A FAMILY OF TQFT’S ASSOCIATED WITH HOMOLOGY THEORY

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Abstract. The purpose of this paper is to give a categorical construction of a homotopy theoretic analogue of \( C_k \)-valued TQFT from a \( \text{Hopf}^{bc}_k \)-valued Brown functor. It is formally given by a projective symmetric monoidal functor from a cospan category of spaces to \( C_k \). Here, \( C_k \) is a category of finite-dimensional, semisimple, cosemisimple, bicommutative Hopf algebras over a field \( k \), which is introduced in this paper. As an analogue of the obstruction class of a projective representation, we construct an obstruction class in the second cohomology of a cospan category of spaces. A \( \text{Hopf}^{bc}_k \)-valued homology theory consists of a sequence of \( \text{Hopf}^{bc}_k \)-valued Brown functors. We show that the obstruction classes induced by some \( \text{Hopf}^{bc}_k \)-valued homology theory vanish in two cases. One is related with the dimension reduction in the literature of topological field theory; and the other one is the case of bounded-below (or bounded-above) homology theories. The latter case gives a generalization of higher abelian Dijkgraaf-Witten-Freed-Quinn TQFT and bicommutative Turaev-Viro-Barrett-Westbury TQFT.

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1. Introduction

The purpose of this paper is to give a categorical construction of a homotopy theoretic analogue of \( C_k \)-valued TQFT from a \( \text{Hopf}^{bc}_k \)-valued Brown functor. Here, \( C_k \) is a symmetric monoidal category of finite-dimensional semisimple cosemisimple bicommutative Hopf algebras over a field \( k \) (see section 3 for definition). The symbol \( \text{Hopf}^{bc}_k \) represents the category of bicommutative Hopf algebras over \( k \) and Hopf homomorphisms. We note that the homotopy theoretic analogue of \( C_k \)-valued TQFT is formally given by a projective symmetric monoidal functor from a cospan category of spaces \( \text{Cospan} \) (instead of a cobordism category) to \( C_k \). It is projective in the sense that it preserves compositions up to a scalar in \( k^* = k \setminus \{0\} \). As
an analogue of the obstruction class of a projective representation, we construct an obstruction class in the second cohomology of a cospan category of spaces with coefficients in the multiplicative group $k^\times$.

A Hopf$^\text{bc}$-valued homotopy theory consists of a sequence of Hopf$^\text{bc}$-valued Brown functors. We show that the obstruction classes induced by some Hopf$^\text{bc}$-valued homotopy theory vanish and obtain two main theorems. In the following statements, the symbol $\text{Cosp}_{<d}^\sim(CW^\text{fin}_*)$ denotes the $d$-dimensional cospan category of spaces with coan analogue of the obstruction class of a projective representation, we construct an obstruction class in the second cohomology of a cospan category of spaces with coefficients in the multiplicative group $k^\times$.

### Theorem 1.1

Let $\tilde{E}_*$ be a Hopf$^\text{bc}$-valued reduced homology theory. Suppose that for any $r \in \mathbb{Z}$, the Hopf algebra $\tilde{E}_r(S^0)$ is finite-dimensional, semisimple and cosemisimple. Let $q \in \mathbb{Z}$. For the induced functor $\tilde{E}_q^\sim : \text{Ho}(CW^\text{fin}_*) \to \text{Hopf}^\text{bc,vol}_k$, there exists a symmetric monoidal functor $Z = \tilde{Z}(\tilde{E}_*; q) : \text{Cosp}_{<\infty}^\sim(CW^\text{fin}_*) \to C_k$ satisfying the following conditions:

1. The following diagram strictly commutes where $W_{T^+}(K) = K \wedge T^+$. In particular, we have $Z(K) = \tilde{E}_q(K \wedge T^+)$ for a space $K$.

\[
\begin{array}{ccc}
\text{Ho}(CW^\text{fin}_*) & \xrightarrow{W_{T^+}} & \text{Ho}(CW^\text{fin}_*) \\
\downarrow & & \downarrow \\
\text{Cosp}_{<\infty}^\sim(CW^\text{fin}_*) & \xrightarrow{Z} & C_k
\end{array}
\]

2. For a pointed finite CW-space $L$, denote by $L$ the cospan diagram $(pt \to L \leftarrow pt)$ with some abuse of notations. The corresponding endomorphism $Z(L)$ on $Z(pt) \cong k$ defines a homotopy invariant valued in $k^\times$. The induced homotopy invariant is given by $Z(L) = \dim \tilde{E}_q(L) \in k^\times$.

Theorem 1.1 implies that the obstruction class induced by the dimension reduction vanishes. In the literature of topological field theory, the cartesian product of manifolds with a circle $T$ induces the dimension reduction (for example see [5]). The smash product with $T^+ = T \sqcup \{pt\}$ gives a pointed version.

On the one hand, the second main theorem is related with bounded homology theories:

### Theorem 1.2

Let $\tilde{E}_*$ be a Hopf$^\text{bc}$-valued reduced homology theory which is bounded below. Suppose that for any $r \in \mathbb{Z}$, the Hopf algebra $\tilde{E}_r(S^0)$ is finite-dimensional, semisimple and cosemisimple. Let $q \in \mathbb{Z}$. For the induced functor $\tilde{E}_q^\sim : \text{Ho}(CW^\text{fin}_*) \to \text{Hopf}^\text{bc,vol}_k$, there exists a symmetric monoidal functor $Z = \tilde{Z}(\tilde{E}_*; q) : \text{Cosp}_{<\infty}^\sim(CW^\text{fin}_*) \to C_k$ satisfying the following conditions:

1. The following diagram strictly commutes. In particular, we have $Z(K) = \tilde{E}_q(K)$ for a space $K$.

\[
\begin{array}{ccc}
\text{Ho}(CW^\text{fin}_*) & \xrightarrow{\tilde{E}_q^\sim} & \text{Hopf}^\text{bc,vol}_k \\
\downarrow & & \downarrow \\
\text{Cosp}_{<\infty}^\sim(CW^\text{fin}_*) & \xrightarrow{Z} & C_k
\end{array}
\]

2. The induced homotopy invariant is given by $Z(L) = \prod_{l>0} \dim \tilde{E}_{q-l}(L)(-1)^l \in k^\times$. 
A refinement of Theorem 1.1 (Theorem 1.2, resp.) is given by Theorem 6.5, 6.6 (Theorem 6.13, resp.). For an arbitrary Hopf\(^{bc}\)-valued homology theory \(\tilde{E}_*\), we introduce a (possibly empty) subset \(\Gamma(\tilde{E}_*) \subseteq \mathbb{Z}\). Then an analogous statement is true for \(q \in \Gamma(\tilde{E}_*)\). In fact, we have \(\Gamma(\tilde{E}_*) = \mathbb{Z}\) under the assumptions in the main theorems.

There are various ways to obtain some nontrivial Hopf\(^{bc}\)-valued homology theories \([12]\). We apply our main theorems to such examples. See Example 6.8, 6.14.

The homotopy-theoretic analogue of TQFT in the main theorems induces an \(n\)-dimensional \(C_k\)-valued TQFT for arbitrary \(n\). In fact, we have a canonical functor from the cobordism category to the cospan category of pointed finite CW-spaces. See Definition 6.3.

The application of Theorem 1.2 to the first singular homology theory gives a generalization of untwisted Dijkgraaf-Witten-Freed-Quinn TQFT \([4, 19, 7, 17, 8, 6, 16]\) of abelian groups and Turaev-Viro-Barrett-Westbury TQFT \([18, 13]\) of bicommutative Hopf algebras. The TQFT’s obtained by DWFQ and TVBW factor through \(Z\) in Theorem 1.2 for an appropriate \(k\).

The path-integral and state-sum in the literature are formulated in different ways although they stem from the same idea. We give a new approach to the path-integral and state-sum by using a notion of integral along bialgebra homomorphisms and our previous results in \([10]\).

The vector spaces assigned to surfaces in TVBW theory are naturally isomorphic to the ground-state spaces in the Kitaev lattice Hamiltonian model (a.k.a. toric code) \([1]\). The reader is referred to \([14, 3]\) for Kitaev lattice Hamiltonian model. In \([9]\), we gave a generalization of the Kitaev lattice Hamiltonian model based on bicommutative Hopf algebras: the singular (co)homology theory of any finite CW-complex is realized as the ground-state space of some lattice Hamiltonian model. As a consequence of this paper and the previous study, the relationship between TVBW theory and Kitaev lattice Hamiltonian model is generalized to an arbitrary ground field \(k\) and pointed finite CW-complexes.

The results in this paper hold for cohomology theories in a dual way.

This paper is organized as follows. In section 2, we give some overviews of our previous studies \([10, 11]\). In section 3, we introduce the symmetric monoidal category \(C_k\). In section 4, we introduce path-integral along cospan diagrams based on the results in \([10]\). In particular, we introduce the path-integral projective functors \(\hat{\mathcal{P}}l_k\) and \(\check{\mathcal{P}}l_k\). In section 5, we apply the path-integral to Hopf\(^{bc,\text{vol}}\)-valued Brown functors. In subsection 5.1, we introduce an obstruction cocycle and class for a Brown functor \(\hat{E}\) to extend to a homotopy-theoretic analogue of \(C_k\)-valued TQFT. In subsection 5.2, we study the obstruction cocycle for the case of homology theory. Especially, in subsection 5.2.3, we prove some inversion formula of the obstruction class. In section 6, we compute the obstruction cocycle and class for some special cases and give concrete examples. In subsection 6.1, we consider the dimension reduction. In subsection 6.2, we consider bounded-below homology theories. In subsection 6.3, we reconstruct the abelian DWFQ TQFT and TVBW TQFT from the results in subsection 6.2. In appendix A, we give an overview of projective symmetric monoidal functors and the obstruction class.

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2. **Overview of previous study**

In this section, we give an overview of our previous study. In subsection 2.1, we give main theorems in \([11]\). In subsection 2.2, we give an overview of the results in \([10]\).
2.1. Some extensions of Brown functors.

Definition 2.1. Let $\mathcal{A}$ be an abelian category. A square diagram is a quadruple $(g, f, g', f')$ of morphisms in $\mathcal{A}$ such that $g, f$ and $g', f'$ are composable respectively. Consider a following square diagram $\square$ in $\mathcal{A}$.

$$
\begin{array}{ccc}
B & \xrightarrow{g} & D \\
\uparrow & & \uparrow \\
A & \xrightarrow{f'} & C
\end{array}
$$

The morphism $f$ induces a morphism $k_\square : \text{Ker}(g') \to \text{Ker}(g)$. The morphism $g'$ induces a morphism $c_\square : \text{Cok}(f') \to \text{Cok}(g)$. The square diagram is exact if the morphism $k_\square$ is an epimorphism and the morphism $c_\square$ is a monomorphism.

The exactness of a square diagram is represented in a familiar way as follows. See [11] for the proof.

Proposition 2.2. Consider the square diagram (3). We define a chain complex $C(\square)$ by

$$
A \xrightarrow{\triangleleft} B \oplus C \xrightarrow{\triangleleft} D
$$

where $u_\square \overset{\text{def}}{=} (f \oplus (-f')) \circ \Delta_A$ and $v_\square \overset{\text{def}}{=} \nabla_D \circ (g \oplus g')$. Then the following conditions are equivalent:

1. The square diagram $\square$ is exact.
2. The induced chain complex $C(\square)$ is exact.

Definition 2.3. For cospan diagrams $\Lambda = \left( A_0 \xleftarrow{f_0} B \xrightarrow{f_1} A_1 \right)$ and $\Lambda' = \left( A'_0 \xleftarrow{f'_0} B' \xrightarrow{f'_1} A'_1 \right)$, we denote by $\Lambda \leq \Lambda'$ if $A_0 = A'_0$, $A_1 = A'_1$ and there exists a monomorphism $g : B \to B'$ in $\mathcal{A}$ such that $g \circ f_0 = f'_0$ and $g \circ f_1 = f'_1$. For such a monomorphism $g$, we say that the monomorphism $g$ gives $\Lambda \preceq \Lambda'$. The relation $\preceq$ gives a preorder of cospan diagrams in $\mathcal{A}$.

Definition 2.4. We define an equivalence relation $\approx$ of cospan diagrams in $\mathcal{A}$. We define $\Lambda \approx \Lambda'$ if there exists an upper bound of $\{\Lambda, \Lambda'\}$ with respect to the preorder $\preceq$ in Definition 2.3. In fact, $\Lambda \approx \Lambda'$ is equivalent with the condition that there exists a lower bound of $\{\Lambda, \Lambda'\}$.

Definition 2.5. The equivalence relation $\approx$ is compatible with the direct sum of cospan diagrams and the composition of cospan diagrams. The equivalence classes form a dagger symmetric monoidal category denoted by $\text{Cosp}^\approx(\mathcal{A})$. It contains $\mathcal{A}$ as a subcategory. In the same manner, one can define a dagger symmetric monoidal category $\text{Sp}^\approx(\mathcal{A})$ consisting of (equivalence classes of) span diagrams in $\mathcal{A}$.

