Free algebras in varieties of Hilbert algebras with supremum generated by finite chains.

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Abstract

Hilbert algebras with supremum, i.e., Hilbert algebras where the associated order is a join–semilattice were first considered by A.V. Figallo, G. Ramón and S. Saad in [11], and independently by S. Celani and D. Montangie in [7].

On the other hand, L. Monteiro introduced the notion of \(n\)--valued Hilbert algebras (see [12]). In this work, we investigate the class of \(n\)--valued Hilbert algebras with supremum, denoted \(H_n^\lor\), i.e., \(n\)--valued Hilbert algebras where the associated order is a join–semilattice. The varieties \(H_n^\lor\) are generated by finite chains. The free \(H_n^\lor\)--algebra \(\text{Free}_{n+1}(r)\) with \(r\) generators is studied. In particular, we determine an upper bound to the cardinal of the finitely generated free algebra \(\text{Free}_{n+1}(r)\).

1 Introduction

The study of Hilbert algebras was initiated by Diego in his important work [9]. It is well–Known that Hilbert algebras constitute an algebraic counter-
part of the implicative fragment of Intuitionistic Propositional Logic (IPL). A topological duality for this algebras was developed in [3].

The class of Hilbert algebras where the associated order is a meet–semilattice was consider in [10] under the name of Hilbert algebras with infimum. On the other hand, Hilbert algebras with supremum, i.e., Hilbert algebras where the associated order is a join–semilattice were first considered by Figallo, Ramón and Saad in [11], and independently by Celani and Montangie in [7]. The latter denoted the class of these algebras by $H^\vee$.

In 1977, L. Monteiro introduced the notion of $n$–valued Hilbert algebras (see [12]). These algebras constitute an algebraic counterpart of the $n$–valued Intuitionistic Implicative Propositional Calculus.

In this work, we investigate the class of $n$–valued Hilbert algebras with supremum (denoted $H^\vee_n$), i.e., $n$–valued Hilbert algebras where the associated order is a join–semilattice. The objects of $H^\vee_n$ are algebraic models for the fragment of intuitionistic propositional calculus in the connectives $\rightarrow$ and $\lor$ and which satisfies the well–known axiom of Ivo Thomas

$$\beta_{n-1} \rightarrow (\beta_{n-2} \rightarrow (\ldots (\beta_0 \rightarrow x_0) \ldots))$$

where $\beta_i = (x_i \rightarrow x_{i+1}) \rightarrow x_0$, $0 \leq i \leq n - 1$.

It was noted in [7] that $H^\vee_n$ constitute a variety. Here we shall exhibit a very simple and natural equational base for $H^\vee_n$ different from the one presented in [7]. The most important contribution of this paper is the study of free algebras in $H^\vee_n$ finitely generated. In particular, we shall describe a formula to determine the cardinal of the free $(n + 1)$–valued Hilbert algebra with supremum, $\text{Free}_{n+1}(r)$, in terms of the finite number of the free generators $r$ and the numbers $\alpha_{k,p+1}$’s of minimal irreducible deductive systems of certain subalgebras of $\text{Free}_{n+1}(r)$. Finally, we shall exhibit a formula to calculate an upper bound to $|\text{Free}_{n+1}(r)|$ in terms of $r$ only.

## 2 Preliminaries

If $\mathcal{K}$ is a variety we will denote by $\text{Con}_\mathcal{K}(A)$, $\text{Hom}_\mathcal{K}(A,B)$ and $\text{Epi}_\mathcal{K}(A,B)$ the set of $\mathcal{K}$–congruences of $A$, $\mathcal{K}$–homomorphisms from $A$ into $B$ and $\mathcal{K}$–epimorphisms from $A$ onto $B$, respectively.
Besides, if $S \subseteq A$ is a $\mathbb{K}$–subalgebra of $A$ we write $S \triangleleft_\mathbb{K} A$. We note by $[G]_{\mathbb{K}}$ the $\mathbb{K}$–subalgebra of $A$ generated by the set $G$. When there is no doubt about what variety we are referring to, the subindex will be omitted.

Recall that a Hilbert algebra is a structure $\langle A, \rightarrow, 1 \rangle$ of type (2,0) that satisfies the following:

(H1) $x \rightarrow (y \rightarrow x) = 1$,

(H2) $(x \rightarrow (y \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = 1$,

(H3) $x \rightarrow y = 1 = y \rightarrow x$, implies $x = y$.

The variety of all Hilbert algebras is denoted by $H$. It is well-known that in every $A \in H$ the following holds.

(H4) $(x \rightarrow x) \rightarrow x = x$,

(H5) $x \rightarrow x = y \rightarrow y$,

(H6) $x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z)$,

(H7) $(x \rightarrow y) \rightarrow ((y \rightarrow x) \rightarrow x) = (y \rightarrow x) \rightarrow ((x \rightarrow y) \rightarrow y)$,

(H8) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,

(H9) $x \rightarrow 1 = 1$,

The relation $\leq$ defined by $x \leq y$ iff $x \rightarrow y = 1$ is a partial order on $A$ and $x \leq 1$ for all $x \in A$. If $a, b \in A$ are such that there exists the supremum of $\{a, b\}$ in $A$, denoted by $a \lor b$, then for every $c \in A$ there exists the infimum of $\{(a \rightarrow c), (a \rightarrow b)\}$, denoted by $(a \rightarrow c) \land (a \rightarrow b)$ and it is verified that

(H10) $(a \lor b) \rightarrow c = (a \rightarrow c) \land (b \rightarrow c)$.

If $a, b \in A$ are such that there exists the supremum $a \lor b$ of $\{a, b\}$, then for every $c \in A$ it is verified that:

(H11) $(a \rightarrow c) \rightarrow ((b \rightarrow c) \rightarrow ((a \lor b) \rightarrow c)) = 1$.
It is said that \( A \in H \) is a \((n+1)\)-valued Hilbert algebra if \( n \) is the least natural number, \( n \geq 2 \), in such a way that

\[
T_{n+1} = T(x_0, \ldots, x_n) = \beta_{n-1} \rightarrow (\beta_{n-2} \rightarrow (\cdots \rightarrow (\beta_0 \rightarrow x_0) \cdots)) = 1,
\]

holds in \( A \), where \( \beta_i = (x_i \rightarrow x_{i+1}) \rightarrow x_0 \) for \( 0 \leq i \leq n - 1 \).

