Optimization of heat engines using different heat transfer laws by means of the method of saving functions

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Abstract. In 2000 Velasco et al [1] introduced a new optimization criterion for the CA-engine model in terms of a profitable type process in the operation of a power plant model. This approach is based on using the so-called saving function as a measure of possible reduction of undesired side effects in heat engine operation. Velasco et al [1] defined two saving functions; one associated with fuel consumption and another associated with thermal pollution, where each saving function takes into account three weight coefficients that measure the participation degree of the corresponding process in the optimization criterion. We made use of this criterion to analyse the Novikov’s power engine model [2] but by using different heat transfer laws: Newtonian and Dulong-Petit (DP) heat transfer laws [3]. We compare our results with those reported by Velasco et al [1]. Our results show a saving in fuel consumption of approximately 44% and a reduction in thermal pollution of 42% with respect to the operating regime of maximum power, a value that is in agreement with some reported in the literature.

1. Introduction

Within the context of Finite time thermodynamics several regimes of performance have been employed to study the well known Curzon-Ahlborn (CA) heat engine model (see Figure 1) [1,4-8]. One of the regimes most studied consists in maximizing the power output. In 2000 Velasco et al [1] introduced a new optimization criterion for the CA-engine model in terms of the profitable type process in the operation of the power plant; that is, the power production and the use of the concept of saving function as a measure of possible reduction of undesired side effect in heat engine operation. When we apply the first law of thermodynamics for a heat engine that absorbs a heat flow \( \dot{Q}_{\text{in}} \) from a heat reservoir with a high temperature \( T_h \) and rejects a heat flow \( \dot{Q}_{\text{out}} \) to a heat reservoir with a low temperature \( T_c \) and provides a power output we obtain,

\[
\dot{Q}_{\text{in}} = W + \dot{Q}_{\text{out}}
\]

If we consider that the operation of the power plant consists of three interdependent processes, one of them of profitable type (energy production) and the other two of undesirable type (the input and output heat flows), it is possible to characterize the reduction in the undesirable process if we define the...
concept of a saving function. Velasco et al [1] considered that a given undesirable process can be mathematically described by a function $F(\{x\};\{\lambda\})$, where $\{x\}$ denotes a set of independent variables and $\{\lambda\}$ denotes a set of controllable parameters, so that the saving function associated with $F(\{x\};\{\lambda\})$ can be written as

$$f(\{x\};\{\lambda\}) = 1 - \frac{F(\{x\};\{\lambda\})}{F_{\text{max}}(\{\lambda\})},$$

(2)

where $F_{\text{max}}(\{\lambda\})$ is the maximum value of $F(\{x\};\{\lambda\})$ for the allowed range of values of $\{x\}$. This maximum value corresponds to the most possible inefficient operation mode of the system, so that, the saving function is smaller as the operation regime of the system becomes more inefficient. Velasco et al [1] defined two saving functions; one associated with fuel consumption, that is, proportional to the input heat flow $\dot{q}_{in}$ defined by [1],

$$q_{in} = 1 - \frac{\dot{q}_{in}}{(\dot{q}_{in})_{\text{max}}},$$

(3)

where $(\dot{q}_{in})_{\text{max}}$ is the maximum heat flow we can extract from the reservoir at high temperature $T_h$ without supplying power. The other saving function associated with thermal pollution is proportional to the heat output flow $\dot{q}_{out}$ defined by [1],

$$q_{out} = 1 - \frac{\dot{q}_{out}}{(\dot{q}_{out})_{\text{max}}},$$

(4)

where $(\dot{q}_{out})_{\text{max}} = (\dot{q}_{in})_{\text{max}}$, this maximum heat flow is considered as a measure of the size of the plant, as it was proposed by De Vos [9]. If we now consider three processes: (1) the production of useful energy defined by $\omega = W/W_{\text{mp}} = (\dot{q}_{in} - \dot{q}_{out})/W_{\text{mp}}$, where $W$ represents the power output and $W_{\text{mp}}$ represents the maximum power output of the engine; (2) the saving of fuel consumption, characterized by the function $q_{in}$; and (3) the reduction on thermal pollution, characterized by the function $q_{out}$. An optimum operation mode of the engine coming from the simultaneous optimization of the three considered processes, could be the optimization of an objective function of the form $\Phi(\omega, q_{in}, q_{out})$. Velasco et al [1] proposed two objective functions, one of them defined on the basis of a linear formalism supported by the first law of thermodynamics, and other objective function which was constructed on the basis of a power-law formalism. These functions are given by,

