Deformations of complex cones and neighborhoods of ample divisors

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Let $X$ be a complex manifold and $S \hookrightarrow X$ be an embedding of complex submanifold. Assuming that the embedding is $(k-1)$-linearizable or $(k-1)$-comfortably embedded, we construct via the deformation to the normal cone a Kähler-Einstein metric $\omega$ on $X \setminus D$ whose tangent cone at infinity is some explicit conical Calabi-Yau metric $\kappa$. Then by Tian-Yau [27], there exists a complete Calabi-Yau metric $\omega_{TY}$ on $M := X \setminus D$ whose tangent cone at infinity is some explicit conical Calabi-Yau metric $\omega_0 = dr^2 + r^2 g_Y$ on the complex cone $C := C(D, N_D)$ which will be recalled in Section 3.1. More precisely, there exists a diffeomorphism $F_K : C(D, N_D) \setminus B_R(g) \to M \setminus K$ for some compact set $K \subset M$ such that the following inequality holds:

$$\|\nabla_{\omega_0}(F_K^*\omega_{TY} - \omega_0)\|_{\omega_{TY}} \leq C r^{-\lambda_1-j} \quad \text{for} \quad j \geq 0,$$

where $\nabla_{\omega_0}$ is the Levi-Civita connection of $\omega_{TY}$. A natural question is what is the optimal value for the convergence rate $\lambda_{\text{max}}$. This is important for example for gluing constructions of Calabi-Yau metrics on compact manifolds. By the work of Cheeger-Tian [8], Coevering [30], Goto [14] and Colon-Hein ([10], [11]), we already have a good understanding of this convergence rate abstractly. Denote by $\mathfrak{t}$ is Kähler class represented by $\omega_{TY}$. Then we have the following estimate of the optimal rate (see [10], [11], and [12, Remark 1.2]):

$$\lambda_{\text{max}} \geq \begin{cases} \min(2n, \lambda_1), & \text{if} \ \mathfrak{t} \in H^2(M) \\
\min(2, \lambda_1), & \text{if} \ \mathfrak{t} \in H^2(M). \end{cases} \quad (1)$$

Here $\lambda_1$ is any number satisfying the following condition: there exists a diffeomorphism $F_K : C(D, N_D) \setminus B_R(g) \to M \setminus K$ such that

$$\|\nabla_{\omega_0}(F_K^*\Omega - \Omega_0)\|_{\omega_{TY}} \leq C r^{-\lambda_1-j} \quad \text{for} \quad j \geq 0, \quad (2)$$

where $\Omega$ (resp. $\Omega_0$) is the meromorphic volume form on $X$ (resp. $\overline{C(D, N_D)}$) that is non-vanishing holomorphic on $M = X \setminus D$ (resp. $C(D, N_D)$) and has pole of order $\alpha$ along $D$. Colon-Hein [10] showed that the condition (2) implies the following condition:

$$\|\nabla_{\omega_0}(F_K^*J - J_0)\|_{\omega_{TY}} \leq C r^{-\lambda_1-j} \quad \text{for} \quad j \geq 0, \quad (3)$$

1 Introduction

Our motivation for this paper is to understand the optimal convergence rate of asymptotically conical (AC) Calabi-Yau Kähler (CY) metrics on non-compact Kähler manifolds. To explain the problem, we consider AC CY metrics constructed by Tian-Yau. Let $X$ be a Fano manifold and $D$ be a smooth divisor such that $-K_X \sim_{\Q} \alpha D$ with $Q \ni \alpha > 1$. Assume that $D$ has a Kähler-Einstein metric. Then by Tian-Yau [27], there exists a complete Calabi-Yau metric $\omega_{TY}$ on $M := X \setminus D$ whose tangent cone at infinity is some explicit conical Calabi-Yau metric $\omega_0 = dr^2 + r^2 g_Y$ on the complex cone $C := C(D, N_D)$ which will be recalled in Section 3.1. More precisely, there exists a diffeomorphism $F_K : C(D, N_D) \setminus B_R(g) \to M\setminus K$ for some compact set $K \subset M$ such that the following inequality holds:

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$$\lambda_{\text{max}} \geq \begin{cases} \min(2n, \lambda_1), & \text{if} \ \mathfrak{t} \in H^2(M) \\
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Here $\lambda_1$ is any number satisfying the following condition: there exists a diffeomorphism $F_K : C(D, N_D) \setminus B_R(g) \to M\setminus K$ such that

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where $\Omega$ (resp. $\Omega_0$) is the meromorphic volume form on $X$ (resp. $\overline{C(D, N_D)}$) that is non-vanishing holomorphic on $M = X \setminus D$ (resp. $C(D, N_D)$) and has pole of order $\alpha$ along $D$. Colon-Hein [10] showed that the condition (2) implies the following condition:

$$\|\nabla_{\omega_0}(F_K^*J - J_0)\|_{\omega_{TY}} \leq C r^{-\lambda_1-j} \quad \text{for} \quad j \geq 0, \quad (3)$$
where $J$ (resp. $J_0$) is the complex structure on $M$ (resp. $C(D, N_D)$). So we see that $\lambda_1$ essentially measures the difference between the complex structure of $M \setminus K$ and $C(D, N_D) \setminus BR(g)$. It’s easy to see that, equivalently we are comparing the complex structure on the (punctured) neighborhood of $D$ inside $X$ and the complex structure of (punctured) neighborhood of $D$ inside $N_D$.

In general, let $S$ be a complex submanifold of an ambient complex manifold $X$. The comparison between neighborhoods of $S$ inside $X$ with neighborhoods of $S$ inside the normal bundle $N_S$ was studied in depth in depth by Grauert ([15]), Griffiths ([16]), and Camacho-Movasati-Sad ([6], [7]). Note that although that in general $N_S$ has a very different holomorphic structure with neighborhoods of $S$ inside $X$, $N_S$ can be viewed as a first order approximation of small neighborhood of $S$. Denote $S(k)$ the ringed analytic space $(S, O_X |_{S^k/N_S^{k+1}})$, which is also called the $k$-th infinitesimal neighborhood of $S$ inside $X$.

Definition 1.1. $S \to X$ is $k$-linearizable if its $k$-th infinitesimal neighbourhood $S(k)$ in $X$ is isomorphic to its $k$-th infinitesimal neighbourhood $S_N(k)$ in $N_S$. Here we identify $S$ with the zero section $S_0$ of $N_S =: N$.

Grauert [15] showed that the obstruction for extending an isomorphism $S(k - 1) \to S_N(k - 1)$ to an isomorphism $S(k) \to S_N(k)$ lies in the cohomology group $H^1(S, \Theta_X |_{S \otimes I^k_S/I^{k+1}_S})$. He also pointed out that this obstruction consists of two parts. To see this, consider the exact sequence:

$$0 \to N_S \otimes I^k_S/I^{k+1}_S \to \Theta_X |_{S \otimes I^k_S/I^{k+1}_S} \to \Theta_S \otimes I^k_S/I^{k+1}_S \to 0.$$ |

from which we get the long exact sequence:

$$\cdots \to H^1(S, N_S \otimes I^k_S/I^{k+1}_S) \to H^1(S, \Theta_X |_{S \otimes I^k_S/I^{k+1}_S}) \to H^1(S, \Theta_S \otimes I^k_S/I^{k+1}_S) \to \cdots$$

So roughly speaking, the obstruction comes from two parts, one from $H^1(S, N_S \otimes I^k_S/I^{k+1}_S)$ and the other from $H^1(S, \Theta_S \otimes I^k_S/I^{k+1}_S)$. In [1], Abate-Bracci-Toneva explicitly described these two cohomological obstruction classes, and introduced the notion of $k$-splitting and $k$-comfortably embedded such that $k$-linearizable = $k$-splitting + $(k-1)$-comfortably embedded. For more details, see Appendix 6.2.

Returning back to our goal of estimating optimal rate, to get condition (2) and (3) we want to compare the two embeddings $S \subset X$ and $S_0 \subset N_S$ by constructing some diffeomorphism which is in some sense the most holomorphic one. This may be known to experts after the celebrated work of Grauert [15]. For example, some rough arguments appeared in [29]. Here we would like to give an almost explicit construction using the work of Abate-Bracci-Toneva [1] together with the deformation to the normal cone construction. We first state a preliminary result. Let $\tilde{g}_0$ be a smooth Riemannian metric on a neighborhood $W_0$ of $S_0$ inside $N_S$. Let $\|\cdot\|_{\tilde{g}_0}$ denote the $C^1$-norm of tensors on $W_0$ with respect to $\tilde{g}_0$ and $F$ be the distance function to $S_0$ with respect to $\tilde{g}_0$.

**Proposition 1.1.** Assume $S \to X$ is $(k - 1)$-linearizable, then there exists a diffeomorphism $F : W_0 \to F(W_0) \subset W$ where $U$ is a small neighborhood of $S \subset X$, such that $F$ satisfies

$$\|\nabla_{\tilde{g}_0}^j (F^*J - J_0)\|_{\tilde{g}_0} \leq r^{k-j}.$$  

Here we point out the norm used in (4) is with respect to $\tilde{g}_0$ while the norms used in (2)-(3) are with respect to the cone metric $\omega_0$ (see Section 3.1 for the comparison between these two Kähler metrics). This discrepancy is explained by the difference between linearizability and comfortably embeddedness. In other words, if we assume $D$ is furthermore $(k-1)$-comfortably embedded, we get exactly the conditions (2)-(3) that we want.

**Proposition 1.2.** Assume that $D$ is $(k - 1)$-comfortably embedded. Then there exists a diffeomorphism away from compact sets $F_K : C(D, N_D) \setminus BR(g) \to (X \setminus D) \setminus K$ such that

$$\|\nabla_{\omega_0}^j (F_K^*\Omega - \Omega_0)\|_{\omega_0} \leq r^{-\frac{j}{k-1} - j}, \quad \text{and} \quad \|\nabla_{\omega_0}^j (F_K^*J - J_0)\|_{\omega_0} \leq r^{-\frac{j}{k-1} - j}.$$  

Here the number $\frac{j}{k-1}$ is the inverse of the exponent in the Calabi-ansatz (see (31) in Section 3.1).

We will see in Theorem 1.2 that estimates in (5) are sharp. As a consequence, we immediately get a lower bound estimate of the optimal rate of some class of AC Calabi-Yau metrics constructed by Tian-Yau [27].
Corollary 1.1. Using the above notation, the Tian-Yau metric $\omega_{TY}$ satisfies:

$$\|\nabla_{\omega}(F_K^*\omega_{TY} - \omega_0)\|_{\omega_0} \leq r^{-\min\{2n, \frac{nk}{2} - 1\} - j} \text{ for } j \geq 0.$$ 

If moreover we assume that the Kähler class is contained in the compactly supported cohomology $H^2_c(X \setminus D)$, then we get:

$$\|\nabla_{\omega}(F_K^*\omega_{TY} - \omega_0)\|_{\omega_0} \leq r^{-\min\{2n, \frac{nk}{2} - 1\} - j} \text{ for } j \geq 0.$$ 

As discussed before, the corollary follows from Proposition 1.2 and regularity theory developed by Conlon-Hein [10]. The last statement also follows implicitly from Proposition 1.2 and Cheeger-Tian [8]. In many cases, Proposition 1.2 improves the regularity in [11] and is essentially sharp (see [12, Remark 1.2]). See Appendix 6.1, in particular, Remark 6.2, for more background details.

As mentioned above, the construction of diffeomorphism $F$ in above Propositions uses the standard construction in algebraic geometry which is called deformation to the normal cone. This is a way to degenerate a neighborhood of $S \hookrightarrow X$ to a neighborhood $S_0 \hookrightarrow N_S$. The construction is simply a blow-up of the submanifold $D \times \{0\} \subset X \times \mathbb{C}$ which we will denote by $\tilde{X} = Bl_{D \times \{0\}}(X \times \mathbb{C})$ with the projection $\pi : \tilde{X} \to X$. The central fibre $\tilde{X}_0 = X \cup E$ is the union of two components. The exceptional divisor $E = \mathbb{P}(N_S \oplus \mathbb{C})$ is the projective compactification of the normal bundle $N_S$ of $S \subset X$. In this way we can view $S \hookrightarrow X$ as a deformation of $S_0 \hookrightarrow N_S$ and try to read out the order of difference from deformation theory.

To move further in this direction, our next result relates the number $k$ in the above proposition to the weight of some deformation of singularity, in some special cases. From now on, we assume $D$ is an ample divisor in $X$. The component $X \subset \tilde{X}_0$ from above can be contracted to a point. See figure 1 for illustration. Under appropriate assumptions (see Remark 2.1), the contracted central fibre $\tilde{X}_0$ is normal and hence coincides with the projective cone $\tilde{C} = \mathbb{C}(D, N_D)$. Then we get a $C^*$-equivariant degeneration $\mathcal{X}$ of $X$ to the projective cone $\mathcal{C} = \mathbb{C}(D, N_D)$ which in general has an isolated singularity $\mathcal{O}$. Note that $\mathcal{O}$ is simply the image of the infinity divisor $D_{\infty}$ of $E = \mathbb{P}(N_D \oplus \mathbb{C})$ under the contraction. Denote $\mathcal{D} \cong D \times \mathbb{C}$ the strict transform of $D \times \mathbb{C}$ in this process. Then the variety $\mathcal{X}^o = \mathcal{X} \setminus \mathcal{D}$ is a degeneration of quasi-projective variety $X \setminus D$ to the affine cone $C = C(D, N_D) = \mathcal{C} \setminus \mathcal{D}$. In other words, $\mathcal{X}^o$ can be viewed as a deformation of the cone $C$. Let $\mathcal{C} \to Def(C)$ be the versal deformation of $C$. Then $\mathcal{X}^o$ is induced from a map $\mathcal{I}_{\mathcal{X}^o} : \mathcal{C} \to Def(C)$. By Kuranishi-Grauert, $Def(C)$ is a locally complete analytic variety in $\mathbb{C}^1$. So $\mathcal{I}_{\mathcal{X}^o}$ is the germ of a vector-valued holomorphic function whose image passes through $0 \in \mathbb{T}_C^1$ and lies in $Def(C) \subset \mathbb{T}_C^1$. Denote $\kappa = ord_0 \mathcal{I}_{\mathcal{X}^o}$ the vanishing order of the $\mathcal{I}_{\mathcal{X}^o}$ at $0 \in \mathcal{C}$. We define the reduced Kodaira-Spencer class to be

$$KS_{\mathcal{X}^o}^{red} = \left. \frac{d^{\kappa}}{dt^\kappa} \right|_{t=0} \mathcal{I}_{\mathcal{X}^o}(t).$$ (6)

By the works of ([25], [24], [28]), we have a good understanding of $\mathbb{T}_C^1$ which is a graded vector space. We denote $w(D, X)$ the weight of $KS_{\mathcal{X}^o}^{red}$. Because $\mathcal{X}^o$ comes from $\mathcal{X}$ that is also a deformation of $\mathcal{C}(D, N_D)\setminus \mathcal{O} = (N_D \to D)$, it’s easy to get that $w(D, X) \leq 0$. On the other hand, we define the integer $m(D, X)$ as in [1, Remark 4.6] to be the maximum positive integer $m$ such that the embedding $D \hookrightarrow X$ is $(m-1)$-comfortably embedded. We then have:

![Figure 1: $\mu : \tilde{X} \longrightarrow X$](image-url)
Theorem 1.1. In the above setup, i.e. we assume $D$ is ample and $X_0$ is normal, assume furthermore that $n \geq 3$. Then $|w(D, X)| = m(D, X)$.

Combining this result with Proposition 1.2, we can give simple algebraic interpretations of ad hoc calculations in [10] on the asymptotically rate of holomorphic volume forms. See Example 3.1.

Note that here we assume $\dim D \geq 2$. So by remark 6.4, $D \subset X$ is $(m(X, D) - 1)$-linearizable but not $m(X, D)$-linearizable. The main idea in the proof of Theorem 1.1 is to use the “holographic principle”: the Hartogs’ type extension theorem of cohomology by Andreotti-Grauert.

Reversely, it’s natural to ask if any deformation of $C$ comes from this construction. We will prove results in both differential geometric set-up and algebraic set-up. On the differential geometric side, we will prove an analytic compactification result. This result might be known to experts. For example, for asymptotically conical Calabi-Yau with fast decay, this result was claimed by Cheeger-Tian in [8, Paragraph after Theorem 0.16]. Here we will try to give a detailed proof. Note that a similar compactification result in the asymptotically cylindrical Calabi-Yau case has recently appeared in [17]. See Remark 4.2 for some comparison. To state this result in a slightly more general form, let $h$ be a Hermitian metric on any negative line bundle $L^{-1} \to D$ with negative Chern curvature. Since $C = C(D, L)$ is obtained from $L^{-1}$ by contracting the zero section, $h$ can be thought as a function on the cone $C$. Fixing any $\delta > 0$, there is a complete Kähler cone metrics on $C(D, L)$ given as (see section 3.1)

$$g_0 = \sqrt{-1} \partial \bar{\partial} \delta = dr^2 + r^2 g_Y.$$  

Let $U_\epsilon$ be a neighborhood of the infinity end of $C(D, L)$. Equivalently $U_\epsilon$ is a punctured neighborhood of the embedding $D \to \overline{C}(D, L)$. Denote $J_0$ the standard complex structure on $C(D, L)$, and $\overline{U}_\epsilon = U_\epsilon \cup D$ the compactification of $U_\epsilon$ in $\overline{C}(D, L)$.

Theorem 1.2. Assume that $J$ is a complex structure on $U_\epsilon = \overline{U}_\epsilon \setminus D$ such that

$$\|\nabla^k g_0 (J - J_0)\|_{g_0} \leq r^{-\lambda - k},$$

for some $\lambda > 0$. Then the complex structure extends to a complex structure $J$ on $\overline{U}_\epsilon$. Moreover, if we denote $m = [\delta \lambda]$ (the minimal integer which is bigger than or equal to $\delta \lambda$), then in the compactification the divisor $D$ is $(m - 1)$-comfortably embedded.

This can be seen as a converse to the Proposition 1.2 and implies the estimate in Proposition 1.2 is sharp.

Remark 1.1. Because our proof uses only locally information near the divisor, the argument in the proof applies in the more general orbifold case. Actually Colon-Hein [12] has recently used the compactification obtained in Theorem 1.2 to prove any AC CY metric with quasi-regular tangent cone at infinity comes from Tian-Yau’s construction.

On the algebraic geometry side, we will show the following theorem which may be of independent interest and can be seen as generalization of one of Pinkham’s results ([24]) from the very ample to the ample line bundle case.

Theorem 1.3. Any infinitesimal deformation of the cone $C(D, L)$ of negative weight comes from an infinitesimal deformation of the $\overline{C}(D, L)$.

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2 Embeddings of submanifolds and deformations

2.1 Construction of comparison diffeomorphism

Under appropriate conditions on the embedding of $S \subset X$, we will construct a diffeomorphism which is holomorphic up to some precise order. The idea is to use the deformation to the normal cone to construct a (local) differentiable family. Then we use the method as in Kodaira’s book [19, Section 2.3] to construct the diffeomorphism. The work of Abate-Bracci-Tovena [1] allows us to read out the order of holomorphicity. We refer to Appendix 6.2 for the definitions and preliminary results from [1] we need for the following discussions. We consider two slightly different conditions.
1. \textit{(k-1)-linearizable)} By Theorem 6.6 in Appendix 6.2 we can find coordinate charts \(\{V_{\alpha}, \{z_{\alpha}\}\}\) of \(X\) near the submanifold \(S\) such that the transition functions on \(V_{\alpha} \cap V_{\beta}\) are given by:

\[
\begin{align*}
  z_{\beta}^{r} &= \sum_{a=1}^{m}(a_{\beta a})_{r}z_{\alpha}^{a} + R_{k}^{r}, & \text{for } r = 1, \ldots, m, \\
  z_{\beta}^{p} &= \phi_{\beta a}^{p}(z_{\alpha}^{a}) + R_{k}^{p}, & \text{for } p = m + 1, \ldots, n.
\end{align*}
\] (7)

such that \(S \cap V_{\alpha} = \{z_{\alpha}^{1} = \cdots = z_{\alpha}^{m} = 0\}\). Here \(R_{k}^{r}, R_{k}^{p} \in \mathbb{T}_{S}^{1}\). We also consider coordinate charts \(\{V_{\alpha} \times \mathbb{C}, \{z_{\alpha}, t\}\}\) on \(X \times \mathbb{C}\) so that \(S \times \{0\} = \{z_{\alpha}^{1} = \cdots = z_{\alpha}^{m} = 0\}\).

