BOUNDARY VALUES OF HOLOMORPHIC DISTRIBUTIONS IN NEGATIVE LIPSCHITZ CLASSES

ANTHONY G. O’FARRELL

Abstract. We consider the behaviour at a boundary point of an open subset $U \subset \mathbb{C}$ of distributions that are holomorphic on $U$ and belong to what are called negative Lipschitz classes. The result explains the significance for holomorphic functions of series of Wiener type involving Hausdorff contents of dimension between 0 and 1. We begin with a survey about function spaces and capacities that sets the problem in context and reviews the relevant general theory.

1. Introduction

1.1. Boundary values. It may happen that all bounded holomorphic functions on an open set $U \subset \mathbb{C}$ admit a ‘reasonable boundary value’ at some boundary point. This was first noted by Gamelin and Garnett [10]. The condition for the existence of such a boundary value is expressed using a series of ‘Wiener type’, and involves the Ahlfors analytic capacity, $\gamma$. The condition is

$$\sum_{n=1}^{\infty} 2^n \gamma(A_n \setminus U) < +\infty.$$ 

Here, if $b$ is the boundary point in question, $A_n$ denotes the annulus

$$A_n(b) := \left\{ z \in \mathbb{C} : \frac{1}{2^{n+1}} \leq |z - b| \leq \frac{1}{2^n} \right\}.$$ 

This condition says that in an appropriate sense the complement of $U$ is very thin at $b$; in particular it implies that $U$ has full area density at $b$, i.e.

$$\lim_{r \downarrow 0} \frac{|B(b, r) \setminus U|}{\pi r^2} = 1,$$

where we denote the area of a set $E \subset \mathbb{C}$ by $|E|$. When the series converges, it is emphatically not the case that the limit

$$\lim_{x \to a, z \in U} f(z)$$

is

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exists for all functions $f$ bounded and holomorphic on $U$ (unless all such functions extend holomorphically to a neighbourhood of $b$). But for each such function there is a (unique) value which we may call $f(a)$, with the property that for some set $E \subset U$ having full area density at the point $a$

$$\lim_{x \to a, z \in E} f(z) = f(a).$$

1.2. Peak points. This result is one of many about the boundary behaviour of analytic and harmonic functions on arbitrary open sets. The original Wiener series (cf. [32] or [1]) involved logarithmic capacity in dimension two, Newtonian capacity in dimension three, and other Riesz capacities in higher dimensions, and characterised boundary points that are regular for the Dirichlet problem. Later these points were recognised as peak points for the space of functions harmonic on an open set $U$ and continuous on its closure. The first person to use such a series with holomorphic functions was Melnikov [9, Theorem VIII.4.5]. He characterised the peak points for the uniform closure on a compact set $X \subset \mathbb{C}$ of the algebra of all rational functions having poles off $X$. He used the Ahlfors capacity, and he showed that a point $b \in X$ is a peak point if and only if

$$\sum_{n=1}^{\infty} 2^n \gamma(\mathring{A}_n \setminus X) < +\infty.$$  

(This was used by Gamelin and Garnett to obtain their above-quoted result.)

For a bounded open set $U \subset \mathbb{C}$, and a point $b \in \partial U$, the condition

$$\sum_{n=1}^{\infty} 2^n \alpha(A_n \setminus U) < +\infty,$$

where $\alpha$ denotes the so-called continuous analytic capacity (introduced by Dolzhenko) is necessary and sufficient for $b$ to be a peak point for the algebra of all continuous functions on $\overline{U}$, holomorphic on $U$ [9].

1.3. Capacities. The vague idea that there is a capacity for every problem has gathered momentum over time. A capacity is a function $c$ that assigns nonnegative extended real numbers to sets, and is nondecreasing:

$$E_1 \subset E_2 \implies c(E_1) \leq c(E_2).$$

Keldysh [16] used Newtonian capacity to solve the problem of stability for the Dirichlet problem. Vitushkin used analytic capacity to solve the problem of uniform rational approximation [9, Chapter VIII]. Vitushkin’s theorem is completely analogous to Keldysh’s: harmonic
functions have been replaced by holomorphic functions, and Newton-
nian capacity by analytic capacity. The same switch relates Wiener’s
regularity criterion and Melnikov’s peak-point criterion.

In an influential little book [4], Carleson explained how other capa-
cities (particularly kernel capacities) could be used to solve problems
about boundary values, convergence of Fourier series, and removable
singularities, and in an appendix (prepared by Wallin) he listed over a
thousand articles from Mathematical Reviews up to 1965 that involve
some combination of these ideas.

1.4. Continuous point evaluations. In relation to $L^p$ holomorphic
approximation, the appropriate capacity is a condenser capacity. The
groundwork on condenser capacities and (generalized) extremal length
had already been laid down by Fuglede [8]. Hedberg [14] (see also
[2]) worked out the analogue of Vitushkin’s theorem for $L^p$ approxima-
tion on compact $X \subset \mathbb{C}$, and [13] proved the analogue of Melnikov’s
theorem. Hedberg’s result is about continuous point evaluations. To
explain this concept, consider a Banach space $F$ of ‘functions’ on some
set $E \subset \mathbb{C}$, where each element $f \in F$ is defined almost-everywhere on
$E$ with respect to area measure $m$. Suppose $b \in \overline{E}$ and the subspace
$F_b$, consisting of those $f \in F$ that extend holomorphically to some
neighbourhood of $b$, is a dense subset of $F$. Then we say that $F$ admits
a continuous point evaluation (cpe) at $b$ if there exists $\kappa > 0$ such that

$$|f(b)| \leq \kappa \|f\|_F, \quad \forall f \in F_b.$$  

This means that the functional $f \mapsto f(b)$ has a continuous extension
from $F_b$ to the whole of $F$. Taking the case where $F$ is the closure
$R^p(X)$ in $L^p(X, m)$ of the rational functions with poles off a compact
$X \subset \mathbb{C}$, Hedberg showed that if $2 < p < +\infty$, then $R^p(X)$ admits a
continuous point evaluation at $b$ if and only if

$$\sum_{n=1}^{\infty} 2^{nq} \Gamma_q(A_n(b) \setminus X) < +\infty.$$  

Here $q = p/(p-1)$ is the conjugate index, and $\Gamma_q$ is a certain con-
denser capacity. When $p < 2$, $R^p(X)$ never admits a continuous point
evaluation at $b$, unless $b$ is an interior point of $X$.

1.5. Continuous point derivations. There are similar results about
the possibility that the $k$-th complex derivative $f \mapsto f^{(k)}$ may have
a continuous extension from $F_b$ to all of $F$. These involve the same
Wiener series as continuous point evaluations, except that the base 2 is
replaced by $2^{k+1}$. For instance, the $R^p(X)$ result (also due to Hedberg) involves the condition
\[ \sum_{n=1}^{\infty} 2^{(k+1)q} \Gamma_q(A_n(b) \setminus X) < +\infty. \]

The earliest such result was for the uniform closure of the rationals, and was due to Hallstrom [12].

1.6. Intrinsic capacities. The present author began to formalize the pairing of problems and capacities in his thesis [20]. He considered the limited context of uniform algebras on compact subsets of the plane. To each functor $X \mapsto F(X)$ that associates a uniform algebra to each compact $X \subset \mathbb{C}$, and subject to certain coherence assumptions, he associated a capacity
\[ \alpha(F, \cdot) : \mathcal{O} \to [0, +\infty), \]
where $\mathcal{O}$ is the topology of $\mathbb{C}$. He then proved a Capacity Uniqueness Theorem, which stated that the map $F \mapsto \alpha(F, \cdot)$ is injective on the set of such functors, i.e. the capacity determines the functor. The Local Capacity Uniqueness Theorem states that two functors $F$ and $G$ have $F(X) = G(X)$ for a given compact set $X$ if and only if the capacities $\alpha(F, \cdot)$ and $\alpha(G, \cdot)$ agree on all open subsets of the complement of $X$.

Thus $F(X) \supseteq G(X)$ is a problem for which there are two capacities, not one! Vitushkin’s theorem on rational approximation is the case when $F(X)$ is the uniform closure of the rational functions having poles off $X$ and $G(X)$ is the algebra of all functions continuous on $X$ and holomorphic on $\hat{X}$. This part of the thesis is unpublished, mainly because the main result is essentially equivalent to a theorem of Davie [6], obtained independently. In another unpublished chapter, the author established that the results of Melnikov and Hallstrom extended to all these $F(X)$, replacing the analytic capacity by $\alpha(F, \cdot)$.

Other work by Wang [31] and the author [21, 22] established a link between equicontinuous pointwise Hölder conditions at a boundary point and series in which 2 is replaced by $2^\lambda$ for a nonintegral $\lambda > 1$.

Moving on from the uniform norm, the author considered parallel questions for Lipschitz or Hölder norms. Building on a result of Dolzhenko, he established that the equivalent of continuous analytic capacity is the lower $\beta$-dimensional Hausdorff content $M_\alpha^\beta$, with $\beta = \alpha + 1$. (For $\beta > 0$, the $\beta$-dimensional Hausdorff content $M_\beta(E)$ of a set $E \subset \mathbb{R}^d$ is defined to be the infimum of the sums $\sum_{n=0}^{\infty} r_n^\beta$, taken over all countable coverings of $E$ by closed balls $(B(a_n, r_n))_n$. If we replace $r_n^\beta$ by $h(r_n)$ for an increasing function $h : [0, +\infty) \to [0, +\infty)$ we get
the Hausdorff $h$-content $M_h(E)$. The lower $\beta$-dimensional Hausdorff content $M^\beta_h(E)$ is defined to be the supremum of $M_h(E)$, taken over all $h$ such that $0 \leq h(r) \leq r^\beta$ for all $r > 0$, and $r^{-\beta/h(r)} \to 0$ as $r \downarrow 0$.) He proved [19] the analogue of Vitushkin’s theorem for rational approximation. Later, Lord and he [17] proved the analogue of Hallstrom’s theorem for boundary derivatives. For the $k$-th derivative, this involved the series condition

$$\sum_{n=1}^{\infty} 2^{(k+1)n} M^{\alpha+1}_h(A_n(b) \setminus X) < +\infty.$$  

1.7. SCS. Moving to a more general context, the author introduced the notion of a Symmetric Concrete Space $F$ on $\mathbb{R}^d$, and considered the relation between problems about a given such space $F$, in combination with an elliptic operator $L$, and an appropriate associated capacity, the $L$-$F$-cap. A Symmetric Concrete Space (SCS) on $\mathbb{R}^d$ is a complete locally-convex topological vector space $F$ over the field $\mathbb{C}$, such that

- $\mathcal{D} \hookrightarrow F \hookrightarrow \mathcal{D}^*$;
- $F$ is a topological $\mathcal{D}$-module under the usual product $\phi \cdot f$ of a test function and a distribution;
- $F$ is closed under complex conjugation;
- The affine group of $\mathbb{R}^d$ acts by composition on $F$, and each compact set of affine maps gives an equicontinuous family of composition operators.

