Models of breakup: a final state interaction problem

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Abstract In this paper, we discuss the evolution of breakup models from fully quantum mechanical, such as the Ichimura–Austern–Vincent model to semiclassical, to eikonal approximations following the insight on the mechanism first proposed by Hussein and McVoy (HM) for the presently called stripping term. In particular we concentrate on, and stress that, the correct implementation of a quantum mechanical model of breakup requires the use of energy dependent interactions and the energy averaging procedure is a key point to understand the difference among models. On the other hand using fixed energy potentials is one of the steps towards the high energy eikonal limit first proposed by HM. However the intermediate semiclassical transfer to the continuum model of Bonaccorso and Brink does use an energy dependent nucleon–target optical potential, while fixing the core–target interaction at the incident energy. The relationship between these methods is clarified.

1 Introduction

In the last 40 years nuclear reaction theory largely benefitted from the activity of Mahir Hussein (MH). Mahir had been a student of A. Kerman and thus he came from one of the world-class top-schools in the subject. In particular MH dedicated a large part of his research activity to the developments of direct reaction theory. Mahir had a gift for physics insight of complicated problems and one of his most notable achievements was to create a tradition of semiclassical nuclear reaction models and a group of experts on it in Brasil.

Among direct reactions, breakup emerged as one of the most important following the seminal papers of a number of authors. They are mostly based on extensions of the Distorted Wave Born Approximation (DWBA) theory to the case of an unbound final state. The problem with DWBA is that it has a complicated formalism and it is difficult to disentangle the various reaction mechanisms which are believed to contribute to the inclusive spectra. The work of Huby et al. [1,2], is similar to our Semiclassical Transfer to the Continuum (STC) [3–7] since in both theories the final state for the unbound nucleon is represented by a scattering state with an appropriate normalization. The problem of normalization is of fundamental importance if one is interested in the absolute value of the breakup cross section and not only in the shape of the spectra. In fact since the advent of exotic nuclei, measurements and calculations of absolute breakup cross sections [8,9] have concentrated on the extraction of spectroscopic factors [10]. Other approaches by Tamura and Udagawa [11,12], Mermaz [13,14] are based on statistical compound nucleus theories. They require quite lengthy numerical calculations and do not determine the absolute normalization. McVoy and Nemes [15], proposed a simple model based on the Plane Wave Born Approximation to calculate both transfer to continuum and break-up. However they obtained only very qualitative results.

Later Ichimura [16], revised the existing approaches to what has been called inclusive break-up, which is the sum of elastic and inelastic breakup (called also diffraction and stripping) corresponding to an inclusive reaction in which only the core is detected when a given projectile fragments on a target. These approaches are due to Udagawa and Tamura [11,12], Austern and Vincent [17], Kasano and Ichimura [18], J. Pam岌us et al. [19], Baur et al. [20–22], and finally to Hussein and McVoy [23]. These works mainly differ in the model wave function used to describe the final state as we shall see in the following.

In fact there are several ways in which a nucleon in a bound state in the projectile can make a transition to a continuum final state. In all cases the nucleon ends up in an
unbound state. The approaches differ in their treatment of the final state interactions (FSI) and of the corresponding continuum nucleon wave function. One possibility is transfer to a continuum state of the target. This means that final state interactions of the nucleon with the target are included. Transfer to resonances states of the target can be described by this approach. A second possibility is to include FSI with the projectile–core and a third is to neglect FSI altogether. It is clear that these processes can not all be independent. They are discussed in detail in a formal way based on the time dependent G-matrix approach in Ref. [4] and in Appendix C of this work.

Then, in the originally complicated scenario, emerged the paper by Hussein and McVoy [23] in which a simple semi-classical reduction of the inclusive breakup formula was proposed in terms of a WKB approximation for the projectile and core distorted waves. Furthermore averaging the energy dependence of the interactions and wave functions on the whole kinematically allowed range and thus taking the relative projectile–target, core–target, and nucleon target momenta and potentials at the value of the incident energy per nucleon an eikonal formula was deduced. Note that the inclusive breakup models contain two terms, one for the elastic breakup and another one for the inelastic breakup. However the eikonal reduction for the EBU term was not attempted by HM. Thus a formal derivation of the eikonal formalism for EBU from one of the fully QM methods is still missing.

On the other hand Fujita and Hufner [24] and Hufner and Nemes [25] started directly from an eikonal formalism to describe breakup at relativistic energies. Later on, in the ‘90s, somewhat independently Yabana et al. [26,27] and Hencken et al. [28] introduced eikonal models to study halo breakup in reactions induced by radioactive nuclei. They obtained in an almost straightforward way the stripping term and derived also the diffraction term but for the integrated cross section they needed to introduce a further hypothesis besides the initial weak binding namely that the projectile should have no bound excited states. For diffraction they derived a formula for the total cross section and the intrinsic momentum distribution. It should be noticed that HM did not use the weakly bound projectile hypothesis to deduce their model and obtained as we said before only the stripping term. Later on an eikonal formalism for core perpendicular and parallel momentum distributions was obtained by Bertulani et al. [29]. Carstoiu et al. [30] used a sudden approximation method to obtain both parallel and transverse distributions and absolute cross sections.

One of the fully quantum mechanical QM models of breakup, introduced by Ichimura–Austen–Vincent (IAV) [31], was originally complicated to understand from the formal point of view and also it was impossible to apply in a numerical calculation unless the projectile was small and the energy low, because computers at that time were not able to handle the large number of partial waves necessary for heavy-ion reactions at high or even intermediate energies. Recently this model has been revised by Jin Lei and Moro [32–36] and other groups [37–39] who have been able to implement it numerically and made several applications, still at low energy and for deuteron or α breakup.

Thus, it is now well understood that considering the FSI of the breakup nucleon with the target means that the nucleon can re-scatter on the target elastically and inelastically. The two possibilities correspond obviously to the elastic and inelastic cross sections of a free-nucleon–target interaction as described by the optical model for example. This suggests that physically the most natural choice of the nucleon–target final wave function is a continuum wave function determined by the optical model. Because the nucleon in the continuum can have an energy from zero to a maximum allowed by kinematical conditions, it is clear that the final wave function must be energy dependent. Thus the correct implementation of the model requires an energy dependent optical potential as stressed by IAV [30]. However an energy dependent potential while applied in the past to low energy reactions and/or small projectile-target combinations [32,38,39], has not been implemented until recently [40] in any of the QM model mentioned above in the case of heavy ion reactions at high energy (>50 A.MeV). On the other hand the standard implementation of the STC method is with an energy dependent potential because it was introduced for heavy-ion reactions at medium to high incident energies where the energy spectra are quite broad [41–43]. Recently results from the IAV and STC models have been compared [40] calculating neutron and proton breakup from 14O and 16C on a 9Be target [44]. This is the first example of an application of the IAV model implemented with an energy dependent potential, for heavy ions at intermediate energies.

In Sect. 2 of this paper we shall begin by giving IAV and HM formulae and linking them to the STC formalism and we shall show explicitly how the choice of an appropriate final state wave function and potential come about. Energy averaging will be also discussed and in Sect. 3 we will give some examples of energy spectra and integrated cross sections according to the STC and eikonal models and clarify the limit of applicability of each of them. Finally in Sect. 4 our conclusions will be drawn.

2 IAV, HM and STC models

2.1 IAV model

In this section, we briefly summarize the model of Ichimura, Austern, and Vincent (IAV), whose original derivation can be found in Refs. [31,45] and has been also recently revisited by several authors [32,33,35–40]. We outline here the main
results of this model and refer the reader to these references for further details on their derivations. The discussion and notation follows that of the recent publication [40] in which the IAV and STC method were compared.