Consider the blow up \(\pi : \tilde{X} := B_{S_{0}}(X \times \mathbb{C}) \to X \times \mathbb{C}\) with the exceptional divisor \(E = \mathbb{P}(N_{S} \oplus \mathbb{C})\). \(E\) is the projective compactification of the normal bundle \(N_{S} \to \mathcal{S}\), and \(S_{0}\) sits inside \(\mathcal{S} \subset E \subset \tilde{X}_{0} \subset \tilde{X}\) as the zero section of \(N_{S} \to \mathcal{S}\). The subset \(\pi^{-1}(V_{\alpha} \times \mathbb{C}) \subset \tilde{X}\) is defined as a subvariety of \(V_{\alpha} \subset \mathbb{C} \times \mathbb{P}^{m+1}\):

\[
\{(z_{\alpha}^{1}, z_{\alpha}^{m}, t, [Z_{\alpha}^{n}, T]); (z_{\alpha}^{1}, z_{\alpha}^{m}) \in V_{\alpha}, t \in \mathbb{C}, z_{\alpha}^{n}Z_{\alpha}^{n} = z_{\alpha}^{m}Z_{\alpha}^{m} = 0, z_{\alpha}^{r} \cdot T - t \cdot Z_{\alpha}^{r} = 0, r = 1, \ldots, m, p = m + 1, \ldots, n\}
\]

where \([Z_{\alpha}^{n}, T]\) are homogenous coordinates on \(\mathbb{P}^{m+1}\). Near \(S_{0}, T \neq 0\), and so we can define new coordinate charts \(\{w_{\alpha}, t\}\) such that the map \(\pi\) is given by:

\[
\begin{align*}
  z_{\alpha}^{1} &= tw_{\alpha}^{1}, \ldots, z_{\alpha}^{m} = tw_{\alpha}^{m}, & z_{\alpha}^{m+1} = w_{\alpha}^{m+1}, \ldots, z_{\alpha}^{n} = w_{\alpha}^{n}; & t &= t.
\end{align*}
\]

Without loss of generality we can assume \(V_{\alpha} = \{z_{\alpha}; |z_{\alpha}| < \epsilon\}\) for sufficiently small \(\epsilon > 0\). Then if we denote the polydisc on the total space:

\[
\mathcal{U}_{\alpha} = \{(t, w_{\alpha}); |t| < 1, |w_{\alpha}| < \epsilon\},
\]

then \(\pi(\mathcal{U}_{\alpha}) \subset V_{\alpha} \times \mathbb{C}\), and when \(t \neq 0, \pi(\mathcal{U}_{\alpha}) \cap \tilde{X}_{0} \cong \{z_{\alpha}; |z_{\alpha}| < \epsilon, |z_{\alpha}''| < \epsilon\}\). See Figure 1 for a schematic illustration of deformation to the normal cone using the graph construction. See also [13, Remark 5.1.1, Section 5.1]. Denote \(\tilde{S}\) the strict transform of \(S \times \mathbb{C}\) on \(\tilde{X}\). Then the collection of open sets \(\{\mathcal{U}_{\alpha}\}\) is a covering of \(\tilde{S}\) inside the total space \(\tilde{X}\) and on \(\mathcal{U}_{\alpha}\), the ideal sheaf \(\mathcal{I}_{S}\) is generated by \(w_{\alpha}^{1}, \ldots, w_{\alpha}^{m}\). Denote \(\mathcal{U} = \bigcup_{\alpha} \mathcal{U}_{\alpha}\). We can find a small neighborhood \(\mathcal{W}\) of \(\tilde{S} \subset \tilde{X}\) such that \(\mathcal{W} \subset \mathcal{U}\). Denote \(w'_{\alpha} = (w'_{\alpha}^{1}, \ldots, w'_{\alpha}^{m}), w''_{\alpha} = (w''_{\alpha}^{m+1}, \ldots, w''_{\alpha}^{n})\) and define

\[
\begin{align*}
  \tilde{R}_{k}(t; w'_{\alpha}, w''_{\alpha}) &= t^{-k}R(tw'_{\alpha}, w''_{\alpha}), & \tilde{R}_{k}^{p}(t; w'_{\alpha}, w''_{\alpha}) &= t^{-k}R(tw'_{\alpha}, w''_{\alpha}).
\end{align*}
\]

Then \(\tilde{R}_{k} \in \mathcal{I}_{\mathcal{S}}, \tilde{R}_{k}^{p} \in \mathcal{I}_{\mathcal{S}}\). Note that \(\{\mathcal{U}_{\alpha} \cap \mathcal{W}_{t}, w_{\alpha}\}\) gives coordinate chart covering of \(\mathcal{W}_{t} := \pi^{-1}(\{t\}) \cap \mathcal{W}\). The transition function on \(\mathcal{U}_{\alpha} \cap \mathcal{W}_{t} \cap (\mathcal{U}_{\beta} \cap \mathcal{W}_{t})\) is given by:

\[
\begin{align*}
  w'_{\beta} &= \sum_{a=1}^{m}(a_{\beta a})_{r}w'_{\alpha}^{a} + t^{k-1}R_{k}', & \text{for } r = 1, \ldots, m, \\
  w''_{\beta} &= \phi_{\beta a}^{p}(w''_{\alpha}^{a}) + t^{k}R_{k}^{p}, & \text{for } p = m + 1, \ldots, n.
\end{align*}
\] (8)
Now we choose a partition of unity \( \{ \rho_\alpha, \tilde{\rho} \} \) subordinate to the covering \( \{ U_\alpha, \tilde{X} \backslash \mathcal{W} \} \). In particular, \( \text{Supp}(\rho_\alpha) \subset U_\alpha \), \( \text{Supp}(\tilde{\rho}) \cap \mathcal{W} = \emptyset \). As in Appendix 6.3, define the differentiable vector field in the small neighborhood \( \mathcal{W} \subset \tilde{X} \):

\[
V = \sum_{\alpha} \rho_\alpha \left( \frac{\partial}{\partial t} \right)_\alpha = \sum_{i=1}^{n} \left( \sum_{\alpha} \rho_\alpha \frac{\partial f_{\beta\alpha}^i(w_{\alpha}, t)}{\partial t} \right) \frac{\partial}{\partial w^\beta} + \left( \frac{\partial}{\partial t} \right)_\beta
\]

Then we can calculate:

\[
\Phi(t) = \sigma(t)^* \mathcal{J} - \mathcal{J}.
\]

Let \( \sigma(t) \) be the flow generated by \( V \) which exists when \( |t| \leq \delta \) for sufficiently small \( \delta \). Note that the vector field \( V \) is tangent to \( \mathcal{S} \) so that \( \sigma(t) \) preserves \( \mathcal{S} \). Denote \( \mathcal{J} \) the complex structure on the total space \( \tilde{X} \) of blow up. Denote

\[
\Phi(t) = \sigma(t)^* \mathcal{J} - \mathcal{J}.
\]

Then we can calculate:

\[
\Phi(t) = \frac{d}{dt} (\sigma(t)^* \mathcal{J}) = \mathcal{L}_V \mathcal{J} = \partial V
\]

Assume \( \tilde{\omega}_0 \) is a smooth Kähler metric on the open set \( \mathcal{W} \). Because both \( \tilde{R}_k^M, \tilde{R}_k^v \in T_{S \times X} \), we get:

\[
|\Phi|_{\tilde{\omega}_0} \leq C t^{\max(0,k-2)} |w'|^k.
\]

So we can integrate to get:

\[
|\Phi(t)|_{\tilde{\omega}_0} = |\sigma(t)^* \mathcal{J} - \mathcal{J}|_{\tilde{\omega}_0} \leq C t^{k-1} |w'|^k.
\]

(9)

When \( 0 < |t| < t_1 \) for \( t_1 \) sufficiently small, we get a map \( \sigma(t) : \mathcal{W} \cap \tilde{X}_0 \to U \cap \tilde{X}_t \) which gives a diffeomorphism to its image. By construction, \( \mathcal{W} \cap \tilde{X}_0 \) is a small neighborhood of \( S_0 \) and \( U \cap \tilde{X}_t \) is a small neighborhood of \( S \subset X = \tilde{X}_t \). (8)

2. \((k-1)\)-comfortably embedded In this case, we can improve the order of some components. This will also be reflected in later discussions. By Theorem 6.5, we can choose the coordinate charts such that the following holds:

\[
\left\{ \begin{array}{l}
\rho_\alpha = \sum_{i=1}^{m} (a_{\beta\alpha})_i (z_{\alpha}^r) z_{\alpha}^r + \tilde{R}_{k+1}^r, \quad \text{for } r = 1, \ldots, m, \\
\rho_\alpha = \delta_{\beta\alpha} (z_{\alpha}^p) + \tilde{R}_v^p, \quad \text{for } p = m + 1, \ldots, n.
\end{array} \right.
\]

where \( \tilde{R}_{k+1}^r \in T_{S \times X} \), \( \tilde{R}_v^p \in T_{S \times X} \). Similarly as before, denote \( \tilde{R}_{k+1}(t; w_{\alpha}^r, w_{\alpha}^p) = t^{-(k+1)} \tilde{R}_{k+1}(t w_{\alpha}^r, w_{\alpha}^p) \) and \( \tilde{R}_v(t; w_{\alpha}^r, w_{\alpha}^p) = t^{k} \tilde{R}_v(t w_{\alpha}^r, w_{\alpha}^p) \). Then \( \tilde{R}_{k+1} \in T_{S \times X} \) and \( \tilde{R}_v \in T_{S \times X} \). On the total space of the deformation to the normal cone, we have

\[
\left\{ \begin{array}{l}
w_{\alpha}^r = \sum_{i=1}^{m} (a_{\beta\alpha})_i (w_{\alpha}^r) w_{\alpha}^r + t^{k} \tilde{R}_{k+1}^r, \quad \text{for } r = 1, \ldots, m, \\
w_{\alpha}^p = \delta_{\beta\alpha} (w_{\alpha}^p) + t^{k} \tilde{R}_v^p, \quad \text{for } p = m + 1, \ldots, n.
\end{array} \right.
\]

Similarly as before the differentiable vector field \( V \) (see Appendix 6.3) becomes

\[
V = \sum_{i=1}^{n} \left( \sum_{\alpha} \rho_\alpha \frac{\partial f_{\beta\alpha}^i(w_{\alpha}, t)}{\partial t} \right) \frac{\partial}{\partial w^\beta} + \left( \frac{\partial}{\partial t} \right)_\beta
\]

\[
= \sum_{i=1}^{m} \sum_{\alpha} \rho_\alpha [\partial_i (t^{k} \tilde{R}_{k+1}^r)] \otimes \frac{\partial}{\partial w^\beta} + \sum_{p=m+1}^{n} \sum_{\alpha} \rho_\alpha [\partial_i (t^{k} \tilde{R}_v^p)] \otimes \frac{\partial}{\partial w^\beta} + \left( \frac{\partial}{\partial t} \right)_\beta.
\]

(12)
Use the same notations $\sigma(t), J, \Phi(t)$ and $\hat{\Phi}(t)$ as before. We have:

$$
\Phi(t) = \frac{d}{dt}(\sigma(t)^* J) = \mathcal{L}_V J = \partial V
$$

$$
= \sum_{r=1}^m \sum_{\alpha} [\theta_r(t^k \hat{R}^r_{k+1})] (\partial \rho_\alpha) \otimes \frac{\partial}{\partial w^\alpha} + \sum_{p=m+1}^n \sum_{\alpha} [\theta_p(t^k \hat{R}^p_k)] (\partial \rho_\alpha) \otimes \frac{\partial}{\partial w^\alpha}.
$$

We assume the index $v \in \{1, \ldots, m, 1, \ldots, \pi\}, h \in \{m+1, \ldots, n, 1, \ldots, \pi\}$. We decompose $\Phi$ into four types of components:

$$
\Phi = \Phi^k_w + \Phi^k_v + \Phi^k_h + \Phi^k_h := \phi^k_w dw^w \otimes \partial_{w^w} + \phi^k_v dw^v \otimes \partial_{w^v} + \phi^k_h dw^h \otimes \partial_{w^h}.
$$

Again we assume $\tilde{\omega}$ is a smooth Kähler metric on $\mathcal{W}$. Since $\hat{R}^k_{k+1} \in I_{\mathcal{S}}^{k+1}, \hat{R}^k \in I_{\mathcal{S}}^k$, it's easy to see that:

$$
|\dot{\phi}^h_k| \leq Ct^{k-1}|w'|^{k+1}, |\dot{\phi}^v_k| \leq Ct^{k-1}|w'|^{k+1}, |\dot{\phi}^h_k| \leq Ct^{k-1}|w'|^k, |\phi^h_k| \leq Ct^{k-1}|w'|^k.
$$

Integrating these, we get:

$$
|\Phi^k_w|_{\tilde{\omega}} \leq Ct^k|w'|^{k+1}, |\Phi^k_v|_{\tilde{\omega}} \leq Ct^k|w'|^{k+1}, |\Phi^k_h|_{\tilde{\omega}} \leq Ct^k|w'|^k, |\Phi^k_h|_{\tilde{\omega}} \leq Ct^k|w'|^k.
$$

Again when $|t|$ is sufficiently small, we get the estimates, which improve the estimates in (9) for the horizontal-to-vertical and vertical-to-horizontal terms.

### 2.2 Deformation weight and Comfortably embeddedness

We will use the notations in the introduction and Appendix 6.3. For a differentiable family of complex manifolds, assume that the transition functions are given by

$$
z^i_a = f^i_{\alpha\beta}(z_{\beta}, t), t|_{u_\alpha} = t|_{u_\beta} \text{ on } U_{\alpha} \cap U_{\beta}.
$$

Assume that the functions $f^i_{\alpha\beta}(z_{\beta}, t) - f^i_{\alpha\beta}(z_{\beta}, 0)$ vanish up to order $(k-1)$ at $t = 0$:

$$
\left. \frac{\partial^l f^i_{\alpha\beta}(z_{\beta}, t)}{\partial t^l} \right|_{t=0} = 0 \quad \text{for } 1 \leq l \leq k.
$$

Using this and the cocycle condition of $\{f_{\alpha\beta}\}$, we deduce that:

$$
\left. \frac{\partial^k f^i_{\alpha\beta}(z_{\beta}, t)}{\partial t^k} \right|_{t=0} = \frac{\partial^k f^i_{\alpha\beta}(z_{\beta}, t)}{\partial t^k} + O(t) + \frac{\partial^k f^i_{\alpha\beta}(z_{\beta}, t)}{\partial t^k} = \frac{\partial^k f^i_{\alpha\beta}(z_{\beta}, t)}{\partial t^k}.
$$

Denoting $\dot{U}_{\alpha} = U_{\alpha} \cap X_0$, we define:

$$
\dot{\theta}^{(k)}_{\alpha\beta} = \sum_{i=1}^n \frac{\partial^k f^i_{\alpha\beta}(z_{\beta}, t)}{\partial t^k} \left. \left|_{t=0} \right. \right. \in H^{0}(\dot{U}_{\alpha} \cap \dot{U}_{\beta}, \Theta_{X_0})
$$

Then $\dot{\theta}^{(k)}$ satisfies the cocycle condition:

$$
\dot{\theta}^{(k)}_{\alpha\gamma} = \dot{\theta}^{(k)}_{\alpha\gamma} - \dot{\theta}^{(k)}_{\alpha\beta}.
$$

We will call $\dot{\theta}^{(k)} = \{\dot{\theta}^{(k)}_{\alpha\beta}\}$ the reduced Kodaira-Spencer cocycle. From now on, we use the same $V_\alpha, U_\alpha$ and $U$ as those in the above section, and denote $\dot{U}_\alpha = U_\alpha \cap X_0$. Then $\{\dot{U}_\alpha\}$ is an open covering.
We can substitute the transition function in (14) into (13) above to get:

\[
\begin{align*}
\left\{
\begin{array}{ll}
w_{\beta}^r &= \sum_{i=1}^{m} (a_{\beta})_i (w_{\alpha}^r) w_{\alpha}^s + t^k \tilde{R}_{k+1}^r, & \text{for } r = 1, \ldots, m, \\
w_{\beta}^p &= \phi_{\beta}^p (w_{\alpha}^r) + t^k \tilde{R}_k^p, & \text{for } p = m + 1, \ldots, n.
\end{array}
\right.
\]
\]

We can substitute the transition function in (14) into (13) above to get:

\[
\theta^{(k)}_{\beta \alpha} = \sum_{i=1}^{n} \left. \frac{\partial f^i_{\beta \alpha}}{\partial t^k} (w_{\alpha}, t) \right|_{t=0} \frac{\partial}{\partial w_{\beta}^s} = \sum_{r=1}^{m} \tilde{R}_{k+1}^r (0; w_{\alpha}) \frac{\partial}{\partial w_{\beta}^r} + \sum_{p=m+1}^{n} \tilde{R}_k^p (0; w_{\alpha}) \frac{\partial}{\partial w_{\beta}^p}.
\]

Note that

\[
V_\alpha \cap S = \{ z_{\alpha}^s = 0, |z_{\alpha}^p| < \epsilon; 1 \leq r \leq m, m + 1 \leq p \leq n \} \cong U_\alpha \cap S_0.
\]

Then we observe that

\[
\theta^{(k)}_{\beta \alpha} \in H^0(\hat{U}_\alpha \cap \hat{U}_\beta, \Theta_{U_\alpha \cap S_0} \otimes \mathcal{I}^k_{S_0})
\]

can be viewed as representing a cocycle in the cohomology group \( H^1(U_\alpha \cap U_\beta, \Theta_{U_\alpha \cap S_0} \otimes \mathcal{I}^k_{S_0}) \). When we restrict to \( S_0 \), we get:

\[
\tilde{\theta}^{(k)}_{\beta \alpha} := \theta^{(k)}_{\beta \alpha} \mid_{S_0} = \sum_{r=1}^{m} \tilde{R}_{k+1}^r (0; 0, w_{\alpha}^s) \frac{\partial}{\partial w_{\beta}^r} + \sum_{p=m+1}^{n} \tilde{R}_k^p (0; 0, w_{\alpha}^r) \frac{\partial}{\partial w_{\beta}^p}.
\]

By Theorem 6.1 we see that

\[
g^{(k)}_{\beta \alpha} = \sum_{p=m+1}^{n} \tilde{R}_k^p (0; 0, w_{\alpha}^r) \otimes \frac{\partial}{\partial w_{\beta}^p} \in H^0(\hat{U}_\alpha \cap \hat{U}_\beta \cap S_0, \Theta_{S_0} \otimes \mathcal{I}^k_{S_0} / \mathcal{I}^{k+1}_{S_0})
\]

is the obstruction to \( k \)-splitting. Furthermore, if this \( g^{(k)}_{\beta \alpha} \) represents a vanishing cohomology class in \( H^1(S_0, \Theta_{S_0} \otimes \mathcal{I}^k_{S_0} / \mathcal{I}^{k+1}_{S_0}) \), then by Theorem 6.6 the transition functions in (14) can be improved to

\[
\begin{align*}
\left\{
\begin{array}{ll}
w_{\beta}^r &= \sum_{i=1}^{m} (a_{\beta})_i (w_{\alpha}^r) w_{\alpha}^s + t^k \tilde{R}_{k+1}^r, & \text{for } r = 1, \ldots, m, \\
w_{\beta}^p &= \phi_{\beta}^p (w_{\alpha}^r) + t^k \tilde{R}_k^p, & \text{for } p = m + 1, \ldots, n.
\end{array}
\right.
\]
\]

Substituting this into (13), \( \theta^{(k)} \) now becomes:

\[
\theta^{(k)}_{\beta \alpha} = \sum_{i=1}^{n} \left. \frac{\partial f^i_{\beta \alpha}}{\partial t^k} (w_{\alpha}, t) \right|_{t=0} \frac{\partial}{\partial w_{\beta}^s} = \sum_{r=1}^{m} \tilde{R}_{k+1}^r (0; w_{\alpha}) \frac{\partial}{\partial w_{\beta}^r}.
\]

When we restrict to \( S_0 \), we get:

\[
h^{(k)}_{\beta \alpha} = \theta^{(k)}_{\beta \alpha} \mid_{S_0} = \sum_{r=1}^{m} \tilde{R}_{k+1}^r (0; 0, w_{\alpha}^s) \frac{\partial}{\partial w_{\beta}^r} \in H^0(\hat{U}_\alpha \cap \hat{U}_\beta \cap S_0, N_{S_0} \otimes \mathcal{I}^{k+1}_{S_0} / \mathcal{I}^{k+2}_{S_0}).
\]

Comparing with (68), we see that \( \{ h^{(k)}_{\beta \alpha} \} \) is nothing but the obstruction to \( k \)-comfortably embedding.

Before we prove Theorem 1.1, we recall the construction in the introduction and make an important remark.

**Remark 2.1.** Denote \( \hat{X} = Bl_{D \times \{0\}}(X \times \mathbb{C}) \). Then \( \hat{X}_0 = X \cup E \) where \( E = \mathbb{P}(N_D \oplus \mathbb{C}) \). We assume \( D \) is an ample divisor and denote by \( L \) the holomorphic line bundle generated by \( D \), it’s easy to see that the line bundle \( L = \pi^* L - E \) is semi-ample and its linear system \( |mL| \) contracts the component \( X \) of the central fibre \( \hat{X}_0 \). In this way we get a degeneration of \( X \) to a singular variety \( X_0 \) which is obtained from \( E \) by contracting the infinity section \( D_{\infty} \). \( \hat{X}_0 \) is very close to being the projective cone \( \mathbb{C}(D, L) \). One delicate point here is that \( \hat{X}_0 \) may not be normal. For example, let \( X \) be any Riemann surface of genus \( \geq 1 \). \( D = \{ p \} \) is any point. Then \( D \) is ample (but not very ample). In this special
case, the central fibre $X_0$ is a rational curve with cusp singularities. H-J. Hein and C. Xu pointed to me the right condition ensuring the normality should be that:

$$\phi_m : H^0(X, mL) \to H^0(D, mL)$$  \hfill (20)

is surjective for any $m \geq 0$. For example, in the above example, the map $\phi_1 = 0 : H^0(\mathbb{P}^1, \{p\}) \to H^1(\mathbb{P}^1, \{p\}) = \mathbb{C}$ is not surjective. On the other hand, from the exact sequence:

$$0 \to H^0(X, (m-1)L_D) \to H^0(X, mL_D) \to H^0(D, mL|_D) \to H^1(X, (m-1)L) \to \cdots,$$

we see that (20) is surjective if $H^1(X, (m-1)L) = 0$ for all $m \geq 0$. In particular, this is satisfied in our setting where $X$ is Fano by the Nakano-Kodaira vanishing theorem.