Here $\mathcal{D} = C^\infty_c(\mathbb{R}^d, \mathbb{C})$, is the space of test functions and $\mathcal{D}^*$ is its dual, the space of distributions, $A \hookrightarrow B$ stands for “$A \subset B$ and the inclusion map is continuous”. (In fact, it is elementary that if $A$ and $B$ are metrizable SCS, then $A \subset B$ implies $A \hookrightarrow B$.)

An SCS is a Symmetric Concrete Banach Space (SCBS) when it is normable and is equipped with a norm.

We shall be concerned only with the case $d = 2$, and we identify $\mathbb{R}^2$ with $\mathbb{C}$.

The various analytic capacities are $\frac{\partial}{\partial z} F$-cap for particular $F$. He planned a book about this subject, but this project has never been completed. Some extracts with useful ideas and results were published.

The most useful ideas concern localness. For SCS $F$ and $G$, We define

$$F_{\text{loc}} := \{ f \in \mathcal{D} : \phi \cdot f \in F, \forall \phi \in \mathcal{D} \},$$
$$F_{\text{cs}} := \{ \phi \cdot f \in F : \phi \in \mathcal{D} \},$$
$$F_{\text{loc}} \hookrightarrow G \iff F_{\text{loc}} \hookrightarrow G_{\text{loc}},$$
$$F_{\text{loc}} = G \iff F_{\text{loc}} = G_{\text{loc}}.$$
and observe that
\[ F_{\text{loc}} = F_{\text{loc}} = F_{\text{cs}}. \]

Published results include the following:

1. A Fundamental Theorem of Calculus for SCS that are weakly-locally invariant under Calderon-Zygmund operators \cite{25, Lemma 12}. This says that
\[ D \int F_{\text{loc}} = \int D F_{\text{loc}} = F, \]
where
\[ D F := D + \text{span} \left\{ \frac{\partial f}{\partial x_j} : 1 \leq j \leq d, f \in F \right\} \]
and
\[ \int F := \left\{ f \in \mathcal{D}' : \frac{\partial f}{\partial x_j} \in F, \text{ for } 1 \leq j \leq d \right\}. \]

2. A 1-reduction principle that allows us to establish equivalences between problems for different operators \( L \) \cite{25, Theorem 1}. The identity operator \( 1 : f \mapsto f \) is elliptic. If \( U \) is open, then the equation \( 1f = 0 \) on \( U \) just means that \( U \cap \text{spt}(f) = \emptyset \). The idea is to reduce questions about \( L \) and some space \( F \) to equivalent problems about \( 1 \) and the space \( LF \).

3. A general Sobolev-type embedding theorem \cite{24} involving the concept of the order of an SCS, and

4. A theorem that says that in dimension two all SCS are essentially (technically, weakly-locally-) \( T \)-invariant \cite{26}, i.e. invariant under the Vitushkin localization operators (see Section 5 below).

In 1990 he circulated a set of notes on the concept of SCS and the main examples. Some ideas from these papers were expounded by Tarkhanov in his book on the Cauchy Problem for Solutions of Elliptic Equations \cite{29, Chapter 1}.

The general point of view raised many particular questions, and some of these have been solved, while other loose ends remain.

2. The Problem

Our objective in the present paper is to address a loose end connected to the results on boundary behaviour of holomorphic functions mentioned above. For \( 0 < \alpha < 1 \), the \( \frac{\partial}{\partial \bar{z}} F \)-cap associated to the Lipschitz class \( \text{Lip}_\alpha \) is \( M^{\alpha+1} \), and that associated to the little Lipschitz class \( \text{lipo} \) is \( M_*^{\alpha+1} \). Kaufmann \cite{15} showed that \( M^1 \) is the \( \frac{\partial}{\partial \bar{z}} \)-BMO-cap
and Verdera \[30\] established that \( M^1 \) is the \( \frac{\partial}{\partial \bar{z}} \)-VMO-cap, the capacity associated to the space of functions of vanishing mean oscillation. Verdera proved the Vitushkin theorem for VMO.

The question is, what do \( M^\beta \) and \( M^\beta_\ast \) have to do with the boundary behaviour of analytic functions when \( 0 < \beta < 1 \)? What is the significance of the condition

\[
\sum_{n=1}^{\infty} 2^n M^\beta(A_n \setminus U) < +\infty,
\]

when \( 0 < \beta < 1 \), where \( U \) is a bounded open subset of \( \mathbb{C} \) and \( b \in \partial U \)?

We are considering a local problem, and it is worth noting that there are several different meanings commonly attached to the global Lipschitz classes, and the little Lipschitz classes. For \( 0 < \alpha < 1 \), we define \( \text{Lip}_\alpha(\mathbb{R}^d) \) to be the space of bounded functions \( f : \mathbb{R}^d \to \mathbb{C} \) that satisfy a Lipschitz-alpha condition:

\[
|f(x) - f(y)| \leq \kappa_f |x - y|^\alpha, \quad \forall x, y \in \mathbb{R}^d.
\]

We would obtain a locally-equivalent Banach SCS if we omit the word ‘bounded’. We would also obtain a locally-equivalent SCBS if we just require the Lipschitz condition for \( |x - y| \leq 1 \). Another locally-equivalent space is obtained by requiring the Lipschitz condition with respect to the spherical metric (associated to the stereographic projection \( S^d \to \mathbb{R}^d \)). We shall shortly meet another locally-equivalent space, defined in terms of the Poisson transform. It makes no difference for our problem which of these versions is used, and we can exploit this fact by choosing whatever version is easiest to use in each context. For this paper, we define \( \text{lip}_\alpha \) to be the closure of \( \mathcal{D} \) in \( \text{Lip}_\alpha \). This space is locally-equivalent to the space of functions that have restriction in \( \text{lip}(\alpha, X) \) for each compact \( X \subset \mathbb{R}^d \), but it has an additional property ‘near \( \infty \)’, irrelevant for our purposes.

3. Results

3.1. The spaces \( T_s \) and \( C_s \). The answer to the problem will not surprise anyone who has studied the paper \[25\], but may be regarded as rather strange by others.

The first step in trying to identify the \( L-F \)-cap for given \( L \) and \( F \) is based on the principle that the compact sets \( X \subset \mathbb{R}^d \) having \( (L-F\text{-cap})(X) = 0 \) should be the sets of removable singularities for solutions of \( Lf = 0 \) of class \( F \). This means that \( (L-F\text{-cap})(X) = 0 \) should be
the necessary and sufficient condition that the restriction map
\[ \{ f \in F : Lf = 0 \text{ on } U \} \to \{ f \in F : Lf = 0 \text{ on } U \setminus X \} \]
be surjective for each open set \( U \subset \mathbb{R}^d \).

In [25, p.140] it was established (as a special case of the 1-reduction principle) that for nonintegral \( \beta \) the set function \( M^\beta \) is zero on the sets of removable singularities for holomorphic functions of a Lipschitz class, but when \( 0 < \beta < 1 \) this is a negative Lipschitz class, there denoted \( T_{\beta - 1} \).

The negative Lipschitz classes can be described in a number of equivalent ways. In informal terms, The basic idea is that the \( T_s \) for \( s \in \mathbb{R} \) form a one-dimensional ‘scale’ of spaces of distributions, i.e. a family of spaces totally-ordered under local inclusion. When \( 0 < s < 1 \), \( T_s \) is locally-equal to Lips. Differentiation takes \( T_s \) down to \( T_{s-1} \), and \( DT_s \) is locally-equal to \( T_{s-1} \). Thus if \( s < 0 \) and \( k \in \mathbb{N} \) has \( s + k > 0 \), then \( T_s \) is locally-equal to \( D^k\text{Lip}(s + k) \). The elements of \( T_s \) having compact support may also be characterised by the growth of the Poisson transform as you approach the plane from the upper half of \( \mathbb{R}^3 \), or by the growth of the convolution with the heat kernel. This idea originated in the work of Littlewood and Paley and was fully developed by Taibleson [28, Chapter 5]. The Poisson kernel is
\[ P_t(z) := \frac{t}{\pi (t^2 + |z|^2)^{\frac{3}{2}}} , \quad (t > 0, z \in \mathbb{C}) . \]
It is real-analytic in \( z \) and \( t \), and is harmonic in \((z,t)\) in the upper half-space
\[ \mathbb{H}^3 := \mathbb{C} \times (0, +\infty) . \]
For a distribution \( f \in \mathcal{E}^* := C^\infty(\mathbb{C}, \mathbb{C})^* \) having compact support, the Poisson transform of \( f \) is the convolution
\[ F(z, t) := (P_t \ast f)(z) \]
where \( P_t \ast f \) denotes the convolution on \( \mathbb{C} = \mathbb{R}^2 \). For \( s < 0 \) we say (following [25]) that \( f \) belongs to the ‘negative Lipschitz space’ \( T_s \) if
\[ \| f \|_s := \sup \{ t^{-s} |F(z, t)| : z \in \mathbb{C}, t > 0 \} < +\infty , \]
and belongs to the ‘small negative Lipschitz space \( C_\beta \) if, in addition,
\[ \lim_{t \downarrow 0} t^{-s} \sup_{\mathbb{C}} |F(z, t)| = 0 . \]

For \( s \geq 0 \), we define \( T_s \) and \( C_s \) by requiring that for \( f \in \mathcal{E}^* \), \( f \in T_s \) (respectively \( C_s \)) if and only if all \( k \)-th order partial derivatives of \( f \) belong to \( T_{k-s} \) (respectively \( C_{s-k} \)) for each (or, equivalently, for some) integer \( k > s \).
The Riesz transforms\footnote{Strictly speaking the order $t$ Riesz transform is the operator $(-\Delta)^{-t/2}$, which for $t > 0$ is convolution with $c_t |z|^{t-d}$, for a certain constant $c_t$ \cite[p 117]{())). map $T_s$ locally into $T_{s+t}$, so behave like ‘fractional integrals’.

The scale corresponding to the little Lipschitz class $C_s$ may be described as the closure of the space $\mathcal{D}$ in $T_s$.

Delicate questions arise at integral values $s$, and we shall not consider such $s$ in this paper.

