BLOW-UP PHENOMENA FOR THE LIOUVILLE EQUATION WITH A SINGULAR SOURCE OF INTEGER MULTIPLICITY

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Abstract. We are concerned with the existence of blowing-up solutions to the following boundary value problem

\[-\Delta u = \lambda a(x) e^u - 4\pi N\delta_0 \quad \text{in} \ \Omega, \quad u = 0 \ \text{on} \ \partial\Omega,\]

where \(\Omega\) is a smooth and bounded domain in \(\mathbb{R}^2\) such that \(0 \in \Omega\), \(a(x)\) is a positive smooth function, \(N\) is a positive integer and \(\lambda > 0\) is a small parameter. Here \(\delta_0\) defines the Dirac measure with pole at \(0\). We find conditions on the function \(a\) and on the domain \(\Omega\) under which there exists a solution \(u_\lambda\) blowing up at \(0\) and satisfying \(\lambda \int_\Omega a(x) e^{u_\lambda} \to 8\pi(N + 1)\) as \(\lambda \to 0^+\).

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1. Introduction

Let \(\Omega\) be a bounded domain in \(\mathbb{R}^2\) with a smooth boundary containing the origin. In this paper we consider the following Liouville equation with Dirac mass measure

\[
\begin{cases}
  -\Delta u = \lambda a(x) e^u - 4\pi N\delta_0 & \text{in} \ \Omega, \\
  u = 0 & \text{on} \ \partial\Omega.
\end{cases}
\]

(1.1)

Here \(\lambda\) is a positive small parameter, \(\delta_0\) denotes Dirac mass supported at \(0\), \(a\) is a smooth function satisfying \(\inf_{\Omega} a(x) > 0\) and \(N\) is a positive integer.

Problem (1.1) is motivated by its links with the modeling of physical phenomena. In particular, (1.1) arises in the study of vortices in a planar model of Euler flows (see [10], [29]). In vortex theory the interest in constructing blowing-up solutions is related to relevant physical properties, in particular the presence of vortices with a strongly localised electromagnetic field.

The asymptotic behaviour of family of blowing up solutions can be referred to the papers [1], [6], [18], [19], [21], [23] for the regular problem, i.e. when \(N = 0\). An extension to the singular case \(N > 0\) is contained in [2], [3].

The analysis of the blowing-up behaviour at points away from \(0\) actually is very similar to the asymptotic analysis arising in the regular case which has been pursued with success and, at the present time, an accurate description of the concentration phenomenon is available. Precisely, the analysis in the above works yields that if \(u_\lambda\) is an unbounded family of solutions of (1.1) for which \(\lambda \int_\Omega a(x) e^{u_\lambda}\) is uniformly bounded and \(u_\lambda\) is uniformly bounded in a neighborhood of \(0\), then, up to a subsequence, there is an integer \(m \geq 1\) such that

\[
\lambda \int_\Omega a(x) e^{u_\lambda} dx \to 8\pi m \text{ as } \lambda \to 0^+.
\]

(1.2)
Moreover there are points $\xi_1^\lambda, \ldots, \xi_m^\lambda \in \Omega$ which remain uniformly distant from the boundary $\partial \Omega$, from 0 and from one another such that

$$\lambda a(x)e^{u_\lambda} - 8\pi \sum_{j=1}^m \delta_{\xi_j^\lambda} \to 0$$

in the measure sense. Also the location of the blowing-up points is well understood when concentration occurs away from 0. Indeed, in [21] and [23] it is established that the $m$-tuple $(\xi_1^\lambda, \ldots, \xi_m^\lambda)$ converges, up to a subsequence, to a critical point of the functional

$$\frac{1}{2} \sum_{j=1}^m H(\xi_j, \xi_j) + \frac{1}{2} \sum_{j,h=1 \atop j \neq h}^m G(\xi_j, \xi_h) - \frac{N}{2} \sum_{j=1}^m G(\xi_j, 0).$$

Here $G(x, y)$ is the Green’s function of $-\Delta$ over $\Omega$ under Dirichlet boundary conditions and $H(x, y)$ denotes its regular part:

$$H(x, y) = G(x, y) - \frac{1}{2\pi} \log \frac{1}{|x - y|}.$$ 

The above description of blowing-up behaviour continues to work if we are in the presence of multiples singularities $\sum_i N_i \delta_{p_i}$ in (1.1), provided that we substitute the term $\frac{N}{2} \sum_j G(\xi_j, 0)$ in (1.4) by $\sum_i \frac{N_i}{2} \sum_j G(\xi_j, p_i)$ in (1.4).

The reciprocal issue, namely the existence of positive solutions with the property (1.3), has been addressed for the regular case $N = 0$ first in [28] in the case of a single point of concentration (i.e. $m = 1$), later generalised to the case of multiple concentration associated to any nondegenerate critical point of the functional (1.4) ([1], [7]) or, more generally, to any topologically nontrivial critical point ([11]-[13]). In particular, for $N = 0$, a family of solutions $u_\lambda$ concentrating at $m$-tuple of points as $\lambda \to 0^+$ has been found in some special cases: for any $m \geq 1$, provided that $\Omega$ is not simply connected ([11]), and for $m \in \{1, \ldots, h\}$ if $\Omega$ is a $h$-dumbell with thin handles ([13]). We mention that functionals similar to (1.4) occur to detect multiple-bubbling solutions in different contexts, see [4], [14], [15], [26] for other related singularly perturbed problems.

In the singular case $N > 0$ solutions which concentrate in the measure sense at $m$ distinct points away from 0 have been built in [11] provided that $m < 1 + N$. This result has been extended in [8] to the case of multiple singular sources: in particular it is showed that, under suitable restrictions on the weights, if several sources exist then the more involved topology generates a large number of blow-up solutions.

We point out that in all the above results concentration occurs at points different from the location of the source. The problem of finding solutions with additional concentration around the source is of different nature. In case they exist, the blowing-up at the singularity provides an additional contribution of $8\pi(1 + N)\delta_0$ in the limit (1.2), see [2], [3], [12], [21], [23]. More precisely the asymptotic analysis in the general case can be formulated as follows: if $u_\lambda$ is an unbounded family of solutions of (1.1) for which $\lambda \int_\Omega a(x)e^{u_\lambda} dx$ is uniformly bounded and $u_\lambda$ is unbounded in any neighborhood of 0, then, up to a subsequence, there is an integer $m \geq 0$ such that

$$\lambda \int_\Omega a(x)e^{u_\lambda} dx \to 8\pi m + 8\pi(N + 1) \quad \text{as } \lambda \to 0^+.$$
Moreover there are $m$ distinct points $\xi_1, \ldots, \xi_m \in \Omega \setminus \{0\}$ such that, up to subsequence,

$$\lambda a(x)e^{u_\lambda} \rightarrow 8\pi \sum_{j=1}^m \delta_{\xi_j} + 8\pi(N+1)\delta_0$$

(1.5)

in the measure sense. We mention that also in this case the analysis can be generalized to any number of sources. Moreover, under some extra assumptions it is possible to define a functional which replaces (1.4) in locating the points $\xi_j$ where the concentration occurs, anyway to avoid technicalities we will not go into any further detail (see [12]).

The question on the existence of solutions to (1.1) concentrating at 0 is far from being completely settled. Indeed only partial results are known: in [12] the construction of solutions concentrating at 0 is carried out provided that $N \in (0, +\infty) \setminus \mathbb{N}$. To our knowledge, the only paper dealing with the case $N \in \mathbb{N}$ is [10], where, for any fixed positive integer $N$, it is proved the existence of a solution to (1.1) with $a = 1$ and $\delta_0$ replaced by $\delta_{p_\lambda}$ for a suitable $p_\lambda \in \Omega$ with $N + 1$ blowing up points at the vertices of a sufficiently tiny regular polygon centered in $p_\lambda$; moreover $p_\lambda$ lies uniformly away from the boundary $\partial \Omega$ but its location is determined by the geometry of the domain in an $\lambda$--dependent way and does not seem possible to be prescribed arbitrarily as in [12].

The case $N \in \mathbb{N}$ is more difficult to treat, and at the same time the most relevant to physical applications. Indeed, in vortex theory the number $N$ represents vortex multiplicity, so that in that context the most interesting case is precisely that in which it is a positive integer. The difference between the case $N \in \mathbb{N}$ and $N \not\in \mathbb{N}$ is analitically essential. Indeed, as usual in problems involving small parameters and concentration phenomena like (1.1), after suitable rescaling of the blowing-up around a concentration point one sees a limiting equation. More specifically, as we will see in Section 2, we can associate to (1.1) the limiting problem of Liouville type (2.4) which will play a crucial role in the construction of solutions blowing up at 0 as $\lambda \rightarrow 0^+$; if $N \in \mathbb{N}$, (2.4) admits a larger class of finite mass solutions with respect to the case $N \not\in \mathbb{N}$ since the family of all solutions extends to one carrying an extra parameter $b \in \mathbb{R}^2$ (see [22]).

In this paper we are interested in finding conditions on the potential $a$ and on the domain $\Omega$ under which there exists a solution $u_\lambda$ blowing up at 0. Even though finding general conditions is a notoriously open issue, our analysis reveals that the interplay between the geometry of the domain, which is described in terms of the Robin function $H(x, x)$, and the potential $a$ plays a crucial role. More specifically our conditions involve the first and the second derivative of $a$ and $H(x, x)$.

Now we pass to provide the exact formulations of our results. In the following we will assume that

$$a \in C(\overline{\Omega}) \cap C^2(\Omega) \quad \text{and} \quad \inf_{\Omega} a(x) > 0.$$

Moreover, after suitably rotating the coordinate system, we may assume that in a small neighborhood of 0 the following expansion holds:

$$a(x) = a(0) + \langle \nabla a(0), x \rangle + \frac{a_{11}x_1^2 + a_{22}x_2^2}{2} + o(|x|^2) \ \text{as} \ x \rightarrow 0,$$

where $a_{ii} = \frac{\partial^2 a}{\partial x_i^2}(0)$. 

[12] 

[10] 

[22] 

[24]
**Theorem 1.1.** Let \( N \geq 2, N \in \mathbb{N} \). Assume that
\[
\nabla a(0) + 4\pi (N+2)a(0)\nabla_x H(0,0) = 0, \quad \Delta a(0) \neq 16\pi^2 (N+2)^2 a(0) |\nabla_x H(0,0)|^2. \tag{1.6}
\]
Then, for \( \lambda \) sufficiently small, the problem (1.1) has a family of solutions \( u_\lambda \) blowing up at 0 as \( \lambda \to 0^+ \). More precisely the following holds:
\[
\lambda a(x)e^{u_\lambda} \, dx \to 8\pi (1+N)\delta_0 \tag{1.7}
\]
in the measure sense. More precisely \( u_\lambda \) satisfies
\[
u_\lambda = 4\pi (N+2)G(x,0) + o(1) \tag{1.8}
\]
away from 0.

In the particular case when 0 is a critical point of the potential \( a \) and of the Robin function, we get the existence of a solution blowing up at 0 provided that \( \Delta a(0) \neq 0 \).

**Corollary 1.2.** Let \( N \geq 2, N \in \mathbb{N} \). Assume that
\[
\nabla a(0) = \nabla_x H(0,0) = 0, \quad \Delta a(0) \neq 0.
\]
Then, for \( \lambda \) sufficiently small the problem (1.1) has a family of solutions \( u_\lambda \) blowing up at 0 as \( \lambda \to 0^+ \). More precisely \( u_\lambda \) satisfies (1.7) - (1.8) of Theorem 1.1.

The case \( N = 1 \) is considered in a separate theorem since the result requires different assumptions; the above result continues to hold in symmetric domains under an additional relation involving the second derivatives of \( a \).

**Theorem 1.3.** Let \( N = 1 \). Assume that \( \Omega \) is \( \ell \)-symmetric for some \( \ell \geq 3 \), i.e.
\[
x \in \Omega \iff e^{i\frac{2\pi}{\ell}} x \in \Omega \tag{1.9}
\]
and, in addition,
\[
\nabla a(0) = 0, \quad \Delta a(0) \neq 0, \quad a_{11} = a_{22}. \tag{1.10}
\]
Then, for \( \lambda \) sufficiently small the problem (1.1) has a family of solutions \( u_\lambda \) blowing up at 0 as \( \lambda \to 0^+ \). More precisely \( u_\lambda \) satisfies (1.7) - (1.8) of Theorem 1.1.

We point out that in symmetric domains the center of symmetry 0 is a critical point of the Robin function, so the condition \( \nabla_x H(0,0) = 0 \) is automatically satisfied. Assumptions (1.9) - (1.10) are obviously satisfied if \( \Omega \) is a ball centered at 0 and \( a \) is a radially symmetric potential with a nondegenerate critical point at 0.

The proofs use singular perturbation methods. Roughly speaking, the first step consists in the construction of an approximate solution, which should turn out to be precise enough. In view of the expected asymptotic behavior, the shape of such approximate solution will resemble a *bubble* of the form (2.5) with a suitable choice of the parameter \( \delta = \delta(\lambda, b) \). Then we look for a solution to (1.1) in a small neighborhood of the first approximation. As quite standard in singular perturbation theory, a crucial ingredient is nondegeneracy of the explicit family of solutions of the limiting Liouville problem (2.4), in the sense that all bounded elements in the kernel of the linearization correspond to variations along the parameters of the family, as established in [10]. This allows us to study the invertibility of the linearized operator associated to the problem (1.1) under suitable orthogonality conditions. Next we introduce an intermediate problem and a fixed point argument will provide a solution for an

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1Here \( \nabla_x H(0,0) \) denotes the gradient of the function \( x \mapsto H(x,0) \) at 0.
auxiliary equation, which turns out to be solvable for any choice of $b$. Finally we test the auxiliary equation on the elements of the kernel of the linearized operator and we find out that, in order to find an exact solution of (1.1), the parameter $b$ should be a zero for a reduced finite dimensional map.

The rest of the paper is organized as follows. Section 2 is devoted to some preliminary results, notation, and the definition of the approximating solution. Moreover, a more general version of Theorems 1.1-1.3 is stated there (see Theorems 2.2-2.3). The error up to which the approximating solution solves problem (1.1) is estimated in Section 3. In Section 4 we prove the solvability of the linearized problem. Section 5 considers the solvability of an auxiliary problem by a contraction argument. Finally, in Section 6, we prove the existence results and we conclude the proof of Theorems 1.1-1.3. In Appendix A and Appendix B we collect some results, most of them well-known, which are usually referred to throughout the paper.

