Construction of Rank 2 Indecomposable Modules in Grassmannian Cluster Categories

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Abstract

The category $\text{CM}(B_{k,n})$ of Cohen-Macaulay modules over a quotient $B_{k,n}$ of a preprojective algebra provides a categorification of the cluster algebra structure on the coordinate ring of the Grassmannian variety of $k$-dimensional subspaces in $\mathbb{C}^n$, [JKS16]. Among the indecomposable modules in this category are the rank 1 modules which are in bijection with $k$-subsets of $\{1,2,\ldots,n\}$, and their explicit construction has been given by Jensen, King and Su. These are the building blocks of the category as any module in $\text{CM}(B_{k,n})$ can be filtered by them. In this paper we give an explicit construction of rank 2 modules. With this, we give all indecomposable rank 2 modules in the cases when $k = 3$ and $k = 4$. In particular, we cover the tame cases and go beyond them. We also characterise the modules among them which are uniquely determined by their filtrations. For $k \geq 4$, we exhibit infinite families of non-isomorphic rank 2 modules having the same filtration.

1 Introduction

An addive categorisation of the cluster algebra structure on the homogeneous coordinate ring of the Grassmannian variety of $k$-dimensional subspaces in $\mathbb{C}^n$ has been given by Geiss, Leclerc, and Schroer [GLS06, GLS08] in terms of a subcategory of the category of finite dimensional modules over the preprojective algebra of type $A_{n-1}$. Jensen, King, and Su [JKS16] gave a new addive categorification of this cluster structure using the maximal Cohen-Macaulay modules over the completion of an algebra $B_{k,n}$ which is a quotient of the preprojective algebra of type $A_{n-1}$. In the category $\text{CM}(B_{k,n})$ of Cohen-Macaulay modules over $B_{k,n}$, among the indecomposable modules are the rank 1 modules which are known to be in bijection with $k$-subsets of $\{1,2,\ldots,n\}$, and their explicit construction has been given in [JKS16]. For a given $k$-subset $I$, the corresponding rank 1 module is denoted by $L_I$. Also, we refer to $k$-subsets as rims, because of the way we use them to visualize rank 1 modules (see Section 2). Rank 1 modules are the building blocks of the category as any module in $\text{CM}(B_{k,n})$ can be filtered by rank 1 modules (the filtration is noted in the profile of a module, [JKS16, Corollary 6.7]). The number of rank 1 modules appearing in the filtration of a given module is called the rank of that module.

The aim of this paper is to explicitly construct rank 2 indecomposable Cohen-Macaulay $B_{k,n}$ modules in the cases when $k = 3$ and $k = 4$. In particular, we construct all indecomposable rank 2 modules in the tame cases (3,9) and (4,8), and more generally, for an arbitrary $k$, we construct all indecomposable modules of rank 2 whose rank 1 filtration layers $L_I$ and $L_J$ satisfy the condition $|I \cap J| \geq k - 4$.

Once we have the construction, we investigate the question of uniqueness. Here, the central notions are that of $r$-interlacing (Definition 2.4) and of the poset of a given rank 2 module (Section 2). If $I$ and $J$ are $k$-subsets of $\{1,\ldots,n\}$, then $I$ and $J$ are said to be $r$-interlacing if there exist subsets $\{i_1, i_3, \ldots, i_{2r-1}\} \subset I \setminus J$ and $\{i_2, i_4, \ldots, i_{2r}\} \subset J \setminus I$ such that $i_1 < i_2 < i_3 < \cdots < i_{2r} < i_1$ (cyclically) and if there exist no larger subsets of $I$ and of $J$ with this property. The filtration layers of a module $M$ give a poset structure. In rank 2, if $I$ and $J$ are $r$-interlacing, the sets $I$ and $J$ form a number $r_1 \leq r$ of boxes in the so-called lattice diagram of $M$ (see Section 2 for details on how we picture $M$ with its filtration layers). The associated poset is $1^{r_1} \mid 2$; the poset consists of a tree with one vertex of degree $r_1$ and $r_1$ leaves, it has dimension 1 at the leaves and dimension 2 at central vertex. See Figure 1.
Theorem 1.2 (Theorem 4.4, Theorem 6.2). The poset of such a module is of the form 1\(r\) ≥ 1. This proposition tells us that when dealing with rank 2 indecomposable modules, we can assume that uniquely determined by its profile if and only if its poset is 1\(r\). More precisely, in the case of r-interlacing, where \(r \geq r_1 \geq 3\). Our main results are the following two theorems.

Theorem 1.2 (Theorem 4.4, Theorem 6.2). An indecomposable rank 2 module \(M \in \text{CM}(B_{k,n})\) is uniquely determined by its profile if and only if its poset is 1\(r\) ≥ 2.

More precisely, in the case of r-interlacing rank 1 layers with poset 1\(r_1\) ≥ 2, where \(r \geq r_1 \geq 4\), we show that there are infinitely many non-isomorphic rank 2 modules with the same profile, e.g. there are infinitely many non-isomorphic indecomposable rank 2 modules with filtration \(\{1,3,5,7\} \setminus \{2,4,6,8\}\) in the tame case (4,8).

Theorem 1.3 (Theorem 6.2). Let \(M\) be an indecomposable rank 2 module with profile I | J, where I and J are r-interlacing with poset 1\(r_1\) ≥ 2, where \(r \geq r_1 \geq 4\). Then there are infinitely many non-isomorphic rank 2 indecomposable modules from \(\text{CM}(B_{k,n})\) with profile I | J.

In the case \(r \geq r_1 = 4\), we show that this infinite family of indecomposable modules with the profile I | J is parameterized by the set \(\mathbb{C} \setminus \{0,1,-1\}\) where two points from this set are identified if their sum is 0.

For the filtration layers I and J of an indecomposable module with profile I | J, we construct all decomposable rank 2 modules that are extensions of these rank 1 modules, i.e. we construct all decomposable modules that appear as middle terms in short exact sequences with I and J as end terms.

The paper is organized as follows. In Section 2 we recall the definitions and key results about Grassmannian cluster categories. In Section 3 we give the construction of rank 2 modules in the case when the layers are tightly 3-interlacing. This covers in particular the tame case (3,9) and almost all rank 2 modules in the tame case (4,8). Section 4 is devoted to the cases of non-tightly 3-interlacing layers. Section 5 is devoted to the case of tightly 4-interlacing layers, which completes the case (4,8). In the last section, we deal with the general case of r-interlacing, when \(r \geq 4\), and we show that there are infinitely many non-isomorphic rank 2 indecomposable modules with the same filtration.

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2 Preliminaries

We follow the exposition from [JKS16, BB16, BBE19] in order to introduce notation and background results. Let $\Gamma_n$ be the quiver of the boundary algebra, with vertices $1, 2, \ldots, n$ on a cycle and arrows $x_i : i - 1 \to i, y_i : i \to i - 1$. We write $\text{CM}(B_{k,n})$ for the category of maximal Cohen-Macaulay modules for the completed path algebra $B_{k,n}$ of $\Gamma_n$, with relations $xy - yx$ and $x^k - y^{n-k}$ (at every vertex). The centre of $B_{k,n}$ is $Z := \mathbb{C}[t]$, where $t = \sum_i x_iy_i$. For example, when $n = 5$ we have the quiver

![Diagram of a quiver with vertices numbered 1 to 5 and arrows labeled accordingly, forming a cycle with a 4-cycle attached at each end.]

The algebra $B_{k,n}$ coincides with the quotient of the completed path algebra of the graph $C$ (a circular graph with vertices $C_0 = \mathbb{Z}_n$, set clockwise around a circle, and with the set of edges, $C_1$, also labeled by $\mathbb{Z}_n$, with edge $i$ joining vertices $i - 1$ and $i$), i.e. the doubled quiver as above, by the closure of the ideal generated by the relations above (we view the completed path algebra of the graph $C$ as a topological algebra via the $m$-adic topology, where $m$ is the two-sided ideal generated by the arrows of the quiver, see [DWZ08, Section 1]). The algebra $B_{k,n}$, that we will often denote by $B$ when there is no ambiguity, was introduced in [JKS16, Section 3]. Observe that $B_{k,n}$ is isomorphic to $B_{n-k,n}$, so we will always take $k \leq \frac{n}{2}$.

The (maximal) Cohen-Macaulay $B$-modules are precisely those which are free as $\mathbb{Z}$-modules. Such a module $M$ is given by a representation $\{M_i : i \in C_0\}$ of the quiver with each $M_i$ a free $\mathbb{Z}$-module of the same rank (which is the rank of $M$).

**Definition 2.1** ([JKS16, Definition 3.5]). For any $B_{k,n}$-module $M$ and $K$ the field of fractions of $\mathbb{Z}$, the rank of $M$, denoted by $\text{rk}(M)$, is defined to be $\text{rk}(M) = \text{len}(M \otimes Z K)$.

Note that $B \otimes \mathbb{Z} K \cong M_n(K)$, which is a simple algebra. It is easy to check that the rank is additive on short exact sequences, that $\text{rk}(M) = 0$ for any finite-dimensional $B$-module (because these are torsion over $\mathbb{Z}$) and that, for any Cohen-Macaulay $B$-module $M$ and every idempotent $e_j$, $1 \leq j \leq n$, $\text{rk}_\mathbb{Z}(e_j M) = \text{rk}(M)$, so that, in particular, $\text{rk}_\mathbb{Z}(M) = n \text{rk}(M)$.

**Definition 2.2** ([JKS16, Definition 5.1]). For any $k$-subset $I$ of $C_1$, we define a rank 1 $B$-module

$$L_I = \{U_i, \ i \in C_0; \ x_i, y_i, \ i \in C_1\}$$

as follows. For each vertex $i \in C_0$, set $U_i = \mathbb{C}[t]$ and, for each edge $i \in C_1$, set

$$x_i : U_{i-1} \to U_i$$

and

$$y_i : U_i \to U_{i-1}$$

to be multiplication by $1$ if $i \in I$, and by $t$ if $i \notin I$.

The module $L_I$ can be represented by a lattice diagram $L_I$ in which $U_0, U_1, U_2, \ldots, U_n$ are represented by columns of vertices (dots) from left to right (with $U_0$ and $U_n$ to be identified), going down infinitely. The vertices in each column correspond to the natural monomial $\mathbb{C}$-basis of $\mathbb{C}[t]$. The column corresponding to $U_{i+1}$ is displaced half a step vertically downwards (respectively, upwards) in relation to $U_i$ if $i + 1 \in I$ (respectively, $i + 1 \notin I$), and the actions of $x_i$ and $y_i$ are shown as diagonal arrows. Note that the $k$-subset $I$ can then be read off as the set of labels on the arrows pointing down to the right which are exposed to the top of the diagram. For example, the lattice diagram $L_{\{1,4,5\}}$ in the case $k = 3$, $n = 8$, is shown in the following picture
We see from Figure 2 that the module $L_I$ is determined by its upper boundary, denoted by the thick lines, which we refer to as the rim of the module $L_I$ (this is why we call the $k$-subset $I$ the rim of $L_I$). Throughout this paper we will identify a rank 1 module $L_I$ with its rim. Moreover, most of the time we will omit the arrows in the rim of $L_I$ and represent it as an undirected graph.

We say that $i$ is a peak of the rim $I$ if $i \notin I$ and $i + 1 \in I$. In the above example, the peaks of $I = \{1, 4, 5\}$ are 3 and 8. We say that $i$ is a valley of the rim $I$ if $i \in I$ and $i + 1 \notin I$. In the above example, the valleys of $I = \{1, 4, 5\}$ are 1 and 5.

**Proposition 2.3** (*JKS16*, Proposition 5.2). Every rank 1 Cohen-Macaulay $B$-module is isomorphic to $L_I$ for some unique $k$-subset $I$ of $C_1$.

Every $B$-module has a canonical endomorphism given by multiplication by $t \in Z$. For $L_I$ this corresponds to shifting $L_I$ one step downwards. Since $Z$ is central, $\text{Hom}_B(M, N)$ is a $Z$-module for arbitrary $B$-modules $M$ and $N$. If $M, N$ are free $Z$-modules, then so is $\text{Hom}_B(M, N)$. In particular, for any two rank 1 Cohen-Macaulay $B$-modules $L_I$ and $L_J$, $\text{Hom}_B(L_I, L_J)$ is a free module of rank 1 over $Z = \mathbb{C}[[t]]$, generated by the canonical map given by placing the lattice of $L_I$ inside the lattice of $L_J$ as far up as possible so that no part of the rim of $L_I$ is strictly above the rim of $L_J$ [*JKS16*, Section 6].

**Definition 2.4** ($r$-interlacing). Let $I$ and $J$ be two $k$-subsets of $\{1, \ldots, n\}$. The sets $I$ and $J$ are said to be $r$-interlacing if there exist subsets $\{i_1, i_2, \ldots, i_{2r-2}\} \subset I \setminus J$ and $\{i_2, i_4, \ldots, i_{2r}\} \subset J \setminus I$ such that $i_1 < i_2 < i_3 < \cdots < i_{2r} < i_1$ (cyclically) and if there exist no larger subsets of $I$ and of $J$ with this property. We say that $I$ and $J$ are tightly $r$-interlacing if they are $r$-interlacing and $|I \cap J| = k - r$.

**Definition 2.5.** A $B$-module is rigid if $\text{Ext}^1_B(M, M) = 0$.

If $I$ and $J$ are $r$-interlacing $k$-subsets, where $r < 2$, then $\text{Ext}^1_B(L_I, L_J) = 0$, in particular, rank 1 modules are rigid (see [*JKS16* Proposition 5.6]).

Every rigid indecomposable $M$ of rank $n$ in $\text{CM}(B)$ has a filtration having factors $L_{I_1}, L_{I_2}, \ldots, L_{I_n}$ of rank 1. This filtration is noted in its profile, $\text{pr}(M) = I_1 \mid I_2 \mid \ldots \mid I_n$, [*JKS16* Corollary 6.7]. In the case of a rank 2 module $M$ with filtration $L_I \mid L_J$ (i.e. with profile $I \mid J$), we picture this module by drawing the rim $J$ below the rim $I$, in such a way that $J$ is placed as far up as possible so that no part of the rim $J$ is strictly above the rim $I$. We refer to this picture of $M$ as its lattice diagram. Note that there is at least one point where the rims $I$ and $J$ meet (see Figure 4 for an example).

**Remark 2.6.** Suppose that the two $k$-subsets $I$ and $J$ are $r$-interlacing and that $M$ is a rank 2 module with profile $I \mid J$. Then the two rims in the lattice diagram of $M$ form a number of regions between the points where the two rims meet but differ in direction before and/or after meeting. We call these regions the boxes formed by the rims or by the profile. The term box is a combinatorial tool which will be very useful in finding conditions for indecomposability. Let us point out, however, that the module $M$ might be a direct sum in which case the lattice diagram is really a pair of lattice diagrams of rank 1 modules. We still view the corresponding diagram as forming boxes. If $I$ and $J$ are $r$-interlacing, then they form exactly $r$-boxes if and only if they are tightly $r$-interlacing. (If we consider the lattice...
diagram as an infinite branched graph in the plane, the boxes are the closures of the finite regions in the complement.) A lattice diagram with three boxes is shown in Figure 1. If $M$ is a rank 2 module with $r_1$ boxes, with $r_1 \leq r$, the poset structure associated with $M$ is $1^{r_1} \rightarrow 2$, see Figure 1.

