Jumping numbers of a unibranch curve on a smooth surface

Daniel Naie

Abstract

A formula for the jumping numbers of a curve unibranch at a singular point is established. The jumping numbers are expressed in terms of the Enriques diagram of the log resolution of the singularity, or equivalently in terms of the canonical set of generators of the semigroup of the curve at the singular point.

The jumping numbers of a curve on a smooth complex surface are a sequence of positive rational numbers revealing information about the singularities of the curve. They extend in a natural way the information given by the log-canonical threshold, the smallest jumping number (see [3] for example). They are periodic, completely determined by the jumping numbers less than 1, but otherwise difficult to compute in general, even if a set of candidates is easy to provide, cf. [9, Lemma 9.3.16].

The aim of this paper is to give a formula for the jumping numbers of a curve unibranch at a singular point. A curve C will be said to be unibranch at a point P, if the analytic germ of C at P is irreducible. The formula is expressed in terms of the Enriques diagram associated to the singularity, or equivalently (see Theorem 3.1) in terms of a minimal set of generators \((β_0, β_1, \ldots, β_g)\) of the semigroup \(S(C, P)\) of C at P:

\[
\{\text{jumping numbers } < 1\} = \bigcup_{j=1}^{g} \frac{1}{m_j, β_j} R^{m_{j+1}} \left( \frac{m_j}{m_{j+1}}, \frac{β_j}{m_{j+1}} \right),
\]

where the \(m_j\) are defined below. Here

\[R^m(p, q) = \bigcup_{k=0}^{m-1} (kpq + R(p, q))\]

and

\[R(p, q) = R^1(p, q) = \{ap + bq \mid a, b \in \mathbb{N}^\ast, ap + bq < pq\}\]  

The semigroup is defined by \(S(C) = \{\text{ord}_P s \mid s \in \mathcal{O}_{C, P}\}\), the order of the local section \(s\) being computed using a normalization of \(C\). It is finitely generated and a minimal set of generators \((β_0, β_1, \ldots, β_g)\) is constructed as follows (see [13, Theorem 4.3.5]): \(β_0\) is the least element of \(S(C)\); set \(m_1 = β_0\); \(β_j\) is the least element of \(S(C)\) not divisible by \(m_j\) and \(m_{j+1} = \gcd(m_j, β_j)\).

To prove (1) we use the notion of relevant divisors of the minimal log resolution of C at P, notion introduced in [12], and previously in [6] from the point of view of valuations corresponding to Puiseux exponents: a relevant divisor is an irreducible exceptional divisor that intersects at least three other components of the total transform of C through the resolution. When C is unibranch at P, we show that the relevant divisors account for all the jumping numbers. This is the content of Proposition 2.5 and represents the key step of the proof.

In
general, if the curve is not unibranch, there are jumping numbers that are not contributed by any relevant divisor (see [12, Example 2.2]). Using Proposition 2.5 and the Enriques diagram associated to the minimal log resolution of $C$ at $P$, we compute the jumping numbers contributed by the relevant divisors, and hence all the jumping numbers in Theorem 2.3. Finally, Theorem 3.1 follows as a consequence of Theorem 2.3 and of the equivalence between the Enriques diagram and the semigroup $S(C)$.

The construction of the Enriques diagram as well as the definition of the jumping numbers are recalled in §1. The proof of Theorem 2.3 is given in §2 together with some necessary technical lemmas, whereas Proposition 2.5 is established in the last section. The explicit equivalence between the Puiseux characteristic, and hence the semigroup, and the Enriques diagram of a unibranch curve is presented in Theorem 3.4.

In [8], Tarmo Järviletho obtained recently an explicit description of the jumping numbers of a simple complete ideal $p$ in a two dimensional regular local ring. The jumping numbers are expressed in terms of the Zariski exponents of the ideal. Moreover (see [9, Proposition 9.2.8] and also [8, Theorem 9.4]) the jumping numbers $< 1$ of the ideal $p$ coincide to those of the unibranch plane curve corresponding to a general element of $p$, and they amount to the jumping numbers given in (1).

If the unibranch curve is characterized by a single Puiseux exponent $q/p$, with $\gcd(p, q) = 1$, or equivalently if the semigroup is generated by $p$ and $q$, the jumping numbers

$$\frac{ap + bq}{pq} < 1, \quad a, b \in \mathbb{N}^*,$$

were computed by L. Ein in [2], or by J. A. Howald in [7] as a particular case of his formula for the multiplier ideals of monomial ideals.

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1 Preliminaries and notation

In this section we recall the definition of the jumping numbers and introduce the Enriques diagram associated to a minimal log resolution of a curve at a singular point. The diagram will be used to perform the calculations of the jumping numbers.

1.1 Log resolutions and Enriques diagrams

Let $C$ be a curve on a smooth surface with an isolated singularity at $P$. A minimal log resolution of $C$ at $P$ is the composition $\mu : Y \to X$ of blowings up such that $\mu$ gives an isomorphism $Y \smallsetminus \mu^{-1}(P) \to X \smallsetminus \{P\}$, the strict transform $\widetilde{C}$ of $C$ is smooth, the support of the total transform $\mu_*C$ has normal crossings and the number of blowings up is minimal with these properties.

Let $Y = Y_{r+1} \to Y_r \to \cdots \to Y_1 = X$ be a decomposition of $\mu : Y \to X$ into successive blowings up with $Y_{\alpha+1} = \text{Bl}_{P_\alpha} Y_\alpha$. Each point $P_\alpha$ is infinitely near to $P = P_1$ and has an
associated exceptional divisor $E_\alpha$ on $Y_{\alpha+1}$. Its strict transform on $Y$ will also be denoted by $E_\alpha$ and its total transform on $Y$ will be denoted by $W_\alpha$. The strict transforms $E_\alpha$ and the the total transforms $W_\alpha$ form two bases of the $\mathbb{Z}$-module $\Lambda_C = \bigoplus_\alpha \mathbb{Z} E_\alpha \subset \text{Pic} Y$. In particular

$$\mu^*C = \tilde{C} + D = \tilde{C} + \sum_\alpha e_\alpha E_\alpha = \tilde{C} + \sum_\alpha w_\alpha W_\alpha.$$  

**Remark 1.1.** The weight $w_\alpha$, the coefficient of the total transform $W_\alpha$ in the divisor $\mu^*C - \tilde{C}$, is the multiplicity of the corresponding strict transform of $C$ at $P_\alpha$.

If the curve $C$ is unibranch, then for any $\alpha$, the strict transform of $C$ on $Y_\alpha$ has a unique singular point $P_\alpha$. In particular there is a unique log resolution of the singular point $C$. In the general case the resolution is not unique; the ordering of the exceptional divisors $E_\alpha$, or equivalently of the points $P_\alpha$ might vary. Nevertheless, the ordering of the points is compatible with the partial order of the infinitely near points. If $\alpha < \beta$, then either $P_\beta$ is infinitely near to $P_\alpha$ or there is $\gamma < \alpha$ such that $P_\gamma$ and $P_\alpha$ are infinitely near to $P_\gamma$.

The combinatorics of the configuration of the exceptional curves $E_\alpha$ on $Y$, or equivalently the geometric relation between the infinitely near points $P_\alpha$, is encoded in the notion of proximity: a point $P_\beta$ is said to be proximate to $P_\alpha$ if $P_\beta$ lies on the strict transform of $E_\alpha$ on $Y_\beta$. A point that is infinitely near is always proximate to at most two other points. A point is said to be free if it is proximate to exactly one other point and satellite if it is proximate to two infinitely near points.

**Remark 1.2.** If $P_\beta$ is proximate to both $P_\alpha$ and $P_\alpha'$ and if $\alpha < \alpha'$, then $P_\alpha'$ is infinitely near to $P_\alpha$. Moreover, if the curve is unibranch, then $P_\beta$ is always proximate to $P_{\beta-1}$.

A convenient way to present the proximity relations is given by the proximity matrix $\Pi = \|p_{\alpha\beta}\|$, where $p_{\alpha\alpha} = 1$ for any $\alpha$ and $p_{\alpha\beta}$ equals $-1$ if $P_\beta$ is proximate to $P_\alpha$ and $0$ if not. The proximity matrix is upper unitriangular by the previous remark and represents at the same time the decomposition matrix of the strict transforms in terms of the total transforms on $Y$. A simple but useful consequence of this remark is the following lemma.

**Lemma 1.3.** Let $D = \sum_\alpha e_\alpha E_\alpha = \sum_\alpha w_\alpha W_\alpha$ be the divisor associated to a log resolution of $C$ at $P$. If $P_\beta$ is a satellite point proximate to $P_\alpha$ and $P_\alpha'$, then

$$e_\beta = e_\alpha' + e_\alpha'' + w_\beta.$$  

**Proof.** Use the relation $E_\alpha = \sum_\beta p_{\alpha\beta} W_\beta = W_\alpha - \sum_\beta p_{\alpha\beta}$ proximate to $P_\alpha$ $W_\beta$ to express the coefficients $w_\beta$ in terms of the coefficients $e_\alpha$. $\square$

The resolution data of a curve $C$ at a singular point $P$ has been encoded by Enriques in an appropriate weighted tree diagram now called the Enriques diagram (see $\mathbb{I}$ $\mathbb{I}$ $\mathbb{I}$ $\mathbb{I}$ $\mathbb{I}$ $\mathbb{I}$ $\mathbb{I}$). The tree graphically represents the proximity relations of the infinitely near points.

