CUBIC POLYNOMIALS: A MEASURABLE VIEW ON PARAMETER SPACE

ROMAIN DUJARDIN

Abstract. We study the fine geometric structure of bifurcation currents in the parameter space of cubic polynomials viewed as dynamical systems. In particular we prove that these currents have some laminar structure in a large region of parameter space, reflecting the possibility of quasiconformal deformations. On the other hand, there is a natural bifurcation measure, supported on the closure of rigid parameters. We prove a strong non laminarity statement relative to this measure.

1. Introduction

The study of closed positive currents associated to bifurcations of holomorphic dynamical systems is a topic under rapidly growing interest [DeM1, DeM2, BB, Ph, DF]. The underlying thesis is that bifurcation currents should contain a lot of information on the geography of parameter spaces of rational maps of the Riemann sphere.

The simplest case where the currents are not just measures is when the parameter space under consideration has complex dimension 2: a classical such example is the space Poly_3 of cubic polynomials. The purpose of the present article is to give a detailed account on the fine structure of bifurcation currents and the bifurcation measure (to be defined shortly) in this setting. We chose to restrict ourselves to cubics because, thanks to the existing literature, the picture is much more complete in this case; nevertheless it is clear that many results go through in a wider context.

The work of Branner and Hubbard [BH1, BH2] is a standard reference on the parameter space of cubic polynomials. A crude way of comparing our approach to theirs would be to say that in [BH1, BH2] the focus was more on the topological properties of objects while here we concentrate on their measurable and complex analytic properties – this is exactly what positive closed currents do.

Let us describe the setting more precisely (see Section 2 for details). Consider the cubic polynomial $f_{c,v}(z) = z^3 - 3c^2z + 2c^3 + v$, with critical points at $\pm c$. Since the critical points play symmetric roles, we can focus on $+c$. We say that $c$ is passive near the parameter $(c_0, v_0)$ if the family of holomorphic functions $(c, v) \mapsto f_{c,v}^n(c)$ is locally normal. Otherwise, $c$ is said to be active. It is classical that bifurcations always occur at parameters with active critical points.

Let $C^+$ be the set of parameters for which $+c$ has bounded orbit (and similarly for $C^-$). It is a closed, unbounded subset in parameter space, and it can easily be seen that the locus where $c$ is active is $\partial C^+$. We also denote by $C = C^+ \cap C^-$ the connectedness locus.

The Green function $z \mapsto G_{f_{c,v}}(z)$ depends plurisubharmonically on $(c,v)$ so the formulas $(c,v) \mapsto G_{f_{c,v}}(\pm c)$ define plurisubharmonic functions in the space $C_{c,v}$ of cubic polynomials.

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We define $T^{\pm} = dd^c G_{f,c}(\pm c)$ as the bifurcation currents respectively associated to $\pm c$. Notice that $T_{\text{bif}} = T^+ + T^-$ is the standard bifurcation current, associated to the variation of the Lyapounov exponent of the maximal entropy measure, as considered in [DeM1, DeM2, BB, Ph]. It is an easy fact that $\text{Supp}(T^\pm)$ is the activity locus of $c$, that is, $\text{Supp}(T^\pm) = \partial C^\pm$.

The geometric intuition underlying positive closed currents is that of analytic subvarieties. Nevertheless, the “geometry” of general positive closed currents can be quite poor. Laminar currents form a class of currents with rich geometric structure, well suited for applications in complex dynamics. We say that a positive closed current is laminar if it can be written as an integral of compatible disks $T = \int [D_\alpha] d\nu(\alpha)$. Here compatible means that two disks do not have isolated intersection points. The local geometry of a laminar current can still be very complicated. On the other hand we say that $T$ is uniformly laminar if the disks $D_\alpha$ are organized as laminations. The local geometry is tame in this case: a uniformly laminar current is locally the “product” of the leafwise integration along the leaves by a transverse measure on transversals.

The main point in our study is to look for some laminar structure for the bifurcation currents in some regions of parameter space. To understand why some laminarity should be expected, it is useful to mention some basics on deformations. We say that two rational maps are deformations of each other if there is a $J$-stable (in the sense of MSS) family connecting them. There is a natural stratification of the space of cubic polynomials, according to the dimension of the space of deformations. The mappings in the bifurcation locus correspond to dimensions 0 and 1. The dimension 1 case occurs for instance when one critical point is active and the other one is attracted by an attracting cycle. The bifurcation locus contains a lot of holomorphic disks near such a parameter, and it is not a surprise that the structure of the bifurcation current indeed reflects this fact. On the other hand we will see that the bifurcation measure $T^+ \wedge T^- = T^2_{\text{bif}}$ concentrates on the “most bifurcating” part of the parameter space, that is, on the closure of the set of parameters with zero dimensional deformation space.

Let us be more specific. Assume first that we are outside the connectedness locus, say $-c$ escapes to infinity (hence is passive). The structure of $C^+ \setminus C$ has been described by Branner-Hubbard [BH1, BH2], and also Kiwi [K] from a different point of view. There is a canonical deformation of the maps outside $C$, called the wringing operation, and $C^+$ locally looks like a holomorphic disk times a closed transversal set which is the union of countably many copies of the Mandelbrot set and uncountably many points. Our first result is as follows (Proposition 3.5 and Theorem 3.12).

**Theorem 1.** The current $T^+$ is uniformly laminar outside the connectedness locus, and the transverse measure gives full mass to the point components.

The delicate part of the theorem is the statement about the transverse measure. It follows from an argument of similarity between the dynamical and parameter spaces at the measurable level, together with a symbolic dynamics construction in the spirit of DDS. We also give a new and natural proof of the continuity of the wringing operation, which is a central result in [BH1].

Assume now that $-c$ is passive but does not escape to infinity (this corresponds to parameters in $\text{Int}(C^-)$). This happens when $-c$ is attracted towards an attracting cycle, and, if the hyperbolicity conjecture holds, this is the only possible case. We have the following result (Theorem 4.1 and Proposition 4.2).
Theorem 2. The current $T^+$ is laminar in $\text{Int}(C^-)$, and uniformly laminar in components where $-c$ is attracted to an attracting cycle.

In particular laminarity holds regardless of the hyperbolic nature of components. The proof is based on a general laminarity criterion due to De Thélin [dT1]. The proof of uniform laminarity in the hyperbolic case is more classical and follows from quasiconformal surgery.

As a corollary of the two previous results, we thus get the following.

Corollary 3. The current $T^+$ is laminar outside $\partial C^+ \cap \partial C^-$. 

In the last part of the paper, we concentrate on the remaining part of the parameter space. More precisely we study the structure of the bifurcation measure $\mu_{\text{bif}} = T^+ \wedge T^-$, which is supported in $\partial C^+ \cap \partial C^-$. Understanding this measure was already one of the main goals in [DF], where it was proved that Supp($\mu_{\text{bif}}$) is the closure of Misiurewicz points (i.e. strictly critically preperiodic parameters), and where the dynamical properties of $\mu_{\text{bif}}$-a.e. parameters were investigated. It is an easy observation that $\mu_{\text{bif}}$-a.e. polynomial has zero dimensional deformation space (Proposition 5.1).

It follows from [DF] that the currents $T^\pm$ are limits, in the sense of currents, of the curves $\text{Per}^\pm(n,k) = \{(c,v) \in C^2, f^n(\pm c) = f^k(\pm c)\}$. These curves intersect at Misiurewicz points –at least the components where $\pm c$ are strictly preperiodic do. The laminarity of $T^\pm$ and the density of Misiurewicz points could lead to the belief that the local structure of $\mu_{\text{bif}}$ is that of “geometric intersection” of the underlying “laminations” of $T^+$ and $T^-$. The next theorem asserts that the situation is in fact more subtle (Theorem 5.6).

Theorem 4. The measure $\mu_{\text{bif}}$ does not have local product structure on any set of positive measure.

Observe that this can also be interpreted as saying that generic wringing curves do not admit continuations through $\partial C$. As a perhaps surprising consequence of this result and previous work of ours [Du1, Du2], we obtain an asymptotic lower bound on the (geometric) genus of the curves $\text{Per}^\pm(n,k(n))$ as $n \to \infty$, where $0 \leq k(n) < n$ is any sequence. Recall that the geometric genus of a curve is the genus of its desingularization; also, we define the genus of a reducible curve as the sum of the genera of its components. Since these curves are possibly very singular (for instance at infinity in the projective plane), the genus cannot be directly read from the degree.

Theorem 5. For any sequence $k(n)$ with $0 \leq k(n) < n$,

$$\frac{1}{3^n} \text{genus}(\text{Per}^\pm(n,k(n))) \to \infty.$$ 

Finally, there is a striking analogy between the parameter space of cubic polynomials with critical points marked and dynamical spaces of polynomial automorphisms of $C^2$. This has already be remarked earlier: for instance, as far as the global topology of the space is concerned, the reader should compare the papers [BH1, HO]. We give in Table 1 a list of similarities and dissimilarities between the two. Of course there is no rigorous link between the two columns, and this table has to be understood more as a guide for the intuition. For notation and basic concepts on polynomial automorphisms of $C^2$, we refer
Parameter space of cubics

| Polynomial automorphisms of \( \mathbb{C}^2 \) |
|---|
| **\( C^+, \partial C^+, C, \partial C^+ \cap \partial C^- \)** | **\( K^+, I^+, K, J \)** |
| **\( T^+, T^-, \mu_{bil} \)** | **\( T^+, T^-, \mu \)** |
| \( \frac{1}{\pi} \lim \{ \text{Per}^+(n, k) \} \to T^+ \) | \( \frac{1}{\pi} \lim \{ (f^{\pm n})^*(L) \} \to T^+ \) |
| \( \text{Supp}(\mu_{bil}) \) is the Shilov boundary of \( C \) | \( \text{Supp}(\mu) \) is the Shilov boundary of \( K \) |
| \( T^+ \) are laminar outside \( \partial \mathcal{C}^+ \cap \partial \mathcal{C}^- \) | \( W^s/\alpha \) are equidistributed |
| \( \text{Supp}(\mu_{bil}) = \{ \text{Misiurewicz pts} \} \) | \( \text{Supp}(\mu) = \{ \text{saddle pts} \} \) |
| \( T^+ \) are laminar | \( T^+ \) are extremal near infinity |
| \( T^+ \) intersects all algebraic subvarieties | \( T^+ \) intersects all algebraic subvarieties |
| Point components have full transverse measure | The same when it makes sense |
| \( T^+ \) intersects all alg. subv. but the \( \text{Per}^+(n) \) (Prop. \[2.8\]) | \( T^+ \) intersects all algebraic subvarieties |
| \( T^\pm \) are not extremal near infinity (Cor. \[3.14\]) | \( T^+ \wedge T^- \) is a geometric intersection |
| \( T^\pm \wedge T^- \) is not a geometric intersection (Th. \[1.4\]) | ?? |
| Uniform laminarity outside \( C \) | ?? |

Table 1. Correspondences between \( \text{Poly}_3 \) and polynomial automorphisms of \( \mathbb{C}^2 \).

to \[\text{BS1, BLS, FS, Si}\]. On the other hand, Milnor’s article \[\text{Mi1}\] exhibits similar structures between parameter spaces of cubic polynomials and quadratic Hénon maps.

We structure the paper is as follows. In Section \[2\] we recall some basics on the dynamics on cubic polynomials and laminar currents. In Section \[3\] we explain and reprove some results of Branner-Hubbard and Kiwi on the escape locus, and prove Theorem \[1\]. Theorem \[2\] is proved in Section \[4\]. Lastly, in \[5\] we discuss a notion of higher bifurcation based on the dimension of the space of deformations, and prove Theorems \[4\] and \[5\].

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## 2. Preliminaries

### 2.1. Stability. Active and passive critical points. The notion of stability considered in this article will be that of \( J \)-stability, in the sense of \[\text{MSS}\].

**Definition 2.1.** Let \( (f_\lambda)_{\lambda \in \Lambda} \) be a family of rational maps, parameterized by a complex manifold \( \Lambda \). We say that \( f_\lambda \) is \( J \)-stable (or simply stable) if there is a holomorphic motion of the Julia sets \( J(f_\lambda) \), compatible with the dynamics.

