A $p$-adically entire function with integral values on $\mathbb{Q}_p$
and entire liftings of the $p$-divisible group $\mathbb{Q}_p/\mathbb{Z}_p$

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with an Appendix by Maurizio Candilera *

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Abstract
We give a self-contained proof of the fact, discovered in [1] and proven in [2] with
the methods of [12], that, for any prime number $p$, there exists a power series
$$\Psi = \Psi_p(T) \in T + T^2\mathbb{Z}[[T]]$$
which trivializes the addition law of the formal group of Witt covectors [12, [10 II.4]], is
$p$-adically entire and assumes values in $\mathbb{Z}_p$ all over $\mathbb{Q}_p$. We actually generalize, following
a suggestion of M. Candilera, the previous facts to any fixed unramified extension $\mathbb{Q}_q$
of $\mathbb{Q}_p$ of degree $f$, where $q = p^f$. We show that $\Psi = \Psi_q$ provides a quasi-finite ramified
covering of the Berkovich affine line $\mathbb{A}^1_{\mathbb{Q}_p}$ by itself and prove new strong estimates on its
growth. These estimates are essential in the application to $p$-adic Fourier expansions
[3].

We reconcile the present discussion (for $q = p$) with the formal group proof given
in [2] which takes place in the Fréchet algebra $\mathbb{Q}_p\{x\}$ of the analytic additive group
over $\mathbb{Q}_p$. We use the addition law of $\Psi$ to exhibit, for any $r \in \mathbb{Z}$, a closed $\mathbb{Z}_p$-Hopf
subalgebra $\mathcal{E}^r_\circ \subset \mathbb{Q}_p\{x\}$ which lifts canonically the $p$-divisible group $\mathbb{Q}_p/p^{r+1}\mathbb{Z}_p$
over $\mathbb{F}_p$. We obtain in particular convergent expansions of continuous $\mathbb{Q}_p$-valued functions
on $\mathbb{Q}_p$ as series of entire functions.

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Conventions  

A prime number $p$ is fixed throughout this paper and $q = p^f$ is a power of $p$. So, $Q_q$ will denote the unramified extension of $Q_p$ of degree $f$, and $Z_q$ will be its ring of integers. Unless otherwise specified, a ring is meant to be commutative with 1.  

Linear topologies  

Let $k$ be a separated and complete linearly topologized ring; we will denote by $P(k)$ the family of open ideals of $k$. We will consider the category $LM_k^u$ of separated and complete linearly topologized $k$-modules $M$ such that the map multiplication by scalars $k × M → M$, $(r, m) ↦ rm$ is uniformly continuous for the product uniformity of $k × M$. This is the classical category of [8, Chap. III, §2]. See [4], [5] for more detail. Again, for any object $M$ of $LM_k^u$, $P(M)$ will denote the family of open $k$-submodules of $M$. The category $LM_k^u$ admits all limits and colimits. The former are calculated in the category of $k$-modules but not the latter. So, a limit will be denoted by $\lim←$ while a colimit will carry an apex ($−u$) as in $\lim−u$. In particular, for any family $M_\alpha, \alpha ∈ A$, of objects of $LM_k^u$, the direct sum and direct product will be denoted by  

$$\bigoplus_{\alpha ∈ A} M_\alpha, \prod_{\alpha ∈ A} M_\alpha,$$

respectively. It will also be useful to introduce the uniform box product of the same family

$$(0.0.1) \prod_{\alpha ∈ A} □_u M_\alpha$$

which, set-theoretically, coincides with $\prod_{\alpha ∈ A} M_\alpha$ but whose family of open submodules consists of all $U := \prod_{\alpha ∈ A} U_\alpha$, with $U_\alpha ∈ P(M_\alpha)$, such that there exists $I_U ∈ P(k)$ such that $I_U M_\alpha ⊂ U_\alpha$, for any $\alpha ∈ A$. The category $LM_k^u$, equipped with the tensor product $⊗_k$ defined in [8] Chap. III, §2, Exer. 28] is a symmetric monoidal category.  

Semivaluations  

We denote by $Z(p) = Q ∩ Z_p$, the localization of $Z$ at $(p)$. Then $C_p$ will be the completion of a fixed algebraic closure of $Q_p$. On $C_p$ we use the absolute value $|x| = |x|_p = p^{−v_p(x)}$, for the $p$-adic valuation $v = v_p$, with $v_p(p) = 1$, and $x ∈ C_p$.  

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Definition 0.1. A semivaluation on a ring \( R \) is a map \( w : R \to \mathbb{R} \cup \{+\infty\} \) such that 
\[ w(0) = +\infty, \quad w(x + y) \geq \min(w(x), w(y)) \quad \text{and} \quad w(xy) \geq w(x) + w(y), \] 
for any \( x, y \in R \). We will say that a semivaluation is 
\[ (0.1.1) \] 
positive semivaluation. \( \) 

The family of 
\[ R_{w,c} := \{ x \in R \mid w(x) \geq c \} \]
for \( c \in \mathbb{R} \) is a fundamental set of open subgroups for a group topology of \( R \). Moreover, \( R_{w,0} \) is a subring of \( R \) and all \( R_{w,c} \) are \( R_{w,0} \)-submodules of \( R \). A semivalued ring \( (R, \{w_\alpha\}_{\alpha \in A}) \) is a ring \( R \) equipped with a family \( \{w_\alpha\}_{\alpha \in A} \) of semivaluations. A semivalued ring is endowed with the topology in which any \( x \in R \) has a fundamental system of neighborhoods consisting of the subsets 
\[ x + \bigcap_{\alpha \in F} R_{\alpha,c_\alpha} \]
where \( F \) varies among finite subsets of \( A \) and, for any \( \alpha \in F, c_\alpha \in \mathbb{R} \). A Fréchet ring (resp. Banach ring) is a ring \( R \) which is separated and complete in the topology induced by a countable family of semivaluations (resp. by a single semivaluation). If the semivaluations \( w_\alpha \) are all positive, the Fréchet (resp. Banach) ring \( (R, \{w_\alpha\}_{\alpha \in A}) \) is linearly topologized. We will call it a linearly topologized Fréchet (resp. Banach) ring. When \( R \) is an algebra over a Banach ring \( (S, v) \), and the valuations \( w_\alpha \) induce \( v \) on \( S \), we also say that \( R = (R, \{w_\alpha\}_{\alpha \in A}) \) is a Fréchet (resp. Banach) \( S \)-algebra. In the particular case when \( (S, v) \) is a complete real-valued field \( (K, v) \) a Fréchet or Banach \( S \)-algebra is a Fréchet or Banach algebra over \( K \) in the classical sense. Notice however that we allow \( v \) to be the trivial valuation of \( S \) or \( K \). We denote by \( \mathcal{LC}_K \) the category of locally convex topological \( K \)-vector spaces of \([13]\) which are moreover separated and complete.

0.3 Tensor products

Let \( (S, v) \) be a complete real-valued ring and let \( R = (R, \{w_\alpha\}_{\alpha \in A}) \) and \( R' = (R', \{w'_\beta\}_{\beta \in B}) \) be two Fréchet \( S \)-algebras. Then we define a Fréchet \( S \)-algebra \( R \hat{\otimes}_S R' \) as the completion of the \( S \)-algebra \( R \otimes_S R' \) in the topology induced by the following semivaluations \([7\ 2.1.7]\), for any \( (\alpha, \beta) \in A \times B \),
\[ w_{\alpha,\beta}(g) = \sup \left( \min_{1 \leq i \leq n} w_\alpha(x_i) + w'_\beta(y_i) \right), \]
where the supremum runs over all possible representations
\[ g = \sum_{i=1}^n x_i \otimes y_i \quad , \quad x_i \in R , \quad y_i \in R'. \]

Notice that

1. if \( R \) and \( R' \) are Fréchet algebras over a complete real-valued field \( (K, v) \), with non-trivial valuation \( v \), \( R \hat{\otimes}_s K R' \) coincides with the completed projective tensor product of \([14\ Lemma 17.2]\);

2. if \( R \) and \( R' \) are linearly topologized Fréchet algebras over a linearly topologized Banach ring \( (S, v) \), \( R \hat{\otimes}_s S R' \) coincides with \( R \hat{\otimes}_s S R' \).
0.4 Entire functions bounded on $\mathbb{Q}_p$

**Definition 0.2.** For any closed subfield $K$ of $\mathbb{C}$, we set $K^0 := \{ x \in K \mid v(x) \geq 0 \}$ and $K^\infty := \{ x \in K \mid v(x) > 0 \}$. We denote by $K[x]$ the ring of entire functions on the $K$-analytic affine space $\mathbb{A}^n_K$. The standard Fréchet topology on the $K$-algebra $K[x]$ is induced by the family $\{ w_n^{(i)} \}_{i \in \mathbb{Z}}$ of valuations 
\[
w_n^{(i)}(f) := \inf_{x \in (p^{-r} \mathbb{Q})^n} v(f(x)) ,
\]
for any $f \in K[x]$.

**Definition 0.3.** We denote by $\mathcal{E}^{(n)}$ the $\mathbb{Q}_p$-subalgebra of $\mathbb{Q}_p[x]$ consisting of functions whose restriction to $\mathbb{Q}_p^n$ is bounded and uniformly continuous. For $f \in \mathcal{E}^{(n)}$ we introduce one further valuation 
\[
(0.3.1) \quad w^{(n)}_\infty(f) := \inf_{x \in \mathbb{Q}_p^n} v(f(x)).
\]

Then $\mathcal{E}^{(n)}$ is a $\mathbb{Q}_p$-Fréchet algebra in the $\mathbb{Q}_p$-uniform topology which is induced by the valuations $\{ w^{(n)}_n \}_{n \in \mathbb{Z}}$ together with the further valuation $w^{(n)}_\infty$. We define an open $\mathbb{Z}_p$-Fréchet subring of $\mathcal{E}^{(n)}$ as 
\[
(0.3.2) \quad \mathcal{E}^{(n)} \cap \mathbb{Z}_p := \{ f \in \mathcal{E}^{(n)} \mid f(\mathbb{Q}_p^n) \subset \mathbb{Z}_p \}.
\]

Notice that the topology of the ring $\mathcal{E}^{(n)} \cap \mathbb{Z}_p$ does not in general admit a basis of open ideals. We have natural identifications 
\[
(0.3.3) \quad \mathbb{Q}_p[x_1, \ldots, x_n] \hat{\otimes} \mathbb{Q}_p \mathbb{Q}_p[y_1, \ldots, y_n] = \mathbb{Q}_p[x_1, \ldots, x_n, y_1, \ldots, y_n],
\]
via $x_i \mapsto x_i$ and $1 \otimes y_j \mapsto y_j$, for $i = 1, \ldots, n, j = 1, \ldots, m$. The previous identifications induce identifications 
\[
(0.3.4) \quad \mathcal{E}^{(n)} \hat{\otimes} \mathbb{Q}_p \mathcal{E}^{(m)} \xrightarrow{\sim} \mathcal{E}^{(n+m)} \text{ and } \mathcal{E}^{(n)} \hat{\otimes} \mathbb{Z}_p \mathcal{E}^{(m)} \xrightarrow{\sim} \mathcal{E}^{(n+m)} \cap \mathbb{Z}_p.
\]

**Remark 0.4.** For $n = 1$, we also drop the apex $(-1)$ in our notation, so that we get the Fréchet $\mathbb{Q}_p$-subalgebra $\mathcal{E}$ of $\mathbb{Q}_p[x]$ consisting of entire functions whose restriction to $\mathbb{Q}_p$ is bounded and uniformly continuous. and its open $\mathbb{Z}_p$-Fréchet subring $\mathcal{E}^\circ$ of functions $f$ such that $f(\mathbb{Q}_p) \subset \mathbb{Z}_p :$
\[
\mathcal{E}^\circ \subset \mathcal{E} \subset \mathbb{Q}_p[x].
\]

Our notation for coproduct (resp. counit, resp. inversion) of an Hopf algebra object $A$ in a symmetric monoidal category with monoidal product $\otimes$ and unit object $I$ is usually $\Psi : A \rightarrow A \otimes A$ (resp. $\varepsilon : A \rightarrow I$, resp. $\rho : A \rightarrow A$).

1 Introduction

1.1 The function $\Psi$

In the paper [2] we introduced, for any prime number $p$, a power series 
\[
\Psi(T) = \Psi_p(T) = T + \sum_{i=2}^\infty a_i T^i \in \mathbb{Z}[T],
\]
which represents an entire $p$-adic analytic function, i.e. is such that 
\[
(1.0.1) \quad \lim_{i \rightarrow \infty} |a_i|_p 1/p^i = 0 .
\]
This function has the remarkable property that \( \Psi_p(Q_p) \subset \mathbb{Z}_p \) and that, for any \( i \in \mathbb{Z} \) and \( x \in Q_p \), if we define

\[
(1.0.2) \quad x_{-i} := \Psi_p(p^i x) \mod p \in \mathbb{F}_p,
\]
then

\[
(1.0.3) \quad x = \sum_{i \gg -\infty}^\infty [x_i]p^i \in W(\mathbb{F}_p)[1/p] = \mathbb{Q}_p,
\]

where \([t]\), for \( t \in \mathbb{F}_p \), is the Teichmüller representative of \( t \) in \( W(\mathbb{F}_p) = \mathbb{Z}_p \). In particular, for any \( i \in \mathbb{Z} \), the function

\[
(1.0.4) \quad x_i : Q_p \longrightarrow \mathbb{F}_p, \quad x \longmapsto x_i
\]
factors through a function, still denoted by \( x_i \),

\[
(1.0.5) \quad x_i : Q_p/p^{i+1}\mathbb{Z}_p \longrightarrow \mathbb{F}_p, \quad h \longmapsto h_i.
\]

We regard the function in (1.0.5) as an \( \mathbb{F}_p \)-valued periodic function of period \( p^{i+1} \) on \( Q_p \).

