SOME ELEMENTARY OBSERVATIONS REGARDING REDUCTIVE CARTAN GEOMETRIES

JACOB W. ERICKSON

Abstract. After defining generalizations of the notions of covariant derivatives and geodesics from Riemannian geometry for reductive Cartan geometries in general, various results for reductive Cartan geometries analogous to important elementary results from Riemannian geometry are proven using these generalizations. In particular, a generalization of the Hopf-Rinow theorem is given with a pleasantly concise proof.

1. Introduction

Cartan geometries provide a unified framework for various branches of differential geometry, including Riemannian geometry, affine geometry, and projective geometry. However, despite their inherent utility and charming simplicity, Cartan geometries appear to be largely overlooked.

There are two main aims of this paper. The first is to record a handful of observations the author has made about reductive geometries. In particular, the paper will aim to show various similarities between some elementary properties of Riemannian manifolds and analogous properties for reductive Cartan geometries in general. The second aim of this paper, largely related to the first, is to encourage the exploration of Cartan geometries by demonstrating how general (reductive) Cartan geometries are not that different from the geometries with which many mathematicians are already familiar.

2. Preliminaries

The following is a brief summary of the terms used in this paper. It is not meant as a particularly cogent introduction to the topic. For a detailed introduction, the author cautiously recommends [4].

Definition 2.1. Suppose $M$ is a smooth manifold, $G$ is a Lie group, and $H$ is an embedded Lie subgroup of $G$. Then, a Cartan geometry of type $(G, H)$ on $M$ is a pair $(G, \omega)$, where $G$ is a principal $H$-bundle over $M$ and $\omega$ is a $g$-valued 1-form on $G$ such that

- For every $p \in G$, $\omega_p : T_p G \to g$ is a linear isomorphism;
- For every $h \in H$, $R^*_h \omega = \text{Ad}_{h^{-1}} \omega$;
- For every $Y \in h$, $\omega_p^{-1}(Y)$ is the tangent vector to the curve $t \mapsto p \exp(tY)$ at $t = 0$.

This definition might seem intimidating, but it can actually be rather intuitive. It essentially says that we have a convenient and consistent method of identifying each tangent space of the principal bundle to a model Lie algebra and that this method of identification shares several useful similarities to the Maurer-Cartan
Definition 2.2. The curvature of a Cartan geometry \((\mathcal{G}, \omega)\) of type \((G, H)\) is given by the \(g\)-valued 2-form \(\Omega = d\omega + \frac{1}{2}[\omega, \omega]\). The Cartan geometry \((\mathcal{G}, \omega)\) is said to have constant curvature if and only if, for all \(X, Y \in \mathfrak{g}\), there exists a constant \(Z \in \mathfrak{g}\) depending only on \(X\) and \(Y\) such that \(\Omega(\omega^{-1}(X) \wedge \omega^{-1}(Y)) = Z\).

Remark. Alternatively, we can define \((\mathcal{G}, \omega)\) to have constant curvature if and only if, for all \(X, Y \in \mathfrak{g}\), there exists a \(Z \in \mathfrak{g}\) such that \([\omega^{-1}(X), \omega^{-1}(Y)] = \omega^{-1}(Z)\).

The portion of the curvature of a Cartan geometry of type \((G, H)\) that lies outside of \(\mathfrak{h}\) is of particular interest, since it generalizes the notion of torsion from affine geometry, as we will see in the next section.

Definition 2.3. Let \(q : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}\) be the quotient map from \(\mathfrak{g}\) to \(\mathfrak{g}/\mathfrak{h}\). The torsion of a Cartan geometry with curvature \(\Omega\) is the 2-form given by \(q \circ \Omega\).

As mentioned before, we shall be focusing specifically on reductive Cartan geometries. If \((\mathcal{G}, \omega)\) is a reductive Cartan geometry of type \((G, H)\), then we are able to identify \(T_p(\mathcal{G}/\mathfrak{h}) \approx \mathfrak{g}/\mathfrak{h}\) with a subspace of \(\mathfrak{g}\) complementary to \(\mathfrak{h}\) and invariant under the adjoint action of \(H\). This allows us to specify a “horizontal” direction.

