The geometric approach to study the dynamics of $U(1)$-invariant membranes is developed. The approach reveals an important role of the Abel nonlinear differential equation of the first type with variable coefficients depending on time and one of the membrane extendedness parameters. The general solution of the Abel equation is constructed. Exact solutions of the whole system of membrane equations in the $D = 5$ Minkowski space-time are found and classified. It is shown that if the radial component of the membrane world vector is only time dependent then the dynamics is described by the pendulum equation.
1 U(1) invariant membranes

The major role of membranes in M-theory poses the problem of their quantization as one of the most fundamental in quantum field theory and strings [1, 2, 3, 4]. In spite of the great progress obtained in studying the classical membrane dynamics the question of its integrability [5, 6, 7] is still open even for the case of a free Dirac membrane in D-dimensional Minkowski space-time $x^m = (x^0, \vec{x})$ given by the Dirac action [8]

$$S = \int \sqrt{|G|} d\xi^3, \quad G_{\alpha\beta} := \partial_\alpha x_m \partial_\beta x^m,$$

where $\xi^a = (\tau, \sigma^r)$ (with $r = 1, 2$) and $G$ are the internal coordinates and the determinant of the induced metric of the membrane world-volume respectively. The principle obstacle is the three dimensional nonlinear nature of its equations of motion

$$\dot{P}_m = -\partial_r (\sqrt{|G|} G^{\alpha\beta} \partial_\alpha x^m), \quad P_m = \sqrt{|G|} G^{\tau\beta} \partial_\beta x^m$$

and the primary constraints

$$P^m \partial_r x_m \approx 0, \quad P^m P_m - \det G_{rs} \approx 0.$$  (2)

A reduction of the 3-dim problem to an effective 2-dim one could simplify the search for the general solution of Eqs. (1). In the paper [9] Hoppe observed that the presence of an additional $U(1)$ global symmetry, characterizing the membrane shape, excludes the $\sigma^2$ parameter from the partial differential equations (PDE’s) (1) and (2). The general static solution of the resulting two dimensional nonlinear problem was found in [10] for $D=2N+1$ dimensional Minkowski space including M-theory case $D=11$.

Here we propose a new approach to study the general time-dependent solutions of the membrane equations of motion for $D=5$. The $U(1)$ membrane is defined by the ansatz for the space components $\vec{x}$ of its world vector

$$\vec{x} = (m_1 \cos \sigma^2, m_1 \sin \sigma^2, m_2 \cos \sigma^2, m_2 \sin \sigma^2),$$

$$m_a = m_a(t, \sigma^1), \quad \vec{m} := (m_1, m_2).$$

In the orthogonal gauge $\tau = x^0, \quad G_{\tau\tau} = -(\vec{x} \cdot \partial_\tau \vec{x}) = 0$ the ansatz (3) results in the dynamics completely determined by the following nonlinear differential equations

$$\dot{\vec{m}} = (\vec{m} \cdot \vec{m}^\prime - \vec{m}^\prime \vec{m}) - \vec{m} \cdot \vec{m}$$

(5)
and the constraints
\[ \dot{m}^2 + m^2 m'^2 - 1 = 0, \quad \dot{mm}' = 0, \]
where \( m' \) and \( \ddot{m} \) are partial derivatives of \( m \) with respect to \( \sigma^1 \) and \( t \) respectively [9] (see also [10]).

It turns out that the static case \( \ddot{m} = 0 \) is exactly solvable and the general solution of (5) and (6) consists of two solutions [10]. The first one is given by
\[ m = \pm \sqrt{2\sigma^1} \left( \cos \psi_0, \sin \psi_0 \right), \quad \psi_0 \in \mathbb{R}, \]
and describes a family of planes going through the origin of the Euclidean subspace \( \mathbb{R}^4 \) of the Minkowski space.

The second solution is as follows
\[ m(\sigma^1) = C_1 \sqrt{\frac{1 + z^2 + 1}{2}} + C_2 \sqrt{\frac{1 + z^2 - 1}{2}}, \]
\[ C_1 = (C, D), \quad C_2 = (D, -C), \quad z = \frac{2(\sigma^1 + \tilde{c})}{C^2 + D^2}, \]
where \( C, D \) and \( \tilde{c} \) are the integration constants, and the vector function \( m \) yields the membrane surface which can be visualized by rotating the hyperbola, lying in the \( x_1x_3 \) plane, simultaneously in \( x_1x_2 \) and \( x_3x_4 \) planes.

