Vector bundles with a fixed determinant on an irreducible nodal curve

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Abstract. Let $M$ be the moduli space of generalized parabolic bundles (GPBs) of rank $r$ and degree $d$ on a smooth curve $X$. Let $M^\overline{L}$ be the closure of its subset consisting of GPBs with fixed determinant $\overline{L}$. We define a moduli functor for which $M^\overline{L}$ is the coarse moduli scheme. Using the correspondence between GPBs on $X$ and torsion-free sheaves on a nodal curve $Y$ of which $X$ is a desingularization, we show that $M^\overline{L}$ can be regarded as the compactified moduli scheme of vector bundles on $Y$ with fixed determinant. We get a natural scheme structure on the closure of the subset consisting of torsion-free sheaves with a fixed determinant in the moduli space of torsion-free sheaves on $Y$. The relation to Seshadri–Nagaraj conjecture is studied.

Keywords. Nodal curves; torsion-free sheaves; fixed determinant.

1. Introduction

Generalized parabolic vector bundles (GPBs) on a smooth curve $X$ are vector bundles on $X$ together with parabolic structures on finitely many disjoint divisors $D_j$, $j = 1, \ldots, m$. There is an open subscheme $M''$ of the moduli space $M$ of GPBs on which one can define a determinant morphism into the moduli space of generalized parabolic line bundles $\overline{L}$, the map does not extend to $M$. Let $M^\overline{L}$ be its locally closed subset consisting of GPBs with a fixed determinant $\overline{L}$. In this note, we define a moduli functor and construct a coarse moduli scheme $M^\overline{L}$ for it. The moduli scheme contains $M''^\overline{L}$ as an open dense subscheme.

Let $Y$ be an irreducible projective nodal curve with nodes $y_j$, $j = 1, \ldots, m$ and $p: X \to Y$ its desingularization with $D_j$ the inverse image of $y_j$. Denote by $U$ the moduli variety of torsion-free sheaves of rank $r$, degree $d$ on $Y$. Let $U'$ be the open subvariety of $U$ corresponding to vector bundles on $Y$. There is a surjective morphism $f$ from $M$ onto $U'$ [1,2]. The restriction of the morphism $f$ to $M' = f^{-1}U'$ is an isomorphism onto the open subvariety $U'$ of $U$. A GPB $L$ gives a torsion-free sheaf $\mathcal{L}$ on $Y$. If $\mathcal{L}$ is locally free, let $U'' = U' \cap \{ \mathcal{L} \}$. The image $f(M^\overline{L})$ can be described (as a

We show that $f(M^\overline{L}) = U''^\overline{L}$, the closure of $U''$ in $U$, thus giving $U''^\overline{L}$ the scheme structure of an image subscheme of $U$. Let $I_j$ denote the ideal sheaf at the node $y_j$. For a torsion-free sheaf $F$ of rank $r$ on $Y$, let $N = N_F/(\text{torsion})$ where (torsion) denotes the torsion subsheaf. Then we show that for any $\overline{L}$, the image $f(M^\overline{L})$ can be described (as a
set) by

\[ f(M_L) = \{ F \in U : I_j^r \subset N \subset \mathcal{L}, \ \forall j \}. \]

This gives a proof of a conjecture by Seshadri and Nagaraj (Conjecture (a), p. 136 of \([3]\)). Proving Seshadri–Nagaraj conjecture was not the aim of this note. The conjecture was proved by Sun \([4]\) by degeneration methods. However he does not get a scheme structure on \(U_{d, \mathcal{L}}\) or a moduli functor (except in some low rank cases). Our aim is to give a moduli functor and an explicit construction of a projective moduli space for it which contains an open subvariety isomorphic to \(U_{d, \mathcal{L}}\) if \(\mathcal{L}\) is a line bundle. We also deal with the case when \(\mathcal{L}\) is torsion-free but not locally free. The construction is much simpler than that of Schmitt \([4]\) and hence the moduli space is easier to study. For example, properties like reduced, irreducible, Cohen–Macaulay follow immediately for our moduli set) by

2. The moduli scheme of GPBs with fixed determinant

2.1.

