Universal holographic hydrodynamics at finite coupling

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We consider thermal plasmas in a large class of superconformal gauge theories described by a holographic dual geometry of the form $AdS_5 \times M_5$. In particular, we demonstrate that all of the thermodynamic properties and hydrodynamic transport parameters for a large class of superconformal gauge theories exhibit a certain universality to leading order in the inverse 't Hooft coupling and $1/N_c$. In particular, we show that independent of the compactification geometry, the leading corrections to any thermal transport parameters for a large class of superconformal gauge theories is universal.

I. INTRODUCTION

The gauge/gravity correspondence presents a powerful tool with which to study strongly coupled gauge theories \cite{1}. One of the most striking new insights is that the ratio of the shear viscosity $\eta$ to the entropy density $s$ is universal with $\eta/s = 1/4\pi$, for any gauge theory with an Einstein gravity dual in the limit of an infinite number of colours and large 't Hooft coupling, i.e., $N_c, \lambda \rightarrow \infty$ \cite{2}. In fact, this result has been conjectured to be a universal lower bound in nature, the KSS bound \cite{2}. Corrections to this result arising for finite $N_c$ and $\lambda$ can be calculated by taking into account higher derivative corrections to the dual gravity action. These corrections were first calculated for $\mathcal{N} = 4$ super-Yang-Mills gauge theory \cite{3}.

\begin{equation}
\frac{\eta}{s} = \frac{1}{4\pi} \left[ 1 + \frac{15}{\lambda^{3/2}} + \frac{5}{16} \frac{\lambda^{1/2}}{N_c^2} + \mathcal{O}(N_c^{-3/2} e^{-c_1 N_c}) \right].
\end{equation}

Recently, it was noted \cite{6} that the same corrections to the ratio of $\eta/s$ appear for a certain $\mathcal{N} = 1$ superconformal $U(N_c) \times U(N_c)$ gauge theory with bifundamental matter \cite{7} — the identification of corrections requires an appropriate interpretation of $\lambda$ and $N_c$ (see below). It was conjectured in \cite{6} that the leading corrections at strong coupling to the shear-viscosity-to-entropy ratio of any four-dimensional conformal gauge theory plasma are universal. In the following, we prove this universality conjecture \cite{6} for a large class of four-dimensional superconformal gauge theories with an $AdS_5 \times M_5$ string theory dual (where $M_5$ is a general Sasaki-Einstein manifold). In this context, the first higher derivative corrections are well understood \cite{8, 9, 10, 11} and appear at order $\alpha'^3$ in the ten-dimensional type IIB supergravity action. Our approach will be to reduce the action including the relevant higher curvature terms down to five-dimensions and to demonstrate that the resulting effective action is completely independent of the internal manifold $M_5$. Hence for all of these theories, the question of determining the effect of the higher derivative corrections to thermal properties of the gauge theory reduces to a common problem of studying the properties of asymptotically $AdS_5$ black hole within a certain five-dimensional gravity action with a universal set of $R^4$ corrections. Hence, written in terms of supergravity expressions, the results for $\eta/s$ will be universal for the class of theories described above. To convert the results to variables of the dual gauge theory, care must be taken to apply the appropriate AdS/CFT dictionary for a certain internal manifold $M_5$. Our discussion shows that this universality of the corrections to leading order in $1/\lambda$ and $1/N_c$ extends beyond the ratio $\eta/s$ and that, in fact, the leading corrections to any thermal or hydrodynamic properties of these plasmas will take a universal form.
II. REDUCTION TO FIVE DIMENSIONS

As our starting point, we begin with a general solution of the leading order type IIb supergravity equations which has the product form $A_5 \times M_5$. Our discussion will be general and we only assume that $A_5$ and $M_5$ are Einstein manifolds with negative and positive curvature, respectively. However, for our application below, we will have in mind that $A_5$ is an asymptotically $AdS_5$ black hole. Beyond the usual choice of $S^5$ for $M_5$, the following discussion will include any compact Sasaki-Einstein manifold, including $L^{p,q,r}$ [12], $Y^{p,q}$ [13] and $T^{1,1}$ as special cases. Implicitly, we also assume that the only nontrivial fields contributing to this solution are the metric and the Ramond-Ramond (RR) five-form. Now we wish to consider the effects of the leading higher-derivative interactions to this solution but in particular on the $A_5$ part of the spacetime. In the ten-dimensional type IIb supergravity, the first nontrivial corrections appear at order $\alpha'^3$ including the celebrated $R^4$ interaction [3], as well as a host of terms involving the RR five-form (and curvatures) [10, 11]. However, it can be shown that the these additional five-form terms make no contributions to the equations of motion when working with a leading order solution of the form $A_5 \times M_5$ [3]. The ten-dimensional action is then given by

$$S_{10} = \frac{1}{16\pi G_{10}} \int d^{10}x \sqrt{-g} \left( \tilde{R} - \frac{1}{4.5!} F_5^2 + \alpha'^3 g_s^{3/2} f^{(0,0)}(\tau, \bar{\tau}) W \right).$$

(2)