We have an isomorphism $T : \text{Cosp}^\approx(\mathcal{A}) \cong \text{Sp}^\approx(\mathcal{A})$. Under the isomorphism, we denote by $\bar{\mathcal{A}}$ one of them. We have a bijective and faithful functor $\iota_{\mathcal{A}} : \mathcal{A} \to \bar{\mathcal{A}}$.

The following theorem is the main theorem in [11].

Theorem 2.6. For $d \in \mathbb{N} \cup \{\infty\}$, let $E : \text{Ho} \left( \text{CW}^\text{fin}_{*, \leq d} \right) \to \mathcal{A}$ be a $d$-dimensional $\mathcal{A}$-valued Brown functor.

1. There exists a unique dagger-preserving symmetric monoidal extension of $\iota_{\mathcal{A}} \circ E \circ \Sigma$ to $\text{Cosp}^\approx_{\leq d}(\text{CW}^\text{fin}_{*})$. Here, $\Sigma : \text{Ho} \left( \text{CW}^\text{fin}_{*, \leq (d-1)} \right) \to \text{Ho} \left( \text{CW}^\text{fin}_{*, \leq d} \right)$ is the suspension
functor. In other words, the following diagram commutes:

\[
\begin{array}{ccc}
\mathrm{Ho}(\mathrm{CW}_{*, \leq (d-1)}) & \xrightarrow{\Sigma} & \mathrm{Ho}(\mathrm{CW}_{*, \leq d}) \\
\downarrow & & \downarrow \\
\mathrm{Cosp}_{\leq d}(\mathrm{CW}^{\text{fin}}) & \xrightarrow{\exists!} & \mathcal{A}
\end{array}
\]

(5)

(2) There exists a unique dagger-preserving symmetric monoidal extension of \( i_{\mathcal{A}} \circ E \circ i \) to \( \mathrm{Cosp}_{\leq d}(\mathrm{CW}^{\text{fin}}) \). Here, \( i : \mathrm{Ho}(\mathrm{CW}^{\text{fin}}_{*, \leq (d-1)}) \to \mathrm{Ho}(\mathrm{CW}^{\text{fin}}_{*, \leq d}) \) is the inclusion functor. In other words, the following diagram commutes:

\[
\begin{array}{ccc}
\mathrm{Ho}(\mathrm{CW}_{*, \leq (d-1)}) & \leftarrow i & \mathrm{Ho}(\mathrm{CW}_{*, \leq d}) \\
\downarrow & & \downarrow \\
\mathrm{Cosp}_{\leq d}(\mathrm{CW}^{\text{fin}}) & \xrightarrow{\exists!} & \mathcal{A}
\end{array}
\]

(6)

2.2. Integrals along bialgebra homomorphisms. In [10], we consider a symmetric monoidal category \( C \) satisfying some assumptions. Here, we apply the results to the case \( C = (\text{Vec}_k, \otimes_k) \), the tensor category of vector spaces over a field \( k \). In this paper, we freely use the results in this subsection.

For a bialgebra homomorphism \( \xi : A \to B \), a normalized generator integral along \( \xi \) is a morphism \( \mu : B \to A \) in \( C \) satisfying some axioms. For the application here, we consider only bicommutative Hopf algebras. We describe a necessary and sufficient condition for the existence of a normalized generator integral by the kernel of \( \xi \) and cokernel of \( \xi \).

**Theorem 2.7.** Let \( A, B \) be bicommutative Hopf algebras and \( \xi : A \to B \) be a Hopf homomorphism. There exists a normalized generator integral \( \mu_\xi \) along \( \xi \) if and only if the following conditions hold:

1. The kernel Hopf algebra \( \text{Ker}(\xi) \) has a normalized integral.
2. The cokernel Hopf algebra \( \text{Cok}(\xi) \) has a normalized cointegral.

Note that if a normalized integral along \( \xi \) exists, then it is unique.

We introduce an invariant of bicommutative Hopf algebras \( A \), called an inverse volume \( \text{vol}^{-1}(A) \). It is defined as a composition \( \sigma^A \circ \sigma_A \in k \) where \( \sigma_A \) is a normalized integral and \( \sigma^A \) is a normalized cointegral.

**Definition 2.8.** A bicommutative Hopf algebra \( A \) has a finite volume \(^1\) if

1. It has a normalized integral \( \sigma_A : k \to A \).
2. It has a normalized cointegral \( \sigma^A : A \to k \).
3. Its inverse volume \( \text{vol}^{-1}(A) = (\sigma^A \circ \sigma_A) \in k \) is invertible.

Denote by \( \text{Hopf}^{bc, \text{vol}}_k \) the category of bicommutative Hopf algebras with a finite volume and Hopf homomorphisms.

As a corollary of Theorem 2.7, we obtain the following statement.

**Corollary 2.9.** Let \( A, B \) be bicommutative Hopf algebras with a finite volume. For any bialgebra homomorphism \( \xi : A \to B \), there exists a unique normalized generator integral \( \mu_\xi \) along \( \xi \).

**Proposition 2.10.** Let \( \xi : A \to B \) be a Hopf homomorphism between bicommutative Hopf algebras with a finite volume.

\(^1\)See Corollary 3.8 for an equivalent description.
(1) If \( \xi \) is an epimorphism in the category \( \text{Hopf}^{bc}_{k} \), then we have \( \xi \circ \mu_{\xi} = \text{id}_{B} \). In other words, \( \mu_{\xi} \) is a section of \( \xi \) in the category \( \text{Vec}_{k} \).

(2) If \( \xi \) is an monomorphism in the category \( \text{Hopf}^{bc}_{k} \), then we have \( \mu_{\xi} \circ \xi = \text{id}_{A} \). In other words, \( \mu_{\xi} \) is a retract of \( \xi \) in the category \( \text{Vec}_{k} \).

**Proof.** It is immediate from Lemma 7.3 [10]. □

The inverse volume induces a volume on the abelian category \( \text{Hopf}^{bc,bs}_{k} \) consisting of bicommutative Hopf algebras with a normalized integral and a normalized cointegral. Here, the volume on the abelian category is a generalization of the dimension of vector spaces and the order of abelian groups, which is also introduced in [10].

**Theorem 2.11.** We regard the field \( k \) as the multiplicative monoid. Then the inverse volume \( \text{vol}^{-1} \) gives an \( k^{*} \)-valued volume on the abelian category \( \text{Hopf}^{bc,bs}_{k} \), i.e. if \( A \to B \to C \) is a short exact sequence in \( \text{Hopf}^{bc,bs}_{k} \), then we have \( \text{vol}^{-1}(B) = \text{vol}^{-1}(A) \cdot \text{vol}^{-1}(C) \).

By Theorem 2.11. \( \text{Hopf}^{bc,vol}_{k} \subset \text{Hopf}^{bc,bs}_{k} \) is closed under short exact sequences. In particular, \( \text{Hopf}^{bc,vol}_{k} \) is also an abelian category. Then the following corollary is immediate from Theorem 2.11.

**Corollary 2.12.** The inverse volume \( \text{vol}^{-1} \) gives an \( k^{*} \)-valued volume on the abelian category \( \text{Hopf}^{bc,vol}_{k} \). Here, we regard \( k^{*} = k \setminus \{0\} \) as the multiplicative group.

**Proposition 2.13.** Consider the exact square diagram (3) for \( \mathcal{A} = \text{Hopf}^{bc,vol}_{k} \). Then we have

\[
\mu_{g'} \circ g = f' \circ \mu_{f}.
\]

**Proof.** It follows from Theorem 3.4 in [10]. Note that an epimorphism \( \pi \) in the category \( \text{Hopf}^{bc,vol}_{k} \) has a section in \( \text{Vec}_{k} \). In fact, the normalized integral \( \mu_{\pi} \) along \( \pi \) is a section of \( \pi \) in \( \text{Vec}_{k} \) by Lemma 9.3 in [10]. Similarly, any monomorphism in the category \( \text{Hopf}^{bc,vol}_{k} \) has a retract in \( \text{Vec}_{k} \). □

The inverse volume of bicommutative Hopf algebras is generalized to Hopf homomorphisms. For a Hopf homomorphism \( \xi : A \to B \), we define \( \langle \xi \rangle = \sigma^{B} \circ \xi \circ \sigma^{A} \in k \). By using this notion, a composition rule of normalized integrals is represented as follows.

**Proposition 2.14.** Let \( \xi : A \to B, \xi' : B \to C \) be morphisms in the category \( \text{Hopf}^{bc,vol}_{k} \). Then for some \( \lambda \in k^{*} \), we have

\[
\mu_{\xi} \circ \mu_{\xi'} = \lambda \cdot \mu_{\xi' \circ \xi}.
\]

Moreover, we have \( \lambda = \langle \text{cok}(\xi) \circ \text{ker}(\xi') \rangle \) where \( \text{ker}(\xi') : \text{Ker}(\xi') \to B \) and \( \text{cok}(\xi) : B \to \text{Cok}(\xi) \) are the canonical morphisms.

**Proof.** It follows from Theorem 3.6. or Theorem 12.1. in [10]. □

3. The category \( \mathcal{C}_{k} \)

In this section, we introduce a symmetric monoidal category \( \mathcal{C}_{k} \) for a field \( k \).

**Definition 3.1.** (1) Consider a cospan diagram \( \Lambda = \left( A_{0} \xrightarrow{\xi_{0}} B \xleftarrow{\xi_{1}} A_{1} \right) \) in the category \( \text{Hopf}^{bc,vol}_{k} \). In other words, \( A_{0}, A_{1}, B \) are bicommutative Hopf algebras with a finite volume and \( \xi_{0}, \xi_{1} \) are bialgebra homomorphisms. We define a linear homomorphism \( \int_{A}^{\Lambda} : A_{0} \to A_{1} \) by

\[
\int_{A}^{\Lambda} = \mu_{\xi_{1}} \circ \xi_{0}.
\]

Here, \( \mu_{\xi_{1}} \) denotes the normalized integral along \( \xi_{1} \). Analogously, for a span diagram \( V \), we define a linear homomorphism \( \int_{V}^{A} \) for a span diagram \( V \) in \( \text{Hopf}^{bc,vol}_{k} \).
Lemma 3.2. Let \( A_0, A_1, A_2 \) be bicommutative Hopf algebras with a finite volume. A linear homomorphism \( \varrho : A_0 \to A_1 \) is realized as a nontrivial integral along a cospan diagram if there exists a cospan diagram \( \Lambda \) in the category \( \text{Hopf}_{bc,vol}^k \) and \( \lambda \in k^* \) such that \( \varrho = \lambda \cdot \int_\Lambda \).

**Proof.** Suppose that \( \varrho = \lambda \cdot \int_\Lambda \) and \( \varrho' : A_1 \to A_2 \) are realized as a nontrivial integral along a cospan diagram, then the composition \( \varrho' \circ \varrho \) is realized as a nontrivial integral along a cospan diagram.

Recall that the composition \( \Lambda' \circ \Lambda \) is defined by \( \left( \Lambda_0 \xrightarrow{\varphi_0} \Lambda' \xleftarrow{\varphi_1} \Lambda_1 \right) \) where \( B' \) is given by the pushout diagram (10). We obtain \( \int_{\Lambda'} \circ \int_\Lambda = \mu_{\varphi_1} \circ \mu_{\varphi} \circ \varrho \circ \varepsilon_0 \). Since we have \( \mu_{\varphi_1} \circ \mu_{\varphi} = \lambda'' \cdot \mu_{\varphi_0} \) for \( \lambda'' = \langle \text{cok}(\varepsilon'_1) \circ \ker(\varrho') \rangle \in k^* \) by Proposition 2.14, we obtain \( \int_{\Lambda'} \circ \int_\Lambda = \lambda'' \cdot \int_{\Lambda' \circ \Lambda} \), hence \( \varrho' \circ \varrho = \lambda'' \cdot \int_{\Lambda' \circ \Lambda} \). By definition, the composition \( \varrho' \circ \varrho \) is realized as a nontrivial integral along a cospan diagram.

\[
\begin{array}{ccc}
B'' & \xrightarrow{\varphi} & B' \\
\downarrow{\varphi} \quad \downarrow{\varphi} & & \downarrow{\varphi} \\
A_1 & & A_1
\end{array}
\]

(10)

\[\square\]

Definition 3.3. We introduce a category \( C_k \) of bicommutative Hopf algebras with a finite volume. Its morphisms consist of morphisms realized as a nontrivial integral along a cospan diagram. The composition is well-defined due to Lemma 3.2. We have an obvious embedding functor \( \text{Hopf}_{bc,vol}^k \hookrightarrow C_k \).

Remark 3.4. Consider a category \( N \) of bicommutative Hopf algebras with a finite volume defined as follows. For two objects \( A, B \) of \( N \), the morphism set \( \text{Mor}_N(A, B) \) consists of linear homomorphisms. Then the category \( C_k \) is the smallest subcategory of \( N \) which contains the following three classes of morphisms:

- a Hopf homomorphism \( \xi : A \to B \) for objects \( A, B \) of \( N \),
- a morphism \( \mu : A \to B \) in \( N \) which is a normalized integral along some Hopf homomorphism \( \xi : B \to A \),
- an automorphism on the unit object \( k \) in \( N \).

