Bayesian Inference of Proportional Hazard Rate Model under Progressively Type-II Censored Sample

Min Hu¹, Haiping Ren²*

¹ School of Materials and Electromechanics, Jiangxi Science and Technology Normal University, Nanchang, Jiangxi, 330038, China
² Teaching Department of Basic Subjects, Jiangxi University of Science and Technology, Nanchang, Jiangxi, 330013, China
*Corresponding author’s e-mail: chinarhp@163.com

Abstract. Statistical inference of the parameter of the proportional hazard rate model is studied based on progressive type-II censored sample. Bayesian and empirical Bayesian estimators are first obtained under a scaled squared error loss function. Then we derived a class of Bayesian shrinkage estimators motivated by the idea of Thompson’s shrinkage algorithm based on obtained estimators. Finally, a practical example and Monte Carlo simulations illustrate that the proposed Bayesian shrinkage estimators are more robust comparing to the Bayesian and empirical Bayesian estimators.

1. Introduction

As an important distributed model in the field of survival analysis, reliability test and quality control, the application and its related statistical inference of the proportional hazard rate model have attracted many scholars’ attention [1, 2]. Ren et al. [3] studied the minimax estimation of parameter of the proportional hazard rate model. They derived Bayesian estimators under squared logarithmic error and MLINEX loss functions using Jeffery’s non-information prior distribution, and they proved these obtained estimators are also minimax estimators. Ren [4] derived the Bayesian estimator of the parameter of proportional hazard rate model and then discussed the admissibility and non-admissibility of a class of linear form estimators. Wang and Shi [5] discussed the empirical Bayesian estimators of the unknown parameter and reliability of the proportional hazard rate model when the loss functions are squared error loss and linear-exponential (LINEX) loss under progressively type-II censored test. Wang et al. [6] studied the estimation of the parameters of proportional hazard rate model under progressively Type II censored test. They introduced the inverse estimation method and derived the point estimators and obtained exact confidence intervals for the parameters. Kızılaslan [7] studied the maximum likelihood, Bayesian, E-Bayesian, and hierarchical Bayesian estimation of the unknown parameter and reliability function of the proportional hazard rate model under the squared error and linear-exponential loss (LINEX) functions. Salehi et al. [8] compared record ranked set sampling with the record values sampling scheme by mean squared error and Pitman measure of closeness criteria when the proportional hazard rate model is considered as a lifetime distribution. Ahmadi et al. [9] derived a class of preliminary test estimators and shrinkage estimators for a class of proportional hazard rate model on the basis of generalized order statistics. Kızılaslan [10] studied the reliability estimation of the proportional hazard rate system by using frequentist and Bayesian methods, respectively.
In parameter estimation, merging prior knowledge about unknown parameters into parameter estimation will improve the original estimation, such as shrinkage estimation. Thompson [11] proposed the following shrinkage estimation for estimating population mean parameters:

\[ \theta_T = k\theta + (1-k)\theta_0, \quad 0 \leq k \leq 1 \]  

(1)

Here \( \theta_0 \) is the parameter prior value, \( \hat{\theta}_T \) is called Thompson-type estimation.

Let \( X \) is a random variable distributed with the proportional hazard rate model with parameter \( \theta \).

The corresponding probability density function and distribution function are respectively:

\[ f(x;\theta) = \theta^{-1}g(x)[G(x)]^{\theta-1}, \quad -\infty < c < x < d \leq \infty \]  

(2)

and

\[ F(x;\theta) = 1 - [G(x)]^{\theta-1}, \quad -\infty < c < x < d \leq \infty \]  

(3)

Here, \( c \) and \( d \) are known constants. Function \( G(x) \) is monotone decreasing and differentiable, and it satisfies \( g(x) = -G'(x) > 0 \), \( G(c) = 1 \) and \( G(d) = 0 \).

Many common life distribution models are special cases of the proportional hazard rate model, such as: exponential distribution \( G(x) = e^{-x} \), Pareto distribution \( G(x) = x^{-1} \).

This paper will study Bayesian shrinkage estimation of the parameters of the proportional hazard rate model under the scaled squared error loss function based on progressively type II censored samples.

