Poisson bivectors and Poisson brackets on affine derived stacks

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Abstract

Let Spec $A$ be an affine derived stack. We give two proofs of the existence of a canonical map from the moduli space of shifted Poisson structures (in the sense of [PTVV]) on Spec $A$ to the moduli space of homotopy (shifted) Poisson algebra structures on $A$. The first makes use of a more general description of the Poisson operad and of its cofibrant models, while the second is more computational and involves an explicit resolution of the Poisson operad.

Introduction

In classical Poisson geometry, one defines a Poisson structure on a smooth manifold to be a Poisson bracket on the algebra of global functions, which is just a Lie bracket compatible with the product of functions. This notion (which is of algebraic nature) has a more geometric version. The geometric analog of skew-symmetric biderivations are bivector fields, and quite expectedly one can define a Poisson structure to be a bivector field satisfying some additional property. The equivalence of the two definitions of Poisson structure is a well-known fact in classical algebraic or differential geometry. It should be noted that the geometric approach becomes obligatory in the case of complex manifolds.

Recently, in their paper [PTVV] Pantev, Toën, Vaquié and Vezzosi introduced the notion of symplectic and Poisson structures in the context of derived algebraic geometry. Informally speaking, derived algebraic geometry is the study of spaces whose local models are derived commutative algebras, that is to say simplicial commutative algebras. If we suppose to be working over a base field of characteristic zero, the local models can also be taken to be non-positively graded commutative dg-algebras. See [To1] for a recent survey, or [HAG-I], [HAG-II], [Lu] for a complete treatment of the subject.

To be a bit more specific, [PTVV] use the bivector approach to define a $n$-Poisson structure on a (nice enough) derived algebraic stack.

**Definition** ( [PTVV], [To1] ). Let $X$ be a (nice enough) derived algebraic stack, and let $n \in \mathbb{Z}$. The space of $n$-shifted Poisson structures on $X$ is the simplicial set

$$\text{Pois}(X, n) := \text{Map}_{\text{dgLie}}(k[-1](2), \text{Pol}(X, n)[n + 1])$$

(explicitly, $X$ has to be a derived Artin stack locally of finite presentation over $\mathbb{k}$. This means in particular that its cotangent complex is perfect.)
where $k[-1](2)$ is concentrated in degree 1, pure of weight 2, and has the trivial bracket. The graded complex $\text{Pol}(X,n)$ is the complex of $n$-shifted polyvector fields.

The purpose of this paper is to show that, at least for a nice enough affine derived stack $\text{Spec } A$ (where $A$ is a derived commutative algebra), the equivalence between Poisson bivectors and Poisson brackets remains true. As we are working in an inherently homotopical context, Poisson brackets have to be intended up to homotopy: these are basically $P_{n,\infty}$-structures on $A$ whose (weakly) commutative product is (equivalent to) the one given on $A$.

With this goal in mind, after having fixed our notational conventions in Section 1, we study in Section 2 the relation between the categories of dg-operads and of graded dg-Lie algebras. In particular, we would like to be able to describe the moduli space of Poisson brackets on a given commutative algebra via a mapping space in the category of graded dg-Lie algebras. This is accomplished in greater generality in Theorem 2.11.

In Section 3, we apply the results of the previous section to derived algebraic geometry, and we eventually obtain the following result

**Theorem.** Let $X = \text{Spec } A$ be an affine derived stack, and let $P_{n+1}^h(A)$ be the homotopy fiber of the morphism of simplicial sets

$$\text{Map}_{\text{dgOp}}(P_{n+1}, \text{End}_A) \longrightarrow \text{Map}_{\text{dgOp}}(\text{Comm}, \text{End}_A)$$

taken at the point $\mu_A$ corresponding to the given (strict) multiplication in $A$.

Then there is a natural map in the homotopy category of simplicial sets

$$\text{Pois}(X,n) \longrightarrow P_{n+1}^h(A).$$

Moreover, this is an isomorphism if $L_X$ is a perfect complex.

Here $\text{End}_A$ is the (linear) endomorphism operad of the dg-module $A$. This is exactly the result we were looking for, since the simplicial set $P_{n+1}^h(A)$ is the natural moduli space of weak Poisson brackets on $A$.

Finally in Section 4 we give an alternative proof of this theorem, which is more computational and uses both an explicit resolution of the strict Poisson operad and the classical concrete definition of $L_{\infty}$-algebra.

As suggested by B. Toën, the argument of Section 3 should also prove our main theorem in the case where $A$ is concentrated in degree $(-\infty, n]$, with $n \geq 0$. We will investigate this question and other possible generalizations in a future version of this paper.

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1 Notations

- \( k \) is the base field, which is of characteristic 0.

- \( \text{cdga} \leq 0 \) denotes the category of (strictly) commutative differential graded algebras, concentrated in non-positive degrees. We adopt the cohomological point of view, and the differential increases the degree by 1. The category \( \text{cdga} \leq 0 \) has the usual model structure for which weak equivalences are quasi-isomorphisms, and fibrations are surjections in negative degrees.

- \( \mathcal{C}(k) \) denotes the category of unbounded cochain complexes over \( k \). Its objects will be called also dg-modules. It has the usual model structure for which weak equivalences are the quasi-isomorphisms and fibrations are surjections. It is also a symmetric monoidal model category for the standard tensor product \( \otimes_k \).

- We will use the term \emph{symmetric sequence} to indicate a collection of dg-modules \( \{ V(m) \}_{m \in \mathbb{N}} \) such that every \( V(m) \) has an action of the symmetric group \( S_m \) on it. Explicitly, \( V(m) \) is a differential graded \( S_m \)-module, meaning that for every \( p \in \mathbb{Z} \) the degree \( p \) component \( V(m)^p \) is an \( S_m \)-module, and that the differential is a map of \( S_m \)-modules. Equivalently, one can say that \( V(m) \) is a differential graded \( k[S_m] \)-module, where \( k[S_m] \) is the group algebra of \( S_m \). In the literature objects of this kind are sometimes called \( S \)-modules, \( \Sigma^\ast \)-objects or also just collections in \( \mathcal{C}(k) \) (see for example [BM] or Chapter 5 in [LV]). If \( V \) is a symmetric sequence and \( f \in V(m) \), we will denote by \( f^\sigma \) the image of \( f \) under the action of a permutation \( \sigma \in S_m \). We will say that \( f \) is \emph{symmetric} if \( f^\sigma = f \) for every \( \sigma \in S_m \). Similarly, we will say that \( f \) is \emph{anti-symmetric} if \( f^\sigma = (-1)^\sigma f \) for every \( \sigma \), where \( (-1)^\sigma \) denotes the sign of \( \sigma \). We will use the notation \( V^S \) for the symmetric sequence of invariants (i.e. of symmetric elements): explicitly, \( V^S(m) = V(m)^{S_m} \). We will allow ourselves to switch quite freely from the point of view of symmetric sequences to the one of graded dg-modules with an action of \( S_m \) on the weight \( m \) component.

- \( \text{dgOp} \) is the category of (monochromatic) operads in the symmetric monoidal category \( \mathcal{C}(k) \) (dg-operads for short). It carries a model structure with componentwise quasi-isomorphisms as weak equivalences and componentwise surjections as fibrations (see [HI]). In particular, every dg-operad is fibrant. If \( \mathcal{P} \) is a dg-operad, we denote by \( \mathcal{P}_\infty \) its cofibrant replacement; then \( \mathcal{P}_\infty \)-algebras are up-to-homotopy \( \mathcal{P} \)-algebras. The operads of commutative algebras, of Lie algebras and of Poisson \( n \)-algebras will be denoted with \( \text{Comm} \), \( \text{Lie} \) and \( \text{Poisson} \) respectively. Our convention is that a Poisson \( n \)-algebra has a Lie bracket of degree \( 1 - n \); with this definition, the homology of an \( E_n \)-algebra is a \( P_n \)-algebra. Notice however that there are other conventions in the literature: for example in [CFL] the authors define a Poisson \( n \)-algebra to have Lie bracket of degree \( -n \).