The pre-factor in front of the higher derivative term $W$ is a modular form $f^{(0,0)}(\tau, \bar{\tau})$ written in terms of the usual axiodilaton field $\tau = a + i e^{-\phi}/14$. Recall that we assume the latter is constant in the leading supergravity solution, i.e., $e^{\phi} = g_s$, and so it will turn out that the kinetic term for this field is not needed for our discussion. Further note that self-duality constraint is imposed on $F_5$ as an additional equation, beyond the equations of motion derived from (1). Now as described above, the only relevant contribution to $W$ is fourth order in curvatures:

$$W(\tilde{C}) = \tilde{C}_{ABCD} \tilde{C}^{AB} \tilde{C}^{CD} - \frac{1}{4} \tilde{C}_{ABCD} \tilde{C}^{AB} \tilde{C}^{CD} \tilde{C}^{EF} \tilde{C}^{EF}$$

(3)

where $\tilde{C}$ is the Weyl tensor in ten dimensions. Above, we have introduced the ‘tilde’ to distinguish ten-dimensional objects, e.g., metric or Ricci scalar, from their five-dimensional counterparts below. Similarly, our notation will be to use a ‘hat’ to denote quantities associated with the internal manifold, e.g., $\tilde{R}$ will be the Ricci scalar of $M_5$. Quantities on $A_5$, the asymptotically $AdS$ space in five dimensions, will remain unadorned. Further, indices on the full ten-dimensional geometry, the $AdS$ space $A_5$ and the internal manifold $M_5$ will be denoted $A, B, C, D, E, \ldots$, and $a, b, c, d, e, \ldots$ and $m, n, p, q, r, \ldots$, respectively.

Now we want to perform a Kaluza-Klein reduction on the $M_5$ to construct the five-dimensional action which reproduces the gravity equations of motion on $A_5$:

$$S_5 = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left( R - \frac{12}{L^2} + \alpha'^3 g_s^{3/2} f^{(0,0)}(\tau, \bar{\tau}) W \right)$$

(4)

where $L$ is the radius of curvature of $AdS_5$. Of course, the reduction of the two-derivative terms is standard but we must take care in reducing the higher curvature contribution $W$. First, note that the formula for the Weyl tensor in $d$ dimensions is given by

$$C_{abcd} = R_{abcd} - \frac{2}{d-2} \left( g_{a[c} R_{d]b} - g_{b[c} R_{d]a} \right) + \frac{2}{(d-2)(d-1)} R g_{a[c} g_{d]b}.$$  

(5)

In the present background with the product form, $A_5 \times M_5$, we have

$$\tilde{C}_{abcd} = C_{abcd} + 10 \left( g_{a[c} Y_{d]b} - g_{b[c} Y_{d]a} \right) + 2X g_{a[c} g_{d]b}$$

$$\tilde{C}_{mnqp} = \tilde{C}_{mnqp} + 2X \hat{g}_{m[p} \hat{g}_{q]} n$$

$$\tilde{C}_{manb} = -3Y_{ab} \hat{g}_{mn} - \frac{4}{5} X \hat{g}_{ab} \hat{g}_{mn}$$

(6)

where we have defined

$$Y_{ab} = \frac{1}{24} \left( R_{ab} - \frac{1}{5} R g_{ab} \right)$$

$$X = \frac{1}{72} (R + \hat{R}).$$

(7)
It will be important in what follows that $Y$ and $X$ vanish when evaluated on the leading order supergravity solution and also that $Y$ is traceless (in general), i.e., $Y_{aa} = 0$.

Given these expressions, we have carefully evaluated (8) in terms of $\hat{C}$, $C$, $Y$ and $X$. Here we will only indicate that this straightforward but somewhat tedious exercise yields an expression which has the schematic form:

$$\hat{C}^4 = C^4 + \hat{C}^4 + \hat{C}^3X + C^3Y + C^2X + O(Y^2, X^2, XY), \tag{8}$$

where any contributions that are quadratic or higher order in $Y$ and $X$ have been left implicit in the final term. Note that the tensor structure of $\hat{C}^4$ precludes the appearance of any terms containing both $\hat{C}$ and $C$ together, e.g., $C^2 \hat{C}^2$. However, the full expression certainly depends on the internal geometry through the appearance of $\hat{C}$. In particular, beyond the $\hat{C}^4$ term, there are several cross terms combining $\hat{C}$ with the tensors $Y$ and $X$. As well as $\hat{C}^3X$, the final term in (8) includes two further terms if the form $\hat{C}^2Y^2$ and $\hat{C}^2X^2$.

Now we wish to consider how the five-dimensional equations of motion are modified when these contributions (8) are included in the effective action (4). A key observation is that for consistency of the expansion in $\alpha'^3$, we can evaluate the contributions of these new terms using only the leading order supergravity solution. In particular, this means that we can dismiss the terms which are quadratic and higher order in $Y$ and $X$. Their contribution to the equations of motion take the form, e.g.,

$$R_{ab} - \frac{1}{2} g_{ab} R - \frac{6}{L^2} g_{ab} \simeq \alpha'^3 \left( 2X \frac{\delta X}{\delta g_{ab}} C^2 + X^2 \frac{\delta C^2}{\delta g_{ab}} + \ldots \right) \tag{9}$$

However, as we observed above $X = 0 = Y_{ab}$ when evaluated on the supergravity solution and hence all of these contributions, which still contain one or more factors of $X$ or $Y$, vanish at this order in the perturbative expansion. Note that it is this same reasoning that allows us to ignore the appearance of $\hat{R}$ in the definition of $X$ given by (7).

In principle, this quantity could depend on the details of the internal geometry, however, the leading supergravity equations dictate that $\hat{R} = 20/L^2$ for any choice of $\mathcal{M}_5$. Hence at this order, $X$ contains no information which distinguishes different internal geometries and our discussion above only considered how $\mathcal{M}_5$ might modify the effective action through the appearance of $\hat{C}$.