2. Preliminaries and statement of the main results

We are going to provide an equivalent formulation of problem (1.1) and Theorems 1.1-1.3. Indeed, let us observe that, setting $v$ the regular part of $u$, namely

$$v = u + 4\pi(\alpha - 1)G(x, 0), \quad \alpha = N + 1,$$

problem (1.1) is then equivalent to solving the following (regular) boundary value problem

$$\begin{cases} -\Delta v = \lambda V(x)|x|^{2(\alpha - 1)}e^v & \text{in } \Omega \\ v = 0 & \text{on } \partial \Omega, \end{cases}$$  

where $V(x)$ is the new potential

$$V(x) = a(x)e^{-4\pi(\alpha - 1)H(x, 0)}.$$  

Here $G$ and $H$ are Green’s function and its regular part as defined in the introduction. This problem is actually variational. Indeed, let us consider the following energy functional associated with (2.2):

$$J(v) = \frac{1}{2} \int_\Omega |\nabla v|^2 dx - \lambda \int_\Omega V(x)|x|^{2(\alpha - 1)}e^v dx, \quad v \in H^1_0(\Omega).$$

Then the following Moser-Trudinger inequality (20, 27) guarantees that $J$ is of class $C^1(H^1_0(\Omega))$ and solutions of (2.2) correspond to critical points of $J$.

**Lemma 2.1.** There exists $C > 0$ such that for any bounded domain $\Omega$ in $\mathbb{R}^2$

$$\int_\Omega e^{\frac{4\pi u^2}{|\Omega|}} dx \leq C|\Omega| \quad \forall u \in H^1_0(\Omega),$$

where $|\Omega|$ stands for the measure of the domain $\Omega$. In particular, for any $q \geq 1$

$$\|e^u\|_q \leq C^\frac{1}{q} |\Omega|^{\frac{1}{q}} e^{\frac{\alpha}{16\pi q} \|u\|^2} \quad \forall u \in H^1_0(\Omega).$$

Theorems 1.1-1.3 will be a consequence of more general results concerning Liouville-type problem (2.2). In order to provide such results (2.2), we now give a construction of a suitable approximate solution for (2.2). In what follows, we identify $x = (x_1, x_2) \in \mathbb{R}^2$ with $x_1 + ix_2 \in \mathbb{C}$. Moreover, $\langle x_1, x_2 \rangle$ stands for the inner product between the vectors $x_1, x_2 \in \mathbb{R}^2$, whereas $x_1 x_2$ will denote the multiplication of the complex numbers $x_1, x_2$. Clearly $\langle x_1, x_2 \rangle = \text{Re}(x_1 \overline{x_2})$. 

For any $\alpha \in \mathbb{N}$, we can associate to (2.2) a limiting problem of Liouville type which will play a crucial role in the construction of the blowing-up solutions:

$$-\Delta w = |x|^{2(\alpha - 1)}e^w \text{ in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} |x|^{2(\alpha - 1)}e^w \, dx < +\infty. \quad (2.4)$$

A complete classification for solutions of (2.4) is due to [22] and corresponds, in complex notation, to the three-parameter family of functions

$$w^{\alpha}_{\delta,b}(x) := \log \frac{8\alpha^2 \delta^{2\alpha}}{\delta^{2\alpha} + |x^{\alpha} - b|^2} \quad \delta > 0, \ b \in \mathbb{C}. \quad (2.5)$$

The following quantization property holds:

$$\int_{\mathbb{R}^2} |x|^{2(\alpha - 1)}e^{w^{\alpha}_{\delta,b}(x)} \, dx = 8\pi \alpha. \quad (2.6)$$

In the following we agree that

$$W_\lambda = w^{\alpha}_{\delta,b}(x),$$

where the value $\delta = \delta(\lambda, b)$ is defined as:

$$\delta^{2\alpha} := \frac{\lambda}{8\alpha^2} V(0)e^{8\pi \sum_{i=1}^\alpha H(0,\beta_i)}. \quad (2.7)$$

To obtain a better first approximation, we need to modify the functions $W_\lambda$ in order to satisfy the zero boundary condition. Precisely, we consider the projections $PW_\lambda$ onto the space $H^1_0(\Omega)$ of $W_\lambda$, where the projection $P : H^1(\mathbb{R}^N) \to H^1_0(\Omega)$ is defined as the unique solution of the problem

$$\Delta Pv = \Delta v \text{ in } \Omega, \quad Pv = 0 \text{ on } \partial \Omega.$$ 

Let us consider $b$ in a small neighborhood of 0 and let us denote by $\beta_0, \ldots, \beta_{\alpha-1}$ the $\alpha$-roots of $b$, i.e., $\beta_\alpha^\alpha = b$ and $\beta_i \neq \beta_h$ for $i \neq h$. Observe that the function $\sum_{i=0}^{\alpha-1} H(x, \beta_i)$ is harmonic in $\Omega$ and satisfies $\sum_{i=0}^{\alpha-1} H(x, \beta_i) = \frac{1}{2\pi} \log |x^{\alpha} - b|$ on $\partial \Omega$. A straightforward computation gives that for any $x \in \partial \Omega$

$$|PW_\lambda - W_\lambda + \log \left(8\alpha^2 \delta^{2\alpha}\right) - 8\pi \sum_{i=0}^{\alpha-1} H(x, \beta_i)| = \left|W_\lambda - \log \left(8\alpha^2 \delta^{2\alpha}\right) + 4\log |x^{\alpha} - b|\right| \leq C\delta^{2\alpha}.$$ 

Since the expressions considered inside the absolute values are harmonic in $\Omega$, then the maximum principle applies and implies the following asymptotic expansion

$$PW_\lambda = W_\lambda - \log \left(8\alpha^2 \delta^{2\alpha}\right) + 8\pi \sum_{i=0}^{\alpha-1} H(x, \beta_i) + O(\delta^{2\alpha})$$

$$= -2\log \left(\delta^{2\alpha} + |x^{\alpha} - b|^2\right) + 8\pi \sum_{i=0}^{\alpha-1} H(x, \beta_i) + O(\delta^{2\alpha}) \quad (2.8)$$

uniformly for $x \in \bar{\Omega}$ and $b$ in a small neighborhood of 0.

We shall look for a solution to (2.2) in a small neighborhood of the first approximation, namely a solution of the form

$$v_\lambda = PW_\lambda + \phi_\lambda,$$

where the rest term $\phi_\lambda$ is small in $H^1_0(\Omega)$-norm.

We are now in the position to state the main theorems of the paper.
Theorem 2.2. Assume that $\alpha \geq 3$ and hypotheses (1.6) hold. Then, for $\lambda$ sufficiently small there exist $\phi_\lambda \in H^1_0(\Omega)$ and $b = b_\lambda = O(\lambda^{-\frac{\alpha+1}{2\alpha}})$ such that the couple $PW_\lambda + \phi_\lambda$ solves problem (2.2). Moreover, for any fixed $\varepsilon > 0$,  
$$
\|\phi_\lambda\|_{H^1_0(\Omega)} \leq \lambda^{\frac{1}{\alpha} - \varepsilon} \text{ for } \lambda \text{ small enough. (2.9)}
$$

Theorem 2.3. Assume that $\alpha = 2$, and hypotheses (1.9)-(1.10) hold. Then, for $\lambda$ sufficiently small there exist $\phi_\lambda \in H^1_0(\Omega)$ and $b = b_\lambda = O(\lambda^{-\frac{1}{2\alpha}})$ such that the couple $PW_\lambda + \phi_\lambda$ solves problem (2.2). Moreover, for any fixed $\varepsilon > 0$, (2.9) holds.

In the remaining part of the paper we will prove Theorems 2.2-2.3 and at the end of the Section 6 we shall see how Theorems 1.1-1.3 follow quite directly as a corollary according to Proposition 2.5.

Proposition 2.5. The embedding $i_\alpha : H_\alpha(\mathbb{R}^2) \hookrightarrow L_\alpha(\mathbb{R}^2)$ is compact.

Proof. See [17, Proposition 6.1]. □

As commented in the introduction, our proof uses the singular perturbation methods. For that, the nondegeneracy of the functions that we use to build our approximating solution is essential. Next proposition is devoted to the nondegeneracy of the finite mass solutions of the Liouville equation (regular and singular).

Proposition 2.4. For any $\alpha \geq 1$ we will make use of the Hilbert spaces

$$
L_\alpha(\mathbb{R}^2) := \left\{ u \in L^2_{\text{loc}}(\mathbb{R}^2) : \frac{|y|^{\alpha-1}}{1 + |y|^{2\alpha}} \frac{u}{u} \right\}, \text{ and } H_\alpha(\mathbb{R}^2) := \left\{ u \in W^{1,2}_{\text{loc}}(\mathbb{R}^2) : \|\nabla u\|_{L^2(\mathbb{R}^2)} + \frac{|y|^{\alpha-1}}{1 + |y|^{2\alpha}} \frac{u}{u} \right\}
$$

endowed with the norms

$$
\|u\|_{L_\alpha} := \left( \frac{|y|^{\alpha-1}}{1 + |y|^{2\alpha}} \right)^{1/2} \text{ and } \|u\|_{H_\alpha} := \left( \frac{|y|^{\alpha-1}}{1 + |y|^{2\alpha}} + \frac{|y|^{\alpha-1}}{1 + |y|^{2\alpha}} \right)^{1/2}
$$

We denote by $\langle u, v \rangle_{L_\alpha}$ and $\langle u, v \rangle_{H_\alpha}$ the natural scalar product in $L_\alpha$ and in $H_\alpha$, respectively.

$\square$
Proof. In [17, Theorem 6.1] it was proved that any solution \( \phi \) of (2.13) is actually a bounded solution. Therefore we can apply the result in [9] to conclude that
\[
\phi = c_0 \phi_0 + c_1 \phi_1 + c_2 \phi_2
\]
for some \( c_0, c_1, c_2 \in \mathbb{R} \).
\[\square\]

In our estimates throughout the paper, we will frequently denote by \( C > 0, c > 0 \) fixed constants, that may change from line to line, but are always independent of the variable under consideration. We also use the notations \( O(1), o(1), O(\lambda), o(\lambda) \) to describe the asymptotic behaviors of quantities in a standard way.

3. Estimate of the error term

The goal of this section is to provide an estimate of the error up to which the function \( W_\lambda \) solves problem (2.2).

Lemma 3.1. Let \( r > 0 \) be a fixed number. Define
\[
R_\lambda := -\Delta PW_\lambda - \lambda V(x) |x|^{2(\alpha-1)} e^{PW_\lambda} = |x|^{2(\alpha-1)} e^{W_\lambda} - \lambda V(x) |x|^{2(\alpha-1)} e^{PW_\lambda}.
\]
For any fixed \( p \geq 1 \) the following holds
\[
|R_\lambda|_p = O(\lambda^{1-p}).
\]
uniformly for \( |b| \leq r \sqrt{\lambda} \). Consequently, for every fixed \( p \geq 1 \),
\[
\|\lambda V(x) |x|^{2(\alpha-1)} e^{PW_\lambda}\|_p = |||x|^{2(\alpha-1)} e^{W_\lambda}\|_p + o(1) = O(\lambda^{\frac{1-p}{\alpha p}}) \tag{3.1}
\]
uniformly for \( |b| \leq r \sqrt{\lambda} \).

Proof. By (2.8) and the choice of \( \delta \) in (2.7) we derive
\[
\lambda V(x) |x|^{2(\alpha-1)} e^{PW_\lambda} = \frac{\lambda}{8 \alpha^2 \delta^{2\alpha}} V(x) |x|^{2(\alpha-1)} e^{W_\lambda} + 8\pi \sum_{i=0}^{\alpha-1} H(x, \beta_i) + O(\delta^{2\alpha})
\]
\[
= |x|^{2(\alpha-1)} e^{W_\lambda} \frac{V(x)}{V(0)} e^{8\pi \sum_{i=0}^{\alpha-1} (H(x, \beta_i) - H(0, \beta_i))} + O(\delta^{2\alpha}) \tag{3.2}
\]
\[
= |x|^{2(\alpha-1)} e^{W_\lambda} \frac{a(x)}{a(0)} e^{-4\pi(\alpha-1)(H(x,0) - H(0,0)) + 8\pi \sum_{i=0}^{\alpha-1} (H(x, \beta_i) - H(0, \beta_i)) + O(\delta^{2\alpha})}.
\]

According to (B.1) we have
\[
H(x,0) - H(0,0) = \Re \left( \frac{dH}{dx}(0,0) x \right) + O(|x|^2)
\]
whereas, by Lemma [B.1]
\[
\sum_{i=0}^{\alpha-1} (H(x, \beta_i) - H(0, \beta_i)) = \alpha \Re \left( \frac{dH}{dx}(0,0) x \right) + O(|x|^2) + O(|b|^2)
\]
Now if we scale $x$ are linear combinations of the functions $\lambda V$ by which we arrive at

$$\lambda V(x)|x|^{2(\alpha - 1)} e^{PW_\lambda}$$

$$= |x|^{2(\alpha - 1)} e^{W_\lambda} a(x) + 4\pi(\alpha + 1)\text{Re}\left(\frac{d^2 H}{dx^2}(0,0)x\right) + O(|x|^2) + O(|b|^2) + O(\delta^2 \alpha)$$

$$= |x|^{2(\alpha - 1)} e^{W_\lambda} \left(1 + 4\pi(\alpha + 1)\text{Re}\left(\frac{d^2 H}{dx^2}(0,0)x\right) + O(|x|^2) + O(|b|^2) + O(\delta^2 \alpha)\right).$$

Taking into account that $\text{Re}(\frac{d^2 H}{dx^2}(0,0)x) = \langle \nabla_x H(0,0), x \rangle$ and using that

$$\langle \nabla a(0), x \rangle a(0) + 4\pi(\alpha + 1)\text{Re}\left(\frac{d^2 H}{dx^2}(0,0)x\right) = \langle \nabla a(0), x \rangle a(0) + 4\pi(\alpha + 1)\langle \nabla_x H(0,0), x \rangle = 0$$

by assumptions (1.6) and (1.9)-(1.10), we arrive at

$$\lambda V(x)|x|^{2(\alpha - 1)} e^{PW_\lambda} = |x|^{2(\alpha - 1)} e^{W_\lambda} + O(|x|^2) + O(|b|^2) + O(\delta^2 \alpha) |x|^{2(\alpha - 1)} e^{W_\lambda}.$$

Now if we scale $x = \delta y$, recalling that $|b| \leq \sqrt{\lambda} \leq C \delta^\alpha$, we get

$$|x|^{2(\alpha - 1)} e^{W_\lambda} = 8\alpha^2 \delta^2 (1 + |y^\alpha - \delta^{-\alpha} b|^2)^2 = O\left(\frac{1}{\delta^2 (1 + |y|^{2\alpha + 2})}\right).$$

and, similarly,

$$|x|^{2\alpha} e^{W_\lambda} = 8\alpha^2 \frac{|y|^{2\alpha}}{(1 + |y^\alpha - \delta^{-\alpha} b|^2)^2} = O\left(\frac{1}{1 + |y|^{2\alpha}}\right).$$

by which

$$||x|^{2(\alpha - 1)} e^{W_\lambda}||_p = O(\delta^{2 - \frac{1}{p}}) \quad |||x|^{2\alpha} e^{W_\lambda}||_p = O(\delta^{2})$$

The thesis is thus proved.

\[\square\]

4. ANALYSIS OF THE LINEARIZED OPERATOR

According to Proposition 25 by the change of variable $x = \delta y$, we immediately get that all solutions $\psi \in H_0(\mathbb{R}^2)$ of

$$-\Delta \psi = 8\alpha^2 \frac{\delta^{2 \alpha} |x|^{2(\alpha - 1)}}{(\delta^{2 \alpha} + |x^\alpha - \delta^{-\alpha} b|^2)^2} \psi = |x|^{2(\alpha - 1)} e^{W_\lambda} \psi \quad \text{in} \quad \mathbb{R}^2$$

are linear combinations of the functions

$$Z_{\delta,b}^{0}(x) = \frac{\delta^{2 \alpha} - |x^\alpha - \delta^{-\alpha} b|^2}{\delta^{2 \alpha} + |x^\alpha - \delta^{-\alpha} b|^2}, \quad Z_{\delta,b}^{1}(x) = \frac{\delta^{\alpha} \text{Re}(x^\alpha - b)}{\delta^{2 \alpha} + |x^\alpha - \delta^{-\alpha} b|^2}, \quad Z_{\delta,b}^{2}(x) = \frac{\delta^{\alpha} \text{Im}(x^\alpha - b)}{\delta^{2 \alpha} + |x^\alpha - \delta^{-\alpha} b|^2}.$$

We introduce their projections $PZ_{\delta,b}^{j}$ onto $H_0^j(\Omega)$. It is immediate that

$$PZ_{\delta,b}^{0}(x) = Z_{\delta,b}^{0}(x) + 1 + O\left(\delta^{2 \alpha}\right) \quad (4.1)$$

and

$$PZ_{\delta,b}^{j}(x) = Z_{\delta,b}^{j}(x) + O\left(\delta^{\alpha}\right), \quad j = 1, 2 \quad (4.2)$$
uniformly with respect to \( x \in \Omega \) and \( b \) in a small neighborhood of 0.

