APPROXIMATION OF CLASSES OF FUNCTIONS DEFINED BY A GENERALIZED $r$-TH MODULUS OF SMOOTHNESS

M. K. POTAPOV AND F. M. BERISHA

Dedicated to Professor P. L. Ul’yanov on the occasion of his 70-th birthday

Abstract. In this paper, a $k$-th generalized modulus of smoothness is defined based on an asymmetric operator of generalized translation and a theorem is proved about the coincidence of class of functions defined by this modulus and a class of functions having given order of best approximation by algebraic polynomials.

Introduction. In paper [4], an asymmetric operator of generalized translation was introduced and by means of it the corresponding generalized modulus of smoothness of first order was defined. Then a theorem was proved about coincidence of the class of functions defined by this modulus with the class of functions having a given order of best approximation by algebraic polynomials.

In the present paper, analogous results are obtained for the generalized modulus of smoothness of order $r$. In addition, the space in which the theorem of coincidence of the corresponding classes of functions holds true is widened.

1. For $1 \leq p < \infty$, as usual, $L_p$ denotes the set all measurable functions $f$ on $[-1, 1]$ for which

$$\|f\|_p = \left( \int_{-1}^{1} |f(x)|^p \, dx \right)^{1/p} < \infty.$$  

For $p = \infty$, $L_\infty$ is the space of all continuous functions $f$ on $[-1, 1]$ with a norm

$$\|f\|_\infty = \max_{-1 \leq x \leq 1} |f(x)|.$$  

Denote by $L_{p, \alpha, \beta}$ the set of functions $f$ such that $f(x)(1-x)^\alpha(1+x)^\beta \in L_p$, and set

$$\|f\|_{p, \alpha, \beta} = \|f(x)(1-x)^\alpha(1+x)^\beta\|_p.$$  

By $E_n(f)_{p, \alpha, \beta}$ we denote the best approximation of $f \in L_{p, \alpha, \beta}$ by algebraic polynomials of degree not greater than $n - 1$ in $L_{p, \alpha, \beta}$ metrics, that is,

$$E_n(f)_{p, \alpha, \beta} = \inf_{P_n \in P_n} \|f - P_n\|_{p, \alpha, \beta},$$

where $P_n$ is the set of algebraic polynomials of degree not greater than $n - 1$.

By $E(p, \alpha, \beta, \lambda)$ we denote the class of functions $f \in L_{p, \alpha, \beta}$ satisfying the condition

$$E_n(f)_{p, \alpha, \beta} \leq Cn^{-\lambda},$$

where $\lambda > 0$ and $C$ is a constant not depending on $n$ ($n \in \mathbb{N}$).

1991 Mathematics Subject Classification. Primary 41A35, Secondary 41A50, 42A16.

Key words and phrases. Generalized modulus of smoothness, asymmetric operator of generalized translation, coincidence of classes, best approximations by algebraic polynomials.

This work was done under the support of the Russian Foundation for Fundamental Scientific Research, Grant #97-01-00010 and Grant #96/97-15-96073.
For functions $f$ we define the operator of generalized translation $\hat{T}_t(f, x)$ by

$$\hat{T}_t(f, x) = \frac{1}{\pi(1-x^2)} \int_0^\pi \left(1 - \left(x \cos t - \sqrt{1-x^2}\sin t\cos \varphi\right)^2 - 2 \sin^2 t \sin^2 \varphi + 4 \left(1-x^2\right) \sin^4 \varphi \right) f(x \cos t - \sqrt{1-x^2}\sin t \cos \varphi) \, d\varphi.$$  

By means of this operator of generalized translation we define the generalized difference of order $r$ by

$$\Delta_1^r(f, x) = \Delta_1(f, x) = \hat{T}_t(f, x) - f(x),$$  

$$\Delta_{t_1, \ldots, t_r}^r(f, x) = \Delta_{t_r} \left(\Delta_{t_1, \ldots, t_{r-1}}^{r-1}(f, x), x\right) \quad (r = 2, 3, \ldots),$$  

and the generalized modulus of smoothness of order $r$ by

$$\hat{\omega}_r(f, \delta)_{p, \alpha, \beta} = \sup_{|t_1| \leq \delta} \left\|\Delta_{t_1, \ldots, t_r}(f, x)\right\|_{p, \alpha, \beta} \quad (r = 1, 2, \ldots).$$

Consider the class $H(p, \alpha, \beta, r, \lambda)$ of functions $f \in L_{p, \alpha, \beta}$ satisfying the condition

$$\hat{\omega}_r(f, \delta)_{p, \alpha, \beta} \leq C\delta^\lambda,$$

where $\lambda > 0$ and $C$ is a constant not depending on $\delta$.

The aim of the present paper is to prove the following statement

**Theorem 1.1.** Let $p, \alpha, \beta$ and $r$ be given numbers such that $1 \leq p \leq \infty$, $r \in \mathbb{N}$;

- $\frac{1}{2} < \alpha \leq 2$, $\frac{1}{2} < \beta \leq 2$ for $p = 1$,
- $1 - \frac{1}{2p} < \alpha < 3 - \frac{1}{p}$, $1 - \frac{1}{2p} < \beta < 3 - \frac{1}{p}$ for $1 < p < \infty$,
- $1 \leq \alpha < 3$, $1 \leq \beta < 3$ for $p = \infty$.

Then, for any $\lambda$ satisfying the condition

$$\lambda_0 = 2 \max \left(\left|\alpha - \beta\right|, \alpha - \frac{3}{2} + \frac{1}{2p}, \beta - \frac{3}{2} + \frac{1}{2p}\right) < \lambda < 2r$$

the class $H(p, \alpha, \beta, r, \lambda)$ coincides with the class $E(p, \alpha, \beta, \lambda)$.

The validity of Theorem 1.1 will follow from the validity of Theorems 4.3 and 4.4 which we are going to prove below.

2. Put $y = \cos t$, $z = -\cos \varphi$ in the definition of $\hat{T}_t(f, x)$ and denote the resulting operator by $T_y(f, x)$. Let us rewrite it in the form

$$T_y(f, x) = \frac{1}{\pi(1-x^2)} \int_{-1}^{1} \left(1 - R^2 - 2 \left(1-y^2\right) (1-z^2) + 4 \left(1-x^2\right) \left(1-y^2\right) (1-z^2)^2\right) f(R) \frac{dz}{\sqrt{1-z^2}}$$

where $R = xy - z \sqrt{1-x^2}\sqrt{1-y^2}$. We define the operator of generalized translation of order $r$ by

$$T_y^1(f, x) = T_y(f, x),$$  

$$T_{y_1, \ldots, y_r}^r(f, x) = T_{y_r} \left(T_{y_1, \ldots, y_{r-1}}^{r-1}(f, x), x\right) \quad (r = 2, 3, \ldots).$$

By $P_{y_1, \ldots, y_r}^{(\alpha, \beta)}(x)$ ($\nu = 0, 1, \ldots$) we denote the Jacobi polynomials, i.e. algebraic polynomials of degree $\nu$ orthogonal on the segment $[-1, 1]$ with a weight $$(1-x)^\alpha (1+x)^\beta$$ and normalized by the condition $P_{y_1, \ldots, y_r}^{(\alpha, \beta)}(1) = 1$ ($\nu = 0, 1, \ldots$).
For any integrable function $f$ on $[-1, 1]$ with a weight $(1 - x^2)^2$, we denote by $a_n(f)$ the Fourier–Jacobi coefficients of $f$ with respect to the system of Jacobi polynomials $\{P_n^{(2,2)}(x)\}_{n=0}^{\infty}$, i.e.

$$a_n(f) = \int_{-1}^{1} f(x)P_n^{(2,2)}(x) \, (1 - x^2)^2 \, dx \quad (n = 0, 1, \ldots).$$

Introduce certain operators which will play an auxiliary role later on. First we set

$$T_{1,y}(f,x) = \frac{1}{\pi} \frac{1}{(1 - x^2)} \int_{-1}^{1} (1 - R^2 - 2(1 - y^2)(1 - z^2)) f(R) \frac{dz}{\sqrt{1 - z^2}},$$

$$T_{2,y}(f,x) = \frac{8}{3\pi} \int_{-1}^{1} (1 - z^2)^2 f(R) \frac{dz}{\sqrt{1 - z^2}},$$

where $R = xy - z\sqrt{1 - x^2}\sqrt{1 - y^2}$, and then define the corresponding operators of order $r$ by

$$T_{k,y}^{1}(f,x) = T_{k,y}^{r}(f,x),$$

$$T_{k,y}^{r}(f,x) = T_{k,y}^{r-1}(T_{k,y}^{r-1}(f,x),x) \quad (r = 2, 3, \ldots)$$

for $k = 1, 2$.

