Diptych varieties. II: Polar varieties

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Abstract

Part I introduced diptych varieties $V_{ABLM}$ and gave a rigorous construction of them in the case $d, e \geq 2$ and $de > 4$. Here we prove the existence of $V_{ABLM}$ in all the cases with $de \leq 4$. At the same time we construct some classes of interesting quasihomogeneous spaces for groups such as $GL(2) \times \mathbb{G}_m^n$ based on the algebra of polars.

In [BR1] we introduced diptych varieties. Each is an affine 6-fold $V_{ABLM}$ arising as a 4-parameter deformation of a reducible Gorenstein toric surface $T = S_0 \cup S_1 \cup S_2 \cup S_3$ that is a cycle of four affine toric components meeting along their 1-dimensional strata, with the four deformation parameters smoothing the four axes of transverse intersections of the cycle. A diptych variety is characterised by three natural numbers $d, e, k$, that arise from the underlying combinatorics as a 2-step recurrent continued fraction $[d, e, d, \ldots]$ to $k$ terms. [BR1], Theorem 1.1 asserts that a diptych variety exists for any choice of $d, e, k$ (with bounds on $k$ when $de \leq 3$), and proves it for $de > 4$ with $d, e \geq 2$. The cases $de > 4$ with $d$ or $e = 1$ are treated by similar methods in [BR3]. This paper constructs diptych varieties in the remaining cases $de \leq 4$, fulfilling the promise of [BR1], Theorem 1.1.

The diptych varieties with $de = 4$ have a beautiful description in terms of key 5-folds $V(k) \subset \mathbb{A}^{k+5}$, that play a principal role in this paper (see Section 1 and especially 1.3). These are quasihomogeneous spaces that are easy to describe based on the algebra of polars, and we offer several alternative approaches. With a final unprojection argument, any of these descriptions is enough to prove the existence of diptych varieties with $de = 4$.

Geometrically, the $V(k)$ are quasihomogeneous spaces for $G = GL(2) \times \mathbb{G}_m$; they are the closure of orbits of a ‘polar’ vector in a reducible representation of $G$, and we refer to them as polar varieties, as yet with no formal definition, but see 1.3. Other diptych varieties also have more symmetry along these lines, and it is an interesting problem to know how far this will go. We would also like to know whether polar varieties such as the $V(k)$ and the $W(d)$ introduced in 3.1 arise naturally in other parts of geometry and representation theory – we see similar phenomena in other calculations in codimension $\geq 4$, and this type of polar geometry should apply more widely.

From the point of view of equations, we express the $V(k)$ using a generalised form of Cramer’s rule. This provides all the equations of $V(k)$ in closed form, in contrast to the
small subset of Pfaffian equations that we get away with in [BR1]. The varieties $V(k)$ are serial unprojections, although that description does not directly provide all the equations.

Section 3 introduces another series of polar varieties, this time quasihomogeneous 7-folds $W(d) \subset \mathbb{A}^{d+9}$, and applies them as models for diptychs with $k = 2$. With a single additional unprojection, they also provide a format for diptychs with $k = 3$ and $e = 1$ involving crazy Pfaffians, reminiscent of Riemenschneider’s ‘quasi-determinants’ [R]; see 3.2 where we discuss the equations in terms of floating factors. Section 4 handles the few remaining cases with $k = 4, 5$ and $de = 3$, where unprojection methods and pentagrams provide the equations directly. Rather than the polar varieties $V(k)$ and $W(d)$, given by serial unprojection, these case are most naturally described as regular pullbacks from a parallel unprojection key variety, a 10-fold $W \subset \mathbb{A}^{16}$.

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1 The polar variety $V(k)$

Gorenstein rings arise naturally in geometry as homogeneous coordinate rings of Fanos, Calabi–Yaus, regular canonical $n$-folds, and other constructions. Thus a supply of model Gorenstein rings, with explicit information about their relations, gradings and so on, is of practical importance. It is hard to construct Gorenstein rings in high codimension in general; there is no classification beyond codimension 3. Grojnowski and Corti and Reid [CR] study quasihomogeneous spaces or closed orbits in highest weight representations of
semisimple algebraic groups, in particular for SL(5) and SO(5, 10); Qureshi and Szendrői [QS] generalise these to more classes of examples.

We describe an infinite family of irreducible Gorenstein 5-folds $V(k) \subset \mathbb{A}^{k+5}$ that are quasihomogeneous spaces under $\text{GL}(2) \times \mathbb{G}_m$ and provide several different constructions of them. Our application to diptych varieties in Section 2 is itself applied in [BR4] to the equations of 3-fold Mori flips, which also have Gorenstein total coordinate rings. Here we treat the $V(k)$ as varieties in their own right from several different points of view.

1.1 The definition by equations

We define 5-folds $V(k) \subset \mathbb{A}^{k+5}$ for each $k \geq 3$. First set up $2 \times k$ and $k \times (k - 2)$ matrices

\[
M = \begin{pmatrix}
x_0 & \ldots & x_{i-1} & \ldots & x_{k-1} \\
x_1 & \ldots & x_i & \ldots & x_k
\end{pmatrix} \quad \text{and} \quad
N = \begin{pmatrix}
a & a & a \\
b & a & a \\
c & b & a \\
\vdots & \vdots & \vdots \\
c & b & a \\
c & b & c
\end{pmatrix}
\]

Our variety $V(k) \subset \mathbb{A}^{k+5}$ is defined by two sets of equations:

(I) $MN = 0 \quad \text{and} \quad \text{(II)} \quad \bigwedge^2 M = z \cdot \bigwedge^{k-2} N. \quad (1.1)$

(I) is a recurrence relation

\[ax_{i-1} + bx_i + cx_{i+1} = 0 \quad \text{for } i = 1, \ldots, k - 1. \quad (1.2)\]

(II) is a $(k - 2) \times k$ adaptation of Cramer’s rule giving the Plücker coordinates of the space of solutions of (I) up to a scalar factor $z$. The order and signs of the minors in (II) is not a problem here, as one sees from the guiding cases

\[x_{i-1}x_{i+1} - x_i^2 = a^{i-1}c^{k-i-1}z \quad \text{and} \quad x_{i-1}x_{i+2} - x_ix_{i+1} = a^{i-1}bc^{k-i-2}z.\]

(However, in subsequent cases, in particular when we work with Pfaffians in [12], we need to fix a convention on their order and signs.) Note that the maximal $(k - 2) \times (k - 2)$ minors of $N$ include $a^{k-2}$ (delete the last two rows) and $c^{k-2}$ (delete the first two). More generally, deleting two adjacent rows $i-1, i$ gives $a^{i-1}c^{k-i-1}$ as a minor (only the diagonal contributes), whereas deleting two rows $i-1, i+1$ gives the minor $a^{i-1}bc^{k-i-2}$.

Thus our second set of equations is

\[x_{i-1}x_{j+1} - x_ix_j = z \det N(i - 1, j).\]
Relations for $x_i x_j - x_k x_l$ for all $i + j = k + l$ are obtained as combinations of these; for example
\[
x_{i-1} x_{j+2} - x_{i+1} x_j = x_{i-1} x_{j+2} - x_i x_{j+1} + x_i x_{j+1} - x_{i+1} x_j
\]
\[
= zN(i - 1, j + 1) + zN(i, j).
\]

**Theorem 1.1** For $k \geq 3$, (I) and (II) define a reduced irreducible Gorenstein 5-fold
\[
V(k) \subset \mathbb{A}^{k+5}_{(x_0, \ldots, x_k, a, b, c, z)}.
\]

This also holds for $k = 2$, with (II) involving the $0 \times 0$ minors interpreted as the single equation $1 \cdot z = x_0 x_2 - x_1^2$.

This theorem follows at once from the following lemma.

**Lemma 1.2**
(i) $z$ is a regular element for $V(k)$.

(ii) The section $z = 0$ of $V(k)$ is the quotient of the hypersurface
\[
\widetilde{W} : (g := au^2 + buv + cv^2 = 0) \subset \mathbb{A}^5_{(a, b, c, u, v)}
\]
by the $\mu_k$ action $\frac{1}{k}(0, 0, 0, 1, 1)$. It is Gorenstein because
\[
\frac{da \wedge db \wedge dc \wedge du \wedge dv}{g} \in \omega_{\mathbb{A}^5}(\widetilde{W}).
\]
is $\mu_k$ invariant.

