The Anarchy-Stability Tradeoff in Congestion Games

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Abstract

This work focuses on the design of incentive mechanisms in congestion games, a commonly studied model for competitive resource sharing. While the majority of the existing literature on this topic focuses on unilaterally optimizing the worst case performance (i.e., price of anarchy), in this manuscript we investigate whether optimizing for the worst case has consequences on the best case performance (i.e., price of stability). Perhaps surprisingly, our results show that there is a fundamental tradeoff between these two measures of performance. Our main result provides a characterization of this tradeoff in terms of upper and lower bounds on the Pareto frontier between the price of anarchy and the price of stability. Interestingly, we demonstrate that the mechanism that optimizes the price of anarchy inherits a matching price of stability, thereby implying that the best equilibrium is not necessarily any better than the worst equilibrium for such a design choice. Our results also establish that, in several well-studied cases, the unincentivized setting does not even lie on the Pareto frontier, and that any incentive with price of stability equal to 1 incurs a much higher price of anarchy.

1 Introduction

We consider the design of systems that involve the interactions of strategic users and an underlying shared technological infrastructure. A major difficulty in designing such systems is that one must account for each user’s decision making process in order to guarantee good overall system performance. The detrimental effects of selfish user behaviour on the performance of these systems have been observed in a variety of contexts, including unfair allocation of essential goods and services [17, 18], overexploitation of natural resources [22, 23] and congestion in internet and road-traffic networks [9, 27]. A widely studied approach for influencing the system performance is the use of incentives, which can come in the form of rewards or penalties. Examples of incentives include taxes levied on users whose decisions have a negative impact on the system performance or rebates given to users for making decisions that are aligned with the greater good.

Many of the aforementioned systems can be suitably modeled as congestion games [33], where users compete over a set of shared resources and the overall objective is to minimize the sum of users’ costs. In this model, users are assumed to be self interested, focused solely on minimizing their own experienced costs. In order to measure the inefficiency of users’ selfish decision making in such settings, several well studied performance metrics have been proposed, including the price of anarchy [26] and the price of stability [3]. Informally, the price of anarchy provides performance guarantees for the worst system outcome, while the price of stability provides guarantees for the best system outcome. A rich body of literature analyzes these two performance metrics in settings spanning far beyond the class of congestion games. More recently, a number of works have focused on deriving incentive mechanisms that optimize the price of anarchy in congestion games as a surrogate for optimizing the system performance [5, 10, 30]. This prompts the following question:
What are the consequences of optimizing for the price of anarchy on other performance metrics such as the price of stability?

Perhaps surprisingly, we prove that there exists a fundamental tradeoff between the price of anarchy and the price of stability in the class of congestion games. Specifically, we characterize upper and lower bounds on the Pareto frontier between these two performance metrics.

1.1 Model

In this work, we consider the class of (atomic) congestion games as defined by Rosenthal [33]. A congestion game consists of a set of users $N = \{1, \ldots, n\}$ and a set of resources $E$. Each user $i \in N$ must select an action $a_i$ from a corresponding set of feasible actions $A_i \subseteq 2^E$. The cost that a user experiences for selecting a given resource $e \in E$ depends only on the total number of users selecting $e$, and is denoted as $\ell_e : \{1, \ldots, n\} \rightarrow \mathbb{R}$. Given an assignment $a = (a_1, \ldots, a_n) \in A$, where $A = \Pi_{i \in N} A_i$, each user $i \in N$ experiences a cost equal to the sum over costs on resources $e \in a_i$. Correspondingly, the system cost is measured by the sum of the users costs, i.e.,

$$SC(a) = \sum_{i \in N} \sum_{e \in a_i} \ell_e(|a|_e)$$

where $|a|_e$ denotes the number of users selecting resource $e$ in assignment $a$. We denote by $\mathcal{G}$ the set of all congestion games with a maximum number of users $n$, where all resource cost functions $\{\ell_e\}_{e \in E}$ belong to a common set of functions $\mathcal{L}$.

Incentive mechanisms. In the study of incentive mechanisms, each resource $e \in E$ is associated with an incentive function $\tau_e : \{1, \ldots, n\} \rightarrow \mathbb{R}$ (positive or negative). In this case, each user $i \in N$ incurs a cost involving both the resource costs it experiences and the imposed incentives, i.e.,

$$C_i(a) = \sum_{e \in a_i} [\ell_e(|a|_e) + \tau_e(|a|_e)].$$

Here, a common assumption is that incentives only influence the users’ costs and do not factor into the social cost. When users selfishly choose their actions to minimize their incurred costs, an emergent outcome is often described by a pure Nash equilibrium. A pure Nash equilibrium is an assignment $a \in A$ such that $C_i(a) \leq C_i(a_i', a_{-i})$ for all $a_i' \in A_i$ and all $i \in N$, where $a_i', a_{-i}$ denotes the assignment obtained when user $i$ plays action $a_i'$ and the remaining users continue to play their actions in $a$. Observe that a system designer can influence the set of pure Nash equilibria by carefully selecting the incentive functions $\tau_e$ for each $e \in E$.

We consider the use of local incentive mechanisms to improve the equilibrium performance. A local incentive mechanism uses only information about the resource cost function $\ell_e$ to compute the incentive function $\tau_e$ on any given resource $e \in E$. The restriction to local incentives is a natural requirement, especially in settings where scalable and computationally simple mechanisms are desirable. This structure is also commonly utilized in the existing literature, e.g., Pigovian taxes [31]. Accordingly, we define a local mechanism as a map from the set of admissible resource costs $\mathcal{L}$ to incentives. Given a local mechanism $T$, the incentive function associated with a resource $e \in E$ is given by $\tau_e = T(\ell_e)$.

Performance metrics. To measure the equilibrium performance of a given mechanism $T$, we use
two commonly studied metrics termed price of anarchy and price of stability defined as follows:

\[
\text{PoA}(T) = \sup_{G \in G} \max_{a \in \text{NE}(G)} \frac{\text{SC}(a)}{\text{MinCost}(G)},
\]

\[
\text{PoS}(T) = \sup_{G \in G} \min_{a \in \text{NE}(G)} \frac{\text{SC}(a)}{\text{MinCost}(G)},
\]

where MinCost\((G)\) denotes the minimum achievable social cost for instance \(G\) as defined in Equation (1) and NE\((G)\) denotes the set of all pure Nash equilibria in \(G\) when employing the mechanism \(T\). Observe that the price of anarchy provides guarantees on the performance of any equilibrium in the set of games while the price of stability offers performance guarantees for the best equilibrium of any instance in the set. By definition, PoA\((T)\) ≥ PoS\((T)\) ≥ 1 for any mechanism \(T\). While we introduce the price of anarchy and price of stability with respect to pure Nash equilibria, we note that all of our results extend to coarse correlated equilibria. Therefore, in the remainder of this work, we use the price of anarchy and price of stability to refer to the efficiency of pure Nash and coarse correlated equilibria equivalently.

1.2 Summary of our contributions

In this work, we demonstrate that there exists an inherent tradeoff between the price of anarchy and the price of stability in congestion games. A discussion of our results is provided below:

*Optimizing for anarchy or stability.* Our first contribution characterizes the resulting price of stability when one optimizes solely for the price of anarchy. The result holds for all congestion games with convex, nondecreasing resource costs, including polynomial congestion games.

**Theorem 1** (Informal). Let \(T\) denote any local mechanism achieving minimum price of anarchy \(\text{PoA}^*\). Then, \(\text{PoS}(T) = \text{PoA}^*\).

Theorem 1 establishes that the incentives minimizing the price of anarchy have corresponding price of stability equal to the price of anarchy. This is perhaps unexpected as, in the uninventivized setting, there exists a significant gap between the price of anarchy and price of stability. For example, in polynomial congestion games, the price of anarchy grows exponentially in the order \(d\) [1] while the price of stability grows only polynomially in \(d\) [12]. Observe that Theorem 1 also implies (by contrapositive) that any improvement in the price of stability necessarily comes at the expense of the price of anarchy, as any mechanism \(T\) with \(\text{PoS}(T) < \text{PoA}^*\) must have price of anarchy satisfying \(\text{PoA}(T) > \text{PoA}^*\).

*The Pareto frontier of inefficiency.* Our second main result demonstrates that there exists a tradeoff between the price of anarchy and price of stability, and characterizes an upper and lower bound on the corresponding Pareto frontier.

**Theorem 2** (Informal). Let \(T\) denote any local mechanism achieving price of anarchy \(\text{PoA}(T) \leq \alpha\). Then, \(\text{PoS}(T) \geq \text{LB}(\alpha)\), where \(\text{LB}(\alpha)\) is nonincreasing in \(\alpha\). Further, the best achievable price of stability among all such mechanisms satisfies \(\text{PoS}(T) \leq \text{UB}(\alpha)\), where \(\text{UB}(\alpha)\) is also nonincreasing. The characterization of \(\text{LB}(\alpha)\) and \(\text{UB}(\alpha)\) is provided in Theorems 4 and 5.

Theorem 2 shows that any local mechanism \(T\) satisfying a price of anarchy requirement \(\text{PoA}(T) \leq \alpha\) cannot achieve a price of stability below a lower bound \(\text{LB}(\alpha)\) that is nonincreasing in \(\alpha\). Nonetheless, the best achievable price of stability among all such mechanisms is guaranteed to be below an
upper bound UB(α), also nonincreasing. Thus, any joint performance guarantee falling strictly below the lower bound curve is unachievable by any local mechanism, while those strictly above the upper bound curve are suboptimal (i.e., there exists another local mechanism providing strict improvement in the price of anarchy while guaranteeing at most the same price of stability or vice versa).

By virtue of the techniques used to derive the two bounds, three important observations can also be made: First, for any set of congestion games with convex, nondecreasing resource costs, our upper and lower bound match at extreme values of α, i.e., LB(α) = UB(α), when α = PoA∗, or min_{T∈T} PoS(T) = 1. Second, we are able to establish that the joint performance guarantees associated with congestion games without incentives are strictly suboptimal in several well-studied sets of congestion games. Third, our bounds extend to more general notions of equilibrium (e.g., coarse-correlated equilibria) and to other performance metrics (e.g., price of stochastic anarchy).

**Figure 1:** The Pareto frontier between the price of anarchy and price of stability in congestion games with affine and quadratic resource costs. The Pareto frontier lies within the region below the upper bound curves (solid black) and above the lower bound curves (dotted black), which were derived with the techniques put forward in Theorems 6 and 5. The joint price of anarchy and price of stability values for the mechanism that minimizes the price of anarchy (in red), no incentive (in green), and the mechanism that minimizes the price of stability (in blue) are reported in the table on the right. Note that all joint performance guarantees falling above the upper bound curves are suboptimal, below the lower bound curves are unachievable by any local mechanism and in the grey region are inadmissible as PoA(T) ≥ PoS(T) must hold. Although the upper and lower bound curves do not always match, we show that they are tight at the endpoints for any set of congestion games with convex, nondecreasing resource costs. Similar bounds on the Pareto frontier can be derived for any set of congestion games using the techniques outlined in Section 4.

(a) Congestion games with affine resource costs.

(b) Congestion games with quadratic resource costs.
In Figure 1, we plot our upper bound curves (solid, black lines) and lower bound curves (dotted, black lines) on the Pareto frontier between the price of anarchy and price of stability in affine and quadratic congestion games. We also provide the price of anarchy and price of stability of the mechanisms that minimize the price of anarchy (red circle), the mechanisms that minimize the price of stability (blue circle) and no incentive (green circle) in the tables on the right. Under local mechanisms, the best achievable price of anarchy PoA* is approximately 2.012 in affine congestion games and 5.101 in quadratic congestion games \[30\], and – in accordance with Theorem 1 – the corresponding price of stability values are also 2.012 and 5.101, respectively.

From the tables in Figure 1, observe that utilizing either no incentive or the mechanism that minimizes the price of stability yields price of stability strictly lower than the best achievable price of anarchy PoA*. However, they also have strictly higher price of anarchy than PoA*. Finally, observe that neither of the mechanisms that unilaterally minimize for either performance metric (red and blue circles) is strictly “better” (Pareto dominant) than the no incentive setting (green circle) as they do not guarantee that both the price of anarchy and price of stability are reduced. Nevertheless, the joint price of anarchy and price of stability of no incentive falls above the upper bound curves in both plots, implying that Pareto dominant mechanisms do exist in this example. Finally, note that any joint price of anarchy and price of stability below the lower bound curves cannot be achieved by any local mechanism.

