A brief review on the Problem of Divergence in Krein Space Quantization

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Abstract

In this paper we have a brief review on the problem of divergence in quantum field theory and its elimination using the method of Krein space quantization. In this method, the auxiliary negative frequency states have been utilized, the modes of which do not interact with the physical states and are not affected by the physical boundary conditions. It is remarkable that Krein space quantization is similar to Pauli-Villars regularization, so we can call it the "Krein regularization". Considering the QED in Krein space quantization, it could be shown that the theory is automatically regularized. Calculation of the three primitive divergent integrals, the vacuum polarization, electron self energy and vertex function using Krein space method leads to finite values, since the infrared and ultraviolet divergencies do not appear. For another example, the Casimir stress on a spherical shell in de Sitter spacetime for a massless scalar field could be calculated using Krein space quantization.

1 Introduction

The historical background of Krein space quantization goes back to the covariant quantization of minimally coupled scalar field in de Sitter spacetime. It has been shown that the linear quantum gravity in the background field method is perturbatively non-renormalizable and also there appears an infrared divergence. This infrared divergence does not manifest itself in the quadratic part of the effective action in the one-loop approximation. This means that the pathological behavior of the graviton propagator may be gauge dependent and so should not appear in an effective way as a physical quantity [1]. The infrared divergence which appears in the linear gravity in de Sitter space is the same as the minimally coupled scalar field in de Sitter space [2,3]. It is shown that one can not construct a covariant quantization of the minimally coupled scalar field with only positive norm states [4]. It has been proved that the use of the two sets of solutions (positive and negative norms states) is an unavoidable feature if one wants to preserve causality (locality), covariance and elimination of the infrared divergence in quantum field theory for the minimally coupled scalar field in de Sitter space [5,6], i.e. Krein space quantization.

The singular behavior of Green function at short relative distances (ultraviolet divergence) or in the large relative distances (infrared divergence) leads to main divergences in the quantum field theory. It was conjectured that quantum metric fluctuations might smear out the singularities of Green functions on the light cone, but it does not remove other ultraviolet divergences [7]. However, it has been shown that quantization in Krein space removes all ultraviolet divergences of quantum field theory (QFT) except the light cone singularity [8]. By using the Krein space quantization and the quantum metric fluctuations in the linear approximation, it has been shown that the infinities in the Green function are disappeared [7,9].

Quantization in Krein space instead of Hilbert space has some interesting features. For example in this method, the vacuum energy becomes zero naturally, so the normal ordering would not be necessary [5,8]. The auxiliary negative norm states, which are used in the Krein space quantization, play the role of regularization in the theory. Since Krein space quantization is similar to Pauli-Villars regularization, so it could be called the "Krein regularization". In the the Pauli-Villars regularization the particles with mass $M$ and negative norms are added to the theory. In

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In Krein space the quantum scalar field is defined as follows [5, 6]: quantization, yielding the standard result obtained [17, 18].

stress on a spherical shell in de Sitter spacetime for a massless scalar field could be calculated using Krein space quadratic in the field, such as the energy density and stresses.

the polarization of the vacuum by boundary conditions or geometry. The presence of reflecting boundaries alters fluctuation of the field operator between two parallel plates. In other words, the Casimir effect can be viewed as effect is a small attractive force acting between two parallel uncharged conducting plates and it is regarded as one work by Casimir in 1948 [16] many theoretical and experimental works have been done on this problem. The Casimir becomes zero naturally, so the normal ordering would not be necessary [5, 8].

space method leads to finite values, since the infrared and ultraviolet divergencies do not appear [10, 11, 12, 13].

considering the QED in Krein space quantization, results in an automatically regularized theory. Calculation of the Krein space the negative norm states with mass $m$ [12]:

One of the interesting features of quantization in Krein space instead of Hilbert space is that the vacuum energy becomes zero naturally, so the normal ordering would not be necessary [5, 8].

