A short proof of some recent results related to Cesàro function spaces

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Abstract

We give a short proof of the recent results that, for every $1 \leq p < \infty$, the Cesàro function space $Ces_p(I)$ is not a dual space, has the weak Banach-Saks property and does not have the Radon-Nikodym property.

The main purpose of this paper is to give a short proof of the following recent results related to the Cesàro function spaces: for every $1 \leq p < \infty$ the space $Ces_p = Ces_p(I)$ has the weak Banach-Saks property [2, Theorem 8], does not have the Radon-Nikodym property and it is not a dual space [10, Corollaries 5.1 and 5.5].

The Cesàro function spaces $Ces_p = Ces_p(I)$ ($1 \leq p < \infty$), where $I = [0, 1]$ or $I = [0, \infty)$, are the classes of all Lebesgue measurable real functions $f$ on $I$ such that

$$\|f\|_{C(p)} = \left[ \int_I \left( \frac{1}{x} \int_0^x |f(t)| \, dt \right)^p \, dx \right]^{1/p} < \infty.$$ 

A Banach space $X$ is said to have the weak Banach-Saks property if every weakly null sequence in $X$, say $(x_n)$, contains a subsequence $(x_{n_k})$ whose first arithmetical means converge strongly to zero, that is, $\lim_{m \to \infty} \frac{1}{m} \left\| \sum_{k=1}^m x_{n_k} \right\|_X = 0$.

It is known that uniformly convex spaces, $c_0$, $l^1$ and $L^1$ have the weak Banach-Saks property, whereas $C[0, 1]$ and $l^\infty$ do not have. We should mention that the result on $L^1$ space, proved by Szlenk [16] in 1965, was a very important break-through in studying of the weak Banach-Saks property.

In 1982, Rakov [15, Theorem 1] proved that a Banach space with non-trivial type (or equivalently B-convex) has the weak Banach-Saks property (cf. also Tokarev [17, Theorem 1]). Recently Dodds-Semenov-Sukochev [8] investigated the weak Banach-Saks property of rearrangement invariant spaces and Astashkin-Sukochev [3] have got a complete description of Marcinkiewicz spaces with the latter property.

The spaces $Ces_p[0, 1]$ for $1 \leq p < \infty$ are neither B-convex (they have trivial type) nor rearrangement invariant. Nevertheless, by studying the dual space $Ces_p[0, 1]^*$, Astashkin-Maligranda [2, Theorem 8] proved that these spaces have the weak Banach-Saks property. Here, we present another simpler proof of this result which does not use any knowledge of the structure of the latter dual space.

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Theorem 1. For every $1 \leq p < \infty$ the Cesàro function space $Ces_p(I)$ has the weak Banach-Saks property.

The proof will be based on the following simple observation. Recall that the space with mixed norm $L^p(I)[L^1[0, 1]]$ consists of all classes of Lebesgue measurable functions on $I \times [0, 1]$ $x(s, t)$ such that for a.e. $s \in I$ the function $x(s, \cdot) \in L^1[0, 1]$ and the function $\|x(s, \cdot)\|_{L^1[0, 1]} \in L^p(I)$ with the norm $\|x\|_{L^p(I)[L^1[0, 1]]} = \|\|x(s, \cdot)\|_{L^1[0, 1]}\|_{L^p(I)}$ (see, for example, [11, § 11.1, p. 400]).

Lemma 2. For every $1 \leq p < \infty$ the space $Ces_p(I)$ is isometric to a closed subspace of the mixed norm space $L^p(I)[L^1[0, 1]]$.

Proof. In fact, the mapping $f(t) \mapsto Sf(x, t) = f(xt)$ is such an isometry from $Ces_p(I)$ into $L^p(I)[L^1[0, 1]]$ since

$$\|f\|_{C(p)} = \left\|\frac{1}{x} \int_0^x |f(t)| \, dt\right\|_{L^p(I)} = \left\|\int_0^1 |f(tx)| \, dt\right\|_{L^p(I)} = \|Sf(x, \cdot)\|_{L^1[0, 1]}\|_{L^p(I)}.$$

Proof of Theorem 1. Firstly, we note that the Bochner vector-valued Banach space $L^p(I, L^1[0, 1])$ coincides with the mixed norm space $L^p(I)[L^1[0, 1]]$ (see [9, Theorem 1.1], [5, Theorem 2.2]; cf. also [13, pp. 282-283]). Moreover, by the Szlenk theorem [10], the space $L^1(I)[L^1[0, 1]] = L^1(I \times [0, 1])$ has the weak Banach-Saks property. Therefore, applying the Cembranos theorem [6, Theorem C] (see also [12, pp. 295-302]), we see that the same is true also for the space $L^p(I)[L^1[0, 1]]$. Since, due to Lemma 2, the Cesàro function space $Ces_p(I)$ is isometric to a closed subspace of $L^p(I)[L^1[0, 1]]$ and any closed subspace inherits the weak Banach-Saks property, then $Ces_p(I)$ has this property as well. The proof is complete.