**Proof of Theorem 1.1.** Assume $D$ is $(k-1)$-comfortably embedded but not $k$-comfortably embedded. By the above calculations, we get a cocycle $\theta^{(k)} = \{\theta^{(k)}_\alpha\} \in H^1(\mathcal{U} \cap \tilde{X}_0, \Theta_{\mathcal{U} \cap \tilde{X}_0} \otimes \mathcal{O}(-kD))$ with $\theta^{(k)}_\alpha \in H^0(\tilde{U}_\alpha \cap \tilde{U}_\beta, \Theta_{\tilde{U}_\alpha \cap \tilde{U}_\beta} \otimes \mathcal{I}_D^{k}) \subset H^0(\tilde{U}_\alpha \cap \tilde{U}_\beta, \Theta_{\tilde{U}_\alpha \cap \tilde{U}_\beta})$. By shrinking $\mathcal{U}$, there is no loss of generality in assuming that:

$L_\epsilon := \mathcal{U} \cap \tilde{X}_0 \subset E := \mathbb{P}(N_D \oplus \mathbb{C}) \subset \tilde{X}_0$

is a neighborhood of the zero section of $N_D = L \to D$. So we have:

$$\theta^{(k)}_\epsilon \in H^1(L_\epsilon, \Theta_{L_\epsilon}(-kD)).$$

We will also denote by $\theta_j^{(k)}$ the image of $\theta^{(k)}_\epsilon$ under the natural maps:

$$H^1(L_\epsilon, \Theta_{L_\epsilon}(-kD)) \to H^1(L_\epsilon, \Theta_{L_\epsilon}(-jD)),$$

for $0 \leq j \leq k$.

Because we assume $L \to D$ is ample, we can assume $L_\epsilon$ is a strictly pseudo-concave domain in the total space of the line bundle $L$. We also denote $U_\epsilon = L_\epsilon \setminus D$, $U = L \setminus D$ and $K = L_\epsilon \setminus U_\epsilon = U \setminus U_\epsilon$. Then we have the exact sequence:

$$\cdots \to H^1_k(L_\epsilon, \Theta_{L_\epsilon}(-jD)) \to H^1(L_\epsilon, \Theta_{L_\epsilon}(-jD)) \to H^1(L_\epsilon, \Theta_{L_\epsilon}(-jD)) \to H^1_{k+1}(L_\epsilon, \Theta_{L_\epsilon}(-jD)) \to \cdots,$$

where $H^1_{k}$ is the cohomology with compact support $K$. By [2, Proposition 25] (or [4, Theorem 3.1]) we know that $H^1_k(L_\epsilon, \Theta_{L_\epsilon}(-jD))$ vanishes if $i \leq n-1$, because $\Theta_{L_\epsilon}(-jD)$ has depth $n$ (locally free) and $K$ is strongly pseudoconvex. From the exact sequence (21), we get we get (see [2, Theorem 15]) (for $0 \leq j \leq k$)

$$H^1(L_\epsilon, \Theta_{L_\epsilon}(-jD)) \overset{F^{AG}_{j}}{\to} H^1(L_\epsilon, \Theta_{L_\epsilon}(-jD))$$

is an isomorphism when $n \geq 3$, and

$$H^1(L_\epsilon, \Theta_{L_\epsilon}(-jD)) \to H^1(L_\epsilon, \Theta_{L_\epsilon}(-jD))$$

is injective when $n = 2$. From now on, we assume $n \geq 3$. Then $\theta_j^{(k)}$ also represents a cohomology class:

$$(F^{AG}_{j})^{-1} \theta_j^{(k)} \in H^1(L_\epsilon, \Theta_{L_\epsilon}(-jD)),$$

for each $0 \leq j \leq k$.

Now, as in the introduction, we assume that the deformation is induced from a map: $\mathbf{I}_{X^0} : \mathbb{C} \to T^1_{L}$. If $\mathbf{I}_{X^0}$ vanishes of order $j$ at $0 \in \mathbb{C}$, then we define the $j$-th order Kodaira-Spencer class as:

$$\mathbf{KS}^{(j)}_{X^0} = \frac{d^j}{dt^j} \mathbf{I}_{X^0} \bigg|_{t=0} \in T^1_{L}.$$ 

Note that by Schlessinger [25] (see Appendix 6.4) $T^1_{L} \subset H^1(U, \Theta_U)$. From the definition of $\mathbf{KS}^{(j)}_{X^0}$, we can get

$$\mathbf{KS}^{(j)}_{X^0} \bigg|_{U_\epsilon} = [\theta^{(j)}_{\epsilon}] \bigg|_{U_\epsilon}.$$ 

9
If \( j < k \), then by the calculation (13)-(18) above, \( \vartheta^{(j)} = 0 \) on \( U_\epsilon \). By Andreotti-Grauert, \( H^1(U, \Theta_U) \rightarrow H^1(U_\epsilon, \Theta_{U_\epsilon}) \) is injective. So \( KS^{(j)}_{X^0} = 0 \). So we get \( \text{ord}_0 L \geq k \).

On the other hand, denote

\[
\vartheta^{(k)} = \left( F^{AG} \right)^{-1}(\vartheta^{(k)}) |_{U} \in H^1(U, \Theta_U).
\]

Then we have:

\[
\vartheta^{(k)} = KS^{(k)}_{X^0} \in T^1_C \subset H^1(U, \Theta_U).
\]

Because \( KS^{(k)}_{X^0} \) comes from the deformation of \( L \), we know that the weight of \( KS^{(k)}_{X^0} \) is non-positive.

To finish the proof, we need to show that \( \pi_k(\vartheta^{(k)}) \neq 0 \in H^1(U, \Theta_U)(-k) \) and \( \pi_j(\vartheta^{(k)}) = 0 \in H^1(U, \Theta_U)(-j) \) for \( 0 \leq j < k \). To prove these, we need to study \( H^1(U, \Theta_U)(-k) \) and compare it with \( H^1(L, \Theta_L)(-k) \).

On the total space \( L \), we have the exact sequence:

\[
0 \rightarrow \pi_L^*L \rightarrow \Theta_L \rightarrow \pi_L^*D \rightarrow 0.
\]

By restricting this exact sequence to \( U = L \setminus D \), we have a similar exact sequence on \( U \). So we get commutative diagram of long exact sequence:

\[
\cdots \rightarrow H^1(L, \pi_L^*L) \rightarrow H^1(L, \Theta_L) \rightarrow H^1(L, \pi_L^*D) \rightarrow \cdots
\]

By projection formula we have (note the range of indices)

\[
H^p(L, \pi_L^*L) = \bigoplus_{j=0}^{+\infty} H^p(D, L^{-j}), \quad H^p(L, \pi_L^*D) = \bigoplus_{j=0}^{+\infty} H^p(D, \Theta_D \otimes L^{-j}).
\]

\[
H^p(U, \pi_U^*L) = \bigoplus_{j=-\infty}^{+\infty} H^p(D, L^{-j}), \quad H^p(U, \pi_U^*D) = \bigoplus_{j=-\infty}^{+\infty} H^p(D, \Theta_D \otimes L^{-j}).
\]

For \( j \geq 0 \), we can extract the weight \((-j)\)-part to get the exact sequences:

\[
\cdots \rightarrow H^1(D, L^{-j}) \rightarrow H^1(L, \Theta_L)(-j) \rightarrow H^1(D, \Theta_D \otimes L^{-j}) \rightarrow \cdots
\]

\[
\cdots \rightarrow H^1(D, L^{-j}) \rightarrow H^1(U, \Theta_U)(-j) \rightarrow H^1(D, \Theta_D \otimes L^{-j}) \rightarrow \cdots
\]

Using this we construct the following commutative diagram for integers \( j \leq k \):

\[
\begin{array}{ccc}
H^1(L, \Theta_L)(-jD) & \xrightarrow{F^{AG}_{|L}} & H^1(L_\epsilon, \Theta_{L_\epsilon})(-jD) \\
\downarrow \pi_j & & \downarrow \pi_j \\
H^1(D, L^{-j}) & \xrightarrow{f_j^U} & H^1(U, \Theta_U)(-j) & \xrightarrow{f_j^U} & H^1(D, \Theta_D \otimes L^{-j}) \\
\downarrow Res_D^j & & \downarrow Res_D^j & & \downarrow Res_D^j \\
H^1(D, L^{-j}) & \xrightarrow{f_j^L} & H^1(L, \Theta_L)(-j) & \xrightarrow{f_j^L} & H^1(D, \Theta_D \otimes L^{-j})
\end{array}
\]

From the above calculation, we have \( \vartheta^{(k)} = Res_U^j \circ (F^{AG})^{-1}(\vartheta^{(k)}) \in H^1(U, \Theta_U) \). Then calculations at the beginning of this section show that:

* \( f_k^U \circ \pi_k(\vartheta^{(k)}) = f_k^L \circ \pi_k \circ (F^{AG})^{-1}(\vartheta^{(k)}) = Res_D^j(\vartheta^{(k)}) = \vartheta^{(k)} \).
• If $f^U_j \circ \pi_j(\theta^{(k)}) = 0$, then $\pi_k(\theta^{(k)}) = \bar{f}^U_j(\mathfrak{h}^{(k)})$.

Because we assume that $D \hookrightarrow X$ is not $k$-comfortably embedded, either $\{g^{(k)}_{\alpha a}\}$ is not zero or $\{\mathfrak{h}^{(k)}_{\alpha a}\}$ is not zero. Note that we are assuming that $\dim D \geq 2$ and $k \geq 1$. So we have $H^1(D, L^{-k}) = 0$ by Kodaira-Nakano vanishing, so that $\mathfrak{h}^{(k)}$ is always zero. So we get $g^{(k)}$ is not zero, hence $\pi_k(\theta^{(k)})$ is nonzero in $H^1(U, \Theta_{D/L})(-k)$.

On the other hand, since the representing cocycle $\theta^{(k)}$ vanishes at order $k$ along the divisor $D$, $\text{Res}^L_D(\theta^{(k)}) = 0$. Using the above commutative diagram and similar argument, we see that:

$$f^U_j \circ \pi_j(\theta^{(k)}) = f^L_j \circ \pi_j \circ (F^AC_j)^{-1}(\theta^{(k)}) = \text{Res}^L_D(\theta^{(k)}) = 0.$$

By discussion above, we can write:

$$\pi_j(\theta^{(k)}) = \bar{f}^U_j(\mathfrak{h}^{(j)}).$$

Because $D$ is $k$-comfortably embedded, we have $\mathfrak{h}^{(j)} = 0$ for $0 \leq j < k$. So $\pi_j(\theta^{(k)}) = 0$. So we have proved that the cohomology class $\theta^{(k)}$ gives rise to $\mathfrak{g}^{(k)} = \text{KS}_{X^k}^{(k)} \neq 0$ which is of weight $k$.

\[
\begin{align*}
\text{3 Applications to AC Kähler metrics} \\
\text{3.1 Rotationally symmetric Kähler metric on the cone}
\end{align*}
\]

We consider the Kähler metric on $C(D, L)$ given by the special Calabi ansatz $\omega_0 = \sqrt{-1} \ddbar h^k$. Then $\omega_0$ is a Riemannian cone metric on $C(D, L)$:

$$g = dr^2 + r^2 g_Y,$$

where $Y$ is the circle bundle over $D$. To see this, we consider the coordinate chart on $\mathbb{P}(L^{-1} \oplus \mathbb{C})$.

Away from the infinity section $D_{\infty}$, we have coordinate chart given by $(z, [z_\alpha e_\alpha], 1) = (z, \alpha, \zeta_\alpha]$]. Let $h = |\mathcal{e}_\alpha|^2 \mathfrak{h}(z) = a(z)|\zeta_\alpha|^2 = (a_+ + a/z)|\zeta_\alpha|^2$. For simplicity, we will denote $\zeta = \zeta_\alpha$, $\xi = \xi_\alpha$, $a = a_+ - a_-$. Then we can calculate:

$$\omega_0 = \sqrt{-1} \ddbar h^k = \delta h^k \omega_D + \delta^2 h^k \frac{\nabla \zeta \wedge \overline{\nabla \zeta}}{|\zeta|^2} = \delta h^k \omega_D + \delta^2 h^k \frac{\nabla \xi \wedge \overline{\nabla \xi}}{|\xi|^2}, \quad (26)$$

where $\omega_D = \sqrt{-1} \ddbar \log h$ is a smooth Kähler metric on $D$, and we have used vertical and horizontal frames:

$$dz^\alpha, \nabla \zeta = d\zeta + \zeta a^{-1} \partial a \overset{\text{dual}}{\longrightarrow} \nabla_z = \frac{\partial}{\partial z^\alpha} - a^{-1} \frac{\partial a}{\partial \zeta} \frac{\partial}{\partial \zeta}.$$

Under the $\{z, \xi\}$ coordinate, we have similarly:

$$dz^\alpha, \nabla \xi = d\xi - \xi a^{-1} \partial a \overset{\text{dual}}{\longrightarrow} \nabla_z = \frac{\partial}{\partial z^\alpha} + a^{-1} \frac{\partial a}{\partial \xi} \frac{\partial}{\partial \xi}.$$

To write the metric into a metric cone, we write $\zeta = \rho e^{i\theta}$, then

$$\nabla \zeta = d\zeta + \zeta a^{-1} \partial a = e^{i\theta}(d\rho + i\rho \partial \theta + \bar{a} \partial a) = e^{i\theta}(d\rho + i\rho (d\theta - ia^{-1} \partial a)).$$

So if we let $r = h^{5/2} = (a(z)|\zeta|^2)^{5/2}$ and $\nabla \theta = d\theta - ia^{-1} \partial a$, then it’s easy to verify that:

$$g_{w_0} = dr^2 + r^2 (\delta g_{\omega_D} + \delta^2 \nabla \theta \otimes \overline{\nabla \theta}).$$

Note that $\nabla \theta$ is nothing but the connection form on the unit $S^1$-bundle in $L$. Now we compare the norm of tensors on $U = L \backslash D$ with respect to two metrics $\omega_0$ and $\tilde{\omega}_0$, where $\tilde{\omega}_0$ is any smooth Kähler metric on a neighborhood of $D$ in $L$. For example, we can take

$$\tilde{\omega}_0 = \pi_k^\perp \omega_D + \epsilon \sqrt{-1} \ddbar (a_+(z)|\xi|^2)$$

for small $\epsilon > 0$. Suppose $\Phi$ is a tensor of type $(p = p_+ + p_-, q = q_+ + q_-)$, i.e.

$$\Phi \in (T^*_a X)^{\otimes p_+} \otimes (T^*_c X)^{\otimes p_-} \otimes (T_a X)^{\otimes q_+} \otimes (T_c X)^{\otimes q_-},$$

$$11$$
Then
\[
|\Phi|_{\omega_0} \sim |\xi|^{\delta p_k + (\delta + 1) p_v - \delta q_k - (\delta + 1) q_v}. \tag{27}
\]
In particular, we get:

**Lemma 3.1.** If $\Phi$ is tensor of type $(1,1)$, then
\[
|\Phi|_{\omega_0} \sim |\Phi|_{\tilde{\omega}_0} \sim |\xi|^\delta, \quad |\Phi|_{\omega_0} \sim |\Phi|_{\omega_0}^{-1}, \quad |\Phi|_{\omega_0} \sim |\Phi|_{\omega_0}, \quad |\Phi|_{\omega_0} = |\Phi|_{\tilde{\omega}_0}.
\]
So if $|\Phi|_{\omega_0} \sim |\xi|^\delta$, then we have
\[
|\Phi|_{\omega_0} \sim |\xi|^{\delta - 1}, \quad |\Phi|_{\omega_0} \sim |\xi|^{\delta + 1}, \quad |\Phi|_{\omega_0} \sim |\xi|^{\delta}. \tag{28}
\]
Next we compare the Christoffel symbols of the two metrics, which will be useful to convert the estimate with respect to $\omega_0$ to that with respect to $\tilde{\omega}_0$. See (32)-(33). To simplify the calculation, we can choose the coordinate and holomorphic frame such that
\[
g_{ij} = \omega_D (\partial_{z_{\alpha}}, \partial_{\bar{z}_{\alpha}})(0) = \delta_{ij}, \quad (\partial_{z_{\alpha}} g_{ij})'(0) = 0; \quad (\partial_{\bar{z}_{\alpha}} a)(0) = 0, \quad (\partial_{z_{\alpha}} \partial_{\bar{z}_{\alpha}} a)(0) = 0.
\]
Denote by the index 0 the coordinate corresponding to $\xi = \xi_0$, then we have the components of the metric tensor associated with $\omega_0$: \[
g_{i0} = \delta_{i0} \delta_{0j} = \delta_{ij}, \quad g_{00} = \delta^2 \delta_{ij} \delta_{0j} = \delta_{ij}, \quad g_{j0} = g_{j0} = 0.
\]
In other words,
\[
\nabla \partial_{\bar{z}_i} = -\frac{\delta}{\xi} d\xi \otimes \partial_{\bar{z}_i}, \quad \nabla \partial_{\bar{z}_k} = -\frac{\delta + 1}{\xi} d\xi \otimes \partial_{\bar{z}_i} - \frac{\delta}{\xi} d\xi \otimes \partial_{\bar{z}_i}.
\]
We conclude this section by recalling the Calabi-Yau cone metric in the case when $K_D^{-1} = \mu L_D = \mu N_D$ for $\mu > 0$ and $D$ has a Kähler-Einstein metric $\omega_D = \omega_D^{KE}$ such that $Ric(\omega_D^{KE}) = \mu \cdot \omega_D^{KE}$. In this case, Note that the Hermitian metric $h$ satisfies $\sqrt{-1} \partial \bar{\partial} \log h = \omega_D^{KE}$. To find the Calabi-Yau cone metric, it’s straightforward to calculate that:
\[
Ric(\omega_0) = -\sqrt{-1} \partial \bar{\partial} \log \omega_0^n = (-n\delta + \mu)\pi^L \omega_D^{KE},
\]
where $n = \dim D + 1$. So we get the exponent for the Calabi-Yau cone metric:
\[
- K_D = \mu N_D \implies \delta = \frac{\mu}{\dim D + 1}. \tag{31}
\]
3.2 Asymptotical rates of Tian-Yau’s Examples

Assume that $X$ is a Fano manifold of dimension $n \geq 3$ and $D$ is a smooth divisor such that $-K_X \sim \alpha D$ is a smooth divisor with $\mathbb{Q} \ni \alpha > 1$. By adjunction formula, we get $-K_D = -K_X|_D - |D| = (\alpha - 1)|D| = (1 - \alpha^{-1})K_X^{-1}$ is still ample, and so $D$ is also a Fano manifold. Assume that $D$ has a Kähler-Einstein metric. Tian-Yau [27] constructed an AC Calabi-Yau metric $\omega_{TY}$ on $X \setminus D$. The tangent cone at infinity is the conical Calabi-Yau metric on $C(D, N_D)$ discussed at the end of last section with

$$\delta = \frac{\alpha - 1}{n}.$$ 

By the work of Conlon-Hein [10] (Theorem 6.3 in the Appendix), to find the convergence rate of $\omega_{TY}$ to the $C(D, N_D)$ at infinity, we need to construct a diffeomorphism $F : C(D, N_D) \setminus B_R(\overline{0}) \to (X \setminus D) \setminus K$ such that

$$\|\nabla_{\omega_0}(F^*(\Omega) - \Omega_0)\|_{\omega_0} \leq C r^{-\lambda - j}.$$ 

We will use the diffeomorphism constructed in Section 2.1. Now assume $D$ is $(k - 1)$-comfortably embedded. If $\dim D \geq 2$, then by Remark 6.4, this is the same as $(k - 1)$-linearizable. By Theorem 6.5, there exist coordinate charts such that:

$$\begin{cases}
  z_b^1 = a_{\beta \alpha} (z_{\alpha}^0) z_{\alpha} + R_{k+1}^i, \\
  z_b^p = \phi_{\beta \alpha} (z_{\alpha}^0) + R_k^p, \\
  \text{for } p = 2, \ldots, n.
\end{cases}$$

The vector field $V$ in (12) becomes:

$$V = \sum_{\alpha} \rho_\alpha [\partial_i (t^k \tilde{R}_{k+1}^i)] \otimes \frac{\partial}{\partial w^\alpha} + \sum_{p=2}^n \sum_{\alpha} \rho_\alpha [\partial_i (t^k \tilde{R}_k^p)] \otimes \frac{\partial}{\partial w^\alpha} + \left( \frac{\partial}{\partial t} \right).$$

On the total space of $\tilde{X}$, the relative holomorphic form with a pole of order $\alpha$ along $D$ can be written locally as:

$$\Omega = f(t, w) \frac{d w_1^\alpha \wedge \cdots \wedge d w_{\alpha}^n}{(w_1^\alpha)^\alpha}$$

with $f(t, w)$ a locally defined nowhere vanishing holomorphic function. We can then calculate:

$$\frac{d}{dt} (\sigma(t)^* \Omega) = \mathcal{L}_V \Omega = div \Omega$$

$$= d \left( f(t, w)(w_1^\alpha)^{-\alpha} \sum_{\alpha} \rho_\alpha [\partial_i (t^k \tilde{R}_{k+1}^i)] \wedge d w_1^\alpha \wedge \cdots \wedge d w_n^\alpha \right)$$

$$+ \sum_{p=2}^n (-1)^{p-1} d \left( f(t, w)(w_1^\alpha)^{-\alpha} \sum_{\alpha} \rho_\alpha [\partial_i (t^k \tilde{R}_k^p)] \right) d w_1^\alpha \wedge \cdots \wedge d w_{p-1}^\alpha \wedge d w_{p+1}^\alpha \wedge \cdots.$$

Note that $\tilde{R}_{k+1}$ and $\tilde{R}_k^p$ are holomorphic functions. From this and (27) we see that

$$\| F^* \Omega - \Omega_0 \|_{\omega_0} = \| F^* \Omega - \Omega_0 \|_{\omega_0} \leq C t^k |w_1^\alpha|^{1+\delta(n-1)} \leq C t^k |w_1^\alpha|^{1-\alpha} |w_1^\alpha|^{n\delta+1}$$

$$= C t^k |w_1^\alpha|^{1-k/\delta}.$$ 

Here we have used the value $\delta = \frac{\alpha - 1}{n}$ in (31). Changing to the cone metric, we have $r = a(z)^{1/2} |\xi|^{-\delta} \sim |\xi|^{-\delta}$. By (29) and (30), we see that:

$$\| \nabla_{\omega_0} (F^* \Omega - \Omega_0) \|_{\omega_0} \leq C t^k \frac{|w_1^\alpha|^{1-k}}{|\xi|^{\delta+1}} = C t^k r^{-\delta-1} = C t^k r^{-\frac{n-k}{n-1}}.$$ 

So when $|t|$ is small, we get the required diffeomorphism in Proposition 1.2.

