Massless particle of any spin in static uncharged spherically symmetric curved spacetimes

Amnon Moalem and Alexander Gersten
Department of Physics, Ben Gurion University of the Negev, Beer-Sheva, Israel

Abstract. Quantum equations for massless particles of any spin in static spherically symmetric spacetimes, namely, the Minkowski, FRW and Schwarzschild spacetimes are considered. The metrics of all three are diagonal and their corresponding $E_2^2$ and $E_3^3$ inverse vierbein fields are identical. Thus the angular wave functions are spherical harmonics spinors corresponding to simple modes specified by orbital angular momenta and their z component. By applying consistently a procedure based on variables separation it is shown that the radial wave functions are second order ordinary differential equations sharing the same homogeneous equation though nonhomogeneous terms are different depending on the characteristic $E_0^0$ and $E_1^1$ of each spacetime.

1. Introduction

Out the boundary of black holes gravity is extremely weak and the known laws of physics should still be valid. In this note we consider quantum theory of massless particles of any spin in the vicinity of massive objects in the outer region of spacetimes. Among these particles one counts photons, gluons and gravitons which are associated with the electromagnetic, strong and gravitational interactions. Specifically, we consider quantum equations in a flat Minkowski spacetime and in the Schwarzschild and Friedman-Walker-Robertson (FWR) spacetimes. The Schwarzschild and FWR metrics are exact solutions of the Einstein’s Equations of general relativity (GR). In GR theory the metric $g_{ij}$ in four dimensional spacetime is the gravitational field and at each point it can be diagonalized as $g_{ij} = diag(1, -1, -1, -1)$ (see for example, [7], [8]). Thus the equations and their solutions in the Schwarzschild and FWR spacetimes are expected to have common features and that asymptotically should converge to those corresponding to particles in the Minkowskian case. Our main objective in what follows is to shed light on these common features. In the next section we consider quantum equations of free massless particle of a given spin in a global spacetime which we separate in two equations, a reduced equation independent of the spin connection [6] and an equation depending on the spin connection. In section 2 we consider the angular and reduced radial wave equations. Second quantization of the wave functions is worked out and the vacuum energy is calculated in section 3. We conclude in section 4.
2. Quantum equations in global spacetime

In previous contributions [1]-[5] we have shown that free massless particles of any spin satisfy the following quantum equation,

\[ E_\alpha^\mu (x) \gamma^\alpha \partial_\mu \left[ \sqrt{e} \Phi^{(4s)} (x) \right] = -\frac{1}{2} \gamma^b \partial_\sigma E_\sigma^b \Phi^{(4s)} (x), \]  

(1)

where \( s \) is the particle spin, \( e = \sqrt{-\det g} \) and \( \det g \) is the determinant of the metric tensor \( g_{\mu\nu} \), \( E_\alpha^\mu (x) \) are inverse vierbein fields, and the gamma are \( 4s \times 4s \) diagonal block matrices \( \gamma^\mu = \text{diag} (\sigma^\mu, \ldots, \sigma^\mu) \) with the \( \sigma^\mu \) being the Pauli matrices and \( \gamma^0 = I^{(4s)} \) is the identity matrix. The wave functions \( \Phi^{(4s)} (x) \) are \( 4s \) column matrices with \( (2s-1) \) components of spin \( (s-1) \) and \( (2s+1) \) components of spin \( s \) which form bases for the \( D^{(s-1/2,1/2)} \) representations of the Lorentz group. The Hermitian conjugate of \( \Phi^{(4s)} \) is taken to be the transpose of the complex conjugate, \( \Phi^{(4s)} |^H = \left[ \Phi^{*(4s)} \right]^T \) so that \( \Phi^H \Phi \) is positive definite and is interpreted as probability density.