3. \textbf{Lemma 3.1. Let $f \in L_{p,\alpha,\beta}$ and let the numbers $p$, $\alpha$, $\beta$, $\rho$, $\sigma$ and $\lambda$ be such that $1 \leq p \leq \infty$, $\rho \geq 0$, $\sigma \geq 0$, $\lambda > \lambda_0 = 2 \max\{\rho, \sigma\}$;}

$$\alpha > -\frac{1}{p}, \quad \beta > -\frac{1}{p} \quad \text{for } 1 \leq p < \infty,$$

$$\alpha \geq 0, \quad \beta \geq 0 \quad \text{for } p = \infty.$$

If there exists a sequence of algebraic polynomials $\{P_{2^n}(x)\}_{n=0}^{\infty}$ such that

$$\|f - P_{2^n}\|_{p,\alpha+\rho,\beta+\sigma} \leq \frac{C_1}{2n\lambda},$$

then the following inequalities also hold true

$$\|f - P_{2^n}\|_{p,\alpha,\beta} \leq \frac{C_2}{2^{n(\lambda - \lambda_0)}} \quad (n = 1, 2, \ldots),$$

where the constants $C_1$ and $C_2$ do not depend on $n$.

Lemma 3.1 was proved in [2].

\textbf{Lemma 3.2. Let $P_n(x)$ be an algebraic polynomial of degree not greater than $n-1$, $1 \leq p \leq \infty$, $\rho \geq 0$, $\sigma \geq 0$. Assume that}

$$\alpha > -\frac{1}{p}, \quad \beta > -\frac{1}{p} \quad \text{for } 1 \leq p < \infty,$$

$$\alpha \geq 0, \quad \beta \geq 0 \quad \text{for } p = \infty.$$

Then

$$\|P_n'(x)\|_{p,\alpha+\frac{1}{2},\beta+\frac{1}{2}} \leq C_1 n \|P_n\|_{p,\alpha,\beta},$$

$$\|P_n\|_{p,\alpha,\beta} \leq C_2 n^{2\max(\rho,\sigma)} \|P_n\|_{p,\alpha+\rho,\beta+\sigma},$$

where the constants $C_1$ and $C_2$ do not depend on $n$.

Lemma was proved in [1].
Lemma 3.3. The operators $T_{1,y}$ and $T_{2,y}$ have the following properties

$$
T_{1,y} \left( P_{\nu}^{(2,2), x} \right) = P_{\nu}^{(2,2)}(x)P_{\nu+2}^{(0,0)}(y),
$$

$$
T_{2,y} \left( P_{\nu}^{(2,2), x} \right) = P_{\nu}^{(2,2)}(x)P_{\nu}^{(2,2)}(y)
$$

for $\nu = 0, 1, \ldots$

Lemma 3.3 was proved in [4].

Lemma 3.4. Let $g(x)T_{k,y}(f, x) \in L_{1,2,2}$ for every $y$. Then for $k = 1, 2$ the following equality holds true

$$
\int_{-1}^{1} f(x)T_{k,y}(g, x) \left( 1 - x^2 \right)^2 \, dx = \int_{-1}^{1} g(x)T_{k,y}(f, x) \left( 1 - x^2 \right)^2 \, dx.
$$

Proof. Let $k = 1$ and

$$
I_1 := \int_{-1}^{1} f(x)T_{1,y}(g, x) \left( 1 - x^2 \right)^2 \, dx
$$

$$
= \frac{1}{\pi} \int_{-1}^{1} \int_{-1}^{1} f(x)g(R) \left( 1 - R^2 - 2 \left( 1 - y^2 \right) \left( 1 - z^2 \right) \right) \left( 1 - x^2 \right) \frac{dz \, dx}{\sqrt{1 - z^2}}.
$$

where $R = xy - z\sqrt{1 - x^2} \sqrt{1 - y^2}$. Performing change of variables in the double integral by the formulas

$$
x = Ry + V \sqrt{1 - R^2} \sqrt{1 - y^2},
$$

$$
z = -\frac{R \sqrt{1 - y^2} - V\sqrt{1 - R^2}}{\sqrt{1 - \left( Ry + V\sqrt{1 - R^2} \sqrt{1 - y^2} \right)^2}},
$$

we get

$$
I_1 = \frac{1}{\pi} \int_{-1}^{1} \int_{-1}^{1} (1 - R^2) f \left( Ry + V \sqrt{1 - R^2} \sqrt{1 - y^2} \right) g(R)
$$

$$
\times \left( 1 - \left( Ry + V \sqrt{1 - R^2} \sqrt{1 - y^2} \right)^2 - 2 \left( 1 - y^2 \right) \left( 1 - V^2 \right) \right) \frac{dV \, dR}{\sqrt{1 - V^2}}
$$

$$
= \int_{-1}^{1} g(R)T_{1,y}(f, R) \left( 1 - R^2 \right)^2 \, dR,
$$

which proves the equality of the lemma for $k = 1$.

Let $k = 2$ and

$$
I_2 := \int_{-1}^{1} f(x)T_{2,y}(g, x) \left( 1 - x^2 \right)^2 \, dx
$$

$$
= \frac{8}{3\pi} \int_{-1}^{1} \int_{-1}^{1} f(x)g(R) \left( 1 - x^2 \right)^2 \left( 1 - z^2 \right)^2 \frac{dz \, dx}{\sqrt{1 - z^2}}.
$$

Performing again the change (3.1) in the double integral we get

$$
I_2 = \frac{8}{3\pi} \int_{-1}^{1} \int_{-1}^{1} f \left( Ry + V \sqrt{1 - R^2} \sqrt{1 - y^2} \right) g(R) \left( 1 - R^2 \right)^2
$$

$$
\times \left( 1 - V^2 \right)^2 \frac{dV \, dR}{\sqrt{1 - V^2}} = \int_{-1}^{1} g(R)T_{2,y}(f, R) \left( 1 - R^2 \right)^2 \, dR.
$$

Lemma 3.4 is proved. □
Corollary 3.1. If \( f \in L_{1,2,2} \), then for every \( r \in \mathbb{N} \) we have \( T_{r_1,\ldots,r_n}^r(f,x) \in L_{1,2,2} \) \((k = 1,2)\).

Proof. Put \( g(x) \equiv 1 \) on \([-1,1]\). Taking into account that by Lemma 3.3
\[
T_{1,y} (1, x) = T_{1,y} \left( P_0^{(2,2)}, x \right) = P_0^{(2,2)}(x)P_2^{(0,0)}(y) = \frac{3}{2} b^2 - \frac{1}{2}, \quad T_{2,y} (1, x) = 1,
\]
we clearly have \( f(x)T_{k,y}(1,x) \in L_{1,2,2} \) \((k = 1,2)\). Hence, applying Lemma 3.4 we derive the relation
\[
\int_{-1}^{1} T_{k,y} (f,x) (1-x^2)^2 \, dx = \int_{-1}^{1} f(x)T_{k,y} (1,x) (1-x^2)^2 \, dx \quad (k = 1,2),
\]
which implies that \( T_{k,y}(f,x) \in L_{1,2,2} \). Now the corollary can be proved by induction. \(\square\)

Lemma 3.5. Let \( f \) be an integrable function on \([-1,1]\) with a weight \((1-x^2)^2\). For every natural number \( n \) the following equality holds true
\[
\int_{-1}^{1} T_{1,y} (f,x) P_n^{(1,1)}(y) \, dy = \sum_{m=0}^{n-2} a_m(f)\gamma_m(x),
\]
where \( \gamma_m(x) \) is an algebraic polynomial of degree not greater than \( n-2 \), and \( \gamma_m(x) \equiv 0 \) for \( n = 0 \) or \( n = 1 \).

Lemma 3.5 was proved in [4].

Lemma 3.6. Let \( q \) and \( m \) be given natural numbers. Let \( f \) be an integrable function on \([-1,1]\) with a weight \((1-x^2)^2\). Then for every natural numbers \( l \) and \( r \) \((l \leq r)\) the function
\[
Q_1^{(l)}(x) = \int_{0}^{\pi} \cdots \int_{0}^{\pi} T_{l_1,\ldots,l_r} (f,x) \prod_{s=1}^{r} \left( \frac{\sin \frac{\pi t_s}{2}}{\sin \frac{\pi}{2}} \right)^{2q+4} \sin^3 t_s \, dt_1 \cdots dt_r
\]
is an algebraic polynomial of degree not greater than \((q+2)(m-1)\).

