Discrete ambiguities in a truncated partial-wave analysis of pseudoscalar meson photoproduction

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The discrete ambiguities appearing in the complete experiment problem for single pseudoscalar meson photoproduction within truncated partial-wave analysis are studied. We show that apart from the double ambiguity, known from the previous works, there always exists another discrete ambiguity, arising from the invariance of single-polarization observables under certain transformation of the helicity amplitudes.

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In recent years there has been renewed interest in the problem of a complete experiment for photoproduction of pseudoscalar mesons on nucleons. Due to a large increase of experimental information an opportunity has appeared to solve this problem via applying the developed theoretical methods to real data. In addition to the standard complete experiment problem in terms of spin amplitudes (as a rule, the CGLN amplitudes $F_i$, $i = 1, \ldots, 4$, are considered), rather powerful method has been developed within the so-called truncated partial-wave analysis (TPWA) [1]. In this case, the expansion of $F_i$ in terms of partial waves is limited by a certain maximum value $L$ of the angular momentum, and the method is aimed at extracting the coefficients of these series, that is, the multipole amplitudes $E_l^\pm, M_l^\pm$. As observables, the expansion coefficients of spin polarization observables in some complete basis, for example, $\cos^n \theta$ or $P_n(\cos \theta)$, $n = 0, 1, 2, \ldots$, are used.

As an important advantage of TPWA over the conventional methods one usually notes simpler requirements to achieve the complete experiment conditions. In particular, the corresponding complete set of measurements includes, as a rule, a smaller number of double polarization observables [1, 2].

As applied to photoproduction of pseudoscalar mesons, the most developed approach is based on the Gersten roots method [3] known from the phase analysis of $\pi N$ scattering. The corresponding formalism was mainly developed in [4] and then supplemented and extended in [5–7]. In the latter works some important aspects related to practical application were also considered in detail.

The mathematical basis of the method is the partial-wave expansion of transversity amplitudes $b_i(\theta)$, $i = 1, \ldots, 4$, in the form

$$
\begin{align}
 b_1(\theta) &= C \frac{e^{i\theta/2}}{(1 + x)^L} A_{2L}(x), \\
 b_2(\theta) &= -b_1(-\theta), \\
 b_3(\theta) &= -C \frac{e^{i\theta/2}}{(1 + x)^L} B_{2L}(x), \\
 b_4(\theta) &= -b_3(-\theta),
\end{align}
$$

where $x = \tan(\theta/2)$ and $A_{2L}$ and $B_{2L}$ are polynomials of the order $2L$

$$
A_{2L}(x) = \sum_{n=0}^{2L} a_n x^n, \quad B_{2L}(x) = \sum_{n=0}^{2L} b_n x^n,
$$

with $L$ being the value of the orbital momentum $l$, at which the multipole expansion is truncated. The polynomial coefficients are linear functions of the multipole amplitudes $E/M_{l\pm}$. The leading terms and the free terms of $A_{2L}$ and $B_{2L}$ are equal, that is

$$
a_{2L} = b_{2L}, \quad a_0 = b_0.
$$

Taking into account the relations (3) the set of $4L$ independent coefficients $a_0, \ldots, a_{2L}, b_1, \ldots, b_{2L-1}$ is mapped onto the set of $4L$ amplitudes $E_{l\pm}, M_{l\pm}$ through a nonsingular linear transformation

$$
(a_n, b_n) \rightarrow (E_{l\pm}, M_{l\pm}).
$$

The constant $C$ in (1) depends on the definition of the amplitudes $b_i$. Its exact value is irrelevant for further discussion, so it will be omitted in the equations to follow.
The second equality in Eq. (3) leads to the relation between the zeros (Gersten roots) \( \alpha_i, \beta_i \) of \( A_{2L} \) and \( B_{2L} \)

\[
\prod_{i=1}^{2L} \alpha_i = \prod_{i=1}^{2L} \beta_i. \tag{5}
\]

Given the complex roots \( \alpha_i \) and \( \beta_i \), the amplitudes \( b_1 \) and \( b_3 \) may be written as (suppressing the irrelevant constant \( C \) in \( \alpha \))

\[
b_1(\theta) = \frac{e^{i\theta/2}}{(1+x)^L} a_{2L} \prod_{i=1}^{2L} (x - \alpha_i), \tag{6a}
\]

\[
b_3(\theta) = -\frac{e^{i\theta/2}}{(1+x)^L} a_{2L} \prod_{i=1}^{2L} (x - \beta_i). \tag{6b}
\]

