Some $q$-Supercongruences for the truncated $q$-trinomial coefficients

Chuanan Wei

School of Biomedical Information and Engineering,
Hainan Medical University, Haikou 571199, China
Email address: weichuanan78@163.com

Abstract. In 1987, Andrews and Baxter introduced six kinds of $q$-trinomial coefficients in exploring the solution of a model in statistical mechanics. In this paper, we give some $q$-supercongruences for the truncated forms of these polynomials.

Keywords: $q$-supercongruence; $q$-trinomial coefficient; the cyclotomic polynomial

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1 Introduction

Define the trinomial coefficient $\binom{n}{m}$ to be

$$(1 + x + x^2)^n = \sum_{m=-n}^{n} \binom{n}{m} x^{m+n}.$$ 

It is well known that there are the following two simple formulas (cf. [18, P. 43]):

$$\binom{n}{m} = \sum_{k=0}^{n} \binom{n}{k} \binom{n-k}{m+k},$$

$$\left( \binom{n}{m} \right) = \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \binom{2n-2k}{n-m-k}.$$ 

In 1819, Babbage [3] proved the interesting congruence: for any prime $p \geq 3$,

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^2}. \quad (1.1)$$

Decades later, Wolstenholme [24] told us that the above formula is true modulo $p^3$ for any prime $p \geq 5$. In 1952, Ljunggren showed the generalization of Wolstenholme’s result (cf. [5]):

$$\binom{ap}{bp} \equiv \left( \frac{a}{b} \right) \pmod{p^3}. \quad (1.2)$$

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Inspired by the works just mentioned, it is natural to consider supercongruence associated with the trinomial coefficient \( \binom{ap}{bp} \).

Let \([r]\) be the \(q\)-integer \((1 - q^r)/(1 - q)\) and define the \(q\)-binomial coefficient \( \binom{n}{m}_q \) by
\[
\binom{n}{m}_q = \begin{cases} 
\frac{(1 - q^n)(1 - q^{n-1}) \cdots (1 - q^{n-m+1})}{(1 - q)(1 - q^2) \cdots (1 - q^m)}, & \text{if } 0 \leq m \leq n; \\
0, & \text{otherwise}.
\end{cases}
\]

All over the paper, let \( \Phi_n(q) \) stand for the \(n\)-th cyclotomic polynomial in \(q\):
\[
\Phi_n(q) = \prod_{1 \leq k \leq n, \gcd(k,n)=1} (q - \zeta_k),
\]
where \(\zeta\) is an \(n\)-th primitive root of unity. It is surprising that Andrews \([1]\) gave a \(q\)-analogue of (1.1):
\[
\begin{align*}
\left[ \binom{2p-1}{p-1} \right] &
\equiv \frac{q^{p(p-1)/2}}{(p^2)} \pmod{p^2},
\end{align*}
\]
where \(p \geq 3\) is any prime. In 2019, Straub \([19]\) discovered a \(q\)-analogue of (1.2):
\[
\begin{align*}
\left[ \binom{an}{bn} \right] &
\equiv \left[ \binom{a}{b} \right]_{q^{n^2}} + (a - b) \binom{a}{b} \frac{1 - n^2}{24} \frac{(1 - q^{n^2})^2}{(\Phi_n(q)^3)},
\end{align*}
\]
where \(a, b\) are nonnegative integers and \(n\) is a positive integer.

