Cosmological Constraints on Nonflat Exponential $f(R)$ Gravity

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Abstract

We explore the viable $f(R)$ gravity models in FLRW backgrounds with a free spatial curvature parameter $\Omega_K$. In our numerical calculation, we concentrate on the exponential $f(R)$ model of $f(R) = R - \lambda R_{ch}(1 - \exp(-R/R_{ch}))$, where $R_{ch}$ is the characteristic curvature scale, which is independent of $\Omega_K$, and $\lambda$ corresponds to the model parameter, while $R_{ch}, \lambda = 2\Lambda$ with $\Lambda$ the cosmological constant. In particular, we study the evolutions of the dark energy density and equation of state for exponential $f(R)$ gravity in open, flat, and closed universes, and compare with those for $\Lambda CDM$. From the current observational data, we find that $\lambda = 0.42927^{+0.39921}_{-0.32927}$ at 68% C.L. and $\Omega_K = -0.00050^{-0.00420\ast}_{+0.00414}$ at 95% C.L. in the exponential $f(R)$ model. By using the Akaïke Information Criterion, Bayesian Information Criterion, and Deviance Information Criterion, we conclude that there is no strong preference between the exponential $f(R)$ gravity and $\Lambda CDM$ models in the nonflat universe.

Unified Astronomy Thesaurus concepts: Dark energy (351); Curved space (345)

1. Introduction

Cosmological observations hint that our universe has been experiencing another accelerating expansion in the recent epoch (Riess et al. 1998; Perlmutter et al. 1999) besides inflation in the very early time. However, the origin of the late-time acceleration remains a mystery. Although the $\Lambda CDM$ model, in which the cosmological constant $\Lambda$ plays the role of dark energy, could give an explanation about this problem, it still suffers from some difficulties, such as the cosmological constant problem (Weinberg 1989; Peebles & Ratra 2003) and Hubble tension (Riess et al. 2019). To describe our accelerating universe, many models with dynamical dark energy (Copeland et al. 2006) beyond $\Lambda CDM$ have been proposed. In particular, there are two representative approaches, in which one is to introduce some unknown matters called “dark energy” in the framework of general relativity (Copeland et al. 2006; Li et al. 2011), and the other is to modify the gravitational theory, e.g., $f(R)$ gravity (Sotiriou & Faraoni 2010; De Felice & Tsujikawa 2010; Nojiri & Odintsov 2011).

It is known that $f(R)$ gravity replaces the Ricci scalar, $R$, in the Einstein–Hilbert action with an arbitrary function of $f(R)$. Several viable models have been constructed in $f(R)$ gravity (De Felice & Tsujikawa 2010; Bamba et al. 2010a), such as Starobinsky (Starobinsky 2007), Hu–Sawicki (Hu & Sawicki 2007), Tsujikawa (Tsujikawa 2008; Cen et al. 2019), and exponential (Linder 2009; Bamba et al. 2010b) models. These models satisfy the following viable conditions (De Felice & Tsujikawa 2010; Bamba et al. 2010a): (1) the positivity of effective gravitational couplings; (2) the stability of cosmological perturbations; (3) the asymptotic behavior to $\Lambda CDM$ in the large curvature regime; (4) the stability of the late-time de Sitter point; (5) constraints from the equivalence principle; and (6) solar system constraints. In this study, to illustrate our numerical results, we concentrate on the exponential $f(R)$ gravity model, which contains only one more parameter than the standard $\Lambda CDM$ model of cosmology.

Recently, the survey of the Planck 2018 CMB data along with $\Lambda CDM$ has suggested that our universe is closed at 99% C.L. (Di Valentino et al. 2020). Motivated by this result, we would like to examine the viable $f(R)$ gravity models without the spatial flatness assumption and explore the constraints on the models from the recent observational data. We would also compare viable $f(R)$ gravity with $\Lambda CDM$ with the spatial curvature parameter $\Omega_K$ set to be free. We note that the study of the viable $f(R)$ gravity models with an arbitrary spatial curvature has not been performed in the literature yet. To illustrate our results, we will concentrate on the viable exponential $f(R)$ model.