1.3 Related works

The price of anarchy was introduced by Koutsoupias and Papadimitriou [26] as a performance metric to characterize equilibrium efficiency in games. The first exact characterization of the price of anarchy in congestion games was derived by Awerbuch et al. [4] and Christodoulou and Koutsoupias [13] for affine congestion games without incentives. These results were later generalized to all polynomial congestion games without incentives by Al and et al. [1].

Characterizations of the price of anarchy without incentives naturally led to the study of incentive mechanisms to improve worst-case efficiency guarantees. The design of incentive mechanisms to optimize equilibrium efficiency guarantees falls under the broader literature on coordination mechanisms introduced by Christodoulou et al. [15]. Within the context of congestion games, Caragiannis et al. [10] derive local and global congestion independent incentive mechanisms that minimize the price of anarchy for linear resource costs. For polynomial congestion games, Bilò and Vinci [5] consider the class of incentive mechanisms that use only information about a social optimum of each instance. Among incentive mechanisms of this specialized class, they derive the best achievable price of anarchy guarantees in polynomial congestion games, as well as a methodology for computing an optimal mechanism. Paccagnan and Gairing [29] generalize these results beyond polynomial congestion games and show, perhaps surprisingly, that the efficiency of optimal, polynomially-computable incentives matches the corresponding bound on the hardness of approximation. Paccagnan et al. [30] derive local incentive mechanisms that minimize the price of anarchy in any class of congestion games, which are shown to have similar efficiency guarantees as the more complex mechanisms considered in [5, 10, 29]. Bjelde et al. [7] derive upper bounds on the price of anarchy associated with the marginal cost mechanism in polynomial congestion games, which were later refined and generalized in [30].

While the characterization and optimization of the price of anarchy has been the subject of extensive analysis in the literature on congestion games, much less is understood about other performance metrics. This may be due to the increased difficulty in deriving bounds on the performance when one must restrict the set of equilibria considered while preserving generality. A particularly interesting and important metric in this respect is the price of stability. The price of stability
was introduced by Anshelevich et al. [3], who provide an exact characterization of this metric for a specialized class of congestion games. The exact price of stability for linear congestion games without incentives was derived by Caragiannis et al. [11] and Christodoulou and Koutsoupias [14], followed by an exact characterization for all polynomial congestion games without incentives by Christodoulou and Gairing [12]. Kleer and Schäfer [24] study a generalized class of affine congestion games and provides exact price of anarchy and price of stability bounds for various mechanisms of interest (e.g., altruism, congestion independent incentives). Though these and many other works on congestion games study the price of anarchy and the price of stability independently, there is no systematic framework for analyzing the concurrent optimization of these two performance metrics. For example, one may wish to identify the incentive mechanism that optimizes the price of stability while guaranteeing that the price of anarchy remains below some acceptable value. The techniques developed in this paper directly address the problem of optimal incentive mechanism design in congestion games while accounting for possible tradeoffs between the worst and best case performance metrics.

A natural concern in the joint study of the price of anarchy and price of stability under incentives is whether optimizing for the worst-case efficiency has negative consequences on the best-case efficiency. To the best of our knowledge, only two prior works have proposed the study of tradeoffs between the price of anarchy and price of stability, albeit in specialized settings. Filos-Ratsikas et al. [20] study the tradeoff between the price of anarchy and the price of stability in the analogous setting of mechanism design, focusing on the problem of unrelated machine scheduling. The existence of a tradeoff between these two performance metrics was also considered by Ramaswamy Pillai et al. [32] for the class of covering games with nonincreasing utility functions [21]. These two prior works both answer the above concern in the affirmative, providing characterizations of the Pareto frontier between the price of anarchy and price of stability within their respective problem settings. Interestingly, though they consider analogous classes of problems, the two analyses show that the mechanisms that optimize the price of anarchy have price of anarchy equal to the price of stability, which is mirrored by the result in Theorem 1. In this manuscript, we investigate the same research direction as these two prior studies, and show that their findings hold more generally in the well-studied class of congestion games.

2 Preliminaries

In the literature on congestion games, the resource cost functions are often taken to be polynomials with nonnegative coefficients [1, 5, 10, 13, 12, 30], i.e.,

$$\ell_e(x) = \alpha_1 + \alpha_2 x + \cdots + \alpha_{d+1} x^d, \quad \forall e \in \mathcal{E},$$

where $\alpha_j \geq 0$, $j = 1, \ldots, d + 1$. Note that one could equivalently consider the class of congestion games with resource cost functions $\ell_e \in \mathcal{L}$ where $\mathcal{L}$ is the set of all linear combinations with nonnegative coefficients, $\ell(x) = \sum_{j=1}^{d+1} \alpha_j b_j(x)$, of the set of polynomial basis functions $b_j(x) = x^{j-1}$, $j = 1, \ldots, d + 1$. Throughout this paper, we consider a broader class of congestion games where the set of resource cost functions $\mathcal{L}$ contains all linear combinations with nonnegative coefficients of a finite set of basis functions $b_1, \ldots, b_m$.

2.1 Minimizing the price of anarchy

Paccagnan et al. [30] provide a linear programming based methodology to compute the local incentive mechanism that minimizes the price of anarchy in a given congestion game. For the reader’s convenience, we reproduce this linear programming methodology in the following proposition:
Proposition 1 (Theorem 2.1 [30]). A local incentive mechanism $T^\text{opt}$ minimizing the price of anarchy over congestion games with at most $n$ agents, resource costs $\ell(x) = \sum_{j=1}^{m} \alpha_j b_j(x)$, $\alpha_j \geq 0$, and basis functions $\{b_1, \ldots, b_m\}$ is given by

$$T^\text{opt}(\ell) = \sum_{j=1}^{m} \alpha_j \tau_j^\text{opt}, \quad \text{where } \tau_j^\text{opt} : \{1, \ldots, n\} \to \mathbb{R}, \quad \tau_j^\text{opt}(x) = F_j^\text{opt}(x) - b_j(x)$$

and $\rho_j^\text{opt} \in \mathbb{R}$, $F_j^\text{opt} : \{1, \ldots, n\} \to \mathbb{R}$ solve the following $m$ linear programs (one for each $b_j$):

$$\max_{\rho, \tau} \rho$$

subject to: $b_j(y) - \rho b_j(x)x + F(x)(x - z) - F(x + 1)(y - z) \geq 0, \ \forall (x, y, z) \in \mathcal{I}(n)$,

where we define $b_j(0) = F(0) = F(n + 1) = 0$. The corresponding price of anarchy is $\text{PoA}(T^\text{opt}) = \max_j \{1/\rho_j^\text{opt}\}$. These results are tight for pure Nash equilibria and extend to coarse correlated equilibria.

2.2 Minimizing the price of stability

Along with the local incentive mechanisms that minimize the price of anarchy, we are naturally also interested in mechanisms that minimize the price of stability. In the next proposition, we provide an intuitive argument to show that the marginal incentive mechanism is the unique local incentive mechanism that minimizes the price of stability.

Proposition 2. Consider congestion games with $n$ users, resource costs $\ell(x) = \sum_{j=1}^{m} \alpha_j b_j(x)$, $\alpha_j \geq 0$, and positive, nondecreasing basis functions $\{b_1, \ldots, b_m\}$. Then, the marginal cost mechanism

$$T^\text{mc}(b_j)(x) = (x - 1)[b_j(x) - b_j(x - 1)], \quad \forall x \in \{1, \ldots, n\}, \forall j \in \{1, \ldots, m\}$$

is the unique local incentive mechanism with $\text{PoS}(T) = 1$.

**Proof.** We prove the claim in two parts: (i) show that any local incentive mechanism $T \neq T^\text{mc}$ has $\text{PoS}(T) > 1$; and (ii) prove that an optimal assignment is an equilibrium under $T^\text{mc}$.

**Part (i):** Assume, by contradiction, that there exists a local incentive mechanism $T$ with $\text{PoS}(T) = 1$ with $T(\ell)(k) > T^\text{mc}(\ell)(k)$ for some integer $1 \leq k \leq n$ and resource cost function $\ell \in \mathcal{L}$. Consider the game $G$ with user set $N = \{1, \ldots, k\}$ and two resources $E = \{e_0, e_1\}$. The resource $e_0$ has resource cost $\ell(x)$ and resource $e_1$ has resource cost $[\ell(k) + \ell(k - 1)] + \ell(x)$ for $x = 1, \ldots, k$ where $0 < \epsilon < T(\ell)(k) - T^\text{mc}(\ell)(k)$. Every user $i \in \{1, \ldots, k\}$ has only one action, $a_i = \{e_0\}$, while user $k$ has action set $A_k = \{a_k, a'_k\}$, where $a_k = \{e_0\}$ and $a'_k = \{e_1\}$. Observe that if $T(\ell)(k) > T^\text{mc}(\ell)(k)$, then the unique pure Nash equilibrium corresponds with when user $k$ selects $a'_k$ resulting in social cost $\ell(k - 1)(k - 1) + \ell(k - 1)$ for $x = 1, \ldots, k$ and $0 < \epsilon < T^\text{mc}(\ell)(k) - T(\ell)(k)$. Thus, the price of stability in this game is $[\ell(k) + \ell(x)]/[\ell(k) + \ell(x)] > 1$, which contradicts $\text{PoS}(T) = 1$. We conclude this part by observing that a similar argument holds for $T(\ell)(k) < T^\text{mc}(\ell)(k)$, when the resource cost of $e_1$ is $[\ell(k) + \ell(k - 1) + \ell(x)]$ for $x = 1, \ldots, k$ and $0 < \epsilon < T^\text{mc}(\ell)(k) - T(\ell)(k)$. In this case, user $k$’s Nash action is $a_k$ and the price of stability is $\ell(k)/[\ell(k) - \epsilon] > 1$.

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1We denote by $\mathcal{I}(n)$ the set of all triplets $x, y, z \in \{0, \ldots, n\}$ that satisfy $1 \leq x + y - z \leq n$ and $z \leq \min\{x, y\}$. Note that further reductions to this set are possible (see, e.g., [30]), but are omitted for conciseness.
Part (ii): Consider an optimal assignment $a_{\text{opt}}$ in a given game $G$. It is straightforward to show that this assignment must be an equilibrium under $T_{\text{mc}}$:

$$C_i(a_{\text{opt}}) - C_i(a_i, a_{\text{opt}}^{-i}) = \sum_{e \in a_i^{\text{opt}}} \left[ \ell_e(|a_{\text{opt}}^i|_e) |a_{\text{opt}}^i|_e - \ell_e(|a_{\text{opt}}^i|_e - 1) (|a_{\text{opt}}^i|_e - 1) \right]$$

$$- \sum_{e \in a_i} \left[ \ell_e(|a_i, a_{\text{opt}}^{-i}|_e) |a_i, a_{\text{opt}}^{-i}|_e - \ell_e(|a_i, a_{\text{opt}}^{-i}|_e - 1) (|a_i, a_{\text{opt}}^{-i}|_e - 1) \right]$$

$$= \sum_{e \in a_i^{\text{opt}} \setminus a_i} \ell_e(|a_{\text{opt}}^i|_e) |a_{\text{opt}}^i|_e + \sum_{e \in a_i \setminus a_i^{\text{opt}}} \ell_e(|a_i, a_{\text{opt}}^{-i}|_e - 1) - \sum_{e \in a_i \setminus a_i^{\text{opt}}} \ell_e(|a_i, a_{\text{opt}}^{-i}|_e - 1)$$

$$= \sum_{e \in E} \ell_e(|a_{\text{opt}}^i|_e) |a_{\text{opt}}^i|_e - \sum_{e \in E} \ell_e(|a_i, a_{\text{opt}}^{-i}|_e) |a_i, a_{\text{opt}}^{-i}|_e$$

$$= \text{SC}(a_{\text{opt}}) - \text{SC}(a_i, a_{\text{opt}}^{-i}),$$

where the third equality holds because we add and subtract $\ell_e(|a_{\text{opt}}^i|_e) |a_{\text{opt}}^i|_e$ for all $e \in E \setminus (a_i^{\text{opt}} \cup a_i)$ and all $e \in a_i^{\text{opt}} \cap a_i$. The final line must be nonpositive for all actions $a_i \in A_i$ and all users $i \in N$ by the optimality of $a_{\text{opt}}$, concluding the proof.