In 1987, Andrews and Baxter \([2]\) introduced six kinds of \(q\)-trinomial coefficients in exploring the solution of a model in statistical mechanics. They can be laid out as follows:
\[
\begin{align*}
\left( \binom{n}{m}_q \right) &
= \sum_{k=0}^{n} q^{k(k+m)} \left[ \binom{n}{k} \right] \left[ \binom{n-k}{m+k} \right],
\end{align*}
\]
\[
\begin{align*}
\tau_0(n, m, q) &
= \sum_{k=0}^{n} (-1)^k q^{nk} \binom{n}{k} \binom{2n-2k}{n-m-k},
\end{align*}
\]
\[
\begin{align*}
T_0(n, m, q) &
= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2n-2k}{n-m-k},
\end{align*}
\]
\[
\begin{align*}
T_1(n, m, q) &
= \sum_{k=0}^{n} (-q)^k \binom{n}{k} \binom{2n-2k}{n-m-k},
\end{align*}
\]
\[
\begin{align*}
t_0(n, m, q) &
= \sum_{k=0}^{n} (-1)^k q^{k^2} \binom{n}{k} \binom{2n-2k}{n-m-k},
\end{align*}
\]
\[
\begin{align*}
t_1(n, m, q) &
= \sum_{k=0}^{n} (-1)^k q^{k(k-1)} \binom{n}{k} \binom{2n-2k}{n-m-k}.
\end{align*}
\]
Recently, Liu [12] determined \( \binom{an}{bn} q \) and \( \binom{an}{bn} q \) modulo \( \Phi_n(q)^2 \). Chen, Xu and Wang [4] further studied \( \binom{an}{an-n} q \) modulo \( \Phi_n(q)^2 \). There are more \( q \)-analogues of supercongruences in the literature. We refer the reader to [6–11, 13, 14, 16, 17, 21–23, 25].

Motivated by the works just mentioned, we shall establish the following theorem.

**Theorem 1.1.** Let \( \binom{an}{bn} \) denote the truncated form of the \( q \)-trinomial coefficient \( \binom{an}{bn} \):

\[
\binom{an}{bn} \equiv \sum_{k=0}^{\lfloor n/2 \rfloor} q^{k(k+bn)} \binom{an-k}{bn+k},
\]

where \( a, b, n \) are positive integers subject to \( a > b \) and \( \lfloor x \rfloor \) is the integral part of a real number \( x \). Then, modulo \( \Phi_n(q)^2 \),

\[
\binom{an}{bn} \equiv \binom{an}{bn} \left\{ 1 - (a-b)(1 - \theta_n(q)) \right\},
\]

where

\[
\theta_n(q) \equiv \begin{cases} 
(-1)^m(1 + q^m)q^{m(3m-1)/2}, & \text{if } n = 3m; \\
(-1)^m q^{m(3m+1)/2}, & \text{if } n = 3m + 1; \\
(-1)^{m+1} q^{m(3m+2)/2}, & \text{if } n = 3m + 2.
\end{cases}
\]

Choosing \( n = p \) and then letting \( q \to 1 \) in the above theorem, we obtain the supercongruence after using (1.2).

**Corollary 1.2.** Let \( \binom{ap}{bp} \) represent the truncated form of the trinomial coefficient \( \binom{ap}{bp} \):

\[
\binom{ap}{bp} \equiv \sum_{k=0}^{(p-1)/2} \binom{ap}{k} \binom{ap-k}{bp+k},
\]

where \( a, b \) are positive integers with \( a > b \) and \( p \) is an old prime. Then

\[
\binom{ap}{bp} \equiv \binom{a}{b} \pmod{p^2}.
\]

Very recently, Liu and Qi [15] made sure of \( \tau_0(an, an-n, q) \), \( T_0(an, an-n, q) \), and \( T_1(an, an-n, q) \) modulo \( \Phi_n(q)^2 \). We shall establish the following five theorems.

**Theorem 1.3.** Let \( \tau_0(an, bn, q) \) stand for the truncated form of the \( q \)-trinomial coefficient \( \tau_0(an, bn, q) \):

\[
\tau_0(an, bn, q) = \sum_{k=an-bn-[n/2]}^{an-bn} (-1)^k q^{an-k} \binom{an}{k} \binom{2an-2k}{an-bn-k},
\]
where $a, b, n$ are positive integers satisfying $a > b$. Then, modulo $\Phi_n(q)^2$,

$$
\tau_0(\alpha, \beta, \gamma) \equiv (-1)^{\alpha - \beta} q^{(\alpha - \beta)(\alpha + \beta + 1)/2} \cdot \frac{\alpha}{\beta} \left\{ 1 - (\alpha - \beta)(2 - \theta_\alpha(q) - \vartheta_\alpha(q)) \right\},
$$

(1.4)

where

$$
\vartheta_\alpha(q) \equiv \begin{cases} 
(-1)^m (1 + q^{2m}) q^{(3m-5)/2}, & \text{if } n = 3m; \\
(-1)^m q^{(3m+1)/2}, & \text{if } n = 3m + 1; \\
(-1)^{m+1} q^{(m-1)(3m+2)/2}, & \text{if } n = 3m + 2.
\end{cases}
$$