Hence we are left to consider the two remaining terms involving $\hat{C}$. The first of these, denoted by $\hat{C}^4$ in (8), is simply the expression $W$ given in (9) but now evaluated with the ten-dimensional Weyl tensor $\hat{C}$ replaced by $C$. However, an explicit computation shows that $W(\hat{C}) = 0$ for $\mathcal{M}_5 = L^{p,q,r} = \text{recall that the latter provide an infinite family of explicit metrics for five-dimensional Sasaki-Einstein manifolds [12].}$ This result can be extended to any general Sasaki-Einstein manifold using the fact that the latter produce a supersymmetric background [13]. It is known [10, 11] that, in supersymmetric backgrounds with only metric and five-form fields, the full set of $\hat{C}^4$ and five-form higher derivative corrections must vanish. Further it can be shown that the higher derivative terms involving the five-form do not contribute in such backgrounds [11] and hence $W(\hat{C})$ must vanish by itself. Now if one focuses on a supersymmetric background with the product form $AdS_5 \times \mathcal{M}_5$, as described above, one finds that this expression splits to yield: $W(\hat{C}) \simeq W(C) + W(\hat{C})$. Again, in this relation, we have discarded the terms proportional to $X$ and $Y$ which vanish when evaluated on the supergravity solution. Further now, the Weyl tensor $C$ on $AdS_5$ vanishes and therefore one must have $W(\hat{C}) = 0$ on the internal space $\mathcal{M}_5$.

III. SCHOUTEN IDENTITIES

The final contribution in (8) which could in principle introduce some dependence of the effective action (4) on the internal manifold is that linear in $X$ with a cubic contraction of $\hat{C}$. This $\hat{C}^3X$ term has the explicit form:

$$4X \left( 2\hat{C}_{mpqs} \hat{C}^{mp} r s \hat{C}^{mrqs} - \hat{C}_{mpqs} \hat{C}^{m} r s \hat{C}^{mrqs} \right). \tag{10}$$

We will argue that these terms vanish on using Schouten identities in five dimensions. Naively there exist two independent contractions that are cubic in Weyl tensor:

$$\hat{C}^3_{(1)} := \hat{C}_{mpqs} \hat{C}^{mp} r s \hat{C}^{mrqs} \quad \hat{C}^3_{(2)} := \hat{C}_{mpqs} \hat{C}^{m} r s \hat{C}^{mrqs}. \tag{11}$$

However, in five-dimensions, these two expressions are related by a Schouten identity:

$$2 \hat{C}^3_{(1)} - \hat{C}^3_{(2)} = 0. \tag{12}$$
Such Schouten identities can be established as a vanishing contraction of tensors which results in an attempt to antisymmetrize over $d + 1$ indices in $d$ dimensions. Here with three Weyl tensors, one has 12 indices and one may antisymmetrize on 6 of them and then contract with the remainder. Of course, however, the resulting expression must vanish if it is evaluated in five dimensions. That is, up to an overall normalization, the left-hand side of (12) is equivalent to

\[ \hat{C}_{[m n} m n \hat{C}_{p q} p q \hat{C}_{r s]} r s \]  

which again vanishes in five dimensions. These calculations are quickly performed using Cadabra [10].

At this point, we have actually done enough to establish that the five-dimensional equations of motion are independent of the internal manifold $M_5$. However, we proceed further here with the application of Schouten identities to eliminate the $C^3 X, C^3 Y$ terms in (8), as well. The latter terms work out to be

\[
4X \left( 2C_{abcd} C^{ac} e_f C^{bedf} - C_{abcd} C^{a} e_f C^{bedf} \right) \\
+ 40Y_{g f} \left( 2C_{abcd} C^{ac} e_f C^{bedg} - C_{abcd} C^{a} e_f C^{bedg} - C_{abcd} C^{acbe} C^{d} f e g \right). 
\]

Now the first two terms proportional to $X$ again cancel by the same Schouten identity given in (12). Using a similar strategy as above, it is possible to show that even though in principle one can build three independent $C^3 Y$ contractions, namely,

\[
(C^3 Y)_{(1)} := C_{abcd} C^{ac} e_f C^{bedg} Y_{g f} \\
(C^3 Y)_{(2)} := C_{abcd} C^{a} e_f C^{bedg} Y_{g f} \\
(C^3 Y)_{(3)} := C_{abcd} C^{acbe} C^{d} f e g Y_{g f} 
\]

there is now a new Schouten identity

\[
2(C^3 Y)_{(1)} - (C^3 Y)_{(2)} - (C^3 Y)_{(3)} = 0. 
\]

To obtain this identity, one can take for instance

\[
Y_{[a} f C^{a d} C^{b e g} C_{e f] g} c 
\]

which must again vanish in five dimensions. In any event, this new identity [13] ensures that the combination of $C^3 Y$ terms in (14) vanishes.

**IV. UNIVERSAL CORRECTIONS**

Our discussion above shows that the five-dimensional Einstein equations on the asymptotically AdS space $A_5$ will not depend on the detailed structure of the compact manifold $M_5$. In fact with the Schouten identities, we were able to show that upon reducing to five dimensions the quartic curvature term in (4) reduces to

\[
W(\hat{C}) = W(C) + O(Y^2, X^2, XY)  
\]

where $W(C)$ is the expression [3] constructed with the five-dimensional Weyl tensor. Further, as discussed at [9], the terms which are quadratic or higher order in $X$ and $Y$ will not contribute to the equations of motion at this order in the $\alpha'$ expansion. That is, the $\alpha'$ corrected equations for the five-dimensional metric give the same result whether one uses the full $W(\hat{C})$ or simply treats this expression a five-dimensional construction $W(C)$.

