Two Models of Nonadaptive Group Testing for Designing Screening Experiments

Abstract

We discuss two non-standard models of nonadaptive combinatorial search which develop the conventional disjunct search model [8] for a small number of defective elements contained in a finite ground set or a population. The first model called a search of defective supersets (complexes) was suggested in [15, 16]. The second model which can be called a search of defective subsets in the presence of inhibitors was introduced for the case of an adaptive search in [10, 12]. For these models, we study the constructive search methods based on the known constructions for the disjunct model from [1, 13, 14].

1 Description of the Models

We use symbol $\triangleq$ to denote definitional equalities.

The standard disjunct search model for designing screening experiments (DSE) [11] has the following form. Let there be a population containing $t$ distinguishable samples. We identify it with the set $[t] \triangleq \{1, 2, \ldots, t\}$. Assume that the population includes an unknown defective subset $p \subset [t]$. We call elements of $p$ defective samples.

Our aim is to detect $p$ using a number of group tests. Each group test is defined by a subset (testing group or pool) $G \subset [t]$. The result $r(G, p)$ of this test assumes a binary value 0 (negative) or 1 (positive) according to the following rule:

$$r(G, p) \triangleq \begin{cases} 1, & G_n \cap p \neq \emptyset, \\ 0, & G_n \cap p = \emptyset. \end{cases}$$

(1)

One can see that a group test detects whether the testing group intersects with the defective subset or not.

In the adaptive disjunct model each group test $G_{n+1}$ is chosen according to the results of the previous tests $G_1, \ldots, G_n$. This model is considered in [8].

We consider the nonadaptive disjunct model, in which all pools must be constructed before any test is performed. Thus, an explorer is not allowed to use the results of the previous tests.

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to construct the next ones. Such model usually occurs in the problems of molecular biology [10, 14] when different tests can be performed simultaneously as one experiment (one step). Each experiment is expensive, so we are interested in detecting $p$ using only one step, i.e., exactly the nonadaptive search strategy.

We also consider the restriction on the number of defective samples: $|p| \leq s$, where $s$ is the given positive integer, $s < t$. Such condition is usually used in the group testing problems. In many cases when such condition does not occur naturally in problems under consideration it can be justified by the probabilistic arguments. If any sample can be defective with some small probability, then one can choose $s$ so that the inequality $|p| \leq s$ holds with high probability.

Note, that for the practical applications of this model we assume that $s \ll t$. If this condition does not hold, then the present model is not suitable.

**Definition 1.** A series of $N$ nonadaptive tests $X \triangleq (G_1, G_2, \ldots, G_N)$ which allows to identify any defective subset $p \subset [t]$, $|p| \leq s$, is called a disjunct $s$-design [8] of length $N$ and size $t$. Note that we suppose no error in the results of these test.

In general case to detect a defective subset $p$ using a disjunct $s$-design one needs to check all possible subsets. Below we introduce an important class of designs called superimposed codes for which the decoding algorithm is much more simple.

This model can be generalized in several ways. We can consider the list decoding procedure in which one should construct a subset $p' \subset [t]$ such that $p \subseteq p'$ and the size of $p' \setminus p$ is not very large. This model was considered in [4] and some other papers. We also can consider designs which correct errors in the results [7].

In the present paper we introduce two generalizations of the disjunct model of DSE.

1. **Nonadaptive search of defective supersets.**

   Let there be a population $[t] = \{1, 2, \ldots, t\}$. Assume that there exists an unknown superset (complex) $p$ which is composed of a number of subsets $P \subset [t]$:

   $$p \triangleq \{P_1, P_2, \ldots, P_k\}, \quad P_i \subset [t], \quad P_i \nsubseteq P_j \text{ for } i \neq j. \quad (2)$$

   To detect $p$ one can use a number of group tests $G \subset [t]$ for which the result $r(G, p)$ is positive if and only if $G$ includes at least one subset $P \subset [t]$ being a member of the complex $p$, and negative otherwise:

   $$r(G, p) \triangleq \begin{cases} 1, & \exists P \in p : P \subseteq G, \\ 0, & \text{otherwise}. \end{cases} \quad (3)$$

   We consider the following restrictions on complexes $p$: the number of elements $P \in p$ does not exceed $s$ and the size $|P| \leq \ell$ for any $P \in p$, where $s$ and $\ell$ are the given positive integers, $s + \ell \leq t$.

