COUNTING ISOLATED ROOTS OF TRINOMIAL SYSTEMS IN THE PLANE AND BEYOND

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Abstract. We prove that any pair of bivariate trinomials has at most 5 isolated roots in the positive quadrant. The best previous upper bounds independent of the polynomial degrees counted only non-degenerate roots and even then gave much larger bounds, e.g., 248832 via a famous general result of Khovanski. Our bound is sharp, allows real exponents, and extends to certain systems of $n$-variate fewnomials, giving improvements over earlier bounds by a factor exponential in the number of monomials. We also derive new sharper bounds on the number of real connected components of fewnomial hypersurfaces.

1. Introduction

Generalizing Descartes’ Rule of Signs to polynomial systems has proven to be a significant challenge. Recall that a weak version of this famous classical result asserts that any real univariate polynomial with exactly $m$ monomial terms has at most $m-1$ positive roots. This bound is sharp and generalizes easily to real exponents (cf. section 3). The original statement in René Descartes’ *La Géométrie* pre-dates 1641. Proofs can be traced back to work of Gauss in 1828 and other authors earlier, but a definitive sharp bound for multivariate polynomial systems seems to have eluded us in the second millennium. This is particularly unfortunate since sparse polynomial systems now occur in applications as diverse as radar imaging [FH95] and chemistry [GH99].

One simple way to generalize the setting of Descartes’ Rule to higher dimensions and real exponents is the following:

Notation. For any $c \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$ and $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, let $x^a := x_1^{a_1} \cdots x_n^{a_n}$ and call $cx^a$ a monomial term. We will refer to $\mathbb{R}^+_n := \{x \in \mathbb{R}^n \mid x_i > 0 \text{ for all } i\}$ as the positive orthant, or quadrant or octant when $n$ is respectively 2 or 3. Henceforth, we will assume that $F := (f_1, \ldots, f_k)$ where, for all $i$, $f_i \in \mathbb{R}[x^n] \mid a \in \mathbb{R}^n$ and $f_i$ has exactly $m_i$ monomial terms. We call $f_i$ an $n$-variate $m_i$-nomial and, when $m_1, \ldots, m_k \geq 1$, we call $F$ a $k \times n$ fewnomial system (over $\mathbb{R}$) of type $(m_1, \ldots, m_k)$. Finally, we say a real root $\zeta$ of $F$ is isolated (resp. non-degenerate) iff the only arc of real roots of $F$ containing $\zeta$ is $\zeta$ itself (resp. the Jacobian of $F$, evaluated at $\zeta$, has full rank).

Generalized Kushnirenko’s Conjecture (GKC). Suppose $F$ is an $n \times n$ fewnomial system of type $(m_1, \ldots, m_n)$. Then the maximum number of non-degenerate roots of $F$ in the positive orthant is $\prod_{i=1}^n (m_i - 1)$.

Remark 1. The polynomial system $\left(\prod_{i=1}^{m_1-1} (x_1 - i), \ldots, \prod_{i=1}^{m_n-1} (x_n - i)\right)$ easily shows that the conjectured maximum can at least be attained (if not exceeded), and integral exponents and coefficients suffice for this to happen. ○

Received by the editors November 20, 2018.

1991 Mathematics Subject Classification. Primary 34C08; Secondary 14P05, 30C15.

Li was partially supported by a Guggenheim Fellowship. Rojas’ work on this paper was partially supported by Hong Kong UGC Grant #9040402-730, Hong Kong/France PROCORE Grant #9050140-730, and a grant from the Texas A&M University Faculty of Science. Some of Wang’s research was done during a stay at the Key Laboratory for Symbolic Computation and Knowledge Engineering of the Ministry of Education, P. R. China. Wang’s research is supported in part by the Visiting Scholar Foundation of Key Labs In Universities, Ministry of Education, P. R. China.

1 Quite naturally, we will also call 2-nomials binomials and 3-nomials trinomials.

2 We use this terminology solely for succinctness. Fewnomial theory [Kho91] is an important related body of work regarding a special class of functions which includes our $m$-nomials.

3 i.e., point or homeomorphic image of the unit circle or (open, closed, or half-open) unit interval...
We can then succinctly state the original Kushnirenko’s Conjecture (formulated in the mid-1970’s by Anatoly G. Kushnirenko) as the special case of GKC where all the exponents of $F$ are non-negative integers. Curiously, Kushnirenko’s Conjecture was open for nearly three decades until Bertrand Haas found a counter-example in the case $(n, m_1, m_2) = (2, 3, 3)$ (see remark 4 below and [Haa00]). So we will derive a correct and sharp extension of Descartes’ bound to this case, as well as certain additional cases with $n \geq 3$, $m_n \geq 4$, and degeneracies allowed. Interestingly, the introduction of real exponents and degeneracies gives us more flexibility than trouble: The proof of our first main result uses little more than exponential coordinates and Rolle’s Theorem from calculus.

**Definition 1.** For any $m_1, \ldots, m_n \in \mathbb{N}$, let $\mathcal{N}(m_1, \ldots, m_n)$ (resp. $\mathcal{N}'(m_1, \ldots, m_n)$) denote the maximal number of isolated (resp. non-degenerate) roots an $n \times n$ fewnomial system of type $(m_1, \ldots, m_n)$ can have in the positive orthant.

**Theorem 1.** For all $m \geq 3$ we have $\mathcal{N}(3, m) \leq 2^m - 2$ and, in particular, $\mathcal{N}(3, 3) = 5$, $\mathcal{N}(3, 4) \leq 14$, and $\mathcal{N}(3, 5) \leq 30$. Furthermore, $\mathcal{N}'(3, m) = \mathcal{N}(3, m)$.

The quantities $\mathcal{N}'(1, m_2, \ldots, m_n)$, $\mathcal{N}(1, m_2, \ldots, m_n)$, $\mathcal{N}'(2, m_2)$, and $\mathcal{N}(2, m_2)$ are much easier to compute than $\mathcal{N}(3, 3)$: explicit formulae for them are stated in theorem 3 of section 2.

**Remark 2.** The value of $\mathcal{N}(3, 3)$ was previously unknown and the authors are unaware of any earlier result implying the equality $\mathcal{N}'(3, m) = \mathcal{N}(3, m)$. In particular, the only other information previously known about $\mathcal{N}'(3, m)$ or $\mathcal{N}(3, m)$ was an upper bound of $3^{m+2}(m+2)(m+1)/2$ for $\mathcal{N}'(3, m)$ (see remark 4 below). For $m = 3, 4, 5$ the latter formula evaluates to 248832, 2388782, and 4586471424 respectively.

**Remark 3.** Note that $\mathcal{N}'(m_1, \ldots, m_n) \leq \mathcal{N}(m_1, \ldots, m_n)$ for all $m_1, \ldots, m_n \in \mathbb{N}$, since non-degenerate roots of $n \times n$ fewnomial systems are always isolated roots. While we do not yet know of any cases where the inequality is strict, it is interesting to note that GKC can not be strengthened to allow degeneracies: For example, the polynomial system

\[ (x_1(x_1 - 1), x_2(x_2 - 1), \prod_{i=1}^{5}(x_1 - i)^2 + \prod_{i=1}^{5}(x_2 - i)^2) \]

is of type $(2, 2, 21)$, has 25 integral roots in the positive octant (all of which have singular Jacobian), but its GKC bound is 20.

**Remark 4.** Haas’ counter-example to the original Kushnirenko’s Conjecture is

\[ (x_1^{108} + 1.1x_2^{54} - 1.1x_2, x_2^{108} + 1.1x_1^{54} - 1.1x_1) \]

which has 5 roots in the positive quadrant, thus contradicting its alleged GKC bound of 4 [Haa00].

Jan Verschelde has also verified numerically that there are exactly $108^2 = 11664$ complex roots, and thus (assuming the floating-point calculations were sufficiently good) each root is non-degenerate by Bézout’s theorem.

The central observation that led to our proof may be of independent interest. We state it as assertion (3) of theorem 6 below. The first two assertions dramatically refine the bounds of Oleinik, Petrovsky, Milnor, Thom, and Basu on the number of connected components of a real algebraic set [OP49, Mil64, Tho65, Bas99] in the special case of a single polynomial and extend to real exponents.

**Theorem 2.** Let $Z$ be the set of roots in $\mathbb{R}^+_n$ of an $n$-variate $m$-nomial. Also let $\mathcal{K}'(n, \mu)$ denote the maximal number of non-degenerate roots in $\mathbb{R}^+_n$ of an $n \times n$ fewnomial system with exactly $\mu$ distinct exponent vectors. Finally, let $P_{\text{comp}}(n, m)$ (resp. $P_{\text{non}}(n, m)$) be the maximal number of compact (resp. non-compact) connected components of any such $Z$. Then...

1. $P_{\text{comp}}(n, m) \leq 2[\mathcal{K}'(n, m)/2]$, the multiple of 2 can be removed in the smooth case, and $P_{\text{comp}}(1, m) = m - 1$.

4Examples of this type were observed earlier by William Fulton around 1984 (see the first edition of [Ful98]) and Bernd Sturmfels around 1997 [Stu98].

5Dima Grigoriev informed the author on Sept. 8, 2000 that Konstantin A. Sevast’yanov, a colleague of Kushnirenko and contemporary of Grigoriev, had found a similar counter-example much earlier. Unfortunately, this counter-example does not seem to have been recorded and, tragically, Sevast’yanov committed suicide some time before 1997.
Corollary 1. Let $F_1, F_2, \ldots, F_m$ be a polynomial system \((1, 2, 3)\), and let $n \geq 1$. Then
\[
P_{\text{non}}(n, m) \leq 2(P_{\text{comp}}(n - 1, m) + P_{\text{non}}(n - 1, m)),\]
and
\[
P_{\text{non}}(2, m) \leq [m/2],\]
and $P_{\text{non}}(1, m)$ is 1 or 0 according as $m$ is 0 or not.

3. $(n, m) = (2, 3) \implies Z$ has no more than 3 inflection points and no more than 1 isolated point of vertical tangency.