The power series \( \Psi(T) \) is defined by the functional relation

\[
(1.0.6) \quad \sum_{j=0}^\infty p^{-j}\Psi(p^j T)^p = T.
\]

Its inverse function \( \beta = \beta_p \in T + T^2 \mathbb{Z}[[T]] \) was shown to converge exactly in the region

\[
(1.0.7) \quad |T|_p < p \quad \text{i.e.} \quad v_p(T) > -1.
\]

One property we had forgotten to explicitly point out in [2] is the following

**Proposition 1.1.** The restriction of the function \( \Psi_p \) to a map \( Q_p \rightarrow \mathbb{Z}_p \) is uniformly continuous. More precisely, for any \( j = 1, 2, \ldots \) and \( a \in Q_p \),

\[
\Psi_p(a + p^j \mathbb{Z}_p) \subset \Psi_p(a) + p^j \mathbb{Z}_p.
\]

**1.2 Our previous approach [2]**

The proof in [2] was based on the Barsotti-Witt algorithms of [12]. We gave a criterion [2, Lemma 1] of simultaneous admissibility in the sense of [12, Ch.1, §1] for a family indexed by \( \alpha \in A \), of sequences \( i \mapsto x_{\alpha,i} \), for \( i = 0, 1, \ldots \) in a Fréchet \( \mathbb{Q}_p \)-algebra. The notion of (simultaneous) admissibility appears in [12] in two variants, depending on whether the topological ring \( R \) is a \( \mathbb{Q}_p \)-algebra or only a \( \mathbb{Z}_p \)-algebra. For clarity, we will call the two notions PD-admissibility and admissibility, respectively. When both notions apply, the former notion is stronger and permits the use of “ghost components”. Our criterion quoted above refers to simultaneous PD-admissibility. In Barsotti’s theory of \( p \)-divisible groups one regards an admissible sequence \( i \mapsto x_{-i} \) as a Witt covector \((\ldots, x_{-2}, x_{-1}, x_0)\) [12, 10] with components \( x_{-i} \in R \).

We only make a short detour on the group functor viewpoint and refer the reader to [10] for precisions. As abelian group functors on a suitable category of topological \( \mathbb{Z}_p \)-algebras the direct limit \( W_n \rightarrow W_{n+1} \) of the Witt vector groups of length \( n \) via the Verschiebung map

\[
V : (x_{-n}, \ldots, x_{-1}, x_0) \rightarrow (0, x_{-n}, \ldots, x_{-1}, x_0)
\]
indeed exists. It the group functor CW of Witt covectors. For a topological \( \mathbb{Z}_p \)-algebra \( R \) on which \( CW(R) \) is defined, it is convenient to denote an element \( x \in CW(R) \) by an inverse sequence

\[
x = (\ldots, x_{-2}, x_{-1}, x_0)
\]

of elements of \( R \), that is a Witt covector with components in \( R \). Two Witt covectors \( x = (\ldots, x_{-2}, x_{-1}, x_0) \) and \( y = (\ldots, y_{-2}, y_{-1}, y_0) \) with components \( R \) can be summed by taking limits of sums of finite Witt vectors. Namely, let

\[
\varphi_i(X_0, X_1, \ldots, X_i; Y_0, Y_1, \ldots, Y_i) 
\]

be the \( n \)-th entry of the Witt vector \((X_0, X_1, \ldots, X_n) + (Y_0, Y_1, \ldots, Y_n)\). Then, \( x + y = z = (\ldots, z_{-2}, z_{-1}, z_0) \) means that, for any \( i = 0, -1, \ldots, \)

\[
(1.1.1) \quad z_i = \lim_{n \to +\infty} \varphi_i(x_{i-n}, x_{i-n+1}, \ldots, x_i; y_{i-n}, y_{i-n+1}, \ldots, y_i)
\]

converges in \( R \). The convergence properties on the Witt covectors \( x \) and \( y \) above for the expressions \( (1.1.1) \) to converge, are dictated by the following

**Lemma 1.2.** (\cite[Teorema 1.11]{}) Notation as above. For \( i = 0, 1, 2, \ldots \), let us attribute the weight \( p^i \) to the variables \( X_i, Y_i \). Then, for any \( i \geq 0 \) the polynomial \( \varphi_i(X_0, X_1, \ldots, X_i; Y_0, Y_1, \ldots, Y_i) \) is isobaric of weight \( p^i \). Moreover, for any \( i \geq 1 \),

\[
(1.2.1) \quad \varphi_i(X_0, X_1, \ldots, X_i; Y_0, Y_1, \ldots, Y_i) = \varphi_{i-1}(X_1, \ldots, X_i; Y_1, \ldots, Y_i) 
\]

So, we equip the polynomial ring \( \mathbb{Z}[X_0, X_1, \ldots, X_{i-1}; Y_0, Y_1, \ldots, Y_{i-1}] \) with the linear topology defined by the powers of the ideals \( I_N := (X_{-N}, X_{-N-1}, \ldots, Y_{-N}, Y_{-N-1}, \ldots) \) and set

\[
\mathcal{P} := \lim_{N \to +\infty} \mathbb{Z}[X_0, X_{-1}, \ldots, X_{-i}; Y_0, Y_{-1}, \ldots, Y_{-i}, \ldots]/I_N^M. 
\]

Then, the sequence

\[
(1.2.2) \quad i \mapsto \varphi_i(X_0, X_{-1}, \ldots, X_{-i}; Y_0, Y_{-1}, \ldots, Y_{-i})
\]

converges to an element

\[
(1.2.3) \quad \Phi(X_0, X_{-1}, \ldots, X_{-i}; Y_0, Y_{-1}, \ldots, Y_{-i}, \ldots) \in \mathcal{P}. 
\]

So, \( (1.1.1) \) is expressed more compactly as

\[
(1.2.4) \quad z_i = \Phi(x_i, x_{i-1}, \ldots, y_i, y_{i-1}, \ldots)
\]

**Remark 1.3.** The projective limit

\[
W_{n+1} \to W_n, \quad (x_0, x_1, \ldots, x_{n+1}) \mapsto (x_0, x_1, \ldots, x_n),
\]

produces instead the algebraic group \( W \) of Witt vectors.

The approach of Barsotti \cite{12} is more flexible and easier to apply to analytic categories. If \( R \) is complete, for two simultaneously admissible Witt covectors \( x = (\ldots, x_{-2}, x_{-1}, x_0) \) and \( y = (\ldots, y_{-2}, y_{-1}, y_0) \) with components \( R \) the expressions \( (1.2.4) \) all converge in \( R \) and define \( (\ldots, z_{-2}, z_{-1}, z_0) = z := x + y \), which is in turn simultaneously admissible with \( x \).
and \( y \). In the \( \mathbb{Q}_p \)-algebra case a Witt covector \( x = (\ldots, x_{-2}, x_{-1}, x_0) \) has ghost components 

\[
(\ldots, x^{(-2)}, x^{(-1)}, x^{(0)})
\]

defined by

\[
x^{(i)} = x_i + p^{-1}x_{i-1} + p^{-2}x_{i-2} + \cdots, \quad i = 0, -1, -2, \ldots
\]

Under very general assumptions \([12, \text{Teorema 1.11}]\), a finite family of sequences \((x_{\alpha, -i})_{i=0,1,\ldots}\), for \( \alpha \in A \) in a \( \mathbb{Q}_p \)-Fréchet algebra are simultaneously PD-admissible iff the same holds for the family of sequences of ghost components \((x_{\alpha, -i}^{(-i)})_{i=0,1,\ldots}\), for \( \alpha \in A \). Under these assumptions, for simultaneously PD-admissible covectors \( x \) and \( y \), \( x + y = z \) is equivalent to

\[
z^{(i)} = x^{(i)} + y^{(i)}, \quad i = 0, -1, -2, \ldots
\]

In the present case, which coincides with the case treated in \([2]\), the sequences \( i \mapsto x_{-i} := p^i x \) and \( i \mapsto y_{-i} := p^i y \) are simultaneously PD-admissible in the standard \( \mathbb{C}_p \)-Fréchet algebra \( \mathbb{C}_p[x, y] \) of entire functions on \( \mathbb{C}_p \) \([2, \text{Lemma 1 and Lemma 3}]\). It follows from the relation \([12, \text{loc.cit.}]\) we conclude that the two sequences \( i \mapsto \Psi(p^i x) \) and \( i \mapsto \Psi(p^i y) \) are simultaneously admissible in \( \mathbb{C}_p[x, y] \), as well. Moreover, by \([12, \text{loc.cit.}]\) and the definition of the addition law of Witt covectors with coefficients in \( \mathbb{C}_p[x, y] \), we have

\[
(\ldots, \Psi(p^2(x + y)), \Psi(p(x + y)), \Psi(x + y)) = \\
(\ldots, \Psi(p^2x), \Psi(px), \Psi(x)) + (\ldots, \Psi(p^2y), \Psi(py), \Psi(y)).
\]

Equivalently, \( \Psi \) satisfies the addition law \([2, (11)]\)

\[
\Psi(x + y) = \Phi(\Psi(x), \Psi(px), \ldots; \Psi(y), \Psi(py), \ldots).
\]

\textbf{Proof.} (Of Proposition \([11]\), \textbf{1.1}) The shape of \( \Psi_p(x) \) indicates that \( v_p(\Psi_p(x)) = v_p(x) \) if \( v_p(x) > 0 \). The claim follows from \([1.3.4] \) and Lemma \([1.2]\).

Notice that \([1.0.3]\) means that \([1.0.3]\) may be restated to say that, for any \( x \in \mathbb{Q}_p \),

\[
x = (\ldots, x_{-2}, x_{-1}; x_0, x_1, \ldots),
\]

where \( x_i \in \mathbb{F}_p \) is given by \([1.0.4]\), as a Witt bivector \([12]\) with coefficients in \( \mathbb{F}_p \).

### 1.3 Our present approach

We present here in section \([2]\), a direct elementary proof of the previous facts, which makes no use of the Barsotti-Witt algorithms of \([12]\). Actually, following a suggestion of M. Candilera, we consider rather than \([1.0.6]\), the more general functional relation for \( \Psi = \Psi_\alpha, q = p^\ell \)

\[
(1.3.5) \quad \sum_{j=0}^{\infty} p^{-j} \Psi(p^j T)^{\ell j} = T.
\]

The result, at no extra work, will then be that \([1.3.5]\) admits a unique solution \( \Psi_\alpha(T) \in T + T^2 \mathbb{Z}[[T]] \). The series \( \Psi_\alpha(T) \) represents a \( p \)-adically entire function such that \( \Psi_\alpha(\mathbb{Q}_p) \subset \mathbb{Z}_q \). In section \([3]\) we describe in the same elementary style the Newton and valuation polygons of the entire function \( \Psi_\alpha \), and obtain new estimates on the growth of \( |\Psi_\alpha(x)| \) as \( |x| \to \infty \), which will be crucial for the sequel. The proofs in both of these sections are based on correspondence with Ph. Robba of 1980. We present in an Appendix some numerical calculations due to M. Candilera, which exhibit the first coefficients of \( \Psi_p \), for small values
of \( p \). These calculations have been useful to us and we believe they may be quite convincing for the reader.

The function \( \Psi_q: A^1_{\mathbb{Q}_p} \to A^1_{\mathbb{Q}_p} \) is a quasi-finite ramified covering of the Berkovich affine line over \( \mathbb{Q}_p \) by itself. Except for \( \Psi_q(0) = 0 \), the zeros of \( \Psi_q \) have negative integral \( p \)-adic valuation. More precisely, for any \( n = 1, 2, \ldots \), \( \Psi_q \) has \( q^n - q^{n-1} \) distinct zeros of \( p \)-adic valuation \(-nf\).

We do not understand the covering \( \Psi_q: A^1_{\mathbb{Q}_p} \to A^1_{\mathbb{Q}_p} \); in particular, we do not know whether it is Galois. Moreover, we have no interpretation for the zeros of \( \Psi_q \).

### 1.4 Reconciliation of approaches and Fourier-type expansions

In sections 4 and 5 we reconcile the elementary discussion of sections 2 and 3 with the formal

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In order to present a lifting of the constant (étale) \( p \)- divisible group \( \mathbb{Q}_p/\mathbb{Z}_p \) over \( \mathbb{F}_p \) to a \( \mathbb{Z}_p \)-formal group whose analytic fiber is the group \( \mathbb{G}_a \).

Section 3 is preliminary. We briefly discuss there continuous and uniformly continuous functions \( f: \mathbb{Q}_p \to \mathbb{Q}_p \), with special emphasis on the \( \mathbb{Z}_p \)-Hopf algebra \( \mathcal{C}_{\text{unif}}(\mathbb{Q}_p, \mathbb{Z}_p) \) of uniformly continuous functions \( \mathbb{Q}_p \to \mathbb{Z}_p \), equipped with the \( p \)-adic topology. These Hopf algebras are related to the constant \( p \)-divisible group \( \mathbb{Q}_p/\mathbb{Z}_p \) over \( \mathbb{Z}_p \) and to its “universal covering” \( \mathbb{Q}_p \). A more complete discussion of these Hopf algebras and of their duality relation with the affine algebra of the \( p \)-divisible torus, interpreted as an algebra of measures, will appear in [3].

In section 5 we study the semivalued rings \( \mathcal{E}, \mathcal{E}^\circ \) of Remark 0.4. It is clear that \( \mathcal{E} \), equipped with the semivaluations \( \{w_r\}_{r \in \mathbb{Z}_p(\infty)} \), is a Fréchet \( \mathbb{Q}_p \)-Hopf subalgebra of \( \mathbb{Q}_p \{x\} \) and that \( \mathcal{E}^\circ \) is an open Fréchet \( \mathbb{Z}_p \)-Hopf subalgebra of \( \mathcal{E} \). In particular, the element \( \Psi(x) \in \mathcal{E}^\circ \) obeys the addition law (1.3.3) of Witt covectors which is a convergent expression in \( \mathcal{E}^\circ \).

