FROBENIUS AMPLITUDE AND STRONG VANISHING
THEOREMS FOR VECTOR BUNDLES

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With Appendices by Dennis S. Keeler

Abstract. The primary goal of this paper is to systematically exploit the
method of Deligne-Illusie to obtain Kodaira type vanishing theorems for vector
bundles and more generally coherent sheaves on algebraic varieties. The key
idea is to introduce a number which provides a cohomological measure of the
positivity of a coherent sheaf called the Frobenius or F-amplitude. The F-
amplitude enters into the statement of the basic vanishing theorem, and this
leads to the problem of calculating, or at least estimating, this number. Most
of the work in this paper is devoted to doing this various situations.

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In [DI], Deligne, Illusie and Raynaud gave a beautiful proof of the Kodaira-
Akizuki-Nakano vanishing theorem for ample line bundles using characteristic $p$
methods. The goal of this paper is to apply these methods to obtain vanishing
theorems for vector bundles and, more generally, sheaves in a systematic fashion.
In order to facilitate this, we introduce a cohomological measure of the positivity of
a coherent sheaf on an algebraic variety that we call the Frobenius or $F$-amplitude.

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We also introduce some variations on this idea, such as the $F$-amplitude relative to a normal crossing divisor. The smaller the amplitude, the more positive it is; when it is zero, we say that the sheaf is $F$-ample. ($F$-ample vector bundles have been called “cohomologically $p$-ample” in [3], [11] and possibly elsewhere, but we prefer the shorter term.) $F$-ampleness for bundles of rank greater than one turns out to be an unreasonably restrictive notion, and it appears more useful to consider the class of bundles with small $F$-amplitude relative to the rank.

As the terminology suggests, the definition of $F$-amplitude makes use of the Frobenius map in an essential way. However, it can be extended into characteristic zero by the usual reduction modulo $p$ tricks. While this leads to a definition, it is one that is not particularly convenient to use in practice. For curves and projective spaces, we can give a reformulation of $F$-amplitude in characteristic free terms. In general, it seems that the best we can hope for are some reasonable bounds on $F$-amplitude, and much of this paper is devoted to finding such bounds. The key result in this direction is theorem 5, which shows that in characteristic zero the $F$-amplitude of an ample vector bundle is bounded above by its rank. The proof relies on some work of Carter and Lusztig in modular representation theory.

The penultimate section contains the main theorem. It gives the vanishing of the cohomology groups of a sheaf on a smooth projective variety tensored with the differentials with logarithmic singularities along a divisor in a range determined by the $F$-amplitude relative to the divisor. A special case of this for $F$-ample bundles had been considered by Migliorini [11]. The vanishing theorem is nominally a characteristic $p$ result; the interesting consequences are in characteristic zero. From this we are able to recover some old results such as Le Potier’s vanishing theorem, and to discover some new ones as well. One corollary that we want to call attention to is the following Kawamata-Viehweg type theorem (cor. 8.9): Let $E$ be a vector bundle on smooth projective variety $X$. Suppose there is an effective fractional $Q$-divisor $\Delta$ with normal crossing support $D$ such that $E(-\Delta)$ is ample, which means that some symmetric power $S^m(E)(-m\Delta)$ is ample in the usual sense. Then $H^i(\Omega_X^i(log D)(-D) \otimes E) = 0$ for $i + j \geq \text{dim } X + \text{rank}(E)$; in particular, $H^i(\omega_X \otimes E) = 0$ for $i \geq \text{rank}(E)$. This result is put to use in the final section to obtain a refinement of the Lefschetz hyperplane theorem and to obtain a Le Potier theorem for noncompact varieties.

The notion of $F$-semipositivity is obtained by relaxing the condition for $F$-ampleness. We show that $F$-semipositive vector bundles are nef. In characteristic 0, more is true, namely $F$-semipositive bundles are “arithmetically nef” which means roughly that it specializes to a nef bundle in positive characteristic. The converse fails in general. However, for line bundles the equivalence of these notions has been established by Dennis Keeler, and included as an appendix. This can be used to slightly extend the aforementioned vanishing theorem.

### 1. Frobenius amplitude

In this section, we define the notion of Frobenius (or simply $F$-) amplitude. This definition is most natural in positive characteristic, and we start with this case. Let $k$ be a field of characteristic $p > 0$, and let $X$ be a variety defined over $k$. $F$, or sometimes $F_X$, will denote the absolute Frobenius of $X$ (i.e. the morphism of schemes which is the identity on the set $X$ and the $p$th power map on $O_X$). The
Lemma 1.1. Any finite diagram of $k$-schemes of finite type and coherent sheaves has an arithmetic thickening. Any two thickenings have a common refinement.

Suppose that $X$ is a quasiprojective $k$-variety with a coherent sheaf $E$. Given a thickening $(\tilde{X}, \tilde{E})$ over $A$, we will write $p(q) = \text{char}(A/q)$, $X_q$ for the fiber and $E_q = \tilde{E}|_{X_q}$ for each closed point $q \in \text{Spec}A$. We will say that a property holds for almost all $q$ if it holds for all $q$ in a nonempty open subset of $\text{Spec}A$. For each closed point $q \in \text{Spec}A$, the fiber $X_q$ is defined over the finite field $A/q$, so that the $F$-amplitude of the restriction $E_q$ can be defined as before. We say that $i \geq \phi(E)$ if and only if $i \geq \phi(E_q)$ holds for all $q$. Equivalently, the $F$-amplitude $\phi(E)$ is obtained by minimizing $\max_q \phi(E_q)$ over all thickenings. Note that there is no (obvious) semicontinuity property for $\phi(E_q)$. So it is not clear if this is the optimal definition, but it is sufficient for the present purposes. Any alternative definition should satisfy the following: for any arithmetic thickening of absolute Frobenius can be factored as:

$$
\begin{array}{c}
\text{spec }k \quad \xrightarrow{F_k} \quad \text{spec }k \\
\downarrow \quad \Downarrow \quad \downarrow \\
F_X \xrightarrow{F'} \quad X' \xrightarrow{F_k} \quad X \\
\end{array}
$$

where the righthand square is cartesian. $F'$ is the relative Frobenius. When $k$ is perfect, $F_k : \text{spec }k \to \text{spec }k$ and its base change $X' \to X$ are isomorphisms of $\mathbb{Z}/p\mathbb{Z}$-schemes. In view of this, the relative Frobenius can be replaced by the absolute Frobenius and $X'$ by $X$ in the statements of [11] 2.1, 4.2.

Given a coherent sheaf $E$, denote $F^{n*}E$ by $E(p^n)$. For a vector bundle $E$ given by a 1-cocycle $g_{ij}$, $E(p^n)$ is given by $g_{ij}^n$. If $I$ is an ideal sheaf on $\mathbb{P}^n$ generated by polynomials $f_i$, then $I(p^n)$ is the ideal sheaf generated by $f_i^n$. Define the $F$-amplitude $\phi(E)$ of a coherent sheaf $E$ to be the smallest integer $l$ such that for any locally free sheaf $F$, there exists an $N$ such that $H^i(X, E(p^n) \otimes F) = 0$ for all $i > l$ and $m > N$. A few words of caution should be added here. We are purposely using the naive definition, but this has reasonable properties only when $X$ is smooth (which implies that $F$ is flat) or $E$ is locally free. In more general situations, $F^{n*}E$ should be replaced by the derived pullback $LF^{n*}E$, at which point $E$ may as well be replaced by an object in $D_{coh}(X)$ (one day, perhaps). We have that $\phi(E)$ is less than or equal to the coherent cohomological dimension of $X$ which is less than or equal to the dim $X$.

Now suppose that $k$ is a field of characteristic 0. By a diagram over a scheme $S$, we will mean a collection of $S$-schemes $X_i$, $S$-scheme morphisms $f_{ij} : X_i \to X_j$, $O_{X_i}$-modules $E_{i,j}$ and morphisms between the pullbacks and pushforwards of these modules. Given a morphism $S' \to S$, and a diagram $D$ over $S$, we can define its fiber product $D \times_S S'$ in the obvious way. Given a diagram $D$ over $\text{Spec }k$, an arithmetic thickening (or simply just thickening) of it is a choice of a finitely generated $\mathbb{Z}$-subalgebra $A \subset k$, and a diagram $\tilde{D}$ over $\text{Spec }A$, so that $\tilde{D}$ is isomorphic to the fiber product over $\text{Spec }k$. Given two thickenings $\tilde{D}_1 \to \text{Spec }A_1$, $\tilde{D}_2 \to \text{Spec }A_2$, we will say the second refines the first if there is a homomorphism $A_1 \to A_2$, and an isomorphism between $D_2$ and $D_{1 \times \text{Spec }A_1} \to \text{Spec }A_2$.

By standard arguments (e. g. [4] sect. 6):

**Lemma 1.1.** Any finite diagram of $k$-schemes of finite type and coherent sheaves has an arithmetic thickening. Any two thickenings have a common refinement.
a finite collection of coherent sheaves $\mathcal{E}_1, \ldots, \mathcal{E}_N$, there is a sequence of closed points $q_j$ with $\text{char} \ (A/q_j) \to \infty$ such that $\phi(\mathcal{E}_i) \geq \phi((\mathcal{E}_i)_{q_j})$.

Let $X$ be a smooth projective variety over $k$. We have an ordering on divisors defined in the usual way: $D \leq D'$ if and only if the coefficients of $D$ are less than or equal to the coefficients of $D'$. Fix a reduced divisor $D \subset X$ with normal crossings. Assume that $\text{char} \ k = p > 0$, then we define the $F$-amplitude of a coherent sheaf $\mathcal{E}$ relative to $D$ as follows

$$\phi(\mathcal{E}, D) = \min \{ \phi(\mathcal{E}^{(n)} (-D')) | n \in \mathbb{N}, 0 \leq D' \leq (p^n - 1)D \}.$$ 

If $D' \leq (p^n - 1)D$ is a divisor for which this minimum is achieved, we will refer to the $\mathbb{Q}$-divisor $\frac{1}{p^n} D'$ as a critical divisor for $\mathcal{E}$ relative to $D$. It will be convenient to introduce the relation on divisors, $A <_{\text{strict}} B$ if the multiplicity of $A$ along any irreducible component $C$ of the union of their supports is less than the multiplicity of $B$ along $C$. Then the upper inequality above is just that $D' <_{\text{strict}} p^n D$. When $\text{char} \ k = 0$, we proceed as above, $\phi(\mathcal{E}, D)$ is the minimum of $\max_q \phi(\mathcal{E}_q, D_q)$ over all thickenings of $(X, D, \mathcal{E})$. We define the generic $F$-amplitude $\phi_{\text{gen}}(\mathcal{E})$ of a locally free sheaf $\mathcal{E}$ to be the infimum of $\phi(f^* \mathcal{E}, D)$ where $f : Y \to X$ varies over all birational maps $f$ with exceptional divisor $D$ such that $Y$ is smooth and $D$ has normal crossings.

In any characteristic, we will define $\mathcal{E}$ to be $F$-ample if and only if $\phi(\mathcal{E}) = 0$. We will see below that a line bundle is ample if and only if it is $F$-ample. However for bundles of higher rank, $F$-ampleness is a stronger condition. In positive characteristic, $F$-ample vector bundles are the same as cohomologically $p$-ample vector bundles as defined in (3).

Most of the work below will be in positive characteristic. The proofs in characteristic zero are handled by standard semicontinuity arguments on a thickening.

Throughout the rest of this paper, unless stated otherwise, $X$ will denote a projective variety over a field $k$, and the symbols $\mathcal{E}, \mathcal{F}, \ldots$ will denote coherent sheaves on $X$.

## 2. Elementary bounds on $F$-amplitude

**Lemma 2.1.** If a sheaf $\mathcal{F}$ on a topological space is quasi-isomorphic to a bounded complex $\mathcal{F}^{\bullet}$ then $H^i(\mathcal{F}) = 0$ provided that $H^a(\mathcal{F}^b) = 0$ for all $a + b = i$.

**Proof.** This follows from the spectral sequence

$$E_1^{ab} = H^b(\mathcal{F}^a) \Rightarrow \mathbb{H}^{a+b}(\mathcal{F}^{\bullet}) \cong H^{a+b}(\mathcal{F}).$$

\[\square\]

**Lemma 2.2.** Suppose $\text{char} \ k = p > 0$ and that $\mathcal{E}$ is a locally free sheaf on $X$. Then for any coherent sheaf $\mathcal{F}$,

$$H^i(X, \mathcal{E}^{(p^m)} \otimes \mathcal{F}) = 0$$

for $i > \phi(\mathcal{E})$ and $m >> 0$.

**Proof.** This will be proved by descending induction starting from $i = \dim X + 1$. Choose an ample line bundle $O(1)$. We can find an exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F} \to 0$$

where $\mathcal{F}'$ is a sum of twists of $O(1)$ (by Serre’s theorems, we can take $\mathcal{F}' = H^0(\mathcal{F}(n)) \otimes O(-n)$ for $n >> 0$). Tensoring this with $\mathcal{E}^{(p^m)}$ and applying the
long exact sequence for cohomology shows that \( H^i(X, \mathcal{E}(p^n) \otimes \mathcal{F}) = 0 \) for \( i > \phi(\mathcal{E}) \) and \( m \gg 0 \).

The proof gives something slightly stronger:

**Corollary 2.3.** Fix an ample line bundle \( O_X(1) \). Then \( \phi(\mathcal{E}) \leq A \) if and only if for any \( b \) there exists \( n_0 \) such that

\[
H^i(\mathcal{E}(p^n)(b)) = 0
\]

for all \( i > A, n \geq n_0 \).