Also, we say that \( f_\lambda \) is a deformation of \( f_\lambda' \) if there exists a \( J \)-stable family connecting them.

It is well known that if \( f_\lambda \) is a deformation of \( f_\lambda' \), then \( f_\lambda \) and \( f_\lambda' \) are quasiconformally conjugate in the neighborhoods of their respective Julia sets. Notice that there is a stronger notion of stability, where the conjugacy is required on the whole Riemann sphere. This is the one considered for instance in \[\text{McMS}\] for the definition of the Teichmüller space of a rational map. The stronger notion introduces some distinctions which are not relevant from our point of view, like distinguishing the center from the other parameters in a hyperbolic component.

It is a central theme in our study to consider the bifurcations of critical points one at a time. This is formalized in the next classical definition.
Definition 2.2. Let \((f_\lambda, c(\lambda))_{\lambda \in \Lambda}\) be a holomorphic family of rational maps with a marked (i.e. holomorphically varying) critical point. The marked critical point \(c\) is passive at \(\lambda_0 \in \Lambda\) if \(\{\lambda \mapsto f_\lambda^n c(\lambda)\}_{n \in \mathbb{N}}\) forms a normal family of holomorphic functions in the neighborhood of \(\lambda_0\). Otherwise \(c\) is said to be active at \(\lambda_0\).

Notice that by definition the passivity locus is open while the activity locus is closed. This notion is closely related to bifurcation theory of rational maps, as the following classical proposition shows.

Proposition 2.3 ([Ly, McM2]). Let \((f_\lambda)\) be a family of rational maps with all its critical points marked (which is always the case, by possibly replacing \(\Lambda\) with some branched cover). Then the family is \(J\)-stable iff all critical points are passive.

2.2. The space of cubic polynomials. It is well known that the parameter space of cubic polynomials modulo affine conjugacy has complex dimension 2. It has several possible presentations (see Milnor [Mi1] for a more complete discussion).

The most commonly used parametrization is the following: for \((a, b) \in \mathbb{C}^2\), put

\[ f_{a,b}(z) = z^3 - 3a^2z + b. \]

The reason for the \(a^2\) is that it allows the critical points (+\(a\) and −\(a\)) to depend holomorphically on \(f\).

Two natural involutions in parameter space are of particular interest. First, the involution \((a, b) \mapsto (−a, b)\) preserves \(f_{a,b}\) but exchanges the marking of critical points. In particular both critical points play the same role.

The other interesting involution is \((a, b) \mapsto (−a, −b)\). It sends \(f_{a,b}\) to \(f_{a,−b}\) which is conjugate to \(f\) (via −id). It can be shown that the moduli space of cubic polynomials modulo affine conjugacy is \(\mathbb{C}^2_{a,b}/((a, b) \sim (−a, −b))\), which is not a smooth manifold.

In the present paper we will use the following parametrization (cf Kiwi [K]): \((c, v) \in \mathbb{C}^2\), we put

\[ f_{c,v}(z) = z^3 - 3c^2z + 2c^3 + v = (z - c)^2(z + 2c) + v. \]

The critical points here are ±\(c\), so they are both marked, and the critical values are \(v\) and \(v + 4c^3\). The involution exchanging the marking of critical points then takes the form \((c, v) \mapsto (−c, v + 4c^3)\).

Notice also that the two preimages of \(v\) (resp. \(v + 4c^3\)) are \(c\) and −2\(c\) (resp. −\(c\) and 2\(c\)). The points −2\(c\) and 2\(c\) are called cocritical.

The advantage of this presentation is that it is well behaved with respect to the compactification of \(\mathbb{C}^2\) into the projective plane. More precisely, it separates the sets \(\mathcal{C}^+\) and \(\mathcal{C}^-\) at infinity (see below Remark 3.10).

Notice finally that in [DF], we used a still different parameterization,

\[ f_{c,a}(z) = \frac{1}{3}z^3 - \frac{1}{2}cz^2 + a^3, \]

where the critical points are 0 and \(c\). In these coordinates, the two currents respectively associated to 0 and \(c\) have the same mass (compare with Proposition 2.6 below).
2.3. Loci of interest in parameter space. We give a list of notation for the subsets in parameter space which will be of interest to us:

- $\mathcal{C}^+$ the set of parameters for which $+c$ has bounded orbit. Similarly $\mathcal{C}^-$ is associated to $-c$.
- $\text{Per}^+(n)$ the set of parameters for which $+c$ has period $n$, and $\text{Per}^+(n,k)$ the set of cubic polynomials $f$ for which $f^k(c) = f^n(c)$. Similarly for $\text{Per}^-(n)$ and $\text{Per}^-(n,k)$.
- $\mathcal{C} = \mathcal{C}^+ \cap \mathcal{C}^-$ the connectedness locus.
- $\mathcal{C}_2 \setminus \mathcal{C}$ the escape locus.
- $\mathcal{C}_2 \setminus (\mathcal{C}^+ \cup \mathcal{C}^-)$ the shift locus.

The sets $\mathcal{C}^\pm$ and $\mathcal{C}$ are closed. Branner and Hubbard proved in [BH1] that the connectedness locus is compact and connected\(^1\).

The $\text{Per}^\pm(n,k)$ are algebraic curves. Since $\text{Per}^+(n) \subset \mathcal{C}^+$, we see that $\mathcal{C}^+$ is unbounded. It is easy to prove that the activity locus associated to $\pm c$ is $\partial \mathcal{C}^\pm$. Consequently, the bifurcation locus is $\partial \mathcal{C}^+ \cup \partial \mathcal{C}^-$.

**Remark 2.4.** Let $\Omega$ be a component of the passivity locus associated to, say, $+c$. We say that $\Omega$ is hyperbolic if $c$ converges to an attracting cycle throughout $\Omega$. As an obvious consequence of the density of stability [MSS], if the hyperbolicity conjecture holds, all passivity components are of this type. It would be interesting to describe their geometry.

2.4. Bifurcation currents. Here we apply the results of [DF] to define various plurisubharmonic functions and currents in parameter space, and list their first properties. We refer to Demailly’s survey article [De] for basics on positive closed currents. The support of a current or measure is denoted by $\text{Supp}(\cdot)$, and the trace measure by $\sigma$. Also $[V]$ denotes the integration current over the subvariety $V$.

We identify the parameter $(c,v) \in \mathbb{C}^2$ with the corresponding cubic polynomial $f$. For $(f,z) = (c,v,z) \in \mathbb{C}^2 \times \mathbb{C}$, the function $(f,z) \mapsto G_f(z) = G_{f(c,v)}(z)$ is continuous and plurisubharmonic. We define

$$G^+(c,v) = G_f(c) \quad \text{and} \quad G^-(c,v) = G_f(-c).$$

The functions $G^+$ and $G^-$ are continuous, nonnegative, and plurisubharmonic. They have the additional property of being pluriharmonic when positive.

We may thus define $(1,1)$ closed positive currents $T^\pm = dd^c G^\pm$, which will be referred to as the **bifurcation currents associated to $\pm c$**. From [DF, §3] we get that $\text{Supp}(T^\pm)$ is the activity locus associated to $\pm c$, that is, $\text{Supp}(T^\pm) = \partial \mathcal{C}^\pm$.

It is also classical to consider the Lyapounov exponent $\chi = \log 3 + G^+ + G^-$ of the equilibrium measure, and the corresponding current $T_{\text{bif}} = dd^c \chi = T^+ + T^-$ known as the **bifurcation current** [DeM1, BB]. Its support is the bifurcation locus.

The following equidistribution theorem was proved in [DF].

**Theorem 2.5.** Let $(k(n))_{n \geq 0}$ be any sequence of integers such that $0 \leq k(n) < n$. Then

$$\lim_{n \to \infty} \frac{1}{3^n} \left[ \text{Per}^\pm(n,k(n)) \right] = T^\pm.$$

We can compute the mass of the currents $T^\pm$. The lack of symmetry is not a surprise since $c$ and $-c$ do not play the same role with respect to $v$.

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\(^1\)Our notation $\mathcal{C}^+$ corresponds to $B^-$ in [BH2], and $E^-$ in [K].
Proposition 2.6. The mass of \( T^+ \) with respect to the Fubini-Study metric on \( \mathbb{C}^2 \) is \( 1/3 \). On the other hand the mass of \( T^- \) is \( 1 \).

Proof. We use the equidistribution theorem. Indeed a direct computation shows that the degree of \( \text{Per}^+(n) \) is \( 3^{n-1} \) while the degree of \( \text{Per}^-(n) \) is \( 3^n \).

We will mainly be interested in the fine geometric properties, in particular laminarity, of the currents \( T^\pm \). A basic motivation for this is the following observation.

Proposition 2.7. Let \( \Delta \) be a holomorphic disk in \( \mathbb{C}^2 \) where \( G^+ \) and \( G^- \) are harmonic. Then the family \( \{f \in \Delta\} \) is \( J \)-stable.

This holds in particular when \( \Delta \subset \mathbb{C}^+ \setminus \partial \mathbb{C}^- \).

Proof. Recall that \( G^+ \) is nonnegative, and pluriharmonic where it is positive. Hence if \( G^+|_{\Delta} \) is harmonic, either \( G^+ \equiv 0 \) (i.e. \( c \) does not escape) on \( \Delta \), or \( G^+ > 0 \) (i.e. \( c \) always escapes) on \( \Delta \). In any case, \( c \) is passive. Doing the same with \( -c \) and applying Proposition 2.3 then finishes the proof.

The laminarity results will tell us that there are plenty of such disks, and how these disks are organized in space: they will be organized as (pieces of) laminations.

We can describe the relative positions of \( T^+ \) and algebraic curves, thus filling a line in Table 1. Notice that \( G^+ \) is continuous so \( T^+ \wedge [V] = \dd c^2(G^+|_{V}) \) is always well defined. Also, if \( V \subset \text{Supp}(T^+) \), then \( G^+|_{V} = 0 \), so \( T^+ \wedge [V] = 0 \).

Proposition 2.8. Let \( V \) be an algebraic curve such that \( T^+ \wedge [V] = 0 \) then \( V \subset \text{Per}^+(n,k) \) for some \((n,k)\). If moreover \( \text{Supp}(T^+) \cap V = \emptyset \), then \( V \subset \text{Per}^+(n) \) for some \( n \).

Proof. If \( T^+ \wedge [V] = 0 \), then \( c \) is passive on \( V \), which is an affine algebraic curve, so by [DF, Theorem 2.5], \( V \subset \text{Per}(n,k) \) for some \((n,k)\).

Assume now that \( \text{Supp}(T^+) \cap V = \emptyset \), and \( V \) is a component of \( \text{Per}^+(n,k) \) where \( c \) is strictly preperiodic at generic parameters (i.e. outside possibly finitely many exceptions). Then there exists a persistent cycle of length \( n - k \) along \( V \), on which \( c \) falls after at most \( k \) iterations. We prove that this leads to a contradiction.

The multiplier of this cycle defines a holomorphic function on \( V \) which is thus constant. If \( f_0 \in V \), \( c \) is passive in the neighborhood, so the multiplier cannot be greater than 1. On the other hand if the multiplier is a constant of modulus \( \leq 1 \) along \( V \), we claim that the other critical point must have bounded orbit: indeed if the cycle is attracting, there is a critical point in the immediate basin of the cycle, which cannot be \( c \) since \( c \) is generically strictly preperiodic. Now, if the multiplier is a root of unity or is of Cremer type, then the cycle must be accumulated by an infinite critical orbit, necessarily that of \( -c \) (see e.g. [MB3]), and if there is a Siegel disk, its boundary must equally lie in the closure of an infinite critical orbit. In any case, both critical points have bounded orbits, so \( K \) is connected. Since \( V \) is unbounded, this contradicts the compactness of the connectedness locus [BH].

2.5. Background on laminar currents. Here we briefly introduce the notions of laminarity that will be considered in the paper. It is to be mentioned that the definitions of flow boxes, laminations, laminar currents, etc. are tailored for the specific needs of this paper, hence not as general as they could be. For more details on these notions, see [BLS, Du1, Du2]; see also [Gh] for general facts on laminations and transverse measures.