2. Preliminary knowledge

2.1 Progressively Type-II Censored Test

With the progress of science and technology, products with long life and high reliability have become the mainstream products in today’s society. From the angle of saving time and cost, traditional life test needs to be replaced by new truncated life test. Gradually increasing type II truncated life test has become a kind of widely concerned and studied in recent ten years because it is more time-saving and cost-saving than traditional truncated life test. The truncation life test is investigated. Progressively type II censored life test is as follows:

The experimenter first places \( n \) units (individual) on test. When he/she observes the first failure at the time \( X_{1:n} \), then he/she randomly selects \( r_1 \) surviving unites and removes them. When he/she observes the second failure at the time \( X_{2:n} \), then he/she randomly selects \( r_2 \) surviving unites and removes them. He/she does not terminate the experiment and works with the similar rules until he/she observes the \( m \)-th failure unit at the time \( X_{m:n} \). Finally \( r_m = n-m-r_1-\cdots-r_{m-1} \) of surviving units are all removed.

Assuming that the product life is a random variable of the proportional hazard rate model, the joint probability density function of \( X = (X_{1:n},X_{2:n},\cdots,X_{m:n}) \) is (Balakrishnan and Aggrwala [14])

\[ f(x_1,x_2,\cdots,x_m) = c \prod_{i=1}^{m} f(x_i;\theta)[1-F(x_i;\theta)]^{c} \]  

(4)

Here \( x_i \equiv x_{i:n} \) is the observation of \( X_{i:n} \) \( (i=1,2,\cdots,m) \), \( c = n( n-r_1-\cdots-r_{m-1}-m+1) \)

and \( r_1,r_2,\cdots,r_m \) denote the removed numbers of units from the test.

Substitute (2) and (3) into (4), then the likelihood function is

\[ L(\theta) = c \prod_{i=1}^{m} \frac{g(x_i)}{G(x_i)} \cdot \theta^{-n} \exp(-t/\theta) \]  

(5)

Where \( t = -\sum_{i=1}^{m}(1+r_i) \ln G(x_i) \).
Then by solving the log-likelihood equation, and the MLE of $\theta$ is solved as
\[
\hat{\theta}_{MLE} = \frac{T}{m},
\]
where $T = \sum_{i=1}^{m} (1 + r_i) \ln X_{i,m,n}$.

Let $Y_i = \theta^{-1} X_{i,m,n}^2$, $i = 1, \ldots, m$, then one can easily prove that $Y_1 < \cdots < Y_m$ is a progressively type II censored sample from the standard exponential distribution $\text{Exp}(1)$.

Let
\[
\begin{align*}
Z_1 &= n Y_1 \\
Z_2 &= (n - r_1)(Y_2 - Y_1) \\
\cdots & \cdots \\
Z_m &= (n - r_1 - \cdots - r_{m-1} - m + 1)(Y_m - Y_{m-1})
\end{align*}
\]

It is easily to prove that $Z_1, Z_2, \ldots, Z_m$ are all independent and identically distributed as $\text{Exp}(1)$. Then
\[
2 \sum_{i=1}^{m} Z_i \sim \chi^2(2m). 
\]

We can see that $T$ is distributed with $\Gamma(m, \theta^{-1})$, and thus we have
\[
ET = m\theta. 
\]

2.2 Prior Distribution and Loss Function
In this paper, we always suppose that the prior distribution of $\theta$ is inverse Gamma distribution, $\text{IG}(\alpha, \beta)$, i.e.
\[
\pi(\theta; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{-\alpha-1} \exp(-\beta / \theta), \quad \theta > 0 
\]
where $\alpha > 0$ and $\beta > 0$.

To study Bayes estimation of a parameter, this article will use the following scaled squared error loss function
\[
L(\tilde{\theta}, \theta) = \frac{(\tilde{\theta} - \theta)^2}{\theta^d} 
\]

Under the loss (10), the unique Bayesian estimator of parameter $\theta$ is
\[
\hat{\theta}_B = \frac{E(\theta^{d-l} | X)}{E(\theta^d | X)} 
\]

3. Bayesian and Bayesian Shrinkage Estimation

3.1 Bayesian Estimation

\textbf{Theorem 1.} Let $X = (X_{1,m,n}, X_{2,m,n}, \ldots, X_{m,m,n})$ be as progressively type-II censored sample coming from proportional hazard rate model (2), $r_1, r_2, \ldots, r_m$ denote the removed numbers of units from the test. Suppose that the prior distribution of $\theta$ is inverse Gamma distribution, $\text{IG}(\alpha, \beta)$, then under the scaled squared error loss function the Bayesian estimator of $\theta$ is
\[
\hat{\theta}_B = \frac{\beta + T}{m + \alpha + l - 1} 
\]

\textbf{Proof.} Let. The posterior probability density function can be obtained as follows:
\[ h(\theta \mid x) \propto L(\theta) \cdot \pi(\theta; \alpha, \beta) \propto \theta^{m+i-1} \exp\left(-\frac{(\beta + t)}{\theta}\right), \]

where \( t = \sum_{i=1}^{n} (1 + r_i) x_i^{2}_{i,m,n}. \)