For another example, the Casimir effect could be investigated using Krein space method, too. Since the original work by Casimir in 1948 [16] many theoretical and experimental works have been done on this problem. The Casimir effect is a small attractive force acting between two parallel uncharged conducting plates and it is regarded as one of the most striking manifestation of quantum fluctuations in quantum field theory. It is due to the quantum vacuum fluctuation of the field operator between two parallel plates. In other words, the Casimir effect can be viewed as the polarization of the vacuum by boundary conditions or geometry. The presence of reflecting boundaries alters the zero-point modes of a quantized field, and results in the shifts in the vacuum expectation values of quantities quadratic in the field, such as the energy density and stresses.

Applying the unphysical negative frequency states and defining the field operator in Krein space, the Casimir stress on a spherical shell in de Sitter spacetime for a massless scalar field could be calculated using Krein space quantization, yielding the standard result obtained [17, 18].

2 A brief review on Krein space quantization

In Krein space the quantum scalar field is defined as follows [5, 6]:

$$\phi(x) = \frac{1}{\sqrt{2}} [\phi_p(x) + \phi_n(x)],$$

where

$$\phi_p(x) = \int d^3k [a(\vec{k})u_p(k, x) + a^\dagger(\vec{k})u_p^*(k, x)],$$

$$\phi_n(x) = \int d^3k [b(\vec{k})u_n(k, x) + b^\dagger(\vec{k})u_n^*(k, x)].$$

$a(\vec{k})$ and $b(\vec{k})$ are two independent operators and

$$u_p(k, x) = \frac{e^{i\vec{k}.\vec{x} - iwt}}{\sqrt{(2\pi)^32w}} = e^{-ik.x},$$

$$u_n(k, x) = \frac{e^{-i\vec{k}.\vec{x} + iwt}}{\sqrt{(2\pi)^32w}} = e^{ik.x}$$

where $w(\vec{k}) = k^0 = (\vec{k}.\vec{k} + m^2)^{1/2} \geq 0$. The positive mode $\phi_p$ is the scalar field operator as was used in the usual QFT and $\phi_n$ plays the role of the regularization field. The time-ordered product is defined as:

$$iG_T(x, x') = <0 | T\phi(x)\phi(x') | 0 >= \Re G_F(x, x'),$$

where $G_F(x, x')$ is the Feynman Green function.

As we know, the origin of divergences in standard quantum field theory lies in the singularity of the Green’s function. The divergence appears in the imaginary part of the Feynman propagator, and the real part is convergent [12]:

$$G_T^F(x, x') = -\frac{1}{8\pi}\frac{\delta(\sigma_0)}{\sigma_0} + \frac{m^2}{8\pi}\theta(\sigma_0)\frac{J_1(\sqrt{2m^2\sigma_0})}{\sqrt{2m^2\sigma_0}} - \frac{im^2}{4\pi^2}\theta(-\sigma_0)\frac{K_1(\sqrt{2m^2(-\sigma_0)}}{\sqrt{2m^2(-\sigma_0)}}$$

Consideration of negative frequency states removes singularity of the Green function with exception of delta function singularity:

$$G_T(x, x') = -\frac{1}{8\pi}\frac{\delta(\sigma_0)}{\sigma_0} + \frac{m^2}{8\pi}\theta(\sigma_0)\frac{J_1(\sqrt{2m^2\sigma_0})}{\sqrt{2m^2\sigma_0}}, \quad \sigma_0 \geq 0$$

However, considering the quantum metric fluctuations removes the latter singularity:
\[ < G_T(x, x') > = -\frac{1}{8\pi} \sqrt{\frac{\pi}{2 < \sigma_1^2 >}} \exp\left(-\frac{\sigma_0^2}{2 < \sigma_1^2 >}\right) + \frac{m^2}{8\pi} \theta(\sigma_0) \frac{J_1(\sqrt{2m^2\sigma_0})}{\sqrt{2m^2\sigma_0}}. \]  

where \( < \sigma_1^2 > \) is related to the density of gravitons. When \( \sigma_0 = 0 \), due to the metric quantum fluctuation \( < \sigma_1^2 > \neq 0 \), and we have

\[ < G_T(0) > = -\frac{1}{8\pi} \sqrt{\frac{\pi}{2 < \sigma_1^2 >}} + \frac{m^2}{16\pi}. \]