We write the reaction under study in the form,

\[ P(= C + n) + T \rightarrow C + B^* , \]  

(1)

where the projectile \( P \), composed of \( C \) and \( n \), collides with a target \( T \), emitting \( C \) fragments and any other fragments. Thus, \( B^* \) denotes any final state of the \( n + T \) system.

This process is described by the effective Hamiltonian

\[ H = K + V_C + U_C(T) + H_T(\xi) + V_{nT}(\xi, r_n) , \]  

(2)

where \( K \) is the total kinetic energy operator, \( V_C \) is the interaction binding the two clusters \( C \) and \( n \) in the initial composite nucleus \( P \), \( H_T(\xi) \) is the Hamiltonian of the target nucleus (with \( \xi \) denoting its internal coordinates), and \( V_{nT} \) and \( U_C(T) \) are the fragment–target interactions. The relevant coordinates are depicted in Fig. 1.

In writing the Hamiltonian of the system in the form (2) we make a clear distinction between the two cluster constituents: the interaction of the fragment \( C \), the one that is assumed to be observed in the experiment, with the target, is described with a (complex) optical potential. Inclusive breakup processes arising from this interaction (e.g., target excitation) are included only effectively through \( U_C(T) \). The particle \( C \) is said to act as spectator. On the other hand, the interaction of the particle \( n \) with the target retains the dependence of the target degrees of freedom (\( \xi \)). In the following this dependence is released by the choice of a nucleon–target optical potential.

Starting from the Hamiltonian of Eq. (2) IAV derived the following expression for the double differential cross section for the inclusive breakup with respect to the angle and energy of the \( C \) fragments:

\[ \frac{d^2 \sigma}{dE_C d\Omega_C} = \frac{2\pi}{\hbar v_p} \rho_C(E_C) \left( \rho_n(E_n) \int |\langle \chi_n^{(-)}|S_n \rangle|^2 d\Omega_n \right) \]  

(3)

where \( v_p \) is the projectile–target relative velocity, \( \rho_C(E_b) = k_C^3 \mu_C / ((2\pi)^3 h^3) \) and \( \rho_n(E_n) = k_n^3 \mu_n / ((2\pi)^3 h^3) \) are the density of states for the particle \( C \) and \( n \) respectively, \( W_n \) is the imaginary part of the optical potential \( U_n \), obtained from the optical reduction of \( V_{nT}(\xi, r_n) \) which describes \( n + T \) elastic scattering, \( G_n = 1/(E_n^{(+)} - U_n - T_n) \) is the Green’s function for the nucleon–target channel, \( E_n \) is the energy in the \( n + T \) channel which satisfies the energy conservation, \( \chi_n \) is the distorted-wave in this channel, and \( S_n \) is the source term which takes the form

\[ S_n(r_n) = \langle r_n | \chi_C^{(-)} | V_{post} | \psi^{3b(+)} \rangle , \]  

(4)

\( \chi_C^{(-)} \) is the distorted-wave describing the scattering of \( C \) in the final channel with respect to the \( n + T \) subsystem, and \( V_{post} = V_{Cn} + U_C(T) - U_C(T) \) is the post form transition operator.

Thus finally the NEB term can be written in a more compact form, which is useful for the comparisons we shall make in the next sections, as:

\[ \frac{d^2 \sigma^{NEB}}{dE_C d\Omega_C} = -\frac{2}{\hbar v_p} \rho_C(E_C) \langle \psi | W_n | \psi \rangle \]  

(5)

where

\[ \psi(r_n) = G_n S_n \]

and \( G_n \) and \( S_n \) are the Green function and source term discussed above.
2.2 From IAV to HM and the eikonal approximation

HM model [23] differs from IAV [40] in the choice of the wave function \( \psi \) in Eq. (5) which they take as

\[
\psi(r_n) = \langle \chi_{PT}(-) | \chi_C^{(+)} \phi_P(r_{CT}) \rangle.
\]  

(6)

A big step forward in the understanding of the QM models of breakup came about when HM made the hypothesis in the formalism that the core and the nucleon breakup could be decoupled by summing the core–target optical potential to the nucleon–target optical potential:

\[
U^{PT} = U^{CT} + U^{nT}
\]

and then assumed eikonal wave functions for the distorted waves in the entrance and exit channel. In this way the entrance wave function factorises into the core and neutron wave functions \( \chi_{PT}^{(+)} = \chi_C^{(+)} \chi_n^{(+)} \) such that the two eikonal phases of the core in the entrance and exit channel sum-up providing the core–target S-matrix. Then in the calculation of Eq. (6) the eikonal S-matrix of the core wave function factorises from the S-matrix of the nucleon wave function. This is indeed from the physics point of view similar to the starting point of the eikonal model of breakup of Yabana et al. and Hencken et al. [26–28] as summarised by Eq. (17) below and it is straightforward to obtain the final cross section formulae, Eqs. (24,25). The great merit of the HM paper was then to make it clear what the core spectator model would be in practice and that the n-target S-matrix

\[
\frac{d^2 \sigma}{dE_C d\Omega_C} = \frac{2 \pi}{\hbar v_P} \rho_C(E_C) \int d(r_r) \psi_{eik}^*(r_{CT}, k) W^{nT} \psi_{eik}(r_{CT}, k)
\]

(8)

2.3 STC model

The semiclassical STC model [3–5,7] is a generalization to final unbound states of the transfer between bound states model of Brink and collaborators [46–50]. Semiclassical methods were very popular in the ’70s as a substitute to full DWBA calculations which, as mentioned in Sect. 1, were very lengthy and computationally expensive. Transfer reactions were a common tool to study single particle characteristics, in particular occupation probabilities but often theoretical calculations gave cross sections much larger than the data and spectroscopic factors different from the shell model values [51–56]. Then an attempt to disentangle the content and the ingredients of the DWBA approach via semiclassical methods which could provide analytical expressions for the cross sections. The procedure followed was first to choose a WKB wave functions for the distorted waves, similarly to HM [23], then the standard reduction of the three dimensional integral for the transfer form factor to a surface integral, similarly to what is done in Refs. [19,20,57–60]. Finally analytical Hankel functions were used for the initial and final states on the surface between the two nuclei. The method is valid for peripheral reactions based on the core spectator model as mentioned above. The use of the Hankel function, which is the asymptotic form of the initial state wave function is an additional approximation in the STC which is not present in the HM formalism. However the advantage is that it allows the calculations to be carried on analytically up to the end allowing for a transparent interpretation of the formalism and of its results, therefore we remind in the following a few steps leading to the final probability and cross section formulae Eqs. (13) and (15). The full formal derivation is given in the Appendices B and C.

The semiclassical transfer to the continuum amplitude [3] is:

\[
A_{fi} = \frac{1}{i \hbar} \int_{-\infty}^{\infty} dt \langle \phi_f(r_n) | U_{nT}(r_n) | \phi_i(r_n - R(t)) \rangle e^{-i \tilde{\omega} t}
\]

(9)

with \( \tilde{\omega} = (\omega t - mvz/\hbar) \). \( R(t) \) is the classical trajectory of relative motion corresponding to the coordinate \( r_P \) in Fig.1. The time dependent nucleon initial and final wave functions in their respective reference frames are

\[
\psi_{i,f}(r_n, t) = \phi_{i,f}(r_n) e^{-iE_{i,f} t/\hbar}
\]

(10)
Initial and final radial wave functions are taken as Hankel functions according to [3] such that we have for the initial state:
\[
\phi_i(r_Cn) = -C_i \gamma h^{(1)}_{\ell i}(i r_C n) Y_{l_{i,m_i}}(\Omega_i).
\]  
(11)

And for the final continuum state a scattering wave function defined as:
\[
\phi_f(r_n) = C_f k_f \frac{i}{2} \left( h^{(e)}_{\ell f}(k_f r_n) - S_f h^{(c)}_{\ell f}(k_f r_n) \right) Y_{l_{f,m_f}}(\Omega_f).
\]  
(12)

\(S_{\ell f}(\varepsilon_f)\) is the n-target S-matrix at energy \(\varepsilon_f\).