3.2. Statements. For an open set $U \subset \mathbb{C}$, and $s \in \mathbb{R}$, let

$$A^s(U) := \{ f \in C_s : f \text{ is holomorphic on } U \},$$

and

$$B^s(U) := \{ f \in T_s : f \text{ is holomorphic on } U \}.$$

We are interested in the range $-1 < s < 0$, and for such $s$ the elements of $A^s(U)$ and $B^s(U)$ are distributions on $\mathbb{C}$ that may fail to be representable by integration against a locally-$L^1$ function, so the definition of continuous point evaluation given above does not apply. However, we can make a straightforward adjustment.

We shall prove the following lemma:

Lemma 1. For each $s \in \mathbb{R}$, each open set $U \subset \mathbb{C}$ and each $b \in \mathbb{C}$, the set $\{ f \in A^s(U) : f \text{ is holomorphic on some neighbourhood of } b \}$ is dense in $A^s(U)$.

Here, when we say that the distribution $f$ on $\mathbb{C}$ is holomorphic on an open set $V$, we mean that its distributional $\bar{\partial}$-derivative has support off $V$, i.e.

$$\left\langle \phi, \frac{\partial f}{\partial \bar{z}} \right\rangle := -\left\langle \frac{\partial \phi}{\partial \bar{z}}, f \right\rangle = 0$$

whenever the test function $\phi$ has support in $V$. Recall that by Weyl’s Lemma this means that the restriction $f|V$ is represented by an ordinary holomorphic function, so that it and all its derivatives have well-defined values throughout $V$.

Let us denote

$$A^s_b(U) := \{ f \in A^s(U) : f \text{ is holomorphic on some neighbourhood of } b \}.$$

Definition 1. We say that $A^s(U)$ admits a continuous point evaluation at a point $b \in \mathbb{C}$ if the functional $f \mapsto f(b)$ extends continuously from $A^s_b(U)$ to the whole of $A^s(U)$.

Our main result is this:
Theorem 1. Let $0 < \beta < 1$ and $s = \beta - 1$. Let $U \subset \mathbb{C}$ be a bounded open set, and $b \in \partial U$. Then $A^s(U)$ admits a continuous point evaluation at $b$ if and only if
\[
\sum_{n=1}^{\infty} 2^n M_\beta(A_n \setminus U) < +\infty,
\]

3.3. Weak-star continuous evaluations. We can also give a result about the big Lipschitz class $B^s(U)$. We cannot replace $A^s(U)$ by $B^s(U)$ in Lemma 1 as it stands. However, the $T_s$ spaces are dual spaces, and so have a weak-star topology (see Subsection 5.2 for details), and restricting this topology gives us a second useful topology on $B^s(U)$. We have the following:

Lemma 2. For each $s \in \mathbb{R}$, each open set $U \subset \mathbb{C}$ and each $b \in \mathbb{C}$, the set $\{ f \in B^s(U) : f$ is holomorphic on some neighbourhood of $b \}$ is weak-star dense in $B^s(U)$.

Denoting
\[ B^s_b(U) := \{ f \in B^s(U) : f$ is holomorphic on some neighbourhood of $b \}, \]
we can then state a definition:

Definition 2. We say that $B^s(U)$ admits a weak-star continuous point evaluation at a point $b \in \mathbb{C}$ if the functional $f \mapsto f(b)$ extends weak-star continuously from $B^s_b(U)$ to the whole of $B^s(U)$.

Our result for $B^s(U)$ is this:

Theorem 2. Let $0 < \beta < 1$ and $s = \beta - 1$. Let $U \subset \mathbb{C}$ be a bounded open set, and $b \in \partial U$. Then $B^s(U)$ admits a weak-star continuous point evaluation at $b$ if and only if
\[
\sum_{n=1}^{\infty} 2^n M_\beta(A_n \setminus U) < +\infty,
\]

3.4. Boundary derivatives. In the same spirit, we get results about boundary derivatives:

Theorem 3. Let $0 < \beta < 1$, $s = \beta - 1$, and let $k \in \mathbb{N}$. Let $U \subset \mathbb{C}$ be a bounded open set, and $b \in \partial U$. Then the functional $f \mapsto f^{(k)}(b)$ has a continuous extension from $A^s_b(U)$ to the whole of $A^s(U)$ if and only if
\[
\sum_{n=1}^{\infty} 2^{(k+1)n} M_\beta(A_n \setminus U) < +\infty,
\]
Theorem 4. Let $0 < \beta < 1$, $s = \beta - 1$, and let $k \in \mathbb{N}$. Let $U \subset \mathbb{C}$ be a bounded open set, and $b \in \partial U$. Then the functional $f \mapsto f^{(k)}(b)$ has a weak-star continuous extension from $B^s_b(U)$ to the whole of $B^s(U)$ if and only if
\[
\sum_{n=1}^{\infty} 2^{(k+1)n} M^\beta(A_n \setminus U) < +\infty.
\]

The spaces $A^s(U)$ are not algebras — essentially SCS are only algebras when they are locally-included in $C^0$ — so we avoid using the term derivation, lest we confuse people.

3.5. Harmonic functions. The foregoing results concern objects $f$ that are not ‘proper functions’. Using the ideas related to 1-reduction, we may derive a theorem about ordinary harmonic functions:

For $0 < \alpha < 1$, let $H^\alpha(U)$ denote the space of (complex-valued) functions that are harmonic on $U$ and belong to the little Lipschitz $\alpha$ class on the closure of $U$ (or, equivalently, have an extension belonging to the global little Lipschitz class). For $b \in \partial U$, let
\[
H^\alpha(U)_b := \{ h \in H^\alpha(U) : h \text{ is harmonic on a neighbourhood of } b \}.
\]

If we denote, as is usual,
\[
\frac{\partial}{\partial z} := \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)
\]
and
\[
\frac{\partial}{\partial \bar{z}} := \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right),
\]
then $\Delta = 4 \frac{\partial}{\partial z} \frac{\partial}{\partial \bar{z}}$.

Theorem 5. Let $0 < \alpha < 1$, let $U \subset \mathbb{C}$ be bounded and open, and $b \in \partial U$. Then
(1) $H^\alpha(U)_b$ is dense in $H^\alpha(U)$.
(2) The functional $h \mapsto \frac{\partial h}{\partial z}(b)$ extends continuously from $H^\alpha(U)_b$ to $H^\alpha(U)$ if and only if
\[
\sum_{n=1}^{\infty} 2^n M^\alpha(A_n \setminus U) < +\infty,
\]
(3) The $C^2$-valued function $h \mapsto (\nabla h)(b)$ extends continuously from $H^\alpha(U)_b$ to $H^\alpha(U)$ if and only if
\[
\sum_{n=1}^{\infty} 2^n M^\alpha(A_n \setminus U) < +\infty,
\]
4. Tools

We abbreviate $\|f\|_{T_s}$ to $\|f\|_s$.

We use $K$ to denote a positive constant which is independent of everything but the value of the parameter $s$, and which may be different at each occurrence.

4.1. The strong module property. For a nonnegative integer $k$, and $\phi \in \mathcal{D}$, we use the notation

$$N_k(\phi) := d(\phi)^k \cdot \sup |\nabla^k(\phi)|,$$

where $d(\phi)$ denotes the diameter of the support of $\phi$. Here, we take the norm $|\nabla^k\phi|$ to be the maximum of all the $k$-th order partial derivatives of $\phi$.

Note that for $\kappa > 0$, $N_k(\kappa \cdot \phi) = \kappa \cdot N_k(\phi)$, that

$$N_0(\phi) \leq N_1(\phi) \leq N_2(\phi) \leq \cdots,$$

and that (by Leibnitz’ formula)

$$N_k(\phi \cdot \psi) \leq 2^k N_k(\phi) N_k(\psi)$$

whenever $\phi, \psi \in \mathcal{D}$ and $k \in \mathbb{N}$.

Also, $N_k(\phi)$ is invariant under rescaling: If $\phi \in \mathcal{D}$, $r > 0$, and $\psi(x) := \phi(r \cdot x)$, then $N_k(\psi) = N_k(\phi)$ for each $k$.

An SCBS $F$ has the (order $k$-) strong module property if there exists $k \in \mathbb{Z}_+$ and $K > 0$ such that

$$\|\phi \cdot f\|_F \leq K \cdot N_k(\phi) \cdot \|f\|_F, \ \forall f \in F.$$ 

Most common SCBS have this property. Note that for every SCBS and each $\phi \in \mathcal{D}$ the fact that $F$ is a topological $\mathcal{D}$-module tells us that there is some constant $K(\phi)$ such that

$$\|\phi \cdot f\|_F \leq K(\phi) \|f\|_F, \ \forall f \in F.$$ 

Thus the strong module property amounts to saying that the least $K(\phi)$ are dominated by some $N_k(\phi)$, up to a fixed multiplicative constant.

It is readily seen that the ordinary Lipschitz spaces have the order 1 strong module property.

4.2. Standard pinchers. If $b \in \mathbb{R}^d$, a standard pincher at $b$ is a sequence of nonnegative test functions $(\phi_n)_n$, such that $\phi_n = 1$ on a neighbourhood of $b$, the diameter of $\text{spt}(\phi_n)$ tends to zero, and for each $k \in \mathbb{N}$, the sequence $(N_k(\phi_n))_n$ is bounded.

The elements $\phi_n$ of a standard pincher make the transition from the value 1 at $b$ down to zero in a reasonably gentle way, so that the
various derivatives are not greatly larger than they have to be in order to achieve the transition.

It is easy to see that such sequences exist. For instance, they may be constructed in the form \( \phi_n(x) = \psi(|x-b|) \), where \( \psi : [0, +\infty) \to [0, 1] \) is \( C^\infty \), has \( \psi = 1 \) near 0 and \( \psi = 0 \) off \([0, 1]\).

4.3. The Cauchy transform. The Cauchy transform is the convolution operator

\[
\mathcal{C}f := \frac{1}{\pi \bar{z}} * f.
\]

It acts (at least) on distributions having compact support, and it almost inverts the \( \partial \bar{z} \) operator:

\[
\frac{\partial}{\partial \bar{z}} \mathcal{C}f = f = \mathcal{C} \frac{\partial f}{\partial \bar{z}},
\]

whenever \( f \in \mathcal{E}^\ast \). It maps \( (T_s)_{cs} \) continuously into \( T_{s+1} \) and \( (C_s)_{cs} \) into \( C_{s+1} \), so in combination with \( \frac{\partial}{\partial \bar{z}} \) it can be used to relate properties of \( T_s \) to properties of \( T_{s+1} \). For our present purposes, this allows us to move from our spaces of distributions corresponding to \(-1 < s < 0\) to spaces of ordinary \( \text{Lip}(s + 1) \) functions.