In order to present a lifting of the \( \mathbb{F}_p \)- group \( \mathbb{Q}_p/\mathbb{Z}_p \) to \( \mathbb{Z}_p \), we set the following

**Definition 1.4.** Let \( r \in \mathbb{Z} \). We define \( \mathcal{E}_r^\circ \) as the set of those \( f \in \mathcal{E}^\circ \) such that

\[
(1.4.1) \quad f(x + p^{r+1} \mathbb{Z}_p) \subset f(x) + p \mathbb{Z}_p, \quad \forall x \in \mathbb{Q}_p.
\]

Notice that \( p^{\mathcal{E}^\circ} \subset \mathcal{E}_r^\circ \), for any \( r \).

Clearly, \( \mathcal{E}_r^\circ \) is a closed \( \mathbb{Z}_p \)-Hopf subalgebra of \( \mathcal{E}^\circ \). Moreover, \( \{\mathcal{E}_r^\circ\}_{r \in \mathbb{Z}} \) is an increasing family of \( \mathbb{Z}_p \)-subalgebras of \( \mathcal{E}^\circ \).

We prove that

**Theorem 1.5.** For any \( r \in \mathbb{Z} \), let \( \pi_r: \mathbb{Q}_p \to \mathbb{Q}_p/p^{r+1} \mathbb{Z}_p \) denote the canonical projection.

1. The Fréchet \( \mathbb{Z}_p \)-Hopf algebra \( \mathcal{E}_r^\circ \) is the completion of the semivalued ring

\[
(\mathbb{Z}_p|\Psi(p^{i-r}x)| i \in \mathbb{Z}_{\geq 0}, \{w_s\}_{s \in \mathbb{Z}_p(\infty)}).
\]

Equivalently, \( \mathcal{E}_r^\circ \) is the closure in \( \mathcal{E} \) of the subring \( \mathbb{Z}_p[\Psi(p^{i-r}x)| i \in \mathbb{Z}_{\geq 0}] \).

2. For any \( f \in \mathcal{E}_r^\circ \) the composition \( \pi_1 \circ (f|_{\mathbb{Q}_p}) \) factors through a map

\[
\text{Res}_{\psi}(f): \mathbb{Q}_p/p^{r+1} \mathbb{Z}_p \to \mathbb{F}_p
\]
such that
\[ \text{Res}^r_s(f) \circ \pi_{r+1} = \pi_1 \circ f|_{\mathbb{Q}_p} . \]
The map $\text{Res}^r_s$ induces an isomorphism
\[ \delta^r/p \delta^r \sim \mathcal{E}(\mathbb{Q}_p/p^{r+1}\mathbb{Z}_p, \mathbb{F}_p) \]
\[ \Psi(p^{i-r}x) \mapsto x_{r-i}, \text{ for any } i \geq 0 , \]
of $\mathbb{F}_p$-Hopf Fréchet algebras, where, for any $j \in \mathbb{Z}_{\leq r}, x_j : \mathbb{Q}_p/p^{r+1}\mathbb{Z}_p \to \mathbb{F}_p$ is induced by the map defined in (1.0.4);

3. $\mathcal{C}_{\text{unif}}(\mathbb{Q}_p, \mathbb{Z}_p)$ is the completion of the valued ring $(\delta^r, w_{\infty})$. More precisely, for any $f \in \mathcal{C}_{\text{unif}}(\mathbb{Q}_p, \mathbb{Z}_p)$ and any $a \in \mathbb{Z}_{\geq 0}$, there is an index $r = r(a) \in \mathbb{Z}$ and a function
\[ \psi_a(x) \in \mathbb{Z}_p[\Psi(p^{i-r}x) \mid i \in \mathbb{Z}_{\geq 0}] \]
such that
\[ w_{\infty}(f - \psi_a) \geq a . \]

Theorem 1.5 shows that the $\mathbb{Z}_p$-Fréchet Hopf algebra $\delta^r$, for any $r \in \mathbb{Z}$, provides a formal convergent lifting of the constant $p$-divisible group $\mathbb{Q}_p/p^{r+1}\mathbb{Z}_p$ over $\mathbb{F}_p$ to a formal group over $\mathbb{Z}_p$ whose analytic fiber is $G_a$.

1.5 Comments

We regard Theorem 1.5 as providing an analog of the classical Féjer theorem which asserts that any continuous function $f : \mathbb{R} \to \mathbb{C}$ periodic of period 1, is the uniform limit on $\mathbb{R}$ of the symmetric averages of the partial sums in the Fourier series of $f$ in terms of the entire functions
\[ e^{2\pi i n x/p} : \mathbb{R} \to \mathbb{C}, \quad n = 0, 1, \ldots . \]
The similarity with classical Fourier expansions will be made more stringent in [3], where we will obtain an extension of the classical Mahler binomial expansions of continuous functions $\mathbb{Z}_p \to \mathbb{Z}_p$ to an expansion of any continuous functions $\mathbb{Q}_p \to \mathbb{Q}_p$ as a series of entire functions of exponential type.

Some (still scarce) numerical evidence based on the calculations in the Appendix, indicates that
\[ v_p(\zeta_{p^n}/p^n) > \frac{p^n-1}{p-1} \]
for any $n \geq 1$, where $\zeta_{p^n}$ is any primitive $p^n$-th root of unity. If this were the case, the non-zero zeros of $\Psi_p$ would be parametrized by $p^\infty$-roots of unity. More precisely, for any primitive $p^n$-th root of unity $\zeta = \zeta_{p^n}$ there would exist exactly one zero $z_\zeta$ of $\Psi_p$ in the disc $D(\zeta p^{-n}, (p^n)^-)$. Moreover, by Newton approximation, we would deduce $z_\zeta \in \mathbb{Q}_p(\zeta)$ and, for any $\sigma \in \text{Gal}(\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p)$,
\[ (z_\zeta)^\sigma = z_{\zeta^\sigma} . \]
It would then be tempting to set, for $n = 1, 2, \ldots$
\[ \psi_n(x) = \prod_\zeta (1 - \frac{x}{z_\zeta}) , \]
where the product runs over the primitive $p^n$-th roots of unity so that
\[ \psi_n(x) \in 1 + (p^n x)^{p^n-1}\mathbb{Z}_p[x] . \]
We would then get the formula

\[(1.5.4) \quad \Psi_p(x) = x \prod_{n=1}^{\infty} \psi_n(x).\]

Unfortunately, none of the above is proven yet.

We expect that a completely analogous theory should exist for any finite extension \(K/\mathbb{Q}_p\). To develop it properly it will be necessary to extend Barsotti covector’s construction to ramified Witt vectors modeled on \(K\) and to relate this construction to Lubin-Tate groups over \(K^\circ\) \[15\].

### 1.6 Acknowledgments

It is a pleasure to acknowledge that the proofs in sections 2 and 3 of this text are based on a discussion with Philippe Robba which took place in April 1980. I am strongly indebted to him for this and for his friendship.

I am indebted to Maurizio Candilera for the idea of replacing \(p\) by \(q = p^f\) in the original draft of this manuscript. He also provided numerical calculations that were extremely useful to clarify our ideas.

The collaboration with Maurizio Cailotto in [4] and [5] has greatly helped us to properly express the subtle properties of topological \(\mathbb{Z}_p\)-algebras related to \(\Psi\).

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### 2 Elementary proofs of the main properties of \(\Psi\)

We prove here the basic properties of the function \(\Psi\). In contrast to [2], the proofs are here completely self-contained.

**Proposition 2.1.** The equation \((1.3.5)\) has a unique solution in \(\Psi = \Psi_q \in T + T^2\mathbb{Z}[[T]]\).

**Proof.** We endow \(\mathbb{Z}[[T]]\) of the \(T\)-adic topology. It is clear that, for any \(\varphi \in T\mathbb{Z}[[T]]\), the series \(\sum_{j=1}^{\infty} p^{-j} \varphi(p^j T)^{q^j} \) converges in \(T\mathbb{Z}[[T]]\). Moreover, the map

\[(2.1.1) \quad \mathcal{L} : \varphi \mapsto T - \sum_{j=1}^{\infty} p^{-j} \varphi(p^j T)^{q^j},\]

is a contraction of the complete metric space \(T + T^2\mathbb{Z}[[T]]\). In fact, let \(\varepsilon(T) \in T^r \mathbb{Z}[[T]]\), with \(r \geq 3\). For any \(\varphi \in T + T^2\mathbb{Z}[[T]]\) we see that

\[\mathcal{L}(\varphi + \varepsilon) - \mathcal{L}(\varphi) \in T^{r(q-1)+q} \mathbb{Z}[[T]].\]

Since \(r(q-1) + q > r\) this shows that \(\mathcal{L}\) is a contraction. So, this map has a unique fixed point which is \(\Psi_q(T)\). \(\square\)

The following proposition, due to M. Candilera, provides an alternative proof of Proposition 2.1 and finer information on \(\Psi_q(T)\).

**Proposition 2.2.** (M. Candilera) The functional equation for the unknow function \(u\)

\[(2.2.1) \quad 1 = \sum_{j=0}^{\infty} p^j(q^j-1)^j T^{q^j-1} u(p^j(q-1) T)^{q^j} - u^q(T)^{q^j} - u^q(T)^{q^j}.\]
admits a unique solution \( u(T) = u_q(T) \in 1 + T\mathbb{Z}[[T]] \). We have

\[
\Psi_q(T) = Tu_q(T^{q-1}).
\]

**Proof.** In this case we consider the \( T \)-adic metric space \( 1 + T\mathbb{Z}[[T]] \) and the map

\[
\mathcal{M} : 1 + T\mathbb{Z}[[T]] \rightarrow 1 + T\mathbb{Z}[[T]]
\]

\[
\varphi \mapsto 1 - \sum_{j=1}^{\infty} p^j(q^{j-1})T^{\frac{j-1}{q-1}} \varphi(p^j(q-1)^j)q^j.
\]

We endow \( \mathbb{Z}[[T]] \) of the \( T \)-adic topology. It is clear that, for any \( \varphi \in T\mathbb{Z}[[T]] \), the series \( \sum_{j=1}^{\infty} p^j \varphi(p^j)q^j \) converges in \( T\mathbb{Z}[[T]] \). If \( \varepsilon(T) \in T^r \mathbb{Z}[[T]] \), with \( r \geq 2 \). For any \( \varphi \in 1 + T\mathbb{Z}[[T]] \) we see that

\[
\mathcal{M}(\varphi + \varepsilon) - \mathcal{M}(\varphi) \in T^{r+1} \mathbb{Z}[[T]].
\]

So, the map \( \mathcal{M} \) is a contraction and its unique fixed point has the properties stated for the series \( u \) in the statement. \( \square \)

**Proposition 2.3.** The series \( \Psi(T) = \Psi_q(T) \) is entire.

**Proof.** Since \( \Psi \in T + T^2 \mathbb{Z}[[T]] \subseteq T\mathbb{Z}[[T]] \), we deduce that \( \Psi \) converges for \( v_p(T) > 0 \). Since the coefficient of \( T \) in \( \Psi(T) \) is 1, whenever \( v_p(T) > 0 \) we have \( v_p(\Psi(T)) = v_p(T) \).

Suppose \( \Psi \) converges for \( v_p(T) > \rho \), for \( \rho \leq 0 \). Then, for \( j \geq 1 \), \( \Psi(p^j T)^q \) converges for \( v_p(T) > \rho - 1 \). Moreover, if \( j > -\rho + 1 \) and \( v_p(T) > \rho - 1 \), we have

\[
v_p(p^{-j} \Psi(p^j T)^q) = -j + q^j(v_p(p^j T)) \geq -j + q^j(j + \rho - 1),
\]

and this last term \( \to +\infty \), as \( j \to +\infty \).

This shows that the series \( T - \sum_{j=1}^{\infty} p^{-j} \Psi(p^j T)^q \) converges uniformly for \( v_p(T) > \rho - 1 \), so that its sum, which is \( \Psi \), is analytic for \( v_p(T) > \rho - 1 \). It follows immediately from this that \( \Psi \) is an entire function. \( \square \)

**Remark 2.4.** Notice that we have proved that, for any \( j = 0, 1, \ldots \) and for \( v_p(T) > -j \),

\[
v_p(p^{-j} \Psi(p^j T)^q) = -j + q^j(j + v_p(T)).
\]

**Proposition 2.5.** For any \( a \in \mathbb{Q}_q, \Psi_q(a) \in \mathbb{Z}_q \).

**Proof.** Let \( a \in \mathbb{Z}_q \). We define by induction the sequence \( \{a_i\}_{i=0,1,\ldots} :\)

\[
a_0 = a, \quad a_i = \sum_{j=0}^{i-1} p^j (a_{q^{j-1}} - a_{q^j}).
\]

Since, for any \( a, b \in \mathbb{Z}_q \), if \( a \equiv b \mod p \), then \( a^n \equiv b^n \mod pq^n \), hence \( \mod p^{n+1} \), while \( a \equiv a^q \mod p \), we see that \( a_i \in \mathbb{Z}_q \), for any \( i \).

We then see by induction that, for any \( i \),

\[
a_i = p^{-i}(a - \sum_{j=0}^{i-1} p^j a_{q^{j-1}}) \quad \text{or, equivalently,} \quad a = \sum_{j=0}^{i} p^j a_{q^{j-1}}.
\]

Explicitly, if we substitute in the formula which defines \( a_i \), namely

\[
p^j a_i = \sum_{j=0}^{i-1} p^j a_{q^{j-1}} - \sum_{j=0}^{i-1} p^j a_{q^{j-1}}.
\]
the \((i-1)\)-st step of the induction, namely, \(a = \sum_{j=0}^{i-1} p^j a_j^{i-j-1}\), we get

\[ p^i a_i = a - \sum_{j=0}^{i-1} p^j a_j^{i-j} , \]

which is precisely the \(i\)-th inductive step.

From the functional equation (1.3.5) and from Remark 2.4 we have, for \(a \in \mathbb{Z}_q\) and \(i = 0, 1, 2, \ldots\),

\[(2.5.3) \quad \Psi(p^{-i} a) \equiv p^{-i} a - \sum_{\ell=1}^{i} p^{-\ell} \Psi(p^{\ell} p^{-i} a)^{q^\ell} = p^{-i} (a - \sum_{j=0}^{i-1} p^j \Psi(p^{-j} a)^{q^{i-j}}) \mod p\mathbb{Z}_q. \]

Notice that \(\Psi(a) \in \mathbb{Z}_q\) and that, modulo \(p\mathbb{Z}_q\), \(\Psi_q(a) \equiv a_0\), defined as in (2.5.1).

