Barbero-Immirzi parameter in Regge calculus

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**Abstract**

We consider Regge calculus in the representation in terms of area tensors and self- and antiselfdual connections generalised to the case of Holst action that is standard Einstein action in the tetrad-connection variables plus topological (on equations of motion for connections) term with coefficient $1/\gamma$, $\gamma$ is Barbero-Immirzi parameter. The quantum measure is shown to exponentially decrease with areas with typical cut-off scales $4\pi G$ and $4\pi G\gamma$ in spacelike and timelike regions, respectively ($G$ is the Newton constant).

PACS numbers: 04.60.-m Quantum gravity
The formal nonrenormalisability of quantum version of general relativity (GR) may cause us to try to find alternatives to the continuum description of underlying space-time structure. An example of such alternative description is given by Regge calculus (RC) suggested in 1961 [1]. It is the exact GR developed in the piecewise flat space-time which is a particular case of general Riemannian spacetime [2]. In its turn, the general Riemannian spacetime can be considered as a limiting case of the piecewise flat spacetime [3]. Any piecewise flat spacetime is simplicial one: it can be represented as a collection of flat 4-dimensional simplices(tetrahedrons), and its geometry is completely specified by a countable number of freely chosen lengths of all edges (or 1-simplices). Thus, RC implies a discrete description alternative to the usual continuum one. For a review of RC and alternative discrete gravity approaches see, e. g., [4].

Since fully discrete theory such as RC does not possess a continuous coordinate playing the role of time, the canonical Hamiltonian formalism and operator quantization are not immediately applicable to it. However, the functional integral approach remains most universal. The functional integral measure in RC was considered in [5, 6]. Our strategy (briefly reviewed in [7]) is based on the requirement for the full discrete measure to result in the canonical Hamiltonian functional integral measure with time \( t \), with some coordinate chosen as \( t \) and made continuous.

Since this strategy implies intermediate use of canonical Hamiltonian measure, it is of importance that we could perform continuous time limit in a nonsingular way. Meanwhile, this limit implies infinitely flattened in some direction simplices, and description of these objects is singular if made in terms of the edge lengths only.

The way to avoid singularities is to extend the set of variables via adding the new ones having the sense of angles and considered as independent variables. Such variables are the finite rotation matrices which are the discrete analogs of the connections in the continuum GR. The situation considered is analogous to that one occurred when recasting the Einstein action in the Hilbert-Palatini form. We consider more general action which differs from the Hilbert-Palatini one by adding term which is topological one on the equations of motion for the connection and thus leads to the same Einstein action. Namely, we consider action introduced by Holst [8]. He has shown that his action leads to the Hamiltonian formalism by Barbero [9] which modifies Ashtekar formalism (see, e.g., review [10]) to the case of real variables. It also incorporates definition by Immirzi [11] who extended definition of [9] to a whole family of quantum
theories specified by parameter $\gamma$ called recently Barbero-Immirzi parameter. The considered action reads

$$\int R\sqrt{g}d^4x \quad \omega^{ab} \omega^{cb}(e^a_\lambda) = \frac{1}{4} \int (\epsilon_{abcd} e^a_\lambda e^b_\mu + \frac{2}{\gamma} e^\lambda_{ac} e^\mu_{bd}) \epsilon^\lambda_{\mu\nu\rho} [\partial_\nu + \omega_\nu, \partial_\rho + \omega_\rho]^{cd} d^4x, \quad (1)$$

where the tetrad $e^a_\lambda$ and connection $\omega^{ab}_\lambda = -\omega^{ba}_\lambda$ are independent variables, the RHS being reduced to LHS in terms of $g_{\lambda\mu} = e^a_\lambda e^a_\mu$ if we substitute for $\omega^{ab}_\lambda$ solution of the equations of motion for these variables in terms of $e^a_\lambda$. The Latin indices $a, b, c, \ldots$ are vector ones with respect to the local Minkowskian frames introduced at each point $x$.

Now in RC the Einstein action in the LHS of (1) becomes the Regge action,

$$2 \sum_{\sigma^2} \alpha_{\sigma^2} |\sigma^2|, \quad (2)$$

where $|\sigma^2|$ is the area of a triangle (the 2-simplex) $\sigma^2$, $\alpha_{\sigma^2}$ is the angle defect on this triangle, and summation run over all the 2-simplices $\sigma^2$. The discrete analogs of the tetrad and connection, edge vectors and finite rotation matrices, were first considered in [12]. The local inertial frames live in the 4-simplices. The analogs of the connection are defined on the 3-simplices $\sigma^3$ and are the matrices $\Omega_{\sigma^3}$ connecting the frames of the pairs of the 4-simplices $\sigma^4$ sharing the 3-faces $\sigma^3$. These matrices are the finite SO(3,1) rotations in the Minkowskian case in contrast with the continuum connections $\omega^{ab}_\lambda$ which are the elements of the Lee algebra so(3,1) of this group. This definition includes pointing out the direction in which the connection $\Omega_{\sigma^3}$ acts (and, correspondingly, the opposite direction, in which the $\Omega_{\sigma^3}^{-1} = \Omega_{\sigma^3}^T$ acts). That is, the connections $\Omega$ are defined on the oriented 3-simplices $\sigma^3$. We have suggested self-dual formulation of RC [13] which easily modifies from Euclidean to Minkowskian case and to include also topological term considered here. Instead of RHS of (1) we write