The Cauchy kernel \( \frac{1}{\pi \bar{z}} \) does not belong to \( L^1 \), but it does belong to \( L^1_{\text{loc}} \), and indeed there is a uniform bound on its norm on discs of fixed radius:

\[
\left\| \frac{1}{\pi \bar{z}} \right\|_{L^1(\mathbb{B}(a, r))} \leq 2r, \forall a \in \mathbb{C}, \forall r > 0.
\]

So if \( F \) a SCBS, and translation acts isometrically on \( F \), then

\[
\|\mathcal{C}f\|_F \leq d\|f\|_F,
\]

whenever \( f \in F_{cs} \) is supported in a disc of radius \( d \).

4.4. Evaluating the Cauchy transform. The value of \( (\mathcal{C}f)(b) \) at a point off the support of the distribution \( f \) may be evaluated in the obvious way:

**Lemma 3.** Let \( f \in \mathcal{E}' \), and \( b \in \mathbb{C} \setminus \text{spt}(f) \). Let \( \chi \in \mathcal{D} \) be any test function having \( \chi(z) = 1/(z-b) \) near \( \text{spt}(f) \). Then

\[
\mathcal{C}(f)(b) = \left\langle \frac{\chi}{\pi}, f \right\rangle.
\]

**Proof.** We take \( b = 0 \), without loss in generality.

For any \( \psi \in \mathcal{D} \) with \( \int \psi dm = 1 \) and \( \text{spt}(\psi) \cap \text{spt}(f) = \emptyset \), we have

\[
\left\langle \psi, \mathcal{C}(f) \right\rangle = -\left\langle \psi * \frac{1}{\pi z}, f \right\rangle.
\]
Let $(\phi_n)_n$ be a standard pincher at 0 and take 
\[ \psi_n = \frac{\phi_n}{\int \phi_n dm}. \]
Then $\psi_n \ast \frac{1}{\pi z} \to \frac{1}{\pi z} = \chi/\pi$ in $C^\infty$ topology on a neighbourhood of $\text{spt}(\phi)$, so 
\[ \langle \frac{\chi}{\pi}, f \rangle = \lim_n \langle \psi_n \ast \frac{1}{\pi z}, f \rangle = -\lim_n \langle \psi_n, \mathcal{C}(f) \rangle = (\mathcal{C}(f)(0)). \]

4.5. **The Vitushkin localization operator.** The Vitushkin localization operator is defined by 
\[ \mathcal{I}_\phi(f) := \mathcal{C} \left( \phi \cdot \frac{\partial f}{\partial \bar{z}} \right). \]
Here $\phi \in D$ and $f \in D^*$. 
In view of the distributional equation 
\[ \frac{\partial}{\partial \bar{z}} \mathcal{I}_\phi(f) = \phi \cdot \frac{\partial f}{\partial \bar{z}}, \]
$\mathcal{I}_\phi(f)$ is holomorphic wherever $f$ is holomorphic and off the support of $\phi$.

It was established in [26] (using soft general arguments) that whenever $F$ is an SCS, $\mathcal{I}_\phi$ maps $F$ continuously into $F_{\text{loc}}$, and that when $F$ is an SCBS, we actually get a continuous map into the Banach subspace $F_\infty \subset F_{\text{loc}}$ normed by 
\[ \|f\|_{F_\infty} := \sup \{ \|f|B\|_{F(B)} : B \text{ is a ball of radius 1} \}, \]
where $f|B$ denotes the restriction coset $f + J(F, B)$, with $J(F, B)$ equal to the space of all elements $g \in F$ that vanish near $B$, and the $F(B)$ norm of a restriction is the infimum of the $F$ norms of all its extensions in $F$, i.e.
\[ \|f|B\|_{F(B)} := \inf \{ \|h\|_F : h \in F, h - f = 0 \text{ near } B \}. \]

When $F$ is an SCBS with the strong module property (of order $k$), and translation acts isometrically on $F$, the identity 
\[ \mathcal{I}_\phi f = \phi \cdot f - \mathcal{C} \left( \frac{\partial \phi}{\partial \bar{z}} \cdot f \right) \]
together with equation (1) yields the more precise estimate (2) 
\[ \|\mathcal{I}_\phi f\|_F \leq KN_{k+1}(\phi) \cdot \|f\|_F, \ \forall \phi \in D, \forall f \in F. \]
4.6. Our spaces $T_s$.

**Lemma 4.** Let $k \in \mathbb{N}$ and $-k - 1 < s < -k$. Then $(T_s)_\infty$ has the order $(k + 1)$ strong module property, and in fact

$$\|\phi \cdot f\|_s \leq K \cdot N_{k+1}(\phi) \cdot \|f\|_s,$$

for all $\phi \in \mathcal{D}$ with $d(\phi) \leq 1$ and all $f \in T_s$, where $K > 0$ is independent of $\phi$ and $f$.

**Proof.** We use induction on $k$, starting with $k = -1$.

For $0 < s < 1$, Lip$(s)$ has the strong module property of order 1, and since $T_s$ is locally-equal to Lip$(s)$, we have the result in this case.

Now suppose it holds for some $k$, and fix $s \in (-k - 2, -k - 1)$.

It suffices to prove the estimate for $\phi$ supported in $B(0, 2)$, and multiplying $f$ by a fixed test function $\psi$ that equals 1 on $B(0, 2)$, we may assume that $f$ has compact support (without changing $\phi \cdot f$ or increasing $\|f\|_s$ by more than a fixed constant that depends only on $\psi$ and $s$).

Then $g := \mathcal{C}f \in T_{s+1}$ has $\frac{\partial g}{\partial \bar{z}} = f$ and $\|g\|_{s+1} \leq K\|f\|_s$. Then

$$\|\phi \cdot f\|_s \leq K\|\mathcal{C}(\phi f)\|_{s+1} = K\|\mathcal{I}_\phi(g)\|_{s+1} \leq K \cdot N_{k+2}\|g\|_{s+1, \infty},$$

since $(T_{s+1})_\infty$ has the strong module property. Thus

$$\|\phi \cdot f\|_s \leq K \cdot N_{k+2}\|g\|_{s+1, \infty} \leq K \cdot N_{k+2} \cdot \|f\|_s$$

Hence the result holds for $k + 1$, completing the induction step. □

**Remark 1.** A similar result holds for all $s$, but for positive $s$ one has to replace $\|g\|$ by $\|g - p\|$, where $p$ is the degree $\lfloor s \rfloor$ Taylor polynomial of $g$ about $a$.

4.7. The $C_s$ norm on small discs. Since $\mathcal{C}$ maps $C_{sts}$ into $C_{s+1}$, induction also gives the following:

**Lemma 5.** Let $s < 0$, $f \in C_s$ and $a \in \mathbb{C}$. Then for each $\epsilon > 0$ there exists $r > 0$ and $g \in C_s$ such that $f = g$ on $\mathbb{B}(a, r)$ and $\|g\|_{T_s} < \epsilon$. □

**Remark 2.** We note that since $\mathcal{I}_\phi(f)$ is holomorphic off the support of $\phi$ and has a zero at $\infty$, $\mathcal{I}_\phi$ maps $C_s$ into $C_s$.

4.8. Estimate for $\langle \phi, f \rangle$. The strong module property gives an estimate for the action of $f \in F$ on a given $\phi \in \mathcal{D}$:

**Lemma 6.** Suppose $F$ is an SCBS with the order $k$ strong module property. Then for each compact $X \subset \mathbb{C}$, there exists $K > 0$ such that

$$|\langle \phi, f \rangle| \leq K \cdot N_k(\phi) \cdot \|f\|_F,$$

whenever $\phi \in \mathcal{D}$, $f \in F$ and $\text{spt}(\phi \cdot f) \subset X$. 

Proof. Fix $\chi \in D$ with $\chi = 1$ near $X$. Then since $f \mapsto \langle \chi, f \rangle$ is continuous there exists $K > 0$ such that

$$|\langle \chi, f \rangle| \leq K \cdot \|f\|_F.$$  

Thus

$$|\langle \phi, f \rangle| = |\langle \chi, \phi \cdot f \rangle| \leq K \cdot N_k(\phi) \cdot \|f\|_F.$$

□

Putting it another way, the order $k$ strong module property says that the operator $f \mapsto \phi \cdot f$ on $F$ has operator norm dominated by $N_k(\phi)$, and this last lemma says that the functional $f \mapsto \langle \phi, f \rangle$ has $F(X)^*$ norm dominated by $N_k(\phi)$.

It turns out that we can improve substantially on this estimate, for our particular spaces. The trick is to pay close attention to the support of $\phi \cdot f$.

4.9. Scaling: better estimate. The action of an affine map $A : \mathbb{R}^d \to \mathbb{R}^d$ on distributions is defined by

$$\langle \phi, f \circ A \rangle := |A|^{-1} \langle \phi \circ A^{-1}, f \rangle, \quad \forall \phi \in D.$$  

In the case of a dilation $A(z) = r \cdot z$ on $\mathbb{C}$, this means that

$$\langle \phi, f \rangle = r^2 \langle \phi \circ A, f \circ A \rangle.$$  

Taking into account the fact that $N_k(\phi) = N_k(\phi \circ A)$ whereas the identity

$$(P_t \ast f)(z) = (P_{t/r} \ast (f \circ A))(rz)$$

gives

$$\|f \circ A\|_s = r^s \|f\|_s,$$

we obtain:

**Lemma 7.** Let $-2 < s < 0$. Then

$$|\langle \phi, f \rangle| \leq K \cdot N_3(\phi) \cdot \|f\|_s \cdot r^{s+2},$$

whenever $\phi \in D$ and $\text{spt}(\phi \cdot f) \in \mathbb{B}(0, r)$. □

Note that for $f \in C_s$, the norm of $f$ in $T_s(\mathbb{B}(a,r))$ tends to zero as $r \downarrow 0$, so we can replace the constant $K$ in the estimate by $\eta(r)$, where $\eta$ depends on $f$, and $\eta(r) \to 0$ as $r \downarrow 0$. 

4.10. **Hausdorff content estimate.** Next, using a covering argument, we can bootstrap the estimate to:

**Lemma 8.** Let $-2 < s < 0$. Then

$$|\langle \phi, f \rangle| \leq K \cdot N_3(\phi) \cdot \|f\|_s \cdot M^{s+2}(\operatorname{spt}(\phi \cdot f)),$$

whenever $\phi \in \mathcal{D}$ and $f \in T_s$. \qed

Before giving the proof of this lemma, we need some preliminaries.