Definition 2.4. A Cartan geometry of type \((G, H)\) on \(M\) is reductive if and only if there is a subspace \(m \subset \mathfrak{g}\) such that \(\mathfrak{g}\) decomposes into \(m \oplus \mathfrak{h}\) as an \(\text{Ad}_H\)-module.\(^2\)

In the reductive case, if we fix a choice of \(m\), then we can always decompose \(\omega = \omega_m + \omega_h\), where \(\omega_m\) and \(\omega_h\) are the projections of \(\omega\) to \(m\) and \(\mathfrak{h}\), respectively.\(^3\)

We can think of \(\omega_m\) as a horizontal projection and \(\omega_h\) as a vertical projection.

We also have a notion of completeness. The intuition behind the definition is the same as the intuition behind the definition of completeness for metric spaces: the geometry is complete if and only if there is nothing “missing.”

Definition 2.5. A Cartan geometry \((\mathcal{G}, \omega)\) of type \((G, H)\) is complete if and only if, for every \(X \in \mathfrak{g}\), the vector field \(\omega^{-1}(X)\) is complete.

In \([3]\), it is stated that this notion of completeness implies completeness in the traditional sense for Riemannian geometries. Later, we shall prove in Theorem \([5, 7]\) both this implication and its converse.

Development is a convenient way to transfer problems on a Cartan geometry to problems on a Lie group.

Definition 2.6. Given a curve \(c : (a, b) \to \mathcal{G}\) on a Cartan geometry \((\mathcal{G}, \omega)\) of type \((G, H)\), a development of \(c\) is a curve \(\tilde{c} : (a, b) \to G\) such that \(c^\ast \omega = \tilde{c}^\ast \omega_{\mathcal{G}}\).

In order to make sure these definitions are clear, we provide the following example.

\(^2\)We shall use the convention that, for reductive Cartan geometries, the complement of the principal Lie subalgebra be denoted by the lowercase Fraktur of the base manifold. That is, given a reductive Cartan geometry of type \((A, B)\) over \(C\), \(a\) will reduce as \(\epsilon \oplus b\). The reasons for this choice should be clear.

\(^3\)We shall always implicitly fix a choice of \(m\).
Example 2.7. Let \( i(n) \) be the Lie algebra of \( I(n) \), the Lie group of isometries of \( n \)-dimensional Euclidean space. Consider the map

\[
\sigma: \mathfrak{o}^+(1, n) \to i(n),
\begin{pmatrix} 0 & v^T \\ v & X \end{pmatrix} \mapsto (v, X).
\]

We can describe the standard Riemannian hyperbolic \( n \)-space as the Cartan geometry \((O^+(1, n), \sigma(\omega_{O^+(1, n)}))\) of type \((I(n), O(n))\).

**Curvature:** The curvature of \((O^+(1, n), \sigma(\omega_{O^+(1, n)}))\) is given by

\[
\Omega = d(\sigma(\omega_{O^+(1, n)})) + \frac{1}{2}[\sigma(\omega_{O^+(1, n)}), \sigma(\omega_{O^+(1, n)})]
= \sigma(d\omega_{O^+(1, n)} + \frac{1}{2}[(\omega_{O^+(1, n)}), \sigma(\omega_{O^+(1, n)})]
= \sigma \left( d\omega_{O^+(1, n)} + \frac{1}{2}(\omega_{O^+(1, n)}), \omega_{O^+(1, n)} \right) - \frac{1}{2} \sigma([\omega_{O^+(1, n)}, \omega_{O^+(1, n)}])
+ \frac{1}{2}[\sigma(\omega_{O^+(1, n)}), \sigma(\omega_{O^+(1, n)})]
= 0 - \frac{1}{2} \sigma([\omega_{O^+(1, n)}, \omega_{O^+(1, n)}]) + \frac{1}{2}[\sigma(\omega_{O^+(1, n)}), \sigma(\omega_{O^+(1, n)})]
= \frac{1}{2} ([\sigma(\omega_{O^+(1, n)}), \sigma(\omega_{O^+(1, n)})] - \sigma([\omega_{O^+(1, n)}, \omega_{O^+(1, n)}])).
\]

This also shows that this Cartan geometry has constant curvature.

