BILINEAR ESTIMATES IN THE PRESENCE OF A LARGE POTENTIAL AND A CRITICAL NLS IN 3D

FABIO PUSATERI AND AVY SOFFER

ABSTRACT. We propose an approach to nonlinear evolution equations with large and decaying external potentials that addresses the question of controlling globally-in-time the nonlinear interactions of localized waves in this setting. This problem arises when studying localized perturbations around (possibly non-decaying) special solutions of evolution PDEs, and trying to control the projection onto the continuous spectrum of the nonlinear radiative interactions.

One of our main tools is the Fourier transform adapted to the Schrödinger operator $H = -\Delta + V$, which we employ at a nonlinear level. As a first step we analyze the spatial integral of the product of three generalized eigenfunctions of $H$, and determine the precise structure of its singularities. This leads to study bilinear operators with certain singular kernels, for which we derive product estimates of Coifman-Meyer type. This analysis can then be combined with multilinear harmonic analysis tools and the study of oscillations to obtain (distorted Fourier space analogues of) weighted estimates for dispersive and wave equations.

As a first application we consider the nonlinear Schrödinger equation in 3d in the presence of large decaying potential with no bound states, and with a $u^2$ non-linearity. The main difficulty is that a quadratic nonlinearity in 3d is critical with respect to the Strauss exponent; moreover, this nonlinearity has non-trivial fully coherent interactions even when $V = 0$. We prove quantitative global-in-time bounds and scattering for small solutions.

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1. INTRODUCTION

This work is motivated by questions on the long-time stability of large, and possibly non-localized, special solutions of nonlinear evolution equations. We propose a systematic approach to the study of equations of the form

$$i\partial_t u + L(D)u + V(x)u = N(u), \quad u(t = 0) = u_0.$$  \hfill (1.1)
where \( u : (t,x) \in \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}, \ d > 1 \), \( L(D) = L(-i\nabla_x) \) is a real-valued dispersion relation, \( \mathcal{N} \) is a nonlinear term in \((u, \pi)\), and \( V \) is a large real-valued decaying potential. The initial data is assumed to be sufficiently regular, small and localized, and we are interested in the global existence and quantitative estimates of solutions. Typical examples are nonlinear Schrödinger equations \((L(D) = -|D|^2 = \Delta)\), nonlinear Klein-Gordon \((L = \sqrt{m^2 + |D|^2})\) and wave equations \((L = |D|)\).

The methods we develop are intended to be most relevant for the stability of special solutions in the following contexts:

(1) localized special solutions (solitons and solitary waves) for equations with nonlinearities of low degree of homogeneity, e.g. quadratic nonlinearities in dimensions 2 and 3, such as those appearing in water waves and plasma models;

(2) non-localized solutions (e.g. topological solitons) for equations with nonlinearities of higher degree; examples here include field theories in dimensions 2 or higher, such as Ginzburg-Landau-type theories and their vortices solutions, and generalizations of \( \phi^4 \)-type field theories with ‘domain walls’ solutions (see [34]).

**Linearized dynamics.** To explain the relevance of (1.1) consider a special solution of a nonlinear evolution equation, such as, for the sake of concreteness, a stationary soliton, \( Q = Q(x) \).

The basic strategy to investigate its stability is to look at solutions of the full nonlinear problem in the form \( Q(x) + v(t,x) \), where \( v \) is small (and localized) at the initial time \( t = 0 \). Disregarding for the sake of exposition the issue of modulating \( Q \) by the symmetries of the equation (such as translations and phase rotations), one can immediately see that understanding the stability for \( Q \) amounts to studying the long-time behavior of \( v \).

The equation for the evolution of the perturbation \( v \) presents two fundamental difficulties:

(A) The linear part of the equation involves an added effective potential coming from \( Q \); we refer to this as the perturbed linear operator, as opposed to the “unperturbed” or “flat” operator, which is one with no potential term. A basic example of such an operator is the operator \( L_V := L(D) + V \) from (1.1).

(B) The equation for the perturbation will typically contain ‘pure’ quadratic nonlinear terms without additional localization, in both scenarios (1) and (2) above.

In many relevant applications the potential part in the perturbed operator is smooth and decaying, so we assume this to be the case in our discussion. Then, it is well established that many quantitative properties of linear homogeneous solutions, such as for example time-decay and Strichartz estimates for solutions of \( i\partial_t u + Lu = 0 \), still hold for solutions of \( i\partial_t u + L_V u = 0 \), when one projects onto the continuous spectrum of \( L_V \). However, applications of classical nonlinear tools such as commuting vectorfields and normal forms transformations are almost immediately ruled out by the presence of the inhomogeneous potential term; Furthermore, the perturbed operator may have eigenvalues below the continuous spectrum, or resonances at the edge. This is an important aspect to consider, but it is not the main focus of the present paper.

The second main difficulty is the presence of quadratic nonlinear terms in the equation for \( v \). Quadratic nonlinearities are the most difficult to control due to the slow decay. Moreover, in both cases (1) and (2) one does not expect localized coefficients in front of these quadratic

\[1\text{In many cases the operators obtained upon linearization are more complicated than } L_V \text{ and are not necessarily scalar or self-adjoint.}\]
terms - unlike in the case of solitons for models with nonlinearities which are at least cubic - that can be leveraged through improved local decay. As a result, tools from the linear theory of perturbed operators and energy methods are ineffective to treat these equations.

In the unperturbed case, $V = 0$, quadratic models like (1.1) present similar issues. Starting with seminal works by various authors in the ‘80s, including [6, 28, 38, 26], many techniques have been developed for the study of these classes of weakly nonlinear equations. Roughly speaking, when dispersive effects are weak, one needs a refined analysis of the nonlinear interactions; this can be done in various ways, including normal form analysis, commuting vectorfields methods, harmonic analysis tools and more. However, these techniques are hard to adapt to the perturbed case $V \neq 0$, and, to our knowledge, alternative systematic and robust approaches have not been developed so far. In fact, very little is known about the long-time behavior of nonlinear equations with external potentials and low power nonlinearities (e.g. at or below the Strauss exponent), especially in comparison to the very rich theory for higher power nonlinearities, or the unperturbed cases. Recently, there have been some results in this direction in particular cases, such as the case of one spatial dimension [36, 9, 15, 5], the case of small potentials [30, 31], and the case of non-resonant nonlinearities [14]; see below for more on [30] and [14].

**Nonlinear evolution with a potential.** The aim of this paper is to initiate a refined study of nonlinear interactions in the perturbed setting, for models like (1.1). In particular, the method developed here addresses the combination of (A) and (B) in the simplest case of a potential such that $L_V$ has no eigenvalues or resonances. The understanding is that the analysis can be used in the cases (1) and (2) above, when one restricts the nonlinear interactions to the continuous spectrum of the relevant operator. The additional interplay with the discrete modes, if any are present, needs to be dealt with separately.

One of the main difficulties in treating problems like (1.1) is to understand how two (or more) localized waves interact in the presence of an external potential. As we will see more in details below, the potential, although smooth and localized, has a “delocalizing” effect: waves moving under its influence have a higher degree of uncertainty compared to ‘flat’ waves and therefore are harder to control precisely. Moreover, the potential “decorrelates” frequencies: the sum of the frequencies of two interacting waves is not the frequency of the product, as it is in the unperturbed case. The method that we are proposing here deals with these issues.

More precisely, Sections 3–6 and Section 8 contain results that are generally applicable to Schrödinger operators $H = -\Delta + V$, for a real, regular and decaying potential $V$. In essence, we show how to use the Fourier transform adapted to $H$ - the so-called distorted/perturbed Fourier transform - to write in a fairly explicit way the product of two functions in distorted frequency space. The key is to identify the singular structure of the product, which then permits a precise analysis of nonlinear oscillations; see Section 2 for a more detailed explanation. These results can in principle be applied as a black-box, or with some modifications, to tackle the nonlinear analysis on the continuous spectrum for several problems in the categories (1) and (2) above.

In Sections 7 and 9 we give a first application of our general approach and study in detail the quadratic nonlinear Schrödinger equation

$$i\partial_t u + (-\Delta + V)u = u^2, \quad u(t = 0) = u_0, \quad x \in \mathbb{R}^3. \quad (1.2)$$
This is a prototypical model for a nonlinear equation with a potential; we have chosen the Schrödinger equation for the simplicity of its dispersion relation, but our analysis could extend to the Klein-Gordon equation, or the wave equation under some additional null form assumptions. Note that a quadratic nonlinearity in 3d is critical with respect to the Strauss exponent. The choice of the nonlinearity $u^2$, as opposed to $\bar{u}^2$ or $|u|^2$ is relevant. The $\bar{u}^2$ nonlinearity was treated in [14], and we can easily included it in our analysis. The important difference is that $\bar{u}^2$ admits a global normal form transformation which effectively makes the equation sub-critical relative to the Strauss exponent. In particular, one does not need a precise analysis of spatially localized interactions. For the $|u|^2$ nonlinearity the existence of global solutions is still open even for $V = 0$. For $V = 0$ the other nonlinearities $au^2 + b\bar{u}^2$ have been treated in [20] and [12].

**Distorted Fourier Transform (dFT) and Nonlinear Spectral Distribution (NSD).** Our general set-up is based on the use of the distorted Fourier Transform (dFT) adapted to the Schrödinger operator $H = -\Delta + V$. For the moment, it suffices to say that under suitable decay and generic spectral assumptions on $V$, the familiar formulas relating the Fourier transform and its inverse (in dimension $d = 3$) hold if one replaces (up to constants) $e^{ikx}$ by the generalized eigenfunctions $\psi(k, x)$, which solve $H\psi(x, k) = |k|^2\psi(x, k)$, for all $k \in \mathbb{R}^3 \setminus \{0\}$. That is, for any $g \in L^2$, there exists a unitary operator $\tilde{F}$ defined by

$$\tilde{F}g(k) := \tilde{g}(k) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \overline{\psi(x, k)} g(x) \, dx,$$

with

$$\tilde{F}^{-1}g(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \psi(x, k) g(k) \, dk,$$

that diagonalizes the Schrödinger operator: $\tilde{F}H = |k|^2 \tilde{F}$. See Theorem 3.1.

For a solution $u$ of (1.2) - with the obvious modifications in the case of other dispersion relations or other nonlinearities - we look at the ‘profile’ or ‘interaction variable’

$$f(t, x) := (e^{-it(-\Delta+V)}u(t, \cdot))(x), \quad \tilde{f}(t, k) = e^{-it|k|^2} \tilde{u}(t, k),$$

which satisfies the equation $	ilde{f}(t, k) = \tilde{u}_0(k) - iD(t)(f, f)$ where

$$D(t)(f, f) := \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{is(-|k|^2+|\ell|^2+|m|^2)} \tilde{f}(s, \ell) \tilde{f}(s, m) \mu(k, \ell, m) \, d\ell \, dm \, ds,$$

with

$$\mu(k, \ell, m) := \frac{1}{(2\pi)^d/2} \int_{\mathbb{R}^d} \overline{\psi(x, k)} \psi(x, \ell) \psi(x, m) \, dx.$$

(1.5) is Duhamel’s formula for (1.2) in distorted Fourier space. The distribution $\mu$ characterizes the interaction between the generalized eigenfunctions, and we call it the “Nonlinear Spectral Distribution” (NSD).

Note that in the unperturbed case $V = 0$ the NSD is just a delta function $\delta(k - \ell - m)$. In contrast with this, in equations (1.5)-(1.6) all frequencies interact with each other without any a priori constraint. Looking at (1.5) we see that the set where the integral has no oscillations in time $s$ is always larger than in the case $V = 0$; this implies that time averaging and the standard theory of normal forms transformations are less efficient. At the same time, as we shall see, $\mu(k, \ell, m)$ is singular on a much larger set compared to $\delta(k - \ell - m)$; for example, it is singular when $|k - \ell| = |m|$ (see (2.9)) or when $|k| = |\ell| + |m|$. Then, even
when the oscillatory exponential factor in (1.5) is non-stationary one cannot directly obtain cancellations; these are only possible if non-stationarity holds in the directions where \( \mu \) is regular. The singular behavior of \( \mu \) is another manifestation of both the ‘uncertainty’ and the ‘loss of invariance properties’ caused by the external potential, and leads to the ineffectiveness of a direct application of methods based on vectorfields.

One of our main ideas is to study precisely the structure of \( \mu \) and its singularities as a distribution of \( \mathbb{R}^9 \). After identifying all the singularities, we are naturally led to look at bilinear operators with kernels that are singular on certain annuli in frequency space. For all the relevant operators that appear we prove suitable bilinear estimates of H"older (Coifman-Meyer) type. See for example (2.10)-(2.11) for an informal statement of this type. In the specific case of the NLS equation (1.2) we can then proceed to analyze in detail the integral (1.5). We point out that the general analysis of the NSD can be used for other equations as a black-box. The ideas for the analysis of the nonlinear model (1.2) are also quite general, and in particular the integration by parts using “good vectorfields” that are tangential to the singularities of \( \mu \); see (2.16)-(2.17) and the last part of 2.1 and for more on this.

Overall, the present approach can be seen as an extension of the analysis put forward by [19, 12, 13] for the case \( V = 0 \), where one uses the regular Fourier transform (in which case, recall, the NSD is a delta) and analyzes the corresponding oscillatory integral (1.5). In recent years, some deep advancements have been made starting from these types of basic ideas, and the resulting methods have been quite successful in the study of global regularity and asymptotics for small solutions of dispersive and wave equations. See for example [22, 23, 17, 10] and references therein where the authors study the stability of ‘trivial’ equilibria (e.g. a flat and still sea in the context of the free-boundary Euler equations, or a neutral plasma in the context of the Euler-Maxwell system). Our long-term hope is that, following the approach in this paper, parallel developments can be made in the context of nonlinear equations with potentials, leading to advances in the study of the long-time dynamics around non-trivial equilibria.

More background and related works. The first question one asks when studying equations with potentials such as (1.1), is how much of the linear theory for solutions of \( i\partial_t u + L(D)u = 0 \) can be carried onto perturbed/inhomogeneous linear solutions. As an example, classical results for linear Schrödinger operators [24, 16], guarantee that pointwise decay estimates like

\[
\|e^{itH} P_c f\|_{L^1_{t,x} L^\infty} \lesssim |t|^{-d/2}, \quad x \in \mathbb{R}^d,
\]

hold under mild assumption on the decay of \( V \) (here \( P_c \) is the projection onto the continuous spectrum). Generalizations to other dispersive and wave equations are also known. For more on dispersive estimates, see the survey of Schlag [37], the book [29] and references therein. Strichartz estimates can also be derived for solutions of the perturbed linear problem under fairly general assumptions on \( V \).

For nonlinear problems such as (1.1), the first attempt is to use linear estimates (and energy estimates) to control the flow for long times. This is in parallel to the classical strategy that one would use without the potential; see the seminal work of Strauss [41] and [15] and references therein. Even in the absence of discrete spectrum this approach works only when the spatial dimension and homogeneity of the nonlinearity are high enough, so that dispersive effects are sufficiently strong. If the perturbed operator has discrete spectrum the situation
is more delicate. The main issue is the presence of linear and nonlinear bound states (time-periodic localized structures), and their interaction with the radiative/dispersive part of the solution. Among the many important works in this direction, we mention Soffer-Weinstein [41, 42], Tsai-Yau [46], Gustafson-Nakanishi-Tsai [18], Bambusi-Cuccagna [3], Kirr-Zarnescu [27]. For more general overviews we refer the reader to the surveys [40, 47] and reference therein.

The works cited above are characterized by strong dispersive effects due to the combination of the large spatial dimension and/or a high power (or highly localized) nonlinearity. The situation is quite different, and much less is known, when one cannot get global existence by means of dispersive estimates or energy methods, even if the perturbed operator has only continuous spectrum.

An interesting work in this direction is the paper of Germain-Hani-Walsh [14] who treated an NLS equation like (1.2) with a large, generic, and decaying potential, and with a \( u^2 \) nonlinearity. In [14] the authors use the dFT and lay some groundwork for understanding the NSD (1.6). In particular, they establish Coifman-Meyer type bilinear estimates for operators whose kernel is given by \( \mu \) times a Coifman-Meyer-type symbol, as well as estimate for their bilinear commutator with \( \partial_k \). Since the \( u^2 \) nonlinearity leads to a factor of \(-(|k|^2 + |\ell|^2 + |m|^2)\) in the exponential phase in [15], in [14] the authors can directly exploit time oscillations and do not need to analyze \( \mu \) and its structure and regularity in further details, nor need to exploit oscillations in distorted frequency space. In general one does not expect a typical scenario to be as favorable and, indeed, this is not the case for (1.2). More recently, Léger [30] was able to treat (1.1) under the assumption that the potential, which can also be mildly time-dependent, is small. The smallness of \( V \) permits a more ‘perturbative’ approach using the regular Fourier transform but the problem is still hard due to the presence of coherent resonant interactions (which are absent for \( \overline{u}^2 \)). We also mention the recent work of Kenig and Mendelson [25] where the authors obtain a soliton stability result with high probability for the focusing energy critical 3d wave using a randomization procedure through distorted Fourier projections.

Finally, we point out that in the one dimensional case the distorted Fourier transform has already been used fairly effectively. Indeed, in 1d the generalized eigenfunctions of \( H \) satisfy ODEs and are given as solutions of much simpler Volterra-type integral equations, as opposed to solutions of the Lippmann-Schwinger integral equation (2.1); in particular, their difference with the standard exponentials decays to zero at infinity as fast as the potential, and the structure of the NSD is more explicit. For the 1d cubic NLS this approach has been used by the author with Germain and Rousset [15] and by the author and Chen [5]. See also the related earlier works by Naumkin [36] and Delort [9] on the same problem, of Cuccagna-Georgiev-Visciglia [8] in the subcritical case; other related works in 1d include [11] on wave equations, [32, 33] on certain Klein-Gordon equations with non-constant coefficients, and [35] on the stability of the \( \phi^4 \) kink.

**Main result for quadratic NLS.** We consider the equation

\[
i\partial_t u + (-\Delta + V)u = u^2
\]  

(1.7)

with an initial data \( u(t = 0) = u_0 \). Under suitable assumptions on \( V \) and \( u_0 \), our main result is the global-in-time existence of solutions, together with quantitative pointwise decay estimates and global bounds on certain weighted-type norms of the solution. More precisely,
let
\[ N = 2000, \quad N_1 = 200, \]
and assume that
\[ V \in H^N, \quad \int_{\mathbb{R}^3} (1 + |x|)^{N_1+10} |\nabla_x \alpha V(x)| \, dx < \infty, \quad 0 \leq |\alpha| \leq N_1 + 10, \]
and that
\[ H = -\Delta + V \text{ has no eigenvalues or resonances.} \] (1.10)
This is our main result for (1.7).

**Theorem 1.1.** Consider (1.7) under the assumption (1.8)-(1.9). Consider an initial data \( u_0 \) satisfying
\[ \|u_0\|_{H^N} + \|\nabla_k \tilde{u}_0\|_{L^2} + \|\nabla_k^2 \tilde{u}_0\|_{L^2} \leq \varepsilon_0, \]
where \( \tilde{g} = \tilde{F}(g) \) denotes the distorted Fourier transform of \( g \) as defined in Theorem 3.1.
Then, there exists \( \varepsilon \) small enough such that, for all \( \varepsilon_0 \leq \varepsilon \), the equation (1.7) admits a unique global solution \( u \in C(\mathbb{R}; H^N(\mathbb{R}^3)) \) with \( u(t = 0) = u_0 \), satisfying
\[ \|u(t)\|_{H^N} + \langle t \rangle^{1+\alpha} \|u(t)\|_{L^\infty} \lesssim \varepsilon_0 \] (1.12)
for some \( \alpha > 0 \).

Here is a few comments about the statement

- **High regularity:** We consider very smooth solutions in a high Sobolev space \( H^N \) (and therefore require the same amount of regularity for the potential). Although (1.7) is a semilinear equation we find it useful to control a large number derivatives in many parts of our analysis; for example, when we perform various expansions that lose derivatives or want to deal with non-standard symbols of multilinear operators that have some losses at high frequencies. Thanks to the \( H^N \) bound we can think of frequencies as being effectively bounded from above by a small power of time, and thus having minimal impact on all our estimates for the evolution. These smoothness and decay hypotheses are clearly not optimal, and the values of \( N \) and \( N_1 \) can certainly be improved.

- **Distorted weighted norm:** The last two norms in (1.11) are distorted Fourier analogues of the more standard \( L^2(\langle x \rangle^4 \, dx) \) norms which are found in the literature on this and similar types of problems.

- **Decay and scattering:** Part of the conclusion of our nonlinear analysis, see the bootstrap Proposition 2.1, is that we can control, up to some small power of time, distorted Fourier analogues of weighted norms along the evolution. In particular we can prove that, for \( f = e^{-itH} u, \partial_k \tilde{f}(t) \) is uniformly bounded in \( L_k^2 \), and \( \partial_k^2 \tilde{f}(t) \) grows in \( L_k^2 \) at most like \( \langle t \rangle^{1+\delta} \) for \( \delta > 0 \) small. These bounds, combined with standard linear decay estimates, interpolation, and the boundedness of wave operators, then imply that \( u(t) \) decays pointwise at an integrable rate of \( \langle t \rangle^{-1-\alpha} \) for some \( \alpha > 0 \). In particular the solution scatters to a (perturbed) linear solution as \( |t| \to \infty \).

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2. Main ideas and strategy

In this section we first give a summary of our strategy pointing out some important elements in our proofs. Then we introduce the necessary notation, define the functional space in which we will work to prove global existence for (1.7), and state the main bootstrap proposition which will imply Theorem 1.1.

2.1. Main steps.

Step 1: Distorted Fourier Transform and the Nonlinear Spectral Distribution. Under our assumptions on the potential $V$ we can define, for $k \in \mathbb{R}^3 \setminus \{0\}$, a family of generalized eigenfunctions associated to $H = -\Delta + V$ as the unique solutions of the problem

$$(-\Delta + V)\psi(x, k) = |k|^2 \psi(x, k), \quad k \in \mathbb{R}^3 \setminus \{0\},$$

with the asymptotic condition $v(x, k) := \psi(x, k) - e^{ix \cdot k} = O(|x|^{-1})$ and verifying the Sommerfeld radiation condition $r(\partial_r - i|k|)v(x, k) \to 0$, for $r = |x| \to \infty$. These satisfy the integral equation

$$\psi(x, k) = e^{ix \cdot k} - \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{i|k||x-y|}}{|x-y|} V(y)\psi(y, k) \, dy. \quad (2.1)$$

The family $\{\psi(\cdot, k)\}$ forms a basis for the absolutely continuous spectrum of $H$ and thanks to classical results (see Theorem 3.1) the familiar formulas relating the Fourier transform and its inverse in dimension $d = 3$ hold if one replaces (up to constants) $e^{ik \cdot x}$ by $\psi(k, x)$.

Recall that, given $u$ solution of (1.7), one has the distorted Duhamel’s formula for the profile (1.4)-(1.5) with the nonlinear spectral distribution (NSD) defined by (1.6).

Step 2: Expansion of the generalized eigenfunctions. To understand the global-in-time properties of solutions through (1.5), one needs a very precise understanding of the NSD, and the ability to exploit generalized frequencies oscillations. We begin by separating the flat and the potential contributions to the generalized eigenfunction by setting

$$\psi(x, k) = e^{ix \cdot k} - e^{ik|x|} \frac{1}{4\pi|x|} \psi_1(x, k),$$

$$\psi_1(x, k) := \int_{\mathbb{R}^3} e^{i|k||x-y|-|x|} \frac{|x|}{|x-y|} V(y)\psi(y, k) \, dy, \quad (2.2)$$

and then expanding $\psi_1$ in negative powers of $|x|$:

$$\psi_1(x, k) = \sum_{j=0}^{n} g_j(\omega, k) r^{-j} (k)^j + R(x, k), \quad r := |x|, \ \omega := \frac{x}{|x|}, \quad (2.3)$$

where $R$ is a sufficiently regular remainder that decays faster than $r^{-n}$ with $n$ large enough, and the coefficients $g_j(\omega, k)$ belong to a suitably defined symbol class whose prototypical element has the form

$$g(\omega, k) = \int_{\mathbb{R}^3} e^{-i|k|\omega \cdot y} f(y) \, dy, \quad (2.4)$$

for a fast decaying $f$. In particular, these are smooth functions of $\omega$ with some singularity in $k$. For full details on the expansion (2.3) see Subsection 3.3 and in particular the statement of Lemma 3.4. Notice that $\omega$-derivatives of (2.4) grow with $|k|$; this causes some technical difficulties in dealing with high frequencies. We resolve this by restricting the nonlinear
analysis for the evolution to frequencies $|k| \leq \langle t \rangle^{\delta}$ for a small $\delta > 0$, and treating high frequencies $|k| \geq \langle t \rangle^{\delta}$ by leveraging the high $H^N$ smoothness of solutions. We do not discuss the estimate for high frequencies in this explanation, but refer the reader to [4.3].

**Step 3: Asymptotics for the NSD.** The next step consists of plugging-in the (linear) expansions (2.2)-(2.3) into the expression (1.6) for $\mu$ to obtain an expansion of the NSD. We see that for large $|x|$

$$
\psi(x, k)\psi(x, \ell)\psi(x, m) = e^{ix(-k+\ell+m)} - \frac{1}{4\pi|x|} e^{i|x|} e^{i\langle x-k+\ell \rangle} g_0(\omega, m) \\
- \frac{1}{4\pi|x|} e^{i|\ell| |x|} e^{i\langle x-k+\ell \rangle} g_0(\omega, \ell) - \frac{1}{4\pi|x|} e^{-i|k| |x|} e^{i\langle \ell+m \rangle} g_0(\omega, k) + O(|x|^{-2}),
$$

(2.5)

so that (up to irrelevant constants)

$$
\mu(k, \ell, m) \approx \delta(k - \ell - m) + \int_{\mathbb{R}^3} \frac{1}{|x|} e^{i|\ell| |x|} e^{i\langle x-k+\ell \rangle} g_0(\omega, m) \, dx \\
+ \text{“similar or better terms”}.
$$

(2.6)

This leads us to study the behavior of oscillatory integrals of the form

$$
\nu(p, q) := \int_{\mathbb{R}^3} \frac{1}{|x|} e^{i|p| |x|} e^{i\langle x-q \rangle} g_0(\omega, p) \, dx.
$$

(2.7)

In Proposition 5.1 we give an expansion for this integral and, in particular, establish that

$$
\nu(p, q) \approx \frac{1}{|q|} \left( \delta(|p| - |q|) + \text{p.v.} \frac{1}{|p| - |q|} \right) + \text{“better terms”}
$$

(2.8)

up to some coefficient involving $g_0(\pm q/|q|, p)$. This gives an explicit expression for the leading order in the expansion of $\mu$: we can essentially think that

$$
\mu(k, \ell, m) \approx \frac{1}{|\ell - k|} \text{p.v.} \frac{1}{|\ell - k| - |m|} + \text{“similar or better terms”}.
$$

(2.9)

**Step 4: Multiplier estimates.** Next, we are going to establish some multiplier estimates for the terms in the expansion of $\mu$. The typical statement will be an Hölder/Coifman-Meyer type estimate for the multiplier appearing on the right-hand side of (2.9). More precisely, if we define non-standard pseudo-product operators of the form

$$
B(g, h)(k) = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \tilde{g}(\ell) \tilde{h}(m) b(k, \ell, m) \frac{1}{|\ell - k|} \text{p.v.} \frac{1}{|\ell - k| - |m|} \, dm \, d\ell,
$$

(2.10)

where $b$ belongs to a suitable symbol class, then, up to some small losses and some less important factors which we do not detail here,

$$
\|B(g, h)\|_{L^2} \lesssim \|\mathcal{W}^* g\|_{L^p} \|\mathcal{W}^* h\|_{L^q}, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{2},
$$

(2.11)

where $\mathcal{W} = \tilde{F}^{-1} \tilde{F}$ is the wave operator, see (3.3). We refer the reader to Section 6 and Theorem 6.1 for precise statements. Similar estimates are also needed for all the other bilinear operators associated to the other terms in the expansion of (2.9). An interesting aspect is how these estimates are obtained by establishing results on bilinear pseudo-product operators supported on thin annuli (see Lemmas 8.4 also 8.5).
Step 5: Set up for the nonlinear analysis. With the precise information obtained on \( \mu \), we can proceed to study our nonlinear equation through the distorted Duhamel’s formula. From (1.5) and (2.6)-(2.9) we see that
\[
\tilde{f}(t) = \tilde{u}_0 + \mathcal{D}(t)(f, f)
\]
where, at leading order, we have
\[
\mathcal{D}(t)(f, f) \approx \int_0^t \int \left[ e^{is(-|k|^2+|\ell|^2+|m|^2)} \tilde{f}(s, \ell) \tilde{f}(s, m) \frac{1}{|\ell-k|^2} \right] \text{p.v.} \frac{1}{|\ell-k|-|m|} d\ell d\ell d\ell,
\]
(2.12)

Our aim is to estimate globally-in-time the solution \( u \) through its representation via \( \tilde{f} \) above. To do this we devise a proper functional framework and place the evolution in a space that is strong enough to guarantee global decay, but also sufficiently weak to allow us the possibility of closing the estimates.

As mentioned after Theorem 1.1, we will prove the following bounds:
\[
\|u(t)\|_{H^N} \lesssim \varepsilon, \quad \|\partial^j_k \tilde{f}(t)\|_{L^2} \lesssim \varepsilon, \quad \|\partial^2_k \tilde{f}(t)\|_{L^2} \lesssim \varepsilon(t)^{1/2+\delta},
\]
(2.13)
for some small \( \delta > 0 \), and all \( t \in \mathbb{R} \); see also the bootstrap Proposition 2.1.

The remaining part of the argument is dedicated to estimating a priori the nonlinear expressions in the right-hand side of (2.12) according to (2.13). The most difficult estimate is the one for \( \partial^2_k \tilde{f}, \) for which we have to allow a certain growth in time. This is ultimately due to the lack of invariances (such as scaling and gauge-invariance) of the equation.

Step 6: Nonlinear estimates and “good directions”. For simplicity, let us concentrate on the leading order term in (2.12), that is
\[
\mathcal{B}(f, f)(t, k) := \int_0^t \int \int e^{is\Phi(k, \ell, m)} \tilde{f}(s, \ell + k) \tilde{f}(s, m) \frac{1}{|\ell|^2} \text{p.v.} \frac{1}{|\ell|-|m|} d\ell d\ell d\ell,
\]
(2.14)
\[
\Phi(k, \ell, m) := |\ell|^2 + 2\ell \cdot k + |m|^2.
\]
\( \Phi \) is the so-called ‘phase’ or ‘modulation’. Recall that we have Hölder-type bilinear estimates for such expressions, see (2.5)-(2.6), and that we may restrict to frequencies \(|k| + |\ell| + |m| \lesssim 1 \). Moreover, we may restrict our attention to the case
\[
||\ell| - |m|| \ll |\ell| \approx |m|
\]
(2.15)
where the p.v. is indeed singular.

The main difficulty is that an application of \( \partial^j_k, \) \( j = 1, 2, \) to (2.14) leads to terms like
\[
\int_0^t \int \int (-2is\ell)^j e^{is\Phi(k, \ell, m)} \tilde{f}(s, \ell + k) \tilde{f}(s, m) \frac{1}{|\ell|^2} \text{p.v.} \frac{1}{|\ell|-|m|} d\ell d\ell d\ell
\]
(2.16)
which contain powers of \( s \) in the integrand. To obtain good bounds on expressions like (2.16) it is necessary to exploit oscillations through integration by parts arguments. This leads to several difficulties:

(1) the amplitudes, i.e. the profiles \( \tilde{f} \), have limited smoothness, according to the a priori assumptions (see (2.13)),
(2) the oscillating factor \( s\Phi(k, \ell, m) \) is stationary in many directions, and
(3) the integral is taken with respect to a singular kernel.

We note that issues similar to (1) and (2) have been handled in various problems for equations without external potential. The third issue, and its combination with (2), is however a new difficulty and, to our knowledge, appears here for the first time.
Due to the singularity of the kernel, several directions of integrations are forbidden, such as, for example, $\partial_t$ and $\partial_n$. However, there is a natural choice of direction along which we are allowed to integrate, that is, the “good direction”

\[ X := \partial_{|\ell|} + \partial_{|m|} \tag{2.17} \]

which is tangential to the singularity. A calculations gives

\[ \Phi(k, \ell, m) = (|m| - |\ell|)^2 + |\ell|X\Phi(k, \ell, m) - 2|\ell|^2, \]

and it follows that, close to the singularity of the kernel,

\[ |X\Phi(k, \ell, m)| \neq |\ell| \implies |\Phi| \gtrsim |\ell| \max(|\ell|, |X\Phi|). \tag{2.18} \]

In other words, one of the following three things happens: either (a) the kernel is not very

(c) the integrand of (2.16) is non-stationary in the $s$ direction, (more precisely, $|\Phi| \gtrsim |\ell|^2$).

These facts turn out sufficient to obtain the desired weighted $L^2$ bounds.

We refer the reader to Section 7 for details of these weighted estimates for the main term (2.14). The lower order terms corresponding to the “similar and better terms” in (2.20) are estimated in Section 9.

**Notation.** We fix $\varphi : \mathbb{R} \to [0, 1]$ an even smooth function supported in $[-8/5, 8/5]$ and equal to 1 in $[-5/4, 5/4]$. For simplicity of notation, we also let $\varphi : \mathbb{R}^n \to [0, 1]$ denote the corresponding radial function on $\mathbb{R}^n$. Let

\[ \varphi_K(x) := \varphi(|x|/2^K) - \varphi(|x|/2^{K-1}) \text{ for any } K \in \mathbb{Z}, \quad \varphi_I := \sum_{M \in I \cap \mathbb{Z}} \varphi_M \text{ for any } I \subseteq \mathbb{R}, \]

\[ \varphi_{\leq B} := \varphi(\infty, B), \quad \varphi_{\geq B} := \varphi(B, \infty), \quad \varphi_{< B} := \varphi(\infty, B), \quad \varphi_{> B} := \varphi(B, \infty). \tag{2.19} \]

For any $A < B \in \mathbb{Z}$ and $J \in [A, B] \cap \mathbb{Z}$ we let

\[ \varphi_{[A,B]}^J := \begin{cases} \varphi_{J} & \text{if } A < J < B, \\ \varphi_{\leq A} & \text{if } J = A, \\ \varphi_{\geq B} & \text{if } J = B. \end{cases} \]

\[ \varphi_{J}^{(A)} := \begin{cases} \varphi_{J} & \text{if } A < J, \\ \varphi_{\leq A} & \text{if } J = A. \end{cases} \tag{2.20} \]

For simplicity of notation, we will also use $\varphi_K$ to denote a generic smooth cutoff function which is one on the support of $\varphi_K$ and is supported in $[c_1 2^K, c_2 2^K]$ for some absolute constants $c_1 < 1 < c_2$. We will also sometimes denote with $\varphi'$ a generic cutoff function with support properties similar to those of $\varphi$, such as derivatives of $\varphi$.

$P_K, K \in \mathbb{Z}$, denotes the Littlewood–Paley projection operator defined by the (flat) Fourier multiplier $\xi \mapsto \varphi_K(\xi)$. $P_{\leq B}$ denotes the operator defined by the Fourier multiplier $\xi \mapsto \varphi_{\leq B}(\xi)$. Similarly we define $P_{< B}, P_{\geq B}$ and so on.

For any $x \in \mathbb{Z}$ let $x_+ = \max(x, 0)$ and $x_- := \min(x, 0)$. For any number $p \in \mathbb{R}$ we will denote with $p^+$, resp. $p^-$, a number which is larger, resp. smaller, than $p$ but can be chosen arbitrarily close to $p$.

We use standard notation for functional spaces such as $L^p, W^{s,p}$ and $H^s$. $S$ denotes the Schwartz class.
2.2. The main bootstrap argument. We place our evolution in the space \(X = A \cap W\) defined by the following norms:
\[
\|u(t)\|_A := \|\langle k \rangle^N \tilde{f}(t)\|_{L^2} + \|\partial_k \tilde{f}(t)\|_{L^2},
\]
\[
\|u(t)\|_W := \|\partial_k^2 \tilde{f}(t)\|_{L^2},
\]
where we recall that \(u\) and \(f\) are related by \(u = e^{uH} f\).

The first norm \(\|\langle k \rangle^N \tilde{f}(t)\|_{L^2}\) is the equivalent of the Sobolev norm \(\|u(t)\|_{H_N}\). Since we will be able to prove integrable-in-time decay for \(u\) the norm \(\|u(t)\|_A\) can be uniformly bounded in \(t\). On the contrary, the highest weighted-type norm \(\|u(t)\|_W\) is allowed to grow at a certain (quite fast) rate of \(\langle t \rangle^{1/2}\). This is still sufficient to obtain a certain control (with growth) of the \(L^2_1\)-norm of \(Wf\), and infer pointwise-in-\(x\) time decay via a standard dispersive estimate.

The proof of Theorem 1.1 relies on the following bootstrap estimates.

**Proposition 2.1** (Main Bootstrap). Let \(u\) be a solution of (1.7) on a time interval \([0, T]\), with initial data satisfying
\[
\|u_0\|_{H_N} + \|\partial_k \tilde{u}_0\|_{L^2} + \|\partial_k^2 \tilde{u}_0\|_{L^2} \leq \varepsilon_0.
\]
With the definitions (2.21) assume the a priori bounds
\[
\sup_{t \in [0, T]} \left( \|u(t)\|_A + \langle t \rangle^{-1/2-\delta} \|u(t)\|_W \right) \leq \varepsilon \leq \varepsilon_0^{2/3},
\]

for some properly chosen \(\delta \in (0, 1/4)\). Then, we have the improved bounds
\[
\sup_{t \in [0, T]} \left( \|u(t)\|_A + \langle t \rangle^{-1/2-\delta} \|u(t)\|_W \right) \leq \frac{\varepsilon}{2}.
\]

Through a standard bootstrap argument this proposition gives us global solutions for (1.7). The Sobolev bound in (1.12) follows from \(\|\langle k \rangle^N \tilde{f}\|_{L^2} = \|\langle k \rangle^N \tilde{u}\|_{L^2} \approx \|u\|_{H_N}\). The bounds (2.21) imply the pointwise decay estimate stated in (1.12) with \(\alpha = 1/4 - \delta/2\), via the linear estimate (4.1) and the interpolation \(\|\tilde{f}\|_{L^2_\infty} \lesssim \|\partial_k \tilde{f}\|_{L^2} \|\partial_k^2 \tilde{f}\|_{L^2}\).

The proof of Proposition 2.1 is performed in 7 (for the leading order terms) and 9 (for all the lower order terms).

3. Linear Spectral Theory

3.1. Generalized eigenfunctions and distorted Fourier transform. Given a potential \(V : \mathbb{R}^3 \to \mathbb{R}\), consider the Schrödinger operator \(H = -\Delta + V\) associated to it. If \(V\) decays fast enough (e.g. it is ‘short range’ in the sense of Agmon [1]) the spectrum of \(H\) consists of the absolutely continuous spectrum \([0, \infty)\) and a countable number of negative eigenvalues \(0 > \lambda_1 > \lambda_2 > \ldots\) with finite multiplicity. One has the orthogonal decomposition \(L^2(\mathbb{R}^3) = L^2_{ac}(\mathbb{R}^3) \oplus L^2_p(\mathbb{R}^3)\) where \(L^2_{ac}(\mathbb{R}^3)\) is the absolutely continuous subspace for \(H\) and \(L^2_p(\mathbb{R}^3)\) is the span of the eigenfunctions corresponding to the negative eigenvalues.

For any \(k \in \mathbb{R}^3 \setminus \{0\}\) we have that \(|k|^2\) is in the continuous spectrum of \(H\) and the associated (generalized) eigenfunctions \(\psi(x, k)\) are defined as solutions of
\[
(-\Delta + V)\psi(x, k) = |k|^2 \psi(x, k), \quad \forall k \in \mathbb{R}^3 \setminus \{0\},
\]

\(^2\text{We can choose } \delta = 240/(N - 5) \text{ for example.}\)
with the asymptotic condition $\psi(x, k) - e^{ix \cdot k} = O(|x|^{-1})$ for $|x| \to \infty$, and the Sommerfeld radiation condition

$$r(\partial_r - i|k|) \psi(x, k) \to 0,$$

as $r = |x| \to \infty$. The functions $\psi(x, k)$ are ‘distorted’ version of the plane waves $e^{ik \cdot x}$. They satisfy the so-called Lippmann-Schwinger equation

$$\psi(x, k) = e^{ix \cdot k} - R_V(|k|^2)(V e^{ix \cdot k}),$$

where $R_V(\lambda) = (H - \lambda)^{-1}$ is the (perturbed) resolvent. For our analysis it will actually be more convenient to write $\psi$ as a solution of the integral equation

$$\psi(x, k) = e^{ix \cdot k} - R_0(|k|^2)(V \psi(\omega, k)),$$

where $R_0(\lambda) = (-\Delta - \lambda)^{-1}$ is the flat/unperturbed resolvent; more explicitly,

$$\psi(x, k) = e^{ix \cdot k} - \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{i|k| |x - y|}}{|x - y|} V(y) \psi(y, k) \, dy. \quad (3.2)$$

The following Theorem guarantees the existence of the Distorted Fourier Transform and its inverse, under suitable assumptions on the potential.

**Theorem 3.1** (Distorted Fourier Transform). Consider the Schrödinger operator $H = -\Delta + V$ with a fast decaying potential $V = O(|x|^{-1})$ in dimension $d = 3$, and assume (1.10). For $g \in S$ define the distorted Fourier Transform (dFT) by

$$\tilde{F}g(k) := \tilde{g}(k) := \frac{1}{(2\pi)^{3/2}} \lim_{R \to \infty} \int_{|x| \leq R} \tilde{\psi}(x, k) g(x) \, dx. \quad (3.3)$$

Then, $\tilde{F}$ extends to an isometric isomorphism of $L^2(\mathbb{R}^3)$ with inverse

$$(\tilde{F}^{-1}g)(x) := \frac{1}{(2\pi)^{3/2}} \lim_{R \to \infty} \int_{|x| \leq R} \psi(x, k) g(k) \, dk. \quad (3.4)$$

Moreover, $\tilde{F}$ diagonalizes the Schrödinger operator: $\tilde{F}H\tilde{F}^{-1} = |k|^2$.

This theorem is due to several authors including Ikebe [21], Alsholm-Schmidt [2], and Agmon [1]. We refer the interested reader to Section 2 of [14] for a more extensive presentation of this topic, and a discussion about the validity of Theorem 3.1 under weaker assumptions on the potential, such as those made in the references cited above. In the case that (1.10) does not hold and $H_V$ has discrete spectrum in $(-\infty, 0)$, then the generalized eigenfunctions will diagonalize $H_V$ restricted to the absolutely continuous subspace $L^2_{ac}$, and the dFT is well-defined and invertible there.

An important object in the study of the flow associated to $H$ is the wave operator defined by

$$W = s - \lim_{t \to \infty} e^{itH} e^{it\Delta}, \quad (3.5)$$

where the limit is in the strong operator topology. The wave operator is unitary on $L^2$ and is connected to the dFT by the formula

$$W = \tilde{F}^{-1} \tilde{F}, \quad (3.6)$$

This can be formally understood as $R_V(\lambda) := \lim_{\epsilon \to 0^+} (H - \lambda + i\epsilon)^{-1}$, where the limit is taken with respect to a proper operator norm topology, say from $\langle x \rangle^{-s}L^2$ to $\langle x \rangle^s H^2$ for some $s > 1/2$. 
where $\tilde{F}$ is the regular/flat Fourier transform. In particular, $W^{-1} = W^* = \tilde{F}^{-1} \tilde{F}$, and one has the following intertwining formulas for $H$ and $H_0 = -\Delta$:

$$a(H) = Wa(H_0)W^*.$$  \tag{3.7}

Under relatively mild decay and regularity assumptions on $V$ (much weaker than our assumption (1.9)) and provided that $V$ is generic, that is, there are no solutions of $H\psi = 0$ in $\langle x \rangle^{1/2+L^2}$ (no resonances), we have that $W$ and $W^*$ are bounded on $W^{k,p}$. See Yajima \cite{48} and the discussion in \cite{14} and reference therein.

