THE F-SIGNATURE FUNCTION ON THE AMPLE CONE

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Abstract. For any fixed globally F-regular projective variety X over an algebraically closed field of positive characteristic, we study the F-signature of section rings of X with respect to the ample Cartier divisors on X. In particular, we define an F-signature function on the ample cone of X and show that it is locally Lipschitz continuous. We further prove that the F-signature function extends to the boundary of the ample cone. We also establish effective comparison between the F-signature function and the volume function on the ample cone. As a consequence, we show that for divisors that are nef but not big, the extension of the F-signature is zero.

1. Introduction

The F-signature is an invariant of the singularities of a Noetherian, F-finite local ring R of prime characteristic. First arising implicitly in [SVdB97] and formally defined in [HL02], this invariant measures the asymptotic growth of the number of Frobenius splittings of R (See Definition 2.2). The positivity of the F-signature corresponds exactly to R being a strongly F-regular singularity [AL03]; only when R is regular, does the F-signature achieve its maximum value of 1. This invariant has also attracted attention as a candidate for the positive characteristic analogue of the normalized volume of a Kawamata log-terminal (klt) singularity, extending the established analogy between strongly F-regular and klt singularities; see [LLX20], [Tay19], [MPST19]. There have been applications of the F-signature to bounding the sizes of the étale fundamental group and the torsion subgroup of the divisor class group; see [CRST18], [Mar22], [CR22] and [Pol22].

In the global setting, globally F-regular varieties (Definition 2.9), introduced in [Smi00] are the positive characteristic analogues of log-Fano type varieties enjoying additional properties such as satisfying a Kawamata-Viehweg vanishing theorem [SS10]. It was shown in [Smi00] that a projective variety X is globally F-regular if and only if the section ring S(X, L) (Definition 2.6) is strongly F-regular for some (equivalently, every) ample invertible sheaf L. Since the F-signature is positive for all strongly F-regular rings, it is natural to ask: How does the F-signature of the section ring S(X, L) vary with L? The purpose of this paper is to answer this question. We prove:

Theorem 1.1. Fix any globally F-regular projective variety X over an algebraically closed field k of positive characteristic p. Then, the F-signature function L ↦→ s_X(L), assigning to any ample Cartier divisor L, the F-signature of the section ring of X with respect to L, satisfies the following properties:

(a) ([VK12], [CR22], Theorem 3.2) The F-signature function s_X naturally extends to a unique, well-defined, real-valued function

s_X : Amp_Q(X) → ℝ

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on the set of rational classes in the ample cone of $X$ satisfying:

$$s_X(\lambda L) = \frac{1}{\lambda} s_X(L) \quad \text{for all ample } \mathbb{Q}\text{-divisors } L \text{ and all } \lambda \in \mathbb{Q}_{>0}.$$ 

(b) (Theorem 4.1) The function $s$ is continuous on the rational ample cone of $X$, with respect to the usual topology on the Néron-Severi space.

(c) (Corollary 4.3) The function $s_X$ extends continuously to all real classes in the ample cone of $X$.

Theorem 1.1 provides us with a new tool for the study of globally $F$-regular varieties. Such varieties have found various applications, for instance, to the three dimensional minimal model program in positive characteristic [HX15] and in the study of Fano type complex varieties [GOST15]. For other investigations regarding globally $F$-regular varieties, see [GLP+15], [GT19] and [Kaw21].

Another motivation for considering the $F$-signature function comes from the volume function on the big cone of a projective variety. On a projective variety $X$ over an algebraically closed field, to any Cartier divisor $D$ on $X$, we can associate a non-negative real number called the volume of $D$, measuring the growth of the global sections of multiples of $D$. A foundational result in the theory of volumes is that the volume of a big divisor $D$ depends only on its numerical equivalence class. Moreover, it extends suitably to all $\mathbb{R}$-divisors and varies continuously as $D$ varies on the Néron-Severi space of $X$. See [Laz04, Section 2.2] and [LM09] for the details. The study of volumes of divisors has been important in birational geometry; for example, see [Laz04, Section 2.2], [LM09], [Bou02], [ELM+05], [HM06], [Tak06], and [KÔ6].

Theorem 1.1 parallels the theory of the $F$-signature function (Definition 3.1) on the ample cone of a globally $F$-regular variety with the volume function on the big cone. This perspective was first considered in [VK12], where Theorem 1.1 was proved in the special case when $X$ is a toric variety.

The ample cone is an open cone in the Néron-Severi space of a projective variety $X$, and its closure is represented by the set of nef divisors on $X$. Hence, it is natural to ask if the $F$-signature function $s_X$ from Theorem 1.1 has a natural extension to the nef cone. We show that this is indeed true:

**Theorem 1.2 (Theorem 5.1).** Suppose $X$ is a globally $F$-regular projective variety. Then the $F$-signature function $s_X$ extends continuously to all non-zero classes of the Nef cone of $X$. Moreover, if $L$ is a nef Cartier divisor which is not big, then $s_X(L) = 0$.

The proof of Theorem 1.1 and Theorem 1.2 consists of several steps. First, we need to verify that the $F$-signature function is well-defined on the rational ample cone of $X$. We do this in Section 3. The main result here is that on globally $F$-regular projective varieties, numerical equivalence and $\mathbb{Q}$-linear equivalence coincide. This is reminiscent of the same fact for log-Fano type varieties over the complex numbers. Once we have this, we prove the continuity of the $F$-signature function in Section 4. This needs several ideas towards analyzing Frobenius splittings of linear systems; a detailed sketch of the proof is presented in Section 4.2. We further utilize these ideas to extend the $F$-signature function to all non-zero nef divisors in Section 5. Lastly, in Theorem 6.1 we prove a local effective upper bound for the $F$-signature function.
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2. Preliminaries

Throughout this paper, unless specified otherwise, all rings are assumed commutative with a unit, and are of positive characteristic $p$. $k$ will denote an algebraically closed field of characteristic $p$. All varieties are assumed to be integral, separated schemes of finite type over $k$.

2.1. $F$-signature. Let $R$ be any ring of prime characteristic $p$. Then $R$ is naturally equipped with the Frobenius morphism, $F : R \to R$ sending $r \mapsto r^p$. Since $R$ has characteristic $p$, $F$ defines a ring homomorphism, allowing us to view $R$ as a new $R$-module obtained via restriction of scalars along $F$. We denote this new $R$-module by $F^\ast R$ and its elements by $F^\ast r$ (where $r$ is an element of $R$). Concretely, $F^\ast R$ is the same as $R$ as an abelian group, but the $R$-module action is given by:

$$r \cdot F^\ast s := F^\ast r^ps \quad \text{for } r \in R \text{ and } F^\ast s \in F^\ast R.$$ 

Throughout, we will assume that $R$ is essentially of finite type over $k$, which also makes it $F$-finite, i.e., $F^\ast R$ is a finitely generated $R$-module. Similarly, for any natural number $e \geq 1$, we have the iterate of the Frobenius, $F^e : R \to R$ sending $r \mapsto r^{p^e}$ and the $R$-module $F^e \ast R$ obtained by restricting scalars along $F^e$. We will be interested in invariants of $R$ defined by analyzing the $R$-module structure of the modules $F^e \ast R$.

Now let $(R, m, k)$ denote a local ring, essentially of finite type over $k$, with maximal ideal $m$ and residue field $k$.

Definition 2.1 (Free rank). Let $M$ be a finitely generated module over a local ring $R$. Consider a decomposition:

$$M \cong R^{a(M)} \oplus N$$

where $N$ has no $R$-summands. Then, since $R$ is local, the number $a(M)$ is independent of the decomposition chosen, and is called the free rank of $M$.

Definition 2.2 ($F$-signature). Let $(R, m, k)$ be a local ring as above, and $a_e(R)$ denote the free rank of $F^e \ast R$ (Definition 2.1). Then the $F$-signature of $R$ is defined to be the limit:

$$s(R) := \lim_{e \to \infty} \frac{a_e(R)}{p^{ed}}$$

where $d$ is the Krull dimension of $R$. This limit exists by [Tuc12].

The $F$-signature of $R$ admits an alternate description as follows: We say a map of $R$-modules $\phi : M \to N$ splits if there exists an $R$-module map $\psi : N \to M$ such that $\psi \circ \phi = \text{id}_M$. Define the subset $I_e \subseteq R$ as

$$I_e = \{ x \in R \mid \text{the map } R \to F^e \ast R \text{ sending } 1 \mapsto F^e x \text{ does not split} \}.$$
Then, we observe that $I_e$ is an ideal of $R$ and by [Tuc12, Proposition 4.5], the free rank of $F^e_* R$ equals $l_R(R/I_e)$, the length of the $R$-module $R/I_e$. Hence, the $F$-signature of $R$ can be defined as the limit:

$$s(R) = \lim_{e \to \infty} \frac{l_R(R/I_e)}{p^{e d}}.$$ 

Though the definition of $F$-signature is given for a local ring $(R, \mathfrak{m}, k)$, we may also work with $\mathbb{N}$-graded rings $(S, \mathfrak{m}, k)$ i.e. $S$ is $\mathbb{N}$-graded with $S_0 = k$ and $\mathfrak{m} = S_{>0}$. We next relate the local and graded situations.

**Definition 2.3** (Graded free rank). Let $(S, \mathfrak{m}, k)$ be an $\mathbb{N}$-graded ring, finitely generated over $k$, with $S_0 = k$ and $M$ a finitely generated $\mathbb{Z}$-graded module over $S$. Then we can decompose $M$ as a graded $S$-module as:

$$M \cong P \oplus N$$

where $P$ is a graded free $S$-module (i.e. a direct sum of $S(j)$, the shifted rank 1 free modules, for various $j \in \mathbb{Z}$) and $N$ is a graded module with no graded free summands. Then the rank of $P$ is independent of the chosen decomposition and we define it to be the graded free rank of $M$ over $S$ (denoted by $\text{gr}r(M)$).