The paper is organized as follows. In Section 2, we review the Friedmann equations in $f(R)$ gravity in the nonflat background. In Section 3, we present the cosmological evolutions of the dark energy density parameter and equation of state in open, flat, and closed exponential $f(R)$ gravity models and constrain the model parameters by using the Markov Chain Monte Carlo (MCMC) method. We summarize our results in Section 4.

2. $f(R)$ Gravity in Spatially NonFlat FLRW Spacetime

The action of $f(R)$ gravity is given by

$$S = \int d^4x \sqrt{-g} \frac{k^2}{2\kappa^2} f(R) + S_M,$$

where $\kappa^2 = 8\pi G$ with $G$ the Newton’s constant, and $S_M$ is the action for both relativistic and nonrelativistic matter. In the viable exponential gravity model, $f(R)$ is given by (Zhang 2006; Tsujikawa 2008; Linder 2009; Bamba et al. 2010b; Yang et al. 2010)

$$f(R) = R - \lambda R_{ch}(1 - e^{-R/R_{ch}}),$$
where \( R_{\text{ch}} \) is related to the characteristic curvature modification scale. Based on the viable \( f(R) \) conditions, one has that, when \( R \to \infty \), \( f(R) \to R - 2\Lambda \), the product of \( \Lambda \) and \( R_{\text{ch}} \) corresponds to the cosmological constant by the relation of \( \lambda R_{\text{ch}} = 2\Lambda \). As a result, there is only one additional model parameter in the exponential gravity model in (2).

By varying the action (1), field equations for \( f(R) \) gravity can be found to be

\[
FR_{\mu\nu} - \frac{1}{2}g_{\mu\nu} f - \nabla_\mu \nabla_\nu F + g_{\mu\nu} \Box F = \kappa^2 T^{(M)}_{\mu\nu},
\]

where \( F \equiv df(R)/dR, \Box \equiv \nabla^{\mu} \nabla_\mu \) is the d’Alembert operator, and \( T^{(M)}_{\mu\nu} \) represents the energy–momentum tensor for relativistic and nonrelativistic matter. The above Equation (3) can also be written as

\[
G_{\mu\nu} = \kappa^2 (T^{(M)}_{\mu\nu} + T^{(de)}_{\mu\nu}),
\]

where \( G_{\mu\nu} = R_{\mu\nu} - (1/2)g_{\mu\nu}R \) is the Einstein tensor and

\[
T^{(de)}_{\mu\nu} = \frac{1}{\kappa^2} \left( F_{\mu\nu} - FR_{\mu\nu} + \frac{1}{2}g_{\mu\nu} f - \nabla_\mu \nabla_\nu F - g_{\mu\nu} \Box F \right),
\]

stands for the energy–momentum tensor for dark energy.

### 2.1. Modified Friedmann Equations

We consider the spatially nonflat Friedmann–Lemaître–Robertson–Walker (FLRW) spacetime, given by

\[
ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{1 - Kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right),
\]

where \( a(t) \) is the scale factor, and \( K = -1, 0, \) and \( 1 \) represent the spatially open, flat, and closed universes, respectively. By applying metric (6) into Equation (3), we obtain the modified Friedmann equations, given by

\[
3FH^2 + \frac{3KF}{a^2} = \frac{1}{2} (FR - f) - 3HF + \kappa^2 \rho_M,
\]

\[
\ddot{F} = H\dot{F} - 2FH + \frac{2KF}{a^2} - \kappa^2 (\rho_M + P_M),
\]

where \( H = \dot{a}/a \) is the Hubble parameter, the dot “\( \cdot \)” denotes the derivative w.r.t the cosmic time \( t \), and the Ricci scalar \( R \) takes the form

\[
R = 12H^2 + 6\dot{H} + \frac{6K}{a^2}.
\]