Similarly, we show by induction on \(i\) that for \(a_1, \ldots, a_i, \ldots\) defined as in (2.5.1),

\[(2.5.4) \quad \Psi(p^{-i} a) \equiv a_i \mod p\mathbb{Z}_q, \]

which proves the statement. In fact, assume \(\Psi(p^{-j} a) \equiv a_j \mod p\mathbb{Z}_q\), for \(j = 0, 1, \ldots, i-1\), and plug this information in (2.5.3). We get

\[(2.5.5) \quad \Psi(p^{-i} a) \equiv p^{-i} a - \sum_{\ell=1}^{i} p^{-\ell} a_\ell^{q^\ell} = p^{-i} (a - \sum_{j=0}^{i-1} p^j a_j^{q^{i-j}}) = a_i \mod p\mathbb{Z}_q, \]

which is the \(i\)-th inductive step.

\[ \square \]

**Remark 2.6.** Notice that from (2.5.3) it follows that, for any \(a \in p^{-n}\mathbb{Z}_q\),

\[ a \equiv \sum_{\ell=0}^{n} p^{-\ell} \Psi_q(p^{\ell} a)^{q^\ell} \mod p\mathbb{Z}_q. \]

The formula can be more precise using the functional equation (1.3.5). We get, for any \(a \in \mathbb{Q}_q\),

\[(2.6.1) \quad a \equiv \sum_{\ell=0}^{-v_p(a)+i} p^{-\ell} \Psi_q(p^{\ell} a)^{q^\ell} \mod p^{i+1}\mathbb{Z}_q, \quad \forall \; i \in \mathbb{Z}_{\geq 0}. \]

that is

\[(2.6.2) \quad a \equiv \sum_{\ell=0}^{i} p^{-\ell} \Psi_q(p^{\ell} a)^{q^\ell} \mod p^{i+v_p(a)+1}\mathbb{Z}_q, \quad \forall \; i \in \mathbb{Z}_{\geq -v_p(a)}. \]

We generalize (1.0.3) as

**Corollary 2.7.** For any \(a \in \mathbb{Q}_q\), let

\[ a_i := \Psi_q(p^{-i} a) \mod p\mathbb{Z}_q \in \mathbb{F}_q. \]

We have

\[(2.7.1) \quad a = \sum_{i \gg -\infty} a_i p^i \in W(\mathbb{F}_q)[1/p] = \mathbb{Q}_q. \]
Proof. Assume first that \( a \in \mathbb{Z}_q \). In this case \( (2.6.2) \) implies

\[
a \equiv \sum_{\ell=0}^{i} p^{-\ell} \Psi_q(p^\ell a)^q \mod p^{i+1} \mathbb{Z}_q , \quad \forall \ i \in \mathbb{Z}_{\geq 0} .
\]

So, the statement follows from the obvious

**Lemma 2.8.** Let \( i \mapsto b_i \) and \( i \mapsto c_i \), for \( i = 0, 1, \ldots \), be two sequences in \( \mathbb{Z}_q \) such that

\[
\sum_{j=0}^{i} p^{j} b_j^{i-j} = \sum_{j=0}^{i} p^{j} c_j^{i-j} \mod p^{i+1} \mathbb{Z}_q , \quad \forall \ i \in \mathbb{Z}_{\geq 0} .
\]

Then

\[
b_i \equiv c_i \mod p \mathbb{Z}_q , \quad \forall \ i \in \mathbb{Z}_{\geq 0} .
\]

In the general case, assume \( a \in p^{-n} \mathbb{Z}_q \). Then

\[
p^n a = \sum_{i=0}^{\infty} [\Psi_q(p^{n-i} a) \mod p \mathbb{Z}_q] p^i \in W(\mathbb{F}_q) .
\]

hence

\[
a = \sum_{i=0}^{\infty} [\Psi_q(p^{n-i} a) \mod p \mathbb{Z}_q] p^{i-n} \in p^{-n} W(\mathbb{F}_q) .
\]

\( \square \)

From the previous corollary, it follows that \( a \in \mathbb{Q}_q \) has the following expression as a Witt bivector with coefficients in \( \mathbb{F}_q \)

\[
a = (\ldots, a^{(q/p)^i}, \ldots, a^{(q/p)^2}, a^{q/p}, a_0, a_1, a_2^2, \ldots) .
\]

which obviously equals \( (\ldots, a^{-i}, \ldots, a^{-2}, a^{-1}; a_0, a_1, a_2, \ldots) \), if \( q = p \).

**Remark 2.9.** We would have liked to provide a simple addition formula for \( \Psi_q \) of the form \( (1.3.4) \), in terms of the same power-series \( \Phi \). We could not do it, nor were we able to establish the relation between \( \Psi_q \) and \( \Psi_p \), for \( q = p^f \). On the other hand it is clear that Barsotti’s construction of Witt bivectors, based on classicals Witt vectors, extends to the \( L \)-Witt vectors of \( [15, \text{Chap. 1}] \), where \( L/\mathbb{Q}_p \) denotes any fixed finite extension. In our case, we would only need the construction of \( \text{loc.cit.} \) in the case of the field \( L = \mathbb{Q}_q \). We believe that the inductive limit of \( \mathbb{Z}_q \)-groups \( W_{\mathbb{Q}_q, n} \rightarrow W_{\mathbb{Q}_q, n+1} \) under Verschiebung

\[
V : (x_{-n}, \ldots, x_{-1}, x_0) \rightarrow (0, x_{-n}, \ldots, x_{-1}, x_0)
\]

is a \( \mathbb{Z}_q \)-formal groups whose addition law is expressed by a power-series \( \Phi_q \) analog to Barsotti’s \( \Phi \). We believe that equation \( (1.3.4) \) still holds true for \( \Psi_q \) if we replace \( \Phi \) by \( \Phi_q \). We also believe that a generalized \( \Psi \) exists for any finite extension \( L/\mathbb{Q}_p \), with analogous properties.
3 Valuation and Newton polygons of $\Psi_q$

This section is dedicated to establishing the growth behavior of $|\Psi_q(x)|$ as $|x| \to \infty$. These results will be essential to get the delicate estimates of [3].

We recall from [11] that the valuation polygon of a Laurent series $f = \sum_{i \in \mathbb{Z}} a_i T^i$ with coefficients $a_i \in \mathbb{C}_p$, converging in an annulus $A := \alpha \leq v_p(T) \leq \beta$, is the graph $\text{Val}(f)$ of the function $\mu \mapsto v(f, \mu) := \inf_i (v_p(a_i) + i\mu)$, which is in fact finite along the segment $\alpha \leq \mu \leq \beta$. The function $\mu \mapsto v(f, \mu)$ is continuous, piecewise affine, and concave on $[\alpha, \beta]$. For any $\mu \in [\alpha, \beta]$, we have $v(f, \mu) = \inf \{v_p(\Psi(x)) \mid v_p(x) = \mu\}$. In the case of $\Psi$, $A = \mathbb{C}_p$ and the segment $[\alpha, \beta]$ is the entire $\mu$-line. For the convenience of the reader we have recalled below the relation between the valuation polygon and the Newton polygon of $f$.

We prove

**Theorem 3.1.** The valuation polygon of $\Psi_q$ goes through the origin, has slope 1 for $\mu > -1$, and slope $q^j$, for $-j - 1 < \mu < -j$, $j = 1, 2, \ldots$ (see Figure 1).

![Figure 1: The valuation polygon of $\Psi_q$.](image)

**Proof.** We recall that if both $f$ and $g$ converge in the annulus $A := \alpha \leq v_p(T) \leq \beta$, $v(f + g, \mu) \geq \inf(v(f, \mu), v(g, \mu))$ and that we have equality at $\mu \in [\alpha, \beta]$, if $v(f, \mu) \neq v(g, \mu)$. Moreover, for any $n \in \mathbb{N}$, $v(f^n, \mu) = n v(f, \mu)$.

In the polygon in Figure 1, for $j = 1, 2, \ldots$, the side of projection $[-j, 1 - j]$ on the $\mu$-axis is the graph of the function

\[
\sigma_j(\mu) := q^{j-1}(\mu + j - 1) - q^{j-2} - \cdots - q - 1 .
\]

Notice that

\[
\sigma_{j+1}(\mu) = -1 + q \sigma_j(\mu + 1) ,
\]

and therefore

\[
\sigma_{j+i}(\mu) = -1 - q - \cdots - q^{i-1} + q^i \sigma_j(\mu + i) ,
\]
for any \( i = 0, 1, 2, \ldots \).

Since \( \Psi \in T\mathbb{Z}[T] \) and since the coefficient of \( T \) is \( 1 \), we have \( v(\Psi, \mu) = \mu \), for \( \mu \geq 0 \). For \( 0 > \mu > -j \), \( j \geq 1 \), we have

\[
(3.1.4) \quad v(p^{-j}\Psi(p^jT)^{q_j}, \mu) = -j + v(\Psi(p^jT)^{q_j}, \mu) = -j + q^j v(\Psi(p^jT), \mu) = -j + q^j v(\Psi(S), j + \mu) = -j + (j + \mu)q^j > \mu = v(T, \mu),
\]

where we have used the variable \( S = p^jT \).

**Remark 3.2.** For \( \mu = -j \) we get equality in the previous formula.

Let us set, for \( j = 0, 1, 2, \ldots \),

\[
\ell_j(\mu) = -j + (j + \mu)q^j,
\]

so that \((3.1.4)\) becomes

\[
(3.2.1) \quad v(p^{-j}\Psi(p^jT)^{q_j}, \mu) = \ell_j(\mu) > \ell_0(\mu) = v(T, \mu),
\]

for \( 0 > \mu > -j \), \( j \geq 1 \), with equality holding if \( \mu = -j \). Notice that

\[
\ell_0(\mu) = \mu = \sigma_1(\mu).
\]

Because of \((3.2.1)\) and \((1.0.6)\), and by continuity of \( \mu \mapsto v(\Psi, \mu) \), we have

\[
(3.2.2) \quad v(\Psi, \mu) = v(T, \mu) = \mu = \sigma_1(\mu), \text{ for } \mu \geq -1.
\]

We now reason by induction on \( n = 1, 2, \ldots \). We assume that, for any \( j = 1, 2, \ldots, n \) the side of projection \([-j, 1 - j]\) on the \( \mu \)-axis of the valuation polygon of \( \Psi \) is the graph of \( \sigma_j(\mu) \). This at least was proven for \( n = 1 \). We consider the various terms in the functional equation

\[
\Psi = T - p^{-1}\Psi(pT)^q - p^{-2}\Psi(p^2T)^{q^2} - \sum_{j=3}^{\infty} p^{-j}\Psi(p^jT)^{q^j}.
\]

We assume \( n > 1 \). For \( j = 1, 2, \ldots, n \), and \(-n - 1 < \mu < -n \), we have

\[
(3.2.3) \quad v(p^{-j}\Psi(p^jT)^{q_j}, \mu) = -j + v(\Psi(p^jT)^{q_j}, \mu) = -j + q^j v(\Psi(p^jT), \mu) = -j + q^j v(\Psi(S), j + \mu) = -j + q^j \sigma_{n-j+1}(\mu + j),
\]

since \( j - n - 1 < j + \mu < j - n \), and therefore the inductive assumption gives \( v(\Psi, j + \mu) = \sigma_{n-j+1}(\mu + j) \) in that interval. For \( j > n \), and \(-n - 1 < \mu \), we have instead, from \((3.2.1)\), \( v(p^{-j}\Psi(p^jT)^{q_j}, \mu) = \ell_j(\mu) \).

**Lemma 3.3.** Let \( n > 1 \).

1. For \( j = 1, 2, \ldots, n \) and for any \( \mu \in \mathbb{R} \),

\[
(3.3.1) \quad \sigma_{n+1}(\mu) < -j + q^l \sigma_{n-j+1}(\mu + j).
\]

2. For \( j > n \) and \( \mu > -n - 1 \), we have

\[
(3.3.2) \quad \sigma_{n+1}(\mu) < \ell_j(\mu).
\]
3. For $-n - 1 < \mu < -n$,

\[
\sigma_{n+1}(\mu) < \mu.
\]

**Proof.** Assertion [3.3.1] is clear, since the two affine functions $\mu \mapsto \sigma_{n+1}(\mu)$ and $\mu \mapsto -j + q^n \sigma_{n-j+1}(\mu + j)$, have the same slope $q^n$, while their values at $\mu = -n$ are $-q^{n-1}$.
\( q^{n-2} - \cdots - q - 1 \) and \(- j - q^{n-1} - q^{n-2} - \cdots - q^j - q^i \), respectively. Notice that
\[-j - q^{n-1} - q^{n-2} - \cdots - q^j - q^i = (q^j - j) - q^{n-1} - q^{n-2} - \cdots - p - 1 > -q^{n-1} - q^{n-2} - \cdots - q - 1, \]
so that the conclusion follows.

We examine assertion (3.3.2), namely that, for \( j > n \) and \( \mu > -n - 1 \), we have
\[ q^n(\mu + n) - q^n - q^{n-1} - q^{n-2} - \cdots - q - 1 < -j + (j + \mu)q^j. \]
The previous inequality translates into
\[ q^n(\mu + n) - q^n - q^{n-1} - q^{n-2} - \cdots - q - 1 < -j + (j - n)q^j + (n + \mu)q^n(j - n), \]
that is
\[ q^{n-1} + q^{n-2} + \cdots + q + 1 - j + (j - n)q^j + (n + \mu)q^n(q^{j-n} - 1) > 0, \]
for \( \mu > -n - 1 \). Since the l.h.s. is an increasing function of \( \mu \), it suffices to show that the inequality hold for \( \mu = -n - 1 \), that is to prove that
\[ q^{n-1} + q^{n-2} + \cdots + q + 1 - j + (j - n)q^j - q^n(q^{j-n} - 1) > 0, \]
for any \( j > n > 1 \). We rewrite the l.h.s. of (3.3.4) as
\[ q^{n-1} + q^{n-2} + \cdots + q + 1 - n + (n - j)q^j - q^j + q^n = (q^{n-1} + q^{n-2} + \cdots + q + 1 - n) + (q^j - 1)(j - n) + (q^n - q^j), \]
where the four terms in round brackets on the r.h.s. are each, obviously, positive numbers. The conclusion follows.