**Definition.** An Enriques tree is a couple $(T, \varepsilon_T)$, where $T = T(\mathfrak{V}, \mathfrak{E})$ is an oriented tree (a graph without loops) with a single root, with $\mathfrak{V}$ the set of vertices and $\mathfrak{E}$ the set of edges, and where $\varepsilon_T$ is a map

$$\varepsilon_T : \mathfrak{E} \to \{ \text{‘slant’}, \text{‘horizontal’}, \text{‘vertical’} \}.$$  

\(^1\)The terms exceptional divisor and exceptional curve will be used indifferently in the sequel.
 fixing the graphical representation of the edges. An *Enriques diagram* is a weighted or labeled Enriques tree.

**Definition.** Let $T$ be an Enriques tree. A horizontal (respectively vertical) $L$-shape branch of $T$ is a path of length $\geq 1$ such that all edges but the first, are horizontal (respectively vertical) through $e_T$. An $L$-shape branch is proper if it contains at least two edges. A maximal $L$-shape branch is an $L$-shape branch that cannot be continued to a longer one.

The construction of the Enriques tree associated to a log resolution of $C$ at $P$ is as follows. The set of vertices is $\mathfrak{V} = \{P_1, \ldots, P_r\}$, i.e. the set of infinitely near points; the root of the tree is the proper point $P$. There is an edge starting at $P_\alpha$ and ending at $P_\beta$ if and only if $P_\beta$ is proximate to $P_\alpha$ and, either $P_\beta$ is free, or $P_\beta$ is satellite, proximate to $P_\alpha$ and $P_\alpha'$ and $\alpha > \alpha'$. There is an $L$-shape branch that starts at $P_\alpha$ and ends at $P_\beta$ if and only if $P_\beta$ is proximate to $P_\alpha$; there is either a horizontal or a vertical edge that ends at $P_\alpha$ if and only if $P_\alpha$ is satellite. To normalize the shape of the tree it is assumed that an edge that starts at a free point and ends at a satellite point is horizontal. The weights $w_\alpha$ are given by the coefficients of the total transforms in $\mu^*C = \tilde{C} + \sum_\alpha w_\alpha W_\alpha$.

It is to be noticed that $E_\alpha$ and $E_\beta$ intersect on $Y$ if and only if there is a maximal $L$-shape branch that starts at $P_\alpha$ and ends at $P_\beta$.

**Example 1.4 (Definition of $T_{p,q}$).** Let $p < q$ be relatively prime positive integers and let $C$ be defined locally at $P$ by $x^p - y^q = 0$. The Enriques tree $T_{p,q}$ associated to the minimal log resolution of $C$ at $P$ is defined as follows. Consider the Euclidean algorithm: $r_0 = a_1 r_1 + r_2$, $\ldots$, $r_m = a_{m-1} r_{m-1} + r_m$ and $r_m = a_m r_m$, with $r_0 = q$ and $r_1 = p$. Set

$$\mathfrak{V} = \{P_\alpha \mid 1 \leq \alpha \leq a_1 + \cdots + a_m = r\}$$

and

$$\mathfrak{E} = \{[P_\alpha P_{\alpha+1}] \mid 1 \leq \alpha \leq a_1 + \cdots + a_m - 1\}.$$ 

The map $\epsilon$ is locally constant on the $a_j$ edges $[P_\alpha P_{\alpha+1}]$ with $a_1 + \cdots + a_{j-1} + 1 \leq \alpha \leq a_1 + \cdots + a_j$. The first constant value of $\epsilon$—on the first $a_1$ edges—is ‘slant’. The other constant values are alternatively either ‘horizontal’ or ‘vertical’, starting with ‘horizontal’. The Enriques trees $T_{5,7}$ is

![Diagram of $T_{5,7}$](image)

and together with the weights 5, 2, 2, 1, 1, it becomes the Enriques diagram $T_{5,7}$ of the minimal log resolution of $x^5 - y^7 = 0$. In general, the Enriques diagram $T_{p,q}$ that encodes the minimal log resolution of the curve $x^p - y^q = 0$ consists of the Enriques tree $T_{p,q}$ together with the corresponding remainders of the Euclidean algorithm as weights.

**Corollary 1.5.** Let $D = \sum_\alpha e_\alpha E_\alpha$ be the divisor associated to the Enriques diagram $T_{p,q}$. If $P_\beta$ is a satellite point proximate to $P_\alpha'$ and $P_\alpha''$, then

$$e_\beta = e_\alpha' + e_\alpha'' + r_\beta.$$

**Proof.** By Lemma 1.3 and the interpretation of the weights of $T_{p,q}$. □


1.2 Multiplier ideals and jumping numbers

We briefly recall the notions of multiplier ideals and jumping numbers. We refer the reader to [9] for the results cited below. In the context of curves on surfaces, we define the relevant divisors following [12].

If \( X \) is a smooth variety, \( D \subset X \) an effective \( \mathbb{Q} \)-divisor and \( \mu: Y \to X \) a log resolution for \( D \), then \( \mathcal{I}(D) = \mu_* \mathcal{O}_Y(K_\mu - [\mu^*D]) \) is an ideal sheaf on \( X \). The divisor \( K_\mu \) is the relative canonical divisor of the map \( \mu \). The ideal sheaf \( \mathcal{I}(D) \) is independent of the choice of the resolution and is called the multiplier ideal of \( D \). When \( \mu \) is the resolution of \( D \) at a singular point \( P \), the multiplier ideal may be denoted by \( \mathcal{I}(D)_p \). The sheaf \( \mathcal{O}_Y(K_\mu - [\mu^*D]) \) computing the multiplier ideal satisfies, for any \( i > 0 \), the local vanishing result

\[
R^i \mu_* \mathcal{O}_Y(K_\mu - [\mu^*D]) = 0.
\]

**Definition-Lemma.** (see [9]) Let \( D \subset X \) be an effective divisor and \( P \in D \) be a fixed point. Then there is an increasing discrete sequence of rational numbers \( \xi_0 = \xi(D,P) \),

\[
0 = \xi_0 < \xi_1 < \cdots
\]

such that \( \mathcal{I}(\xi D)_P = \mathcal{I}(\xi_1 D)_P \) for every \( \xi \in [\xi_0, \xi_1) \), and \( \mathcal{I}(\xi_1 D)_P \subsetneq \mathcal{I}(\xi_2 D)_P \). The rational numbers \( \xi_i \) are called the jumping numbers of \( D \) at \( P \).

The jumping numbers of \( D \) at \( P \) are periodic (see [9, Theorem 9.3.24]) and that they are completely determined by the ones that are less than 1. Therefore, in the sequel, we will talk about the jumping numbers \( \xi < 1 \).

We have anticipated in the introduction that a set of candidates for the jumping numbers is easy to provide in case \( C \) is a curve singular at \( P \) on the smooth surface \( X \). Indeed, let \( \mu: Y \to X \) be a log resolution of \( C \) with \( \mu^{-1}(P) = \bigcup_{\alpha=1}^r E_\alpha \). Then \( K_\mu = \sum_{\alpha=1}^r W_\alpha = \sum_{\alpha=1}^r k_\alpha E_\alpha \), with \( k_\alpha > 0 \). Writing \( \mu^*C = \bar{C} + \sum_{\alpha=1}^r \epsilon_\alpha E_\alpha \), form the proof of the above lemma it follows that the set of jumping numbers must be contained in the set of the rational numbers \( (k_\alpha + n)/\epsilon_\alpha \), where \( 1 \leq \alpha \leq r \) and \( n \) is a positive integer.

**Definition 1.6** (see [12]). Let \( \xi = (k_\alpha + n)/\epsilon_\alpha \) be a jumping number of \( C \) at \( P \). The exceptional divisor \( E_\alpha \) is said to contribute the jumping number \( \xi \) if

\[
\mathcal{I}(\xi \cdot C) \subsetneq \mu_* \mathcal{O}_Y(K_\mu - [\xi \mu^*C] + E_\alpha).
\]

If the above inclusion is satisfied for \( \xi = (k_\alpha + n)/\epsilon_\alpha \), then \( \xi \) is a jumping number, since for any sufficiently small \( \varepsilon > 0 \),

\[
\mu_* \mathcal{O}_Y(K_\mu - [\xi \mu^*C] + E_\alpha) \subset \mu_* \mathcal{O}_Y(K_\mu - [(\xi - \varepsilon) \mu^*C]) = \mathcal{I}((\xi - \varepsilon) \cdot C).
\]

In [12], on the one hand, Theorem 3.1 shows that a divisor \( E_\alpha \) contributes to the sequence of jumping numbers if and only if \( E_\alpha \) has non-trivial intersection with at least three of the other components of the total transform \( \mu^*C \). On the other hand, it is also shown that not all jumping numbers are contributed by exceptional divisors. For example no \( E_\alpha \) contributes

\[\footnote{The integral part or round-down \([D]\) of \( D \) is the integral divisor \([D] = \sum_{\alpha} [c_\alpha] D_\alpha \), where for \( c \in \mathbb{Q} \), \([c]\) denotes the greatest integer \( \leq c \).}

\[\footnote{The same characterization was established in [9] Lemma 2.11 in an analytical context.}
the log-canonical threshold \(1/2\) of the curve defined by \((x^2 - y^3)(x^3 - y^2) = 0\). It is to be noticed that this curve is not unibranch at the origin. The essential step in the computation of the jumping numbers of a unibranch curve is that each jumping number is contributed by an exceptional divisor \(E_\alpha\). This is the content of the forthcoming Proposition 2.5.