In order to study the behavior of dark energy and the effects of spatial curvature, we rewrite the modified Friedmann equations in (7) and (8) as

\[
H^2 = \frac{\kappa^2}{3}(\rho_M + \rho_{de} + \rho_K),
\]

\[
\dot{H} = -\frac{\kappa^2}{2}(\rho_M + \rho_{de} + \rho_K + P_M + P_{de} + P_K),
\]

where \( \rho_M = \rho_m + \rho_r \) is the density of nonrelativistic matter and radiation, while the dark energy density and pressure are given by

\[
\rho_{de} = \frac{3}{\kappa^2} \left( H^2 (1 - F) - \frac{1}{6}(f - FR) - HF - \frac{K}{a^2} (1 - F) \right),
\]

\[
P_{de} = \frac{1}{\kappa^2} \left( \dot{F} + 2HF + \frac{1}{2} (f - FR) - (1 - F) \left( 3H^2 + 2H + \frac{K}{a^2} \right) \right),
\]

respectively. Here, the effects of spatial curvature in the modified Friedmann equations can be described by the effective energy density and pressure, written as

\[
\rho_K = -\frac{3K}{\kappa^2 a^2},
\]

\[
P_K = \frac{K}{\kappa^2 a^2},
\]

respectively. Note that the energy density and pressure for nonrelativistic matter, radiation, dark energy, and spatial curvature satisfy the continuity equation

\[
\frac{d\rho_i}{dt} + 3H (1 + w_i) \rho_i = 0,
\]

where \( w_i \) with \( i = (m, r, de, K) \) represent the corresponding equations of state, defined by

\[
w_i \equiv \frac{P_i}{\rho_i},
\]

respectively. By rewriting the Friedmann equation (10) in terms of observational parameters, we have

\[
1 = \Omega_m + \Omega_r + \Omega_{de} + \Omega_K,
\]

where \( \Omega_i \) are the corresponding density parameters, defined by

\[
\Omega_i = \frac{\kappa^2 \rho_i}{3H^2}.
\]

From (14), we have that \( \Omega_K = -K/(aH)^2 \) with \( \Omega_K > 0, = 0 \) and <0 for open, flat, and closed universes, respectively.

In order to solve the modified Friedmann equations numerically, we define the dimensionless parameters \( y_H \) and \( y_R \) to be

\[
y_H \equiv \frac{\rho_{de}}{\rho_m^{(0)}} = \frac{H^2}{m^2} - a^{-3} - \chi a^{-4} - \beta a^{-2},
\]

\[
y_R = \frac{R}{m^2} - 3a^{-3},
\]

where \( m^2 = \kappa^2 \rho_m^{(0)}/3, \chi = \rho_r^{(0)}/\rho_m^{(0)} \), and \( \beta = \rho_K^{(0)}/\rho_m^{(0)} \) with \( \rho_r^{(0)} \equiv \rho_r(z = 0) \). It can be found from (7) that these two parameters obey the equations

\[
\frac{dy_H}{d \ln a} = \frac{y_R}{3} - 4y_H,
\]

\[
\frac{dy_R}{d \ln a} = \frac{1}{m^2} \frac{dR}{d \ln a} + 9a^{-3}.
\]
As a result, one is able to combine these two first-order differential equations into a single second-order equation
\[ y''_H + J_1y'_H + J_2y_H + J_3 = 0, \tag{24} \]
where the prime "\( \prime \)" denotes the derivative w.r.t \( \ln a \), and
\[ J_1 = 4 + \frac{1}{y_H + a^{-3} + \chi a^{-4} + \beta a^{-2} - \frac{1}{6m^2F_{-R}}} - F, \tag{25} \]
\[ J_2 = \frac{1}{y_H + a^{-3} + \chi a^{-4} + \beta a^{-2} - \frac{2}{3m^2F_{-R}}}, \tag{26} \]
\[ J_3 = -3a^{-3} - \frac{(1 - F)(a^{-3} + 2\chi a^{-4}) + (R - f)/3m^2}{y_H + a^{-3} + \chi a^{-4} + \beta a^{-2}} \times \frac{1}{6m^2F_{-R}}. \tag{27} \]

With the differential equation in (24), the cosmological evolution can be calculated through the various existing programs in the literature.

### 3. Numerical Calculations

In this section, we study the background evolutions of the dark energy density parameter and equation of state for the exponential \( f(R) \) model without the spatial flatness assumption. We modify the CAMB (Lewis et al. 2000) program at the background level (Dalal et al. 2001; Liddle 2007; Chen et al. 2010; Rezaei & Malekjani 2021; Zheng et al. 2021) and use the CosmoMC (Lewis & Bridle 2002) package, which is an MCMC engine, to explore the cosmological parameter space and constrain the exponential \( f(R) \) model from the observational data.