**Example 3.1.** Assume $D^{n-1} \subset \mathbb{P}^{N-1}$ is a smooth complete intersection:

$$D = \{ F_i(Z_1, \cdots, Z_N) = 0 \} \subset \mathbb{P}^{N-1}.$$
where $m = N - n$ and $F_i(Z_1, \cdots, Z_N)$ is a homogeneous polynomial of degree $d_i$. Denote the affine cone and projective cone over $D$ inside $\mathbb{P}^N$ by

$$C(D, H) = \bigcap_{i=1}^{m} \{F_i(z_1, \cdots, z_N) = 0\} \subset \mathbb{C}^N.$$ 

$$\overline{C}(D, H) = \bigcap_{i=1}^{m} \{F_i(Z_1, \cdots, Z_N)\} \subset \mathbb{P}^N.$$

Now assume $G_i(Z_0, Z_1, \cdots, Z_N)$ is a homogeneous polynomial such that $G_i(1, z_1, \cdots, z_N)$ a polynomial of degree $e_i$, for each $i = 1, \cdots, m$. We construct a degeneration:

$$\mathcal{X} = \bigcap_{i=1}^{m} \{F_i(Z_1, \cdots, Z_N) + (tZ_0)^{d_i - \deg G_i} G_i(tZ_0, Z_1, \cdots, Z_N) = 0\} \subset \mathbb{P}^N \times \mathbb{C}.$$

This degenerates the variety $\mathcal{X}_1 \subset \mathbb{P}^N$ to $\overline{C}(D, H)$. In fact, $\mathcal{X}$ is a degeneration of $\mathcal{X}_1$ generated by the transformation:

$$[Z_0, Z_1, \cdots, Z_N] \rightarrow [t^{-1}Z_0, Z_1, \cdots].$$

Away from $\{Z_0 = 0\}$, we have the deformation of $C(D, H)$:

$$\mathcal{X} = \bigcap_{i=1}^{m} \{F_i(z_1, \cdots, z_N) + t^{d_i - \deg G_i} G_i(t, z_1, \cdots, z_N) = 0\} \subset \mathbb{C}^N \times \mathbb{C}.$$

By adjunction formula, we know that $-K_{\mathcal{X}_1} = (N + 1 - \sum_{i=1}^{m} d_i)H$ and $-K_D = (N - \sum_{i=1}^{m} d_i)H$. Consider $D = D_1 = \mathcal{X}_1 \cap \{Z_0 = 0\} \subset \mathcal{X}_1$. Then if we assume $\sum_{i=1}^{m} d_i \leq N - 1$, we are in the above setting with $\alpha = N + 1 - \sum_{i=1}^{m} d_i \geq 2$.

By Appendix 6.4, $T_C$ can be calculated as a quotient ring. Furthermore if we assume the image of $\mathcal{G} = \{G_i(1, z_1, \cdots, z_N)\}$ in $T_C^1$ is nonzero, then the weight of deformation $w(X, D)$ is equal to the weight of $[\mathcal{G}]$. If $e_i \leq d_i - 2$, then we get $w = -\min_{i=1}^{m} \{d_i - e_i\}$ (see Example 6.1 in Appendix 6.4). Now, we assume $n \geq 3$. Then by Theorem 1.1, we know that the divisor $D$ is $(|w| - 1)$-comfortably embedded into $X$. So by the above calculation, we see that the asymptotic rate of holomorphic form is given by

$$\lambda = \frac{|w|}{\delta} = \frac{n|w|}{\alpha - 1}.$$

If furthermore $e_i \leq d_i - 2$, then

$$\lambda = \frac{|w|}{\delta} = \frac{n|w|}{\alpha - 1} = \frac{n \cdot \min_{i=1}^{m} \{d_i - e_i\}}{N - \sum_{i=1}^{m} d_i}.$$

In this way, we can give an algebraic interpretation of the corresponding calculations in [10].

1. [10, Example 1]. Smoothing of the cubic cone:

$$C = \left\{ z \in \mathbb{C}^4; \sum_{i=1}^{4} z_i^3 = 0 \right\} \sim M = \left\{ z \in \mathbb{C}^4; \sum_{i=1}^{4} z_i^3 = \sum_{i,j} a_{ij} z_i z_j + \sum_k a_k z_k + \epsilon \right\}.$$

where $a_{ij}$, $a_k$, $\epsilon$ are small (generic) constants. We have

$$T_C^1 = \mathbb{C}[z_1, \cdots, z_4] / (z_1^3, \cdots, z_4^3).$$

With the earlier notation, $G(Z_0, \cdots, Z_4) = \sum_{i,j} a_{ij} Z_i Z_j + \sum_k a_k Z_k Z_0 + \epsilon Z_0^2$ with

$$[\mathcal{G}] = \sum_{i,j} a_{ij} z_i z_j + \sum_k a_k z_k + \epsilon \in T_C^1(-1) + T_C^1(-2) + T_C^1(-3).$$

Note that we assume $a_{ij}$ are generic if it’s not zero. So we get

$$\lambda = \left\{ \begin{array}{ll}
\frac{3}{3} = 9, & \text{if } a_{ij} = a_k = 0 \\
\frac{3}{2} = 6, & \text{if } a_{ij} = 0, a_k \neq 0 \\
\frac{3}{1} = 3, & \text{if } a_{ij} \neq 0.
\end{array} \right.$$
2. \[10, \text{Example 2}\]. Smoothing of the complete intersection:

\[ C = \left\{ z \in \mathbb{C}^5; f_1 = \sum_{i=1}^{5} z_i^2 = 0, f_2 = \sum_{i=1}^{5} \lambda_i z_i^2 = 0 \right\} \sim M = \left\{ z \in \mathbb{C}^5; f_1(z) = f_2(z) = \epsilon \right\}. \]

Here \( \lambda_i \) are distinct complex numbers. We have:

\[ T_C^1 = \frac{C[z_1, \ldots, z_5]}{\text{Im} \begin{pmatrix} z_1 & \cdots & z_5 \\ \lambda_1 z_1 & \cdots & \lambda_5 z_5 \end{pmatrix}}. \]

Because the images of \( \mathcal{G} = (-\epsilon, -\epsilon) \) is not zero inside \( T_C^1 \), we have \( \lambda = \frac{3-2}{5-2-2} = 6 \).

3. \[10, \text{Example 3}\]. Smoothing of the ordinary double point:

\[ C = \left\{ z \in \mathbb{C}^{n+1}; \sum_{i=1}^{n+1} z_i^2 = 0 \right\} \sim M = \left\{ z \in \mathbb{C}^{n+1}; \sum_{i=1}^{n+1} z_i^2 = \sum_{i=1}^{n+1} a_i z_i + \epsilon \right\}. \]

\[ T_C^1 = \frac{C[z_1, \ldots, z_{n+1}]}{\{z_1, \ldots, z_{n+1}\}}. \]

\[ G(Z_0, \ldots, Z_{n+1}) = \sum_{i=1}^{n+1} a_i Z_i + \epsilon Z_0. \] So \([G(1, z_1, \ldots, z_n)] = \sum_{i=1}^{n+1} a_i z_i + \epsilon = |\epsilon| \] is of weight \(-2\). So we have \( \lambda = \frac{\text{deg} G}{n-2} = \frac{n-2}{n-1} \).

4 Analytic compactification

4.1 Sketch of the proof

As before, denote \( U = L \setminus D \). Denote the standard complex structure on \( U \) by \( J_0 \). Assume that we have a complex structure \( J \) on some neighborhood \( U_\epsilon \) of \( D \). Denote \( \Phi = J - J_0 \). We assume the index \( \nu \in \{1, 2\} \) associates to the fiber variable \( \xi = z_\nu \), \( h \in \{2, \ldots, n, \overline{2}, \ldots, \overline{n}\} \) associates to the base variables \( \{z_2^2, \ldots, z_n^2\} \). By abuse of notations, we decompose \( \Phi \) into four types of components:

\[ \Phi = \Phi_v^h + \Phi_v^h + \Phi_v^h + \Phi_v^h = \phi_v^h dz^v \otimes \partial_{z^h} + \phi_v^h dz^v \otimes \partial_{z^h} + \phi_v^h dz^v \otimes \partial_{z^h} + \phi_v^h dz^v \otimes \partial_{z^h}. \]

We assume \( \Phi \) satisfies \( |\nabla_\nu \Phi|_{\omega_0} \leq C|r|^{\alpha - \lambda - \frac{j}{2}} \sim |\xi|^\delta \). We first need to transform this estimate to the corresponding estimate with respect to \( \omega_0 \). For this, note that we know the basic tensors satisfy (29) and (30). So we can equivalently assume \( \Phi \) satisfies:

\[ |(\partial_{z^h}^\nu \partial_{z^h}^\nu \Phi)(dz^v) \otimes (dz^v)_{\otimes j} |_{\omega_0} \leq C|r|^{\alpha - \lambda - j} = C|\xi|^{\delta \lambda - j}. \] (32)

Recall the norm in Section 3.1:

\[ |dz^v|_{\omega_0} \leq C|\xi|^{\delta + 1}, |dz^h|_{\omega_0} \leq C|\xi|^{\delta} \Rightarrow |(dz^v) \otimes (dz^h)_{\otimes j}|_{\omega_0} \leq |\xi|^{\delta + 1 + j} \leq C. \]

Also we have:

\[ |dz^v \otimes \partial_{z^h}|_{\omega_0} \leq |\xi|, |dz^h \otimes \partial_{z^h}|_{\omega_0} \leq |\xi|^{-1}, |dz^v \otimes \partial_{z^h}|_{\omega_0} \leq C, |dz^h \otimes \partial_{z^h}|_{\omega_0} \leq C. \]

By these inequalities, it’s easy to see that:

\[ |\partial_{z^h}^\nu \partial_{z^h}^\nu \phi_v^h| \leq |\xi|^{\delta - 1 - j}, |\partial_{z^h}^\nu \partial_{z^h}^\nu \phi_v^h| \leq |\xi|^{\delta-j}, |\partial_{z^h}^\nu \partial_{z^h}^\nu \phi_v^h| \leq |\xi|^{\delta-j-1}, |\partial_{z^h}^\nu \partial_{z^h}^\nu \phi_v^h| \leq |\xi|^{\delta-j} \leq |\xi|^{\delta-j}. \] (33)

The \((0,1)\) vector under the new complex structure \( J \) is given by

\[ \frac{1}{2}(1 + \sqrt{-1} J) \frac{\partial}{\partial z^\nu} = \frac{\partial}{\partial z^\nu} + \frac{\sqrt{-1}}{2} \phi_v^h \frac{\partial}{\partial z^\nu} + \frac{\sqrt{-1}}{2} \phi_v^h \frac{\partial}{\partial z^\nu}. \]

Denote \( \eta = \lambda \delta \) and \( \rho = |\xi| = |z^1|. \) Then from (33), we can write:

\[ \left( \frac{\partial}{\partial z^\nu} \right) \sim \begin{pmatrix} O(\rho^{\eta})_{1 \times 1} & O(\rho^{\eta+1})_{1 \times (n-1)} \\ O(\rho^{\eta-1})_{(n-1) \times 1} & O(\rho^{\eta})_{(n-1) \times (n-1)} \end{pmatrix} \sim \left( \frac{\partial}{\partial z^\nu} \right). \] (34)
Lemma 4.1. When \( \rho \) is sufficiently small, the matrix \( \begin{pmatrix} \frac{\partial^2}{\partial z^2} + \frac{\sqrt{-1}}{2} \partial \bar{\partial} & \frac{\sqrt{-1}}{2} \partial \end{pmatrix} \) is invertible. We have:

\[
\begin{pmatrix} a^k_+ \end{pmatrix} := \left( \frac{\partial^2}{\partial z^2} + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \right)^{-1} \left( \frac{\sqrt{-1}}{2} \partial \right) \sim \begin{pmatrix} O(\rho^0)_{1 \times 1} & O(\rho^{q+1})_{1 \times (n-1)} \\ O(\rho^{q-1})_{(n-1) \times 1} & O(\rho^q)_{(n-1) \times (n-1)} \end{pmatrix}.
\]

Proof. First we can eliminate the lower left part:

\[
\begin{pmatrix} 1 & 0 \\ -1 + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \end{pmatrix}^{-1} \begin{pmatrix} I_{(n-1) \times (n-1)} \\ \frac{\sqrt{-1}}{2} \partial \bar{\partial} \end{pmatrix} \sim \begin{pmatrix} O(\rho^0)_{1 \times 1} & O(\rho^{q+1})_{1 \times (n-1)} \\ O(\rho^{q-1})_{(n-1) \times 1} & O(\rho^q)_{(n-1) \times (n-1)} \end{pmatrix}.
\]

This implies:

\[
\begin{pmatrix} 1 + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \end{pmatrix}^{-1} \sim \begin{pmatrix} O(1)_{1 \times 1} & O(\rho^{q+1})_{1 \times (n-1)} \\ O(\rho^{q-1})_{(n-1) \times 1} & O(1)_{(n-1) \times (n-1)} \end{pmatrix}.
\]

Multiplying this by \( \begin{pmatrix} \phi^k_+ \end{pmatrix} \) in (34), we get the lemma. \( \square \)

To get an analytic compactification of the complex structure \( J \), we want to solve a map \( z : D_0^R \to \mathbb{D}^2 \subset \mathbb{C}^n \) where \( D_0^R = \{ (z^1, \ldots, z^n) \in \mathbb{C}^n ; |z^1| \leq R \} \), such that \( z \) is a homeomorphism onto the image and is holomorphic with respect to \( J_0 \) and \( J \). For holomorphicity, \( z^i = z^i(\zeta) \) must satisfy the following equations:

\[
\frac{\partial z^i}{\partial \zeta^j} + \sum_{p=1}^n a^i_p(z) \frac{\partial z^p}{\partial \zeta^j} = 0, \quad i, l = 1, \ldots, n. \tag{35}
\]

We can write these out into components: \( 1 < j, k, m \leq n \)

\[
\begin{align*}
\frac{\partial a^i_+}{\partial \zeta^j} &+ (a^1_+ \sim \rho^0) \frac{\partial a^m_+}{\partial \zeta^k} + (a^1_+ \sim \rho^{q+1}) \frac{\partial a^m_+}{\partial \zeta^k} = 0 \\
\frac{\partial a^i_+}{\partial \zeta^j} &+ (a^1_+ \sim \rho^0) \frac{\partial a^m_+}{\partial \zeta^k} + (a^1_+ \sim \rho^{q+1}) \frac{\partial a^m_+}{\partial \zeta^k} = 0 \\
\frac{\partial a^i_+}{\partial \zeta^j} &+ (a^1_+ \sim \rho^{q-1}) \frac{\partial a^m_+}{\partial \zeta^k} + (a^1_+ \sim \rho^q) \frac{\partial a^m_+}{\partial \zeta^k} = 0 \\
\frac{\partial a^i_+}{\partial \zeta^j} &+ (a^1_+ \sim \rho^{q-1}) \frac{\partial a^m_+}{\partial \zeta^k} + (a^1_+ \sim \rho^q) \frac{\partial a^m_+}{\partial \zeta^k} = 0.
\end{align*}
\]

Remark 4.1. The existence of complex analytic coordinate system for any integrable almost complex structures \( J \) is a classical result in complex geometry. If the complex structure is analytic, the existence follows from the Frobenius theorem. When \( J \) is \( C^{\infty} \) this was the celebrated Newlander-Nirenberg theorem [21]. Nijenhuis-Woolf [22] proved the existence when \( J \) is only \( C^{1+\alpha} \). Later Malgrange [20] gave a short proof of the \( C^{1+\alpha} \)-case by using some gauge fixing to reduce the existence to the analytic case. More recently, Hill-Taylor [18] generalized Malgrange’s method to deal with the case when \( J \) is only \( C^{1} \) which seems to be the weakest assumption on the regularity of complex structures in the literature.

By the above remark, if we assume \( \eta > 1 \), then the existence of solutions to the system (36) follows from the work of [18]. We want also to deal with the case when we only assume \( \eta \) when the component \( a^i_+ \) might blows up if \( \eta < 1 \). Since \( J \) is assumed to be smooth outside \( D \), this can also be seen as some removable singularity problem. We will solve the system (36) following the work of Newlander-Nirenberg [21]. One should also be able to adapt the work of Nijenhuis-Woolf [22], Malgrange [20] to the current setting to prove the compactification (extension) of the complex structures considered here. See also Remark 4.2.

We first recall the important homotopy operator in [21]. For a vector of \( n \) complex-valued functions \( F = (f_1, \cdots, f_n) \), denote \((21, (2.5)))

\[
\mathcal{T} F = \sum_{s=0}^{n-1} (-1)^s \sum_{\ell=1}^s T^{j_1} \bar{\partial} \cdots T^{j_s} \bar{\partial} \cdot T^k f_\tau.
\]
where $\sum'$ denote the summation over all $(s+1)$-tuples with $j_1, \ldots, j_s, k$ distinct, and

$$T^1 f(\zeta) = \frac{1}{2\pi i} \int_{0<|\tau|<R} \frac{f(\tau, \zeta^1, \ldots, \zeta^n)}{\zeta^1 - \tau} d\tau d\zeta,$$

$$T^j f(\zeta) = \frac{1}{2\pi i} \int_{0<|\tau|<R} \frac{f(\zeta^1, \ldots, \zeta^{j-1}, \tau, \zeta^j, \ldots, \zeta^n)}{\zeta^j - \tau} d\tau d\zeta, \text{ for } j \geq 2.$$

For fit our setting, we need to modify this by defining:

$$\tilde{T}^1 f(\zeta) = T^1 f(\zeta^1, \zeta^2, \ldots, \zeta^n) - T^1 f(0, \zeta^2, \ldots, \zeta^n), \quad \tilde{T}^j f(\zeta) = T^j f(\zeta), \text{ if } j \geq 2.$$

Then by Lemma 4.3 and Lemma 4.4, these operators are well defined for functions $f$ such that $f \sim O(|\zeta|^{|n-1|})$ and satisfy (see [9, (18)]) the following identities on $D^R_1 \times D^R_n$:

$$\partial_j \tilde{T}^j f = f, j = 1, \ldots, n; \quad \text{and} \quad \partial_j \tilde{T}^k f = \tilde{T}^k \partial_j f, \text{ for } j \neq k. \quad (37)$$

Then we define

$$\tilde{T} F(\zeta) = \sum_{s=0}^{n-1} \frac{(-1)^s}{(s+1)!} \sum_{j} \tilde{T}^{j s} \partial_{j_1} \partial_{j_2} \cdots \partial_{j_s} f_{j_1} \cdots f_{j_s}.$$

Then using relation (37) to manipulate, we can easily get the following formula which is a variation of the formula in cf. [21, 6] by replacing the operator $T^j$ by $\tilde{T}^j$.

$$\partial_j \tilde{T} F - f_j = \sum_{s=0}^{n-2} \frac{(-1)^s}{(s+2)!} \sum_{j} \tilde{T}^{j s} \partial_{j_1} \partial_{j_2} \cdots \partial_{j_s} f_{j_1} \cdots f_{j_s} \tilde{T}^k (\partial_k f_k - \tilde{T}^k f_k). \quad (38)$$

where $\sum'$ denotes the summation over all $(s+1)$-tuples with $j_1, \ldots, j_s, k$ distinct and different from $j$. From (35), we will denote

$$f^j_t = -\sum_{p=1}^{n} a^p(t) \frac{\partial^p}{\partial \zeta^p}, \quad F^j = (f^1_t, f^2_t, \ldots, f^n_t) = \sum_{l=1}^{n} f^j_t d\zeta^l.$$

Denote also $z^j(\zeta) = z^j(\zeta) - \zeta^j$. We want to transform equations (35) into:

$$z^j = \zeta^j + \tilde{T}(F^j(z)) \iff z^j = \tilde{T}(F^j(\zeta + z)) \iff z = \tilde{T}(z). \quad (39)$$

We will show in Lemma 4.9 that the solution to this equation with the appropriate control is indeed the solution to (35). To get compatible solution to the system (36), we prescribe asymptotically behaviors:

$$z^1 = \zeta^1 + O(\rho^{1+\eta}), \quad z^j = \zeta^j + O(\rho^{n}), \quad \iff \quad z^1 \sim O(\rho^{1+\eta}), \quad z^j \sim O(\rho^{n}). \quad (40)$$

Here and in the following, we still denote $\rho = |\zeta^1|$ since $|\zeta^1|$ and $|z^1|$ is comparable with this prescription. If we denote $h$ the index $\{2, \ldots, n\}$, then the precise meaning of (40) is the following

$$|\partial_{\zeta^1} \partial_{\zeta^h} (z^1 - \zeta^1)| \leq C |\zeta^1|^{1+\eta-1}, \quad |\partial_{\zeta^1} \partial_{\zeta^h} (z^h - \zeta^h)| \leq C |\zeta^1|^{n-1}.$$

Under this prescription, by (36), we can show (Lemma 4.8) that

$$(f^1_t, f^2_t) \sim (O(\rho^{n} + \rho^{2n}), O(\rho^{2n+1} + \rho^{n+1})), \quad (f^j_t, f^h_t) \sim (O(\rho^{n-1} + \rho^{2n-1}), O(\rho^{2n} + \rho^n)) \sim (\rho^{n-1}, \rho^n).$$

Then we can show that (Lemma 4.6):~

$$\tilde{T}[F^1] \sim O(\rho^{n+1}), \quad \tilde{T}[F^k] \sim O(\rho^n) \text{ for } k \geq 2.$$

This is compatible with the prescription in (40) and should allow us to use the arguments in [21] to solve the system (39). However, to use the contraction-iteration principle, we have to relax asymptotic behaviors in (40) a little bit by replacing $\eta$ by any $\nu$ such that $0 < \nu < \eta$ (See Lemma 4.7). This might seem a loss. But actually later we will gain this $\epsilon$ back using the analyticity of transition functions.

