ON BIFURCATIONS FROM A TYPICAL CYCLE IN NONSMOOTH DYNAMICAL SYSTEMS

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ABSTRACT. We study the existence of periodic solutions in a class of nonsmooth differential systems obtained from nonautonomous periodic perturbations of reversible piecewise smooth differential systems. It is assumed that the unperturbed system presents a simple two-fold cycle, which is characterized by a closed trajectory connecting a visible two-fold singularity to itself. It is shown that under certain generic conditions the perturbed problem has sliding and crossing periodic solutions. In order to get our results, Melnikov’s ideas were applied together with tools from the geometric singular perturbation theory. Finally, a study of a perturbed piecewise Hamiltonian model is performed.

1. INTRODUCTION

Over the last decade, the theory of nonsmooth dynamical systems has been developed at a very fast pace, with growing importance at the frontier between mathematics, physics, engineering, and the life sciences (see, for instance, [5, 8, 21], and references therein). The study of such systems goes back to the work of Andronov et. al [3] in 1937. A rigorous mathematical formalization of this theory was provided by Filippov [10] in 1988, who used the theory of differential inclusions for establishing the definition of trajectory for nonsmooth differential systems. Nowadays, such systems are called Filippov systems.

In 1981, motivated by the work of Ekeland [9] on discontinuous Hamiltonian vector fields, Teixeira [25] studied generic singularities of refractive nonsmooth vector fields. It was performed a qualitative analyses of two-fold singularities appearing in these systems. Later, the generic classification of such singularities has been approached in several works [13, 16, 18].

Recently, many efforts have been dedicated to understand some typical global minimal sets (see, for instance, [1, 2, 19, 22, 23]) in Filippov systems. In particular, Novaes et al. [24] studied the unfolding of a Simple Two-Fold Cycle (see Figure 1) inside the class of piecewise smooth vector fields. A Simple Two-Fold Cycle is characterized by a closed trajectory connecting a two-fold singularity to itself (see Figure 1).

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The present material focuses on understanding how a simple two-fold cycle unfolds under small nonautonomous perturbations. More specifically, we are concerned with sliding and crossing periodic solutions bifurcating from periodic nonautonomous perturbation of the following class of $R$-reversible Filippov system

\[
Z_0(x, y) = \begin{cases} 
F^+(x, y) & \text{if } y > 0, \\
F^-(x, y) & \text{if } y < 0.
\end{cases}
\]

By $R$-reversibility, we mean $F^+(x, y) = -RF^-R(x, y)$, where $R : \mathbb{R}^2 \to \mathbb{R}^2$ is an involution for which $y = 0$ is the set of fixed points (see [14]). Here, we take $R(x, y) = (x, -y)$. In addition, we shall assume that system (1) admits a Simple Two-Fold Cycle enclosing an annulus $\mathcal{A}$ of crossing periodic solutions (see Figure 1). Usually, the above system is concisely denoted by $Z_0 = (F^+, F^-)$.

As examples of a piecewise smooth differential systems satisfying the above hypotheses, we have $Z_{\alpha}^0(x, y) = \left((1, x^2 - \alpha), (-1, x^2 - \alpha)\right)$, for $\alpha > 0$ (see Figure 1). The vector field $Z_{\alpha}^0$ contains a simple two-fold cycle $\mathcal{S}$ connecting the visible two-fold singularity $\left(\sqrt{\alpha}, 0\right)$ to itself. This cycle encloses an annulus $\mathcal{A}$ fulfilled with crossing periodic orbits.

![Figure 1](image-url). Phase space of the piecewise smooth differential system $(\dot{x}, \dot{y})^T = Z_{\alpha}^0(x, y) = \left((1, x^2 - \alpha), (-1, x^2 - \alpha)\right)$. The points $(-\sqrt{\alpha}, 0)$ and $(\sqrt{\alpha}, 0)$ represent the invisible and visible two-fold singularities, respectively. The bold line represents the simple two-fold cycle $\mathcal{S}$, which encloses a period annulus $\mathcal{A}$ of crossing periodic orbits.

In our setting, the construction of a suitable displacement function and its related Melnikov function are the central mechanisms behind our study. As it is fairly known
in Melnikov theory, the existence of periodic solutions is associated with simple zeros of a certain bifurcation function, called Melnikov Function. Such a function is obtained through the analysis of the perturbed parameterized system via expansion in Taylor series. Indeed, in the smooth case, the displacement function (equivalently, the Poincaré Map) is smooth in the parameter of perturbation, consequently, the Melnikov function is obtained by expanding the displacement function in Taylor series. The same procedure is used in nonsmooth systems for detecting crossing periodic solutions (see, for instance, [4, 7, 12, 20], and the references therein). Here, we also use such criterion to detect crossing periodic orbits. However, a regular perturbation of a Filippov system produces a singular perturbation problem in the sliding dynamics. Thus, the procedure for the detection of sliding periodic orbits is rather different and tools from singular perturbation theory must be employed.

This paper is organized as follows. First, in Section 2, we present the basic notions and results needed to state our main Theorems. More specifically, in Section 2.1, we recall the basic definitions about piecewise smooth differential systems, and in Section 2.2 we give some basic concepts and results concerning the reversible unperturbed problem. In Section 3, we state our main results, Theorems A and B. They deal with non-autonomous perturbations of $R$-reversible piecewise smooth differential systems admitting a simple two-fold cycle. More specifically, we provide a Melnikov function which determines the existence of crossing and sliding periodic solutions for such systems. In Theorem A, it is shown that this function determines the existence of crossing periodic solutions bifurcating from orbits of the period annulus $A$. In Corollary 3, we also consider autonomous perturbations. In Theorem B, it is shown that the same Melnikov function also determines, with additional hypotheses, the existence of both sliding and crossing periodic solutions bifurcating from the simple two-fold cycle $S$. In Section 4, we apply our results to study non-autonomous perturbations of the piecewise Hamiltonian differential system $Z^0_\alpha(x, y) = ((1, x^2 - \alpha), (-1, x^2 - \alpha))$. Finally, Section 5 is devoted to prove our main results.

2. Basic concepts and preliminary results

In this section, we recall the basic concepts and definitions from the theory of non-smooth dynamical systems as well as some preliminary results needed to state our main theorems.

2.1. Piecewise dynamical system. The content of this Section is standard and can be found in several other works (see for instance [11]).

Let $U$ be an open bounded subset of $\mathbb{R}^2$. We denote by $C^r(U, \mathbb{R}^2)$ the set of all $C^r$ vector fields $X : U \rightarrow \mathbb{R}^n$. Given $h : U \rightarrow \mathbb{R}$ a differentiable function having 0 as a regular value, we denote by $\Omega^r_h(U, \mathbb{R}^2)$ the space of piecewise smooth differential
systems $Z$ in $\mathbb{R}^2$ such that

$$Z(x,y) = \begin{cases} X^+(x,y), & \text{if } h(x,y) > 0, \\ X^-(x,y), & \text{if } h(x,y) < 0, \end{cases}$$

with $X^+, X^- \in C^r(\overline{U}, \mathbb{R}^2)$. As usual, system (2) is denoted by $Z = (X^+, X^-)$ and the switching surface $h^{-1}(0)$ by $\Sigma$.

The points on $\Sigma$ where both vectors fields $X^+$ and $X^-$ simultaneously point outward or inward from $\Sigma$ define, respectively, the escaping $\Sigma^e$ or sliding $\Sigma^s$ regions, and the interior of its complement in $\Sigma$ defines the crossing region $\Sigma^c$. The complementarity of the union of those regions are the tangency points between $X^+$ or $X^-$ with $\Sigma$.

The points in $\Sigma^e$ satisfy $X^+h(p) \cdot X^-h(p) > 0$, where $Xh$ denotes the derivative of the function $h$ in the direction of the vector $X$, that is $Xh(p) = \langle \nabla h(p), X(p) \rangle$. The points in $\Sigma^s$ (resp. $\Sigma^c$) satisfy $X^+h(p) < 0$ and $X^-h(p) > 0$ (resp. $X^+h(p) > 0$ and $X^-h(p) < 0$). Finally, the tangency points of $X^+$ (resp. $X^-$) satisfy $X^+h(p) = 0$ (resp. $X^-h(p) = 0$). For points $p \in \Sigma^s \cup \Sigma^c$, we define the sliding vector field

$$\tilde{Z}(p) = \frac{X^-h(p)X^+(p) - X^+h(p)X^-(p)}{X^-h(p) - X^+h(p)}.$$

A tangency point $p \in \Sigma$ is called a visible fold of $X^+$ (resp. $X^-$) if $(X^+)^2h(p) > 0$ (resp. $(X^-)^2h(p) < 0$). Analogously, reversing the inequalities, we define an invisible fold.

2.2. Preliminary results. Consider the involution $R(x,y) = (x,-y)$ and denote by $\text{Fix}(R) = \{(x,0), x \in \mathbb{R}\}$ its set of fixed points. For a $C^2$ function $F : D \to \mathbb{R}^2$, defined on an open bounded subset $D$ of $\mathbb{R}^2$, we consider the following $R$-reversible discontinuous piecewise smooth differential system with two zones separated by the straight line $\Sigma = \text{Fix}(R),$

$$\begin{align*}
(x', y')^T &= Z_0(x,y) = \begin{cases} 
F^+(x,y), & \text{if } y > 0, \\
F^-(x,y), & \text{if } y < 0,
\end{cases}
\end{align*}$$

where

$$F^-(x,y) = F(x,y), \quad F^+(x,y) = -RF(R(x,y)).$$

For $z = (x,y)^T$, we denote by $\Gamma^\pm(t,z) = (\Gamma_1^\pm(t,z), \Gamma_2^\pm(t,z))^T$ the solutions of systems $(x', y')^T = F^\pm(x,y)$ such that $\Gamma^\pm(0,z) = z$. Let

$$Y^\pm(t,z) = D_2\Gamma^\pm(t,z) = \begin{pmatrix} \frac{\partial \Gamma^\pm}{\partial x}(t,z) & \frac{\partial \Gamma^\pm}{\partial y}(t,z) \end{pmatrix}$$

be a Fundamental Matrix Solution of the variational equations

$$\frac{\partial Y^\pm}{\partial t}(t,z) = DF^\pm(\Gamma^\pm(t,z)) Y^\pm(t,z),$$

with initial condition $Y^\pm(0,z) = I_2 (2 \times 2 \text{ identity matrix}).$
The following result is a straightforward consequence of the reversibility property of the solution, $\Gamma^+(t, z) = R\Gamma^-(t, Rz)$.

**Lemma 1.** The equality $Y^{-}(t, z) = RY^{+}(-t, Rz)R$ holds.

As a consequence of the above lemma, we get

$$Dz\Gamma^{-1}(t, z) = Dz\Gamma^{+1}(-t, Rz)R$$

and

$$Dz\Gamma^{-2}(t, z) = -Dz\Gamma^{+2}(-t, Rz)R.$$

Let $F = (F_1, F_2)^T$. In order to assure that system (3) has a simple two-fold cycle (see Figure 2), we have to assume the following hypotheses.