**Torsion:** Since the curvature maps into \( \mathfrak{o}(n) \), \((O^+(1, n), \sigma(\omega_{O^+(1, n)}))\) is torsion-free.

**Reductive?** Since \( I(n) \simeq \mathbb{R}^n \rtimes O(n) \), \( i(n) \) reduces as \( \mathbb{R}^n \oplus \mathfrak{o}(n) \) (as an \( \text{Ad}_{O(n)} \) module).

**Complete?** For \( X \in i(n), \sigma(\omega_{O^+(1, n)}))^{-1}(X) = \omega_{O^+(1, n)}^{-1}(\sigma^{-1}(X)) \), so we must have that \((O^+(1, n), \sigma(\omega_{O^+(1, n)}))\) is complete because left-invariant vector fields are complete.

**Development:** Consider the curve \( \gamma: t \mapsto \exp(tX) \) on \( O^+(1, n) \), where we have \( X \in \mathfrak{o}^+(1, n) \). Then, \( \gamma \) develops to the curve \( \tilde{\gamma}: t \mapsto \exp(t\sigma(X)) \) on \( I(n) \).

3. **Examples of reductive geometries**

As stated in the beginning, Riemannian geometries are a specific type of reductive Cartan geometry. Specifically, an \( n \)-dimensional **Riemannian geometry** is a torsion-free Cartan geometry of type \((I(n), O(n))\).

It should be noted, as it is in [4], that a Riemannian geometry on a smooth manifold \( M \) only specifies a Riemannian metric on \( M \) up to scale. With a bit of thought, this should be clear simply from the origin of the structure: a Cartan geometry of type \((I(n), O(n))\) is, in some sense, based on the Klein geometry given by the pair \((I(n), O(n))\), which itself does not induce a specific notion of length on \((I(n))/O(n) \cong \mathbb{R}^n\), though it does determine a notion of length up to scale.

We also have affine geometries, which are just Cartan geometries of type \((\mathbb{R}^n \rtimes GL_n\mathbb{R}, GL_n\mathbb{R})\), or more generally, any \( H \)-structure as a Cartan geometry of type \((\mathbb{R}^n \rtimes H, H)\).

Of course, reductive geometries do not have to have Lie algebras that reduce to \( \mathfrak{m} \supset \mathfrak{h} \). For example, a hyperbolic geometry is a Cartan geometry of type \((O^+(1, n), O(n))\). Since \( O(1,n)/O(1) \times O(n) \) is a symmetric space, in this case we have \([\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h} \).
4. Covariant derivative

Traditionally, the purpose of a covariant derivative is to show change in one vector field on the base manifold along another vector field on the base manifold. However, the structural information explicitly given by a Cartan geometry is in the principal bundle over the base manifold, so in order to generalize the notion of a covariant derivative to arbitrary reductive Cartan geometries, we need to lift the vectors on the base manifold up to the principal bundle.

With vector fields X and Y lifted to the principal bundle as $\tilde{X}$ and $\tilde{Y}$, respectively, change in Y with respect to X, viewed using the perspective of the Lie algebra, can be given by simply applying $\tilde{X}$ as a differential operator to the “horizontal” part of $\tilde{Y}$; for a reductive Cartan geometry $(G, \omega)$ of type $(G, H)$ on $M$, this would just be $\tilde{X} \omega_{\tilde{m}}(\tilde{Y})$. However, when moving along $\tilde{X}$, part of that movement might be “vertical,” so that there is additional change in $\tilde{Y}$ along $\tilde{X}$: if $Z \in \omega^{-1}(h)$ for some $p \in G$, then

$$Z \omega_{\tilde{m}}(\tilde{Y}) = \frac{d}{dt} \bigg|_0 R_{\exp(t\omega(Z))} \omega_{\tilde{m}}(\tilde{Y})$$

$$= \frac{d}{dt} \bigg|_0 \text{Ad}_{\exp(-t\omega(Z))} \omega_{\tilde{m}}(\tilde{Y}) = \text{ad}_{-\omega(Z)} \omega_{\tilde{m}}(\tilde{Y})$$

$$(*)$$

Removing this extra change gives $\tilde{X} \omega_{\tilde{m}}(\tilde{Y}) = [\omega_{\tilde{m}}(\tilde{Y}), \omega_{\tilde{h}}(\tilde{X})]$, which shows the “actual” change in Y along X.