(iii) Also $z, a, c$ is a regular sequence, and the section $z = a = c = 0$ of $V(k)$ is the toric Gorenstein surface (three-sided tent) consisting of $\frac{1}{k}(1, 1)$ with coordinates $x_0, \ldots, x_k$ and two copies of $\mathbb{A}^2$ with coordinates $x_0, b$ and $x_k, b$.

**Proof** First, if $c \neq 0$ then $a, b, c, x_0, x_1$ are free parameters, and the recurrence relation (I) gives $x_2, \ldots, x_k$ as rational function of these. One checks that the first equation in (II) gives $z = \frac{ax_0^2 + bx_0x_1 + cx_1^2}{c}$ and the remainder follow. Similarly if $a \neq 0$.

If $a = c = 0$ and $b \neq 0$ then one checks that $x_0, x_k, b$ are free parameters, $x_i = 0$ for $i = 1, \ldots, k - 1$ and $z = \frac{x_0 x_k}{x_1^2}$. Finally, if $a = b = c = 0$ then $x_0, \ldots, x_k$ and $z$ obviously parametrise $\frac{1}{k}(1, 1) \times \mathbb{A}^1$.

Therefore, no component of $V(k)$ is contained in $z = 0$, which proves (i).

After we set $z = 0$, the equations (II) become $\bigwedge^2 M = 0$, and define the cyclic quotient singularity $\frac{1}{k}(1, 1)$ (the cone over the rational normal curve). Introducing $u, v$ as the roots of $x_0, \ldots, x_k$, with $x_i = u^{k-i} v^i$, boils the equations $MN = 0$ down to the single equation $g := au^2 + buv + cv^2 = 0$. This proves (ii). (iii) is easy.
Alternative proof of Theorem 1  We can start with any of the codimension 2 complete intersections
\[
\begin{aligned}
(x_{i-1}x_{i+1} &= x_i^2 + a^{i-1}c^{k-i-2} \\
ax_{i-1} + bx_i + cx_{i+1} &= 0)
\end{aligned}
\]⊂ \mathbb{A}^7_{\langle x_{i-1}, x_i, x_{i+1}, a, b, c, z \rangle}
\]and add the remaining variables one at a time by unprojection. This is now a standard application of serial unprojection (see [PR], [Ki], [TJ] and [BR1]) and we omit the details.

1.2 The equations as Pfaffians

The equations of \(V(k)\) fit together as \(4 \times 4\) Pfaffians of a skew matrix. For this, edit \(M\) and \(N\) to get two new matrixes,
\[
M' = \begin{pmatrix}
x_0 & \cdots & x_{i-1} & x_i & \cdots & x_{k-2} \\
x_1 & \cdots & x_i & x_{i+1} & \cdots & x_{k-1} \\
x_2 & \cdots & x_{i+1} & x_{i+2} & \cdots & x_k
\end{pmatrix}
\]which is \(3 \times (k - 1)\) and \(N'\), the \((k - 1) \times (k - 3)\) matrix with the same display as \(N\) (that is, delete the first (or last) row and column of \(N\)). Equations (I) can be rewritten \((a, b, c)M' = 0\).

Now all of the equations (1.1) can be written as the \(4 \times 4\) Pfaffians of the \((k+2) \times (k+2)\) skew matrix
\[
\begin{pmatrix}
c & -b \\
a & M' \\
z & \wedge^{k-3} N'
\end{pmatrix}
\]The Pfaffians Pf\(_{12,3(i+3)}\) give the recurrence relation (1.2), while the remaining Pfaffians give (II). In more detail, the big matrix is
\[
\begin{pmatrix}
c & -b & x_0 & \cdots & x_{i-1} & x_i & \cdots & x_{k-2} \\
a & x_1 & \cdots & x_i & x_{i+1} & \cdots & x_{k-1} \\
x_2 & \cdots & x_{i+1} & x_{i+2} & \cdots & x_k \\
z c^{k-3} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
z c^{k-i-2}a^{i-2} & -zbc^{k-i-2}a^{i-2} & \cdots & \cdots & \cdots & \cdots & \cdots \\
z c^{k-i-2}a^{i-1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
z a^{k-3}
\end{pmatrix}
\]with bottom right \((k - 1) \times (k - 1)\) block equal the \((k - 3)rd\) wedge of \(N'\) (with signs).
Small values of $k$ Our family starts with $k \geq 3$; the case $k = 2$ would give the hypersurface $ax_0 + bx_1 + cx_2 = 0$, with $z := x_0x_2 - x_1^2$. The first regular case is $k = 3$, which gives the $5 \times 5$ skew determinantal

$$
\begin{pmatrix}
c & -b & x_0 & x_1 \\
a & x_1 & x_2 \\
x_2 & x_3 \\
z
\end{pmatrix}
$$

a regular section of the affine Grassmannian $\text{aGr}(2, 5)$. The case $k = 4$ is

$$
\begin{pmatrix}
c & -b & x_0 & x_1 & x_2 \\
a & x_1 & x_2 & x_3 \\
x_2 & x_3 & x_4 \\
zc & -zb & za
\end{pmatrix},
$$

an easy case of the standard extrasymmetric $6 \times 6$ determinantal of Dicks and Reid, [TJ], 9.1, equation (9.4).

The first really new case is $k = 5$, with equations the $4 \times 4$ Pfaffians of the $7 \times 7$ skew matrix

$$
\begin{pmatrix}
c & -b & x_0 & x_1 & x_2 & x_3 & x_4 \\
a & x_1 & x_2 & x_3 & x_4 & x_5 \\
zc^2 & -zb & z(b^2 - ac) \\
zc & -za & za^2
\end{pmatrix}
$$

We first arrived at this matrix by guesswork (with the $z$ floated over), determining the superdiagonal entries $c^2, ac, a^2$ and those immediately above $-bc, -ac$ by eliminating variables to smaller cases; the entry $b^2 - ac$ is then fixed so that the bottom $4 \times 4$ Pfaffian vanishes identically.

1.3 The variety $V(k)$ by apolarity

We can treat $V(k)$ as an almost homogeneous space under $\text{GL}(2) \times \mathbb{G}_m$. For this, view $x_0, \ldots, x_k$ as coefficients of a binary form and $a, b, c$ as coefficients of a binary quadratic form in dual variables, so that the equations $MN = 0$ or $(a, b, c)M' = 0$ are the apolarity relations. In general terms, polarity can be described as a choice of splitting of maps such as $\text{Sym}^{d-1} U \otimes U \rightarrow \text{Sym}^d U$ (here $U = \mathbb{C}^2$ is the given representation of $\text{GL}(2)$), or more
vaguely as a way of viewing the $2 \times d$ matrix $(y_0 \ldots y_{d-1})$ or his bigger cousin (1.3) as a single object in determinantal constructions.

More formally, write

$$q = a\check{u}^2 + 2b\check{u}\check{v} + c\check{v}^2 \in \text{Sym}^2 U^\vee$$

and

$$f = x_0u^k + kx_1u^{k-1}v + \binom{k}{2}x_2u^{k-2}v^2 + \cdots + x_kv^k \in \text{Sym}^k U.$$ 

Including the factor $\binom{k}{i}$ in the coefficient of $u^iv^{k-i}$ is a standard move in this game.

The second polar of $f$ is the polynomial

$$\Phi(u, v, u', v') = \frac{1}{k(k-1)} \left( \frac{\partial^2 f}{\partial u^2} \otimes u'^2 + 2 \frac{\partial^2 f}{\partial u \partial v} \otimes u'v' + \frac{\partial^2 f}{\partial v^2} \otimes v'^2 \right)$$

$$= \sum_{i=0}^{k-2} \binom{k-2}{i}x_iu^{k-i-2}v^i \otimes u'^2 + 2 \sum_{i=1}^{k-1} \binom{k-2}{i-1}x_iu^{k-i-1}v^{i-1} \otimes u'v'$$

$$+ \sum_{i=2}^{k} \binom{k-2}{i-2}x_iu^{k-i}v^{i-2} \otimes v'^2$$

$$= \sum_{i=0}^{k-2} \binom{k-2}{i}u^{k-2-i}v^i \otimes (x_iu'^2 + 2x_{i+1}u'v' + x_{i+2}v'^2) \in \text{Sym}^{d-2} U \otimes \text{Sym}^2 U.$$ 

We apply $q \in \text{Sym}^2 U^\vee$ to the second factor and equate to zero to obtain the recurrence relation $(a, b, c)M = 0$. In other words, substitute $u'^2 \mapsto a$, $u'v' \mapsto \frac{1}{2}b$, and $v'^2 \mapsto c$ in $\Phi$.