**Lemma 2.4.** Let $(S, \mathfrak{m}, k)$ and $M$ be as above. Then the free rank of $M_\mathfrak{m}$ over the local ring $S_\mathfrak{m}$ is the same as the graded free rank of $M$.

**Proof.** It is enough to show that if $N$ is a finitely generated $\mathbb{Z}$-graded $S$-module, then $N_\mathfrak{m}$ has a free $S_\mathfrak{m}$-summand if and only if $N$ has a graded free $S$-summand. Clearly any graded $S$-summand of $N$ localizes to give a $S_\mathfrak{m}$-summand of $N_\mathfrak{m}$. For the converse, we observe that for any $j$, the module of (ungraded) maps $\text{Hom}_S(N, S(j))$ is a graded $S$-module with the property that any $\phi \in \text{Hom}_S(N, S(j))$ can be written as a sum of homogeneous elements $\phi = \sum \phi_i$ where each $\phi_i$ sends elements of degree $d$ to elements of degree $d + i$. Further, since $\text{Hom}_S(N_\mathfrak{m}, S(j)_\mathfrak{m}) = \text{Hom}_S(N, S(j))_\mathfrak{m}$, we see that a map $\phi : N_\mathfrak{m} \to S_\mathfrak{m}$ sending a non-zero element $n \in N$ to a unit in $S_\mathfrak{m}$ (corresponding to an $S_\mathfrak{m}$-summand of $M_\mathfrak{m}$) gives us a homogeneous element $u \in N$ of degree $-j$ and a homogeneous map $\phi' : N \to S(j)$ sending $u$ to 1. This realizes $S(j)$ as a graded free summand of $N$. \qed

Now, we describe the $F$-signature of $\mathbb{N}$-graded rings, relating it to the (local) $F$-signature at the vertex. For similar discussions relating the local and global situations, see [Smi00, Section 3], [Smi97, Section 4], and [VK12, Section 2.2].

Let $S$ be an $\mathbb{N}$-graded ring. Then $F^e_* S$ is also naturally an $\frac{1}{p^e} \mathbb{N}$-graded $S$-module by taking

$$(F^e_* S)_{\frac{i}{p^e}} = F^e_* S_i.$$ 

This gives rise to the $\mathbb{N}$-grading on $F^e_* S$ given by

$$(F^e_* S)_n = \bigoplus_{0 \leq i \leq p^e - 1} (F^e_* S)_{\frac{i + np^e}{p^e}}.$$ 

Thus, $F^e_* S$ decomposes as

$$F^e_* S = \bigoplus_{0 \leq i \leq p^e - 1} \bigoplus_{j \geq 0} F^e_* S_{i + jp^e}$$

as an $\mathbb{N}$-graded $S$-module.
Definition 2.5 (F-signature of \(N\)-graded rings). Let \((S, m, k)\) be an \(N\)-graded, finitely generated \(k\)-algebra, with \(S_0 = k\). Then, we define the F-signature of \(S\) to be the limit:

\[
\lim_{e \to \infty} \frac{a_{e,\text{gr}}(S)}{p^d}
\]

where \(a_{e,\text{gr}}(S)\) is the graded free rank of \(F^e S\) and \(d\) denotes the Krull dimension of \(S\). We note that by Lemma 2.4, the F-signature of \(S\) coincides with the F-signature of \(S_m\), the localization of \(S\) at the maximal ideal \(m\).

2.2. Section Rings and Cones: The \(N\)-graded rings we will be interested in arise as the section rings of projective varieties over \(k\) with respect to some ample divisor. In this subsection, we will review this and other related constructions.

Definition 2.6 (Section Rings and Modules). Let \(X\) be a projective variety over \(k\), \(L\) an ample invertible sheaf on \(X\) and \(F\) a coherent sheaf on \(X\). Then the \(N\)-graded ring \(S\) defined by

\[
S = S(X, L) := \bigoplus_{n \geq 0} H^0(X, L^n)
\]

is called the section ring of \(X\) with respect to \(L\). The affine scheme \(\text{Spec}(S)\) is called the cone over \(X\) with respect to \(L\). The section module of \(F\) with respect to \(L\) is a \(Z\)-graded \(S\)-module \(M\) defined by

\[
M = M(X, L) := \bigoplus_{n \in \mathbb{Z}} H^0(X, F \otimes L^n).
\]

Similarly, the sheaf corresponding to \(M\) on \(\text{Spec}(S)\) is called the cone over \(F\) with respect to \(L\).

Lemma 2.7. Let \(X\) be a projective variety over \(k\) and \(L\) an ample invertible sheaf over \(X\).

(a) The section ring \(S\) of \(X\) with respect to \(L\) is always finitely generated over \(k\) and hence, is Noetherian. If \(X\) is normal, then the section ring is also characterized as the unique normal \(N\)-graded ring \(S\) such that \(\text{Proj}(S)\) is isomorphic to \(X\) and the corresponding \(\mathcal{O}_X(1)\) is isomorphic to \(L\).

(b) The section module of any coherent sheaf over \(X\) with respect to \(L\) is finitely generated over \(S\). It is also characterized as the unique saturated (with respect to the homogeneous maximal ideal) \(S\)-module \(M\) such that the associated coherent sheaf \(\tilde{M}\) on \(X\) is isomorphic to \(F\).

(c) For two coherent sheaves \(F\) and \(G\), we have a natural isomorphism:

\[
\text{Hom}_{\mathcal{O}_X}(F, G) \cong \text{Hom}_S^g(M(F, L), M(G, L))
\]

where \(\text{Hom}_S^g(, , )\) denotes the set of grading preserving \(S\)-module maps between two graded \(S\)-modules.

Proof. See [Sta, Tag 0BXF].

2.3. \(F\)-regularity:

Definition 2.8 (Strong \(F\)-regularity). [HH89] Let \(R\) be a Noetherian \(F\)-finite ring of characteristic \(p\). Then, \(R\) is said to be strongly \(F\)-regular if for any element \(c \in R\) that is not contained in any minimal prime of \(R\), there exists an integer \(e \gg 0\), such that, the following map
splits as a map of $R$-modules.

**Definition 2.9** (Global $F$-regularity). [SS10, Definition 3.2] Let $X$ be a normal variety over $k$ and $\Delta \geq 0$ be an effective $\mathbb{Q}$-divisor. The pair $(X, \Delta)$ is said to be **globally $F$-regular** if for any effective Weil divisor $D$ on $X$, there exists an integer $e \gg 0$, such that, the natural map

$$\mathcal{O}_X \to F^e_*\mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil + D)$$

splits as a map of $\mathcal{O}_X$-modules. A normal variety $X$ is said to globally $F$-regular if the pair $(X, 0)$ is globally $F$-regular.

**Remark 2.10.** When $X = \text{Spec}(R)$ is an affine variety, $X$ being globally $F$-regular is equivalent to $R$ being strongly $F$-regular [Smi00].

**Remark 2.11.** A local ring $R$ is strongly $F$-regular if and only if its $F$-signature $s(R)$ is positive [AL03].

**Theorem 2.12.** [Smi00, Theorem 3.10] Let $X$ be a projective variety over $k$. Then, $X$ is globally $F$-regular if and only if the section ring $S(X, \mathcal{L})$ (Definition 2.6) with respect to some (equivalently, every) ample invertible sheaf $\mathcal{L}$ is strongly $F$-regular.

Combining with Part (b) of Remark 2.11, $X$ is globally $F$-regular if and only if the $F$-signature $s(S(X, \mathcal{L}))$ is positive for some (equivalently, every) ample invertible sheaf $\mathcal{L}$ on $X$.

Globally $F$-regular varieties enjoy plenty of nice properties: For example,

- As proved in [SS10, Theorem 4.3], they are **log-Fano type**. More precisely, there exists an effective divisor $\Delta \geq 0$ such that the pair $(X, \Delta)$ is globally $F$-regular and $-K_X - \Delta$ is ample.
- A version of the Kawamata-Viehweg vanishing theorem holds on all globally $F$-regular varieties [SS10, Theorem 6.8].

**Theorem 2.13** ([Smi00], Corollary 4.3). Let $X$ be a projective, globally $F$-regular variety over $k$. Suppose $\mathcal{L}$ is a nef invertible sheaf over $X$. Then,

$$H^i(X, \mathcal{L}) = 0 \quad \text{for all } i > 0.$$
Definition 2.16 (Néron-Severi space). Let $X$ be a projective variety over $k$. Let $\text{Div}(X)$ be the group of integral Cartier divisors on $X$. The Néron-Severi group of $X$ is the set of Cartier divisors modulo numerical equivalence. i.e.,
\[ N^1(X) := \text{Div}(X)/\text{Num}(X). \]

Let $\text{Div}_\mathbb{Q}(X)$ (resp. $\text{Div}_\mathbb{R}(X)$) be the set of $\mathbb{Q}$ (resp. $\mathbb{R}$)-Cartier divisors on $X$. Then, there is an isomorphism $\text{Div}_\mathbb{Q}(X)/\text{Num}(X) \cong N^1(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ (similarly, $\text{Div}_\mathbb{R}(X)/\text{Num}(X) \cong N^1(X) \otimes_{\mathbb{Z}} \mathbb{R}$). Then, we define:
- the rational Néron-Severi space $N^1_{\mathbb{Q}}(X)$ to be the $\mathbb{Q}$-vector space $N^1(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.
- the real Néron-Severi space $N^1_{\mathbb{R}}(X)$ to be the $\mathbb{R}$-vector space $N^1(X) \otimes_{\mathbb{Z}} \mathbb{R}$.