It should be noted that the choice of lifts for X and Y does not change the result.

**Proposition 4.1.** Suppose $(G, \omega)$ is a reductive Cartan geometry of type $(G, H)$ on $M$. Let $X, Y \in \Gamma(TM)$, $\tilde{X}$ and $\tilde{X}$ be lifts of X, and $\tilde{Y}$ and $\tilde{Y}$ be lifts of Y. Then,

$$\tilde{X} \omega_{\tilde{m}}(\tilde{Y}) - [\omega_{\tilde{m}}(\tilde{Y}), \omega_{\tilde{h}}(\tilde{X})] = \tilde{X} \omega_{\tilde{m}}(\tilde{Y}) - [\omega_{\tilde{m}}(\tilde{Y}), \omega_{\tilde{h}}(\tilde{X})].$$

**Proof.** Since $\tilde{Y}$ and $\tilde{Y}$ are both lifts of Y, we know that $\omega_{\tilde{m}}(\tilde{Y}) = \omega_{\tilde{m}}(\tilde{Y})$. Thus, we need only show that

$$(\tilde{X} - \tilde{X}) \omega_{\tilde{m}}(\tilde{Y}) - [\omega_{\tilde{m}}(\tilde{Y}), \omega_{\tilde{h}}(\tilde{X} - \tilde{X})] = 0.$$
In the interest of brevity and uncluttered notation, for a reductive Cartan geometry \((\mathcal{G}, \omega)\) on \(M\) with projection \(\pi : \mathcal{G} \to M\) and \(Z : \mathcal{G} \to \mathfrak{m}\), we shall write \(\nabla_X Z\) for \(\nabla_X \circ \omega^{-1}(Z)\).

**Theorem 4.3.** Suppose \((\mathcal{G}, \omega)\) is a reductive Cartan geometry of type \((G, H)\) on \(M\) with curvature \(\Omega = \Omega_m + \Omega_h\), where \(\Omega_m\) and \(\Omega_h\) are the \(m\)-component and the \(h\)-component, respectively. Let \(X, Y, Z \in \Gamma(TM)\) with lifts \(\hat{X}, \hat{Y}, \hat{Z} \in \Gamma(T\mathcal{G})\), respectively. Then,

\[
\nabla_X Y - \nabla_Y X - \omega_m([\hat{X}, \hat{Y}]) = \Omega_m(\hat{X} \wedge \hat{Y}) - [\omega_m(\hat{X}), \omega_m(\hat{Y})]_m
\]

and

\[
\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = \left[\omega_m(\hat{Z}), \Omega_h(\hat{Y} \wedge \hat{X}) - [\omega_m(\hat{Y}), \omega_m(\hat{X})]_h\right],
\]

where \([\omega_m(\hat{X}), \omega_m(\hat{Y})]_m\) is the \(m\)-component of \([\omega_m(\hat{X}), \omega_m(\hat{Y})]\) and \([\omega_m(\hat{X}), \omega_m(\hat{Y})]_h\) is the \(h\)-component.

**Proof.**

\[
\nabla_X Y - \nabla_Y X = \hat{X} \omega_m(\hat{Y}) - \hat{Y} \omega_m(\hat{X}) + [\omega_m(\hat{X}), \omega_h(\hat{Y})] - [\omega_m(\hat{Y}), \omega_h(\hat{X})]
\]

\[= d \omega_m(\hat{X} \wedge \hat{Y}) + [\omega_m(\hat{X}), \omega_h(\hat{X} \wedge \hat{Y})]
\]

\[= \Omega_m(\hat{X} \wedge \hat{Y}) - [\omega_m(\hat{X}), \omega_m(\hat{Y})]_m + [\omega_m(\hat{X}), \omega_m(\hat{Y})]_h.
\]