More precisely, in the next subsection, we will introduce weighted multiple Hölder norm $\| \cdot \|_{n+\alpha,(\nu+1,\nu)}$ and show in Theorem 4.1 that, for any $\lambda, \bar{\lambda}$ satisfying $\|\bar{\lambda}\|_{n+\alpha,(\nu+1,\nu)} \leq 1$, $\|\bar{\lambda}\|_{n+\alpha,(\nu+1,\nu)} \leq 1$, the following estimates hold:
1. \[ \|3[3]\|_{n+a,\nu+1,\nu} \leq 1. \] (41)

2. \[ \|3[3] - 3[3]\|_{n+a,\nu+1,\nu} \leq \frac{1}{2} \|3\|_{n+a,\nu+1,\nu}. \] (42)

By standard iteration, there is a unique solution to the system (39) such that:

\[ y^1 = O(\rho^{1+\nu}), \quad y^i = O(\rho^\nu), \] or equivalently \[ z^1 = \zeta^1 + O(\rho^{1+\nu}), \quad z^i = \zeta^i + O(\rho^\nu). \] (43)

In the following \( D_R = \{ \zeta \in \mathbb{C}; |\zeta| \leq R \} \) denotes the closed disc of radius \( R \) with center \( 0 \), and \( D_R^c = \{ \zeta \in \mathbb{C}; 0 < |\zeta| \leq R \} \) denotes the punctured closed disc. We need to show that the map \( \zeta \mapsto z \) gives a coordinate chart for \( \zeta \in D_R^c \) when \( R \) is sufficiently small. First note that \{\( z^i(\zeta) \)\} is identity for \( \zeta^1 = 0 \) and is Hölder continuous on \{\( \zeta^1 = 0 \)\}. Secondly on \( U_R = D_R^c \times \mathbb{D}^{-1}_R \), consider the Jacobian

\[ J = \begin{pmatrix} \partial(z^i, z^j) \\ \partial(\zeta^i, \zeta^j) \end{pmatrix}. \]

By the similar argument as that in the proof Lemma 34, it’s easy to see that \( J \) is invertible if \( R \) is very small. So on \( U_R, \zeta \mapsto z \) is a local diffeomorphism to its image. We just need to show that it’s an injective map and hence a homeomorphism.

To do this, we decompose the coordinate change in (40) into two steps. First we let

\[ y^i = z^1(\zeta) = \zeta^i + O(|\zeta^1|^{1+\nu}), \quad y^k = \zeta^k \] for \( k \geq 2. \] (44)

Since the Jacobian matrix is invertible and \( C^\nu \), the map is a \( C^{1,\nu} \)-diffeomorphism and is clearly a change of coordinates. We can express \( \zeta \) in terms of \( y \) to get:

\[ \zeta^1 = y^1 + O(|y^1|^{1+\nu}), \quad \zeta^k = y^k \] for \( k \geq 2. \]

Now we can write the map in (40) as:

\[ z^1 = y^1, \quad z^k = y^k + O(|y^1|^{1+\nu}) \] for \( k \geq 2. \]

We just need to show this is injective. We assume \( z(y) = z(\tilde{y}) \). Then \( y^1 = \tilde{y}^1 \), and \( z^i(y) = z^i(\tilde{y}) \). On the slice \( y^1 = \tilde{y}^1 \), we connect \( y \) and \( \tilde{y} \) by \( y_t = (1-t)y + t\tilde{y} \), then we have

\[ 0 = ||z(\tilde{y}) - z(y)|| = \sum_{j=1}^n \left| \int_0^1 \sum_{k=1}^n (\partial_{y^k} z^j)(y_t) \cdot (\tilde{y}^k - y^k)dt \right| = \sum_{j=2}^n \left| \int_0^1 \sum_{k=2}^n (\delta_j^k + O(|y^1|^{1+\nu}))(\tilde{y}^k - y^k)dt \right| \geq C(1 - R^\nu)||\tilde{y} - y||. \]

So if \( R \) is sufficiently small, we indeed have \( \tilde{y} = y \).

By the similar argument in [21], we can show in the present more technical set-up (see Lemma 4.9) that the \{\( z^i = \zeta^i + y^i \)\}_{i=1}^n \) are indeed solutions to the system (35).

To see the last statement in Theorem 1.2, note that the transition function on the bundle \( N_D \rightarrow D \) in terms of \{\( z_{\alpha}^i \)\} are standard ones:

\[ z_{\beta}^1 = a_{\beta\alpha}(z_{\alpha}^1), \quad z_{\beta}^k = \phi_{\beta\alpha}^k(z_{\alpha}^k) \] for \( k \geq 2. \]

By the asymptotical behavior (43) and its inverse, we see that the transition functions in the \( \zeta \)-coordinates have the shape:

\[ \zeta_{\beta}^1 = a_{\beta\alpha}(\zeta_{\alpha}^1), \quad \zeta_{\beta}^k = \phi_{\beta\alpha}^k(\zeta_{\alpha}^k) + O(|\zeta_{\alpha}^1|^{1+\nu}) \] for \( k \geq 2. \]

We know that \( \zeta_{\alpha}^i \), for any \( 1 \leq i \leq \alpha \), is a holomorphic function of \( \zeta_{\alpha} \) outside \( D \), and from above expressions it’s Hölder continuous across \( D = \{\zeta_{\alpha}^1 = 0 \} \). So we see that \( \zeta_{\beta}^i \) is holomorphic across \( D \) and hence is a holomorphic function of \( \zeta_{\alpha} \). Denote \( m = [\nu] = [\eta] = [\lambda\delta] \) (Recall that \( \eta = \lambda\delta \) and
\( \nu = \eta - \epsilon \) for small \( \epsilon \). Then the analyticity of holomorphic functions clearly implies that we must have the following improved transition:

\[
\zeta_1^1 = a_{\beta\alpha}(\zeta_2^1) \zeta_1^\alpha + R_{m+1}^1, \quad \zeta_2^k = \phi_{\beta\alpha}(\zeta_1^\alpha) + R_m^k.
\]

where \( R_{m+1}^1 \in \mathcal{I}_D^{m+1} \), \( R_m^k \in \mathcal{I}_D^m \), where \( \mathcal{I}_D \) is the ideal sheaf of \( D \) generated by \( \{ \zeta_1^1 \} \). By Theorem 6.5 (see also (10)), we see that in the compactification, the divisor \( D \) is indeed \((m-1)\)-comfortably embedded. In this way, we prove theorem 1.2.

**Remark 4.2.** In [17], the authors proved an analytic compactification result in the asymptotically cylindrical Calabi-Yau case. Their compactification result depends on classifying asymptotical models of the asymptotically cylindrical Calabi-Yau metrics. In the asymptotically conical case, the classification of models at infinity is not clear at present. So here we just concentrate in a case when the model at infinity is known. In this sense, the result obtained here is a counterpart of [17, Theorem 3.1] in our different setting. In the proof of [17, Theorem 3.1], the authors used gauge fixing and used result of Nijenhuis-Woolf [22].

The method used here is technically different and we aim to give a detailed proof by following the fundamental work of Newlander-Nirenberg. In other words, we decide to solve the system (36) altogether to get the coordinate charts. In the next section, we will write down in details the required estimates for \( \overline{\partial} \) equations and iteration processes.

### 4.2 Integral operator on weighted multiple Hölder space

Suppose \( f \) is a complex-valued function defined on \( \mathbb{D}_R^2 \times \mathbb{D}_R^{n-1} \). Denote \( D_i \) either of the differential operators \( \frac{\partial}{\partial \tau_i}, \frac{\partial}{\partial \xi_i} \). \( D^k \) will denote a general \( k \)-th order derivative \( D^k = D_{i_1} \ldots D_{i_k} \) with \( i_1, \ldots, i_k \) distinct (i.e. we consider only “mixed” derivatives). \( D^{k,j} = D_{i_1} \ldots D_{i_k} \) (resp. \( D^{k,(1,j)} \)) will denote such a derivative with the \( i_1, \ldots, i_k \) distinct and different from \( j \) (resp. \( \{1, j\} \)). For a fixed positive \( \alpha < 1 \), we denote the difference quotient operators:

\[
\delta_1 f = \frac{f(\zeta^1, \zeta^2, \ldots, \zeta^n) - f(\zeta^1, \zeta^2, \ldots, \zeta^n)}{\zeta^1 - \zeta_1^1} \quad \text{for} \quad 0 < |\zeta^1| \leq R, 0 < |\zeta^1| \leq R, \zeta^1 \neq \zeta_1^1.
\]

\[
\delta_i f = \frac{f(\zeta^1, \ldots, \zeta^i, \ldots, \zeta^n) - f(\zeta^1, \ldots, \zeta^i, \ldots, \zeta^n)}{|\zeta^i - \zeta_1^i|} \quad \text{for} \quad i > 1, |\zeta^i| < R, |\zeta^i| < R, \zeta^i \neq \zeta_1^i.
\]

Denote \( \delta_1 = \delta_1 \ldots \delta_{j_m} \) for \( 0 \leq m \leq n \) and \( j_1, \ldots, j_m \) distinct; \( \delta^0 \) will denote the identity operator; \( \delta^{m,1} \) will denote such a difference quotient with \( j_1, \ldots, j_m \) distinct and different from \( 1 \). The following is the standard Schauder estimate for the elliptic operator \( \partial \) for a single variable.

**Lemma 4.2.** Assume \( \alpha \in (0, 1) \) is fixed. There exists a constant \( c > 0 \) such that, if \( w \in C^{1,\alpha}(\mathbb{D}_1(0)) \) satisfies \( \frac{\partial w}{\partial \xi} = f \) in \( \mathbb{D}_1 \) and if \( f \in C^{0,\alpha}(\mathbb{D}_1(0)) \), then

\[
\|w\|_{C^{1,\alpha}(\mathbb{D}_1/2)} \leq c \left( \|w\|_{L^\infty(\mathbb{D}_1)} + \|f\|_{C^{0,\alpha}(\mathbb{D}_1)} \right). \tag{45}
\]

**Proof.** Denote operators:

\[
T f(\zeta) = \frac{1}{2\pi i} \int_{\mathbb{D}_1} \frac{f(\tau)}{\tau - \zeta} d\tau \wedge d\zeta, \quad S w(\zeta) = \frac{1}{2\pi i} \int_{\mathbb{D}_1} \frac{w(\tau)}{\tau - \zeta} d\tau.
\]

Then \( w \in C^{1,\alpha}(\mathbb{D}_1) \) satisfies:

\[
w = T \partial w + S w = T f + S w.
\]

By Chern [9, Main Lemma], we have

\[
\|T f\|_{C^{1,\alpha}(\mathbb{D}_1)} \leq \|f\|_{C^{0,\alpha}(\mathbb{D}_1)}
\]

On the other hand, because \( S w = w - T f \) is holomorphic, we have:

\[
\|S w\|_{C^{1,\alpha}(\mathbb{D}_1/2)} \leq \|S w\|_{L^\infty(\mathbb{D}_1)} \leq \|w\|_{L^\infty(\mathbb{D}_1)} + \|T f\|_{L^\infty(\mathbb{D}_1)} \leq \|w\|_{L^\infty(\mathbb{D}_1)} + \|f\|_{L^\infty(\mathbb{D}_1)}.
\]

\[\blacksquare\]
We need to extend the above Schauder estimate to the weighted Hölder space. We follow [23, Chapter 2] to define the weighted Hölder norm for functions on the punctured disks. Note that this definition of weighted norm is slightly different from the definition used in for example [17] and [10]. Although two norms may be equivalent in some sense, the norm used here following [23] only uses the Hölder norm for $x$ and $y$ with comparable distances to the puncture. To the author’s understanding, this constraint is well adapted to the rescaling argument. For any $s > 0$, denote the annulus $\{ \zeta \in \mathbb{C}; s < |\zeta| < 2s \}$ by $A(s, 2s)$. First we define the norm on the annulus:

$$[w]_{1, \alpha, s} := \sup_{A(s, 2s)} |w| + s \sup_{A(s, 2s)} \left| D_1 w \right| + s^\alpha \sup_{x, y \in A(s, 2s)} \frac{|w(x) - w(y)|}{|x - y|^\alpha}.$$

The following is the rescaling invariant weighted Hölder norm for functions on the punctured disk of radius $R$:

$$\|w\|_{\tilde{C}^1, \alpha, \nu}(D_R(0)) = R^\nu s^{-\nu}[w]_{1, \alpha, s},$$

As pointed out in [23, Corollary 2.1], the following Lemma is important for deriving the rescaled Schauder estimate in Lemma 4.4.

**Lemma 4.3.** Assume $f \in \tilde{C}^{1, \alpha}_{\nu^{-1}}(D_R)$, then we have

$$\|\rho^{-\nu} \tilde{T} f\|_{L^\infty(D_R)} \leq C \|\rho^{-\nu} f\|_{L^\infty(D_R)}.$$  \hspace{1cm} (46)

**Proof.** We can first estimate:

$$\begin{align*}
|\rho^{-\nu} \tilde{T} f| &= |\zeta|^{-\nu} \left| \int_{D_R} \left( \frac{f(\tau)}{\tau - \zeta} - \frac{f(\tau)}{\tau} \right) dV \right| = |\zeta|^{-\nu} \left| \int_{D_R} \frac{f(\tau) \zeta}{(\tau - \zeta) \tau} dV \right| \\
&\leq \|\rho^{-\nu} f\|_{L^\infty} \int_{D_R} \left| \frac{1}{|\tau - \zeta|^2} \right| dV.
\end{align*}$$

We split the integral into three parts:

$$\int_{D_R(0)} = \int_{D_{R/2}(0)} + \int_{D_{R/2}(\zeta)} + \int_{D_{R}(0) \setminus (D_{R/2}(0) \cup D_{R/2}(\zeta))} = I + II + III.$$

The inequality (46) follows from the following estimates:

$$I \leq C \int_0^{R^2} \frac{ds}{s^{1-\nu} R^{2/2}} \leq C R^{1-\nu}, \quad II \leq C \int_0^{R^2} \frac{ds}{s^{2-\nu}} \leq C R^{1-\nu},$$

$$III \leq C \int_{R/2}^R \frac{ds}{s^{2-\nu}} \leq C \left( \left( \frac{R^2}{2} \right)^{\nu-1} - R^{\nu-1} \right).$$



**Lemma 4.4 (Potential lemma).** If $f \in \tilde{C}^{0, \alpha}_{\nu^{-1}}(D_R)$, then $\tilde{T} f \in \tilde{C}^{1, \alpha}_{\nu}(D_R)$ and satisfies:

$$\|\tilde{T} f\|_{\tilde{C}^{1, \alpha}_{\nu}(D_R)} \leq C \|f\|_{\tilde{C}^{0, \alpha}_{\nu^{-1}}(D_R)}.$$  \hspace{1cm} (47)

**Proof.** Let $F(\zeta) = \tilde{T} f(\zeta)$. Let $\rho = |\zeta|$. By Lemma 4.2, Lemma 4.3 and standard rescaling argument as in [23, Corollary 2.1], we have:

$$\|\tilde{T} f\|_{\tilde{C}^{1, \alpha}_{\nu}(D_R)} \leq C \|f\|_{\tilde{C}^{0, \alpha}_{\nu^{-1}}(D_R)}.$$

To get estimate on $D_R \setminus D_{R/2}$, we use the explicit formula of $\tilde{T}$. As in [9, (18), (26)], we have:

$$F_\tau = f(\zeta), \quad F_\zeta = \frac{1}{2\pi \sqrt{-1}} \int_{D_R(0)} \frac{f(\tau) - f(\zeta)}{(\tau - \zeta)^2} d\tau dr.$$
So that
\[
\left| \frac{|F_\xi|}{|\zeta|^{\nu-1}} \right| \leq \frac{1}{2\pi} \frac{1}{|\zeta|^{\nu-1}} \int_{\mathbb{D}_R(0)} \frac{|f(\tau) - f(\xi)|}{|\tau - \zeta|^2} dV(\tau)
\]

We can assume \( R/8 \leq |\zeta| \leq R \). To estimate the integrals, we split it into two parts:
\[
\int_{\mathbb{D}_R(0)} = \int_{\mathbb{D}_{R/2}(0)} + \int_{\mathbb{D}_R(0) \setminus \mathbb{D}_{R/2}(0)} = \mathbf{I} + \mathbf{II}.
\]
We then estimate:
\[
\mathbf{I} \leq C\|\rho^{-\nu} f\|_{L^\infty(\mathbb{D}_R)} \frac{1}{R^2} \int_0^{R/8} s^{\nu-1} ds \leq CR^{\nu-1}\|\rho^{-\nu} f\|_{L^\infty(\mathbb{D}_R(0))}.
\]
\[
\mathbf{II} \leq C \int_{\mathbb{D}_R(0) \setminus \mathbb{D}_{R/2}(0)} \frac{|f|_{\mathcal{C}^1,\alpha,1}^{\nu}|\tau - \zeta|^\alpha R^{-\alpha}}{|\tau - \zeta|^2} dV \leq C\|f\|_{\mathcal{C}^1,\alpha,1}^\nu R^{-\alpha} \int_0^{2R} s^{\alpha-2+1} ds \leq C\|f\|_{\mathcal{C}^1,\alpha,1}.
\]
So we get \( \|\rho^{-\nu} D_1 \hat{T} f\|_{L^\infty} \leq \|f\|_{\mathcal{C}^1,\alpha,1}^\nu \). Similarly, one can prove that:
\[
R^\alpha \sup_{x,y \in A(R/8, R)} \frac{|w(x) - w(y)|}{|x - y|^\alpha} + R^1 + \alpha \sup_{x,y \in A(R/8, R)} \frac{|D_1 w(x) - D_1 w(y)|}{|x - y|^\alpha} \leq \|f\|_{\mathcal{C}^0,\alpha,1}.
\]
with \( w = \hat{T}(f) \). In fact, we can prove the inequality as in [22, Section 6.1e], the only difference is that we need to separate the integral over \( \mathbb{D}_{R/2}(0) \) from each estimate since we only have Hölder estimate for \( x \) and \( y \) of comparable lengths.

Similarly to [21, (3.1)-(3.3)], we introduce the weighted multiple-Hölder space by incorporating the weighted 1st order Hölder space for \( \zeta^1 \) and the usual 1st order Hölder spaces for the other variables. Formally, we define:

1. (Integral part)
\[
\|u\|_{n,\nu} = R^n \sum_{k=0}^{n-1} \left( \frac{R^k}{k!} \sup_{x,y \in A(0, R)} \left( \frac{|D_1^{k,1} u|}{|\zeta|^\nu} \right) + \frac{R^k}{k!} \sup_{x,y \in A(0, R)} \left( \frac{|D_1^{1,1} u|}{|\zeta|^\nu} \right) \right).
\]

2. (Fractional part i.e. difference quotient part):
\[
[u]_{n,\alpha,\nu} = R^n \sum_{m=1}^{n-1} \frac{R^m}{m!} \sup_{s \in (0, 1/2)} \left( \frac{|D_1^{m,1} u|}{|\zeta|^\nu} \right) + \frac{R^m}{m!} \sup_{s \in (0, 1/2)} \left( \frac{|D_1^{1,1} u|}{|\zeta|^\nu} \right).
\]

3. (0th-order weighted multiple Hölder norm)
\[
\|u\|_{n,\alpha,\nu} = \tilde{H}_{n,\alpha}[u] = R^n \sup_{|\zeta|^\nu} \frac{|u|}{|\zeta|^\nu} + [u]_{n,\alpha,\nu}
\]

4. (1st-order weighted multiple Hölder norm)
\[
\|u\|_{n+\alpha,\nu} = \|u\|_{n,\nu} + \sum_{k=0}^{n-1} \left( \frac{R^k}{k!} [D_1^{k,1} u]_{n,\alpha,\nu} + \frac{R^k}{k!} [D_1^{1,1} u]_{n,\alpha,\nu-1} \right)
\]
\[
= \sum_{k=0}^{n-1} \left( \frac{R^k}{k!} \tilde{H}_{n,\alpha}[D_1^{k,1} u] + \frac{R^k}{k!} \tilde{H}_{n,\alpha-1}[D_1^{1,1} u] \right).
\]
5. (Partial 1st-order weighted multiple Hölder norm)

\[
\|u\|_{n-1+n\alpha,\nu}^1 = \sum_{k=0}^{n-1} \frac{R^k}{k!} \sup \hat{H}_{\alpha,\nu}[D^{k,1}u].
\]

\[
\|u\|_{n-1+n\alpha,\nu}^n = \sum_{k=0}^{n-2} \left( \frac{R^k}{k!} \sup \hat{H}_{\alpha,\nu}[D^{k,1}\cdot]u] + \frac{R^k}{(k+1)!} \hat{H}_{\alpha,\nu-1}[D_1D^{k,1}\cdot]u] \right) \text{ for } j \geq 2.
\]

6. (Anisotropically-weighted norm for vector of functions) Denote \( \mathfrak{f} = (\mathfrak{f}^1, \cdots, \mathfrak{f}^n) \), \( F = (f_T, \cdots, f_\nu) \). Denote:

\[
\|\mathfrak{f}\|_{n+n\alpha,\nu+1} = \|\mathfrak{f}^1\|_{n+n\alpha,\nu+1} + \sum_{j=2}^{n} \|\mathfrak{f}^j\|_{n+n\alpha,\nu+1}.
\]

\[
\|F\|_{n-1+n\alpha,\nu+1} = \|f_T\|_{n-1+n\alpha,\nu+1} + \sum_{j=2}^{n} \|f_T^j\|_{n-1+n\alpha,\nu+1}.
\]

Now we come back to solve the system (39) which is equivalent to:

\[
\mathfrak{f} = \widehat{T}(F^{i}(\zeta + \mathfrak{f})) = \mathfrak{f}^i[\mathfrak{f}], \text{ where } F^i = \left( \sum_{j=1}^{n} \frac{\partial}{\partial \zeta_j} \right).
\]  

(47)

Arguing as in [21], the following lemma is a consequence of Lemma 4.4.