Proof. In this paper we denote, for simplicity,
\[
A(t) := \left( \frac{\sin \frac{\pi t}{2}}{\sin \frac{\pi}{2}} \right)^{2q+4}.
\]
Since
\[
A(t_s) = \sum_{k=0}^{(q+2)(m-1)} a_k \cos k t_s = \sum_{k=0}^{(q+2)(m-1)} b_k (\cos t_s)^k,
\]
it follows that
\[
A(t_s) \sin^2 t_s = \sum_{k=0}^{(q+2)(m-1)} b_k (\cos t_s)^k \left( 1 - \cos^2 t_s \right) = \sum_{k=0}^{(q+2)(m-1)+2} c_k (\cos t_s)^k = \sum_{k=0}^{(q+2)(m-1)+2} \alpha_k P_k^{(1,1)}(\cos t_s) \quad (s = 1,2,\ldots,r).
\]
Hence we have

\[
Q_{1}^{(l)}(x) = \sum_{k=0}^{(q+2)(m-1)+2} \alpha_k \int_0^{\pi} \cdots \int_0^{\pi} \prod_{s \neq 1}^{r} A(t_s) \sin^3 t_1 \cdots dt_{l-1} dt_{l+1} \cdots dt_r \\
\times \int_0^{\pi} T_{l,\cos t_1, \ldots, \cos t_l}^{(l)}(f, x) P_{k}^{(l,1)}(\cos t_l) \sin t_l dt_l.
\]

Let

\[
\varphi_{l,k}(x) := \int_0^{\pi} T_{l,\cos t_1, \ldots, \cos t_l}^{(l)}(f, x) P_{k}^{(l,1)}(\cos t_l) \sin t_l dt_l \\
= \int_0^{\pi} T_{1,\cos t_1, \ldots, \cos t_l}^{(l-1)}(f, x, x) P_{k}^{(l,1)}(\cos t_l) \sin t_l dt_l.
\]

Substituting \( y = \cos t_l \) we obtain

\[
\varphi_{l,k}(x) = \int_{-1}^{1} T_{1,y}^{(l-1)}(T_{1,\cos t_1, \ldots, \cos t_{l-1}}^{(l-1)}(f, x), x) P_{k}^{(l,1)}(y) dy.
\]

Then, by Lemma 3.3

\[
\varphi_{l,k}(x) = \sum_{m=0}^{k-2} \gamma_m(x) \int_{-1}^{1} T_{1,\cos t_1, \ldots, \cos t_{l-1}}^{(l-1)}(f, R) P_{m}^{(2,2)}(R) (1 - R^2)^2 dR.
\]

On the bases of Corollary 3.1 we conclude that \( T_{l,\cos t_1, \ldots, \cos t_{l-1}}^{(l-1)}(f, R) \in L_{1,2,2} \).

Applying now \( l - 1 \) times Lemma 3.3 we obtain

\[
\varphi_{l,k}(x) = \sum_{m=0}^{k-2} \gamma_m(x) \int_{-1}^{1} T_{1,\cos t_1, \ldots, \cos t_{l-1}}^{(l-2)}(f, R) T_{1,\cos t_{l-1}}^{(l-1)}(P_m^{(2,2)}, R) (1 - R^2)^2 dR
\]

\[
= \sum_{m=0}^{k-2} \gamma_m(x) P_{m+2}^{(0,0)}(\cos t_{l-1}) \int_{-1}^{1} T_{1,\cos t_1, \ldots, \cos t_{l-2}}^{(l-2)}(f, R) P_{m+2}^{(2,2)}(R) (1 - R^2)^2 dR
\]

\[
= \sum_{m=0}^{k-2} \gamma_m(x) P_{m+2}^{(0,0)}(\cos t_1) \cdots P_{m+2}^{(0,0)}(\cos t_{l-1}) \int_{-1}^{1} f(R) P_{m+2}^{(2,2)}(R) (1 - R^2)^2 dR
\]

\[
= \sum_{m=0}^{k-2} \gamma_m(x) a_m(f) \prod_{s=1}^{l-1} P_m^{(0,0)}(\cos t_s),
\]

where \( a_m(f) \) is the Fourier–Jacobi coefficient of the function \( f \) with respect to the system \( \{P_m^{(2,2)}(x)\}_{m=0}^{\infty} \). Substituting the last expression of \( \varphi_{l,k}(x) \) in the formula above for \( Q_{1}^{(l)}(x) \) we get

\[
Q_{1}^{(l)}(x) = \sum_{k=0}^{(q+2)(m-1)+2} \alpha_k \sum_{m=0}^{k-2} \beta_m \gamma_m(x).
\]

Since \( \gamma_m(x) \) is from \( P_{k-1} \) for \( k \geq 2 \) and \( \gamma_m(x) \equiv 0 \) for \( k = 0 \) and \( k = 1 \), then the last equality yields that \( Q_{1}^{(l)}(x) \) is an algebraic polynomial of degree not greater than \( (q + 2)(m - 1) \).

Lemma 3.6 is proved. \( \square \)
Lemma 3.7. Let \( q \) and \( m \) be given natural numbers. Let \( f \) be an integrable function on \([-1,1]\) with a weight \((1 - x^2)^2\). For every natural numbers \( l \) and \( r \) (\( l \leq r \)) the function

\[
Q^{(l)}_2(x) = \int_0^\pi \ldots \int_0^\pi T_{2,\cos t_1,\ldots,\cos t_l}(f,x) \prod_{s=1}^r A(t_s) \sin^5 t_s \, dt_1 \ldots dt_r
\]

is an algebraic polynomial of degree not greater than \((q + 2)(m - 1)\).

Proof. As shown in Lemma 3.6

\[
A(t_s) = \sum_{k=0}^{(q+2)(m-1)} b_k \cos t_s)^k = \sum_{k=0}^{(q+2)(m-1)} \beta_k P_k^{(2,2)}(\cos t_s) \quad (s = 1, 2, \ldots, r).
\]

Hence

\[
Q^{(l)}_2(x) = \sum_{k=0}^{(q+2)(m-1)} \beta_k \int_0^\pi \ldots \int_0^\pi \prod_{s=1}^r A(t_s) \sin^5 t_s \, dt_1 \ldots dt_{l-1} \, dt_l \ldots dt_r
\]

\[
\times \int_0^\pi T_{2,\cos t_1,\ldots,\cos t_l} \left( f(x) P_k^{(2,2)}(\cos t_l) \sin^5 t_l \, dt_l. \right.
\]

Let

\[
\psi_{l,k}(x) := \int_0^\pi T_{2,\cos t_1,\ldots,\cos t_l} \left( f(x) P_k^{(2,2)}(\cos t_l) \sin^5 t_l \, dt_l \right.
\]

\[
= \int_0^\pi T_{2,\cos t_l} \left( T_{2,\cos t_1,\ldots,\cos t_{l-1}} \left( f(x), x \right) P_k^{(2,2)}(\cos t_l) \sin^5 t_l \, dt_l. \right.
\]

Substituting \( y = \cos t_l \) we obtain

\[
\psi_{l,k}(x) = \int_{-1}^1 T_{2,y} \left( T_{2,\cos t_1,\ldots,\cos t_{l-1}} \left( f(x), x \right) P_k^{(2,2)}(y) \left(1 - y^2\right)^2 \right) dy.
\]

Since the operator \( T_{2,y} \left( f(x), x \right) \) is symmetric with respect to \( x \) and \( y \) (i.e. \( T_{2,y} \left( g, x \right) = T_{2,x} \left( g, y \right) \) for every function \( g \)), we have

\[
\psi_{l,k}(x) = \int_{-1}^1 T_{2,x} \left( T_{2,\cos t_1,\ldots,\cos t_{l-1}} \left( f(y), y \right) P_k^{(2,2)}(y) \left(1 - y^2\right)^2 \right) dy.
\]

Note that, in view of Corollary 3.1, \( T_{2,x} \left( f(y), y \right) \in L_{1,2,2} \). Then, by Lemma 3.3,

\[
\psi_{l,k}(x) = \int_{-1}^1 T_{2,\cos t_1,\ldots,\cos t_{l-1}} \left( f(y), y \right) T_{2,y} \left( P_k^{(2,2)}(y) \left(1 - y^2\right)^2 \right) dy.
\]

Using the property of the operator \( T_{2,y} \) described in Lemma 3.3 we get

\[
\psi_{l,k}(x) = P_k^{(2,2)}(x) \int_{-1}^1 T_{2,\cos t_1,\ldots,\cos t_{l-1}} \left( f(y), P_k^{(2,2)}(y) \left(1 - y^2\right)^2 \right) dy.
\]

Now we apply \( l - 1 \) times Lemma 3.3 and arrive at the expression

\[
\psi_{l,k}(x) = P_k^{(2,2)}(x) P_k^{(2,2)}(\cos t_1) \ldots P_k^{(2,2)}(\cos t_{l-1})
\]

\[
\times \int_{-1}^1 f(y) P_k^{(2,2)}(y) \left(1 - y^2\right)^2 \, dy = P_k^{(2,2)}(x) a_k(f) \prod_{s=1}^{l-1} P_k^{(2,2)}(\cos t_s).
\]
Lemma 3.9. If \( n \) and for \( 1 \)