We will denote by \( H_{n+1} \) the variety of \((n+1)\)-valued Hilbert algebras (see \[12\]).

**Example 2.1** Let \( C_{n+1} = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\} \) and let \( \rightarrow \) defined as

\[
x \rightarrow y = \begin{cases} 
1 & \text{if } x \leq y, \\
y & \text{if } x > y.
\end{cases}
\]

Then \( (C_{n+1}, \rightarrow, 1) \) is a \((n+1)\)-valued Hilbert algebra.

Recall that if \( A \in H \), \( D \subseteq A \) is a deductive system (d.s.) of \( A \) iff \( 1 \in D \) and \( D \) is closed by modus ponens, i.e., if \( x, x \rightarrow y \in D \) then \( y \in D \). A d.s. \( D \) is said to be irreducible (i.d.s.) iff \( D \) is a proper d.s. and \( D = D_1 \cap D_2 \) implies \( D = D_1 \) or \( D = D_2 \) for any d.s. \( D_1 \) and \( D_2 \). It is said that the proper d.s. \( D \) is fully irreducible (f.i.d.s) iff

\[
D = \bigcap_{i \in I} D_i \text{ implies } D = D_i \text{ for some } i \in I. \quad \text{(see [9])}
\]

Besides, \( D \) is said to be prime (p.d.s.) iff it is a proper d.s. and for any \( a, b \in A \) we have \( a \lor b \in D \) implies \( a \in D \) or \( b \in D \).

We denote by \( \mathcal{D}(A) \) and \( \mathcal{E}(A) \) the set of all d.s and f.i.d.s. of a given Hilbert algebra \( A \), respectively. On the other hand, \( D \) is minimal iff it is a minimal element of the ordered set \( (\mathcal{D}(A), \subseteq) \), i.e., if \( D' \in \mathcal{D}(A) \) is such that \( D' \subseteq D \) then \( D' = D \). By \( \mathcal{M}(A) \) we denote the set of all minimal elements of the ordered set \( (\mathcal{E}(A), \subseteq) \), i.e., \( \mathcal{M}(A) = \{ D \in \mathcal{E}(A) : \text{ if } D' \in \mathcal{E}(A) \text{ is such that } D' \subseteq D \text{ then } D' = D \} \).

The following results are well-known and will be used in the next sections.

**Theorem 2.2** ( \[9\]) Let \( A \in H \) and \( D \in \mathcal{D}(A) \). The following conditions are equivalents:

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(i) \( D \in \mathcal{E}(A) \),
(ii) if \( a, b \in A \setminus D \) then there is \( c \in A \setminus D \) such that \( a \leq c \) and \( b \leq c \),

**Theorem 2.3** ([9]) Let \( A \in H \) and \( D \in \mathcal{D}(A) \). The following conditions are equivalents:

(i) \( D \) is a f.i.d.s.,

(ii) there is \( a \in A \setminus D \) and \( D \) is a maximal d.s. among all d.s. of \( A \) that do not contain the element \( a \),

(iii) there is \( a \in A \setminus D \) such that \( x \to a \in D \) for all \( x \notin D \).

Besides,

**Theorem 2.4** ([9]) Let \( A \in H \). For every \( D \in \mathcal{D}(A) \) and \( a \notin D \), there exists \( M \in \mathcal{D}(A) \) such that \( M \) is maximal among all d.s. that contain \( D \) but do not contain \( a \).

The family of all f.i.d.s (minimal i.d.s) of a given Hilbert algebra \( A \) is a splitting set, i.e.,

\[
\bigcap_{D \in \mathcal{E}(A)} D = \{1\} \quad \bigcap_{D \in \mathcal{M}(A)} D = \{1\}.
\]

Recall that if \( A, B \in H \) and \( h \in \text{Hom}_H(A, B) \) then \( \text{Con}_H(A) = \{ R(D) : D \in \mathcal{D}(A) \} \) where \( R(D) = \{ (x, y) \in A^2 : x \to y, y \to x \in D \} \). If \( R = R(D) \) we shall denote by \( A/D \) the quotient algebra determined by \( R \). Also, it is well-known that the kernel of \( h, \text{Ker}(h) \), is a d.s. of \( A \) where \( \text{Ker}(h) = \{ x \in A : h(x) = 1 \} \). Also, if \( z \in A \) the segment \([z] = \{ x \in A : z \leq x \} \) is a d.s. of \( A \).

**Definition 2.5** ([12]) Let \( A \in H \). \( D \in \mathcal{D}(A) \) is said to be \( (p + 1) \)-valued if \( A/D \cong C_{p+1} \) (see Example 2.1).

We shall denote by \( \mathcal{E}_{p+1}(A) \) and \( \mathcal{M}_{p+1}(A) \) the sets of all \( (p + 1) \)-valued d.s. of \( A \) included in \( \mathcal{E}(A) \) and \( \mathcal{M}(A) \), respectively. Then,
Theorem 2.6 ([12]) The following conditions are equivalents:

(i) \( A \in H_{n+1} \),

(ii) \( E(A) = \bigcup_{p=1}^n E_{p+1}(A) \) and \( E_{n+1}(A) \neq \emptyset \).

It is worth mentioning that in \( H_n \) the notions of i.d.s and f.i.d.s coincide (see [12]). Let \( A \in H \) and \( X \subseteq A \), we denote by \( \mu(X) \) the set of all minimal elements of \( X \). Then,

Lemma 2.7 ([9]) If \([X]_H = A\) then \( \mu(X) = \mu(A) \).

As a consequence of this lemma we have the following corollary.

Corollary 2.8 Let \( A \in H \) and \( X \subseteq A \) such that \([X]_H = A\). Then, \( X = \mu(A) \) iff \( X \) is an antichain.

On the other hand, it can be proved that:

Lemma 2.9 The only \( H \)-automorphism of \( C_{n+1} \) is the identity.