For the second part, note that we can choose \([\hat{X}, \hat{Y}]\) to be the lift of \([X, Y]\) because the choice of lift does not change the covariant derivative. Thus,

\[
\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = \nabla_X (\hat{Y} \omega_m(\hat{Z}) - [\omega_m(\hat{Z}), \omega_h(\hat{Y})])
\]

\[- \nabla_Y (\hat{X} \omega_m(\hat{Z}) - [\omega_m(\hat{Z}), \omega_h(\hat{X})])
\]

\[- [X, Y] \omega_m(\hat{Z}) + [\omega_m(\hat{Z}), \omega_m([X, Y])] \]

\[= (\hat{X} \hat{Y} - \hat{Y} \hat{X} - [\hat{X}, \hat{Y}]) \omega_m(\hat{Z})
\]

\[- \hat{X} [\omega_m(\hat{Z}), \omega_h(\hat{Y})] + \hat{Y} [\omega_m(\hat{Z}), \omega_h(\hat{X})]
\]

\[+ [\hat{X} \omega_m(\hat{Z}), \omega_h(\hat{Y})] - [\hat{Y} \omega_m(\hat{Z}), \omega_h(\hat{X})]
\]

\[+ [[\omega_m(\hat{Z}), \omega_h(\hat{Y})], \omega_h(\hat{X})]
\]

\[- [[\omega_m(\hat{Z}), \omega_h(\hat{X})], \omega_h(\hat{Y})]
\]

\[+ [[\omega_m(\hat{Z}), \omega_h(\hat{X})], \omega_h(\hat{Y})]_h]
\]

\[= \omega_m(\hat{Z}) \hat{Y} \omega_h(\hat{X}) - \hat{X} \omega_h(\hat{Y}) - \omega_h([\hat{Y}, \hat{X}])
\]

\[+ [\omega_h(\hat{Y}), \omega_h(\hat{X})]
\]

\[= \omega_m(\hat{Z}) \Omega_h(\hat{Y} \wedge \hat{X}) - [\omega_m(\hat{Y}), \omega_m(\hat{X})]_h.\] □

We also get an obvious notion of parallel transport.

**Definition 4.4.** Suppose \((\mathcal{G}, \omega)\) is a reductive Cartan geometry of type \((G, H)\) on \(M\). Let \(\gamma\) be a curve in \(M\) and let \(v\) be a tangent vector at \(\gamma(t_0)\) for some \(t_0\) in the domain of \(\gamma\). There exists a unique \(X_\gamma\) along \(\gamma\) such that \(X_\gamma(t_0) = v\) and \(\nabla_\gamma X_\gamma = 0\). The **parallel transport** of \(v\) along \(\gamma\) from \(t_0\) to \(t_1\), where \(t_1\) is in the domain of \(\gamma\), is given by \(\|^{t_1}_{t_0}(\gamma)v = X_\gamma(t_1)\).

We omit the proof that there exists a unique vector field with a given initial value parallel along a given curve.
5. Geodesics

With a notion of covariant derivative, we also get a way to define geodesics.

**Definition 5.1.** Suppose $(G, \omega)$ is a reductive Cartan geometry of type $(G, H)$ on $M$. Then, a curve $\gamma$ on $M$ is a geodesic of $(G, \omega)$ if and only if $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$. Equivalently, we could define $\gamma$ to be a geodesic if and only if it lifts to a curve $\hat{\gamma}$ on $G$ satisfying

$$\omega_m(\hat{\gamma})' = [\omega_m(\hat{\gamma}), \omega_h(\hat{\gamma})].$$

Intuitively, this definition says that $\hat{\gamma}$ is the lift of a geodesic if and only if any change in the horizontal velocity of $\hat{\gamma}$ is only there to offset the vertical velocity. This is sensible, since we want geodesics to “not accelerate.”

Since the specific lift does not matter, we get the following.

**Proposition 5.2.** Suppose $(G, \omega)$ is a reductive Cartan geometry on $M$. Then, a curve $\gamma$ on $M$ is a geodesic of $(G, \omega)$ if and only if its horizontal lift $\hat{\gamma}$ to $G$ satisfies $\omega_m(\hat{\gamma})' = 0$.

We also get the following familiar definition.

**Proposition 5.3.** Suppose $(G, \omega)$ is a reductive Cartan geometry on $M$. Then, a curve $\gamma$ on $M$ is a geodesic of $(G, \omega)$ if and only if $\gamma$ lifts to a curve $\hat{\gamma}$ such that $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$. Thus, the result follows by the definition of parallel transport.