$h_1$) There exist $x_i < x_v$ such that

$$F_2(p_v) = F_2(p_i) = 0, \quad \frac{\partial F_2}{\partial x}(p_v)F_1(p_v) < 0, \quad \text{and} \quad \frac{\partial F_2}{\partial x}(p_i)F_1(p_i) > 0.$$

where $p_v = (x_v, 0) \in \Sigma$, $p_i = (x_i, 0) \in \Sigma$, and $F_2(x, 0) \neq 0$ for $x_i < x < x_v$.

$h_2$) For each $x_i < x \leq x_v$, the solution $\Gamma^{-}(t, x, 0)$ reaches transversely the line of discontinuity $\Sigma$ for $t = \tilde{\sigma}(x) > 0$, that is

$$\Gamma^{-}(\tilde{\sigma}(x), x, 0) = 0 \quad \text{and} \quad F_2(\Gamma^{-}(\tilde{\sigma}(x), x, 0)) \neq 0.$$

![Figure 2](image-url)

**Figure 2.** Periodic orbits of system (3) surrounding the invisible two-fold point $p_i$ and fulfilling an annulus enclosed by the simple two-fold cycle $S$.

From the reversibility property of the vector field $Z_0$, hypothesis $(h_1)$ implies that the points $p_v, p_i \in \Sigma$ are, respectively, visible–visible and invisible–invisible folds.

Hypothesis $(h_2)$ fixes the orientation of the flow, which implies that

$$F_1(p_v) < 0, \quad \frac{\partial F_2}{\partial x}(p_v) > 0, \quad \text{and} \quad \frac{\partial F_2}{\partial x}(p_i) < 0.$$

Hypothesis $(h_1)$ also leads to the next result, which allows us to make explicit the first column of the matrix $Y^-(t, p_v)$, see (5).
Lemma 2. For every $t \in \mathbb{R}$, the following equality holds

$$\frac{\partial \Gamma^-}{\partial x}(t, p_v) = \frac{F(\Gamma^- (t, p_v))}{F_1(p_v)}.$$  

Proof. First we note that, as $F^- = F$, the function $w(t) = \frac{\partial \Gamma^-}{\partial x}(t, p_v)$ is a solution of the differential equation $\dot{w} = D_x F(\Gamma^- (t, p_v)) w$ with the initial condition $w(0) = (1, 0)$. Now, take

$$\overline{w}(t) = \frac{\partial \Gamma^-}{\partial t}(t, p_v) = F(\Gamma^- (t, p_v)) = \frac{F(\Gamma^- (t, p_v))}{F_1(p_v)}.$$  

Computing its derivative with respect to the variable $t$ we have

$$\frac{\partial \overline{w}}{\partial t}(t) = D_x F(\Gamma^- (t, p_v)) \frac{\partial \Gamma^-}{\partial t}(t, p_v) = D_x F(\Gamma^- (t, p_v)) \overline{w}(t).$$

Moreover, hypothesis $(h_1)$ implies that

$$\overline{w}(0) = \frac{F(p_v)}{F_1(p_v)} = \left( \frac{F_1(p_v)}{F_1(p_v)} \right) = (1, 0) = w(0).$$

Hence, we conclude that $w(t) = \overline{w}(t)$. \hfill \Box

Hypothesis $(h_2)$, together with the reversibility property, imply that for each $x_1 < x \leq x_0$ the function

$$\gamma(t, x) = (\gamma_1(t, x), \gamma_2(t, x)) = \begin{cases} 
\Gamma^-(t, x, 0) & \text{if } 0 \leq t \leq \overline{\sigma}(x), \\
R\Gamma^-(t, x, 0) & \text{if } -\sigma(x) \leq t \leq 0 
\end{cases}$$

is a $2\sigma(x)$–periodic solution of system (3) such that $\gamma(0, x) = (x, 0) \in \Sigma$. Consequently, the invisible two fold $p_i$ behaves as a center having an annulus of periodic orbits ending at the simple two-fold cycle $S = \{ \gamma(t, x_0) : -\sigma(x_0) \leq t \leq \sigma(x_0) \}$ (see Figure 2). Notice that

$$S \cap \Sigma = \{ p_v, q_v \}, \quad \text{where} \quad q_v = \Gamma^-(\overline{\sigma}(x_v), p_v).$$

From now on, when it is convenient, we shall denote $\Gamma^-$ and $\Gamma^-$, only by $\Gamma$ and $\gamma$, respectively.

We note that, by hypothesis $(h_2)$, the function $\overline{\sigma}(x)$ is differentiable on the interval $(x_i, x_v]$. Indeed, it is a solution of the implicit equation $\Gamma_2(\overline{\sigma}(x), x, 0) = 0$. Differentiating this last equality implicitly in the variable $x$ we obtain, for each $x_1 < x \leq x_v$, the following relation

$$\overline{\sigma}'(x) = -\frac{\partial \Gamma_2}{\partial x}(\overline{\sigma}(x), x, 0) \frac{1}{\Gamma_2(\overline{\sigma}(x), x, 0)}.$$

(10)
Furthermore, since $p_i = (x_i, 0)$ is an invisible-invisible fold, then $\inf\{\sigma(x) : x_i < x \leq x_\sigma\} = 0$, and $0 \leq \sigma_M = \sup\{\sigma(x) : x_i < x \leq x_\sigma\} < \infty$. Accordingly, we fix the interval $T = [0, \sigma_M]$.

3. Statement of the main results

We consider the following perturbation of system (3).

$$
(x', y')^T = Z_\epsilon(t, x, y) = \begin{cases} 
X^+(t, x, y; \epsilon) & \text{if } y > 0, \\
X^-(t, x, y; \epsilon) & \text{if } y < 0,
\end{cases}
$$

where

$$
X^\pm(t, x, y; \epsilon) = F^\pm(t, x, y) + \epsilon G^\pm(t, x, y) + \epsilon^2 H^\pm(t, x, y; \epsilon).
$$

We assume that $G^\pm(t, x, y)$ and $H^\pm(t, x, y; \epsilon)$ are smooth functions in $\mathbb{R} \times D$ and $\mathbb{R} \times D \times (-\epsilon_0, \epsilon_0)$, respectively, and $2\sigma$–periodic in the variable $t$ for some $\sigma \in T = [0, \sigma_M]$.

We want to detect, for $\epsilon > 0$ small enough, the existence of $2\sigma$–periodic solutions of system (11). Accordingly, let $S^1_\sigma \equiv \mathbb{R}/(2\sigma \mathbb{Z})$ and define the function $M : S^1_\sigma \times (x_i, x_\sigma] \to \mathbb{R}$ as

$$
M(\theta, x) = F(\gamma(\sigma(x), x)) \wedge \left( Y(\sigma(x), x, 0) \int_0^{\sigma(x)} Y(t, x, 0)^{-1} \{G^-, G^+\}_\theta(t, \gamma(t, x)) \, dt \right).
$$

where $\gamma$ is given in (8), $Y$ is the fundamental matrix given in (5). Here, the wedge product is defined by $(a_1, a_2) \wedge (b_1, b_2) = ((-a_2, a_1), (b_1, b_2))$, and, for $z \in D$,

$$
\{G^-, G^+\}_\theta(t, z) = G^-(t + \theta, z) + RG^+(t - \theta, Rz).
$$

Observe that if we consider the involution $\tilde{R}(t, x, y) = (-t, x, -y)$, then system (11) in the extended phase space

$$
\theta' = 1 \\
(x', y')^T = Z_\epsilon(\theta, x, y)
$$

is reversible if, and only if, $\{G^-, G^+\}_0 = 0$.

Our first main result is concerned about the existence of subharmonic crossing periodic solutions of system (11) when the perturbation is $2\sigma$–periodic in the variable $t$, $\sigma \in T$.

**Theorem A.** Take $\sigma \in T = [0, \sigma_M]$ and $x_\sigma \in (x_i, x_\sigma)$ such that $\sigma(x_\sigma) = \sigma$, where $\sigma(x)$ is given in (7). Assume that the vector field $Z_\epsilon$ in (11) is $2\sigma$–periodic in the variable $t$. If there exists $\theta^* \in S^1_\sigma$ such that

$$
M(\theta^*, x_\sigma) = 0, \quad \text{and } \frac{\partial M}{\partial \theta}(\theta^*, x_\sigma) \neq 0,
$$

then for $\epsilon > 0$ sufficiently small there exists a crossing $2\sigma$–periodic solution of system (11) with initial condition, in $S^1_\sigma \times D$, $\epsilon$–close to $(t_0, z_0) = (\theta^*, (x_\sigma, 0))$. 
The next result is obtained as a consequence of Theorem A and deals with the continuation problem of subharmonic crossing periodic solutions of system (11) when it is autonomous.

**Corollary 3.** Assume that the vector field $Z_\epsilon$ in (11) is autonomous and denote $M(x) = M(\theta, x)$. If there exists $x^* \in (x_1, x_2)$ such that $M(x^*) = 0$ and $M'(x^*) \neq 0$ then, for $\epsilon > 0$ sufficiently small, there exists a crossing periodic solution of system (11) with initial condition, in $D$, $\epsilon$-close to $(x^*, 0)$.

Our second main result is concerned about the bifurcation of periodic solutions in the special case that system (11) is perturbed by $2\sigma_\epsilon = 2\sigma(x_\epsilon)$–periodic functions. We shall see that, in this case, either a crossing or a sliding periodic solution can appear. For each $\theta \in S^1_{\epsilon_\sigma}$, we define the number $g_{\theta} \in \mathbb{R}$ as

$$g_{\theta} = \left( D_z \Gamma_2(\sigma_\epsilon, p_{\epsilon}), \int_{\sigma_\epsilon}^{\sigma_\epsilon} \langle Y(t, p_{\epsilon}), -G^- \{ t, \Gamma(t, p_{\epsilon}) \} \rangle dt \right)$$

$$= \left( D_z \Gamma_2(\sigma_\epsilon, p_{\epsilon}), \int_{\sigma_\epsilon}^{\sigma_\epsilon} \langle Y(t, p_{\epsilon}), (G^- (t + \theta, \Gamma(t, p_{\epsilon}))) - R G^+ (-t + \theta, R \Gamma(t, p_{\epsilon})) \rangle dt \right).$$

In the above expression, the inner product notation $\langle *, * \rangle$ is actually an abuse of notation. Indeed, the left and right factors are expressed as row and column vectors, respectively. Thus, the matrix product between them results in a scalar. Nevertheless, due to the amount of computations involving matrices, we decide to consider the inner product notation to emphasize that the result is in fact a scalar, avoiding then any possible misunderstanding.

**Theorem B.** Suppose that the vector field $Z_\epsilon$ in (11) is $2\sigma_\epsilon$–periodic in the variable $t$ and assume that there exists $\theta^* \in S^1_{\epsilon_\sigma}$ such that $M(\theta^*, x_\epsilon) = 0$ and $(\partial M / \partial \theta)(\theta^*, x_\epsilon) \neq 0$.