Proof. Let \( h \in \text{Hom}_H(C_{n+1}, C_{n+1}) \) such that \( h \) is a bijection. If \( h \) is not the identity map, then there is \( x_0 \in C_{n+1} \) such that \( h(x_0) \neq x_0 \). Then, \( x_0 < h(x_0) \) or \( h(x_0) < x_0 \). In the first case we have that \( x_0 \leq h(x_0) \leq h^2(x_0) \leq \cdots \leq h^p(x_0) \leq \cdots \). If there is \( p \) such that \( h^p(x_0) = h^{p+1}(x_0) \) then \( x_0 = (h^{-1})^p(h^p(x_0)) = (h^{-1})^p(h^{p+1}(x_0)) = h(x_0) \), a contradiction. So, the only possibility is that \( x_0 < h(x_0) < h^2(x_0) < \cdots < h^p(x_0) < \cdots \). But this contradicts the finiteness of the algebra \( C_{n+1} \). \( \square \)

Finally, we have:

Corollary 2.10 Let \( A \in H \) and \( h_1, h_2 \in \text{Epi}_H(A, C_{n+1}) \). Then, \( h_1 = h_2 \) iff \( \text{Ker}(h_1) = \text{Ker}(h_2) \).
3  \((n + 1)\)-valued Hilbert algebras with supremum

A Hilbert algebra with supremum (or \(H^\vee\)-algebra), as it was defined in [7], is an algebra \(\langle A, \to, \vee, 1 \rangle\) of type \((2, 2, 0)\) such that

- \(\langle A, \to, 1 \rangle\) is a Hilbert algebra,
- \(\langle A, \vee, 1 \rangle\) is a join-semilattice, and
- for all \(a, b \in A\), \(a \to b = 1 \iff a \vee b = b\).

In the next theorem, we exhibit a simple and natural equational base for \(H^\vee\) different from the one showed in [7].

**Theorem 3.1** Let \(\langle A, \to, \vee, 1 \rangle\) be an algebra of type \((2, 2, 0)\). The following conditions are equivalent.

(i) \(\langle A, \to, \vee, 1 \rangle\) is a \(H^\vee\)-algebra,

(ii) \(\langle A, \to, 1 \rangle\) is a Hilbert algebra and the following equations hold:

   (a) \(x \to (x \vee y) = 1\),
   (b) \(y \to (x \vee y) = 1\),
   (c) \((x \to z) \to ((y \to z) \to ((x \vee y) \to z)) = 1\).

**Proof.** It is routine. \(\square\)

Next, we introduce the notion of \((n + 1)\)-valued Hilbert algebra with supremum.

**Definition 3.2** \(A = \langle A, \to, \vee, 1 \rangle \in H^\vee\) is a \((n + 1)\)-valued Hilbert algebra with supremum, or \(H^\vee_{n+1}\)-algebra, if \(\langle A, \to, \vee, 1 \rangle \in H^\vee\) and \(\langle A, \to, 1 \rangle \in H_{n+1}\).

**Example 3.3** Let \(\langle C_{n+1}, \to, 1 \rangle\) as in the Example 2.1 and let \(\vee\) the operation defined by \(x \vee y = \text{Sup}\{x, y\}\). Then, \(J_{n+1} = \langle C_{n+1}, \to, \vee, 1 \rangle\) is a \(H^\vee_{n+1}\)-algebra. Besides, if \(n \leq m\) then \(J_n\) is isomorphic to some subalgebra of \(J_m\).
It is clear that $H_n^\lor$ are varieties of Hilbert algebras with supremum generated by finite chains. If fact, $H_n^\lor$ is generated by the algebra $J_n$.

All d.s. of a Hilbert algebra with supremum are prime.

**Lemma 3.4** Let $A \in H^\lor$ and $D \in \mathcal{E}(A)$. Then $D$ is prime.

**Proof.** Let $D \in \mathcal{E}(A)$ and $a, b \in A$ such that (1) $a \lor b \in D$. Suppose that $a \notin A$ and $b \notin A$, by Theorem 2.1, there exists (2) $c \in A \setminus D$ such that $a \leq c$ and $b \leq c$. Then, $1 = (a \rightarrow c) \rightarrow ((b \rightarrow c) \rightarrow ((a \lor b) \rightarrow c)) = 1 \rightarrow (1 \rightarrow ((a \lor b) \rightarrow c)) = (a \lor b) \rightarrow c$. That is, $a \lor b \leq c$ and, by (1), $c \in D$ which contradicts (2). \hfill \square

**Lemma 3.5** Let $h \in \text{Hom}_{H^\lor}(A, B)$. Then,

(i) If $h \in \text{Epi}_{H^\lor}(A, B)$ then $h([z]) = [h(z)]$ for all $z \in A$.

(ii) $\text{Ker}(h) \in \mathcal{E}(A)$, for all $h \in \text{Epi}_{H^\lor}(A, J_{r+1})$.

(iii) If $A \in H_{n+1}^\lor$ and $B = J_{r+1}$ then $\text{Ker}(h) \in \mathcal{E}_{p+1}(A)$ for some $p$, $1 \leq p \leq n$.

**Proof.**

(i) Since $h$ is isotonic we have $h([z]) \subseteq [h(z)]$. Besides, for all $y \in [h(z)]$ there exists $x \in A$ such that $y = h(x)$. Let $u = z \lor x \in [z]$ then $h(u) = h(z) \lor h(x) = h(z) \lor y = y$. Therefore, $y \in h([z])$.

(ii) Let $a \in A$ such that $h(a) = \frac{r-1}{r}$. Then, $a \notin \text{Ker}(h)$ and for all $x \notin \text{Ker}(h)$ we have that $h(x \rightarrow a) = h(x) \rightarrow h(a) = h(x) \rightarrow \frac{r-1}{r} = 1 \in \text{Ker}(h)$. Then, $x \rightarrow a \in \text{Ker}(h)$ and, by Theorem 2.3, $\text{Ker}(h)$ is fully irreducible.

(iii) By (ii) and Theorem 2.6 \hfill \square

Theorems 3.6 and 3.9 extend to $H_{n+1}^\lor$ the corresponding results for $H_{n+1}$ proved in [12].

