(Y') \oplus \Gamma(Y'').$$

Every exact sequence of vector bundles admits a splitting.

Let $\pi : \overline{Y} \to X$ be a vector bundle with a typical fibre $\overline{V}$. An affine bundle modelled over the vector bundle $\overline{Y} \to X$ is a fibre bundle $\pi : Y \to X$ whose typical fibre $V$ is an affine space modelled over $\overline{V}$, while the following conditions hold. (i) All the fibres $Y_x$ of $Y$ are affine spaces modelled over the corresponding fibres $\overline{Y}_x$ of the vector bundle $\overline{Y}$. (ii) There is a bundle atlas $\Psi = \{(U_\alpha, \psi_\alpha)\}$ of $Y \to X$ whose trivializations morphisms $\psi_\zeta(x) : Y_x \to V, \ x \in U_\zeta$, are affine isomorphisms. The associated bundle coordinates $(y^i)$ have affine transition functions. There are the bundle morphisms

$$Y \times \overline{Y} \xrightarrow{X} Y, \quad (y^x, \overline{y}) \mapsto y^x + \overline{y},$$

$$Y \times Y \xrightarrow{X} \overline{Y}, \quad (y^x, y'^x) \mapsto y^x - y'^x,$$

where $(\overline{y})$ are linear coordinates on the vector bundle $\overline{Y}$.

In particular, every vector bundle has a natural structure of an affine bundle.

By a morphism of affine bundles is meant a bundle morphism whose restriction to each fibre of $Y$ is an affine map. Every affine bundle morphism $\Phi : Y \to Y'$ from an affine bundle $Y$ modelled over a vector bundle $\overline{Y}$ to an affine bundle $Y'$ modelled over a vector bundle $\overline{Y}'$ determines uniquely the linear bundle morphism

$$\overline{\Phi} : \overline{Y} \to \overline{Y}', \quad \overline{y}^i \circ \overline{\Phi} = \frac{\partial \Phi^i}{\partial y^j} \overline{y}^j,$$

called the linear derivative of $\Phi$.

By virtue of Theorem 6.1, affine bundles have global sections. Let $\pi : Y \to X$ be an affine bundle modelled over a vector bundle $\overline{Y} \to X$. Every global section $s$ of an affine bundle $Y \to X$ yields the bundle morphism

$$D_s : Y \ni y \to y - s(\pi(y)) \in \overline{Y}.$$
Tangent and cotangent bundles

Tangent and cotangent bundles exemplify vector bundles. The fibres of the tangent bundle $\pi_Z : TZ \to Z$ of a manifold $Z$ are tangent spaces to $Z$. Given an atlas $\Psi_Z = \{(U_\xi, \phi_\xi)\}$ of a manifold $Z$, the tangent bundle is provided with the holonomic atlas

$$\Psi = \{U_\xi, \psi_\xi = T\phi_\xi\},$$ (6.8)

where by $T\phi_\xi$ is meant the tangent map to $\phi_\xi$. The associated linear bundle coordinates are coordinates with respect to the holonomic frames $\{\partial_\lambda\}$ in tangent spaces $T_zZ$. They are called holonomic coordinates, and are denoted by $(\dot{z}^\lambda)$ on $TZ$. The transition functions of holonomic coordinates read

$$\dot{z}'^\lambda = \frac{\partial z^\lambda}{\partial z'\mu} \dot{z}'^\mu.$$

Every manifold map $f : Z \to Z'$ generates the linear bundle morphism of the tangent bundles

$$Tf : TZ \to TZ', \quad \dot{z}'^\lambda \circ Tf = \frac{\partial f^\lambda}{\partial z^\mu} \dot{z}^\mu,$$

called the tangent map to $f$.

The cotangent bundle of a manifold $Z$ is the dual $\pi^*_Z : T^*Z \to Z$ of the tangent bundle $TZ \to Z$. It is equipped with the holonomic coordinates $(z^\lambda, \dot{z}_\lambda)$ with respect to the coframes $\{dz^\lambda\}$ dual of $\{\partial_\lambda\}$. Their transition functions read

$$\dot{z}'^i = \frac{\partial z'^i}{\partial z^\lambda} \dot{z}^\lambda.$$

Various tensor products

$$T = (\otimes TZ) \otimes (\otimes T^*Z)$$ (6.9)

over $Z$ of tangent and cotangent bundles are called tensor bundles.

Let $\pi_Y : TY \to Y$ be the tangent bundle of a fibre bundle $\pi : Y \to X$. Given bundle coordinates $(x^\lambda, y^i)$ on $Y$, the tangent bundle $TY$ is equipped with the holonomic coordinates $(x^\lambda, y^i, \dot{x}^\lambda, \dot{y}^i)$. The tangent bundle $TY \to Y$ has the vertical tangent subbundle $VY = \text{Ker} T\pi$, given by the coordinate relations $\dot{x}^\lambda = 0$. This subbundle consists of the vectors tangent to fibres of $Y$. The vertical tangent bundle $VY$ is provided with the holonomic coordinates $(x^\lambda, y^i, \dot{y}^i)$ with respect to the frames $\{\partial_i\}$.

Let $T\Phi$ be the tangent map to a fibred morphism $\Phi : Y \to Y'$. Its restriction

$$V\Phi = T\Phi|_{VY} : VY \to VY', \quad \dot{y}'^i \circ V\Phi = \partial_i \Phi^i = \dot{y}^j \partial_j \Phi^i,$$
to $VY$ is a linear bundle morphism of the vertical tangent bundle $VY$ to the vertical tangent bundle $VY'$, called the vertical tangent map to $\Phi$.