**TABLE I.** Polarization observables in terms of transversity amplitudes, classified into the four groups: \( S \) (single polarization), \( BT \) (beam-target polarization), \( BR \) (beam-recoil polarization), and \( TR \) (target-recoil polarization). The notations in the last column indicate, whether the observable is invariant (+), or changes sign (−) under the transformations (15) to (17).

| Observable Transversity representation | Group |
|---------------------------------------|-------|
| Im \([-b_1 b_3^* - b_2 b_4^*]\) | \( BT \) |
| Re \([b_1 b_5^* - b_2 b_6^*]\) | + |
| \(-Re[b_1 b_3^* + b_2 b_4^*]\) | + |
| \(Im[b_1 b_5^* - b_2 b_6^*]\) | − |
| Im \([-b_1 b_4^* + b_2 b_5^*]\) | \( BR \) |
| \(-Re[b_1 b_2^* - b_2 b_4^*]\) | − |
| \(-Im[b_1 b_2^* - b_2 b_4^*]\) | − |
| \(-Re[b_1 b_2^* + b_2 b_4^*]\) | + |
| \(Re[-b_1 b_2^* + b_2 b_4^*]\) | + |
| \(-Im[b_1 b_2^* - b_2 b_4^*]\) | − |
| \(Re[-b_1 b_2^* - b_2 b_4^*]\) | + |

In Table I expressions for the observables are presented in transversity basis. The complex conjugation of the polynomial coefficients in Eqs. (2)

\[
a_n \rightarrow a_n^*, \quad b_n \rightarrow b_n^*, \quad n = 0, \ldots, 2L, \tag{7}
\]

leads to the transformation of the transversity amplitudes

\[
b_i(\theta) \rightarrow e^{i\theta} b_i^*(\theta), \quad i = 1, 3, \tag{8a}
\]

\[
b_i(\theta) \rightarrow e^{-i\theta} b_i^*(\theta), \quad i = 2, 4, \tag{8b}
\]

which, as may readily be seen from Table I leaves the observables of the group \( S \) unchanged.

In view of the one-to-one relationship between the coefficients \( (a_n, b_n) \) and the multipole amplitudes \( (E_{l\pm}, M_{l\pm}) \), the operation (7) generate the corresponding transformation

\[
E/M_{l\pm} \rightarrow \tilde{E}/\tilde{M}_{l\pm} = E/M_{l\pm}(a_n^*, b_n^*), \tag{9}
\]

where \( E/M_{l\pm}(a_n^*, b_n^*) \) are the multipoles calculated according to the same rules [4] but with complex conjugated \( a_n \) and \( b_n \). The invariance with respect to [9] results in the discrete ambiguity (called double ambiguity in [1, 3]) of the group \( S \) observables within TPWA.

Instead of the coefficients \( a_n, b_n \), one can use the roots \( \alpha_i, \beta_i \), as was done in [4, 5]. In this case, as follows directly from (6), the multipole transformation which do not change the group \( S \) observables, reads

\[
E/M_{l\pm} \rightarrow \tilde{E}/\tilde{M}_{l\pm} = E/M_{l\pm}(\alpha_n^*, \beta_n^*). \tag{10}
\]
It is evident that (9) and (10) give the same set of the transformed amplitudes $\tilde{E}/\tilde{M}_{l\pm}$ apart from an overall phase. Indeed, from the Vieta’s formulas (taking into account (3))

\[
\sum_{i=1}^{2L} \alpha_i = -\frac{a_{2L-1}}{a_{2L}}, \quad \sum_{i=1}^{2L} \beta_i = -\frac{b_{2L-1}}{a_{2L}}, \quad \sum_{i<j=1}^{2L} \alpha_i \alpha_j = \frac{a_{2L-2}}{a_{2L}}, \quad \sum_{i<j=1}^{2L} \beta_i \beta_j = \frac{b_{2L-2}}{a_{2L}}, \quad \alpha_1 \alpha_2 \ldots \alpha_{2L} = \beta_1 \beta_2 \ldots \beta_{2L} = \frac{a_0}{a_{2L}}
\]

immediately follows that $\alpha_i, \beta_i \rightarrow \alpha_i^*, \beta_i^*$ leads to $a_n, b_n \rightarrow a_n^*, b_n^*$ so that both methods are equivalent.

Note, that the double discrete ambiguity discussed above has global character in the sense that it is present at any energy $W$. In addition, the so-called accidental ambiguities may arise when the equality (5) still holds if complex conjugation is applied to only some of the roots on the right and the left hand sides. It is clear, however, that such ambiguities, appearing accidentally at isolated energies $W$, cannot generate branches of solutions. Therefore in a real energy-dependent analysis, such “point-like” degeneracies should not pose principal difficulties. Situations in which they become dangerous are rather exotic [7]. For this reason, we will focus only on the double discrete ambiguities of the type (9).

