WHAT IS BEING CALCULATED WITH THERMAL FIELD THEORY? *

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ABSTRACT
Thermal field theory is reviewed briefly. It is noted that until recently it was not known what type of real-time Green function was being calculated in the Euclidean approach. The formal answer to this question is then given and the unexpectedly complicated answer is discussed. The physical implications of the results are then considered.

1. Introduction
Thermal field theory, or finite temperature field theory as it is also called, has been around for at least forty years¹. The resulting tools, Feynman rules etc., are very similar to those found in ‘normal’ (zero temperature) quantum field theory, and which has been used very successfully in particle physics and elsewhere. However thermal field theory encodes additional physics not present in zero temperature field theory, and the two are quite different in many ways.

Thus, the key message of this work is that one should not be seduced into thinking thermal field theory is ‘just’ like ordinary field theory. Hence one must think carefully about the link between a calculation of Green functions etc. and a given physical problem and one must not rely upon the superficial similarities between the zero-temperature and thermal cases.

1.1. What is thermal field theory?
Thermal field theory is the combination of two theories. The first element is quantum field theory which is used to describe the behaviour of small, often fundamental, particles. One uses it to describe the dynamics of just of few such particles e.g. as in $e^+e^-$ collisions at LEP. A fundamental aspect of the physics encoded in this theory is the idea of quantum fluctuations, as characterised by Heisenberg’s uncertainty relation. The effects of such quantum fluctuations on physical processes is represented by Feynman diagrams with internal lines such as in figure 1. The second theory is thermodynamics/statistical mechanics, which is used when one wants to study many body problems. This involves the ideas of statistical fluctuations, as seen in the appearance of Bose-Einstein, $n_b$, or Fermi-Dirac, $n_f$, distributions.

The combination of quantum field theory and statistical mechanics is called thermal field theory. It is therefore used in many-body problems where the underlying dynamics is de-

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scribed by quantum fields. It is describing quite a different physical situation as compared to that appropriate for normal quantum field theory. Both quantum and statistical fluctuations must be accounted for at the same time. It is therefore not surprising if thermal field theory has some quite different properties from standard quantum field theory. What is amazing is that formally the two can be described in a very similar way, as I will briefly outline below.

As an example of the different physics, consider an electron propagating through a vacuum and through a QED plasma. The usual quantum fluctuations are always present and must be accounted for. The first Feynman diagram of figure 1 is one such contribution and the internal lines represent interactions of the electron with virtual particles present in the vacuum. For the many-body problem only, one must also consider interactions with the real physical particles present in the plasma. The last diagram of figure 1 represents such an example, while the middle two diagrams represent the effects of a mixture of quantum and statistical fluctuations. The graphical representation of figure 1 is closely linked to terms seen when the Feynman integral representation of the electron self-energy for Minkowski time-ordered Green functions in thermal field theory is examined.

1.2. An example of the problem

Consider the example of a decay rate of a particle in a many-body situation, say a baryon number violating process in the early universe. Typically in a one-loop Feynman diagram calculation, the decay rate at high temperatures is found to be of order $g^2 T$ where $g$ is the gauge coupling and $T$ the temperature.

However we know from our experience with zero-temperature field theory that the coupling we measure at lab scales is not always a good measure of the coupling strength relevant for processes at other scales. One excellent solution is to use renormalisation group which gives a link between the best values to use for the coupling constant at different scales. It is therefore vital to supplement the original perturbative calculation with what is non-perturbative information obtained from renormalisation group analysis.

In a many-body problem, it is therefore essential to use the renormalisation group to ‘run’ the coupling from the known lab (zero temperature) value, to a value appropriate for a hot plasma, effectively $g \rightarrow g(T)$. This can be done in principle and one needs to know how the quantum and statistical corrections to the coupling. This usually involves calculating one-loop three-point diagrams.