Vertical tangent bundles of many fibre bundles are equivalent to the pull-backs

$$VY = Y \times X \overrightarrow{Y}$$

where $\overrightarrow{Y} \to X$ is some vector bundle. This means that the transition functions of the holonomic coordinates $\dot{y}^i$ on $VY$ are independent of $y^i$. In particular, every affine bundle $Y \to X$ modelled over a vector bundle $\overrightarrow{Y} \to X$ admits the canonical vertical splitting

$$VY \cong Y \times X \overrightarrow{Y}$$

(6.10)

because the coordinates $\dot{y}^i$ on $VY$ have the same transformation law as the linear coordinates $\overrightarrow{y}^i$ on the vector bundle $\overrightarrow{Y}$. If $Y$ is a vector bundle, the splitting (6.10) reads

$$VY \cong Y \times Y.$$  

(6.11)

The vertical cotangent bundle $V^*Y \to Y$ of a fibre bundle $Y \to X$ is defined as the vector bundle dual of the vertical tangent bundle $VY \to Y$. There is the canonical projection

$$\zeta : T^*Y \to V^*Y, \quad \zeta : \dot{x} \lambda dx^\lambda + \dot{y}^i dy^i \mapsto \dot{y}^i dy^i,$$

(6.12)

where $\{dy^i\}$ are the bases for the fibres of $V^*Y$, which are dual of the holonomic frames $\{\partial_i\}$ for the vertical tangent bundle $VY$.

With $VY$ and $V^*Y$, we have the following two exact sequences of vector bundles over $Y$:

$$0 \to VY \hookrightarrow TY \xrightarrow{\pi_T} Y \times TX \to 0,$$

(6.13a)

$$0 \to Y \times T^*X \hookrightarrow T^*Y \xrightarrow{\zeta} V^*Y \to 0.$$ 

(6.13b)

Their splitting corresponds to the choice of a connection on the fibre bundle $Y \to X$.

**Vector fields**

A vector field on a manifold $Z$ is defined as a global section of the tangent bundle $TZ \to Z$. The set $T(Z)$ of vector fields on $Z$ is a real Lie algebra with respect to the Lie bracket

$$[v, u] = (v^\lambda \partial_\lambda u^\mu - u^\lambda \partial_\lambda v^\mu) \partial_\mu, \quad u = u^\lambda \partial_\lambda, \quad v = v^\lambda \partial_\lambda.$$
Every vector field on a manifold $Z$ is the generator of a local 1-parameter group of local diffeomorphisms of $Z$. A vector field $u$ on a manifold $Z$ is called complete if it is induced by a 1-parameter group of diffeomorphisms of $Z$.

A vector field $u$ on a fibre bundle $Y \to X$ is said to be projectable if it projects over a vector field $\tau$ on $X$, i.e., $\tau \circ \pi = T\pi \circ u$. It has the coordinate expression

$$u = u^\lambda(x^\mu)\partial_\lambda + u^i(x^\mu, y^j)\partial_i, \quad \tau = u^\lambda\partial_\lambda.$$  

A projectable vector field $u = u^i\partial_i$ on a fibre bundle $Y \to X$ is said to be vertical if it projects over the zero vector field $\tau = 0$ on $X$.

A vector field $\tau = \tau^\lambda\partial_\lambda$ on a base $X$ of a fibre bundle $Y \to X$ gives rise to a projectable vector field on the total space $Y$ by means of some connection on this fibre bundle.

Nevertheless, every tensor bundle (6.9) admits the canonical lift $\tilde{\tau} = \tau^\mu\partial_\mu + \partial^\nu\tau^\alpha x^\nu_{\beta_1...\beta_k} - ...$, $\partial x^\alpha_{\beta_1...\beta_k}$, $\partial = \frac{\partial}{\partial \dot{x}^\lambda}$, $\tilde{x}^\alpha_{\beta_1...\beta_k}$ (6.14)

of any vector field $\tau$ on $X$. In particular, we have the canonical lift

$$\tilde{\tau} = \tau^\mu\partial_\mu + \partial^\alpha\tau^\nu \dot{x}^\nu\dot{\partial}_\alpha, \quad \dot{\partial}_\lambda = \frac{\partial}{\partial \dot{x}^\lambda}, \quad (6.15)$$

of $\tau$ onto the tangent bundle $TX$.

**Exterior forms**

An exterior $r$-form on a manifold $Z$ is a section

$$\phi = \frac{1}{r!}\phi_{\lambda_1...\lambda_r}dz^{\lambda_1} \wedge ... \wedge dz^{\lambda_r}$$

of the exterior product $\wedge T^*Z \to Z$. All exterior forms on $Z$ constitute the exterior $Z$-graded algebra $\mathfrak{O}^*(Z)$ with respect to the exterior product. It is provided with exterior differential

$$d : \mathfrak{O}^r(Z) \to \mathfrak{O}^{r+1}(Z), \quad d\phi = \frac{1}{r!}\partial_\mu\phi_{\lambda_1...\lambda_r}dz^\mu \wedge dz^{\lambda_1} \wedge ... dz^{\lambda_r},$$

which obeys the relations

$$d(\phi \wedge \sigma) = d(\phi) \wedge \sigma + (-1)^{|\phi|}\phi \wedge d(\sigma), \quad d \circ d = 0.$$  

Given a morphism $f : Z \to Z'$, by $f^*\phi$ is meant the pull-back on $Z$ of an $r$-form $\phi$ on $Z'$ by $f$, which is defined by the condition

$$f^*\phi(v^1, ..., v^r)(z) = \phi(Tf(v^1), ..., Tf(v^r))(f(z)), \quad \forall v^1, ..., v^r \in T_zZ,$$
and obeys the relations
\[ f^*(\phi \wedge \sigma) = f^*\phi \wedge f^*\sigma, \quad df^*\phi = f^*(d\phi). \]

Let \( \pi : Y \to X \) be a fibre bundle with bundle coordinates \((x^\lambda, y^i)\). The pull-back on \( Y \) of exterior forms on \( X \) by \( \pi \) provides the inclusion \( \pi^* : \Omega^*(X) \to \Omega^*(Y) \). Elements of its image are called basic forms. Exterior forms

\[ \phi : Y \to \wedge^r T^*X, \quad \phi = \frac{1}{r!} \phi_{\lambda_1...\lambda_r} dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_r}, \]
on \( Y \) such that \( \vartheta \rfloor \phi = 0 \) for arbitrary vertical vector field \( \vartheta \) on \( Y \) are said to be horizontal forms.