(a) If $G_2^+(\theta^*, p_{\epsilon}) \neq G_2^-(\theta^*, p_{\epsilon})$ and

$$g_{\theta^*} > \frac{2 F_2(q_{\epsilon})}{F_1(p_{\epsilon})} \frac{\partial F_2}{\partial x}(p_{\epsilon}) \max \{ G_2^+(\theta^*, p_{\epsilon}) \},$$

then, for $\epsilon > 0$ sufficiently small, there exists a sliding $2\sigma_\epsilon$–periodic solution of system (11) with initial condition, in $S^1_{\epsilon_\sigma} \times D$, $\epsilon$-close to $(t_0, z_0) = (\theta^*, p_{\epsilon})$. Moreover, this solution slides either on $\Sigma^+$ or $\Sigma^-$ provided that $G_2^+(\theta^*, p_{\epsilon}) < G_2^-(\theta^*, p_{\epsilon})$ or $G_2^+(\theta^*, p_{\epsilon}) > G_2^-(\theta^*, p_{\epsilon})$, respectively.

(b) If

$$g_{\theta^*} < \frac{2 F_2(q_{\epsilon})}{F_1(p_{\epsilon})} \frac{\partial F_2}{\partial x}(p_{\epsilon}) \max \{ G_2^+(\theta^*, p_{\epsilon}) \},$$

then, for $\epsilon > 0$ sufficiently small, there exists a crossing $2\sigma_\epsilon$–periodic solution of system (11) with initial condition, in $S^1_{\epsilon_\sigma} \times D$, $\epsilon$-close to $(t_0, z_0) = (\theta^*, p_{\epsilon})$.  

4. A PIECEWISE HAMILTONIAN MODEL

In this section, we apply the previous results to study the crossing and sliding periodic solutions of nonautonomous perturbations of a piecewise Hamiltonian model. This kind of problem had been previously addressed in [17], where the authors applied KAM theory to prove that, under certain conditions, a piecewise Hamiltonian model has infinitely many periodic solutions.

Consider the following continuous Hamiltonian function

\[ H(x, y) = |y| - \frac{x^3}{3} + ax, \quad \text{where} \quad a > 0. \]

As usual, \(| \cdot |\) denotes the absolute value of a real number. The above Hamiltonian gives rise to the following discontinuous piecewise Hamiltonian differential system

(15) \[ (x', y')^T = Z_0^\sigma(x, y) = (\text{sign}(y), x^2 - a) = \begin{cases} (1, x^2 - a) & \text{if } y > 0, \\ (-1, x^2 - a) & \text{if } y < 0, \end{cases} \]

The switching surface is given by \( \Sigma = \{(x, 0), x \in \mathbb{R}\} \). Its phase space is depicted in Figure 1. Following the notation of the previous Section we take

\[ F^-(x, y) = (-1, x^2 - a), \quad F^+(x, y) = (1, x^2 - a). \]

Notice that the above piecewise Hamiltonian differential system \( Z_0^\sigma \) is \( R \)-reversible with \( R(x, y) = (x, -y) \). In addition, it has two two-fold singularities, one invisible \( p_i = (x_i, 0) = (-\sqrt{a}, 0) \) and other visible \( p_v = (x_v, 0) = (\sqrt{a}, 0) \).

The solution \( \Gamma^-(t, x, y) \) of \( (x', y')^T = F^-(x, y) \) can be easily computed as

\[ \Gamma^-(t, x, y) = \left( -t + x, \frac{1}{3}(t^3 - 3t^2x + 3tx^2 + 3y - 3ta) \right). \]

Furthermore, for each \(-\sqrt{a} < x \leq \sqrt{a}\), it is straightforward to see that \( \Gamma^-(t, x, 0) \) reaches transversely \( \Sigma \) for \( t = \sigma(x) = \frac{1}{2}(3x + \sqrt{3\sqrt{-x^2 + 4a}}) \). Hence, for \(-\sqrt{a} < x \leq \sqrt{a}\), the reversibility property implies that the solution \( \gamma(t, x) \) of (15), satisfying \( \gamma(0, x) = (x, 0) \), is given by

\[ \gamma(t, x) = \begin{cases} \left( -t + x, \frac{1}{3}(t^3 - 3t^2x + 3tx^2 - 3ta) \right) & \text{if } 0 \leq t \leq \sigma(x), \\ \left( t + x, \frac{1}{3}(t^3 + 3t^2x + 3tx^2 - 3ta) \right) & \text{if } -\sigma(x) \leq t \leq 0, \end{cases} \]

From the formula of \( \sigma(x) \), one obtains an explicit expression for the point \((x_\sigma, 0)\) satisfying \( \sigma(x_\sigma) = \sigma \),

(16) \[ x_\sigma = \frac{1}{6}(3\sigma - \sqrt{3\sqrt{12a - \sigma^2}}) \in (-\sqrt{a}, \sqrt{a}). \]
Accordingly, $Z_0^\sigma$ satisfies hypotheses $(h_1)$ and $(h_2)$. Furthermore, since $\sigma(\sqrt{\alpha}) = 3\sqrt{\alpha}$, we get $S = \{ \gamma(t, \sqrt{\alpha}) : -3\sqrt{\alpha} \leq t \leq 3\sqrt{\alpha} \}$ (see Figure 1). Clearly $S \cap \Sigma = \{ p_v, q_v \}$, with $q_v = \Gamma^-(\sqrt{\alpha}, \sqrt{\alpha}, 0) = (-2\sqrt{\alpha}, 0)$.

4.1. Nonautonomous perturbation. Now, in order to illustrate the application of Theorems A and B, we consider the following nonautonomous perturbation of (15).

\begin{equation}
(x', y')^T = Z_\epsilon(x, y) = \begin{cases} F^+(x, y) + \epsilon G^+(t, x, y) & \text{if } y > 0, \\
F^-(x, y) + \epsilon G^-(t, x, y) & \text{if } y < 0,
\end{cases}
\end{equation}

where

$$G^+(t, x, y) = \left( 0, \lambda \sin \frac{\pi t}{\sigma} \right) \quad \text{and} \quad G^-(t, x, y) = \left( 0, \sin \frac{\pi t}{\sigma} \right),$$

for some $\lambda \in \mathbb{R}$. Notice that $G^\pm(t, x, y)$ are $2\sigma$-periodic in the variable $t$. We shall see that, for convenient values of $\lambda$, system (17) satisfies the hypotheses either of Theorem A or Theorem B.

The fundamental matrix solution $Y(t, x, y) = Y^-(t, x, y)$, defined in (5), is given by

$$Y(t, x, y) = \begin{pmatrix} 1 & 0 \\ -t^2 + 2tx & 1 \end{pmatrix}.$$

Thus, we compute the function (12) as

$$M(\theta, x) = \frac{(1 + \lambda)\sigma}{\pi} \left( \cos \left[ \frac{\pi(3x + \sqrt{3}\sqrt{4\alpha - x^2} + 2\theta)}{2\sigma} \right] - \cos \left[ \frac{\pi\theta}{\sigma} \right] \right).$$

In the next result, as an application of Theorem A, we show that system (17) has two crossing periodic solutions, provided that the period of the perturbation is strictly less than $6\sqrt{\alpha}$.

**Proposition 4.** Assume that $\lambda \neq -1$. Then, for each $\sigma \in (0, 3\sqrt{\alpha})$ and for $\epsilon > 0$ sufficiently small, there exist two crossing $2\sigma$-periodic solutions of system (17) with initial conditions $\epsilon$-close to $(3\sigma/2, (x, 0))$ and $(\sigma/2, (x, 0))$, respectively.

**Proof.** Given $\sigma \in (0, 3\sqrt{\alpha})$, notice that $\sigma(x, \sigma) = \sigma$ if, and only if, $x = \frac{1}{6}(3\sigma - \sqrt{3}\sqrt{12\alpha - \sigma^2}) \in (-\sqrt{\alpha}, \sqrt{\alpha})$ (see (16)). Then,

$$M(\theta, x) = -\frac{2(1 + \lambda)\sigma}{\pi} \cos \left( \frac{\pi\theta}{\sigma} \right),$$

where we used the following relation

$$\sqrt{36\alpha + 2\sigma \left( \sqrt{36\alpha - 3\sigma^2} - \sigma \right)} = \sigma + \sqrt{36\alpha - 3\sigma^2},$$

for every $\alpha > 0$ and $\sigma \in (0, 2\sqrt{3\alpha}]$.

Solving $M(\theta, x) = 0$, for $\theta \in S_0^*$, we get $\theta_1^* = 3\sigma/2$ and $\theta_2^* = \sigma/2$. Moreover,

$$\frac{\partial M}{\partial \theta}(\theta_2^*, x) = -\frac{\partial M}{\partial \theta}(\theta_1^*, x) = 2(1 + \lambda) \neq 0.$$
Proposition 5. Assume that \( \sigma = 3\sqrt{\alpha} \) and \( \lambda \neq -1 \). Then, for \( \varepsilon > 0 \) sufficiently small, the following statement holds

(i) for \( \lambda \neq 1 \), there exists a sliding \( 6\sqrt{\alpha} \)-periodic solution of system \((17)\) with initial condition \( \varepsilon \)-close to \( (3\sqrt{\alpha}/2, (x_v, 0)) \), which slides either on \( \Sigma^s \) or \( \Sigma^c \) provided that \( \lambda < 1 \) or \( \lambda > 1 \);
(ii) for \( \lambda < 0 \), there exists a sliding \( 6\sqrt{\alpha} \)-periodic solution of system \((17)\) with initial conditions \( \varepsilon \)-close to \( (9\sqrt{\alpha}/2, (x_v, 0)) \), which slides on \( \Sigma^c \);
(iii) for \( \lambda > 0 \), there exists a crossing \( 6\sqrt{\alpha} \)-periodic solution of system \((17)\) with initial conditions \( \varepsilon \)-close to \( (9\sqrt{\alpha}/2, (x_v, 0)) \).

Remark 6. Notice that, from Proposition 5, sliding and crossing periodic solutions may coexist. More specifically, comparing the statements (i) and (ii) we get

- For \( \lambda \in (-\infty, 0) \setminus \{ -1 \} \), there exists two sliding \( 6\sqrt{\alpha} \)-periodic solutions. One with initial condition \( \varepsilon \)-close to \( (3\sqrt{\alpha}/2, (x_v, 0)) \), which slides on \( \Sigma^s \), and another with initial conditions \( \varepsilon \)-close to \( (9\sqrt{\alpha}/2, (x_v, 0)) \), which slides on \( \Sigma^c \).
- For \( \lambda \in (0, 1) \), there exist a crossing \( 6\sqrt{\alpha} \)-periodic solution with initial conditions \( \varepsilon \)-close to \( (9\sqrt{\alpha}/2, (x_v, 0)) \) and a sliding \( 6\sqrt{\alpha} \)-periodic solution with initial condition \( \varepsilon \)-close to \( (3\sqrt{\alpha}/2, (x_v, 0)) \), which slides on \( \Sigma^s \).
- For \( \lambda \in (1, +\infty) \), there exist a crossing \( 6\sqrt{\alpha} \)-periodic solution with initial conditions \( \varepsilon \)-close to \( (9\sqrt{\alpha}/2, (x_v, 0)) \) and a sliding \( 6\sqrt{\alpha} \)-periodic solution with initial condition \( \varepsilon \)-close to \( (3\sqrt{\alpha}/2, (x_v, 0)) \), which slides on \( \Sigma^c \).