**Proof.** By the definition of geodesic, $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$. Thus, the result follows by the definition of parallel transport. □

Using developments of curves, we can also provide an equivalent definition based on the exponential map on $G$.

**Proposition 5.4.** Suppose $(G, \omega)$ is a reductive Cartan geometry of type $(G, H)$ on $M$. Then, a curve $\gamma$ on $M$ is a geodesic of $(G, \omega)$ if and only if $\gamma$ lifts to a curve $\hat{\gamma}$ that develops into a curve $\beta : t \mapsto \exp(tX)$ on $G$ for some $X \in \mathfrak{m}$.

**Proof.** Suppose $\hat{\gamma}$ develops to the curve $\beta : t \mapsto \exp(tX)$. Then, since by definition $\hat{\gamma}^*\omega = \beta^*\omega_G$, our geodesic equation can be rewritten as $\omega_M(\dot{\beta})' = [\omega_M(\dot{\beta}), \omega_H(\dot{\beta})]$, with $\omega_M$ the projection of $\omega_G$ to $\mathfrak{m}$. Since this equation is clearly satisfied by $\beta$, $\hat{\gamma}$ must be the lift of a geodesic.

Conversely, suppose $\gamma$ is a geodesic and $\hat{\gamma}$ is the horizontal lift of $\gamma$. Let $\beta$ be a development of $\hat{\gamma}$ to $G$. Since $\beta$ satisfies $\hat{\gamma}^*\omega = \beta^*\omega_G$ by definition, and $\hat{\gamma}$ is the horizontal lift, we have $\omega_M(\dot{\beta})' = 0$ by Proposition 5.2. Additionally, since $\hat{\gamma}$ is the horizontal lift, we know that $\omega_h(\hat{\gamma}) = \omega_H(\dot{\beta}) = 0$. Thus, $\omega_G(\dot{\beta}) = \omega_M(\dot{\beta}) = X$, for some fixed $X \in \mathfrak{m}$, so $\dot{\beta} = L_{\dot{\beta}}(X)$, implying that $\beta$ is a left translate of $t \mapsto \exp(tX)$. □

Note that Proposition 5.3 is probably equivalent to the definition of geodesics from [1], and shows that geodesics in this case are distinguished curves as described by [1].

**Corollary 5.5.** Suppose $(G, \omega)$ is a reductive Cartan geometry on $M$, with projection map $\pi : G \to M$. Let $\hat{p} \in G$ be a lift of $p \in M$. Then, the curve $t \mapsto \pi(\exp(t\omega^{-1}(X))\hat{p})$ is a geodesic through $p$ for each $X \in \mathfrak{m}$, and every geodesic through $p$ is of this form.

---

4The language in [1] is somewhat ambiguous. He describes geodesics as the “image” of the curves, and further specifies that he is describing “unparametrized” geodesics later in the book. However, the definition as curves is so obvious from what is written that it would be nonsensical to think that he meant something else.
REDUCTIVE CARTAN GEOMETRIES

With Corollary 5.8 in mind, we can prove an analogue of the Hopf-Rinow theorem for all reductive Cartan geometries.

Definition 5.6. A reductive Cartan geometry \((G, \omega)\) of type \((G, H)\) on \(M\) is geodesically complete if and only if every geodesic \(\gamma : (a, b) \to M\) can be extended to a geodesic \(\tilde{\gamma} : [0, 1] \to M\) such that \(\tilde{\gamma}(b) = \gamma(b)\).

Theorem 5.7. Suppose \((G, \omega)\) is a reductive Cartan geometry of type \((G, H)\) on \(M\). Then, \((G, \omega)\) is complete if and only if it is geodesically complete.

Proof. Suppose \((G, \omega)\) is complete. Then, for all \(X \in \mathfrak{g}\), \(\omega^{-1}(X)\) is complete. Thus, by Corollary 5.8, \((G, \omega)\) is geodesically complete.