For background on the poset associated to an indecomposable module or to its profile, we refer to [JKS16, Section 6] and to [BBE19, Section 2].

Consider the tame cases $(k, n) = (3, 9)$ or $(k, n) = (4, 8)$ and let $M$ be a rigid indecomposable rank 2 module of $CM(B_{k,n})$. Then $M \cong L_I \mid L_J$ where $I$ and $J$ are 3-interlacing, [BBE19, Proposition 5.5]. Furthermore, we also know that if $I$ and $J$ are tightly 3-interlacing and if $M \cong L_I \mid L_J$, then $M$ is indecomposable, [BBE19, Lemma 5.11].

We therefore start studying pairs of tightly 3-interlacing $k$-subsets in order to construct indecomposable rank 2 modules and will later consider higher interlacing.

Throughout the paper, our strategy to prove a module is indecomposable is to show that its endomorphism ring does not have non-trivial idempotent elements. When we deal with a decomposable rank 2 module, in order to determine the summands of this module, we construct a non-trivial idempotent in its endomorphism ring, and then find corresponding eigenvectors at each vertex of the quiver and check the action of the morphisms $x_i$ on these eigenvectors.

### 3 Tight 3-interlacing

In this section we give a construction of rank 2 indecomposable modules with the profile $I \mid J$ in the case when $I$ and $J$ are tightly 3-interlacing $k$-subsets, i.e. when $|I \setminus J| = |J \setminus I| = 3$ and non-common elements of $I$ and $J$ interlace. This covers all indecomposable rank 2 modules in the tame case $(3, 9)$ and almost all indecomposable rank 2 modules in the tame case $(4, 8)$.

We want to define a rank 2 module $M(I, J)$ with filtration $L_I \mid L_J$ in a similar way as rank 1 modules are defined in $CM(B_{k,n})$. Let $V_i := \mathbb{C}[t_i] \oplus \mathbb{C}[t_i]$, $i = 1, 2, \ldots, n$. The module $M(I, J)$ has $V_i$ at each vertex $1, 2, \ldots, n$ of $\Gamma_n$. In order to have a module structure for $B_{k,n}$, for every $i$ we need to define $x_i : V_{i-1} \to V_i$ and $y_i : V_i \to V_{i-1}$ in such a way that $x_i y_i = t \cdot \text{id}$ and $x^k = y^{n-k}$.

Since $L_J$ is a submodule of a rank 2 module $M(I, J)$, and $L_I$ is the quotient, if we extend the basis of $L_J$ to the basis of the module $M(I, J)$, then with respect to that basis all the matrices $x_i, y_i$ must be upper triangular with diagonal entries from the set $\{1, t\}$. More precisely, the diagonal of $x_i$ (resp. $y_i$) is $(1, t)$ (resp. $(t, 1)$) if $i \in J \setminus I$, it is $(1, 1)$ (resp. $(1, t)$) if $i \in I \setminus J$, (t, t) (resp. $(1, 1)$) if $i \in I \cap J$, and $(1, 1)$ (resp. $(1, t)$) if $i \in I \cap J$. The only entries in all these matrices that are left to be determined are the ones in the upper right corner.

Let us assume that $n = 6$, $I = \{1, 3, 5\}$, and $J = \{2, 4, 6\}$. In the general case, the arguments are the same. Denote by $b_i$ the upper right corner element of $x_i$. From $x_i y_i = t \cdot \text{id}$, we have that the upper right corner element of $y_i$ is $-b_i$. From the relation $x^k = y^{n-k}$ it follows that $\sum_{i=1}^{6} b_i = 0$. Thus, our module $M(I, J)$ is

$$
\begin{array}{ccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\begin{pmatrix} t & b_1 \\ 0 & 1 \end{pmatrix} & 1 & 2 & 3 & 4 & 5 & \begin{pmatrix} 1 & b_6 \\ 0 & t \end{pmatrix} \\
\begin{pmatrix} 1 & b_5 \\ 0 & 1 \end{pmatrix} & 1 & 2 & 3 & 4 & \begin{pmatrix} 1 & b_4 \\ 0 & 1 \end{pmatrix} \\
\begin{pmatrix} t & b_3 \\ 0 & 1 \end{pmatrix} & 1 & 2 & 3 & \begin{pmatrix} 1 & b_2 \\ 0 & 1 \end{pmatrix} \\
\begin{pmatrix} t & b_1 \\ 0 & 1 \end{pmatrix} & 1 & 2 & \begin{pmatrix} 1 & b_4 \\ 0 & 1 \end{pmatrix} \\
\begin{pmatrix} 1 & b_3 \\ 0 & 1 \end{pmatrix} & 1 & \begin{pmatrix} 1 & b_2 \\ 0 & 1 \end{pmatrix} \\
\begin{pmatrix} t & b_5 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & b_6 \\ 0 & 1 \end{pmatrix} \\
\end{array}
$$

with $\sum b_i = 0$. We say that $M(I, J)$ is determined by the 6-tuple $(b_1, b_2, b_3, b_4, b_5, b_6)$.

#### 3.1 Divisibility conditions for (in)decomposability

Let $I, J$ be two $k$-subsets and $M(I, J)$ be given by the tuple $(b_1, b_2, b_3, b_4, b_5, b_6)$ with $\sum b_i = 0$. The question is how to determine the $b_i$'s so that the module $M(I, J)$ is indecomposable. Assume first that
\(M(I, J)\) is decomposable and that \(L_J\) is a direct summand of \(M(I, J)\). Then there exists a retraction 
\[ \mu = (\mu_i)_{i=1}^6 \] such that \(\mu_i \circ \theta_i = \text{id}\), where \((\theta_i)_{i=1}^6\) is the natural injection of \(L_J\) into \(M(I, J)\). Using the same basis as before, we can assume that \(\mu_i = [1 \alpha_i]\) for some \(\alpha_i \in \mathbb{C}\). From the commutativity relations we have \(\text{id} \circ \mu_i = \mu_{i+1} \circ x_{i+1}\) for \(i\) odd, and \(t \cdot \text{id} \circ \mu_i = \mu_{i+1} \circ x_{i+1}\) for \(i\) even. It follows that \(\alpha_i = b_{i+1} + t\alpha_{i+1}\) for \(i\) odd, and \(t\alpha_i = b_{i+1} + \alpha_{i+1}\) for \(i\) even. From this we have

\[
\begin{align*}
    t(\alpha_2 - \alpha_4) &= b_3 + b_4, \\
    t(\alpha_4 - \alpha_6) &= b_5 + b_6, \\
    t(\alpha_6 - \alpha_2) &= b_1 + b_2.
\end{align*}
\]

Thus, if \(L_J\) is a direct summand of \(M(I, J)\), then \(t|b_3 + b_4, t|b_5 + b_6, \) and \(t|b_1 + b_2\) (and we can easily find elements \(\alpha_i, i = 1, \ldots, 6,\) satisfying previous equations). If only one of these conditions is not met, then \(L_J\) is not a direct summand of \(M\). For example, if we choose \(b_2 = -b_3 = 0, b_4 = -b_5 = 1,\) and \(b_6 = -b_1 = 2\) in the construction of the module \(M(I, J)\), then \(L_J\) is not a direct summand of \(M(I, J)\). Our aim is to study the structure of the module \(M(I, J)\) in terms of the divisibility conditions
the coefficients \(b_i\) satisfy.

**Remark 3.1.** If \(L_J\) is not a summand of \(M\), it does not mean that \(M\) is indecomposable (cf. Theorem 3.12 in [BBE19]).

Let us now consider the general case, i.e. let \(M(I, J)\) be the module as defined above, but in general terms when \(I\) and \(J\) are tightly 3-interlacing. Write \(I \setminus J\) as \(\{i_1, i_2, i_3\}\) and \(J \setminus I = \{j_1, j_2, j_3\}\) so that \(1 \leq i_1 < j_1 < i_2 < j_2 < i_3 < j_3 \leq n\). Define

\[
\begin{align*}
    x_{i_r} &= \begin{pmatrix} t & b_{2r-1} \\ 0 & 1 \end{pmatrix}, & x_{j_r} &= \begin{pmatrix} 1 & b_{2r} \\ 0 & t \end{pmatrix}, \\
    y_{i_r} &= \begin{pmatrix} 1 & -b_{2r-1} \\ 0 & t \end{pmatrix}, & y_{j_r} &= \begin{pmatrix} t & -b_{2r} \\ 0 & 1 \end{pmatrix},
\end{align*}
\]

for \(r = 1, 2, 3\) (see the previous figure for \(n = 6\)). For \(i \in I \cap J\), we set \(x_i = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}\) and \(y_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\). For \(i \in I \cap J\), we set \(x_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\) and \(y_i = \begin{pmatrix} 0 & 0 \\ 0 & t \end{pmatrix}\). Also, we assume that \(\sum_{r=1}^6 b_r = 0\). Note that for \(i \in (I^c \cap J^c) \cup (I \cap J)\) we define the matrices \(x_i\) and \(y_i\) to be diagonal, i.e. we assume that the upper right corner of \(x_i\) and \(y_i\) is 0 if \(i \in (I^c \cap J^c) \cup (I \cap J)\). We can achieve this under a suitable base change of \(V_i\).

By construction it holds that \(xy = yx\) and \(x^k = y^{n-k}\) at all vertices, and that \(M(I, J)\) is free over the centre of the boundary algebra. Hence, the following proposition holds.

**Proposition 3.2.** The module \(M(I, J)\) as constructed above is in \(CM(B_{k,n})\).

For the remainder of the paper, if \(w = t^av\), for some positive integer \(a\), then \(t^{-a}w\) stands for \(v\).

**Proposition 3.3.** Let \(I, J\) be tightly 3-interlacing, \(n \geq 6\) arbitrary, \(I \setminus J = \{i_1, i_2, i_3\}\), and \(J \setminus I = \{j_1, j_2, j_3\}\), where \(1 \leq i_1 < j_1 < i_2 < j_2 < i_3 < j_3 \leq n\). If \(\varphi = (\varphi_i)_{i=1}^n \in \text{Hom}(M(I, J), M(I, J))\),
\[ \varphi_{j_3} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \]
\[ \varphi_{j_1} = \begin{pmatrix} a + b t^{-1} c & t b + (d - a) b_1 - b_2^2 t^{-1} c \\ t^{-1} c & d - b_1 t^{-1} c \end{pmatrix}, \]
\[ \varphi_{j_2} = \begin{pmatrix} a + (b_1 + b_2) t^{-1} c & b + t^{-1} ((d - a) (b_1 + b_2) - (b_1 + b_2)^2 t^{-1} c) \\ t^{-1} c & d - (b_1 + b_2) t^{-1} c \end{pmatrix}, \]
\[ \varphi_{j_2} = \begin{pmatrix} a + (b_1 + b_2 + b_3) t^{-1} c & t b + (d - a) (b_1 + b_2 + b_3) - (b_1 + b_2 + b_3)^2 t^{-1} c \\ t^{-1} c & d - (b_1 + b_2 + b_3 + b_4) t^{-1} c \end{pmatrix}, \]
\[ \varphi_{j_3} = \begin{pmatrix} a - b a t^{-1} c & t b - (d - a) b_0 - b_2^2 t^{-1} c \\ t^{-1} c & d + b_0 t^{-1} c \end{pmatrix}, \]
\[ \varphi_i = \varphi_{i-1}, \text{ for } i \in (I^c \cap J^c) \cup (I \cap J), \]

with \( a, b, c, d \in \mathbb{C}[|t|]. \) Furthermore, \( t \mid c, t \mid (d - a) (b_1 + b_2) - (b_1 + b_2)^2 t^{-1} c, \) and \( t \mid (d - a) (b_1 + b_2 + b_3 + b_4) - (b_1 + b_2 + b_3 + b_4)^2 t^{-1} c. \)

**Proof.** First we consider the case \( n = 6. \) Let \( \varphi = (\varphi_1, \ldots, \varphi_6) \) be an endomorphism of \( \mathbb{M}(I, J) \), where each \( \varphi_i \) is an element of \( \mathbb{M}_2(\mathbb{C}[|t|]) \) (matrices over the centre).

\[
\begin{array}{cccccc}
\begin{pmatrix} (t & b_1) \\
0 & 1 \end{pmatrix} & \begin{pmatrix} (t & b_2) \\
0 & 1 \end{pmatrix} & \begin{pmatrix} (t & b_3) \\
0 & 1 \end{pmatrix} & \begin{pmatrix} (t & b_4) \\
0 & 1 \end{pmatrix} & \begin{pmatrix} (t & b_5) \\
0 & 1 \end{pmatrix} & \begin{pmatrix} (t & b_6) \\
0 & 1 \end{pmatrix} \\
\begin{pmatrix} (1 & -b_1) \\
0 & t \end{pmatrix} & \begin{pmatrix} (t & -b_2) \\
0 & 1 \end{pmatrix} & \begin{pmatrix} (1 & -b_3) \\
0 & t \end{pmatrix} & \begin{pmatrix} (t & -b_4) \\
0 & 1 \end{pmatrix} & \begin{pmatrix} (1 & -b_5) \\
0 & t \end{pmatrix} & \begin{pmatrix} (t & -b_6) \\
0 & 1 \end{pmatrix} \\
\begin{pmatrix} (t & b_1) \\
0 & 1 \end{pmatrix} & \begin{pmatrix} (t & b_2) \\
0 & 1 \end{pmatrix} & \begin{pmatrix} (t & b_3) \\
0 & 1 \end{pmatrix} & \begin{pmatrix} (t & b_4) \\
0 & 1 \end{pmatrix} & \begin{pmatrix} (t & b_5) \\
0 & 1 \end{pmatrix} & \begin{pmatrix} (t & b_6) \\
0 & 1 \end{pmatrix} \\
\begin{pmatrix} (1 & -b_1) \\
0 & t \end{pmatrix} & \begin{pmatrix} (t & -b_2) \\
0 & 1 \end{pmatrix} & \begin{pmatrix} (1 & -b_3) \\
0 & t \end{pmatrix} & \begin{pmatrix} (t & -b_4) \\
0 & 1 \end{pmatrix} & \begin{pmatrix} (1 & -b_5) \\
0 & t \end{pmatrix} & \begin{pmatrix} (t & -b_6) \\
0 & 1 \end{pmatrix}
\end{array}
\]

We use commutativity relations \( x_{i+1} \varphi_i = \varphi_{i+1} x_i, \) i.e. we check the relations:

\[
(i) \quad x_3 \varphi_1 = \varphi_1 x_3, \quad (ii) \quad x_3 \varphi_3 = \varphi_3 x_3, \\
(iii) \quad x_4 \varphi_3 = \varphi_3 x_4, \quad (iv) \quad x_5 \varphi_4 = \varphi_4 x_5, \\
(v) \quad x_6 \varphi_5 = \varphi_5 x_6, \quad (vi) \quad x_1 \varphi_6 = \varphi_6 x_1.
\]

Let \( \varphi_0 = \begin{pmatrix} a & b \\
c & d \end{pmatrix} \) and \( \varphi_1 = \begin{pmatrix} c & f \\
g & h \end{pmatrix}. \) From (vi), we get \( \begin{pmatrix} a & b t + b_1 d \\
c & d \end{pmatrix} = \begin{pmatrix} e & b_1 + f \\
g t + h + g b_1 \end{pmatrix}, \)

and \( a t + b_1 c = e t, b t + b_1 d = e b_1 + f, c = g t, \) and \( d = g b_1 + h. \) It follows that \( t \mid c, \) and that

\[ \varphi_1 = \begin{pmatrix} a + b_1 t^{-1} c & t b + (d - a) b_1 - b_2^2 t^{-1} c \\ t^{-1} c & d - b_1 t^{-1} c \end{pmatrix}. \]

Similarly, if \( \varphi_2 = \begin{pmatrix} c & f \\
g & h \end{pmatrix}, \) then equality \( x_2 \varphi_1 = \varphi_2 x_2 \) yields

\[ \varphi_2 = \begin{pmatrix} a + (b_1 + b_2) t^{-1} c & b + t^{-1} ((d - a) (b_1 + b_2) - (b_1 + b_2)^2 t^{-1} c) \\ c & d - (b_1 + b_2) t^{-1} c \end{pmatrix}. \]
and $t \mid (d - a)(b_1 + b_2) - (b_1 + b_2)^2t^{-1}c$.

