SCATTERING ON PERIODIC METRIC GRAPHS

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Abstract. We consider the Laplacian on a periodic metric graph and obtain its decomposition into a direct fiber integral in terms of the corresponding discrete Laplacian. Eigenfunctions and eigenvalues of the fiber metric Laplacian are expressed explicitly in terms of eigenfunctions and eigenvalues of the corresponding fiber discrete Laplacian and eigenfunctions of the Dirichlet problem on the unit interval. We show that all these eigenfunctions are uniformly bounded. We apply these results to the periodic metric Laplacian perturbed by real integrable potentials. We prove the following: a) the wave operators exist and are complete, b) the standard Fredholm determinant is well-defined and is analytic in the upper half-plane without any modification for any dimension, c) the determinant and the corresponding S-matrix satisfy the Birman-Krein identity.

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1. INTRODUCTION AND MAIN RESULTS

1.1. Introduction. We consider a Schrödinger operator \( H = H_0 + Q \) on a periodic metric graph \( \mathcal{G} \) with the same edge lengths, i.e., on the so-called periodic equilateral graph. Here \( H_0 = \Delta_M \) is a positive metric Laplace operator with Kirchhoff vertex conditions and \( Q \) is a real integrable potential. Differential operators on metric graphs arise naturally as simplified models in mathematics, physics, chemistry, and engineering.

It is well-known that \( \Delta_M \) is self-adjoint and its spectrum consists of an absolutely continuous part and an infinite number of flat bands (i.e., eigenvalues with infinite multiplicity). The absolutely continuous spectrum is a union of an infinite number of spectral bands separated by gaps. Note that the flat bands always exist, mainly due to the existence of the so-called "Dirichlet-eigenvalues" (the corresponding eigenfunctions vanish on all vertices). The spectrum of metric Laplacians has deep relationships with the spectrum of discrete Laplacians \( \Delta \) on the corresponding discrete graphs, see [B85], [BGP08], [C97] and references therein.

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There are a lot of papers, and even books, on the spectrum of discrete and metric Laplacians on finite and infinite graphs (see \cite{BK13}, \cite{Ch97}, \cite{CDS95}, \cite{CDGT88}, \cite{P12} and references therein).

It is well known that periodic operators can be decomposed into a direct integral. The existence of the direct integral for the discrete Schrödinger operator with periodic potentials on periodic graphs was discussed in many papers (see, e.g., \cite{HS99}, \cite{KSS98}, \cite{SS92}). In the case of a concrete periodic graph it is not difficult to write down an explicit expression for the fiber operator. In the case of an arbitrary periodic graph explicit forms of the fiber operators were given in \cite{KS14a}, \cite{S13}. In particular, from this direct integral decomposition it follows that the spectrum of the Schrödinger operator on periodic discrete graphs consists of an absolutely continuous part and a finite number of flat bands (i.e., eigenvalues with infinite multiplicity). The absolutely continuous spectrum consists of a finite number of bands (intervals) separated by gaps. In \cite{KS14a}, \cite{KS19} the authors estimated the Lebesgue measure of the spectrum of the discrete Schrödinger operator in terms of geometric parameters of the graph and the potentials. Furthermore, they estimated a global variation of the Lebesgue measure of the spectrum and a global variation of gap-length in terms of the potentials and geometric parameters of the graph. In \cite{KS16a} the authors considered Laplacians on periodic equilateral metric graphs and estimated the Lebesgue measure of the bands and gaps on a finite interval in terms of geometric parameters of the graph.

A localization of spectral bands of Laplacians both on metric and discrete periodic graphs in terms of eigenvalues of the operator on finite graphs (the so-called eigenvalue bracketing) was described in \cite{KS14b}, \cite{KS15}, \cite{KS19}, \cite{LP08}.

Effective masses for Laplacians on periodic discrete and metric equilateral graphs were studied in \cite{KS16b}. The authors estimated effective masses associated with the ends of each spectral band in terms of geometric parameters of the graphs. Moreover, at the beginning of the spectrum they obtained two-sided estimates of the effective mass in terms of geometric parameters of the graphs.

Results about Laplacians on periodic graphs are used in spectral analysis of the Schrödinger operator with a decaying potential and also for the study of the Laplacians on periodic graphs with various defects. We briefly describe these works. The scattering problem for the Schrödinger operator with a decaying potential on the lattice $\mathbb{Z}^d$, $d > 1$, was considered in the papers \cite{BS99}, \cite{IK12}, \cite{IM14}, \cite{Ko10}, \cite{KM17}, \cite{Na14}, \cite{RoS09}, \cite{SV01}, see also references therein. Inverse scattering theory for the discrete Schrödinger operators with finitely supported potentials was considered in \cite{IK12} for the case of the lattice $\mathbb{Z}^d$ and in \cite{A12} for the case of the hexagonal lattice. The discrete Schrödinger operator with decaying potentials on arbitrary periodic graphs was studied in \cite{KS20}, \cite{PR18}. The Schrödinger operator with a potential periodic in some directions and finitely supported in others on arbitrary periodic graphs was investigated in the article \cite{KS17}. In papers \cite{AIM16}, \cite{AIM18}, \cite{KS17a}, \cite{Ku14}, \cite{SS17} the Laplace and Schrödinger operators on periodic graphs with different defects were considered.

We describe the main results of this paper:

i) The direct integral for the metric Laplacian $\Delta_M$ is described in terms of the discrete Laplacian $\Delta$.

ii) Eigenfunctions of fiber operators for the metric Laplacian with eigenvalues from the set $\sigma(\Delta_M) \setminus \sigma_D$, where $\sigma_D = \{(\pi j)^2 : j \in \mathbb{N}\}$ is the so-called Dirichlet spectrum, are described in terms of eigenfunctions of fiber operators for the discrete Laplacian $\Delta$. 
iii) We compute eigenfunctions of fiber operators for the metric Laplacian corresponding to the Dirichlet spectrum $\sigma_D$ and determine the multiplicity of this spectrum.

iv) We describe eigenfunctions of the absolutely continuous spectrum of the metric Laplacian $\Delta_M$ and show that all eigenfunctions of $\Delta_M$ are uniformly bounded.

v) We consider scattering for the Schrödinger operator $H = \Delta_M + Q$, where the potential $Q \in L^1(\mathcal{G})$ is real. In particular, we obtain

- the existence and completeness of the wave operators;
- the standard Fredholm determinant (without any modification for any dimension) is well defined and is analytic in the upper half-plane and its main properties are discussed;
- the difference of the resolvents for the Schrödinger operator and for the Laplace operator belongs to the trace class for any dimension. Note that for the corresponding Schrödinger operators on $\mathbb{R}^d$, $d \geq 2$, it does not hold true.

The proof of these results is based on a detailed analysis of eigenfunctions of fiber operators for the metric Laplacian obtained in the present paper. Note that for short range potentials one needs to develop Mourre’s or Enss’s approaches on periodic metric graphs.

1.2. Metric Laplacians. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a connected infinite graph, possibly having loops and multiple edges and embedded into the space $\mathbb{R}^d$. Here $\mathcal{V}$ is the set of its vertices and $\mathcal{E}$ is the set of its unoriented edges. We identify each edge $e \in \mathcal{E}$ with the segment $[0, 1]$. This introduces an orientation on the edge set $\mathcal{E}$. An edge starting at a vertex $u$ and ending at a vertex $v$ from $\mathcal{V}$ will be denoted as the ordered pair $(u, v) \in \mathcal{E}$ and is said to be incident to the vertices. Vertices $u, v \in \mathcal{V}$ will be called adjacent and denoted by $u \sim v$, if $(u, v) \in \mathcal{E}$.

Denote by $I(v)$ the set of all edges from $\mathcal{E}$ incident to the vertex $v \in \mathcal{V}$.

Denote by $\Gamma$ be a lattice of rank $d$ in $\mathbb{R}^d$ with a basis $A = \{a_1, \ldots, a_d\}$, i.e.,

$$\Gamma = \left\{ a : a = \sum_{s=1}^{d} n_s a_s, \ n_s \in \mathbb{Z}, \ s \in \mathbb{N}_d \right\}, \quad \mathbb{N}_d = \{1, \ldots, d\},$$

and let

$$\Omega = \left\{ x \in \mathbb{R}^d : x = \sum_{s=1}^{d} t_s a_s, \ 0 \leq t_s < 1, \ s \in \mathbb{N}_d \right\} \quad (1.1)$$

be the fundamental cell of the lattice $\Gamma$. We define the equivalence relation on $\mathbb{R}^d$: $x \equiv y \pmod{\Gamma} \iff x - y \in \Gamma \quad \forall x, y \in \mathbb{R}^d$.

Below we consider locally finite $\Gamma$-periodic metric equilateral graphs $\mathcal{G}$, i.e., graphs satisfying the following conditions:

1) $\mathcal{G} = \mathcal{G} + a$ for any $a \in \Gamma$;
2) the quotient graph $\mathcal{G}_\ast = \mathcal{G}/\Gamma$ is compact;
3) all edges of the graph $\mathcal{G}$ have the same length.

The basis $a_1, \ldots, a_d$ of the lattice $\Gamma$ is called the periods of $\mathcal{G}$. We also call the quotient graph $\mathcal{G}_\ast = \mathcal{G}/\Gamma$ the fundamental graph of the periodic graph $\mathcal{G}$. The fundamental graph $\mathcal{G}_\ast$ is a graph on the $d$-dimensional torus $\mathbb{R}^d/\Gamma$. The graph $\mathcal{G}_\ast = (\mathcal{V}_\ast, \mathcal{E}_\ast)$ has the vertex set $\mathcal{V}_\ast = \mathcal{V}/\Gamma$ and the set $\mathcal{E}_\ast = \mathcal{E}/\Gamma$ of oriented edges which are finite. Denote by $I_\ast(v)$ the set of all edges from $\mathcal{E}_\ast$ incident to the vertex $v \in \mathcal{V}_\ast$. 


Example 1.1. We consider the periodic graph $G$ shown in Fig. 1b. The periods $a_1, a_2$ of the graph and the fundamental cell $\Omega$ are also shown in the figure. The fundamental graph $G_*$ is a graph on the two-dimensional torus $\mathbb{R}^2/\Gamma$, where $\Gamma$ is the lattice generated by the vectors $a_1, a_2$. The torus is obtained from the fundamental cell $\Omega$ by identification of its opposite edges. The fundamental graph $G_*$ (Fig. 1b) consists of four vertices $v_1, v_2, v_3, v_4$ and six edges

$$e_1 = (v_1, v_2), \quad e_2 = (v_2, v_3), \quad e_3 = (v_3, v_4), \quad e_4 = (v_4, v_1), \quad e_5 = (v_1, v_3), \quad e_6 = (v_2, v_4).$$

Recall that each edge $e \in E$ is identified with the segment $[0, 1]$ along $e$. For each function $y$ on $G$ we define a function $y_e = y|_e$, $e \in E$. We identify each function $y_e$ on $e$ with a function on $[0, 1]$ by using the local coordinate $t \in [0, 1]$. Let $L^2(G) = \oplus_{e \in E} L^2(e)$ be the Hilbert space of all functions $y = (y_e)_{e \in E}$, where each $y_e \in L^2(e) = L^2(0, 1)$, equipped with the norm

$$\|y\|_{L^2(G)}^2 = \sum_{e \in E} \|y_e\|_{L^2(0, 1)}^2 < \infty.$$

We define the positive metric Laplacian $\Delta_M$ on $y = (y_e)_{e \in E} \in L^2(G)$ by

$$(\Delta_M y)_e = -y''_e,$$

where $(y''_e)_{e \in E} \in L^2(G)$,

and $y$ satisfies the Kirchhoff conditions:

$$y \text{ is continuous on } G, \quad \sum_{e \in I(v)} (-1)^{\delta(e,v)} y'_e(\delta(e,v)) = 0, \quad \forall v \in V; \quad (1.2)$$

$$\delta(e,v) = \begin{cases} 1, & \text{if } v \text{ is the terminal vertex of the edge } e \\ 0, & \text{if } v \text{ is the initial vertex of the edge } e. \end{cases} \quad (1.3)$$

Remark 1.2. The spectrum of the metric Laplacian $\Delta_M$ on $G = (V,E)$ does not depend on the orientation of the edges $e \in E$. 

**Figure 1.** a) A periodic graph $G$, the vectors $a_1, a_2$ are the periods of the graph, the fundamental cell $\Omega$ is shaded; b) the fundamental graph $G_*$, the opposite edges of $\Omega$ are identified.
1.3. **Discrete Laplacians.** We introduce the inverse edge $\mathbf{e} = (v, u)$ for each oriented edge $(u, v) \in \mathcal{E}$. Let $\mathcal{A}$ and $\mathcal{A}_*$ denote the sets of all edges from $\mathcal{E}$ and $\mathcal{E}_*$, respectively, and their inverse edges. We introduce the Hilbert space

$$\ell^2(\mathcal{V}) = \left\{ f : \mathcal{V} \to \mathbb{C}, \sum_{v \in \mathcal{V}} \nu_v |f(v)|^2 < \infty \right\},$$

equipped with the inner product

$$(f, g)_\mathcal{V} = \sum_{v \in \mathcal{V}} \nu_v f(v) \overline{g(v)}.$$

We define the discrete normalized Laplace operator $\Delta$ on $\ell^2(\mathcal{V})$ by

$$(\Delta f)(v) = -\frac{1}{\nu_v} \sum_{(v, u) \in \mathcal{A}} f(u), \quad f = (f(v))_{v \in \mathcal{V}} \in \ell^2(\mathcal{V}),$$

where $\nu_v$ is the degree of the vertex $v \in \mathcal{V}$ and the sum is taken over all oriented edges from $\mathcal{A}$ starting at the vertex $v \in \mathcal{V}$. It is well known (see, e.g., Proposition 4.2 in [HN09]) that $\Delta$ is self-adjoint and the point $-1$ belongs to its spectrum $\sigma(\Delta)$ containing in $[-1, 1]$, i.e.,

$$-1 \in \sigma(\Delta) \subset [-1, 1].$$

This Laplacian $\Delta$ on $\ell^2(\mathcal{V})$ has the standard decomposition into a constant fiber direct integral for some unitary operator $U : \ell^2(\mathcal{V}) \to \mathcal{H}$:

$$\mathcal{H} = \int_{\mathbb{T}^d} \ell^2(\mathcal{V}_\vartheta) \frac{d\vartheta}{(2\pi)^d}, \quad U \Delta U^{-1} = \int_{\mathbb{T}^d} \Delta(\vartheta) \frac{d\vartheta}{(2\pi)^d},$$

where $\mathbb{T}^d = \mathbb{R}^d/(2\pi \mathbb{Z})^d$, the fiber Laplacian $\Delta(\vartheta)$ on the fiber space $\ell^2(\mathcal{V}_\vartheta) = \mathbb{C}^\nu$ is given by (2.8), and $\nu = \# \mathcal{V}_\vartheta$ is the number of the fundamental graph vertices. The parameter $\vartheta$ is called the quasimomentum. For each $\vartheta \in \mathbb{T}^d$ the fiber operator $\Delta(\vartheta)$ has $\nu$ eigenvalues. It is known (see, e.g., Proposition 4.2 in [HN09]) that $\nu_v$ is an eigenvalue of $\Delta$ with infinite multiplicity iff $\lambda_\nu$ is an eigenvalue of $\Delta(\vartheta)$ for any $\vartheta \in \mathbb{T}^d$. We call $\{\lambda_\nu\}$ a flat band. We define the multiplicity of flat bands as follows: a flat band $\{\lambda_\nu\}$ of $\Delta$ has multiplicity $m$ iff $\lambda_\nu$ is an eigenvalue of $\Delta(\vartheta)$ with multiplicity $m$ for almost all $\vartheta \in \mathbb{T}^d$. Thus, if the operator $\Delta$ has $r \geq 0$ flat bands, then we denote the corresponding eigenvalues of $\Delta(\vartheta)$ (counting multiplicities) by

$$\lambda_n = \lambda_n(\vartheta) = \text{const}, \quad \forall \nu - r < n \leq \nu = \# \mathcal{V}_\vartheta, \quad \forall \vartheta \in \mathbb{T}^d.$$  

All other eigenvalues $\lambda_1(\vartheta), \ldots, \lambda_{\nu-r}(\vartheta)$ are not constant. They can be enumerated in increasing order (counting multiplicities) by

$$\lambda_1(\vartheta) \leq \lambda_2(\vartheta) \leq \ldots \leq \lambda_{\nu-r}(\vartheta), \quad \forall \vartheta \in \mathbb{T}^d.$$  

Since $\Delta(\vartheta)$ is self-adjoint and analytic in $\vartheta \in \mathbb{T}^d$, each $\lambda_n(\cdot), n \in \mathbb{N}_\nu = \{1, \ldots, \nu\}$, is a real and piecewise analytic function on the torus $\mathbb{T}^d$. Letting $P_n(\vartheta)$ be the associated projection, we have

$$\Delta(\vartheta) = \sum_{n=1}^{\nu} \lambda_n(\vartheta) P_n(\vartheta), \quad \vartheta \in \mathbb{T}^d.$$  

We define the spectral bands $\sigma_n(\Delta)$ by

$$\sigma_n(\Delta) = [\lambda_n^-, \lambda_n^+] = \lambda_n(\mathbb{T}^d), \quad n \in \mathbb{N}_\nu.$$  

Thus, the spectrum of the Laplacian $\Delta$ on $\mathcal{G}$ has the form $\sigma(\Delta) = \bigcup_{n=1}^{\nu} \sigma_n(\Delta)$ and we get

\[
\sigma(\Delta) = \sigma_{ac}(\Delta) \cup \sigma_{fb}(\Delta), \quad \sigma_{ac}(\Delta) = \bigcup_{n=1}^{\nu-r} \sigma_n(\Delta), \quad \sigma_{fb}(\Delta) = \{\lambda_{\nu-r+1}, \ldots, \lambda_{\nu}\}. \quad (1.13)
\]

Here and below $\sigma_{ac}(\Delta)$ is the absolutely continuous spectrum of $\Delta$, which is a union of non-degenerate intervals from $[\lambda_{\nu-r+1}, \lambda_{\nu}]$, and $\sigma_{fb}(\Delta)$ is the set of all flat bands (eigenvalues of infinite multiplicity). An open interval between two neighboring non-degenerate spectral bands is called a spectral gap. If $\lambda_{\nu-r}^+ < 1$, then it is convenient for us to also call an open interval $(\lambda_{\nu-r}^+, 1)$ a gap of the operator $\Delta$.

**Remark 1.3.** It is known that the first spectral band $\sigma_1(\Delta) = [-1, \lambda_1^+]$ of the Laplacian $\Delta$ is non-degenerate. The proof is quite simple. Indeed, assume that the first spectral band $\sigma_1(\Delta)$ is degenerate, i.e., the point $-1$ (which belongs to the spectrum of $\Delta$) is an eigenvalue of the operator $\Delta$ with infinite multiplicity. Then (see, e.g., Theorem 4.5.2 in [BK13]) there exists an eigenfunction $0 \neq f \in \ell^2(\mathcal{V})$, corresponding to this eigenvalue and having finite support. By a discrete analogue of the maximum principle (see, e.g., Theorem 7.7 in [S13]), we conclude that $f = 0$. We get a contradiction.

### 1.4. Direct integral of metric Laplacians.

We introduce the Hilbert space (a constant fiber direct integral)

\[
\mathcal{H} = L^2(\mathbb{T}^d, \frac{d\vartheta}{(2\pi)^d}); \mathcal{H} = \int_{\mathbb{T}^d} \mathcal{H} \frac{d\vartheta}{(2\pi)^d}; \quad \mathcal{H} = L^2(\mathcal{G}_s),
\]
equipped with the norm

\[
\|g\|^2_{\mathcal{H}} = \int_{\mathbb{T}^d} \|g(\vartheta, \cdot)\|^2 \frac{d\vartheta}{(2\pi)^d},
\]
where the function $g(\vartheta, \cdot) \in L^2(\mathcal{G}_s)$ for all $\vartheta \in \mathbb{T}^d$. We have the preliminary standard result about a direct integral, see, e.g., [RS78]. Recall that $\{a_1, \ldots, a_d\}$ is the basis of the lattice $\Gamma$.

**Theorem 1.4.** The metric Laplacian $\Delta_M$ on $L^2(\mathcal{G})$ has the following decomposition into a constant fiber direct integral

\[
\mathcal{U} \Delta_M \mathcal{U}^{-1} = \int_{\mathbb{T}^d} \Delta_M(\vartheta) \frac{d\vartheta}{(2\pi)^d}, \quad (1.14)
\]
for the unitary operator $\mathcal{U} : L^2(\mathcal{G}) \to \mathcal{H}$ defined by

\[
(\mathcal{U} h)(\vartheta, x) = \sum_{m=(m_1, \ldots, m_d) \in \mathbb{Z}^d} e^{-i(m, \vartheta)} h(x + m a_1 + \ldots + m a_d), \quad (\vartheta, x) \in \mathbb{T}^d \times \mathcal{G}_s. \quad (1.15)
\]

Here the fiber operator $\Delta_M(\vartheta)$ acts on $y = (y_e)_{e \in \mathcal{E}_s} \in L^2(\mathcal{G}_s)$ by

\[
(\Delta_M(\vartheta)y)_e = -y''_e, \quad (y''_e)_{e \in \mathcal{E}_s} \in L^2(\mathcal{G}_s), \quad (1.16)
\]
and the function $y$ satisfies the so-called quasi-periodic conditions at any $v \in \mathcal{V}_s$:

\[
e^{-i\delta(e_1, v)(\tau(e_1), \vartheta)} y_{e_1}(\delta(e_1, v)) = e^{-i\delta(e_2, v)(\tau(e_2, \vartheta)} y_{e_2}(\delta(e_2, v)), \quad \forall e_1, e_2 \in I_v(v), \quad (1.17)
\]

\[
\sum_{e \in I_v(v)} (-1)^{\delta(e, v)} e^{-i\delta(e, v)(\tau(e, \vartheta)} y'_e(\delta(e, v)) = 0, \quad (1.18)
\]
where $\delta(e, v)$ is defined by (1.2), $\tau(e) \in \mathbb{Z}^d$ is the index of the edge $e \in E_s$, defined in subsection 2.1, and $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^d$.