It is worth pointing out that while we use standard results on the $L^p$ boundedness of wave operators, we do not rely on any specific structural property about them, such as those found in the literature\footnote{For convenience we use a result of \cite{14}, Proposition 4.5 below, which relies on the structure of $W$; but it is should be possible to use our approach to obtain this independently.}, see for example \cite{48,14,4}. On the other hand, our approach does rely on analyzing the structure of ‘wave operator’- like quantities at a nonlinear level.

### 3.2. Bounds on $\psi(x, k)$

We begin our analysis by establishing some basic estimates on $\psi$ and its derivatives in $x$ and $k$.

**Lemma 3.2 (Basic properties of $\psi$).** Let $\psi$ be defined as in (3.2) with $V$ satisfying (1.9). Then:

$$|\partial_\alpha x \partial_\beta k \psi(x, k)| \lesssim (\langle k \rangle^{|\alpha|} + (|k|/\langle k \rangle)^{1-|\beta|}\langle x \rangle^{|\beta|} \quad 0 \leq |\alpha|, |\beta| \leq N_1$$  \tag{3.8}

**Proof.** For $f \in L^\infty_{x,k}$ let us define the operator

$$(T_k f(\cdot, k))(x) := -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{ik|x-y|}}{|x-y|} V(y)f(y, k) \, dy.$$  \tag{3.9}

Let us write

$$v(x, k) := \psi(x, k) - e^{ix\cdot k},$$  \tag{3.10}

so that (3.2) implies

$$v(x, k) = (T_k v(\cdot, k))(x) + (T_k e^{ix\cdot k})(x).$$  \tag{3.11}

Note that $T_k$ is a compact operator from $L^\infty$ to $C_0$, where $C_0$ is the space of bounded continuous functions decaying to 0 at infinity. In particular, for any $g \in C_0$ there exists a unique $C_0$ solution to the integral equation $f = g + T_k f$ if and only if $f = T_k f$ admits only the trivial solution; this is indeed the case since $T_k f = -R_0(V f)$, where $R_0$ is the flat resolvent $(-\Delta - |k|^2)^{-1}$ and we are assuming absence of eigenvalues and resonances for $-\Delta + V$. This and (3.11) imply (3.8) for $\alpha = \beta = 0$.

Let us define

$$v_{\alpha\beta} := \langle k \rangle^{-|\alpha|}(|k|/\langle k \rangle)^{|\beta|-1}\langle x \rangle^{-|\beta|}\partial_\alpha x \partial_\beta k v, \quad |\beta| \geq 1.$$  \tag{3.12}

The conclusion (3.8) will follow from uniform bounds on $v_{\alpha\beta}$. To obtain these bounds we will show, by induction, that $v_{\alpha\beta}$ satisfies an integral equation similar to the one satisfied $v$ (3.11), up to lower order terms.
To formalize this, let us define the class $\mathcal{T}^N$ of $k$ dependent operators as follows:

$$
T_k \in \mathcal{T}^N \iff T_k f := \int_{\mathbb{R}^3} e^{i|k||x-y|} a(x, y) f(y) \, dy,
$$

with

$$
\int_{\mathbb{R}^3} \langle y \rangle^N \left( \langle \partial_x \rangle^N + \langle \partial_y \rangle^N \right) |a(x, y)| \, dy \lesssim 1.
$$

We think of operators in $\mathcal{T}^N$ as acting on function $f = f(x, k)$. The operator $T_k$ in (3.9) belongs to $\mathcal{T}^{N_1}$ by the assumption (1.9). The following properties hold:

(i) For $N > 3$, operators in $\mathcal{T}^N$ are compact from $L^\infty$ to $C_0$.

(ii) We have

$$
\mathcal{T}^N \subset \mathcal{T}^{N'}, \quad N' \leq N,
$$

$$
\langle x \rangle^{-\ell} \mathcal{T}^N \subset \mathcal{T}^N, \quad \forall \ell \geq 0,
$$

$$
\mathcal{T}^N(\langle y \rangle^\cdot), \quad \mathcal{T}^N(y^\cdot) \subset \mathcal{T}^{N-1},
$$

$$
\langle x \rangle^{-1} \partial_k \mathcal{T}^N \subset \frac{k}{|k|} \mathcal{T}^{N-1}.
$$

Here, for $T_k \in \mathcal{T}_k$ we denote $\partial_k T_k$ the operator obtained by differentiating in $k$ the term $e^{i|k||x-y|}$ in (3.13).

(iii) We have

$$
[\partial_x, T_k] := \partial_x T_k - T_k \partial_x \subset \mathcal{T}^{N-1}.
$$

This can be seen by applying directly $\partial_x$ to (3.13), converting it into $-\partial_y$ and integrating by parts.

Claim. Let $v_{\alpha\beta}$ be defined as in (3.12) for $|\alpha| + |\beta| = N \leq N_1$. The following identity holds true:

$$
v_{\alpha\beta} + T_{0,\beta}(v_{\alpha\beta}) = G_{\alpha\beta}
$$

where

$$
T_{0,\beta}(f) := \frac{1}{\langle x \rangle^{|\beta|}} \frac{1}{4\pi} \int_{\mathbb{R}^3} e^{i|k||x-y|} V(y) \langle y \rangle^{|\beta|} f(y) \, dy,
$$

and $G_{\alpha\beta}$ is a linear combination of the form

$$
G_{\alpha\beta} = \sum a_{\ell}(k) T_{\ell}(v_{\gamma\delta}) + \sum a'_{\ell}(k) T'_{\ell}(e^{ix-k}), \quad T_{\ell}, T'_{\ell} \in \mathcal{T}^{N_1-N-1},
$$

where the sums run over finitely many indexes $\ell$ and

$$
|\gamma| \leq |\alpha|, \quad |\delta| \leq |\beta|, \quad |\gamma| + |\delta| \leq N - 1,
$$

and with coefficients $a_{\ell}(k), a'_{\ell}(k)$ that are either (a) smooth and bounded with all their derivatives, or (b) 0-homogeneous for $|k| \ll 1$ and otherwise smooth and bounded with all their derivatives.

Proof of the Claim. We proceed by induction on $N$. The case $N = 0$ is given by (3.11). Let us assume that the claimed identity is true for $v_{\alpha\beta}$ with $|\alpha| + |\beta| = N$. In order to prove it for $N + 1$ we derive the corresponding identity for $v_{\alpha\beta'}$ with $\beta' = \beta + \beta_0$, $|\beta_0| = 1$. This will suffice since the case of $v_{\alpha'\beta}$ with $\alpha' = \alpha + \alpha_0$, $|\alpha_0| = 1$, is simpler and follows more directly by applying (3.15).
Assuming without loss of generality $\beta' = \beta + (1, 0, 0)$, from (3.12) and (3.16), denoting $T = T_{0,\beta}$, we have

$$v_{\alpha\beta'} = \frac{1}{\langle x \rangle} \frac{|k|}{\langle k \rangle} \partial_k v_{\alpha\beta} = -\frac{1}{\langle x \rangle} \frac{|k|}{\langle k \rangle} \partial_k T(v_{\alpha\beta}) + \frac{1}{\langle x \rangle} \frac{|k|}{\langle k \rangle} \partial_k G. \quad (3.20)$$

First we calculate

$$\frac{1}{\langle x \rangle} \frac{|k|}{\langle k \rangle} \partial_k T(v_{\alpha\beta}) = \frac{1}{\langle x \rangle} T\left(\frac{|k|}{\langle k \rangle} \partial_k v_{\alpha\beta}\right) + \frac{1}{\langle x \rangle} \frac{|k|}{\langle k \rangle} (\partial_k T)(v_{\alpha\beta})$$

$$= \frac{1}{\langle x \rangle} T\left(\mathcal{I}_{\langle y \rangle} v_{\alpha\beta'}\right) + \frac{1}{\langle x \rangle} \frac{|k|}{\langle k \rangle} (\partial_k T)(v_{\alpha\beta}).$$

Since $T \in \mathcal{T}_{N_1-N}$, in view of the properties (3.14), this is of the form

$$T_1(v_{\alpha\beta'}) + \frac{k_1}{\langle k \rangle} T_2(v_{\alpha\beta}), \quad T_1, T_2 \in \mathcal{T}_{N_1-(N+1)},$$

which is consistent with (3.16). Next, we look at the second term in (3.20), and consider the first contribution to $G$ from (3.18). For $|\gamma| + |\delta| \leq N - 1$ and $\delta' = \delta + (1, 0, 0)$, proceeding similarly as above we have

$$\frac{1}{\langle x \rangle} \frac{|k|}{\langle k \rangle} \partial_k \left[ a_\ell(k) T_\ell(v_{\gamma\delta}) \right]$$

$$= a_\ell(k) \frac{1}{\langle x \rangle} T_\ell(\langle y \rangle v_{\gamma\delta'}) + \left(\frac{|k|}{\langle k \rangle} \partial_k a_\ell(k)\right) \frac{1}{\langle x \rangle} T_\ell(v_{\gamma\delta}) + \left(\frac{|k|}{\langle k \rangle} a_\ell(k)\right) \frac{1}{\langle x \rangle} (\partial_k T_\ell)(v_{\gamma\delta})$$

which, using the properties (3.14), is of the form

$$a(k) T_1(v_{\gamma\delta'}) + b(k) T_2(v_{\gamma\delta}) + c(k) T_3(v_{\gamma\delta})$$

$$T_1, T_2, T_3 \in \mathcal{T}_{N_1-(N+1)},$$

for some coefficients $a, b, c$ with the same properties of $a_\ell$. This is consistent with (3.18) with $N$ and $\beta$ replaced by $N + 1$ and $\beta'$ as desired. We can deal similarly with the second sum in (3.18), thus obtaining our induction step.

**Conclusion.** From (3.16)-(3.20) we can deduce inductively that $v_{\alpha\beta}$ is the unique bounded solution of the equation (3.16). Indeed, the existence of $v_{\alpha\beta} \in L_{x,k}^\infty$ is given by the fact that $T_{0,\beta}$ is compact, and $G_{\alpha\beta} \in C_0$. Moreover, $v_{\alpha\beta}$ is the unique solution of (3.16) if and only if the equation $f + T_{0,\beta}f = 0$ admits only the trivial solution $f \equiv 0$. To verify that this is the case, we notice that if $f + T_{0,\beta}f = 0$ for a bounded $f$, then $g = \langle x \rangle^{2\beta} f$ is a polynomially bounded solution of $g = T_k g$; this means that $g$ is in the spectrum of $-\Delta + V$ and thus has to be trivial [39].

From (3.10) we see that

$$|\partial^2_x \partial_k^2 \psi(x, k)| \lesssim |\partial^2_x \partial_k^2 e^{ix\cdot k}| + \langle k \rangle^{|\alpha|} (\langle k \rangle / \langle k \rangle)^{1-|\beta|} |\langle x \rangle^{\beta}|$$

$$\lesssim \left( |\langle k \rangle^{\alpha}| + (\langle k \rangle / \langle k \rangle)^{1-|\beta|} \right) |\langle x \rangle^{\beta}|$$

which proves the estimates (3.8).
3.3. Expansion of \( \psi \). From the formula (3.21) we write
\[
\psi(x, k) = e^{ix \cdot k} - e^{i|k||x|} \frac{1}{4\pi|x|} \psi_1(x, k),
\]
\[
\psi_1(x, k) := \int_{\mathbb{R}^3} e^{i|k||x-y|-|x|} \frac{|x|}{|x-y|} V(y) \psi(y, k) \, dy.
\]
(3.21)

\( \psi_1 \) is the key linear object that we want to study, and for which we want to obtain precise asymptotic expansions.

In the following Lemma we summarize some basic properties of \( \psi_1 \).

**Lemma 3.3** (Basic properties of \( \psi_1 \)). Under the assumption (1.3), the function \( \psi_1 \) defined by (3.21) satisfies, for \(|x| \gtrsim 1\),
\[
|\langle k \rangle^{-|\alpha|}|x|^{|\alpha|} \nabla_x^\alpha \psi_1(x, k)| \leq c_\alpha, \quad |\alpha| \leq N_1,
\]
\[
|\langle k \rangle^{-|\alpha|}|x|^{|\alpha|} \nabla_x^\alpha \nabla_k^\beta \psi_1(x, k)| \leq c_{\alpha, \beta} \max(1, |k|^{-|\beta|}), \quad |\alpha| + |\beta| \leq N_1, \quad \beta \neq 0.
\]
(3.22)

In particular, if we define the angular derivative vectorfields
\[
\Omega_x := x \wedge \nabla_x = (x_2 \partial_{x_3} - x_3 \partial_{x_2}, x_3 \partial_{x_1} - x_1 \partial_{x_3}, x_1 \partial_{x_2} - x_2 \partial_{x_1}) =: (\Omega_1, \Omega_2, \Omega_3)
\]
(3.23)

one has
\[
|\langle k \rangle^{-|\alpha|} \Omega_x^\alpha \psi_1(x, k)| \leq c_\alpha, \quad |\alpha| \leq N_1,
\]
\[
|\langle k \rangle^{-|\alpha|} \Omega_x^\alpha \nabla_k^\beta \psi_1(x, k)| \leq c_{\alpha, \beta} \max(1, |k|)^{1-|\beta|}, \quad |\alpha| + |\beta| \leq N_1, \quad \beta \neq 0.
\]
(3.24)

**Proof.** Let us decompose
\[
\psi_1(x, k) = \psi_1^-(x, k) + \psi_1^+(x, k),
\]
\[
\psi_1^-(x, k) := \int_{\mathbb{R}^3} e^{i|k||x-y|-|x|} \frac{|x|}{|x-y|} V(y) \psi(y, k) \varphi_{\leq -10(|y|/|x|)} \, dy,
\]
\[
\psi_1^+(x, k) := \int_{\mathbb{R}^3} e^{i|k||x-y|-|x|} \frac{|x|}{|x-y|} V(y) \psi(y, k) \varphi_{> -10(|y|/|x|)} \, dy;
\]
(3.25)

recall the notation for cutoffs from (2.19).

**Estimate of \( \psi_1^- \).** From Faá di Bruno’s formula we see that \( \partial_{x_1}^\alpha e^{i|k||x-y|-|x|} \) is bounded by a linear combination of terms of the form
\[
\sum_{\sum_{a \geq 1} \alpha_a = -\alpha_1} \prod_{b=1}^{\alpha_1} (|k| \partial_{x_1}^b (|x-y|-|x|))^p_b.
\]
(3.26)

On the support of \( \psi_1^- \) we have \(|\partial_{x_1}^b (|x-y|-|x|)| \lesssim |y||x|^{-b}\) and therefore we see that
\[
|\partial_{x_1}^\alpha e^{i|k||x-y|-|x|}| \lesssim (|k| + |k|^{\alpha_1}) \cdot (|y| + |y|^{\alpha_1}) |x|^{-\alpha_1}.
\]
(3.27)

Since we also have
\[
\left| \partial_{x_1}^\alpha \frac{|x|}{|x-y|} \right| + |\partial_{x_1}^\alpha \varphi_{\leq -10(|y|/|x|)}| \lesssim |x|^{-\alpha_1}
\]
the first inequality in (3.22) follows for the term \( \psi_1^- \).
To deal with derivatives in \( k \) we apply again Faá-di Bruno and estimate
\[
|\partial_{k_1}^{\beta_1} e^{i[k||x-y|-|x||]}| \lesssim \sup_{\sum_{\beta_1 \geq 1} b_{\beta_1} = \beta_1} \prod_{b=1}^{\beta_1} |(\partial_{k_1}^{b})(|x-y|-|x|)|^{p_b} \lesssim (1 + |k|^{1-\beta_1}) \cdot |y|/|x|
\]

(3.28)

Arguing as before for the \( x \)-derivatives, and using the estimates (3.28) to bound \( \partial_{k_1}^{\beta_1} \psi \), with the assumptions on \( V \), we obtain the second inequality in (3.22).

Estimate of \( \psi_1^+ \). First notice that since the potential satisfies (1.9), \( \psi_1^+ \) decays very fast in \( x \):
\[
|\psi_1^+(x, k)| \lesssim |x|^{-N_1-5} \int_{\mathbb{R}^3} \frac{|x|}{|x-y|} |y|^{N_1+5} |V(y)| \varphi_{>10}(|y|/|x|) \, dy \\
\lesssim |x|^{-N_1-5} \| \langle x \rangle^{N_1+6} V \|_{L^\infty \cap L^1}.
\]

To prove estimates on several \( x \) derivatives however we need to take care of the singularity arising when differentiating the integrand. Let us write
\[
\psi_1^+(x, k) = a(x, k) + b(x, k),
\]
\[
a(x, k) = \int_{\mathbb{R}^3} e^{i[k||x-y|-|x||]} \frac{|x|}{|x-y|} V(y) \psi(y, k) \varphi_{>10}(|y|/|x|) \varphi_{>0}(|x-y|) \, dy,
\]
\[
b(x, k) = \int_{\mathbb{R}^3} e^{i[k||x-y|-|x||]} \frac{|x|}{|x-y|} V(y) \psi(y, k) \varphi_{<10}(|y|/|x|) \varphi_{<0}(|x-y|) \, dy.
\]

Arguing as in the proof of (3.27) above, on the support of \( a(x, k) \), where \(|y| \gtrsim |x| \gtrsim 1 \) and \(|y-x| \gtrsim 1 \), we have
\[
|\partial_{x_1}^\alpha e^{i[k||x-y|-|x||]}| \lesssim |k| + |k|^\alpha,
\]

(3.30)
as well as
\[
|\partial_{x_1}^\alpha \frac{|x|}{|x-y|}| \lesssim 1 + |y|.
\]

(3.31)

Then we can estimate \( |\partial_{x_1}^{\alpha_1} a(x, k)| \) by a linear combination of terms of the form
\[
I_{\alpha_2 \alpha_3} = \int_{\mathbb{R}^3} \left| \partial_{x_1}^{\alpha_2} \frac{|x|}{|x-y|} \right| V(y) \psi(y, k) \varphi_{>10}(|y|/|x|) \varphi_{>0}(|x-y|) \, dy
\]

(3.32)

with \( \alpha_2 + \alpha_3 = \alpha_1 \), plus easier terms arising when derivatives hit the cutoffs, which we disregard. We then see that
\[
I_{\alpha_2 \alpha_3} \lesssim (|k| + |k|^{\alpha_2}) \int_{\mathbb{R}^3} (1 + |y|)|V(y)| \varphi_{>10}(|y|/|x|) \, dy \lesssim (1 + |k|)^{\alpha_1} (1 + |x|)^{-N_1}
\]

which is consistent with the right-hand side of (3.22). To deal with the derivatives in \( k \) we use
\[
|\partial_{k_1}^{\beta_1} e^{i[k||x-y|-|x||]}| \lesssim (1 + |k|^{1-\beta_1})(1 + |y|)^3.
\]
see (3.28), and obtain that $\partial_{k_1}^{\beta} a(x, k)$ is bounded by a linear combination of terms of the form

$$J_{\beta_2\beta_3} = \int_{\mathbb{R}^3} \left| \partial_{k_1}^{\beta_2} e^{ik|x-y|}|x| \right| V(y) \left| \partial_{k_1}^{\beta_3} \psi(y, k) \right| \varphi_{>-\alpha}(|y|/|x|) \, dy \right|$$

$$\lesssim \int_{\mathbb{R}^3} (1 + |k|^{1-\beta_2})(1 + |y|^{\beta_2}(1 + |y|)|V(y)|(1 + |y|)\beta_3(1 + |k|^{1-\beta_3}) \varphi_{>-\alpha}(|y|/|x|) \, dy$$

for $\beta_2 + \beta_3 = \beta$, having used (3.8). In view of (1.9) and $\beta_1 \leq N_1$, we have

$$|J_{\beta_2\beta_3}| \lesssim (1 + |x|)^{-N_1+1}(1 + |k|^{1-\beta_1})$$

which is sufficient for the second inequality in (3.22) when $\alpha = 0$. The same arguments can be used to obtain the full bound for $(x, k)$-derivatives.

To estimate the term $b(x, k)$ in (3.29) we need to take care of the singularity of high derivatives of $|x-y|^{-1}$. We first rewrite

$$b(x, k) = E_k^{-1}(x) \int_{\mathbb{R}^3} E_k(x-y)V(y)\psi(y, k) \varphi_{>-\alpha}(|y|/|x|) \, dy, \quad E_k(z) = \frac{e^{i|k|z}}{|z|} \varphi_{\leq 0}(z).$$

When applying $k$ derivatives we can use the same arguments as above. For the spatial derivatives instead, the desired estimates can be easily seen to hold when derivatives hit $E_k^{-1}(x)$. We may then just look at the cases when derivatives hit the integrand. For such terms we convert $\partial_z$ hitting $E_k(x-y)$ into $-\partial_y$ and integrate by parts onto $V\psi$. Using the assumptions (1.9) and (3.8) we arrive at (3.22). □

The next lemma gives an expansion for $\psi_1$ in powers of $|x|^{-1}$.

**Lemma 3.4.** Let $N_2 \in [1, N_1] \cap \mathbb{Z}$ where $N_1$ is as in (1.9). Denoting $r = |x|$ and $\omega = x/|x|$, we have the expansion

$$\psi_1(x, k) = \sum_{j=0}^{N_2-1} g_j(\omega, k) r^{-j}\langle k \rangle^j + R_{N_2}(x, k),$$

for $r \geq 1$, where

$$g_0(\omega, k) := -\frac{1}{4\pi} \int_{\mathbb{R}^3} e^{-i|k|\omega y}V(y)\psi(y, k) \, dy,$$

the coefficients $g_j$, $j = 0, 1, \ldots, N_2 - 1$, satisfy

$$\left| \partial_\alpha \partial_\beta g_j(\omega, k) \right| \lesssim \langle k \rangle^{|\alpha| + (|k|/\langle k \rangle)^{1-|\beta|}}, \quad |\alpha| + |\beta| \leq N_1 - N_2,$$

and

$$\left| \partial_\beta R_{N_2}(x, k) \right| \lesssim r^{-N_2}(\langle k \rangle^{N_2} + (|k|/\langle k \rangle)^{1-|\beta|}), \quad |\beta| \leq N_1 - N_2 - 1.$$

**Proof.** From the definition (3.21), writing $x = r\omega$, $r = |x|$, we have

$$\psi_1(r\omega, k) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} e^{i|k|(\omega-y/r)\omega} \frac{1}{|\omega - y/r|}V(y)\psi(y, k) \, dy$$

$$= g_0(\omega, k) + I_1(x, k) + I_2(x, k) + I_3(x, k),$$

(3.38)
where $g_0$ is defined in (3.35) and
\begin{align}
I_1 &:= -\frac{1}{4\pi} \int_{\mathbb{R}^3} \left[ e^{i|kr|(|\omega - y|/r| - 1)} - e^{-i|kr|y} \right] \frac{1}{|\omega - y/r|} V(y) \psi(y, k) \varphi_{\leq 10}(y/|x|) \, dy, \\
I_2 &:= -\frac{1}{4\pi} \int_{\mathbb{R}^3} e^{-i|kr|y} \left[ \frac{1}{|\omega - y/r|} - 1 \right] V(y) \psi(y, k) \varphi_{\leq 10}(y/|x|) \, dy, \\
I_3 &:= -\frac{1}{4\pi} \int_{\mathbb{R}^3} \left[ e^{i|kr|(|\omega - y|/r| - 1)} - e^{-i|kr|y} \right] V(y) \psi(y, k) \varphi_{>10}(y/|x|) \, dy.
\end{align}

We will expand the integrands in the first two terms in powers of $y/r$, while the third term is a remainder that can be absorbed into $R_{N_2}$ directly.

**Estimate of (3.39).** Observe that, for $|y| \leq r/2$ and arbitrary $n$ we can expand
\begin{equation}
|\omega - y/r| = \sqrt{1 + |y|^2 r^{-2} - 2 \omega \cdot y/r} = 1 - \omega \cdot y/r + \sum_{j=2}^{n-1} r^{-j} \sum_{j_1 + j_2 = j} a_{j_1 j_2} |y|^{j_1} (\omega \cdot y)^{j_2} + R_n(x, y),
\end{equation}
for some coefficients $a_{j_1 j_2} \in \mathbb{C}$, with
\begin{equation}
R_n(x, y) = (|y|/r)^n (1 + a(\omega, y)), \quad |\partial_\omega^n a(\omega, y)| \lesssim 1.
\end{equation}

A similar expansion holds for $|\omega - y/r|^{-1}$. Then, we can write
\begin{equation}
X := r(|\omega - y/r| - 1) + \omega \cdot y = \sum_{j=1}^{n-1} r^{-j} a_j(\omega, y) + R_n(x, y),
\end{equation}
with
\begin{equation}
|\partial_\omega^n a_j(\omega, y)| \lesssim (y)^{j+1}, \quad |\partial_\omega^n R_n(x, y)| \lesssim (y)^{n+1} r^{-n}.
\end{equation}

We look at the factor in the integrand of (3.39) and write
\begin{align}
\left[ e^{i|kr|(|\omega - y|/r| - 1)} - e^{-i|kr|y} \right] \frac{1}{|\omega - y/r|} = e^{-i|kr|y} \frac{1}{|\omega - y/r|} \left[ e^{i|kr|X} - 1 \right] \\
= e^{-i|kr|y} \left[ \sum_{j=1}^{n-1} r^{-j} \sum_{1 \leq \ell \leq j} |k|^{\ell} a_{j, \ell}(\omega, y) + \sum_{1 \leq \ell \leq n} |k|^{\ell} R_{n, \ell}(x, y) \right],
\end{align}
where the coefficients and remainder terms satisfy
\begin{equation}
|\partial_\omega^n a_{j, \ell}(\omega, y)| \lesssim (y)^{j+1}, \quad |R_{n, \ell}(x, y)| \lesssim (y)^{n+1} r^{-n}.
\end{equation}

Using the definition (3.39) and the expansion (3.45) we have
\begin{equation}
I_1(x, k) = \sum_{j=1}^{N_2-1} b_j(\omega, k) (k)^j r^{-j} + R_{N_2}^1(x, k),
\end{equation}

having defined
\begin{align}
b_j(\omega, k) &:= \int_{\mathbb{R}^3} e^{-i|kr|y} \frac{1}{(k)^j} \sum_{1 \leq \ell \leq j} |k|^{\ell} a_{j, \ell}(\omega, y) V(y) \psi(y, k) \, dy, \\
R_{N_2}^1(x, k) &:= \int_{\mathbb{R}^3} e^{-i|kr|y} \sum_{1 \leq \ell \leq N_2} |k|^{\ell} R_{N_2, \ell}(x, y) V(y) \psi(y, k) \, dy.
\end{align}
In view of the first estimate of (3.46), the integrability assumptions on $V$ from (1.9), the constraints $|\alpha| + |\beta| \leq N_1 - N_2$, and the estimates (3.8) giving $|\partial_k^\beta \psi(y,k)| \lesssim \langle y \rangle^{\beta} (|k| / \langle k \rangle)^{1-|\beta|}$ for $\beta \neq 0$, we see that the coefficients $b_j$ satisfy estimates as in (3.36). Similarly the remainder $R_{N_2}$ satisfies estimates as in (3.37). This gives an expansion of the desired form (3.34) for $I_1$.

Estimate of (3.40). The term $I_2$ is similar to (3.39) so we can skip the details.

Estimate of (3.41). Since on the support of $I_3$ we have $|y| \gtrsim |x|$, we can use the weighted integrability of $V$ in (1.9) to show that this term is a remainder as in (3.37). Using that for $|x| \lesssim |y|$, $\beta \neq 0$, we have the bounds

$$
|\partial_k^\beta e^{i|k| r (|\omega| - |r|) - 1}| + |\partial_k^\beta e^{-i|k||\omega|} | + |\partial_k^\beta \psi(y,k)| \lesssim \langle y \rangle^\beta (|k| / \langle k \rangle)^{1-|\beta|},
$$

see (3.8), we have, for $|\beta| \leq N_1 - N_2 - 1$

$$
|\partial_k^\beta I_3(x,k)| \lesssim (|k| / \langle k \rangle)^{1-|\beta|} \int_{\mathbb{R}^3} \frac{|x|}{|x-y|} \langle y \rangle^{|\beta|} V(y) \varphi_{>-10} (y/r) \, dy,
$$

$$
\lesssim (|k| / \langle k \rangle)^{1-|\beta|} r^{-N_2} \| \langle x \rangle^{N_1} V \|_{L^\infty \cap L^1}.
$$

This concludes the proof of the Lemma.

Motivated by (3.22) and the expansion (3.34) we define the following classes of symbols:

**Definition 3.5.** For $N \in \mathbb{Z}_+$ we let $\mathcal{G}^N$ be the class of $L^\infty_{x,k}$ functions $f : \mathbb{S}^2 \times \mathbb{R}^3 \mapsto \mathbb{C}$ such that

$$
|\nabla_\omega^\alpha \nabla_k^\beta f(\omega,k)| \leq c_{\alpha,\beta} \langle \langle k \rangle \rangle^{\alpha} + (|k| / \langle k \rangle)^{1-|\beta|} \quad 1 \leq |\alpha| + |\beta| \leq N. \tag{3.49}
$$

To fix ideas one can think of functions in $\mathcal{G}^N$ as functions of the form $\exp(i |k|x_1 / |x|)$. This is essentially how $\psi_1$ looks like, with the exception that its differentiability in $k$, is limited by the integrability of $V$. More precisely, one should think of the class $\mathcal{G}^N$ as functions of the form

$$
\int_{\mathbb{R}^3} e^{i|k|x_1 / |x|} y f(y) \, dy, \quad (1 + |y|)^N f(y) \in L^1. \tag{3.50}
$$

Compare this with the formula for $g_0$ in (3.35). Functions in $\mathcal{G}^N$ will often appear in the expressions for symbols of bilinear operators in our applications. As symbols these are not standard ones (e.g., of bilinear Mihlin-Hörmander type), for example because of losses when $k$ is large.

**4. Preliminary bounds: Linear estimates and high frequencies**

In this section we first state some decay estimates for the linear evolution and then show how to obtain the bootstrap estimate on the standard Sobolev norms in (2.24) using the decay and the a priori assumptions. The rest of the section is then dedicated to a priori bounds for the nonlinear evolution when one restricts the analysis to high frequencies that are large compared to time.
4.1. Linear Estimates. We start by collecting some dispersive estimates for Schrödinger operators.

**Lemma 4.1.** Under the assumptions \((1.8)-(1.9)\) on the potential \(V\), with \(\tilde{f}\) defined as in Theorem 3.1, we have

\[
\|e^{it(\Delta+V)}f\|_{L^\infty} \lesssim \frac{1}{|t|^{3/2}} \|\tilde{f}\|_{L^\infty} + \frac{1}{|t|^{7/4}} \|\partial_k^2 \tilde{f}\|_{L^2},
\]

(4.1)

and

\[
\|e^{it(\Delta+V)}f\|_{L^6} \lesssim \frac{1}{|t|} \|\partial_k \tilde{f}\|_{L^2}.
\]

(4.2)

Interpolating (4.2) with the \(L^2\) conservation we have

\[
\|e^{it(\Delta+V)}f\|_{L^p} \lesssim \frac{1}{|t|^{(3/2)(1-2/p)}} \|\tilde{f}\|_{H_k^1}; \quad 2 \leq p \leq 6.
\]

(4.3)

Moreover, for all \(6 < p < \infty\),

\[
\|e^{it(\Delta+V)}f\|_{L^p} \lesssim \frac{1}{|t|^{3/2(1-2/p)}} \|\partial_k \tilde{f}\|_{L^2}^{1-\theta} \|\partial_k^2 \tilde{f}\|_{L^2}^\theta, \quad \theta = \frac{1}{2} - \frac{3}{p}.
\]

(4.4)

**Proof.** All these linear estimates can be deduced from the corresponding estimates involving the flat Fourier transform, and using the boundedness of the wave operator. We recall that, under our assumptions, the wave operators, defined by

\[
W_\pm := \lim_{t \to \pm \infty} e^{it(\Delta+V)} e^{-it \Delta},
\]

are bounded on Sobolev spaces; see for example Yajima [48]. Moreover, as in see (3.5),

\[
W := W_+ = \tilde{F} - \hat{F}.
\]

To prove (4.1), recall first that

\[
\|e^{-it\Delta}f\|_{L^\infty} \lesssim \frac{1}{|t|^{3/2}} \|\tilde{f}\|_{L^\infty} + \frac{1}{|t|^{7/4}} \|\partial_k \tilde{f}\|_{L^2},
\]

(4.5)

see, for example, [13]. Then it suffices to write

\[
e^{it(\Delta+V)}f = We^{-it\Delta}W^* f
\]

(4.6)

and, by the boundedness of \(W\) on \(L^p\) spaces, (4.5), and the fact that \(\hat{F}\) and \(\tilde{F}\) are unitary on \(L^2\), we obtain (4.1).

Similarly, (4.2) can be obtained using the standard Klainerman-Sobolev type embedding

\[
\|e^{-it\Delta}f\|_{L^6} \lesssim \frac{1}{|t|} \|xf\|_{L^2} \lesssim \frac{1}{|t|} \|\partial_k \tilde{f}\|_{L^2}
\]

and (4.4) using, for \(q > 6\) with \(1/q + 1/q' = 1\), and \(\theta = 1/2 - 3/p\), that

\[
\|e^{-it\Delta}f\|_{L^q} \lesssim \frac{1}{|t|^{3/2(1-2/q)}} \|f\|_{L^{q'}} \lesssim \frac{1}{|t|^{(3/2)(1-2/q)}} \|xf\|_{L^2}^{1-\theta} \|x^2 f\|_{L^2}^{\theta}.
\]

□

Next, we use Lemma 4.1 to obtain some a priori decay bounds as direct consequences of the a priori assumptions (2.23).
Lemma 4.2. Let \( u = e^{it(-\Delta + V)} f \) and assume the bounds (2.23) hold with the definitions in (2.21). Then,
\[
\|e^{it(-\Delta + V)} f\|_{L^p} \lesssim \varepsilon(t)^{-3/2(1-2/p)}, \quad 2 \leq p \leq 6, \tag{4.7}
\]
\[
\|e^{it(-\Delta + V)} f\|_{L^p} \lesssim \varepsilon(t)^{-5/4+3/(2p)+\delta}, \quad p > 6, \quad \theta = \frac{1}{2} - \frac{3}{p}. \tag{4.8}
\]

Proof. For \(|t| \leq 1\) the estimates follow from the boundedness of wave operators, Sobolev’s embedding, and the a priori bound (2.23):
\[
\|e^{it(-\Delta + V)} f\|_{L^p} \lesssim \|e^{-it\Delta W} f\|_{L^p} \lesssim \|e^{-it\Delta W^*} f\|_{H^2} \lesssim \|\tilde{f}\|_{L^2} \lesssim \varepsilon.
\]

For \(|t| \geq 1\) the estimate (4.7), resp. (4.8), is a direct consequences of (4.3), resp. (4.4), and the bounds on the weighted norms in (2.23). \(\square\)

4.2. Sobolev estimates. We now prove the bootstrap estimate (2.24) for the Sobolev-type norm using energy estimates and the pointwise decay from Lemma 4.2.

Proposition 4.3. Under the a priori assumptions (2.23) we have
\[
\|u(t)\|_{H^N} + \|\langle k \rangle^N \tilde{f}\|_{L^2} \leq \varepsilon_0 + C\varepsilon^2. \tag{4.9}
\]

Proof. First notice that
\[
\|\langle k \rangle^j \tilde{f}\|_{L^2} = \|\langle k \rangle^j \tilde{u}\|_{L^2} = c\|(-\Delta + V)^{j/2} u\|_{L^2}. \tag{4.10}
\]

Moreover, by direct estimates (or also using the boundedness of wave operators) for any \(j \leq N/2\)
\[
\|(-\Delta + V)^j g\|_{L^2} \lesssim \|g\|_{H^{2j}} \lesssim \sum_{\ell=0}^j \|(-\Delta + V)^\ell g\|_{L^2}. \tag{4.11}
\]

In particular, the two norms in (4.9) are equivalent so it suffices to bound the first one. We use a standard energy estimate. We let
\[
u^j := (-\Delta + V)^j u,
\]
for \(j = 0, \ldots, N\), and differentiate the equation (1.7) using \(-\Delta + V\) to obtain
\[
i \partial_t \nu^j + (-\Delta + V) \nu^j = (-\Delta + V)^j u^2.
\]

Therefore, using (4.11) and standard product estimates,
\[
\frac{d}{dt}\|\nu^j\|_{L^2} \lesssim \|(-\Delta + V)^j u^2\|_{L^2} \lesssim \|u\|_{H^{2j}} \|u\|_{L^\infty}.
\]

Using the apriori assumption (2.23) and the decay estimate (4.8) we get
\[
\|\nu^j(t)\|_{L^2} - \|\nu^j(0)\|_{L^2} \lesssim \int_0^t \|u(s)\|_{H^{2j}} \|u(s)\|_{L^\infty} ds \lesssim \int_0^t \varepsilon(s)^{-5/4+\delta/2} ds \lesssim \varepsilon^2.
\]

Summing over \(j \leq N/2\) gives the desired conclusion. \(\square\)
4.3. **Weighted estimates for high frequencies.** Recall that we define the profile of a solution $u$ of (1.7) by

$$f(t, x) := (e^{-i(t-\Delta+V)u(t, \cdot)})(x), \quad \tilde{f}(t, k) = e^{-i|k|^2} \tilde{u}(t, k).$$

(4.12)

and that this satisfies the equation

$$\tilde{f}(t, k) = \tilde{u}_0(k) - iD(t)(f, f)$$

(4.13)

where

$$\mu(k, \ell, m) := (2\pi)^{-9/2} \int \psi(x, k) \psi(x, \ell) \psi(x, m) dx$$

(4.14)

We want to estimate the weighted norms in (2.21) as in Proposition 2.1 when frequencies are large relative to a (small) power of time. This will be helpful later on in the analysis of the nonlinear spectral distribution and its asymptotic expansion. More precisely, let us restrict (4.13) to high frequencies by considering

$$\mathcal{D}_{HF}(t)(f, f) := \int_0^t \int e^{is(-|k|^2+|\ell|^2+|m|^2)} \tilde{f}(s, \ell) \tilde{f}(s, m) \mu(k, \ell, m) d\ell dm ds,$$

(4.15)

where

$$\delta_N := \frac{3}{N - 5}$$

(4.16)

with $N$ the Sobolev regularity of our solution, see (1.8) and (2.21)-(2.23). This is our main Proposition in this section:

**Proposition 4.4** (High frequencies estimates). **Under the a priori assumptions** (2.23) we have

$$\|\partial_k \mathcal{D}_{HF}(t)(f, f)\|_{L^2} + (t)^{-1/2 - \delta} \|\partial^2_k \mathcal{D}_{HF}(t)(f, f)\|_{L^2} \leq \varepsilon_0 + C\varepsilon^2.$$ 

(4.17)

To prove Proposition 4.4 we are going to make use, among other things, of Proposition 4.5 below, which can be deduced from [14]. For convenience, and only for the purpose of stating Proposition 4.5 below and applying it to the proof of Proposition 4.4 we introduce a Coifman-Meyer type norm for symbols as in [14]:

$$\|n\|_{CM_\delta} := \sup_{0 \leq |a| \leq 10} \left| (|k| + |\ell| + |m|)^{\delta + |a|} \nabla^a n(k, \ell, m) \right|, \quad \delta > 0.$$ 

(4.18)

**Proposition 4.5** (Germain-Hani-Walsh [14]). **Consider the bilinear operator**

$$\mathcal{B}_n(g, h)(k) := \int \int \tilde{g}(\ell) \tilde{h}(m) n(k, \ell, m) \mu(k, \ell, m) d\ell dm,$$

(4.19)

where $n$ is a symbol verifying the estimate

$$\|n\|_{CM_\delta} \leq A,$$

(4.20)

for some $\delta > 0$. Then:
(i) **(Hölder estimates)** For any \(1 < p, q, r, p', q' < \infty\), the following estimate holds

\[
\| \tilde{F}^{-1} B_n(g, h) \|_{L^r} \lesssim A \left( \| g \|_{L^p} \| h \|_{L^q} + \| g \|_{L^{p'}} \| h \|_{L^{q'}} \right), \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} < \frac{1}{p'} + \frac{1}{q'}.
\]  

(4.21)

In particular, for \(p' \in (1, p)\), we have

\[
\| \tilde{F}^{-1} B_n(g, h) \|_{L^r} \lesssim A \| g \|_{L^{p'}}, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r},
\]

and a similar estimate exchanging the roles of \(g\) and \(h\).

(ii) **(Algebraic identity for the weights)** The following identities hold:

\[
\partial_k B_n(g, h)(k) = B_{\partial_n}(g, h)(k) + B_n' \left( \tilde{F}^{-1} \partial_k \tilde{g}, h \right) + B_{\partial_n}(g, h)(k) + B_n'(g, h)(k)
\]

\[
= B_{\partial_n}(g, h)(k) + B_n' \left( \tilde{F}^{-1} \partial_m \tilde{h} \right) + B_{\partial_n}(g, h)(k) + B_n'(g, h)(k),
\]

where we use the ‘prime’ notation \(B_n'\) to denote a generic operator of the form \(4.19\), with \(\mu\) replaced by a slightly different expression \(\mu'\), and which satisfies the same Hölder bounds \(4.21\)–\(4.22\) satisfied by \(B_n\).

Moreover, one can iterate formula \(4.23\) to obtain

\[
\partial_k^2 B_n(g, h)(k) = \sum_{a, b \geq 0, a + b \leq 2} B_{\partial_{n}(k, \ell)} \left( \tilde{F}^{-1} \partial_k \tilde{g}, h \right) = \sum_{a, b \geq 0, a + b \leq 2} B_{\partial_{n}(k, m)}'(g, \tilde{F}^{-1} \partial_m \tilde{h})(k),
\]

\[4.24\]

where \(B'\) are operators as above, and where we denote \(\partial_{(k, \ell)}\) a generic derivative in \(k\) and/or \(\ell\), and similarly for \(\partial_{(k, m)}\).

Let us make a few remarks:

- Proposition \(4.5\) is contained in \(14\). Below we give some more precise references and a few elements of the proof for completeness. In \(14\) the analysis is actually carried out for a slightly different \(\mu\), with \(\psi(x, k)\) instead of \(\psi(x, k)\) in \(11.14\). However, this has no impact on the structure of \(\mu\) that is used in the proofs and on the final estimates stated in the theorem.

- Part (i) gives product Hölder-type estimates for \(\tilde{F}^{-1} B_n(g, h)\). When \(\mu(k, \ell, m) = \delta(k - \ell - m)\) the expression \(\tilde{F}^{-1} B_n(g, h)\) is usually called a pseudo-product, and Hölder-type estimates under conditions similar to \(4.20\) (with \(\delta = 0\)) are due to Coifman-Meyer \(7\).

- Part (ii) is a commutation formula which essentially states that the bilinear commutator between \(B_n(g, h)(k)\) and \(\partial_k\) is given by pseudo-products of the same form as \(B_n\) and where the symbol gets differentiated.

- The assumption on the symbols \(4.20\) is probably not optimal, but it suffices for our purposes.

