Gleason’s theorem for composite systems

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Received 10 February 2023; revised 20 August 2023
Accepted for publication 20 September 2023
Published 12 October 2023

Abstract
Gleason’s theorem (Gleason 1957 J. Math. Mech. 6 885) is an important result in the foundations of quantum mechanics, where it justifies the Born rule as a mathematical consequence of the quantum formalism. Formally, it presents a key insight into the projective geometry of Hilbert spaces, showing that finitely additive measures on the projection lattice \( P(\mathcal{H}) \) extend to positive linear functionals on the algebra of bounded operators \( B(\mathcal{H}) \). Over many years, and by the effort of various authors, the theorem has been broadened in its scope from type I to arbitrary von Neumann algebras (without type I\(_2\) factors). Here, we prove a generalisation of Gleason’s theorem to composite systems. To this end, we strengthen the original result in two ways: first, we extend its scope to dilations in the sense of Naimark (1943 Dokl. Akad. Sci. SSSR 41 359) and Stinespring (1955 Proc. Am. Math. Soc. 6 211) and second, we require consistency with respect to dynamical correspondences on the respective (local) algebras in the composition (Alfsen and Shultz 1998 Commun. Math. Phys. 194 87). We show that neither of these conditions changes the result in the single system case, yet both are necessary to obtain a generalisation to bipartite systems.

Keywords: Gleason’s theorem, quantum foundations, quantum correlations, von Neumann algebras, Stinespring dilation
1. Gleason’s theorem

Gleason’s theorem is a landmark result in quantum theory. It justifies the Born rule as a mathematical fact of the projective geometry of Hilbert spaces\(^3\).

**Theorem 1. (Gleason [4])** Let \( \mathcal{H} \) be a separable Hilbert space, \( \dim(\mathcal{H}) \geq 3 \). Then every countably additive probability measure \( \mu : \mathcal{P}(\mathcal{H}) \to [0,1] \) over the projections on \( \mathcal{H} \) is of the form \( \mu(p) = \text{tr}[\rho_p p] \) for all \( p \in \mathcal{P}(\mathcal{H}) \), with \( \rho_p : \mathcal{H} \to \mathcal{H} \), \( \rho_p \geq 0 \), \( \text{tr}[\rho_p] = 1 \) a density operator.

Here, a countably additive probability measure is a map \( \mu : \mathcal{P}(\mathcal{H}) \to [0,1] \), \( \mu(1) = 1 \) and such that \( \mu(\bigcup_{i=1}^{\infty} p_i) = \sum_{i=1}^{\infty} \mu(p_i) \) for all \( p_i \in \mathcal{P}(\mathcal{H}) \) such that \( p_i p_j = 0 \) whenever \( i \neq j \). Note that when \( \mathcal{N} = M_2(\mathbb{C}) \), there exist measures that fail to extend to linear functionals.

While the original argument was for type I factors only, theorem 1 was later extended to type II and type III von Neumann algebras in [5–7] (see also [8]). In this setting, \( \mu \) is further called completely additive if \( \mu(\bigcup_{i \in I} p_i) = \sum_{i \in I} \mu(p_i) \) for every family of pairwise orthogonal projections \( (p_i)_{i \in I}, p_i \in \mathcal{P}(\mathcal{N}) \).\(^4\) Recall that a state \( \sigma \in \mathcal{S}(\mathcal{N}) \) is a positive, normalised linear functional, it is normal if \( \sigma(\bigcup_{i \in I} p_i) = \sum_{i \in I} \sigma(p_i) \) for all families of pairwise orthogonal projections \( (p_i)_{i \in I}, p_i \in \mathcal{P}(\mathcal{N}) \) (theorem 7.1.12 in [9]). Clearly, a (normal) state \( \sigma \) defines a finitely (completely) additive probability measure on \( \mathcal{P}(\mathcal{N}) \), conversely:

**Theorem 2 (Gleason-Christensen-Yeadon [5–7]).** Let \( \mathcal{N} \) be a von Neumann algebra with no summand of type I\(_2\) and let \( \mu : \mathcal{P}(\mathcal{N}) \to \mathbb{R} \) be a finitely additive probability measure on the projections of \( \mathcal{N} \). There exists a unique state \( \sigma_\mu \in \mathcal{S}(\mathcal{N}) \) such that \( \mu(p) = \sigma_\mu(p) \) for all \( p \in \mathcal{P}(\mathcal{N}) \). If \( \mu \) is completely additive then \( \sigma_\mu \) is normal and of the form \( \mu(p) = \sigma_\mu(p) = \text{tr}[\rho_\mu p] \) for all \( p \in \mathcal{P}(\mathcal{N}) \) with \( \rho_\mu \) a positive trace-class operator.

2. Gleason’s theorem in context

2.1. Partial order of contexts and probabilistic presheaf

Note that finite (complete) additivity only assumes that \( \mu : \mathcal{P}(\mathcal{N}) \to [0,1] \) is quasi-linear, i.e. linear in commuting von Neumann subalgebras. Remarkably, theorem 1 and its generalisation theorem 2 show that this already implies linearity on all of \( \mathcal{N} \).

We emphasise the passage from quasi-linearity to linearity as follows. Note that \( \mu \) restricts to a probability measure \( \mu_V = \mu|_V \) in every commutative von Neumann subalgebra \( V \subset \mathcal{N} \). \( \mu \) thus defines a collection of probability measures \( (\mu_V)_{V \subset \mathcal{N}} \), one for every commutative von Neumann subalgebra, and such that whenever \( V \subset W \) is a von Neumann subalgebra, the respective probability measures are related by restriction, \( \mu_W = \mu|_W \) (viz. marginalisation).

From this perspective, the constraints on measures \( \mu : \mathcal{P}(\mathcal{N}) \to [0,1] \) in Gleason’s theorem arise via the inclusion relations between commutative von Neumann subalgebras. This motivates the definition of the following partially ordered set (poset).

**Definition 1.** Let \( \mathcal{N} \) be a von Neumann algebra. The poset of commutative von Neumann subalgebras of \( \mathcal{N} \) is called the context category of \( \mathcal{N} \) and is denoted by \( \mathcal{V}(\mathcal{N}) \).

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\(^3\) Gleason’s theorem reduces traditional axiomatic systems. We note, however, that many modern interpretations of quantum mechanics do not rely on projective Hilbert space as a primitive, see e.g. [1–3].

\(^4\) For \( \mathcal{N} = \mathcal{B}(\mathcal{H}) \) with \( \mathcal{H} \) separable, complete additivity reduces to countable additivity.
The name ‘context category’ is motivated as follows. First, note that every poset can be regarded as a category, whose objects are the elements of the poset and whose arrows are defined by $a \rightarrow b \iff a \leq b$. Second, in quantum physics contextuality refers to the fact that not all observables, represented by the self-adjoint operators in $\mathcal{N}$, can be measured jointly in an arbitrary state. Only commuting operators can be measured jointly, i.e. those that are contained in a commutative von Neumann subalgebra—a context [12]. The constraints in Gleason’s theorem are therefore noncontextuality constraints: a projection $p \in \mathcal{P}(\mathcal{N})$ is assigned a probability $\mu(p)$ independent of the context that $p$ lies in: $\mu_V(p) = \mu_{\tilde{V}}(p) = \mu(p)$ whenever $p \in \tilde{V}, V$. Next, we formalise the idea that a measure $\mu : \mathcal{P}(\mathcal{N}) \rightarrow [0,1]$ corresponds to a collection of probability measures over $\mathcal{V}(\mathcal{N})$.