Moreover, the second set of equations follows from the first by substitution, provided (say) that $c \neq 0$ and we fix the value of $x_0x_2 - x_1^2$; for example, in

$$x_ix_{i+2} - x_{i+1}^2$$

substituting $x_{i+2} = -\frac{a}{c}x_i - \frac{b}{c}x_{i+1}$ gives

$$x_i \left( -\frac{a}{c}x_i - \frac{b}{c}x_{i+1} \right) - x_{i+1}^2 = -\frac{a}{c}x_i^2 - \left( \frac{b}{c}x_i + x_{i+1} \right)x_{i+1},$$

and we can substitute $-\frac{2}{c}x_{i-1}$ for the bracketed expression, to deduce that

$$x_ix_{i+2} - x_{i+1}^2 = \frac{a}{c} \left( x_{i-1}x_{i+1} - x_i^2 \right), \quad \text{etc.}$$

A normal form for a quadratic form under $\text{GL}(2)$ is $uv$, so that a typical solution to the equations is

$$(a, b, c) = (0, 1, 0), \quad (x_{0..k}) = (1, 0, \ldots, 1), \quad z = 1.$$ 

in the representation $\text{Sym}^2 U^\vee \oplus \text{Sym}^k U \oplus \mathbb{C}^1$ of $\text{GL}(2) \times \mathbb{G}_m$, where the final $\mathbb{G}_m$ acts by homotheties on $U^\vee$, so acts on $q \in \text{Sym}^2 U^\vee$ by $q \mapsto \lambda^2 q$ and on $z$ by $z \mapsto \lambda^2 z$. Then $V(k)$ is the closure of the orbit of this typical apolar vector.
2 Application to diptych varieties with \( de = 4 \)

We construct diptych varieties \( V_{ABLM} \) as unprojections of pullbacks of \( V(k) \). In each case we construct almost all of the ring of \( V_{ABLM} \) by a regular pullback from the key variety \( V(k) \) of Section 1. We then adjoin the remaining few variables by an unprojection argument using the ideas of [BR1]. The proofs here are self contained, although we make free use of the definitions and notation of [BR1] (Example 1.2 contains all the main ideas).

2.1 Case \([2, 2]\)

We first construct the diptych variety \( V_{ABLM} \) with the monomial cones \( \sigma_{AB} \) and \( \sigma_{LM} \) of Figure 2.1. It has variables \( x_{0..k} \) on the left against \( y_{0..2} \) on the right, tagged as in Figure 2.1 together with \( A, B, L, M \). Although we do not yet own \( V_{ABLM} \), we know some of its equations: following the model calculation of [BR1], 1.2, we find the two bottom equations:

\[
x_1y_0 = A^{k-1}B^k + x_0^2L \quad \text{and} \quad x_0y_1 = ABx_1 + y_0M.
\]

(2.1)

Then the pentagram \( y_1, y_0, x_0, x_1, x_2 \) adjoins \( x_2 \), and \( x_3, \ldots, x_k \) are adjoined by a long rally of flat pentagrams \( y_1, x_{i-1}, x_i, x_{i+1}, x_{i+2} \) with matrixes

\[
\begin{pmatrix}
y_1 & x_1 & M & x_2 \\
y_0 & AB & x_0L \\
x_0 & A^{k-2}B^{k-1} \\
& & & x_1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
y_1 & x_{i+1} & LM & x_{i+2} \\
x_{i-1} & AB & x_i \\
x_i & (AB)^{k-i-2}(LM)^{i-1}BM & x_{i+1}
\end{pmatrix}
\]

(2.2)

giving the Pfaffian equations

\[
y_1x_i = ABx_{i+1} + LMx_{i-1}, \quad x_{i-1}x_{i+1} = x_i^2 + (AB)^{k-i-1}(LM)^{i-1}BM
\]

and \( x_{i-1}x_{i+2} = x_ix_{i+1} + (AB)^{k-i-2}(LM)^{i-1}BMy_1. \)

We see that these are the equations of \( V(k) \) after the substitution

\[
(a, b, c, z) \mapsto (LM, -y_1, AB, BM).
\]

(2.3)
Thus to construct our diptych variety, we pull back $V(k) \subset \mathbb{A}^{k+5}$ by (2.3), then adjoin the two corners $y_0, y_2$ as unprojection variables. Adjoining either of these is easy, but adjoining the second then requires a simple application of some of the main ideas of proof in Sections 4–5 of [BR1].

**Lemma 2.1** Define $W_0 \subset \mathbb{A}^{k+6}_{(x_0, \ldots, x_k, y_1, A, B, L, M)}$ as the pullback of $V(k)$ under the morphism $\mathbb{A}^{k+6} \to \mathbb{A}^{k+5}$ given by (2.3).

(i) $W_0 \subset \mathbb{A}^{k+6}$ is an irreducible 6-fold.

(ii) $D_0 = \{x_1 = \cdots = x_k = M = 0\}$ is contained in $W_0$ as a divisor.

(iii) The unprojection $W_1 \subset \mathbb{A}^{k+6} \times \mathbb{A}^1_0$ of $D_0 \subset W_0$ with unprojection variable $y_0$ includes the equations (2.1) as generators of its defining ideal.

**Proof** (ii) is immediate from the defining equations (1.1) of $V(k)$: setting $x_1 = \cdots = x_k = 0$ leaves only terms divisible by $M$. (iii) follows from the Pfaffians of the first matrix of (2.2), that express the unprojection variable $y_0$ as a rational function in $x_0, x_1, y_1, A, B, L, M$ with a simple pole on $D$. This includes the equations (2.1). Q.E.D.

Once we own $y_0 \in k[W_1]$, we have to establish that the unprojection divisor of $y_2$ is contained in the variety $W_1$. The detailed statement is Theorem 2.3. (This is the same as the key point of the proof of [BR1], but our case here is much easier.) To prove it, we work with the $\mathbb{T}$-weights of each homogeneous polynomial in $x_0, \ldots, y_2, A, B, L, M$, written in terms of the so-called impartial basis dual to the monomials $L, M, A, B$; compare [BR1], Proposition 4.1. These base a slightly larger lattice, giving some of the impartial coordinates of monomials little denominators $d$ or $e$. The tag equations of $V_{AB}$ and $V_{LM}$ from Figure 2.1 determine the impartial coordinates, as follows.

**Lemma 2.2** In the impartial basis $L, M, A, B$, the monomials $x_0, \ldots, y_2$ have $\mathbb{T}$-weights:

\[
\begin{array}{cccc}
L & M & A & B \\
\hline
x_0 &=& \left( -\frac{1}{2}, 0, \frac{k-1}{2}, \frac{k}{2} \right) \\
x_1 &=& \left( 0, \frac{1}{2}, \frac{k-2}{2}, \frac{k-1}{2} \right) \\
x_2 &=& \left( \frac{1}{2}, 1, \frac{k-3}{2}, \frac{k-2}{2} \right) \\
&\vdots& \\
x_i &=& \left( \frac{i-1}{2}, \frac{i}{2}, \frac{k-i-1}{2}, \frac{k-i}{2} \right) \\
&\vdots& \\
x_{k-1} &=& \left( \frac{k-2}{2}, \frac{k-1}{2}, 0, \frac{1}{2} \right) \\
x_k &=& \left( \frac{k-1}{2}, \frac{k}{2}, -\frac{1}{2}, 0 \right)
\end{array}
\]

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The odd numbered diptych variety has variables \( x \) required. Q.E.D.

The following statement specifies the unprojection divisor \( D_1 \subset W_1 \) of \( y_2 \), completing our construction.

**Theorem 2.3** In the notation of Lemma 2.1, define

\[
D_1 = (x_0 = \cdots = x_{k-1} = y_0 = B = 0) \subset A^{k+7}_{(x_0, \ldots, x_k, y_0, y_1, A, B, L, M)}.
\]

Then \( D_1 \subset W_1 \), and the unprojection of \( D_1 \) in \( W_1 \) is the diptych variety \( V_{ABLM} \) on the pair of long rectangles of Figure 2.1.

**Proof** Most of the generators of \( I_{W_1} \) are already in the ideal of \( I_{W_0} \), and so lie in the ideal \( I_{D_1} \) by the argument of Lemma 2.1 applied to \( y_2 \) rather than \( y_0 \). The equation (2.1) of the form \( x_0 y_1 = \cdots \) is known by Lemma 2.1(iii), and also lies in \( I_{D_1} \).