A **closed dyadic square** is a set of the form $I_{m,n} \times I_{r,n}$ (for integers $n, m, r$) where

$$I_{m,n} := \left\{ x \in \mathbb{R} : \frac{m}{2^n} \leq x \leq \frac{m+1}{2^n} \right\}.$$

Let $S_2$ denote the family of closed dyadic squares. For $\beta > 0$, the $\beta$-dimensional dyadic content of a set $E \subset \mathbb{R}^2$ is

$$M_2^\beta(E) := \inf \left\{ \sum_{n=1}^{\infty} \text{side}(S_n)^\beta : E \subset \bigcup_{n=1}^{\infty} S_n, S_n \in S_2 \right\}.$$

This content is comparable to $M^\beta$:

$$M^\beta(E) \leq 2^{\beta/2} M_2^\beta(E), \quad M_2^\beta(E) \leq 2^{\beta+2} M^\beta(E),$$

for all bounded sets $E$.

Thus it suffices to prove Lemma 8 with $M^{s+2}$ replaced by $M_2^{s+2}$ in the statement.

Also, since both sides change by a factor $r^2$ when $f$ and $\phi$ are replaced by $f(r \cdot)$ and $\phi(r \cdot)$, it suffices to consider the case when $E := \operatorname{spt}(\phi \cdot f)$ has diameter at most 1. Given $\epsilon > 0$, we may cover such an $E$ by a countable sequence $(S_n)_n$ of dyadic squares, of side at most 1, with

$$\sum_{n=1}^{\infty} \text{side}(S_n)^\beta < M_2^\beta + \epsilon.$$

We now state the key partition-of-identity lemma:

**Lemma 9.** Let $k \in \mathbb{N}$ be given. There exist positive constants $K$ and $\Lambda$ such that whenever $E \in \mathbb{R}^2$ is compact, and

$$E \subset \bigcup_{n=1}^{\infty} S_n,$$

where the $S_n$ are dyadic squares of side at most 1, there exists a sequence of test functions $(\phi_n)_n$ such that

1. $\phi_n = 0$ except for finitely many $n$;
2. $\sum_n \phi_n = 1$ on a neighbourhood of $E$;
Here ΛS denotes the square with the same centre as S and Λ times
the side. We shall show that Λ may be taken equal to 5, although we
do not claim this is sharp.

**Proof.** Rearrange the $S_n$ in nonincreasing order of size. The interiors of
the $\frac{5}{4}S_n$ form an open covering of $E$, so we may select a finite subcover,
$$
F := \left\{ \frac{5}{4}S_n : 1 \leq n \leq N \right\}.
$$
Remove all squares from the sequence $(S_n)_n$ that are contained in
$(\frac{5}{4}S_1) \setminus S_1$, re-number the remaining squares, and adjust $n$; then re-
move all in $(\frac{5}{4}S_2) \setminus S_2$, and so on. Now no element $S_n \in F$ is contained
in any square ‘cordon’ $(\frac{5}{4}S_m) \setminus S_m$.

Group the squares of $F$ into generations
$$
G_m := \{ S \in F : \text{side}(S) = 2^{-m} \}
$$
for $m = 0, 1, 2, \ldots$.

Each (finite) generation $G_m$ forms part of the tesselation $T_m$ of the
whole plane by dyadic squares of side $2^{-m}$. We can construct a uniform
partition of unity on the whole plane subordinate to the covering by
the open squares $\frac{5}{4}S$, with $S \in T$, as follows:

Choose $\rho \in C^\infty([0, +\infty))$ such that $\rho$ is nonincreasing, $\rho(r) = 1$ for
$0 \leq r \leq \frac{5}{8}$ and $\rho(r) = 0$ for $r \geq \frac{3}{4}$. Then define
$$
\theta(x, y) := \rho(x)\rho(y).
$$
For a dyadic square $S$ having centre $(a, b)$ and side 1, define $\theta_S(x, y) :=
\theta(x - a, y - b)$, and
$$
\tau := \sum_{S \in T_1} \theta_S.
$$
Then $\theta_S = 1$ on $\frac{5}{4}S$ and is supported on $\frac{3}{2}S$, so that $1 \leq \tau \leq 4$. Let
$$
\psi_S := \frac{\theta_S}{\tau}.
$$
Then the test functions $\psi_S$, for $S \in T_1$ form a partition of unity, and
c_k := N_k(\psi_S) is independent of $S$. (This partition is invariant under
translation by Gaussian integers.)

For general $m \in \mathbb{N}$, and a dyadic square $S$ of side $2^{-m}$, define
$\psi_S(z) := \psi_{2^mS}(2^m z)$. Then the $\psi_S$, for $S \in T_m$ also form a nonnegative
smooth partition of unity, $N_k(\psi_S) = c_k$ is independent of $S$ (and $m$),
the support of $\psi_S$ is contained in $\frac{3}{2}S$, hence at most 4 $\psi_S$ are nonzero
at any given point.
Note that
\[ |\nabla^k \psi_S| \leq \left( \text{diam} \frac{3}{2} S \right)^k c_k = \left( \frac{3}{\sqrt{2}} \right)^k \cdot (\text{side } S)^k \cdot c_k \]
for each \( k \in \mathbb{N} \).

For a dyadic square \( S \), let \( S^+ \) denote the set of 9 dyadic squares of the same size that meet \( S \). For any family \( \mathcal{H} \) of dyadic squares, let
\[ \mathcal{H}^+ := \bigcup \{ S^+ : S \in \mathcal{H} \}. \]
Thus \( S^{++} \) is a family of 25 squares. Observe that the smooth function
\[ \sum_{T \in S^+} \psi_T \]
is supported in \( \bigcup S^{++} = 5S \) and has sum identically 1 on \( \frac{3}{2} S \).

We now proceed to construct the desired collection of functions \((\phi_n)\).

Let
\[ \sigma_m := \sum_{S \in \mathcal{G}^+_m} \psi_S. \]
Then \( \sigma_m \) is supported in \( \bigcup \mathcal{G}^{++}_m \) and
\[ \sigma_m = 1 \text{ on } K_m := \bigcup_{S \in \mathcal{G}^*_m} \frac{3}{2} S. \]

Since at most 4 \( \psi_S \) are nonzero at any one point, we have
\[ |\nabla^k \sigma_m| \leq 4c_k \left( \frac{3}{\sqrt{2}} \right)^k \cdot 2^{km}, \forall k \in \mathbb{N}. \]

Now take the squares \( S \in \mathcal{G}^+_0 \), and allocate each one to a ‘nearest’ square \( n(S) \in \mathcal{G}^*_0 \) so that:
1. If \( S \in \mathcal{G}^*_0 \), take \( n(S) = S \);
2. If \( S \not\in \mathcal{G}^*_0 \), pick \( n(S) \) with \( S \in n(S)^+ \). (There may be up to eight ways to pick \( n(S) \). It does not matter which one you choose.)

Next, let
\[ \phi_T := \sum_{n(S)=T} \psi_S, \forall T \in \mathcal{G}_0. \]
Then each \( \phi_T \) is supported on \( 5T \), and
\[ \sigma_0 = \sum_{T \in \mathcal{G}_0} \phi_T. \]
Since the sum defining \( \phi_T \) has at most 9 terms, and its support has diameter at most 5 times that of \( T \), we have
\[ N_k(\phi_T) \leq 9 \cdot 5^k \cdot c_k. \]
Let \( \tau_0 := \sigma_0 \).
Next, consider $G_1$. As before, allocate each square $S \in G_1^+$ to a nearest square $n(S) \in G_1$, but this time let
\[ \phi_T := (1 - \tau_0) \sum_{n(S) = T} \psi_S, \quad \forall T \in G_1. \]
Then
\[ \sum_{T \in G_1} \phi_T = (1 - \tau_0) \sigma_1. \]
and
\[ \tau_1 := \tau_0 + (1 - \tau_0) \sigma_1 \]
is supported in $\bigcup (G_0^+ \cup G_1^+)$ and is identically equal to 1 on $K_0 \cup K_1$.

Continuing in this way, for $m \geq 1$ we allocate each square $S \in G_{m+1}^+$ to a nearest square $n(S) \in G_{m+1}$, and let
\[ \phi_T := (1 - \tau_m) \sum_{n(S) = T} \psi_S, \quad \forall T \in G_{m+1}, \]
and
\[ \tau_{m+1} := \tau_m + (1 - \tau_m) \sigma_{m+1}. \]
Then
\[ \sum_{T \in G_{m+1}} \phi_T = (1 - \tau_m) \sigma_{m+1} \]
and $\tau_{m+1} = 1$ on $K_0 \cup \cdots \cup K_{m+1}$.

When we have worked through all the nonempty generations $G_m$, we will have defined $\phi_{S_n}$ for each $S_n$, and (renaming $\phi_{S_n}$ as $\phi_n$) we have $\sum_n \phi_n = 1$ on the union of all the $\frac{3}{2}S_n$, and hence on a neighbourhood of $E$. Since $\phi_n$ is supported on $\frac{5}{2}S_n$, it remains to prove the estimate (3) of the statement, i.e. to prove that $\sup_n N_k(\phi_n) < +\infty$.

This amounts to showing that there is a constant $K > 0$ (depending on $k$) such that for $0 \leq m \in \mathbb{Z}$ and $S \in G_m$, we have
\[ |\nabla^k \phi_S| \leq K 2^{km}. \]
To do this, we start by proving that for each $k$ there exists $C = C_k > 0$ such that
\[ |\nabla^k \tau_m| \leq C \cdot 2^{km}, \quad \forall k \in \mathbb{N}. \]
for all $m \geq 0$.

To see this, we use induction on $k$ and on $m$, and the identity
\[ \tau_{m+1} = \sigma_{m+1} + (1 - \sigma_{m+1}) \tau_m, \]
together with the bound (3).

Let us write
\[ d_k := 4c_k \left( \frac{3}{\sqrt{2}} \right)^k, \]
so that (3) becomes $|\nabla^k \sigma_m| \leq d_k 2^{km}$. Since $\tau_0 = \sigma_0$, then for any $k$, we know that (4) holds for $m = 0$ as long as $C$ is at least $d_k$.

Take the case $k = 1$, and proceed by induction on $m$. If (4) holds for $k = 1$ and some $m$, then the identity (5) gives

$$|\nabla \tau_{m+1}| \leq d_1 2^{m+1} + |\nabla \tau_m| + |\nabla \sigma_{m+1}| \leq \left( d_1 + \frac{C}{2} + d_1 \right) 2^{m+1}.$$ 

Thus we get (4) with $m$ replaced by $m+1$, as long as $C \geq 4d_1$. This proves the case $k = 1$, with $C_1 = 4d_1$.

Now suppose that $k > 1$, and we have (4) with $k$ replaced by any number $r$ from 1 to $k - 1$ and $C$ replaced by some $C_r$. We proceed by induction on $m$. We have the case $m = 0$, with any constant $C \geq d_k$. Suppose we have the case $m$, with a constant $C$.