We agree that \( Z^{j}_{\lambda} := Z^{j}_{\delta,b} \) for any \( j = 0, 1, 2 \), where \( \delta \) is defined in terms of \( \lambda \) and \( b \) according to (2.7).

Let us consider the following linear problem: given \( h \in H^{1}_{0}(\Omega) \), find a function \( \phi \in H^{1}_{0}(\Omega) \) satisfying

\[
\begin{align*}
-\Delta \phi - \lambda V(x)|x|^{2(\alpha-1)}e^{PW_{\lambda}}\phi &= \Delta h \\
\int_{\Omega} \nabla \phi \nabla PZ^{j}_{\lambda} &= 0 \quad j = 1, 2
\end{align*}
\]  

(4.3)

Before going on, we recall the following identities which follow by straightforward computations using Lemma A.1 for every \( \xi \in \mathbb{R}^{2} \)

\[
\begin{align*}
\int_{\mathbb{R}^{2}} |y|^{2(\alpha-1)} \log(1 + |y^{\alpha} - \xi|^{2}) & \frac{1 - |y^{\alpha} - \xi|^{2}}{(1 + |y^{\alpha} - \xi|^{2})^{3}} \, dy = \frac{1}{\alpha} \int_{\mathbb{R}^{2}} \log(1 + |y|^{2}) \frac{1 - |y|^{2}}{(1 + |y|^{2})^{3}} \, dy \\
&= -\frac{\pi}{2\alpha},
\end{align*}
\]

(4.4)

\[
\begin{align*}
\int_{\mathbb{R}^{2}} \frac{|y|^{2(\alpha-1)} - 1 - |y^{\alpha} - \xi|^{2}}{1 + |y^{\alpha} - \xi|^{2}} \, dy &= \frac{1}{\alpha} \int_{\mathbb{R}^{2}} \frac{1 - |y|^{2}}{1 + |y|^{2}} \, dy = 0,
\end{align*}
\]

(4.5)

\[
\begin{align*}
\int_{\mathbb{R}^{2}} \frac{|y|^{2(\alpha-1)}(\text{Re}(y^{\alpha} - \xi))^{2}}{(1 + |y^{\alpha} - \xi|^{2})^{4}} \, dy &= \int_{\mathbb{R}^{2}} \frac{|y|^{2(\alpha-1)}(\text{Im}(y^{\alpha} - b))^{2}}{(1 + |y^{\alpha} - \xi|^{2})^{4}} \, dy \\
&= \frac{1}{2\alpha} \int_{\mathbb{R}^{2}} \frac{|y|^{2}}{1 + |y|^{2}} \, dy = \frac{\pi}{12\alpha}.
\end{align*}
\]

(4.6)

**Proposition 4.1.** Let \( r > 0 \) be fixed. There exist \( \lambda_{0} > 0 \) and \( C > 0 \) such that for any \( \lambda \in (0, \lambda_{0}) \), any \( b \in \mathbb{R}^{2} \) with \( |b| < r\sqrt{\lambda} \) and any \( h \in H^{1}_{0}(\Omega) \), if \( \phi \in H^{1}_{0}(\Omega) \) solves (4.3), then the following holds

\[
\|\phi\| \leq C|\log \lambda||h||.
\]

**Proof.** We argue by contradiction. Assume that there exist sequences \( \lambda_{n} \to 0 \), \( h_{n} \in H^{1}_{0}(\Omega) \), \( |b_{n}| \leq r\sqrt{\lambda_{n}} \) and \( \phi_{n} \in H^{1}_{0}(\Omega) \) which solve (4.3) and

\[
\|\phi_{n}\| = 1, \quad |\log \lambda_{n}|\|h_{n}\| \to 0.
\]

(4.7)

Let \( \delta_{n} > 0 \) be the value associated to \( \lambda_{n} \) according to (2.7). Then we may assume

\[
\delta_{n}^{-\alpha}b_{n} \to b_{0}.
\]

We define \( \tilde{\Omega}_{n} := \frac{\Omega}{\delta_{n}} \) and

\[
\tilde{\phi}_{n}(y) := \begin{cases} 
\phi_{n}(\delta_{n}y) & \text{if } y \in \tilde{\Omega}_{n} \\
0 & \text{if } y \in \mathbb{R}^{2} \setminus \tilde{\Omega}_{n}
\end{cases}
\]

In what follows at many steps of the arguments we will pass to a subsequence, without further notice. We split the remaining argument into five steps.

**Step 1. We will show that**

\( \tilde{\phi}_{n} \) is bounded in \( H_{\alpha}(\mathbb{R}^{2}) \).
It is immediate to check that
\[ \int_{\mathbb{R}^2} |\nabla \tilde{\phi}_n|^2 dy = \int_{\Omega} |\nabla \phi_n|^2 dx \leq 1. \quad (4.8) \]

Next, we multiply the equation in (4.3) by \( \phi_n \); then we integrate over \( \Omega \) to obtain
\[ \lambda_n \int_{\Omega} V(x)|x|^{2(\alpha-1)} e^{P W_{\lambda_n}} \phi_n^2 dx = \int_{\Omega} |\nabla \phi_n|^2 dx + \int_{\Omega} \nabla h_n \nabla \phi_n dx \]
which implies, by (4.7),
\[ \lambda_n \int_{\Omega} V(x)|x|^{2(\alpha-1)} e^{P W_{\lambda_n}} \phi_n^2 dx \leq C. \quad (4.9) \]

So, Lemma 3.1 gives \( \int_{\Omega} |x|^{2(\alpha-1)} e^{W_{\lambda_n}} \phi_n^2 \leq C \) or, equivalently,
\[ \int_{\mathbb{R}^2} \frac{|y|^{2(\alpha-1)}}{(1 + |y^\alpha - \delta_n b_n|^2)^2} \phi_n^2 dy \leq C. \]

Combining this with (4.8), we deduce that \( \tilde{\phi}_n \) is bounded in the space \( H_\alpha(\mathbb{R}^2) \).

**Step 2.** We will show that, for some \( \gamma_0 \in \mathbb{R} \),
\[ \tilde{\phi}_n \to \gamma_0 \frac{1 - |y^\alpha - b_0|^2}{1 + |y^\alpha - b_0|^2} \quad \text{weakly in } H_\alpha(\mathbb{R}^2) \text{ and strongly in } L_\alpha(\mathbb{R}^2). \]

Step 1 and Proposition 2.4 give
\[ \tilde{\phi}_n \to f \quad \text{weakly in } H_\alpha(\mathbb{R}^2) \text{ and strongly in } L_\alpha(\mathbb{R}^2). \]

Let \( \tilde{\psi} \in C_c^\infty(\mathbb{R}^2) \) and set \( \psi_n = \tilde{\psi}(\frac{\cdot}{\alpha}) \in C_c^\infty(\Omega) \), for large \( n \). We multiply the equation in (4.3) by \( \psi_n \), we integrate over \( \Omega \) and we get
\[ \int_{\Omega_n} \nabla \tilde{\phi}_n \nabla \tilde{\psi} dy - \lambda_n \int_{\Omega} V(x)|x|^{2(\alpha-1)} e^{P W_{\lambda_n}} \phi_n \psi_n dx = - \int_{\Omega} \nabla h_n \nabla \psi_n dx. \quad (4.10) \]

According to Lemma 3.1 we have
\[ \lambda_n \int_{\Omega} V(x)|x|^{2(\alpha-1)} e^{P W_{\lambda_n}} \phi_n \psi_n dx = \int_{\Omega} |x|^{2(\alpha-1)} e^{W_{\lambda_n}} \phi_n \psi_n dx + o(1) \]
\[ = 8\alpha^2 \int_{\mathbb{R}^2} \frac{|y|^{2(\alpha-1)}}{(1 + |y^\alpha - \delta_n b_n|^2)^2} \tilde{\phi}_n \tilde{\psi} dy + o(1) \]
\[ = 8\alpha^2 \int_{\mathbb{R}^2} \frac{|y|^{2(\alpha-1)}}{(1 + |y^\alpha - b_0|^2)^2} f \tilde{\psi} dy + o(1). \]

Finally, by (4.7), using that \( \int_{\Omega} |\nabla \psi_n|^2 = \int_{\mathbb{R}^2} |\nabla \tilde{\psi}|^2 \),
\[ \int_{\Omega} \nabla h_n \nabla \psi_n dx = O(\|h_n\|) = o(1). \quad (4.11) \]

Therefore, we may pass to the limit in (4.10) to obtain
\[ \int_{\mathbb{R}^2} \nabla f \nabla \tilde{\psi} dy = 8\alpha^2 \int_{\mathbb{R}^2} \frac{|y|^{2(\alpha-1)}}{(1 + |y^\alpha - b_0|^2)^2} f \tilde{\psi} dy. \]
Thus \( f \) solves the equation

\[ -\Delta f = 8\alpha^2 \frac{|y|^{2(\alpha-1)}}{(1 + |y^\alpha - b_0|^2)^2} f. \]

Proposition 2.5 gives

\[ f = \gamma_0 Z_0 + \gamma_1 Z_1 + \gamma_2 Z_2 \]

for some \( \gamma_0, \gamma_1, \gamma_2 \in \mathbb{R} \). It remains to show that \( \gamma_1 = \gamma_2 = 0 \). Indeed, we compute

\[ 0 = \int_\Omega \nabla \phi_n \nabla PZ^{1}_{\lambda_n} \, dx = \int_\Omega |x|^{2(\alpha-1)} e^{W_{\lambda_n} \phi_n} Z^{1}_{\lambda_n} \, dx = 8\alpha^2 \int_{\mathbb{R}^2} \frac{|y|^{2(\alpha-1)}}{(1 + |y^\alpha - \delta_n^{-\alpha} b_n|^2)^2} \tilde{\phi}_n Z_1 \, dy \]

\[ = 8\alpha^2 \int_{\mathbb{R}^2} \frac{|y|^{2(\alpha-1)}}{(1 + |y^\alpha - b_0|^2)^2} Z_1 \, dy + o(1) = \int_{\mathbb{R}^2} \nabla f \nabla Z_1 \, dy + o(1). \]

We get \( \int_{\mathbb{R}^2} \nabla f \nabla Z_1 = 0 \), by which, taking into account that \( \int_{\mathbb{R}^2} \nabla Z_1 \nabla Z_0 = \int_{\mathbb{R}^2} \nabla Z_1 \nabla Z_2 = 0 \),

\[ \gamma_1 \int_{\mathbb{R}^2} |\nabla Z_1|^2 \, dy = 0. \]

So \( \gamma_1 = 0 \) and, similarly, \( \gamma_2 = 0 \).

**Step 3.** We will show that

\[ \int_{\mathbb{R}^2} \frac{|y|^{2(\alpha-1)}}{(1 + |y^\alpha - \delta_n^{-\alpha} b_n|^2)^2} \tilde{\phi}_n \, dy = o\left(\frac{1}{\log \lambda_n}\right). \]

We multiply the equation in (4.3) by \( PZ^{0}_{\lambda_n} \), we integrate over \( \Omega \) and we get

\[ \int_\Omega \nabla \phi_n \nabla PZ^{0}_{\lambda_n} \, dx - \lambda_n \int_\Omega V(x)|x|^{2(\alpha-1)} e^{PW_{\lambda_n} \phi_n} PZ^{0}_{\lambda_n} \, dx = - \int_\Omega \nabla h_n \nabla PZ^{0}_{\lambda_n} \, dx. \]  

(4.13)

We are now concerned with the estimates of each term of the above expression. First, we compute

\[ \int_\Omega \nabla \phi_n \nabla PZ^{0}_{\lambda_n} \, dx = \int_\Omega |x|^{2(\alpha-1)} e^{W_{\lambda_n} \phi_n} Z^{0}_{\lambda_n} \, dx. \]  

(4.14)

Using Lemma 3.1 (with \( p = 2 \)) and (4.1), we obtain

\[ \lambda_n \int_\Omega V(x)|x|^{2(\alpha-1)} e^{PW_{\lambda_n} \phi_n} PZ^{0}_{\lambda_n} \, dx = \int_\Omega |x|^{2(\alpha-1)} e^{W_{\lambda_n} \phi_n} (Z^{0}_{\lambda_n} + 1) \, dx + O\left(\lambda_n^{\frac{1}{2}}\right) \]

\[ = \int_\Omega |x|^{2(\alpha-1)} e^{W_{\lambda_n} \phi_n} Z^{0}_{\lambda_n} \, dx + 8\alpha^2 \int_{\mathbb{R}^2} \frac{|y|^{2(\alpha-1)}}{(1 + |y^\alpha - \delta_n^{-\alpha} b_n|^2)^2} \tilde{\phi}_n \, dy + O\left(\lambda_n^{\frac{1}{2}}\right). \]  

(4.15)

Finally, since \( PZ^{0}_{\lambda} = O(1) \), we have \( \int_\Omega |\nabla PZ^{0}_{\lambda}|^2 \leq \|h_n\| \|PZ^{0}_{\lambda}\| = o\left(\frac{1}{\log \lambda_n}\right) \).

(4.16)

We now multiply (4.13) by \( \log \lambda_n \) and pass to the limit: inserting (4.14), (4.15), (4.16), we obtain the thesis of the step.