$$S(v, \Omega) = \sum_{\sigma^2} \left( 1 + \frac{i}{\gamma} \right) \sqrt{2 + v_{\sigma^2} \circ + v_{\sigma^2}} \arcsin \frac{+ v_{\sigma^2} \circ + v_{\sigma^2}}{\sqrt{2 v_{\sigma^2} \circ + v_{\sigma^2}}} + \left( 1 - \frac{i}{\gamma} \right) \sqrt{2 - v_{\sigma^2} \circ - v_{\sigma^2}} \arcsin \frac{- v_{\sigma^2} \circ + v_{\sigma^2}}{\sqrt{2 - v_{\sigma^2} \circ - v_{\sigma^2}}}, \quad (3)$$

where we have defined $A \circ B = \frac{1}{2} A^{ab} B_{ab}$ for the two tensors $A, B; \{ \ldots \}$ means ”the set of . . . ”; $v_{\sigma^2}$ is the dual tensor of the triangle $\sigma^2$ in terms of the vectors of its edges $l_i^a$,

$$v_{\sigma^2}^{ab} = \frac{1}{2} \epsilon_{abcd} l_i^c l_j^d \quad (4)$$

(in some 4-simplex frame containing $\sigma^2$). The curvature matrix $R_{\sigma^2}$ on the 2-simplex $\sigma^2$ is the path ordered product of the connections $\Omega_{\sigma^3}^{-1}$ on the 3-simplices $\sigma^3$ sharing $\sigma^2$.
along the contour enclosing $\sigma^2$ once and contained in the 4-simplices sharing $\sigma^2$,

$$R_{\sigma^2} = \prod_{\sigma^3 \supset \sigma^2} \Omega_{\sigma^3}^{\pm 1}. \quad (5)$$

The $\pm(\ldots)$-notations (self- and antiselfdual parts) are as follows. For SO(3,1) matrix

$$\Omega = \exp (\varphi^k E^a_{kb} + \psi^k L^a_{kb})$$

with generators

$$E_{kab} = -\epsilon_{kab}, \quad L_{kab} = g_{ka}g_{0b} - g_{0a}g_{kb} \quad (g_{ab} = \text{diag}(-1, 1, 1, 1), \epsilon_{123} = +1)$$

we define

$$\mp\Sigma_{kab} = -\epsilon_{kab} \pm i(g_{ak}g_{0b} - g_{a0}g_{kb})$$

so that

$$\mp\Sigma_{-kb} \mp\Sigma_{lc} = -\delta_{kl}\delta_{c}^{a} + \epsilon_{klm}^{\mp}\Sigma_{mc}^{a}$$

and then

$$\Omega = \mp\Omega \cdot \Omega, \quad \pm\Omega = \exp \left( \frac{\varphi^k \mp i\varphi^k}{2} \mp\Sigma_{kb}^{a} \right).$$

Area tensor $\nu$ splits additively,

$$\nu^{ab} = \mp\nu^{ab} + \nu^{ab}, \quad \mp\nu^{ab} = \frac{1}{2}\nu^{ab} \pm i\frac{1}{4}\epsilon_{cd}^{ab}\nu^{cd} \quad (\epsilon_{0123} = +1)$$

so that

$$\star\nu^{ab} \equiv \frac{1}{2}\epsilon_{cd}^{ab} \mp\nu^{cd} = \mp i\mp\nu^{ab}. \quad \star\nu^{ab} \equiv \nu \circ \pm \nu \pm i\nu \ast \nu.$$

In particular,

$$2 \mp\nu \circ \pm\nu = \nu \circ \nu \pm i\nu \ast \nu.$$

The $\pm$-parts map to 3d vectors $\pm\nu$,

$$\pm\nu_{ab} \equiv \frac{1}{2}\pm\nu^{k} \mp\Sigma_{kab}, \quad 2\pm\nu_{k} = -\epsilon_{klm}\nu^{lm} \pm i(v_{k0} - v_{0k}).$$

In particular,

$$\pm\nu^{2} = 2\pm\nu \circ \pm\nu.$$

For $v_{\sigma^2}$ given by (4) the $\pm\nu_{\sigma^2}^{2}$ is $(-4)$ times square of (real for spacelike $\sigma^2$) area.