We first recall the notion of direct integral of a family of positive closed currents. Assume that \( (T_a)_{a \in A} \) is a measurable family of positive closed currents in some open subset \( \Omega \subset \mathbb{C}^2 \),
and $\nu$ is a positive measure on $\mathcal{A}$ such that (reducing $\Omega$ is necessary) $\alpha \mapsto \text{Mass}(T_\alpha)$ is $\nu$-integrable. Then we can define a positive closed current $T = \int T_\alpha d\nu(\alpha)$ by the obvious pairing with test forms

$$\langle T, \varphi \rangle = \int \langle T_\alpha, \varphi \rangle d\nu(\alpha).$$

Let now $T$ and $S$ are positive closed currents in a ball $\Omega \subset \mathbb{C}^2$. We say that the wedge product $T \wedge S$ is admissible if for some (hence for every) potential $u_T$ of $T$, $u_T$ is locally integrable with respect to the trace measure $\sigma_S$ of $S$. In this case we may classically define $T \wedge S = dd^c(u_T S)$ which is a positive measure. The next lemma shows that integrating families of positive closed currents is well behaved with respect to taking wedge products.

**Lemma 2.9.** Let $T = \int T_\alpha d\nu(\alpha)$ as above, and assume that $S$ is a positive closed current such that the wedge product $T \wedge S$ is admissible. Then for $\nu$-a.e. $\alpha$, $T_\alpha \wedge S$ is admissible and

$$T \wedge S = \int \langle T_\alpha \wedge S \rangle d\nu(\alpha).$$

**Proof.** The result is local so we may assume $\Omega$ is the unit ball. Assume for the moment that there exists a measurable family of nonpositive psh functions $u_\alpha$, with $dd^cu_\alpha = T_\alpha$ and $\|u_\alpha\|_{L^1(\Omega')} \leq C \text{Mass}(T_\alpha)$, for every $\Omega' \Subset \Omega$. In particular we get that $\alpha \mapsto \|u_\alpha\|_{L^1(\Omega')}$ is $\nu$-integrable, and the formula $u_T = \int u_\alpha d\nu(\alpha)$ defines a non positive potential of $T$. Then since $u_T \in L^1(\sigma_S)$ and all potentials are non positive, we get that for $\nu$-a.e. $\alpha$, $u_\alpha \in L^1(\sigma_S)$, and $u_T S = \int \langle u_\alpha S \rangle d\nu(\alpha)$ by Fubini’s Theorem.

It remains to prove our claim. Observe first that if we are able to find a potential $u_\alpha$ of $T_\alpha$, with $\|u_\alpha\|_{L^1(\Omega')} \leq C \text{Mass}(T_\alpha)$, then, by slightly reducing $\Omega'$, we can get a non positive potential with controlled norm by substracting a constant, since by the submean inequality, $\sup_{\Omega'} u_\alpha \leq C(\Omega'') \|u_\alpha\|_{L^1(\Omega')}$. Another observation is that it is enough to prove it when $T_\alpha$ is smooth and use regularization.

The classical way of finding a potential for a positive closed current of bidegree (1,1) in a ball is to first use the usual Poincaré lemma for $d$, and then solve a $\overline{\partial}$ equation for a $(0,1)$ form (see [De]). The Poincaré lemma certainly preserves the $L^1$ norm (up to constants) because it boils down to integrating forms along paths. Solving $\overline{\partial}$ with $L^1$ control in a ball is much more delicate but still possible (see for instance [Ra, p.300]). This concludes the proof.

By flow box, we mean a closed family of disjoint holomorphic graphs in the bidisk. In other words, it is the total space of a holomorphic motion of a closed subset in the unit disk, parameterized by the unit disk. It is an obvious consequence of Hurwitz’ Theorem that the closure of any family of disjoint graphs in $\mathbb{D}^2$ is a flow box. Moreover, the holonomy map along these family of graphs is automatically continuous. This follows for instance from the $\Lambda$-lemma of [MSS].

Let $\mathcal{L}$ be a flow box, written as the union of a family of disjoint graphs as $\mathcal{L} = \bigcup_{\alpha \in \tau} \Gamma_\alpha$, where $\tau = \mathcal{L} \cap \{(0) \times \mathbb{D}\}$ is the central transversal. To every positive measure $\nu$ on $\tau$, there corresponds a positive closed current in $\mathbb{D}^2$, defined by the formula

$$T = \int_{\tau} [\Gamma_\alpha] d\nu(\alpha).$$

A lamination in $\Omega \subset \mathbb{C}^2$ is a closed subset of $\Omega$ which is locally biholomorphic to an open subset of a flow box $\mathcal{L} = \bigcup \Gamma_\alpha$. A positive closed current supported on a lamination is said to be uniformly laminar and subordinate to the lamination if it is locally of the form (2). Not
every lamination carries a uniformly laminar current. This is the case if and only if there exists an invariant transverse measure, that is, a family of positive measures on transversals, invariant under holonomy. Conversely, a uniformly laminar current induces a natural measure on every transversal to the lamination.

We say that two holomorphic disks $D$ and $D'$ are compatible if $D \cap D'$ is either empty or open in the disk topology. A positive current $T$ in $\Omega \subset \mathbb{C}^2$ is laminar if there exists a measured family $(A, \nu)$ of holomorphic disks $D_\alpha \subset \Omega$, such that for every pair $(\alpha, \beta)$, $D_\alpha$ and $D_\beta$ are compatible and

$$T = \int_A [D_\alpha]d\nu(\alpha).$$

The difference with uniform laminarity is that there is no (even locally) uniform lower bound on the size of the disks in (3). In particular there is no associated lamination, and the notion is strictly weaker. Notice also that the integral representation (3) does not prevent $T$ from being closed, because of boundary cancellation.

Equivalently, a current is laminar if it is the limit of an increasing sequence of uniformly laminar currents. More precisely, $T$ is laminar in $\Omega$ if there exists a sequence of open subsets $\Omega^i \subset \Omega$, such that $\|T\|_{\partial\Omega^i} = 0$, together with an increasing sequence of currents $(T^i)_{i \geq 0}$, $T^i$ uniformly laminar in $\Omega^i$, converging to $T$.

In the course of section 5, we will be led to consider woven currents. The corresponding definitions will be given at that time.

3. Laminarity outside the connectedness locus

In this section we give a precise description of $T^+$ outside the connectedness locus. Subsections 3.1 and 3.2 are rather of expository nature. We first recall the wringing construction of Branner-Hubbard, and how it leads to uniform laminarity. Then, we explain some results of Kiwi [K] on the geometry of $\mathcal{C}^+$ at infinity. In §3.3, based on an argument of similarity between the dynamical and parameter spaces and a theorem of [DDS], we prove that the transverse measure induced by $T^+$ gives full mass to the point components. The presentation is as self-contained as possible, only the results of [BH2] depending on the combinatorics of tableaux are not reproved.

3.1. Wringing and uniform laminarity. We start by defining some analytic functions in part of the parameter space, by analogy with the definition of the functions $G^\pm$. Recall that for a cubic polynomial $f$, the Böttcher coordinate $\varphi_f$ is a holomorphic function defined in the open neighborhood of infinity

$$U_f = \{ z \in \mathbb{C}, G_f(z) > \max(G^+, G^-) \},$$

and semiconjugating $f$ to $z^3$ there, i.e. $\varphi_f \circ f = (\varphi_f)^3$. Also $\varphi_f = z + O(1)$ at infinity.

Assume $-c$ is the fastest escaping critical point, i.e. $G_f(c) < G_f(-c)$. The corresponding critical value $v + 4c^3$ has two distinct preimages $-c$ and $2c$ (because $c \neq 0$), and it turns out that

$$\varphi_f(2c) := \lim_{U_f \ni z \rightarrow 2c} \varphi_f(z)$$

is always well defined (whereas there would be an ambiguity in defining $\varphi_f(-c)$). We put $\varphi^-(f) = \varphi_f(2c)$, so that $\varphi^-$ is a holomorphic function in the open subset $\{G^+ < G^-\}$ satisfying $G^-(f) = \log |\varphi^-(f)|$. 

Remark 3.1. We notice for further use that it is possible to continue \((\varphi^-)^3\) to the larger subset \(\{G^+ < 3G^-\}\) by evaluating \(\varphi_f\) at the critical value, and similarly for \((\varphi^-)^9\), etc. So locally, it is possible to define branches of \(\varphi^-\) in larger subsets of parameter space by considering inverse powers.

Following Branner and Hubbard [BH1] (we reproduce Branner’s exposition [Br]) we now define the basic operation of *wringing the complex structure*. This provides a holomorphic 1-parameter deformation of a map in the escape locus, which has the advantage of being independent of the number and relative position of escaping critical points. This is in contrast with the full deformation theory [McMS] which is sensitive to critical orbit relations, etc.

Let \(\mathbb{H}\) be the right half plane, endowed with the following group structure (with 1 as unit element)

\[
u_1 * \nu_2 = (s_1 + i t_1) * (s_2 + i t_2) = (s_1 + i t_1)s_2 + it_2.
\]

For \(\nu \in \mathbb{H}\) we define the diffeomorphism \(g_\nu : \mathbb{C} \setminus \mathbb{D} \to \mathbb{C} \setminus \mathbb{D}\) by \(g_\nu(z) = z |z|^\nu-1\). Notice that \(g_\nu\) commutes with \(z^3\), and \(g_{\nu_1 * \nu_2} = g_{\nu_1} \circ g_{\nu_2}\), i.e. \(\nu \mapsto g_\nu\) is a left action on \(\mathbb{C} \setminus \mathbb{D}\). Since \(g_\nu\) commutes with \(z^3\) we can define a new invariant almost complex structure on \(U_f = \{G_f(z) > \max(G^+, G^-)\}\) by replacing it with \(g_\nu(\sigma_0)\) in the Böttcher coordinate. Here \(\sigma_0\) is the standard complex structure on \(\mathbb{C}\). More precisely define \(\sigma_u\) as the unique almost complex structure on \(\mathbb{C}\), invariant by \(f\), and such that

\[
\sigma_u = \begin{cases} 
\varphi_f^* g_\nu(\sigma_0) & \text{on } U_f \\
\sigma_0 & \text{on } K_f
\end{cases}
\]

Of course, viewed as a Beltrami coefficient, \(\sigma_u\) depends holomorphically on \(\nu\), and it can be straightened by a quasiconformal map \(\phi_\nu\) such that

- \(\phi_\nu \circ f \circ \phi_\nu^{-1}\) is a monic centered polynomial,
- \(g_\nu \circ \varphi_f \circ \phi_\nu^{-1}\) is tangent to identity at infinity.

Under these assumptions \(\phi_\nu\) is unique and depends holomorphically on \(\nu\) [BH1 §6].

We put \(w(f, \nu) = f_\nu = \phi_\nu \circ f \circ \phi_\nu^{-1}\). This is a holomorphic disk through \(f = f_1\) in parameter space. Using the terminology of [DH], \(f_\nu\) is hybrid equivalent to \(f\), so by the uniqueness of the renormalization for maps with connected Julia sets, if \(f \in \mathbb{C}\), \(f_\nu \equiv f\). On the other hand the construction is non trivial in the escape locus, as the next proposition shows.

**Proposition 3.2.** Wringing yields a lamination of \(\mathbb{C}^2 \setminus \mathcal{E}\) by Riemann surfaces, with leaves isomorphic to the disk or the punctured disk.

This provides in particular a new proof of the following core result of [BH1].

**Corollary 3.3.** The wring mapping \(w : \mathbb{C}^2 \setminus \mathcal{E} \to \mathbb{C}^2 \setminus \mathcal{E}\) is continuous.

The proof of the proposition is based on the following simple but useful lemma.

**Lemma 3.4.** Let \(\Omega \subset \mathbb{C}^2\) be a bounded open set, and \(\pi : \Omega \to \mathbb{C}\) be a holomorphic function. Assume that there exists a closed subset \(\Gamma\) and a positive constant \(c\) so that

- For every \(p \in \Gamma\), there exists a holomorphic map \(\gamma_p : \mathbb{D} \to \Gamma\), with \(\gamma_p(0) = p\) and \(|(\pi \circ \gamma_p)'(0)| \geq c\).
- the disks are compatible in the sense that \(\gamma_p(\mathbb{D}) \cap \gamma_q(\mathbb{D})\) is either empty or open in each of the two disks.
Then $\bigcup_{p \in \Gamma} \gamma_p(\mathbb{D})$ is a lamination of $\Gamma$ by holomorphic curves.