The rest of the proof for $\phi_3, \phi_4,$ and $\phi_5$ is analogous. We omit the details of elementary, but tedious computation.

In the general case of arbitrary $n$, the proof is almost the same as for $n = 6$. The only thing left to note is that if $i \in (I \cap J) \cup (I \cap J)$, then $x_i$ is a scalar matrix (either identity or $t$ times identity), so from $x_i\phi_{i-1} = \phi_i x_i$, it follows immediately that $\phi_{i-1} = \phi_i$. \hfill \Box

**Remark 3.4.** Take $\phi$ as in Proposition 3.3. The morphism $\phi$ also satisfies the other six relations $\phi_1(y_{i+1}) = y_{i+1}\phi_1$. Indeed, if $x_{i+1}\phi_i = \phi_{i+1}x_{i+1}$, then if we multiply this equality by $y_{i+1}$ both from the left and right, we obtain $t \cdot \phi_i y_{i+1} = t \cdot y_{i+1}\phi_i$. Since $t$ is a regular element in $\mathbb{C}[t]$, after cancellation by $t$ we obtain $\phi_i y_{i+1} = y_{i+1}\phi_i$. Also, note that in order to prove that $\phi$ is idempotent we only need to make sure that only one $\phi_i$ is idempotent. If $\phi_i$ is idempotent, then from $x_{i+1}\phi_i = \phi_{i+1}x_{i+1}$ when we multiply by $\phi_{i+1}$ from the left, we have $\phi_{i+1}x_{i+1}\phi_i = \phi_{i+1}^2x_{i+1}$, and multiplying with $\phi_i$ from the right we get $\phi_{i+1}x_{i+1}\phi_i = x_{i+1}\phi_i^2$. Then $x_{i+1}\phi_i^2 = \phi_{i+1}^2x_{i+1}$ and $x_{i+1}\phi_i = \phi_{i+1}^2x_{i+1}$ as $\phi_i$ is idempotent, and subsequently, $\phi_{i+1}x_{i+1} = \phi_{i+1}^2x_{i+1}$, yielding $\phi_{i+1} = \phi_{i+1}^2$ after multiplication by $y_{i+1}$ from the right and cancellation by $t$.

We now give necessary and sufficient conditions for the module $\mathbb{M}(I, J)$ to be indecomposable.

**Theorem 3.5.** Let $I$ and $J$ be tightly 3-interlacing. The module $\mathbb{M}(I, J)$ is indecomposable if and only if $t \nmid b_1 + b_2$, $t \nmid b_3 + b_4$, and $t \nmid b_5 + b_6$. Furthermore, if $t \nmid b_1 + b_2$, $t \nmid b_3 + b_4$, and $t \nmid b_5 + b_6$, then $\mathbb{M}(I, J)$ is isomorphic to $L_{(i_1, j_1, i_2)} \cup (I \cap J)$ $\oplus L_{(i_2, j_2, j_3)} \cup (I \cap J)$.

**Proof.** As before, it is sufficient to consider the case $n = 6$, so we can assume that $I = \{1, 3, 5\}$ and $J = \{2, 4, 6\}$. Let $\phi = (\phi_i)_{i=1}^6 \in \text{Hom}(\mathbb{M}(I, J), \mathbb{M}(I, J))$ be an idempotent homomorphism and assume that $\phi_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. From the previous proposition we know that

1. $t \mid c$,
2. $t \mid (d - a)(b_1 + b_2) - (b_1 + b_2)^2t^{-1}c$,
3. $t \mid (d - a)(b_1 + b_2 + b_3 + b_4) - (b_1 + b_2 + b_3 + b_4)^2t^{-1}c$.

Assume that $t \nmid b_1 + b_2$, $t \nmid b_3 + b_4$, and $t \nmid b_5 + b_6$. Since $t \nmid b_1 + b_2$, it follows from relation (2) that

$t \mid d - a - (b_1 + b_2)t^{-1}c$.

Similarly, since $t \nmid b_5 + b_6 = -(b_1 + b_2 + b_3 + b_4)$, from relation (3) follows that

$t \mid d - a - (b_1 + b_2 + b_3 + b_4)t^{-1}c$.

Thus, it must hold that

$t \mid (b_3 + b_4)t^{-1}c$,

and since $t \nmid b_3 + b_4$, it must be that $t \mid t^{-1}c$, and subsequently that $t \mid d - a$.

From the fact that $\phi_0$ is idempotent and $t \mid c$ it follows that $t \mid a - a^2$ and $t \mid d - d^2$. Also, from $\phi_0^2 = \phi_0$ it follows that either $a = d$ or $a + d = 1$. If $a = d$, then $b = c = 0$ (otherwise $a = d = \frac{1}{2}$ and $\frac{1}{2} = bc$, which is not possible as $c$ is divisible by $t$), and $a = d = 1$ or $a = d = 0$ giving us the trivial idempotents. If $a + d = 1$, then $t \mid a$ or $t \mid d$. Taking into account that $t \mid d - a$, we conclude that $t \mid a$ and $t \mid d$. This implies that $1 = a + d$ is divisible by $t$, which is not true. Thus, the only idempotent homomorphisms of $\mathbb{M}(I, J)$ are the trivial ones. Hence, $\mathbb{M}(I, J)$ is indecomposable.

Assume now that $t$ divides at least one of the elements $b_1 + b_2, b_3 + b_4, b_5 + b_6$. If $t$ divides all three of them, we have seen before that $\mathbb{M}(I, J)$ is the direct sum of $L_I$ and $L_J$, hence it is a decomposable module. We can assume that one of $b_1 + b_2, b_3 + b_4, b_5 + b_6$ is divisible by $t$ and that the other two are not divisible by $t$. Note that it is not possible that two of them are divisible by $t$, and one is not, because they sum up to zero. So assume that $t \mid b_5 + b_6$, $t \nmid b_1 + b_2$, and $t \nmid b_3 + b_4$. In order to find a non-trivial idempotent $\phi$, note that the relation (3) holds because $t \mid b_5 + b_6 = -(b_1 + b_2 + b_3 + b_4)$. 

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Hence, we only need to find elements $a$, $b$, $c$, and $d$ in such a way that $t \mid c$ and $t \mid d-a-(b_1+b_2)t^{-1}c$. Recall that if $a=d$, then we only obtain the trivial idempotents because $t \mid c$. So it must be $a+d=1$ if we want to find a non-trivial idempotent. If we choose $a=1$, $d=0$, then $t \mid 1 + (b_1 + b_2)^{-1}c$. Thus $(b_1 + b_2)^{-1}c = -1 + tg$, for some $g$. We can take $g=0$, i.e. $c = t(b_1 + b_2)^{-1}$ (recall that $b_1 + b_2$ is invertible because $t \nmid b_1 + b_2$) giving us the idempotent $(b=0$ since $a-a^2=bc$ and $c \neq 0$)

$$
\phi_0 = \begin{pmatrix} 1 \\ t(b_1 + b_2)^{-1} \end{pmatrix}.
$$

Its orthogonal complement is the idempotent

$$
\left( 0 \\ 1 \\ -t(b_1 + b_2)^{-1} \right).
$$

Since these are non-trivial idempotents, it follows that the module $M(I, J)$ is decomposable, what we needed to prove.

It remains to show that $M(I, J) \cong L_{\{i_1, j_1, i_3\}} \oplus L_{\{i_2, j_2, j_3\}}$. We know that $M(I, J)$ is the direct sum of rank 1 modules $L_X$ and $L_Y$ for some $X$ and $Y$. Let us determine $X$ and $Y$. For this, we take, at vertex $i$, eigenvectors $v_i$ and $w_i$, corresponding to the eigenvalue 1 of the idempotents $\phi_i$ and $1-\phi_i$, respectively. For example, $v_0 = [1, t(b_1 + b_2)^{-1}]^t$ $w_0 = [0, 1]^t$, $v_1 = [1 - b_1(b_1 + b_2)^{-1}, -(b_1 + b_2)^{-1}]^t$ and $w_1 = [b_1, 1]^t$, and so on. A basis for $L_X$ is $\{v_i \mid i = 0, \ldots, 5\}$, and a basis for $L_Y$ is $\{w_i \mid i = 0, \ldots, 5\}$.

Direct computation gives us that $x_1 v_0 = t v_1$, $x_2 v_1 = t v_2$, $x_3 v_2 = v_3$, $x_4 v_3 = v_4$, $x_5 v_4 = t v_5$, and $x_6 v_5 = v_0$. Thus, $X = \{3, 4, 6\}$. Analogously, $Y = \{1, 2, 5\}$. In the general case, this means that $X = \{i_2, j_2, j_3\} \cup (I \cap J)$ and $Y = \{i_1, j_1, i_3\} \cup (I \cap J)$.

**Example 3.6.** Let $k = 3$, $n = 6$, $I = \{1, 3, 5\}$ and $J = \{2, 4, 6\}$. If $b_1 = -2$, $b_2 = 0$, $b_3 = 0$, $b_4 = 1$, $b_5 = -1$, and $b_6 = 2$, then it is easily checked that $t \nmid b_i + b_{i+1}$, $i = 1, 3, 5$, and that $\sum_{i=1}^{6} b_i = 0$, thus giving us a $B_{k,n}$-module structure:

$$
\begin{array}{cccccc}
6 & (t & 2) & 0 & 1 & 1 \\
 & (1 & 0 & t) & 0 & 1 \\
 & (0 & 2 & t) & 0 & 1 \\
\hline
1 & (t & 0) & 1 & 0 & 1 \\
2 & (t & 0) & 1 & 0 & 1 \\
3 & (t & 0) & 1 & 0 & 1 \\
\hline
4 & (t & -1) & 0 & 1 & 0 \\
5 & (t & -1) & 0 & 1 & 0 \\
6 & (t & -2) & 0 & 1 & 0 \\
\end{array}
$$

The lattice diagram (showing only the rims) of $M = L_{135} \, | \, L_{246}$ is

$$
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
$$

**Remark 3.7.** If $n = 6$, in the case when $t \nmid b_1 + b_2$, $t \nmid b_3 + b_4$, and $t \nmid b_5 + b_6$, if we just rename the vertices of the quiver by adding 2 (modulo $n$) to every vertex or by adding 4 to every vertex, we obtain two modules that correspond to the cases when $t \nmid b_1 + b_2$, $t \nmid b_3 + b_4$, $t \nmid b_5 + b_6$ and $t \nmid b_1 + b_2$, $t \nmid b_3 + b_4$, $t \nmid b_5 + b_6$. In the general case, these two modules are direct sums $L_{\{i_1, i_2, j_2\} \cup (I \cap J)} \oplus L_{\{i_1, i_3, j_3\} \cup (I \cap J)}$ and $L_{\{i_2, i_3, j_3\} \cup (I \cap J)} \oplus L_{\{i_2, j_1, j_3\} \cup (I \cap J)}$.

**Example 3.8.** When $n = 6$, $I = \{1, 3, 5\}$, and $J = \{2, 4, 6\}$, an indecomposable module which has $L_J$ as a submodule and $L_I$ as a quotient module is given in Example 3.6. Also, there are four different decomposable modules appearing as the middle term in a short exact sequence that has $L_I$ (as a quotient) and $L_J$ (as a submodule) as end terms:

$$
0 \rightarrow L_J \rightarrow L_{\{1,3,5\}} \oplus L_{\{2,4,6\}} \rightarrow L_I \rightarrow 0,
$$

$$
0 \rightarrow L_J \rightarrow L_{\{1,2,5\}} \oplus L_{\{3,4,6\}} \rightarrow L_I \rightarrow 0,
$$

$$
0 \rightarrow L_J \rightarrow L_{\{1,3,4\}} \oplus L_{\{2,5,6\}} \rightarrow L_I \rightarrow 0,
$$

$$
0 \rightarrow L_J \rightarrow L_{\{1,2,4\}} \oplus L_{\{3,5,6\}} \rightarrow L_I \rightarrow 0.
$$
The profiles of the four modules that appear in the middle in these short exact sequences are illustrated in Figure 3. Note that the two rims now stand for two rank 1 modules which are direct summands of the module. The pictures each show two lattice diagrams which are overlaid so we can compare the positions of the peaks. In particular, the two lattice diagrams in (a) look like the lattice diagram of the indecomposable extension described in Example 3.6.

![Diagram of lattice diagrams](image)

Figure 3: The pairs of lattice diagrams of decomposable extensions between $L_{\{1,3,5\}}$ and $L_{\{2,4,6\}}$.

Remark 3.9. Note that for $I = \{1, 3, 5\}$ and $J = \{2, 4, 6\}$ there is a non-trivial extension

$$0 \longrightarrow L_J \xrightarrow{[a_i,f_i]} L_{\{1,3,5\}} \oplus L_{\{2,4,6\}} \xrightarrow{[f_2,f_3]} L_I \longrightarrow 0,$$

where $f_i, i = 1, 2, 3$, is the canonical map between rank 1 modules (see Section 2, this is a homomorphism of minimal codimension). The middle term is equal to the direct sum of the end terms, but the maps make this short exact sequence not isomorphic to the trivial sequence.