Because of the scaling invariance \( \tilde{\sigma}^1 = b\sigma^1, \tilde{m} = \sqrt{bm} \) of the static problem, we write out the general solutions for the case \( (C_1 C_2) = 0 \) fixing the second integration constant \( \delta := (C_1 C_2)/c^2 = 0 \) and simplifying the expression of general solution with arbitrary \( \delta \).

There are also two time dependent, physically interesting solutions [10]. The first one corresponds to a contracting, flat torus described by elliptic cosine
\[ m = \rho(t) \left( \cos n\sigma^1, \sin n\sigma^1 \right), \]
\[ \rho(t) = \frac{1}{\sqrt{n}} \left( \sqrt{2nt}, \frac{1}{\sqrt{2}} \right), \quad n \in \mathbb{Z}_+, \]
and the second corresponds to a spinning, flat torus
\[ m = \rho(\sigma^1) \left( \cos \omega t, \sin \omega t \right), \]
\[ \rho(\sigma^1)^2 = \frac{1}{\omega^2} - (\pi - \sigma^1)^2 \omega^2, \]
rotating with the frequency \( \Omega \leq \Omega_{\text{max}} = \frac{T^{1/3}}{\sqrt{\pi}} \), where \( T \) is the membrane tension.
2 The geometric approach and the Abel equation

The case of the $U(1)$-invariant membranes (3), embedded in the $D = 5$ Minkowski space, is special since in that case $\mathbf{m}'$ and $\dot{\mathbf{m}}$ form a local basis attached to the effective two-dimensional $\mathbf{m}$-surface (because of their linear independence provided by the orthogonality constraint (6)). An important consequence of the independence is that the constraints (6) generate the dynamical equations (5). To see it one can, following [9], differentiate Eqs. (6) and compare them with Eqs. (5) multiplied by $\dot{\mathbf{m}}$ and $\mathbf{m}'$.

These observations also imply that the dynamical equations (5) preserve the constraints (6) which may be treated as the initial data constraints for Eqs. (5) selecting a closed sector of the non-static solutions. To analyze the sector we need to study solutions of the constraints (6).

To this end it is convenient to introduce a local moving frame in the two dimensional $\mathbf{m}$-space formed by the unit orthogonal 2-vectors $\mathbf{n}_0$ and $\mathbf{n}_1$

\[ \mathbf{n}_0 = \frac{\dot{\mathbf{m}}}{\sqrt{1 - \mathbf{m}'^2 \mathbf{m}^2}}, \quad \mathbf{n}_1 = \frac{\mathbf{m}'}{\sqrt{\mathbf{m}'^2}} , \quad \mathbf{n}_i \cdot \mathbf{n}_j = \delta_{ij}. \tag{9} \]

One can see that the constraints (6) are equivalent to the orthonormality conditions (9) for the local frame vectors $\mathbf{n}_i(t, \sigma^1)$ (where $i, k = 0, 1$).

The introduction of the moving frame (or reper) opens a way to apply the famous Cartan approach [11] to an alternative formulation of our problem. The geometry of curved spaces [12] in the Cartan approach is described not in terms of the original world coordinates $x^m$, but in terms of the differential forms, associated with the vielbeins and connections characterizing the spaces and defined by the Maurer-Cartan equations. In string theory the Cartan approach has been firstly applied by Regge and Lund [13] (see also [14], [15]) and was called the geometric approach, or the embedding approach, because it describes the string worldsheet as a two-dimensional surface embedded in the target space-time.

In the case at hand the above mentioned Cartan one-forms $\omega_i = \omega_{\mu i} d\sigma^\mu$ and $\mathcal{A} = A_\mu d\sigma^\mu$, ($\sigma^\mu = t, \sigma^1$) are defined by the following equations

\[ \partial_\mu \mathbf{m} = \omega_{\mu i} \mathbf{n}_i, \quad \partial_\mu \mathbf{n}_i = -\epsilon_{ik} A_\mu \mathbf{n}_k, \tag{10} \]

where $\epsilon_{ik}$ is the 2d antisymmetric Levi-Civita tensor ($\epsilon_{01} = -\epsilon_{10} = 1$), which encode the membrane evolution. The connection $A_\mu$ is a $U(1)$ gauge field
belonging to the set of gauge fields previously considered in the gauge reformulation of the Regge-Lund approach [16, 17, 18, 19, 20, 21].