- \( \text{dgLie}^\mathbb{R} \) is the category whose objects are graded dg-Lie algebras, that is to say graded dg-modules \( L \) together with an antisymmetric binary operation \( [\cdot, \cdot] : L \otimes L \to \)
who satisfies the (graded) Jacobi identity. The bracket must be of cohomological degree 0 and of weight −1. Notice thus that these are not algebras for the trivial graded version of the Lie operad, since we are asking for the bracket to have weight −1.

• Given a dg-module V, one defines its suspension V[1] to be the cochain complex V ⊗ k[1], where k[1] is the complex who is k in degree −1 and 0 elsewhere. If we do the same on operads, we should be a bit more careful. In fact, given an operad O, the symmetric sequence O′(m) = O(m)[1] does not inherit an operad structure. Instead, one defines the suspension of O to be the symmetric sequence whose terms are sO(m) = O(m)[1 − m], together with the natural operadic structure on it. A little more abstractly, sO is just O ⊗_H End_k[1], where ⊗_H denotes the Hadamard tensor product of operads (see [LV, Section 5.3.3]). Note that the arity p component of End_k[1] is k[1 − p]; as a S_p-module, it is just the sign representation. This operadic suspension is an auto equivalence of the category dgOp, its inverse being a desuspension functor denoted O ↦→ s^{−1}O, and which sends O to O ⊗_H End_k[−1].

2 Operads and graded Lie algebras

In this section we study the dg-operad Lie and its cofibrant resolutions. Namely, we describe what it means to have a map from any of these dg-operads to another dg-operad O.

We start by recalling how we can obtain a graded dg-Lie algebra L(O) in a natural way starting with a dg-operad O. These are classical results in operad theory, and they play a very important role in the remainder of the paper.

**Proposition 2.1.** Let O be a dg-operad. Then the graded dg-module L(O) = ⊕_p O(p) has a natural structure of a graded dg-Lie algebra, where the Lie bracket is induced by the following pre-Lie product

\[ f \star g = \sum_{i=1}^{p} \sum_{\sigma \in S_{p,q}^i} (f \circ_i g)^{\sigma} \]

where f and g are of weight p and q respectively, and where S_{p,q}^i is the set of permutations of p + q − 1 elements such that

\[ \sigma^{-1}(1) < \sigma^{-1}(2) < \cdots < \sigma^{-1}(i) < \sigma^{-1}(i+q) < \cdots < \sigma^{-1}(p+q-1) \]

and

\[ \sigma^{-1}(i) < \sigma^{-1}(i+1) < \cdots < \sigma^{-1}(i+q-1) . \]

Recall that one obtains a Lie bracket starting from a pre-Lie structure in a natural way: in our case, \[ [f,g] = f \star g - (-1)^{|f||g|} g \star f . \] One has of course to check that the \star operation defines a pre-Lie product (and therefore a Lie bracket): this is done by direct
computation, showing that the so called associator \( f \star (g \star h) - (f \star g) \star h \) is (graded) symmetric on \( g \) and \( h \) (see [LV], Section 5.4.6).

The Lie bracket defined above has a first nice property: the following lemma is a straightforward consequence of the definition of the pre-Lie product.

**Lemma 2.2.** Let \( \mathcal{O} \) be a dg-operad, and let \( f, g \in \mathcal{L}(\mathcal{O}) \) be two symmetric elements. Then their bracket in \( \mathcal{L}(\mathcal{O}) \) remains symmetric.

In particular, \( \mathcal{L}(\mathcal{O}) \) has a sub-Lie algebra of symmetric elements \( \mathcal{L}(\mathcal{O})^S \).

Our first goal is to use the construction of \( \mathcal{L}(\mathcal{O}) \) to find an alternative description to the set \( \text{Hom}_{\text{dgOp}}(\text{Lie}, \mathcal{O}) \).

In Section 1, we defined the operadic suspension, which is an auto-equivalence of the category of dg-operads. The operad \( s\text{Lie} \) has one generator in arity 2 of degree 1, which is now symmetric; the Jacobi relation still holds in the same form, since it only involves even permutations. Note that algebras for this operad are just dg-Lie algebras whose bracket is of degree 1, or equivalently dg-modules \( V \) with a dg-Lie algebra structure on \( V[-1] \).

The operadic suspension being an equivalence, we have in particular \( \text{Hom}_{\text{dgOp}}(\text{Lie}, \mathcal{O}) \cong \text{Hom}_{\text{dgOp}}(s\text{Lie}, s\mathcal{O}) \). Maps from the operad \( s\text{Lie} \) have a nice description in terms of maps of graded dg-Lie algebras.

**Proposition 2.3.** Let \( \mathcal{O} \) be a dg-operad. Then we have

\[
\text{Hom}_{\text{dgOp}}(s\text{Lie}, \mathcal{O}) \cong \text{Hom}_{\text{dgLie}^{gr}}(k[-1](2), \mathcal{L}(\mathcal{O})^S)
\]

where \( k[-1](2) \) is the graded dg-Lie algebra who has just \( k \) in degree 1 and weight 2, with zero bracket, while \( \mathcal{L} \) is the functor \( \text{dgOp} \to \text{dgLie}^{gr} \) defined at the beginning of this section.

**Proof.** It follows from the explicit presentation of \( s\text{Lie} \) given before that

\[
\text{Hom}_{\text{dgOp}}(s\text{Lie}, \mathcal{O}) = \{x \in \mathcal{O}(2)_1|x^{(12)} = x \text{ and } x \circ_1 x + (x \circ_1 x)^{(123)} + (x \circ_1 x)^{(132)} = 0\}
\]

so that in order to prove the lemma we are led to show that the Jacobi relation is equivalent to the condition \([x, x] = 0\) in \( \mathcal{L}(\mathcal{O}) \). This is done by direct calculation, since for any symmetric \( x \in \mathcal{O}(2) \) we have

\[
x \star x = x \circ_1 x + (x \circ_1 x)^{(123)} + (x \circ_2 x) = x \circ_1 x + (x \circ_1 x)^{(123)} + (x \circ_1 x)^{(132)}
\]
where we just use the general identities that describe the relationship between partial composition and the action of the symmetric groups. More specifically, take $f \in \mathcal{O}(p)$ and $g \in \mathcal{O}(q)$. Then for every $\sigma \in S_p$ one has
\[ f \circ_1 g^\sigma = (f \circ_1 g)^{\sigma'} \]
where $\sigma' \in S_{p+q-1}$ acts as $\sigma$ on the block $\{i, i+1, \ldots, i+q-1\}$ and as the identity elsewhere. Moreover, for every $\tau \in S_p$, one has
\[ f^\tau \circ_1 g = (f \circ_\tau(i) g)^{\tau'} \]
where $\tau' \in S_{p+q-1}$ acts as the identity on the block $\{i, i+1, \ldots, i+q-1\}$ with values in $\{\tau(i), \tau(i)+1, \ldots, \tau(i)+q-1\}$ and as $\tau$ elsewhere (sending $\{1, \ldots, p+q-1\} \setminus \{i, \ldots, i+q-1\}$ to $\tau(i),$ $\ldots,$ $\tau(i)+q-1\})$.

The lemma now follows from the observation that for an element $x \in L(P)$ of degree 1, one has $[x, x] = 2(x \star x)$. \[\square\]

One immediately has the following consequence.

**Corollary 2.4.** For any dg-operad $\mathcal{O}$, we have

\[ \text{Hom}_{\text{dgOp}}(\mathcal{O}, \text{Lie}) \cong \text{Hom}_{\text{dgOp}}(\text{sLie}, s\mathcal{O}) \cong \text{Hom}_{\text{dgLiegr}}(k[-1](2), L(s\mathcal{O})^S) \].

Next we try to find a result analogous to the last corollary for a cofibrant resolution of the dg-operad Lie.