**Definition 2.** Let $\mathcal{N}$ be a von Neumann algebra with context category $\mathcal{V}(\mathcal{N})$. The (normal) probabilistic presheaf $\prod$ of $\mathcal{N}$ over $\mathcal{V}(\mathcal{N})$ is the presheaf given

(i) on objects: for all $V \in \mathcal{V}(\mathcal{N})$, let

$$ \prod_V := \{ \mu_V : \mathcal{P}(V) \rightarrow [0,1] \mid \mu_V \text{is a finitely (completely) additive probability measure} \} , $$

(ii) on arrows: for all $\tilde{V}, V \in \mathcal{V}(\mathcal{N})$ with $\tilde{V} \subset V$, let $\prod(i_{\tilde{V}V}) : \prod_V \rightarrow \prod_{\tilde{V}}$ with $\mu_V \mapsto \mu_{\tilde{V}}|_V$, where $i_{\tilde{V}V} : \tilde{V} \hookrightarrow V$ denotes the inclusion map between contexts $V \subset \tilde{V}$.

We remark that the study of presheaves over the partial order of contexts is at the heart of the topos approach to quantum theory [14–23]. For an introduction, see e.g. [24]. For the intimate relationship between contextuality (in the sense of definition 1) and various key theorems in quantum theory, see [12].

### 2.2. Gleason’s theorem in presheaf form

We have seen that a finitely (completely) additive probability measure over the projections of a von Neumann algebra $\mathcal{N}$ can be regarded as a collection of finitely (completely) additive probability measures over contexts $(\mu_V)_{V \in \mathcal{V}(\mathcal{N})}$, which satisfy the constraints $\mu_{\tilde{V}} = \mu_V|_{\tilde{V}}$ whenever $\tilde{V}, V \in \mathcal{V}(\mathcal{N})$ with $\tilde{V} \subset V$. In terms of definition 2, $\mu$ thus becomes a global section of the (normal) probabilistic presheaf $\prod$. We denote the set of global sections of $\prod$ by $\Gamma[\prod]$. In this terminology, we obtain the following reformulation of Gleason’s theorem [25–27].

**Theorem 3 (Gleason in presheaf form (I)).** Let $\mathcal{N}$ be a von Neumann algebra with no summand of type $I_2$. There is a bijective correspondence between (normal) states on $\mathcal{N}$ and global sections of the (normal) probabilistic presheaf $\prod$ of $\mathcal{N}$ over $\mathcal{V}(\mathcal{N})$.

**Proof.** Note that every (normal) state $\sigma : \mathcal{N} \rightarrow \mathbb{C}$ defines a finitely (completely) additive probability measure over the projections of $\mathcal{N}$ (cf theorem 7.1.12 in [9]), which corresponds with

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5 This is in contrast to classical physics, where observables are represented by elements in a (commutative) algebra of functions on a (locally) compact Hausdorff space. The Bell-Kochen-Specker theorem shows that in the noncommutative case such a description is impossible [10, 11].

6 We remark that while the notion of joint measurability coincides with commutativity for observables (equivalently, for projection-valued measures), it differs for generalised measurements (that is, for positive operator-valued measures) [13].

7 A presheaf $\mathcal{P}$ over the category $\mathbb{C}$ is a functor $\mathcal{P} : \mathbb{C}^{op} \rightarrow \text{Set}$. We denote presheaves with an underscore.

8 Here, ‘global’ refers to the fact that $\mu$ satisfies the restriction constraints in $\prod$ over all of $\mathcal{V}(\mathcal{N})$. In contrast, a local section satisfies the constraints only on a sub-poset of $\mathcal{V}(\mathcal{N})$. 
a global section of $\Pi$ by the preceding discussion. By the same correspondence, each global section of the (normal) probabilistic presheaf $\gamma \in \Gamma(\Pi(N))$ extends to a (normal) state $\sigma \in S(N)$ as a consequence of theorem 2.

Our generalisation of Gleason’s theorem to composite systems (theorem 7 below) will similarly be phrased in presheaf form.

2.3. Linearity without positivity

The canonical product on partial orders, denoted $\mathcal{V}_1 \times \mathcal{V}_2$, is the Cartesian product with elements $(V_1, V_2)$ for $V_1 \in \mathcal{V}_1$, $V_2 \in \mathcal{V}_2$ and order relations such that for all $V_1, V_2, V_1', V_2' \in \mathcal{V}_2$:

$$(\bar{V}_1, \bar{V}_2) \subseteq (V_1, V_2) :\leftrightarrow \bar{V}_1 \subseteq V_1 \text{ and } \bar{V}_2 \subseteq V_2.$$ 

It is interesting to ask whether the state space of the composite system can be recovered from product contexts $V \in \mathcal{V}(N_1) \times \mathcal{V}(N_2) \subseteq \mathcal{V}(N_1 \otimes N_2)$ only. For this to be possible states on the subsystem algebras need to define unique states on the composite algebra and vice versa. For this reason we will consider the spatial tensor product between von Neumann algebras $\otimes$ (for details, see section 11.2 in [9]). In particular, recall that given normal states $\sigma_1 \in (N_1)_+$ and $\sigma_2 \in (N_2)_+$, where $N_\sigma$ denotes the predual (similarly, $N^\ast$ denotes the dual) of $N$, there exists a unique normal product state $\sigma = \sigma_1 \otimes \sigma_2 \in (N_1 \otimes N_2)_+$ (cf proposition 11.2.7 in [9]). Conversely, every normal linear functional $\sigma \in N_\sigma$ on the spatial tensor product $N = N_1 \otimes N_2$ is the norm limit of normal product states $\sigma_1 \otimes \sigma_2$ (proposition 11.2.8 in [9]). It follows that we can identify the product context $V = (V_1, V_2) \in \mathcal{V}(N_1) \times \mathcal{V}(N_2)$ with the commutative von Neumann subalgebra $\mathcal{V} = V_1 \otimes V_2 \subset N_1 \otimes N_2$ for every $V_1 \in N_1$ and $V_2 \in N_2$.

**Theorem 4.** Let $N_1$, $N_2$ be von Neumann algebras with no summand of type $I_2$ and let $N = N_1 \otimes N_2$. There is a bijective correspondence between global sections of the probabilistic presheaf $\Pi$ of $N$ over $\mathcal{V}(N_1) \times \mathcal{V}(N_2)$ and linear functionals $\sigma : N \to \mathbb{C}$ such that $\sigma(1) = 1$ and $\sigma(a \otimes b) \geq 0$ for all $a \in (N_1)_+$ and $b \in (N_2)_+$.

**Proof.** Clearly, every state $\sigma \in S(N)$ yields a global section $\gamma_\sigma \in \Gamma(\Pi(\mathcal{V}(N_1) \times \mathcal{V}(N_2)))$, defined by $\gamma_\sigma(p, q) = \sigma(p \otimes q)$ for all $p \in \mathcal{P}(N_1)$ and $q \in \mathcal{P}(N_2)$. In particular, $\gamma_\sigma(1, 1) = \sigma(1) = 1$, and $\sigma(a \otimes b) \geq 0$ for all $a \in (N_1)_+$ and $b \in (N_2)_+$ by positivity of $\sigma$.