Remark 5. Note that a non-compact component of $Z$ can actually have compact closure, since $\mathbb{R}^m$ is not closed in $\mathbb{R}^n$, e.g., \(\{(x_1, x_2) \in \mathbb{R}^2_+ \mid x_1^2 + x_2^3 = 1\}\). Also, Bertrand Haas has pointed out that the bound on $P_{\text{non}}(2, m)$, at least in the case of integral exponents, may date back to work of Isaac Newton in the 17th century on power series. 

While the above bounds on the number of connected components are non-explicit, they are stated so they can immediately incorporate any advance in computing $K'(n, m)$. So for a general and explicit bound independent of the underlying polynomial degrees now, one could, for instance, simply insert
\[
\text{the explicit upper bound for } F_{\text{poly}} \text{ in the number of monomial terms of } \mathcal{F}_{\text{poly}}.
\]

The bound above is already significantly sharper than an earlier bound of $2^{m-1}(m-2)/2^{2m-2}n(n+1)^{m-1}$, which held only for the smooth case, following from \([Kho91, \text{sec. 3.14, cor. 5}]\). The bounds of theorem 5 are further refined in theorem 6 of section 3, and these additional bounds also improve an earlier result of the author on smooth algebraic hypersurfaces \([Ro00a, \text{cor. 3.1}]\).

1.1. Important Related Results.

It is interesting to note that the best current general bounds in the direction of GKC are exponential in the number of monomial terms of $F$, even for fixed $n$. Observe one of the masterpieces of real algebraic geometry.

Khovanski’s Theorem on Real Fewnomials (Special Case). \((\text{See also } [Kho91, \text{cor. 7, sec. 3.12}].)\) Let $F$ be an $n \times n$ fewnomial system and $\mu$ the total number of distinct exponent vectors of $F$. Then $F$ has no more than $(n + 1)^{\mu/2}$ non-degenerate roots in the positive orthant, i.e.,
\[
K'(n, \mu) \leq (n + 1)^{\mu/2}.
\]

Remark 6. In the case $(n, m_1, m_2) = (2, 3, m)$, one can divide both equations by suitable monomials to obtain $\mu = m + 2$ and thus $N'(3, m) \leq K'(3, m + 2)$. So Khovanski’s bound implies $N'(3, m) \leq 3^{m+2}(2m+2)(m+1)/2$. It is also very easy to see that a simple application of Gaussian elimination yields $K'(n, m) \leq \sum_{j=1}^{n-m} m_j - 1$. \(\diamondsuit\)

Non-trivial lower bounds on even $N'(3, m)$ are scarce and surprisingly little else is known about what an optimal version of Khovanski’s Theorem on Fewnomials should resemble. For example, an earlier (conjectural) polyhedral generalization of Descartes’ Rule to multivariate systems of equations proposed by Itenberg and Roy in 1996 \([R96]\) (based on a famous construction of Oleg Viro from 1989 and extensions by Bernd Sturmfels \([Stu94]\)) was recently disproved \([W98]\). Also, a bit earlier, Bernd Sturmfels bet (and unfortunately lost) US$500 on a challenge problem involving a family of polynomial systems of type \((4, 4)\) \([R97]\).

To the best of the authors’ knowledge, all other general bounds on the number of real roots depend strongly on the individual exponents of $F$ and are actually geared more toward counting complex roots, e.g., \([BKK76, \text{Kaz81, BLR91, Roj99, Roj00a}]\). So even proving $N(3, 3) < \infty$ already requires a different approach. Nevertheless, the aforementioned bounds can be quite practical when the exponents are integral and the degrees of the polynomials are small.

In any event, it still remains unknown whether $K'(n, m)$ is polynomial in $m$ for $n$ fixed. (The polynomial system $(x_1 - 3x_1^2 + 2, \ldots, x_n^2 - 3x_n + 2)$ shows us that fixing $n$ is necessary.) Even the case of a trinomial and an $m$-nomial, in two variables, remains open. More to the point, it is also
unknown whether a simple modification (e.g., increasing the original GKC bound by a constant power or a factor exponential in \( n \)) changes the status of GKC from false to true. The \( 2k \times 2k \) fewnomial system \((x_1^{106} + 1.1y_1^{54} - 1.1y_1, y_1^{108} + 1.1x_1^{54} - 1.1x_1, \ldots, x_k^{106} + 1.1y_k^{54} - 1.1y_k, y_k^{108} + 1.1x_k^{54} - 1.1x_k)\), thanks to Haas’ counter-example (cf. remark \( \frac{3}{2} \)), easily shows that the GKC bound now needs at least an extra multiple no smaller than \((\sqrt[n]{\frac{3}{2}})^n\) if it is to be salvaged.

**Remark 7.** Domenico Napoletani has recently shown that to calculate \( N'(m_1, \ldots, m_n) \) for any given \( (m_1, \ldots, m_n) \), it suffices to restrict to the case of integral exponents \( \text{[Nap01]} \). Here, we will bound \( N(3, m) \) directly without using this reduction. 

Let us also make a related number-theoretic observation: Hendrik W. Lenstra has shown that for any fixed number field \( L \), the maximal (finite) number of roots in \( L \) of a univariate \( m \)-nomial, with integral exponents and coefficients in \( L \), is quasi-quadratic in \( m \) and independent of the degree of the polynomial \( \text{[Len99]} \). Thus an immediately corollary of theorem \( \frac{3}{2} \) (and theorem \( 3 \) and remark \( \frac{3}{2} \) of section \( \frac{3}{2} \)) is that Lenstra’s result can be effectively extended to certain families of fewnomial systems, provided we fix \( n \) and restrict to real algebraic number fields. (Fixing \( n \) is necessary for the same reason as in the last paragraph.)

Whether Lenstra’s result can be more fully extended to polynomial systems is also an open question, even in the case of two bivariate trinomials. However, it is at least now known that the number of **geometrically isolated** roots in \( L^n \) of any \( k \times n \) polynomial system can be bounded above by some function depending only on \( L, n, \) and the total number of distinct exponent vectors \( \text{[Roj00]} \). 

1.2. **Organization of the Proofs.**

Section \( \frac{3}{2} \) provides some background and unites some simple cases where GKC in fact holds. We then prove theorems \( \frac{3}{2} \) and \( \frac{3}{2} \) in sections \( \frac{3}{2} \) and \( \frac{3}{2} \), respectively. Proving the upper bound on \( N(3, m) \) turns out to be surprisingly elementary, but lowering the bound on \( N(3, 3) \) to 5 then becomes a more involved case by case analysis.

Section \( \frac{3}{2} \) then gives an alternative geometric proof that \( N(3, 3) \leq 6 \). We include this second proof for motivational purposes, since it was essentially the first improvement we found over \( N'(3, 3) \leq 248832 \). We then derive bounds for the number of isolated singularities and inflection points of an \( m \)-nomial, and discuss how the underlying **Newton polygons** (cf. the next section) strongly control how \( N(3, 3) \) can exceed 4 (cf. corollary \( \frac{3}{2} \) of section \( \frac{3}{2} \)). Roughly speaking, we show that if a fewnomial system of type \( (3, 3) \) has maximally many roots in the positive quadrant, then its underlying exponent vectors must be in “general position.” In particular, just like Haas’ counter-example, the underlying Newton polygons of any counter-example to this case of GKC must have Minkowski sum a hexagon (cf. sections \( \frac{3}{2} \) and \( \frac{3}{2} \)).

2. **The Pyramidal, Simplicial, and Zero Mixed Volume Cases**

Consider the following constructions.

**Definition 2.** For any \( S \subseteq \mathbb{R}^n \), let \( \text{Conv}(S) \) denote the smallest convex set containing \( S \). Also, for any \( m \)-nomial of the form \( f := \sum_{a \in A} c_a x^a \), we call \( \text{Supp}(f) := \{ a | c_a \neq 0 \} \) the support of \( f \), and define \( \text{Newt}(f) := \text{Conv}(\text{Supp}(f)) \) to be the **Newton polytope** of \( f \). More generally, a **polytope** is simply the convex hull of any finite point set in \( \mathbb{R}^n \). □

**Definition 3.** Let \( F = (f_1, \ldots, f_n) \) be a fewnomial system and for all \( i \) let \( L_i \) be the linear subspace affinely generated by \( \text{Supp}(f_i) \). We call \( F \) **pyramidal** iff the following condition holds for all \( i \): either \( L_i \supseteq L_j \) for all \( j \neq i \), or there is a \( j \) such that \( L_i \oplus L_j = L_i \oplus L \) for some line \( L \not\subset L_i \) with \( 0 \in L \). Finally,

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6 A root is geometrically isolated iff it is a zero-dimensional component of the underlying zero set in \( \mathbb{L}^n \), where \( \mathbb{L} \) is the algebraic closure of \( \mathbb{L} \).

7 In fact, as was done more explicitly in \( \text{[Len99]} \) for the univariate case, one can also allow \( \mathbb{L} \) to be any finite extension of the \( p \)-adic rationals. The latter setting is perhaps closer to our current focus since \( \mathbb{R} \), like the \( p \)-adics, is a metrically complete field.
we call any change of variables of the form \((x_1, \ldots, x_n) \mapsto y^A := (y_1^{a_1}, \ldots, y_n^{a_n}),\) with \(A := [a_{ij}]\) a real \(n \times n\) matrix, a monomial change of variables. ■

For example, the polynomial systems from remark \([1]\) are all pyramidal, but the systems from remarks \([3]\) and \([2]\) are not pyramidal (cf. section 1). Pyramidal systems are a simple generalization of the so-called “triangular” systems popular in Gröbner-basis papers on computer algebra. The latter family of systems simply consists of those \(F\) for which the equations and variables can be reordered so that for all \(i, f_i\) depends only on \(x_1, \ldots, x_i\). Put another way, pyramidal systems are simply the image of a triangular system (with real exponents allowed) after multiplying the individual equations by arbitrary monomials, shuffling the equations, and then performing a monomial change of variables. In particular, we note the following elementary fact on monomial changes of variables.

