Kaluza-Klein towers for spinors in warped spaces

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Abstract

All the boundary conditions compatible with the reduction of a five dimensional spinor field of bulk mass $M$ in a compactified warped space to a four dimensional brane are derived from the hermiticity conditions of the relevant operator. The possible presence of metric singularities is taken into account. Examples of resulting Kaluza-Klein spinor towers are given for a representative set of values for the basic parameters of the model and of the parameters describing the allowed boundary conditions, within the hypothesis that there exists one-mass-scale-only, the Planck mass. In many cases, the lowest mass in the tower is small and very sensitive to the parameters while the other masses are much higher and become more regularly spaced. In these cases, if a basic fermion of the standard model (lepton or quark) happens to be the lowest mass of a Kaluza-Klein tower, the other masses would be much larger and weakly dependent on the fermion which defines the tower.

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1 Introduction

In recent articles, we have analysed the generation of Kaluza-Klein mass towers in five dimensional theories with a fifth dimension compactified either to a strip or on a circle. This study was carried out considering a scalar field propagating freely in the bulk, at first in a flat space [1], then in a warped space [2] without metric singularities [3] and finally in a warped space with metric singularities [4]. The approach relies on a careful study of the hermiticity properties of the operators which arise in the Kaluza-Klein reduction equations and which are of second order in the derivatives. This lead us, considering different five-dimensional metric configurations (flat and warped), to classify all the sets of allowed boundary conditions and, from them, the corresponding mass equations leading to the construction of the so-called Kaluza-Klein mass towers [6]. Remember that the consideration of warp spaces offers the possibility to solve in an elegant way, with only one extradimension, the hierarchy problem, in the sense that starting from the Planck mass as the only fundamental mass of the model, the observable low lying Kaluza-Klein masses can be made of the order of TeV without fine-tuning.

Having in mind that the future high energy colliders are expected to look for the possible appearance of Kaluza-Klein mass towers which could be of non zero spin as evidence for the existence of fields with spin propagating in higher dimensions, we were led to extend our work to spinor fields. In a previous article [5], we restricted ourselves, as a first step in a more general approach, to a five-dimensional flat space. Requesting the Dirac operator be a symmetric operator and taking into account the underlying symmetries of the Dirac equation in five dimensions, in particular covariance and parity invariance in the brane, the whole set of allowed boundary conditions has been established leading to the mass equations from which the Kaluza-Klein mass towers are built.

In our preceding papers, illustrative numerical examples of Kaluza-Klein mass towers were given for the different configurations we considered.

In this article, we extend our study of Dirac fields by the consideration of five dimensional compactified warp spaces, first without, and then with metric singularities. The article is organized as follows. In Sect.(2), we consider the case of a warp space with no metric singularity. We establish the specific form of the Dirac equation and proceed with the Kaluza-Klein reduction. The whole set of allowed boundary conditions are obtained from the hermiticity
of the Dirac operator. The solutions for a free field with an arbitrary mass $M$ propagating in the bulk are given explicitly. In Sect. (3), we extend the same considerations to the case of warp spaces with an arbitrary number of singularities. Sect. (4) is devoted to various physical considerations concerning the determination and the interpretation of the Kaluza-Klein mass eigenstates, in particular considerations about the possible choices of boundary conditions, the closure of the extradimension strip to a circle, the mass scales of the model, the relation between Kaluza-Klein eigenmasses and observable masses and finally the mass state probability densities. In Sect. (5), the general procedure adopted for the determination of the Kaluza-Klein mass towers is elaborated from the boundary conditions and the analytical expressions of the field. In Sect. (6), some illustrative numerical examples of Kaluza-Klein mass towers are given for specific boundary conditions, in the cases without and with metric singularities.

In App. (A), we show that the boundary relations derived from the application of the least action principle are identical to those we deducted from the symmetry of the Dirac operator. In App. (B), we developed some examples of boundary conditions in the general case of $N$ metric singularities.
2 The Dirac equation in a warped space. No metric singularities

2.1 The Dirac equation. Invariances

We study the Dirac equation (see App. (A)) with a bulk mass \( M \)

\[
\left( i \gamma^A D_A - M \right) \Psi = 0
\]  

(1)

and with the invariant scalar product between the spinors \( \Phi \) and \( \Psi \)

\[
(\Phi, \Psi) = \int \Phi(x) \Psi(x) \sqrt{g} \, d^4x
\]  

(2)

in a five dimensional warped space. With the following notation for the indices

\[
\begin{align*}
\text{warp space} & : \{A\} \equiv \{\Sigma, S\} \equiv \{0, I, S\} , \quad I = 1, 2, 3 \quad, \quad S = 5 \\
\text{local space} & : \{a\} \equiv \{\sigma, s\} \equiv \{0, i, s\} , \quad i = 1, 2, 3 \quad, \quad s = 5
\end{align*}
\]  

(3)

the warped metric is

\[
dS^2 = g_{AB} dx^A dx^B = g_{\Sigma \Theta} dx^\Sigma dx^\Theta - ds^2 = \lambda^2 e^{-2\epsilon k s} \eta_{\sigma \theta} dx^\Sigma dx^\Theta - ds^2 .
\]  

(5)

In this equation, \( \eta_{\sigma \theta}, \eta_{ss} \) are the components of the five dimensional flat space metric with signature \((+, -, -, -, -)\), \( \lambda \) is an arbitrary positive constant, introduced for later convenience, while, with \( k \) defined to be positive, the warp factor \( \epsilon k \), \( \epsilon = \pm 1 \) can be chosen to be positive or negative. As in the four-dimensional space, the Dirac spinor is four-dimensional \( \Psi_\alpha \), \( \alpha = 1, \ldots, 4 \) in a five dimensional space.

We now compute explicitly the vielbein and the related quantities (\( \gamma \)-matrices and covariant derivatives) needed to put the Dirac equation in the form (16) suitable for its application to the case of interest.

The non zero elements of the vielbein \( e^A_c \) defined as usual by

\[
g_{AB} = e^A_c \eta_{cd} e^d_B
\]  

(6)

are chosen as

\[
\begin{align*}
e^\Sigma_\sigma = e^0_\sigma = e^1_\sigma = e^2_\sigma = e^3_\sigma & = \lambda e^{-\epsilon k s} , \quad e^s_s = 1 \\
e^\Sigma_\sigma = e^0_\sigma = e^1_\sigma = e^2_\sigma = e^3_\sigma & = \lambda^{-1} e^{\epsilon k s} , \quad e^s_s = 1
\end{align*}
\]  

(7)
We take local $\gamma^a$ as those of the flat five-dimensional space ($[\gamma_a, \gamma_b]_+ = \eta_{ab}$). They can be built from a set $\gamma_\sigma$, $\sigma = 0, \ldots, 3$ matrices of the four dimensional flat space. In particular one has $\gamma_s = \gamma_0 \gamma_1 \gamma_2 \gamma_3$. The warped $\gamma^A$ are given by
\[ \gamma^A = e_a^A \gamma^a. \] (9)

The Dirac equation is covariant under the diffeomorphisms
\[
\begin{align*}
    x'^A &= x'^A(x^B) \\
    dx'^A &= \frac{\partial x'^A}{\partial x^B} dx^B \\
    e_a^A(x') &= \frac{\partial x^A}{\partial x^B} e_a^B(x) \\
    \Psi'(x') &= \Psi(x)
\end{align*}
\] (10)

and under the local SO(4,1) local transformations $T^a_b(x)$ ($T^a_t \eta T = \eta$)
\[ \begin{align*}
    \hat{e}_a^A &= T^a_b e_A^b \\
    \hat{\Psi}_\alpha &= S_{\alpha\beta}^T \Psi_\beta
\end{align*} \] (11)

where $S_{\alpha\beta}^T$ is the spinor transformation corresponding to the vector transformation $T^a_b$, in particular $S^{-1}_{\alpha\gamma}^A S = T^a_{\gamma} \gamma^b$.

The non zero elements of the covariant derivatives of the vielbein are computed to be
\[ (D_0 e)_5^o = (D_I e)_5^i = \epsilon \lambda k e^{-\epsilon k s} \]
(12)

The covariant derivative of the four-component spinor field $\Psi_\alpha$ is given by
\[ (D_A \Psi)_\alpha = \partial_A \Psi_\alpha + (G_A^s)_\alpha_\beta \Psi_\beta \] (13)

where the spinor connection $(G_A^s)_\alpha_\beta$, written in general
\[ G_A^s = -\frac{i}{2} g^{BC} e_B^a (D_A e)_C^b \sigma_{ab}, \] (14)

reduces here to
\[ G_\Sigma^s = -\frac{1}{2} \epsilon \lambda k e^{-\epsilon k s} \gamma_\sigma \gamma^5. \] (15)
Collecting the terms, one finds from (11) the specific Dirac equation of the warped space (5)

\[ \left( \frac{e^{cks}}{\lambda} (i\gamma^a \partial_\Sigma) + (i\gamma^5) (\partial_5 - 2k\epsilon) \right) \Psi = M \Psi. \] (16)

The five-dimensional mass \( M \), the mass in the bulk, is an arbitrary parameter of the model.

### 2.2 Symmetry of the Dirac operator

In order to have real \( M \), the Dirac operator \( D \) in (11)

\[ D = i\gamma^A D_A \] (17)

should be symmetric for the hermitian scalar product (2), namely

\[ (\Phi, D\Psi) = (D\Phi, \Psi). \] (18)

Using the identity

\[ \frac{1}{2} \sqrt{g} (g^{AB} \eta_{ac} - e_a^B e_c^A) (D_A e^c_B) - \partial_A \left( \sqrt{g} e_a^A \right) = 0 \] (19)

which can be proved using

\[ \partial_A (\sqrt{g}) = \frac{1}{2} g^{BC} (\partial_A g_{BC}) \] (20)

the equation (18) reduces (up to a factor \( i \)) to the integral of a divergence

\[ \int \partial_A \left( \Phi \gamma^A \Psi \sqrt{g} \right) d^d x = 0 \] (21)

meaning that this operator is formally symmetric. This equation determines the boundary conditions which must be satisfied by \( \Phi \) and \( \Psi \) in order for the Dirac operator to be fully symmetric. In this Section, the discussion is carried on in a warped space without metric singularities and in Section (3) with an arbitrary number \( N \) of metric singularities.
2.3 Kaluza-Klein reduction. No metric singularity

We adopt the following Kaluza-Klein separation of variables

\[ \Psi(x^\mu, s) = \sum_n (F[n](s) + iG[n](s)\gamma^5)\psi[n](x^\mu) \]  

(22)

assuming that \( F[n](s) \) and \( G[n](s) \) are complex functions depending on \( s \) only while \( \psi[n](x^\mu) \) is an \( x^\mu \) dependent spinor.

In this form we have made the most general choice compatible with an SO(3,1) spinor invariance in the sense that \( \Psi \) and \( \psi[n] \) are supposed to transform in the same way under this subgroup of the spinor SO(4,1) transformations (11).

2.4 Boundary relation and conditions for the spinor fields. No metric singularity

For each variable \( x^A \) with range \([-\infty, \infty]\), the integration in (21) is identically zero for \( \Psi \) and \( \Phi \) in the spinor Hilbert space (sufficient decrease of the fields at \( \pm \infty \)). For the variables \( s \) which has a finite range \([0, 2\pi R]\), the boundary relation is

\[ \int d^4x \left[ \Phi \gamma^5 \sqrt{g} \right]_{s=2\pi R} = \int d^4x \left[ \Phi \gamma^5 \sqrt{g} \right]_{s=0} \]  

(23)

where the integration is carried on all the variables \( x^\mu \). In turn, the relation (23) implies conditions between the fields evaluated at \( s = 2\pi R \) and \( s = 0 \). These boundary conditions will be written explicitly below for the case of a five dimensional warped space without metric singularity or in Section (3) for the case with metric singularities.

We do not consider here the variable \( s \) with a semi-infinite range \([0, \infty]\) (up to a transformation \( s' = \pm s + \beta \)) which require a special treatment.

In order to obtain the boundary conditions which must be satisfied by the components \( F[n,\Phi] \) and \( G[n,\Phi] \) of the field (with identical relations for \( F[n,\Phi] \) and \( G[n,\Phi] \)), one introduces the reduction ansatz (22) in the boundary relation (23).