Note that by the [Kle66, Ch.4, Sec. 1, Prop. 4], $N^1(X)$ is a finitely generated abelian group. Hence, $N^1(X)_{\mathbb{Q}}$ and $N^1(X)_{\mathbb{R}}$ are finite dimensional vector spaces. The dimension of $N^1_{\mathbb{Q}}(X)$ (equivalently, $N^1_{\mathbb{R}}(X)$) is called the Picard number of $X$, denoted by $\rho(X)$.

Definition 2.17 (Ample cone and Nef cone). Let $X$ be a projective variety over $k$. Then,
- The rational ample cone $\text{Amp}_\mathbb{Q}(X) \subseteq N^1_{\mathbb{Q}}(X)$ of $X$ is the set of all ample $\mathbb{Q}$-classes on $X$.
- The real ample cone $\text{Amp}_\mathbb{R}(X) \subseteq N^1_{\mathbb{R}}(X)$ of $X$ is the set of all ample $\mathbb{R}$-classes on $X$.

Similarly, we have,
- The rational nef cone $\text{Nef}_\mathbb{Q}(X) \subseteq N^1_{\mathbb{Q}}(X)$ of $X$ is the set of all nef $\mathbb{Q}$-classes on $X$.
- The real nef cone $\text{Nef}_\mathbb{R}(X) \subseteq N^1_{\mathbb{R}}(X)$ of $X$ is the set of all nef $\mathbb{R}$-classes on $X$.

Theorem 2.18. [Laz04, Theorem 1.4.23] Let $X$ be a projective variety over $k$.
(a) The rational (or real) ample cone is an open cone in the rational (or real) Néron-Severi space, with respect to the usual topology on $N^1_{\mathbb{Q}}(X)$ (or $N^1_{\mathbb{R}}(X)$).
(b) The real nef cone is the closure of the real ample cone.
(c) The real ample cone is the interior of the real nef cone.

Definition 2.19 (Volume of a divisor). [Laz04, Definition 2.2.31] Let $X$ be an irreducible projective variety of $\dim X = d$ over $k$. Let $L$ be an invertible sheaf on $X$. The volume of $L$ is defined as
\[ \text{vol}(L) := \limsup_{m \to \infty} \frac{h^0(X, L^\otimes m)}{m^d/d!}. \]

If $D$ is a Cartier divisor on $X$, then the volume of $D$ is defined to be the volume of the corresponding invertible sheaf.

Remark 2.20. The limsup in Definition 2.19 is actually a limit, see [Laz04, Remark 2.2.50].

Remark 2.21. When $D$ is a nef Cartier divisor, the volume of $D$ has a simpler interpretation as the top self-intersection [Laz04, Section 2.2.C]:
\[ \text{vol}(D) = (D)^d. \]

Lemma 2.22. [LM09, Corollary 4.12] Let $X$ be a projective variety over $k$. The volume of Cartier divisors extends to a well-defined, real-valued continuous function, called the volume function on the Néron-Severi space of $X$, satisfying the identity
\[ \text{vol}(\lambda D) = \lambda^d \text{vol}(D) \]
for all divisors $D$ and real numbers $\lambda$. Here $d$ denotes the dimension of $X$. 

Lemma 2.23. [Laz04, Theorem 2.2.44] Let $X$ be a projective variety of dimension $d$ over $k$. Fix a norm $\|\|$ on the real Nérond-Severi space. Then, there exists a positive constant $C > 0$ such that for any two real ample classes $\xi$ and $\xi'$, we have:

$$|\text{vol}(\xi) - \text{vol}(\xi')| \leq C \max(\|\xi\|, \|\xi'\|)^{d-1} \|\xi - \xi'\|.$$  

Proof. Since the volume function coincides with the intersection form on the real Nef cone, it is given by a polynomial $P$ of degree $d$ once we choose a basis for $N^1_\mathbb{R}(X)$. Hence, there exists a constant $C$ (depending only on $X$), such that

$$\|P'(x_1, \ldots, x_\rho)\| \leq C \|(x_1, \ldots, x_\rho)\|^{d-1}$$

for any vector $(x_1, \ldots, x_\rho) \in \text{Nef}_\mathbb{R}(X)$. With this observation, the Lemma follows from an application of the mean-value theorem. 

Remark 2.24. The volume function is known to satisfy further properties:

(a) At least if $k$ has characteristic zero, Lemma 2.23 holds for any two (not necessarily ample) classes in $N^1_\mathbb{R}(X)$, [Laz04, Theorem 2.2.44].

(b) Again, when $k$ has characteristic zero, the volume function is in fact, differentiable see [LM09] and [BFJ09].

Definition 2.25 (Big divisors). A Cartier divisor $D$ on a projective variety is said to be big if the volume of $D$ is positive. Similarly, an $\mathbb{R}$-Cartier divisor is said to be big, its volume (as obtained by the continuous extension above) is positive.

Lemma 2.26. [Laz04, Corollary 2.2.10] If $D$ is a big Cartier divisor, then for all sufficiently large integers $m$, $H^0(X, \mathcal{O}_X(mD))$ is non-zero.

Similar to the set of ample classes, the set of big divisor classes forms a cone in $N^1_\mathbb{R}(X)$. See [Laz04, Definition 2.25] for the details.

Definition 2.27 (Big cone and Pseudoeffective cone). Let $X$ be a projective variety over $k$. Then,

- The big cone $\text{Big}(X) \subset N^1_\mathbb{R}(X)$ is the set of all big $\mathbb{R}$-classes on $X$.
- The pseudoeffective cone $\text{Eff}(X) \subset N^1_\mathbb{R}(X)$ of $X$ is the closure of the set of all effective $\mathbb{R}$-classes on $X$.

In other words, a divisor $D$ is pseudoeffective if $[D]$ is a limit of effective divisor classes. We note that every effective divisor is pseudoeffective.

Theorem 2.28. [Laz04, Theorem 2.26] Let $X$ be a projective variety over $k$.

(a) The big cone is an open cone in the real Nérond-Severi space, with respect to the usual topology on $N^1_\mathbb{R}(X)$.

(b) The pseudoeffective cone is the closure of the big cone.

(c) The big cone is the interior of the pseudoeffective cone.

3. Definition of the $F$-signature Function

In this section, we will define an $F$-signature function on the rational ample cone of a globally $F$-regular projective variety.
Definition 3.1. Let $X$ be a globally $F$-regular projective variety over $k$. The $F$-signature function

$$s_X : \text{Amp}_Q(X) \to \mathbb{R}$$

on the rational ample cone of $X$ (Definition 2.17), is defined as follows:

(a) If the class $[L] \in \text{Amp}_Q(X)$ is defined by an integral Cartier divisor $L$, then we define $s_X([L])$ to be the $F$-signature (Definition 2.2) of the section ring $S(X, L)$ (Definition 2.6) of $L$:

$$s_X([L]) := s(S(X, L)).$$

(b) If the class $[L]$ is defined by a rational multiple of an integral Cartier divisor i.e. $L = \frac{a}{b}D$ where $D$ is an integral Cartier divisor on $X$, then we define:

$$s_X([L]) := \frac{b}{a} s_X([D]) = \frac{b}{a} s(S(X, D)).$$

The rest of this section is devoted to checking that the function $s$ is indeed well-defined.

Theorem 3.2. Let $X$ be a globally $F$-regular projective variety over $k$. Then, Definition 3.1 gives a well-defined $F$-signature function $s$ on the rational ample cone of $X$, satisfying the identity:

$$s_X\left(\frac{a}{b}L\right) = \frac{b}{a} s_X(L)$$

for any two non-zero natural numbers $a$ and $b$ and any ample $\mathbb{Q}$-divisor $L$.

Proof. We need to check that the function $s_X$ as defined in Definition 3.1 is well-defined. There are two issues:

(a) The first arising from the choice of a $\mathbb{Q}$-divisor representing a numerical equivalence class (Theorem 3.3).

(b) Having chosen a $\mathbb{Q}$-divisor $L$ representing a numerical class, there is still ambiguity in choosing a representation of $L$ as a rational multiple of an integral Cartier divisor (Theorem 3.5).

We address the first ambiguity by proving that on a globally $F$-regular variety, numerical equivalence and linear equivalence are the same conditions. This is an analog of the same result for $\log$-$Fano$ varieties over the complex numbers, a well-known consequence of the Kawamata-Viehweg vanishing theorem. The following theorem maybe well-known to experts, but we do not know a reference.

Theorem 3.3. Let $X$ be a projective, globally $F$-regular variety over $k$. Suppose $\mathcal{L}$ is a numerically trivial invertible sheaf on $X$, i.e. $\deg(\mathcal{L}|_C) = 0$ for all curves $C$ on $X$. Then, $\mathcal{L}$ is isomorphic to the trivial invertible sheaf $\mathcal{O}_X$.

Proof. First, we note that by [Kle66, Ch. 2, Section 2, Corollary 1], some power $\mathcal{L}^m$ of $\mathcal{L}$ is algebraically equivalent to $\mathcal{O}_X$ i.e. $\mathcal{L}^m$ deforms to $\mathcal{O}_X$. Since the Euler-characteristic (for sheaf-cohomology) is invariant in flat families [Har77, Ch. III, Theorem 9.9], we get that

$$\chi(\mathcal{O}_X) = \chi(\mathcal{L}^m) \ (3.1)$$

for some natural number $m$. Now, by Theorem 2.13, since $\mathcal{O}_X$ and $\mathcal{L}^m$ are both nef invertible sheaves, we get that

$$H^i(X, \mathcal{O}_X) = H^i(X, \mathcal{L}^m) = 0 \ \text{for all} \ i > 0. \ (3.2)$$
Hence, we get
\[ 1 = h^0(X, \mathcal{O}_X) = \chi(\mathcal{O}_X) = \chi(\mathcal{L}^m) = h^0(X, \mathcal{L}^m). \tag{3.3} \]

Hence, we have shown that $\mathcal{L}^m$ has a non-zero global section. But, since it is also numerically trivial, it must indeed be trivial (since an effective divisor can’t be numerically trivial unless it is the zero divisor). Therefore, $\mathcal{L}^m \cong \mathcal{O}_X$.