Of course, we must now consider the implications of this result for the dual gauge theory. In particular, we are interested in the thermodynamic and hydrodynamic properties of the gauge theory. In this case, we will take $A_5$ to be an asymptotically $AdS_5$ black hole. The result of our above discussion is that for all of the gauge theories (defined by different $M_5$), the dual five-dimensional gravity action is universal. Therefore determining the effect of the higher derivative gravity corrections reduces to a common problem of studying the properties of asymptotically $AdS_5$ black holes within the same five-dimensional effective theory with a universal set of $R^4$ corrections. Hence the corrections to all of the thermodynamic properties (e.g., entropy density or equation of state) and hydrodynamic parameters (e.g., shear viscosity or relaxation time) of the gauge theories will have a universal form. In particular then, we have proven the universality of corrections to the quasinormal spectrum, as conjectured in [8]. Of course, the latter spectrum captures a great deal of information about the thermal transport coefficients. However, we emphasise that our result extends this universal feature of the $\alpha'^3$ corrections to the full set of higher order coefficients recently explored in
including those for terms nonlinear in the local four-velocity. Again our present discussion indicates that it is sufficient to consider the effective action \( \mathcal{O} \) with \( \mathcal{W} \) term constructed with five-dimensional Weyl tensors to calculate the \( \alpha'^3 \) corrections to all of these transport coefficients.

We must add that the above conclusions rely on the physical quantities of interest not being modified by new fields which are trivial in the original background. For example, we know that although the dilaton and the warp factor are trivial in the AdS\( _5 \) black hole solution of the leading order supergravity equations, both of these fields are sourced by the higher curvature corrections in this background \( \mathcal{O} \). Further, our analysis is restricted to the \( \alpha'^3 \) terms in the ten-dimensional action, which include only the curvatures and the RR five-form \( \mathcal{O} \). However, we know that there exist a host of additional higher derivative interactions at order \( \alpha'^3 \) and in principle, even more additional type IIB fields could be sourced by these terms. For definiteness, consider the RR axion \( \alpha' \) which vanishes at lowest order. There might still be \( \alpha'^3 \) terms which are linear in \( \alpha' \), e.g., \( C^2 \nabla F_5 \nabla^2 \alpha' \). The corrected solution would then also include an axion of order \( \alpha'^3 \). However, in five-dimensional Einstein’s equations, \( \alpha' \) will only appear quadratically or in terms with an \( \alpha'^3 \) factor. Hence, its effects in, e.g., the quasi-normal spectrum will only be felt at order \( \alpha'^6 \) and therefore it can be neglected here. The same reasoning can be made for all other fields, including the dilaton and warp factor.

At this point, we have reduced the determination of higher order corrections in a large number of CFT’s down to the study of a common five-dimensional gravity theory. The latter gives a universal set of corrections in terms of supergravity expressions. As an example, let us consider the ratio of shear viscosity to entropy density:

\[
\frac{\eta}{s} = \frac{1}{4\pi} \left[ 1 + 15 \frac{\alpha'^3}{\pi^6} \left( \zeta(3) + \frac{\pi^2}{3} g_s^2 + \tilde{f}_{\text{NP}} \right) \right].
\]

This result is produced by expanding the modular formula \( f^{(0,0)}(\tau, \tau) \) in a regime of small string coupling \( g_s \). As \( \mathcal{O} \) shows, in this regime, the modular form can be interpreted in terms of a tree-level term, a one string-loop contribution and a series of nonperturbative corrections captured by \( \tilde{f}_{\text{NP}} \). Again with small \( g_s \), the leading nonperturbative contribution is \( \tilde{f}_{\text{NP}} \approx 4\pi g_s^{3/2} e^{-2\pi/g_s} \). Now in the case of \( N = 4 \) SYM for which \( \mathcal{M}_5 = S^5 \), we have the standard AdS/CFT dictionary which includes \( \lambda = L^4/\alpha'^2 \) and \( g_{\text{YM}}^2 = \lambda/N_c = 4\pi g_s \). Applying these two expressions converts the above supergravity expression \( \mathcal{O} \) to the dual gauge theory expression \( \mathcal{O} \) given in the introduction.

Now in the case where \( \mathcal{M}_5 \) is replaced by a more general Sasaki-Einstein manifold, one must take care in interpreting \( \mathcal{O} \) with the correct AdS/CFT dictionary. In this case, the dual gauge theory corresponds to a quiver theory containing a number of \( U(N_c) \) gauge groups coupled to certain bifundamental matter fields \( \mathcal{O} \). Hence with a product of \( n \) gauge groups, there are \( n \) independent gauge couplings \( (g_{\text{YM}}^2)_i \). The string coupling is related to the following combination \( \mathcal{O} \):

\[
\frac{1}{g_s} = \sum_i \frac{4\pi}{(g_{\text{YM}}^2)_i},
\]

while the remaining independent linear combinations of \( (g_{\text{YM}}^2)_i \) are related to various form fields on \( \mathcal{M}_5 \). It will prove useful to define a “collective” \( \lambda \) Hooft coupling for the quiver theory with an averaged gauge coupling:

\[
\lambda_{\text{CFT}} \equiv g_{\text{YM}}^2 N_c \quad \text{where} \quad \frac{1}{g_{\text{YM}}^2} = \frac{1}{n} \sum_i \frac{1}{(g_{\text{YM}}^2)_i}. \]

