Multi-object searching algorithm using subgrouped oracles

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We present in this work, if a set of well organized suboracles is available, a quantum algorithm for multiobject search with certainty in an unsorted database of \( N \) items. Depending on the number of the objects, the technique of phase tuning is included in the algorithm. If one single object is to be searched, this algorithm performs a factor of two improvement over the best algorithm for a classical sorted database. While if the number of the objects is larger than one, the algorithm requires slightly less than \( \log_4 N \) queries, but no classical counterpart exists since the resulting state is a superposition of the marked states.

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Quantum mechanical algorithms are much more powerful than their classical counterparts because they are quantum physical processes possessing the unique feature of quantum parallelism due to superpositions and entanglements of quantum states. The quantum parallelism potentially can bring about an exponential speedup in running time over the classical computation processing. A typical example is the factorizing algorithm discovered by Shor\textsuperscript{[1]} . The running time required in Shor’s algorithm is proportional to \( O(n^3) \) for the factorization of a number registered in \( n \) bits, while the running time for the classical algorithm would take \( O(\exp(n^{1/3})) \). This is the first quantum algorithm that reduces a classical NP problem to a P problem, showing a quantum computer, if it can ever be built, can transcend any classical one. One the other hand, Grover\textsuperscript{[2,3]} proposed the quantum searching algorithm, which performs a quadratic speedup over its classical counterpart. If there is an unsorted database of \( N \) items, and out of which only one marked item satisfies a given condition, then using Grover’s algorithm one will find the object in \( O(\sqrt{N}) \) quantum mechanical steps, instead of \( O(N) \) classical steps. The Grover algorithm can be extended to find \( M \) objects out of a database of size \( N \). It would then required \( O(\sqrt{N}/M) \) iterations to search the superposition of these \( M \) marked states. The superposition of the marked states is an unique feature that only quantum algorithm can search and therefore no classical algorithms can ever do. It has been shown that Grover’s original algorithm is optimal\textsuperscript{[1,2,4]}, in the sense it requires the minimal queries to undertake the search task. Although the quantum searching algorithm did not provide an exponential speedup, it indeed greatly improved in reducing the exhausting effort needed in the classical counterpart. Up to date the searching problem remains an NP problem.

Grover’s algorithm utilizes a designed unitary operator, the Grover operator, successively on a uniform superposition of all the possible states to enhance the probability of the marked state. The Grover operator is a product of two operators with specific functions, namely, the selective inversion operator \( G_\tau \) and the inverse-about-average operator \(-G_\tau\). Expressed mathematically, the Grover operator is given by \( G = -G_\tau G_\tau \). Iterating \( G \) about \( (\pi/4)\sqrt{N} \) times on the initial state and then measuring, one will obtain the marked state. Alternatively, the searching problem can be rephrased in terms of the action of an oracle. To carry out the search, one requires an oracle \( f(x) \) with output 0 or 1 and, during the processing, is assigned to find a specific marked item \( \tau \) corresponding to the query \( f(x) = 1 \), for \( x = \tau \) and \( f(x) = 0 \), for \( x \neq \tau \). In the Grover algorithm each action of \( G_\tau \) needs one oracle call, so the quantum searching task consume at least \( (\pi/4)\sqrt{N} \) queries.

Instead of using one single oracle in Grover’s algorithm for the search in an unsorted database, Patel\textsuperscript{[5]} recently suggested that if a set of factorized oracles is available, one can locate a desired item in an unsorted database using \( O(\log_4 N) \) queries, which is a factor of two improvement over the best search algorithm for a classical sorted database. Patel’s scenario, in terms of the action of oracle calls, is to search a marked state

\[
|\tau\rangle = |(\tau^{(n/2)}) \rangle \otimes ... \otimes |(\tau^{(2)}) \rangle \otimes |(\tau^{(1)}) \rangle
\]

(1)

satisfying the factorizable function

\[
f(\tau) = f_{n/2}(\tau^{(n/2)})...f_2(\tau^{(2)})f_1(\tau^{(1)}) = 1,
\]

(2)

provided that the state \( |\tau\rangle \) is registered in \( n(= \log_2 N) \) qubits while each of the factorized state \( |\tau^{(j)}\rangle \), \( j = 1, ..., n/2 \), is in two qubits. The outline of this factorized search algorithm is arranged to successively find the factorized state \( |\tau^{(j)}\rangle \) in the corresponding four-dimensional subspace in a single iteration according to the corresponding factorized oracle \( f_j(x), x \in \{0,1\}^2 \), so the total queries are \( n/2 \). The factorized search algorithm is successful in searching one single marked state since the single state is natively factorizable. It turns out to be, however, that this algorithm will not be applicable to a multi-solution searching problem. As mentioned, if \( M \) marked states are to be searched, a quantum algorithm is designed to enhance the probability amplitude of the superposition of these objects. The superposition is an
entangled state, so the factorized search algorithm developed by Patel fails to solve the multi-solution searching problem. In this work, we are thus motivated to show how to search the superposition of the M marked states in $O(\log_2 N)$ iterations provided that a specific set of subgrouped oracles is available.