Suppose we have a semi-free resolution $Q(k[-1](2))$ of $k[-1](2)$ as a graded dg-Lie algebra. This means that if we forget the differential $Q(k[-1](2))$ is a free graded Lie algebra, say with generators $\{p_i\}_{i \in I}$, homogeneous of degree $d_i$ and of weight $w_i$. Then there are of course relations that specify the value of $d(p_i)$, where $d$ is the differential. We can now use this resolution to build a dg-operad $\text{sLie}_Q$.

Concretely, for every $i \in I$, take a symmetric generator $\tilde{p}_i \in \mathcal{O}(w_i)$ of degree $d_i$. As for relations, we take the same relations defining $Q(k[-1](2))$; this means that whenever such a relation contains a bracket, we reinterpret it as the bracket (introduced at the beginning of this section) of elements of an operad. The definition of the operad $\text{sLie}_Q$ allows us to describe quite naturally the set of morphism $\text{Hom}_{\text{dgOp}}(\text{sLie}_Q, \mathcal{O})$ for an arbitrary operad $\mathcal{O}$. One can in fact prove the following result.

**Proposition 2.5.** Let $Q(k[-1](2))$ be a semi-free resolution of the graded dg-Lie algebra $k[-1](2)$, and let $\text{sLie}_Q$ be the operad defined above, which has the same generators and relations of $Q(k[-1](2))$, and such that all generators are symmetric. Then for every dg-operad $\mathcal{O}$ we have

\[ \text{Hom}_{\text{dgOp}}(\text{sLie}_Q, \mathcal{O}) \cong \text{Hom}_{\text{dgLie}^S}(Q(k[-1](2)), L(\mathcal{O})^S) \].
Proof. This follows from the definition of \( \tilde{s}
abla\text{Lie}_Q \) in terms of generators and relations. Just like what we said before for \( s\text{Lie} \), morphisms form \( \tilde{s}
abla\text{Lie}_Q \) are completely determined by the images of the generators \( \tilde{p}_i \), provided that they satisfy the relations defining \( \tilde{s}
abla\text{Lie}_Q \). Every relation can be expressed inside \( L(s\nabla\text{Lie}_Q) \), since they only specify the differentials of the \( \tilde{p}_i \) in terms of their brackets. And by definition these relations of course coincide with those of \( Q(k[-1](2)) \), giving the desired result.

In particular, we observe that \( \tilde{s}
abla\text{Lie}_Q \) is a cofibrant approximation of \( s\text{Lie} \): the weak equivalence \( \tilde{s}
abla\text{Lie}_Q \to s\text{Lie} \) is induced by the weak equivalence \( Q(k[-1](2)) \to k[-1](2) \). The fact that it is cofibrant follows from the very definition of cofibrations in the model category of dg-operads, given in \[HI\].

The operad \( \tilde{s}
abla\text{Lie}_Q \) is therefore weakly equivalent to the cofibrant replacement of \( s\text{Lie} \). This is just a consequence of the existence of the dotted arrow in the following commutative diagram

\[
\begin{array}{ccc}
\emptyset & \longrightarrow & (s\text{Lie})_\infty \\
\downarrow & & \downarrow \\
\tilde{s}\nabla\text{Lie}_Q & \longrightarrow & s\text{Lie} \\
\end{array}
\]

Since the operadic suspension preserves weak equivalences and fibrations, the map \( s(Lie_\infty) \to s\text{Lie} \) is a trivial fibration, where \( Lie_\infty \) is the standard minimal model of the operad \( Lie \), studied for example by Markl in \[Mar\]. In particular, it follows that \( \tilde{s}\nabla\text{Lie}_Q \) is weakly equivalent to \( s(Lie_\infty) \). Once again this is just a consequence of the existence of a model category structure on \( dgOp \), which assures that the dotted arrow in the following diagram

\[
\begin{array}{ccc}
\emptyset & \longrightarrow & s(Lie)_\infty \\
\downarrow & & \downarrow \\
\tilde{s}\nabla\text{Lie}_Q & \longrightarrow & s\text{Lie} \\
\end{array}
\]

exists, and that it is a weak equivalence.

Note that this does not imply that

\[
\text{Hom}_{dgOp}(Lie_\infty, O) \cong \text{Hom}_{dgOp}(sLie_\infty, sO) \cong \text{Hom}_{dlie}(Q(k[-1](2)), L(sO)^S)
\]

since \( \tilde{s}\nabla\text{Lie}_Q \) and \( s\nabla\text{Lie}_\infty \) are not isomorphic in general.

Recall that for a commutative dg-algebra \( A \) we have a standard notion of multi-derivation. Namely one says that a linear map \( \phi : A^\otimes p \to A \) is a multi-derivation if for every \( i = 1, \ldots, p \) and for every choice of \( a_1, \ldots, a_i, \ldots a_p \in A \) the induced linear map

\[
A \to A \\
x \mapsto \phi(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots a_p)
\]
is a (graded) derivation of $A$. More generally, for every operadic morphism $\mu : \text{Comm} \to \mathcal{O}$, we can say what it means for any element of $\mathcal{O}$ to be a derivation with respect to $\mu$. Notice that the map $\mu$ is completely determined by the image in $\mathcal{O}(2)$ of the generator of the operad $\text{Comm}$; in order to simplify the notation, we will also use the letter $\mu$ to denote the image of the generator.

**Definition 2.6.** Let $\mathcal{O}$ be a dg-operad, and let $\mu : \text{Comm} \to \mathcal{O}$ be a morphism of dg-operads. Suppose $f \in \mathcal{O}(p)$ is an element of $\mathcal{O}$ of arity $p \in \mathbb{N}$. We say that $f$ is a $p$-derivation with respect to $\mu$ if we have

$$f \circ_i \mu = (\mu \circ_1 f)(p+1 \ p \ldots \ i+2 \ i+1) + (\mu \circ_2 f)(1 \ 2 \ldots \ i-1 \ i)$$

for every $i = 1, \ldots, p$. The symmetric sub-sequence of $\mathcal{O}$ formed by $p$-derivations will be denoted by $\mathcal{MD}(\mathcal{O}, \mu)$, and its elements will just be called multi-derivations with respect to $\mu$. If the morphism $\mu$ is clear from the context, we will just write $\mathcal{MD}(\mathcal{O})$.

The definition is coherent with the classical case of derivations of an algebra: if $\mathcal{O}$ is the endomorphism operad of a dg-module $V$ and $\mu$ is an actual commutative product on $V$ (so that $(V, \mu)$ is just a commutative dg-algebra), then multi-derivations in our sense are exactly multi-derivations in the standard sense.

Let us remark that one could give a definition analogous to Definition 2.6 that works for every element $\mu \in \mathcal{O}(2)$, of any degree, and without making any assumption on the symmetry of $\mu$. For example, a derivation with respect to such a $\mu$ is just an element $f \in \mathcal{O}(1)$ such that

$$f \circ \mu = (-1)^{|\mu||f|} \mu \circ_1 f + (-1)^{|\mu||f|} \mu \circ_2 f .$$

In order to generalize this to multi-derivations, one should keep track of the signs.

**Definition 2.7.** Let $\mathcal{O}$ be a dg-operad, and let $\mu \in \mathcal{O}(2)$. An element $f \in \mathcal{O}(p)$ is called a $p$-derivation with respect to $\mu$ if for every $i = 1, 2, \ldots, p$ we have

$$f \circ_i \mu = (-1)^{|\mu||f|} (\mu \circ_1 f)(i+1 \ p \ldots \ i+2) + (-1)^{|\mu||f|} (\mu \circ_2 f)(1 \ 2 \ldots \ i-1 \ i)$$

The dg-module of derivations of an algebra $A$ is known to be a dg-Lie algebra in a natural way: the (graded) commutator of two derivations is in fact still a derivation. Derivations thus form a sub-Lie algebra of $\text{Hom}_{\text{dgMod}}(A, A)$. More can be said, since actually the graded module of multi-derivations of $A$ is a graded sub-Lie algebra of $\mathcal{L}(\text{End}_A)$. The following lemma tells us that the same remains true in the world of operads.