Now the problems begin to appear. The calculations have been done for QCD but the results are generic. There are two distinct calculational schemes used in thermal field theory, RTF (the Real-Time Formalism) and ITF (the Imaginary Time Formalism). In this case the calculations using RTF suggested that such diagrams could grow as fast as $T^3$ at high temperatures for the case where the external legs of the diagram were at zero energy. On the other hand, it was known that if one used ITF to do the calculation, with zero energy on external legs, the diagrams can not grow faster than $T$. In solving this apparent disparity, a proper understanding of the ITF calculations in particular emerges.
2. A general approach to thermal field theory

In order to provide a precise answer to the question of what is being calculated in the different thermal field theory schemes, I will first outline what these schemes are. I will use a path ordered approach to thermal field theory as opposed to the alternative operator style used in Thermo Field Dynamics. The results hold in either case. The great advantage of this path ordered approach is that we can obtain different formalisms through different choices of a curve $C$ in the complex time plane. The derivation will show that they must contain the same physics, they merely encode it in different ways.

The starting point is the generating functional $Z[j]$ for the example of a single scalar field. The $T_c$ indicates some sort of time ordering of the operators, and the time integration is along some path $C$, both to be discussed later. Note that the statistical mechanics element is suggesting the $\text{Tr}\{e^{-\beta H}\ldots\}$, so that $Z[j = 0]$ is the partition function of statistical mechanics. The quantum field theory comes in through the path ordered exponential. This means that the Green functions, the key tools of Quantum field theory, are generated by taking functional derivatives with respect to the sources of the generating functional and then setting these unphysical sources to zero.

\[
\frac{\partial^N Z[j]}{\partial j_1(\tau_1)\partial j_2(\tau_2)\ldots\partial j_N(\tau_N)}\bigg|_{j = 0} = \text{Tr}\{e^{-\beta H}T_c\phi_1(\tau_1)\phi_2(\tau_2)\ldots\phi_N(\tau_N)\} \quad (2)
\]

Now the problem in trying to link the TFT expression (1) to those normally encountered in regular QFT is the thermal weight $e^{-\beta H}$. The way this is formally accounted for is the key trick of thermal field theory, and that is to regard the thermal weight as a time evolution operator. We have

\[
\text{Tr}\{e^{-\beta H}\ldots\} = \sum_n \langle n;\tau_m|e^{-\beta H}\ldots|n;\tau_m\rangle = \sum_n \langle n;\tau_m - i\beta|\ldots|n;\tau_m\rangle. \quad (3)
\]

Thus we are treating this thermal weight as if it was a time translation of amount $\beta$ in the Euclidean direction. Note that temperature has not replaced time in any sense, the dots in (3) represent combinations of field operators which still have any time lying on the path $C$.

This can now be compared with the expressions usually encountered in quantum field theory which are of the form $\langle n_{out};\tau_{out}|\ldots|n_{in};\tau_{in}\rangle$. One can then just use the standard tricks of quantum field theory to proceed. For instance, the path integral can be used and this gives

\[
Z[j] = \sum_{\Phi_{in}} \langle \Phi_{in}(\vec{x});\tau_m - i\beta|\exp\{T_c\int_C d\tau \int d^3\vec{x} j(\tau,\vec{x})\hat{\phi}(\tau,\vec{x})\}|\Phi_{in}(\vec{x});\tau_m\rangle
= \int_{PBC} D\Phi \exp\{i \int_C d\tau \int d^3\vec{x}(\mathcal{L} + j\Phi)\} \quad (4)
\]

where $C$ must now run from $\tau_m$ to $\tau_{out} = \tau_m - i\beta$. The $PBC$ on the path integral stands for the periodic boundary conditions which must be enforced, namely that $\Phi(\vec{x},\tau_m) = \Phi(\vec{x},\tau_m - i\beta) = \Phi_{in}(\vec{x})$.

Equation (4) is a remarkably compact result. It is also clear that temperature only appears as a boundary condition, viz we only consider fields configurations which are periodic in time over $-i\beta$. This does not stop us from having field configurations with any sort of time
dependence in the real physical time direction or over the unphysical range of imaginary times up to $-i\beta$.

For the purposes of the problem at hand, the most important point is that all times must lie on a curve $C$ which runs from $\tau_{in}$ to $\tau_{out} = \tau_{in} - i\beta$. The fields in this formalism, e.g. the operators in Green functions generated from $Z[j]$, only take times whose values lie on $C$. Further, working from the definition of the path integral, we see that the path ordering in (1) must be path ordering with respect to this same curve $C$. Thus the Green functions generated from $Z[j]$ (2) are ones where the field operators are ordered according to the position of their time arguments on $C$, the closer to $\tau_{in}$ ($\tau_{out}$) the closer that field is placed to the left (right) in the thermal expectation value. Thus we have identified for the general case exactly what is being calculated in thermal field theories.