Definition 3.5. Let \( A \) be a bicommutative Hopf algebra with a finite volume. Let \( \Lambda \) be a cospan diagram of Hopf algebras \( \left( k \xrightarrow{\eta} A \xleftarrow{\varepsilon} A \otimes A \right) \). We define morphisms \( i_A : k \to A \otimes A \) and \( e_A : A \otimes A \to k \) in \( C_k \) by

\[
i_A \overset{\text{def.}}{=} \int_\Lambda,
\]

(11)

\[
e_A \overset{\text{def.}}{=} a^{-1} \cdot \int_{\Lambda^1}.
\]

(12)

Here, \( a = \text{vol}^{-1}(A) \in k^* \) denotes the inverse volume of \( A \).

Proposition 3.6. The morphisms \( i_A, e_A \) give a symmetric self-duality of \( A \) in \( C_k \).
Proof. Since $A$ is bicommutative, we have $\tau \circ i_A = i_A, e_A \circ \tau = e_A$ where $\tau : A \otimes A \to A \otimes A; x \otimes y \mapsto y \otimes x$. All that remain is to prove that $i_A, e_A$ form a duality. Let $e'_A = \int_A$. Then a zigzag diagram is computed as Figure 1. The third equality holds since $\mu \nabla$ is an integral along the multiplication $\nabla$. Note that the normalized cointegral $\sigma^A$ is a normalized integral along the unit $\eta : k \to A$. The last morphism $(id_A \otimes \sigma_A) \circ \mu \nabla : A \to A$ is proportional to a normalized integral along the identity on $A$. The proportional factor coincides with $\langle id_A \rangle = \text{vol}^{-1}(A)$ due to Proposition 2.14. It completes the proof since $i_A, e_A$ are symmetric.

\[ \begin{align*}
\text{Figure 1.}
\end{align*} \]

**Corollary 3.7.** A bicommutative Hopf algebra with a finite volume is finite-dimensional. Moreover, we have $\text{vol}^{-1}(A)^{-1} = e_A \circ i_A$ so that $\text{vol}^{-1}(A)^{-1}$ coincides with the dimension of $A$ modulo the characteristic of $k$.

**Proof.** Since a vector space with duality is finite-dimensional, we obtain the first claim by Proposition 3.6. We prove the second claim. We use the fact that for a morphism $\xi : A \to B$ in $\text{Hopf}_{\text{bc,vol}}$, if $\xi$ is an epimorphism, then we have $\xi \circ \mu_{\xi} = id_B$. It follows from Lemma 9.3. in [10]. Then we obtain,

\begin{align*}
\text{vol}^{-1}(A) \cdot (e_A \circ i_A) &= \sigma^A \circ \nabla \circ \mu \nabla \circ \eta, \\
&= \sigma^A \circ \eta, \\
&= 1.
\end{align*}

**Corollary 3.8.** A bicommutative Hopf algebra $A$ has a finite volume if and only if it is finite-dimensional, semisimple and cosemisimple.

**Proof.** A finite-dimensional Hopf algebra is semisimple if and only if it has a normalized integral [15]. In the same manner, the cosemisimplicity is equivalent with an existence of a normalized cointegral. In Theorem 3.3 [2], it is proved that the composition of a left (right) $\text{Int}_A$-valued integral and a left (right) $\text{Int}_A$-based integral of finite-dimensional Hopf algebra is invertible. The other part follows from Corollary 3.7.

### 4. Path-integral along (co)span diagrams

In this section, we formulate the integral in Definition 3.1 as a projective symmetric monoidal functor valued in $C_k$. For the definition of projective symmetric monoidal functor, see the appendix. We give some basic properties of the induced obstruction class. From the observation, we also show some nontriviality of the second cohomology theory of the symmetric monoidal category $\text{Hopf}_{\text{bc,vol}}^\text{dc}$. 


Lemma 4.1. If $\Lambda \leq \Lambda'$, then we have $\int_{\Lambda} = \int_{\Lambda'}$. In particular, the equivalence $\Lambda \cong \Lambda'$ implies $\int_{\Lambda} = \int_{\Lambda'}$.

Proof. We use the notations in Definition 2.3. The inverse volume $\langle \text{cok}(f) \circ \text{ker}(g) \rangle = \sigma^\text{cok}(f) \circ \text{cok}(f) \circ \text{ker}(g) \circ \sigma^\text{ker}(g)$ is $1 \in k$ since $\text{ker}(g) \cong k$ and $\sigma^\text{cok}(f)$ is a normalized cointegral. By Proposition 2.14, we have $\mu_k \circ \mu_k = \mu^\sigma f_1$. Hence, we obtain $\int_{\Lambda'} = \mu_k \circ \mu_k \circ g \circ f_0 = \mu_k \circ g \circ f_0$. By the second part in Proposition 2.10, we have $\mu_k \circ g = \text{Id}_k$ so that we obtain $\int_{\Lambda'} = \mu_k \circ f_0 = \int_{\Lambda}$. It completes the proof. □

Definition 4.2. We define the path-integral projective functor $\hat{P}l_k$. It is a projective symmetric monoidal functor $\hat{P}l_k : \text{Cosp}^\times(\text{Hopf}_{b,\text{vol}}^k) \rightarrow C_k$ which is the identity on objects and assigns $\hat{P}l_k([\Lambda]) = \int_{\Lambda}$. The compositions are preserved up to a scalar due to Proposition 2.14. It is a well-defined projective functor by Lemma 4.1. Analogously, we define a projective symmetric monoidal functor $\hat{P}l_k : \text{Sp}^\times(\text{Hopf}_{b,\text{vol}}^k) \rightarrow C_k$ by using the path-integral along spans in Definition 3.1.

Recall Definition 2.5. Under the identification of $\text{Cosp}^\times(\text{Hopf}_{b,\text{vol}}^k)$ with $\text{Sp}^\times(\text{Hopf}_{b,\text{vol}}^k)$, the above projective functors $\hat{P}l_k$, $\hat{P}l_k$ induce the same projective functor which we denote by $\bar{P}l_k : \text{Hopf}_{b,\text{vol}}^k \rightarrow C_k$.

Proposition 4.3. The following conditions are equivalent.

1. The obstruction cocycle $\omega(\bar{P}l_k)$ vanishes.
2. The obstruction class $O(\bar{P}l_k)$ vanishes.
3. The categorical dimension of any bicommutative Hopf algebra with a finite volume is $1 \in k$.

Proof. It suffices to prove the statement for $\bar{P}l_k = \hat{P}l_k$.

(1) $\Rightarrow$ (2): It is obvious.

(2) $\Rightarrow$ (3): Suppose that $O(\bar{P}l_k) = 1$. By Proposition A.7, there exists a symmetric monoidal functor $F : \text{Cosp}^\times(\text{Hopf}_{b,\text{vol}}^k) \rightarrow C_k$ such that $F \cong_{\text{proj}} \bar{P}l_k$. Let $A$ be a bicommutative Hopf algebra with a finite volume. Note that $A$ is self-dualizable in $\text{Cosp}^\times(\text{Hopf}_{b,\text{vol}}^k)$. Let $d$ be the categorical dimension of $A$ in $\text{Cosp}^\times(\text{Hopf}_{b,\text{vol}}^k)$. Then $d$ is the identity on the unit object $k$ since the endomorphism set of $k$ in $\text{Cosp}^\times(\text{Hopf}_{b,\text{vol}}^k)$ has only the identity. Since $F$ is a symmetric monoidal functor, $F(A) = A$ has a trivial categorical dimension in $C_k$.

(3) $\Rightarrow$ (1): Suppose that every bicommutative Hopf algebra with a finite volume has a trivial categorical dimension. Let $A$ be a bicommutative Hopf algebra with a finite volume. Then $A$ is dualizable and its categorical dimension coincides with the inverse of the inverse volume $\text{vol}^{-1}(A)$ by Corollary 3.7. By the assumption, the inverse volume $\text{vol}^{-1}(A)$ is trivial. In other words, we have $\text{vol}^{-1}(A) = 1 \in k$ for any bicommutative Hopf algebra $A$ with a finite volume. By Proposition 11.9 in [10], we have $\langle \xi \rangle = 1 \in k$ for any homomorphism $\xi : A \rightarrow B$ between bicommutative Hopf algebras with a finite volume. Therefore, the cocycle $\omega(\bar{P}l_k)$ vanishes by definitions. □

Corollary 4.4. Let $p$ be the characteristic of the ground field $k$.

1. The obstruction class $O(\bar{P}l_k)$ vanishes if and only if $p = 2$.
2. If $p \neq 2$, then the second cohomology group $H^2(\text{Hopf}_{b,\text{vol}}^k; k^*)$ is not trivial.

Proof. Note that if $p \neq 2$, then there exists a bicommutative Hopf algebra with a finite volume whose categorical dimension is not $1 \in k$. Such examples could be obtained from group Hopf algebras. If $p = 2$, then the categorical dimension of any bicommutative Hopf algebra with a finite volume is $1 \in k$ since the dimension should be invertible in $k$ by Corollary 3.7. It proves the first claim.
By the first claim, the class $O(\Pi_k) \neq 1$ if $p \neq 2$. It proves the second claim. \qed

**Corollary 4.5.** Let $p$ be the characteristic of the ground field $k$.

1. If $p \neq 0, 2$, then the second cohomology group $H^2(\text{Hopf}_{k}^{bc, \text{vol}}; \mathbb{F}_p)$ is not trivial.

2. If $p = 0$, then the second cohomology group $H^2(\text{Hopf}_{k}^{bc, \text{vol}}; \mathbb{Q}_{>0})$ is not trivial.

**Proof.** Let $G$ be the multiplicative group $\mathbb{F}_p^*$ if $p \neq 0, 2$ or $\mathbb{Q}_{>0}$ if $p = 0$. Then the obstruction cocycle $\omega(\Pi_k)$ has coefficients in $G$ due to Corollary 3.7. It induces a class $[\omega(\Pi_k)] \in H^2(\text{Hopf}_{k}^{bc, \text{vol}}; G)$. The induced map $H^2(\text{Hopf}_{k}^{bc, \text{vol}}; G) \to H^2(\text{Hopf}_{k}^{bc, \text{vol}}; k^*)$ assigns $O(\Pi_k)$ to the class $[\omega(\Pi_k)] \neq 1$. By Corollary 4.4, $[\omega(\Pi_k)] \in H^2(\text{Hopf}_{k}^{bc, \text{vol}}; G)$ is nontrivial. It completes the proof. \qed

5. **Applications of the Path-integral**

In this section, we apply the path-integral projective functor to $\text{Hopf}_{k}^{bc, \text{vol}}$-valued Brown functors. Roughly speaking, Brown functors induce a homotopy-theoretic analogue of $C_k$-valued TQFT. In general, the obtained TQFT preserves compositions up to a scalar in $k^*$. For homology theories, we deduce some formulas to compute the induced obstruction classes. See subsection 5.2.

5.1. **Brown functor.** Let $E : \text{Ho}(\text{CW}_{*, \leq d}^{\text{fin}}) \to \text{Hopf}_{k}^{bc, \text{vol}}$ be a $d$-dimensional $\text{Hopf}_{k}^{bc, \text{vol}}$-valued Brown functor where $d \in \mathbb{N} \cup \{\infty\}$.

**Definition 5.1.** Let $\hat{E}$ be the cospanical extension of $E \circ i$ by the second part of Theorem 2.6. We define a projective symmetric monoidal functor $\hat{\Pi}_k(E) \overset{\text{def}}{=} \hat{\Pi}_k \circ \hat{E}$.

\begin{equation}
\text{Cosp}_{\leq d}(\text{CW}_{*}^{\text{fin}}) \overset{\hat{E}}{\longrightarrow} \text{Cosp}(\text{Hopf}_{k}^{bc, \text{vol}}) \overset{\hat{\Pi}_k}{\longrightarrow} C_k.
\end{equation}

Analogously, we define a projective symmetric monoidal functor $\hat{\Pi}_k(E) \overset{\text{def}}{=} \hat{\Pi}_k \circ \hat{E}$ where $\hat{E}$ is the spanical extension of $E \circ \Sigma$ by the first part of Theorem 2.6.

\begin{equation}
\text{Cosp}_{\leq d}(\text{CW}_{*}^{\text{fin}}) \overset{\hat{E}}{\longrightarrow} \text{Sp}(\text{Hopf}_{k}^{bc, \text{vol}}) \overset{\hat{\Pi}_k}{\longrightarrow} C_k.
\end{equation}

**Remark 5.2.** The projective symmetric monoidal functor $\hat{\Pi}(E) : \text{Cosp}_{\leq d}(\text{CW}_{*}) \to C_k$ satisfies the following (strictly) commutative diagram by definitions.

\begin{equation}
\begin{array}{ccc}
\text{Ho}(\text{CW}_{*, \leq (d-1)}) & \overset{i}{\longrightarrow} & \text{Ho}(\text{CW}_{*, \leq d}) \overset{E}{\longrightarrow} \text{Hopf}_{k}^{bc, \text{vol}} \\
\downarrow & & \downarrow \\
\text{Cosp}_{\leq d}(\text{CW}_{*}) & \overset{\hat{\Pi}(E)}{\longrightarrow} & C_k
\end{array}
\end{equation}

In fact, the commutativity of the diagram follows from Theorem 2.6 and definitions. The analogous statement for $\hat{\Pi}(E)$ is true. In that case, consider the suspension functor instead of the inclusion functor $i$.