**Proposition 1.** If \(A\) is a real non-singular \(n \times n\) matrix, then \((x^A)^{A^{-1}} = x\) and the map \(x \mapsto x^A\) is an analytic automorphism of the positive orthant. In particular, such a map preserves smooth points, singular points, and the number of compact and non-compact connected components, of analytic subvarieties of the positive orthant. Furthermore, this invariance also holds for fewnomial zero sets in the positive orthant. ■

The assertion on analytic subvarieties follows easily from an application of the chain rule from calculus, and noting that such monomial maps are also diffeomorphisms. That the same invariance holds for fewnomial zero sets follows immediately upon observing that the substitution \((x_1, \ldots, x_n) = (e^{t_1}, \ldots, e^{t_n})\) maps any \(n\)-variate real \(m\)-nomial to a real analytic function, and noting that \((t_1, \ldots, t_n) \mapsto (e^{t_1}, \ldots, e^{t_n})\) is a diffeomorphism from \(\mathbb{R}^n\) to \(\mathbb{R}_+^n\).

**Remark 8.** The zero set of \(x_1 + x_2 - 1\), and the change of variables \((x_1, x_2) \mapsto (\frac{y_1}{y_2}, y_1 y_2)\), show that the number of isolated inflection points need not be preserved by such a map: the underlying curve goes from having no isolated inflection points to having one in the positive quadrant. ◦

We will also need the following analogous geometric extension of the concept of an over-determined system.

**Definition 4.** Given polytopes \(P_1, \ldots, P_n \subset \mathbb{R}^n\), we say that they have mixed volume\(^8\) zero if there exists a \(d\)-dimensional subspace of \(\mathbb{R}^n\) containing translates of \(P_i\) for at least \(d + 1\) distinct \(i\). ■

A simple special case of an \(n\)-tuple of polytopes with mixed volume zero is the \(n\)-tuple of Newton polytopes of an \(n \times n\) fewnomial system where, say, the variable \(x_i\) does not appear. Indeed, by multiplying the individual \(m\)-nominals by suitable monomials, and applying a suitable monomial change of variables, the following corollary of proposition \([1]\) is immediate.

**Corollary 2.** Suppose \(F\) is a fewnomial system, with only finitely many roots in the positive orthant, whose \(n\)-tuple of Newton polytopes has mixed volume zero. Then \(F\) has no roots in the positive orthant. ■

Indeed, modulo a suitable monomial change of variables, one need only observe that the existence of a single root in the positive orthant implies the existence of an entire ray of roots (parallel to some coordinate axis) in the positive orthant.

We will also need the following elegant extension of Descartes' Rule to real exponents. It’s proof involves a very simple induction using Rolle’s Theorem (cf. the next section) and dividing by suitable monomials \([Kho91]\) — tricks we will build upon in the next section.

**Definition 5.** For any sequence \((c_1, \ldots, c_m) \in \mathbb{R}^m\), it’s number of sign alternations is the number of pairs \((j, j') \in \{1, \ldots, m\}\) such that \(j < j'\), \(c_j c_{j'} < 0\), and \(c_i = 0\) when \(j < i < j'\). ■

Univariate Generalized Descartes’ Rule of Signs (UGDRS). Let \(c_1, a_1, \ldots, c_m, a_m\) be any real numbers with \(a_1 < \cdots < a_m\). Then the number of positive roots of \(\sum_{i=1}^m c_i x_1^{a_i}\) is at most the number of sign alternations in the sequence \((c_1, \ldots, c_m)\). In particular, \(N'(m) = N(m) = m - 1\). ■

\(^8\)The reader curious about mixed volumes of polytopes in this context can consult \([BZ88, Roj99]\) for further discussion.
As a warm-up, we can now prove a stronger version of GKC for the following families of special cases.

**Theorem 3.** Suppose $F$ is an $n \times n$ fewnomial system of type $(m_1, \ldots, m_n)$ (so $m_1, \ldots, m_n \geq 1$) and we restrict to those $F$ which also satisfy one of the following conditions:

(a) The $n$-tuple of Newton polytopes of $F$ has mixed volume zero.
(b) All the supports of $F$ can be translated into a single set of cardinality $\leq n + 1$.
(c) $F$ is pyramidal.

Then, following the notation of theorem 1...

- $N(m_1, \ldots, m_n)$ is respectively $0, 1, \text{or } \prod_{i=1}^{n} (m_i - 1)$ in case (a), (b), or (c).
- In cases (a), (b), and (c), $F$ has infinitely many roots $\implies F$ has no isolated roots.
- In general, $N'(1, m_2, \ldots, m_n) = N(1, m_2, \ldots, m_n) = 0$, $N'(2, m_2, \ldots, m_n) = N'(m_2, \ldots, m_n)$, and $N'(2, m_2, \ldots, m_n) = N(m_2, \ldots, m_n)$.

3. $N'(2, m) = N(2, m) = m - 1$.

**Proof:** First note that the Newton polytopes must all be nonempty. The case (a) portion of assertions (0) and (1) then follows immediately from corollary 2. Note also that the case (a) portion of assertion (0) immediately implies our formula for $N'(1, m_2, \ldots, m_n)$ (and thus $N'(1, m_2, \ldots, m_n)$ as well) in assertion (2), since the underlying $n$-tuple of polytopes clearly has mixed volume zero.

The case (b) portion of assertions (0) and (1) follows easily upon observing that $F$ is a linear system of $n$ equations in $n$ monomial terms, after multiplying the individual equations by suitable monomial terms. We can then finish by proposition 1.

To prove the case (c) portion of assertions (0) and (1), note that the case $n = 1$ follows immediately from UGDRS. For $n > 1$, we have the following simple proof by induction: Assuming GKC holds for all $(n - 1) \times (n - 1)$ pyramidal systems, consider any $n \times n$ pyramidal system $F$. Then, via a suitable monomial change of variables, multiplying the individual equations by suitable monomials, and possibly reordering the $f_i$, we can assume that $f_1$ depends only on $x_1$. (Otherwise, $F$ wouldn’t be pyramidal.) We thus obtain by UGDRS that $f_1$ has at most $m_1 - 1$ positive roots. By back-substituting these roots into $F' := (f_2, \ldots, f_n)$, we obtain a new $(n' - 1) \times (n' - 1)$ pyramidal fewnomial system of type $(m_2', \ldots, m_n')$ with $n' \leq n$ and $m_2' \leq m_2, \ldots, m_n' \leq m_n$. By our induction hypothesis, we obtain that each such specialized $F'$ has at most $\prod_{i=2}^{n'} (m_i' - 1)$ isolated roots in the positive orthant, and thus $F$ has at most $\prod_{i=2}^{n'} (m_i' - 1)$ isolated roots in the positive orthant. (Remark 1 from the introduction shows us that this bound can indeed be attained.)

Our recursive formulae for $N'(2, m_2, \ldots, m_n)$ and $N'(2, m_2, \ldots, m_n)$ from assertion (2) then follow by applying just the first step of the preceding induction argument, and noting that proposition 1 tells us that our change of variables preserves non-degenerate roots.

Assertion (3) follows immediately from assertion (2) via UGDRS.

**Remark 9.** One can of course combine and interweave families (a), (b), and (c) to obtain less trivial examples where GKC is true. More generally, one can combine theorems 1 and 3 to obtain bounds significantly sharper than Khovanski’s Theorem on Real Fewnomials, free from Jacobian assumptions, for additional families of fewnomial systems.

3. Substitutions and Calculus: Proving Theorem 1

Let us preface our first main proof with some useful basic results.

**Lemma 1.** For $m_1 = 1 + \dim \text{Newt}(f_1)$, the computation of $N'(m_1, \ldots, m_n)$ and $N(m_1, \ldots, m_n)$ can be reduced to the case where $f_1 := 1 \pm x_1 \pm \cdots \pm x_{m_1-1}$ (with the signs in $f_1$ not all “+”) and, for all $i$, $f_i$ has 1 as one of its monomial terms. In particular, for $m_1 = 3$, we can assume further that $f_1 := 1 - x_1 - x_2$.

**Proof:** By dividing each $m_i$-nomial by a suitable monomial term, we can immediately assume that all the $f_i$ possess the monomial term 1. In particular, we can also assume that the origin $O$ is a vertex of $\text{Newt}(f_1)$. Note also that the sign condition on $f_1$ must obviously hold, for otherwise the
value of $f_1$ would be positive on the positive orthant. (The refinement for $m = 3$ then follows by picking the monomial term one divides $f_1$ by a bit more carefully.) So we now need only check that the desired canonical form for $f_1$ can be attained.

Suppose $f_1 := 1 + c_1 x_1^2 + \cdots + c_{m-1} x_1^{2m-2}$. By assumption, Newt($f_1$) is an $m_1$-simplex with vertex set $\{O, a_1, \ldots, a_{m_1-1}\}$, so $a_1, \ldots, a_{m_1-1}$ are linearly independent. Now pick any $a_{m_1}, \ldots, a_n \in \mathbb{R}^n$ so that $a_1, \ldots, a_n$ are linearly independent. The substitution $x \mapsto x^{A-1}$ (with $A$ the matrix whose columns are $a_1, \ldots, a_n$) then clearly sends $f_1 \mapsto 1 + c_1 x_1 + \cdots + c_{m_1-1} x_1^{m_1-1}$, and proposition 1 tells us that this change of variables preserves degenerate and non-degenerate roots in the positive orthant. Then, via the change of variables $(x_1, \ldots, x_{m_1-1}) \mapsto (x_1/c_1, \ldots, x_{m_1-1}/c_{m_1-1})$, we obtain that $f_1$ can indeed be placed in the desired form. (The latter change of variables preserves degenerate and non-degenerate roots in the positive orthant for even more obvious reasons.)

Recall that a polynomial $p \in \mathbb{C}[x_1, \ldots, x_n]$ is homogeneous iff $p(ax_1, \ldots, ax_n) = a^d p(x_1, \ldots, x_n)$ for some non-negative integer $d$.

**Proposition 2.** Suppose $p \in \mathbb{R}[x_1, x_2]$ is homogeneous and has degree $d \geq 0$. Also let $\alpha, \beta \in \mathbb{R}$. Then there is a homogeneous $q \in \mathbb{R}[x_1, x_2]$, either identically zero or of degree $d+1$, such that $\frac{d}{dt} \left( t^\alpha (1-t)^\beta p(t, 1-t) \right) = t^\alpha (1-t)^\beta q(t, 1-t)$. In particular, $\frac{d}{dt} \left( t^\alpha (1-t)^\beta p(t, 1-t) \right)$ is identically zero iff $p(x_1, x_2) = x_1^{-\alpha} x_2^-\beta$, $\alpha - \beta = d$, and $\alpha, \beta \in \mathbb{Z}$ with $\alpha \geq 0$.

