Steady state entanglement, cooling, and tristability in a nonlinear optomechanical cavity

S. Shahidani 1, M. H. Naderi 1, 2, M. Soltanolkotabi 1, 2, and Sh. Barzanjeh 1, 3

1 Department of Physics, Faculty of Science, University of Isfahan, Hezar Jerib, 81746-73441, Isfahan, Iran
2 Quantum Optics Group, Department of Physics, Faculty of Science, University of Isfahan, Hezar Jerib, 81746-73441, Isfahan, Iran
3 School of Science and Technology, Physics Division, Universita di Camerino, I-62032 Camerino (MC), Italy

(Dated: October 25, 2013)

The interaction of a single-mode field with both a weak Kerr medium and a parametric nonlinearity in an intrinsically nonlinear optomechanical system is studied. The nonlinearities due to the optomechanical coupling and Kerr-down conversion lead to the bistability and tristability in the mean intracavity photon number. Also, our work demonstrates that the lower bound of the resolved sideband regime and the minimum attainable phonon number can be less than that of a bare cavity by controlling the parametric nonlinearity and the phase of the driving field. Moreover, we find that in the system under consideration the degree of entanglement between the mechanical and optical modes is dependent on the two stability parameters of the system. For both cooling and entanglement, while parametric nonlinearity increases the optomechanical coupling, the weak Kerr nonlinearity is very useful for extending the domain of the stability region to the desired range in which the minimum effective temperature and maximal entanglement are attainable. Also, as shown in this paper, the present scheme allows to have significant entanglement in the tristable regime for the lower and middle branches which makes the current scheme distinct from the bare optomechanical system.

PACS numbers: 42.50.Pq, 42.65.Lm

I. INTRODUCTION

Considerable interest has recently been focused on the optomechanical system as an excellent candidate for studying the transition of a macroscopic degree of freedom from the classical to the quantum regime. This system also provides novel routes for practical applications such as detection and interferometry of gravitational waves [1] and quantum limited displacement sensing [2]. The standard and simplest setup of this system is a Fabry-Perot cavity in which one of the mirrors is much lighter than the other, so that it can move under the effect of the radiation pressure force.

State-of-the-art technology allows experimental demonstration of cooling of the vibrational mode of the mechanical oscillator to its ground state [3–5] and strong optomechanical coupling between the vibrational mode of the mechanical oscillator and cavity field [6–9]. This coupling is intrinsically nonlinear since the length of the cavity depends upon the intensity of the field in an analogous way to the optical length of a Kerr material [10]. Therefore it enables pondermotive squeezing of the field [11], photon blockade [12], generation of nonclassical states of the mechanical and optical mode [13], optical bistability [14] and phonon-photon entanglement in the bistable regime [15]. Besides this intrinsic nonlinearity, the presence of an optical parametric amplifier (OPA) [16, 17] or the optical Kerr medium [18] inside the cavity has opened up a new domain for combining nonlinear optics and optomechanics towards the enhancement of quantum effects. It has been predicted [16] that the presence of an OPA in a single-mode Fabry-Perot cavity causes a strong coupling between the oscillating mirror and the cavity mode resulting from increasing the intracavity photon number. Also, when the optomechanical cavity contains an optical Kerr medium with strong \( \chi^{(3)} \) nonlinearity, the photon-photon repulsion and the reduction of the cavity photon fluctuations provide a feasible route towards controlling the dynamics of the micromirror [18].

On the other hand, the interaction of a single-mode field with both a Kerr medium and a parametric nonlinearity is a well-known quantum optical model which has been proposed for the generation of nonclassical states of the cavity field [19–21].

Here, we consider the interaction of a single-mode field with both a weak Kerr medium and a parametric nonlinearity in an intrinsically nonlinear optomechanical system. In particular, we investigate the multistability, intensity, back-action cooling and stationary optomechanical entanglement. It turns out that the mean intracavity photon number, in addition to bistability, exhibits tristability for a certain range of the parameters which can be controlled by the intrinsic and external nonlinearities in the system. Then, we investigate the effect of Kerr-down conversion nonlinearity on the back-action ground state cooling of the mirror based on the covariance matrix and identify the modified optimal regime for cooling. We will show that the lower bound of the resolved sideband regime and the minimum attainable phonon number can
be less than that of a bare cavity by controlling the parametric nonlinearity and the phase of the field driving the OPA. Also, the weak Kerr nonlinearity can be used to extend the domain of the stability to the desired range of the effective detuning in which the effective temperature of the system minimizes. Then we show that for a fixed effective detuning, in spite of the bare cavity, one of the stability parameters (the counterpart of the bistability parameter) is a nonlinear function of the input power, allows to approach significant entanglement simultaneously with the ground state cooling of the mirror. In the last part of the paper we shall focus on the generation of stationary entanglement in the presence of the nonlinearity and show that not only the entanglement is not a monotonic function of the optomechanical coupling strength, but also the two stability parameters of the system are the key parameters for maximizing the degree of entanglement. Based on these results we show that in the tristable regime, for the first and second branches the degree of entanglement is maximum at the end of the branches, while for the third branch the phonon-photon entanglement is null.