Now, by [Kle66, Ch. 1, Section 1], we have that the function $\chi(\mathcal{L}^n)$ is a polynomial function of $n$ (as $n$ varies over all integers). Since $\mathcal{L}^m \cong \mathcal{O}_X$, we must have that $\chi(\mathcal{L}^n) = 1$ for all $n \in \mathbb{Z}$. But, again, since $\mathcal{L}^n$ is nef for all $n \geq 0$, by Theorem 2.13 we have
\[ h^0(X, \mathcal{L}) = \chi(\mathcal{L}) = 1. \tag{3.4} \]

Hence, $\mathcal{L} \cong \mathcal{O}_X$ as well because $\mathcal{L}$ is numerically trivial and has a non-zero global section.

\[ \square \]

Remark 3.4. It was proved in [CR22] that torsion divisors (i.e., $L$ such that $nL \sim 0$ for some $n$) are themselves linearly equivalent to 0. Hence, the last part of the proof above follows from this fact, but we include a proof for the convenience of readers.

Next, we address the second kind of ambiguity in Definition 3.1. For this, we note the following scaling property for $F$-signature of section rings under taking Veronese subrings, first observed in [VK12].

**Theorem 3.5 ([VK12]).** Let $X$ be a projective variety over $k$ and $\mathcal{L}$ an ample invertible sheaf on $X$. Let $S(\mathcal{L})$ and $S(\mathcal{L}^n)$ denote the section rings with respect to $\mathcal{L}$ and $\mathcal{L}^n$ respectively, where $n$ is any positive natural number. Then, we have the following relation between their $F$-signatures:
\[ s(S(\mathcal{L})) = n s(S(\mathcal{L}^n)). \tag{3.5} \]

Note that in the Theorem above, $n$ may be divisible by $p$, the characteristic of the ground field. Indeed, this follows from a theorem of Carvajal-Rojas:

**Theorem 3.6 ([CR22], Theorem 4.8).** Let $(R, m) \subset (S, n)$ be a finite local extension of $F$-finite Noetherian domains of characteristic $p > 0$. Suppose there exists a surjective $\Psi \in \text{Hom}_R(S, R)$ that freely generates $\text{Hom}_R(S, R)$ as an $S$-module. Suppose, further, that $\Psi(n) \subset m$. Then,
\[ [S/n : R/m] s(S) = [K(S) : K(R)] s(R), \tag{3.6} \]

where $K$ denotes the fraction field of the ring.

Setting $S = S(\mathcal{L})$ and $R = S(\mathcal{L}^n)$ yields the desired formula (3.5).

Remark 3.7. An alternate proof of Theorem 3.5 can obtained by considering an analogue of the Hilbert-Kunz density function (see [Tri18]) for the $F$-signature of section rings. The analogues of the Hilbert-Kunz density function and the related Frobenius-Poincaré function [Muk22] for the $F$-signature will be considered in Alapan Mukhopadhyay’s thesis.

**Example 3.8.** When $X$ is a toric variety, [VK12] describes the $F$-signature function of $X$ in terms of the volumes of some naturally defined polytopes. For example, when $X = \mathbb{P}^1 \times \mathbb{P}^1$, it is computed that
\[ s_X(a, b) = \begin{cases} \frac{a^2}{b^2}, & \text{if } 0 \leq 2a \leq b \\ \frac{(b-2a)^3+3a^3}{3a^2b^2}, & \text{if } 0 \leq a \leq b \leq 2a \end{cases} \]

where $s(a, b)$ denotes the $F$-signature of the (ample, when $a, b > 0$) invertible sheaf $p_1^* \mathcal{O}_{\mathbb{P}^1}(a) \otimes p_2^* \mathcal{O}_{\mathbb{P}^1}(b)$ where $p_1$ and $p_2$ are the two projections onto the two $\mathbb{P}^1$ factors.
4. Continuity of the F-signature function

In this section, we prove that the F-signature function (Definition 3.1) varies continuously on the ample cone. Throughout, we fix a globally F-regular projective variety $X$ over $k$.

**Theorem 4.1.** The F-signature function is continuous at each rational class in the ample cone of $X$.

In fact, much more is true: the F-signature function is locally Lipschitz around any real class in the ample cone $\operatorname{Amp}_R(X)$, with respect to any norm chosen on the Néron-Severi space. More precisely, we prove:

**Theorem 4.2.** Fix any norm $\|\|$ on the Néron-Severi space $N^1(X)$ of a projective globally F-regular variety $X$. Then for each real class $D \in \operatorname{Amp}_R(X)$, there exist positive real numbers $C(D)$ and $r(D)$ (depending only on $D$ and the norm $\|\|$), such that for any two ample $\mathbb{Q}$-divisors $L$, $L'$ contained in the ball $B_r(D)(D) := \{D' \in \operatorname{Amp}_R(X) : \|D - D'\| < r(D)\}$, we have

$$|s_X(L) - s_X(L')| \leq C(D)\|L - L'\|. \quad (4.1)$$

We will say that the F-signature function $s$ is locally Lipschitz at a real class $D$ with Lipschitz constant $C(D)$ if the inequality (4.1) is satisfied for all ample $\mathbb{Q}$-divisors $L$, $L'$ that are sufficiently close to $D$.

As an immediate corollary of Theorem 4.2, we obtain:

**Corollary 4.3.** The F-signature function $s$ extends to a well-defined, continuous, locally Lipschitz function on the real ample cone $\operatorname{Amp}_R(X)$ of $X$ satisfying the identity:

$$s_X(\lambda L) = \frac{1}{\lambda} s_X(L) \quad \text{for all } \lambda \in \mathbb{R}_{>0} \text{ and all } L \in \operatorname{Amp}_R(X).$$

**Proof.** Let $D \in \operatorname{Amp}_R(X)$ be a real ample class on $X$. The Lipschitz inequality (4.1) implies that for any sequence of ample $\mathbb{Q}$-divisors $L_n$ converging to $D$, the sequence $s_X(L_n)$ is Cauchy, hence converges to a unique real number. This gives a well-defined extension of $s_X$ to the real ample cone $\operatorname{Amp}_R(X)$, that remains locally Lipschitz. Hence, $s$ is continuous on $\operatorname{Amp}_R(X)$. Finally, the identity $s_X(\lambda L) = \frac{1}{\lambda} s_X(L)$ follows by continuity, since it already holds for all rational $L$ and $\lambda$. \qed

4.1. Informal sketch of the proof of Theorem 4.2: The proof of Theorem 4.2 consists of several steps. We summarize the ideas in this subsection.

There are two main differences from the case of the volume function. First is that if we have two Cartier divisors $L$ and $L'$ such that $L' - L$ is effective (or even ample), it is not clear that the F-signature of $\mathcal{O}_X(L)$ is greater than (or less than) that of $\mathcal{O}_X(L')$. Another difficulty is that we can not run inductive arguments since any divisors we construct may not be globally F-regular. However, we may still use comparisons to the volume function as indicated below:

**Step 1:** First, in Lemma 4.6, we prove a formula for calculating the F-signature of an ample Cartier divisor $L$, in terms of Frobenius splittings of the linear systems $|mL|$ for $m \gg 0$. This gives us a tool to compare $s_X(L)$ and $s_X(L')$ whenever we have a non-zero map $\mathcal{O}_X(mL) \to \mathcal{O}_X(m'L')$ for $m \gg 0$ (Lemma 4.11).

**Step 2:** Given two ample $\mathbb{Q}$-divisors $L$ and $L'$, we first consider the case when $L' - L$ is big. Since $L' - L$ is big, for $m \gg 0$, we have $|mL' - mL| \neq \phi$ allowing us to compare
Step 3: In this step, we estimate the difference in the $F$-signatures by comparing it to the difference in volumes. Here, we encounter the key difficulty, which is that we don’t know the sign of the difference between $s_X(L')$ and $s_X(L)$, even if $L' - L$ is effective, which we have already assumed. This is overcome by introducing the difference between $s_X(L)$ and $s_X(\alpha L)$ (where $\alpha$ is as in Step 3), along with comparisons to the volume function to estimate the difference between $s_X(L)$ and $s_X(L')$. These estimates are the contents of Lemma 4.13 and Lemma 4.14.

Step 4: To control the difference in the volumes (from Step 4), we need an additional ingredient: For any $e \geq 1$, we need effective bounds for the degrees $m$ that contribute Frobenius splittings to the $e^{th}$ free-rank for $S(X, L)$ and $S(X, L')$ (Theorem 4.8).

Step 5: The steps so far give us an inequality of the form
\[ |s_X(L) - s_X(L')| \leq C(L)\|L - L'\| \tag{4.2} \]
for a fixed $L$ and all $L'$ sufficiently close to $L$ and for some constant $C(L)$ depending on $L$ (Lemma 4.10). One result required here is the (Lipschitz) continuity of the volume function on the ample cone (Lemma 2.23).

Step 6: Though (4.2) proves continuity of $s_X$ at a fixed $\mathbb{Q}$-divisor $L$, it does not prove that $s_X$ is locally Lipschitz, since the constant $C(L)$ depends on $L$. So, in Proposition 4.16 and Lemma 4.17, we track the constant $C(L)$ and examine the variation with $L$. This involves carefully choosing the scalar $\alpha$ from Step 3.

Step 7: As a result, we see that we may pick the constants $C(L)$ such that $C(L) = O\left(\frac{1}{\|L\|^2}\right)$ as $\|L\| \to \infty$. Now, since $s_X(rL) = \frac{1}{r} s_X(L)$ by Theorem 3.2, we see that for a $\mathbb{Q}$-divisor $L$, we may pick $C(L) = r^2 C(rL)$ for any $r > 0$. This shows that we may pick uniform Lipschitz constants on compact subsets of the ample cone.