**Theorem 1.4.** Let $T_0(\alpha, \beta, \gamma)'$ denote the truncated form of the $q$-trinomial coefficient $T_0(\alpha, \beta, \gamma)$:

$$
T_0(\alpha, \beta, \gamma)' = \sum_{k=\alpha - \beta - \lceil n/2 \rceil}^{\alpha - \beta} (-1)^k \frac{\alpha}{\beta} \left[ \frac{\alpha}{\beta} q^2 \left[ \frac{2\alpha - 2k}{\alpha - \beta - k} \right] \right],
$$

where $a, b, n$ are positive integers subject to $a > b$. Then, modulo $\Phi_n(q)^2$,

$$
T_0(\alpha, \beta, \gamma)' \equiv (-1)^{\alpha - \beta} \frac{\alpha}{\beta} q^2 \left\{ 1 - 2(\alpha - \beta)(1 - \theta_\alpha(q)) \right\}.
$$

(1.5)

**Theorem 1.5.** Let $T_1(\alpha, \beta, \gamma)'$ represent the truncated form of the $q$-trinomial coefficient $T_1(\alpha, \beta, \gamma)$:

$$
T_1(\alpha, \beta, \gamma)' = \sum_{k=\alpha - \beta - \lceil n/2 \rceil}^{\alpha - \beta} (-1)^k \frac{\alpha}{\beta} \left[ \frac{\alpha}{\beta} q^2 \left[ \frac{2\alpha - 2k}{\alpha - \beta - k} \right] \right],
$$

where $a, b, n$ are positive integers with $a > b$. Then, modulo $\Phi_n(q)^2$,

$$
T_1(\alpha, \beta, \gamma)' \equiv (-1)^{\alpha - \beta} \frac{\alpha}{\beta} q^2 \left\{ 1 - 2(\alpha - \beta)(1 - \theta_\alpha(q)) \right\}.
$$

(1.6)

**Theorem 1.6.** Let $t_0(\alpha, \beta, \gamma)'$ stand for the truncated form of the $q$-trinomial coefficient $t_0(\alpha, \beta, \gamma)$:

$$
t_0(\alpha, \beta, \gamma)' = \sum_{k=\alpha - \beta - \lceil n/2 \rceil}^{\alpha - \beta} (-1)^k q^k \left[ \frac{\alpha}{\beta} q^2 \left[ \frac{2\alpha - 2k}{\alpha - \beta - k} \right] \right],
$$

where $a, b, n$ are positive integers satisfying $a > b$. Then, modulo $\Phi_n(q)^2$,

$$
t_0(\alpha, \beta, \gamma)' \equiv (-1)^{\alpha - \beta} q^{\alpha - \beta} \left[ \frac{\alpha}{\beta} q^2 \left\{ 1 - 2(\alpha - \beta)(1 - \theta_\alpha(q^{-1})) \right\} \right].
$$

(1.7)
Theorem 1.7. Let \( t_1(an, bn, q)' \) denote the truncated form of the \( q \)-trinomial coefficient \( t_1(an, bn, q) \):

\[
t_1(an, bn, q)' = \sum_{k=an-bn-[n/2]}^{an-bn} (-1)^k q^{k(k-1)} \left[ \begin{array}{c} an \\ k \end{array} \right] q^{2[an-2k]} \left[ \begin{array}{c} 2an-2k \\ an-bn-k \end{array} \right],
\]

where \( a, b, n \) are positive integers subject to \( a > b \). Then, modulo \( \Phi_n(q)^2 \),

\[
t_1(an, bn, q)' \equiv (-1)^{an-bn} q^{(an-bn)(an-bn-1)} \left[ \begin{array}{c} an \\ bn \end{array} \right] q^{2\left\{ 1-2(a-b)(1-v_n(q^{-1})) \right\}}. \tag{1.8}
\]

Setting \( n = p \) and then letting \( q \to 1 \) in Theorems 1.3-1.7, we get the supercongruence after utilizing (1.2).

