General Forms of a $\mathcal{N}$-fold Supersymmetric Family

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Abstract

We report general forms of one family of the $\mathcal{N}$-fold supersymmetry in one-dimensional quantum mechanics. The $\mathcal{N}$-fold supersymmetry is characterized by the supercharges which are $\mathcal{N}$-th order in differential operators. The family reported here is defined as a particular form of the supercharges and is referred to as “type A”. We show that a quartic and a periodic potentials, which were previously found to be $\mathcal{N}$-fold supersymmetric by the authors, are realized as special cases of this type A family.

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I. INTRODUCTION

Recently, much attention have been paid to the $\mathcal{N}$-fold supersymmetry in quantum mechanics as one of the most fruitful generalization of the supersymmetry $^{[1,2]}$. The $\mathcal{N}$-fold supersymmetry is characterized by a non-linear superalgebra among the supercharges and the Hamiltonian; the anticommutator of the supercharges is a polynomial of the Hamiltonian. The coordinate representation of the supercharges involves $\mathcal{N}$-th order derivative.

There are several ways to construct the $\mathcal{N}$-fold supersymmetric models. If one has a Hamiltonian for which the exact $\mathcal{N}$ eigenfunctions are known, the $\mathcal{N}$-th order supercharge is given by the $\mathcal{N}$-th order Darboux transformation and can be represented in the form known as Crum-Krein formula $^{[5,6]}$. The formal expressions for the partner Hamiltonian and the anticommutator of the supercharge are also known $^{[5,6]}$. However, the applicability of this approach will be, in practice, quite limited since we rarely have exact solutions of a Hamiltonian under consideration. In addition, serious difficulties will be expected when one intends to construct a model in which dynamical SUSY breaking takes place. In this case, the prepotential is related to (the logarithmic derivative of) the perturbative ground-state eigenfunction which can be solved analytically, but not to the exact ground-state eigenfunction. Actually, we have already known a $\mathcal{N}$-fold supersymmetric model in which the purely nonperturbative effect breaks the $\mathcal{N}$-fold supersymmetry and only the perturbative $\mathcal{N}$ non-degenerate states can be obtained analytically $^{[11]}$.

Two $\mathcal{N}$-fold supersymmetric models reported in Ref. $^{[11]}$ and Ref. $^{[12]}$ have the common significant features. The one is the simplicity of the form of the potentials; in spite of the fact that the higher order Darboux transformations generally lead to a quite complicated form of the partner potential $^{[7,13–16]}$. The other is that the $\mathcal{N}$-fold supersymmetry for any $\mathcal{N}$ are realized only through the specific values of a parameter, say $\epsilon = \mathcal{N}$, involved in one Hamiltonian. These examples show the existence of $\mathcal{N}$-fold supersymmetric family in which a pair of the specific Hamiltonians possesses any $\mathcal{N}$-fold supersymmetry via one (or more) parameter(s) involved in the Hamiltonians.

In this letter, we report the general forms and conditions of a system to be $\mathcal{N}$-fold supersymmetric family with respect to a particular form of the supercharges, without any recourse to the information on eigenfunctions. In section $^{[1]}$, we review the $\mathcal{N}$-fold supersymmetry including ordinary one. In section $^{[11]}$, we define a particular class of the $\mathcal{N}$-fold supersymmetry, which will be referred to as type A. We then give the conditions of type A $\mathcal{N}$-fold supersymmetry for arbitrary $\mathcal{N}$. In the case of $\mathcal{N} = 2$, the results reduces to just the ones reported in Refs. $^{[2–4]}$. Section $^{[14]}$ is devoted to illustrations of special cases of type A, including the quartic and periodic potential cases. In section $^{[14]}$, we reexamine the factorized intertwining approach previously done in Ref. $^{[2]}$ and compare the results with those in section $^{[11]}$. We will see that novel intermediate relations, which were not considered in Ref. $^{[2]}$ at all, are allowed in order to get the $\mathcal{N}$-fold superpartner. Concluding remarks are in the last section.