II. THE PHYSICAL MODEL

As is shown in Fig. 1, we consider a Kerr-down conversion optomechanical system composed of a degenerate OPA and a nonlinear Kerr medium placed within a Fabry-Perot cavity formed by a fixed partially transmitting mirror and one movable perfectly reflecting mirror in equilibrium with a thermal bath at temperature $T_0$. The movable mirror is free to move along the cavity axis and is treated as a quantum mechanical harmonic oscillator with effective mass $m$, frequency $\omega_m$ and energy decay rate $\gamma_m = \omega_m/Q$ where $Q$ is the mechanical quality factor. The cavity field is coherently driven by an input laser field with frequency $\omega_L$ and amplitude $\varepsilon$ through the fixed mirror. Furthermore, the system is pumped by a coupling field to produce parametric oscillation and induce the Kerr nonlinearity in the cavity. In our investigation, we restrict the model to the case of single-cavity and mechanical modes\textsuperscript{[21][23]}. The single cavity-mode assumption is justified in the adiabatic limit, i.e., $\omega_m \ll \pi c/L$ in which $c$ is the speed of light in vacuum and $L$ is the cavity length in the absence of the cavity field. We also assume that the induced resonance frequency shift of the cavity and the Kerr medium are much smaller than the longitudinal-mode spacing in the cavity. Furthermore, one can restrict to a single mechanical mode when the detection bandwidth is chosen such that it includes only a single, isolated, mechanical resonance and mode-mode coupling is negligible.

Under these conditions, the total Hamiltonian of the system in a frame rotating at the laser frequency $\omega_L$ can be written as

$$H = H_0 + H_1,$$

(1)

where

$$H_0 = \hbar(\omega_0 - \omega_L)a^{\dagger}a + \frac{\hbar \omega_m}{2}(q^2 + p^2) - \hbar g_0 a^{\dagger}aq + i\hbar \varepsilon(a^{\dagger} - a),$$

$$H_1 = i\hbar G(e^{i\theta}a^{\dagger}a^{\dagger} - e^{-i\theta}a^2) + \hbar \chi a^{\dagger}a^2.$$  

(2a)

$$H_1 = i\hbar G(e^{i\theta}a^{\dagger}a^{\dagger} - e^{-i\theta}a^2) + \hbar \chi a^{\dagger}a^2.$$  

(2b)

The first two terms in $H_0$ are, respectively, the free Hamiltonians of the cavity field with annihilation(creation) operator $a(a^{\dagger})$ and the movable mirror with resonance frequency $\omega_m$ and dimensionless position and momentum operators $q$ and $p$. The third term describes the optomechanical coupling between the cavity field and the mechanical oscillator due to the radiation pressure force with coupling constant $g_0 = \frac{\omega_0}{L} \sqrt{\frac{\hbar}{m\omega_m}}$, and the last term in $H_0$ describes the driving of the intracavity mode with the input laser. Also, the two terms in $H_1$ describe, respectively, the coupling of the intracavity field with the OPA and the Kerr medium; $G$ is the nonlinear gain of the OPA which is proportional to the pump power driving amplitude, $\theta$ is the phase of the field driving the OPA, and $\chi$ is the anharmonicity parameter proportional to the third order nonlinear susceptibility $\chi^{(3)}$ of the Kerr medium. The input laser field populates the intracavity mode through the partially transmitting mirror, then the photons in the cavity will exert a radiation pressure force on the movable mirror. In a realistic treatment of the dynamics of the system, the cavity-field damping due to the photon-leakage through the incomplete mirror and the Brownian noise associated with the coupling of the oscillating mirror to its thermal bath should be considered. Using the input-output formalism of quantum optics\textsuperscript{[24]}, we can consider the effects of these sources of noise and dissipation in the quantum Langevin equations of motion. For the given Hamiltonian (1), we
obtain the following nonlinear equations of motion
\begin{align}
\dot{q} &= \omega_m p, \\
\dot{p} &= -\omega_m q + \sqrt{g_0 a^1} a - \gamma_m p + \xi, \\
\dot{a} &= -i(\omega_0 - \omega_L) a + i\sqrt{g_0 q} a + \varepsilon - 2i\chi a^1 a^2 + 2Ga e^{i\theta} - \kappa a + \sqrt{2}\kappa a_{in},
\end{align}
(3a-b-c)
where \(\kappa\) is the cavity decay rate coming by the movable mirror, \(a_{in}\) is the input vacuum noise operator characterized by the following correlation functions [24-26]
\[<a_{in}(t)a^\dagger_{in}(t')> = \delta(t-t'), \quad <a_{in}(t)a_{in}(t')> = <a^1_{in}(t)a^1_{in}(t')> = 0.\]
(4a-b)
The Brownian noise operator \(\xi\) describes the heating of the mirror by the thermal bath at temperature \(T_0\) and is characterized by the following correlation function [24-26]
\[<\xi(t)\xi(t')> = \frac{\hbar\gamma_{in} m}{2\pi} \int \omega e^{-i\omega(t-t')} [\coth(\frac{\omega}{2k_B T_0}) + 1] d\omega,\]
(5)
where \(k_B\) is the Boltzmann constant.