Corollary 1.8. Let \( {\left( \begin{array}{c} ap \\ bp \end{array} \right)}^* \) represent the truncated form of the trinomial coefficient \( {\left( \begin{array}{c} ap \\ bp \end{array} \right)} \):

\[
{\left( \begin{array}{c} ap \\ bp \end{array} \right)}^* = \sum_{k=ap-bp-(p-1)/2}^{ap-bp} (-1)^k \left( \begin{array}{c} ap \\ k \end{array} \right) \left( \frac{2ap-2k}{ap-bp-k} \right),
\]

where \( a, b \) are positive integers with \( a > b \) and \( p \) is an old prime. Then

\[
{\left( \begin{array}{c} ap \\ bp \end{array} \right)}^* \equiv (-1)^{ap-bp} \left( \begin{array}{c} a \\ b \end{array} \right) \pmod{p^2}.
\]

The rest of the paper is arranged as follows. We shall prove Theorems 1.1 and 1.3 in Sections 2 and 3, respectively. The proof of Theorems 1.4-1.7 will be displayed in Section 4.

2 Proof of Theorem 1.1

In order to prove Theorem 1.1 we require the following two lemmas (cf. [20, Lemma 3.3] and [15, Lemma 2.1]).

Lemma 2.1. Let \( k, n \) be positive integers satisfying \( 1 \leq k \leq n-1 \). Then

\[
\left[ \frac{2k-1}{k} \right] \equiv (-1)^k q^{k(k-1)/2} \left[ \frac{n-k}{k} \right] \pmod{\Phi_n(q)}.
\]

Lemma 2.2. Let \( n \) be a nonnegative integer. Then

\[
(1 - q^n) \sum_{k=0}^{[n/2]} \frac{(-1)^k q^{k(k-1)/2}}{1 - q^{n-k}} \left[ \frac{n-k}{k} \right] = \theta_n(q).
\]

Now we start to prove Theorem 1.1.
Proof of Theorem 1.1. It is routine to see that

\[
\left( \begin{array}{c}
\binom{an}{bn}
\end{array} \right)_q' = \left[ \begin{array}{c}
an
\end{array} \right] + \sum_{k=1}^{\lfloor n/2 \rfloor} q^{k(k+bn)} \left[ \begin{array}{c}
an
\end{array} \right] \left[ \begin{array}{c}
an - k
\end{array} \right].
\]

(2.1)

Noticing the relation

\[ 1 - q^{an} = (1 - q^n)(1 + q^n + \cdots + q^{an-n}) \equiv a(1 - q^n) \pmod{\Phi_n(q)^2}, \]

we can proceed as follows:

\[
\left[ \begin{array}{c}
an
\end{array} \right] = \frac{(1 - q^{an})(1 - q^{an-1}) \cdots (1 - q^{an-k+1})}{(1 - q)(1 - q^2) \cdots (1 - q^k)}
\equiv \frac{a(1 - q^n)(1 - q^{-1}) \cdots (1 - q^{-k+1})}{(1 - q^k)(1 - q)(1 - q^2) \cdots (1 - q^{k-1})}
= \frac{a(1 - q^n)(-1)^{k-1}q^{-k(k-1)/2}}{1 - q^k}
\equiv \frac{a(1 - q^n)(-1)^kq^{-k(k+1)/2}}{1 - q^{n-k}} \pmod{\Phi_n(q)^2}.
\]

(2.2)

With the help of Lemma 2.1, there holds

\[
\left[ \begin{array}{c}
an - k
\end{array} \right] \left[ \begin{array}{c}
bn + k
\end{array} \right] = \left[ \begin{array}{c}
an
\end{array} \right] \frac{(1 - q^{an-bn})(1 - q^{an-bn-1}) \cdots (1 - q^{an-bn-2k+1})}{(1 - q^{an})(1 - q^{an-k+1})(1 - q^{bn+1}) \cdots (1 - q^{bn+k})}
\equiv \left[ \begin{array}{c}
an
\end{array} \right] \frac{a - b}{a} \frac{(1 - q^{-1})(1 - q^{-2}) \cdots (1 - q^{-2k+1})}{(1 - q^{-1}) \cdots (1 - q^{-k+1})(1 - q) \cdots (1 - q^k)}
= \left[ \begin{array}{c}
an
\end{array} \right] \frac{a - b}{a} (-1)^{k}q^{-k(3k-1)/2} \left[ \begin{array}{c}
2k - 1
\end{array} \right]
\equiv \left[ \begin{array}{c}
an
\end{array} \right] \frac{a - b}{a} \left[ \begin{array}{c}
n - k
\end{array} \right] \pmod{\Phi_n(q)}.
\]