Now we turn our attention to the question of uniqueness of the constructed indecomposable module. If we choose a different set of values for $b_i$, i.e. we choose a 6-tuple different from $(-2,0,0,1,-1,2)$, so that the conditions $t \nmid b_3 + b_4$, $t \nmid b_5 + b_6$, and $t \nmid b_1 + b_2$ are fulfilled, then we obtain a module which is not the same as the above constructed module $M(I,J)$ in Example 3.6. In the next theorem, we will show directly that for different choices of 6-tuples giving us indecomposable modules with the same filtration $L_I \mid L_J$ we obtain isomorphic modules. Thus, there is a unique indecomposable module with filtration $L_I \mid L_J$.

Theorem 3.10. Let $(b_1, b_2, b_3, b_4, b_5, b_6)$ and $(c_1, c_2, c_3, c_4, c_5, c_6)$ be different 6-tuples corresponding to indecomposable modules $M_1$ and $M_2$, respectively, as constructed in Theorem 3.5. Then the modules $M_1$ and $M_2$ are isomorphic.

Proof. As before, it is sufficient to consider the $n = 6$ case. We will explicitly construct an isomorphism $\varphi = (\varphi_i)_{i=0}^5$ between the two modules, where $\varphi_i : V_i \rightarrow W_i$, and $V_i$ and $W_i$ are the vector spaces at vertex $i$ of the modules $M_1$ and $M_2$ respectively. Also, each $\varphi_i$ is invertible. Recall that $b_i$ (resp. $c_i$) is the right upper corner element of $x_i$ for the module $M_1$ (resp. $M_2$):
Let us assume that $\varphi_i = \begin{pmatrix} \alpha_i \\ \delta_i \end{pmatrix}$, for $i = 0, \ldots, 5$. Then from $\varphi_1 \begin{pmatrix} t \\ 1 \end{pmatrix} = \begin{pmatrix} b_1 \\ 1 \end{pmatrix}$, we obtain $t \mid \gamma_0$, $t \gamma_1 = \gamma_0$, $a_1 = a_0 + c_1 t^{-1} \gamma_0$, $b_1 = b_0 t - a_0 b_1 + c_1 \delta_0 - b_1 c_1 t^{-1} \gamma_0$, and $\delta_1 = \delta_0 - b_1 t^{-1} \gamma_0$. Hence,

$$\varphi_1 = \begin{pmatrix} a_0 + c_1 t^{-1} \gamma_0 & b_0 t - a_0 b_1 + c_1 \delta_0 - b_1 c_1 t^{-1} \gamma_0 \\ \delta_0 - b_1 t^{-1} \gamma_0 \end{pmatrix}.$$

Since $t \mid \gamma_0$ and we would like $\varphi_0$ to be invertible, then it must be that $t \not\mid a_0$ and $t \not\mid \delta_0$. Then the inverse of $\varphi_0$ is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

From $\varphi_2 \begin{pmatrix} 1 \\ -b_2 \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ c_2 \\ t \end{pmatrix}$, we obtain that

$$\varphi_2 = \begin{pmatrix} a_0 + (c_1 + c_2) t^{-1} \gamma_0 & b_0 t - a_0(b_1 + b_2) + (c_1 + c_2) \delta_0 - (b_1 + b_2)(c_1 + c_2)t^{-1} \gamma_0 \\ \delta_0 - (b_1 + b_2)t^{-1} \gamma_0 \end{pmatrix},$$

where $t \mid -a_0(b_1 + b_2) + (c_1 + c_2) \delta_0 - (b_1 + b_2)(c_1 + c_2)t^{-1} \gamma_0$.

Analogously, it is easily computed that

$$\varphi_3 = \begin{pmatrix} a_0 + (c_1 + c_2 + c_3) t^{-1} \gamma_0 & b_0 t - a_0(\sum_{i=1}^{3} b_i) + (\sum_{i=1}^{3} c_i) \delta_0 - (\sum_{i=1}^{3} b_i)(\sum_{i=1}^{3} c_i)t^{-1} \gamma_0 \\ \delta_0 - (b_1 + b_2 + b_3)t^{-1} \gamma_0 \end{pmatrix},$$

$$\varphi_4 = \begin{pmatrix} a_0 - (c_5 + c_6) t^{-1} \gamma_0 & b_0 t - a_0(b_5 + b_6) - (c_5 + c_6) \delta_0 - (b_5 + b_6)(c_5 + c_6)t^{-1} \gamma_0 \\ \delta_0 - (b_5 + b_6)t^{-1} \gamma_0 \end{pmatrix},$$

$$\varphi_5 = \begin{pmatrix} a_0 - c_6 t^{-1} \gamma_0 & b_0 t + a_0 b_6 - c_6 \delta_0 - b_6 c_6 t^{-1} \gamma_0 \\ \delta_0 + b_6 t^{-1} \gamma_0 \end{pmatrix},$$

where $t \mid -a_0(b_5 + b_6) - (c_5 + c_6) \delta_0 - (b_5 + b_6)(c_5 + c_6)t^{-1} \gamma_0$.

In order to find an isomorphism $\varphi$, we must determine $a_0$, $b_0$, $\gamma_0$, and $\delta_0$ satisfying the following conditions: $t \mid \gamma_0$, $t \not\mid a_0$, $t \not\mid \delta_0$, and

$$\begin{align*}
t &\mid -a_0(b_1 + b_2) + (c_1 + c_2) \delta_0 - (b_1 + b_2)(c_1 + c_2)t^{-1} \gamma_0, \\
t &\mid a_0(b_5 + b_6) - (c_5 + c_6) \delta_0 - (b_5 + b_6)(c_5 + c_6)t^{-1} \gamma_0.
\end{align*}$$

Note that there are no conditions attached to $b_0$ so we set it to be 0. If we set

$$-a_0(b_1 + b_2) + (c_1 + c_2) \delta_0 - (b_1 + b_2)(c_1 + c_2)t^{-1} \gamma_0 = 0,$$

$$a_0(b_5 + b_6) - (c_5 + c_6) \delta_0 - (b_5 + b_6)(c_5 + c_6)t^{-1} \gamma_0 = 0,$$

then we get

$$a_0(b_5 + b_6) \left[ 1 + \frac{c_5 + c_6}{c_1 + c_2} \right] - \delta_0(c_5 + c_6) \left[ 1 + \frac{b_5 + b_6}{b_1 + b_2} \right] = 0.$$
If \( t \mid 1 + \frac{c_5 + c_6}{c_1 + c_2} \), then from \( \sum_{i=1}^{6} c_i = 0 \), we get \( t \mid c_3 + c_4 \), which is not true. The same holds for \( 1 + \frac{b_3 + b_6}{b_1 + b_2} \), so both of these elements are invertible. Thus, if we set \( \delta_0 = 1 \), then we get

\[
\alpha_0 = \frac{(c_1 + c_2)(b_3 + b_4)(c_5 + c_6)}{(b_1 + b_2)(c_3 + c_4)(b_5 + b_6)},
\]

and

\[
\gamma_0 = t\frac{(c_3 + c_4)(b_5 + b_6) - (b_3 + b_4)(c_5 + c_6)}{(b_1 + b_2)(c_3 + c_4)(b_5 + b_6)}.
\]

Hence,

\[
\varphi_0 = \begin{pmatrix}
\frac{(c_1 + c_2)(b_3 + b_4)(c_5 + c_6)}{(b_1 + b_2)(c_3 + c_4)(b_5 + b_6)} & 0 \\
0 & \frac{(c_3 + c_4)(b_5 + b_6) - (b_3 + b_4)(c_5 + c_6)}{(b_1 + b_2)(c_3 + c_4)(b_5 + b_6)}
\end{pmatrix}.
\]

The other invertible matrices \( \varphi_i \) are now determined from the above equalities. Note that all of them are invertible because their determinant is equal to \( \alpha_0\delta_0 - \beta_0\gamma_0 \) which is an invertible element. Also, \( \beta_0 = \beta_2 = \beta_4 = 0 \), and \( \gamma_0 = \gamma_2 = \gamma_4 = t\gamma_1 = t\gamma_3 = t\gamma_5 \).

**Example 3.11.** Assume that \( n = 6 \). We use the previous theorem to construct an isomorphism between the modules corresponding to the 6-tuples \( (b_1, b_2, b_3, b_4, b_5, b_6) = (-2,0,0,1,-1,2) \) and \( (c_1, c_2, c_3, c_4, c_5, c_6) = (0,1,-1,2,-2,0) \). From the previous theorem we get that \( \varphi_0 = \begin{pmatrix} 1 & 0 \\ -\frac{3}{2}t & 1 \end{pmatrix} \), \( \varphi_1 = \varphi_3 = \varphi_5 = \begin{pmatrix} 1 & 2 \\ -\frac{3}{2} & -2 \end{pmatrix} \), \( \varphi_2 = \begin{pmatrix} -\frac{1}{2} & 0 \\ -\frac{3}{4t} & -2 \end{pmatrix} \), \( \varphi_4 = \begin{pmatrix} 0 & -2 \\ 0 & -\frac{1}{2} \end{pmatrix} \).

**Remark 3.12.** In the case when \( t \nmid b_1 + b_2 \), \( t \nmid b_3 + b_4 \), and \( t \mid b_5 + b_6 \), we have seen that the module in question is isomorphic to the direct sum \( L_{\{i_1,j_1,i_2\} \cup \{i_1,j_1,i_2\}} \oplus L_{\{i_1,j_1,i_2\} \cup \{i_1,j_1,i_2\}} \). This means that regardless of the choice of the elements \( b_i \) that satisfy these conditions, we get a module that is isomorphic to the same direct sum of rank 1 modules. Obviously, the same holds when \( t \mid b_1 + b_2 \), \( t \mid b_3 + b_4 \), and \( t \mid b_5 + b_6 \), in which case we get the direct sum \( L_I + L_J \). Thus, once we know which divisibility conditions our coefficients fulfil, we immediately know which module we are dealing with.

### 4 Non-tight 3-interlacing

In the tame case, besides the indecomposable modules of rank 2 that we have already constructed, i.e. the modules whose layers \( I \) and \( J \) are 3-interlacing and satisfy the condition \( |I \cap J| = k - 3 \), there are also the cases of non-tightly 3-interlacing layers with poset \( 1^3 \mid 2 \) and of 4-interlacing layers (with poset \( 1^4 \mid 2 \)). In this section, we deal with the former case. Recall that the two rings form three boxes in this case (Remark 2.6). This happens exactly for pairs of subsets \( J = \{i, i+2, i+4, i+5\} \), \( I = \{i+1, i+3, i+6, i+7\} \), e.g. for the profile \( 2478 \mid 1356 \).

We prove that, for general \( k \), if \( I \) and \( J \) are \( r \)-interlacing with poset \( 1^3 \mid 2 \), where \( r \geq 3 \), then up to isomorphism there exists a unique indecomposable module with profile \( I \mid J \), and we construct such a module, i.e. we find the conditions on the parameters \( b_i \) of the construction. We will mimic the procedure we used for the module with the profile \( 135 \mid 246 \).

So we assume for now that \( I \) and \( J \) are \( r \)-interlacing for some \( r \geq 3 \) and that \( I \mid J \) forms three boxes, with poset \( 1^3 \mid 2 \). We construct an indecomposable rank 2 module \( M(I,J) \) with \( L_J \) as submodule and \( L_I \) as quotient.

As before, we define \( x_i = \begin{pmatrix} t & b_i \\ 0 & 1 \end{pmatrix} \) and \( y_i = \begin{pmatrix} 1 & -b_i \\ 0 & t \end{pmatrix} \), if \( i \in I \setminus J \), and \( x_i = \begin{pmatrix} 1 & b_i \\ 0 & t \end{pmatrix} \) and \( y_i = \begin{pmatrix} t & -b_i \\ 0 & 1 \end{pmatrix} \) if \( i \in J \setminus I \). For any other \( i \) we define \( x_i \) to be the identity matrix and \( y_i \) to be \( t \cdot \text{id} \) if \( i \in I \cap J \) and \( x_i = t \cdot \text{id} \), \( y_i = \text{id} \) if \( i \in I^c \cap J^c \). This gives an element of \( \text{CM}(B_{k,n}) \), Proposition 4.1, as we will explain now.
In order to have a representation for $B_{k,n}$, our matrices have to satisfy the relation $x^k = y^{n-k}$. After multiplication by $x^{n-k}$ from the left, we conclude that this is equivalent to $x^n = t^{n-k} \cdot \text{id}$ because $x_i y_i = t$. When we compute such a product $x_n x_{n-1} \cdots x_1$ we get the matrix \[
abla_{n-k} \begin{pmatrix} t^{n-k} & z \\ 0 & t^{n-k} \end{pmatrix},\] where $z$ is a linear combination of the coefficients $b_i$ over the centre $Z$. This linear combination must be zero if we want to have a $B_{k,n}$-module structure.

Note that for all $i \in (I \cap J) \cup (I^c \cap J^c)$, $x_i$ is equal to $\text{id}$ or $t \cdot \text{id}$, so in the product $x^n$, any such $x_i$ does not contribute to $z$, or more precisely, it cancels out in $a^n = t^{n-k} \cdot \text{id}$. Therefore, in finding conditions for $z = 0$, we will assume $(I \cap J) \cup (I^c \cap J^c) = \emptyset$, i.e. that the rims of $L_I$ and of $L_J$ have no parallel segments and that $(k, n)$ are modified accordingly to $(k', n')$ for some $k' \leq k$, $n' \leq n$. This implies in particular that the boxes are symmetric around horizontal axes and that $n' = 2k'$.

We consider the product of all $x_i$ appearing in a single box. Since we have removed all common elements of $I$ and $J$ and of $I^c$ and $J^c$, the boxes are separated by the three points where the rims meet.

We call them branching points of $I \mid J$. In other words, let $\{i_1, i_2, i_3\} \subset I$ be the positions where the arrow $x_{i_m}$ in the rim of $L_I$ ends at the position/height (in the lattice diagram) where the arrow $y_{i_m}$ starts in the rim of $L_J$, $m = 1, 2, 3$. And let $\{j_1, j_2, j_3\} \subset J$ be defined as $j_m = i_m + 1$. We call the $i_m$'s the branching points for $I$ and the $j_m$'s the branching points for $J$. (We will give a definition of branching points in the general setting later, cf. Definition 4.2)

The profile $I \mid J$ of a rank 2 module with 6-interlacing layers, with poset $1^3 \mid 2$, and with $I \cap J = I^c \cap J^c = \emptyset$ is given in Figure 4 for $(k, n) = (8, 16)$.

![Figure 4: The profile of a module with 3 branching points and 3 junctions.](image)

Note that all the boxes now fit between consecutive branching points $i_m, i_{m+1} \in I$, but that there might be some points, along the boundary of a box where the rims deviate from forming a square: considering the rim of $L_I$, these points are precisely the valleys of $I$ (and by symmetry, the peaks of $J$), i.e. points $i \in I$ (and thus $i \notin J$) such that $i + 1 \notin I$ (and thus $i + 1 \in J$). We call such a point a junction of $I$ (of $J$, by the symmetry). In Figure 4, the first box has two junctions at 4 and at 6 in $I$. By definition, branching points are not junctions.