**Example.** In [7] it is shown that if the Puiseux exponent of \(C\) at \(P\) is \(p/q\), with gcd\((p, q) = 1\), then the jumping numbers less than 1 of \(C\) are \(a/p + b/q < 1\) with \(a\) and \(b\) positive integers. There is only one divisor that contributes all these jumping numbers, namely the last exceptional divisor.

**Definition 1.7.** An exceptional divisor \(E_\rho\) is said to be a relevant divisor, or \(\rho\) is said to be a relevant position of \(C\) at \(P\), if \(E_\rho \cdot E_\rho^0 \geq 3\), where \(E^0_\rho = (\mu^* C)_{\text{red}} - E_\rho\). The set of relevant positions of \(C\) at \(P\) will be denoted by \(\mathfrak{R}_P\).

**Remark 1.8.** A relevant position \(\rho\) is easy to identify on the Enriques tree. Either it corresponds to a satellite point from which a 'slant' edge starts, or it corresponds to a non-zero coefficient in the expression of \(\mu^* C - \tilde{C}\) in the branch basis. To define this basis, let \(\Pi\) be the proximity matrix. The intersection matrix of the curves \(W_\alpha\) is minus the identity. It follows that there exists effective\(^4\) divisors \(B_\alpha\) that form the dual basis to \((-E_\alpha)\) with respect to the intersection form for the lattice \(\Lambda_C = \bigoplus_\alpha \mathbb{Z}E_\alpha\). This basis is the branch basis\(^5\).

Going back to the relevant position \(\rho\) corresponding to a non-zero coefficient in the branch basis, we notice that this position might be represented on the tree using arrowhead vertices, the number of arrows being given by the coefficient \(b_\rho\) in \(\mu^* C - \tilde{C} = \sum_\alpha b_\alpha B_\alpha\). We would obtain in this way an augmented Enriques tree, equivalent to the Enriques diagram \(T\).

### 2 Jumping numbers of a unibranch curve

The aim of this section is to prove Theorem 2.3. To formulate it we need to make some considerations about the unibranch trees. A unibranch tree is an Enriques tree having out-valence 1 for any of its vertices. The trees \(T_{p,q}\) introduced in Example 1.4 are unibranch and represent the simplest such trees in the sense that there is no slant edge starting at a satellite point. If a curve is unibranch at \(P\), then the Enriques diagram is given by a unibranch Enriques tree with the last element in the branch basis \((B_\alpha)\) as the associated divisor. The next definition allows us to see a unibranch tree as being constructed from \(T_{p,q}\) trees.

**Definition 2.1.** Let \(T\) and \(T'\) be unibranch Enriques trees with \(\mathfrak{S}(T) = \{P_1, \ldots, P_r\}\) and \(\mathfrak{S}(T') = \{P'_1, \ldots, P'_s\}\). The connected sum of \(T\) and \(T'\) is the Enriques tree \(T \# T'\) with the set of vertices \(\mathfrak{S}(T \# T') = \mathfrak{S}(T) \cup \mathfrak{S}(T') / \{P_r = P'_1\}\), the set of edges \(\mathcal{E}(T \# T') = \mathcal{E}(T) \cup \mathcal{E}(T')\) and the map \(\varepsilon_{T \# T'}\) defined by \(\varepsilon_T\) and \(\varepsilon_{T'}\) through the natural restrictions.

\(^4\)\(|B_1 \ldots B_s| = \Pi^{-1}|W_1 \ldots W_s|\) and the matrix \(\Pi^{-1}\) has non negative entries since it decomposes the \(W_\alpha\) in terms of the \(E_\alpha\).

\(^5\)The divisor \(\mu^* C - \tilde{C}\) may be expressed in three ways. Its expression in the branch basis reflects the branches of \(\tilde{C}\)—the analytically irreducible components of \(\tilde{C}\) above \(P\). Its expression in the basis of total transforms \((W_\alpha)\) reflects the multiplicities of \(C\) and the multiplicities of its strict transforms along the resolution process, as it has been noticed in Remark 1.1. Finally, its expression in the basis \((E_\alpha)\), the basis of strict transforms, gives the coefficients necessary to compute the multiplier ideals associated to \(C\).

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Example 2.2. The minimal log resolution of \((x^3 - y^2)^2 - 4x^5y - x^7 = 0\) needs five blowings up with the following Enriques diagram.

The Enriques tree is the connected sum \(T_{2,3} \# T_{2,3}\).

Theorem 2.3. Let \(C\) be a curve unibranch at \(P\) with the Enriques tree of the minimal log resolution \(S = T_{p_1,q_1} \# \cdots \# T_{p_g,q_g}\). Set \(\overline{q}_1 = q_1\) and

\[
\overline{q}_j = \frac{m_{j-1}}{m_j+1} \overline{q}_{j-1} - p_j + q_j
\]

for any \(2 \leq j \leq g\), where \(m_j = p_j \cdots p_g\) for any \(1 \leq j \leq g\), and \(m_{g+1} = 1\). Then the jumping numbers less that 1 of \(C\) at \(P\) are given by

\[
\bigcup_{j=1}^{g} \frac{1}{m_j \overline{q}_j} R^{m_{j+1}}(p_j, \overline{q}_j),
\]

where

\[
R^{m_{j+1}}(p_j, \overline{q}_j) = \bigcup_{k=0}^{m_{j+1}-1} (kp_j \overline{q}_j + \{ap_j + b\overline{q}_j \mid a, b \in \mathbb{N}^*, ap_j + b\overline{q}_j < p_j \overline{q}_j\}).
\]

We begin by describing the sets \(R^m(p, q)\) as they will appear in the proof of Theorem 2.3.

For the purposes of this section we denote by \(\lceil x \rceil\) the round-up of \(x\), i.e. the least integer \(\geq x\), and by \(\langle x \rangle = x - \lfloor x \rfloor\) the fractional part of \(x\).

Let \(2 \leq p < q\) be relatively prime integers and let \(m\) be a positive integer. Setting \(q'\) to be the positive integer that satisfies \(q' < p\) and \(qq' = -1 \mod p\), we define \(R^m(p, q)\) as the set of integers \(k\), \(1 \leq k < mpq\), such that

\[
\left\langle \frac{k}{pq} \right\rangle + \left\langle \frac{q'k}{p} \right\rangle > 1.
\]

If \(m = 1\) we shall denote the set \(R^1(p, q)\) by \(R(p, q)\). Clearly

\[
R^m(p, q) = \bigcup_{j=0}^{m-1} (jpq + R(p, q))
\]

and \(R(p, q)\) is computed in the following Proposition.

Proposition 2.4.

\[
R(p, q) = \{ap + bq \mid a, b \in \mathbb{N}^*, ap + bq < pq\}
\]
Proof. If \( k_0 \in R(p, q) \) then
\[
\frac{k_0 + p}{pq} + \left\langle \frac{q'(k_0 + p)}{p} \right\rangle = \frac{k_0}{pq} + \frac{1}{q} + \left\langle \frac{q'k_0}{p} \right\rangle > 1,
\]
hence \( k_0 + p \in R(p, q) \). It follows that to determine \( R(p, q) \) it is sufficient to determine the first element belonging to \( R(p, q) \) in each equivalence class mod \( p \). Clearly the multiples of \( p \) do not belong to \( R(p, q) \). So such an element is of the form \( jq + Np \), with \( j \in \{1, \ldots, p-1\} \) and \( N \) a positive integer. Using the hypothesis \( qq' = -1 \mod p \),
\[
\frac{jq + Np}{pq} + \left\langle \frac{q'(jq + Np)}{p} \right\rangle = \frac{j}{p} + \frac{N}{q} + 1 - \frac{j}{p} = 1 + \frac{N}{q}.
\]
So the minimal element in each equivalence class different from zero is \( jq + p \). The result follows. \( \square \)

As we have anticipated in \( \S 1 \), the computation of the jumping numbers depends on the fact that each one of them is contributed by an irreducible exceptional divisor. More precisely we have the following proposition whose proof will be given in the last section.

