Quantum Battle of the Sexes Revisited

João José de Farias Neto
Informatics Division
Institute for Advanced Studies
Centro Técnico Aeroespacial, Brazil
joaojfn@ieav.cta.br

March 3, 2022

Abstract

Several quantum versions of the battle of the sexes game are analyzed. Some of them are shown to reproduce the classical game. In some, there are no Nash quantum pure equilibria. In some others, the payoffs are always equal to each other. In others still, all equilibria favor Alice or Bob depending on a phase shift of the initial state of the system. Explicit detailed calculations are for the first time exhibited.

1 Introduction

Since Meyer’s [1] idea of quantizing classical games, a flood of papers have been published on this theme. Although that author’s motivation was to explore the mathematical resemblance between mixed strategies and quantum superposition, the strongest argument in favor of this line of research is the pending question about the possibility of existence of quantum algorithms computationally more efficient then classical ones for the solution of large sequential games (like chess, for instance); research in this line is described in [2], [3], [4], [5] and [6]. If such algorithms do exist, they must be applied to a quantum version of the classical game, thus requiring the fidelity of the former to the later one.

Several quantum versions for the battle of the sexes game have been proposed [7], [8], [9], [10]. Benjamin and Hayden [11] recommend Eisert’s [12] quantization scheme, which computes the final state as $|E_f\rangle = J^\dagger (U_A \otimes U_B) J |E_i\rangle$, because it generalizes correctly the classical game. Here, this last framework is used. Firstly, the operator $J$ is set equal to Eisert’s with $\gamma = \frac{\pi}{2}$ (which totally entangles the qubits) and four cases are analyzed: (3,3), meaning unrestricted $U_A$ and $U_B$ (each one thus defined by three real numbers); (2,2), which is the restricted version used by Eisert; (2,1), in which one of the players uses a restricted quantum operator and the other can only use classical ones; and (1,1), which corresponds to the classical game. Then, $J$ is set to the identity and the results for some initial states $|E_i\rangle$ are obtained; the interesting cases here are when $|E_i\rangle$ is an entangled state combining the two pure equilibria of the classical game.

2 The classical game

Alice and Bob are dating and want to go out together on Saturday night. But Bob prefers to go to the football, whilst Alice prefers to go to the ballet. Each one prefers to go with the other to the show (s)he doesn’t like instead of going alone to the show (s)he likes. The following table may represent the personal satisfaction (payoff) of each player in the four possible situations:

|       | Football(F) | Ballet(B) |
|-------|-------------|-----------|
| Alice | (1,2)       | (0,0)     |
| Bob   |             | (2,1)     |

Lines represent Alice’s choices and columns, Bob’s. The first entry in each ordered pair is Alice’s payoff and the second one, Bob’s. The decisions have to be made simultaneously,
independently and thoroughly followed. This game has two pure equilibria: FF and BB and a mixed one, in which Alice chooses ballet with probability \( \frac{1}{3} \) and Bob chooses football with probability \( \frac{2}{3} \).

3 Quantization

The two possible choices (F or B) can be associated with the direction (up or down) of the spin of the nucleus of a carbon 13 atom, for instance, embedded in a magnetic field. Suppose Alice and Bob control, each, one atom. Two initial states are considered here (of the four possible ones): the nucleus of a carbon 13 atom, for instance, embedded in a magnetic field. Suppose Alice and Bob control, each, one atom. Two initial states are considered here (of the four possible ones): the nucleus of a carbon 13 atom, for instance, embedded in a magnetic field. Suppose Alice and Bob control, each, one atom. Two initial states are considered here (of the four possible ones): the nucleus of a carbon 13 atom, for instance, embedded in a magnetic field. Suppose Alice and Bob control, each, one atom. Two initial states are considered here (of the four possible ones): the nucleus of a carbon 13 atom, for instance, embedded in a magnetic field. Suppose Alice and Bob control, each, one atom.

Now, Alice chooses a quantum operator \( U_A \) and Bob, \( U_B \), each one acting only upon his own atom (qubit). As stated in the introduction, the final state is computed by \( |E_f \rangle \). However, with the aid of the operator

\[
J = e^{i\frac{\pi}{4}C \otimes C} = e^{i\frac{\pi}{4}C \otimes C} = e^{i\frac{\pi}{4}} \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix} \otimes \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]

\[
J = e^{i\frac{\pi}{4}C \otimes C} = e^{i\frac{\pi}{4}C \otimes C} = e^{i\frac{\pi}{4}} \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix} \otimes \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]

\[
= e^{\cos \frac{\pi}{4}} \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix} + e^{i \sin \frac{\pi}{4}} \begin{bmatrix}
0 & i \\
0 & 0
\end{bmatrix} = \begin{bmatrix}
\frac{\sqrt{2}}{2} & 0 \\
0 & \frac{\sqrt{2}}{2}
\end{bmatrix}
\]

Defining the basis of the system's state space as \( \{ |FF \rangle, |FB \rangle, |BF \rangle, |BB \rangle \} \), it follows that

\[
|FF \rangle = (1, 0, 0, 0)
\]

\[
|FB \rangle = (0, 1, 0, 0)
\]

\[
|BF \rangle = (0, 0, 1, 0)
\]

\[
|BB \rangle = (0, 0, 0, 1)
\]

Thus, \( J|FF \rangle = (\frac{\sqrt{2}}{2}, 0, 0, \frac{\sqrt{2}}{2}) \) and \( J|BB \rangle = (\frac{\sqrt{2}}{2}, 0, 0, \frac{\sqrt{2}}{2}) \).

Notice that \( J|FF \rangle \) and \( J|BB \rangle \) seem to be fair in the sense that they assign equal probabilities \( \left( \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) \) to both pure equilibria (FF and BB) of the classical game.