   **Definition 2.** A series of $N$ nonadaptive tests $X = (G_1, G_2, \ldots, G_N)$ which allows to identify any such complex $p$ is called a superset $(s, \ell)$-design of length $N$ and size $t$.

   Obviously, for $\ell = 1$ this model is identical to the conventional disjunct model, because each subset $P \in p$ is composed of exactly one element.

   One can easily understand the necessity of the additional condition in (2): if $P_i \subset P_j$, then we cannot detect the defective property of $P_j$. 

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Consider the following simple example of such application. We have some chemical (or medical) substances. We assume that some combinations of them may be dangerous. To detect these combinations we can perform group tests, i.e., put some of the samples together and test whether the obtained substance is dangerous. The result is positive if the group contains one or more dangerous combinations, that leads to the given model.
2. Nonadaptive search of defective subsets in the presence of inhibitors.

Return to the base disjunct model and assume that along with defective subset \( p \subset [t] \) there exists a subset \( I \subset [t], p \cap I = \emptyset \). We call samples from this set inhibitors.

Inhibitors make the result of a group test negative despite the existence of defective elements in testing group \( G \subset [t] \):

\[
\text{\( r(G, p, I) \triangleq \{ \begin{array}{ll} 1, & G \cap p \neq \emptyset \quad \text{and} \quad G \cap I = \emptyset, \\ 0, & \text{otherwise.} \end{array} \))}
\]

 Definition 3. A series of \( N \) nonadaptive tests \( X = (G_1, G_2, \ldots, G_N) \) which allows to identify any defective subset \( p, |p| \leq s \), in the presence of not more then \( i \) inhibitors is called an inhibitory \((s, i)\)-design of length \( N \) and size \( t \), where \( s \geq 1 \) and \( i \geq 0 \) are the given integers, \( s + i \leq t \).

Obviously for \( i = 0 \) this model is identical to the conventional disjunct model of DSE.

The current model arises in applications in which some samples in a population are not defective, but affect the testing result, namely, make it always negative. If a test has a form of some chemical reaction, then it may stop due to the inhibitor. This illustration gives the name to the current model.

The rest part of the present paper is organized as follows. In section 2 we consider important classes of designs for the models under consideration. The detecting algorithm for these types of designs are simple. In section 3 we consider a constructive method for such designs. Section 4 contains several examples.

2 Superimposed codes

2.1 Notations

Let \([t]\) be a population of samples. Consider an arbitrary series of \( N \) testing groups \( X = (G_1, G_2, \ldots, G_N) \), \( G_n \subset [t] \) for \( n = 1, 2, \ldots, N \). Following [8], we encode it by the binary \( N \times t \) incidence matrix \( X = [x_n(u)] \), where index \( n = 1, 2, \ldots, N \) denotes a row number and index \( u = 1, 2, \ldots, t \) denotes a column number. An element \( x_n(u) \) of this matrix has the form

\[
\text{\( x_n(u) \triangleq \{ \begin{array}{ll} 1, & u \in G_n, \\ 0, & \text{otherwise.} \end{array} \))}
\]

The \( n \)-th row \( x_n \triangleq (x_n(1), x_n(2), \ldots, x_n(t)) \) of matrix \( X \) encodes the \( n \)-th test \( G_n \). The \( u \)-th column \( x(u) \triangleq (x_1(u), x_2(u), \ldots, x_N(u)) \) is called the \( u \)-th codeword.