It is straightforward to verify that the expression above can be derived from the Lagrangian density,

\[ \mathcal{L} = \Phi^H (E_\alpha^\mu \gamma^\alpha \partial_\mu \left( \sqrt{e} \Phi \right) + \frac{1}{2} \gamma^b \partial_\sigma E_\sigma^b \Phi), \]

(2)

The variation w.r.t. \( \Phi^H \) obviously yields Eq.(1), while variation w.r.t. \( \Phi \) gives the Hermitian conjugate equation, i.e.,

\[ E_\alpha^\mu \gamma^\alpha \partial_\mu \left( \sqrt{e} \Phi^* \right) = -\frac{1}{2} \gamma^b \partial_\sigma E_\sigma^b \Phi^*. \]

(3)

From the Lagrangian Eq.(2) and using Noether’s Theorem the conserved current density is given by,

\[ j^\mu = \frac{\delta \mathcal{L}}{\delta (\partial_\mu \Phi^b)} \frac{\delta \Phi^b}{\delta \alpha} - J_\mu^0 = \Phi^H \gamma^\alpha E_\alpha^b \gamma^b = \Phi^H \gamma^\mu \Phi. \]

(4a)

Note that \( j^0 = \Phi^H \gamma^0 \Phi = \Phi^*_\psi \Phi^b \) is probability density. The expression above does not depend on the spin connection and satisfies the continuity equation, i.e.,

\[ \partial_\mu j^\mu = \partial_\mu \left[ \Phi^H \gamma^\mu \Phi \right]. \]

(5)

Equation (1) is not mathematically soluble in the general case. However we demonstrate below that for the static spherically symmetric Minkowski, FRW and Schwarzschild spacetimes we may replace Eq.(1) by a pair of equations which are both mathematically manageable. To this aim we make strongly simplifying assumption that the wave function \( \Phi \) factorizes as,

\[ \Phi (t, r, \theta, \phi) = \psi (t, r, \theta, \phi) f (r, \theta), \]

(6)

and require that the function \( \psi \) satisfies an equation independent of the spin connection, i.e.,

\[ E_\alpha^\mu \gamma^\alpha \partial_\mu \psi (t, r, \theta, \phi) = 0. \]

(7)
In what follows we refer to this equation as the reduced equation and to its solutions as reduced wave functions. Inserting Eq. (6) in Eq. (1) and using Eq. (7) one finds that the function \( f(r, \theta) \) satisfies the equation,

\[
\partial_\mu \ln (f(r, \theta) \sqrt{e}) = -\frac{1}{2} e^\mu_\nu \partial_\nu E^\nu_\mu. \tag{8}
\]

A similar approach was applied in Ref. [9, 10] to deal with the Dirac equation. Note that all complexities due to the spin connection are now content in the expression above. Yet for axially symmetric static spacetime \( f(r, \theta) \) does not depend neither on time nor on the azimuthal coordinate and Eq. (8) becomes rather simple,

\[
\partial_\mu \ln (f(r, \theta) \sqrt{e}) = -\frac{1}{2} e^r_\nu \partial_\nu E^r_\nu. \tag{9}
\]

The equation above is integrable and \( f(r, \theta) \) assumes a simple analytic form. The results of the integration for the Minkowski, Schwarzschild and FWR are listed in Table 1. To determine the wave equation \( \Phi \) it remains to solve the reduced Eq. (7) to obtain the reduced wave function \( \psi \) . This task however is simpler since Eq. (7) is rather similar to the wave equation in the Minkowskian case.

### Table 1: The Function \( f(r, \theta) \) for different spacetimes

| Spacetime    | Inverse vierbeins \( E^\mu_v \) | \( f(r, \theta) \) |
|--------------|----------------------------------|-------------------|
| Minkowski    | \( \text{diag} (1, 1, 1/r, 1/(r \sin \theta)) \) | \( r \sin^{1/2} \theta \) |
| FWR\(^{\ast}\) | \( \text{diag} (a, F, 1/r, 1/(r \sin \theta)) \) | \( a^{3/2} r^{1/2} \sin^{1/2} \theta \) |
| Schwarzschild\(^{\ast\ast}\) | \( \text{diag}(1/\sqrt{F}, \sqrt{F}, 1/r, 1/(r \sin \theta)) \) | \( F^{1/4} r \sin^{1/2} \theta \) |

\(^{\ast}\) \( F = \sqrt{1 - kr^2}, k = \pm 1 \), \( a \) is the size of the universe, \( r \) is dimensionless parameter.

\(^{\ast\ast}\) \( F = \left(1 - \frac{2GM}{r}\right), G \) is the gravitation constant, \( M \) is a constant with the dimension of mass.