We finally show (3.3.3), namely that for \(-n - 1 < \mu < -n\),
\[ q^n(\mu + n) - q^n - q^{n-1} - q^{n-2} - \cdots - p - 1 < \mu. \]
It suffices to compare the values at \( \mu = -n - 1 \) and at \( \mu = -n \). We get
\[ -q^n - q^{n-1} - q^{n-2} - \cdots - q - 1 < -n - 1, \]
and
\[ -q^{n-1} - q^{n-2} - \cdots - q - 1 < -n, \]
respectively, both obviously true. \( \Box \)

The previous calculation shows that the side of projection \([-n - 1, -n]\) on the \( \mu \)-axis of the valuation polygon of \( \Psi \) is the graph of \( \sigma_{n+1}(\mu) \). We have then crossed the inductive step Case \( n \Rightarrow \) Case \( n + 1 \), and Theorem 3.1 is proven. \( \Box \)

**Corollary 3.4.** For any \( i = 1, 2, \ldots \), and \( v_p(x) \geq -i \) (resp. \( v_p(x) > -i \)), we have \( v_p(\Psi_q(x)) \geq -q-1 \) (resp. \( v_p(\Psi_q(x)) > -q-1 \)). If \( v_p(x) > -1 \), we have \( v_p(\Psi_q(x)) = v_p(x) \).

**Proof.** The last part of the statement is a general fact for automorphisms of an open \( k \)-analytic disk \( D \) with one \( k \)-rational fixed point \( a \in D(k) \) (the disk \( v_p(x) > -1 \) and \( x(a) = 0 \), in the present case) \( \Box \).

We now recall that to a Laurent series \( f = \sum_{i \in \mathbb{Z}} a_i T^i \) with coefficients \( a_i \in C_p \), converging in an annulus \( A := \alpha \leq v_p(T) \leq \beta \), one associates two, dually related, polygons. The valuation polygon \( \mu \mapsto v(f, \mu) \), was recalled before. The **Newton polygon** \( \text{Nw}(f) \) of \( f \) is the convex closure in the standard affine plane \( \mathbb{R}^2 \) of the points \((-i, v(a_i))\) and \((0, +\infty)\). If
If \(a_i = 0\), then \(v(a_i)\) is understood as \(= +\infty\). We define \(s \mapsto \text{Nw}(f, s)\) to be the function whose graph is the lower-boundary of \(\text{Nw}(f)\). The main property of \(\text{Nw}(f)\) is that the length of the projection on the \(X\)-axis of the side of slope \(\sigma\) is the number of zeros of \(f\) of valuation \(= \sigma\). The formula

\[v(f, \mu) = \inf_{i \in \mathbb{Z}} i \mu + v(a_i)\]

indicates (cf. [11]) that the relation between \(\text{Nw}(f)\) and \(\text{Val}(f)\) coincides with the duality formally described in the following lemma.

**Lemma 3.5. (Duality of polygons)** In the projective plane \(\mathbb{P}^2\), with affine coordinates \((X, Y)\), we consider the polarity with respect to the parabola \(X^2 = -2Y\)

\[\mathbb{P}^2 \to (\mathbb{P}^2)^* \to \mathbb{P}^2,\]

point \((\sigma, \tau) \mapsto \text{line} (Y = -\sigma X - \tau) \mapsto \text{point} (\sigma, \tau)\).

Assume the graph \(\Gamma\) of a continuous convex piecewise affine function has consecutive vertices

\[\ldots, (-i_0, \varphi_0), (-i_1, \varphi_1), (-i_2, \varphi_2), (-i_3, \varphi_3), \ldots\]

joined by the lines

\[\ldots, Y = \sigma_1 X + \tau_1, Y = \sigma_2 X + \tau_2, Y = \sigma_3 X + \tau_3, \ldots .\]

Then, the lines joining the points

\[\ldots, (\sigma_1, \tau_1), (\sigma_2, \tau_2), (\sigma_3, \tau_3), \ldots\]

are

\[\ldots, Y = i_1 X - \varphi_1, Y = i_2 X - \varphi_2, \ldots ,\]

and the polarity transforms these back into

\[\ldots, (-i_1, \varphi_1), (-i_2, \varphi_2), \ldots .\]

We say that the graph \(\Gamma^*\) joining the vertices \((\sigma_i, \tau_i), (\sigma_{i+1}, \tau_{i+1})\) by a straight segment is the dual graph of \(\Gamma\). It is clear that the relation is reciprocal, that is \((\Gamma^*)^* = \Gamma\) and that \(\Gamma^*\) is a continuous concave piecewise affine function.

**Proof.** It is the magic of polarities. \(\square\)

We now apply the previous considerations to the two polygons associated to the function \(\Psi_q\).

**Corollary 3.6.** The Newton polygon \(\text{Nw}(\Psi_q)\) has vertices at the points

\[V_i := (-q^i, i q^i - \frac{q^i - 1}{q^{-1}}) = (-q^i, i q^i - q^{i-1} - \cdots - q - 1).\]

The equation of the side joining the vertices \(V_i\) and \(V_{i-1}\) is

\[Y = -i X - \frac{q^i - 1}{q - 1};\]

its projection on the \(X\)-axis is the segment \([-q^i, -q^{i-1}]\). So, \(\text{Nw}(\Psi)\) has the form described in Figure 4.
Corollary 3.7. For any $i = 0, 1, \ldots$, the map $\Psi = \Psi_q$ induces coverings of degree $q^i$.

(3.7.1) \[ \Psi : \{ x \in \mathbb{C}_p \mid v_p(x) > -i - 1 \} \longrightarrow \{ x \in \mathbb{C}_p \mid v_p(x) > -\frac{q^{i+1} - 1}{q - 1} \}, \]

(in particular, an isomorphism

(3.7.2) \[ \Psi : \{ x \in \mathbb{C}_p \mid v_p(x) > -1 \} \longrightarrow \{ x \in \mathbb{C}_p \mid v_p(x) > -1 \}, \]

for $i = 0$, finite maps of degree $q^i$

(3.7.3) \[ \Psi : \{ x \in \mathbb{C}_p \mid -(i + 1) < v_p(x) < -i \} \longrightarrow \{ x \in \mathbb{C}_p \mid -\frac{q^{i+1} - 1}{q - 1} < v_p(x) < -\frac{q^i - 1}{q - 1} \}, \]

and finite maps of degree $q^{i+1} - q^i$

(3.7.4) \[ \Psi : \{ x \in \mathbb{C}_p \mid v_p(x) = -i - 1 \} \longrightarrow \{ x \in \mathbb{C}_p \mid -\frac{q^{i+1} - 1}{q - 1} \leq v_p(x) \}. \]

Proof. The shape of the Newton polygon of $\Psi$ indicates that, for any $a \in \mathbb{C}_p$, with $v_p(a) > -1$, the side of slope $= \nu_p(a)$ of the Newton polygon of $\Psi - a$ has projection of length 1 on the $X$-axis. So, $\Psi : \{ x \in \mathbb{C}_p \mid v_p(x) > -1 \} \rightarrow \{ x \in \mathbb{C}_p \mid v_p(x) > -1 \}$ is bijective, hence biholomorphic. Now we recall from Corollary 3.4 that for any given $i \geq 1$,

(3.7.5) \[ \Psi(\{ x \in \mathbb{C}_p \mid v_p(x) > -i - 1 \}) = \{ x \in \mathbb{C}_p \mid v_p(x) > -\frac{q^{i+1} - 1}{q - 1} \}. \]

So, let $a$ be such that $-\frac{q^{i+1} - 1}{q - 1} < v_p(a) \leq -\frac{q^{i+1} - 1}{q - 1}$, say $v_p(a) = -\frac{q^{i+1} - 1}{q - 1} + \varepsilon$, with $\varepsilon \in [0, q^i)$. Then, the Newton polygon of $\Psi - a$ has a single side of slope $> -i - 1$, which has precisely slope $= -\varepsilon q^{-i} > -i$, and has projection of length $q^i$ on the $X$-axis. So, the equation $\Psi(x) = a$ has precisely $q^i$ solutions $x$ in the annulus $-i - 1 < v_p(x) \leq -i$. If, for the same $i$, $-\frac{q^{i} - 1}{q - 1} < v_p(a) \leq -\frac{q^{i} - 1}{q - 1}$, the Newton polygon of $\Psi - a$ has a side of slope $-i$, whose projection on the $X$-axis has length $q^{i+1} - q^i$, and a side of slope $\sigma$, $1 - i \geq \sigma > -i$, whose
projection on the $X$-axis has length $q^{-i-1}$. So again $\Psi^{-1}(a)$ consists of $q^i$ distinct points. We go on, for $a$ in an annulus of the form $-\frac{q^{i+1}}{q-1} < v_p(a) \leq -\frac{q^{i+1}}{q-1}$, up to $j = i - 2$, i.e. to $-\frac{q^{i+1}}{q-1} < v_p(a) \leq -\frac{q^{i+1}}{q-1}$. In that case, the Newton polygon of $\Psi - a$ has a side of slope $-i$ of projection $q^i - q^{i+1}$, a side of slope 1 of projection $q^{i+1} - q^{i+2}$, ..., a side of slope $j - i$ of projection $q^{j-i} - q^{j-i+1}$ on the $X$-axis, ... up to a side of slope $-1$ of projection $q - 1$ on the $X$-axis. Finally, for $v_p(a) > -1$, there is still exactly one solution of $\Psi(x) = a$, with $v_p(x) > -1$. This means that $\Psi$ induces a (ramified) covering of degree $q^i$ in (3.7.1). 

Corollary 3.8. For any $i = 1, 2, \ldots$, the map $\Psi_q$ has $q^i - q^{i-1}$ simple zeros of valuation $-i$. The inverse function $\beta(T) = \beta_q(T)$ of $\Psi_q(T)$ (i.e. the power series such that, in $T\mathbb{Z}[[T]]$, $\Psi_q(\beta_q(T)) = T = \beta_q(\Psi_q(T))$) belongs to $T + T^2\mathbb{Z}[[T]]$. Its disk of convergence is exactly $v_p(T) > -1$.

Proof. The fact that $\beta_q$ belongs to $T + T^2\mathbb{Z}[[T]]$ is obvious. The convergence of $\beta_q$ for $v_p(T) > -1$ follows from (3.7.2). The fact that it cannot converge in a bigger disk is a consequence of the fact that $\Psi_q$ has $q - 1$ zeros of valuation $-1$.

4 Rings of continuous functions on $\mathbb{Q}_p$

We consider here a linearly topologized separated and complete ring $k$, whose family of open ideals we denote by $\mathcal{P}(k)$. In practice $k = \mathbb{Z}_p$ or $= \mathbb{F}_p$, or $= \mathbb{Z}_p/p^r\mathbb{Z}_p$, for any $r \in \mathbb{Z}$. More generally, $A$ will be a complete and separated topological ring equipped with a $\mathbb{Z}$-linear topology, defined by a family of open additive subgroups of $A$. In particular we have in mind $A = k$ a fixed finite extension $K$ of $\mathbb{Q}_p$, whose topology is $K$-linear but not $K$-linear. Again, a possible $k$ would be $K^a$ or any $K^a/\langle \pi_K \rangle^r$, for a parameter $\pi = \pi_K$ of $K$, and $r$ as before.

We will express our statements for an abelian topological group $G$, which is separated and complete in the $\mathbb{Z}$-linear topology defined by a countable family of profinite subgroups $G_r$, with $G_r \supset G_{r+1}$, for any $r \in \mathbb{Z}$. So,

$$G = \lim_{r \to \infty} G/G_r = \lim_{r \to -\infty} G_r,$$

where $G/G_r$ is discrete, $G_r$ is compact, and limits and colimits are taken in the category of topological abelian groups separated and complete in a $\mathbb{Z}$-linear topology. We denote by $\pi_r : G \to G/G_r$ the canonical projection. Then, $G$ is canonically a uniform space in which a function $f : G \to A$ is uniformly continuous iff, for any open subgroup $J \subset A$, the induced function $G \to A/J$ factors via $\pi_r$, for some $r = r(J)$. A subset of $G$ of the form $\pi_r^{-1}(h)) = g + G_r$, for $g \in G$ and $h = \pi_r(g)$ is sometimes called the ball of radius $r$ and center $g$. In particular, $G$ is a locally compact, paracompact, 0-dimensional topological space. A general discussion of the duality between $k$-valued functions and measures on such a space, will appear in [5]. In practice here $G = \mathbb{Q}_p$ or $\mathbb{Q}_p/p^r\mathbb{Z}_p$ or $p^r\mathbb{Z}_p$, with the obvious uniform and topological structure.

Definition 4.1. Let $G$ and $A$ be as before. We define $\mathcal{C}(G, A)$ (resp. $\mathcal{C}_{unif}(G, A)$) as the $A$-algebra of continuous (resp. bounded and uniformly continuous) functions $f : G \to A$. We equip $\mathcal{C}(G, A)$ (resp. $\mathcal{C}_{unif}(G, A)$) with the topology of uniform convergence on compact subsets of $X$ (resp. on $X$). For any $r \in \mathbb{Z}$ and $g \in G$, we denote by $\chi_{g+G_r}$ is the characteristic function of $g + G_r \subset G/G_r$. If $G$ is discrete and $h \in G$, by $\epsilon_h : G \to k$ we denote the function such that $\epsilon_h(h) = 1$, while $\epsilon_h(x) = 0$ for any $x \neq h$ in $G$.