Using the identity, we can estimate $|\nabla^k \tau_{m+1}|$ by

$$d_k 2^{k(m+1)} + C \cdot 2^{km} + \sum_{j=1}^{k} \binom{k}{j} |\nabla^j \sigma_{m+1}| \cdot |\nabla^{k-j} \tau_m|.$$ 

This is no greater than

$$\left( \frac{C}{2} + R \right) \cdot 2^{k(m+1)},$$

where $R$ is an expression involving $d_1, \ldots, d_k$ and $C_1, \ldots, C_{k-1}$. So as long as $C > 2R$, we get (4) with $m$ replaced by $m+1$, and the induction goes through.

So we have (4) for all $k$ and $m$. It follows easily that for some $C > 0$ (depending on $k$) and for each $S \in G^+_m$ we have

$$|\nabla^k (1 - \tau_m) \cdot \psi_S| \leq C \cdot 2^{km},$$

and this gives

$$|\nabla^k \phi_S| \leq 9C \cdot 2^{km}$$

whenever $S \in G_m$, as required. \(\square\)

**Proof of Lemma 8.** With $E = \text{spt}(\phi \cdot f)$, take the partition of the identity $(\phi_n)$ just constructed, and note that $\phi = \sum_n \phi \cdot \phi_n$ on a neighbourhood of $E$. Thus

$$\langle \phi, f \rangle = \sum_{n=1}^{N} \langle \phi \cdot \phi_n, f \rangle.$$ 

Now apply Lemma 7 with $\phi$ replace by $\phi \cdot \phi_n$. The fact that $N_k$ is submultiplicative implies that $N_3(\phi \cdot \phi_n) \leq KN_3(\phi)$, so we get the stated result at once. \(\square\)
Further, using the remark about $\eta(r) \downarrow 0$, we get a stronger statement for elements of $C_s$:

**Lemma 10.** Let $-2 < s < 0$. Then

$$|\langle \phi, f \rangle| \leq K \cdot N_3(\phi) \cdot \|f\|_s \cdot M_s^{s+2}(\text{spt}(\phi \cdot f)),$$

whenever $\phi \in \mathcal{D}$ and $f \in C_s$. \hfill \Box

5. Proofs of preliminary lemmas

5.1. Proof of Lemma [1]

*Proof.* Fix $f \in A^s(U)$. Take some $\psi \in \mathcal{D}$ having $\psi = 1$ near $\overline{U}$. Then $f_1 := \psi \cdot f \in A^s(U)$, so we may write $f = f_1 + f_2$, where $f_2 \in C_s$ vanishes near $\overline{U}$, and hence is holomorphic near $b$. So it remains to show that we can approximate $f_1$ by elements of $A^s(U)_b$.

Now $f_1$ has compact support. Take a standard pincher $(\phi_n)_n$ at $b$. Take $g_n := T_{\phi_n}(f_1)$. Then $\|g_n\|_s \leq K \|f_1\|_s$. Since $T_{\phi_n}$ depends only on the restriction of $f_1$ to the support of $\phi_n$, an application of Lemmas 5 and 4 shows that $\|g_n\|_s \to 0$ as $n \uparrow \infty$. Thus $f_1 - g_n \to f_1$ in $T_s$ norm. Finally,

$$\frac{\partial}{\partial \overline{z}}(f_1 - g_n) = (1 - \phi_n) \frac{\partial f_1}{\partial \overline{z}},$$

so $f_1 - g_n$ is holomorphic on a neighbourhood of $b$, and so belongs to $A^s(U)_b$. \hfill \Box

5.2. Proof of Lemma [2] First, we have to explain about the weak-star topology.

The fact is that $T_s$ is essentially the double dual of $C_s$. More, it is a **concrete dual**: An SCS $F$ is called small if it is the closure of $\mathcal{D}$. If $F$ is a small SCS, then its dual $F^*$ is naturally isomorphic to an SCS, where the isomorphism is the restriction map $L \mapsto L|\mathcal{D}$. We call this SCS the concrete dual of $F$, and denote it by the same symbol $F^*$. Also $F_{loc}$ and $F_{cs}$ are also small,

$$(F_{loc})^* = (F^*)_{cs},$$

$$(F_{cs})^* = (F^*)_{loc},$$

and so

$$(F^*)^{\text{loc}} = (F_{loc})^{\text{loc}} = (F_{cs})^*.$$ 

In the case of $C_s$, for $0 < s < 1$, the concrete dual $C_s^*$ is also small, and we have

$$(C_s^*)_{\text{loc}} = T_s.$$ 

This fact is basically due to Sherbert, who observed the isomorphism

$$\text{lip}(\alpha, K)^{**} = \text{Lip}(\alpha, K)$$
for all compact metric spaces $K$. The key to this is the fact that for each $L \in \text{lip}(\alpha, K)^*$ that annihilates constants there exists a measure $\mu$ on $K \times K$, having no mass on the diagonal, such that

$$\int_{K \times K} \frac{f(x) - f(y)}{\text{dist}(x, y)^\alpha} d\mu(x, y),$$

whenever $f \in \text{lip}(\alpha, K)$. In particular, each point-mass at an off-diagonal point $(z, w) \in \mathbb{C} \times \mathbb{C}$ gives an element of the dual of $\text{lip}_\alpha(\mathbb{C})$:

$$Q(z, w)(f) := \frac{f(z) - f(w)}{|z - w|^{\alpha}}, \forall f \in \text{lip}_\alpha.$$

This might lead you to suspect that the dual is non-separable, but the norm topology on these functionals is not discrete. In fact, the map

$$P : \mathbb{C}^2 \setminus \text{diagonal} \to \text{lip}_\alpha^*$$

is continuous, and indeed locally Hölder-continuous: one may show that

$$||Q(z, w) - Q(z', w)||_{\text{lip}_\alpha^*} \leq \frac{4|z - z'|^{\alpha}}{|z - w|},$$

whenever $|z = z'| < \frac{1}{4}|z - w|$. Hence the functional $L$ on $\text{lip}_{\alpha\text{loc}}$ represented by a given measure $\mu$ may be approximated in the dual norm by finite linear combinations of elements from

$$\mathcal{P} := \{Q(z, w) : z \neq w\}.$$

By smearing the point masses, each functional $Q(z, w)$ may be approximated in the dual norm by functionals $\int_{\mathbb{C}} Q(z + \zeta, w + \zeta) \phi(\zeta) dm(\zeta)$, where $\phi \in \mathcal{D}$ has $\int \phi dm = 1$, which send an element $f \in \text{lip}(\alpha)$ to

$$\int_{\mathbb{C}} Q(z + \zeta, w + \zeta)(f) \cdot \phi(\zeta) dm(\zeta)$$

$$= \int_{\mathbb{C}} f(\omega) \left( \frac{\phi(\omega - z) - \phi(\omega - w)}{|z - w|^{\alpha}} \right) dm(\omega),$$

and the function

$$\omega \mapsto \frac{\phi(\omega - z) - \phi(\omega - w)}{|z - w|^{\alpha}}$$

is a test function, so the functional $L$ may be approximated by test functions. Thus $\text{lip}_{\alpha\text{loc}}^*$ has a concrete dual, and by Sherbert’s result this can only be $\text{Lip}_\alpha^{cs}$.

Moreover, it follows that a sequence in $\text{Lip}_\alpha$ is weak-star convergent to zero if and only if it is bounded in $\text{Lip}_\alpha$ norm and converges pointwise to zero on the span of $\mathcal{P}$. So in fact it suffices to show that it is bounded in norm and converges pointwise on $\mathbb{C}$. But we already know that if $(\phi_n)$ is a standard pincher, then, for $f \in \text{Lip}_\alpha$, $T_{\phi_n} f$ is bounded in $\text{Lip}_\alpha$.
norm and converges uniformly to zero, hence we conclude that $T_{\phi_n}f$ is weak-star convergent to zero.

This proves the lemma in case $0 < s < 1$.

For other nonintegral $s$, we obtain it by applying the Fundamental Theorem of Calculus. In particular, for the case of immediate interest, $-1 < s < 0$, we have that

$$(C_s)_{\text{loc}}^{**} \equiv (DC_{s+1})_{\text{loc}}^{**} \equiv \int T_{s+1}^{\text{loc}} = T_s,$$

so to show that, for $f \in T_s$, the sequence $(T_{\phi_n}f)$ converges weak-star in $T_s$, it suffices to show that $(T_{\phi_n}f)$ converges weak-star in $T_{s+1}$. We may assume that $f$ has compact support, since $(T_{\phi_n}f)$ depends only on the restriction of $f$ to a neighbourhood of $\text{spt} \phi$, and then taking $g = \mathcal{C}f \in T_{s+1}$, it suffices to show that $\mathcal{C}T_{\phi_n} \frac{\partial g}{\partial \overline{z}}$ converges weak-star to zero. But

$$\mathcal{C}T_{\phi_n} \frac{\partial g}{\partial \overline{z}} = \mathcal{C}^2 \left( \phi_n \cdot \frac{\partial^2 g}{\partial z^2} \right),$$

so we are just dealing with the equivalent of $\mathcal{F}_\phi$ for the d-bar-squared operator instead of the d-bar operator, so it is bounded on $\text{Lip} \alpha$ and on $C^0$, independently of $n$, and thus we have the desired weak-star convergence.

5.3. Remark. We expect that the argument of Subsection 5.2 may be used more generally, i.e we conjecture the following:

Let $F$ be a small SCBS, such that $F^*$ is also small, $F^{**} \hookrightarrow C^0$, and the span of the point evaluations is dense in $F^*$, and $F^{**}$ has the strong module property. Then whenever $(\phi_n)$ is a standard pincher, and $L$ is an elliptic operator with smooth coefficients,

$$L^{-1}(\phi_n \cdot Lf) \to 0 \text{ weak-star } \forall f \in F^{**}.$$  

Here, $L^{-1}$ denotes some suitably-chosen parametrix for $L$.

6. PROOFS OF THEOREMS

6.1. Proof of Theorem I. We fix $\beta \in (0, 1)$ and $s = \beta - 1$, and without loss in generality we assume that the boundary point $b = 0$.