Denote by \( x \lor y \) (\( x \land y \)) the componentwise disjunction (conjunction) of binary vectors \( x \) and \( y \) of the same length. We say that a vector \( x \) covers a vector \( y \) if \( x \lor y = x \). For a matrix \( X \) and subset of columns \( \tau \subset [t] \) consider the disjunction and conjunction

\[
V(X, \tau) \triangleq \bigvee_{u \in \tau} x(u), \quad \Lambda(X, \tau) \triangleq \bigwedge_{u \in \tau} x(u).
\]

If \( \tau = \emptyset \) then we put \( V(X, \emptyset) \triangleq 0 = (0, 0, \ldots, 0) \).


2.2 Disjunct model of DSE

Let \( p \subset [t] \) be a defective subset. Denote by \( r(X, p) \) the binary vector of results of all \( N \) tests: \( r(X, p) = (r(G_1, p), r(G_2, p), \ldots, r(G_N, p)) \). From definition (1) one can see that it has the form of the disjunction of codewords:

\[
 r(X, p) = V(X, p) = \bigvee_{u \in p} x(u) .
\]

**Definition 1'.** \( X \) is a disjunct \( s \)-design iff for any two different subsets \( p_1, p_2 \subset [t] \), \( |p_1| \leq s \), \( |p_2| \leq s \), the result vectors \( r(X, p_1) \) and \( r(X, p_2) \) are different.

**Definition 4 [1].** Let \( s \) be an integer, \( 0 < s < t \). A binary \( N \times t \) matrix \( X \) is called a superimposed \( s \)-code of length \( N \) and size \( t \) if for any subset \( p \subset [t] \), \( |p| \leq s \), and any sample \( u \in [t] \setminus p \) the codeword \( x(u) \) is not covered by the disjunction \( r(X, p) \) (5). One can see that it is equivalent to the following condition: for the given pair \((p, u)\) there exists a row number \( n \in [N] \) such that \( x_n(u) = 1 \) and \( x_n(u') = 0 \) for all \( u' \in p \).

**Lemma 1 [1].** Any superimposed \( s \)-code is a disjunct \( s \)-design.

**Proof.** Obviously, for any sample \( u \in p \) the disjunction \( r(X, p) \) (5) covers the codeword \( x(u) \). If \( X \) is a superimposed \( s \)-code and \( |p| \leq s \), then from definition 4 it follows that for any sample \( u \notin p \) this disjunction does not cover the codeword \( x(u) \). Thus, one can easily detect \( p \) and \( X \) is a disjunct \( s \)-design.

If \( X \) is a disjunct \( s \)-design, then the complexity of the trivial algorithm of detecting \( p \) is \( \sim \binom{t}{s} \) because we need to perform an exhaustive search over all subsets \( p \subset [t] \), \( |p| \leq s \). The superimposed \( s \)-code condition provides the simple decoding algorithm: given the vector \( r(X, p) \) and any sample \( u \in [t] \) one can easily detect whether \( u \) is defective or not. The complexity of this algorithm is \( \sim t \).

Superimposed \( s \)-codes were introduced in [1] and studied in many papers [4, 7, 11, 13, 14]. Below we consider the similar types of designs for the models of searching supersets and searching subsets in the presence of inhibitors.

2.3 Search of defective supersets

Let \( X \) be a binary \( N \times t \) matrix which encodes a search strategy as described before. Let \( p \) be a defective superset (complex) composed of a number of subsets \( P \subset [t] \) (2). Using the definition (3) one can easily prove that the result vector \( r(X, p) = (r(G_1, p), r(G_2, p), \ldots, r(G_N, p)) \) in this model has the form

\[
 r(X, p) = \bigvee_{P \in p} A(X, P) = \bigvee_{P \in p} \bigwedge_{u \in P} x(u) .
\]

**Definition 2'.** \( X \) is a superset \((s, \ell)\)-design (see Def. 2) iff for any two different complexes \( p_1 \) and \( p_2 \) composed of not more then \( s \) subsets which sizes do not exceed \( \ell \) the results \( r(X, p_1) \neq r(X, p_2) \).