**Proof.** We may assume $c = 1$. First, since $(\pi \circ \gamma_p)'(0)$ never vanishes, $\pi$ is a submersion in the neighborhood of $\Gamma$. Hence locally near $p \in \Gamma$, by choosing adapted coordinates $(z, w)$ we may suppose that $\pi$ is the first projection, i.e. $\pi(z, w) = z$.

The second claim is that $f$ is holomorphic and bounded by $R$ on $\mathbb{D}$, with $f(0) = 0$ and $|f'(0)| \geq 1$, then there exists a constant $\varepsilon(R)$ depending only on $R$ so that $f$ is injective on $D(0, \varepsilon(R))$. Then by the Koebe Theorem, the image univalently covers a disk of radius $\varepsilon(R)/4$.

Now observe that since $\Omega$ is bounded, if $\Gamma_1 \subset \Gamma$ is relatively compact, the family of mappings \{\gamma_p, p \in \Gamma_1\} is locally equicontinuous. Hence in the local coordinates $(z, w)$ defined above, we can apply the previous observation to the family of functions $\pi \circ \gamma_p$. The claim asserts that given any $z_0$ and any point $p$ in the fiber $\Gamma \cap \pi^{-1}(z_0)$, the disk $\gamma_p(\mathbb{D})$ contains a graph for the projection $\pi$ over the disk $D(z_0, \varepsilon/4)$, for some uniform $\varepsilon$.

So all components of $\bigcup_{p \in \Gamma} \gamma_p(\mathbb{D})$ over the disk $D(z_0, \varepsilon/8)$ are graphs over that disk and they are disjoint. Moreover they are uniformly bounded in the vertical direction by equicontinuity. As observed in [25] from Hurwitz’ Theorem and equicontinuity, we conclude that $\bigcup_{p \in \Gamma} \gamma_p(\mathbb{D})$ is a lamination.

**Proof of the proposition.** The idea is to use the previous lemma with $\varphi = \varphi^\pm$. We need to understand how $\varphi^\pm$ evolve under wringing. We prove that the wringing disks form a lamination in the open set $\{G^+ < G^-\}$ where $G^-$ is defined —what we only need in the sequel is a neighborhood of $C^+$. The case of $\{G^- < G^+\}$ is of course symmetric, and remark 3.1 allows to extend the reasoning to a neighborhood of $\{G^+ = G^-\}$.

Recall that $w(f, u) = f_u = \phi_u \circ f \circ \phi_u^{-1}$, so if $f = f_{(c, u)}$, the critical points of $f_u$ are $\phi_u(\pm c)$. Also an explicit computation shows that $g_u \circ \varphi_f \circ \phi_u^{-1}$ semiconjugates $f_u$ and $z^3$ near infinity, so by the normalization done, this is the Böttcher function $\varphi_{f_u}$. In particular, $\varphi_{f_u}(\phi_u(z)) = g_u(\varphi_f(z))$. At the level of the Green function, we thus get the identity

$$G_{f_u}(\phi_u(z)) = sG_f(z) \quad \text{where} \quad u = s + it. \quad (4)$$

Assume that $G^+ < G^-$ so that the fastest escaping critical point is $-c$. By (3), the fastest critical point of $f_u$ is $\phi_u(-c)$ so

$$\varphi^-(f_u) = g_u \circ \varphi_f \circ \phi_u^{-1}(\phi_u(-c)) = g_u(\varphi^-(f)) = \varphi^-(f) \frac{1}{u^{u-1}} \quad (5)$$

This equation tells us two things: first, that $(f, u) \mapsto w(f, u)$ is locally uniformly bounded, and next, that $f \mapsto \frac{d\varphi^-(w(f, u))}{du}|_{u=0}$ is locally uniformly bounded from below in parameter space.

Moreover the wringing disks are compatible because of the group action: $w(w(f, v), u) = w(f, u \ast v)$. By using lemma 3.4 we conclude that they fit together in a lamination.

Now if $L$ is a leaf of the lamination, and $f \in L$ is any point, $w(f, \cdot) : \mathbb{H} \to L$ is a universal cover. Indeed it is clearly onto because of the group action, and since both the leaf and the map $w(f, \cdot)$ are uniformly transverse to the fibers of $\varphi^-$, it is locally injective.

There are two possibilities: either it is globally injective or not. In the first case $L \simeq \mathbb{D}$. In the second case, if $w(f, s_1 + it_1) = w(f, s_2 + it_2)$, then by (4), $s_1 = s_2$. Since near $w(f, s_1 + it_1)$, the leaf is a graph with uniform size for the projection $\varphi^-$, there exists a minimal $T > 0$ such that $w(f, s_1 + it_1 + iT) = w(f, s_1 + it_1)$. The group structure on $\mathbb{H}$ has the property that
\[ u \ast (v + iT) = (u \ast v) + iT, \] from which we deduce that for every \( u \in \mathbb{H}, w(f, u + iT) = w(f, u). \]

In particular, \( L \) is conformally isomorphic to the punctured disk.

**Proof of Corollary 3.3.** By definition, the holonomy of a lamination is continuous. We only have to worry about the compatibility between the natural parametrization of the leaves and the lamination structure. Here, by [3], the correspondence between the parameter \( u \) on the leaves and the parameter induced by the transversal fibration \( \varphi^- = \text{cst} \) is clearly uniformly bicontinuous –by uniform we mean here locally uniform with respect to the leaf. This proves the corollary.

**Proposition 3.5.** The current \( T^+ \) is uniformly laminar in the escape locus, and subordinate to the lamination by wringing curves.

**Proof.** The limit of a converging sequence of uniformly laminar currents, all subordinate to the same lamination, is itself subordinate to this lamination: indeed locally in a flow box this is obvious. So by the convergence Theorem 2.5, to get the uniform laminarity of \( T^+ \), it is enough to prove that the curves \( \{\text{Per}^+(n)\} \) are subordinate to the wringing lamination in \( \mathbb{C}^2 \setminus \mathcal{E} \). But again this is obvious: \( w(f, u) \) is holomorphically conjugate to \( f \) on \( \text{Int}(K_f) \), so if \( f \) has a superattracting cycle of some period, then so does \( w(f, u) \).

3.2. **Geometry of \( C^+ \) at infinity.** In this paragraph we follow Kiwi [K]. For the sake of convenience, we include most proofs.

The following lemma is [K, Lemma 7.2], whose proof relies on easy explicit estimates. Besides notation, the only difference is the slightly more general hypothesis \( |v| < 3|c| \). We leave the reader check that this more general assumption is enough to get the conclusions of the lemma.

**Lemma 3.6.** If \( |v| < 3|c| \) and \( c \) is large enough, then the following hold:
- \( |f^n(-c)| \geq |c|^{3n} (\sqrt{2})^{-3n-1-1} \);
- \( G^-(c, v) = \log |c| + O(1) \);
- \( G^+(c, v) \leq \frac{1}{3} \log |c| + O(1) \).

As a consequence we get an asymptotic expansion of \( \varphi^- \) (compare [K, Lemma 7.10]). Notice that under the assumptions of the previous lemma, \( G^+(c, v) < G^-(c, v) \) so \( \varphi^- \) is well defined.

**Lemma 3.7.** When \( |v| < 3|c| \) and \((c, v) \to \infty\), then
\[ \varphi^-(c, v) = 2^{2/3} c + o(c). \]

**Proof.** Recall that
\[ \varphi^-(c, v) = \varphi_{f(c,v)}(2c) = \lim_{n \to \infty} 2c \left( \frac{f(2c)}{(2c)^3} \right)^1 \cdots \left( \frac{f^n(2c)}{(f^{n-1}(2c))^3} \right)^{1/n}. \]

For large \( z \), \( f(z)/z^3 = 1 + O(1/z) \), hence since \( f(2c) = f(-c) \), by lemma 3.6, we get that for \( n > 1 \),
\[ \frac{f^n(2c)}{(f^{n-1}(2c))^3} = 1 + O \left( \frac{2^{3n/2}}{|c|^{3n}} \right), \]
and the product converges uniformly for large \( c \). Moreover, \( f(2c)/(2c)^3 \) converges to 1/2 when \((c, v) \to \infty\) and \( |v| \leq 3|c| \), whereas, for \( n \geq 2 \), \( f^n(2c)/(f^{n-1}(2c))^3 \) converges to 1. This gives the desired estimate.
We now translate these results into more geometric terms. See Figure 1 for a synthetic picture. Compactify \( \mathbb{C}^2 \) as the projective plane \( \mathbb{P}^2 \), and choose homogeneous coordinates \([C : V : T]\) such that \([c : v : 1], (c, v) \in \mathbb{C}^2\) is our parameter space. Consider the coordinates \( x = \frac{1}{c} = \frac{T}{c}, y = \frac{v}{c} = \frac{V}{c} \) near infinity, and the bidisk \( B = \{ |x| \leq x_0, |y| \leq 1/3 \} \). The line at infinity becomes the vertical line \( x = 0 \) in the new coordinates.

**Proposition 3.8** (Kiwi [K]).

i. For \( k_0 \) large enough, the level sets \( \{ \varphi^- = k, |k| \geq k_0 \} \), are vertical holomorphic graphs in \( B \). They fit together with the line at infinity \( x = 0 \) as a holomorphic foliation.

ii. For small enough \( x_0 \), the leaves of the wringing lamination are graphs over the first coordinate in \( B \setminus \{ x = 0 \} \).

**Proof.** In the new coordinates, we have

\[
\varphi^-(x, y) = \frac{2^{2/3}}{x}(1 + \delta(x, y)),
\]

where \( \delta(x, y) \) is a holomorphic function outside \( \{ x = 0 \} \). Since \( \delta \to 0 \) as \( x \to 0 \), \( \delta \) extends holomorphically to \( B \). Let \( k' = 1/k \); the equation \( \varphi^- = k \) rewrites as

\[
x = 2^{2/3}(1 + \delta(x, y))k'.
\]

For small enough \( k' \) and fixed \( y \), this equation has exactly one solution in \( x \), depending holomorphically on \( y \), which means that \( \{ \varphi^- = k \} \) is a vertical graph close to \( x = 0 \). So
clearly,
\[ \{x = 0\} \cup \bigcup_{|k| \geq k_0} \{\varphi^{-} = k\} \]
is a lamination near \(\{x = 0\}\). If we fix two small holomorphic transversals to \(\{x = 0\}\), the holonomy map is holomorphic outside the origin, so it extends holomorphically. We conclude that this lamination by vertical graphs extends as a holomorphic foliation across \(\{x = 0\}\).

On the other hand we know that the wringing curves are transverse to the fibers of \(\varphi^{-}\), i.e. they are graphs for the projection \(\varphi^{-}\). So by using the estimate of lemma 3.7 again and Rouché’s Theorem, we obtain that the wringing leaves are graphs over the \(c\) coordinate for large \(c\). Hence the second part of the proposition.

The next proposition asserts that the wringing curves contained in \(C^{+}\) cluster only at two points at infinity. In particular, these points are singular points for the wringing lamination. The results of [Ki] §7 may be thought as the construction of an abstract “desingularization” of this lamination.

**Proposition 3.9** (Kiwi [Ki]). *The closure of \(C^{+}\) in \(\mathbb{P}^2\) is \(\overline{C^{+}} = C^{+} \cup \{[1 : 1 : 0], [1 : -2 : 0]\}\). Likewise \(\overline{C^{-}} = C^{-} \cup \{[0 : 1 : 0]\}\).

**Remark** 3.10. The choice of the \((c, v)\) parametrization is precisely due to this proposition. By an easy computation, the reader may check that in \((\varphi^{-})\) coordinates of \(\mathbb{P}^2\), \(C^{+}\) and \(C^{-}\) both cluster at the same point \([0 : 1 : 0]\) at infinity, an unwelcome feature.