### 3. Free particle equations

It is instructive to introduce the procedure to be applied for the Schwarzschild and FRW spacetimes by first considering the equations (1) in the Minkowskian case where the metric, \( g_{\mu\nu} = \text{diag}(1, -1, -1, -1) \), \( e = \sqrt{-\det g} = 1 \). Assuming \( \Phi \sim \exp(-i\omega t) \exp(i m \phi) \), the Eqs. (1) reduces to,

\[
\omega E^0_0 \Phi = -i \gamma \cdot \nabla \Phi = H_0 \Phi, \tag{10}
\]

\[
\omega E^0_0 \Phi^* = i \gamma \cdot \nabla \Phi^* = H_0 \Phi^*. \tag{11}
\]
Here $\nabla$ is the del operator and $H_0 = -i \gamma \cdot \nabla$ is the Hamiltonian of the unperturbed free states. Naturally the equations above do not depend on the spin connection. In spherical coordinates

$$\nabla = \hat{r} \partial_r + \hat{\theta} \partial_\theta + \hat{\phi} \partial_\phi,$$  \hspace{1cm} (12)

and the free Hamiltonian reads,

$$H_0 = -i\gamma \cdot \nabla = -i\gamma^r \partial_r + \gamma^\theta \partial_\theta + \gamma^\phi \partial_\phi.$$  \hspace{1cm} (13)

For diagonal metrics, as is the case for the Minkowski, FWR and Schwarzschild spacetimes, the matrices $\gamma^t, \gamma^r, \gamma^\theta, \gamma^\phi$ are given via the transformation,

$$\begin{pmatrix} \gamma^t \\ \gamma^r \\ \gamma^\theta \\ \gamma^\phi \end{pmatrix} = \begin{pmatrix} E^0_0 & 0 & 0 & 0 \\ 0 & E^1_1 & 0 & 0 \\ 0 & 0 & E^2_2 & 0 \\ 0 & 0 & 0 & E^3_3 \end{pmatrix} \begin{pmatrix} \gamma^0 \\ \gamma^1 \\ \gamma^2 \\ \gamma^3 \end{pmatrix}.$$  

The inverse vierbein fields $E^2_2 = 1/r$ and $E^3_3 = 1/r \sin \theta$ are common to all spherically symmetric spacetimes, but the $E^0_0$ and $E^1_1$ fields are characteristics of a spacetime which as will be shown below are responsible for departures of the radial function from the Minkowskian case. Thus the angular parts of $\Phi$ and the corresponding reduced wave function $\psi$ are the same though the radial functions and the reduced radial functions are different depending on what radial dependence $E^0_0$ and $E^1_1$ may have.

3.1. The angular wave function

By definition the angular momentum operator is defined as,

$$\hat{L} = -i\hat{r} \times \nabla.$$  

Using Eq.(12) one obtains,

$$\hat{r} \times \hat{L} = i \left( \hat{\theta} \partial_\theta + \hat{\phi} \partial_\phi \right),$$

so that,

$$\gamma \cdot \hat{r} \times \hat{L} = \frac{1}{r} i \left( \gamma^\theta \partial_\theta + \gamma^\phi \partial_\phi \right) = \frac{1}{r} i \left( E^2_2 \gamma^2 \partial_\theta + E^3_3 \gamma^3 \partial_\phi \right).$$  \hspace{1cm} (14)

We use this expression to define an angular operator $K$. Based on the properties of the gamma matrices, any three vectors $A$ and $B$ satisfy the identity,
\[(\gamma \cdot A)(\gamma \cdot B) = \gamma^0 A \cdot B + \frac{i}{2} \gamma \cdot (A \times B).\]  

(15)

Then for \(A = \hat{r}\) and \(B = L\), one obtains,

\[\gamma \cdot \hat{r} \times L = -\frac{i}{r} (\gamma \cdot \hat{r}) (2\gamma \cdot L).\]  

(16)

With this relation we can rewrite the Hamiltonian in the form,

\[H_0 = -i\gamma \cdot \nabla = -i\gamma \cdot \hat{r} \left[ \partial_r - \frac{1}{r} \left( 2\gamma \cdot L \right) \right].\]  

(17)

Finally we identify the spin \(S\) with \(\gamma\) and define \(K\) as,

\[K = 2\gamma \cdot L + s = 2S \cdot L + s = J^2 - L^2 - S^2 + s,\]  