**Definition 5.3.** We define two cohomology classes in $H^2(\text{Cosp}_{\leq d}(\text{CW}_{*}^{\text{fin}}); k^*)$ by,

\begin{equation}
\begin{aligned}
\tilde{O}(E) & \overset{\text{def}}{=} O(\hat{\Pi}_k(E)), \\
\hat{O}(E) & \overset{\text{def}}{=} O(\hat{\Pi}_k(E)).
\end{aligned}
\end{equation}

In the following theorem, for $\Theta$, see Definition A.5.
Theorem 5.4. The cohomology class \( \hat{O}(E) \) vanishes if and only if there exists \( \theta \in \Theta(\hat{P}l(E)) \) such that the symmetric monoidal functor \( \theta^{-1} \cdot \hat{P}l(E) : \text{Cosp}^\simeq_{\leq d}(\text{CW}_*^\text{fin}) \to C_k \) satisfies the following (strictly) commutative diagram.

\[
\begin{array}{ccc}
\text{Ho}(\text{CW}_*^\text{fin}_{\leq (d-1)}) & \xrightarrow{i} & \text{Ho}(\text{CW}_*^\text{fin}_{\leq d}) & \xrightarrow{E} & \text{Hopf}_{k}^{\text{bc, vol}} \\
\downarrow & & \downarrow & & \\
\text{Cosp}^\simeq_{\leq d}(\text{CW}_*^\text{fin}) & \xrightarrow{\theta^{-1} \cdot \hat{P}l(E)} & C_k
\end{array}
\]

(21)

The analogous statement for \( \hat{O}(E) \) is true. In that case, consider the suspension functor instead of the inclusion functor \( i \).

The proof follows from the following lemma.

Definition 5.5. Let \( \theta \) be a 1-cochain of \( \text{Cosp}^\simeq_{\leq d}(\text{CW}_*^\text{fin}) \) with coefficients in \( k^* \). The cochain \( \theta \) is good if \( i^* \theta \) is trivial where \( i : \text{Ho}(\text{CW}_*^\text{fin}_{\leq (d-1)}) \to \text{Cosp}^\simeq_{\leq d}(\text{CW}_*^\text{fin}) \) is the embedding functor.

Lemma 5.6. If the obstruction class \( \hat{O}(E) \) vanishes, then there exists \( \theta \in \Theta(\hat{P}l(E)) \) which is good.

Proof. By the assumption, we have \( \Theta(\hat{P}l(E)) \neq \emptyset \). We prove that there exists \( \theta \in \Theta(\hat{P}l(E)) \) such that \( \theta(i(f)) = 1 \) for \( f : K \to L \). Choose any \( \theta' \in \Theta(\hat{P}l(E)) \). By direct calculation, we have \( \omega(i(f), i(g)) = 1 \). Hence, we obtain

\[
\theta'(i(f))\theta'(i(g)) = \theta'(i(g \circ f)).
\]

(22)

For \( \Lambda = \left( K_0 \xrightarrow{f_0} L \xrightarrow{f_1} K_1 \right) \), we define

\[
\theta(\Lambda) = \theta'(i(f_0))^{-1}\theta'(\Lambda)\theta'(i(f_1)).
\]

(23)

By direct calculation, we obtain \( \delta^i \theta = \omega \) where we use (22). In other words, \( \theta \in \Theta(\hat{P}l(E)) \). Then for \( f : K \to L \), we have

\[
\theta(i(f)) = \theta'(i(f))^{-1}\theta'(i(f))\theta'(i(Id_L)),
\]

(24)

\[
= 1.
\]

(25)

\[\square\]

Proof of Theorem 5.4 The obstruction class of a projective symmetric monoidal functor vanishes if and only if it is naturally isomorphic to a symmetric monoidal functor. For example, see Proposition 5.7. We call such a symmetric monoidal functor by a lift of the projective symmetric monoidal functor. The nontrivial part for the proof of the theorem is to verify whether there exists a lift extending the functor \( E \circ i \). See Remark 5.2.

First suppose that the obstruction \( \hat{O}(E) \) vanishes. Then there exists \( \theta \in \Theta(\hat{P}l(E)) \) such that \( \theta(i(f)) = 1 \) for \( f : K \to L \) by Lemma 5.6. For such \( \theta \), the lift \( \theta^{-1} \cdot \hat{P}l(E) \) satisfies the claim by definition. The converse is obvious. It completes the proof of Theorem 5.4.

5.2. Homology theory.

5.2.1. Definitions. Let \( \tilde{E} \) be a \( \text{Hopf}_{k}^{\text{bc}} \)-valued reduced homology theory.

Proposition 5.7. For \(-\infty \leq q_0 \leq q_1 \leq \infty\), the following conditions are equivalent:

1. Let \( q \) be an integer such that \( q_0 \leq q \leq q_1 \). For any pointed finite CW-space \( K \) such that \( \dim K \leq (q - q_0) \), the Hopf monoid \( \tilde{E}_q(K) \) has a finite volume. In other words, the restriction \( \tilde{E}_q : \text{Ho}(\text{CW}^\text{fin}_{*,\leq (q-q_0)}) \to \text{Hopf}_{k}^{\text{bc}} \) factors through \( \text{Hopf}_{k}^{\text{bc, vol}} \).
(2) The $q$-th coefficient $\tilde{E}_q(S^0)$ has a finite volume for any integer $q$ such that $0 \leq q \leq q_1$. Here, $S^0$ denotes the pointed 0-dimensional sphere.

(3) Let $r$ be any integer such that $0 \leq r < (q_1 - q_0)$ and. If $q$ is an integer such that $q_0 + r \leq q \leq q_1$, then the Hopf algebra $\tilde{E}_q(K)$ has a finite volume for any pointed $r$-dimensional finite CW-space $K$.

**Proof.** (1) obviously implies (2).

We prove (3) from (2). If $r = 0$, then $K$ is 0-dimensional so that by (2), the Hopf algebra $\tilde{E}_q(K)$ has a finite volume for $0 \leq q \leq q_1$. Hence (3) holds for $r = 0$. Let $r$ be an integer such that $0 < r < (q_1 - q_0)$. Suppose that if $q$ is an integer such that $q_0 + r < q \leq q_1$, then the Hopf algebra $\tilde{E}_q(K)$ has a finite volume for a pointed $r$-dimensional finite CW-space $K$. Let $L$ be a $(r+1)$-dimensional finite CW-complex. Let $q$ be an integer such that $q_0 + r + 1 \leq q \leq q_1$. Consider the long exact sequence associated with the pair $(L, L^{(r)})$ where $L^{(r)}$ is the $r$-skeleton of $L$.

\begin{equation}
\tilde{E}_{q+1}(L/L^{(r)}) \to \tilde{E}_q(L^{(r)}) \to \tilde{E}_q(L) \to \tilde{E}_q(L/L^{(r)}) \to \tilde{E}_{q-1}(L^{(r)})
\end{equation}

By the assumption, the Hopf algebras $\tilde{E}_q(L^{(r)})$, $\tilde{E}_{q-1}(L^{(r)})$ have a finite volume. Moreover the quotient complex $L/L^{(r)}$ is homeomorphic to a finite bouquet $\bigvee S^{r+1}$ of the pointed $(r+1)$-dimensional spheres. From the isomorphism $\tilde{E}_{q+1}(\bigvee S^{r+1}) \cong \tilde{E}_q(\bigvee S^r)$ and $\tilde{E}_q(\bigvee S^{r+1}) \cong \tilde{E}_{q-1}(\bigvee S^r)$ and the assumption, the Hopf algebras $\tilde{E}_{q+1}(\bigvee S^{r+1})$ and $\tilde{E}_q(\bigvee S^{r+1})$ have a finite volume. The Hopf algebra $\tilde{E}_q(L)$ has a finite volume since $\text{Hopf}_{k, \text{vol}}^{d, \text{vol}} \subset \text{Hopf}_{k}^{d, \text{vol}}$ is closed under short exact sequences. It proves (3).

We prove (1) from (3). Let $q$ be an integer such that $0 \leq q \leq q_1$. Let $K$ be a pointed finite CW-space with $\dim K \leq (q-q_0)$. Put $r = \dim K$. Since $0 \leq r \leq (q_1 - q_0)$ and $q_0 + r \leq q \leq q_1$ by definitions, the Hopf algebra $\tilde{E}_q(K)$ has a finite volume by (3). It completes the proof.

**Definition 5.8.** Denote by $\Gamma(\tilde{E}_\ast)$ the set of integer $q \in \mathbb{Z}$ such that the $q$-th coefficient $\tilde{E}_q(S^0)$ has a finite volume. For $q \in \Gamma(\tilde{E}_\ast)$, we define $d(\tilde{E}_\ast; q) \overset{\text{def}}{=} (q - m(\tilde{E}_\ast; q)) \geq 0$ where

\begin{equation}
m(\tilde{E}_\ast; q) \overset{\text{def}}{=} \inf \{ r \in \Gamma(\tilde{E}_\ast) ; \ r \leq r' \leq q \Rightarrow r' \in \Gamma(\tilde{E}_\ast) \} \geq -\infty.
\end{equation}

**Corollary 5.9.** For $q \in \Gamma(\tilde{E}_\ast)$, the restriction $\tilde{E}_q : \text{Ho}(\text{CW}_{\ast, \leq d}) \to \text{Hopf}_{k, \text{vol}}^{d, \text{vol}}$ factors through $\text{Hopf}_{k, \text{vol}}^{d, \text{vol}}$ where $d = d(\tilde{E}_\ast; q)$. The induced symmetric monoidal functor $\tilde{E}_q^\natural : \text{Ho}(\text{CW}_{\ast, \leq d}) \to \text{Hopf}_{k, \text{vol}}^{d, \text{vol}}$ is a $d$-dimensional Hopf$^{d, \text{vol}}_{k, \text{vol}}$-valued Brown functor.

**Proof.** It is immediate from Proposition 5.7

\[\square\]

5.2.2. **Isomorphism between path-integrals along spans and cospans.**

**Lemma 5.10.** Let $\mathcal{A}$ be a small abelian category and $\mathcal{B}$ be an abelian subcategory of $\mathcal{A}$. Let $\tilde{E}_\ast$ be an $\mathcal{A}$-valued reduced homology theory. Suppose that $\tilde{E}_{q+1}(K)$ lies in $\mathcal{B}$ for a pointed finite CW-space $K$ with $\dim K \leq (d + 1)$. Denote by $\tilde{E}_{q+1}^\prime : \text{Ho}(\text{CW}_{\ast, \leq d+1}) \to \mathcal{B}$ the induced functor. In particular, by the suspension isomorphism, $\tilde{E}_{q+1}^\prime$ induces a functor $\tilde{E}_q : \text{Ho}(\text{CW}_{\ast, \leq d}) \to \mathcal{B}$ such that $\tilde{E}_q(K) = \tilde{E}_q(K)$. Denote $\hat{\Sigma}$ the cospans of $E \circ \Sigma$ for $E = \tilde{E}_q$ in Theorem 2.6 Denote by $\hat{\Sigma}$ the spans of $E \circ \Sigma$ for $E = \tilde{E}_q$ in the first part of Theorem 2.6 Then the following diagram commutes up to $a$
natural isomorphism.

\[
\begin{array}{ccc}
\Cosp^\sim_{ \leq d}(\CW_{s}^{\text{fin}}) & \xrightarrow{\hat{\chi}} & \Cosp^\sim(\mathcal{B}) \\
\downarrow & & \uparrow \\
\Cosp^\sim_{ \leq (d+1)}(\CW_{s}^{\text{fin}}) & \xrightarrow{\hat{\eta}} & \Sp^\sim(\mathcal{B})
\end{array}
\]  

(28)

**Proof.** We show that the dagger-preserving symmetric monoidal functor \( T \circ \hat{Y} \) is a cospanical extension of the symmetric monoidal functor \( \tilde{E}'_q : \CW_{s}^{\text{fin}}_{ \leq d} \to \mathcal{B} \). In fact, for a cospan \( \Lambda = (K_0 \to L \leftarrow K_1) \) in the category \( \CW_{s}^{\text{fin}}_{ \leq d} \), the square diagram (29) is exact by Proposition 2.2. Hence, we obtain \( T(\tilde{E}'_{q+1}(\Sigma(\Lambda))) \leq \tilde{E}'_q(\Lambda) \) by the uniqueness in Theorem 2.6.

\[
\begin{array}{ccc}
\tilde{E}_q(K_0) & \xrightarrow{\hat{\eta}} & \tilde{E}_q(L) \\
\uparrow & & \uparrow \\
\tilde{E}_{q+1}(C(f_0 \vee f_1)) & \xrightarrow{\hat{\eta}} & \tilde{E}_q(K_1)
\end{array}
\]  

(29)

By the uniqueness in Theorem 2.6, we obtain a natural isomorphism \( \hat{X} \cong T \circ \hat{Y} \). \( \square \)