**Lemma 2.4.** A line bundle is \( F \)-ample if and only if it is ample.

**Proof.** First assume that we are over a field of characteristic \( p > 0 \). Then we have \( L^{(p^n)} = L^n \). Therefore if \( L \) is ample, it is \( F \)-ample by Serre’s vanishing theorem. Suppose \( L \) is \( F \)-ample. Choose \( x_0 \in X \), then by lemma 2.2 \( H^1(X, m_{x_0} \otimes L^n) = 0 \) for \( x_0 \) and some \( n_0 \) a power of \( p \). Therefore \( L^n \) has a global section \( s_0 \) which is nonzero at \( x_0 \). Let \( U_0 \) be the complement of the zero set of \( s_0 \). If \( U_0 \neq X \), we can choose \( x_1 \) in the complement and arrange that \( H^1(X, m_{x_1} \otimes L^n) = 0 \) for some power \( n_1 \) of \( p \). Therefore \( L^n \) has a section \( s_1 \) not vanishing at \( x_1 \). If \( U_0 \cup U_1 \neq X \), then we can choose \( x_2 \) in the complement and proceed as above. Eventually this process has to stop, because \( X \) is noetherian. Therefore \( L^{n_1n_2...} \) is generated by the sections \( s_0^{n_1n_2...} \). Repeating the same line of reasoning with the sheaves \( m_{x_0} \otimes L^n \) shows that some power of \( L \) is very ample.

Choose a thickening of \((\tilde{X}, \tilde{L})\) over \( A \). If \( L \) is ample, then we can assume \( \tilde{L} \) is ample by shrinking \( \text{Spec} \, A \) if necessary. Consequently \( \tilde{L}_q \) is \( F \)-ample for each closed point \( q \in \text{Spec} \, A \) by the previous paragraph. Therefore \( L \) is \( F \)-ample. Now suppose that \( L \) is \( F \)-ample. As above, it suffices to show that for any ideal sheaf \( I \), \( H^1(I \otimes L^n) = 0 \) for some \( n > 0 \). But this is easily seen by choosing a thickening of \((X, L, I)\) applying the previous case on a closed fiber, using semicontinuity to deduce this for the generic fiber, then flat base change to deduce the vanishing for \( X \).

**Theorem 1.** Let \( \mathcal{E}, \mathcal{E}_0 \ldots \) be coherent sheaves on a projective variety \( X \). Assume either that \( X \) is smooth or that these sheaves are locally free. Then the following statements hold.

1. Given an exact sequence \( 0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0 \), \( \phi(\mathcal{E}_2) \leq \max(\phi(\mathcal{E}_1), \phi(\mathcal{E}_3)) \).
2. Let

\[
0 \to \mathcal{E}_n \to \mathcal{E}_{n-1} \to \ldots \mathcal{E}_0 \to \mathcal{E} \to 0
\]

be an exact sequence such that for each \( i \), \( \phi(\mathcal{E}_i) \leq i + l \), then \( \phi(\mathcal{E}) \leq l \).
3. Let \( 0 \to \mathcal{E} \to \mathcal{E}^0 \to \mathcal{E}^1 \to \ldots \mathcal{E}^n \to 0 \) be an exact sequence such that for each \( i \), \( \phi(\mathcal{E}^i) \leq i - l \), then \( \phi(\mathcal{E}) \leq l \).
4. Let \( f : Y \to X \) be a proper morphism of projective varieties such that \( d \) is the maximum dimension of the closed fibers. If \( \mathcal{E} \) is locally free then \( \phi(f^* \mathcal{E}) \leq \phi(\mathcal{E}) + d \). In particular, if \( f \) is a closed immersion, then \( \phi(f^* \mathcal{E}) \leq \phi(\mathcal{E}) \).
5. If \( f : Y \to X \) is an étale morphism of smooth projective varieties, then \( \phi(f_* \mathcal{E}) = \phi(\mathcal{E}) \).

**Proof.** The first statement is obvious, and second and third follow from lemma 2.3.
For the remaining statements, we will assume that $\text{char} k = p > 0$, the characteristic 0 case is a straightforward semicontinuity argument. There is a commutative diagram

\begin{equation}
\begin{array}{ccc}
Y & \xrightarrow{F_P} & Y \\
\downarrow & & \downarrow \\
X & \xrightarrow{F_G} & X
\end{array}
\end{equation}

Suppose that $f$ is proper with fibers of dimension $\leq d$. If $\mathcal{E}$ is a locally free $O_X$-module, and $\mathcal{F}$ a coherent $O_Y$-module, then $H^i(\mathcal{E}(p^m) \otimes R^1f_*\mathcal{F}) = 0$ for $i > \phi(\mathcal{E})$ and $m >> 0$. Therefore the Leray spectral sequence implies

$$H^i((f^*\mathcal{E})(p^m) \otimes \mathcal{F}) = 0$$

for $i > \phi(\mathcal{E}) + d$ and $m >> 0$.

If $f : Y \to X$ is étale, then the above diagram is cartesian. Furthermore, $F_X$ and $F_Y$ are both flat when $X$ and $Y$ are smooth. Cohomology commutes with flat base change, therefore for any coherent $O_Y$-module $\mathcal{E}$ and locally free $O_X$-module $\mathcal{F}$,

$$H^i(X, (f_*\mathcal{E})(p^m) \otimes \mathcal{F}) \cong H^i(Y, \mathcal{E}(p^m) \otimes f^*\mathcal{F}),$$

and this implies the equality of amplitudes.

These result easily extend to the case of $F$-amplitude relative to a divisor. Here we just treat one case that will be needed later.

**Lemma 2.5.** Let $Y \to X$ be a morphism of varieties with $Y$ smooth. Suppose that $D = \sum D_i$ is a divisor with normal crossings on $Y$, such that there exist $a_i \geq 0$ for which $L = O_Y(-\sum a_i D_i)$ is relatively ample. Then for any locally free sheaf $\mathcal{E}$ on $X$, $\phi(f^*\mathcal{E}, D) \leq \phi(\mathcal{E})$.

**Proof.** The proof is very similar case (4), of the previous theorem. Assume $\text{char} k = p$, and choose $n_0$ such that $p^{n_0} > a_i$. Set $\mathcal{E}' = f^*\mathcal{E}(p^{n_0}) \otimes L$ and let $\mathcal{F}$ be another coherent sheaf on $Y$. Since $L$ is relatively ample, the higher direct images of $\mathcal{F} \otimes L^N$ vanish for $N >> 0$. Therefore, the spectral sequence

$$H^p(\mathcal{E}(p^{n+n_0}) \otimes R^q f_* (\mathcal{F} \otimes L^{p^n})) \Rightarrow H^i((\mathcal{E}')(p^n) \otimes \mathcal{F})$$

yields the vanishing of the abutment for $i > \phi(\mathcal{E})$ and $n >> 0$.

**Corollary 2.6.** Suppose that $k$ has characteristic 0. If $f : Y \to X$ is a resolution of singularities such that the exceptional divisor $D$ has normal crossings, then $\phi(f^*\mathcal{E}, D) \leq \phi(\mathcal{E})$. If $g : Z \to X$ is any resolution of singularities, then $\phi_{\text{gen}}(g^*\mathcal{E}) \leq \phi(\mathcal{E})$.

**Proof.** Since $f$ can be realized as the blow up of $X$ along an ideal, it follows that we can find a relatively ample divisor of the form $-\sum a_i D_i$, with $a_i \geq 0$ where the $D_i$ are the irreducible components of $D_{\text{red}}$. The first assertion clearly implies that second, since $Z$ can be blown up further.

**Corollary 2.7.** Suppose that $k$ has characteristic 0. If $g : Z \to X$ is a surjective morphism with $Z$ smooth, then $\phi_{\text{gen}}(g^*\mathcal{E}) \leq \phi(\mathcal{E}) + d$, where $d$ is the dimension of the generic fiber.
Proof. Construct the following commutative diagram:

\[
\begin{array}{c}
Y & \xrightarrow{\alpha} & Z \\
\downarrow{\gamma} & & \downarrow{\delta} \\
Y' & \xrightarrow{\beta} & X
\end{array}
\]

where \(\gamma\) is a generically finite rational map (which exists by Noether’s normalization lemma), \(\delta\) is the projection, \(\alpha\) is a resolution of the indeterminacy locus of \(\gamma\), and \(Y \to Y' \to \mathbb{P}^d \times X\) the Stein factorization of \(\beta\). By theorem \(\text{(4)}\) and the previous corollary (applied to \(\delta \circ \kappa\) and \(\epsilon\) respectively),

\[
\phi_{\text{gen}}(g^*\mathcal{E}) \leq \phi_{\text{gen}}(\alpha^*g^*\mathcal{E}) \leq \phi((\delta \circ \kappa)^*\mathcal{E}) \leq \phi(\mathcal{E}) + d.
\]

\[\square\]

3. Asymptotic regularity

Fix a very ample line bundle \(O_X(1)\) on a projective variety \(X\). Recall that a coherent sheaf \(\mathcal{F}\) on \(X\) is \(m\)-regular \([\text{M, p. 100}]\) provided that \(H^i(\mathcal{F}(m - i)) = 0\) for \(i > 0\). The regularity \(\text{reg}(\mathcal{F})\) of a sheaf \(\mathcal{F}\) is the least \(m\) such that \(\mathcal{F}\) is \(m\)-regular. Let \(\text{Reg}(X) = \max(1, \text{reg}(O_X))\). Although we will not need this, it is worth remarking that \(\text{Reg}(X) = \text{reg}(O_X)\) unless \((X, O_X(1))\) is a projective space with an ample line bundle of degree 1.

Lemma 3.1. Let \(\mathcal{F}\) be 0-regular coherent sheaf, then it is globally generated and

\[
\ker[H^0(\mathcal{F}) \otimes O_X \to \mathcal{F}]
\]

is \(\text{Reg}(X)\)-regular.

Proof. The global generation of 0-regular sheaves is due to \([\text{M}, \text{p. 100}]\). Let \(R = \text{Reg}(X)\) and \(K = \ker[H^0(\mathcal{F}) \otimes O_X \to \mathcal{F}]\). By definition \(R \geq 1\). We have an exact sequence

\[
0 \to K(R - i) \to H^0(\mathcal{F}) \otimes O_X(R - i) \to \mathcal{F}(R - i) \to 0.
\]

From the long exact of cohomology groups and \(R + 1\)-regularity of \(O_X\) and the \(R\)-regularity of \(\mathcal{F}\) [loc. cit.], we can conclude that \(H^i(K(R - i)) = 0\) for \(i > 1\). As the multiplications

\[
H^0(\mathcal{F}) \otimes H^0(O_X(i)) \to H^0(\mathcal{F}(i))
\]

are surjective for \(i \geq 0\) [loc. cit.], \(H^1(K(R - 1))\) injects into \(H^1(O_X(R - 1)) = 0\). Therefore \(K\) is \(R\)-regular.

\[\square\]

Corollary 3.2. Let \(\mathcal{F}\) be \(m\)-regular, then for any \(N \geq 0\), there exist vector spaces \(V_i\) and a resolution

\[
V_N \otimes O_X(-m - NR) \to \ldots \to V_1 \otimes O_X(-m - R) \to V_0 \otimes O_X(-m) \to \mathcal{F} \to 0
\]

where \(R = \text{Reg}(X)\).

Proof. After replacing \(\mathcal{F}\) by \(\mathcal{F}(m)\), we may assume that \(m = 0\). Therefore \(\mathcal{F}\) is generated by its global sections \(V_0 = H^0(\mathcal{F})\). Let \(K_0 = \mathcal{F}(-R)\) and \(K_1\) be the kernel of the surjection \(V_0 \otimes O_X \to K_0(R)\). Then \(K_1(R)\) is 0-regular, so we can continue the above process indefinitely and define vector spaces \(V_i\) and sheaves \(K_i\) which fit into exact sequences

\[
0 \to K_{i+1} \to V_i \otimes O_X \to K_i(R) \to 0.
\]
After tensoring these with $O_X(-iR)$, we can splice these sequences together to obtain the desired resolution.

Lemma 3.3. Let $\mathcal{E}$ be a coherent sheaf on $X$, and let $n$ be the greatest integer strictly less than $-\text{areg} (\mathcal{E})/\text{Reg}(X)$. Then

$$\phi(\mathcal{E}) \leq \max(\dim X - n - 1, 0).$$

In particular, $\mathcal{E}$ is $F$-ample if $\text{areg}(\mathcal{E}) < -\text{Reg}(X)(\dim X - 1)$.

Proof. Let $m = \text{areg}(\mathcal{E})$, $R = \text{Reg}(X)$ and $d = \dim X$. We may assume $d - n - 1 > 0$, otherwise the lemma is trivially true. By corollary 3.2, there exists a resolution

$$0 \to \mathcal{E}_{n+1} \to \mathcal{E}_n \to \ldots \mathcal{E}_0 \to \mathcal{E} \to 0$$

where $\mathcal{E}_i = V_i \otimes O_X(-m - iR)$ for $i \leq n$, and

$$\mathcal{E}_{n+1} = \ker [V_n \otimes O_X(-m - nR) \to V_{n-1} \otimes O_X(-m - (n - 1)R)].$$

When $i \leq n$, we have $-m - iR > 0$, therefore $\phi(\mathcal{E}_i) = 0 \leq i + d - n - 1$ by lemma 2.4. Also $\phi(\mathcal{E}_{n+1}) \leq d = (n + 1) + d - n - 1$. Consequently the lemma follows from theorem 3.2.

Suppose that $\text{char } k = p > 0$. Let

$$\text{minreg}(\mathcal{E}) = \inf_n \{\text{areg}(\mathcal{E}^{(p^n)})\}.$$

Corollary 3.4. For any coherent sheaf $\mathcal{E}$,

$$\phi(\mathcal{E}) \leq \max(\dim X - n - 1, 0),$$

where $n$ is the greatest integer strictly less than $-\text{minreg}(\mathcal{E})/\text{Reg}(X)$.

Proof. Apply the lemma to all powers $\mathcal{E}^{(p^n)}$.

When $\text{char } k = p > 0$, we define the asymptotic regularity

$$\text{areg}(\mathcal{E}) = \limsup_n \text{areg}(\mathcal{E}^{(p^n)}).$$

Of course $\text{minreg}(\mathcal{E}) \leq \text{areg}(\mathcal{E})$, but equality will usually fail. For example, $\text{minreg}(O_X(-1)) < \text{areg}(O_X(-1)) = \infty$. When $\text{char } k = 0$, define $\text{areg}(\mathcal{E})$ to be the infimum of $\sup_q \text{areg}(\mathcal{E}^{(q)})$ over all thickenings of $(X, \mathcal{E}, O_X(1))$. In other words, $\text{areg}(\mathcal{E}) \leq m$ if and only if $\text{areg}(\mathcal{E}^{(q)}) \leq m$ for almost all $q$ for a given thickening.