(18)

where \(J = L + S\) is the total angular momentum and \(s\) the spin of a particle. Substituting this in Eq.(17) yields,

\[H_0 = -i\gamma \cdot \hat{r} \left[ \partial_r + \frac{s}{r} - \frac{1}{r} K \right].\]  

(19)

This expression of the free Hamiltonian suggests that we may construct the solution \(\Phi\) by variables separation. Indeed \(H_0, J^2, J^3, K\) and the parity \(P\) form a complete set of commuting operators. The angular part of the wave function \(\Phi\) is a spinor of spherical harmonics which we may write as,

\[\Psi_{lm}^j(\Omega) = \sum_{m_i=-s,s} C(l, s, j; m_j - m_s, m_s, m_j) Y_{l,m_j}^m(\theta, \phi) \chi_{s}^{m_i},\]  

(20)

where \(C(l, s, j; m_j - m_s, m_s, m_j)\) is a Clebsch-Gordan coefficient for combining orbital angular momentum \(l\) and spin \(s\) to a total angular momentum \(j\) with magnetic quantum numbers \(m_j - m_s, m_s, m_j\), respectively, \(Y_{l,m}^m(\theta, \phi)\) are the usual spherical harmonics which are eigen functions of \(L^2, L_3\), and \(\chi_{s}^{m_i}\) are eigen functions of \(S^2\) and \(S_3\). For massless particles \(m_s\) can have only two values \(s\) and \(-s\). The spherical harmonic spinors are orthonormal, i.e.,

\[\int d\Omega \left( \Psi_{lm,j}^j(\Omega) \right)^\dagger \Psi_{lm,j}^j(\Omega) = \delta_{l'j} \delta_{m'j} \delta_{m_j, m_j}.\]  

(21)

Further more, the function \(\Phi\) is an eigen function of the operators \(J^2, J_3\), angular operator \(K\) and parity \(P\),

\[J^2 \Phi_{l,m,j}^j = j (j + 1) \Phi_{l,m,j}^j,\]  

(22)

\[J_3 \Phi_{l,m,j}^j = m_j \Phi_{l,m,j}^j,\]  

(23)

\[K \Phi_{l,m,j}^j = \kappa_j \Phi_{l,m,j}^j.\]  

(24)
To calculate the eigenvalues of the angular operator $K$ for $j = l \pm s$ we consider the operator $K^2 = (2L \cdot S + s)^2$. From addition of angular momenta one finds that,

$$K^2 = (2S \cdot L + s)^2 = (j^2 - L^2 - S^2 + s)^2 = s^2 (2l + 1)^2.$$  

Equation (25)

So that for both $j = l + s$ and $j = l - s$ the eigenvalues of $K$ are $\kappa_j = \pm s (2l + 1)$. This means that for each of the cases $j = l + s$ and $j = l - s$ there exist two solutions $R_1 (r) \phi^+ (\theta, \varphi)$ and $R_2 (r) \phi^- (\theta, \varphi)$ corresponding to positive and negative $\kappa_j$, respectively. The eigenvalues of $\kappa_j$ span all values $-s (2l + 1), -s (2l + 1) + 1, \cdots, s (2l + 1) - 1, s (2l + 1)$.

3.2. The radial wave function

We may now turn to consider the radial function. Taking the Hamiltonian as in Eq. (19) we write Eq. (10) as,

$$iE^0_0 \omega \psi = E^1_1 \gamma^1 [\partial_r + \frac{s}{r} - \frac{1}{r} K] \psi.$$  

Equation (26)

Then substituting $\gamma^\mu = \text{diag} (\sigma^\mu, \cdots, \sigma^\mu)$ the above expression yields linear non-autonomous systems,

$$iE^0_0 \omega I^2 \psi = E^1_1 \sigma^1 [\partial_r + \frac{s}{r} - \frac{1}{r} K] \psi.$$  

Equation (27)

To resolve this expression we substitute $\psi \sim \exp (-i \omega t) \left( \begin{array}{c} R_1 (r) \phi^+ (\theta, \varphi) \\ R_2 (r) \phi^- (\theta, \varphi) \end{array} \right)$ with $K \Phi = \kappa \left( \begin{array}{c} R_1 (r) \phi^+ (\theta, \varphi) \\ -R_2 (r) \phi^- (\theta, \varphi) \end{array} \right)$. This gives a pair of coupled equations,

$$i \omega E^0_0 R_1 \phi^+ = E^1_1 [\partial_r + \frac{s + \kappa}{r}] R_2 \phi^-,$$  