Now, Alice chooses a quantum operator \( U_A \) and Bob, \( U_B \), each one acting only upon his own atom (qubit). As stated in the introduction, the final state is computed by \( |E_f \rangle = J(U_A \otimes U_B) |E_i \rangle \) and the qubits are measured, thus collapsing to one of the four basis states, corresponding to the four possible outcomes of the classical game. If \( |E_f \rangle = (r, s, t, u) \), meaning \( |E_f \rangle = r|FF \rangle + s|FB \rangle + t|BF \rangle + u|BB \rangle \), the expected payoffs are then:

\[
S_A = |r|^2 + 2|u|^2
\]

\[
S_B = 2|r|^2 + |u|^2
\]

General quantum unitary operators acting over one qubit can be defined by

\[
U_A = \begin{bmatrix}
a & b \\
-b^* & a^*
\end{bmatrix}
\]

\[
U_B = \begin{bmatrix}
c & d \\
-d^* & c^*
\end{bmatrix}
\]

with \( |a|^2 + |b|^2 = |c|^2 + |d|^2 = 1 \) (see [10]).}

Now,

\[
U_A \otimes U_B = \begin{bmatrix}
a & b \\
-b^* & a^*
\end{bmatrix} \otimes \begin{bmatrix}
c & d \\
-d^* & c^*
\end{bmatrix} = \begin{bmatrix}
a & c & d & c \\
-c & d & -c & d \\
b & -b^* & a & -b^* \\
-a^* & -a^* & b & -a^*
\end{bmatrix}
\]

\[
= \begin{bmatrix}
a & c & d & c \\
-c & d & -c & d \\
b & -b^* & a & -b^* \\
-a^* & -a^* & b & -a^*
\end{bmatrix}
\]
The function of these angles become:

\[
\begin{bmatrix}
ac & ad & bc & bd \\
-\alpha d^* & \alpha c^* & -b^* c & b^* d \\
b^* c & -b^* d & a^* c & a^* d \\
b^* d^* & -b^* c^* & -a^* d^* & a^* c^*
\end{bmatrix}
\]

So

\[
(U_A \otimes U_B) J_{FF} = \begin{bmatrix}
\frac{1}{2}ac\sqrt{2} + \frac{1}{2}ibd\sqrt{2} \\
-\frac{1}{2}ad^*\sqrt{2} + \frac{1}{2}ibc^*\sqrt{2} \\
-\frac{1}{2}b^* c\sqrt{2} + \frac{1}{2}ia^* d\sqrt{2} \\
\frac{1}{2}b^* d\sqrt{2} + \frac{1}{2}ia^* c\sqrt{2}
\end{bmatrix}
\]

\[
(U_A \otimes U_B) J_{BB} = \begin{bmatrix}
-\frac{1}{2}iac\sqrt{2} + \frac{1}{2}ibd\sqrt{2} \\
-\frac{1}{2}iad^*\sqrt{2} + \frac{1}{2}ibc^*\sqrt{2} \\
-\frac{1}{2}ib^* c\sqrt{2} + \frac{1}{2}ia^* d\sqrt{2} \\
\frac{1}{2}ib^* d\sqrt{2} + \frac{1}{2}ia^* c\sqrt{2}
\end{bmatrix}
\]

The Hermitian conjugate of \( J \) is

\[
J^\dagger = \begin{bmatrix}
\frac{1}{2}\sqrt{2} & 0 & 0 & -\frac{1}{2}i\sqrt{2} \\
0 & \frac{1}{2}\sqrt{2} & \frac{1}{2}i\sqrt{2} & 0 \\
0 & \frac{1}{2}i\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \\
-\frac{1}{2}i\sqrt{2} & 0 & 0 & \frac{1}{2}\sqrt{2}
\end{bmatrix}
\]

As a consequence,

\[
J^\dagger (U_A \otimes U_B) J_{FF} = \begin{bmatrix}
\frac{1}{2}ac + \frac{1}{2}ibd - \frac{1}{2}b^* d^* + \frac{1}{2}a^* c^* \\
-\frac{1}{2}ad^* + \frac{1}{2}ib^* c - \frac{1}{2}a^* d \\
-\frac{1}{2}iad^* - \frac{1}{2}ib^* c + \frac{1}{2}a^* d \\
-\frac{1}{2}iac + \frac{1}{2}bd + \frac{1}{2}b^* d^* + \frac{1}{2}a^* c^*
\end{bmatrix}
\]

= \begin{bmatrix}
\text{Re}(ac) - \text{Im}(bd) \\
\text{Im}(bc) + \text{Re}(ad^*) \\
\text{Im}(ad^*) - \text{Re}(bc^*) \\
\text{Re}(bd) + \text{Im}(ac)
\end{bmatrix}

And

\[
J^\dagger (U_A \otimes U_B) J_{BB} = \begin{bmatrix}
\frac{1}{2}iac + \frac{1}{2}bd + \frac{1}{2}b^* d^* - \frac{1}{2}a^* c^* \\
-\frac{1}{2}iad^* + \frac{1}{2}ib^* c + \frac{1}{2}a^* d \\
\frac{1}{2}ad^* + \frac{1}{2}ib^* c - \frac{1}{2}a^* d \\
\frac{1}{2}ac - \frac{1}{2}ibd + \frac{1}{2}b^* d^* + \frac{1}{2}a^* c^*
\end{bmatrix}
\]

= \begin{bmatrix}
\text{Re}(bd) - \text{Im}(ac) \\
\text{Re}(bd^*) + \text{Im}(ac^*) \\
\text{Re}(bd^*) - \text{Im}(ac^*) \\
\text{Re}(ac) + \text{Im}(bd)
\end{bmatrix}

The most general form of \( a, b, c, d \) is:

\[
a = e^{i\alpha} \cos \frac{1}{2}\theta \\
b = e^{i\beta} \sin \frac{1}{2}\theta \\
c = e^{i\gamma} \cos \frac{1}{2}\omega \\
d = e^{i\delta} \sin \frac{1}{2}\omega
\]

which gives the operators

\[
U_A = \begin{bmatrix}
e^{i\alpha} \cos \frac{1}{2}\theta & e^{i\beta} \sin \frac{1}{2}\theta \\
-e^{i\beta} \sin \frac{1}{2}\theta & e^{-i\alpha} \cos \frac{1}{2}\theta
\end{bmatrix}
\]

\[
U_B = \begin{bmatrix}
e^{i\gamma} \cos \frac{1}{2}\omega & e^{i\delta} \sin \frac{1}{2}\omega \\
-e^{i\delta} \sin \frac{1}{2}\omega & e^{-i\gamma} \cos \frac{1}{2}\omega
\end{bmatrix}
\]