Whenever we have a junction $i$ in $I$, we proceed as follows. If $x_{i+1} = \begin{pmatrix} 1 & b_{i+1} \\ 0 & t \end{pmatrix}$, then we change the basis of the vector space $V_i$ at vertex $i$ by changing the second basis element to a new basis element so that the matrix of $x_i$ with respect to the new basis is $\begin{pmatrix} t & -b_{i+1} \\ 0 & 1 \end{pmatrix}$ (we start from the rightmost branching point of the box and move to the left in this process). Now, $x_{i+1} x_i = t \cdot \text{id}$, so $x_{i+1} x_i$ does not contribute to the element $z$ in the upper right corner. For this reason we can assume that the rim of $L_I$ (and thus the rim of $L_J$) does not have any junctions when studying the element $z$, further reducing $k$ and $n$ accordingly. So, without loss of generality, one can assume that all boxes in our $1^3 \mid 2$ poset are squares, see Figure 5. This is illustrated in Figure 5 with $(k, n) = (5, 10)$.

We denote the element $i_m + 1$ by $j_m$, $m = 1, 2, 3$, this is an element of $J \setminus I$ which plays a dual role to the element $i_m$. 
Consider the first box in the reduced setting (no parallel segments, only square boxes), i.e. the square box with starting/ending points $i_1, i_2$. The size of the set $I \cap (i_1, i_2]$ is the same as the size of the set $J \cap (i_1, i_2]$. We call this number the size of the box, and denote it by $l_1 := \frac{1}{t} \{i_1 + 1, \ldots, i_2\}$, $l_2$ and $l_3$ are defined accordingly. Then the product of the matrices $x_{i_2} x_{i_2-1} \cdots x_{i_1+i_1+1}$ on the second half of the first box is 

$$
\begin{pmatrix}
1 & b_{i_2} + b_{i_2-1}t + \cdots + b_{i_1+i_1+1} t^{i_1-1} \\
0 & 1
\end{pmatrix}.
$$

In the linear combination $b_{i_2} + b_{i_2-1}t + \cdots + b_{i_1+i_1+1} t^{i_1-1}$ there is only one term that is potentially not divisible by $t$, the one corresponding to the branching point $i_2$. Denote the sum $b_{i_2} + b_{i_2-1}t + \cdots + b_{i_1+i_1+1} t^{i_1-1}$ by $B_{i_2}$. On the first half of the box, the product of matrices $x_{i_1+i_1} x_{i_1+i_1+1} \cdots x_{i_1+i_1+1}$ is 

$$
\begin{pmatrix}
1 & b_{j_1} + b_{j_1+1}t + \cdots + b_{j_1+i_1+j_1-1} t^{j_1-1} \\
0 & t^{j_1}
\end{pmatrix}.
$$

Note that $b_{j_1}$ is the only term potentially not divisible by $t$ in the sum $b_{j_1} + b_{j_1+1}t + \cdots + b_{j_1+i_1+j_1-1}$, which we denote by $B_{j_1}$. The product $x_{i_2} x_{i_2-1} \cdots x_{j_1}$ is thus equal to 

$$
\begin{pmatrix}
1 & t^{l_1} (B_{j_1} + B_{i_2}) \\
0 & t^{l_1}
\end{pmatrix}.
$$

Similarly, for the remaining two boxes we get that the corresponding products are (with the obvious analogous notation): 

$$
\begin{pmatrix}
1 & t^{l_2} (B_{j_2} + B_{i_3}) \\
0 & t^{l_2}
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & t^{l_3} (B_{j_3} + B_{i_2}) \\
0 & t^{l_3}
\end{pmatrix}.
$$

Therefore, the whole product $x^n$ is 

$$
\begin{pmatrix}
1 & t^{n-k} (B_{j_3} + B_{i_1} + B_{i_2} + B_{i_3} + B_{j_2} + B_{j_3}) \\
0 & t^{n-k}
\end{pmatrix}.
$$

The following proposition is now obvious.

**Proposition 4.1.** If $B_{i_1} + B_{i_2} + B_{i_3} + B_{j_1} + B_{j_2} + B_{j_3} = 0$, then $\mathcal{M}(I, J)$ is a rank 2 Cohen-Macaulay module.

From now on we assume that $B_{i_1} + B_{i_2} + B_{i_3} + B_{j_1} + B_{j_2} + B_{j_3} = 0$.

Let $I$ and $J$ be $r$-interlacing and form three boxes. In general, $I \cap J$ and $I^c \cap J^c$ are non-empty. We have to modify our definition of branching points $i_m$ and associated points $j_m$ for the general setting.

**Definition 4.2.** Let $I$ and $J$ be two $k$-subsets such that their lattice diagram forms $r_1$ boxes. The **branching points of the lattice diagram $I \mid J$** are defined to be the points where the boxes end, i.e. $i \in I \setminus J$ is a branching point if $i+1 \notin I$ and the two rims meet at $i$. We denote them by $\{i_1, i_2, \ldots, i_{r_1}\}$.

In addition, we define the points $\{j_1, j_2, \ldots, j_{r_1}\}$ at the beginning of the boxes as the set of $j \in J \setminus I$ such that $j-1 \notin J$ and such that $j_m$ is minimal in $\{i_m, i_m+1, \ldots, i_{m+1}\}$ (cyclically) with this property. The size of the box ending at $i_m$ is defined to be the number of elements of $I \setminus J$ for that box.
Theorem 4.3. The above module $\mathbb{M}(I, J)$ is indecomposable if and only if $t \nmid b_{i_1} + b_{j_1}$, $t \nmid b_{i_2} + b_{j_2}$, $t \nmid b_{i_3} + b_{j_3}$. Furthermore, if $t \mid b_{i_1} + b_{j_1}$ while $t \nmid b_{i_2} + b_{j_2}$ and $t \nmid b_{i_3} + b_{j_3}$, then $\mathbb{M}(I, J)$ is isomorphic to $L_X \oplus L_Y$ where $X = (J \cup (I \cap (i_3, i_1))) \setminus (J \cap (i_3, i_1))$ and $Y = (I \cup (J \cap (i_3, i_1))) \setminus (I \cap (i_3, i_1))$.

Proof. Let $\varphi = (\varphi_i)_{i=1}^n$ be an idempotent endomorphism of $\mathbb{M}(I, J)$ and let $\varphi_{i_3+i_3} = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$ (see Figure 6). From $x_{i_1+i_1}x_{i_2+i_2-1} \cdots x_{i_3+i_3-1}x_{i_4} \cdots x_{i_3+i_3+1} = \varphi_{i_3+i_3} x_{i_1+i_1} x_{i_2+i_2-1} \cdots x_{i_3+i_3+1}$ follows that

$$\varphi_{i_1+i_1} = \left( \begin{array}{c} a + (B_{i_1} + B_{j_1}) t^{-l_3}c \\ t^{l_3} \end{array} \right) = \left( \begin{array}{c} bt^{l_3} + (d-a)(B_{i_1} + B_{j_1}) - (B_{i_3} + B_{j_3})^2 t^{-l_3}c \\ d - (B_{i_3} + B_{j_3})t^{-l_3}c \end{array} \right),$$

where $t^{l_3} \mid (d-a)(B_{i_1} + B_{j_1}) - (B_{i_3} + B_{j_3})^2 t^{-l_3}c$. Here, we assume, without loss of generality, that $l_3$ is the largest amongst $l_1, l_2, l_3$. If $t \mid b_{i_1} + b_{j_1}$, $t \mid b_{i_2} + b_{j_2}$, $t \mid b_{i_3} + b_{j_3}$, then the last relation is equivalent to $t^{l_3} \mid (d-a)(b_{i_1} + b_{j_1}) - (b_{i_3} + b_{j_3})^2 t^{-l_3}c$.

From $x_{i_3+i_3} \cdots x_{i_3+i_3+1} = \varphi_{i_3+i_3} x_{i_1+i_1} x_{i_2+i_2-1} x_{i_3+i_3} \cdots x_{i_4} \cdots x_{i_3+i_3+1+1}$ follows that

$$\varphi_{i_2+i_2} = \left( \begin{array}{c} a -(B_{i_3} + B_{j_3}) t^{-l_3}c \\ t^{l_3} \end{array} \right) = \left( \begin{array}{c} Mt^{l_3} - (d-a)(B_{i_3} + B_{j_3}) - (B_{i_3} + B_{j_3})^2 t^{-l_3}c \\ d + (B_{i_3} + B_{j_3})t^{-l_3}c \end{array} \right),$$

where $t^{l_3} \mid (d-a)(B_{i_3} + B_{j_3}) + (B_{i_3} + B_{j_3})^2 t^{-l_3}c$. The last relation is equivalent to $t^{l_3} \mid (d-a)(b_{i_3} + b_{j_3}) + (b_{i_3} + b_{j_3})^2 t^{-l_3}c$.

It follows that $t \mid (d-a)-(b_{i_1} + b_{j_1}) t^{-l_3}c$ and $t \mid (d-a)+(b_{i_3} + b_{j_3}) t^{-l_3}c$, implying that $t \mid (b_{i_3} + b_{j_3}) t^{-l_3}c$ and that $t \mid t^{-l_3}c$. Then $t \mid d-a$ and it must be that $a = d$ and $b = c = 0$, giving us the trivial idempotents. Hence, the module $\mathbb{M}(I, J)$ is indecomposable.

If $t \mid b_{i_1} + b_{j_1}$, $t \mid b_{i_2} + b_{j_2}$, and $t \mid b_{i_3} + b_{j_3}$, then the module is isomorphic to the direct sum $L_J \oplus L_J$.

If $t \mid b_{i_1} + b_{j_1}$, $t \mid b_{i_2} + b_{j_2}$, and $t \mid b_{i_3} + b_{j_3}$, then we repeat the same procedure as in the previous section to construct a non-trivial idempotent and its eigenvectors. The only divisibility conditions that the elements of the matrix $\varphi_{i_3+i_3}$ has to fulfill are $t^{l_3} \mid c$ and $t \mid d-a-(b_{i_3} + b_{j_3}) t^{-l_3}c$.

Recall that if $a = d$, then we only obtain the trivial idempotents because $t \mid c$. So it must be $a + d = 1$.

If we set $a = 1$, $d = 0$, $c = t^{l_3}(b_{i_1} + b_{j_1})^{-1}$, and $b = 0$, we get the idempotent

$$\varphi_{i_3+i_3} = \left( \begin{array}{cc} t^{l_3}(b_{i_1} + b_{j_1})^{-1} & 0 \\ 0 & 1 \end{array} \right).$$

Its orthogonal complement is the idempotent

$$\left( \begin{array}{cc} 0 & t^{l_3}(b_{i_1} + b_{j_1})^{-1} \\ t^{l_3}(b_{i_1} + b_{j_1})^{-1} & 0 \end{array} \right).$$

From $x_{i_3+i_3} = \varphi_i x_i$, we easily determine idempotents $\varphi_i$, for all $i$. Since these are non-trivial idempotents, it follows that the module $\mathbb{M}(I, J)$ is decomposable, which we needed to prove. It remains to determine the summands of this modules. We know that $\mathbb{M}(I, J)$ is the direct sum of
rank 1 modules $L_X$ and $L_Y$ for some $X$ and $Y$. Let us determine $X$ and $Y$. For this, we take, at vertex $i$, eigenvectors $v_i$ and $w_i$ corresponding to the eigenvalue 1 of the idempotents $\varphi_i$ and $1 - \varphi_i$ respectively. For example, $v_{i_3+i_5} = [1, -t^{l_3}(b_{i_3} + b_{i_5})]$, $w_{i_5+i_7} = [0, 1]^t$, $v_{i_3+i_5+1} = [1 - t^{l_3}(b_{i_3} + b_{i_5}) - t^{l_3}(b_{i_3} + b_{i_5}) - b_{i_3} - b_{i_5}]$ and $w_{i_5+i_7+1} = [b_{i_3}, 1]$, and so on. A basis for $L_X$ is $\{v_i \mid i = 0, \ldots, n - 1\}$, and a basis for $L_Y$ is $\{w_i \mid i = 0, \ldots, n - 1\}$. Direct computation gives us that $x_{i_3+i_5+1} = t v_{i_3+i_5+1}$, $x_{i_5+i_7+1} = w_{i_5+i_7+1}$, so that $i_3 + i_5 + 1 \in X$ and $i_5 + i_7 + 1 \notin X$, and so on for other vertices. It is easy to conclude that $X = (J \cup (I \cap (i_3, i_5))) \setminus (J \cap (i_3, i_5))$ and $Y = (I \cup (J \cap (i_3, i_5))) \setminus (I \cap (i_3, i_5))$.

**Theorem 4.4.** Let $(b_i)$ and $(c_i)$ be different $n$-tuples corresponding to indecomposable modules $M_1$ and $M_2$ with filtration $L_1 \mid L_1$ satisfying the indecomposables conditions of Theorem 4.3. Then the modules $M_1$ and $M_2$ are isomorphic.

**Proof.** We explicitly construct an isomorphism between these modules as in the proof of Theorem 3.10 for the module $135 \mid 246$, so we only give some details. If $\varphi_{i_3+i_5} = \begin{pmatrix} \alpha \\ \gamma \\ \delta \end{pmatrix}$, then $\varphi_{i_3+i_5}$ is equal to

$$(\alpha + (C_{i_3} + C_{i_5})t^{l_3} \gamma) t^{-l_3} \beta - \alpha(B_{i_3} + B_{i_5}) + (C_{i_3} + C_{i_5})\delta - (B_{i_3} + B_{i_5})(C_{i_3} + C_{i_5})t^{-l_3} \gamma),$$

and $\varphi_{i_5+i_7}$ is equal to

$$(\alpha - (C_{i_5} + C_{i_7})t^{l_5} \gamma) t^{-l_5} \beta + \alpha(B_{i_5} + B_{i_7}) - (C_{i_5} + C_{i_7})\delta - (B_{i_5} + B_{i_7})(C_{i_5} + C_{i_7})t^{-l_5} \gamma),$$

where $t^{l_i} \mid c, t \mid -\alpha(B_{i_3} + B_{i_5}) + (C_{i_3} + C_{i_5})\delta - (B_{i_3} + B_{i_5})(C_{i_3} + C_{i_5})t^{-l_3} \gamma$, and $t^{l_5} \mid \alpha(B_{i_5} + B_{i_7}) - (C_{i_5} + C_{i_7})\delta - (B_{i_5} + B_{i_7})(C_{i_5} + C_{i_7})t^{-l_5} \gamma$. The last two conditions are equivalent to the conditions $t^{l_i} \mid -\alpha(b_{i_3} + b_{i_5}) + (c_{i_3} + c_{i_5})\delta - (b_{i_3} + b_{i_5})(c_{i_3} + c_{i_5})t^{-l_3} \gamma$ and $t^{l_5} \mid \alpha(b_{i_5} + b_{i_7}) - (c_{i_5} + c_{i_7})\delta - (b_{i_5} + b_{i_7})(c_{i_5} + c_{i_7})t^{-l_5} \gamma$. Now we proceed as in the case of the module $135 \mid 246$ with exactly the same calculations to obtain the isomorphism

$$\varphi_{i_3+i_5} = \begin{pmatrix} (c_{i_3} + c_{i_5})(b_{i_3} + b_{i_5}) & (c_{i_3} + c_{i_5}) \\ (b_{i_3} + b_{i_5})(c_{i_3} + c_{i_5}) & (b_{i_3} + b_{i_5})(c_{i_3} + c_{i_5}) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$ 

Once we know $\varphi_{i_3+i_5}$, $\varphi_{i_5+i_7}$, and $\varphi_{i_3+i_5}$, we easily compute $\varphi_i$, for all $i$, again by using relations $x_i \varphi_i = \varphi_i x_i$.