The posterior distribution of \( \theta \) is also a Gamma distribution \( \text{IG}(m + \alpha, \beta + t) \). Then

\[
\begin{align*}
E(\theta^{-1} \mid x) &= \int_{0}^{\beta} \theta^{-1} h(\theta \mid x) d\theta = \frac{\Gamma(m + \alpha + l - 1)}{\Gamma(m + \alpha)} \cdot \frac{\Gamma(m + \alpha + l - 1)}{(\beta + t)^{m+i-1}} \\
E(\theta^{-1} \mid x) &= \int_{0}^{\beta} \theta^{-1} h(\theta \mid x) d\theta = \frac{\Gamma(m + \alpha + l)}{\Gamma(m + \alpha)} \cdot \frac{\Gamma(m + \alpha + l)}{(\beta + t)^{m+i-1}}
\end{align*}
\]

According to equation (12), under the scaled squared error loss function the Bayesian estimator of \( \theta \) is

\[
\hat{\theta}_B = \frac{E(\theta^{-1} \mid X)}{E(\theta^{-1} \mid X)} = \frac{\beta + T}{m + \alpha + l - 1}.
\]

3.2 Bayesian shrinkage Estimation

Assume that the prior estimate of the known parameter \( \theta \) is \( \theta_0 \) based on the existing engineering experience, and then we adopt the following methods to determine the hyper-parametric values in the prior distribution:

Calculate the values of prior hyper-parameters \( \alpha \) and \( \beta \) such that

\[
E(\hat{\theta}_B) = \theta_0 \quad (13)
\]

Substitute (12) into (13), we have

\[
E(\theta) = (m + \alpha + l - 1)\theta_0 - \beta = m\theta_0
\]

Then

\[
\beta = (\alpha + l - 1)\theta_0 \quad (14)
\]

Substitute the value of \( \beta \) into (13), we can get

\[
\hat{\theta}_{SB} = \frac{mT}{m + \alpha + l - 1} + \frac{\alpha + l - 1}{m + \alpha + l - 1} \theta_0
\]

where \( k = \frac{m}{m + \alpha + l - 1} \) and \( \bar{T} = \frac{1}{m} \sum_{i=1}^{m} (1 + r_i) \ln X_{i,m,n}. \)

The estimator \( \hat{\theta}_{SB} \) has a form of the shrinkage estimator (1), and here we call it the Bayesian shrinkage estimator.

4. Applied example

In this section, a practical example adopted from Nelson [15] is used to compare the performance of MLE, Bayesian and Bayesian shrinkage estimators. The example is a an experiment testing the voltage withstanding strength of an electronic insulating fluid at 34 KV voltage and it is measured the breakdown time of 19 samples of the insulating fluid. Balakrishnan and Lin [16] used this data set to produce a progressively increasing type II truncated sample. The data set is shown in Table 1.
Table 1. Progressively type-II censored sample

| $i$ | $x_i$ | $r_i$ |
|-----|-------|-------|
| 1   | 0.19  | 0     |
| 2   | 0.78  | 0     |
| 3   | 0.96  | 0     |
| 4   | 1.31  | 3     |
| 5   | 2.78  | 0     |
| 6   | 4.85  | 3     |
| 7   | 6.50  | 0     |
| 8   | 7.35  | 5     |

Balakrishnan and Lin [16] has proved that it is appropriate to fit the time distribution of voltage withstanding strength of Nelson experimental electronic insulating fluid with single parameter exponential distribution (i.e. the probability density function is $f(x; \theta) = \theta^{-1} \exp(-x / \theta)$, $x > 0, \theta > 0$) by goodness-of-fit test. This is a concrete example of the ratio hazard rate model. The maximum likelihood estimation of parameter $\theta$ is $T = \sum_{i=1}^{m}(1 + r_i)x_i = 72.69$, and the Bayesian and Bayesian shrinkage estimates of parameter are shown in Table 2.