For proton breakup we use the same type of wave functions. For the initial state we first calculate the exact proton bound state wave function and then we fit to it a neutron wave function. Finally we use such a neutron wave function and the corresponding, effective separation energy. This method was checked and found very accurate in Ref. [61].

Using the above definitions in the amplitude Eq. (9) and then taking the modulus square, the breakup probability reads:
\[
\frac{dP_{-n}}{d\varepsilon_f} \approx \frac{1}{2} \Sigma_{\ell f} (2 j_f + 1) |1 - S_{\ell f}|^2 (1 + R_{\ell f})
\]
\[\times \left[ \frac{\hbar}{mv} \right]^2 \frac{m}{\hbar^2 k_f} |C_i|^2 e^{-2\hbar h c \frac{2\eta b_c}{2\eta b_c}} M_{l_{j_{1f}}}, \]
(13)

Note that up to this stage the practical choice of the n-target S-matrix has not been made yet, we have only required that the final n-target state be a scattering state with respect to the target describing the n-target FSI. The above formula is very general and as such it contains all possibilities discussed in the models of Sect. 1, when the interest is not on the n-core FSI. As it was shown in Ref. [3], the following choices can be made for the model potential and its possible energy dependence, describing different physical situations:

- An energy independent real potential and just one angular momentum state \(j_f\) would describe a low energy single particle resonance in the target holding the correct normalization (c.f. Eq.(3.2) of [3]).
- An energy independent real potential and the sum over final spin states of the target \(j_f\) would describe elastic breakup including the sum of narrow resonances.
- An energy dependent optical potential would describe compound nucleus resonances, both at low and high n-target energies. At low energies the resonances will be narrow, at high energies they will be broad and overlapping. In this case one needs to keep the sum over final states in Eq. (13) and the formula needs energy averaging, according to the optical model, under the hypothesis that only the term \(|1 - S|^2\) has a strong energy dependence. Thus one obtains two terms representing the so-called elastic and inelastic breakup.
\[
\frac{dP_{-n}}{d\varepsilon_f} \approx \frac{1}{2} \Sigma_{\ell f} (2 j_f + 1) (|1 - S_{\ell f}|^2 + 1 - |S_{\ell f}|^2) B_{\ell f}
\]  
(14)

where \(B_{\ell f}\) contains all smoothly varying energy dependent factors of Eq. (13). Then apart from the choice of the potential one can further discuss the energy averaging procedure and the method to calculate the S-matrix.

- Suppose the incident energy is high enough for the matching conditions to favour high energy states of the nucleon in the continuum in which the potential can be considered smoothly varying and no resonances are present. In this situation an eikonal model calculation of the S-matrix can be acceptable and thus taking the classical limit in Eq. (13) \(\Sigma_{\ell f} \rightarrow \int d^2b_n\) the eikonal elastic breakup formula could be obtained but only in presence of a real potential.

- If absorption is present then the energy averaging is to be done according to Eq. (14) but still the S-matrix can be calculated in the eikonal approximation. In this way elastic and inelastic scattering would be obtained consistently in the eikonal approximation. From the nucleon energy distribution one can get the nucleon momentum distribution by using Eq. (16) below and the appropriate Jacobian. Note that in this case the eikonal limit taken from Eq. (13) would satisfy the kinematical condition \(\varepsilon_{\ell f}^{min} = 0\) translating into \(k_{1f}^{min} = -(\varepsilon_i + \frac{1}{2}m v^2)/(\hbar v)\).

To summarise: in Eq. (14) \(S_{\ell f}\) are nucleon–target S-matrices calculated for each nucleon final energy according to the optical model in an energy dependent optical potential, including the spin-orbit term of the nucleon–target optical potential. The sum of the two terms \((1 - |S_{\ell f}|^2 + 1 - |S_{\ell f}|^2)\) is obtained automatically [3] as a result of using an unitary energy averaged optical model S-matrix thus including non elastic and elastic breakup (stripping and diffraction). These two terms correspond to the first and second term of Eq. (3). The sum over partial waves in Eq. (13) is a sum over total nucleon–target angular momenta. \(C_i\) is the initial wave-function asymptotic normalization constant. It is obtained as the ratio between the numerically calculated single particle wave function and the Hankel function. The form factor \(e^{-2\hbar h c \eta b_c}/2\eta b_c\) is due to the combined effects of the initial and final wave-function Fourier transforms, while \(M_{l_{j_{1f}}}\) is due to the overlap of the angular parts. \(R_{\ell f}\) are spin-coupling coefficients. Further definitions and discussion can be found in Refs. [6,7] and in Appendix B.

In the core spectator model the breakup cross section is obtained by integrating the differential breakup probability on the core–target impact parameter by weighting it with the probability \(|S_{\ell f}(b_c)|^2\) that the measured core has survived
“intact” the scattering. This term in the HM model comes from the eikonal choice of the projectile and core distorted waves exactly as in the STC [47,48]. Finally if a shell model Woods–Saxon wave functions is used one multiplies by \( C^2 S \) the spectroscopic factor of the initial state

\[
\frac{d\sigma_{STC}}{d\xi} = C^2 S \int_0^\infty db_c|S_{CT}(b_c)|^2 \frac{dP_{n}}{db_c}. \tag{15}
\]

In Eq. (15) the variable \( \xi \) can be the nucleon final energy in the continuum \( \epsilon_f \) as in Eq. (14) and/or the nucleon momentum with respect to the core or target given in Eq. (16) below, or by using 4-energy momentum conservation (see for example [62]) and the relative Jacobian, the differential \( d\sigma_{STC}/d\xi \) cross section becomes directly comparable to the measured momentum distributions function of \( P_{\perp} \) the core parallel momentum.

This formalism does not include Coulomb recoil effects of the core because it does not distinguish the center of mass of the core–target system from the center of mass of the projectile–target, c.f. the coordinates \( r_{CT} \) and \( r_P \) in Fig. 1. Core recoil effects give rise to the so called Coulomb breakup which is important for heavy targets and very weakly bound initial states. It can be treated together with the nuclear breakup according to [61,63–66]. On the other hand, STC contains the energy recoil effect of the nucleon via the definitions of

\[
k_{1,2} = \left( \epsilon_f - \epsilon_i \mp \frac{1}{2}mv^2 \right) / (\hbar \nu)
\]

which can be interpreted as the z-components of the nucleon momentum in the initial (core) and final (target) reference frames respectively, see Appendix B. Those can be sampled in a breakup reaction according to the kinematical constraints.

### 3 Comparison of the models

In this section we make a critical comparison of the IAV, HM and STC model described before. In general it is clear that any proper theory of, at least, one nucleon breakup, should consistently provide the elastic and inelastic breakup terms. It should also be general enough to be applicable in a large range of incident energies, to any projectile–target combination and with no constraint on the initial nucleon separation energy and/or the number of possible projectile excited states. The IAV model satisfies those requirements while the eikonal HM model is a high energy model.