As in the flat case for spinors [5], there are two sets of possible boundary conditions. The first set

Set BC1 : \[ \left( \begin{array}{c} F[n](2\pi R) \\ G[n](2\pi R) \end{array} \right) = B \left( \begin{array}{c} F[n](0) \\ G[n](0) \end{array} \right) \]  

(24)
where $B$ is a complex $2 \times 2$ matrix. After some computation one finds that $B$ must be of the form

$$B = e^{4\epsilon \pi k R} e^{i \rho} \begin{pmatrix} \cosh(\omega) & \sinh(\omega) \\ \sinh(\omega) & \cosh(\omega) \end{pmatrix}$$  \hspace{1cm} (25)$$

where $\rho$ is a real parameter with range $0 \leq \rho < 2\pi$ and $\omega$ is an arbitrary real parameter. Compared to the flat case there is simply an extra $e^{4\epsilon \pi k R}$ factor.

The second set

Set BC2 : \hspace{1cm} \left\{ \begin{array}{l}
G^{[n]}(0) = \epsilon_0 F^{[n]}(0) , \quad \epsilon_0^2 = 1 \\
G^{[n]}(2\pi R) = \epsilon_R F^{[n]}(2\pi R) , \quad \epsilon_R^2 = 1
\end{array} \right. \hspace{1cm} (26)$$

where $\epsilon_0$ and $\epsilon_R$ are two arbitrary signs is identical to the corresponding set in the flat case.

One supposes that the fields satisfy the $SO(3,1)$ invariant boundary conditions

2.5 Solutions. No metric singularity

Introducing the reduction ansatz (22) in the five dimensional Dirac equation (11) and postulating that $\psi^{[n]}$ satisfies the four dimensional parity invariant Dirac equation

$$(i\gamma^\mu \partial_\mu - mn)\psi^{[n]} = 0$$  \hspace{1cm} (27)$$

one finds from (16) the two coupled equations for the components of the field

$$\partial_s G^{[n]} = \left( M - e^{\epsilon ks} \frac{m_n}{\lambda} \right) F^{[n]} + 2\epsilon k G^{[n]}$$

$$\partial_s F^{[n]} = 2\epsilon k F^{[n]} + \left( M + e^{\epsilon ks} \frac{m_n}{\lambda} \right) G^{[n]} .$$  \hspace{1cm} (28)$$

In terms of the following constants, variable and functions

$$\begin{align*}
\tilde{M} &= \frac{\epsilon M}{k} \\
\tilde{m}_n &= \frac{\epsilon m_n}{\lambda k} \\
z &= \tilde{m}_n e^{\epsilon ks} \\
\tilde{F}^{[n]}(z) &= F^{[n]}(s(z)) \\
\tilde{G}^{[n]}(z) &= G^{[n]}(s(z))
\end{align*}$$  \hspace{1cm} (29)$$
one obtains the simplified equations
\[ z \partial_z \bar{F}^{[n]} = 2 \bar{F}^{[n]} + \left( \bar{M} + z \right) \bar{G}^{[n]} \]
\[ z \partial_z \bar{G}^{[n]} = 2 \bar{G}^{[n]} + \left( \bar{M} - z \right) \bar{F}^{[n]} \]  \hspace{1cm} (30)

Defining
\[ \bar{F}_+^{[n]} = \bar{F}^{[n]} + \bar{G}^{[n]} \]
\[ \bar{F}_-^{[n]} = \bar{F}^{[n]} - \bar{G}^{[n]} \]  \hspace{1cm} (31)

one equation gives \( \bar{F}_-^{[n]} \) in terms of \( \bar{F}_+^{[n]} \)
\[ \bar{F}_-^{[n]} = z^{\frac{3}{2}} \left( -z \partial_z \bar{F}_+^{[n]} + \left( \bar{M} - \frac{1}{2} \right) \bar{F}_+^{[n]} \right) \]  \hspace{1cm} (32)

while the second equation, of second order, leads to
\[ z^2 \partial_z^2 \bar{F}_+^{[n]} - 4z \partial_z \bar{F}_+^{[n]} + \left( z^2 + 6 - \bar{M}^2 + \bar{M} \right) \bar{F}_+^{[n]} = 0 \]  \hspace{1cm} (33)

The function \( \mathcal{F}_+^{[n]} = z^{-\frac{3}{2}} \bar{F}_+^{[n]} \) satisfies the following equation
\[ z^2 \partial_z^2 \mathcal{F}_+^{[n]} + z \partial_z \mathcal{F}_+^{[n]} + \left( z^2 - \left( \bar{M} - \frac{1}{2} \right)^2 \right) \mathcal{F}_+^{[n]} = 0 \]  \hspace{1cm} (34)

which is of Bessel type. The final solution for \( \bar{F}_+^{[n]}(s) \) is the following linear superposition of Bessel functions.
\[ \bar{F}_+^{[n]}(s) = \left( \frac{m_n e^{\epsilon ks}}{\lambda k} \right)^{\frac{3}{4}} \left( \sigma_n J_{\frac{\epsilon kM}{m_n}} - \frac{m_n e^{\epsilon ks}}{\lambda k} \right) + \tau_n Y_{\frac{\epsilon kM}{m_n}} - \frac{1}{2} \left( \frac{m_n e^{\epsilon kM}}{\lambda k} \right) \]  \hspace{1cm} (35)

with two arbitrary constants while
\[ \bar{F}_-^{[n]}(s) = \frac{\lambda e^{-\epsilon ks}}{m_n} \left( -\partial_s \bar{F}_+^{[n]}(s) + (M + 2\epsilon k) \bar{F}_+^{[n]}(s) \right) \]  \hspace{1cm} (36)

and
\[ F^{[n]}(s) = \frac{1}{2} \left( \bar{F}_+^{[n]}(s) + \bar{F}_-^{[n]}(s) \right) \]
\[ G^{[n]}(s) = \frac{1}{2} \left( \bar{F}_+^{[n]}(s) - \bar{F}_-^{[n]}(s) \right) \]  \hspace{1cm} (37)

The constants \( \sigma_n \) and \( \tau_n \) of (35) are determined by the boundary conditions (24) or (26).
3 Application to a five dimensional warped space with metric singularities

3.1 The five-dimensional warped space with $N$ metric singularities

The extension of the preceding arguments to a warped space with an arbitrary number $N$ of metric singularities situated at the points $s_i$, $i = 1, N$ with $s_0 = 0 < s_1 < s_2, \ldots, s_N < s_{N+1} = 2\pi R$ on the strip is straightforward. By definition, the metric is of the general form (5) with $\epsilon = 1$ for some (possibly non connected) region of $s$ and $\epsilon = -1$ for the complementary region. A singular point is a point which joins two regions of opposite values of $\epsilon$. We moreover postulate, for physical reasons, that all the components of the metric are continuous at the singular points.

There are $N + 1$ intervals $I_i$, $i = 0, \ldots, N$

$$I_0 = [0, s_1], \ I_1 = [s_1, s_2], \ldots, I_{N-1} = [s_{N-1}, s_N], \ I_N = [s_N, 2\pi R] \quad (38)$$

of respective length

$$l_0 = s_1, \ l_1 = s_2 - s_1, \ l_2 = s_3 - s_2, \ldots, l_N = 2\pi R - s_N \quad (39)$$

Defining

$$r_i = -2(-1)^{i+1} \left( \sum_{j=0}^{i-1} (-1)^j s_{i-j} \right) \quad (40)$$

(note $r_0 = 0$) equivalent to

$$r_{2i} = 2 \sum_{j=1}^{i} l_{2j-1}$$

$$r_{2i+1} = -2 \sum_{j=0}^{i} l_{2j} \quad (41)$$

the metric takes the explicit form

for $s \in I_i : \ dS^2 = e^{-2k\epsilon((-1)^i s - r_i)} dx_\mu dx^\mu - ds^2 \quad (i = 0, \ldots, N) \quad (42)$

chosen to be normalised to one at $s = 0$. The sign of the coefficient of $s$ in the exponent alternates between $\epsilon$ and $-\epsilon$ for the intervals $I_i$ with even
and odd $i$. The end points of each interval are thus singular points, except $s = 0$ and $s = 2\pi R$ (see however the special case of a closure to a circle in Sec. 4.2).

The solution for the non-zero elements of the vielbein, of the vector connection, of the covariant derivatives of the vielbein and of the spinor connection depends on the intervals. For $s \in I_i$

$$
e_{\Sigma}^{\sigma} = e_{0}^{0} = e_{1}^{1} = e_{2}^{2} = e_{3}^{3} = e^{-\epsilon k((-1)^{i}s-r_i)}$$
$$
e_{\sigma}^{\Sigma} = e_{0}^{0} = e_{1}^{1} = e_{2}^{2} = e_{3}^{3} = e^{\epsilon k((-1)^{i}s-r_i)}$$

$$G_{500} = G_{050} = -G_{005} = -G_{5II}$$
$$=-G_{I5I} = G_{II5} = (-1)^i \epsilon k e^{-2\epsilon k((-1)^i s-r_i)}$$

$$G_{50}^{50} = G_{05}^{0} = G_{5I}^{I} = G_{I5}^{I} = (-1)^i \epsilon k$$

$$G_{00}^{5} = -G_{II}^{5} = (-1)^i \epsilon k e^{-2\epsilon k((-1)^i s-r_i)}$$

$$\left( D_{0}e \right)^{5} = \left( D_{I}e \right)^{5} = (-1)^i \epsilon k e^{-2\epsilon k((-1)^i s-r_i)}$$

$$\left( D_{0}e \right)^{5} = -\left( D_{I}e \right)^{5} = (-1)^i \epsilon k e^{-2\epsilon k((-1)^i s-r_i)}$$

$$G_{\Sigma}^{[5]} = -\frac{1}{2}(-1)^i \epsilon k e^{-\epsilon k((-1)^i s-r_i)} \gamma_{\sigma}\gamma^{5}.$$  (43)

In each subspace, the Dirac equation derived from (1) assumes the form

$$\left( e^{\epsilon k((-1)^i s-r_i)} (i \gamma^5 D_{\Sigma}) + (i \gamma^5) \left( \partial^5 - 2(-1)^i \epsilon k e \right) \right) \Psi = M \Psi.$$  (44)

### 3.2 Kaluza-Klein reduction with metric singularities

We adopt the following Kaluza-Klein separation of variables analogous to the no singularity case (22)

$$\Psi(x^\mu, s) = \sum_n \psi^{[n]}(x^\mu, s)$$

$$= \sum_n \left( F^{[n]}(s) + i G^{[n]}(s) \gamma^5 \right) \psi^{[n]}(x^\mu)$$  (45)

assuming $\psi^{[n]}(x^\mu)$ to be a spinor depending on $x^\mu$ only, and independent of the interval $I_i$ to which $s$ belongs. The complex functions $F^{[n]}(s)$ and $G^{[n]}(s)$
are functions depending on $s$ only. They are supposed to be smooth within the intervals $I_i$, where they may take different analytical forms, respectively $F^{[n,i]}(s)$ and $G^{[n,i]}(s)$.

Introducing the reduction (45) into the Dirac equation (44) in each subspace $I_i$ and postulating that the $\psi^{[n]}(x_\mu)$ satisfies the four dimensional Dirac equation (27), we find the two coupled equations

$$
\begin{aligned}
\partial_s G^{[n,i]} &= \left( M - \epsilon^k((-1)^i s - r_i) m_n \right) F^{[n,i]} + 2((-1)^i \epsilon k G^{[n,i]}) \\
\partial_s F^{[n,i]} &= 2((-1)^i \epsilon k F^{[n,i]} + \left( M + \epsilon^k((-1)^i s - r_i) m_n \right) G^{[n,i]}.
\end{aligned}
$$

(46)

3.3 Solutions with metric singularities

Following the same procedure as in the case without singularity (2.5) we find that in the interval $I_i$, the solution is

$$
\begin{aligned}
F^{[n,i]}(s) &= \frac{1}{2} \left( F^{[n,i]}_+(s) + F^{[n,i]}_-(s) \right) \\
G^{[n,i]}(s) &= \frac{1}{2} \left( F^{[n,i]}_+(s) - F^{[n,i]}_-(s) \right).
\end{aligned}
$$

(47)

The function $F^{[n,i]}_+(s)$ is the following linear superposition of Bessel functions.

$$
F^{[n,i]}_+(s) = \left( \frac{m_ne^{\epsilon k((-1)^i s - r_i)}}{k} \right)^{\frac{1}{2}} \left( \sigma_{n,i} J_{\epsilon((-1)^i s - r_i)} - \frac{1}{2} \left( \frac{m_ne^{\epsilon k((-1)^i s - r_i)}}{k} \right) \right)
$$

(48)

depending on two arbitrary constants $\sigma_{n,i}, \tau_{n,i}$ and with

$$
F^{[n,i]}_-(s) = \frac{e^{-\epsilon k((-1)^i s - r_i)}}{m_n} \left( -\partial_s F^{[n,i]}_+(s) + \left( M + 2\epsilon k((-1)^i) \right) F^{[n,i]}_+(s) \right).
$$

(49)

The constants $\sigma_{n,i}$ and $\tau_{n,i}$ (altogether $2(N+1)$ parameters) must satisfy 2$(N+1)$ homogeneous linear boundary relations expressing the boundary conditions (see (66)). For given boundary conditions, in order to obtain a non trivial solution for the $\sigma_{n,i}$ and $\tau_{n,i}$, the related 2$(N+1) \times 2(N+1)$ determinant must vanish, leading to a mass eigenvalue equation for the $m_n$. 