Conversely, we show that every global section $\gamma \in \Gamma(\Pi(\mathcal{V}(N_1) \times \mathcal{V}(N_2)))$ defines a unique linear functional $\gamma_\sigma$ on $N = N_1 \otimes N_2$, which restricts to $\gamma$ on $\mathcal{V}(N_1) \times \mathcal{V}(N_2)$, i.e. such that $\gamma = (\sigma, (p, q))_{p \in \mathcal{P}(N_1) \times \mathcal{P}(N_2)}$. Fix a context $V_1 \in \mathcal{V}(N_1)$ and consider the corresponding partial order of contexts under inclusion, inherited from $V_{1\otimes 2} := \mathcal{V}(N_1) \times \mathcal{V}(N_2)$ by restriction,

$$V_{1\otimes 2}(V_1) := \{ V_1 \times V_2 \mid V_2 \in \mathcal{V}(N_2) \}.$$ 

In every context $V = V_1 \otimes V_2 \in V_{1\otimes 2}$, the probability measure $\mu_{\gamma_\sigma} \in \mathcal{P}(V_{1\otimes 2})$ corresponding to the global section $\gamma$ can be written as follows:

$$\forall p \in \mathcal{P}(V_1), q \in \mathcal{P}(V_2) : \mu_{\gamma_\sigma}(p, q) = \mu_{\gamma_\sigma}(p) \mu_{\gamma_\sigma}(q | p) = \mu_{\gamma_\sigma}(p) \gamma_{\sigma}^p(q) = \mu_{\gamma_\sigma}(p) \sigma_{\gamma}^p(q). \quad (1)$$

Here, $\mu_{\gamma_\sigma}(q | p)$ denotes the conditional probability distribution of $q$ given $p$. Note that $(\mu_{\gamma_\sigma}(\cdot | p))_{p \in \mathcal{P}(N_1)} := \gamma_{\sigma}^p \in \Gamma(\Pi(\mathcal{V}_{1\otimes 2}(V_1)))$ defines a global section of the probabilistic presheaf $\Pi(\mathcal{V}_{1\otimes 2}(V_1))$, which also depends on $p \in \mathcal{P}(V_1)$. Since $\Pi_{1\otimes 2}(V_1) \cong \mathcal{V}(N_2)$, such global sections correspond with states on $N_2$ by theorem 3, hence, $\sigma_{\gamma}^p \in S(N_2)$ for all $p \in \mathcal{P}(V_1)$. Moreover, since $V_1 \in \mathcal{V}(N_1)$ was arbitrary, equation (1) holds for all $p \in \mathcal{P}(N_1)$. 

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Define a map \( \tilde{\phi}_\gamma : \mathcal{P}(\mathcal{N}_1) \to \mathcal{N}_2^* \) by \( \tilde{\phi}_\gamma(p) := \mu_{\gamma_1}(p)\sigma_2^\gamma \). Clearly, \( \tilde{\phi}_\gamma \) is bounded and \( \mathcal{N}_2^* \) is a Banach space. Moreover, for \( p = p_1 + p_2 \) with \( p_1, p_2 \in \mathcal{P}(\mathcal{N}_1) \) orthogonal, i.e. \( p_1 p_2 = 0 \),
\[
\tilde{\phi}_\gamma(p) = \mu_{\gamma_1}(p)\sigma_2^\gamma = \mu_{\gamma_1}(p_1)\sigma_2^{\gamma_1} + \mu_{\gamma_1}(p_2)\sigma_2^{\gamma_2} = \tilde{\phi}_\gamma(p_1) + \tilde{\phi}_\gamma(p_2),
\]
by additivity of \( \gamma \). By theorem A in [28], \( \tilde{\phi}_\gamma \) uniquely extends to a linear map \( \tilde{\phi}_\gamma : \mathcal{N}_1 \to \mathcal{N}_2^* \).

Furthermore, \( \sigma_\gamma(a \otimes b) := \tilde{\phi}_\gamma(a)(b) \) (linearly) extends to a linear functional such that \( \sigma_\gamma(1) = \gamma(1, 1) = 1 \). Finally, \( \sigma_\gamma(a \otimes b) \geq 0 \) for all \( a \in (\mathcal{N}_1)_+, b \in (\mathcal{N}_2)_+ \), using the spectral decomposition of \( a, b \) together with positivity of the (product) measures \( \mu_{\gamma_i} \) for all \( V \in \mathcal{V}(\mathcal{N}_1) \times \mathcal{V}(\mathcal{N}_2) \).

We emphasise that the reduction to products of commutative von Neumann algebras in \( \mathcal{V}(\mathcal{N}_1) \times \mathcal{V}(\mathcal{N}_2) \) is already sufficient to lift quasi-linear measures to linear functionals. In finite dimensions this has been observed before [29, 30]. Yet, unlike theorem 3 the linear functionals thus obtained are not necessarily positive, hence, theorem 4 does not provide a classification of states on \( \mathcal{N} = \mathcal{N}_1 \otimes \mathcal{N}_2 \). From this perspective, it seems unjustified to call theorem 4 a generalisation of Gleason’s theorem (theorem 2) to composite systems. In contrast, in [31] we do obtain a classification of bipartite states for finite dimensional, simple algebras. Here, we will extract and extend the relevant arguments from [31] to von Neumann algebras, which will enable us to prove a substantially stronger composite Gleason theorem in theorem 7 below.

### 3. A composite Gleason theorem

Theorem 4 highlights that in order to obtain a generalisation of Gleason’s theorem, which singles out the state spaces of composite systems, one needs to strengthen definition 2. In particular, one needs to characterise positivity of the linear functional \( \sigma : \mathcal{N}_1 \otimes \mathcal{N}_2 \to \mathbb{C} \) in theorem 4.

We will derive positivity of \( \sigma_\gamma \), from complete positivity of an associated map \( \phi_\gamma : \mathcal{N}_1 \to \mathcal{N}_2 \). In order to motivate this map, recall that in the proof of theorem 4 we constructed a map \( \tilde{\phi}_\gamma : \mathcal{N}_1_+ \to (\mathcal{N}_2)_+^* \), into positive linear functionals on \( \mathcal{N}_2 \). If we further assume \( \gamma \in \mathcal{P}_+(\mathcal{V}(\mathcal{N}_1) \times \mathcal{V}(\mathcal{N}_2)) \) to be completely additive, it is easy to see that we instead obtain a map \( \phi_\gamma : (\mathcal{N}_1)_+ \to (\mathcal{N}_2)_+ \) into the predual of \( \mathcal{N}_2 \), i.e. into positive normal linear functionals. In this case, we immediately identify \( \phi_\gamma(p) \in (\mathcal{N}_2)_+ \) for every \( p \in \mathcal{P}(\mathcal{N}_1) \) with a positive trace-class operator in \( \mathcal{N}_2 \). More precisely, consider a faithful representation of the von Neumann algebra \( \mathcal{N}_2 \) as bounded operators on a Hilbert space \( \mathcal{H}_2 \). With respect to the latter, every normal state \( \sigma \in (\mathcal{N}_2)_+ \), is of the form \( \sigma(b) = \text{tr}_{\mathcal{N}_2}[\rho_{\sigma}b] = \text{tr}_{\mathcal{N}_2}[\rho_{\sigma}b] \) for all \( b \in \mathcal{N}_2 \) (theorem 7.1.9 in [9]) and where \( \rho_{\sigma} \) is a bounded, positive (self-adjoint) operator of unit trace.

Based on this identification we will seek conditions that ensure that \( \phi_\gamma \) is not only positive but also completely positive. We will achieve this in two steps: first, we extend the constraints in Gleason’s theorem to dilations in section 3.1; second, we enforce a consistency condition with respect to dynamical correspondences in section 3.2. Finally, in section 3.3 we prove our main result: a generalisation of Gleason’s theorem to composite systems. Along the way we show that neither of these additional constraints jeopardises applicability in the single system case. Indeed, we prove sharper versions of Gleason’s theorem, thereby addressing a number of subtleties that only arise for composite systems.

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9 We may take \( \mathcal{H}_2 \) to be the unique Hilbert space defined by the standard form of \( \mathcal{N}_2 \) [32].