**Theorem 5.11.** Let \( \hat{E}_* \) be a Hopf\(_k^{bc}\) -valued reduced homology theory. Let \( q \in \Gamma(\tilde{E}_*) \) such that \((q + 1) \in \Gamma(\hat{E}_*)\). Denote by \( i : \Cosp^\sim_{ \leq d}(\CW_{s}^{\text{fin}}) \to \Cosp^\sim_{ \leq (d+1)}(\CW_{s}^{\text{fin}}) \) the inclusion functor where \( d = d(\tilde{E}_*; q) \). Then we have a natural isomorphism of projective symmetric monoidal functors in the strong sense,

\[
i^* (\hat{\Pi}(\tilde{E}_q)) \cong \hat{\Pi}(\tilde{E}_q^{\Sigma}).
\]

(30)

**Proof.** Let \( \hat{F} \) be the cospanical extension of \( \tilde{E}_q^{\Sigma} \circ i \) and \( \hat{G} \) be the spanial extension of \( \tilde{E}_q^{\Sigma} \circ \Sigma \). Consider the following diagram of functors. By considering \( \mathcal{A} = \text{Hopf}_k^{bc}, \mathcal{B} = \text{Hopf}_k^{bc, \text{vol}}, \tilde{E}'_{q+1} = \tilde{E}_q^{\Sigma} \) and \( \tilde{E}'_q = \tilde{E}_q^{\Sigma} \) in Lemma 5.10, the left diagram commutes up to the suspension isomorphism. Furthermore, the right diagram commutes up to a natural isomorphism in the strong sense. By composing the natural isomorphisms, we obtain the results.

\[
\begin{array}{ccc}
\Cosp^\sim_{ \leq d}(\CW_{s}^{\text{fin}}) & \xrightarrow{\hat{F}} & \Cosp^\sim(\text{Hopf}_k^{bc, \text{vol}}) \\
\downarrow & & \downarrow \\
\Cosp^\sim_{ \leq (d+1)}(\CW_{s}^{\text{fin}}) & \xrightarrow{\hat{G}} & \Sp^\sim(\text{Hopf}_k^{bc, \text{vol}})
\end{array}
\]  

(31)

By definition of the obstruction class, we obtain the following formula.

**Corollary 5.12.** Let \( q \in \Gamma(\tilde{E}_*) \) such that \((q + 1) \in \Gamma(\hat{E}_*)\). Then we have

\[
i^* (\hat{\Omega}(\tilde{E}_q)) = \hat{\Omega}(\tilde{E}_q^{\Sigma}).
\]

(32)

5.2.3. Inversion formula of obstruction class.

**Definition 5.13.** For \( q \in \Gamma(\tilde{E}_*) \), we define a normalized 1-cochain \( \theta_q(\tilde{E}_*) \) of the symmetric monoidal category \( \Cosp^\sim_{ \leq d}(\CW_{s}^{\text{fin}}) \) with coefficients in \( k^s \). Here, \( d = d(\tilde{E}_*; q) \). Let \( [\Lambda] \) be a morphism in \( \Cosp^\sim_{ \leq d}(\CW_{s}^{\text{fin}}) \) where \( \Lambda = (K_0 \xrightarrow{h_0} L \xleftarrow{h_1} K_1) \). Then,

\[
\theta_q(\tilde{E}_*)([\Lambda]) \overset{\text{def.}}{=} vol^{-1}(\tilde{E}_q(C(f_1))) \in k^s.
\]

(33)
Here, $C(f_1)$ denotes the mapping cone of the pointed map $f_1$. Since $C(f_1)$ is a complex with the dimension lower than $d$, the bicommutative Hopf monoid $\tilde{E}_q(C(f_1))$ has a finite volume so that (33) is well-defined. Note that the 1-cochain $\theta_q(\tilde{E}_*)$ is good in the sense of Definition 5.5.

**Remark 5.14.** By Corollary 3.7, the definition is equivalent with

$$\theta_q(\tilde{E}_*)([\Lambda]) \overset{\text{def}}{=} \dim(\tilde{E}_q(C(f_1)))^{-1} \in k^*.$$  

**Lemma 5.15.** Suppose that we have an exact sequence in the abelian category $\text{Hopf}_{k}^{\text{bc,vol}}$:

$$\newcommand{\map}{\rightarrow} C_1 \map A_0 \map B_0 \map C_0 \map A_{-1}.\tag{36}$$

Then we have

$$\langle \check{\alpha}_0 \rangle \cdot \langle \check{\alpha}_1 \rangle = \text{vol}^{-1}(A_0) \cdot \text{vol}^{-1}(B_0) \cdot \text{vol}^{-1}(C_0).\tag{37}$$

**Proof.** Note that the exact sequence induces the following exact sequence:

$$k \map \text{Im}(\check{\alpha}_1) \map A_0 \map B_0 \map C_0 \map \text{Coim}(\check{\alpha}_0) \map k.\tag{38}$$

Since the inverse volume is a volume on the abelian category $\text{Hopf}_{k}^{\text{bc,vol}}$ by Corollary 2.12, we obtain the following equation:

$$\text{vol}^{-1}(\text{Im}(\check{\alpha}_1)) \cdot \text{vol}^{-1}(B_0) \cdot \text{vol}^{-1}(\text{Coim}(\check{\alpha}_0)) = \text{vol}^{-1}(A_0) \cdot \text{vol}^{-1}(C_0).\tag{39}$$

Since $\text{vol}^{-1}(\text{Im}(\check{\alpha}_1)) = \langle \check{\alpha}_1 \rangle$ and $\text{vol}^{-1}(\text{Coim}(\check{\alpha}_0)) = \langle \check{\alpha}_0 \rangle$ by Proposition 9.9 in [10], the claim is proved. \hfill \Box

**Lemma 5.16.** Let $\tilde{E}_*$ be a $\text{Hopf}_{k}^{\text{bc}}$-valued homotopy theory. For $q \in \Gamma(\tilde{E}_*)$, we have

$$\omega(\tilde{\Pi}(\tilde{E}^3_q)) \cdot \omega(\tilde{\Pi}(\tilde{E}^2_q)) = \delta^1(\theta_q(\tilde{E}_*)).\tag{40}$$

**Proof.** Let $d = d(\tilde{E}_*; q)$. Consider composable morphisms $[\Lambda], [\Lambda']$ in $\text{Cosp}_{\text{bc}}^\geq (\text{CW}_{*}^\text{in})$ with the representatives,

$$\Lambda = \left( K_0 \overset{f_0}{\map} L \overset{f_1}{\map} K_1 \right), \quad \Lambda' = \left( K_1 \overset{f_1'}{\map} L' \overset{f_2}{\map} K_2 \right).\tag{41}$$

We introduce notations of maps associated with the composition $\Lambda' \circ \Lambda$ and $\Sigma(\Lambda), \Sigma(\Lambda'), \Sigma(\Lambda' \circ \Lambda)$ following Figure 2.3 For simplicity, denote by $\alpha = \omega(\tilde{\Pi}(\tilde{E}^3_q))$ and $\beta = \omega(\tilde{\Pi}(\tilde{E}^2_q))$.

Recall the definition, $\alpha([\Lambda], [\Lambda']) = \langle \text{cok}(\tilde{E}_q(f_2')) \circ \ker(\tilde{E}_q(g')) \rangle$. Let $\tilde{E}_{q+1}(C(g')) \map \tilde{E}_q(C(f_2'))$ be the connecting morphism in the long exact sequence associated with the mapping cone sequence $C(f_2') \map C(g' \circ f_2') \map C(g')$. Note that the images of $\text{cok}(\tilde{E}_q(f_2')) \circ \ker(\tilde{E}_q(g'))$ and $\tilde{E}_{q+1}(C(g'))$ are canonically isomorphic with each other. In fact, each long exact sequence associated with $g'$ and $f_2'$ respectively implies that the image of $\text{cok}(\tilde{E}_q(f_2')) \circ \ker(\tilde{E}_q(g'))$ coincides with the image of the composition $\tilde{E}_{q+1}(C(g')) \overset{\tilde{E}_{q+1}(C(g'))}{\map} \tilde{E}_q(L') \overset{\tilde{E}_{q}(f_2')}{\map} \tilde{E}_q(L')$. Then we have $\alpha([\Lambda], [\Lambda']) = \langle \text{cok}(\tilde{E}_q(f_2')) \circ \ker(\tilde{E}_q(g')) \rangle = \langle \tilde{E}_{q+1}(C(g')) \rangle$ by Proposition 9.9 [10].

In the same manner, we calculate $\beta([\Lambda], [\Lambda'])$ to obtain $\beta([\Lambda], [\Lambda']) = \langle \tilde{E}_{q+1}(C(g')) \rangle$. Note that the suspension of the mapping cone sequence $C(f_2') \map C(g' \circ f_2') \map C(g')$ is homotopy equivalent with the mapping cone sequence $C(k) \map C(h_0 \circ k) \map C(h_0)$. It implies that $\langle \tilde{E}_{q+1}(C(g')) \rangle = \beta([\Lambda], [\Lambda'])$. Above all, we obtain $\alpha([\Lambda], [\Lambda']) \cdot \beta([\Lambda], [\Lambda']) = \langle \tilde{E}_{q+1}(C(g')) \rangle$. By applying Lemma 5.15, it is verified that $\alpha([\Lambda], [\Lambda']) \cdot \beta([\Lambda], [\Lambda'])$ coincides with $\delta^1(\theta_q(\tilde{E}_*))([\Lambda], [\Lambda'])$. It completes the proof.
Theorem 5.17. Let \( \tilde{E}_\bullet \) be a Hopf \( \mathcal{H} \)-valued reduced homology theory. For \( q \in \Gamma(\tilde{E}_\bullet) \), we have
\[
\hat{O}(\tilde{E}_q) = \tilde{O}(\tilde{E}_q)^{-1}.
\]
\[\text{(41)}\]

Proof. It follows from Lemma 5.16. \( \square \)

Corollary 5.18. If \( q, (q + 1) \in \Gamma(\tilde{E}_\bullet) \), then we have
\[
i^*(\hat{O}(\tilde{E}_{q+1})) = \tilde{O}(\tilde{E}_q)^{-1}.
\]
\[\text{(42)}\]

Proof. It is immediate from Corollary 5.12 and Theorem 5.17. \( \square \)

6. Vanishing of the obstruction classes

In this section, we give our main results which imply the main theorems. We verify that some obstruction classes vanish mainly by using the formulas in the previous section. One class of examples is obtained from a dimension reduction of homology theories, and the other class is from bounded homology theories. The latter examples contain a generalization of DWFQ and TVBW TQFT’s.

6.1. Dimension reduction.

Definition 6.1. Let \( X \) be a pointed finite CW-space. For a pointed finite CW-space \( K \), we define \( W_X(K) \) by
\[
W_X(K) = K \wedge X.
\]

By the functoriality of \( W_X \), an \( \mathcal{A} \)-valued reduced homology theory \( \tilde{E}_\bullet \) induces a homology theory. We denote it by \( W_X^*\tilde{E}_\bullet \). In particular, \( W_X^*\tilde{E}_q(K) = \tilde{E}_q(K \wedge X) \).
Lemma 6.2. Let $\tilde{E}_\ast$ be an $\mathcal{A}$-valued homology theory. We have an isomorphism of homology theories, $W^*_{\mathcal{T}, \tilde{E}_\ast} \cong \tilde{E}_\ast \oplus \tilde{E}_{\ast-1}$.

Proof. Denote a pointed $n$-sphere by $S^n$. Denote $T$ a 1-sphere without basepoint. By the mapping cone sequence $S^0 \to T^+ \to S^1$, we obtain a split exact sequence for a pointed finite CW-space $K$:

$$0 \to W^*_{S^0, \tilde{E}_q(K)} \to W^*_{T^+, \tilde{E}_q(K)} \to W^*_{S^1, \tilde{E}_q(K)} \to 0.$$  

Furthermore, we obtain an isomorphism of homology theories, $W^*_{\mathcal{T}, \tilde{E}_\ast} \cong W^*_{S^0, \tilde{E}_\ast} \oplus W^*_{S^1, \tilde{E}_\ast}$. Here, $\oplus$ denotes the biproduct in $\mathcal{A}$. By the natural isomorphism $X \cong X \wedge S^0$, we obtain an isomorphism of homology theories, $W^*_{S^0, \tilde{E}_\ast} \cong \tilde{E}_\ast$. With some careful treatment of signs, we also obtain an isomorphism, $W^*_{S^1, \tilde{E}_\ast} \cong \tilde{E}_{\ast-1}$ by using the suspension isomorphism. It completes the proof. \hfill $\square$

From now on, we consider $\mathcal{A} = \text{Hopf}_k^\text{bc}$ the category of bicommutative Hopf algebras. Fix a $\text{Hopf}_k^\text{bc}$-valued reduced homology theory $\tilde{E}_\ast$. We put $\tilde{F}_\ast = W^*_k \tilde{E}_\ast$.