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3.4 Boundary relation and conditions for the spinor fields with metric singularities

In the boundary relation (21), the total derivative terms in Σ vanish since the fields are supposed to decrease sufficiently fast at infinity in the Σ directions. The fields are in general discontinuous at the metric singularity point. We define

\[
\Psi_l(s_i) = \lim_{\eta \to 0^+} \Psi(x^\mu, s_i - \eta)
\]

\[
\Psi_r(s_i) = \lim_{\eta \to 0^+} \Psi(x^\mu, s_i + \eta)
\]

Noting that \(\gamma^5 = \gamma^5 \) and (43), the boundary relation (21), after integration over \(s\), becomes

\[
\int d^4x \left( \sum_{i=1}^{N+1} \overline{\Phi}(s_i) \gamma^5 \Psi_l(s_i) \sqrt{g(s_i)} - \sum_{i=0}^{N} \overline{\Phi}(s_i) \gamma^5 \Psi_r(s_i) \sqrt{g(s_i)} \right) = 0 .
\]

Expanding Ψ and Φ according to the Kaluza-Klein reduction (45), leading to

\[
\overline{\Phi}(x^\mu, s) = \sum_n \phi^{[n]}(x^\mu) \left( C^{[n]}(s) - iD^{[n]}(s)\gamma^5 \right),
\]

one finds after some algebra

\[
\sum_{i=0}^{N} \left( D^{[m]sr}(s_i) F^[n]r(s_i) - C^{[m]sr}(s_i) G^[n]r(s_i) \right) \sqrt{g(s_i)}
\]

\[
- \sum_{i=1}^{N+1} \left( D^{[m]st}(s_i) F^[n]t(s_i) - C^{[m]st}(s_i) G^[n]t(s_i) \right) \sqrt{g(s_i)} = 0
\]

\[
\sum_{i=0}^{N} \left( C^{[m]sr}(s_i) F^[n]r(s_i) - D^{[m]sr}(s_i) G^[n]r(s_i) \right) \sqrt{g(s_i)}
\]

\[
- \sum_{i=1}^{N+1} \left( C^{[m]st}(s_i) F^[n]t(s_i) - D^{[m]st}(s_i) G^[n]t(s_i) \right) \sqrt{g(s_i)} = 0 .
\]

In terms of the left and right boundary values, we define the \(4(N+1)\)
dimensional vectors

\[
\Phi = \begin{pmatrix}
\sqrt{g(s_0)} C[n]I(s_0) \\
\sqrt{g(s_0)} D[n]I(s_0) \\
\sqrt{g(s_1)} C[n]I(s_1) \\
\sqrt{g(s_1)} D[n]I(s_1) \\
\vdots \\
\sqrt{g(s_N)} C[n]I(s_N) \\
\sqrt{g(s_N)} D[n]I(s_N) \\
\sqrt{g(s_{N+1})} C[n]I(s_{N+1}) \\
\sqrt{g(s_{N+1})} D[n]I(s_{N+1})
\end{pmatrix}, \quad \Psi = \begin{pmatrix}
\sqrt{g(s_0)} F[n]I(s_0) \\
\sqrt{g(s_0)} G[n]I(s_0) \\
\sqrt{g(s_1)} F[n]I(s_1) \\
\sqrt{g(s_1)} G[n]I(s_1) \\
\vdots \\
\sqrt{g(s_N)} F[n]I(s_N) \\
\sqrt{g(s_N)} G[n]I(s_N) \\
\sqrt{g(s_{N+1})} F[n]I(s_{N+1}) \\
\sqrt{g(s_{N+1})} G[n]I(s_{N+1})
\end{pmatrix}. \tag{55}
\]

The two boundary relations (53), (54) can be written in matrix form

\[
\Phi^* S_j^{[4(N+1)]} \Psi = 0, \quad j = 1, 2 \tag{56}
\]

where \( S_j \) are square matrices with upper index \([4(N+1)]\) referring to their size. For (53), the antisymmetric matrix \( S_1^{[4(N+1)]} \) has the following form

\[
S_1^{[4(N+1)]} = \begin{pmatrix}
S_1^{[2(N+1)]} & 0^{[2(N+1)]} \\
0^{[2(N+1)]} & -S_1^{[2(N+1)]}
\end{pmatrix} \tag{57}
\]

with the zero matrix \( 0^{[2(N+1)]} \) and the antisymmetric block diagonal matrix \( S_1^{[2(N+1)]} \)

\[
S_1^{[2(N+1)]} = \begin{pmatrix}
-i\sigma_2 & 0^{[2]} & \cdots \\
0^{[2]} & -i\sigma_2 & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix}. \tag{58}
\]

For (54), the matrix \( S_2^{[4(N+1)]} \) is block diagonal

\[
S_2^{[4(N+1)]} = \begin{pmatrix}
S_2^{[2(N+1)]} & 0^{[2(N+1)]} \\
0^{[2(N+1)]} & -S_2^{[2(N+1)]}
\end{pmatrix} \tag{59}
\]

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with the diagonal matrix $S_2^{[2(N+1)]}$

$$S_2^{[2(N+1)]} = \begin{pmatrix} \sigma_3 & 0^{[2]} & \ldots \\ 0^{[2]} & \sigma_3 & \ldots \\ \vdots & \vdots & \ddots \end{pmatrix}. \quad (60)$$

The allowed sets of boundary conditions can be obtained from the boundary relations (56), by the following general procedure. The boundary conditions are expressible in terms of $2(N+2)$ independent homogeneous linear relations among the components of the matrix $\Psi$ (55) chosen in such a way as to guarantee the two boundary relations (53) and (54). The components of $\Phi$ have of course to satisfy the same linear relations. The boundary conditions are written

$$M_{BC} \Psi = 0 \quad (61)$$

where $M_{BC}$ is a $2(N+1) \times 4(N+1)$ matrix of rank $2(N+1)$. For any such $M_{BC}$, there exists a $4(N+1) \times 4(N+1)$ permutation matrix $P$ such that, defining

$$\Psi_P \equiv P \Psi, \quad (62)$$

the $2(N+1)$ boundary conditions are equivalent to

$$\Psi_P = V_P^{[4(N+1)]} \Psi_P \quad (63)$$

with the matrix $V_P^{[4(N+1)]}$ written in terms of a matrix $V_P^{[2(N+1)]}$ (depending on $P$) and the unit matrix $1^{[2(N+1)]}$ as

$$V_P^{[4(N+1)]} = \begin{pmatrix} 1^{[2(N+1)]} & 0^{[2(N+1)]} \\ V_P^{[2(N+1)]} & 0^{[2(N+1)]} \end{pmatrix}. \quad (64)$$

Writing $\Psi_P$ in terms of its $2(N+1)$ upper elements $\Psi_P^u$ and its down elements $\Psi_P^d$,

$$\Psi_P = \begin{pmatrix} \Psi_P^u \\ \Psi_P^d \end{pmatrix} \quad (65)$$

one finds that the $2(N+1)$ first equations are trivial while the last $2(N+1)$ equations express the boundary conditions equivalent to (61)

$$\Psi_P^d = V_P^{[2(N+1)]} \Psi_P^u \quad (66)$$

This is in agreement with the observation that, from (61), there exists always a permutation $P$ of the component of $\Psi$ such that the $2(N+1)$ components ($\Psi_P^d$) are linear functions of the $2(N+1)$ other components ($\Psi_P^u$).
Writing $S^{[4(N+1)]}_{Pj}$ (j = 1, 2) the transformed of $S_j^{[4(N+1)]}$ under $P$

$$S^{[4(N+1)]}_{Pj} = P S_j^{[4(N+1)]} P^{-1},$$

(67)

the matrix $V^{[4(N+1)]}_P$ expressing the allowed boundary conditions (63), must satisfy the two matrix equations

$$\left( V^{[4(N+1)]}_P \right)^+ S^{[4(N+1)]}_{Pj} V^{[4(N+1)]}_P = 0. \tag{68}$$

This follows from the fact that the boundary relations (56) then depend on $\Psi^u_P$ and $\Phi^u_P$ only which are all independent and arbitrary.

With the four $2(N+1) \times 2(N+1)$ matrices $S^{[2(N+1)],r}_{Pj}$, r = 1, ..., 4, j = 1, 2 defined for each $S^{[4(N+1)]}_{Pj}$ as

$$S^{[4(N+1)]}_{Pj} = \begin{pmatrix} S^{[2(N+1)],1}_{Pj} & S^{[2(N+1)],2}_{Pj} \\ S^{[2(N+1)],3}_{Pj} & S^{[2(N+1)],4}_{Pj} \end{pmatrix}, \tag{69}$$

the boundary relations (56) lead explicitly to two equations for the matrix $V^{[2(N+1)]}_P$

for $j = 1, 2$

$$S^{[2(N+1)],1}_{Pj} + \left( V^{[2(N+1)]}_P \right)^+ S^{[2(N+1)],3}_{Pj} + S^{[2(N+1)],2}_{Pj} V^{[2(N+1)]}_P + \left( V^{[2(N+1)]}_P \right)^+ S^{[2(N+1)],4}_{Pj} V^{[2(N+1)]}_P = 0. \tag{70}$$

It should be stressed that different choices of $P$ may lead to equivalent, differently expressed, boundary conditions, in particular by multiplying $P$ by further permutations within the elements of $\Psi^u$ or within the elements of $\Psi^d$. A few examples of boundary conditions are given in App. (B).

4 Physical considerations

In our previous article [4], we have given a detailed discussion of the physical relevance of the main aspects underlying the Kaluza-Klein construction for scalars. We summarize here the points which apply to the spinor case.
4.1 Physical discussion of the generalized boundary conditions

It happens that the boundary conditions (66) may connect the values of the components $F$ and $G$ of the field (not their derivatives as in the scalar case) at different points of the $s$-domain i.e. at the $N$ metric singular points and at the two edges. In this case, the field explores, in fact, its full domain at once. This is tantamount to action at a distance or non locality, which we argued in [4] not to be in contradiction with quantum mechanics.

In our numerical applications (6) however, we restrict ourselves either to fully local boundary conditions (locality at the metric singular points as well as at the edges (App.B.2)) or to partially local boundary conditions (excluding locality at the edges (App.B.1)).

4.2 Closure to a circle

The strip could be closed in a circle by identifying the points $s = 0$ and $s = 2\pi R$ under the following requirements.

There must be an even number $2p$ ($p > 0$) of singularities. By rotation around the circle, the first singularity can always be placed at the closure point $s = 0$. Then $N \equiv 2p-1$. The total range where the sign of $s$ in the exponential in the metric is positive must be equal to the total range where it is negative (38), (39)

$$
\sum_{j=0}^{j=p-1} l_{2j} = \sum_{j=0}^{j=p-1} l_{2j+1} = \pi R .
$$

4.3 The ”one-mass-scale-only” hypothesis

By assumption, there is only one high mass scale in the theory which is chosen to be the Planck mass

$$
M_{Pl} \approx 1.22 \times 10^{16} \text{ TeV} .
$$

The dimensionfull parameters $k$, $R$ and $M$ can be written in terms of reduced parameters $\bar{k}$, $\bar{R}$ and $\bar{M}$

$$
k = \bar{k} M_{Pl}
$$

$$
R = \bar{R} (M_{Pl})^{-1}
$$

$$
M = \bar{M} M_{Pl} .
$$
We call the assumption that the reduced parameters are neither large nor small numbers (except 0) the “one-mass-scale-only” hypothesis. The parameter $\overline{k}R = kR$ governs the reduction from the high mass scale to the TeV scale of the low lying masses in the Kaluza-Klein towers.