**Proof of Proposition 4.5** (i) The estimate \(4.21\) is the content of Theorem 1.1 in \(14\).

(ii) To explain \(4.24\), let us first introduce some notation. Let \(W = \tilde{F}^{-1} \tilde{F}\) be the wave operator as in \(4.6\), let \(R_k := \partial_k / |\nabla_k|, i = 1, 2, 3,\) be the standard Euclidean Riesz transform and denote

\[
E := \tilde{F} \cdot |x|, W \cdot \tilde{F}^{-1}.
\]

\[4.25\]
$E_k$ will be used to denote the operator $E$ acting on the variable $k$. With our notation (4.14) for $\mu$, the formula (3.57) derived on pages 8523-8524 of [14], to be understood in the sense of distributions, reads

$$\partial_k \mu(k, \ell, m) = R_k R_\ell \cdot \partial_\ell \mu(k, \ell, m) + R_k E^*_\ell \mu(k, \ell, m) - R_k E^*_k \mu(k, \ell, m),$$  \hspace{1cm} (4.26)$$
or, equivalently,

$$|\nabla_k \mu(k, \ell, m)| = |\nabla_\ell \mu(k, \ell, m) + E^*_\ell \mu(k, \ell, m) - E^*_k \mu(k, \ell, m)|$$

$$|\nabla_m \mu(k, \ell, m) + E^*_m \mu(k, \ell, m) - E^*_k \mu(k, \ell, m)|.$$  \hspace{1cm} (4.27)$$

Applying (4.26) to an expression like (4.19) and integrating by parts in $\ell$ gives

$$\partial_k B_n(g, h) = B_{\partial_k n}(g, h) + A + B + C + D,$$

$$A := -\int \int \tilde{g}(\ell) \tilde{h}(m) \partial_\ell n(k, \ell, m) \cdot R_\ell R_k \mu(k, \ell, m) \partial \ell \partial m,$$

$$B := -\int \int \partial_\ell \tilde{g}(\ell) \tilde{h}(m) n(k, \ell, m) \cdot R_\ell R_k \mu(k, \ell, m) \partial \ell \partial m,$$

$$C := \int \int \tilde{g}(\ell) \tilde{h}(m) n(k, \ell, m) R_k E^*_\ell \mu(k, \ell, m) \partial \ell \partial m,$$

$$D := -\int \int \tilde{g}(\ell) \tilde{h}(m) n(k, \ell, m) R_k E^*_k \mu(k, \ell, m) \partial \ell \partial m.$$  \hspace{1cm} (4.28)$$

One is then led to study the operators

$$\Lambda_1(g, h) := \int \int \tilde{g}(\ell) \tilde{h}(m) n(k, \ell, m) R_\ell R_k \mu(k, \ell, m) \partial \ell \partial m,$$

$$\Lambda_2(g, h) := \int \int \tilde{g}(\ell) \tilde{h}(m) n(k, \ell, m) R_k E^*_\ell \mu(k, \ell, m) \partial \ell \partial m,$$

$$\Lambda_3(g, h) := \int \int \tilde{g}(\ell) \tilde{h}(m) n(k, \ell, m) R_k E^*_k \mu(k, \ell, m) \partial \ell \partial m.$$  \hspace{1cm} (4.29)$$
corresponding to the three different bilinear operators in (4.28), where $n$ denotes a generic symbol. Theorem 3.13 of [14], gives Hölder estimates, with small losses in the Lebesgue exponents as in (4.21), on the operators (4.29); more precisely, under the assumption (4.20) on the symbol $n$, one has, for $j = 1, 2, 3,$

$$\|\tilde{\mathcal{F}}^{-1} \Lambda_j(g, h)\|_{L^r} \lesssim \|g\|_{L^p} \|h\|_{L^q} + \|g\|_{L^{p'}} \|h\|_{L^{q'}},$$  \hspace{1cm} (4.30)$$

This estimate and (4.28) give the claimed identity (4.23).

By iterating the application of (4.26) to the expressions in (4.28), we can derive the second identity (4.24).

\textbf{Proof of Proposition 4.4}  By symmetry we may assume $|m| \geq |\ell|$ on the support of (4.15).

\textbf{Case} $|k| \lesssim |m|$. In this case we have $|m| = |m| + |\ell| + |k| \gtrsim \langle s \rangle^{\delta_N}$ on the support of $D_{HF}$. We want to apply the identity (4.23) to (4.15). To shorten our formulas we will often omit
the argument of the cutoff \(\varphi_{\geq 0}\) or the measure \(\mu\), or other arguments when there is no risk of confusion. Formula (4.23) applied to (4.15) gives
\[
\partial_k \mathcal{D}_{HF} = \mathcal{D}_{HF,1} + \mathcal{D}_{HF,2} + \mathcal{D}_{HF,3},
\]
where \(\mathcal{D}_{HF,1}(f, f)(t) = \int_0^t \int \int e^{is\Phi(k, t, m)} \tilde{f}(s, \ell) \tilde{f}(s, m) \varphi_{\geq 0}(\cdot - isk\mu + is\ell\mu' + \mu') \, d\ell dm \, ds,
\]
\(\mathcal{D}_{HF,2}(f, f)(t) = \int_0^t \int \int e^{is\Phi(k, t, m)} \partial_{t}\tilde{f}(s, \ell) \tilde{f}(s, m) \varphi_{\geq 0} \mu' \, d\ell dm \, ds,
\]
\(\mathcal{D}_{HF,3}(f, f)(t) = \int_0^t \int \int e^{is\Phi(k, t, m)} \tilde{f}(s, \ell) \tilde{f}(s, m) (\partial_k \varphi_{\geq 0} \mu + \partial_t \varphi_{\geq 0} \mu') \, d\ell dm \, ds,
\]
where \(\mu'\) is as in the statement of Proposition 4.3(ii). Notice that thanks to the two identities in (4.23) the above expressions do not involve \(\partial_m \tilde{f}\), which is the function with largest frequency.

For the first term in (4.31) it suffices to show how to estimate the contribution involving \(\mu\), since the other two contributions involving \(\mu'\) are similar. We rewrite it as
\[
\mathcal{D}'_{HF,1} := \int_0^t \int_{\Omega} \int_{\Omega} \int_{\Omega} \int_{\Omega} e^{is|k|^2} \tilde{f}(s, \ell) \tilde{f}(s, m) n_1(k, \ell, m) \mu(k, \ell, m) \, d\ell dm \, ds,
\]
\(n_1(k, \ell, m) := -k(\ell)^2 |m|^4 \varphi_{\geq 0}((|\ell|^2 + |m|^2 + |k|^2)(\langle s \rangle)^{-2\delta N}).\)

It is easy to check that since \(|m| \gtrsim \max(\langle s \rangle^{\delta N}, |\ell|, |k|)\), we have, see (4.18),
\[\|n_1\|_{CM^3} \lesssim 1.\]

The assumptions (4.20) is then verified with \(A \lesssim 1\). For small \(\gamma\) we let
\[2+ := (1/2 - \gamma)^{-1}, \quad M := \gamma^{-1},\]
and apply (4.22) to estimate
\[\|\mathcal{D}'_{HF,1}(f, f)(t)\|_{L^2} \lesssim \int_0^t s \cdot \|(-\Delta + V + 1)^{-1} u(s)\|_{L^{M,\gamma} L^{M-\gamma}} \cdot \|\tilde{F}^{-1}(\langle m \rangle^4 \varphi_{\geq 10}(|m| \langle s \rangle^{-\delta N}) \tilde{u}(s))\|_{L^{2,\gamma}} \, ds.
\]

Using Sobolev’s embeddings, and the apriori Sobolev bound (2.23) we deduce
\[\|(-\Delta + V + 1)^{-1} u(s)\|_{L^{M,\gamma} L^{M-\gamma}} \lesssim \|u(s)\|_{L^2} \lesssim \epsilon,
\]
and, using also the boundedness of \(\tilde{F}^{-1} \tilde{F}\) and see (1.16),
\[
\|\tilde{F}^{-1}(\langle m \rangle^4 \varphi_{\geq 0}(|m| \langle s \rangle^{-\delta N}) \tilde{u}(s))\|_{L^{2+}} \lesssim \|\langle m \rangle^5 \varphi_{\geq 0}(|m| \langle s \rangle^{-\delta N}) \tilde{f}(s)\|_{L^2}
\lesssim \|\langle m \rangle^N \tilde{f}(s)\|_{L^2} \langle s \rangle^{-\delta N(N-5)} \lesssim \epsilon \langle s \rangle^{-3}.
\]

It follows that
\[\|\mathcal{D}_{HF,1}(f, f)(t)\|_{L^2} \lesssim \int_0^t s \cdot \epsilon \cdot \epsilon \langle s \rangle^{-3} \, ds \lesssim \epsilon^2.
\]

The terms \(\mathcal{D}_{HF,2}\) and \(\mathcal{D}_{HF,3}\) are easier and can be dealt with similarly, using also the a priori bound (2.23) for \(\partial_t \tilde{f}\).
To estimate $\partial^2_k \mathcal{D}_{HF}$ we use the identity (4.24). This leads to many terms, but since all of them can be treated similarly, we only give details for some. Among the many terms generated by applying (4.24) the main ones are

$$
\mathcal{D}_{HF,4}(f, f)(t) = \int_0^t \int s^2 e^{is|k|^2} \int \langle \ell \rangle^{-2} \overline{u}(s, \ell) \left| m \right|^5 \overline{u}(s, m) n_4(k, \ell, m) \varphi \mu'(k, \ell, m) \ d\ell d m \ ds,
$$

$$
\mathcal{D}_{HF,5}(f, f)(t) = \int_0^t \int s \ell e^{is|k|^2} \partial \tilde{f}(s, \ell) \tilde{f}(s, m) \varphi \mu'(k, \ell, m) \ d\ell d m \ ds,
$$

$$
\mathcal{D}_{HF,6}(f, f)(t) = \int_0^t \int e^{is|k|^2} \partial^2 \tilde{f}(s, \ell) \tilde{f}(s, m) \varphi \mu'(k, \ell, m) \ d\ell d m \ ds.
$$

Similarly to (4.32) we rewrite

$$
\mathcal{D}_{HF,4} := \int_0^t s^2 e^{is|k|^2} \int \langle \ell \rangle^{-2} \overline{u}(s, \ell) \left| m \right|^5 \overline{u}(s, m) n_4(k, \ell, m) \mu'(k, \ell, m) \ d\ell d m \ ds,
$$

$$
n_4(k, \ell, m) := \frac{-\ell^2 \langle \ell \rangle^2}{|m|^5} \varphi \mu \left( \left| \ell \right|^2 + |m|^2 + |k|^2 \right) \left( s \right)^{-2\delta_N},
$$

and, using $\|n_4\|_{CMs} \lesssim 1$ and the same notation above for the indexes, we can estimate

$$
\|\mathcal{D}_{HF,4}(f, f)(t)\|_{L^2} \lesssim \int_0^t s^2 \cdot \|(-\Delta + V + 1)^{-1} u(s)\|_{L^0 \cap L^\infty} \cdot \|\tilde{F}^{-1}(\left| m \right|^5 \varphi \mu \left( \left| \ell \right|^2 \right) \overline{u}(s))\|_{L^2+} \ ds
\lesssim \int_0^t s^2 \cdot \varepsilon \cdot \varepsilon \left( s \right)^{-\delta N(N-6)} \ ds \lesssim \varepsilon^2 \left( s \right)^{1/2}.
$$

Similarly, using the apriori bounds (4.23)

$$
\|\mathcal{D}_{HF,5}(f, f)(t)\|_{L^2} \lesssim \int_0^t s \cdot \|(-\Delta + V + 1)^{-1} \tilde{F}^{-1} e^{it|\ell|^2} \partial \tilde{f}(s)\|_{L^0 \cap L^\infty} \cdot \|\tilde{F}^{-1}(\left| m \right|^4 \varphi \mu \left( \left| \ell \right|^2 \right) \overline{u}(s))\|_{L^2+} \ ds
\lesssim \int_0^t s \cdot \|\partial \tilde{f}(s)\|_{L^2} \cdot \varepsilon \left( s \right)^{-\delta N(N-5)} \left| m \right|^N \overline{u}(s)\|_{L^2} \ ds
\lesssim \int_0^t s \cdot \varepsilon \cdot \varepsilon \left( s \right)^{-3} \ ds \lesssim \varepsilon^2,
$$

and

$$
\|\mathcal{D}_{HF,6}(f, f)(t)\|_{L^2} \lesssim \int_0^t \|(-\Delta + V + 1)^{-1} \tilde{F}^{-1} e^{it|\ell|^2} \partial^2 \tilde{f}(s)\|_{L^0 \cap L^\infty} \cdot \|\tilde{F}^{-1}(\left| m \right|^3 \varphi \mu \left( \left| \ell \right|^2 \right) \overline{u}(s))\|_{L^2+} \ ds
\lesssim \int_0^t \|\partial^2 \tilde{f}(s)\|_{L^2} \cdot \varepsilon \left( s \right)^{-5} \left| m \right|^N \overline{u}(s)\|_{L^2} \ ds
\lesssim \int_0^t \varepsilon \left( s \right)^{1/2+\delta} \cdot \varepsilon \left( s \right)^{-3} \ ds \lesssim \varepsilon^2.
$$

Case $|k| \gg |m|$. In this case $|k| \gg |m| + |\ell|$ and $|k| \gtrsim \left( s \right)^{\delta N}$ on the support of $\mathcal{D}_{HF}$. Our strategy is to first integrate by parts in $s$ and then estimate the weighted norms of the
resulting expression. More precisely, we use $e^{i\phi} = (i\Phi)^{-1} \partial_x e^{i\phi}$ and write

$$
D_{HF}(t)(f,f) = A_{HF}(t)(f,f) - A_{HF}(t)(f,f)(0) - \int_0^t A_{HF}(s)(\partial_x f, f)\, ds
$$

(4.34)

where

$$
A_{HF}(s)(g,h) := \int \int e^{i\Phi(k,\ell,m)}g(s,\ell)\tilde{h}(s,m)\frac{1}{\Phi(k,\ell,m)} \mu(k,\ell,m) \varphi_{\geq 0} \, d\ell dm
$$

(4.35)

and the “easier terms” are those where $\partial_x$ hits the cutoff which gives $\partial_x \varphi_{\geq 0} = s^{-1} \varphi_{\sim 0}$ and an easier term to treat.

We apply $\partial_k$ and $\partial_k^2$ to the terms in (4.34) using the identities (4.23)-(4.24) obtaining many different contributions. In the case of one derivative, we have (omitting irrelevant constants)

$$
\partial_k D_{HF}(f,f) = A_1 + A_2 + A_3 + \text{easier terms}
$$

$$
A_1 := tk \int \int e^{i\Phi(k,\ell,m)} \tilde{f}(t,\ell)\tilde{f}(t,m)\frac{1}{\Phi(k,\ell,m)} \mu(k,\ell,m) \varphi_{\geq 0} \, d\ell dm,
$$

$$
A_2 := \int_0^t sk \int \int e^{i\Phi(k,\ell,m)} \partial_s \tilde{f}(s,\ell)\tilde{f}(s,m)\frac{1}{\Phi(k,\ell,m)} \mu(k,\ell,m) \varphi_{\geq 0} \, d\ell dm ds,
$$

(4.36)

$$
A_3 := \int_0^t \int \int e^{i\Phi(k,\ell,m)} \partial_s \tilde{f}(s,\ell) \partial_m \tilde{f}(s,m)\frac{1}{\Phi(k,\ell,m)} \mu(k,\ell,m) \varphi_{\geq 0} \, d\ell dm ds.
$$

To estimate these terms we can use that $\|k/\Phi\|_{C^M} \lesssim 1$, and

$$
\tilde{F}^{-1}e^{is|k|^2} \partial_s \tilde{F}(s) = u^2(s).
$$

(4.37)

Using the Hölder estimate (4.22), and the decay estimate (4.7) in Lemma 4.2 we can bound

$$
\|A_1\|_{L^2} \lesssim t\|u(t)\|_{L^6 \cap L^\infty} \|u(t)\|_{L^3} \lesssim t \cdot \varepsilon \langle t \rangle^{-(1-)} \cdot \varepsilon \langle t \rangle^{-1/2} \lesssim \varepsilon^2.
$$

Similarly, using also (4.37) above, we can estimate

$$
\|A_2\|_{L^2} \lesssim \int_0^t s \cdot \|u^2(s)\|_{L^3} \|u(s)\|_{L^6 \cap L^\infty} \, ds \lesssim \int_0^t s \cdot \varepsilon^2 \langle s \rangle^{-2} \cdot \varepsilon \langle s \rangle^{-(1-)} \, ds \lesssim \varepsilon^2,
$$

and

$$
\|A_3\|_{L^2} \lesssim \int_0^t s \cdot \|u^2(s)\|_{L^3 \cap L^\infty} \cdot \|\tilde{F}^{-1}e^{is|m|^2} \partial_m \tilde{f}(s)\|_{L^2} \, ds
$$

$$
\lesssim \int_0^t s \cdot \varepsilon^2 \langle s \rangle^{-2+} \cdot \varepsilon s^{-1} \langle s \rangle^{1/2+\delta} \, ds \lesssim \varepsilon^2
$$

having used also (4.2) for the second inequality.
When applying $\partial^2_k$ we again obtain several terms. Omitting irrelevant constants, we can write schematically

$$\partial^2_k \mathcal{D}_{HF}(f, f) = B_1 + B_2 + B_3 + \text{easier terms}$$

$$B_1 = t^2 \int e^{i\Phi(k, \ell, m)} \tilde{f}(t, \ell) \tilde{f}(t, m) \frac{k^2}{\Phi(k, \ell, m)} \mu(k, \ell, m) \varphi_0 \, dt \, dm,$$

$$B_2 = t \int e^{i\Phi(k, \ell, m)} \tilde{f}(t, \ell) \partial_m \tilde{f}(t, m) \frac{k}{\Phi(k, \ell, m)} \mu(k, \ell, m) \varphi_0 \, dt \, dm,$$

$$B_3 = \int_0^t s^2 \int e^{i\Phi(k, \ell, m)} \partial_s \tilde{f}(s, \ell) \tilde{f}(s, m) \frac{k^2}{\Phi(k, \ell, m)} \mu(k, \ell, m) \varphi_0 \, dt \, dm \, ds,$$

$$B_4 = \int_0^t s \int e^{i\Phi(k, \ell, m)} \partial_s \tilde{f}(s, \ell) \partial_m \tilde{f}(s, m) \frac{k}{\Phi(k, \ell, m)} \mu(k, \ell, m) \varphi_0 \, dt \, dm \, ds.$$ (4.38)

The “easier terms” contain those terms where two derivatives fall on the profile $f$, which can be treated by an Hölder type inequality using the integrable-in-time $L^p$ decay, $p > 6$, and other terms where derivatives hit the cutoff.

In $B_1$ we write $k^2/\Phi(k, \ell, m) = -1 + (|k|^2 + |\ell|^2)/\Phi(k, \ell, m)$ The contribution corresponding to the symbol $-1$ is estimated using (4.22):

$$\|u(t)\|_{L^6 \cap L^6^-} \lesssim t^2 \cdot \varepsilon(t)^{-1} \cdot \varepsilon(t)^{-1/2} \lesssim \varepsilon^2(t)^{1/2+},$$

where $1/2+$ denotes here a number larger, but arbitrarily close to, $1/2$. This is consistent with (4.17). For the other contribution we note that $\|1/\Phi(k, \ell, m)\|_{C_{M\beta}} \lesssim 1$, $\beta > 0$, and estimate the bilinear term by

$$t^2 \|u(t)\|_{L^6 \cap L^6^-} \|\Delta u(t)\|_{L^3} \lesssim t^2 \cdot \varepsilon(t)^{-1} \cdot \varepsilon(t)^{-1/2} \lesssim \varepsilon^2(t)^{1/2+\rho+}$$ (4.39)

having used

$$\|\Delta u(t)\|_{L^3} \lesssim \|P_{\leq K_0} \Delta u(t)\|_{L^3} + \|P_{> K_0} \Delta u(t)\|_{L^3} \lesssim 2^{2K_0} \|u(t)\|_{L^3} + \|u(t)\|_{HF} 2^{-K_0(N-3)} \lesssim \varepsilon(2^{2K_0} |t|^{-1/2} + 2^{-K_0(N-3)})$$

with $2^{K_0} = |t|^{\rho/2}$ and $\rho = 1/(N - 3)$. Imposing that $\rho < \delta$, the bound (4.39) is consistent with the desired (4.17).

For the second term in (4.38) we use $\|k/\Phi(k, \ell, m)\|_{C_{M\beta}} \lesssim 1$, the decay estimate (4.2), and the apriori bounds, to obtain

$$\|B_2\|_{L^2} \lesssim t \|u(t)\|_{L^3 \cap L^3^-} \|\tilde{f}^{-1} e^{i|\ell| \varphi} \partial_m \tilde{f}(t)\|_{L^6} \lesssim t \cdot \varepsilon(t)^{-(1/2-)} \cdot |t|^{-1} \|\partial_m \tilde{f}(t)\|_{L^2} \lesssim t \cdot \varepsilon(t)^{-(1/2-)} \cdot \varepsilon(t)^{-1} \lesssim \varepsilon^2(t)^{2\delta}.$$ (4.37)

The term $B_3$ is similar to the term $B_1$ as it contains a time integration but one profile is differentiated in time. One can then proceed similarly with an $L^3 \times L^6^- \times L^6$ estimate using, see (4.37),

$$\|\tilde{f}^{-1} e^{i|k| \varphi} \partial_s \tilde{f}(s)\|_{L^3} \lesssim \|u(s)\|_{L^6}^2 \lesssim \varepsilon^2(s)^{-2}.$$ (4.36)

The term $B_4$ is easier and can be treated similarly to the previous ones by an $L^3 \cap L^3^- \times L^6$ estimate, so we skip it.
5. Analysis of the NSD I: structure of the leading order

5.1. Expansion of $\mu$ and leading order nonlinear terms. We expand the integrand in (4.14) in negative powers of $|x|$ according to the relation between $\psi$ and $\psi_1$ in (3.21), and write

$$(2\pi)^{9/2} \mu(k, \ell, m) = (2\pi)^{3/2} \delta_0(k - \ell - m)$$

$$- \frac{1}{4\pi} \mu_1(k, \ell, m) + \frac{1}{(4\pi)^2} \mu_2(k, \ell, m) - \frac{1}{(4\pi)^3} \mu_3(k, \ell, m),$$

(5.1)

where the integrand in $\mu_\rho$ is $O(|x|^{-\rho})$; more precisely

- The distribution $\mu_1$ is given by

$$\mu_1(k, \ell, m) = \nu_1(-k + \ell, m) + \nu_1(-k + m, \ell) + \nu_1(-\ell - m, k)$$

(5.2)

where

$$\nu_1(p, q) := \int e^{ix.p} \frac{e^{i|q||x|}}{|x|} \psi_1(x, q) \, dx;$$

(5.3)

- The measure $\mu_2$ is given by

$$\mu_2(k, \ell, m) = \nu_2^1(k, \ell, m) + \nu_2^2(k, \ell, m) + \nu_2^3(k, \ell, m)$$

(5.4)

where

$$\nu_2^1(k, \ell, m) := \int e^{-ix.k} \frac{e^{i(|q|+|m|)|x|}}{|x|^2} \psi_1(x, \ell) \psi_1(x, m) \, dx,$$

$$\nu_2^2(k, a, b) := \int e^{ix.a} \frac{e^{i|q|(|k|+|b|)}}{|x|^2} \psi_1(x, k) \psi_1(x, b) \, dx;$$

(5.5)

- The measure $\mu_3$ is given by

$$\mu_3(k, \ell, m) = \int \frac{e^{i(|q|+|m|)|x|}}{|x|^3} \psi_1(x, k) \psi_1(x, \ell) \psi_1(x, m) \, dx.$$  

(5.6)

We are interested in the regularity of these distributions, and therefore are mostly concerned with the behavior for large $|x|$ of the integrands.

According to (5.1) we decompose the nonlinear interaction in Duhamel’s formula (4.13) as

$$(2\pi)^{9/2} D(t)(f, f) = (2\pi)^{3/2} D_0(t)(f, f)$$

$$- \frac{1}{4\pi} D_1(t)(f, f) + \frac{1}{(4\pi)^2} D_2(t)(f, f) - \frac{1}{(4\pi)^3} D_3(t)(f, f),$$

(5.7)

$$D_0(t)(f, f) := \int_0^t \int e^{i\sigma(-|k|^2 + |\ell|^2 + |k-\ell|^2)} \tilde{f}(s, \ell) \tilde{f}(s, k - \ell) \, d\ell \, ds,$$

$$D_*(t)(f, f) := \int_0^t \int e^{i\sigma(-|k|^2 + |\ell|^2 + |m|^2)} \tilde{f}(s, \ell) \tilde{f}(s, m) \mu_*(k, \ell, m) \, d\ell \, dm \, ds.$$

In this section we will analyze in details $\mu_1$, and postpone the analysis of the lower order terms (5.4)-(5.6) to Subsection 8.1 and 8.3. We write more explicitly the leading order terms
in (5.7) using (5.2)
\[
D_1(t)(f, f) = \int_0^t \int_0^t e^{is(-|k|^2 + |\ell|^2 + |m|^2)} \tilde{f}(s, \ell) \tilde{f}(s, m) \nu_1(-k + m, \ell) \, d\ell \, dm \, ds
+ \int_0^t \int_0^t e^{is(-|k|^2 + |\ell|^2 + |m|^2)} \tilde{f}(s, \ell) \tilde{f}(s, m) \nu_1(-k + \ell, m) \, d\ell \, dm \, ds
+ \int_0^t \int_0^t e^{is(-|k|^2 + |\ell|^2 + |m|^2)} \tilde{f}(s, \ell) \tilde{f}(s, m) \nu_1(-\ell - m, k) \, d\ell \, dm \, ds \tag{5.8}
\]
which, using the symmetry in \( \ell \) and \( m \) and changing variables, we may rewrite as
\[
D_1(t)(f, f) = 2 \int_0^t \int_0^t e^{is(|\ell|^2 + 2k - |m|^2)} \tilde{f}(s, \ell + k) \tilde{f}(s, m) \nu_1(\ell, m) \, d\ell \, dm \, ds
+ \int_0^t \int_0^t e^{is(-|k|^2 + |\ell|^2 + 2m + 2|k|^2)} \tilde{f}(s, -\ell - m) \tilde{f}(s, m) \nu_1(\ell, k) \, d\ell \, dm \, ds. \tag{5.9}
\]
Note that the two integrals in (5.9) are somewhat similar but have slightly different structure as, for example, the measure in the second one is \( k \) dependent. This will require a slightly different treatment in some of the estimates that will follow in Sections 6 and 7.

5.2. Structure and properties of \( \nu_1 \). Motivated by (5.9) we need to study \( \nu_1 \) as in (5.3). Our main aim in this section is to prove that a precise version of the following approximate identity:
\[
\nu_1(p, q) = \frac{1}{|p|} \left[ \delta(|p| - |q|) + \text{p.v.} \frac{1}{|p| - |q|} \right] m_0(p, q) + R(p, q)
\]
where \( m_0 \) is a “nice” symbol of Coifman-Meyer type up to some losses, and \( R \) is a better behaved remainder. This is the content of the following main proposition:

**Proposition 5.1** (Structure of \( \nu_1 \)). Let \( \nu_1 \) be the distribution defined in (5.3), with \( \psi_1 \) defined by (3.21). Fix \( N_2 \in [5, N_1/4) \cap \mathbb{Z} \). Fix \( p, q \in \mathbb{R}^3 \) with \(|p| \approx 2^P, |q| \approx 2^Q\), and assume that \( P, Q \leq A \) for some \( A > 0 \). Then we can write
\[
\nu_1(p, q) = \nu_0(p, q) + \nu_L(p, q) + \nu_R(p, q), \tag{5.10}
\]
where:

1. The leading order is
   \[
   \nu_0(p, q) := \frac{b_0(p, q)}{|p|} \left[ i\pi \delta(|p| - |q|) + \text{p.v.} \frac{1}{|p| - |q|} \right] \tag{5.11}
   \]
   with \( b_0 \) satisfying the bounds
   \[
   \left| \varphi_P(p) \varphi_Q(q) \nabla_P \nabla_Q b_0(p, q) \right| \lesssim 2^{-|\alpha|P} \left( 2^{\alpha|Q|} + 2^{(1 - |\beta|)Q} \right) \cdot 1_{|P - Q| < 5}, \tag{5.12}
   \]
   for all \( P, Q \leq A, |\alpha| + |\beta| \leq N_1 \). Recall our notation \( Q_- = \min(Q, 0) \).
2. The lower order terms \( \nu_L(p, q) \) can be written as
   \[
   \nu_L(p, q) = \frac{1}{|p|} \sum_{a=1}^{N_2} \sum_{j \in \mathbb{Z}} b_{a,j}(p, q) \cdot 2^{j} K_a(2^{j}(|p| - |q|)) \tag{5.13}
   \]
with $K_a \in \mathcal{S}$ and $b_{a,J}$ satisfying
\begin{equation}
\sum_{J \in \mathcal{Z}} |\varphi_{p}(p)\varphi_{q}(q)\nabla_{p}^{\alpha}\nabla_{q}^{\beta}b_{a,J}(p,q)| \lesssim 2^{-|\alpha|P} \left(2^{(|\alpha|Q+2(1-|\beta|)Q_0)} \cdot 1_{|P-Q|<5}\right),
\end{equation}
for all $P, Q \leq A, |\alpha| + |\beta| \leq N_2$.

(3) The remainder term $\nu_R$ satisfies the estimates
\begin{equation}
|\varphi_{p}(p)\varphi_{q}(q)\nabla_{p}^{\alpha}\nabla_{q}^{\beta}\nu_{R}(p,q)| \lesssim 2^{-2\max(P,Q)} \cdot 2^{-((|\alpha|+|\beta|)\max(P,Q))} \cdot 2^{(|\alpha|+|\beta|+2)A}
\end{equation}
for $|P-Q| < 5$, and
\begin{equation}
|\varphi_{p}(p)\varphi_{q}(q)\nabla_{p}^{\alpha}\nabla_{q}^{\beta}\nu_{R}(p,q)| \lesssim 2^{-2\max(P,Q)} \cdot 2^{-|\alpha|\max(P,Q)} \max(0, 2^{(1-|\beta|)Q_0}) \cdot 2^{(|\alpha|+|\beta|+2)A}
\end{equation}
for $|P-Q| \geq 5$, for all $|\alpha| + |\beta| \leq N_2/2 - 3$.

Here are some comments

- One should think of Proposition 5.1 as the statement that
\begin{equation}
\mu(p,q) \approx \nu_{1}(p,q) \approx \frac{1}{|p|} \delta(|p| - |q|) + \frac{1}{|p|} \text{ p.v.} \frac{1}{|p| - |q|},
\end{equation}
up to small losses. This is clearly most relevant when $||p| - |q|| \ll |p| \approx |q|$ (note the indicator functions in (5.12) and (5.14)) and is essentially exact when $|p| \approx |q| \lesssim 1$. It is important to notice how Proposition 5.1 singles out the singularity of $\nu_1$, its strength and its structure up to a sufficiently high order, after which the measure is essentially smooth.

- There are some losses in our estimates when frequencies are large, see the factors of $2^A$ in (5.15)- (5.16). These are coming from the various expansions, such as the one in (5.34), where we allow growing factors of the frequency. In the evolution problem we will handle these by comparing the size of frequencies and time, using the high Sobolev regularity; see Proposition 4.4 where ‘large’ frequencies are treated, leaving us only with frequencies of size $\lesssim (s)^{N_2}$, see (4.16), in the integrals in (5.9).

- In the estimates (5.12) and (5.14) we have $|p| \approx |q|$ so the factors on the right-hand sides could be simplified a bit. Nevertheless, we have decided to leave the explicit dependence on $2^P$ and $2^Q$ to highlight the different roles played by the two variables, such as the fact that the integrand in $\nu_1$ is smooth in $p$ but not in $q$.

Proposition 5.1 is proven in Subsection 5.4 using as key step Lemma 5.2 below, which gives asymptotics for a basic “building block” (5.19).

**Lemma 5.2** (Asymptotic expansion for the “building block”). Let $p, q \in \mathbb{R}^3$ with $|p| \approx 2^P, |q| \approx 2^Q$ and $|P-Q| < 5$. Assume that for some for some $A > 0$ we have $P, Q \leq A$, and let
\begin{equation}
\mathcal{J}(A, P, Q) := \mathcal{J} := \{J \geq -\min(P, Q, 0) + 4A\}.
\end{equation}
Consider the function
\begin{equation}
K_J(p, q) := \int_{\mathbb{R}^3} e^{ix \cdot p} e^{i|x| \omega} g(\omega, q) \varphi(x2^{-J}) \, dx, \quad J \in \mathcal{J},
\end{equation}
where \( \omega := x/|x| \), with \( g \in G^N \) (see Definition 3.3) and some smooth compactly supported \( \varphi \). Then, for any fixed \( M \in (10, N) \cap \mathbb{Z} \) we have the expansion

\[
\mathcal{K}_J(p, q) = \frac{a_0(p, q)}{|p|} 2^J \chi_0(2^J (|p| - |q|)) + \sum_{\ell=1}^{M-1} \frac{a_\ell(p, q)}{|p|} \cdot 2^J \chi_\ell(2^J (|p| - |q|)) \cdot C_J + R_{J,M}(p, q) \cdot C_J^R,
\]

where:

- \( \chi_\ell \) are Schwartz functions;
- \( a_0, a_1, \ldots \) are smooth functions of \( (p, q) \neq (0, 0) \) with
  \[
a_0 := 2\pi i g(-p/|p|, q),
\]
  and satisfying
  \[
  |\nabla_p^a \nabla_q^\beta a_\ell(p, q)| \lesssim 2^{-|\alpha|p} (2^Q|\alpha| + 2^{(1-|\beta|)Q-}) \cdot 1_{\{|p-Q|<5\}}
  \]
  for \( |\alpha| + |\beta| \leq N - \ell \).
- The remainder satisfies
  \[
  |\nabla_p^a \nabla_q^\beta R_{J,M}(p, q)| \lesssim 2^{-2 \max(P, Q)} \cdot 2^{-(|\alpha|+|\beta|) \max(P, Q)} \cdot 2^{A(|\alpha|+|\beta|+2)}
  \]
  for \( |\alpha| + |\beta| \leq \min(N - M - 1, M/4 - 2) \);
- The coefficients \( C_J, C_J^R \geq 0 \) satisfy
  \[
  \sum_{J \in \mathcal{J}} C_J + C_J^R \leq 1.
  \]

**Remark 5.3.** Notice that (5.22) implies the symbol-type bound (with losses)

\[
|\varphi_P(p) \varphi_Q(q) \nabla_p^a \nabla_q^\beta [\nabla_q a_\ell(p, q)]| \lesssim 2^{-|\alpha|p} 2^{-|\beta|Q} 2^{A(|\alpha|+|\beta|)},
\]

for \( |\alpha| + |\beta| \leq N - \ell - 1 \). This is an estimate for \( \nabla_q a_\ell \) as well. This symbol-type bounds will be more convenient to use in some cases than (5.22).

### 5.3. Proof of Lemma 5.2

Recall that by Definition 3.5 we have

\[
\|\partial_q^\alpha \partial_q^\beta g(\cdot, q)\|_{L^\infty} \lesssim_{|\alpha|, |\beta|} (|q| + (|q|/\langle q \rangle)^{1-|\beta|})
\]

for all \( 1 \leq |\alpha| + |\beta| \leq N \). Note that, using the assumption \( |q| \approx 2^Q \lesssim 2^A \), \( A > 0 \) we also have the (slightly less precise) bound

\[
\|\partial_q^\alpha \partial_q^\beta g(\cdot, q)\|_{L^\infty} \lesssim 2^{-Q|\beta|} 2^{A(|\alpha|+|\beta|)}.
\]

We write in polar coordinates, \( x = r\omega \),

\[
4\pi \mathcal{K}_J(p, q) = \int_0^\infty \left( \int_{\mathbb{S}^2} e^{ir\omega \cdot p} g(\omega, q) d\omega \right) e^{i|q| r} \varphi(r 2^{-J}) r dr.
\]
Asymptotics in the angular variable. We look at the spherical integral in (5.28) and, using the rotation invariance, without loss of generality we reduce matters to considering \( p = |p| e_3 \) and the integral

\[
I(X; q) := \int_{S^2} e^{iX \omega \cdot e_3} g(\omega, q) d\omega, \quad X := r |p|, \tag{5.29}
\]

where \( X \approx 2^J |p| \gg 1 \) by assumption. Writing in standard spherical coordinates \( \theta \in [0, 2\pi], \phi \in [0, \pi], \omega = (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi) \), we see that

\[
I = \int_0^{2\pi} \int_0^\pi e^{iX \cos \phi} g(\omega, q) \sin \phi \, d\phi \, d\theta = I_0 + II
\]

\[
I_0 = \frac{2\pi}{iX} \left[ e^{iX} g(e_3, q) - e^{-iX} g(-e_3, q) \right]
\]

\[
II = \frac{1}{iX} \int_0^{2\pi} \int_0^\pi e^{iX \cos \phi} \partial_\phi g(\omega, q) \, d\phi \, d\theta.
\]

The contribution of the leading order term \( I_0 \) to (5.28) is

\[
\int_0^\infty \frac{2\pi}{iX} \left[ e^{iX} g(e_3, q) - e^{-iX} g(-e_3, q) \right] e^{ir|q|} \varphi(r 2^{-J}) \, r \, dr
\]

\[
= \frac{2\pi}{i|p|} g(p/|p|, q) \int_0^\infty e^{i(1/|q| + |p|)} \varphi(r 2^{-J}) \, dr - \frac{2\pi}{i|p|} g(-p/|p|, q) \int_0^\infty e^{i(|q| - |p|)} \varphi(r 2^{-J}) \, dr
\]

\[
= \frac{2\pi}{i|p|} g(p/|p|, q) 2^J \varphi(2^J (|q| + |p|)) - \frac{2\pi}{i|p|} g(-p/|p|, q) 2^J \varphi(2^J (|q| - |p|)). \tag{5.31}
\]

The second term in (5.31) coincides with the first term on the right-hand side of (5.20) with \( a_0 \) in (5.21). The first term of (5.31) can instead be absorbed into the remainder \( R_{I,M} \) as we explain below. In view of (5.26), we have

\[
|\partial_\alpha^\alpha \partial_\beta^\beta g(p/|p|, q)| \lesssim_{\alpha, \beta} |p|^{-|\alpha|} (|q|/|\langle q \rangle|)^{1-|\beta|} \lesssim |p|^{-\min(P,Q)} 2^{-|\alpha|+|\beta|}. \tag{5.32}
\]

Given our localization of the variables and (5.18) we have \( 2^{-\min(P,Q)} \lesssim 2^J \), so that for arbitrarily large \( p \),

\[
(2^J (|q| + |p|))^{\rho} |\partial_\alpha^\alpha \partial_\beta^\beta 2^J \varphi(2^J (|q| + |p|))| \lesssim_{\alpha, \beta} 2^{-\max(P,Q) (1+|\alpha|+|\beta|)}. \tag{5.33}
\]

Together with (5.32) this gives

\[
|\partial_\alpha^\alpha \partial_\beta^\beta \left( \frac{1}{|p|} g(p/|p|, q) 2^J \varphi(2^J (|p| + |q|)) \right)| \lesssim 2^{-\max(P,Q)} \cdot 2^{-\min(P,Q) (|\alpha|+|\beta|)} \cdot c_J \tag{5.34}
\]

with \( c_J := 2^{-(P+J)} \). Thus the property (5.23) with (5.24) holds true for this term.

Next we analyze the contribution from the lower order term \( II \) in (5.30), that is,

\[
\int_0^\infty II(X; q) e^{ir|q|} \varphi(r 2^{-J}) \, r \, dr
\]

\[
= \frac{1}{i|p|} \int_0^\infty \left( \int_0^{2\pi} \int_0^\pi e^{iX \cos \phi} \partial_\phi g(\omega, q) \, d\phi \, d\theta \right) e^{ir|q|} \varphi(r 2^{-J}) \, dr. \tag{5.35}
\]
To deal with the innermost integral we want to apply the following stationary phase expansion:

**Claim:** Suppose \( f \) is a smooth function with \( f(x_0) = f'(x_0) = 0 \) and \( f''(x_0) \neq 0 \), and let \( F \) be a function supported in a sufficiently small neighborhood of \( x_0 \) where \( f \) does not have any other critical point. Define

\[
I(\lambda) := \int e^{i\lambda f(x)} F(x) \, dx. \tag{5.36}
\]

Then, for \( \lambda \gg 1 \), there exist coefficients \( a_\ell \) such that

\[
\left| \left( \frac{d}{d\lambda} \right)^r [I(\lambda) - \lambda^{-1/2} \sum_{\ell=0}^M a_\ell \lambda^{-\ell/2}] \right| \lesssim \lambda^{-r-(M+1)/2}. \tag{5.37}
\]

The coefficients \( a_\ell \) depend linearly on the first \( \ell \) derivatives of \( F \) at the point \( x_0 \); they also depend on a lower bound for \( f' \) on the support of \( F \), and on higher-order derivatives of \( f \).

In our application, we will have that all the quantities involving the phase \( f \) are uniformly bounded by some absolute constant. Then, we slightly abuse notation and let \( a_\ell = c_\ell \partial_\ell^\xi F(x_0) \), for some \( c_\ell \in \mathbb{R} \), while this coefficient should technically be of the form \( \sum_{k=0}^\ell c_k \partial_\ell^k F(x_0) \).

Finally, the implicit constant in (5.37) is upperbounded by the \( \ell^\infty \) norm of at most \( M+1 \) derivatives of \( F \); we will similarly abuse notation and assume the constant is \( C \| \partial_\ell^{M+1} F \|_{\ell^\infty} \).

The above claim is a classical statement, see for example Stein’s book \[43, Proposition 3, p. 334\].

We isolate the stationary points \( x_0 = 0 \) and \( \pi \) in the \( d\phi \) integral in (5.35) by defining

\[
2 \int_0^\pi e^{iX \cos \phi} \partial_\phi g(\omega, q) \, d\phi = J_+(X, \theta, q) + J_-(X, \theta, q),
\]

\[
J_+ := \int_{-\pi}^\pi e^{iX \cos \phi} \partial_\phi g(\omega, q) \varphi_1(\phi) \, d\phi,
\]

\[
J_- := \int_{-\pi}^\pi e^{iX \cos \phi} \partial_\phi g(\omega, q) \varphi_2(\phi) \, d\phi,
\]

where \( 0 \leq \varphi_1 \leq 1 \) is a smooth cutoff around \( \phi = 0 \) and \( \varphi_2 = 1 - \varphi_1 \).