The great advantage of this path ordered approach to thermal field theory is that we are free to choose $C$ as we wish so long as it runs from $\tau_{in}$ to $\tau_{out} = \tau_{in} - i\beta$. Thus we can obtain the two main formalisms through different choices of $C$. The derivation shows that all formalisms must contain the same physics, they merely encode it in different ways as they generate different types of Green function, path ordered with respect to different curves.

### 3. The Real-Time Formalism

The RTF (real-time formalism) can be generated by choosing the curve of figure 2, and then taking the limit of $t_{in} \to \infty$ to give the second version in figure 2. The curve of figure 2 avoids various problems found when alternative curves are used. The parameter $\alpha$ is unphysical and does not appear in any physical quantity.

The RTF is designed to be very like standard Minkowskii zero temperature field theory. When a field lies on the $C_1$ section it is just like those encountered in Minkowskii field theory. Even when the field lies on $C_2$, the field still depends only on a real-time parameter which varies between $-\infty$ and $+\infty$. The formalism contains fields at real physical times at the expense of having to give every physical field (the bits associated with $C_1$ time arguments, ‘type 1’) a thermal partner (associated with the $C_2$ section, ‘type 2’). With experience, it is easy to cope with this doubling.

The important element for the problem at hand is that this means that if we look at diagrams with type 1 external legs this corresponds to calculating

$$\text{Tr}\{e^{-\beta H}Tc_1(\tau_1)\phi_2(\tau_2)...\phi_N(\tau_N)\} = \text{Tr}\{e^{-\beta H}T\phi_1(t_1)\phi_2(t_2)...\phi_N(t_N)\},$$

where $\tau_i = t_i \in \mathbb{R}$. RTF was intended to look like zero temperature field theory, so it is not surprising to see that it is calculating time-ordered Green functions of fields with real physical times. This is what is calculated in most RTF calculations.
4. The Imaginary-Time Formalism

It turns out that the older ITF\textsuperscript{1,4,5,6} (Imaginary-Time Formalism) is the one which had not been studied in depth. The path chosen for ITF in path ordered approach just runs down the imaginary-time axis as shown in figure 3. This means that only Euclidean times appear directly in this formalism. This is fine for time independent quantities such as specific heats and other thermodynamic things.

However, when one wants to learn about real dynamics we need physical, that is real, times. ITF, however, gives $N$-point Green functions which are functions of $N-1$ independent Euclidean times. Just like any other function one may attempt to analytically continue it to real values. This is usually straightforward given an explicit expression. One question that immediately comes to mind is what type of Green function does one obtain after such a continuation? After all one starts with a Green function which is the expectation value of fields ordered according to their Euclidean time arguments. In what order are the fields when all their values have been continued to real times? The answer is surprising and not very obvious.

4.1. The analytic continuation of Euclidean Green functions

Such a fundamental question for ITF has been studied for the most important case, that of the two-point function. This was done by Baym and Mermin in 1961\textsuperscript{8,6}. There they showed that one obtains the retarded/advanced two-point Green functions. Further it is easy to see that the real part of these Green functions, which is related to the effective thermal mass, is the same as the real part of the time-ordered Green function. The two different types of Green functions do differ by $O(T)$ but only in their imaginary part.

What is surprising is that this work had not been extended. The example of the running coupling constant, given in section 1.2, shows that understanding higher Green functions is important. It was as if it was assumed that the same results (real parts the same, imaginary parts slightly different) held for higher Green functions.

It is not trivial to generalise the work of Baym and Mermin to $N$-point functions. Further, the results for two-point thermal Green functions do not generalise in a very obvious way either. However their basic approach still works. One uses three ideas. First the definitions of the various types of Green function are written down for a general case, the Euclidean time-ordered Green functions in particular. Next the boundary conditions are encoded in terms of Green functions with fields in a fixed order, the thermal Wightman functions. This is a generalisation of the KMS\textsuperscript{12,6} (Kubo-Martin-Schwinger) condition from two- to $N$-point functions. One simply uses the definition of a thermal Green function as trace, uses the cyclicity and then interprets the $\exp\{-\beta H\}$ factor as a time translation in the usual way. Lastly, one finds that one must make some assumption about the behaviour of the Green functions at large real times.
One then obtains a spectral representation of the Green functions that are calculated in ITF. This is\(^{13}\)