Calling \( |E_{FF} \rangle = J^\dagger (U_A \otimes U_B) J_{FF} \) and \( |E_{BB} \rangle = J^\dagger (U_A \otimes U_B) J_{BB} \), the final states in function of these angles become:

\[
|E_{FF} \rangle = \begin{bmatrix}
\cos \frac{1}{2}\theta \cos \frac{1}{2}\omega \cos (\alpha + \gamma) - \sin \frac{1}{2}\theta \sin \frac{1}{2}\omega \sin (\beta + \delta) \\
\sin \frac{1}{2}\theta \cos \frac{1}{2}\omega \sin (-\beta + \gamma) - \cos \frac{1}{2}\theta \sin \frac{1}{2}\omega \cos (\alpha - \delta) \\
- \sin \frac{1}{2}\theta \cos \frac{1}{2}\omega \cos (-\beta + \gamma) + \cos \frac{1}{2}\theta \sin \frac{1}{2}\omega \sin (\alpha - \delta) \\
\cos \frac{1}{2}\theta \cos \frac{1}{2}\omega \sin (\alpha + \gamma) + \sin \frac{1}{2}\theta \sin \frac{1}{2}\omega \cos (\beta + \delta)
\end{bmatrix}
\]

\[
|E_{BB} \rangle = \begin{bmatrix}
\cos \frac{1}{2}\theta \cos \frac{1}{2}\omega \sin (\alpha + \gamma) + \sin \frac{1}{2}\theta \sin \frac{1}{2}\omega \cos (\beta + \delta) \\
\sin \frac{1}{2}\theta \cos \frac{1}{2}\omega \sin (-\beta + \gamma) + \cos \frac{1}{2}\theta \sin \frac{1}{2}\omega \sin (\alpha - \delta) \\
\sin \frac{1}{2}\theta \cos \frac{1}{2}\omega \cos (-\beta + \gamma) - \cos \frac{1}{2}\theta \sin \frac{1}{2}\omega \cos (\alpha - \delta) \\
\cos \frac{1}{2}\theta \cos \frac{1}{2}\omega \sin (\alpha + \gamma) - \sin \frac{1}{2}\theta \sin \frac{1}{2}\omega \cos (\beta + \delta)
\end{bmatrix}
\]

Defining

\[
x = \cos \frac{1}{2}\theta \\
y = \cos \frac{1}{2}\omega \\
z = \sin (\alpha + \gamma) \\
w = \sin (\beta + \delta)
\]
it is possible to write

$$|E_{FF}| = \begin{bmatrix}
-\sqrt{(1-x^2)y \sin(\beta - \gamma)} - x\sqrt{(1-y^2)\cos(\alpha + \delta)} \\
-\sqrt{(1-x^2)y \sin(\beta - \gamma)} + x\sqrt{(1-y^2)\sin(\alpha + \delta)} \\
xyz + \sqrt{(1-x^2)\sqrt{(1-y^2)\sqrt{(1-w^2)}}}
\end{bmatrix}$$

$$|E_{BB}| = \begin{bmatrix}
\sqrt{(1-x^2)y \cos(\beta - \gamma)} - x\sqrt{(1-y^2)\sin(\alpha + \delta)} \\
-\sqrt{(1-x^2)y \sin(\beta - \gamma)} + x\sqrt{(1-y^2)\cos(\alpha + \delta)} \\
xyz + \sqrt{(1-x^2)\sqrt{(1-y^2)\sqrt{(1-w^2)}}}
\end{bmatrix}$$

The restriction $\beta = \delta = 0$, used by some authors results on:

$$|E_{FF}| = \begin{bmatrix}
-\sqrt{(1-x^2)y \sin(\gamma)} - x\sqrt{(1-y^2)\cos(\alpha)} \\
-\sqrt{(1-x^2)y \cos(\gamma)} - x\sqrt{(1-y^2)\sin(\alpha)} \\
xyz + \sqrt{(1-x^2)\sqrt{(1-y^2)\sqrt{(1-w^2)}}}
\end{bmatrix}$$

$$|E_{BB}| = \begin{bmatrix}
\sqrt{(1-x^2)y \sin(\gamma)} - x\sqrt{(1-y^2)\cos(\alpha)} \\
-\sqrt{(1-x^2)y \cos(\gamma)} + x\sqrt{(1-y^2)\sin(\alpha)} \\
xyz + \sqrt{(1-x^2)\sqrt{(1-y^2)\sqrt{(1-w^2)}}}
\end{bmatrix}$$

Further restriction $\alpha = \gamma = 0$ reduces to the classical case

$$|E_{FF}| = \begin{bmatrix}
-x\sqrt{(1-y^2)} \\
-\sqrt{(1-x^2)y} \\
\sqrt{(1-x^2)\sqrt{(1-y^2)}}
\end{bmatrix}$$

$$|E_{BB}| = \begin{bmatrix}
\sqrt{(1-x^2)y} - x\sqrt{(1-y^2)} \\
xyz \\
\sqrt{(1-x^2)\sqrt{(1-y^2)}}
\end{bmatrix}$$

The asymmetric game, in which Alice can use restricted quantum operators and Bob, only classical ones, is obtained by imposing $\gamma = 0$ on the 2 × 2 quantum case, resulting on:

$$|E_{FF}| = \begin{bmatrix}
\cos \frac{1}{2}\theta \cos \frac{1}{2}\omega \cos \alpha \\
-\cos \frac{1}{2}\theta \sin \frac{1}{2}\omega \cos \alpha \\
-\sin \frac{1}{2}\theta \cos \frac{1}{2}\omega \sin \alpha + \sin \frac{1}{2}\theta \sin \frac{1}{2}\omega \\
\cos \frac{1}{2}\theta \cos \frac{1}{2}\omega \sin \alpha + \sin \frac{1}{2}\theta \sin \frac{1}{2}\omega \\
xyz + \sqrt{(1-x^2)\sqrt{(1-y^2)\sqrt{(1-w^2)}}}
\end{bmatrix}$$

$$|E_{BB}| = \begin{bmatrix}
-\cos \frac{1}{2}\theta \cos \frac{1}{2}\omega \sin \alpha + \sin \frac{1}{2}\theta \sin \frac{1}{2}\omega \\
\sin \frac{1}{2}\theta \cos \frac{1}{2}\omega + \cos \frac{1}{2}\theta \sin \frac{1}{2}\omega \sin \alpha \\
\cos \frac{1}{2}\theta \sin \frac{1}{2}\omega \cos \alpha \\
\cos \frac{1}{2}\theta \cos \frac{1}{2}\omega \cos \alpha \\
xyz + \sqrt{(1-x^2)\sqrt{(1-y^2)\sqrt{(1-w^2)}}}
\end{bmatrix}$$