**Step 4.** We will show that \( \gamma_0 = 0 \).

We multiply the equation in (4.3) by \( PW_{\lambda_n} \), we integrate over \( \Omega \) and we get

\[ \int_\Omega \nabla \phi_n \nabla PW_{\lambda_n} \, dx - \lambda_n \int_\Omega V(x)|x|^{2(\alpha-1)} e^{PW_{\lambda_n} \phi_n} PW_{\lambda_n} \, dx = - \int_\Omega \nabla h_n \nabla PW_{\lambda_n} \, dx. \]  

(4.17)
Let us estimate each of the terms above. Let us begin with:
\[
\int_{\Omega} \nabla \phi_n \nabla PW_{\lambda_n} dx = \int_{\Omega} |x|^{2(\alpha-1)} e^{W_{\lambda_n}} \phi_n dx = 8\alpha^2 \int_{\mathbb{R}^2} \frac{|y|^{2(\alpha-1)}}{(1 + |y^\alpha - \delta_n^{-\alpha} b_n|^2)^2} \tilde{\phi}_n dy = o(1)
\]
by step 3. By Lemma 3.1 and (4.7), using that \( |PW_{\lambda_n}| = O(|\log \lambda_n|) \), we get
\[
\lambda_n \int_{\Omega} V(x)|x|^{2(\alpha-1)} e^{PW_{\lambda_n}} \phi_n PW_{\lambda_n} dx = \int_{\Omega} |x|^{2(\alpha-1)} e^{W_{\lambda_n}} \phi_n PW_{\lambda_n} dx + o(1)
\]
\[
= 8\alpha^2 \int_{\mathbb{R}^2} \frac{|y|^{2(\alpha-1)}}{(1 + |y^\alpha - \delta_n^{-\alpha} b_n|^2)^2} \tilde{\phi}_n PW_{\lambda_n} (\delta_n y) dy + o(1).
\]
Observe that by (2.8)
\[
PW_{\lambda_n}(\delta_n y) = -2 \log(1 + |y^\alpha - \delta_n^{-\alpha} b_n|^2) + 8\pi \alpha H(\delta_n y, 0) - 4\alpha \log \delta_n + O(\sqrt{\lambda_n})
\]
by which
\[
PW_{\lambda_n}(\delta_n y) + 4\alpha \log \delta_n \to -2 \log(1 + |y^\alpha - b_0|^2) + 8\pi \alpha H(0, 0) \text{ uniformly in } \mathbb{R}^2.
\]
Using this convergence in (4.19), and recalling step 2, we obtain
\[
\lambda_n \int_{\Omega} V(x)|x|^{2(\alpha-1)} e^{PW_{\lambda_n}} \phi_n PW_{\lambda_n} dx
\]
\[
= -16\alpha^2 \gamma_0 \int_{\mathbb{R}^2} \log(1 + |y^\alpha - b_0|^2) \frac{|y|^{2(\alpha-1)}}{(1 + |y^\alpha - b_0|^2)^2} 1 - |y^\alpha - b_0|^2 dy
\]
\[
+ 64\pi \alpha^3 H(0, 0) \gamma_0 \int_{\mathbb{R}^2} \frac{|y|^{2(\alpha-1)}}{(1 + |y^\alpha - b_0|^2)^2} 1 - |y^\alpha - b_0|^2 dy
\]
\[
- 32\alpha^3 \log \delta_n \int_{\mathbb{R}^2} \frac{|y|^{2(\alpha-1)}}{(1 + |y^\alpha - \delta_n^{-\alpha} b_n|^2)^2} \tilde{\phi}_n dy + o(1).
\]
Then by step 3, (4.21),
\[
\lambda_n \int_{\Omega} V(x)|x|^{2(\alpha-1)} e^{PW_{\lambda_n}} \phi_n PW_{\lambda_n} dx = 8\alpha^2 \gamma_0 + o(1).
\]
Finally, taking into account that \( PW_{\lambda} = O(|\log \lambda|) \), we have
\[
\int_{\Omega} |\nabla PW_{\lambda_n}|^2 = \int_{\Omega} |x|^{2(\alpha-1)} e^{W_{\lambda_n}} PW_{\lambda} = O(|\log \lambda|),
\]
by which, owing to (4.7),
\[
\int_{\Omega} |\nabla h_n| |\nabla PW_{\lambda_n}| dx \leq \|h_n\| \|PW_{\lambda_n}\| = o(1).
\]
By inserting (4.18), (4.20), (4.21) into (4.17) and passing to the limit we deduce \( \gamma_0 = 0 \).

**Step 5. End of the proof.**

We will show that a contradiction arises. According to Step 2 and Step 4 we have
\[
\tilde{\phi}_n \to 0 \text{ weakly in } H_\alpha(\mathbb{R}^2) \text{ and strongly in } L_\alpha(\mathbb{R}^2).
\]
By Lemma 3.1
\[
\lambda_n \int_{\Omega} V(x)|x|^{2(\alpha-1)} e^{PW_{\lambda_n}} \phi_n^2 dx = \int_{\Omega} |x|^{2(\alpha-1)} e^{W_{\lambda_n}} \phi_n^2 dx + o(1) \leq C\|\tilde{\phi}_n\|_\alpha^2 + o(1) = o(1).
\]
Moreover, by (4.7),
\[ \int_{\Omega} \nabla h_n \nabla \phi_n dx = o(1). \]

We multiply the equation in (4.3) by \( \phi_n \), we integrate over \( \Omega \) and we obtain
\[ \int_{\Omega} |\nabla \phi_n|^2 dx = \lambda_n \int_{\Omega} V(x)|x|^{2(\alpha-1)} e^{PW_{\lambda}} \phi_n^2 dx - \int_{\Omega} \nabla h_n \nabla \phi_n dx = o(1), \]
in contradiction with (4.7).

In addition to (4.3), let us consider the following linear problem: given \( h \in H^1_0(\Omega) \), find a function \( \phi \in H^1_0(\Omega) \) and constants \( c_1, c_2 \in \mathbb{R} \) satisfying
\[
\begin{cases}
- \Delta \phi - \lambda V(x)|x|^{2(\alpha-1)} e^{PW_{\lambda}} \phi = \Delta h + \sum_{j=1,2} c_j Z^j_\lambda |x|^{2(\alpha-1)} e^{W_{\lambda}} \\
\int_{\Omega} \nabla \phi \nabla PZ^j_\lambda dx = 0 \quad j = 1, 2
\end{cases}
\tag{4.22}
\]

In order to solve problem (4.22), we need to establish an a priori estimate analogous to that of Proposition 4.1.

**Proposition 4.2.** Let \( r > 0 \) be fixed. There exist \( \lambda_0 > 0 \) and \( C > 0 \) such that for any \( \lambda \in (0, \lambda_0) \), any \( b \in \mathbb{R}^2 \) with \( |b| < r \sqrt{\lambda} \) and any \( h \in H^1_0(\Omega) \), if \( (\phi, c_1, c_2) \in H^1_0(\Omega) \times \mathbb{R}^2 \) solves (4.3), then the following holds
\[ \|\phi\| \leq C \log \lambda \|h\|. \]

**Proof.** First observe that by (4.2)
\[
\int_{\Omega} \nabla PZ^j_\lambda \nabla PZ^j_\lambda dx = \int_{\Omega} |x|^{2(\alpha-1)} e^{W_{\lambda}} Z^1_\lambda PZ^2_\lambda dx = \int_{\mathbb{R}^2} |x|^{2(\alpha-1)} e^{W_{\lambda}} Z^1_\lambda Z^2_\lambda dx + o(1)
\]
\[ = \int_{\mathbb{R}^2} \nabla PZ^1_\lambda \nabla PZ^2_\lambda dy + o(1) = o(1). \]

Moreover
\[
\|PZ^j_\lambda\|^2 = \int_{\Omega} |x|^{2(\alpha-1)} e^{W_{\lambda}} Z^1_\lambda PZ^j_\lambda dx = \int_{\Omega} |x|^{2(\alpha-1)} e^{W_{\lambda}} (Z^1_\lambda)^2 dx + o(1)
\]
\[ = 8\alpha^2 \int_{\mathbb{R}^2} |y|^{2(\alpha-1)} \frac{|\text{Re}(y^\alpha - \delta^{-\alpha} b)|^2}{(1 + |y^\alpha - \delta^{-\alpha} b|^2)^4} dy + o(1) = \frac{2}{3} \pi \alpha + o(1) \tag{4.24}
\]
where we have used (4.3). Similarly
\[ \|PZ^2_\lambda\|^2 = \frac{2}{3} \pi \alpha + o(1). \tag{4.25} \]

Then, taking into account that \(- \Delta PZ^j_\lambda = |x|^{2(\alpha-1)} e^{W_{\lambda}} Z^j_\lambda\), according to Proposition 4.1 we have
\[ \|\phi\| \leq C \log \lambda (\|h\| + |c_1| + |c_2|). \tag{4.26} \]

Hence it suffices to estimate the values of the constants \( c_j \). We multiply the equation in (4.22) by \( PZ^j_\lambda \) and we find
\[
\int_{\Omega} \phi |x|^{2(\alpha-1)} e^{W_{\lambda}} Z^j_\lambda dx - \lambda \int_{\Omega} V(x) |x|^{2(\alpha-1)} e^{PW_{\lambda}} \phi PZ^j_\lambda dx = \frac{2}{3} \pi \alpha c_1 + o(c_1) + o(c_2) + O(\|h\|). \tag{4.27}
\]
Let us fix $p \in (1, +\infty)$ sufficiently close to 1. Then, by (1.22) and (3.1) we may estimate
\[
\int_{\Omega} |\phi| |x|^{2(\alpha-1)} e^{W_{\lambda}} |PZ_{\lambda}^{1} - Z_{\lambda}| dx \leq C \sqrt{\lambda} \int_{\Omega} |\phi| |x|^{2(\alpha-1)} e^{W_{\lambda}} dx \leq C \sqrt{\lambda} \|\phi\| \| |x|^{2(\alpha-1)} e^{W_{\lambda}}\|_{p}
\]
and, since $PZ_{\lambda}^{1} = O(1)$, using Lemma 3.1,
\[
\int_{\Omega} |\phi||x|^{2(\alpha-1)} e^{W_{\lambda}} - \lambda V(x)|x|^{2(\alpha-1)} e^{PW_{\lambda}}|PZ_{\lambda}^{1}| dx \leq C \int_{\Omega} |\phi||x|^{2(\alpha-1)} e^{W_{\lambda}} - \lambda V(x)|x|^{2(\alpha-1)} e^{PW_{\lambda}}| dx \leq C \lambda^{-\frac{1}{\alpha p}} \|\phi\|.
\]
By inserting the above two estimates into (4.27) we obtain
\[
|c_1| + o(c_2) \leq C \|h\| + C \lambda^{-\frac{1}{\alpha p}} \|\phi\|.
\]
We multiply the equation in (4.22) by $PZ_{\lambda}^{2}$ and, by a similar argument as above, we find
\[
|c_2| + o(c_1) \leq C \|h\| + C \lambda^{-\frac{1}{\alpha p}} \|\phi\|,
\]
and so
\[
|c_1| + |c_2| \leq C \|h\| + C \lambda^{-\frac{1}{\alpha p}} \|\phi\|.
\]
Combining this with (4.26) we obtain the thesis.

\(
\square
\)

5. The nonlinear problem: a contraction argument

In order to solve (1.1), let us consider the following intermediate problem:
\[
\begin{cases}
-\Delta (PW_{\lambda} + \phi) - \lambda V(x)|x|^{2(\alpha-1)} e^{PW_{\lambda}} + \phi = \sum_{j=1,2} c_j Z_{\lambda}^{j}|x|^{2(\alpha-1)} e^{W_{\lambda}}, \\
\phi \in H_{0}^{1}(\Omega), \quad \int_{\Omega} \nabla \phi \nabla PZ_{\lambda}^{j} dx = 0 \quad j = 1, 2.
\end{cases}
\]

Then it is convenient to solve as a first step the problem for $\phi$ as a function of $b$. To this aim, first let us rewrite problem (5.1) in a more convenient way.

For any $p > 1$, let
\[
i_p^s : L^p(\Omega) \rightarrow H_{0}^{1}(\Omega)
\]
be the adjoint operator of the embedding $i_p : H_{0}^{1}(\Omega) \hookrightarrow L^p(\Omega)$, i.e. $u = i_p^s(v)$ if and only if $-\Delta u = v$ in $\Omega$, $u = 0$ on $\partial \Omega$. We point out that $i_p^s$ is a continuous mapping, namely
\[
\|i_p^s(v)\| \leq c_p \|v\|_p, \text{ for any } v \in L^p(\Omega),
\]
for some constant $c_p$ which depends on $\Omega$ and $p$. Next let us set
\[
K := \text{span} \{PZ_{\lambda}^{1}, PZ_{\lambda}^{2}\}
\]
and
\[
K_{\perp} := \left\{ \phi \in H_{0}^{1}(\Omega) : \int_{\Omega} \nabla \phi \nabla PZ_{\lambda}^{1} dx = \int_{\Omega} \nabla \phi \nabla PZ_{\lambda}^{2} dx = 0 \right\}
\]
and denote by
\[
\Pi : H_{0}^{1}(\Omega) \rightarrow K, \quad \Pi_{\perp} : H_{0}^{1}(\Omega) \rightarrow K_{\perp}
\]
the corresponding projections. Let \( L : K^\perp \to K^\perp \) be the linear operator defined by

\[
L(\phi) := \Pi^\perp \left( i^*_p \left( \lambda V(x)|x|^{2(\alpha-1)}e^{PW\lambda \phi} \right) \right) - \phi.
\]

Notice that problem (4.22) reduces to

\[
L(\phi) = \Pi^\perp h, \quad \phi \in K^\perp.
\]

As a consequence of Proposition 4.2, we derive the invertibility of \( L \).

**Proposition 5.1.** Let \( r > 0 \) be a fixed number. For any \( p > 1 \) there exist \( \lambda_0 > 0 \) and \( C > 0 \) such that for any \( \lambda \in (0, \lambda_0) \), any \( b \in \mathbb{R}^2 \) with \( \|b\| < r\sqrt{\lambda} \) and any \( h \in K^\perp \) there is a unique solution \( \phi \in K^\perp \) to the problem

\[
L(\phi) = h.
\]

In particular, \( L \) is invertible; moreover,

\[
\|L^{-1}\| \leq C|\log \lambda|.
\]

**Proof.** Observe that the operator \( \phi \mapsto \Pi^\perp \left( i^*_p \left( \lambda V(x)|x|^{2(\alpha-1)}e^{PW\lambda \phi} \right) \right) \) is a compact operator in \( K^\perp \). Let us consider the case \( h = 0 \) and take \( \phi \in K^\perp \) with \( L(\phi) = 0 \). In other words, \( \phi \) solves the system (4.22) with \( h = 0 \) for some \( c_1, c_2 \in \mathbb{R} \). Proposition 4.2 implies \( \phi = 0 \). Then, Fredholm’s alternative implies the existence and uniqueness result.