With \( \lambda = X \gg 1 \), the non-degenerate phase \( \cos \phi - 1 \), the stationary point \( x_0 = 0 \), we deduce from the claim above and (5.37) that

\[
\left| \left( \frac{d}{dX} \right)^\alpha \partial^\beta \left[ J_+(X) - e^{iX X^{-1/2}} \sum_{\ell=0}^{M-1} b_\ell^+ X^{-\ell/2} \right] \right| \lesssim X^{-M/2 - \alpha} \sup_{\phi} |\partial_{\phi}^{M+1} \partial^\beta g(\omega, q)| \tag{5.38}
\]

and, similarly, with the phase \( \cos \phi + 1 \) and the stationary point \( x_0 = \pi \), we get

\[
\left| \left( \frac{d}{dX} \right)^\alpha \partial^\beta \left[ J_-(X) - e^{-iX X^{-1/2}} \sum_{\ell=0}^{M-1} b_\ell^- X^{-\ell/2} \right] \right| \lesssim X^{-M/2 - \alpha} \sup_{\phi} |\partial_{\phi}^{M+1} \partial^\beta g(\omega, q)| \tag{5.39}
\]

where we have defined

\[
b_\ell^\pm(p, q) := c_\ell \partial_{\phi}^{\ell+1} g(\pm p/|p|, q), \tag{5.40}
\]

for some absolute constants \( c_\ell \).
Plugging the asymptotics \((5.38)-(5.39)\) into \((5.35)\) we see that

\[
\int_0^\infty II(X; q)e^{ir|q|}\varphi(r2^{-J}) r\, dr
\]

\[
= \sum_{\ell=0}^{M-1} \frac{2\pi b^\pm_{\ell}(p, q)}{|p|^{3/2+\ell/2}} \int_0^\infty e^{-ir|q|} e^{ir|q|} r^{-(\ell+1)/2} \varphi(r2^{-J}) \, dr
\]

\[
+ \sum_{\ell=0}^{M-1} \frac{2\pi b^\pm_{\ell}(p, q)}{|p|^{3/2+\ell/2}} \int_0^\infty e^{ir|q|} e^{ir|q|} r^{-(\ell+1)/2} \varphi(r2^{-J}) \, dr
\]

\[
+ \frac{1}{|p|} R^\pm_{j, M}(p, q) + \frac{1}{|p|} R^\pm_{j, M}(p, q)
\]

with

\[
R^\pm_{j, M}(p, q) = \int_0^\infty e^{\pm ir|q|} B^\pm(r|p|; q) \, dr,
\]

where

\[
B^\pm(X; q) := J_\pm(X) - e^{\pm iX} X^{-1/2} \sum_{\ell=0}^{M-1} b^\pm_{\ell} X^{-\ell/2}
\]

satisfies

\[
\left| \left( \frac{d}{dX} \right)^\alpha \partial_\beta B^\pm(X; q) \right| \lesssim X^{-M/2-\alpha} \sup_{\omega \in \mathbb{S}^2} |\partial_\phi^{M+1} \partial_\beta g(\omega, q)|.
\]

Moreover, we can see that

\[
\left| \partial_\beta \partial_\beta B(r|p|; q) \right| \lesssim \sup_{\alpha_1 + \alpha_2 = \alpha} X^{-M/2-|\alpha_1||r|^{\alpha_1}} \cdot |p|^{-|\alpha_2|} \sup_{|\beta'| \leq |\alpha_2|, \omega \in \mathbb{S}^2} |\partial_\phi^{M+1} \partial_\omega \partial_\beta g(\omega, q)|
\]

\[
\lesssim X^{-M/2-P|\alpha|} \cdot (2^{Q(M+1+|\alpha|)} + 2^{-(|\beta|-1)Q}).
\]

**Contribution of \((5.41)\).** The sum in \((5.20)\), with the claimed properties, arises from the sum in \((5.41)\). To see this we write it as

\[
\sum_{\ell=1}^{M} \frac{b_{\ell-1}(p, q)}{|p|^{1+\ell/2}} \int_0^\infty e^{-ir|q|} e^{ir|q|} r^{-\ell/2} \varphi J(r) \, dr
\]

\[
= \sum_{\ell=1}^{M} \frac{b_{\ell-1}(p, q)}{|p|} (2^\ell |p|)^{-\ell/2} \cdot 2^\ell (r^{-\ell/2} \varphi J(2^\ell |p| - |q|))
\]

\[
= \sum_{\ell=1}^{M} \frac{a_\ell(p, q)}{|p|} - C_J 2^\ell \chi_\ell(2^\ell |q| - |p|)
\]

where we define, according to the notation in \((3.20)\) (see also \((5.40)\))

\[
a_\ell(p, q) := (2^{J+P})^{-\ell/2+\delta} (2^{-|p|} |p|)^{-\ell/2} c_{\ell-1} \partial_\phi g(-p/|p|, q),
\]

\[
\chi_\ell := r^{-\ell/2} \varphi,
\]

\[
C_J := (2^{J+P})^{-\delta}.
\]
for some small fixed \( \delta > 0 \). To verify the estimates \((5.22)\) on the coefficients \( a_\ell \) we use the fact that our parameters satisfy \( J + P \geq 4A \geq 4Q \), see \((5.18)\), and the estimates \((5.26)\) on \( g \): for \(|p| \approx 2^P, |q| \approx 2^Q\),

\[
|\nabla_p^\alpha \nabla_q^\beta \varrho_\ell(p, q)| \lesssim (2^{J+P} - \ell/2 + \delta) \sup_{\alpha_1 + \alpha_2 = \alpha} |p|^{-|\alpha_2|} |\nabla_p^{\alpha_1} \nabla_q^\beta \partial_r^\ell g(-p/|p|, q)|
\]

\[
\lesssim (2^{4A} - \ell/2 + \delta) \sup_{\alpha_1 + \alpha_2 = \alpha} 2^{-|\alpha_2|P} 2^{-|\alpha_1|P} \sup_{\rho \leq |\alpha_1| + \ell} \| \nabla_q^\beta \nabla_r^\rho g(\cdot, q) \|_{L^\infty}
\]

\[
\lesssim (2^{4A} - \ell/2 + \delta) 2^{-|\alpha|P}(\langle q \rangle^{|\alpha|+\ell} + \langle |q|/\langle q \rangle \rangle^{1-|\beta|}).
\]

For \( Q \geq 0 \) this gives the bound \( 2^{-|\alpha|P}\langle q \rangle^{|\alpha|} \lesssim 2^{-|\alpha|P}2^{A|\alpha|} \) while for \( Q \leq 0 \) it implies a bound of \( 2^{-|\alpha|P}(1 + 2^{(1-|\beta|)Q^-}) \); these are consistent with the desired bounds \((5.22)\). The definition of \( C_J \) in \((5.49)\) gives the property \((5.24)\), since \( J + P \gg 1 \) in the set \( J \).

To complete the proof of the lemma it suffices to show how the three remaining terms in \((5.42)-(5.43)\) satisfy the estimates \((5.23)\) and can therefore be absorbed into the remainders \( R_{J,M} \).

**Contribution of \((5.42)\).** Similar to \((5.48)\), the sum in \((5.42)\) is given by

\[
\sum_{\ell=1}^M b_{\ell-1}^+(p, q) \int_0^\infty e^{ir|p|} e^{ir|q|} r^{-\ell/2} \varphi(r2^{-J}) \, dr
\]

\[
= \frac{1}{i|p|(|p| + |q|)} \sum_{\ell=1}^M d_\ell(p, q) 2^J(|q| + |p|)(r^{-\ell/2}) \varphi(2^J(|q| + |p|) \cdot C_J)
\]  

having defined \( d_\ell := (2^{P+J} - \ell/2 + \delta) (2^P |p|)^{\ell/2} c_{\ell-1} \partial_r^\ell g(p/|p|, q) \), similarly to \( a_\ell \), and \( C_J \) as in \((5.49)\). Since \( 2^{P+J} \geq 2^{4A} \), and \( r^{-\ell/2} \varphi \) is a Schwartz function, it follows that

\[
\left| \sum_{\ell=1}^M b_{\ell-1}^+(p, q) \int_0^\infty e^{ir|p|} e^{ir|q|} r^{-\ell/2} \varphi(r2^{-J}) \, dr \right|
\]

\[
\lesssim \left( |p| + |q| \right)^2 \sup_{\ell=1, \ldots, M} \| d_\ell \|_{L^\infty} \cdot C_J
\]

\[
\lesssim \left( \frac{1}{22^\max(P,Q)} \right) \cdot 2^{3A(-\ell/2 + \delta)} \sup_{\ell=1, \ldots, M} \sup_{\omega \in \mathbb{S}^2} |\partial_\omega^\ell g(\omega, q)| \cdot C_J,
\]

having used the definitions \((5.40)\). In view of the inequality \((5.26)\) and \(|q| \lesssim 2^A\), we see the validity of \((5.23)\) for \( \alpha = \beta = 0 \). The general estimate \((5.23)\) for \( |\alpha| + |\beta| \geq 1 \) follows similarly after differentiating the first line of \((5.50)\) and using as before \((5.40)\), the estimate \((5.26)\), and our localization and restrictions on the parameters \((5.18)\). Note in particular how each differentiation of the integral in \( p \) or \( q \) can cost a potentially dangerous factor of \( 2^J \), which can be traded for a factor of \((|p| + |q|)^{-1} \) since \( \varphi \) is Schwartz.

**Contribution of the remainders \((5.43)\).** Since \(|P - Q| \leq 5\), we see from \((5.44)\) and \((5.46)\) that

\[
\left| \frac{1}{|p|} |R_{J,M}^\pm(p, q)| \right| \lesssim 2^{-P} \sup_{\omega \in \mathbb{S}^2} |\partial_\omega^{M+1} g(\omega, q)| \int_0^\infty \varphi(r2^{-J})(r|p|)^{-M/2} \, dr
\]

\[
\lesssim 2^{-2P}(1 + 2^{(M+1)Q}) \cdot 2^{-(J+P)(M/2-1)}
\]

\[
\lesssim 2^{-2P}2^{-\delta(P+J)},
\]
for $\delta > 0$ small enough, since $P + J \geq 4A \geq 4Q$ and $M > 5$. This is consistent with (5.23) when $\alpha = \beta = 0$. The general estimate (5.23) follows by differentiating the formula (5.44):

$$
\frac{1}{|p|} \left| \nabla_p \nabla_q^\beta R_{J,M}^\pm (p, q) \right| \\
\lesssim \sup_{\alpha_1 + \alpha_2 = \alpha \atop \beta_1 + \beta_2 = \beta} \left| \int_0^\infty \nabla_p^{\alpha_1} e^{\pm i r |p|} \nabla_q^{\beta_1} e^{i r |q|} \nabla_p^{\alpha_2} \nabla_q^{\beta_2} B(r |p|; q) \phi (r 2^{-J}) \, dr \right| .
$$

On the support of the integral, since $r \approx 2^J \gtrsim 2^{-P} \approx 2^{-Q}$, we have

$$
\left| \nabla_p^{\alpha_1} e^{\pm i r |p|} \nabla_q^{\beta_1} e^{i r |q|} \right| \lesssim 2^{(\alpha_1 + |\beta_1|) J}.
$$

Using (5.47) we obtain

$$
\frac{1}{|p|} \left| \nabla_p \nabla_q^\beta R_{J,M}^\pm (p, q) \right| \\
\lesssim 2^{-2P} \sup_{\alpha_1 + \alpha_2 = \alpha \atop \beta_1 + \beta_2 = \beta} 2^{(\alpha_1 + |\beta_1|) J} \cdot (2^J + P - M/2 + 1) 2^{-P |\alpha_2|} \cdot (2Q(M + 1 + |\alpha_2|) + 2^{-|\beta_2| - 1}) Q
$$

(5.53)

If $Q \leq 0$ for each term above we have a bound of

$$
2^{-2P} 2^{-(J + P)(M/2 - 1 - |\alpha_1| - |\beta_1|)} \cdot 2^{-|\alpha| + |\beta|} P_2 - |\beta_2| Q
$$

which is consistent with (5.23). For $Q \geq 0$ instead each term in (5.53) is bounded by

$$
2^{-2P} \cdot 2^{(\alpha_1 + |\beta_1|) J} \cdot (2^J + P - M/2 + 1) 2^{-P |\alpha_2|} \cdot 2Q(M + 1 + |\alpha_2|)
\lesssim 2^{-2P} \cdot (2^J + P - M/2 + 1) 2^{(\alpha_1 + |\beta_1|) J} \cdot 2Q(M + 1) 2^{\alpha_1 |A|} \cdot 2^{-|\beta_1|} Q 2^{-P |\alpha_1 + |\alpha_2||}
\lesssim 2^{-2P} \cdot 2^{-P |\alpha| + 2|Q| |\beta|} \cdot 2A(|\alpha| + |\beta|) \cdot 2^{-\delta(J + P)}
$$

having used $J + \min(P, Q) \geq 4A \geq 4Q$ and $|\alpha| + |\beta| \leq M/4 - 2$. This concludes the proof of (5.23) and the Lemma.

We now combine Lemma 5.2 and the asymptotic expansion (3.34) for $\psi_1$ in Lemma 3.4 to prove Proposition 5.1.

5.4. Proof of Proposition 5.1. We consider $|p| \approx 2^P$ and $|q| \approx 2^Q$ with $P, Q \leq A$, $A \gg 1$ and write

$$
\nu (p, q) = \nu^+ (p, q) \mathbf{1}_{|p - Q| < 5} + \nu^- (p, q), \quad \nu^+ (p, q) := \sum_{J \in \mathcal{J}} \nu^J (p, q),
$$

(5.54)

$$
\nu^J (p, q) := \int e^{ix \cdot p} e^{i |q| |x|} \psi_1 (x, q) \varphi_J (0) (x) \, dx ,
$$

where the cutoff $\varphi_J (0)$ is defined in (2.19)-(2.20) and we recall

$$
\mathcal{J} := \{ J \geq 4A, J \geq - \min(P, Q) + 4A \}.
$$

(5.55)

The term $\nu_- \mathbf{1}_{|p - Q| < 5}$ will give rise to the leading order terms (and some remainder terms), while all the terms in $\nu_-$ are lower order remainders.

Analysis of $\nu_+ \mathbf{1}_{|p - Q| \leq 5}$. For $g \in \mathcal{G}^N$, $N > N_2$, see (3.49), let us denote

$$
K_{J,n}(p, q) [g] := \int_{\mathbb{R}^3} e^{ix \cdot p} e^{i |x| |q|} g (\omega, q) (| \cdot |^{-n} \varphi (x/2^J)) \, dx .
$$

(5.56)
From the definition of $\nu^+$ in (5.54), and using the expansion (3.34) from Lemma 3.4 we can write

$$\nu^+(p, q) = \sum_{J \in \mathcal{J}} \sum_{n=0}^{N_{2,n}} K_{J,n}(p, q)[g_n] 2^{-J_n}(q)^n + R(p, q)$$  \hspace{1cm} (5.57)

where

$$R(p, q) = \sum_{J \in \mathcal{J}} \int_{\mathbb{R}^3} e^{i x \cdot p} \frac{\hat{\phi}(|x| q)}{|x|} R_{N_2}(x, q) \varphi(x/2^J) \, dx$$  \hspace{1cm} (5.58)

for $R_{N_2}(x, q)$ satisfying estimates as in (3.37).

**Leading order contribution.** Let us look first at the term with $n = 0$ in (5.57), that is $\sum_{J \in \mathcal{J}} K_{J,0}(p, q)[g_0]$. Since $|P - Q| < 5$, in view of the result of Lemma 5.2 and adopting the same notation, we have that

$$\sum_{J \in \mathcal{J}} K_{J,0}(p, q)[g_0] = A + \sum_{\ell = 1}^{M-1} B_\ell + C,$$  \hspace{1cm} (5.59)

$$A := \frac{a_0(p, q)}{|p|} \sum_{J \in \mathcal{J}} 2^J \hat{\varphi}(2^J(|p| - |q|)),$$  \hspace{1cm} (5.60)

$$B_\ell := \frac{a_\ell(p, q)}{|p|} \sum_{J \in \mathcal{J}} 2^J \chi_\ell(2^J(|p| - |q|)) \cdot C_J,$$  \hspace{1cm} (5.61)

$$C := \sum_{J \in \mathcal{J}} R_{J,M}(p, q) \cdot C_J^R,$$  \hspace{1cm} (5.62)

where the coefficients $a_\ell$ satisfy the estimates (5.22), $\sum_{J \in \mathcal{J}} |C_J| + |C_J^R| < \infty$, and the estimates (5.23) hold for the remainder $R_{J,M}$.

Using the properties of the standard cutoff $\varphi$, see (2.19), we can define

$$J_0 := \max(4A, -\min(P, Q) + 4A) = 4A - \min(0, P, Q)$$  \hspace{1cm} (5.63)

and rewrite the term (5.60) as

$$A = \frac{a_0(p, q)}{|p|} \sum_{J \geq J_0} \varphi(2^{-J})(|p| - |q|) = \frac{a_0(p, q)}{|p|} \varphi_{\geq 1}(2^{-J_0})(|p| - |q|).$$  \hspace{1cm} (5.64)

Writing $\varphi_{\geq 1}(x) = \int_{-\infty}^{x} \psi(y) \, dy = 1_{\{x > 0\}} * \psi$, for a smooth $\psi \geq 0$ which is compactly supported in $[1/8, 2]$ and with integral 1, we deduce the formula

$$\varphi_{\geq 1}(\xi) = \mathcal{F}((1 + \text{sign}x)/2 * \psi)(\xi) = \sqrt{2\pi} \mathcal{F}((1 + \text{sign}x)/2)(\xi) \cdot \hat{\psi}(\xi) = \sqrt{\frac{\pi}{2}} \delta_0(\xi) + \text{p.v.} \frac{\hat{\psi}(\xi)}{i \xi}.$$  \hspace{1cm} (5.65)

It follows that

$$A = \frac{a_0(p, q)}{|p|} 2^{J_0} \varphi_{\geq 1}(2^{-J_0}((|p| - |q|)2^{J_0}))$$

$$= \frac{a_0(p, q)}{|p|} \frac{1}{i \sqrt{2\pi}} \left[ i \pi \delta(|p| - |q|) + \sqrt{2\pi} \text{p.v.} \frac{\hat{\psi}(2^{J_0}(|p| - |q|))}{|p| - |q|} \right].$$  \hspace{1cm} (5.65)
Up to slightly redefining $a_0$, this gives us the first terms in the right hand-side of (5.10) with (5.11)-(5.12), provided we show that the term (we use that $\hat{\psi}(0) = 1/\sqrt{2\pi}$)

$$A_R := \frac{a_0(p,q)}{|p|} \text{p.v.} \frac{\sqrt{2\pi} \hat{\psi}(2^{J_0}(|p| - |q|)) - 1}{|p| - |q|}$$

(5.66)

satisfies estimates as in (5.15). To see that this is the case, let us consider first the case $J_0 = 4A$, $\min(P,Q) \geq 0$. It is not hard to see, using the estimates for $a_0$ from (5.24), and $2^{\max(P,Q)} \leq 2^A \leq 2^{J_0}$, then

$$|\nabla_p \nabla_q A_R| \lesssim 2^{J_0} 2^{-P} \cdot 2^{J_0(|\alpha| + |\beta|)}$$

$$\lesssim 2^{-2\max(P,Q)} 2^{2A} \cdot 2^{-|\alpha|P_2(|\alpha| - 1)A} \cdot 2^{4A(|\alpha| + |\beta| + 1)}$$

which is acceptable. The case $J_0 = 4A - \min(P,Q)$ (i.e. $\min(P,Q) \leq 0$) is similar, using again that $\hat{\psi}$ is Schwartz, and that $2^P \approx 2^Q$.

To verify that the terms $B_\ell$ in (5.61) are of the form (5.13)-(5.14), it suffices to recall that $a_\ell$ satisfies (5.22) for $1 \leq \ell < M$, and $M \leq N_2$.

For the term in (5.62) we can use directly the estimate (5.23)-(5.24), to see that this satisfies bounds like those in (5.15).

**Lower order contributions.** Let us consider the contribution to the sums in (5.57) with $n = 1, \ldots, N_2$. We consider a term of the form

$$I_n = \sum_{J \in \mathcal{J}} K_{J,n}(p,q)[g_n] \cdot 2^{-Jn} \langle q \rangle^n, \quad g_n \in \mathcal{G}^{N_1-n},$$

and want to apply Lemma 5.2 with $M \leq N_1 - n$. Since $|q| \approx 2^Q \lesssim 2^A$ and for $J \in \mathcal{J}$ we must have $(1 + |q|)2^{-J} \leq 2^{-J/2}$, the conclusion of Lemma 5.2 gives

$$I_n = \sum_{\ell=0}^{M-1} B_\ell^{(n)} + C^{(n)},$$

where

$$B_\ell^{(n)} := \sum_{J \in \mathcal{J}} \frac{a_\ell^{(n)}(p,q)}{|p|} \cdot 2^J \chi_\ell^{(n)}(2^J(|p| - |q|)) \cdot 2^{-Jn/2},$$

(5.67)

$$C^{(n)} := \sum_{J \in \mathcal{J}} R_\ell^{(n)}(p,q) \cdot 2^{-Jn/2},$$

(5.68)

and we have that

1. $a_\ell^{(n)}(p,q) \in \mathcal{G}^{N_1-n-\ell} \subset \mathcal{G}^{N_1-N_2-M}$, for $0 \leq \ell < M$, and
2. $R_\ell^{(n)}(p,q)$ satisfy estimates like those in (5.23) with $|\alpha| + |\beta| \leq \min(N_1 - n - M - 1, M/4 - 2)$.

In particular, we see that the terms $B_\ell^{(n)}$ are of the form (5.13)-(5.14). The remainder term (5.68) satisfies estimates which are consistent with (5.15) since we can choose $M = N_1 - N_2 - n$. 


The remainder $\nu$ in (5.58). From (3.37) we know that, for $|x| \approx 2^J$ and $|q| \approx 2^Q$
\[
|\nabla^\beta q R_{N_2}(x, q)| \lesssim 2^{-N_2 J} (2^{N_2 Q} + 2^{(1-|\beta|)Q^+})
\]
(5.69)
for all $|\beta| \leq N_1 - N_2$. Differentiating (5.58), using (5.69), and $Q \leq J/4$, we see that, as long as $|\alpha| + |\beta| \leq N_2/2 - 3$,
\[
|\nabla^\alpha_q \nabla^\beta q R(p, q)| \lesssim 2^{(1-|\beta|)Q^+}.
\]
(5.70)
This is upper bounded by the right-hand side of (5.15). We can therefore absorb the term $R$ into $\nu_R$.

The remainder $\nu^-$ in (5.54). Finally we show that the term $\nu_-$ can also be absorbed into the remainder $\nu_R$. We look at the case $\min(P, Q) \leq 0$, since the other case is easier. By definition
\[
\nu^-(p, q) = \sum_{J \in \mathcal{J}^c} \nu_J(p, q) 1_{\{|P - Q| < 5\}} + \sum_{J \in \mathbb{Z}^+} \nu_J(p, q) 1_{\{|P - Q| \geq 5\}},
\]
(5.71)
$\mathcal{J}^c = \{ J \in \mathbb{Z}^+ : J < -\min(P, Q) + 4A \}$.

Let us look first at the term with $|P - Q| < 5$ and $J \in \mathcal{J}^c$. We inspect the formula (5.54) for $\nu_J$ to see that $\partial_p^\alpha \partial_q^\beta \nu^-(p, q)$ is a linear combination of terms of the form
\[
I_{\alpha, \beta_1, \beta_2} := \int_{\mathbb{R}^3} (ix)^\alpha e^{ixp} \partial_q^{\beta_2} \left( e^{i|x||q|} \frac{1}{|x|} \partial_q^{\beta_1} \psi_1(x, q) \right) \varphi(x/2^J) \, dx
\]
(5.72)
for $\beta_1 + \beta_2 = \beta$. The estimates (3.22) for $\psi_1$ give us
\[
|\nabla_q^{\beta_1} \psi_1(x, q)| \lesssim 1 + 2^{(1-|\beta_1|)Q^+}.
\]
(5.73)
Since we also have
\[
|\nabla_q^{\beta_2} e^{i|x||q|}| \lesssim \sum_{a + b = |\beta_2| - 1} 2^J \cdot 2^J a \cdot 2^{-bQ}
\]
(5.74)
we see that
\[
|I_{\alpha, \beta_1, \beta_2}| \lesssim 2^{2J} \cdot 2^{J|\alpha|} \cdot \left( 2^J 2^{(1-|\beta_2|)Q^+} + 2^{J|\beta_2|} \right) \cdot (1 + 2^{(1-|\beta_1|)Q^+})
\]
(5.75)
\[
\lesssim 2^{-2 \max(P, Q)} \cdot 2^{-\max(P, Q)(|\alpha| + |\beta|)} \cdot 2^{4A(|\alpha| + |\beta| + 2)},
\]
since $J \leq -\max(P, Q) + 4A + 5$. This is consistent with the desired bound (5.15).

For the elements in the second sum in (5.71) we can resort to an integration by parts argument using that $||p| - |q|| \gtrsim \max(|p|, |q|)$. Notice that, for any integer $\rho > 0$, we can write
\[
e^{i(xp + |x||q|)} = T^\rho e^{i(xp + |x||q|)}, \quad T := \frac{p + (x/|x|)|q|}{i|p + (x/|x|)|q|^2} \cdot \nabla_x.
\]
(5.76)
Since $|p + (x/|x|)|q| \gtrsim 2^{\max(P, Q)}$, and for any $|\gamma| \geq 1$ we have $|\nabla_x^\gamma (p + (x/|x|)|q|)| \lesssim 2^{-|\gamma|J} 2^{Q}$
and
\[
|\nabla_x^\gamma \psi_1(x, q)| \lesssim 2^{-|\gamma|J} 2^{Q},
\]

see (3.22), we obtain
\[
|\nu_f(p, q)| = \left| \int e^{i(x-p+|q||x|)}(T^*)^{\rho} \left[ \frac{1}{|x|} \psi_1(x, q) \varphi_j^{(0)}(x) \right] dx \right|
\leq 2^{2J} 2^{-J \rho 2^{-\max(P,Q) \rho} 2^A \rho}. \tag{5.77}
\]

With $\rho = 2$ this gives us (5.15) for $\alpha = \beta = 0$. For $|\alpha| + |\beta| \geq 1$ we apply derivatives obtaining terms as in (5.72) and then use integration by parts as above. Using also the first line of (5.73), we get the following improvement of (5.75): $|I_{\alpha, \beta_1, \beta_2}|$ is bounded by a linear combination of terms of the form
\[
2^{2J} 2^{-J \rho 2^{-\max(P,Q) \rho} 2^A \rho} \cdot 2^J |\alpha| \cdot (2^J 2^A 2^{bQ}) \cdot (1 + 2^{(1-|\beta_1|)Q_-}). \tag{5.78}
\]
for $a + b = |\beta_2| - 1$ (with the understanding that if $|\beta_2| = 0, 1$ then the whole term involving $a$ and $b$ is absent). We then use (5.78) with $\rho = |\alpha| + a + 3$ to get that $|I_{\alpha, \beta_1, \beta_2}|$ is bounded by a linear combination of factors
\[
2^{-2 \max(P,Q) 2^{-|\alpha| \max(P,Q)} 2^{-|\beta_1| \max(P,Q)}} \cdot 2^{Q(a+b+1)} \cdot 2^{A(|\alpha|+a+3)} \cdot (1 + 2^{(1-|\beta_1|)Q_-} 2^A |\alpha|+a+3)
\leq 2^{-2 \max(P,Q) 2^{-|\alpha| \max(P,Q)} 2^{-|\beta_1| \max(P,Q)}} \cdot 2^{Q(a+b+1)} \cdot 2^{A(|\alpha|+a+3)} \cdot 2^{A(|\alpha|+|\beta|)+3}
\]
consistently with (5.16).

\section{Bilinear estimates for the leading order of the NSD}

In this section we prove several bilinear estimate for the (singular) multipliers appearing in our problem, such as those arising from the asymptotic formulas of Proposition 5.1. The bilinear operators that we need to look at have the form
\[
T_{\mu_j}(g, h)(x) := \mathcal{F}_{k \to x}^{-1} \int_{\mathbb{R}^3 \times \mathbb{R}^3} g(\ell) h(m) \mu_j(k, \ell, m) \, d\ell dm, \quad j = 1, 2, 3. \tag{6.1}
\]
see (5.1). We will often need to consider these operators with additional symbols $b$ with suitable properties to be specified below, that is, we will look at
\[
T_{\mu_j}[b](g, h)(x) := \mathcal{F}_{k \to x}^{-1} \int_{\mathbb{R}^3 \times \mathbb{R}^3} g(\ell) h(m) b(k, \ell, m) \mu_j(k, \ell, m) \, d\ell dm, \quad j = 1, 2, 3. \tag{6.2}
\]

Our results will be a series of Hölder type estimates with some (small) losses and up to suitable remainders. These estimates will then be used to establish the nonlinear bounds of Section 7.

\subsection{Bilinear estimates for $\mu_1$}

The most important operator is the one corresponding to the leading order term $\mu_1$, see (5.1)-(5.2). Using the notation (6.1), the formula (5.2), and the symmetry in exchanging $\ell$ and $m$, we see that,
\[
T_{\mu_1}(g, h) = 2T_1(g, h)(k) + T_2(g, h)(k), \tag{6.3}
\]
where
\[
T_1(g, h)(x) := \mathcal{F}_{k \to x}^{-1} \int_{\mathbb{R}^3 \times \mathbb{R}^3} g(k - \ell) h(m) \nu_1(\ell, m) \, d\ell dm, \tag{6.4}
\]
\[
T_2(g, h)(x) := \mathcal{F}_{k \to x}^{-1} \int_{\mathbb{R}^3 \times \mathbb{R}^3} g(-m - \ell) h(m) \nu_1(\ell, k) \, d\ell dm, \tag{6.5}
\]
where $\nu_1$ is defined in \cite{5.3} and satisfies the formulas of Proposition \cite{5.1}. To allow for additional symbols we define

\[
T_1[b](g, h)(x) := \mathcal{F}_{k \to x}^{-1} \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} g(k - \ell)h(m) b(k, \ell, m) \nu_1(\ell, m) \, d\ell \, dm, \tag{6.6}
\]

\[
T_2[b](g, h)(x) := \mathcal{F}_{k \to x}^{-1} \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} g(-\ell - m)h(m) b(k, \ell, m) \nu_1(\ell, k) \, d\ell \, dm. \tag{6.7}
\]

\textbf{Theorem 6.1} (Bilinear bounds 1). Let $T_1[b]$ and $T_2[b]$ be the bilinear operators defined in \eqref{6.6} and \eqref{6.7}. Assume that:

- The symbol $b$ is such that
  \[
  \supp(b) \subseteq \{(k, \ell, m) \in \mathbb{R}^3 : |k| + |\ell| + |m| \leq 2^A, |\ell| \approx 2^L, |m| \approx 2^M\}, \tag{6.8}
  \]
  for some $A \geq 1$.
- For all $|k| \approx 2^K$, $|\ell| \approx 2^L$ and $|m| \approx 2^M$
  \[
  |\nabla_k^\alpha \nabla_\ell^\beta \nabla_m^\gamma b(k, \ell, m)| \lesssim 2^{-K|\alpha|} 2^{-L|\beta|} 2^{-M|\gamma|} \cdot 2^{(|\alpha| + |\beta| + |\gamma|)A}, \quad |\alpha|, |\beta|, |\gamma| \leq 5. \tag{6.9}
  \]
- There is $10A \leq D \leq 2^{A/10}$ such that
  \[
  \mathcal{D}(g, h) := \|g\|_{L^2} \|h\|_{L^2} + \min\left(\|\partial_k g\|_{L^2} \|h\|_{L^2}, \|g\|_{L^2} \|\partial_k h\|_{L^2}\right) \leq 2^D. \tag{6.10}
  \]

Then, the following estimates hold:

(i) For any $p, q \in (1, \infty)$ and $r > 1$ with
  \[
  \frac{1}{p} + \frac{1}{q} > \frac{1}{r}, \tag{6.11}
  \]
  we have
  \[
  \|T_1[b](g, h)\|_{L^r} \lesssim \|\hat{g}\|_{L^p} \|\hat{h}\|_{L^q} \cdot 2^{\max(L, M)} \cdot 2^{C_0 A} + 2^{-D} \mathcal{D}(g, h), \tag{6.12}
  \]
  and
  \[
  \|P_K T_2[b](g, h)\|_{L^r} \lesssim \|\hat{g}\|_{L^p} \|\hat{h}\|_{L^q} \cdot 2^{\max(L, K)} \cdot 2^{C_0 A} + 2^{-D} \mathcal{D}(g, h), \tag{6.13}
  \]
  where $C_0 := 65$. Recall the notation after \eqref{2.20} for the projection $P_K$.

(ii) Define the “good vector-field”
  \[
  X = \partial_\ell + \partial_m \tag{6.14}
  \]
  and, for $a \leq 2$, define the operators
  \[
  T X^a[b](g, h)(k) := \mathcal{F}_{k \to x}^{-1} \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} g(k - \ell)h(m) b(k, \ell, m) X^a \nu_1(\ell, m) \, d\ell \, dm. \tag{6.15}
  \]
  Then
  \[
  \|T X^a[b](g, h)\|_{L^r} \lesssim \|\hat{g}\|_{L^p} \|\hat{h}\|_{L^q} \cdot 2^{-a \min(L, M)} \cdot 2^{\max(L, M)} \cdot 2^{(C_0 + 12) A} + 2^{-D} \mathcal{D}(g, h). \tag{6.16}
  \]

\footnote{This is a convenient value of the absolute constant $C_0$ that we can choose in our proof, but it can certainly be improved. In the nonlinear estimates for the evolution equation (Sections \cite{7} and \cite{8} we are going to impose conditions on the smallness of $C_0 N$ (or similar quantities), see the condition \eqref{7.11} for example, and recall the definition \eqref{8.11}. Then, a smaller value of $C_0$ would reduce the total number of derivatives $N$ required for our initial data.}
Let us make a few comments about the statement of the theorem and its uses:

- Note that our operators are localized according to (6.8) and that factors of $2^{\max(L,M)}$ and $2^A$ enter the final bound (6.12). The power of $2^A$ represents a loss for high frequencies, due to the fact that we allow multipliers $b$ which are not standard ones, and satisfies estimates with losses (6.9). Even when $b = 1$, our proof would give similar types of losses coming from the contribution of large frequencies. The factor of $2^{\max(L,M)}$ is consistent with the homogeneity of $\nu_1$ and gives a useful gain for small frequencies.

- The choice of $D$ and $A$: in our application of the bilinear estimate (6.12) to the nonlinear evolution, a typical choice of the parameters will be, see (4.16),
  
  $$2^A \approx \langle t \rangle^{6/N}, \quad 2^D \approx \langle t \rangle^3.$$  

  With this choice of $A$, there are only very small losses in the estimate (6.12). The choice of $D$ allows us to: (a) comfortably verify (6.10) for the various arguments $g$ and $h$ that we will encounter, (b) treat the $2^{-D}$ factor in (6.12) as a remainder which decays fast in time and can always be disregarded. Moreover, with the choice (6.17) the technical restriction $D \leq 2^A$ is clearly satisfied.

- Note the compatibility of (6.9) with the properties of the coefficients $b_0$ and $b_{a,J}$ in (5.12) and (5.14).

- The estimates for the operators $T_{X^a}$ where the good vectorfield (6.14) is applied to $\nu_1$ follow from the structural Proposition 5.1, and the proof for the case $a = 0$. The key point is that $X f (|\ell| - |m|) = 0$ for all $f$.

- The analogue of the good vectorfield (6.14) for the operator $T_2$, is the derivative $\partial_m$, so this does not require a separate estimate as (6.15)-(6.16).

Theorem 6.1 is proved in Subsection 6.3. Its proof will be done in several steps using as key ingredients the decomposition and the asymptotic formulas for $\nu_1$ in Proposition 5.1 and Lemma 6.4 below. In certain frequency configuration we will need to differentiate $\nu_1$ in directions other than $X$. The next Theorem establishes bilinear bounds for the relevant operators that will appear in the nonlinear analysis in Section 7.

**Theorem 6.2** (Bilinear bounds with vectorfields 1). Under the assumption and notation of Theorem 6.1 the following additional bilinear estimates hold:

1. For $a = (a_1, a_2)$, $1 \leq |a| \leq 2$ define the operators

   $$T_{\nabla a}[b](g, h)(k) := F_{k \rightarrow x}^{-1} \int_{\mathbb{R}^3 \times \mathbb{R}^3} g(k - \ell) h(m) b(k, \ell, m) \times \nabla^{a_1}_\ell \nabla^{a_2}_m [\nu_1(\ell, m) \chi_+(\ell, m)] \, d\ell dm$$

   where

   $$\chi_+(\ell, m) := \varphi_{2^{\max(L,M)-10}}(|\ell| - |m|);$$

   recall the notation (2.19) for the cutoffs.

2. Then

   $$\| T_{\nabla a}[b](g, h) \|_{L^r} \lesssim \| g \|_{L^p} \| h \|_{L^q} \cdot 2^{(1-|a|)M2^{(C_0+12)A}} + 2^{-D} \mathcal{D}(g, h).$$
For $1 \leq |a| \leq 2$ as above, and $K \in \mathbb{Z}$, define the operators (we omit the $K$ dependence)

$$
T_{\partial}[b](g, h)(k) := \mathcal{F}_{k \to x}^{-1} \int_{\mathbb{R}^3 \times \mathbb{R}^3} g(-m - \ell) h(m) b(k, \ell, m) \times \left[ \varphi_K(k) \nabla^a \nabla^2 \nu_1(\ell, k) \chi_+(\ell, k) \right] \, d\ell dm,
$$

where (with a slight abuse of notation)

$$
\chi_+(\ell, k) := \varphi_{\geq \max(L, K) - 10}(|\ell| - |k|).
$$

Then

$$
\|T_{\partial}[b](g, h)\|_{L^r} \lesssim \|\hat{g}\|_{L^p} \|\hat{h}\|_{L^q} \cdot 2^{(1-|a|)K} \cdot 2^{(C_0 + 12)A} + 2^{-D} D(g, h). \quad (6.23)
$$

(iii) Let

$$
Y = \partial_k + \frac{k}{|k|} \left( \frac{\ell}{|\ell|} \cdot \partial_\ell \right)
$$

and, for $a = 1, 2$, $K \in \mathbb{Z}$, define the operators

$$
T_Y[a][b](g, h)(k) := \mathcal{F}_{k \to x}^{-1} \int_{\mathbb{R}^3 \times \mathbb{R}^3} g(-m - \ell) h(m) b(k, \ell, m) \times \left[ \varphi_K(k) Y^a \nu_1(\ell, k) \right] \, d\ell dm.
$$

Then

$$
\|T_Y[a][b](g, h)\|_{L^r} \lesssim \|\hat{g}\|_{L^p} \|\hat{h}\|_{L^q} \cdot 2^{(1-a)K} \cdot 2^{(C_0 + 12)A} + 2^{-D} D(g, h). \quad (6.26)
$$

Theorem 6.2 is proved in Subsection 6.4. Let us explain how we are going to use these estimates:

- Part (i) is used to prove bilinear bounds for operators of the $T_1$-type when the support is restricted away from the singularity of $\nu_1(\ell, m)$; see 7.1.4.
- Part (ii) is used similarly to (i) when dealing with operators of $T_2$-type away from the singularity of $\nu_1(\ell, k)$; see 7.3.1.
- The bounds in part (iii) are used to estimate $\partial_k^n T_2$; see Subsection 7.3. In particular, we are going to use (6.24)-(6.26) to transform $k$ derivatives of $\nu_1(\ell, k)$ into $T_Y$ operators plus operators involving more manageable $\ell$ derivatives. Notice that the operator in (6.25) has no restriction on the support in terms of $(|\ell| - |k|)^{-1}$, so we are dealing with the full singular kernel. The main point of (6.26) is that using the vectorfield $Y$ does not increase the singularity in terms of the size of $|\ell| - |k|$.
- Note how the bounds (6.20) and (6.26) have a certain gain in terms of factors of $2^{-K}$: the application of $Y^a$ only gives a factor $2^{-(a-1)K}$ instead of a more singular $2^{-aK}$. This type of gain is consistent with the estimate (5.16), where a $\nabla^\beta$-derivative costs $2^{(1-|\beta|)Q}$. These bounds will be helpful in the nonlinear estimates of Section 7.
6.2. Bilinear operators supported on thin annuli. Let us first state a Lemma for bilinear operators with “regular” symbols.

**Lemma 6.3 (Bounds for regular symbols).** For $L, M \in \mathbb{Z}$, consider the bilinear operator

$$B[b](g, h)(x) = \mathcal{F}^{-1}_{k \to x} \int_{\mathbb{R}^3 \times \mathbb{R}^3} g(\ell - k)h(m) b(k, \ell, m) \, dk \, dm,$$

under the assumptions

- For some $A \geq 1$
  
  \[ \text{supp}(b) \subseteq \{(k, \ell, m) \in \mathbb{R}^3 : |k| + |\ell| + |m| \lesssim 2^A, |k| \approx 2^K, |\ell| \approx 2^L, |m| \approx 2^M\}; \]  

- The following estimate holds
  
  \[ |\nabla^a_\ell \nabla^\alpha_\ell \nabla^\beta_m b(k, \ell, m)| \lesssim 2^{-|\alpha|K - |\alpha|L - |\beta|M} \cdot 2^{|\alpha| + |\alpha| + |\beta|} A, \quad |\alpha|, |\alpha|, |\beta| \leq 4. \]  

Then, for $p, q, r \in [1, \infty]$, we have

$$\|B[b](g, h)\|_{L^r} \lesssim 2^{3\max(L, M)} \cdot 2^{2A} \cdot \|\mathcal{F}g\|_{L^p} \|\mathcal{F}h\|_{L^q}, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r}. \]  

The proof of Lemma 6.3, which is more standard than that of Lemma 6.4 below, is given at the end of the section.

**Lemma 6.4 (Bilinear operators restricted to small annuli 1).** Let $j \geq 1$, consider the bilinear operator

$$B_j[b](g, h)(x) = \mathcal{F}^{-1}_{k \to x} \int_{\mathbb{R}^3 \times \mathbb{R}^3} g(\ell - k)h(m) b(k, \ell, m) \chi(2^j (|\ell| - |m|)) \, dk \, dm,$$

where $\chi$ is a Schwartz function. Assume:

- For some $A \geq 1$ and $L \gg -j$ we have
  
  \[ \text{supp}(b) \subseteq \{(k, \ell, m) \in \mathbb{R}^3 : |k| + |\ell| + |m| \lesssim 2^A, |k| \approx 2^K, |\ell| \approx 2^L\}; \]  

- The following estimate holds
  
  \[ |\nabla^a_\ell \nabla^\alpha_\ell \nabla^\beta_m b(k, \ell, m)| \lesssim 2^{-|\alpha|K - |\alpha|L - |\beta|M} \cdot 2^{|\alpha| + |\alpha| + |\beta|} A, \quad |\alpha|, |\alpha|, |\beta| \leq 4. \]  

Then, for $p, q, r \in [1, \infty]$, we have

$$\|B_j[b](g, h)\|_{L^r} \lesssim 2^{-j} \cdot 2^{2L} \cdot 2^{8A} \cdot \|\mathcal{F}g\|_{L^p} \|\mathcal{F}h\|_{L^q}, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r}. \]  

The main conclusion of Lemma 6.4 in the final bound (6.34) is the $2^{-j+2L}$ factor which gives a gain proportional to the volume of the annulus in which the support of the operator lies, up to some small losses due to the presence of the multiplier $b$. The proof of Lemma 6.4 is given in Subsection 6.5.
6.3. **Proof of Theorem 6.1.** Let us begin by estimating the operator $T_1$ in (6.4).

**Frequency localized estimate.** We first claim that we can reduce the proof of the main conclusion (6.12) to the following slightly stronger localized version:

$$
\| P_K T_1[b](g, h) \|_{L^r} \leq \| \hat{g} \|_{L^{p'}} \cdot \| \hat{h} \|_{L^q} \cdot 2^{\max(L, M)} \cdot 2^{(C_0 - 1)A + 2^{-D'}}
$$

(6.35)

where $A, L, M, D$ are as in the statement of the theorem. Assume let $(p, q, r)$ be such that $1/p + 1/q > 1/r$ and, for $\delta \ll 1$ as above, let $1 < r - \delta < r' < r$ be such that $1/p + 1/q = 1/r'$. Using Bernstein’s inequality, and (6.35) with exponents $r', p$ and $q$, we have

$$
\| T_1[b](g, h) \|_{L^r} \leq \sum_{K \leq A} \| P_K T_1[b](g, h) \|_{L^r} \approx \sum_{K \leq A} 2^{3(\frac{1}{r'} - \frac{1}{r})K} \| P_K T_1[b](g, h) \|_{L^r'}
$$

$$
\lesssim \sum_{K \leq A} 2^{3(\frac{1}{r'} - \frac{1}{r})K} \left( \| \hat{g} \|_{L^{p'}} \cdot \| \hat{h} \|_{L^q} \cdot 2^{\max(L, M)} \cdot 2^{(C_0 - 1)A + 2^{-D'}} \right)
$$

$$
\lesssim 2^{\delta A} \| \hat{g} \|_{L^{p'}} \| \hat{h} \|_{L^q} \cdot 2^{\max(L, M)} \cdot 2^{(C_0 - 1)A + 2^{-D'}}.
$$

This implies the main estimate (6.12). In the rest of the proof we then concentrate on the estimate (6.35).