**Lemma 4.5** (cf. [21, Lemma 4.1, Lemma 4.3]).

\[
\left\| \widehat{T}^{j}D_j f \right\|_{n-1+n\alpha,\nu} \leq \|f\|_{n-1+n\alpha,\nu}, \quad j, l = 1, \cdots, n, j \neq l.
\]

\[
\left\| \widehat{T}^{j} f \right\|_{n+n\alpha,\nu+1} \leq c \|f\|_{n-1+n\alpha,\nu},
\]

\[
\left\| \widehat{T}^{j} f \right\|_{n+n\alpha,\nu} \leq c \|f\|_{n-1+n\alpha,\nu}, \text{ for } j \geq 2.
\]

(48)

Note that, the operator \( \widehat{T}^{1} \) improves the weight by 1. Packing these estimates for components of \( F^1, F^j \), the above Lemma implies:

**Lemma 4.6** (cf. [21, Theorem 4.1]).

\[
\left\| \widehat{T}(F^1) \right\|_{n+n\alpha,\nu+1} \leq c \|F^1\|_{n+n\alpha,\nu+1}; \quad \left\| \widehat{T}(F^j) \right\|_{n+n\alpha,\nu} \leq c \|F^j\|_{n-1+n\alpha,\nu}, \text{ for } j \geq 2.
\]

The following Lemma shows the reason to relax the asymptotics by replacing \( \eta \) by \( \nu \).

**Lemma 4.7** (cf. [21, Lemma 3.1]). Suppose \( \|\mathfrak{f}\|_{n+n\alpha,\nu+1} \leq 1 \), then

\[
\|a^{(1)}_{\nu}(\zeta + \mathfrak{f})\|_{n+n\alpha,\nu} \leq KR^{\nu-\nu} \|\mathfrak{f}\|_{n+n\alpha,\nu+1} \leq KR^{\nu-\nu} \|\mathfrak{f}\|_{n+n\alpha,\nu+1};
\]

\[
\|a^{(2)}_{\nu}(\zeta + \mathfrak{f})\|_{n+n\alpha,\nu} \leq KR^{\nu-\nu} \|\mathfrak{f}\|_{n+n\alpha,\nu+1} \leq KR^{\nu-\nu} \|\mathfrak{f}\|_{n+n\alpha,\nu+1}.
\]

**Lemma 4.8** (cf. [21, Lemma 5.1]). If \( \|\mathfrak{f}\|_{n+n\alpha,\nu+1} \leq 1 \). Then

\[
\|F^1\|_{n-1+n\alpha,\nu+1} \leq CR^{\nu-\nu} \|\mathfrak{f}\|_{n+n\alpha,\nu+1}; \quad \|F^1[\mathfrak{f}] - F^1[\mathfrak{f}]\|_{n+n\alpha,\nu+1} \leq CR^\nu \|\mathfrak{f} - \mathfrak{f}\|_{n+n\alpha,\nu+1}.
\]

(49)

For \( j \geq 2 \), we have:

\[
\|F^j\|_{n-1+n\alpha,\nu} \leq CR^{\nu-\nu} \|\mathfrak{f}\|_{n+n\alpha,\nu+1}; \quad \|F^j[\mathfrak{f}] - F^j[\mathfrak{f}]\|_{n+n\alpha,\nu} \leq CR^\nu \|\mathfrak{f} - \mathfrak{f}\|_{n+n\alpha,\nu+1}.
\]

(50)
Proof. $L_1^n \sim O(\rho^n + \rho^{2n})$:

\[
\left\| \frac{a_1^\top \partial^{1 \top}}{\partial \zeta} \right\|_{n+1+n\alpha, \nu} \leq \| a_1^\top \|_{n+1+n\alpha, \nu} \left( 1 + \left\| \frac{\partial^1}{\partial \zeta} \right\|_{n+1+n\alpha, 0} \right)
\]

\[
\leq KR^{n-\nu} \| \delta \|_{n+\alpha, (\nu+1, \nu)} (1 + \| \delta \|_{n+\alpha, (\nu+1, \nu)} R^n).
\]

\[
\left\| a_1^\top (\zeta + \tilde{\bar{z}}) \frac{\partial^{1 \top}}{\partial \zeta} + a_1^\top (\zeta + \bar{\bar{\tilde{z}}}) \frac{\partial^{1 \top}}{\partial \zeta} \right\|_{n+1+n\alpha, \nu} \leq \| a_1^\top (\zeta + \bar{\bar{\tilde{z}}}) - a_1^\top (\zeta + \bar{\bar{\tilde{z}}}) \|_{n+1+n\alpha, \nu} \left\| \frac{\partial^{1 \top}}{\partial \zeta} \right\|_{n+1+n\alpha, 0}
\]

\[
+ \| a_1^\top (\zeta + \bar{\bar{\tilde{z}}}) \|_{n+1+n\alpha, \nu} \left\| \frac{\partial^{1 \top} - \tilde{\bar{z}} \tilde{\bar{\bar{z}}}}{\partial \zeta} \right\|_{n+1+n\alpha, 0}
\]

\[
\leq KR^{\nu} \| \delta \|_{n+\alpha, (\nu+1, \nu)}.
\]

Note that, similar to the method in our proof that $\zeta \mapsto z$ gives coordinate charts, in the above estimates, we can estimate the difference of $a_1^\top(z) - a_1^\top(\tilde{\bar{z}})$ by decomposing into two parts and then uses mean value theorem to get the above estimate:

\[
a_1^\top(z + \bar{\bar{\tilde{z}}}) - a_1^\top(z + \bar{\bar{\tilde{z}}}) = \left[ a_1^\top(z + \bar{\bar{\tilde{z}}}) - a_1^\top(z + \bar{\bar{\tilde{z}}}) \right] + \left[ a_1^\top(z + \bar{\bar{\tilde{z}}}) - a_1^\top(z + \bar{\bar{\tilde{z}}}) \right].
\]

\[
\left\| a_1^\top (\zeta + \tilde{\bar{z}}) \frac{\partial^{1 \top}}{\partial \zeta} \right\|_{n+1+n\alpha, \nu} \leq \left\| a_1^\top (\zeta + \bar{\bar{\tilde{z}}}) \frac{\partial^{1 \top}}{\partial \zeta} \right\|_{n+1+n\alpha, 0} \leq KR^{\nu} \| \delta \|_{n+\alpha, (\nu+1, \nu)} R^n.
\]

$L_2^n \sim O(\rho^{2n+1} + \rho^{3n})$:

\[
\left\| \frac{a_2}{\partial \zeta} \right\|_{n+1+n\alpha, \nu} \leq \left\| a_2^\top \|_{n+1+n\alpha, \nu} \left\| \frac{\partial^2}{\partial \zeta} \right\|_{n+1+n\alpha, 1} \leq KR^{n-\nu} \| \delta \|_{n+\alpha, (\nu+1, \nu)} R^n.
\]

\[
\left\| a_2 (\zeta + \tilde{\bar{z}}) \frac{\partial^{1 \top}}{\partial \zeta} + a_2 (\zeta + \bar{\bar{\tilde{z}}}) \frac{\partial^{1 \top}}{\partial \zeta} \right\|_{n+1+n\alpha, \nu} \leq \| a_2 (\zeta + \bar{\bar{\tilde{z}}}) - a_2 (\zeta + \bar{\bar{\tilde{z}}}) \|_{n+1+n\alpha, \nu} \left\| \frac{\partial^{1 \top}}{\partial \zeta} \right\|_{n+1+n\alpha, 1}
\]

\[
+ \| a_2 (\zeta + \bar{\bar{\tilde{z}}}) \|_{n+1+n\alpha, \nu} \left\| \frac{\partial^{1 \top} - \tilde{\bar{z}} \tilde{\bar{\bar{z}}}}{\partial \zeta} \right\|_{n+1+n\alpha, 1}
\]

\[
\leq KR^{\nu} \| \delta \|_{n+\alpha, (\nu+1, \nu)}.
\]

\[
\left\| a_2^\top \frac{\partial^{1 \top}}{\partial \zeta} \right\|_{n+1+n\alpha, \nu} \leq \left\| a_2^\top \|_{n+1+n\alpha, \nu} \left\| \frac{\partial^1}{\partial \zeta} \right\|_{n+1+n\alpha, 0} \leq KR^{n-\nu} \| \delta \|_{n+\alpha, (\nu+1, \nu)} R^n.
\]

\[
\left\| a_2 (\zeta + \tilde{\bar{z}}) \frac{\partial^{1 \top}}{\partial \zeta} + a_2 (\zeta + \bar{\bar{\tilde{z}}}) \frac{\partial^{1 \top}}{\partial \zeta} \right\|_{n+1+n\alpha, \nu} \leq \| a_2 (\zeta + \bar{\bar{\tilde{z}}}) - a_2 (\zeta + \bar{\bar{\tilde{z}}}) \|_{n+1+n\alpha, \nu} \left\| \frac{\partial^{1 \top}}{\partial \zeta} \right\|_{n+1+n\alpha, 1}
\]

\[
+ \| a_2 (\zeta + \bar{\bar{\tilde{z}}}) \|_{n+1+n\alpha, \nu} \left\| \frac{\partial^{1 \top} - \tilde{\bar{z}} \tilde{\bar{\bar{z}}}}{\partial \zeta} \right\|_{n+1+n\alpha, 1}
\]

\[
\leq KR^{\nu} \| \delta \|_{n+\alpha, (\nu+1, \nu)}.
\]
\[ \left\| a_\nu^1 (\xi + \zeta) \frac{\partial \sigma^m}{\partial \xi} - a_\nu^1 (\zeta + \tilde{\zeta}) \right\|_{n-1+n_0, \nu+1} \leq \left\| a_\nu^1 (\xi + \zeta) - a_\nu^1 (\zeta + \tilde{\zeta}) \right\|_{n-1+n_0, \nu+1} \left\| \frac{\partial \sigma^m}{\partial \xi} \right\|_{n-1+n_0, \nu+1} + \left\| a_\nu^1 (\zeta + \tilde{\zeta}) \right\|_{n-1+n_0, \nu+1} \left\| \frac{\partial (\sigma^m - \tilde{\sigma}^m)}{\partial \xi} \right\|_{n-1+n_0, \nu+1} \leq KR^n \left\| \frac{\partial}{\partial \xi} \right\|_{n+n_0, \nu+1}. \]

Similarly, one can verify the inequality for \( F^k = (f^k_1, f^k_2) \sim (O(p^{\nu-1} + p^{2\nu-1}), O(p^{2\nu} + p^\nu)). \)

Combining Lemma 4.5 and 4.8, we get:

**Theorem 4.1.** For any \( \bar{\zeta}, \bar{\theta} \) satisfying \( \| \bar{\zeta} \|_{(\nu+1), \nu} \leq 1, \| \bar{\theta} \|_{(\nu+1), \nu} \leq 1 \), we have

\[ \| \bar{\zeta} \|_{n+n_0, \nu+1} \leq c R^{\nu-\nu'} \| \bar{\theta} \|_{n+n_0, \nu+1}, \]

\[ \| \bar{\zeta} - \bar{\theta} \|_{n+n_0, \nu+1} \leq c R^n \| \bar{\theta} - \bar{\zeta} \|_{n+n_0, \nu+1}. \]

So for \( R \) sufficiently small, we indeed get the desired inequalities (41) and (42) to apply the contraction-iteration principle to get a solution to the system (47).

**Lemma 4.9.** If \( \bar{\zeta} \) is a solution to the system (47), then \( \bar{\theta} \) is a solution to (35), i.e.

\[ g^i = \frac{\partial z^i}{\partial \zeta_k} + \sum_{\nu=1}^m \alpha^i_{\nu}(z) \frac{\partial \sigma^m}{\partial \xi_k} = 0, \quad i, l = 1, \ldots, n. \]  

**Proof.** We follow the argument in [21]. Using the formula (38) and calculating as in [21, (2.11-2.12)] (see also [22, 4.1.2]) we get the following identity

\[ g^i = \sum_{j=0}^{n-2} \frac{-1}{(s+2)!} \sum_{j} T^i_j \bar{\zeta}_j \cdots T^i_{n-1} \bar{\zeta}_{n-1} \left( \partial_\nu a^0_\nu(\zeta) \frac{\partial \sigma^m}{\partial \xi_k} - \bar{\zeta}_k \frac{\partial \sigma^m}{\partial \xi_k} \right) \]

where \( \sum_{j} \) denotes the summation over all \( (s+1) \)-tuples with \( j_1, \ldots, j_s, k \) distinct and different from \( j \). We claim that from (52) the following holds:

\[ \| G^1 \|_{n+n_0, \nu+1} + \| G^2 \|_{n+n_0, \nu+1} \leq CR^n \| G^1 \|_{n+n_0, \nu+1} + \| G^2 \|_{n+n_0, \nu+1}. \]

**Proof.**

1. \( (i = 1, j = 1) \) In this case \( k \geq 2 \) (since \( k \neq j \) in \( \sum_{j} \)).
   - \( (p = 1, m = 1) \) \( \Theta_{kX}^1 \sim \rho^{\nu-1}(\rho^{2\nu+1} + \rho^{\nu+1}) \sim \rho^\nu \rho^{\nu'} \)
   - \( (p \geq 2, m = 1) \) \( \Theta_{kX}^1 \sim \rho^{\nu}(\rho^{2\nu} + \rho^{\nu+1}) \sim \rho^\nu \rho^{\nu'} \)
   - \( (p = 1, m \geq 2) \) \( \Theta_{kX}^1 \sim \rho^{\nu}(\rho^{2\nu-1} + \rho^{\nu-1}) \sim \rho^\nu \rho^{\nu'} \)
   - \( (p \geq 2, m \geq 2) \) \( \Theta_{kX}^1 \sim \rho^{\nu+1}(\rho^{2\nu-1} + \rho^{\nu-1}) \sim \rho^\nu \rho^{\nu'} \)

2. \( (i = 1, j \geq 2) \) In this case \( k \) can be 1.
   - \( (k = 1) \)
     - \( (p = 1, m = 1) \) \( \Theta_{kX}^1 \sim \rho^{\nu}(\rho^{2\nu} + \rho^{\nu+1}) \sim \rho^\nu \rho^{\nu'} \)
     - \( (p = 1, m \geq 2) \) \( \Theta_{kX}^1 \sim \rho^{\nu+1}(\rho^{2\nu} + \rho^{\nu-1}) \sim \rho^\nu \rho^{\nu'} \)
   - \( (k \geq 2) \)
     - \( (p = 1, m = 1) \) \( \Theta_{kX}^1 \sim \rho^{\nu}(\rho^{2\nu} + \rho^{\nu+1}) \sim \rho^\nu \rho^{\nu'} \)
     - \( (p = 1, m \geq 2) \) \( \Theta_{kX}^1 \sim \rho^{\nu+1}(\rho^{2\nu} + \rho^{\nu-1}) \sim \rho^\nu \rho^{\nu'} \)
iii. \((p = 1, m \geq 2)\) \(\mathcal{O}_{1l}^{|l|} \sim \rho^n(\rho^{0+\nu}) \sim \rho^n\rho^{\nu+1}\).

iv. \((p \geq 2, m \geq 2)\) \(\mathcal{O}_{1l}^{|l|} \sim \rho^{0+\nu}(\rho^{0+\nu}) \sim \rho^{2\nu+1}\).

3. \((i \geq 2, j = 1)\) In this case \(k \geq 2\). From the expression of \(\mathcal{O}_{1l}^{|l|}\), we see that the only difference from the case \(i = 1, j = 1\) lies in the term \(\partial_\rho a_m^{\alpha m}\). We just need to decrease each order by 1 to get

\[ \mathcal{O}_{1l}^{|l|} \sim \rho^n\rho^{\nu-1}. \]

4. \((i \geq 2, j \geq 2)\) In this case, \(k\) can be 1. Again, we see that the only difference with the case \(i = 1, j \geq 2\) lies in the term \(\partial_\rho a_m^{\alpha m}\). So we just need to decrease each order by 1 to get

\[ \mathcal{O}_{1l}^{|l|} \sim \rho^n\rho^{\nu-1}, \quad \mathcal{O}_{1l}^{|l|} \sim \rho^n\rho^{\nu}. \]

Now from item 1, we have that:

\[
\|g_1^{|l|}\|_{n-1+n\alpha,\nu} \leq C \sum_{k \geq 2} \|\hat{T}^k \mathcal{O}_{1l}^{|l|}\|_{n-1+n\alpha,\nu} \\
\leq CR^3(||G^1||_{n-1+n\alpha,\nu} + ||G^j||_{n-1+n\alpha,\nu+1} + ||G^j||_{n-1+n\alpha,\nu-1,\nu}).
\]

From item 2, we have for \(j \geq 2\),

\[
\|g_1^{|l|}\|_{n-1+n\alpha,\nu} \leq C(||\hat{T}^1 \mathcal{O}_{1l}^{|l|}\|_{n-1+n\alpha,\nu+1} + \sum_{k \geq 2} \|\hat{T}^k \mathcal{O}_{1l}^{|l|}\|_{n-1+n\alpha,\nu+1}) \\
\leq CR^3(||G^1||_{n-1+n\alpha,\nu+1} + ||G^j||_{n-1+n\alpha,\nu+1} + ||G^j||_{n-1+n\alpha,\nu-1,\nu}).
\]

Note that we have used the fact from (48) that the operator \(\hat{T}^1\) improves the weight from \(\nu\) to \(\nu+1\). The same argument applies to item 3 and 4 too. So we get the estimate (53).

\[ \square \]

5  Algebraic counterpart

5.1 Jet bundle and normal bundle of projective manifolds

The basic reference here is [28, Section 3]. Suppose \(H \to X\) is a very ample bundle over a projective manifold \(X\). \(S_1, \cdots, S_{N+1}\) form a basis of \(H^0(X, H)\), and the associated embedding \(X \subset \mathbb{P}^N\) is projectively normal. Choose a coordinate atlas \(\{V_\alpha, z_\alpha\}\) and a local holomorphic frame \(e_\alpha\) on \(V_\alpha\).