Once we have existence, the norm estimate follows directly from Proposition 4.2. \( \Box \)

Now we come back to our goal of finding a solution to problem (5.1). In what follows we denote by \( N : K^\perp \to K^\perp \) the nonlinear operator

\[
N(\phi) = \Pi^\perp \left( \left( i^*_p \left( \lambda V(x)|x|^{2(\alpha-1)}e^{PW\lambda (e^{\phi} - 1)} \right) \right) \right).
\]

Therefore problem (5.1) turns out to be equivalent to the problem

\[
L(\phi) + N(\phi) = \tilde{R}, \quad \phi \in K^\perp
\]

where, recalling Lemma 3.1,

\[
\tilde{R} = \Pi^\perp \left( \left( i^*_p (R_\lambda) \right) \right) = \Pi^\perp \left( PW\lambda - i^*_p \left( \lambda|x|^{2(\alpha-1)}e^{PW\lambda} \right) \right).
\]

We need the following auxiliary lemma.

**Lemma 5.2.** Let \( r > 0 \) be a fixed number. For any \( p > 1 \) there exists \( \lambda_0 > 0 \) such that for any \( \lambda \in (0, \lambda_0) \), any \( b \in \mathbb{R}^2 \) with \( |b| \leq r\sqrt{\lambda} \) and any \( \phi_1, \phi_2 \in H^1_0(\Omega) \) with \( \|\phi_1\|, \|\phi_2\| < 1 \) the following holds

\[
\|e^{\phi_1} - \phi_1 - e^{\phi_2} + \phi_2\|_p \leq C \left( \|\phi_1\| + \|\phi_2\| \right) \|\phi_1 - \phi_2\|, \quad (5.5)
\]

\[
\|N(\phi_1) - N(\phi_2)\| \leq C \lambda^{\frac{1-\frac{2}{p}}{2}} \left( \|\phi_1\| + \|\phi_2\| \right) \|\phi_1 - \phi_2\|. \quad (5.6)
\]

**Proof.** A straightforward computation give that the inequality \( |e^a - a - e^b + b| \leq e^{|a|+|b|}(|a| + |b|)|a-b| \) holds for all \( a, b \in \mathbb{R} \). Then, by applying Hölder’s inequality with \( \frac{1}{q} + \frac{1}{r} + \frac{1}{t} = 1 \), we derive

\[
\|e^{\phi_1} - \phi_1 - e^{\phi_2} + \phi_2\|_p \leq C \|e^{\phi_1+|\phi_2|}\|_p \left( \|\phi_1\|_{pr} + \|\phi_2\|_{pr} \right) \|\phi_1 - \phi_2\|_{pt}
\]

and (5.5) follows by using Lemma 2.1 and the continuity of the embeddings \( H^1_0(\Omega) \subset L^{pr}(\Omega) \) and \( H^1_0(\Omega) \subset L^d(\Omega) \). Let us prove (5.6). According to (5.2) we get

\[
\|N(\phi_1) - N(\phi_2)\| \leq C \|\lambda V(x)|x|^{2(\alpha-1)}e^{PW\lambda (e^{\phi_1} - \phi_1 - e^{\phi_2} + \phi_2)}\|_p.
\]
and by Hölder’s inequality with \( \frac{1}{p} + \frac{1}{q} = 1 \) we derive
\[
\|N(\phi_1) - N(\phi_2)\| \leq C\|\lambda x|^{2(\alpha - 1)}e^{PW}\|_p^2\|e^{\phi_1} - e^{\phi_2} + \phi_2\|_p q
\]
\[
\leq C\|\lambda x|^{2(\alpha - 1)}e^{PW}\|_p^2(\|\phi_1\| + \|\phi_2\|)\|\phi_1 - \phi_2\|
\]
by (5.5), and the conclusion follows recalling (3.1).

Problem (5.1) or, equivalently, problem (5.4), turns out to be solvable for any choice of point \( b \) with \( |b| \leq r\sqrt{\lambda} \), provided that \( \lambda \) is sufficiently small. Indeed we have the following result.

**Proposition 5.3.** Let \( r > 0 \) be fixed. For any \( \varepsilon \in (0, \frac{1}{\alpha}) \) there exists \( \lambda_0 > 0 \) such that for any \( \lambda \in (0, \lambda_0) \) and any \( b \in \mathbb{R}^2 \) with \( |b| < r\sqrt{\lambda} \) there is a unique \( \phi_\lambda = \phi_{\lambda,b} \in K^\perp \) satisfying (5.1) for some \( c_1, c_2 \in \mathbb{R} \) and

\[ \|\phi_\lambda\| \leq \lambda^{\frac{1}{\alpha} - \varepsilon}. \]

**Proof.** Since, as we have observed, problem (5.4) is equivalent to problem (5.1), we will show that problem (5.4) can be solved via a contraction mapping argument. Indeed, in virtue of Proposition 5.1, let us introduce the map

\[ T := L^{-1}(\tilde{R} - N(\phi)), \quad \phi \in K^\perp. \]

Let us fix
\[ 0 < \eta < \min\left\{ \varepsilon, \frac{1}{\alpha} - \varepsilon \right\} \]
and \( p > 1 \) sufficiently close to 1. According to (5.2) and Lemma 3.1 we have

\[ \|\tilde{R}\| = O(\lambda^{\frac{1}{\alpha} - \eta}). \tag{5.7} \]

Similarly, by (5.6), choosing \( p > 1 \) sufficiently close to 1, we get

\[ \|N(\phi_1) - N(\phi_2)\| \leq C\lambda^{-\eta}(\|\phi_1\| + \|\phi_2\|)\|\phi_1 - \phi_2\| \quad \forall \phi_1, \phi_2 \in H^1_0(\Omega), \|\phi_1\|, \|\phi_2\| < 1. \tag{5.8} \]

In particular, by taking \( \phi_2 = 0 \),

\[ \|N(\phi)\| \leq C\lambda^{-\eta}\phi^2 \quad \forall \phi \in H^1_0(\Omega), \|\phi\| < 1. \tag{5.9} \]

We claim that \( T \) is a contraction map over the ball

\[ \left\{ \phi \in K^\perp \left| \|\phi\| \leq \lambda^{\frac{1}{\alpha} - \varepsilon} \right. \right\} \]

provided that \( \lambda \) is small enough. Indeed, combining Proposition 5.1, (5.7), (5.8), (5.9) with the choice of \( \eta \), we have

\[ \|T(\phi)\| \leq C|\log \lambda|(\lambda^{\frac{1}{\alpha} - \eta} + \lambda^{-\eta}\|\phi\|^2) < \lambda^{\frac{1}{\alpha} - \varepsilon}, \]

\[ \|T(\phi_1) - T(\phi_2)\| \leq C|\log \lambda||N(\phi_1) - N(\phi_2)|| \leq C\lambda^{-\eta}|\log \lambda|(\|\phi_1\| + \|\phi_2\|)\|\phi_1 - \phi_2\| \]

\[ < \frac{1}{2}\|\phi_1 - \phi_2\|. \]

\[ \square \]
6. Proof of Theorems 1.1-1.3 and Theorem 2.2-2.3

After problem (5.1) has been solved according to Proposition 5.3, then we find a solution to the original problem (2.2) if \( b \) is such that

\[
c_j = 0 \text{ for } j = 1, 2.
\]

Let us find the condition satisfied by \( b \) in order to get the \( c_j \)'s equal to zero.

**Proof of Theorems 2.2-2.3** We multiply the equation in (5.1) by \( PZ_j^\lambda \) and integrate over \( \Omega \):

\[
\int_\Omega \nabla(PW_\lambda + \phi_\lambda) \nabla PZ_j^\lambda dx - \lambda \int_\Omega V(x)|x|^{2(\alpha - 1)} e^{PW_\lambda + \phi_\lambda} PZ_j^\lambda dx = \sum_{h=1,2} c_h \int_\Omega Z_h(x)|x|^{2(\alpha - 1)} e^{W_\lambda} PZ_j^\lambda dx.
\]

The object is now to expand each integral of the above identity and analyze the leading term. Let us begin by observing that the orthogonality in (5.1) gives

\[
\int_\Omega \nabla \phi_\lambda \nabla PZ_j^\lambda dx = \int_\Omega |x|^{2(\alpha - 1)} e^{W_\lambda} \phi_\lambda Z_j^\lambda dx = 0
\]

and, by (4.23)-(4.24),

\[
\int_\Omega Z_h(x)|x|^{2(\alpha - 1)} e^{W_\lambda} PZ_j^\lambda dx = \int_\Omega \nabla PZ_h^\lambda \nabla PZ_j^\lambda dx = \begin{cases} \frac{2}{3} \pi \alpha + o(1) & \text{if } h = j \\ o(1) & \text{if } h \neq j \end{cases}
\]

Using the expansion (3.2) we get

\[
\int_\Omega \nabla PW_\lambda \nabla PZ_j^\lambda dx - \lambda \int_\Omega V(x)|x|^{2(\alpha - 1)} e^{PW_\lambda} PZ_j^\lambda dx = \int_\Omega |x|^{2(\alpha - 1)} e^{W_\lambda} PZ_j^\lambda dx - \lambda \int_\Omega V(x)|x|^{2(\alpha - 1)} e^{PW_\lambda} PZ_j^\lambda dx = \int_\Omega |x|^{2(\alpha - 1)} e^{W_\lambda} \left( 1 - \frac{a(x)}{a(0)} e^{-4\pi(\alpha - 1)(H(x,0) - H(0,0))} + 8\pi \delta \sum_{i=0}^{\alpha - 1} (H(x,\beta_i) - H(0,\beta_i)) + O(\delta^{2\alpha}) \right) PZ_j^\lambda dx.
\]

Recalling (3.1) and Lemma B.1 we deduce

\[
- 4\pi(\alpha - 1)(H(x,0) - H(0,0)) + 8\pi \sum_{i=0}^{\alpha - 1} (H(x,\beta_i) - H(0,\beta_i))
\]

\[
= 4\pi(\alpha + 1) \sum_{k=1}^{2} \frac{1}{k!} \text{Re} \left( \frac{d^k \tilde{H}(0,0)x^k}{dx^k} \right) + \frac{8\pi}{(\alpha - 1)!} \text{Re} \left( \frac{\partial^{\alpha+1} \tilde{H}(0,0)x}{\partial p^{\alpha} \partial x} \right) + O(|b||x|^2) + O(|b|^2|x|) + O(|x|^3).
\]
Consequently using the Taylor expansion $e^y = 1 + y + \frac{y^2}{2} + O(|y|^3)$,

\[
e^{-4\pi(\alpha-1)(H(x,0) - H(0,0)) + 8\pi \sum_{i=0}^{\alpha-1} (H(x,\beta_i) - H(0,\beta_i)) + O(\delta^{2\alpha})}
\]
\[
= 1 + 4\pi(\alpha + 1) \sum_{k=1}^{2} \frac{1}{k!} \text{Re} \left( \frac{d^k \tilde{H}}{dx^k} (0,0) x^k \right) + \frac{8\pi}{(\alpha - 1)!} \text{Re} \left( \frac{\partial^\alpha \tilde{H}}{\partial p^\alpha \partial x} (0,0) bx \right)
\]
\[
+ \frac{1}{2} \left( 4\pi(\alpha + 1) \sum_{k=1}^{2} \frac{1}{k!} \text{Re} \left( \frac{d^k \tilde{H}}{dx^k} (0,0) x^k \right) \right)^2
\]
\[
+ O(|b||x|^2) + O(|b|^2|x|) + O(|x|^3) + O(\delta^{2\alpha})
\]
\[
= 1 + 4\pi(\alpha + 1) \sum_{k=1}^{2} \frac{1}{k!} \text{Re} \left( \frac{d^k \tilde{H}}{dx^k} (0,0) x^k \right) + \frac{8\pi}{(\alpha - 1)!} \text{Re} \left( \frac{\partial^\alpha \tilde{H}}{\partial p^\alpha \partial x} (0,0) bx \right)
\]
\[
+ 8\pi^2(\alpha + 1)^2 \left( \text{Re} \left( \frac{d\tilde{H}}{dx} (0,0) x \right) \right)^2
\]
\[
+ O(|b||x|^2) + O(|b|^2|x|) + O(|x|^3) + O(\delta^{2\alpha}).
\]

By assumptions (1.6) and (1.9)-(1.10) in Theorems 1.1 and 1.3 respectively, taking into account that $\text{Re} \left( \frac{d\tilde{H}}{dx} (0,0) x \right) = \langle \nabla x H(0,0), x \rangle$, we get

\[
\frac{a(x)}{a(0)} = 1 + \frac{\langle \nabla a(0), x \rangle}{a(0)} + \frac{1}{2a(0)} \left( a_{11} x_1^2 + a_{22} x_2^2 \right) + O(|x|^3)
\]
\[
= 1 - 4\pi(\alpha + 1) \text{Re} \left( \frac{d\tilde{H}}{dx} (0,0) x \right) + \frac{1}{2a(0)} \left( a_{11} (\text{Re} x)^2 + a_{22} (\text{Im} x)^2 \right) + O(|x|^3),
\]

and then we derive

\[
\frac{a(x)}{a(0)} e^{-4\pi(\alpha-1)(H(x,0) - H(0,0)) + 8\pi \sum_{i=0}^{\alpha-1} (H(x,\beta_i) - H(0,\beta_i)) + O(\delta^{2\alpha})}
\]
\[
= 1 + 2\pi(\alpha + 1) \text{Re} \left( \frac{d^2 \tilde{H}}{dx^2} (0,0) x^2 \right) + \frac{8\pi}{(\alpha - 1)!} \text{Re} \left( \frac{\partial^\alpha + 1 \tilde{H}}{\partial p^\alpha \partial x} (0,0) bx \right)
\]
\[
- 8\pi^2(\alpha + 1)^2 \left( \text{Re} \left( \frac{d\tilde{H}}{dx} (0,0) x \right) \right)^2
\]
\[
+ \frac{1}{2a(0)} \left( a_{11} (\text{Re} x)^2 + a_{22} (\text{Im} x)^2 \right) + O(|b||x|^2) + O(|b|^2|x|) + O(|x|^3) + O(\delta^{2\alpha}).
\]

(6.5)

First let us assume that $\alpha \geq 3$: let us insert the above expansion into (6.4) and, using Lemma 6.1 and next Corollary 6.2 we get
\[ \int \nabla PW_\lambda \nabla Z_\lambda^j dx - \lambda \int V(x) |x|^{2(\alpha - 1)} e^{PW_\lambda} P Z_\lambda^j dx \]
\[ \begin{align*}
&= 8\pi^2 (\alpha + 1)^2 \int |x|^{2(\alpha - 1)} e^{W_\lambda} P Z_\lambda^j \left( \text{Re} \left( \frac{d\tilde{H}}{dx}(0,x) \right) \right)^2 dx \\
&\quad - \frac{1}{2a(0)} \int |x|^{2(\alpha - 1)} e^{W_\lambda} P Z_\lambda^j \left( a_{11} (\text{Re} x)^2 + a_{22} (\text{Im} x)^2 \right) dx \\
&\quad + O(\delta^3) + O(\delta^2 |b|) + O(\delta |b|^2) \\
&= \delta^2 \left( 8\pi^2 (\alpha + 1)^2 \left| \frac{d\tilde{H}}{dx}(0,0) \right|^2 - \frac{a_{11} + a_{22}}{2a(0)} \right) \int |x|^{2\alpha} e^{W_\lambda} Z_\lambda^j dx \\
&\quad + O(\delta^3) + O(\delta^2 |b|) + O(\delta |b|^2).
\end{align*} \] (6.6)

We have thus obtained that if \( \alpha \geq 3 \) then
\[ \begin{align*}
\int \nabla PW_\lambda \nabla Z_\lambda^j dx - \lambda \int V(x) |x|^{2(\alpha - 1)} e^{PW_\lambda} P Z_\lambda^j dx \\
&= \alpha^2 F_j (\delta - \alpha b) + O(\delta^3) + O(\delta^2 |b|) + O(\delta |b|^2)
\end{align*} \] (6.7)

where
\[ A := 4\pi^2 (\alpha + 1)^2 \left| \frac{d\tilde{H}}{dx}(0,0) \right|^2 - \frac{a_{11} + a_{22}}{4a(0)} \neq 0 \]

thanks to assumptions \((\text{i.6})\) in Theorem \(\text{i.1}\) and \(F = (F_1, F_2)\) is the map defined in Lemma \(\text{i.6}\).