**Theorem 3.6** Let $A \in H_{n+1}^\lor$ and $M \in \mathcal{D}(A)$. The following conditions are equivalent.

(i) $M \in \mathcal{E}(A)$,
(ii) there exists $h \in \text{Hom}_{H_n^\vee}(A, J_{r+1})$ such that $\text{Ker}(h) = M$.

(iii) $A/M$ is isomorphic to some subalgebra of $J_{n+1}$.

Note that if $A \in H^\vee$ and $c \in A$ then $[c] \triangleleft_{H^\vee} A$. Then,

**Lemma 3.7** Let $A \in H_{n+1}^\vee$ and $c \in A$. For every $D \in \mathcal{E}([c])$ there is a unique $M \in \mathcal{E}(A)$ such that $D = M \cap [c]$.

**Proof.** Let $D \in \mathcal{E}([c])$. By Theorem 2.3 there is $a \in [c]$ such that $D$ is a maximal d.s. among all d.s. of $A$ that do not contain the element $a$. Then, by Theorem 2.3 there exists $M \in D(A)$ such that (1) $M$ is maximal among all d.s. of $A$ that verifies: $a \notin M$ and $D \subseteq M$. Let $P = M \cap [c]$. Then, $D \subseteq P$.

On the other hand, $P \in D([c])$ and $a \notin P$, so, by hypothesis on $D$, $P \subseteq D$. Therefore, $D = P = M \cap [c]$. Besides, suppose that $M = \bigcap \{M_i \mid M_i \in D(A)\}$, then there is $i \in I$ such that $a \notin M_i$. Then, $D \subseteq M \subseteq M_i$ and, by (1), $M = M_i$. So, $M \in \mathcal{E}(A)$.

Now, suppose that there are $M_1, M_2 \in \mathcal{E}(A)$ such that $M_1 \cap [c] = M_2 \cap [c] = D$. Since $D \neq [c]$, there is (2) $z \in [c] \setminus D$. Then, $z \notin M_1 \cup M_2$. If $z \in M_1 \setminus M_2$, by (2), we have $x \vee z \in M_1 \cap [c] \subseteq M_2$. By Lemma 3.3 $x \in M_2$ or $z \in M_2$ and both cases lead to a contradiction. Therefore, $M_1 \subseteq M_2$. Analogously, it can be proved that $M_2 \subseteq M_1$.

In what follows, we shall denote by $M_D$ the only d.s. of $A$ associated to $D \in \mathcal{E}([c])$.

**Lemma 3.8** Let $A \in H_{n+1}^\vee$, $c \in A$, $D \in \mathcal{E}_{p+1}([c])$ and $M_D \in \mathcal{E}_{q+1}(A)$. Then, $p \leq q$. Besides, if $M_D \in \mathcal{M}(A)$ then $D \in \mathcal{M}([c])$.

**Proof.** By Theorem 3.6 there exists $h \in \text{Hom}_{H_{n+1}^\vee}(A, J_{n+1})$ such that $\text{Ker}(h) = M$. Let $h' = h|_{[c]}$, then $\text{Ker}(h') = \text{Ker}(h) \cap [c] = M_D \cap [c] = D$ and $p + 1 = |[c]/D| = |h'( [c])| = |h([c])| \leq |h(A)| = |A/M| = q + 1$.

Suppose that $M_D \in \mathcal{M}(A)$ and that there is $D' \in \mathcal{E}([c])$ such that $D' \subseteq D$. Then, $M_D \cap [c] = D' \subseteq D = M_D \cap [c]$. If there is $x \in M_D \setminus M_D$ then
$x \lor z \in M_{D'} \cap [c] = D'$ para cada $z \in [c] \setminus D$, and $x \lor z \in D \subseteq M_{D}$. By Lemma 3.3, $x \in M_{D}$ or $z \in M_{D}$ and both cases lead to a contradiction. Then, $M_{D'} \subseteq M_{D}$ and since $M$ is minimal $M_{D'} = M_{D}$ and therefore $D' = D$. □

Finally,

**Theorem 3.9** Let $A \in H_{n+1}^\forall$ be a non-trivial algebra. Then, $A$ is isomorphic to a subalgebra of $P = \prod_{M \in \mathcal{L}(A)} A/M$.

4  Finitely generated free $H_{n+1}^\forall$—algebras

Let $r$ be an arbitrary cardinal number, $r > 0$. We say that $\text{Free}(r)$ is the free $H_{n+1}^\forall$—algebra with $r$ free generators if:

(L1) There exists $G \subseteq \text{Free}(r)$ such that $|G| = r$ and $[G]_{H_{n+1}^\forall} = \text{Free}(r)$,

(L2) Any function $f : G \to A$, $A \in H_{n+1}^\forall$, can be extended to a unique homomorphisms $h : \text{Free}(r) \to A$.

Since the class of $H_{n+1}^\forall$—algebras is equationally definable, we know that $\text{Free}(r)$ is unique up to isomorphisms. If we want to stress that $\text{Free}(r) \in H_{n+1}^\forall$, we shall write $\text{Free}_{n+1}(r)$.

**Lemma 4.1** Let $G$ be a set of free generators of $\text{Free}(r)$. Then, $G = \mu(\text{Free}(r))$.

**Proof.** If $G = 1$, then $G = \mu(G)$ and, by Lemma 2.7, $G = \mu(\mathcal{L}(1))$. Suppose now that $|G| > 1$ and let $g, g' \in G$. If $g < g'$, let $f : G \to J_{n+1}$ the function defined by $f(g) = 1$ and $f(t) = 0$ if $t \neq g$. By (L2), there exists $h \in \text{Hom}_{H_{n+1}^\forall}(\text{Free}(r), J_{n+1})$ that extends $f$. Since $h$ is isotonic, $1 = h(g) \leq h(g') = 0$. Then $g \not< g'$. Analogously, we have that $g' \not< g$. Then, $g$ and $g'$ are incomparable elements and, therefore, $G = \mu(G) = \mu(\text{Free}(r))$. □

As an immediate consequence we have:
Corollary 4.2 \( \text{Free}(r) = \bigcup_{g \in G} [g] \).