Remark 4.2. It is clear that if $A = k$ is a linearly topologized ring any subset of $k$ and therefore any function $f : G \to k$, is bounded. So, we write $\mathcal{C}_{unif}(G, k)$ instead of $\mathcal{C}_{unif}(G, k)$.
in this case. If \( G \) is discrete, any function \( G \rightarrow k \) is (uniformly) continuous; still, the bijective map \( C_{\text{unif}}(G,k) \rightarrow \mathcal{C}(G,k) \) is not an isomorphism, so we do keep the difference in notation. If \( G \) is compact, any continuous function \( G \rightarrow k \) is uniformly continuous, but \( C_{\text{unif}}(G,k) \rightarrow \mathcal{C}(G,k) \) is an isomorphism, so there is no need to make any distinction.

**Lemma 4.3.** Notation as above, but assume \( G \) is discrete (so that the \( G_r \)'s are finite). Then \( \mathcal{C}(G,k) \) (resp. \( C_{\text{unif}}(G,k) \)) is the \( k \)-module of functions \( f : G \rightarrow k \) endowed with the topology of simple (resp. of uniform) convergence on \( G \). So

\[
\mathcal{C}(G,k) = \lim_{r \to -\infty} \mathcal{C}(G_r,k) = \prod_h k e_h , \quad h \in G.
\]

Similarly,

\[
C_{\text{unif}}(G,k) = \lim_{r \to +\infty} \prod_{I \in \mathcal{P}(k) \text{ finite}} (k/I) e_h = \prod_{h \in G} k e_h ,
\]

where \( \prod_{h \in G} (k/I) e_h \) carries the discrete topology.

**Proof.** Clear from the definitions.

The next lemma is a simplified abstract form, in the framework of linearly topologized rings and modules, of the classical wavelet decomposition of a continuous function (see for example Colmez [9, §1.3.1]).

**Lemma 4.4.** Notation as above but assume \( G \) is compact (so that the \( G/G_r \)'s are finite). Then

\[
\mathcal{C}(G,k) = C_{\text{unif}}(G,k) = \lim_{r \to +\infty} \mathcal{C}(G/G_r,k) = \lim_{r \to +\infty} \bigoplus_{g + G_r \in G/G_r} k \chi_{g+G_r}.
\]

For any \( r \), the canonical morphism \( \mathcal{C}(G/G_r,k) \rightarrow \mathcal{C}(G,k) \) is injective.

**Proof.** This is also clear from the definitions.

**Remark 4.5.** We observe that the inductive limit appearing in the formula hides the complication of formulas of the type

\[
\chi_{g+G_r} = \sum_i \chi_{g_i+G_{r+1}} \quad \text{if} \quad g + G_r = \bigcup g_i + G_{r+1}
\]

which we do not need to make explicit for the present use (see [5] for a detailed discussion).

**Proposition 4.6.** Notation as above, with \( G \) general. Then in the category \( \mathcal{LM}_k^w \) we have :

1. \( \mathcal{C}(G,k) = \lim_{r \to -\infty} \mathcal{C}(G_r,k) \) for the restrictions \( \mathcal{C}(G_r,k) \to \mathcal{C}(G_{r+1},k) \).

In particular, for any fixed \( r \in \mathbb{Z} \),

\[
\mathcal{C}(G,k) = \prod_{g+G_r \in G/G_r} \mathcal{C}(g+G_r,k).
\]
2. \( C_{\text{unif}}(G, k) = \lim_{\rightarrow} u \rightarrow +\infty C_{\text{unif}}(G/G_r, k) \)

for the embeddings \( C_{\text{unif}}(G/G_r, k) \hookrightarrow C_{\text{unif}}(G/G_{r+1}, k) \)

3. The natural morphism
\[
C_{\text{unif}}(G, k) \rightarrow C(G, k)
\]
is injective and has dense image.

**Proof.** The first two parts follow from the universal properties of limits and colimits. The morphism in part 3 comes from the injective morphisms, for \( r \in \mathbb{Z} \),
\[
C_{\text{unif}}(G/G_r, k) \rightarrow C_{\text{unif}}(G/G_{r+1}, k)
\]
and the universal property of colimits. The inductive limit of these morphisms in the category \( \mathcal{LM}_k \) is a completion of the inductive limit taken in the category \( \text{Mod}_k \) of \( k \)-modules equipped with the \( k \)-linear inductive limit topology. Since the latter is separated and since the axiom AB5 holds for the abelian category \( \text{Mod}_k \), we deduce that the morphism in part 3 is injective. The morphism has dense image because, for any \( r \in \mathbb{Z} \) and for any \( s \in \mathbb{Z} \geq 0 \), the composed morphism
\[
C_{\text{unif}}(G_r/G_{r+s}, k) \rightarrow C_{\text{unif}}(G_{r+s}, k) \rightarrow C(G, k) \rightarrow C(G_r, k)
\]
is the canonical map of Lemma 4.4.

**Remark 4.7.** For \( A = k \), a linearly topologized ring, the topologies in Definition 4.1 are \( k \)-linear. The explicit description of Proposition 4.6 and of its lemmas shows that we have a natural identification
\[
(4.7.1) \quad C(G, k) \hat{\otimes}_k C(G, k) \xrightarrow{\sim} C(G \times G, k),
\]
where \( \hat{\otimes}_k \) is the tensor product of [8 Chap. III, §2, Exer. 28]. Similarly for \( C_{\text{unif}}(G, k) \).

We are especially interested in

**Corollary 4.8.**

1. For any \( r \in \mathbb{Z} \),
\[
(4.8.1) \quad C_{\text{unif}}(\mathbb{Q}_p/p^{r+1}\mathbb{Z}_p, \mathbb{Z}_p) = \lim_{s \rightarrow +\infty} C(\mathbb{Q}_p/p^{r+1}\mathbb{Z}_p, \mathbb{Z}_p/p^s\mathbb{Z}_p)
\]
where \( C(\mathbb{Q}_p/p^{r+1}\mathbb{Z}_p, \mathbb{Z}_p/p^s\mathbb{Z}_p) \) is equipped with the discrete topology. It is the \( \mathbb{Z}_p \)-Hopf algebra of all maps \( \mathbb{Q}_p/p^{r+1}\mathbb{Z}_p \rightarrow \mathbb{Z}_p \) equipped with the \( p \)-adic topology;

2. \[
(4.8.2) \quad C_{\text{unif}}(\mathbb{Q}_p, \mathbb{Z}_p) = \lim_{r \rightarrow +\infty} C_{\text{unif}}(\mathbb{Q}_p/p^{r+1}\mathbb{Z}_p, \mathbb{Z}_p).
\]
Remark 4.9. We point out a tautological, but useful, formula which holds in \( \mathcal{C}(\mathbb{Q}_p/p^{r+1}\mathbb{Z}_p, \mathbb{F}_p) \).

For any \( h \in \mathbb{Q}_p/p^{r+1}\mathbb{Z}_p \), let \( e_h \) denote as before the function \( \mathbb{Q}_p/p^{r+1}\mathbb{Z}_p \rightarrow \mathbb{F}_p \) such that \( e_h(h) = 1 \) while \( e_h(x) = 0 \), if \( x \in \mathbb{Q}_p/p^{r+1}, x \neq h \). For any \( i \leq r \), the function

\[
x_i : \mathbb{Q}_p/p^{r+1}\mathbb{Z}_p \rightarrow \mathbb{F}_p,
\]

was introduced in (1.0.5). We then have

\[
(4.9.1)\]

\[
 x_i = \sum_{h \in \mathbb{Q}_p/p^{r+1}\mathbb{Z}_p} h_i e_h,
\]

where \( h_i = x_i(h) \).

Lemma 4.10. Let \( G \) and \( K \) be as above but assume \( G \) is discrete. Then in the category \( \mathcal{LC}_K \)

1. \[
\mathcal{C}(G, K) = \prod_{g \in G} Ke_g
\]

is a Fréchet \( K \)-algebra.

2. \[
\mathcal{C}^{\text{bd unif}}(G, K) = \ell_\infty(G, K)
\]

is the Banach \( K \)-algebra of bounded sequences \( (a_g)_{g \in G} \) of elements of \( K \), equipped with the componentwise sum and product and with the supnorm.

Proof. Obvious from the definitions. \( \square \)

Lemma 4.11. Let \( G \) and \( K \) be as above, but assume \( G \) is compact. Then in the category \( \mathcal{LC}_K \)

\[
\mathcal{C}(G, K) = \mathcal{C}^{\text{bd unif}}(G, K) = \ell_\infty(G, K)
\]

is the Banach \( K \)-algebra of sequences \( (a_g)_{g \in G} \), with \( a_g \in K \), such that \( a_g \rightarrow 0 \) along the filter of cofinite subsets of \( G \), equipped with componentwise sum and product and with the supnorm.

Proof. This is the classical wavelet decomposition. See Colmez loc.cit. \( \square \)

Proposition 4.12. Let \( G \) and \( K \) be as above. Then in the category \( \mathcal{LC}_K \) we have :

1. \[
\mathcal{C}(G, K) = \lim_{r \rightarrow -\infty} \mathcal{C}(G_r, K) \text{ for the restrictions } \mathcal{C}(G_r, K) \rightarrow \mathcal{C}(G_{r+1}, K).
\]

In particular, \( \mathcal{C}(G, K) \) is a Fréchet \( K \)-algebra.

2. \[
\mathcal{C}^{\text{bd unif}}(G, K) = \lim_{r \rightarrow +\infty} \mathcal{C}^{\text{bd unif}}(G/G_r, K)
\]

for the embeddings

\[
\mathcal{C}^{\text{bd unif}}(G/G_r, K) \hookrightarrow \mathcal{C}^{\text{bd unif}}(G/G_{r+1}, K),
\]

where the inductive limit of Banach \( K \)-algebras is strict. In particular, \( \mathcal{C}^{\text{bd unif}}(G, K) \) is a complete bornological \( K \)-algebra.

3. The natural morphism

\[
\mathcal{C}^{\text{bd unif}}(G, K) \rightarrow \mathcal{C}(G, K)
\]

is injective and has dense image.

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**Remark 4.13.** In case \( A = K \), the explicit description of Proposition \([4.12]\) and of its lemmas shows that we have

\[
(4.13.1) \quad \mathcal{C}(G, K) \hat{\otimes}_{\pi, K} \mathcal{C}(G, K) \rightarrow \mathcal{C}(G \times G, K),
\]

for the completed projective tensor product of locally convex topological \( K \)-vector spaces \([14, \S 17]\). In the case of \( G \) compact this is detailed in the Example after Prop. 17.10 of loc. cit. Similarly for \( \mathcal{C}_{\text{uni}}(G, K) \).

We point out that \((\mathcal{L}C_K, \hat{\otimes}_{\pi, K})\) is a \( K \)-linear symmetric monoidal category.

From Remarks \([4.7, 4.13]\) we conclude

**Proposition 4.14.** Let \( A \) be either \( k \) or \( K \) as before, and regard \((\mathcal{LM}_K^\infty, \hat{\otimes}_k)\) and \((\mathcal{LC}_K, \hat{\otimes}_{\pi, K})\) as symmetric monoidal categories. The coproduct, counit, and inversion

\[
\mathbb{P}(f)(x, y) = f(x + y), \quad \varepsilon(f) = f(0_G), \quad \rho(f)(x) = f(-x),
\]

for any \( f \) in one of the rings of Definition \([4.1]\) and any \( x, y \in G \), define a structure of topological \( A \)-Hopf algebra in the sense of the previous monoidal categories, in any such ring.

The following result describes the structure of the Hopf algebras of functions

\[
\mathbb{Q}_p/p^{r+1}\mathbb{Z}_p \rightarrow \mathbb{Z}_p/p^{a+1}\mathbb{Z}_p,
\]

for any \( r, a \in \mathbb{Z} \) and \( a \geq 0 \) in terms of the functions \( x_i \)

\[
x_i : \mathbb{Q}_p/p^{i+1}\mathbb{Z}_p \rightarrow \mathbb{F}_p
\]

introduced in \([1.0.5]\). See also Remark \([4.9]\)

**Proposition 4.15.** For any \( i \in \mathbb{Z} \), let \( x_i \) be as in \([1.0.5]\) and let \( X_i \) be indeterminates. For \( r \in \mathbb{Z} \) and \( i \in \mathbb{Z}_{\geq 0} \), let \( \mathbb{F}_p(r, i) \) denote the \( \mathbb{F}_p \)-algebra

\[
\mathbb{F}_p[X_r, X_{r-1}, X_{r-2}, \ldots, X_{r-i}]/(1 - X_r^{p-1}, 1 - X_{r-1}^{p-1}, \ldots, 1 - X_{r-i}^{p-1}).
\]

The dimension of \( \mathbb{F}_p(r, i) \) as a \( \mathbb{F}_p \)-vector space is \((p-1)^{i+1}\). Let \( X_{r,i} := (X_{r-i}, X_{r-i+1}, \ldots, X_{r-1}, X_r) \) be viewed as a Witt vector of length \( i + 1 \) with coefficients in \( \mathbb{F}_p(r, i) \). We make \( \mathbb{F}_p(r, i) \) into an \( \mathbb{F}_p \)-Hopf algebra by setting

\[
\mathbb{P}X_{r,i} = X_{r,i} \otimes_{\mathbb{F}_p} 1 + 1 \otimes_{\mathbb{F}_p} X_{r,i}.
\]

For any \( i = 0, 1, \ldots \), the map \( \mathbb{F}_p \)-algebra map \( \mathbb{F}_p(r, i + 1) \rightarrow \mathbb{F}_p(r, i) \) sending \( X_{r-j} \) to \( X_{r-j} \) if \( 0 \leq j \leq i \) and \( X_{r-i-1} \) to 0 is a homomorphism of \( \mathbb{F}_p \)-Hopf algebras. Then, in the category \( \mathcal{LM}_K^\infty \)

1. The map

\[
(4.15.1) \quad \mathbb{F}_p(r, i) \rightarrow \mathcal{C}(p^{r-i}\mathbb{Z}_p/p^{r+1}\mathbb{Z}_p, \mathbb{F}_p)
\]

\[
X_j \mapsto x_j, \quad \text{for } r - i \leq j \leq r,
\]

is an isomorphism of \( \mathbb{F}_p \)-Hopf algebras.
2. the $\mathbb{F}_p$-Hopf algebra $\mathcal{C}(\mathbb{Q}_p/p^r\mathbb{Z}_p, \mathbb{F}_p)$ equals

$$\mathbb{F}_p(r, \infty) := \lim_{i \to +\infty} \mathbb{F}_p(r, i)$$

with the prodiscrete topology;

3. the $\mathbb{F}_p$-Hopf algebra $\mathcal{C}_{\text{unif}}(\mathbb{Q}_p/p^r\mathbb{Z}_p, \mathbb{F}_p)$ equals $\mathbb{F}_p(r, \infty)$ equipped with the discrete topology.