**Remark 1.5.** In (1.15) we identify an edge $e_s \in E_s$ of the fundamental graph $G_s$ with an edge $e \in E$ of the periodic graph $G$ (for more details see subsection 2.2).

### 1.5. Spectra of the metric Laplacian $\Delta_M$ and its fiber operators $\Delta_M(\vartheta)$

Consider the eigenvalues problem on the unit interval with Dirichlet boundary conditions

$$-y'' = Ey, \quad y(0) = y(1) = 0.$$  

(1.19)

It is known that $(\pi j)^2$, $j \in \mathbb{N}$, are the so-called Dirichlet eigenvalues of the problem (1.19). Thus the spectrum of this problem is given by

$$\sigma_D = \{(\pi j)^2 : j \in \mathbb{N}\}.$$  

(1.20)

We denote by $\beta$ the Betti number of the fundamental graph $G_s = (V_s, E_s)$:

$$\beta = \#E_s - \#V_s + 1,$$  

(1.21)

where $\#A$ is the number of elements in a set $A$. The Betti number $\beta$ of a finite connected graph $G_s$ can also be defined as the dimension of the cycle space $C$ of the graph $G_s$, i.e.,

$$\beta = \dim C.$$  

(1.22)

Due to the connectivity of the $\Gamma$-periodic graph $G$, the fundamental graph $G_s$ contains at least $d$ linearly independent cycles, where $d$ is the rank of the lattice $\Gamma$. Then, by (1.22), the Betti number of the fundamental graph $G_s$ satisfies the inequality $\beta \geq d$.

We describe our first result about eigenfunctions of the fiber operators $\Delta_M(\vartheta)$, $\vartheta \neq 0$, corresponding to the eigenvalues $(\pi j)^2$, $j \in \mathbb{N}$. Note that in Theorem 1.6 we do not consider the fiber operator $\Delta_M(0)$ which is just the metric Laplacian on the fundamental graph $G_s$ and is well studied (see Remarks 1.7, 2.3 and 2.1).

**Theorem 1.6.** Let $\Delta_M(\vartheta)$, $\vartheta \in \mathbb{T}^d \setminus \{0\}$, be the fiber operator defined by (1.16) - (1.18) on the fundamental graph $G_s = (V_s, E_s)$. Then for all $j \in \mathbb{N}$ the following statements hold true:

i) $(\pi j)^2$ is an eigenvalue of the operator $\Delta_M(\vartheta)$.

ii) Each eigenvalue $(\pi j)^2$ has multiplicity $\beta - 1$, where $\beta$ is the Betti number of the fundamental graph $G_s$ defined by (1.22).

iii) The corresponding normalized eigenfunctions $\Psi^0_{j,s} = (\Psi^0_{j,s,e})_{e \in E_s}$, $s \in \mathbb{N}_{\beta-1}$, have the form:

$$\Psi^0_{j,s,e}(\vartheta, t) = X_{j,s,e}(\vartheta) \sqrt{2} \sin(\pi j t), \quad t \in [0, 1],$$

$$\| \Psi^0_{j,s}(\vartheta, \cdot) \|_{L^2(G_s)}^2 = \sum_{e \in E_s} |X_{j,s,e}(\vartheta)|^2 = 1,$$  

(1.23)

where $(X_{j,s,e}(\vartheta))_{e \in E_s}$ is a normalized solution of the system of $\nu$ equations

$$\sum_{e \in E_s, \text{starting at } v} x_e - (-1)^j \sum_{e \in E_s, \text{ending at } v} e^{-i(\tau(e), \vartheta)} x_e = 0, \quad \forall v \in V_s,$$  

(1.24)

where $\nu = \#V_s$, one equation for each $v \in V_s$. This system has rank $\nu$ and there exist $\beta - 1$ linearly independent normalized solutions $X_{j,s,e}(\vartheta)$, $s \in \mathbb{N}_{\beta-1}$. Moreover, $X_{j,s,e}(\vartheta) =$
Remark 1.7. 1) In order to determine system (1.24) of \((s, t)\) for \(\beta - 1\) with the vertex set \(V\) indeed, if \(G\) is obtained by identification of \(e\) in (1.23), for each \(j \in \mathbb{N}\) vanish at each vertex of the fundamental graph \(G_s\).

2) The eigenfunctions \(\Psi_{j,s}^0 = (\Psi_{j,s}^0(e))_{e \in E_s}, (j, s) \in \mathbb{N} \times \mathbb{N}_{\beta - 1}\), defined by (1.23) vanish at each vertex of the fundamental graph \(G_s\).

3) The fiber operator \(\Delta_M(0)\) is the metric Laplacian on the fundamental graph \(G_s\). Moreover, it is known (see e.g., [BS5, C97, LP08]) that

- \((\pi j)^2, j \in \mathbb{N}\), is an eigenvalue of the operator \(\Delta_M(0)\);
- each eigenvalue \((2\pi j)^2\) of \(\Delta_M(0)\) has multiplicity \(\beta + 1\);
- each eigenvalue \((2j + 1)^2\pi^2\) of \(\Delta_M(0)\) has multiplicity \(\beta + 1\), if \(G_s\) is bipartite and multiplicity \(\beta - 1\), if \(G_s\) is not bipartite;
- the corresponding eigenfunctions are also constructed in the papers mentioned above (for more details see Remark 3.1).

From item i) and Theorem 1.6 i) it follows that \((\pi j)^2, j \in \mathbb{N}\), is an eigenvalue of \(\Delta_M(\vartheta)\) for all \(\vartheta \in \mathbb{T}^d\).

Theorem 1.6 allows us to decompose the fiber operator \(\Delta_M(\vartheta)\) in the following form

\[
\Delta_M(\vartheta) = \Delta_{MD}(\vartheta) \oplus \Delta_{MV}(\vartheta), \quad \text{where} \quad \Delta_{MD}(\vartheta) = \sum_{j \geq 1} (\pi j)^2 \mathcal{P}_j^0(\vartheta),
\]

\(\mathcal{P}_j^0(\vartheta)\) is the projection associated with the Dirichlet eigenvalue \((\pi j)^2\) of \(\Delta_M(\vartheta)\). The operator \(\Delta_{MD}(\vartheta)\) corresponds to the Dirichlet spectrum \(\sigma_D\). The other part \(\Delta_{MV}(\vartheta)\) is associated with the vertex set \(\mathcal{V}_s\) and it is discussed in the next theorem. Without loss of generality we may assume that for a bipartite periodic graph \(G\) the fundamental graph \(G_s\) is also bipartite. Indeed, if \(G_s\) is non-bipartite, then we just need to go to the vectors \(2a_1, \ldots, 2a_d\) as the periods of the graph \(G\), gluing together several copies of \(G_s\) to a new bipartite fundamental graph. For example, if \(G = \mathbb{Z}\), then the bipartite fundamental graph \(G_s\) is obtained by identification of every second vertex of \(G\).

We describe the “vertex fiber Laplacian” \(\Delta_{MV}(\vartheta)\) and its eigenfunctions. These eigenfunctions are expressed in terms of eigenfunctions of the discrete fiber Laplacians.

Theorem 1.8. The operator \(\Delta_{MV}(\vartheta)\) defined by (1.26) has the following form

\[
\Delta_{MV}(\vartheta) = \sum_{(j, n) \in \mathbb{Z}_0 \times \mathbb{N}_p} z_{j,n}^2(\vartheta) \mathcal{P}_{j,n}(\vartheta), \quad \text{for all} \quad \vartheta \in \mathbb{T}^d \setminus \{0\},
\]

where \(\mathcal{P}_{j,n}(\vartheta)\) is the projection associated with the eigenvalue \(z_{j,n}^2(\vartheta)\) of \(\Delta_{MV}(\vartheta)\) given by

\[
z_{j,n}(\vartheta) = \begin{cases} z_n(\vartheta) + \pi j, & \text{if } j \text{ is even} \\ (\pi - z_n(\vartheta)) + \pi j, & \text{if } j \text{ is odd} \end{cases}, \quad z_n(\vartheta) = \arccos(-\lambda_n(\vartheta)) \in [0, \pi],
\]

\(\mathbb{Z}_0 = \{0, 1, 2, \ldots\}\). The corresponding normalized eigenfunctions \(\Psi_{j,n}(\vartheta) = (\Psi_{j,n,e}(\vartheta, t))_{e \in E_s}\) satisfy:

\[
\Psi_{j,n,e}(\vartheta, t) = \frac{\sqrt{2}}{\sin z_n(\vartheta)} \left(\psi_n(\vartheta, u) \sin(z_{j,n}(\vartheta) (1 - t)) + \psi_n(\vartheta, v) e^{i(\tau(e), \vartheta)} \sin(z_{j,n}(\vartheta) t)\right),
\]
where $\mathbf{e} = (u, v)$, and $\psi_n(\vartheta, \cdot) \in \ell^2(V_n)$ is the normalized eigenfunction of the discrete fiber Laplacian $\Delta(\vartheta)$ given by (2.3), corresponding to the eigenvalue $\lambda_n(\vartheta)$.

**Remark 1.9.** 1) The eigenvalue $z^2_{j,n}(\vartheta) \neq (\pi j)^2$ for all $\vartheta \in \mathbb{T}^d \setminus \{0\}$ and all $(j, n) \in \mathbb{Z}_0 \times \mathbb{N}_\nu$.

2) Recently, Pankrashkin (see Proposition 1 in [Pa12], Theorem 14 in [Pa13]) showed the following result. Let $\eta(z) = -\cos \sqrt{z}$ and $\chi_\nu$ be the characteristic function of the set $\omega \subset \mathbb{R}$ and let an interval $J \subset \mathbb{R} \setminus \sigma_D$. Then the operator $A = \Delta_M \chi_J(\Delta_M V)$ is unitarily equivalent to the operator $B = \eta^{-1}(\Delta \chi_{\nu}(\Delta))$, i.e., the identity $A = UBU^{-1}$ holds true for some unitary operator $U$.

3) We do not know any paper about (1.29) – (1.30). These results are important to study the Fredholm determinant in Theorem 1.14.

4) The explicit form of the constant $C$ in (1.30) is defined in Propositions 3.3, 3.4. Results about the effective masses for Laplacians [KS16b] are essential in the proof of (1.30).

Theorems 4.4, 4.6 and 4.8 imply the following well-known results, see, e.g., [BK13], [BGP08], [P12]:

**Corollary 1.10.** i) The spectrum of the Laplacian $\Delta_M$ on $L^2(\mathcal{G})$ is given by

$$\sigma(\Delta_M) = \sigma_{ac}(\Delta_M) \cup \sigma_{fb}(\Delta_M), \quad \sigma_{ac}(\Delta_M) = \bigcup_{(j, n) \in \mathbb{Z}_0 \times \mathbb{N}_{\nu-r}} \sigma_{j,n}(\Delta_M), \quad \sigma_{j,n}(\Delta_M) = [E^-_{j,n}, E^+_{j,n}],$$

$$E^\pm_{j,n} = \left\{ \left( z_n^\pm + \pi j \right)^2, \quad \text{if } j \text{ is even} \right\}, \quad \left\{ \left( \pi - z_n^\pm + \pi j \right)^2, \quad \text{if } j \text{ is odd} \right\}, \quad z_n^\pm = \arccos(-\lambda_n^\pm) \in [0, \pi], \quad n \in \mathbb{N}_{\nu-r},$$

and

$$\sigma_{fb}(\Delta_M) = \sigma_{fb}(\Delta_{MD}) \cup \sigma_{fb}(\Delta_{MV}), \quad \sigma_{fb}(\Delta_{MD}) = \sigma_D = \{(\pi j)^2 : j \in \mathbb{N}\},$$

$$\sigma_{fb}(\Delta_{MV}) = \bigcup_{j=0}^{\infty} \{E_{j,\nu-r+1}, \ldots, E_{j,\nu}\}, \quad E_{j,\nu} = \left\{ \left( z_n^\pm + \pi j \right)^2, \quad \text{if } j \text{ is even} \right\} \cup \left\{ \left( \pi - z_n^\pm + \pi j \right)^2, \quad \text{if } j \text{ is odd} \right\},$$

$$z_n = \arccos(-\lambda_n) \in (0, \pi), \quad n = \nu - r + 1, \ldots, \nu,$$

where $\lambda_n^\pm$, $n \in \mathbb{N}_{\nu-r}$, are defined in (1.12), and $r \geq 0$ is the number of the flat bands $\{\lambda_{\nu-r+1}, \ldots, \lambda_{\nu}\}$ of the discrete Laplacian $\Delta$ (counting multiplicities).

ii) $\sigma(\Delta_M) = [0, +\infty)$ iff $\sigma(\Delta) = [-1, 1]$.

iii) Both the sets $\sigma_{ac}(\sqrt{\Delta_M})$ and $\sigma_{fb}(\sqrt{\Delta_M})$ are 2$\pi$-periodic on the half-line $(0, +\infty)$ and are symmetric on the interval $(0, 2\pi)$ with respect to the point $\pi$.

iv) The spectrum of the operator $\Delta_M$ has exactly $k$ gaps on the interval $[0, \pi^2]$ iff $\sigma(\Delta)$ has exactly $k$ gaps. The number of flat bands of $\Delta_M$ on $[0, \pi^2]$ is $r + 1$.

**Remark 1.11.** 1) Recall that Brüning-Geyler-Pankrashkin [BGP08] proved the following identities

$$
\sigma_\alpha(\Delta_M) \setminus \sigma_D = \{ E \in \mathbb{R} \setminus \sigma_D : -\cos \sqrt{E} \in \sigma_\alpha(\Delta) \}, \quad \alpha \in \{\text{fb, ac}\},
$$

where $\sigma_D = \{(\pi j)^2 : j \in \mathbb{N}\}$. The relation between the spectra of the operators $\Delta$ and $\sqrt{\Delta_M}$ is shown in Fig.2. The spectrum of the discrete Laplacian $\Delta$ is along the vertical axis $\lambda$. It consists of an absolutely continuous part which is a union of a finite number of the
non-degenerate spectral bands ($\sigma_1(\Delta)$ and $\sigma_2(\Delta)$ in the figure) and a finite number of the flat bands ($\{\lambda_3\}$ in the figure). The spectrum of $\sqrt{\Delta_M}$ is along the horizontal axis $z$. In order to obtain $\sigma(\sqrt{\Delta_M})$ one needs to find the preimage of $\sigma(\Delta)$ under the function $\lambda = - \cos z$. Each spectral band and each flat band of the discrete Laplacian $\Delta$ generate infinitely many spectral bands and flat bands, respectively, for the operator $\sqrt{\Delta_M}$. For example, in the figure the flat band $\{\lambda_3\}$ of the discrete Laplacian $\Delta$ generates the eigenvalues $z_3, 2\pi - z_3, 2\pi + z_3, 4\pi - z_3, \ldots$ with infinite multiplicity. Moreover, $\sqrt{\Delta_M}$ has the additional eigenvalues $\pi, 2\pi, 3\pi, \ldots$ with infinite multiplicity. We note that the spectrum of $\sqrt{\Delta_M}$ is $2\pi$-periodic on $(0, +\infty)$ and is symmetric with respect to the point $\pi$ on the interval $(0, 2\pi)$.

2) If $\sigma(\Delta) \neq [-1, 1]$, then in the spectrum of $\Delta_M$ there exist infinitely many gaps $\gamma_1, \gamma_2, \ldots$ and $|\gamma_n| \to \infty$ as $n \to \infty$.

3) The identities (1.31) show that spectral properties of the discrete Laplacian $\Delta$ are important to study spectral properties of the metric Laplacian $\Delta_M$.

Figure 2. Relation between the spectra of $\Delta$ and $\sqrt{\Delta_M}$.

Now we discuss the connection between generalized eigenfunctions of the absolutely continuous spectrum of the discrete and metric Laplacians and eigenfunctions of their fiber operators. Let $L^2_{\mathrm{loc}}(\mathcal{V})$ be the set of all sequences on $\mathcal{V}$, and let $L^2_{\mathrm{loc}}(\mathcal{G})$ be the set of all functions which are square integrable on any finite subset in $\mathcal{E}$. Recall that $\{a_1, \ldots, a_d\}$ is the basis of the lattice $\Gamma$.

**Proposition 1.12.** Let $\vartheta \in \mathbb{T}^d \setminus \{0\}$, $j \in \{0, 1, 2, \ldots\}$, $n \in \{1, 2, \ldots, \nu - r\}$. Let $\psi_n(\vartheta, \cdot) \in \ell^2(\mathcal{V}_j)$ be the normalized eigenfunction of the fiber discrete Laplacian $\Delta(\vartheta)$ corresponding to the eigenvalue $\lambda_n(\vartheta) \in \sigma_{\text{ac}}(\Delta)$ and let $\Psi_{j,n}(\vartheta, \cdot) \in L^2(\mathcal{G}_j)$ be the normalized eigenfunction of the fiber metric Laplacian $\Delta_M(\vartheta)$ corresponding to the eigenvalue $z_{j,n}^2(\vartheta) \in \sigma_{\text{ac}}(\Delta_M)$, where $z_{j,n}^2(\vartheta)$ is defined by (1.28). Then the following statements hold true.
i) The generalized eigenfunction \( \tilde{\psi}_n(\vartheta, \cdot) \in L^2_{\text{loc}}(\mathcal{V}) \) of the discrete Laplacian \( \Delta \) corresponding to \( \lambda_n(\vartheta) \in \sigma_{ac}(\Delta) \) has the form
\[
\tilde{\psi}_n(\vartheta, v + m_1 a_1 + \ldots + m_d a_d) = e^{i(m, \vartheta)} \psi_n(\vartheta, v),
\]
\[\forall v \in \mathcal{V}, \quad \forall m = (m_1, \ldots, m_d) \in \mathbb{Z}^d. \tag{1.34}\]

ii) The generalized eigenfunction \( \tilde{\Psi}_{j,n}(\vartheta, \cdot) \in L^2_{\text{loc}}(\mathcal{G}) \) of the metric Laplacian \( \Delta_M \) corresponding to \( z_{j,n}(\vartheta) \in \sigma_{ac}(\Delta_M) \) has the form
\[
\tilde{\Psi}_{j,n}(\vartheta, x + m_1 a_1 + \ldots + m_d a_d) = e^{i(m, \vartheta)} \Psi_{j,n}(\vartheta, x),
\]
\[\forall x \in \mathcal{G}, \quad \forall m = (m_1, \ldots, m_d) \in \mathbb{Z}^d. \tag{1.35}\]

iii) The generalized eigenfunctions from (1.34), (1.35) satisfy
\[
\tilde{\psi}_n(\vartheta, v) = e^{i(v, \vartheta)} \varphi_n(\vartheta, v), \quad \forall v \in \mathcal{V}; \tag{1.36}
\]
\[
\tilde{\Psi}_{j,n}(\vartheta, x) = e^{i(x, \vartheta)} \Phi_{j,n}(\vartheta, x), \quad \forall x \in \mathcal{G}; \tag{1.37}
\]
where the functions \( \varphi_n(\vartheta, v) \) and \( \Phi_{j,n}(\vartheta, x) \) are \( \Gamma \)-periodic with respect to \( v \) and \( x \), respectively, and \( x_\Lambda \in \mathbb{R}^d \) is the coordinate vector of \( x \in \mathbb{R}^d \) with respect to the basis \( \Lambda = \{a_1, \ldots, a_d\} \) of the lattice \( \Gamma \) (see (2.2)).

Remark 1.13. 1) In (1.34), (1.35) we identify the vertices and edges of the fundamental graph \( \mathcal{G}_* \) with some vertices and edges of the periodic graph \( \mathcal{G} \). For more details see the proof of this proposition.

2) From (1.35) and (1.30) it follows that all generalized eigenfunctions of the absolutely continuous spectrum of the Laplacian on the periodic metric graph \( \mathcal{G} \) are uniformly bounded in \( L^\infty(\mathcal{G}) \).

3) Formulas (1.29), (1.34) and (1.35) give that the eigenfunctions of the absolutely continuous spectrum of the discrete and metric Laplacians are connected by the same relation (1.29) as the eigenfunctions of their fiber operators.

4) Eigenfunctions of the discrete and metric Laplacians corresponding to flat bands are connected with eigenfunctions of their fiber operators by the Floquet transforms (for more details see Proposition 3.6).