**Definition 5 [6].** Let \( s \) and \( \ell \) be positive integers, \( s + \ell \leq t \). A binary \( N \times t \) matrix \( X \) is called a superimposed \((s, \ell)\)-code if for any subsets \( S, L \subset [t] \), \( |S| \leq s \), \( |L| \leq \ell \) and \( S \cap L = \emptyset \) there exists a row number \( n \in [N] \) for which \( x_n(u) = 1 \) for any \( u \in L \) and \( x_n(u') = 0 \) for any \( u' \in S \).

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One can see that for $\ell = 1$ this definition coincides with Def. 4 of superimposed $s$-codes. Superimposed $(s, \ell)$-codes were first introduced in [6] for cryptography applications. It the present paper we study them for the search problems. Below we introduce some simple results which are going to be published in a more detailed form in [16].

**Lemma 2.** Any superimposed $(s, \ell)$-code is a superset $(s, \ell)$-design.

**Proof.** Let $p$ be a superset (2), where $k \leq s$ and $|P_i| \leq \ell$ for any $i$. Let $X$ be a superimposed $(s, \ell)$-code. Our aim is to detect $p$ given the vector $r(X, p)$ (6).

Any subset $P \subset [t]$, $|P| \leq \ell$, belongs to one of the following two types:

(α) there exists $P_i \in p$ such that $P_i \subseteq P$;

(β) $P$ does not contain any defective subset $P_i \in p$.

Let us show that for any subset $P$ we can detect whether it belongs to the class (α) or (β) given the result $r(X, p)$. Indeed, if $P$ satisfies (α), then from (6) it follows that the vector $r(X, p)$ covers the conjunction $\Lambda(X, P)$.

Assume that $P$ satisfies condition (β). Then all $k$ sets $P_i \setminus P$ for $1 \leq i \leq k$ are not empty. Take a sample from each of these sets and construct a set $S \subset [t]$ containing all these samples, $|S| \leq k \leq s$. Put $L \triangleq P$, $|L| \leq \ell$. For this pair of sets $(S, L)$ take a row number $n \in [N]$ according to definition 5. One can easily prove that $n$-th components of all vectors $\Lambda(X, P_i)$, $1 \leq i \leq k$, are zeros, and thus the $n$-th component of $r(X, p)$ (6) is zero. And the $n$-th component of the vector $\Lambda(X, P)$ is 1. This proves that for the case (β) the vector $r(X, p)$ does not cover $\Lambda(X, P)$.

The class (α) contains both defective subsets $P_i \in p$ and subsets $P$ such that $P_i \not\subseteq P$ for some $i$. To separate these cases note that if $P \in p$, then all subsets of $P$ satisfy (β). And for the second case there exists a subset of $P$ which satisfies (α).

We showed that given the vector $r(X, p)$ it is possible to detect $p$, that proves the statement. Moreover, we obtained the algorithm of such detection. Given a subset $P \subset [t]$ this algorithm detects whether $P \in p$ or not.

If $X$ is a superset $(s, \ell)$-design, then the complexity of the trivial algorithm of detecting a complex $p$ is $\sim \binom{|s|}{s}$ because we need to perform an exhaustive search over all supersets $p$. The superimposed $(s, \ell)$-code condition provides the simple decoding algorithm: given the vector $r(X, p)$ and any subset $P \subset [t]$, $|P| \leq \ell$, one can easily detect whether $P \in p$ or not. The complexity of this algorithm is $\sim \binom{|s|}{s}$.