**Proof:** By the chain-rule, $\frac{d}{dt} \left( t^\alpha (1-t)^\beta p(t, 1-t) \right)$ is simply

$$
\alpha t^{\alpha-1}(1-t)^{\beta-1} p(t, 1-t) + \beta t^\alpha (1-t)^{\beta-1} p(t, 1-t) + t^\alpha (1-t)^\beta \left( p_1(t, 1-t) - p_2(t, 1-t) \right),
$$

where $p_i$ denotes the partial derivative of $p$ with respect to $x_i$. Factoring out a multiple of $t^\alpha (1-t)^{\beta-1}$ from the preceding expression, we then easily obtain that we can in fact take

$$
q(x_1, x_2) = (\alpha x_2 + \beta x_1) p(x_1, x_2) + x_1 x_2 \left( p_1(x_1, x_2) - p_2(x_1, x_2) \right).
$$

The final assertion of our proposition then follows immediately.

**Rolle’s Theorem.** Let $g : [a, b] \longrightarrow \mathbb{R}$ be any function with a well-defined derivative $g'$ defined on $(a, b)$. Then $g$ has $r$ roots in $[a, b] \implies g'$ has at least $r-1$ roots in $(a, b)$.

**Lemma 2.** Let $k \geq 2$. Then for any real $c_1, a_1, b_1, \ldots, c_k, a_k, b_k$, the function $f(t) := 1 + c_1 t^{a_1} (1-t)^{b_1} + \cdots + c_k t^{a_k} (1-t)^{b_k}$ has at most $2k+1-2$ roots in the open interval $(0, 1)$. Furthermore, $f$ has exactly $r$ roots in $(0, 1) \implies$ there exist $\tilde{c}_1, \ldots, \tilde{c}_k \in \mathbb{R}$ such that

$$
\tilde{f}(t) := 1 + \tilde{c}_1 t^{a_1} (1-t)^{b_1} + \cdots + \tilde{c}_k t^{a_k} (1-t)^{b_k}
$$

has at least $r$ roots in $(0, 1)$, and no root of $\tilde{f}$ is degenerate.

**Proof:** Henceforth, let us assume all roots lie in the open interval $(0, 1)$. Assume $f$ has exactly $r$ roots. Then by Rolle’s Theorem, $f'$ has at least $r-1$ roots. Since

$$
f'(t) = \sum_{i=1}^{k} c_i t^{a_i-1} (1-t)^{b_i-1} (a_i (1-t) + b_i t),
$$

and since $t^{a_k-1} (1-t)^{b_k-1}$ never vanishes in $(0, 1)$, the function $g_1(t) := c_k (a_k (1-t) + b_k t) + \sum_{i=1}^{k-1} c_i t^{a_i-a_k} (1-t)^{b_i-b_k} (a_i (1-t) + b_i t)$ has at least $r-1$ roots.

---

9For a simple proof, note that the special case $r = 2$ follows immediately from the Mean Value Theorem of calculus (see, e.g., [Rud76, thm. 5.10, pg. 107]), since we can replace $[a, b]$ by a sub-interval whose end-points are the roots of $g$. The general case then follows by replacing $a$ (resp. $b$) by the smallest (resp. largest) root of $g$, and then subdividing $[a, b]$ into $r-1$ sub-intervals whose endpoints consist of the roots of $g$. 

By Rolle’s Theorem again, $g''_k$ has at least $r - 3$ roots. By proposition \[, g''_k(t) \] will then be of the form $\sum_{i=1}^{k-1} c_i t^{a_i-a_k-2}(1-t)^{b_i-b_k-2} q_{i,1}(t, 1-t)$, where the $q_{i,1} \in \mathbb{R}[x_1, x_2]$ are homogeneous polynomials, which are either identically zero or of degree 3. In particular, we can assume that at least one $q_{i,1}$ must be different from the zero polynomial. (For otherwise we would obtain that $g''_k = 0$ identically, which would in turn imply that $g_1$ is a linear function, and thus $r \leq 3 < 2^{k+2} - 2$.) By again dividing by a suitable monomial in $t$ and $1-t$, we then see that $g''_k$ has the same number of roots as

$$g_2(t) := q_{k-1,1}(t, 1-t) + \sum_{i=1}^{k-2} c_i t^{a_i-a_k-1}(1-t)^{b_i-b_k-1} q_{i,1}(t, 1-t).$$

Thus $g_2$ has at least $r - 3$ roots.

By induction, we then easily obtain a sequence of polynomials $g_1, \ldots, g_j$, where $j \leq k$ and $g_j = q_{j,j-1}(t, 1-t)$ for some homogeneous $q \in \mathbb{R}[x_1, x_2]$ of degree $2^j - 1$ having at least $r - (2^j - 1)$ roots. So by Rolle’s Theorem one last time, $r \leq (2^j - 1) + (2^j - 1) \leq 2^{k+1} - 2$ and we are done with the first part of our lemma.

To prove the second part, note that the first part of our lemma implies that $f$ has only finitely many critical values — no more than $2^{k+1} - 2$, in fact. So for all $\delta \in \mathbb{R}^*$ with $|\delta|$ sufficiently small, $f - \delta$ will have no degenerate roots. We can in fact guarantee that $f - \delta$ will also have at least $r$ non-degenerate roots in $(0,1)$ as follows: Let $n_+$ (resp. $n_-$) be the number of roots $t$ of $f$ with $f'(t) = 0$ and $f''(t) > 0$ (resp. $f''(t) < 0$). Clearly then, for all $\delta \in \mathbb{R}^*$ with $|\delta|$ sufficiently small, $f - \delta$ will have exactly $r + n_--n_+$ or $r + n_+ - n_-$ roots, according as $\delta > 0$ or $\delta < 0$. (The analogous statement for roots in $(0,1)$ holds as well, since $\delta > 0$ is open.) So let $\delta$ be sufficiently small, and of the correct sign, so that $f - \delta$ has at least $r$ roots in $(0,1)$ and no degenerate roots.

To conclude, simply let $\tilde{c}_i := \frac{c_i}{1-\delta}$ for all $i$. Since $\tilde{f}$ is thus $\frac{f - \delta}{1-\delta}$, we are done. \[\square\]

**Proof of Theorem \[** First note that by lemma \[, we can immediately reduce to the case of a fewnomial system of the form $F := (1 - x_1 - x_2, 1 + c_1 x_1^{a_1} x_2^{b_1} + \cdots + c_{m-1} x_1^{a_{m-1}} x_2^{b_{m-1}})$, and this reduction preserves the degeneracy or non-degeneracy of any root of $F$. We can then simply solve for $x_2$ via the first equation and then substitute into the second equation to obtain a bijection between the roots of $F$ in the positive quadrant and the roots of $f(t) := 1 + c_1 t^{a_1}(1-t)^{b_1} + \cdots + c_{m-1} t^{a_{m-1}}(1-t)^{b_{m-1}}$ with $0 < t < 1$. A simple Jacobian calculation yields that $(\zeta_1, \zeta_2)$ is a degenerate root of $F \iff \sum_{i=1}^{m-1} c_i \zeta_1^{a_i-1}(1-\zeta_1)^{b_i-1}(a_i(1-\zeta_1) - b_i \zeta_1) = 0$ and $f(\zeta_1) = 0 \iff f'(\zeta_1) = f(\zeta_1) = 0$. So degenerate roots of our univariate reduction correspond bijectively to degenerate roots of $F$.

By lemma \[, and the fact that $\mathcal{N}(3, m) \leq \mathcal{N}(3, m)$, we immediately obtain $\mathcal{N}(3, m) \leq 2^m - 2$ and $\mathcal{N}(3, m) = \mathcal{N}(3, m)$. Our upper bounds on $\mathcal{N}(3, 4)$ and $\mathcal{N}(3, 5)$ are then simply specializations of our new upper bound for $\mathcal{N}(3, m)$.

To prove that $\mathcal{N}(3, 3) = 5$, thanks to Haas’ counter-example from remark \[, it suffices to show that $\mathcal{N}(3, 3) < 6$. To do this, let us specialize our preceding notation to $m = 3$, $(c_1, c_2) = (-A, -B)$, and $(a_1, b_1, a_2, b_2) = (a, b, c, d)$, for some $a, b, c, d \in \mathbb{R}$ and positive $A$ and $B$. (Restricting to positive $A$ and $B$ can easily be done simply by dividing $f_2$ by a suitable monomial term, à la the proof of lemma \[.)

By symmetry we can then clearly reduce to the following cases:

**A:** $a, b, c > 0$ and $d < 0$  \hspace{1cm} **B:** $a, c > 0$ and $b, d < 0$

**C:** $a, b > 0$ and $c, d < 0$  \hspace{1cm} **D:** $a, b, c, d > 0$

**E:** $a > 0$ and $b, c, d < 0$  \hspace{1cm} **F:** $a, b, c, d < 0$

**G:** $a, d > 0$, $b, c < 0$  \hspace{1cm} **H:** At least one of the numbers $a, b, c, d$ is zero.

In particular, our earlier substitution trick tells us that it suffices to show that any

$$f(t) := 1 - At^{a}(1-t)^{b} - Bt^{c}(1-t)^d,$$

with all roots non-degenerate, always has strictly less than 6 roots in the open interval $(0, 1)$. So let $r$ be the number of roots of any such non-degenerate $f$ in $(0, 1)$.

Let us now prove $r < 6$ in all 8 cases:
A. \(a, b, c > 0, \ d < 0:\)

Let \(Q(x) = 1 - Ax^a(1 - x)^b\) and \(R(x) = Bx^c(1 - x)^d\). The roots of \(f\) may be regarded as the intersections of \(y = Q(x)\) and \(y = R(x)\) in the positive quadrant. Since \(\lim_{x \to 0^+} Q(x) = 1\), \(\lim_{x \to 1^-} Q(x) = 1\), \(\lim_{x \to 0^+} R(x) = 0\), and \(\lim_{x \to 1^-} R(x) = \infty\), it is easy to see via the Intermediate Value Theorem of calculus that the number of intersections must be odd. (One need only note that \(f = Q - R\) and that the signs of \(f'\) at the ordered roots of \(f\) alternate.)