10 In physics parlance, \( \rho_{\sigma} \) is called a density matrix.
3.1. Dilations

First, we require that measures $\mu V$ admit dilations in every commutative von Neumann subalgebra $V \in \mathcal{V}(\mathcal{N})$. More precisely, by Gelfand duality we may identify every context $V \in \mathcal{V}(\mathcal{N})$ with a compact Hausdorff space $X$ such that $\mu V$ becomes a measure on $X$. In particular, we identify the projections $P(V)$ with the clopen subsets of the Gelfand spectrum of $V$. In this way, we can interpret $\mu V$ as a $B(\mathcal{H})$-valued measure for $\mathcal{H} = \mathbb{C}$ (thus also $B(\mathcal{H}) \cong \mathbb{C}$). By Naimark’s theorem [33, 34], the latter admits a spectral dilation of the form $\mu V = v^* \varphi V v,$ where $\varphi_V : P(V) \rightarrow \mathcal{P}(\mathcal{K})$ is an embedding for some Hilbert space $\mathcal{K}$ and $v : \mathbb{C} \rightarrow \mathcal{K}$ is a linear map (equivalently, a vector $v \in \mathcal{K}$). By choosing $\mathcal{K}$ sufficiently large, we can choose $\mathcal{K}$ independent of contexts $V \in \mathcal{V}(\mathcal{N})$. Moreover, we take $v$ to be independent of contexts $V \in \mathcal{V}(\mathcal{N})$, since we can absorb any context dependence on $v$ into $\varphi_V$.

Specifically, by a finitely (completely) additive embedding $\varphi_V$ we mean that (i) $\varphi_V$ is an orthomorphism, that is, $\varphi_V(0) = 0$, $\varphi_V(1 - p) = 1 - \varphi_V(p)$ for all $p \in P(V)$, and $\varphi_V(p_1 + p_2) = \varphi_V(p_1) + \varphi_V(p_2)$ for all $p_1, p_2 \in P(V), p_1 p_2 = 0, (\varphi_V(\sum_{i \in I} p_i)) = \sum_{i \in I} \varphi_V(p_i)$ for every family of pairwise orthogonal projections $(p_i)_{i \in I}$, $p_i \in P(V)$; and (ii) we further require that embeddings preserve spatial tensor products, i.e., whenever $V = V_1 \otimes V_2$, then $\varphi_V = \varphi_{V_1} \otimes \varphi_{V_2}$, where $\varphi_{V_1} : P(V_1) \rightarrow \mathcal{P}(\mathcal{K}_1)$ and $\varphi_{V_2} : P(V_2) \rightarrow \mathcal{P}(\mathcal{K}_2)$ are orthomorphisms and $\mathcal{K} = \mathcal{K}_1 \otimes \mathcal{K}_2$. Taken together, this suggests to strengthen definition 2 as follows.

**Definition 3.** Let $\mathcal{N}$ be a von Neumann algebra with context category $\mathcal{V}(\mathcal{N})$. The (normal) dilated probabilistic presheaf $\Pi_D$ of $\mathcal{N}$ over $\mathcal{V}(\mathcal{N})$ is the presheaf given

(i) on objects: for all $V \in \mathcal{V}(\mathcal{N})$, let

$$\Pi_D(V) := \{ \mu_V : P(V) \rightarrow [0, 1] \mid \mu_V \text{ a probability measure of the form } \mu_V = v^* \varphi_V v \text{ with } v : \mathbb{C} \rightarrow \mathcal{K} \text{ linear, and } \varphi_V : P(V) \rightarrow \mathcal{P}(\mathcal{K}) \text{ a finitely (completely) additive embedding preserving spatial tensor products, that is, } \varphi_V = \varphi_{V_1} \otimes \varphi_{V_2} \text{ and } \mathcal{K} = \mathcal{K}_1 \otimes \mathcal{K}_2, \text{ where } \varphi_{V_i} : P(V_i) \rightarrow \mathcal{P}(\mathcal{K}_i), \; i = 1, 2 \text{ are orthomorphisms whenever } V = V_1 \otimes V_2 \}.$$

(ii) on arrows: for all $V, \tilde{V} \in \mathcal{V}(\mathcal{N})$, if $\tilde{V} \subseteq V$, let

$$\Pi_D(i) : \Pi_D(V) \rightarrow \Pi_D(\tilde{V}) \text{ with } \mu_V \mapsto \varphi_V|_{\tilde{V}}.$$

As in definition 2, the condition that $\varphi_V$ preserves orthogonality is required in commutative von Neumann subalgebras only. Consequently, definition 3 requires global sections of $\Pi_D$ to be quasi-linear only. The key difference to the probabilistic presheaf $\Pi$ in definition 2 is that we require the constraints to hold also with respect to dilations.

If definition 3 is to generalise the single system case, it should be consistent with it. We therefore start by analysing global sections of the dilated probabilistic presheaf in the original setting. We have the following refined version of theorem 3.

**Theorem 5. (Gleason in contextual form (II))** Let $\mathcal{N}$ be a von Neumann algebra with no summand of type I$_2$. There is a bijective correspondence between (normal) states on $\mathcal{N}$ and global sections of the (normal) dilated probabilistic presheaf $\Pi_D$ of $\mathcal{N}$ over $\mathcal{V}(\mathcal{N})$.

**Proof.** Let $\sigma \in \mathcal{S}(\mathcal{N})$ be (normal) state and consider the corresponding (normal) representation $\pi : \mathcal{N} \rightarrow B(\mathcal{K})$ in the GNS construction of $\sigma$. In this representation, $\sigma$ is a vector state $\sigma(a) = (\pi(a)v, v)$ with $v \in \mathcal{K}$ for all $a \in \mathcal{N}$. Interpreting $v : \mathbb{C} \rightarrow \mathcal{K}$ as a linear map, this reads
\[ \sigma(a) = v^* \pi(a) v \] for all \( a \in \mathcal{N} \), and since \( \pi \) is a (normal) representation, \( \varphi := \pi|_{\mathcal{P}(\mathcal{N})} \) is a completely additive \( \Phi \)-representation. It follows that \( \{ \sigma|_{\mathcal{P}(\mathcal{V})} \}_{\mathcal{V} \in \mathcal{V}(\mathcal{N})} \) defines a global section of the (normal) dilated probabilistic probabilistic \( \Pi_2(\mathcal{V}(\mathcal{N})) \).

Conversely, by the proof of theorem 2, \( \gamma = (v^* \varphi v)_{v \in \mathcal{V}(\mathcal{N})} \in \Gamma[\Pi(\mathcal{V}(\mathcal{N}))] \) defines a (normal) state \( \gamma \in \mathcal{S}(\mathcal{N}) \). In fact, the embeddings \( (\varphi v)_{v \in \mathcal{V}(\mathcal{N})} \) define a (completely additive) \( \mathcal{V}(\mathcal{N}) \)-homomorphism \( \varphi : \mathcal{P}(\mathcal{N}) \to \mathcal{P}(\mathcal{K}) \), which by corollary 2 in [35] extends to a (normal) Jordan \( * \)-homomorphism \( \Phi : \mathcal{N} \to \mathcal{B}(\mathcal{K}) \) such that \( \sigma_\gamma = v^* \Phi v \) and \( \varphi_\gamma = \Phi|_{\mathcal{P}(\mathcal{N})} \). \( \square \)

Despite the fact that, by theorems 3 and 5, global sections of probabilistic and dilated probabilistic presheaf correspond with states on \( \mathcal{N} \), the dilated probabilistic presheaf encodes more constraints than the probabilistic presheaf. To see this, note that we deduced linearity in theorem 4 from a theorem due to Bunce and Wright in [28]. Under the conditions of definition 3 a stronger version applies [35], leading to the following refinement of theorem 4.