Proof of Proposition 5 If \( \sigma = 3\sqrt{\alpha} \), then \( G^\pm(t, x, y) \) are \( 6\sqrt{\alpha} \)-periodic in the variable \( t \), and

\[
M(\theta, \sqrt{\alpha}) = \frac{-6\sqrt{\alpha}(1 + \lambda)}{\pi} \cos \left( \frac{\pi \theta}{3\sqrt{\alpha}} \right).
\]

Solving \( M(\theta^*, \sqrt{\alpha}) = 0 \), for \( \theta^* \in [0, 6\sqrt{\alpha}] \), we get \( \theta^*_1 = 3\sqrt{\alpha}/2 \) and \( \theta^*_2 = 9\sqrt{\alpha}/2 \).

Moreover,

\[
\frac{\partial M}{\partial \theta}(\theta^*_2, \sqrt{\alpha}) = -\frac{\partial M}{\partial \theta}(\theta^*_1, \sqrt{\alpha}) = 2(1 + \lambda) \neq 0,
\]

and \( g_\theta = \frac{6\sqrt{\alpha}(1 - \lambda)}{\pi} \cos \left( \frac{\pi \theta}{3\sqrt{\alpha}} \right) \). Thus, \( g_{\theta^*2} = 0 \). Furthermore, \( G^+ (\theta^*_n, p_v) = (-1)^{1+n}\lambda \), \( G^- (\theta^*_n, p_v) = (-1)^{1+n} \), for \( n = 1, 2 \), and

\[
\frac{2F_2(q_v)}{F_1(p_v) \frac{\partial F_2}{\partial x}(p_v)} = -3\sqrt{\alpha}.
\]
To obtain statement (i) notice that, for \( \lambda \neq 1 \), \( G^+_2(\theta^*_1, p_\psi) \neq G^-_2(\theta^*_1, p_\psi) \). In this case, \( g_\theta^1 = 0 > -3\sqrt{\lambda} \max \{ G^\pm_2(\theta^*_1, p_\psi) \} = -3\sqrt{\lambda} \max \{ 1, \lambda \} \).

Therefore, from statement (a) of Theorem B there exists a sliding \( 6\sqrt{\lambda} \)-periodic solution with initial condition \( \varepsilon \)-close to \( (3\sqrt{\lambda}/2, p_\psi) \). Moreover, for \( \lambda > 1 \), we have \( G^+_2(\theta^*_1, p_\psi) > G^-_2(\theta^*_1, p_\psi) \), which implies that this periodic solution slides on \( \Sigma^\varepsilon \). Analogously, for \( \lambda < 1 \), we have \( G^+_2(\theta^*_1, p_\psi) < G^-_2(\theta^*_1, p_\psi) \), which implies that this periodic solution slides on \( \Sigma^\varepsilon \).

To obtain statement (ii) notice that, for \( \lambda < 0 \), we have \( g_\theta^2 = 0 > -3\sqrt{\lambda} \max \{ G^\pm_2(\theta^*_2, p_\psi) \} = -3\sqrt{\lambda} \max \{ -1, -\lambda \} = 3\sqrt{\lambda} \lambda \).

Therefore, from statement (a) of Theorem B there exists a sliding \( 6\sqrt{\lambda} \)-periodic solution with initial condition \( \varepsilon \)-close to \( (9\sqrt{\lambda}/2, p_\psi) \). Moreover, in this case, \( G^+_2(\theta^*_2, p_\psi) > G^-_2(\theta^*_2, p_\psi) \), which implies that this periodic solution slides on \( \Sigma^\varepsilon \).

Finally, to obtain statement (iii) notice that, for \( \lambda > 0 \), we have \( g_\theta^3 = 0 < -3\sqrt{\lambda} \max \{ G^\pm_2(\theta^*_3, p_\psi) \} = -3\sqrt{\lambda} \max \{ -1, -\lambda \} \).

Therefore, from statement (b) of Theorem B there exists a crossing \( 6\sqrt{\lambda} \)-periodic solution with initial condition \( \varepsilon \)-close to \( (9\sqrt{\lambda}/2, p_\psi) \). \( \square \)

5. Proofs of Theorems A and B

Recall that to study a nonautonomous periodic differential equation \( w' = f(t, w) \), \( (t, z) \in S^1 \times D \) we can work in the extended phase space adding time as a variable \( \theta' = 1 \) and \( v' = f(\theta, v) \). If \( (\theta(t), v(t)) \) is a solution of the autonomous system such that \( (\theta(0), v(0)) = (\bar{\theta}, \bar{v}) \), then \( v'(t) = f(\theta + t, v(t)) \) and \( w(t) := v(t - \bar{\theta}) \) is the solution of the nonautonomous system such that \( w(\bar{\theta}) = \bar{v} \).

Accordingly, we study system (II) in the extended phase space

\[
(18) \quad \theta' = 1, \quad (x', y)^T = Z_{\epsilon}(x, y),
\]

where \( (\theta, x, y) \in S^1 \times D, D \subseteq \mathbb{R}^2 \) being \( S^1 = \mathbb{R}/(2\pi \mathbb{Z}) \). We note that (18) is also a Filippov system having \( \tilde{\Sigma} = S^1 \times \Sigma \) as its discontinuous manifold. Moreover, \( \tilde{\Sigma} = h^{-1}(0) \) for \( h(\theta, x, y) = y \).

Let \( z \in D \), the solutions \( \Phi^\pm(t, \theta, z; \varepsilon) \) of (18), restricted to \( y \geq 0 \), such that \( \Phi^\pm(0, \theta, z; \varepsilon) = (\theta, z) \) are given as

\[
\Phi^\pm(t, \theta, z; \varepsilon) = (t + \theta, \zeta^\pm(t + \theta, \theta, z; \varepsilon)),
\]

where \( \zeta^\pm(t, \theta, z; \varepsilon) \) are solutions of

\[
(19) \quad \zeta' = X^\pm(t + \theta, \zeta; \varepsilon), \quad \zeta(0) = z, \quad \zeta \in D.
\]

Lemma 7. Fix \( T > 0, \theta \in S^1, z_0 \in D, \) and \( z_1 \in \mathbb{R}^2 \). Let

\[
(20) \quad \psi^\pm(t, \theta, z_0, z_1) = Y^\pm(t, z_0) \left( z_1 + \int_0^t Y^\pm(s, z_0)^{-1} G^\pm(s + \theta, \Gamma^\pm(s, z_0)) \, ds \right),
\]
where \( Y^\pm \) are the fundamental solutions of the variational equations. Then, for \( \varepsilon > 0 \) small enough, \( z_0 + \varepsilon z_1 \in D \) and the next equality holds

\[
\xi^\pm (t, \theta_0 + \varepsilon \theta_1, z_0 + \varepsilon z_1; \varepsilon) = \Gamma^\pm (t, z_0) + \varepsilon \psi^\pm (t, \theta_0, z_0, z_1) + \mathcal{O}(\varepsilon^2), \ t \in [-T, T]
\]

**Proof.** Computing the derivative in the variable \( t \) in both sides of the equality \( \xi^\pm (t, \theta_0 + \varepsilon \theta_1, z_0 + \varepsilon z_1; \varepsilon) = \Gamma^\pm (t, z_0) + \varepsilon \psi^\pm (t) + \mathcal{O}(\varepsilon^2) \) we obtain

\[
F^\pm (\xi^\pm (t, \theta_0 + \varepsilon \theta_1, z_0 + \varepsilon z_1; \varepsilon)) + \varepsilon G^\pm (t + \theta_0 + \varepsilon \theta_1, \xi^\pm (t, \theta_0 + \varepsilon \theta_1, z_0 + \varepsilon z_1; \varepsilon)) =
\]

\[
F (\Gamma^\pm (t, z_0)) + \varepsilon \frac{d \psi^\pm}{dt} (t) + \mathcal{O}(\varepsilon^2).
\]

Expanding in Taylor series the lefthand side of the above equation around \( \varepsilon = 0 \), and comparing the coefficient of \( \varepsilon \) in the both sides, we conclude that

\[
\frac{d \psi^\pm}{dt} (t, \theta_0, z_0, z_1) = DF^\pm (\Gamma^\pm (t, z_0)) \psi^\pm (t) + G^\pm (t + \theta_0, \Gamma^\pm (t, z_0)).
\]

Moreover, \( \psi^\pm (0, \theta_0, z_0, z_1) = z_1 \). Hence, the solution of the above differential equation is given by \( (20) \). We observe that \( \psi^\pm (t) \) depends on \( \theta_0, z_0, z_1 \), then we denote \( \psi^\pm (t) = \psi^\pm (t, \theta_0, z_0, z_1) \).

Applying Lemma [1] to the fundamental matrices \( Y^\pm \) (see [5]) in the expression (20) we get

\[
\psi^-(t, \theta, z_0, z_1) = Y(t, z_0) \left( z_1 + \int_0^t Y(s, z_0)^{-1} G^- (s + \theta, \Gamma(s, z_0)) \, ds \right),
\]

\[
\psi^+(t, -t, Rz_0, Rz_1) = RY(t, z_0) \left( z_1 - \int_0^t Y(s, z_0)^{-1} R G^+ (-s + \theta, R \Gamma(s, z_0)) \, ds \right).
\]

Moreover, using that \( Y(t, z) = D_z \Gamma(t, z) \) in the first part of the above expressions we have

\[
\psi^-_i (t, \theta, z_0, z_1) = \left\langle D_z \Gamma_i(t, z_0), z_1 + \int_0^t Y(s, z_0)^{-1} G^- (s + \theta, \Gamma(s, x_0)) \, ds \right\rangle,
\]

\[
\psi^+_i (-t, \theta, Rz_0, Rz_1) = \left\langle (-1)^{i+1} D_z \Gamma_i(t, z_0), z_1 - \int_0^t Y(s, z_0)^{-1} R G^+ (-s + \theta, R \Gamma(s, x_0)) \, ds \right\rangle.
\]

for \( i = 1, 2 \).

Observe that for \( \varepsilon = 0 \) system (18) has two lines of two-fold points, one invisible \((\theta, x_i, 0)\) and one visible \((\theta, x_v, 0)\) (see Figure 3). Moreover, for each \( \theta \in S_\nu^1 \) and \( x_i < x \leq x_v \) there exists a \( 2\pi(x) \)–periodic solution \( \Gamma(t, \theta, x) = (t + \theta, \gamma(t, x)) \), where \( \gamma \) is given in (5).

Notice that studying the bifurcation of the fold lines of system (18), for \( \varepsilon > 0 \), is equivalent to study the zeros of the functions

\[
\langle \nabla h(x, 0), X^\pm (\theta, (x, 0); \varepsilon) \rangle = X^\pm_2 (\theta, (x, 0); \varepsilon) = F_2^\pm (x, 0) + \varepsilon G_2^\pm (\theta, x, 0) + \mathcal{O}(\varepsilon^2).
\]
Thus, by hypothesis (h1), we obtain that for $\varepsilon > 0$ sufficiently small each one of the lines of visible-visible fold points $(\theta, x_v, 0)$ bifurcates in two lines $(\theta, \ell_\pm^v(\theta; \varepsilon), 0)$, one of visible fold points for $X^+$ and another of visible fold points for $X^-$. Analogously, the line of invisible-invisible fold points $(\theta, x_i, 0)$ bifurcates in two lines $(\theta, \ell_\pm^i(\theta; \varepsilon), 0)$, one of invisible fold points for $X^+$ and another of invisible fold points for $X^-$. Furthermore,

\begin{align*}
\ell_\pm^v(\theta; \varepsilon) &= x_v - \varepsilon \frac{G_2^\pm(\theta, p_v)}{\partial x} + O(\varepsilon^2) = x_v + \varepsilon v_\pm^v(\theta) + O(\varepsilon^2), \quad \text{and} \\
\ell_\pm^i(\theta; \varepsilon) &= x_i - \varepsilon \frac{G_2^\pm(\theta, p_i)}{\partial x} + O(\varepsilon^2) = x_i + \varepsilon v_\pm^i(\theta) + O(\varepsilon^2).
\end{align*}

In what follows, $\pi_\theta, \pi_{x_v}$ and $\pi_{x_i}$ will denote the projections, defined on $S^1_v \times D$, onto the first, second, and third coordinates, respectively.