II. REVIEW OF THE $\mathcal{N}$-FOLD SUPERSYMMETRY

First of all, we review the $\mathcal{N}$-fold supersymmetry in one-dimensional quantum mechanics $^{[1,2,5,6,11,12]}$ including the ordinary supersymmetric case $^{[17,20]}$. The $\mathcal{N}$-fold supercharges
are generally defined in matrix form by the following;

\[ Q_N = \begin{pmatrix} 0 & 0 \\ P_N^\dagger & 0 \end{pmatrix}, \quad Q_N^\dagger = \begin{pmatrix} 0 & P_N \\ 0 & 0 \end{pmatrix}, \tag{1} \]

where \( P_N \) is a differential operator of order \( N \). The general form of \( P_N \) is thus given by

\[ P_N = p_N + w_N(q)p_N^{N-1} + \cdots + w_1(q)p + w_0(q). \tag{2} \]

where \( p = -i(d/dq) \). Clearly \( Q_N \) and \( Q_N^\dagger \) are nilpotent, or equivalently,

\( \{Q_N, Q_N\} = \{Q_N^\dagger, Q_N^\dagger\} = 0. \tag{3} \)

The Hamiltonian \( H_N \) defined in matrix form as

\[ H_N = \begin{pmatrix} H_{+N} & 0 \\ 0 & H_{-N} \end{pmatrix}, \tag{4} \]

is said to be \( N \)-fold supersymmetric if it commutes with the \( N \)-fold supercharges;

\[ [Q_N, H_N] = [Q_N^\dagger, H_N] = 0. \tag{5} \]

The components of the above relations \( \text{(5)} \) are

\[ P_N H_{-N} - H_{+N} P_N = 0 \tag{6} \]

and its hermitian conjugate. The anticommutator of \( Q_N^\dagger \) and \( Q_N \) now becomes a differential operator of order \( 2N \). Therefore, if the component Hamiltonians of \( H_N \) are given by the following ordinary Schrödinger type;

\[ H_{\pm N} = \frac{1}{2} \left( p^2 + W(q)^2 + V_{\pm N}(q) \right), \tag{7} \]

the anticommutator can be generally expressed by a \( N \)th order polynomial \( P_N \) of the Hamiltonian \( H_N \);

\[ \mathcal{H}_N \equiv \frac{1}{2} \{Q_N^\dagger, Q_N\} = P_N(H_N). \tag{8} \]

The operator \( \mathcal{H}_N \) defined above is called the *Mother Hamiltonian* and satisfies the following commutation relations;

\( [Q_N, \mathcal{H}_N] = [Q_N^\dagger, \mathcal{H}_N] = 0. \tag{9} \)

In the case of \( N = 1 \), the \( N \)-fold supersymmetry defined above reduces to the ordinary supersymmetric quantum mechanics \[17-20\]. Explicitly if we put,

\[ P_1 \equiv D = p - iW(q), \quad P_1^\dagger \equiv D^\dagger = p + iW(q), \tag{10} \]

we immediately get the ordinary superalgebra;
\[ \{Q_1, Q_1\} = \{Q_1^\dagger, Q_1^\dagger\} = 0, \quad (11a) \]
\[ \{Q_1^\dagger, Q_1\} = 2H_1, \quad (11b) \]
\[ [Q_1, H_1] = [Q_1^\dagger, H_1] = 0. \quad (11c) \]

The component Hamiltonians of \( H_1 \) are given by
\[ H_{\pm 1} = \frac{1}{2} \left( p^2 + W^2(q) \pm W'(q) \right). \quad (12) \]

Comparing with the expressions (7) and (8) we yield the relations;
\[ V_{\pm 1}(q) = \pm W'(q), \quad H_1 = P_1(H_1) = H_1. \quad (13) \]

### III. TYPE A \( \mathcal{N} \)-FOLD SUPERSYMMETRY

In the previous paper \([12]\), it was proved that if the \( \mathcal{N} \)-fold supercharges are limited to the form:
\[ P_\mathcal{N} = D^\mathcal{N}, \quad D = p - iW(q), \quad (14) \]
the \( \mathcal{N} \)-fold supersymmetry can be realized only for quadratic \( W(q) \). It was also shown that for a periodic \( W(q) \) with periodicity \( 2\pi/g \), the system can possess \( \mathcal{N} \)-fold supersymmetry with respect to the following form of the \( \mathcal{N} \)-fold supercharge:
\[ P_\mathcal{N} = \prod_{k=-(\mathcal{N}-1)/2}^{(\mathcal{N}-1)/2} (D + kg). \quad (15) \]