Suppose we have an unsorted database of $N$ items at hand and according to its size we need $n = \log_2 N$ qubits to register the state in the database. The $n$ qubits are divided into $\eta + 1$ subgroups in which all are of two qubits except one $n_0$-qubit subgroup, so $n = 2\eta + n_0$, where $n_0$ is chosen depending on the number of the desired items $M$ and will be defined in what follows. The marked states then can be expressed as

$$|\tau_j\rangle = \left| \tau_j^{(1)} \cdots \tau_j^{(2)} \tau_j^{(1)} \rightangle,$$

where

$$\tau_j^{(1)} \in \{1,0\}^{n_0}, \quad n_0 = \lfloor \log_2 (4M) \rfloor \geq 2,$$

(4)

$$\tau_j^{(k)} \in \{1,0\}^2, \quad k = 2, \ldots, \eta + 1.$$  

(5)

The ”floor” symbol $\lfloor \rfloor$ in (4) denotes the largest integer smaller than the value in it. Although the ordering of the $n_0$-qubit factorized states can be arbitrary specified, we have designated them as $\tau_j^{(1)}$ in the first subspace for convenience. As mentioned, our purpose is to outlet the superposition of the marked states, namely,

$$|\tau\rangle = \frac{1}{\sqrt{M}} \sum_{j=1}^{M} |\tau_j\rangle = \frac{1}{\sqrt{M}} \sum_{j=1}^{M} \left| \tau_j^{(1)} \cdots \tau_j^{(2)} \tau_j^{(1)} \right\rangle.$$  

(6)

The initial state is prepared in the uniform superposition of all the possible states, given by

$$|s\rangle = W_n |\bar{0}\rangle = \frac{1}{\sqrt{N}} \sum_{y \in \{0,1\}^n} |y\rangle,$$

(7)

where $W_n$ is the $n$-qubit Walsh-Hadamard transformation and $|\bar{0}\rangle$ is the standard state with 0 in every qubit.

By expression (3) we realize that the global, $2^n$-dimensional Hilbert space $H$ for the search is the tensor product of $\eta + 1$ subspaces, expressed as $H = H_2^{\otimes \eta} \otimes H_{n_0}$, where the subspace $H_{n_0}$ is $2^{n_0}$-dimensional and the subspace $H_2$ is four-dimensional. The scenario of the algorithm contains $\eta + 1$ sequential stages. It begins with finding the $M$ factorized states $|\tau_j^{(1)}\rangle$ in the subspace $H_{n_0}$ in just one searching step. Knowing that Grover’s algorithm promises to search $M$ objects with certainty in a database containing $4M$ items by applying the Grover operator once, we therefore choose $n_0 = \log_2 (4M)$ for the first subgrouped qubits. Even if $\log_2 (4M)$ is not an integer, we can also search the objects with certainty in just one searching step by choosing $n_0 = |\log_2 (4M)|$ and using the technique of phase tuning [3]. In the $2^{n_0}$-dimensional subspace $H_{n_0}$, we apply the subgrouped Grover operator $G_1$, given by

$$G_1 = -(I + (e^{i\phi} - 1) |s\rangle\langle s|) (I + (e^{i\phi} - 1) |\tau\rangle\langle \tau|),$$

(8)

where

$$|\tau\rangle = \frac{1}{\sqrt{\eta}} \sum_{j=1}^{\eta} |\tau_j\rangle,$$

(9)

and the phase angle, according to Hsieh and Li [3], is given by

$$\phi = \phi(M) = 2 \sin^{-1}\left(\frac{2^{n_0}}{4M}\right).$$

(10)