Corollary 3.2. for \( \varepsilon \)

Let \( \alpha, \beta \) and \( \gamma \) be given numbers such that \( 1 \leq p \leq \infty \), \( \gamma = \min\{\alpha, \beta\} \), and

Let \( \varepsilon \), \( 0 < \varepsilon < \frac{1}{2} \), be an arbitrary number. Define

and for \( 1 < p \leq \infty \) let

where \( a_k(f) \) is the Fourier–Jacobi coefficient of the function \( f \) with respect to the system \( \left\{ p_k^{(2,2)}(x) \right\}_{k=0}^{\infty} \). Substituting the last expression of \( \psi_{\lambda, k}(x) \) into the formula for \( Q_2^{(l)}(x) \) above, we finally get

Since \( p_k^{(2,2)}(x) \) belongs to \( P_{k+1} \), it is seen from the last identity that \( Q_2^{(l)}(x) \) is an algebraic polynomial of degree not greater than \( (q + 2)(m - 1) \). The lemma is proved. \( \square \)

Lemma 3.8. The operator \( T_y \) has the following properties

(1) \( T_y(f, x) = f(x) \);
(2) \( T_y(P_n, x) = P_n(y) \) \( (n = 0, 1, \ldots) \),
where \( R_n(y) = P_{n+2}(y) + \frac{1}{2} (1 - y^2) P_n^{(2)}(y) \);
(4) \( T_y(1, x) = 1 \);
(5) \( a_k(T_y(f, x)) = R_k(y)a_k(f) \) \( (k = 0, 1, \ldots) \).

Lemma 3.8 was proved in [3].

Corollary 3.2. If \( P_n(x) \) is an algebraic polynomial of degree not greater than \( n - 1 \), then for every natural number \( r \) and any fixed \( y_1, y_2, \ldots, y_r \), the functions \( T_{y_1, y_2, \ldots, y_r}(P_n, x) \) are algebraic polynomials of \( x \) of degree not greater than \( n - 1 \).

Lemma 3.9. If \( -1 \leq x \leq 1, -1 \leq z \leq 1, 0 \leq t \leq \pi \) and \( R = x \cos t - z \sqrt{1 - x^2} \times \sin t \), then \( -1 \leq R \leq 1 \) and

\[
(1 - x^2)(1 - z^2) \leq (1 - R^2),
(1 - y^2)(1 - z^2) \leq (1 - R^2),
(x \sqrt{1 - y^2} + yz \sqrt{1 - x^2})^2 \leq (1 - R^2),
1 - x^2 \leq C (1 - R^2 + t^2),
1 - x \leq C (1 - R + t^2),
1 + x \leq C (1 + R + t^2),
\]

where \( y = \cos t \) and \( C \) is an absolute constant.

Lemma 3.9 was proved in [3] and [4].

Lemma 3.10. Let \( p, \alpha, \beta \) and \( \gamma \) be given numbers such that \( 1 \leq p \leq \infty \), \( \gamma = \min\{\alpha, \beta\} \), and

\[
\gamma > 1 - \frac{1}{2p} \quad \text{for } 1 \leq p < \infty,
\gamma \geq 1 \quad \text{for } p = \infty.
\]

Let \( \varepsilon \), \( 0 < \varepsilon < \frac{1}{2} \), be an arbitrary number. Define

and for \( 1 < p \leq \infty \) let

\[
\gamma_3 = \begin{cases} 
\gamma - \frac{1}{2} + \frac{1}{2p} + \varepsilon, & \text{if } \gamma \geq \frac{1}{2} - \frac{1}{2p} \\
0, & \text{if } \gamma < \frac{1}{2} - \frac{1}{2p},
\end{cases}
\]
while, for \( p = 1 \),
\[
\gamma_3 = \begin{cases} 
\gamma - 1, & \text{if } \gamma \geq 1 \\
0, & \text{if } \gamma < 1.
\end{cases}
\]

Let \( R = x \cos t - z \sqrt{1 - x^2} \sin t \). Then, for every measurable function \( f \) on \([-1, 1]\), the following inequality holds
\[
\left\| \frac{1}{1 - x^2} \int_{-1}^{1} (1 - R^2) |f(R)| \frac{dz}{\sqrt{1 - z^2}} \right\|_{p, \alpha, \beta} 
\leq C \left( \|f\|_{p, \alpha, \beta} + t^{2(\gamma_1 + \gamma_2)} \|f\|_{p, \alpha - \gamma_1, \beta - \gamma_2} + t^{2\gamma_3} \|f\|_{p, \alpha - \gamma_3, \beta - \gamma_3} 
+ t^{2(\gamma_1 + \gamma_2 + \gamma_3)} \|f\|_{p, \alpha - \gamma_3, \beta - \gamma_2 - \gamma_3} \right),
\]
where the constant \( C \) does not depend on \( f \) and \( t \).

**Proof.** If at least one of the terms on the right-hand side of the inequality is not finite, then the lemma is obvious.

Suppose now that all the terms on the right-hand side of the inequality are finite. Let \( \alpha \geq \beta \). We first consider the case \( 1 \leq p < \infty \). Clearly
\[
I := \left\| \frac{1}{1 - x^2} \int_{-1}^{1} (1 - R^2) |f(R)| \frac{dz}{\sqrt{1 - z^2}} \right\|^p_{p, \alpha, \beta}
= \int_{-1}^{1} \left| \int_{-1}^{1} |f(R)| (1 - z^2)^{-1/2} (1 - R^2) \, dz \right|^p (1 - x)^{p(\alpha - 1)} (1 + x)^{p(\beta - 1)} \, dx.
\]
If \( p = 1 \), then
\[
I \leq \int_{-1}^{1} \int_{-1}^{1} |f(R)| \delta \, dz \, dx,
\]
where
\[
\delta = (1 - z^2)^{-1/2} (1 - R^2) (1 - x^2)^{\beta - 1} (1 - x)^{\alpha - \beta}.
\]
Let \( \beta < 1 \). Then, in view of Lemma 3.3,
\[
\delta = (1 - z^2)^{\frac{1}{2} - \beta} \left( (1 - z^2) (1 - x^2) \right)^{\beta - 1} (1 - R^2) (1 - x)^{\alpha - \beta}
\leq C_1 (1 - z^2)^{-1/2} \left( (1 - z^2) (1 - x^2) \right)^{\beta - 1} (1 - R^2) (1 - R + t^2)^{\alpha - \beta}
= C_1 \delta_1 (x, z, R).
\]
Suppose that \( \beta \geq 1 \). Making use of Lemma 3.3 we see that
\[
\delta \leq C_2 (1 - z^2)^{-1/2} (1 - R^2) (1 - R^2 + t^2)^{\beta - 1} (1 - R + t^2)^{\alpha - \beta}
= C_2 \delta_2 (x, z, R).
\]
Incorporating these estimates for \( \delta \) we get the inequality
\[
I \leq C_3 \int_{-1}^{1} \int_{-1}^{1} |f(R)| \delta_k (x, z, R) \, dz \, dx \quad (k = 1, 2),
\]
which, after the change of variables \( 3.1 \), takes the form
\[
I \leq C_3 \int_{-1}^{1} \int_{-1}^{1} |f(R)| \delta_k (R, V, R) \, dV \, dR \quad (k = 1, 2).
\]
Set
\[
\delta_1 (R) := (1 - R^2)^{\beta} (1 - R + t^2)^{\alpha - \beta}
\leq C_4 \left( (1 - R)^{\alpha} (1 + R)^{\beta} + t^{2(\alpha - \beta)} (1 - R^2)^{\beta} \right),
\]
\[ \hat{\delta}_2(R) := (1 - R^2) (1 - R^2 + t^2)^{\beta - 1} (1 - R + t^2)^{\alpha - \beta} \]
\[ \leq C_6 \left( (1 - R)^\alpha (1 + R)^\beta + t^2(\alpha - \beta) (1 - R^2)^\beta \right. \]
\[ \left. + t^2(\beta - 1) (1 - R)^{\alpha - \beta + 1} (1 + R) + t^2(\alpha - 1) (1 - R^2) \right) . \]

Then clearly
\[ I \leq C_6 \int_{-1}^{1} |f(R)| \hat{\delta}_k(R) dR \quad (k = 1, 2). \]

The last inequality and the estimates for \( \hat{\delta}_k(R) \), given above, yield
\[ I \leq C_7 \left( \|f\|_{1, \alpha, \beta} + t^2(\gamma) \|f\|_{1, \alpha - \gamma_1, \beta} + t^2(\gamma) \|f\|_{1, \alpha - \gamma_3, \beta - \gamma_3} \right. \]
\[ \left. + t^2(\gamma + \gamma_3) \|f\|_{1, \alpha - \gamma_1 - \gamma_3, \beta - \gamma_3} \right), \]
where the constant \( C_7 \) does not depend on \( f \) and \( t \). Hence the lemma is true in the case \( p = 1 \).