**Proposition 2.5.** Let \( C \) be a unibranch curve at \( P \) and let \( \mu \) be a log resolution such that \( \mu^*C = \tilde{C} + \sum_{\alpha \in \mathcal{V}} e_\alpha E_\alpha \). If \( \xi \) is a jumping number of \( C \) at \( P \), then there exists \( \beta \) a relevant position such that \( \lfloor \xi e_\beta \rfloor = \xi e_\beta \) and such that \( E_\beta \) contributes \( \xi \).

**Proof of Theorem 2.3.** By Proposition 2.5 each jumping number is contributed by a relevant divisor and it is sufficient to compute the jumping numbers contributed by each relevant divisor. Remember that a relevant divisor is an exceptional divisor that satisfies \( E_\rho \cdot E_0 \geq 3 \).

If \( \rho \) is a relevant position, then tensoring the exact sequence
\[
0 \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_Y(E_\rho) \longrightarrow \mathcal{O}_{E_\rho}(E_\rho|E_\rho) \longrightarrow 0
\]
with \( \mathcal{O}_Y(K_\mu - [\xi \mu^*C]) \) and pushing it down to \( X \) give
\[
0 \longrightarrow \mu_*\mathcal{O}_Y(K_\mu - [\xi \mu^*C]) \longrightarrow \mu_*\mathcal{O}_Y(K_\mu - [\xi \mu^*C] + E_\rho) \longrightarrow \mu_*\mathcal{O}_{E_\rho}(K_{E_\rho} - [\xi \mu^*C]|_{E_\rho}) \longrightarrow 0
\]
thanks to the local vanishing. Since \( E_\rho \) is a projective line, \( \xi \) is a jumping number contributed by \( E_\rho \) if and only if \( \xi e_\rho \) is an integer and
\[
- [\xi \mu^*C] \cdot E_\rho \geq 2. \tag{3}
\]
Assume that \( \xi e_\rho \) is an integer. Let \( r_j \) be the number of vertices of the Enriques tree \( T_{p, q_j} \) and set \( s = r_1 + \cdots + r_g - (g - 1) \), the number of vertices of \( S \). There are two cases to be considered: either \( \rho = s \), i.e. \( \rho \) is the highest point of the Enriques tree, or \( \rho \) is a relevant position different from \( s \). Whatever the case, the study of the numbers \( [\xi \mu^*C] \cdot E_\rho \) depends on the control of the coefficients \( e_\alpha \) in \( \mu^*C = \tilde{C} + \sum_{\alpha = 1}^s e_\alpha E_\alpha \). The following technical lemmas 2.6, 2.7, 2.8 and 2.9 give explicit formulae for these coefficients and also, some other useful numerical relations.
Some notation is in order. Let $T$ be an Enriques tree. Denote by $E^T_\alpha$ the elements of the basis of strict transforms, by $W^T_\alpha$ the elements of the basis of total transforms and by $B^T_\alpha$ the elements of the branch basis that has been introduced in Remark 1.8. If $\Lambda = \bigoplus_\alpha E^T_\alpha$, then $(e^T_\alpha)$ will denote the basis for $\Lambda^*$, dual to $(E^T_\alpha)$. Similarly, $(w^T_\alpha)$ will denote the dual basis to the basis of total transforms and $(b^T_\alpha)$ the dual basis to the branch basis.

Let $p < q$ be two relatively prime positive integers. Consider the Euclidean algorithm
\[r_0 = a_1 r_1 + r_2, \ldots, r_{m-2} = a_{m-1} r_{m-1} + r_m\text{ and } r_{m-1} = a_m r_m,\] with $r_0 = q$ and $r_1 = p$. Define as in [10, Lemma A.8] two finite sequences $(f_j)_{1 \leq j \leq m}$ and $(\delta_j)_{1 \leq j \leq m+1}$ by
\[
f_j = f_{j-2} + a_j \delta_j, \quad \text{for any } 1 \leq j \leq m,
\]
\[
\delta_j = \delta_{j-2} + a_{j-1} f_{j-2}, \quad \text{for any } 2 \leq j \leq m + 1,
\]
such that $f_{-1} = f_0 = 0$ and $\delta_0 = \delta_1 = 1$. It is easy to show that the remainder $r_j$ in the Euclidean algorithm is given by $-f_{j-1} q + \delta_j p$ if $j$ is odd and by $\delta_j q - f_{j-1} p$ if $j$ is even.

Furthermore, if $m$ is odd, then $f_m = q$ and $\delta_{m+1} = p$, and if $m$ is even, then $f_m = p$ and $\delta_{m+1} = q$. We have the following lemma that computes various coefficients for the Enriques tree $T_{p,q}$.

**Lemma 2.6.** Let $T = T_{p,q}$. Then for any $0 \leq j \leq m-1$ and any $1 \leq k \leq a_{j+1}$,
\[
e^T_{a_1 + \ldots + a_j + k}(B^T_r) = \begin{cases} (f_{j-1} + k\delta_{j+1})p & \text{if } j \text{ is even} \\
(f_{j-1} + k\delta_{j+1})q & \text{if } j \text{ is odd} \end{cases}
\]
and
\[
e^T_{a_1 + \ldots + a_j + k}(W^T_1) = \begin{cases} \delta_j + kf_j & \text{if } j \text{ is even} \\
(f_{j-1} + k\delta_{j+1}) & \text{if } j \text{ is odd.} \end{cases}
\]

**Proof.** We proceed by induction using Corollary 1.5 the relations (1) and the relations quoted after them. For the computation of $e^T_{a_1 + \ldots + a_j + k}(B^T_r)$, we suppose that $j$ is even, the case $j$ odd being similar. If $k = 1$, then
\[
e^T_{a_1 + \ldots + a_j + 1}(B^T_r) = e^T_{a_1 + \ldots + a_j - 1}(B^T_r) + e^T_{a_1 + \ldots + a_j}(B^T_r) + r_{j+1}
= f_{j-1} p + f_j q + (-f_j q + \delta_{j+1} p)
= (f_{j-1} + \delta_{j+1}) p.
\]

Now, if $1 < k \leq a_{j+1}$, then
\[
e^T_{a_1 + \ldots + a_j + k}(B^T_r) = e^T_{a_1 + \ldots + a_j}(B^T_r) + e^T_{a_1 + \ldots + a_j + k-1}(B^T_r) + r_{j+1}
= f_j q + (f_{j-1} + (k-1)\delta_{j+1}) p + (-f_j q + \delta_{j+1} p)
= (f_{j-1} + k\delta_{j+1}) p.
\]

As for the second equality, if we suppose again that $j$ is even, we get
\[
e^T_{a_1 + \ldots + a_j + 1}(W^T_1) = e^T_{a_1 + \ldots + a_j - 1}(W^T_1) + e^T_{a_1 + \ldots + a_j}(W^T_1) = \delta_j + f_j
\]
for $k = 1$, and
\[
e^T_{a_1 + \ldots + a_j + k}(W^T_1) = e^T_{a_1 + \ldots + a_j}(W^T_1) + e^T_{a_1 + \ldots + a_j + k-1}(W^T_1) = f_j + (\delta_j + (k-1)f_j)
\]
for $1 < k \leq a_{j+1}$. \qed
The remaining lemmas will allow us to compute the $e_\alpha$ coefficients for $\mu^*C - \tilde{C}$ for a unibranch curve.

**Lemma 2.7.** Let $S = T\#T'$ be a unibranch Enriques tree. If $r$ is the number of vertices of $T$ and $s$ is the number of vertices of $S$, then

$$e_s^S(B_s^S) = w_r^S(B_s^S) e_r^T(B_r^T)$$

for any $1 \leq \alpha \leq r$.

**Proof.** Write $B_s^S = \sum_{\beta} w_s^S(B_s^S) W_\beta^S$ using the basis of total transforms. Since the proximity matrix in upper unitriangular, if $1 \leq \alpha \leq r$, then only the divisors $W_\beta$ with $\beta \leq r$ count when computing $e_s^S(B_s^S)$. For $\beta \leq r$, we have

$$w_\beta^S(B_s^S) = w_\beta^S(w_r^S(B_s^S) B_r^S) = w_r^S(B_s^S) w_\beta^S(B_r^S) = w_r^S(B_s^S) w_\beta(T_r^T).$$

So

$$e_s^S(B_s^S) = w_r^S(B_s^S) \sum_{\beta \leq r} w_\beta(T_r^T) e_\alpha(W_\beta^S)$$

$$= w_r^S(B_s^S) e_r^T(\sum_{\beta \leq r} w_\beta(T_r^T) W_\beta^S),$$

since $e_s^S(W_\beta^S) = e_r^T(W_\beta^S)$, and the result follows. \qed

**Lemma 2.8.** Let

$$S = T_{p_1,q_1} \# T_{p_2,q_2} \# \cdots \# T_{p_g,q_g}$$

with $r_j$ the number of vertices of the tree $T_{p_j,q_j}$, and let $s = r_1 + \cdots + r_g - (g - 1)$ be the number of vertices of $S$. If $\bar{q}_1 = q_1$ and

$$\bar{q}_j = p_{j-1}p_j \bar{q}_{j-1} - p_j + q_j$$

for any $2 \leq j \leq g$, then

$$w_{r_1+\cdots+r_j-(j-1)}(B_s^S) = p_j \cdots p_g$$

$$e_{r_1+\cdots+r_j-(j-1)}(B_s^S) = p_j \cdots p_g \bar{q}_j$$

for any $1 \leq j \leq g$.