The interior product of a vector field \( u = u^\mu \partial_\mu \) and an exterior \( r \)-form \( \phi \) is given by the coordinate expression

\[ u \rfloor \phi = \sum_{k=1}^r \frac{(-1)^{k-1}}{r!} u^\lambda \phi_{\lambda \lambda_1...\lambda_k} dz^{\lambda_1} \wedge \cdots \wedge \widehat{dz}^{\lambda_k} \wedge \cdots dz^{\lambda_r}. \]

It satisfies the relations
\[ \phi(u_1, \ldots, u_r) = u_r \rfloor \cdots u_1 \rfloor \phi, \quad u \rfloor (\phi \wedge \sigma) = u \rfloor \phi \wedge \sigma + (-1)^|\phi| \phi \wedge u \rfloor \sigma. \]

The Lie derivative of an exterior form \( \phi \) along a vector field \( u \) is
\[ \mathbf{L}_u \phi = u \rfloor d\phi + d(u \rfloor \phi). \]

**Tangent-valued forms**

A tangent-valued \( r \)-form on a manifold \( Z \) is a section
\[ \phi = \frac{1}{r!} \phi_{\lambda_1...\lambda_r} dz^{\lambda_1} \wedge \cdots \wedge dz^{\lambda_r} \otimes \partial_\mu \]
of the tensor bundle \( \wedge T^*Z \otimes TZ \to Z \).

There is one-to-one correspondence between the tangent-valued 1-forms \( \phi \) on a manifold \( Z \) and the linear bundle endomorphisms
\[ \hat{\phi} : TZ \to TZ, \quad \hat{\phi} : T_z Z \ni v \mapsto \phi(z) \in T_z Z, \]
\[ \hat{\phi}^* : T^*_z Z \to T^*_z Z, \quad \hat{\phi}^* : T^*_z Z \ni v^* \mapsto \phi(z) \rfloor v^* \in T^*_z Z. \]  

(6.16) (6.17)

In particular, the canonical tangent-valued 1-form
\[ \theta_Z = dz^\lambda \otimes \partial_\lambda \]

(6.18)
on $Z$ corresponds to the identity morphisms (6.16) and (6.17).

The space $\mathcal{D}^*(M) \otimes \mathcal{T}(M)$ of tangent-valued forms is provided with the Frölicher–Nijenhuis bracket generalizing the Lie bracket of vector fields as follows:

$$[\phi, \sigma]_{FN} : \mathcal{D}^r(M) \otimes \mathcal{T}(M) \times \mathcal{D}^s(M) \otimes \mathcal{T}(M) \to \mathcal{D}^{r+s}(M) \otimes \mathcal{T}(M),$$

$$[\phi, \sigma]_{FN} = \frac{1}{r!s!}(\phi^\nu_{\lambda_1 \ldots \lambda_r} \partial_\nu \sigma^\mu_{\lambda_{r+1} \ldots \lambda_{r+s}} - \sigma^\nu_{\lambda_{r+1} \ldots \lambda_{r+s}} \partial_\nu \phi^\mu_{\lambda_1 \ldots \lambda_r} -$$

$$r \phi^\nu_{\lambda_1 \ldots \lambda_{r-1}} \partial_\nu \sigma^\mu_{\lambda_{r+1} \ldots \lambda_{r+s}} + s \sigma^\mu_{\nu \lambda_{r+2} \ldots \lambda_{r+s}} \partial_\nu \phi^\nu_{\lambda_1 \ldots \lambda_r}) dz^\lambda_1 \wedge \ldots \wedge dz^\lambda_{r+s} \otimes \partial_\mu,$$

$$\phi \in \mathcal{D}^r(M) \otimes \mathcal{T}(M), \quad \sigma \in \mathcal{D}^s(M) \otimes \mathcal{T}(M).$$

There are the relations

$$[\phi, \psi]_{FN} = (-1)^{\phi||\psi+1}[\psi, \phi]_{FN}, \quad (6.19)$$

$$[\phi, [\psi, \theta]_{FN}]=[[\phi, \psi]_{FN}, \theta]_{FN} + (-1)^{\phi||\psi}[\psi, [\phi, \theta]_{FN}]. \quad (6.20)$$

Given a tangent-valued form $\theta$, the Nijenhuis differential on $\mathcal{D}^*(M) \otimes \mathcal{T}(M)$ is defined as the morphism

$$d_\theta : \sigma \mapsto d_\theta \sigma = [\theta, \sigma]_{FN}, \quad \forall \sigma \in \mathcal{D}^*(M) \otimes \mathcal{T}(M).$$

By virtue of (6.20), it has the property

$$d_\phi[\psi, \theta]_{FN} = [d_\phi \psi, \theta]_{FN} + (-1)^{\phi||\psi}[\psi, d_\phi \theta]_{FN}.$$

Let $Y \to X$ be a fibre bundle. One consider the following classes of tangent-valued forms on $Y$:

- **tangent-valued horizontal forms**

  $$\phi : Y \to \tilde{T}^*X \otimes TY,$$

  $$\phi = dx^{\lambda_1} \wedge \ldots \wedge dx^{\lambda_r} \otimes [\phi^\mu_{\lambda_1 \ldots \lambda_r}(y) \partial_\mu + \phi^i_{\lambda_1 \ldots \lambda_r}(y) \partial_i],$$

- **vertical-valued horizontal forms**

  $$\phi : Y \to \tilde{T}^*X \otimes VY,$$

  $$\phi = \phi^i_{\lambda_1 \ldots \lambda_r}(y) dx^{\lambda_1} \wedge \ldots \wedge dx^{\lambda_r} \otimes \partial_i,$$

- **vertical-valued horizontal 1-forms** $\sigma = \sigma^\lambda(y) dx^\lambda \otimes \partial_i$, called soldering forms,

- **basic soldering forms** $\sigma = \sigma^\lambda(x) dx^\lambda \otimes \partial_i$. 