\[
\Gamma^{(N)}(\{z\}) = \left(\frac{-1}{2\pi}\right)^{N-1} \int_{-\infty}^{\infty} \frac{dk_1 \ldots dk_N}{i} \delta(k_1 + \ldots + k_N) \cdot \left[ \rho_{123\ldots N}(\{k\}; \beta, \ldots) \right] \\
\times \left[ (z_2 + z_3 + \ldots + z_N) - (k_2 + k_3 + \ldots + k_N) \right] \times \left[ (z_3 + \ldots + z_N) - (k_3 + \ldots + k_N) \right] \\
\times \ldots \times \frac{i}{z_N - k_N} \right] + (\text{ALL permutations of } (123 \ldots N)) \tag{6}
\]

The dependence on the three-momentum or space variables has not been written explicitly. The \(k\)'s are real energy variables and the \(z\)'s are complex energy variables. One of the \(k\)'s is clearly redundant, and an \(N\)-th \(z\) complex variable has likewise been introduced through the complex equation \(\sum z_j = 0\). This has been done to make the expressions symmetric and easy to write down.

All the individual details of the theory (masses, coupling constants etc.) and the approximation used to calculate the Green function are contained in the spectral densities, \(\rho\). The dots in the arguments represent dependence on these factors. The spectral densities are the difference of two thermal Wightman functions,

\[
\rho_{123\ldots N}(\{k\}; \beta, \ldots) = \text{Tr}\left\{ e^{-\beta H} \phi_1(k_1)\phi_2(k_2)\phi_3(k_3) \ldots \phi_N(k_N) \right\} / \text{Tr}\left\{ e^{-\beta H} \right\} \\
- (-1)^N \text{Tr}\left\{ e^{-\beta H} \phi_N(k_N) \ldots \phi_3(k_3)\phi_2(k_2)\phi_1(k_1) \right\} / \text{Tr}\left\{ e^{-\beta H} \right\} \tag{7}
\]

for the case of pure bosonic fields. For instance in the case of two-point functions one might finds there is just one spectral density and in simple cases this has the form \((\theta(k) - \theta(-k))\delta(k^2 - \omega^2)\). The \(\omega\) is a three-momentum dependent dispersion relation, \(\sqrt{\vec{k}^2 + m^2}\) for relativistic free fields. In general the dispersion relation and the spectral function’s form are more complicated\(^{14,15}\).

The aim is to extract real dynamics and therefore to look at the real energy limit of the \(z\)'s. However, it is clear that the expression (6) has cuts on the real \(z\) axes. Further, there are also cuts at real values of \(z_1 + z_2, z_1 + z_2 + z_3\), and indeed in all possible sums of these complex energies. One must therefore specify from which side of the cut one is approaching by leaving in a small imaginary part,

\[
z_j \rightarrow E_j + i\epsilon_j \tag{8}
\]

where \(E_j\) is real, \(\epsilon_j\) is real and infinitesimal, and both can be either positive or negative. Note that it is the sign of the \(\epsilon_j\)'s, and the sign of all possible sums of the \(\epsilon_j\)'s, which tell us which side of each cut we are looking at. For \(N \geq 4\), the sign of the single \(\epsilon_j\)'s does not always specify the signs of all possible \(\epsilon_j\) sums\(^{13}\).

Now we are at last in a position to answer the question exactly what sort of real-time Green function has ITF calculated, in particular what operator ordering has we been left with after analytic continuation. Every distinct way of approaching all these cuts corresponds to a different operator ordering and so a different type of Green function. To see exactly what type of Green function has emerged, one merely does a Fourier transform of (6) from \(E\)'s to real times \(t\), for a given analytic continuation (8). However, the resulting expression can be further manipulated by using identities such as \(\theta(t) + \theta(-t) = 1\). This makes it highly non trivial to see if an expression is equivalent to something familiar or not.

The results\(^{13}\) can be summarised as follows:-
• The analytic continuation of ITF Green functions produces the thermal Generalised Retarded Functions. Eqs (6) and (8) form a definition of these functions c.f. at zero temperature\textsuperscript{16,17}.