where $z = \sin \alpha$

4 Analysis

4.1 Case 3×3

As remarked by Du et. al. [7], the totally entangled 3 × 3 case doesn’t have pure quantum equilibria (mixed quantum equilibria, defined by way of probability measures over the set of admissible density matrices, may exist - in fact, Lee and Jonhson [13] seem to have proved they always do). The argument here goes as follows: define $u = xy = \cos \frac{1}{2}\theta \cos \frac{1}{2}\omega$ and $v = \sqrt{(1-x^2)\sqrt{(1-y^2)}} = \sin \frac{1}{2}\theta \sin \frac{1}{2}\omega$; then, the payoffs corresponding to $|E_{BB}|$ are

$$S_A = \left(-uz + v\sqrt{(1-w^2)}\right)^2 + 2\left(u\sqrt{(1-z^2)} + vw\right)^2$$

$$S_B = 2\left(-uz + v\sqrt{(1-w^2)}\right)^2 + \left(u\sqrt{(1-z^2)} + vw\right)^2$$

Let $z = \sin \rho$ and $w = \sin \sigma$. Then
\[ S_A = \left( -u \sin \rho + v \sqrt{(1 - \sin^2 \sigma)} \right)^2 + 2 \left( u \sqrt{(1 - \sin^2 \rho)} + v \sin \sigma \right)^2 = \frac{3}{2} u^2 + \frac{1}{2} u^2 \cos 2\rho + u \sin (\rho + \sigma) - 3uv \sin (\rho - \sigma) - \frac{1}{2} v^2 \cos 2\sigma + \frac{3}{2} v^2, \]

whose only candidates for extrema on the variables \( \rho \) and \( \sigma \) are 2 \((u \pm v)^2\), at \((\rho, \sigma) \in \{0, k\pi, k \in \mathbb{Z}\}\). These values for \((\rho, \sigma)\) correspond to \((z, w) \in \{0, \pm 1\}\). With such values of \(z\) and \(w\), the results are always \(S_B = 2S_A\) or \(S_A = 2S_B\), where \(\min(S_A, S_B) \in \{(u \pm v)^2\}\). As neither Alice nor Bob can control \(z\) or \(w\) independently, whatever be the triplet \((\theta, \alpha, \beta)\) chosen by Alice, Bob can always change \((\gamma, \delta)\) so as to set \(z\) and \(w\) to values that give him \(S_B = \max \left\{ (2(u + v)^2, 2(u - v)^2) \right\}\) and, whatever be the triplet \((\omega, \gamma, \delta)\) chosen by Bob, Alice can always change \((\alpha, \beta)\) so as to set \(z\) and \(w\) to values that give her \(S_A = \max \left\{ 2(u + v)^2, 2(u - v)^2 \right\}\). And both can escape the case \(S_B = S_A = 0\) (when \(u = v = 0\)) by changing unilaterally the parameters that they really control \((\theta\) for Alice, \(\omega\) for Bob\), as \(u = v = 0\) requires \((z = 1, w = 0)\) or \((z = 0, w = 1)\).

The same analysis applies to \(|E_{FF}|\).

### 4.2 Case 2 \times 2

The payoffs in this case are:

\[
\begin{align*}
S_A(FF) &= (xy \sqrt{(1 - z^2)})^2 + 2 \left( xyz + \sqrt{(1 - x^2)} \sqrt{(1 - y^2)} \right)^2 \\
S_B(FF) &= 2 \left( xy \sqrt{(1 - z^2)} \right)^2 + \left( xyz + \sqrt{(1 - x^2)} \sqrt{(1 - y^2)} \right)^2 \\
S_A(BB) &= (-xyz + \sqrt{(1 - x^2)} \sqrt{(1 - y^2)})^2 + 2 \left( xy \sqrt{(1 - z^2)} \right)^2 \\
S_B(BB) &= 2 \left(-xyz + \sqrt{(1 - x^2)} \sqrt{(1 - y^2)} \right)^2 + \left( xy \sqrt{(1 - z^2)} \right)^2
\end{align*}
\]

where \((FF)\) and \((BB)\) indicate the initial state \(|E_i|\) of the system. It is sufficient to notice that, in the case FF, Alice can always annulate the first term by setting \(x = 0\), which results in \(S_A = 2S_B\), giving Bob no other choice then agreeing to maximize the second term, which is accomplished by setting \(y = 0\), resulting in \(S_B = 1\). As Alice can never expect to get a payoff bigger than 2, the resulting profile gives no incentive to change for the members of the couple, thus being a Nash equilibrium. In the case BB, it is Bob who has the couteau and the fromage, being able to annulate the second term, by setting \(y = 0\), leaving Alice no choice other then choosing \(x = 0\) to maximize her payoff.

Reminding that \(x = \cos \frac{1}{2} \theta\) and \(y = \cos \frac{1}{2} \omega\), in all of the above equilibria the players use the flipping operators:

\[
U_A = \begin{bmatrix} 
 e^{i\omega} \cos \frac{1}{2} \theta & \sin \frac{1}{2} \theta \\
 -\sin \frac{1}{2} \theta & e^{-i\omega} \cos \frac{1}{2} \theta
\end{bmatrix} = \begin{bmatrix} 
 0 & \pm 1 \\
 \mp 1 & 0
\end{bmatrix}
\]

\[
U_B = \begin{bmatrix} 
 e^{i\omega} \cos \frac{1}{2} \omega & \sin \frac{1}{2} \omega \\
 -\sin \frac{1}{2} \omega & e^{-i\omega} \cos \frac{1}{2} \omega
\end{bmatrix} = \begin{bmatrix} 
 0 & \pm 1 \\
 \mp 1 & 0
\end{bmatrix}
\]

Indeed all equilibria of this case favor Alice, when \(|E_i| = |FF\) and Bob, when \(|E_i| = |BB|\).