Now the AdS radius of curvature on \( A_5 \) is determined by \( \mathcal{O} \):

\[
\frac{L^4}{\alpha'^2} = 4\pi g_s N_c \frac{\pi^3}{\text{Vol}(\mathcal{M}_5)},
\]

where the internal volume is defined for \( \mathcal{M}_5 \) with unit curvature, i.e., with \( \tilde{R} = 20 \). Further the volume of the internal space is related to the central charge of the dual quiver theory as \( \mathcal{O} \):

\[
\frac{c_{\text{CFT}}}{c_{N=4}} = \frac{\pi^3}{\text{Vol}(\mathcal{M}_5)},
\]

where \( c_{N=4} = (N_c^2 - 1)/4 \) is the central charge of \( N = 4 \) super-Yang-Mills with gauge group \( SU(N_c) \). Combining these last three equations then yields

\[
\frac{L^4}{\alpha'^2} = \frac{\lambda_{\text{CFT}}}{n} \frac{c_{\text{CFT}}}{c_{N=4}}.
\]
Combining these above expressions allows us to translate (18) to the following gauge theory result

$$\frac{\eta}{s} = \frac{1}{4\pi} \left[ 1 + \left( \frac{n c_{N_c=4}}{c_{\text{CFT}}} \right)^{3/2} \frac{15 \zeta(3)}{\lambda_{\text{CFT}}^{3/2}} \right] + \frac{n c_{N_c=4}}{c_{\text{CFT}}} \frac{5 \lambda_{\text{CFT}}^{1/2}}{16n^2 N_c^2} + O \left( N_c^{-3/2} e^{-8\pi^2 \eta / \lambda_{\text{KW}}} \right).$$  \tag{24}

As we can see while the basic structure of the corrections in (11) and (24) are the same, the precise numerical coefficients differ in a way dictated by the distinct gauge/gravity duality for each choice of the internal geometry. As an illustrative example, let us consider the case where $M_5 = T^{1,1}$. With this internal space, the dual quiver theory has $n = 2$ nodes and corresponds to an $\mathcal{N} = 1$ superconformal $U(N_c) \times U(N_c)$ gauge theory coupled to four chiral superfields, two each in the $(N_c, \bar{N}_c)$ and $(\bar{N}_c, N_c)$ representations, as originally elucidated by Klebanov and Witten [7]. Further Vol$(M_5) = 16\pi^3/27$ and so for the Klebanov-Witten theory, (24) yields

$$\frac{\eta}{s} \bigg|_{\text{KW}} = \frac{1}{4\pi} \left[ 1 + \left( \frac{32}{27} \right)^{3/2} \frac{15 \zeta(3)}{\lambda_{\text{KW}}^{3/2}} \right] + \frac{32}{27} \frac{5 \lambda_{\text{KW}}^{1/2}}{64 N_c^2} + O \left( N_c^{-3/2} e^{-16\pi^2 N_c / \lambda_{\text{KW}}} \right).$$  \tag{25}

V. DISCUSSION

In this note, we have extended the universality of $\eta/s$ to the regime of a large but finite ’t Hooft coupling and finite number of colours for the class of superconformal field theories described by a holographic dual geometry of the form $\text{AdS}_5 \times M_5$. Our proof followed by constructing the effective five-dimensional gravity action which results from compactifying with type IIb supergravity action supplemented by the $C^4$ term in 10 dimensions on some geometry $M_5$. We demonstrated that the resulting five-dimensional equations of motion are independent of $M_5$ and in fact, the latter equations can be derived from an effective action with the same quartic curvature term (3) constructed from the five-dimensional Weyl tensor. This shows that the leading corrections to $\eta/s$ are derived for a large class of theories by studying the same problem, i.e., determining the effect of the higher derivative gravity corrections to an asymptotically $\text{AdS}_5$ black hole within the same five-dimensional effective action with a universal set of $C^4$ corrections. Hence, written in terms of string theory variables (string tension and string coupling), the result (18) for $\eta/s$ will be universal for a large class of theories. To convert the result to variables of the dual gauge theory, one must apply the $\text{AdS/CFT}$ dictionary that is appropriate for a given internal manifold $M_5$. This universality was first conjectured for the corrections to the quasinormal spectrum of the dual $\text{AdS}_5$ black holes [8]. Our proof of universality encompasses this conjecture but also shows that this universal behaviour extends to all other thermodynamic or hydrodynamic quantities e.g., the entropy density [18] and the relaxation time [24], in the same way as described above.

One step in our analysis was to show the quartic curvature term (3) vanishes when evaluated for the internal Weyl tensor $\hat{C}$. We presented a general argument that relied on supersymmetry and so required choosing $M_5$ to be a five-dimensional Sasaki-Einstein space. It would be interesting to generalise this discussion beyond supersymmetric compactifications. One infinite family of Einstein spaces can be constructed as coset spaces $(SU(2) \times SU(2))/U(1)$. The resulting manifolds, denoted $T^{p,q}$, have been considered for type IIB compactifications in [22, 25, 26]. Of these only $T^{1,1}$ provides a supersymmetric background [23] and in fact, with $p \neq q$, the compactifications contain tachyons violating the Breitenlohner-Freedman bound [27]. In any event, evaluating $W(\hat{C})$ for these spaces, we find that it only vanishes if $p = q$. Even though the case $p = q > 1$ is not supersymmetric, one should expect the quartic curvature term to vanish on these spaces since it vanishes on the supersymmetric $T^{1,1} \times \mathbb{Z}_p$ geometry and the spaces $T^{p,p}$ are $\mathbb{Z}_p$ orbifolds of $T^{1,1}$. Hence it seems that the vanishing of $W(\hat{C})$ is closely related to supersymmetry but it would be interesting to examine this question further.