So \(r < 6\).

B. \(a, c > 0, \ b, d < 0:\)

By an argument similar to that of case A, \(r\) is odd and thus less than 6.

C. \(a, b > 0, \ c, d < 0:\)

See lemma 4 below.

D. \(a, b, c, d > 0:\)

See lemma 5 below.

E. \(a > 0, \ b, c, d < 0:\)

See lemma 6 below.

F. \(a, b, c, d < 0:\)

Multiplying \(f(t)\) by \(t^\max\{-a, -c\}(1 - t)^\max\{-b, -d\}\), we can immediately reduce to case D.

G. \(a, d > 0, \ b, c < 0:\)

See lemma 7 below.

H. At least one of the numbers \(a, b, c, d\) is zero:

Use lemma 8 below, noting that our hypotheses here imply that either \(F\) or \(\hat{F}\) is a quadratic polynomial.

This concludes the proof of theorem 1. ■

We now detail the lemmata we cited above.

**Lemma 3.** Following the notation of the proof of theorem 1, recall that \(r\) is the number of roots of \(f(t) := 1 - At^a(1 - t)^b - Bt^c(1 - t)^d\) in the open interval \((0, 1)\), where \(f\) has no degenerate roots. Also let \(g(t) := \frac{B}{A}t^{a-c}(1 - t)^b - c(1 - t) + dt\), \(F(u) := -a(a-c)(a-1-c)u^3 + (a-c)2u(b+1) + b(a-c+1)u^2 + (d-b)[a(b-d+1)+2b(a-c+1)]u+b(b-d)(b-d-1)\), and \(\hat{F}(u) := -c(a-c)(a-1-c)u^3 + (c-a)2u(d-b+1) + d(a-c+1)u^2 + (b-d)[c(d-b+1)+2d(a-c+1)]u+d(d-b)(d-b-1)\).

Finally, let \(N\) (resp. \(M\)) be the number of roots in \((0, 1)\) of \(g\) (resp. the maximum of the number of positive roots of \(F\) and \(\hat{F}\)). Then \(r - 3 \leq N - 2 \leq M \leq 3\).

**Proof:** Just as in the proof of lemma 2, we easily see by Rolle’s Theorem and division by suitable monomials in \(t\) and \(1 - t\) that \(r - 1\) is no more than the number of roots in \((0, 1)\) of \(g\). So \(r - 1 \leq N\). Note also that, in a similar way, \(r - 1\) is no more than the number of roots of \(\hat{g}(t) := \frac{B}{A}t^{a-c}(1 - t)^d - b\) \(g(t)\) in \((0, 1)\), and the latter function has the same number of roots in \((0, 1)\) as \(g\).

To conclude, simply note that for suitable \(a, b, \gamma, \delta \in \mathbb{R}\), we have that \(F(\frac{1-t}{t}) = t^a(1 - t)^\beta g''(t)\) and \(\hat{F}(\frac{1-t}{t}) = t^\gamma(1 - t)^\delta \hat{g}''(t)\). So, by our preceding trick again, \(N = 2 \leq M\), and thus \(r - 3 \leq M\).

That \(M \leq 3\) is clear from the fundamental theorem of algebra. ■

**Lemma 4.** Following the notation of lemma 2, let

\[T(x) := \frac{A}{B}x^{a-c}(1 - x)^{-b}(a(1 - x) + bx), \quad S(x) := c - (c + d)x,\]

and

\[\hat{T}(x) := \frac{B}{A}x^{c-a}(1 - x)^{d-b}(-c(1 - x) + dx), \quad \hat{S}(x) := a - (a + b)x.\]

Then \([a, b > 0 \text{ and } c, d < 0] \implies r < 6\).

**Proof:** By lemma 3 we are done if \(M < 3\) or \(N < 5\). So let us assume \(M = 3\) to derive a contradiction. By Descartes’ Rule of Signs (see section 2 for a generalization), the coefficients of \(F(u)\) or \(\hat{F}(u)\) (ordered by exponent) must have alternating signs. Thus, since \(a, a-c, b, b-d > 0\), we have that \(a - c - 1\) and \(b - d - 1\) must have the same sign. We then need to discuss two cases:
• $a - c - 1 < 0$ and $b - d - 1 < 0$:
  This implies $c - a + 1 > 0$ and $d - b + 1 > 0$. Consequently, coefficients of $u^3$ and $u^2$ in $\tilde{F}(u)$ and $F(u)$ are all positive — a contradiction.
• $a - c - 1 > 0$ and $b - d - 1 > 0$:
  The roots of $g$ in $(0, 1)$ can be regarded as intersections of $y = T(x)$ and $y = S(x)$, for $0 < x < 1$. Since $T(x) < 0$ for $0 < x < 1$ and $T(x) > 0$ for $0 < 1 - x < 1$, there is a smallest positive local minimum $c_0$ of $T$ with $T(c_0) < 0$. Thus for $x$ near $c_0$, $T'(x) > 0$. Since $T''(x) < 0$ for $0 < x < 1$, there is $c^* \in (0, c_0)$ such that $T''(c^*) = 0$. Let $(x_1, y_1), \ldots, (x_K, y_K)$ be the intersection points of $y = T(x)$ and $y = S(x)$ with $x_1 < x_2 < \cdots < x_K$, where a tangent point is counted twice. Then for all $i \in \{1, \ldots, K - 1\}$ there is a $c_i \in (x_i, x_{i+1})$ with $T'(c_i) = -(c + d) > 0$, and for all $i \in \{1, \ldots, K - 2\}$ there is a $d_i \in (c_i, c_{i+1})$ with $T''(d_i) = 0$. Note that $c_0 < c_1$. Thus $c^* < d_1$ and therefore $T''(x) = 0$ has at least $K - 1$ solutions. Since $T''$ and $F$ have the same number of positive roots (observing that $T''(u)/F(u)$ is a monomial in $u$ and $1 - u$), we have $N - 1 \leq K - 1 \leq 3.$

**Lemma 5.** Following the notation of lemma [4], $a, b, c, d > 0 \Rightarrow r < 6$.

**Proof:** Again, by lemma [3], we need only show that $M < 3$ or $N < 5$. So let us assume $M = 3$. Then by Descartes’ Rule of Signs, $(a - c)(a - c - 1)$ and $(b - d)(b - d - 1)$ in the coefficients of $u^3$ and $u^0$ in $F(u)$ must have the same sign. There are now four cases to be examined.
• The signs of $a - c, a - c - 1, b - d,$ and $b - d - 1$ are respectively $+, -, +,$ and $-$:
  This makes the signs of coefficients of $u^3$ and $u^2$ of $\tilde{F}(u)$ both positive.
• The signs of $a - c, a - c - 1, b - d,$ and $b - d - 1$ are respectively $-, -, +,$ and $+$:
  Since $b - d > 0$, we have $d - b < 0$ and $d - b - 1 < 0$. This makes the constant term of $\tilde{F}(u)$ positive, and hence, the coefficients of $u$ and $u^2$ of $\tilde{F}(u)$ must respectively be negative and positive. That is, $c(d - b + 1) + 2d(c - a + 1) < 0$ and $2c(d - b + 1) + d(c - a + 1) > 0$. Thus, $-c(d - b + 1) + d(c - a + 1) < 0$. This is false, since $b - d - 1 > 0$ and $a - c - 1 < 0$.
• The signs of $a - c, a - c - 1, b - d,$ and $b - d - 1$ are all negative:
  By Descartes’ rule of signs, $d - b - 1$ and $c - a - 1$ in the coefficients of $y^3$ and $y^0$ of $\tilde{F}(y)$ must have the same sign. If both are negative, then coefficients of $u^3$ and $u^2$ of $F(u)$ would both be negative. Thus $d - b - 1 > 0$ and $c - a - 1 > 0$. It is easy to see that $\hat{T}(x) < 0$ for $0 < x < 1$ and $\hat{T}(x) > 0$ for $0 < 1 - x < 1$ and $\lim_{x \to 0^+} \hat{T}(x) = \lim_{x \to 1^-} \hat{T}(x) = 0$. Now let $L_0 = \min\{c \mid 1 > c > 0\}$, $\hat{T}(c) < 0$ and $c$ is a local minimum and $U_0 = \max\{c \mid 1 > c > L_0\}$, $\hat{T}(c)$ is a local maximum. Then for $x$ near $L_0$, $\hat{T}''(x) > 0$. Since $\hat{T}''(x) < 0$ for $0 < x < 1$, there exists $0 < L_1 < L_0$ such that $\hat{T}''(L_1) = 0$. Similarly, there is a $U_1 \in (U_0, 1)$ such that $\hat{T}''(U_1) = 0$.
  The roots of $\hat{g} \hat{g} - (1 - t)^{d - b}y$ can be regarded as the intersections of $y = \hat{T}(x)$ and $y = \hat{S}(x)$, for $0 < x < 1$. Let $(x_1, y_1), \ldots, (x_k, y_k)$ be the intersection points with $x_1 < x_2 < \cdots < x_k$, where a tangent point is counted twice. Then there exist $x_i < c_i < x_{i+1}$ such that $\hat{T}'(c_i) = -(a + b) < 0$, $i = 1, \ldots, k - 1$ and $c_i < d_i < c_{i+1}$ such that $\hat{T}''(d_i) = 0$, $i = 1, \ldots, k - 2$. If $x_1 > L_0$, then $L_1 < d_i$. If $x_1 < L_0$, then $T(x_1) < 0$. This implies $T(x_i) < 0$ for all $i = 1, \ldots, k - 2$, since the slope $-(a + b)$ of $\hat{S}(x)$ is negative. Therefore, $x_{k-1} < U_0$ and hence $d_{k-2} < U_1$. So $\hat{T}''(x) = 0$ has at least $k - 1$ solutions. Since $\hat{T}''(x) = 0$ and $\hat{F}(y) = 0$ have the same number of solutions, we have $N - 1 \leq k - 1 \leq M = 3$.
• The signs of $a - c, a - c - 1, b - d,$ and $b - d - 1$ are all positive:
  Since $a - c - 1 > 0$ and $b - d - 1 > 0$, the proof follows the same line of arguments as the last case by considering the intersections of $T(x)$ and $S(x)$ instead.