**Proof.** The starting point is the fact that \(C^{+} + G^{-}\) is proper in \(C^{2}\); see [BH] §3. Consider now the following open neighborhood of \(C^{+}\):
\[ \mathcal{V} = \left\{(c, v) \in C^{2}, \; G_f(v) = 3G_f(c) < \frac{1}{3}G_f(-c)\right\} = \left\{3G^{+} < \frac{1}{3}G^{-}\right\}. \]
We will show that \(\overline{\mathcal{V}}\) intersects the line at infinity in \([1 : 1 : 0]\) and \([1 : -2 : 0]\), by proving that when \(G_f(-c)\) tends to infinity in \(\mathcal{V}\), \(v/c\) converges to either 1 or −2. The fact that both points are actually reached is easy: this is already the case for the \(\text{Per}^{+}(2)\) curve.

Let us introduce some standard concepts. If \((c, v) \in \mathcal{V}\), in the dynamical plane, the level curves \(\{G_f(z) = r\}\) are smooth Jordan curves for \(r > G_f(-c)\), and \(\{G_f(z) = G_f(-c)\}\) is a figure eight curve with self crossing at \(-c\). By disk at level \(n\), we mean a connected component of \(\{z, \; G_f(z) < 3^{-n+1}G_f(-c)\}\). If \(z\) is a point deeper than level \(n\), we define \(D_{f}^{n}(z)\) as the connected component of \(\{G_f(z) < 3^{-n+1}G_f(-c)\}\) containing \(z\). Also
\[ A_{f}^{0} = \{z, \; G_f(-c) < G_f(z) < 3G_f(-c)\} \]
is an annulus of modulus
\[ \frac{1}{2\pi} \log \frac{e^{3G_f(-c)}}{e^{G_f(-c)}} = \frac{1}{\pi}G_f(-c). \]
If \((c, v) \in \mathcal{V}\), then \(v, c\), and \(-2c\) (the cocritical point) are points of level \(\geq 2\). There are two disks of level 1 corresponding to the two inner components of the figure eight: \(D_{f}^{1}(c)\) and \(D_{f}^{1}(-2c)\). In particular \(D_{f}^{1}(v) \setminus D_{f}^{2}(v)\) is an annulus of modulus \(\geq \frac{1}{2}\) modulus \(A_{f}^{0}\) that
- either separates \(c\) and \(v\) from \(-2c\)
- or separates \(-2c\) and \(v\) from \(c\),

depending on which of the two disks contains $v$.

When $(c, v)$ tends to infinity in $V$, modulus($A_f^0$) = $G_f(-c)$ → $\infty$, and standard estimates in conformal geometry imply that $|v - c| = o(c)$ in the first case and $|v + 2c| = o(c)$ in the second, that is, $v/c$ respectively converges to 1 or $-2$. To prove this, we may for instance use the fact that an annulus with large modulus contains an essential round annulus with almost the same modulus [McM1].

From this we easily get the corresponding statement for $C^-$, by noting that the involution exchanging the markings $(c, v)$ $\mapsto (-c, v + 4c^3)$ contracts the line at infinity with $[1 : 0 : 0]$ deleted onto the point $[0 : 1 : 0]$.

3.3. Transverse description of $T^+$. Branner and Hubbard gave in [BH2] a very detailed picture of $C^+ \setminus C$ both from the point of view of the dynamics of an individual mapping $f \in C^+ \setminus C$ and the point of view of describing $C^+ \setminus C$ as a subset in parameter space. Roughly speaking, a map in $C^+ \setminus C$ can be of two different types: quadratically renormalizable or not. If $f \in C^+ \setminus C$, $K_f$ is disconnected so it has infinitely many components. We denote by $C(+c)$ the connected component of the non escaping critical point. Then

**Renormalizable case:** If $C(+c)$ is periodic, then $f$ admits a quadratic renormalization, and $C(+c)$ is qc-homeomorphic to a quadratic Julia set. Moreover a component of $K(f)$ is not a point if and only if it is a preimage of $C(+c)$.

**Non renormalizable case:** If $C(+c)$ is not periodic, then $K_f$ is a Cantor set.

There is a very similar dichotomy in parameter space. Notice that near infinity, $C^+ \subset B$, where $B$ is the bidisk of proposition 3.8. For large $k$, consider a disk $\Delta$ of the form $\{f^- = k\} \cap B$. Then $C^+ \cap \Delta$ is a disconnected compact subset of $\Delta$. The connected components are of two different types

- **Point components:** corresponding to non quadratically renormalizable maps.
- **Copies of M:** the quadratically renormalizable parameters are organized into Mandelbrot-like families [DH], giving rise to countably many homeomorphic copies of the Mandelbrot set in $\Delta$.

**Remark 3.11.** These results imply that the (passivity) components of $\text{Int}(C^+) \setminus C$ are exactly the open subsets obtained by moving holomorphically (under wringing) the components of the Mandelbrot copies. In particular, if the quadratic hyperbolicity conjecture holds, all these components are hyperbolic.

Since $T^+$ is uniformly laminar and subordinate to the lamination by wringing curves, to have a good understanding of $T^+$, it is enough to describe the transverse measure $T^+ \wedge [\Delta]$. 

**Theorem 3.12.** The transverse measure induced by $T^+$ on a transversal gives full mass to the point components.

In particular, point components are dense in $\partial C^+ \cap \Delta$. Notice that the Mandelbrot copies are also dense, since they contain the points $\text{Per}^+(n) \cap \Delta$.

**Proof.** The proof of the theorem will follow from similarity between the dynamical and parameter spaces, and a symbolic dynamics argument in the style of [DDS].

In the domain of parameter space under consideration, $G^+ < G^-$, so in the dynamical plane, the open set $\{G_f(z) < G^-(f)\}$ is bounded by a figure eight curve (see figure2). Let $U_1$ and $U_2$ be its two connected components, and assume that $c \in U_2$, so that $f|_{U_i} : U_i \to \{G_f < 3G^-\}$
is proper with topological degree \( i \). We denote by \( f_i \) the restriction \( f|_{U_i} \), \( d_i = \deg(f_i) \) and by \( U \) the topological disk \( \{ G_f < 3G^- \} \).

Thus, classically, we get a decomposition of \( K(f) \) in terms of itineraries in the symbol space \( \Sigma := \{1, 2\}^\mathbb{N} \). More precisely, for a sequence \( \alpha \in \Sigma \), let
\[
K_\alpha = \{ z \in \mathbb{C}, f^i(z) \in U_{\alpha(i)} \}.
\]
For every \( \alpha \), \( K_\alpha \) is a nonempty closed subset, and the \( K_\alpha \) form a partition of \( K \).

For example, for \( \alpha = \overline{2} = (2222\cdots) \), \( K_2 \) is the filled Julia set of the quadratic-like map \( f|_{U_2} : U_2 \to U \). On the other hand \( K_1 \) is a single repelling fixed point. Notice also that \( K_{12} \) is not a single point, even if the sequence \( \overline{12} \) contains infinitely many 1’s.

We now explain how the decomposition \( K = \bigcup_{\alpha \in \Sigma} K_\alpha \) is reflected on the Brolin measure \( \mu \). Let \( p \) be any point outside the filled Julia set and write
\[
\frac{1}{3^n}(f^n)^*\delta_p = \frac{1}{3^n} \sum_{(\alpha_0,\ldots,\alpha_{n-1}) \in \{1,2\}^n} f_{\alpha_0}^* \cdots f_{\alpha_{n-1}}^* \delta_p.
\]
In [DDS], we proved that for any sequence \( \alpha \in \Sigma \), and for any \( p \) outside the Julia set, the sequence of measures
\[
\frac{1}{d_{\alpha_0} \cdots d_{\alpha_{n-1}}} f_{\alpha_0}^* \cdots f_{\alpha_{n-1}}^* \delta_p
\]
converges to a probability measure \( \mu_\alpha \) independent of \( p \), and supported on \( \partial K_\alpha \). The measure \( \mu_\alpha \) is the analogue of the Brolin measure for the sequence \( (f_\alpha) \). This was stated as Theorem 4.1 and Corollary 4.6 in [DDS] for sequences of horizontal-like in the unit bidisk but it is easy to translate it in the setting of polynomial-like maps in \( \mathbb{C} \).

Let \( \nu \) be the shift invariant measure on \( \Sigma \), giving mass \( (d_{\alpha_0} \cdots d_{\alpha_{n-1}})/3^n \) to the cylinder of sequences starting with \( \alpha_0,\ldots,\alpha_{n-1} \) the \( (1/3, 2/3) \) measure on \( \Sigma \). Then (6) rewrites as
\[
\frac{1}{3^n}(f^n)^*\delta_p = \sum_{(\alpha_0,\ldots,\alpha_{n-1}) \in \{1,2\}^n} \frac{d_{\alpha_0} \cdots d_{\alpha_{n-1}}}{3^n} \left[ \frac{1}{d_{\alpha_0} \cdots d_{\alpha_{n-1}}} f_{\alpha_0}^* \cdots f_{\alpha_{n-1}}^* \delta_p \right],
\]
so at the limit we get the decomposition \( \mu = \int_{\Sigma} \mu_\alpha d\nu(\alpha) \).
We fix a transversal disk \( \Delta \) as in Proposition 3.8. If \( \Delta \) close enough to the line at infinity, \( G^+ > 0 \) on \( \partial \Delta \). Observe that the measure \( T^+ \wedge \Delta \) induced by \( T^+ \) on \( \Delta \) is the “bifurcation current” of the family of cubic polynomials parameterized by \( \Delta \). We denote it by \( T^+|_\Delta \).

Following [DF] \( \S 3 \) consider \( \hat{\Delta} = \Delta \times \mathbb{C} \) and the natural map \( \hat{f} : \hat{\Delta} \to \hat{\Delta} \), defined by \( \hat{f}(\lambda, z) = (\lambda, f_\lambda(z)) \). Consider any graph \( \Gamma_\gamma = \{ z = \gamma(\lambda) \} \) over the first coordinate such that \( \gamma(\lambda) \) escapes under iteration by \( f_\lambda \) for \( \lambda \in \Delta \) (for instance \( \gamma(\lambda) = -c(\lambda) \)). Then the sequence of currents \( \frac{1}{3^n} (\hat{f}^n)^*[\Gamma] \) converges to a current \( \hat{T} \) such that \( T^+|_\Delta = (\pi_1)_* (\hat{T}|_{\Gamma^+}) \).

Here \( \pi_1 : \hat{\Delta} \to \Delta \) is the natural projection. Now there is no choice involved in the labelling of the \( U_1 \) and \( U_2 \), so since \( G^+ < G^- \) on \( \Delta \), the decomposition \( f_\lambda^{-1} U = U_1 \cup U_2 \) can be followed continuously throughout \( \Delta \). So with obvious notation, we get for \( i = 1, 2 \) a map \( \hat{f}_i : \hat{U}_i \to \hat{U} \). This gives rise to a coding of \( \hat{f} \) orbits, in the same way as before. We thus get a decomposition of \( \frac{1}{3^n} (\hat{f}^n)^*[\Gamma] \) as

\[
\frac{1}{3^n} (\hat{f}^n)^*[\Gamma] = \frac{1}{3^n} \sum_{(\alpha_0, \ldots, \alpha_{n-1}) \in \{1, 2\}^n} \hat{f}_{\alpha_0}^* \cdots \hat{f}_{\alpha_{n-1}}^* [\Gamma].
\]

The convergence theorem of [DDS] implies that for any \( \alpha \in \Sigma \), the sequence of currents

\[
\hat{T}_{\alpha, n} = \frac{1}{d_{\alpha_0} \cdots d_{\alpha_{n-1}}} \hat{f}_{\alpha_0}^* \cdots \hat{f}_{\alpha_{n-1}}^* [\Gamma]
\]

converges to a current \( \hat{T}_\alpha \) in \( \Delta \times \mathbb{C} \). Indeed, for every \( \lambda \in \Delta \), the sequence of measures \( \frac{1}{d_{\alpha_0} \cdots d_{\alpha_{n-1}}} (f_\lambda)^{\alpha_0} \cdots (f_\lambda)^{\alpha_{n-1}} \delta_{\gamma(\lambda)} \) converges by [DDS]. On the other hand the currents \( \hat{T}_{\alpha, n} \) are contained in a fixed vertically compact subset of \( \Delta \times \mathbb{C} \), and all cluster values of this sequence have the same slice measures on the vertical slices \( \{ \lambda \} \times \mathbb{C} \). Currents with horizontal support being determined by their vertical slices, we conclude that the sequence converges (see [DDS] \( \S 2 \) for basics on horizontal currents).