Let \(X\) be a nonsingular projective curve over an algebraically closed base field \(k\). Let \(D_j, j = 1, \ldots, m\) be disjoint divisors on \(X\) with \(D_j = x_j + x_j'\), where \(x_j, x_j'\) are distinct closed points. We recall here some basics on generalized parabolic bundles (GPBs), details may be found in \([12]\).

**DEFINITION 2.1.**

A generalized parabolic bundle (GPB, in short) of rank \(r\) and degree \(d\) on \(X\) is a vector bundle \(E\) of rank \(r\) and degree \(d\) on \(X\) together with \(r\)-dimensional vector subspaces \(F_j(E)\) of \(E_{x_j} \oplus E_{x_j}'\). For a subbundle \(N\) of \(E\), define \(F_j(N) = F_j(E) \cap (N_{x_j} \oplus N_{x_j}')\) and \(f_j(N) = \dim F_j(N)\).

**DEFINITION 2.2.**

Fix a rational number \(\alpha \in (0, 1]\). A GPB \((E, F_j(E))\) is \(\alpha\)-stable (resp. \(\alpha\)-semistable) if for every proper subbundle \(N\) of \(E\), one has \((d(N) + \alpha \Sigma_j f_j(N))/r(N) < \alpha \Sigma_j f_j(E)(d(E) + \alpha \Sigma_j f_j(E))/r(E)\). 

**DEFINITION 2.3.**

Let \(p_j : F_j(E) \to E_{x_j}, p_j' : F_j(E) \to E_{x_j}'\) be the projections. Assume that for each \(j\), at least one of \(p_j, p_j'\) is an isomorphism. The subspace \(F_j(E)\) determines an element \(F_j(E)\) of \(\text{Gr}(r, E_{x_j} \oplus E_{x_j}') \subset \mathbb{P}(N' (E_{x_j} \oplus E_{x_j}'))\). One has a (rational) morphism \(\delta : \mathbb{P}(N' (E_{x_j} \oplus E_{x_j}')) \to \mathbb{P}(N' E_{x_j} \oplus N' E_{x_j}').\) Let \(\det F_j(E)\) denote the one-dimensional subspace of \(N' E_{x_j} \oplus N' E_{x_j}'\) determined by \(\delta(F_j(E))\). Define the determinant of \((E, F_j(E))\) to be the generalized parabolic line bundle \((\det E, \det F_j(E))\).

**DEFINITION 2.4.**

A family of GPBs of rank \(r\), degree \(d\) parametrized by a scheme \(T\) is a tuple \((\mathcal{E}, F_j(\mathcal{E}))_j\) where \(\mathcal{E} \to T \times X\) is a family of vector bundles of rank \(r\), degree \(d\) on \(X\) which is flat.
over $T$ and $F_j(\mathcal{E})$ is a rank $r$ subbundle of $\mathcal{E}|_{T \times x_j} \oplus \mathcal{E}|_{T \times x_j'}$. The notion of equivalence of families is the obvious one.

We fix a generalized parabolic line bundle $L := (L, F_j(L))$. Fix isomorphisms $h_j : L_{x_j} \to k$, $h_j' : L_{x_j'} \to k$. Then $F_j(L)$ can be identified to a point $F_j(L)$ of $\mathbf{P}^1$ of the form $(1 : 0), (0 : 1)$ or $(1 : \lambda_j)$, $\lambda_j \in k^*$.

2.2 The moduli functor

For simplicity, let us assume that there is only one divisor $D = x_1 + x_2$. Let $(\mathcal{E}, F(\mathcal{E})) \to T \times X$ be a family of GPBs of rank $r$, degree $d$ on $X$ with $\mathcal{E}_t, t \in T$, of fixed determinant $L$. For $i = 1, 2$ we have vector bundles