Proposition 6.3. We have

$$\Gamma(\tilde{F}_\ast) = \Gamma(\tilde{E}_\ast) \cap \left( \Gamma(\tilde{E}_\ast) + 1 \right).$$

Proof. Note that the biproduct in the abelian category $\mathcal{A} = \text{Hopf}_k^\text{bc}$ is the tensor product of Hopf algebras. By Lemma 6.2, it suffices to prove that $\Gamma(\tilde{E}_\ast \otimes \tilde{E}_{\ast-1}) = \Gamma(\tilde{E}_\ast) \cap \Gamma(\tilde{E}_{\ast-1})$. $\Gamma(\tilde{E}_\ast) \cap \Gamma(\tilde{E}_{\ast-1}) \subset \Gamma(\tilde{E}_\ast \otimes \tilde{E}_{\ast-1})$ is clear. Let $q \in \Gamma(\tilde{E}_\ast \otimes \tilde{E}_{\ast-1})$, i.e. the Hopf algebra $\tilde{E}_q(S^0) \otimes \tilde{E}_{q-1}(S^0)$ has a finite volume. We claim that $\tilde{E}_q(S^0)$ and $\tilde{E}_{q-1}(S^0)$ have a finite volume. More generally, for a bicommutative Hopf algebras $A, B$, if the tensor product $A \otimes B$ has a finite volume, then $A$ has a finite volume. In fact, the composition of the inclusion $i : A \to A \otimes B$ and the normalized cointegral on $A \otimes B$ induces a normalized cointegral on $A$. In the same manner, $A$ is proved to have a normalized integral by using the projection $A \otimes B \to A$. Likewise, the Hopf algebra $B$ has a normalized integral and a normalized cointegral. In particular, the inverse volume of $A, B$ are defined. By Theorem 2.11, we obtain $\text{vol}^{-1}(A) \cdot \text{vol}^{-1}(B) = \text{vol}^{-1}(A \otimes B)$. The inverse volume $\text{vol}^{-1}(A \otimes B)$ is invertible so that $\text{vol}^{-1}(A)$ is invertible. It proves that $A$ has a finite volume. \hfill $\square$

Proposition 6.4. Let $q \in \Gamma(\tilde{F}_\ast)$.

1. The obstruction classes $\tilde{O}(\tilde{F}_q), \tilde{O}(\tilde{E}_q)$ vanish.
2. $\omega(\tilde{P}(\tilde{F}_q)) = \delta^1(\theta_{q-1}(\tilde{E}_\ast))$.
3. $\omega(\tilde{P}(\tilde{F}_q)) = \delta^1(i^*(\theta_q(\tilde{E}_\ast)))$. Here, $i$ is the inclusion from the $(d - 1)$-dimensional cospan category of spaces to the $d$-dimensional one.

Proof. The first part follows from the last two statements, but here we give a way to compute the classes by using formulas of obstruction classes. We first prove that the class $\tilde{O}(\tilde{F}_q)$ vanishes. By Proposition 6.3, $q \in \Gamma(\tilde{F}_\ast)$ implies $q \in \Gamma(\tilde{E}_\ast) \cap \left( \Gamma(\tilde{E}_\ast) + 1 \right)$ so that the obstruction classes $\tilde{O}(\tilde{E}_{q-1}), \tilde{O}(\tilde{E}_q)$ are defined. By Lemma 6.2, we have $\tilde{O}(\tilde{F}_q) = i^*(\tilde{O}(\tilde{E}_q)) \cdot \tilde{O}(\tilde{E}_{q-1})$. By Corollary 5.18, we obtain $\tilde{O}(\tilde{F}_q) = 1$. By Theorem 5.17, the obstruction class $\tilde{O}(\tilde{F}_q)$ also vanishes.

Note that the Brown functor $\tilde{F}_q$ is isomorphic to $i^*\tilde{E}_q \otimes \tilde{E}_{q-1}$ by definitions where $i : \text{Ho}(\text{CW}_{\ast, \leq (d-1)}) \to \text{Ho}(\text{CW}_{\ast, \leq d})$ is the inclusion. Hence, $\tilde{P}(\tilde{F}_q) \cong \tilde{P}(i^*\tilde{E}_q) \otimes \tilde{P}(\tilde{E}_{q-1}) \cong i^*\tilde{P}(\tilde{E}_q) \otimes \tilde{P}(\tilde{E}_{q-1})$ where the last $i$ denotes the inclusion from $(d - 1)$-dimensional cospan
category of spaces to $d$-dimensional one. By Theorem 5.11 we obtain an isomorphism of projective symmetric monoidal functors in the strong sense:

\begin{equation}
\mathcal{P}l(\tilde{F}_q^\circ) \cong \mathcal{P}l(\tilde{E}_{q-1}^\circ) \otimes \mathcal{P}l(\tilde{E}_{q-1}^\circ).
\end{equation}

By Lemma 5.16 we obtain $\omega(\mathcal{P}l(\tilde{F}_q)) = \delta^i(\theta_{q-1}(\tilde{E}_*)).$

We compute $\omega(\mathcal{P}l(\tilde{F}_q^{\circ}))$ as follows. We have $\omega(\mathcal{P}l(\tilde{F}_q^{\circ})) = \delta^i(\theta_{q}(\tilde{F}_*)) \cdot \omega(\mathcal{P}l(\tilde{F}_q))^{-1}$ by Lemma 5.16. By the previous result, we obtain $\omega(\mathcal{P}l(\tilde{F}_q)) = \delta^i(\theta_{q}(\tilde{F}_*) \cdot \theta_{q-1}(\tilde{E}_*)^{-1}).$ By Lemma 6.2 we have $\theta_q(\tilde{F}_*) = \iota^*(\theta_q(\tilde{E}_*)) \cdot \theta_{q-1}(\tilde{E}_*)$ so that we obtain the claim.

**Theorem 6.5.** For $q \in \Gamma(\tilde{F}_*)$, let $d = d(\tilde{E}_*; q) = d(\tilde{F}_*; q) + 1$. Then there exists a canonical symmetric monoidal functor $Z = \tilde{Z}(\tilde{F}_*; q) : \text{Cosp}^\lessapprox(d-1)(\text{CW}_{\text{fin}}^\lessapprox) \to C_k$ satisfying the following conditions:

1. The diagram below commutes strictly.

\begin{equation}
\begin{array}{ccc}
\text{Ho}(\text{CW}_{\text{fin}}^\lessapprox(d-2)) & \xleftarrow{i} & \text{Ho}(\text{CW}_{\text{fin}}^\lessapprox(d-1)) \\
\downarrow & & \downarrow \\
\text{Cosp}^\lessapprox(d-1)(\text{CW}_{\text{fin}}^\lessapprox) & \xrightarrow{Z} & C_k
\end{array}
\end{equation}

2. The induced homotopy invariant is given by

\begin{equation}
Z(L) = \dim(\tilde{E}_{q-1}(L)) \in k.
\end{equation}

**Proof.** The existence of such $Z$ satisfying only the first condition is easily deduced from the first part of Proposition 6.4. To construct such $Z$ also satisfying the second condition, we need to choose some concrete lift of the projective functor obtained from path-integral. In fact, by the second part of Proposition 6.4, we could choose $\theta_{q-1}(\tilde{E}_*)$ as a canonical complementary 1-cochain. Then $Z = \theta_{q-1}(\tilde{E}_*)^{-1} \cdot \mathcal{P}l(\tilde{F}_q)^{\circ}$ satisfies all the conditions by definitions.

In a parallel way, one can use the path-integral along cospan diagrams. In that case, we use the third part of Proposition 6.4 instead of the second part. We give the result without proof as follows.

**Theorem 6.6.** For $q \in \Gamma(\tilde{F}_*)$, let $d = d(\tilde{E}_*; q) = d(\tilde{F}_*; q) + 1$. Then there exists a canonical symmetric monoidal functor $Z = \tilde{Z}(\tilde{E}_*; q) : \text{Cosp}^\lessapprox(d-1)(\text{CW}_{\text{fin}}^\lessapprox) \to C_k$ satisfying the following conditions:

1. The diagram below commutes strictly.

\begin{equation}
\begin{array}{ccc}
\text{Ho}(\text{CW}_{\text{fin}}^\lessapprox(d-2)) & \xleftarrow{i} & \text{Ho}(\text{CW}_{\text{fin}}^\lessapprox(d-1)) \\
\downarrow & & \downarrow \\
\text{Cosp}^\lessapprox(d-1)(\text{CW}_{\text{fin}}^\lessapprox) & \xrightarrow{Z} & C_k
\end{array}
\end{equation}

2. The induced homotopy invariant is given by

\begin{equation}
Z(L) = \dim(\tilde{E}_{q}(L)) \in k.
\end{equation}
Proposition 6.7. If \( q, (q + 1) \in \Gamma(\tilde{F}_*) \), then the restriction of \( \tilde{Z}(\tilde{E}_*; q + 1) \) coincides with \( \tilde{Z}(\tilde{E}_*; q) \). In other words, if \( d = d(\tilde{E}_*; q) \), then the diagram below commutes strictly:

\[
\begin{array}{ccc}
\text{Cosp}^\cong_{\leq (d-1)}(\text{CW}^{\text{fin}}_\ast) & \xrightarrow{\tilde{Z}(\tilde{E}_*; q)} & C_k \\
\downarrow & & \downarrow \\
\text{Cosp}^\cong_{\leq d}(\text{CW}^{\text{fin}}_\ast) & \xrightarrow{\tilde{Z}(\tilde{E}_*; q+1)} & C_k
\end{array}
\]

(50)

Proof. It follows from Theorem 5.11 and definitions of \( \tilde{Z}(\tilde{E}_*; q) \), \( \tilde{Z}(\tilde{E}_*; q + 1) \). \( \square \)

Example 6.8. In [12], we give three ways to construct some nontrivial class of Hopf-k-valued homology theories. For such homology theories \( \tilde{E}_* \), one can obtain homotopy-theoretic analogue of TQFT’s by Theorem 6.5, 6.6. In particular, they are parametrized by the set \( \Gamma(\tilde{F}_*) \).

1. Let \( \tilde{D}_* \) be a generalized homology theory. The group Hopf algebra functor with coefficients in \( k \) induces a Hopf-k-valued homology theory \( \tilde{E}_* = k\tilde{D}_* \). Then \( \Gamma(\tilde{F}_*) \) consists of \( q \in \mathbb{Z} \) such that \( \tilde{D}_q(S^0) \) is finite and its order is coprime to the characteristic of \( k \). For example, let \( k \) be a field with characteristic zero. Consider the generalized reduced homology theory \( \tilde{D}_* = \tilde{\pi}_* \) induced by the sphere spectrum. It is well-known that the \( q \)-th coefficient \( \tilde{\pi}_q(S^0) \) is finite if \( q \neq 0 \). Hence, the obtained Hopf-k-valued homology theory \( \tilde{E}_* = k\tilde{D}_* \) satisfies \( \Gamma(\tilde{E}_*) = \mathbb{Z}\{0\} \). By Proposition 6.3, we obtain \( \Gamma(\tilde{F}_*) = \Gamma(\tilde{E}_*) \cap (\Gamma(\tilde{E}_*) + 1) = \mathbb{Z}\{0, 1\} \). For an integer \( q \geq 2 \), we have \( d(\tilde{E}_*; q) = (q - 2) \) by definitions. The symmetric monoidal functors \( \tilde{Z}(\tilde{E}_*; q) \) in Theorem 6.5 and \( \tilde{Z}(\tilde{E}_*; q) \) in 6.6 are defined on \( (q - 3) \)-dimensional cospan category of spaces. For a pointed finite CW-space \( K \) with \( \dim K \leq (q - 4) \), we have

\[
\tilde{Z}(\tilde{E}_*; q)(K) = \tilde{F}_{q-1}(K) \cong k (\tilde{\pi}_{q-1}^0(K) \times \tilde{\pi}_{q-2}^0(K)),
\]

(51)

\[
\tilde{Z}(\tilde{E}_*; q)(K) = \tilde{F}_q(K) \cong k (\tilde{\pi}_q^0(K) \times \tilde{\pi}_{q-1}^0(K)).
\]

(52)

Furthermore, for a pointed finite CW-space \( L \) with \( \dim L \leq (q - 3) \), we have \( \tilde{Z}(\tilde{E}_*; q)(L) = |\tilde{\pi}_{q-1}^0(L)| \) and \( \tilde{Z}(\tilde{E}_*; q)(L) = |\tilde{\pi}_q^0(L)| \).