Classically, we can write equations of motion for $\mp\Omega_{\sigma^3}$, that is, for the corresponding parameters $\varphi, \psi$. This results in equations for $\varphi - i\psi$ and $\varphi + i\psi$ separately for $+$- and $-$-parts as for holomorphic functions. Take $+$-part. Dependence on $\mp\Omega_{\sigma^3}$ is due to contributions from the faces $\sigma^2$ of the tetrahedron $\sigma^3$; take certain such face with
tensor $\tau^a\tau_2 \equiv \tau^\nu$. Consider $U \equiv \tau^\Omega_3$ as a priori arbitrary $4 \times 4$ matrix but add Lagrange multiplier terms taking into account orthogonality of $U$ and its self-duality (equivalent to self-duality of its antisymmetric part). The dependence on $U$ in action is

$$S \propto \sqrt{2 \tau^\nu \tau^\nu \arcsin \left( \frac{\tau^\nu \circ (\Gamma_1 U T_2)}{\sqrt{2 \tau^\nu \tau^\nu}} \right)} + (U^T \circ U - 1) \circ \lambda + (\ast U + i U) \circ \mu,$$

$$\Gamma_1 U T_2 = R(\{\ast \Omega\}) \text{ or } R^T(\{\ast \Omega\}) \quad (6)$$

Here symmetric $\lambda$ and antisymmetric $\mu$ matrices are Lagrange multipliers. Let us form combination of the equations of motion

$$i \epsilon_{fg} U^a_c \frac{\partial S}{\partial U^{bc}} + U_f^e \frac{\partial S}{\partial U^{gc}} - U_g^c \frac{\partial S}{\partial U^f c} = 0 \quad (7)$$

(in fact, $+$-part of $U^T \partial S/\partial U$), thereby $\lambda$- and $\mu$-terms are cancelled. We are left with the $+$-part of some product of $+$-matrices which coincides with this product itself (antisymmetrised),

$$\Gamma_1^T R(\{\ast \Omega\}) + \tau^\nu R^T(\{\ast \Omega\}) \quad (8)$$

Take as $\{\Omega\}$ the set of compatible with edge lengths connections so that $R_{\sigma^2}(\{\Omega\})$ really rotates around $\sigma^2$ by the defect angle. In correspondence with Minkowskian metric signature, there are two types of the rotations.

i). Euclidean rotation around timelike area. In certain frame we have for triangle of area 1/2 (in module),

$$v_{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad 2 \tau^\nu v_{ab} = \begin{pmatrix} 0 & \pm i & 0 & 0 \\ \mp i & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \tau^\nu = (1, 0, 0),$$

$$\tau^a \tau^b = \begin{pmatrix} -\cos \frac{\varphi}{2} & \mp i \sin \frac{\varphi}{2} & 0 & 0 \\ \pm i \sin \frac{\varphi}{2} & \cos \frac{\varphi}{2} & 0 & 0 \\ 0 & 0 & \cos \frac{\varphi}{2} & -\sin \frac{\varphi}{2} \\ 0 & 0 & \sin \frac{\varphi}{2} & \cos \frac{\varphi}{2} \end{pmatrix}, \quad \sqrt{\tau^\nu \tau^\nu} \arcsin \frac{\mp \nu \circ \tau^\nu \circ \nu}{\sqrt{\tau^\nu \tau^\nu}} = \frac{\varphi}{2}. \quad (9)$$

ii). Rotation around spacelike area, i.e. Lorentz boost,

$$v_{ab} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad 2 \tau^\nu v_{ab} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \mp i & 0 \\ 0 & 0 & \pm i & 0 \end{pmatrix}, \quad \tau^\nu = (\pm i, 0, 0),$$
In both cases the $^+$- and $^-$-parts contribute the same half of the action \( (2) \) and reproduce it while topological term is cancelled. What is important, in \( (3) \) we have \( R(\{ ^+\Omega \})^+v + R^T(\{ ^+\Omega \}) = 2^+v \cos ^+\alpha \) and proportional to

\[
\Gamma_1^T v_{\sigma_2} \Gamma_1 \tag{11}
\]

contribution of the given 2-face $\sigma^2$ of $\sigma^3$ into equations of motion for $^+\Omega_{\sigma^3}$. Here matrices $\Gamma$ serve to transform $^+v_{\sigma_2}$ to the same local frame for all faces of the given $\sigma^3$. As a result, we get simply the closure condition for the surface of $\sigma^3$ fulfilled identically by construction of the manifold. Thus equations of motion for connections really lead to the original Regge action.

Can we fix full discrete measure from requirement to result in the Hamiltonian path integral measure in the continuous time limit whatever coordinate is chosen as time? This strategy has solution in 3 dimensions. A specific feature of the 3D case important here is commutativity of the dynamical constraints in the (continuous time) Hamiltonian formulation leading to a simple form of the functional integral. In 3D case edge vectors $l_{\sigma_1}$ are considered instead of area tensors $v_{\sigma_2}$. Edge vectors are independent of each other, at least locally. In 4 dimensions, the variables $v_{\sigma_2}$ are not independent but obey a set of (bilinear) intersection relations. For example, tensors of the two triangles $\sigma_1^2, \sigma_2^2$ sharing an edge satisfy the relation

\[
\epsilon_{abcd} v_{\sigma_1^2}^{ab} v_{\sigma_2^2}^{cd} = 0. \tag{12}
\]

These purely geometrical relations define a hypersurface in the configuration superspace of the formally independent area tensors. The idea is to construct quantum measure first in this superspace. At the second stage the measure should be projected onto this hypersurface: geometrical relations of the type \( (12) \) are taken into account by inserting certain $\delta$-function-like factors in the measure. Note that the RC with formally independent (scalar) areas has been considered in the literature \[4, 14\].