Next assume that \( \alpha = 2 \). If \( \Omega \) is \( \ell \)-symmetric for some \( \ell \geq 3 \) in the sense of \(\text{i.9}\), then \( \tilde{H}(x,0) \) is 3-symmetric too:
\[ \tilde{H}(e^{\frac{i\pi}{2}} x,0) = \tilde{H}(x,0) \quad \forall x \in \Omega; \]

this implies that its Taylor expansion at 0 involves only the powers corresponding to integers multiples of \( \ell \) and, consequently,
\[ \frac{d\tilde{H}}{dx}(0,0) = \frac{d^2\tilde{H}}{dx^2}(0,0) = 0. \]

Then let us insert \(\text{6.5}\) into \(\text{6.4}\) and, using Lemma \(\text{6.1}\) and next Corollary \(\text{6.3}\) we get for \( j = 1 \)
\[ \begin{align*}
\int \nabla PW_\lambda \nabla Z_\lambda dx - \lambda \int V(x) |x|^{2(\alpha - 1)} e^{PW_\lambda} P Z_\lambda^j dx \\
&= \frac{1}{2a(0)} \int |x|^{2(\alpha - 1)} e^{W_\lambda} P Z_\lambda \left( a_{11} (\text{Re} x)^2 + a_{22} (\text{Im} x)^2 \right) dx + O(\delta^3) + O(\delta^2 |b|) + O(\delta |b|^2) \\
&\quad - \delta^2 \frac{a_{11} + a_{22}}{4a(0)} \int \int |x|^{2\alpha} e^{W_\lambda} Z_\lambda^j dx - \pi \alpha^2 \delta^2 \frac{a_{11} - a_{22}}{a(0)} + O(\delta^3) + O(\delta^2 |b|) + O(\delta |b|^2)
\end{align*} \]
and, similarly for $j = 2$
\[
\int_{\Omega} \nabla P \nabla P^{\lambda} \lambda^{j} dx = \lambda \int_{\Omega} V(x)|x|^{2(\alpha-1)} e^{\lambda P} P^{j} dx = -\frac{2^{(2(\alpha-1) e^{\lambda P} P^{j} dx}
\]
\[
= -\frac{2^{(2(\alpha-1) e^{\lambda P} P^{j} dx + O(\lambda^3) + O(\delta^2 |b|) + O(\delta^2 |b|^2)}
\]

Therefore, using (1.1) we conclude that (6.7) holds for any $\alpha \geq 2$ for some $A \neq 0$.

Finally let us fix $\varepsilon > 0$ sufficiently small and $p > 1$ sufficiently close to 1. Next let $1 < q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, recalling that $\delta^{2\alpha} \sim \lambda$ according to (2.7), (3.9) with $\phi_2 = 0$ and Proposition 3.1 imply
\[
\|e^{\phi_{\lambda}} - 1 - \phi_{\lambda} \|_q \leq C\|\phi_{\lambda}\|^{2} \leq \delta^{4-4\alpha\varepsilon}
\]
and, consequently,
\[
\|e^{\phi_{\lambda}} - 1\|_q \leq C\|\phi_{\lambda}\| \leq \delta^{2-2\alpha\varepsilon}. (6.8)
\]
Therefore, the orthogonality (6.2) and Lemma 3.1 imply
\[
\int_{\Omega} |x|^{2(\alpha-1)} e^{\lambda P} (e^{\phi_{\lambda}} - 1)Z_{\lambda}^{j} dx = \int_{\Omega} |x|^{2(\alpha-1)} e^{\lambda P} (e^{\phi_{\lambda}} - 1 - \phi_{\lambda})Z_{\lambda}^{j} dx
\]
\[
= O(\|\lambda^{2(\alpha-1)} e^{\lambda P} (e^{\phi_{\lambda}} - 1 - \phi_{\lambda})\|_1)
\]
\[
= O(\|e^{\lambda P} |x|^{2(\alpha-1)} \|_p \|e^{\phi_{\lambda}} - 1 - \phi_{\lambda}\|_q)
\]
\[
= O(\delta^{\frac{2}{\lambda} - 2\delta^{4-4\alpha\varepsilon}})
\]
and, by using again Lemma 3.1 and (4.2),
\[
\lambda \int_{\Omega} V(x)|x|^{2(\alpha-1)} e^{\lambda P} (e^{\phi_{\lambda}} - 1)P^{j} dx = \int_{\Omega} |x|^{2(\alpha-1)} e^{\lambda P} (e^{\phi_{\lambda}} - 1 - \phi_{\lambda})Z_{\lambda}^{j} dx + O(\delta^{\frac{2}{\lambda} + 2-2\alpha\varepsilon})
\]
\[
= O(\delta^{\frac{2}{\lambda} - 2\delta^{4-4\alpha\varepsilon}}) + O(\delta^{\frac{2}{\lambda} + 2-2\alpha\varepsilon}) = o(\delta^3)
\]
provided that $\varepsilon$ is chosen sufficiently close to 0 and $p$ sufficiently close to 1.

In order to conclude, combining (6.2), (6.3), (6.7), (6.9), the identities (6.1) turn out to be equivalent to the system
\[
\delta^{2} A(F_{1}(\delta^{\alpha}b) + O(\delta)) = \frac{2}{3} \pi \alpha c_{1} + o(c_{1}) + o(c_{2}),
\]
\[
\delta^{2} A(F_{2}(\delta^{\alpha}b) + O(\delta)) = \frac{2}{3} \pi \alpha c_{2} + o(c_{1}) + o(c_{2})
\]
uniformly for $|b| \leq \delta^{\alpha}$. According to Lemma 6.4 we have $F(0,0) = (0,0)$ and $det F(0,0) \neq 0$.

Then the local invertibility theorem assures that $F$ is invertible in a small ball $B_{r}$ with center 0 or, equivalently, $F(\delta^{\alpha}b)$ is invertible in a the ball $B_{r\delta^{\alpha}}$, and hence $\deg(F(\delta^{\alpha}b), B_{r\delta^{\alpha}}, 0) = 1$. Taking into account that $|F(\delta^{\alpha}b)| \geq c$ for $|b| = r\delta^{\alpha}$, the continuity property of the topological degree gives that $\deg(F(\delta^{\alpha}b) + O(\delta), B_{r\delta^{\alpha}}, 0) > 0$ for $\delta$ (hence $\lambda$) small enough. Then for such $\delta$ there exists $b \in B_{r}$ such that
\[
F(\delta^{\alpha}b) + O(\delta) = 0.
\]
and so the linear system (6.10) has only the trivial solution $c_{1} = c_{2} = 0$. Finally $\delta^{\alpha}|b| \sim |F(\delta^{\alpha}b)| = O(\delta)$, hence $|b| = O(\delta^{\alpha + 1})$. That concludes the proof of Theorems 2.2, 2.3.
Proof of Theorems 1.1, 1.3. Theorems 2.2, 2.3 provide a solution to the problem (2.2) of the form

\[ v_\lambda = PW_\lambda + \phi_\lambda \]

for some \( b = b_\lambda \) with \(|b_\lambda| = O(\delta^{\alpha+1})\). So we have

\[ \sum_{k=0}^{\alpha-1} H(x, \beta_\lambda) = \alpha H(x, 0) + O(\delta), \tag{6.11} \]

\[ \log(\delta^{2\alpha} + |x^\alpha + b_\lambda|^2) = \log(\delta^{2\alpha} + |x|^{2\alpha}) + O(\delta) \tag{6.12} \]

uniformly for \( x \in \Omega \). Clearly, by (2.1),

\[ u_\lambda = v_\lambda - 4\pi(\alpha - 1)G(x, 0) \]

solves equation (1.1) and (1.7) of Theorem 1.1 follows from (2.8) and (6.11) (6.12). Moreover, using (3.1) and (6.8), by Hölder’s inequality with \( \frac{1}{p} + \frac{1}{q} = 1 \) we get

\[ \lambda ||x|^{2(\alpha-1)}V(x)(e^{u_\lambda} - e^{PW_\lambda})||_1 = \lambda ||x|^{2(\alpha-1)}V(x)e^{PW_\lambda}(e^{\phi_\lambda} - 1)||_1 \]

\[ \leq \lambda ||x|^{2(\alpha-1)}V(x)e^{PW_\lambda}||_p ||e^{\phi_\lambda} - 1||_q \]

\[ = O(\lambda^{\frac{1}{mp} + \frac{1}{q} - \varepsilon}) = o(1), \]

if \( p \) is chosen sufficiently close to 1 and \( \varepsilon \) sufficiently close to 0. Then, by (2.6) and Lemma 3.1

\[ \lambda \int_{\Omega} a(x)e^{u_\lambda}dx = \lambda \int_{\Omega} |x|^{2(\alpha-1)}V(x)e^{u_\lambda}dx = \lambda \int_{\Omega} |x|^{2(\alpha-1)}V(x)e^{PW_\lambda}dx + o(1) \]

\[ = \int_{\Omega} |x|^{2(\alpha-1)}e^{W_\lambda}dx + o(1) = \int_{\mathbb{R}^2} |x|^{2(\alpha-1)}e^{W_\lambda}dx + o(1) = 8\pi\alpha + o(1). \]

Similarly for every neighborhood \( U \) of 0

\[ \lambda \int_{U} a(x)e^{u_\lambda}dx \to 8\pi\alpha. \]

So (1.8) is verified and Theorem 1.1 is thus completely proved.

Lemma 6.1. Let \( \alpha \geq 2 \) and \( \xi \in \mathbb{C} \). For any \( \gamma = 0, 1, \ldots, \alpha - 1 \) the following holds:

\[ \int_{\Omega} |x|^{2(\alpha-1)}e^{W_\lambda}PZ^1_{\lambda}\text{Re}(\xi x^\gamma)dx = O(\delta^{\alpha+\gamma}), \int_{\Omega} |x|^{2(\alpha-1)}e^{W_\lambda}PZ^1_{\lambda}\text{Im}(\xi x^\gamma)dx = O(\delta^{\alpha+\gamma}) \]

and

\[ \int_{\Omega} |x|^{2(\alpha-1)}e^{W_\lambda}PZ^1_{\lambda}\text{Re}(\xi x^\alpha)dx = 4\pi\alpha^2\delta^{\alpha}\text{Re}(\xi) + O(\delta^{2\alpha}) \]

\[ \int_{\Omega} |x|^{2(\alpha-1)}e^{W_\lambda}PZ^1_{\lambda}\text{Im}(\xi x^\alpha)dx = 4\pi\alpha^2\delta^{\alpha}\text{Im}(\xi) + O(\delta^{2\alpha}) \]

\[ \int_{\Omega} |x|^{2(\alpha-1)}e^{W_\lambda}PZ^2_{\lambda}\text{Re}(\xi x^\alpha)dx = -4\pi\alpha^2\delta^{\alpha}\text{Im}(\xi) + O(\delta^{2\alpha}) \]

\[ \int_{\Omega} |x|^{2(\alpha-1)}e^{W_\lambda}PZ^2_{\lambda}\text{Im}(\xi x^\alpha)dx = 4\pi\alpha^2\delta^{\alpha}\text{Re}(\xi) + O(\delta^{2\alpha}) \]

uniformly for \( b \) in a small neighborhood of 0.
Proof. Let us first show the identities for \( j = 1 \) and \( \xi = 1 \). By (4.2) for \( \gamma = 0, 1, \ldots, \alpha \) we compute

\[
\int_{\Omega} |x|^{2(\alpha-1)} e^{W_\lambda} PZ_\lambda^1 \text{Re}(x^\gamma) dx = 8\alpha^2 \gamma \int_{\mathbb{R}} \frac{|y|^{2(\alpha-1)}}{(1 + |y^\alpha - \delta^{-\alpha} b|^2)^3} \text{Re}(y^\alpha - \delta^{-\alpha} b) \text{Re}(y^\gamma) dy + O(\delta^{\alpha+\gamma})
\]

\[
= 8\alpha^2 \gamma \int_{\mathbb{R}^2} \frac{|y|^{2(\alpha-1)}}{(1 + |y^\alpha - \delta^{-\alpha} b|^2)^3} \text{Re}(y^\alpha - \delta^{-\alpha} b) \text{Re}(y^\gamma) dy + O(\delta^{\alpha+\gamma}).
\]