5.1. Proof of Theorem A. The idea of this proof is to define a function $F : S^1_v \times (x_v, x_i) \to \mathbb{R}^2$ which allows us to determine the existence of crossing periodic solutions. Given $\theta \in S^1_v$ and $x \in (x_v, x_i)$, we consider the flows $\Phi^-(t, \theta, (x, 0); \varepsilon)$ and $\Phi^+(t, \theta + 2\sigma, (x, 0); \varepsilon)$ of (18). If for some $\theta_\ast \in S^1_v$ and $x_\ast \in (x_v, x_i)$ there exist $t^-_\ast \geq 0$ and $t^+_\ast \leq 0$ such that

\begin{equation}
\Phi^-(t^-_\ast, \theta_\ast, (x_\ast, 0); \varepsilon) = \Phi^+(t^+_\ast, \theta_\ast + 2\sigma, (x_\ast, 0); \varepsilon) \in \tilde{\Sigma}.
\end{equation}
then \( t^+_* = t^-_* - 2\sigma \) and therefore

\[
\Phi(t, \theta_*, (x_*, 0); \epsilon) = \begin{cases} 
\Phi^+(t, \theta_*, 2\sigma, (x_*, 0); \epsilon) & \text{if } t^+_* = t^-_* - 2\sigma \leq t \leq 0, \\
\Phi^-(t, \theta_*, (x_*, 0); \epsilon) & \text{if } 0 \leq t \leq t^-_* \n\end{cases}
\]

is a \( 2\sigma \)--periodic crossing solution of system \((18)\). Indeed, this solution is well defined because

\[
\Phi^+(0, \theta_*, 2\sigma, (x_*, 0); \epsilon) = (\theta_*, 2\sigma, \xi^+(0, \theta_*, 2\sigma, (x_*, 0); \epsilon)) = (\theta_*, 2\sigma, x_*, 0)
\]

and, as we are working in the cylinder \( S^1_\sigma \times D \), these two points are the same.

In what follows, we show the existence of \( \theta_* \) and \( x_* \) satisfying \((23)\). For \( \epsilon = 0 \) we know that (see \((8)\))

\[
\pi_y \Phi^- (\overline{\sigma}(x), \theta, (x, 0); 0) = \xi^-_2 (\overline{\sigma}(x), \theta, (x, 0); 0) = \Gamma_2 (\overline{\sigma}(x), x, 0) = 0.
\]

Since, by hypothesis \((h2)\), this flow reaches transversally the set of discontinuity \( \Sigma \) we can apply the implicit function theorem to obtain a time \( t^- (\theta, x; \epsilon) = \overline{\sigma}(x) + \epsilon t^-_1 (\theta, x) + \mathcal{O}(\epsilon^2) > 0 \) such that

\[
\pi_y \Phi^- (t^- (\theta, x; \epsilon), \theta, (x, 0); \epsilon) = \xi^-_2 (t^- (\theta, x; \epsilon), \theta, (x, 0); \epsilon) = 0.
\]

Analogously,

\[
\pi_y \Phi^+ (-\overline{\sigma}(x), \theta + 2\sigma, (x, 0); 0) = \xi^+_2 (-\overline{\sigma}(x), \theta + 2\sigma, (x, 0); 0) = -\Gamma_2 (\overline{\sigma}(x), x, 0) = 0,
\]

therefore there exists \( t^+ (\theta, x; \epsilon) = -\overline{\sigma}(x) + \epsilon t^+_1 (\theta, x) < 0 \) such that

\[
\pi_y \Phi^+ (t^+ (\theta, x; \epsilon), \theta + 2\sigma, (x, 0); \epsilon) = \xi^+_2 (t^+ (\theta, x; \epsilon), \theta + 2\sigma, (x, 0); \epsilon) = 0.
\]

Using the expression for \( \xi^\pm_2 \) given in Lemma\((7)\) we can easily obtain that

\[
t^-_1 (\theta, x) = -\frac{\psi^+_2 (+\overline{\sigma}(x), \theta_*, (x, 0), (0, 0))}{F_2 (\gamma(\overline{\sigma}(x), x, 0))}, \tag{24}
\]

and

\[
t^+_1 (\theta, x) = -\frac{\psi^+_2 (-\overline{\sigma}(x), \theta + 2\sigma, (x, 0), (0, 0))}{F_2 (\gamma(\overline{\sigma}(x), x, 0))} = -\frac{\psi^+_2 (-\overline{\sigma}(x), \theta, (x, 0), (0, 0))}{F_2 (\gamma(\overline{\sigma}(x), x, 0))}, \tag{25}
\]
where \( \gamma \) is defined in (8). Moreover, from (21), we get

\[
\psi_i^- (\bar{\sigma}(x), \theta, (x, 0), (0, 0)) = \\
\left< D_2 \Gamma_i(\bar{\sigma}(x), x, 0), \int_0^{\bar{\sigma}(x)} Y(s, x, 0)^{-1} G^- (s + \theta, \gamma(s, x)) \, ds \right>,
\]

\[
(26) \quad \psi_i^+ (-\bar{\sigma}(x), \theta + 2\sigma, (x, 0), (0, 0)) = \\
(-1)^i \left< D_2 \Gamma_i(\bar{\sigma}(x), x, 0), \int_0^{\bar{\sigma}(x)} Y(s, x, 0)^{-1} R G^+ (-s + \theta, R \gamma(s, x)) \, ds \right>,
\]

for \( i = 1, 2 \).

Accordingly, define \( \mathcal{F}(\theta, x; \varepsilon) = (F_1(\theta, x; \varepsilon), F_2(\theta, x; \varepsilon)) \) as

\[
\mathcal{F}_1(\theta, x; \varepsilon) = \pi \Phi^- (t^- (\theta, x; \varepsilon), \theta, (x, 0); \varepsilon) - \pi \Phi^+ (t^+ (\theta, x; \varepsilon), \theta + 2\sigma, (x, 0); \varepsilon) \\
= t^- (\theta, x; \varepsilon) - t^+ (\theta, x; \varepsilon) - 2\sigma = 2(\bar{\sigma}(x) - \sigma) + \mathcal{O}(\varepsilon),
\]

\[
\mathcal{F}_2(\theta, x; \varepsilon) = \pi_x \Phi^- (t^- (\theta, x; \varepsilon), \theta, (x, 0); \varepsilon) - \pi_x \Phi^+ (t^+ (\theta, x; \varepsilon), \theta + 2\sigma, (x, 0); \varepsilon) \\
= \xi_+^- (t^- (\theta, x; \varepsilon), \theta, (x, 0); \varepsilon) - \xi_+^+ (t^+ (\theta, x; \varepsilon), \theta + 2\sigma, (x, 0); \varepsilon).
\]

From Lemma 7, expressions (24) and (25), the reversibility condition (4), and using that \( \gamma(-\bar{\sigma}(x), x) = \gamma(\bar{\sigma}(x), x) \) for \( x_1 < x < x_v \), we get

\[
\xi_+^- (t^+ (\theta, x; \varepsilon), \theta + 2\sigma, (x, 0); \varepsilon) \\
= \gamma_1(\bar{\sigma}(x), x) + \varepsilon \left( F_1^- (\gamma(\bar{\sigma}(x), x)) t_1^- (\theta, x) + \psi_1^- (-\bar{\sigma}(x), \theta, (x, 0), (0, 0)) \right) + \mathcal{O}(\varepsilon^2) \\
= \gamma_1(\bar{\sigma}(x), x) + \varepsilon \left( \left. \frac{F_1(\gamma(\bar{\sigma}(x), x)) - \frac{F_2(\gamma(\bar{\sigma}(x), x))}{\bar{\sigma}(x)} \phi_2^-(\bar{\sigma}(x), \theta, (x, 0), (0, 0))}{\phi_2^-(\bar{\sigma}(x), \theta, (x, 0), (0, 0))} \right) \right) + \mathcal{O}(\varepsilon^2), \text{ and}
\]

\[
\xi_1^- (t^- (\theta, x; \varepsilon), \theta, (x, 0); \varepsilon) \\
= \gamma_1(\bar{\sigma}(x), x) + \varepsilon \left( F_1^- (\gamma(\bar{\sigma}(x), x)) t_1^- (\theta, x) + \psi_1^- (-\bar{\sigma}(x), \theta, (x, 0), (0, 0)) \right) + \mathcal{O}(\varepsilon^2) \\
= \gamma_1(\bar{\sigma}(x), x) + \varepsilon \left( \left. \frac{F_1(\gamma(\bar{\sigma}(x), x)) - \frac{F_2(\gamma(\bar{\sigma}(x), x))}{\bar{\sigma}(x)} \phi_2^-(\bar{\sigma}(x), \theta, (x, 0), (0, 0))}{\phi_2^-(\bar{\sigma}(x), \theta, (x, 0), (0, 0))} \right) \right) + \mathcal{O}(\varepsilon^2).
Therefore,
\[
\frac{F_2(\theta, x; \epsilon)}{\epsilon} = \psi_1^- (\varphi(x), \theta, (x, 0), (0, 0)) - \psi_1^+ (-\varphi(x), \theta + 2\sigma, (x, 0), (0, 0)) \\
- \frac{F_1(\gamma(\varphi(x), x))}{F_2(\gamma(\varphi(x), x))} \left( \psi_2^- (-\varphi(x), \theta + 2\sigma, (x, 0), (0, 0)) + \psi_2^- (\varphi(x), \theta, (x, 0), (0, 0)) \right) + O(\epsilon).
\]

Now, from (26) we have that
\[
\psi_1^- (\varphi(x), \theta, (x, 0), (0, 0)) - \psi_1^+ (-\varphi(x), \theta + 2\sigma, (x, 0), (0, 0)) = \\
\left\langle D_2 \Gamma_1(\varphi(x), x, 0), \int_{\varphi(x)}^{\varphi(x)} Y(t, x, 0)^{-1} \{G^-, G^+\}_\theta (t, \gamma(t, x)) dt \right\rangle,
\]
and
\[
\psi_2^- (\varphi(x), \theta, (x, 0), (0, 0)) + \psi_2^- (-\varphi(x), \theta + 2\sigma, (x, 0), (0, 0)) = \\
\left\langle D_2 \Gamma_2(\varphi(x), x, 0), \int_{\varphi(x)}^{\varphi(x)} Y(t, x, 0)^{-1} \{G^-, G^+\}_\theta (t, \gamma(t, x)) dt \right\rangle,
\]
where \(\{G^-, G^+\}_\theta (t, z) = G^-(t + \theta, z) + RG^+(-t + \theta, Rz)\), see (13).