1.6. Scattering on periodic metric graphs. We consider the Schrödinger operator \( H = H_0 + Q \) on \( L^2(\mathcal{G}) \), where \( H_0 = \Delta_M \) and the potential \( Q \in L^2(\mathcal{G}) \cap L^1(\mathcal{G}) \) is real. Here \( L^1(\mathcal{G}) \) is the space of all functions \( f = (f_\alpha)_{\alpha \in \mathcal{E}} \) on \( \mathcal{G} \) equipped with the norm \( \|f\|_{L^1(\mathcal{G})} = \sum_{\alpha \in \mathcal{E}} \|f_\alpha\|_{L^1(\mathcal{E})} \). Let \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) be the trace and the Hilbert-Schmidt class equipped with the norm \( \|\cdot\|_{\mathcal{B}_1} \) and \( \|\cdot\|_{\mathcal{B}_2} \), respectively.

In Theorem 1.14 we show that \( QR_0(k) \in \mathcal{B}_2 \) for all \( k \in \mathbb{C}_+ \). This implies that the operator \( H \) is self-adjoint on \( \mathcal{D}(H_0) \). For any \( k \in \mathbb{C}_+ \) we put
\[
R_0(k) = (H_0 - k^2)^{-1}, \quad R(k) = (H - k^2)^{-1}, \tag{1.38}
\]
\[
Y_0(k) = |Q|^{1/2} R_0(k) Q^{1/2}, \quad Q^{1/2} = |Q|^{1/2} \text{ sign } Q.
\]
Below we show that the operator \( Y_0(k) \) belongs to the trace class. Thus, we can define the Fredholm determinant \( D \) by
\[
D(k) = \det(I + Y_0(k)), \quad k \in \mathbb{C}_+. \tag{1.39}
\]
We describe our third main result about scattering on metric graphs.
**Theorem 1.14.** Let $Q \in L^2(\mathcal{G})$ be real. Then $QR_0(k) \in \mathcal{B}_2$ for all $k \in \mathbb{C}_+$ and the operator $H = H_0 + Q$ is self-adjoint on $\mathcal{D}(H_0)$. Let, in addition, $Q \in L^1(\mathcal{G})$. Then

$$R(k) - R_0(k), \quad Y_0(k) \in \mathcal{B}_1 \quad \forall k \in \mathbb{C}_+, \quad (1.40)$$

and the determinant $D(k) = \det(I + Y_0(k))$ is well-defined and analytic in $\mathbb{C}_+$ and the limit $D(k + i0)$ exists for almost all $k \in \mathbb{R}$. Furthermore, the wave operators

$$W_\pm = \text{s.-lim } e^{iHt} e^{-iH_0 t} P_{ac}(H_0) \quad \text{as} \quad t \to \pm \infty, \quad (1.41)$$

exist and are complete, i.e., the range of $W_\pm$ is equal to $\mathcal{H}_{ac}(H)$ and $\sigma_{ac}(H) = \sigma_{ac}(H_0)$. Moreover, the $S$-operator given by

$$S = W^*_+ W_- \quad (1.42)$$

is unitary on $\mathcal{H}_{ac}(H_0)$ and the corresponding $S$-matrix $S(k)$ (defined by (1.45)) for almost all $k^2 \in \sigma_{ac}(H_0)$ satisfies

$$S(k) = I_k - 2\pi iA(k), \quad A(k) \in \mathcal{B}_1, \quad (1.43)$$

$$\det S(k) = \frac{\widetilde{D}(k + i0)}{D(k + i0)}. \quad (1.44)$$

**Remark 1.15.** 1) Consider the Schrödinger operator $\widetilde{H} = -\Delta + \widetilde{Q}$ on $L^2(\mathbb{R}^d)$, $d \geq 1$, where $-\Delta$ is the standard Laplacian in $\mathbb{R}^d$ and $\widetilde{Q} \in C_0^\infty(\mathbb{R}^d)$ is a real potential. Define the operator $\widetilde{Y}_0(k) = |\widetilde{Q}|^{1/2}(-\Delta - k^2)^{-1}\widetilde{Q}^{1/2}, \quad k \in \mathbb{C}_+$.

Firstly, if $d = 1$, then it is well known that the corresponding operator $\widetilde{Y}_0(k) \in \mathcal{B}_1$ and so the Fredholm determinant $\det(I + \widetilde{Y}_0(k))$ is well defined. Recall that the Fredholm determinant is the basic function to study trace formulas, spectral shift functions, etc. in the general framework.

Secondly, if $d \geq 2$, then it is well known that the corresponding operator $\widetilde{Y}_0(k)$ is not trace class and so the Fredholm determinant $\det(I + \widetilde{Y}_0(k))$ is not defined. Thus, we need an essential modification. For example, Newton [NT77] defined the modified Fredholm determinant by

$$\widetilde{D}(k) = \det \left[ (I + \widetilde{Y}_0(k)) e^{-\widetilde{Y}_0(k)} \right], \quad k \in \mathbb{C}_+, \quad \text{if} \quad d = 3.$$

Note that the case $d > 3$ is more complicated.

Thus, in contrast to the Schrödinger operator $\widetilde{H}$ on $\mathbb{R}^d$ in the case of the Schrödinger operator $H = \Delta_M + Q$ on metric graphs the operator $Y_0(k)$ is trace class and the corresponding Fredholm determinant $D(k) = \det(I + Y_0(k))$ is well defined for all $k \in \mathbb{C}_+$ and for any dimension $d \geq 1$. Then the metric case (for any dimension $d \geq 1$) is very similar to the case of the Schrödinger operator $\widetilde{H}$ on $\mathbb{R}^1$. This fact is very important for spectral theory of Schrödinger operators on periodic metric graphs.

2) In fact the identity (1.44) is the so-called Birman-Krein [BK62] formula for the $S$-matrix in the case of scattering on metric graphs.

3) Recall that if the scattering operator $S = W^*_+ W_-$ is unitary on $\mathcal{H}_{ac}(H_0)$, then the operators $H_0$ and $S$ commute and thus are simultaneously diagonalizable:

$$\mathcal{H}_{ac}(H_0) = P_{ac}(H_0)L^2(\mathcal{G}) = \int_\sigma \mathcal{H}_\lambda d\lambda, \quad H_0 = \int_\sigma \lambda I_\lambda d\lambda, \quad S = \int_\sigma S(\lambda) d\lambda; \quad (1.45)$$

here $\sigma = \sigma_{ac}(H_0)$, $I_\lambda$ is the identity in the fiber space $\mathcal{H}_\lambda$, and $S(\lambda)$ is the scattering matrix acting in $\mathcal{H}_\lambda$ for the pair $H_0, H$. 


The paper is organized as follows. In Section 2 we prove Theorem 1.4 about the direct integral decomposition for the metric Laplacian $\Delta_M$ on periodic graphs. Section 3 is devoted to the eigenfunctions of the metric Laplacian $\Delta_M(\vartheta)$, where $\vartheta$ is the Dirichlet eigenvalue and Theorem 1.8 about the connection between the eigenfunctions of the fiber metric Laplacian $\Delta_M(\vartheta)$ and discrete $\Delta(\vartheta)$ Laplacians. Here we also prove Proposition 1.12 about the connection between generalized eigenfunctions of the absolutely continuous spectrum of the discrete and metric Laplacians and the eigenfunctions of their fiber operators. Section 4 deals with the periodic metric Laplacian perturbed by real integrable potentials. Here we prove Theorem 1.14 about scattering on metric graphs. In Sections 5–7 we consider some examples of periodic graphs: the $d$-dimensional lattice, the hexagonal lattice and the stanene lattice and give detailed descriptions of the spectrum of the metric Laplacian on these graphs including descriptions of the eigenfunctions of the fiber operators.

2. Direct integral for metric Laplacians

2.1. Edge indices. We define an edge index, which was introduced in [KS14a]. The indices are important to study the spectrum of the Laplacians and Schrödinger operators on periodic graphs, since the fiber operators are expressed in terms of the indices of the fundamental graph edges (see (2.8) for the discrete fiber Laplacian and (1.16)–(1.18) for the metric one).

For any vertex $v \in V$ of the $\Gamma$-periodic graph $G$ the following unique representation holds true:

$$v = v_0 + [v], \quad \text{where} \quad v_0 \in V_0 = V \cap \Omega, \quad [v] \in \Gamma,$$

(2.1)

$\Omega$ is the fundamental cell of the lattice $\Gamma$ defined by (1.1). In other words, each vertex $v \in V$ can be obtained from a vertex $v_0 \in V_0$ by the shift by a vector $[v] \in \Gamma$. We call $[v]$ the integer part of the vertex $v$.

We recall that $\{a_1, \ldots, a_d\}$ is the basis of the lattice $\Gamma$. For each $x \in \mathbb{R}^d$ we introduce the vector $x_\Lambda \in \mathbb{R}^d$ by

$$x_\Lambda = (t_1, \ldots, t_d), \quad \text{where} \quad x = \sum_{s=1}^d t_s a_s.$$

(2.2)

In other words, $x_\Lambda$ is the coordinate vector of $x$ with respect to the basis $\Lambda = \{a_1, \ldots, a_d\}$ of the lattice $\Gamma$.

For any oriented edge $e = (u, v) \in A$ we define the edge index $\tau(e)$ as the vector of the lattice $\mathbb{Z}^d$ given by

$$\tau(e) = [v]_\Lambda - [u]_\Lambda \in \mathbb{Z}^d,$$

(2.3)

where $[v] \in \Gamma$ is the integer part of the vertex $v$, and the vector $[v]_\Lambda \in \mathbb{Z}^d$ is defined by (2.2). Edge indices depend on the embedding of $G$ into $\mathbb{R}^d$ and on the choice of the basis $a_1, \ldots, a_d$ of the lattice $\Gamma$. We note that edges connecting vertices inside the fundamental cell $\Omega$ have zero indices.

Example 2.1. We consider the periodic graph $G$ shown in Fig.1a. The periods $a_1, a_2$ of the graph and the fundamental cell $\Omega$ are also shown in the figure. The index of the edge $(v_1, v_3 + a_1)$ is equal to $(1, 0)$, since $[v_1] = 0$ and $[v_3 + a_1] = a_1$. The edge $(v_1, v_4)$ has zero index.

On the $\Gamma$-periodic graph $G = (V, E)$ we define two surjections

$$f_V : V \to V_\Gamma = V / \Gamma, \quad f_A : A \to A_\Gamma = A / \Gamma,$$

(2.4)
which map each vertex \( v \in \mathcal{V} \) and each oriented edge \( e \in \mathcal{A} \) of \( \mathcal{G} \) to their equivalence classes \( f_{\mathcal{V}}(v) \) and \( f_{\mathcal{A}}(e) \), respectively, which are a vertex and an oriented edge of the fundamental graph \( \mathcal{G}_* = (\mathcal{V}_*, \mathcal{E}_*) \).

For each oriented edge \( e_* \in \mathcal{A}_* \) of the fundamental graph \( \mathcal{G}_* \) we define the edge index \( \tau(e_*) \in \mathbb{Z}^d \) by:

\[
\tau(e_*) = \tau(e) \quad \text{for some } e \in \mathcal{A} \text{ such that } e_* = f_{\mathcal{A}}(e), \quad e_* \in \mathcal{A}_*.
\] (2.5)

Indices of the fundamental graph edges are induced by indices of the periodic graph edges and uniquely determined by (2.5), since

\[
\tau(e + a) = \tau(e), \quad \forall (e, a) \in \mathcal{A} \times \Gamma.
\]

Denote by \( \mathcal{C} \) the cycle space of the fundamental graph \( \mathcal{G}_* \). We define the vector-valued flux function \( \Phi_{\tau} : \mathcal{C} \rightarrow \mathbb{R}^d \) as follows:

\[
\Phi_{\tau}(c) = \sum_{e \in c} \tau(e), \quad c \in \mathcal{C}.
\] (2.6)

It is known (see Propositions 4.1.i and 4.2 in [KS19]) that the image of the flux function \( \Phi_{\tau} \) is the lattice \( \mathbb{Z}^d \):

\[
\Phi_{\tau}(\mathcal{C}) = \mathbb{Z}^d.
\] (2.7)

### 2.2. Fiber Laplacians

The normalized Laplacian \( \Delta \), given by (1.6), on the periodic graph \( \mathcal{G} \) has the standard decomposition into the constant fiber direct integral (1.8), where the fiber Laplacian \( \Delta(\vartheta) \) has the form

\[
(\Delta(\vartheta)f)(v) = -\frac{1}{\kappa_v} \sum_{e = (v, u) \in \mathcal{A}_*} e^{i\langle \tau(e), \vartheta \rangle} f(u), \quad f \in \ell^2(\mathcal{V}_*), \quad v \in \mathcal{V}_*,
\] (2.8)

see Theorem 2.2 in [KS16b]. We recall that \( \kappa_v \) is the degree of the vertex \( v \), \( \langle \cdot, \cdot \rangle \) denotes the standard inner product in \( \mathbb{R}^d \), and \( \tau(e) \) is the index of the edge \( e \in \mathcal{A}_* \) defined by (2.3), (2.5).

We identify the vertices of the fundamental graph \( \mathcal{G}_* = (\mathcal{V}_*, \mathcal{E}_*) \) with the corresponding vertices of the periodic graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) from the set \( \mathcal{V}_0 = \mathcal{V} \cap \Omega \). We also identify an edge \( e_* = (u_*, v_*) \in \mathcal{E}_* \) of the fundamental graph \( \mathcal{G}_* \) with an index \( \tau(e_*) \) with the edge \( e = (u, v + \tau(e_*)) \in \mathcal{E} \) of the periodic graph \( \mathcal{G} \), where \( u, v \in \mathcal{V}_0 \) such that \( u_* = f_{\mathcal{V}}(u) \), \( v_* = f_{\mathcal{V}}(v) \), and \( f_{\mathcal{V}} \) is defined in (2.4). We note that \( \tau(e) = \tau(e_*) \).

Let \( a(m) \in \Gamma \), \( m \in \mathbb{Z}^d \), be defined by

\[
a(m) = \sum_{s=1}^{d} m_s a_s, \quad m = (m_1, \ldots, m_d) \in \mathbb{Z}^d.
\] (2.9)

**Proof of Theorem 1.4.** Denote by \( L^2_{\text{com}}(\mathcal{G}) \) the set of all compactly supported functions \( h \in L^2(\mathcal{G}) \). Standard arguments (see pp. 290–291 in [RS78]) yield that \( \mathcal{W} \) given by (1.13) is well defined on \( L^2_{\text{com}}(\mathcal{G}) \) and has a unique extension to a unitary operator. For \( h \in L^2_{\text{com}}(\mathcal{G}) \)
the sum (1.15) is finite and using the identity $E = \{e_s + a : (e_s, a) \in \mathcal{E}_s \times \Gamma\}$ we have

$$\|\mathcal{U}h\|^2_{\mathcal{H}} = \int_{T^d} \|(\mathcal{U}h)(\vartheta, \cdot)\|^2_{L^2(\mathcal{G})} \frac{d\vartheta}{(2\pi)^d}$$

$$= \sum_{e_s \in \mathcal{E}_s} \int_0^1 \int_{T^d} \left( \sum_{m \in \mathbb{Z}^d} e^{-im\vartheta} h_{e_s + a(m)}(t) \right) \left( \sum_{m' \in \mathbb{Z}^d} e^{im'\vartheta} h_{e_s + a(m')}^*(t) \right) \frac{d\vartheta}{(2\pi)^d} dt$$

$$= \sum_{e_s \in \mathcal{E}_s} \int_0^1 \int_{T^d} \left( h_{e_s + a(m)}(t) \bar{h}_{e_s + a(m')}^*(t) \right) \frac{d\vartheta}{(2\pi)^d} dt$$

$$= \sum_{e_s \in \mathcal{E}_s} \int_0^1 \int_{T^d} |h_{e_s + a}(t)|^2 dt = \sum_{e_s \in \mathcal{E}_s} \int_0^1 |h_{e}(t)|^2 dt = \|h\|^2_{L^2(\mathcal{G})}.$$ 

Thus, $\mathcal{U}$ is well defined on $L^2_{\text{com}}(\mathcal{G})$ and has a unique isometric extension. In order to prove that $\mathcal{U}$ is onto $\mathcal{H}$ we compute $\mathcal{U}^*$. Let $g = (g(\cdot, x_s))_{x_s \in \mathcal{G}_s} \in \mathcal{H}$, where $g(\cdot, x_s) : T^d \to \mathbb{C}$. We define

$$(\mathcal{U}^*g)(x) = \int_{T^d} e^{im\vartheta} g(\vartheta, x_s) \frac{d\vartheta}{(2\pi)^d}, \quad x = x_s + a(m) \in \mathcal{G}, \quad (2.10)$$

where $(x_s, m) \in \mathcal{G}_s \times \mathbb{Z}^d$ are uniquely defined. A direct computation gives that this is indeed the formula for the adjoint of $\mathcal{U}$. Moreover, the Parseval identity for Fourier series gives

$$\|\mathcal{U}^*g\|^2_{L^2(\mathcal{G})} = \sum_{e \in \mathcal{E}} \|(\mathcal{U}^*g)_e\|^2_{L^2(0,1)} = \sum_{e \in \mathcal{E}} \sum_{a \in \mathbb{Z}^d} \int_0^1 |(\mathcal{U}^*g)_{e + a}(t)|^2 dt$$

$$= \sum_{e \in \mathcal{E}} \sum_{a \in \mathbb{Z}^d} \int_0^1 \left| \int_{T^d} e^{im\vartheta} g_{e_s}(\vartheta, t) \frac{d\vartheta}{(2\pi)^d} \right|^2 dt$$

$$= \sum_{e \in \mathcal{E}} \int_0^1 \int_{T^d} |g_{e_s}(\vartheta, t)|^2 \frac{d\vartheta}{(2\pi)^d} dt = \int_{T^d} \|g(\cdot, \cdot)\|^2_{L^2(\mathcal{G})} \frac{d\vartheta}{(2\pi)^d} = \|g\|^2_{\mathcal{H}}.$$ 

Further, for $h \in L^2_{\text{com}}(\mathcal{G})$ and $x \in e \in \mathcal{E}_s$, $x \notin \mathcal{V}_s$ we obtain

$$(\mathcal{U} \Delta_M h)(\vartheta, x) = \sum_{m \in \mathbb{Z}^d} e^{-im\vartheta} (\Delta_M h)(x + a(m))$$

$$= \sum_{m \in \mathbb{Z}^d} e^{-im\vartheta} h_{e + a(m)}^n(x + a(m)) = \Delta_M(\vartheta)(\mathcal{U} h)(\vartheta, x). \quad (2.11)$$
Let \( v \in V_* \) and \( e_1, e_2, e \in I_*(v) \). Denote by \( \delta_1 = \delta(e_1, v) \), \( \delta_2 = \delta(e_2, v) \) and \( \delta = \delta(e, v) \), where \( \delta \) is defined by \((1.3)\). Then, using \((1.2)\), we have

\[
\begin{align*}
e^{-i\delta_1(\tau(e_1), \vartheta)} (G h)_{e_1}(\vartheta, \delta_1) &= e^{-i\delta_1(\tau(e_1), \vartheta)} \sum_{m \in \mathbb{Z}^d} e^{-i(m, \vartheta)} h_{e_1+a(m)}(\delta_1) \\
&= e^{-i\delta_2(\tau(e_2), \vartheta)} \sum_{m \in \mathbb{Z}^d} e^{-i(m+\delta_1(\tau(e_1)-\delta_2(\tau(e_2), \vartheta)) h_{e_1+a(m)}(\delta_1) \\
&= e^{-i\delta_2(\tau(e_2), \vartheta)} \sum_{m \in \mathbb{Z}^d} e^{-i(m+\delta_1(\tau(e_1)-\delta_2(\tau(e_2), \vartheta)) h_{e_2+a(m+\delta_1(\tau(e_1)-\delta_2(\tau(e_2), \vartheta))}(\delta_2) \\
&= e^{-i\delta_2(\tau(e_2), \vartheta)} \sum_{m \in \mathbb{Z}^d} e^{-i(m, \vartheta)} h_{e_2+a(m)}(\delta_2) = e^{-i\delta_2(\tau(e_2), \vartheta)} (G h)_{e_2}(\vartheta, \delta_2),
\end{align*}
\]

and

\[
\sum_{e \in I_*(v)} (-1)^\delta e^{-i\delta(\tau(e), \vartheta)} (G h)'_e(\vartheta, \delta) = \sum_{e \in I_*(v)} (-1)^\delta e^{-i\delta(\tau(e), \vartheta)} \sum_{m \in \mathbb{Z}^d} e^{-i(m, \vartheta)} h'_{e+a(m)}(\delta) \\
= \sum_{e \in I_*(v)} (-1)^\delta \sum_{m \in \mathbb{Z}^d} e^{-i(m+\delta_2(\tau(e), \vartheta), \vartheta)} h'_{e+a(m)}(\delta) = \sum_{e \in I_*(v)} (-1)^\delta \sum_{m \in \mathbb{Z}^d} e^{-i(m, \vartheta)} h'_{e+a(m-\delta_2(\tau(e), \vartheta))}(\delta) \\
= \sum_{m \in \mathbb{Z}^d} (-1)^\delta e^{-i(m, \vartheta)} h'_{e+a(m-\delta_2(\tau(e), \vartheta))}(\delta) = \sum_{m \in \mathbb{Z}^d} (-1)^\delta h'_{e+a(m)} \sum_{e \in I_*(v)} (-1)^\delta h'_{e+a(m)}(\delta) = 0.
\]

\[
(2.13)
\]