We are interested in the steady-state regime and small fluctuations with respect to the steady state. Thus we obtain the steady-state mean values of \(p, q\) and \(a\) as
\[p_s = 0, \quad q_s = \frac{g_0}{\omega_m} a_s^2, \quad a_s = \frac{\varepsilon}{\sqrt{(\Delta - 2G\sin\theta)^2 + \kappa_-^2}},\]
(6-7)
where \(q_s\) denotes the new equilibrium position of the movable mirror, \(\kappa_- = \kappa - 2G \cos\theta\), and \(\Delta = \omega_0 - \omega_L - g_0 q_s + 2\chi a^2_s = \Delta_0 + (2\chi - g_0^2/\omega_m)a^2_s\) is the effective detuning of the cavity which includes both the radiation pressure and the Kerr medium effects. It is obvious that the optical path and hence the cavity detuning are modified in an intensity-dependent way. The first modification which is a mechanical nonlinearity, arises from the radiation pressure-induced coupling between the movable mirror and the cavity field and the second modification comes from the presence of the nonlinear Kerr medium in the optomechanical system. Since the mean intracavity photon number in the steady state \(I_s(=a_s^2)\) satisfies a third-order equation, it can have three real solutions and hence the system may exhibit multistability for a certain range of parameters. The multisolition region exists for intracavity intensity values between \(I_-\) and \(I_+\) with
\[I_\pm = \frac{(4G\sin\theta - 2\Delta_0) \pm \sqrt{(2G\sin\theta - \Delta_0)^2 - 3\kappa_-^2}}{3(2\chi - g_0^2/\omega_m)}.\]
(8)
The multistability of the solution fails if \(\chi = g_0^2/2\omega_m\) and requires \(2G\sin\theta - \Delta_0 > \sqrt{3}\kappa_-\).

To study the effect of the presence of the Kerr-down conversion nonlinearity on the steady-state response of the optomechanical system we consider a cavity with length \(L = 1\)mm and decay rate \(\kappa = 0.9\omega_m\) which is driven by a laser with \(\lambda = 810\)nm. The mass, the mechanical frequency and the damping rate of the oscillating end mirror are \(m = 5\)ug, \(\omega_m/2\pi = 10\)MHz, \(\gamma_m = 100\) Hz and the environment temperature is \(T_0 = 400\)K. This set of parameters is close to several optomechanical experiments [7-27-29]. In Fig. 2 we plot the mean intracavity photon number as a function of the bare detuning \(\Delta_0\) for input power \(P = 15\)mW for various values of \(\chi\) (Fig. 2a), \(G\) (Fig. 2b), and \(\theta\) (Fig. 2c). Figure 2a shows that for \(\chi < g_0^2/2\omega_m\) (\(\chi = 0.01\)) the presence of both intrinsically optomechanical and Kerr nonlinearities shift the center of the resonance while the curve is nearly Lorentzian. However, for \(\chi > g_0^2/2\omega_m\) the resonance frequency of the cavity shifts to the lower values and the third order polynomial equation for \(I_s\) has three real roots for \(\Delta_0 < 0\). The frequency shift of the cavity mode and mult solutions due to the Kerr nonlinearity can be reduced or compensated by the term \(2Ga e^{i\theta}\) which acts to shift the cavity resonant frequency to the right for \(\theta \geq \pi/2\) (Figs. 2c).

### III. DYNAMICS OF SMALL FLUCTUATIONS

In order to investigate the dynamics of the system, we need to study the dynamics of small fluctuations near the steady state. We assume that the nonlinearity in the system is weak and decompose each operator in Eq. \(4\) as the sum of its steady-state value and a small fluctuation with zero mean value,
\[a = a_s + \delta a, \quad q = q_s + \delta q, \quad p = p_s + \delta p.\]
(9)
Inserting the above linearized forms of the system operators into Eq. \(4\), and defining the cavity field quadratures \(\delta x = (\delta a + \delta a^\dagger)/\sqrt{2}\) and \(\delta y = i(\delta a - \delta a^\dagger)/\sqrt{2}\) and the input noise quadratures \(\delta x_{in} = (\delta a_{in} + \delta a_{in}^\dagger)/\sqrt{2}\) and \(\delta y_{in} = i(\delta a_{in}^\dagger - \delta a_{in})/\sqrt{2}\), the linearized quantum Langevin equations for the fluctuation operators can be written in the compact matrix form
\[\dot{u} = Mu(t) + n(t),\]
(10)
where \(u(t) = (\delta q, \delta p, \delta x, \delta y)^T\) is the vector of quadratures, \(n(t) = (0, \xi, \sqrt{2}k\delta x_{in}, \sqrt{2}k\delta y_{in})^T\) is the vector of noise sources and the matrix \(M\) is given by
\[M = \begin{pmatrix}
0 & \omega_m & 0 & 0 \\
-\omega_m & -\gamma_m & g_1 & 0 \\
0 & 0 & -\kappa + \Gamma_r & \Delta_1 + \Gamma_i \\
g_1 & 0 & -\Delta_1 + \Gamma_i & -\kappa - \Gamma_r
\end{pmatrix},\]
(11)
where \(g_1 = \sqrt{2}\sqrt{g_0 a_s}\) is the enhanced optomechanical coupling rate, \(\Gamma_r, \Gamma_i\) is the real (imaginary) part of \(\Gamma = 2Ge^{i\theta} - 2i\chi a_s^2\), and \(\Delta_1 = \Delta + 2\chi a_s^2\).
FIG. 2. (Color online) The mean intracavity photon number as a function of normalized bare detuning $\Delta_0/\omega_m$ : (a) for different values of the anharmonicity parameter $\chi$ with $G = 0.6k$ and $\theta = \pi/2$, (b) for different values of the parametric nonlinearity $G$ with $\chi = 0.1$ Hz and $\theta = \pi/2$, and (c) for different values of $\theta$ with $G = 1.1k$ and $\chi = 0.04$ Hz.