(2.3)

The combination of (2.1)–(2.3) gives

\[
\left( \begin{array}{c}
\binom{an}{bn}
\end{array} \right)_q' \equiv \left[ \begin{array}{c}
an
\end{array} \right] + \left[ \begin{array}{c}
an
\end{array} \right] (a - b)(1 - q^n) \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{(-1)^kq^{-k(k-1)/2}}{1 - q^{n-k}} \left[ \begin{array}{c}
n - k
\end{array} \right] \pmod{\Phi_n(q)^2}.
\]

Evaluating the series on the right-hand side by Lemma 2.2, we arrive at (1.3). \qed
3 Proof of Theorem 1.3

For proving Theorem 1.3, we draw support from Lemmas 2.1 and 2.2 and the following lemma (cf. [15, Lemma 2.2]).

Lemma 3.1. Let $n$ be a nonnegative integer. Then

$$
(1 - q^n) \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k q^{-k(k-3)/2}}{1 - q^{n-k}} \left\lfloor \frac{n - k}{k} \right\rfloor = q_n(q).
$$

Now we start to prove Theorem 1.3.

Proof of Theorem 1.3. Replacing $k$ by $an - bn - k$ in $\tau_0(an, bn, q)'$, we have

$$
\tau_0(an, bn, q)' = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^{an - bn - k} q^{(an - bn - k)(an + bn + k + 1)/2} \left[ \begin{array}{c} an \\ bn \end{array} \right] \left[ \begin{array}{c} 2bn + 2k \\ k \end{array} \right].
$$

It is not difficult to verify that

$$
\left[ \begin{array}{c} an \\ bn \end{array} \right] = \left[ \begin{array}{c} an \\ bn \end{array} \right] \frac{(1 - q^{an - bn})(1 - q^{an - bn - 1}) \cdots (1 - q^{an - bn - k + 1})}{(1 - q^{bn + 1})(1 - q^{bn + 2}) \cdots (1 - q^{bn + k})}

\equiv \left[ \begin{array}{c} an \\ bn \end{array} \right] \frac{(a - b)(1 - q^n)(1 - q^{-1}) \cdots (1 - q^{-k + 1})}{(1 - q^k)(1 - q)(1 - q^2) \cdots (1 - q^{k-1})}

= \left[ \begin{array}{c} an \\ bn \end{array} \right] \frac{(a - b)(1 - q^n)(-1)^{k-1}q^{-k(k-1)/2}}{1 - q^k}

\equiv \left[ \begin{array}{c} an \\ bn \end{array} \right] \frac{(a - b)(1 - q^n)(-1)^{k-1}q^{-k(k+1)/2}}{1 - q^{n-k}} \pmod{\Phi_n(q)^2}. \tag{3.2}
$$

In terms of Lemma 2.1 there is

$$
\left[ \begin{array}{c} 2bn + 2k \\ k \end{array} \right] = \frac{(1 - q^{2bn + 2k})(1 - q^{2bn + 2k - 1}) \cdots (1 - q^{2bn + k + 1})}{(1 - q)(1 - q^2) \cdots (1 - q^k)}

\equiv (1 + q^k) \left[ \begin{array}{c} 2k - 1 \\ k \end{array} \right]

\equiv (-1)^k q^{k(3k-1)/2}(1 + q^k) \left[ \begin{array}{c} n - k \\ k \end{array} \right] \pmod{\Phi_n(q)}. \tag{3.3}
$$
The combination of (3.1)-(3.3) produces
\[
\tau_0(an, bn, q)^{'} \equiv (-1)^{an-bn} q^{(an-bn)(an+bn+1)/2} \left[ \begin{array}{c} an \\ bn \end{array} \right] + (-1)^{an-bn} q^{(an-bn)(an+bn+1)/2} \left[ \begin{array}{c} an \\ bn \end{array} \right] \\
\times (a-b)(1-q^n) \sum_{k=1}^{[n/2]} \frac{(-1)^k q^{k(k-3)/2}(1+q^k)}{1-q^{n-k}} \left[ \begin{array}{c} n-k \\ k \end{array} \right] \pmod{\Phi_n(q)^2}.
\]
Calculating the series on the right-hand side by Lemmas 2.2 and 3.1 we deduce (1.4). □

4 Proof of Theorems 1.4-1.7

Firstly, we shall prove Theorem 1.4.