These facts indicate that the allowed \( \mathcal{N} \)-fold supersymmetric systems are characterized and limited by the form of the \( \mathcal{N} \)-fold supercharges. Motivated by this observation, we investigate a particular class of the \( \mathcal{N} \)-fold supercharges, which is called type \( A \). The form of the type \( A \) \( \mathcal{N} \)-fold supercharges \( P_\mathcal{N}^{(A)} \) is defined as follows;
\[ P_\mathcal{N}^{(A)} = (D + i(\mathcal{N} - 1)E(q))(D + i(\mathcal{N} - 2)E(q)) \cdots (D + iE(q))D \]
\[ \equiv \prod_{k=0}^{\mathcal{N}-1} (D + ikE(q)). \quad (16) \]

We will prove that the conditions of the Hamiltonian (3) with (7) to be type \( A \) \( \mathcal{N} \)-fold supersymmetric, that is, to satisfy the relation (5), are as the following;
\[ V_{\pm \mathcal{N}}(q) = -\mathcal{N}^2 E'(q) \pm \mathcal{N} \left( W'(q) - \frac{\mathcal{N}-1}{2} E'(q) \right), \quad (17a) \]
\[ W(q) = \frac{E(q)}{2} + C e^{-\int dqE(q)} \int dq \left( e^{\int dqE(q)} \int dq e^{\int dqE(q)} \right) \quad (\mathcal{N} \geq 2), \quad (17b) \]
\[ E''(q) + E(q)E''(q) + 2E'(q)^2 - 2E(q)^2 E'(q) = 0 \quad (\mathcal{N} \geq 3). \quad (17c) \]
We can prove the above conditions (17) by induction. For \( \mathcal{N} = 1 \), the above (17) (actually, only the Eq. (17a) is applied) reads \( V_{\pm 1}(q) = \pm W'(q) \), which is the ordinary supersymmetric case.

Suppose the relation (18) holds for an integer \( \mathcal{N} \). Then, if we put

\[
H_{\pm(\mathcal{N}+1)} = H_{\pm\mathcal{N}} \pm h_{\pm\mathcal{N}},
\]

and use the relation (18) for this \( \mathcal{N} \), we obtain,

\[
P^{(A)}_{\mathcal{N}+1}H_{-(\mathcal{N}+1)} - H_{+(\mathcal{N}+1)}P^{(A)}_{\mathcal{N}+1} = [D + i\mathcal{N}E, H_{\pm\mathcal{N}}]P^{(A)}_{\mathcal{N}} - h_{\pm\mathcal{N}}P^{(A)}_{\mathcal{N}+1} - P^{(A)}_{\mathcal{N}+1}h_{-\mathcal{N}}. \tag{19}
\]

To facilitate the calculation, we make use of a similarity transformation by \( U \), which is defined by

\[
U = e^{\int dqW(q)}. \tag{20}
\]

The transformation of (20) is then calculated as

\[
I_{\mathcal{N}+1} \equiv 2i^{\mathcal{N}+1}U(P^{(A)}_{\mathcal{N}+1}H_{-(\mathcal{N}+1)} - H_{+(\mathcal{N}+1)}P^{(A)}_{\mathcal{N}+1})U^{-1}
\]

\[
= \left[ \partial - \mathcal{N}E, -\partial^2 + 2W\partial + W' + V_{\mathcal{N}} \right] \tilde{P}^{(A)}_{\mathcal{N}} - 2h_{\mathcal{N}}\tilde{P}^{(A)}_{\mathcal{N}+1} - 2\tilde{P}^{(A)}_{\mathcal{N}+1}h_{-\mathcal{N}}
\]

\[
= 2(W' - \mathcal{N}E' - h_{\mathcal{N}} - h_{-\mathcal{N}})\partial \tilde{P}^{(A)}_{\mathcal{N}}
\]

\[
+ (V'_{\mathcal{N}} + W'' - \mathcal{N}E'' + 2\mathcal{N}E'W + 2\mathcal{N}(h_{\mathcal{N}} + h_{-\mathcal{N}})E)\tilde{P}^{(A)}_{\mathcal{N}} - 2[\tilde{P}^{(A)}_{\mathcal{N}+1}, h_{-\mathcal{N}}], \tag{21}
\]

where

\[
\tilde{P}^{(A)}_{\mathcal{N}} \equiv i^{\mathcal{N}+1}U P^{(A)}_{\mathcal{N}} U^{-1}
\]

\[
= \left( \partial - (\mathcal{N} - 1)E(q) \right) \left( \partial - (\mathcal{N} - 2)E(q) \right) \cdots \left( \partial - E(q) \right) \partial
\]

\[
\equiv \prod_{k=0}^{\mathcal{N}-1} \left( \partial - kE(q) \right). \tag{22}
\]