In the above discussion, we identified incentive mechanisms that unilaterally minimize the price of anarchy and the price of stability. In the next section, we consider the consequences of the unilateral minimization of either of these performance metrics on the value of the other.

### 3 Optimizing for anarchy or stability

We first seek to investigate whether the performance of the best case equilibria suffers when we optimize for the worst case equilibrium performance. In the next result, we prove that the price of stability is equal to the price of anarchy for any local incentive mechanism $T_{\text{opt}}$ that minimizes the price of anarchy in congestion games with convex, nondecreasing resource cost functions. Additionally, we show that linear mechanisms are optimal over all possible local mechanisms.

**Theorem 3.** Consider the set of congestion games with at most $n$ users, resource costs $\ell(x) = \sum_{j=1}^{m} \alpha_j b_j(x)$, $\alpha_j \geq 0$, and convex, nondecreasing functions $b_1, \ldots, b_m$. The following statements hold:

i) Let $T_{\text{lin}}$ denote a Pareto optimal mechanism in the set of all linear mechanisms, i.e., all mechanisms $T$ that satisfy $T(\sum_{j=1}^{m} \alpha_j b_j) = \sum_{j=1}^{m} \alpha_j T(b_j)$. Then, $T_{\text{lin}}$ is Pareto optimal over all mechanisms.

ii) Let $T_{\text{opt}}$ denote the mechanism that minimizes the price of anarchy as defined in Equation (5) where $(\ell_j^{\text{opt}}, \rho_j^{\text{opt}})$, $j = 1, \ldots, m$, are solutions to the $m$ linear programs in Equation (6) (one for each $b_j$). The corresponding price of stability is $\text{PoS}(T_{\text{opt}}) = \text{PoA}(T_{\text{opt}}) = \max_j \{1/\rho_j^{\text{opt}}\}$. Furthermore, the functions $\ell_j^{\text{opt}}$, $j = 1, \ldots, m$, are nondecreasing and unique up to rescaling.

The proof is presented in Appendix A. Alternatively, one might wish to understand how unilaterally minimizing the price of stability impacts the price of anarchy. In the previous section, we showed that the marginal cost mechanism is
the unique local mechanism that achieves the minimum price of stability of 1 in congestion games. Paccagnan et al. [30] provide a tractable linear program for computing the price of anarchy in congestion games under the marginal cost mechanism. For the reader’s convenience, we reproduce their result in the following proposition:

**Proposition 3** (Corollary 6.1 [30]). Consider congestion games with $n$ users, resource costs $\ell(x) = \sum_{j=1}^{m} \alpha_j b_j(x)$, $\alpha_j \geq 0$, and positive, convex basis functions $\{b_1, \ldots, b_m\}$. Then, the marginal cost mechanism

$$ T^{mc}(b_j)(x) = (x-1)[b_j(x) - b_j(x-1)], \quad \forall x \in \{1, \ldots, n\}, \forall j \in \{1, \ldots, m\} $$

has price of anarchy $\text{PoA}(T^{mc}) = 1/\rho^{mc}$, where $\rho^{mc}$ is the optimal value of the following:

$$ \max_{\rho, \nu \geq 0} \rho \text{ subject to:} $$

$$ b_j(y) - \rho b_j(x) + \nu[(x^2 - xy)b_j(x) - x(x-1)b_j(x-1) - y(x+1)b_j(x+1)] \geq 0, $$

$$ x, y \in \{0, \ldots, n\}, \quad x + y \leq n, \quad \forall j \in \{1, \ldots, m\}, $$

$$ b_j(y) - \rho b_j(x) \geq \nu[xb_j(x)(2n - x - y) + (x-1)b_j(x-1)(y - n - (x+1)b_j(x+1)(x-n)] \geq 0, $$

$$ x, y \in \{0, \ldots, n\}, \quad x + y > n, \quad \forall j \in \{1, \ldots, m\}, $$

where we set $b_j(-1) = b_j(0) = b_j(n+1) = 0$.

**4 The Pareto frontier of inefficiency**

In the previous sections, we showed that the unique local incentive mechanism that minimizes the price of anarchy has price of stability equal to the price of anarchy. Furthermore, the well-known marginal cost mechanism is the unique local incentive mechanism that minimizes the price of stability, always achieving a price of stability of 1. Thus, a local incentive mechanism $T$ simultaneously minimizes the price of anarchy and the price of stability if and only if $\text{PoA}(T) = \text{PoS}(T) = 1$. As the minimum achievable price of anarchy is strictly greater than 1 for all nondecreasing, convex resource costs (except constant) [29], it immediately follows that there exists a tradeoff between the two metrics. In this section, we derive upper and lower bounds on the Pareto frontier between the price of anarchy and price of stability in congestion games, which permit us to better understand the tradeoff between these two metrics.

**4.1 The upper bound**

Before presenting our upper bound, we introduce a modified version of the smoothness argument in Christodoulou and Gairing [12] that provides upper bounds on the price of stability of $\mathcal{G}$ in the next proposition. Note that congestion games admit the Rosenthal potential function [33], defined as $\Phi(a) = \sum_{e \in \cup_{A_i} \sum_{k=1}^{\vert e \vert} \ell_e(k)}$, which we will use throughout this manuscript.

**Proposition 4.** Let $\mathcal{G}$ denote a set of games defined for a maximum number of users $n$. Suppose there exist $\zeta > 0$, $\lambda > 0$ and $\mu < 1$ such that, for every game $G \in \mathcal{G}$ and any two assignments $a, a' \in A$, it holds that

$$ \text{SC}(a) + \sum_{i=1}^{n} [C_i(a'_i, a_{-i}) - C_i(a)] + \zeta [\Phi(a') - \Phi(a)] \leq \lambda \text{SC}(a') + \mu \text{SC}(a). \quad (8) $$

Then, the price of stability satisfies $\text{PoS}(\mathcal{G}) \leq \lambda/(1 - \mu)$.
Proof. Consider any game $G \in \mathcal{G}$ and let $a_{\text{opt}} \in \mathcal{A}$ denote an optimal assignment, i.e., $\text{SC}(a_{\text{opt}}) = \text{MinCost}(G)$. Recall that all congestion games admit an exact potential function \cite{23}. Thus, let $a_{\text{ne}} \in \text{NE}(G)$ denote a pure Nash equilibrium that satisfies $\Phi(a_{\text{ne}}) \leq \Phi(a_{\text{opt}})$. Since $C_i(a_{\text{opt}}, a_{\text{ne}}) \geq C_i(a_{\text{ne}})$ for all $i \in N$ and $\Phi(a_{\text{ne}}) \leq \Phi(a_{\text{opt}})$, it follows from Equation (8) that

$$\text{SC}(a_{\text{ne}}) \leq \lambda \text{SC}(a_{\text{opt}}) + \mu \text{SC}(a_{\text{ne}}).$$

Rearranging the above inequality gives us that $\text{SC}(a_{\text{ne}})/\text{SC}(a_{\text{opt}}) \leq \lambda/(1 - \mu)$. Since $a_{\text{ne}}$ is not necessarily the pure Nash equilibrium with minimum social cost, $\text{PoS}(G) \leq \lambda/(1 - \mu)$ also holds. \qed

The above smoothness argument provides an upper bound on the price of stability by bounding the efficiency of all pure Nash equilibria with potential lower than the potential at the optimum. Note that the potential of the worst case ratio between the system cost at the potential minimizer and the system cost at optimum. We continue our discussion on the price of stochastic anarchy in Section 5.

\begin{theorem} \label{thm:smoothness_argument}
Consider the set of congestion games $\mathcal{G}$ with at most $n$ agents, resource costs $\ell(x) = \sum_{j=1}^n \alpha_j b_j(x)$, $\alpha_j \geq 0$, and basis functions $\{b_1, \ldots, b_n\}$. Further, consider a maximum allowable price of anarchy $\text{PoA}^*$ greater than or equal to the minimum achievable price of anarchy in $\mathcal{G}$. Let $\{F_1^{\text{opt}}, \ldots, F_m^{\text{opt}}\}, \nu^{\text{opt}}, \rho^{\text{opt}}, \gamma^{\text{opt}}, \kappa^{\text{opt}}$ be solutions to the following bilinear program:

\begin{equation}
\begin{aligned}
\text{maximize} \quad & \{F_j, \nu^{-1}, \nu^{-1} \rho, \gamma, \kappa\} \quad \text{subject to:} \\
\nu^{-1} \rho \geq & \frac{1}{\text{PoA}^*}, \nu^{-1} \\
\nu^{-1} b_j(y) y - & \nu^{-1} \rho b_j(x) x + (x - z) F_j(x) - (y - z) F_j(x + 1) \geq 0, \\
& \forall (x, y, z) \in \mathcal{I}(n), \forall j \in \{1, \ldots, m\}, \\
\nu^{-1} b_j(y) y - & \gamma b_j(x) x + (x - z) F_j(x) - (y - z) F_j(x + 1) + \kappa \left[ \sum_{k=1}^x F_j(k) - \sum_{k=1}^y F_j(k) \right] \geq 0, \\
& \forall (x, y, z) \in \mathcal{I}(n), \forall j \in \{1, \ldots, m\}.
\end{aligned}
\end{equation}

Then, the local incentive mechanism $T^{\text{opt}}$ defined as $T^{\text{opt}}(b_j) = F_j^{\text{opt}}(x) - b_j(x)$, $j = 1, \ldots, m$, achieves price of anarchy $\text{PoA}(T^{\text{opt}}) = 1/\rho^{\text{opt}} \leq \text{PoA}^*$ and price of stability $\text{PoS}(T^{\text{opt}}) \leq 1/\gamma^{\text{opt}}$.
\end{theorem}

The proof is presented in Appendix B. The optimization problem in Equation (9) is a bilinear program with a single bilinearity, since $\kappa$ is multiplied with $F$ in the final set of constraints. Such programs can be solved efficiently using the method of bisections, which involves solving a finite number of linear programs for appropriate guesses of the value $\kappa^{\text{opt}}$.

A possible interpretation of the above result is that the incentive mechanism $T^{\text{opt}}$ guarantees that every game $G \in \mathcal{G}$ has at least one pure Nash equilibrium with social cost at most $1/\gamma^{\text{opt}}$ times greater than $\text{MinCost}(G)$. This equilibrium may not represent the best performing equilibrium of $G$, so this is only an upper bound on the price of stability, in general. Recall from the proof of Proposition 4 that our upper bound also applies to the efficiency of any pure Nash equilibrium that has potential less than or equal to the potential at the optimum. Note that the potential minimizer of the game is one such pure Nash equilibrium. Thus, the upper bound on the price of stability metric we have derived also applies to the price of stochastic anarchy \cite{16}, which measures the worst case ratio between the system cost at the potential minimizer and the system cost at optimum. We continue our discussion on the price of stochastic anarchy in Section 5.
4.2 The lower bound

The following theorem states our lower bound on the best achievable price of stability for a maximum allowable price of anarchy PoA*:

**Theorem 5.** Consider the set of congestion games \( \mathcal{G} \) with at most \( n \) agents, resource costs \( \ell(x) = \sum_{j=1}^{m} \alpha_j b_j(x), \alpha_j \geq 0, \) and basis functions \( \{b_1, \ldots, b_m\} \). Further, consider a maximum allowable price of anarchy PoA* greater than or equal to the minimum achievable price of anarchy in \( \mathcal{G} \). Let \( F_{j}^{\text{opt}}, \nu_j, \rho_j \) be optimal values that solve the following \( m \) linear programs (one for each \( j \)):

\[
\begin{align*}
\text{maximize} & \quad \sum_{x=1}^{n} F(x) \\
\text{subject to:} & \quad \rho \nu - 1 \geq \frac{1}{\text{PoA}^*} \nu - 1, \quad F(1) = 1, \\
& \quad \nu \nu - 1 b_j(y) y - \rho \nu - 1 b_j(x) x + (x - z) F(x) - (y - z) F(x + 1) \geq 0, \forall (x, y, z) \in I(n),
\end{align*}
\]

and define \( F_j^{(u,v)}(k) = \max_{v+k \leq x \leq u} F_j^{\text{opt}}(x) \), for \( k = 1, \ldots, u - v \). Then, the best price of stability that can be achieved by any local incentive mechanism \( T \) satisfies \( \text{PoS}(T) \geq \max_j \{1/\gamma_j^{\text{opt}}\} \), where

\[
\gamma_j^{\text{opt}} = \min_{0 \leq v < u \leq n} \frac{b(v) + \sum_{k=1}^{u-v} F_j^{(u,v)}(k)}{b(u)u}.
\]

The proof is presented in Appendix C. We highlight some important observations regarding the above result in the discussion below:

Note that the linear program in Equation (10) must be feasible for all values PoA* greater than or equal to the minimum achievable price of anarchy in the set \( \mathcal{G} \), as there exists at least one set of feasible values \( F, \nu, \rho \). Furthermore, the linear program must provide a (tight) lower bound of \( \text{PoS}(T) \geq 1 \) for any PoA* greater than the price of anarchy of the marginal cost mechanism since the price of stability of the marginal cost mechanism is 1. When the basis functions are convex and nondecreasing, the linear program must also provide a (tight) lower bound \( \text{PoS}(T) \geq \text{PoA}^* \) when PoA* is equal to the minimum achievable price of anarchy for \( n \) users, since we showed in Part (ii) of the proof of Theorem 3 that a worst case game in this setting has the same structure as the construction we use to obtain this lower bound.