The integrability condition for the surface vector \( \mathbf{m} \)
\[
\frac{d}{d^2} \mathbf{m} = 0 \Rightarrow \partial_{[\mu} \omega_{\nu]} + \epsilon_{ik} A_{[\mu} \omega_{\nu]k} = 0,
\]
gives the zero torsion conditions
\[
\partial_{\sigma} \sqrt{1 - \mathbf{m}^2} - A_{\tau} \sqrt{\mathbf{m}^2} = 0,
\]
\[
\partial_{\tau} \sqrt{\mathbf{m}^2} + A_{\sigma} \sqrt{1 - \mathbf{m}^2} = 0
\]
from which the connection components \( A_{\mu} \) can be read off explicitly.

The integrability condition \( \frac{d}{d^2} n_{i} = 0 \) for \( n_{i} \) yields the Maurer-Cartan equations for the connection \( A_{\mu}(t, \sigma^1) \)
\[
F_{\mu\nu} = \partial_{[\mu} A_{\nu]} = 0 \Rightarrow A_{\mu} = \partial_{\mu} \phi
\]
equivalent to the pure gauge condition for the abelian gauge field \( A_{\mu} \) associated with the surface \( \mathbf{m}(t, \sigma^1) \). The scalar function \( \phi \) is defined by Eqs. (11) that may be presented in the form
\[
\frac{A'}{B} = \dot{\phi}, \quad \frac{\dot{B}}{A} = -\phi',
\]
where \( A := \sqrt{1 - \mathbf{m}^2} \), \( B := \sqrt{\mathbf{m}^2} \). Using (9) and (10) we find the equations of motion (5) to be equivalent to the following expansion of \( \mathbf{m} \)
\[
\mathbf{m} = \frac{\rho \dot{n}_1}{B} + \frac{\rho \dot{n}_0}{A}, \quad \rho := \sqrt{\mathbf{m}^2} = \sqrt{x^2}
\]
in the moving basis \( n_i \). Thus, the components of \( \mathbf{m} \) are expressed in terms of its length \( \rho \), \( A \) and \( B \). It is easy to see that the projections of Eqs. (13) on the linear independent 2-vectors \( \mathbf{m}' \) and \( \dot{\mathbf{m}} \) result in identities, so that Eqs. (13) turn out to be equivalent to the only one ellipse like relation
\[
\left( \frac{\rho}{B} \right)^2 + \left( \frac{\dot{\rho}}{A} \right)^2 = 1
\]
which is the square of (13). The relation (14) connects the functions \( \rho, A, B \) and is additional to the first of the constraints (6)

\[ A^2 + \rho^2 B^2 = 1. \]  

(15)

Then the problem is shifted to finding the solution of Eqs. (12).

The first of Eqs. (12) is presented in the simple form

\[ \frac{(B\rho)'}{\sqrt{1 - (B\rho)^2}} = -\frac{\dot{\phi}}{\rho}. \]  

(16)

and its integration yields the following representations

\[ B = \frac{1}{\rho} \sin \lambda, \quad A = \cos \lambda, \]  

(17)

\[ \lambda = -\int d\sigma \frac{\dot{\phi}}{\rho} + C_0(t), \]  

(18)

for the functions \( B \) and \( A \). The substitution of the expressions for \( A \) and \( B \) (17) to the constraint (15) yields an identity, so that Eq. (15) can be omitted. On the other hand the substitution of \( A \) and \( B \) to the second of Eqs. (12) and also in Eq. (14) transforms them into

\[ \dot{\lambda} - \frac{\dot{\rho}}{\rho} \tan \lambda + \rho \phi' = 0, \]  

(19)

\[ \left( \frac{\rho \rho'}{\sin \lambda} \right)^2 + \left( \frac{\dot{\rho}}{\cos \lambda} \right)^2 = 1. \]  

(20)

As a result, the membrane dynamics in the geometric approach is encoded by the system of three coupled equations (18), (19) and (20) for the three functions \( \lambda, \rho \) and \( \phi \). In this system the equation of motion (5) is represented by the differential equation (20) for the length \( \rho \) of the 2-vector \( m \). The rest of the system, i.e. equations (18) and (19), represent the constraints (6) accompanied by the equations of the moving reper.

There are a few ways to try to solve the nonlinear system (18-20). At first one can observe that Eq. (20) does not include the function \( \phi \) and is the exactly solvable biquadratic (or quadratic if \( \dot{\rho} = 0 \) equation for \( \Lambda := \tan \lambda \)

\[ \rho^2 \Lambda^4 - (1 - \rho^2 - \rho^2 \rho^2) \Lambda^2 + (\rho \rho')^2 = 0 \]  

(21)
which presents Λ (and respectively λ) as the function of ρ and its partial derivatives

\[ \Lambda_{\pm}^2 = \left| (1 - \dot{\rho}^2 - \rho^2 \rho'^2) \pm \sqrt{(1 - \dot{\rho}^2 - \rho^2 \rho'^2)^2 - 4(\rho \dot{\rho} \rho')^2} \right| \frac{1}{2\rho^2}. \]  

(22)

The substitution of the solution (22) in the remaining Eqs. (18), (19) transforms them into the second order differential equation for the function φ with variable coefficients depending on ρ and its derivatives.