The STC formalism described in the previous section partially satisfies all the above requirements. It treats on the same footing both the elastic and inelastic breakup. It can be applied with energy dependent or independent potentials, real and/or complex and with different energy averaging procedures. Besides, fully analytical formulae can be obtained which help disentangling the complicated physics involved. Both the eikonal and STC formalism need an energy averaging with respect to the original IAV model but while the eikonal averages over the whole energy range spanned by the nucleon in the continuum, the STC averages the nucleon–target interaction over small intervals, according to the optical potential that fits the nucleon–target free particle cross section. The IAV model when implemented with an energy dependent optical potential contains the same energy averaging. The STC can treat also cases in which for a heavy target, the n-target system has high lying resonances in the continuum [41,43]. At present, due to the high number of partial waves required, the numerical implementation of the NEB by the IAV model can treat only low-energy and small projectile transfer to target resonances. The eikonal model cannot treat resonances. Finally it should be noted that the STC does not need to average the initial and final n-core and n-target momenta, thus naturally satisfying energy conservation for the nucleon, even when the S-matrix is calculated in the eikonal model. However in the STC the core–target S-matrix is treated in the eikonal approximation similarly to HM. For this reason the STC model can be defined as an intermediate model, because it treats the core–target relative motion in the eikonal approximation while the n-target motion and interaction is treated quantum mechanically. Energy conservation and correct kinematics within the eikonal model have been discussed also by Fujita and Hufner [24] and Ogata et al. [67].

Recently it has been shown that the IAV and STC methods lead to very close results [40]. However from the formal point of view they look quite different if one compares Eq. (3) with Eqs. (13) and (15). The main differences are that in the IAV method the core–target S-matrix is included in the source term Eq. (4) and and n-target S-matrix is considered as solution of an inhomogeneous equation while in STC they are solutions of homogeneous equations. In practice this means that the STC method decouples the core–target scattering from the nucleon–target scattering, considering them as independent. This corresponds to consider off-shell effects negligible and thus it calculates on-shell S-matrices. The historical origin of this difference lies in the fact that IAV approach originated as a method to calculate light nuclei (d) breakup at low energy while STC was developed to treat heavy-ion reactions at intermediate to high energies where surface approximation and thus decoupling of core-valence-particle degrees of freedom appear as the natural choice. Indeed one can see that such a decoupling could be done also in the IAV model by approximating the distorted wave for the projectile–target relative motion in the same way as done by HM, with the core–target distorted wave in the entrance channel multiplied by the nucleon distorted wave \( \chi_C T^{(+)} \approx \chi_C^{(+)} \chi_n^{(+)} \). In fact,
in the case of heavy-ions the projectile and core have nearly the same mass, the core is much heavier than the nucleon and the relative motion trajectory of the projectile and core are nearly the same, as assumed by the STC model. This again can be called the no-recoil approximation.

### 3.1 Numerical results

In this section we present results of numerical calculations for the neutron and proton breakup reactions $^9\text{Be}^{(14}\text{O},^{13}\text{O},X)$, $^9\text{Be}^{(14}\text{O},^{13}\text{N},X)$ at 53 A.MeV. The neutron breakup reaction has become a test case for breakup models since the data and theoretical analysis by STC were presented in [44]. In fact the experimental final core spectrum has shown the strongest deviation from the eikonal predictions, among the cases in literature, highlighting the role of kinematics and FSI effects. The reason is that the initial neutron separation energy is 23.2 MeV while the incident energy per nucleon was 53.2 A.MeV. Thus the medium available energy in the neutron continuum was only about 30 MeV. Recently the model and calculations by STC have been benchmarked by comparing them with the IA V model [40], resulting in excel-
Fig. 4 Proton breakup momentum distributions in the n-Core reference frame for the reaction $^{9}$Be($^{14}$O,$^{13}$N,X) at 53 A.MeV. Red and orange curves are from the STC and eikonal models respectively. Full lines for NEB, dashed lines for EBU.

The goal of this section is to compare the results of the STC with the eikonal model and to clarify the correct procedure to perform such comparison.

First it must be stressed, as argued earlier on in this paper, that at energies above the threshold for the first excited state in a n-target (n-T) interaction, the optical model requires an energy dependent strength for the complex potential. The IAV model and STC require the same for the potential providing the final state of the breakup nucleon. The eikonal model of breakup requires a complex potential but averages over the whole energy range, which means its implementation is with the n-T potential calculated at the incident energy per nucleon.

The IAV and STC contain the correct kinematics and energy conservation. The standard eikonal model of breakup neglects both, however it was shown in [68] that the kinematical cutoff could be easily implemented in the eikonal model. Effects of this correction, linking the nucleon separation energy with the incident energy were numerically predicted in [68] and verified in [40,44].

In both figures the notation is the same and in the legends the integrated cross sections corresponding to each curve are given. The full red curves are obtained by the STC model, the blue long dashed curves are obtained setting the strength of the real and imaginary parts of the n-T optical potential constant and equal to the value at 53 MeV, as it is done in the eikonal calculation, the green short dashed curves are obtained by setting the real part of the n-T potential equal to zero, as in the stripping calculation by the eikonal model. On the other hand the orange thin full curves show the standard eikonal calculation, while the brown double-dotted-dashed curves show the eikonal results in which the kinematical cutoff has been implemented.

The correct comparison requires that each method be implemented according to the physics that it represents, thus we should look first of all to the red curves vs. the orange curves corresponding to the STC and eikonal. It is clear that the curves are rather different and the integrated cross sections differ by about 20% for NEB.
One might naively imagine that the differences are due to the different potentials and/or to the kinematical cut off. To clarify this point we look at the blue long dashed curves and at the green short curve. Both curves and integrated cross sections are different from the standard STC and eikonal results. Thus there is no way to reconcile the STC, which is equivalent to a QM method such as the IAV, with the eikonal. One might also wonder on the role of the kinematical cutoff. Even the kinematically-corrected eikonal (double-dotted-dashed brown curve) does not fit any of the other spectra, nor it does the integrated cross section. Thus it is clear that for NEB cross sections a typical 20% difference is expected between STC results and the eikonal. The spectra look also different but probably in general not distinguishable in comparison to the data, apart from the kinematical cutoff.

The differences in shape are even more noticeable for the spectra of the EBU, Fig. 3 but the STC results and the eikonal results can be reconciled for the total cross section if one implements the kinematical cutoff in the eikonal. Otherwise the cross sections differ by about 30% the eikonal giving the larger value. There is no green short dashed curve in this case because for EBU the real potential cannot be set to zero. In this case the eikonal cross section is larger than the STC. Finally summing up NEB and EBU as given by their correct implementations, we obtain \( \sigma_{\text{tot}} = 13.15 \text{ mb} \) from STC (red curves) and \( \sigma_{\text{tot}} = 12.86 \text{ mb} \) (orange curves) from eikonal.

We note that the STC results show a spike close to the neutron threshold. It is due to the excitation of low energy resonances in the \( n+^9\text{Be} \) systems discussed in Ref. [44]. They are seen in the \( n+^9\text{Be} \) data and reproduced by our n-T optical potential [69]. Their presence in the STC results is a proof that this method can indeed be used to study transfer to target resonances as mentioned in Sect. 2.3 while the eikonal can be used only to get information on the projectile initial state.