A. Stability analysis of the solutions

Here, we concentrate on the stationary properties of the system. For this purpose, we should consider the steady-state condition governed by Eq. (10). The system is stable only if the real part of all eigenvalues of the matrix $M$ are negative, which is also the requirement of the validity of the linearized method. The parameter region in which the system is stable can be obtained from the Routh-Hurwitz criterion [30], which gives the following three independent conditions:

$$s_1 = \kappa^2 + \Delta_1^2 - |\Gamma|^2 > 0, \quad (12a)$$

$$s_2 = \omega_m(\kappa^2 + \Delta_1^2 - |\Gamma|^2) - g_0^2(\Delta_1 + \Gamma_i) > 0, \quad (12b)$$

$$s_3 = 2\kappa\gamma_m\{((s_1 - \omega_m^2)^2 + (\gamma_m + 2\kappa)(\gamma_m s_1 + 2\kappa\omega_m^2)) + g_0^2(\Delta_1 + \Gamma_i)\omega_m(2\kappa + \gamma_m)^2 > 0. \quad (12c)$$

The violation of the third condition, $s_3 < 0$, indicates instability in the region $\Delta_1 + \Gamma_i < 0$. For the bare cavity ($G = \chi = 0$) this condition reduces to the instability in the domain of blue-detuned laser. Within this frequency range, the effective mechanical damping rate becomes negative and self-sustained oscillations set in [31, 32]. The violation of the second condition, $s_2 < 0$, indicates instability in the region $\Delta_1 + \Gamma_i > 0$. For a bare cavity this condition cause a bistability of the system [14]. There is an additional stability condition given by $s_1$, which is always satisfied for the bare cavity, and gives the condition for the threshold for parametric oscillation. Accordingly, for $\Delta_1 + \Gamma_i > 0$ we can define the following stability parameters

$$\eta_1 = 1 - \frac{g_0^2(\Delta_1 + \Gamma_i)}{\kappa^2 + \Delta_1^2 - |\Gamma|^2}, \quad (13)$$

$$\eta_2 = 1 - \frac{|\Gamma|^2}{\kappa^2 + \Delta_1^2}. \quad (14)$$

For $G = \chi = 0$, $\eta_1$ reduces to the well known “bistability parameter” [33].

One of the main features arising from the Kerr-down conversion nonlinearity is the appearance of three stable states for the mirror. Figure 3 shows the hysteresis loop for the intracavity mean photon number when $\Delta_0 < 0$ (blue-detuned laser). In this figure the unstable solutions are represented by dashed lines. It shows that depending on the value of $\theta$ the steady-state response of the mirror can be monostable, bistable and tristable. From an experimental point of view, controllable triple-state switching is possible practically by adding a pulse sequence onto the input field [34]. Such tristability can be used for all-optical switching purposes function as memory devices for optical computing and quantum information processing.

B. Correlation matrix of the quantum fluctuations of the system

Due to the linearization method and the Gaussian nature of the noise operators the asymptotic steady state of the quantum fluctuations is a zero mean Gaussian state. As a consequence, the steady state can be fully characterized by the covariance matrix (CM) $V$. This formalism provides a unified framework for exploring both cooling of the mechanical oscillator and phonon-photon entanglement.

When the stability conditions of Eq. (12) are fulfilled, we can solve Eq. (10) for the $4 \times 4$ stationary correlation
A mechanical oscillator is given by its steady state. In particular, the mean energy of the motional degrees of freedom is therefore interesting to investigate these quantum features, including cooling of the resonator towards the system into a stationary state with genuine quantum states.