An alternative way to derive the equation for φ is to change Eq. (18) by the equivalent differential equation and to unify it together with Eq. (19) in the form of system

\[ \dot{\lambda}' = -\frac{\dot{\phi}}{\rho}, \quad \dot{\lambda} = \frac{\dot{\rho}}{\rho} \tan \lambda - \rho \phi', \]  

(23)

expressing the partial derivatives \( \dot{\lambda} \) and \( \lambda' \) as the functions of \( \lambda, \rho, \dot{\rho} \) and the partial derivatives of \( \phi \). Then the integrability condition of the system (23) yields the 2-dim linear hyperbolic equation of the second order for \( \phi \)

\[ \ddot{\phi} - \rho^2 \phi'' - \rho^{-1} \dot{\rho} \dot{\phi} - \rho \rho' \phi' + \rho(\rho^{-1} \dot{\rho} \Lambda)' = 0 \]  

(24)

with variable coefficients and \( \Lambda \) depending on \( \rho, \rho', \dot{\rho} \), as it follows from Eqs. (22). Of course, the explicit form of the \( \Lambda \)-roots Eq. (21) includes the square roots and yields a rather complicated dependence of the variable coefficients of the wave-like equation (24) on \( \rho \) and its derivatives. As a result, it complicates the solution of the wave equation (24). On the other hand a linear character of Eq. (24) opens a way to apply the general methods of the theory of PDE to seek the solution of this equation.

An alternative way to solve the geometric system (18), (19), (20), encoding the membrane dynamics, is to present it in terms of the above introduced function \( \Lambda := \tan \lambda \) as follows

\[ \Lambda = \tan \left( -\int d\sigma \frac{\dot{\phi}}{\rho} + C_0(t) \right), \]  

(25)

\[ \dot{\Lambda} = (1 + \Lambda^2) \left( \frac{\dot{\rho}}{\rho} \Lambda - \rho \phi' \right), \]  

(26)

\[ \dot{\rho}^2 + \left( \frac{\rho \phi'}{\Lambda} \right)^2 = \frac{1}{1 + \Lambda^2}, \]  

(27)
where the representations \( A = \frac{1}{\sqrt{1+\Lambda}} \), \( B = \frac{\Lambda}{\rho \sqrt{1+\Lambda}} \) were used. Note that the existing solutions (7) or (8) are consistent with Eqs. (25 - 27). To see this consider e.g. solution (7) which implies that \( \rho' = 0 \) and \( \phi' = n \) so that Eq. (27) gives \( \Lambda = \pm \sqrt{1-\dot{\rho}^2} \). Choosing the \( - \) branch and substituting \( \Lambda \) to (26) one finds a differential equation consistent with (7). The choice of the branch is important, i.e. if we take the \( + \) branch we will not find the consistency.

An interesting property of Eqs. (25 - 27) is the coincidence of Eq. (26) with the well known Abel equation having the cubic nonlinearity in \( \Lambda \). Originally the Abel equation of the first kind has its variable coefficients depending only on the variable \( t \). But, in our case the variable coefficients of (26) also depend on the second variable \( \sigma^1 \). Such dependence appears in the theory of extended objects similar to strings/membranes and yields information about their dynamics. Thus, the derived generalization of the Abel equation, arising from the geometric approach opens a new way for studying the nonlinear system of equations (25), (26), (27) and the \( U(1) \) membrane physics.

As a result, our statement is that the original description of the membrane dynamics, given by the Eqs. (5) and (6), is reformulated to the geometrical system of three equations (25), (26), (27) for functions \( \Lambda, \rho \) and \( \phi \), which describe the effective 2-dimensional world sheet in terms of its Cartan moving reper, metrics and connection fields. The knowledge of these functions enables to restore the effective membrane’s world vector \( \mathbf{m} \) given by the expansion (13). The second statement is a partial encoding of the membrane dynamics by the \emph{generalized} Abel equation (26) with the \( t \) and \( \sigma^1 \) dependent coefficients. The appearance of Abel equation gives new insights to the membrane dynamics analysis. The third result of this geometric approach is the derivation of the \emph{linear} wave-like equation (24) associated with the \( U(1) \) membrane. The analysis based on this linear equation seems to be an alternative and promising. However, in this paper we focus our discussion on studying the system (25-27) including the Abel equation (26).