Taking all this into account we can prove that the variety \( H_{n+1}^\vee \) is locally finite.

**Lemma 4.3** \( \text{Free}(r) \) is finite, for any natural number \( r \).

**Proof.** By Theorems 3.6 and 3.9 it is enough to prove that \( \mathcal{E}(\text{Free}(r)) \) is a finite set. For every \( h \in \text{Hom}_{H_{n+1}^\vee}(\text{Free}(r), J_{n+1}) \) we know that \( \text{Ker}(h) \in \mathcal{E}(\text{Free}(r)) \) and that the correspondence that maps \( h \) into \( \text{Ker}(h) \) is surjective. Since \( \text{Hom}(\text{Free}(r), J_{n+1}) = \bigcup_{S \in J_{n+1}} \text{Epi}(\text{Free}(r), S) \), \( \{ f = h|_G : h \in \text{Epi}(\text{Free}(r), S) \} \subseteq S^G \) and \( |S^G| < \infty \) we have that \( |\text{Epi}(\text{Free}(r), S)| < \infty \). Since \( J_{n+1} \) has a finite number of subalgebras, \( \text{Free}(r) \) is finite. \( \square \)

Let \( G_k \) be a subset of \( G \) with \( k \) elements. Then,

**Lemma 4.4** Let \( g_k^* = \bigvee_{g \in G_k} g \). Then, \( [g_k^*] = \bigcap_{g \in G_k} [g] \).

**Proof.** It is clear that, if \( g \in G_k \) then \( g \leq g_k^* \) and \( [g_k^*] \subseteq \bigcap_{g \in G_k} [g] \).

Let \( x \in \bigcap_{g \in G_k} [g] \), then \( g \leq x \) for all \( g \in G_k \). Then, \( g_k^* \leq x \) and therefore \( x \in [g_k^*] \). \( \square \)

**Lemma 4.5** Let \( h \in \text{Epi}(\text{Free}(r), J_{q+1}) \). Then, \( C_{q+1} \setminus \{ 1 \} \subseteq h(G) \).

**Proof.** Let \( z \in C_{q+1} \setminus \{ 1 \} \) and let \( x \in h^{-1}(z) \subseteq \text{Free}(r) \). Then, we can express \( x \) in terms of the generators using the operations \( \rightarrow \) and \( \vee \). We will call \textit{length} of \( x \in \text{Free}(r) \) the least natural number \( m \) such that there exists an expression for \( x \) in terms of the generators which is constructed with \( m \) applications of the operations \( \rightarrow \) or \( \vee \) in it.
If \( m = 1 \), then (1) \( x = g_1 \lor g_2 \) or (2) \( x = g_1 \rightarrow g_2 \). In case (1), \( h(x) = h(g_1 \lor g_2) = h(g_1) \lor h(g_2) = z \). Since, (3) \( h(g_1) \leq h(g_2) \) or \( h(g_2) \leq h(g_1) \), we have that \( h(g_2) = z \) or \( h(g_1) = z \), therefore, \( z \in h(G) \). In case (2), if \( h(g_1) \leq h(g_2) \) then \( h(g_1) \rightarrow h(g_2) = 1 \neq z \) and so this case is discarded. Then, \( h(g_2) \leq h(g_1) \) and if \( h(g_1) \rightarrow h(g_2) = h(g_2) = z \) and so \( z \in h(G) \).

Suppose that the theorem is true for every formula which its expression in terms of the generators has length \( m - 1 \). Let \( x \in \text{Free}(r) \) be a formula of length \( m \). Then, (4) \( x = x_1 \lor x_2 \) or (5) \( x = x_1 \rightarrow x_2 \) with \( x_1, x_2 \in \text{Free}(r) \).

Clearly, the length of \( x \) and \( x_2 \) is \( m - 1 \) and, therefore, there exist \( g_1, g_2 \in G \) such that \( h(g_1) = h(x_1) \) and \( h(g_2) = h(x_2) \). In case (4), \( h(x) = h(x_1) \lor h(x_2) = h(g_1) \lor h(g_2) = z \) and using the same reasoning above, \( h(g_1) = z \) or \( h(g_2) = z \). Analogously, in case (5), it must be the case \( h(x_2) \leq h(x_1) \) and therefore \( h(x_2) = h(x_1) \rightarrow h(x_2) = z \) and by induction hypothesis there exists \( g \in G \) such that \( h(g) = h(x_2) = z \).

\[ \text{Lemma 4.6} \quad \text{Let } g_k^* \text{ be as in Lemma 4.4. Then } [g_k^*] \text{ is a } H_{m+1}^\lor \text{subalgebra of } \text{Free}_{n+1}(r) \text{ where } m \leq n. \]

**Proof.** For every \( x, y \in [g_k^*] \), \( g_k^* \leq y \leq x \rightarrow y \) by (H1), and then \( x \rightarrow y \in [g_k^*] \). Besides, \( g_k^* \leq y \leq x \lor y \) and so \( x \lor y \in [g_k^*] \). Then, \([g_k^*] \triangleleft \text{Free}_{n+1}(r)\).

Let \( m = \min\{q : \text{ the equation } T_{q+1} \approx 1 \text{ holds in } [g_k^*]\} \). Then, \( m \leq n \) and \([g_k^*] \in H_{m+1}^\lor \). Let’s denote by \([r]\) the set of the first \( r \) natural numbers, i.e., \( r = \{1, \ldots, r\} \). As an immediate consequence of Corollary 4.2 and the inclusion-exclusion principle we have:

\[ \text{Lemma 4.7} \quad |\text{Free}(r)| = \sum_{k=1}^{r} (-1)^{k+1} |\bigcap_{i \in I, J \subseteq [r], |J|=k} [g_i]|. \]

By Lemma 4.7 and Lemma 4.4 we conclude:

\[ \text{Lemma 4.8} \quad |\text{Free}(r)| = \sum_{k=1}^{r} (-1)^{k+1} \binom{r}{k} |[g_k^*]|. \]

Let \( \mathcal{E}_{k,p+1} = \mathcal{E}_{p+1}([g_k^*]) \) and \( \mathcal{M}_{k,p+1} = \mathcal{M}_{p+1}([g_k^*]) \).
Remark 4.9 Let \( D \in \mathcal{E}_{p+1}([g]) \) and \( h : [g] \longrightarrow J_{p+1} \) be the composition of the canonical map with the isomorphism that there is between \( [g]/D \) and \( J_{p+1} \). Since \( D \) is a \((p+1)\)-valued d.s., the family \( \mathcal{S}(D) \) of all d.s of \([g]\) that contain \( D \) is a chain whose elements are principal prime d.s. \( \mathcal{S}(D) = \{ [a_j] : j \in \{ 0, \ldots, p \} \} \). So \( D = [a_0] \subseteq [a_1] \subseteq \cdots \subseteq [a_j] \subseteq \cdots \subseteq [a_p] = [g] \) and \( h(a_j) = \frac{p+1}{j} \) for all \( j \in \{ 0, \ldots, p \} \) (see [12]).