The theory with formally independent area tensors can be called area tensor RC.
The completely discrete quantum measure reads

\[
< \Psi(\{\pi\}, \{\Omega\}) > = \int \Psi(\{\pi\}, \{\Omega\}) \exp \left\{ i \sum_{\sigma^2} \left[ \left(1 + \frac{i}{\gamma} \right)^{+\tau_{\sigma^2} \circ R_{\sigma^2}(\{^+\Omega\})} + \left(1 - \frac{i}{\gamma} \right)^{-\tau_{\sigma^2} \circ R_{\sigma^2}(\{-\Omega\})} \right] \right\} \prod_{\sigma^2} d^6\pi_{\sigma^2} \prod_{\sigma^3} D\Omega_{\sigma^3}
\]

\[
\equiv \int \Psi(\{\pi\}, \{\Omega\}) d\mu_{\text{area}}(\{\pi\}, \{\Omega\})
\]

where \(D\Omega_{\sigma^3}\) is the Haar measure on the group \(SO(3,1)\) of connection matrices \(\Omega_{\sigma^3}\). Appearance of some set \(F\) of triangles \(\sigma^2\) integration over area tensors of which is omitted (denoted as "t-like" in (13)) is connected with that integration over all area tensors is generally infinite, in particular, when normalizing measure (finding \(< 1 >\)). Indeed, different \(R_{\sigma^2}\) for \(\sigma^2\) meeting at a given link \(\sigma^1\) are connected by Bianchi identities [1]. Therefore the product of \(\delta^6(R_{\sigma^2} - R_{\sigma^2}^T)\) for all these \(\sigma^2\) which follows upon integration over area tensors for these \(\sigma^2\) contains singularity of the type of \(\delta\)-function squared. To avoid this singularity we should confine ourselves by only integration over area tensors on those \(\sigma^2\) on which \(R_{\sigma^2}\) are independent. The complement \(F\) to this set of \(\sigma^2\) are those \(\sigma^2\) on which \(R_{\sigma^2}\) are dependent, that is, expressible by the Bianchi identities in terms of independent \(R_{\sigma^2}\). Let us adopt regular way of constructing 4D simplicial structure of the 3D simplicial geometries (leaves) of the same structure. A \(n\)-simplex \(\sigma^n\) is denoted by the set of its \(n + 1\) vertices in round brackets (unordered sequence), \((A_1A_2...). The \(i, k, l, ...\) are vertices of the current leaf, \(i^+, k^+, l^+, ...\) and \(i^-, k^-, l^-, ...\) are corresponding vertices of the nearest future and past in \(t\) leaves. Or we shall speak of the "upper" and "lower" leaves, respectively. See fig[1] Each vertex is connected by links (edges) with its \(\pm\)-images. These links (of the type of \((ii^+), (ii^-)\)) will be called

Figure 1: Fragment of the \(t\)-like 3-prism.
\textit{t-like} ones (do not mix with the term "timelike" which is reserved for the local frame components). The \textit{leaf} links \((ik)\) are completely contained in the 3D leaf. There may be \textit{diagonal} links \((ik^+), (ik^-)\) connecting a vertex with the ±-images of its neighbors. We call arbitrary simplex \textit{t-like} one if it has \textit{t-like} edge, the \textit{leaf} one if it is completely contained in the 3D leaf and \textit{diagonal} one in other cases. It can be seen that the set of the \textit{t-like} triangles is fit for the role of the above set \(\mathcal{F}\). In the case of general 4D simplicial structure we can deduce that the set \(\mathcal{F}\) of the triangles with the Bianchi-dependent curvatures pick out some one-dimensional field of links. We can take this field as definition of the coordinate \(t\) direction so that \(\mathcal{F}\) be just the set of the \textit{t-like} triangles. Also existence of the set \(\mathcal{F}\) naturally fits our above requirement that limiting form of the full discrete measure (when any one of the coordinates, not necessarily \(t\), is made continuous) should coincide with Hamiltonian path integral. Namely, in the Hamiltonian formalism absence of integration over area tensors of triangles which pick out some coordinate \(t\) (\textit{t-like} ones) corresponds to some gauge fixing.

There is the invariant (Haar) measure \(\mathcal{D}\Omega\) in (13) which looks natural from symmetry considerations. But it also arises from the formal point of view within our strategy. If one makes a coordinate \(t\) continuous and takes it as a time he finds kinetic terms of the type \(\pi_\sigma \sigma^2 \circ \Omega_\sigma^T \hat{\Omega}_\sigma^2\) (here \(\Omega_\sigma^2\) serves to parameterize limiting form of \(\Omega_\sigma^3\) with \(\sigma^3\) filling up infinitesimal \(t\)-like prism with the base \(\sigma^2\)). Then standard Hamiltonian path integral just has \(\mathcal{D}\Omega_\sigma^2\) as a measure. To reproduce the latter in the continuous \(t\) limit the full discrete measure should also include Haar measure \(\mathcal{D}\Omega_\sigma^3\).