**Proposition 2.8.** Let $\mathcal{O}$ be a dg-operad, and let $\mu \in \mathcal{O}(2)$ be a binary operation. The (graded module associated to the) symmetric sequence of multi-derivations with respect to $\mu$ of Definition 2.7 is closed under the Lie bracket of $\mathcal{L}(\mathcal{O})$. 

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Proof. This is just a tedious and straightforward computation. Let us give the main ideas without going into all the details. We will suppose that $\mu$ is of even degree in order to avoid keeping track of too many signs. The proof for $\mu$ of odd degree is exactly the same, with additional signs of course.

Let $f \in \mathcal{O}(p)$ and $g \in \mathcal{O}(q)$ be two multi-derivations with respect to $\mu$. We have

$$[f, g] \circ_i \mu = (f \ast g) \circ_i \mu + (-1)^{|f||g|}(g \ast f) \circ_i \mu,$$

and we would like to show that this is equal to

$$(\mu \circ_1 [f, g])(p+q\ldots i+1) + (\mu \circ_2 [f, g])(1 \ldots i) = (\mu \circ_1 (f \ast g))(p+q\ldots i+1) +$$

$$+ (-1)^{|f||g|}(\mu \circ_1 (g \ast f))(p+q\ldots i+1) +$$

$$+ (\mu \circ_2 (f \ast g))(1 \ldots i) +$$

$$+ (-1)^{|f||g|}(\mu \circ_2 (g \ast f))(1 \ldots i).$$

Notice that $f \ast g$ and $g \ast f$ have no hope of being multi-derivations themselves, and one really has to develop the sums in order to prove the result. Using the relations between partial compositions and the action of the symmetric groups, we may write

$$(f \ast g) \circ_i \mu = \sum_{j=1}^{p} \sum_{\sigma \in S_{p,q}} (f \circ_j g)^{\sigma} \circ_i \mu = \sum_{j=1}^{p} \sum_{\sigma \in S_{p,q}} ((f \circ_j g) \circ_{\mu} \mu^{\sigma}).$$

We now observe that if $\sigma(i) \notin \{j, j+1, \ldots, j+q-1\}$, then we can just use the fact that $f$ is a derivation, and we are done. A similar reasoning applies to $(g \ast f) \circ_i \mu$. When $\sigma(i) \in \{j, j+1, \ldots, j+q-1\}$, it gets a bit more complicated. With some care, we can write down what it is left to prove, that is

$$\sum_{j=1}^{p} \sum_{\sigma \in S_{p,q}} ((\mu \circ_2 f) \circ_1 g)^{\varphi(j+q, j+q-1 \ldots \sigma(i)+1 \ldots p') \varphi(i) \varphi(i')} =$$

$$= (-1)^{|f||g|} \sum_{k=1}^{q} \sum_{\tau \in S_{p+q}} ((\mu \circ_1 g) \circ_{q+1} f)^{\psi(k+1 \ldots \tau(i) \ldots p+q)},$$

where $\varphi \in S_{p+q}$ is the permutation that exchanges the blocks $\{1, \ldots, j-1\}$ and $\{j, \ldots, j+q-1\}$, and $\psi \in S_{p+q}$ is the permutation that exchanges the blocks $\{k+1, \ldots, k+p\}$ and $\{k+p+1, \ldots, p+q\}$. This last equation is true by direct verification: both sides are equal to the sum of all possible “products” of the form $\mu \circ (f, g)$.

Recall (see Section 8.6 of [LY] and references therein) that given two operads $\mathcal{P}$ and $\mathcal{Q}$, if we dispose of a morphism of symmetric sequences $\Lambda : \mathcal{Q} \circ \mathcal{P} \rightarrow \mathcal{P} \circ \mathcal{Q}$ (satisfying a series of axioms), then we can put an operad structure on the composite of the underlying symmetric sequences $\mathcal{P} \circ \mathcal{Q}$. The idea is that in order to define a composition $(\mathcal{P} \circ \mathcal{Q}) \circ (\mathcal{P} \circ \mathcal{Q})$, we use the following rule:

$$((f \circ_i g) \circ_j h) = ((f \circ_1 h) \circ_2 g) \circ_i \mu.$$
\( \Lambda \rightarrow P \circ Q \) we can use the morphism \( \Lambda \) followed by the given compositions \( P \circ P \rightarrow P \) and \( Q \circ Q \rightarrow Q \), coming from the operad structures on \( P \) and \( Q \). Informally speaking, \( \Lambda \) specifies how the operations encoded by the operad \( P \) interact with those encoded by \( Q \). Such a \( \Lambda \) is called a distributive law, because of the motivating example of the relation between the sum and the multiplication in a ring. When \( P \) and \( Q \) have a nice presentation in terms of generators and relations, we only need a rewriting rule for the generators \( [LV] \).

We now let \( Q(k[-1](2)) \) be again a semi-free resolution of the graded dg-Lie algebra \( k[-1](2) \): as before, we can associate to it an operad \( \tilde{sLie}_{Q} \), which is quasi-isomorphic to \( sLie_{\infty} \). The operad introduced in the following definition will play a central role in the remainder of the paper.

**Definition 2.9.** Let \( Q(k[-1](2)) \) be again a semi-free resolution of the graded dg-Lie algebra \( k[-1](2) \), and let as before \( sLie_{Q} \) be the operad of Proposition 2.5. We define the operad \( \tilde{P}_{n,Q} \) to be the operad obtained by means of a rewriting rule out of \( s^{-n}sLie_{Q} \) and \( Comm \), with the condition that every generator of \( sLie_{Q} \) is a multi-derivation with respect to the generator of \( Comm \).

It is clear from the definition that \( \tilde{P}_{n,Q} \)-algebras are commutative dg-algebras \( A \) with a compatible \( sLie_{Q} \)-structure on \( A[n] \). This operad is obviously weakly equivalent to the dg-operad obtained in a similar way out of \( Comm \) and \( sLie_{\infty} \), which is the minimal model of the operad \( Lie \). Let us call \( \tilde{P}_{n} \) this latter operad. More specifically, \( \tilde{P}_{n} \)-algebras are commutative dg-algebras \( A \) together with a \( Lie_{\infty} \)-structure on \( A[n-1] \). The two structures are compatible, meaning that the multi-brackets defining the \( Lie_{\infty} \)-structure are multi-derivations on the algebra \( A \).

**Remark.** The operad \( \tilde{P}_{n} \) (actually a non-shifted version of it) has appeared for instance in [CT], where the authors called its algebras flat \( P_{n,\infty} \)-algebras. However, as Cattaneo and Felder correctly remarked in their paper, their notation is a bit misleading, because \( \tilde{P}_{n} \) is not a cofibrant replacement of the operad \( P_{n} \): in particular the product encoded in \( \tilde{P}_{n} \) is strictly commutative. One could see \( \tilde{P}_{n} \) as an operad standing between the original \( P_{n} \) and its cofibrant replacement \( P_{n,\infty} \). For this reason, algebras for \( \tilde{P}_{n} \) will be called semi-strict \( P_{n} \)-algebras. For an explicit definition of \( P_{n,\infty} \)-algebras in terms of generators and relations in the case \( n = 2 \) (corresponding to homotopy Gerstenhaber algebras), one can look at [Gi].