### 2.4 Search of defective subsets in the presence of inhibitors

Let $X$ be a binary $N \times t$ matrix which encodes a search strategy as described before. Let $s \geq 1$ and $t \geq 0$ be integers, $s + t \leq t$. Denote by $\pi(t, s, i)$ the set of all possible pairs $(p, I)$, where $p$ is a defective set and $I$ is a set of inhibitors:

$$\pi(t, s, i) \triangleq \left\{ (p, I) : p \subseteq [t], 1 \leq |p| \leq s, 0 \leq |p| \leq i, p \cap I = \emptyset \right\}.$$

For a pair of binary symbols $x, y \in \{0, 1\}$ we define the inhibition of $x$ due to $y$ operation:

$$x \setminus y \triangleq \begin{cases} 1, & \text{if } x = 1 \text{ and } y = 0, \\ 0, & \text{otherwise}. \end{cases}$$
For a pair of binary vectors $x$ and $y$ we denote by $x \setminus y$ the componentwise inhibition operation. Note that $x \setminus y = 0$ if $x$ is covered by $y$, and $x \setminus y = x$ if the conjunction $x \setminus y = 0$.

Let $(p, I) \in \pi(s, t, l)$ be a pair of defective set and inhibitor set. Using the definition (4) one can easily prove that the result vector $r(X, p, I) = (r(G_1, p, I), r(G_2, p, I), \ldots, r(G_N, p, I))$ in this model has the form

$$r(X, p) = V(p) \setminus V(I) = \bigvee_{u \in p} x(u) \setminus V(I) = \bigvee_{u \in p} [x(u) \setminus V(I)]. \quad (7)$$

**Definition 3'.** $X$ is an inhibitory $(s, t)$-design (see Def. 3) iff for any two pairs $(p, I), (p', I') \in \pi(t, s, i)$, such that $p \neq p'$, the result vectors $r(X, p, I) \neq r(X, p', I')$. Note that this condition allows to detect the defective subset $p$ but not the inhibitory set $I$.

**Definition 6.** Let $s \geq 1$ and $i \geq 0$ be integers, $s + i \leq t$. A binary $N \times t$ matrix $X$ is called an inhibitory $(s, t)$-code if it is a superimposed $(s + i)$-code (see Def. 4).

**Lemma 3.** Any inhibitory $(s, t)$-code is an inhibitory $(s, i)$-design.

**Proof.** Let $(p, I) \in \pi(t, s, i)$ be a pair of defective and inhibitory sets and $X$ be an inhibitory $(s, i)$-code, i.e., a superimposed $(s + i)$-code. We should detect $p$ given the vector $r(X, p, I)$ (7).

Let us call a sample $u \in [t]$ $i$-acceptable due to the vector $r(X, p, I)$ if there exists a subset $I' \subset [t]$ such that $u \notin I'$, $|I'| \leq i$ and the vector $r(X, p, I)$ covers $x(u) \setminus V(I')$.

If $u \in p$, then $u$ is acceptable because all conditions hold for $I' = I$. Assume that $u \notin p$. Then for any subset $I' \subset [t]$, $u \notin I'$, $|I'| \leq i$, consider a pair $(p \cup I', u)$ and take a row number $n \in [N]$ according to the definition of superimposed $(s + i)$-code (Def. 4). One can easily prove that the $n$-th component of the vector $r(X, p, I)$ is zero and the $n$-th component of $x(u) \setminus V(I')$ is 1. So the first vector does not cover the second one. Since this is true for any $I'$, the sample $u$ is not acceptable.

We proved that a sample is defective iff it is $i$-acceptable due to $r(X, p, I)$. Obviously, one can check this given only the result $r(X, p, I)$. This completes the proof of the statement and also gives the decoding algorithm. Note that one can consider only such subsets $I'$ for which $V(I') \setminus r(X, p, I) = 0$. ■

The complexity of the trivial algorithm of detecting a subset $p$ for an arbitrary inhibitory $(s, i)$-design is $\sim \binom{t}{s+i} \cdot \binom{s+i}{i}$ because we should perform an exhaustive search over all pairs from the set $\pi(t, s, i)$. The inhibitory $(s, i)$-code condition provides the simple decoding algorithm: given the vector $r(X, p)$ and a sample $u \in [t]$ we need to check all subsets $I'$. The complexity of this algorithm is $\sim t \cdot \binom{t}{i}$.