To see this, write

\[
\begin{align*}
S_A(FF) &= u^2 (1 - z^2) + 2 (uz + v)^2 = u^2 z^2 + 4vuz + (u^2 + 2v^2)
\]

\[
S_B(FF) = 2u^2 (1 - z^2) + (uz + v)^2 = -u^2 z^2 + 2vuz + 2u^2 + v^2
\]

with the same definitions of \(u\) and \(v\) as before. Then

\[
\begin{align*}
\frac{\partial}{\partial z} S_A(FF) &= 2u^2 z + 4vu \\
\frac{\partial}{\partial z} S_B(FF) &= -2u^2 z + 2vu \\
\frac{\partial^2}{\partial z^2} S_A(FF) &= 2u^2 \\
\frac{\partial^2}{\partial z^2} S_B(FF) &= -2u^2
\end{align*}
\]

Alice’s payoff is maximized at the extremities \(|z| = 1\); Bob’s, on \(z = \frac{u}{v}\). As neither can control \(z\) independently, whatever value one sets to \(z\) the other one can unset. As a consequence, in any equilibrium, they must agree as to the best value of \(z\). This will be the case only if \(|\frac{u}{v}| = 1\). Then, the payoffs are reduced to \(S_A(FF) = 2S_B(FF) = 2(uz + v)^2\), obliging them to coordinate to make
$|uz + v| = 1$, thus jointly maximizing their payoffs, but maintaining the bias towards Alice, who gets 2, while Bob gets 1. Besides, Bob must not have incentive to choose $y$ such that $|\frac{v}{u}| \neq 1$.

Equilibria of the kind $|z| = 1$, $|u| = |v|$ reduce the payoffs to

$S_A = 2(uz + v)^2$

$S_B = (uz + v)^2$

If $z = 1$, then $|u + v| = 1$ and $v = u = \pm \frac{1}{2}$. If $z = -1$, then $|u + v| = 1$ and $v = -u = \pm \frac{1}{2}$.

Now,

$u = xy = \frac{1}{2}$ implies $y = \frac{1}{2}$

$v = \sqrt{1 - x^2} \sqrt{1 - y^2} = \frac{1}{2} \sqrt{(1 - x^2)(1 - \frac{1}{4})} = \frac{1}{2} \sqrt{(1 - x^2)} \sqrt{(4x^2 - 1)}$

$\frac{1}{2} \sqrt{(1 - x^2)} \sqrt{(4x^2 - 1)} = \frac{1}{2}$, which results on $x = \frac{1}{2} \sqrt{2}$ and, thus, $y = \frac{1}{2} \sqrt{2}$.

As a consequence, these equilibria require $|z| = 1$, $|x| = \frac{1}{2} \sqrt{2} = |y| = \frac{1}{2} \sqrt{2}$. The condition $|x| = \frac{1}{2} \sqrt{2}$ is translated for $\theta$ as:

$|\cos \frac{1}{2} \theta| = \frac{1}{2} \sqrt{2}$, which is satisfied by $\theta = \pm \left(\frac{1}{2} \pi + 2k\pi\right)$

By the same token, $\omega = \pm \left(\frac{1}{4} \pi + 2k\pi\right)$

The corresponding operators are of the type:

$U_A = \frac{1}{2} \sqrt{2} \begin{bmatrix} e^{i\alpha} & 1 & 1 & e^{-i\alpha} \\ 1 & e^{i\alpha} & e^{-i\alpha} & 1 \\ e^{i\beta} & e^{-i\gamma} & 1 & e^{i\beta} & e^{-i\gamma} & 1 \\ e^{i\gamma} & e^{-i\gamma} & 1 & e^{i\beta} & e^{-i\gamma} & 1 \\ 1 & e^{i\alpha} & e^{-i\alpha} & 1 \\ e^{i\beta} & e^{-i\gamma} & 1 & e^{i\beta} & e^{-i\gamma} & 1 \\ e^{i\gamma} & e^{-i\gamma} & 1 & e^{i\beta} & e^{-i\gamma} & 1 \\ e^{i\gamma} & e^{-i\gamma} & 1 & e^{i\beta} & e^{-i\gamma} & 1 \end{bmatrix}$

recalling that these equilibria, as the former ones, give 2 to Alice and 1 to Bob.

Given the symmetry, the analysis of $|E_{BB}|$ follows the same lines. In this case, all equilibria give 2 to Bob and 1 to Alice.

What is new here, besides the explicitation of the analysis, is the comparison between the two cases, FF and BB, showing that the simple interchange of $i = e^{i\pi}$ in the components of the state $J|E_i\rangle$, changes completely the balance of a game otherwise symmetrical. That is: define the system being controlled by the players by the pair $(J, |G_i\rangle)$, where $|G_i\rangle = J|E_i\rangle$; then, a change from $|G_i\rangle = (\frac{1}{2} \sqrt{2}, 0, 0, \frac{1}{2} i \sqrt{2})$ to $|G_i\rangle = (\frac{1}{2} i \sqrt{2}, 0, 0, \frac{1}{2} \sqrt{2})$ turns a game totally favorable to Alice to one totally favorable to Bob.

### 4.3 Case $2 \times 1$

In this case, the operators available for the players are:

$U_A = \begin{bmatrix} e^{i\alpha} \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\ -\sin \frac{\theta}{2} & e^{-i\alpha} \cos \frac{\theta}{2} \\ \cos \frac{1}{2} \omega & \sin \frac{1}{2} \omega \\ -\sin \frac{1}{2} \omega & \cos \frac{1}{2} \omega \end{bmatrix}$

$U_B = \begin{bmatrix} e^{i\gamma} & 1 & 1 & e^{-i\gamma} \\ 1 & e^{i\gamma} & e^{-i\gamma} & 1 \\ e^{i\beta} & e^{-i\alpha} & 1 & e^{i\beta} & e^{-i\alpha} & 1 \\ e^{i\alpha} & e^{-i\gamma} & 1 & e^{i\beta} & e^{-i\gamma} & 1 \\ 1 & e^{i\gamma} & e^{-i\gamma} & 1 \\ e^{i\beta} & e^{-i\alpha} & 1 & e^{i\beta} & e^{-i\alpha} & 1 \\ e^{i\gamma} & e^{-i\alpha} & 1 & e^{i\beta} & e^{-i\alpha} & 1 \\ e^{i\gamma} & e^{-i\alpha} & 1 & e^{i\beta} & e^{-i\alpha} & 1 \end{bmatrix}$

Then,

$S_A(FF) = (\cos \frac{1}{2} \theta \cos \frac{1}{2} \omega \cos \alpha)^2 + 2 (\cos \frac{1}{2} \theta \cos \frac{1}{2} \omega \sin \alpha + \sin \frac{1}{2} \theta \sin \frac{1}{2} \omega)^2$