By using the Fourier transformation, we obtain \[19\]

\[ < \tilde{G}_T(p) > = \tilde{G}_T(p) + PP \frac{m^2}{p^2(p^2 - m^2)} \]

However, in the one-loop approximation, the contribution of delta function is negligible and the Green function in Krein space quantization appearing in the transition amplitude is

\[ < \tilde{G}_T(p) >_{\text{one-loop}} = \tilde{G}_T(p) \equiv PP \frac{m^2}{p^2(p^2 - m^2)} \]

where \( \tilde{G}_1(p) \) is the Fourier transformation of the first part of the Green function (1) and its explicit form is not needed for our discussion here. In a previous paper, it has proved that for the \( \lambda \phi^4 \) theory in the one-loop approximation, the Green function in Krein space quantization, which appear in the s-channel contribution of transition amplitude, is the second part of (1) \[6\]. That means in this approximation, the contribution of the first part (i.e. quantum metric fluctuation) is negligible. It is worth mentioning that in order to improve the UV behavior in relativistic higher-derivative correction theories, the second part of equation (1) has been used by some authors \[20, 21\]. This part also appears in the super-symmetry theory \[22\].

The time-order product of the spinor field is:

\[ < S^T_T(x - x') > \equiv (i \partial + m) < G_T(x, x') > \]

And the time-ordered product propagator in the Feynman gauge for the vector field in Krein space is given by:

\[ < D^T_{\mu\nu}(x, x') > = -\eta_{\mu\nu} < G_T(x, x') > . \]

### 3 Essential graphs of QED in Krein space quantization

In the standard QED the divergent quantities are found in the electron self-energy, the vacuum polarization and the vertex graphs. In the standard QED, we have \[23\]:

\[ \Sigma_{H_1}(p) = \frac{e^2}{8\pi^2} \left\{ \ln \left( -\frac{\Lambda^2}{m^2} \right) \left( 2m - \frac{\hat{p}}{2} \right) + \left( 2m - \frac{3}{4} \hat{p} \right) \right. \]

\[ -\frac{\hat{p}}{2} \left[ \frac{m^4 - (p^2)^2}{(p^2)^2} \ln \left( 1 - \frac{p^2}{m^2} \right) \right] + 2m \left[ \frac{m^2 - p^2}{p^2} \ln \left( 1 - \frac{p^2}{m^2} \right) \right] \}

and

\[ \Pi_{H_1}(k^2) = \frac{e^2}{12\pi^2} \ln \left( \frac{\Lambda^2}{m^2} \right) - \frac{e^2}{2\pi^2} \int_0^1 dx (1 - x) \ln \left( 1 - x (1 - x) \frac{k^2}{m^2} \right). \]

and

\[ F_{1h}^T(q^2)_{q^2 \to 0} = -\frac{e^2}{16\pi^2} \ln \left( \frac{\Lambda^2}{m^2} \right) - \frac{e^2q^2}{12\pi^2m^2} \left( \ln \frac{m}{\mu} - \frac{3}{8} \right). \]

Calculating in Krein space, we get:
\[
\Sigma_{kr}(p) = \frac{e^2}{8\pi^2} \left\{ \ln \left( \frac{-p^2}{m^2} \right) \left( 2m - \frac{\not{p}k}{2} \right) - \frac{\not{p}}{2} \left( \frac{m^2}{p^2} \right) \right. \\
- \frac{\not{p}}{2} \left[ m^4 - \frac{(p^2)^2}{2} \ln \left( 1 - \frac{p^2}{m^2} \right) \right] \left. + 2m \left[ \frac{m^2}{p^2} - \frac{m^2}{p^2} \ln \left( 1 - \frac{p^2}{m^2} \right) \right] \right\} ,
\]