Lemma 3.5. (char $k = p$) Let $\mathcal{E}$ be a coherent sheaf. The following statements are equivalent

1. $\mathcal{E}$ is $F$-ample.
2. $\text{areg}(\mathcal{E}) = -\infty$.
3. $\text{minreg}(\mathcal{E}) < \text{Reg}(X)(\dim X - 1)$

Proof. If $\mathcal{E}$ is $F$-ample, then clearly $\text{reg}(\mathcal{E}^{(p^n)}) \to -\infty$ which is the content of 2. The implication 2 $\Rightarrow$ 3 follows from the inequality $\text{minreg}(\mathcal{E}) \leq \text{areg}(\mathcal{E})$. The implication 3 $\Rightarrow$ 1 follows from corollary 3.4.

Corollary 3.6. Conditions (1) and (2) are equivalent in characteristic 0.

In any characteristic, call $\mathcal{E}$ $F$-semipositive (with respect to $O_X(1)$) if and only if $\text{areg}(\mathcal{E}) < -\infty$. We will see, shortly, that this notion is independent of the choice of $O_X(1)$. The previous lemma shows that an $F$-ample sheaf is $F$-semipositive.
Lemma 3.7. Let \( N + 1 \geq d = \dim X \). If
\[
\mathcal{E}_N \to \mathcal{E}_{N-1} \to \ldots \mathcal{E}_0 \to \mathcal{E} \to 0
\]
is an exact sequence of coherent sheaves on \( X \), then
\[
\reg(\mathcal{E}) \leq \max\{\reg(\mathcal{E}_0), \reg(\mathcal{E}_1) - 1, \ldots \reg(\mathcal{E}_{d-1}) - d\}
\]

Proof. Extend this to a sequence
\[
0 \to \mathcal{E}_{N+1} \to \mathcal{E}_N \to \ldots \mathcal{E}_0 \to \mathcal{E} \to 0.
\]
The regularity estimate follows from lemma 2.1 and the fact that \( m \)-regular sheaves are \( m' \)-regular for all \( m' \geq m \) [M, p. 100].

Proposition 3.8. Let \( f : X \to Y \) be a morphism of projective varieties. Assume that \( Y \) is equipped with a very ample line bundle \( O_Y(1) \). Let \( \mathcal{E} \) be a coherent sheaf on \( Y \) which is \( F \)-semipositive with respect to \( O_Y(1) \). If \( \text{Tor}_{f^{-1}O_Y}^i(\mathcal{O}_X, \mathcal{E}) = 0 \) for all \( i > 0 \) (e. g. if \( f \) is flat, or \( \mathcal{E} \) is locally free), then \( f^* \mathcal{E} \) is \( F \)-semipositive with respect to \( O_X(1) \).

Proof. We give the proof in positive characteristic. By hypothesis and corollary 3.2, there exists a resolution.
\[
V_N \otimes O_Y(-m - NR) \to \ldots V_1 \otimes O_Y(-m - R) \to V_0 \otimes O_Y(-m) \to \mathcal{E}(p^n) \to 0
\]
where the constants \( m, R, N \gg 0 \) can be chosen independently of \( n \). This stays exact after applying \( f^* \) by our assumptions. Therefore the regularity of \( f^*(\mathcal{E}(p^n)) = (f^*\mathcal{E})(p^n) \) stays bounded as \( n \to \infty \) by lemma 3.7.

Corollary 3.9. Let \( O_X(1)' \) be another very ample line bundle on \( X \), then a sheaf \( \mathcal{E} \) is \( F \)-semipositive with respect to \( O_X(1) \) if and only if it is \( F \)-semipositive with respect to \( O_X(1)' \).

Proof. Apply the proposition to the identity map.

Recall that a locally free sheaf \( \mathcal{E} \) on \( X \) is nef (or numerically semipositive) if for any curve \( f : C \to X \), any quotient of \( f^*\mathcal{E} \) has nonnegative degree. In characteristic 0, it is convenient to introduce an ostensibly stronger property: \( \mathcal{E} \) is arithmetically nef if there is a thickening \( (\tilde{X}, \tilde{\mathcal{E}}) \) over \( \text{Spec } A \) such that the restriction of \( \tilde{\mathcal{E}} \) to the fibers are nef. To simplify the statements, we define arithmetically nef to be synonymous with nef in positive characteristic. Further discussion of these matters can be found in the appendix. The name \( F \)-semipositive stems from the following:

Lemma 3.10. If \( \mathcal{E} \) is an \( F \)-semipositive locally free sheaf, then it is arithmetically nef.

Proof. By definition, we may work over a field of characteristic \( p > 0 \). Suppose that \( \mathcal{F} \) is a quotient of \( f^*\mathcal{E} \) with negative degree. This implies that \( \deg(f(\mathcal{F}(p^n))) \to -\infty \) as \( n \to \infty \). By proposition 3.8, \( f^*\mathcal{E} \) is \( F \)-semipositive, which implies that there is a fixed line bundle \( L \) such that \( \mathcal{E}(p^n) \otimes L \) is globally generated for all \( n \). Therefore \( \mathcal{F}(p^n) \otimes L \) is globally generated for all \( n \) which implies that \( \deg(\mathcal{F}(p^n)) \) is bounded below. This is a contradiction.

For line bundles, the converse is given by proposition 3.1. However, it fails for higher rank (example 5.8).
4. Tensor Products

\textbf{Theorem 2.} Let $\mathcal{E}$ and $\mathcal{F}$ be two vector bundles on a smooth projective variety $X$, then

$$\phi(\mathcal{E} \otimes \mathcal{F}) \leq \phi(\mathcal{E}) + \phi(\mathcal{F}).$$

\textit{Proof.} Assume that $k$ is a field of characteristic $p > 0$. Let $Y = X \times X$ and let $p_i : Y \to X$ denote the projections. Given two coherent sheaves $\mathcal{E}_i$ on $X$, let $\mathcal{E}_1 \boxtimes \mathcal{E}_2 = p_1^*\mathcal{E}_1 \otimes p_2^*\mathcal{E}_2$. Choose a very ample line bundle $O_X(1)$ on $X$, then $L = O_X(1) \boxtimes O_X(1)$ is again very ample. Let $\Delta \subset X$ be the diagonal. Choose $\nu >> 0$.

By corollary 3.2, we can construct a resolution

\begin{equation}
0 \to \mathcal{G}_{\nu+1} \to \mathcal{G}_\nu \to \ldots \to \mathcal{G}_0 \to O_\Delta \to 0
\end{equation}

where $\mathcal{G}_i = V_i \otimes L^\otimes a_i$, for $i \leq \nu$.

The Frobenius map $F_Y = F_X \times F_X$. Thus using Künneth’s formula [EGA, III, 6.7.8], for any $b$ we get

$$H^i(\mathcal{G}_j \otimes L^b \otimes F_1^{N*}(\mathcal{E} \boxtimes \mathcal{F})) = V_j \otimes H^i(\mathcal{E}^{(p^N)}(b + a_j)(\mathcal{F}^{(p^N)}(b + a_j)))$$

$$= V_j \otimes \bigoplus_{c + d = i} H^c(\mathcal{E}^{(p^N)}(b + a_j)) \otimes H^d(\mathcal{F}^{(p^N)}(b + a_j))$$

$$= 0$$

for $i > \phi(\mathcal{E}) + \phi(\mathcal{F})$, $j \leq \nu$ and $N >> 0$. Tensoring (2) by $L^b \otimes F_1^{N*}(\mathcal{E} \boxtimes \mathcal{F})$ and applying lemma 2.4 shows that

$$H^i((\mathcal{E} \otimes \mathcal{F})^{(p^N)}(2b)) = H^i(O_\Delta \otimes L^b \otimes F_1^{N*}(\mathcal{E} \boxtimes \mathcal{F}))$$

$$= 0$$

for $i > \phi(\mathcal{E}) + \phi(\mathcal{F})$ and $N >> 0$. Thus corollary 2.3 gives the desired bound on $\phi(\mathcal{E} \otimes \mathcal{F})$.

If $\text{char } k = 0$, then we can carry out the above argument on the fiber of some thickening. \hfill \Box

\textbf{Corollary 4.1.} The tensor product of two $F$-ample bundles is $F$-ample.

\textbf{Corollary 4.2.} Let $D$ and $E$ be reduced effective divisors such that $D + E$ has normal crossings. Suppose that $\mathcal{E}$ and $\mathcal{F}$ are a pair of vector bundles with critical divisors $\Delta$ and $\Xi$ along $D$ and $E$ respectively. If $\Delta + \Xi$ is strictly fractional, i.e. has all its multiplicities less then 1, then

$$\phi(\mathcal{E} \otimes \mathcal{F}, D + E) \leq \phi(\mathcal{E}, D) + \phi(\mathcal{F}, E).$$

\textit{Proof.} We can find $n, m$ such that $\phi(\mathcal{E}, D) = \phi(\mathcal{E}^{(p^n)}(-D'))$ and $\phi(\mathcal{F}, E) = \phi(\mathcal{F}^{(p^m)}(-E'))$ where $D' = p^n\Delta$ and $E' = p^m\Xi$. After replacing $E'$ by $p^{n-m}E'$, or the other way around, we can assume that $m = n$. Therefore

$$(\mathcal{E} \otimes \mathcal{F})^{(p^n)}(-D' - E') = \mathcal{E}^{(p^n)}(-D') \otimes \mathcal{F}^{(p^n)}(-E')$$

has $F$-amplitude bounded by the sum. \hfill \Box

\textbf{Remark 4.3.} If $D$ and $E$ are disjoint, then the conditions on the critical divisors are automatic.
Theorem 3. Let $\mathcal{E}$ and $\mathcal{F}$ be two coherent sheaves on $X$ such that one of them is locally free and $\mathcal{E}$ is $F$-semipositive, then

$$\phi(\mathcal{E} \otimes \mathcal{F}) \leq \phi(\mathcal{F}).$$

Proof. Assume that $\text{char } k = p > 0$. Let $m = \text{areg}(\mathcal{E}), N >> 0$ and $R = \text{Reg}(X)$. Then $\text{reg}(\mathcal{E}(p^\mu)) \leq m$ for all but finitely many $\mu$. Given a locally free sheaf $\mathcal{G}$, choose $\mu_0$, so that $H^i(F(p^\mu) \otimes \mathcal{G}) = 0$ for all $\mu > \mu_0$, $i > \phi(F)$. By increasing $\mu_0$, if necessary, we can assume that $\mathcal{E}(p^\mu)$ is $m$-regular when $\mu > \mu_0$. From corollary 3.2, we obtain a resolution

$$0 \to \mathcal{E}_{N+1} \to \mathcal{E}_N \to \ldots \mathcal{E}_0 \to \mathcal{E} \to 0$$

where $\mathcal{E}_i = V_i \otimes O_X(-m - iR)$ for $i \leq N$. Tensoring this by $F(p^\mu) \otimes \mathcal{G}$ and applying lemma 2.1 shows that

$$H^i((\mathcal{E} \otimes F)(p^\mu) \otimes \mathcal{G}) = 0$$

when $\mu > \mu_0$ and $i > \phi(F)$.

If $\text{char } k = 0$, then we can carry out the above argument on the fiber of some thickening.

We can refine corollary 4.4.

Corollary 4.4. The tensor product of an $F$-ample vector bundle and an $F$-semipositive vector bundle is $F$-ample.

5. Characterization of $F$-ample sheaves on special varieties

It is possible to give an elementary characterization of $F$-ampleness for curves and projective spaces. Recall that a vector bundle $\mathcal{E}$ over a variety $X$ defined over a field $k$ of characteristic $p$ is $p$-ample if for any coherent sheaf $\mathcal{F}$ there exists $n_0$ such that $\mathcal{E}(p^n) \otimes \mathcal{F}$ is globally generated for all $n \geq n_0$. Since we will see the converse to both statements fail in general.

Lemma 5.1. An $F$-ample vector bundle $\mathcal{E}$ is $p$-ample.

Proof. Suppose $\mathcal{F}$ is a coherent sheaf. Since the regularity of the sheaves $\mathcal{E}(p^n) \otimes \mathcal{F} \to -\infty$, these sheaves are globally generated for $n >> 0$.

Corollary 5.2. An $F$-ample vector bundle $\mathcal{E}$ is ample.

Proof. [6, 6.3]

As we will see the converse to both statements fail in general.

Lemma 5.3. If $\mathcal{E}$ is a $p$-ample vector bundle on a projective variety $X$, then $\phi(\mathcal{E}) < \dim X$.

Proof. Choose a coherent sheaf $\mathcal{F}$. Let $L$ to be the $N$th power of an ample line bundle, chosen large enough so that $H^i(\mathcal{F} \otimes L) = 0$ for $i > 0$. Then for all $n >> 0$, $\mathcal{E}_n = \mathcal{E}(p^n) \otimes L^{-1}$ is globally generated. Therefore

$$H^0(\mathcal{E}_n) \otimes \mathcal{F} \otimes L \to \mathcal{E}(p^n) \otimes \mathcal{F}$$

is surjective. It follows that the top degree cohomology of the right hand vanishes for $n >> 0$. 

\[ \square \]
This leads to a complete characterization for curves.

**Proposition 5.4.** Let $E$ be a coherent sheaf over a smooth projective curve $X$ defined over a field $k$, then the following are equivalent

1. $E$ is $F$-ample.
2. $E/\text{torsion}$ is $p$-ample when $\text{char} \ k = p$.
3. $E/\text{torsion}$ is ample.

**Proof.** Since $E$ is a direct sum of $E/\text{torsion}$ with the torsion part, we may assume that $E$ is a vector bundle. Suppose $\text{char} \ k = p$. Then the equivalence of the first two statements follows from lemmas 5.1 and 5.3. The equivalence of the last two from [H1, 6.3, 7.3].