However, the general exact expression is too cumbersome for the steady state CM can be straightforwardly solved.

where $V$ is the diagonal diffusion matrix in which we used $\Delta = \omega m, \gamma_m (2n + 1), \kappa, \kappa$]

$V_i = \frac{\hbar \omega_i}{2k_B T_0} \approx \omega_i (2\tilde{n} + 1), \quad \Delta_A V_i = \omega_m k_B T_0$, \quad \Delta_B V_i = \omega_m k_B T_0,$

$\Delta_C V_i = \omega_m k_B T_0$, \quad \Delta_D V_i = \omega_m k_B T_0,$

$\Delta_{\ell} V_i = \omega_m k_B T_0$, \quad \Delta_{\ell} V_i = \omega_m k_B T_0,$

$\Delta_{\ell} V_i = \omega_m k_B T_0$, \quad \Delta_{\ell} V_i = \omega_m k_B T_0,$

$\Delta_{\ell} V_i = \omega_m k_B T_0$, \quad \Delta_{\ell} V_i = \omega_m k_B T_0,$

$\Delta_{\ell} V_i = \omega_m k_B T_0$, \quad \Delta_{\ell} V_i = \omega_m k_B T_0,$

$\Delta_{\ell} V_i = \omega_m k_B T_0$, \quad \Delta_{\ell} V_i = \omega_m k_B T_0,$

$\Delta_{\ell} V_i = \omega_m k_B T_0$, \quad \Delta_{\ell} V_i = \omega_m k_B T_0,$

$\Delta_{\ell} V_i = \omega_m k_B T_0$, \quad \Delta_{\ell} V_i = \omega_m k_B T_0,$

$\Delta_{\ell} V_i = \omega_m k_B T_0$, \quad \Delta_{\ell} V_i = \omega_m k_B T_0,$

$\Delta_{\ell} V_i = \omega_m k_B T_0$, \quad \Delta_{\ell} V_i = \omega_m k_B T_0,$

$\Delta_{\ell} V_i = \omega_m k_B T_0$, \quad \Delta_{\ell} V_i = \omega_m k_B T_0,$

$\Delta_{\ell} V_i = \omega_m k_B T_0$, \quad \Delta_{\ell} V_i = \omega_m k_B T_0,$

$\Delta_{\ell} V_i = \omega_m k_B T_0$, \quad \Delta_{\ell} V_i = \omega_m k_B T_0,$

$\Delta_{\ell} V_i = \omega_m k_B T_0$, \quad \Delta_{\ell} V_i = \omega_m k_B T_0,$

$\Delta_{\ell} V_i = \omega_m k_B T_0$, \quad \Delta_{\ell} V_i = \omega_m k_B T_0,$

$\Delta_{\ell} V_i = \omega_m k_B T_0$, \quad \Delta_{\ell} V_i = \omega_m k_B T_0,$

$\Delta_{\ell} V_i = \omega_m k_B T_0$, \quad \Delta_{\ell} V_i = \omega_m k_B T_0,$

$\Delta_{\ell} V_i = \omega_m k_B T_0$, \quad \Delta_{\ell} V_i = \omega_m k_B T_0,$

$\Delta_{\ell} V_i = \omega_m k_B T_0$, \quad \Delta_{\ell} V_i = \omega_m k_B T_0,$

$\Delta_{\ell} V_i = \omega_m k_B T_0$, \quad \Delta_{\ell} V_i = \omega_m k_B T_0,$

$\Delta_{\ell} V_i = \omega_m k_B T_0$, \quad \Delta_{\ell} V_i = \omega_m k_B T_0,$

$\Delta_{\ell} V_i = \omega_m k_B T_0$, \quad \Delta_{\ell} V_i = \omega_m k_B T_0,$

$\Delta_{\ell} V_i = \omega_m k_B T_0$, \quad \Delta_{\ell} V_i = \omega_m k_B T_0,$

$\Delta_{\ell} V_i = \omega_m k_B T_0$, \quad \Delta_{\ell} V_i = \omega_m k_B T_0,$

$\Delta_{\ell} V_i = \omega_m k_B T_0$, \quad \Delta_{\ell} V_i = \omega_m k_B T_0,$

$\Delta_{\ell} V_i = \omega_m k_B T_0$, \quad \Delta_{\ell} V_i = \omega_m k_B T_0,$

$\Delta_{\ell} V_i = \omega_m k_B T_0$, \quad \Delta_{\ell} V_i = \omega_m k_B T_0,$

$\Delta_{\ell} V_i = \omega_m k_B T_0$, \quad \Delta_{\ell} V_i = \omega_m k_B T_0,
at $\Delta \simeq 0.8\omega_m$. Also, for $G = 0.8\kappa$ and $\theta = 0.71\pi$ we have $\kappa_+ \simeq 8 \times 10^{-6}\kappa$ and the minimum effective temperature is about $0.16\text{mK}$ at $\Delta \simeq 0.6\omega_m$.