By construction, the operad \( \tilde{P}_{n,Q} \) has a natural map from the commutative dg-operad \( Comm \). If \( O \) is any dg-operad, we now describe the fiber of the induced morphism \( \text{Hom}_{dgOp}(\tilde{P}_{n,Q},O) \rightarrow \text{Hom}_{dgOp}(Comm,O) \) at a point \( \mu \). In particular, if we take \( O \) to be the endomorphism operad of a dg-module \( V \), we are studying the possible ways in which a given commutative structure on \( V \) can be extended to a \( \tilde{P}_{n,Q} \)-structure. From the very definition of \( \tilde{P}_{n,Q} \), it is clear that what we are missing is a shifted \( sLie_{Q} \)-structure made out of multi-derivations. Luckily the preceding results give us exactly a way to compute those structures.
Proposition 2.10. Let $\mathcal{O}$ be a dg-operad, and let $\mu : \text{Comm} \to \mathcal{O}$ be a map of operads. The fiber at $\mu$ of the map

$$\text{Hom}_{dg\text{Op}}(\hat{P}_{n,Q}, \mathcal{O}) \to \text{Hom}_{dg\text{Op}}(\text{Comm}, \mathcal{O})$$

is the set $\text{Hom}_{dg\text{Lie}^\varphi}(Q(k[-1](2)), \mathcal{L}(s^n\mathcal{MD}(\mathcal{O}))^\varphi)$.

Proof. By definition of the operad $\hat{P}_{n,Q}$, the fiber we are trying to compute is a subset of $\text{Hom}_{dg\text{Op}}(s\text{Lie}_Q, s^n\mathcal{O})$: in fact, it is composed of morphisms $s^{-n}s\text{Lie}_Q \to \mathcal{O}$. The condition they must satisfy is that the image of the generators must be multi-derivations with respect to $\mu$. It follows that our fiber is the subset of maps $s\text{Lie}_Q \to s^n\mathcal{O}$ which send generators to suspensions of multi-derivations. Using Proposition 2.5, we thus get that the fiber is exactly $\text{Hom}_{dg\text{Lie}^\varphi}(Q(k[-1](2)), \mathcal{L}(s^n\mathcal{MD}(\mathcal{O}))^\varphi)$. Notice that it may seem that we are being a bit inaccurate here, as it is not entirely obvious that the (operadic) suspensions of elements of the sub-Lie algebra $\mathcal{MD}(\mathcal{O})$ are still a sub-Lie algebra of $\mathcal{L}(s^n\mathcal{O})$. This is nonetheless true, and it follows from the observation that elements in $s^n\mathcal{MD}(\mathcal{O})$ are exactly multi-derivations with respect to the $n$-suspension of the commutative product $\mu$. To see this, take $f$ a multi-derivation of $\mathcal{O}$ of arity $p$. We want to show that image under the operadic suspension of $f$ is a multi-derivation of $\mathcal{O}$ with respect to the suspension of $\mu$. This would easily imply our claim, and therefore the theorem.

Recall that the component of arity $p$ of $s\mathcal{O}$ is $\mathcal{O}(p) \otimes k[1-p]$, where $k[1-p]$ is the signature representation of $S_p$ put in degree $p-1$. We denote the generator of $k[1-p]$ by $x_{p-1}$, so that $|x_{p-1}| = p-1$. By definition of the compositions in $s\mathcal{O}$, we have

$$(f \otimes x_{p-1}) \circ_i (\mu \otimes x_1) = (f \circ_i \mu) \otimes (x_{p-1} \circ_i x_1)$$

$$= (\mu \circ_i f)^{(i+1)p+1,i+2} \otimes (x_{p-1} \circ_i x_1) +$$

$$+ (\mu \circ_i f)^{(i+1)p+1,i+1} \otimes (x_{p-1} \circ_i x_1)$$

$$= (-1)^{p-i}((\mu \circ_i f) \otimes (x_{p-1} \circ_i x_1))^{(i+1)p+1,i+2} +$$

$$+ (-1)^{p+1-i}((\mu \circ_i f) \otimes (x_{p-1} \circ_i x_1))^{(i+1)p+1,i+1}$$

Now observe that

$$(\mu \circ_i f) \otimes (x_{p-1} \circ_i x_1) = (-1)^{i-1}(\mu \circ_i f) \otimes (x_1 \circ_i x_{p-1})$$

$$= (-1)^{i-1}(-1)^{i|f|}(\mu \otimes x_1) \circ_1 (f \otimes x_{p-1})$$

so that we have

$$(f \otimes x_{p-1}) \circ_i (\mu \otimes x_1) = (-1)^{p-i}(-1)^{i|f|}((\mu \otimes x_1) \circ_1 (f \otimes x_{p-1}))^{(i+1)p+1,i+2} -$$

$$- (-1)^{p-1}(-1)^{i|f|}((\mu \otimes x_1) \circ_1 (f \otimes x_{p-1}))^{(i)p+1,i+1}$$

which tells us exactly that $f \otimes x_{p-1}$ is a multi-derivation with respect to the binary operation $\mu \otimes x_1$. \qed
We are now ready to prove our first main result. Given two dg-operads \( \mathcal{P} \) and \( \mathcal{Q} \), one can form a simplicial space of morphisms from \( \mathcal{P} \) to \( \mathcal{Q} \), which we will denote by \( \text{Hom}_{\text{dgOp}}(\mathcal{P}, \mathcal{Q}) \). Namely, we can construct a simplicial resolution \( \mathcal{Q}_\bullet \) of \( \mathcal{Q} \) and consider the simplicial set whose \( n \)-simplices are \( \text{Hom}(\mathcal{P}, \mathcal{Q}_n) \) and whose face and degeneracy maps are the ones induced by the simplicial structure of \( \mathcal{Q}_\bullet \). Notice that this is not the derived mapping space between \( \mathcal{P} \) and \( \mathcal{Q} \) in the model category of dg-operads, since we are not replacing \( \mathcal{P} \) with a cofibrant model. If the operad \( \mathcal{P} \) is cofibrant, then \( \text{Hom}_{\text{dgOp}}(\mathcal{P}, \mathcal{Q}) \) is isomorphic to the mapping space \( \text{Map}_{\text{dgOp}}(\mathcal{P}, \mathcal{Q}) \) in the homotopy category of simplicial sets.

The simplicial set \( \text{Hom}_{\text{dgOp}}(\mathcal{P}, \mathcal{Q}) \) has a nice interpretation if we put \( \mathcal{Q} = \text{End}_V \), where \( V \) is a dg-module. In this case, \( \text{Hom}_{\text{dgOp}}(\mathcal{P}, \text{End}_V) \) can be thought as a sort of moduli space of \( \mathcal{P} \)-algebra structures on \( V \).

We can ask whether Proposition 2.10 remains true at the level of simplicial sets. First of all, we remark that the question makes sense: for every operad \( \mathcal{O} \), we have a naturally induced map \( \text{Hom}_{\text{dgOp}}(\tilde{\mathcal{P}}_n, \mathcal{Q}, \mathcal{O}) \to \text{Hom}_{\text{dgOp}}(\text{Comm, } \mathcal{O}) \) (induced by the natural morphism of operads \( \text{Comm} \to \tilde{\mathcal{P}}_n, \mathcal{Q} \)) that forgets the additional structure, and we could wonder if we can describe the fiber of a 0-simplex \( \mu \in \text{Hom}_{\text{dgOp}}(\text{Comm, } \mathcal{O}) \) in terms of some simplicial set of morphisms in the category \( \text{dgLie}^{gr} \). The following theorem answers this question affirmatively.

**Theorem 2.11.** Let \( \mathcal{O} \) be a dg-operad, and let \( \mu : \text{Comm} \to \mathcal{O} \) be a map of operads. The (strict) fiber at \( \mu \) of the morphism of simplicial sets

\[
\text{Hom}_{\text{dgOp}}(\tilde{\mathcal{P}}_n, \mathcal{Q}, \mathcal{O}) \to \text{Hom}_{\text{dgOp}}(\text{Comm, } \mathcal{O})
\]

is the simplicial set \( \text{Hom}_{\text{dgLie}}^{gr}(Q(k[-1](2)), \mathcal{L}(s^n, \mathcal{MD}(\mathcal{O}))^S \otimes \Omega_*) \), which is a right homotopy function complex from \( k[-1](2) \) to \( \mathcal{L}(s^n, \mathcal{MD}(\mathcal{O}))^S \) in the model category of graded dg-Lie algebras.

Before proving the theorem, we give explicit ways to compute simplicial resolutions and mapping spaces in both model categories \( \text{dgOp} \) and \( \text{dgLie}^{gr} \).