$$\Phi_A = a_1 \omega + a_2 q_{in} + a_3 q_{out},$$

(5)

$$\ln\Phi_B = b_1 \ln\omega + b_2 \ln q_{in} + b_3 \ln q_{out},$$

(6)

where $a_i$ and $b_i$ are weight coefficients measuring the participation degree of the corresponding process in the optimization criterion. In Ref. [1] were studied the cases when the three processes involved are considered without discrimination; that is, for $a_1 = a_2 = a_3 = a$ and for $b_1 = b_2 = b_3 = b$. Velasco et al [1] made an optimization of the Eqs. (5) and (6) by considering the CA engine model (see figure 1) and taken the efficiency $\eta(\{x\})$ as the independent variable and $\tau(\{\lambda\})$ as the controllable parameter of the system. From de CA engine model it is possible to obtain [8],

$$\dot{q}_{in} = A \frac{(1-\tau-\eta)}{(1-\eta)},$$

(7)

where $A$ is an $\eta$-independent constant and $\tau = T_c/T_h$. Also these authors considered the maximum power output for the CA engine given by $(\dot{q}_{out})_{\text{max}} = A(1 - \sqrt{\tau})^2$, taking into account the fact that the maximum value of the equation (7) is $(\dot{q}_{in})_{\text{max}} = A(1 - \tau)$, and $\dot{q}_{out} = (1 - \eta)\dot{q}_{in}$ [4,8].
In this work, we study the Velasco et al objective functions for the cases \( a_1 \neq a_2 \neq a_3 \) and \( b_1 \neq b_2 \neq b_3 \), and we also study the cases considering different heat transfer laws in the Novikov heat engine model (see figure 2) [2].

2. Optimization of the saving functions with different values for the weight coefficients

The optimization presented in the previous section was made with the weight coefficients \( a_i \) and \( b_i \), (for \( i = 1, 2, 3 \)) constants and equal between them in equation (5); in this section we present the optimization of the Novikov heat engine when the three processes involved are considered with different degree of participation, i.e. when we consider different values in the weight coefficients. If we consider a Newton heat transfer law for the engine in figure 2 and taking constant the weight coefficient \( a_1 \), we obtain the objective function,

\[
\phi_{AN1} = \frac{\eta \left[ (1+\beta_2)(-1+\eta)-(1+\beta_1)(-1+\eta)\sqrt{\tau}+\tau-a_1\tau+(1+a_1)\tau^{3/2} \right]}{(-1+\eta)(1+\sqrt{\tau})},
\]

where \( \alpha_1 = \frac{a_2}{a_1}, \beta_1 = \frac{a_3}{a_1} \) and \( \tau = T_c/T_h \). From figure 3 is easy to show that the function (8) displays a maximum (\( d\phi_{AN1}/d\eta = 0 \) at,

\[
\tilde{\eta}_{AN1} = 1 - \frac{\sqrt{-(-1+\alpha_1)(1+\beta_1)+2(1+\alpha_1\beta_1)\sqrt{-(-1+\alpha_1)(1+\beta_1)\tau}}}{1+\beta_1-(\beta_1-1)\sqrt{\tau}}.
\]

In a similar way, if we consider constant the weight coefficients \( a_2 \) and \( a_3 \) respectively, it’s possible to obtain the following expressions,

\[
\phi_{AN2} = \frac{\eta \left[ a_2(1+\sqrt{\tau})(-1+\eta+\tau)+(1+\sqrt{\tau})(\beta_2+\eta\beta_2+\tau) \right]}{(-1+\eta)(1+\sqrt{\tau})(1+\sqrt{\tau})},
\]

\[
\phi_{AN3} = \frac{\eta \left[ a_3(1+\sqrt{\tau})(-1+\eta+\tau)+(1+\sqrt{\tau})(1-\eta+\beta_3\tau) \right]}{(-1+\eta)(1+\sqrt{\tau})(1+\sqrt{\tau})},
\]