**Decomposition of $T_1$.** Using the decomposition (5.10) and defining

$$
\nu_\delta(\ell, m) = \frac{b_0(\ell, m)}{|\ell|} \delta_0(|\ell| - |m|), \quad \nu_{p.v.}(\ell, m) = \frac{b_0(\ell, m)}{|\ell|} \frac{1}{|\ell| - |m|},
$$

(6.36)

we reduce to proving the desired bound (6.35) for the operators

$$
T_\nu[b](g, h)(x) := F_{k \to x}^{-1} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} g(k - \ell) h(m) b(k, \ell, m) \nu(\ell, m) \, d\ell dm,
$$

(6.37)

$$
\nu \in \{ \nu_\delta, \nu_{p.v.}, \nu_L, \nu_R \}.
$$

Recall that, in view of the support restrictions on $b$, we have $|\ell| \approx 2^L$ and $|m| \approx 2^M$, and that $\nu_\delta, \nu_{p.v.}$ and $\nu_L$ are non-zero only when $|L - M| < 5$.

**Estimate of $T_{\nu_\delta}$.** By definition

$$
\mathcal{F}(T_{\nu_\delta}[b](g, h))(k) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} T_\varepsilon(g, h)(k),
$$

$$
T_\varepsilon(g, h)(k) := \iint_{\mathbb{R}^3 \times \mathbb{R}^3} g(k - \ell) h(m) b(k, \ell, m) \frac{b_0(\ell, m)}{|\ell|} \phi \left( \frac{|\ell| - |m|}{\varepsilon} \right) \, d\ell dm,
$$

(6.38)

where $\phi$ is a smooth, even, positive and radially decreasing cutoff which equals 1 close to the origin, and whose integral is 1. In view of the properties of $b$, see (6.6)-(6.7) and of $b_0$, see (5.12), we may let $b_0 \equiv 1$. Moreover, we may consider $\varepsilon \ll 2^L$ and write (recall the notation for $\sim$ after (2.20))

$$
T_\varepsilon(g, h) = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} g(k - \ell) h(m) \frac{b(k, \ell, m)}{|\ell|} \phi_{\sim L}(\ell) \varphi_{L}(m) \phi \left( \frac{|\ell| - |m|}{\varepsilon} \right) \, d\ell dm.
$$

(6.39)

\(^6\)Please note that this index $L$ has no relation with the size of $\ell$. 

Using the notation of Lemma 6.4 and changing slightly the definition of $\varphi_{\sim L}(\ell)$, we have, for $K \in \mathbb{Z}$,

$$\hat{F}^{-1}(\varphi_K(k) T_\epsilon(g,h)) = 2^{-L} T_j[c](g,h), \quad 2^j = \epsilon^{-1},$$

with

$$c(k, \ell, m) := \varphi_K(k) \varphi_{\sim L}(\ell) \varphi_{\sim L}(m) b(k, m, \ell)$$

The assumptions (6.32)-(6.33) of Lemma 6.4 hold true for such $c$, and applying the conclusion (6.34) we obtain

$$\frac{1}{\epsilon} \| \hat{F}^{-1}(\varphi_K(k) T_\epsilon(g,h)) \|_{L^r} \lesssim 2^L \cdot \| \hat{g} \|_{L^p} \| \hat{h} \|_{L^q} \cdot 2^{12A}. \quad (6.40)$$

This gives the desired bound (6.35) for $\| P_K T_\epsilon \nu \delta [b](g, \varphi_M h) \|_{L^r}$.

**Estimate of $T_{\epsilon,v}$**. Recall the definition (6.36)-(6.37). As above we may disregard the symbol $b_0$. We define (omitting the dependence on fixed $K, L$ and $M$)

$$T_{\epsilon,B}(g, h) := \frac{\hat{F}^{-1}}{k \rightarrow x} \int_{||\ell|-|m|| \geq \epsilon} g(k-\ell) h(m) b(k, \ell, m) \frac{1}{|\ell|} \varphi_B(|\ell|-|m|) \, d\ell \, dm, \quad (6.41)$$

$$b(k, \ell, m) := \varphi_K(k) \varphi_M(m) \varphi_{\sim L}(\ell) b(k, \ell, m).$$

Note that these are trivial if $B > L + 10$. We then decompose according to the size of the singularity:

$$P_K T_{\epsilon,v}(g, \varphi_M h) = \lim_{\epsilon \to 0} \left[ T_{\epsilon,\text{low}}(g, h) + T_{\epsilon,\text{med}}(g, h) + T_{\epsilon,\text{high}}(g, h) \right], \quad (6.42)$$

$$T_{\epsilon,\text{low}} := \sum_{B \leq B_0} T_{\epsilon,B}, \quad B_0 := \min(-50(D' + 10A), L - 10, 0) \quad (6.43)$$

$$T_{\epsilon,\text{med}} := \sum_{B_0 < B < L - 10} T_{\epsilon,B}, \quad (6.44)$$

$$T_{\epsilon,\text{high}} := \sum_{B \geq L - 10} T_{\epsilon,B}. \quad (6.45)$$

**Estimate of (6.43)**. On the support of (6.43) we have $||\ell|-|m|| \lesssim 2^{B_0} \ll \min(2^{-D'}, 2^L)$ and we estimate it using the principal value and the regularity of the inputs. In particular, let us assume that the assumption (6.10) is fulfilled by

$$D(g, h) = (\|g\|_{L^2} + \|\partial_k g\|_{L^2}) \|h\|_{L^2} \leq 2^D. \quad (6.46)$$

We split the function $g$ which appears as an argument in each term $T_{\epsilon,B}(g, h)$ of the sum (6.43) as

$$g = g_1 + g_2, \quad g_1 := P_{>X} g, \quad g_2 := P_{\leq X} g, \quad X := 2D' + 20A - (1/100)B. \quad (6.47)$$
The pieces corresponding to the high frequency part of $g$ are estimated using Lemma 6.4:

\[
\sum_{B \leq B_0} \| T_{\epsilon, B}(g_1, h) \|_{L^1} \lesssim \sum_{B \leq B_0} 2^L \cdot \| \tilde{g}_1 \|_{L^2} \cdot \| \tilde{h} \|_{L^2} \cdot 2^{12A} \\
\lesssim \sum_{B \leq B_0} 2^L \cdot 2^{-X} \| \nabla g_1 \|_{L^2} \cdot \| \tilde{h} \|_{L^2} \cdot 2^{12A} \\
\lesssim \sum_{B \leq B_0} 2^{-2D' + (1/100)B} \cdot 2^D \\
\lesssim 2^{-D'}.
\]

The estimate for $L'$, $r > 1$, instead of $L^1$, is obtained by first applying Bernstein and then estimating as above.

The contribution with $g_2$, see (6.47), is handled using the principal value. We begin by writing

\[
T_{\epsilon, B}(g_2, h) = T_B^{(1)}(g_2, h) + T_B^{(2)}(g_2, h) + T_B^{(3)}(g_2, h),
\]

where

\[
\begin{align*}
\hat{T}_B^{(1)}(g, h)(k) &:= \int_{|\ell| - |m| \geq \epsilon} g(k - \ell) h(m) \frac{b(k, \ell, m)}{|\ell|^2} \varphi_B(|\ell| - |m|) \, d\ell dm, \\
\hat{T}_B^{(2)}(g, h)(k) &:= \int_{|\ell| - |m| \geq \epsilon} g(k - \ell) \left[ b(k, \ell, m) - b(k, |m| |\ell| / |m|, m) \right] \\
&\quad \times h(m) \frac{|m|}{|\ell|^2} \frac{1}{|\ell| - |m|} \varphi_B(|\ell| - |m|) \, d\ell dm, \\
\hat{T}_B^{(3)}(g, h)(k) &:= \int_{|\ell| - |m| \geq \epsilon} \left[ g(k - \ell) - g(k - |m| |\ell| / |m|) \right] b(k, |m| |\ell| / |m|, m) \\
&\quad \times h(m) \frac{|m|}{|\ell|^2} \frac{1}{|\ell| - |m|} \varphi_B(|\ell| - |m|) \, d\ell dm,
\end{align*}
\]

having used that

\[
\begin{align*}
\int \int_{|\ell| - |m| \geq \epsilon} g_2(k - \ell|m|/|\ell|) h(m) \frac{|m|}{|\ell|^2} \frac{b(k, |m| |\ell| / |m|, m)}{|\ell| - |m|} \varphi_B(|\ell| - |m|) \, d\ell dm \\
= \int \| h(m) \|_{S^2_\theta} \int_{S^2_\theta} g_2(k - \theta |m|) b(k, |m| \theta, m) \\
&\quad \times \left[ \int_{|\rho - |m| |\ell| / |m|} \frac{1}{\rho - |m|} \varphi_B(\rho - |m|) \, d\rho \right] \, d\theta dm = 0.
\end{align*}
\]

For the first term in (6.49) we use Lemma 6.4 to estimate

\[
\| T_B^{(1)}(g_2, h) \|_{L^r} \lesssim \| \tilde{g} \|_{L^p} \| \tilde{h} \|_{L^q} \cdot 2^{12A} \cdot 2^B
\]

Summing this bound over $B \leq B_0 \leq L$ suffices.

The term $T_B^{(2)}(g_2, h)$ is similar by noticing that

\[
\left| \frac{|m|}{|\ell| - |m|} [b(k, \ell, m) - b(k, |m| |\ell| / |m|, m)] \right| \lesssim \min \left( \frac{|m|}{|\ell|}, \frac{|m|}{|\ell| - |m|} \right) \lesssim 1,
\]
with estimates for the derivatives matching the assumption \(6.33\) of Lemma \(6.4\), see \(6.9\), so that the same argument above applies.

To estimate the contribution from the last term in \(6.49\) we take advantage of the restriction of \(g_2\) to “not too large” frequencies. Again, it suffices to prove a (slightly stronger) \(L^1\) bound and the \(L^\infty\) bounds follow similarly. We first rewrite

\[
\widetilde{T}_B^{(3)}(g_2, h)(k) = \sum_{N \in \mathbb{Z}} \int_{|\ell| - |m| \geq \epsilon} \varphi_N(k - \ell) h(m) c(k, \ell, m) \frac{1}{|\ell|} \varphi_B(|\ell| - |m|) \, d\ell \, dm, \tag{6.51}
\]

where the symbol, which now involves \(g_2\), is given by

\[
c(k, \ell, m) := [g_2(k - \ell) - g_2(k - |m|\ell/|\ell|)] \delta(k, |m|\ell/|\ell|, m). \tag{6.52}
\]

The key observation is that \(c\) satisfies good symbol bounds, up to the usual small losses, plus losses in terms of the parameter \(X\) defined in \(6.47\). More precisely, using that, with \(\theta := \ell/|\ell|\),

\[
g_2(k - \ell) - g_2(k - |m|\theta) = \int_0^1 \nabla g_2(k - t\ell - (1 - t)|m|\theta) \, dt \cdot \theta (|\ell| - |m|),
\]

we can rewrite

\[
\widetilde{T}_B^{(3)}(g_2, h)(k) = \sum_{N \in \mathbb{Z}} \int_{|\ell| - |m| \geq \epsilon} \varphi_N(k - \ell) h(m) d(k, \ell, m) \frac{1}{|\ell|} \varphi_B(|\ell| - |m|) \, d\ell \, dm, \tag{6.53}
\]

\[
d(k, \ell, m) := \int_0^1 \nabla g_2(k - t\ell + (1 - t)|m|\theta) \, dt \cdot \ell/|\ell| \, \delta(k, |m|\ell/|\ell|, m)
\]

In view of the assumptions on \(b\), see \(6.8\)-\(6.9\) and \(g_2 = P_{\leq X} g\), we can see that

\[
|\nabla^a \nabla^a_{\ell} \nabla^\beta_{m} d(k, \ell, m)| \lesssim 2^{-K|a|} 2^{-|a|L^2 - |\beta| M} \cdot 2^{(|a| + |a| + |\beta|) A} \cdot 2^{(|a| + |a| + |\beta| + 2) X} \|\nabla g\|_{L^2} \tag{6.54}
\]

for \(|a|, |a|, |\beta| \leq 4\). Using \(6.54\) to apply Lemma \(6.4\) to \(6.53\), we obtain

\[
\|T_B^{(3)}(g_2, h)\|_{L^1} \lesssim \sum_{N \in \mathbb{Z}} 2^{15X} \|\nabla g\|_{L^2} \cdot 2^B 2^{12A_2 L} \cdot \|\varphi_N\|_{L^2} \|h\|_{L^2} \\
\lesssim 2^{15X} \cdot 2^B \cdot 2^D \cdot 2^{14A}
\]

having used the assumption \(6.46\), \(\|\varphi_N\|_{L^2} \approx 2^{3N/2}\) and \(N \leq A + 5\). Recalling the definition of \(X\) from \(6.47\), and summing over \(B \leq B_0\), where \(B_0\) is given in \(6.43\), we get

\[
\sum_{B \leq B_0} \|T_B^{(3)}(g_2, h)\|_{L^1} \lesssim 2^D \cdot 2^{14A} \sum_{B \leq B_0} 2^{15X} \cdot 2^B \\
\lesssim 2^{D + 30D'} + 320A \cdot 2^5B_0/6 \\
\lesssim 2^{-D'}.
\]

This bound completes the estimate for the term \(6.43\) provided \(6.46\) holds. If the assumption \(6.10\) is fulfilled with \(\|g\|_{L^2}(\|h\|_{L^2} + \|\partial h\|_{L^2}) \leq 2^{-D}\) instead, we can use a similar
argument exchanging the roles of $g$ and $h$ and using the following replacement of (6.50):

$$
\int_{||\ell|-|m|| \geq \epsilon} \int g(k - \ell) h(m|\ell|/|m|) \frac{|\ell|}{|m|^2} \frac{b(k, \ell, m|\ell|/|m|)}{|\ell| - |m|} \varphi_{\leq B_0}(|\ell| - |m|) \, d\ell \, dm
$$

$$
= \int_{\mathbb{R}^3} g(k - \ell) |\ell| \int_{\mathbb{S}_0^2} h(\theta|\ell|) b(k, \ell, |\ell|\theta) \times \left[ \int_{||\ell| - \rho| \geq \epsilon} \frac{1}{|\ell| - \rho} \varphi_{\leq B_0}(|\ell| - \rho) \, d\rho \right] \, d\ell \, d\theta = 0.
$$

**Estimate of (6.44).** For this term we can use directly Lemma 6.4. Since by assumption $2^{B_0} \lesssim ||\ell| - |m|| \approx 2^B \ll |\ell| \approx 2^L \approx |m| \lesssim 2^A$ on the support of the integral, there are at most $\sim (A + D)$ possible indexes $B$, see (6.43). Using (6.34) we can estimate

$$
\sum_{B_0 \leq B \leq L - 10} \|T_{\epsilon, B}(g, h)\|_{L^p} \lesssim (|A| + |D|) \cdot 2^L \cdot 2^{12A} \cdot \|\hat{g}\|_{L^p} \|\hat{h}\|_{L^q}
$$

$$
\lesssim 2^L \cdot 2^{13A} \cdot \|\hat{g}\|_{L^p} \|\hat{h}\|_{L^q}
$$

having used also $D \lesssim 2^{A/10}$.

**Estimate of (6.45).** To estimate this term we notice that on the support of the integral we have $2^B \approx ||\ell| - |m|| \gtrsim |\ell| \approx 2^L$ so that the kernel is not singular. In particular, we have

$$
T_{\epsilon, B}(g, h) = 2^{-L} 2^{-\max(L, M)} \int_{||\ell|-|m|| \geq \epsilon} g(k - \ell) h(m) c(k, \ell, m) \, d\ell \, dm,
$$

with

$$
c(k, \ell, m) := b(k, \ell, m) \varphi_K(k) \varphi_M(m) \varphi_{\sim L}(\ell) 2^{\max(L, M)} |\ell| - |m| \varphi_{\sim B}(|\ell| - |m|), \quad B \geq L - 10.
$$

(6.56)

We verify that for all $|k| \approx 2^K$, and $|M - L| < 5$

$$
|\nabla_k^a \nabla_{\ell}^\alpha \nabla_m^\beta c(k, \ell, m)| \lesssim 2^{-K|a| - L|\alpha|} 2^{-M|\beta|} 2^{||a| + |\alpha| + |\beta|} A, \quad |a|, |\alpha|, |\beta| \leq 4.
$$

(6.57)

Using Lemma 6.3 we can then estimate, for each fixed $B \in [L - 10, L + 10]$

$$
\|T_{\epsilon, B}(g, h)\|_{L^r} \lesssim 2^{-2\max(L, M)} \cdot \|\hat{g}\|_{L^p} \|\varphi_{\sim M} h\|_{L^n} \cdot 2^{3\max(L, M)} \cdot 2^{12A}.
$$

(6.58)

which gives the desired (6.35).

**Estimate of $T_{\nu_L}$.** We now want to estimate by the right hand-side of (6.12) the term $T_{\nu_L}$, see (6.37). Since $\nu_L$ satisfies (5.13)–(5.14), we can write $T_{\nu_L}[b](g, h)$ as a finite sum of terms of the form

$$
\sum_{J \in \mathcal{J}} T_J(g, h)[b](k),
$$

$$
\hat{F}(T_J(g, h)[b])(k) := \int_{\mathbb{R}^3 \times \mathbb{R}^3} g(k - \ell) h(m) b(k, \ell, m) \frac{1}{|\ell|} b_J(\ell, m) \cdot 2^J K(2^J(|\ell| - |m|)) \, d\ell \, dm
$$

(6.59)
where $K$ is a Schwartz function and, for all $|\ell| \approx 2^L$ and $|m| \approx 2^M$, the symbols satisfy

$$|\nabla_\ell \nabla_m^\beta b_j(\ell, m)| \lesssim 2^{-|\alpha|L} 2^{-|\beta|M} \cdot 2^{(|\alpha|+|\beta|)A} \cdot C_J, \quad |\alpha|, |\beta| \leq 5,$$

(6.60)

For each term $T_J$ in (6.59) we can apply the same arguments used to estimate $T_{p.v.}$, see (6.36)-(6.37), based on Lemma 6.4 and, in view of the bounds on the symbols in (6.60), we can obtain

$$\|T_J(g, h)[b]\|_{L^r} \lesssim 2^L \cdot \|\hat{g}\|_{L^p} \|\hat{h}\|_{L^q} \cdot 2^{12A} \cdot C_J.$$

Summing over $J$ using (6.60) we obtain (6.12).

**Estimate of $T_{\nu_R}$**. Recall the notation (6.36)-(6.37) and the definition and properties of $\nu_R$ in Proposition 5.1. In both cases $|L - M| < 5$ and $|L - M| \geq 5$, we use (5.15) and (5.16) respectively, to write

$$\hat{F}(T_{\nu_R}[b](g, h))(k) = 2^{-2\max(L, M)} \int_{\mathbb{R}^3 \times \mathbb{R}^3} g(k - \ell) h(m) b(k, \ell, m) d(\ell, m) \, d\ell dm \quad \text{(6.61)}$$

with

$$|\nabla_\ell \nabla_m^\beta d(\ell, m)| \lesssim 2^{-|\alpha|L} 2^{-|\beta|M} \cdot 2^{(|\alpha|+|\beta|+2)A} \cdot 2^A, \quad |\alpha|, |\beta| \leq 4,$$

for $|\ell| \approx 2^L$ and $|m| \approx 2^M$. Then, the bilinear term in (6.61) is similar to the one in (6.35)-(6.37), up to the different power of $2^A$. From the same argument above, using Lemma 6.3 it then follows that

$$\|T_{\nu_R}[b](g, h)\|_{L^r} \lesssim \|\hat{g}\|_{L^p} \|\hat{h}\|_{L^q} \cdot 2^{\max(L, M)} \cdot 2^{60A} \quad \text{(6.62)}$$

which gives (6.12).

**Estimate of $T_2$.** To prove that the operator $T_2$ defined in (6.7) also satisfies the bound (6.12), one can use a similar proof to the one above done for $T_1$. The main ingredient needed is a version of Lemma 6.4 and the main bound (6.34) adapted to the kernel in the operator $T_2$. This can be seen by duality. More precisely, in analogy with the notation for the operator $B_j[b](g, h)(x)$ in (6.31) we can define

$$B_{j,2}[b](g, h)(x) = \mathcal{F}_{k \rightarrow x}^{-1} \int_{\mathbb{R}^3 \times \mathbb{R}^3} g(-m - \ell) h(m) b(k, \ell, m) \chi(2^j(|\ell| - |k|)) \, d\ell dm. \quad \text{(6.63)}$$

Using Plancharel (omitting the irrelevant factors of $\pi$), we have the following identities

$$\langle B_{j,2}[b](g, h), f \rangle_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3 \times \mathbb{R}^3} g(-m - \ell) h(m) b(k, \ell, m) \chi(2^j(|\ell| - |k|)) \, d\ell dm \, \overline{\hat{f}(k)} \, dk$$

$$= \int_{\mathbb{R}^3} \left[ \int_{\mathbb{R}^3 \times \mathbb{R}^3} g(k - \ell) \overline{\hat{f}(-m)} b(m, \ell, -k) \chi(2^j(|\ell| - |m|)) \, d\ell dm \right] h(-k) \, dk$$

$$= \langle \hat{F} B_j[b'][\overline{\hat{g}}, \overline{\hat{f}}], h(-\cdot) \rangle_{L^2} = \langle B_j[b'][\overline{\hat{g}}, \overline{\hat{f}}], \overline{\hat{h}} \rangle_{L^2} \quad \text{(6.64)}$$

where, according to the notation (6.31), $b'(k, \ell, m) := \overline{b(m, \ell, -k)}$. Then, assumptions analogous to (6.32)-(6.33) hold for $b'$. From (6.34) it follows that for any $1/r = 1/p + 1/q$,
$g \in L^p, h \in L^q$ and $f \in L^{r'}$

\[
\left| \langle B_{j,2}[b](g,h), f \rangle_{L^2(\mathbb{R}^4)} \right| \lesssim \|h\|_{L^r} \|B_{j}[b'](\mathcal{T})\|_{L^{r'}} \\
\lesssim \|h\|_{L^r} \cdot 2^{-j} 2^{L/2} 2^A \|\mathcal{T}\|_{L^p} \|f\|_{L^{r'}}.
\]

This gives the desired bound by the right-hand side of (6.34) for the operator $B_{j,2}$.

**Proof of (6.16).** To conclude we show how to estimate the operators (6.15), where the “good vectorfield” (6.14) acts on the measure $\nu_1$. By the statement of Proposition 5.1 and since $X_{\ell,m} f(|\ell| - |m|) = 0$ for any $f$, an application of $X^a$ to $\nu_1$ gives:

\[
X^a \nu_1(\ell, m) = X^a \nu_0(\ell, m) + X^a \nu_L(\ell, m) + X^a \nu_R(\ell, m)
= \nu_{0,a}(\ell, m) + \nu_{L,a}(\ell, m) + \nu_{R,a}(\ell, m),
\]

where:

1. The leading order has the form

\[
\nu_{0,a}(\ell, m) = \frac{b_{0,a}(\ell, m)}{|\ell|} \left[ \delta(|\ell| - |m|) + \text{p.v.} \frac{1}{|\ell| - |m|} \right]
\]

with

\[
| \nabla_\ell^a \nabla_m^\beta (\varphi_L(\ell) \varphi_M(m) b_{0,a}(\ell, m)) | \lesssim 2^{-a \min(L,M)} \cdot 2^{-|a|L} 2^{-|\beta|M} \cdot 2^{(|a| + |\beta| + a)A} \cdot 1_{\{L-M < 5\}}
\]

for all $L, M \leq A, |a| + |\beta| + a \leq N_1$.

2. $\nu_{L,a}(\ell, m)$ can be written as

\[
\nu_{L,a}(\ell, m) = \frac{1}{|\ell|} \sum_{i=1}^{N_2} \sum_{J \in \mathcal{J}} b_{L,i}(\ell, m) \cdot 2^i K_i \left( 2^i (|\ell| - |m|) \right)
\]

with $K_i \in \mathcal{S}$ and

\[
\sum_{J \in \mathcal{J}} | \nabla_\ell^a \nabla_m^\beta (\varphi_L(\ell) \varphi_M(m) b_{L,i}(\ell, m)) | \lesssim 2^{-a \min(L,M)} 2^{-|\alpha|L} 2^{-|\beta|M} \cdot 2^{(|\alpha| + |\beta| + a)A} \cdot 1_{\{L-M < 5\}},
\]

for all $L, M \leq A, |\alpha| + |\beta| + a \leq N_1 - N_2$.

3. The remainder term satisfies

\[
| \nabla_\ell^a \nabla_m^\beta (\varphi_L(\ell) \varphi_M(m) \nu_{R,a}(\ell, m)) | \lesssim 2^{-a \min(L,M)} \cdot 2^{-2 \max(L,M)} \cdot 2^{-|\alpha|L} 2^{-|\beta|M} \cdot 2^{(|\alpha| + |\beta| + 2a)5A} 2^A
\]

for all $L, M \leq A$ and $|\alpha| + |\beta| + a \leq N_2/2 - 3$. Note how we do not use here the improvement of (6.15) given by (6.16) here.

Therefore the components of $X^a \nu_1$ in (6.65) have the same structure of the components of $\nu_1$, where the bounds (6.66)-(6.68) on the coefficients are like the bounds in the case of $\nu_1$ times a factor of $2^{-a \min(L,M)}$. Then, the same proof used to obtain (6.12) can be applied and (6.16) follows.
6.4. Proof of Theorem 6.2. Proof of (i). To prove (6.20) it suffices to examine the structure and bounds satisfied by $\nabla^{a_1}_\ell \nabla^{a_2}_m \nu_1(\ell, m) \chi_{-}(\ell, m)$. First note that when we differentiate the cutoff the behavior is

$$\nabla^{b_1}_\ell \nabla^{b_2}_m \chi_{-}(\ell, m) \approx 2^{-(b_1+b_2) \max(L,M)} y^{a, b}_\max(L,M)-10(|\ell| - |m|).$$

The bilinear terms corresponding to these symbols are easier to treat than those where the derivatives hit $\nu_1$, so we concentrate on these latter ones. According to the decomposition (5.10) from Proposition 5.1 we write

$$2^{-2A_2([\alpha]-1)M} \nabla^{a_1}_\ell \nabla^{a_2}_m \nu_1(\ell, m) \chi_{-}(\ell, m) = \left[ \nu^0_\ell(\ell, m) + \nu^a_\ell(\ell, m) + \nu^a_2(\ell, m) \right] \chi_{-}(\ell, m)$$

with

$$\nu^0_\ell(\ell, m) := 2^{-2A_2([\alpha]-1)M} \nabla^{a_1}_\ell \nabla^{a_2}_m \left( \frac{b_0(\ell, m)}{|\ell|} \frac{1}{|\ell| - |m|} \right) 1_{\left\{ \ell - M < 5 \right\}},$$

$$\nu^a_\ell(\ell, m) := 2^{-2A_2([\alpha]-1)M} \nabla^{a_1}_\ell \nabla^{a_2}_m \nu_L(\ell, m) 1_{\left\{ \ell - M < 5 \right\}},$$

$$\nu^a_2(\ell, m) := 2^{-2A_2([\alpha]-1)M} \nabla^{a_1}_\ell \nabla^{a_2}_m \nu_R(\ell, m).$$

It will suffice to show that the terms in (6.70) satisfy

$$|\nabla^a_\ell \nabla^\beta_m \nu^a_\ell(m)| \lesssim 2^{-3 \max(L,M)} \cdot 2^{-[\alpha]L} 2^{-\max([\alpha]+|\beta|+4)}, \quad \ast \in \{0, 1, 2\},$$

and then apply Lemma 6.3 to get the desired bound (6.20).

To verify (6.71) for $\ast = 0$, we recall the estimates (5.12) for the coefficient $b_0$, which in particular imply

$$|\nabla^a_\ell \nabla^\beta_m b_0(\ell, m)| \lesssim 2^{-[\alpha]L} 2^{-\max(|\alpha| + |\beta|)}. $$

Moreover, since on the support of $\nu^0_\ell$ we have $||\ell| - |m|| \approx \max(|m|, |\ell|)$, it follows that

$$|\nabla^a_\ell \nabla^\beta_m \frac{1}{|\ell|} \frac{1}{|\ell| - |m|}| \lesssim 2^{-L} 2^{-\max(L,M)} \cdot 2^{-[\alpha]L} 2^{-\max(L,M)} 2^{-|\beta|}. $$

Combining the last two inequalities, the estimate (6.71) for $\nu^0_\ell$ follows.

The case of $\nu_\ell^a$ can be treated similarly by differentiating the formula for $\nu_L$ in (5.13), using the symbol bounds (5.14) in the form used just above, and the fact that for any Schwartz function $K$ we have, on the support of $\chi_{-}(\ell, m)$,

$$|\nabla^a_\ell \nabla^\beta_m 2^J K(2^J(|\ell| - |m|))| \lesssim 2^{-\max(L,M)} \cdot 2^{-[\alpha]L} 2^{-|\beta|}. $$

The bound (6.71) for $\nu_\ell^a$ follows from the estimates (5.15) and (5.16) for $\nu_R$.

Proof of (ii). The proof of (6.23) can be done similarly to part (i) above, so we skip it.

Proof of (iii). Our aim is to prove that $Y^a \nu_1(\ell, k)$ has a similar structure and enjoys similar bilinear estimates as those satisfied by $\nu_1$ multiplied by a factor of $2^{12A_2([\alpha]-1)K} 2^{-\max(K,L)}$. This will then imply the desired bilinear bound (6.26) by means of an application of the estimate (6.13) to $2^{-12A_2([\alpha]-1)K} 2^{\max(K,L)} Y^a \nu_1$: compare the right-hand sides of (6.12) and (6.26).

To prove the desired property we again look at the decomposition of $\nu_1$ in Proposition 5.1. The point of using the vectorfield $Y$ is that for any function $f$ we have $Y f(|\ell| - |k|) = 0$. In particular we have, for $|a| = 1, 2$,

$$Y^a \nu_0(\ell, k) = Y^a \left[ \frac{b_0(\ell, k)}{|\ell|} \right] \left[ i \pi \delta(|\ell| - |k|) + \text{p.v.} \frac{1}{|\ell| - |k|} \right].$$

(6.72)
We now want to argue that the differentiated coefficient $Y^a b_0(\ell, k)$ behave like $b_0$ up to the correct factor. In view of the estimates (5.12), and treating $Y^a$ as a regular $\nabla^a_\ell \nabla^a_k$-derivative, we have

$$|Y^a b_0(\ell, k)| \lesssim \sup_{a_1 + a_2 = |a|} 2^{-a_1 L - \ell} (2^{a_1 K_+} + 2^{(a_2 - 1)K_-})$$

so that (recall that $|K - L| < 5$) we have $|Y b_0(\ell, k)| \lesssim 2^{-L/2} A$, and for $|a| = 2$

$$|Y^a b_0(\ell, k)| \lesssim 2^{-2L/2} 2^A + 2^{-K} \lesssim 2^{-2 \max(L, K) 2^2 A}.$$

These bounds are consistent with the desired factor of $2^{12A} 2^{-(a-1)K} 2^{-\max(L, K)}$.

The term $Y^a \nu_L(\ell, k)$, where $\nu_L$ is given by (5.13)–(5.14), can be treated similarly to show that, up to the proper factor, $Y^a \nu_L(\ell, k)$ has the same structure of $\nu_L$.

For the remainder term $\nu_R$ we treat the application of $Y$ as a general derivative $\nabla(\ell, k)$, and distinguish two cases. When $|L - K| < 5$, (5.15) gives immediately that the $\nabla^a_\ell \nabla^a_k$-derivatives of $\nu_R(\ell, k)$ behave like those of $\nu_R$ times a factor of $2^{-a_1 L} 2^{-a_2 K} 2^{10A}$ which is more than sufficient when applying Lemma 6.3 to bound the associated bilinear operator.

When $|K - L| \geq 5$ we use instead (5.16). Writing $Y^a$ as a combination of $\nabla^a_\ell \nabla^a_k$ gives the bound

$$|\nabla^a_\ell \nabla^a_k Y^a \nu_R| \lesssim 2^{-2 \max(K, L)} \cdot 2^{-|a| \max(L, K)} 2^{-|\beta| K} \cdot 2^{(|a| + |\beta| + 5)5A} \times \sup_{a_1 + a_2 = a} 2^{-a_1 \max(L, K)} \max(0, 2^{-(a_2 - 1)K_-}) 2^{5A}.$$

The first line gives bounds of a regular symbol type times $2^{-2 \max(K, L)}$; the second line is a factor that, in the worst case $a_1 = 0, a_2 = 2$, is bounded by $2^{-K} 2^{10A}$. This is consistent with what we want and completes the proof of (6.26). 

6.5. Proof of Lemma 6.4

In view of the assumption (6.32) we have

$$B_j[b](g, h)(x) = \hat{F}_{k \to x}^{-1} \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} g(\ell - k) h(m) \varphi_{\sim L}(\ell) \varphi_{\sim L}(m) b(k, \ell, m) \times \chi(2^{i(|\ell| - |m|)}) d\ell dm,$$

having used $|\ell| \approx 2^L \approx |m| \gg 2^{-j}$ to insert the cutoffs for $\ell$ and $m$. Upon taking the inverse Fourier transform (disregarding factor of $2\pi$) we may rewrite (6.73) as

$$B(x) = \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} e^{-i(z - x) \cdot k} H(z, k) \hat{g}(z) d z d k,$$

where (we omit the dependence on $L$)

$$H(z, k) := \int_{\mathbb{R}^3} A_j(z, y, k) \hat{h}(y) dy$$

$$A_j(z, y, k) := \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} e^{i z \cdot \ell} e^{i y \cdot m} b(k, \ell, m) \varphi_{\sim L}(\ell) \varphi_{\sim L}(m) \chi(2^{i(|\ell| - |m|)}) d \ell d m.$$
As a first step we integrate by parts in $k$ in the first formula of (6.74):

$$B(x) = \int \int \frac{1}{(1 + 2^{2K}|x - z|^2)^2} (1 - 2^{2K} \Delta_k)^2 e^{-i(z-k) \cdot x} \varphi_k(k) H(z, k) \widehat{g}(z) \, dz \, dk$$

and deduce, using Hölder’s and Haussdorf-Young’s inequalities, that

$$\|B(x)\|_{L^p} \lesssim \int \int \frac{2^{3K}}{(1 + 2^{2K}|x - z|^2)^2} \sup_k (1 - 2^{2K} \Delta_k)^2 (\varphi_k(k) H(z, k)) \|\widehat{g}(z)\|_{L^2}$$

$$\lesssim \sup_k (1 - 2^{2K} \Delta_k)^2 (\varphi_k(k) H(z, k)) \|\widehat{g}\|_{L^p}.$$  

From the definition (6.75) we see that it suffices to prove that

$$\left\| \sup_k A'_j(z, y, k) \left| \widehat{h}(y) \right| \right\|_{L^{q}_y} \lesssim 2^{2L} \cdot 2^{-j} \cdot 2^{8A} \cdot \|\widehat{h}\|_{L^q}$$  

where $A'_j$ is defined in the same way as $A_j$ in (6.75) but with the symbol

$$b'(k, \ell, m) := (1 - 2^{2K} \Delta_k)^2 (\varphi_k(k) b(k, \ell, m))$$

instead of $b$:

$$A'_j(z, y, k) := \int \int e^{iz \cdot \ell} e^{iy \cdot m} b'(k, \ell, m) \varphi_{\sim L}(\ell) \varphi_{\sim L}(m) \chi(2^j(|\ell| - |m|)) \, d\ell \, dm.$$  

To estimate this kernel we observe that for fixed $m$, the integral over $\ell$ is supported on an annular region $C_{j, m} = \{ \ell : ||\ell| - |m|| \lesssim 2^{-j} \}$, which has volume $\approx 2^{2L} 2^{-j}$. We then pick points $p_r, r = 1, \ldots, 2^{2(j+L)}$, uniformly distributed on the sphere of radius $|m|$, of the form $p_r = R_r m$ for suitable rotations $R_r$, with $R_1 = \text{id}$. We cover the annular region by $2^{2(j+L)}$ balls or radius $\approx 2^{-j}$ centered at the points $p_r$, and, with a partition of unity, write

$$1_{C_{j, m}} = \sum_{r=1}^{2^{2(j+L)}} \chi_r(2^j(\ell - p_r)), \quad p_r = R_r m,$$  

for smooth compactly supported radial functions $\chi_r$. We then write

$$A'_j(z, y, k) = \sum_{r=1}^{2^{2(j+L)}} A_{j, r}(z, y, k).$$  

$$A_{j, r}(z, y, k) := \int \int e^{iz \cdot \ell} e^{iy \cdot m} \varphi_{\sim L}(\ell) \varphi_{\sim L}(m) b'(k, \ell, m) \chi(2^j(|\ell| - |m|)) \chi_r(2^j(\ell - p_r)) \, d\ell \, dm.$$
We bound the term with $r = 1$, that is $p_r = m$. After changing variables $\ell$ to $\ell + m$, we write

$$
(1 + 2^{2L}|z + y|^2)|A_{j,1}(z, y, k)| = \left| \int_{\mathbb{R}^3} e^{ix \cdot \ell} (1 - 2^{2L}\Delta_m)^2 e^{i(z-y) \cdot m} \varphi_{\sim L}(\ell + m) \varphi_{\sim L}(m) \times b'(k, \ell + m, m) \chi(2^j(|\ell + m| - |m|)) \chi_r(2^j \ell) \, d\ell dm \right|
$$

$$
\lesssim \int_{\mathbb{R}^3} \left| (1 - 2^{2L}\Delta_m)^2 [\varphi_{\sim L}(\ell + m) \varphi_{\sim L}(m) b'(k, \ell + m, m) \times \chi(2^j(|\ell + m| - |m|))] \chi_r(2^j \ell) \, d\ell dm. \right|
$$

Using the hypothesis (6.33) we see that, on the support of the integral,

$$
|\partial_m^a [\varphi_{\sim L}(\ell + m) \varphi_{\sim L}(m)]| \lesssim 2^{-|a|},
$$

$$
|\partial_m^a \chi(2^j(|\ell + m| - |m|))| \lesssim 2^{-|a|},
$$

$$
|\partial_m^a b'(k, \ell + m, \ell)| \lesssim 2^{-|a|} 2^{|a|}, \quad |a| \leq 4.
$$

Thus, we obtain

$$
|A_{j,1}(z, y, k)| \lesssim \frac{2^{3L}}{(1 + 2^{2L}|z + y|^2)^2} \cdot 2^{-3j} \cdot 2^{8A}. \quad (6.79)
$$

Note that by changing variables $\ell \mapsto R_r \ell$ in the integral in (6.78), the same argument above shows that, for any $r = 1, \ldots, 2^{2(j+L)}$,

$$
|A_{j,r}(z, y, k)| \lesssim \frac{2^{3L}}{(1 + 2^{2L}|z + R_r y|^2)^2} \cdot 2^{-3j} \cdot 2^{8A}. \quad (6.80)
$$

Going back to the left-hand side of (6.76) and using (6.78) and (6.80) we can estimate

$$
\left\| \int_{\mathbb{R}^3} \sup_k |A_j'(z, y, k)| |\hat{h}(y)| \, dy \right\|_{L^q_x}
$$

$$
\lesssim 2^{2(j+L)} \left\| \int_{\mathbb{R}^3} \sup_k \sup_{r=1, \ldots, 2^{2(j+L)}} |A_{j,r}(z, y, k)| |\hat{h}(y)| \, dy \right\|_{L^q_x}
$$

$$
\lesssim 2^{2(j+L)} \cdot 2^{-3j} \cdot 2^{8A} \sup_{r=1, \ldots, 2^{2(j+L)}} \left\| \int_{\mathbb{R}^3} \frac{2^{3L}}{(1 + 2^{2L}|z + y|^2)^2} |\hat{h}(R_r^{-1} y)| \, dy \right\|_{L^q_y}
$$

$$
\lesssim 2^{2L} \cdot 2^{-j} \cdot 2^{8A} \cdot \left\| \hat{h} \right\|_{L^q_y}. \quad \square
$$

This gives (6.76) and concludes the proof.

**Proof of Lemma 6.3.** We insert frequency localization according to (6.28) and the notation after (2.20), and write

$$
B[b](g, h)(x) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} A(x, y, z) \hat{g}(y) \hat{h}(z) \, dydz \quad (6.81)
$$

$$
A(x, y, z) := \int_{\mathbb{R}^3} e^{ik \cdot (x-y)} e^{iz \cdot \ell} e^{iy \cdot m} b(k, \ell, m) \varphi_{\sim K}(k) \varphi_{\sim L}(\ell) \varphi_{\sim M}(m) \, dk d\ell dm.
$$
Assume without loss of generality that \( L \geq M \). changing variables \( \ell \mapsto \ell + m \) and integrating by parts in \( k \) and \( m \) we get
\[
(1 + 2^{2K} |x - y|^2) (1 + 2^{2M} |y + z|^2) |A(x, y, z)| \leq \left| \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} \left( 1 - \frac{2^{2K} \Delta_k}{2^{2M} \Delta_m} \right)^2 \left( 1 - \frac{2^{2M} \Delta_m}{2^{2M} \Delta_m} \right)^2 b(k, \ell + m, m) \times \varphi_{\sim K}(k) \varphi_{\sim L}(\ell + m) \varphi_{\sim M}(m) \right| dk d\ell dm \leq 2^{8A} \cdot 2^{3(K + L + M)},
\]
having used the estimates (6.29) for the symbol. The conclusion follows from (6.81) and the Hausdorff-Young inequality. \( \square \)

7. Weighted estimates for leading order terms

Recall Duhamel's formula (5.7) according to the decomposition (5.1)- (5.6). Our main aim in this section is to estimate the weighted norms of the leading order term \( D_1 \) as written in (5.3).

For convenience we first localize the integrals dyadically in time \( s \approx 2^S \) by introducing a partition of \( [0, t] \) associated to cutoff functions \( \tau_S \), \( S = 0, \ldots \log_2 t \), with \( \int_0^t |\tau'_S(s)| ds \lesssim 1 \).

More precisely we write
\[
D_1(t)(f, f) = \sum_{S=0}^{\log_2 t} N_{1,S}(t)(f, f) + 2N_{2,S}(t)(f, f),
\]
\[
N_{1,S}(t)(f, f) := \int_0^t \int \int e^{is(|\ell|^2 + 2k|\ell| + |m|^2)} \tilde{f}(s, \ell + k) \tilde{f}(s, m) \nu_1(\ell, m) d\ell dm \tau_S(s) ds,
\]
\[
N_{2,S}(t)(f, f) = \int_0^t \int \int e^{is(-|\ell|^2 + |\ell|^2 - 2\ell \cdot m + |m|^2)} \tilde{f}(s, m - \ell) \tilde{f}(s, m) \nu_1(\ell, k) d\ell dm \tau_S(s) ds.
\]

(7.1)

For lighter notation, we will often omit the dependence on \( S \) in what follows; see for example (7.3) where we omit the dependence on \( S \) of the terms \( N_{L,M} \).

In view of Proposition 4.4 see (4.15)–(4.16), we assume that on the support of the integral in (7.1) we have
\[
|\ell| + |m| + |k| \lesssim 2^A := 2^{\delta N_S}, \quad \delta_N = \frac{3}{N - 5}.
\]

(7.2)

This is our main Proposition:

**Proposition 7.1.** With the definitions (7.1)-(7.2), under the a priori assumptions (2.23), we have
\[
\|\partial_k N_{1,S}(t)(f, f)\|_{L^2} \lesssim \varepsilon^2 2^{-\delta'S},
\]
for some \( \delta' > 0 \), and
\[
\|\partial^2_k N_{1,S}(t)(f, f)\|_{L^2} \lesssim 2^{(1/2 + \delta) S} \varepsilon^2.
\]

(7.3)
The estimate (7.3) is proven in Subsection 7.1 while (7.4) is proven in Subsection 7.2. In Subsection 7.3 we discuss how to obtain the same bounds for \( N_{2,s} \), and therefore conclude the desired bootstrap estimate for \( D_1 \) upon summing over \( S \).