**Lemma 6.** Following the notation of lemma [3], $a > 0$ and $b, c, d \leq 0 \Rightarrow r < 6$.

**Proof:** Once again, by lemma [3], it suffices to show that $M < 3$ or $N < 5$. So let us assume that $M = 3$. By checking coefficients of $u^3$ and $u^0$ in $F(u)$, Descartes’ Rule of Signs tells us that $a - c - 1$ and $(b - d)(b - d - 1)$ must have different signs. There are now three cases to be examined.
• $a - c - 1$, $b - d$, and $b - d - 1$ are all negative.

Then the signs of the coefficients of both $u^3$ and $u^2$ in $F(u)$ will all be positive.

• The signs of $a - c - 1$, $b - d$, and $b - d - 1$ are respectively $-,+,$ and $+$. 

Multiplying $f$ by $x^{c-1}(1-x)^d$ yields $u(x) := x^{-c}(1-x)^{-d} - Ax^{a-c}(1-x)^{b-d} - B$, where $-c > 0$, $a - c > 0$, $-d > 1$, and $-d + b > 1$. The roots of $u$ in $(0,1)$ can be regarded as the intersections of the curves $y = v(x) = x^{-c}(1-x)^{-d} - Ax^{a-c}(1-x)^{b-d}$ and $y = B$. Let $(x_1, y_1), \ldots, (x_n, y_n)$ be the intersection points of $y = v(x)$ and $y = B$ with $x_1 < x_2 < \cdots < x_n$, where a tangent point is counted twice. Then there exist $x_i < c_i < x_{i+1}$ such that $v'(c_i) = 0$, $i = 1, \ldots, n - 1 = N$. Thus $v'$ has at least $N$ roots in $(0, 1)$. A straightforward computation then yields,

$$v'(x) = Ax^{a-c-1}(1-x)^{b-d-1}(-(a-c)(1-x) + (b-d)x) + x^{-c-1}(1-x)^{-d-1}(-c(1-x) + dx),$$

which clearly has the same number of roots in $(0,1)$ as

$$t(x) := Ax^a(1-x)^b(-(a-c)(1-x) + (b-d)x) - c(1-x) + dx.$$

Thus $t''$ has at least $\hat{N} - 2$ roots in $(0, 1)$. Since

$$t''(x)/A = x^{a-2}(1-x)^{b-2}(-(a-c)a(a-1)(1-x)^3 + a((a+1)(b-d) + 2(b+1)(a-c))x(1-x)^2$$

$$-b((b+1)(a-c) + 2(b-d)(a+1))x^2(1-x) + (b-d)b(b-1)x^3],$$

$t''$ has as many roots in $(0,1)$ as

$$P(u) = -(a-c)a(a-1)u^3 + a((a+1)(b-d) + 2(b+1)(a-c))u^2$$

$$-b((b+1)(a-c) + 2(b-d)(a+1))u + (b-d)b(b-1)$$

has positive roots. Since $a - 1 < a - c - 1 < 0$, the coefficients of $u^3$ and $u^0$ in $P(u)$ are both positive. Thus $P$ has at most 2 positive roots and we obtain $\hat{N} - 2 \leq 2$.

• The signs of $a - c - 1$, $b - d$, and $b - d - 1$ are respectively $+,+,+$ and $-:

Since $a - c - 1 > 0$ and $b - d > 0$, it is easy to see that $T(x) < 0$ for $0 < x < 1$ and $\lim_{x \to 1^-} T(x) = -\infty$. If $T(x)$ has no local minimum, then $y = T(x)$ and $y = S(x)$ have at most one intersection point. Otherwise, let $c_0 = \min\{c \mid 1 > c > 0\}$, $c$ is a local minimum of $T$.

The rest of the proof is similar to that of lemma \[3\] \[3\]

**Lemma 7.** Following the notation of lemma \[3\] \[3\], $[a, d > 0 \text{ and } b, c < 0] \implies r < 6$.

**Proof:** One last time, lemma \[3\] \[3\] tells us that it suffices to prove that $M < 3$ or $N < 5$. So let’s assume that $M = 3$. Checking signs of coefficients of $u^3$ and $u^0$ of both $F(u)$ and $\hat{F}(u)$, Descartes' Rule of Signs tells us that $a - c - 1 < 0$ and $d - b - 1 < 0$. On the other hand, the alternating signs of coefficients of $u^2$ and $u^1$ of $F(u)$ yield

$$2a(b - d + 1) + b(a - c + 1) < 0 \quad \text{and} \quad a(b - d + 1) + 2b(a - c + 1) > 0.$$ 

Thus,

$$-a(b - d + 1) + b(a - c + 1) = a(d - 1) + b(1 - c) > 0.$$ 

This is impossible, since $d - 1 < d - 1 - b < 0$, $1 - c > 0$, $a > 0$, and $b < 0$. \[3\]

**Remark 10.** When $A = 1.12$, $B = 0.71$, $a = 0.5$, $b = 0.02$, $c = -0.05$, and $d = 1.8$,

$$f(x) = 1 - Ax^a(1-x)^b - Bx^c(1-x)^d = 0, \quad 0 < x < 1$$

(1)

has 5 solutions. They are, approximately, \{0.00396494, 0.02986317, 0.4354707, 0.7252344, 0.99620026\}. \[3\]
4. A Simple Geometric Approach

Let us begin with an extension of Rolle’s Theorem to smooth curves in the plane.

**Lemma 8.** Suppose \( C \subset \mathbb{R}^2 \) is an arc (i.e., image of an interval or circle under a continuous map) with

1. A unique well-defined tangent line for each \( x \in C \).
2. At most \( I \) isolated inflection points.
3. At most \( V \) isolated points of vertical tangency.

Then the maximum finite number of intersections of any line with \( C \) is \( I + V + 2 \).

**Proof:** Let \( S^1 \) be the realization of the circle obtained by identifying 0 and \( \pi \) in the closed interval \([0, \pi]\). Consider the natural map \( \phi : C \longrightarrow S^1 \) obtained by \( x \mapsto \theta_x \) where \( \theta_x \) is the angle the normal line of \( x \) forms with the \( x_1 \)-axis. We claim that any \( \theta \in S^1 \) has at most \( I + V + 1 \) pre-images under \( \phi \).

To see why, note that by assumption we can express \( C \) as the union of no more than \( I + V + 1 \) arcs where (a) any distinct pair of arcs is either disjoint or meets at \( \leq 2 \) end-points, and (b) every end-point is either an isolated point of inflection or vertical tangency of \( C \). Calling these arcs **basic arcs**, it is then clear that the interior of any basic arc is homeomorphic (via \( \phi \)) to a connected subset of \( S^1 \setminus \{0\} \). Furthermore, by construction, the cardinality of \( \phi^{-1}(0) \) is exactly \( V \). So we indeed obtain that any \( \theta \in S^1 \) has at most \( I + V + 1 \) pre-images under \( \phi \).

Now note that any line \( \{x \mid m_1 x_1 + m_2 x_2 = m_0 \} \) normal to \( C \) forms an acute angle of \( \arctan \left( \frac{m_2}{m_1} \right) \) with the \( x_1 \)-axis. Thus, the number of contact points \( C \) has with the differential system

\[
\frac{\partial x_1}{\partial t} = m_2, \quad \frac{\partial x_2}{\partial t} = -m_1
\]

is at most \( I + V + 1 \). By Rolle’s Theorem for Dynamical Systems in the Plane (see, e.g., [Kho91, corollary, pg. 23]), we then obtain that the number of intersections of \( \{x \mid m_1 x_1 + m_2 x_2 = m_0 \} \) with \( C \) is at most \( I + V + 2 \), for any real \((m_0, m_1, m_2)\). So we are done. ■

**Remark 11.** The bound from lemma 8 is tight in all cases. This is easily revealed by the following examples and their obvious extensions:

\[\text{Figure 1. Lemma 8 gives a tight bound for } (I, V) \in \{(0,0), (3,1), (4,1), (3,2), (7,5)\} \text{ and this generalizes easily to arbitrary } (I, V).\]

The authors do not presently know whether this bound remains tight when restricted to fewnomial zero sets. ♦

\(^{10}\) Relative to the locus of inflection points.

\(^{11}\) i.e., the number of points at which some solution of the differential system has a tangent line in common with \( C \) is...
We are now ready to give a quick geometrically motivated proof of the nearly optimal bound \( \mathcal{N}(3, 3) \leq 6 \). This “second” proof of \( \mathcal{N}(3, 3) \leq 6 \) was actually the original motivation behind this paper.

**Short Geometric Proof of \( \mathcal{N}(3, 3) \leq 6 \):** Theorem \( \boxed{3} \) implies that we can assume that \( f_1 \) and \( f_2 \) have Newton polygons that are each triangles. Letting \( Z \) denote the zero set of \( f_2 \) in \( \mathbb{R}^2 \), lemma \( \boxed{8} \) of the last section tells us that we can assume that \( f_1 = 1 \pm x_1 \pm x_2 \); and by proposition \( \boxed{3} \) the underlying change of variables also implies that \( Z \) is diffeomorphic to a line. So \( Z \) is smooth and theorem \( \boxed{2} \) tells us that \( Z \) has no more than 3 inflection points and 1 vertical tangent. So we now need only check how many intersections \( Z \) will have with the line \( \{ x \mid 1 \pm x_1 \pm x_2 = 0 \} \). By lemma \( \boxed{8} \) we are done.

It turns out that inflection points for \( m \)-nomial curves are easy to describe in a \( m \)-nomial way. Let \( \partial_i := \frac{\partial}{\partial x_i} \).

**Lemma 9.** Suppose \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) is analytic and \( Z \) is the real zero set of \( f \). Then \( z \) is an inflection point or a singular point of \( Z \) if and only if \( (\partial_1 f \cdot \partial_2 f)^2 - 2\partial_1 \partial_2 f \cdot \partial_1 f \cdot \partial_2 f + \partial_1^2 f \cdot (\partial_1 f)^2 \big|_{x_1 = x_2} = 0 \). In particular, in the case \( f := 1 + c_1 x^{a_1} + \cdots + c_m x^{a_m} \), the preceding polynomial in derivatives is, up to a multiple which is a monomial in the \( x^{a_i} \), a cubic polynomial homogeneous in the \( c_i x^{a_i} \).