#### 3.1. Cosmological Evolution

To examine the cosmological evolution of dark energy for the viable exponential gravity model, we plot the density parameter \( \Omega_{DE} \) and equation of state \( w_{DE} \) of the model in Figures 1 and 2, respectively. From the previous studies of the exponential \( f(R) \) gravity model, the model and spatial curvature density parameters are constrained to be \( 0.392 < \lambda^{-1} < 0.851 \) (Chen et al. 2019) and \( -0.0011 < \Omega_{DE} < 0.0027 \) at 68% C.L. (Vagnozzi et al. 2021), respectively. In this work, we choose \( \lambda^{-1} = 0.5 \) and \( \Omega_{DE} = \pm 0.001 \) to see the behavior of exponential \( f(R) \) gravity. The initial conditions are set to be \( (\Omega_m^0, \Omega_{DE}^0) = (0.2998, 1.5 \times 10^{-3}), (\Omega_{DE}^0, \Omega_K^0) = (0.001, 0, -0.001) \) for the (open, flat, and closed) universe models, and \( H_0 = 67 \text{ km s}^{-1} \text{ Mpc}^{-1} \).

Figure 1 shows the evolutions of dark energy for exponential \( f(R) \) gravity with \( \lambda^{-1} = 0.5 \) and \( \Lambda \text{CDM} \) with \( \Omega_{DE}^0 = (0.001, 0, -0.001) \). From the figures, we see that the dark energy density parameter for exponential \( f(R) \) gravity is slightly larger (smaller) than that of \( \Lambda \text{CDM} \) when \( 10^{-1} < z < 10^0 \) and \( z < 10^0 \). It approaches the cosmological constant in the high-redshift region as a characteristic of the viable \( f(R) \) gravity models. It is clear that the deviation for \( f(R) \) gravity from flat \( \Lambda \text{CDM} \) is small, with \( |(\Omega_{DE} - \Omega_{CDM,flat})|/\Omega_{CDM,flat} | < 5\% \). Note that \( \Omega_{DE} \) for \( f(R) \) gravity in the closed universe has a larger value in
comparison with the other cases. Figure 2 illustrates equation of state \( w_{DE} \) for dark energy as a function of \( z \). We can see that \( w_{DE} \) evolves from the phantom phase \((w_{DE} < -1)\) to the nonphantom phase \((w_{DE} > -1)\) for a fixed value of \( \Omega_K^0 \).

With the initial conditions and

\[
t_{\text{age}} = \frac{1}{H_0} \int_0^1 \frac{da}{a\sqrt{\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_K a^{-2} + \Omega_{de}(a)}},
\]

we can calculate the age of the universe. Consequently, we obtain that

\[
t_{\text{age}}^{\text{open}} = 14.021, 14.045 \text{ Gyr} \quad (\Omega_K^0 = 0.001) \quad (29)
\]

\[
t_{\text{age}}^{\text{flat}} = 14.025, 14.049 \text{ Gyr} \quad (\Omega_K^0 = 0) \quad (30)
\]

\[
t_{\text{age}}^{\text{closed}} = 14.028, 14.054 \text{ Gyr} \quad (\Omega_K^0 = -0.001), \quad (31)
\]

for the exponential \( f(R) \) and \( \Lambda \text{CDM} \) models in the open, flat, and closed universes, respectively. From (29), (30), and (31), we see that the age of the universe for exponential \( f(R) \) gravity is shorter than the corresponding one for \( \Lambda \text{CDM} \). Note that the larger value of \( t_{\text{age}} \) is related to the longer growth time of the large-scale structure (LSS) and larger matter-density fluctuations.

### 3.2. Global Fitting

In this section, we present constraints on the cosmological parameters in the exponential \( f(R) \) model without the spatial flatness assumption. With the modifications of CAMB at the background level and the CosmoMC package, we perform the MCMC analysis.

To break the geometrical degeneracy (Efstathiou & Bond 1999; Howlett et al. 2012), we fit the model with the combinations of the observational data, including CMB temperature and polarization angular power spectra from Planck 2018 with high-l TT, TE, EE, low-l TT, EE, CMB lensing from SMICA (Planck Collaboration et al. 2020a, 2020b, 2020c, 2020d), BAO observations from 6\(^\circ\) Field Galaxy Survey (6dF) (Beutler et al. 2011), SDSS DR7 Main Galaxy Sample (MGS) (Ross et al. 2015), and BOSS Data Release 12 (DR12) (Alam et al. 2017), and supernova (SN) data from the Pantheon compilation (Scolnic et al. 2018). As we set the density parameter of curvature and the neutrino mass sum to be free, our fitting for the exponential \( f(R) \) model contains nine free parameters, where the priors are listed in Table 1.