According to Eq. [23], it seems that $n_{\min}$ does not depend on the anharmonicity parameter $\chi$, but it should be noted that even if the effect of Kerr nonlinearity on $\eta_1$ can be eliminated, the above analysis is carried out for the limit of low-temperature environment ($\kappa \gg \bar{n}\gamma_m$). In a realistic case the thermal noise will also be present, which in turn modifies the minimum attainable temperature. Figures (a) and (b) show the variation of effective temperature with $\Delta$ for different values of $\chi$ and for two different environment temperatures $T_0 = 400\text{mK}$ and 25nK. For $G = 0.8\kappa$ and $\theta = 3\pi/4$ the minimum temperature is achieved for the optimal detuning $\Delta \simeq 0.66\omega_m$. As can be seen, the system is unstable in this range for $\chi = 0$ and $\chi = 0.1$. For $\chi = 0.03$ and $\chi = 0.06$ the stability domain is extended to the desired range and the effective temperature is increased with increasing $\chi$. As shown in Fig. (b) this heating effect is smaller for lower environment temperature $T_0 = 25\text{mK}$.

Also, in Fig. the dependence of effective temperature on the Kerr nonlinearity has been illustrated for fixed values of stability parameter $\eta_1$. It should be emphasized, however, that the controllable ground state cooling of the vibrational mode is possible only if the limit $\eta_1 \simeq 1$ is reachable in the presence of the nonlinear medium. Figure 6 shows the variation of the parameter $\eta_1$ versus the input power $P$ and the normalized effective detuning $\Delta/\omega_m$ for the data of Fig. (a) and for $\chi = 0.03$. As can be seen, $0.8 < \eta_1 < 1$ and the required condition holds actually very well. Also, Fig. shows that for a fixed effective detuning, in contrast to the case of bare cavity $13$, $\eta_1$ is not a linear function of the input power $P$. This feature arises from the Kerr nonlinearity (the term $2\chi \alpha_\Sigma^2$ in $\Delta_\lambda$ and $\Gamma$ in Eq. [13]) and allows to approach significant entanglement simultaneously with the ground state cooling of the mirror (since one can enhance optomechanical coupling by increasing the input power while $\eta_1$ remains approximately unaffected). As an example, Fig. 7 represents the optomechanical entanglement and the effective temperature for $G = 1.3\kappa$, $\chi = 0.05$, and $\Delta = 0.5\omega_m$ as functions of the input power. It shows that the minimum value of the effective temperature $T$ and the maximum value of entanglement is achieved in
the same range of the input power.

Nonetheless, we will show that as in the case of bare cavity, entanglement and cooling are different phenomena and generally are optimized in different regimes. We study the effects of \( \theta \), \( G \) and \( \chi \) on \( E_N \) separately to find the regime of maximal phonon-photon entanglement.

We first study the behavior of \( E_N \) when the phase of the field driving the OPA, i.e., \( \theta \), is varied. Figure 8 shows that entanglement increases with decreasing the phase of the field driving the OPA. This entanglement increment is related to the increasing of the photon number and the optomechanical coupling rate \( g_1 \) (see Fig. 2(c)).

![Figure 7](image1)

**FIG. 7.** (Color online) Plot of (a) the effective temperature \( T \) and (b) the logarithmic negativity versus the input power \( P \) for \( \Delta = 0.5\omega_m \), \( G = 1.3\kappa \), \( \theta = 0.67\pi \) and \( \chi = 0.05 \). Other parameters are the same as those in Fig. 4.

![Figure 8](image2)

**FIG. 8.** (Color online) Plot of the logarithmic negativity versus the normalized effective detuning \( \Delta/\omega_m \) for different values of \( \theta \). The parameters are: \( P = 3\) mW, \( G = 1.3\kappa \), \( \chi = 0.05 \). Other parameters are the same as those in Fig. 4.

Figure 9 shows the entanglement as a function of the normalized effective detuning \( \Delta/\omega_m \) and laser power \( P \) for (a) \( G = 0.6\kappa \) and (b) \( G = \kappa \). The parameters are: \( \theta = 0.67\pi \), \( \chi = 0.05 \). Other parameters are the same as those in Fig. 4.

![Figure 9](image3)

**FIG. 9.** (Color online) Contour plot of the logarithmic negativity versus the normalized effective detuning \( \Delta/\omega_m \) and input power \( P \) for (a) \( G = 0.6\kappa \) and (b) \( G = \kappa \). The parameters are: \( \theta = 0.67\pi \), \( \chi = 0.05 \). Other parameters are the same as those in Fig. 4.
TABLE I. Calculated logarithmic negativities, normalized effective detunings, normalized optomechanical couplings, and the two stability parameters for the input powers $P = 2.5\text{mW}$ and $P = 5\text{mW}$ in Fig. [2]

| $G/\kappa$ | $P$ (mW) | $E_N$ | $\Delta/\omega_m$ | $\eta_1$ | $\eta_2$ | $\eta_1/\omega_m$ |
|-----------|---------|-------|-------------------|---------|---------|------------------|
| 0.6       | 2.5     | 0.20  | 0.53              | 0.77    | 0.93    | 0.56             |
| 1         | 2.5     | 0.21  | 0.43              | 0.67    | 0.87    | 0.47             |
| 0.6       | 5       | 0.18  | 0.68              | 0.77    | 0.87    | 0.69             |
| 1         | 5       | 0.23  | 0.30              | 0.66    | 0.88    | 0.66             |

It shows that for the first and second branches the entanglement is maximum at the end of the branches, while for the third branch the phonon-photon entanglement is null. This result might be interpreted as arising from the Kerr-induced shift of the resonance frequency of the cavity (for the third branch this shift is more than $1.7\omega_m$). Also, it can be seen that at the end of each stable branch $\eta_1 \neq 0$ and the entanglement is found only in the region with $\eta_1 < 0.65$.