$S_B(FF) = 2 (\cos \frac{1}{2} \theta \cos \frac{1}{2} \omega \cos \alpha)^2 + (\cos \frac{1}{2} \theta \cos \frac{1}{2} \omega \sin \alpha + \sin \frac{1}{2} \theta \sin \frac{1}{2} \omega)^2$

$S_A(BB) = -(\cos \frac{1}{2} \theta \cos \frac{1}{2} \omega \sin \alpha + \sin \frac{1}{2} \theta \sin \frac{1}{2} \omega)^2 + 2 (\cos \frac{1}{2} \theta \cos \frac{1}{2} \omega \cos \alpha)^2$

$S_B(BB) = 2 (\cos \frac{1}{2} \theta \cos \frac{1}{2} \omega \sin \alpha + \sin \frac{1}{2} \theta \sin \frac{1}{2} \omega)^2 + (\cos \frac{1}{2} \theta \cos \frac{1}{2} \omega \cos \alpha)^2$

With the same definitions as before, the only difference being that now $z = \sin \alpha$, which gives it total control by Alice, the analysis is the following:

The first and fourth components of the final quantum state $|E_{FF}\rangle$ are:

$\begin{bmatrix} xy \sqrt{1 - z^2} \\ xz \sqrt{(1 - x^2)(1 - y^2)} \end{bmatrix}$

The payoffs become, with the definitions of $u$ and $v$ already given:
$A = u^2(1 - z^2) + 2(uz + v)^2 = u^2z^2 + 4uvz + u^2 + 2v^2$
$B = 2u^2(1 - z^2) + (uz + v)^2 = -u^2z^2 + 2uvz + 2u^2 + v^2$

As $A(z)$ is a parable upwards, Alice always chooses $|z| = 1$. Then,

$A = 2(uz + v)^2$
$B = (uz + v)^2$

Alice and Bob coordinate to maximize $|uz + v|$. 
Because the minimum of $A(z)$ is at $z = -\frac{2u}{v}$, if $u$ and $v$ have opposite signs, Alice chooses $z=-1$; otherwise, $z=1$. Now,

$u = \cos \frac{1}{2} \theta \cos \frac{1}{2} \omega$
$v = \sin \frac{1}{2} \theta \sin \frac{1}{2} \omega$

Suppose $c$ and $d$ have equal signs. Then, $z=1$ and

$|uz + v| = |u + v| = |\cos \frac{1}{2} \theta \cos \frac{1}{2} \omega + \sin \frac{1}{2} \theta \sin \frac{1}{2} \omega|$

The maximum value of this expression is 1, attained when $\omega = \theta + 2k\pi$, $k \in \mathbb{Z}$.

Suppose $c$ and $d$ have opposite signs. Then, $z=-1$ and

$|uz - v| = |u - v| = |\cos \frac{1}{2} \theta \cos \frac{1}{2} \omega - \sin \frac{1}{2} \theta \sin \frac{1}{2} \omega|$

The maximum value of this expression is 1, attained when $\omega = -\theta + 2k\pi$, $k \in \mathbb{Z}$.

Particularly, for $\theta, \omega \in \{ \pi + 2k\pi | k \in \mathbb{Z} \}$, $U_A$ and $U_B$ become independent from $\alpha$, so that this parameter is free.

When $|E_i| = |BB|$, the final state has the following first and fourth components:

$[-xyz + \sqrt{(1 - x^2)} \sqrt{(1 - y^2)} ]$

The payoffs are:

$A = 2u^2(1 - z^2) + (uz + v)^2 = -u^2z^2 - 2uvz + 2u^2 + v^2$
$B = u^2(1 - z^2) + 2(uz + v)^2 = u^2z^2 + 4uvz + u^2 + 2u^2$

$\frac{\partial}{\partial z} P_A = -2u^2z - 2uv$

$A(z)$ is a downwards parable, with its maximum at $z = \frac{1}{u} v$.
So, if $\frac{1}{u} v \in [-1, 1]$, Alice chooses $z = -\frac{1}{u} v$; otherwise, $|z| = 1$.

Suppose $\frac{1}{u} v \in [-1, 1]$. Then,

$A = 2u^2(1 - \frac{1}{u^2} v^2) + 4v^2 = 2u^2 + 2v^2$
$B = u^2(1 - \frac{1}{u^2} v^2) = u^2 - v^2$

To maximize his payoff, Bob annihilates $v$, by setting $|y| = 1$. The resulting payoffs are:

$A = 2u^2$
$B = u^2$

and Alice maximizes $u$ by choosing $|x| = 1$. As $v = 0$ and $u \neq 0$, the hypothesis $\frac{1}{u} v \in [-1, 1]$ is still valid.

This leaves Bob with no other choice than setting $|y| = 1$. The result is 2 to Alice and 1 to Bob.

Suppose that $\frac{1}{u} v \notin [-1, 1]$. Then Alice chooses $|z| = 1$ and

$A = -(uz + v)^2$
$B = 2(uz + v)^2$

Thus, Bob tries to maximize $|uz + v|$ and Alice, $|uz + v|$. Both are simultaneously maximized only when $uz = 0$. As $z \neq 0$, $v$ must be annihilated, which any player has the power to do unilaterally.

So, in the equilibria, $u=0$ and the payoffs become

$A = v^2$
$B = 2v^2$

The players coordinate to maximize $|v|$ by setting $|x| = |y| = 1$. As $v \neq 0$ and $u = 0$, the hypothesis $\frac{1}{u} v \notin [-1, 1]$ is still valid.

As $u=0$, $z$ becomes irrelevant, which frees $\alpha$. The result is 2 to Bob and 1 to Alice.

This case, having only three real parameters, allows the graphical illustration below, showing the sets of Nash equilibria in the parameter space.

Figure 1: Quantum Alice versus classical Bob - points of the parameter space belonging to the bold straight segments are the Nash equilibria of this quantum version of the battle of the sexes game. The initial state is FF. All equilibria favor Alice ($S_A = 2$, $S_B = 1$). Vertical axis is $\omega$, horizontal is $\theta$ and the one perpendicular to the page is $\alpha$.

Figure 2: Quantum Alice versus classical Bob - points of the parameter space belonging to the
Fig. 1.