Assume now that \( 1 < p < \infty \). Applying Hölder’s inequality to the inside integral in (3.2) we get
\[ I = \int_{-1}^{1} \int_{-1}^{1} |f(R)| (1 - z^2)^{-\frac{1}{2} - \frac{p}{2p} + \frac{1}{b}} (1 - R^2)^{p} (1 - x^2)^{p(\beta - 1)} (1 - x)^{p(\alpha - \beta)} dz dx \leq C_8 \int_{-1}^{1} |f(R)|^p \kappa dx dz dx, \]
where
\[ \kappa = (1 - z^2)^{-1 + p\left(\frac{1}{2} - \frac{1}{b}\right)} (1 - R^2)^p (1 - x^2)^{p(\beta - 1)} (1 - x)^{p(\alpha - \beta)}, \]
b is an arbitrary positive number, the constant \( C_8 \) does not depend on \( t \) and the function \( f \).

Let \( \beta < \frac{4}{2} - \frac{1}{2p} \). Put \( b = \frac{2}{2} - \frac{1}{2p} - \beta \). Applying Lemma 3.9 we derive the estimate
\[ \kappa \leq C_9 \left( (1 - z^2)^{-1/2} \left( (1 - z^2) (1 - x^2) \right)^{p(\beta - 1)} (1 - R^2)^p (1 - R + t^2)^{p(\alpha - \beta)} \right) \]
\[ = C_{9,1} \kappa(x, z, R). \]

Let \( \beta \geq \frac{4}{2} - \frac{1}{2p} \). Put \( b = \varepsilon \), where \( \varepsilon \) is an arbitrary number belonging to the interval \( 0 < \varepsilon < \frac{1}{2} \). Again by Lemma 3.9 we see that
\[ \kappa \leq C_{10} \left( (1 - z^2)^{-1/2} \left( (1 - z^2) (1 - x^2) \right)^{\frac{1}{p} + \frac{1}{p} - \varepsilon} (1 - R^2)^p \right. \]
\[ \times (1 - R^2 + t^2)^{p(\beta - 1 + \frac{1}{p} + \frac{1}{p} + \varepsilon)} (1 - R + t^2)^{p(\alpha - \beta)} = C_{10,2} \kappa_2(x, z, R). \]

Using these estimates for \( \kappa \) we get the inequality
\[ I \leq C_{11} \int_{-1}^{1} \int_{-1}^{1} |f(R)|^p \kappa_2(x, z, R) dz dx \quad (k = 1, 2), \]
and consequently (after the changes of variables (3.1)),
\[ I \leq C_{11} \int_{-1}^{1} \int_{-1}^{1} |f(R)|^p \kappa_2(R, V, R) dV dR \quad (k = 1, 2). \]

Set
\[ \tilde{\kappa}_1(R) := (1 - R^2)^{p\beta} (1 - R + t^2)^{p(\alpha - \beta)} \]
\[ \leq C_{12} \left( (1 - R)^{p\alpha} (1 + R)^{p\beta} + t^{2p(\alpha - \beta)} (1 - R^2)^{p\beta} \right), \]
\[ \hat{z}_2(R) := \left(1 - R^2 \right)^{-\frac{1}{2} + p(\frac{1}{2} - \varepsilon)} \left(1 - R^2 + t^2 \right)^{p(\beta - \frac{1}{2} + \frac{2}{b} + \varepsilon)} \left(1 - R + t^2 \right)^{p(\alpha - \beta)} \]

\[ \leq C_{13} \left( (1 - R)^{p\alpha} (1 + R)^{p\beta} + t^{2p(\alpha - \beta)} (1 - R^2)^{p\beta} \right. \]

\[ + t^{2p(\beta - \frac{1}{2} + \frac{2}{b} + \varepsilon)} (1 - R)^{p(\alpha - \beta + \frac{1}{2} - \frac{1}{b} - \varepsilon)} (1 + R)^{p(\frac{1}{2} - \frac{1}{b} - \varepsilon)} \]

Then clearly

\[ I \leq C_{14} \int_{-1}^{1} |f(R)|^p \hat{z}_k(R) \, dR \quad (k = 1, 2). \]

From the last inequality and the estimates of \( \hat{z}_k(R) \) we obtain

\[ I \leq C_{15} \left( \|f\|_{p, \alpha, \beta}^p + t^{2p\gamma_1} \|f\|_{p, \alpha - \gamma_1, \beta}^p + t^{2p\gamma_3} \|f\|_{p, \alpha - \gamma_3, \beta - \gamma_3}^p \right. \]

\[ + t^{2p(\gamma_1 + \gamma_3)} \|f\|_{p, \alpha - \gamma_1 - \gamma_3, \beta - \gamma_3}^p \right), \]

where the constant \( C_{15} \) does not depend on \( f \) and \( t \). This shows that the lemma is true in the case \( 1 < p < \infty \) as well.

Now let \( p = \infty \). Consider the integral

\[ J := \int_{-1}^{1} |f(R)| \left(1 - z^2\right)^{-1/2} \left(1 - R^2\right) (1 - x)^{\alpha - 1} (1 + x)^{\beta - 1} \, dz \]

\[ = \int_{-1}^{1} |f(R)| \lambda \, dz, \]

where

\[ \lambda = \left(1 - z^2\right)^{-\frac{1}{2} - b} \left(1 - R^2\right) (1 - x^2)^{\beta - 1} (1 - x)^{\alpha - \beta} (1 - z^2)^{\frac{1}{2} - b} \]

and \( b \) is an arbitrary positive number.

Let \( \beta < \frac{3}{2} \). Put \( b = \frac{3}{2} - \beta \). Applying the estimate from Lemma \( 3.9 \) we get

\[ \lambda = \left(1 - z^2\right)^{\frac{1}{2} - \beta} \left(1 - R^2\right) \left(1 - x^2\right)^{\beta - 1} (1 - x)^{\alpha - \beta} \]

\[ \leq C_{16} \left(1 - z^2\right)^{\frac{1}{2} - \beta} (1 - R^2)^{\beta} (1 - R + t^2)^{\alpha - \beta} = C_{16} \left(1 - z^2\right)^{\frac{3}{2} - \beta} \lambda_1(R). \]

Let \( \beta \geq \frac{3}{2} \). Put \( b = \varepsilon \), where \( \varepsilon \) is an arbitrary number from the interval \( 0 < \varepsilon < \frac{1}{2} \). Applying again Lemma \( 3.9 \) we see that

\[ \lambda = \left(1 - z^2\right)^{-1 + \varepsilon} \left(1 - z^2\right)^{\frac{3}{2} - \varepsilon} (1 - R^2)^{\beta - \frac{3}{2} - \varepsilon} (1 - x)^{\alpha - \beta} \]

\[ \leq C_{17} \left(1 - z^2\right)^{-1 + \varepsilon} (1 - R^2)^{\frac{3}{2} - \varepsilon} (1 - R^2 + t^2)^{\beta - \frac{3}{2} + \varepsilon} (1 - R + t^2)^{\alpha - \beta} \]

\[ = C_{17} \left(1 - z^2\right)^{-1 + \varepsilon} \lambda_2(R). \]

Using these estimates for \( \lambda \) and taking into account the relations

\[ \lambda_1(R) \leq C_{18} \left(1 - R\right)^{\alpha} (1 + R)^{\beta} + t^{2(\alpha - \beta)} (1 - R^2)^{\beta} \right), \]

\[ \lambda_2(R) \leq C_{19} \left(1 - R\right)^{\alpha} (1 + R)^{\beta} + t^{2(\alpha - \beta)} (1 - R^2)^{\beta} \]

\[ + t^{2(\beta - \frac{3}{2} + \varepsilon)} (1 - R)^{\alpha - \beta + \frac{3}{2} - \varepsilon} (1 + R)^{\frac{3}{2} - \varepsilon} + t^{2(\alpha - \frac{3}{2} + \varepsilon)} (1 - R^2)^{\frac{3}{2} - \varepsilon} \right), \]
for $k = 1, 2$ we obtain

\[
J \leq C_{20} \max_{-1 \leq R \leq 1} |f(R)| \lambda_k(R) \leq C_{21} \left( \|f\|_{\infty, \alpha, \beta} + t^{2 \gamma_1} \|f\|_{\infty, \alpha-\gamma_1, \beta}
\right.
\]
\[
+ t^{2 \gamma_2} \|f\|_{\infty, \alpha-\gamma_2, \beta} + t^{2(\gamma_1+\gamma_2)} \|f\|_{\infty, \alpha-\gamma_1-\gamma_2, \beta-\gamma_3},
\]

where the constant $C_{21}$ does not depend on $f$ and $t$. This proves the lemma for $p = \infty$.