The identities \((2.11) - (2.13)\) yield \((1.14), (1.17), (1.18)\). \(\blacksquare\)

**Corollary 2.2.** The Laplacian \( \Delta_M \) on \( L^2(G) \) has a decomposition into a constant fiber direct integral of the form \((1.14)\), where the fiber operator \( \widetilde{\Delta}_M(\vartheta) \) on \( L^2(G_v) \) is defined by

\[
(\widetilde{\Delta}_M(\vartheta) \gamma)_v = -\left( \frac{d}{dt} + i \langle \tau(e), \vartheta \rangle \right)^2 \gamma_v, \quad \gamma = (\gamma_v)_{e \in \mathcal{E}_*},
\]

where \((\gamma_v)_{e \in \mathcal{E}_*} \in L^2(G_v)\), \( y \) is continuous on \( G_* \) and satisfies the boundary conditions:

\[
\sum_{e \in I_*(v)} (-1)^\delta(\vartheta, \vartheta) \left( \frac{d}{dt} + i \langle \tau(e), \vartheta \rangle \right) y_v(\delta(e, v)) = 0 \quad \forall v \in V_*,
\]

or, that is the same,

\[
\sum_{e \in I_*(v)} (-1)^\delta(\vartheta, \vartheta) y_v'(\delta(e, v)) = y(v) i \sum_{e \in I_*(v)} (-1)^\delta(\vartheta, \vartheta) \langle \tau(e), \vartheta \rangle, \quad \forall v \in V_*,
\]

\[
y(v) = y_v(\delta(e, v)), \quad \forall e \in I_*(v).
\]

**Proof.** After the gauge transformation \( y_v(t) = e^{-it \langle \tau(e), \vartheta \rangle} y_v(t) \), for all \( e \in \mathcal{E}_* \), identities \((1.16) - (1.18)\) take the form \((2.11) - (2.13)\). \(\blacksquare\)

**3. Eigenfunctions of metric Laplacians**

In this section we prove Theorems \((1.6)\) and \((1.8)\) about eigenfunctions of the fiber metric Laplacian \( \Delta_M(\vartheta) \) and Proposition \((1.12)\) about the connection between eigenfunctions of the absolutely continuous spectrum of the discrete and metric Laplacians and the eigenfunctions of their fiber operators.
Proof of Theorem [1.6, i) – iii)] Let \( \vartheta \in \mathbb{T}^d \setminus \{0\} \) and \( j \in \mathbb{N} \). We construct all eigenfunctions \( \Psi_j^0 = (\Psi_j^0)_{e \in E^*} \) of the operator \( \Delta_M(\vartheta) \) corresponding to the eigenvalue \( (\pi j)^2 \). These eigenfunctions have the form

\[
\Psi_j^0(\vartheta, t) = \Psi_j^0(\vartheta, 0) \cos(\pi j t) + \Psi_j^0(\vartheta, 0) \frac{\sin(\pi j t)}{\pi j}, \quad \forall t \in [0, 1],
\]

which yields

\[
\Psi_j^0(\vartheta, 1) = (-1)^j \Psi_j^0(\vartheta, 0), \quad \Psi_j^0(\vartheta, 1) = (-1)^j \Psi_j^0(\vartheta, 0).
\]

Let \( c \in \mathcal{C} \), where \( \mathcal{C} \) is the cycle space of the fundamental graph \( G_* \). Then, using the quasi-periodic condition (1.17) and the first identity in (3.2), we obtain

\[
\Psi_j^0(\vartheta, 0)\left((-1)^j e^{-i(\Phi_r(c), \vartheta)} - 1\right) = 0, \quad \forall e \in c,
\]

where \( l_c \) is the length of the cycle \( c \), i.e., the number of its edges, and \( \Phi_r(c) \) is defined by (2.6). Since \( \Phi_r(c) = \mathbb{Z}^d \) and \( \vartheta \neq 0 \), from (3.3) it follows that \( \Psi_j^0(\vartheta, 0) = 0 \) for each \( e \in c \). Due to the connectivity of the fundamental graph, the quasi-periodic condition (1.17) and the first identity in (3.2) give \( \Psi_j^0(\vartheta, 0) = 0 \) for all \( e \in E_* \). Thus, the eigenfunctions (3.1) take the form

\[
\Psi_j^0(\vartheta, t) = \Psi_j^0(\vartheta, 0) \frac{\sin(\pi j t)}{\pi j}, \quad \forall e \in E_*.
\]

These functions must satisfy the quasi-periodic condition (1.18), which, using the second identity in (3.2), can be rewritten in the form

\[
\sum_{e=(v,u) \in E_*} \Psi_j^0(\vartheta, 0) - (-1)^j \sum_{e=(u,v) \in E_*} e^{-i(\tau(e), \vartheta)} \Psi_j^0(\vartheta, 0) = 0, \quad \forall v \in V_*.
\]

This is a homogeneous system of \( \nu = \#V_* \) linear equations with \( \nu_1 = \#\mathcal{E}_* \) variables \( (\Psi_j^0(\vartheta, 0))_{e \in E_*} \). The \( \nu \times \nu_1 \) coefficient matrix \( D_j(\vartheta) = \{D_j^{(v)}(\vartheta)\}_{v \in V_*} \) of the system is given by

\[
D_j^{(v)}(\vartheta) = \begin{cases} 
(\pi j)^2 e^{-i(\tau(e), \vartheta)}, & \text{if } v \text{ is the terminal vertex of } e \\
1, & \text{if } v \text{ is the initial vertex of } e \\
1 - (\pi j)^2 e^{-i(\tau(e), \vartheta)}, & \text{if } e \text{ is a loop in the vertex } v \\
0, & \text{otherwise}
\end{cases}
\]

The number of linearly independent solutions of the system (3.5) is \( (\nu_1 - \text{rank} D_j(\vartheta)) \). We will show that \( \text{rank} D_j(\vartheta) = \nu \).

Let \( D_j^{(v)}(\vartheta) \) denote the row of the matrix \( D_j(\vartheta) \), corresponding to the vertex \( v \in V_* \). In order to show that the rows of this matrix are linearly independent, we consider their linear combination with coefficients \( \alpha_v, v \in V_* \):

\[
\sum_{v \in V_*} \alpha_v D_j^{(v)}(\vartheta) = 0.
\]

From this, using the form (3.6) of the matrix \( D_j(\vartheta) \), we obtain

\[
\alpha_v = \alpha_v (-1)^j e^{-i(\tau(e), \vartheta)}, \quad \forall e = (u, v) \in E_*.
\]

Let \( c \in \mathcal{C} \). Then for each vertex \( v \) of the cycle \( c \), the identities (3.8) give

\[
\alpha_v (-1)^j e^{i(\Phi_r(c), \vartheta)} - 1 = 0.
\]

Since \( \vartheta \neq 0 \) and \( \Phi_r(c) = \mathbb{Z}^d \), we conclude that \( \alpha_v = 0 \) for each vertex \( v \) of each cycle \( c \). Due to the connectivity of \( G_* \) and the identity (3.8), all coefficients \( \alpha_v = 0 \) in (3.7). Thus, the
rows of the matrix $D_j(\vartheta)$ are linearly independent and rank $D_j(\vartheta) = \nu$. Then the number of linearly independent solutions of the system (3.5) is equal to $\nu_1 - \nu$. Using this and the identity (1.21), we obtain that the eigenvalue $(\pi j)^2$ has multiplicity $\beta - 1$.

From (3.4) it follows that

$$\|\Psi_j^0(\vartheta, \cdot)\|^2_{L^2(\mathcal{G}_s)} = \frac{1}{2(\pi j)^2} \sum_{e \in \mathcal{E}_s} |\Psi_{j,e}^0(\vartheta, 0)|^2. \quad (3.10)$$

Choosing the normalized solutions $(X_{j,s,e}(\vartheta))_{e \in \mathcal{E}_s}$, $s \in \mathbb{N}_{\beta - 1}$, of the system (3.5) or, equivalently, of the system (1.24), we obtain that the normalized eigenfunctions $\Psi_{j,s}^0 = (\Psi_{j,s,e}^0)_{e \in \mathcal{E}_s}$, $s \in \mathbb{N}_{\beta - 1}$, have the form (1.23).

The identity $X_{j,s,e}(\vartheta) = X_{j+2,s,e}(\vartheta)$, $(s,e) \in \mathbb{N}_{\beta - 1} \times \mathcal{E}_s$, follows from the form of the system (1.24). The inequality (1.25) is a direct consequence of (1.24). \[ \square \]

**Remark 3.1.** The Floquet operator $\Delta_M(0)$ is the metric Laplacian on the fundamental graph $\mathcal{G}_s = (\mathcal{V}_s, \mathcal{E}_s)$. It is known (see [BS85], [CL97], [LP08], that

1) the normalized eigenfunctions $\Psi_{j,s} = (\Psi_{j,s,e})_{e \in \mathcal{E}_s}$, $s \in \mathbb{N}_\beta$ of the operator $\Delta_M(0)$ corresponding to the eigenvalue $(\pi j)^2$, $j$ is even, have the form (1.23) as $\vartheta = 0$, where $(X_{j,s,e}(0))_{e \in \mathcal{E}_s}$ is a normalized solution of the system of $\nu$ equations

$$\sum_{e \in \mathcal{E}_s, \text{starting at } v} x_e - \sum_{e \in \mathcal{E}_s, \text{ending at } v} x_e = 0, \quad \forall v \in \mathcal{V}_s, \quad (3.11)$$

where $\nu = \# \mathcal{V}_s$, one equation for each $v \in \mathcal{V}_s$. This system has rank $\nu - 1$ and thus there exist $\beta$ linearly independent normalized solutions $(X_{j,s,e}(0))_{e \in \mathcal{E}_s}$, $s \in \mathbb{N}_\beta$, where $\beta = \# \mathcal{E}_s - \# \mathcal{V}_s + 1$ is the Betti number of the fundamental graph $\mathcal{G}_s$. In this case there exists an additional normalized eigenfunction $\Psi_j^0 = (\Psi_{j,e}^0)_{e \in \mathcal{E}_s}$, having the form

$$\Psi_{j,e}^0(0, t) = \sqrt{\frac{2}{\nu_1}} \cos(\pi j t), \quad (3.12)$$

where $\nu_1 = \# \mathcal{E}_s$. This additional eigenfunction is related to the eigenfunction of the discrete Laplacian $\Delta(0)$ with the eigenvalue $-1$.

2) the normalized eigenfunctions $\Psi_{j,s} = (\Psi_{j,s,e})_{e \in \mathcal{E}_s}$ corresponding to the eigenvalue $(\pi j)^2$, $j$ is odd, also have the form (1.23) as $\vartheta = 0$, where $(X_{j,s,e}(0))_{e \in \mathcal{E}_s}$ is a normalized solution of the system of $\nu$ equations

$$\sum_{e \in \mathcal{E}_s, \text{starting at } v} x_e + \sum_{e \in \mathcal{E}_s, \text{ending at } v} x_e = 0, \quad \forall v \in \mathcal{V}_s. \quad (3.13)$$

If $\mathcal{G}_s$ is bipartite, then this system has rank $\nu - 1$. If $\mathcal{G}_s$ is not bipartite, this system has rank $\nu$. For a bipartite graph there exists an additional normalized eigenfunction $\Psi_j^0 = (\Psi_{j,e}^0)_{e \in \mathcal{E}_s}$, having the form

$$\Psi_{j,e}^0(0, t) = \pm \sqrt{\frac{2}{\nu_1}} \cos(\pi j t), \quad (3.14)$$

where the sign is chosen such that the eigenfunction has alternative sign in adjacent vertices. This additional eigenfunction is related to the eigenfunction of the discrete Laplacian $\Delta(0)$ with the eigenvalue $1$. 
In the following proposition we describe the connection between eigenfunctions and corresponding eigenvalues (except for the Dirichlet eigenvalues $\sigma_D$) of the Floquet operators for metric and discrete Laplacians.

**Proposition 3.2.** Let $\vartheta \in \mathbb{T}^d$. Then the following statements hold true.

i) If $\Psi_e(\vartheta) = (\Psi_e(\vartheta,t))_{t \in \mathbb{R}} \in L^2(\mathcal{G}_e)$ is an eigenfunction of the operator $\Delta_M(\vartheta)$ with an eigenvalue $E(\vartheta) \notin \sigma_D$, then the function $\psi(\vartheta, \cdot) \in L^2(V_e)$, defined by

$$
\psi(\vartheta, v) = e^{-i\vartheta(z(e,v))} \Psi_e(\vartheta, \delta(e, v)), \quad e \in I_e(v),
$$

is an eigenfunction of the operator $\Delta(\vartheta)$ with the eigenvalue $\lambda(\vartheta) = -\cos \sqrt{E(\vartheta)}$.

ii) Conversely, if $\psi(\vartheta, \cdot) \in L^2(V_e)$ is an eigenfunction of $\Delta(\vartheta)$ with an eigenvalue $\lambda(\vartheta) \in (-1, 1)$, then for each $j \in \mathbb{N}$ the function $\Psi_j(\vartheta) = (\Psi_j(e, \vartheta,t))_{t \in \mathbb{R}} \in L^2(\mathcal{G}_e)$, defined by

$$
\Psi_j(e, \vartheta,t) = \frac{1}{\sin j(\vartheta)} \left( \psi(\vartheta, u) \sin(z_j(\vartheta)(1-t)) + \psi(\vartheta, v)e^{i(z(\vartheta, \vartheta,v))} \sin(z_j(\vartheta)) \right), \quad e = (u, v),
$$

is an eigenfunction of $\Delta_M(\vartheta)$ with the eigenvalue $E_j(\vartheta) = z^2_j(\vartheta)$.

**Proof.** i) Let $\Psi(\vartheta)$ be an eigenfunction of the operator $\Delta_M(\vartheta)$ with an eigenvalue $E \notin \sigma_D$, $E \neq 0$, $\vartheta \equiv \vartheta(\vartheta)$. The proof for the case $E = 0$ is similar. Then

$$
\Psi_e(t) = \Psi_e(0) \cos(zt) + \Psi'_e(0) \frac{\sin(zt)}{z}, \quad z = \sqrt{E}, \quad \Psi_e(t) \equiv \Psi_e(\vartheta, t),
$$

which yields

$$
\Psi_e(1) = \Psi_e(0) \cos z + \Psi'_e(0) \frac{\sin z}{z}, \quad \Psi'_e(1) = -\Psi_e(0) z \sin z + \Psi'_e(0) \cos z.
$$

From \[3.18\] we obtain

$$
\Psi'_e(0) = \frac{z}{\sin z} \left( \Psi_e(1) - \Psi_e(0) \cos z \right), \quad \Psi'_e(1) = \frac{z}{\sin z} \left( \Psi_e(1) \cos z - \Psi_e(0) \right).
$$

Using the formulas \[3.19\], we rewrite the left-hand side of the condition \[1.18\] in the form:

$$
\sum_{e \in I_e(v)} (-1)^{\vartheta(z(e,v))} e^{-i\vartheta(z(e,v))} \Psi_e(\delta(e, v)) = \sum_{e=(u,v) \in E_e} \Psi'_e(0) - \sum_{e=(u,v) \in E_e} e^{-i z(\vartheta, e, v)} \Psi'_e(1)
$$

$$
= \sum_{e=(u,v) \in E_e} \frac{z}{\sin z} \left( \Psi_e(1) - \Psi_e(0) \cos z \right) - \sum_{e=(u,v) \in E_e} e^{-i z(\vartheta, e, v)} \frac{z}{\sin z} \left( \Psi_e(1) \cos z - \Psi_e(0) \right),
$$

for all $v \in V_e$. Thus, after dividing by $\frac{z}{\sin z}$, the condition \[1.18\] takes the form

$$
\sum_{e=(u,v) \in E_e} \left( \Psi_e(1) - \Psi_e(0) \cos z \right) - \sum_{e=(u,v) \in E_e} e^{-i z(\vartheta, e, v)} \left( \Psi_e(1) \cos z - \Psi_e(0) \right) = 0,
$$

for all $v \in V_e$. Using the formulas \[1.17\] and \[3.15\], we write the left-hand side of the identity \[3.20\] in the form

$$
-\psi(\vartheta, v) \beta_v \cos z + \sum_{e=(u,v) \in E_e} \Psi_e(1) + \sum_{e=(u,v) \in E_e} e^{-i z(\vartheta, e, v)} \Psi_e(0)
$$

$$
= -\psi(\vartheta, v) \beta_v \cos z + \sum_{e=(u,v) \in E_e} e^{i z(\vartheta, e, v)} \psi(\vartheta, u) + \sum_{e=(u,v) \in E_e} e^{-i z(\vartheta, e, v)} \psi(\vartheta, u)
$$

$$
= -\psi(\vartheta, v) \beta_v \cos z + \sum_{e=(u,v) \in A_e} e^{i z(\vartheta, e, v)} \psi(\vartheta, u).$$
Thus, the condition \((3.20)\) takes the form
\[
\psi(\vartheta, v) \cos z = \frac{1}{\nu} \sum_{e=(v,u)\in A} e^{i(\tau(e), \vartheta)} \psi(\vartheta, u),
\]
or, using the formula \((2.8)\),
\[
- \psi(\vartheta, v) \cos z = (\Delta(\vartheta)\psi(\vartheta, \cdot))(v).
\]
Thus, \(\psi(\vartheta, \cdot)\) is an eigenfunction of the operator \(\Delta(\vartheta)\) with the eigenvalue \((- \cos z)\).

\(ii)\) Conversely, let \(\psi(\vartheta, \cdot)\) be an eigenfunction of the operator \(\Delta(\vartheta)\) with the eigenvalue \(\lambda(\vartheta) \in (-1, 1)\). For the function \(\Psi_j(\vartheta) = (\Psi_j, e(\vartheta, t))_{e \in E}\) defined by \((3.10)\) we have
\[
\Psi''_{j,e}(t) = \frac{z_j}{\sin z_j} \left( - \psi(u) \cos(z_j(1-t)) + \psi(v)e^{i(\tau(e), \vartheta)} \cos(z_j t) \right),
\]
where for the shortness
\[
\Psi_{j,e}(t) = \Psi_{j,e}(\vartheta, t), \quad z_j = z_j(\vartheta), \quad \psi(u) = \psi(\vartheta, u).
\]
Thus, on each edge \(e \in E\) the function \(\Psi_j(\vartheta)\) satisfies the equation \(-\Psi''_{j,e}(t) = z_j^2\Psi_{j,e}(t)\). From \((3.10)\) we obtain that
\[
\Psi_{j,e}(0) = \psi(u), \quad \Psi_{j,e}(1) = \psi(v)e^{i(\tau(e), \vartheta)}, \quad e = (u, v),
\]
which yields
\[
e^{-i\delta(e,v)(\tau(e), \vartheta)}\Psi_{j,e}(\delta(e, v)) = \psi(v), \quad \forall v \in V_*, \quad \forall e \in I_*(v),
\]
i.e., the condition \((1.17)\) holds true. Similarly, from \((3.22)\) we have
\[
\Psi''_{j,e}(0) = \frac{z_j}{\sin z_j} \left( - \psi(u) \cos z_j + \psi(v)e^{i(\tau(e), \vartheta)} \right),
\]
\[
\Psi''_{j,e}(1) = \frac{z_j}{\sin z_j} \left( - \psi(u) + \psi(v)e^{i(\tau(e), \vartheta)} \cos z_j \right), \quad e = (u, v).
\]
Using \((3.23)\), the left-hand side of the condition \((1.18)\) takes the form
\[
\sum_{e \in I_*(v)} (-1)^{\delta(e,v)} e^{-i\delta(e,v)(\tau(e), \vartheta)} \Psi''_{j,e}(\delta(e, v)) = \sum_{e=(v,u)\in E} \Psi''_{j,e}(0) - \sum_{e=(u,v)\in E} e^{-i(\tau(e), \vartheta)} \Psi''_{j,e}(1)
\]
\[
= \sum_{e=(v,u)\in E} \frac{z_j}{\sin z_j} \left( \psi(u)e^{i(\tau(e), \vartheta)} - \psi(v) \cos z_j \right) + \sum_{e=(u,v)\in E} \frac{z_j}{\sin z_j} \left( \psi(u)e^{-i(\tau(e), \vartheta)} - \psi(v) \cos z_j \right)
\]
\[
= -\frac{z_j \cos z_j}{\sin z_j} \varkappa_v \psi(v) + \frac{z_j}{\sin z_j} \sum_{e=(v,u)\in E} \psi(u)e^{i(\tau(e), \vartheta)}, \quad \forall v \in V_*,
\]
This, the definition \((2.8)\) of the operator \(\Delta(\vartheta)\) and the fact that \(\psi(\vartheta, \cdot)\) is an eigenfunction of \(\Delta(\vartheta)\) with the eigenvalue \(\lambda(\vartheta) = -\cos z_j\) yield
\[
\sum_{e \in I_*(v)} (-1)^{\delta(e,v)} e^{-i\delta(e,v)(\tau(e), \vartheta)} \Psi''_{j,e}(\delta(e, v)) = -\frac{z_j \varkappa_v}{\sin z_j} \psi(\vartheta, v) \cos z_j + (\Delta(\vartheta)\psi(\vartheta, \cdot))(v) = 0
\]
for all \(v \in V_*,\) i.e., the condition \((1.18)\) also holds true. Thus, \(\Psi_j(\vartheta)\) is an eigenfunction of the operator \(\Delta_M(\vartheta)\) with the eigenvalue \(z_j^2\). \(\blacksquare\)