One else specific feature of the quantum measure is the absence of the inverse trigonometric function ‘arcsin’ in the exponential, whereas the Regge action \([3]\) contains such functions. This is connected with using the canonical quantization at the intermediate stage of derivation: in gravity this quantization is completely defined by the constraints, the latter being equivalent to those ones without arcsin (in some sense on-shell).

Consider averaging functions of only area tensors \(\pi_\sigma^2\). By the properties of invariant measure, integrations over \(\prod \mathcal{D}\Omega_\sigma^3\) in (13) reduce to integrations over \(\prod \mathcal{D}R_\sigma^2\) with independent \(R_\sigma^2\) (i.e. at not \textit{t-like} \(\sigma^2\)) and some number of connections \(\prod \mathcal{D}\Omega_\sigma^3\) which we can call gauge ones. The expectation value of any field monomial, \(<\pi_{a_1 b_1}^{\alpha_1} \ldots \pi_{a_n b_n}^{\alpha_n}\>\) reduces to the (derivatives of) \(\delta\)-functions \(\delta^3((+R_{\sigma_1^2}^{a_1} - +R_{\sigma_1^2}^{T}) + (-R_{\sigma_1^2}^{a_1} - -R_{\sigma_1^2}^{T})) \propto \delta^3(\Re(+R_{\sigma_1^2}^{a_1} - +R_{\sigma_1^2}^{T}))\) and \(\delta^3(i[+R_{\sigma_1^2}^{a_1} - +R_{\sigma_1^2}^{T}) - (-R_{\sigma_1^2}^{a_1} - -R_{\sigma_1^2}^{T})]) \propto \delta^3(3(+R_{\sigma_1^2}^{a_1} - +R_{\sigma_1^2}^{T}))\)
which are then integrated out over $\mathcal{D}R_{\sigma^2}$ giving finite nonzero answer. This finiteness means that the result of integrations over connections should be function of $\pi_{\sigma^2}$ sufficiently rapidly decreasing at infinity. If we try to get monotonic exponent from oscillating one by moving integration contour over curvature to complex plane, we should eventually have decreasing exponent. Consider typical integral

$$
\int \exp \left[ i \left( 1 + \frac{i}{\gamma} \right)^{\pi} + R + i \left( 1 - \frac{i}{\gamma} \right)^{-\pi} \right] \mathcal{D}R.
$$

The $R$ is parameterized by 6 variables $\varphi, \psi$. Let us deform integration contour over $\varphi, \psi$ in the complex space $\mathbb{C}^6$ in the following way.

$$
\frac{1}{2} (\varphi \mp i \psi) \Rightarrow \left[ \begin{array}{c}
- \frac{i}{\gamma} \left( 1 \pm \frac{i}{\gamma} \right)^{\pm \pi} \text{ch}^{\mp \zeta} \\
\sqrt{- 1 + \frac{i}{\gamma}}^{\pm \pi^2} \\
+ i (\text{sh}^{\mp \zeta} (\mp e_1 \cos^{\pm} \chi + \mp e_2 \sin^{\pm} \chi)) \left( \frac{\pi}{2} + i \mp \eta \right).
\end{array} \right]
$$

Here $\mp e_1^2 = 1 = \mp e_2^2$, $\mp e_1 \cdot \mp e_2 = 0$, $\mp e_1 \cdot \mp \pi = 0 = \mp e_2 \cdot \mp \pi$ (that is, orthonormal to $\mp \pi$ the double). The $\sqrt{z}$ is defined in the plane $C$ with cut off along negative real half-axis $\Im z = 0, \Re z \leq 0$ so that $\sqrt{1} = 1$. Integral becomes that over $\mp \eta, \mp \zeta, \mp \chi$.

$$
(4\pi)^2 \int \exp \left[ - \sqrt{- 1 + \frac{i}{\gamma}}^{\pm \pi^2} \text{ch}^{\mp \eta} \text{ch}^{\mp \chi} - \sqrt{- 1 - \frac{i}{\gamma}}^{\pm \pi^2} \text{ch}^{-\eta} \text{ch}^{-\chi} \right] \\
\cdot \text{ch}^{\mp \eta} \text{d}^{\mp \eta} \text{d} \text{ch}^{\mp \zeta} \text{d}^{\pm} \chi \cdot \text{ch}^{-\eta} \text{d}^{-\eta} \text{d} \text{ch}^{-\zeta} \text{d}^{-\chi}
$$

$$
= \frac{K_1 \left[ \sqrt{- 1 + \frac{i}{\gamma}}^{\pm \pi^2} \right]}{\sqrt{- 1 + \frac{i}{\gamma}}^{\pm \pi^2}} \cdot \frac{K_1 \left[ \sqrt{- 1 - \frac{i}{\gamma}}^{\pm \pi^2} \right]}{\sqrt{- 1 - \frac{i}{\gamma}}^{\pm \pi^2}}.
$$