Lemma 4.10 Let \( D \in \mathcal{M}_{p+1}([g]), D' \in \mathcal{M}_{q+1}([g]) \), \( \mathcal{S}(D) = \{ [a_j] : j \in \{ 0, \ldots, p \} \}, \mathcal{S}(D') = \{ [b_i] : i \in \{ 0, \ldots, q \} \} \) with \( a_0 \neq b_0 \). Also, let \( h_{D'} \) be the canonical epimorphism associated to \( D' \) (see Remark 4.9). Then,

(i) \( a_j \in [g] \setminus [b_0] \) for all \( j \in \{ 0, \ldots, p \} \),

(ii) \( h_{D'}(a_j) = \begin{cases} \frac{p+1}{j} & \text{if } D' = D, \\ 0 & \text{if } D' \neq D. \end{cases} \)

Proof. (i) If \( a_j \in [b_0], b_0 \leq a_j \leq a_0 \) and \( a_0 \in [b_0] \). Then, \([a_0] \subseteq [b_0]\) and since \( a_0 \neq b_0, [b_0] \) is not a minimal f.i.d.s. of \([g]\). So, \( a_j \in [g] \setminus [b_0]\).

(ii) It is immediate from (i).

Theorem 4.11 \([g_k] \simeq \prod_{p=1}^{n} J_{p+1}^{k_p + 1} \) where \( k_{p+1} = |\mathcal{M}_{k,p+1}| \).

Proof. For the sake of simplicity we shall identify \([g_k]/D\) with \( J_{p+1} \) for all \( D \in \mathcal{M}_{p+1}([g_k]) = \mathcal{M}_{p+1} \). Let \( \mathcal{M} = \bigcup_{p=1}^{n} \mathcal{M}_{p+1} \) and \( h_D \in Epi_{H_{p+1}}([g_k], J_{q+1}) \) the canonical epimorphism associated to \( D \).

Let \( h' : [g_k] \longrightarrow \prod_{p=1}^{n} J_{p+1}^{k_p + 1} \) be the function defined as \( h'(x) = (h_D(x))_{D \in \mathcal{M}} \).

It is clear that \( h_D \) is a \( H_{p+1} \)-homomorphism. Besides, \( \text{Ker}(h') = \{ x \in [g_k] : h'(x) = 1 \} = \bigcap_{D \in \mathcal{M}} \{ x \in [g_k] : h_D(x) = 1 \} = \bigcap_{D \in \mathcal{M}} D = \{ 1 \} \), since \( \mathcal{M} \) is a splitting set. Then \( h' \) is injective.

Let \( y = (y_D)_{D \in \mathcal{M}} \in \prod_{p=1}^{n} J_{p+1}^{k_p + 1} \). For every \( D \in \mathcal{M}, y_D = \frac{p+1}{j}, \) for some \( j \in \{ 0, \ldots, p \} \). By Remark 4.9 there exists \( a_D \in [g_k] \) such that \([a_D]\) is a
minimal element in the chain $S(D)$ and by Lemma 4.10, $h_F(a_D) = 0$ for all $F \in \mathcal{M}, F \neq D$, and $h_D(a_D) = \frac{p-j}{p}$.

Let $z = \bigvee_{F \in \mathcal{M}} a_F$. Then, $h'(z) = (h_D(z))_{D \in \mathcal{M}} = (h_D(\bigvee_{F \in \mathcal{M}} a_F))_{D \in \mathcal{M}} = (\bigvee_{F \in \mathcal{M}} h_D(a_F))_{D \in \mathcal{M}} = (h_D(a_D))_{D \in \mathcal{M}} = (y)_{D \in \mathcal{M}} = y$. So $h'$ is surjective. 

\[\blacksquare\]

**Corollary 4.12** Let $G$ a set of free generators. Then,

$$\text{Free}_{n+1}(r) = \bigcup_{g \in G} [g] \text{ where } [g] \simeq \prod_{p=1}^{n} J_{p+1}^{n_1,p+1}.$$ 

**Corollary 4.13** $|\text{Free}_{n+1}(r)| = \sum_{k=1}^{r} (-1)^{k+1} \binom{r}{k} \prod_{p=1}^{n} (p+1)^{a_k,p+1}$

\[\text{5 Computing an upper bound to } \alpha_{k,p+1}\]

In this section, compute an upper bound to $\alpha_{k,p+1}$. Since $\mathcal{M}_{k,p+1} \subseteq \mathcal{E}_{k,p+1}$, $|\mathcal{M}_{k,p+1}| \leq |\mathcal{E}_{k,p+1}|$ and so we shall compute the number $\eta_{k,p+1} = |\mathcal{E}_{k,p+1}|.$ That is to say, we shall determine how many $(p+1)$-valued i.d.s. the subalgebra $[g^*_k]$ has.

Let $p \leq q \leq n$ and let $\mathcal{F}_{k,p}(q)$ be the set of all functions $f : G \rightarrow C_{q+1}$ that satisfies the following conditions:

(F1) $C_{q+1} \setminus \{1\} \subseteq f(G)$,

(F2) $f(g) \leq \frac{q-p}{q}$ for all $g \in G_k$,

(F3) there exists $g_p \in G_k$ such that $f(g_p) = \frac{q-p}{q}$.