**Proof of Theorem 1.8.** For all \(\vartheta \in T^d \setminus \{0\}\) the fiber Laplacian \(\Delta(\vartheta)\) has \(\nu\) real eigenvalues \(-1 < \lambda_1(\vartheta) \leq \ldots \leq \lambda_\nu(\vartheta) < 1\). Due to Proposition 3.2 all eigenvalues \(E_{j,n}(\vartheta) = z_j^2\) of the operator \(\Delta_M(\vartheta)\) are given by \((1.28)\) and the eigenfunction \(\Psi_{j,n}(\vartheta) \) corresponding to the eigenvalue \(E_{j,n}(\vartheta)\) (defined up to a constant factor) has the form \((1.29)\).
Now we will show that $\Psi_{j,n}(\vartheta)$ is a normalized eigenfunction. Let for the shortness \[ c_{j,n} = \cos z_{j,n}(\vartheta), \quad s_{j,n} = \sin z_{j,n}(\vartheta), \quad z_{j,n} = z_{j,n}(\vartheta), \quad \psi_n(u) = \psi_n(\vartheta, u). \]

Direct integration yields
\[
\| \Psi_{j,n}(\vartheta) \|^2_{L^2(G_s)} = \sum_{e \in \mathcal{E}_s} \int_0^1 |\Psi_{j,n,e}(\vartheta, t)|^2 \, dt = \frac{1}{z_{j,n}^2 s_{j,n}^2} \left( (z_{j,n} - c_{j,n}s_{j,n})(|\psi_n(u)|^2 + |\psi_n(v)|^2) + (s_{j,n} - z_{j,n}c_{j,n}) (e^{-i(\tau(e),0)}\psi_n(u)\overline{\psi_n}(v) + e^{i(\tau(e),0)}\overline{\psi_n}(u)\psi_n(v)) \right). \tag{3.24}
\]

The definition (2.3) of the operator $\Delta(\vartheta)$ and the fact that $\psi_n(\vartheta, \cdot)$ is an eigenfunction of $\Delta(\vartheta)$ corresponding to $\lambda_n(\vartheta)$ yield
\[
\sum_{e = (u,v) \in \mathcal{A}_s} (e^{-i(\tau(e),0)}\psi_n(u)\overline{\psi_n}(v) + e^{i(\tau(e),0)}\overline{\psi_n}(u)\psi_n(v))
= \sum_{e = (u,v) \in \mathcal{A}_s} e^{-i(\tau(e),0)}\psi_n(u)\overline{\psi_n}(v) = \sum_{u \in \mathcal{V}_s} \psi_n(u) \sum_{e = (u,v) \in \mathcal{A}_s} e^{-i(\tau(e),0)}\overline{\psi_n}(v)
= -\lambda_n(\vartheta) \sum_{u \in \mathcal{V}_s} \mathcal{X}_n\overline{\psi_n}(u)\overline{\psi_n}(u) = c_{j,n} \|\psi_n(\cdot)\|^2_{L^2(V_s)}. \tag{3.25}
\]
Substituting the identities
\[
\sum_{(u,v) \in \mathcal{E}_s} (|\psi_n(u)|^2 + |\psi_n(v)|^2) = \sum_{v \in \mathcal{V}_s} \mathcal{X}_n|\psi_n(v)|^2 = \|\psi_n(\vartheta, \cdot)\|^2_{L^2(V_s)}
\]
and (3.25) into (3.24), we obtain

\[
\| \Psi_{j,n}(\vartheta) \|^2_{L^2(G_s)} = \frac{(z_{j,n} - c_{j,n}s_{j,n}) + (s_{j,n} - z_{j,n}c_{j,n})c_{j,n}}{z_{j,n}^2 s_{j,n}^2} \|\psi_n(\vartheta, \cdot)\|^2_{L^2(V_s)} = \|\psi_n(\vartheta, \cdot)\|^2_{L^2(V_s)} = 1.
\]

The identity (1.30) follows from Propositions 3.3 and 3.4. \[ \square \]

The first eigenvalue $\lambda_1(\vartheta)$ and the corresponding normalized eigenfunction $\psi_1(\vartheta, \cdot)$ of the fiber Laplacian $\Delta(\vartheta)$ have asymptotics as $\vartheta = \varepsilon \omega, \varepsilon = |\vartheta| \to 0, \omega \in S^{d-1}$:
\[
\lambda_1(\vartheta) = \lambda_1(0) + \varepsilon^2 \mu(\omega) + O(\varepsilon^3), \quad \lambda_1(0) = -1, \quad \mu(\omega) = \frac{1}{2} \lambda_1(\varepsilon \omega)|_{\varepsilon = 0},
\]
\[
\psi_1(\vartheta, \cdot) = \psi_1(0, \cdot) + \varepsilon \psi_1^{(1)} + O(\varepsilon^2), \quad \psi_1(0, v) = \varepsilon^{-1/2} \mathcal{X}, \quad \mathcal{X} = \sum_{v \in \mathcal{V}_s} \mathcal{X}_v,
\tag{3.26}
\]
where $\mathcal{X} = \frac{\partial \omega}{\partial \mathcal{V}}$ and $S^d$ is the $d$-dimensional sphere.

**Proposition 3.3.** i) Each eigenfunction \(\Psi_{j,1}(\vartheta) = \left(\Psi_{j,1,e}(\vartheta, t)\right)_{e \in \mathcal{E}_s}, j \in \mathbb{Z}_0,\) of the operator \(\Delta_M(\vartheta), \vartheta \in \mathbb{T}^d \setminus \{0\},\) defined by \(\mathcal{L}_{\mathcal{A}}\), satisfies as \(e = (u, v)\) and \(\varepsilon = |\vartheta| \to 0\)
\[
\Psi_{j,1,e}(\vartheta, t) = \left(-1\right)^j \frac{1}{\sqrt{2\pi}} \cos(\pi j, t) - \frac{\sin(\pi j, t)}{\sqrt{\mu(\omega)}} \left(\psi_1^{(1)}(\omega, u) - \psi_1^{(1)}(\omega, v) - i\frac{\tau(e), \omega}{\sqrt{\mu(\omega)}}\right) + O(\varepsilon),
\tag{3.27}
\]
where \( j_* = j \), if \( j \) is even, and \( j_* = j + 1 \) if \( j \) is odd.

ii) Let \( \vartheta \in \mathbb{T}^d \setminus \{0\} \) and \( z_1(\vartheta) \leq \frac{\pi}{2} \). Then all eigenfunctions \( \Psi_{j,1}(\vartheta), j \in \mathbb{Z}_0, \) of \( \Delta_M(\vartheta) \) satisfy

\[
\| \Psi_{j,1}(\vartheta) \|_{L^\infty(G_\vartheta)} < \sqrt{2} \left( 2 + \frac{\pi}{\lambda} + M \frac{|\vartheta|}{\sin z_1(\vartheta)} \right),
\]

where

\[
M = \max_{e \in A_*} |\tau(e)| + \frac{2}{\lambda} \max_{\omega \in \mathbb{V}_*} \sum_{e=(u,v) \in A} |\tau(e)|,
\]

\( \Lambda \) is the distance between \( \lambda_1(0) = -1 \) and the set \( \sigma(\Delta(0)) \setminus \{ \lambda_1(0) \} \).

iii) Let \( \vartheta \in \mathbb{T}^d \setminus \{0\} \) and \( z_1(\vartheta) > \frac{\pi}{2} \). Then all eigenfunctions \( \Psi_{j,1}(\vartheta), j \in \mathbb{Z}_0, \) of \( \Delta_M(\vartheta) \) satisfy

\[
\| \Psi_{j,1}(\vartheta) \|_{L^\infty(G_\vartheta)} < \frac{2\sqrt{2}}{\sin z_1^+} < \infty,
\]

where \( z_1^+ \) is defined in (1.31).

iv) There exists a constant \( C_0 > 0 \) such that

\[
\sup_{\vartheta \in \mathbb{T}^d \setminus \{0\}} \frac{|\vartheta|}{\sin z_1(\vartheta)} = C_0 < \infty.
\]

**Proof.** Let for the shortness

\[
z_{j,1} = z_{j,1}(\vartheta), \quad z_1 = z_1(\vartheta), \quad \psi_1(u) = \psi_1(\vartheta, u).
\]

i) For the eigenfunction \( \Psi_{j,1}(\vartheta) = (\Psi_{j,1,e}(\vartheta, t))_{e \in A_*} \), defined by (1.29), we have

\[
\Psi_{j,1,e}(\vartheta, t) = \frac{\sqrt{2}}{\sin z_1} \left[ \psi_1(u) \left( \sin z_{j,1} \cos(z_{j,1} t) - \cos z_{j,1} \sin(z_{j,1} t) \right) \right.
\]

\[+ \psi_1(v) \epsilon^{i(\tau(e), \vartheta)} \sin(z_{j,1} t) \left] = (-1)^j \sqrt{2} \psi_1(u) \cos(z_{j,1} t) - \sqrt{2} \sin(z_{j,1} t) A(\vartheta), \right. \]

where

\[
A(\vartheta) = \frac{1}{\sin z_1} \left( \psi_1(u) \cos z_1 - \psi_1(v) \epsilon^{i(\tau(e), \vartheta)} \right).
\]

Using (1.28) and (3.26), we have the following asymptotics

\[
\cos z_1 = -\lambda_1(\vartheta) = 1 - \epsilon^2 \mu(\omega) + O(\epsilon^3),
\]

\[
\sin z_1 = (1 - \lambda_1^2(\vartheta))^{1/2} = (2\epsilon^2 \mu(\omega) + O(\epsilon^3))^{1/2} = \epsilon \sqrt{2\mu(\omega)} (1 + O(\epsilon)).
\]

Substituting (3.26) and (3.33) into (3.32), we obtain

\[
A(\vartheta) = \frac{(1 + O(\epsilon))}{\epsilon \sqrt{2\mu(\omega)}} \left[ \psi_1(0,u) + \epsilon \psi_1^{(1)}(\omega,u) - \psi_1(0,v)(1 + \epsilon i(\tau(e), \omega)) \right.
\]

\[- \epsilon \psi_1(\omega,v) + O(\epsilon^2) \right] = \frac{1}{\sqrt{2\mu(\omega)}} \left( \psi_1^{(1)}(\omega,u) - \psi_1^{(1)}(\omega,v) - \frac{i}{\sqrt{2}} \langle \tau(e), \omega \rangle \right) + O(\epsilon).
\]

Substituting (3.35) into (3.32) and using (3.26) and

\[
\sin(z_{j,1} t) = \sin(\pi j_* t) + O(\epsilon), \quad \cos(z_{j,1} t) = \cos(\pi j_* t) + O(\epsilon),
\]

we get (3.27).
ii) The identity \( (3.32) \) gives
\[
|\Psi_{3,1,e}(\vartheta, t)| < \sqrt{2}(1 + |A(\vartheta)|). \tag{3.36}
\]
We have the decomposition
\[
\psi_1(\vartheta, v) = \psi_1(0, v) + \tilde{\psi}_1(\vartheta, v) = \varpi^{-1/2} + \tilde{\psi}_1(\vartheta, v).
\]
We rewrite the difference on the right-hand side of \( (3.33) \) in the form
\[
\psi_1(u) \cos z_1 - \psi_1(v) e^{i(\vartheta, \varrho)} = \psi_1(u) - \psi_1(v) + \psi_1(u)(\cos z_1 - 1) - \psi_1(v)(e^{i(\vartheta, \varrho)} - 1)
\]
\[
= \tilde{\psi}_1(u, v) - \tilde{\psi}_1(\vartheta, v) - 2\psi_1(u) \sin^2 \frac{z_1}{2} - \psi_1(v)(e^{i(\vartheta, \varrho)} - 1),
\]
which yields
\[
|\psi_1(u) \cos z_1 - \psi_1(v) e^{i(\vartheta, \varrho)}| < |\tilde{\psi}_1(\vartheta, u)| + |\tilde{\psi}_1(\vartheta, v)| + 2 \sin^2 \frac{z_1}{2} + |\langle \vartheta, \varrho \rangle| \tag{3.37}
\]
Now we estimate \( \tilde{\psi}_1(\vartheta, \cdot) \). Let
\[
\Delta(\vartheta) = \Delta(0) + \tilde{\Delta}(\vartheta), \quad \lambda_1(\vartheta) = \lambda_1(0) + \tilde{\lambda}_1(\vartheta).
\]
Then we can rewrite the equation \( \Delta(\vartheta) \psi_1(\vartheta, \cdot) = \lambda_1(\vartheta) \psi_1(\vartheta, \cdot) \) in the form
\[
(\Delta(0) + \tilde{\Delta}(\vartheta)) \left( \psi_1(0, \cdot) + \tilde{\psi}_1(\vartheta, \cdot) \right) = (\lambda_1(0) + \tilde{\lambda}_1(\vartheta)) \left( \psi_1(0, \cdot) + \tilde{\psi}_1(\vartheta, \cdot) \right) \tag{3.38}
\]
or, since \( \Delta(0) \psi_1(0, \cdot) = \lambda_1(0) \psi_1(0, \cdot) \),
\[
(\Delta(0) - \lambda_1(0) \mathbb{I}) \tilde{\psi}_1(\vartheta, \cdot) = -(\tilde{\Delta}(\vartheta) - \tilde{\lambda}_1(\vartheta) \mathbb{I}) \psi_1(\vartheta, \cdot), \tag{3.39}
\]
where \( \mathbb{I} \) is the identity operator. Let \( P \) be the orthogonal projection onto the subspace of \( \ell^2(V_\vartheta) \) orthogonal to \( \psi_1(0, \cdot) \). Then from \( (3.39) \) we obtain
\[
\tilde{\psi}_1(\vartheta, \cdot) = -D^{-1}P(\tilde{\Delta}(\vartheta) - \tilde{\lambda}_1(\vartheta) \mathbb{I}) \psi_1(\vartheta, \cdot), \quad \text{where} \quad D = P(\Delta(0) - \lambda_1(0) \mathbb{I}). \tag{3.40}
\]
This yields
\[
\|\tilde{\psi}_1(\vartheta, \cdot)\| \leq \|D^{-1}P\| \cdot \|(\tilde{\Delta}(\vartheta) - \tilde{\lambda}_1(\vartheta) \mathbb{I}) \psi_1(\vartheta, \cdot)\| \leq \frac{1}{\Lambda} \|\tilde{\Delta}(\vartheta) - \tilde{\lambda}_1(\vartheta) \mathbb{I}\|, \tag{3.41}
\]
where \( \Lambda \) is the distance between \( \lambda_1(0) = -1 \) and \( \sigma(\Delta(0)) \setminus \{\lambda_1(0)\} \). Due to \( (2.3) \), we have
\[
\|\tilde{\Delta}(\vartheta) - \tilde{\lambda}_1(\vartheta) \mathbb{I}\| \leq \tilde{\lambda}_1(\vartheta) + \max_{u \in V_\vartheta} \frac{1}{x_u} \sum_{e \in (u,v) \in A_\vartheta} |\langle \vartheta, \varrho \rangle| \leq \tilde{\lambda}_1(\vartheta) + |\vartheta| T, \tag{3.42}
\]
where \( T = \max_{u \in V_\vartheta} \frac{1}{x_u} \sum_{e \in (u,v) \in A_\vartheta} |\tau(\varrho)|. \) Substituting \( (3.42) \) into \( (3.41) \), we obtain
\[
\|\tilde{\psi}_1(\vartheta, \cdot)\| \leq \frac{1}{\Lambda} \left( \tilde{\lambda}_1(\vartheta) + |\vartheta| T \right). \tag{3.43}
\]
Using \( (3.37), (3.43) \), for \( A(\vartheta) \) defined by \( (3.33) \), we have
\[
|A(\vartheta)| \leq \frac{1}{\sin z_1} \left( \frac{2}{\Lambda} \left( \tilde{\lambda}_1(\vartheta) + |\vartheta| T \right) + 2 \sin^2 \frac{z_1}{2} + |\langle \vartheta, \varrho \rangle| \right)
\]
\[
\leq \tan \frac{z_1}{2} + \frac{2}{\Lambda} \frac{\tilde{\lambda}_1(\vartheta)}{\sin z_1} + \frac{|\vartheta|}{\sin z_1} \left( \frac{2}{\Lambda} T + \max_{e \in A_\vartheta} |\tau(\varrho)| \right). \tag{3.44}
\]
Let $z_1 \in (0, \frac{\pi}{2}]$. Then $\sin z_1 \geq \frac{2}{\pi} z_1$. This and (3.44) give

$$|A(\vartheta)| \leq 1 + \frac{\pi}{\Lambda} \cdot \tilde{\lambda}_1(\vartheta) + M \frac{|\vartheta|}{\sin z_1(\vartheta)},$$

(3.45)

where $M$ is defined by (3.29). We have the simple inequality

$$\tilde{\lambda}_1(\vartheta) = \lambda_1(\vartheta) - \lambda_1(0) = -\cos z_1(\vartheta) + 1 = \int_0^{z_1(\vartheta)} \sin t \, dt \leq z_1(\vartheta),$$

which yields that $\frac{\tilde{\lambda}_1(\vartheta)}{z_1(\vartheta)} \leq 1$. Then the estimate (3.45) has the form

$$|A(\vartheta)| \leq 1 + \frac{\pi}{\Lambda} \cdot \tilde{\lambda}_1(\vartheta) + M \frac{|\vartheta|}{\sin z_1(\vartheta)}.$$

(3.46)

Combining (3.36) and (3.46), we obtain (3.28).

$iii$ We have $\frac{\pi}{2} < z_1(\vartheta) \leq z_1^+ < \pi$. This yields $0 < \sin z_1^+ \leq \sin z_1(\vartheta)$. We note that $z_1^+ \neq \pi$, since we assume that for a bipartite periodic graph $G$ its fundamental graph $G^*$ is also bipartite and, consequently, has more than 1 vertex. Then the estimate (3.30) follows directly from the identity (1.29).

$iv$ Using the second formula in (3.34), we obtain

$$\frac{|\vartheta|}{\sin z_1(\vartheta)} = \frac{1}{\sqrt{2\mu(\omega)}} + O(|\vartheta|) \quad \text{as} \quad |\vartheta| \to 0.$$  

(3.47)

The following estimate holds true (see Theorem 1.2 in [KS16b]):

$$\mu(\omega) \geq T_1 = \frac{1}{\kappa \nu d} \left( \frac{d - 1}{C} \right)^{d-1} > 0, \quad \forall \omega \in S^{d-1},$$

(3.48)

where $e_1, \ldots, e_d \in A_*$ are fundamental graph edges with linearly independent indices $\tau(e_1), \ldots, \tau(e_d) \in \mathbb{Z}^d$. Substituting the estimate (3.48) into (3.47), we get

$$\frac{|\vartheta|}{\sin z_1(\vartheta)} \leq \frac{1}{\sqrt{2T_1}} + O(|\vartheta|).$$

Then there exists a constant $C_1 > 0$ such that

$$\frac{|\vartheta|}{\sin z_1(\vartheta)} < \frac{1}{\sqrt{2T_1}} + C_1, \quad \text{for all} \quad |\vartheta| < R,$$

(3.49)

where $R$ is some positive number. Now let $|\vartheta| \geq R$. Then $\sin z_1(\vartheta) \geq \alpha$ for some constant $\alpha > 0$ and

$$\frac{|\vartheta|}{\sin z_1(\vartheta)} \leq \frac{\pi \sqrt{d}}{\alpha} < \infty.$$  

(3.50)

Combining (3.49) and (3.50), we obtain (3.31).  

Proposition 3.4. i) All eigenfunctions $\Psi_{j,n}(\vartheta), (j, n) \in \mathbb{Z}_0 \times (\mathbb{T}^d \setminus \{0\}), n = 2, \ldots, \nu - 1,$ of $\Delta_M(\vartheta)$ satisfy

$$\|\Psi_{j,n}(\vartheta)\|_{L^\infty(G_\vartheta)} < \frac{2\sqrt{2}}{\alpha}, \quad \alpha = \inf_{\vartheta \in T^d} \sin z_n(\vartheta) > 0.$$  \hfill (3.51)

ii) Let $\vartheta \in \mathbb{T}^d \setminus \{0\}$ and let $G_\ast$ be non-bipartite. Then all eigenfunctions $\Psi_{j,\nu}(\vartheta), j \in \mathbb{Z}_0,$ of $\Delta_M(\vartheta)$ satisfy

$$\|\Psi_{j,\nu}(\vartheta)\|_{L^\infty(G_\vartheta)} < \frac{2\sqrt{2}}{\alpha_\nu}, \quad \alpha_\nu = \inf_{\vartheta \in \mathbb{T}^d} \sin z_\nu(\vartheta) > 0.$$  \hfill (3.52)

iii) Let $\vartheta \in \mathbb{T}^d \setminus \{0\}$ and let $G_\ast$ be bipartite. Then all eigenfunctions $\Psi_{j,\nu}(\vartheta), j \in \mathbb{Z}_0,$ of $\Delta_M(\vartheta)$ satisfy

$$\|\Psi_{j,\nu}(\vartheta)\|_{L^\infty(G_\vartheta)} < \sqrt{2} \left( 2 + \frac{\pi}{\Lambda} + M \frac{|\vartheta|}{\sin z_1(\vartheta)} \right),$$  \hfill (3.53)

where $M$ and $\Lambda$ are defined in Proposition 3.3 ii.