Fig. 2.
bold straight segments are Nash equilibria of this quantum version of the battle of the sexes game. The initial state is BB. Most equilibria favor Bob ($$A = 1, B = 2$$). In the isolate points shown, the equilibrium favors Alice ($$A = 2, B = 1$$). Vertical axis is $$\omega$$, horizontal is $$\theta$$ and the one parallel to the bold straight lines is $$\alpha$$.

4.4 Some simpler models

Marinatto and Weber [11] analyzed the non-entangled case, where $$J = I$$ (the identity operator, obtained by setting $$\gamma = 0$$ in J’s formula) with initial state $$|BB\rangle$$, without restrictions on the parameters set. The final state is given by

$$|E_{BB}\rangle = (U_A \otimes U_B)|BB\rangle = \begin{bmatrix} ac & ad & bc & bd \\ -ad^* & ac^* & -bd^* & bc^* \\ -b^*c & -b^*d & a^*c & a^*d \\ b^*d^* & -b^*c^* & -a^*d^* & a^*c^* \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} bd \\ bc^* \\ a^*d \\ a^*c^* \end{bmatrix}$$

which results in the payoffs:

$$A = |bd|^2 + 2|a^*c^*|^2 = |b|^2|d|^2 + 2|a|^2|c|^2 = \left(1 - x^2\right)\left(1 - y^2\right) + 2x^2y^2$$

$$B = 2|bd|^2 + |a^*c^*|^2 = 2|b|^2|d|^2 + |a|^2|c|^2 = \left(1 - x^2\right)\left(1 - y^2\right) + 2x^2y^2$$

If the initial state is $$|FF\rangle$$, the situation changes to:

$$|E_{FF}\rangle = (U_A \otimes U_B)|FF\rangle = \begin{bmatrix} ac & ad & bc & bd \\ -ad^* & ac^* & -bd^* & bc^* \\ -b^*c & -b^*d & a^*c & a^*d \\ b^*d^* & -b^*c^* & -a^*d^* & a^*c^* \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} ac \\ -ad^* \\ -b^*c \\ b^*d^* \end{bmatrix}$$

which results in the payoffs:

$$A = 2|b^*d^*|^2 + |ac|^2 = 2|b|^2|d|^2 + |a|^2|c|^2 = \left(1 - x^2\right)\left(1 - y^2\right) + 2x^2y^2$$

$$B = |b^*d^*|^2 + 2|ac|^2 = |b|^2|d|^2 + 2|a|^2|c|^2 = \left(1 - x^2\right)\left(1 - y^2\right) + 2x^2y^2$$

Both cases above reproduce the classical game, sufficing to identify $$x$$ and $$y$$ with the probabilities of Alice and Bob, respectively, choosing ballet, in the first case ($$|E_{BB}\rangle$$) and choosing football, in the second one ($$|E_{FF}\rangle$$). Thus, the equilibria are $$(x, y) \in \{(1, 1), (0, 0), \left(\frac{1}{2}, \frac{1}{2}\right)\}$$, in the first case, and $$(x, y) \in \{(1, 1), (0, 0), \left(\frac{1}{2}, \frac{1}{2}\right)\}$$, in the second case.

Now, suppose the initial state fed to the system is already entangled.

First, let $$|E_i\rangle = \frac{x^2}{2}(|FF\rangle + |BB\rangle)$$.

In the basis here used, this is represented by the vector $$\frac{x^2}{2}(1001)$$. So

$$|E_f\rangle = (U_A \otimes U_B)|E_i\rangle = \frac{x^2}{2} \begin{bmatrix} ac & ad & bc & bd \\ -ad^* & ac^* & -bd^* & bc^* \\ -b^*c & -b^*d & a^*c & a^*d \\ b^*d^* & -b^*c^* & -a^*d^* & a^*c^* \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sqrt{2ac + \sqrt{2bd}} \\ -\frac{1}{2}\sqrt{2ad^* + \sqrt{2bc^*}} \\ -\frac{1}{2}\sqrt{2b^*c + \frac{2}{2}\sqrt{2a^*d}} \\ \frac{1}{2}\sqrt{2b^*d^* + \sqrt{2a^*c^*}} \end{bmatrix}$$

The resulting payoffs are:

$$A = \frac{1}{2} \left(\frac{\sqrt{a^2 + b^2d^2} + 2|a^*c^* + b^*d^*|^2}{2}\right) = \frac{1}{2} |ac + bd|^2$$

$$B = \frac{1}{2} \left(2|ac + bd|^2 + |a^*c^* + b^*d^*|^2\right) = \frac{3}{2} |ac + bd|^2$$

So, in this case, $$A = B$$ whatever are the values of the parameters and this is a game of coordination, with both players being interested in raising $$|ac + bd|^2$$ to its maximal attainable value, which is 1. When they get this, the expected payoffs are $$\frac{1}{2}$$ for each one. This is equivalent to, in the classical game, the couple making a decision like a single entity, by flipping a coin and deciding to go to football, if it turns out as head, and to ballet otherwise, which is basically a one player (the couple) game.

Is there an asymmetry between $$|E_i\rangle = \frac{x^2}{2}(|FF\rangle + |BB\rangle)$$ and $$|E_j\rangle = \frac{x^2}{2}(|FF\rangle + i|BB\rangle)$$, as in the case when J was chosen with $$\gamma = \frac{\pi}{2}$$?

This case is equivalent to defining the quantum game by $$|E_f\rangle = (U_A \otimes U_B)J|BB\rangle$$ and
$|E_f\rangle = (U_A \otimes U_B) |FF\rangle$, respectively, with $J = J\ (\gamma = \frac{n}{2})$, that is, skipping the post multiplication by $J^1$.