Step 8: Given two ample $\mathbb{Q}$-divisors $L$ and $L'$, we may consider a small perturbation $\lambda L'$ of $L'$ (i.e. $\lambda \approx 1$) so that $\lambda L' - L$ is big (or even ample). Using the transformation rule as in Theorem 3.2, we may replace $L'$ by $\lambda L'$ and reduce to the case when $L' - L$ is big, concluding the proof.

4.2. Proof of Theorem 4.2. The rest of this section is dedicated to a detailed proof of Theorem 4.2.

Notation 4.4. For any Cartier divisor $D$, we use the notation $H^0(D)$ to denote the space of global sections $H^0(X, \mathcal{O}_X(D))$.

Definition 4.5. For any Cartier divisor $D$ on $X$, define the $k$-vector subspace $I_\epsilon(D)$ of $H^0(D)$ as follows:
\[ I_\epsilon(D) := \{ f \in H^0(D) \mid \varphi(F^*_e f) = 0 \text{ for all } \varphi \in \text{Hom}_{\mathcal{O}_X}(F^*_e \mathcal{O}_X(D), \mathcal{O}_X) \}. \]

That is, $I_\epsilon(D)$ is the set of global sections $f$ of $\mathcal{O}_X(D)$ such the map $\mathcal{O}_X \to F^*_e \mathcal{O}_X(D)$ sending $1 \mapsto F^*_e f$ does not split. Hence, the maximum number of $\mathcal{O}_X$-summands of $F^*_e \mathcal{O}_X(D)$ is equal to the dimension of the vector space $H^0(D)/I_\epsilon(D)$. A section $f \in H^0(D)$ that is not contained in $I_\epsilon(D)$ along with a map $\varphi : F^*_e \mathcal{O}_X(D) \to \mathcal{O}_X$ sending $F^*_e f$ to $1$ is called an $e^{th}$-Frobenius splitting of the linear system $|D|$. 
Lemma 4.6. Let $L$ be an ample Cartier divisor and $S$ denote the section ring of $X$ with respect to $L$. Then, for any $e \geq 1$, if $a_e(L)$ denotes the free-rank of $F^e_S$ as an $S$-module, then $a_e(L)$ is computed by the following formula:

$$a_e(L) = \sum_{m=0}^{\infty} \dim_k \frac{H^0(mL)}{I_e(mL)}.$$  \hfill (4.3)

Hence, the $F$-signature of $L$ can be computed as

$$s_X(L) = \lim_{e \to \infty} \frac{\sum_{m=0}^{p^e-1} \dim_k \frac{H^0(mL)}{I_e(mL)}}{p^e(\dim(X)+1)}.$$  \hfill (4.4)

Proof. Let $\mathcal{L}$ denote the invertible sheaf $O_X(L)$. We note that the $S$-module $F^e_S$ naturally decomposes as an $\mathbb{N}$-graded module as (see the discussion preceding Definition 2.5):

$$F^e_S = \bigoplus_{n=0}^{p^e-1} \bigoplus_{i \geq 0} H^0(X, \mathcal{L}^i \otimes F^e_{n,L}),$$

where $M_{e,n} := \bigoplus_{i \geq 0} H^0(X, \mathcal{L}^i \otimes F^e_{n,L})$ is naturally an $\mathbb{N}$-graded $S$-module. Note also that since $H^0(X, \mathcal{L}^{n+i} \otimes F^e_{n,L}) = 0$ for $i < 0$ and $0 \leq n \leq p^e - 1$, the module $M_{e,n}$ is the section module of the sheaf $F^e_{n,L}$ with respect to $\mathcal{L}$. We recall that by Lemma 2.4, $a_e(L)$ can be calculated as the graded free-rank of $F^e_S$ i.e.

$$a_e(L) = \max\{r \mid F^e_S \cong \bigoplus_{r=1}^{r} S(-j_t) \bigoplus_{t=1}^{r} N \text{ as graded } S\text{-modules} \}$$

for some $j_t \in \mathbb{Z}$ and some graded $S$-module $N$.

Since $F^e_S$ is $\mathbb{N}$-graded, we note that each integer $j_t$ occurring in any decomposition of $F^e_S$ as above is non-negative. Sheaf theoretically, we have

$$a_e(L) = \max\{r \mid \widehat{F^e_S} \cong \bigoplus_{0 \leq n \leq p^e-1} F^e_{n,L} \cong \bigoplus_{t=1}^{r} \mathcal{L}^{-j_t} \bigoplus N \}$$

as $O_X$-modules for some $j_t \in \mathbb{N}$ and some sheaf $N$.

For any $0 \leq n \leq p^e - 1$, and $j \geq 0$, the maximum number of $\mathcal{L}^{-j}$ summands of $F^e_{n,L} \mathcal{L}^n \otimes F^e_{n,L}$ is the same as the maximum number of $O_X$ summands of $F^e_{n,L} \mathcal{L}^n \otimes F^e_{n,L}$. Running over all $0 \leq n \leq p^e - 1$ and $j \geq 0$, we get the desired formula (4.3) for $a_e(L)$, since the maximum number of $O_X$ summands of any $F^e_{n,L}$ is exactly given by the dimension of $H^0(mL)/I_e(mL)$.

Remark 4.7. Since the free-rank of $F^e_S$ is bounded by its generic rank (which is exactly $p^e(\dim(X)+1)$), the sum in equation (4.3) is indeed finite. Next, we will find effective bounds for the number of terms in this sum.

Theorem 4.8. Let $X$ be a globally $F$-regular projective variety over $k$. Fix a norm $\| \|$ on the Néron-Severi space $N^1_\mathbb{R}(X)$. There exists a constant $C_1 := C_1(X)$ (depending only on $X$, and the norm $\| \|$) such that, whenever $L$ and $H$ are any two effective Cartier divisors on $X$, we have:

(a) $$I_e(mL) = H^0(mL) \text{ for } m > \frac{C_1}{\|L\|} p^e,$$ and,
(b) For all $n > \frac{2\|H\|}{\|L\|}$,
\[ I_\varepsilon(m(nL + H)) = H^0(m(nL + H)) \text{ for all } m > \frac{C_1p^e}{n\|L\|}. \]

Proof. Since $X$ is normal, we can consider the canonical (Weil) divisor on $X$ (denoted by $K_X$), by extending the canonical divisor on the non-singular locus of $X$. Choosing an ample divisor $A$ such that $A + K_X$ is effective, we may write $A \sim -K_X + E$ for some effective (Weil) divisor $E$. Let $[A]$ denote the class of $A$ in the ample cone of $X$.

Let $L$ be any effective Cartier divisor on $X$. By applying duality for the Frobenius map, we have,
\[ \mathcal{H}om_{\mathcal{O}_X}(F^e_*\mathcal{O}_X(mL), \mathcal{O}_X) \cong F^e_*\mathcal{O}_X(-(p^e - 1)K_X - mL). \]
See [SS10, Section 4.1] for a detailed discussion regarding duality for the Frobenius map. Hence, we have,
\[ \text{Hom}_{\mathcal{O}_X}(F^e_*\mathcal{O}_X(mL), \mathcal{O}_X) \cong F^e_*H^0(X, \mathcal{O}_X(-(p^e - 1)K_X - mL)). \tag{4.5} \]
This shows that to prove that $H^0(mL) = I_\varepsilon(mL)$ for any given $m$, it suffices to show the right hand side in (4.5) is zero.

Claim: There exists a positive constant $C_1'$ (depending only on $X$, the choice of $A$ and the norm $\|\|$), such that for any effective divisor $D$ with $\|D\| > C_1'$, we have $-K_X - D$ is not an effective divisor, i.e., $-K_X - D$ is not $\mathbb{R}$-linearly equivalent to any effective divisor.

Proof of the claim. Since the pseudoeffective cone (Definition 2.27) $\overline{\text{Eff}}(X)$ is a closed strongly convex cone (i.e. there is no non-zero class $\nu \in \overline{\text{Eff}}(X)$ such that $-\nu \in \overline{\text{Eff}}(X)$), the set
\[ \kappa := \overline{\text{Eff}}(X) \cap ([A] - \overline{\text{Eff}}(X)) \]
is a compact subset of $\overline{\text{Eff}}(X)$. Since, the norm function achieves a maximum on $\kappa$, we may choose $C_1'$ to be bigger than the norm of any class in $\kappa$.
\[ C_1' > \max\{\|\xi\| \mid \xi \in \kappa\}. \]
Note that $C_1'$ depends only on the choice of $A$ and the norm $\|\|$.

Recall that (the class of) every effective divisor on $X$ is contained in the pseudoeffective cone of $X$. Now, if $D$ is an effective divisor with $\|D\| > C_1'$, then $D$ can not belong to $\kappa$ by the definition of $C_1'$. Hence, we see that $A - D$ is not effective. Since $A = -K_X + E$ for an effective divisor $E$, this means that $-K_X - D$ is not effective. This proves the claim. \qed

Continuation of the proof of Theorem 4.8: For any effective Cartier divisor $L$, if $m > \frac{C_1p^e}{\|L\|}$, we have $\|\frac{m}{p^e - 1}L\| > C_1'$, hence, applying the claim above, we conclude that $-K_X - \frac{m}{p^e - 1}L$ is not effective. Therefore, the divisor
\[ -(p^e - 1)K_X - mL \]
is not effective. By (4.5), this gives us $H^0(mL) = I_\varepsilon(mL)$ as required. This proves part (a).