Using the expressions in Table I one can show that the ambiguity (9) (or (10)) can be resolved by measuring one additional observable $G$ or $F$, or any of the observables from the sets $BR$ and $TR$. Indeed, as may be seen, the transformation (7) leads to a sign change of $G$ with respect to (9). For example, for $\tilde{O}_x$ we will have

\[
\tilde{O}_x \rightarrow \text{Re} \left[ -b_1^* b_4 e^{i \theta} + b_2^* b_3 e^{-i \theta} \right] \neq \pm \tilde{O}_x.
\]

Most of the above results were obtained in earlier works. In the present paper, we will show that, in addition to the ambiguity (7), there is another double ambiguity of the $S$ group, which has not yet been discussed in the literature. The corresponding symmetry arises when the partial-wave expansion is truncated in the total angular momentum $j = l \pm \frac{1}{2}$ and is not equivalent to the symmetry (7) associated with the truncation in the orbital momentum $l$. To explain its mechanism, it is convenient to use the expansion of the helicity amplitudes in Wigner rotation matrices $d_{\lambda \mu}^j$

\[
H_{\mu \lambda} (\theta) = \sum_{j} h_{\mu \lambda}^j d_{\lambda \mu}^j (\theta),
\]

where $j$ is the total angular momentum. Following [8], the four independent amplitudes with $\lambda = \frac{1}{2}, \frac{3}{2}$ and $\mu = \pm \frac{1}{2}$ will be numbered by $i = 1, \ldots, 4$ (see Table I of Ref. [8]). Taking into account the relationship between the amplitudes $b_i$ and $H_i$

\[
b_1 = \frac{1}{2} [H_1 + H_4 + i(H_2 - H_3)], \quad b_2 = \frac{1}{2} [H_1 + H_4 - i(H_2 - H_3)], \quad b_3 = \frac{1}{2} [H_1 - H_4 - i(H_2 + H_3)], \quad b_4 = \frac{1}{2} [H_1 - H_4 + i(H_2 + H_3)],
\]

it is clear that the replacement

\[
H_1 \rightarrow H_1^*, \ H_2 \rightarrow -H_2^*, \ H_3 \rightarrow -H_3^*, \ H_4 \rightarrow H_4^*,
\]
leading to

\[ b_i \rightarrow b_i^*, \quad i = 1, \ldots, 4, \]  

(16)

leaves all observables of the group \( S \) invariant. The symmetry \( (15) \) was apparently first noted in Ref. [9]. Note that in contrast to \( (8) \) all amplitudes \( b_i \) in \( (16) \) are transformed according to the same rule.

Using the definition \( (13) \) and taking into account the realness of the Wigner rotation matrices, the transformation \( (15) \) results in the corresponding replacement of the partial amplitudes

\[ h_1^j \rightarrow h_1^{*j}, \quad h_2^j \rightarrow -h_2^{*j}, \quad h_3^j \rightarrow -h_3^{*j}, \quad h_4^j \rightarrow h_4^{*j}. \]  

(17)

From this it immediately becomes clear that this ambiguity can be resolved by adding to the group \( S \) one of those observables which are not invariant (change sign) under the replacement \( (15) \) (see the last column in Table I), that is, one of

\[ \hat{G}, \hat{F} \]  

(18)

from the group \( BT \), or one of

\[ \hat{O}_z, \hat{C}_x, \hat{T}_z, \hat{L}_x \]  

(19)

from the groups \( BR \) and \( TR \).

We emphasize that the symmetry \( (17) \) occurs when the partial-wave expansion is truncated in \( j \) and does not occur when one cuts off in \( l \). Similarly, \( (7) \) arises only when the truncation is in the orbital momentum \( l \) and, in the strict sense, vanishes when one cuts off in the total momentum \( j \). At the same time, it is clear that as \( L = l_{\text{max}} \) and, accordingly, \( J = j_{\text{max}} \) increase (at fixed \( W \)), both symmetries should manifest themselves, regardless of the truncation schemes. To show how this happens, we derive, by analogy with \( (1) \) similar representation of \( b_j \), or one of \( BR \) is, one of the observables which are not invariant (change sign) under the replacement \( (15) \) (see the last column in Table I), that is, one of

\[ \hat{G}, \hat{F} \]  

(18)

from the group \( BT \), or one of

\[ \hat{O}_z, \hat{C}_x, \hat{T}_z, \hat{L}_x \]  

(19)

from the groups \( BR \) and \( TR \).