### 3 Exact solutions and the pendulum equation

Equations (26) and (27) may be easily solved for the case of \( \rho \) independent of time, i.e. \( \dot{\rho} = 0 \), because then the Abel equation (26) transforms into the
Riccati equation that has the general solution for $\lambda$ in the form

$$\lambda = -\rho \int dt \phi' + C_1(\sigma^1). \quad (28)$$

Then Eq. (27) shows that $\dot{\lambda} = 0$ which implies $\phi' = 0$, and $C_0(t) = 0$, $\dot{\phi} = \omega = const$ because of (18). As a result, we obtain

$$\phi(t) = \omega t + b, \quad \lambda = C_1(\sigma^1), \quad (\omega, b, \tilde{c} \in \mathbb{R}),$$

$$\rho^2(\sigma^1) = \pm 2 \int d\sigma^1 \sin C_1(\sigma^1) + \tilde{c}, \quad \rho \dot{C}_1 = -\omega. \quad (29)$$

The solution of Eqs. (29) gives for $\rho(\sigma^1)$ and $C_1(\sigma^1)$

$$\rho(\sigma^1)^2 = \frac{1}{\omega^2} - (\omega \sigma^1 + c)^2, \quad (30)$$

$$C_1(\sigma^1) = \pm \arcsin \sqrt{1 - (\omega \rho)^2}.$$

The solution (30) coincides with the solution (8) after using the periodicity condition. It proves that (8) is the general solution of the toric membrane equations in the case $\partial \mathbf{x}^2 / \partial t = 0$, i.e. $\mathbf{x}^2 = m^2 = \rho^2$ preserved in time.

The next solvable case of the Abel equation corresponds to $\rho$ independent of $\sigma^1$, i.e. $\rho' = 0$ that is equivalent to the case $\partial \mathbf{x}^2 / \partial \sigma = 0$. In this case Eq. (27) shows that $\lambda' = 0$, which implies $\dot{\phi} = 0$ because of Eq. (18), and $\phi' = const = \omega$ because of (26), so that

$$\phi = \omega \sigma^1 + a, \quad \Lambda(t) = \tan C_0(t), \quad (\omega, a \in \mathbb{R}). \quad (31)$$

As a result, Eqs. (26,27) take the form

$$\rho \dot{C}_0 = (\pm \sin C_0 - \omega \rho^2), \quad (32)$$

$$\rho = \pm \cos C_0. \quad (33)$$

The differentiation of Eq. (32) results in the pendulum equation for the function $C_0(t)$ coinciding with $\lambda$ (18)

$$\ddot{C}_0 \pm 2 \omega \cos C_0 = 0. \quad (34)$$

Using Eq. (33) one can present the pendulum equation (34) in the form of the total derivative

$$\frac{d}{dt}(\dot{C}_0 + 2 \omega \rho) = 0 \Rightarrow \dot{C}_0 + 2 \omega \rho = \ddot{a}. \quad (35)$$
Then the substitution of $\dot{C}_0$ (32) to (35) in combination with (33) gives the equation for the radial component $\rho(t)$ of the membrane world vector

$$\dot{\rho}^2 = 1 - \rho^2(\tilde{\alpha} - \omega \rho)^2.$$ (36)

The integration constant $\tilde{\alpha}$ can be chosen to be equal zero because for $\dot{\rho}(t_0) = 1$, corresponding to the membrane motion with the velocity of light ($\tilde{m}^2(t_0) = 1$ at the moment $t_0$), the radial component $\rho(t_0)$ vanishes, as it follows from Eqs. (32,33). The general solution of Eq. (36) with $\tilde{\alpha} = 0$ is presented by the elliptic integral

$$t = \int \frac{d\rho}{\sqrt{1 - \omega^2 \rho^4}} + \text{const.},$$ (37)

where the chosen sign plus in front of the integral correlates with the opposite sign in the solution $\Lambda = -\sqrt{1 - \dot{\rho}^2}$. The solution (37) coincides with the Jacobi elliptic solution (7), found in [10], and describes the toric membrane after taking into account the periodicity condition for $\sigma_1$ implying $\omega = n$.

We see that the geometric approach to the solution of the nonlinear system (5-6), reformulating it into the system of Eqs. (25-27), yields the same particular time dependent solutions as Eqs. (5-6) [10]. The coincidence of the geometrical approach, presented by the system (25-27), shows its equivalence to the standard description of $U(1)$ membranes.