From Eq. (21), we see that \( I_{\mathcal{N}+1} \) contains up to \( (\mathcal{N} + 1) \)-th derivative. Therefore, \( I_{\mathcal{N}+1} = 0 \) if and only if all the coefficients of \( \partial^k \) \( (k = 0, 1, \ldots, \mathcal{N} + 1) \) vanish. The \( \partial^{\mathcal{N}+1} \) term comes only from the first term of the r.h.s. of (21) and thus,

\[
h_{\mathcal{N}} + h_{-\mathcal{N}} = W' - \mathcal{N}E'. \tag{23}
\]

When this condition (23) satisfied, the difference \( I_{\mathcal{N}+1} \) now reads

\[
I_{\mathcal{N}+1} = (V'_{\mathcal{N}} + W'' - \mathcal{N}E'' + 2\mathcal{N}E'W + 2\mathcal{N}E'W' - 2\mathcal{N}E'\partial E + \mathcal{N}E)\tilde{P}^{(A)}_{\mathcal{N}} - 2[\tilde{P}^{(A)}_{\mathcal{N}+1}, h_{-\mathcal{N}}]. \tag{24}
\]

The second term of the r.h.s. of (24) is calculated as follows;

\[
[\tilde{P}^{(A)}_{\mathcal{N}+1}, h_{-\mathcal{N}}] = h'_{-\mathcal{N}}\tilde{P}^{(A)}_{\mathcal{N}} + (\partial - \mathcal{N}E)[\tilde{P}^{(A)}_{\mathcal{N}}, h_{-\mathcal{N}}]
\]

\[
= h'_{-\mathcal{N}}\tilde{P}^{(A)}_{\mathcal{N}} + (\partial - \mathcal{N}E)\left[ \mathcal{N}h'_{-\mathcal{N}}\partial^{\mathcal{N}-1}
\right.
\]

\[
+ \frac{\mathcal{N}(\mathcal{N} - 1)}{2} \left( h''_{-\mathcal{N}} - (\mathcal{N} - 1)Eh'_{-\mathcal{N}} \right) \partial^{\mathcal{N}-2} + \cdots \right], \tag{25}
\]
where \( \cdots \) denotes the terms which contain up to the \((\mathcal{N} - 2)\)-th order derivative. From the \(\partial \mathcal{N}\) and \(\partial^{\mathcal{N}-1}\) terms, we obtain the following conditions respectively;

\[
2(\mathcal{N} + 1)h'_{-\mathcal{N}} = V'_{+\mathcal{N}} + W'' - \mathcal{N}E'' + 2\mathcal{N}E'W + 2\mathcal{N}EW' - 2\mathcal{N}^2EE',
\]

\[
h''_{-\mathcal{N}} - E h'_{-\mathcal{N}} = 0.
\]

The condition (26) can be easily integrated, and with the condition (23) we get

\[
\pm h_{\pm\mathcal{N}} = \frac{1}{2} \left[ -EW + \frac{4\mathcal{N} - 1}{6}E^2 - \frac{2\mathcal{N} + 1}{6}E' \pm (W' - \mathcal{N}E') \right].
\]

Here we omit the irrelevant integral constants. Therefore, we finally yield

\[
V_{\pm(\mathcal{N}+1)} = V_{\pm\mathcal{N}} \pm 2h_{\pm\mathcal{N}}
= -\mathcal{N}EW + \frac{\mathcal{N}(2\mathcal{N} + 1)}{6}E^2 - \frac{\mathcal{N}(\mathcal{N} + 2)}{6}E' \pm (\mathcal{N} + 1) \left( W' - \frac{\mathcal{N}}{2}E' \right),
\]

which are nothing but the assumed forms of the potential (17a) with \(\mathcal{N}\) replaced by \(\mathcal{N} + 1\).