Expressions \( (20) \) can also be obtained directly from \( (11) \) by eliminating the contribution of the multipoles \( E_{l+}, M_{l+} \) with \( l = L \). Then the remaining polynomials in \( b_1 \) and \( b_2 \) have a constant root \( x = -i \). This feature, which is valid for any \( J \), was noted in \( [8] \) for the simplest particular case \( J = 1/2 \). Taking into account the identity

\[ e^{i\theta/2}(1 - ix) = \left( \cos \frac{\theta}{2} \right)^{-1} \]  

(22)

we come to Eqs. \( (20) \).

The operation \( (15) \) (or \( (17) \)), leading to \( (16) \) and resulting in the corresponding double ambiguity, means complex conjugation of all coefficients \( c_n \) and \( d_n \) of the polynomials \( C_{2J} \) and \( D_{2J} \). As already noted above, all 16 observables, including those from the groups \( TR \) and \( BR \) have definite parity (are invariant or change sign) with respect to this transformation.

In order to return from \( (20) \) to \( (11) \) we add the contribution of \( E_{L+}, M_{L+} \). It is clear that if the expansions \( (20) \) with the chosen \( J \) provide the necessary accuracy, then the added multipoles are small. This change leads to

\[ b_1(\theta) \rightarrow \tilde{b}_1(\theta) = b_1(\theta) + \frac{e^{i\theta/2}}{(1 + x^2)^L} \tilde{A}_{2L}(x), \]  

(23a)

\[ b_2(\theta) \rightarrow \tilde{b}_2 = b_2(\theta) - \frac{e^{-i\theta/2}}{(1 + x^2)^L} \tilde{A}_{2L}(-x), \]  

(23b)

and the analogous expressions for \( b_3 \) and \( b_4 \). Here the polynomial \( \tilde{A}_{2L}(x) \) contains only \( E_{L+} \) and \( M_{L+} \).
FIG. 1. Double polarization observables for $\gamma p \to \pi^0 p$ at $W = 1300$ MeV. Calculations are performed with the MAID2007 multipole amplitudes [10]. Only the $l = 0, 1, 2$ multipoles (from $E_0^+, E_2^+, M_2^+$) are included. The solid lines are the starting solution. The dash-dotted lines are obtained by the transformation (7). The dashed lines correspond to the transformation (16) of the amplitudes $b_i$ on the r.h.s. of (23).

The terms proportional to $e^{i\theta/2}$ and $e^{-i\theta/2}$ lead to different transformation rules for $b_1$ and $b_2$ (and, respectively, for $b_3$ and $b_4$) under the complex conjugation of all coefficients on the r.h.s. in Eqs. (23), so that the previous symmetric rule (10) no longer holds. However, since these "symmetry-breaking" terms contain only the small amplitudes $E_{L^+}, M_{L^+}$, their contribution is comparable to the error caused by the truncation of the multipole expansion series.

Thus, as we see, within a given cutoff scheme, one of the two ambiguities is always approximate. At the same time, the corresponding numerical error is proportional to the contribution of the last small amplitudes. Therefore, with increasing $L$ (and $J$), both ambiguities gradually become more and more exact.

Fig. 1 demonstrates the degree of accuracy, to which the symmetry (17) holds, if the expansion is truncated in the orbital momentum at $l_{\text{max}} = L$. Here the double polarization observables are calculated at $W = 1300$ MeV with $L = 2$. The dashed lines are obtained with the amplitudes (23) in which the first terms are transformed according to (10), and the last term proportional $A_{2L}$ is not transformed. Recall that the observables $H, E, O_x, C_z, T_x, L_z$ remain invariant and $G, F, O_z, C_x, T_z, L_z$ change sign if $A_{2L} = 0$ (see Table 1). The deviation from this rule (resulting, for instance, in the difference between the solid and the dashed curves for $H, E, O_x, C_z, T_x, L_z$) demonstrates the degree of the symmetry breaking, caused by the last terms in (23).

One important feature must be emphasized. While the ambiguity (9) may be resolved by adding any observable from the groups $BR$ or $TR$, in the case of (10), the observables

$$O_x, C_z, T_x, L_z$$

belonging to these groups remain invariant. Therefore, in order to eliminate both ambiguities simultaneously, only the observables marked with "−" in Table 1 should be included into the complete set.

In summary, apart from the double discrete ambiguity, noted in Refs. [1, 4–7] there is always another ambiguity arising from the invariance of the group $S$ observables under the complex conjugate transformation (17) of the helicity amplitudes. The conditions at which these additional uncertainties occur are most naturally controlled by using the partial wave expansion (13) in the helicity representation.
This result means that the rules for choosing the complete set are more complicated than those given in the above cited works. In particular, both types of the ambiguities can be eliminated simultaneously only by choosing the observables marked by the minus sign in Table I. On the contrary, taking any observable from (24) will leave the ambiguity (17) unresolved.

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