First, consider the ‘only if’ direction. Suppose the series diverges:

$$\sum_{n=1}^{\infty} 2^n M^n_{s}(A_n \setminus U) = +\infty.$$  

We wish to show that there exist $f \in A^s(U)_b$ having $\|f\|_s \leq 1$ and $|f(0)|$ arbitrarily large.
Since $M^\beta$ is subadditive, there exists at least one of the four right-angle sectors

$$S_r := \{ z \in \mathbb{C} : |\arg(\bar{r} z)| < \frac{\pi}{4} \}$$

(for $r \in \{0, 1, 2, 3\}$) such that

$$\sum_{n=1}^{\infty} 2^n M^\beta_s((S_r \cap A_n) \setminus U) = +\infty.$$ 

We may assume that this happens for $r = 0$, and we may assume further that $U$ contains the whole complement of $S_0$ and the whole exterior of the unit disc. So we may select closed sets $E_n \subset S_0 \cap A_n$ such that $U \cap E_n = \emptyset$ and

$$\sum_{n=1}^{\infty} 2^n M^\beta_s(E_n) = +\infty.$$ 

We may select numbers $\lambda_n > 0$ such that the individual terms $\lambda_n 2^n M^\beta_s(E_n) \leq 1$, and yet

$$\sum_{n=1}^{\infty} \lambda_n 2^n M^\beta_s(E_n) = +\infty.$$ 

For each $n$, by Frostman’s Lemma, we may select a positive Radon measure supported on $E_n$ such that (1) $\mu_n(\mathbb{B}(a, r)) \leq r^\beta$ for all $a \in \mathbb{C}$ and all $r > 0$ (— i.e. $\mu_n$ ‘has growth $\beta$’—), (2) the total variation $||\mu_n|| \geq K \cdot M^\beta_s(E_n)$, and (3) $\mu_n(\mathbb{B}(a, r))/r^\beta \to 0$ uniformly in $a$ as $r \downarrow 0$. Then $\lambda_n 2^n ||\mu_n|| \leq 1$ and

$$\sum_{n=1}^{\infty} \lambda_n 2^n ||\mu_n|| = +\infty.$$ 

Let $h_n := \lambda_n \mathcal{C}(\mu_n)$. Then $h_n \in C_s$, $h_n$ is holomorphic off $\text{spt}(\mu_n)$, hence $h_n \in A^s(U)_0$. Also $\Re(h_n(0)) \geq \lambda_n \frac{2^n}{\sqrt{2 ||\mu_n||}}$. Hence

$$\left| \sum_{n=1}^{N} h_n(0) \right| \geq \frac{1}{\sqrt{2}} \sum_{n=1}^{N} \lambda_n 2^n ||\mu_n|| \to +\infty$$

as $N \uparrow \infty$. So now it suffices to show that $f_N := \sum_{n=1}^{N} h_n$ is bounded in $T_s$ norm, independently of $N \in \mathbb{N}$.

For this, it suffices to show that the $\partial / \partial \bar{z}$-derivatives

$$g_N := \frac{\partial}{\partial \bar{z}} f_N = \sum_{n=1}^{N} \lambda_n \mu_n$$
are bounded in $T_{k-1} = T_{\beta-2}$, i.e. that for some $K > 0$ we have
\[
\sum_{n=1}^{N} \frac{\lambda_n}{\pi} \cdot \int \frac{t \, d\mu_n(\zeta)}{(t^2 + |z - \zeta|^2)^\frac{\beta}{2}} \leq K t^{\beta-2},
\]
whenever $z \in \mathbb{C}$ and $t > 0$.

When $t \geq 1$, we have the trivial estimate (independent of $z$)
\[
\lambda_n(P_t * \mu_n)(z, t) \leq \frac{\lambda_n M^\beta(E_n)}{\pi t^2} \leq \frac{1}{\pi 2^n t^2},
\]
so this gives $|P_t * g_N| \leq K t^{-2} \leq K t^{\beta-2}$.

So to finish, fix $t \in (0, 1)$, and choose $m \in \mathbb{N}$ such that $2^{-m-1} t \leq 2^{-m}$, take the $n$-th term in the sum, and consider separately the ranges of $n$:
case 1°: $n > m - 2$, and case 2°: $n < m - 2$,
and the possible positions of $z$ in relation to $A_n$.

Case 1°:
The trivial estimate also gives
\[
\lambda_n (P_t * \mu_n)(z, t) \leq \frac{M^\beta(A_n)}{\pi t^2} \leq \frac{(2^{-n})^\beta}{\pi t^2},
\]
so we get an estimate for the total contribution from all the Case 1° terms:
\[
t^{2-\beta} \sum_{n=m-2}^{\infty} \lambda_n (P_t * \mu_n)(z, t) \leq \frac{1}{\pi} \sum_{n=m-2}^{\infty} 2^{(m+1-n)\beta} = \frac{8^\beta}{\pi (1 - 2^{-\beta})},
\]
Case 2°:
To deal with this we have to consider the position of $z$ in relation to $A_n$.

There are at most three $n$ such that the distance from $z$ to $A_n$ is less than $2^{-n-1}$. For these we can use the uniform estimate
\[
t^{2-\beta} (P_t * \mu_n)(z, t) \leq K,
\]
which follows from the fact that $\mu_n$ has growth $\beta$. (— just write the value of $P_t * \mu_n(w)$ as a sum of the integrals over the annuli
\[\{z \in \mathbb{C} : 2^m t < |z - w| \leq 2^{m+1} t\}
\]
from 0 to $-\log_2 t$ plus the integral over the disc $\mathbb{B}(w, t)$ —).

For the remaining $n \in \{1, \ldots, m - 3\}$, the estimate
\[
(P_t * \mu_n)(z, t) \leq \frac{t \cdot M^\beta(A_n)}{\pi \cdot \text{dist}(z, A_n)^3}
\]
gives
\[
t^{2-\beta} (P_t * \mu_n)(z, t) \leq (2^{\beta-3})^{m+1-n},
\]
so
\[ t^{2-\beta} \sum_{n=1}^{m-3} \lambda_n \cdot (P_t * \mu_n)(z, t) \leq 3K + \sum_{n=1}^{m-3} (2^{\beta-3})^{m+1-n} = K, \]

another constant (depending on \( \beta \)), and we are done.

Now consider the converse. Suppose \( \sum_n 2^\alpha M^\beta(A_n \setminus U) < +\infty \). We want to show that \( A^s(U) \) admits a cpe at 0.

If \( V \) is an open subset of \( U \), then \( A^s(U) \) is a subset of \( A^s(V) \), so it suffices to prove the result for \( U \) that are contained in \( \overline{B}(0, 1/2) \). We assume this is the case.

We may choose radial functions \( \psi_n \in \mathcal{D} \) such that \( \psi_n = 1 \) on \( A_n \), \( \psi_n = 0 \) off \( A_{n-1} \cup A_n \cup A_{n+1} \), and for each \( k \) the sequence \( (N_k(\psi_n))_n \) is bounded. Let
\[ \phi_n := \frac{\psi_n}{\sum_{m=1}^{\infty} \psi_m} \]
on the complement of \( \{0\} \), and \( \phi_n(0) = 0 \). Then each \( \phi_n \in \mathcal{D} \), is zero off \( A_{n-1} \cup A_n \cup A_{n+1} \), the sequences \( (N_k(\phi_n))_n \) are all bounded, and \( \sum_n \phi_n = 1 \) on the union of all the \( A_n \).

Fix a test function \( \chi \) that equals 1 on \( \overline{B}(0, 1/2) \) and is supported on \( B(0, 1) \).

Fix \( f \in A^s_0 \). We want to prove that \( |f(0)| \leq K \|f\|_s \), where \( K > 0 \) does not depend on \( f \).

We have \( f(0) = (\chi \cdot f)(0) \), \( \chi \cdot f \in A^s(U) \), and \( \|\chi \cdot f\|_s \leq K\|f\|_s \), so it suffices to prove the estimate for \( f \in A^s_0 \) having support in \( B(0, 1) \).

Choose \( N \in \mathbb{N} \) such that \( f(z) \) is holomorphic for \( |z| < 2^{2-N} \). Define \( \phi_0(z) \) to be 1 on \( \phi^N(z) \) when \( |z| < 2^{-N-1} \) and 0 otherwise. Then \( \phi_0 \in \mathcal{D} \), \( N_k(\phi_0) = N_k(\phi^N) \), and the test function
\[ \phi := \phi_0 + \sum_{n=1}^{N} \phi_n \]
is equal to 1 near \( B(0, 1) \).

We have
\[ f = \phi \cdot f = \mathcal{E} \left( \frac{\partial}{\partial \overline{z}} (\phi \cdot f) \right). \]

Since \( \frac{\partial}{\partial \overline{z}} \phi = 0 \) on the support of \( f \), this equals
\[ \mathcal{E} \left( \phi \frac{\partial f}{\partial \overline{z}} \right) = \mathcal{E} \left( \phi_0 \frac{\partial f}{\partial \overline{z}} \right) + \sum \mathcal{E} \left( \phi_n \frac{\partial f}{\partial \overline{z}} \right) = \sum \mathcal{E}(\phi_n \frac{\partial f}{\partial \overline{z}}), \]
since \( f \) is holomorphic on \( \text{spt}(\phi_0) \).

Take a test function \( \psi \) that equals \( 1/z \) for \( 2^{-N} < |z| < 2 \).
Applying Lemma 3, we have
\[ C\left(\frac{\phi_n \cdot \partial f}{\partial \bar{z}}\right)(0) = -\left\langle \frac{\psi}{\pi}, \phi_n \cdot \frac{\partial f}{\partial \bar{z}} \right\rangle = -\left\langle \frac{\psi \cdot \phi_n}{\pi}, \frac{\partial f}{\partial \bar{z}} \right\rangle. \]

Thus
\[ f(0) = -\sum_{n=1}^{N} \left\langle \frac{\phi_n}{\pi z}, \frac{\partial f}{\partial \bar{z}} \right\rangle. \]

(Here, by \( \phi_n/z \) we understand the test function that equals 0 at the origin and \( \phi_n/z \) everywhere else in \( \mathbb{C} \).)

Applying the Hausdorff content estimate from Lemma 10, we have
\[ |f(0)| \leq K \cdot \sum_{n=1}^{N} N_3\left(\frac{\phi_n}{z}\right) \cdot M^\beta_\ast(\text{spt}(\phi_n \cdot \frac{\partial f}{\partial \bar{z}})) \cdot \left\| \frac{\partial f}{\partial \bar{z}} \right\|_{s-1}, \]
\[ \leq K \cdot \sum_{n=1}^{N} 2^n M^\beta_\ast((A_{n-1} \cup A_n \cup A_{n+1}) \setminus U) \cdot \left\| \frac{\partial f}{\partial \bar{z}} \right\|_{s-1}, \]

since \( N_3(\phi_n/z) \leq 2^{n+1} N_3(\phi_n) \leq K2^n \).