$$\mathcal{E}_{x_i} = \mathcal{E}|_{T \times x_i} \to T.$$ 

Let $\mathcal{G} \to T$ denote the Grassmannian bundle of rank $r$ subbundles of $\mathcal{E}_{x_1} \oplus \mathcal{E}_{x_2}$. It is embedded as a closed subvariety in $\mathbf{P}(\Lambda'(\mathcal{E}_{x_1} \oplus \mathcal{E}_{x_2}))$ by Plücker embedding. Note that $F(\mathcal{E})$ defines a section of $\mathcal{G}$. Since $\det \mathcal{E}|_{T \times X} = L$, it follows that $\det \mathcal{E} = p^*_T N \otimes p^*_X L$ for some line bundle $N$ on $T$. Hence for $i = 1, 2$ one has $\det \mathcal{E}_{x_i} = N \otimes L_{x_i} = N$, using the isomorphism $h_i$. Let $q_i : \Lambda'(\mathcal{E}_{x_i} \oplus \mathcal{E}_{x_2}) \to \det \mathcal{E}_{x_i} = N$ be the projections, $i = 1, 2$. Define a hyperplane subbundle $\mathcal{H}$ of $\mathbf{P}(\Lambda'(\mathcal{E}_{x_1} \oplus \mathcal{E}_{x_2}))$ by $q_2 = 0$ if $F(L) = (1 : 0)$, by $q_1 = 0$ if $F(L) = (0 : 1)$ and by $q_2 - \lambda_j q_1 = 0$ if $F(L) = (1 : \lambda)$. Let $H_T := \mathcal{G} \cap \mathcal{H}$. It is a closed reduced subscheme of $\mathcal{G} \subset \mathbf{P}(\Lambda'(\mathcal{E}_{x_1} \oplus \mathcal{E}_{x_2}))$. Note that $H_T$ is independent of the choice of $h_1, h_2$.

More generally, if we consider parabolic structures over finitely many disjoint divisors $D_j = x_j + x'_j$, for each $j$ one constructs the hyperplane bundle $\mathcal{H}_j$ and Grassmannian bundle $\mathcal{G}_j$ over $T$. Let $\mathcal{G}$ be the fibre product of $\mathcal{G}_j$ over $T$ and $H_T$ the fibre product of $\mathcal{G}_j \cap \mathcal{H}_j$ over $T$.

DEFINITION 2.5.

Let $F^S_L$ be the functor $F^S_L : \text{Schemes} \to \text{Sets}$ which associates to a scheme $T$ the set of equivalence classes of families $(\mathcal{E}, F(\mathcal{E})) \to T \times X$ of $\alpha$-semistable GPBs of rank $r$ and degree $d$ with $\det \mathcal{E}_t \cong L$ for all $t \in T$ such that the section of $\mathcal{G}$ defined by $(F_j(\mathcal{E}))$ maps into $H_T$.

One similarly defines a full subfunctor $F^S_L$ of $F^S_L$ with semistable replaced by stable.

2.3 Construction of the moduli space

Let $S$ denote the set of semistable GPBs $(E, F(E))$, where $E$ is a vector bundle of rank $r$, degree $d$ with fixed determinant $L$ and $F(E)$ is a subspace of $E_{x_j} \oplus E_{x_j'}$ of dimension $r$ with fixed weights $(0, \alpha), 0 < \alpha < 1$. For $m \gg 0$, all GPBs in $S$ satisfy the condition

$$(*) \quad H^1(E(m)) = 0, H^0(E(m)) \cong \mathbf{C}^r, H^0(E(m)) \to H^0(E(m) \otimes (\oplus_j \mathcal{O}_{D_j}))$$

is surjective.

Let $Q$ denote the quot scheme of coherent quotients of $\mathcal{O}_X^r \otimes \mathcal{O}_X(-m)$ with fixed Hilbert polynomial determined by $r, d$. Let $R$ be the nonsingular subvariety of $Q$ corresponding to quotient vector bundles $\bar{E}$ satisfying condition $(*)$. Denote by $R_0$ the nonsingular closed subvariety in $R$ corresponding to $\bar{E}$ with $\Lambda'(E) = L$. Let $\mathcal{E} \to R \times X$ be the universal
quote

PGL vector bundles on

3.1. The projective scheme

Remark

if and only if

moduli space of torsion-free sheaves of rank

PGL

phism

the coarse moduli space for GPBs [1,2]. It is a normal projective variety. Since

open, is dense in it. It follows that

3.1. Application to nodal curves

the closure of

The correspondence induces a surjective morphism

GP quotient

functor

Theorem 1. Let

Then there is a coarse moduli space

(resp. 