For part (b), we use part (a) of the Theorem by replacing $L$ by $nL + H$, which gives us that $H^0(m(nL + H)) = I_\varepsilon(m(nL + H))$ for $m > \frac{C_1p^e}{\|nL + H\|}$. Since by assumption $\|H\| \leq \frac{1}{2}\|nL\|$, we have, $\|nL + H\| \geq \|nL\| - \|H\| \geq \frac{1}{2}\|nL\|$. Therefore,
\[ \frac{2C_1p^e}{n\|L\|} \geq \frac{C_1p^e}{\|nL + H\|}. \]
using which we see that \( C_1 = 2C_1' \) works for both part (b). This completes the proof of Theorem 4.8.

\[ \square \]

**Remark 4.9.** For a more effective, but less uniform version of Theorem 4.8, see Lemma 6.2.

Next, we prove Theorem 4.2 in the special case when the divisor \( L \) is fixed and the difference \( L' - L \) is big.

**Lemma 4.10 (Key Lemma).** Let \( L \) be an integral ample divisor on \( X \). Then, there exists a constant \( C(L) \) (depending only on \( L \) and the norm \( \| \| \) ) such that for any other ample \( \mathbb{Q} \)-divisor \( L' \) sufficiently close to \( L \), and for which \( L' - L \) is big, we have:

\[
|s_X(L) - s_X(L')| \leq C(L)\|L - L'\|.
\]

**Proof of Lemma 4.10.** Throughout the proof, we fix the following set-up: Fixing the ample, integral divisor \( L \) on \( X \), we pick an arbitrary ample \( \mathbb{Q} \)-divisor \( L' \) such that \( L' - L \) is big. Then, we may write \( L' = L + \frac{1}{n}H \), for some \( n \gg 0 \) and an effective and big Cartier divisor \( H \).

**Lemma 4.11.** For effective divisors \( D_1 \) and \( D_2 \), consider the natural inclusion:

\[
\phi : F^e_\ast \mathcal{O}_X(D_1) \subset F^e_\ast \mathcal{O}_X(D_1 + D_2).
\]

Then,

\[
\phi(I_e(D_1)) \subset I_e(D_1 + D_2). \tag{4.6}
\]

Equivalently, viewing \( H^0(X, \mathcal{O}_X(D_1)) \) as a subset of \( H^0(X, \mathcal{O}_X(D_1 + D_2)) \) through the map \( \phi \), we have:

\[
\phi(I_e(D_1)) \subset I_e(D_1 + D_2) \cap H^0(D_1) = \{ x \in H^0(D_1) \mid \phi(x) \in I_e(D_1 + D_2) \}. \tag{4.7}
\]

**Proof.** This follows from the definitions once we observe that for every map \( \varphi \) in \( \text{Hom}_{\mathcal{O}_X}(F^e_\ast \mathcal{O}_X(D_1 + D_2), \mathcal{O}_X) \), we get a map \( \tilde{\varphi} \) in \( \text{Hom}_{\mathcal{O}_X}(F^e_\ast \mathcal{O}_X(D_1), \mathcal{O}_X) \) by pre-composing with the map \( \phi \).

\[
\begin{array}{ccc}
F^e_\ast \mathcal{O}_X(D_1) & \xrightarrow{\phi} & F^e_\ast \mathcal{O}_X(D_1 + D_2) \\
\downarrow \tilde{\varphi} & & \downarrow \varphi \\
\mathcal{O}_X & & \\
\end{array}
\]

\[ \square \]

**Lemma 4.12.** With \( L \) an ample Cartier divisor and \( H \) an effective big divisor on \( X \), and any natural number \( n \), suppose that we have a natural number \( b := b(n) \), such that \( nL - bH \) is big. Consequently, there is a \( C_2 \gg 0 \) (Lemma 2.26) such that for all \( m \geq C_2 \), we have

\[
H^0(m(nL - bH)) \neq 0.
\]

Then, for \( m \geq C_2 \) and all \( e \geq 1 \), there is a factorization of inclusions:

\[
\begin{array}{ccc}
F^e_\ast \mathcal{O}_X(mnbL) & \xrightarrow{F^e_\ast \mathcal{O}_X(mnbH)} & F^e_\ast \mathcal{O}_X(mb(nL + H)) \\
\downarrow F^e_\ast e & & \downarrow F^e_\ast d \\
F^e_\ast \mathcal{O}_X(mb(nL - mbH)) & & \end{array}
\]

given by a choice of a section \( d \in H^0(X, \mathcal{O}_X(mb(nL - mbH))) \).
Proof. Given a section $d \in H^0(X, \mathcal{O}_X(mnL - mbH))$, let $D_1$ be the corresponding effective divisor. Then, we have $D_2 = mbH + D_1 \sim mbH + mnL - mbH = mnL$. Then, we get inclusions

$$
\mathcal{O}_X(mnbL) \rightarrow \mathcal{O}_X(mb(nL + H)) \rightarrow \mathcal{O}_X(mnbL + D_2)
$$

since $D_1$ was effective. We get the required factorization by applying $F^e_*$ to the above diagram and taking $c$ to be the section corresponding to the divisor $D_2$. □

Lemma 4.13. Fix an ample Cartier divisor $L$ and an effective Cartier divisor $H$ on $X$. For any fixed $n > \frac{2\|H\|}{\|L\|}$, suppose there is a natural number $b := b(n) > 0$ depending on $n$ such that $nL - bH$ is big. Further, let $C_1 := C_1(X)$ be the constant as obtained in Theorem 4.8. Then, we have the following inequality (here, $d = \dim(X)$):

$$
|s_X(L) - s_X(L + \frac{1}{n}H)| \leq \frac{C_1^{d+1}}{\|L\|^{d+1}(d+1)!} \left( 2 \text{vol}(L) \left( (b+1)^d - b^d \right) \frac{\|L\|^d}{b^d} \right)
$$

$$+
\left( \text{vol}(L + \frac{1}{n}H) - \text{vol}(L) \right)
+ 2\frac{s_X(L)}{b+1}
$$

(4.8)

Proof. First, fixing $n$ and $b$, there is a $C_2 \gg 0$ such that $H^0(m(nL - bH)) \neq 0$ for all $m \geq C_2$. Using Lemma 4.6 and Theorem 4.8, we have the following formulas for the $F$-signatures:

$$
s_X(nbL) = \lim_{e \to \infty} \frac{1}{p^{e(d+1)}} \sum_{m=C_2}^{\frac{C_1p^e}{\|L\|^d}} \dim_k \frac{H^0(mnbL)}{I_e(mnbL)}
$$

(4.9)

and similarly,

$$
s_X(b(nL + H)) = \lim_{e \to \infty} \frac{1}{p^{e(d+1)}} \sum_{m=C_2}^{\frac{C_1p^e}{\|L\|^d}} \dim_k \frac{H^0(mb(nL + H))}{I_e(mb(nL + H))}.
$$

(4.10)

Note that even though the formula from Lemma 4.6 requires us to begin the sums (4.9) and (4.10) at $m = 0$, we may begin the sums at $C_2$ since changing finitely many terms does not alter the limit.

According to formulas (4.9) and (4.10), to compare $s_X(nbL)$ with $s_X(b(nL + H))$, we need to understand the difference

$$
\dim_k \frac{H^0(mbnL)}{I_e(mbnL)} - \dim_k \frac{H^0(mb(nL + H))}{I_e(mb(nL + H))}.
$$

We have an inclusion

$$
\frac{H^0(mbnL)}{H^0(mbnL) \cap I_e(mb(nL + H))} \hookrightarrow \frac{H^0(mb(nL + H))}{I_e(mb(nL + H))}
$$

(4.11)

coming from the inclusion of $H^0(mbnL) \hookrightarrow H^0(mb(nL + H))$. 


Let \( J_e(mbnL) = H^0(mbnL) \cap I_e(mb(nL + H)) \). Then using (4.11), we have:

\[
\dim_k \frac{H^0(mb(nL + H))}{I_e(mb(nL + H))} = \dim_k \frac{H^0(mbnL)}{J_e(mbnL)} + \dim_k \frac{H^0(mb(nL + H))}{H^0(mbnL) + I_e(mb(nL + H))}. \tag{4.12}
\]

Then, using (4.12) and the triangle inequality, we get

\[
\left| \sum_{m=\mathcal{C}_2} \frac{C_1 \rho^e}{\|L\|^{n_b}} \dim_k \frac{H^0(mbnL)}{I_e(mbnL)} - \sum_{m=\mathcal{C}_2} \frac{C_1 \rho^e}{\|L\|^{n_b}} \dim_k \frac{H^0(mb(nL + H))}{I_e(mb(nL + H))} \right| \leq \left| \sum_{m=\mathcal{C}_2} \frac{C_1 \rho^e}{\|L\|^{n_b}} \dim_k \frac{H^0(mbnL)}{I_e(mbnL)} - \sum_{m=\mathcal{C}_2} \frac{C_1 \rho^e}{\|L\|^{n_b}} \dim_k \frac{H^0(mbnL)}{J_e(mbnL)} \right| + \sum_{m=\mathcal{C}_2} \frac{C_1 \rho^e}{\|L\|^{n_b}} \dim_k \frac{H^0(mb(nL + H))}{H^0(mbnL)}
\]

\[
\leq \left| \sum_{m=\mathcal{C}_2} \frac{C_1 \rho^e}{\|L\|^{n_b}} \dim_k \frac{H^0(mb(nL + H))}{I_e(mb(nL + H))} \right| + \left| \sum_{m=\mathcal{C}_2} \frac{C_1 \rho^e}{\|L\|^{n_b}} \dim_k \frac{H^0(m(b + 1)nL)}{I_e(m(b + 1)nL)} \right| + \left| \sum_{m=\mathcal{C}_2} \frac{C_1 \rho^e}{\|L\|^{n_b}} \dim_k \frac{H^0(mb(nL + H))}{I_e(m(b + 1)nL)} \right|
\]

\[
+ \left| \sum_{m=\mathcal{C}_2} \frac{C_1 \rho^e}{\|L\|^{n_b}} \dim_k \frac{H^0(mb(nL + H))}{J_e(mbnL)} \right| + \sum_{m=\mathcal{C}_2} \frac{C_1 \rho^e}{\|L\|^{n_b}} \dim_k \frac{H^0(mb(nL + H))}{H^0(mbnL)}
\]

where in the last inequality, we use the triangle inequality again after adding and subtracting the term \( \sum_{m=\mathcal{C}_2} \frac{C_1 \rho^e}{\|L\|^{n_b}} \dim_k \frac{H^0(mb(nL + H))}{I_e(m(b + 1)nL)} \).