Since \(Y^T\left(-F_2, F_1\right)^T = F_1 D_2 \Gamma_2 - F_2 D_2 \Gamma_1\), we obtain
\[
\langle F_1 D_2 \Gamma_1 - F_2 D_2 \Gamma_2, V \rangle = \langle \left(-F_2, F_1\right), YV \rangle = F^V V.
\]

Hence, we conclude that
\[
(27) \quad -F_2(\gamma(\varphi(x), x))F_2(\theta, x; \epsilon) = \epsilon M(\theta, x) + O(\epsilon^2).
\]

where \(M(\theta, x)\) is defined in (12).

From the construction of \(F\) it is clear that a subharmonic crossing periodic solution of system (11) exists, for \(\epsilon > 0\) sufficiently small, if, and only if, there are \(\theta_\epsilon \in S_\theta^1\) and \(x_\epsilon \in (x_i, x_v)\) such that \(F(\theta_\epsilon, x_\epsilon) = (0, 0)\).

By hypothesis \(F_1(\theta, x_\epsilon; 0) = 0\) and, from (10),
\[
\frac{\partial F_1}{\partial x}(\theta, x_\epsilon; 0) = 2\varphi'(x_\epsilon) = \frac{\partial \varphi'(x_\epsilon)}{\partial x}(\varphi(x_\epsilon), x_\epsilon) \neq 0.
\]

Thus, by the Implicit Function Theorem there exists \(x(\theta; \epsilon)\) such that \(F_1(\theta, x(\theta; \epsilon); \epsilon) = 0\) and \(x(\theta; \epsilon) \to x_\epsilon\) when \(\epsilon \to 0\) for every \(\theta \in S_\theta^1\).

Now, we take
\[
\bar{F}(\theta; \epsilon) = -\frac{F_2(\gamma(\varphi(x(\theta; \epsilon))), x(\theta; \epsilon)))}{\epsilon} F_2(\theta, x(\theta; \epsilon); \epsilon).
\]

From (27) the above equation reads
\[
\bar{F}(\theta; \epsilon) = M(\theta, x(\theta; \epsilon)) + O(\epsilon) = M(\theta, x_\epsilon) + O(\epsilon),
\]
By hypothesis there exists \( \theta^* \in S_1^l \) such that \( \tilde{F}(\theta^*, 0) = M(\theta^*, x_r) = 0 \) and \((\partial \tilde{F} / \partial \theta)(\theta^*, 0) = (\partial M / \partial \theta)(\theta^*, x_r) \neq 0 \). Thus, applying again the Implicit Function Theorem we conclude that, for \( \varepsilon > 0 \) sufficiently small, there exists \( \theta_\varepsilon \in S_1^l \) such that \( \tilde{F}(\theta_\varepsilon; \varepsilon) = 0 \). Moreover, \( \theta_\varepsilon \to \theta^* \) as \( \varepsilon \to 0 \). This concludes the proof of Theorem A. \( \Box \)

5.2. **Proof of statement (a) of Theorem B** Since \( G_2^+(\theta^*, p_\sigma) \neq G_2^- (\theta^*, p_\sigma) \), we can assume that there exists \( a, b \in [0, 2\sigma] \) with \( a < b \) and \( \theta^* \in (a, b) \) such that \( G_2^+(t, p_\sigma) \neq G_2^- (t, p_\sigma) \) for every \( t \in [a, b] \). Without loss of generality, we suppose that \( G_2^+(t, p_\sigma) < G_2^- (t, p_\sigma) \) for every \( t \in [a, b] \). At the end of the proof we shall comment the case when \( G_2^+(t, p_\sigma) > G_2^- (t, p_\sigma) \) for every \( t \in [a, b] \).

The above assumption and expression (22) imply that \( \ell^+_\nu (\theta; \varepsilon) > \ell^-_\nu (\theta; \varepsilon) \) for every \( \theta \in [a, b] \) and \( \varepsilon > 0 \) sufficiently small.

Let \( \mathcal{R}_\varepsilon \) be the region on \( \Sigma \subset S_1^l \times \mathbb{R} \) delimited by the graphs \( \ell^+_\nu (\theta; \varepsilon) \) for \( \theta \in [a, b] \), that is \( \mathcal{R}_\varepsilon = \{(\theta, x, 0) : \theta \in [a, b], \ell^-_\nu (\theta; \varepsilon) < x < \ell^+_\nu (\theta; \varepsilon)\} \). A straightforward computation shows that this is a region of sliding type. Moreover, the autonomous vector field (18) is \( 2\sigma \)-periodic in the variable \( \theta \), so the regions \( \mathcal{R}_\varepsilon^n = \{(\theta + 2n\sigma, x) : (\theta, x) \in \mathcal{R}_\varepsilon\} \) for \( n \in \mathbb{N} \) are of sliding type.

The expression of the sliding vector field for each region \( \mathcal{R}_\varepsilon^n, n \in \mathbb{N} \), is

\[
\theta' = u(\theta, x; \varepsilon) = 1,
\]

\[
\varepsilon x' = v(\theta, x; \varepsilon) = \frac{f_0(x)}{G_2^+(\theta, x, 0) - G_2^- (\theta, x, 0)} + \varepsilon \left( \frac{f_1(\theta, x)}{G_2^+(\theta, x, 0) - G_2^- (\theta, x, 0)} + f_0(x) \frac{H^-_2(\theta, (x, 0); \varepsilon) - H^+_2(\theta, (x, 0); \varepsilon)}{(G_2^+(\theta, x, 0) - G_2^- (\theta, x, 0))^2} \right) + O(\varepsilon^2),
\]

where

\[
f_0(x) = 2F_1(x, 0)F_2(x, 0),
\]

\[
f_1(\theta, x) = F_2(x, 0) \left(G_1^- (\theta, x, 0) - G_1^+ (\theta, x, 0)\right) + F_1(x, 0) \left(G_2^+ (\theta, x, 0) + G_2^- (\theta, x, 0)\right).
\]

System (28) can be studied using singular perturbation theory (see for instance [10, 15]). In this theory, system (28) is known as a slow system. Doing \( \varepsilon = 0 \) we can find the critical manifold as

\[
\mathcal{M}_0 = \{(\theta, x) \in \mathcal{R}_\varepsilon^0 : f_0(x) = 0\} = \{(\theta + 2n\sigma, x, \varepsilon) : \theta \in [a, b]\}.
\]

Now, doing the time rescaling \( t = \varepsilon t \), system (28), for \( \varepsilon > 0 \), becomes

\[
\dot{\theta} = \varepsilon u(\theta, x; \varepsilon) = \varepsilon,
\]

\[
\dot{x} = v(\theta, x; \varepsilon),
\]
which is known as fast system. Computing the derivative with respect to the variable $x$ of the function $v$ for $\varepsilon = 0$ at the points of $(\theta + 2n\sigma_v, x_v) \in M_0$ we obtain

\begin{equation}
\frac{\partial v}{\partial x}(\theta + 2n\sigma_v, x_v; 0) = p(\theta) = \frac{2F_1(p_v) \frac{\partial F_2}{\partial x}(p_v)}{G_2^+(\theta, p_v) - G_2^-(\theta, p_v)} > 0,
\end{equation}

for every $\theta \in [a, b]$, by hypothesis (h2) and the assumption $G_2^+(\theta, p_v) < G_2^-(\theta, p_v)$. Therefore, for $\varepsilon = 0$, $M_0$ is a normally hyperbolic repelling critical manifold for the vector field (29) and also for the sliding vector field (28).

Applying Fenichel’s theorem we conclude that there exists a normally hyperbolic repelling locally invariant manifold $M_\varepsilon = \{(\theta + 2n\sigma_v, m(\theta; \varepsilon)) : \theta \in [a, b]\}$ of the system (30) and (29), which is $\varepsilon$–close to $M_0$:

\[ m(\theta; \varepsilon) = x_v + \varepsilon m_1(\theta) + \mathcal{O}(\varepsilon^2). \]

Notice that $(\theta(t) + 2n\sigma_v, m(\theta(t); \varepsilon))$ is a trajectory of system (28), so $v(\theta + 2n\sigma_v, m(\theta; \varepsilon); \varepsilon) = \varepsilon (\partial m / \partial \theta)(\theta; \varepsilon)$. Accordingly, for $\varepsilon > 0$ small enough, we may compute

\begin{equation}
\partial \varepsilon \partial \varepsilon
\end{equation}

therefore $M_\varepsilon \subset \mathcal{R}_\varepsilon$.

Since the Fenichel’s manifold is repelling we have that for a given point $(\theta_0, \ell^- (\theta_0; \varepsilon)) \in \partial \mathcal{R}^u_\varepsilon$ there exists a orbit $\delta(\theta_0; \varepsilon)$ of the sliding vector field (28) reaching the point $(\theta_0, \ell^- (\theta_0; \varepsilon))$ (see Figure 4). In the sequel, we shall parametrize this orbit.

Given $N > 0$, we want to compute the solution of system (28) starting at $(\theta, \ell^- (\theta; \varepsilon))$, for $-N \varepsilon < t < 0$. equivalently, we compute the solution of system (29) starting at the same point but for $-N < t < 0$.

We denote by $(\theta_\varepsilon(t, \theta; \varepsilon), x_\varepsilon(t, \theta; \varepsilon))$ the solution of (29) with initial condition: $(\theta_\varepsilon(0, \theta; \varepsilon), x_\varepsilon(0, \theta; \varepsilon)) = (\theta, \ell^- (\theta; \varepsilon))$. Clearly $\theta_\varepsilon(t, \theta; \varepsilon) = \theta + \varepsilon t$. Take $x_\varepsilon(t, \theta; \varepsilon) = x_v + \varepsilon k(t, \theta) + \mathcal{O}(\varepsilon^2)$. Expanding the both sides of the equality

\[ \frac{\partial x}{\partial \varepsilon}(t, \theta; \varepsilon) = v(\theta + \varepsilon t, x_\varepsilon(t, \theta; \varepsilon); \varepsilon) \]

in Taylor series respect to $\varepsilon$ we derive the following differential equation

\begin{equation}
\frac{\partial k}{\partial \varepsilon}(t, \theta) = p(\theta)k(t, \theta) + F_1(p_v) \left( \frac{G_2^+(\theta, p_v) + G_2^-(\theta, p_v)}{G_2^+(\theta, p_v) - G_2^-(\theta, p_v)} \right)
= p(\theta)k(t, \theta) - m_1(\theta)p(\theta),
\end{equation}

(32)

\[ k(0, \theta) = v^-_\varepsilon(\theta) = -\frac{G_2^-(\theta, p_v)}{\frac{\partial F_2}{\partial x}(p_v)} = m_1(\theta) + \frac{F_1(p_v)}{p(\theta)}, \]

where $v^-_\varepsilon(\theta), p(\theta)$, and $m_1(\theta)$ are defined in (22), (30), and (31), respectively. The relation $p(\theta)(v^-_\varepsilon(\theta) - m_1(\theta)) = F_1(p_v)$ has been used in order to get the above equalities.
Solving the initial value problem (32) we obtain

\[ k(\tau, \theta) = m_1(\theta) + \frac{F_1(p_v)}{p(\theta)}e^{\tau p(\theta)}. \]

We have then found a set
\[ \tilde{\delta}(\theta; \varepsilon) = \{ (\theta_s(\tau, \theta; \varepsilon), x_s(\tau, \theta; \varepsilon)) : -N < \tau < 0 \} \]
parameterized by \( \tau \), which is contained in the orbit \( \delta(\theta; \varepsilon) \).