where \( \alpha_2 = \frac{a_1}{a_2}, \beta_2 = \frac{a_3}{a_2} \) and \( \alpha_3 = \frac{a_1}{a_3}, \beta_3 = \frac{a_2}{a_3} \) in equations (10) and (11) respectively.
In figure 3 we show the behaviour of the objective function (equation (5)) when the three processes involved are considered with different degree of participation; that is, the case for different values of the weight coefficients. We can observe from figure 4 that, when the three processes involved in the optimization have different degree of participation or when the weight coefficients are not equal, the efficiency of the power plant model engine decreases, and the same for the maximum value of the objective function. For the case of the objective functions \( \phi_{038} \) and \( \phi_{038} \), these functions have a similar behavior that the objective function \( \phi_{038} \) (see figure 3). The objective functions previously obtained display a maximum \( \frac{d \phi_{AN2}}{d \eta} = 0 \) respectively at,

\[
\eta_{AN2} = 1 - \frac{(1-\alpha_2-(1+\alpha_2)\sqrt{\tau})}{\sqrt{\tau}(1-\alpha_2)(\alpha_2+\beta_2)+2(\alpha_2^2+\beta_2)\sqrt{\tau}+(1+\alpha_2)(\alpha_2+\beta_2)\tau},
\]

(12)

\[
\eta_{AN3} = 1 - \frac{\sqrt{\tau}(1+\alpha_3)(\alpha_3+\beta_3)+2(\alpha_3^2+\beta_3)\sqrt{\tau}+(1+\alpha_3)(\alpha_3+\beta_3)\tau}{1+\alpha_3+(-1+\alpha_3)\sqrt{\tau}}.
\]

(13)

We can observe from equations (8), (12) and (13) for the cases \( \alpha_1 = \beta_1 \), \( \alpha_2 = \beta_2 \) and \( \alpha_3 = \beta_3 \) respectively, that the results of Velasco et al are recovered [1]. In figure 4, we also show the Carnot efficiency with \( \eta_C = 1 - \tau \), the Curzon-Ahlborn efficiency [4] with \( \eta_{CA} = 1 - \sqrt{\tau} \) and the optimal efficiencies for different values of the constants \( \alpha_i \) and \( \beta_i \), for \( i = 1, 2, 3 \). As it was previously mentioned, we can observe how the optimal efficiencies defined by the equations (9), (12) and (13) decrease as the three processes considered in the optimization have different degree of participation.

3. Optimization of the saving functions with Dulong-Petit heat transfer law

Now, we consider the heat transfer of the Dulong-Petit type given by the expression [3],

\[
\frac{1}{A} \frac{dQ}{dt} = \alpha (T - T_a)^n,
\]

(14)

where \( \frac{1}{A} \frac{dQ}{dt} \) is the rate of heat loss per unit area from a body at temperature \( T \), \( \alpha \) is the heat transfer coefficient [3], \( T_a \) is the temperature of the fluid surrounding the body. The case \( n = 5/4 \), allows us to describe the case of a combined conductive-convective and radiative cooling by a power-law relationship and is given by Dulong-Petit law of cooling [3], which is
\[
\frac{1}{A} \frac{dQ}{dt} = \alpha (T - T_a)^{5/4}.
\]  
(15)

From figure 2, if we consider a Dulong-Petit transfer law we obtain the following expression,

\[
\dot{Q}_{in} = A \left( \frac{1 - \tau - \eta}{1 - \eta} \right)^{5/4},
\]  
(16)

where \(A\) is an \(\eta\)-independent constant. It is easy to verify that the power output \(W = \eta \dot{Q}_{in}\), displays a maximum at \(W_{mp} = A \frac{1}{B} (8 + \tau - \sqrt{\tau(80 + \tau)}\) \[10\]. If we consider that the maximum value of the input heat flow occurs at \(\eta = 0\), \((\dot{Q}_{in})_{max} = a T_h^{5/4} (1 - \tau)^{5/4}\) and by considering the same degree of participation of the three processes involved the objective function (5), for this engine we get,

\[
\phi_{ADP} = 2 + \frac{(-1 + \eta + \tau) \left(\frac{1 + \frac{\tau}{1 - \eta}}{1 - \eta} \right)^{1/4}}{(1 - \eta)^{5/4}} + \frac{\eta \left(1 + \frac{\tau}{1 - \eta} \right)^{5/4}}{\left(1 - \frac{\tau}{1 - \eta} \right)^{5/4}} - \frac{\left(\frac{1 + \frac{\tau}{1 - \eta}}{1 - \frac{\tau}{1 - \eta}} \right)^{5/4}}{\left(1 - \frac{\tau}{1 - \eta} \right)^{5/4}},
\]  
(17)