Equation (28)

$$i \omega E^0_0 R_2 \phi^- = E^1_1 [\partial_r + \frac{s - \kappa}{r}] R_1 \phi^+.$$  

Equation (29)

In the Minkowskian case $E^0_0 = 1$ and $E^1_1 = -1$ and $R_1 (r)$ and $R_2 (r)$ satisfy the following radial equations,

$$\frac{d^2 R_1}{dr^2} + \frac{1}{r} \frac{dR_1}{dr} - \frac{(\kappa - 1) \kappa}{r^2} R_1 + \frac{s (s - 1)}{r^2} R_1 + \omega^2 R_1 = 0,$$  

Equation (30)

$$\frac{d^2 R_2}{dr^2} + \frac{1}{r} \frac{dR_2}{dr} - \frac{\kappa (\kappa + 1)}{r^2} R_2 + \frac{s (s - 1)}{r^2} R_2 + \omega^2 R_2 = 0.$$  

Note that by replacing $\kappa$ by $-\kappa$ in the equation for $R_2 / R_1$ we obtain the equation for $R_1 / R_2$. Also note that for spin $s = 1$ these are spherical Bessel’s equations and their solutions are spherical Bessel’s functions $j_\kappa (r)$ with $\kappa = 1, 3, 5, \cdots$ for $R_1 (r)$ and $j_{-\kappa - 1} (r)$ with $-\kappa - 1 = -2, -4, -6, \cdots$ for $R_2 (r)$. For other values of the spin the forth term on the l.h.s contributes a pure spin dependent potential.
More generally for non-constant $E_0^1$ and $E_1^1$ fields Eqs. (27, 28) give the following reduced radial wave equations,

$$\left( \frac{E_1^1}{E_0^1} \right)^2 \frac{d^2 R_1}{dr^2} + \left[ \left( \frac{E_1^1}{E_0^1} \right) \frac{d \left( \frac{E_1^1}{E_0^1} \right)}{dr} + \left( \frac{E_1^1}{E_0^1} \right)^2 \frac{s}{r} \right] \frac{dR_1}{dr} +$$

$$\left[ \left( \frac{E_1^1}{E_0^1} \right) \frac{d \left( \frac{E_1^1}{E_0^1} \right)}{dr} \right] \frac{s - \kappa}{r} + \frac{1}{r^2} \left[ s (s - 1) - (\kappa - 1) \kappa \right] \right] R_1 = -\omega^2 R_1,$$

(31)

and,

$$\left( \frac{E_1^1}{E_0^1} \right)^2 \frac{d^2 R_2}{dr^2} + \left[ \left( \frac{E_1^1}{E_0^1} \right) \frac{d \left( \frac{E_1^1}{E_0^1} \right)}{dr} + \left( \frac{E_1^1}{E_0^1} \right)^2 \frac{s}{r} \right] \frac{dR_2}{dr} +$$

$$\left[ \left( \frac{E_1^1}{E_0^1} \right) \frac{d \left( \frac{E_1^1}{E_0^1} \right)}{dr} \right] \frac{s + \kappa}{r} + \frac{1}{r^2} \left[ s (s - 1) - (\kappa + 1) \right] \right] R_2 = -\omega^2 R_1 (r).$$

(32)

Clearly setting $E_0^1 = 1$ and $E_1^1 = -1$ the expressions above reduce to Eqs. (29, 30) for a Minkowski spacetime. Taking $E_0^1 = a, E_1^1 = \sqrt{1 - kr^2}$ for the FRW spacetime one obtains,

$$\frac{d^2 R_1}{d(ar)^2} + 2 \frac{s}{(ar)} \frac{dR_1}{d(ar)} + \frac{1}{(ar)^2} \left[ s (s - 1) - (\kappa - 1) \kappa \right] R_1 + \omega^2 R_1$$

$$= \left[ \frac{k (ar)}{\sqrt{a^2 - k (ar)^2}} \right] \frac{dR_1}{d(ar)} + \frac{k (s - \kappa)}{\sqrt{a^2 - k (ar)^2}} - \frac{k (ar)^2}{(a^2 - k (ar)^2)^2} \omega^2 \right] R_1,$$