Notice that if $\log_2 (4M)$ is an integer, $\phi = \pi$ and $G_1$ reduces to the original Grover operator defined in $H_{n_0}$, while if $\log_2 (4M)$ is not an integer, the choice $n_0 = |\log_2 (4M)|$ leads to the smallest deviation of the phase $\phi$ form $\pi$. The oracle used in the first subgrouped Grover operator $G_1$ should be

$$f_1(y) = 1, \quad y = \tau_j^{(1)}, \quad j = 1, \ldots, M,$$

(11)

$$f_1(y) = 0, \quad y \neq \tau_j^{(1)}, \quad j = 1, \ldots, M.$$  

In the first stage the searching operator that we apply on the initial state $|s\rangle$ is

$$S_1 = I_2^{\otimes \eta} \otimes G_1,$$

(12)

where $I_2$ denotes the identity operator in the subspace $H_2$. The outlet at the end of this stage then becomes

$$|t_1\rangle = S_1 |s\rangle = \left( \frac{1}{\sqrt{2^{n_0-n_0}}} \sum_{y \in \{0,1\}^{n-n_0}} |y\rangle \right) \otimes |\tau^{(1)}\rangle.$$  

(13)

The outlet state $|t_1\rangle$ can be rewritten, for convenience in the depiction of the next stage,

$$|t_1\rangle = \left( \frac{1}{\sqrt{2^{n_0-n_0-2}}} \sum_{y \in \{0,1\}^{n-n_0-2}} |y\rangle \right) \otimes |\sigma^{(2)}\rangle,$$  

(14)

where

$$|\sigma^{(2)}\rangle = \frac{1}{\sqrt{4M}} \sum_{j=1}^{M} \sum_{x \in \{0,1\}^2} |x\rangle \otimes |\tau_j^{(1)}\rangle.$$  

(15)
is located in the subspace $\mathcal{H}_2 \otimes \mathcal{H}_{n_0}$. Now, observing that in the subspace $\mathcal{H}_2 \otimes \mathcal{H}_{n_0}$ the state $|s^{(2)}\rangle$ is a superposition of $4M$ nonvanishing orthonormal states $|x^{(1)}_j\rangle$, $x \in \{0,1\}^2$, $j = 1,\ldots,M$, and in them the $M$ factorized marked states $|r^{(2)}_j\rangle$ are embedded. We then once again can apply a second subgrouped Grover operator $G_2$ to search these factorized marked states with certainty in just one searching step. The second subgrouped Grover operator $G_2$ is defined in the subspace $\mathcal{H}_2 \otimes \mathcal{H}_{n_0}$, given by

$$G_2 = -(I - 2|s^{(2)}\rangle \langle s^{(2)}|)(I - 2|\tau^{(2)}\rangle \langle \tau^{(2)}|), \quad (16)$$

where

$$|\tau^{(2)}\rangle = \frac{1}{\sqrt{M}} \sum_{j=1}^{M} |r^{(2)}_j\rangle |r^{(1)}_j\rangle. \quad (17)$$

The oracle implemented in $G_2$ is of course defined by

$$f_2(y) = 1, \quad y = r^{(2)}_j, \quad j = 1,\ldots,M, \quad (18)$$

$$f_2(y) = 0, \quad y \neq r^{(2)}_j, \quad j = 1,\ldots,M.$$ 

The searching operator that we apply in the second stage then is

$$S_2 = I_2^{(n-1)} \otimes G_2, \quad (19)$$

and the outlet at the end of the second stage is

$$|t_2\rangle = S_2 |t_1\rangle = S_2 S_1 |s\rangle \quad (20)$$

$$= \frac{1}{\sqrt{2^{n-n_0-2}}} \sum_{y \in \{0,1\}^{n-n_0-2}} |y\rangle \otimes |\tau^{(2)}\rangle$$

$$= \frac{1}{\sqrt{2^{n-n_0-4}}} \sum_{y \in \{0,1\}^{n-n_0-4}} |y\rangle \otimes |s^{(3)}\rangle,$$

where the state $|s^{(3)}\rangle$ is written

$$|s^{(3)}\rangle = \frac{1}{\sqrt{4M}} \sum_{j=1}^{M} \left( \sum_{y \in \{0,1\}^{2}} |x\rangle \right) \otimes |r^{(2)}_j\rangle |r^{(1)}_j\rangle, \quad (21)$$

which locates in the third subspace $\mathcal{H}_2^{(n-2)} \otimes \mathcal{H}_{n_0}$ and is also a superposition of the $4M$ nonvanishing orthonormal states $|x^{(2)}_j r^{(1)}_j\rangle$, $x \in \{0,1\}^2$, $j = 1,\ldots,M$, with the $M$ factorized marked states $|r^{(3)}_j r^{(2)}_j r^{(1)}_j\rangle$ embedded in them. So the scenario obviously can proceed straightforward to the final stage following the same way as depicted from the first to the second stage. That is, in a next stage we add two more qubits to the former subgrouped qubits, so a next subspace is the one enlarged by multiplying the dimensions of the former subspace by four, and apply a searching operator like those given by (12) and (19) and a corresponding subgrouped oracle like those expressed in (11) and (18).