For completeness in Fig. 4 the spectra of the proton breakup reaction are presented. The red curves represent the SCT, full line is the NEB while dashed line is the EBU. The orange curves give the eikonal results. In this case kinematical effects are less important because the proton separation energy is only 4.63 MeV. In fact the maxima of the spectra are closer to each other and to the \( k_1 = 0 \) point where the maximum would be expected if the momentum distributions after breakup represented exactly the nucleon momentum distributions in the initial states. In this case the eikonal gives larger cross sections than the STC, the difference being 20% for NEB and 10% for EBU. The summed cross sections are \( \sigma_{\text{tot}} = 30.35 \text{ mb} \) from STC (red curves) and \( \sigma_{\text{tot}} = 33.19 \text{ mb} \) (orange curves) from eikonal.

The conclusion of this section is that in both neutron and proton breakup case and from a very deep and less deep initial bound state the difference between the eikonal results and the STC results for the total cross sections are about 20% in agreement with our previous findings [68]. However the shape of the spectra can be very different demonstrating the effects of kinematics and FSI with the target. It appears that the use of an energy dependent FSI is more consistent with a QM model.

4 Conclusions

In this paper we have reviewed some of the models of breakup that at present are able to calculate heavy ion reactions and have been applied to exotic nuclei studies. We have concentrated in particular on the inelastic breakup mechanism. In principle the IAV is the most accurate method on the market and it allows also the use of FSI potentials with energy dependent strengths [40]. It is also very successful in describing transfer to continuum at low energy [32-36]. At the moment it cannot deal with high number of partial waves, which means it cannot calculate breakup on heavy targets. Also in the version with DWBA final wave functions it cannot calculate halo breakup because the process is not perturbative. However this is just a technical problem which hopefully will be solved in a near future. IAV integrated cross sections and the shape of the spectra from breakup on a small \(^9\text{Be}\) target [40] at intermediate energies are in agreement with STC values thus providing a cross-check for both models.

The eikonal model for stripping originated from HM seminal paper [23]. A derivation of the eikonal diffraction term from IAV does not exist but from the STC elastic breakup formula taking the eikonal limit one can obtain an expression consistent with the diffraction no recoil approximation [74]. The eikonal model obviously cannot describe transfer to resonances or detailed structures of inclusive spectra originating from FSI effects and/or kinematics. This is because it assumes an energy averaging procedure over the whole energy spectrum. However kinematics can be implemented in it at various levels [24,67,68]. Total cross sections differ typically by 20% with respect to QM models but this effect is not systematic.

The STC originated [3,4] from the need to interpret the physical content of all QM models available in the ’70s and ’80s which we have reviewed in Sect. 1. Its derivation is completely different from them but it assumes the core spectator model as in the HM model and the method is basically an extension of previous semiclassical transfer models to the case of a continuum final state. In this sense it is quite similar to [19–22]. Having been implemented with energy dependent FSI potentials it has been able to reproduce a quite large number of different reaction data over more than 30 years. Besides it is numerically rather trivial which allows for a multitude of applications. It is accurate to describe low and high lying resonances [41-43] as well as the three-body elastic breakup background [42] and breakup from weakly and
An interesting result of the earlier applications to heavy targets such as $^{208}\text{Pb}$ was to show that single particle states in the continuum were reproduced in position and width thanks to the use of the accurate Mahaux and Sartor n-Pb potential \[75\]. Thus it is clear that the early problematic distinction between breakup fusion in the elastic and/or inelastic channel, the question of fluctuations in the cross section vs direct part, etc., they are all solved by the use of an accurate energy dependent FSI potential. This was true for lead and other heavy targets and it is true nowadays for the $^9\text{Be}$ target. Furthermore there is no limitation on the initial state angular momentum that STC can treat because the formulae are analytical. In the Appendices B and C we provide the most formal derivation of it which allows, see in particular Appendix C, a systematic inclusion of FSI to all orders and helps clarifying the content of other models.

To study properties of the last nucleon in short-lived, exotic nuclei, one-nucleon breakup at intermediate energies has been used in the last 15 years and has largely contributed to establishing the picture of shell structure away from stability by extracting the spectroscopic factors \[10\] for the initial state wave function from the comparison of experimental data to the reaction theory predictions. A recent compilation of experimental integrated knockout cross sections at intermediate energies has shown a systematic trend when compared to theoretical calculations based on shell-model predictions for shell occupancy and eikonal approximation for the nucleon removal reactions \[9\]. However, this marked dependence does not seem to be supported by the results obtained with transfer reactions \[76–78\] and quasifree scattering with $(p, 2p), (p, pn),$ and $(e, e’p)$ reactions \[8\]. It is worth noting that the theoretical cross section depends on the description of the reaction mechanism but also on the choice of the initial state wave function. In this paper we have shown that detailed features of the spectra which are not represented by the eikonal procedure are also somewhat washed out by the energy integration of QM results when the same FSI and/or inelastic channel, the question of fluctuations in the cross section vs direct part, etc., they are all solved by the use of an accurate energy dependent FSI potential. This was true for lead and other heavy targets and it is true nowadays for the $^9\text{Be}$ target. Furthermore there is no limitation on the initial state angular momentum that STC can treat because the formulae are analytical. In the Appendices B and C we provide the most formal derivation of it which allows, see in particular Appendix C, a systematic inclusion of FSI to all orders and helps clarifying the content of other models.

When comparing results from two models (eikonal and STC or IA V), nucleus by nucleus, state by state, differentiating EBU and NEB and results from two models (eikonal and STC or IA V), nucleus by nucleus, state by state, differentiating EBU and NEB and always cross-checking the spectra. If the results of \[9\] would be confirmed, together with the recent work on knockout at the proton drip line \[80\] the logical conclusion would be that the shell model concepts have to be completely revised at and across the driplines.

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A Reminder of eikonal formulae

Following Refs. \[26–28,81\] we consider a single-particle model for a halo nucleus and introduce the eikonal approximation to study its scattering on another target nucleus. The ground state is described by a wave function $\phi_0(\mathbf{r}_{\text{CN}})$ which depends on the relative coordinate $\mathbf{r}_{\text{CN}}$, Fig. 1. between the nucleon and the core. After interacting with the target the eikonal wave-function of the halo nucleus in its rest frame has the form

$$\Psi (\mathbf{r}_{\text{CN}}, \mathbf{r}_p) = S_n (\mathbf{b}_n) S_{CT} (\mathbf{b}_c) \phi_0 (\mathbf{r}_{\text{CN}}) \quad (17)$$

where $\mathbf{r}_p$ and $\mathbf{r}_{\text{CN}}$ are the coordinates of the center-of-mass of the projectile consisting of the core plus one nucleon, and of the nucleon with respect to the core respectively, see Fig. 1. The vectors

$$\mathbf{b}_n = \mathbf{r}_p + \beta_2 \mathbf{r}_{\text{CN}} \quad \text{and} \quad \mathbf{b}_c = \mathbf{r}_p - \beta_1 \mathbf{r}_{\text{CN}} \quad (18)$$

are the impact parameters of the nucleon and the core with respect to the target nucleus. Thus $\beta_1 = m_n/m_p$, $\beta_2 = m_c/m_p = 1 - \beta_1$, where $m_n$ is the nucleon mass, $m_c$ is

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the mass of the projectile core and \( m_p = m_n + m_c \) is the projectile mass. The two profile functions \( S_n \) and \( S_{CT} \) are defined in terms of the corresponding potentials by

\[
S (b) = \exp \left( -\frac{i}{\hbar v} \int dz V (b, z) \right)
\]

(19)

where \( v \) is the beam velocity. The breakup amplitude generated from the eikonal wave function (17) has a direct contribution from the nucleon–target optical potential \( V_{NT} \) represented by nucleon–target profile function \( S_n \) and a core-recoil contribution from the core–target interaction \( V_{CT} \) represented by the profile function \( S_{CT} \) [63,64]. The recoil contribution depends on the ratio \( \beta_1 \) of the nucleon mass to the projectile mass and goes to zero in the limit \( \beta_1 \to 0 \). The potential \( V_{CT} \) includes the core–target Coulomb potential and the real and imaginary parts of the nuclear potential. The Coulomb part of \( V_{CT} \) is responsible for Coulomb breakup. Using the approximate form of the wave function (17) with (19) implies the “frozen halo” approximation; the nucleon velocity relative to the core is slow compared with the incident velocity \( v \).