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The tangent bundle $TX$ is provided with the canonical soldering form
\[ \theta_J = dx^\lambda \otimes \hat{\partial}_\lambda. \] (6.21)
Due to the canonical vertical splitting
\[ VTX = TX \times TX, \] (6.22)
the canonical soldering form (6.21) on $TX$ defines the canonical tangent-valued form $\theta_X$ (6.18) on $X$. By this reason, tangent-valued 1-forms on a manifold $X$ are also called soldering forms.

**First order jet manifolds**

Given a fibre bundle $Y \rightarrow X$ with bundle coordinates $(x^\lambda, y^i)$, let us consider the equivalence classes $j^1_x s, \ x \in X$, of its sections $s$, which are identified by their values $s^i(x)$ and the values of their first order derivatives $\partial_\mu s^i(x)$ at points $x \in X$. The equivalence class $j^1_x s$ is called the first order jet of sections $s$ at the point $x \in X$. The set $J^1Y$ of first order jets is provided with a manifold structure with respect to the adapted coordinates $(x^\lambda, y^i, y^i_\lambda)$ such that
\[
(x^\lambda, y^i, y^i_\lambda)(j^1_x s) = (x^\lambda, s^i(x), \partial_\lambda s^i(x)), \]
\[ y^i_\lambda = \frac{\partial x^\mu}{\partial x^\nu}(\partial_\mu + y^j_\mu \partial_j)y^i. \] (6.23)
It is called the jet manifold of the fibre bundle $Y \rightarrow X$. It has the natural fibrations
\[ \pi^1 : J^1Y \ni j^1_x s \mapsto x \in X, \quad \pi^0_0 : J^1Y \ni j^1_x s \mapsto s(x) \in Y, \]
where the latter is an affine bundle modelled over the vector bundle
\[ T^*X \otimes VY \rightarrow Y. \] (6.24)

The jet manifold $J^1Y$ admits the canonical imbedding
\[ \lambda_1 : J^1Y \ni j^1_x s \mapsto T^*X \otimes TY, \quad \lambda_1 = dx^\lambda \otimes (\partial_\lambda + y^i_\lambda \partial_i). \] (6.25)
Therefore, one usually identifies $J^1Y$ with its image under this morphism, and represents jets by the tangent-valued form
\[ dx^\lambda \otimes (\partial_\lambda + y^i_\lambda \partial_i). \]
Any bundle morphism $\Phi : Y \to Y'$ over a diffeomorphism $f$ is extended to the morphism of the corresponding jet manifolds

$$J^1\Phi : J^1Y \to J^1Y', \quad y'^i \circ J^1\Phi = \frac{\partial(f^{-1})^\mu}{\partial x'^\lambda} \partial_\mu \Phi^i.$$  

It is called the jet prolongation of the morphism $\Phi$. Each section $s$ of a fibre bundle $Y \to X$ has the jet prolongation to the section

$$(J^1s)(x) \overset{\text{def}}{=} j^1_x s, \quad (y^i, y'^i) \circ J^1s = (s^i(x), \partial_\lambda s^i(x)),$$

of the jet bundle $J^1Y \to X$.

## 7 Appendix. Connections

A connection on a fibre bundle $Y \to X$ is defined as a linear bundle monomorphism

$$\Gamma : Y \times TX \to TY, \quad \Gamma : \dot{x}^\lambda \partial_\lambda \mapsto \dot{x}^\lambda (\partial_\lambda + \Gamma^i_\lambda \partial_i),$$

over $Y$ which splits the exact sequence (6.13a) and provides the horizontal decomposition

$$TY = \Gamma(Y \times TX) \oplus VY,$$

$$\dot{x}^\lambda \partial_\lambda + y^i \partial_i = \dot{x}^\lambda (\partial_\lambda + \Gamma^i_\lambda \partial_i) + (y^i - \dot{x}^\lambda \Gamma^i_\lambda) \partial_i,$$

of the tangent bundle $TY$. A connection always exists.

The morphism $\Gamma$ (7.1) yields uniquely the horizontal tangent-valued 1-form

$$\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^i_\lambda \partial_i)$$

on $Y$ such that

$$\Gamma : \partial_\lambda \mapsto \partial_\lambda]\Gamma = \partial_\lambda + \Gamma^i_\lambda \partial_i.$$

One can think of it as being another definition of a connection on a fibre bundle $Y \to X$.

Given a connection $\Gamma$ on a fibre bundle $Y \to X$, every vector field $\tau$ on its base $X$ gives rise to the projectable vector field

$$\Gamma\tau = \tau^\lambda(\partial_\lambda + \Gamma^i_\lambda \partial_i)$$

on $Y$, called the horizontal lift of $\tau$ by the connection $\Gamma$.

### Connections as sections of the jet bundle
Let \( Y \to X \) be a fibre bundle, and \( J^1Y \) its first order jet manifold. Given the canonical morphism (6.23), we have the corresponding morphism
\[
\hat{\lambda}_1 : J^1Y \times TX \ni \partial \lambda \mapsto d\lambda = \partial_\lambda \lambda \in J^1Y \times TY. \tag{7.5}
\]
This morphism yields the canonical horizontal splitting of the pull-back
\[
J^1Y \times TY = \hat{\lambda}_1(TX) \oplus VY; \tag{7.6}
\]
\[
\ddot{x}^\lambda \partial_\lambda + \ddot{y}^i \partial_i = \ddot{x}^\lambda (\partial_\lambda + y^i_\lambda \partial_i) + \left( \dot{y}^i - \dot{x}^\lambda y^i_\lambda \right) \partial_i.
\]
Let \( \Gamma \) be a global section of \( J^1Y \to Y \). Substituting the tangent-valued form
\[
\lambda_1 \circ \Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^i_\lambda \partial_i)
\]
in the canonical splitting (7.6), we obtain the familiar horizontal splitting (7.2) of \( TY \) by means of a connection \( \Gamma \) on \( Y \to X \). Then one can show that there is one-to-one correspondence between the connections on a fibre bundle \( Y \to X \) and the global sections of the affine jet bundle \( J^1Y \to Y \).