Conversely, suppose all geodesics of \((G, \omega)\) can be extended to have domain \(\mathbb{R}\). Then, again by Corollary 5.8, \(\omega^{-1}(X)\) is complete for all \(X \in \mathfrak{m}\). By the definition of a Cartan geometry, \(\omega^{-1}(Y)_p = \frac{d}{dt} |_{t=0} R_{\exp(Y)p} \) for all \(Y \in \mathfrak{h}\), so \(\omega^{-1}(Y)\) is complete for all \(Y \in \mathfrak{h}\). Thus, since we can write any \(Z \in \mathfrak{g}\) as \(Z = X + Y\) with \(X \in \mathfrak{m}\) and \(Y \in \mathfrak{h}\), and

\[
\exp(t\omega^{-1}(Z))p = \lim_{n \to \infty} \left( \exp \left( \frac{t}{n} \omega^{-1}(X) \right) \exp \left( \frac{t}{n} \omega^{-1}(Y) \right) \right)^n p
\]

is defined for all \(t \in \mathbb{R}\) and \(p \in G\) by Theorem 2.1 of [3], \((G, \omega)\) is complete. \(\square\)

Corollary 5.8. If \((G, \omega)\) is a (reductive) Cartan geometry and \(G\) is compact\(^6\), then \((G, \omega)\) is geodesically complete.

Remark. The Hopf-Rinow theorem in Riemannian geometry is often associated with connecting points by geodesic segments\(^5\). That is, on a (connected) Riemannian manifold, geodesic completeness implies that any two points on the manifold can be connected by a geodesic. This is not true on general reductive Cartan geometries. For example, for any Lie group \(H\), the Cartan geometry \((\text{SL}_2 \mathbb{R} \times H, \omega_{\text{SL}_2 \mathbb{R} \times H})\) of type \((\text{SL}_2 \mathbb{R} \times H, H)\) over \(\text{SL}_2 \mathbb{R}\) is reductive, but the usual exponential map is not surjective on \(\text{SL}_2 \mathbb{R}\). Thus, the point given by the coset of \([-1, 1, 1, 1]\), for example, will not be connected to the point given by the coset of the identity by a geodesic.

Additionally, note that Corollary 5.8 applies when \(G\) is compact, but not necessarily when the underlying manifold is compact. For example, since \(O(n)\) is compact, compact Riemannian manifolds must be geodesically complete, but since \(O^+(1, n)\) is not compact, compact Lorentzian manifolds need not be geodesically complete, as is the case for the Clifton-Pohl torus.

We can also give an analogue of Beltrami’s theorem.

Definition 5.9. Given a reductive Cartan geometry of type \((G, H)\) on \(M\) and a reductive Cartan geometry of type \((Q, H)\) over \(N\), a geodesic map from \(M\) to \(N\) is a diffeomorphism \(\varphi : M \to N\) such that, if \(\gamma\) is a geodesic on \(M\), then \(\varphi(\gamma)\) is a geodesic on \(N\), and if \(\beta\) is a geodesic on \(N\), then \(\varphi^{-1}(\beta)\) is a geodesic on \(M\).

Theorem 5.10. Suppose \((G, \omega)\) is a reductive Cartan geometry of type \((G, H)\) on \(M\) and \((Q, \eta)\) is a reductive Cartan geometry of type \((Q, H)\) on \(N\). If there exists a geodesic map \(\varphi : M \to N\) and \((G, \omega)\) has constant curvature, then so does \((Q, \eta)\).

Proof. To simplify, we shall identify \(M = G/H\) and \(N = Q/H\). Suppose that \(\varphi : M \to N\) is a geodesic map and \(\tilde{\varphi} : G \to Q\) is a lift of \(\varphi\). Further, let \(p \in \mathcal{G}\) and \(X \in \mathfrak{m}\). Then, by definition,

\[
\varphi(\exp(\omega^{-1}(X))pH) = \exp(\tilde{\varphi} \cdot \omega^{-1}(X)) \varphi(pH) = \exp(\eta^{-1}(\phi(X))) \varphi(pH),
\]

\(^5\)Note that all compact Cartan geometries (of type \((G, H)\)) must be reductive since compactness of the bundle implies compactness of \(H\).