Thus, the lemma is proved for $\alpha \geq \beta$. The case $\alpha \leq \beta$ goes similarly. We omit the details. The proof is complete. \hfill \Box

4.

**Theorem 4.1.** Let $p$, $\alpha$, $\beta$ and $\gamma$ be given numbers such that $1 \leq p \leq \infty$, $\gamma = \min\{\alpha, \beta\}$. Assume that

\[
\gamma > 1 - \frac{1}{2p} \quad \text{for} \quad 1 \leq p < \infty,
\]
\[
\gamma \geq 1 \quad \text{for} \quad p = \infty.
\]

Let $\varepsilon$ be an arbitrary number belonging to the interval $0 < \varepsilon < \frac{1}{2}$. Let

\[
\gamma_1 = \begin{cases} 
\alpha - \beta, & \text{if} \ \alpha > \beta \\
0, & \text{if} \ \alpha \leq \beta
\end{cases}, \quad \gamma_2 = \begin{cases} 
0, & \text{if} \ \alpha > \beta \\
\beta - \alpha, & \text{if} \ \alpha \leq \beta
\end{cases},
\]

Set for $1 < p \leq \infty$

\[
\gamma_3 = \begin{cases} 
\gamma - \frac{3}{2} + \frac{1}{2p} + \varepsilon, & \text{for} \ \gamma \geq \frac{3}{2} + \frac{1}{2p} \\
0, & \text{for} \ \gamma < \frac{3}{2} + \frac{1}{2p}
\end{cases},
\]

and

\[
\gamma_3 = \begin{cases} 
\gamma - 1, & \text{for} \ \gamma \geq 1 \\
0, & \text{for} \ \gamma < 1
\end{cases},
\]

for $p = 1$. Then the following inequality holds true

\[
\left\|\mathcal{T}_t (f, x)\right\|_{p, \alpha, \beta} \leq C \left( \|f\|_{p, \alpha, \beta} + t^{2(\gamma_1+\gamma_2)} \|f\|_{p, \alpha-\gamma_1, \beta-\gamma_2}
\right.
\]
\[
+ t^{2 \gamma_3} \|f\|_{p, \alpha-\gamma_2, \beta-\gamma_3} + t^{2(\gamma_1+\gamma_2+\gamma_3)} \|f\|_{p, \alpha-\gamma_1-\gamma_2, \beta-\gamma_3},
\]

where the constant $C$ does not depend on $f$ and $t$.

**Proof.** We have

\[
\left\|\mathcal{T}_t (f, x)\right\|_{p, \alpha, \beta} \leq \frac{1}{\pi} \left\| \frac{1}{1-x^2} \int_{-1}^{1} A(R) \frac{dz}{\sqrt{1-z^2}} \right\|_{p, \alpha, \beta},
\]

where $R = x \cos t - z \sqrt{1-x^2} \sin t$,

\[
A = 1 - R^2 - 2 \left( 1 - x^2 \right) \sin^2 t + 4 \left( 1 - x^2 \right) \left( 1 - z^2 \right)^2 \sin^2 t.
\]

Using Lemma 3.9 we get

\[
A \leq 1 - R^2 + 2 \left( 1 - R^2 \right) + 4 \left( 1 - R^2 \right)^2 \leq 7 \left( 1 - R^2 \right).
\]

Hence

\[
\left\|\mathcal{T}_t (f, x)\right\|_{p, \alpha, \beta} \leq \frac{7}{\pi} \left\| \frac{1}{1-x^2} \int_{-1}^{1} (1 - R^2) |f(R)| \frac{dz}{\sqrt{1-z^2}} \right\|_{p, \alpha, \beta}.
\]

Now the theorem follows from Lemma 3.10. \hfill \Box
Theorem 4.2. Let \( q, m \) and \( r \) be given natural numbers and let \( f \in L_{1,2,2} \). The function

\[
Q(x) = \frac{1}{(\gamma_m)^r} \int_0^\pi \cdots \int_0^\pi (\Delta_t^{r_1}, \ldots, \Delta_t^{r_r} f(x) - (-1)^r f(x))
\times \prod_{s=1}^r A(t_s) \sin^3 t_s dt_1 \cdots dt_r,
\]

where

\[
\gamma_m = \int_0^\pi A(t) \sin^3 t dt,
\]

is an algebraic polynomial of degree not greater than \((q + 2)(m - 1)\).

Proof. To prove the theorem it is sufficient to show that for every \( l = 1, \ldots, r \) the function

\[
Q^{(l)}(x) = \frac{1}{(\gamma_m)^r} \int_0^\pi \cdots \int_0^\pi \Delta_t^{l_1}, \ldots, \Delta_t^{l_l} f(x) \prod_{s=1}^r A(t_s) \sin^3 t_s dt_1 \cdots dt_r
\]

is an algebraic polynomial of degree not greater than \((q + 2)(m - 1)\).

It is obvious that the function \( Q^{(l)}(x) \) can be written in the form

\[
Q^{(l)}(x) = \frac{1}{(\gamma_m)^r} \left( Q_1^{(l)}(x) + \frac{3}{2} Q_2^{(l)}(x) \right),
\]

where \( Q_1^{(l)}(x) \) and \( Q_2^{(l)}(x) \) are the functions from Lemmas 3.6 and 3.7, respectively. But then it follows from Lemmas 3.6 and 3.7 that \( Q^{(l)}(x) \) is an algebraic polynomial of degree not greater than \((q + 2)(m - 1)\).

The theorem is proved. \( \square \)

Theorem 4.3. Let \( p, \alpha, \beta, r \) and \( \lambda \) be given numbers such that \( 1 \leq p \leq \infty, \lambda > 0, r \in \mathbb{N} \). Assume that

\[
\alpha \leq 2, \quad \beta \leq 2 \quad \text{for } p = 1,
\]

\[
\alpha < 3 - \frac{1}{p}, \quad \beta < 3 - \frac{1}{p} \quad \text{for } 1 < p \leq \infty.
\]

Let \( f \in L_{p,\alpha,\beta} \) and

\[
\hat{\omega}_r(f, \delta)_{p,\alpha,\beta} \leq M\delta^\lambda.
\]

Then

\[
E_n(f)_{p,\alpha,\beta} \leq CMn^{-\lambda},
\]

where the constant \( C \) does not depend on \( f, M \) and \( n \in \mathbb{N} \).

Proof. Under the conditions of the theorem, if \( f \in L_{p,\alpha,\beta} \), then \( f \in L_{1,2,2} \). Indeed, for \( p = 1 \) we have

\[
\|f\|_{1,2,2} = \int_{-1}^1 |f(x)|(1 - x)^\alpha(1 + x)^\beta(1 - x)^{2-\alpha}(1 + x)^{2-\beta} dx \leq C_1 \|f\|_{1,\alpha,\beta}
\]

provided \( \alpha \leq 2 \) and \( \beta \leq 2 \). For \( 1 < p < \infty \), by Hölder’s inequality,

\[
\|f\|_{1,2,2} \leq \left\{ \int_{-1}^1 |f(x)|^p(1 - x)^{\alpha p}(1 + x)^{\beta p} dx \right\}^{1/p}
\times \left\{ \int_{-1}^1 (1 - x)^{(2-\alpha)\frac{p}{p-1}}(1 + x)^{(2-\beta)\frac{p}{p-1}} dx \right\}^{1-p} = C_2 \|f\|_{p,\alpha,\beta}
\]
for \( \alpha < 3 - \frac{1}{p} \) and \( \beta < 3 - \frac{1}{p} \). For \( p = \infty \) we have
\[
\|f\|_{1,2,2} \leq \|f\|_{\infty, \alpha, \beta} \int_{-1}^{1} (1-x)^{2-\alpha} (1+x)^{2-\beta} \, dx = C_3 \|f\|_{\infty, \alpha, \beta}
\]
provided \( \alpha < 3 \) and \( \beta < 3 \).