Proof of Theorem 1.4. Replace k by an − bn − k in \( T_0(an, bn, q)^{'} \) to find
\[
T_0(an, bn, q)^{'} = \sum_{k=0}^{[n/2]} (-1)^{an-bn-k} \left[ \begin{array}{c} an \\ bn \end{array} \right] \left[ \begin{array}{c} 2bn+2k \\ k \end{array} \right] q^k
\]
\[
= (-1)^{an-bn} \left[ \begin{array}{c} an \\ bn \end{array} \right] q^k + \sum_{k=1}^{[n/2]} (-1)^{an-bn-k} \left[ \begin{array}{c} an \\ bn \end{array} \right] \left[ \begin{array}{c} 2bn+2k \\ k \end{array} \right] q^k \pmod{\Phi_n(q)^2}. (4.1)
\]

Similar to the derivation of (3.2), we obtain
\[
\left[ \begin{array}{c} an \\ bn \end{array} \right] q^k = \left[ \begin{array}{c} an \\ bn \end{array} \right] \frac{(1-q^{2an-2bn})(1-q^{2an-2bn-2}) \cdots (1-q^{2an-2bn-2k+2})}{(1-q^{2bn+2})(1-q^{2bn+4}) \cdots (1-q^{2bn+2k})}
\]
\[
\equiv \left[ \begin{array}{c} an \\ bn \end{array} \right] \frac{2(a-b)(1-q^n)(1-q^{-2}) \cdots (1-q^{-2k+2})}{(1-q^{2k})(1-q^2)(1-q^4) \cdots (1-q^{2k-2})}
\]
\[
= \left[ \begin{array}{c} an \\ bn \end{array} \right] \frac{2(a-b)(1-q^n)(-1)^{k-1} q^{-k(1-1)}}{1-q^{2k}} \pmod{\Phi_n(q)^2}. (4.2)
\]

Substituting (3.3) and (4.2) into (4.1), we get
\[
T_0(an, bn, q)^{'} \equiv (-1)^{an-bn} \left[ \begin{array}{c} an \\ bn \end{array} \right] q^k + (-1)^{an-bn} \left[ \begin{array}{c} an \\ bn \end{array} \right] 2(a-b)(1-q^n)
\]
\[
\times \sum_{k=1}^{[n/2]} \frac{(-1)^k q^{k(k-1)/2}}{1-q^{n-k}} \left[ \begin{array}{c} n-k \\ k \end{array} \right] \pmod{\Phi_n(q)^2}.
\]
Evaluating the series on the right-hand side by Lemma 2.2 we catch hold of (1.5). □
Secondly, we shall prove Theorem 1.5.

**Proof of Theorem 1.5.** Replace $k$ by $an - bn - k$ in $T_1(an, bn, q)'$ to derive

$$T_1(an, bn, q)' = \sum_{k=0}^{\lfloor n/2 \rfloor} (-q)^{an-bn-k} \binom{an}{bn+k}_{q^2} q^{2bn+2k}$$

$$= (-q)^{an-bn} \binom{an}{bn}_{q^2} + \sum_{k=1}^{\lfloor n/2 \rfloor} (-q)^{an-bn-k} \binom{an}{bn+k}_{q^2} q^{2bn+2k}$$

$$\equiv (-q)^{an-bn} \binom{an}{bn}_{q^2} + (q)^{an-bn} \binom{an}{bn}_{q^2} 2(a-b)(1-q^n)$$

$$\times \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^k q^{k(k-3)/2} \frac{n-k}{q^{n-k}} \binom{n-k}{k} \pmod{\Phi_n(q)^2},$$

where we have employed (3.3) and (4.2). Calculating the series on the right-hand side by Lemma 3.1, we are led to (1.6).

For the aim to prove Theorem 1.6, we need the following lemma.