(33)

and

$$\frac{d^2 R_2}{d(ar)^2} + 2 \frac{s}{(ar)} \frac{dR_2}{d(ar)} + \frac{1}{(ar)^2} \left[ s (s - 1) - (\kappa + 1) \right] R_2 + \omega^2 R_2$$

$$= \left[ \frac{k (ar)}{\sqrt{a^2 - k (ar)^2}} \right] \frac{dR_2}{d(ar)} + \frac{k (s + \kappa)}{\sqrt{a^2 - k (ar)^2}} - \frac{k (ar)^2}{(a^2 - k (ar)^2)^2} \omega^2 \right] R_2.$$

(34)

Similarly for the Schwarzschild spacetime, $E_0^1 = 1 / (1 - 2MG/r)^{1/2}$ and $E_1^1 = (1 - 2MG/r)^{1/2}$ one obtains,

$$\frac{d^2 R_1}{d r^2} + 2 \frac{s}{r} \frac{dR_1}{dr} + \frac{1}{r^2} \left[ s (s - 1) - (\kappa - 1) \kappa \right] R_1 + \omega^2 R_1$$
\[ = - \frac{2GM}{r^2} \left( \frac{dR_1}{dr} + \frac{s - \kappa}{r} R_1 \right) - \omega^2 \frac{4GM (1 - \frac{GM}{r})}{r (1 - \frac{2GM}{r})^2} R_1, \tag{35} \]

and

\[ \frac{d^2 R_2}{dr^2} + \frac{2s dR_2}{r dr} + \frac{1}{r^2} \left[ s (s - 1) - \kappa (\kappa + 1) \right] R_2 + \omega^2 R_2 (r) \]

\[ = - \frac{2GM}{r^2} \left( \frac{dR_2}{dr} + \frac{s + \kappa}{r} R_2 \right) - \omega^2 \frac{4GM (1 - \frac{GM}{r})}{r (1 - \frac{2GM}{r})^2} R_2. \tag{36} \]

Thus the homogeneous equations (the r.h.s of the equations above) for the Schwarzschild and FRW spacetimes are the same as the homogeneous equations Eqs.(29,30) derived in the Minkowskian case. All dependence on characteristic \( E_0^0 \) and \( E_1^1 \) fields appears in the nonhomogenous terms (r.h.s). Clearly in the Schwarzschild and FWR cases the nonhomogenous terms on the l.h.s vanish as \( r \) (or \( ar \)) goes to infinity. This ascertains that the solutions for the equations above must converge asymptotically to the Minkowskian solutions.

4. Second quantization

4.1. Second quantization of \( \Phi \) in Minkowski spacetime

The free particle solutions of Eqs.(1-11) correspond respectively to forward and backward helicity. Each of these has two energy solutions one positive and one negative where the negative energy forward helicity solution of Eq.(1) being identical to the positive energy backward solution of Eq.(11), and likewise the negative backward solution of Eq.(11) is the same as the positive energy forward helicity solution of Eq.(1). With this in mind, we expand the wave function \( \Phi \) in terms of positive energy eigenstates. Let \( u(p,s) \exp (-ipx) \) denote a positive energy forward helicity state, and let \( v(p,s) \exp (ipx) \) denote a positive energy backward helicity state. Inserting these into Eqs.(10-11) gives,

\[ (\omega_p E_0^0 - \gamma \cdot p) u(p) = 0, \quad \text{and} \quad (\omega_p E_0^0 + \gamma \cdot p) v(p) = 0. \tag{37} \]

Here \( \omega_p = p_0 = \pm \sqrt{p^+ p^-} \). The spinors \( u(p) \) and \( v(p) \) are orthogonal so that,

\[ u(p)^H v(p) = v^H (p) u(p) = 0. \tag{38} \]

We normalize them according to,

\[ u^a (p) u^*_a (p') = \omega_p; \quad v^a (p) v^*_a (p') = \omega_p. \tag{39} \]