Another interesting direction would be to see if our findings extend to the case of non-zero chemical potential. In this case, the gravity analysis would require turning on background Kaluza-Klein gauge fields and so the resulting background would not have the product form $A_5 \times M_5$ assumed throughout the above discussion. In particular, we expect that the higher derivative terms involving the RR five-form will play a crucial role [3]. Further, different choices of $M_5$ will in general lead to different gauge fields in the effective five-dimensional action and so we cannot expect universality to extend to any general chemical potential in the dual gauge theory. However, it was recently shown that any supersymmetric compactification on a Sasaki-Einstein manifold yields a consistent reduction to minimal five-dimensional supergravity [27]. This construction relies on using the Killing symmetry of $M_5$ which is dual to the $U(1)$ R-symmetry in the dual $\mathcal{N} = 1$ superconformal gauge theory. Hence, with an infinite ‘t Hooft coupling and an infinite number of colours, the thermal and hydrodynamic properties of the gauge theory plasma will be universal in the presence of this corresponding chemical potential. It is likely that this universality could also be extended to finite $\lambda$ and $N_c$ in this case.

Above we observed that even though the $\alpha'^3$ corrections to the supergravity action may source a nontrivial dilaton, warp factor and other new fields, these new fields will not modify the thermal and hydrodynamic properties of the
dual gauge theory at this order. The appearance of additional fields at the $\alpha'^3$ order would indicate that various operators acquire a nonvanishing expectation value in the gauge theory plasma, but that these expectation values are suppressed by inverse powers of the 't Hooft coupling and the number of colours. While certain fields, e.g., the dilaton, may exhibit universal behaviour, this could not be expected to apply for all of the new operators acquiring an expectation value. In particular, the precise family of operators available would certainly depend on the details of the gauge theory or alternatively the compactification manifold.

The warp factor deserves further attention as we would like to relate our conclusions to the discussion in \cite{6}. Specifically, it was shown in \cite{6} that the equation of motion for the shear quasinormal mode obtained in ten-dimensional using $\alpha'\hat C^4$ higher derivative metric corrections differed from the corresponding equation derived in five-dimensional effective action with $C^4$ term only. While the wave-functions of the lowest (hydrodynamic) shear quasinormal modes were different, hydrodynamic shear dispersion relation was found to be the same \cite{6}. We point out here that a simple $O(\alpha'^3)$ rescaling of the five-dimensional shear quasinormal wave-function $Z_{\text{shear,5D}}$

\begin{equation}
Z_{\text{shear,5D}} \rightarrow Z_{\text{shear,5D}} \cdot \left(1 - \frac{3}{10} \alpha'^3 \nu \right) = Z_{\text{shear,10D}}
\end{equation}

identified it with the ten-dimensional shear quasinormal wave-function $Z_{\text{shear,10D}}$. In \cite{20} $\nu$ is a warp factor modifying the direct product $A_5 \times M_5$ due to higher derivative corrections \cite{18}.

\begin{equation}
ds^2_{A_5} \times ds^2_{M_5} \rightarrow \left\{ds^2_{A_5} \cdot e^{-\frac{4}{3} \alpha'^3 \nu} \right\} \times \left\{ds^2_{M_5} \cdot e^{2\alpha'^3 \nu} \right\}
\end{equation}

Since the warp factor $\nu$ is non-singular (and momenta independent), a rescaling \cite{20} changes the quasinormal wave-function while keeping the spectrum of the quasinormal modes invariant. Such a rescaling is simply an artifact of the (arbitrary) normalization used in \cite{28} (and later in \cite{6}) — a ten-dimensional shear quasinormal wave-function was normalized as

\begin{equation}
Z_{\text{shear,10D}} \sim g_{5D}^{xx} \cdot \delta g
\end{equation}

where $\delta g$ is the corresponding shear metric fluctuation. If instead the wave-functions are normalized as

\begin{equation}
Z \sim g^{xx} \cdot \delta g
\end{equation}

in five and ten dimensions, correspondingly, there is no need for a rescaling \cite{20}.

As displayed in \cite{17}, the ten-dimensional quartic curvature corrections reduce down to $W(C)$ constructed from the five-dimensional Weyl tensor, plus a number of terms that are quadratic or higher order in $X, Y$. While these terms only contribute to the equations of motion with expressions that vanish when evaluated on the leading order supergravity solution, in appendix \ref{appA} we show that we can use field redefinitions to remove these terms and so the resulting equations of motion would identical independent of the choice of $M_5$. However, we note that such field redefinitions would make it difficult to realize the supersymmetry of the various $AdS_5 \times M_5$ backgrounds. Of course, the latter is not central to the present investigation but remarkably we were able to demonstrate universality without resorting to such field redefinitions. Note that this would not have been possible if the terms linear in $X$ and $Y$ had not vanished by the Schouten identities.