### 4 On a general solution of the geometric approach equations

Having the elliptic cosine solution or different particular solutions for the case $\dot{\rho} \neq 0$, could we use them to construct the general solution of Eqs. (25-27)? We do not know the answer for this question for the time being. However, there is some information connected with such a possibility which comes from the generalized Abel equation (26). Studying this question in case of the Abel equation we find that a sufficient condition for a restoration of its general solution in terms of three known particular solutions is

$$\frac{\rho^2 \dot{\phi}}{\dot{\rho}^2} = \chi(\sigma^1), \quad \dot{\rho} \neq 0,$$ (38)

$^2$The Referee is acknowledged for sharpening this question.
where $\chi(\sigma^1)$ is an arbitrary function of the $\sigma^1$ membrane’s parameter. This observation follows from interesting properties of solutions of the original Abel equation of the first kind

$$\dot{y}(t) = f_3(t)y^3 + f_2(t)y^2 + f_1(t)y + f_0(t)$$

(39)

with variable coefficients $f_\nu(t)$, ($\nu = 0, 1, 2, 3$) depending on the argument $t$. Because the r.h.s. of (39) is a cubic polynomial, one can find its roots $y_i(t)$, depending on $f_\nu(t)$, and present (39) in the form

$$\dot{y} = f_3(y - y_1)(y - y_2)(y - y_3), \quad f_3(t) \neq 0.$$  

(40)

It is clear that the roots $y_i(t)$ are particular solutions of the Abel equation, if its variable coefficients $f_\nu(t)$ are connected by the conditions

$$\dot{y}_i(t) = 0$$  

(41)

which means that $y_i = \text{const}$. For such coefficients $f_\nu(t)$, the general solution of the Abel equation (39) is given by the expression

$$\prod_{i=1}^{3}(y - y_i)^{\alpha_i} = \kappa e^{\int dt f_3(t)},$$  

(42)

where $\kappa$ is an arbitrary integration constant and the constants $\alpha_i$ are defined by the roots $y_i$. We find the constants solving the set of algebraic equations

$$\alpha_1 + \alpha_2 + \alpha_3 = 0, \quad \alpha_1 y_2 y_3 + \alpha_2 y_3 y_1 + \alpha_3 y_1 y_2 = 1,$$

$$\alpha_1 (y_2 + y_3) + \alpha_2 (y_3 + y_1) + \alpha_3 (y_1 + y_2) = 0.$$  

(43)

One can try to use the representation (42) to construct the general solution of (26) despite the fact that its coefficients depend not only on $t$ but also on the membrane extendedness coordinate $\sigma^1$. A necessary condition to preserve the representation (42) is to treat the constants $\alpha_i$ as the functions $\alpha_i(\sigma^1)$ satisfying the same conditions (43). Moreover, Eqs. (43) for the coefficients in Eq. (26) have to be consistent with the remaining equations (25) and (27).

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3The case $\dot{\phi} = 0$ reduces the Abel equation to the Riccati equation with the general solution (28) considered in the previous section.

4Take into account misprints in the representation of the general solution in [24].
We observe that the two roots of the cubic polynomial in the r.h.s. of the Abel equation (26)

\[ P(\Lambda) = (1 + \Lambda^2) \left( \frac{\dot{\Lambda}}{\rho} - \rho \phi' \right) \]  

are the complex conjugate numbers

\[ \Lambda_{\pm} = \pm i \]  

which actually are particular solutions of (26), and they do not create any additional conditions on the functions \( \rho \) and \( \phi \). However, the third root \( \Lambda_3 \) of the polynomial \( P(\Lambda) \) (44)

\[ \Lambda_3 = \frac{\rho^2 \phi'}{\dot{\rho}}, \quad \dot{\rho} \neq 0 \]  

yields the new additional condition \( \dot{\Lambda}_3 = 0 \), connecting \( \rho \) and \( \phi \)

\[ \dot{\Lambda}_3 = \frac{\partial}{\partial t} \left( \frac{\rho^2 \phi'}{\dot{\rho}} \right) = 0 \quad \Rightarrow \quad \frac{\rho^2 \phi'}{\dot{\rho}} = \chi(\sigma^1), \]  

where the integration "constant" \( \chi(\sigma^1) \) is the previously mentioned function in (38). Because the fulfilment of the condition (47) is the necessary condition for the presentation of the general solution in the form (42), this condition has to be considered as a new constraint for \( \rho \) and \( \phi \) additional to Eqs. (25, 27).