Let \( L \in \text{dgLie}^{gr} \). We can construct new graded dg-Lie algebras from \( L \) by extension of scalars to \( k \) to any \( k \)-dg-algebra. Let us define \( \Omega_n \) to be the dg-algebra of algebraic differential forms on \( \text{Spec}(k[t_0, \ldots, t_n]/t_0 + \cdots + t_n = 1) \). As an algebra, we have

\[
\Omega_n = k[t_0, \ldots, t_n, dt_0, \ldots, dt_n]/(1 - \sum t_i, \sum dt_i)
\]

where the generators \( t_i \) have degree 0 and the \( dt_i \) have degree 1. The algebras \( \Omega_n \) define a simplicial object in the category of commutative dg-algebras in a natural way. Then the simplicial graded dg-Lie algebra \( L \otimes \Omega_* \) is a simplicial resolution of \( L \). Hence in \( \text{dgLie}^{gr} \), the mapping space between two objects \( L \) and \( M \) has an explicit representative. Its \( n \)-simplices are

\[
\text{Map}_{\text{dgLie}^{gr}}(L, M)_n = \text{Hom}_{\text{dgLie}^{gr}}(Q(L), M \otimes_k \Omega_n),
\]

12
where $Q$ is a cofibrant replacement of $L$.

Just as for graded dg-Lie algebras, given an operad $O$ we can construct new operads by extension of scalars.

**Proposition 2.12.** For a dg-operad $O$, the simplicial object $O \otimes_k \Omega_\ast$ (defined as above) gives a fibrant simplicial framing of the operad $O$ (i.e. a fibrant replacement of $O$ in the Reedy model category of simplicial objects in $dgOp$).

This follows directly from [FF], Part II, Chapter 7 (in particular Theorem 7.3.5).

**Proof of Theorem 2.11.** By construction, the $m$-simplices of the fiber are the $m$-simplices of the simplicial set $\text{Hom}_{dgOp}(\tilde{P}_{n,Q}, O)$ that are sent to $\mu$, viewed as a degenerate $m$-simplex of $\text{Hom}_{dgOp}(\text{Comm}, O)$. Therefore we can use Proposition 2.10 in order to compute them: they are the fiber of the function

$$\text{Hom}_{dgOp}(\tilde{P}_{n,Q}, O \otimes \Omega_m) \to \text{Hom}_{dgOp}(\text{Comm}, O \otimes \Omega_m)$$

taken at the point $\mu$. Notice that we are being a bit sloppy in order to keep notation as simple as possible, as we are identifying $\mu : \text{Comm} \to O$ with the composition $\text{Comm} \to O \to O \otimes \Omega_n$. So Proposition 2.10 tells us that the $m$-simplices of the fiber are $\text{Hom}_{dgLie}(Q(k[-1](2)), L(s^n MD(O \otimes \Omega_m))^{\mathbb{S}})$.

Observe now that multi-derivations of $O \otimes \Omega_n$ are just multi-derivations of $O$ with respect to $\mu$, considered over the dg-algebra $\Omega_n$. Concretely, this means $MD(O \otimes \Omega_n) = MD(O) \otimes \Omega_n$ as graded dg-Lie algebras. Moreover, the operadic suspension commutes with extension of scalars, as does taking invariants. It follows that the graded dg-Lie of $n$-simplices is

$$\text{Hom}_{dgLie}(Q(k[-1](2)), L(s^n MD(O))^{\mathbb{S}} \otimes \Omega_m) = \text{Map}_{dgLie}(k[-1](2), L(s^n MD(O))^\mathbb{S})_m$$

where the Map on the right is computed by means of the right homotopy function complex described before. This isomorphisms organize in a natural way to give an isomorphism of simplicial set between the fiber at $\mu$ and a right homotopy function complex $\text{Map}_{dgLie}(k[-1](2), L(s^n MD(O))^\mathbb{S})$, and this proves the theorem. \qed

### 3 Applications to derived algebraic geometry

Let again $Q(k[-1](2))$ be a semi-free resolution of the dg-Lie algebra $k[-1](2)$. In this section we apply Theorem 2.11 to the context of derived Poisson geometry. In particular, we will show in Theorem 3.1 that a $n$-Poission structure in the sense of [PTVV] on an affine derived stack $\text{Spec} A$ gives rise to a $\tilde{P}_{n+1,Q}$-structure on $A$.

Recall from [PTVV] that for a derived Artin stack $X$ which is locally of finite presentation, the space $\text{Pois}(X, n)$ of $n$-Poisson structures on $X$ is by definition the mapping
Theorem 3.1. Let $A$ be a cofibrant object in $\cdga_{\leq 0}$. Suppose that $A$, viewed as an affine derived stack, admits a $n$-shifted Poisson structure in the sense of [PTVV]. Then $A$ has a structure of an $\tilde{P}_{n+1, Q}$-algebra, whose commutative product coincide with the given multiplication in $A$. More precisely, let $\mu_A$ be the multiplication in $A$, and let $\tilde{P}_{n+1, Q}(A)$ be the fiber of the map of simplicial sets

$$\text{Hom}_{\text{dgOp}}(\tilde{P}_{n+1, Q}, \text{End}_A) \rightarrow \text{Hom}_{\text{dgOp}}(\text{Comm}, \text{End}_A)$$

at the point $\mu_A$.

We have a natural map of simplicial set

$$\text{Pois}(\text{Spec } A, n) \rightarrow \tilde{P}_{n+1, Q}(A)$$

Moreover, this map is a quasi-isomorphism if the cotangent complex $L_A$ is perfect.

Proof. The simplicial set $\tilde{P}_{n+1, Q}(A)$ has an equivalent description given by Theorem 2.11 namely we can rewrite it as $\text{Hom}_{\text{dgLie}^\text{gr}}(Q[k[-1]/2], L(s^{n+1}MD(A))^S \otimes \Omega_*)$, where $MD(A)$ is the Lie algebra of multi-derivations of the operad $\text{End}_A$, with respect to the natural multiplication $\mu_A : \text{Comm} \rightarrow \text{End}_A$ (see Definition 2.6); that is to say, the classically defined multi-derivations of the algebra $A$. By functoriality, in order to prove the theorem it will suffice to build up a map of graded dg-Lie algebras

$$\text{Sym}_A(\mathbb{T}_A[-n-1])[n+1] \rightarrow L(s^{n+1}MD(A))^S.$$ 

To construct this morphism, notice that since $A$ is cofibrant, $L_A$ is just the standard module of Kähler differentials, and multi-derivations of $A$ of arity $p$ are by definition the $A$-module $\text{Hom}_A(L_A^{S^p}, A)$. Hence the weight $p$ component of the graded dg-Lie algebra $L(s^{n+1}MD(A))^S$ is precisely given by the symmetric elements inside $\text{Hom}_A(L_A^{S^p}, A) \otimes k[1-p]^\otimes(n+1)$, where $k[1-p]$ is the signature representation of $S_p$ concentrated in degree $p-1$. As an $S_p$-module, $k[1-p]^\otimes(n+1)$ can be either a trivial or a signature representation, depending on the parity of $n$. Concretely, we have

$$k[1-p]^\otimes(n+1) = \begin{cases} 
\text{the trivial representation of } S_p & \text{if } n \text{ is odd} \\
\text{the signature representation of } S_p & \text{if } n \text{ is even}
\end{cases}$$

\footnote{See [162] for a proof of the existence of the Poisson algebra structure on $\text{Pol}(X, n)$, in the case where $X$ is of the form $[Y/G]$, where $Y$ is quasi-projective derived scheme and $G$ is a reductive smooth group acting on $Y$.}
where the $S_p$-modules are always concentrated in degree $(n+1)(p-1)$. It follows that as a dg-module, the weight $p$ part of $L(s^{n+1} MD(A))^S$ is isomorphic to $\text{Hom}_A(\text{Sym}^p A[L_A, A][(n+1)(1-p)]$ if $n$ is odd, and to $\text{Hom}_A(\Lambda^p A[L_A, A][(n+1)(1-p)]$ if $n$ is even.