M

F

is a projective (irreducible) variety, containing

M

as an open subvariety.

Let

(resp. 

M

be the open subscheme of

(resp. 

M

consisting of GPBs

such that the projections

are isomorphisms for all

Then

corresponds to GPBs in

with fixed determinant

with

Then

is the closure of

3. Application to nodal curves

3.1.

Let

be an irreducible projective nodal curve with nodes

and

X its desingularization with

the inverse image of

Then there is a correspondence from GPBs on

of rank

degree

torsion-free sheaves on

of the same rank and degree

The correspondence induces a surjective morphism

from

onto

where

is the moduli space of torsion-free sheaves of rank

degree

on

The restriction of the morphism

to

is an isomorphism onto the open subvariety

of

corresponding to vector bundles on

One has

For

a GPB

corresponds to a torsion-free sheaf

on

Then

is a line bundle if and only if

Suppose that

is a line bundle and let

be the closed subset of

corresponding to vector bundles with fixed determinant

Then

and the morphism

maps

isomorphically onto

Hence, if

is a line bundle, then

contains

as an open subset. Since

is an irreducible, closed subscheme of

the image

is a closed, irreducible subscheme of

and

being open, is dense in it. It follows that

is the closure of

Remark 3.1. The projective scheme

can be regarded as the compactified moduli space of vector bundles of rank

denominator

L on the nodal curve

Remark 3.2. In fact, for any torsion-free sheaf

of rank 1, the image

is a closed, irreducible subscheme of

containing the subset of

consisting of torsion-free sheaves with fixed determinant

as an open dense set.

3.2 Relation to Seshadri–Nagaraj conjecture

For a torsion-free sheaf

of rank

on

let

where

denotes the maximum subsheaf with proper support. Denote by

the ideal sheaf of the
node \( y_j \). Define \( U_\mathcal{L} \) as the set
\[
U_\mathcal{L} = \{ F \in U : I_j \mathcal{L} \subset N \subset \mathcal{L}, \quad \forall j \}.
\]
Seshadri and Nagaraj had defined this set for \( Y \) with one node and conjectured that if \( \mathcal{L} \) is a line bundle, then \( U_\mathcal{L} \) is the closure of \( U'_\mathcal{L} \) (Conjecture (a), page 136 of [3]). We prove this conjecture.

Let \( \tilde{R}_0^{1-ss} \) denote the subset consisting of 1-semistable points, then \( \tilde{R}_0^{ss} \subset \tilde{R}_0^{1-ss} \). Let \( P \) be the moduli space of 1-semistable GPBs [5]. One has morphisms \( f : \tilde{R}_0^{ss} \to U \) inducing \( f : M_L \to U \) and \( f_1 : \tilde{R}_0^{1-ss} \to U \) inducing \( f_1 : P \to U \).

**Proposition 3.3.**

Let \( L \) be any GBP of rank 1 on \( X \) and \( \mathcal{L} \) the corresponding torsion-free sheaf of rank 1 on \( Y \).

1. If \( (E, F_j(E)) \in \tilde{R}_0^{1-ss} \), then \( F = F_1((E, F_j(E))) \in U_\mathcal{L} \) and hence \( f(M_L) \subset U_\mathcal{L} \).
2. The morphism \( \tilde{R}_0^{1-ss} \to U \) surjects onto \( U_\mathcal{L} \).
3. \( f(M_L) = U_\mathcal{L} \) for \( \alpha \) sufficiently close to 1.

**Proof.** We may assume that \( Y \) has only one node \( y \). It is easy to see (from the proof) that the general case follows exactly on same lines. Consider a GBP \( (E, F_j(E)) \). Let \( p_i : E \to E_i \), \( i = 1, 2 \) be the projections and \( a_i = \dim \ker p_i \). Let \( E_0 = p^*(F)/\text{torsion} \), then \( E_0 \subset E \). Since \( F|_{Y-y} = p_*E|_{Y-y} \), one has \( N|_{Y-y} = (p_*L)|_{Y-y} = \mathcal{L}|_{Y-y} \). Hence to check that \( \mathcal{L}_L \subset N \subset \mathcal{L} \), we have only to check it locally at the node \( y \).