To find the best-fit results, we minimize the \( \chi^2 \) function, which is given by

\[
\chi^2 = \chi^2_{\text{CMB}} + \chi^2_{\text{BAO}} + \chi^2_{\text{Pan}}.
\]

Explicitly, we take

\[
\chi^2_{\text{CMB}} = \sum_{l,l'} (C_l^\text{obs} - C_l^\text{th}) M_{ll'}^{-1} (C_l^\text{obs} - C_l^\text{th}),
\]

where \( C_l^{\text{obs}} \) corresponds to the observational (theoretical) value of the related power spectrum, and \( M \) is the covariance matrix for the CMB data (Planck Collaboration et al. 2020a, 2020e).

For the BAO data, we adopt the data set from the 6\(^\circ\) Field Galaxy Survey (6dF) at \( z_{\text{eff}} = 0.106 \) (Beutler et al. 2011), the SDSS DR7 Main Galaxy Sample (MGS) at \( z_{\text{eff}} = 0.15 \) (Ross et al. 2015), and BOSS Data Release 12 (DR12) at \( z_{\text{eff}} = 0.38, 0.51, 0.61 \) (Alam et al. 2017). As a result, we have

\[
\chi^2_{\text{BAO}} = \chi^2_{6\text{dF}} + \chi^2_{\text{MGS}} + \chi^2_{\text{DR12}}.
\]

For the uncorrelated data points, such as 6dF and MGS, \( \chi^2 \) is given by

\[
\chi^2 (p) = \sum_{i=1}^{N} \frac{[A_{\text{th}} (z_i) - A_{\text{obs}} (z_i)]^2}{\sigma_i^2},
\]

where \( A_{\text{th}} (z_i) \) is the predicted value computed in the model under consideration, and \( A_{\text{obs}} (z_i) \) denotes the measured value at \( z_i \) with the standard deviation \( \sigma_i \). Note that, in our study, we adopted 6dF and MGS, whose standard deviations are given by \( \sigma_{6\text{dF}} = 0.015 \) and \( \sigma_{\text{MGS}} = 0.168 \), respectively. The data points from DR12 are correlated. In this case, \( \chi^2 \) is given by

\[
\chi^2_{\text{DR12}} = [A_{\text{th}} - A_{\text{obs}}] \cdot C^{-1} \cdot [A_{\text{th}} - A_{\text{obs}}]_t,
\]

where \( C^{-1} \) is the inverse of the covariance matrix, which is a 6 \( \times \) 6 matrix given in Equation (20) of Ryan et al. (2019).

For the Pantheon SN Ia samples, there are 1048 data points scattering between 0.01 \( \leq z \leq 2.3 \), with the observable to be the distance modulus \( \mu \) defined in Ref. (Scolnic et al. 2018). We

### Table 1

| Parameter                  | Priors                                                                 |
|---------------------------|------------------------------------------------------------------------|
| \( f(R) \) model parameter \( \lambda^{-1} \) | \( 0.1 \leq \lambda^{-1} \leq 1 \)                                    |
| Curvature parameter \( \Omega_K \) | \(-0.1 \leq \Omega_K \leq 0.1 \)                                      |
| Baryon density            | \( 0.5 \leq \Omega_b h^2 \leq 10 \)                                   |
| CDM density               | \( 0.1 \leq \Omega_c h^2 \leq 99 \)                                   |
| Optical depth             | \( 0.01 \leq \tau \leq 0.8 \)                                        |
| Neutrino mass sum         | \( 0 \leq \Sigma m_\nu \leq 2 \text{ eV} \)                           |
| Angular size of the sound horizon | \( 0.5 \leq 100 h_{\text{BAO}} \leq 10 \)                             |
| Scalar power spectrum amplitude | \( 1.61 \leq \ln (10^3 \Delta A) \leq 3.91 \)                          |
| Spectral index            | \( 0.8 \leq n_s \leq 1.2 \)                                           |