### 3 Concatenated construction for superimposed codes

In the present section we consider a concatenated construction for superimposed $(s, \ell)$-codes. For special case $\ell = 1$ this leads to the construction for superimposed $s$-codes. For this case the similar method was suggested in [1] and developed in [13, 14].

**Definition 7 [2].** Let $q \geq 2$ be an integer and $X = ||x_n(u)||$ be an $N \times t$ $q$-ary matrix: $n \in [N], u \in [t], x_n(u) \in [q]$. Let $s$ and $t$ be positive integers, $s + \ell \leq t$. Then $X$ is called a $q$-ary separating $(s, \ell)$-code if for any two subsets $S, L \subset [t], |S| \leq s, |L| \leq \ell, S \cap L = \emptyset$, there exists
a row number \( n \in [N] \) such that the corresponding coordinate sets \( S_n \) and \( L_n \) do not intersect, where
\[
L_n \triangleq \{ x_n(u) : u \in L \} \subset [q], \quad S_n \triangleq \{ x_n(u') : u' \in S \} \subset [q].
\]

Note that for \( q = 2 \) definition 7 does not coincide with definition 5 of binary superimposed \((s, \ell)\)-code. Binary separating \((2, 2)\)-codes were studied before for certain applications [5, 9].

**Lemma 4.** Let \( X(q) \) be a \( q \)-ary separating \((s, \ell)\)-code of size \( t(q) \) and length \( N(q) \). Let \( X' \) be a binary superimposed \((s, \ell)\)-code of size \( q \) and length \( N' \). Then there exists a binary superimposed \((s, \ell)\)-code \( X \) of size \( t = t(q) \) and length \( N = N(q) \cdot N' \).

**Proof.** Consider the code \( X \) obtained by the concatenation of codes \( X(q) \) and \( X' \), i.e., each \( q \)-ary symbol \( \theta \in [q] \) in matrix \( X(q) \) is replaced by the \( \theta \)-th codeword from \( X' \):
\[
X(q) = \begin{array}{|c|c|c|}
\hline
\theta & \cdots & \cdots \\
\hline
\end{array}
\quad \begin{array}{|c|c|}
\hline
1 &\theta & q \\
\hline
\end{array}
\quad \begin{array}{|c|c|}
\hline
\cdots & \cdots \\
\hline
\end{array}
\]
\[
X' = \left\{ \begin{array}{c}
\cdots \\
\end{array} \right\}
\]
\[
N(q) = \begin{array}{|c|c|}
\hline
\cdots & \cdots \\
\hline
\end{array}
\quad \begin{array}{|c|c|}
\hline
1 &\theta & q \\
\hline
\end{array}
\quad \begin{array}{|c|c|}
\hline
\cdots & \cdots \\
\hline
\end{array}
\quad \begin{array}{|c|c|}
\hline
\cdots & \cdots \\
\hline
\end{array}
\]
\[
N' = \left\{ \begin{array}{c}
\cdots \\
\end{array} \right\}
\]

One can easily prove the superimposed \((s, \ell)\)-code property for \( X \). ■

A \( q \)-ary code \( X(q) \) is called an external code, and a binary code \( X' \) is called an internal code. To construct concatenated codes with large sizes we need \( q \)-ary external codes with large sizes and binary internal codes with small sizes. Below we discuss some simple methods for constructing them.