Let \((A, m)\) be the local ring at \( y \). Its normalization \( \bar{A} \) is a semilocal ring with two maximal ideals \( m_1, m_2 \). The inclusion
\[
F_y \subset (p_*E)_y
\]
may be identified with the inclusion
\[
(r - a_1 - a_2)A + a_1 m_1 + a_2 m_2 \subset r\bar{A}
\]
(Proposition 4.2 of [2]).

1. We consider the following cases separately.

   **Case (i).** Suppose that \( p_1, p_2 \) are both isomorphisms. Then \( \det(E, F_j(E)) = L \) corresponds to a torsion-free sheaf \( \mathcal{L} \) which is locally free at \( y \). In this case, \( F \) is locally free at \( y \) with \( \det F_y = \mathcal{L}_y \) so that \( N = \mathcal{L} \supset \mathcal{L}_L \).

   **Case (ii).** Assume that \( p_1 \) is an isomorphism, \( p_2 \) is not an isomorphism (the opposite case can be dealt with similarly). Then \( \det(E, F_j(E)) = (L, L_{a_1}) = \mathcal{L} \) corresponds to \( \mathcal{L} \) which is not locally free at \( y \). One always has \( N \subset p_*L \) and \( N_y \subset \mathcal{L}_y \) if and only if \( N_y \otimes k(y) \subset F(L) \). Locally, \( a_1 = 0 \) so that \( F_y = (r - a_2)A + a_2 m_2 \). Then \( N_y = (m_2)^{a_2} \) so that \( N_y \otimes k(y) \subset L_{a_1} = F(L) \). Hence \( N \subset \mathcal{L} \). Since \( m' \subset m_2^{a_2} \), it follows that \( \mathcal{L}_L \subset N \subset \mathcal{L} \).

   **Case (iii).** If both \( p_1 \) and \( p_2 \) are not isomorphisms, one has \( a_1 \geq 1, a_2 \geq 1 \). Then locally, \( N_y = m_1^{a_1} m_2^{a_2} \subset m_1 m_2 = m \). It follows that \( N_y \) maps to zero in \( p_*L \otimes k(y) \) so that \( N \subset L \mathcal{L}_L \subset \mathcal{L} \). Since \( m' \subset m_1^{a_1} m_2^{a_2} \), one has \( \mathcal{L}_L \subset N \subset \mathcal{L} \).
Note that any $(E, F(E)) \in M_L$ with $F(L) = (1 : \lambda), \lambda \in k^*$ (i.e. $\mathcal{L}$ locally free at $y$) occurs only in cases (i) or (iii). For $(E, F(E)) \in M_L$ with $F(L) = (1 : 0)$ (or $F(L) = (0 : 1)$) only cases (ii) and (iii) occur. Part (1) now follows. Note that $(E, F_j(E))$ is 1-semistable if and only if $F$ is semistable \[12\].

(2) Let $F \in U_{\mathcal{L}}$. Since $I' \mathcal{L} \subset N \subset \mathcal{L}$ it follows that $L|_{X - D} = p^*N|_{X - D}$ where $D = \sum_j(x_j + x_j')$. Since $\det E_0 = p^*N$ outside $D$, one has $L = \det E_0$ outside $D$. It follows that $L = \det E_0 \otimes \mathcal{O}_X(\sum_j(a_jx_j + a_j'x_j'))$, $a_j + a_j' \leq r$. Let $E$ be given by an extension

$$0 \to E_0 \to E \to \oplus_j(k(x_j)^{a_j} \oplus k(x_j')^{a_j'}) \to 0.$$ 

The composite $F \leftarrow p_*E_0 \to p_*E$ induces a linear map $F \otimes k(y_j) \to p_*E \otimes k(y_j)$. Let $F_j(E)$ be the image of this linear map. Then $(E, F_j(E))$ maps to $F$ and it is 1-semistable as $F$ is semistable \[12\]. By construction, $\det E = \det E_0 \otimes \mathcal{O}_X(\sum_j(a_jx_j + a_j'x_j')) = L$. It follows that $(E, F_j(E)) \in \bar{R}_{1-ss}^{1}$. 