Finally, let us note that by rescaling the parameter $\overline{k}$ can always be chosen to be equal to one
\[ \overline{k} = 1 . \] (74)
Since the mass eigenvalue equation are covariant under a rescaling of all the reduced parameters $p$ according to their energy dimension $d_p$
\[ p \rightarrow \lambda^d p, \] (75)
one finds that the mass eigenvalues for a given $\overline{k}$ can be obtained from eigenvalues corresponding to our choice $\overline{k} = 1$ (using $\lambda = 1/\overline{k}$) by
\[ m_n\left( \frac{\overline{k}, R, M}{1, \overline{k}R, \frac{M}{\overline{k}}} \right) = \overline{k}m_n\left( \frac{1, kR, \frac{M}{k}}{1} \right) . \] (76)

4.4 The physical masses

For a four-dimensional observer supposed to be sitting at $s = s_{\text{obs}}$ in a given $I_i$ interval (38), the metric (42)
\[ dS^2 = e^{-2\bar{c}((-1)^i s_{\text{obs}} - r_i)} dx_\mu dx^{\mu} - ds^2 \] (77)
can be transformed in canonical form
\[ dS^2 = d\bar{x}_\mu d\bar{x}^{\mu} - ds^2 \] (78)
by the following rescaling
\[ \bar{x}_\mu = e^{-\bar{c}((-1)^i s_{\text{obs}} - r_i)} x_\mu . \] (79)
According to (27), we have
\[ i\gamma^\mu \overline{\partial}_\mu \psi[n] = e^{\bar{c}((-1)^i s_{\text{obs}} - r_i)} (i\gamma^\mu \partial_\mu \psi[n]) = e^{\bar{c}((-1)^i s_{\text{obs}} - r_i)} m_n \psi[n] . \] (80)
The mass as seen in by the observer in the brane at $s = s_{\text{obs}} \in I_i$ is thus related to the mass eigenvalue $m_n$ by
\[ m_{\text{obs}}^n = e^{\bar{c}((-1)^i s_{\text{obs}} - r_i)} m_n . \] (81)
For $s_{\text{obs}} = 0$, the physical mass is just equal to the mass eigenvalue.
4.5 Probability density

Once all the parameters defining a specific model are chosen and the mass eigenvalue tower is determined, there exists a unique field $\psi[n](x^\mu, s)$ (see (15)) for each mass eigenvalue leading to a naive probability density field distribution $D[n](x^\mu, s)$ which depends both on $x^\mu$ and $s$

$$D[n](x^\mu, s) = \sqrt{g} \left( \bar{\psi}[n](x^\mu, s) \psi[n](x^\mu, s) \right).$$

(82)

Note that the shape of this density distribution depends in general on the interval $I_i$ to which $s$ belongs. As observed and discussed in [3], these probability densities are fast varying functions of $s$. The total normalized probability density for a Kaluza-Klein particle to be present in a brane situated at $s = s_{\text{obs}}$ is

$$D[n](s_{\text{obs}}) = \frac{\int d^4x \ D[n](x^\mu, s_{\text{obs}})}{\int d^4x \ ds \ D[n](x^\mu, s)}.$$ 

(83)

Remember that the physical mass as seen by the observer is also a function of the $s_{\text{obs}}$ position (81).

5 Towers

In the absence of metric singularities, the two arbitrary parameters $\sigma$ and $\tau$ which appear in the solution (37), (35), (36) of the Dirac equation (11) in the five dimensional space after the KK reduction (22) have to satisfy two homogeneous linear equations expressing an allowed set of boundary conditions, belonging either to the set BC1 (24) or to the set BC2 (26). The condition for the existence of a non trivial $\sigma, \tau$ solution is the vanishing of the related determinant. This leads in each case to a mass equations from which the KK mass towers can be derived. In Sect.(6.1), numerical examples of KK mass towers are given for each of the two sets of boundary conditions, for different values of the basic parameters of the model, i.e. the warp factors $\epsilon, k, kR$, the bulk mass $M$, as well as for different values of the parameters $\rho, \omega$ or $\epsilon_0, \epsilon_B$ defining the boundary conditions considered.

In the general case, when there are $N$ metric singularities, there are $N+1$ parameters $\sigma_{n,i}$ and $N+1$ parameters $\tau_{n,i}$ appearing in the solution (37), (48), (49) of the Dirac equation after the Kaluza-Klein reduction (15). These parameters have to satisfy the $2(N+1)$ homogeneous linear equations (66).
resulting from the imposition of the $2(N+1)$ boundary conditions on the $2(N+1)$ values of the fields at the edges of the $N+1$ intervals $I_i$ in the $s$-range (50). Indeed, for a given singularity configuration, there exists a set of $2(N+1)$ boundary conditions resulting from the two boundary relations (56) expressing the condition of hermiticity of the Dirac operator. As in the preceding case, the requested vanishing of the determinant of the coefficients of the $2(N+1)$ boundary conditions with respect to the $2(N+1)$ parameters $(\sigma_{n,i}, \tau_{n,i})$ leads to the corresponding KK mass equation. In Sect. (6.2), a few examples of towers are given when there is one singularity.

For completeness, let us list all the parameters. They are the basic parameters of the warp model $k, \epsilon, kR$, the bulk mass $M$, the positions $s_i$ of the $N$ metric singularities and the boundary parameters defining the matrix $V_p^{[2(\sigma_{n,i})]}$ subject to the two conditions (70). Once all these parameters are chosen, the vanishing of the above determinant is generally a transcendental function of the eigenvalues $m_n$.

6 Examples of towers

For an illustration of the types of spinor towers which appear in warped spaces, we construct examples of the eight lowest mass eigenvalues for simple specific boundary conditions. We first discuss the case when there is no metric singularity, then when there is one metric singularity. We would like to stress that, in order to perform the numerical computations, high precision is mandatory.

6.1 Examples of towers. No metric singularity

In this subsection, a few illustrative numerical examples of Kaluza-Klein spinor towers in warped spaces are presented for each of the two sets BC1 (24) and BC2 (26) of boundary conditions and for some chosen values of the bulk mass $M$ and of the parameters fixing the boundary conditions. In general, the Kaluza-Klein mass eigenvalues are irregularly spaced. With the adopted values of the basic parameters of the warp model, i.e. $k$ arbitrarily normalized to the Planck mass ($\bar{k} = 1$, see (73)) and $kR \approx 6.3$, all the low lying Kaluza-Klein masses are of the order TeV. In the tables, $kr$ is fixed to

$$kR = 6.3$$ (84)
and the Kaluza-Klein tower masses denoted with $\tilde{m}_i$ are given in TeV. As a general rule, the values of $\tilde{m}_i$ decrease (exponentially) when $kR$ increases, hence fixing the overall scale of the masses in the tower. It should be noted that choosing the value of the bulk mass $M$ to zero or to values of the order of the Planck mass, within the one-mass-scale-only (4.3), does not lead to substantially different Kaluza-Klein towers.

In Table(1), the eight low lying mass eigenvalues ($\tilde{m}_i$, $i = 1, \ldots, 8$) of Kaluza-Klein towers are given in the case of boundary conditions BC1 (24) for zero bulk mass $M$, for different values of the parameter $\rho$, and for each of them, for different values of the parameter $\omega$ (25). One observe that the first mass of the towers, $\tilde{m}_1$, is relatively sensitive to the value of $\omega$, particularly for small values of the parameter $\rho$. For $\omega = \rho = 0$, the Kaluza-Klein tower exhibits some characteristic features: it is the only tower to possess a zero mass state while the higher masses are doubly degenerate. Indeed, one sees that, for $\rho = 0$, when $\omega$ decreases toward zero, pairs of adjacent masses in the towers are getting closer and closer and take the same value when $\omega$ reaches the value zero.

In Table(2), Table(3) and Tables(4)-(5), the Kaluza-Klein mass towers are similarly presented for a representative choice of bulk mass values, respectively $M = 0.01$, $M = 0.1$ and $M = 1$ (73). It appears, as a general rule, that the lowest lying mass of the towers $\tilde{m}_1$ vanishes when $\rho = 0$ and when the parameter $\omega$ takes exactly the value $\omega_M$:

$$\rho = 0 \text{ and } \omega = \omega_M = 2\pi M \implies m_1 = 0.$$  

(85)

This agrees with the analogous result for $M = 0$ as seen in Table(1). Moreover, for $\rho = 0$ and for any $M$ of the order 1, the value of $\tilde{m}_1$ depends almost exactly linearly on the value of the parameter $\omega$ from about $\omega = \omega_M - 1$ up to values very close to $\omega_M$ and on the other side from $\omega$ very close to $\omega_M$ up to about $\omega_M + 1$.

In general, except for the first mass of the towers in the case $\rho = 0$, all the other masses in the towers do not show a strong dependence on the value of the $\omega$ parameter. The fact that the first mass of the tower can take values between 0 and about 0.1 TeV, and hence can be small when $\rho$ is not large, allows one, by a suitable choice of the parameters $kR$, $M$, $\omega$ to associate a tower to a particular fermion of the Standard Model be it a lepton or a quark and assuming it to be the lowest state of a Kaluza-Klein tower in a five
dimensional warped space. From the second mass on, the intervals between successive masses are generally much larger and more regular.

In Table 6, Kaluza-Klein towers are presented for the set of boundary conditions BC2 (set (26)) for the two possible choices of the product $\epsilon_0 \epsilon_R$ of the boundary condition parameters, and for each of them, for some values of the reduced bulk mass $\overline{M}$. In general, the tower masses have a fairly mild dependence with respect to the bulk mass, with the exception of the first mass in the towers when $\epsilon_0 \epsilon_R$ is equal to -1, in which case, starting from a value of about 0.1 TeV for $\overline{M} = 0$, it falls to less than $10^{-10}$ TeV when $\overline{M}$ is equal to one or higher. This feature is again of importance in view of practical applications of the warp model to fermions, either to the leptons or to the quarks of the Standard Model. Indeed, by an adequate choice $kR$ and of the bulk mass $M$, the first mass of a Kaluza-Klein tower could be made equal to the mass of a given lepton, for example the muon, leading to identify the tower as associated to that lepton. Considering for example $kR = 6.3$ and the case of the muon, with mass equal to $1.057 \times 10^{-4}$ TeV as the lowest mass in a tower, the associated Kaluza-Klein tower would result from adopting a value around 0.65 for the reduced bulk mass of the five-dimensional fermion $\overline{M}_\mu$ associated to the muon. A not very different reduced bulk mass $\overline{M}_e = 0.8$ would produce the electron of mass equal to $5.11 \times 10^{-7}$ TeV as its first mass. It is interesting to remark that bulk fermions with rather close reduced bulk masses (0.65, 0.8) would lead to the observed fermions with masses in the large ratio $m_\mu/m_e = 206.8$. It should be noted that the Kaluza-Klein tower masses associated to either of these two leptons would be hardly distinguishable beyond the first mass. One should also be aware that the Kaluza-Klein towers associated to a given fermion would be of a different structure depending on the set (BC1 or BC2) of boundary conditions considered.

### 6.2 Examples of towers. One metric singularity. Semi-local boundary conditions

When there is one metric singularity, the number of arbitrary parameters increases. Besides $kR$ and $M$, the position of the singularity on the strip $[0, 2\pi R]$ appears as a new parameter

$$s_1 = (2\pi R) \overline{s}_1 , \quad 0 \leq \overline{s}_1 \leq 1 . \quad (86)$$

which is complemented by the boundary condition parameters.
In order to keep the mass eigenvalues roughly of the order of TeV, we are led to adapt the value of $kR$ to the value chosen for $\varphi_1$. Satisfactory choices are

$$\begin{align*}
\varphi_1 &= 1 \quad \longleftrightarrow \quad kR = 6.3 \\
\varphi_1 &= 0.9 \quad \longleftrightarrow \quad kR = 6.9 \\
\varphi_1 &= 0.75 \quad \longleftrightarrow \quad kR = 8.3 \\
\varphi_1 &= 0.5 \quad \longleftrightarrow \quad kR = 12.5 .
\end{align*}$$

(87)

There are many possible sets of allowed boundary conditions as seen in the discussion of the appendix App. [13]. To build our examples, we have limited ourselves to what we call semi-local boundary conditions: the fields on the left and on the right of the singularity are related and, separately, the fields at $s = 0$ are related to the fields at $s = 2\pi R$. Both boundary conditions are taken of the form BC1 (24), (25) and hence are defined by four parameters

- $BC1$ with parameters $\omega_b, \rho_b$ at the edges $0$ and $2\pi R$
- $BC1$ with parameters $\omega_s, \rho_s$ on both sides of the singularity $s_1$. (88)

The conditions for $m_1 = 0$ are analogous to the conditions in the case of no metric singularity (85)

$$\left\{ \rho_b - \rho_s = 0 \text{ and } \omega_b - \omega_s = \omega_M = 2\pi RM \right\} \quad \longleftrightarrow \quad m_1 = 0 .$$

(89)

In Table(7), for $M = 1$, numerical examples of towers are given for some arbitrarily chosen positions $s_1$ of the metric singularity and some values of the boundary parameters (85). Similar results, respectively for $M = 0.1$ and $M = 0$, are presented in Tables (8) and (9).