The remaining generators of \( I_{W_1} \) have leading terms \( x_i y_0 \) for \( i = 1, \ldots, k \). To prove that each of these lies in \( I_{D_1} \), we prove a stronger statement: every monomial in it is divisible by one of \( x_0, \ldots, x_k, y_0 \) or \( B \). In other words, as in [BR1], 5.1, rather than working directly with these generators, we work with their \( \mathbb{T} \)-weights.

For monomials \( m, n \), write \( m \sim n \) if \( m \) and \( n \) have the same \( \mathbb{T} \)-weight, or equivalently, the same impartial coordinates. Suppose \( m \in k[W_1] \) is a monomial with \( m \sim x_i y_0 \) for some \( i = 1, \ldots, k \). (Any term in the equation having leading term \( x_i y_0 \) satisfies this equivalence, so if each such monomial lies in \( I_{D_1} \), then certainly the generator itself does.)

We may assume that the monomial \( m \) is of the form \( x_i^e y_0^f L^\lambda M^\mu A^\alpha B^\beta \), since the other variables already lie in \( I_{D_1} \). We may assume further that \( \xi = 0 \), otherwise the variable \( y_0 \) has the same \( \mathbb{T} \)-weight as monomial in the other variables, which contradicts Lemma 2.2 as \( y_0 \) is the only variable whose \( M \)-coordinate is negative.

Now compare \( x_i y_0 \) and \( m = y_0^\eta L^\lambda M^\mu A^\alpha B^\beta \); their impartial coordinates are

\[
x_i y_0 = \left( \frac{i-1}{2}, \frac{i-1}{2}, \frac{2k-i-1}{2}, \frac{2k-1+1}{2} \right) \quad y_0^\eta L^\lambda M^\mu A^\alpha B^\beta = \left( \frac{\eta}{2} + \lambda, \frac{\eta}{2} + \mu, \frac{\eta}{2} + \alpha, \frac{\eta}{2} + \beta \right).
\]

Since \( \lambda, \mu, \alpha, \beta \geq 0 \), it follows that \( \eta \leq \min \{ i-1, 2k-i-1 \} \). In particular, \( \eta/2 \leq (2k-i-1)/2 \), so \( \beta \geq 1 \). In other words, \( B \) divides the monomial \( m \), and \( m \in I_{D_1} \) as required. Q.E.D.

### 2.2 Case [4, 1] with even \( l = 2k \)

The odd numbered \( x_i \) are redundant generators, and omitting them gives Figure 2.2. The diptych variety has variables \( x_0, \ldots, x_k, y_0, y_1, A, B, L, M \) with the two bottom equations

\[
x_1 y_0 = A^{k-1} B^{2k-1} y_1 + x_0^3 L \quad \text{and} \quad x_0 y_1 = A^k B^{2k+1} + y_0 M.
\]

We adjoin \( y_2 \), then \( x_2, \ldots, x_k \) by a game of pentagrams centred on a long rally of flat
pentagrams, with $y_2$ against $x_{i-1}, x_i, x_{i+1}, x_{i+2}$ and Pfaffian equations

$$y_2 x_i = AB^2 x_{i+1} + LM^2 x_{i-1}, \quad x_{i-1} x_{i+1} = x_i^2 + (AB^2)^{k-i-1}(LM^2)^{i-1}BM$$

and $x_{i-1} x_{i+2} = x_i x_{i+1} + (AB^2)^{k-i-2}(LM^2)^{i-1}BM y_2$

These are the equations of $V(k)$ after the substitution

$$(a, b, c, z) \mapsto (LM^2, -y_2, AB^2, BM). \quad (2.4)$$

**Lemma 2.4** In the impartial basis $L, M, A, B$, the monomials $x_0, \ldots, y_4$ have $T$-weights:

\[
\begin{align*}
x_0 &= \left( \begin{array}{cccc}
  -\frac{1}{4} & 0 & 2k-1 & k \\
  \frac{1}{4} & 1 & 2k-3 & k-1 \\
  \frac{3}{4} & 2 & 2k-5 & k-2 \\
  \vdots \\
  \frac{2i-1}{4} & i & 2k-2i-1 & k-i \\
  \vdots \\
  \frac{2k-3}{4} & k-1 & \frac{1}{4} & 1 \\
  \frac{2k-1}{4} & k & -\frac{1}{4} & 0 
\end{array} \right) \\
y_0 &= \left( \begin{array}{cccc}
  0 & -1 & k & 2k+1 \\
  \frac{1}{4} & 0 & 2k+1 & k+1 \\
  \frac{1}{2} & 1 & 1 & 1 \\
  \frac{2k+1}{4} & k+1 & \frac{1}{4} & 0 \\
  k & 2k+1 & 0 & -1 
\end{array} \right)
\end{align*}
\]

**Proof** Once more, either observe that these vectors satisfy all the tag relations of the pair of long rectangles, or plug in the formulas from [BR1], Proposition 4.1, then delete every alternate $x$ variable (the ones tagged with a 1) and relabel to get these $x_0, \ldots, k$. Q.E.D.

The proof is similar to that of Theorem 2.3, so we restrict ourselves to sketching the steps and indicating how to modify them for this case.

**Theorem 2.5** The diptych variety on the pair of long rectangles of Figure 2.2 exists.

**Proof** First construct the 6-fold $W_0 \subset A_{(x_0, \ldots, k, y_2, A, B, L, M)}^{k+6}$ as the pullback of $V(k)$ by the morphism (2.4). From the equations (1.1) of $V(k)$, one sees that $D_0 \subset W_0$, where $I_{D_0} = (x_1, \ldots, k, M)$, and we can unproject this to construct $W_1$ with new ambient variable $y_1$. Q.E.D.
We define $D_1 \subset \mathbb{A}^{k+7}_{(x_0,...,x_k,y_0,y_2,A,B,L,M)}$. To show that $D_1 \subset W_1$ we check that any monomial $m$ with the same $T$-weight as a generator of $I_{W_1}$ that has not already been considered is already in $I_{D_1}$. For example, if $m \sim x_i y_1$, for any $i = 1, \ldots, k$, then we can suppose without loss of generality that $m = x_0^2 L^A M^B A^\alpha B^\beta$. By Lemma 2.4, in impartial $L, M, A, B$ coordinates we see that

$$x_i y_1 = (\frac{i}{2}, i, k - \frac{i}{2}, 2k - i + 1).$$

The $M$-coordinate of this term is $i \geq 1$, and since $x_0 = (-1/4, 0, (2k - 1)/4, k)$, the only contributor to the $M$-coordinate on the righthand side is $M^\mu$, so $\mu \geq 1$. In other words, $M$ divides $m$, so $m \in I_{D_1}$ as required.

The only other equation to check has leading term $x_0 y_2 \sim m = x_0^2 y_0^2 L^A M^B A^\alpha B^\beta$. Since both $x_0$ and $y_1$ have zero $M$ coefficient, the same argument works again. Thus $D_1 \subset W_1$, and we can unproject with new variable $y_0$ to obtain $W_2 \subset \mathbb{A}^{k+8}_{(x_0,...,k,y_0,...,2,A,B,L,M)}$. The pentagrams confirm the tag equations at the bottom corners.

We continue to unproject $y_3$ and then $y_4$ to conclude. For the first of these, define $D_2 \subset \mathbb{A}^{k+8}$ by the ideal $I_{D_2} = (x_0,...,k-1,y_0,...,1,B)$ and check that $D_2 \subset W_2$. We check the critical equations (those that are not automatically in $I_{D_2}$ as a corollary of previous checks). First suppose that $x_k y_0 \sim m = y_2^2 L^A M^B A^\alpha B^\beta$. Since

$$x_k y_0 = (\frac{2k-1}{4}, k-1, k - \frac{1}{4}, 2k + 1) \quad \text{and} \quad y_2 = (\frac{1}{2}, 1, \frac{1}{2}, 1)$$

consideration of the $A$-coordinate shows that $\eta < 2k$, so the $B$-coordinate shows that $\beta \geq 2$; in particular, $m \in I_{D_2}$ as required.

Now consider $y_0 y_2 \sim m = x_0^2 y_0^2 L^A M^B A^\alpha B^\beta$. We have $y_0 y_2 = (1/2, 0, k + 1/2, 2(k + 1))$ and $x_k = ((2k - 1)/4, k, -1/4, 0)$, so $\beta \geq 2(k + 1)$, whence $B$ divides $m$ and $m \in I_{D_2}$.