Assuming that Eq. (47) is actually consistent with the remaining equations (25, 27), one can solve the algebraic system (43) and obtain

\[ \alpha_+ = \frac{i}{2(\Lambda_3 - i)}, \quad \alpha_- = (\alpha_+)^* = \frac{-i}{2(\Lambda_3 + i)}, \]

\[ \alpha_3 = - (\alpha_+ + \alpha_-) = \frac{1}{1 + \Lambda_3^2}. \]  

The substitution of the solution (48) in the representation (42) yields the desired expressions for the general solution \( \Lambda(t, \sigma^1) \) of the membrane Abel equation (26) in the complex form

\[ \frac{\Lambda - \Lambda_3}{\sqrt{1 + \Lambda^2}} \left( \frac{\Lambda + i}{\Lambda - i} \right)^{\frac{i\Lambda_3}{2}} = (\kappa \rho)^{(1 + \Lambda_3^2)} \]
where \( \kappa \) is now an arbitrary function of \( \sigma^1 \). Taking into account that the general physical solution implies the reality of \( \Lambda \) in (49) and using the relation

\[
\frac{\Lambda + i}{\Lambda - i} = e^{2i \arctan(\Lambda^{-1})}
\]

one can rewrite the expression (49) in the real form

\[
\frac{\Lambda - \Lambda_3}{\sqrt{1 + \Lambda^2}} = (\kappa \rho)^{(1 + \Lambda_3^2)} e^{-\Lambda_3 \arctan(\Lambda^{-1})},
\]

where \( \Lambda_3 = \frac{\rho^2 \dot{\phi}}{\rho} \) in accordance with (46). So, we obtain the desired general solution of Eq. (26), accompanying the remaining Eqs. (25, 27), in the form of the non-algebraic functional relation connecting the functions \( \Lambda, \rho \) and \( \phi \).

It proves that the general solution of the Abel equation (26) is expressed in terms of the three particular solutions, if the constraint (38) is satisfied.

An example of an explicit solution, encoded by the representation (51) and satisfying the constraint (38), is given by the choice

\[
\Lambda_3 = 0 \quad \rightarrow \quad \phi' = 0, \quad \chi = 0
\]

which reduces the implicit equation (51) to a simple explicit form

\[
\Lambda = \pm \frac{\kappa \rho}{\sqrt{1 - (\kappa \rho)^2}}.
\]

Unfortunately, the condition (38) is very bounding and excludes the elliptic solution (37). It follows from the constraint (38) matching with the conditions

\[
\dot{\rho} = \Lambda' = \dot{\phi} = 0,
\]

fixing the elliptic solution, that yields either solution (52) or the solution

\[
\frac{d(\rho^{-1})}{dt} = -\frac{\phi'}{\chi} = \text{const.},
\]

both of which are different from the elliptic cosine solution.

Thus, the problem appears in the realization of the formulated way to construct the general solution of the geometric system (25, 27). However, in this way we observe another interesting property of the Abel equation (26) which might be characterized as its duality with the Riccati equation. This property is discussed in the next section.
5 A duality between the Abel and Ricatti membrane equations

The observation is that Eq. (26), considered as the Abel equation for $\Lambda$ with the coefficients depending on $\rho$ and $\phi$, transforms to the Ricatti equation for the function $\rho$ with the coefficients depending on $\Lambda$ and $\phi'$

$$\dot{\rho} = \rho \Lambda^{-1} \left( \frac{\dot{\Lambda}}{\sqrt{1 + \Lambda^2}} + \rho \phi' \right). \quad (56)$$

After the substitution $r = 1/\rho$ the Ricatti equation (56) transforms into the linear differential equation (cp. (23))

$$\frac{\partial (r \sin \lambda)}{\partial t} = -\phi' \cos \lambda \quad (57)$$

and its general solution is obtained by the quadrature

$$r = \frac{1}{\sin \lambda} \left( \bar{C}(\sigma^1) - \int dt \phi' \cos \lambda \right), \quad (58)$$

where $\bar{C}(\sigma^1)$ is the integration constant. Thus, the general solution $\Lambda$ of the Abel equation studied above is encoded in the integral relation

$$\frac{\Lambda}{\rho \sqrt{1 + \Lambda^2}} = -\int dt \frac{\phi'}{\sqrt{1 + \Lambda^2}} + \bar{C}(\sigma^1) \quad (59)$$

instead of the discussed non-algebraic functional equation (51). The extension captures the whole space of the Abel equation solutions. One can see, e.g. that the particular solution (53) of Eq. (51) is extracted from the general solution (59) by the condition $\phi' = 0$ and the identification $\bar{C}(\sigma^1) = \kappa(\sigma^1)$. The desired elliptic solution (37) is respectively extracted by the conditions $\phi' = \omega$, $\Lambda = -\sqrt{1 - \dot{\rho}^2}/\dot{\rho}$, $\sqrt{1 + \Lambda^2} = 1/\dot{\rho}$, whose substitution into (59) transforms it into the equation (56)

$$\dot{\rho}^2 = 1 - \rho^2 (\bar{C} - \omega \rho)^2 \quad (60)$$

which results in the elliptic cosine solution (37) after identification $\bar{C} = \bar{a} = 0$.