So again at the limit we get a decomposition of \( \hat{T} \) as an integral of positive closed currents, \( \hat{T} = \int_\Sigma \hat{T}_\alpha d\nu(\alpha) \), where \( \nu \) is the \( (\frac{3}{5}, \frac{2}{5}) \) measure on \( \Sigma \). Furthermore, for every \( \alpha \), the wedge product \( \hat{T}_\alpha \wedge [\Gamma_c] \) is well defined. This follows for instance from a classical transversality argument: recall that on \( \partial \Delta \), \( c \) escapes so \( \text{Supp} \hat{T}_\alpha \cap \Gamma_c \subset \Gamma_c \). In particular any potential \( \hat{G}_\alpha \) of \( \hat{T}_\alpha \) is locally integrable on \( \Gamma_c \). Hence by taking the wedge product with the graph \( \Gamma_c \) and projecting down to \( \Delta \), we get a decomposition

\[
T^+|_\Delta = (\pi_1)_* (\hat{T} \wedge [\Gamma_c]) = \int_\Sigma (\pi_1)_* (\hat{T}_\alpha \wedge [\Gamma_c]) d\nu(\alpha) = \int_\Sigma T^+_\alpha|_\Delta d\nu(\alpha).
\]

The second equality is justified by Lemma 2.29.

To fix the ideas, \( T^+_\alpha|_\Delta \) should be understood as the contribution of the combinatorics \( \alpha \) to the measure \( T^+|_\Delta \) in parameter space, and is supported on the set of parameters for which \( c \in K_\alpha \). Notice also that since by definition \( c \in U_2 \), for \( T^+_\alpha|_\Delta \) to be non zero, it is necessary that \( \alpha \) starts with the symbol 2.

Furthermore, since there are no choices involved in the labelling of the \( U_i \), the decompositions \( K = \bigcup K_\alpha \) and \( \mu = \int \mu_\alpha d\nu(\alpha) \) are invariant under wringing. Hence the decomposition \( T^+ = \int T^+_\alpha d\nu(\alpha) \), which was defined locally, makes sense as a global decomposition of \( T^+ \) in \( \mathbb{C}^2 \setminus C \).

\[\text{The results in [DDS] actually imply that } \hat{T}_\alpha \text{ has continuous potential for } \nu\text{-a.e. } \alpha\]
We summarize this discussion in the following proposition.

**Proposition 3.13.** There exists a decomposition of $T^+$ as an integral of positive closed currents in $\mathbb{C}^2 \setminus \mathcal{C}$,

$$T^+ = \int_\Sigma T^+_\alpha d\nu(\alpha),$$

where $\Sigma = \{1, 2\}^\mathbb{N}$ and $\nu$ is the $(\frac{1}{3}, \frac{2}{3})$-measure on $\Sigma$. Moreover $T^+_\alpha$ is supported in the set of parameters where $c$ has itinerary $\alpha$ with respect to the natural decomposition $\{G_f(z) < G^-\} = U_1 \cup U_2$, with $c \in U_2$.

We now conclude the proof of Theorem 3.12. We combine the previous discussion with the results of Corollary 3.14. The current $T^+$ is not extremal in a small neighborhood of the line at infinity because of Proposition 3.13. Notice that the two branches of $T^+$ at $[1 : 1 : 0]$ (resp. $[1 : -2 : 0]$) correspond to sequences starting with the symbol 21 (resp. 22). More generally we have the following corollary, which fills a line in Table 1.

**Corollary 3.14.** The current $T^+$ is not extremal any open subset of the escape locus $\mathbb{C}^2 \setminus \mathcal{C}$.

**Proof.** Since the measure $\nu$ on $\Sigma$ does not charge points, and the currents $T^+_\alpha$ have disjoint support, the decomposition of Proposition 3.13 is non trivial in any open subset of $\mathbb{C}^2 \setminus \mathcal{C}$. □

### 4. Laminarity at the boundary of the connectedness locus

We start by a general laminarity statement in $\text{Int}(\mathcal{C}^-)$ which, together with Proposition 3.5, completes the proof of the laminarity of $T^+$ outside $\partial \mathcal{C}^-$. Notice that since $\text{Supp}(T^+) = \partial \mathcal{C}^+$, this is a really a statement about the boundary of the connectedness locus.

**Theorem 4.1.** The current $T^+$ is laminar in $\text{Int}(\mathcal{C}^-)$.

**Proof.** Consider a parameter $f_0 \in \text{Int}(\mathcal{C}^-) \cap \text{Supp}(T^+) = \text{Int}(\mathcal{C}^-) \cap \partial \mathcal{C}^+ \subset \partial \mathcal{C}$. The critical point $+c$ is active at $f_0$, whereas $-c$ is passive. Let $U \ni f_0$ be a small ball where $-c$ remains passive. There are nearby parameters where $+c$ escapes, and we find that parameters in $U \setminus \mathcal{C}$ have a stable quadratic renormalization (see remark 3.11).

Moreover $G^+$ is pluriharmonic where it is positive, so $dd^c \max(G^+, \varepsilon)$ has laminar structure. We will study the topology of the leaves in $U$ and conclude by using a result by de Thélin 3.11.

In the open set $U$, $-c$ never escapes, so the Böttcher function $\varphi^+(f) = \varphi_f(-2c)$ is well defined, and $G^+ = \log |\varphi^+|$. The real hypersurface $\{G^+ = \varepsilon\}$ is foliated by the holomorphic
curves \( \{ \varphi^+ = \exp(\epsilon + i \theta) \} \), and it is well known (see e.g. [De]) that \( d\nu \max(G^+, \epsilon) \) is uniformly laminar and subordinate to this foliation—from the fact that both \( G^+ = \epsilon \) and the leaves are non singular will be a consequence of the wringing argument below.

The point is that in \( U \) the leaves of this foliation are closed and biholomorphic to the disk. Indeed, recall that the wring deformation \( f \mapsto w(f, u) \) is a homeomorphism in \( \mathbb{C}^2 \setminus \mathcal{C} \) for fixed \( u \); it can be inverted by using the group action. In particular \( w(\cdot, s) \) maps \( U \cap \{ G^+ = \epsilon \} \) homeomorphically into a neighborhood of infinity in \( \text{Int}(\mathcal{C}^-) \), where the foliation \( \varphi^+ = \text{cst} \) is well understood. By using proposition \[ \text{3.3} \], with + and − swapped, we get that the leaves are planar Riemann surfaces. We conclude that the leaves are disks. Indeed if \( L = \{ \varphi^+ = \exp(\epsilon + i \theta) \} \) is such a leaf in \( U \), \( w(\cdot, s)(L) \) is an open subset in a vertical disk \( \Delta \) (see proposition \[ \text{3.3} \]), hence \( w(\cdot, s)^{-1}(\Delta) \) is a disk traced on \( \{ \varphi^+ = \exp(\epsilon + i \theta) \} \), containing \( L \). So its intersection with the ball \( B \) is a disk by the maximum principle.

De Thélin’s Theorem \[ \text{3.11} \] asserts that if \( \Delta_n \) is a sequence of (possibly disconnected) submanifolds of zero genus in the unit ball, then any cluster value of the sequence of currents \( [\Delta_n]/\text{Area}(\Delta_n) \) is a laminar current. Pick a sequence \( \epsilon_n \to 0 \). In our situation, the approximating currents \( T^{\epsilon_n}_\epsilon \) are integrals of families of holomorphic disks. So, for every \( n \), we first approximate \( T^{\epsilon_n}_\epsilon \) by a finite combination of leaves and denote the resulting approximating sequence of currents by \( (S_{n,j})_{j \geq 0} \). For every \( j \), \( S_{n,j} \) is a normalized finite sum of currents of integration along disks. Hence by choosing an appropriate subsequence we can ensure that \( S_{n,j(n)} \to T^\epsilon_\epsilon \), and applying de Thélin’s result finishes the proof.

Using quasiconformal deformations, we obtain a much more precise result in hyperbolic (hence conjecturally all) components. Notice that due to Remark \[ \text{3.11} \] the quadratic hyperbolicity conjecture is enough to ensure uniform laminarity of \( T^\epsilon_\epsilon \) outside \( \partial \mathcal{C}^+ \cap \partial \mathcal{C}^- \).

**Proposition 4.2.** The current \( T^\epsilon_\epsilon \) is uniformly laminar in hyperbolic components of \( \text{Int}(\mathcal{C}^-) \).

**Proof.** The proof will use a very standard quasiconformal surgical argument (cf. e.g. [CG, VIII.2]). The major step is to prove that the foliation by disks of the form \( \{ \varphi^+ = \epsilon \} \) considered in the previous proposition does not degenerate in the neighborhood of \( \partial \mathcal{C}^+ \).

Before going into the details, we sketch the argument. Assume \( f_0 \in \partial \mathcal{C}^+ \cap \Omega \), where \( \Omega \) is a hyperbolic component of \( \text{Int}(\mathcal{C}^-) \). The multiplier of the attracting cycle defines a natural holomorphic function in \( \Omega \). Using quasiconformal surgery, we construct a transverse section of this function through \( f_0 \), which is a limit of sections of the form \( \{ \varphi^+ = \epsilon \} \), uniform in the sense of Lemma \[ \text{3.4} \]. Then by Lemma \[ \text{3.4} \] we conclude that the foliation \( \{ \varphi^+ = \epsilon \} \) extends as a laminating to \( \partial \mathcal{C}^+ \). Then, approximating \( T^\epsilon_\epsilon \) by \( T^{\epsilon_n}_\epsilon \) as before implies that \( T^\epsilon_\epsilon \) is uniformly laminar.

So let \( \Omega \) be a hyperbolic component of \( \text{Int}(\mathcal{C}^-) \) and \( f_0 \in \Omega \cap \mathcal{F} \). The critical point \( -c \) is attracted by an attracting periodic point of period \( m \) that persists through \( \Omega \). Let \( \rho \) be the multiplier of the cycle; \( \rho \) is a holomorphic function on \( \Omega \). Let \( U_0, \ldots, U_m = U_0 \) be the cycle of Fatou components containing the attracting cycle. Since \( f_0 \) is in the escape locus or at its boundary, \( +c \) is not attracted by the attracting cycle., so its orbit does not enter \( U_0, \ldots, U_m \).

The map \( f_0^m : U_0 \to U_0 \) is conjugate to a Blaschke product of degree 2: there exists a conformal map \( \varphi : U_0 \to \mathbb{D} \) such that

\[
\varphi \circ f_0^m \circ \varphi^{-1} = \frac{z + \lambda_0}{1 + \lambda_0 z} = B_{\lambda_0}(z),
\]
where $\lambda_0 = \rho(f_0)$ is the multiplier of the cycle. For simplicity we assume that $m = 1$, see 
\[\text{for the adaptation to the general case and more details.}\]

Choose a small $\varepsilon$ and then $r$ so that $|\lambda_0| < 1 - \varepsilon < r < 1$, and for every $\lambda \in D(0, 1 - \varepsilon)$ we have
\[D(0, r) \subseteq B_{\lambda}^{-1}(D(0, r)).\]

For $\lambda \in D(0, 1 - \varepsilon)$, choose a smoothly varying family of diffeomorphisms
\[\psi_\lambda : f^{-1} \varphi^{-1}(D(0, r)) \to B_{\lambda}^{-1}(D(0, r))\]
with $\psi_{\lambda_0} = \varphi$ and such that
- $B_\lambda \circ \psi_\lambda = \varphi \circ f$ on $\partial f^{-1} \varphi^{-1}(D(0, r))$,
- $\psi_\lambda = \varphi$ in $\varphi^{-1}(D(0, r))$.