It is important to remark that in this general construction we can set all upper right corner elements to be 0 for the points which are not branching or junctions. When these elements are not zero, then the divisibility conditions become more complicated as we have seen, but the module we get is isomorphic to the one where upper right corner elements are set to zero for all vertices which are not branching nor junctions. Also, as we have seen in the discussion before Proposition 4.1, we can assume that the $x_i$ at the junctions are also triangular, after a suitable change of the second basis element for the corresponding vector space $V_i$.

**Example 4.5.** In the same case $(4, 8)$, there is only one type of a profile that is non-tightly 3-interlacing. Such a profile is $2478 \mid 1356$ (and all profiles obtained from this profile by adding $a$ to each element of both 4-subsets, for $a = 1, \ldots, 7$). To construct modules with this profile we define $x_i = \begin{pmatrix} t \\ b_i \end{pmatrix}$, $y_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, for $i = 2, 4, 7, 8$ and $x_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $y_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, for $i = 1, 3, 5, 6$. Note that the $x_i$’s are almost the same as for the module with the profile $2468 \mid 1357$ constructed in the next section, with only the ones at vertices 6 and 7 changing places. In order for this to be a module we assume that $b_1 + b_2 + b_3 + b_4 + b_5 + t(b_6 + b_7) = 0$. Denote this module again by $M(I, J)$. As for the module $135 \mid 246$, it is easily seen that $L_J$ is a summand of $M(I, J)$ if and only if $t \mid b_8 + b_1$, $t \mid b_2 + b_3$, and $t \mid b_4 + b_5$. The module $M(I, J)$ is indecomposable if and only if $t \mid b_2 + b_3$, $t \mid b_4 + b_5$, $t \mid b_8 + b_1$. If
With respect to the divisibility conditions of the coefficients $x$, the pairs of profiles of the four modules that appear in the middle in these short exact sequences can be pictured as follows (similarly as in Example 3.8). The construction is the same as for the module $M$ (as a submodule) as end terms:

$$
0 \rightarrow L_J \rightarrow L_{\{1,3,5,6\}} \oplus L_{\{2,4,7,8\}} \rightarrow L_I \rightarrow 0,
0 \rightarrow L_J \rightarrow L_{\{1,4,7,8\}} \oplus L_{\{2,3,5,6\}} \rightarrow L_I \rightarrow 0,
0 \rightarrow L_J \rightarrow L_{\{1,3,7,8\}} \oplus L_{\{2,4,5,6\}} \rightarrow L_I \rightarrow 0,
0 \rightarrow L_J \rightarrow L_{\{2,3,7,8\}} \oplus L_{\{1,4,5,6\}} \rightarrow L_I \rightarrow 0.
$$

The pairs of profiles of the four modules that appear in the middle in these short exact sequences can be pictured as follows (similarly as in Example 3.8).

![Diagram of pairs of profiles](image)

**Figure 7:** The pairs of profiles of decomposable extensions between $L_{\{1,3,5,6\}}$ and $L_{\{2,4,7,8\}}$.

## 5 Tight 4-interlacing

In the tame case $(4,8)$, there is only one type of configuration of layers with 4-interlacing, $1357 \mid 2468$ (and the one obtained by adding 1 to each element of the two 4-subsets). We study this now. Let $I = \{1,3,5,7\}$ and $J = \{2,4,6,8\}$. The construction is the same as for the module $135 \mid 246$, we just have two more vertices. So assume that $x_i = \begin{pmatrix} t & b_i \\ 0 & 1 \end{pmatrix}$ for odd $i$ and $x_i = \begin{pmatrix} 1 & b_i \\ 0 & t \end{pmatrix}$ for even $i$. From $x^k = y^{n-k}$ it follows that $\sum_{i=1}^{8} b_i = 0$. We denote the constructed module by $\mathbb{M}$. We will study the structure of this module with respect to the divisibility conditions of the coefficients $b_i$.

Just as in the case of the module $135 \mid 246$ (Section 3.1), one can argue that $\mathbb{M} = L_I \oplus L_J$ if and only if $t \mid b_1 + b_2$, $t \mid b_3 + b_4$, $t \mid b_5 + b_6$, $t \mid b_7 + b_8$. In order to determine the structure of the module $\mathbb{M}$ when these four divisibility conditions are not fulfilled, first we determine the structure of an endomorphism of this module.
If $\varphi = (\varphi_i)_{i=0}^7$ is an endomorphism of $M$ and $\varphi_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$\varphi_{2i+1} = \begin{pmatrix} a + (b_1 + \cdots + b_{2i+1})t^{-1}c \\ t^{-1}c \end{pmatrix} \left( b + (d-a)(b_1 + \cdots + b_{2i+1}) - (b_1 + \cdots + b_{2i+1})^2t^{-1}c \right),$$

$$\varphi_{2i} = \begin{pmatrix} a + (b_1 + \cdots + b_{2i})t^{-1}c \\ c \end{pmatrix} \left( b + t^{-1}((d-a)(b_1 + \cdots + b_{2i}) - (b_1 + \cdots + b_{2i})^2t^{-1}c) \right),$$

where $t \mid c$, and

$$t \mid (d-a)(b_1 + b_2) - (b_1 + b_2)^2t^{-1}c,$$

$$t \mid (d-a)(b_1 + b_2 + b_3 + b_4) - (b_1 + b_2 + b_3 + b_4)^2t^{-1}c,$$

$$t \mid (d-a)(b_1 + b_2 + b_3 + b_4 + b_5 + b_6) - (b_1 + b_2 + b_3 + b_4 + b_5 + b_6)^2t^{-1}c.$$

We distinguish between different cases depending on whether the sums $b_1 + b_2$, $b_3 + b_4$, $b_5 + b_6$, and $b_7 + b_8$ are divisible by $t$ or not. We will call these the four divisibility conditions $t \mid b_1 + b_2$, $t \mid b_3 + b_4$, $t \mid b_5 + b_6$ and $t \mid b_7 + b_8$, and write $(\text{div})$ to abbreviate. There are three base cases: one of the sums is divisible by $t$ and three are not, two are divisible by $t$ and two are not, and none of the sums is divisible by $t$. We will see that $M$ is indecomposable in the first case and partly in the third case. We will explain how the module decomposes in the other cases. Furthermore, we will also show that there are infinitely many non-isomorphic modules with the same filtration for the indecomposable case when none of the sums is divisible by $t$.

### 5.1 Only one of the sums is not divisible by $t$ (Case 1)

We first assume that $t \mid b_1 + b_2$, $t \nmid b_3 + b_4$, $t \mid b_5 + b_6$, and $t \mid b_7 + b_8$.

**Theorem 5.1.** The above defined module $M$ is indecomposable if $t \mid b_1 + b_2$, $t \mid b_3 + b_4$, $t \mid b_5 + b_6$, and $t \mid b_7 + b_8$.

**Proof.** As in the proof of Theorem 3.5 for the module $135 \mid 246$, we repeat the same arguments using the divisibility conditions

$$t \mid (d-a)(b_1 + b_2) - (b_1 + b_2)^2t^{-1}c,$$

$$t \mid (d-a)(b_1 + b_2 + b_3 + b_4) - (b_1 + b_2 + b_3 + b_4)^2t^{-1}c,$$

to conclude that the only possible idempotent endomorphisms of $M$ are the trivial ones. We only note that $t \mid b_1 + b_2 + b_3 + b_4$ because if it were not so, then $t \mid b_5 + b_6$ which is not true. \qed

In the previous theorem it suffices to choose $b_1 = 0$, $b_2 = 1$, $b_3 = 2$, $b_4 = 0$, $b_5 = 0$, $b_6 = -3$, $b_7 = -1$, and $b_8 = 1$ in order to fulfil the conditions of the theorem and to have an indecomposable module.

In the next theorem we show that this module only depends on the divisibility conditions of the coefficients $b_i$, so if we have two different 8-tuples satisfying the same divisibility conditions, then they give rise to isomorphic modules.

**Theorem 5.2.** Let $(c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8)$ be an 8-tuple such that $t \mid c_1 + c_2$, $t \mid c_3 + c_4$, $t \mid c_5 + c_6$, and $t \mid c_7 + c_8$. If $M'$ is the module determined by this 8-tuple, then the modules $M'$ and $M$ are isomorphic.

**Proof.** As in the proof of Theorem 3.10 for the module $135 \mid 246$, we will explicitly construct an isomorphism $\varphi = (\varphi_i)_{i=0}^7$ between the two modules, where $\varphi_i : V_i \to W_i$, and $V_i$ and $W_i$ are the vector spaces at vertex $i$ of the modules $M$ and $M'$ respectively.
Let us assume that $\varphi_0 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Then by repeating the same calculations as for the module 135 | 246 we get

$$
\varphi_{2i+1} = \begin{pmatrix} \alpha + (c_1 + \cdots + c_{2i+1})t^{-1} + \beta t - \alpha \sum_{j=1}^{2i+1} b_j + \delta \sum_{j=1}^{2i+1} c_j - \frac{(\sum_{j=1}^{2i+1} b_j)(\sum_{j=1}^{2i+1} c_j) t^{-1}}{\gamma} \\
\gamma \end{pmatrix} + \frac{t^{-1}}{\gamma} \left( \begin{array}{c} \sum_{j=1}^{2i+1} b_j \\
\sum_{j=1}^{2i+1} c_j \end{array} \right),
$$

$$
\varphi_{2i} = \begin{pmatrix} \alpha + (c_1 + \cdots + c_{2i+1})t^{-1} + \beta t - \alpha \sum_{j=1}^{2i} b_j + \delta \sum_{j=1}^{2i} c_j - \frac{(\sum_{j=1}^{2i} b_j)(\sum_{j=1}^{2i} c_j) t^{-1}}{\gamma} \\
\gamma \end{pmatrix} + \frac{t^{-1}}{\gamma} \left( \begin{array}{c} \sum_{j=1}^{2i} b_j \\
\sum_{j=1}^{2i} c_j \end{array} \right),
$$

where $t \mid \gamma$ and

$$
t | -\alpha (b_1 + b_2) + (c_1 + c_2) \delta - (b_1 + b_2)(c_1 + c_2)t^{-1}\gamma,
$$

$$
t | -\alpha (b_1 + b_2 + b_3 + b_4) + (c_1 + c_2 + c_3 + c_4) \delta - (b_1 + b_2 + b_3 + b_4)(c_1 + c_2 + c_3 + c_4)t^{-1}\gamma,
$$

$$
t | -\alpha (b_1 + b_2 + b_3 + b_4 + b_5 + b_6) + (c_1 + c_2 + c_3 + c_4 + c_6) \delta - (b_1 + b_2 + b_3 + b_4 + b_5 + b_6)(c_1 + c_2 + c_3 + c_4 + c_6)t^{-1}\gamma.
$$

Since $t \mid \gamma$ and we would like $\varphi$ to be invertible, then it must be that $t \nmid \alpha$ and $t \nmid \delta$. Then the inverse of $\varphi_0$ is

$$
\frac{1}{\alpha \delta - \beta \gamma} \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}.
$$

In order to find an isomorphism $\varphi$, note that the last divisibility condition is fulfilled because $t \mid b_7 + b_8$ and $t \mid c_7 + c_8$. Also note that the condition $t \mid -\alpha (b_1 + b_2 + b_3 + b_4) + (c_1 + c_2 + c_3 + c_4) \delta - (b_1 + b_2 + b_3 + b_4)(c_1 + c_2 + c_3 + c_4)t^{-1}\gamma$ is equivalent to the condition $t \mid \alpha (b_5 + b_6) - (c_5 + c_6) \delta - (b_5 + b_6)(c_5 + c_6)t^{-1}\gamma$. Now we repeat the same calculations as for the module 135 | 246.

Note that there are no conditions attached to $\beta$ so we set it to be 0. If we set

$$
-\alpha (b_1 + b_2) + (c_1 + c_2) \delta - (b_1 + b_2)(c_1 + c_2)t^{-1}\gamma = 0,
$$

$$
\alpha (b_5 + b_6) - (c_5 + c_6) \delta - (b_5 + b_6)(c_5 + c_6)t^{-1}\gamma = 0,
$$

then we get

$$
\alpha (b_5 + b_6) \left[ 1 + \frac{c_5 + c_6}{c_1 + c_2} \right] - \delta (c_5 + c_6) \left[ 1 + \frac{b_5 + b_6}{b_1 + b_2} \right] = 0.
$$

If $t \mid 1 + \frac{c_3 + c_4}{c_1 + c_2}$, then from $\sum_{i=1}^{n} c_i = 0$, we get $t \mid c_3 + c_4$, which is not true. The same holds for $1 + \frac{c_3 + c_4}{b_1 + b_2}$, so both of these elements are invertible. Thus, if we set $\delta = 1$, then we get

$$
\alpha = \frac{(c_1 + c_2)(b_3 + b_4)(c_5 + c_6)}{(b_1 + b_2)(c_1 + c_2)(b_5 + b_6)},
$$

and

$$
\gamma = \frac{(c_3 + c_4)(b_5 + b_6) - (b_3 + b_4)(c_5 + c_6)}{(b_1 + b_2)(c_3 + c_4)(b_5 + b_6)}.
$$

Hence,

$$
\varphi_0 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.
$$

The other invertible matrices $\varphi_i$ are now determined from the above equalities. Note that all of them are invertible because their determinant is equal to $\alpha \delta - \beta \gamma$, which is an invertible element.

We denote the unique module (up to isomorphism) from Theorem 5.1 by $M_{1, 8}$. It is obvious, due to the symmetry of the arguments, that there are also modules $M_{1, 2}$, $M_{3, 4}$ and $M_{5, 6}$ that correspond to the remaining three possible divisibility conditions for Case 1, e.g. $M_{1, 2}$ corresponds to the case when $t \mid b_1 + b_2$, $t \nmid b_3 + b_4$, $t \mid b_5 + b_6$, and $t \nmid b_7 + b_8$. In the next statement we prove that no two of these modules are isomorphic to each other.
Proposition 5.3. There are no isomorphic modules amongst $M_{1,2}$, $M_{3,4}$, $M_{5,6}$ and $M_{7,8}$.