Again in view of applications to leptons and quarks, it should be noted that, when the parameters almost satisfy the mass zero conditions (89), the tower consists of a low mass $\tilde{m}_1$ accompanied, as a signature, by almost regularly separated doublets of higher masses with $\tilde{m}_2$ much larger than $\tilde{m}_1$. 

22
7 Conclusions

In this article, we have extended our previous study of the generation of Kaluza-Klein mass towers for spinor fields propagating in a five dimensional flat space with the fifth dimension compactified either on a strip or on a circle. We have now studied spinor fields propagating in five dimensional compactified warp spaces.

We first considered the case of a warp space without metric singularity. We established the specific Dirac equation in the relevant five dimensional warp space for a spinor field with an arbitrary bulk mass $M$ and proceeded with the Kaluza-Klein reduction considering the most general choice of separation of variables compatible with a SO(3,1) spinor covariance. The reduced components of the Dirac fields are found to satisfy a system of two coupled equations for which the most general solutions for a four dimensional mass $m$ are given in terms of Bessel functions.

From the requirement of hermiticity of the Dirac operator, we have established all the allowed sets of boundary conditions which have to be imposed on the fields. We found that these boundary conditions belong to two essentially different sets $BC1$ and $BC2$, leading to the mass equations from which the Kaluza-Klein mass towers can be built. The same considerations have been extended to the case of warp spaces with an arbitrary number of metric singularities.

In view of the interpretation of the Kaluza-Klein mass eigenstates, specific physical considerations have been made about the possible choices of boundary conditions, about the particular situation in which the extradimension strip could be closed to a circle, about the mass scale of the model, about the relation between the Kaluza-Klein masses and the physical masses as observed in a brane and also about the mass state probability densities. In particular, all the parameters with energy dimension are scaled to the Plank mass within the only-one-mass-scale hypothesis.

Finally, illustrative numerical examples of Kaluza-Klein mass towers are given when there is no metric singularity for each of the two sets of boundary conditions, $BC1$ and $BC2$. When there is one metric singularity we have exemplified towers for some boundary conditions belonging to what we call the semi-local set. With $kR = 6.3$ or around, it happens that the low lying masses are of the order of TeV, thereby solving the hierarchy problem without fine tuning.

In the different situations considered, the towers have been established
for several choices of the basic parameters of the warp model, i.e. the mass reduction parameter $kR = 6.3$ (suitably readjusted to the value of $s_1$), the bulk mass $M$, the position $s_1$ of the singularity on the extradimension if any, as well as of the parameters defining the boundary conditions. In general, the mass towers are irregularly spaced, and a zero mass state or a small mass state exists which depends on the boundary parameters and on the value given to the bulk mass $M$. This situation allows one, by a suitable choice of the parameters of the model, to associate a mass tower to any particular fermion of the standard model whose mass would be the smallest in the tower. In the assumption that the known leptons and quarks propagate in the bulk under consideration, one would expect to observe the next low lying masses at high energy colliders, in particular at the LHC.

**Acknowledgment:** The authors would like to thank Professor Yves Brihaye for an important suggestion.
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A Least action principle

The most general invariant action linear in $\Psi$ and $\overline{\Psi}$ and of first order in their derivatives is, using (13),

$$\mathcal{A} = i\alpha \int \Psi \gamma^A (\overline{D}_A \Psi) \sqrt{g} d^5 x + i\beta \int (\overline{\Psi} \overline{D}_A) \gamma^A \Psi \sqrt{g} d^5 x$$

$$- m \int \overline{\Psi} \Psi \sqrt{g} d^5 x .$$  

(90)

The underlying Lagrangian is hermitian if

$$\beta = -\alpha^*, \quad m = m^*$$  

(91)

and we choose

$$\alpha = \frac{\alpha_1 + i\alpha_2}{2}, \quad \beta = \frac{-\alpha_1 + i\alpha_2}{2}.$$  

(92)

Let us note the useful identity

$$\overline{\Psi} \gamma^A (\overline{D}_A \Psi) \sqrt{g} + (\overline{\Psi} \overline{D}_A) \gamma^A \Psi \sqrt{g} = \partial_A \left( \overline{\Psi} \gamma^A \Psi \sqrt{g} \right).$$  

(93)

Requesting then the variation of the action (90)

$$\delta \mathcal{A} = \int \delta \overline{\Psi} \left( \alpha_1 i\gamma^A \overline{D}_A - m \right) \Psi \sqrt{g} d^5 x$$

$$- \int \overline{\Psi} \left( \alpha_1 i\overline{D}_A \gamma^A + m \right) \delta \Psi \sqrt{g} d^5 x$$

$$+ i \frac{\alpha_1 + i\alpha_2}{2} \int \partial_A \left( \overline{\Psi} \gamma^A \delta \Psi \sqrt{g} \right) d^5 x$$

$$+ i \frac{-\alpha_1 + i\alpha_2}{2} \int \partial_A \left( \delta \overline{\Psi} \gamma^A \Psi \sqrt{g} \right) d^5 x.$$  

(94)

to vanish for arbitrary variations $\delta \Psi$ and $\delta \overline{\Psi}$ of the fields $\Psi$ and $\overline{\Psi}$, one finds the Dirac equations provided that

$$\alpha_1 \neq 0 \quad \text{conveniently normalised to: } \alpha_1 = 1.$$  

(95)

They are

$$i\gamma^A (\overline{D}_A \Psi) - m \Psi = 0$$

$$i(\overline{\Psi} \overline{D}_A) \gamma^A + m \overline{\Psi} = 0.$$  

(96)
independently of the boundary conditions. Indeed, since the action is linear in $\Psi$, if it is extremal for two $\delta \Psi$ with the same boundary conditions, it is also extremal for their difference which is automatically zero at the boundaries. Hence, the field equations are those obtained from the usual least action principle i.e. with vanishing variations at the boundaries (96).

However, further attention has to be devoted to the variation of the action arising from the boundary terms (third and fourth term in (94)). Suppose that there are $N$ metric singularities located at the points $s_i$, $i = 1, N$ in the $s$ space extending from $s_0 = 0$ to $s_{N+1} = 2\pi R$. Denote by $\Psi^l(s_i)$ and $\Psi^r(s_i)$ the values of the fields at the left and at the right of the metric singularities, and similarly for their variations. The boundary relations expressed from the boundary terms in (94) become

\[
\sum_{i=1}^{N+1} \Psi^l(s_i) \gamma^s \delta \Psi^l(s_i) \sqrt{g(s_i)} - \sum_{i=0}^{N} \Psi^r(s_i) \gamma^s \delta \Psi^r(s_i) \sqrt{g(s_i)} = 0 \quad (97)
\]

\[
\sum_{i=1}^{N+1} \delta \Psi^l(s_i) \gamma^s \Psi^l(s_i) \sqrt{g(s_i)} - \sum_{i=0}^{N} \delta \Psi^r(s_i) \gamma^s \Psi^r(s_i) \sqrt{g(s_i)} = 0. \quad (98)
\]

It is natural to suppose that the variations $\delta \Psi$, $\delta \overline{\Psi}$ and the fields $\Psi$ and $\overline{\Psi}$ belong to the same Hilbert space i.e. satisfy the same boundary conditions. The relations (97) and (98) then imply boundary conditions which turn out to be identical to those obtained in the main part of the article from (21) which resulted from the requirement of symmetry of the Dirac operator (17).
B Examples of boundary relations

There are many inequivalent sets of boundary conditions related to various choices of the permutation \( P \) in (62). Let us give a few.

B.1 \( P=1 \). Local boundary conditions at the metric singular points. Non local conditions at the edges of the \( s \)-domain

With \( P = 1 \), one can obtain boundary conditions which satisfy the locality conditions (see Sect.(4.1)) at the singular points but not at the edges. The form of \( V_P^{2(N+1)} \) compatible with this partial-locality is

\[
V_P^{2(N+1)} = \begin{pmatrix}
0^2 & V_1^{[2]} & 0^2 & 0^2 & \ldots & 0^2 \\
0^2 & 0^2 & V_2^{[2]} & 0^2 & \ldots & 0^2 \\
0^2 & 0^2 & 0^2 & V_3^{[2]} & \ldots & 0^2 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0^2 & 0^2 & 0^2 & 0^2 & \ldots & V_N^{[2]} \\
V_{N+1}^{[2]} & 0^2 & 0^2 & 0^2 & \ldots & 0^2 \\
\end{pmatrix}.
\]  

Introducing this form of the matrix in the equations (70), one finds for all \( j \) \((j = 1, \ldots, N+1)\)

\[
V_j^{[2]+(i\sigma_2)V_j^{[2]} = i\sigma_2} \tag{100}
\]
\[
V_j^{[2]+(\sigma_3)V_j^{[2]} = \sigma_3} \tag{101}
\]

From (100), all the \( V_j^{[2]} \) are complex symplectic 2 \( \times \) 2 matrices restricted by the further condition (101). Their resulting general form is

\[
V_j^{[2]} = e^{i\rho_j} \begin{pmatrix}
\cosh(\omega_j) & \sinh(\omega_j) \\
\sinh(\omega_j) & \cosh(\omega_j) \\
\end{pmatrix} , \quad j = 1, \ldots, N+1 \tag{102}
\]

and depends on 2\((N+1)\) arbitrary parameters. Hence the explicit boundary conditions at the metric singularities \( j = 1, \ldots, N \) are

\[
\begin{pmatrix} F[n]^l(s_j) \\ G[n]^l(s_j) \end{pmatrix} = e^{i\rho_j} \begin{pmatrix} \cosh(\omega_j) & \sinh(\omega_j) \\ \sinh(\omega_j) & \cosh(\omega_j) \end{pmatrix} \begin{pmatrix} F[n]^r(s_j) \\ G[n]^r(s_j) \end{pmatrix} . \tag{103}
\]
At the edges, the boundary conditions are non local

\[
\begin{pmatrix} F^{[n]}(2\pi R) \\ G^{[n]}(2\pi R) \end{pmatrix} = \frac{4}{g(2\pi R)} e^{i\rho_{N+1}} \begin{pmatrix} \cosh(\omega_{N+1}) & \sinh(\omega_{N+1}) \\ \sinh(\omega_{N+1}) & \cosh(\omega_{N+1}) \end{pmatrix} \begin{pmatrix} F^{[n]}(0) \\ G^{[n]}(0) \end{pmatrix}.
\]

since they connect the values of the fields at \( s = 2\pi R \) to the values of the fields at \( s = 0 \) (a long distance effect). When the conditions (71) for the closure of the strip to a circle are met, these would also be local boundary conditions.

### B.2 Local boundary conditions both at the metric singular points and at the edges of the \( s \)-domain

A way to obtain fully local boundary conditions is to perform the following permutation

\[
P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 \end{pmatrix}.
\]

and to take \( V^{[2(N+1)]} \) of the form (99) with \( V^{[2]}_{N+1} \) diagonal. This leads, on one side to a relation between \( G^{[N]}(s_{N+1}=2\pi R) \) and \( F^{[N]}(s_{N+1}=2\pi R) \), on the other side to a relation between \( G^{[N]}(s_0=0) \) and \( F^{[N]}(s_0=0) \). The conditions for \( V^{[N]}_j \) (\( j = 1, \ldots, N \)) are the same as in the preceding case (100), (101), leading to the same conditions at each of the singular points (102). For the diagonal \( V^{[2]}_{N+1} \),

\[
V^{[2]}_{N+1} \sigma_3 = \sigma_3 V^{[2]}_{N+1}
\]

(106)

\[
V^{[2]}_{N+1} \sigma_3 V^{[2]}_{N+1} = \sigma_3.
\]

(107)

Introducing (106) in (107) on see that \( (V^{[2]}_{N+1})^2 = 1^{[2]} \).