**Lemma 1.** Let \( \mathcal{N}_1, \mathcal{N}_2 \) be von Neumann algebras with no summand of type II, \( \mathcal{N} = \mathcal{N}_1 \oplus \mathcal{N}_2 \), and let \( \Pi_2(\mathcal{V}(\mathcal{N}_1) \times \mathcal{V}(\mathcal{N}_2)) \) be the (normal) dilated probabilistic presheaf over the composite context category \( \mathcal{V}(\mathcal{N}_1) \times \mathcal{V}(\mathcal{N}_2) \). Then every global section \( \gamma \in \Gamma[\Pi_2(\mathcal{V}(\mathcal{N}_1) \times \mathcal{V}(\mathcal{N}_2))] \) uniquely extends to a linear functional \( \sigma_\gamma \), given for all \( a \in \mathcal{N}_1 \) and \( b \in \mathcal{N}_2 \) by

\[
\sigma_\gamma(a \otimes b) = \delta_{a,0}(\sigma_2(b)) =: \text{tr}_{\mathcal{N}_1} \left[ \delta(\sigma_1(a) \sigma_2(b)) \right],
\]

where \( \delta = * \circ \varphi : \mathcal{N}_1 \to \mathcal{N}_2 \) is of the form \( \varphi = w^* \Phi_\gamma w \) with \( \Phi_\gamma : \mathcal{N}_1 \to \mathcal{B}(\mathcal{K}) \) a (normal) Jordan \( * \)-homomorphism, \( \mathcal{V}_1 = \mathcal{B}(\mathcal{H}_1) \) and \( \mathcal{V}_2 = \mathcal{B}(\mathcal{H}_2) \) a (normal) faithful \( * \)-representation.

**Proof.** As in the proof of theorem 4, we obtain a map \( \hat{\varphi}_\gamma : \mathcal{P}(\mathcal{N}_1) \to \mathcal{P}(\mathcal{N}_2) \), which by the correspondence between normal states and positive trace-class operators (theorem 7.1.9 in [9]) yields a map \( \hat{\varphi}_\gamma : \mathcal{P}(\mathcal{N}_1) \to (\mathcal{N}_2)_+ \) into positive trace-class operators (viz. density matrices), and which further lifts to a positive bounded linear map \( \varphi_\gamma : \mathcal{N}_1 \to \mathcal{N}_2 \) (cf [28]).

Furthermore, by assumption there exist a Hilbert space \( \mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2 \), a linear map \( v : \mathcal{C} \to \mathcal{K} \) and orthomorphisms \( \varphi_1 : \mathcal{P}(\mathcal{N}_1) \to \mathcal{P}(\mathcal{K}_1) \) and \( \varphi_2 : \mathcal{P}(\mathcal{N}_2) \to \mathcal{P}(\mathcal{K}_2) \) (uniquely defined by \( \varphi_1|_{\mathcal{V}_1} = \varphi|_{\mathcal{V}_1} \) and \( \varphi_2|_{\mathcal{V}_2} = \varphi|_{\mathcal{V}_2} \) for all \( \mathcal{V}_1 \in \mathcal{V}(\mathcal{N}_1) \), \( \mathcal{V}_2 \in \mathcal{V}(\mathcal{N}_2) \), respectively) such that

\[
\sigma_\gamma(p \otimes q) = v^* (\varphi_1(p) \otimes \varphi_2(q)) v
\]

for all \( p \in \mathcal{P}(\mathcal{N}_1) \) and \( q \in \mathcal{P}(\mathcal{N}_2) \). Without loss of generality, we may identify \( \mathcal{K}_2 = \mathcal{H}_2 \) (for \( \mathcal{N}_2 \subset \mathcal{B}(\mathcal{H}_2) \)) and assume that \( \varphi_2(q) = \pi_2(q) \) lifts to a faithful \( * \)-representation of \( \mathcal{N}_2 \) (cf corollary 1 in [35]). Let \( v = \sum_i c_i v_{1i} \otimes v_{2i} \in \mathcal{K} \), where \( c_i \in \mathcal{C} \) and \( \{v_{11}, v_{12}, v_{21}, v_{22}\} \) are orthonormal bases in \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \), respectively. Then for all \( p \in \mathcal{P}(\mathcal{N}_1) \) and \( q \in \mathcal{P}(\mathcal{N}_2) \) we write

\[
\sigma_\gamma(p \otimes q) = \sum_{ij} c_i^* v_{1i}^* v_{2j} (\varphi_1(p) \otimes \pi_2(q)) v_{1j}^* v_{2i}^* (\varphi_2(q) \otimes \pi_1(p)) v_{1i} v_{2j}
\]

Note that two global sections \( \gamma, \gamma' \in \Gamma[\Pi_2(\mathcal{V}(\mathcal{N}))] \) are identified if \( \mu_\gamma = \mu_{\gamma'} \) for all \( \mu \in \mathcal{V}(\mathcal{N}) \).

11 See section 3.2 below for a definition and more details on Jordan algebras.

12 In order to avoid clutter, here and below we leave \( \gamma \)-subscripts on \( v, \phi \) and \( \varphi, \gamma \), \( i, j = 1, 2 \) implicit.
Note that under the Hermitian adjoint in the last line the inner product on $\mathcal{N}_1$ is changed to its complex conjugate. To reflect this, let $\tilde{K}_1 \cong K_1$ be such that for all pairs $v_{1i}, v_{ik} \in \tilde{K}_1$, $\tilde{v}_{1i}, \tilde{v}_{ik} \in \tilde{K}_1$ and $a \in \mathcal{B}(K_1)$, $\tilde{v}_{1k}^* \tilde{v}_{1i} = (\tilde{v}_{1k}, \tilde{v}_{1i}) := (v_{1i}, v_{1k}) = (v_{1k}, v_{1i})^*$, and define an action of $\mathcal{B}(\mathcal{K})$ on $\tilde{K}_1$ by

$$
(\tilde{v}_{1k}^* \tilde{v}_{1i}) := (v_{1k}^* v_{1i})^*.
$$

Next, define a bounded linear map $w : \mathcal{K}_2 \to \tilde{K}_1 \otimes \mathcal{K}_2$ by $w = \sum_{ijkl} c_i^j c_k^l (\tilde{v}_{1j}^* \tilde{v}_{1l} \otimes v_{20} v_{2j}^*)$ with $v_{20} \in \mathcal{K}_2$ a unit vector, and define an orthomorphism $\varphi' : \mathcal{P}(\mathcal{N}_1) \to \mathcal{P}(\mathcal{K})$ by $\varphi' = \varphi \otimes 1_{\mathcal{B}(\mathcal{K}_2)}$ such that

$$
w^* \varphi'^*(p) w = \sum_{ijkl} c_i^j c_k^l (\tilde{v}_{1j}^* \tilde{v}_{1l} (\varphi_1^*)(p)) v_{20} v_{2j}^*
$$

$$
= \sum_{ijkl} c_i^j c_k^l (\tilde{v}_{1j}^* \tilde{v}_{1l} (\varphi_1^*)(p)) v_{20} v_{2j}^*
$$

$$
= (\sum_{ijkl} c_i^j c_k^l (\tilde{v}_{1j}^* \tilde{v}_{1l} (\varphi_1^*)(p)) v_{20} v_{2j}^*)^*,
$$

where we changed labels $i \leftrightarrow k, j \leftrightarrow l$ in the third line, and used equation (5) in the last line. Combining equations (4) and (6) we thus obtain

$$
\tilde{\varphi}_1^*(p) = \text{tr}_{\mathcal{K}_1^*} [w^* \varphi'^*(p) w \pi_2(q)].
$$

With corollary 1 in [35] the finitely (completely) additive orthomorphism $\varphi' : \mathcal{P}(\mathcal{N}_1) \to \mathcal{P}(\mathcal{K})$ extends to a (normal) Jordan $^*$-homomorphism $\Phi_{\gamma} : J(\mathcal{N}_1) \to \mathcal{B}(\mathcal{K})$ (and $\pi_2$ to a normal $^*$-representation). Finally, we have $\Phi_{\gamma}^* = w^* \Phi_{\gamma}^* w$, hence, $\phi_{\gamma} = w^* \Phi_{\gamma} w$. \hfill $\square$

Recall that a map $\phi : \mathcal{N} \to \mathcal{B}(\mathcal{H})$ is called decomposable if it is of the form $\phi = v^* \Phi v$ for $v : \mathcal{N} \to \mathcal{K}$ a bounded linear map and $\Phi : \mathcal{N} \to \mathcal{B}(\mathcal{K})$ a Jordan $^*$-homomorphism. Such maps satisfy a weaker notion of positivity than complete positivity [36], owing to their close resemblance with completely positive maps $\phi = v^* \Phi v$, for which $\Phi$ is a $C^*$-homomorphism by the classification due to Stinespring [34]. By lemma 2 below, completely positive maps $\phi_{\gamma}$ correspond with positive linear functionals under equation (2). In contrast, linear functionals $\sigma_{\gamma}$ corresponding to decomposable maps under equation (2) are generally not positive. We infer that for $\sigma_{\gamma}$ to be a state, $\phi_{\gamma}$ in lemma 1 further needs to lift to a completely positive map, equivalently the Jordan $^*$-homomorphism in lemma 1 needs to lift to a $C^*$-homomorphism.