As a result, we can thus derive the general formula for the present searching algorithm. Excluding the first stage, where as we can apply expressions (8)-(13), in the $k$-th stage, $2 \leq k \leq \eta + 1$, the searching operator is

$$S_k = I_2^{(\eta-k+1)} \otimes G_k, \quad (22)$$

and the $k$-th subgrouped Grover operator $G_k$ is defined in the subspace $\mathcal{H}_2^{(k-1)} \otimes \mathcal{H}_{n_0}$, given by

$$G_k = -(I - 2|s^{(k)}\rangle \langle s^{(k)}|)(I - 2|\tau^{(k)}\rangle \langle \tau^{(k)}|), \quad (23)$$

where

$$|\tau^{(k)}\rangle = \frac{1}{\sqrt{M}} \sum_{j=1}^{M} |r^{(k)}_j r^{(2)}_j r^{(1)}_j\rangle, \quad (24)$$

and

$$|s^{(k)}\rangle = \frac{1}{\sqrt{4M}} \sum_{j=1}^{M} \left( \sum_{y \in \{0,1\}^{2}} |x\rangle \right) \otimes |r^{(k-1)}_j r^{(2)}_j r^{(1)}_j\rangle. \quad (25)$$

The corresponding subgrouped oracle required in the $k$-th stage is defined by

$$f_k(y) = 1, \quad y = r^{(k)}_j r^{(2)}_j r^{(1)}_j, \quad j = 1,\ldots,M, \quad (26)$$

$$f_k(y) = 0, \quad y \neq r^{(k)}_j r^{(2)}_j r^{(1)}_j, \quad j = 1,\ldots,M.$$ 

The outlet state at the end of the $k$-th stage then is

$$|t_k\rangle = S_k |t_{k-1}\rangle = S_k \ldots S_2 S_1 |s\rangle \quad (27)$$

$$= \left( \frac{1}{\sqrt{2^{n-n_0-2(k-1)}}} \sum_{y \in \{0,1\}^{n-n_0-2(k-1)}} |y\rangle \right) \otimes |\tau^{(k)}\rangle.$$ 

As $k = \eta + 1$, i.e., as we arrive at the final stage, we will reach the final state at the end,

$$|t_{\eta+1}\rangle = S_{\eta+1} |t_{\eta}\rangle = S_{\eta+1} \ldots S_2 S_1 |s\rangle \quad (28)$$

$$= |\tau^{(\eta+1)}\rangle = \frac{1}{\sqrt{M}} \sum_{j=1}^{M} |r^{(\eta+1)}_j r^{(2)}_j r^{(1)}_j\rangle$$

$$= |\tau\rangle,$$

which is exactly the object that we search for. It should be mentioned that a projected / measurement operator, although not shown, has been combined within the
searching operator used in each of the stages to make sure the eliminated states do not take part in the subsequent stages of the algorithm. To summarize, totally $\eta + 1$ queries are required in our subgrouped searching algorithm since in each stage of the scenario only one query is utilized. The number $\eta + 1$, being equal to $(n - n_0 + 2)/2$, is in fact smaller than $n/2$ for $M > 1$. As $M = 1$, our algorithm will reduce to the factorized algorithm presented by Patel. A circuit representation of the present algorithm is shown in Fig. 1 to schematically depict the whole scenario.

In this work, we have shown a subgrouped algorithm using only $O(\log_4 N)$ queries in searching $M$ objects out of an unsorted database of $N$ items, if in advance a set of subgrouped oracles is available. The subgrouped oracles required in the algorithm are expressed in (11) and (26), while the searching operator are defined by (12) and (22), with the aides of (8)-(10) and (23)-(25), respectively. The first searching operator $S_1$ is more particular than the others because in it the technique of phase tuning is necessary for the action of the subgrouped Grover operator $G_1$ especially when $\log_2(4M)$ is not an integer. Using this subgrouped searching algorithm, one will require $O(\log_4 N)$ queries in the search of multi-object. The resulting state at the end of the algorithm is the superposition of the $M$ marked states, which is an unique feature of the quantum algorithm and can not be performed by any classical algorithm. Difficulties, however, may arise from how to physically implement the black boxes associated with the subgrouped oracles, but these are not of concern in the complexity analysis of this work and should be studied in a further work.

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