Note. This table contains priors of cosmological parameters for exponential \( f(R) \) and \( \Lambda \text{CDM} \) models. Note that the angular size of the sound horizon is defined as \( \theta_{\text{BAO}} = r_s/\Delta A \), where \( r_s \) and \( \Delta A \) represent the sound horizon and the angular diameter distance at recombination, respectively.
have that

$$\chi^2_{\text{Pan}} = (\mathbf{\mu}_{\text{obs}} - \mathbf{\mu}_{\text{th}})^T \cdot \mathbf{Cov}^{-1} \cdot (\mathbf{\mu}_{\text{obs}} - \mathbf{\mu}_{\text{th}}),$$

(37)

where $\mathbf{Cov}^{-1}$ is the inverse covariance matrix (Scolnic et al. 2018) of the sample including the contributions from both the statistical and systematic errors. The covariance matrix of Pantheon samples can also be found in the website.4

The global fitting results of the exponential $f(R)$ model without the spatial flatness assumption with CMB+BAO+SN data sets are shown in Figure 3 and listed in Table 2. We note that the exponential $f(R)$ model is barely distinguishable from $\Lambda$CDM. This statement is in agreement with the previous work in the flat universe (Geng et al. 2015). From Table 2, we see that the model and curvature density parameters are constrained

4 http://supernova.lbl.gov/Union/

Table 2

| Model        | Exp $f(R)$ | $\Lambda$CDM |
|--------------|------------|---------------|
| $\Omega_b h^2$ | 0.0224 $^{+0.00032}_{-0.00031}$ | 0.02242 $^{+0.00031}_{-0.00031}$ |
| $\Omega_c h^2$ | 0.1195 $^{+0.00270}_{-0.00266}$ | 0.11948 $^{+0.00268}_{-0.00263}$ |
| $\tau$ | 0.0553 $^{+0.01518}_{-0.01484}$ | 0.05558 $^{+0.01567}_{-0.01507}$ |
| $\Omega_K$ | 0.00050 $^{+0.000420}_{-0.000414}$ | 0.00050 $^{+0.000400}_{-0.000403}$ |
| $\Sigma m_\nu$ [eV] | $<0.06816$ | $<0.06121$ |
| $\lambda^{-1}$ | 0.4292 $^{+0.19927}_{-0.19297}$ | ... |
| $H_0$ [km/s/Mpc] | 67.73 $^{+1.1593}_{-1.4549}$ | 67.95 $^{+1.23489}_{-1.22664}$ |
| Age/Gyr | 13.75 $^{+0.1588}_{-0.1585}$ | 13.75 $^{+0.1576}_{-0.1554}$ |

$\chi^2_{\text{min}}$ 3821.50 3821.84

Note. The constraints of cosmological parameters for exponential $f(R)$ and $\Lambda$CDM models fitted with CMB+BAO+SN data sets, where the cosmological parameters are given at 95% C.L, while $\lambda^{-1}$ and $\Sigma m_\nu$ are given at 68% C.L.
to be $\chi^2 = 0.42927^{+0.39921}_{-0.32921}$ at 68% C.L. and $\Omega_K = -0.00050^{+0.00420}_{-0.00414}$ at 95% C.L. for the exponential $f(R)$ model, respectively. Note that the flat $\Lambda$CDM model is recovered when $\chi^2 = 0$ and $\Omega_K = 0$. We also obtain that $\chi^2 = 3821.50\ (3821.84)$ for $f(R)$ ($\Lambda$CDM) with $x_{f(R)}^2 \lesssim x_{\Lambda\text{CDM}}^2$, indicating that exponential $f(R)$ is consistent with $\Lambda$CDM. The neutrino mass sum is evaluated to be $\Sigma m_\nu < 0.06816\ (0.06121)$ for $f(R)$ ($\Lambda$CDM) at 68% C.L., which is relaxed at 11% comparing with $\Lambda$CDM. This phenomenon is caused by the shortened age of the universe in the exponential $f(R)$ model, in which the matter-density fluctuation is suppressed as discussed in (Chen et al. 2019) and Section 3.1.