If $\text{char} \ k = 0$, we can deduce the equivalence of (1) and (3) from the previous cases, because ampleness is an open condition. 

Now, we turn to projective space. For integers $a \leq b$, let $[a,b] = \{a,a+1, \ldots , b\}$.

**Theorem 4.** Let $E$ be a coherent sheaf on the projective space $\mathbb{P}^n_k$, then

$$\phi(E) = \min\{i_0 \mid H^i(\mathbb{P}^n_k, E(j)) = 0, \forall i, \forall j \in [-n-1,0]\}$$

In particular, $E$ is $F$-ample if and only if $H^i(E(j)) = 0$ for all $j \in [-n-1,0]$ and $i > 0$.

This leads to a characterization of $F$-ample bundles on $\mathbb{P}^n = \mathbb{P}^n_k$. For a slightly different characterization, see [M, sect. 4]. The key step in the proof of theorem 4 is the following proposition.

**Proposition 5.5.** Let $\pi : \mathbb{P}^n \to \mathbb{P}^n$ be a finite morphism, and let $d$ be the degree of $\pi^*O(1)$.

1. For each $i$, $\pi^*O(i)$ is a direct sum of line bundles.
2. If $-d-n-1 < i < d$ then $\pi^*O(i)$ is a sum of line bundles of the form $O(l)$ with $l \in [-n-1,0]$.
3. There exists a constant $C$ depending only on $n$, such that if $d > C$, then for each $O(l)$ with $l \in [-n-1,0]$, $O(l)$ occurs as a component of $\pi^*O(i)$ for some $i \in [-n-1,0]$.

**Proof.** Since $\pi$ is finite,

$$H^a((\pi^*O(i)) \otimes O(j)) = H^a(O(i + dj))$$

for all $a, i$ and $j$. In particular, these groups vanish for all $0 < a < n$ and all $j$. Therefore $\pi^*O(i)$ splits into a sum of line bundles by a theorem of Horrocks [Ho].

If $-d-n-1 < i < d$, then

$$H^n((\pi^*O(i)) \otimes O(1)) = H^n(O(i + d)) = 0$$

and

$$H^0((\pi^*O(i)) \otimes O(-1)) = H^0(O(i - d)) = 0.$$ 

This implies the second statement.

Let $p_m(x) = \left( \frac{x + m}{m} \right)$ and $(\Delta_x p)(x,y,\ldots) = p(x,y,\ldots) - p(x-1,y,\ldots)$. Note that $\Delta_x p_m(x) = p_{m-1}(x)$.

Choose $-n-1 \leq i \leq 0$. Let us write

$$\pi^*O(i) = \bigoplus_i O(l) \oplus f(l,i).$$
By comparing cohomology of $\pi_*O(i)$ and $O(i)$, we see that $S = \{l \mid f(l, i) \neq 0\}$ is contained in $[-n, 0]$ if $i = 0$, $S$ is contained in $[-n, -1]$ if $-n - 1 < i < 0$, and $S$ is contained in $[-n - 1, -1]$ if $i = -n - 1$. Furthermore $f(0, 0) = f(-n - 1, -n - 1) = 1$, and this shows that the proposition holds true for $l = 0, -n - 1$. We now assume that $-n \leq l \leq 0$. Tensoring $\pi_*O(i)$ by $O(x)$ and computing Euler characteristics yields:

$$\sum_i f(l, i)p_n(x + l) = p_n(dx + i).$$

Setting $x = 0$ yields

$$f(0, i) = p_n(i).$$

Applying $\Delta_x$ to equation (3) and setting $x = 0$ yields

$$(-1)^{n-1}f(-n, i) = (p_n(i) - f(0, i)) - p_n(i - d)$$

hence

$$f(-n, i) = (-1)^np_n(i - d).$$

Applying $\Delta_x^2$ to equation (3) and setting $x = 0$ yields

$$(-1)^{n-2}(n-1)f(-n, i) + (-1)^{n-2}f(-n+1, i) = (p_n(i) - f(0, i)) - 2p_n(i - d) + p_n(i - 2d)$$

hence

$$f(-n + 1, i) = (-1)^{n-1}(n+1)p_n(i - d) + (-1)^{n-2}p_n(i - 2d).$$

Doing this repeatedly gives a formula for each $f(l, i)$, with $-n - 1 < l < 0$, which is a nonzero polynomial of degree at most $n$ in $i$ and $d$. Choosing a specific $d >> 0$ (for $n$ fixed) forces $f(l, i)$ to be a nonzero polynomial in $i$ of degree at most $n$. Therefore $f(l, i) \neq 0$ for some $i$ in the range $-n \leq i \leq 0$.

We need the following (presumably well known) version of the projection formula.

**Lemma 5.6.** If $\pi : X \to Y$ is a finite map of quasiprojective schemes, then $\pi_*(\pi^*E \otimes F) \cong E \otimes \pi_*F$.

**Proof.** Choose a resolution $E_1 \to E_0 \to E \to 0$ by vector bundles $E_i$. There is a diagram

$$\begin{array}{ccc}
E_1 \otimes \pi_*F & \to & E_0 \otimes \pi_*F \\
\downarrow & & \downarrow \\
(\pi_*(\pi^*E_1 \otimes F)) & \to & (\pi_*(\pi^*E_0 \otimes F))
\end{array}$$

where the first two vertical arrows are isomorphisms by the usual projection formula. This implies that the third arrow is also an isomorphism.

**Proof of theorem.** As usual, we prove the result in positive characteristic; the characteristic zero case is a formal consequence. To begin with, we show $H^i(\mathcal{E}(j)) = 0$ for $i > \phi(\mathcal{E})$ and $j \in [-n - 1, 0]$. First assume $\text{char } k = p > 0$. Choose $m >> 0$, then $O(j)$ is a direct summand of some $F^m_*O(l)$ for $l \in [-n - 1, 0]$ by the previous proposition. Therefore by the projection formula (lemma 5.6),

$$H^i(\mathcal{E}(j)) \subseteq H^i(\mathcal{E} \otimes F^m_*O(l)) = H^i(\mathcal{E}^{(p^m)} \otimes O(l)) = 0.$$  

Conversely, suppose that $H^i(\mathcal{E}(j)) = 0$ for all $i > i_0$ and $j \in [-n - 1, 0]$. For any integer $l$, we can choose $m >> 0$ so that $F^m_*O(l)$ is a direct of line bundles $O(j)$ with $j \in [-n - 1, 0]$. Therefore

$$H^i(\mathcal{E}^{(p^m)} \otimes O(l)) = H^i(\mathcal{E} \otimes F^m_*O(l)) = 0.$$
for \(i > i_0\). Since any coherent sheaf \(\mathcal{F}\) on \(\mathbb{P}^n\) has a finite resolution by direct sums of line bundles, this shows that \(H^i(\mathcal{E}^{(r^n)} \otimes \mathcal{F}) = 0\) for \(m \gg 0\) and \(i > i_0\) (by the same argument as in the proof of corollary 5.3).

The theorem yields improvements on the regularity estimates of the previous section.

**Corollary 5.7.** An \(F\)-ample sheaf \(\mathcal{E}\) on \(\mathbb{P}^n\) is \((-1)\)-regular; in particular \(\mathcal{E}(-1)\) is globally generated.

**Example 5.8.** The tangent bundle \(\mathcal{T}\) of \(\mathbb{P}^n\) is ample and in fact \(p\)-ample in positive characteristic, but \(\mathcal{T}\) is not \(F\)-ample if \(n \geq 2\), because \(H^{n-1}(\mathbb{T}(-n-1)) = H^1(\Omega^1)^* \neq 0\). The bundle \(\mathcal{T}(-1)\) is globally generated, and therefore nef. However, it cannot be \(F\)-semipositive, since otherwise \(\mathcal{T}\) would be \(F\)-ample by corollary 5.7.

**Example 5.9.** Let \(X\) be a projective variety with an ample line bundle \(L\). Embed \(i: X \hookrightarrow \mathbb{P}^n\) using a large multiple of \(L\). Then \(L\) is \(F\)-ample but \(i_*L\) is not, because the conclusion of the above corollary fails. Therefore theorem 5 (5) fails for nonétale finite maps.

**Corollary 5.10.** A vector bundle on \(\mathbb{P}^2\) is \(F\)-ample if and only if it is isomorphic to a sum of the form \(E \oplus O(1)^{\oplus N}\) where \(E\) is \((-2)\)-regular.

**Proof.** A direct sum of a \((-2)\)-regular sheaf and a bunch of \(O(1)\)'s satisfies the conditions of the theorem by [M, p. 100].

Now suppose that \(V\) is a \(F\)-ample vector bundle. It is \((-1)\)-regular by the previous corollary, and therefore \(V(-1)\) is generated by global sections. Suppose that \(H^2(V(-4)) \neq 0\). Then by Serre duality, there is a nonzero morphism \(V(-1) \to O\) and let \(V'\) be the kernel twisted by \(O(1)\). Since the map \(H^0(V(-1)) \otimes O \to O\) must split, it follows that the map \(V(-1) \to O\) also splits. Therefore \(V'\) is again \((-1)\)-regular, so we can continue splitting off copies of \(O(1)\) from \(V\) until we arrive at a summand \(E\) with \(H^2(E(-4)) = 0\). Since we also have \(H^1(E(-3)) = 0\), it follows that \(E\) is \((-2)\)-regular.

6. \(F\)-Amplitude of Ample Bundles

As we have seen, ample vector bundles need not be \(F\)-ample. However, we do have an estimate on their amplitude, at least in characteristic 0.

**Theorem 5.** Let \(X\) be a projective variety over a field of characteristic 0 and let \(\mathcal{E}\) be an ample vector bundle of rank \(r\) on \(X\). Then \(\phi(\mathcal{E}) < r\).

Keeler has found that the inequality \(\phi(\mathcal{E}) < dim X\) also holds for ample vector bundles (proposition 6.3).

Before giving the proof, we need to review some results from (modular) representation theory. We will choose our notations consistent with those of [A2]. Let \(A\) be a commutative ring and \(E = A^r\). Fix a partition \(\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots)\) of weight \(|\lambda| = \sum \lambda_i\). The Schur power \(S^\lambda(E)\) can be constructed as the space of global sections of a line bundle associated to \(\lambda\) over the scheme \(Flag(E)\) of flags on \(E\). A more elementary construction is possible; \(S^\lambda(E)\) can be defined as a quotient of \(\mathcal{E}^{\otimes |\lambda|}\) by an explicit set of relations involving \(\lambda\) [F, 8.1]. This quotient map can be split using a Young symmetrizer when \(A\) contains \(\mathbb{Q}\), but not in general. The paper
Corollary 6.1. If \( p > \) 0, then there exists an integer \( N \) such that we can assume that \( E \) will write \( \tilde{E} \) instead of \( E \). With notation and assumptions as above (specifically that Lemma 6.2. Let \( \pi_k : \text{Flag}(E) \rightarrow \text{Grass}_k(E) \) be the canonical map to the Grassmannian of \( k \)-dimensional quotients of \( E \), and \( i_k \) its Plücker embedding. Then

\[
S^\lambda(E) = H^0(\text{Flag}(E), L_\lambda)
\]

where \( a_i = \lambda_i - \lambda_{i+1} \) and

\[
L_\lambda = \bigotimes_k \pi_k^* i_k^* O_{F^\lambda(\lambda^* E)}(a_k).
\]

At the two extremes, \( S^{(n)}(E) = S^n(E) \) and \( S^{(1,1,...,1)}(E) = \wedge^i(E) \) where \( i \) is the length of the string. These Schur powers turn out to be free \( A \)-modules, and since the constructions are functorial, they carry \( GL_r(A) \) actions. When \( A = k \) is a field of characteristic 0, the \( GL_r(k) \)-modules \( S^\lambda(E) \) are all irreducible. This is no longer true when \( A = k \) is a field of characteristic \( p > 0 \). For example, the symmetric power \( S^p(E) \) contains a nontrivial submodule \( E^{(p)} \) which is the representation associated to the \( p \)th power map \( GL_r(k) \rightarrow GL_r(k) \). This inclusion can be extended to a resolution:

**Theorem 6.** (Carter-Lusztig [CL, pg 235]) If \( k \) is a field of characteristic \( p > 0 \), then there exists an exact sequence of \( GL_r(k) \)-modules

\[
0 \rightarrow E^{(p)} \rightarrow S^{(p)}(E) \rightarrow S^{(p-1,1)}(E) \rightarrow S^{(p-2,1,1)}(E) \rightarrow \ldots S^\lambda(E) \rightarrow 0
\]

where \( \lambda = (p - \min(p - 1, r - 1), 1, 1, \ldots) \).

These constructions are easy to globalize to the case where \( E \) is replaced by a vector bundle \( E \). In this case, the two meanings of \( E^{(p)} \) agree.

**Corollary 6.1.** If \( E \) is a vector bundle over a scheme \( X \) defined over a field \( k \) of characteristic \( p > 0 \), then there exists a resolution of \( E^{(p)} \) as above.

Suppose we are in the situation of theorem 6. Choose a line bundle \( M \) on \( X \). Let \( (\tilde{X}, \tilde{E}, \tilde{M}) \) be an arithmetic thickening over \( \text{Spec} A \). After shrinking \( \text{Spec} A \) we can assume that \( E \) is locally free. Then we have vector bundles \( S^\lambda(\tilde{E}) \) over \( \tilde{X} \). Fix a partition \( \lambda \). Then for any natural number \( N \), we get a new partition \( (N) + \lambda = (N + \lambda_1, \lambda_2, \ldots) \).

**Lemma 6.2.** With notation and assumptions as above (specifically that \( E \) is ample), there exists an integer \( N_0 \) such that

\[
H^i(\tilde{X}, S^{(N) + \lambda}(\tilde{E}) \otimes \tilde{M}) = 0
\]

for \( N \geq N_0 \).