**Proof:** In the case of a singular point, the first assertion is trivial. Assuming \( \partial_2 f \neq 0 \) at an inflection point then a straightforward computation of \( \partial_2^2 f \) (via implicit differentiation and the chain rule) proves the first assertion. If \( \partial_2 f = 0 \) at an inflection point then we must have \( \partial_1 f \neq 0 \). So by computing \( \partial_2^2 f \) instead, we arrive at the remaining case of the first assertion. The second assertion also follows routinely.

Let us now reveal the hardest case of our result for pairs of trinomials. First note that while one can naturally associate a pair of polygons to \( F \) when \( n = 2 \), we can also associate a **single** polygon by forming the **Minkowski sum** \( P_F := \text{Newt}(f_1) + \text{Newt}(f_2) \). We can then give the following addendum to theorem \( \boxed{3} \) (with an independent proof).

**Corollary 3.** Following the notation of GKC and theorem \( \boxed{3} \), consider the case \( (n, m_1, m_2) = (2, 3, 3) \). Then \( \mathcal{N}(3, 3) \) is respectively 0, 2, or 4, according as we restrict to those \( F \) with \( P_F \) a line segment, triangle, or \( \ell \)-gon with \( \ell \in \{4, 5\} \).

**Proof:** The segment case follows immediately from corollary \( \boxed{2} \). For the remaining cases, proposition \( \boxed{3} \) implies that we can assume \( f_1 := 1 - x_1 - x_2 \) and \( f_2 := 1 + Ax_1^2x_2^2 + Bx_1^3x_2 \). In particular, it is easily verified that the underlying monomial change of variables preserves the positivity of angles between lines (in exponent space), so the number of edges of \( P_F \) is unchanged.

Let \( S_1 := Ax_1^2x_2^2 \), \( S_2 := Bx_1^3x_2 \), and let \( Z \) denote the zero set of \( f_2 \). Now observe that \( \text{lemma } \boxed{3} \) tells us that we can bound the number of inflection points of \( Z \) by analyzing the roots of a homogeneous polynomial in \( (S_1, S_2) \) of degree \( \leq 3 \). So let us now explicitly examine this polynomial in our polygonally defined cases.

Clearly then, the triangle case corresponds to setting \( a = d > 0 \) and \( b = c = 0 \). We then obtain that \( \{ x \in Z : \text{inflection point or a singular point of } Z \} \implies 1 + S_1 + S_2 = 0 \) and \( S_1 + S_2 = 0 \). So \( Z \) has no inflection points (or singularities). It is also even easier to see that \( Z \) has no vertical tangents. So by lemma \( \boxed{8}, \mathcal{N}(3, 3) \leq 2 \) in this case. To see that equality can hold in this case, simply consider \( F := (x_1^2 + x_2^2 - 25, x_1 + x_2 - 7) \), which has \( P_F = \text{Conv}(\{(0, 0), (3, 0), (0, 3)\}) \) and root set \( \{(3, 4), (4, 3)\} \).

Similarly, the quadrilateral case corresponds to setting \( b = c = 0 \) and \( a, d > 0 \). We then get the pair of equations \( 1 + S_1 + S_2 = 0 \) and \( ad - d - 2c \). If \( \{a, d\} \cap \{0, 1\} \neq \emptyset \) then \( F \), or a suitable pair of linear combination of \( F \), would be pyramidal and we would be done by theorem \( \boxed{3} \) So \( Z \) can have at most 1 inflection point. It is also even easier to see that \( Z \) has no vertical tangents. So by another application of lemma \( \boxed{8}, \mathcal{N}(3, 3) \leq 4 \) in this case. To see that equality can hold in this case, simply consider the system \( (x_1^2 - 3x_1 + 2, x_2^2 - 3x_2 + 2) \), which has \( P_F = \text{Conv}(\{(0, 2), (2, 2), (0, 2)\}) \) and root set \( \{(1, 1), (1, 2), (2, 1), (2, 2)\} \).

Finally, the pentagonal case corresponds to setting \( b = 0 \) and \( a, c, d > 0 \). We then get the pair of equations \( 1 + S_1 + S_2 = 0 \) and \( a^2(d-1)S_1^2 + (a(d-2)c) S_1S_2 - c(d+c)S_2^2 = 0 \), with \( ac(d-1)(c+d) \neq 0 \).
(Similar to the last case, it is easily checked that if the last condition were violated, then we would be back in one of our earlier solved cases.) However, a simple check of the discriminant of the above quadratic form in \((S_1, S_2)\) shows that there is at most 1 root, counting multiplicities, in any fixed quadrant. So, similar to the last case, we obtain \(N(3, 3) \leq 4\) in this case. To see that the equality can hold in this case, simply consider the system \((x_1^2 - 7x_2 + 12, -1 + x_1x_2 - x_1^2)\), which has \(P_r = \text{Conv}(\{O, (2, 0), (2, 2), (1, 3), (0, 2)\})\) and root set \((3, 2 + \sqrt{3}), (4, 2 \pm \sqrt{3})\).

5. Monomial Morse Functions and Connected Components: Proving Theorem

A construction which will prove quite useful when we count connected components via critical points of maps is to find a monomial which is a Morse function relative to a given fewnomial zero set.

Remark 12. In what follows, we will always understand \(\dim\) (resp. \(\dim_{\C}\)) to mean real (resp. complex) dimension. Also, unless otherwise noted, “dimension” will be understood to mean real dimension. Q.E.D.

Lemma 10. Suppose \(Z\) is the zero set in \(\mathbb{R}_+^n\) of an \(n\)-variate monomial \(f\). Then there exists a finite union of hyperplanes \(H_Z \subset \mathbb{R}^n\) such that for all \(a \in \mathbb{R}^n \setminus H_Z\) we have...

1. Every critical point of the restriction of \(x^a\) to \(Z\) is non-degenerate.
2. The level set in \(Z\) of any regular value of \(x^a\) has dimension \(\leq n - 2\).
3. No connected component of \(Z\) (other than an isolated point) is contained in any level set of \(x^a\).
4. Every unbounded connected component of \(Z\) has unbounded values of \(x^a\).

Proof: Let us prove the last two assertion first: Since the number of connected components of \(Z\) is finite\(^{12}\) we can temporarily assume that \(Z\) consists of a single connected component. Then, if we could find \(n\) linearly independent \(a\) with \(Z \cap \{x \in \mathbb{R}_+^n \mid x^a = c_n\}\) for some \(c_n\), proposition \(^{12}\) would immediately imply that \(Z\) is contained in a point. Similarly, if we could find \(n\) linearly independent \(a\) for which the restriction of \(x^a\) to \(Z\) is bounded, then we would obtain by proposition \(^{12}\) again that \(Z\) is bounded — a contradiction.

To prove the rest of our lemma, let us return to general \(Z\) and consider the substitution \(x_i = e^{z_i}\). A simple derivative computation (noting that \(x \mapsto (e^{z_1}, \ldots, e^{z_n})\) is a diffeomorphism between \(\mathbb{R}_+^n\) and \((\mathbb{R}^*)^n\)) then shows that it suffices to instead prove the analogous statement where \(f\) is replaced by a real exponential sum (a real analytic function in any event) and \(x^a\) is replaced by the linear form \(a_1 z_1 + \cdots + a_n z_n\). The latter analogue is then nothing more than an application of [Ko91, sec. 3.14], combined with Khovanski’s Theorem on Fewnomials to ensure that \(H_Z\) is finite instead of countable.

We will also need the following useful perturbation result, which can be derived via a simple homotopy argument. (See, e.g., Bas99 lemma 2 for even stronger results of this form in the case of integral exponents.)

Lemma 11. Following the notation of lemma \([12]\), let \(Z_\delta\) denote the solution set of \(|f| \leq \delta\) in \(\mathbb{R}_+^n\) and \(\bar{Z}_\delta\) its boundary. Then for \(\delta > 0\) sufficiently small, \(\bar{Z}_\delta\) and its closure are smooth, and there is a bijection between the connected components of \(Z\) and \(Z_\delta\) which preserves compact and non-compact components.

Finally, we will need the following two results (the latter dating back to an analogous result of Giusti and Heintz [GH98, sec. 3.4.1] in the complex algebraic case, if not earlier) for dealing with over-determined fewnomial systems.

Real Dimension Lemma. Suppose \(U\) is an open subset of \(\mathbb{R}^n\), \(W\) is an irreducible real analytic subvariety of \(U\), and \(g : U \to \mathbb{R}\) is a real analytic function with \(g(w) \neq 0\) for some \(w \in W\). Then \(\dim W \cap \{z \in U \mid g(z) = 0\} < \dim W\).

\(^{12}\)The smooth case is detailed in [Ko91, sec. 3.14] and the case of integral exponents (allowing degeneracy) is a special case of Roj00a lemma 3.2. In any event, the proof of the latter lemma extends easily to real exponents.
**Proof:** Let $d := \dim W$ and let $W_C$ be the complexification of $W$. Then $W_C$ is an irreducible analytic subvariety of $U'$ where $U' \subseteq \mathbb{C}^n$ is an open subset containing $U$ and $\dim_C W_C \geq d$. Furthermore, by [GR84, Active Lemma, pg. 100] we have $\dim_C W_C \cap \{ z \in U' \mid g(z) = 0 \} = \dim_C W_C - 1$. So, $W \cap \{ z \in U \mid g(z) = 0 \}$ (the real part of $W_C \cap \{ z \in U' \mid g(z) = 0 \}$) must have strictly smaller real dimension than $W$. 

**Lemma 12.** Suppose $k \geq n$ and that $F := (f_1, \ldots, f_k)$ is a $k \times n$ fewnomial system. Assume further that there are at most $m$ distinct exponent vectors in $F$. Then there exist real numbers $a_{ij}$ such that

1. the real zero set of $G := (a_{11}f_1 + \cdots + a_{1k}f_k, \ldots, a_{n1}f_1 + \cdots + a_{nk}f_k)$ is the union of the real zero set of $(f_1, \ldots, f_k)$ and a finite (possibly empty) set of points.
2. $G$ is of type $(m - 1, \ldots, m - 1)$ and has no more than $m$ distinct exponent vectors.