**Proof.** Parts 1 and 2 are [13, Teorema 3.31]. Part 3 follows by forgetting the topology. □

**Remark 4.16.** Notice that the $\mathbb{F}_p$-algebras $\mathbb{F}_p(r, i)$ are perfect.

**Corollary 4.17.** For $r \in \mathbb{Z}$ and $i, a \in \mathbb{Z}_{\geq 0}$

1. the $\mathbb{Z}_p/p^{a+1}\mathbb{Z}_p$-Hopf algebra $\mathcal{C}_{\text{unif}}(\mathbb{Q}_p/p^r\mathbb{Z}_p, \mathbb{Z}_p)$ equals

$$W_a(\mathcal{C}_{\text{unif}}(\mathbb{Q}_p/p^r\mathbb{Z}_p, \mathbb{F}_p)) = W_a(\mathbb{F}_p(r, \infty))$$

equipped with the discrete topology. Therefore,

$$\mathcal{C}_{\text{unif}}(\mathbb{Q}_p/p^r\mathbb{Z}_p, \mathbb{Z}_p) = W(\mathcal{C}_{\text{unif}}(\mathbb{Q}_p/p^r\mathbb{Z}_p, \mathbb{F}_p)) = W(\mathbb{F}_p(r, \infty))$$

equipped with the $p$-adic topology.

2. the $\mathbb{Z}_p/p^{a+1}\mathbb{Z}_p$-Hopf algebra $\mathcal{C}(\mathbb{Q}_p/p^r\mathbb{Z}_p, \mathbb{Z}_p)$ equals

$$W_a(\mathcal{C}(\mathbb{Q}_p/p^r\mathbb{Z}_p, \mathbb{F}_p)) = W_a(\mathbb{F}_p(r, \infty))$$

with the prodiscrete topology. Therefore,

$$\mathcal{C}(\mathbb{Q}_p/p^r\mathbb{Z}_p, \mathbb{Z}_p) = W(\mathcal{C}(\mathbb{Q}_p/p^r\mathbb{Z}_p, \mathbb{F}_p)) = W(\mathbb{F}_p(r, \infty))$$

equipped with the product topology of the prodiscrete on $\mathbb{F}_p(r, \infty)$.

**Definition 4.18.** For any $r \in \mathbb{Z}$ and $a \in \mathbb{Z}_{\geq 0}$, we set

$$\mathcal{C} = \mathcal{C}_{\text{unif}}(\mathbb{Q}_p, \mathbb{Z}_p),$$

$$\mathcal{C}_{r,a} := \{ f \in \mathcal{C} \mid f(x + p^r\mathbb{Z}_p) \subset f(x) + p^a\mathbb{Z}_p \ , \forall x \in \mathbb{Q}_p \} .$$

In particular, for any $r$ and $a$, $\mathcal{C}_{r,a}$ is a $\mathbb{Z}_p$-Hopf Banach subalgebra of $\mathcal{C}$, and

$$p^{a+1}\mathcal{C} \subset \mathcal{C}_{r,a}$$

is an open ideal of $\mathcal{C}_{r,a}$. Let $F$ be the set-theoretic map

$$F : \mathcal{C} \longrightarrow \mathcal{C}$$

$$f \longmapsto f^p .$$

The map $F$ induces maps

$$F : \mathcal{C}_{r,a} \longrightarrow \mathcal{C}_{r,a+1}$$

for any $r, a$ as before. In particular, for $a \geq 1$

$$F(\mathcal{C}_{r,a-1}) + p\mathcal{C}_{r,a-1} \subset \mathcal{C}_{r,a} .$$

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Remark 4.19. Let $r, a$ be as before. There exists a canonical map

\[ R_{r,a} : \mathcal{C}_{r,a} \rightarrow \mathcal{C}_{\text{unif}}(Q_p/p^{r+1}Z_p, Z_p/p^{a+1}Z_p) \]

such that

\[ \pi_{a+1} \circ f = R_{r,a}(f) \]

which sits in the exact sequence

\[ 0 \rightarrow p^{a+1}\mathcal{E} \rightarrow \mathcal{C}_{r,a} \xrightarrow{R_{r,a}} \mathcal{C}_{\text{unif}}(Q_p/p^{r+1}Z_p, Z_p/p^{a+1}Z_p) = W_a(F_p(r, \infty)) \rightarrow 0 \]

We conclude

Proposition 4.20. For any $r \in \mathbb{Z}$ and any $a \in \mathbb{Z}_{\geq 1}$, the map $f \mapsto \pi_1 \circ f$ induces an isomorphism

\[ \mathcal{C}_{r,a}/p\mathcal{C}_{r,a-1} \xrightarrow{\sim} \mathcal{C}_{\text{unif}}(Q_p/p^{r+1}Z_p, F_p) . \]

For $a = 0$ we similarly have

\[ \mathcal{C}_{r,0}/p\mathcal{C}_{r,0} \xrightarrow{\sim} \mathcal{C}_{\text{unif}}(Q_p/p^{r+1}Z_p, F_p) . \]

The inverse of the isomorphism of discrete $F_p$-Hopf algebras

\[ \mathcal{C}_{r,a}/p\mathcal{C}_{r,a-1} \xrightarrow{\sim} \mathcal{C}_{r,0}/p\mathcal{E} \]

is provided by the map

\[ F^a : \mathcal{C}_{r,0}/p\mathcal{E} \xrightarrow{\sim} \mathcal{C}_{r,a}/p\mathcal{C}_{r,a-1} \]

\[ f \mapsto f^a . \]

Proof. The first formula follows from (4.18.1) and (4.19.2). In fact,

\[ \mathcal{C}_{r,a}/p\mathcal{C}_{r,a-1} = \mathcal{C}_{r,a}/p^a\mathcal{C}_{r,0} \]

\[ = W_a(F_p(r, \infty))/pW_a-1(F_p(r, \infty)) = F_p(r, \infty) . \]

Similarly for the other formulas. \qed

By iteration, we get

Corollary 4.21.

\[ \mathcal{C}_{r,a}/p^{a+1}\mathcal{E} \xrightarrow{\sim} \mathcal{C}_{\text{unif}}(Q_p/p^{r+1}Z_p, Z_p/p^{a+1}Z_p) = W_a(F_p(r, \infty)) . \]

For any $f \in \mathcal{C}_{r,a}$ there exist $f_0, f_1, \ldots, f_a \in \mathcal{C}_{r,0}$, well determined modulo $p\mathcal{C}_{r,0}$, such that

\[ f \equiv f_0^a + pf_1^{a-1} + p^2f_2^{a-2} + \cdots + p^af_a \mod p^{a+1}\mathcal{E} . \]

5 \ $p$-adically entire functions bounded on $Q_p$

We prove here the statements we made in subsection 1.4 and in particular Theorem 1.5. We assume $q = p$ from now on, so in particular $\Psi$ stands for $\Psi_p$.

Notice that any function $f \in Q_p\{x\}$ induces a continuous function $f_{|Q_p} : Q_p \rightarrow Q_p$. We get a morphism

\[ \text{Res} : Q_p\{x\} \rightarrow \mathcal{C}(Q_p, Q_p) , \ f \mapsto f_{|Q_p} , \]

where $Q_p\{x\}$ is equipped with the standard Fréchet topology. We recall that $\mathcal{C} \subset Q_p\{x\}$ is the $Q_p$-Hopf subalgebra of the $f \in Q_p\{x\}$ such that

\[ \text{Res}(f) \in \mathcal{C}_{\text{unif}}(Q_p, Q_p) . \]
Definition 5.1. We regard \( \mathcal{E} \) as a Fréchet \( \mathbb{Q}_p \)-algebra with the Fréchet structure induced by the family of semivaluations \( \{ w_r \}_{r \in \mathbb{Z} \cup \{ \infty \}} \), which we call the uniform Fréchet structure.

Remark 5.2. We recall that the standard Fréchet structure of \( \mathbb{Q}_p \{ x \} \) is induced by the family of semivaluations \( \{ w_r \}_{r \in \mathbb{Z} \cup \{ \infty \}} \). So, \( (\mathcal{E}, \{ w_r \}_{r \in \mathbb{Z} \cup \{ \infty \}}) \) is a closed Fréchet subalgebra of \( (\mathbb{Q}_p \{ x \}, \{ w_r \}_{r \in \mathbb{Z}}) \), but the closed embedding \( \mathcal{E} \subset \mathbb{Q}_p \{ x \} \) is not strict.

Then, \( \text{Res} \) induces a morphism of topological \( \mathbb{Q}_p \)-algebras
\[
\text{Res} : \mathcal{E} \longrightarrow \mathcal{E}_{\text{bd}}^\text{unif}(\mathbb{Q}_p, \mathbb{Q}_p) \ , \ f \longmapsto f|_{\mathbb{Q}_p}
\]
and a morphism of topological \( \mathbb{Z}_p \)-algebras
\[
\text{Res}^\circ : \mathcal{E}^\circ \longrightarrow \mathcal{E}^\text{unif}(\mathbb{Q}_p, \mathbb{Z}_p) \ , \ f \longmapsto f|_{\mathbb{Z}_p}
\]

Remark 5.3. The morphisms \( \text{Res} \) and \( \text{Res}^\circ \) are injective, since two entire functions which coincide on \( \mathbb{Q}_p \) necessarily coincide, but not surjective. For example, any function \( f : \mathbb{Q}_p \rightarrow \mathbb{Z}_p \) which is constant on balls of radius \( p^r \) is uniformly continuous. But, unless it is globally constant, \( f \) cannot possibly extend to an entire function. We are going to show that the set-theoretic image of \( \text{Res} \) (resp. of \( \text{Res}^\circ \)) is dense in \( \mathcal{E}_{\text{bd}}^\text{unif}(\mathbb{Q}_p, \mathbb{Q}_p) \) (resp. in \( \mathcal{E}^\text{unif}(\mathbb{Q}_p, \mathbb{Z}_p) \)). This should be remindful of the Theorem of Féjer in classical Fourier analysis. Our story will actually become even closer to the classical one in [3], when we will shift our considerations from \( \mathbb{G}_a \) to \( \mathbb{G}_m \).

Definition 5.4. For any \( r \in \mathbb{Z} \) and any \( a \in \mathbb{Z}_{\geq 0} \), we consider the closed \( \mathbb{Z}_p \)-Hopf Fréchet subalgebra of \( \mathcal{E}^\circ \)
\[
\mathcal{E}^\circ_{r,a} = (\text{Res}^\circ)^{-1}(\mathcal{E}_{r,a}) .
\]

Remark 5.5. Notice that \( \mathcal{E}^\circ_{r,0} = \mathcal{E}^\circ_r \) cf. Definition 1.4

Remark 5.6. Let \( r \in \mathbb{Z} \).

1. The filtration by strictly closed \( \mathbb{Z}_p \)-Hopf Banach subalgebras
\[
\cdots \subset \mathcal{E}_{r,a+1} \subset \mathcal{E}_{r,a} \subset \cdots \subset \mathcal{E}_{r,1} \subset \mathcal{E}_{r,0} = \mathcal{E}_r .
\]
induces a filtration by strictly closed \( \mathbb{Z}_p \)-Hopf Fréchet subalgebras
\[
\cdots \subset \mathcal{E}^\circ_{r,a+1} \subset \mathcal{E}^\circ_{r,a} \subset \cdots \subset \mathcal{E}^\circ_{r,1} \subset \mathcal{E}^\circ_{r,0} = \mathcal{E}^\circ_r .
\]

2. For \( F \) defined as in (4.18.2), we have the analog of (4.18.3). The exact sequence of (4.19.2) becomes
\[
0 \longrightarrow p^{a+1} \mathcal{E}^\circ \longrightarrow \mathcal{E}^\circ_{r,a} \longrightarrow \mathcal{E}^\text{unif}(\mathbb{Q}_p/p^{a+1}\mathbb{Z}_p, \mathbb{Z}_p/p^{a+1}\mathbb{Z}_p) \equiv \mathbb{W}_a(\mathbb{F}_p(r, \infty)) \longrightarrow 0
\]
Corollary 4.21 now becomes

Corollary 5.7.
\[
\mathcal{E}^\circ_{r,a}/p^{a+1} \mathcal{E}^\circ \sim \mathcal{E}^\text{unif}(\mathbb{Q}_p/p^{a+1}\mathbb{Z}_p, \mathbb{Z}_p/p^{a+1}\mathbb{Z}_p) = \mathbb{W}_a(\mathbb{F}_p(r, \infty)) .
\]
For any \( f \in \mathcal{E}^\circ_{r,a} \) there exist \( f_0, f_1, \ldots, f_a \in \mathcal{E}^\circ_{r,0} \), well determined modulo \( p\mathcal{E}^\circ \), such that
\[
f \equiv f_0^{p^a} + p^{a+1} f_1^{p^{a-1}} + p^2 f_2^{p^{a-2}} + \cdots + p^a f_a \mod p^{a+1} \mathcal{E}^\circ .
\]

The main result of this section is Theorem 1.5 of the Introduction. We first summarize what we have proven in the following

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Lemma 5.8. Let \( r,a \) be as before.

1. For any \( i \in \mathbb{Z}_{\geq 0} \) and \( \ell = 0,1,\ldots,p-1 \), the function \( \Psi(p^{\ell-r}x)^{p^n} \) belongs to \( \mathcal{E}_{r,a}^a \).