Thus we obtain $W_3 \subset \mathbb{A}^{k+9}_{(x_0,...,k-1,y_0,...,2,A,B,L,M)}$ by unprojecting $D_2 \subset W_2$. Finally we observe that $D_3 \subset W_3$, where $I_{D_3} = (x_0,...,k-1,y_0,...,2,B)$ for similar reasons. For example, if $y_0 y_3 \sim m = x_0^2 y_0^2 L^A M^B A^\alpha B^\beta$, then $y_0 y_3 = (\frac{2k+1}{4}, k, k + 1/4, 2k + 1)$ and $x_k = (\frac{2k-1}{4}, k, -1/4, 0)$ shows that $\beta \geq k + 1$, so again $B$ divides $m$ and so $m \in I_{D_3}$. Unprojecting $D_3 \subset W_3$ gives the diptych variety we seek. Q.E.D.

### 2.3 Case [1, 4] with even $l = 2k$

Omit the even numbered $x_i$, giving Figure 2.3. The diptych variety has variables $x_0,...,k$, $y_0,...,2$, $A, B, L, M$ with the two bottom equations

$$x_1 y_0 = A^{2k-1} B^k + x_0 L \quad \text{and} \quad x_0 y_1 = x_1^2 A^2 B + y_0^2 M.$$

As before, adjoining $x_2, \ldots, x_k$ features a long rally of flat pentagrams, with $y_1$ against $x_{i-1}, x_i, x_{i+1}, x_{i+2}$ and Pfaffian equations

$$y_1 x_i = A^2 B x_{i+1} + L^2 M x_{i-1}, \quad x_{i-1} x_{i+1} = x_i^2 + (A^2 B)^{k-i-1} (L^2 M)^{i-1} A L$$

and

$$x_{i-1} x_{i+2} = x_i x_{i+1} + (A^2 B)^{k-i-2} (L^2 M)^{i-1} B M y_2.$$
These are the equations of $V(k)$ after the substitution

$$(a, b, c, z) \mapsto (L^2 M, -y_1, A^2 B, BM).$$

We omit the formal statement and proof of the analogue of Theorem 2.5: the diptych variety on the pair of long rectangles of Figure 2.3 exists, and after the substitution the proof unprojects $y_0$ and $y_2$ by similar arguments in impartial coordinates.

### 2.4 Case $[1, 4]$ with odd $l = 2k + 1$

This is $[1, 4]$ read from the top, but $[4, 1]$ read from the bottom, so is a mix of the two preceding cases. Omit the odd numbered $x_i$, giving Figure 2.4. The diptych variety has

$$x_1 y_0 = y_1 A^{2k-3} B^{k-1} + x_0^3 L \quad \text{and} \quad x_0 y_1 = A^{2k-1} B^k + y_0 M.$$ 

Adjoin $y_2$ then $x_2$ by

$$
\begin{pmatrix}
  y_1 & A^2 B & M & y_2 \\
  y_0 & A^{2k-3} B^{k-1} & x_0^2 L & x_2 \\
  x_0 & y_1 & x_1 &
\end{pmatrix}
$$

then

$$
\begin{pmatrix}
  y_2 & x_1 & M & x_2 \\
  y_1 & A^2 B & x_0 L M & x_0 A^{2k-5} B^{k-2} \\
  x_0 & y_2 A^{2k-5} B^{k-2} & x_1 &
\end{pmatrix}
$$

Figure 2.3: Case $[4, 1]$ with even $l = 2k$

Figure 2.4: Case $[1, 4]$ with odd $l = 2k + 1$
After this, adjoining \( x_3, \ldots, x_{k-1} \) is the usual long rally of flat pentagrams, with \( y_2 \) against \( x_{i-1}, x_i, x_{i+1}, x_{i+2} \) and

\[
\begin{pmatrix}
  y_2 & x_{i+1} & LM^2 & x_{i+2} \\
  x_{i-1} & A^2B & x_i \\
  x_i & (A^2B)^{k-i-3}(LM^2)^{i-1}ABM y_2 \\
  x_{i+1}
\end{pmatrix}
\]

and the Pfaffian equations

\[
y_2x_i = A^2Bx_{i+1} + LM^2x_{i-1}, \quad x_{i-1}x_{i+1} = x_i^2 + (A^2B)^{k-i-2}(LM^2)^{i-1}ABM y_2
\]

and

\[
x_{i-1}x_{i+2} = x_i x_{i+1} + (A^2B)^{k-i-3}(LM^2)^{i-1}ABM y_2^2.
\]

These are the equations of \( V(k-1) \) after the substitution

\[
(a, b, c, z) \mapsto (LM^2, -y_2, A^2B, BM).
\]

We again omit the formal statement and proof: the diptych variety on the pair of long rectangles of Figure 2.4 exists, and after the substitution the proof unprojects \( y_3, y_1 \) and \( y_0 \) by arguments in impartial coordinates.

3 The polar varieties \( W(d) \) and diptychs with \( k \leq 3 \)

By [BR1], Classification Theorem 3.3, (3.7), when \( de < 3 \), the cases to treat are

\[
\begin{align*}
(d, e) &= (1, 1), \quad k \leq 2 \\
(d, e) &= (1, 2), \quad k \leq 3 \\
(d, e) &= (1, 3), \quad k \leq 5 \\
(d, e) &= (2, 1), \quad k \leq 3 \\
(d, e) &= (2, 3), \quad k \leq 5 \\
(d, e) &= (3, 1), \quad k \leq 5 \\
(d, e) &= (3, 3), \quad k \leq 5
\end{align*}
\]

We discuss \( k = 1 \) and \( k = 2 \) as specialisations of varieties with arbitrary \( d, e \). The cases with \( k \geq 3 \) have edge variables with tags = 1, that are therefore redundant generators. Eliminating them leaves a variety in low codimension that we can specify by equations. For \( k \geq 3 \), the reduced models are as follows (for odd \( k \), top-to-bottom symmetry swaps \( d, e \); we only list the cases with \( d = 1 \)):

\[
\begin{array}{|c|c|c|c|}
\hline
k & V_{AB} \text{ tags} & \text{codim as given} & \text{reduced codim} \\
\hline
3 & [1, 2, 1, (0)] & 4 & 2 \\
3 & [1, 3, 1, (0)] & 5 & 4 \\
4 & [1, 3, 1, 3, (0)] & 5 & 4 \\
4 & [3, 1, 3, 1, (0)] & 6 & 3 \\
5 & [1, 3, 1, 3, 1, (0)] & 6 & 2 \\
\hline
\end{array}
\]
Eliminating the redundant generators is convenient to establish that the varieties exist, but leaving them in has its own advantages. It allows us to write their equations more naturally (in fact, usually as Tom unprojections), sometimes in closed Pfaffian formats. In addition, we can put an extra deformation parameter as coefficient in front of each variable tagged with 1, thus exhibiting the variety as a section of a bigger key variety.

The case \( k = 1 \) is already in [BR1], (3.9). For any values of \( d, e \) we get the codimension 2 complete intersection

\[
x_1y_0 = B + Lx_0^e, \quad x_0y_1 = Ax_1^d + M \quad \text{in } \mathbb{A}^8_{(x_0,x_1,y_0,y_1,A,B,L,M)}.
\]

3.1 Case \( k = 2, \) any \( d, e; \) the polar variety \( W(d) \)

For any \( d, e \geq 1 \), the variables and tags on \( V_{AB} \) are as follows: going up the lefthand side we have \( x_0, x_1, x_2 \) tagged with \((0), e, d\), against \( y_0...d \) tagged with \((-e + 1), 2, \ldots, 2, 1\). In \( V_{AB} \) the projection sequence first eliminates the variables \( y_d, y_{d-1}, \ldots, y_2 \), and then the top left corner \( x_2 \); in \( V_{LM} \) the sequence of projections is \( y_0, y_1, \ldots, y_{d-2} \), then the bottom left corner \( x_0 \). Following [BR1], 1.2, one sees as usual that the bottom cross is

\[
x_1y_0 = AB^d + Lx_0^d \quad \text{and} \quad x_0y_1 = -x_1^{e-1}AB^{d-1} + y_0M.
\]

The equations form a single vertebra given by the \( 4 \times 4 \) Pfaffians of the \((d + 4) \times (d + 4)\) skew matrix

\[
\begin{pmatrix}
C & -x_0 & B & y_0 & y_1 & \cdots & y_{d-1} \\
-M & x_2 & y_1 & y_2 & \cdots & y_d \\
x_1 & AB^{d-1} & AB^{d-2}x_2 & \cdots & Ax^{d-1}_2 \\
Lx_0^{d-1} & LMx_0^{d-2} & \cdots & LM^{d-1}
\end{pmatrix}
\]

\tag{3.2}

in which we have replaced \( x_1^{e-1} \) by the token \( C \) in \( m_{12} \); the bottom right entries are

\[
m_{i+5,j+5} = ALC(x_0B)^{d-j-1}(x_2M)^i \cdot \frac{(x_0x_2)^{j-i} - (BM)^{j-i}}{x_0x_2 - BM} \tag{3.3}
\]

for \( 0 \leq i < j \leq d - 1 \).