**Proof.** Set $T_j = T_{p_j,q_j}$ for any $j$. The first identity is clear by induction since using the proximity relations,

$$w_{r_1+\cdots+r_j-(j-1)}(B_s^S) = w_{T_j}^{T_{j+1}#\cdots#T_g}(B_s^S) e_{T_j}^{T_{j+1}#\cdots#T_g(r_j+\cdots+r_g-(j-1))}.$$

As for the second, by Lemma 2.7

$$e_{r_1+\cdots+r_j-(j-1)}(B_s^S) = w_{r_1+\cdots+r_j-(j-1)}(B_s^S) e_{r_1+\cdots+r_j-(j-1)}(B_r^T)$$

$$= p_j \cdots p_g \bar{q}_j.$$

\qed
Lemma 2.9. Let $S = T'\#T$ be a unibranch Enriques tree with $T = T_{p,q}$. Let $r'$ be the number of vertices of $T'$ and $r$ the number of vertices of $T$. Then

$$e_{r'-1+\alpha}(B_{r'-1+r}^S) = (e_{r'}^T(B_{r'}^T) - 1) e_{\alpha}^T(W_1^T)p + e_{\alpha}^T(B_r^T)$$

for any $1 \leq \alpha \leq r$.

Proof. Set $s = r' + r - 1$, the number of vertices of $S$. Using that

$$w_{r'-1+\beta}(B_s^S) = w_{\beta}^T(B_r^T)$$

for any $2 \leq \beta \leq r$ and discarding the exponent $S$, we have

$$e_{r'-1+\alpha}(B_s) = e_{r'-1+\alpha} \left( \sum_{\gamma=1}^{r'} w_{\gamma}(B_s) W_\gamma + \sum_{\beta=2}^{r} w_{r'-1+\beta}(B_s) W_{r'-1+\beta} \right)$$

$$= e_{r'-1+\alpha} \left( w_{r'}(B_s) B_s - w_{r'}(B_s) W_{r'} + \sum_{\beta=1}^{r} w_{r'-1+\beta}(B_s) W_{r'-1+\beta} \right)$$

$$= e_{r'-1+\alpha} \left( w_1^T(B_r^T) B_s - w_1^T(B_r^T) W_{r'} + e_{\alpha}^T \left( \sum_{\beta=1}^{r} w_{\beta}^T(B_r^T) W_{\beta} \right) \right)$$

$$= w_1^T(B_r^T) e_{r'-1+\alpha} \left( e_{r'}^T(B_{r'}^T) W_{r'} - W_r \right) + e_{\alpha}^T(B_r^T),$$

and hence the result, since $e_{r'-1+\alpha}(W_{r'}) = e_{\alpha}^T(W_1^T)$ and $w_1^T(B_r^T) = p$. \hfill \qed

End of the proof of Theorem 2.3. We need to study $[\xi_\mu^* C] \cdot E_\rho$ when $\rho$ is a relevant position and $q_\rho$ is an integer. We have already noticed that there are two cases to be considered: either $\rho = s$ or $\rho \neq s$, where $s$ is the number of vertices of $S = T_{p_1,q_1}\# \cdots \# T_{p_s,q_s}$.

In the former case, set $T = T_{p_s,q_s}$. Then

$$[\xi_\mu^* C] \cdot E_s^S = \left( [\xi_{e_{s-a_m}}] E_{r_g-a_m}^T + [\xi_{e_{s-1}}] E_{r_g-1}^T + [\xi_{e_s}] E_{r_g}^T \right) \cdot E_{r_g},$$

where $a_m$ is the last quotient in the Euclidean algorithm for $q_g$ and $p_g$. By Corollary 1.5, $e_s = e_{s-a_m} + e_{s-1} + 1$. By Lemma 2.8, $e_s = p_g q_g$ with $p_g \leq q_g$, relatively prime integers. Then, by Lemma 2.9

$$e_{s-\alpha} = (p_g-1q_g-1 - 1) p_g e_{r_g-\alpha}(W_1^T) + e_{r_g-\alpha}(B_{r_g}^T)$$

for any $1 \leq \alpha < r_g$. One of the two positions $r_g - m$ and $r_g - 1$ must belong to a proper horizontal $L$-shape branch. We suppose that $r_g - 1$ satisfies this, the argument being similar in the other case. By Lemma 2.6

$$e_{r_g-1}(B_{r_g}) = e_{r_g-1}(W_1^T) q_g,$$

i.e.

$$e_{s-1} = (p_g-1q_g-1 - 1) p_g e_{r_g-1}(W_1^T) + e_{r_g-1}(B_{r_g}) = e_{r_g-1}(W_1^T) q_g,$$

and $p_g$ divides $e_{s-m}$. Set $M = e_{r_g-1}(W_1^T)$. From $p_g q_g = e_{s-m} + M q_g + 1$ it follows that

$$M q_g = -1 \mod p_g.$$
Set \( x = \xi e = \xi p_y q_g \). It is an integer satisfying \( 1 \leq x < p_y q_g \). Then putting everything together,

\[
- [\xi \mu^* C] \cdot E_s = - \left( \left\lfloor \frac{x}{p_y q_g} \right\rfloor (p_y q_g - M q_g - 1) \right) - \left( \left\lfloor \frac{x}{p_y q_g} \right\rfloor M q_g \right) + x
\]

\[
= \left\lceil \frac{M x}{p_g} + \frac{x}{p_y q_g} \right\rceil - \left\lfloor \frac{M x}{p_g} \right\rfloor - \left\lfloor \frac{x}{p_y q_g} \right\rfloor.
\]

Hence the inequality (3) is satisfied, i.e., \( - \left\lfloor \xi \mu^* C \right\rfloor \cdot E_\rho \geq 2 \), if and only if

\[
\left\lceil \frac{M x}{p_g} + \frac{x}{p_y q_g} \right\rceil > 1.
\]

By Proposition 2.4 this is equivalent to \( x \in R(p_y, q_g) \).

In the second case, if \( \rho = r_1 + \cdots + r_j - (j - 1) \) is a relevant position different from \( s \), the highest one, set \( r = r_j \), \( T = T_{p_j, q_j} \), and \( S = T^* \# T^\# T^" \). When computing \( [\xi \mu^* C] \cdot E_\rho \) we distinguish two situations according to whether \( a''_1 = 1 \) or not, where \( a'_1, a''_1, \ldots \) are the quotients in the Euclidean algorithm for \( q_j + 1 \) and \( p_j + 1 \). In case \( a''_1 \neq 1 \),

\[
[\xi \mu^* C] \cdot E_\rho = \left( \left\lfloor \xi e_{\rho - a_m} \right\rfloor E_{\rho - a_m} + \left\lfloor \xi e_{\rho - 1} \right\rfloor E_{\rho - 1} + \xi e_{\rho - 1} E_\rho + \left\lfloor \xi e_{\rho + 1} \right\rfloor E_{\rho + 1} \right) \cdot E_\rho
\]

\[
= \left\lfloor \xi e_{\rho - a_m} \right\rfloor + \left\lfloor \xi e_{\rho - 1} \right\rfloor - 2 \xi e_{\rho} + \left\lfloor \xi e_{\rho + 1} \right\rfloor,
\]

whereas in case \( a''_1 = 1 \),

\[
[\xi \mu^* C] \cdot E_\rho = \left\lfloor \xi e_{\rho - a_m} \right\rfloor + \left\lfloor \xi e_{\rho - 1} \right\rfloor - (2 + a''_1) \xi e_{\rho} + \left\lfloor \xi e_{\rho + a''_1 + 1} \right\rfloor.
\]

By Lemma 1.3 \( e_{\rho} = e_{\rho - a_m} + e_{\rho - 1} + m_{j+1} \), and by Lemma 2.8 \( e_{\rho} = p_j q_j m_{j+1} \). As before, either \( m \) is odd and \( e_{\rho - a_m} = M \tilde{q}_j m_{j+1} \), or \( m \) is even and \( e_{\rho - 1} = M \tilde{q}_j m_{j+1} \) with \( M \tilde{q}_j = -1 \) mod \( p_j \). Set \( x = \xi e_{\rho} \). The integer \( x \) satisfies \( 1 \leq x < e_{\rho} = p_j q_j m_{j+1} \). We claim that independently of \( a''_1 \),

\[
- [\xi \mu^* C] \cdot E_\rho = \left\lfloor \frac{M x}{p_j} \right\rfloor + \frac{x}{p_j q_j} - \left\lfloor \frac{M x}{p_j} \right\rfloor - \left\lfloor \frac{x}{p_j q_j} \right\rfloor.
\] (5)