The $K_1$ is the modified Bessel function. Transform (15) has simple sense if divided into two stages. i) Make $\psi$ imaginary so that $(\varphi \mp i \psi)/2$ become independent 3-vector real variables $\pm \phi$. Then it becomes possible to split the measure

$$
\mathcal{D}R = \mathcal{D}^{\pm} \mathcal{D} \mathcal{D}^{\mp} R.
$$

ii) Transform spherical components of $\pm \phi$: move $\sqrt{\pm \phi^2}$ to $\pi/2 + i \pm \eta$, $-\infty < \pm \eta < +\infty$, move the azimuthal angle $\pm \theta$ of $\pm \phi$ w.r.t. $\pm \pi$ to $i \pm \zeta$, $0 \leq \pm \zeta < +\infty$, the polar angle $\pm \chi$ remaining the same.

On physical hypersurface $\mathbb{H}$ $\pm \pi^2 = - \pi^2 \equiv \pi^2$. For spacelike (i.e. usual) areas $\pi^2 < 0$ and we have

$$
\left( 1 + \frac{1}{\gamma} \right)^{-1} (- \pi^2)^{-1} K_1 \left[ 1 + \frac{i}{\gamma} \sqrt{- \pi^2} \right] K_1 \left[ 1 - \frac{i}{\gamma} \sqrt{- \pi^2} \right]
$$

\[ (18) \]
and for timelike areas \((\pi^2 > 0)\)

\[
\left(1 + \frac{1}{\gamma^2}\right)^{-1} (\pi^2)^{-1} K_1 \left[\left(\frac{1}{\gamma} - i\right) \sqrt{\pi^2}\right] K_1 \left[\left(\frac{1}{\gamma} + i\right) \sqrt{\pi^2}\right].
\]  

(19)

Both these have the same asymptotic form at \(\pi^2 \to 0\),

\[
\left(1 + \frac{1}{\gamma^2}\right)^{-2} (\pi^2)^{-2},
\]

but differ at \(|\pi^2| \to \infty\),

\[
\frac{\pi}{2} \left(1 + \frac{1}{\gamma^2}\right)^{-3/2} |\pi^2|^{-3/2} \cdot \left\{ \begin{array}{ll}
\exp \left(-2\sqrt{|\pi^2|}\right), & \text{at } \pi^2 < 0, \\
\exp \left(-\frac{2}{\gamma} \sqrt{|\pi^2|}\right), & \text{at } \pi^2 > 0.
\end{array} \right.
\]

(21)

In usual units (coefficient \((16\pi G)^{-1}\) at the action) and taking into account that \(|\pi^2|^{-1/2}\) is twice \(A\), module of the triangle area, we find exponential decrease proportional to \(\exp(-A(4\pi G)^{-1})\) and \(\exp(-A(4\pi G\gamma)^{-1})\) in the spacelike and timelike regions, respectively.

In reality \(\tau_{\sigma^2} \neq 0\). The \(R_{\sigma^2}\) in the terms \(\mp R_{\sigma^2}\) are by Bianchi identities functions of \(\{R_{\sigma^2}|\sigma^2\text{ not }t\text{-like}\}\). But any given \(R_{\sigma^2}\) for not \(t\)-like \(\sigma^2\) enters these terms only linearly (or does not enter at all). Therefore for any given \(R_{\sigma^2}\) at not \(t\)-like \(\sigma^2\) we have integral of the type \((14)\) but \(\pm \pi_{\sigma^2}\) should be replaced by some matrix \(\pm m_{\sigma^2}\). Here \(\pm m_{\sigma^2}\) differs from \(\pm \pi_{\sigma^2}\) by a sum of (multiplied by products of some \(\pm R_{\sigma^2}\)) some \(\pm \tau_{\sigma^2}\) for the \(\sigma^2\)s forming lateral surface of 3-prism considered below. Area tensor of one of the bases of this prism is just this \(\pm \pi_{\sigma^2}\).

The \(\pm m_{\sigma^2}\) has not only antisymmetric but also trace part,

\[
\pm m = \frac{1}{2} \pm m_k \cdot \pm \Sigma^k + \frac{1}{2} \pm m_0 \cdot 1, \quad 2 \pm m \circ \pm m = \pm m^2 + \pm m_0^2.
\]

(22)

(Here notations \(\pm m, \pm m_0\) do not mean \(\pm\)-parts of anything.) The integral \((14)\) generalises to

\[
(4\pi)^{-2} \int \exp \left[\left(1 + \frac{i}{\gamma}\right) \pm m \circ + R + \left(1 - \frac{i}{\gamma}\right) \pm m \circ - R\right] DR
\]

\[
= K_1 \left[\sqrt{-\left(1 + \frac{i}{\gamma}\right)^2 \cdot 2 \pm m \circ + m}\right] \cdot K_1 \left[\sqrt{-\left(1 - \frac{i}{\gamma}\right)^2 \cdot 2 \pm m \circ - m}\right].
\]