**Theorem 5.1** $|\mathcal{E}_{k,p+1}| = \sum_{q=p}^{n} |\mathcal{F}_{k,p}(q)|$. 

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Proof. Let \( f \in \bigcup_{q=p}^{n} \mathcal{F}_{k,p}(q) \). Then, there exists \( q \in \{p, \ldots, n\} \) such that \( f \) verifies (F1)-(F3). We extend \( f \) to a unique \( h \in Epi_{H_{n+1}}^r(\text{Free}(r), J_{q+1}) \). From (F2) and F(3) we have that \( h_f(g_k) = \bigvee_{g \in G_k} f(g) = \bigvee_{g \in G_k} f(g) = \frac{q-p}{q} \). By Lemma 3.5 (i), \( h_f([g_k]) = \{h_f(g_k)\} = \{\frac{q-p}{q}\} \). Let \( D^k_f = \text{Ker}(h_f) \cap [g_k^*]. \) By Theorem 3.6, \( \text{Ker}(h_f) \in \mathcal{E}(\text{Free}(r)) \). Then, \( |\{g_k^*\}/D_f^k| = |h_f([g_k^*])| = p + 1 \) and so \( D^k_f \in \mathcal{E}_{k,p+1} \).

Let \( \psi : \bigcup_{q=p}^{n} \mathcal{F}_{k,p}(q) \longrightarrow \mathcal{E}_{k,p+1} \) be the map defined by \( \psi(f) = D_f^k \). By Theorem 3.6 \( \psi \) is well-defined.

For every \( D \in \mathcal{E}_{k,p+1} \), by Lemma 3.7 there is a unique \( M \in \mathcal{E}_{q+1}(\text{Free}(r)) \) such that \( D = M \cap [g_k^*] \). Let \( h \in Epi_{H_{n+1}}^r(\text{Free}(r), J_{q+1}) \) such that \( \text{Ker}(h) = M \) (see Theorem 3.6) and let \( f \) be the restriction of \( h \) to \( G \). To see that \( f \in \mathcal{F}_{k,p}(q) \) it is enough to prove that \( h(g_k^*) = \frac{q-p}{q} \). If \( h(g_k^*) = \frac{i}{q} \) then \( p + 1 = |\{g_k^*\}/D| = |\frac{i}{q}| = q - i + 1 \). So, \( i = q - p \). On the other hand, \( h(g_k^*) = h(\bigvee_{g \in G_k} g) = \bigvee_{g \in G_k} h(g) = \bigvee_{g \in G_k} f(g) = \frac{q-p}{q} \). Then, it is clear that condition (F2) and (F3) are fulfilled. On the other hand, (F1) holds by Lemma 4.5. Then, \( \psi \) is surjective.

Suppose now that there are \( f_1, f_2 \in \bigcup_{q=p}^{n} \mathcal{F}_{k,p}(q) \) with extensions to \( \text{Free}(r) \) \( h_1 \) and \( h_2 \), respectively, such that \( D = \text{Ker}(h_1) \cap [g_k^*] = \psi(f_1) = \psi(f_2) = \text{Ker}(h_2) \cap [g_k^*] \). By the uniqueness of the d.s. \( M \) associated to \( D \) (Lemma 3.7) we have that \( \text{Ker}(h_1) = \text{Ker}(h_2) \). By Corollary 2.10, \( h_1 = h_2 \) and so \( f_1 = f_2 \). Then, \( \psi \) is injective. Taking into account that if \( q_1 \neq q_2 \) then \( \mathcal{F}_{k,p}(q_1) \cap \mathcal{F}_{k,p}(q_2) = \emptyset \) we conclude that \( |\mathcal{E}_{k,p+1}| = |\sum_{q=p}^{n} \mathcal{F}_{k,p}(q)| = \sum_{q=p}^{n} |\mathcal{F}_{k,p}(q)| \).

\( \square \)

Next we shall introduce some notation.

\[
\epsilon_{d,a} = \begin{cases} 
\sum_{j=0}^{a-1} (-1)^j \binom{a}{j} (a-j)^d & \text{if } 1 \leq a \leq d, \\
0 & \text{if } d < a \text{ or } a \leq 0.
\end{cases}
\] (2)
\[ u(q, t, b) = e_{r-k,q+1-t+b} + e_{r-k,q-t+b} \]  
\[ u(q, t) = \sum_{b=0}^{t} \binom{t}{b} u(q, t, b) \]  
\[ \beta(k, p) = \sum_{q=p}^{n} \sum_{t=1}^{q-p+1} \binom{q-p}{t-1} e_{k,t} \cdot u(q, t) \]  

**Lemma 5.2** If \([g_k^p] \in H_{n+1}^r, m \leq n\), for all \(1 \leq k \leq r, 1 \leq p \leq n\), then the number \(\eta_{k,p+1}\) of all \((p+1)\)-valued i.d.s. of \([g_k^p]\) verifies:

\[
\eta_{k,p+1} = \begin{cases} 
\beta(k, p) & \text{if } p \leq m, \ k < r, \\
0 & \text{if } p > m, k < r, \\
\sum_{q=1}^{r} e_{r,q} & \text{if } k = r,
\end{cases}
\]

**Proof.** By Theorem 5.1 we know \(\eta_{k,p+1} = |E_{k,p+1}| = \sum_{q=p}^{n} |F_{k,p}(q)|\) for all \(1 \leq k \leq r\) and \(1 \leq p \leq n\). On the other hand, it is well-known that if \(W(A, A')\) is the set of all functions from \(A\) into \(A'\) is given by

\[
|W(A, A')| = \sum_{j=0}^{a-1} (-1)^j \binom{a}{j}(a-j)^d
\]

where \(a = |A|\) and \(d = |A'|\).