\(^6\)For example, it is included as part of the theorem statement in [2].
where \( \phi : m \to n \) is a linear isomorphism determined by the above relation. From this, we get that \( \eta = \phi \circ \omega \circ \hat{\phi}^{-1} = \phi(\hat{\phi}^{-1} \omega) \). Thus,
\[
d\eta + \frac{1}{2} [\eta, \eta] = \hat{\phi}^{-1*} \left( \phi(d\omega) + \frac{1}{2} [\phi(\omega), \phi(\omega)] \right) \\
= \hat{\phi}^{-1*} \left( \phi(\Omega) + \frac{1}{2} ([\phi(\omega), \phi(\omega)] - \phi([\omega, \omega])) \right).
\]
Thus, if \((G, \omega)\) has constant curvature, then so must \((Q, \eta)\). \( \square \)

In order to demonstrate a couple of these theorems, let us return to the previous example.

**Example 5.11.** Consider again the hyperbolic \( n \)-space from Example 2.7 determined by \((O^+(1, n), \sigma(\omega_{O^+(1, n)}))\). Then, by using any of the various methods given above, we see that the geodesics of \((O^+(1, n), \sigma(\omega_{O^+(1, n)}))\) are of the form
\[
t \mapsto p \exp \left( t \begin{bmatrix} 0 & v^T \\ v & 0 \end{bmatrix} \right) O(n) = \begin{bmatrix} \cosh(t \sqrt{v^T v}) & \sinh(t \sqrt{v^T v}) v^T \\ \sinh(t \sqrt{v^T v}) v & \cosh(t \sqrt{v^T v}) - 1 \end{bmatrix} O(n)
\]
with \( p \in O^+(1, n) \).

Since this is defined for all \( t \in \mathbb{R} \), the geometry is geodesically complete. As shown above, this Cartan geometry is complete as well, demonstrating Theorem 5.7.

Since the Cartan geometry \((O^+(1, n), \omega_{O^+(1, n)})\) of type \((O^+(1, n), O(n))\) over \( O^+(1, n)/O(n) \) has the same geodesics as \((O^+(1, n), \sigma(\omega_{O^+(1, n)}))\), the identity map is a geodesic map between these two. Since the former is a Klein geometry, it is flat, so it must have constant curvature. As shown above, the latter also has constant curvature, demonstrating Theorem 5.10.

### 6. Further Exploration

Jacobi fields for general reductive Cartan geometries can still be defined, though without a notion of distance they lack the usefulness that they have in Riemannian geometry. The relevant equations are also more difficult: for a reductive Cartan geometry of type \((G, H)\) over \( M \), the usual equation from Riemannian geometry to define a Jacobi field \( Z \) along a geodesic \( \gamma \) becomes
\[
\nabla_\gamma \nabla_\gamma Z + [\omega_m(\hat{\gamma}), \Omega_h(\hat{\gamma} \wedge \hat{Z})] = \hat{\gamma} [\omega_m(\hat{Z}), \omega_m(\hat{\gamma})]_m - [\omega_m(\hat{Z}), \omega_m(\hat{\gamma})]_{\omega h(\hat{\gamma})} - [\omega_m(\hat{\gamma}), [\omega_m(\hat{Z}), \omega_m(\hat{\gamma})]]_b.
\]
Clearly, this equation is unwieldy. On the other hand, if we can concoct a suitable generalization or analogue of the \( \text{Ad}_{O(n)} \)-invariant inner product in the Riemannian case, analogues of various comparison theorems might arise.

### References

1. A. Čap and J. Slovák, *Parabolic Geometries I: Background and General Theory*, Mathematical Surveys and Monographs, vol. 154, American Mathematical Society, 2009.
2. J. Cheeger and D. G. Ebin, *Comparison Theorems in Riemannian Geometry*, American Mathematical Society, 1975.
3. A. J. Chorin, T. J. R. Hughes, M. F. McCracken, and J. E. Marsden, *Product Formulas and Numerical Algorithms*, Communications on Pure and Applied Mathematics **31** (1978), 209-210.
4. R. W. Sharpe, *Differential Geometry: Cartan’s Generalization of Klein’s Erlangen Program*, Graduate Texts in Mathematics, vol. 166, Springer-Verlag New York, 1997.