Proof. Due to the symmetry of the arguments, we will only show that $M_{7,8}$ is not isomorphic to any of the other modules. Assume otherwise, that $M_{7,8}$ is isomorphic to $M_{i,i+1}$, where $i$ is 1, 3 or 5. Then there is an isomorphism between these two modules. Keeping the same notation from the proof of the Theorem 5.2 we have that this isomorphism has to satisfy the following divisibility conditions:

$$t | -\alpha(b_1 + b_2) + (c_1 + c_2)\delta - (b_1 + b_2)(c_1 + c_2)t^{-1}\gamma,$$

$$t | -\alpha(b_1 + b_2 + b_3 + b_4) + (c_1 + c_2 + c_3 + c_4)\delta - (b_1 + b_2 + b_3 + b_4)(c_1 + c_2 + c_3 + c_4)t^{-1}\gamma,$$

$$t | -\alpha(b_1 + b_2 + b_3 + b_4 + b_5 + b_6) + (c_1 + c_2 + c_3 + c_4 + c_5 + c_6)\delta - t^{-1}\gamma \sum_{i=0}^{6} b_i \sum_{i=0}^{6} c_i.$$ 

Here, the coefficients $b_i$ correspond to $M_{7,8}$ and $c_j$ correspond to $M_{i,i+1}$. Since $t | b_1 + b_2 + b_3 + b_4 + b_5 + b_6 = -(b_7 + b_8)$, from the last condition it follows that $t | \delta(c_1 + c_2 + c_3 + c_4 + c_5 + c_6)$. Since $t \nmid \delta$, it must be $t | (c_1 + c_2 + c_3 + c_4 + c_5 + c_6)$. Then $t | c_7 + c_8 = -(c_1 + c_2 + c_3 + c_4 + c_5 + c_6)$ which is in contradiction with our assumption that $t \nmid c_7 + c_8$.

It follows from the previous proposition that we have now constructed four non-isomorphic rank 2 modules whose filtration is $L_{1357} | L_{2468}$. Before we show that in fact there are infinitely many, we consider the other two cases for the divisibility conditions.

5.2 Exactly two of the sums are divisible by $t$ (Case 2)

There are two subcases. The first subcase is when the divisible sums are consecutive, e.g. when $t | b_1 + b_2$, $t | b_3 + b_4$, $t \nmid b_5 + b_6$ and $t \nmid b_7 + b_8$. The second subcase is when the divisible sums are not consecutive, e.g. when $t | b_1 + b_2$, $t | b_5 + b_6$, $t \nmid b_3 + b_4$ and $t \nmid b_7 + b_8$.

Assume first that $t | b_1 + b_2$, $t | b_3 + b_4$, $t \nmid b_5 + b_6$, and $t \nmid b_7 + b_8$.

Theorem 5.4. If $t | b_1 + b_2$, $t | b_3 + b_4$, $t \nmid b_5 + b_6$, and $t \nmid b_7 + b_8$, then the module $\mathbb{M}$ is isomorphic to $L_{(1,3,5,6)} \oplus L_{(2,4,7,8)}$.

Proof. We will show that $\mathbb{M}$ is decomposable by constructing a non-trivial idempotent endomorphism of $\mathbb{M}$. Recall that an endomorphism $\varphi = (\varphi_i)_{i=0}^{n}$ of $\mathbb{M}$, where $\varphi_0 = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right)$, satisfies the divisibility conditions. Since $t | b_1 + b_2$, $t | b_3 + b_4$, these conditions reduce to a single condition $t | (d-a)(b_5 + b_6) - (b_5 + b_6)^2t^{-1}c$. From $t \nmid b_5 + b_6$ we conclude that $t | (d-a) - (b_5 + b_6)t^{-1}c$.

To construct a non-trivial idempotent homomorphism, as in the case $n = 6$, we set $\alpha = 1$, $\delta = 0 = \beta$, and $\gamma = -t(b_5 + b_6)^{-1}$. Thus,

$$\varphi_0 = \frac{1}{-t(b_5 + b_6)^{-1}} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

Its orthogonal complement is the idempotent

$$\frac{0}{t(b_5 + b_6)^{-1}} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

Since these are non-trivial idempotents, it follows that the module $\mathbb{M}$ is decomposable. It remains to show that $\mathbb{M} \cong L_{(2,4,7,8)} \oplus L_{(1,3,5,6)}$. We know that $\mathbb{M}$ is the direct sum of rank 1 modules $L_X$ and $L_Y$ for some $X$ and $Y$. Let us determine $X$ and $Y$. For this, we take, at vertex $i$, eigenvectors $v_i$ and $w_i$ corresponding to the eigenvalue 1 of the idempotents $\varphi_i$ and $1 - \varphi_i$, respectively. For example, $v_0 = [1, -t(b_5 + b_6)^{-1}]^t$, $w_0 = [0, 1]^t$, $v_1 = [1 - b_1(b_5 + b_6)^{-1}, -(b_5 + b_6)^{-1}]^t$ and $w_1 = [b_1, 1]^t$, and so on. A basis for $L_X$ is $\{v_i | i = 0, \ldots, 7\}$, and a basis for $L_Y$ is $\{w_i | i = 0, \ldots, 7\}$. Direct computation gives us that $x_1v_1 = tv_1$, $x_2v_1 = v_2$, $x_3v_2 = tv_3$, $x_4v_3 = v_4$, $x_5v_4 = tv_5$, $x_6v_5 = tv_6$, $x_7v_6 = v_7$, $x_8v_7 = v_0$. Thus, $X = \{2, 4, 7, 8\}$. Analogously, $Y = \{1, 3, 5, 6\}$. \qed

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Remark 5.5. In the case when \( t \mid b_1 + b_2 \), \( t \mid b_3 + b_4 \), \( t \mid b_5 + b_6 \), and \( t \mid b_7 + b_8 \), \( M \) is the direct sum \( L_{\{3,5,7,8\}} \oplus L_{\{1,2,4,6\}} \). Similarly, by suitable renaming of the vertices of the quiver, we obtain two more direct sums \( L_{\{1,2,5,7\}} \oplus L_{\{3,4,6,8\}} \) and \( L_{\{1,3,4,7\}} \oplus L_{\{2,5,6,8\}} \) that have \( L_J \) as a submodule and \( L_I \) as a quotient module, and there are short exact sequences with \( L_I \) and \( L_J \) as end terms:

\[
\begin{align*}
(a) & \quad 0 \to L_J \to L_{\{1,3,5,6\}} \oplus L_{\{2,4,7,8\}} \to L_I \to 0, \\
(b) & \quad 0 \to L_J \to L_{\{3,5,7,8\}} \oplus L_{\{1,2,4,6\}} \to L_I \to 0, \\
(c) & \quad 0 \to L_J \to L_{\{1,2,5,7\}} \oplus L_{\{3,4,6,8\}} \to L_I \to 0, \\
(d) & \quad 0 \to L_J \to L_{\{1,3,4,7\}} \oplus L_{\{2,5,6,8\}} \to L_I \to 0.
\end{align*}
\]

Here, (a) is the case where \( t \mid (b_1 + b_2) \) and \( t \mid (b_3 + b_4) \), (b) the case where \( t \mid (b_3 + b_4) \) and \( t \mid (b_5 + b_6) \), (c) the case where \( t \mid (b_5 + b_6) \) and \( t \mid (b_7 + b_8) \), and (d) the case \( t \mid (b_7 + b_8) \) and \( t \mid (b_1 + b_2) \).

The pairs of profiles of the four decomposable modules that appear in the middle in these short exact sequences can be pictured as follows:

![Figure 8: The pairs of profiles of decomposable extensions between \( L_{\{1,3,5,7\}} \) and \( L_{\{2,4,6,8\}} \).](image)

Assume now that \( t \mid b_1 + b_2 \), \( t \mid b_3 + b_4 \), \( t \mid b_5 + b_6 \), and \( t \mid b_7 + b_8 \).

Theorem 5.6. If \( t \mid b_1 + b_2 \), \( t \mid b_3 + b_4 \), \( t \mid b_5 + b_6 \), and \( t \mid b_7 + b_8 \), then the module \( M \) is isomorphic to \( L_{\{1,3,4,6\}} \oplus L_{\{2,5,7,8\}} \).

Proof. The only difference from the proof of the previous statement is that the divisibility conditions are now reduced to the condition \( t \mid (d - a) - (b_3 + b_4)t^{-1}c \). To construct a non-trivial idempotent homomorphism, we set \( \alpha = 1 \), \( \delta = 0 = \beta \), and \( \gamma = -t(b_3 + b_4)^{-1} \). Thus,

\[
\varphi_0 = \begin{pmatrix}
1 \\
-t(b_3 + b_4)^{-1} & 0 \\
0 & 0
\end{pmatrix},
\]

and the rest of the proof is analogous to the proof of the previous statement.

Remark 5.7. In the case when \( t \mid b_1 + b_2 \), \( t \mid b_3 + b_4 \), \( t \mid b_5 + b_6 \), and \( t \mid b_7 + b_8 \), \( M \) is the direct sum \( L_{\{1,2,4,7\}} \oplus L_{\{3,5,6,8\}} \). The pairs of profiles of these two modules can be pictured as follows:

![Figure 9: The pairs of profiles of decomposable extensions between \( L_{\{1,3,5,7\}} \) and \( L_{\{2,4,6,8\}} \).](image)
5.3 None of the four sums is divisible by $t$ (Case 3)

There are two subcases we have to consider. The first subcase is when all sums $b_i + b_{i+1} + b_{i+2} + b_{i+3}$, for $i = 1, 3, 5, 7$, are divisible by $t$, the second subcase is when at least one of these sums is not divisible by $t$. In the latter case, we get infinitely many non-isomorphic indecomposable modules as we will show. In this subsection, we always assume that none of the four divisibility conditions $t \mid b_i + b_{i+1}$, $i$ odd, which we continue to abbreviate as $(\text{div})$, is satisfied.

We first consider the case where all sums $b_i + b_{i+1} + b_{i+2} + b_{i+3}$ are divisible by $t$.

**Theorem 5.8.** Assume that the $(b_i)$, satisfy none of the four divisibility conditions $(\text{div})$ but that $t \mid b_i + b_{i+1} + b_{i+2} + b_{i+3}$, for $i = 1, 3, 5, 7$. Then $\mathcal{M} \cong L_{\{1,2,5,6\}} \oplus L_{\{3,4,7,8\}}$.

**Proof.** As before, we will construct a non-trivial idempotent endomorphism of $\mathcal{M}$ to prove that it is a decomposable module. Also, as before, our endomorphism has to satisfy the conditions $(\text{div})$.

From $t \mid b_1 + b_2 + b_3 + b_4$ it follows that the conditions $(\text{div})$ reduce to a single condition $t \mid (d - a) - (b_1 + b_2)t^{-1}c$. Note that this condition is equivalent to the condition $t \mid (d - a) - (b_5 + b_6)t^{-1}c$. As before, we obtain a non-trivial idempotent

$$\varphi_0 = \begin{pmatrix} 1 \\ -t(b_1 + b_2)^{-1} \\ 0 \end{pmatrix}. $$

The rest of the proof is analogous to the proof of the previous two statements.

**Remark 5.9.** The pairs of profiles of $L_{\{1,2,5,6\}} \oplus L_{\{3,4,7,8\}}$ can be pictured as follows:

![Figure 10: The pairs of profile of a decomposable extension between $L_{\{1,3,5,7\}}$ and $L_{\{2,4,6,8\}}$.](image)

Assume now that for the tuple $(b_i)$, one of the consecutive sums of four entries is not divisible by $t$.

**Proposition 5.10.** If none of the four divisibility conditions $(\text{div})$ holds for $(b_i)$, and if there exists an $i \in \{1, 3, 5, 7\}$ such that $t \nmid b_i + b_{i+1} + b_{i+2} + b_{i+3}$, then $\mathcal{M}$ is indecomposable.

**Proof.** Assume, without loss of generality, that $t \nmid b_1 + b_2 + b_3 + b_4$. As in the proof of indecomposability of the module $135 \mid 246$ the endomorphism of $\mathcal{M}$ has to satisfy the conditions

$$t \mid (d - a)b_1 + (d - a)b_2 - (b_1 + b_2)^2t^{-1}c,$$

$$t \mid (d - a)(b_1 + b_2 + b_3 + b_4) - (b_1 + b_2 + b_3 + b_4)^2t^{-1}c,$$

$$t \mid (d - a)(b_1 + b_2 + b_3 + b_4 + b_5 + b_6) - (b_1 + b_2 + b_3 + b_4 + b_5 + b_6)^2t^{-1}c.$$

Now we repeat the same arguments as for the module $135 \mid 246$. From the first two conditions we get that $t \mid (b_3 + b_4)t^{-1}c$. Since $t \nmid b_3 + b_4$, it follows that $t \mid t^{-1}c$ and that $t \mid d - a$. If $a + d = 1$, then, because $t \mid a$ or $t \mid 1 - a$, it must be that $t \mid 1$, which is not true. Thus, it must be $a = d$ and $c = b = 0$, giving us only trivial idempotents.

Now that we know that whenever $t \nmid b_i + b_{i+1} + b_{i+2} + b_{i+3}$, for at least one $i$, the module $\mathcal{M}$ with the given $(b_i)$, is indecomposable, we would like to know if the constructed modules are isomorphic. Since $\sum b_i = 0$, we have $t \nmid b_i + b_{i+1} + b_{i+2} + b_{i+3}$ for some $i$ and if only if $t \nmid b_{i+4} + b_{i+5} + b_{i+6} + b_{i+7}$. Therefore, these conditions come in pairs of “complementary” sums. Thus we have to distinguish between the cases when two of the sums $b_i + b_{i+1} + b_{i+2} + b_{i+3}$ are divisible by $t$ and two are not, and when none...
of these sums of four consecutive $b_i$'s is divisible by $t$. We will see that in the first case, we only get two indecomposable modules (up to isomorphism) while in the latter case, we get infinitely many.

Let us assume that $(c_1)_i$ is another 8-tuple satisfying none of the divisibility conditions (div) and such that $t \nmid c_i + c_{i+1} + c_{i+2} + c_{i+3}$, for some $i \in \{1, 3, 5, 7\}$. Denote the module given by these $(c_i)_i$ by $M'$.

In the following two propositions we consider the case when both $b_i$'s and $c_i$'s have two of the above mentioned sums divisible by $t$, and two sums not divisible by $t$.

**Proposition 5.11.** If $t \mid b_1 + b_2 + b_3 + b_4, t \mid b_3 + b_4 + b_5 + b_6, t \mid c_1 + c_2 + c_3 + c_4$, and $t \mid c_3 + c_4 + c_5 + c_6$, then $M$ and $M'$ are isomorphic.