and for the objective function in the case of the power law formalism (see equation (8)) we obtain,

\[
\phi_{BDP} = \left[ \eta \left(1 + \frac{\tau}{1 - \eta} \right)^{5/4} \left(1 + \frac{(-1 + \eta + \tau) \left(\frac{1 + \frac{\tau}{1 - \eta}}{1 - \eta} \right)^{1/4}}{(1 - \eta)^{5/4}} \right) \left(1 - \frac{\left(\frac{1 + \frac{\tau}{1 - \eta}}{1 - \frac{\tau}{1 - \eta}} \right)^{5/4}}{\left(1 - \frac{\tau}{1 - \eta} \right)^{5/4}} \right) \right]^{1 - (1 - \sqrt{\tau})^{5/4}}.  
\]  
(18)

Finally, by means of the maximization of the equation (18) we get the following optimal efficiency,

\[
\bar{\eta}_{ADP} = 1 - \frac{(1 - \sqrt{\tau})^{9/4} + (1 - \tau)^{5/4} - \sqrt{(1 - \sqrt{\tau})^{9/4} + (1 - \tau)^{5/4} (1 - \sqrt{\tau})^{9/4} + (1 - \tau)^{5/4} (1 - \sqrt{\tau})^{9/4} + (1 - \tau)^{5/4} (1 - \sqrt{\tau})^{9/4} + (1 - \tau)^{5/4} (1 - \sqrt{\tau})^{9/4} + (1 - \tau)^{5/4} (1 - \sqrt{\tau})^{9/4} + (1 - \tau)^{5/4} (1 - \sqrt{\tau})^{9/4} + (1 - \tau)^{5/4}}{8 (1 - \sqrt{\tau})^{9/4} + (1 - \tau)^{5/4}}.  
\]  
(19)

In our study we also analysed the input heat flow given by equation (16) and the optimal efficiency \(\bar{\eta}_{ADP}\), and we compared it with the input heat flow at maximum power by means of a Dulong-Petit heat transfer law, where the optimal efficiency is given by \(\eta_{MP} = 1 - (\tau(80 + \tau) - 8)/8\). In a similar way to the objective function (17), the Eq. (18) displays a maximum \((d\phi_{BDP}/d\eta) = 0\), but due to the mathematical cumbersome the calculations were made
numerically. Figure 5 shows the optimal efficiencies $\bar{\eta}_{ADP}$ and $\bar{\eta}_{BDP}$, together with the Carnot efficiency, $\eta_C = 1 - \tau$, the CA efficiency, $\eta_{CA} = 1 - \sqrt{\tau}$ and the ecological efficiency, $\bar{\eta}_{CA} = 1 - \sqrt{(1 + \tau)\tau/2}$ [7]. We can observe from figure 5, that the inequality $\eta_{CA} < \eta_E < \eta_{ADP} < \eta_C$ is fulfilled; that is, for the case of the Dulong-Petit heat transfer law, also when we consider a power law formalism the efficiency of the heat engine is slightly larger than that with the linear formalism. In figure 6 we show the comparison of optimal ratios between saving functions conditions and maximum power conditions for the quotients of the power output $W$, that is, $W^p/W_{MP}$ and $W^{DP}/W_{MP}$. We also show the optimal ratios for the input heat flows $\dot{Q}_{in}$ and the output heat flows $\dot{Q}_{out}$. In all cases we can observe a slightly difference between the linear heat transfer law and the non-linear heat transfer law.

4. Concluding Remarks
In this work we apply the saving function method to analyze the Novikov’s power engine model [2] by using a heat transfer law of Dulong-Petit [3]. For the case of a Newtonian heat transfer law, we consider different degrees of participation of each weight coefficient and the optimal efficiency was obtained. For the other two cases, the weight coefficients were considered with the same degree of participation. Besides, for the DP heat transfer law case, we also obtain the optimal efficiencies for both formalisms and we compare them with the results of Velasco et al [1]. Our results show a saving in fuel consumption of approximately 44% and a reduction in thermal pollution of 42% with respect to the operating regime of maximum power (see figure 6), values that are in agreement with some reported in the literature [10-12].

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