Our analysis has demonstrated that the thermal properties of a large class of gauge theories can be derived from a universal holographic framework. Of course, one is tempted to consider the application of our results to the strongly coupled quark-gluon plasma under study in experiments at RHIC and soon at the LHC. A key difference between $\mathcal{N} = 4$ SYM and QCD is the number of degrees of freedom that are active in the strongly coupled plasma \cite{2, 30}. Our analysis captures the effect of the latter on thermal properties and, in particular, on the $1/N_c$, $1/\lambda$ corrections with the dependence on the central charge, e.g., with the factors proportional to $c_{\mathcal{N}=4}/c_{\text{CFT}}$ in \cite{24}. Imagining that the QCD plasma is described by a CFT, we can proceed by treating its central charge to be a phenomenological parameter. Let us then consider the energy density and shear viscosity arising from the present holographic model:

\begin{equation}
\varepsilon = \frac{3}{4} \left(1 + \frac{\Delta}{8}\right) \varepsilon_0 \quad \text{and} \quad \frac{\eta}{s} = \frac{1}{4\pi} (1 + \Delta).
\end{equation}

where

\begin{equation}
\Delta \equiv 5 \left(\frac{c_{\mathcal{N}=4}}{c_{\text{QCD}}} \right)^{3/2} \left(\frac{3 \zeta(3)}{\lambda^{3/2}} + \frac{1}{16} \frac{\lambda^{1/2}}{N_c^2} \right).
\end{equation}

Here $\varepsilon$ and $\varepsilon_0$ denote the energy density of the conformal plasma and that in limit of a free theory. We have set $n = 1$ in \cite{31} and recall that $c_{\mathcal{N}=4} = 2$ with an $SU(3)$ gauge group. Lattice QCD results can provide insight into the energy
density and recent studies seem to indicate that energy density should be in the range $\varepsilon/\varepsilon_0 \approx 0.85 - 0.90$ \cite{31}. In this case, \cite{30} yields $\Delta \approx 1.07 - 1.60$ and hence

$$\left. \eta \right|_{s_{\text{QCD}}} \approx 0.16 - 0.21 . \quad (32)$$

We must observe that the corrections here are not small. These ‘corrected’ values for $\eta/s$ are significantly larger than leading result, which corresponds to the conjectured KSS bound $\eta/s_{KSS} = 1/4\pi \approx 0.08$ \cite{2}. However, these results \cite{34} are still consistent with values emerging from the analysis of RHIC data \cite{32}. Of course, our analysis must be regarded with a highly skeptical eye. We are assuming that that the QCD plasma is described by a supersymmetric conformal field theory. In fact for the QCD plasma, we certainly have no supersymmetry and at RHIC temperatures, we should expect that it is only approximately conformal.

Another interesting comparison can be made for pure $SU(3)$ Yang-Mills, again using lattice results \cite{32}. While lattice results for the thermodynamics of the pure gauge theory have long been available \cite{33}, reliable results for $\eta/s$ have only been established very recently \cite{34}:

$$\eta/s \approx 0.10 - 0.17 \quad \text{at } T = 1.65 T_c . \quad (33)$$

Examining the thermodynamic results \cite{33} at this temperature and applying the same analysis as above, one finds

$$T = 1.65 T_c : \varepsilon/\varepsilon_0 \approx 0.81 - 0.84 , \quad \Delta \approx 0.64 - 0.96 , \quad \eta/s \approx 0.13 - 0.16 . \quad (34)$$

Hence the holographic formulae yield results in good agreement with those from the lattice \cite{33} — however, the errors are still relatively large. We should mention that \cite{34} also presents a result for $T = 1.24 T_c$. However, this close to the critical temperature, one finds that $\varepsilon/\varepsilon_0 \approx 0.71 - 0.74$. Achieving $\varepsilon/\varepsilon_0 < 3/4$ in \cite{30} would require $\Delta < 0$ and so one cannot apply the present holographic model because our formula \cite{31} always gives $\Delta > 0$. Of course, one should also note that the lattice results indicate that the interaction measure, i.e., $(\varepsilon - 3p)/T^4$, peaks just below $T = 1.24 T_c$ \cite{32} and so it seems the Yang-Mills plasma is out of the conformal regime at this temperature. For comparison purposes, we also apply our calculations for $T = 4 T_c$, where the plasma appears to be well into the conformal regime:

$$T = 4 T_c : \varepsilon/\varepsilon_0 \approx 0.83 - 0.86 , \quad \Delta \approx 0.88 - 1.20 , \quad \eta/s \approx 0.15 - 0.17 . \quad (35)$$

Of course, just as for the QCD plasma, this holographic analysis should be considered in a skeptical light.

Note that the results in \cite{32}, \cite{34} and \cite{35} rely simply on the form in \cite{30} arising from the holographic model and are actually independent of any microscopic details i.e., $N_c$ or $c_{\text{QCD}}$. If we adopt $\lambda = 6\pi$ (i.e., $\alpha_s = 0.5$) along with $N_c = 3$ for the QCD plasma, we may use \cite{31} to derive the effective central charge: $c_{\text{QCD}} \approx 0.75 - 0.90$. A general observation is that this reduced central charge (compared to $N = 4$ SYM) should reflect the reduction in degrees of freedom for QCD and is responsible for the enhancement of the the $1/N_c$, $1/\lambda$ corrections. For example, with the same ‘t Hooft coupling, the corrected result for $N = 4$ SYM in \cite{11} yields $\eta/s|_{N=4} \approx 0.11$ \cite{2}.