The justification is more complicated if \( a''_1 = 1 \). The Enriques tree is shown below in case \( a''_1 = 1 \) and \( m \) even—the last proper L-shape branch of \( T_{p_j, q_j} \) is horizontal.
By Lemma \textit{2.6},
\[
e_{\rho + a''_j + 1} = (a''_j + 1)e_{\rho} + a''_jr''_2m_j + a''_r(m_j + 1),
\]

hence \(-[\xi\mu^*C] \cdot E_{\rho}\) is given by
\[
-\left[\frac{x}{p_jq_jm_j + 1} - \left(\frac{x}{p_jq_jm_j + 1} - M\right)\right] + (2 + a'_j)x - \left[\frac{x}{p_jq_jm_j + 1}((a'_j + 1)p_jq_jm_j + m_j + 1)\right]
\]
and the equality follows establishing the equality (5). Finally, \(-[\xi\mu^*C] \cdot E_{\rho} \geq 2\) is equivalent to
\[
\left\langle\frac{Mx}{p_j}\right\rangle + \left\langle\frac{x}{p_jq_j}\right\rangle > 1
\]
with \(x = \xi e_{\rho} < p_jq_jm_j + 1\), i.e. to \(x \in R^{m_j+1}(p_j, q_j)\), again by Proposition \textit{2.4}. \(\Box\)

Example. A jumping number might be contributed by more than one relevant divisor. For example if the Enriques diagram associated to the minimal log resolution of \(C\) is given by
\(T = T_{2,3} \# T_{5,11}\) —a tree with nine vertices—and \(\mu^*C = \tilde{C} + B^\mu_0\), then \(\xi = 11/30\) is a jumping number contributed by either \(E_3\) or \(E_9\).

The first jumping numbers of a unibranch curve can be obtained by inspecting the jumping numbers of the term ideal associated to \(C\) in a suitable coordinate system.

**Corollary 2.10.** Let \(C\) be a curve unibranch at \(P\) with the Enriques tree of the minimal log resolution \(T_{p_1,q_1} \# \cdots \# T_{p_g,q_g}\). Fixing \(P\) an allowable system of local parameters, let \(a_{C,P}\) be the term ideal of an equation of \(C\) in \(P\). Then the first card\((R(p_1,q_1))\) jumping numbers of \(C\) at \(P\) coincide with the first jumping numbers of \(a_{C,P}\).

**Proof.** Set \(\pi = p_2 \cdots p_g\). If \(T\) is the Enriques diagram defined by the tree \(T_{p_1,q_1}\) and the weights corresponding to the last \(B_0\) considered with multiplicity \(\pi\), then
\[
\tilde{a}_{C,P} = \mu^*O_Y(-D_T).
\]
Notice that \(\tilde{a}_{C,P}\) is the smallest integrally closed monomial ideal containing an equation of \(C\). The jumping numbers of \(a_{C,P}\), or equivalently the jumping numbers of \(\tilde{a}_{C,P}\), less than 1 are given by
\[
ap_1 + bq_1 < \frac{1}{p_1q_1\pi},
\]
with \(a, b\) positive integers. Then the first card\((R(p_1,q_1))\) of them, i.e. those for which \(ap_1 + bq_1 < p_1q_1\) are among the jumping numbers of \(C\), in the subset \(R^\pi(p_1,q_1)\), by Theorem \textit{3.1}. It is sufficient to show that these jumping numbers are also the first ones of \(C\). But they all satisfy
\[
ap_1 + bq_1 < \frac{1}{p_1q_1\pi} < \frac{1}{\pi}
\]
and \(1/\pi\) is bigger that any element of any set \(1/m_jq_jR^{m_j+1}(p_j, q_j)\), with \(j \geq 2\). \(\Box\)
3 Reformulation of Theorem 2.3 in terms of the semigroup of the singularity

Let $S(C)$ be the semigroup of the curve $C$ unibranch at $P$. It is defined by
\[ S(C) = \{ \text{ord}_P s \mid s \in \mathcal{O}_{C,P} \} \]
where the order of the local section $s$ is computed using a normalization $\widetilde{C} \to C$. If the Puiseux characteristic of $C$ at $P$ is $(m; \beta_1, \ldots, \beta_g)$, the first part of Theorem 4.3.5 in [13] states that the integers $\overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_g$ generate $S(C)$, where $\overline{\beta}_j$ are defined by $\overline{\beta}_0 = m = m_1$, $\overline{\beta}_1 = \beta_1$ and then inductively by
\[ \overline{\beta}_{j+1} = \frac{m_j}{m_{j+1}} \overline{\beta}_j + \beta_{j+1} - \beta_j, \quad (6) \]
for any $1 \leq j < g$, and where $m_{j+1} = \gcd(m_j, \beta_j)$ for any $1 \leq j \leq g$. The second part shows how to recover the generators $\overline{\beta}_j$ once the semigroup $S(C)$ given: $\overline{\beta}_0 = m_1$ is the least non-zero element of $S(C)$, and inductively $\overline{\beta}_j$ is the least element non divisible by $m_j$, and $m_{j+1} = \gcd(m_j, \overline{\beta}_j)$. We can state the main result of the paper.

**Theorem 3.1.** Let $C$ be a curve unibranch at $P$ and let $(\overline{\beta}_0, \overline{\beta}_1, \ldots, \overline{\beta}_g)$ be the canonical minimal set of generators of the semigroup $S(C)$. Then the jumping numbers of $C$ at $P$ less that 1 are given by
\[ \bigcup_{j=1}^{g} \frac{1}{[m_j, \overline{\beta}_j]} \mathbb{R}^{m_{j+1}} \left( \frac{m_j}{m_{j+1}}, \frac{\overline{\beta}_j}{m_{j+1}} \right), \]
where $m_1 = \overline{\beta}_0$ and $m_{j+1} = \gcd(m_j, \overline{\beta}_j)$ for any $j \geq 1$, and where $[m_j, \overline{\beta}_j]$ denotes the least common multiple of the two integers.

For the proof we will use Theorem 2.3 and need Enriques’ equivalence between the Puiseux characteristic of $C$ at $P$ and the Enriques diagram associated to the minimal log resolution of $C$ at $P$, equivalence that we present next.

Let $(x, y)$, be a system of local parameters. If $x = 0$ is not tangent to $C$ at $P$, there exists a good parametrization (see [13, Chapter 2]) of $C$, $x = t^m$ and $y = \sum_{k=m}^{\infty} c_k t^k$. The Puiseux characteristic of $C$ is the sequence of integers $(m; \beta_1, \ldots, \beta_g)$ defined as follows: $\beta_1$ is the exponent of the first term in the power series which is not a power of $t^m$. Set $m_1 = m$ and $m_2 = \gcd(m_1, \beta_1)$. Inductively, $\beta_j$ is the exponent of the first term which is not a power of $m_j$ and $m_{j+1} = \gcd(m_j, \beta_j)$. The construction stops when $m_{g+1} = 1$ is reached. The integers $\beta_j$ are also called the Puiseux characteristic exponents and the Puiseux exponents are just the rationals $\beta_1/m, \beta_2/m, \ldots, \beta_g/m$. Note that
\[ \frac{\beta_1}{m} < \frac{\beta_2}{m} < \cdots < \frac{\beta_g}{m}. \]

**Proposition 3.2** (see [13, Theorem 3.5.5]). Let $C \subset X$ be a unibranch curve at $P$ on the smooth surface $X$. If the Puiseux characteristic of $C$ at $P$ is $(m; \beta_1, \ldots, \beta_g)$, then the Puiseux characteristic of the curve obtained by blowing up $X$ at $P$ is given by
\[
\begin{align*}
(m; \beta_1 - m, \ldots, \beta_g - m) \quad & \text{if } \beta_1 > 2m \\
(\beta_1 - m; \beta_2 - \beta_1 + m, \ldots, \beta_g - \beta_1 + m) \quad & \text{if } \beta_1 < 2m, (\beta_1 - m)|m \\
(\beta_1 - m; \beta_2 - \beta_1 + m, \ldots, \beta_g - \beta_1 + m) \quad & \text{if } (\beta_1 - m)|m.
\end{align*}
\]
Corollary 3.3. Let $C$ be a curve unibranch at $P$ whose Puiseux characteristic is $(m; \beta_1, \ldots, \beta_g)$. Consider the sequence of the strict transforms of $C$ constructed by the successive blowings up that desingularize $C$. The first Puiseux characteristic in this sequence with exactly $g - j + 1$ characteristic exponents is $(m_j; \beta_j^{(j-1)}, \ldots, \beta_g^{(j-1)})$, where $m_j = \gcd(m_{j-1}, \beta_{j-1})$ as before, and for any $k \geq j$,

$$
\beta_k^{(j-1)} = \beta_k - \beta_{j-1} + a_{j-1}m_j
$$

with $a_{j-1}$ the last quotient in the Euclidean algorithm for $m_{j-1}$ and $\beta_{j-1}$.