3.2. Dynamical correspondences

We recall some basic facts about Jordan algebras. A Jordan algebra $\mathcal{J}$ is a commutative algebra, which satisfies the characteristic equation $(a^2 \circ b) \circ a = a^2 \circ (b \circ a)$ for all $a, b \in \mathcal{J}$. $\mathcal{J}$ is called a JB algebra if it is also a Banach space such that $||a \circ b|| \leq ||a|| \cdot ||b||$ and $||a^2|| = \text{tr} \left[ \sum_{ijkl} c_i^j c_k^l (\tilde{v}_{1j}^* \tilde{v}_{1l} (\varphi_1^*)(p)) v_{20} v_{2j}^* \right].$
A JBW algebra isomorphic to a subalgebra of $\mathcal{J}(\mathcal{N})$. Finally, $\mathcal{J}$ is called a JBW algebra if it is a JB algebra with a (unique) predual. A Jordan homomorphism $\Phi : \mathcal{J}_1 \to \mathcal{J}_2$ is a linear map such that $\Phi(a \circ b) = \Phi(a) \circ \Phi(b)$ for all $a, b \in \mathcal{J}_1$. For more details on Jordan algebras, see [37].

Note that the self-adjoint part of a von Neumann algebra $\mathcal{N}$ naturally gives rise to a real JBW-algebra under the symmetrised product $a \circ b = \frac{1}{2}(a, b) = \frac{1}{2}(ab + ba)$. We denote this algebra by $\mathcal{J}(\mathcal{N})_{sa} = (\mathcal{N}_{sa}, \circ)$ and by $\mathcal{J}(\mathcal{N})$ its complexification[14]. In this case we speak of Jordan *-homomorphisms if $\Phi$ is a Jordan homomorphism and $\Phi^*(a) = \Phi(a^*)$.

In general, $\mathcal{J}(\mathcal{N})$ does not determine $\mathcal{N}$ completely, as it lacks compatibility with the antisymmetric part or commutator of the associative product in $\mathcal{N}$, $[a, b] = ab - ba$. In particular, for the Jordan *-homomorphism in Lemma 1 to lift to a $C^*$-homomorphism it needs to preserve commutators. On the level of JBW algebras, this can be expressed in terms of one-parameter groups of Jordan automorphisms $\mathbb{R} \ni t \mapsto e^{\delta t}(a) \in \text{Aut}(\mathcal{J})$, where $\delta \in \mathcal{OD}_+(\mathcal{J})$ is a skew order derivation, i.e. a bounded linear map $\delta : \mathcal{J}_+ \to \mathcal{J}_+$ acting on the positive cone of $\mathcal{J}$ such that $\delta(1) = 0$ (for details, see [38]). For $\mathcal{J} = \mathcal{J}(\mathcal{N})_{sa}$, these are of the form $\delta_{ia}(b) = \frac{1}{2}[a, b]$ for all $a, b \in \mathcal{J}(\mathcal{N})_{sa}$. This yields a canonical map $\psi_N : \mathcal{J}(\mathcal{N})_{sa} \to \mathcal{OD}_+(\mathcal{J}(\mathcal{N})_{sa})$, $\psi_N : a \mapsto \delta_{ia}$, which gives expression to the double role of self-adjoint operators as observables and generators of symmetries [39–41]. In particular, $e^{\psi_N(a)(t)}(b) = e^{\delta_{ia}t}be^{-\delta_{ia}t}$ expresses the unitary evolution with Hamiltonian $a$ and time parameter $t$ of the system $\mathcal{J}(\mathcal{N})_{sa}$. We also define the map $\psi_N^* := \circ \circ \psi_N : a \mapsto \delta_{ia}$ for all $a \in \mathcal{J}(\mathcal{N})_{sa}$ (cf proposition 15 in [38]).

Note that $\psi_N(a)(a) = 0$ and $[\delta_{ia}, \delta_{ib}] = -[\psi_N(a), \psi_N(b)]$ for all $a, b \in \mathcal{J}(\mathcal{N})_{sa}$, where $\delta_{ia}(b) := a \circ b$. More generally, Alfsen and Shultz define a dynamical correspondence to be any map $\psi : \mathcal{J} \to \mathcal{OD}_+(\mathcal{J})$ satisfying these properties and prove that a JBW algebra $\mathcal{J}$ is a JW algebra if and only if $\mathcal{J}$ admits a dynamical correspondence[15]. In this case, associative products bijectively correspond with dynamical correspondences (theorem 23 in [40]).

We conclude that a von Neumann algebra $\mathcal{N}$ is determined as the pair $(\mathcal{J}(\mathcal{N}), \psi_N)$ with the canonical dynamical correspondence $\psi_N : a \mapsto \delta_{ia} = \frac{1}{2}[a, \cdot]$ for all $a \in \mathcal{J}(\mathcal{N})_{sa}$. It follows that for a Jordan *-homomorphism $\Phi : \mathcal{J}(\mathcal{N}_1) \to \mathcal{J}(\mathcal{N}_2)$ to lift to a homomorphism between the respective von Neumann algebras $\mathcal{N}_1$ and $\mathcal{N}_2$ it is for $\mathcal{J}$ to preserve the respective dynamical correspondences $\psi_{N_1}$ and $\psi_{N_2}$. Returning to the Jordan *-homomorphism $\Phi_\gamma$ in Lemma 1, we need to lift the dynamical correspondence $\psi_{N_1}$ to (a subalgebra of) $\mathcal{B}(\mathcal{K})$. To do so, note first that the argument reduces to factors in $\mathcal{N}_2$ [40], and further that we may choose $\nu$ in definition 3 so that it preserves the factor decomposition. With this choice, let $\mathcal{N}_2 \subseteq \mathcal{B}(\mathcal{K})$ be the largest von Neumann algebra which restricts to $\mathcal{N}_2$ under $\nu$. Clearly, $\Phi(\mathcal{N}_1) \subseteq \mathcal{N}_2 \subseteq \mathcal{B}(\mathcal{K})$, and it is easy to see that in this case $\psi_{N_2}$ defines a unique dynamical correspondence $\psi_{N_2}^\nu$ on $\mathcal{N}_2$ via the completely positive map $\nu \cdot \nu^* : \mathcal{N}_2 \to \mathcal{N}_2$. This entitles us to the following key definition (cf [31]).

**Definition 4.** Let $\mathcal{N}_1 = (\mathcal{J}(\mathcal{N}_1), \psi_{N_1})$, $\mathcal{N}_2 = (\mathcal{J}(\mathcal{N}_2), \psi_{N_2})$ be von Neumann algebras. A global section of the dilated probabilistic presheaf $\gamma \in \Gamma[\mathcal{H}_0(\mathcal{V}(\mathcal{N}_1) \times \mathcal{V}(\mathcal{N}_2))]$ is called time-oriented if the Jordan *-homomorphism $\Phi_\gamma$ in Lemma 1 preserves dynamical correspondences,

$$\Phi_\gamma \circ \psi_{N_1} = \psi_{N_2}^\nu \circ \Phi_\gamma,$$

where $\psi_{N_2}^\nu$ is the dynamical correspondence on $\mathcal{N}_2 \subseteq \mathcal{B}(\mathcal{K})$ uniquely defined by $\psi_{N_2}$ and $\nu$.