The following results were proved in [10] (see Corollaries 5.1 and 5.5) by using an isometric representation of the dual space of $Ces_p(I)$, $1 \leq p < \infty$. Here, we show that they are rather simple consequences of well-known classical theorems.

Theorem 3. Let $1 \leq p < \infty$. Then

(a) $Ces_p(I)$ is not a dual space;

(b) $Ces_p(I)$ does not have the Radon-Nikodym property.

Firstly, we prove the following auxiliary statement.

Lemma 4. For every $1 \leq p < \infty$ there is a norm $\| \cdot \|_{C(p)}$ equivalent to the usual norm in $Ces_p(I)$ such that the space $(Ces_p(I), \| \cdot \|_{C(p)})$ contains a closed subspace isometric to the space $L^1[0, 1]$.

Proof. For arbitrary $f \in Ces_p := Ces_p[0, 1]$ we set

$$\|f\|_{C(p)} := \| f \cdot \chi_{[0,1/4]} \|_{C(p)} + \| f \cdot \chi_{(1/4,3/4]} \|_{L^1[0, 1]}.$$
Since

\[ \| f \cdot \chi_{(1/4,3/4)} \|_{C(p)}^p = \int_{1/4}^{3/4} \left( \frac{1}{x} \int_{1/4}^{x} |f(s)| \, ds \right)^p \, dx + \int_{3/4}^{1} \left( \frac{1}{x} \int_{1/4}^{3/4} |f(s)| \, ds \right)^p \, dx \]

\[ \leq \left( \frac{1}{2} 4^{p} + \frac{1}{4} \left( \frac{4}{3} \right)^p \right) \left( \int_{1/4}^{3/4} |f(s)| \, ds \right)^p \leq 4^p \| f \cdot \chi_{(1/4,3/4)} \|_{L^1}^p, \]

we have

\[ \| f \|_{C(p)} \leq 4 \| f \|_{C(p)}^* \quad (f \in Ces_p). \]

Conversely,

\[ \| f \cdot \chi_{(1/4,3/4)} \|_{C(p)}^p \geq \int_{3/4}^{1} \left( \frac{1}{x} \int_{1/4}^{3/4} |f(s)| \, ds \right)^p \, dx \geq \frac{1}{4} \| f \cdot \chi_{(1/4,3/4)} \|_{L^1}^p, \]

whence for every \( f \in Ces_p \)

\[ \| f \|_{C(p)} \leq \| f \|_{C(p)} + \| f \cdot \chi_{(1/4,3/4)} \|_{L^1} \leq 5 \| f \|_{C(p)}. \]

Therefore, the norms \( \| \cdot \|_{C(p)} \) and \( \| \cdot \|_{C(p)}^* \) are equivalent on \( Ces_p := Ces_p[0, 1] \). Since the mapping

\[ f(t) \mapsto \mathcal{H}f(t) := \begin{cases} 2f(2t - 1/2) & \text{if } 1/4 < t < 3/4, \\ 0 & \text{if } 0 \leq t \leq 1/4 \text{ or } 3/4 \leq t \leq 1 \end{cases} \]

is a linear isometry from \( L^1[0,1] \) onto the subspace of \( (Ces_p[0,1], \| \cdot \|_{C(p)}^*) \) consisting of all functions with support from the interval \((1/4,3/4)\), we obtain the result in the case \( I = [0,1] \). If \( I = [0, \infty) \) the proof follows in the same way.

**Proof of Theorem** Assume that \( (a) \) is not valid and \( Ces_p \) is a dual space. Since this property is invariant with respect isomorphisms, we see that \( (Ces_p(I), \| \cdot \|_{C(p)}^*) \), where \( \| \cdot \|_{C(p)}^* \) is the norm from Lemma \( \[4 \] \) is also a dual space. Then, since it is separable, by the classical Bessaga-Pelczyński result \( \[4 \] \), the space \( (Ces_p(I), \| \cdot \|_{C(p)}^*) \) has the Krein-Milman property (i.e., every closed bounded set in this space is the closed convex hull of its extreme points). Since \( (Ces_p(I), \| \cdot \|_{C(p)}^*) \) contains a closed subspace isometric to the space \( L^1[0,1] \), the latter contradicts to the fact that the closed unit ball in \( L^1[0,1] \) has no extreme points. Therefore, \( (a) \) is proved.

It is well-known that every Banach space which has the Radon-Nikodým property possesses also the Krein-Milman property \( \[7 \] \) (see also \( \[10 \] \) p. 118, \( \[14 \] \) p. 229 and the references given there). Thus, we obtain \( (b) \), and the proof is complete.

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