• For the special case where one epsilon is positive, say $\epsilon_j > 0$, and all the others are negative (which fixes all possible sums of epsilons too), one obtains the $j$-th retarded function, $R_j(t_j; \{t_{\text{others}}\})$\textsuperscript{13,18,19}. The advanced functions are obtained when the signs of the epsilons are the opposite way round. These functions are discussed below.

• No one analytic continuation of an ITF N-point Green function produces the time-ordered Green function\textsuperscript{13,18}.

The retarded and advanced functions are very useful. They appear in the linear response approximation which tries to describe small perturbations to a plasma\textsuperscript{5,6}. The retarded functions are usually written as expectation values of multiple commutators, with theta functions. One field is picked out as being special, they are symmetric only in the remaining fields. That one special field must have the latest time if the retarded function is to be non-zero. This means there are $N$ N-point retarded functions. The advanced functions are the same but with all the opposite theta functions $\theta(t) \rightarrow \theta(-t)$ and a possible overall minus sign. For instance, for pure bosonic fields,

$$R_1(t_1; t_2, t_3, t_4) = \theta(t_1 - t_2)\theta(t_2 - t_3)\theta(t_3 - t_4)\text{Tr}\{e^{-\beta H}[[[\phi_1, \phi_2], \phi_3], \phi_4]\}/\text{Tr}\{e^{-\beta H}\}$$

More recently, Baier and Niegawa have derived the link between ITF and the retarded/advanced functions only, in a much simpler way\textsuperscript{19}.

One interesting sideline is that I do not know how many of these generalised functions ITF produces. There are $N$ retarded and $N$ advanced functions, but they are a small subset of the total number of generalised retarded functions. The results to date are shown in table (1). The results for $N=2-6$ were produced elsewhere\textsuperscript{13} whereas the $N = 7, 8$ results were found first by van Eijck using a different method. I have cross checked the $N = 7$ result. The results up to $N = 6$ were independently derived in the zero temperature case by Araki\textsuperscript{16} but the $N = 6$ result quoted there seems to be incorrect.

5. Physical Implications

The most important physical implication comes when we try to compare time-ordered functions and retarded/advanced functions. This is because the former are most easily produced in RTF whereas ITF is almost always calculating the latter. One finds for truncated/1PI functions\textsuperscript{18}

$$\Gamma_{t.o.}^{1\text{PI}}(E_1, E_2, E_3) = n_2n_3(\sigma_1 R_1^{1\text{PI}} - e^{-\beta E_1}A_1^{1\text{PI}}) + (\text{All (123) permutations})$$

Table 1. Numbers of $N$-point functions. The rows are- $N$, number of external legs; $R, A$ numbers of retarded and advanced functions; RTF, numbers of distinct RTF functions; TWF, the number of thermal Wightman functions; ITF\&GRF, the number of generalised retarded functions.
where \( n_j = (\exp\{\beta E_j\} - \sigma_j)^{-1} \) and \( \sigma_j \) is +1 (−1) if the \( j \)-th field is bosonic (fermionic). \( A_j \) is the \( j \)-th advanced thermal Green function. For the case of purely bosonic fields, we find that the real parts of the Green functions in the limit of \( T \gg E_j \) satisfy

\[
Re\{\Gamma^{\text{1PI}}_{\text{t.o.}}(E_1, E_2, E_3)\} = T^2 \left( Re\{R^{\text{1PI}}_{1}(E_1)\} \frac{E_1}{E_1 E_2 E_3} + (\text{All other (123) permutations})\right) + O(1)
\]

(11)

Roughly speaking then, we find that the time-ordered (\( \Gamma^{\text{1PI}}_{\text{t.o.}} \)) and retarded/advanced 1PI functions differ by a factor of the square of the temperature. This accounts for the differences in the results of the calculations of QCD three-point diagrams\(^2\) described in section 1.2.

From this work we see that the same Feynman diagram represents different types of thermal Green function for different formalisms. Further, different types of thermal Green function differ by large amounts. Therefore it is essential that one understands quite carefully which type of Green function is required for a given physical problem. Only then, using the results described here, can one choose the calculational scheme best suited for your purpose. The similarity in the Feynman rules of thermal and normal zero temperature field theory obscures the fundamental differences between the physics of the two theories. These differences arise because thermal field theory is modelling a very different physical situation, a quantum plasma not a quantum vacuum.

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