**Proof.** Keeping the same notation as before when constructing isomorphisms, it must hold:

$$
\begin{align*}
&\text{If } t \mid -t \sum b_i c_i, \\
&\text{then the isomorphism is given by } (b_1, b_2, b_3, b_4, c_1, c_2, c_3, c_4) \rightarrow (b_3 + b_4 + b_5 + b_6, c_3 + c_4 + c_5 + c_6),
\end{align*}
$$

Since $t \mid b_3 + b_4 + b_5 + b_6$ and $t \mid c_3 + c_4 + c_5 + c_6$, the above conditions reduce to the first two conditions. Now, we set $-\alpha(b_1 + b_2) + (c_1 + c_2) \cdot -\alpha((b_1 + b_2)(c_1 + c_2)) = 0$ and $-\alpha(b_1 + b_2 + b_3 + b_4) + (c_1 + c_2 + c_3 + c_4) \cdot -\alpha((b_1 + b_2 + b_3 + b_4) = 0$. Subsequently, $\alpha(b_1 + b_2 + b_3 + b_4)(c_1 + c_2 + c_3 + c_4) = 0$. By setting $\alpha = 1, \beta = 0$, we get that $\delta = [(b_1 + b_2)(b_3 + b_4)](c_1 + c_2) = 0$. This $\delta$ is another isomorphism between $M$ and $M'$.

**Proposition 5.12.** If $t \mid b_1 + b_2 + b_3 + b_4, t \mid b_3 + b_4 + b_5 + b_6, t \mid c_1 + c_2 + c_3 + c_4$, and $t \mid c_3 + c_4 + c_5 + c_6$, then $M$ and $M'$ are not isomorphic.

**Proof.** If there were an isomorphism between $M$ and $M'$, then its coefficients would satisfy

$$
\begin{align*}
&\text{If } t \mid \alpha(b_1 + b_2 + b_3 + b_4), \\
&\text{then the isomorphism is given by } (b_1, b_2, b_3, b_4, c_1, c_2, c_3, c_4) \rightarrow (b_3 + b_4 + b_5 + b_6, c_3 + c_4 + c_5 + c_6),
\end{align*}
$$

From the second condition we obtain $t \mid \alpha(b_1 + b_2 + b_3 + b_4).$ But $t \nmid b_1 + b_2 + b_3 + b_4$ and $t \nmid \alpha$, which is a contradiction.

**Remark 5.13.** The same arguments used in the proof of the previous proposition tell us that the two modules $M$ and $M'$ from Proposition 5.12 are two new non-isomorphic indecomposable modules which are not isomorphic to any of the modules $M_{1,2}, M_{3,4}, M_{5,6}$ and $M_{7,8}$ constructed before. For example, if the $b_i$'s correspond to the module $M_{1,2}$ and the $c_i$'s correspond to the module $M$, then from the third relation in the proof of Proposition 5.11 we obtain that $t \mid \delta(c_1 + c_2 + c_3 + c_4 + c_5 + c_6)$, yielding $t \mid \delta(c_1 + c_2)$, which is not true since $t \nmid \delta$ and $t \nmid (c_1 + c_2)$.

We are only left to examine if, in the case when none of the sums $b_i + b_{i+1} + b_{i+2} + b_{i+3}$ is divisible by $t$, for two different tuples we obtain isomorphic modules. In the following theorem we assume that $b_i$'s correspond to the module $M$ and $c_i$'s correspond to the module $M'$. Also, we assume that $t \mid b_i + b_{i+1}$ $t \nmid c_i + c_{i+1}$ for odd $i$.

**Theorem 5.14.** If $t \mid b_i + b_{i+1} + b_{i+2} + b_{i+3}$ and $t \mid c_i + c_{i+1} + c_{i+2} + c_{i+3}$, for $i = 1, 3, 5, 7$, then the modules $M$ and $M'$ are isomorphic if and only if

$$
\begin{align*}
\text{If } t \mid (b_1 + b_2)(c_3 + c_4)(b_5 + b_6)(c_7 + c_8) + (c_1 + c_2)(b_3 + b_4)(c_5 + c_6)(b_7 + b_8),
\end{align*}
$$

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**Proposition 5.17.** From the proof of the previous corollary it follows that the two modules

\[ t \mid -\alpha(b_1 + b_2) + (c_1 + c_2)\delta - (b_1 + b_2)(c_1 + c_2)t^{-1}\gamma, \]

\[ t \mid -\alpha(b_1 + b_2 + b_3 + b_4) + (c_1 + c_2 + c_3 + c_4)\delta - (b_1 + b_2 + b_3 + b_4)(c_1 + c_2 + c_3 + c_4)t^{-1}\gamma, \]

\[ t \mid \alpha(b_7 + b_8) - (c_7 + c_8)\delta - (b_7 + b_8)(c_7 + c_8)t^{-1}\gamma. \]

From these we get that

\[ t \mid \alpha(c_3 + c_4)[(c_1 + c_2)(c_1 + c_2 + c_3 + c_4)]^{-1} - \delta(b_3 + b_4)[(b_1 + b_2)(b_1 + b_2 + b_3 + b_4)]^{-1}, \]

\[ t \mid \alpha(c_5 + c_6)[(c_7 + c_8)(c_1 + c_2 + c_3 + c_4)]^{-1} - \delta(b_5 + b_6)[(b_7 + b_8)(b_1 + b_2 + b_3 + b_4)]^{-1}. \]

Finally, from the last two relations we get

\[ t \mid \alpha((b_1 + b_2)(c_3 + c_4)(b_5 + b_6)(c_7 + c_8) - (c_1 + c_2)(b_3 + b_4)(c_5 + c_6)(b_7 + b_8)). \]

If \( t \mid (b_1 + b_2)(c_3 + c_4)(b_5 + b_6)(c_7 + c_8) - (c_1 + c_2)(b_3 + b_4)(c_5 + c_6)(b_7 + b_8) \), then there is no isomorphism between \( M' \) and \( M \). If \( t \mid (b_1 + b_2)(c_3 + c_4)(b_5 + b_6)(c_7 + c_8) - (c_1 + c_2)(b_3 + b_4)(c_5 + c_6)(b_7 + b_8) \), then we simply set \( \alpha = 1 \), and compute \( \delta \) and \( \gamma \) from the above relations (as before, we set \( \beta = 0 \)).

**Remark 5.15.** It is easily shown that none of the indecomposable modules from Theorem 5.14 is isomorphic to any of the indecomposable modules from the previous cases. To prove this, we use the same arguments as in Remark 5.13.

We will now parametrize the non-isomorphic indecomposable modules from Theorem 5.14.

Let \( \beta \in \mathbb{C} \setminus \{0, 1, -1\} \). Keeping the notation from the theorem, choose the parameters \( b_i \) in the following way: \( b_1 + b_2 = -(b_5 + b_6) = 1 \), \( b_3 + b_4 = -(b_7 + b_8) = \beta \). Then, \( t \mid b_i + b_{i+1} \) and \( t \mid b_i + b_{i+1} + b_{i+2} + b_{i+3} \), for odd \( i \). What is the indecomposable module that corresponds to these coefficients by \( \gamma = \gamma_i \)?

**Corollary 5.16.** There are infinitely many non-isomorphic rank 2 indecomposable modules in \( CM(B_{4,8}) \) with profile 1357 | 2468.

**Proof.** Let \( \alpha \in \mathbb{C} \setminus \{0, 1, -1\} \), with \( \beta \neq \pm \alpha \), and \( M_\alpha \) be the corresponding indecomposable module. Then \( M_\alpha \) and \( M_\beta \) are not isomorphic. Indeed, assuming that the coefficients \( c_i \) correspond to \( M_\alpha \), we have that

\[ t \mid (b_1 + b_2)(c_3 + c_4)(b_5 + b_6)(c_7 + c_8) - (c_1 + c_2)(b_3 + b_4)(c_5 + c_6)(b_7 + b_8) = \alpha^2 - \beta^2, \]

so \( \alpha \neq \pm \beta \), by the previous theorem the corresponding modules are not isomorphic.

From the proof of the previous corollary it follows that the two modules \( M_\alpha \) and \( M_\beta \) are isomorphic if and only if \( \alpha = \pm \beta \). Thus, the non-isomorphic indecomposable modules of this form are parameterized by \( \mathbb{C} \setminus \{0, 1, -1\} \), where two points in this set are identified if they sum up to 0. In the next proposition we show that every indecomposable module in \( M_\beta \). Let \( \beta \in \mathbb{C} \setminus \{0, 1, -1\} \), for odd \( i \). Then there exists \( \beta \in \mathbb{C} \setminus \{0, 1, -1\} \) such that \( M \) is isomorphic to \( M_\beta \) for some \( \beta \).

**Proposition 5.17.** Let \( M \) be a rank 2 indecomposable module with the corresponding coefficients \( c_i \) satisfying \( t \mid c_i + c_{i+1} \), \( t \mid c_i + c_{i+1} + c_{i+2} + c_{i+3} \), for odd \( i \). Then there exists \( \beta \in \mathbb{C} \setminus \{0, 1, -1\} \) such that \( M \) is isomorphic to \( M_\beta \).

**Proof.** Let \( c_i + c_{i+1} = C_i \), for \( i = 1, 3, 5, 7 \). Since the coefficients of \( M_\beta \) satisfy \( b_1 + b_2 = -(b_5 + b_6) = 1 \), \( b_3 + b_4 = -(b_7 + b_8) = \beta \), it follows from Theorem 5.14 that we need to find \( \beta \) satisfying \( t \mid \beta^2 C_1 C_5 - C_3 C_7 \). If \( \gamma_i \) is the constant term of \( C_i \), then we choose \( \beta \) to be a square root of \( (\gamma_3 \gamma_5)(\gamma_5 \gamma_7)^{-1} \).

Note that by the divisibility conditions (div), \( \gamma_1 \gamma_5 \neq 0 \) and \( \gamma_3 \gamma_7 \neq 0 \). In particular, \( \beta \neq 0 \). If \( \beta = \pm 1 \), then \( \gamma_1 \gamma_5 = \gamma_3 \gamma_7 \). From \( C_1 + C_3 + C_5 + C_7 = 0 \), we get after multiplying by \( C_5 \) that \( (\gamma_3 + \gamma_5)(\gamma_5 + \gamma_7) = 0 \) which is not possible. Hence, \( \beta \neq \pm 1 \).
6 The general case: $r$-interlacing, $r \geq 4$

In this section we generalize the results from the previous two sections. We deal with the case of $r$-interlacing, where $r \geq 4$ and at least 4 boxes (i.e. $r$-interlacing rims where $r \geq r_1 > 3$). We prove that if $I$ and $J$ are $r$-interlacing with poset $1^{r_1} | 2$, where $r \geq 4$ and $r \geq r_1 > 3$, then there exist non-isomorphic indecomposable modules with the given filtration $L_I \mid L_J$. It follows that in this case, as in Subsection 5, the profile of a module does not uniquely determine the module.

Let $r \geq 4$. If $k$ is arbitrary and $I$, $J$ are such that $I$ and $J$ are $r$-interlacing and $I \mid J$ has poset of the form $1^{r_1} | 2$, where $r \geq r_1 > 3$, then we are able to construct more than one non-isomorphic indecomposable rank 2 module which has $L_J$ as a submodule and $L_I$ as the quotient as follows. Denote this module by $\mathbb{M}(I, J)$. We mimic the same procedure as for the module $1357 \mid 2468$.

![Figure 11: The profile of a module with 4-interlacing layers and with poset 1^3 | 2, for (k, n) = (4, 10).](image)

The two rims form $r_1$ boxes. Denote the $r_1$ branching points from $I$ where the boxes of the two rims end by $i_1, \ldots, i_{r_1}$ and their counterparts in $J$ by $j_1, \ldots, j_{r_1}$, as in Definition 4.2.

For these branching points, we set $x_i = \begin{pmatrix} t & b_i \\ 0 & 1 \end{pmatrix}$ and $x_{j_l} = \begin{pmatrix} 1 & b_{j_l} \\ 0 & t \end{pmatrix}$ for $l = 1, \ldots, r_1$. For all other vertices $i$ we define $x_i$ (resp. $y_i$) to be diagonal matrices as follows: the diagonal of $x_i$ (resp. $y_i$) is $(1, t)$ (resp. $(t, 1)$) if $i \in J \setminus I$, it is $(t, 1)$ (resp. $(1, t)$) if $i \in I \setminus J$, $(t, t)$ (resp. $(1, 1)$) if $i \in I^c \cap J^c$, and $(1, 1)$ (resp. $(t, t)$) if $i \in I \cap J$. We also assume that $\sum_{i=1}^{r_1} (b_{i_l} + b_{j_l}) = 0$ so that we have a module structure.

Now we assume that the following divisibility conditions hold for the $b_i$'s at the first three branching points, as in Theorem 5.1 (recalling from its proof that $t \mid b_1 + b_2 + b_3 + b_4$) for the module $1357 \mid 2468$: $t \mid b_{i_1} + b_{j_1}$, $t \mid b_{i_2} + b_{j_2}$, $t \mid b_{i_3} + b_{j_3}$, with $t \mid b_{i_l} + b_{j_l} + b_{i_{l+1}} + b_{j_{l+1}}$, and $t \mid b_{i_l} + b_{j_l}$ for $l \leq 4$. As in the previous sections, it is now easy to prove that this module is indecomposable by invoking the same divisibility arguments as before. Obviously, we could start at any branching point in order to obtain additional $r_1 - 1$ indecomposable modules. As in Proposition 5.3 one can easily prove that no two of these $r_1$ indecomposable rank 2 modules are isomorphic.

Therefore, we have the following proposition.

**Proposition 6.1.** If $I$ and $J$ are $r$-interlacing and $I \mid J$ has the poset $1^{r_1} | 2$, where $r \geq r_1 > 3$, then there are more than one indecomposable rank 2 modules with the profile $I \mid J$.

Furthermore, it is easy to adopt the proof of Theorem 5.14 and Corollary 5.10 to the general case when $r \geq 4$ in order to obtain the following theorem (we omit the proof).

**Theorem 6.2.** Let $I$ and $J$ be $r$-interlacing with poset $1^{r_1} | 2$, where $r \geq r_1 > 3$. There are infinitely many non-isomorphic rank 2 indecomposable modules in $\mathbb{CM}(B_{k,n})$ with profile $I \mid J$.

For given $r$-interlacing $k$-subsets $I$ and $J$ with the poset $1^{r_1} | 2$, where $r \geq r_1 > 3$, we note that as the number $r_1$ increases the parameterization of non-isomorphic indecomposable rank 2 modules
with filtration $L_I | L_J$ becomes more complicated. In the case $r = r_1 = 4$, we have seen that the family of non-isomorphic indecomposable modules with filtration $L_I | L_J$ is parameterized by the set $\mathbb{C} \setminus \{0, 1, -1\}$ up to sign (if $\alpha = -\beta$, then $M_{\alpha} \cong M_{\beta}$). Here, we do not pursue the classification of these non-isomorphic indecomposable modules, but it would be nice to have this sort of classification for general $r \geq r_1$.

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