Additionally, the game construction from which we obtain this lower bound on the price of stability has a unique, dominant pure Nash equilibrium \( a^{\text{ne}} \) in the sense that each user \( i \in N \) strictly prefers to play \( a^{\text{ne}}_i \) as long as users \( 1, \ldots, i-1 \) play their respective actions in \( a^{\text{ne}} \). It is straightforward to verify that \( a^{\text{ne}} \) is also the unique coarse correlated equilibrium of the game and, thus, our lower bound extends to the best case efficiency of coarse correlated equilibria in \( \mathcal{G} \).

4.3 Arbitrary number of users

In this section, we seek to investigate the system performance that can be obtained using a local incentive mechanism in settings with an arbitrary (possibly infinite) number of users. We first observe that a lower bound on the Pareto curve follows immediately from the linear program in Section 4, as the price of anarchy and price of stability metrics are both nondecreasing in the number of users. As the same argument cannot be applied to upper bounds on the two metrics, here we develop a method for computing an upper bound on the Pareto curve between the price of anarchy and price of stability in polynomial congestion games with any number of users.
Before presenting our next result, we define several parameters that are necessary for extending the solution of the upcoming finite dimensional bilinear program to an arbitrary number of users. Given integer \( d \in \mathbb{N}_{\geq 1} \), even integer \( \bar{n} \in \mathbb{N}_{\geq 2} \), optimal values \( \nu^{\text{opt}}, \rho^{\text{opt}}, \gamma^{\text{opt}}, \kappa^{\text{opt}} \) and function \( F^{\text{opt}} : \{1, \ldots, n\} \rightarrow \mathbb{R} \), we define:

\[
F^{\infty}(x) := \begin{cases} 
F^{\text{opt}}(x) & \text{if } 0 \leq x \leq \bar{n}/2, \\
\beta d^{d+1} - (x-1)^{d+1} & \text{if } x > \bar{n}/2
\end{cases}
\]

\[
\rho^{\infty} := \min \left\{ \rho^{\text{opt}}, \beta\nu^{\text{opt}} \frac{n}{2} \left[ 1 - \left( \frac{2}{n} \right)^{d+1} \right] - d \left[ \beta\nu^{\text{opt}} - \frac{n}{2} \left( \frac{2}{n} + 1 \right)^{d+1} - 1 \right] \right\}^{1+1/d}
\]

\[
\gamma^{\infty} := \min \{ \gamma^{\text{opt}}, \gamma_1, \gamma_2 + \beta\kappa^{\text{opt}}, \gamma_3 \}
\]

where \( \beta := F^{\text{opt}}(\bar{n}/2)/[(\bar{n}/2)^{d+1} - (\bar{n}/2 - 1)^{d+1}] \),

\[
\gamma_1 := \min_{x \in \{1, \ldots, \bar{n}/2-1\}} \frac{1}{x^{d+1}} \left[ \gamma^{d+1}_1(x) + F^{\text{opt}}(x) - F^{\text{opt}}(x+1) \gamma_1(x) - \kappa^{\text{opt}} \sum_{k=x+1}^{\bar{n}/2} F^{\text{opt}}(k) - \beta\kappa^{\text{opt}} \left[ \gamma^{d+1}_1(x) - \left( \frac{n}{2} \right)^{d+1} \right] \right]
\]

where \( \gamma_1(x) = \max\{\bar{n}/2 + 1, [F^{\text{opt}}(x+1)/(d+1)/(1 - \beta\kappa^{\text{opt}})]^{1/d}\} \),

\[
\gamma_2 := \min_{y \in \{0, \ldots, \bar{n}/2-1\} \cap \mathbb{N}} \min_{r \geq \bar{n}/2} \beta y + \frac{1}{r^{d+1}} \left[ \gamma^{d+1}_2(y) + \beta \left[ \gamma^{d+1}_2(y) - (r-1)^{d+1} \right] r - \beta (r+1)^{d+1} y + \kappa^{\text{opt}} \sum_{k=y+1}^{\bar{n}/2-1} F^{\text{opt}}(k) - \beta\kappa^{\text{opt}} \left( \frac{n}{2} - 1 \right)^{d+1} \right]
\]

and

\[
\gamma_3 := \min_{x \geq \bar{n}/2} \beta\kappa^{\text{opt}} + \frac{1}{x^{d+1}} \left[ (1 - \beta\kappa^{\text{opt}}) \gamma^{d+1}_3(x) + \beta [x^{d+1} - (x-1)^{d+1}] x - \beta [(x+1)^{d+1} - x^{d+1}] \gamma_3(x) \right]
\]

where \( \gamma_3(x) = \max\{\bar{n}/2, [\beta ((x+1)^{d+1} - x^{d+1})/(d+1)/(1 - \beta\kappa^{\text{opt}})]^{1/d}\} \).

The following theorem presents our upper bound on the Pareto frontier between the price of anarchy and price of stability metrics in polynomial congestion games for any number of users:

**Theorem 6.** Consider congestion games \( G \) with polynomial resource costs \( \ell(x) = \alpha x^d, \alpha \geq 0 \), of order \( d \in \mathbb{N}_{\geq 1} \). Further, consider a maximum allowable price of anarchy \( \text{PoA}^x \), greater than or equal to the minimum achievable price of anarchy in \( G \). Let \( F^{\text{opt}}, \nu^{\text{opt}}, \rho^{\text{opt}}, \gamma^{\text{opt}}, \kappa^{\text{opt}} \) be optimal values
that solve the following bilinear program for some even integer \( \bar{n} \in \mathbb{N}_{\geq 2} \) and \( \epsilon > 0 \):

\[
\begin{align*}
\text{maximize} & \quad \gamma \\
\text{subject to:} & \quad F_{\geq 0, \nu^{-1} \geq 0, \nu^{-1} \rho, \gamma, \kappa \geq 0} \\
\nu^{-1} \rho & \geq (\text{PoA}^*)^{-1} \nu^{-1} \\
F(x + 1) & \geq F(x), \quad \forall x \in \{1, \ldots, n - 1\} \\
F\left(\frac{\bar{n}}{2}\right) & \leq \nu^{-1}\left[\left(\frac{\bar{n}}{2} + 1\right)^{d+1} - \left(\frac{\bar{n}}{2}\right)^{d+1}\right] \\
F(1) + \frac{(d + 1)\bar{n}/2 + 1)^d \kappa}{(\bar{n}/2)^{d+1} - (\bar{n}/2 - 1)^{d+1}} F\left(\frac{\bar{n}}{2}\right) & \leq (d + 1)\left(\frac{\bar{n}}{2} + 1\right)^d \\
1 - \frac{\kappa}{\nu^{-1}} & \geq \epsilon \\
\nu^{-1} y^{d+1} - \nu^{-1} \rho x^{d+1} + xF(x) - yF(x + 1) & \geq 0, \quad \forall (x, y) \in \mathcal{I}_{\leq \bar{n}}(\bar{n}) \\
y^{d+1} - \gamma x^{d+1} + xF(x) - yF(x + 1) + \kappa \left[\sum_{k=1}^{x} F(k) - \sum_{k=1}^{y} F(k)\right] & \geq 0, \quad \forall (x, y) \in \mathcal{I}_{\leq \bar{n}}(\bar{n})
\end{align*}
\]

where \( \mathcal{I}_{\leq \bar{n}}(n) \) is the set of all pairs \( x, y \in \{0, \ldots, n\} \) such that \( 1 \leq x + y \leq n \). Then, for \( F^\infty : \mathbb{N} \rightarrow \mathbb{R} \), \( \rho^\infty \) and \( \gamma^\infty \) defined as in Equation (11), the local incentive mechanism \( T^\infty \) with \( T^\infty(x^n)(x) = F^\infty(x) - x^d \) satisfies PoA\((T^\infty) \leq 1/\rho^\infty \) and PoS\((T^\infty) \leq 1/\gamma^\infty \).

Though the formal proof of Theorem 6 is presented in Appendix D of the supplementary, here we provide an informal outline for the reader’s convenience. A similar approach can be used to compute upper bounds for any class of congestion games. For a given number of users \( n \) (possible infinite), the proof amounts to showing that the values \( \gamma^\infty, \rho^\infty, \nu^\text{opt}, \kappa^\text{opt} \) from the bilinear program in Theorem 6 satisfy the constraints of two linear programs governing upper bounds on the price of anarchy and the price of stability, respectively. The constraints of these two linear programs are parameterized by each pair \( x, y \in \{0, \ldots, n\} \). We divide the proof as follows:

- **Upper bound on the price of anarchy:** In the first part, we show that the values \( \rho^\infty, \nu^\text{opt} \) are feasible points of the linear program in Equation (6) for the function \( F^\infty \) and \( n \) users. We first focus on the values \( x, y \) such that \( 1 \leq x + y \leq n \). We show that the constraints parameterized by \( 0 \leq x < \bar{n}/2 \) and \( y \geq 0 \) are equivalent to constraints from the bilinear program in Theorem 6, leveraging the fact that \( F^\infty(x) = F^\text{opt}(x) \) and \( F^\infty(x + 1) = F^\text{opt}(x + 1) \). Then, for \( x \geq \bar{n}/2 \) and \( y \geq 0 \), we prove that all the constraints are satisfied if \( \rho^\infty \) is less than or equal to the second expression in the minimum that governs the definition of \( \rho^\infty \) in Equation (11). Finally, we show that the constraints with \( x + y > n \) are redundant, as they are less strict than those with \( 1 \leq x + y \leq n \).

- **Upper bound on the price of stability:** Next, we show that the values \( \gamma^\infty, \kappa^\text{opt} \) are feasible points of the linear program in Section 4.1 for \( \nu = 1 \), the function \( F^\infty \) and \( n \) users. We first focus on the values \( x, y \) such that \( 1 \leq x + y \leq n \). We show that the constraints with \( 0 \leq x < \bar{n}/2 \) and \( 0 \leq y \leq \bar{n}/2 \), and the constraints with \( x = 0 \) and \( y > \bar{n}/2 \), are equivalent to constraints from the bilinear program in Theorem 6. Then, we prove that \( \gamma^\infty, \kappa^\text{opt} \) satisfy the constraints with \( 1 \leq x < \bar{n}/2 \) and \( y > \bar{n}/2 \) because \( \gamma^\infty \leq \gamma_1 \), the constraints with \( x \geq \bar{n}/2 \) and \( 0 \leq y < \bar{n}/2 \) because \( \gamma^\infty \leq \gamma_2 \), and the constraints with \( x, y \geq \bar{n}/2 \) because \( \gamma^\infty \leq \gamma_3 \), for \( \gamma_1, \gamma_2, \gamma_3 \) defined as in Equations (12)–(14). Finally, we show that the constraints with \( x + y > n \) are less strict than those with \( 1 \leq x + y \leq n \).