If we treat \( C \) as an independent variable, then the Pfaffians of (3.2) generate the ideal of a 7-fold

\[
W(d) \subset \mathbb{A}^{d+9}_{(x_0, x_1, y_0 \ldots d, A, B, L, M, C)}.
\]

It can be realised by serial unprojection following [BR1], 1.2: the equations appearing in pentagrams are

\[
\begin{align*}
x_0x_2 &= -x_1C + BM \\
y_{i-1}y_{i+1} &= y_i^2 + ALC^2(x_0B)^{d-i-1}(x_2M)^{i-1} \\
x_0y_i &= -x_1^{i-1}AB^{d-i}C + y_{i-1}M \\
x_1y_i &= Ax_1^d B^{d-i} + Lx_0^{d-i}M^i \\
x_2y_i &= y_{i+1}B - x_0^{d-i-1}CLM^i
\end{align*}
\]
The equation for \( x_0 x_2 \) and for all \( x_i y_j \) are contained among the Pfaffians of the first 4 rows of (3.2). Beyond the 4th row, each entry \( m_{i+5,j+5} \) of (3.3) appears in just one generating relation, namely

\[
Pf_{2,3,i+5,j+5} = Cm_{i+5,j+5} - y_i y_{j+1} + y_{i+1} y_j.
\] (3.4)

These varieties are interesting in several ways. Replacing \( x_1^{e-1} \) by the token \( C \) in \( m_{12} \) displays \( V_{ABLM} \) as the section \( C = x_1^{e-1} \) of the 7-fold \( W(d) \), that is a quasihomogeneous variety under \( \operatorname{GL}(2) \times \mathbb{G}_m^3 \). Setting \( C = 0 \) or \( C = 1 \) gives invariant 6-fold sections that are also quasihomogeneous. The case \( d = 1 \) is just the affine cone \( W(1) = a\operatorname{Gr}(2,5) \) on \( \operatorname{Gr}(2,5) \).

**Exercise 3.1** Write \( U \) for the given representation of \( \operatorname{GL}(2) \). Use \( y_0...d \) as coefficients of a binary form \( f = \sum \binom{d}{i} y_i u^{d-i} v^i \in \operatorname{Sym}^d U \) and \((B, x_2), (x_0, M)\) as those of two linear forms \( g = Bu + x_2 v, h = x_0 u + Mv \in U \). Then the \( 4 \times 4 \) Pfaffians of (3.2) take the form

\[
\begin{align*}
x_1 f & = Ag^d + Lh^d, \\
M f_u - x_0 f_v & = dAC g^{d-1}, \\
-x_2 f_u + B f_v & = dLC h^{d-1}, \\
C f_1 & = \det \begin{pmatrix} B & x_0 \\ x_2 & M \end{pmatrix} = \frac{g \wedge h}{u \wedge v}, \\
(\text{3.5})
\end{align*}
\]

where of course \( f_u = \frac{\partial f}{\partial u} \) and \( f_v = \frac{\partial f}{\partial v} \). As we saw in (3.3), \( g^{d-1} \wedge h^{d-1} \) written out as \( 2 \times 2 \) minors is identically divisible by \( BM - x_0 x_2 \), so the final set of equations give (3.4). This form of the equations is manifestly \( \operatorname{GL}(2) = \operatorname{GL}(U) \) invariant. A typical solution of (3.5) is \( x_0 = x_2 = 0 \) and \( x_1 = A = C = L = B = M = 1 \), giving \( g = u, h = v, f = u^d + v^d \), and one sees that \( W(d) \) is the orbit closure of this typical solution under \( \operatorname{GL}(2) \times \mathbb{G}_m^3 \).

At the level of the matrix (3.2), the \( \operatorname{GL}(2) \) action replaces rows 1 and 2 by their general linear combinations, and the \( d \) rows-and-columns 5,6,...,d+4 by the linear combinations corresponding to the (\( d-1 \))st symmetric power. For example, adding \( \lambda \) times row 2 to row 1 (and the same for the columns to preserve skew symmetry), and adding \( \lambda^j \times \binom{d}{i} \times \text{column} 5+j \) to column 5+i for \( j = i+1, \ldots, d \) does \( x_0 \mapsto x_0 + \lambda M, B \mapsto B + \lambda x_2 \) and \( y_i \mapsto \sum \lambda^{i+j} y_j + (d-i) \lambda y_{i+1} + \text{etc.} \), meaning \( f(u,v) \mapsto f(u+\lambda v,v) \).

### 3.2 Cases \( k = 3, e = 1 \); floating factors and crazy Pfaffians

We do \( e = 1 \), since this also covers \( d = 1 \) after top-to-bottom reflection. The case \( e = 1 \) differs from \( e \geq 2 \) in the order of elimination in \( V_{AB} \), as we discuss systematically in [BR3]: projecting \( V_{AB} \) from the top, we eliminate \( x_2 \) and all the \( y_i \) for \( i = d - 1, d - 2, \ldots, 2 \) before eliminating \( x_3 \) becomes possible. This qualitative change prevents us from treating cases with \( e = 1 \) as a limit of \( e \geq 2 \).

Consider the general case \( k = 3, d \geq 2 \). In \( V_{AB} \) we have \( x_{0..3} \) tagged with \((0), d, 1, d \) against \( y_{0,..,d-1} \) tagged with \((-d+2), 2, \ldots, 2, 1 \). The equations of \( V_{ABLM} \) not involving
$x_0$ are those of a single vertebra, and we can see them as the $4 \times 4$ Pfaffians of the $(d + 3) \times (d + 3)$ matrix

$$
\begin{pmatrix}
-C & x_1 & B & y_0 & y_1 & \ldots & y_{d-2} \\
LM & x_3 & y_1 & y_2 & \ldots & y_{d-1} \\
x_2 & x_3 AB^{d-2} & x_2^2 AB^{d-3} & \ldots & x_3^{d-1} A \\
x_1^{d-2} L & x_1^{d-3} L^2 M & \ldots & L^{d-1} M^{d-2}
\end{pmatrix}
$$

(3.6)

with

$$
m_{i+5,j+5} = x_3 ALC(x_1 B)^{d-2-j}(x_3 LM)^i \frac{(x_1 x_3)^{j-i} - (BLM)^{j-i}}{x_1 x_3 - BLM}.
$$

(3.7)

For general $d$, this is the regular pullback of the polar $7$-fold $W(d - 1)$ constructed in 3.11 under the substitution

$$(x_{0..2}, y_{0..d-1}, A, B, L, M, C) \mapsto (-x_1, x_2, x_3, y_{0..d-1}, x_3 A, B, L, LM, -C).$$

The diptych variety $V_{ABL}M$ comes from this pullback on adjoining $x_0$ by unprojection of the divisor

$$D_0 = A^6_{(x_0, y_0, A, B, M, C)} = (x_2 = x_3 = y_{1..d-1} = L = 0) \subset A^{d+8}_{(x_1, y_0, A, B, L, M, C)}.$$

The Pfaffians of (3.6) clearly vanish on $D_0$, so $D_0$ is contained in the pullback and we can unproject it to get $V_{ABL}M$.

For our application, this proves that $V_{ABL}M$ exists (for any $d \geq 2$), and we could stop there. However, this case still has a general point to teach us: namely, how the Pfaffians of (3.6) fit together with the unprojection equations of $x_0$.

Starting from the other end, we see as in [BR1], 1.2 that $V_{ABL}M$ has bottom cross

$$x_1 y_0 = AB^{d-1} C^2 + L x_0 \quad \text{and} \quad x_0 y_1 = x_1^{d-2} AB^{d-2} C + M y_0^2.$$  

(We add a variable $C$ as annotation on $x_2$, making its tag equation $C x_2 = x_1 x_3$ in $V_{AB}$ and $V_{LM}$.) It contains the unprojection divisor $D : (x_0 = y_0 = AB^{d-2} C = 0)$, leading to the pentagram $x_1, y_0, y_0, y_1, \xi$ and the $4 \times 4$ Pfaffians of

$$
\begin{pmatrix}
x_1 & BC & -L & -\xi \\
x_0 & AB^{d-2} C & -M y_0 & x_1^{d-2} \\
y_0 & y_1 & x_1^{d-2} & y_1
\end{pmatrix}
$$

(3.8)

The unprojection variable $\xi$ here must be $x_3$ (rather than $x_2$ with the tag $e = 1$), as one sees for example from the Pfaffian $\text{Pf}_{12,35} = x_1^{d-1} - x_0 \xi + BMC y_0$.