From here, the idea of the proof is analogous to the proof of Theorem A, which consists in defining a function \( F : (a, b) \times (-N, 0) \rightarrow \mathbb{R}^2 \) that allows us to determine the existence of sliding periodic solutions of system (11). Given \( \theta \in (a, b) \), we consider the flows
\[ \Phi^{-}(t, \theta, \ell_v^{-}(\theta; \varepsilon), 0; \varepsilon) \text{ and } \Phi^{+}(t, \theta, \ell_v^{+}(\theta; \varepsilon), 0; \varepsilon). \]

The vector field (18) is \( 2\sigma_v \)-periodic in the variable \( \theta \), which means that \( \theta \equiv \theta + 2\sigma_v \).

Thus, if for some \( \theta_s \in [0, 2\sigma_v] \) and \( \tau_* \in (-N, 0) \) there exist \( s_*^- \geq 0 \) and \( s_*^+ \leq 0 \) such that
\[ \Phi^{-}(s_*^-, \theta_s, \ell_v^{-}(\theta_s; \varepsilon), 0; \varepsilon) = \Phi^{+}(s_*^+, \theta_s, \theta_s + 2\sigma_v; \varepsilon), x_s(\tau, \theta + 2\sigma_v; \varepsilon), 0; \varepsilon) \in \Sigma, \]
then there exists a sliding \( 2\sigma_v \)-periodic solution of system (18) and, consequently, of system (11) (see Figure 4).

**Figure 4.** Methodology for constructing the displacement function to detect sliding \( 2\sigma_v \)-periodic solutions of system (11), for \( \varepsilon > 0 \) sufficiently small.
Moreover, from (21), we get
\[ \xi^{-}_2(s^-(\theta;\epsilon), \theta, \ell^{-}_v(\theta;\epsilon), 0; \epsilon) = 0 \quad \text{and} \quad \xi^{+}_2(s^+(\tau, \theta;\epsilon), \theta, s_0(\tau, \theta + 2\sigma_v;\epsilon), x_0(\tau, \theta + 2\sigma_v;\epsilon), 0; \epsilon) = 0. \]

Moreover, using the expression for \( \xi^{-}_2 \) given in Lemma 2 with \( \theta = \theta, z_0 = p_v, \) and \( z_1 = (\nu^{-}_v(\theta), 0) \), where \( \nu^{-}_v(\theta) \) is given in (22), we obtain that \( s^-(\theta;\epsilon) = \sigma_v + \epsilon s^{-}_1(\theta) + \mathcal{O}(\epsilon^2) \) provided that
\[
(34) \quad s^-_1(\theta) = -\frac{\psi^{-}_2(\sigma_v, \theta, p_v, (\nu^{-}_v(\theta), 0))}{\overline{F}_2(q_v)},
\]
where \( q_v \) is defined in (9). Analogously, using the expression for \( \xi^{+}_2 \) given in Lemma 7 with \( \theta = \theta + 2\sigma_v, z_0 = p_v, \) and \( z_1 = (k(\tau, \theta), 0) \), where \( k(\tau, \theta) \) is given in (33), we obtain \( s^+(\tau, \theta;\epsilon) = -\sigma_v + \epsilon s^{+}_1(\tau, \theta) + \mathcal{O}(\epsilon^2) \) provided that
\[
(35) \quad s^+_1(\tau, \theta) = -\frac{\psi^{+}_2(-\sigma_v, \theta, p_v, (k(\tau, \theta), 0))}{\overline{F}_2(q_v)}.
\]

Moreover, from (21), we get
\[
\psi^{-}_i(\sigma_v, \theta, p_v, (\nu^{-}_v(\theta), 0)) = \frac{\partial \Gamma_i}{\partial \sigma_v}(\sigma_v, p_v)\nu^{-}_v(\theta) + \left( D_2\Gamma_i(\sigma_v, p_v), \int_0^{\nu^{-}_v(\theta)} \bar{Y}(s, p_v)^{-1} G^{-}(s + \theta, \gamma(s, x_v)) ds \right),
\]
\[
(36) \quad \psi^{+}_i(-\sigma_v, \theta, p_v, (k(\tau, \theta), 0)) = (-1)^{i+1} \frac{\partial \Gamma_i}{\partial \sigma_v}(\sigma_v, p_v)k(\tau, \theta) + \left( D_2\Gamma_i(\sigma_v, p_v), (-1)^i \int_0^{\nu^{-}_v(\theta)} \bar{Y}(s, p_v)^{-1} R G^+(s + \theta, R \gamma(s, x_v)) ds \right).
\]

for \( i = 1, 2. \)

Accordingly, define \( G(\tau, \theta;\epsilon) = (G_1(\tau, \theta;\epsilon), G_2(\tau, \theta;\epsilon)) \) as
\[
G_1(\tau, \theta;\epsilon) = \pi_\theta \Phi^{-}(s^-(\theta;\epsilon), \theta, (\ell^{-}_v(\theta;\epsilon), 0); \epsilon)
\]
\[ -\pi_v \Phi^{+}(s^+(\tau, \theta;\epsilon), \theta, (s_0(\tau, \theta + 2\sigma_v;\epsilon), x_0(\tau, \theta + 2\sigma_v;\epsilon), 0); \epsilon) = s^-(\theta;\epsilon) + \theta - s^+(\tau, \theta;\epsilon) - \theta_0(\tau, \theta + 2\sigma_v;\epsilon)
\]
\[ = \epsilon(s^{-}_1(\theta) - s^{-}_1(\tau, \theta) + 2\sigma_v;\epsilon) + \mathcal{O}(\epsilon^2),
\]
\[
G_2(\tau, \theta;\epsilon) = \pi_\theta \Phi^{-}(s^-(\theta;\epsilon), \theta, (\ell^{-}_v(\theta;\epsilon), 0); \epsilon)
\]
\[ -\pi_v \Phi^{+}(s^+(\tau, \theta;\epsilon), \theta, (s_0(\tau, \theta + 2\sigma_v;\epsilon), x_0(\tau, \theta + 2\sigma_v;\epsilon), 0); \epsilon) = \xi^{-}_1(s^-(\theta;\epsilon), \theta, (x_v + \epsilon\nu^{-}_v(\theta), 0); \epsilon)
\]
\[ -\xi^{+}_1(s^+(\tau, \theta;\epsilon), \theta, (x_v + \epsilon\nu^{-}_v(\theta), 0); \epsilon).\]
To compute the function $G_2$, first we see that
\[
F_2(q_v)(s_1^-(\theta) - s_1^+(\tau, \theta))
= \psi_2^+(-\sigma_v, \theta, p_v, (k(\tau, \theta), 0)) - \psi_2^- (\sigma_v, \theta, p_v, (v_\nu^-(\theta), 0))
\]
\[
= -\frac{\partial F_2}{\partial x}(\sigma_v, p_v, (k(\tau, \theta) + v_\nu^-(\theta)) - g_\theta
\]
\[
= -\frac{F_2(q_v)}{F_1(q_v)}(k(\tau, \theta) + v_\nu^-(\theta)) - g_\theta,
\]
where $g_\theta$ is defined in (14). To obtain the above expression we have used Lemma 2 and expression (36). Therefore,
\[
G_1(\tau, \theta; \epsilon) = -\frac{\epsilon}{F_2(q_v)} \left( F_2(q_v) \tau + \frac{F_2(q_v)}{F_1(q_v)}(k(\tau, \theta) + v_\nu^-(\theta)) + g_\theta \right) + O(\epsilon^2).
\]
We compute $G_2(\tau, \theta; \epsilon)$. From Lemma 7 and expressions (34), (35), and (36) we get
\[
\xi_1^{-} (s^-(\theta; \epsilon), \theta, (x_v + \epsilon v_\nu^-(\theta), 0); \epsilon)
= \gamma_1(\sigma_v, x_v) + \epsilon \left( \frac{F_1(q_v)}{F_2(q_v)} \psi_1^-(\sigma_v, \theta, p_v, (v_\nu^-(\theta), 0)) \right) + O(\epsilon^2)
\]
\[
= \gamma_1(\sigma_v, x_v) + \epsilon \left( -\frac{F_1(q_v)}{F_2(q_v)} \psi_2^- (\sigma_v, \theta, p_v, (v_\nu^-(\theta), 0)) + \psi_1^- (\sigma_v, \theta, p_v, (v_\nu^-(\theta), 0)) \right) + O(\epsilon^2),
\]
where $\gamma$ is given in [5], and
\[
\xi_1^+(s^+(\tau, \theta; \epsilon), \theta + 2\sigma_v + \epsilon \tau, (x_v + \epsilon k(\tau, \theta), 0); \epsilon)
= \gamma_1(\sigma_v, x_v) + \epsilon \left( \frac{F_1(q_v)}{F_2(q_v)} \psi_1^+(\sigma_v, \theta, k(\tau, \theta), 0)) \right) + O(\epsilon^2)
\]
\[
= \gamma_1(\sigma_v, x_v) + \epsilon \left( F_1(q_v) \psi_2^+ (-\sigma_v, \theta, p_v, (k(\tau, \theta), 0)) \right) + O(\epsilon^2).
\]
Thus,
\[
\frac{G_2(\tau, \theta; \epsilon)}{\epsilon} = \psi_1^- (\sigma_v, \theta, p_v, (v_\nu^-(\theta), 0)) - \psi_1^+ (\sigma_v, \theta, p_v, (k(\tau, \theta), 0))
\]
\[
-\frac{F_1(q_v)}{F_2(q_v)} \left( \psi_2^- (\sigma_v, \theta, p_v, (v_\nu^-(\theta), 0)) + \psi_2^+ (\sigma_v, \theta, p_v, (k(\tau, \theta), 0)) \right) + O(\epsilon).
\]
From (36) we have
\[
\psi^-(\sigma_v, \theta, p_v, (v^-(\theta), 0)) - \psi^+(\sigma_v, \theta, p_v, (k(\tau, \theta), 0)) = \\
\frac{\partial \Gamma_1}{\partial x}(\sigma_v, p_v)(v^-(\theta) - k(\tau, \theta)) + \\
\left\langle D_2 \Gamma_1(\sigma_v, p_v), \int_0^{\sigma_v} Y(t, p_v)^{-1}\{G^-, G^+\}_{\theta}(t, \gamma(t, x_v))dt \right\rangle,
\]
and
\[
\psi^-(\sigma_v, \theta, p_v, (v^-(\theta), 0)) + \psi^+(\sigma_v, \theta, p_v, (k(\tau, \theta), 0)) = \\
\frac{\partial \Gamma_2}{\partial x}(\sigma_v, p_v)(v^-(\theta) - k(\tau, \theta)) + \\
\left\langle D_2 \Gamma_2(\sigma_v, p_v), \int_0^{\sigma_v} Y(t, p_v)^{-1}\{G^-, G^+\}_{\theta}(t, \gamma(t, x_v))dt \right\rangle.
\]