2. An ordinary homology theory with coefficients in a Hopf algebra provides a class of examples. Let \( A \) be a bicommutative Hopf algebra with a finite volume. Then there exists an ordinary homology theory with coefficients in \( A \) which we denote by \( \tilde{E}_*(-) = \tilde{H}_*(-; A) \). By definitions, we have \( \Gamma(\tilde{E}_*) = \mathbb{Z} \) so that \( \Gamma(\tilde{F}_*) = \mathbb{Z} \) by Proposition 6.3. For \( q \in \Gamma(\tilde{F}_*) \), we have \( d(\tilde{E}_*; q) = \infty \). The symmetric monoidal functors \( \tilde{Z}(\tilde{E}_*; q) \) in Theorem 6.5 and \( \tilde{Z}(\tilde{E}_*; q) \) in 6.6 are defined on \( \infty \)-dimensional cospan category of spaces. For a pointed finite CW-space \( K \), we have

\[
\tilde{Z}(\tilde{E}_*; q)(K) \cong \tilde{H}_{q-1}(K; A) \otimes \tilde{H}_{q-2}(K; A),
\]

(53)

\[
\tilde{Z}(\tilde{E}_*; q)(K) \cong \tilde{H}_q(K; A) \otimes \tilde{H}_{q-1}(K; A).
\]

(54)

3. An exponential functor gives a class of examples. Let \( h \) be another field. Consider a bicommutative Hopf algebra \( A \) over \( k \) with an \( h \)-action \( \alpha \). Then there exists an assignment of a Hopf-h-valued homology theory \( \tilde{E}_* = (A, \alpha)\tilde{D}_* \) to a \( \text{Vec}_h^{\text{fin}} \)-valued homology theory \( \tilde{D}_* \). Note that a \( \text{Vec}_h^{\text{fin}} \)-valued homology theory is nothing but a generalized homology theory such that \( \tilde{D}_*(K) \) are finite-dimensional vector spaces over \( h \). If \( A \) is not a group Hopf algebra, then the obtained homology theory \( \tilde{E}_* \) is not induced by the group Hopf algebra functor as the first part above. By definitions, we have \( \Gamma(\tilde{E}_*) = \mathbb{Z} \) and \( d(\tilde{E}_*; q) = \infty \) for any \( q \in \Gamma(\tilde{E}_*) \). The symmetric monoidal
functors $\tilde{Z}(\tilde{E}_*:q)$ in Theorem 6.5 and $\check{Z}(\check{E}_*:q)$ in 6.6 are defined on $\infty$-dimensional cospan category of spaces. For a pointed finite CW-space $K$, we have isomorphisms of Hopf algebras

\begin{align}
\tilde{Z}(\tilde{E}_*:q)(K) &\cong \bigotimes_{b_q(K)} A \otimes \bigotimes_{b_q(K)} A, \\
\check{Z}(\check{E}_*:q)(K) &\cong \bigotimes_{b_q(K)} A \otimes \bigotimes_{b_q(K)} A.
\end{align}

Here, we fix a basis $b_q(K)$ of the vector space $\tilde{D}_q(K)$. Furthermore, for a pointed finite CW-space $L$ with $\dim L \leq (q - 3)$, we have $\tilde{Z}(\tilde{E}_*:q)(L) = (\dim A)^{\dim D_q(L)}$ and $\check{Z}(\check{E}_*:q)(L) = (\dim A)^{\dim D_q(L)}$.

**Remark 6.9.** Note that the set $\Gamma(\tilde{F}_*)$ could be empty in general even if $\Gamma(\check{F}_*)$ is not. For example, consider reduced K-theory $\tilde{K}_*$. Then the group Hopf algebra functor over a field $k$ induces a Hopf $k^\text{bc}$-valued homology theory $\tilde{E}_* = k\tilde{K}_*$. Then $\Gamma(\tilde{F}_*) = \mathbb{ZZ} + 1$ since the coefficient Hopf algebra $k\tilde{K}_q(S^0)$ has a finite volume if and only if $q$ is odd by definitions. By Proposition 6.3, we obtain $\Gamma(\tilde{F}_*) = \emptyset$.

**Remark 6.10.** In [6][8][16], gauge fields in DWFQ theory are described by classifying maps. In this sense, DWFQ theory is a sigma-model whose target space is the classifying space. The first part of Example 6.8 or 6.14 gives more examples of possible sigma-models which naturally have a quantization by path-integral. In fact, a spectrum in algebraic-topological sense plays a role of the target space. It is well-known that a spectrum induces a generalized cohomology theory of CW-spaces. Such generalized cohomology theory is constructed by a homotopy set of maps from (the suspension spectrum of) spaces to the spectrum.

### 6.2. Bounded-below homology theory

Let $\tilde{E}_*$ be a Hopf $k^\text{bc}$-valued reduced homology theory which is bounded below. In other words, there exists $q_0 \in \mathbb{Z}$ such that $q < q_0$ implies $\tilde{E}_q(K) \cong k$ for any pointed finite CW-space $K$. In this subsection, we prove that the obstruction class $\tilde{O}(\tilde{E}_q)$ vanishes for $q \in \Gamma(\tilde{E}_*)$ such that $m(\tilde{E}_*:q) = -\infty$ (see Definition 5.8). More strongly, we give a canonical complementary 1-cochain for the projective functor $\tilde{P}(\tilde{E}_q)$.

There is an analogous result for bounded above homology theory and appropriate degree $q$. In that case, the obstruction class $\tilde{O}(\check{E}_q) = O(\check{P}(\check{E}_q))$ vanishes due to an analogue of Proposition 6.12.

**Definition 6.11.** We define a normalized 1-cochain $\theta_{q}(\tilde{E}_*)$ of the symmetric monoidal category $\text{Cosp}^\text{bc}_{\infty}(\text{CW}^\text{in})$ with coefficients in the multiplicative group $k^*$ by

\begin{equation}
\theta_{q}(\tilde{E}_*) \overset{\text{def.}}{=} \prod_{i \geq 0} \theta_{q-i}(\tilde{E}_*)(-1)^i
\end{equation}

Here, the normalized 1-cochains $\theta_{q}(\tilde{E}_*)$ are defined in Definition 5.13. Note that the 1-cochain $\theta_{q}(\tilde{E}_*)$ is good in the sense of Definition 5.5 since so does $\theta_{q}(\check{E}_*)$.

**Proposition 6.12.** Let $\tilde{E}_*$ be a Hopf $k^\text{bc}$-valued reduced homology theory which is bounded below. For $q \in \Gamma(\tilde{E}_*)$ such that $m(\tilde{E}_*:q) = -\infty$, we have

\begin{equation}
\omega(\tilde{P}(\tilde{E}_q)) = \delta^1 \theta_{q}(\tilde{E}_*)
\end{equation}

In particular, the obstruction class $\tilde{O}(\tilde{E}_q)$ vanishes.

**Proof.** By Lemma 5.16, we obtain $\omega(\tilde{P}(\tilde{E}_q)) = \omega(\tilde{P}(\tilde{E}_q))^{-1} \cdot \delta^1(\theta_{q}(\check{E}_*))$. By Theorem 5.11, we have a natural isomorphism $\tilde{P}(\tilde{E}_q) \cong \check{P}(\check{E}_q)$ in the strong sense so that we obtain
\[ \omega(\widetilde{\mathcal{P}}(\widetilde{E}_{q}^{\circ})) = \omega(\widetilde{\mathcal{P}}(\widetilde{E}_{q-1}^{\circ}))^{-1} \cdot \delta^1(\theta_q(\widetilde{E}_{q}^{\circ})). \] We repeat this formula until the integer \( q_0 \). Since the homology theory \( \widetilde{E}_{\bullet} \) is assumed to be bounded below, we obtain the claim. \( \square \)

**Theorem 6.13.** Let \( q \in \Gamma(\widetilde{E}_{\bullet}) \) such that \( m(\widetilde{E}_{q}^{\circ}; q) = -\infty \). Then there exists a canonical symmetric monoidal functor \( Z = \widetilde{Z}(\widetilde{E}_{\bullet}; q) : \text{Cosp}^{\infty}_{\leq \infty}(\text{CW}_{\bullet}^{\text{fin}}) \rightarrow C_k \) satisfying the following conditions:

1. The diagram below commutes strictly.

\[
\begin{array}{ccc}
\text{Ho}(\text{CW}_{\bullet}^{\text{fin}}) & \rightarrow & \text{Hopf}^{\text{bc,vol}}_k \\
\downarrow & & \downarrow \\
\text{Cosp}^{\infty}_{\leq \infty}(\text{CW}_{\bullet}^{\text{fin}}) & \rightarrow & C_k
\end{array}
\]

(59)

2. The induced homotopy invariant is given by

\[
Z(L) = \prod_{l \geq 0} \dim(\widetilde{E}_{q-1}^{-l}(L))^{-1} \in k.
\]

**Proof.** Note that \( d(\widetilde{E}_{q}^{\circ}; q) = q - m(\widetilde{E}_{q}^{\circ}; q) = \infty \). We choose some concrete lift of the projective functor obtained from path-integral. In fact, by Proposition 6.4 we could choose \( \theta_{\leq q}(\widetilde{E}_{\bullet}) \) as a canonical complementary 1-cochain. Then \( \widetilde{Z}(\widetilde{E}_{q}^{\circ}; q) = \theta_{\leq q}(\widetilde{E}_{\bullet})^{-1} \cdot \widetilde{\mathcal{P}}(\widetilde{E}_{q}^{\circ}) \) satisfies all the conditions by definitions. \( \square \)

**Example 6.14.** Recall Example 6.8

1. Let \( \widetilde{D}_{\bullet} \) be a generalized homology theory which is bounded below. Then the induced Hopf\( k_{\text{bc}} \)-valued homology theory \( \widetilde{E}_{\bullet} = k\widetilde{D}_{\bullet} \) is bounded below. Let \( q \in \mathbb{Z} \) such that \( q' < q \) implies that the \( q' \)-th coefficient \( \widetilde{D}_{q'}(S^0) \) is finite and its order is coprime to the characteristic of \( k \). Then \( q \in \Gamma(\widetilde{E}_{\bullet}) \) satisfies \( m(\widetilde{E}_{q}^{\circ}; q) = -\infty \) by definitions. Hence, for such \( q \in \Gamma(\widetilde{E}_{\bullet}) \) we obtain a symmetric monoidal functor \( Z = \widetilde{Z}(\widetilde{E}_{q}^{\circ}; q) \) satisfying the conditions in Theorem 6.13. In particular, the induced homotopy invariant is given by \( Z(L) = \prod_{l \geq 0} \dim(\widetilde{D}_{q-1}^{-l}(L))^{-1} \in k. \)

2. Let \( A \) be a bicommutative Hopf algebra over \( k \). Let \( \widetilde{E}_{\bullet}(\dashv) \equiv \widetilde{H}_{\bullet}(\dashv; A) \) be the reduced ordinary homology theory with coefficients in \( A \). The homology theory \( \widetilde{E}_{\bullet} \) is bounded below. Suppose that the Hopf algebra \( A \) has a finite volume. Then we have \( \Gamma(\widetilde{E}_{\bullet}) = \mathbb{Z} \) and \( m(\widetilde{E}_{q}^{\circ}; q) = -\infty \) for any \( q \in \Gamma(\widetilde{E}_{\bullet}) \) by definitions. The application of Theorem 6.13 gives a generalization of Dijkgraaf-Witten-Freed-Quinn TQFT for an abelian groups and Turaev-Viro-Barrett-Westbury TQFT for a bicommutative Hopf algebra. See subsection 6.3.

3. Let \( h \) be another field. Consider a bicommutative Hopf algebra \( A \) over \( k \) with an \( h \)-action \( \alpha \). Then there exists an assignment of a Hopf\( h_{\text{bc}} \)-valued homology theory \( \widetilde{E}_{\bullet} = (A, \alpha)^{\hat{D}_{\bullet}} \) to a Vec\( h_{\text{bc}} \)-valued homology theory \( \hat{D}_{\bullet} \). By definitions, we have \( \Gamma(\widetilde{E}_{\bullet}) = \mathbb{Z} \) and \( m(\widetilde{E}_{q}^{\circ}; q) = -\infty \) for any \( q \in \Gamma(\widetilde{E}_{\bullet}) \). If \( \hat{D}_{\bullet} \) is bounded below, then so is \( \widetilde{E}_{\bullet} \). The application of Theorem 6.13 gives a symmetric monoidal functor \( Z = \widetilde{Z}(\widetilde{E}_{q}^{\circ}; q) \) for \( q \in \Gamma(\widetilde{E}_{\bullet}) \). In particular, we have \( Z(L) = (\dim A)^{\chi_{\leq q}(L, \hat{D}_{\bullet})} \) where \( \chi_{\leq q}(L, \hat{D}_{\bullet}) = \sum_{l \geq 0} (-1)^l \cdot \dim \hat{D}_{q-1}^{-l}(L). \)

6.3. **DWFQ TQFT and TVBW TQFT.**

**Definition 6.15.** Let \( d \in \mathbb{N} \cup \{\infty\} \). For \( n \in \mathbb{N} \) such that \( n \leq d \), let \( \text{Cob}_n \) be the \( n \)-dimensional cobordism category of oriented smooth compact manifolds. We define a symmetric monoidal functor \( \Phi_{n,d} : \text{Cob}_n \rightarrow \text{Cosp}_{\leq d}(\text{CW}_{\bullet}^{\text{fin}}) \). It assigns \( M^+ = M \sqcup \{\text{pt}\} \)
We sketch the proof. Note that the set of isomorphism classes of principal bundles
(62) \( \Phi_{n,d}([N; M_0, M_1]) \overset{\text{def}}{=} [M_0^+ \hookrightarrow N^+ \hookrightarrow M_1^+] \).
It is obviously well-defined. Especially, the composition is preserved since boundaries have a
collar neighborhood.