On the other hand, the weight $p$ component of $\text{Sym}_A(\Lambda_n[-n-1])[n+1]$ is just $\text{Sym}_A^p(\Lambda_n[-n-1])[n+1]$, and we have a natural map of $k$-dg-modules (actually of $A$-dg-modules)

$$\text{Sym}_A^p(\Lambda_n[-n-1])[n+1] \longrightarrow \text{Hom}_A(\text{Sym}_A^p(\Lambda_n[n+1]), A)[n+1]$$

induced by the fact that $\Lambda_n$ is by definition the dual of $L_A$. Notice that this map is not an equivalence in general: it becomes an equivalence however if we suppose that $L_A$ is perfect. Observe next that we have

$$\text{Sym}_A^p(\Lambda_n[n+1]) = \begin{cases} \text{Sym}_A^p(\Lambda_n)[n(p-1)] & \text{if } n \text{ is odd} \\ \Lambda_A^p(\Lambda_n)[n(p-1)] & \text{if } n \text{ is even} \end{cases}$$

so that for every $n$, $\text{Hom}_A(\text{Sym}_A^p(\Lambda_n[n+1]), A)[n+1]$ is isomorphic as a dg-module to the weight $p$ component of $L(s^{n+1} MD(A))^S$.

Putting all this together, we do get a map of graded dg-modules

$$\text{Sym}_A(\Lambda_n[-n-1])[n+1] \longrightarrow L(s^{n+1} MD(A))^S.$$  

The point is to check that this map is compatible with the two Lie brackets: on the left hand side, we have the Schouten bracket, induced by the natural Lie structure on $\Lambda_n$, while on the right hand side we have the bracket of the Lie algebra associated to the operad $s^{n+1}\text{End}_A = \text{End}_A[n+1]$.

This can be done by direct calculation, since both brackets have a known explicit expression. One has just to check that the signs coincide.

More abstractly, we can also observe that there is an adjunction

$$\begin{cases} A\text{-dg-modules } X \text{ with a} \\ \text{k-linear dg-Lie structure on } X[m] \end{cases} \dashv \begin{cases} \text{commutative } A\text{-dg-algebras } X \text{ with a} \\ \text{compatible k-linear dg-Lie structure on } X[m] \end{cases}$$

where on the left hand side, compatible means that if we forget the $A$-action we are left with a $P_{m+1}$-algebra. The right adjoint is the forgetful functor, while the left adjoint sends $X$ to $\text{Sym}_A(X)$. In particular this implies that if we were able to show that $L(s^{n+1} MD(A))^S[1][n-1]$ has a compatible $A$-algebra structure, then the existence of a Lie algebra map

$$\text{Sym}_A(\Lambda_n[-n-1])[n+1] \longrightarrow L(s^{n+1} MD(A))^S.$$  

would follow from the existence of a morphism of Lie algebras (and of $A$-modules)

$$\Lambda_n \longrightarrow L(s^{n+1} MD(A))^S.$$  

But it follows from the definitions that the weight one component of $L(s^{n+1} MD(A))^S$ is precisely $\Lambda_n$, and that the restriction of the bracket of $L(s^{n+1} MD(A))^S$ to $\Lambda_n$ is the natural one (that is to say the graded commutator).
We are thus left to define an appropriate degree zero product on \( L(s^{n+1}MD(A))^{\mathbb{S}}[-n-1] \). It turns out that it is induced by the natural shuffle product on the multilinear morphisms from \( A[n+1] \) to itself, which has the following explicit description. Denote by \( \mu \) the multiplication of \( A \); for \( f \in \text{End}_{A[n+1]}(p) \) and \( g \in \text{End}_{A[n+1]}(q) \), we pose
\[
f \cdot g = \sum_{\sigma \in \text{Sh}_{p,q}} (s^{n+1} \mu(f, g))^{\sigma}
\]
where the sum is taken over all permutations \( \sigma \in S_{p+q} \) such that \( \sigma^{-1}(1) < \cdots < \sigma^{-1}(p) \) and \( \sigma^{-1}(p+1) < \cdots < \sigma^{-1}(p+q) \). It easy to check that this defines a degree \( m \) product, which becomes commutative if regarded on \( L(\text{End}_{A[n+1]})[-n-1] \). Moreover, if \( f \) and \( g \) are symmetric multi-derivations, then \( f \cdot g \) is again a symmetric multi-derivation. Finally, the graded Leibniz identity
\[
[f, g \cdot h] = [f, g] \cdot h + (-1)^{|g||f|+n+1} g \cdot [f, h]
\]
for \( f, g, h \in L(s^{n+1}MD(A))^{\mathbb{S}}[-n-1] \) should be checked to be true. Notice that here the product \( g \cdot h \) denotes the operation induced by the shuffle product defined above: this means that there are other signs involved, due to the so-called décalage isomorphism. The verification of the identity is a long but straightforward computation, and we omit the details. To summarize, \( L(s^{n+1}MD(A))^{\mathbb{S}}[-n-1] \) is an \( A \)-algebra with a \( k \)-linear compatible Lie bracket of degree \( -n-1 \), and by the discussion above this proves the theorem.

We can rephrase the results of Theorem 3.1 in a different way: we constructed a map of simplicial sets
\[
\text{Map}_{dgLie}^{\text{dgOp}}(k[-1](2), \text{Sym}_A(T_A[-n-1])[n+1]) \longrightarrow \text{Hom}_{dgOp}(\tilde{P}_{n+1}, Q, \text{End}_A)
\]
that fits in the following diagram
\[
\begin{array}{ccc}
\text{Pois}(\text{Spec} A, n) & \longrightarrow & \tilde{P}_{n+1, Q}(A) \\
\phi \downarrow & & \downarrow \mu_A \\
\text{Hom}_{dgOp}(\tilde{P}_{n+1}, Q, \text{End}_A) & \longrightarrow & \text{Hom}_{dgOp}((\text{Comm}, \text{End}_A)
\end{array}
\]
where the square on the right is a pullback of simplicial sets.

Let us weaken a bit our results in order to express them in a more homotopical language. The following theorem is the main result of this text.

**Theorem 3.2.** Let \( A \in \text{cdga}^{\leq 0} \), and let \( X = \text{Spec} A \) be an affine derived stack. Let \( P^h_{n+1}(A) \) be the homotopy fiber of the morphism of simplicial sets
\[
\text{Map}_{dgOp}(P_{n+1}, \text{End}_A) \longrightarrow \text{Map}_{dgOp}((\text{Comm}, \text{End}_A)
\]
taken at the point \( \mu_A \) corresponding to the given (strict) multiplication in \( A \).

Then there is a natural map in the homotopy category of simplicial sets

\[
Pois(X, n) \rightarrow P^h_{n+1}(A).
\]

Moreover, this is an isomorphism if \( L_X \) is a perfect complex.

**Proof.** As already mentioned towards the end of Section 2, the mapping space between two operads \( P \) and \( Q \) can be computed by taking a cofibrant replacement of the first one and a simplicial resolution of the second one. Let us denote by \( C \) the cofibrant replacement functor in the model category of dg-operads. In particular, one has

\[
\text{Map}_{dgOp}(P, Q) \cong \text{Hom}_{dgOp}(C(P), Q).
\]

Notice that we don’t replace \( Q \) with a fibrant model, since all operads are fibrant.