**Proof.** Let \( \pi : \text{Flag}(E) \rightarrow X \) be the bundle of flags on \( E \). To simplify notation, we will write \( \tilde{M} \) instead of \( \pi^* \tilde{M} \). The fibers of \( \pi \) are partial flag varieties. The higher cohomology groups of \( L_{(N) + \lambda} \) along these fibers are zero by Kempf’s vanishing theorem (see for example [H, II, 4.5]). Therefore the higher direct images vanish, and consequently the Leray spectral sequence yields isomorphisms

\[
H^i(\tilde{X}, S^{(N) + \lambda}(\tilde{E}) \otimes \tilde{M}) \cong H^i(\text{Flag}(E), L_{(N) + \lambda} \otimes \tilde{M}).
\]
Let $\pi_1 : \text{Flag}(E) \to \mathbb{P}(E)$ be the canonical projection. For reasons similar to those above, there are isomorphisms

$$H^i(\text{Flag}(E), L_{(n)+\lambda} \otimes \tilde{M}) \cong H^i(\mathbb{P}(E), \pi_1^* L_{(n)+\lambda} \otimes \tilde{M}).$$

By the projection formula, the right hand side is the cohomology of $O_p(E)(N) \otimes \pi_1^* (L_{\lambda}) \otimes \tilde{M}$. Since $O(1)$ is ample, these groups vanish for $N \gg 0$ and $i > 0$. □

**Corollary 6.3.** With the notation of section 1, given $a > 0$, there exists $N_0$

$$H^i (X_q, S^{(N) + \lambda}(E_q) \otimes M_q^\otimes n) = 0$$

for all $i > 0$, $0 \leq n \leq a$, $N \geq N_0$ and closed points $q \in \text{Spec } A$.

Proof. [H, III. 12.9] □

**Proof of theorem 3.** Choose $M = O_X(-1)$ with $O_X(1)$ very ample. Let $C << 0$ be a constant. By corollary 3.3, there exists a $N_0$ such that the sheaves $S^{(N-i,\ldots,1)}(E_q)$ ($0 \leq i < r$) have regularity less than $C$ for all $N \geq N_0$ and all closed points $q \in \text{Spec } A$. In particular, there exits a nonempty open set $U \subset \text{Spec } A$ such that

$$\text{reg}(S^{(p(q)-i,1,\ldots,1)}(E_q)) < C$$

for all closed $q \in U$ ($p(q) = \text{char } A/q$). By lemma 3.3, these sheaves are $F$-ample. Then the Carter-Lusztig resolution (which has length bounded by $r = \text{rank}(E)$) together with lemma 2.1 shows that $\phi(E) < r$. □

**Corollary 6.4.** Let $E_i$ be ample vector bundles. Then

$$\phi(E_1 \otimes E_2 \otimes \ldots E_m) < \text{rank}(E_1) + \text{rank}(E_2) + \ldots \text{rank}(E_m).$$

The analogue for a pair is the following:

**Theorem 7.** Let $X$ be a smooth projective variety defined over a field of characteristic $0$, and let $E$ be a vector bundle of rank $r$ on $X$. Suppose there exists a reduced normal crossing divisor $D$, a positive integer $n$, and a divisor $0 \leq D' <_{\text{strict}} nD$ such that $S^n(E)(-D')$ is ample. Then $\phi(E, D) < r$.

**Remark 6.5.** The hypothesis amounts to the condition that the “vector bundle” $E(-\Delta)$ is ample for some fractional effective $\mathbb{Q}$-divisor $\Delta = \frac{1}{n}D'$ supported on $D$.

Proof. The proof is very similar to the previous one, so we will just summarize the main points. Let $M = O_X(-1)$ with $O_X(1)$ very ample. Choose a thickening of $(X, E, D, M)$. A small modification of corollary 6.3 shows that the regularity of the sheaves

$$S^{(Nn+j-i,1,\ldots,1)}(E_q)(-ND_q'), 0 \leq i < r, 0 < j < n$$

can be made less than a given $C$ for all $N$ greater than some $N_0$ depending on $C$. All but finitely primes are of the form $Nn+j$ for $N$ and $j$ as above. Thus the above sheaves will be $F$-ample for almost all $q$. The Carter-Lusztig resolution shows that $\phi(E^{(p(q))}(-ND')) < r$ which implies the theorem. □
7. An \( F \)-ampleness criterion

The notion of geometric positivity was introduced in [A2]. Although the methods are very different, there appear to be some parallels between \( F \)-ampleness and geometric positivity. The following result is an analogue of [loc. cit., cor. 3.10].

**Theorem 8.** Let \( E \) be a rank \( r \) vector bundle on a smooth projective variety \( X \) such that \( \det(E) \) is ample and \( S^rN(E) \otimes \det(E)^{-N} \) is globally generated for some \( N > 0 \) prime to char \( k \). Then \( E \) is \( F \)-ample.

**Remark 7.1.** The hypothesis that \( S^rN(E) \otimes \det(E)^{-N} \) is globally generated implies that \( E \) is strongly semistable in the sense of [A2, p. 247]. We leave it as an open problem to determine whether \( F \)-ampleness follows only assuming strong semistability of \( E \) and ampleness of \( \det(E) \).

As usual all the work will be in characteristic \( p > 0 \). Let \( q = p^n \) for some \( n > 0 \). In this section we will modify our previous conventions, and write \( F_X : X \rightarrow X \) for the absolute \( q \)th power Frobenius. Let \( P = P(E) \) and \( P' = P(E^q) \) with canonical projections denoted by \( \pi \) and \( \pi' \). Consider the commutative diagram

\[
\begin{array}{c}
\Phi \quad \Phi^q \quad \Phi^q \\
\downarrow \quad \downarrow \quad \downarrow \\
\pi' \quad \pi' \quad \pi \\
\downarrow \quad \downarrow \quad \downarrow \\
X \quad F_X \quad X
\end{array}
\]

where the right hand square is cartesian. \( \Phi \) is the relative \( q \)th power Frobenius associated to \( \pi \). Let \( \Sigma_N = S^rN(E) \otimes \det(E)^{-N} \).

**Proposition 7.2.** If \( \Sigma_{q-1} \) is globally generated, then \( O_{P'} \rightarrow \Phi_*O_P \) splits.

**Proof.** By Grothendieck duality for finite flat maps [H3, ex. III 6.10, 7.2], the proposition is equivalent to the splitting of the trace map

\[
\text{tr} : \Phi_*\omega_{P'/P} \rightarrow O_{P'}
\]

We have

\[
F_P^*O_P(1) = O_P(q) \\
\phi^*O_P(1) = O_{P'}(1),
\]

therefore

\[
\Phi^*O_{P'}(1) = O_P(q).
\]

Also

\[
\omega_{P'/X} = O_{P'}(-r) \otimes (\pi')^*\det(E(q)) = O_{P'}(-r) \otimes (\pi')^*\det(E)^q
\]

\[
\omega_{P'/X} = O_{P'}(-r) \otimes \pi^*\det(E)
\]

[H3, ex. III 8.4]. Therefore

\[
\Phi^*\omega_{P'/X} = O_P(-qr) \otimes \pi^*\det(E)^q
\]

\[
\omega_{P'/P'} = O_{P'}((q-1)r) \otimes \pi^*\det(E)^{1-q}.
\]

Observe that

\[
(5) \quad \pi_*\omega_{P'/P'} = \Sigma_{q-1}.
\]
Suppose that $0 < i < r$, then using the projection formula and the previous computations
\[
R^i\pi'_*[(\Phi_*\omega_{P'/P})(-i)] = R^i\pi_*\omega_{R^i}(q-i)
= R^i\pi_*\omega_{r-q-1} \otimes \pi^*\det(E)^{1-q}
= 0
\]
Thus $\Phi_*\omega_{P'/P}$ is regular relative to $\pi'$, and it follows that the canonical map
\[(\pi')^*\pi'_*\Phi_*\omega_{P'/P} \to \Phi_*\omega_{P'/P}
\]
is surjective [FL, V, 2.2]. By (5), this gives a surjection
\[(\pi')^*\Sigma_{q-1} \to \Phi_*\omega_{P'/P}
\]
Composing this with the Grothendieck trace (4), gives a surjection
\[
(\pi')^*\Sigma_{q-1} \to O_{P'}
\]
Since $\Sigma_{q-1}$ is globally generated, there exists a morphism $s : O_{P'} \to (\pi')^*\Sigma_{q-1}$ such that the composite $O_{P'} \to O_{P'}$ is nonzero. This corresponds to an element $a \in k'$ with (4) gives a splitting of (4).

**Corollary 7.3.** With the same assumptions as in the proposition, $E^{(q)}$ is a direct summand of $S^q(E)$.

**Proof.** By the projection formula, the canonical map
\[(O_{P'}(1) \to \Phi_*\Phi^*O_{P'}(1) = \Phi_*O_{P'}(q)
\]
can be identified with
\[
O_{P'}(1) \to O_{P'}(1) \otimes \Phi_*O_{P'}
\]
This splits. Applying $\pi'_*$ to (7) yields a split injection $E^{(q)} \to S^q(E)$.

**Proof of theorem 8.** Choose $q = p^n \equiv 1 (mod N)$; $q$ can be chosen arbitrarily large. Then $\Sigma_{q-1}$ is globally generated since it is a quotient of $S^q(\Sigma_N)$. It follows that $E^{(q)}$ is a direct summand of $S^q(E)$. Since $S^N(E) = \Sigma_N \otimes det(E)^N$ is ample, the same is true for $E$. Thus $\text{reg}(E^{(q)}) = \text{reg}(S^q(E)) \to -\infty$ as $q \to \infty$.

This finishes the proof in characteristic $p$, the remaining case is handled as usual.

8. THE MAIN VANISHING THEOREM

As a warm up to the main theorem, we will extend some of the conclusions of theorem 4 to a more general class of spaces called Frobenius split varieties [MR]. This means that the map $O_X \to F_*O_X$ splits (actually, we only need the ostensibly weaker property that this map splits in the derived category). Proposition 7.2 implies that projective spaces are Frobenius split. Other examples of Frobenius split varieties include quotients of semisimple groups by parabolic subgroups [loc. cit], and most mod $p$ reductions of a smooth Fano variety [5. 4.11].

**Proposition 8.1.** Suppose that $X$ is a smooth projective variety such that $O_X \to F_*O_X$ splits in the derived category of $O_X$-modules. Then $H^i(X, E) = 0$ for $i > \phi(E)$. If $E$ is locally free, then $H^i(X, \omega_X \otimes E) = 0$ for $i > \phi(E)$. 

Proof. For the first statement, use the fact there is an injection
\[ H^i(X, \mathcal{E}) \hookrightarrow H^i(X, \mathcal{E} \otimes F_o \mathcal{O}_X) \]
because it splits by hypothesis. On the other hand the projection formula gives
\[ H^i(X, \mathcal{E} \otimes F_o \mathcal{O}_X) \cong H^i(X, \mathcal{E}^{(p)}) \]
By iterating we get a sequence of injections
\[ H^i(\mathcal{E}) \hookrightarrow H^i(\mathcal{E}^{(p)}) \hookrightarrow \ldots H^i(\mathcal{E}^{(p^n)}) = 0 \]
for \( i > \phi(\mathcal{E}) \) and \( n \gg 0 \).
We can replace \( \mathcal{E} \) by \( \mathcal{E}^* \) and \( i \) by \( \text{dim}X - i \) in the above sequence of injections. This together with Serre duality yields the result. \( \square \)

When \( \mathcal{E} \) is a vector bundle on \( \mathbb{P}^n \), the proposition yields the vanishing
\[ H^i(\mathcal{E}(j - n - 1)) = 0 \]
for \( j \geq 0, i > \phi(\mathcal{E}) \geq \phi(\mathcal{E}(j)) \), obtained earlier.

When \( k \) is a perfect field, let \( W(k) \) be the ring of Witt vectors over \( k \), and \( W_2(k) = W(k)/p^2W(k) \). It is helpful to keep the following example in mind: if \( k = \mathbb{Z}/p\mathbb{Z} \), then \( W(k) \) is the ring of \( p \)-adic integers, so that \( W_2(k) \cong \mathbb{Z}/(p^2) \).

**Theorem 9.** Let \( k \) be a perfect field of characteristic \( p > n \), and let \( X \) be a smooth \( n \)-dimensional projective \( k \)-variety with a reduced normal crossing divisor \( D \). Suppose that \( \mathcal{E} \) is a coherent sheaf on \( X \). If \( (X, D) \) can be lifted to a pair over \( \text{Spec} W_2(k) \). Then
\[ H^i(X, \Omega^j_X(\log D)(-D) \otimes \mathcal{E}) = 0 \]
for \( i + j > n + \phi(\mathcal{E}, D) \).

**Remark 8.2.** Note that \( \mathcal{E} \) is not required to lift. It is possible to obtain a weaker statement when \( p \leq n \), but we won’t need it.

The proof is based on the following lemmas.