This leads to boundary conditions at the singularities \( (s_j, j=1, \ldots, N) \) as
above (103) and to
\[
\begin{align*}
G^{[n]}(0) &= \epsilon_0 F^{[n]}(0) , & \epsilon_0^2 &= 1 \\
G^{[n]}(2\pi R) &= \epsilon_R F^{[n]}(2\pi R) , & \epsilon_R^2 &= 1
\end{align*}
\tag{108}
\]
at the edges.

## B.3 General boundary conditions for \( P = 1 \)

When \( P = 1 \), i.e. when the boundary conditions express the values at the left of the exceptional points (singularities and edges) in terms of the values at the right, the two equations \((j=1, 2) (70)\) take the simplified form with
\[
V^{[2(N+1)]} \equiv V_{P=1}^{[2(N+1)]} , \quad S_1^{[2(N+1)]} \quad \text{from (58)} \quad \text{and} \quad S_2^{[2(N+1)]} \quad \text{from (60)}
\]
\[
(V^{[2(N+1)]})^+ S_j^{[2(N+1)]} V^{[2(N+1)]} = S_j^{[2(N+1)]} .
\tag{109}
\]
The matrix \( V^{[2(N+1)]} \) must be an element in the intersection of the complex sympletic group \( Sp(2(N+1)) \) (from the relation for \( j = 1 \)) and of the pseudo-unitary group \( U(N+1, N+1) \) (from the relation \( j = 2 \)). The dimension of the parameter space can be obtained by writing \( V^{[2(N+1)]} \) infinitesimally close to the identity \( 1^{[2(N+1)]} \)
\[
V^{[2(N+1)]} = 1^{[2(N+1)]} + i\eta H^{[2(N+1)]} , \quad \eta \to 0 \tag{110}
\]
in 2 \( \times \) 2 blocks. One finds that there are, for \( H^{[2(N+1)]} \), \( N+1 \) diagonal 2 \( \times \) 2 blocks \( H_{jj} \) \((j = 1, \ldots , N+1)\), each depending on two real parameters
\[
H_{jj} = \begin{pmatrix} p_j & iq_j \\ iq_j & p_j \end{pmatrix} , \quad p_j, q_j \quad \text{real} \tag{111}
\]
and \( N(N+1) \) independent non diagonal 2 \( \times \) 2 blocks \( H_{jk} \), \( j < k = 1, \ldots , N+1 \), each depending on two complex (four real) parameters \( p_{jk} \) and \( q_{jk} \)
\[
H_{jk} = \begin{pmatrix} p_{jk} & q_{jk} \\ q_{jk} & p_{jk} \end{pmatrix} , \quad H_{kj} = \begin{pmatrix} p_{jk}^* & -q_{jk}^* \\ -q_{jk}^* & p_{jk}^* \end{pmatrix} .
\tag{112}
\]
Hence, the set of boundary conditions for \( P = 1 \) is indexed by \( 2(N+1)^2 \) real parameters.

Let us finally remark that contrary to what happens for the scalar fields where the boundary conditions relate the fields and their derivatives, the boundary parameters have zero energy dimension (73) and, hence, there is no need to introduce reduced parameters in the spinor case.
Table 1: Mass towers for $M = 0$ and for the boundary conditions BC1 (no metric singularity) (24), (25). The towers are symmetric under $\omega \leftrightarrow -\omega$. The mass eigenvalues $\tilde{m}_i$ are in TeV.

| $\rho$ | $\omega$ | $m_1$ | $m_2$ | $m_3$ | $m_4$ | $m_5$ | $m_6$ | $m_7$ | $m_8$ |
|--------|----------|-------|-------|-------|-------|-------|-------|-------|-------|
| $\pi/10$ | 0 | 0.24681 | 0.46894 | 0.5184 | 0.9626 | 1.1194 | 1.456 | 1.5055 | 1.9497 |
| 0.01 | 0.024693 | 0.46893 | 0.51832 | 0.96256 | 1.0119 | 1.4562 | 1.5056 | 2.0004 |
| 0.1 | 0.025858 | 0.46777 | 0.51948 | 0.96139 | 1.0131 | 1.455 | 1.5067 | 1.9486 |
| 1 | 0.071234 | 0.42239 | 0.56486 | 0.91602 | 1.0585 | 1.4096 | 1.5521 | 1.9033 |
| 5 | 0.1224 | 0.37123 | 0.61603 | 0.86485 | 1.097 | 1.3585 | 1.6033 | 1.8521 |
| 100 | 0.12341 | 0.37022 | 0.61703 | 0.86384 | 1.1107 | 1.3575 | 1.6043 | 1.8511 |
| $\pi/4$ | 0 | 0.061703 | 0.43192 | 0.5553 | 0.92555 | 1.049 | 1.4192 | 1.5426 | 1.9128 |
| 0.01 | 0.061707 | 0.43192 | 0.55534 | 0.92554 | 1.049 | 1.4192 | 1.5426 | 2.0004 |
| 0.1 | 0.062094 | 0.43153 | 0.55572 | 0.92516 | 1.0493 | 1.4188 | 1.543 | 1.9124 |
| 1 | 0.08601 | 0.40762 | 0.57963 | 0.90124 | 1.0733 | 1.3949 | 1.5669 | 1.8885 |
| 5 | 0.12266 | 0.37097 | 0.61628 | 0.86459 | 1.1099 | 1.3582 | 1.6035 | 1.8518 |
| 100 | 0.12341 | 0.37022 | 0.61703 | 0.86384 | 1.1107 | 1.3575 | 1.6043 | 1.8511 |
| $\pi/2$ | 0.12341 | 0.37022 | 0.61703 | 0.86384 | 1.1107 | 1.3575 | 1.6043 | 1.8511 |
| 0.01 | 0.18451 | 0.30852 | 0.67874 | 0.80214 | 1.172 | 1.2958 | 1.665 | 1.7894 |
| 0.1 | 0.18472 | 0.30891 | 0.67834 | 0.80253 | 1.172 | 1.2962 | 1.6656 | 1.7898 |
| 1 | 0.1608 | 0.33282 | 0.65443 | 0.82645 | 1.1481 | 1.3201 | 1.6417 | 1.8137 |
| 5 | 0.12415 | 0.36947 | 0.61778 | 0.8631 | 1.1114 | 1.3567 | 1.605 | 1.8503 |
| 100 | 0.12341 | 0.37022 | 0.61703 | 0.86384 | 1.1107 | 1.3575 | 1.6043 | 1.8511 |
| $3\pi/4$ | 0 | 0.18551 | 0.30852 | 0.67874 | 0.80214 | 1.172 | 1.2958 | 1.665 | 1.7894 |
| 0.01 | 0.18472 | 0.30891 | 0.67834 | 0.80253 | 1.172 | 1.2962 | 1.6656 | 1.7898 |
| 0.1 | 0.1608 | 0.33282 | 0.65443 | 0.82645 | 1.1481 | 1.3201 | 1.6417 | 1.8137 |
| 1 | 0.12415 | 0.36947 | 0.61778 | 0.8631 | 1.1114 | 1.3567 | 1.605 | 1.8503 |
| 5 | 0.12415 | 0.36947 | 0.61778 | 0.8631 | 1.1114 | 1.3567 | 1.605 | 1.8503 |
| 100 | 0.12341 | 0.37022 | 0.61703 | 0.86384 | 1.1107 | 1.3575 | 1.6043 | 1.8511 |
| $\pi$ | 0.12341 | 0.37022 | 0.61703 | 0.86384 | 1.1107 | 1.3575 | 1.6043 | 1.8511 |
| 0.01 | 0.2397 | 0.25466 | 0.73259 | 0.74829 | 1.2262 | 1.2419 | 1.7198 | 1.7355 |
| 0.1 | 0.1788 | 0.31483 | 0.67242 | 0.80846 | 1.166 | 1.3021 | 1.6507 | 1.7957 |
| 5 | 0.12447 | 0.36916 | 0.61809 | 0.86279 | 1.1117 | 1.3564 | 1.6053 | 1.85 |
| 100 | 0.12341 | 0.37022 | 0.61703 | 0.86384 | 1.1107 | 1.3575 | 1.6043 | 1.8511 |
Table 2: Mass towers for $\overline{M} = 0.01$ and for the boundary conditions BC1 (no metric singularity) (24), (25). Here $\omega_{0.01} = 0.3958406 \ldots$. The mass eigenvalues $\tilde{m}_i$ are in TeV.

| $\overline{M}$ | $\rho$ | $\omega$ | $m_1$ | $m_2$ | $m_3$ | $m_4$ | $m_5$ | $m_6$ | $m_7$ | $m_8$ |
|--------------|-------|---------|------|------|------|------|------|------|------|------|
| 0.01         | -100  | 0.12486 | 0.37153 | 0.01832 | 0.86511 | 1.1119 | 1.3587 | 1.6055 | 1.8523 |
|              | -90   | 0.12413 | 0.37227 | 0.01675 | 0.86855 | 1.1121 | 1.3595 | 1.6048 | 1.8531 |
|              | -1    | 0.09617 | 0.41059 | 0.57881 | 0.90447 | 1.0722 | 1.3884 | 1.6567 | 1.902 |
|              | 0     | 0.09042 | 0.46598 | 0.5221 | 0.96011 | 1.0152 | 1.454 | 1.5085 | 1.9479 |
|              | 0.01  | 0.02069 | 0.4667 | 0.52135 | 0.96984 | 1.0145 | 1.4548 | 1.5078 | 1.9486 |
|              | 0.05  | 0.02672 | 0.46964 | 0.51834 | 0.96377 | 1.0115 | 1.4577 | 1.5048 | 1.9515 |
|              | 0.3   | 0.0075237 | 0.48849 | 0.49891 | 0.98262 | 0.99202 | 1.4765 | 1.4853 | 1.9704 |
| $\omega_{0.01} - 10^{-5}$ | 7.8549 $10^{-5}$ | 0.49095 | 0.49623 | 0.98406 | 0.99035 | 1.4774 | 1.4843 | 1.9704 |
| $\omega_{0.01} + 10^{-7}$ | 7.8546 $10^{-5}$ | 0.4908 | 0.49063 | 0.98398 | 0.99043 | 1.4773 | 1.4843 | 1.9704 |
| 1            | 0.0456 | 0.44608 | 0.53983 | 0.93924 | 1.0339 | 1.4326 | 1.5278 | 1.926 |
| 2            | 0.09136 | 0.5382 | 0.56746 | 0.88909 | 1.0634 | 1.3812 | 1.5583 | 1.8738 |
| 5            | 0.1204 | 0.37043 | 0.61423 | 0.86409 | 1.1079 | 1.3577 | 1.6015 | 1.8513 |
| 10           | 0.12194 | 0.3689 | 0.61574 | 0.86258 | 1.1094 | 1.3562 | 1.603 | 1.8499 |
| 100          | 0.12195 | 0.3689 | 0.61575 | 0.86257 | 1.1094 | 1.3562 | 1.603 | 1.8498 |

BC1 (no singularity). Case $\overline{M} = 0.01$ and $kR = 6.3$
Table 3: Mass towers for $\overline{M} = 0.1$ (73) and for the boundary conditions BC1 (no metric singularity) (24), (25). Here $\omega_{0.1} = 3.958406 \ldots$ (85). The mass eigenvalues $\tilde{m}_i$ are in TeV (25).