To proceed, we need to understand the difference between the spaces \( \frac{H^0(mb(nL + H))}{I_e(mb(nL + H))} \) and \( \frac{H^0(mbnL)}{I_e(mbnL)} \). To this end, we prove the following:

**Lemma 4.14.** Suppose, as in Lemma 4.12, \( b \) is such that \( nL - bH \) is big and \( \mathcal{C}_2 \) is such that for all \( m \geq \mathcal{C}_2 \), we have \( H^0(m(nL - bH)) \neq 0 \). Then, for \( m \geq \mathcal{C}_2 \) and all \( e \geq 1 \), choosing a non-zero global section \( d \in H^0(m(nL - bH)) \) and setting \( c = d \otimes h^{mn} \), where \( h \) is the section of \( \mathcal{O}_X(H) \) that corresponds to the rational function 1, we have the inclusions

\[
I_e(mbnL) \subset J_e(mbnL) \subset \{ x \in H^0(mbnL) \mid cx \in I_e(mn(b + 1)L) \}. \tag{4.14}
\]

Moreover, we have the following inequality (with \( C_1 \) being the constant from Theorem 4.8):
Before proving Lemma 4.14, we note that putting (4.15) together with (4.13), we obtain:

\[
\left| s_X(nbL) - s_X(b(nL + H)) \right| = \lim_{e \to \infty} \frac{1}{e^{d+1}} \left( \sum_{m=C_2}^{C_1 \rho^e} \dim_k \frac{H^0(mb(nL + H))}{I_e(mb(nL + H))} - \sum_{m=C_2}^{C_1 \rho^e} \dim_k \frac{H^0(mbL)}{I_e(mbL)} \right)
\]

\[
\leq \lim_{e \to \infty} \frac{1}{e^{d+1}} \left( \sum_{m=C_2}^{C_1 \rho^e} 2 \dim_k \frac{H^0(mbL)}{H^0(mb(nL + H))} + \sum_{m=C_2}^{C_1 \rho^e} \dim_k \frac{H^0(mbL)}{H^0(mb(nL + H))} \right)
\]

\[
+ 2 \sum_{m=C_2}^{C_1 \rho^e} \dim_k \frac{H^0(mbL)}{I_e(mbL)} - \sum_{m=C_2}^{C_1 \rho^e} \dim_k \frac{H^0(mb(nL + H))}{I_e(mb(nL + H))}
\]

\[
\leq \lim_{e \to \infty} \frac{C^{d+1} \rho^{(d+1)}}{\|L\|^{d+1}n^{d+1}b^{d+1}e^{(d+1)}(d + 1)!} \left( 2n^d \text{vol}(L) \left( (b + 1)^d - b^d \right) + b^d n^d \left( \text{vol}(L + \frac{1}{n}H) - \text{vol}(L) \right) \right)
\]

\[
+ 2 \left| s_X(nbL) - s_X(n(b + 1)L) \right|
\]

\[
= \frac{C^{d+1} \rho^{d+1}n^d b^d}{\|L\|^{d+1}n^{d+1}b^{d+1}(d + 1)!} \left( 2\text{vol}(L) \left( \frac{(b + 1)^d - b^d}{b^d} \right) + \left( \text{vol}(L + \frac{1}{n}H) - \text{vol}(L) \right) \right)
\]

\[
+ \left| s_X(nbL) - s_X(n(b + 1)L) \right|
\]
Finally, using the scaling property for $s_X$ (Theorem 3.2), we get:

$$\left| s_X(L) - s_X(L + \frac{1}{n}H) \right|$$

$$= nb \left| s_X(nbL) - s_X(b(nL + H)) \right|$$

$$\leq \frac{C_1^{d+1}}{\|L\|^{d+1}(d+1)!} \left( 2 \emph{vol}(L) \frac{(b+1)^d - b^d}{b^d} + \left( \emph{vol}(L + \frac{1}{n}H) - \emph{vol}(L) \right) \right)$$

$$+ 2nb \left| \frac{s_X(L)}{nb} - \frac{s_X(L)}{n(b+1)} \right|$$

$$= \frac{C_1^{d+1}}{\|L\|^{d+1}(d+1)!} \left( 2 \emph{vol}(L) \frac{(b+1)^d - b^d}{b^d} + \left( \emph{vol}(L + \frac{1}{n}H) - \emph{vol}(L) \right) \right) + 2 \frac{s_X(L)}{b+1}$$

This completes the proof of Lemma 4.13, pending the proof of Lemma 4.14, which we prove next. \hfill \square

**Notation 4.15.** Recall that $L$ and $H$ are fixed integral Cartier divisors, with $L$ ample and $H$ effective and $n \geq 1$ is any natural number. For any natural number $k \in \mathbb{N}$, we define:

$$I_\epsilon(k) := I_\epsilon(kL),$$

$$J_\epsilon(kn) := H^0(knL) \cap I_\epsilon(k(nL + H)), $$

where we view $H^0(knL)$ as a subspace of $H^0(k(nL + H))$ via the inclusion map $\mathcal{O}_X(nkL) \subset \mathcal{O}_X(knL + kH)$.

**Proof of Lemma 4.14.** The first inclusion in (4.14) follows from Lemma 4.11 by taking $D_1 = mnbL$ and $D_2 = mnbH$. The second inclusion follows from Lemma 4.12 and the second part of Lemma 4.11, by taking $D_1 = mb(nL + H)$ and $D_2$ to be the effective divisor corresponding...
to \( d \in H^0(mnL - mbH) \). Hence, we get

\[
\sum_{m=0}^{C_1 p^e_{L/n}} \dim_k \frac{H^0(mn(b+1)L)}{I_e(mn(b+1))} - \sum_{m=C_2}^{\infty} \dim_k \frac{H^0(mnbL)}{I_e(mnb)}
\]

\[
= \sum_{m=0}^{C_1 p^e_{L/n}} \dim_k \frac{H^0(mn(b+1)L)}{cH^0(mnbL)} - \sum_{m=C_2}^{\infty} \dim_k \frac{I_e(mn(b+1))}{cI_e(mnb)}
\]

\[
\leq \sum_{m=0}^{C_1 p^e_{L/n}} \dim_k \frac{H^0(mn(b+1)L)}{cH^0(mnbL)} + \sum_{m=C_2}^{\infty} \dim_k \frac{I_e(mn(b+1))}{cI_e(mnb)}
\]

\[
\leq \sum_{m=0}^{C_1 p^e_{L/n}} \dim_k \frac{H^0(mn(b+1)L)}{cH^0(mnbL)} + \sum_{m=C_2}^{\infty} \dim_k \frac{H^0(mn(b+1)L)}{I_e(mn(b+1))}
\]

where in the second-last step, we just rearrange terms of the sum, and in the last step use the triangle inequality again. This completes the proof of the lemma. \(\square\)

**Completion of the proof of Lemma 4.10:** Recall that \( L \) is a fixed ample divisor on \( X \) (in particular, \( L \) is big). Suppose \( L' \) is an ample \( \mathbb{Q} \)-divisor such that \( L' - L \) is big. Further assume that \( ||L' - L|| < \frac{||L||}{2} \). Then, we may write \( L' = L + \frac{1}{n}H \) for a suitable effective Cartier divisor \( H \) and some natural number \( n \geq 1 \).

We would like to apply Lemma 4.13 to this choice of \( L, H \) and \( n \). For this, we need to choose a natural number \( b \) such that \( nL - bH \) is big. We note that we may choose \( b \) in the following way: Since \( L \) is big, by openness of the big cone of \( X \), there exists a constant \( C_4 > 0 \) (depending only on \( L \)) such that any \( \mathbb{Q} \)-divisor \( D \) satisfying \( ||L - D|| \leq C_4 \) is also big. Since we need \( L - \frac{b}{n}H \) to be big, it is sufficient that \( \frac{b}{n}H \leq C_4 \). So we may choose \( b(n) = \lceil \frac{nc_4}{||H||} \rceil \) so that \( b(n) \to \infty \) as \( n \to \infty \).
Now, applying Lemma 4.13 to this choice of $n$ and $b$, we get:

$$|s_X(L) - s_X(L + \frac{1}{n} H)| \leq \frac{C_1^{d+1}}{\|L\|^{d+1}(d+1)!} \left( 2 \text{vol}(L) \frac{(b+1)^d - b^d}{b^d} \right)$$

$$+ \left( \text{vol}(L + \frac{1}{n} H) - \text{vol}(L) \right) + 2 \frac{s_X(L)}{b+1} \tag{4.16}$$

Further, we have

$$\frac{(b+1)^d - b^d}{b^d} \leq \frac{2^d}{b}$$

and by Lemma 2.23, there is a positive constant $C_3$, depending only on $X$ and the norm $\|\|$, such that for any two ample classes $\xi_1, \xi_2 \in N_1^Q(X)$,

$$|\text{vol}(\xi_1) - \text{vol}(\xi_2)| \leq C_3 \left( \max(\|\xi_1\|, \|\xi_2\|) \right)^{d-1} \|\xi_1 - \xi_2\|.$$