It follows that connections on a fibre bundle \( Y \to X \) make up an affine space modelled over the vector space of soldering forms on \( Y \to X \), i.e., sections of the vector bundle (6.24). One deduces immediately from (6.23) the coordinate transformation law
\[
\Gamma^i_\lambda = \frac{\partial x^\mu}{\partial x'^\lambda_\mu} (\partial_\mu + \Gamma^i_\mu \partial_j) y^i_j.
\]

Every connection \( \Gamma \) on a fibre bundle \( Y \to X \) yields the first order differential operator
\[
D_\Gamma : J^1Y \to T^*X \otimes VY, \quad D_\Gamma = \lambda_1 - \Gamma \circ \pi_0 = (y^i_\lambda - \Gamma^i_\lambda) dx^\lambda \otimes \partial_i, \tag{7.7}
\]
called the covariant differential relative to the connection \( \Gamma \). If \( s : X \to Y \) is a section, one obtains from (7.7) its covariant differential
\[
\nabla_\Gamma s = D_\Gamma \circ J^1s : X \to T^*X \otimes VY,
\]
\[
\nabla_\Gamma s = (\partial_\lambda s^i - \Gamma^i_\lambda \circ s) dx^\lambda \otimes \partial_i,
\]
and the covariant derivatives \( \nabla_\tau^\Gamma = \tau \nabla_\Gamma \) along a vector field \( \tau \) on \( X \). A section \( s \) is said to be an integral section of a connection \( \Gamma \) if \( \nabla_\Gamma s = 0 \).

Let us recall the following standard constructions of new connections from the old ones. Given a fibre bundle \( Y \to X \), let \( f : X' \to X \) be a map and \( f^*Y \to X' \) the pull-back of \( Y \) by \( f \). Any connection \( \Gamma \) on \( Y \to X \) induces the pull-back connection
\[
f^*\Gamma = (dy^i - (\Gamma \circ f)_\lambda \partial f^\lambda_{\partial x'^\mu} dx'^\mu) \otimes \partial_i \tag{7.9}
\]
on $f^*Y \to X'$.

Let $Y$ and $Y'$ be fibre bundles over the same base $X$. Given a connection $\Gamma$ on $Y \to X$ and a connection $\Gamma'$ on $Y' \to X$, the fibre bundle $Y \times_X Y' \to X$ is provided with the product connection

$$\Gamma \times \Gamma' : Y \times Y' \to J^1(Y \times Y') = J^1Y \times J^1Y',$$

$$\Gamma \times \Gamma' = dx^\lambda \otimes (\partial_\lambda + \Gamma^i_\lambda \partial_{y^i} + \Gamma^j_\lambda \partial_{y'^j}). \quad (7.10)$$

Let $i_Y : Y \to Y'$ be a subbundle of a fibre bundle $Y' \to X$ and $\Gamma'$ a connection on $Y' \to X$. If there exists a connection $\Gamma$ on $Y \to X$ such that the diagram

\[
\begin{array}{ccc}
Y' & \xrightarrow{\Gamma'} & J^1Y \\
Y' \downarrow i_Y & & \downarrow J^1i_Y \\
Y & \xrightarrow{\Gamma} & J^1Y'
\end{array}
\]

commutes, $\Gamma'$ is said to be reducible to the connection $\Gamma$.

The curvature and the torsion

Let $\Gamma$ be a connection on a fibre bundle $Y \to X$. Given vector fields $\tau$, $\tau'$ on $X$ and their horizontal lifts $\Gamma\tau$ and $\Gamma\tau'$ (7.4) on $Y$, let us compute the vector field

$$R(\tau, \tau') = -\Gamma[\tau, \tau'] + [\Gamma\tau, \Gamma\tau'] \quad (7.11)$$

on $Y$. It is readily observed that this is the vertical vector field

$$R(\tau, \tau') = \tau^\lambda \tau'^\mu R^i_{\lambda\mu} \partial_i,$$

$$R^i_{\lambda\mu} = \partial_\lambda \Gamma^i_\mu - \partial_\mu \Gamma^i_\lambda + \Gamma^j_\lambda \partial_j \Gamma^i_\mu - \Gamma^j_\mu \partial_j \Gamma^i_\lambda. \quad (7.12)$$

The $VY$-valued horizontal 2-form on $Y$

$$R = \frac{1}{2} R^i_{\lambda\mu} dx^\lambda \wedge dx^\mu \otimes \partial_i \quad (7.13)$$

is called the curvature of the connection $\Gamma$. In an equivalent way, the curvature (7.13) is defined as the Nijenhuis differential

$$R = \frac{1}{2} d_{\Gamma} \Gamma = \frac{1}{2} [\Gamma, \Gamma]_{\text{FN}} : Y \to \wedge^2 T^*X \otimes VY. \quad (7.14)$$
Given a connection \( \Gamma \) and a soldering form \( \sigma \), the torsion of \( \Gamma \) with respect to \( \sigma \) is defined as
\[
T = d_\sigma \sigma = \frac{2}{\lambda} \tilde{T}^* \mathcal{X} \otimes \mathcal{V},
\]
\[
T = (\partial_\lambda \sigma_\mu^i + \Gamma^i_\lambda \partial_\mu \sigma^j_i - \partial_j \Gamma^j_\lambda \sigma_\mu^i) dx^\lambda \wedge dx^\mu \otimes \partial_i. \tag{7.15}
\]

**Linear and affine connections**

A connection \( \Gamma \) on a vector bundle \( Y \to X \) is said to be a linear connection if the section
\[
\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^i_\lambda y^j_i \partial_i)
\]
(7.16)
of the affine jet bundle \( J^1 Y \to Y \) is a linear bundle morphism over \( X \).