Before investigating the condition (27), we return to the difference \(I_{\mathcal{N}+1}\) under the condition (26), which now reads

\[
I_{\mathcal{N}+1} = 2\mathcal{N}h'_{-\mathcal{N}}\hat{P}_{\mathcal{N}}^{(A)} - 2(\partial - \mathcal{N}E)[\hat{P}_{\mathcal{N}}^{(A)}, h_{-\mathcal{N}}].
\]

It is easy to see that under the condition (27), the following relation holds;

\[
[\hat{P}_{\mathcal{N}}^{(A)}, h_{-\mathcal{N}}] = Mh'_{-\mathcal{N}}\hat{P}_{\mathcal{N}-1}^{(A)} + \left[ \prod_{k=M}^{\mathcal{N}-1} (\partial - kE), h_{-\mathcal{N}} \right] \hat{P}_{M}^{(A)} \quad (0 \leq M \leq \mathcal{N}).
\]

Applying this relation (with \(M = \mathcal{N}\)) to Eq.(30), we immediately find \(I_{\mathcal{N}+1} = 0\). That is, no additional conditions are needed for satisfying the relation (8) with \(\mathcal{N} + 1\). So, all that remains to be investigated is the condition (27). From Eq.(28), this condition reads,

\[
(W'_{\mathcal{N}} + EW_{\mathcal{N}})' - E(W'_{\mathcal{N}} + EW_{\mathcal{N}})' = 0,
\]

where

\[
W_{\mathcal{N}}(q) = W(q) - \frac{4\mathcal{N} - 1}{6}E(q).
\]

In the case of \(\mathcal{N} = 1\), the condition (32) gives the relation between \(W(q)\) and \(E(q)\);

\[
\left( \frac{W - E}{2} \right)' + E \left( \frac{W - E}{2} \right)'' = E \left[ \left( \frac{W - E}{2} \right)' + E \left( \frac{W - E}{2} \right) \right]' = 0.
\]

Equation (34) can be integrated for \(W(q)\) in terms of \(E(q)\), which leads to the condition (17b). In the case of \(\mathcal{N} \geq 2\), the condition (32) should be compatible with that for \(\mathcal{N} = 1\) by the inductive assumption. This immediately leads to
\[(E' + E^2)'' - E(E' + E^2)' = 0,\]  
\[(35)\]

which is equivalent to the last condition (17c) and the proof is completed.

It is tempting from the potential form (17a) to redefine the prepotential as

\[\tilde{W}(q) \equiv W(q) - \frac{N - 1}{2} E(q).\]  
\[(36)\]

From the conditions (34) and (35), \(\tilde{W}\) should satisfy

\[(\tilde{W}' + E\tilde{W})'' - E(\tilde{W}' + E\tilde{W})' = 0 \quad \text{for} \quad N \geq 2.\]  
\[(37)\]

With this \(\tilde{W}(q)\), we obtain another general form of type A \(N\)-fold supersymmetry;

\[P_{N}^{(A)} = \prod_{k=-(N-1)/2}^{(N-1)/2} (\tilde{D} + ikE(q)), \quad \tilde{D} = p - i\tilde{W}(q),\]  
\[(38a)\]

\[2H_{\pm N} = p^2 + \tilde{W}(q)^2 + \frac{N^2 - 1}{12} \left( E(q)^2 - 2E'(q) \right) \pm N\tilde{W}'(q),\]  
\[(38b)\]

\[\tilde{W}(q) = Ce^{-\int dqE(q)} \int dq \left( e^{\int dqE(q)} \int dq e^{\int dqE(q)} \right) \quad (N \geq 2),\]  
\[(38c)\]

\[E'''(q) + E(q)E''(q) + 2E'(q)^2 - 2E(q)^2 E'(q) = 0 \quad (N \geq 3).\]  
\[(38d)\]

Furthermore, we can express the Hamiltonians (38b) solely in terms of the prepotential \(\tilde{W}(q)\). From the condition (37), an useful relation holds;

\[[\tilde{W}^2(E^2 - 2E')]' = 2\tilde{W}\tilde{W}'''.\]  
\[(39)\]

Using this equality we yield, instead of Eq.(38b),

\[2H_{\pm N} = p^2 + \tilde{W}(q)^2 + \frac{N^2 - 1}{12} \left( \frac{2\tilde{W}'''(q)}{\tilde{W}(q)} - \frac{\tilde{W}'(q)^2}{\tilde{W}(q)^2} + \frac{A}{\tilde{W}(q)^2} \right) \pm N\tilde{W}'(q),\]  
\[(40)\]

where \(A\) is an arbitrary constant. In the case of \(N = 2\), the above (40) is reduced to the result obtained in Ref. [2] for the second order Darboux transformation.