**Lemma 8.3.** Suppose that \( 0 \leq D' \leq pD \) is a divisor such that
\[ H^i(\Omega^j_X(\log D)(-D') \otimes \mathcal{E}^{(p)}) = 0 \]
for all \( i + j > N \), then
\[ H^i(\Omega^j_X(\log D)(-D'_{\text{red}}) \otimes \mathcal{E}) = 0 \]
for all \( i + j > N \). 
Proof. Set \( D_1 = D'_{\text{red}} \) and \( B = pD_1 - D' \). To avoid confusion, we will say few words about our conventions. The differentials on \( \Omega^*_X(\log D)(B) \) and \( \Omega^*_X(\log D) \) are inherited from the complex of meromorphic forms. All other differentials are induced from these using tensor products and pushforwards. There is a quasi-isomorphism
\[ \Omega^*_X(\log D) \cong \Omega^*_X(\log D)(B) \]
where \( B = pD_1 - D' \) by [Ha, 3.3] (see also [MS, 4.1]). Tensoring both sides with \( \mathcal{E}^{(p)}(-D') \) yields
\[ \Omega^*_X(\log D) \otimes \mathcal{E}^{(p)}(-D') \cong \Omega^*_X(\log D) \otimes [\mathcal{E}(-D_1)]^{(p)} \]
This implies that
\[ F_*(\Omega^*_X(\log D)(-D') \otimes \mathcal{E}^{(p)}) \cong F_*(\Omega^*_X(\log D) \otimes F^*(\mathcal{E}(-D_1))). \]
Lemma 5.6 shows that the right side is quasi-isomorphic to

\[ [F, \Omega^*_X(\log D)] \otimes \mathcal{E}(\mathcal{-D_1}) \]

By [DI, 4.2] (and the remarks of section 1), this is quasi-isomorphic to

\[ \left( \bigoplus_j \Omega^*_X(\log D)[-j] \right) \otimes \mathcal{E}(\mathcal{-D_1}). \]

The spectral sequence

\[ H^i(\Omega^*_X(\log D)(-D') \otimes \mathcal{E}(\mathcal{p^a})) \Rightarrow H^{i+j}(\Omega^*_X(\log D)(-D') \otimes \mathcal{E}(\mathcal{p^a})) \]

together with the hypothesis shows that the abutment vanishes for \( i + j > N \). Therefore

\[ H^i(\Omega^*_X(\log D)(-D') \otimes \mathcal{E}(\mathcal{p^a})) \cong \bigoplus_j H^{i-j}(\Omega^*_X(\log D) \otimes \mathcal{E}(\mathcal{-D_1})) \]

vanishes for \( i > N \).

**Lemma 8.4.** Suppose that \( 0 \leq D' \leq p^a D \) is a divisor such that

\[ H^i(\Omega^*_X(\log D)(-D') \otimes \mathcal{E}(\mathcal{p^a})) = 0 \]

for all \( i + j > N \), then

\[ H^i(\Omega^*_X(\log D)(-D_{red}' \otimes \mathcal{E}) = 0 \]

for all \( i + j > N \).

**Proof.** We prove this by induction on \( a \). The case where \( D' = 0 \) is straightforward, so we assume that \( D' \neq 0 \). The initial case \( a = 1 \) is the previous lemma. Suppose that the lemma holds for \( a \), and suppose that \( (\mathcal{E}, D') \) satisfies the hypothesis of the lemma with \( a \) replaced by \( a + 1 \). Since \( \{1, \ldots, p\} \) forms a set of representatives of \( \mathbb{Z}/p\mathbb{Z} \), we can decompose \( D' = pD_1 + D_2 \) such that \( D_{red}' = D_{red}' \), \( 0 < D_2 \leq pD \) and \( 0 \leq D_1 <_{\text{strict}} p^a D \). These assumptions guarantee that the divisor \( D_{red}' + D_1 \) is less than or equal to \( p^a D \) and has the same support as \( D' \). Then

\[ \mathcal{E}(\mathcal{p^{a+1}})(-D') = \mathcal{E}(\mathcal{p}(-D_2)) \]

where \( \mathcal{E}_1 = \mathcal{E}(\mathcal{p}(-D_1)) \). By assumption,

\[ H^i(\Omega^*_X(\log D)(-D') \otimes \mathcal{E}(\mathcal{p^{a+1}})) = H^i(\Omega^*_X(\log D)(-D_2) \otimes \mathcal{E}_1) = 0 \]

for \( i + j > N \). Lemma 8.3 implies

\[ H^i(\Omega^*_X(\log D)(-D_{red}' \otimes \mathcal{E}_1) = H^i(\Omega^*_X(\log D)(-D_{red}' - D_1) \otimes \mathcal{E}(\mathcal{p^{a+1}}) = 0 \]

for \( i + j > N \). Induction yields the desired conclusion.

**Proof of theorem.** By definition, \( \phi(\mathcal{E}, D) = \phi(\mathcal{E}(\mathcal{p}(-D'))) \) for some \( 0 \leq D' <_{\text{strict}} p^a D \). We can assume \( a > 0 \) since we can replace \( \mathcal{E}(\mathcal{p}(-D')) \) by a Frobenius power. Therefore

\[ H^i(\Omega^*_X(\log D)(-D' \otimes \mathcal{E}(\mathcal{p^{b+1}})) = H^i(\Omega^*_X(\log D)(-D) \otimes (\mathcal{E}(\mathcal{p}(-D'))^{(p^b)}) = 0 \]

for \( b > 0 \) and all \( i > \phi(\mathcal{E}, D) \) and all \( j \). Since the support of \( p^b D' + D \) is \( D \) and the coefficients of this divisor are less than or equal to \( p^{a+b} \), lemma 8.4 implies the theorem.
Corollary 8.5. Suppose that \((X, D, \mathcal{E})\) is as above and \(\text{char } k = 0\), then

\[ H^i(X, \Omega^j_X \log D ^{-D}) \otimes \mathcal{E} = 0 \]

for \(i + j > n + \phi(\mathcal{E}, D)\). In particular,

\[ H^i(X, \Omega^j_X \otimes \mathcal{E}) = 0 \]

for \(i + j > n + \phi(\mathcal{E})\)

Proof. Choose an arithmetic thickening of \((X, D, \mathcal{E})\). Almost all of the closed fibers satisfies the conditions of the theorem. Therefore the corollary follows by semicontinuity.

Corollary 8.6. Suppose that \((X, D, \mathcal{E}, k)\) is as in theorem or corollary 8.5, and that \(L\) is an arithmetically nef line bundle on \(X\). Then

\[ H^i(X, \Omega^j_X \log D ^{-D}) \otimes \mathcal{E} \otimes L = 0 \]

for \(i + j > n + \phi(\mathcal{E}, D)\).

Proof. By proposition B.1, \(L\) is \(F\)-semipositive, hence \(\phi(\mathcal{E} \otimes L, D) = \phi(\mathcal{E}, D)\). So the corollary follows from theorem 9 in positive characteristic, or the previous corollary in characteristic 0.

Corollary 8.7. Suppose that \((X, D, \mathcal{E}, k)\) satisfy the conditions of the theorem or the corollary 8.5, then

\[ \text{Ext}^i(\mathcal{E}, \Omega^j_X \log D) = 0 \]

for \(i + j < n - \phi(X, D)\). If \(\mathcal{E}\) is locally free, then

\[ H^i(X, \Omega^j_X \log D \otimes \mathcal{E}^*) = 0 \]

for \(i + j < n - \phi(X, D)\).

Proof. This is a consequence of Serre duality.

Corollary 8.8. (Le Potier) Suppose that \(\text{char } k = 0\) and \(\mathcal{E}_i\) are ample locally free sheaves on a smooth variety \(X\), then

\[ H^i(X, \Omega^j_X \otimes \mathcal{E}_1 \otimes \ldots \otimes \mathcal{E}_m) = 0 \]

for \(i + j \geq n + \text{rank}(\mathcal{E}_1) + \ldots + \text{rank}(\mathcal{E}_m)\)

Proof. Follows from corollary 8.5 and corollary 8.4.

The next result is a generalization of the Kawamata-Viehweg vanishing theorem [Ka, 3]. (To obtain the statement given in the introduction, set \(\Delta = \frac{1}{m}D'\).)

Corollary 8.9. With \((X, D, \mathcal{E}, k)\) as in the corollary 8.5. Suppose that there is a positive integer \(m\), and a divisor \(0 \leq D' \leq (m - 1)D\) such that \(S^m(\mathcal{E})(-D')\) is ample. Then

\[ H^i(X, \Omega^j_X \log D ^{-D}) \otimes \mathcal{E} = 0 \]

for \(i + j \geq n + \text{rank}(\mathcal{E})\). In particular,

\[ H^i(X, \omega_X \otimes \mathcal{E}) = 0 \]

for \(i \geq \text{rank}(\mathcal{E})\).

Proof. This follows from theorem 7 and corollary 8.5.
Corollary 8.10. Suppose that char \( k = 0 \) and \( \mathcal{E} \) is a locally free sheaf on a projective \( k \)-variety \( Z \) with rational singularities, then
\[
H^i(Z, \omega_Z \otimes \mathcal{E}) = 0
\]
for \( i > \phi_{\text{gen}}(\mathcal{E}) \).

Proof. By the previous corollary, \( H^i(Y, \omega_Y \otimes f^*\mathcal{E}) \) vanishes for \( i > \phi_{\text{gen}}(\mathcal{E}) \), for some resolution of singularities \( f : Y \to Z \). We have \( Rf_*\omega_Y = \omega_Z \) because \( Z \) has rational singularities, and so the corollary follows.

Corollary 8.11. Suppose that char \( k = 0 \) and \( \mathcal{E} \) is the pull back of an ample vector bundle under a surjective morphism \( f : X \to Y \) with \( X \) smooth, then
\[
H^i(X, \omega_X \otimes \mathcal{E}) = 0
\]
for \( i \geq \text{rank}(\mathcal{E}) + d \) where \( d \) is the dimension of the generic fiber of \( f \).

Proof. Follows from corollary 2.7 and corollary 8.10.

9. Some applications

In this section, we work over \( \mathbb{C} \). We start with a refinement of the Lefschetz hyperplane theorem (which corresponds to case B with \( \mathcal{E} = O_X \) and \( r = m = 1 \)).

Proposition 9.1. Let \( D \subset X \) be a smooth divisor on a smooth \( n \)-dimensional projective variety. Suppose that \( m > 0 \).

A. If \( S^m(\mathcal{E})(rD) \) is ample for some \( -m < r \leq 0 \), then
\[
H^i(X, \Omega^j_X \otimes \mathcal{E}) \to H^i(D, \Omega^j_D \otimes \mathcal{E})
\]
is bijective if \( i + j \geq n + \text{rank}(\mathcal{E}) \) and surjective if \( i + j = n + \text{rank}(\mathcal{E}) - 1 \).

B. If \( S^m(\mathcal{E})(rD) \) is ample for some \( 0 < r \leq m \), then
\[
H^i(X, \Omega^j_X \otimes \mathcal{E}^*) \to H^i(D, \Omega^j_D \otimes \mathcal{E}^*)
\]
is bijective if \( i + j \leq n - \text{rank}(\mathcal{E}) - 1 \) and injective if \( i + j = n - \text{rank}(\mathcal{E}) \).

Proof. We have an exact sequence
\[
0 \to \Omega^i_X(logD)(-D) \to \Omega^j_X \to \Omega^j_D \to 0.
\]
Tensoring this with \( \mathcal{E} \) and applying corollary 8.9 proves A. For B, tensor this with \( \mathcal{E}^* \) and observe that by Serre duality and corollary 8.9
\[
H^i(X, \Omega^j_X(logD)(-D) \otimes \mathcal{E}^*) \cong H^{n-i}(X, \Omega^{n-j}_X(logD)(-D) \otimes \mathcal{E}(D))^* = 0
\]
when \( i + j \leq n - \text{rank}(\mathcal{E}) \).

Given an algebraic variety \( Y \) and algebraic coherent sheaf \( \mathcal{F} \) over \( Y \), we denote the corresponding analytic objects by \( Y^\text{an} \) and \( \mathcal{F}^\text{an} \). Given a closed subvariety \( Z \subset Y \), let
\[
codim(Z, Y) = \begin{cases} 
\text{the codimension of } Z \text{ if } Z \neq \emptyset \\
\dim Y + 1 \text{ otherwise}
\end{cases}
\]

Our goal is to prove a vanishing theorem for ample vector bundles over quasiprojective varieties. This generalizes some results for line bundles due to Bauer, Kosarew [BK2] and the author [A1].
Theorem 10. Suppose that $U$ is a smooth quasiprojective variety with a possibly singular projective compactification $Y$. Let $E$ be the restriction to $U$ of an ample vector bundle on some compactification of $U$ (possibly other than $Y$). Then

$$H^i(U^{an}, (\Omega_U^i \otimes E^*)^{an}) = H^i(U, \Omega_U^i \otimes E^*) = 0$$

for $i + j \leq \text{codim}(Y - U, Y) - \text{rank}(E) - 1$.

As a first step, we need a generalization of Steenbrink’s vanishing theorem [St].

Proposition 9.2. Suppose that $f : X \to Y$ is a desingularization of an $n$-dimensional projective variety such that $X$ possesses a reduced normal crossing divisor $D$ containing the exceptional divisor. If $E$ is a nef (e.g. globally generated) vector bundle, then

$$R^if_*[\Omega^i_X((logD)(-D)) \otimes E] = 0$$

for $i + j \geq n + \text{rank}(E)$. For $N >> 0$, the Leray spectral sequence and Serre vanishing yields

$$H^r(U, \Omega^j_U((logD)(-D)) \otimes E \otimes f^*L^j) = 0$$

for $i + j \geq n + \text{rank}(E)$. We can assume that these sheaves are all globally generated by increasing $N$ if necessary. Thus they must vanish.

Corollary 9.3. Let $Y$ be a projective variety, $Z \subset Y$ a closed subvariety containing the singular locus, and $f : X \to Y$ a resolution of singularities which is an isomorphism over $Y - Z$ such that $D = f^{-1}(Z)_{red}$ is a divisor with normal crossings. Then for any nef vector bundle $E$ on $X$,

$$H^i_D(X, \Omega^i_X((logD)) \otimes E^*) = 0$$

for $i + j \leq \text{codim}(Y, Z) - \text{rank}(E)$.

Proof. This is a generalization of [A1, thm 1]. A proof of this corollary can be obtained by simply replacing Steenbrink’s theorem with proposition [2] in the proof given there.

Proof of theorem [1]. The groups $H^i(U, F)$ and $H^i(U^{an}, F^{an})$ are isomorphic for any coherent sheaf $F$ on $Y$ and $i < \text{codim}(Y - U, Y) - 1$ by [H2, IV 2.1]. So it suffices to prove the vanishing in the algebraic category. Let $Z = Y - U$ and let $f : X \to Y$ be a desingularization satisfying the assumptions of corollary 9.3. We can assume that $X$ also dominates the compactification where $E$ extends to an ample bundle. We use the same symbol for this extension, and its pullback to $X$. Corollary [8] and theorem [9] implies that $\phi(E, D) < \text{rank}(E)$. Corollary [9] shows that

$$H^{i+1}_D(X, \Omega^i_X((logD)) \otimes E^*) = 0.$$ 

Then the exact sequence for local cohomology yields a surjection

$$H^i(X, \Omega^i_X((logD)) \otimes E^*) \to H^i(U, \Omega^i_U((logD)) \otimes E^*) = H^i(U, \Omega_U^i \otimes E^*).$$
The cohomology group on the left vanishes as a consequence of corollary 8.7.