To compare exponential $f(R)$ gravity with $\Lambda$CDM, we introduce the Akaike Information Criterion (AIC) (Akaike 1974), Bayesian Information Criterion (BIC) (Schwarz 1978), and Deviance Information Criterion (DIC) (Spiegelhalter et al. 2002). The AIC is defined through the maximum likelihood $L_{\text{max}}$ (satisfying $-2 \ln L_{\text{max}} \propto x_{\text{min}}^2$ under the Gaussian likelihood assumption) and the number of model parameters, $d$:

$$AIC = -2 \ln L_{\text{max}} + 2d = x_{\text{min}}^2 + 2d.$$  

(38)

The BIC is defined as

$$BIC = -2 \ln L_{\text{max}} + d \ln N = x_{\text{min}}^2 + d \ln N,$$

(39)

where $N$ is the number of data points. The DIC is determined by the quantities obtained from posterior distributions, given by

$$DIC = D(\bar{\theta}) + 2p_D,$$

(40)

where $D(\theta) = -2 \ln L(\theta) + C$, $p_D = D(\bar{\theta}) - D(\bar{\theta})$, $C$ is a constant, and $p_D$ represents the effective number of parameters in the model.

We now compute the AIC, BIC, and DIC values from CMB +BAO+SN samples described above for both models, with the difference given by $\Delta AIC = AIC_{f(R)} - AIC_{\Lambda\text{CDM}} = 1.66$, $\Delta BIC = BIC_{f(R)} - BIC_{\Lambda\text{CDM}} = 7.85$, and $\Delta DIC = DIC_{f(R)} - DIC_{\Lambda\text{CDM}} = 1.49$, respectively. The results are summarized in Table 3, where the differences are residuals with respect to the $\Lambda$CDM model. Due to $\Delta AIC$, $\Delta DIC < 2$ being small, there is no strong preference between the exponential $f(R)$ and $\Lambda$CDM models in terms of AIC and DIC (Rezaei & Malekjani 2021). However, as $\Delta BIC < 10$, there is strong evidence against the exponential $f(R)$ model (Liddle 2007).

### 4. Conclusions

We have considered the exponential $f(R)$ gravity model without the spatial flatness assumption. We have derived the energy density ($\rho_{\text{de}}$) and pressure ($p_{\text{de}}$) of dark energy, and simplified the modified Friedmann equations into a second-order differential equation in Equations (24)–(27) with the involvement of the spatial curvature $K$. In our numerical calculations, by modifying the CAMB program for the exponential $f(R)$ model in open, flat, and closed universes, we have studied the cosmological evolutions of the dark energy density parameter and equation of state. We have found that exponential $f(R)$ has a shortened age of the universe comparing with $\Lambda$CDM. To constrain the cosmological parameters of the exponential $f(R)$ model, we have used the CosmoMC package to explore the parameter space. In particular, we have obtained that $\chi^2 = 0.42927^{+0.39921}_{-0.32921}$ at 68% C.L. and $\Omega_K = -0.00050^{+0.00420}_{-0.00414}$ at 95% C.L. In addition, we have gotten that $\chi^2 = 3821.50\ (3821.84)$ for $f(R)$ ($\Lambda$CDM) with $x_{f(R)}^2 \lesssim x_{\Lambda\text{CDM}}^2$, which matches the previous work in the flat universe (Yang et al. 2010). We have also evaluated the AIC, BIC, and DIC values for the exponential $f(R)$ and $\Lambda$CDM models. We have shown that the $\Lambda$CDM model is slightly more preferable in terms of BIC, but such a preference has not been found based on the AIC and DIC results.

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### References

Akaike, H. 1974, *ITAC*, 19, 716

Alam, S., Ata, M., Bailey, S., et al. 2017, *MNRAS*, 470, 2617

Bamba, K., Geng, C.-Q., & Lee, C.-C. 2010a, *J. Cosmol. Astropart. Phys.*, 2010, 001

Bamba, K., Geng, C.-Q., & Lee, C.-C. 2010b, *J. Cosmol. Astropart. Phys.*, 2010, 021

Beutler, F., Blake, C., Colless, M., et al. 2011, *MNRAS*, 416, 3017

Cen, J.-Y., Chien, S.-Y., Geng, C.-Q., & Lee, C.-C. 2019, *PDU*, 26, 100375

Dalal, N., Abazajian, K., Jenkins, E., & Manohar, A. V. 2001, *PhRvL*, 87, 141302