We choose a natural number \( q \) such that \( 2q > \lambda \), and for each \( n \in \mathbb{N} \) we choose a number \( m \in \mathbb{N} \) satisfying the condition
\[
(4.1) \quad \frac{n-1}{q+2} < m \leq \frac{n-1}{q+2} + 1.
\]
For these \( q \) and \( m \) the polynomial \( Q(x) \) defined in Theorem 4.2 is from \( \mathcal{P}_n \). Hence
\[
E_n(f)_{p, \alpha, \beta} \leq \|f(x) - (-1)^{r+1}Q(x)\|_{p, \alpha, \beta}
\]
Applying the generalized inequality of Minkowski we obtain
\[
E_n(f)_{p, \alpha, \beta} \leq \frac{1}{(\gamma_m)^r} \int_{0}^{\pi} \cdots \int_{0}^{\pi} \sup_{|u_i| \leq \sum_{j=1}^{r} t_j} \|\Delta_{u_1, \ldots, u_r}^r (f, x)\|_{p, \alpha, \beta} \prod_{s=1}^{r} A(t_s) \sin^3 t_s \, dt_1 \cdots dt_r
\]
\[
\leq \frac{1}{(\gamma_m)^r} \int_{0}^{\pi} \cdots \int_{0}^{\pi} \prod_{s=1}^{r} A(t_s) \sin^3 t_s \, dt_1 \cdots dt_r
\]
Hence, taking into account the assumptions of the theorem, we have
\[
E_n(f)_{p, \alpha, \beta} \leq \frac{M}{(\gamma_m)^r} \int_{0}^{\pi} \cdots \int_{0}^{\pi} \left( \sum_{j=1}^{r} t_j \right)^\lambda \prod_{s=1}^{r} A(t_s) \sin^3 t_s \, dt_1 \cdots dt_r
\]
\[
\leq C_4 M \sum_{j=1}^{r} \frac{1}{(\gamma_m)^r} \int_{0}^{\pi} \cdots \int_{0}^{\pi} t_j^\lambda \prod_{s=1}^{r} A(t_s) \sin^3 t_s \, dt_1 \cdots dt_r.
\]
Applying the standard evaluation of Jackson’s kernel and making use of inequality (4.1) we obtain
\[
E_n(f)_{p, \alpha, \beta} \leq C_5 M n^{-\lambda} \leq C_6 M n^{-\lambda}.
\]

Theorem 4.3 is proved. \( \square \)

**Theorem 4.4.** Let \( p, \alpha, \beta, r \) and \( \lambda \) be given numbers such that \( 1 \leq p \leq \infty \), \( r \in \mathbb{N} \). Assume that
\[
\alpha > 1 - \frac{1}{2p}, \quad \beta > 1 - \frac{1}{2p} \quad \text{for} \quad 1 \leq p < \infty,
\]
\[
\alpha \geq 1, \quad \beta \geq 1 \quad \text{for} \quad p = \infty;
\]
\[
\lambda_0 = 2 \max \left\{ |\alpha - \beta|, \alpha - \frac{3}{2} + \frac{1}{2p}, \beta - \frac{3}{2} + \frac{1}{2p} \right\} < \lambda < 2r.
\]
If \( f \in L_{p, \alpha, \beta} \) and
\[
E_n(f)_{p, \alpha, \beta} \leq \frac{M}{n^\lambda},
\]
then
\[ \hat{\varphi}_r(f, \delta)_{p,\alpha,\beta} \leq CM\delta^\lambda, \]
where the constant \( C \) does not depend on \( f, M \) and \( \delta \).

Proof. Let \( P_n(x) \) be the polynomial from \( \mathcal{P}_n \) for which
\[ \|f - P_n\|_{p,\alpha,\beta} = E_n(f)_{p,\alpha,\beta} \quad (n = 1, 2, \ldots). \]
We construct the polynomials \( Q_k(x) \) by
\[ Q_k(x) = P_{2k}(x) - P_{2k-1}(x) \quad (k = 1, 2, \ldots) \]
and \( Q_0(x) = P_1(x) \). Since for \( k \geq 1 \) we have
\[ \|Q_k\|_{p,\alpha,\beta} = \| P_{2k} - P_{2k-1} \|_{p,\alpha,\beta} \leq \| P_{2k} - f \|_{p,\alpha,\beta} + \| f - P_{2k-1} \|_{p,\alpha,\beta} = E_{2k}(f)_{p,\alpha,\beta} + E_{2k-1}(f)_{p,\alpha,\beta}, \]
then it follows from the assumptions of the theorem that
\[ \|Q_k\|_{p,\alpha,\beta} \leq C_1 M 2^{-k\lambda}. \]

It is obvious that without lost of generality we may assume that \( t_s \neq 0 \) (\( s = 1, \ldots, r \)). Next we estimate the quantity
\[ I = \| \Delta_{t_1, \ldots, t_r}(f, x) \|_{p,\alpha,\beta} \]
for \( 0 < |t_s| < \delta \) (\( s = 1, \ldots, r \)). For every natural number \( N \), taking into account that the linearity of the operator \( \hat{T}_{t_1}(f, x) \) implies the linearity of \( \hat{T}_{t_1, \ldots, t_r}(f, x) \), i.e. the linearity of the difference \( \Delta_{t_1, \ldots, t_r}(f, x) \), we have
\[ I \leq \| \Delta_{t_1, \ldots, t_r}(f - P_{2N}, x) \|_{p,\alpha,\beta} + \| \Delta_{t_1, \ldots, t_r}(P_{2N}, x) \|_{p,\alpha,\beta}. \]
Since \( P_{2N}(x) = \sum_{k=0}^{N} Q_k(x) \), we get
\[ I \leq \| \Delta_{t_1, \ldots, t_r}(f - P_{2N}, x) \|_{p,\alpha,\beta} + \sum_{k=0}^{N} \| \Delta_{t_1, \ldots, t_r}(Q_k, x) \|_{p,\alpha,\beta} =: A + \sum_{k=1}^{N} I_k. \]

Let \( N \) be chosen so that
\[ \frac{\pi}{2^N} < \delta \leq \frac{\pi}{2^{N-1}}. \]
We shall show that
\[ A \leq C_2 M \delta^\lambda \]
and
\[ I_k \leq C_3 M \delta^{2r-2k(2r-\lambda)}. \]
Consider first the quantity \( A \). Assume that \( r = 1 \). An application of Theorem 4.11 to the function \( \varphi(x) = f(x) - P_{2N}(x) \) gives
\[ \| \Delta_{t_1}(f - P_{2N}, x) \|_{p,\alpha,\beta} = \| \hat{T}_{t_1}(\varphi, x) - \varphi(x) \|_{p,\alpha,\beta} \leq \| \hat{T}_{t_1}(\varphi, x) \|_{p,\alpha,\beta} + \| \varphi(x) \|_{p,\alpha,\beta} \leq C_4 \left( \| \varphi \|_{p,\alpha,\beta} + \delta^{2(\gamma_1 + \gamma_2)} \| \varphi \|_{p,\alpha - \gamma_1, \beta - \gamma_2} + \delta^{2\gamma_3} \| \varphi \|_{p,\alpha - \gamma_1, \beta - \gamma_2 - \gamma_3} + \delta^{2(\gamma_1 + \gamma_2 + \gamma_3)} \| \varphi \|_{p,\alpha - \gamma_1, \beta - \gamma_2 - \gamma_3} \right) \]
for \( |t_1| \leq \delta \), where the numbers \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) are chosen as in Theorem 4.11. Hence, by Lemma 5.9.1
\[ \| \Delta_{t_1}(f - P_{2N}, x) \|_{p,\alpha,\beta} \leq C_5 M \left( 2^{-N \lambda} + \delta^{2(\gamma_1 + \gamma_2)} 2^{-N(\lambda - 2\gamma_1 - 2\gamma_2)} + \delta^{2\gamma_3} 2^{-N(\lambda - 2\gamma_3)} + \delta^{2(\gamma_1 + \gamma_2 + \gamma_3)} 2^{-N(\lambda - 2\gamma_1 - 2\gamma_2 - 2\gamma_3)} \right) \]
for $\lambda > \lambda_0 + \varepsilon$, where the constant $C_5$ does not depend on $f$, $M$ and $\delta$. Here $\varepsilon$ is either equal to 0 or is an arbitrary number belonging to the interval $0 < \varepsilon < \frac{1}{7}$. Therefore, this inequality holds for any $\lambda > \lambda_0$. Finally, applying inequality (4.2), we get

$$\|\Delta_t (f - P_{2^N}, x)\|_{\alpha, \beta} \leq C_6 M 2^{N \lambda} \leq C_7 M \lambda.$$

Thus inequality (4.3) is proved for $r = 1$.