and

\[
\Pi_{\mu
u}^{kr}(k^2) = (k^2 \gamma^\mu - k^\nu k^\rho) \Pi_{kr}(k^2) ,
\]

where

\[
\Pi_{kr}(k^2) = -\frac{e^2}{12\pi^2} \ln \left( -\frac{k^2}{m^2} \right) - \frac{e^2 k^2}{6\pi^2 m^2} - \frac{e^2}{2\pi^2} \int_0^1 dx (1-x) x \ln \left( 1 - x(1-x) \frac{k^2}{m^2} \right) .
\]

and

\[
\Lambda^\mu_{kr}(p', p) = \frac{e^2}{8\pi} \int \frac{d^4k}{(2\pi)^4} \gamma^\nu \left( p' - k + m \right) \gamma^\mu \left( p - k + m \right) \gamma_\nu PP \frac{1}{k^2 - \mu^2} \\
PP \left( \frac{m^2}{(p' - k)^2 - m^2} \right) PP \left( \frac{m^2}{(p - k)^2 - m^2} \right) = F_1^{kr}(q^2) \gamma^\mu + \frac{i\sigma^{\mu\nu} q_\nu}{2m} F_2^{kr}(q^2) .
\]

\(F_2^{kr}(q^2)\) in the two different method is the same and \(F_1^{kr}(q^2)\) in the Krein regularization is:

\[
F_1^{kr}(q^2)_{q^2 \rightarrow 0} = -\frac{e^2 q^2}{16\pi^2 m^2} + \frac{3e^2 q^2}{64\pi^2 m^2} - \frac{e^2 q^2}{12\pi^2 m^2} \left( \ln \frac{m}{\mu} - \frac{3}{8} \right) ,
\]

where \(q^2 = (p - p')^2\).

The singular terms of 3 standard graphs of QED are replaced with the two first terms in the resulted graphs in Krein space quantization [24, 25]. By using the value of \(F_1(q^2)\) and the photon self energy in Krein space, the value of Lamb Shift is calculated to be 1018.19 MHz, whereas in standard QED it is 1052.1 MHz; and its experimental value has been given as 1057.8 MHz. The small differences may be because of neglecting the linear quantum gravitational effect and working in the one-loop approximation [20].

4 Scalar Casimir effect for a sphere in de Sitter space

The Casimir force due to fluctuations of a free massless scalar field satisfying Dirichlet boundary conditions on a spherical shell in Minkowski space-time has been studied in [27]. Doing the calculations in Krein space, the two-point Green’s function \(G_K(x, t; x', t')\) is defined as the vacuum expectation value of the time-ordered product of two fields (\(K\) and \(T\) stand for quantities in Krein space and the time ordered product, respectively):

\[
G_K(x, t; x', t') \equiv -i < 0 | T \Phi_K(x, t) \Phi_K(x', t') | 0 > .
\]

\[
\Phi_K(x, t) = \sum_k [(a_{\vec{k}} + b_{\vec{k}}^\dagger) u_p(k, x) + (a_{\vec{k}}^\dagger + b_{\vec{k}}) u_n(k, x)]
\]

The operators \(a^\dagger(\vec{k})\) and \(a(\vec{k})\) create and destroy respectively the mode \(u_p(k, x)\) with positive energy \((k^0 = \omega_{\vec{k}})\), which may be considered as the operators of creation and annihilation of a particle and the operators \(b^\dagger(\vec{k})\) and \(b(\vec{k})\) create and destroy respectively the mode \(u_n(k, x)\) with negative energy \((-k^0 = -\omega_{\vec{k}})\), which may be considered as the operators of creation and annihilation of an “anti-particle” in the inverse time direction. The two sets of modes do not affect on each other and in the standard QFT, the negative energy states are eliminated in the quantum field operators, which is the origin of the appearance of divergence. On the contrary, the divergence disappears by taking these states into account.