Note that \( \widetilde{C}P_{n+1, Q} \) is a cofibrant model for \( P_{n+1} \). By functoriality of the cofibrant replacement, we get a commutative square

\[
\begin{array}{ccc}
\text{Hom}_{dgOp}(\widetilde{C}P_{n+1, Q}, \text{End}_A) & \longrightarrow & \text{Hom}_{dgOp}(C\widetilde{P}_{n+1, Q}, \text{End}_A) \\
\downarrow & & \downarrow \\
\text{Hom}_{dgOp}(\text{Comm}, \text{End}_A) & \longrightarrow & \text{Hom}_{dgOp}(\text{Comm}_\infty, \text{End}_A)
\end{array}
\]

We know from Theorem 3.1 that there is a natural map of simplicial sets from \( \text{Pois}(X, n) \) to the fiber of the left vertical morphism, and the last diagrams gives us immediately a map in \( \text{Ho}(sSet) \) from \( \text{Pois}(X, n) \) to the fiber of

\[
\text{Hom}_{dgOp}(C\widetilde{P}_{n+1, Q}, \text{End}_A) \longrightarrow \text{Hom}_{dgOp}(\text{Comm}_\infty, \text{End}_A)
\]

taken at the point \( \mu_A \) (viewed as homotopy commutative product).

If we compose with the general map from a fiber to the corresponding homotopy fiber, we obtain a map in \( \text{Ho}(sSet) \)

\[
\text{Pois}(X, n) \longrightarrow \text{hofib}(\text{Hom}_{dgOp}(C\widetilde{P}_{n+1, Q}, \text{End}_A) \longrightarrow \text{Hom}_{dgOp}(\text{Comm}_\infty, \text{End}_A) : \mu_A).
\]

But as already observed \( C\widetilde{P}_{n+1, Q} \) is weakly equivalent to \( P_{n+1, \infty} \), which allows us to compute the homotopy fiber with \( P_{n+1, \infty} \) instead of \( C\widetilde{P}_{n+1, Q} \). This gives us a morphism in \( \text{Ho}(sSet) \)

\[
\text{Pois}(X, n) \longrightarrow \text{hofib}(\text{Hom}_{dgOp}(P_{n+1, \infty}, \text{End}_A) \longrightarrow \text{Hom}_{dgOp}(\text{Comm}_\infty, \text{End}_A) : \mu_A)
\]

which is exactly what we wanted.

We conclude by observing that the last statement of the theorem is a direct consequence of the analogous statement in Theorem 3.1. \( \square \)
4 Another proof of the main result

In this last section we will give a more explicit description of our results: we take a particular resolution of the graded dg-Lie algebra \( k[-1](2) \) and study the induced resolution of the Lie operad. We check that its algebras are just Lie\(_\infty\)-algebras in the standard sense, see for example [HS]. These concrete computations also give an alternative proof of Theorem 3.2.

The graded dgLie algebra \( k[-1](2) \) has a cofibrant resolution \( L_0 \) given by the free Lie algebra generated by elements \( p_i \) for \( i = 2, 3, \ldots \), such that \( p_i \) sits in weight \( i \) and in cohomological degree 1; the differential in \( L_0 \) is defined as to satisfy

\[
d p_n = -\frac{1}{2} \sum_{i+j=n+1} [p_i, p_j]
\]

Notice that in particular that we have \( dp_2 = 0 \). The map \( L_0 \to k[-1](2) \) sends \( p_2 \) to the generator of \( k[-1](2) \) and the other \( p_i \) to zero.

By definition the points of the moduli space of \( n \)-shifted Poisson structures on \( A \) are just elements in

\[
\text{Hom}_{dg\text{Lie}^\text{gr}}(L_0, \text{Sym}_A(T_A[-n-1])[n+1]).
\]

By the same argument used in the proof of Theorem 3.1, this set maps into

\[
\text{Hom}_{dg\text{Lie}^\text{gr}}(L_0, \bigoplus_{i \in \mathbb{N}} \text{Hom}_k(\text{Sym}_k^i(A[n+1]), A[n+1])).
\]

So at the level of the vertices, an \( n \)-Poisson structure on \( A \) gives a sequence of symmetric multilinear maps \( q_i \) (the images of the \( p_i \)) on \( A[n+1] \), such that every \( q_i \) is an \( i \)-linear map of degree 1.

One of the possible definitions (see for example [Man]) of a \( L_\infty \)-structures is the following.

**Definition 4.1.** If \( V \) is a graded vector space, an \( L_\infty \)-structure on \( V \) is a sequence of symmetric maps of degree 1

\[
l_n : \text{Sym}^n V[1] \to V[1], \ n > 0
\]

such that for every \( n > 0 \) we have

\[
\sum_{i+j=n+1} [l_i, l_j] = 0,
\]

where the bracket is the Lie bracket we defined before on \( \bigoplus_{i \in \mathbb{N}} \text{Hom}_k(\text{Sym}_k^i(V[1]), V[1]) \).

So if we want to prove that (still at the level of the vertices) an \( n \)-Poisson structure gives us an \( L_\infty \)-structure on \( A[n] \), we could try to find such \( l_n \) on \( A[n+1] \). Natural candidates are the \( q_i \) that come directly from the shifted Poisson structure; these are...
given for $i > 1$. Notice that our brackets satisfy the graded antisymmetry relation $[x, y] = -(-1)^{|x||y|}[y, x]$; in particular, this relation does not involve the weights of $x$ and $y$. In our case $|p_i| = |q_i| = 1$, and so it follows $[p_i, p_j] = [q_i, q_j] = [p_j, p_i] = [q_j, q_i]$. Let us take $q_1 = d$, the differential of $A[n + 1]$. We should now verify that the symmetric maps $q_i$ satisfy

$$
\sum_{i+j=n+1} [q_i, q_j] = 0 .
$$

The other observation we need to make is that for every multilinear map $f \in \text{Hom}_k(\text{Sym}_k^i(A[n + 1]), A[n + 1])$, we have $[q_1, f] = [f, q_1] = d(f)$, where $d$ here is the differential of multilinear maps on $A[n + 1]$. So using these facts we have

$$
\sum_{i+j=n+1} [q_i, q_j] = 2d(q_n) + \sum_{i+j=n+1, i,j>1} [q_i, q_j] = 0 ,
$$

which is what we wanted. To summarize, an $n$-Poisson structure induces an $L_\infty$-structure on $A[n]$. Now we need to show that the induced $L_\infty$-structure on $A[n]$ is compatible with the algebra structure on $A$, that is to say that $A$ becomes a semi-strict $P_{n+1}$-algebra. But the $q_i$ we constructed in the previous step are (by definition) derivations of the given commutative product on $A$; this gives $A$ precisely the structure of a semi-strict $P_{n+1}$-algebra.

The upshot of this discussion is the fact that we got a map

$$
\text{Hom}_{\text{dgLie}^{gr}}(L_0, \text{Sym}_A(T[-n-1])[n+1]) \rightarrow \text{Hom}_{\text{dgOp}}(\hat{P}_{n+1}, \text{End}_A)
$$

for which the image is contained in the $\hat{P}_{n+1}$-structures whose commutative product is the one given on $A$. Equivalently, we get a function from $\text{Hom}_{\text{dgLie}^{gr}}(L_0, \text{Sym}_A(T[-n-1])[n+1])$ to the (non-homotopical) fiber product of the following diagram of sets

$$
\text{Hom}_{\text{dgOp}}(\hat{P}_{n+1}, \text{End}_A) \quad \text{pt} \quad \nu_A \quad \mu_A 
$$

where $\mu_A$ denotes the given commutative product of $A$. From here one can proceed in the exact same way as done towards the end of Section 2: namely, we can use Theorem 2.11 (and the explicit descriptions of the simplicial framings in $\text{dgOp}$ and $\text{dgLie}^{gr}$) in order to prove that we have a map of simplicial sets from $\text{Hom}_{\text{dgLie}^{gr}}(L_0, \text{Sym}_A(T[-n-1])[n+1])$ to the (strict) fiber of the natural map $\text{Hom}_{\text{dgOp}}(\hat{P}_{n+1}, \text{End}_A) \rightarrow \text{Hom}_{\text{dgOp}}(\text{Comm}, \text{End}_A)$, taken at $\mu_A$. 

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Now the same arguments used at the end of Section 3 allow to obtain a map in the homotopy category of simplicial sets

$$\text{Pois}(X, n) \longrightarrow P^h_{n+1}(A)$$

giving a more concrete proof of Theorem 3.2.

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