**Proof.** i) This estimate follows from the identity (1.29).

ii) Since $G$ is non-bipartite, using (1.29), we also have (3.52).

iii) Let for the shortness

$z_{j,\nu} = z_{j,\nu}(\vartheta), \quad z_\nu = z_\nu(\vartheta), \quad \psi_\nu(u) = \psi_\nu(\vartheta, u).$

For the eigenfunction $\Psi_{j,\nu}(\vartheta) = \left(\Psi_{j,\nu,\psi}(\vartheta, t)\right)_{e \in E_\ast}$ defined by (1.29), we have

$$\Psi_{j,\nu,\psi}(\vartheta, t) = (-1)^j \sqrt{2} \psi_\nu(u) \cos(z_{j,\nu}t) - \sqrt{2} \sin(z_{j,\nu}t) A_\nu(\vartheta),$$  \hfill (3.54)

where

$$A_\nu(\vartheta) = \frac{1}{\sin z_\nu} \left( \psi_\nu(u) \cos z_\nu - \psi_\nu(v) e^{i(\pi(e), \vartheta)} \right).$$

For the bipartite graph $G_\ast$ with parts $V_1$ and $V_2$ we have

$$\sin z_\nu = \sin z_1, \quad \cos z_\nu = - \cos z_1, \quad \psi_\nu(v) = \begin{cases} \psi_1(v), & \text{if } v \in V_1 \\ -\psi_1(v), & \text{if } v \in V_2. \end{cases}$$

This yields that $A_\nu(\vartheta) = A(\vartheta)$, where $A(\vartheta)$ is defined by (3.53). Combining this, (3.54) and (3.40), we obtain (3.53). \hfill ■

**Remark 3.5.** The estimates (3.23), (3.29), (3.31) - (3.53) give that the eigenfunctions $\Psi_{j,n}(\vartheta), (j, n, \vartheta) \in \mathbb{Z}_0 \times \mathbb{N}_\nu \times \mathbb{T}^d \setminus \{0\},$ are uniformly bounded with respect to $j, n, \vartheta.$

**Proof of Corollary 1.10.** i) Recall that $\Delta_M = \Delta_{MD} \oplus \Delta_{MV}$ and Theorem 1.6 gives

$$\sigma(\Delta_{MD}) = \sigma_{fb}(\Delta_{MD}) = \sigma_D.$$  \hfill (3.55)

Thus, we need to consider the operator $\Delta_{MV}$. Since each $\lambda_n(\cdot)$ is a real and piecewise analytic function on the torus $\mathbb{T}^d,$ each $z_{j,n}(\cdot), (j, n) \in \mathbb{Z}_0 \times \mathbb{N}_\nu$, defined by the formula (1.28), is also a real and piecewise analytic function on $\mathbb{T}^d$. Then (1.28) yields

$$\sigma(\Delta_{MV}) = \bigcup_{(j,n) \in \mathbb{Z}_0 \times \mathbb{N}_\nu} \sigma_{j,n}(\Delta_M), \quad \sigma_{j,n}(\Delta_M) = [E_{j,n}^-, E_{j,n}^+] = z_{j,n}^2(\mathbb{T}^d).$$  \hfill (3.56)

Since $\lambda_n(\cdot) = \text{const},$ for $\nu - r < n \leq \nu$ and all other eigenvalues $\lambda_1(\cdot), \ldots, \lambda_{\nu-r}(\cdot)$ are not constants (in any small ball), we obtain (1.31), (1.32).
Proposition 3.6. Let \( \psi_n(\vartheta, \cdot) \in \ell^2(\mathcal{V}_n) \), \( \vartheta \in \mathbb{T}^d \setminus \{0\} \), be the family of the normalized eigenfunctions of the fiber operators \( \Delta(\vartheta) \) corresponding to the eigenvalue \( \lambda_n \in \sigma_{fb}(\Delta) \) for some \( n = \nu - r + 1, \ldots, \nu \) and let \( \Psi_{j,n}(\vartheta, \cdot) \in \ell^2(\mathcal{G}_n) \) be the family of the normalized eigenfunctions
of the fiber operators $\Delta_M(\vartheta)$ corresponding to the eigenvalue $z_{j,n}^2 \in \sigma_{fb}(\Delta_M)$ defined by (1.28), $j = 0, 1, 2, \ldots$. Then the following statements hold true.

i) The eigenfunction $\tilde{\psi}_n(\cdot) \in \ell^2(V)$ of the discrete Laplacian $\Delta$ corresponding to the eigenvalue $\lambda_n$ has the form

$$
\tilde{\psi}_n(v + a(m)) = \int_{\mathbb{T}^d} e^{i(m,\vartheta)} \psi_n(\vartheta, v) \frac{d\vartheta}{(2\pi)^d}, \quad \forall (v, m) \in V_x \times \mathbb{Z}^d,
$$

(3.59)

where $a(m) \in \Gamma$ is defined by (2.2).

ii) The eigenfunction $\tilde{\Psi}_{j,n}(\cdot) \in L^2(G)$ of the metric Laplacian $\Delta_M$ corresponding to the eigenvalue $z_{j,n}^2$ has the form

$$
\tilde{\Psi}_{j,n}(x + a(m)) = \int_{\mathbb{T}^d} e^{i(m,\vartheta)} \Psi_{j,n}(\vartheta, x) \frac{d\vartheta}{(2\pi)^d}, \quad \forall (x, m) \in G_x \times \mathbb{Z}^d.
$$

(3.60)

Remark 3.7.

1) The right-hand sides of (3.59) and (3.60) are the inverse operators of the unitary operators $(U)$ and $V$ from (1.8) and (1.15), respectively.

2) The eigenfunctions defined by (3.59) and (3.60) are compactly supported.

4. SCHRODINGER OPERATORS ON PERIODIC METRIC GRAPHS

4.1. Trace class estimates. We consider the Schrödinger operator $H$ on the periodic graph $G$ given by

$$
H = H_0 + Q, \quad H_0 = \Delta_M.
$$

We assume that the potential $Q = (Q_e)_{e \in E}$ is real and belongs to the space $L^1(G) \cap L^2(G)$. Recall that $R_0(k) = (H_0 - k^2)^{-1}$, $R(k) = (H - k^2)^{-1}$ and

$$
Y_0(k) = |Q|^{1/2} R_0(k) Q^{1/2}, \quad Y(k) = |Q|^{1/2} R(k) Q^{1/2}, \quad Q^{1/2} = |Q|^{1/2} \text{sign } Q, \quad k \in \mathbb{C}_+.
$$

(4.1)

Let $C_g := \sup_{(j,n) \in G_x \times \mathbb{N}_0 \times \mathbb{T}^d} \|\Psi_{j,n}(\vartheta)\|_{L^\infty(G_1)} < \infty$.

(4.2)

Let $\mathcal{R}_k(x,x',\vartheta), x, x' \in G_x$, be the kernel of the integral operator $(\Delta_{MV}(\vartheta) - k^2)^{-1}$ on $L^2(G_x)$. Due to Theorem 1.8 we have

$$
\mathcal{R}_k(x,x',\vartheta) = \sum_{j=0}^{\infty} \sum_{n=1}^{\nu} \frac{\Psi_{j,n}(\vartheta, x) \bar{\Psi}_{j,n}(\vartheta, x')}{z_{j,n}^2 - k^2}, \quad x, x' \in G_x.
$$

(4.3)

Theorem 4.1.

i) Let $(Q,k) \in L^1(G) \times \mathbb{C}_+$ and let $\Delta_M^{1/2} \geq 0$. Then $(\Delta_M^{1/2} - k)^{-1}|Q|^{1/2} \in B_2$ and $Y_0(k) \in B_1$ and they satisfy

$$
\|\left(\Delta_M^{1/2} - k\right)^{-1}|Q|^{1/2}\|_{B_2} \leq C_0 \|Q\|_{L^1(G)} \left(\frac{2}{|\text{Im } k|^2} + \frac{1}{|\text{Im } k|}\right),
$$

(4.4)

$$
\|Y_0(k)\|_{B_1} \leq C(Q,k),
$$

where $C(Q,k) = C_0 \|Q\|_{L^1(G)} \left(\frac{2}{|\text{Im } k|^2} + \frac{1}{|\text{Im } k|}\right)^{1/2} \left(\frac{2}{|k|^2} + \frac{1}{|k|}\right)^{1/2}$,

(4.5)

$$
C_0 = 2(\nu C_0^2 + \nu_1 - \nu), \quad \nu = \#\mathcal{V}_s, \quad \nu_1 = \#\mathcal{E}_s,
$$

$$
\|Y_0(k)\|_{B_1} = \frac{O(1)}{|k| \text{Im } k|^2} \quad \text{as } \text{Im } k \to \infty.
$$

(4.6)
Moreover, if \( \text{Re} \, k \leq 0 \), then
\[
\| (\Delta_{M^2}^{1/2} - k)^{-1} |Q|^{1/2} \|_{B_2}^2 \leq C_0\|Q\|_{L^1(G)} \left( \frac{2}{|k|^2 + 1} \right).
\] (4.7)

ii) Let, in addition,
\[
\sup_{x \in \mathcal{G}_*} |Q(x + a)| = \gamma_a, \quad a \in \Gamma, \quad \sum_{a \in \Gamma} \gamma_a = \gamma < \infty.
\] (4.8)

Then
\[
\| (\Delta_{M^2}^{1/2} - k)^{-1} |Q|^{1/2} \|_{B_2}^2 \leq \gamma (\nu_1 + \nu) \left( \frac{2}{|k|^2 + 1} \right).
\] (4.9)

Moreover, if \( \text{Re} \, k \leq 0 \), then
\[
\| (\Delta_{M^2}^{1/2} - k)^{-1} |Q|^{1/2} \|_{B_2}^2 \leq \gamma (\nu_1 + \nu) \left( \frac{2}{|k|^2 + 1} \right).
\] (4.10)

**Proof.** i) Due to Theorem 1.4, we have
\[
\left( \mathcal{U} (\Delta_{M^2}^{1/2} - k)^{-1} h \right) (\vartheta, \cdot) = \left( (\Delta_{M^2}^{1/2} - k)^{-1} \right) \left( \mathcal{U} h \right) (\vartheta, \cdot),
\] (4.11)
where the unitary operator \( \mathcal{U} : L^2(\mathcal{G}) \to \mathcal{H} \) is defined by (1.15), \( h \in L^2(\mathcal{G}) \) is compactly supported and \( \vartheta \in \mathbb{T}^d \). We consider the second term on the right-hand side of (4.11). Using Theorem 1.8 and denoting \( \alpha = (j, n) \in \mathbb{Z}_0 \times \mathbb{N}_\nu \), \( \mathbb{Z}_0 = \{0, 1, 2, \ldots\} \), we obtain
\[
(\Delta_{M^2}^{1/2} (\vartheta) - k)^{-1} (\mathcal{U} h) (\vartheta, x) = \sum_{\alpha \in \mathbb{Z}_0 \times \mathbb{N}_\nu} \frac{1}{z_\alpha (\vartheta) - k} \mathcal{P}_\alpha (\vartheta) \mathcal{U} h (\vartheta, x)
\]
\[
= \sum_{\alpha \in \mathbb{Z}_0 \times \mathbb{N}_\nu} \frac{1}{z_\alpha (\vartheta) - k} \Psi_\alpha (\vartheta, x) \int_{\mathcal{G}_*} (\mathcal{U} h) (\vartheta, x') \overline{\Psi}_\alpha (\vartheta, x') \, dx'.
\]

Using \( Q \in L^1(\mathcal{G}) \), the normalized condition \( \int_{\mathbb{G}_*} |\Psi_\alpha (\vartheta, x)|^2 \, dx = 1 \) and (4.12), this gives
\[
\| (\Delta_{M^2}^{1/2} (\cdot) - k)^{-1} \mathcal{U} |Q|^{1/2} \|_{B_2}^2
\]
\[
= \frac{1}{2\pi^d} \sum_{\alpha \in \mathbb{Z}_0 \times \mathbb{N}_\nu} \sum_{\vartheta \in \mathbb{T}^d} \int_{\mathbb{G}_*} \frac{1}{z_\alpha (\vartheta) - k} \left| \Psi_\alpha (\vartheta, x) \right|^2 \, dx \, d\vartheta
\]
\[
\leq \frac{C_0^2}{2\pi^d} \|Q\|_{L^1(\mathcal{G})} \sum_{\alpha \in \mathbb{Z}_0 \times \mathbb{N}_\nu} \int_{\mathbb{T}^d} \frac{d\vartheta}{|z_\alpha (\vartheta) - k|^2}.
\] (4.12)

Let \( k = p + iq \in \mathbb{C}, \, q \neq 0 \). We assume that the following inequality holds true
\[
\sum_{j=0}^{\infty} \frac{1}{|z_{j,n}(\vartheta) - k|^2} \leq \frac{4}{q^2 + 2/|q|} \quad \text{for each} \quad n \in \mathbb{N}_\nu.
\] (4.13)

Substituting (4.13) into (4.12), we obtain
\[
\| (\Delta_{M^2}^{1/2} (\cdot) - k)^{-1} \mathcal{U} |Q|^{1/2} \|_{B_2}^2 \leq 2\nu C_0^2 \|Q\|_{L^1(\mathcal{G})} \left( \frac{2}{q^2 + 1/|q|} \right).
\] (4.14)
Now we prove (4.13). Due to (1.28), we have the following identity
\[
\sum_{j=0}^{\infty} \frac{1}{|z_{j,n}(\vartheta)| - k^2} = \sum_{j=0}^{\infty} \frac{1}{(z_n(\vartheta) + \pi j - p)^2 + q^2} + \sum_{j=1}^{\infty} \frac{1}{(\pi j + 1 - z_n(\vartheta) - p)^2 + q^2}.
\]

Let
\[ p - z_n(\vartheta) = \pi(m + b), \quad p + z_n(\vartheta) = \pi(m_1 + b_1) \]
for some \( m, m_1 \in \mathbb{Z} \), and \( b = b(\vartheta), b_1 = b_1(\vartheta) \in (0, 1) \). Then
\[
\sum_{j=0}^{\infty} \frac{1}{|z_{j,n}(\vartheta)| - k^2} \leq \sum_{j=-\infty}^{\infty} \frac{1}{\pi^2(j - b)^2 + q^2} + \sum_{j=-\infty}^{\infty} \frac{1}{\pi^2(j - b_1)^2 + q^2} \leq \frac{4}{q^2} + \sum_{j=1}^{\infty} \frac{4}{\pi^2 j^2 + q^2} \leq \frac{4}{q^2} + 4 \int_{0}^{\infty} \frac{dt}{\pi^2 t^2 + q^2} = \frac{4}{q^2} + \frac{2}{|q|}.
\]

Now we consider the first term on the right-hand side of (4.11). Similar arguments yield
\[
\| (\Delta_{MV}^{1/2}(-) - k)^{-1} \mathcal{U} |Q|^{1/2} \|^2_{B_2} \leq 2(\nu_1 - \nu) \| Q \|_{L^1(\mathcal{G})} \left( \frac{2}{q^2} + \frac{1}{|q|} \right). \tag{4.15}
\]

Summing (4.14) and (4.15) and using (4.11), we get
\[
\| (\Delta_{M}^{1/2} - k)^{-1} |Q|^{1/2} \|^2_{B_2} = \| \mathcal{U} (\Delta_{M}^{1/2} - k)^{-1} |Q|^{1/2} \|^2_{B_2}
\leq 2(\nu C_{\mathcal{G}}^2 + \nu_1 - \nu) \| Q \|_{L^1(\mathcal{G})} \left( \frac{2}{q^2} + \frac{1}{|q|} \right) = C_0 \| Q \|_{L^1(\mathcal{G})} \left( \frac{2}{|\text{Im } k|} + \frac{1}{|\text{Im } k|} \right). \tag{4.16}
\]

Thus, (4.4) has been proved.

Let, in addition, \( \text{Re } k \leq 0 \). Then
\[
|z_{\alpha}(\vartheta) - k|^2 = (z_{\alpha}(\vartheta) - p)^2 + q^2 \geq z_{\alpha}^2(\vartheta) + p^2 + q^2 = z_{\alpha}^2(\vartheta) + |k|^2 \tag{4.17}
\]
and using the above arguments we obtain (4.7).

For any \( k \in \mathbb{C}_+ \) we have the following factorization
\[
Y_0(k) = |Q|^{1/2} R_0(k) |Q|^{1/2} = (|Q|^{1/2} (\Delta_{M}^{1/2} + k)^{-1})( (\Delta_{M}^{1/2} - k)^{-1} |Q|^{1/2} ).
\]

Then
\[
\| Y_0(k) \|_{B_1} \leq \| |Q|^{1/2} (\Delta_{M}^{1/2} + k)^{-1} \|_{B_2} \| (\Delta_{M}^{1/2} - k)^{-1} |Q|^{1/2} \|_{B_2},
\]
and, applying the estimates (4.16) and (4.7), we obtain (4.5), which yields asymptotics (4.6).

ii) Let the potential \( Q \) satisfy (4.8). Then using (4.12) and (4.13), we obtain
\[
\| (\Delta_{MV}^{1/2}(-) - k)^{-1} \mathcal{U} |Q|^{1/2} \|^2_{B_2}
\leq \frac{1}{(2\pi)^d} \sum_{a \in \mathbb{Z}_0 \times \mathbb{N}_v} \int_{T^d} |z_{\alpha}(\vartheta) - k|^2 \int_{\mathcal{G}_*} |Q(x + a)| \cdot |\Psi_{\alpha}(\vartheta, x)|^2 \, dx \, d\vartheta
\leq \frac{1}{(2\pi)^d} \sum_{a \in \mathbb{Z}_0 \times \mathbb{N}_v} \gamma_a \sum_{a \in \mathbb{Z}_0 \times \mathbb{N}_v} \int_{T^d} \frac{d\vartheta}{|z_{\alpha}(\vartheta) - k|^2} \leq 2\gamma \nu \left( \frac{2}{|\text{Im } k|} + \frac{1}{|\text{Im } k|} \right). \tag{4.18}
\]
Similar arguments yield
\[
\left\| (\Delta_{MV}^{1/2} - k^2)^{-1} f, g \right\|^2_{L^2(G)} \leq \gamma (\nu_1 - \nu) \left( \frac{2}{|\text{Im} k|^2} + \frac{1}{|\text{Im} k|} \right). 
\]
(4.19)
Summing (4.18) and (4.19) and using (4.11), we get (4.9).