**Lemma 4.1.** Let $n$ be a nonnegative integer. Then

$$(1 - q^n) \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k q^{k(3k-1)/2} \binom{n-k}{k} \equiv \theta_n(q^{-1}) \pmod{\Phi_n(q)^2}. \quad (4.3)$$

**Proof.** Performing the replacement $q \rightarrow q^{-1}$ in Lemma 2.2, there holds

$$(1 - q^n) \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k q^{k(3k-1)/2} \binom{n-k}{k} = \theta_n(q^{-1}).$$

Considering that $q^n \equiv 1 \pmod{\Phi_n(q)}$, it is ordinary to achieve (4.3).

Thirdly, we shall prove Theorem 1.6.

**Proof of Theorem 1.6.** Replacing $k$ by $an - bn - k$ in $t_0(an, bn, q)'$ and using (3.3) and (4.2), we have

$$t_0(an, bn, q)' = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^{an-bn-k} q^{(an-bn-k)^2} \binom{an}{bn+k}_{q^2} q^{2bn+2k}$$

$$= (-1)^{an-bn} q^{(an-bn)^2} \binom{an}{bn}_{q^2}$$

$$+ \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^{an-bn-k} q^{(an-bn-k)^2} \binom{an}{bn+k}_{q^2} q^{2bn+2k}$$
\[
\equiv (-1)^{an-bn}q^{(an-bn)^2} \left[ \begin{array}{c} an \\ bn \end{array} \right] q^2 + (-1)^{an-bn}q^{(an-bn)^2} \left[ \begin{array}{c} an \\ bn \end{array} \right] 2(a-b)(1-q^n)
\]
\[
\times \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{(-1)^k q^{k(3k-1)/2}}{1-q^{n-k}} \left[ \begin{array}{c} n-k \\ k \end{array} \right] \left( \mod \Phi_n(q)^2 \right).
\]

Via Lemma 4.1 and the last relation, we can deduce (1.7). \qed

For the sake of proving Theorem 1.7, we demand the following lemma.

**Lemma 4.2.** Let \( n \) be a nonnegative integer. Then
\[
(1-q^n) \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k q^{k(3k+1)/2}}{1-q^{n-k}} \left[ \begin{array}{c} n-k \\ k \end{array} \right] \equiv \nu_n(q^{-1}) \pmod{\Phi_n(q)^2}.
\] (4.4)

**Proof.** Performing the replacement \( q \to q^{-1} \) in Lemma 3.1 there is
\[
(1-q^n) \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k q^{k(3k+1)/2}}{1-q^{n-k}} \left[ \begin{array}{c} n-k \\ k \end{array} \right] = \nu_n(q^{-1}).
\]
Observing that \( q^n \equiv 1 \pmod{\Phi_n(q)} \), it is regular to attain (4.4). \qed

Finally, we shall prove Theorem 1.7

**Proof of Theorem 1.7.** Replacing \( k \) by \( an-bn-k \) in \( t_1(an,bn,q)' \) and utilizing \( 3.3 \) and \( 4.2 \), we catch hold of
\[
t_1(an,bn,q)' = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^{an-bn-k}q^{(an-bn-k)(an-bn-k-1)} \left[ \begin{array}{c} an \\ bn+k \end{array} \right] q^2 \left[ \begin{array}{c} 2bn+2k \\ k \end{array} \right]
\]
\[
= (-1)^{an-bn}q^{(an-bn)(an-bn-1)} \left[ \begin{array}{c} an \\ bn \end{array} \right] q^2
\]
\[
+ \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^{an-bn-k}q^{(an-bn-k)(an-bn-k-1)} \left[ \begin{array}{c} an \\ bn+k \end{array} \right] q^2 \left[ \begin{array}{c} 2bn+2k \\ k \end{array} \right]
\]
\[
\equiv (-1)^{an-bn}q^{(an-bn)(an-bn-1)} \left[ \begin{array}{c} an \\ bn \end{array} \right] q^2
\]
\[
+ (-1)^{an-bn}q^{(an-bn)(an-bn-1)} \left[ \begin{array}{c} an \\ bn \end{array} \right] 2(a-b)(1-q^n)
\]
\[
\times \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{(-q)^k q^{k(3k+1)/2}}{1-q^{n-k}} \left[ \begin{array}{c} n-k \\ k \end{array} \right] \left( \mod \Phi_n(q)^2 \right).
\]
Through Lemma 4.2 and the last formula, we are led to (1.8). \qed
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