For \(f \in F_{k,p}(q)\) let \(f_k = f|_{G_k}\) and \(f_{r-k} = f|_{G_r/G_k}\). By (F2) and (F3), \(f_k(G_k)\) verifies that \(\frac{q-p}{q} \in f_k(G_k) \subseteq [0, \frac{q-p}{q}]\). Let \(\mathcal{H} = \{T : T \subseteq [0, \frac{q-p}{q}] \text{ such that } \frac{q-p}{q} \in T\}\). Then

\[ f_k \in W(G_k, T) \text{ for some } T \in \mathcal{H} \]
We want to determine the family of all surjective functions which $f_{r-k}$ belongs to. Let us consider $G_{r-k} = G \setminus G_k$, $T = f_k(G_k)$ and $B = f_{r-k}(G_{r-k}) \cap T$. Then, $f_{r-k}(G_{r-k}) = \begin{cases} V_B = (C_{q+1} \setminus T) \cup B & \text{if } 1 \in f(G_{r-k}), \\ V_B' = (C_{q+1} \setminus (\{1\} \cup T)) \cup B & \text{if } 1 \not\in f(G_{r-k}). \end{cases}$

So, $f_{r-k} \in W(G_{r-k}, V_B) \cup W(G_{r-k}, V_B')$ (8)

for some $B \subseteq T$ and $T \in H$. Let now,

$$\mathcal{U} = \bigcup_{T \in \mathcal{T}} \bigcup_{B \subseteq T} (W(G_k, T) \times (W(G_{r-k}, V_B) \cup W(G_{r-k}, V_B')))$$

and

$$\eta : \mathcal{F}_{k,p}(q) \longrightarrow \mathcal{U} \text{ defined by } \eta(f) = (f_k, f_{r-k}).$$

It is clear that $\eta$ is injective. On the other hand, let $(h_1, h_2) \in \mathcal{U}$ and consider

$$h(g) = \begin{cases} h_1(g) & \text{if } g \in G_k, \\ h_2(g) & \text{if } g \in G_{r-k}. \end{cases}$$

If $h_2 \in W(G_{r-k}, V_B)$ then

$$h(G) = h_1(G_k) \cup h_2(G_{r-k}) = T \cup V_B = C_{q+1}$$

If $h_2 \in W(G_{r-k}, V_B')$ then

$$h(G) = h_1(G_k) \cup h_2(G_{r-k}) = T \cup V_B' = C_{q+1} \setminus \{1\}$$

Besides, if $g \in G_k$, $h(g) \in T \subseteq [0, \frac{q-p}{q}]$ and

$$h(g) \leq \frac{q-p}{q} \text{ for every } g \in G_k.$$  

(11)

Since $h_1(g_p) = \frac{q-p}{q}$ for some $g_p \in G_k$ it is verified that

$$h(g_p) = \frac{q-p}{q} \text{ for some } g_p \in G_k.$$  

(12)
From equations (9), (10), (11) and (12), we have that \( f = h|_G \in \mathcal{F}_{k,p}(q) \) and \( \eta \) is surjective.

Then,

\[
|\mathcal{F}_{k,p}(q)| = |U| = \left| \bigcup_{T \in T} \bigcup_{B \subseteq T} (W(G_k, T) \times (W(G_{r-k}, V_B) \cup W(G_{r-k}, V'_B))) \right| \tag{13}
\]

Observe that if \( \{ A_U \}_{U \in \mathcal{U}} \) is a family of pairwise disjoint sets such that \( \mathcal{U} = \{ U : U \subseteq [0, \frac{m}{q}] \}, |A_U| = |A_U'| \) iff \( |U| = |U'| \). If \( u = |U| \) then

\[
| \bigcup_{U \in \mathcal{U}} A_U | = \sum_{u=0}^{m+1} \left( \binom{m+1}{u} \right) |A_U|
\tag{14}
\]

Then, if \( p \leq m \) and \( k \leq r \)

\[
\eta_{k,p+1} = \sum_{q=p}^{n} |\mathcal{F}_{k,p}(q)|
\]

\[
= \sum_{q=p}^{n} \sum_{l=1}^{q-p+1} \binom{m+1}{u} \cdot W(G_k, T) \cdot |(W(G_{r-k}, V_B) \cup W(G_{r-k}, V'_B))|
\]

\[
= \sum_{q=p}^{n} \sum_{l=1}^{q-p+1} \binom{m+1}{u} \cdot \epsilon_{k,t} \cdot \left( \sum_{b=0}^{t} \binom{t}{b} (e_{r-k,q+1-t+b} + e_{r-k,q-t+b}) \right)
\]

\[
= \sum_{q=p}^{n} \sum_{l=1}^{q-p+1} \binom{m+1}{u} \cdot \epsilon_{k,t} \cdot \left( \sum_{b=0}^{t} \binom{t}{b} u(q, t) \right)
\]

\[
= \beta(k, p)
\]

If \( k = r \) then \( G_{r-k} = \emptyset \) and \( \mathcal{F}_{k,p}(q) = \{ f : G \rightarrow C_{q+1} : f(G) = [0, \frac{q-p}{q}] = [0, \frac{q}{q}] \} \). So, \( p = 1 \) and \( \mathcal{F}_{k,p}(q) = W(G, [\frac{q-1}{q}]) \). Then, \( \eta_{r,2} = \sum_{q=1}^{r} e_{r,q} \). \( \square \)

From Lemma 4.8 and Theorem 4.11 we have

\[
|\text{Free}(r)| \leq \sum_{k=1}^{r} (-1)^{k+1} \binom{r}{k} \prod_{p=1}^{n} (p + 1)^{\eta_{k,p+1}}
\]

where \( \eta_{k,p+1} \) is as in Lemma 5.2

**Example 5.3**
(i) For $n = 1$ and $r = 1$. We have that $\alpha_{1,2} = 1$ and

$$\text{Free}_2(1) \simeq [g] \simeq J_2.$$ 

(ii) For $n = 1$ and $r = 2$, $p = q = 1$ and $G = \{g_1, g_2\}$.

$$\text{Free}_{1+1}(2) \simeq [g_1] \cup [g_2] \text{ where } [g_1] \simeq [g_2] = J^{a_{1,2}}_2 = J_2^2$$

$\alpha_{1,2} = 2$ then $[g_i] \simeq J_2^2$. For $k = 2$, $\alpha_{2,2} = 1$. On the other hand, $[g_1] \cap [g_2] = [g_1 \lor g_2] \simeq J^{a_{2,2}}_2 = J_2$. Then,

$$|\text{Free}_2(2)| = 6$$

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