In the study of polynomial congestion games, we are often interested in the setting where resource cost functions have the form \( \ell(x) = \sum_{j=0}^{d} \alpha_j x^j \) for \( \alpha_j \geq 0 \) for given order \( d \geq 1 \). As it is
stated, the result in Theorem 6 can only accommodate resource cost functions corresponding to a single monomial basis function. However, consider the scenario where we solve the bilinear program in Theorem 6 for monomial basis functions $b_1, \ldots, b_m$, some fixed values $\kappa = \kappa^*$ and $\nu = \nu^*$, and PoA$^*$ greater than or equal to the best achievable price of anarchy for the corresponding set of games $G$. Then, the values $\text{PoA}(T) \leq \max_j \{1/\rho_j^\infty\}$ and $\text{PoS}(T) \leq \max_j \{1/\gamma_j^\infty\}$ must be valid upper bounds on the Pareto frontier between the price of anarchy and price of stability in $G$, where $\rho_j^\infty$ and $\gamma_j^\infty$ are derived as in Equation 11 for all $j = 1, \ldots, m$. We state this consequence of Theorem 6 in the following corollary:

**Corollary 1.** Consider congestion games $G$ with resource costs $\ell(x) = \sum_{j=1}^m \alpha_j b_j(x)$ where $\alpha_j \geq 0$, for monomial basis functions $b_j(x) = x^{d_j}$ of order $d_j \in \mathbb{N}_{\geq 1}$. Further, consider a maximum allowable price of anarchy $\text{PoA}^*$ greater than or equal to the minimum achievable price of anarchy in $G$. For fixed $\kappa \geq 0$, fixed $\nu \geq 0$, even integer $\bar{n} \in \mathbb{N}_{\geq 2}$ and $\epsilon > 0$, let $F_j^{\text{opt}}, \rho_j^{\text{opt}}, \gamma_j^{\text{opt}}$ be optimal values corresponding with a solution to the bilinear program in Theorem 6 for each basis functions $b_j$, $j = 1, \ldots, m$.

Then, for $F_j^\infty : \mathbb{N} \to \mathbb{R}$, $\rho_j^\infty$ and $\gamma_j^\infty$ defined as in Equation (11), the local incentive mechanism $T^\infty$ with $T^\infty(b_j)(x) = F_j^\infty(x) - b_j(x)$ satisfies $\text{PoA}(T^\infty) \leq \max_j \{1/\rho_j^\infty\}$ and $\text{PoS}(T^\infty) \leq \max_j \{1/\gamma_j^\infty\}$.

5 **Remarks**

The results we presented in the previous sections have several important consequences, some of which we highlight in the discussion below:

5.1 **Equilibrium efficiency in congestion games without incentives**

Many works have focused on identifying tight bounds on the price of anarchy and price of stability, particularly for congestion games without incentives. In this respect, Aland et al. [1] put forward an expression for the price of anarchy in polynomial congestion games without incentives. Meanwhile, Christodoulou and Gairing [12] provide exact bounds on the price of stability in polynomial congestion games without incentives. As both these values are known exactly, we can compare the equilibrium performance in the absence of incentives against upper bounds on the Pareto frontier we derive using the methodology we developed in Section 4.3.

In Columns 2 and 3 of Table 1, we summarize the price of anarchy and price of stability bounds from the literature on polynomial games without incentives. In Columns 4 and 5, we provide the upper bounds on the best achievable price of stability while guaranteeing price of anarchy no greater than that of polynomial congestion games without incentives. Conversely, in Columns 6 and 7, we provide the upper bounds on the best achievable *price of anarchy* without exceeding the *price of stability* in polynomial congestion games without incentives. From the values in Table 1, one can easily verify that using no incentive in the polynomial congestion games considered is not Pareto optimal. In fact, significant improvements can be achieved in terms of price of anarchy while guaranteeing the same price of stability. For the specific case of polynomial congestion games of degree $d = 4$, we observe that the price of anarchy can be reduced by more than 66.7% with no increase in the price of stability. Demonstrating the suboptimality of no incentive more generally remains an interesting open problem and should be considered in future work.
Table 1: Comparison of price of anarchy and price of stability values for no incentives and Pareto optimal incentive mechanisms in congestion games with polynomial resource costs of degree \( d = 1, \ldots, 4 \). The price of anarchy and price of stability values for no incentive are summarized in Columns 2 and 3 and were derived by Aland et al. [1] and Christodoulou and Gairing [12], respectively. Columns 4–7 report values on the Pareto frontier between the price of anarchy and price of stability metrics corresponding with the price of anarchy and price of stability without incentives. These were computed using the technique in Theorem 6.

| \( d \) | No incentive | PoA Requirement | PoS Requirement |
|-------|--------------|----------------|----------------|
|       | PoA          | PoS            | PoA            | PoS            |
| 1     | 2.500        | 1.577          | 2.500          | 1.418          |
| 2     | 9.583        | 2.361          | 9.583          | 1.156          |
| 3     | 41.536       | 3.322          | 41.536         | 1.290          |
| 4     | 267.643      | 4.398          | 267.643        | 1.135          |

5.2 Bounds on the price of stochastic anarchy

The price of stochastic anarchy measures the efficiency of the potential minimizer within a given set of games. For a given set of games \( G \) and incentive mechanism \( T \), the price of stochastic anarchy is formally defined as follows:

\[
\text{PoSA}(T) = \sup_{G \in \mathcal{G}} \frac{\text{SC}(a_{\text{pm}})}{\text{MinCost}(G)},
\]

where \( a_{\text{pm}} \) is the potential minimizing assignment of the game \( G \), i.e., \( a_{\text{pm}} \) satisfies \( \Phi(a_{\text{pm}}) \leq \Phi(a) \) for all \( a \in \mathcal{A} \). Observe that \( a_{\text{pm}} \) must be a pure Nash equilibrium of the game, so it always holds that \( \text{PoA}(T) \geq \text{PoSA}(T) \geq \text{PoS}(T) \geq 1 \). The assignment \( a_{\text{pm}} \) is sometimes called the stochastically stable state, since any sequence of sufficiently noisy best responses will converge to this assignment with probability 1. Thus, the price of stochastic anarchy is particularly relevant when considering the performance of noisy learning dynamics such as logit response dynamics [2, 8, 28, 35].

In Section 4, our upper bound on the price of stability was obtained by computing the worst case equilibrium performance with the additional requirement that the potential at the equilibrium is less than or equal to the potential at the optimum. Since the potential minimizer of the game satisfies this requirement the upper bound we derive for the price of stability also applies to the price of stochastic anarchy. Furthermore, the lower bound we derive corresponds with the efficiency of the unique equilibrium of the constructed congestion game, which must be the potential minimizing assignment. Thus, both the upper and lower bounds we have characterized for the tradeoff between the price of anarchy and price of stability are also valid bounds on the tradeoff between the price of anarchy and the price of stochastic anarchy. In fact, all of our results on the Pareto frontier between the price of anarchy and price of stability directly apply to a Pareto frontier between the price of anarchy and price of stochastic anarchy.

5.3 Features of the optimal tradeoff curve

In the tradeoff curves for affine and quadratic congestion games in Figure 1, observe that the joint performance guarantees of mechanisms that minimize the price of anarchy are not equal to those of mechanisms that minimize the price of stability, which implies that there exists a tradeoff between price of anarchy and price of stability. However, it remains to be shown whether this strict separation between the extreme points of the Pareto frontier exists more generally beyond the two sets of congestion games we considered. In particular, if there is an incentive that guarantees a price of anarchy equal to 1 for a given set of congestion games, then there can be no tradeoff. Following
this line of reasoning, there must exist a tradeoff between price of anarchy and price of stability in all sets of congestion games with convex, nondecreasing latency functions (except constant), as it was recently shown by Paccagnan and Gairing [29] that the best achievable price of anarchy among all incentives (local or otherwise) is strictly greater than 1. A further consequence of the result in [29] is that the marginal cost mechanism, which we showed is the unique mechanism that achieves price of stability of 1 within this setting, never provides optimal price of anarchy.

### 6 Conclusions and Future Directions

In this paper, we investigated the consequences of minimizing the price of anarchy on the price of stability in congestion games. Our first set of results showed that the local incentive mechanism that minimizes the price of anarchy has price of stability equal to the price of anarchy. As the marginal cost mechanism always achieves a price of stability of 1, it followed that a tradeoff exists between the price of anarchy and the price of stability, as the best achievable price of anarchy is generally strictly greater than 1. We then developed techniques for deriving upper and lower bounds on the Pareto frontier between the price of anarchy and price of stability. All of our results extend to the efficiency of coarse correlated equilibria and to the Pareto frontier between the price of anarchy and the price of stochastic anarchy.

A parallel research effort has investigated the design of global mechanisms where the design of incentive functions is conditioned on all parameters of a game instance. Interestingly, recent results have suggested that transitioning from local to global incentive mechanisms may not provide significant reductions of the achievable price of anarchy [30]. Whether a similar tradeoff between the price of anarchy and the price of stability exists in settings with global mechanisms is an open and interesting problem.

The price of anarchy and price of stability are only two of the many metrics used to evaluate the performance of distributed algorithms. In this respect, our contributions represent preliminary results toward the broader research agenda of identifying tradeoffs in optimal mechanism design. Future work should focus on understanding the impact of minimizing the price of anarchy on the algorithmic performance with respect to other metrics, including the rate of convergence to an equilibrium [19] and the transient system performance [6, 25].

Filos-Ratsikas et al. [20] and Ramaswamy Pillai et al. [32] investigate tradeoffs between the price of anarchy and price of stability in distinct classes of problems. However, they all report findings analogous to our first main result: when the price of anarchy is optimized, the price of anarchy and the price of stability are equal. While it is obvious that the price of stability can never exceed the price of anarchy, it is unclear whether these two metrics must always be in tension with one another. A relevant research direction is to understand the class of problems for which the price of anarchy can only be optimized to the detriment of the price of stability.

### References

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A Proof of Theorem 3

We prove statements i) and ii) separately, below:

Proof of statement i). Given any mechanism $T$ (not necessarily linear), we show that there exists some linear mechanism $T_{\text{lin}}$ satisfying $\text{PoA}(T_{\text{lin}}) \leq \text{PoA}(T)$. The linear mechanism we consider is generated from the incentives $T(b_j)$, $j = 1, \ldots, m$, as follows: $T_{\text{lin}}(\sum_{j=1}^{m} \alpha_j b_j) = \sum_{j=1}^{m} \alpha_j T(b_j)$. As the same set of arguments also hold for the price of stability, the statement follows.

Let $\mathcal{G}_b$ be the restricted class of congestion games in which every resource $e$ has resource cost $\ell_e \in \{b_1, \ldots, b_m\}$. Within this restricted class of games, the price of anarchy of $T$ must be equal to that of $T_{\text{lin}}$ since the resulting incentives are equivalent. For linear mechanisms such as $T_{\text{lin}}$, one can show that for any congestion game $G \in \mathcal{G}$ there is another game $G' \in \mathcal{G}_b$ (possibly with many more edges) that has arbitrarily close price of anarchy following the proof of Theorem 5.6 in Roughgarden [34]. In other words, the price of anarchy that the mechanism $T_{\text{lin}}$ achieves in $G$ is equal to the price of anarchy it achieves in $\mathcal{G}_b$. We observe that for general mechanisms such as $T$, the price of anarchy achieved within the restricted class of games $\mathcal{G}_b$ can only be said to be less than or equal to the price of anarchy achieved within $\mathcal{G}$. It then follows that $\text{PoA}(T_{\text{lin}}) \leq \text{PoA}(T)$.

Proof of statement ii). Given the maximum number of users $n$ and basis functions $b_1, \ldots, b_m$, consider the following $m$ (relaxed) linear programs:

$$\begin{align*}
\text{maximize} & \quad \rho \\
\text{subject to:} & \quad b_j(y)y - \rho b(x)x + \min\{x, n - y\} F(x) - \min\{y, n - x\} F(x + 1) \geq 0 \\
& \quad \forall (x, y) \in \{0, \ldots, n\} \times \{1, \ldots, n\} \cup (n, 0). 
\end{align*}$$

(23)

Observe that the above linear program is a relaxation of the linear program in Equation (6) where we only consider the constraints $(x, y, z) \in \mathcal{I}(n)$ such that $(x, y) \in \{0, \ldots, n\} \times \{1, \ldots, n\} \cup (n, 0)$ and $z = \max\{0, x + y - n\}$. Reference [30] provides an expression for a set of optimal solutions $(F_{\text{opt}}^j, \rho_{\text{opt}}^j)$, $j = 1, \ldots, m$, to the $m$ linear programs above and show that these are also optimal solutions of the $m$ linear programs in Equation (6). As part of their proof, they show that the functions $F_{\text{opt}}^j$ must be nondecreasing.