**IV. SPECIAL CASES OF TYPE A**

In this section, we illustrate some special cases of the type A \(N\)-fold supersymmetry by using the general results obtained in the previous section. We will see that the quadratic and the periodic \(W(q)\)s which were earlier found to possess the \(N\)-fold supersymmetry [11,12] can be obtained in this way.

First of all, we set \(E(q) = 0\). This is a trivial solution of Eq.(17c). From Eq.(17b) we yield,

\[W(q) = C_1 q^2 + C_2 q + C_3,\]  
\[(41)\]

that is, quadratic \(W(q)\). In this case, the Hamiltonians and the supercharge are given by
\[ 2H_{\pm N} = p^2 + W(q)^2 \pm NW'(q), \quad P_N^{(A)} = D^N. \] (42)

The special choices \( C_1 = -g, C_2 = 1 \) and \( C_3 = 0 \) correspond to just the case in Ref. [11].

In the next, we set \( E(q) = E_0 (\text{non-zero constant}) \). This is also a trivial solution of Eq.(38d). From Eq.(38c) we yield,

\[ W(q) = C_1 e^{E_0 q} + C_2 e^{-E_0 q} + C_3, \] (43)

that is, exponential \( W(q) \). In this case, the Hamiltonians and the supercharge are given by

\[ 2H_{\pm N} = p^2 + W(q)^2 \pm NW'(q), \quad P_N^{(A)} = \prod_{k=-(N-1)/2}^{(N-1)/2} \left(D + i k E_0\right). \] (44)

The special choices \( E_0 = ig, C_1 = 1/2ig, C_2 = -1/2ig \) and \( C_3 = 0 \) correspond to the periodic case in Ref. [12].

Next, we set \( E(q) = (\nu - 1)/q \) with \( \nu \neq 1 \). It is easy to see that Eq.(38d) is satisfied when \( \nu = \pm 2 \). In both the cases we get from Eq.(38d),

\[ W(q) = C_1 q^3 + C_2 q + C_3 \frac{1}{q}. \] (45)

In these cases, the Hamiltonians are

\[ 2H_{\pm N} = p^2 + W(q)^2 + \frac{N^2 - 1}{4q^2} \pm NW'(q), \] (46)

and the supercharges are given by

\[ P_N^{(A)} = \prod_{k=0}^{N-1} \left(D + i \frac{k}{q}\right) \quad (\nu = +2), \quad \text{or} \quad P_N^{(A)} = \prod_{k=0}^{N-1} \left(D - 3i \frac{k}{q}\right) \quad (\nu = -2). \] (47)

This cubic type \( W(q) \) is a new form of the \( N \)-fold supersymmetry. It should be noted that the \( N \)-fold supercharges for one Hamiltonian pair permits different factorized forms in general, as is the case above (47), owing to the fact that \( E(q) \) satisfies a differential equation (17c).

V. FACTORIZED INTERTWINING APPROACH

We note the type A \( N \)-fold supercharges belong to reducible \( N \)-th order intertwiners \( L_N \), which can be factorized as a product of \( N \) first order differential operators \( L^{(k)} \); \[ L_N = L^{(N)} \cdots L^{(1)}. \] (48)

For such a reducible operator, the factorized intertwining technique [2] can be applicable. In this approach, a \( N \)-fold supersymmetric model is constructed by introducing a sequence of intermediate Hamiltonians \( H^{(k)} \), which satisfy the ordinary supersymmetric relations;