**Proof:** Let us first make the substitution $x_i = e^{x_i}$, noting that $x \mapsto (e^{x_1}, \ldots, e^{x_n})$ is a diffeomorphism between $\mathbb{R}^n$ and $\mathbb{R}_+^n$ which preserves the dimension of the underlying subanalytic varieties. Now pick $a_{11}, \ldots, a_{1k} \in \mathbb{R}$ so that $a_{11}f_1 + \cdots + a_{1k}f_k$ is not identically zero. Fix a set of points $\{w_i\}$, one lying in each irreducible component of the zero set $Z_1$ of $a_{11}f_1 + \cdots + a_{1k}f_k$ in $\mathbb{R}^n$. Let us then pick $a_{21}, \ldots, a_{2k}$ so that $a_{21}f_1 + \cdots + a_{2k}f_k$ does not vanish at any $\{w_i\}$. By the Real Dimension Lemma we then obtain that the zero set $Z_2$ of $(a_{11}f_1 + \cdots + a_{1k}f_k, a_{21}f_1 + \cdots + a_{2k}f_k)$ in $\mathbb{R}^n$ is the union of a diffeomorphic copy of $Z$ and a real analytic variety of dimension $n - 2$. Continuing this construction inductively, and then changing variables back again, we easily obtain assertion (1).

An application of Gaussian elimination to eliminate one monomial from each of the polynomials of $G$ then gives us assertion (2). 

To finally prove theorem 3, let us make one last definition.

**Definition 6.** Letting $f$ be a bivariate $m$-nomial and $Z$ the zero set of $f$ in the positive orthant, define...

- $S(m) :=$ The maximal number of isolated singular points of such a $Z$.
- $I(m) :=$ The maximal number of isolated\footnote{Relative to the locus of inflection points.} inflection points of such a $Z$.
- $V(m) :=$ The maximal number of isolated\footnote{Relative to the locus of points of vertical tangency.} points of vertical tangency of $Z$. 

Theorem 3 is then an immediate corollary of theorem 3 below.

**Definition 7.** Let $\mathcal{K}(n, \mu)$ be the maximal number of isolated roots in $\mathbb{R}_+^n$ of an $n \times n$ fewnomial system with exactly $\mu$ distinct exponent vectors. (So $\mathcal{K}(n, \mu) \leq \mathcal{K}(n, \mu)$.)

**Theorem 4.** Theorem 3 is true. Furthermore, defining $\mathcal{K}(n, 0) := 0$ and following the notation of definition 3, we also have the following inequalities:

4. $S(m), V(m) \leq \mathcal{K}(n, m)$
5. $S(m) + I(m) \leq 3\mathcal{K}(n, m)$ for $m \leq 3$

**Proof:** Let us focus first on proving theorem 3. To prove assertions (1) and (2), note that we can divide by a suitable monomial so that $f$ has a nonzero constant term. By lemma 3, we have that for $\delta > 0$ sufficiently small, it suffices to bound the number of compact and non-compact connected components of $Z_\delta$ (a “thickening” of $Z$). In particular, $\mathring{Z}_\delta$, the boundary of $Z_\delta$, and its closure, can be assumed to be smooth. Noting that every connected component of $\mathring{Z}_\delta$ is contained in some connected component of $Z_\delta$, it then suffices to bound the number of connected components of $\mathring{Z}_\delta$.

By proposition 3 and lemma 4, we can pick an $n \times n$ matrix $A$ so that, after we make the change of variables $x \mapsto x^A$, the number of compact and non-compact real connected components of $Z_\delta$ is preserved and no connected component of $Z_\delta$ of positive dimension is contained in a hyper-plane parallel to the $x_1$-coordinate hyperplane. Furthermore, we can also assume that every non-compact component of $\mathring{Z}_\delta$ has unbounded values of $x_1$. So we are now ready to use critical points to count connected components.
Consider then the system of equations $G_{\pm} := (f \pm \delta, x_2 \partial_2 f, \ldots, x_n \partial_n f)$, where $\partial_i$ denotes the operator $\frac{\partial}{\partial x_i}$. By construction, every compact connected component of $Z_\delta$ results in at least two extrema of the function $x_1$, i.e., $P_{\text{comp}}(n, m)$ is bounded above by an integer no more than half of the total number of roots of $G_+$ and $G_-$. (In particular, if $Z$ were smooth to begin with, then it would suffice to count the isolated roots of $G := (f, x_2 \partial_2 f, \ldots, x_n \partial_n f)$ instead and omit the use of $Z_\delta$ and $G_{\pm}$.) Note also that by construction, all the roots of $G_{\pm}$ (or $G$) are non-degenerate. Furthermore, by a simple application of Gaussian Elimination, we obtain that $G$ is of type $(m-1, \ldots, m-1)$ (and there are no more than $m$ distinct monomial terms occurring in $G_+$ or $G$), so assertion (1) follows immediately. (The bound for $P_{\text{comp}}(1, m)$ follows immediately from UGDRS.)

To prove assertion (2) of theorem 2, another application of lemma 14 (and our much used proposition 1) tells us that we can assume that every unbounded connected component of $Z$ has arbitrarily large values of $x_1$. For $\varepsilon > 0$ sufficiently small, we then observe that every such component induces at least one connected component of the intersection $Z' := Z \cap \{x_1 = \frac{\varepsilon}{2}\}$. So fix an $\varepsilon > 0$ sufficiently small so that this holds for all unbounded components. (Recall that there are only finite many, cf. the proof of lemma 14) Then, by substituting $x_1 = \frac{\varepsilon}{2}$ into $f$, we obtain a new fewnomial hypersurface $Z'' \subseteq \mathbb{R}^{n-1}$, also defined by an $m$-nomial, with at least as many connected components as $Z$ has unbounded components. To conclude, note that under the change of variables $x \mapsto (x_1^{-1}, \ldots, x_n^{-1})$, the bounded non-compact components of $Z$ are injectively embedded into the unbounded components of a new $m$-nomial hypersurface. So by what we’ve already proved for our unbounded components, we at last obtain $P_{\text{non}}(n, m) \leq 2(P_{\text{comp}}(n-1, m) + P_{\text{non}}(n-1, m))$.

The bound for $P_{\text{non}}(2, m)$ then follows from the now classical moment map. That is, given any $n$-dimensional convex compact polytope $P \subseteq \mathbb{R}^n$, there is a real analytic diffeomorphism $\psi : \mathbb{R}^n_+ \rightarrow \text{Int}(P)$, where $\text{Int}(P)$ denotes the interior of $P$ [Ful93, sec. 4.2]. In particular, if one picks $P$ to be the Newton polygon of $f$ then there is a bijection between (a) the intersections of $\psi(Z)$ with the interior of an edge of $P$ with inner normal $w$, and (b) the roots of the initial term polynomial $\text{in}_w(f) := \sum a_0 x^a$ in $(\mathbb{R}^*)^n \times \{1\}$, where the sum is over all $a \in \text{Supp}(f)$ with minimal inner product with $w$. Since any non-compact component $U$ of $Z$ results in $\psi(U)$ having at least 2 intersections with the edges of $P$, UGDRS immediately implies our bound for $P_{\text{non}}(2, m)$, not to mention our bound for $P_{\text{non}}(1, m)$. (In fact, in our bound for $P_{\text{non}}(2, m)$, we can even replace $m$ by the number of monomials corresponding to points on the boundary of $P$.)

Assertion (3) of theorem 2 follows immediately from assertion (4), which we will now prove. First note that the singular points of $Z$ are exactly the roots of the over-determined fewnomial system $F := (f, x_j \partial_j f, x_2 \partial_2 f)$. By lemma 12 the singular points of $Z$ are also contained in the roots of the system $G := (g_1, g_2)$, where $G$ is of type $(m-1, m-1)$, has no more than $m$ distinct exponent vectors, and each $g_i$ is a suitable linear combination of $f, x_1 \partial_1 f$, and $x_2 \partial_2 f$. Furthermore, the real zero set of $G$ is the union of the real zero set of $F$ and a (possibly empty) finite set of points. This proves the bound on $S(m)$, and the bound on $V(m)$ is proved in almost exactly the same way, starting with the polynomial system $(f, x_2 \partial_2 f)$ instead. So assertion (4) is proved.

To prove assertion (5), note that by lemmata 1 and 14 $(x_1, x_2)$ is an inflection point or a singular point of $Z \implies f = q = 0$, where $q$ is a homogeneous polynomial, in the non-constant monomials terms of $f$, of degree at most 3. Letting $S_1, \ldots, S_{m-1}$ denote the non-constant monomials terms of $f$, note that each complex factor $q' := \alpha_1 S_1 + \cdots + \alpha_{m-1} S_{m-1}$ of $q$ is a $j$-nomial for some $j \leq m - 1$. (Note that the fundamental theorem of algebra tells us that $q'$ indeed splits completely over $\mathbb{C}[S_1, \ldots, S_{m-1}]$, provided $m \leq 3$.) Also note that if $\alpha_i \neq 0$, the fewnomial systems $(1 + S_1 + \cdots + S_{m-1}, q')$ and $G := (1 + S_1 + \cdots + S_{m-1} - q'/\alpha_i, q')$ have the same zero set, and $\alpha_i$ must be nonzero for some $i$. However, $G$ is of type $(m-1, m-1)$, has no more than $m$ distinct exponent vectors, and has no degenerate roots. So the system $(f, q)$ has at most $3V(m-1, m-1)$ isolated roots in the positive quadrant of the $(x_1, x_2)$-plane. So assertion (5) is proved.

**Remark 13.** The equality $K(n, m) = \mathcal{K}(n, m)$ appears to be known only for $(n, m) \in (1 \times \mathbb{N}) \cup \{(2,2), (2,3), (2,4)\}$ and $m = n+1$. These few cases follow easily from theorems 1 and 4, via remark 4.
Acknowledgements

The authors thank Alicia Dickenstein and Bernd Sturmfels for pointing out Haas’ counter-example. Special thanks also go to Bertrand Haas for pointing out an error in an earlier version of Lemma 3, an anonymous referee for giving many nice corrections, and to Felipe Cucker, Jesus Deloera, Paulo Lima-Filho, and Steve Smale for some nice conversations.

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