Proof. Considering the Euclidean algorithm for $\beta_1$ and $m_1 = m$ and using the previous proposition, it is easy to see that the first element in the second Puiseux characteristic is $m_2 = \gcd(m_1, \beta_1)$. Computing $\beta_j - \beta_j'$ we get that that $\beta_j' = \beta_j - \beta_1 + a_1m_2$, for any $j \geq 2$. Notice that

$$
\gcd(m_2, \beta_2) = \gcd(m_2, \beta_2')
$$

and that furthermore, the last quotients in the Euclidean algorithms for $\beta_2$ and $m_2$, and $\beta_2'$ and $m_2$ coincide. The result follows by induction on $j$. \hfill \Box

Enriques established the equivalence between the Puiseux characteristic and the Enriques diagram associated to a curve unibranch and singular at $P$. We refer the reader to [4], [1] and especially [11] Theorem XI § 6.1.3 for further details. What we want to state now is just a more concise way to present this equivalence.

Theorem 3.4 (Enriques). Let $(m; \beta_1, \ldots, \beta_g)$ be the Puiseux characteristic of $C$ at $P$. Set

$$
p_j = \frac{m_j}{m_{j+1}} \quad \text{and} \quad q_j = \frac{\beta_j - \beta_{j-1} + m_j}{m_{j+1}}
$$

for any $1 \leq j \leq g$, with $\beta_0 = m_1$. Then, the corresponding Enriques diagram $T$ is given by the Enriques tree

$$
T = T_{p_1,q_1} \# T_{p_2,q_2} \# \cdots \# T_{p_g,q_g}.
$$

Sketch of proof. We will use the notation from Corollary 3.3. The first part of the Enriques tree associated to $C$ coincides with the Enriques tree associated to a curve having the Puiseux characteristic $(m_1; \beta_1)$. Such a curve is desingularized by the blowings up corresponding to the whole Enriques tree $T_{p_1,\beta_1/m_2} = T_{p_1,a_1}$ except for the last stretch. It is noteworthy that the length of this stretch equals the last quotient in the Euclidean algorithm for $\beta_1/m_2$ and $p_1$, i.e. $a_1$, and that the blowings up of this last stretch are needed to obtain a log-resolution for such a curve. Now, if $C_2$ denotes the strict transform of $C$ before the blowings up of the last stretch of $T_{p_1,\beta_1/m_2}$, then $C_2$ is the first strict transform among the strict transforms of $C$ given by the log resolution, having exactly $g - 1$ Puiseux exponents. The Puiseux characteristic is $(m_2; \beta_2', \ldots, \beta_g')$. What we have previously said for $C$ is true for $C_2$. So the Enriques tree associated to $C_2$ starts with $T_{p_2,n_2}$, where

$$
n_2 = \beta_2'/m_3.
$$

It follows that the part of the Enriques tree associated to $C$ and corresponding to the first two Puiseux exponents is

$$
T_{p_1,q_1} \# T_{p_2,n_2-(a_1-1)p_2}.
$$
But
\[ n_2 - (a_1 - 1)p_2 = \frac{\beta'_2}{m_3} - (a_1 - 1)m_2 = \frac{\beta_2 - \beta_1 + a_1 m_2}{m_3} - (a_1 - 1)m_2 = q_2. \]
The proof is finished by induction on the number of Puiseux exponents. \[\square\]

**Example.** The curve given in Example 2.2 has the good parametrization \( t \mapsto (t^4, t^6 + t^7) \). It follows that the Puiseux characteristic is \((4; 6, 7)\), with \( \frac{3}{2} \) the first Puiseux exponent, and after one blow-up, the strict transform has the unique Puiseux exponent \( \frac{5}{2} \). We recover the fact that the Enriques tree of the minimal log resolution is \( T_{2,3}\#T_{2,3} \).

**Corollary 3.5.** Let \( C \) be a curve unibranch at \( P \). In the notation of Theorem 3.4, if \( w_\alpha \) are the weights of the Enriques diagram associated to the minimal log resolution of \( C \) at \( P \), and if \( r_j \) is the number of vertices of the tree \( T_{p_j,q_j} \) for any \( 1 \leq j \leq g \), then
\[ w_{r_1 + \ldots + r_j - (j-1)} = m_{j+1} = p_{j+1}p_{j+2} \cdots p_g \]
for any \( 1 \leq j \leq g \).

**Proof.** Follows from Theorem 3.4, Proposition 3.2, Corollary 3.3 and Remark 1.1. \[\square\]

**Proof of Theorem 3.1.** By Theorem 2.3, the jumping numbers are given by
\[ g \bigcup_{j=1}^g \frac{1}{m_j \bar{q}_j} R^{m_j+1}(p_j, \bar{q}_j). \]
where the Enriques tree of the minimal log resolution is \( T_{p_1,q_1} \# \cdots \# T_{p_g,q_g} \). Furthermore \( m_j = p_j \cdots p_g \) and
\[ \bar{q}_j = \frac{m_{j-1}}{m_{j+1}} \bar{q}_{j-1} - p_j + q_j \]
for any \( 1 \leq j \leq g \), with \( m_{g+1} = 1 \) and \( \bar{q}_1 = q_1 \). Clearly \( p_j = m_j/m_{j+1} \). As for \( \bar{q}_j \), using Theorem 3.4 and the identity (6), we have
\[ \frac{m_{j-1}}{m_{j+1}} \bar{q}_{j-1} - p_j + q_j = \frac{m_{j-1}}{m_{j+1}} \bar{B}_{j-1} - \frac{m_j}{m_{j+1}} + \frac{\beta_j - \beta_{j-1} + m_j}{m_{j+1}} \]
\[ = \frac{1}{m_{j+1}} \left( \frac{m_{j-1}}{m_j} \bar{B}_{j-1} + \beta_j - \beta_{j-1} \right) \]
\[ = \frac{\bar{B}_j}{m_{j+1}}. \]
The result follows. \[\square\]

### 4 The proof of Proposition 2.5

Let \( \xi \) be a jumping number and set \( E = \sum_{\beta \in \mathcal{B}} E_\beta \) with \( \mathcal{B} \subset \mathcal{V} \) the subset of all vertices such that \( [\xi e_\beta] = \xi e_\beta \). Then
\[ \mu_* \mathcal{O}_Y(K_\mu - [\xi \mu^* C]) \subset \mu_* \mathcal{O}_Y(K_\mu - [\xi \mu^* C] + E). \] (7)
We will show that the right hand side ideal can be computed using only \( \mathfrak{R} \)-chains contained in \( E \). A chain \( \Gamma \) of exceptional divisors \( E_\beta \) in the dual graph of the log resolution is called an \( \mathfrak{R} \)-chain if its both extremities are relevant divisors. A relevant divisor will be considered as an improper \( \mathfrak{R} \)-chain.

**Claim.**

\[ \mu_* \mathcal{O}_Y(K_\mu - [\xi \mu^* C] + E) = \mu_* \mathcal{O}_Y(K_\mu - [\xi \mu^* C] + \sum \Gamma). \]  \hspace{1cm} (8)

where \( \Gamma \) runs through all the maximal \( \mathfrak{R} \)-chains contained in \( E \).

Indeed, suppose that \( E \) is a connected subgraph of the dual graph that contains at least two irreducible components. Let \( E_0 \) be an irreducible component which is not a relevant divisor and which is an extremity for \( E \). The intersection of \( E_0 \) with \( E' = E - E_0 \) consists of exactly one point \( P \) that will be seen as a divisor of \( E_0 \). Notice that

\[ \deg(\tilde{C} - E)|_{E_0} \leq 1. \] \hspace{1cm} (9)

Since \( (K_\mu + E)|_{E_0} \sim K_{E_0} + (E - E_0)|_{E_0} = K_{E_0} + P \), we have the commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathcal{O}_Y(\Delta) & \rightarrow & \mathcal{O}_Y(\Delta + E') & \rightarrow & \mathcal{O}_{E'}(K_{E'} - [\xi \mu^* C]|_{E'}) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \mathcal{O}_Y(\Delta) & \rightarrow & \mathcal{O}_Y(\Delta + E) & \rightarrow & \mathcal{O}_E(K_E - [\xi \mu^* C]|_E) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{O}_{E_0}(\Delta_0 + P) & \rightarrow & \mathcal{O}_{E_0}(\Delta_0 + P) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\]

where \( \Delta = K_\mu - [\xi \mu^* C] \) and \( \Delta_0 = K_{E_0} - [\xi \mu^* C]|_{E_0} \). The last entry in the middle vertical short exact sequence is given by the snake lemma. Pushing down to \( X \) this exact sequence and using the local vanishing, we get that

\[ \mu_* \mathcal{O}_Y(K_\mu - [\xi \mu^* C] + E) = \mu_* \mathcal{O}_Y(K_\mu - [\xi \mu^* C] + E') \]

if and only if \( h^0(E_0, K_{E_0} + P - [\xi \mu^* C]|_{E_0}) = 0 \). But this is true since \( E_0 \) is an extremity of \( E \) and

\[ [\xi \mu^* C] \cdot E_0 = \sum_{\beta \in \mathfrak{B}} \xi e_\beta E_\beta \cdot E_0 + \sum_{\alpha \notin \mathfrak{B}} [\xi e_\alpha] E_\alpha \cdot E_0 \]

\[ > \sum_{\beta \in \mathfrak{B}} \xi e_\beta E_\beta \cdot E_0 + \sum_{\alpha \notin \mathfrak{B}} (\xi e_\alpha - 1) E_\alpha \cdot E_0 + (\xi - 1) \tilde{C} \cdot E_0 \]

\[ = \xi \mu^* C \cdot E_0 - (\tilde{C} + \sum_{\alpha \notin \mathfrak{B}} E_\alpha) \cdot E_0 \]

\[ = -(\tilde{C} - E) \cdot E_0, \]
and with \( \text{Eq.} \) yield
\[
\deg P - [\xi \mu^* C] \cdot E_0 < 1 + (\tilde{C} - E) \cdot E_0 \leq 2.
\]

Hence
\[
\deg(K_{E_0} + P - [\xi \mu^* C]|_{E_0}) \leq -1.
\]

Repeated use of this argument shows that we can eliminate from \( E \) one component at a time replacing \( E \) by \( E' \) as long as \( E_0 \) is not a relevant divisor. Hence if \( E \) is connected, only its maximal \( \mathfrak{A} \)-chain counts in computing the ideal \( \mathcal{O}_Y(K_\mu - [\xi \mu^* C] + E) \). The above argument, applied to each connected part of \( E \), justifies the claim.