Definition 6.16. Consider \( k = \mathbb{C} \), the complex number field. Note that any bicommutative
Hopf algebra \( A \) with a finite volume is a function Hopf algebra by Wedderburn’s theorem. In
other words, there naturally exists a group with an isomorphism \( A \cong \mathbb{C}^G \). Via the isomor-
phism, the Hopf algebra \( A \) has a unique Hilbert inner product whose orthonormal basis is
given by \( \delta_g \in \mathbb{C}^G, g \in G \), i.e. the delta functionals. It induces a symmetric monoidal functor
\( U_{Hil} : \mathbb{C}_C \to \text{Hil}^{\text{fin}} \) where \( \text{Hil}^{\text{fin}} \) is the category of finite-dimensional Hilbert spaces.

Proposition 6.17. Let \( n \in \mathbb{N} \) such that \( n \leq d \). Let \( k = \mathbb{C} \). Every symmetric monoidal functor
from \( \text{Cosp}_{\mathbb{C}^d}(\mathbb{C}W_{\mathbb{C}^d}^*) \) to \( \mathbb{C}_k \) induces a \( \text{Hil}^{\text{fin}} \)-valued \( n \)-dimensional TQFT by \( \Phi_{n,d} \) and \( U_{Hil} \).

Remark 6.18. Even if the ground field \( k \) is general, there is an assignment of nondegenerate
pairing to bicommutative Hopf algebras with a finite volume. Denote by \( \text{Vec}_{\text{ssd}} \) the category
of vector spaces \( (V, i : k \to V \otimes V, e : V \otimes V \to k) \) equipped with a symmetric self-duality.
Recall Definition 3.5. We define a symmetric monoidal functor \( U_{\text{ssd}} : \mathbb{C}_k \to \text{Vec}_{\text{ssd}} \) by
\( U_{\text{ssd}}(A) \overset{\text{def}}{=} (U(A), i_A, e_A) \) where \( U(A) \) is the underlying vector space of \( A \). By Proposition
3.6, it is well-defined. Analogously to Proposition 6.17, every symmetric monoidal functor
from \( \text{Cosp}_{\mathbb{C}^d}(\mathbb{C}W_{\mathbb{C}^d}^*) \) to \( \mathbb{C}_k \) induces a \( \text{Vec}_{\text{ssd}} \)-valued \( n \)-dimensional TQFT by \( \Phi_{n,d} \) and \( U_{\text{ssd}} \).

6.3.1. DWFQ TQFT for abelian groups. For a finite abelian group \( G \), let \( A = \mathbb{C}^G \) be the
function Hopf algebra. Note that the Hopf algebra \( A \) has a finite volume.

Proposition 6.19. Put \( Z = Z(\mathbb{E}_*, 1) = \theta_{\leq 1}^{-1}(\mathbb{E}_*) \cdot \hat{\Pi}(\mathbb{E}_d) \) in Theorem 6.13. Then the composit-
on \( U_{Hil} \circ Z \circ \Phi_{\pi, r} : \text{Cob}_d \to \text{Hil}^{\text{fin}} \) is the untwisted DWFQ TQFT associated with \( G \). Here,
\( U_{Hil}, \Phi_{\pi, r} \) are defined in Definition 6.13, 6.16.

Proof. We sketch the proof. Note that the set of isomorphism classes of principal bundles
over a pointed space \( K \) is given by \( \tilde{H}^1(K; G) \). For a cobordism \( B \) from \( M_0 \) to \( M_1 \), we have the
following cospan diagram (62). Then the path-integral along this cospan is nothing but
the finite path-integral in the literature. Moreover, we have a natural isomorphism \( \mathbb{C}^{\tilde{H}^1(K; G)} \cong
\tilde{H}^1(K; \mathbb{C}^G) \) as Hopf algebras.

(62)

In the same manner, the untwisted higher abelian Dijkgraaf-Witten (unextended) TQFT
is reproduced by our result. The proof is similar with that of the above proposition. Note
that the TQFT in [16] is extended to codimension two.

Proposition 6.20. Put \( Z = Z(\mathbb{E}_q; g) = \theta_{\leq q}^{-1}(\mathbb{E}_*) \cdot \hat{\Pi}(\mathbb{E}_q) \). The composition \( U_{Hil} \circ Z \circ \Phi_{\pi, q} \) is
the untwisted higher abelian Dijkgraaf-Witten TQFT obtained by \( q \)-form gauge fields.
6.3.2. TVBW TQFT for bicommutative Hopf algebras. Let $k$ be an algebraically closed field of characteristic zero. Let $A$ be a bicommutative Hopf algebra over $k$ with a finite volume. Then $A$ is finite-dimensional by Corollary 3.7. Moreover, $A$ is involutory since it is bicommutative. Hence, the category of finite-dimensional left $A$ modules $\text{Rep}(A)$ is spherical fusion category in the sense of [12].

**Proposition 6.21.** For $Z = \tilde{Z}(\tilde{E}_r; 1) = \theta^{-1}_r(\tilde{E}_r) \cdot \tilde{P}(\tilde{E}_r)$ in Theorem 6.13, the composition $U \circ Z \circ \Phi_{x, 3}$ coincides with the TVBW TQFT associated with the category $\text{Rep}(A)$. Here, $U : C_k \to \text{Vec}_{fin}$ denotes the forgetful functor.

**Proof.** We sketch the proof. If we denote by $G$ the set of group-like elements of $A$, then we have an isomorphism $A \cong kG$ since $k$ is algebraically closed and $A$ is finite-dimensional, semisimple. The construction of TVBW TQFT implies that it is isomorphic to DWFQ TQFT associated with $\text{Hom}(G, k^*) \cong \hat{G}$, i.e. the Pontryagin dual of $G$. By subsubsection 6.3.1, it is isomorphic to our TQFT induced by the first singular homology theory with coefficients $H_1(\cdot; A)$ since we have $k\hat{G} \cong kG \cong A$. $\square$

**A. Projective symmetric monoidal functor**

Fix a symmetric monoidal category $\mathcal{E}$ satisfying the following assumptions:

1. The endomorphism monoidal set of the unit object $\text{End}_\mathcal{E}(\mathbb{1})$ consists of automorphisms, i.e. $\text{End}_\mathcal{E}(\mathbb{1}) = \text{Aut}_\mathcal{E}(\mathbb{1})$.
2. For any morphism $f : x \to y$ in $\mathcal{E}$, the scalar-multiplication $\lambda \in \text{End}_\mathcal{E}(\mathbb{1}) \mapsto \lambda \cdot f \in \text{Mor}_\mathcal{E}(x, y)$ (induced by the monoidal structure) is injective.

The typical example of such $\mathcal{E}$ in this paper is the symmetric monoidal category $\text{Hopf}_{k}^{\text{bc,vol}}$ defined in subsubsection 2.2.

**Definition A.1.** Let $\mathcal{D}$ be a symmetric monoidal category. Consider a triple $F = (F_o, F_m, \Psi)$ satisfying followings:

1. $F_o$ assigns an object $F_o(x)$ of $\mathcal{E}$ to an object $x$ of $\mathcal{D}$.
2. $F_m$ assigns a morphism $F_m(f) : F_o(x) \to F_o(y)$ of $\mathcal{E}$ to a morphism $f : x \to y$ of $\mathcal{D}$.
3. $F_m(Id_x) = Id_{F_o(x)}$.
4. For composable morphisms $f, g$ of $\mathcal{D}$, there exists $\lambda \in \text{End}_\mathcal{E}(\mathbb{1}_\mathcal{E})$ such that $(F_m(g) \circ F_m(f)) = \lambda \cdot F_m(g \circ f)$.
5. $\Psi$ is a natural isomorphism $\Psi_{x, x'} : F_o(x) \otimes F_o(x') \to F_o(x \otimes x')$ which is compatible with the unitors, associators and symmetry of $\mathcal{D}, \mathcal{E}$.

We refer to such an assignment $F$ as a projective symmetric monoidal functor from $\mathcal{D}$ to $\mathcal{E}$. As a notation, we denote by $F : \mathcal{D} \to \mathcal{E}$. By an abuse of notations, we also denote by $F_o(x) = F(x)$ and $F_m(f) = F(f)$.

**Definition A.2.** Let $F, F' : \mathcal{D} \to \mathcal{E}$ be projective symmetric monoidal functors. Then a natural isomorphism $\Phi : F \to F'$ in the projective sense is given as follows:

1. For any object $x$ of $\mathcal{D}$, we have an isomorphism $\Phi(x) : F(x) \to F'(x)$ in $\mathcal{E}$.
2. For a morphism $f : x \to y$ of $\mathcal{D}$, there exists $\lambda, \lambda' \in \text{End}_\mathcal{E}(\mathbb{1}_\mathcal{E})$ such that $\lambda' \cdot F'(f) \circ \Phi(x) = \lambda \cdot \Phi(y) \circ F(f)$.

If there exists a projective natural isomorphism from $F$ to $F'$, we denote by $F \cong_{\text{proj}} F'$. A natural isomorphism $\Phi : F \to F'$ of projective symmetric monoidal functors gives a natural isomorphism in the strong sense if the second condition holds for $\lambda = \lambda' = Id_{\mathcal{E}}$. In that case, we write $F \cong F'$. 
Definition A.3. Let $F : \mathcal{D} \to \mathcal{E}$ be a projective symmetric monoidal functor. We define a 2-cochain $\omega(F) \in C^2(\mathcal{D}; \text{Aut}_\mathcal{E}(1))$ as follows. For a 2-simplex $(f, g)$ of $\mathcal{D}$, we define $\omega(F)(f, g) \in \text{Aut}_\mathcal{E}(1)$ by

$$F(g) \circ F(f) = \omega(F)(f, g) \cdot F(g \circ f).$$

Then the assignment $\omega(F)$ is a well-defined 2-cochain with coefficients in $\text{Aut}_\mathcal{E}(1)$.

Proposition-Definition A.4. Let $F : \mathcal{D} \to \mathcal{E}$ be a projective symmetric monoidal functor. Then the 2-cochain $\omega(F)$ is a normalized 2-cocycle. We define

$$\Theta(F) \stackrel{\text{def}}{=} \{ \theta \in C^1(\mathcal{D}; \text{Aut}_\mathcal{E}(1)) \mid \delta^1 \theta = \omega(F) \}.$$

An element $\theta \in \Theta(F)$ is called a complementary 1-cochain for a projective symmetric monoidal functor $F$. Note that any $\theta \in \Theta(F)$ is normalized. It is obvious that $\Theta(F) = 1 \in H^2_\text{nor}(\mathcal{D}; \text{Aut}_\mathcal{E}(1))$ if and only if $\Theta(F) \neq \emptyset$.

Definition A.5. Suppose that the automorphism group of the unit object $\text{Aut}_\mathcal{E}(1)$ is an abelian group. Let $F : \mathcal{D} \to \mathcal{E}$ be a projective symmetric monoidal functor. We define a set $\Theta(F)$ as follows:

$$\Theta(F) \stackrel{\text{def}}{=} \{ \theta \in C^1(\mathcal{D}; \text{Aut}_\mathcal{E}(1)) \mid \delta^1 \theta = \omega(F) \}.$$

An element $\theta \in \Theta(F)$ is called a complementary 1-cochain for a projective symmetric monoidal functor $F$. Note that any $\theta \in \Theta(F)$ is normalized. It is obvious that $\Theta(F) = 1 \in H^2_\text{nor}(\mathcal{D}; \text{Aut}_\mathcal{E}(1))$ if and only if $\Theta(F) \neq \emptyset$.

Definition A.6. Let $F : \mathcal{D} \to \mathcal{E}$ be a projective symmetric monoidal functor. For $\theta \in \Theta(F)$, we define a lift of $F$ by a complementary 1-cochain $\theta$ as a symmetric monoidal functor $(\theta^{-1} \cdot F) : \mathcal{D} \to \mathcal{E}$ given by

$$\begin{align*}
(\theta^{-1} \cdot F)(x) & \stackrel{\text{def}}{=} F(x) \\
(\theta^{-1} \cdot F)(f) & \stackrel{\text{def}}{=} \theta(f)^{-1} \cdot F(f).
\end{align*}$$

The assignment $(\theta^{-1} \cdot F)$ is verified to be a symmetric monoidal functor by definitions.

Proposition A.7. Let $F : \mathcal{D} \to \mathcal{E}$ be a projective symmetric monoidal functor. The induced obstruction is trivial, i.e., $\Theta(F) = 1$ if and only if there exists a symmetric monoidal functor $F' : \mathcal{D} \to \mathcal{E}$ such that $F \cong_{\text{proj}} F'$.

Proof. Suppose that $\Theta(F) = 1 \in H^2(\mathcal{D}; \text{Aut}_\mathcal{E}(1))$. By definition of $\Theta(F)$, we choose a normalized 1-cochain $\theta \in \Theta(F)$. We define a symmetric monoidal functor $F' = (\theta^{-1} \cdot F)$ We have a natural isomorphism $F \to F'$ between projective symmetric monoidal functors. In fact, the identity $Id_{F(x)} : F(x) \to F(x) = F'(x)$ for any object $x$ of $\mathcal{D}$ gives a natural isomorphism between projective symmetric monoidal functors. It completes the proof. \qed
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