Since \( M^\beta_\ast \) is subadditive and \( \left\| \frac{\partial f}{\partial \bar{z}} \right\|_{s-1} \leq K\|f\|_s \leq K \), we get
\[ |f(0)| \leq K \cdot \sum_{n=1}^{\infty} 2^n M^\beta_\ast(A_n \setminus U) \cdot \|f\|_s. \]

This completes the proof.

We remark that the proof actually shows that the sum of the series is the dual norm of the point evaluation \( f \mapsto f(b) \), up to multiplicative constants that depend only on \( \beta \).

6.2. Proof of Theorems 2, 3 and 4. To prove Theorem 2, one can use exactly the same argument, just replacing \( M^\beta_\ast \) by \( M^3 \), and using Lemma 8 instead of Lemma 10.

To prove the other two theorems, one just uses the corresponding Cauchy-Pompeiu formulas for derivatives:
\[ C(\mu_n)^{(k)}(0) = \frac{k!}{\pi} \int \frac{d\mu_n(z)}{z^{k+1}} \]
for the ‘only if’ direction, and
\[ f^{(k)}(0) = -\sum_{n=1}^{N} \left\langle \frac{k! \phi_n}{\pi z^{k+1}}, \frac{\partial f}{\partial \bar{z}} \right\rangle \]
for the ‘if’ direction.
6.3. **Proof of Theorem 5.** The point is that a distribution \( f \in C_s \) satisfies \( \Delta f = 0 \) on the open set \( U \) if and only if \( \frac{\partial f}{\partial z} \) is holomorphic on \( U \), and the operator \( \frac{\partial}{\partial z} \) maps \( T_s \) into \( T_{s-1} \), and is inverted on \( (T_{s-1})_{cs} \) by the ‘anti-Cauchy’ transform. So the results are just reformulations of Theorem 1.

6.4. **Remark.** We remark that this is an example of 1-reduction, and one could also formulate equivalent results about other elliptic operators. In particular, \( M^\beta \) is also the capacity for \( T_\beta \) and the operator \( \left( \frac{\partial}{\partial z} \right)^2 \), which is associated to complex elastic potentials, and it is the capacity for \( T_{\beta+2} \) (a space of functions that are twice differentiable, but may have discontinuities in the third derivative) and the operator \( \Delta^2 \), associated to elastic plates.

7. **Examples**

7.1. **Smooth boundary.** If \( U \) is smoothly-bounded, then there are no bounded point evaluations at any boundary point on \( A_s(U) \) for any \( s < 0 \). Indeed, if \( b \) belongs any to any nontrivial continuum \( K \subset \mathbb{C} \setminus U \), then no such bpd exists at \( b \).

7.2. **Multiple components.** If \( b \) belongs to the boundary of two (or more) connected components of the open set \( U \), then no such bpd exists at \( b \).

To see this, note that the assumptions imply that for all small enough \( r \), the circle \( |z - b| = r \) meets the complement of \( U \), and this implies that for large enough \( n \), the \( M^\beta \) content of \( A_n \setminus U \) is at least \( 2^{-n\beta} \). Hence the series in Theorem 1 diverges for all \( s \in (-1, 0) \).

This contrasts with the behaviour found in [17] for ordinary Lipschitz classes, for which interesting behaviour is possible at the boundary of Jordan domains with piecewise-smooth boundary, and at common boundary points of two components.

7.3. **Slits.** Let \( a_n \downarrow 0, r_n \downarrow 0 \) and

\[
a_{n+1} + r_{n+1} < a_n - r_n, \quad \forall n \in \mathbb{N}.
\]

Then 0 is a boundary point of the slit domain

\[
U := \hat{B}(0, a_1 + r_1) \setminus \bigcup_{n=1}^{\infty} [a_n - r_n, a_n + r_n].
\]
For a line segment $I$ of length $d$, we have

$$M^\beta(I) = M^\beta_*(I) = d^\beta,$$

for $0 < \beta < 1$. Then for $-1 < s < 0$, $A^s(U)$ admits a cpd at 0 if and only if $B^s(U)$ admits a weak-star cpd at 0, and if and only if

$$(6) \sum_{n=1}^\infty \frac{a_n}{r_n^{s+1}} < +\infty.$$ 

This follows at once from Theorems 1 and 2 in case $a_n = 2^{-n}$. In the general case, one obtains it by imitating the proofs, using contours that pass between the slits.

For example, if $a_n = 2^{-n}$ and $r_n = 4^{-n}$, then there is a cpd at 0 on $A^s(U)$ if and only if $s > -\frac{1}{2}$.

7.4. Road-runner sets. The $M^\beta$ content of a disc and of one of its diameters are both fixed multiples of $(\text{radius})^\beta$, and the lower content is the same, so the same condition (6) is necessary and sufficient for the existence of a cpd on $A^s(U)$ at 0 on the so-called road-runner set

$$U := \tilde{B}(0, a_1 + r_1) \setminus \bigcup_{n=1}^\infty \mathbb{B}(a_n, r_n),$$

when the $a_n$ and $r_n$ are as in the last subsection.

7.5. Below minus 1. The $L - F - \text{cap}$ capacity of a singleton is positive as soon as there are distributions $f$ of class $F$ having $LF = 0$ on a deleted neighbourhood of 0. In the case $L = \frac{\partial}{\partial \bar{z}}$, this happens when the distribution represented by the $L^1_{\text{loc}}$ function $\frac{1}{z}$ belongs to $F_{\text{loc}}$.

That explains why, in the case of $L^p$ holomorphic functions, there is a major transition at $p = 2$; the function $\frac{1}{z}$ belongs to $L^p_{\text{loc}}$ when $1 \leq p < 2$. (The ‘smoothness’ of $L^p(\mathbb{R}^d)$ is $-d/p$. This can be extended below $p = 1$ by using Hardy spaces $H^p$ instead of $L^p$.)

The ‘delta-function’, $\delta_0$, the unit point mass at the origin, is the d-bar (distributional) derivative of $1/z$. More precisely

$$\frac{\partial \frac{1}{z}}{\partial \bar{z}} = \delta_0.$$ 

The Poisson transform of $\delta_0$ is just the Poisson kernel $t/\pi(|z|^2 + t^2)\frac{\bar{z}}{z}$, so grows no faster than $1/t^2$ as $t \downarrow 0$. Thus $\delta_0$ belongs to $T_{-2}$ and $\frac{1}{z} \in T_{-1}$. 

Hence, $\frac{1}{z}$ is a boundary point of the Trudinger class $T_{-2}$.

The use of $\frac{1}{z}$ to convey boundedness of the $L^p$ norm of holomorphic functions is due to Hörmander.
7.6. **Remark.** In [3, p.311] Carleson proved that for $0 < \beta < 1$ the $M^\beta$-null sets are the removable singularities for the class of $\text{Lip}\beta$ "multiple-valued holomorphic functions having single-valued real part." The expression in quotation marks is really code for "harmonic functions", so this is really the first version of the fact that $M^\beta$ is the $\Delta - \text{Lip}\beta - \text{cap}$.

In the same paper, Carleson proved a precursor to Dolzhenko’s theorem [7] about removable singularities for $\text{Lip}\beta$ holomorphic functions. He left a little gap, between the Hausdorff content and the nearby Riesz capacity, and Dolzhenko closed the gap.

7.7. **Conjecture.** Recently [27] the author showed that the existence of a continuous point derivation on $A^s(U)$ at $b$, for some positive $s < 1$, implies that the value of the derivation may be calculated by taking limits from a subset $E \subset U$ having full area density at $b$. It seems reasonable to hope that for negative $s$, if $A^s(U)$ admits a continuous point evaluation at $b$, then the value can be calculated in a similar way, as

$$\lim_{z \to b, z \in E} f(z)$$

for some $E \subset U$ having full area density at $b$.

7.8. **Question.** Suppose $F$ is an SCBS on $\mathbb{R}^d$ having the strong module property

$$\|\phi \cdot f\|_F \leq K \cdot N_k(\phi) \cdot \|f\|_F, \ \forall \phi \in \mathcal{D} \forall f \in F,$$

for some positive constant $K$ and some nonnegative integer $k$. Define an inner capacity $c = c_{F,k}$ by the rule that for compact $E \subset \mathbb{R}^d$ the value $c_F(E)$ is the least nonnegative number such that

$$|\langle \phi, f \rangle| \leq N_k(\phi) \cdot \|f\|_F \cdot c(\text{spt}(\phi \cdot f))$$

whenever $\phi \in \mathcal{D}$ and $f \in F$. For example, if $F = L^\infty$, it is easy to see that $c_{F,0}(E)$ is the $d$-dimensional Lebesgue measure of $E$, whereas for $F = L^1$, $c_{F,0}(E) = 1$ for all $E$.

The question is this: *For which $F$ and $k$ is it the case that $c_F \leq K \cdot (1-F\text{-cap})$ for some constant $K$?*

Recall that for compact $E \subset \mathbb{R}^d$,

$$1-F\text{-cap}(E) := \inf\{||\chi, f|| : \|f\|_F \leq 1, \text{spt}(f) \subset E\},$$

where $\chi \in \mathcal{D}$ is any fixed test function such that $\chi = 1$ on $E$.

We have seen that this holds for $F = T_s$, $s \in \mathbb{R}$. Does it hold for all SCBS having the strong module property?
7.9. **Question.** If an SCBS $F$ has the order $k$ strong module property, when is there an SCBS locally-equal to $DF$ that has the strong order $k + 1$ module property? And what about $\int F$?

$DF_{\text{loc}}$ is the Frechet space topologised by the seminorms defined by

$$
\|f\|_n = \inf \left\{ \|g_1 + \cdots + g_d\|_{F(B(0, n))} : g_j \in F, f = \frac{\partial g_1}{\partial x_1} + \cdots + \frac{\partial g_d}{\partial x_d} \right\}.
$$

Note that if $d = 2$, and $F$ is weakly-locally invariant under C-Z operators (or just under the Beurling transform), then $DF_{\text{loc}}$ is topologised by the seminorms

$$
\|f\|_n = \inf \left\{ \|g\|_{F(B(0, n))} : g \in F_{cs}, f = \frac{\partial g}{\partial \bar{z}} \right\},
$$

and this implies that each $\|\phi \cdot f\|_n$ is dominated by $N_k(\phi) \cdot \|f\|_n$, because

$$
\phi \cdot \frac{\partial g}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}} (\phi \cdot g) - \left( \frac{\partial \phi}{\partial \bar{z}} \right) \cdot g.
$$

This property is a kind of local version of the strong module property.

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E-mail address: anthony.ofarrell@mu.ie

MATHEMATICS AND STATISTICS, NUI, MAYNOOTH, CO KILDARE, W23 HW31, IRELAND