7.1. **Proof of (7.3).** In order to apply Theorem 6.1 we begin by localizing to \(|\ell| \approx 2^\ell \) and \(|m| \approx 2^M \), and write

\[
N_{1,s}(f, f) = \sum_{L, M \in (-\infty, A] \cap \mathbb{Z}} N_{L,M}(f, f),
\]

\[
N_{L,M}(g, h) := \int_0^t \int \int e^{i\Phi(k,\ell,m)} \bar{g}(s, \ell + k) \bar{h}(s, m) \varphi_L(\ell, m) \varphi_M(m) \nu_1(\ell, m) \, dl \, dm \, \tau_S(s) \, ds,
\]

(7.5)

\[
\Phi(k, \ell, m) := |\ell|^2 + 2k \cdot \ell + |m|^2.
\]

Taking \( \partial_k \) we get

\[
\partial_k N_{L,M}(f, f) = M_{1,L,M}(f, f) + 2M_{2,L,M}(f, f),
\]

(7.6)

where

\[
M_{1,L,M}(f, f) = \int_0^t \int \int e^{i\Phi(k,\ell,m)} \partial_k \bar{f}(s, \ell + k) \bar{f}(s, m) \varphi_L(\ell, m) \varphi_M(m) \nu_1(\ell, m) \, dl \, dm \, \tau_S(s) \, ds,
\]

(7.7)

\[
M_{2,L,M}(f, f) = \int_0^t \int \int is\ell e^{i\Phi(k,\ell,m)} \bar{f}(s, \ell + k) \bar{f}(s, m) \varphi_L(\ell, m) \varphi_M(m) \nu_1(\ell, m) \, dl \, dm \, \tau_S(s) \, ds.
\]

(7.8)

We proceed to estimate the terms above with the understanding that all the bounds will need to be summed over \( L, M \in (-\infty, A] \cap \mathbb{Z} \).

7.1.1. **Estimate of (7.7).** This term can be treated almost directly by applying the bilinear estimate (6.12) from Theorem 6.1. Define the distorted Littlewood-Paley projection \( g_M := \tilde{F}^{-1} \varphi_M(m) \bar{g} \), and let

\[
b = \varphi_\sim(\ell) \varphi_\sim M(m),
\]

(recall the notation for \( \varphi_\sim \) after (2.20)). Using the notation in (6.6), we can write

\[
M_{1,L,M}(f, f)(t) = \int_0^t e^{-is|k|^2} \tilde{F}_{x-k} T_1[b] \left( e^{is|k|^2} \partial_k \bar{f}(s), \bar{F} u_M(s) \right) \tau_S(s) \, ds.
\]

(7.9)

In view of the apriori assumptions (2.23) we have \( \| \partial_k \bar{f} \|_{L^2} \| \bar{f} \|_{L^2} \lesssim \varepsilon^2 \), so that, choosing \( 2^D = \langle s \rangle^2 \approx 2^{2s} \), the hypothesis (6.10) is verified. Let us assume that \( M \leq L \). An application of (6.12) then gives, for arbitrary large \( q \),

\[
\| M_{1,L,M}(f, f) \|_{L^2} \lesssim \int_0^t \| T_1[b] \left( e^{is|k|^2} \partial_k \bar{f}(s), \bar{F} u_M(s) \right) \|_{L^2} \tau_S(s) \, ds
\]

\[
\lesssim \int_0^t \left[ \| \tilde{F} \partial_k \bar{f}(s) \|_{L^2} \cdot 2^{(0+)^M} \| u(s) \|_{L^{q^-}} \cdot 2^{\max(L, M)} \cdot 2C_\varepsilon A + \langle s \rangle^{-2} \varepsilon^2 \right] \tau_S(s) \, ds.
\]

(7.10)

For the second inequality we have used

\[
\| \tilde{F} \bar{F} u_M(s) \|_{L^{q^-}} \lesssim 2^{(0+)^M} \| \tilde{F} \bar{F} u(s) \|_{L^{q^-}},
\]
which follows from Bernstein’s inequality (for flat Littlewood-Paley projections) and the boundedness of wave operators. If we have $L \leq M$, a similar $L^2 \times L^q$ application of (6.12) would give the same conclusion (7.10) with a factor of $2^{(0+)}L$ instead of $2^{(0+)}M$.

Using that $2^A \approx 2^\delta \nu_1$ and the a priori dispersive estimate (4.8), we obtain

$$\sum_{L,M \leq A} \|M_{1,L,M}(t)\|_{L^2} \lesssim \sum_{L,M \leq A} \int_0^t \varepsilon \cdot \varepsilon(s)^{-5/4+\delta} \cdot 2^{(0+)} \min(L,M)^{(C_0+1)\delta_N} \tau_s(s)ds + \varepsilon^2 2^{-S}$$

provided

$$\delta + (C_0 + 1)\delta_N \leq 1/4 - \delta';$$ (7.11)

this last inequality holds with our choice of $\delta_N$ in (7.2), $N$ in (1.8), and $C_0 = 65$.

7.1.2. Estimate of (7.8): Set-up. This is the main term in the estimate of the first weight (7.3). Here we need to exploit the oscillations given by $e^{is\Phi}$, by integrating by parts in the distorted frequency space and in time. Due to the singularity of $\nu_1$ we can only integrate by parts in the direction of the “good vectorfield”

$$X := \partial_{|\ell|} + \partial_{|m|}. \quad (7.12)$$

In order to be able to exploit the oscillations efficiently, we first split (7.8) into a region where $||\ell| - |m|| \ll |\ell|$ and the complement one, by defining:

$$\chi_+(\ell, m) := \varphi_{-10} \left( \frac{|m| - |\ell|}{|m| + |\ell|} \right), \quad \chi_-(\ell, m) := \varphi_{-10} \left( \frac{|m| - |\ell|}{|m| + |\ell|} \right) = 1 - \chi_+(\ell, m), \quad (7.13)$$

and the corresponding terms

$$M_2^- := \int_0^t \int \int is\ell e^{is\Phi} \tilde{f}(s, \ell + k) \tilde{f}(s, m) \varphi_L(\ell) \varphi_L(m) \chi_-(\ell, m) \nu_1(\ell, m) d\ell dm \tau_s(s)ds,$$

$$M_2^+ := \int_0^t \int \int is\ell e^{is\Phi} \tilde{f}(s, \ell + k) \tilde{f}(s, m) \varphi_L(\ell) \varphi_M(m) \chi_+(\ell, m) \nu_1(\ell, m) d\ell dm \tau_s(s)ds. \quad (7.14)$$

Notice how we have dropped the dependence on the indexes $L, M$ for brevity, and replaced $\varphi_M(m)$ with $\varphi_{L}(m)$ since on the support of $\chi_-, |m|$ and $|\ell|$ are comparable.

For (7.3) it will suffice to prove the following bounds:

$$\sum_{L \leq A} \|M_2^-(t)\|_{L^2} \lesssim \varepsilon^2 2^{-S\delta'}, \quad (7.16)$$

$$\sum_{L, M \leq A} \|M_2^+(t)\|_{L^2} \lesssim \varepsilon^2 2^{-S\delta'}. \quad (7.17)$$
7.1.3. Proof of \((7.16)\). We calculate
\[
X\Phi := 2k \cdot \frac{\ell}{|\ell|} + 2|\ell| + 2|m|, \quad \Phi(k, \ell, m) = |\ell|^2 + 2k \cdot \ell + |m|^2
\] (7.18)
and see that the following identity holds:
\[
|\ell|X\Phi(k, \ell, m) - \Phi(k, \ell, m) = |\ell|^2 + 2|m||\ell| - |m|^2
= |m|^2 + |\ell|^2 - 2|m|(|m| - |\ell|) =: c(\ell, m).
\] (7.19)
This implies
\[
\frac{1}{c(\ell, m)} \left( \frac{1}{i\ell} |\ell|X + i\partial_s \right) e^{is\Phi(k,\ell,m)} = e^{is\Phi(k,\ell,m)},
\]
which we can use to integrate by parts in \((7.14)\). Note that, since \(||\ell| - |m|| \ll |\ell| \approx |m|\), then \(1/c(\ell, m)\) behaves like a multiplier of the form \(1/(|m|^2 + |\ell|^2)\): for all \(\alpha, \beta \in \mathbb{Z}^3\)
\[
|\nabla_\ell^\alpha \nabla_m^\beta \frac{1}{c(\ell, m)}| \lesssim \frac{1}{|\ell|^2 + |m|^2} |\ell|^{-|\alpha||m|^{-|\beta|}}.
\] (7.21)
Also, we can freely insert a cutoff in the size of \(|m| \approx 2^L\) in \((7.14)\). We then have to estimate the \(L^2\)-norm of the term
\[
\mathcal{M}_L(f, g)(t) := \int_0^t \int \frac{d\ell}{\ell} e^{is\Phi} \tilde{f}(s, \ell + k) \tilde{f}(s, m) a_L(\ell, m) \nu_1(\ell, m) \, df \, dm \tau_S(s) \, ds,
\]
\[
a_L(\ell, m) := \varphi_L(m) \varphi_L(\ell) \chi_- (\ell, m).
\] (7.22)

- The case \(L \leq L_0 := -S/3\). First, we show that when \(L\) is sufficiently small we can obtain \((7.16)\) directly using the bilinear estimates \((6.12)\) of Theorem \(6.1\) with
\[
b = 2^{-L} \varphi_{\sim L}(m) \varphi_{\sim L}(\ell) \chi_-(\ell, m)
\]
and using the a priori decay estimates \((4.17)\), we have
\[
\|\mathcal{M}_L(t)\|_{L^2} \lesssim \int_0^t \int s 2^{L} \|T_1[\nu](\tilde{F}u(s), \xi_F u(s))\|_{L^2} \tau_S(s) \, ds
\]
\[
\lesssim \int_0^t \left[ \|\tilde{F}u(s)\|_{L^3} \|\xi_F u(s)\|_{L^{6_1}} \cdot 2^L \cdot 2^{C_0 \alpha} + \langle s \rangle^{-\frac{3}{2}} \right] 2^{L_0} 2^S \tau_S(s) \, ds.
\]
\[
\lesssim \int_0^t \varepsilon 2^{-S/2} \cdot \varepsilon 2^{-(1-\delta) S} \cdot 2^{2L} \cdot 2^{C_0 \alpha} \cdot 2^S \tau_S(s) \, ds + \varepsilon 2^{-S}
\]
\[
\lesssim \varepsilon 2^{(1/2 + (C_0 + \delta_N) \delta_N) S} \cdot 2^{2L} + \varepsilon 2^{-S}
\]
Since \(L \leq L_0 := -S/3\), and our choice of parameters guarantees \((C_0 + 1) \delta_N + \delta' \leq 1/6\) (see \((4.16)\) and \((1.8)\)), we have
\[
\sum_{L \leq L_0} \|\mathcal{M}_L(t)\|_{L^2} \lesssim \varepsilon 2^{-S \delta'}
\] (7.23)
consistently with \((7.16)\). Note that, similarly to the estimate of \(\mathcal{M}_1\) above, we have used that the assumption \((6.10)\) is fulfilled by choosing \(2^D = \langle s \rangle^3 \approx 2^{3S}\).

**Notation.** For simplicity, in what follows we will often disregard the extra lower order terms coming from the \(2^{-D}\) factor on the right-hand side of \((6.12)\), since we will always be able to guarantee \((6.10)\) with large enough \(2^D\) of the form \(\langle s \rangle^n\), under the apriori assumptions
where \( T_1[b] \) when this will be clear from the context.

- Integration by parts. For \( L > L_0 \), using (7.20), we write \( \mathcal{M}_L \) in (7.22) as:

\[
\mathcal{M}_L = A_L + B_L, 
\]

\[
A_L = \int_0^t \int \left( \frac{\ell|\ell|}{c(\ell, m)} (Xe^{i\Phi}) \tilde{f}(s, \ell + k) \tilde{f}(s, m) a_L(\ell, m) \nu_1(\ell, m) \right) d\ell dm \tau_S(s) ds, 
\]

\[
B_L = \int_0^t \int \frac{-\ell}{c(\ell, m)} s (\partial_s e^{i\Phi}) \tilde{f}(s, \ell + k) \tilde{f}(s, m) a_L(\ell, m) \nu_1(\ell, m) \right) d\ell dm \tau_S(s) ds. 
\]

Integrating by parts in \( X \) we get

\[
-A_L = \int_0^t \int e^{i\Phi} X \left[ \frac{\ell|\ell|}{c(\ell, m)} a_L(\ell, m) \right] \tilde{f}(s, \ell + k) \tilde{f}(s, m) \nu_1(\ell, m) \right) d\ell dm \tau_S(s) ds, 
\]

\[
= A^1_L + A^2_L + A^3_L, 
\]

where

\[
A^1_L = \int_0^t \int e^{i\Phi} X \left[ \frac{\ell|\ell|}{c(\ell, m)} a_L(\ell, m) \right] \tilde{f}(s, \ell + k) \tilde{f}(s, m) \nu_1(\ell, m) \right) d\ell dm \tau_S(s) ds, 
\]

\[
A^2_L = \int_0^t \int e^{i\Phi} \frac{\ell}{c(\ell, m)} a_L(\ell, m) X \left[ \tilde{f}(s, \ell + k) \tilde{f}(s, m) \right] \nu_1(\ell, m) d\ell dm \tau_S(s) ds, 
\]

\[
A^3_L = \int_0^t \int e^{i\Phi} \frac{\ell}{c(\ell, m)} a_L(\ell, m) \tilde{f}(s, \ell + k) \tilde{f}(s, m) X \left[ \nu_1(\ell, m) \right] d\ell dm \tau_S(s) ds. 
\]

Integrating by parts in \( s \) instead we can write

\[
B_L(t) = -B^1_L + B^2_L + B^3_L + B^4_L, 
\]

where

\[
B^1_L = t \int e^{i\Phi} \frac{\ell}{c(\ell, m)} a_L(\ell, m) \tilde{f}(t, \ell + k) \tilde{f}(t, m) \tau_S(t) \nu_1(\ell, m) d\ell dm \tau_S(t), 
\]

\[
B^2_L = \int_0^t \int e^{i\Phi} \frac{\ell}{c(\ell, m)} a_L(\ell, m) (\partial_s \tilde{f}(s, \ell + k)) \tilde{f}(s, m) \nu_1(\ell, m) d\ell dm \tau_S(s) ds, 
\]

\[
B^3_L = \int_0^t \int e^{i\Phi} \frac{\ell}{c(\ell, m)} a_L(\ell, m) \tilde{f}(s, \ell + k) (\partial_s \tilde{f}(s, m)) \nu_1(\ell, m) d\ell dm \tau_S(s) ds, 
\]

\[
B^4_L = \int_0^t \int e^{i\Phi} \frac{\ell}{c(\ell, m)} a_L(\ell, m) \tilde{f}(s, \ell + k) \tilde{f}(s, m) \nu_1(\ell, m) d\ell dm \partial_s \tau_S(s) ds. 
\]

Estimate of (7.28). With the notation of Theorem 6.1 we have

\[
\|A^1_L\|_{L^4_t} \lesssim \int_0^t 2^{-L} \|T_1[b](\tilde{F}u(s), \tilde{F}u(s))\|_{L^2} \tau_S(s) ds 
\]

\[
\text{with} \quad b(\ell, m) := 2^L X \left[ \frac{\ell|\ell|}{c(\ell, m)} a_L(\ell, m) \right] 
\]
Since \( |m| - |\ell| \ll |\ell| \) on the support of the integral (7.28), it is easy to check that \( b \) satisfies the symbol bounds (6.8)-(6.9). Using the bilinear estimate (6.12), and the apriori decay estimates (4.7), we have

\[
\|A^1_L\|_{L^2} \lesssim \int_0^t 2^{-L} \|\hat{F}\hat{u}(s)\|_{L^\infty} \|\hat{F}\hat{u}(s)\|_{L^6} \cdot 2^L \cdot 2^{C_0 A} + \langle s \rangle^{-3} \|\tau_S(s)\|ds \\
\lesssim \int_0^t \varepsilon^2 2^{-S/2} \cdot \varepsilon 2^{-(1-S)} \cdot 2^{C_0 A} \|\tau_S(s)\|ds + \varepsilon^2 2^{-S} \\
\lesssim \varepsilon^2 2^{-S/4}.
\]

Summing over \( L \in [-L_0, A] \cap \mathbb{Z} \) at the expense of an \( O(S) \) factor gives the desired bound on the right-hand side of (7.16).

**Estimate of (7.29).** This term consists of two similar terms (corresponding to one of the two profiles being hit by \( X \)), so we only estimate one of them. We denote

\[
A^2_L = \int_0^t \int e^{is\Phi} b(\ell, m) \varphi_L(\ell) \partial_{\ell\ell} \tilde{f}(s, \ell + k) \tilde{f}(s, m) \nu_1(\ell, m) \, d\ell dm \|\tau_S(s)\|ds
\]

and estimate

\[
\|A^2_L\|_{L^2} \lesssim \int_0^t \|T_1[b](e^{is|k|^2} \partial_k \tilde{f}(s), \tilde{F}\hat{u}(s))\|_{L^2} \|\tau_S(s)\|ds.
\]

Note that we have made a little abuse of notation converting \( \partial_{\ell\ell} \) into \( \partial_\ell \); this can be done without loss of generality by slightly redefining the symbol \( b \). Using that \( b \) satisfies the hypotheses (6.8)-(6.9), and applying (6.12) together with the apriori bound (4.8), we obtain

\[
\|A^2_L\|_{L^2} \lesssim \int_0^t \|\hat{F}e^{is|k|^2} \partial_k \tilde{f}(s)\|_{L^2} \|\hat{F}\hat{u}(s)\|_{L^\infty} \cdot 2^L \cdot 2^{C_0 A} \|\tau_S(s)\|ds \\
\lesssim \int_0^t \varepsilon \cdot \varepsilon 2^{-\left(5/4-\delta\right)S} \cdot 2^{(C_0+1)A} \|\tau_S(s)\|ds \lesssim \varepsilon^2 2^{-2\delta^S/4},
\]

see (7.11), and note that we have now disregarded the fast decaying remainder term from \( 2^{-D}D \).

**Estimate of (7.30).** Using the notation (6.15) we see that

\[
A^3_L = \int_0^t e^{-is|k|^2} \mathcal{F}_{x+k}(T_X[b](\tilde{u}, \tilde{u})) \tau_S(s)ds
\]

where \( b = (|\ell|/c(\ell, m))a_L(\ell, m) \). Using the bounds (6.16) from Theorem 6.1 we see that (7.30) enjoys a similar bound (up a different power of \( 2^A \)) to that of (7.28). We skip the details.

**Estimate of (7.32).** With the notation of Theorem 6.1 we have

\[
\|B^1_L\|_{L^2} \lesssim t 2^{-L} \|T_1[b](\tilde{F}\hat{u}(s), \tilde{F}\hat{u}(s))\|_{L^2}, \quad b(\ell, m) = \frac{\ell^2 L a_L(\ell, m)}{c(\ell, m)} \quad (7.37)
\]
Similarly to before we can apply (6.12) (again disregarding the remainder term on the right-hand side of the inequality)
\[ \| B_L \|_{L^2_k} \lesssim t \ 2^{-L} \| \hat{F}\bar{u}(t) \|_{L^3} \| \hat{F} \bar{u}(t) \|_{L^6} \cdot 2^L \cdot 2^{C_0 A} \tau_S(t) \]
\[ \lesssim 2^S \cdot \varepsilon \ 2^{-S/2} \cdot \varepsilon \ 2^{-(1-\delta)S} \cdot 2^{C_0 A} \]
\[ \lesssim \varepsilon \ 2^{-S/2} (C_0 + 1) \delta_N \]

which is more than sufficient.

**Estimate of (7.33) and (7.34).** The terms (7.33) and (7.34) are similar to each other, so it suffices to show how to estimate the first one. With the usual notation we have
\[ \| B^2_L \|_{L^2_k} \lesssim \int_0^t 2^{-L} \| T_1 [b] (\varepsilon \ | k |^2 \partial_s \bar{f}(s), \hat{F} \bar{u}(s)) \|_{L^2} \tau_S(s) \, ds, \]
\[ b = 2^L \ell \, a \ell \ell / c \ell \ell, \]

Recall that \( \varepsilon \ | k |^2 \partial_s \bar{f} = \hat{F} (u^2) \), see (1.31), and therefore
\[ \| \hat{F} \varepsilon \varepsilon \ | k |^2 \partial_s \bar{f} \|_{L^p} \lesssim \| u \|_{L^2_p}^2. \]

(7.38)

Applying (6.12, 7.38) with \( p = 3 \), the apriori decay estimates (4.7), we obtain
\[ \| B^2_L \|_{L^2_k} \lesssim \int_0^t 2^{-L} \| \hat{F} \varepsilon \varepsilon \ | k |^2 \partial_s \bar{f} \|_{L^3} \| \hat{F} \bar{u}(s) \|_{L^6} \cdot 2^L \cdot 2^{C_0 A} 2^S \tau_S(s) \, ds \]
\[ \lesssim \int_0^t (\varepsilon \ 2^{-S}) \cdot \varepsilon \ 2^{-(1-\delta)S} \cdot 2^{C_0 A} \cdot 2^S \tau_S(s) \, ds \]
\[ \lesssim \varepsilon \ 2^{-2S/3} \]

provided \( \delta_N \) is small enough as usual.

**Estimate of (7.35).** This term is almost identical to (7.32) by noticing that \( \partial_s (s \tau_S(s)) \approx \tau_S(s) \), and that the time integration is equivalent to the factor of \( t \) in front of the expression in (7.32). We can then skip the details. This concludes the estimate of \( B_L \), hence of \( M_L \), see (7.24), and gives us (7.16).

**7.1.4. Proof of (7.17).** We now look at the term (7.15). In this case, the integrals are supported on a region where \( |\ell| - |m| \gtrsim |m| + |\ell| \) so that \( \nu_1 \) is not really singular. In particular, we can differentiate it in other directions besides the direction of \( X \) and use the bilinear bounds from Theorem (6.2(i)). This makes the estimates more straightforward.

More precisely, we have the identity
\[ \sqrt{|\ell|^2 + |m|^2} X_+ \Phi = \Phi + |\ell|^2 + |m|^2, \quad X_+ := \frac{(\ell, m)}{\sqrt{|\ell|^2 + |m|^2}} \cdot \nabla (\ell, m), \]
(7.39)

and, consequently,
\[ \frac{1}{|\ell|^2 + |m|^2} \left( \frac{1}{i \varepsilon} \sqrt{|\ell|^2 + |m|^2} X_+ + i \partial_s \right) e^{i \varepsilon \Phi(k, \ell, m)} = e^{i \varepsilon \Phi(k, \ell, m)}. \]

(7.40)

Note how (a) \( \sqrt{|\ell|^2 + |m|^2} X_+ \) plays the role of \( |\ell| \Xi \) in (7.19), (b) the identity (7.40) is analogous to the identity (7.20), and (c) integration by parts in the \( X_+ \) direction is possible thanks to the bilinear estimates (6.20), which are analogous to the estimate (6.16) that we have used before when we integrated by parts in the \( X \) direction. Then, the terms \( M^2_2 \) in
have the same structure of the terms $\mathcal{M}_2^-$ in (7.3). We can then skip the details for the proof of (7.17).

7.2. **Proof of (7.4).** We now estimate the highest weighted norm. Recall the notation (7.5) and the formulas (7.6)-(7.8) for the first weight. Taking an extra derivative gives:

$$\|\partial_k^2 N_L(f, f)\| \lesssim \sum_{j=1}^3 \|A_j(f, f)\|_{L^2}$$

(7.41)

where

$$A_1(f, f) = \int_0^t \int \int e^{i s \Phi(k, \ell, m)} \partial_k^2 f(s, \ell + k) \tilde{f}(s, m) \varphi_L(\ell) \nu_1(\ell, m) \, dl \, dm \, \tau_S(s) \, ds,$$

(7.42)

$$A_2(f, f) = \int_0^t \int \int s^2 |\ell|^2 e^{i s \Phi(k, \ell, m)} \tilde{f}(s, \ell + k) \tilde{f}(s, m) \varphi_{-L}(\ell) \nu_1(\ell, m) \, dl \, dm \, \tau_S(s) \, ds,$$

(7.43)

$$A_3(f, f) = \int_0^t \int \int is \ell e^{i s \Phi(k, \ell, m)} \partial_k \tilde{f}(s, \ell + k) \tilde{f}(s, m) \varphi_L(\ell) \nu_1(\ell, m) \, dl \, dm \, \tau_S(s) \, ds.$$ 

(7.44)

Note that in (7.43) we have used the shorthand notation $|\ell|^2$ in place of the expressions $\ell^2 = \partial_k (k \cdot \ell) \partial_k (k \cdot \ell)$.

7.2.1. **Estimate of (7.42).** This term can be treated with a direct application of Theorem 6.1 with an $L^2 \times L^{3-}$ estimate, and using the decay estimate from (4.8), in the same way that we estimated (7.7) in (7.1).

7.2.2. **Estimate of (7.43).** This is the most complicated term due to the presence of an $s^2$ factor. Similarly to (7.14)-(7.15), and using the notation

$$a_L(\ell, m) := a(\ell, m) = \varphi_{-L}(m) \varphi_L(\ell) \chi_-(\ell, m)$$

from (7.22) and (7.13), we split the integral in two pieces by defining

$$A_2^- := \int_0^t \int \int s^2 |\ell|^2 e^{i s \Phi} \tilde{f}(s, \ell + k) \tilde{f}(s, m) a(\ell, m) \nu_1(\ell, m) \, dl \, dm \, \tau_S(s) \, ds.$$ 

(7.45)

and

$$A_2^+ = \int_0^t \int \int s^2 |\ell|^2 e^{i s \Phi} \tilde{f}(s, \ell + k) \tilde{f}(s, m) \varphi_L(\ell) \chi_+(\ell, m) \nu_1(\ell, m) \, dl \, dm \, \tau_S(s) \, ds.$$ 

(7.46)

As before, the most difficult term will be (7.45).

To start, note that we may again assume $L \geq L_0 := -S/3$ by an $L^6 \times L^{3-}$ estimate similar to the one that led to (7.23). Next, recall the identity (7.21)

$$\frac{1}{c(\ell, m)} \left( \frac{1}{i s} |\ell| X + i \partial_s \right) e^{i s \Phi} = e^{i s \Phi}, \quad |\nabla_{\ell}^\alpha \nabla_{m}^\beta c(\ell, m)| \lesssim |\ell|^{-2-|\alpha|-|\beta|},$$

(7.47)

where the estimates hold on the support of $A_2^-$ where $2^L \sim |\ell| \sim |m| \gg |\ell| - |m|$. The general idea is similar to the one used before, based on integration by parts using (7.47). This procedure will generate many terms. We will give full details for the ones that need to be analyzed more carefully, and skip some details for the easier ones.
Using (7.47) in (7.45) and doing some algebra one sees that
\[ \|A_2\|_{L^2_t} \lesssim \|B_1\|_{L^2} + \|B_2\|_{L^2} + \|C_1\|_{L^2} + \|C_2\|_{L^2} + \|D\|_{L^2} + \cdots \]  
(7.48)
where the first terms are those obtained integrating by parts in $s$:
\[ B_1 = t^2 \int_0^t \int e^{i\varphi} \frac{|\ell|^2}{c(\ell, m)} \tilde{f}(t, \ell + k) \tilde{f}(t, m) a(\ell, m) \nu_1(\ell, m) \frac{\dd \ell \dd m \tau_S(t)}{S}, \]  
(7.49)
\[ B_2 = \int_0^t s^2 \int \frac{|\ell|^2}{c(\ell, m)} e^{is\varphi} (\partial_{s} \tilde{f}(s, \ell + k)) \tilde{f}(s, m) a(\ell, m) \nu_1(\ell, m) \frac{\dd \ell \dd m \tau_S(s) \dd s}{S}, \]  
(7.50)
other terms arise when integrating by parts in the $X$ direction without hitting one of the profiles
\[ C_1 = \int_0^t s \int e^{is\varphi} X \left( \frac{|\ell|^3}{c(\ell, m)} a(\ell, m) \right) \tilde{f}(s, \ell + k) \tilde{f}(s, m) \nu_1(\ell, m) \frac{\dd \ell \dd m \tau_S(s) \dd s}{S}, \]  
(7.51)
\[ C_2 = \int_0^t s \int e^{is\varphi} \frac{|\ell|^3}{c(\ell, m)} a(\ell, m) \tilde{f}(s, \ell + k) \tilde{f}(s, m) (X \nu_1(\ell, m)) \frac{\dd \ell \dd m \tau_S(s) \dd s}{S}, \]  
(7.52)
and
\[ D := \left( \int_0^t s \int e^{is\varphi} \frac{|\ell|^3}{c(\ell, m)} a(\ell, m) (\partial_{s} \tilde{f}(s, \ell + k)) \tilde{f}(s, m) \nu_1(\ell, m) \frac{\dd \ell \dd m \tau_S(s) \dd s}{S} \right), \]  
(7.53)
in (7.48) the “…” denote all the other terms that are similar or easier to treat than those in (7.49)–(7.53). The term (7.53) requires a further use of (7.47), which gives
\[ \|D\|_{L^2} \lesssim \|D_1\|_{L^2} + \|D_2\|_{L^2} + \|D_3\|_{L^2} + \|D_4\|_{L^2} + \cdots \]  
(7.54)
where
\[ D_1 := \int_0^t \int e^{is\varphi} \frac{|\ell|^4}{c^2(\ell, m)} a(\ell, m) (\partial_{s}^2 \tilde{f}(s, \ell + k)) \tilde{f}(s, m) \nu_1(\ell, m) \frac{\dd \ell \dd m \tau_S(s) \dd s}{S}, \]  
(7.55)
\[ D_2 := \int_0^t \int e^{is\varphi} \frac{|\ell|^4}{c^2(\ell, m)} a(\ell, m) (\partial_{s}^3 \tilde{f}(s, \ell + k)) \tilde{f}(s, m) \nu_1(\ell, m) \frac{\dd \ell \dd m \tau_S(s) \dd s}{S}, \]  
(7.56)
\[ D_3 := \int_0^t s \int e^{is\varphi} \frac{|\ell|^3}{c^2(\ell, m)} a(\ell, m) (\partial_{s} \tilde{f}(s, \ell + k)) \tilde{f}(s, m) \nu_1(\ell, m) \frac{\dd \ell \dd m \tau_S(s) \dd s}{S}, \]  
(7.57)
\[ D_4 := \int_0^t s \int e^{is\varphi} \frac{|\ell|^3}{c^2(\ell, m)} a(\ell, m) (\partial_{s} \partial_{m} \tilde{f}(s, \ell + k)) \tilde{f}(s, m) \nu_1(\ell, m) \frac{\dd \ell \dd m \tau_S(s) \dd s}{S}. \]  
(7.58)
and “…” denote terms that are easier to estimate.

**Estimate of** (7.49)–(7.50). The boundary integral $B_1$ is the term that causes the growth in time of the weighted norm (7.4). With the notation of Theorem 6.1 we have
\[ \|B_1\|_{L^2_t} \lesssim t^2 \cdot \|T_1[b](\tilde{F}u(t), \tilde{F}u(t))\|_{L^2}, \quad b(\ell, m) = \frac{|\ell|^2 a(\ell, m)}{c(\ell, m)}. \]
Applying (6.12) (disregarding the remainder term on the right-hand side as usual), and the decay estimates (1.7), we get, for \( t \approx 2^S \),
\[
\|B_1\|_{L^2_k} \lesssim t^2 \cdot \left[ \|\hat{\mathcal{F}}u(t)\|_{L^3} \|\hat{\mathcal{F}}\bar{u}(t)\|_{L^6} \cdot 2^L \cdot 2^{C_0 A} \right] \\
\lesssim 2^S \cdot \varepsilon 2^{-S/2} \cdot \varepsilon 2^{-(1-)}S \cdot 2^{(C_0+1)A} \\
\lesssim \varepsilon^2 2^{S/2-2(C_0+1)\delta_N}.
\]

This is bounded as in (7.4) provided we choose \( \delta \) depending on \( N \) so that \((C_0 + 1)\delta_N < \delta\), see (1.16).

The term (7.50) can be seen to satisfy an even stronger bound with a similar application of (6.12) and using (6.38).

**Estimate of (7.51)–(7.52).** We have
\[
\|C_1\|_{L^2_k} \lesssim \int_0^t s \|T_1[b](\hat{\mathcal{F}}u(s), \hat{\mathcal{F}}\bar{u}(s))\|_{L^2} \tau_S(s) ds, \quad b(\ell, m) = X\left(\frac{|\ell|^3}{c(\ell, m)^{A(\ell, m)}}\right)
\]
so that (6.12) and (1.7) give us
\[
\|C_1\|_{L^2_k} \lesssim \int_0^t 2^S \cdot \|\hat{\mathcal{F}}\bar{u}(s)\|_{L^3} \|\hat{\mathcal{F}}\bar{u}(s)\|_{L^6} \cdot 2^L \cdot 2^{C_0 A} \tau_S(s) ds \\
\lesssim \int_0^t 2^S \cdot \varepsilon 2^{-S/2} \cdot \varepsilon 2^{-(1-)}S \cdot 2^{(C_0+1)A} \tau_S(s) ds \\
\lesssim \varepsilon^2 2^{S/2-2(C_0+1)\delta_N} S
\]
which suffices. The term (7.52) can be estimated similarly using the bilinear estimate (6.16).

**Estimate of (7.55)–(7.57).** The first term (7.55) can be handled directly with an \( L^2 \times L^\infty \) estimate, using that the symbol satisfies the hypothesis (6.9) of Theorem 6.1 and the \( L^\infty \) decay from (1.8).

For the second term we have
\[
\|D_2\|_{L^2_k} \lesssim \int_0^t \|T_1[b](e^{is|k|^2} \partial |\tilde{\mathcal{F}}(s), e^{is|k|^2} \partial |\tilde{\mathcal{F}}(s))\|_{L^2} \tau_S(s) ds, \quad b(\ell, m) = \frac{|\ell|^4 a(\ell, m)}{c^2(\ell, m)}.
\]
We then use (6.12) with an \( L^6 \times L^3 \) estimate, and interpolation, to get
\[
\|D_2\|_{L^2_k} \lesssim \int_0^t \|\hat{\mathcal{F}} e^{is|k|^2} \partial |\tilde{\mathcal{F}}(s)\|_{L^2} \|\hat{\mathcal{F}} e^{is|k|^2} \partial |\tilde{\mathcal{F}}(s)\|_{L^3} \cdot 2^L \cdot 2^{C_0 A} \tau_S(s) ds \\
\lesssim \int_0^t \varepsilon 2^{-(1-C_0+1)\delta_N} S \|\partial^2 f(s)\|_{L^2} \cdot \|\hat{\mathcal{F}} e^{is|k|^2} \partial |\tilde{\mathcal{F}}(s)\|_{L^2}^{1/2} \|\partial \tilde{\mathcal{F}}(s)\|_{L^2}^{1/2+} \tau_S(s) ds \\
\lesssim \int_0^t \varepsilon 2^{-(1-C_0+1)\delta_N} S \|\partial^2 f(s)\|_{L^2} \cdot 2^{-(1-2S)} \|\partial^2 \tilde{\mathcal{F}}(s)\|_{L^2}^{1/2} \|\partial f(s)\|_{L^2}^{1/2+} \tau_S(s) ds \\
\lesssim \int_0^t \varepsilon 2^{-(1-C_0+1)\delta_N} S \varepsilon 2^{(1/2-\delta S)} \cdot 2^{-(1-2S)} \varepsilon 2^{(1/2-\delta S)} \|\partial f(s)\|_{L^2} \tau_S(s) ds \\
\lesssim \varepsilon^2 2^{S/2},
\]
having used, see (4.2), that \( \|\hat{\mathcal{F}} e^{is|k|^2} \tilde{\mathcal{F}}(s)\|_{L^6} \lesssim 2^{-S} \|\partial \tilde{\mathcal{F}}(s)\|_{L^2} \) in the second and third inequalities.
The term (7.57) can be treated similarly with an $L^6 \times L^3$ estimate and using (7.38). We skip the details.

**Estimate of (7.58).** This is the last term arising from the double integration by parts argument in the main term (7.45). To bound it we need an additional bound for $\partial_k |\partial_s \tilde{f}|$. In particular, let us assume for the moment that

\[ \| \varphi_{\leq A}(k) \partial_k \partial_t \tilde{f}(t) \|_{L^2_k} \lesssim 2^{-S/2} 2^{2\delta N S}, \quad t \approx 2^S. \]  

(7.59)

Then we can bound

\[ \| D_4 \|_{L^2_k} \lesssim \int_0^T s 2^{-L} \| T_1[b] (e^{i|k|^2} \partial_k \partial_s \tilde{f}(s), \tilde{u}(s)) \|_{L^2} \tau_S(s) ds, \quad b(\ell, m) = 2^L \frac{|\ell|^3 a(\ell, m)}{c^2(\ell, m)}, \]

and using (6.12) for an $L^2 \times L^{\infty}$ estimate, with the decay bound (4.8) and (7.59), we get:

\[ \| D_4 \|_{L^2_k} \lesssim \int_0^T 2^S \cdot \| \hat{F} e^{i|k|^2} \partial_k \partial_s \tilde{f}(s) \|_{L^2} \| \hat{F} \tilde{u}(s) \|_{L^{\infty}} \cdot 2^{C_0A} \tau_S(s) ds \]

\[ \lesssim \int_0^T 2^S \cdot 2^{-S/2 + 2\delta N S} \cdot \varepsilon 2^{-(5/4 - \delta) S} \cdot 2^{C_0 \delta N S} \tau_S(s) ds \]

\[ \lesssim \varepsilon 2^{S/2} \]

since we have $(C_0 + 2)\delta_N + \delta \leq 1/4$. The desired estimate (7.47) for (7.45) is concluded once we verify (7.59).

To prove (7.59) we start from (4.13) in the form

\[ \partial_t \tilde{f}(t, k) = \int \int e^{it(-|k|^2 + |\ell|^2 + |m|^2)} \tilde{f}(t, \ell) \tilde{f}(t, m) \mu(k, \ell, m) d\ell dm, \]

and look back at the “commutation formula” (4.23) for $\partial_k$ and the operator (4.19) in Proposition 4.5. Using this and the same notation in that proposition, we obtain (compare with the similar terms obtained in (4.31)):

\[ \partial_k \partial_t \tilde{f} = E_1 + E_2 + E_3, \]

\[ E_1(f, f)(t, k) = \int \int e^{it\Phi(k, \ell, m)} \tilde{f}(t, \ell) \tilde{f}(t, m) (-itk\mu + \mu') d\ell dm, \]  

(7.60)

\[ E_2(f, f)(t, k) = \int \int e^{it\Phi(k, \ell, m)} \tilde{f}(t, \ell) \tilde{f}(t, m) it\ell \mu(k, \ell, m) d\ell dm, \]  

(7.61)

\[ E_3(f, f)(t, k) = \int \int e^{it\Phi(k, \ell, m)} \partial_\ell \tilde{f}(t, \ell) \tilde{f}(t, m) \mu(k, \ell, m) d\ell dm. \]  

(7.62)

Notice that since $E_3(f, f) = B'(\partial_k f, f)$ this term can be easily bounded using an $L^2 \times L^{\infty}$ Hölder estimate (which holds for the bilinear operator $B'$).

For the term (7.60) we have, for $t \approx 2^S$,

\[ \| \varphi_{\leq A}(\cdot) E_1(t, \cdot) \|_{L^2_k} \lesssim 2^A \cdot 2^S \cdot \| \hat{F} \tilde{u}(t) \|_{L^2} \| \hat{F} \tilde{u}(s) \|_{L^{\infty}} \lesssim \varepsilon 2^{2\delta N S} 2^{-S/2} \]

which is more than sufficient. For the remaining term we can use a similar estimate when $|\ell| \leq 2^A$. If instead $|\ell| \geq 2^A$ we can use an $L^2 \times L^{\infty}$ bound and the control on the high
Sobolev norm:
\[ \| E_2(t) \|_{L^2} \lesssim 2^S \cdot \| \mathcal{F}_{\geq A} \tilde{u}(t) \|_{H^1} \cdot \| \mathcal{F}_{\leq 1} \tilde{u}(s) \|_{L^\infty} \]
\[ \lesssim 2^S \cdot 2^{-A(N-1)} \| \mathcal{F}_{\leq 1} \tilde{u}(t) \|_{H^N} \cdot \varepsilon 2^{-S} \lesssim \varepsilon^2 2^{-S}. \]

Estimate of (7.46). This term is easier to treat than the previous one (7.45). We can proceed similarly to the case of the corresponding term (7.15) in the estimate of the first weight, see (7.1.4). In particular, we can use the identities (7.39)-(7.40) as explained before, to integrate by parts (once or twice) as done for the term (7.45) just above.

7.2.3. Estimate of (7.44). By using the same splitting depending on the size of \(||\ell| - |m|| \) relative to \(|\ell|\) as before, and integrating by parts once using the same formula (7.47) (and (7.39)) above, we can see that (7.44) gives rise to terms of the same form as those treated before. We can therefore skip the details. The proof of (7.4) and Proposition 7.1 is concluded.

7.3. Estimates for \(N_{2,S}\). We now prove the weighted bounds (7.3)-(7.4) for the term \(N_{2,S}\), see (7.1). We can use a similar strategy to the one used for \(N_{1,S}\). To implement such a strategy we will need: (1) a similar structure including a “good vectorfield” and algebraic relations like (7.20), and (2) proper multiplier estimates, like those of Theorem 6.1(i) and Theorem 6.2(ii)-(iii).

As before we will sometimes drop some of the indexes (for example the index \(S\) in \(N_{2,S}\)) when this causes no confusion. For convenience let us recall the definition from (7.1) here:
\[
N_{2,S}(t)(f, f) = \int_0^t \int e^{i\Psi(k, \ell, m)} \tilde{f}(s, -\ell - m) \tilde{f}(s, m) \nu_1(\ell, k) \, d\ell dm \tau_S(s) ds,
\]
\[
\Psi(k, \ell, m) := -|k|^2 + |\ell|^2 + 2\ell \cdot m + 2|m|^2. \tag{7.63}
\]

We want to estimate \(\partial_k N_2\) and \(\partial_k^2 N_2\) as in Proposition 7.1 and show that, under the apriori assumptions (\ref{2.23}), we have
\[
\| \partial_k N_{2,S}(t)(f, f) \|_{L^2} \lesssim \varepsilon^2 2^{-\delta' S}, \quad \delta' > 0 \tag{7.64}
\]
\[
\| \partial_k^2 N_{2,S}(t)(f, f) \|_{L^2} \lesssim 2^{(1/2 + \delta') S} \varepsilon^2. \tag{7.65}
\]

7.3.1. Proof of (7.64). We concentrate only on the singular region where \(||\ell|| \approx |k|\). The non-singular region can be treated more easily as in the case of \(\nu_1(\ell, m)\) in Subsection 7.1.4 and using part (ii) of Theorem 6.2 to absorb derivatives in \(\ell\); also notice that \(\partial_m\) is also always a “good direction” for integration since \(\nu_1(\ell, k)\) is independent of \(m\).

We restrict close to the singularity of \(\nu_1(\ell, k)\) by inserting localization in \(||\ell|| \approx 2^L, |k| \approx 2^K\) and a cutoff \(\chi_+(\ell, k)\) as in (6.22). For lighter notation, we do not display the cutoff and assume that \(\nu_1 = \nu_1 \chi_+\) (plus we disregard terms where derivatives hit the cutoff).