We may then write,

\[ \Phi (x) = \int \frac{d^3 p}{(2\pi)^{3/2}} \sqrt{\frac{1}{E_0^0 \omega_p}} \left[ b(p) u(p) \exp (-ipx) + d^t (p) v(p) \exp (ipx) \right], \tag{40} \]

\[ \Phi^H (x) = \int \frac{d^3 p}{(2\pi)^{3/2}} \sqrt{\frac{1}{E_0^0 \omega_p}} \left[ b^t (p) u^t (p) \exp (ipx) + d (p) v^t (p) \exp (-ipx) \right], \tag{41} \]

Thus the homogeneous equations (the r.h.s of the equations above) for the Schwarzschild and FRW spacetimes are the same as the homogeneous equations Eqs.(29,30) derived in the Minkowskian case. All dependence on characteristic \( E_0^0 \) and \( E_1^1 \) fields appears in the nonhomogenous terms (r.h.s). Clearly in the Schwarzschild and FWR cases the nonhomogenous terms on the l.h.s vanish as \( r \) (or \( ar \)) goes to infinity. This ascertains that the solutions for the equations above must converge asymptotically to the Minkowskian solutions.
where $b(p)$ and $b^\dagger(p)$ are annihilation and creation operators for a positive energy forward helicity, and $d(p)$, $d^\dagger(p)$ are annihilation and creation operators for a positive energy backward helicity. In order to quantize $\Phi$ we impose the following commutators,

$$\left[\Phi(x), \Phi(x')^H\right] = i\delta^3(x - x'),$$

$$\left[\Phi(x), \Phi(x')\right] = 0,$$

$$\left[\Phi^H(x), \Phi^H(x')\right] = 0. \quad (42)$$

These are equivalent to the following commutators of creation and annihilation operators,

$$\left[b(p), b^\dagger(p')\right] = \left[d(p), d^\dagger(p')\right] = \delta_{pp'} \delta_{\sigma\sigma'}, \quad (43)$$

$$\left[b^\dagger(p), b^\dagger(p')\right] = \left[d^\dagger(p), d^\dagger(p')\right] = \left[b(p), b(p')\right] = \left[d(p), d(p')\right] = 0, \quad (44)$$

$$\left[b(p), b^\dagger(p')\right] = \left[d(p), b^\dagger(p')\right] = \left[d^\dagger(p), b(p')\right] = \left[d^\dagger(p), d^\dagger(p')\right] = 0, \quad (45)$$

where $\sigma$ stands for helicity. This can be shown rather straightforward by inserting Eq.(40) and Eq.(41) in Eqs.(42). Then substituting in the double integral obtained the normalization conditions Eq.(39), the orthogonality conditions Eq.(38), the commutators Eqs.(43-45), and by integrating Eqs.(42) are recovered.

### 4.2. Quantization of the reduced function $\psi$

Quantization of the Reduced Function $\psi$ can be performed following the same procedure as applied above for $\Phi$ in the Minkowskian case. In fact the reduced Eqs. (7) of the function $\psi$ and its complex are quite similar to Eqs.(10-11), i.e.,

$$iE_0^0 \partial_t \psi = -i\gamma \cdot \nabla \psi, \quad (46)$$

$$iE_0^0 \partial_t \psi^* = i\gamma \cdot \nabla \psi^*, \quad (47)$$

Then we may expand $\psi$ as,

$$\psi(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{1}{E_0^0 \omega_p}} [b(p) \tilde{u}(p) \exp(-ipx) + d^\dagger(p) \tilde{v}(p) \exp(ipx)], \quad (48)$$

$$\psi^H(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{1}{E_0^0 \omega_p}} \left[b^\dagger(p) \tilde{u}^\dagger(p) \exp(ipx) + d(p) \tilde{v}^\dagger(p) \exp(-ipx)\right]. \quad (49)$$

In fact by normalizing the spinors as,

$$\tilde{u}^a(p) \tilde{u}_b^\dagger(p') = \omega_p / f(r, \theta) ; \quad \tilde{v}^a(p) \tilde{v}_b^\dagger(p') = \omega_p / f(r, \theta), \quad (50)$$
and requiring the commutators Eqs.(43,44,45) we guarantee that the reduced function $\psi$ also satisfies,

\[
[\psi(x),\psi(x')^H] = i\delta^3(x-x'),
\]

\[
[\psi(x),\psi(x')] = 0,
\]

\[
[\psi^H(x),\psi^H(x')] = 0.
\]

These are the same as those obtained for the function $\Phi$, Eqs.(42).