We can thus define a map $g_\lambda : \mathbb{C} \to \mathbb{C}$ by
\[
\begin{cases}
  g_\lambda = \varphi^{-1} \circ B_\lambda \circ \psi_\lambda \text{ in } f^{-1} \varphi^{-1}(D(0, r)) \\
  g_\lambda = f \text{ outside } f^{-1} \varphi^{-1}(D(0, r))
\end{cases}
\]

By definition of $\psi_\lambda$, the two definitions match on $\partial f^{-1} \varphi^{-1}(D(0, r))$.

Now define an almost complex structure $\sigma_\lambda$ by declaring that $\sigma_\lambda = \psi_\lambda^* \sigma_0$ on $f^{-1} \varphi^{-1}(D(0, r))$, extending it by invariance under $g_\lambda$, and $\lambda_0 = \sigma_0$ outside the grand orbit (under $f_0$) of the attracting Fatou component. By construction, any orbit under $g_\lambda$ hits the region where $g_\lambda$ is not holomorphic at most once. So we can straighten $\sigma_\lambda$, we obtain that $g_\lambda$ is quasiconformally conjugate on $\mathbb{C}$ to a cubic polynomial $f_\lambda$, which varies continuously with $\lambda \in D(0, 1 - \varepsilon)$. By construction, $f_{\lambda_0} = f_0$ and $\rho(f_\lambda) = \lambda$ (because the conjugacy is holomorphic in the neighborhood of the attracting fixed point).

The family of maps $f_\lambda$ that we have constructed defines a continuous section of $\rho$ through $f_0$. Now, if $f_0 \not\in \mathcal{C}$, $\varphi^+(f_\lambda)$ is constant because $f_0$ and $f_\lambda$ are holomorphically conjugate outside the Julia set. So $f_\lambda$ is constrained to stay in the one dimensional local leaf $L = \{\varphi^+ = \varphi^+(f_0)\}$. Since $\rho$ is holomorphic on $L$ and $f_\lambda$ is a section of $\rho$, we obtain that $f_\lambda$ depends holomorphically on $\lambda$, and that $\rho$ is a local biholomorphism on $L$.

Now assume $f_0$ is at the boundary of the escape locus (that is at the boundary of $\mathcal{C}^+$). Let $\{f_n\}$ be a sequence of maps in $\mathcal{E}$ converging to $f_0$, say with $\rho(f_n) = \lambda_0$. Attached to each $f_n$, there is a holomorphic disk $\Delta_n = \bigcup_{\lambda \in D(0, 1 - \varepsilon)} f_n(\lambda)$ with $\rho(f_n(\lambda)) = \lambda$. Notice that any two such disks are equal or disjoint. Let $\Delta_0$ be any normal limit of the sequence of disks $(\Delta_n)$. This disk defines a section of $\rho$ through $f_0$ so in particular we deduce that $\rho$ is submersive at $f_0$.

We claim that $\Delta_0$ does not depend on the sequence $(\Delta_n)$. Indeed $\rho$ is a submersion at $f_0$, hence a fibration in the neighborhood of $f_0$, so the conclusion follows from Hurwitz’ Theorem.

By construction, this holomorphic disk is contained in $\partial \mathcal{C}^+$, because along $\Delta_n$, $G^+$ is constant and tends to zero as $n \to \infty$. So both critical points are passive on $\Delta_0$, and the dynamics along $\Delta_0$ is $J$-stable.

In conclusion we have constructed a family of holomorphic sections of $\rho$ in $\Omega \cap (\mathcal{E} \cup \partial \mathcal{C}^+)$. In $\mathcal{E}$, the sections are compatible in the sense of lemma 3.4 because they are subordinate to the foliation by $\{\varphi^+ = \text{cst}\}$. As seen before, since the sections on $\partial \mathcal{C}^+$ are obtained by taking limits, they are also compatible by Hurwitz’ Theorem. Hence lemma 3.4 applies and tells us
that these disks laminate \( \Omega \cap (\mathcal{E} \cup \partial \mathcal{C}^+) \). Hence we have proved that the natural foliation of \( \Omega \cap \mathcal{E} \) admits a continuation, as a laminating, to the boundary of \( \mathcal{E} \).

It is now clear that \( T^+ \) is uniformly laminar in \( \Omega \), because \( T^+ = \lim T^+_\varepsilon = \lim \frac{d\mathcal{F} \max(G^+, \varepsilon)}{d\varepsilon} \), and, as we have seen in the proof of the previous theorem, \( T^+_\varepsilon \) is subordinate to the natural foliation of \( \Omega \cap \mathcal{E} \).

\[
\square
\]

5. Rigidity at the boundary of the connectedness locus

5.1. Higher order bifurcations and the bifurcation measure. So far we have seen that many parameters in the bifurcation locus admit a one parameter family of deformations. Due to laminarity, this holds for \( T_{\text{bif}} \text{-a.e.} \) parameter outside \( \partial \mathcal{C}^+ \cap \partial \mathcal{C}^- \). In this section we will concentrate on rigid parameters, that is, cubic polynomials with no deformations.

Let \( \mu_{\text{bif}} \) be the positive measure defined by \( \mu_{\text{bif}} = \frac{d\mathcal{F} G}{d\varepsilon} \). An easy calculation shows that \( T^2_{\text{bif}} = (T^+ + T^-)^2 = 2\mu_{\text{bif}} \). One major topic in [DF] was to study the properties of \( \mu_{\text{bif}} \), which is in many respects the right analogue for cubics of the harmonic measure of the Mandelbrot set. In particular it was proved that \( \text{Supp}(\mu_{\text{bif}}) \) is the closure of Misiurewicz points, and is a proper subset of \( \partial \mathcal{C}^+ \cap \partial \mathcal{C}^- \).

We define the second bifurcation locus \( \text{Bif}_2 \) as the closure of the set of rigid parameters, that is, parameters that do not admit deformations. By the density of Misiurewicz points, it is clear that \( \text{Supp}(\mu_{\text{bif}}) \subset \text{Bif}_2 \). In the next proposition we get a considerably stronger statement. Notice that the result is valid for polynomials of all degrees.

**Proposition 5.1.** The set of parameters \( f \) for which there exists a holomorphic disk with \( f \in \Delta \subset \mathcal{C} \) has zero \( \mu_{\text{bif}} \)-measure.

In view of Proposition 2.7, this can be understood as a generic rigidity result.

**Corollary 5.2.** A \( \mu_{\text{bif}} \) generic polynomial admits no deformations.

**Proof of Proposition 5.1.** Recall from [DF] §6 that \( \mu_{\text{bif}} = (d\mathcal{F} G)^2 \), where \( G = \max(G^+, G^-) \). Now if \( \Delta \) is a holomorphic disk contained in \( \mathcal{C} \), \( G = 0 \) on \( \Delta \). It is known that if \( E \) is a subset of \( \text{Supp}(d\mathcal{F} G)^2 \) such that through every point in \( E \) there is a holomorphic disk on which \( G \) is harmonic, then \( E \) has zero \( (d\mathcal{F} G)^2 \) measure (see [SI] Corollary A.10.3].

Another argument goes as follows: by [DE] Theorem 10, \( \mu_{\text{bif}} \text{-a.e.} \) polynomial \( f \) satisfies the Topological Collet Eckmann (TCE) property. Also both critical points are on the Julia set so the only possible deformations come from invariant line fields [McMS]. On the other hand the Julia set of a TCE polynomial has Hausdorff dimension strictly less than 2 [PR] so it has no invariant line fields.

The next result gives some rough information on \( \text{Bif}_2 \). We do not know whether the first inclusion is an equality or not.

**Proposition 5.3.** The second bifurcation locus satisfies

\[
\text{Supp}(\mu_{\text{bif}}) \subset \text{Bif}_2 \subset (\partial \mathcal{C}^+ \cap \partial \mathcal{C}^-) \cup \mathcal{E} \subset \partial \mathcal{C},
\]

where the set \( \mathcal{E} \) is empty if the hyperbolicity conjecture holds.

**Proof.** The relationship of \( \text{Bif}_2 \) with active and passive critical points is as follows:
If one critical point is passive while the other is passive, it is expected that the passive critical point will give rise to a modulus of deformation. This is the case when it is attracted by a (super-)attracting cycle. Otherwise, we do not know how to prove it, nevertheless it is clear that there are nearby parameters with deformations, and moreover, these will be generic in the measure theoretic sense (see Theorem 4.1). The existence of such parameters contradicts the hyperbolicity conjecture anyway.

When the two critical points are active, there may exist deformations. The list of possibilities is as follows (see [McMS]):

1. There is a parabolic cycle attracting both critical points, and their grand orbits differ.
2. There is a Siegel disk containing a postcritical point.
3. There is an invariant line field –this is of course conjectured not to happen.

In [DF], we gave an example (due to Douady) of a cubic polynomial $f_0 \in (\partial C^+ \cap \partial C^-) \setminus \text{Supp}((\mu_{\text{bif}})$. This map has the property of having a parabolic fixed point attracting both critical points, as in case (1) above. Furthermore, every nearby parameter either has a parabolic point, and is in fact conjugate to $f_0$, or has an attracting point. In particular locally there is a disk of such parameters, conjugate to $f_0$. It is then clear that such a parameter is not approximated by rigid ones, so $f_0 \notin \text{Bif}_2$. □

We have no precise information on the location of parameters in case (2) above. Here is a specific question: let $\text{Per}_1(\theta)$ be the subvariety of parameters with a fixed point of multiplier $e^{2\pi i \theta}$. Assume that $\theta$ satisfies a Diophantine condition so that every $f \in \text{Per}_1(\theta)$ has a Siegel disk. For some parameters (see Zakeri [Z]), it happens that one of the critical point falls into the Siegel disk after finitely many iterations (the other one is necessarily in the Julia set). In parameter space, this gives rise to disks contained in $\partial C^+ \cap \partial C^-$. Are these disks contained in $\text{Supp}(\mu_{\text{bif}})$? In $\text{Bif}_2$?

5.2. Woven currents. In the same way as foliations can be generalized to webs, there is a class of woven currents, extending laminar currents. They were introduced by T.C. Dinh [Di].

**Definition 5.4.** Let $\Omega \subset C^2$ be an open subset. A web in $\Omega$ is any family of analytic subsets of $\Omega$, with volume bounded by some constant $c$.

A positive closed current $T$ in $\Omega$ is uniformly woven iff there exists a constant $c$, and a web $\mathcal{W}$ as above, with leaves of volume bounded by $c$, and endowed with a positive measure $\nu$, such that

$$T = \int_{\mathcal{W}} [V] d\nu(V).$$

Notice that by Bishop’s Theorem, any family of subvarieties with uniformly bounded volume is relatively compact for the Hausdorff topology on compact subsets of $\Omega$. In analogy with the laminar case, we also define general woven currents.

**Definition 5.5.** A current $T$ in $\Omega$ is said to be woven if there exists a sequence of open subsets $\Omega_i \subset \Omega$ such that $T$ gives zero mass to $\partial \Omega_i$, and an increasing sequence of currents $(T_i)$, converging to $T$, and such that $T_i$ is uniformly woven in $\Omega_i$.

5.3. Non geometric structure of $\mu_{\text{bif}}$. In this paragraph, we investigate the structure of the bifurcation measure $\mu_{\text{bif}}$, and prove that it is not the “geometric intersection” of the laminar currents $T^\pm$. As a consequence we will derive an asymptotic estimate on the genera of the $\text{Per}(n, k)$ curves.
It is natural to wonder whether generic wringing curves in \(\partial C^\pm\) admit continuations across \(\partial C\) (compare with the work of Willumsen \[W\]). Many wringing curves are subordinate to algebraic subsets so the continuation indeed exists: this is the case for the Per\((n,k)\) curves as well as the subsets in parameter space defined by the condition that a periodic cycle of given length has a given indifferent multiplier. Similarly, we may wonder whether the disks constituting the laminar structure of \(T^\pm\) on \(\partial C \setminus (\partial C^+ \cap \partial C^-)\) admit continuations across \(\partial C^+ \cap \partial C^-\).