![FIG. 10. (Color online) The logarithmic negativity versus the normalized effective detuning $\Delta/\omega_m$ for $\chi = 0$, $\chi = 0.03$, and $\chi = 0.05$. The parameters are: $\theta = 0.67\pi$, $G = \kappa$, and $P = 6\text{mW}$. Other parameters are the same as those in Fig. [4].](image1)

Until now we have studied the entanglement in the good cavity limit ($\kappa < \omega_m$) and have chosen the values of the input power and the detuning far from the multistability region. It is also interesting to examine the entanglement in the bad cavity limit and in the tristable regime. Figure 11 shows the entanglement and the stability parameter $\eta_1$ as functions of the laser input power $P$ in the tristable regime for the data of Fig. [3]. It

![FIG. 11. (Color online) (a) The logarithmic negativity and (b) the stability parameter $\eta_1$ as functions of the input power $P$ corresponding to the three stable branches for $\theta = 0.57\pi$ in Fig. [3]. The dashed line corresponds to the unstable region.](image2)

It should be noted that in order to stay within the range of validity of the linearization approximation, we have been assured that the condition $< \delta a \dagger \delta a >_{ss} \ll \alpha_s^2$ is satisfied in all the results given above.

V. CONCLUSIONS

In conclusion, we have studied the interaction of a single-mode field with both a weak Kerr medium and a parametric nonlinearity in an intrinsically nonlinear optomechanical system. We have investigated the stability behavior of the intracavity mean photon number, the intensity, cooling and stationary entanglement.

We have found that the combination of the nonlinearities leads to a shift of the resonance frequency of the cavity toward the lower values and appearance of three real roots for the steady-state response of the system.
have derived the general condition for tristability in the system. Furthermore, we have studied the cooling and stationary entanglement by using the covariance matrix formalism. In particular, we have shown that the lower bound of the resolved sideband regime and the minimum attainable phonon number can be less than that of a bare cavity by controlling the parametric nonlinearity and the phase of the field driving the OPA. The weak Kerr nonlinearity can be used to extend the domain of the stability to the desired range of the effective detuning in which the effective temperature of the system is minimized. Also, the Kerr nonlinearity modifies the behavior of the stability parameter and allows to approach significant entanglement simultaneously with the ground state cooling of the mirror. In the investigation of the degree of stationary entanglement between the cavity and the mechanical modes we have identified four key parameters that affect the behavior of the entanglement. They are effective detuning of the cavity, the optomechanical coupling and the two stability parameters of the system. Also, as shown in this paper the present scheme allows to have significant entanglement in the tristable regime for the lower and middle branches which makes the current scheme distinct from the bare optomechanical system.

ACKNOWLEDGEMENTS

S.Sh. is grateful to M. Xiao and R.Ghobadi for useful discussions. The authors wish to thank The Office of Graduate Studies of The University of Isfahan for their support.