Taking into account that the integral relation (59) may be considered as the general solution for the membrane length $\rho$, expressed in terms of $\lambda$...
and $\phi'$, one can present the geometric system (25-27) in the $\lambda$-representation ($\lambda = \text{arctan} \Lambda$) as

$$
(cos \lambda)' = \dot{\phi} \left( \dot{\bar{C}}(\sigma^1) - \int dt \phi' \cos \lambda \right),
$$

(61)

$$
\frac{1}{\rho} = \frac{1}{\sin \lambda} \left( \dot{\bar{C}}(\sigma^1) - \int dt \phi' \cos \lambda \right),
$$

(62)

$$
\left( \frac{\rho \rho'}{\sin \lambda} \right)^2 + \left( \frac{\dot{\rho}}{\cos \lambda} \right)^2 = 1.
$$

(63)

A new property of the representation is that Eq. (61) takes the form of the integro-differential equation for $\cos \lambda$, which defines $\lambda$ as a function of only the partial derivatives of $\phi$ and the integration constant $\dot{\bar{C}}(\sigma^1)$. The substitution of the supposed $\lambda$-solution into Eq. (62) will also express the length $\rho$ as a function of only the derivatives of $\phi$ and $\bar{C}$. The substitution of these $\lambda$ and $\rho$ to the last Eq. (63) will yield the equation for the last unknown function of the geometrical approach, the phase $\phi$ (cp. Eq. (24)).

The system (61-63) is a final point in our discussion of the geometric formulation in this paper. It leaves the problem of the general solution of the $U(1)$ membrane equations still open. Nevertheless, the system (61-63) seems a promising starting point for new attempts to solve the problem.

6 Conclusion

The investigation of classical and quantum dynamics of the non-linear sigma-models created a powerful group theory methods for their study [22, 23]. The Regge-Lund geometric approach [13, 14, 15], based on the classical theory of embedded surfaces [12], and its reformulation in terms of gauge fields [16, 17, 18, 19, 20, 21] were applied here in search of the general solution of the $U(1)$ membrane equations in the five dimensional Minkowski space. We revealed that the membrane nonlinearities are partially encoded by the cubic polynomial of the known Abel differential equation of the first type, but with the variable coefficients depending on the membrane’s extendedness parameter $\sigma^1$ in addition to the original time dependence. We found that the cases when the radial component $\rho$ of the membrane world vector depends only on the time $t$ or only on the extendedness parameter $\sigma^1$, the membrane equations are exactly solvable. The obtained solutions are proven to be the
general ones for such types of the $\rho$ dependencies, and they coincide with the previously found solutions [10]. For the case of $\rho$ dependent only on time the dynamics of membranes turns out to be connected with the well known nonlinear differential pendulum equation.

In search of an alternative approach to the solution of the $U(1)$ membrane equations we also derived a 2-dim linear hyperbolic equation of the second order (for a phase $\phi$) with variable coefficients, depending on the membrane length $\rho$ and its partial derivatives $\rho', \dot{\rho}$. The linear character of this equation allows us to apply the general methods of the PDE theory to seek the desired general solution.

Some attempts were undertaken here to find the general solution of the whole system of nonlinear equations in the considered geometric approach. The sufficient criteria to restore the general solution of the membrane Abel equation in terms of its three particular solutions was formulated. Using this criteria we present the general solution of the membrane Abel equation in the form of a non-algebraic functional equation, connecting the geometric approach functions. However, this criteria forbids the elliptic solution of the Abel equation. Then we observed that the Abel equation, interpreted as an equation for the membrane length, coincides with the Ricatti equation and we found its general solution by quadrature. The general solution of the Ricatti equation encodes the whole space of the Abel equation solutions. It seems that the derivation of the integral representation for the membrane length partially simplifies the whole system of membrane equations and gives new tools to solve the problem. Thus, the presented geometrical reformulation of the $U(1)$ membrane mechanics gives a new promising information which will help in the search for its solvability.

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