The eikonal breakup amplitude is defined by [28]

\[
A (K, k) = \int d^2 r_{\perp} \rho_{\perp} e^{-\i \cdot (k_\perp \cdot r_{\perp})} 
\times \int d^3 r_C \phi_k (r_C) (S_{CT} (b_C) S_n (b_n) - 1) \phi_0 (r_C).
\]

(20)

The impact parameters \( b_n \) and \( b_C = b_n + r_\perp \) are defined in Eq. (18). The quantities \((K, k)\) are the momenta conjugate to the coordinates \((r_{\perp}, r_C)\). They are related to the final momenta of the core, nucleon and target by

\[
k_C = -k + \beta_2 K, \quad k_n = k + \beta_1 K, \quad k_T = -K. \quad (21)
\]

The wave function \( \phi_k (r_C) \) is the final continuum wave function of the nucleon relative to the core. The complete differential cross-section is

\[
\frac{d\sigma}{d^2 K d^3 k} = \frac{1}{(2\pi)^5} |A (K, k)|^2
\]

(22)

Equation (20) can also be written as

\[
A (K, k) = \int d^2 r_{\perp} \rho_{\perp} e^{-\i \cdot (k_\perp \cdot r_{\perp})} 
\times \int d^3 r_C \phi_k (r_C) S_{CT} (b_C) S_n (b_n) \phi_0 (r_C).
\]

(23)

because of the orthogonality of \( \phi_k (r_C) \) and \( \phi_0 (r_C) \) (cf. Eq. (8) of Ref. [28]). Equations (23) is a general eikonal expression which has been used in [28] and by many other authors.

Finally following the derivation in [74] the EBU eikonal cross section in the no-recoil approximation is

\[
\frac{d\sigma_{EBU}}{dk_1} = \int d^2 b_C |S_{CT} (b_C)|^2 
\times \int d^2 r_{CT} \left( 1 - |S_n (b_n)|^2 \right) |\tilde{\phi}_0 (r_{CT}, k_1)|^2.
\]

(24)

and the NEB formula [27,28]

\[
\frac{d\sigma_{NEB}}{dk_1} = \int d^2 b_C |S_{CT} (b_C)|^2 
\times \int d^2 r_{CT} \left( 1 - |S_n (b_n)|^2 \right) |\phi_0 (r_{CT})|^2.
\]

(25)

**B Derivation of the STC transfer amplitude**

The derivations of Eq. (9) is given in these appendices in a systematic approach which is useful for making different approximations to the FSI. The core spectator model is at the basis of semiclassical methods of breakup and transfer as shown by Eq. (15). To simplify the notation in this section the coordinate \( r \) and \( K \) are used instead of \( r_n \) and \( r_p \) of Eqs. (9, 10) and Fig. 1.

Semi-classical formulae for transfer amplitudes in heavy ion transfer reactions which incorporate the kinematical conditions are developed following the approach of Brink [46], Hasan and Brink [47,48], Lo Monaco and Brink [49] and Bonaccorso et al. [3,50]. The method of Broglia et al. [51,52] is very similar to ours. The theory is semi-classical in the sense that the nuclei involved in the reaction are assumed to follow classical trajectories but the transfer is calculated by quantum mechanics. The wave function \( \psi \) of the transferred particle satisfies a time-dependent Schrödinger equation

\[
i \hbar \frac{\partial \psi}{\partial t} = (T + V_1 (r_1, t) + V_2 (r_2, t)) \psi.
\]

(26)

Here \( T = -(\hbar^2 / 2m) \nabla^2 \) is the kinetic energy operator for the transferred particle, while the potentials \( V_1 \) and \( V_2 \) represent its interaction with the projectile and target. The potentials are time dependent and move along classical trajectories \( s_1 (t) \) and \( s_2 (t) \) describing the motion of the projectile and target during the collision.

\[
V_1 (r_1, t) = V_1 (r - s_1 (t)), \quad V_2 (r_2, t) = V_2 (r - s_2 (t)).
\]

(27)

The initial state \( \psi_1 (r, t) \) of the transferred particle satisfies the time-dependent Schrödinger equation for the potential \( V_1 \)
with a correction $\Delta V_2$ which takes into account some of the effects of $V_2$ a

$$i\hbar \frac{\partial \psi_1}{\partial t} = (T + V_1(r, t) + \Delta V_2(r, t)) \psi_1. \quad (28)$$

In the case of neutron transfer we can choose $\Delta V_2 = 0$, but for charged particle transfer it is non zero and takes into account the long range effects of the Coulomb field. The exact form of $\Delta V_2$ will be specified later. The final state wave-function satisfies a similar equation

$$i\hbar \frac{\partial \psi_2}{\partial t} = (T + V_2(r, t) + \Delta V_1(r, t)) \psi_2. \quad (29)$$

A very similar starting point as the present one was used in Ref. [26] where however the eikonal formalism was eventually followed. During transfer the particle is affected by both the potentials $V_1$ and $V_2$ and a perturbation formula [46, 51] can be derived for the transfer amplitude $A_{21}$ between the initial state of the particle $\psi_1$ in the projectile and the final state $\psi_2$ in the target. The amplitude is

$$A_{21} \approx \frac{1}{i\hbar} \int dt \int d\mathbf{r} \psi_2^*(\mathbf{r}, t)(V_1(\mathbf{r}) - \Delta V_1(\mathbf{r})) \psi_1(\mathbf{r}, t). \quad (30)$$

Equation (30) can be obtained from the following argument. From Eqs. (26) and (29) we have

$$i\hbar \frac{\partial}{\partial t} \langle \psi_2(t) | \psi(t) \rangle = \langle \psi_2(t) | (T + V_1 + V_2 - (T + V_2 + \Delta V_1)) \psi \rangle = \langle \psi_2(t) | (V_1 - \Delta V_1) \psi \rangle. \quad (31)$$

The state $\psi(t)$ satisfies the initial condition $\psi(t) \rightarrow \psi_1(t)$ as $t \rightarrow -\infty$. Integrating both sides of Eq. (31) gives

$$A_{21} = \lim_{t \rightarrow -\infty} \frac{1}{i\hbar} \int dt \langle \psi_2(t) | (V_1 - \Delta V_1) \psi(t) \rangle. \quad (32)$$

Equation (30) is obtained by making the approximation $\psi(t) \approx \psi_1(t)$ in the integral on the right hand side of Eq. (32). An alternative derivation of Eq. (30) is given in the following.

It was shown by Lo Monaco and Brink [49] and Stancu and Brink [82] that this perturbation integral can be transformed to a surface integral over a surface $\Sigma$ drawn between the two nuclei perpendicular to the line joining their centers at the point of closest approach. This surface is at a distance $d_1$ from the center of the first nucleus and $d_2$ from the second and $d_1 + d_2 = d$. The surface integral formula is

$$A_{21} = \frac{\hbar}{2mi} \int dt \int dS \cdot (\psi_2^*(\mathbf{r}, t) \nabla \psi_1(\mathbf{r}, t) - \psi_1(\mathbf{r}, t) \nabla \psi_2^*(\mathbf{r}, t)). \quad (33)$$

Equation (33) is derived from Eq. (30) in Appendix C. The two equations are exactly equivalent if $V_2 - \Delta V_2 = 0$ in the region $R_1$ above the surface $\Sigma$ and $V_1 - \Delta V_1 = 0$ in $R_2$ below $\Sigma$. Otherwise Eq. (33) is an approximation to Eq. (30).