Suppose that

$$\|\Delta_{t_1, \ldots, t_{r-1}} (f - P_{2^N}, x)\|_{\alpha, \beta} \leq C_8 M \lambda.$$

Then inequality (4.2) yields

$$\|\Delta_{t_1, \ldots, t_{r-1}} (f - P_{2^N}, x)\|_{\alpha, \beta} = \|\Delta_{t_1, \ldots, t_{r-1}} (f, x) - \Delta_{t_1, \ldots, t_{r-1}} (P_{2^N}, x)\|_{\alpha, \beta} \leq C_9 M 2^{N \lambda}.$$

Reasoning as above, i.e. applying first Theorem 4.1 to the function

$$\Delta_{t_1, \ldots, t_{r-1}, t_r} (f - P_{2^N}, x),$$

taking into account that by Corollary 3.2 $\Delta_{t_1, \ldots, t_{r-1}} (P_{2^N}, x)$ is an algebraic polynomial of degree not greater than $2^N - 1$, applying Lemma 3.1 and finally inequality (4.2), we obtain that

$$A = \|\Delta_{t_1, \ldots, t_r} (f - P_{2^N}, x)\|_{\alpha, \beta} \leq C_{10} \delta \lambda.$$

Inequality (4.3) is proved.

Now we prove inequality (4.2). Let

$$\psi_k (x) = \Delta_{t_1, \ldots, t_r} (Q_k, x).$$

It can be shown that

$$\psi_k (x) = \frac{1}{2\pi (1 - x^2)} \int_0^{t_r} \int_{-u}^u \int_0^\pi \left( A(v)(R_0')^2 \frac{d^2}{dR_0^2} \Delta_{t_1, \ldots, t_r} (Q_k, R_0) - A(v)R_0 - 2A'(v)R_0' \frac{d}{dR_0} \Delta_{t_1, \ldots, t_r} (Q_k, R_0) + A''(v) \Delta_{t_1, \ldots, t_r} (Q_k, R_0) \right) d\varphi dv du,$$

where $R_0 = x \cos v - \sqrt{1 - x^2} \cos \varphi \sin v$,

$$A(v) = 1 - R_0^2 - 2 \sin^2 v \sin^2 \varphi + 4 (1 - x^2) \sin^2 v \sin^4 \varphi.$$

Applying the estimates from Lemma 3.1 and performing the change of variables $z = \cos \varphi$ we obtain

$$|\psi_k (x)| \leq \frac{C_{11}}{1 - x^2} \int_0^{t_r} \int_{-u}^u \int_1^1 B(R_0) \frac{dz}{\sqrt{1 - z^2}} dv du,$$

where

$$B(R_0) = (1 - R_0^2)^2 \left| \frac{d^2}{dR_0^2} \Delta_{t_1, \ldots, t_r} (Q_k, R_0) \right| + (1 - R_0^2) \left| \frac{d}{dR_0} \Delta_{t_1, \ldots, t_r} (Q_k, R_0) \right| + \left| \Delta_{t_1, \ldots, t_r} (Q_k, R_0) \right| = B_1(R_0) + B_2(R_0) + B_3(R_0).$$
Therefore, using the generalized Minkowski inequality, we get

\begin{equation}
I_k = \| \psi_k(x) \|_{p,\alpha,\beta} \leq C_{11} \int_0^{t_r} \int_{-u}^u \left\| \frac{1}{1-x^2} \int_{-1}^1 B(R_v) \frac{dz}{\sqrt{1-z^2}} \right\|_{p,\alpha,\beta} \, dv \, du
\end{equation}

\begin{align*}
&\leq C_{12} t_r^2 \sup_{|v| \leq t_r} \left\| \frac{1}{1-x^2} \int_{-1}^1 B(R_v) \frac{dz}{\sqrt{1-z^2}} \right\|_{p,\alpha,\beta} \\
&\leq C_{12} t_r^2 \sup_{|v| \leq t_r} \left\| \frac{1}{1-x^2} \int_{-1}^1 B(R_v) \frac{dz}{\sqrt{1-z^2}} \right\|_{p,\alpha,\beta}.
\end{align*}

Next, applying first Lemma 3.10 to the function $B_1(R_v)$, then Lemma 3.2 and finally inequality (4.2), we get for $|v| \leq |t_r| \leq \delta$ and $k \leq N$

\begin{align*}
&\left\| \frac{1}{1-x^2} \int_{-1}^1 B_1(R_v) \frac{dz}{\sqrt{1-z^2}} \right\|_{p,\alpha,\beta} \\
&\leq C_{13} \left( |v|^{2(\gamma_1+\gamma_2)} \left\| \frac{1}{1-x^2} \frac{d^2}{dx^2} \Delta_{t_1,\ldots,t_{r-1}}^{r-1} (Q_k, x) \right\|_{p,\alpha,\beta} \\
&\quad + |v|^{2\gamma_1} \left\| \frac{1}{1-x^2} \frac{d^2}{dx^2} \Delta_{t_1,\ldots,t_{r-1}}^{r-1} (Q_k, x) \right\|_{p,\alpha-\gamma_1,\beta-\gamma_2} \\
&\quad + |v|^{2\gamma_2} \left\| \frac{1}{1-x^2} \frac{d^2}{dx^2} \Delta_{t_1,\ldots,t_{r-1}}^{r-1} (Q_k, x) \right\|_{p,\alpha-\gamma_2,\beta-\gamma_3} \right)
\end{align*}

\begin{align*}
&\leq C_{14} \left( 1 + |v|^{2(\gamma_1+\gamma_2)} 2^{k(\gamma_1+\gamma_2)} + |v|^{2\gamma_1} 2^{2k\gamma_1} + |v|^{2(\gamma_1+\gamma_2)} 2^{2k(\gamma_1+\gamma_2)} \right) \\
&\quad \times \left\| \frac{1}{1-x^2} \frac{d^2}{dx^2} \Delta_{t_1,\ldots,t_{r-1}}^{r-1} (Q_k, x) \right\|_{p,\alpha,\beta} \\
&\leq C_{15} \left\| \frac{d^2}{dx^2} \Delta_{t_1,\ldots,t_{r-1}}^{r-1} (Q_k, x) \right\|_{p,\alpha+1,\beta+1}.
\end{align*}

Similarly, applying first Lemma 3.10 then Lemma 3.2 and finally inequality (4.2) we obtain

\begin{align*}
&\left\| \frac{1}{1-x^2} \int_{-1}^1 B_2(R_v) \frac{dz}{\sqrt{1-z^2}} \right\|_{p,\alpha,\beta} \leq C_{16} \left\| \frac{d}{dx} \Delta_{t_1,\ldots,t_{r-1}}^{r-1} (Q_k, x) \right\|_{p,\alpha,\beta}
\end{align*}

and

\begin{align*}
&\left\| \frac{1}{1-x^2} \int_{-1}^1 B_3(R_v) \frac{dz}{\sqrt{1-z^2}} \right\|_{p,\alpha,\beta} \leq C_{17} \left\| \Delta_{t_1,\ldots,t_{r-1}}^{r-1} (Q_k, x) \right\|_{p,\alpha-1,\beta-1}.
\end{align*}

Now, from inequality (4.5) and the fact that $|t_r| \leq \delta$ we derive the estimate

\begin{align*}
I_k &\leq C_{18} \delta^2 \left( \left\| \frac{d^2}{dx^2} \Delta_{t_1,\ldots,t_{r-1}}^{r-1} (Q_k, x) \right\|_{p,\alpha+1,\beta+1} \right. \\
&\quad + \left. \left\| \frac{d}{dx} \Delta_{t_1,\ldots,t_{r-1}}^{r-1} (Q_k, x) \right\|_{p,\alpha,\beta} + \left\| \Delta_{t_1,\ldots,t_{r-1}}^{r-1} (Q_k, x) \right\|_{p,\alpha-1,\beta-1} \right).
\end{align*}

Applying twice Lemma 3.2 we obtain the recurrence relation

\begin{align*}
I_k &= \left\| \Delta_{t_1,\ldots,t_r}^{r} (Q_k, x) \right\|_{p,\alpha,\beta} \leq C_{18} \delta^2 2^{2k} \left\| \Delta_{t_1,\ldots,t_{r-1}}^{r-1} (Q_k, x) \right\|_{p,\alpha,\beta},
\end{align*}

which yields

\begin{align*}
I_k &\leq C_{19} \delta^2 2^{2kr} \| Q_k \|_{p,\alpha,\beta} \leq C_{20} M \delta^2 2^{k(2r-\lambda)}.
\end{align*}
Inequality (4.4) is proved. Now combining (4.3), (4.4) and (4.2) we finally get

\[ I \leq C_{21} M \left( \delta^\lambda + \delta^{2r} \sum_{k=1}^{N} 2^{k(2r-\lambda)} \right) \leq C_{22} M \left( \delta^\lambda + \delta^{2r} 2^{N(2r-\lambda)} \right) \leq C_{23} M \delta^\lambda. \]

The proof of Theorem 4.4 is completed. \qed

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M. K. Potapov, Department of Mechanics and Mathematics, Moscow State University, Moscow 117234, Russia

F. M. Berisha, Faculty of Mathematics and Sciences, University of Prishtina, Nëna Terezi 5, 38000 Prishtina, Kosovo

E-mail address: faton.berisha@uni-pr.edu