Consider \(X \to H\) as the embedding of \(X\) as the zero section. Denote \(M_{X,H} = \Theta_H(-\log X)|_X\). In local coordinates, \(M_{X,H}\) is a vector bundle spanned locally by the basis \(\{D_{\alpha}, \xi_\alpha D_{\xi_\alpha}\}\). Assume transition functions between two trivializations are given by

\[ z_\beta = z_\beta(z_\alpha), \xi_\beta = g_{\beta\alpha}(z_\alpha)\xi_\alpha. \]

Denote

\[ J_{\alpha\beta} = \left( \frac{\partial z_\beta}{\partial z_\alpha} \right)_{n \times n}, \quad \hat{D}_{\beta} = \left( D_{\beta}^1, \cdots, D_{\beta}^N \right)^t, \quad \hat{\partial}_\beta = \left( \partial_{\beta}^1, \cdots, \partial_{\beta}^N \right)^t. \]

Then by the chain rule, we get

\[
D_{\beta}^i = \frac{\partial z_\beta^i}{\partial z_\alpha^j} D_{\alpha}^j + \frac{\partial g_{\beta\alpha}}{\partial z_\alpha^j} \xi_\alpha D_{\xi_\alpha} = \left( J_{\alpha\beta} \hat{D}_{\beta} \right)_i - \sum_{\alpha} \left( (J_{\alpha\beta} \hat{\partial}_\beta) g_{\alpha\beta} \right) \xi_\alpha D_{\xi_\alpha}. \quad (54)
\]

\[ \xi_\alpha D_{\xi_\alpha} = \xi_\beta D_{\xi_\beta}. \quad (55) \]
The fact important to us is that we have the commutative diagram:

\[
\begin{array}{cccc}
0 & \rightarrow & \Omega_X & \rightarrow \mathcal{O}_X(H) \oplus (N + 1) & \rightarrow \Theta_{X}|_X & \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & \\
0 & \rightarrow & M_{X,H} & \rightarrow \mathcal{O}_X(H) \oplus (N + 1) & \rightarrow N_{X|_X} & \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & \\
\Theta_X & & 0 & & 0 & \\
\end{array}
\]

The morphisms are defined as follows.

\[\iota(z,1) = (z, -\xi_\alpha D_{\xi_\alpha}); \quad pr_1(D_{z_\alpha}) = \partial_{z_\alpha}; \quad pr_1(\xi_\alpha D_{\xi_\alpha}) = 0.\]

\[\phi_1((z;1)) = ((z;S_1, \ldots, S_{N+1})) = (a_\alpha|_{p=1}; (a_\alpha e_\alpha)|_{p=1}).\]

\[\psi_1([a_\alpha e_\alpha], (f^p e_\alpha)) = [a_\alpha^p \rightarrow f^p] = (C(a_\alpha^p))^* \otimes [(f^p)] \in \mathcal{O}_p(1) \otimes \left(C^{N+1}/\mathcal{O}_p(-1)\right).\]

\[\phi_2(D_{z_\alpha}) = \left(\partial_{z_\alpha} a_\alpha^p e_\alpha\right), \quad \phi_2(\xi_\alpha D_{\xi_\alpha}) = (-S_\rho = -a_\rho e_\alpha).\]

Note that the morphism \(\phi_2\) is well defined because:

\[\phi_2(D_{z_\alpha}) = \left(\partial_{z_\alpha} a_\alpha^p e_\alpha\right) = \frac{\partial z_\alpha^p}{\partial z_\alpha^p} \frac{\partial}{\partial z_\alpha^p} (g_{\alpha\beta} a_\beta^p e_\alpha) = \frac{\partial z_\alpha^p}{\partial z_\alpha^p} \frac{\partial a_\beta^p}{\partial z_\alpha^p} g_{\alpha\beta} e_\alpha + \frac{\partial z_\alpha^p}{\partial z_\alpha^p} \frac{\partial g_{\alpha\beta}}{\partial z_\alpha^p} a_\beta^p g_{\alpha\beta} e_\alpha = \left(J_{\alpha\beta}\frac{\partial}{\partial z_\alpha} (g_{\alpha\beta})\right) \phi_2(\xi_\alpha D_{\xi_\alpha}).\]

\(\psi_2\) is defined as the composite: \(\psi_2 = pr_2 \circ \psi_1\). To see that the diagram is commutating, it suffices to show that the following holds: \(\Theta_{pN}|_X \supset \text{Im}(\psi_1 \circ \phi_2(M_{X,H}) = \Theta_X\). This is clear by the definition:

\[\psi_1 \circ \phi_2(D_{z_\alpha}) = \psi_1([a_\alpha e_\alpha], (\partial_{z_\alpha} a_\alpha^p e_\alpha) = (C(a_\alpha^p))^* \otimes [\partial_{z_\alpha} a_\alpha^p] = (\text{emb}_X)_* \theta_{z_\alpha}.\]

**Remark 5.1.** The 1st vertical exact sequence in the diagram (56) is dual to the familiar jet bundle sequence:

\[0 \rightarrow \Omega^2_X \otimes H \rightarrow J_1(Z) \rightarrow \mathcal{O}_X(H) \rightarrow 0.\]

In other words, \(M_{X,H} = J_1(Z)^\vee \otimes H\).

### 5.2 Proof of Theorem 1.3

Fix an ample line bundle \(L \rightarrow D\). As abstract varieties we have,

\[\text{Spec} \left( \bigoplus_{m=0}^{+\infty} H^0(D, L^m) \right) = C(D, L) \subset \overline{C}(D, L) = \text{Proj} \left( \bigoplus_{m=0}^{+\infty} \bigoplus_{r=0}^m (H^0(D, L^r) \cdot \sum_{r=0}^m \cdot x_{m+1}^r) \right).\]

In the following, we will denote \(\overline{C} = \overline{C}(D, L), \quad L = \mathcal{O}_p(1) = [D]. \quad H^0(\overline{C}, L^m)\) has a basis:

\[S_{m+1}^1, S_{m-1}^1 \cdot x_{m+1}, S_{m-2}^2 \cdot x_{m+1}, \ldots, S_{m-n}^n \cdot x_{m+1}, \ldots, S_m^m \cdot x_{m+1}, \ldots, x_{m+1} \cdot x_{m+1}.\]

where \(\{S_m^m, 1 \leq m \leq N_m\}\) form a basis of \(H^0(D, L^m)\). When \(m\) is sufficiently big, the basis (57) induces an embedding into the projective space \(\mathbb{P}^N\), where we denote \(N_m = N_1 + \cdots + N_m\). We will denote \(H = mL\) the hyperplane bundle of this embedding. In this embedding, \(C(D, L)\) lies in the affine space

\[\mathbb{C}^N = \{x_{m+1}^r \neq 0\}.
\]
In fact, by choosing $m$ sufficiently large, we can assume that the sections $\{S^{(m)}, S^{(m-1)}, \ldots, S^{(1)}\}$ generate the section ring $R(D, L)$. Now we apply the construction in section 5.1 to both $X = L$, which is the total space of the line bundle $\pi_L : L \to D$, and $U = L - D$. To study $M_{X, H}$, we consider transition functions:

$$z_\beta = z_\beta(z_\alpha), t_\beta = f_{\alpha \beta}(z_\alpha) t_\alpha, \xi_\beta = f_{\alpha \beta}^m(z_\alpha) \xi_\alpha.$$ 

The Jacobian matrix from $\{z_\alpha, t_\alpha\}$ to $\{z_\beta, t_\beta\}$ has shape:

$$\frac{\partial (z_\beta, t_\beta)}{\partial (z_\alpha, t_\alpha)} = \left( \begin{array}{cc} \frac{\partial z_\beta}{\partial z_\alpha} & \frac{\partial f_{\alpha \beta}(z_\alpha)}{\partial t_\alpha} \\ 0 & \frac{f_{\alpha \beta}(z_\alpha)}{f_{\alpha \beta}(z_\alpha)} \end{array} \right) = \left( \begin{array}{cc} J_{\alpha \beta} & -f_{\alpha \beta}^{-1}(\partial_{\alpha \beta}) f_{\alpha \beta} t_\beta \\ 0 & f_{\beta \alpha} \end{array} \right).$$

So it's easy to verify that the formulas (54) and (55) have the following form now:

$$D_{z_\alpha} = (J_{\alpha \beta} D_\beta) t_\beta - f_{\alpha \beta}^{-1} \left[(J_{\alpha \beta} \partial_{\alpha \beta}) f_{\alpha \beta}\right] t_\beta D_{t_\beta} - f_{\alpha \beta}^{-1} \left[(J_{\alpha \beta} \partial_{\alpha \beta}) f_{\alpha \beta}\right] \xi_\beta D_{\xi_\beta},$$

$$D_{t_\alpha} = f_{\beta \alpha} D_{t_\beta}, \quad \xi_\alpha D_{\xi_\alpha} = \xi_\beta D_{\xi_\beta}.$$ 

Note that the sub-basis $\{D_{z_\alpha} : \xi_\alpha D_{\xi_\alpha}\}$ form a basis of $\pi_L^* M_{D, H}$. It's an easy exercise to verify that following:

**Lemma 5.1.** We have the following commutating exact diagram:

$$\xymatrix{ \mathcal{Q}_L \ar[r] & \pi_L^* \mathcal{O}_D \\
0 \ar[r] & \pi_L^* L \ar[r] & M_{L, H} \ar[r] & \pi_L^* M_{D, H} \ar[r] & 0 \ar[d] \\
0 \ar[r] & \pi_L^* L \ar[r] & \Theta_L \ar[r] & \pi_L^* \Theta_D \ar[r] & 0 \\
}$$

(58)

**Corollary 5.1.** We have the following exact sequence:

$$H^{i-1}(L, \pi_L^* M_{D, H}) \to H^i(L, \pi_L^* L) \to H^i(L, M_{L, H}) \to H^i(L, \pi_L^* M_{D, H}) \to H^{i+1}(L, \pi_L^* L),$$

(59)

where the vertical morphisms are given by restrictions.

By projection formula, we know that for any coherent sheaf $F$ on $D$, the following identities hold:

$$H^i(L, \pi^* F) = \bigoplus_{j=0}^{+\infty} H^i(D, F \otimes L^{-j}), \quad H^i(U, \pi^* F) = \bigoplus_{j=-\infty}^{+\infty} H^i(D, F \otimes L^{-j}).$$

(60)

So when $j \geq 0$, we can extract the weight $(-j)$-part of the exact sequence (59) to get:

**Corollary 5.2.** We have the commutating exact sequences:

$$H^{i-1}(D, M_{D, H} \otimes L^{-j}) \to H^i(D, L^{-j}) \to H^i(D, M_{L, H}(L^{-j}) \to H^i(D, M_{D, H} \otimes L^{-j}) \to H^{i+1}(D, L^{-j})$$

(61)

As a consequence of this and the 5-Lemma, we have the isomorphism:

$$H^i(L, M_{L, H})(-j) \cong H^i(U, M_{L, H}|_U)(-j) \quad \text{for } j \geq 0.$$ 

(62)

On the other hand, from the 2nd horizontal exact sequence in the diagram (56), we have the following exact sequence:

$$H^i(L, M_{L, H}) \to H^i(L, \pi_L^* H)^{\oplus (\bar{N}_m + 1)} \to H^i(L, N_L) \to H^{i+1}(L, M_{L, H}) \to H^{i+1}(L, \pi_L^* H)^{\oplus (\bar{N}_m + 1)}$$

$$H^i(U, M_{L, H}|_U) \to H^i(U, \pi_U^* H)^{\oplus (\bar{N}_m + 1)} \to H^i(U, N_U) \to H^{i+1}(U, M_{L, H}|_U) \to H^{i+1}(U, \pi_U^* H)^{\oplus (\bar{N}_m + 1)}.$$ 

(63)
There were many beautiful works on this subject. Tian-Yau [27] (see also Bando-Kobayashi [5]) proved that a Ricci-flat complete manifold with maximal volume growth and a diffeomorphism \( g \) to \( \text{AC} \) of order \( \eta \) is both Kähler and Ricci-flat. If this is the case, \( \| \nabla_{g_0} (\phi^*_{X} (g_0) - g_0) \|_{C^0} \leq C r^{-\lambda - j} \) for \( j \geq 0 \).

Using (60), we can again extract the weight \((-j)\)-part:

\[
\begin{align*}
H^j(L,M,L,H)(\cdot,\cdot,\cdot)(-j) & \rightarrow H^j(D,L,m^{-j})(\cdot,\cdot,\cdot)(-j) \rightarrow H^j(L,M,L,H)(\cdot,\cdot,\cdot)(-j) \\
& \rightarrow H^j(L,M,L,H)(\cdot,\cdot,\cdot)(-j) \\
& \rightarrow H^j(D,L,m^{-j})(\cdot,\cdot,\cdot)(-j).
\end{align*}
\]

Using 5-Lemma and (62), we see that for \( j \geq 0 \),

\[
H^j(L,M,L,H)(\cdot,\cdot,\cdot)(-j) \cong H^j(U,\mathcal{N}_U)(-j).
\]

Now because the sheaf \( \mathcal{N}_\Theta \) is reflexive, so by [26, Lemma 1] it has depth \( \geq 2 \) at the vertex \( \Theta \) so we have

\[
H^0(\Theta,\mathcal{N}_\Theta) = H^0(L,M,L), \quad \text{and } H^1(\Theta,\mathcal{N}_\Theta) \hookrightarrow H^1(L,M,L) \quad \text{for } j \geq 0.
\]

Combing (65) and (66),

\[
H^0(\Theta,\mathcal{N}_\Theta)(-j) \cong H^0(U,\mathcal{N}_U)(-j), \quad \text{and } H^1(\Theta,\mathcal{N}_\Theta)(-j) \hookrightarrow H^1(U,\mathcal{N}_U)(-j) \quad \text{for } j \geq 0.
\]

So we see that the map \( H^0(\Theta,\mathcal{N}_\Theta)(-j) \rightarrow T^1_{\mathcal{C}}(-j) \) is surjective. Since this map factors through \( T^1_{\mathcal{L}(m)} \rightarrow T^1_{\mathcal{L}}, \) the latter map is also surjective. We illustrate these arguments by the following commutative diagram, in which we assume \( j \geq 0 \).

\[
\begin{array}{ccc}
H^0(L,M,L)(-j) & \rightarrow & H^1(L,\Theta)(-j) \\
\downarrow \cong & & \downarrow \cong \\
H^0(\Theta,\mathcal{N}_\Theta)(-j) & \rightarrow & T^1_{\mathcal{C}}(-j) \\
\downarrow & & \downarrow \\
H^0(U,\mathcal{N}_U)(-j) & \rightarrow & H^1(U,\Theta)(-j) \\
\downarrow & & \downarrow \\
T^1_{\mathcal{L}}(-j) & & T^1_{\mathcal{L}}(-j)
\end{array}
\]

We conclude this section by the following remark. Using the same argument as in [24] (see also [3, Section 23]), we can show the following generalization of Pinkham’s result.

**Proposition 5.1.** Assume \( \mathcal{C} \) is embedded into \( \mathbb{P}^N \) as above. Suppose \( T^1_{\mathcal{L}} \) is negatively graded, i.e. \( T^1_{\mathcal{L}(m)} = 0 \) for \( m > 0 \). Then the functor:

\[
\text{Hilb}(\mathcal{C}) \rightarrow \text{Def}(\mathcal{L})
\]

is formally smooth.

6 Appendixes

6.1 Appendix I: AC Calabi-Yau metric of Tian-Yau

Let \((M,g)\) be a non-compact complete Riemannian manifold. \((M,g)\) will be called asymptotically conical (AC) of order \( \eta \) if there exists a metric cone \((C(Y),g)\) with the cone metric \( g_0 = dt^2 + r^2 g_Y \) and a diffeomorphism \( \phi_K : C(Y) \setminus B_R(\rho) \rightarrow M \setminus K \) such that

\[
\| \nabla_{g_0} (\phi^*_{K}(g_0) - g_0) \|_{C^0} \leq \rho^e r^{-\lambda - j} \quad \text{for } j \geq 0.
\]

Here \( K \) is a compact set in \( M \) and \( B_R(\rho) \) is the ball of radius \( R \) around the vertex \( \rho \) of the metric cone. Cheeger-Tian [8] proved that a Ricci-flat complete manifold with maximal volume growth and satisfying suitable integral bounds on curvature tensors is indeed an asymptotically conical Ricci-flat manifold. We will be interested in the case when \( g \) is both Kähler and Ricci-flat. If this is the case, we denote \( \omega_g \), or simply \( \omega \), the Kähler form of \( g \) and will call \( g \) or \( \omega \) AC Ricci-flat Calabi-Yau metric. There were many beautiful works on this subject. Tian-Yau [27] (see also Bando-Kobayashi [5])
constructed a class of such AC Calabi-Yau manifold. This work is generalized later by van Coevering [29] in a series of papers, and also refined and clarified by Conlon-Hein ([10], [11]) in detail.

A natural question is to determine the optimal order of such AC Calabi-Yau metric. This issue was studied in detail in Cheeger-Tian [8] and in Conlon-Hein ([10], [11]). The fact important to us is that this optimal rate is related to the rate of convergence of complex structures (see [8, Section 7]). More precisely, we would like to determine the largest number $\lambda_1 > 0$ such that there exists a diffeomorphism $\phi_K$ as above and the following inequality holds:

$$\|\nabla^j_{g_0}(\phi_K^*J - J_0)\|_{g_0} \leq C r^{-\lambda_1-j}$$

for all $j \geq 0$.

Now assume that $X$ is a Fano manifold, i.e. $-K_X$ is an ample line bundle. Assume $D$ is a smooth divisor such that $\alpha D \sim K_X^{-1}$ is a smooth divisor with $\mathbb{Q} \ni \alpha > 1$. By adjunction formula, we get $-K_D = -K_X - [D] = (1 - \alpha^{-1})K_X^{-1}$ is still ample, and so $D$ is also a Fano manifold. We also assume that $D$ has a Kähler-Einstein metric, then Tian-Yau [27] constructed an AC Calabi-Yau metric on $M = X\setminus D$.

**Theorem 6.1** ([27]). Under the above assumptions, $X\setminus D$ admits a complete Ricci-flat Kähler metric $g$ with Euclidean volume growth. Further more, if we denote $R(g)$ the curvature tensor of $g$ and by $\rho(\cdot)$ the distance function on $X\setminus D$ from some fixed point with respect to $g$, then $R(g)$ decays at the order of exactly $\rho^{-2}$ with respect to the $g$-norm.

This construction can be viewed as a generalization of the basic example of Eguchi-Hanson metric $\omega_{EH}$ in which case we have:

$$X = \mathbb{P}^1 \times \mathbb{P}^1, D = \Delta(\mathbb{P}^1) \cong \mathbb{P}^1,$$

where $\Delta : \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ is the diagonal embedding. Note that $M = \mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta(\mathbb{P}^1)$ is isomorphic to the deformed conifold $\{z_1^2 + z_2^2 + z_3^2 = 1\} \subset \mathbb{C}^3$ which via the hyperKähler rotation becomes the ALE metric on the crepant resolution of $\mathbb{C}^3/\mathbb{Z}_2$.

**Remark 6.1.** The assumptions for the existence to hold can be weakened to the following items: 1. $X$ is a Kähler manifold; 2. $-K_X = \alpha D$ with $\mathbb{Q} \ni \alpha > 1$; 3. Either “almost ample” in the sense of Tian-Yau in [27], or $N_D = \alpha^{-1}K_X^{-1}|_D$ is ample. 4. $D$ has Kähler-Einstein metric. For these technical details, see the nice explanation in [11].

The tangent cone at infinity of this Tian-Yau metric is the conical Calabi-Yau metric on $C(D, N_D)$ discussed in Section 3.1. Since the normal bundle is an approximation of $X$ in a small neighborhood of $D$, the idea is to prescribe the metric on $X\setminus D$ to be asymptotically equivalent to $\omega_0$ near $D$ and solve the Monge-Ampère equation for solutions with this behavior at infinity. Conlon-Hein [11] refined Tian-Yau's construction and studied the asymptotical order of the AC Ricci-flat Kähler metrics.

**Theorem 6.2** ([11]). In each Kähler class of $X\setminus D$ and for every $c > 0$, there exists a unique AC Calabi-Yau metric $\omega_c$ on $X\setminus D$ satisfying

$$\exp^*(\omega_c) - c\omega_0 = O(r^{-\min(2^{-c}, \frac{\alpha}{\alpha - 1})})$$

with $g_0$ derivatives, for any $\delta > 0$.

Here, $\exp : N_D \to X$ denotes the restriction to the normal bundle $N_D$ of $D$ in $X$ of the exponential map of any background Kähler metric on $X$, $g_0$ denotes the pullback to $N_D\setminus\{0\}$ of Calabi ansatz Ricci-flat Kähler cone metric on $-K_D$.

To obtain this, they used the following general existence and regularity result from their first paper [10].

**Theorem 6.3** ([10]). Let $M$ be an open complex manifold of complex dimension $n \geq 3$ such that $K_M$ is trivial. Let $\Omega$ be a holomorphic volume form on $M$ and let $L$ be Sasaki-Einstein with associated Calabi-Yau cone $(C, \Omega_0, \omega_0)$ and radius function $r$. Suppose that there exists $\lambda_1 < 0$, a compact subset $K \subset M$, and a diffeomorphism $F_K : (1, \infty) \times L \to M\setminus K$ such that

$$F_K^*\Omega - \Omega_0 = O(r^{-\lambda_1})$$

with $g_0$ derivatives, where $g_0$ is the Kähler metric associated to $\omega_0$. Let $\mu < 0$ and assume $\nu := \max\{\lambda_1, \mu\} \notin \{-2n, -2, \nu_0 - 2\}$, where $\nu_0 \geq 1$ denotes the smallest growth rate of a pluriharmonic function on $C$. Then for every $c > 0$, there exists, in each $\mu$-almost compactly supported Kähler class, a unique AC Calabi-Yau metric $\omega_c$ satisfying

$$F^*\omega_c - c\omega_0 = O\left(r^{\max\{-2n, \mu\}}\right)$$

with $g_0$ derivatives.
Remark 6.2. These results are summarized in the estimate (1). Again the important thing for
us is that the estimate of asymptotical rate of convergence depends on the construction of some
diffeomorphism. For example, to get Theorem 6.2, Colon-Hein [11] just used smooth exponential
map with respect to any smooth Kähler metric on X as the comparing diffeomorphism F_X, which
can be seen as a first order approximation. In Proposition 1.2 and Corollary 1.1 we get better rate
(essentially optimal one) by constructing diffeomorphisms which are more adapted to the embedding
of D inside X. Also for special examples in [10], Conlon-Hein constructed the diffeomorphisms in
a somehow ad hoc way. Proposition 1.2 and Proposition 1.1 together provide a uniform algebraic
interpretation and generalization of their constructions.

6.2 Appendix II: Neighborhoods of complex submanifold

Recall from the introduction, we have defined:

Definition 6.1. S is k-linearizable if its k-th infinitesimal neighbourhood (S(k), O_X/I_S^{k+1}) in X is
isomorphic to its k-th infinitesimal neighbourhood (S_N(k), O_N/I_N^{k+1}) in N_S. Here we identify S with
the zero section S_0 of N_S := N.