The curvature \( R \) (7.13) of a linear connection \( \Gamma \) (7.16) reads
\[
R = \frac{1}{2} R_{\lambda \mu}^i j (x) y^j_i dx^\lambda \xi dx^\mu \otimes \partial_i,
\]
\[
R_{\lambda \mu}^i j = \partial_\lambda \Gamma^i_\mu^j - \partial_\mu \Gamma^i_\lambda^j + \Gamma^i_\lambda^h \Gamma^h_\mu^j - \Gamma^i_\mu^h \Gamma^h_\lambda^j. \tag{7.17}
\]

Due to the vertical splitting (6.11), it can also be represented by the vector-valued form
\[
R = \frac{1}{2} R_{\lambda \mu}^i j dx^\lambda \wedge dx^\mu \otimes e_i. \tag{7.18}
\]

There are the following standard operations with linear connections.

(i) Let \( Y \to X \) be a vector bundle and \( \Gamma \) a linear connection (7.16) on \( Y \). It defines the dual linear connection
\[
\Gamma^* = dx^\lambda \otimes (\partial_\lambda - \Gamma^i_\lambda y^j_i \partial^j)
\]
(7.19)
on the dual bundle \( Y^* \to X \).

(ii) Let \( Y \to X \) and \( Y' \to X \) be vector bundles with linear connections \( \Gamma \) and \( \Gamma' \), respectively. Then the product connection (7.10) is the direct sum connection \( \Gamma \oplus \Gamma' \) on the Whitney sum \( Y \oplus Y' \).

(iii) Let \( Y \to X \) and \( Y' \to X \) be vector bundles with linear connections \( \Gamma \) and \( \Gamma' \), respectively. They define the tensor product connection
\[
\Gamma \otimes \Gamma' = dx^\lambda \otimes (\partial_\lambda + (\Gamma^i_\lambda y^j_i + \Gamma^i_\lambda y^j_k + \Gamma^i_\lambda y^j_\nu \partial_k) \partial_i)
\]
(7.20)
on the tensor product \( Y \otimes Y' \to X \).

For instance, given a linear connection \( K \) (1.4) on the tangent bundle \( TX \to X \), the dual connection on the cotangent bundle \( T^* X \) is
\[
K^* = dx^\lambda \otimes (\partial_\lambda - K^\mu_\nu x^\lambda \partial^\nu).
\]
(7.21)
Then, using the construction of the tensor product connection (7.20), one can introduce the corresponding linear connection on an arbitrary tensor bundle $T_\lambda^\mu(6.9)$.

It should be emphasized that the expressions (1.4) and (7.21) for a world connection differ in a minus sign from those usually used in the physical literature.

The curvature of a world connection is defined as the curvature $R(7.18)$ of the connection $K(1.4)$ on the tangent bundle $TX$. It reads

$$R = \frac{1}{2} R_{\lambda\mu}^\alpha_\beta \dot{x}^\alpha d\dot{x},$$

$$R_{\lambda\mu}^\alpha_\beta = \partial_\lambda K_{\mu}^\alpha_\beta - \partial_\mu K_{\lambda}^\alpha_\beta + K_{\lambda}^\gamma_\beta K_{\mu}^\alpha_\gamma - K_{\mu}^\gamma_\beta K_{\lambda}^\alpha_\gamma.$$  (7.22)

By the torsion of a world connection is meant the torsion (7.15) of the connection $\Gamma(1.4)$ on the tangent bundle $TX$ with respect to the canonical soldering form $\theta_J(6.21)$:

$$T = \frac{1}{2} T_{\mu}^\nu_\lambda dx^\lambda \otimes \partial^\nu, \quad T_{\mu}^\nu_\lambda = \Gamma_{\mu}^\nu_\lambda - \Gamma_{\lambda}^\nu_\mu.$$  (7.23)

**Affine connections**

Let $Y \to X$ be an affine bundle modelled over a vector bundle $\bar{Y} \to X$. A connection $\Gamma$ on $Y \to X$ is said to be an affine connection if the section $\Gamma$ is an affine bundle morphism over $X$.

For any affine connection $\Gamma : Y \to J^1Y$, the corresponding linear derivative $\Gamma : \bar{Y} \to J^1\bar{Y}$ (6.7) defines uniquely the associated linear connection on the vector bundle $\bar{Y} \to X$. Since every vector bundle has a natural structure of an affine bundle, any linear connection on a vector bundle is also an affine connection.

Using affine bundle coordinates $(x^\lambda, y^i)$ on $Y$, an affine connection $\Gamma$ on $Y \to X$ reads

$$\Gamma_{\lambda}^i = \Gamma_{\lambda}^i(x)y^i + \sigma_{\lambda}^i(x).$$  (7.24)

The coordinate expression of the associated linear connection is

$$\bar{\Gamma}_{\lambda}^i = \Gamma_{\lambda}^i(x)\bar{y}^i,$$  (7.25)

where $(x^\lambda, \bar{y}^i)$ are the associated linear bundle coordinates on $\bar{Y}$.

Affine connections on an affine bundle $Y \to X$ constitute an affine space modelled over the soldering forms on $Y \to X$. In view of the vertical splitting (6.10), these soldering forms can be seen as global sections of the vector bundle $T^*X \otimes \bar{Y} \to X$. If $Y \to X$ is a vector bundle, both the affine connection $\Gamma(7.24)$ and the associated linear connection $\Gamma$ are connections on the same vector bundle $Y \to X$, and their difference is a basic soldering form on $Y$. Thus, every affine connection on a vector bundle $Y \to X$ is the sum of a linear connection and a basic soldering form on $Y \to X$. 