The rest of the proof is shown in two steps: (i) we show that for the solutions $(F_{\text{opt}}^j, \rho_{\text{opt}}^j)$, $j = 1, \ldots, m$, to the $m$ linear programs in Equation (6), the functions $F_1^\text{opt}, \ldots, F_m^\text{opt}$ are unique (up to rescaling); (ii) leveraging the fact that the functions $F_1^\text{opt}, \ldots, F_m^\text{opt}$ are nondecreasing, we construct a congestion game $G$ that has $\text{PoS}(G) = \max_j\{1/\rho_j^\text{opt}\}$.

Part (i) – Proof that $F_{\text{opt}}^j$ is the unique optimal solution. We must show that there is no other function $F$ that yields a value of $\rho = \rho_{\text{opt}}$. By contradiction, let us assume that there exists a function $\tilde{F}$ different from $F_{\text{opt}}^j$ that also achieves $\rho_{\text{opt}}$. Let $k + 1$ be the first index at which $\tilde{F}(k + 1) \neq F_{\text{opt}}^j(k + 1)$. If $k = 0$, due to the constraint corresponding to $(x = 0, y = 1)$ in the linear program in Equation (23), it holds that $\tilde{F}(1) \leq b(1) = F_{\text{opt}}^j(1)$. Since $\tilde{F}(1) \neq F_{\text{opt}}^j(1)$, it must hold that $\tilde{F}(1) < F_{\text{opt}}^j(1)$. A similar argument holds for $k > 0$, since

$$\tilde{F}(k + 1) \leq \max_{y \in \{1, \ldots, n\}} \frac{b(y)y - \rho_{\text{opt}} b(k)k + \min\{k, n - y\} \tilde{F}(k)}{\min\{y, n - k\}} = F_{\text{opt}}^j(k + 1),$$

where the equality holds since $\tilde{F}(k) = F_{\text{opt}}^j(k)$, by assumption. In short, at the first $k + 1$ where $\tilde{F}$ does not equal $F_{\text{opt}}^j$, the former is always strictly lower than the latter or else a constraint
in the linear program would be violated. The contradiction follows from the constraints with 
\((x = n, y = y^*_n)\):  
\[
\rho^\text{opt}_j \leq \frac{b(y^*_n)g_n + (n - y^*_n)\bar{F}(n)}{b(n)n} < \frac{b(y^*_n)g_n + (n - y^*_n)\bar{F}_j^\text{opt}(n)}{b(n)n} = \rho^\text{opt}_j,
\]
where the strict inequality holds since \(n - y^*_n > 0\). Observe that if \(n - y^*_n = 0\), it holds that \(\rho^\text{opt}_j = 1\), which violates the stated conditions for uniqueness in the theorem statement.

**Part (ii) – Game construction.** Here we show that for each function in \(\{F_j^\text{opt}\}\), we can construct a congestion game that has price of stability equal to \(1/\rho^{\text{opt}}_j\). Without loss of generality, we assume that all basis functions \(b_1, \ldots, b_m\) are scaled such that \(F_j(1) = 1\). Consider the active constraint corresponding to \(x = n\) for each \(F_j^\text{opt}\), which – after some rearrangement – appears as follows:
\[
\frac{1}{\rho^\text{opt}_j} = \max_{y \in \{0,1,\ldots,n\}} \frac{b_j(n)n}{(n - y)F_j^\text{opt}(n) + b_j(y)y}.
\]

Define \(y^*_n \in \{0,1,\ldots,n\}\) as an argument that maximizes the right-hand side in the above expression. Consider a congestion game \(G\) with a set of \(n\) users \(N = \{1, \ldots, n\}\) and \(n - y^*_n + 1\) resources \(\mathcal{E} = \{e_0, e_1, \ldots, e_{n-y^*_n}\}\). The users’ action sets are defined as follows: each user \(i \in \{1, \ldots, n - y^*_n\}\) has action set \(\mathcal{A}_i = \{a^{pe}_i, a^{oe}_i\}\) where \(a^{pe}_i = \{e_0\}\) and \(a^{oe}_i = \{e_i\}\); and, each user \(i \in \{n - y^*_n + 1, \ldots, n\}\) has \(\mathcal{A}_i = \{a_i = \{e_0\}\}\). The cost on resource \(e_0\) is \(b_j\), whereas each \(e \in \{e_1, \ldots, e_{n-y^*_n}\}\) has cost \([F_j^\text{opt}(n) + \epsilon]b_j\) for some \(\epsilon > 0\). Since \(F_j^\text{opt}(1) = 1\) and \(F_j^\text{opt}\) is nondecreasing, it is straightforward to verify that the assignment \((a^{pe}_1, \ldots, a^{pe}_{n-y^*_n}, a^{oe}_{n-y^*_n+1}, \ldots, a_n)\) is the unique pure Nash equilibrium of the game. Simply observe that for any assignment \(a \in \Pi_i \mathcal{A}_i\), any user \(i \in \{1, \ldots, n - y^*_n\}\) selecting its action \(a_i^\text{opt}\) can decrease its cost by selecting its action \(a_i^{pe}\) instead, since \(F_j^\text{opt}(|a|_{e_0}) < F_j^\text{opt}(n) + \epsilon\). Thus, the constructed game has a unique pure Nash equilibrium, with system cost \(b_j(n)n\). The assignment \((a^{pe}_1, \ldots, a^{pe}_{n-y^*_n}, a^{oe}_{n-y^*_n+1}, \ldots, a_n)\) has system cost \((n - y^*_n)[F_j^\text{opt}(n) + \epsilon] + b_j(y^*_n)y^*_n\). Thus, taking the limit as \(\epsilon \to 0^+\), the price of stability of the constructed game satisfies \(\text{PoS}(G) \geq b_j(n)n/[(n - y^*_n)\bar{F}(n) + b_j(y^*_n)y^*_n] = 1/\rho^\text{opt}_j\). Since the function \(F_j^\text{opt}\) has corresponding price of anarchy guarantee of \(1/\rho^{\text{opt}}_j\), the price of stability must also be upper-bounded by \(1/\rho^{\text{opt}}_j\). Thus, \(\text{PoS}(T) = \max_j \{1/\rho^\text{opt}_j\} = \text{PoA}(T)\), concluding the proof.

## B Proof of Theorem 4

Observe that the optimal upper bound achievable from the smoothness argument in Proposition 4 can be computed as the solution to the following fractional program:
\[
\inf_{\zeta > 0, \lambda > 0, \mu} \left\{ \frac{\lambda}{1 - \mu} \right\} \text{ s.t. } (\zeta, \lambda, \mu) \text{ satisfy } (8) \forall a, a' \in \mathcal{A}, \forall G \in \mathcal{G} \right\}.
\]

To reduce the number of parameters, we introduce the following parameterization of any pair of assignments \(a, a' \in \mathcal{A}\) in a game \(G \in \mathcal{G}\): Consider each resource \(e \in \mathcal{E}\) and recall that \(\ell_e(x) = \sum_{j=1}^n \alpha_{e,j} \cdot b_j(x)\) with \(\alpha_{e,j} \geq 0\) for all \(j\). Let \(x_e = |a|_e, y_e = |a'|_e\) and \(z_e = |\{i \in N : e \in a_i\}\} \cap \{i \in N : e \in a'_i\}\}. It follows that \((x_e, y_e, z_e)\) belongs to the set \(\mathcal{I}(n)\) of all triplets \((x, y, z)\) \(\in \mathbb{N}^3\) that satisfy \(1 \leq x + y - z \leq n\) and \(z \leq \min\{x, y\}\). We define parameters \(\theta(x, y, z, j) = \sum_{e \in E_{x,y,z}} \alpha_{e,j}\) where \(E_{x,y,z} = \{e \in \mathcal{E} \text{ s.t. } (x_e, y_e, z_e) = (x, y, z)\}\). Under this parameterization, observe that the
inequality in Equation (8) can be rewritten as
\[
\sum_{j=1}^{m} \sum_{x,y,z} \left[ b_j(x)x + (y - z)F_j(x + 1) - (x - z)F_j(x) + \zeta \left( \sum_{k=1}^{x} F_j(k) - \sum_{k=1}^{y} F_j(k) \right) \right] \theta(x, y, z, j) \\
\leq \sum_{j=1}^{m} \sum_{x,y,z} [\lambda b_j(y)y + \mu b_j(x)x] \theta(x, y, z, j).
\]

We note that any \((\zeta, \lambda, \mu)\) that satisfies the above constraint for each individual summand corresponding with the triplets \((x, y, z)\) \(\in I(n)\) and \(j = 1, \ldots, n\) must satisfy the smoothness definition in Proposition 4 since we have shown that the inequalities governing the smoothness definition are a linear combination over the \(|I(n)| \times m\) summands with nonnegative coefficients \(\theta(x, y, z, j)\). Based on this observation, we obtain the following linear program for computing an upper bound on the price of stability after the change of variables \(\gamma = (1 - \mu)/\lambda\), \(\nu = 1/\lambda\) and \(\kappa = \zeta/\lambda\):

\[
\begin{align*}
\max_{\gamma, \nu, \kappa \geq 0} & \quad \gamma \\
\text{subject to:} & \quad b_j(y)y - \gamma b_j(x)x + \nu [(x - z)F_j(x) - (y - z)F_j(x + 1)] + \kappa \left( \sum_{k=1}^{x} F_j(k) - \sum_{k=1}^{y} F_j(k) \right) \geq 0, \\
& \quad \forall (x, y, z) \in I(n), \forall j \in \{1, \ldots, m\}.
\end{align*}
\]

Under this change of variables, it holds that \(\text{PoS}(G) \leq 1/\gamma_{\text{opt}}\) for optimal solutions \((\gamma^\text{opt}, \nu^\text{opt}, \kappa^\text{opt})\).

We note that the ‘inf’ objective can now be written as a ‘maximize’ since \(\gamma \in [0, 1]\) must hold.

We are interested in obtaining an upper bound on the best price of stability that can be achieved by introducing incentive mechanisms. By including the functions \(F_j\), \(j = 1, \ldots, m\), as decision variables in the dual program, we obtain the following bilinear program for computing a local incentive mechanism that minimizes the upper bound on the price of stability:

\[
\begin{align*}
\max_{\{F_j\}, \gamma, \kappa \geq 0} & \quad \gamma \\
\text{subject to:} & \quad b_j(y)y - \gamma b_j(x)x + (x - z)F_j(x) - (y - z)F_j(x + 1) + \kappa \left( \sum_{k=1}^{x} F_j(k) - \sum_{k=1}^{y} F_j(k) \right) \geq 0, \\
& \quad \forall (x, y, z) \in I(n), \forall j \in \{1, \ldots, m\}.
\end{align*}
\]

Then, for optimal solution \((\{F_j^\text{opt}\}, \gamma^\text{opt}, \kappa^\text{opt})\), the incentive mechanism \(T^\text{opt}\) defined as \(T^\text{opt}(b_j)(x) = F_j^\text{opt}(x) - b_j(x)\) for all \(j\) and \(x\) satisfies \(\text{PoS}(T^\text{opt}) \leq 1/\gamma^\text{opt}\). In the above bilinear program, we have imposed \(\nu = 1\), which removes one bilinearity in the constraints. The only remaining bilinearity involves the decision variable \(\kappa\) and the functions \(F_1, \ldots, F_m\).

To obtain the local incentive mechanism \(T^\text{opt}\) that guarantees a particular price of anarchy \(\text{PoA}^*\) while minimizing our upper bound on the price of stability, we add the constraints for the price of anarchy from Equation (6) to our bilinear program. We require that \(\text{PoA}^*\) be greater than or equal to the minimum achievable price of anarchy in the set \(G\) for feasibility. We can then simultaneously minimize the upper bound on the price of stability while guaranteeing the desired price of anarchy. After some rearrangement of decision variables, we obtain the bilinear program in the claim.