Let, in addition, Re \( k \leq 0 \). Then using (4.11) and the above arguments we obtain (4.10).

\[ \triangledown \]

**Lemma 4.2.** Let \((Q, k) \in L^1(G) \times \mathbb{C}_+\). Then

\[
\text{Tr} \, Y_0(k) = \frac{1}{(2\pi)^{d}} \int_{T^d} \left( \sum_{j=0}^{\infty} \sum_{n=1}^{\nu} \frac{Q_{j,n}(\vartheta)}{z_{j,n}^2(\vartheta) - k^2} + \sum_{j=1}^{\infty} \sum_{s=1}^{\beta-1} \frac{Q_{j,s}^0(\vartheta)}{(\pi j)^2 - k^2} \right) d\vartheta, 
\]
(4.20)
where \( \beta = \#\mathcal{E} - \#\mathcal{V} + 1 \) is the Betti number of the fundamental graph \( \mathcal{G}_* = (\mathcal{V}_*, \mathcal{E}_*), \nu = \#\mathcal{V}_* \), and

\[
Q_{j,n}(\vartheta) = \int_{\mathcal{G}_*} \sum_{a \in \Gamma} Q(x + a) |\Psi_{j,n}(\vartheta, x)|^2 dx, \\
Q_{j,s}^0(\vartheta) = \int_{\mathcal{G}_*} \sum_{a \in \Gamma} Q(x + a) |\Psi_{j,s}^0(\vartheta, x)|^2 dx. 
\]
(4.21)

**Proof.** Due to Theorem 1.8 and (1.3) we have

\[
( (\Delta_{MV} - k^2)^{-1} f, g ) = \int_{\mathcal{G}_*} dx \int_{\mathcal{G}_*} dx' \int_{T^d} d\vartheta \sum_{m, m' \in \mathbb{Z}^d} e^{-i(m - m', \vartheta)} R_k(x, x', \vartheta)f(x + a(m))g(x' + a(m')), 
\]

where \( f, g \in L^2(G) \) are compactly supported functions, and \( a(m) \in \Gamma \) is defined by (2.9). This yields

\[
\text{Tr} \left( |Q|^{1/2}(\Delta_{MV} - k^2)^{-1}Q^{1/2} \right) = \int_{\mathcal{G}_*} dx \int_{T^d} d\vartheta \sum_{a \in \Gamma} R_k(x, x, \vartheta)Q(x + a) \\
= \sum_{j=0}^{\infty} \sum_{n=1}^{\nu} \int_{\mathcal{G}_*} dx \int_{T^d} d\vartheta \sum_{a \in \Gamma} \frac{Q(x + a)|\Psi_{j,n}(\vartheta, x)|^2}{z_{j,n}^2(\vartheta) - k^2} = \frac{1}{(2\pi)^d} \int_{T^d} \sum_{j=0}^{\infty} \sum_{n=1}^{\nu} \frac{Q_{j,n}(\vartheta)}{\pi^2 j^2 - k^2} d\vartheta. 
\]
Similar arguments yield

\[
\text{Tr} \left( |Q|^{1/2}(\Delta_{MD} - k^2)^{-1}Q^{1/2} \right) \\
= \sum_{j=1}^{\infty} \sum_{s=1}^{\beta-1} \int_{\mathcal{G}_*} dx \int_{T^d} d\vartheta \sum_{a \in \Gamma} \frac{Q(x + a)|\Psi_{j,s}^0(\vartheta, x)|^2}{(\pi j)^2 - k^2} = \frac{1}{(2\pi)^d} \int_{T^d} \sum_{j=1}^{\infty} \sum_{s=1}^{\beta-1} \frac{Q_{j,s}^0(\vartheta)}{(\pi j)^2 - k^2} d\vartheta. 
\]
Substituting these formulas into the identity

\[
\text{Tr} \, Y_0(k) = \text{Tr} \left( |Q|^{1/2}(\Delta_{MV} - k^2)^{-1}Q^{1/2} \right) + \text{Tr} \left( |Q|^{1/2}(\Delta_{MD} - k^2)^{-1}Q^{1/2} \right), 
\]
we obtain (4.20).
4.2. Fredholm determinants. In order to discuss Fredholm determinants we recall well-known facts \[\text{[S05]}\]. Let \(\mathcal{H}\) be a Hilbert space endowed with an inner product \((\cdot, \cdot)\) and a norm \(\|\cdot\|\). Let \(\mathcal{B}\) denote the class of bounded operators. Let \(\mathcal{B}_1\) be the set of all trace class operators on \(\mathcal{H}\) equipped with the trace norm \(\|\cdot\|_{\mathcal{B}_1}\).

- Let \(A, B \in \mathcal{B}\) and \(AB, BA \in \mathcal{B}_1\). Then
  \[
  \text{Tr} \, AB = \text{Tr} \, BA, \quad (4.22)
  \]
  \[
  \det(I + AB) = \det(I + BA). \quad (4.23)
  \]
- Let \(A, B \in \mathcal{B}_1\). Then
  \[
  |\det(I + A)| \leq e^{\|A\|_{\mathcal{B}_1}}, \quad (4.24)
  \]
  \[
  |\det(I + A) - \det(I + B)| \leq \|A - B\|_{\mathcal{B}_1} e^{1+\|A\|_{\mathcal{B}_1}+\|B\|_{\mathcal{B}_1}}. \quad (4.25)
  \]

Moreover, \(I + A\) is invertible if and only if \(\det(I + A) \neq 0\).
- Suppose for a domain \(\mathcal{D} \subset \mathbb{C}\), a function \(\Omega(\cdot) - I : \mathcal{D} \to \mathcal{B}_1\) is analytic and is invertible for any \(z \in \mathcal{D}\). Then \(F(z) = \det \Omega(z)\) is analytic and satisfies
  \[
  F'(z) = F(z) \text{Tr} \Omega^{-1}(z) \Omega'(z). \quad (4.26)
  \]

By Theorem 4.1, each operator \(Y_0(k) \in \mathcal{B}_1, k \in \mathbb{C}_+\), and we can define the determinant
  \[
  D(k) = \det(I + Y_0(k)), \quad k \in \mathbb{C}_+. \quad (4.27)
  \]

**Lemma 4.3.** Let \(Q \in L^1(\mathcal{G})\). Then the determinant \(D(k), k \in \mathbb{C}_+, \) is analytic in \(k \in \mathbb{C}_+.\) If, in addition, \(C(Q,k) < 1,\) where the constant \(C(Q,k)\) is defined by \(4.5\), then
  \[
  \log D(k) = -\sum_{n=1}^{\infty} \frac{1}{n} \text{Tr} \left(-Y_0(k)\right)^n, \quad (4.28)
  \]

where the series converges absolutely and uniformly, and for any \(N \geq 0\) we have
  \[
  \left| \log D(k) + \sum_{n=1}^{N} \frac{\text{Tr}(-Y_0(k))^n}{n} \right| \leq \frac{\|Y_0(k)\|_{\mathcal{B}_1}^{N+1}}{(N+1)(1 - C(Q,k))}. \quad (4.29)
  \]

**Proof.** By Theorem 1.1 the \(\mathcal{B}_1\)-valued function \(Y_0(k)\) is analytic in \(\mathbb{C}_+.\) The series \(4.28\) is standard, see \[\text{[S05]}\]. Under the condition \(\|Y_0(k)\|_{\mathcal{B}_1} \leq C(Q,k)\) the identity \(4.28\) gives \(4.29\).

**Proof of Theorem 1.14.** Due to \(4.4, 4.5\) we have \(4.10\). Then it is well known that there exist the wave operators \(W_\pm\) and they are complete, see \[\text{[RS79]}\]. Moreover, the S-operator given by \(4.32\) is unitary on \(\mathcal{H}_{ac}(H_0)\) and the corresponding S-matrix \(S(k)\) for almost all \(k^2 \in \sigma_{ac}(H_0)\) satisfies \(4.43\) and \(4.44\).

5. \(d\)-DIMENSIONAL LATTICE

We consider the lattice graph \(\mathbb{L}^d = (\mathcal{V}, \mathcal{E})\), where the vertex set and the edge set are given by
  \[
  \mathcal{V} = \mathbb{Z}^d, \quad \mathcal{E} = \{(m, m + a_1), \ldots, (m, m + a_d), \quad \forall m \in \mathbb{Z}^d\}, \quad (5.1)
  \]

and \(a_1, \ldots, a_d\) is the standard orthonormal basis, see Fig.3. The "minimal" fundamental graph \(\mathbb{L}^d_0 = (\mathcal{V}_*, \mathcal{E}_*)\) of the lattice \(\mathbb{L}^d\) consists of one vertex \(v\) and \(d\) loop edges \(e_1 = \ldots = e_d = \ldots = e_{d-1} = \ldots = e_d = \ldots = e_{d^2} \)
For each $\vartheta = (\vartheta_1, \ldots, \vartheta_d) \in \mathbb{T}^d$ the fiber operator $\Delta(\vartheta)$ degenerates to the scalar function
\[
\Delta(\vartheta) = -\frac{1}{d} (\cos \vartheta_1 + \ldots + \cos \vartheta_d),
\]
which yields that the unique eigenvalue of $\Delta(\vartheta)$ and the corresponding normalized eigenfunction are given by
\[
\lambda(\vartheta) = \Delta(\vartheta), \quad \psi(\vartheta, v) = (2d)^{-1/2}.
\]
The fiber operator $\Delta_M(\vartheta)$ acts on functions $y = (y_e)_{e \in \mathcal{E}} \in L^2(L^*_d)$ by $(\Delta_M(\vartheta)y)_e = -y''_e$, where $(y''_e)_{e \in \mathcal{E}} \in L^2(L^*_d)$ and $y$ satisfies the quasi-periodic conditions:
\[
y_{e_1}(0) = \ldots = y_{e_d}(0) = e^{-i\vartheta_1} y_{e_1}(1) = \ldots = e^{-i\vartheta_d} y_{e_d}(1), \quad (5.4)
y'_{e_1}(0) + \ldots + y'_{e_d}(0) - e^{-i\vartheta_1} y'_{e_1}(1) - \ldots - e^{-i\vartheta_d} y'_{e_d}(1) = 0. \quad (5.5)
\]
Indeed, substituting the indices $a_1, \ldots, a_d$ of the fundamental graph edges into the formulas $(1.17), (1.18)$, we obtain the conditions $(5.4) – (5.5)$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{a) The lattice $\mathbb{L}^d$; b) the fundamental graph $\mathbb{L}_d^*$.}
\end{figure}

**Proposition 5.1.** Let $j \in \mathbb{N}$. Then for each $\vartheta \in \mathbb{T}^d \setminus \{0\}$ the operator $\Delta_M(\vartheta)$ has the eigenvalue $(\pi j)^2$ of multiplicity $d - 1$ and the corresponding normalized eigenfunctions $\Psi_{j,s}^0 = (\Psi_{j,s,e}^0)_{e \in \mathcal{E}}$, $s = 1, \ldots, d - 1$, have the form
\[
\Psi_{j,s}^0(\vartheta, t) = \frac{\sqrt{2} J_p \sin(\pi j t)}{\sqrt{1 + |C_{j,s}(\vartheta)|^2}}, \quad \left\{ \begin{array}{l}
J_d = 1, \quad J_s = -C_{j,s}(\vartheta) \\
J_p = 0, \quad \forall p \neq s, d
\end{array} \right., \quad t \in [0,1],
\]
where
\[
C_{j,s}(\vartheta) = \frac{1 - (-1)^j e^{-i\vartheta_d}}{1 - (-1)^j e^{-i\vartheta_s}}.
\]

**Proof.** The Betti number $\beta$ of the fundamental graph $\mathbb{L}_d^*$, defined by $\beta_G$, is equal to $d$. Then, due to Theorem 1.6 for all $\vartheta \in \mathbb{T}^d \setminus \{0\}$ the eigenvalue $(\pi j)^2$ has multiplicity $d - 1$ and all normalized eigenfunctions corresponding to the eigenvalue $(\pi j)^2$, have the form
\[
\Psi_{j,s}^0 = (\Psi_{j,s,e}^0)_{e \in \mathcal{E}}, \quad \Psi_{j,s,e}^0(\vartheta, t) = X_{j,s,e}(\vartheta) \sqrt{2} \sin(\pi j t), \quad s = 1, \ldots, d - 1.
\]
where \( (X_{j,s,e}(\vartheta))_{e \in E^*} \) is a normalized solution of the equation

\[
\sum_{p=1}^{d} x_p (1 - (-1)^j e^{-i \vartheta p}) = 0.
\]

This equation has \( d - 1 \) linearly independent solutions

\[
\begin{align*}
X_{j,1,e_1}(\vartheta) &= -C_{j,1} X_{j,1,e_2}(\vartheta), \quad X_{j,1,e_2}(\vartheta) = 0, \\
\vdots \quad X_{j,1,e_{d-1}}(\vartheta) &= 0, \quad X_{j,1,e_2}(\vartheta); \\
X_{j,2,e_1}(\vartheta) &= 0, \quad X_{j,2,e_2}(\vartheta) = -C_{j,2} X_{j,2,e_1}(\vartheta), \\
\vdots \quad X_{j,2,e_{d-1}}(\vartheta) &= 0, \quad X_{j,2,e_2}(\vartheta); \\
\cdots \\
X_{j,d-1,e_1}(\vartheta) &= 0, \quad X_{j,d-1,e_2}(\vartheta) = 0, \quad \cdots, \\
X_{j,d-1,e_{d-1}}(\vartheta) &= -C_{j,d-1} X_{j,d-1,e_2}(\vartheta), \quad X_{j,d-1,e_2}(\vartheta),
\end{align*}
\]

(5.8)

where \( C_{j,s} = C_{j,s}(\vartheta) \) are defined in (5.6). Each constant \( X_{j,s,e_2}(\vartheta), s \in \mathbb{N}_{d-1} \), is determined by the condition \( \|\Psi_{j,s}^{0}(\vartheta, \cdot)\|_{L^2(\mathbb{L}_d)} = 1 \). Then (5.7), (5.8) give

\[
1 = \|\Psi_{j,s}^{0}(\vartheta, \cdot)\|_{L^2(\mathbb{L}_d)}^2 = \sum_{p=1}^{d} \int_{0}^{1} |\Psi_{j,s,e_p}(\vartheta, t)|^2 dt = \int_{0}^{1} |\Psi_{j,s,e_1}(\vartheta, t)|^2 dt + \int_{0}^{1} |\Psi_{j,s,e_2}(\vartheta, t)|^2 dt
\]

\[
= 2\left( |X_{j,s,e_1}(\vartheta)|^2 + |X_{j,s,e_2}(\vartheta)|^2 \right) \int_{0}^{1} \sin^2(\pi j t) dt = (1 + |C_{j,s}|^2) |X_{j,s,e_2}(\vartheta)|^2,
\]

which yields that the constant \( X_{j,s,e_2}(\vartheta) \) may be chosen in the form \( X_{j,s,e_2}(\vartheta) = \frac{1}{\sqrt{1 + |C_{j,s}|^2}} > 0 \). Substituting this and (5.8) into (5.7), we obtain (5.6). \( \blacksquare \)

From (5.3) and Theorem 1.8 we deduce that the operator \( \Delta_{MV}(\vartheta) \) on the metric graph \( \mathbb{L}_d \), defined by (1.20), has the spectral representation

\[
\Delta_{MV}(\vartheta) = \sum_{j=0}^{\infty} z_j^2(\vartheta) P_j(\vartheta), \quad \text{for all } \vartheta \in \mathbb{T}_d \setminus \{0\},
\]

where its eigenvalues \( z_j^2(\vartheta) \) and the corresponding normalized eigenfunctions \( \Psi_{j,e}(\vartheta) = (\Psi_{j,e}(\vartheta, t))_{e \in E^*} \) are given by:

\[
z_j(\vartheta) = \begin{cases} 
z(\vartheta) + \pi j, & j \text{ is even} \\
(\pi - z(\vartheta)) + \pi j, & j \text{ is odd}
\end{cases}, \quad z(\vartheta) = \arccos \left( -\lambda(\vartheta) \right) \in [0, \pi],
\]

(5.9)

\[
\Psi_{j,e}(\vartheta, t) = \frac{1}{\sqrt{d \sin z(\vartheta)}} \left( \sin(z_j(\vartheta)(1-t)) + e^{i \vartheta t} \sin(z_j(\vartheta) t) \right), \quad t \in [0, 1].
\]

Since \( \lambda(\vartheta) = -\frac{1}{d} \left( \cos \vartheta_1 + \ldots + \cos \vartheta_d \right) \), we have

\[
[\lambda^-, \lambda^+] = [-1, 1], \quad z^- = \arccos(-\lambda^-) = 0, \quad z^+ = \arccos(-\lambda^+) = \pi.
\]
Recall that $\mathbb{Z}_0 = \{0, 1, 2, \ldots \}$. Then, due to the formulas (1.31), (1.32), the Laplacian $\Delta_M$ on $L^2(\mathbb{L}^d)$ has the spectrum given by

$$
\sigma(\Delta_M) = \sigma_{ac}(\Delta_M) \cup \sigma_{fb}(\Delta_M), \quad \sigma_{fb}(\Delta_M) = \sigma_{fb}(\Delta_{MD}) = \sigma_D. 
$$

$$
\sigma_{ac}(\Delta_M) = \bigcup_{j \in \mathbb{Z}_0} \sigma_j(\Delta_M) = [0, +\infty), \quad \sigma_j(\Delta_M) = [(\pi j)^2, (\pi + \pi j)^2]. 
$$

(5.10)

6. **Graphene lattice**

We consider the hexagonal lattice $\mathbf{G}$ shown in Fig. 4a. The periods $a_1, a_2$ of the lattice $\mathbf{G}$ are also shown in the figure. The fundamental graph $\mathbf{G}_* = (\mathcal{V}_*, \mathcal{E}_*)$, where $\mathcal{V}_* = \{v_1, v_2\}$, consists of two vertices and three *multiple* edges $e_1, e_2, e_3$, having the form $(v_1, v_2)$ (Fig. 4b), with the indices $\tau(e_1) = (1, 0)$, $\tau(e_2) = (0, 1)$, $\tau(e_3) = (0, 0)$.

The fiber Laplacian $\Delta(\vartheta)$, $\vartheta = (\vartheta_1, \vartheta_2) \in \mathbb{T}^2$, in the standard orthonormal basis of the space $\ell^2(\mathcal{V}_*)$ has the form

$$
\Delta(\vartheta) = \begin{pmatrix} 0 & \varphi(\vartheta) \\ -\varphi(\vartheta)^* & 0 \end{pmatrix}, \quad \varphi(\vartheta) = -\frac{1}{2} (1 + e^{-i\vartheta_1} + e^{-i\vartheta_2}).
$$

(6.1)

Then the eigenvalues of $\Delta(\vartheta)$ are given by

$$
\lambda_1(\vartheta) = -\lambda_2(\vartheta) = -|\varphi(\vartheta)|.
$$

(6.2)

**Figure 4.** a) Graphene $\mathbf{G}$; b) the fundamental graph $\mathbf{G}_*$ of graphene.

**Proposition 6.1.** i) The fiber operator $\Delta_M(\vartheta)$ acts on $y = (y_e)_{e \in \mathcal{E}_*} \in L^2(\mathbf{G}_*)$ by $(\Delta_M(\vartheta)y)_e = -y''_e$, where $(y''_e)_{e \in \mathcal{E}_*} \in L^2(\mathbf{G}_*)$ and $y$ satisfies the quasi-periodic conditions:

$$
y_{e_1}(0) = y_{e_2}(0) = y_{e_3}(0), \quad e^{-i\vartheta_1} y_{e_1}(1) = e^{-i\vartheta_2} y_{e_2}(1) = y_{e_3}(1),
$$

(6.3)

$$
y'_{e_1}(0) + y'_{e_2}(0) + y'_{e_3}(0) = 0, \quad e^{-i\vartheta_1} y'_{e_1}(1) + e^{-i\vartheta_2} y'_{e_2}(1) + y'_{e_3}(1) = 0.
$$

(6.4)

ii) Let $j \in \mathbb{N}$. Then for each $\vartheta \in \mathbb{T}^2 \setminus \{0\}$ the operator $\Delta_M(\vartheta)$ has the simple eigenvalue $(\pi j)^2$ and the corresponding normalized eigenfunction $\Psi^0_j$ has the form

$$
\Psi^0_j = (\Psi^0_{j, e})_{e \in \mathcal{E}_*}, \quad \Psi^0_{j, e}(\vartheta, t) = X_e(\vartheta) \sqrt{2} \sin(\pi j t), \quad t \in [0, 1],
$$

(6.5)

where

$$
X_{e_1}(\vartheta) = -X_{e_3}(\vartheta) \frac{1 - e^{-i\vartheta_2}}{e^{-i\vartheta_1} - e^{-i\vartheta_2}}, \quad X_{e_2}(\vartheta) = X_{e_3}(\vartheta) \frac{1 - e^{-i\vartheta_1}}{e^{-i\vartheta_1} - e^{-i\vartheta_2}},
$$

(6.6)

$$
X_{e_3}(\vartheta) = \left( \frac{1 - \cos(\vartheta_1 - \vartheta_2)}{3 - \cos \vartheta_1 - \cos \vartheta_2 - \cos(\vartheta_1 - \vartheta_2)} \right)^{1/2}.
$$

(6.7)
Proposition 6.2. The operator $\Delta_{MV}(\vartheta)$ on $G_*$, defined by (1.27), has the form

$$\Delta_{MV}(\vartheta) = \sum_{j=0}^{\infty} z^2_{j,1}(\vartheta) P_{j,1}(\vartheta) + \sum_{j=0}^{\infty} z^2_{j,2}(\vartheta) P_{j,2}(\vartheta), \quad \text{for all } \vartheta \in \mathbb{T}^2 \setminus \{0\},$$

where its eigenvalues $z^2_{j,n}(\vartheta)$ and the corresponding normalized eigenfunctions $\Psi_{j,n}(\vartheta) = (\Psi_{j,n}(\vartheta, t))_{e \in \mathcal{E}}$ have the form:

$$z_{j,n}(\vartheta) = \begin{cases} z_n(\vartheta) + \pi j, & j \text{ is even}, \\ (\pi - z_n(\vartheta)) + \pi j, & j \text{ is odd}, \end{cases} \quad z_n(\vartheta) = \arccos \left( (-1)^n + 1 \right) \sin(\vartheta), \quad \vartheta \in [0, \pi].$$

$$\Psi_{j,n,e}(\vartheta, t) = \frac{1}{\sin z_n(\vartheta)} \left( (-1)^n \phi(\vartheta) \sin(z_{j,n}(\vartheta)(1-t)) + e^{i(\pi e_e, \vartheta)} \sin(z_{j,n}(\vartheta) t) \right),$$

$\phi(\vartheta)$ is defined in (6.7).

Proof. Using (1.27), (1.28) and (6.2), we obtain (6.9), (6.10). From (6.1) it follows that the normalized eigenfunction $\psi_n(\vartheta, \cdot)$, $n = 1, 2$, corresponding to the eigenvalue $\lambda_n(\vartheta)$ satisfies

$$-\lambda_n(\vartheta) \psi_n(\vartheta, v_1) + \phi(\vartheta) \psi_n(\vartheta, v_2) = 0,$$

$$3 \left( |\psi_n(\vartheta, v_1)|^2 + |\psi_n(\vartheta, v_2)|^2 \right) = 1,$$

which yields

$$\psi_n(\vartheta, v_1) = \frac{\phi(\vartheta)}{\lambda_n(\vartheta)} \psi_n(\vartheta, v_2) = \frac{\phi(\vartheta)}{(-1)^n |\phi(\vartheta)|} \psi_n(\vartheta, v_2).$$

and we may choose $\psi_n(\vartheta, v_2) = \frac{1}{\sqrt{6}}$. Substituting this and (6.13) into (1.29) of Theorem 1.8 we obtain (6.11).