REFERENCES

[A1] D. Arapura, Local cohomology of sheaves of differential forms and Hodge theory, J. Reine Angew. Math. 409, (1990)

[A2] D. Arapura, A class of sheaves satisfying Kodaira’s vanishing theorem, Math. Ann. 318 (2000)

[BK1] I. Bauer, S. Kosarew, On the Hodge spectral sequence for some classes of non-complete algebraic varieties, Math. Ann. 284 (1989)

[BK2] I. Bauer, S. Kosarew, Some aspects of Hodge theory on noncomplete algebraic manifolds Prospects in complex geometry (Katata and Kyoto, 1989), 281–316, Lect. Notes in Math., 1468, Springer-Verlag (1991).

[CL] R. Carter, G. Lusztig, On modular representations of the general linear group and symmetric groups, Math. Z. 136 (1974), 193–242

[C] M. de Cataldo, Vanishing via lifting to second Witt vectors and a proof of an isotriviality result, J. Algebra 219 (1999), 255-265

[D1] P. Deligne, L. Illusie, Relevements modulo $p^2$ et decomposition du complexe de de Rham, Inv. Math. 89 (1987), 247-280

[EGA] A. Grothendieck, J. Dieudonné, Éléments de géométrie algébrique Publ. IHES (1960-1967)

[EV] H. Esnault, E. Viehweg, Lectures on vanishing theorems, Birkhäuser (1993)

[F] W. Fulton, Young Tableaux Cambridge U. Press (1997)

[FL] W. Fulton, S. Lang, Riemann-Roch Algebra, Springer-Verlag (1985)

[G] D. Gieseker, P-ample bundles and their Chern classes, Nagoya Math. J 43 (1971), 91-116

[Ha] N. Hara, A characterization of rational singularities in terms of injectivity of Frobenius maps Amer. J. Math. 120 (1998), 981-996

[H1] R. Hartshorne, Ample vector bundles, Publ. IHES 29 (1966), 63-94

[H2] R. Hartshorne, Ample subvarieties of algebraic varieties, Lect. Notes in Math. 156, Springer-Verlag (1970)

[H3] R. Hartshorne, Algebraic geometry, Springer-Verlag (1977)

[Ho] G. Horrocks, Vector bundles on the punctured spectrum of a local ring Proc. Lond. Math. Soc. 14 (1964), 689-713

[I] L. Illusie, Réduction semi-stable et décomposition de complexes de de Rham à coefficients Duke Math. J. 60 (1990)

[I2] L. Illusie, Frobenius et dégénérescence de Hodge Introduction à la Théorie de Hodge, Soc. Math. Frances (1996)

[J] J. C. Jantzen, Representations of algebraic groups, Academic Press (1987)

[Ka] Y. Kawamata, A generalization of Kodaira-Ramanujam’s vanishing theorem, Math. Ann. 261 (1982) 43-46

[L] J. Le Potier, Annulation de la cohomologie à valeurs dans un fibré vectoriel holomorphe positif de rang quelconque Math. Ann. 218 (1975)

[MR] V. Mehta, A. Ramanathan, Frobenius splitting and cohomology vanishing for Schubert varieties Ann. Math. 122 (1985)

[MS] V. Mehta, V. Srinivas, A characterization of rational singularities, Asian J. Math. 1 (1997), 249-271

[Mi] L. Migliorini, Some observations on cohomologically p-ample bundles. Ann. Mat. Pura Appl. 164 (1993), 89–102

[M] D. Mumford, Lectures on curves on an algebraic surface. Princeton Univ. Press (1966)

[SS] B. Shiffman, A. Sommese, Vanishing theorems on complex manifolds, Birkhäuser (1985)

[S] K. Smith, Vanishing, singularities and effective bounds via prime characteristic local algebra, Algebraic Geometry, Santa Cruz 1995, AMS.

[St] J. Steenbrink, Vanishing theorems for singular spaces, Astérisque 130 (1984), 330-341

[V] E. Viehweg, Vanishing theorems, J. f. Reine Angew. Math. 335 (1982), 1-8
Let $Y$ be a noetherian scheme, let $f: X \to Y$ be a proper morphism, and let $E$ be a vector bundle on $X$. For each $y \in Y$, let $E_y$ be the restriction of $E$ to the fiber $X_y$. Recall that $E$ is $f$-nef if $E_y$ is nef for every closed $y \in Y$ (see, for instance, [Ke, Definition 2.9]). If $Y$ is affine, then the property of $E$ being $f$-nef does not depend on $f$, so we may simply say that $E$ is nef [Ke, Proposition 2.15].

**Definition 1.** Let $X$ be a proper scheme over a field $k$, and let $E$ be a vector bundle on $X$. If $\text{char } k = 0$, then $E$ is arithmetically nef if there exists a thickening $(\tilde{X} \to \text{Spec } A, \tilde{E})$ such that $\tilde{E}$ is nef. For convenience, if $\text{char } k = p > 0$, we say that $E$ is arithmetically nef if $E$ is nef.

Note that if $E$ is arithmetically nef, then $\tilde{E}$ will be nef on every fiber of a certain thickening [Ke, Lemma 2.18]. In particular, if $E$ is arithmetically nef, then $E$ is nef.

Like nefness, the property of being arithmetically nef behaves well under pullbacks.

**Lemma A.1.** Let $E$ be a vector bundle on a proper scheme $X$ over a field $k$, and let $f: X' \to X$ be a proper morphism.

1. If $E$ is arithmetically nef, then $f^*E$ is arithmetically nef, and
2. if $f$ is surjective and $f^*E$ is arithmetically nef, then $E$ is arithmetically nef.

**Proof.** By replacing $X$ with $\mathbb{P}(E)$ and $X'$ with $\mathbb{P}(f^*E)$, we may assume that $E$ equals a line bundle $L$.

If $\text{char } k = 0$, then choose a thickening $(\tilde{f}: \tilde{X}' \to \tilde{X}, \tilde{L})$ such that $\tilde{f}$ is proper and also surjective if $f$ is surjective [EGA, IV$_3$, 8.10.5]. If $L$ is arithmetically nef, then upon further shrinking the thickening we may assume $\tilde{L}$ is nef, and so $\tilde{f}^*\tilde{L}$ is nef [Ke, Lemma 2.17]. Hence $f^*L$ is arithmetically nef. Now if $f^*L$ is arithmetically nef, then $L$ is arithmetically nef by a similar argument [Ke, loc. cit.]. If $\text{char } k = p > 0$, then these are just statements about nef line bundles [Ke, loc. cit.].

Being arithmetically nef also behaves well under tensor product.

**Lemma A.2.** Let $L$ and $M$ be line bundles on a proper scheme $X$. Then

1. $L$ is arithmetically nef, if and only if $L^n$ is arithmetically nef for all $n > 0$, if and only if $L^n$ is arithmetically nef for some $n > 0$, and
2. if $L$ and $M$ are arithmetically nef, then $L \otimes M$ is arithmetically nef.

**Proof.** These statements follow immediately from the definition of nef.

It is natural to conjecture that if $L$ is nef, then $L$ is arithmetically nef, but we have been unable to prove this. However, we do have a non-trivial collection of examples.

**Proposition A.3.** Let $L$ be a line bundle on a proper scheme $X$ over a field $k$. If $L$ is semi-ample (i.e., there exists $n > 0$ such that $L^n$ is generated by global sections) or $L$ is numerically trivial, then $L$ is arithmetically nef.
Proof. The case \( \text{char } k = p > 0 \) is trivial, so assume \( \text{char } k = 0 \). First suppose that \( L \) is semi-ample. We may replace \( L \) with \( L^n \) and assume that \( L \) is generated by global sections. Then \( L \) defines a \( k \)-morphism \( f : X \to \mathbb{P}^m \) for some \( m \), and \( L = f^*O_{\mathbb{P}^m}(1) \) [EGA, II 7.1]. Now \( f \) is proper [EGA, II 4.8e], so we may replace \( L \) with \( O_{\mathbb{P}^m}(1) \) by lemma A.1. Now since \( O(1) \) is ample, there exists a thickening such that \( O_{\mathbb{P}^m}(1) \) is ample [EGA, III, 4.7.1]. Hence \( O_{\mathbb{P}^m}(1) \) is arithmetically nef.

Now suppose that \( L \) is numerically trivial. Any pullback of \( L \) is also numerically trivial [EGA, Lemma 2.17]. Thus by lemma A.2 we may replace \( X \) with a Chow cover and thus assume that \( X \) is projective. We may also replace \( X \) with the disjoint union \( \coprod_i X_i \) where the \( X_i \) are the reduced, irreducible components of \( X \). Thus we may assume that \( X \) is integral. There exists a projective, surjective morphism \( X' \to X \) such that \( X' \) is geometrically integral [EGA, Lemma 3.3], and thus we may assume that \( X \) is geometrically integral.

Let \( H \) be an ample divisor on \( X \). We may choose a thickening \( (\tilde{\pi} : \tilde{X} \to \text{Spec } A, \tilde{H}, \tilde{L}) \) such that \( \tilde{H} \) is ample [EGA, III, 4.7.1] and \( \tilde{\pi} \) is flat [EGA, IV, 8.9.4]. Further, we may assume that all fibers of \( \tilde{\pi} \) are geometrically integral [EGA, IV, 12.2.4].

For any line bundles \( N, M \) on \( \tilde{X} \) and \( s \in \text{Spec } A \), let \( N_s, M_s \) be the restriction of \( N, M \) to the fiber \( \tilde{X}_s \). Then since \( \tilde{\pi} \) is flat, the intersection numbers \( (N_s, M_s \dim X - r) \) are independent of \( s \) [EGA, Remark 3.5]. Now \( \tilde{L}_s \) is numerically trivial if and only if

\[
(L_{\tilde{\pi}}, H_s \dim X - 1) = (\tilde{L}_s, H_s \dim X - 2) = 0
\]

by [EGA, p. 305, Corollary 3]. But since these intersection numbers are 0 at the generic point, they are 0 at each \( s \in \text{Spec } A \). Thus \( L \) is arithmetically nef.

\begin{corollary}
Let \( X \) be a projective scheme with \( \dim X \leq 1 \). If \( L \) is a nef line bundle, then \( L \) is arithmetically nef.
\end{corollary}

Proof. Using lemma A.2 we may assume that \( X \) is integral. Then \( L \) is either numerically trivial or ample, and hence arithmetically nef by proposition A.3.

We now consider arithmetically nef line bundles on a surface \( X \), that is, an integral scheme of dimension 2.

\begin{corollary}
Let \( X \) be a projective surface, and let \( L \) be a nef line bundle such that \( L^n \) is effective for some \( n > 0 \) (for example, if \( L \) is big). Then \( L \) is arithmetically nef.
\end{corollary}

Proof. We assume that the characteristic of the ground field is 0. Let \( H \) be an ample line bundle. By lemma A.2 we may replace \( L \) by \( L^n \) and hence assume that \( L \cong O(D) \) for an effective divisor \( D \). Since \( L|_D \) is arithmetically nef by corollary A.4, we may choose an arithmetic thickening \( (\tilde{X}, \tilde{H}, \tilde{L} \cong \tilde{O}(D)) \) such that \( \tilde{L}|_D \) is nef and \( \tilde{H} \) is ample [EGA, III, 4.7.1].

Consider the short exact sequences

\[
H^i(\tilde{X}, \tilde{H}^n \otimes \tilde{L}^m) \to H^i(\tilde{X}, \tilde{H}^n \otimes \tilde{L}^{m+1}) \to H^i(\tilde{D}, \tilde{H}^n|_D \otimes \tilde{L}^{m+1}_D)
\]

with \( i > 0, m \geq 0, n > 0 \). We may fix \( n \) sufficiently large so that the leftmost group vanishes for \( m = 0 \) and the rightmost vanishes for all \( m \geq 0 \) [EGA, Theorem 1.5]. But then by induction on \( m \), the middle vanishes for all \( m \geq 0 \). Since any coherent sheaf \( \mathcal{F} \) on \( \tilde{X} \) is a quotient of a finite direct sum of \( \tilde{H}^{-\ell} \), we have that for any \( \mathcal{F} \),
there exists \( n \) such that \( H^i(\tilde{X}, \mathcal{F} \otimes \tilde{H}^m \otimes \tilde{L}^m) = 0 \) for \( i > 0, m \geq 0 \), and so \( \tilde{L} \) is nef [Ke, Proposition 5.18]. Thus \( L \) is arithmetically nef.

\section*{Appendix B. Arithmetically Nef Line Bundles Are \( F \)-Semi-positive}

In this section, we characterize \( F \)-semi-positive line bundles. Lemma 2.4 states that a line bundle \( L \) is \( F \)-ample if and only if it is ample. We later see in lemma 3.10 that any \( F \)-semi-positive vector bundle is arithmetically nef. Given these two facts, it is natural to conjecture that a line bundle \( L \) is \( F \)-semi-positive if and only if it is arithmetically nef, and indeed this is the case.

**Proposition B.1.** Let \( X \) be a projective variety over a field \( k \) and let \( L \) be a line bundle on \( X \). Then \( L \) is \( F \)-semi-positive if and only if \( L \) is arithmetically nef.

**Proof.** Given lemma 3.10, we need only show that an arithmetically nef line bundle is \( F \)-semi-positive. Let \( O_X(1) \) be a very ample line bundle. If \( \text{char } k = p > 0 \), there exists \( m \) such that \( H^i(X, O_X(m - i) \otimes L^n) = 0 \) for \( i > 0, n \geq 0 \) [Ke, Theorem 1.5]. Thus \( \text{areg}(L) \leq m \), so \( L \) is \( F \)-semi-positive.