C Proof of Theorem 5

Consider the set of games \(G\) with at most \(n\) users, basis functions \(b_1, \ldots, b_m\) and local incentive mechanism \(T\). Define \(F_j(x) = b_j(x) + T(b_j)(x)\) for \(x = 1, \ldots, n, j = 1, \ldots, m\). Without loss of
on the price of stability, i.e., maximum value of this lower bound over all also derived the lower bound on the price of stability for each of these games. Observe that the note that, starting from any assignment $a_k = \{a_{1}, \ldots, a_{u-v}\}$. Resource $e_{0}$ has cost function $\ell_{0}(x) = b_{j}(x)$, while each resource $e_{k}$, $k = 1, \ldots, u - v$, has cost function $\ell_{k}(x) = \alpha_{k}b_{j}(x)$ where

$$\alpha_{k} = \max_{v+k \leq x \leq u} F_{j}(x) + \epsilon \text{ for } \epsilon > 0.$$  

Next, we prove that the game $G$ as defined above has a unique pure Nash equilibrium which corresponds with the assignment $a_{\text{ne}} = (a_{1}^{\text{ne}}, \ldots, a_{u-v}^{\text{ne}}, a_{u-v+1}, \ldots, a_{u})$. Consider the choices of user $k = 1, \ldots, u - v$ with respect to any assignment in which all users $i \in \{1, \ldots, k - 1\}$ play the action $a_{i}^{\text{ne}}$. The remaining users $i \in \{k + 1, \ldots, u - v\}$ play either of their actions in $A_{i}$. Observe that user $k$ must select either the resource $e_{0}$ which is currently selected by at least $k + l - 1$ users, or the resource $e_{k}$ which is currently not selected by any other user. It follows that user $k$ selects $a_{i}^{\text{ne}} = \{e_{0}\}$ in this scenario, since $F_{j}(y) < \max_{v+k \leq x \leq u} F_{j}(x) + \epsilon$ with $\epsilon > 0$, for $y = v + k, \ldots, u$.

Note that, starting from any assignment $a \in A$, one can repeat this argument from user $k = 1$ to user $k = u - v$ to show that any sequence of best responses will settle on the assignment $a_{\text{ne}}$ and, thus, that this is the unique pure Nash equilibrium. Note that the system cost associated with this assignment is $SC(a_{\text{ne}}) = b_{j}(u)u$. Meanwhile, the system cost of the assignment $a^{\text{opt}} = (a_{1}^{\text{opt}}, \ldots, a_{u-v}^{\text{opt}}, a_{u-v+1}, \ldots, a_{u})$ is $SC(a^{\text{opt}}) = b_{j}(v)v + \sum_{k=1}^{u-v}\max_{v+k \leq x \leq u} F_{j}(x) + \epsilon$. Furthermore, it holds that $\text{MinCost}(G) \leq SC(a^{\text{opt}})$. Thus, for $\epsilon \rightarrow 0^{+}$, the price of stability satisfies

$$\text{PoS}(T) \geq \frac{b_{j}(u)u}{\text{MinCost}(G)} \geq \frac{b_{j}(u)u}{b_{j}(v)v + \sum_{k=1}^{u-v}\max_{v+k \leq x \leq u} F_{j}(x)}.$$  

We have shown that within the set of games $G$, there exists a singleton game with a unique pure Nash equilibrium for any $b_{j}$, $j = 1, \ldots, m$, and any pair $(u, v)$ such that $0 \leq v < u \leq n$. We also derived the lower bound on the price of stability for each of these games. Observe that the maximum value of this lower bound over all $b_{j}$ and all valid pairs $(u, v)$ represents a lower bound on the price of stability, i.e.,

$$\text{PoS}(T) \geq \max_{j} \max_{0 \leq v < u \leq n} \frac{b_{j}(u)u}{b_{j}(v)v + \sum_{k=1}^{u-v}\max_{v+k \leq x \leq u} F_{j}(x)}$$  

$$= \max_{j} \max_{0 \leq v < u \leq n} \frac{b_{j}(u)u}{b_{j}(v)v + \sum_{k=1}^{u-v} F_{j}^{(u,v)}(k)},$$  

where we define $F_{j}^{(u,v)}(k) := \max_{v+k \leq x \leq u} F_{j}(x)$, for $k = 1, \ldots, u - v$, for conciseness. It follows that, given a set of congestion games $G$ with $n$ users, basis functions $b_{1}, \ldots, b_{m}$ and local incentive mechanism $T$, a lower bound on the price of stability can be computed as $\text{PoS}(T) \geq \max_{j}\{1/\gamma_{j}^{\text{opt}}\}$, where $\gamma_{j}^{\text{opt}}$, $j = 1, \ldots, m$, is the optimal value of the following linear program:

maximize $\gamma$ subject to:

$$\gamma b_{j}(u) \leq b_{j}(v)v + \sum_{k=1}^{u-v} F_{j}^{(u,v)}(k), \quad \forall (u, v) \in \{(u, v) \in \mathbb{N}^{2} \text{ s.t. } 0 \leq v < u \leq n\},$$  

$$F_{j}^{(u,v)}(k) = \max_{v+k \leq x \leq u} F_{j}(x), \quad \forall k \in \{1, \ldots, u - v\}, \forall (u, v) \in \{(u, v) \in \mathbb{N}^{2} \text{ s.t. } 0 \leq v < u \leq n\}. $$  

It is critical to note that we assumed $F_{j}(1) = 1$, for $j = 1, \ldots, m$ in the derivation of this program.
By including the functions $F_j$, $j = 1, \ldots, m$, as decision variables in the above linear program, we obtain a (not necessarily convex) program for minimizing the lower bound on PoS($T$). We can then write the following $m$ programs (one for each $b_j$) for computing the minimum lower bound on the price of stability achievable for a maximum allowable price of anarchy PoA$^*$ greater than or equal to the minimum achievable price of anarchy in $G$, where we include the price of anarchy constraints from the linear program in Equation (6):

$$\max_{F, \nu^{-1}, \gamma} \gamma \quad \text{subject to:}$$

$$\nu^{-1} \geq \frac{1}{\text{PoA}^* \nu^{-1}}, \quad F(1) = 1,$$
$$\nu^{-1} b_j(y) y - \rho \nu^{-1} b_j(x)x + (x - z)F(x) - (y - z)F(x + 1) \geq 0, \quad \forall (x, y, z) \in I(n),$$
$$\gamma b_j(u) u \leq b_j(v) v + \sum_{k=1}^{u-v} F(u,v)(k), \quad \forall (u, v) \in \{(u, v) \in \mathbb{N}^2 \text{ s.t. } 0 \leq v < u \leq n\},$$
$$F(u,v)(k) = \max_{v + k \leq x \leq u} F(x), \quad \forall k \in \{1, \ldots, u - v\}, \forall (u, v) \in \{(u, v) \in \mathbb{N}^2 \text{ s.t. } 0 \leq v < u \leq n\}.$$ (25)

Let $F_j^{\text{opt}}, \nu_j^{\text{opt}}, \rho_j^{\text{opt}}, \gamma_j^{\text{opt}}$ be the optimal values that solve the above program. The price of anarchy achieved by the corresponding local incentive mechanism is $\text{PoA}(T^{\text{opt}}) = \max_j \{1/\rho_j^{\text{opt}}\}$ and the resulting optimal lower bound on the price of stability is $\text{PoS}(T^{\text{opt}}) \geq \max_j \{1/\gamma_j^{\text{opt}}\}$.

Note that the above program is not a convex program because of the equality constraints for the values $F(u,v)(k)$. We show here that solving the above program is equivalent to the problem of maximizing $\sum_{x=1}^{u} F(x)$ subject to the price of anarchy constraints. First, for each basis function $b_j$, observe that maximizing the value of $\gamma$ is equivalent to maximizing the sum over values $F(u,v)(k)$, for $k = 1, \ldots, u - v$, for all $(u, v)$ such that $0 \leq v < u \leq n$. Second, observe that the upper bound on $F(x + 1)$ imposed by $F(x)$ and $\rho$ in the constraints corresponding to the price of anarchy is increasing in the value of $F(x)$, i.e.,

$$(y - z)F(x + 1) \leq (x - z)F(x) - \rho \nu^{-1} b_j(x)x + \nu^{-1} b_j(y)y, \quad \forall (y, z) \in \{(y, z) \in \mathbb{N}^2 \text{ s.t. } (x, y, z) \in I(n)\},$$

since it always holds that $z \leq \min\{x, y\}$ in the definition of $I(n)$. Thus, maximizing the value of $F(x)$ corresponds with maximizing the value of $F(x)$. Finally, maximizing the sum over values $F(u,v)(k)$, for $k = 1, \ldots, u - v$, for all $(u, v)$ such that $0 \leq v < u \leq n$, is equivalent to maximizing the sum over values $\max_{v + k \leq x \leq u} F(x)$, by definition. We showed above that maximizing the feasible value of any given decision variable $F(\hat{x})$, $1 \leq \hat{x} \leq n$, corresponds with maximizing the values of $F(x)$, for $x = 1, \ldots, \hat{x} - 1$. Thus, maximizing $\max_{v + k \leq x \leq u} F(x)$, for $k = 1, \ldots, u - v$, for all $(u, v)$ such that $0 \leq v < u \leq n$, is equivalent to maximizing $F(k)$, $k = 1, \ldots, n$, which is equivalent to maximizing $\sum_{k=1}^{u} F(k)$. It follows that the nonconvex program above yields the same solution as the linear program in Equation (10), concluding the proof.

### D Proof of Theorem 6

It is important to note that $F^\infty$ is a nondecreasing function by Constraint (16) and by its definition in Equation (11). Furthermore, there is always a feasible point in the bilinear program since PoA$^*$ is greater than or equal to the minimum achievable price of anarchy in $G$ and because monomials of order $d \geq 1$ are all convex and nondecreasing. One feasible point corresponds with $\kappa = 0$, $\nu = 1$, and $F$ and $\rho$ solving the linear program in Proposition 1 for $n$ users. Observe that all the constraints in the bilinear program are satisfied because the function $F$ is unique and nondecreasing.
by Theorem 3, $\rho \geq 1/\text{PoA}^*$ since $1/\rho$ is the minimum achievable price of anarchy (which is strictly greater than 1 for polynomials $[5]$) and because the incentive $T(x^d)$ must be lower (pointwise) than the marginal contribution. The last statement can be shown by virtue of our result on the best achievable lower bound on the price of stability in Theorem 5 which showed that the lower bound decreases for larger $F$. Since the price of stability of $T$ according to the lower bound will be $1/\rho > 1$ for $\bar{n} \geq 2$ and the price of stability of marginal contribution is 1, our statement must hold.

The following inequalities are useful for the proof:

Observe that, for any $x \geq \bar{n}/2$, it holds that

$$(x + 1)^d = \sum_{k=0}^{d} \binom{d}{k} x^{d-k} \leq x^d + x^{d-1} \sum_{k=1}^{d} \binom{d}{k} \left(\frac{\bar{n}}{2}\right)^{k-1} = x^d + x^{d-1} \bar{n} \left[\left(\frac{2}{\bar{n}} + 1\right)^d - 1\right].$$  \tag{26}$$

We will also use the following two inequalities for sums of polynomials with $d \geq 1$ and $x \geq y > 0$:

$$\sum_{k=y+1}^{x} k^d \geq \frac{1}{d + 1}(x^{d+1} - y^{d+1}) + \frac{1}{2}(x^d - y^d)$$ \tag{27}$$

$$\sum_{k=y+1}^{x} k^d \leq (x-y)x^d \leq x^{d+1} - y^{d+1}. \tag{28}$$

The remainder of the proof is divided into two parts as in the informal outline in Section 4.3:

- **Upper bound on the price of anarchy:** Consider the following linear program for computing the price of anarchy for any arbitrary number of users $n$ given a nondecreasing function $F$:

  maximize $\rho$ subject to:

  $y^{d+1} - px^{d+1} + \nu [xF(x) - yF(x + 1)] \geq 0, \quad \forall x, y \in \{0, \ldots, n\}$ s.t. $1 \leq x + y \leq n,$

  $y^{d+1} - px^{d+1} + \nu [(n-y)F(x) - (n-x)F(x + 1)] \geq 0, \quad \forall x, y \in \{0, \ldots, n\}$ s.t. $x + y > n.$ \tag{29}$$

First, we show that the above linear program is identical to the linear program in Equation (6) for $F$ nondecreasing. Observe that the constraints from the linear program in Section 4.1 are

$$y^{d+1} - \gamma x^{d+1} + (x-z)F(x) - (y-z)F(x+1)$$

$$= y^{d+1} - \gamma x^{d+1} + xF(x) - yF(x+1) + z[F(x+1) - F(x)].$$

Since $F(x+1) - F(x) \geq 0$, the above expression is minimized for the smallest value of $z$. If follows that $z = 0$ when $x + y \leq n$ and $z = x + y - n$ when $x + y > n$, since the triplets $(x, y, z) \in \mathcal{I}(n)$ satisfy $1 \leq x + y - z \leq n$ and $z \leq \min\{x, y\}$. Thus, for $x, y \in \{0, \ldots, n\}$, $x - z = x$ and $y - z = y$ when $x + y \leq n$, while $x = n - y$ and $y - z = n - x$ when $x + y > n$.