We recall the definition of \(Y\) from (6.24):
\[
Y = \partial_k + \frac{k}{|k|} \left( \frac{\ell}{|\ell|} \cdot \partial_\ell \right), \tag{7.66}
\]
and use this to compute

\[
\partial_{k}N_2(f, f) = \int_0^t \int \left( i s \partial_k \Psi \right) e^{i s \Psi(k, \ell, m)} \tilde{f}(s, -\ell - m) \tilde{f}(s, m) \nu_1(\ell, k) \, d\ell dm \tau_S(s) \, ds \\
+ \int_0^t \int \frac{k}{|k|} \text{div}_\ell \left( \frac{\ell}{|\ell|} e^{i s \Psi(k, \ell, m)} \tilde{f}(s, -\ell - m) \tilde{f}(s, m) \nu_1(\ell, k) \right) \, d\ell dm \tau_S(s) \, ds \\
= M_3(f, f) + M_4(f, f) + M_5(f, f)
\]

(7.67)

where

\[
M_3(f, f) := \int_0^t \int \frac{k}{|k|} e^{i s \Psi(k, \ell, m)} \partial_\ell \tilde{f}(s, -\ell - m) \tilde{f}(s, m) \nu_1(\ell, k) \, d\ell dm \tau_S(s) \, ds,
\]

(7.68)

\[
M_4(f, f) := \int_0^t \int i s Z \Psi(k, \ell, m) e^{i s \Psi(k, \ell, m)} \tilde{f}(s, -\ell - m) \tilde{f}(s, m) \nu_1(\ell, k) \, d\ell dm \tau_S(s) \, ds,
\]

\[
Z \Psi := (\partial_k + \frac{k}{|k|} \ell \cdot \partial_\ell) \Psi = 2 \left[ - k + \frac{k}{|k|} (|\ell| + \frac{\ell}{|\ell|} \cdot m) \right],
\]

(7.69)

and

\[
M_5(f, f) := \int_0^t \int e^{i s \Psi(k, \ell, m)} \tilde{f}(s, -\ell - m) \tilde{f}(s, m) Y' \nu_1(\ell, k) \, d\ell dm \tau_S(s) \, ds.
\]

(7.70)

We discuss briefly the (7.68) and (7.70) and give full details for the treatment of the harder term (7.69).

Notice that (7.68) is essentially the same as (7.71), and therefore can be estimated in the same way as we did in 7.1.1 with a direct application of the bilinear bound for the \(T_2\)-type operator in Theorem 6.1.

In (7.70) note that \(Y' = Y + (k/|k|) \text{div}_\ell (\ell/|\ell|)\). The piece with \(Y \nu_1(\ell, k)\) is an operator of the form (6.25) with \(a = 1\), and applying (6.26) (after localization) for an \(L^6 / L^6\) estimate is more than sufficient. Let us call \(M'_5\) the piece in (7.70) with symbol \((k/|k|) \text{div}_\ell (\ell/|\ell|) \nu_1(\ell, k)\). After being localized to \(|\ell| \approx 2^L\) and \(|k| \approx 2^K\), we see that, as a bilinear operator, it satisfies the same estimates of the bilinear operator with symbol \(2^{-L} b(k, \ell, m) \nu_1(\ell, k)\), for a standard \(b\) as in Theorem 6.1. Then we can bound it using (6.13):

\[
\| P_K \hat{F}^{-1} M'_5(f, f) \|_{L^2} \lesssim 2^S \cdot 2^{-L} \cdot \| \varphi_K T_2(b)(f, f) \|_{L^2} \\
\lesssim 2^S \cdot 2^{c_0 A} \cdot \| \hat{F} \hat{u}(t) \|_{L^6} \| \hat{F} \hat{u}(s) \|_{L^3} \lesssim \varepsilon^2 \cdot 2^{-S/4}.
\]
Estimate of (7.69). We now show
\[ \| \mathcal{M}_4(t)(f,f) \|_{L^2} \lesssim \varepsilon^2 2^{-\delta' S}, \quad \delta' > 0. \] (7.71)
This term is similar to (7.8) (it has a growing factor of $s$ and no differentiation of the profiles $\tilde{f}$). We will need to distinguish different cases depending on whether we are close to the singularity of $\nu_1(\ell, k)$ or not (see (7.1.2) for the similar splitting in the case of (7.8)) and use a good vectorfield to integrate by parts (see the relations at the beginning of 7.1.3).

First of all, notice that the singular kernel $\nu_1(\ell, k)$ does not depend on $m$, so that one can use $X := \partial_m$ as a “good direction” here. We then calculate
\[ X \psi := 2\ell + 4m, \quad X := \partial_m, \]
\[ \Psi(k, \ell, m) - m \cdot X \psi = (|\ell| - |k|)(|\ell| + |k|) - 2|m|^2 \] (7.72)
This last identity is the analogue of (7.19), and an identity similar to (7.20), see (7.78) below, follows from it.

To obtain (7.71) it suffices to show that
\[ \| \mathcal{M}_{K,L,M}(t)(f,f) \|_{L^2} \lesssim \varepsilon^2 2^{-2\delta' S}, \quad \delta' > 0 \] (7.73)
where
\[ \mathcal{M}_{K,L,M}(f,f) := \varphi_K(k) \int_0^t \int \int s Z \psi e^{is \psi} \varphi_L(\ell) \varphi_M(m) \]
\[ \times \tilde{f}(s,-\ell - m) \tilde{f}(s,m) \nu_1(\ell, k) \, d\ell \, dm \, \tau_S(s) \, ds, \] (7.74)
is a localized version of (7.69) at scales $|\ell| \approx 2^L$, $|m| \approx 2^M$ and $|k| \approx 2^K$. The reduction to (7.73) can be done without loss of generality by estimating the contribution of very small frequencies first, say for example $\min(K,L,M) \leq -10S$, using Bernstein and Theorem (6.1(i)); summing over the remaining dyadic scales can be done at the expense of a small loss of $O(S^3) + (A^3)$.

In order to use efficiently (7.72) we need to further split (we omit the dependence on the indexes $K, L, M$)
\[ \mathcal{M}_{K,L,M} = I_1 + I_2 + I_3 \]
where the following conditions on the support are imposed by inserting smooth cutoffs:
\[ I_1 \quad \text{is supported on } |\ell + 2m| \geq 2^{\max(L,M) - 10} \]
\[ I_2 \quad \text{is supported on } |\ell + 2m| \leq 2^{\max(L,M) - 9} \quad \text{and} \quad ||\ell| - |k|| \leq 2^{\max(L,K) - 9} \] (7.75)
\[ I_3 \quad \text{is supported on } |\ell + 2m| \leq 2^{\max(L,M) - 9} \quad \text{and} \quad ||\ell| - |k|| \geq 2^{\max(L,K) - 10} \]

Estimate of $I_1$. Here we have $|\ell + 2m| \geq 2^{\max(L,M) - 10}$ and therefore $|X \psi| \gtrsim 2^{\max(L,M) - 10}$. In particular we can use the identity
\[ \frac{1}{is} \frac{X \psi}{|X \psi|^2} \cdot X e^{is \psi} = e^{is \psi} \] (7.76)
to recover the factor of $s$ in (7.74). Also notice that $X \psi/|X \psi|^2$ behaves like the symbol $1/|\ell + 2m|$ which, on the support of $I_1$, gives
\[ |\nabla_X^\alpha \nabla_m^\beta X \psi/|X \psi|^2| \lesssim \frac{1}{\sqrt{|\ell|^2 + |m|^2}} |\ell|^{-|\alpha|} |m|^{-|\beta|}. \] (7.77)
Then, when multiplied by $2^{\max(M,L)}$ this is a symbol like those allowed by (6.9) in the assumptions of Theorem 6.1. An $L^6 \times L^3$ application of (6.13) to (7.71) gives a bound of $\varepsilon^2 2^{-S/2}$ up to powers of $2^A$.

**Estimate of $I_2$.** In this case we have
\[(19/20)|m| \leq |\ell| \leq (21/20)|m|, \quad (19/20)|k| \leq |\ell| \leq (21/20)|k|\]
and we can use efficiently the identity (7.72). We write
\[d(k, \ell, m) := -(|\ell| - |k|)(|\ell| + |k|) + 2|m|^2,\]
and notice that, thanks to the current frequency restrictions, we have
\[|\nabla_k a \nabla^\alpha \nabla^\beta m| \lesssim |\ell|^2 + |m|^2 + |k|^2|k|^{-|\alpha|}|m|^{-|\beta|},\]
consistently with the assumption (6.9). One can then use (7.78) to integrate by parts in (7.69), and arrive at the estimate (7.73) through applications of the bilinear estimate (6.13).

Since this argument is similar to what was done before for the term (7.8) in 7.1.2-7.1.3, we can skip the details.

**Estimate of $I_3$.** In this last case $\nu_1(\ell, k)$ is not singular and we can integrate by parts also in $\ell$ through the identity
\[\frac{1}{i\Delta} \nabla_k \nabla^\beta m |\nabla_k \nabla^\beta m| \lesssim |\ell|^2 + |m|^2 + |k|^2|k|^{-|\alpha|}|m|^{-|\beta|} \lesssim \frac{1}{\sqrt{|\ell|^2 + |m|^2}}|\ell|^{-|\alpha|}|m|^{-|\beta|}.\]

**7.3.2. Proof of (7.65).** To complete the bounds on $D_1$, see (7.1), we are left with showing that the second derivative of $N_2$, see (7.63), can be estimated as in (7.65). Once again we restrict our analysis to the singular region $|k| - |\ell| \ll |\ell| \approx |k|$. Applying $\partial_k$ to (7.67) we obtain
\[\partial_k^2 N_2(f, f) = \partial_k M_3(f, f) + \partial_k M_4(f, f) + \partial_k M_5(f, f)\]
see (7.68)-(7.70). As in the analysis of $\partial_k^2 N_1$ in Subsection 7.2.4 this generates a large number of terms. It is not hard to see that many of them will be similar to those treated previously and can be handled by similar arguments, through the opposite bilinear estimates for $\nu_1(\ell, k)$ in (6.13) and in Theorem 6.2(ii)-(iii), and the identities (7.70) with (7.77), and (7.78) with (7.79), and (7.81) with (7.81). Notice that the factors of $|k|/|k|$ in (7.68) and (7.70) do not constitute any additional difficulty since $\|\partial_k(k/|k|)F\|_{L^2} \lesssim \|\partial_k F\|_{L^2}$ by Hardy’s inequality.
For completeness we discuss the details for the hardest terms which are coming from \( \partial_k \mathcal{M}_4(f, f) \). Applying \( \partial_k \) to the formula (7.69), and the same algebra used to obtain (7.67), we get
\[
\| \partial_k \mathcal{M}_4(f, f) \| \lesssim \sum_{j=1}^4 \| \mathcal{M}_{4,j}(f, f) \|_{L^2} \tag{7.82}
\]
where
\[
\mathcal{M}_{4,1} := \int_0^t \int_0^t s Z \Psi e^{is\Psi} \tilde{f}(s, \ell - m) \tilde{f}(s, m) \nu_1(\ell, k) d\ell dm \tau_S(s) ds,
\]
\[
\mathcal{M}_{4,2} := \int_0^t \int_0^t s Z^2 \Psi e^{is\Psi} \tilde{f}(s, \ell - m) \tilde{f}(s, m) \nu_1(\ell, k) d\ell dm \tau_S(s) ds,
\]
\[
\mathcal{M}_{4,3} := \int_0^t \int_0^t s Z \Psi e^{is\Psi} \partial_\ell \tilde{f}(s, \ell - m) \tilde{f}(s, m) \nu_1(\ell, k) d\ell dm \tau_S(s) ds,
\]
\[
\mathcal{M}_{4,4} := \int_0^t \int_0^t s^2 (Z\Psi)^2 e^{is\Psi} \tilde{f}(s, \ell - m) \tilde{f}(s, m) \nu_1(\ell, k) d\ell dm \tau_S(s) ds.
\]
Recall that the goal is to bound the \( L^2 \)-norms of these terms by \( \varepsilon^2 s^{(1/2+\delta)S} \).

The first two terms are relatively easy to handle. (7.83) can be bounded with an \( L^{6^-} \times L^3 \) estimates using (6.13) and (6.23) with \( a = 1 \). For (7.84) some attention needs to be paid to the fact that \( Z^2 \Psi \) has a \( 1/|k| \) type singularity. More precisely, from (7.69) we see that \( Z^2 \Psi \) is made of harmless terms plus the symbol
\[
z(k, \ell, m) := \partial_k(k/|k|)(\ell/|\ell|) \cdot (\ell + m).
\]
The \( 1/|k| \) factor is only problematic away from the singularity of \( \nu_1(\ell, k) \) when \( |k| \ll |\ell| \). But in this case we can integrate by parts in \( \ell \) using that \( z(k, \ell, m) = (1/2) \partial_\ell(k/|k|)(\ell/|\ell|) \cdot \partial_\ell \Psi \).

This gains back a factor of \( s^{-1} \). With Hardy’s inequality \( \| |k|^{-1} F \|_{L^2} \lesssim \| F \|_{L^{4/5}} \) and an \( L^2 \times L^{6^-} \) application of (6.13) (when \( \partial_\ell \) hits the profile) or (6.23) (when \( \partial_\ell \) hits \( \nu_1(\ell, k) \)), we get the desired bound.

The remaining two terms (7.85)-(7.86) are similar to (7.43)-(7.44) and require integration by parts arguments using a splitting similar to (7.69) and the formulas (7.70)-(7.84). We concentrate on the hardest contribution, which is (7.86).

**Estimate of (7.86).** As before, without loss of generality we may assume that the integral (7.86) is localized to \( |\ell| \approx 2^L, |m| \approx 2^M, \) and \( |k| \approx 2^K \). Moreover, we may reduce to the case \( K \leq \max(L, M) + 10 \). Indeed, if \( |k| \gg |\ell|, |m| \) we have that \( |\Psi| \gtrsim |k|^2 \approx (|k| + |\ell| + |m|)^2 \) and an integration by parts in \( s \) will give us the desired bound; a similar argument was already used in the proof of Proposition 4.4 see (4.24) and the arguments that follow.

We then split
\[
\mathcal{M}_{4,4} = J_1 + J_2 + J_3
\]
where we may assume that
\[
J_1 \text{ is supported on } |\ell + 2m| \geq 2^{\max(L, M) - 10}
\]
\[
J_2 \text{ is supported on } |\ell + 2m| \leq 2^{\max(L, M) - 9} \quad \text{and} \quad ||\ell| - |k|| \leq 2^{\max(L, K) - 9}
\]
\[
J_3 \text{ is supported on } |\ell + 2m| \leq 2^{\max(L, M) - 9} \quad \text{and} \quad ||\ell| - |k|| \geq 2^{\max(L, K) - 10}
\]
To estimate $J_1$ we integrates by parts twice in $m$ using (7.70)-(7.74). Notice how the factor $X\Psi$ helps canceling the mild singularity introduced by the symbol $X\Psi/|X\Psi|^2$.

The term $J_2$ in (7.88) is the one which requires the most algebraic manipulations to integrate by parts using (7.78)-(7.79). However this term is almost identical to (7.45), so the estimates can be done as in 7.2. Notice that since $|\ell| \approx |m| \approx |k|$ on the support of $J_2$, then the term $(Z\Psi)^2$ plays a role analogous to the factor $|\ell|^2$ in (7.45), and helps to cancel the mild singularities introduced by the division by $d$, see (7.79).

Finally, for $J_3$ we can use integration in $\ell$ through (7.80)-(7.81), and applications of Theorem 6.2(ii). This concludes the proof of (7.65) and the weighted estimates of $D_1$.

8. Analysis of the NSD II: Lower order terms

In this section we analyze all remaining terms in the nonlinear equations. We begin by studying the lower order terms $\mu_2$ and $\mu_3$ from the expansion of the distribution $\mu$, defined in (5.4)-(5.6). We will make use of the “building block” Lemma 5.2 and establish:

1. structural propositions for $\mu_2$ and $\mu_3$ analogous to Proposition 5.1 for $\mu_1$ (in fact $\nu_1$, see (5.2)), and

2. bilinear multiplier estimates for $\mu_2$ and $\mu_3$ which are the analogues of Theorem 6.1. We will then show how to use these in Subsection 9 to establish the desired a priori bounds on the nonlinear terms from (5.7) corresponding to $\mu_2$ and $\mu_3$. We conclude with a discussion of the nonlinear estimates for the “flat” nonlinear part $D_0$, see (5.7).

Recall from (5.4)-(5.7) the nonlinear contribution to Duhamel’s formula (4.13):

$$D(t)(f,f) = D_0(t)(f,f) + D_1(t)(f,f) + D_2(t)(f,f) + D_3(t)(f,f),$$

$$D_0(t)(f,f) := \int_0^t \int e^{is(-|k|^2+|\ell|^2+|k-\ell|^2)} f(s,\ell) \bar{f}(s,k-\ell) \, d\ell \, ds,$$

$$D_s(t)(f,f) := \int_0^t \int e^{is(-|k|^2+|\ell|^2+|m|^2)} f(s,\ell) \bar{f}(s,m) \mu_s(k,\ell,m) \, d\ell \, dm \, ds,$$

where $\mu_1$ is given in (5.2)-(5.3), and, as in (5.4)-(5.6),

$$\mu_2(k,\ell,m) := \nu_2^1(k,\ell,m) + \nu_2^2(k,\ell,m) + \nu_2^3(k,m,\ell),$$

$$\nu_2^1(k,\ell,m) := \int e^{-ix \cdot k} e^{i|\ell|+|m| |x|} |x|^2 \psi_1(x,\ell) \psi_1(x,m) \, dx,$$

$$\nu_2^2(k,a,b) := \int e^{ix \cdot a} e^{i|\ell|+|b| |x|} |x|^2 \psi_1(x,k) \psi_1(x,b) \, dx,$$

and

$$\mu_3(k,\ell,m) := \int e^{i(-|k|+|\ell|+|m|) |x|} |x|^2 \psi_1(x,k) \psi_1(x,\ell) \psi_1(x,m) \, dx.$$  

8.1. Analysis of $\mu_2$: Structure. Let us begin by looking at $\nu_2^1(k,\ell,m)$. We have the following analogue of Proposition 5.1.
Proposition 8.1 (Structure of $\nu_2^1$). Let $\nu_2^1$ be the measure defined in \((8.2)\), with $\psi_1$ defined by \((3.21)\). Fix $N_2 \in [5, N_1/4] \cap \mathbb{Z}$. Let $k, \ell, m \in \mathbb{R}^3$ with $|k| \approx 2^K$, $|\ell| \approx 2^L$, and $|m| \approx 2^M$, and assume that $K, L, M \leq A$ for some $A > 0$. Then we can write

$$\nu_2^1(k, \ell, m) = \nu_{2,0}^1(k, \ell, m) + \nu_{2,R}^1(k, \ell, m), \quad (8.4)$$

where:

1. $\nu_{2,0}^1(k, \ell, m)$ can be written as

$$\nu_{2,0}^1(k, \ell, m) = \frac{1}{|k|} \sum_{i=1}^{N_2} \sum_{J \in \mathbb{Z}} b_{i,J}(k, \ell, m) \cdot K_i(2^J(|k| - |\ell| - |m|)) \quad (8.5)$$

with $K_i \in \mathcal{S}$ and the symbols $b_{i,J}$ satisfy

$$\left| \varphi_k(k) \varphi_j(\ell) \varphi_\ell(m) \nabla_k^a \nabla_\ell^b \nabla_m^c \nu_{2,0}^1(k, \ell, m) \right| \lesssim 2^{-|a|K} \cdot (2^{|a| \max(K, L)} + 2(1-|\alpha|L^2(1-|\beta|)^M) \cdot 1_{\{|K-\max(K, L)| < 5\}}, \quad (8.6)$$

for all $K, L, M \leq A$, and $|a| + |\alpha| + |\beta| \leq N_2$.

2. For $M \leq L$, the remainder term $\nu_{2,R}^1$ satisfies

$$\left| \nabla_k^a \nabla_\ell^b \nabla_m^c \nu_{2,R}^1(k, \ell, m) \right| \lesssim 2^{-2 \max(K, L)} \cdot 2^{-|a| \max(K, L)} \cdot 2^{-|\alpha| \max(K, L)} \cdot 2^{(|a|+|\alpha|+|\beta|+2)A} \quad (8.7)$$

for all $K, L, M \leq A$ and $|a| + |\alpha| + |\beta| \leq N_2/2 - 2$. A similar statement holds when $L \leq M$ exchanging the roles of $L$ and $M$ (and $\alpha$ and $\beta$).

**Proof.** We proceed in two main steps. In the first step we analyze a building block analogous to the one in \((5.19)\) of Lemma 5.2, the second step uses the first step, the expansion of $\psi$ from Lemma 3.4, and arguments similar to those in the proof of Proposition 5.1 to obtain the final statement.

With $|k| \approx 2^K$, $|\ell| \approx 2^L$ and $|m| \approx 2^M$, we assume without loss of generality $L \geq M$ and write

$$\nu_2^1(k, \ell, m) = (\nu^+(k, \ell, m) + \nu^-(k, \ell, m)) 1_{\{|K-L| < 5\}} + \nu_2^1(k, \ell, m) 1_{\{|K-L| \geq 5\}}$$

$$\nu^+(k, \ell, m) := \sum_{J \in \mathcal{J}} \nu^J(k, \ell, m) \quad (8.8)$$

$$\nu^J(k, \ell, m) := \int_{\mathbb{R}^3} e^{-ix \cdot k} \frac{e^{i(|\ell|+|m|)|x|}}{|x|^2} \psi_1(x, \ell) \psi_1(x, m) \varphi_J(x) \, dx,$$

where

$$\mathcal{J} := \{J + \min(K, 0) \geq 4A\}. \quad (8.9)$$

The “building block”. Similarly to \((5.19)\) we can define

$$\mathcal{K}_J(k, \ell, m) := \int_{\mathbb{R}^3} e^{-ix \cdot k} \frac{g_1(\omega, \ell) g_2(\omega, m) \varphi(x^{-J})}{|x|^2} \, dx,$$  

$$J \in \mathcal{J}, \quad g_i \in \mathcal{G}^{a_i} \quad (8.10)$$
as our building block, for some $N_2 \leq s_i \leq N_1$, and a generic compactly supported function $\varphi$. We then write the integrand in polar coordinates $x = r \omega$

$$K_J(k, \ell, m) = \int_0^\infty e^{ir(|\ell|+|m|)} \left( \int_{\mathbb{R}^2} e^{-ir\omega \cdot k} g_1(\omega, \ell) g_2(\omega, m) \, d\omega \right) \varphi(r 2^{-J}) \, dr$$

$$= \int_0^\infty e^{ir(|\ell|+|m|)} (I_0 + II) \varphi(r 2^{-J}) \, dr,$$  \hspace{1cm} (8.11)

where, with $X := r|k|$, we define

$$I_0 := \frac{2\pi}{i X} \left[ e^{-iX} g_1(\frac{k}{|k|}, \ell) g_2(\frac{k}{|k|}, m) - e^{iX} g_1(-\frac{k}{|k|}, \ell) g_2(-\frac{k}{|k|}, m) \right]$$  \hspace{1cm} (8.12)

and

$$II := \frac{1}{iX} \int_0^2 \int_0^\pi e^{-iX \cos \phi} \partial_\phi (g_1(\omega, \ell) g_2(\omega, m)) \, d\phi d\theta.$$  \hspace{1cm} (8.13)

Compare with (5.28)–(5.30). The contribution to (8.11) from $I_0$ is, up to omitting irrelevant constants, a sum of the terms

$$I_0^{\pm} := \int_0^\infty \frac{1}{r|k|} e^{ir(|\ell|+|m|\pm|k|)} g_1(\mp k/|k|, \ell) g_2(\mp k/|k|, m) \varphi(r 2^{-J}) \, dr$$

$$= \frac{1}{|k|} g_1(\mp k/|k|, \ell) g_2(\mp k/|k|, m) \chi (2^J(|\ell| + |m| \pm |k|))$$  \hspace{1cm} (8.14)

where $\chi := \varphi/r$.

For the term (8.13) we want apply the stationary phase estimates (5.35)–(5.37) to the integral in $d\phi$ (recall $2^J|k| \gg 1$) as we did for (5.35) in the proof of Lemma 5.2. We can use the same arguments there and obtain analogous formulas by replacing the quantities in (5.35) as follows:

$$g(\omega, q) \mapsto g_1(\omega, \ell) g_2(\omega, m), \quad e^{ir|q|} \mapsto e^{ir(|\ell|+|m|)}, \quad |p| \mapsto |k|, \quad \varphi \mapsto 2^{-J}(\varphi/r).$$  \hspace{1cm} (8.15)

This gives an expansion with properties similar to (5.41)–(5.46) as follows:

$$\int_0^\infty e^{ir(|\ell|+|m|)} II(X; \ell, m) \varphi(r 2^{-J}) \, dr$$

$$= \sum_{j=0}^{M'-1} b_j^-(k, \ell, m) \int_0^\infty e^{ir(-|k|+|\ell|+|m|)} r^{-(j+1)/2} 2^{-J}(\varphi/r)(r 2^{-J}) \, dr$$

$$+ \sum_{j=0}^{M'-1} b_j^+(k, \ell, m) \int_0^\infty e^{ir(|k|+|\ell|+|m|)} r^{-(j+1)/2} 2^{-J}(\varphi/r)(r 2^{-J}) \, dr$$

$$+ \frac{1}{|k|} R_{J,M'}^-(k, \ell, m) + \frac{1}{|k|} R_{J,M'}^+(k, \ell, m),$$  \hspace{1cm} (8.16)

where

$$b_j^\pm(k, \ell, m) := c_j \partial_\phi^{j+1} (g_1(\omega, \ell) g_2(\omega, m)) \bigg|_{\omega = \pm k/|k|},$$  \hspace{1cm} (8.19)
for some constants $c_\ell$ (we are using here the same convention adopted in the proof of Lemma 3.2 as per the discussion following (5.37)) and

$$R_{J,M'}^\pm (k, \ell, m) := \int_0^\infty e^{\pm i|\ell| |r|} e^{i r(|\ell|+|m|)} 2^{-J} (\varphi/r) (r 2^{-J}) B^\pm (r|k|; \ell, m) \, dr,$$

where $B^\pm (X; \ell, m)$ satisfy

$$\left| \left( \frac{d}{dX} \right)^{a} \partial_{\ell}^a \partial_m^g B^\pm (X; \ell, m) \right| \lesssim X^{-M'/2-a} \sup_{\omega \in S^2} \left| \partial_{\ell}^{M'+1} \partial_m^g g_1 (\omega, \ell) \partial_m^g g_2 (\omega, m) \right|. \quad (8.21)$$

**Expansion of $\nu_1^J$.** From the definition (8.8) and using the expansion of $\psi_1$ from Lemma 3.4 we can write (omitting the dependence on some of the indexes)

$$\nu^J (k, \ell, m) = \sum_{j_1, j_2=0}^{N_2-1} \nu_{j_1j_2}^J (k, \ell, m) + R_1 (k, \ell, m) + R_2 (k, \ell, m), \quad (8.22)$$

where

$$\nu_{j_1j_2}^J (k, \ell, m) := \int_{\mathbb{R}^3} e^{-i x \cdot k} e^{i(|\ell|+|m|)|x|} \frac{1}{|x|^2} g_{j_1} (x, \ell) g_{j_2} (x, m) |x|^{-j_1-j_2} \langle \ell \rangle^{j_1} \langle m \rangle^{j_2} \varphi_J (x) \, dx, \quad (8.23)$$

and

$$R_1 (k, \ell, m) := \int_{\mathbb{R}^3} e^{-i x \cdot k} e^{i(|\ell|+|m|)|x|} \frac{1}{|x|^2} (\psi_1 (x, \ell) - R_{N_2} (x, \ell)) R_{N_2} (x, m) \varphi_J (x) \, dx, \quad (8.24)$$

$$R_2 (k, \ell, m) := \int_{\mathbb{R}^3} e^{-i x \cdot k} e^{i(|\ell|+|m|)|x|} \frac{1}{|x|^2} R_{N_2} (x, \ell) \psi_1 (x, \ell) \varphi_J (x) \, dx. \quad (8.25)$$

For each of the terms in (8.23) we use the result in the first step above. Indeed we can write

$$\nu_{j_1j_2}^J (k, \ell, m) = \frac{\langle \ell \rangle^{j_1} \langle m \rangle^{j_2}}{2^{J(j_1+j_2)}} \mathcal{K}_J (k, \ell, m) \quad (8.26)$$

where $\mathcal{K}_J (k, \ell, m)$ is a building block as in (8.10) with $g_{j_1} \in \mathcal{G}^{N_2-j_1}$. Then, each one of the terms $\nu_{j_1j_2}^J$ admits an expansion as in (8.11)-(8.21), up to a multiplication by the factor $\langle \ell \rangle^{j_1} \langle m \rangle^{j_2} 2^{-J(j_1+j_2)}$. We now analyze the terms in such expansions.

**The leading order terms (8.5)-(8.6).** According to (8.14), the first term in the expansion of $\nu_{j_1j_2}^J (k, \ell, m)$ has the form

$$\frac{\langle \ell \rangle^{j_1} \langle m \rangle^{j_2}}{2^{J(j_1+j_2)}} \frac{1}{|k|} g_{j_1} (+k/|k|, \ell) g_{j_2} (+k/|k|, m) \chi (2^J (|\ell| + |m| \pm |k|)) \quad (8.27)$$

With the choice of the $-$ in the argument of $\chi$ this is a term as in (8.5)-(8.6). When the argument of $\chi$ is $2^J (|k| + |\ell| + |m|)$ we obtain a remainder term that can be absorbed in $\nu_{2,R}^1$ consistently with (8.7).

The next terms in the expansion, corresponding to (8.16)-(8.17) have the form

$$\sum_{j=0}^{M'-1} \frac{b_{j}^\pm (k, \ell, m)}{|k|^{3/2+j/2}} 2^{-J(j+3/2)} \chi_j (2^J (|\ell| + |m|)) \quad (8.28)$$
for some Schwartz functions \( \chi_j \), and with coefficients of the form, see (8.19),

\[
b_j^{\pm}(k, \ell, m) = \frac{(\ell)_{j_1}(m)_{j_2}}{2^{j_1+j_2}} c_j \partial_{\phi}^{j+1} \left( g_{j_1}(\omega, \ell) g_{j_2}(\omega, m) \right) |_{\omega=\pm k/|k|}.
\]

When the choice of the sign is ‘\(-\)’ in the argument of \( \hat{\chi}_j \), these terms belong to the sum in (8.5)–(8.6), provided we impose \( |\alpha|, |\alpha|, |\beta| \leq N_1 - N_2 - M' \). With the opposite choice of the sign we obtain smooth remainder terms satisfying the bounds (8.7).

**Remainder terms.** We have several types of remainders: those coming from the “building block” expansion of (8.23), the remainders (8.24)–(8.25), the measure \( \nu^+ \) in (8.8), and the full \( \nu_2^{+} \) when \( |K - L| \geq 5 \).

According to (8.18) and (8.20)–(8.21) in the expansion of the building block, each of the terms \( \nu^{+}_{j_1j_2}(k, \ell, m) \) in (8.23) has a remainder of the form

\[
R_{j,M',j_1,j_2}^{+}(k, \ell, m) := \frac{1}{|k|} \int_{0}^{\infty} e^{ir|k|} e^{ir(|\ell|+|m|)} 2^{-J} (\varphi/r)(r2^{-J}) B^{\pm}(r|k|; \ell, m) \, dr, \tag{8.29}
\]

\[
\left| \left( \frac{d}{dX} \right)^{\alpha} \partial_{\phi}^{\beta} B^{\pm}(X; \ell, m) \right| \lesssim X^{-M'/2-a} \sup_{\omega \in S^2} \left| \partial_{\phi}^{M'+1} \left( \partial_{\ell}^{\alpha} g_{j_1}(\omega, \ell) \partial_{m}^{\beta} g_{j_2}(\omega, m) \right) \right|. \tag{8.30}
\]

To see how these terms satisfy the desired estimate (8.7), we can use arguments similar to those given at the end of the proof of Lemma 5.2, see (5.52), so we will only give some of the details.

Using (8.29)–(8.30) we see that

\[
\left| R_{j,M',j_1,j_2}^{+}(k, \ell, m) \right| \lesssim 2^{-K} \cdot 2^{(J+K)(-M'/2)} \cdot 2^{(M'+1)A} \lesssim 2^{-K} \cdot 2^{(J+K)(-M'/4)}, \tag{8.31}
\]

which gives (8.7) with \( a = \alpha = \beta = 0 \) since \( K \geq \max(K, L, M) - 10 \) and \( J + K \geq 4A \). For the general estimates we apply derivatives and notice that, compared to the right-hand side of (8.23), each \( \nabla_{(k,\ell)} \) -derivative is going to cost an additional factor of \( \approx 2^{-K} + 2^{J} \lesssim 2^{J} \), and each \( \nabla_{m} \) -derivative is going to cost \( 2^{J} + 2^{-M} \). Then, for \( |\beta| \geq 1 \), we get

\[
\left| \nabla_{k}^{\alpha} \nabla_{\ell}^{\alpha} \nabla_{m}^{\beta} R_{j,M',j_1,j_2}^{+}(k, \ell, m) \right| \lesssim 2^{-K} \cdot 2^{(J+K)(-M'/2)} \cdot 2^{(M'+1)A} \cdot 2^{J(|\alpha|+|\alpha|) + J(2|\beta|-1)J + 2(1-|\beta|)M)}. \tag{8.32}
\]

When \( J \leq -M \) we get (8.7) since \( J + K \geq 4A, |K - L| < 5 \), and we can choose \( M' \) large enough. For \( J \geq -M \), and \( |\alpha| + |\alpha| + |\beta| \leq M'/4 - 2 \) we get an even better bound

\[
\left| \nabla_{k}^{\alpha} \nabla_{\ell}^{\alpha} \nabla_{m}^{\beta} R_{j,M',j_1,j_2}^{+}(k, \ell, m) \right| \lesssim 2^{-K} \cdot 2^{-K(|\alpha|+|\alpha|+|\beta|)}. \tag{8.32}
\]

Next, we analyze the remainders (8.24)–(8.25). These are similar to the remainder (5.58) in the proof of Proposition 5.1. The argument is similar to those given after (5.69). We only analyze (8.24), because (8.24) can be handled in the same way since each term in the expansion of \( \psi_1 - R_{N_2} \) has properties similar to \( \psi_1 \). We calculate

\[
\nabla_{k}^{\alpha} \nabla_{\ell}^{\alpha} \nabla_{m}^{\beta} R_{2}(k, \ell, m) = \int_{\mathbb{R}^3} \frac{(-ix)^{\alpha}}{|x|^2} e^{-ix \cdot k} \nabla_{\ell}^{\alpha} \left( e^{i|x||\ell|} R_{N_2}(x, \ell) \right) \times \nabla_{m}^{\beta} \left( e^{i|x||m|} \psi_1(x, m) \right) \varphi_{J}(x) \, dx, \tag{8.33}
\]
and use the estimate (3.22) for $\psi_1$ and (3.37) for $R_{N_2}$ to see that
\[ |\nabla_k^a \nabla_\ell^\beta \nabla_m^n R_2(k, \ell, m)| \lesssim 2^J \cdot 2^{[a]J} \cdot 2^{[\alpha]J} \cdot 2^{-N_2} 2^{N_2} 2^{(1-|\beta|)M}. \] (8.34)

In the case $J \leq -M, |\beta| \geq 1$, this gives
\[ |\nabla_k^a \nabla_\ell^\beta \nabla_m^n R_2(k, \ell, m)| \lesssim 2^J \cdot 2^{[a]J} \cdot 2^{[\alpha]J} \cdot 2^{-N_2} 2^{N_2} 2^{(1-|\beta|)M} \]
which is better than (8.7) since we are considering $J \geq 4A$, $\max(K, L, M) \leq A$, and $|a|+|\alpha|+|\beta| \leq N_2/2-2$.

Let us now look at $\nu^-$. From (8.8)-(8.9), and looking at $K \leq 0$ (the other case is similar), we write
\[ \nu^-(k, \ell, m) = \sum_{J \in J^c} \nu^J(k, \ell, m), \quad J^c = \{ J < -K + 4A \} \]
The arguments here are similar to the ones that follow (5.72). Using the estimates (3.22) we can see that, for $J \in J^c$
\[ |\nabla_k^a \nabla_\ell^\beta \nabla_m^n \nu^J(k, \ell, m)| = \left| \int_{\mathbb{R}^3} x^\alpha e^{-ix-k} \frac{1}{|x|^2} \nabla_\ell^\beta (e^{i|\ell||x|} \psi_1(x, \ell)) \nabla_m^n (e^{i|m||x|} \psi_1(x, m)) \varphi_J(x) \, dx \right| \]
\[ \lesssim 2^J \cdot 2^{[a]J} \cdot 2^{[\beta]J} \cdot 2^{[1-|\alpha|L]} \cdot 2^{(1-|\beta|)M} \]
\[ \lesssim 2^{-K} 2^{4A} \cdot 2^{-|\alpha|K} \cdot 2^{-|\beta|L} \cdot 2^{-K} \cdot (1 + 2^{1-|\beta|} \cdot 2^{4A(|a|+|\alpha|+|\beta|+2)}), \]

having used $J \leq -K + 4A \leq -M + 4A + 5$; this is consistent with (8.7) since $|K - L| \leq 5$.

Finally, we analyze the case $|K - L| \geq 5$. Similarly to (5.70), we can use integration by parts through the identity
\[ e^{i(-x-k+|x|(|\ell|+|m|))} = T^\rho e^{i(-x-k+|x|(|\ell|+|m|))}, \quad T := \frac{k - (x/|x|)(|\ell|+|m|)}{|k - (x/|x|)(|\ell|+|m|)|^2} \cdot i \nabla_x, \]

since $|k+(x/|x|)(|\ell|+|m|)| \gtrsim 2^{\max(K,L)}$. Using also the estimates $|\nabla_\ell^\gamma (k+(x/|x|)(|\ell|+|m|))| \lesssim 2^{-|\gamma|/2L}$ for $|\gamma| \geq 1$, and, see (3.22),
\[ |\nabla_\ell^\gamma (\psi_1(x, \ell) \psi_1(x, m))| \lesssim 2^{-|\gamma|/2L}, \]

(with the proper version of the estimates for the derivatives in $\ell$ and $m$), we see that each integration by parts through $T$ gives a gain of $2^{-J} \cdot 2^{-\max(K,L)/2}$. Applying derivatives as in (8.35) and then integrating by parts leads to the bound
\[ |\nabla_k^a \nabla_\ell^\beta \nabla_m^n \nu^J(k, \ell, m)| \lesssim 2^J \cdot 2^{[a]J} \cdot 2^{[\beta]J} \cdot 2^{[1-|\alpha|L]} \cdot 2^{(1-|\beta|)M} \cdot 2^{-|\rho|} \cdot 2^{-\max(K,L)2\rho}. \]

In the case $J \leq -L$ we can use (8.36) with $\rho = |a|+3$ to obtain (8.7). If $-L < J \leq -M$ we take instead $\rho = |a|+|\alpha|+2$. When $J > -M$ is suffices to let $\rho = |a|+|\alpha|+|\beta|+1$. This concludes the proof of the Proposition. \qed

We also have the following analogue of Proposition 8.1 for the measure $\nu_2^2$.

**Proposition 8.2** (Structure of $\nu_2^2$). Let $\nu_2^2$ be the measure defined in (8.2), with $\psi_1$ defined by (3.21). Fix $N_2 \in (5, N_1/4] \cap \mathbb{Z}$ and let $k, \ell, m \in \mathbb{R}^3$ with $|k| \approx 2^K$, $|\ell| \approx 2^L$ and $|m| \approx 2^M$, and $K, L, M \leq A$ for some $A > 0$. Then we can write
\[ \nu_2^2(k, \ell, m) = \nu_{2,+}^2(k, \ell, m) + \nu_{2,-}^2(k, \ell, m) + \nu_{2,R}^2(k, \ell, m). \] (8.37)
where:

- The leading order is
  \[
  \nu^2_{2,\pm}(k, \ell, m) = \frac{1}{|\ell|} \sum_{i=1}^{N_2} \sum_{j \in \mathbb{Z}} b^\pm_{i,j}(k, \ell, m) \cdot K_i \left( 2^{J_1(|k| + |\ell| - |m|)} \right)
  \]  
  (8.38)

with

\[
|\varphi_K(k)\varphi_L(\ell)\varphi_M(m)\nabla_k^\alpha \nabla_\ell^\beta \nabla_m^\gamma b^\pm_{i,j}(k, \ell, m)| \\
\lesssim 2^{-|\alpha|L} \cdot \left( 2^{\max(\alpha, \beta)} + 2^{(1-|a|)K_2(1-|\beta|)M} \right) \mathbf{1}_{\{\max(K, M, L) - \text{med}(K, M, L) < 5\}},
\]  
  (8.39)

for all \(K, L, M \leq A\), and \(|a| + |\alpha| + |\beta| \leq N_2\).

- The remainder term satisfies, for \(K \leq M\),
  \[
  |\nabla_k^\alpha \nabla_\ell^\beta \nabla_m^\gamma \nu^2_{2,\pm}(k, \ell, m)| \lesssim 2^{-\max(L, M, K)} \cdot 2^{-((|a|+|\alpha|)+|\beta|)\max(L, M, K)} \\
  \times \max(1, 2^{-(|\alpha|-1)K}) \cdot 2^{(|a|+|\alpha|+|\beta|+2)A}
  \]  
  (8.40)

for all \(K, L, M \leq A\) and \(|a| + |\alpha| + |\beta| \leq N_2/2 - 2\). A similar estimate holds when \(M \leq K\) by exchanging the roles of \(M\) and \(K\) (and \(a, \beta\)).

A similar statement holds for \(\nu^2_{2}(k, m, \ell)\) by exchanging the roles of \(\ell\) and \(m\). Notice how this leaves unchanged the structure of the singularities in (8.38).

Since the proof is similar to the ones above we can skip the details.

8.2. Analysis of \(\mu_2\): Bilinear estimates. Let us define, for a general measure \(\nu\) and symbol \(b\),

\[
T_\nu[b](g, h)(x) = T[\nu; b](g, h)(x) := \mathcal{F}_{k \rightarrow x}^{-1} \int_{\mathbb{R}^3} g(\ell) h(m) b(k, \ell, m) \nu(k, \ell, m) \, dk \, dm.
\]  
  (8.41)

Using Proposition 8.2 and Lemma 8.5 we can establish Hölder-type bilinear bounds for pseudo-products involving the measure \(\mu_2\) in (8.2) (Theorem 8.3) and vectorfields acting on its components (Theorem 8.4).

**Theorem 8.3** (Bilinear bounds 2). Consider the operator \(T_{\mu_2}[b]\), defined according to (8.41) and (8.2), and assume that:

- The symbol \(b\) is such that
  \[
  \text{supp} \, b \subseteq \{(k, \ell, m) \in \mathbb{R}^9 : |k| + |\ell| + |m| \leq 2^A, |\ell| \approx 2^L, |m| \approx 2^M\},
  \]  
  (8.42)

for some \(A \geq 1\).

- For all \(|k| \approx 2^K, |\ell| \approx 2^L\) and \(|m| \approx 2^M\)
  \[
  |\nabla_k^\alpha \nabla_\ell^\beta \nabla_m^\gamma b(k, \ell, m)| \lesssim 2^{-K|a|} 2^{-|\alpha|L} 2^{-|\beta|M} \cdot 2^{(|a|+|\alpha|+|\beta|)A},
  \]  
  (8.43)

\(|a|, |\alpha|, |\beta| \leq 5\).

Then, for any \(p, q \in [2, \infty)\)

\[
\frac{1}{p} + \frac{1}{q} > \frac{1}{2},
\]  
(8.44)
the following estimate holds:

\[ \| P_K T \mu_2[b](g, h) \|_{L^2} \lesssim \| \hat{g} \|_{L^p} \| \hat{h} \|_{L^q} \cdot 2^{\max(K, L, M)} \cdot 2C_0 A \]  \hspace{1cm} (8.45) 

for some sufficiently large \( C_0 \).