If \( \text{char } k = 0 \), then choose a thickening \((\tilde{X}, \tilde{O}_X(1), \tilde{L})\) over a finitely generated \( \mathbb{Z} \)-algebra \( \mathcal{A} \subset k \) such that \( \tilde{O}_X(1) \) is very ample [EGA, III, 4.7.1] and \( \tilde{L} \) is nef. Again there exists \( m \) such that \( H^i(X, \tilde{O}_X(m - i) \otimes L^n) = 0 \) for \( i > 0, n \geq 0 \) [Ke, loc. cit.]. So by semicontinuity, for each closed point \( q \) of \( \text{Spec } \mathcal{A} \), we have \( \text{areg}(L_q) \leq m \). Thus \( \text{areg}(L) \leq m \) and \( L \) is again \( F \)-semi-positive.

\section*{Appendix C. Base Change}

Most of the seminal works on ample vector bundles, such as [H1, B, G], assumed that the base field \( k \) was algebraically closed and some of their proofs use this assumption. However, since this paper’s “reduction to characteristic \( p \)” methods require the non-algebraically closed case, we now prove a few standard lemmas which will allow the application of “algebraically closed results” to the general case.

Since we do not assume that our varieties are geometrically integral, we must also allow \( X \) to be a projective scheme. For a coherent sheaf \( \mathcal{F} \) we keep the same definitions of \( \phi(\mathcal{F}) \), \( F \)-ample, and \( F \)-semi-positive as given in sections 2 and 3, just with \( X \) as a scheme, projective over a field \( k \).

**Lemma C.1.** Let \( X \) be a projective scheme over a field \( k \) of characteristic \( p > 0 \), let \( \mathcal{E} \) be a vector bundle on \( X \), and let \( \mathcal{E}_{\text{red}} \) be the restriction of \( \mathcal{E} \) to the reduced scheme \( X_{\text{red}} \). Then \( \phi(\mathcal{E}) = \phi(\mathcal{E}_{\text{red}}) \).

**Proof.** This follows from a standard argument as in [H1, ex. III 3.1, 5.7], using the commutative diagram (1) with \( Y = X_{\text{red}} \). To see this, let \( f : X_{\text{red}} \to X \) be the natural immersion, let \( N \) be the nilradical of \( O_X \), and let \( \mathcal{F} \) be a coherent sheaf on \( X \). Then \( N^{r_0} = 0 \) for \( r_0 \) sufficiently large and we have short exact sequences

\[
0 \to N^{r+1} \mathcal{F} \to N^r \mathcal{F} \to N^r \mathcal{F}/N^{r+1} \mathcal{F} \to 0
\]

with \( 0 \leq r < r_0 \). Since \( f_*O_X = O_X/N \), there exist coherent \( \mathcal{G}_r \) on \( X_{\text{red}} \) such that \( f_* \mathcal{G} = N^r \mathcal{F}/N^{r+1} \mathcal{F} \). Thus, if \( i > \phi(\mathcal{E}_{\text{red}}) \), we have

\[
H^i(X_{\text{red}}, \mathcal{G}_r \otimes f^*\mathcal{E}(p^n)) = H^i(X, N^r \mathcal{F}/N^{r+1} \mathcal{F} \otimes \mathcal{E}(p^n)) = 0
\]

for \( 0 \leq r < r_0 \) and \( n \gg 0 \). Descending induction on \( r \) then gives \( H^i(X, \mathcal{F} \otimes \mathcal{E}(p^n)) = 0 \) for \( n \gg 0 \).
Lemma C.2. Let $X$ be a reduced projective scheme over a field $k$ of characteristic $p > 0$, let $X_i$ be the irreducible components of $X$, $i = 1, \ldots, r$, let $\mathcal{E}$ be a vector bundle on $X$, and let $\mathcal{E}_i$ be the restriction of $\mathcal{E}$ to $X_i$. Then $\phi(\mathcal{E}) = \max_i \phi(\mathcal{E}_i)$.

Proof. This follows from a standard argument as in [H3, ex. III 3.2, 5.7], using the commutative diagrams (1) for each $i = 1, \ldots, r$ with $Y = X_i$. Let $\mathcal{F}$ be a coherent sheaf on $X$ and let $I_j$ be the sheaf of ideals of $X_j$. Then $I_j \mathcal{F}$ has support on the $X_i$ with $i \neq j$ and $\mathcal{F}/I_j \mathcal{F}$ has a natural $\mathcal{O}_{X_j}$-module structure. So by induction on the number of irreducible components $r$, for $i > \max_i \phi(\mathcal{E}_i) \geq \phi(\mathcal{E}_j)$, we have short exact sequences

$$0 = H^i(I_j \mathcal{F} \otimes \mathcal{E}(p^n)) \rightarrow H^i(\mathcal{F} \otimes \mathcal{E}(p^n)) \rightarrow H^i(\mathcal{F}/I_j \mathcal{F} \otimes \mathcal{E}(p^n)) = 0$$

for $n \gg 0$. Thus $\phi(\mathcal{E}) \leq \max_i \phi(\mathcal{E}_i)$, and the reverse inequality is trivial.

Lemma C.3. Let $f: Y \rightarrow X$ be a finite, surjective morphism (not necessarily a $k$-morphism) of projective $k$-schemes, and let $\mathcal{E}$ be a vector bundle on $X$. Then $\phi(\mathcal{E}) = \phi(f^* \mathcal{E})$.

Proof. When $\text{char } k = p > 0$, this again follows from a standard argument as in [H3, ex. III 4.2, 5.7], using the commutative diagram (1). By lemmas C.1 and C.2, we may assume that $X$ and $Y$ are integral schemes. Then one may follow the argument outlined in [H3, ex. III 4.2].

When $\text{char } k = 0$, we can choose a thickening $(\tilde{f}: \tilde{Y} \rightarrow \tilde{X}, \tilde{\mathcal{E}})$ and assume that $\tilde{f}$ is finite and surjective [EGA IV, 8.10.5]. The claim then follows from the positive characteristic case.

Corollary C.4. Let $X$ be a projective scheme over a field of arbitrary characteristic. Then the conclusions of lemmas C.1 and C.2 remain true.

Proof. Let $\bigsqcup_i X_i$ be the disjoint union of the irreducible components of $X$ if $X$ is reduced. Both claims follow from lemma C.3 because the maps $X_{\text{red}} \rightarrow X$ and $\bigsqcup_i X_i \rightarrow X$ are finite and surjective.

We now show that all of our concepts of ampleness behave well under base change.

Lemma C.5. Let $k \subseteq k'$ be fields, let $X$ be a projective scheme over $k$, and let $\mathcal{E}$ be a vector bundle on $X$. Then $\phi(\mathcal{E}) = \phi(\mathcal{E} \otimes_k k')$. Also, each of the following properties hold for $\mathcal{E}$ if and only if they hold for $\mathcal{E} \otimes_k k'$ on $X \times_k k'$:

1. $F$-ample,
2. $F$-semipositive,
3. $p$-ample, if $\text{char } k = p > 0$,
4. ample,
5. nef,
6. arithmetically nef.

Proof. Let $f: X \times_k k' \rightarrow X$ be the base change, and let $O_X(1)$ be a very ample line bundle for $X$. Then $f^* O_X(1)$ is very ample on $X \times k'$ [EGA II, 4.4.10]. If $\text{char } k = p > 0$, then we have a commutative diagram (1) with $Y = X \times k'$.
Note that this diagram is not cartesian because $F_Y \neq F_X \times id_{k'}$. However, the commutivity of (3) gives
\[
H^i(Y, (f^*E)^{(p^n)} \otimes f^*O_X(b)) = H^i(Y, F_{X\to k'}^p f^*E \otimes f^*O_X(b)) = H^i(Y, f^*(E^{(p^n)}) \otimes f^*O_X(b))
\]
for $i \geq 0, n \geq 0, b \in \mathbb{Z}$. Now since $k \to k'$ is a flat morphism [H3, III 9.3],
\[
H^i(Y, (f^*E)^{(p^n)} \otimes f^*O_X(b)) = H^i(X, E^{(p^n)} \otimes O_X(b)) \otimes k' k'.
\]
Then $\phi(E) = \phi(f^*E)$ follows by corollary 2.3 since $k \to k'$ is faithfully flat. If $char k = 0$, it is clear by the definition of $\phi$ that $\phi(E) = \phi(f^*E)$ because an arithmetic thickening of $X$ is an arithmetic thickening of $X \times k'$. Now (1) is immediate and a similar proof gives (3).

For (1), consider the exact sequence
\[
H^0(X, E^{(p^n)} \otimes O_X(b)) \otimes_k O_X \to E^{(p^n)} \otimes O_X(b) \to C_n \to 0
\]
for fixed $b \in \mathbb{Z}$. Since $k \to k'$ is faithfully flat, we obtain a similar map of global sections of $(f^*E)^{(p^n)} \otimes f^*O_X(b)$ by tensoring the exact sequence with $- \otimes_k k'$ (or equivalently, pulling back by $f$). Then the cokernel $C_n = 0$ if and only if $C_n \otimes_k k' = 0$, so $E$ is $p$-ample if and only if $f^*E$ is $p$-ample.

Finally, (1)–(3) can be reduced to the case of a line bundle by working on $\mathbb{P}(E)$ and $\mathbb{P}(f^*E)$. The cases (1) and (3) are given by [EGA IV, 2.7.2] and [Ko]. Lemma 2.18. If $char k = 0$, then (1) is clear because an arithmetic thickening of $X$ is an arithmetic thickening of $X \times k'$.

**APPENDIX D. DIMENSIONAL BOUND ON $F$-AMPLITUDE OF AMPLE BUNDLES**

Let $E$ be an ample vector bundle. We have seen that if $char k = 0$, then $\phi(E) < \text{rank}(E)$ (theorem 3). We will now derive another bound on $\phi(E)$, which is independent of the characteristic of $k$. First, we need some lemmas.

**Lemma D.1.** Let $X$ be a projective scheme over a field $k$ of characteristic $p > 0$, let $O_X(1)$ be a very ample invertible sheaf, and let $E$ be a vector bundle. Then for any $b \in \mathbb{Z}$, there exists $n_0$ such that
\[
H^i(O_X(b + m) \otimes E^{(p^n)}) = 0
\]
for all $i > \phi(E), n \geq n_0, m \geq 0$.

**Proof.** We induct on $dim X$; the case of $dim X = 0$ is trivial. Since $O_X(1)$ is very ample, we may choose an effective Cartier divisor $H$ with $O_X(H) \cong O_X(1)$ and $dim H = dim X - 1$. By theorem 3 (3) generalized to the case of schemes, $\phi(E_H) \leq \phi(E)$. So the claim follows from the short exact sequences
\[
H^i(O_X(b + m) \otimes E^{(p^n)}) \to H^i(O_X(b + m + 1) \otimes E^{(p^n)}) \to H^i(O_H(b + m + 1) \otimes E_H^{(p^n)}),
\]
where the case $m = 0$ follows from the fact that $\phi(E) < i$.

**Lemma D.2.** Let $X$ be a projective scheme, let $H$ be a very ample Cartier divisor, and let $E$ be a vector bundle. Then
\[
\phi(E_H) \leq \phi(E) \leq \phi(E_H) + 1.
\]
The first inequality is just theorem 1 (4), generalized to schemes.

If \( \text{char } k = p > 0 \), then applying lemma D.1 to \( H \), for any \( b \in \mathbb{Z} \) there exists \( n_0 \) such that there exist exact sequences

\[
0 = H^i(O_H(b + m + 1) \otimes \mathcal{E}^{(p^n)}) \to H^{i+1}(O_X(b + m) \otimes \mathcal{E}^{(p^n)}) \\
\to H^{i+1}(O_X(b + m + 1) \otimes \mathcal{E}^{(p^n)}) \to H^{i+1}(O_H(b + m + 1) \otimes \mathcal{E}_H^{(p^n)}) = 0
\]

for \( i > \phi(\mathcal{E}_H), n \geq n_0, m \geq 0 \). By Serre Vanishing, \( H^{i+1}(O_X(b + m + 1) \otimes \mathcal{E}^{(p^n)}) = 0 \) for \( m \gg 0 \). So by descending induction on \( m \) and corollary 2.3, \( \phi(\mathcal{E}) \leq \phi(\mathcal{E}_H) + 1 \). The case of \( \text{char } k = 0 \) is then immediate.

It is now an easy matter to obtain our bound on \( \phi(\mathcal{E}) \) for ample \( \mathcal{E} \). This generalizes lemma 5.3 and proposition 5.4.

**Proposition D.3.** Let \( X \) be a projective scheme over a field \( k \) with \( \dim X > 0 \), and let \( \mathcal{E} \) be an ample vector bundle. Then \( \phi(\mathcal{E}) < \dim X \).

**Proof.** We may assume that \( k \) is algebraically closed (C.5) and that \( X \) is reduced, irreducible (C.4), and normal (2.3). If \( \dim X = 1 \), then the claim is proposition 5.4. If \( \dim X > 1 \), then induction on \( \dim X \) and lemma D.2 yields the result. \( \square \)

**References**

[B] C. M. Barton, *Tensor products of ample vector bundles in characteristic p*, Amer. J. Math. 93 (1971), 429–438.

[G] D. Gieseker, *p-ample bundles and their Chern classes*, Nagoya Math. J. 43 (1971), 91–116.

[EGA] A. Grothendieck, *Éléments de géométrie algébrique*, Inst. Hautes Études Sci. Publ. Math. (1961, 1966), no. 8, 11, 28.

[H1] R. Hartshorne, *Ample vector bundles*, Inst. Hautes Études Sci. Publ. Math. (1966), no. 29, 63–94.

[H3] ———, *Algebraic geometry*, Graduate Texts in Math., no. 52, Springer-Verlag, New York, 1977.

[Ke] D. S. Keeler, *Ample filters of invertible sheaves*, arXiv:math.AG/0108068, J. Algebra, to appear, 2001.

[Kl] S. L. Kleiman, *Toward a numerical theory of ampleness*, Ann. of Math. (2) 84 (1966), 293–344.

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