If \( \gamma = 1, \ldots, \alpha - 1 \) the thesis follows from Lemma A.2. If \( \gamma = 0 \), then by applying Lemma A.1

\[
\int_{\mathbb{R}^2} \frac{|y|^{2(\alpha-1)}}{(1 + |y^\alpha - \delta^{-\alpha} b|^2)^3} \text{Re}(y^\alpha - \delta^{-\alpha} b) dy = \frac{1}{\alpha} \int_{\mathbb{R}^2} \frac{1}{(1 + |y - \delta^{-\alpha} b|^2)^3} \text{Re}(y - \delta^{-\alpha} b) dy
\]

\[
= \frac{1}{\alpha} \int_{\mathbb{R}^2} \frac{y_1}{(1 + |y|^2)^3} dy = 0
\]

and we get the first estimate for \( \xi = 1 \). The second estimate with \( \xi = 1 \) is analogous. Next, if \( \gamma = \alpha \) then again by Lemma A.1

\[
\int_{\mathbb{R}^2} \frac{|y|^{2(\alpha-1)}}{(1 + |y^\alpha - \delta^{-\alpha} b|^2)^3} \text{Re}(y^\alpha - \delta^{-\alpha} b) Re(y^\gamma) dy
\]

\[
= \frac{1}{\alpha} \int_{\mathbb{R}^2} \frac{1}{(1 + |y - \delta^{-\alpha} b|^2)^3} Re(y - \delta^{-\alpha} b) Re(y) dy
\]

\[
= \frac{1}{\alpha} \int_{\mathbb{R}^2} \frac{1}{(1 + |y|^2)^3} Re(y) Re(y + \delta^{-\alpha} b) dy
\]

\[
= \frac{1}{\alpha} \int_{\mathbb{R}^2} \frac{1}{(1 + |y|^2)^3} y_1(y_1 - \delta^{-\alpha} Re(b)) dy
\]

\[
= \frac{1}{\alpha} \int_{\mathbb{R}^2} \frac{(y_1)^2}{(1 + |y|^2)^3} dy - \delta^{-\alpha} Re(b) \frac{1}{\alpha} \int_{\mathbb{R}^2} \frac{y_1}{(1 + |y|^2)^3} dy
\]

\[
= \frac{\pi}{2}
\]

since \( \int_{\mathbb{R}^2} \frac{(y_1)^2}{(1 + |y|^2)^3} dy = \frac{\pi}{2} \) and \( \int_{\mathbb{R}^2} \frac{y_1}{(1 + |y|^2)^3} dy = 0 \). Similarly

\[
\int_{\mathbb{R}^2} \frac{|y|^{2(\alpha-1)}}{(1 + |y^\alpha - \delta^{-\alpha} b|^2)^3} Re(y^\alpha - \delta^{-\alpha} b) Im(y^\gamma) dy
\]

\[
= \frac{1}{\alpha} \int_{\mathbb{R}^2} \frac{1}{(1 + |y|^2)^3} Re(y) Im(y + \delta^{-\alpha} b) dy
\]

\[
= \frac{1}{\alpha} \int_{\mathbb{R}^2} \frac{1}{(1 + |y|^2)^3} y_1(y_2 + \delta^{-\alpha} Re(b)) dy
\]

\[
= 0.
\]

Taking into account that

\[
\text{Re}(\xi x^\gamma) = \text{Re}(\xi) \text{Re}(x^\gamma) - \text{Im}(\xi) \text{Im}(x^\gamma), \quad \text{Im}(\xi x^\gamma) = \text{Re}(\xi) \text{Im}(x^\gamma) + \text{Im}(\xi) \text{Re}(x^\gamma)
\]

we obtain the thesis for \( j = 1 \) and any \( \xi \in \mathbb{C} \). The remaining estimates with \( j = 2 \) are analogous. \( \square \)
Corollary 6.2. Let $\alpha \geq 3$ and $\xi_1, \xi_2 \in \mathbb{C}$. Then
\[
\int_{\Omega} |x|^{2(\alpha-1)} e^{W_\lambda} PZ^1_\lambda \text{Re}(\xi_1 x) \text{Re}(\xi_2 x) dx = \frac{\delta^2}{2} \langle \xi_1, \xi_2 \rangle \int_{\mathbb{R}^2} |x|^{2\alpha} e^{W_\lambda} Z^1_\lambda dx + O(\delta^{\alpha+2}),
\]
\[
\int_{\Omega} |x|^{2(\alpha-1)} e^{W_\lambda} PZ^1_\lambda \text{Im}(\xi_1 x) \text{Im}(\xi_2 x) dx = \frac{\delta^2}{2} \langle \xi_1, \xi_2 \rangle \int_{\mathbb{R}^2} |x|^{2\alpha} e^{W_\lambda} Z^1_\lambda dx + O(\delta^{\alpha+2})
\]
uniformly for $b$ in a small neighborhood of $0$.

Proof. Since $(\text{Re}(x))^2 = \frac{|x|^2}{2} + \frac{\text{Re}(x^2)}{2}$, according to Lemma [6.1] we have
\[
\int_{\Omega} |x|^{2(\alpha-1)} e^{W_\lambda} PZ^1_\lambda (\text{Re}(x))^2 dx
\]
\[
= \frac{1}{2} \int_{\Omega} |x|^{2(\alpha-1)} e^{W_\lambda} PZ^1_\lambda |x|^2 dx + O(\delta^{\alpha+2})
\]
\[
= 4\alpha^2 \delta^2 \int_{\mathbb{R}^2} \frac{|y|^{2\alpha}}{1 + |y^\alpha - \delta^{-\alpha} b|^2} \text{Re}(y^\alpha - \delta^{-\alpha} b) dy + O(\delta^{\alpha+2})
\]
\[
= 4\alpha^2 \delta^2 \int_{\mathbb{R}^2} \frac{|y|^{2\alpha}}{1 + |y^\alpha - \delta^{-\alpha} b|^2} \text{Re}(y^\alpha - \delta^{-\alpha} b) dy + O(\delta^{\alpha+2})
\]
\[
= \frac{\delta^2}{2} \int_{\mathbb{R}^2} |x|^{2\alpha} e^{W_\lambda} Z^1_\lambda dx + O(\delta^{\alpha+2}).
\]

Similarly, using now $(\text{Im}(x))^2 = \frac{|x|^2}{2} - \frac{\text{Re}(x^2)}{2}$,
\[
\int_{\Omega} |x|^{2(\alpha-1)} e^{W_\lambda} PZ^1_\lambda (\text{Im}(x))^2 dx = \frac{\delta^2}{2} \int_{\mathbb{R}^2} |x|^{2\alpha} e^{W_\lambda} Z^1_\lambda dx + O(\delta^{\alpha+2}).
\]

Moreover, since $\text{Re}(x)\text{Im}(x) = \frac{\text{Im}(x^2)}{2}$, by Lemma [6.1]
\[
\int_{\Omega} |x|^{2(\alpha-1)} e^{W_\lambda} PZ^1_\lambda \text{Re}(x)\text{Im}(x) dx = O(\delta^{\alpha+2}).
\]

The thesis follows for $j = 1$ since $\text{Re}(\xi x) = \text{Re}(\xi)\text{Re}(x) - \text{Im}(\xi)\text{Im}(x)$. The proof for $j = 2$ follows analogously. \hfill \Box

Corollary 6.3. Let $\alpha = 2$ and $\xi_1, \xi_2 \in \mathbb{C}$. Then
\[
\int_{\Omega} |x|^{2(\alpha-1)} e^{W_\lambda} PZ^1_\lambda \text{Re}(\xi_1 x) \text{Re}(\xi_2 x) dx = \frac{\delta^2}{2} \langle \xi_1, \xi_2 \rangle \int_{\mathbb{R}^2} |x|^{2\alpha} e^{W_\lambda} Z^1_\lambda dx
\]
\[+ 2\pi \alpha^2 \delta^2 \text{Re}(\xi_1 \xi_2) + O(\delta^{\alpha+2}),
\]
\[
\int_{\Omega} |x|^{2(\alpha-1)} e^{W_\lambda} PZ^2_\lambda \text{Re}(\xi_1 x) \text{Re}(\xi_2 x) dx = \frac{\delta^2}{2} \langle \xi_1, \xi_2 \rangle \int_{\mathbb{R}^2} |x|^{2\alpha} e^{W_\lambda} Z^2_\lambda dx
\]
\[+ 2\pi \alpha^2 \delta^2 \text{Im}(\xi_1 \xi_2) + O(\delta^{\alpha+2}),
\]
\[
\int_{\Omega} |x|^{2(\alpha-1)} e^{W_\lambda} PZ^1_\lambda \text{Im}(\xi_1 x) \text{Im}(\xi_2 x) dx = \frac{\delta^2}{2} \langle \xi_1, \xi_2 \rangle \int_{\mathbb{R}^2} |x|^{2\alpha} e^{W_\lambda} Z^1_\lambda dx
\]
\[+ 2\pi \alpha^2 \delta^2 \text{Re}(\xi_1 \xi_2) + O(\delta^{\alpha+2}).
\]
\[
\int_\Omega |x|^{2(\alpha-1)} e^{Wx} PZ_\lambda^2 \text{Im}(\xi_1 x) \text{Im}(\xi_2 x) dx = \frac{\delta^2}{2} \langle \xi_1, \xi_2 \rangle \int_\Omega |x|^{2\alpha} e^{Wx} Z_\lambda^3 dx + 2\pi \alpha^2 \delta^2 \text{Im}(\xi_1 \xi_2) + O(\delta^{\alpha+2})
\]

uniformly for \( b \) in a small neighborhood of 0.

**Proof.** Since \( (\text{Re}(x))^2 = \frac{|x|^2}{2} + \frac{\text{Re}(x^2)}{2} \), according to Lemma 6.1 we have

\[
\int_\Omega |x|^{2(\alpha-1)} e^{Wx} PZ_\lambda^1 (\text{Re}(x))^2 dx = \frac{1}{2} \int_\Omega |x|^{2(\alpha-1)} e^{Wx} PZ_\lambda^1 |x|^2 dx + 2\pi \alpha^2 \delta^2 + O(\delta^{\alpha+2})
\]

\[
= 4\alpha^2 \delta^2 \int_\Omega \frac{|y|^{2\alpha}}{(1 + |y^\alpha - \delta^{-\alpha} b|)^3} \text{Re}(y^\alpha - \delta^{-\alpha} b) dy + 2\pi \alpha^2 \delta^2 + O(\delta^{\alpha+2})
\]

\[
= 4\alpha^2 \delta^2 \int_{\mathbb{R}^2} \frac{|y|^{2\alpha}}{(1 + |y^\alpha - \delta^{-\alpha} b|)^3} \text{Re}(y^\alpha - \delta^{-\alpha} b) dy + 2\pi \alpha^2 \delta^2 + O(\delta^{\alpha+2})
\]

\[
= \frac{\delta^2}{2} \int_\Omega |x|^{2\alpha} e^{Wx} Z_\lambda^3 dx + 2\pi \alpha^2 \delta^2 + O(\delta^{\alpha+2}).
\]

Similarly, using now \( (\text{Im}(x))^2 = \frac{|x|^2}{2} - \frac{\text{Re}(x^2)}{2} \),

\[
\int_\Omega |x|^{2\alpha} e^{Wx} PZ_\lambda^1 (\text{Im}(x))^2 dx = \frac{\delta^2}{2} \int_\Omega |x|^{2\alpha} e^{Wx} Z_\lambda^3 dx - 2\pi \alpha^2 \delta^2 + O(\delta^{\alpha+2}).
\]

Moreover, since \( \text{Re}(x) \text{Im}(x) = \frac{\text{Im}(x^2)}{2} \), by Lemma 6.1

\[
\int_\Omega |x|^{2\alpha} e^{Wx} PZ_\lambda^1 \text{Re}(x) \text{Im}(x) dx = O(\delta^{\alpha+2}).
\]

The first estimate follows since \( \text{Re}(\xi x) = \text{Re}(\xi) \text{Re}(x) - \text{Im}(\xi) \text{Im}(x) \) and \( \text{Im}(\xi x) = \text{Re}(\xi) \text{Im}(x) + \text{Im}(\xi) \text{Re}(x) \). The remaining estimates follow analogously. \(\square\)

**Lemma 6.4.** Let \( \alpha \geq 2 \) be an integer and let \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) be defined by

\[
F(B) = \left( \frac{\int_{\mathbb{R}^2} |y|^{2\alpha} \text{Re}(y^\alpha - B) dy}{\int_{\mathbb{R}^2} (1 + |y^\alpha - B|^3)^3} \right),
\]

\[
\text{det}(DF(0)) \neq 0.
\]

**Proof.** According to Lemma A.1, we have

\[
\int_{\mathbb{R}^2} \frac{|y|^{2\alpha}}{(1 + |y^{2\alpha}|)^3} \text{Re}(y^\alpha) dy = \frac{1}{\alpha} \int_{\mathbb{R}^2} \frac{|y|^{2/\alpha}}{(1 + |y|^3)^3} \text{Re}(y) dy = \frac{1}{\alpha} \int_{\mathbb{R}^2} \frac{|y|^{2/\alpha}}{(1 + |y|^3)^3} y_1 dy = 0.
\]

Similarly

\[
\int_{\mathbb{R}^2} \frac{|y|^{2\alpha}}{(1 + |y^{2\alpha}|)^3} \text{Im}(y^\alpha) dy = \frac{1}{\alpha} \int_{\mathbb{R}^2} \frac{|y|^{2/\alpha}}{(1 + |y|^3)^3} y_2 dy = 0.
\]
by which we immediately get $F(0, 0) = 0$. Moreover

$$\frac{\partial F_1}{\partial B_2}(0, 0) = 6 \int_{\mathbb{R}^2} \frac{|y|^{2\alpha}}{(1 + |y|^2)^4} \text{Im}(y^\alpha) \text{Re}(y^\alpha) dy = \frac{6}{\alpha} \int_{\mathbb{R}^2} \frac{|y|^{2\alpha}}{(1 + |y|^2)^4} \text{Im}(y) \text{Re}(y) dy$$

$$= \frac{6}{\alpha} \int_{\mathbb{R}^2} \frac{|y|^{2\alpha}}{(1 + |y|^2)^4} y_1 y_2 dy = 0.$$ 

Similarly $\frac{\partial F_2}{\partial B_1}(0, 0) = 0$. So $DF$ is a diagonal matrix. We compute

$$\frac{\partial F_1}{\partial B_1}(0, 0) = -\int_{\mathbb{R}^2} |y|^{2\alpha} \frac{1 + |y|^{2\alpha} - 6(\text{Re}(y^\alpha))^2}{(1 + |y|^2)^4} dy$$

$$= -\frac{1}{\alpha} \int_{\mathbb{R}^2} |y|^2 \frac{1 + |y|^2 - 6y_1^2}{(1 + |y|^2)^4} dy.$$ 

Using that $\int_{\mathbb{R}^2} |y|^2 \frac{y_1^2}{(1 + |y|^2)^4} = \frac{1}{2} \int_{\mathbb{R}^2} |y|^{2\alpha} \frac{|y|^2}{(1 + |y|^2)^4}$ we get

$$\frac{\partial F_1}{\partial B_1}(0, 0) = \frac{1}{\alpha} \int_{\mathbb{R}^2} |y|^{2\alpha} \frac{2|y|^2 - 1}{(1 + |y|^2)^4} dy.$$ 

Proceeding similarly as above we get

$$\frac{\partial F_2}{\partial B_2}(0, 0) = \frac{\partial F_1}{\partial B_1}(0, 0) = \frac{1}{\alpha} \int_{\mathbb{R}^2} |y|^{2\alpha} \frac{2|y|^2 - 1}{(1 + |y|^2)^4} dy.$$ 

Then the thesis will follow once we have proved the nonvanishing of the above integral:

$$\int_{\mathbb{R}^2} |y|^{2\alpha} \frac{2|y|^2 - 1}{(1 + |y|^2)^4} dy \neq 0. \quad (6.13)$$

To see this, let us first compute

$$\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{2|y|^2 - 1}{(1 + |y|^2)^4} dy = \int_0^{+\infty} \frac{2\rho^2 - 1}{(1 + \rho^2)^4} d\rho$$

$$= 2 \int_0^{+\infty} \frac{1}{(1 + \rho^2)^3} d\rho - 3 \int_0^{+\infty} \frac{\rho}{(1 + \rho^2)^4} d\rho = 0$$

by direct integration. So, using that $(\sqrt{2} |y|)^{\frac{2\alpha}{\alpha}} \leq 1$ if $2|y|^2 - 1 \leq 0$ and $(\sqrt{2} |y|)^{\frac{2\alpha}{\alpha}} > 1$ if $2|y|^2 - 1 > 0$ and

$$\int_{\mathbb{R}^2} |y|^{2\alpha} \frac{2|y|^2 - 1}{(1 + |y|^2)^4} dy = (\sqrt{2})^{-\frac{2\alpha}{\alpha}} \int_{\mathbb{R}^2} (\sqrt{2} |y|)^{\frac{2\alpha}{\alpha}} \frac{2|y|^2 - 1}{(1 + |y|^2)^4} dy > (\sqrt{2})^{-\frac{2\alpha}{\alpha}} \int_{\mathbb{R}^2} \frac{2|y|^2 - 1}{(1 + |y|^2)^4} dy = 0$$

and (6.13) follows. \qed

**Appendix A.**

This appendix is devoted to deduce some integral identities associated to the change of variables: $x \mapsto x^\alpha$.