(23)

(Appropriate complex deformation of integration contours reads

\[
\frac{1}{2} (\varphi \pm i\psi) \implies \left[\frac{-i(1 \pm \frac{i}{\gamma}) \pm m}{\sqrt{-\left(1 \pm \frac{i}{\gamma}\right)^2 \pm m^2}} \text{ch} \pm \zeta
\]

\[
+ i(\text{sh} \pm \zeta) \left(\pm e_1 \cos \pm \chi + \pm e_2 \sin \pm \chi\right)\left(\pm \frac{\pi}{2} + i \pm \eta - \pm \beta\right)
\]

(24)
where
\[
(\cos \pm \beta; \sin \pm \beta) = \frac{\left( \sqrt{-\left(1 + \frac{i}{\gamma}\right)^2} \pm m^2 \text{ch} \pm \zeta; \quad -i \left(1 + \frac{i}{\gamma}\right) \mp m_0 \right)}{\sqrt{-\left(1 + \frac{i}{\gamma}\right)^2} \left(\pm m^2 \text{ch} \pm \zeta + \pm m_0^2\right)}. \tag{25}
\]

The idea is to try to find some set of the 2-simplices \(M\) so that exponential in \(\Omega\) be representable in the form
\[
i \sum_{\sigma^2 \in M} \left[ \left(1 + \frac{i}{\gamma}\right) \mp m_{\sigma^2} \circ R_{\sigma^2}(\{+\Omega\}) + \left(1 - \frac{i}{\gamma}\right) \mp m_{\sigma^2} \circ R_{\sigma^2}(\{-\Omega\}) \right] \\
+i \sum_{\sigma^2 \notin M} \left[ \left(1 + \frac{i}{\gamma}\right) \mp \pi_{\sigma^2} \circ R_{\sigma^2}(\{+\Omega\}) + \left(1 - \frac{i}{\gamma}\right) \mp \pi_{\sigma^2} \circ R_{\sigma^2}(\{-\Omega\}) \right] \tag{26}
\]
where \(\pm m_{\sigma^2} = \pm \pi_{\sigma^2} + \) (linear in \(\pm \pi_{\sigma^2}\) terms). The set \(\{\pm m_{\sigma^2}\}\) depend on \(\{\pm \pi_{\sigma^2}\}\) and on \(\{\pm R_{\sigma^2}\}\), but not on \(\{\pm R_{\sigma^2}\} \in M\). Then integrations over \(\{\pm R_{\sigma^2}\} \in M\) can be explicitly performed according to eq. (23). For other \(\{\pm R_{\sigma^2}\} \notin M\) deformation of integration contours according to eq. (15) is made. The result reads
\[
d\mu_{\text{area}} \equiv \mathcal{N}^{-1} \mathcal{N} \prod_{\text{not t-like}} \sigma^2 \left(\pm m_{\sigma^2}\right); \quad \mathcal{N} = \int \frac{\prod_{\sigma^2 \in M} K_1 \left[ \sqrt{-\left(1 + \frac{i}{\gamma}\right)^2} \mp m_{\sigma^2} \circ \mp m_{\sigma^2} \right]}{\sqrt{-\left(1 + \frac{i}{\gamma}\right)^2} \left(\pm m_{\sigma^2} \circ \pm m_{\sigma^2}\right)} \\
\cdot \left[ \sum_{\text{not t-like}} \sigma^2 \notin M \sqrt{-\left(1 + \frac{i}{\gamma}\right)^2} \mp \pi_{\sigma^2} \text{ch} \pm \eta_{\sigma^2} \text{ch} \mp \eta_{\sigma^2} \right] \prod_{\text{not t-like}} \sigma^2 \notin M \left(\text{ch} \mp \eta_{\sigma^2} \text{d} \mp \eta_{\sigma^2} \text{d} \mp \eta_{\sigma^2} \text{ch} \mp \eta_{\sigma^2} \text{d} \mp \eta_{\sigma^2}\right) \tag{27}
\]
where \(\{\pm m_{\sigma^2}\} \in M\) depend on \(\{\pm \eta_{\sigma^2}, \mp \eta_{\sigma^2}, \pm \chi_{\sigma^2}\} \notin M\) through \(R_{\sigma^2}\) parameterized by these,
\[
\pm R_{\sigma^2} = -i \text{sh} \pm \eta_{\sigma^2} + \pm \Sigma \cdot \pm \sigma_{\sigma^2} \text{ch} \mp \eta_{\sigma^2}, \\
\pm \sigma_{\sigma^2} = \frac{-i \left(1 + \frac{i}{\gamma}\right) \pm \pi_{\sigma^2}}{\sqrt{-\left(1 + \frac{i}{\gamma}\right)^2} \mp \pi_{\sigma^2} \text{ch} \pm \eta_{\sigma^2} + i \left(\text{sh} \pm \eta_{\sigma^2}\right) \left(\pm \epsilon_{1\sigma^2} \cos \pm \chi_{\sigma^2} + \pm \epsilon_{2\sigma^2} \sin \pm \chi_{\sigma^2}\right)} \tag{28}
\]
This looks as product of exponentially dumped at large areas multipliers (note that \(\Re \sqrt{z} \geq 0\) for our choice of the branch of function \(\sqrt{z}\) with cut along negative real half-axis in the complex plane of \(z\) such that \(\sqrt{T} = 1\)).