---

14 A JBW algebra isomorphic to a subalgebra of $\mathcal{J}(\mathcal{N})_{sa}$ is also called a JW algebra.

15 We add that, unlike a JBW algebra, a von Neumann algebra is generally not anti-isomorphic to itself [42].
The terminology reflects the fact that the one-parameter groups \( \mathbb{R} \ni t \mapsto e^{it\mathcal{N}} \) express unitary dynamics in quantum theory. Consequently, they give physical meaning to \( t \) as a time parameter. In particular, \( \psi_N^t : a \to \delta_{t a} \) fixes the (canonical) forward time direction of the system described by \( \mathcal{N} \).

Once again, we show that definition 4 poses no additional constraint in the single system case.

**Theorem 6 (Gleason in contextual form (III)).** Let \( \mathcal{N} = (\mathcal{J}(\mathcal{N}), \psi_N) \) be a von Neumann algebra with no summand of type I\( _2 \). There is a bijective correspondence between (normal) states on \( \mathcal{N} \) and time-oriented global sections of the (normal) dilated probabilistic presheaf \( \Pi_D \) of \( \mathcal{N} \) over \( \mathcal{V}(\mathcal{N}) \).

**Proof.** By theorem 5, states on \( \mathcal{N} \) correspond with global sections of \( \Pi_D(\mathcal{V}(\mathcal{N})) \). It is thus sufficient to show that every state is already time-oriented. This follows from the fact that every positive linear functional is automatically completely positive [34]. More precisely, recall that the GNS construction for \( \sigma \in \mathcal{S}(\mathcal{N}) \) reads \( \sigma = v^* \pi v \) where \( v : \mathcal{C} \to \mathcal{K} \) and \( \pi : \mathcal{N} \to \mathcal{B}(\mathcal{K}) \) a (normal) \( \ast \)-representation of \( \mathcal{N} \). This yields a dilation for \( \sigma^* \in \mathcal{S}(\mathcal{N}) \) of the form in lemma 1 (and vice versa for \( \sigma \) from the GNS construction of \( \sigma^* \)), namely \( \sigma^*(a) = (v^* \pi(a) v)^* = \sigma^*(a \otimes 1_\mathcal{C}) = \text{tr}[v^* \pi(a) v^* 1_\mathcal{C}] \) for all \( a \in \mathcal{N} \). In particular, \( \pi \) preserves the dynamical correspondences \( \psi_{N_1} \) and \( \psi_{N_2} \) in the algebras \( N_1 = \mathcal{N} \) and \( N_2 = \mathcal{C} \).

The proof of theorem 6 shows that states \( \sigma \in \mathcal{S}(\mathcal{N}) \) are time-oriented with respect to both time orientations on \( \mathcal{N} \). As with the existence of dilations in theorem 5, we therefore find that this additional property, as captured by definition 4, is hidden in the single system case.

### 3.3. Gleason’s theorem for composite systems

The refinements of theorem 2 in theorems 5 and 6 both reveal extra structure in the characterisation of states in the single system case. In turn, having identified this additional structure in definitions 3 and 4, we may impose it in order to generalise Gleason’s theorem to composite systems. Then, every global section \( \gamma \in \Gamma[\Pi_D(\mathcal{V}(\mathcal{N}_1) \times \mathcal{V}(\mathcal{N}_2))] \) yields a map \( \phi_{\gamma} : a \mapsto v^{\ast} \Phi_{\gamma}(a) \Phi_{\gamma}(v) \) with \( \Phi_{\gamma} \) a \( \mathcal{C}^\ast \)-homomorphism, which implies that \( \phi_{\gamma} \) is completely positive by Stinespring’s theorem. In the final step, we deduce from this the existence of a positive linear functional \( \sigma_{\gamma} : \mathcal{N} \to \mathcal{C} \) with \( \mathcal{N} = \mathcal{N}_1 \otimes \mathcal{N}_2 \). In finite dimensions, this follows from Choi’s theorem [44]. For generalisations of Choi’s theorem to infinite dimensions, see also [45, 46].

**Lemma 2.** The linear functional in lemma 1, \( \sigma_{\gamma}(a \otimes b) = \tilde{\phi}_{\gamma}(a)(\pi_2(b)) = \text{tr}_{\mathcal{N}_2}[\phi_{\gamma}^*(a)\pi_2(b)] \) for all \( a \in \mathcal{N}_1, b \in \mathcal{N}_2 \), is positive and normal if and only if the linear map \( \phi_{\gamma} : \mathcal{N}_1 \to \mathcal{N}_2 \) is completely positive and both \( \phi_{\gamma} \) as well as the faithful \( \ast \)-representation \( \pi_2 \) are normal.

**Proof.** Let \( \sigma_{\gamma} \) be positive. Then for all \( n \in \mathbb{N} \) and \( a_1, \ldots, a_n \in \mathcal{N}_1, \xi_1, \ldots, \xi_n \in \mathcal{H}_2 \) we have

\[
\sum_{i,j=1}^n (\xi_i^* \phi_{\gamma}(a_i^* a_j) \xi_j) = \sum_{i,j=1}^n \text{tr}_{\mathcal{N}_2}[\phi_{\gamma}^*(a_i^* a_j) \pi_2(\xi_i^* \xi_j)] = \sum_{i,j=1}^n \sigma_{\gamma}(a_i^* a_j \otimes \xi_i^* \xi_j) \geq 0, \tag{8}
\]

---

\(^{16}\) We point out that definition 4 differs from definition 2 in [31] in the finite-dimensional simple case, which replaces the canonical time orientation \( \psi_{N_1} \) in equation (7) with the reverse time orientation \( \psi_{N_1}^* \). The reason is that \( \Phi_{\gamma} \) in lemma 1 differs from \( \Phi_{\gamma} \) in theorem 2 in [31] by composition with the Hermitian adjoint \( \ast \) (cf [43]).
Since \( A_\mu = (a_\mu \otimes \xi_0^* \xi_0)^* \in \mathcal{M}_\mu(\mathcal{N}_1 \otimes \mathcal{N}_2)_+ \) with \( \xi_0 \in \mathcal{H}_2 \) any unit vector is non-negative and \( \sigma \) is completely positive (cf theorem 3 in [34]). But equation (8) implies that \( \phi_\gamma \) is completely positive (cf proposition II.6.9.8 in [47]). Moreover, \( \phi_\gamma \) is normal since for any bounded increasing net \( (a_i) \) in \( (\mathcal{N}_1)_+ \) with \( a = \sup_i a_i \) and \( b \in (\mathcal{N}_2)_+ \),

\[
\sup_i \tilde{\phi}_\gamma (a_i) (\pi_2 (b)) = \sup_i \sigma_\gamma (a_i \otimes b) = \sigma_\gamma \left( \sup_i a_i \otimes b \right) = \sigma_\gamma (a \otimes b) = \tilde{\phi}_\gamma (a) (\pi_2 (b)),
\]

since \( (a_i \otimes b) \) is a bounded increasing net in \( (\mathcal{N}_1 \otimes \mathcal{N}_2)_+ \) with \( a \otimes b = \sup_i (a_i \otimes b) \) and \( \sigma_\gamma \) is normal by assumption. Similarly, one shows that \( \pi_2 \) is normal from normality of \( \sigma_\gamma \).