**Lemma A.1.** For any $f \in L^1(\mathbb{R}^2)$ we have that $|y|^{2(\alpha - 1)} f(y^\alpha) \in L^1(\mathbb{R}^2)$ and

$$\int_{\mathbb{R}^2} |y|^{2(\alpha - 1)} f(y^\alpha) dy = \frac{1}{\alpha} \int_{\mathbb{R}^2} f(y) dy. \quad (A.1)$$
Proof. It is sufficient to prove the thesis for a smooth function \( f \). Using the polar coordinates \((\rho, \theta)\) and then applying the change of variables \((\rho', \theta') = (\rho^\alpha, \alpha \theta)\) we get

\[
\int_{\mathbb{R}^2} |y|^{2(\alpha-1)} f(y^\alpha) \, dy = \int_0^{+\infty} d\rho \int_0^{2\pi} \rho^{2\alpha-1} f(\rho^\alpha e^{i\alpha \theta}) \, d\theta
\]

\[
= \frac{1}{\alpha^2} \int_0^{+\infty} d\rho' \int_0^{2\alpha \pi} \rho' f(\rho' e^{i\theta'}) \, d\theta'
\]

\[
= \frac{1}{\alpha} \int_0^{+\infty} d\rho' \int_0^{2\pi} \rho' f(\rho' e^{i\theta'}) \, d\theta' = \frac{1}{\alpha} \int_{\mathbb{R}^2} f(y) \, dy.
\]

\[\blacksquare\]

Lemma A.2. Let \( \gamma = 1, \ldots, \alpha - 1 \) and let \( f \) be such that \( |f(y)|y^\frac{\gamma}{\alpha} \in L^1(\mathbb{R}^2) \). Then

\[
\int_{\mathbb{R}^2} |y|^{2(\alpha-1)} f(y^\alpha) \text{Re}(y^\gamma) \, dy = \int_{\mathbb{R}^2} |y|^{2(\alpha-1)} f(y) \text{Im}(y^\gamma) \, dy = 0.
\]

Proof. Observe first that according to Lemma A.1 we have \(|y|^{2(\alpha-1)} f(y^\alpha) \text{Re}(y^\gamma) \in L^1(\mathbb{R}^2)\). Suppose that \( f \) is a smooth function. Using the polar coordinates \((\rho, \theta)\) we get

\[
\int_{\mathbb{R}^2} |y|^{2(\alpha-1)} f(y) \text{Re}(y^\gamma) \, dy = \int_0^{+\infty} |\rho|^{2\alpha-1+\gamma} \, d\rho \int_0^{2\pi} \cos(\gamma \theta) f(\rho^\alpha e^{i\alpha \theta}) \, d\theta.
\]

On the other hand

\[
\int_0^{2\pi} \cos(\gamma \theta) f(\rho^\alpha e^{i\alpha \theta}) \, d\theta = \sum_{k=0}^{\alpha-1} \int_0^{2\pi} e^{i\frac{2\pi}{\alpha} k} \cos(\gamma \theta) f(\rho^\alpha e^{i\alpha \theta}) \, d\theta
\]

\[
= \sum_{k=0}^{\alpha-1} \int_0^{2\pi} \cos \left( \gamma \left( \theta + \frac{2\pi}{\alpha} k \right) \right) f(\rho^\alpha e^{i\alpha \theta}) \, d\theta
\]

\[
= \sum_{k=0}^{\alpha-1} \cos \left( \gamma \frac{2\pi}{\alpha} k \right) \int_0^{2\pi} \cos(\gamma \theta) f(\rho^\alpha e^{i\alpha \theta}) \, d\theta
\]

\[
- \sum_{k=0}^{\alpha-1} \sin \left( \gamma \frac{2\pi}{\alpha} k \right) \int_0^{2\pi} \sin(\gamma \theta) f(\rho^\alpha e^{i\alpha \theta}) \, d\theta.
\]

The well known identity \( \sum_{k=0}^{\alpha-1} e^{i\frac{2\pi}{\alpha} k} = 0 \) for all \( \gamma = 1, \ldots, \alpha - 1 \) implies

\[
\sum_{k=0}^{\alpha-1} \cos \left( \gamma \frac{2\pi}{\alpha} k \right) = \sum_{k=0}^{\alpha-1} \sin \left( \gamma \frac{2\pi}{\alpha} k \right) = 0.
\]

and the first identity follows. The second identity is analogous..

\[\blacksquare\]

Appendix B.

In this appendix we carry out some asymptotic expansions involving the regular part \( H(x, y) \) of the Green’s function. Recalling that for any fixed \( p \in \Omega \) the function \( H_p : x \mapsto H(x, p) \) is harmonic in \( \Omega \), then it admits a holomorphic extension

\[
\hat{H}_p(x) = H_p(x) + i h_p(x) \quad \text{in} \ U, \quad h_p(p) = 0 \quad x \approx x_1 + i x_2,
\]
Proof. According to (B.1) we compute

$$\tilde{H}(x, p) = \tilde{H}_p(x) \quad \forall x, p \in U$$

by the symmetry $H(x, p) = H(p, x)$ we also deduce the analogous following symmetry for $\tilde{H}$:

$$\tilde{H}(x, p) = \tilde{H}(p, x) \quad \forall x, p \in U.$$ 

In the following we denote by $\frac{d}{dx}$ and $\frac{d}{dp}$ the (complex) derivative with respect to the first and the second variable of the function $\tilde{H}(\cdot , \cdot)$, respectively. Therefore the Taylor expansion of $H_p(x)$ up to the order $m$ takes the form:

$$H(x, p) - H(0, p) = H_p(x) - H_p(0) = \sum_{k=1}^{m} \frac{1}{k!}\Re\left(\frac{d^k \tilde{H}}{dx^k}(0, p)x^k\right) + O(|x|^{m+1}) \quad (B.1)$$

uniformly for $x \in \Omega$ and $p \in U$, where

$$\frac{d \tilde{H}}{dx}(0, p) = \frac{\partial H_p}{\partial x_1}(0) - i\frac{\partial H_p}{\partial x_2}(0),$$

$$\frac{d^k \tilde{H}}{\partial x^k}(0, p) = \frac{\partial^k H_p}{\partial x_1^k}(0) - i\frac{\partial^k H_p}{\partial x_2 \partial x_1^{k-1}}(0) \quad \forall k \geq 2.$$

Lemma B.1. Using the same notation $b, \beta_i$ of the introduction, the following holds:

$$\sum_{i=0}^{\alpha-1} (H(x, \beta_i) - H(0, \beta_i)) = \alpha \sum_{k=1}^{\alpha} \frac{1}{k!}\Re\left(\frac{d^k \tilde{H}}{dx^k}(0, 0)x^k\right) + \frac{1}{(\alpha - 1)!}\sum_{k=1}^{\alpha} \frac{1}{k!}\Re\left(\frac{\partial^{k+\alpha} \tilde{H}}{\partial p^\alpha \partial x^k}(0, 0)b x^k\right)$$

$$+ O(|b|^2|x|) + O(|x|^\alpha+1).$$

uniformly for $b \in U$ and $x \in \Omega$.

Proof. According to (B.1) we compute

$$\sum_{i=1}^{\alpha} (H(x, \beta_i) - H(0, \beta_i)) = \sum_{k=1}^{\alpha} \frac{1}{k!}\Re\left(\sum_{i=1}^{\alpha} \frac{d^k \tilde{H}}{dx^k}(0, \beta_i)x^k\right) + O(|x|^\alpha+1). \quad (B.2)$$

Let us expand the complex function $\frac{d^k \tilde{H}}{dx^k}(0, \beta_i)$:

$$\frac{d^k \tilde{H}}{dx^k}(0, \beta_i) = \frac{d^k \tilde{H}}{dx^k}(0, 0) + \sum_{h=1}^{2\alpha-1} \frac{1}{h!}\frac{\partial^{k+h} \tilde{H}}{\partial p^h \partial x^k}(0, 0)\beta_i^h + O(|\beta_i|^{2\alpha}).$$

Next we use that $\sum_{i=0}^{\alpha-1} \beta_i^h = 0$ for any $h$ which is not an integer multiple of $\alpha$, whereas $\beta_i^\alpha = b^j$ for any integer $j$, by which

$$\sum_{i=0}^{\alpha-1} \frac{d^k \tilde{H}}{dx^k}(0, \beta_i) = \alpha \frac{d^k \tilde{H}}{dx^k}(0, 0) + \frac{1}{(\alpha - 1)!}\frac{\partial^{k+\alpha} \tilde{H}}{\partial p^\alpha \partial x^k}(0, 0)b + O(|b|^2).$$

By inserting the last identity into (B.2) we obtain the thesis. □
LIOUVILLE EQUATION WITH A SINGULAR SOURCE

References

[1] S. Baraket, F. Pacard. Construction of singular limits for a semilinear elliptic equation in dimension 2, Calc. Var. Partial Differential Equations 6 (1998), 1–38.

[2] D. Bartolucci, C.-C. Chen, C.-S. Lin, G. Tarantello. Profile of blow-up solutions to mean field equations with singular data, Comm. Partial Differential Equations 29 (2004), 1241–1265.

[3] D. Bartolucci, G. Tarantello. Liouville type equations with singular data and their application to periodic multivortices for the electroweak theory, Comm. Math. Phys. 229 (2002), 3–47.

[4] T. Bartsch, A. Pistoia, T. Weth. N-vortex equilibria for ideal fluids in bounded planar domains and new nodal solutions of the sinh-Poisson and the Lane-Emden-Fowler equations, Commun. Math. Phys. 297 (2010), 653–686.

[5] F. Bethurel, H. Brezis, F. Helein. Ginzburg-Landau vortices, Birkhäuser, 1994.

[6] H. Brezis, F. Merle. Uniform estimates and blow-up behavior for solutions of −Δu = V(x)e^u in two dimensions, Comm. Partial Differential Equations 16 (1991), 1223–1253.

[7] W. Chen, C. Li. Classification of solutions of some nonlinear elliptic equations, Duke Math. J. 63 (1991), 615–623.

[8] T. D’Aprile, Multiple blow-up solutions for the Liouville equation with singular data, Commun. Partial Differential Equations 38 (2013), 1409–1436.

[9] M. Del Pino, P. Esposito and M. Musso, Nondegeneracy of entire solutions of a singular Liouville equation, Proc. Am. Math. Soc. 140 (2012), 581–588.

[10] M. Del Pino, P. Esposito, M. Musso. Two dimensional Euler flows with concentrated vorticities, Trans. Amer. Math. Soc. 362 (2010), 6381–6395.

[11] M. Del Pino, M. Kowalczyk, M. Musso. Singular limits in Liouville-type equation, Calc. Var. Partial Differential Equations 24 (2005), 47–81.

[12] P. Esposito. Blow up solutions for a Liouville equation with singular data, SIAM J. Math. Anal. 36 (2005), 1310–1345.

[13] P. Esposito, M. Grossi, A. Pistoia. On the existence of blowing-up solutions for a mean field equation, Ann. Inst. H. Poincaré Anal. Non Linéaire 22 (2005), 227–257.

[14] P. Esposito, M. Musso, A. Pistoia. On the existence and profile of nodal solutions for a two-dimensional elliptic problem with large exponent in nonlinearity Proc. Lond. Math. Soc. 94 (2007), 497–519.

[15] P. Esposito, M. Musso, A. Pistoia. Concentrating solutions for a planar elliptic problem involving nonlinearities with large exponent, J. Differential Equations 227 (2006), 29–68.

[16] P. Esposito, A. Pistoia, J. Wei. Concentrating solutions for the Hénon equation in R^2, J. Anal. Math. 100 (2006), 249–280.

[17] M. Grossi and A. Pistoia, Multiple blow-up phenomena for the sinh-Poisson equation, Arch. Ration. Mech. Anal. 209 (2013), 287–320.

[18] Y.Y. Li, I. Shafrir. Blow-up analysis for solutions of −Δu = Ve^u in dimension two, Indiana Univ. Math. J. 43 (1994), 1255–1270.

[19] L. Ma, J. Wei. Convergence for a Liouville equation, Comment. Math. Helv. 76 (2001), 506–514.

[20] J. Moser, A sharp form of an inequality by N.Trudinger, Indiana Univ. Math. J. 20 (1970/71), 1077–1092.

[21] K. Nagasaki, T. Suzuki. Asymptotic analysis for two-dimensional elliptic eigenvalue problems with exponentially dominated nonlinearities, Asymptotic Anal. 3 (1990), 173–188.

[22] J. Prajapat; G. Tarantello. On a class of elliptic problem in R^2: Symmetry and uniqueness results, Proc. Roy. Soc. Edinburgh Sect. A 131 (2001), 967–985.

[23] T. Suzuki. Two-dimensional Emden-Fowler equation with exponential nonlinearity, Nonlinear diffusion equations and their equilibrium states, 3 (Gregynog, 1989), 493–512. Progr. Nonlinear Differential Equations Appl., 7, Birkhäuser Boston, Boston, MA, 1992.

[24] G. Tarantello. Analytical aspects of Liouville-type equations with singular sources. Stationary partial differential equations. Vol. I, 491–592. Handbook Differ. Equ., North-Holland, Amsterdam, 2004.

[25] G. Tarantello. A quantization property for blow up solutions of singular Liouville-type equations, J. Funct. Anal. 219 (2005), 368–399.

[26] J. Wei, D. Ye, F. Zhou. Bubbling solutions for an anisotropic Emden-Fowler equation, Calc. Var. Partial Differential Equations 28 (2007), 217–247.

[27] N. S. Trudinger, On imbeddings into Orlicz spaces and some applications, J. Math. Mech. 17 (1967), 473–483.
[28] V.H. Weston. *On the asymptotic solution of a partial differential equation with an exponential nonlinearity*, SIAM J. Math. Anal. 9 (1978), 1030–1053.

[29] Y. Yang. *Solitons in Field Theory and Nonlinear Analysis*, Springer-Verlag, New York, 2001.

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