Definition 6.2 ([1, Definition 2.1.2.2]). 1. S is k-splitting into X (for some k ≥ 1) if the exact
sequence

\[ 0 \rightarrow I_S/I_S^{k+1} \rightarrow O_X/I_S^{k+1} \rightarrow O_S \rightarrow 0 \]

split as sequence of sheaves of rings.

2. A k-splitting atlas for S ⊂ X is an atlas \{(V_α, z_α)\} of X adapted to S (that is, V_α ∩ S ≠ ∅ implies
V_α ∩ S = \{z_i = \cdots = z_m = 0\}) such that

\[ \frac{\partial^k z_{i_1} \cdots z_{i_k}}{\partial z_{i_1} \cdots z_{i_k}} \bigg|_S = 0, \]

for all r_1, ..., r_k = 1, ..., m, all p = m + 1, ..., n, and all indices α, β such that V_α ∩ V_β ∩ S ≠ ∅.

In the following, if S is k-splitting, we will always fix a lifting: \( ρ_k : O_S \rightarrow O_X/I_S^{k+1} \). We also
denote \( ϕ_{h,k} \) the natural map

\[ ϕ_{h,k} : O_X/I_S^{h+1} \rightarrow O/I_S^{k+1}, \text{ for } h ≥ k. \]

Proposition 6.1 ([1, Proposition 2.2]). Assume that S is (k−1)-splitting in X; let \( ρ_{k−1} : O_S \rightarrow O_X/I_S^{k+1} \) be a (k−1)-th order lifting, and \( \mathfrak{G} = \{ (V_α, ϕ_{h,α}) \} \) a (k−1)-splitting atlas adapted to \( ρ_{k−1} \). Let \( g_k \in H^1(S, Hom(Ω_S, I_S^{k}/I_S^{k+1})) \) be the cohomology class represented by a 1-cocycle \( \{(g_k)_{βα}\} \in H^1(Ω_S, Hom(Ω_S, I_S^{k}/I_S^{k+1})) \) given by

\[ (g_k)_{βα} = -\frac{1}{k!} \left. \frac{\partial^k z_{i_1} \cdots z_{i_k}}{\partial z_{i_1} \cdots z_{i_k}} \right|_S \left[ z_{i_1} \cdots z_{i_k} \right]_{k+1} ∈ H^0(V_α ∩ V_β ∩ S, Ω_S ⊗ I_S^k/I_S^{k+1}). \]  

(68)

Then there exists a k-th order lifting \( ρ_k : O_S \rightarrow O_X/I_S^{k+1} \) such that \( ρ_{k−1} = ϕ_{k,k−1} ∘ ρ_k \) if and only if \( g_k = 0 \).

Proposition 6.2 ([1, Proposition 3.2]). Assume S is k-splitting in X and let \( ρ : O_S \rightarrow O_X/I_S^{k+1} \) be a k-th order lifting, with k ≥ 0. Then for any \( 1 ≤ h ≤ k + 1 \), the lifting \( ρ \) induces a structure of locally \( O_S \)-free module on \( I_S^h/I_S^{h+1} \) for \( 1 ≤ h ≤ k + 1 \) in such a way that the sequences

\[ 0 \rightarrow I_S^h/I_S^{h+1} \rightarrow I_S/I_S^{h+1} \rightarrow I_S/I_S^h \rightarrow 0 \]  

(69)

becomes exact sequences of \( O_S \)-free modules.

Definition 6.3 ([1, Definition 3.1, 3.2]). 1. If S is k-splitting in X and the sequence (69) split
for \( 1 ≤ h ≤ k + 1 \), S is called to be k-comfortably embedded in X. Denote \( ν_{h−1,k} : I_S/I_S^{h+1} \rightarrow I_S/I_S^h \) the splitting \( O_S \)-morphism of the sequence (69) and the comfortable splitting sequence

\[ ν_k = (ν_{0,1}, ..., ν_{h,k+1}). \]
2. A $k$-comfortable atlas is an atlas $\{(V_\alpha, z_\alpha)\}$ adapted to $S$ such that

$$\frac{\partial^k z_\alpha^p}{\partial z_{\alpha_1}^{s_1} \cdots \partial z_{\alpha_k}^{s_k}} \bigg|_S = 0,$$

for all $r_1, \ldots, r_k = 1, \ldots, m$, all $p = m+1, \ldots, n$, and all indices $\alpha, \beta$ such that $V_\alpha \cap V_\beta \cap S \neq \emptyset$.

**Remark 6.3.** Any submanifold $S$ is always $0$-comfortably embedded. If $S$ is $k$-comfortably embedded, then $S$ is also $k$-splitting.

**Theorem 6.4** ([1, Corollary 3.6]). Assume there exists a $k$-th order lifting $\rho_k : \mathcal{O}_S \to \mathcal{O}_X/T_{S}^{k+1}$ such that $S$ is $(k-1)$-comfortably embedded in $X$ with respect to $\rho_{k-1} = \phi_{k,k-1} \circ \rho_k$. Fix a $(k-1)$-comfortable pair $(\rho_{k-1}, \nu_{k-1})$, and let $\mathcal{V} = \{(V_\alpha, z_\alpha)\}$ be a projectable atlas adapted to $\rho_k$ and $(\rho_{k-1}, \nu_{k-1})$. Then the cohomology class $h_{\rho_k}$ associated to the exact sequence (69) is represented by $1$-cocycle $\{h_{\rho_k}^{\alpha} \in H^1(\mathcal{V}_{\alpha}, \mathcal{N}_S \otimes \mathcal{T}_{S}^{k+1}/\mathcal{T}_{S}^{k+2})\}$ given by

$$h_{\rho_k}^{\alpha} = \frac{1}{(k+1)!} \frac{\partial z_{\alpha,0}^{r_1}}{\partial z_{\alpha_1}^{s_1} \cdots \partial z_{\alpha_k}^{s_k}} \bigg|_{S} \frac{\partial^{k+1} z_\alpha^p}{\partial z_{\alpha_1}^{s_1} \cdots \partial z_{\alpha_k}^{s_k}} \bigg|_{S}.$$

**Remark 6.4.** If $D$ is a divisor, then the obstruction to $k$-comfortably embeddedness lies in $H^1(D, N_D \otimes \mathcal{T}_{S}^{k+1}/\mathcal{T}_{S}^{k+2}) = H^1(D, (N_D)^{-k+1})$. If we assume the normal bundle $N_D$ is ample on $D$ and $n-1 = \dim D \geq 2$, then the Kodaira-Nakano vanishing theorem gives $H^1(D, (N_D)^{-k}) = 0$ for any $k \geq 1$. So in this case, there is no obstruction to passing from $(k-1)$-comfortably embedding to $k$-comfortably embedding (assuming $k$-splitting). Note that $D$ is always $0$-comfortably embedded. We get that, if $N_D$ is ample on $D$ and $\dim X \geq 3$, then $D$ is $k$-comfortably embedded, if and only if $D$ is $k$-splitting, and if and only if $D$ is $k$-linearizable (see Theorem 6.6). This is a simple but important remark for us.

**Theorem 6.5** ([1, Theorem 2.1, Theorem 3.5]). $S$ is $k$-splitting in $X$ if and only if there is a $k$-splitting atlas $\mathcal{V} = \{(V_\alpha, z_\alpha)\}$ of $X$, that is an atlas adapted to $S$ such that

$$\begin{align*}
\frac{\partial z_{\alpha}^p}{\partial z_{\alpha_1}^{s_1} \cdots \partial z_{\alpha_k}^{s_k}} \bigg|_S &= \sum_{r=1}^m (a_{\alpha,0})^r(z_\alpha)z_{\alpha}^r, \\
\frac{\partial^k z_{\alpha}^p}{\partial z_{\alpha_1}^{s_1} \cdots \partial z_{\alpha_k}^{s_k}} \bigg|_S &= \phi_{\alpha,0}^p(z_\alpha) + R^p_{k+1},
\end{align*}$$

where $z_\alpha^r = (z_\alpha^{r_1}, \ldots, z_\alpha^{a_n})$ are local coordinates on $S$, and $R^p_{k+1}$ denotes a term belong to $\mathcal{T}_{S}^{k+1}$. Furthermore, $S$ is $k$-comfortably embedded in $X$ if and only if there is a $k$-comfortable atlas $\mathcal{V} = \{(V_\alpha, z_\alpha)\}$, that is an atlas adapted to $S$ such that

$$\begin{align*}
\frac{\partial z_{\alpha}^p}{\partial z_{\alpha_1}^{s_1} \cdots \partial z_{\alpha_k}^{s_k}} \bigg|_S &= \sum_{r=1}^m (a_{\alpha,0})^r(z_\alpha)z_{\alpha}^r + R^r_{k+2}, \\
\frac{\partial^k z_{\alpha}^p}{\partial z_{\alpha_1}^{s_1} \cdots \partial z_{\alpha_k}^{s_k}} \bigg|_S &= \phi_{\alpha,0}^p(z_\alpha) + R^p_{k+1},
\end{align*}$$

where $R^r_{k+2} \in \mathcal{T}_{S}^{k+2}$ and $R^p_{k+1} \in \mathcal{T}_{S}^{k+1}$.

**Theorem 6.6** ([1, Theorem 4.1]). $S$ is $k$-linearizable if and only if $S$ is $k$-splitting into $X$ and $(k-1)$-comfortably embedded, if and only if there is an atlas $\mathcal{V}$ such that the changes of coordinates are of the form:

$$\begin{align*}
\frac{\partial z_{\alpha}^p}{\partial z_{\alpha_1}^{s_1} \cdots \partial z_{\alpha_k}^{s_k}} \bigg|_S &= \sum_{r=1}^m (a_{\alpha,0})^r(z_\alpha)z_{\alpha}^r + R^r_{k+1}, \\
\frac{\partial^k z_{\alpha}^p}{\partial z_{\alpha_1}^{s_1} \cdots \partial z_{\alpha_k}^{s_k}} \bigg|_S &= \phi_{\alpha,0}^p(z_\alpha) + R^p_{k+1},
\end{align*}$$

where $R^r_{k+1}, R^p_{k+1} \in \mathcal{T}_{S}^{k+1}$.

### 6.3 Appendix III: Kodaira-Spencer’s deformation theory

In this appendix, we recall the construction of Kodaira-Spencer class for a differentiable family using via the variation of holomorphic transition functions. Suppose we have differentiable family $X \to D$.

Suppose we have a collection of coordinate charts $\mathcal{U} = \{U_\alpha, \{z_{\alpha,0}^r(t)\}\}$ such that $\{U_\alpha\}$ is a covering of the neighborhood of $x_0$ and on each $U_\alpha$, $x_0 \cap U_\alpha = \{t = 0\}$. For simplicity, we can assume each $U_\alpha$ is polydisk $D^n$. We can write down the transition function:

$$z_{\alpha}^r = f_{\alpha,0}^r(z_{\alpha,0}^r, t), \quad t|_{U_\alpha} = t|_{U_\beta}.$$
The Kodaira-Spencer class is defined as follows:

\[ f^{i}_{\alpha \beta}(f_{\gamma}(z_{\gamma}, t), t) = f^{i}_{\alpha \gamma}(z_{\gamma}, t) \Rightarrow \frac{\partial f_{\alpha \beta}(z_{\beta}, t)}{\partial z_{\beta}} \frac{\partial f^{i}_{\gamma}(z_{\gamma}, t)}{\partial t} + \frac{\partial f_{\alpha \beta}(z_{\beta}, t)}{\partial t} \bigg|_{t=0} = \frac{\partial f_{\alpha \gamma}(z_{\gamma}, t)}{\partial t} \bigg|_{t=0} \]

\[ \Rightarrow \theta_{\beta \gamma} = \theta_{\alpha \gamma} - \theta_{\alpha \beta}, \quad \theta_{\beta \gamma} = \sum_{i=1}^{n} \frac{\partial f_{\alpha \beta}(z_{\beta}, t)}{\partial t} \frac{\partial}{\partial z_{\beta}}. \]

So we have a cocycle \( \{\theta_{\alpha \beta}\} \in H^{1}(|U_{\alpha}|, \Theta_{X_{0}}) \) where \( U_{\alpha} = U_{\alpha} \cap X_{0} \). Now assume that \( \{\rho_{\alpha}\} \) is a partition of unity for the covering \( \{U_{\alpha}\} \). Then we can define

\[ \xi_{\alpha} = \sum_{i=1}^{n} \sum_{\gamma} \rho_{\gamma} \frac{\partial f^{i}_{\alpha}(z_{\gamma}, t)}{\partial t} \bigg|_{t=0} \frac{\partial}{\partial z_{\alpha}}. \]

It’s easy to verify that \( \theta_{\alpha \beta} = \xi_{\beta} - \xi_{\alpha} \), so that \( KS_{X} = \bar{\partial} \xi_{\alpha} = \bar{\partial} \xi_{\beta} \) is a globally defined \( \Theta_{X_{0}} \)-valued closed \((0,1)\)-form. \( KS_{X} \) represents the Kodaira-Spencer class of the deformation given by \( X \). On the other hand, we have, by the chain rule

\[ \left( \frac{\partial}{\partial t} \right)^{i}_{\beta} = \sum_{i=1}^{n} \sum_{\gamma} \rho_{\gamma} \frac{\partial f^{i}_{\alpha \beta}(z_{\beta}, t)}{\partial t} \frac{\partial}{\partial z_{\alpha}} + \left( \frac{\partial}{\partial t} \right)^{i}_{\alpha}, \quad \frac{\partial}{\partial z^{i}_{\alpha}} = \sum_{i=1}^{n} \sum_{\beta} \rho_{\beta} \frac{\partial f^{i}_{\alpha \beta}(z_{\beta}, t)}{\partial t} \frac{\partial}{\partial z^{i}_{\beta}}. \]

We define the differentiable vector field locally by:

\[ V = \sum_{\gamma} \rho_{\beta} \left( \frac{\partial}{\partial t} \right)^{i}_{\beta} = \sum_{\beta} \rho_{\beta} \sum_{i=1}^{n} \frac{\partial f^{i}_{\alpha \beta}(z_{\beta}, t)}{\partial t} \frac{\partial}{\partial z_{\alpha}} + \left( \frac{\partial}{\partial t} \right)^{i}_{\alpha} \]

\[ = \sum_{i=1}^{n} \left( \sum_{\beta} \rho_{\beta} \frac{\partial f^{i}_{\alpha \beta}(z_{\beta}, t)}{\partial t} \right) \frac{\partial}{\partial z^{i}_{\alpha}} + \left( \frac{\partial}{\partial t} \right)^{i}_{\alpha}. \]

Then \( V \) is a globally defined vector field in the neighborhood of \( X_{0} \). Let \( \sigma(t) \) be the flow associated with \( V \) which exists for sufficiently small \( t \). We have the identity:

\[ \frac{d}{dt} (\sigma(t)^{*} J) = (L_{V} J)(\sigma(t)) d\bar{z} = \partial \bar{V}. \]

Note that \( \partial \bar{V}|_{t=0} = \partial \bar{\xi}_{\alpha} = KS_{X}. \)

### 6.4 Appendix IV: Deformation of complex cones

Here we recall Schlessinger’s work in \([25],[26]\) on the deformation of normal isolated singularities. Assume \( D \) is a projective Kähler manifold with a positive line bundle \( L \rightarrow D \). Consider the affine cone:

\[ C = \text{Spec}^{+\infty}_{k=0} H^{0}(D, L^{\otimes k}). \]

\( C \) has a normal isolated singularity at the vertex \( o \).

**Proposition 6.3** \([25],[26]\). Assume \( C \) is embedded into \( C^{N} \). We have an exact sequence:

\[ H^{0}(U, \Theta_{C^{N}}|U) \rightarrow H^{0}(U, N_{U}) \rightarrow T^{1}_{C} \rightarrow 0 \] (70)

\[ 0 \rightarrow T^{1}_{C} \rightarrow H^{1}(U, \Theta_{U}) \rightarrow H^{1}(U, \Theta_{C^{N}}|U) \] (71)

**Proof.** Then we have the conormal exact sequence:

\[ I_{C}/I_{C}^{2} \rightarrow \Omega_{C^{N}}|_{C} \rightarrow \Omega_{C} \rightarrow 0, \]

whose dual defines the sheaf \( T^{1} \):

\[ 0 \rightarrow \Theta_{C} \rightarrow \Theta_{C^{N}}|_{C} \rightarrow N_{C} \rightarrow T^{1} \rightarrow 0. \]
In particular, the first three sheaves are reflexive. Because $C$ is affine, by definition we get the exact sequence:

$$0 \to H^0(C, \Theta_C) \to H^0(C, \Theta_{C\mid C}) \to H^0(C, N_C) \to T^1_C \to 0. \quad (72)$$

Because $C$ is normal, by Serre’s criterion for normality, $C_D$ has depth depth $C \geq 2$ at its vertex. Because the first three sheaves are reflexive, by [26, Lemma 1], the depth of each is $\geq 2$. So in (72) we can replace $H^0(X, \cdot)$ by $H^0(U, \cdot)$ to get:

$$0 \to H^0(U, \Theta_U) \to H^0(U, \Theta_{C\mid U}) \to H^0(U, N_U) \to T^1_C \to 0, \quad (73)$$

On the other hand, we have

$$0 \to \Theta_U \to \Theta_{C\mid U} \to N_U \to 0,$$

which gives us the exact sequence:

$$0 \to H^0(U, \Theta_U) \to H^0(U, \Theta_{C\mid U}) \to H^1(U, \Theta_U) \to H^1(U, \Theta_{C\mid U}). \quad (74)$$

Combining (73) and (74), we get (70) and (71).

As an example of the above general theory, consider a projective manifold $D \subset P^{N-1}$. We assume that $D$ is projectively normal in $P^{N-1}$ so that the affine cone over $D$ is normal and is equal to $C_D = C(D, H)$ where $H$ is the hyperplane bundle of $P^{N-1}$. Then it’s easy to verify that (see [25], [3]):

$$H^0(U, \Theta_{C\mid U}) = \sum_{j=-\infty}^{+\infty} H^0(D, \mathcal{O}_D(j+1)), \quad H^0(U, N_U) = \sum_{j=-\infty}^{+\infty} H^0(D, N_D(j)).$$

Decompose $T^1_C = \sum_{j=-\infty}^{+\infty} T^1_C(j)$ into weight spaces. Then by (70) we have the exact sequence:

$$H^0(D, \mathcal{O}_D(j+1))^N \xrightarrow{\text{Jac}} H^0(D, N_D(j)) \to T^1_C(j) \to 0. \quad (75)$$

Example 6.1 (cf. [3, Section 4]). Assume $D^{n-1} \subset P^{N-1}$ is a complete intersection

$$D = \bigcap_{i=1}^{N-n} \{F_i = 0\} \subset P^{N-1},$$

where $F_i$ is a homogeneous polynomial of degree $d_i$. We assume $\{Z_1, \ldots, Z_N\}$ are homogeneous coordinates of $P^{N-1}$ and denote

$$R(D, H) = \bigoplus_{m=0}^{+\infty} H^m(D, mH) \cong \mathbb{C}[Z_1, \ldots, Z_N]/\langle F_1, \ldots, F_{N-n} \rangle.$$

Note that this is nothing but the affine coordinate ring of $C(D, H)$. Then

$$H^0(D, \mathcal{O}_D(j+1)) = H^0(D, (j+1)H) = R(D, H)(j+1);$$

$$H^0(D, N_D(j)) = \bigoplus_{i=1}^{N-n} H^0(D, (d_i + j)H) = \bigoplus_{i=1}^{N-n} R(D, H)(d_i + j).$$

The map

$$\text{Jac} : R(D, H)(j+1)^N \to \bigoplus_{i=1}^{N-n} R(D, H)(d_i + j)$$

is given by the Jacobian matrix $\left( \partial F_k/\partial Z^l \right)_{k=1, \ldots, N-n, i=1}^{i=1, \ldots, N}$, with the quotient:

$$T^1_C(j) = \bigoplus_{i=1}^{N-n} R(D, H)(d_i + j) \big/ \text{Jac}(R(D, H)(j+1)^N),$$

Now assume $G = \{g_i = g_i(z_1, \ldots, z_N), i = 1, \ldots, N-n \}$ consists of (not necessarily homogeneous) polynomials. We can consider the deformation of $C(D, H) \subset \mathbb{C}^N$ given by:

$$C_t = \bigcap_{i=1}^{N-n} \{F_i(z_1, \ldots, z_N) + t g_i = 0\} \subset \mathbb{C}^N.$$
If we assume image $[G]$ in $T^0_C$ is not zero, then by (75), we see that the weight of this deformation is the weight of $[G]$. Note that the polynomials in the image of Jac have degree $\geq d_i - 1$. So if $g_i$ is of degree $e_i \leq d_i - 2$, it’s easy to see that the $[G]$ is indeed not zero and the weight is equal to $\max\{e_i - d_i\} = -\min\{d_i - e_i\}$.

**Remark 6.5.** The reason that we assume the non vanishing of $[G]$ is to guarantee the induced map $C \to T^1_C$ does not have a vanishing 1st order derivative. Otherwise, we need to consider higher derivatives (reduced Kodaira-Spencer class) as the following example shows:

$\{z_1^2 + z_2^3 + z_3^3 = 0\} = \{z_1^2 + z_2^3 + (z_3 + t/2)^2 - \frac{t^2}{4} = 0\} \cong \{z_1^2 + z_2^3 + z_3^3 - \frac{t^2}{4} = 0\}$.

So the induced map $C \to T^0_C$ vanishes to the 2nd order at 0 and the weight of the deformation is actually equal to $-2$.

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