**Theorem 8.4** (Bilinear bounds with vectorfields 2). Let \( \nu_1^2 \) be given as in Proposition 8.7 and \( \nu_2^2, \nu_2^2 \) as in Proposition 8.2. With the same notation and assumptions on \( b \) and \( (p, q) \) as in Theorem 8.3, the following hold:

(i) Let

\[ X_\pm = \pm \partial_{|\ell|} + \partial_{|m|}, \]  \hspace{1cm} (8.46) 

Then, for \( a = 1, 2 \),

\[ \left\| P_K T[X_\pm \nu_2^2; b](g, h) \right\|_{L^2} \lesssim \left\| \hat{g} \right\|_{L^p} \left\| \hat{h} \right\|_{L^q} \cdot 2^{(1-a) \max(K, L, M)} \cdot 2(C_0+12)A, \]  \hspace{1cm} (8.47) 

and

\[ \left\| P_K T[X_\pm \nu_2^2, \nu_2^2; b](g, h) \right\|_{L^2} \lesssim \left\| \hat{g} \right\|_{L^p} \left\| \hat{h} \right\|_{L^q} \cdot 2^{-a \min(L, M)} \cdot 2^{\med(K, L, M)} \cdot 2(C_0+12)A. \]  \hspace{1cm} (8.48) 

(ii) Let

\[ Y_\pm = \partial_k \pm \frac{k}{|k|} \left( \frac{\ell}{|\ell|} \cdot \partial_k \right). \]  \hspace{1cm} (8.49) 

Then, for \( a = 1, 2 \), we have

\[ \left\| T[Y_\pm \nu_1^2; b](g, h) \right\|_{L^2} \lesssim \left\| \hat{g} \right\|_{L^p} \left\| \hat{h} \right\|_{L^q} \cdot 2^{(1-a) \max(K, L, M)} \cdot 2(C_0+12)A, \]  \hspace{1cm} (8.50) 

The same estimate holds for \( T[Y_\pm \nu_2^2; b]. \)

Moreover, we have

\[ \left\| T[Y_\pm \nu_2^2, \nu_2^2; b](g, h) \right\|_{L^2} \lesssim \left\| \hat{g} \right\|_{L^p} \left\| \hat{h} \right\|_{L^q} \cdot 2^{-a L} \cdot 2^{\max(K, L, M)} \cdot 2(C_0+12)A; \]  \hspace{1cm} (8.51) 

the same estimate holds for \( T[Y_\pm \nu_2^2, \nu_2^2; b]. \)

(iii) Define the cutoffs

\[ \chi^\pm(k, \ell, m) := \varphi_{\geq -10} \left( \frac{|k| \pm |\ell| - |m|}{|k| + |\ell| + |m|} \right). \]  \hspace{1cm} (8.52) 

Then, for \( a = (a_1, a_2) \) with \( 1 \leq |a| \leq 2 \), we have

\[ \left\| T[\nabla^a_{(\ell, m)}(v \chi^-); b](g, h) \right\|_{L^2} \lesssim \left\| \hat{g} \right\|_{L^p} \left\| \hat{h} \right\|_{L^q} \cdot 2^{-|a_1| L^2 - |a_2| M} \cdot 2^{\max(K, L, M)} \cdot 2(C_0+12)A, \]  \hspace{1cm} (8.53) 

for \( v \in \{ \nu_1^2, \nu_2^2 \} \).

Theorem 8.3 gives bilinear estimates for operators involving \( \mu_2 \) which are analogous to the estimates of Theorems 6.1 and 6.2 for \( \mu_1 \).

To prove Theorem 8.3, the key ingredient is the following:
Lemma 8.5 (Bilinear operators restricted to small annuli 2). Let \( j \geq 1, \sigma_1, \sigma_2 \in \{+, -\} \) and consider the bilinear operator

\[
B_j^{\sigma_1, \sigma_2}[b](g, h)(x) = \mathcal{F}_{k-l}^{-1} \int \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} \hat{g}(\ell) \hat{h}(m) b(k, \ell, m) \chi(2^j (|k| + |\sigma_1| |\ell| + |\sigma_2|m|)) \, d\ell \, dm,
\]

where \( \chi \) is a Schwartz function and

- The support of \( b \) satisfies

\[
supp(b) \subseteq \{(k, \ell, m) \in \mathbb{R}^3 : |k| \approx 2^K, |\ell| \approx 2^L, |m| \approx 2^M \},
\]

with \(-j \ll \max(K, L, M) \leq A \) for some \( A \geq 1 \);

- The following estimates hold

\[
|\nabla_k \nabla_\ell \nabla_m^\beta b(k, \ell, m)| \leq 2^{-|\alpha|} 2^{-|\alpha| L} 2^{-|\beta|M} 2^{(|\alpha| + |\alpha| + |\beta|)A}, \quad |\alpha|, |\alpha|, |\beta| \leq 4.
\]

Then

\[
\|B_j^{\sigma_1, \sigma_2}[b](g, h)\|_{L^2} \lesssim 2^{-j} \cdot 2^{\min(K, L, M) + \text{med}(K, L, M)} \cdot 2^{16A} \cdot \|g\|_{L^p} \|h\|_{L^q},
\]

where \( \hat{f} = \mathcal{F}(f) \) denotes the (flat) Fourier transform of \( f \). Moreover, for all \( p, q \in [2, \infty] \) with \( 1/p + 1/q = 1/2 \),

\[
\|B_j^{\sigma_1, \sigma_2}[b](g, h)\|_{L^2} \lesssim 2^{-j/2} \cdot 2^{\min(K, L, M) + (3/2) \text{med}(K, L, M)} \cdot 2^{16A} \cdot \|g\|_{L^p} \|h\|_{L^q}.
\]

Lemma 8.5 is the analogue of Lemma 6.4 which was used to prove the bilinear estimates for \( \mu_1 \) in Theorems 6.1 and 6.2. The proof is in a similar spirit as the one of Subsection 6.5 but it is more involved, so we give the details below.

Note how, exactly as in (6.34), we gain a factor of \( 2^{-j} \) in (8.57). However, in this case, such a gain is not necessary to obtain the boundedness of the bilinear operator associated to \( \mu_2 \), see (8.1)–(8.2). Indeed, from (8.5) in Proposition 8.1 (and (8.38) in Proposition 8.2) we see that \( \mu_2 \) is a linear combination of operators like those in Lemma 8.5 with coefficients that are uniformly controlled independently of \( j \), in contrast with the case of the more singular distribution \( \mu_1 \), see (5.2) and (5.3).

The estimate (8.57) is sharp relative to the dependence on \( j \) but we only allow the pair of Hölder exponents \((2, \infty)\) there. Since in the nonlinear estimates for the evolution equation we also need different Hölder pairs \( (p, q) \), we will end up using only (8.58) in the proof of Theorems 8.3, 8.4, and 8.7. The smaller gain of \( 2^{-j/2} \) is still more than sufficient to deal with \( \mu_2 \) and \( \mu_3 \).

Proof of Lemma 8.5. We only consider the case \( \sigma_1 = \sigma_2 = - \), since the cases \( \sigma_1 \sigma_2 = - \) are similar, and the case \( \sigma_1 = \sigma_2 = + \) is empty. We denote \( B_j[b](g, h)(x) = B_j^{-}[b](g, h)(x) \), insert cutoffs according to the support restriction (8.55) and write

\[
\langle B_j[b](g, h)(x), f \rangle = \int \int \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} A(x, y, z) f(x) g(y) h(z) \, dx \, dy \, dz,
\]

\[
A(x, y, z) := \int \int \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} e^{ix_k} e^{-iy_\ell} e^{-iz_m} \varphi_K(k) \varphi_L(\ell) \varphi_M(m) b(k, \ell, m) \chi(2^j (|k| - |\ell| - |m|)) \, d\ell \, dm \, dk.
\]
Without loss of generality, by symmetry we may assume $|\ell| \geq |m|$ (or $L \geq M$). In view of $-j \leq \max(K, L) - 10$ and the fact that $\chi$ is Schwartz, we may also assume $|K - L| \leq 5$, for otherwise the kernel is a regular one and the desired bound follows more easily. We also set $b \equiv 1$ for convenience; it will be clear to the reader what minor modification in the arguments are needed for a general $b$ satisfying the assumptions in the statement.

Given a parameter $\lambda$, we introduce an angular partition of unity of $\mathbb{S}^2$ adapted to polar caps of aperture $\lambda$. We consider a family $E_{\lambda}$ of unit vectors $\{e_i\}_{i=1,\ldots,N}$, $N \approx \lambda^{-2}$, uniformly spaced, and associated cutoffs

$$q_{e,\lambda}(k) := \frac{1}{\sum_{e'} \phi_{\leq 0}(|\frac{k}{|k|} - \frac{e'}{|e'|}|^2 \frac{1}{\lambda^2})} \phi_{\leq 0}(|\frac{k}{|k|} - \frac{e}{|e|}|^2 \frac{1}{\lambda^2}), \quad e \in E_{\lambda}$$

and projections given by

$$\widehat{Q_{e,\lambda}} \hat{f}(k) := q_{e,\lambda}(k) \hat{f}(k).$$

Let

$$\lambda_B := 2^{-j/2} 2^{-B/2}$$

for $B = L, M$ or $K$, and define $E_{\lambda,L,M,K} := E_{\lambda_K} \times E_{\lambda_L} \times E_{\lambda_M}$. We write

$$\langle B_j[1](g, h)(x), f \rangle = \sum_{(e_0,e_1,e_2) \in E_{\lambda,L,M,K}} \langle B_j,e_0,e_1,e_2(Q_{e_1,5\lambda_L g}, Q_{e_2,5\lambda_M h})(x), Q_{e_0,5\lambda_K f} \rangle$$

where

$$\langle B_j,e_0,e_1,e_2(f_1, f_2), f_0 \rangle = \int \int \int_{\mathbb{R}^3} A_{e_0,e_1,e_2}(x, y, z) f_0(x) f_1(y) f_2(z) \, dx \, dy \, dz,$$

$$A_{e_0,e_1,e_2}(x, y, z) := \int \int \int_{\mathbb{R}^3} e^{i x \cdot k} e^{-iy \cdot \ell} e^{-iz \cdot m} \phi_K(k) \phi_L(\ell) \phi_M(m)$$

$$\times q_{e_0,\lambda_k}(k) q_{e_1,\lambda_L}(\ell) q_{e_2,\lambda_M}(m) \chi(2^j |k| - |\ell| - |m|) \, dl \, dm \, dk.$$ 

Note that there are $\approx \lambda_{M}^{-2} \cdot \lambda_{K}^{-2}$ elements in the sum over the parameters $e_0, e_2$, and therefore

$$|\langle B_j[1](g, h)(x), f \rangle| \lesssim 2^{2j} 2^{K+M}$$

$$\times \sup_{e_0 \in E_{\lambda_K}, e_2 \in E_{\lambda_M}} \sum_{e_1 \in E_{\lambda_L}} |\langle B_j,e_0,e_1,e_2(Q_{e_1,5\lambda_L g}, Q_{e_2,5\lambda_M h})(x), Q_{e_0,5\lambda_K f} \rangle|$$

We thus fix $e_0, e_2$ and claim that it suffices to prove that the kernel satisfies

$$|A_{e_0,e_1,e_2}(x, y, z)| \lesssim \frac{2^{-j+L}}{(1 + 2^{-j+L}|x - y|^2)^2} \frac{2^{-j+M}}{(1 + 2^{-j+M}|x - z|^2)^2} \frac{2^L}{1 + (2^L e_1 \cdot (x - y))^2} \frac{2^M}{1 + (2^M e_2 \cdot (x - z))^2} \cdot 2^{-3j}.$$  

(8.64)
Let us assume (8.64) and show how it implies the desired conclusions. Up to rotating the variables \(k\) (and \(x\)) in (8.62) we may assume that \(e_0 = e_1\). Then we estimate
\[
\left| \langle B_j, e_1, e_2 \rangle (Q_{e_1, 5} g, Q_{e_2, 5} h) (x), Q_{e_1, 5} f \rangle \right| 
\lesssim 2^{-3j} \int_{\R^3} \int_{\R^3} \left| Q_{e_1, 5} f (x) \right| \left| Q_{e_1, 5} g (y + x) \right| \left| Q_{e_2, 5} h (z + x) \right| 
\times \frac{2^{-j+L}}{(1 + 2^{-j+L} |y|^2)^q} \frac{2^{L}}{1 + (2^L e_1 \cdot y)^2} \frac{2^M}{1 + (2^M e_2 \cdot z)^2} \, dx dy dz 
\lesssim 2^{-3j} \left\| Q_{e_1, 5} f \right\|_{L^2} \left\| Q_{e_1, 5} g \right\|_{L^p} \left\| Q_{e_2, 5} h \right\|_{L^q},
\]
for \(1/p + 1/q = 1/2\). Summing over \(e_1 \in E_\lambda\), using Cauchy-Schwartz and orthogonality, we have
\[
\sum_{e_1 \in E_\lambda} \left| \langle B_j, e_1, e_2 \rangle (Q_{e_1, 5} g, Q_{e_2, 5} h) (x), Q_{e_1, 5} f \rangle \right| 
\lesssim 2^{-3j} \left[ \sum_{e_1 \in E_\lambda} \left\| Q_{e_1, 5} f \right\|_{L^2}^2 \right]^{1/2} \left[ \sum_{e_1 \in E_\lambda} \left\| Q_{e_1, 5} g \right\|_{L^p}^2 \right]^{1/2} \left\| h \right\|_{L^\infty} 
\lesssim 2^{-3j} \cdot \left\| f \right\|_{L^2} \left\| g \right\|_{L^p} \left\| h \right\|_{L^\infty}.
\]
Together with (8.63) which gives a \(2^{2j} 2^{L+M}\) factor we obtain (8.57). For (8.58) we only need to slightly modify the last estimate above using
\[
\sum_{e_1 \in E_\lambda} \left\| Q_{e_1, 5} f \right\|_{L^2} \left\| Q_{e_1, 5} g \right\|_{L^p} \left\| Q_{e_2, 5} h \right\|_{L^q} 
\lesssim \left[ \sum_{e_1 \in E_\lambda} \left\| Q_{e_1, 5k} f \right\|_{L^2}^2 \right]^{1/2} \left[ \sum_{e_1 \in E_\lambda} \left\| Q_{e_1, 5\lambda_l} g \right\|_{L^p}^2 \right]^{1/2} \left\| h \right\|_{L^q} 
\lesssim \left\| f \right\|_{L^2} \left\| g \right\|_{L^p} \left\| E_\lambda \right\|_{L^q} \left\| h \right\|_{L^q} 
\lesssim 2^{j/2} 2^{L/2} \cdot \left\| f \right\|_{L^2} \left\| g \right\|_{L^p} \left\| h \right\|_{L^q},
\]

**Proof of (8.64).** Rotating \(k\) and \(m\), we may assume \(e_0 = e_1 = e_2\). Changing variables \(k \mapsto k + \ell + m\), it suffices to look at
\[
A_{e_1, e_1, e_1} (x, Y, Z) = \int_{\R^3} \int_{\R^3} \int_{\R^3} e^{ix \cdot k} e^{-iY \cdot \ell} e^{-iZ \cdot m} \varphi_K (k + \ell + m) \varphi_L (\ell) \varphi_M (m) 
\times q_{e_1, \lambda_k} (k + \ell + m) q_{e_1, \lambda_L (\ell)} q_{e_1, \lambda_M (m)} (m) \chi (2^j (|k + \ell + m| - |\ell| - |m|)) \, d\ell dm dk
\]
and prove
\[
\left| A_{e_1, e_1, e_1} (x, Y, Z) \right| \lesssim \frac{2^{-j+L}}{(1 + 2^{-j+L} Y^2)^q} \frac{2^{L}}{1 + (2^L e_1 \cdot Y)^2} \frac{2^M}{1 + (2^M e_2 \cdot Z)^2} \cdot 2^{-3j}.
\]
The idea is to integrate by parts using the vectorfields \(2^{-j} \Delta \ell\) and \(2^{-j} \Delta m\) as well as the scaling vectorfields \(\ell \cdot \nabla \ell\) and \(m \cdot \nabla m\), taking advantage of the fact that the three frequencies...
\( \ell, m \) and \( k + \ell + m \) are essentially aligned. Observe that
\[
\left| 2^{-j} \Delta_{\ell} \left[ q_{e_1,\lambda_K}(k + \ell + m)q_{e_1,\lambda_L}(\ell) \chi(2^j(|k + \ell + m| - |\ell| - |m|)) \right] \right| \\
\lesssim 2^{-j2^{-2k}\lambda_K^{-2} + 2^{-j}2^{-2L}\lambda_L^{-2} + 2^j(\lambda_K^2 + \lambda_L^2)} \\
\approx 2^{-L},
\] (8.67)

having used that, on the support of the integral,
\[
|(k + \ell + m)\ell/k + \ell + m| - \ell_i/|\ell| \lesssim \lambda_K + \lambda_L
\]
and that \(-j/2 \leq K, L\). A similar estimate holds replacing \( \ell \) by \( m \) and \( L \) by \( M \).

We also have
\[
\ell \cdot \nabla_\ell q_{e_1,\lambda_L}(\ell) = 0,
\]
\[
\left| \ell \cdot \nabla_\ell \left[ q_{e_1,\lambda_K}(k + \ell + m)\chi(2^j(|k + \ell + m| - |\ell| - |m|)) \right] \right| \lesssim 1,
\] (8.68)

with similar estimates for \( m \cdot \nabla_m \). To see this last inequality, recall the definition \( q_{e,\lambda} \) and calculate
\[
\ell \cdot \nabla_\ell \frac{(k + \ell + m)a}{|k + \ell + m|} = \ell \cdot \frac{\delta_{ia}}{|k + \ell + m|} - \ell \cdot \frac{(k + \ell + m)_a}{|k + \ell + m|^3}
\]
\[
= \ell_a \frac{\delta_{ia}}{|k + \ell + m|} - \frac{\ell \cdot \ell_a \ell_i}{|k + \ell + m||\ell|^2} + O((\lambda_K + \lambda_L) \cdot 2^{L-K}),
\]

Therefore
\[
|\ell \cdot \nabla_\ell q_{e_1,\lambda_K}(k + \ell + m)| \lesssim 1 + \lambda_L \lambda_K^{-1} \approx 1.
\]

Similarly, we calculate
\[
\ell \cdot \nabla_\ell (|k + \ell + m| - |\ell| - |m|) = |\ell| \left[ \frac{\ell}{|\ell|} \cdot \frac{k + \ell + m}{|k + \ell + m|} - 1 \right]
\]
\[
= O(2^L) \sin^2 \angle(k + \ell + m, \ell) = O(2^L(\lambda_K + \lambda_L)^2) = O(\lambda_K),
\]

and see that
\[
|\ell \cdot \nabla_\ell \chi(2^j(|k + \ell + m| - |\ell| - |m|))| \lesssim 2^j \cdot 2^L(\lambda_K + \lambda_L)^2 \approx 1.
\]

Using integration by parts through the identities \((1 - 2^{-j}2^L \Delta_\ell) e^{Y_\ell} = (1 + 2^{-j}2^L|Y|^2) e^{iY_\ell} \) and \(|1 - (2^L e_1 \cdot \nabla_\ell)^2| e^{Y_\ell} = |1 + (2^L e_1 \cdot Y)^2| e^{iY_\ell} \) (and similarly for \( e^{iz_m} \)) together with (8.67)-(8.68), we obtain
\[
|A_{e_1,e_1,e_1}(x,Y,Z)| \lesssim \frac{1}{(1 + 2^{-j+L}|Y|^2)(1 + 2^{-j+M}|Z|^2)^2} \frac{1}{1 + (2^L e_1 \cdot Y)^2} \frac{1}{1 + (2^M e_1 \cdot Z)^2}
\]
\[
\times \int \int \int_{\mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^4} \varphi_K(k + \ell + m) \varphi_L(\ell) \varphi_M(m) q_{e_1,\lambda}(k + \ell + m) q_{e_1,\lambda}(m)
\]
\[
\times \chi(2^j(|k + \ell + m| - |\ell| - |m|)) \, d\ell \, dm \, dk.
\] (8.69)
We then notice that, at fixed \( \ell \) and \( m \), the integral over \( k \) is taken over a region where \( |k| \lesssim 2^{-j} \), since
\[
\left| |\ell| + |m| - |\ell + m| \right| = O((\lambda_M + \lambda_L)^2 2^M) \lesssim 2^{-j},
\]
\[
\left| k + \ell + m - |\ell + m| - |k| \right| = O((\lambda_L + \lambda_K)^2 2^K) \lesssim 2^{-j}.
\]
Finally, since the integrals in \( \ell \) and \( m \) contribute the volumes of integration \( \lambda_L^2 2^{3L} = 2^{-j} 2^{3L} \) and \( \lambda_M^2 2^{3M} = 2^{-j} 2^{3M} \) respectively, we arrive at (8.69).

**Proof of Theorems 8.3 and 8.4.** The proof of Theorem 8.3 follows from the structural Propositions 8.1 and 8.2 and Lemma 8.5. We can use the same arguments as those in Subsections 6.3 and 6.4; note that, actually, the situation here is substantially better since (8.4) and (8.37) do not contain any singular contribution like \( \nu_0 \) in (5.10)-(5.12); moreover, the summations in (8.5) and (8.38) have coefficients uniformly bounded in \( J \), unlike (5.13).

### 8.3. Analysis of \( \mu_3 \): structure and bilinear estimates.
Recall the definition
\[
\mu_3(k, \ell, m) = \int \frac{e^{i(|k|+|\ell|+|m|)x}}{|x|^2} \psi_1(x, k)\psi_1(x, \ell)\psi_1(x, m) \, dx. \tag{8.70}
\]
We have the following analogues of Propositions 5.1 and 8.1.

**Proposition 8.6 (Structure of \( \mu_3 \)).** Let \( \mu_3 \) be defined as in (8.70) with \( \psi_1 \) defined by (3.21). Fix an integer \( N_2 \in [10, N_1/4] \). Let \( k, \ell, m \in \mathbb{R}^3 \) with \( |k| \approx 2^K, |\ell| \approx 2^L \) and \( |m| \approx 2^M \), and assume that \( K, L, M \leq A \) for some \( A > 0 \). Then we can write
\[
\mu_3(k, \ell, m) = \mu_{3,0}(k, \ell, m) + \mu_{3,R}(k, \ell, m), \tag{8.71}
\]
where:

1. The leading order has the form
\[
\mu_{3,0}(k, \ell, m) = \sum_{i=0}^{N_2} \sum_{J \in \mathbb{Z}} b_{i,J}(k, \ell, m) \cdot K_i(2^J(|k| - |\ell| - |m|)) \tag{8.72}
\]
with \( K_i \in \mathcal{S} \), and
\[
|\varphi_K(k)\varphi_L(\ell)\varphi_M(m)\nabla^a
abla^\beta
\nabla_\ell \nabla_m b_{i,J}(k, \ell, m) |
\lesssim \max(1, 2^{-(|a|-1)K}) \max(1, 2^{-(|\alpha|-1)L}) \max(1, 2^{-(|\beta|-1)M}) 1_{|K - \max(L, M)| < 5} \tag{8.73}
\]
for all \( K, L, M \leq A \), and \( |a| + |\alpha| + |\beta| \leq N_2/2 \).

2. The remainder term satisfies
\[
|\nabla^a_k \nabla^\beta_\ell \nabla_m \mu_{3,R}(k, \ell, m) | \lesssim \max(1, 2^{-(|a|-1)K}) \max(1, 2^{-(|\alpha|-1)L}) \max(1, 2^{-(|\beta|-1)M}) \cdot 2^{2A(|a|+|\alpha|+|\beta|+1)} \tag{8.74}
\]
for all \( K, L, M \leq A \) and \( |a| + |\alpha| + |\beta| \leq N_2/2 \).
Proof. Notice how (8.70) is easier to treat than $\mu_1$ and $\mu_2$ since the exponential factor is just a radial function and there is no need to apply stationary phase arguments on the sphere. We let $\mu_3 = \mu_+^3 + \mu_3^-$, where

$$
\mu_3^+(k, \ell, m) := \sum_{J \in [4A, \infty) \cap \mathbb{Z}} \mu_{3, J}(k, \ell, m), \quad \mu_3^-(k, \ell, m) := \sum_{J \in [0, A) \cap \mathbb{Z}} \mu_{3, J}(k, \ell, m),
$$

$$
\mu_{3, J}(k, \ell, m) := \int_{0}^{\infty} e^{i(-|k|+|\ell|+|m|)r} \left( \int_{S^2} \psi_1(r\omega, k) \psi_1(r\omega, \ell) \psi_1(r\omega, m) \, d\omega \right) \varphi^{(0)}_J(r) r^{-1} \, dr.
$$

To isolate the leading order $\mu_{3, 0}(k, \ell, m)$ within $\mu_3^+$, we expand the three $\psi_1$ functions in negative powers of $r$ through Lemma 3.1, writing, according to (3.31),

$$
\psi_1 = \psi_{N_2} + R_{N_2}.
$$

All the contributions that do not contain a reminder term $R_{N_2}$, give rise to linear combinations of terms of the form

$$
\langle k \rangle^{j_1} (\ell) j_2 (m) j_3 \int_{0}^{\infty} e^{i(-|k|+|\ell|+|m|)r} \left( \int_{S^2} g_{j_1}(\omega, k) g_{j_2}(\omega, \ell) g_{j_3}(\omega, m) \, d\omega \right) r^{-j_1-j_2-j_3-1} \varphi(2^{-j}r) \, dr
$$

$$
= \left( \int_{S^2} g_{j_1}(\omega, k) g_{j_2}(\omega, \ell) g_{j_3}(\omega, m) \, d\omega \right) \langle k \rangle^{j_1} (\ell) j_2 (m) j_3 \frac{2^{j_1+j_2+j_3}}{2^{j_1+j_2+j_3}} \cdot \chi (2^j (|k| - |\ell| - |m|))
$$

(8.75)

where $g_{j_i} \in L^{N_1-j_i}$ and $\chi = \chi_{j_1+j_2+j_3} = \mathcal{F}(r^{-j_1-j_2-j_3-1}\varphi)$ is a Schwartz function. We then let

$$
b_i(k, \ell, m) = \sum_{j_1+j_2+j_3 = i} \frac{\langle k \rangle^{j_1} (\ell) j_2 (m) j_3}{2^{j_1+j_2+j_3}} \int_{S^2} g_{j_1}(\omega, k) g_{j_2}(\omega, \ell) g_{j_3}(\omega, m) \, d\omega
$$

(8.76)

for $i = 0, \ldots, N_2$, recall that $K, L, M \leq A \leq J/4$, and use the estimate (3.31) for the $g_j$ factors, to obtain (8.72) and (8.73). All the terms of the form (8.75) with $j_1 + j_2 + j_3 > N_2$ can be absorbed into the remainder terms, since the losses of $2^j$ factors coming from differentiating $\chi$ can be compensated by the factor $2^{-j(j_1+j_2+j_3)}$.

The other terms remaining in $\mu_3^+$ are those containing at least an $R_{N_2}(x, \cdot, \cdot)$ function, such as

$$
\int e^{i(-|k|+|\ell|+|m|)|x|} |x|^3 R_{N_2}(x, k) \psi_1(x, \ell) \psi_1(x, m) \varphi(2^{-j}x) \, dx
$$

(8.77)

and similar terms obtained by exchanging the role of the frequencies, or putting $\psi_{N_2}$ instead of $\psi_1$. Using the estimate for $R_{N_2}$ in (3.31), and the estimates for $\psi_1$ in (3.22), we can directly see that (8.77) satisfies the bounds (8.74).

Finally, we look at $\mu_3^-$. The estimate for $a = \alpha = \beta = 0$ is obvious since $|\mu_{3, J}| \lesssim 1$. The estimates with derivatives follow directly from the estimates (3.22) for $\psi_1$ and the bound $|\nabla q e^{irq}| \lesssim 2^A (2^A|q|-1) + |q|^{-|q|+1})$, which holds for $|r| \lesssim 2^{2A}$.

\[ \square \]

Using Proposition 8.6 and Lemma 8.5 we can obtain:

**Theorem 8.7 (Bilinear bounds for $\mu_3$).** Let $T[\mu_3; b]$ be the operator defined according to the notation (3.31), with $b$ satisfying the same assumptions in Theorem 8.3. Then, with the same assumptions and notation of Theorem 8.3, we have

$$
\| T[\mu_3; b] (g, h) \|_{L^r} \lesssim \| \hat{g} \|_{L^p} \| \hat{h} \|_{L^q} \cdot 2^{C_{bA}}.
$$

(8.78)
Moreover, $T[\mu_3; b]$ satisfies the same estimates satisfied by $\nu_2^1$ in Theorem 8.4, namely, the bound (8.47) for $T[X^a_\mu; b]$, the bound (8.50) for $T[Y^a_\mu; b]$, and the bound (8.53) for $T[\nabla_\ell^m(\chi_{-\mu_3}); b]$.

The proof Theorem 8.7 is essentially the same of Theorem 8.3 so we can skip it.

9. Weighted estimates for lower order terms

In this last section we establish a priori nonlinear estimate for the components of Duhamel’s formula involving $\mu_2$ and $\mu_3$, that is $D_2$ and $D_3$ as defined by (8.1)-(8.3); at the end we also discuss the estimates for $D_0$. We want to show that, under the a priori assumptions (2.23),

$$\|\partial_k D_1(t)\|_{L_2^k} \lesssim \varepsilon_1^2, \quad i = 0, 2, 3,$$

(9.1)

and

$$\|\partial_k^2 D_1(t)\|_{L_2^k} \lesssim \varepsilon_1^2(t)^{1/2+\delta}, \quad i = 0, 2, 3. \quad (9.2)$$

In view of the bilinear bounds from Theorems 8.3 and 8.7, to treat $D_2$ and $D_3$ we can use arguments similar to those we have used in Section 7 for the leading order term $D_1$. It also suffices to highlight the key steps for $D_2$.

9.1. Estimates for $D_2$. We write

$$D_2(t)(f, f) = D_{2,1}(t)(f, f) + D_{2,2}(t)(f, f)$$

$$D_{2,1}(t)(f, f) := \int_0^t \int e^{is\Phi(k, \ell, m)} \tilde{f}(s, \ell) \tilde{f}(s, m) \nu_2^1(k, \ell, m) d\ell dm ds, \quad (9.3)$$

$$D_{2,2}(t)(f, f) := 2 \int_0^t \int e^{is\Phi(k, \ell, m)} \tilde{f}(s, \ell) \tilde{f}(s, m) \nu_2^2(k, \ell, m) d\ell dm ds, \quad (9.4)$$

$$\Phi(k, \ell, m) := -|k|^2 + |\ell|^2 + |m|^2.$$

9.1.1. Estimates for (9.3). Proposition 8.4 and Theorem 8.4 show that $\nu_2^1(k, \ell, m)$ is regular in the direction $\partial_\ell - \partial_m$. We then let $X_1 := (1/2)(\partial_\ell - \partial_m)$ and calculate

$$(|\ell| - |m|) X_1 \Phi = (|\ell| - |m|)^2 = \Phi + |k|^2 - 2|m||\ell|$$

$$\quad = \Phi + |\ell|^2 + |m|^2 + (|k| - |\ell| - |m|)(|k| + |\ell| + |m|) \quad (9.5)$$

This identity allows us to integrate by parts in (9.3) close to the singularity of the kernel when $||k| - |\ell| - |m|| \ll |\ell| + |m| \approx |k|$ using

$$\frac{1}{c_1(k, \ell, m)} \left( \frac{1}{i s} (|\ell| - |m|) X_1 + i\partial_s \right) e^{is\Phi(k, \ell, m)} = e^{is\Phi(k, \ell, m)}, \quad (9.6)$$

$$c_1(k, \ell, m) := |k|^2 - 2|\ell||m|,$$

which is analogous to (7.20) with the estimates (7.21) replaced by

$$|\nabla_\ell^a \nabla_\ell^\alpha \nabla_\ell^\beta | \frac{1}{c_1(\ell, m)} \lesssim \frac{1}{|\ell|^2 + |m|^2 + |k|^2} |k|^{-|\alpha|} |\ell|^{-|\beta|} |m|^{-|\beta|}. \quad (9.7)$$

We can then proceed similarly to in 7.1.3 after observing that: (1) the factor $|\nabla_\ell \Phi| = |2k| \approx |\ell| + |m|$ can be used to cancel part of the mild singularities introduced by the $1/c_1$ factors,
and (2) $\partial_k \nu^1_2$ can be handled through the estimates of part (ii) in Theorem 8.4 using the vectorfield $\dot{Y}_+$, similarly to how this was handled in 7.3.1 using $Y'$.

Next, we look at the region away from the singularity, that is, we analyze \textup{(9.3)} after inserting the cutoff $\chi^+$ defined in \textup{(8.52)}. Recall that, without loss of generality, we may assume $|\ell| \geq |m|$, and note that if $|k| \not\approx |\ell|$ then $|\Phi| \gtrsim \max(|k|^2, |\ell|^2)$, with $1/\Phi$ a nice symbol; then integration by parts in time suffices. For $|k| \approx |\ell| \gtrsim (s)^{-1/2+}$, instead we can integrate by parts in $\nabla_{\ell}$ using

\begin{equation}
\begin{aligned}
e^{is\Phi} &= \frac{1}{2is} |\ell|^2 \cdot \nabla_{\ell} e^{is\Phi}.
\end{aligned}
\end{equation}

With the Hölder estimates in part (iii) of Theorem 8.3 and using that $\partial_k \Phi \ell/|\ell|^2$ is a bounded admissible symbol, we can close our estimates. When instead $|k| \approx |\ell| \lesssim (s)^{-1/2+}$ we can use an $L^6 \times L^3$ bound, and the smallness of $|\partial_k \Phi| \lesssim |k|$ to obtain the bound \textup{(9.2)} for $D_2^1$.

9.1.2. \textbf{Estimates for \textup{(9.4)}}. Recall that we split $\nu_2 = \nu^2_{2,+} + \nu^2_{2,-} + \nu_{2,R}$; see Proposition 8.4. It suffices to treat the leading orders $\nu^2_{2,+}$ and $\nu^2_{2,-}$.

\textbf{The case of $\nu^2_{2,-}$}. When we consider $\nu^2_{2,-}$ the singularity is relative to $|\ell| + |m| - |k|$, see \textup{(8.38)}, and we would like proceed as in the case of $\nu^2_1$ above. We need however to be more careful to treat some of the terms coming from the integration by parts using \textup{(9.6)} because of the less effective bilinear estimates that involve derivatives of $\nu^2_{2,-}$, see \textup{(8.48)} and \textup{(8.51)}, compared to those for derivatives of $\nu^2_1$, see \textup{(6.16)} and \textup{(6.26)}.

More precisely, let us look at $D^2_{2,-}$, defined in the natural way similarly to \textup{(9.4)}. Applying $\partial_k^2$ to it, using \textup{(9.6)} to integrate by parts, and $X_1 = -(1/2)X_-$, see \textup{(8.46)}, gives

\begin{equation}
\|D^2_{2,-}(t)\|_{L^2_k} \lesssim \|A_1(t)\|_{L^2_k} + \|A_2(t)\|_{L^2_k} + \|A_3(t)\|_{L^2_k} + \cdots
\end{equation}

with

\begin{align}
A_1(t)(f,f) &= \int_0^t \int \int e^{is\Phi(k,\ell,m)} b(k,\ell,m) \partial_{|\ell|} \tilde{f}(s,\ell) \tilde{f}(s,m) X_- \nu^2_{2,-}(k,\ell,m) \, d\ell dm ds, \quad \text{\textup{(9.10)}} \\
A_2(t)(f,f) &= \int_0^t \int \int e^{is\Phi(k,\ell,m)} b(k,\ell,m) \tilde{f}(s,\ell) \partial_{|m|} \tilde{f}(s,m) X_- \nu^2_{2,-}(k,\ell,m) \, d\ell dm ds, \quad \text{\textup{(9.11)}} \\
A_3(t)(f,f) &= \int_0^t \int \int e^{is\Phi(k,\ell,m)} b(k,\ell,m) \tilde{f}(s,\ell) \tilde{f}(s,m) X^2 \nu^2_{2,-}(k,\ell,m) \, d\ell dm ds, \quad \text{\textup{(9.12)}}
\end{align}

where

\begin{equation}
b(k,\ell,m) := \frac{|k|^2(|\ell| - |m|)^2}{c^2(k,\ell,m)}.
\end{equation}

The ‘\cdots’ in \textup{(9.9)} are terms easier to bound or similar to the ones obtained when dealing with $D^1_2$ and $D_1$ before. Note how we have included in the ‘\cdots’ also bilinear terms with kernel measure $Y^2_+ \nu^2_{2,-}$, that are coming from $\partial_k$ hitting $\nu^2_{2,-}$; comparing the estimates \textup{(8.51)} and \textup{(8.48)} we see that these are not worse than \textup{(9.10)}-\textup{(9.12)}. Moreover, note that ‘\cdots’ do not contain terms with a measure $Y^2_+ X_- \nu^2_{2,-}$.

Since $b$ is an admissible symbol, we may disregard it in what follows and just assume $b \equiv 1$ for convenience. We may also assume that the integrals \textup{(9.10)}-\textup{(9.12)} are localized as usual to $|k| \approx 2^K$, $|\ell| \approx 2^L$, $|m| \approx 2^M$ and $s \approx 2^S$ (omitting some of these localization for lighter
We observe that the following analogue of (9.5) holds: let the “good direction” be \( \nu \), having used the same input function:

\[
\|A_1(t)(f, f)\|_{L^2_k} \lesssim \int_0^t \|P_k T[X_{-\nu^2_{2,-}}, b(e^{is|k|^2} \partial_k \tilde{f}(s), \tilde{F} u(s))\|_{L^2} \tau_S(s) ds.
\]

Applying (8.48), followed by Bernstein, together with the a priori decay bound (1.7), we obtain

\[
\|A_1(t)(f, f)\|_{L^2_k} \lesssim \int_0^t \|P_{-\nu} \tilde{F} e^{is|k|^2} \partial_k \tilde{f}(s)\|_{L^6} \|\tilde{F} \tilde{u}(s)\|_{L^3} \cdot 2^{-L - 2(C_0 + 13)},\tau_S(s) ds
\]

\[
\lesssim \int_0^t \|P_{-\nu} \tilde{F} e^{is|k|^2} \partial_k \tilde{f}(s)\|_{L^2} \cdot \varepsilon 2^{-S/2} \cdot 2^{(C_0 + 14)\tau_S(s)} ds
\]

\[
\lesssim \varepsilon 2^{S/2 + \delta}.
\]

having used \( 2^A \leq 2^{\delta N} \) with \((C_0 + 14)\delta_N < \delta\).

The term (9.11) is estimated using again (8.48), and then Bernstein followed by (4.7) on the same input function:

\[
\|A_2(t)(f, f)\|_{L^2_k} \lesssim \int_0^t \|P_{-\nu} \tilde{F} \tilde{u}(s)\|_{L^6} \|\tilde{F} e^{is|k|^2} \partial_k \tilde{f}(s)\|_{L^2} \cdot 2^{-L - 2(C_0 + 13)}\tau_S(s) ds
\]

\[
\lesssim \int_0^t \|P_{-\nu} \tilde{F} \tilde{u}(s)\|_{L^3} \cdot \varepsilon \cdot 2^{(C_0 + 13)}\tau_S(s) ds
\]

\[
\lesssim \int_0^t \varepsilon 2^{-S/2} \cdot \varepsilon \cdot 2^{(C_0 + 14)\tau_S(s)} ds \lesssim \varepsilon 2^{S/2 + \delta}.
\]

The last term can be bounded using (8.48), Hardy’s inequality \( 2^{-L} \|P_{-\nu} g\|_{L^2} \lesssim \|\partial_k \tilde{F}^{-1} g\|_{L^2} \) and the usual a priori bounds:

\[
\|A_3(t)(f, f)\|_{L^2_k} \lesssim \int_0^t \|P_{-\nu} \tilde{F} \tilde{u}(s)\|_{L^6} \|\tilde{F} \tilde{u}(s)\|_{L^3} \cdot 2^{-2L} \cdot 2^{(C_0 + 13)}\tau_S(s) ds
\]

\[
\lesssim \int_0^t 2^{-L} \|P_{-\nu} \tilde{F} \tilde{u}(s)\|_{L^2} \cdot \varepsilon 2^{-S/2} \cdot 2^{(C_0 + 14)}\tau_S(s) ds
\]

\[
\lesssim \varepsilon 2^{S/2 + \delta}.
\]

The case of \( \nu^2_{2,+} \). Finally, we consider \( \nu^2_{2,+} \) whose kernel is singular relative to \(-|\ell| + |m| - |k|\).

We observe that the following analogue of (9.5) holds: let the “good direction” be \( X_2 := (1/2)(\partial_{|\ell|} + \partial_{|m|}) \), then

\[
\frac{1}{2}(|\ell| + |m|)X_2 \Phi = \Phi + |k|^2 + 2|\ell||m|
\]

\[
= \Phi + |\ell|^2 + |m|^2 + (|k| + |\ell| - |m|)(|k| - (|\ell| - |m|)).
\]

This gives us an identity as in (9.6) with the slightly different coefficient \( c_2(k, \ell, m) := |k|^2 + 2|\ell||m| \), which still satisfies proper symbol estimates as in (1.7) in the region \(|k| + |\ell| - |m| \ll |m| \approx |k| + |\ell|\). We may again assume \(|k| \lesssim \max(|\ell|, |m|)\) for otherwise integration by parts in \( s \) suffices; then, the factor \( \nabla_k \Phi = 2k \) which appears when we differentiate in \( k \) the exponential \( e^{is\Phi} \), can again be used to cancel part of the singularities introduced by factors of \( 1/c_2 \approx \max(|\ell|, |m|)^{-2} \). This takes care of the singular region.
In the region away from the singularity of $\nu_{2,+}$, when $||k| + |\ell| - |m|| \gtrsim \max(|m|, |k| + |\ell|)$, we can proceed exactly as in the case of $\nu_{1,2}$ above, relying on (9.8) with $m$ instead of $\ell$, and the bilinear estimate (8.53) of Theorem 8.4.

9.2. Nonlinear Estimates for $D_0$. At last, we discuss how to estimate the nonlinear term $D_0$ corresponding to the $\delta$ interaction in the expansion (5.1) of $\mu$. This is the nonlinearity in the “flat” equation, that is, (1.7) with $V = 0$, which has been treated in the works [20] and [13]. We want to show that, under the a priori assumptions (2.2 3),

$$\|\partial_k D_0(t)\|_{L^2} + (t)^{-1/2-\delta} \|\partial_k^2 D_2(t)\|_{L^2} \lesssim \varepsilon_1^2.$$ (9.14)

These bounds are obviously easier than those for the other terms we have analyzed. However, we cannot directly utilize the bounds of [20] and [13] because of the different functional spaces we are considering here, and, more precisely, the different time decay rate of our solution (which is weaker) and time growth rates of our norms. Nevertheless, bounds for $D_0$ follow from simple adaptations of the arguments in Section 7, so we just give a sketch of the proof.

Let us look at the bounds for the highest weighted norm. Applying $\Delta^2_k$ to the expression for $D_0$ in (5.7) gives the term

$$\int_0^t \int \int 4s^2 |\ell|^2 e^{i s \Phi(k,\ell)} \widetilde{f}(s, \ell) \widetilde{f}(s, k-\ell) d\ell ds,$$

$$\Phi(k, \ell) := -|k|^2 + |\ell|^2 + |k-\ell|^2 = -2k \cdot \ell + 2|\ell|^2,$$

up to simpler terms where the derivatives hit the profile $\widetilde{f}$. We notice that $(3\ell - k) \cdot \nabla_k \Phi = 2(|\ell|^2 + |k|^2) + 5\Phi$, and therefore

$$e^{is\Phi(k,\ell)} = \frac{1}{2(|\ell|^2 + |k|^2)} \left( \frac{1}{is} (3\ell - k) \cdot \nabla_k + 5i \partial_s \right) e^{is\Phi(k,\ell)}.$$ (9.16)

This identity plays the same role as the identity (7.20) or (9.6) and allows us to integrate by parts in $\ell$ or $s$. With this, the proof can proceed as for the term (7.45) in 7.2.2 with the distribution $\nu_1$ substituted by a delta measure. We first integrate by parts to obtain terms similar to (7.48)-(7.58) and then estimate all these through standard product estimate of Coifman-Meyer type (see for example Proposition 4.5) in place of the bilinear estimates for $\nu_1$.

This concludes the proof of the a priori bounds (9.1)-(9.2) and gives us the main Proposition 2.1 which implies Theorem 1.1.
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University of Toronto
\textit{E-mail address:} fabiop@math.toronto.edu

Rutgers University
\textit{E-mail address:} soffer@math.rugers.edu