[1] V. Braginsky and S. P. Vyatchanin, Phys. Lett. A 293, 228 (2002).
[2] D. Rugar, R. Budakian, H. J. Mamin, and B. W. Chui., Nature (London) 430, 329 (2004).
[3] A. D. O’Connell, M. Hofheinz, M. Ansmann, R. C. Bialczak, M. Lenander, E. Lucero, M. Neeley, D. Sank, H. Wang, M. Weides, J. Wenner, J. M. Martinis, and A. N. Cleland, Nature (London) 464, 697 (2010).
[4] J. D. Teufel, T. Donner, D. Li, J. W. Harlow, M. S. Allman, K. Cicak, A. J. Sirois, J. D. Whittaker, K. W. Lehnert, and R. W. Simmonds, Nature (London) 475, 359 (2011).
[5] J. Chan, T. P. M. Alegre, A. H. Saiifai-Naeini, J. T. Hill, A. Krause, S. Groblacher, M. Aspelmeyer, and O. Painter, Nature (London) 478, 89 (2011).
[6] O. Arcizet, P. F. Cohadon, T. Briant, M. Pinard, and A. Heidmann, Nature (London) 444, 71 (2006).
[7] S. Gigan, H. R. Bohm, M. Paternostro, F. Blaser, G. Langer, J. B. Hertzberg, K. C. Schwab, D. Bauerle, M. Aspelmeyer, and A. Zeilinger, Nature (London) 444, 67 (2006).
[8] C. H. Metzger and K. Karrai, Nature (London) 432, 1002 (2004).
[9] S. Groblacher, K. Hammerer, M.R. Vanner, and M. Aspelmeyer, Nature (London) 460, 724 (2009).
[10] Z. R. Gong, H. Ian, Yu-xi Liu, C. P. Sun, and F. Nori, Phys. Rev. A 80, 665801 (2009).
[11] C. Fabre, M. Pinard, S. Bourzeix, A. Heidmann, E. Giacobino, and S. Reynaud, Phys. Rev. A 49, 1337 (1994); S. Mancini and P. Tombesi, Phys. Rev. A 49, 4055 (1994).
[12] F. Rabl, Phys. Rev. Lett. 107, 063601 (2011).
[13] A. Nunnenkamp, K. Borkje, and S. M. Girvin, Phys. Rev. Lett. 107, 063602 (2011).
[14] A. Dorsel, J. D. McCullen, P. Meystre, E. Vignes, and H. Walthier, Phys. Rev. Lett. 51, 1550 (1983); A. Gozzini, F. Maccarone, F. Mango, I. Longo, and S. Barbarino, J. Opt. Soc. Am. B 2, 1841 (1985).
[15] R. Gobadi, A. R. Bahrampour, and C. Simon, Phys. Rev. A 84, 033846(2011).
[16] S. Huang and G. S. Agarwal, Phys. Rev. A 79, 013821 (2009).
[17] A. Xuereb, M. Barbier, and M. Paternostro, Phys. Rev. A 86, 013809 (2012).
[18] T. Kumar, A. Bhattacherjee, and ManMohan, Phys. Rev. A 81, 013835 (2010).
[19] B. Wielinga and G. J. Milburn, Phys. Rev. A 48, 2494 (1993).
[20] W. Leon’ski, Phys. Rev. A 54, 3396 (1996).
[21] C. K. Law, Phys. Rev. A 21, 2537 (1985).
[22] S. Mancini, V. I. Manko, and P. Tombesi, Phys. Rev. A 55, 3042 (1997).
[23] S. Bose, K. Jacobs, and P. L. Knight, Phys. Rev. A 56,4175 (1997).
[24] C. W. Gardiner and P. Zoller, Quantum Noise (Springer-Verlag, Berlin, 1991).
[25] V. Giovannetti and D. Vitali, Phys. Rev. A 63, 023812 (2001).
[26] L. Landau and E. Lifshitz, Statistical Physics (Pergamon, New York, 1958).
[27] D. Kleckner and D. Bouwmeester, Nature(London) 444, 75 (2006).
[28] D. Kleckner et al., Phys. Rev. Lett. 96, 173901 (2006).
[29] T. Carmon, H. Rokhsari, L. Yang, T. J. Kippenberg, and K. J. Vahala, Phys. Rev. Lett. 94, 223902 (2005).
[30] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series and Products (Academic Press, Orlando, 1980); A. Hurwitz, Selected Papers on Mathematical Trends in Control Theory, edited by R. Bellman and R. Kabala (Dover, New York, 1964).
[31] F. Marquardt, J. G. Harris, and S. M. Girvin , Phys. Rev. Lett. 96, 103901(2006).
[32] M. Ludwig, C. Neuenhahn, C. Metzger, A. Ortlieb, I. Favero, K. Karrai, and F. Marquardt, Phys. Rev. Lett. 101, 133903(2008).
[33] C. Genes, D. Vitali, P. Tombesi, S. Gigan, and M. Aspelmeyer, Phys.Rev.A 77 , 033804 (2008).
[34] J. Sheng, U. Khadka, and M. Xiao, Phys. Rev. Lett. 109, 223006 (2012).
[35] D. Vitali, S. Gigan, A. Ferreira, H. R. Bohm, P. Tombesi, A. Guerreiro, V. Vedral, A. Zeilinger, and M. Aspelmeyer, Phys. Rev. Lett. 98, 030405 (2007).
[36] C. Genes, A. Mari, P. Tombesi, and D. Vitali, Phys. Rev. A 78, 032316 (2008).
[37] A. Mari and J. Eisert, Phys. Rev. Lett. 103, 213603 (2009).
[38] F. Marquardt, J. P. Chen, A. A. Clerck, and S. M. Girvin, Phys. Rev. Lett. 99, 093902 (2007).
[39] I. Wilson-Rae, N. Nooshi, W. Zwerger, and T. J. Kippenberg, Phys. Rev. Lett. 99, 093901 (2007).
[40] M. Paternostro, D. Vitali, S. Gigan, M. S. Kim, C. Brukner, J. Eisert, and M. Aspelmeyer, Phys. Rev. Lett. 99, 250401 (2007).
[41] J. M. Dobrindt, I. Wilson-Rae, and T. J. Kippenberg, Phys. Rev. Lett. 101, 263602 (2008).
[42] C. Genes, A. Mari, D. Vitali, and P. Tombesi, Adv. At. Mol. Opt. Phys. 57, 33 (2009).
[43] G. Adesso, A. Serafini, and F. Illuminati, Phys. Rev. A 70, 022318 (2004).