Similar to the proof of Theorem A we obtain that
\[-F_2(q_v)G_2(\tau, \theta; \epsilon) = \epsilon(v^-(\theta) - k(\tau, \theta))F(q_v) \wedge \frac{\partial \Gamma}{\partial x}(\sigma_v, p_v) + \epsilon M(\theta, x_v) + \mathcal{O}(\epsilon^2),
\]
where \(M(\theta, x)\) is defined in (12). As a direct consequence of Lemma 2 we have that the above wedge product vanishes. Therefore,
\[
G_2(\tau, \theta; \epsilon) = -\frac{\epsilon}{F_2(q_v)}M(\theta, x_v) + \mathcal{O}(\epsilon^2).
\]

Now, consider the function
\[
\bar{G}(\tau, \theta; \epsilon) = -\frac{F_2(q_v)}{\epsilon}G(\tau, \theta; \epsilon) = \left(\bar{G}_1(\tau, \theta), \bar{G}_2(\theta)\right) + \mathcal{O}(\epsilon).
\]

Thus,
\[
\bar{G}_1(\tau, \theta) = F_2(q_v)\tau + \frac{F_2(q_v)}{F_1(p_v)}(k(\tau, \theta) + v^-(\theta)) + g_\theta,
\]
\[
\bar{G}_2(\theta) = M(\theta, x_v).
\]

By hypothesis there exists \(\theta^* \in \mathbb{S}^1_{\mathcal{L}_v}\) such that \(M(\theta^*, x_v) = 0\) and \((\partial M/\partial \theta)(\theta^*, x_v) \neq 0\). Now, we note that the equation \(\bar{G}_1(\tau, \theta^*) = 0\) is equivalent, using (33), to the equation
\[
\tau + \frac{1}{p(\theta^*)}e^{\tau p(\theta^*)} + A(\theta^*) = 0 \quad \text{where} \quad A(\theta) = \frac{m_1(\theta) + v^-_v(\theta)}{F_1(p_v)} + \frac{g_\theta}{F_2(q_v)},
\]
where \(p(\theta)\) and \(m_1(\theta)\) are defined in (30) and (31), respectively. Since \(p(\theta^*) > 0\), equation (37) becomes
\[
\tau^* = -A(\theta^*) - \frac{1}{p(\theta^*)}W(e^{-A(\theta^*)p(\theta^*)})
\]
and\(w(\tau) = \gamma(t, x_v)\).
Here, $W$ denotes the Lambert $W$–function ($x = W(y)$ gives the solution of $xe^x = y$, for a definition see [6]). From the properties of the $W$–function, we know that $W(e^\beta) > \beta$ if, and only if, $\beta < 1$. Then, we obtain that $\tau^* < 0$ if, and only if, $\Lambda(\theta^*) p(\theta^*) > -1$. This follows from the hypothesis (a) of the theorem, which reads

$$g_{\theta^*} > \frac{2F_2(q_v)}{F_1(p_v)} \frac{\partial F_2}{\partial x}(p_v).$$

Accordingly, we take $N = -2\tau^*$ in order to have $(\tau^*, \theta^*) \in (-N, 0) \times (a, b)$ and $\tilde{G}(\tau^*, \theta^*, 0) = 0$. Moreover,

$$\det \left( \frac{\partial \tilde{G}}{\partial (\tau, \theta)}(\tau^*, \theta^*, 0) \right) = \begin{vmatrix} F_2(q_v) \left(1 + e^{\tau^* p(\theta^*)}\right) & \# & \partial M(\theta^*, x_v) \\ 0 & \frac{\partial M(\theta^*, x_v)}{\partial \theta} \end{vmatrix} = F_2(q_v) \left(1 + e^{\tau^* p(\theta^*)}\right) \frac{\partial M(\theta^*, x_v)}{\partial \theta} \neq 0.$$

Thus, applying the implicit function theorem we conclude that for $\varepsilon > 0$ sufficiently small there exist $\theta_\varepsilon \in (a, b)$ and $\tau_\varepsilon \in (-N, 0)$ such that $\tilde{G}(\tau_\varepsilon, \theta_\varepsilon, \varepsilon) = 0$ and $\theta_\varepsilon \bigto \theta^*$ and $\tau_\varepsilon \bigto \tau^*$ when $\varepsilon \bigto 0$.

For the case when $G_2^+(t, p_v) > G_2^-(t, p_v)$ for every $t \in [a, b]$ the same argument works reversing time. Therefore, in this case, the obtained sliding periodic solutions slide on $\Sigma^\varepsilon$. This concludes the proof of item (a) of the Theorem [8] $\square$

5.3. **Proof of statement (b) of Theorem B.** Let $K \subseteq \mathbb{S}_{\tau_v}^1 \times \mathbb{R}$ be the set of pairs $(\theta, \chi)$ such that $\chi < \min\{v^\pm(\theta)\}$. Clearly, in this case, $\xi_\varepsilon = \chi_0 + \varepsilon \chi < \ell_0^\pm(\theta; \varepsilon)$, and therefore the solutions of system (19) cross the set of discontinuity $\Sigma$ at the points $(\theta, (\xi_\varepsilon, 0))$ when $(\theta, \chi) \in K$.

In what follows, we define a function $H : K \times (0; \varepsilon_0) \to \mathbb{R}^2$ such that its zeros determine the existence of crossing periodic solutions near the separatrix $S$. Given $(\theta, \xi_\varepsilon) \in K$, we consider the flows $\Phi^- (t, \theta, (\xi_\varepsilon, 0); \varepsilon)$ and $\Phi^+ (t, \theta + 2\sigma_v, (\xi_\varepsilon, 0); \varepsilon)$. The existence of times $r^- = r^-(\theta, \chi; \varepsilon) > 0$ and $r^+ = r^+(\theta, \chi; \varepsilon) < 0$ such that

$$\xi_2^- (r^-, \theta, (\xi_\varepsilon, 0); \varepsilon) = 0, \quad \xi_2^+ (r^+, \theta + 2\sigma_v, (\xi_\varepsilon, 0); \varepsilon) = 0$$

is guaranteed by the implicit function theorem. These times can be computed analogously to (34) and (35) as

$$r^-(\theta, \chi; \varepsilon) = \sigma_v + \varepsilon r_1^- (\theta, \chi) + \mathcal{O}(\varepsilon^2), \quad r^+(\theta, \chi; \varepsilon) = -\sigma_v + \varepsilon r_1^+ (\theta, \chi) + \mathcal{O}(\varepsilon^2),$$

where

$$r_1^+ (\theta, \chi) = -\frac{\psi_2^+ (\sigma_v, \theta, p_v, (\chi, 0))}{F_2(q_v)},$$

but here we have used the formula of Lemma [7] for $z_0 = p_v$, and $z_1 = (\chi, 0)$. 

---

[6] refers to a previous work or another reference in the cited text, and [8] is the reference to the particular theorem or statement being proven or discussed. This format helps in maintaining clarity and coherence in the exposition, allowing readers to follow the logical flow of the argument without needing to refer externally to the referenced works.
Accordingly, define $H(\theta, \chi; \epsilon) = (H_1(\theta, \chi; \epsilon), H_2(\theta, \chi; \epsilon))$ as

$$H_1(\theta, \chi; \epsilon) = \pi_\theta \Phi^-(r^- (\theta, \chi; \epsilon), \theta, (\zeta_\epsilon, 0); \epsilon) - \pi_\theta \Phi^+(r^+ (\theta, \chi; \epsilon), \theta + 2\sigma_v (\zeta_\epsilon, 0); \epsilon)$$

$$= r^-(\theta, \chi; \epsilon) - r^+(\theta, \chi; \epsilon) - 2\sigma_v$$

$$= \epsilon (r_1^- (\theta, \chi) - r_1^+(\theta, \chi)) + O(\epsilon^2),$$

$$H_2(\theta, \chi; \epsilon) = \pi_\chi \Phi^-(r^- (\theta, \chi; \epsilon), \theta, (\zeta_\epsilon, 0); \epsilon) - \pi_\chi \Phi^+(r^+ (\theta, \chi; \epsilon), \theta + 2\sigma_v (\zeta_\epsilon, 0); \epsilon)$$

$$= \xi_1^- (r^- (\theta, \chi; \epsilon), \theta, (x_v + \epsilon \chi, 0); \epsilon)$$

$$- \xi_1^+ (r^+ (\theta, \chi; \epsilon), \theta + 2\sigma_v, (x_v + \epsilon \chi, 0); \epsilon) + O(\epsilon^2).$$

From the construction of $H$ it is clear that a crossing $2\sigma_v$–periodic solution of system (11) exists if, and only if, we find $(\theta_\epsilon, \chi_\epsilon) \in K$ such that $H(\theta_\epsilon, \chi_\epsilon; \epsilon) = (0, 0)$. To compute $H$ we proceed analogously to the proof of statement (a) of Theorem B but now using the expressions just obtained for $r_1^\pm$ and again Lemma 7 with $z_0 = p_v$, and $z_1 = (\chi, 0)$, obtaining

$$H_1(\theta, \chi; \epsilon) = -\epsilon \left( \frac{2\chi}{F_1(p_v)} + \frac{g_\theta}{F_2(q_v)} \right) + O(\epsilon^2),$$

$$H_2(\theta, \chi; \epsilon) = -\epsilon \frac{M(\theta, x_v)}{F_2(q_v)} + O(\epsilon^2).$$

We define

$$\tilde{H}(\theta, \chi; \epsilon) = -\frac{F_2(q_v)}{\epsilon} H(\theta, \chi; \epsilon).$$

By hypothesis there exists $\theta^* \in S^1$ such that $M(\theta^*, x_v) = 0$ and $(\partial M \partial \theta)(\theta^*, x_v) \neq 0$. Therefore, for $\chi^* = -F_1(p_v)g_{\theta^*}/(2F_2(q_v))$ we get

$$\det \left( \frac{\partial \tilde{H}}{\partial (\theta, \chi)}(\theta^*, \chi^*, 0) \right) = \left| \begin{array}{c} 2F_2(q_v) \\ \frac{\partial M}{\partial \theta}(\theta^*, x_v) \\ 0 \end{array} \right|$$

$$= -\frac{2F_2(q_v)}{F_1(p_v)} \frac{\partial M}{\partial \theta}(\theta^*, x_v) \neq 0.$$

Applying the implicit function theorem it follows that, for $\epsilon > 0$ sufficiently small, there exist $\theta_\epsilon = \theta^* + O(\epsilon) \in (a, b)$ and $\chi_\epsilon = \chi^* + O(\epsilon)$ such that $H(\theta_\epsilon, \chi_\epsilon; \epsilon) = \tilde{H}(\theta_\epsilon, \chi_\epsilon; \epsilon) = 0$. Furthermore, $(\theta_\epsilon, \chi_\epsilon) \in K$. Indeed, the hypothesis (b) of the theorem, which reads

$$g_{\theta^*} < \frac{2F_2(q_v)}{F_1(p_v)} \frac{\partial F_2}{\partial x}(p_v) \max \{ C_2^+ (\theta^*, p_v) \}$$

and (22) implies that $\chi^* < \min \{ v^+ (\theta^*) \}$. Consequently, $\chi_\epsilon < \min \{ v^+ (\theta_\epsilon) \}$ for $\epsilon > 0$ small enough. This concludes the proof of statement (b) of Theorem B. □
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