Table 2. Bayesian and Bayesian shrinkage estimates of parameters under different prior distributions

| $\alpha$ | $\hat{\theta}_{H_1}$ ($\beta = 2$) | $\hat{\theta}_{SB_1}$ ($\theta_0 = 9.0$) | $\hat{\theta}_{SB_1}$ ($\theta_0 = 9.5$) | $\hat{\theta}_{H_2}$ ($\beta = 2$) | $\hat{\theta}_{SB_2}$ ($\theta_0 = 9.0$) | $\hat{\theta}_{SB_2}$ ($\theta_0 = 9.5$) |
|----------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1.1      | 9.2210          | 9.0852          | 9.0914          | 7.3950          | 9.0683          | 9.1723          |
| 1.5      | 8.7871          | 9.0812          | 9.1106          | 7.1133          | 9.0657          | 9.1848          |
| 2.0      | 8.2989          | 9.0767          | 9.1322          | 6.7900          | 9.0627          | 9.1991          |
| 2.5      | 7.8621          | 9.0726          | 9.1516          | 6.4948          | 9.0600          | 9.2122          |

From Table 2 and a large number of numerical simulation experiments, we draw the following conclusions:

(i) The Bayesian shrinkage estimation is affected by two hyperparameters, but Bayesian shrinkage estimation is influenced by only one hyperparameter. If the parameter prior value is close to the true value in practice, Bayesian shrinkage estimation is more robust than Bayesian estimation, so Bayesian shrinkage estimation method is recommended for parameter estimation.

(ii) When the sample size $n$ is large and $n-m$ is small, all kinds of estimates are close to the real value.

5. Conclusions

In this paper, we discuss a new type of censored life test, i.e. progressively type-II censored life test. The maximum likelihood estimates of unknown parameters are obtained under the assumption that the life distribution obeys the proportional hazard rate model, and Bayesian estimators and Bayesian shrinkage estimators of parameters are obtained under the scaled squared error loss function. Ordinary squared error loss function and scale invariant squared error loss function are both special cases of this loss function. The results obtained in this paper can be directly applied some detail distribution, such as Rayleigh distribution, Weibull distribution and Burr Type XII distribution.

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References

[1] Crescenzo, A. D. (2000) Some results on the proportional reversed hazards model. Statistics and Probability Letters, 50: 313-321.

[2] Burkschat, M., Torrado, N. (2014) On the reversed hazard rate of sequential order statistics. Statistics & Probability Letters, 85: 106-113.

[3] Ren H. P., Yang, L. W., Liao L. (2009) Minimax estimation of parameter of a class distributions under the squared log error and MLINEX loss functions. Journal of Jiangxi Normal University (Natural Science Edition). 33: 326-330.
[4] Ren, H. P. (2010) Admissibility of estimation for parameter of a general class of distributions under entropy loss function. Journal of Northwest Normal University (Natural Science), 46: 19-22.

[5] Wang L., Shi Y. M. (2011) Reliability analysis of proportional hazard rate model under progressively type-II censored samples. Journal of Applied Statistics & Management, 30: 315-321.

[6] Wang B. X., Yu K., Jones M. C. (2010) Inference under progressively type II right-censored sampling for certain lifetime distributions. Technometrics, 2: 453-460.

[7] Kızılaslan, F. (2017) The E-Bayesian and hierarchical Bayesian estimations for the proportional reversed hazard rate model based on record values. Journal of Statistical Computation & Simulation, 87: 2253-2273.

[8] Salehi M., Ahmadi J., Dey S. (2016) Comparison of two sampling schemes for generating record-breaking data from the proportional hazard rate models. Communications in Statistics-Theory and Methods, 45: 3721-3733.

[9] Ahmadi, J., Mirfarah, E., Parsian, A. (2017) Comparison of preliminary test estimators based on generalized order statistics from proportional hazard family using Pitman measure of closeness. Communications in Statistics, 46: 3200-3216.

[10] Kızılaslan, F. (2017) Classical and Bayesian estimation of reliability in a multicomponent stress-strength model based on the proportional reversed hazard rate mode. Mathematics & Computers in Simulation, 136: 36-62.

[11] Thompson, J. R. (1968) Some shrinkage techniques for estimating the mean. Journal of American Statistical Association, 63: 113-122.

[12] Xiao, X., Xie, M. (2016) A shrinkage approach for failure rate estimation of rare events. Quality and Reliability Engineering International, 32: 123-132.

[13] Prakash, G., Singh, D. C. (2010) Bayesian shrinkage estimation in a class of life testing distribution,” Data Science Journal, 8: 243-258.

[14] Balakrishnan, N., Aggarwala, R. (2000) Progressive Censoring: Theory, Method and Applications. Birkhauser Publishers, Boston.

[15] Nelson, W. (1982) Applied Life Data Analysis. John Wiley, New York.

[16] Balakrishnan, N., Lin, C. T. On the distribution of a test for exponentiality based on progressively type-II right censored spacings. Journal of Statistical Computation and Simulation, 73: 277-283.