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Given an affine connection $\Gamma$ on a vector bundle $Y \to X$, let $R$ and $\mathcal{R}$ be the curvatures of the connection $\Gamma$ and the associated linear connection $\bar{\Gamma}$, respectively. It is readily observed that $R = \mathcal{R} + T$, where the $VY$-valued 2-form

$$T = \frac{1}{2} T^i_{\lambda \mu} dx^\lambda \wedge dx^\mu \otimes \partial_i,$$

$$T^i_{\lambda \mu} = \partial_\lambda \sigma^i_\mu - \partial_\mu \sigma^i_\lambda + \sigma^h_\lambda \Gamma^i_{\mu h} - \sigma^h_\mu \Gamma^i_{\lambda h},$$

(7.26)

is the torsion (7.15) of the connection $\Gamma$ with respect to the basic soldering form $\sigma$.

In particular, let us consider the tangent bundle $TX$ of a manifold $X$. We have the canonical soldering form $\sigma = \theta_J = \theta_X$ (6.21) on $TX$. Given an arbitrary world connection $K$ (1.4) on $TX$, the corresponding affine connection

$$A = K + \theta_X, \quad A^\mu_\lambda = K^\mu_{\nu \lambda} \dot{x}^\nu + \delta^\mu_\lambda,$$

(7.27)

on $TX$ is called the Cartan connection. The torsion of the Cartan connection coincides with the torsion $T$ (7.23) of the world connection $K$, while its curvature is the sum $R + T$ of the curvature and the torsion of $K$.

**Composite connections**

Let us consider the composition

$$Y \to \Sigma \to X,$$

(7.28)

of fibre bundles

$$\pi_{Y \Sigma} : Y \to \Sigma, \quad \pi_{\Sigma X} : \Sigma \to X.$$  

(7.29)  

(7.30)

It is called a composite fibre bundle, and is provided with an atlas of bundle coordinates $(x^\lambda, \sigma^m, y^i)$, where $(x^\mu, \sigma^m)$ are bundle coordinates on the fibre bundle (7.30) and the transition functions $\sigma^m \to \sigma^m(x^\lambda, \sigma^k)$ are independent of the coordinates $y^i$. The following two assertions make composite fibre bundles useful for physical applications.

**Proposition 7.1.** Given a composite fibre bundle (7.28), let $h$ be a global section of the fibre bundle $\Sigma \to X$. Then the restriction

$$Y_h = h^* Y$$

(7.31)

of the fibre bundle $Y \to \Sigma$ to $h(X) \subset \Sigma$ is a subbundle $i_h : Y_h \hookrightarrow Y$ of the fibre bundle $Y \to X$. □
Proposition 7.2. Given a section $h$ of the fibre bundle $\Sigma \to X$ and a section $s_\Sigma$ of the fibre bundle $Y \to \Sigma$, their composition
\[ s = s_\Sigma \circ h \] (7.32)
is a section of the composite fibre bundle $Y \to X$ (7.28). Conversely, every section $s$ of the fibre bundle $Y \to X$ is the composition (7.32) of the section $h = \pi_{Y\Sigma} \circ s$ of the fibre bundle $\Sigma \to X$ and some section $s_\Sigma$ of the fibre bundle $Y \to \Sigma$ over the closed submanifold $h(X) \subset \Sigma$. □

Let
\[ A_\Sigma = dx^\lambda \otimes (\partial_\lambda + A^i_\lambda \partial_i) + d\sigma^m \otimes (\partial_m + A^i_m \partial_i) \] (7.33)
be a connection on the fibre bundle $Y \to \Sigma$. Let $h$ be a section of the fibre bundle $\Sigma \to X$ and $Y_h$ the subbundle (7.31) of the composite fibre bundle $Y \to X$, which is the restriction of the fibre bundle $Y \to \Sigma$ to $h(X)$. Every connection $A_\Sigma$ (7.33) induces the pull-back connection
\[ A_h = i^*_h A_\Sigma = dx^\lambda \otimes [\partial_\lambda + ((A^i_m \circ h)\partial_i h^m + (A \circ h)^i_\lambda)\partial_i] \] (7.34)
on $Y_h \to X$.

Given a composite fibre bundle $Y$ (7.28), there is the following exact sequences of vector bundles over $Y$:
\[ 0 \to V_\Sigma Y \hookrightarrow VY \to Y \times V\Sigma \to 0, \] (7.35)
where $V_\Sigma Y$ is vertical tangent bundle of the fibre bundle $Y \to \Sigma$. Every connection $A$ (7.33) on the fibre bundle $Y \to \Sigma$ provides the splitting
\[ VY = V_\Sigma Y \oplus A_\Sigma (Y \times V\Sigma), \] (7.36)
of the exact sequence (7.35). Using this splitting (7.36), one can construct the first order differential operator
\[ \tilde{D} : J^1 Y \to T^* Y \otimes V_\Sigma Y, \quad \tilde{D} = dx^\lambda \otimes (y^i_\lambda - A^i_\lambda - A^i_m \sigma^m) \partial_i, \] (7.37)
called the vertical covariant differential, on the composite fibre bundle $Y \to X$. It possesses the following important property. Let $h$ be a section of the fibre bundle $\Sigma \to X$ and $Y_h$ the subbundle (7.31) of the composite fibre bundle $Y \to X$, which is the restriction of the fibre bundle $Y \to \Sigma$ to $h(X)$. Then the restriction of the vertical covariant differential $\tilde{D}$ (7.37)
to $J^1i_h(J^1Y_h) \subset J^1Y$ coincides with the familiar covariant differential on $Y_h$ relative to the pull-back connection $A_h$ (7.34).