To construct the set \(M\), note that due to the Bianchi identities dependence on the matrix \(R_{\sigma^2}\) on the given leaf/diagonal triangle \(\sigma^2\) in the exponential of (13) comes from all the triangles constituting together with this \(\sigma^2\) a closed surface. This is surface of the \(t\)-like 3-prism, one base of which is just the given \(\sigma^2\), the lateral surface consists
of \( t \)-like triangles and goes to infinity. In practice, replace this infinity by some lowest (initial) leaf where another base \( \sigma_0^2 \) is located the tensor of which \( \pi_\sigma \sigma_0^2 \) is taken as boundary value (it is implied \( \pi_\sigma \sigma_0^2 = 0 \) above). Consider a variety of such prisms with upper bases \( \sigma^2 \) placed in the uppest (final) leaf such that any link in this leaf belongs to one and only one of these bases. That is, lateral surfaces of different prisms do not have common triangles. Then the terms \( \pm m_\sigma \odot \pm R_\sigma \) in (26) represent contribution from these prisms, \( M \) being the set of their bases in the uppest leaf.

To really reduce the measure to such form, we should express the curvature matrices on the \( t \)-like triangles in terms of those on the leaf/diagonal ones. The curvature on a leaf/diagonal triangle \( \sigma^2 \) as product of \( \Omega \)s includes the two matrices \( \Omega \) on the \( t \)-like tetrahedrons \( \sigma^3 \) adjacent to \( \sigma^2 \) from above and from below. Knowing curvatures on the set of leaf/diagonal triangles inside any \( t \)-like 3-prism allows to successively express matrix \( \Omega \) on any \( t \)-like tetrahedron inside the prism in terms of matrix \( \Omega \) on the uppest \( t \)-like tetrahedron in this prism taken as boundary value. Expressions for the considered curvatures look like (fig.1)

\[
R_{(ikl)} = \ldots \Omega^T_{(i-ikl)} \ldots \Omega_{(ik+kl)} \ldots \\
R_{(ik+)} = \ldots \Omega^T_{(ik+kl)} \ldots \Omega_{(ik+t+l)} \ldots \tag{29}
\]

\[
R_{(ik+t+)} = \ldots \Omega^T_{(ik+t+l)} \ldots \Omega_{(i+ik+t+)} \ldots
\]

The dots in expressions for \( R \) mean matrices \( \Omega \) on the leaf/diagonal tetrahedrons which can be considered as gauge ones. We can step-by-step express \( \Omega \)s as follows: \( \Omega_{(i-ikl)} \rightarrow \Omega_{(ik+kl)} \rightarrow \Omega_{(ik+t+l)} \rightarrow \Omega_{(i+ik+t+)} \rightarrow \ldots \) where the arrow means ”in terms of”. Knowing \( \Omega \)s on \( t \)-like tetrahedrons we can find the curvatures on \( t \)-like triangles, the products of these \( \Omega \)s,

\[
R_{(i+ikl)} = \Omega^\epsilon_{(ikn)} \ldots \Omega^\epsilon_{(ikl_n)} \ldots
\]

Here \( \epsilon_{(ikl)}m = \pm 1 \) is some sign function. Thereby we find contribution of the \( t \)-like triangles and thus \( \pm m_\sigma \) in terms of independent (i.e. leaf/diagonal) curvature matrices.

Thus, in order to represent exponential in (13) in the form (26) and thus the measure in the form (27) it is sufficient to divide the whole set of links in the uppest 3D leaf into triples forming the triangles (that is, triangles do not have common edges) and take this set of triangles as \( M \) in (26). In fig.2 example of periodic such set \( M \) (shaded triangles) is shown for periodic simplicial 3D leaf.
Figure 2: Example of the set $\mathcal{M}$ (shaded triangles) with periodic structure such that any edge does belong to one and only one triangle of $\mathcal{M}$. The periodic cell consists of 8 building blocks of the two types C0, C1 alternating in all three directions.

Barbero-Immirzi parameter defines area quantization in Loop Quantum Gravity and, in particular, the number of states on the horizon of black hole. Therefore its possible value was considered in a number of papers from analysis of the black hole entropy [15]-[22]. Suggested values range from 0.127 [17] to 0.274 [19][22]. We see that our measure results in asymmetric picture with timelike areas considerably stronger suppressed than the spacelike ones.

Thus, quantum measure is exponentially suppressed at large areas. The Barbero-Immirzi parameter proves to play significant role in RC for it provides good behavior of the measure on areas in the timelike region.

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