$\blacksquare$
Corollary 6.3. The Laplacian $\Delta_M$ on $L^2(G)$ has the spectrum given by

$$\sigma(\Delta_M) = \sigma_{ac}(\Delta_M) \cup \sigma_{fb}(\Delta_M),$$

$$\sigma_{ac}(\Delta_M) = [0, +\infty), \quad \sigma_{fb}(\Delta_M) = \sigma_{fb}(\Delta_{MD}) = \sigma_D.$$  \hfill (6.14)

**Proof.** The eigenvalues of $\Delta(\vartheta)$ are given by $\lambda_n(\vartheta) = \frac{(-1)^n}{3} |1 + e^{-i\vartheta} + e^{-i\vartheta}|$, $n = 1, 2$, which yields

$$[\lambda_1^-, \lambda_1^+] = \lambda_1(T^2) = [-1, 0], \quad [\lambda_2^-, \lambda_2^+] = \lambda_2(T^2) = [0, 1]$$

and

$$z_1^- = \arccos(-\lambda_1^-) = 0, \quad z_1^+ = \arccos(-\lambda_1^+) = \frac{\pi}{2},$$
$$z_2^- = \arccos(-\lambda_2^-) = \frac{\pi}{2}, \quad z_2^+ = \arccos(-\lambda_2^+) = \pi.$$

Then, using the formulas (1.31), (1.32), we obtain

$$\sigma_{j,1}(\Delta_M) = [E_{j,1}^-, E_{j,1}^+] = \left\{ \begin{array}{ll}
[(\pi j)^2, (\frac{\pi}{2} + \pi j)^2], & j \text{ is even} \\
[(\frac{\pi}{2} + \pi j)^2, (\pi + \pi j)^2], & j \text{ is odd}
\end{array} \right.,$$

$$\sigma_{j,2}(\Delta_M) = [E_{j,2}^-, E_{j,2}^+] = \left\{ \begin{array}{ll}
[\left(\frac{\pi}{2} + \pi j\right)^2, (\pi + \pi j)^2], & j \text{ is even} \\
[(\pi j)^2, (\frac{\pi}{2} + \pi j)^2], & j \text{ is odd}
\end{array} \right., \quad j \in \mathbb{Z}_0 = \{0, 1, 2, \ldots\},$$

$$\sigma_{ac}(\Delta_M) = \bigcup_{j \in \mathbb{Z}_0} \bigcup_{n=1}^2 \sigma_{j,n}(\Delta_M) = [0, +\infty), \quad \sigma_{fb}(\Delta_{MV}) = \emptyset, \quad \sigma_{fb}(\Delta_M) = \sigma_{fb}(\Delta_{MD}) = \sigma_D.$$  

Thus, (6.14) has been proved. \hfill \blacksquare

**Remark 6.4.** The spectrum of the Schrödinger operator $H = \Delta_M + Q$ with a periodic potential $Q$ on the hexagonal lattice was studied in [KP07].

7. Stanene lattice

Stanene is a topological insulator, theoretically predicted by Prof. Shoucheng Zhang’s group at Stanford, which may display dissipationless current at its edges above room temperature [Z13]. It is composed of tin atoms arranged in a single layer, in a manner similar to graphene. The addition of fluorine atoms to the tin lattice could extend the critical temperature up to 100°C. This would make it practical for use in integrated circuits to make smaller, faster and more energy efficient computers. Stanene has a band gap, it is a semiconducting material. That makes it useful as material for use in a transistor, which must have a component that turns on and off. For more details see [Z13] and references therein.
A vertex of degree one is called an end vertex. An edge incident to an end vertex is called a pendant edge. The stanene lattice $S$ is obtained from the hexagonal lattice $G$ by adding a pendant edge at each vertex of $G$ (see Fig.5a). We choose the orientations of the edges as shown in Fig.5b. We consider the discrete and metric Laplacians on the stanene lattice $S$. The fundamental graph $S_k$ consists of 4 vertices $v_1, v_2, v_3, v_4$ with degrees $\kappa_{v_1} = \kappa_{v_2} = 4$, $\kappa_{v_3} = \kappa_{v_4} = 1$ and 5 oriented edges

$$e_1 = (v_1, v_2), \quad e_2 = (v_1, v_3), \quad e_3 = (v_1, v_2), \quad e_4 = (v_1, v_3), \quad e_5 = (v_4, v_2).$$

The indices of the fundamental graph edges are given by

$$\tau(e_1) = (1, 0), \quad \tau(e_2) = (0, 1), \quad \tau(e_3) = \tau(e_4) = \tau(e_5) = (0, 0).$$

For each $\vartheta = (\vartheta_1, \vartheta_2) \in \mathbb{T}^2$ the fiber Laplacian $\Delta(\vartheta)$ in the standard orthonormal basis of the space $\ell^2(V_k)$ has the form

$$\Delta(\vartheta) = -\frac{1}{2}
\begin{pmatrix}
0 & b(\vartheta) & 1 & 0 \\
-\overline{b(\vartheta)} & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}, \quad b(\vartheta) = \frac{1}{2} (1 + e^{-i\vartheta_1} + e^{-i\vartheta_2}).$$

A direct calculation gives

$$\det (\Delta(\vartheta) - \lambda \mathbb{I}_4) = \lambda^4 - (\frac{1}{4} + \frac{|b(\vartheta)|^2}{4}) \lambda^2 + \frac{1}{16},$$

where $\mathbb{I}_4$ is the identity $4 \times 4$ matrix. The eigenvalues of the matrix $\Delta(\vartheta)$ are given by

$$\lambda_{1,2,3,4}(\vartheta) = \pm \frac{|b(\vartheta)|}{4} \pm \sqrt{\frac{|b(\vartheta)|^2 + 4}{4}}.$$

**Proposition 7.1.** The spectrum of the discrete Laplacian $\Delta$ on the stanene lattice $S$ has the form

$$\sigma(\Delta) = \sigma_{ac}(\Delta) = \bigcup_{n=1}^{4} \sigma_n(\Delta) = \left[ -1; -\frac{1}{4} \right] \cup \left[ \frac{1}{4}; 1 \right],$$

$$\sigma_1(\Delta) = [-1, -\frac{1}{2}], \quad \sigma_2(\Delta) = [-\frac{1}{2}, -\frac{1}{4}], \quad \sigma_3(\Delta) = \left[ \frac{1}{4}, \frac{1}{2} \right], \quad \sigma_4(\Delta) = \left[ \frac{1}{2}, 1 \right].$$

**Proof.** We need the following statement (see Theorem 4.3.8 in [HJ85]).

Let $B = \begin{pmatrix} A & y \\ y^* & a \end{pmatrix}$ be a self-adjoint $(\nu + 1) \times (\nu + 1)$ matrix for some self-adjoint $\nu \times \nu$ matrix $A$, some real number $a$ and some vector $y \in \mathbb{C}^\nu$. Denote by $\lambda_1(A) \leq \ldots \leq \lambda_\nu(A)$ the eigenvalues of $A$, arranged in increasing order, counting multiplicities. Then

$$\lambda_1(B) \leq \lambda_1(A) \leq \lambda_2(B) \leq \ldots \leq \lambda_\nu(B) \leq \lambda_\nu(A) \leq \lambda_{\nu+1}(B).$$

Applying this statement to the matrix $\Delta(\vartheta)$, we obtain

$$\lambda_1(\vartheta) \leq -\frac{1}{2} \leq \lambda_2(\vartheta) \leq 0 \leq \lambda_3(\vartheta) \leq \frac{1}{2} \leq \lambda_4(\vartheta), \quad \forall \vartheta \in \mathbb{T}^2.$$  

Since

$$\lambda_1(0) = -1, \quad \lambda_1 \left( \frac{2\pi}{3}, -\frac{2\pi}{3} \right) = -\frac{1}{2},$$

using (7.6) and the fact that the spectrum of a bipartite graph is symmetric with respect to the point 0, we have

$$\sigma_1 = \lambda_1(\mathbb{T}^2) = [-1, -\frac{1}{2}], \quad \sigma_4 = \lambda_4(\mathbb{T}^2) = -\lambda_1(\mathbb{T}^2) = \left[ \frac{1}{2}, 1 \right].$$

(7.7)
From (7.3) and (7.6) it follows that
\[ \lambda_1(\vartheta)\lambda_2(\vartheta) = \frac{1}{4}. \] (7.8)
This and (7.7) yield
\[ \sigma_2 = \lambda_2(T^2) = \left[-\frac{1}{2}, -\frac{1}{4}\right], \quad \sigma_3 = \lambda_3(T^2) = -\lambda_2(T^2) = \left[\frac{1}{4}, \frac{1}{2}\right]. \] (7.9)
Thus, (7.5) has been proved. ■

**Proposition 7.2.** i) The fiber operator \( \Delta_M(\vartheta) \) acts on \( y = (y_e)_{e \in E_s} \in L^2(S_s) \) by \( (\Delta_M(\vartheta)) y = -y''_e \), where \( (y''_e)_{e \in E_s} \in L^2(S_s) \) and \( y \) satisfies the quasi-periodic conditions:

\[
y_e(0) = y_{e2}(0) = y_{e3}(0) = y_{e4}(0),
\]
\[
e^{-i\vartheta} y_{e1}(1) = e^{-i\vartheta} y_{e2}(1) = y_{e3}(1) = y_{e5}(1),
\]
\[
y'_{e1}(0) + y'_{e2}(0) + y'_{e3}(0) + y'_{e4}(0) = 0,
\]
\[
e^{-i\vartheta} y'_{e1}(1) + e^{-i\vartheta} y'_{e2}(1) + y'_{e3}(1) + y'_{e4}(1) = 0,
\]
\[
y'_{e5}(1) = 0, \quad y'_{e5}(0) = 0.
\] (7.10)

ii) Let \( j \in \mathbb{N} \). Then for each \( \vartheta \in T^2 \setminus \{0\} \) the operator \( \Delta_M(\vartheta) \) has the simple eigenvalue \( (\pi j)^2 \) and the corresponding normalized eigenfunction \( \Psi^0_{j,e} = (\Psi^0_{j,e})_{e \in E_s} \) has the form

\[
\Psi^0_{j,e}(\vartheta, t) = X_{e5}(\vartheta)\sqrt{2} \sin(\pi jt), \quad s = 1, 2, 3,
\]
\[
\Psi^0_{j,e_1}(\vartheta, t) = \Psi^0_{j,e_5}(\vartheta, t) \equiv 0,
\] (7.11)
where the constants \( X_{e_s}(\vartheta) \), \( s = 1, 2, 3 \), are defined by (6.6), (6.7).

**Proof.** i) Substituting the indices (7.1) of the fundamental graph edges into the formulas (1.17), (1.18), we obtain the conditions (7.10) – (7.11).

ii) The Betti number \( \beta \) of the fundamental graph \( S_s \) is equal to 2. Then, due to Theorem 1.19 for all \( \vartheta \in T^2 \setminus \{0\} \) the eigenvalue \( (\pi j)^2 \) is simple and the unique normalized eigenfunction, corresponding to the eigenvalues \( (\pi j)^2 \), has the form

\[
\Psi^0_{j,e}(\vartheta, t) = X_{e}(\vartheta)\sqrt{2} \sin(\pi jt),
\] (7.12)
where \( (X_{e}(\vartheta))_{e \in E_s} \) is the normalized solution of the system of 4 equations

\[
x_{e1} + x_{e2} + x_{e3} + x_{e4} = 0,
\]
\[
e^{-i\vartheta} x_{e1} + e^{-i\vartheta} x_{e2} + x_{e3} + x_{e5} = 0, \quad x_{e4} = 0, \quad x_{e5} = 0,
\] (7.13)
which yields the system (6.8) for \( X_{e_s}, s = 1, 2, 3 \). The unique normalized solution of (6.8) is given by (6.6), (6.7). Thus, (7.12) has been proved. ■

**Proposition 7.3.** The operator \( \Delta_{MV}(\vartheta) \) on \( S_s \), defined by (7.20), has the form

\[
\Delta_{MV}(\vartheta) = \sum_{j=0}^{\infty} \sum_{n=1}^{4} z^2_{j,n}(\vartheta) P_{j,n}(\vartheta), \quad \text{for all} \quad \vartheta \in T^2 \setminus \{0\},
\] (7.15)
where its eigenvalues \( z^2_{j,n}(\vartheta) \) and the corresponding normalized eigenfunctions \( \Psi_{j,n}(\vartheta) = (\Psi_{j,n,e}(\vartheta, t))_{e \in E_s} \) have the form:

\[
z_{j,n}(\vartheta) = \begin{cases} z_n(\vartheta) + \pi j, & j \text{ is even} \\ (\pi - z_n(\vartheta)) + \pi j, & j \text{ is odd} \end{cases}, \quad z_n(\vartheta) = \arccos(-\lambda_n(\vartheta)) \in [0, \pi],
\] (7.16)
\[ \Psi_{j,n,e_0}(\vartheta, t) = \frac{\sqrt{2}}{\sin z_n(\vartheta)} \left( \psi_n(\vartheta, v_1) \sin(z_{j,n}(\vartheta) (1 - t)) + \psi_n(\vartheta, v_2) e^{i(\tau(e_0), \vartheta)} \sin(z_{j,n}(\vartheta) t) \right), \]

\[ s = 1, 2, 3, \]

\[ \Psi_{j,n,e_2}(\vartheta, t) = \frac{\sqrt{2}}{\sin z_n(\vartheta)} \left( \psi_n(\vartheta, v_1) \sin(z_{j,n}(\vartheta) (1 - t)) + \psi_n(\vartheta, v_3) \sin(z_{j,n}(\vartheta) t) \right), \]

\[ \Psi_{j,n,e_3}(\vartheta, t) = \frac{\sqrt{2}}{\sin z_n(\vartheta)} \left( \psi_n(\vartheta, v_4) \sin(z_{j,n}(\vartheta) (1 - t)) + \psi_n(\vartheta, v_2) \sin(z_{j,n}(\vartheta) t) \right), \]

\[ s = 1, 2, 3, \]}

(7.17)

\[ \psi_n(\vartheta, v_1) = \left( \frac{2\lambda_n^2(\vartheta)}{16\lambda_n^2(\vartheta) + 1} \right)^{1/2}, \quad \psi_n(\vartheta, v_2) = -\frac{1}{2\lambda_n(\vartheta)} \left( 2\lambda_n(\vartheta) - \frac{1}{2\lambda_n(\vartheta)} \right) \psi_n(\vartheta, v_1), \]

\[ \psi_n(\vartheta, v_3) = -\frac{1}{2\lambda_n(\vartheta)} \psi_n(\vartheta, v_1), \quad \psi_n(\vartheta, v_4) = \frac{1}{2\lambda_n(\vartheta) \sin(\vartheta, v_1)}, \]

\[ \psi_n(\vartheta, v_1), \quad \psi_n(\vartheta, v_2), \]}

(7.18)

where \( b(\vartheta) \) is defined in (7.2).

**Proof.** Using (7.27), (7.28), (7.29), we obtain (7.15), (7.16), (7.17).

For each \( \vartheta \in \mathbb{T}^2 \) the fiber Laplacian \( \Delta(\vartheta) \) has the form (7.2) and the eigenvalues of \( \Delta(\vartheta) \) are given by (7.4). Then the eigenfunction \( \psi_n(\vartheta, \cdot) \), corresponding to the eigenvalue \( \lambda_n(\vartheta) \) satisfies

\[ \lambda_n(\vartheta) \psi_n(\vartheta, v_1) + \frac{1}{2} b(\vartheta) \psi_n(\vartheta, v_2) + \frac{1}{2} \psi_n(\vartheta, v_3) = 0, \quad \lambda_n(\vartheta) \psi_n(\vartheta, v_3) + \frac{1}{2} \psi_n(\vartheta, v_1) = 0, \]

\[ \lambda_n(\vartheta) \psi_n(\vartheta, v_2) + \frac{1}{2} b(\vartheta) \psi_n(\vartheta, v_1) + \frac{1}{2} \psi_n(\vartheta, v_4) = 0, \quad \lambda_n(\vartheta) \psi_n(\vartheta, v_4) + \frac{1}{2} \psi_n(\vartheta, v_2) = 0, \]

(7.19)

which are normalized by \( \sum_{s=1}^{4} \psi_n(\vartheta, v_s)^2 = 1 \). Using (7.3), we obtain that

\[ \left| b(\vartheta) \right| = \left| 2\lambda_n(\vartheta) - \frac{1}{2\lambda_n(\vartheta)} \right|. \]

(7.20)

Solving the system of the equations (7.19) and applying (7.20), we have (7.18). \( \Box \)

**Corollary 7.4.** The Laplacian \( \Delta_M \) on \( L^2(S) \) has the spectrum given by

\[ \sigma(\Delta_M) = \sigma_{ac}(\Delta_M) \cup \sigma_{fb}(\Delta_M), \quad \sigma_{fb}(\Delta_M) = \sigma_{fb}(\Delta_{MD}) = \sigma_D, \]

\[ \sigma_{ac}(\Delta_M) = \left[ 0, r_+^2 \right] \cup_{j \in \mathbb{Z}} \left[ (r_- + \pi j)^2, (r_+ + \pi j + 1)^2 \right], \]

(7.21)

where \( r_\pm = \arccos \left( \pm \frac{1}{2} \right) \).

**Proof.** Due to Proposition 7.1, the bands \( [\lambda_n^{-1}, \lambda_n^+] = \lambda_n(\mathbb{T}^2) \) satisfy

\( [\lambda_1^{-1}, \lambda_1^+] = [-1, -\frac{1}{2}], \quad [\lambda_2^{-1}, \lambda_2^+] = [-\frac{1}{2}, -\frac{1}{3}], \quad [\lambda_3^{-1}, \lambda_3^+] = [\frac{1}{2}, \frac{1}{3}], \quad [\lambda_4^{-1}, \lambda_4^+] = [\frac{1}{2}, 1], \)

and

\[ z_1^- = \arccos(-\lambda_1^-) = 0, \quad z_1^+ = \arccos(-\lambda_1^+) = \frac{\pi}{3}, \]

\[ z_2^- = \arccos(-\lambda_2^-) = \frac{\pi}{3}, \quad z_2^+ = \arccos(-\lambda_2^+) = \arccos \left( \frac{1}{4} \right), \]

\[ z_3^- = \arccos(-\lambda_3^-) = \arccos \left( \frac{3}{4} \right), \quad z_3^+ = \arccos(-\lambda_3^+) = \frac{2\pi}{3}, \]

\[ z_4^- = \arccos(-\lambda_4^-) = \frac{2\pi}{3}, \quad z_4^+ = \arccos(-\lambda_4^+) = \pi. \]
Then, using the formulas (1.31), we obtain

$$\sigma_{j,1}(\Delta_M) = [E_{j,1}^-, E_{j,1}^+] = \begin{cases} [\arccos\left(\frac{j}{4}\right) + \pi j]^2, & j \text{ is even} \\ \left(\frac{3}{2} + \pi j\right)^2, (\pi j)^2, & j \text{ is odd} \end{cases}$$

$$\sigma_{j,2}(\Delta_M) = [E_{j,2}^-, E_{j,2}^+] = \begin{cases} \left(\frac{1}{2} + \pi j\right)^2, (\arccos\left(\frac{j}{4}\right) + \pi j)^2, & j \text{ is even} \\ \left(\arccos\left(-\frac{j}{4}\right) + \pi j\right)^2, \left(\frac{3}{2} + \pi j\right)^2, & j \text{ is odd} \end{cases}$$

$$\sigma_{j,3}(\Delta_M) = [E_{j,3}^-, E_{j,3}^+] = \begin{cases} \left(\arccos\left(-\frac{j}{4}\right) + \pi j\right)^2, \left(\frac{3}{2} + \pi j\right)^2, & j \text{ is even} \\ \left(\frac{1}{2} + \pi j\right)^2, (\arccos\left(\frac{j}{4}\right) + \pi j)^2, & j \text{ is odd} \end{cases}$$

$$\sigma_{j,4}(\Delta_M) = [E_{j,4}^-, E_{j,4}^+] = \begin{cases} \left(\frac{1}{2} + \pi j\right)^2, (\pi j)^2, & j \text{ is even} \\ \left(\frac{3}{2} + \pi j\right)^2, (\pi j)^2, & j \text{ is odd} \end{cases}$$

for all $j \in \mathbb{Z}_0 = \{0, 1, 2, \ldots\}$. Substituting these identities into the formula

$$\sigma_{ac}(\Delta_M) = \bigcup_{j \in \mathbb{Z}_0} \bigcup_{n=1}^4 \sigma_{j,n}(\Delta_M),$$

we obtain the third identity in (7.21). Since $\sigma_{fb}(\Delta_{MV}) = \emptyset$, we have the second formula in (7.21). \[\square\]

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