**Lemma 5** (trivial code). For any positive integers \( s, \ell \) and \( t \), \( s + \ell \leq t \), there exists a superimposed \((s, \ell)\)-code of size \( t \) and length
\[
N = \min \left\{ \binom{t}{s}, \binom{t}{\ell} \right\}.
\]

**Proof.** To obtain a code of length \( N_1 = \binom{t}{s} \) take the binary \( N_1 \times t \) matrix which rows are all possible binary vectors of length \( t \) having exactly \( s \) zeros. To obtain a code of length \( N_2 = \binom{t}{\ell} \) take the binary \( N_2 \times t \) matrix which rows are all possible binary vectors of length \( t \) having exactly \( \ell \) ones. Obviously, both these matrixes satisfy the \((s, \ell)\)-code property. ■

This trivial construction allow to construct superimposed codes for all possible values of \( s, \ell \) and \( t \). But it is reasonable to use it only for small values of \( t \). Note that for \( t = s + \ell \) the trivial code is optimal (has the smallest possible length).

Several methods exist for constructing small superimposed codes for some special values of \( s \) and \( \ell \). Some tables of codes for \( \ell = 1 \) can be found in [13, 14]. The values \( s = \ell = 2 \) are considered in [9, 15, 16] and below.

Finally we discuss a method of constructing large \( q \)-ary separating codes to be used in concatenated construction. It is based on MDS-codes. Since these codes are well-known we do not consider their properties in details.
Definition 8 [3]. Any $q$-ary code of size $t = q^k$, length $n$ and the Hamming distance $d = n - k + 1$ is called a maximal distance separable code (MDS-code) with parameters $(q, k, n)$.

Lemma 6 [15, 16]. If $n \geq s\ell(k - 1) + 1$ and $q^k \geq s + \ell$, then any MDS-code with parameters $(q, k, n)$ is a $q$-ary separating $(s, \ell)$-code.

Lemma 7 [3]. For any positive integer $\lambda$ and any prime power $q \geq \lambda$ there exists an MDS-code with parameters $(q, \lambda + 1, q + 1)$ called the Reed–Solomon code.

Using Reed–Solomon code for the concatenation construction, we obtain

Lemma 8 [15, 16]. Let $s, \ell, \lambda$ be positive integers, and $q \geq s\ell\lambda$ be a prime power. Assume that there exists a binary superimposed $(s, \ell)$-code of size $q$ and length $N_1$. Then there exists a binary superimposed $(s, \ell)$-code of size $t = q^{\lambda+1}$ and length $N = N_1(s\ell\lambda + 1)$.

4 Examples

1. The best known superimposed 2-code of size $t = 12$ has length $N = 9$:

$$X = \begin{pmatrix}
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0
\end{pmatrix}.$$ (8)

It is the first known code for which $N < t$. For all sizes $t_1 < t$ the smallest known codes are trivial. Note that the trivial superimposed $s$-code of size $t$ is the identity $t \times t$ matrix.

Note also that the matrix (8) is an inhibitory $(1, 1)$-code, see Def. 6.

2. Some examples of small superimposed $(2, 2)$-codes are known. For $t = 4$, the optimal code is trivial and has length $N = \binom{4}{2} = 6$. For $t = 5$, the optimal code is also trivial and has length $N = \binom{5}{2} = 10$. For $t = 6$, $t = 7$ and $t = 8$ the optimal code has length $N = 14$. It can be obtained by the concatenated method from the following $3 \times 8$ quaternary matrix:

$$C^{(4)} = \begin{pmatrix}
4 & 2 & 3 & 1 & 2 & 4 & 1 & 3 \\
2 & 4 & 1 & 3 & 2 & 4 & 1 & 3 \\
1 & 1 & 2 & 2 & 3 & 3 & 4 & 4
\end{pmatrix},$$

which is a separating $(2, 2)$-code. It can be concatenated with the trivial superimposed $(2, 2)$-code of size 4 and length 6. This leads to the superimposed $(2, 2)$-code of size $t = 8$ and length $N = 18$. Examining this code, one can see, that there are two rows in it, which are repeated three times. Removing the copies, we obtain the binary superimposed $(2, 2)$-code of length $N = 14$.

The following table gives several numerical values of the known superimposed $(2, 2)$-codes. Some of them were obtained with the help of V.S. Lebedev.
Some of these codes were known before [9]. Most of them are new. In the paper [16] an improved table of codes is going to be published.

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