Suppose $|E_i\rangle = \frac{\sqrt{2}}{2} (i|FF\rangle + |BB\rangle)$. Then

$$|E_f\rangle = (U_A \otimes U_B) |E_i\rangle = \frac{\sqrt{2}}{2} \left[ \begin{array}{cccc} ac & ad & bc & bd \\ -ad^* & ac^* & -bd^* & bc^* \\ -b^*c & -b^*d & a^*c & a^*d \\ b^*d^* & -b^*e^* & -a^*d^* & a^*e^* \end{array} \right] \left[ \begin{array}{c} i \\ 0 \\ 0 \\ 1 \end{array} \right]$$

$$= \left[ \begin{array}{c} \frac{1}{2}i\sqrt{2ac + \frac{1}{2}\sqrt{2bd}} \\ \frac{1}{2}i\sqrt{2ad^* + \frac{1}{2}\sqrt{2bc^*}} \\ -\frac{1}{2}i\sqrt{2b^*c + \frac{1}{2}\sqrt{2a^*d}} \\ \frac{1}{2}i\sqrt{2b^*d^* + \frac{1}{2}\sqrt{2a^*e^*}} \end{array} \right].$$

The resulting payoffs are:

$$\$A = \frac{1}{2} \left( |iac + bd|^2 + |a^*c^* + ib^*d^*|^2 \right)$$

$$\$B = \frac{1}{2} \left( 2|iac + bd|^2 + |a^*c^* + ib^*d^*|^2 \right)$$

But $iac + bd = i(a^*c^* + ib^*d^*)^*$

So $|iac + bd|^2 = |a^*c^* + ib^*d^*|^2$

As a result, $\$A = $\$B = \frac{3}{2}|iac + bd|^2$

and, again, the game is one of coordination, with all equilibria at $|iac + bd| = 1$ and $\$A = $\$B = \frac{3}{2}.$

Suppose $|E_i\rangle = \frac{\sqrt{2}}{2} (|FF\rangle + i|BB\rangle)$. Then

$$|E_f\rangle = (U_A \otimes U_B) |E_i\rangle = \frac{\sqrt{2}}{2} \left[ \begin{array}{cccc} ac & ad & bc & bd \\ -ad^* & ac^* & -bd^* & bc^* \\ -b^*c & -b^*d & a^*c & a^*d \\ b^*d^* & -b^*e^* & -a^*d^* & a^*e^* \end{array} \right] \left[ \begin{array}{c} 1 \\ 0 \\ 0 \\ i \end{array} \right]$$

$$= \left[ \begin{array}{c} \frac{1}{2}i\sqrt{2ac + \frac{1}{2}\sqrt{2ibd}} \\ \frac{1}{2}i\sqrt{2ad^* + \frac{1}{2}\sqrt{2ibc^*}} \\ -\frac{1}{2}i\sqrt{2b^*c + \frac{1}{2}\sqrt{2ia^*d}} \\ \frac{1}{2}i\sqrt{2b^*d^* + \frac{1}{2}\sqrt{2ia^*e^*}} \end{array} \right].$$

The resulting payoffs are:

$$\$A = \frac{1}{2} \left( |ac + ibd|^2 + |ia^*c^* + b^*d^*|^2 \right)$$

$$\$B = \frac{1}{2} \left( 2|ac + ibd|^2 + |ia^*c^* + b^*d^*|^2 \right)$$

But $ac + ibd = i(a^*c^* + b^*d^*)^*$

So $|ac + ibd|^2 = |ia^*c^* + b^*d^*|^2$

As a result, $\$A = $\$B = \frac{3}{2}|ac + ibd|^2$

and, again, the game is one of coordination, with all equilibria at $|ac + ibd| = 1$ and $\$A = $\$B = \frac{3}{2}.$

Paraphrasing Benjamin and Hayden [11], in these coherent equilibria, entanglement shared among the players enables different kinds of cooperative behavior: indeed it can act as a contract, in the sense that it obliges the players to coordinate with each other.

5 Conclusion

Some versions of the quantum battle of the sexes game reproduce the classical one, in the sense that equal proportions of the equilibria favor Alice ($\$A = 2, $\$B = 1$) and Bob ($\$A = 1, $\$B = 2$) and another set gives them both an expected payoff of $\frac{2}{3}$. In other versions, all equilibria favor Alice or all favor Bob, the difference between one type and the other being just a phase in the initial state of the system. In a third kind of versions, the payoffs are equal to each other independently of the operator that each player chooses, all equilibria giving both an expected payoff of $\frac{2}{3}$. In a fourth kind, there are no Nash pure quantum equilibria at all.

Quantum games, as any other game, can be viewed as a dispute between two or more players
to control a system; thus, they are defined by the system and the way the players are connected to it; if any of these change, the game changes; as a consequence, it is arguable if its name should be kept. To name "battle of the sexes" games with completely different qualitative features may not be reasonable.

References

[1] Meyer, David A. Quantum Games and Quantum Algorithms. ArXiv:quant-ph/0004092 v2. (2000)
[2] N. J. Cerf (Caltech), L. K. Grover (Bell labs), C. P. Williams (JPL). Nested quantum search and NP-complete problems. Phys.Rev. A, vol. 61p. 032303. (2000).
[3] E. Farhi and S. Gutmann, Quantum computation and decision trees, Los-Alamos e-print quant-ph/9706062 (1998).
[4] E. Farhi and S. Gutmann, Quantum mechanical square root speedup in a structured search problem, Los-Alamos e-print quant-ph/9711033 (1997).
[5] T. Hogg, Highly structured searches with quantum computers, Phys. Rev. Lett. 80, 2473 (1998).
[6] T. Hogg, A framework for structured quantum search, Los-Alamos e-print quant-ph/9701013 (1998).
[7] Du,Xu,Li,Shi,Zhou,Han Nash. Nash Equilibrium in the Quantum Battle of Sexes Game. China’s Univ. of Science and Technology. xxx.lanl.gov/abs/quant-ph/0010050. (2001).
[8] Du et. All. Remark on Quantum Battle of the Sexes. xxx.lanl.gov/abs/quant-ph/0103004. (2001).
[9] Nawaz, A. and Toor, A. H. Worst-case Payoffs in Quantum Battle of Sexes Game. ArXiv:quant-ph/0110006 v2. (2001)
[10] Marinatto, L. and Weber, T. A Quantum Approach to Static Games of Complete Information. Physics Letters A 277, 183-184. (2000).
[11] Benjamin, S. C. and Hayden, P. M. Multi-Player Quantum Games. Phys. Rev. A 64, 030301(R) (2001)
[12] Eisert, Jens, Wilkens, Martin e Lewenstein, Maciej. Quantum Games and Quantum Strategies. Physical Review Letters, vol 83, 3077.(1999).
[13] Lee, C. F. and Johnson, N.F. . Efficiency and formalism in quantum games. Phys. Rev. A, 67, 022311. (2003).
This figure "FIG1.JPG" is available in "JPG" format from:

http://arxiv.org/ps/quant-ph/0408019v1
This figure "FIG2.JPG" is available in "JPG" format from:

http://arxiv.org/ps/quant-ph/0408019v1