### BC1 (no singularity). Case $\overline{M} = 0.1$ and $kR = 6.3$

| $\overline{M}$ | $\rho$ | $\omega$ | $m_1$ | $m_2$ | $m_3$ | $m_4$ | $m_5$ | $m_6$ | $m_7$ | $m_8$ |
|----------------|-------|----------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0.1            | -100  | 0.13756  | 0.38327 | 0.62981 | 0.87551 | 1.1232 | 1.37 | 1.6168 | 1.8636 |
| 0              | 0.13405 | 0.3924 | 0.56746 | 0.88909 | 1.0634 | 1.3812 | 1.5853 | 1.8738 |
| 0.1            | 0.13368 | 0.38575 | 0.62526 | 0.88417 | 1.1184 | 1.3749 | 1.6118 | 1.8864 |
| 2              | 0.11163 | 0.41108 | 0.59908 | 0.90652 | 1.0906 | 1.4013 | 1.5829 | 1.8958 |
| 3.5            | 0.035687 | 0.47868 | 0.51167 | 0.97459 | 1.01 | 1.4687 | 1.4939 | 1.962 |
| $\omega_{0.1}^{-10^{-3}}$ | 7.6983 $10^{-5}$ | 0.4667 | 0.52135 | 0.96084 | 1.0145 | 1.4548 | 1.5078 | 1.9486 |
| $\omega_{0.1}^{-10^{-3}}$ | 7.6983 $10^{-5}$ | 0.48849 | 0.49891 | 0.98262 | 0.99202 | 1.4765 | 1.4853 | 1.9704 |
| 5              | 0.09129 | 0.3982 | 0.56746 | 0.88909 | 1.0634 | 1.3812 | 1.5853 | 1.8738 |
| 100            | 0.10836 | 0.35695 | 0.60413 | 0.85111 | 1.098 | 1.3449 | 1.5917 | 1.8386 |
| 8/10           | -100  | 0.13756 | 0.38327 | 0.62981 | 0.87551 | 1.1232 | 1.37 | 1.6168 | 1.8636 |
| -5             | 0.13422 | 0.38697 | 0.6259 | 0.88051 | 1.1191 | 1.3742 | 1.6125 | 1.8679 |
| 2.5            | 0.09733 | 0.42499 | 0.58206 | 0.92123 | 1.0726 | 1.4164 | 1.5644 | 1.9112 |
| 3.5            | 0.04326 | 0.46584 | 0.54711 | 0.96012 | 1.0162 | 1.4537 | 1.5089 | 1.9471 |
| $\omega_{0.1}^{-10^{-3}}$ | 0.024178 | 0.4536 | 0.52631 | 0.94278 | 1.0228 | 1.4338 | 1.5181 | 1.9255 |
| $\omega_{0.1}^{-10^{-3}}$ | 0.02418 | 0.45366 | 0.52627 | 0.94284 | 1.0227 | 1.4338 | 1.5181 | 1.9256 |
| 5.5            | 0.082178 | 0.3808 | 0.58249 | 0.871 | 1.0776 | 1.3658 | 1.572 | 1.8588 |
| 9              | 0.10756 | 0.35766 | 0.60346 | 0.85176 | 1.0974 | 1.3455 | 1.5911 | 1.8392 |
| 15             | 0.10836 | 0.35695 | 0.60413 | 0.85111 | 1.098 | 1.3449 | 1.5917 | 1.8386 |
| 8/2            | -100  | 0.13756 | 0.38327 | 0.62981 | 0.87551 | 1.1232 | 1.37 | 1.6168 | 1.8636 |
| -5             | 0.13405 | 0.3924 | 0.56746 | 0.88909 | 1.0634 | 1.3812 | 1.5853 | 1.8738 |
| $\omega_{0.1}$ | 0.12055 | 0.36645 | 0.61279 | 0.85929 | 1.1059 | 1.3525 | 1.5991 | 1.8458 |
| $[8,100]$      | 0.10837 | 0.35695 | 0.60413 | 0.85111 | 1.098 | 1.3449 | 1.5917 | 1.8386 |
| $\pi$          | -100  | 0.13756 | 0.38327 | 0.62981 | 0.87551 | 1.1232 | 1.37 | 1.6168 | 1.8636 |
| -5             | 0.13405 | 0.3924 | 0.56746 | 0.88909 | 1.0634 | 1.3812 | 1.5853 | 1.8738 |
| 2              | 0.14106 | 0.37936 | 0.6339 | 0.87225 | 1.1276 | 1.3656 | 1.6213 | 1.859 |
| 3.5            | 0.12296 | 0.35411 | 0.65892 | 0.84499 | 1.154 | 1.3367 | 1.6486 | 1.8292 |
| $\omega_{0.1}$ | 0.22176 | 0.26629 | 0.70634 | 0.76627 | 1.1958 | 1.2629 | 1.6868 | 1.7584 |
| 4.5            | 0.18156 | 0.29965 | 0.6679 | 0.79554 | 1.1854 | 1.2915 | 1.6501 | 1.7867 |
| 5              | 0.15425 | 0.31828 | 0.64339 | 0.81563 | 1.1351 | 1.3109 | 1.6275 | 1.8056 |
| 100            | 0.10836 | 0.35695 | 0.60413 | 0.85111 | 1.098 | 1.3449 | 1.5917 | 1.8386 |
| 8/2            | 0              | 0.13756 | 0.38327 | 0.62981 | 0.87551 | 1.1232 | 1.37 | 1.6168 | 1.8636 |
| 8              | 0.12055 | 0.36645 | 0.61279 | 0.85929 | 1.1059 | 1.3525 | 1.5991 | 1.8458 |
| 8              | 0.10837 | 0.35695 | 0.60413 | 0.85111 | 1.098 | 1.3449 | 1.5917 | 1.8386 |
Table 4: Mass towers for $M = 1$ (73) as a function of $\omega$ for $\rho = 0$ and for the boundary conditions BC1 (no metric singularity) (24), (25). Here $\omega_1 = 39.58406 \ldots$ (85)). The masses $m_2$ to $m_8$ are essentially independent of $\rho$. The mass $m_1$ is also independent of $\rho$ except for $\omega$ in a range close to $\omega_1$, approximatively in the range $[\omega_1-10, \omega_1+10]$. In this range, the variation of $m_1$ as a function of $\rho$ is given in Table(5). The mass tower is symmetric under $(\rho) \leftrightarrow (2\pi-\rho)$. The mass eigenvalues $\tilde{m}_i$ are in TeV.

| $\omega$ | $m_1$ | $m_2$ | $m_3$ | $m_4$ | $m_5$ | $m_6$ | $m_7$ | $m_8$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 0.24681 | 0.49359 | 0.74044 | 0.98725 | 1.2341 | 1.4808 | 1.7275 | 1.9741 |
| 15 | 0.2468 | 0.49359 | 0.74039 | 0.98718 | 1.234 | 1.4807 | 1.7276 | 1.9744 |
| 19 | 0.20997 | 0.41953 | 0.65138 | 0.88974 | 1.1313 | 1.3746 | 1.6189 | 1.8638 |
| 20 | 0.10356 | 0.36447 | 0.61361 | 0.86141 | 1.1088 | 1.3559 | 1.603 | 1.85 |
| 25 | 7.4472 $10^{-4}$ | 0.35302 | 0.60692 | 0.85666 | 1.1051 | 1.3529 | 1.6004 | 1.8478 |
| 30 | 5.0175 $10^{-6}$ | 0.35302 | 0.60692 | 0.85666 | 1.1051 | 1.3529 | 1.6004 | 1.8478 |
| 39 | 3.8500 $10^{-8}$ | 0.35302 | 0.60692 | 0.85666 | 1.1051 | 1.3529 | 1.6004 | 1.8478 |
| $\omega_1+10^{-3}$ | 3.4548 $10^{-13}$ | 0 | 0.35302 | 0.60692 | 0.85666 | 1.1051 | 1.3529 | 1.6004 | 1.8478 |
| $\omega_1+10^{-3}$ | 3.4514 $10^{-13}$ | 0.35302 | 0.60692 | 0.85666 | 1.1051 | 1.3529 | 1.6004 | 1.8478 |
| 41 | 2.6150 $10^{-10}$ | 0.35302 | 0.60692 | 0.85666 | 1.1051 | 1.3529 | 1.6004 | 1.8478 |
| 50 | 3.4531 $10^{-10}$ | 0.35302 | 0.60692 | 0.85666 | 1.1051 | 1.3529 | 1.6004 | 1.8478 |
Table 5: The lowest mass eigenvalue $\tilde{m}_1$ in the towers for $\mathcal{M} = 1$ (23), as a function of $\rho$ and of $\omega$ in the range $[\omega_1-10,\omega_1+10]$, and for the boundary conditions BC1 (no metric singularity) (24), (25). Here $\omega_1 = 39.58406\ldots$ (18)). The mass $\tilde{m}_1$ (in TeV) is symmetric under $(\rho \leftrightarrow 2\pi - \rho)$.

| $\rho$   | $\omega$ | 30 | 36 | 38 | 39 | $\omega_1-10^{-3}$ | $\omega_1+10^{-3}$ | 41 | 42 | 44 | 50 |
|----------|----------|----|----|----|----|------------------|------------------|----|----|----|----|
| 0        | 0        | 5.02 $10^{-6}$ | 1.21 $10^{-8}$ | 1.34 $10^{-9}$ | 2.74 $10^{-10}$ | 3.46 $10^{-13}$ | 0   | 3.46 $10^{-13}$ | 2.62 $10^{-10}$ | 3.15 $10^{-10}$ | 3.41 $10^{-10}$ | 3.45 $10^{-10}$ |
| $10^{-10}\rho$ | 5.02 $10^{-6}$ | 1.21 $10^{-8}$ | 1.34 $10^{-9}$ | 2.74 $10^{-10}$ | 3.45 $10^{-13}$ | 1.08 $10^{-14}$ | 3.45 $10^{-13}$ | 2.62 $10^{-10}$ | 3.14 $10^{-10}$ | 3.41 $10^{-10}$ | 3.45 $10^{-10}$ |
| $10^{-8}\rho$ | 5.02 $10^{-6}$ | 1.21 $10^{-8}$ | 1.34 $10^{-9}$ | 2.74 $10^{-10}$ | 3.45 $10^{-13}$ | 1.08 $10^{-14}$ | 3.45 $10^{-13}$ | 2.62 $10^{-10}$ | 3.14 $10^{-10}$ | 3.41 $10^{-10}$ | 3.45 $10^{-10}$ |
| $0.1\pi$ | 5.02 $10^{-6}$ | 1.21 $10^{-8}$ | 1.36 $10^{-9}$ | 3.10 $10^{-10}$ | 1.08 $10^{-10}$ | 1.08 $10^{-10}$ | 2.67 $10^{-10}$ | 3.16 $10^{-10}$ | 3.41 $10^{-10}$ | 3.45 $10^{-10}$ |
| $0.2\pi$ | 5.02 $10^{-6}$ | 1.22 $10^{-8}$ | 1.42 $10^{-9}$ | 3.96 $10^{-10}$ | 2.14 $10^{-10}$ | 2.13 $10^{-10}$ | 2.82 $10^{-10}$ | 3.21 $10^{-10}$ | 3.42 $10^{-10}$ | 3.45 $10^{-10}$ |
| $0.3\pi$ | 5.02 $10^{-6}$ | 1.22 $10^{-8}$ | 1.51 $10^{-9}$ | 5.01 $10^{-10}$ | 3.14 $10^{-10}$ | 3.13 $10^{-10}$ | 3.04 $10^{-10}$ | 3.28 $10^{-10}$ | 3.43 $10^{-10}$ | 3.45 $10^{-10}$ |
| $0.4\pi$ | 5.02 $10^{-6}$ | 1.23 $10^{-8}$ | 1.61 $10^{-9}$ | 6.09 $10^{-10}$ | 4.06 $10^{-10}$ | 4.06 $10^{-10}$ | 3.29 $10^{-10}$ | 3.37 $10^{-10}$ | 3.44 $10^{-10}$ | 3.45 $10^{-10}$ |
| $0.5\pi$ | 5.02 $10^{-6}$ | 1.24 $10^{-8}$ | 1.72 $10^{-9}$ | 7.09 $10^{-10}$ | 4.89 $10^{-10}$ | 4.88 $10^{-10}$ | 3.55 $10^{-10}$ | 3.47 $10^{-10}$ | 3.45 $10^{-10}$ | 3.45 $10^{-10}$ |
| $0.6\pi$ | 5.02 $10^{-6}$ | 1.25 $10^{-8}$ | 1.82 $10^{-9}$ | 7.97 $10^{-10}$ | 5.59 $10^{-10}$ | 5.59 $10^{-10}$ | 3.80 $10^{-10}$ | 3.56 $10^{-10}$ | 3.47 $10^{-10}$ | 3.45 $10^{-10}$ |
| $0.7\pi$ | 5.02 $10^{-6}$ | 1.26 $10^{-8}$ | 1.91 $10^{-9}$ | 8.68 $10^{-10}$ | 6.16 $10^{-10}$ | 6.15 $10^{-10}$ | 4.00 $10^{-10}$ | 3.64 $10^{-10}$ | 3.48 $10^{-10}$ | 3.45 $10^{-10}$ |
| $0.8\pi$ | 5.02 $10^{-6}$ | 1.27 $10^{-8}$ | 1.97 $10^{-9}$ | 9.21 $10^{-10}$ | 6.57 $10^{-10}$ | 6.56 $10^{-10}$ | 4.16 $10^{-10}$ | 3.71 $10^{-10}$ | 3.49 $10^{-10}$ | 3.45 $10^{-10}$ |
| $0.9\pi$ | 5.02 $10^{-6}$ | 1.28 $10^{-8}$ | 2.01 $10^{-9}$ | 9.54 $10^{-10}$ | 6.82 $10^{-10}$ | 6.82 $10^{-10}$ | 4.26 $10^{-10}$ | 3.75 $10^{-10}$ | 3.49 $10^{-10}$ | 3.45 $10^{-10}$ |
| $\pi$   | 5.02 $10^{-6}$ | 1.28 $10^{-8}$ | 2.03 $10^{-9}$ | 9.65 $10^{-10}$ | 6.91 $10^{-10}$ | 6.91 $10^{-10}$ | 4.29 $10^{-10}$ | 3.76 $10^{-10}$ | 3.49 $10^{-10}$ | 3.45 $10^{-10}$ |
Table 6: Mass towers for the boundary conditions BC2 (no metric singularity). The mass eigenvalues $\tilde{m}_i$ are in TeV