We show that the values $(\rho, \nu) = (\rho^\infty, \nu^\text{opt})$ are feasible in the above linear program for $F = F^\infty$ as defined in the claim and arbitrary $n$. We dispense with the case where $x = 0$ as, in this case, the strictest constraint on $F^\infty(1) = F^\text{opt}(1)$ is at $(x, y) = (0, 1)$, which is already included in Constraints (20). We first consider the constraints with $1 \leq x + y \leq n$.

- In the region where $1 \leq x < \bar{n}/2$ and $0 \leq y \leq \bar{n}/2$, observe that $x + y < \bar{n}$, $F^\infty(x) = F^\text{opt}(x)$ and $F^\infty(x + 1) = F^\text{opt}(x + 1)$. Then, the values $(\rho^\infty, \nu^\text{opt})$ are feasible in the linear program in Equation (29) for $F = F^\infty$ because $(F^\text{opt}, \rho^\text{opt}, \nu^\text{opt})$ satisfy Constraints (20), $F^\infty(x) = F^\text{opt}(x)$ for $x \leq \bar{n}/2$ and $\rho^\infty \leq \rho^\text{opt}$.
Observe that, in the region where $1 \leq x < \bar{n}/2$ and $y > \bar{n}/2$, the constraint are less strict than when $y = \bar{n}/2$ if it holds that

$$\left(\frac{\bar{n}}{2}\right)^{d+1} - \frac{\bar{n}}{2}^{\nu + 1} F^{\text{opt}}(x + 1) \leq y^{d+1} - y^{\nu + 1} F^{\text{opt}}(x + 1)$$

$$\iff \frac{y^{d+1} - (\bar{n}/2)^{d+1}}{y - \bar{n}/2} \geq \nu^{\text{opt}} F^{\text{opt}}(x + 1).$$

The left-hand side of the last line is minimized for $y = \bar{n}/2 + 1$ by convexity and is most constraining for $x = \bar{n}/2 - 1$ since $F^{\text{opt}}$ is nondecreasing. Observe that this condition on $F^{\text{opt}}(\bar{n}/2)$ holds by Constraint (17).

Consider the region where $x \geq \bar{n}/2$ and $y \geq 0$. In this scenario, the constraints read as

$$y^{d+1} - \rho^\infty x^{d+1} + \beta^{\nu^{\text{opt}}}[x^{d+1} - (x - 1)^{d+1}] x - \beta^{\nu^{\text{opt}}}[x^{d+1} - x^{d+1}] y \geq 0.$$

Observe that the left-hand side in the above is convex in $y$ and that it is minimized over the nonnegative reals $y \geq 0$ at $\bar{y} = [\beta^{\nu^{\text{opt}}}[x^{d+1} - x^{d+1}] / (d + 1)]^{1/d}$. Thus, it is sufficient to show that the following holds:

$$\rho^\infty \leq \beta^{\nu^{\text{opt}}} \left[ x - \frac{(x - 1)^{d+1}}{x^d} \right] - d \frac{1}{(d + 1)^{1 + \frac{1}{d}}} \beta^{\nu^{\text{opt}}}[x^{d+1} - x^{d+1}]^{1 + \frac{1}{d}} \frac{1}{x^{d+1}}$$

$$\iff \rho^\infty \leq \beta^{\nu^{\text{opt}}} x \left[ 1 - \left( 1 - \frac{1}{x} \right)^{d+1} \right] - d \frac{\beta^{\nu^{\text{opt}}} \bar{n}}{d + 1} \left( \left( 1 - \frac{\bar{n}}{2} \right)^{d+1} - 1 \right)^{1 + \frac{1}{d}},$$

where the implication holds by Identity (26). The above inequality is strictest for $x = \bar{n}/2$ and is satisfied by the definition of $\rho^\infty$ in Equation (11).

Note that, in the above, we have shown that for any $x, y \geq 0$, the linear program constraints corresponding with $1 \leq x + y \leq n$ are satisfied, without ever explicitly using the fact that $1 \leq x + y \leq n$. Here, we show that $(\rho^\infty, \nu^{\text{opt}})$ is feasible for the constraints with $x + y > n$ by observing that these are less strict than the constraints we have already shown to be satisfied. Observe that this amounts to showing that

$$\nu x F^\infty(x) - \nu y F^\infty(x + 1) \leq \nu(n - y) F^\infty(x) - \nu(n - x) F^\infty(x + 1) \iff \nu(x + y - n) [F(x) - F(x + 1)] \leq 0.$$

This must hold, since $\nu \geq 0$, $x + y - n > 0$ and $F^\infty$ is nondecreasing.

- **Upper bound on the price of stability**: We continue by proving that $(\gamma, \nu, \kappa) = (\gamma^\infty, 1, \kappa^{\text{opt}})$ are feasible in the following linear program for $F = F^\infty$ as defined in the claim and arbitrary $n$:

$$\begin{align*}
\text{maximize} & \quad \gamma \\
\text{subject to:} & \quad y^{d+1} - x^{d+1} + \nu [xF(x) - yF(x + 1)] + \kappa \left[ \sum_{k=1}^{x} F(k) - \sum_{k=1}^{y} F(k) \right] \geq 0, \forall (x, y) \in I_{\leq n}(n), \\
& \quad y^{d+1} - x^{d+1} + \nu [(n - y)F(x) - (n - x)F(x + 1)] + \kappa \left[ \sum_{k=1}^{x} F(k) - \sum_{k=1}^{y} F(k) \right] \geq 0, \forall (x, y) \in I_{> n}(n).
\end{align*}$$

where $I_{\leq n}(n)$ and $I_{> n}(n)$ are the sets of pairs $x, y \in \{0, \ldots, n\}$ such that $x + y \leq n$ and $x + y > n$, respectively. Following an identical set of arguments as the ones we used to show the equivalence of the linear programs in Equation (6) and Equation (29) for $F$ nondecreasing, one can verify that the linear program in Section 4.1 is identical to the above linear program for $F$ nondecreasing.
We first show that that $(\gamma, \nu, \kappa) = (\gamma^\infty, 1, \kappa^{\text{opt}})$ is feasible for constraints with $1 \leq x + y \leq n$.

– In the region where $0 \leq x < \bar{n}/2$ and $0 \leq y \leq \bar{n}/2$, observe that $x + y < \bar{n}$, $F^\infty(x) = F^{\text{opt}}(x)$ and $F^\infty(x + 1) = F^{\text{opt}}(x + 1)$. Then, the values $(\gamma^\infty, 1, \kappa^{\text{opt}})$ are feasible in the linear program in Equation (30) because $(F^{\text{opt}}, \gamma^{\text{opt}}, 1, \kappa^{\text{opt}})$ satisfy Constraints (21) and $\gamma^\infty \leq \gamma^{\text{opt}}$.

– In the setting where $x = 0$ and $y \geq \bar{n}/2$, the following must hold:

$$(1 - \beta \kappa^{\text{opt}})y^{d+1} - F^{\text{opt}}(1)y - \kappa^{\text{opt}} \sum_{k=1}^{\bar{n}/2} F^{\text{opt}}(k) + \beta \kappa^{\text{opt}} \left( \frac{\bar{n}}{2} \right)^{d+1} \geq 0.$$ 

Since $1 - \beta \kappa^{\text{opt}} > 0$, it follows that the left-hand side is convex and is minimized over nonnegative real values $y \geq \bar{n}/2$ by

$$\hat{y} = \max \left\{ \frac{\bar{n}}{2}, \left[ \frac{F^{\text{opt}}(1)}{(d + 1)(1 - \beta \kappa^{\text{opt}})} \right]^{1/d} \right\}.$$ 

By Constraint 18, it holds that $\hat{y} = \bar{n}/2$, and the corresponding linear program condition is covered in Constraint 21.

– Consider the scenario where $1 \leq x \leq \bar{n}/2 - 1$ and $y \geq \bar{n}/2 + 1$, where we require that

$$(1 - \beta \kappa^{\text{opt}})y^{d+1} - \gamma^\infty x^{d+1} + F^{\text{opt}}(x)x - F^{\text{opt}}(x + 1)y - \kappa^{\text{opt}} \sum_{k=x+1}^{\bar{n}/2} F^{\text{opt}}(k) + \beta \kappa^{\text{opt}} \left( \frac{\bar{n}}{2} \right)^{d+1} \geq 0.$$ 

Observe that the left-hand side is convex since $1 - \beta \kappa^{\text{opt}} > 0$ by Condition (19) and is minimized for nonnegative real values $y \geq \bar{n}/2 + 1$ at

$$\hat{y} = \max \left\{ \frac{\bar{n}}{2} + 1, \left[ \frac{F^{\text{opt}}(x + 1)}{(d + 1)(1 - \beta \kappa^{\text{opt}})} \right]^{1/d} \right\}.$$ 

Observe that the resulting linear program conditions are satisfied for $y = \hat{y}$ and for all $1 \leq x \leq \bar{n}/2$ since $\gamma^\infty \leq \gamma_1$, for $\gamma_1$ as defined in Equation (12).

– We now consider the setting where $x \geq \bar{n}/2$ and $0 \leq y < \bar{n}/2$. Here we require:

$$\gamma^\infty x^{d+1} \leq y^{d+1} + \beta[x^{d+1} - (x - 1)^{d+1}]x - \beta[(x + 1)^{d+1} - x^{d+1}]y$$

$$+ \kappa^{\text{opt}} \sum_{k=y+1}^{\bar{n}/2} F^{\text{opt}}(k) + \beta \kappa^{\text{opt}} \left( \frac{\bar{n}}{2} - 1 \right)^{d+1}.$$ 

Observe that the resulting linear program constraints are satisfied for all $0 \leq y < \bar{n}/2$ since $\gamma^\infty \leq \gamma_2$ as defined in Equation (13).

– For $x, y \geq \bar{n}/2$, we require:

$$\gamma^\infty x^{d+1} \leq y^{d+1} + \beta[x^{d+1} - (x - 1)^{d+1}]x - \beta[(x + 1)^{d+1} - x^{d+1}]y + \beta \kappa^{\text{opt}} (x^{d+1} - y^{d+1}).$$

The left-hand side is convex in $y$ as $1 - \beta \kappa^{\text{opt}} > 0$ and is minimized over the nonnegative reals $y \geq \bar{n}/2$ by

$$\hat{y} = \max \left\{ \frac{\bar{n}}{2}, \left[ \frac{\beta[(x + 1)^{d+1} - x^{d+1}]}{(d + 1)(1 - \beta \kappa^{\text{opt}})} \right]^{1/\beta} \right\}.$$ 

The resulting linear program constraints are satisfied as $\gamma^\infty \leq \gamma_3$ as defined in Equation (14).

Observe that in the above, we have not explicitly used the fact that $1 \leq x + y \leq n$. Thus, as we did for the upper bound on the price of anarchy, here we prove that the constraints with $x + y > n$
are less strict that the constraints with $1 \leq x + y \leq n$ for $(\gamma^{\text{opt}}, 1, \kappa^{\text{opt}})$. In fact, this amounts to showing once more that

$$\nu x F^\infty(x) - \nu y F^\infty(x+1) \leq \nu(n-y) F^\infty(x) - \nu(n-x) F^\infty(x+1) \iff \nu(x+y-n)[F(x) - F(x+1)] \leq 0.$$ 

This must hold, since, $\nu = 1 > 0$, $x + y - n > 0$ and $F^\infty$ is nondecreasing.