(3) Let $P_L$ denote the closure of $\bar{R}_{1-ss}/\text{PGL}(n)$ in $P$. For $\alpha$ close to 1, there is a surjective birational morphism $\phi: M \to P$ with $f = f_1 \circ \phi$. It maps $M_L$ birationally into $P_L$. Since both these spaces are irreducible and of the same dimension, it follows that $\phi(M_L) = P_L$. Since $f(\bar{R}_{1-ss})$ surjects onto $U_{\mathcal{L}}$, it follows that $f_1(P_L) \supset U_{\mathcal{L}}$ and hence $f(M_L) \supset U_{\mathcal{L}}$. From (1), it follows that $U_{\mathcal{L}} = f(M_L) = f_1(P_L)$. \[\square\]

**COROLLARY 3.4.**

If $\mathcal{L}$ is a line bundle, then $U_{\mathcal{L}}$ is the closure of $U'_{\mathcal{L}}$.

**Proof.** From Proposition 3.3, one has (as sets) $f(M_L) = U_{\mathcal{L}}$. Since $f(M_L)$ is the closure of $U'_{\mathcal{L}}$ if $\mathcal{L}$ is a line bundle, the result follows. \[\square\]

**Remark 3.5.** The proof of Proposition 3.3(1) easily generalizes to $(E, F_j(E))$ replaced by a family $(\mathcal{E}, F_j(\mathcal{E})) \to T \times X$.

**Remark 3.6.** Sun \[9\] had proved the conjecture (a) of Seshadri–Nagaraj by considering a smooth curve $X$ degenerating to an irreducible nodal curve $Y$. However he does not get a moduli functor or a scheme structure on $U_{\mathcal{L}}$ except in some cases (e.g. one node, rank 2, degree 1).

**Remark 3.7.** Schmitt \[4\] has constructed a moduli space $\mathcal{M}$ of $\alpha$-semistable descending singular SL$(r)$-bundles $(A, q, \tau)$ where $(A, q)$ is a GPB on $X$ and $\tau$: $\text{Sym}^n(A \otimes C^r)^{\text{SL}(r,E)} \to \mathcal{O}_X$ a nontrivial homomorphism. It is shown that $\det A = \mathcal{O}_X$ and for $\alpha \in (0, 1) \cap \mathbb{Q}$, there is a forgetful morphism $h: \mathcal{M} \to M$ (§5.1 of \[4\]). For $\alpha$ close to 1, one has a forgetful morphism $\mathcal{M} \to U = U(r, 0)$ whose set theoretic image is $U_{\mathcal{L}}$ (Proposition 5.1.1 of \[4\]). Then, since $\det A = \mathcal{O}_X$, it follows that $h(\mathcal{M}) = M_L$ (as sets).

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References

[1] Bhosle Usha N, Generalised parabolic bundles and applications to torsion-free sheaves on nodal curves. *Arkiv for Matematik* **30**(2) (1992) 187–215

[2] Bhosle Usha N, Generalised parabolic bundles and applications II. *Proc. Indian Acad. Sci. (Math. Sci.)* **106**(4) (1996) 403–420

[3] Nagaraj D S and Seshadri C S, Degenerations of the moduli spaces of vector bundles on curves I. *Proc. Indian Acad. Sci. (Math. Sci.)* **107**(2) (1997) 101–137

[4] Schmitt A, Singular principal $G$-bundles on nodal curves. *J. Eur. Math. Soc.* **7** (2005) 215–251

[5] Sun X, Degeneration of moduli spaces and generalized theta functions. *J. Algebraic Geom.* **9** (2000) 459–527

[6] Sun X, Moduli spaces of $SL(r)$-bundles on singular irreducible curves. *Asian J. Math.* **7**(4) (2003) 609–625. [math.AG/0303198]