**Principal connections**

Let $\pi_P : P \to X$ be a principal bundle whose structure group is a real Lie group $G$. By definition, $P \to X$ is provided with the free transitive action of $G$ on $P$ on the right:

$$R_G : P \times G \to P, \quad R_g : p \mapsto pg, \quad p \in P, \quad g \in G.$$ (7.38)

A $G$-principal bundle $P$ is equipped with a bundle atlas $\Psi_P = \{(U_\alpha, \psi^P_\alpha), \rho_{\alpha\beta}\}$ whose trivialization morphisms

$$\psi^P_\alpha : \pi_P^{-1}(U_\alpha) \to U_\alpha \times G$$

obey the condition

$$\text{pr}_2 \circ \psi^P_\alpha \circ R_g = g \circ \text{pr}_2 \circ \psi^P_\alpha, \quad \forall g \in G.$$  

Due to this property, every trivialization morphism $\psi^P_\alpha$ determines a unique local section $z_\alpha : U_\alpha \to P$ such that $\text{pr}_2 \circ \psi^P_\alpha \circ z_\alpha = \text{Id}$. The transformation rules for $z_\alpha$ read

$$z_\beta(x) = z_\alpha(x)\rho_{\alpha\beta}(x), \quad x \in U_\alpha \cap U_\beta.$$ (7.39)

Conversely, the family $\{(U_\alpha, z_\alpha)\}$ of local sections of $P$ which obey (7.39) uniquely determines a bundle atlas $\Psi_P$ of $P$.

The pull-back $f^*P$ (6.5) of a principal bundle is also a principal bundle with the same structure group.

Taking the quotient of the tangent bundle $TP \to P$ and the vertical tangent bundle $VP \to P$ of $P$ by $G$, we obtain the vector bundles

$$T_GP = TP/G, \quad V_GP = VP/G$$ (7.40)

over $X$. Sections of $T_GP \to X$ are $G$-invariant vector fields on $P$, while sections of $V_GP \to X$ are $G$-invariant vertical vector fields on $P$. Hence, the typical fibre of $V_GP \to X$ is the right Lie algebra $g_r$ of the group $G$. The group $G$ acts on this typical fibre by the adjoint representation.

Let $J^1P$ be the first order jet manifold of a $G$-principal bundle $P \to X$. Bearing in mind that $J^1P \to P$ is an affine bundle modelled over the vector bundle

$$T^*X \otimes VP \to P,$$
let us consider the quotient of the jet bundle $J^1 P \to P$ by the jet prolongation of the canonical action \((7.38)\). We obtain the affine bundle

$$C = J^1 P / G \to X$$

modelled over the vector bundle

$$\tilde{C} = T^* X \otimes V_G P \to X.$$  

Turn now to connections on a principal bundle $P \to X$. In this case, the exact sequence \((6.13a)\) can be reduced to the exact sequence

$$0 \to V_G P \hookrightarrow T_G P \to TX \to 0 \quad (7.42)$$

by taking the quotient with respect to the action of the group $G$. A principal connection $A$ on a principal bundle $P \to X$ is defined as a section $A : P \to J^1 P$ which is equivariant under the action \((7.38)\) of the group $G$ on $P$, i.e.,

$$J^1 R_g \circ A = A \circ R_g, \quad \forall g \in G. \quad (7.43)$$

Such a connection defines the splitting of the exact sequence \((7.42)\), and can be represented by the $T_G P$-valued form

$$A = dx^\lambda \otimes (\partial_\lambda + A_3^q \varepsilon_q), \quad (7.44)$$

where \(\{\varepsilon_q\}\) is the basis for the Lie algebra $g_r$.

On the other hand, due to the property \((7.43)\), there is obvious one-to-one correspondence between the principal connection on a principal bundle $P \to X$ and the global sections of the fibre bundle $C \to X \ (7.41)$, called the bundle of principal connections. Given a bundle atlas of $P$, the fibre bundle $C$ is equipped with the associated bundle coordinates $(x^\lambda, a^q_3)$ such that, for any section $A$ of $C \to X$, the local functions $A_3^q = a_3^q \circ A$ are coefficients of the connection form \((7.44)\). One can show that they coincide with coefficients of the familiar local connection form \[32\] and, therefore, can be treated as gauge potentials in gauge theory on a $G$-principal bundle $P$.

There are both pull-back and push-forward operations of principal connections \[32\].

**Theorem 7.3.** Let $P$ be a principal fibre bundle and $f^* P \ (5.5)$ the pull-back principal bundle with the same structure group. If $A$ is a principal connection on $P$, then the pull-back connection $f^* A \ (7.9)$ on $f^* P$ is a principal connection. □

**Theorem 7.4.** Let $P' \to X$ and $P \to X$ be principle bundles with structure groups $G'$ and $G$, respectively. Let $\Phi : P' \to P$ be a principal bundle morphism over $X$ with the
corresponding homomorphism $G' \to G$. For every principal connection $A'$ on $P'$, there exists a unique principal connection $A$ on $P$ such that $T\Phi$ sends the horizontal subspaces of $A'$ onto the horizontal subspaces of $A$. □

Let the structure group $G$ of a principal bundle $P$ acts on some manifold $V$ on the left. Let us consider the quotient

$$Y = (P \times V)/G$$

(7.45)

by identification of the elements $(p, v)$ and $(pg, g^{-1}v)$ for all $g \in G$. It is a fibre bundle over $X$ called a $P$-associated fibre bundle. Every atlas $\Psi_P = \{(U_a, z_a)\}$ of $P$ determines the associated atlas $\Psi = \{(U_a, \psi_a(x) = [z_a(x)]^{-1})\}$ of $Y$. Any automorphism $\Phi$ of a principal bundle $P$ yields the automorphism

$$\Phi_Y : (P \times V)/G \to (\Phi(P) \times V)/G$$

of the $P$-associated fibre bundle (7.45).

Every principal connection on $P \to X$ induces canonically the corresponding connection on the $P$-associated fibre bundle (7.45) as follows. Given a principal connection $A$ (7.44) on $P$ and the corresponding horizontal splitting of the tangent bundle $TP$, the tangent map to the canonical morphism

$$P \times V \to (P \times V)/G$$

defines the horizontal splitting of the tangent bundle $TY$ and the corresponding connection on the $P$-associated fibre bundle $Y \to X$. The latter is called the associated principal connection or simply a principal connection on $Y \to X$. If $Y$ is a vector bundle, this connection takes the form

$$A = dx^\lambda \otimes (\partial_\lambda - A^p_\lambda I_p \partial_i),$$

where $I_p$ are generators of the representation of the Lie algebra $g_r$ in $V$.

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