The two point Green function has to satisfy the Dirichlet boundary conditions on the shell:
\[ G_K(x, t; x', t') \big|_{|x|=a} = 0, \]  

(4)

where \( a \) is radius of the spherical shell. The stress-energy tensor in Krein space \( T_K^{\mu

(\mu} (x, t) \) is given by

\[ T_K^{\mu\nu} (x, t) \equiv \partial_\mu \Phi_K(x, t) \partial^\nu \Phi_K(x, t) - \frac{1}{2} \eta^{\mu\nu} \partial_\lambda \Phi_K(x, t) \partial^\lambda \Phi_K(x, t). \]  

(5)

The radial Casimir force per unit area \( F_A \) on the sphere, called Casimir stress, is obtained from the radial-radial component of the vacuum expectation value of the stress-energy tensor:

\[ F_A = \langle 0 | T_{rr}^{in} - T_{rr}^{out} | 0 \rangle \big|_{r=a}. \]  

(6)

Taking into account the relation (1) between the vacuum expectation value of the stress-energy tensor \( T_K^{\mu\nu} (x, t) \) and the Green’s function at equal times \( G_K(x, t; x', t) \) we obtain

\[ F_A = \frac{i}{2} \[ \partial \partial_{r'} G_K(x, t; x', t)_{in} - \partial \partial_{r'} G_K(x, t; x', t)_{out} \big|_{x=x', |x|=a} \] \]  

(7)

One may use of the above flat space calculation in de Sitter space-time by taking the de Sitter metric in conformally flat form

\[ ds^2 = \Omega(\eta) \left[ d\eta^2 - \sum_{i=1}^3 (dx^i)^2 \right], \]  

(8)

where \( \Omega(\eta) = \frac{a}{\eta} \) and \( \eta \) is the conformal time

\[ -\infty < \eta < 0. \]  

(9)

Assuming a canonical quantization of the scalar field in Krein space, the conformally transformed quantized scalar field in de Sitter spacetime is given by

\[ \Phi_K(x, \eta) = \sum_k \left[ (a_k^+ + b_k^+) \tilde{u}_k(\eta, x) + (a_k^+ + b_k^+) \tilde{u}_k^*(\eta, x) \right], \]  

(10)

where \( a_k^+ \) and \( a_k \) are creation and annihilation operators respectively and the vacuum states associated with the physical modes \( \tilde{u}_k \) defined by \( a_k |0\rangle = 0 \), are called conformal vacuum. Given the flat space Green’s function(1), we obtain

\[ \tilde{G}_K = -i \langle 0 | T \tilde{\Phi}_K(x, \eta) \tilde{\Phi}_K(x', \eta') | 0 \rangle = \Omega^{-1}(\eta) \Omega^{-1}(\eta') G_K, \]  

(11)

where \( \tilde{\Phi}_K(x, \eta) = \Omega^{-1}(\eta) \Phi_K(x, \eta) \) has been used. Therefore, using Eqs. (6), (7) and (11) we obtain the total stress on the sphere in de Sitter spacetime and using Krein space quantization as

\[ \frac{\tilde{F}}{A} = \frac{\eta^2 F}{a^2 A}. \]  

(12)

in accordance with the standard result [18].

## 5 Conclusion

In this paper we had a brief review on Krein space quantization. For a first example, it was deduced that the 3 main divergent graphs of standard QED are automatically regularized in Krein space and the values of magnetic anomaly and Lamb shift in the one-loop approximation are identical to the corresponding results in standard calculations [26]. Due to the appearance of negative norm states this method is similar to the Pauli-Villars regularization, so it could be considered as a new method of regularization called "Krein regularization".

For a second example, it was shown that using Krein space quantization method of Scalar Casimir effect for a sphere in de Sitter space leads to the same standard result. This method could be easily generalized to non-Abelian
gauge theory and quantum gravity in the background field method, and could be used as an alternative way for solving the non-renormalizability of quantum gravity in the linear approximation.

Consequently for QED, the Krein space calculations just eliminates the singularity in the theory without changing the physical contents, and may provide an answer to the Feynman reply: “A Nobel prize for hiding the rushes under the carpet?”.

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