De Felice, A., & Tsujikawa, S. 2010, *LRR*, 13, 3

Di Valentino, E., Melchiorri, A., & Silk, J. 2020, *NatAs*, 4, 196

Efstathiou, G., & Bond, J. R. 1999, *MNRAS*, 304, 75

Geng, C.-Q., Lee, C.-C., & Shen, J.-L. 2015, *PhLB*, 740, 285

Howlett, C., Lewis, A., Hall, A., & Challinor, A. 2012, *J. Cosmol. Astropart. Phys.*, 2012, 027

Hu, W., & Sawicki, I. 2007, *PhRvD*, 76, 064004

Lewis, A., & Bridle, S. 2002, *PhRvD*, 66, 103511

Lewis, A., Challinor, A., & Lasenby, A. 2000, *ApJ*, 538, 473

Li, M., Li, X.-D., Wang, S., & Wang, Y. 2011, *CPTPh*, 56, 525

Liddle, A. R. 2007, *MNRAS*, 377, L74

Linder, E. V. 2009, *PhRvD*, 80, 123528

Nojiri, S., & Odintsov, S. D. 2011, *PhR*, 505, 59

Peebles, P. J., & Ratra, B. 2003, *RvMP*, 75, 559

Perlmuter, S., Aldering, G., Goldhaber, G., et al. 1999, *ApJ*, 517, 565

Planck Collaboration, Aghanim, N., Akrami, Y., et al. 2020a, *A&A*, 641, A6

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**Table 3**

| Model       | $\chi^2_{\text{min}}$ | AIC     | $\Delta$AIC | BIC     | $\Delta$BIC | DIC     | $\Delta$DIC |
|-------------|-----------------------|---------|-------------|---------|-------------|---------|------------|
| $\Lambda$CDM| 3821.84               | 3837.84 | 0           | 3887.35 | 0           | 3850.38 | 0          |
| Exp($f(R)$)| 3821.50               | 3839.50 | 1.66        | 3895.20 | 7.85        | 3851.87 | 1.49       |

**Note.** The AIC, BIC, and DIC values are computed from the sample we use for both $\Lambda$CDM and exponential $f(R)$ models. The differences are calculated with respect to $\Lambda$CDM model.
Planck Collaboration, Aghanim, N., Akrami, Y., et al. 2020b, A&A, 641, A8
Planck Collaboration, Akrami, Y., Arroja, F., et al. 2020c, A&A, 641, A9
Planck Collaboration, Aghanim, N., Akrami, Y., et al. 2020d, A&A, 641, A5
Planck Collaboration, Aghanim, N., Akrami, Y., et al. 2020e, A&A, 641, A8
Rezaei, M., & Malekjani, M. 2021, EPJP, 136, 219
Riess, A. G., Casertano, S., Yuan, W., Macri, L. M., & Scolnic, D. 2019, ApJ, 876, 85
Riess, A. G., Filippenko, A. V., Challis, P., et al. 1998, AJ, 116, 1009
Ross, A. J., Samushia, L., Howlett, C., et al. 2015, MNRAS, 449, 835
Ryan, J., Chen, Y., & Ratra, B. 2019, MNRAS, 488, 3844
Schwarz, G. 1978, AnSta, 6, 461
Scolnic, D. M., Jones, D. O., Rest, A., et al. 2018, ApJ, 859, 101
Sotiriou, T. P., & Faraoni, V. 2010, RvMP, 82, 451
Spiegelhalter, D. J., Best, N. G., Carlin, B. P., & Van Der Linde, A. 2002, J. R. Stat. Soc. Series B Stat. Methodol., 64, 583
Starobinsky, A. A. 2007, JETPL, 86, 157
Tsujikawa, S. 2008, PhRvD, 77, 023507
Vagnozzi, S., Di Valentino, E., Gariazzo, S., et al. 2021, PDU, 33, 100851
Weinberg, S. 1989, RvMP, 61, 1
Yang, L., Lee, C.-C., Luo, L.-W., & Geng, C.-Q. 2010, PhRvD, 82, 103515
Zhang, P. 2006, PhRvD, 73, 123504
Zheng, J., Chen, Y., & Zhu, Z.-H. 2021, arXiv:2107.08916