We link the equations together by adding a final $(d + 4)$th column to (3.6):

$$
\begin{pmatrix}
-C & x_1 & B & y_0 & y_1 & \ldots & y_{d-2} & x_0 \\
LM & x_3 & y_1 & y_2 & \ldots & y_{d-1} & y_0 M \\
x_2 & x_3 AB^{d-2} & x_2^2 AB^{d-3} & \ldots & x_3^{d-1} A & AB^{d-1} M \\
x_1^{d-2} L & x_1^{d-3} L^2 M & \ldots & L^{d-1} M^{d-2} & x_1^{d-1}
\end{pmatrix}
$$

(3.9)
with the same lower right entries \( m_{i+5,j+5} \) as (3.7), and the last column ending in

\[
m_{4+i,4+d} = -AC(Bx_1)^{d-1-i} \times \frac{(x_1x_3)^i - (BLM)^i}{x_1x_3 - BLM} \text{ for } i = 1, \ldots, d - 1.
\]

The \( 4 \times 4 \) Pfaffians of (3.9) provide all but one of the equations of \( V_{ABLM} \). Comparing (3.8) with (3.9), we see that the equation \( x_1y_0 = -AB^{d-1}C^2 + x_0L \) is missing, although \( M \) times it is the Pfaffian \( Pf_{12,4(d+4)} \) (in fact its multiples by \( x_1, x_2, x_3, y_1, \ldots, y_{d-1} \) are also in the ideal of Pfaffians of (3.9)).

The little problem we face is how to cancel the common factor \( M \) in the entries \( m_{2,1}, m_{2,2}, m_{3,1}, m_{3,2} \) of (3.9), or in the \( 3 \times 3 \) submatrix \( \begin{pmatrix} LM & y_0M \\ AB^{d-1}M \end{pmatrix} \) formed by rows and columns 2, 3, \( d+4 \), without spoiling the other Pfaffians. We do this by floating \( M \) from the entries with indices 2, 3, \( d+4 \) to the complementary entries with 1, 4, \ldots, \( d+3 \), adding the \( 4 \times 4 \) Pfaffians of the floated matrix, including the equation for \( x_1y_0 \), to those of (3.9).

The full set of equations is a mild form of crazy Pfaffian, analogous to Riemenschneider’s quasi-determinantal \( [R] \): rather than floating \( M \) as a factor in two matrices, we can view it as a multiplier between entries with indices 2, 3, \( d+4 \) and those with 1, 4, \ldots, \( d+3 \); when evaluating a crazy Pfaffian, we include \( M \) as a factor whenever a product crosses between these two regions. Thus the factors \( M \) in the triangle \( m_{2,3}, m_{2,4}, m_{3,4} \) of (3.9) of (3.9) appear as before in most Pfaffians, but not in \( Pf_{12,3(d+4)} \) or \( Pf_{23,4(d+4)} \) for \( i = 4, \ldots, d+3 \).

We discussed a case of floating in [TJ], 9.1, especially around (9.4), but the present instance displays the phenomenon in a particularly clear form. This type of crazy Pfaffians or floating factors occur frequently in our experience of working with Gorenstein rings of codimension \( \geq 4 \), and seem to be a basic device in understanding how one vertebra links to the next. We expect to return to this in future publications.

4 The cases \( de = 3 \) and parallel unprojection

In 4.1 we construct all remaining cases \( de = 3 \) with \( k = 4 \) or 5 of (3.1) to complete the construction of all diptych varieties with \( de \leq 4 \). Finally, in 4.2 we observe that each of these can be realised as a regular pullback from a single key variety, a 10-fold \( W \subset \mathbb{A}^{16} \).

4.1 Small diptychs by pentagrams

When \( k = 4 \), the cases \( (d, e) = (1, 3) \) or (3,1) are distinct. In each case, we pass to the reduced model, which is isomorphic to the diptych variety we seek but easier to treat because it has lower codimension, and then adjoin the redundant generators using pentagrams, much as in [BR1], Example 1.2.

**Case** [3, 1, 3, 1] Write \( x_0, x_1, x_2, x_3, x_4 \) with \( V_{AB} \) tags [(0), 1, 3, 1, 3] opposite \( y_0, y_1, y_2 \). We work up from the reduced model, that has only \( x_0, x_4 \) against \( y_0, y_1, y_2 \); we eliminate
y_2 from this getting the codimension 2 complete intersection

\[ x_0y_1 = AB + My_0 \quad \text{and} \quad x_4y_0 = By_1^2 + Lx_0, \]

and adjoin \( y_2 \) by the pentagram \( x_4, x_0, y_0, y_1, y_2 \) and its Pfaffian matrix

\[
M_1 = \begin{pmatrix}
x_4 & y_1^2 & -L & -y_2 \\
x_0 & B & -M \\
y_0 & A \\
y_1 \\
\end{pmatrix}
\]

\[
x_0y_2 = x_4A + My_1^2 \\
y_0y_2 = y_1^2 + AL \\
x_4y_1 = y_2B + LM
\]

These five Pfaffian equations define the reduced model in codimension 3.

We recover the full set of equations by adjoining the redundant \( x_2 \), then \( x_1 \) and \( x_3 \) in either order. Adjoin \( x_2 \) by the pentagram \( x_0, x_0, y_0, y_1, x_2 \):

\[
M_2 = \begin{pmatrix}
x_0 & AB & -M & -x_2 \\
y_0 & 1 & -y_1B \\
y_1 & Lx_0 \\
x_4 \\
\end{pmatrix}
\]

\[
x_2 = x_0x_4 - y_1BM \\
\text{and} \quad x_2y_0 = y_1AB^2 + Lx_0^2 \\
x_2y_1 = x_4AB + LMx_0
\]

Adjoin \( x_1 \) by the pentagram \( x_0, y_1, x_4, x_2, x_1 \):

\[
M_3 = \begin{pmatrix}
x_0 & x_2 & -BM & -x_1 \\
y_1 & 1 & -AB \\
x_4 & L_2x_0 \\
x_2 \\
\end{pmatrix}
\]

\[
x_1 = x_0x_2 - AB^2M \\
\text{and} \quad x_1x_4 = x_2^2 + x_0BLM^2 \\
x_1y_1 = x_2AB + LMx_0^2
\]

Finally adjoin \( x_3 \) by the pentagram \( x_2, x_0, y_1, x_4, x_3 \):

\[
M_4 = \begin{pmatrix}
x_2 & x_4AB & -LM & -x_3 \\
x_0 & 1 & -BM \\
y_1 & x_2 \\
x_4 \\
\end{pmatrix}
\]

\[
x_3 = x_2x_4 - BLM^2 \\
\text{and} \quad x_0x_3 = x_2^2 + x_4AB^2M \\
x_3y_1 = x_2^2AB + LMx_2
\]

The five Pfaffians of \( M_1 \) together with the three equations for \( x_1, x_2, x_3 \) define \( V_{ABLM} \subset \mathbb{A}_{\langle x_0, x_0, y_0, 1, A, B, L, M \rangle}^{11} \).