\[ H^{(k)} L^{(k)} = L^{(k)} H^{(k-1)} \quad (k = 1, 2, \ldots, N). \] (49)
Apparently, the following $\mathcal{N}$-fold supersymmetric relation
\[ H_{+\mathcal{N}}L_{\mathcal{N}} = L_{\mathcal{N}}H_{-\mathcal{N}}, \] (50)
holds if we set,
\[ H_{+\mathcal{N}} = H^{(\mathcal{N})}, \quad H_{-\mathcal{N}} = H^{(0)}. \] (51)
In this section we reexamine the conditions of type A $\mathcal{N}$-fold supersymmetry by this intertwining approach and compare the results with those obtained in section [1]. The type A $\mathcal{N}$-fold supercharge is realized if we set the each factor of a intertwiner as
\[ L^{(k)} \equiv D + i(k - 1)E(q) = p - i \left( W(q) - (k - 1)E(q) \right). \] (52)
Each of the above $L^{(k)}$ can be regarded as an ordinary supercharge with prepotential $W - (k - 1)E$. Therefore, if we introduce Hamiltonians $H^{(k)}_{\geq}$ and $H^{(k)}_{\leq}$ as,
\[ 2H^{(k)}_{\geq} = L^{(k)}L^{(k)\dagger} + 2C(k) \]
\[ = p^2 + \left( W - (k - 1)E \right)^2 + \left( W - (k - 1)E \right)' + 2C(k), \] (53a)
\[ 2H^{(k-1)}_{\leq} = L^{(k)}L^{(k)} + 2C(k) \]
\[ = p^2 + \left( W - (k - 1)E \right)^2 - \left( W - (k - 1)E \right)' + 2C(k), \] (53b)
where $C(k)$’s are arbitrary constants, these Hamiltonians satisfy the supersymmetric relation for each $k$;
\[ H^{(k)}_{\geq}L^{(k)} = L^{(k)}H^{(k-1)}_{\leq}. \] (54)
For the above supersymmetric Hamiltonians constructed in each $k$ together to construct the $\mathcal{N}$-fold supersymmetry, the following conditions should be satisfied;
\[ H^{(k)}_{\geq} = H^{(k)}_{\leq} \quad (k = 1, \ldots, \mathcal{N} - 1). \] (55)
This kind of intermediate relations were actually considered in Ref. [2]. Explicitly, this condition is expressed as
\[ \left( W - \frac{E}{2} \right)' + E \left( W - \frac{E}{2} \right) - c_1 = (k - 1) \left( E' + E^2 - c(k) \right) \quad (k = 1, \ldots, \mathcal{N} - 1), \] (56)
where we put $C(k+1) - C(k) = c_1 - (k - 1)c(k)$. For $\mathcal{N} = 2$, the above condition (56) reads
\[ \left( W - \frac{E}{2} \right)' + E \left( W - \frac{E}{2} \right) = c_1. \] (57)
For $\mathcal{N} \geq 3$, to fulfill Eq.(56) for arbitrary $k$, $c(k)$ should not depend on $k$ and thus we put $c(k) \equiv c$, and the following is needed,
\[ E' + E^2 = c, \] (58)
in addition to Eq.(57). Comparing these results (57) and (58) with the conditions obtained in section [1], we see that the results (57) and (58) are sufficient conditions for satisfying Eq.(54) and Eq.(55), respectively. Conversely, $c_1$ and $c$ are not necessarily constant but can be functions of $q$, which satisfy,
\[ c''_{(1)}(q) - E(q) c'_{(1)}(q) = 0. \] (59)
This result indicates that even in the reducible cases there may be wider class of $\mathcal{N}$-fold supersymmetric models than that can be obtained by the factorized intertwining technique.
VI. CONCLUDING REMARKS

In this letter, we have shown the general forms and conditions of a $\mathcal{N}$-fold supersymmetric family. Using the results, one can easily obtain a $\mathcal{N}$-fold supersymmetric model for arbitrary $\mathcal{N}$ with or without dynamical SUSY breaking. If dynamical SUSY breaking takes place or not depends on the domain in which the system is defined and on the asymptotic behavior of $W(q)$. Though the specific type investigated in this paper is quite general, it will be an interesting problem to find another type of family which does not belong to type A.

Finally, we will mention about the non-renormalization theorem. This theorem is one of the most notable properties that the supersymmetric models possess. However, little has been discussed about the theorem in the case of the $\mathcal{N}$-fold supersymmetry. As far as we know, only Ref. [11] investigated the non-renormalization nature for the quartic $W(q)$ case. We have found the same property for the other $\mathcal{N}$-fold supersymmetric models such as the periodic and the cubic $W(q)$s illustrated in section IV. These results will be reported in the near future.

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