Conversely, by lemma 1 \( \phi_\gamma : \mathcal{N}_1 \rightarrow \mathcal{N}_2, \phi_\gamma = w^* \Phi \gamma w \) is a decomposable map (between Jordan algebras \( \mathcal{J}(\mathcal{N}_1) \) and \( \mathcal{J}(\mathcal{N}_2) \)) [36]. Assume further that \( \phi_\gamma \) is completely positive. Then \( \Phi \gamma \) lifts to a (normal) \(*\)-representation \( \tilde{\phi}_\gamma \) (cf theorem III.2.1.4 in [47]) such that \( \sigma_\gamma (a \otimes b) = \gamma^*(\Phi \gamma (a) \otimes \pi_2 (b)) \forall a \in \mathcal{N}_1, b \in \mathcal{N}_2 \) in equation (3) becomes a vector state, hence, is positive. Moreover, \( \sigma_\gamma \) is normal if \( \Phi \gamma \) and \( \pi_2 \) are normal (cf theorem 11.2.9 in [9]).

Finally, we arrive at a generalisation of Gleason’s theorem to composite systems.

**Theorem 7.** Let \( \mathcal{N}_1 = (\mathcal{J}(\mathcal{N}_1), \psi_{\mathcal{N}_1}) \), \( \mathcal{N}_2 = (\mathcal{J}(\mathcal{N}_2), \psi_{\mathcal{N}_2}) \) von Neumann algebras with no summands of type I, and let \( \mathcal{N} = \mathcal{N}_1 \otimes \mathcal{N}_2 \). Then there is a bijective correspondence between the set of (normal) states on \( \mathcal{N} \) and the set of time-oriented global sections of the (normal) dilated probabilistic presheaf \( \mathcal{P}_{\mathcal{H}} \) over \( \mathcal{V}(\mathcal{N}_1) \times \mathcal{V}(\mathcal{N}_2) \).

**Proof.** With theorem 4, every (normal) state \( \sigma \in (\mathcal{N}_1 \otimes \mathcal{N}_2)_+ \) restricts to a global section of the (normal) dilated probabilistic presheaf, \( \gamma_\sigma = \sigma |_{\mathcal{P}(\mathcal{V}(\mathcal{N}_1) \times \mathcal{V}(\mathcal{N}_2))} \in \Gamma[\mathcal{P}_{\mathcal{H}}(\mathcal{V}(\mathcal{N}_1) \times \mathcal{V}(\mathcal{N}_2))] \). Moreover, since \( \phi \sigma (a \otimes b) = \tilde{\phi}_\gamma (a) (\pi_2 (b)) = \operatorname{tr}_{\mathcal{N}_1} [\hat{\gamma}^*(a) \pi_2 (b)] \) for \( a \in \mathcal{N}_1, b \in \mathcal{N}_2 \) (from lemma 1) is completely positive by lemma 2, \( \Phi \gamma : \mathcal{N}_1 \rightarrow \mathcal{B}(\mathcal{K}) \) in \( \phi = \gamma^* \Phi \gamma \) preserves dynamical correspondences \( \psi_{\mathcal{N}_1} \) and \( \psi_{\mathcal{N}_2} \), hence, \( \gamma \) is time-oriented by definition 4.

Conversely, let \( \gamma \in \Gamma[\mathcal{P}_{\mathcal{H}}(\mathcal{V}(\mathcal{N}_1) \times \mathcal{V}(\mathcal{N}_2))] \). By lemma 1, \( \gamma \) corresponds to a linear functional \( \sigma_\gamma (a \otimes b) = \hat{\phi}_\gamma (a) (\pi_2 (b)) \) for \( a \in \mathcal{N}_1, b \in \mathcal{N}_2 \) (by linear extension), \( \pi_2 \) a (normal) \(*\)-representation of \( \mathcal{N}_2 \) and \( \phi_\gamma \) a (normal) decomposable map (between Jordan algebras \( \mathcal{J}(\mathcal{N}_1) \) and \( \mathcal{J}(\mathcal{N}_2) \)), i.e. \( \phi_\gamma (a) = w^* \Phi \gamma (a) w \) for \( w : \mathcal{H}_2 \rightarrow \mathcal{K} \) with \( \mathcal{N}_2 \subseteq \mathcal{B}(\mathcal{H}_2) \) and \( \Phi \gamma : \mathcal{J}(\mathcal{N}_1) \rightarrow \mathcal{J}(\mathcal{B}(\mathcal{K})) \) a (normal) Jordan \(*\)-homomorphism. Since \( \gamma \) is time-oriented with respect to dynamical correspondences \( \psi_{\mathcal{N}_1} \) and \( \psi_{\mathcal{N}_2} \) (cf definition 4), \( \Phi \gamma : \mathcal{N}_1 \rightarrow \mathcal{N}_2 \subseteq \mathcal{B}(\mathcal{K}) \) is also a \(*\)-homomorphism, and \( \phi_\gamma \) thus a (normal) completely positive map by Stinespring’s theorem [34] (see also theorem III.2.2.4 in [47]). Finally, lemma 2 asserts that \( \sigma_\gamma \) is positive and (normal), hence, a (normal) state by normalisation \( \sigma_\gamma (1) = \gamma (1, 1) = 1 \).

We finish with a few remarks. Note that the key steps in the argument leading to theorem 7 apply to \( C^* \)-algebras. The crucial exception is lifting quasi-linearity to linearity in theorem 4. As such it might be possible to further extend our result from von Neumann algebras to, for instance, \( AW^* \)-algebras, for which Gleason’s theorem holds by [48].

Moreover, we observe that theorem 7 can be formulated in an intrinsically Jordan-algebraic setting. Bunce and Wright prove a generalisation of Gleason’s theorem for JBW-algebras in [49]. The case considered here is the one where JBW algebras arise as self-adjoint parts of von Neumann algebra, hence, are JW algebras. Alfsen and Shultz show that this is the case if and only if the JBW algebra admits a dynamical correspondence [50].

In turn, in the context of general JBW algebras, the concept of state requires a weaker notion of positivity, reflecting the generalisation from completely positive to decomposable maps [36]. Clearly, the relation with (preservation of) dynamical correspondences in definition 4
is lost in this case. We point out that this is closely analogous to the limited applicability of Tomita-Takesaki theory in JBW algebras [51].

4. Conclusion

We proved a generalisation of Gleason’s theorem to composite systems. An ad hoc attempt establishes a correspondence with linear functionals on the composite algebra that are positive on product operators, but fails to single out those that are positive (theorem 4). We remedy this by strengthening the additivity constraints on measures in commutative von Neumann subalgebras to dilated systems, together with a consistency condition between dynamical correspondences on the respective subalgebras (theorem 7). Neither of these conditions changes the result in the single system case (theorems 5 and 6), since positive linear functionals are completely positive as a consequence of Stinespring’s theorem [34].

Apart from its mathematical value, our result also carries physical significance. Gleason’s theorem (theorem 2) plays a crucial role in the foundations of quantum theory, where it justifies Born’s rule. It is all the more interesting that a similar result extends to composite systems. For instance, we note that within the framework of algebraic quantum field theory the observable algebra is naturally composed of local algebras [52].

Theorem 7 was foreshadowed for finite type I factors, and with a focus on classifying quantum from non-signalling correlations in [31]. Its close connection with entanglement classification is discussed in [53]. For a broader perspective on the significance of contextuality in the foundations of quantum mechanics we refer to [12].

Data availability statement

No new data were created or analysed in this study.

Acknowledgments

This work is supported through a studentship in the Centre for Doctoral Training on Controlled Quantum Dynamics at Imperial College funded by the EPSRC, by Grant Number FQXi-RFP-1807 from the Foundational Questions Institute and Fetzer Franklin Fund, a donor advised fund of Silicon Valley Community Foundation, and ARC Future Fellowship FT180100317.

Conflict of interest

The authors declare no conflict of interest.

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