We may also calculate the effective central charge for the pure Yang-Mills plasma at $T = 1.65 T_c$ or $4 T_c$. This requires that we also use the lattice results \cite{33} for the Yang-Mills coupling at these temperatures, which yields: $c_{1,65 T_c} \approx 2.34 - 3.23$ and $c_{4 T_c} \approx 1.84 - 2.39$. This is somewhat surprising since here this effective central charge for the pure gauge theory is of the same order as that for the $N = 4$ SYM, i.e., $c_{N=4} = 2$ but one’s intuition would be that the SYM plasma should contain more degrees of freedom than that for the pure gauge theory. In any event, these results emphasize that $c$ is only a phenomenological parameter in our holographic model.

Finally we close by observing that four-dimensional supersymmetric CFT’s are characterised by two central charges $a$ and $c$ defined in terms of the trace anomaly — see \cite{36} for a detailed discussion in the context of gauge/gravity correspondence. All of the AdS/CFT dualities considered here were based on type IIB compactifications on a smooth five-dimensional manifold $M_5$ and the leading higher derivative corrections appeared at order $a^3$. Hence an implicit feature common to all of the dual conformal gauge theories is that the $a$ and $c$ central charges are equal. Of course, it is possible to construct AdS/CFT dualities in which $a \neq c$ but such F-theory constructions typically require the introduction of various D-branes and O-planes \cite{37}. Hence a detailed analysis of such ten-dimensional constructions would require more than just using the type IIB supergravity action, as was done above. A key feature of the resulting five-dimensional effective action is that the leading corrections are now curvature-squared terms with a coefficient proportional to $a - c$ \cite{36}. Therefore such theories are beyond the scope of the present analysis and the universality of the shear viscosity (or any other thermal properties) found here does not apply to such conformal gauge theory plasmas with $a \neq c$. In fact, these theories can violate the conjectured KSS bound \cite{39}, in contrast to the present case where $1/\lambda$ and $1/N_c$ corrections are always positive and so still respect the KSS bound, as has been noted in \cite{4}. 

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APPENDIX A: FIELD REDEFINITIONS

Here we will show that using field redefinitions all of the terms proportional to $X$ or $Y$ in (8) can be set to zero. Suppose the effective five-dimensional gravity action contains two terms

$$M^{ab} Y_{ab} + N X$$

(A1)

where $M_{ab}$ and $N$ are some functionals of the five-dimensional curvature (and perhaps other fields). Now we will show that these two terms can be eliminated by a field redefinition. First of all note that in the supergravity background $\hat{R} = 20/L^2$ and therefore we can write the leading five-dimensional supergravity action as

$$I_{\text{sugra}} = \frac{1}{16\pi G_5} \int d^5 x \sqrt{-g} \left[ R + \frac{12}{L^2} \right] = \frac{1}{16\pi G_5} \int d^5 x \sqrt{-g} \left[ R + \frac{3}{5} \hat{R} \right].$$

(A2)

Now considering a field redefinition of the five-dimensional metric, $g_{ab} \rightarrow g_{ab} + \delta g_{ab}$, the change in the five-dimensional supergravity action is (up to total derivatives):

$$\delta I_{\text{sugra}} = \frac{1}{16\pi G_5} \int d^5 x \sqrt{-g} \left[ -R^{ab} + \frac{1}{2} \left( R + \frac{3}{5} \hat{R} \right) g^{ab} \right] \delta g_{ab}$$

$$= \frac{1}{16\pi G_5} \int d^5 x \sqrt{-g} \left[ - \left( R^{ab} - \frac{R}{5} g^{ab} \right) + \frac{1}{2} \left( -\frac{2}{5} R + R + \frac{3}{5} \hat{R} \right) g^{ab} \right] \delta g_{ab}$$

$$= \frac{1}{16\pi G_5} \int d^5 x \sqrt{-g} \left[ -24 Y^{ab} + \frac{108}{5} X g^{ab} \right] \delta g_{ab}.$$  

(A3)

Now making the specific choice

$$\delta g_{ab} = \alpha M_{ab} + (\beta M^{c}_{\ c} + \gamma N) g_{ab}$$

(A4)

then (A3) yields

$$\delta I_{\text{sugra}} = \frac{1}{16\pi G_5} \int d^5 x \sqrt{-g} \left[ -24 \alpha M_{ab} Y^{ab} + 108 \left( \beta + \frac{\alpha}{5} \right) X M^{c}_{\ c} + 108 \gamma N X \right]$$

$$= \frac{1}{16\pi G_5} \int d^5 x \sqrt{-g} \left[ -M_{ab} Y^{ab} - N X \right]$$

(A5)

where in the last line, we chose $\alpha = 1/24$, $\beta = -\alpha/5$ and $\gamma = -1/108$. Hence by tuning the parameters appropriately we can cancel the extra order $\alpha^3$ terms (A1) from the effective five-dimensional action. Note that this is quite general and these calculations show that one can eliminate any term containing an $X$ or $Y$ – in particular, such a term need not be linear in $X$ and $Y$. Hence all the higher order terms found above in (8) can in principle be completely eliminated leaving only the $C^4$ and $\hat{C}^4$ terms. Hence one is lead to conclude that a field redefinition can be used to bring the five- and ten-dimensional equations into precisely the same form, i.e., not just the same up to terms which vanish when evaluated for the leading order supergravity solution.

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