We will look at the continuation property from the point of view of the structure of the bifurcation measure. If \(E\) is a closed subset in \(\text{Supp}(\mu)\), we say that \(\mu_{\text{bif}}\) has local product structure on \(E\) if there exists uniformly laminar currents \(S^\pm \leq T^\pm\) defined in a neighborhood of \(E\) such that \(\mu_{\text{bif}}|_E = S^+ \wedge S^-\). Abusing conventions, we will extend this definition to the case where \(S^+\) and \(S^-\) are merely uniformly woven currents and \(0 < S^+ \wedge S^- \leq \mu_{\text{bif}}|_E\).

This terminology is justified by the fact that the wedge product of uniformly laminar currents coincides with the natural geometric intersection. More specifically, if \(S^+\) and \(S^-\) are uniformly laminar currents in \(\Omega\), written as integral of disks of the form

\[ S^\pm = \int [\Delta^\pm] d\mu^\pm, \]

then if the wedge product \(S^+ \wedge S^-\) is well defined,

\[ S^+ \wedge S^- = \int [\Delta^+ \cap \Delta^-] d\mu^+ d\mu^-, \]

where \([\Delta^+ \cap \Delta^-]\) is the sum of point masses at isolated intersections of \(\Delta^+\) and \(\Delta^-\). We refer the reader to \[BLS, Du2\] for more details on these notions. Analogous results hold for uniformly woven currents.

The next theorem shows that the structure of the bifurcation measure is somewhat more complicated. Loosely speaking, this means that, in the measure theoretic sense, neither wringing curves nor the disks of Section 4 admit continuations across \(\text{Supp}(\mu_{\text{bif}})\).

**Theorem 5.6.** The measure \(\mu_{\text{bif}}\) does not have local product structure on any set of positive measure.

**Proof.** Assume \(\mu_{\text{bif}}\) has local product structure on a set \(E\) of positive measure. Let \(S^\pm = \int [\Delta^\pm] d\mu^\pm\) be uniformly woven currents as in the previous discussion, with \(0 < S^+ \wedge S^- \leq \mu_{\text{bif}}|_E\). It is classical (see e.g. \[BLS\] Lemma 8.2) that the currents \(S^\pm\) have continuous potentials. Here the \(\Delta^\pm\) are possibly singular curves of bounded area. By lemma \[24\] \(S^+ \wedge S^-\) admits a decomposition as \(\int [\Delta^+] \wedge S^- d\mu^+.\) Since each \(\Delta^+\) has finitely many singular points, the induced measure \([\Delta^+] \wedge S^-\) gives full mass to the regular part of \(\Delta^+\), so we get that the variety \(\Delta^+\) is smooth around \(S^+ \wedge S^-\)-a.e. point.

In particular, by Proposition \[5.1\] above, discarding a set of varieties of zero \(\mu^+\) measure if necessary, we may assume that for every \(\Delta^+,\ \Delta^+ \setminus C \neq \emptyset\). Outside \(C\), \(T^+\) is uniformly laminar and subordinate to the lamination by wringing curves, so \(S^+\) itself is subordinate to this lamination and the transverse measure of \(S^+\) is dominated by that of \(T^+\) (see \[BLS, \S 6\], \[Du3\]).

From this discussion and Theorem \[3.12\] we conclude that for \(\mu^+\)-almost every \(\Delta^+,\ \Delta^+ \setminus C\) is contained in at most countably many wringing curves, and the polynomials in \(\Delta^+ \setminus C\) have Cantor Julia sets. Removing a set of zero \(\mu^+\)-measure once again, we may assume that this property holds for all \(\Delta^+\).
By [DF] Corollary 11, there exists a set of parameters of full $\mu_{\text{bif}}$-measure $E' \subset E$ for which the orbits of both critical points are dense in the Julia set (this is a consequence of the TCE property). By Fubini’s Theorem, we may further assume that every $\Delta^+$ intersects $E'$.

Now since any variety $\Delta^+$ is contained in $C^+$, the point $+c$ is passive on $\Delta^+$. By [DF] Theorem 4], if there exists a parameter $f \in \Delta^+$ for which $+c$ is preperiodic, then

- it is persistently preperiodic
- it is persistently attracted by an attracting periodic point or lies in a persistent Siegel disk.

The first property only occurs on countably many varieties while the second contradicts the genericity assumptions made on $\Delta^+$. So we infer that $+c$ is never preperiodic on $\Delta^+$, in which case its orbit can be followed by a holomorphic motion. But on $E'$, the orbit of $+c$ is dense on the Julia set, which is connected, while on $\Delta \setminus C$ it is contained in a Cantor set. We have reached a contradiction.

Not much seems to be known about the geometry of the $\text{Per}^+(n)$ and $\text{Per}^+(n,k)$ curves. An unpublished result of Milnor’s [Mi2] asserts that the $\text{Per}^+(n)$ are smooth in $C^2$. For every $0 \leq k < n$, $\text{Per}(n,k)$ is an algebraic curve of degree $3^n$. The geometry of such curves is unknown. In particular these curves might be very singular (this can also happen for the $\text{Per}^+(n)$ at infinity) and have many irreducible components, so their genus cannot be directly estimated.

As a consequence of Theorem 5.6 and previous work of ours [Du1, Du2], we get a rough asymptotic estimate on the genus of these curves. Recall that the geometric genus of a compact singular Riemann surface is the genus of its desingularization. In the next theorem, genus means geometric genus and, as usual, $\text{Per}^+$ may be replaced by $\text{Per}^-$.

**Theorem 5.7.** Let $k(n)$ be any sequence satisfying $0 \leq k(n) < n$, and write the decomposition of $\text{Per}^+(k(n), n)$ into irreducible components as $\text{Per}^+(k(n), n) = \sum m_{i,n} C_{i,n}$. Then

$$\frac{1}{3^n} \sum_i m_{i,n} \cdot \text{genus}(C_{i,n}) \to \infty.$$  

Consequently $\max_i \text{genus}(C_{i,n}) \to \infty$.

**Proof.** In [Du1], we proved that if a sequence of algebraic curves $C_n$ in $\mathbb{P}^2$ satisfies certain geometric estimates, then the cluster values of the sequence of currents $[C_n]/\deg(C_n)$ are laminar. In [Du2] we proved that if $T_1$ and $T_2$ are two such laminar currents with continuous potentials, then the wedge product measure $T_1 \wedge T_2$ has local product structure. By Theorem 5.6 we know that $\mu_{\text{bif}}$ does not have product structure on any set of positive measure. Therefore, the geometric estimates of [Du1] are not satisfied. By inspecting the geometry of the $\text{Per}^+(n, k(n))$ curves, we will see that this leads to [8]. A delicate issue is that [Du1] requires an assumption on the singularities which may not be satisfied when $k(n) \neq 0$, so we need to generalize it slightly. This is where woven currents are needed.

We start with the simpler case where $k(n) = 0$. Then, by Milnor’s result, the curves $\text{Per}^+(n)$ are smooth in $C^2$, so in $\mathbb{P}^2$ they can have singular points only at $[1 : 1 : 0]$ and $[1 : -2 : 0]$. Let $\text{Per}^+(n) = \sum m_{i,n} C_{i,n}$ be the decomposition into irreducible components. We have to count the number of "bad components" when projecting $C_{i,n}$ from a generic point in $\mathbb{P}^2$. Remark that the $C_{i,n}$ do not intersect in $C^2$, so no additional bad components arise from intersecting branches of different irreducible components of $\text{Per}^+(n)$. Also, the number of local irreducible components at every singular point of $C_{i,n}$ is bounded by $\deg(C_{i,n})$, since a nearby line cannot
have more than \( \deg(C_{i,n}) \) intersection points with \( C_{i,n} \). By [Du1] Prop. 3.3, the number of bad components for each \( C_{i,n} \) is thus bounded by \( 4\text{genus}(C_{i,n}) + 6\deg(C_{i,n}) \). By summing this estimate for all irreducible components, we get that the number (with multiplicity) of bad components is bounded by \( 4\sum m_{i,n} \cdot \text{genus}(C_{i,n}) + O(3^n) \), and by [Du1] Prop. 3.4, if \( S \) does not hold for a certain subsequence \( n_i \to \infty \), we get that \( T^+ \) is laminar everywhere, and strongly approximable in the sense of [Du2] Def. 4.1. The same holds for \( T^- \), since \( +c \) and \( -c \) are exchanged by a birational involution. In particular, by [Du2], \( T^+ \wedge T^- \) must have local product structure, which contradicts Theorem 5.6.

The proof in the general case involves a modification of [Du1] [Du2], which was already considered in [tT2] (see also [D1]). The required modification is contained in the following proposition. We define the genus of a reducible Riemann surface as the sum of the genera of its components.

**Proposition 5.8.** Let \( C_n \) be a sequence of (singular) algebraic curves in \( \mathbb{P}^2 \), with \( \deg(C_n) \to \infty \). Assume that \( \text{genus}(C_n) = O(\deg(C_n)) \). Then the cluster values of the sequence \( [C_n]/\deg(C_n) \) are woven currents.

**Proof.** This is just a careful inspection of [Du1]. We fix a generic point in \( \mathbb{P}^2 \), and consider the central projection \( \pi_p : \mathbb{P}^2 \setminus \{p\} \to \mathbb{P}^1 \). For a subdivision \( Q \) of \( \mathbb{P}^1 \) by squares, and \( Q \in Q \), we look for good components, that is, irreducible components of \( C_n \cap \pi^{-1}_p(Q) \) which are graphs over \( Q \). The difference with [Du1] is that two intersecting graphs are now considered to yield two good components here. For each irreducible component \( C_{i,n} \) of \( C_n \), the number of bad components is controlled by using the Riemann-Hurwitz formula for the natural map \( \tilde{C}_{i,n} \to \mathbb{P}^1 \), where \( \tilde{C}_{i,n} \) is the normalization of \( C_{i,n} \). A straightforward adaptation of [Du1] Prop. 3.3 shows that the number of bad components is bounded by \( O(\text{genus}(C_{i,n})) + O(\deg(C_{i,n})) \). By summing for all \( C_{i,n} \) and noting that any union of good components is good, we get that under the assumption of the lemma, the total number of bad components is \( O(\deg(C_n)) \).

We can now proceed with the proof of [Du1], by replacing “laminar” by “woven” everywhere, and get that the cluster values are woven currents. \( \square \)

The woven currents we obtain in this way satisfy explicit estimates which allow to adapt [Du2] and study their intersection. This was already observed in [tT2] §3 –be careful that woven currents are called geometric there.

**Proposition 5.9.** Let \( T_1, T_2 \), be woven currents obtained as cluster values of curves satisfying the assumption of Proposition 5.8. Assume further that \( T_1 \) and \( T_2 \) have continuous potentials in some open set \( \Omega \). Then the wedge product \( T_1 \wedge T_2 \) has local product structure in \( \Omega \).

**Proof.** The scheme is as follows: consider two distinct linear projections \( \pi_1 \) and \( \pi_2 \) in \( \mathbb{C}^2 \), and subdivisions by squares of size \( r \) of the projection bases. This gives rise to a subdivision by cubes of size \( O(r) \) in \( \mathbb{C}^2 \), which we denote by \( Q \). If \( T \) is a woven current, as given by Proposition 5.8, then for a generic such subdivision \( Q \) and every \( Q \in Q \), there exists a uniformly woven current \( T_Q \) in \( Q \) such that the mass of \( T - \sum_{Q \in Q} T_Q \) is \( O(r^2) \). If now \( T_1 \) and \( T_2 \) are two such currents, with continuous potentials, it is then easy to adapt [Du2] Theorem 4.1) and get that the wedge product \( T_1 \wedge T_2 \) is approximated by \( \sum_{Q \in Q} T_1, Q \wedge T_2, Q \), which has geometric interpretation by Lemma 2.9. We conclude that the measure \( T_1 \wedge T_2 \) has local product structure. \( \square \)
The conclusion if the proof of the theorem is now clear. If for some sequence $\text{Per}^+(n, k(n))$, the estimate $[\text{5}]$ is violated, then by the previous proposition, the measure $\mu = T^+ \wedge T^-$ would have product structure, which is not the case due to Theorem $[\text{5.6}]$. □

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Université Paris 7 et Institut de Mathématiques de Jussieu Équipe Géométrie et Dynamique, Case 7012, 2 place Jussieu, 75251 Paris Cedex 05 France

E-mail address: dujardin@math.jussieu.fr