4.3. Vacuum energy

Let us evaluate first the vacuum energy in a Minkowski spacetime. From Eq.(2) and Eqs.(40,41) the Hamiltonian density and Hamiltonian are given by,

\[
H = \Phi^H E_0^0 \omega_k \Phi,
\]

\[
H = \int d^3x \Phi^H E_0^0 \omega_k \Phi = \int \frac{d^3p}{(2\pi)^3} \omega_k \left[ b^\dagger(p) b(p') + d(p) d^\dagger(p') \right],
\]

where we have used the orthogonality and normalization conditions Eqs.(38), (39). Finally, with the vacuum state denoted by $|0\rangle$ and the familiar matrix elements $\langle 0 | b^\dagger(p) b(p') | 0 \rangle = 0, \langle 0 | d(p) d^\dagger(p') | 0 \rangle = 1$ we obtain,

\[
\langle 0 | H | 0 \rangle = \frac{V}{(2\pi)^3} \int d^3p \omega_k \rightarrow \sum_k \omega_k.
\]

This same result one obtains by substituting $\Phi(x) = f(x_1,x_2) \psi(x)$ in Eq.(53), taking $\psi(x)$ and $\psi^H(x)$ from Eqs.(48,49) and using the normalization conditions Eq.(50). The factors $E_0^0(x)$ and $f^H(x_1,x_2)f(x_1,x_2)$ cancel out in any case so that the vacuum energy for the Minkowski, FRW and Schwarzschild are the same as in Eq.(54).

5. Concluding remarks

We have considered quantum equations for free massless particles of any spin in static spherically symmetric spacetimes, namely, the Minkowski, FRW and Schwarzschild spacetimes. The metrics for all three are diagonal and share the same $E_2^2$ and $E_3^3$ inverse vierbein fields. Thus the angular wave functions are spherical harmonics spinors corresponding to simple modes specified by the particle spin and its z components $(s, m_s = -s, s)$, the orbital angular momentum $(l = 0, 1, 2, \cdots)$ and its z components $(m_l = -l, -l+1, \cdots, l, l-1)$, the total angular momentum $(j = l + s)$ and its z components $(m_j = -j, \cdots, j)$, and the eigenvalues $\kappa_j = \pm s(2l+1)$ of the angular operator $K$. To completely specify the solution $\Phi$ we have to add of course the radial quantum number $(n_r = 0, 1, 2, \cdots)$. Using the anzats that the wave function factorizes as in Eq.(6) we derived radial equations. These depend on $E_0^0$ and $E_1^1$ which are
characteristic of a spacetime. For the flat Minkowski spacetime the radial equations obtained are homogeneous 2nd order differential equations amenable to exact solution. For the FRW and Schwarzschild spacetimes the equations are nonhomogeneous with the same homogeneous equations as in the Minkowskian case and nonhomogeneous extra terms depending on $E_0^0$ and $E_1^1$. Mathematically the solutions in the FWR and Schwarzschild spacetimes should be combinations of the Minkowskian solutions and private solutions of the nonhomogeneous terms and must converge out the boundary of massive objects to those of the Minkowskian case.

References

[1] A. Gersten and A. Moalem, J. Phys. Conf. Series, 330(2011),012010.
[2] A. Gersten and A. Moalem, J. Phys. Conf. Series, 437(2013),012019.
[3] A. Gersten and A. Moalem, J. Phys. Conf. Series, 615(2015),012011.
[4] A. Gersten and A. Moalem, J. Phys. Conf. Series, 845(2017), 012011.
[5] A. Moalem and A. Gersten, J. Phys. Conf. Series, 845(2017), 012029.
[6] See for example Jeffery Yepez, arXiv:1106.3037,(2011)10.
[7] See for example, Robert M. Wald, General Relativity , University of Chicago Press(1984).
[8] C.I.S. Clarke, Relativity on Curved Manifolds, Cambridge University Press(1990).
[9] Dag Morten Sjostrom , Boson and Fermions in Curved Spacetime, Thesis, Norwegian University of Science and Technology,2013.
[10] Seyd Alwi B. Ahmad, Proceedings of the Conference in Honor of the 90th Birthday of Freeman Dyson, Edited by Phua K K et al. Published by World Scientific Publications Co. Ltd., 2014,p 419-424.