We want to evaluate the amplitude Eq. (33) when the relative motion of the two nuclei is a Coulomb orbit. If the scattering angle is small then such an orbit can be replaced by a constant velocity orbit tangential to it at the point of closest approach. This is a reasonable approximation because the transfer takes place near the point of closest approach and because the acceleration in the Rutherford orbit is small. It would not be a good approximation for large-angle scattering. The transfer amplitude depends only on the relative velocity so we can assume that $V_2$ is at rest and $V_1$ has a velocity $v$ which is the tangential relative velocity in the Rutherford orbit at the distance of closest approach $d$. We write the equation of the orbit relative to the center of $V_2$ as

$$\mathbf{R}(t) = \mathbf{d} + v t. \quad (34)$$

We will choose a coordinate system so that, at the point of closest approach between the two nuclei, the $z$-axis is parallel to the velocity of the first nucleus relative to the second and the $x$-axis is in the reaction plane directed from the center of second nucleus towards the center of the first. The $y$-axis is perpendicular to the reaction plane.

### C Green’s function formalism

We consider the unitary time dependent propagator $G(t_2, t_1)$ which transforms the solution of the Schrödinger Eq. (26) at time $t_1$ into the solution at time $t_2$.

$$\psi(t_2) = G(t_2, t_1) \psi(t_1). \quad (35)$$

This propagator satisfies the integral equation

$$G(t_2, t_1) = G_0(t_2, t_1) + \frac{1}{i\hbar} \int_{t_1}^{t_2} dt \int d\mathbf{r} G_0(t_2, t) \times (V_1(t) + V_2(t)) G(t, t_1) \quad (36)$$

which we abbreviate by

$$G = G_0 + G_0(V_1 + V_2) G. \quad (37)$$

In Eqs. (36) and (37) $G_0$ is the free propagator. Both $G$ and $G_0$ satisfy the boundary condition

$$G(t, t) = G_0(t, t) = 1. \quad (37)$$

First we consider the case of a transition from an initial state $\psi(t_1)$ which is a bound state in the potential $V_1$ to a final state
ψ(t₂) which is a bound state in V₂. The transition amplitude is given by the matrix element
\[ A = \lim_{t_1 \to -\infty} \langle \psi_2(t_2) | G(t_2, t_1) | \psi_1(t_1) \rangle \]  
(38)

where \( \psi_1(t) \) and \( \psi_2(t) \) propagate in time according to the Schrödinger equations (28) and (29) with potentials \( V_1 + \Delta V_2 \) and \( V_2 + \Delta V_1 \) respectively. All the multiple scattering effects are included in the propagator \( G \). The integral equation (36) can be solved by iteration to give a multiple scattering series. Various partial summations can be made by introducing the propagators \( G_1 \) and \( G_2 \) for the potentials \( V_1 + \Delta V_2 \) and \( V_2 + \Delta V_1 \) respectively.

\[ G_1 = G_0 + G_0(V_1 + \Delta V_2)G_1 = G_0 + G_0(V_1 + \Delta V_2)G_0 + G_0(V_1 + \Delta V_2)G_0(V_1 + \Delta V_2)G_0 + \cdots \]  
(39)

\[ G_2 = G_0 + G_0(V_2 + \Delta V_1)G_2 = G_0 + G_0(V_2 + \Delta V_1)G_0 + G_0(V_2 + \Delta V_1)G_0(V_2 + \Delta V_1)G_0 + \cdots \]  
(40)

The operator \( G_1 \) is the exact propagator for a nucleon interacting with the potential \( V_1 + \Delta V_2 \) and similarly for \( G_2 \). Thus

\[ \psi_1(t_2) = G_1(t_2, t_1) \psi_1(t_1); \quad \psi_2(t_2) = G_2(t_2, t_1) \psi_2(t_1). \]  
(41)

The exact propagator \( G \) can be written in terms of \( G_2 \) or \( G_1 \) as

\[ G = G_2 + G_2(V_1 - \Delta V_1)G = G_2 + G(V_1 - \Delta V_1)G_2, \]  
(42)

\[ G = G_1 + G_1(V_2 - \Delta V_2)G = G_1 + G(V_2 - \Delta V_2)G_1. \]  
(43)

The transition amplitude equation (38) can be calculated in different approximations. Using Eqs. (41) and (43) we get

\[ A = \langle \psi_2(t_2) | G_2(t_2, t_1) | \psi_1(t_1) \rangle \]

\[ + \langle \psi_2(t_2) | G_2(V_1 - \Delta V_1)G | \psi_1(t_1) \rangle. \]  
(44)

The first term in Eq. (44) tends to zero as \( t_2 \to -\infty \) because of Eq. (41). In fact using Eq. (41) the first term in Eq. (44) can be written as \( \langle \psi_2(t_2) | \psi_1(t_2) \rangle \). This tends to zero as \( t_2 \to -\infty \) because \( \psi_1 \) is a bound state in the potential \( V_1 \) which moves away from the target as \( t \) increases. In the second term in Eq. (44) \( G \) can be written in terms of its multiple scattering series. Each partial summation represents an approximation to Eq. (44). The simplest approximation is obtained by replacing \( G \) by \( G_1 \)

\[ A_{21} = \langle \psi_2(t_2) | G_2(V_1 - \Delta V_1)G_1 | \psi_1(t_1) \rangle, \]  
(45)

using Eq. (41) this can be written as

\[ A_{21} = \frac{1}{2\hbar} \int_{-\infty}^{\infty} dt \langle \psi_2(t) | (V_1 - \Delta V_1) | \psi(t) \rangle. \]  
(46)

This is the same as Eq. (30) above. An equivalent formula can be obtained by using Eq. (42) and approximating \( G \) by \( G_2 \)

\[ A_{21} = \frac{1}{2\hbar} \int_{-\infty}^{\infty} dt \langle \psi_2(t) | (V_2 - \Delta V_2) | \psi(t) \rangle. \]  
(47)

The argument is slightly different if the final state is a continuum state. We introduce an asymptotic wave function \( \psi_{2\text{out}}(t) \) which propagates in time according to the free particle Schrödinger equation (Newton [83]) and satisfies the boundary condition \( \psi_{2\text{out}}(t) \to \psi_2(t) \) as \( t \to \infty \). The transition amplitude (38) can be written in terms of the asymptotic wave function as

\[ A = \lim_{t_1 \to -\infty} \langle \psi_{2\text{out}}(t_2) | G(t_2, t_1) | \psi_{1i}(t_1) \rangle \]  
(48)

It is important to note that \( \psi_{2\text{out}}(t) \) is a free nucleon wave function. All the FSI of the nucleon are included in the propagator \( G \).