2. Any series of functions of the form

\[
\sum_{\ell=0}^{p-1} \sum_{i=0}^\infty c_{\ell,a,i} (p^{\ell-r}x)^{p^n}, \quad c_{\ell,a,i} \in \mathbb{Z}_p,
\]

converges in the standard Fréchet topology of \( \mathbb{Q}_p\{x\} \) to an element of \( (\mathcal{E}_{r,a}^a)^p \subset \mathcal{E}_{r,a}^a \) along the filter of cofinite subsets of \( \{0,1,\ldots,p-1\} \times \mathbb{Z}_{\geq 0} \).

3. Let \( r \in \mathbb{Z} \) and \( a \in \mathbb{Z}_{\geq 0} \) be fixed. For any element \( f \in \mathcal{E}_{r,a}^a \) there exist uniquely determined elements \( c_{\ell,b,i} = c_{\ell,b,i} \in \mathbb{Z}_p \), such that

\[
f(x) \equiv \sum_{b=0}^a \sum_{\ell=0}^{p-1} \sum_{i=0}^\infty c_{\ell,b,i} p^a b^\ell (p^{\ell-r}x)^{y^b} \mod p^{a+1} \mathcal{E}_{r,0}^a.
\]

Proof. The first two parts are clear. As for the last part, we observe that the isomorphism \((\ref{5.7.1})\) transforms the function \( p^a b^\ell (p^{\ell-r}x)^{y^b} \) into the Witt vector

\[
(0,\ldots,0,w_{a-b},x^\ell_{r-1},0,\ldots,0) \in \mathcal{W}_a(\mathbb{F}_p(r,\infty)),
\]

where \( x^\ell_{r-1} \) is placed at the \((a-b)\)-th level. Since any \( y \in \mathbb{F}_p(r,\infty) \) admits a unique expression as a sum

\[
y = \sum_{\ell=0}^{p-1} \sum_{i=0}^\infty \gamma_{\ell,i} x^\ell_{r-1}, \quad \gamma_{\ell,i} \in \mathbb{F}_p,
\]

is clear that any \( w = (w_0,w_1,\ldots,w_a) \in \mathcal{W}_a(\mathbb{F}_p(r,\infty)) \) admits a unique expression as a sum

\[
\sum_{b=0}^a \sum_{\ell=0}^{p-1} \sum_{i=0}^\infty [\gamma_{\ell,b,i}] (0,\ldots,0,w_{a-b},x^\ell_{r-1},0,\ldots,0)
\]

The next corollary emphasizes how the main point (part 3) of Theorem \ref{1.5} follows from the previous discussion.

Corollary 5.9. For any \( f \in \mathcal{E}_{\text{unif}}(\mathbb{Q}_p,\mathbb{Z}_p) \) and any \( a \in \mathbb{Z}_{\geq 0} \), there exists \( r = r(a) \in \mathbb{Z} \) such that \( f \in \mathcal{E}_{r,a} \). Then, \( f \) can be approximated as in \((\ref{5.8.2})\). In particular, the completion of the valued ring \( (\mathcal{E}^a,w^\infty) \) coincides with \( \mathcal{E}_{\text{unif}}(\mathbb{Q}_p,\mathbb{Z}_p) \).

6 Appendix: Numerical Calculations by M. Candilera

The following calculations were performed with Mathematica. We computed the first coefficients of the series \( \Psi_p(T) = \sum_{n=0}^{\infty} b_n T^n \), for \( p = 2 \), up to the term of degree \( 2^5 \), and for \( p = 3 \), up to degree \( 3^4 \). We also evaluated the a few coefficients of \( \Psi_5(T) \) and \( \Psi_7(T) \). We give here tables of the \( p \)-adic orders of the coefficients \( b_n \) for \( p = 2,3 \). For those values of \( p \), we also draw the graph of the function \( n \mapsto v_p(b_n) \) and compare it with the Newton polygon of \( \Psi_p \) (flipped around the \( y \)-axis). We also confirmed experimentally the calculation of the corresponding valuation polygons.
6.1 Very first coefficients

1. \( p = 2 \)

\[
\Psi_2(T) = T - 2 \cdot T^2 + 2^4 \cdot T^3 - 11 \cdot 2^5 \cdot T^4 + 7 \cdot 2^{11} \cdot T^5 - 7 \cdot 37 \cdot 2^{12} \cdot T^6 + 3 \cdot 751 \cdot 2^{16} \cdot T^7 - 301627 \cdot 2^{17} \cdot T^8 + 308621 \cdot 2^{26} \cdot T^9 + 2^{27} \cdot T^{10} \cdot u(T),
\]

for a unit \( u(T) \in \mathbb{Z}_2[[T]]^\times \).

2. \( p = 3 \)

\[
\Psi_3(T) = T - 3^2 \cdot T^3 + 3^7 \cdot T^5 - 2^2 \cdot 7 \cdot 3^{11} \cdot T^7 + 2 \cdot 7 \cdot 13 \cdot 3^{14} \cdot T^9 - 5 \cdot 89 \cdot 1249 \cdot 3^{22} \cdot T^{11} + 5 \cdot 117 \cdot 217667 \cdot 3^{28} \cdot T^{13} + \ldots.
\]

3. \( p = 5 \)

\[
\Psi_5(T) = T - 5^4 \cdot T^5 + 5^{13} \cdot T^9 - 53 \cdot 5^{21} \cdot T^{11} + 3 \cdot 11 \cdot 97 \cdot 1123 \cdot 1699 \cdot 5^{29} \cdot T^{17} + 5^{37} \cdot T^{21} \cdot u(T),
\]

for a unit \( u(T) \in \mathbb{Z}_5[[T]]^\times \).

4. \( p = 7 \)

\[
\Psi_7(T) = T - 7^6 \cdot T^7 + 7^{19} \cdot T^{13} - 2 \cdot 31 \cdot 37 \cdot 359 \cdot 7^{31} \cdot T^{19} + 7^{43} \cdot T^{25} \cdot u(T),
\]

for a unit \( u(T) \in \mathbb{Z}_7[[T]]^\times \).
6.2  First 24 coefficients of $\Psi_2(t)$ and 2-adic order of the 32 first

$$\Psi_2(t) = \sum_{n \geq 1} b_n t^n$$

| $b_1$ | $b_2$ | $b_3$ | $b_4$ | $b_5$ | $b_6$ | $b_7$ | $b_8$ |
|------|------|------|------|------|------|------|------|
| 0    | 0    | 26   | 61   | 101  | 102  | 110  | 111  |
| 1    | 4    | 5    | 11   | 12   | 11   | 16   | 17   |
| 10   | 11   | 12   | 13   | 14   | 15   | 16   | 17   |

2-adic valuation of the coefficients of $\Psi_2$

- $b_1 = 1$,  $b_2 = -2$,  $b_3 = 16 = 2^4$,  $b_4 = -352 = -2^7 \cdot 11$
- $b_5 = 14336 = 2^{11} \cdot 7$,  $b_6 = -1060864 = -2^{12} \cdot 7 \cdot 37$
- $b_7 = 147652608 = 2^{16} \cdot 3 \cdot 751$
- $b_8 = -395448544144 = -2^{27} \cdot 3^2 \cdot 13 \cdot 701 \cdot 1949$
- $b_9 = 20711204716544 = 2^{26} \cdot 308621$
- $b_{10} = -2145485589845824 = -2^{27} \cdot 3^2 \cdot 13 \cdot 701 \cdot 1949$
- $b_{11} = 4419570051654413808 = 2^{23} \cdot 5145056699$
- $b_{12} = -18155440787932198423040 = -2^{31} \cdot 5 \cdot 41 \cdot 2273 \cdot 22679509$
- $b_{13} = 14984690158521411090694498960 = 2^{32} \cdot 5 \cdot 6773209816823$
- $b_{14} = -244213198442213051286386644146944 = -2^{43} \cdot 3^2 \cdot 8179 \cdot 37716952983613$
- $b_{15} = 8005307469807464997623203521363968 = 2^{48} \cdot 31 \cdot 71 \cdot 1619 \cdot 826201 \cdot 966018887$
- $b_{16} = -52473187145753996332767403649796036743168 = -2^{48} \cdot 31 \cdot 397 \cdot 13687 \cdot 2882489 \cdot 191972726039$
- $b_{17} = 68789352048480570511228427181753513990427152184 = 2^{61} \cdot 3 \cdot 47 \cdot 59 \cdot 919 \cdot 24709 \cdot 1579121645921333$
- $b_{18} = -180321257856882570455986338710346864852507172012032$
- $= 2^{62} \cdot 3^5 \cdot 19 \cdot 97 \cdot 173 \cdot 1605967 \cdot 581220517 \cdot 140723269997$
- $b_{19} = 95442435439381709201817074474135442710753588534117924864$
- $= 2^{70} \cdot 7^2 \cdot 23 \cdot 15973 \cdot 44485316159805664956515547941$
- $b_{20} = -991360662278906630563330625219579052848307308474498666240$
- $= 2^{71} \cdot 5 \cdot 167 \cdot 14503 \cdot 1545577654440991 \cdot 2244675152281633991$
- $b_{21} = 20709045800971981246184722972326557390898717022368004517660824096997376$
- $= 2^{81} \cdot 109 \cdot 23549 \cdot 16744236921 \cdot 2000645152343730624209879183$
- $b_{22} = -87201751894674086963422057530532712162136715200267189551905031269057044832$
- $= 2^{82} \cdot 47 \cdot 867 \cdot 105323 \cdot 2115951 \cdot 80618233393589 \cdot 1141865160972250490671$
- $b_{23} = 7315023597673084112644956884904164755656347795586370441676376428384144588800$
- $= 2^{89} \cdot 3^3 \cdot 5^2 \cdot 17508240991713184384874362658178089332605816887352831773$
- $b_{24} = -1227258187586066935505303554739880208883482157583428276444414675621211077011846045664083968$
- $= 2^{90} \cdot 54617 \cdot 76121647308197 \cdot 238451637287968840726339672350427699951944293$
Figure 6: The valuation and valuation polygons of $\Psi_3$.

6.3 3-adic values of the first 81 coefficients of $\Psi_3(T)$

| $b_1$ | 2 | $b_{21}$ | 58 | $b_{43}$ | 135 | $b_{63}$ | 213 |
|-------|---|----------|----|----------|----|----------|----|
| $b_5$ | 7 | $b_{25}$ | 64 | $b_{45}$ | 141 | $b_{65}$ | 223 |
| $b_7$ | 11 | $b_{27}$ | 68 | $b_{47}$ | 151 | $b_{67}$ | 231 |
| $b_9$ | 14 | $b_{29}$ | 79 | $b_{49}$ | 159 | $b_{69}$ | 238 |
| $b_{11}$ | 22 | $b_{31}$ | 87 | $b_{51}$ | 166 | $b_{71}$ | 247 |
| $b_{13}$ | 28 | $b_{33}$ | 94 | $b_{53}$ | 175 | $b_{73}$ | 255 |
| $b_{15}$ | 33 | $b_{35}$ | 103 | $b_{55}$ | 183 | $b_{75}$ | 262 |
| $b_{17}$ | 40 | $b_{37}$ | 111 | $b_{57}$ | 190 | $b_{77}$ | 271 |
| $b_{19}$ | 46 | $b_{39}$ | 118 | $b_{59}$ | 199 | $b_{79}$ | 279 |
| $b_{21}$ | 51 | $b_{41}$ | 127 | $b_{61}$ | 207 | $b_{81}$ | 284 |

3-adic valuation of the coefficients of $\Psi_3$

The actual coefficients grow fast in real absolute value and it does not seem useful to write them explicitly here; e.g. the coefficient of $T^{83}$, of 3-adic value 298, is
References

[1] Francesco Baldassarri. Interpretazione funzionale di certe iperalgebre e dei loro anelli di bivettori di Witt. Tesi di laurea, Padova, 1974.

[2] Francesco Baldassarri. Una funzione intera $p$-adica a valori interi. Ann. Scuola Norm. Sup. Pisa, Ser. IV Vol. II (2):321–331, 1975.

[3] Francesco Baldassarri. The Artin-Hasse isomorphism of perfectoid open unit disks and a Fourier-type theory for continuous functions on $\mathbb{Q}_p$ preprint

[4] Francesco Baldassarri and Maurizio Cailotto Functional analysis over generalized adic rings preprint

[5] Francesco Baldassarri and Maurizio Cailotto Duality for functions and measures on locally compact 0-dimensional spaces with values in generalized adic rings preprint

[6] Vladimir G. Berkovich. Spectral theory and analytic geometry over non-Archimedean fields, volume 33 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1990.

[7] Siegfried Bosch, Ulrich Güntzer, and Reinhold Remmert, Non-Archimedean analysis, Grundlehren der Mathematischen Wissenschaften 261, Springer-Verlag, Berlin 1984, pp. xii+436

[8] Nicolas Bourbaki. Éléments de Mathématiques. Algèbre Commutative, Chapitres 1 à 4. Springer, 2006 (Masson, Paris 1985).

[9] Pierre Colmez Fonctions d’une variable $p$-adique Astérisque 330, 13–59, 2010.

[10] Jean-Marc Fontaine. Groupes $p$-divisibles sur les corps locaux, Astérisque, No. 47-48, Société Mathématique de France, Paris, 1977.

[11] Michel Lazard. Les zéros d’une fonction analytique d’une variable sur un corps valué complet Publ. Math. Inst. Hautes Études Sci. 14: 47–75, 1962.

[12] Iacopo Barsotti. Metodi analitici per varietà abeliane in caratteristica positiva. Capitoli 1,2. Ann. Scuola Norm. Sup. Pisa, Ser. III Vol. XVIII (1):1–25, 1964.

[13] Iacopo Barsotti. Metodi analitici per varietà abeliane in caratteristica positiva. Capitoli 3,4. Ann. Scuola Norm. Sup. Pisa, Ser. III Vol. XIX (3): 277–330, 1965.
[14] Peter Schneider. Nonarchimedean functional analysis *Springer Monographs in Mathematics*, Springer-Verlag, Berlin vi+156 pp., 2002.

[15] Peter Schneider. Galois representations and \((\varphi, \Gamma)\)-modules *Cambridge Studies in advanced mathematics*, 164 147 pp., 2017.