To end the proof of the proposition it is sufficient to show that the proper \( \mathfrak{A} \)-chains can be discarded in the equality \( \text{Eq.} \). This is done next.

**Claim.** If \( \Gamma \) is a proper \( \mathfrak{A} \)-chain, then \( h^0(\Gamma, K_\Gamma - [\xi \mu^* C]|_{\Gamma}) = 0 \).

Indeed, let us suppose that the \( \mathfrak{A} \)-chain \( \Gamma \) connects \( E_\rho \) and \( E_{\rho'} \), with \( \rho' > \rho \). The first \( E_\gamma \neq E_\rho \) belonging to \( \Gamma \) that \( E_\rho \) intersects is either \( E_{\rho+1} \) if \( a_1' \geq 2 \), or \( E_{\rho+a_2'+1} \) if not. In the former case \( e_{\rho+1} = e_\rho + \pi \) and in the latter we have \( e_{\rho+a_2'+1} = (a_2' + 1)e_\rho + \pi \). Here \( \pi \) indicates a certain \( m_j \), depending on \( \rho \). Since \( \pi e_\gamma \) is an integer for any \( E_\gamma \subset \Gamma \), these equalities imply that \( \pi \xi \) is an integer. Then
\[
[\xi \mu^* C] \cdot E_\rho = [\xi \mu^* C] \cdot E_\rho - \langle \xi \mu^* C \rangle \cdot E_\rho = - \langle \xi e_{\rho-a_m} \rangle E_{\rho-a_m} + \langle \xi e_{\rho-1} \rangle E_{\rho-1} \cdot E_\rho = 0
\]
as \( \pi \) divides both \( e_{\rho-a_m} \) and \( e_{\rho-1} \) by Lemma \( \text{[2.7]} \). Now, if \( \Gamma' = \Gamma - E_\rho \) and \( P \) is the intersection point of \( E_\rho \) with \( \Gamma' \), from the short exact sequence
\[
0 \rightarrow \mathcal{O}_{\Gamma'}(K_{\Gamma'} - [\xi \mu^* C]|_{\Gamma'}) \rightarrow \mathcal{O}_{\Gamma}(K_\Gamma - [\xi \mu^* C]|_{\Gamma}) \rightarrow \mathcal{O}_{E_\rho}(K_{E_\rho} + P - [\xi \mu^* C]|_{E_\rho}) \rightarrow 0
\]
we obtain that
\[
h^0(\Gamma, K_\Gamma - [\xi \mu^* C]|_{\Gamma}) = h^0(\Gamma', K_{\Gamma'} - [\xi \mu^* C]|_{\Gamma'}).
\]

Inductively we cut off from the new chain the lowest extremity \( E_\gamma \) and keep denoting the resulting chain by \( \Gamma' \). We eventually arrive at
\[
0 \rightarrow \mathcal{O}_{E_{\rho'}}(K_{E_{\rho'}} - [\xi \mu^* C]|_{E_{\rho'}}) \rightarrow \mathcal{O}_{\Gamma'}(K_{\Gamma'} - [\xi \mu^* C]|_{\Gamma'}) \rightarrow \mathcal{O}_{E_\gamma}(K_{E_\gamma} + P - [\xi \mu^* C]|_{E_\gamma}) \rightarrow 0,
\]
where \( \gamma \) equals either \( \rho' - 1 \) or \( \rho' - a_m' \), i.e. \( E_\gamma \) is the last exceptional divisor in \( \Gamma \) different from \( E_{\rho'} \) and \( P \) is the intersection point between \( E_\gamma \) and \( E_{\rho'} \). Arguing as before \( \deg [\xi \mu^* C]|_{E_\gamma} = 0 \).

As for the computation of \( \deg [\xi \mu^* C]|_{E_{\rho'}} \), we have
\[
[\xi \mu^* C] \cdot E_{\rho'} = - \langle \xi e_{\rho'-a_m'} \rangle E_{\rho'-a_m'} + \langle \xi e_{\rho'-1} \rangle E_{\rho'-1} + \langle \xi e_{\rho'+\alpha} \rangle E_{\rho'+\alpha} \cdot E_{\rho'}.
\]
Since, as it has been said, \( \gamma \) equals either \( \rho' - 1 \) or \( \rho' - a_m' \), and since \( e_{\rho'} = e_{\rho'-a_m'} + e_{\rho'-1} + \pi' \) with \( \pi'|\pi \),
\[
[\xi \mu^* C] \cdot E_{\rho'} = \langle \xi e_{\rho'+\alpha} \rangle = 0.
\]
Hence \( h^0(\Gamma', K_{\Gamma'} - [\xi \mu^* C]|_{\Gamma'}) = 0 \) and finally, \( h^0(\Gamma, K_\Gamma - [\xi \mu^* C]|_{\Gamma}) = 0 \) for the proper \( \mathfrak{A} \)-chain \( \Gamma \), justifying the claim.
Now, the short exact sequence

\[ 0 \rightarrow \mu_*\mathcal{O}_Y(K_\mu - [\xi\mu^*C]) \rightarrow \mu_*\mathcal{O}_Y(K_\mu - [\xi\mu^*C] + \sum_{\beta \in \mathfrak{B}} E_\beta) \rightarrow \bigoplus_{\rho \in (\mathfrak{B} \cap \mathfrak{R})'} H^0(E_\rho, K_{E_\rho} - [\xi\mu^*C] |_{E_\rho}) \rightarrow 0 \]

shows that \( \xi \) must be contributed by a relevant divisor. The set \((\mathfrak{B} \cap \mathfrak{R})'\) is the set of relevant positions in \( \mathfrak{B} \) that do not define proper \( \mathfrak{R} \)-chains contained in \( \mathfrak{B} \).

**References**

[1] E. Casas-Alvero, Infinitely near imposed singularities and singularities of polar curves. *Math. Ann.* 287 (1990), 429–454.

[2] Ein, L. Multiplier ideals, vanishing theorems and applications. *Algebraic geometry—Santa Cruz 1995*, 203–219.

[3] Ein, L., R. Lazarsfeld, K. E. Smith and D. Varolin. Jumping coefficients of multiplier ideals. *Duke Math. J.* 123 no. 3 (2004), 469-506.

[4] Enriques, F. and Chisini, O. *Lezioni Sulla Teoria Geometrica Delle Equazioni e Delle Funzioni Algebraiche*. N. Zanichelli, Bologna, 1915.

[5] Evain, L. La fonction de Hilbert de la réunion de 4\(^h\) gros points génériques de \( \mathbb{P}^2 \) de même multiplicité. *J. Algebraic Geom.* 8 (1999), 787–796.

[6] Ch. Favre, M. Jonsson Valuations and multiplier ideals, *J. Amer. Math. Soc.* 18 (2005) no. 3, 655–684.

[7] J. A. Howald, Multiplier ideals of monomial ideals. *Trans.Amer.Math.Soc.* 353 (2001), 2665–2671.

[8] T. Järvelätho, *Jumping numbers of a simple complete ideal in a two-dimensional regular local ring*. Ph. D. Thesis, University of Helsinki (2007)

[9] R. Lazarsfeld, *Positivity in algebraic geometry*. A Series of Modern Surveys in Mathematics, Springer-Verlag, Berlin, 2004.

[10] Naie, D. Irregularity of cyclic multiple planes after Zariski. *L’enseignement mathématique* 53 (2008), 265-305.

[11] Semple, J. G. and Kneebone, G. T. *Algebraic curves*. Oxford University Press, London-New York 1959.

[12] K. E. Smith, H. M. Thompson, Irrelevant exceptional divisors for curves on a smooth surface. *Preprint*, [arXiv:math/0611765](http://arxiv.org/abs/math/0611765).
[13] C. T. C. Wall, *Singular points of plane curves*. London Mathematical Society Student Texts, 63. Cambridge University Press, Cambridge, 2004.

Daniel Naie  
Département de Mathématiques  
Université d’Angers  
F-40045 Angers  
France  
Daniel.Naie@univ-angers.fr