The transition amplitude equation (48) can be calculated in different approximations. Using Eqs. (41) and (43) we get

\[ A = \langle \psi_{2\text{out}}(t_2) | G_1(t_2, t_1) | \psi_{1i}(t_1) \rangle 
\]

\[ + \langle \psi_{2\text{out}}(t_2) | G_2 V_1 G_1 | \psi_{1i}(t_1) \rangle \]  
(49)

As in the case of Eq. (44) the first term in equation (49) tends to zero as \( t_2 \to \infty \). In the second term in Eq. (49) \( G \) can be written in terms of its multiple scattering series. Each partial summation represents an approximation to Eq. (49). We consider two cases for the purpose of studying transfer to the continuum.

i. The propagator \( G \) in Eq. (49) is approximated by \( G_2 \). Then the transition amplitude can be written as

\[ A_{21} = \langle \psi_{2\text{out}}(t_2) | G_2 V_2 G_1 | \psi_{1i}(t_1) \rangle \]  
(50)

when \( t_2 \to \infty \). In this matrix element the propagator \( G_2 \) acting on \( \psi_{2\text{out}} \) introduces FSI with \( V_2 + \Delta V_1 \). The combination

\[ \langle \psi_{2\text{out}}(t_2) | G_2(t_2, t) = \langle \psi_2(t) | \]  
(51)

is a time dependent wave function for a nucleon propagating in the potential \( V_2 + \Delta V_1 \) of the target. Using Eqs. (41) and (51) and (50) can be written as

\[ A_{21} = \frac{1}{2\hbar} \int_{-\infty}^{\infty} dt \langle \psi_2(t) | V_2 | \psi_{1i}(t) \rangle \]  
(52)
This is the case discussed by Bonaccurso and Brink [3]. ii. The propagator \( G \) in Eq. (49) is approximated by \( G_1 \). Then the transition amplitude can be written as

\[
A_{21} = \langle \psi_{2\text{out}}(t_2) | G_1(V_2 - \Delta V_2) G_1 | \psi_{1\text{in}}(t_1) \rangle. \tag{53}
\]

This case corresponds to inelastic excitations of the projectile.

iii. The propagator \( G \) is approximated by the free particle propagator \( G_0 \), then one would calculate

\[
A_{11} = \langle \psi_{2\text{out}}(t_2) | G_0 V_2 G_1 | \psi_{1\text{in}}(t_1) \rangle. \tag{54}
\]

Written more explicitly this is

\[
A_{01} = \int_{-\infty}^{\infty} dt \langle \psi_0(V_2 - \Delta V_2) | \psi_1 \rangle. \tag{55}
\]

This form corresponds to the break-up of the nucleon in the projectile due to the interaction with the target.

Case i) includes the FSI of the nucleon with the potential \( V_2 \). It can be used to study the influence of nucleon resonances in the target nucleus. Similarly case ii) includes the final state interaction of the neutron with the potential of the projectile and can be used to study effect of resonances in the projectile. There are no FSI in approximation iii).

Equation (46) can be transformed into a simpler form in the case of a peripheral collision. Let \( \Sigma \) be a surface which lies between the two potentials \( V_1 \) and \( V_2 \) which divides the space into regions \( R_1 \) and \( R_2 \). In the following we assume that

\[
V_1 - \Delta V_1 = 0 \quad \text{in} \quad R_2; \quad V_2 - \Delta V_2 = 0 \quad \text{in} \quad R_1. \tag{56}
\]

Then the matrix element \( \langle \psi_2 | (V_1 - \Delta V_1) | \psi_1 \rangle \) in Eq. (46) can be written as a sum of integrals over the regions \( R_1 \) and \( R_2 \). The integral over \( R_2 \) is zero if \( V_1 - \Delta V_1 = 0 \) in \( R_2 \). Using Eq. (28) the first integral can be written as

\[
\int_{R_1} \psi_2^*(V_1 - \Delta V_1) \psi_1 d^3r = \int_{R_1} \psi_2^* \left( i\hbar \partial_{t} + \frac{\hbar^2}{2m} \nabla^2 - \Delta V_2 - \Delta V_1 \right) \psi_1 d^3r. \tag{57}
\]

Applying Green’s theorem reduces this to

\[
\int_{R_1} \psi_2^* (V_1 - \Delta V_1) \psi_1 d^3r = \int_{S} dS \cdot \left( \psi_2^* \nabla \psi_1 - \psi_1 \nabla \psi_2^* \right) + i\hbar \partial_{t} \int_{R_1} \psi_2^* \psi_1 d^3r
\]

\[+ \int_{R_1} \left( i\hbar \partial_{t} \psi_2^* + \frac{\hbar^2}{2m} \nabla^2 \psi_2 - \Delta V_2 \psi_2 - \Delta V_1 \psi_2 \right) \psi_1 d^3r. \tag{58}
\]

where \( dS \) is a surface element normal to \( \Sigma \) directed out of \( R_1 \). If Eq. (58) is integrated over time between \( t = -\infty \) and \( t = \infty \) the second term vanishes because the potentials \( V_1 \) and \( V_2 \) are very far away from each other as \( t \rightarrow \pm \infty \) and \( \psi_1 \) and \( \psi_2 \) have no overlap in that limit. If \( V_2 - \Delta V_2 = 0 \) in \( R_1 \) then the third term in Eq. (58) is zero because \( \psi_2 \) satisfies the Schrödinger equation (29). This proves Eq. (33). The next step is to obtain our final formula Eq. (63) from (33).

First the wave functions \( \psi_1 \) and \( \psi_2 \) are expressed in terms of \( \phi_1 \) and \( \phi_2 \) using Eq. (10). They can be written as inverse transforms of momentum distributions

\[
\phi_1(x, y, z) = \frac{1}{(2\pi)^2} \int \int dk_{1x} dk_{1y} \exp(i(k_{1x} + z k_{1z}) \phi(x, k_{1y}, k_{1z}), \tag{59}
\]

and similarly for \( \phi_2(x, y, z) \). Substituting into Eq. (33) yields a 7-dimensional integral over \( y, z, t \) and \( k_{1y}, k_{1z}, k_{2y}, k_{2z} \). This integral contains a phase factor

\[
\exp \left[ m v z / h + \left( \frac{v^2}{2} - \frac{1}{2} m v^2 \right) t / h \right]
\]

\[+ (z - v t) k_{1z} + y k_{1y} - z k_{2z} - y k_{2y} \] \( \tag{60} \)

which contains all the dependence on \( y, z \) and \( t \). Hence the integration over those variables gives a product of three \( \delta \)-functions

\[
\frac{(2\pi)^3}{v} \delta(k_{1z} - k_{2z})
\]

\[+ m v / h \delta \left( k_{1z} - \left( \frac{v^2}{2} - \frac{1}{2} m v^2 \right) / h v \right) \delta(k_{1y} - k_{2y}). \tag{61} \]

Finally Eq. (33) reduces to a 1-dimensional integral over \( k_y = k_{1y} = k_{2y} \). The remaining two \( \delta \)-functions in (61) give

\[
k_{1z} = k_1 = \left( \frac{v^2}{2} - \frac{1}{2} m v^2 \right) / h v,
\]

\[
k_{2z} = k_2 = \left( \frac{v^2}{2} - \frac{1}{2} m v^2 \right) / h v. \tag{62} \]

These are the values of \( k_1 \) and \( k_2 \) used in the text following Eqs. (9) and (16). After evaluation the derivatives normal to the surface \( \Sigma \) in (33) one obtains a factor \( i(\gamma_{1z} + \gamma_{2z}) = 2i\sqrt{y^2 + k_z^2} \). When these results are collected the final form of the transfer (breakup) amplitude is obtained as

\[
A_{21} = \frac{\hbar}{2\pi m v} \int dk_y \sqrt{y^2 + k_z^2} \phi_1^* (d_1, k_y, k_z) \phi_1 (-d_1, k_y, k_z). \tag{63}
\]

Note that the amplitude \( A_{21} \) of this section was indicated as \( A_{f_1} \) in the main text.