**Case** \([1, 3, 1, 3] \) Write \( x_0, x_1, x_2, x_3, x_4 \) with \( V_{AB} \) tags \([0, 3, 1, 3, 1] \) against \( y_0, y_1 \). The reduced model is in codimension 4 on variables \( x_0, x_1, x_3, x_4, y_0, y_1 \); eliminating \( x_4 \) then \( x_3 \) from this leaves two equations

\[
x_0y_1 = Ax_1 + y_0^2M \quad \text{and} \quad x_1y_0 = A^2B + Lx_0
\]

To recover the reduced model, we adjoin \( x_3 \) and then \( x_4 \). Adjoin \( x_3 \) by the pentagram \( x_1, x_0, y_0, y_1, x_3 \):

\[
M_1 = \begin{pmatrix}
x_1 & AB & -L & -x_3 \\
x_0 & A & -My_0 \\
y_0 & x_1 \\
y_1 \\
\end{pmatrix}
\]

\[
x_0x_3 = x_1^2 + y_0ABM \\
x_3y_0 = y_1AB + x_1L \\
x_1y_1 = x_3A + LMy_0
\]
The unprojection divisor of \( x_4 \) is \( \langle x_0 = x_1 = y_0 = A \rangle \), so that the reduced model exists. We adjoin \( x_4 \) by the pentagram \( x_3, x_1, y_0, y_1, x_4 \):

\[
M_2 = \begin{pmatrix} x_3 & y_1 B & -L & -x_4 \\ x_1 & A & -LM \\ y_0 & x_3 \\ y_1 \\ \end{pmatrix} \begin{pmatrix} x_1x_4 \\ x_3y_1 \\ x_4y_0 \\ \end{pmatrix} = \begin{pmatrix} x_3 + y_1 BLM \\ x_4A + L^2 M \\ y_1^2 B + x_3L \\ \end{pmatrix}
\]

These 8 equations define the reduced model in codimension 4 together with a residual \( \mathbb{A}_4^4(x_0, x_4, B, M) \). Calculating with syzygies or saturating against \( y_0 \) (say) recovers the “long equation”:

\[
x_0x_4 = x_1x_3 + y_0y_1BM + ABLM.
\]

Finally, we adjoin the redundant generator \( x_2 \) by the pentagram \( x_1, y_0, y_1, x_3, x_2 \):

\[
M_3 = \begin{pmatrix} x_1 & x_3 A & -LM & -x_2 \\ y_0 & 1 & -AB \\ y_1 & x_1 L \\ x_3 \\ \end{pmatrix} \begin{pmatrix} x_1x_3 \\ x_2y_0 \\ x_2y_1 \\ \end{pmatrix} = \begin{pmatrix} x_2 + ABLM \\ x_3A^2B + x_1^2L \\ x_3^2A + x_1L^2M \\ \end{pmatrix}
\]

Thus the diptych in this case is the graph of \( x_2 = x_1x_3 - ABLM \) over its reduced model, with \( 10 \times 25 \) resolution.

Remark 4.1 Since \( x_2 \) has tag 1, it makes sense to give him annotation \( C \); in the pentagram equations above, this can be done simply by replacing the 1 in \( M_3 \) by \( C \). Computer algebra experiments (after saturating these pentagram equations against \( y_0LM \)) show that this gives a 7-fold \( V_{ABCLM} \) in codimension 5 with \( 14 \times 35 \) resolution and serial unprojection form. (See [Dip], [Ma] for Magma code that computes this.)

Case \([1, 3, 1, 3, 1]\) When \( k = 5 \), we consider tags \([1, 3, 1, 3, 1]\) on \( V_{AB} \); this also covers the case \([3, 1, 3, 1, 3]\) by top-to-bottom reflection. Write \( x_0, x_1, x_2, x_3, x_4, x_5, y_0, y_1 \) with \( V_{AB} \) tags \([0, 1, 3, 1, 3, 1]\). The reduced model has only \( x_0, x_5 \) against \( y_0, y_1 \), with two equations

\[
x_0y_1 = A + y_0M \quad \text{and} \quad x_5y_0 = y_1^3B + L.
\]

The diptych variety is isomorphic to \( \mathbb{A}^6_6(x_0, x_5, y_0, y_1, B, M) \), and is the graph over it of \( A, L, x_4, x_2, x_1, x_3 \) expressed as functions by

\[
\begin{align*}
A &= x_0y_1 - y_0M \\
L &= x_5y_0 - y_1^3B \\
x_1 &= x_0x_2 - A^2BM \\
x_2 &= x_0x_4 - y_1ABM \\
x_3 &= x_2x_4 - ABLM^2 \\
x_4 &= x_0x_5 - y_1^2BM.
\end{align*}
\]

It is a fun exercise to compute all of this with magic pentagrams as in previous cases.
4.2 A key variety by parallel unprojection

There is a uniform treatment of the cases $k = 4$ and $5$ and $de = 3$ as regular pullbacks of a key 10-fold $W$ that is given by a parallel unprojection construction similar to that of Papadakis and Neves [PN]. We start from the codimension 2 complete intersection $W_0 \subset \mathbb{A}_{(u_1, \ldots, u_4, v_1, \ldots, v_4)}^{12}$ given by

$$u_1u_3 = a_2s_1s_2u_2 + a_4s_3^3u_4,$$
$$u_2u_4 = a_1s_1s_4u_1 + a_3s_2s_3u_3,$$

which is a normal 10-fold containing as divisors the four codimension 3 complete intersections

$$(s_1, u_3, u_4), \quad (s_2, u_4, u_1), \quad (s_3, u_1, u_2), \quad (s_4, u_2, u_3).$$

Parallel unprojection of these four divisors gives a codimension 6 Gorenstein subvariety $W \subset \mathbb{A}_{(u_1, \ldots, u_4, v_1, \ldots, v_4)}^{16}$ with a $20 \times 66$ resolution, by standard application of the Kustin–Miller unprojection theorem. The full set of equations is obtained as follows.

Each individual unprojection variable $v_i$ is adjoined by a pentagram, giving three linear unprojection equations such as

$$\begin{pmatrix}
  u_2 & a_1s_4u_1 & -a_3s_2s_3 & -v_1 \\
  u_3 & s_1 & -a_4s_3s_4 \\
  u_4 & a_2s_2u_2 \\
  u_1 
\end{pmatrix}
\begin{pmatrix}
  s_1v_1 = u_1u_2 - a_3a_4s_2s_3^2s_4, \\
  u_4v_1 = a_1s_4u_1^2 + a_2a_3s_2^2s_3u_2, \\
  u_3v_1 = a_1a_4s_3^3s_4u_1 + a_2s_2u_2^2.
\end{pmatrix}
\tag{4.1}$$

In addition, there are 6 bilinear equations for $v_i v_j$, making $2 + 4 \times 3 + 6 = 20$ equations.

Four of these also come from pentagrams, the first of which gives

$$v_1v_2 = a_2u_2^3 + a_1a_3a_4s_3^3s_4^3, \tag{4.2}$$

whereas the remaining two are “long equations”

$$v_1v_3 = a_1a_4s_3^3v_4 + a_2a_3s_2^2v_2 + 3a_1a_2a_3a_4s_1s_2s_3s_4,$$
$$v_2v_4 = a_1a_2s_1^3v_1 + a_3a_4s_2^3v_3 + 3a_1a_2a_3a_4s_1^2s_2s_3s_4,$$

that can be computed using syzygies.

The construction has 4-fold cyclic symmetry (1234), apparent in the picture
We view the $v_i$ as tagged by 1 and annotated by $s_i$ (by the first equation of (4.1)), and the $u_i$ as tagged by 3 and annotated by $a_i$ (by (4.2)). We get Gorenstein projections on eliminating any subset of the $v_i$, but we can only eliminate $u_i$ after projecting out the neighbouring $v_{i-1}$ and $v_i$.

We use this variety as a model for diptych varieties. The diptychs with $dc = 3$ and $k = 4, 5$ of $\mathbb{C}$ arise by pullback from $W$ on making the following substitutions:

**Case $[3, 1, 3, 1]$**:

\[
\begin{align*}
v_1 &= x_1 & u_1 &= x_0 & a_1 &= L & s_1 &= 1 \\
v_2 &= x_3 & u_2 &= x_2 & a_2 &= 1 & s_2 &= 1 \\
v_3 &= y_2 & u_3 &= x_4 & a_3 &= A & s_3 &= B \\
v_4 &= y_0 & u_4 &= y_1 & a_4 &= 1 & s_4 &= M
\end{align*}
\]

**Case $[1, 3, 1, 3]$**:

\[
\begin{align*}
v_1 &= x_2 & u_1 &= x_1 & a_1 &= 1 & s_1 &= 1 \\
v_2 &= x_4 & u_2 &= x_3 & a_2 &= 1 & s_2 &= A \\
v_3 &= z & u_3 &= y_1 & a_3 &= B & s_3 &= 1 \\
v_4 &= x_0 & u_4 &= y_0 & a_4 &= M & s_4 &= L
\end{align*}
\]

**Case $[3, 1, 3, 1, 3]$**:

\[
\begin{align*}
v_1 &= x_1 & u_1 &= x_0 & a_1 &= L & s_1 &= 1 \\
v_2 &= x_3 & u_2 &= x_2 & a_2 &= 1 & s_2 &= 1 \\
v_3 &= x_5 & u_3 &= x_4 & a_3 &= 1 & s_3 &= A \\
v_4 &= y_0 & u_4 &= y_1 & a_4 &= B & s_4 &= M
\end{align*}
\]

where, in the second case, $z = y_0y_1 - AL$ is a redundant generator.

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