| $\kappa R$ | $\tilde{m}_1$ | $\tilde{m}_2$ | $\tilde{m}_3$ | $\tilde{m}_4$ | $\tilde{m}_5$ | $\tilde{m}_6$ | $\tilde{m}_7$ | $\tilde{m}_8$ |
|-----------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| 1         | 0.24681       | 0.49363       | 0.74044       | 0.98725       | 1.2341       | 1.4809       | 1.7277       | 1.9745       |
| 0.1       | 0.25789       | 0.5053        | 0.75233       | 0.99925       | 1.2361       | 1.493        | 1.7398       | 1.9867       |
| 0.3       | 0.27967       | 0.52839       | 0.77592       | 1.0231       | 1.2701       | 1.5171       | 1.764        | 2.0109       |
| 0.6       | 0.31158       | 0.56244       | 0.81084       | 1.0585       | 1.3058       | 1.553        | 1.8001       | 2.0471       |
| 0.7       | 0.32204       | 0.57366       | 0.82237       | 1.0702       | 1.3177       | 1.5649       | 1.812        | 2.0591       |
| 1         | 0.35302       | 0.60692       | 0.85666       | 1.1051       | 1.3529       | 1.6004       | 1.8478       | 2.095        |
| 1.5       | 0.40347       | 0.66128       | 0.91289       | 1.1624       | 1.411        | 1.659        | 1.9067       | 2.1542       |
| 2         | 0.45279       | 0.71453       | 0.96813       | 1.2189       | 1.4681       | 1.7169       | 1.9651       | 2.2129       |
| 5         | 0.73502       | 1.0187        | 1.2849        | 1.544        | 1.7995       | 2.0527       | 2.3045       | 2.5552       |
| -1        | 0.12344       | 0.34022       | 0.61065       | 0.86384       | 1.1109       | 1.3675       | 1.6243       | 1.8811       |
| 0.1       | 0.10836       | 0.35695       | 0.60413       | 0.85111       | 1.098        | 1.3449       | 1.5917       | 1.8386       |
| 0.3       | 0.3598 10^{-2}| 0.32963       | 0.57794       | 0.82537       | 0.7255       | 1.3195       | 1.5665       | 1.8134       |
| 0.50627   | 1.5682 10^{-2}| 0.30426       | 0.55445       | 0.80258       | 0.5051       | 1.2974       | 1.5445       | 1.7915       |
| 0.6       | 9.9522 10^{-4}| 0.31159       | 0.56246       | 0.81086       | 1.0585       | 1.3059       | 1.553        | 1.8001       |
| 0.7       | 2.8062 10^{-5}| 0.32204       | 0.57366       | 0.82237       | 1.0702       | 1.3177       | 1.5649       | 1.812        |
| 0.8       | 6.8302 10^{-7}| 0.33244       | 0.5848        | 0.83385       | 0.8189       | 1.3295       | 1.5768       | 1.824        |
| 0.9       | 1.5627 10^{-8}| 0.34276       | 0.59589       | 0.84528       | 0.8935       | 1.3412       | 1.5886       | 1.8359       |
| 1         | 3.4531 10^{-10}| 0.35302       | 0.60692       | 0.85666       | 1.051       | 1.3529       | 1.6004       | 1.8478       |
| 1.1       | 7.4593 10^{-12}| 0.36321       | 0.61789       | 0.86799       | 1.1166      | 1.3646       | 1.6122       | 1.8596       |
| 1.2       | 1.5857 10^{-13}| 0.37335       | 0.62881       | 0.87928       | 1.1281      | 1.3762       | 1.624        | 1.8714       |
Table 7: Mass towers for $M = 1$ (73) and for the semi-local boundary conditions (one metric singularity) (88). Here $\omega_1 = 2\pi(kR)$ (85). The mass eigenvalues $\tilde{m}_i$ are in TeV.

| $\bar{M}$ | $\bar{\tau}$ | $kR$ | $\omega_b$ | $\omega_s$ | $\rho_b$ | $\rho_s$ | $\tilde{m}_1$ | $\tilde{m}_2$ | $\tilde{m}_3$ | $\tilde{m}_4$ | $\tilde{m}_5$ | $\tilde{m}_6$ | $\tilde{m}_7$ | $\tilde{m}_8$ |
|----------|---------|------|-----------|-----------|--------|--------|------------|------------|------------|------------|------------|------------|------------|------------|------------|
| 1        | 0.9     | 6.9  | 2         | 0         | 0      | 0      | 0.28251    | 0.52712    | 0.74625    | 0.96836    | 1.1863     | 1.40558    | 1.6232     | 1.8416     |
| 15       | 0       | 0    | 0.28251   | 0.52712   | 0.74625 | 0.96836 | 1.1863     | 1.40558    | 1.6232     | 1.8416     |
| 20.6     | 0       | 0    | 0.26044   | 0.49672   | 0.73557 | 0.97614 | 1.1951     | 1.4157     | 1.6346     |
| 20.9     | 0       | 0    | 0.27248   | 0.31334   | 0.53237 | 0.75223 | 0.97546    | 1.1947     | 1.4153     | 1.6342     |
| 21.1     | 0       | 0    | 0.24109   | 0.29079   | 0.53166 | 0.75187 | 0.97522    | 1.1945     | 1.4151     | 1.6341     |
| 21.5     | 0       | 0    | 0.1648    | 0.28597   | 0.53108 | 0.75152 | 0.97497    | 1.1943     | 1.4150     | 1.6339     |
| 22       | 0       | 0    | 0.10037   | 0.28481   | 0.53068 | 0.75127 | 0.97478    | 1.1941     | 1.4148     | 1.6338     |
| 25       | 0       | 0    | 0.20070   | 0.53068   | 0.75127 | 0.97478 | 1.1941     | 1.4148     | 1.6338     |
| 28       | 0       | 0    | 0.20070   | 0.53068   | 0.75127 | 0.97478 | 1.1941     | 1.4148     | 1.6338     |
| 35       | 0       | 0    | 0.20070   | 0.53068   | 0.75127 | 0.97478 | 1.1941     | 1.4148     | 1.6338     |
| 42       | 0       | 0    | 0.20070   | 0.53068   | 0.75127 | 0.97478 | 1.1941     | 1.4148     | 1.6338     |
| $\omega_1 - 10^{-3}$ | 0 | 0 | 0 | 5.355 10^{-14} | 0.28481 | 0.53068 | 0.75127 | 0.97478 | 1.1941 | 1.4148 | 1.6338 |
| $\omega_1 - 10^{-3}$ | 0 | 0 | 0 | 5.349 10^{-14} | 0.28481 | 0.53068 | 0.75127 | 0.97478 | 1.1941 | 1.4148 | 1.6338 |
| 80       | 0       | 0    | 0.20070   | 0.53068   | 0.75127 | 0.97478 | 1.1941     | 1.4148     | 1.6338     |

Semi-local BC (one singularity at $s_1 = 2\pi R \bar{s}_1$). Case $\bar{M} = 1$.
Table 8: Mass towers for $\bar{M} = 0.1$ \((73)\) and for the semi-local boundary conditions (one metric singularity) \((88)\). Here $\omega_{0.1} = 0.2\pi(kR)$ \((85)\). The mass eigenvalues $\tilde{m}_i$ are in TeV

| $\bar{M}$ | $\tau_1$ | $kR$ | $\omega_b$ | $\omega_s$ | $\rho_b$ | $\rho_s$ | $\tilde{m}_1$ | $\tilde{m}_2$ | $\tilde{m}_3$ | $\tilde{m}_4$ | $\tilde{m}_5$ | $\tilde{m}_6$ | $\tilde{m}_7$ | $\tilde{m}_8$ |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| 0.1 | 0.9 | 6.9 | 2 | 0 | 0 | 0 | 0.1108 | 0.3680 | 0.5441 | 1.246 | 1.413 | 1.684 | 1.849 | 2.121 |
| $\omega_{0.1} = 10^{-3}$ | $-\omega_{0.1} = 10^{-3}$ | 0 | 0 | 0 | 0 | 6.711 | 0.3962 | 0.4625 | 0.8241 | 0.906 | 1.257 | 1.347 | 1.691 |
| $\omega_{0.1} = 4.34\ldots$ | $\omega_{0.1} = 10^{-3}$ | 0 | 0 | 0 | 0 | 0 | 0.3963 | 0.4625 | 0.8242 | 0.9060 | 1.257 | 1.347 | 1.691 |
| $\omega_{0.1} = 10^{-3}$ | 0 | 0 | 0 | 0 | 6.708 | 0.3962 | 0.4625 | 0.8241 | 0.906 | 1.257 | 1.347 | 1.691 |
| $\omega_{0.1}$ | 0 | $\pi/3$ | 0 | 0.1189 | 0.3579 | 0.5549 | 0.7854 | 0.9903 | 1.233 | 1.426 | 1.670 |
| $\omega_{0.1}$ | 2 | $\pi/3$ | 0 | 0.1189 | 0.3579 | 0.5549 | 0.7954 | 0.9903 | 1.233 | 1.426 | 1.670 |
| $\omega_{0.1}$ | 2 | $\pi/2$ | 0 | 0.1271 | 0.3481 | 0.5656 | 0.7840 | 1.106 | 1.126 | 1.541 | 1.559 |

Table 9: Mass towers for $M = 0$ and for the semilocal boundary conditions (one metric singularity) \((88)\). Here $\omega_0 = 0$ \((85)\). The mass eigenvalues $\tilde{m}_i$ are in TeV

| $\bar{M}$ | $\tau_1$ | $kR$ | $\omega_b$ | $\omega_s$ | $\rho_b$ | $\rho_s$ | $\tilde{m}_1$ | $\tilde{m}_2$ | $\tilde{m}_3$ | $\tilde{m}_4$ | $\tilde{m}_5$ | $\tilde{m}_6$ | $\tilde{m}_7$ | $\tilde{m}_8$ |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| 0 | 0.9 | 6.9 | -2 | 0 | 0 | 0 | 0.90961 | 0.3467 | 0.5279 | 0.7841 | 0.9653 | 1.221 | 1.403 | 1.659 |
| $-10^{-3}$ | 0 | 0 | 0 | 0 | 0 | 6.966 | 0.4373 | 0.4374 | 0.8746 | 0.8747 | 1.312 | 1.312 | 1.749 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0.4373 | 0.4373 | 0.8746 | 0.8746 | 1.312 | 1.312 | 1.749 |
| $10^{-3}$ | 0 | 0 | 0 | 0 | 0 | 6.961 | 0.4373 | 0.4374 | 0.8746 | 0.8747 | 1.3119 | 1.3121 | 1.7492 |
| 2 | 0 | 0 | 0 | 0.90961 | 0.3467 | 0.5279 | 0.7841 | 0.9653 | 1.221 | 1.403 | 1.659 |
| $2\pi/10^{-3}$ | 2 | $\pi/3$ | $\pi/3$ | 1.849 | 0.4346 | 0.4401 | 0.8691 | 0.8803 | 1.304 | 1.320 | 1.738 |

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