Putting these together, along with (4.16), and using that $\|\frac{1}{n} H\| = \|L - L'\|$, we get

$$|s_X(L) - s_X(L')| \leq \frac{C_1^{d+1}}{(d+1)!} \left( 2 \text{vol}(L) \frac{2^d}{b} \right)$$

$$+ C_3 \|L'\|^{d-1} \|L - L'\| + 2 \frac{s_X(L)}{b} \tag{4.17}$$

Next, using the fact $b$ was chosen to be $b(n) = \lfloor \frac{nC_3}{\|H\|} \rfloor$, we have $b \geq \frac{nC_3}{2\|H\|}$, using which we get

$$|s_X(L) - s_X(L')| \leq \frac{C_1^{d+1}}{\|L\|^{d+1}(d+1)!} \left( 2 \text{vol}(L) \frac{2^{d+1}}{C_4} \|L - L'\| \right)$$

$$+ C_3 \left( \frac{2^d}{C_4} \right) \frac{4}{s_X(L)} \|L - L'\| \tag{4.18}$$

Lastly, since $\|L - L'\| < \frac{\|L\|}{2}$ we have $\|L'\| < 2\|L\|$. Hence, we have

$$|s_X(L) - s_X(L')| \leq \frac{C_1^{d+1}}{\|L\|^{d+1}(d+1)!} \left( 2 \text{vol}(L) \frac{2^{d+1}}{C_4} \|L - L'\| \right)$$

$$+ 2^{d-1} C_3 \|L\|^{d-1} \|L - L'\| + 4 \frac{4}{C_4} s_X(L) \|L - L'\| \tag{4.19}$$

Hence, we see that for any ample, integral divisor $L$, we have

$$|s_X(L) - s_X(L')| \leq C(L) \|L - L'\|$$

for all ample $\mathbb{Q}$-divisors $L'$ such that $L' - L$ is big and $\|L - L'\| < \frac{\|L\|}{2}$ where $C(L)$ is given by

$$C(L) = \frac{C_1^{d+1}}{\|L\|^{d+1}(d+1)!} \left( \text{vol}(L) \frac{2^{d+1}}{C_4} + 2^{d-1} C_3 \|L\|^{d-1} \right) + 4 \frac{4}{C_4} s_X(L).$$

This completes the proof of the Key Lemma 4.10. \qed
The proof of the Key Lemma 4.10 actually shows a stronger and more explicit statement that will be useful to us. We record it in the following Proposition.

**Proposition 4.16.** For any ample, integral divisor $L$, we have $$|s_X(L) - s_X(L')| \leq C(L)\|L - L'\|$$ for all ample $\mathbb{Q}$-divisors $L'$ such that $L' - L$ is big and $\|L - L'\| < \frac{\|L\|}{2}$, where $C(L)$ maybe chosen to be of the form

$$C(L) = \frac{C_1^{d+1}}{\|L\|^{d+1}(d+1)!} \left( \text{vol}(L) \frac{2^{d+1}}{C_4} + 2^{d-1}C_3\|L\|^{d-1} \right) + \frac{4}{C_4}s_X(L).$$

Here, $C_1 := C_1(X)$ is the constant (depending only on $X$) obtained in Lemma 6.2, $C_3$ depends only on $X$, and $C_4 := C_4(L)$ is any constant (depending on $L$) with the property that the closed ball $B = \{ D \in N^+_\mathbb{Q}(X) \mid \|D - L\| \leq C_1 \}$ is contained in the big cone of $X$.

Next, we examine how the constant $C(L)$ in Proposition 4.16 varies with $L$.

**Lemma 4.17.** Let $X$ be projective variety and $\mathcal{C}$ be a closed cone contained in the big cone of $X$. Then, there exists a constant $\tilde{C}_4$ (depending only on $\mathcal{C}$) such that for any non-zero class $D \in \mathcal{C}$, the closed ball $$B(D) = \{ \xi \in N^1_R(X) \mid \|\xi - D\| < C_4\|D\| \}$$ is contained in $\text{Big}(X)$.

**Proof.** Consider the set $$\kappa := \{ D' \in \mathcal{C} \mid \|D'\| = 1 \}.$$ Since $\mathcal{C}$ is a closed cone, $\kappa$ is a compact subset of $\mathcal{C}$. Moreover, since $\mathcal{C}$ is contained in the big cone of $X$ and because $\text{Big}(X)$ is an open subset of $N^1_R(X)$, there exists a positive real number $\tilde{C}_4 > 0$ such that the ball $B_{\tilde{C}_4}(D) = \{ \xi \mid \|D - \xi\| \leq \tilde{C}_4 \}$ is contained in the big cone for all $D \in \kappa$. Now the lemma follows by considering $\frac{1}{\|D\|}D \in \kappa$ whenever $D$ is a non-zero class in $\mathcal{C}$.

**Lemma 4.18.** Given any two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on the vector space $\mathbb{R}^N$, we have positive constants $\mu_1$ and $\mu_2$ such that for any vector $\nu \in \mathbb{R}^N$, $$\mu_1\|\nu\|_1 \leq \|\nu\|_2 \leq \mu_2\|\nu\|_1.$$ 

**Proof.** See [Fol99, Section 5.1, Ex. 6].

**Lemma 4.19.** Let $e_1, \ldots, e_\rho$ be a basis for the Néron-Severi space of $X$, where each $e_i$ corresponds to a big divisor. Let $\mathcal{C}$ denote the closed cone generated by the $e_i$'s and $\|\cdot\|$ denote the sup-norm with respect to the basis $\{e_i\}$. For any $L$ in $\mathcal{C}$, let $\lambda_i(L)$ denote the $i^{th}$-coordinate of $L$ with respect to the basis $\{e_i\}$. Suppose we have two positive numbers $0 < A_1 < A_2$ and a compact subset $\kappa$ of $\mathcal{C}$ defined by $$\kappa = \{ \xi \in \mathcal{C} \mid A_1 \leq \|\xi\| \leq A_2 \}.$$ In this situation, for every $D$ in the interior of $\kappa$, there exists a positive real number $r(D)$ such that the following three conditions are satisfied:

(a) $r(D) \leq \frac{A_1}{2}$.
(b) The closed ball

\[ B_{r(D)} := \{ D' \mid \| D' - D \| \leq r(D) \} \]

is contained in the interior of \( \kappa \).

(c) For any two \( \mathbb{Q} \)-divisors \( L \) and \( L' \) in \( B_{r(D)} \), setting \( \lambda = \max \{ \frac{\lambda_i(L)}{\lambda_i(L')} \} \), we have

\[ A_1 < \lambda \| L' \| < A_2 \]

and

\[ \| \lambda L' - L \| < \frac{A_1}{2}. \]

**Proof.** First, pick any positive number \( r < \frac{A_1}{4} \) such that \( B_r \), the closed ball of radius \( r \) around \( D \) is contained in \( \kappa \) (this is possible since \( D \) is contained in the interior of \( \kappa \)). Now, there exists a positive number \( \varepsilon \) such that for any \( L \) in \( B_{r/2} \), both \((1 - \varepsilon)L \) and \((1 + \varepsilon)L \) are contained in \( B_r \). Finally pick \( 0 < r(D) < r/2 \) so small that for each \( i \), we have

\[ 1 - \frac{\lambda_i(L)}{\lambda_i(L')} < \varepsilon \]

for all \( L, L' \) in \( B_{r(D)} \). This is possible due to the local uniform continuity of the function \( \lambda_i(L) \) as \( L \) varies. By construction, for any \( L, L' \in B_{r(D)} \), we have

\[ 1 - \varepsilon < \lambda = \max_i \left\{ \frac{\lambda_i(L)}{\lambda_i(L')} \right\} < 1 + \varepsilon. \]

This ensures that \( \lambda L' \) is in \( B_r \) and since \( r < \frac{A_1}{4} \), also that

\[ \| \lambda L' - L \| \leq \| \lambda L' - D \| + \| D - L \| < \frac{3r}{2} < \frac{A_1}{2}. \]

Finally, we can now prove Theorem 4.2.

**Completion of the proof of Theorem 4.2:** Fix a real class \( D \) in the ample cone \( \text{Amp}_\mathbb{R}(X) \). Then, to prove that \( s_X \) is locally Lipschitz around \( D \), by Lemma 4.18, we may pick a suitable norm depending on \( D \). Since the ample cone \( \text{Amp}_\mathbb{R}(X) \) is an open subset of \( N^1_\mathbb{R}(X) \), given \( D \) in \( \text{Amp}_\mathbb{R}(X) \), we may pick a basis \( e_1, \ldots, e_\rho \) for \( N^1_\mathbb{R}(X) \) such that each \( e_i \) is the class of an ample invertible sheaf and such that \( D \) in contained in the interior of the cone generated by the \( e_i \)'s (equivalently, \( D = \sum a_i e_i \) with each \( a_i > 0 \)). Let \( \mathcal{C} = \{ a_i e_i \mid a_i \geq 0 \} \) denote the closed cone generated by the \( e_i \)'s and \( \| \| \) denote the sup-norm with respect to the basis \( \{ e_i \} \).

Pick two positive real numbers \( A_1 \) and \( A_2 \) such that \( 0 < A_1 < \| D \| < A_2 \). Let \( \kappa = \{ D' \mid A_1 \leq \| D' \| \leq A_2 \} \). We will first consider the case of any two \( \mathbb{Q} \)-divisors \( L \) and \( L' \) in \( \kappa \) such that \( L' - L \) is big and \( \| L' - L \| < \| L \| / 4 \). Choose an integer \( r \gg 0 \) such that \( rL \) is integral. Then, we may apply Proposition 4.16 to \( rL \) and \( rL' \), to get

\[ |s_X(rL) - s_X(rL')| \leq C(rL)\| rL - rL' \| \]

where

\[ C(rL) = \frac{C_1^{d+1}}{\| rL \|^{d+1} (d + 1)!} \left( \text{vol}(rL) \frac{2^{d+1}}{C_4(rL)} + 2^{d-1}C_3\| rL \|^{d-1} \right) + \frac{4}{C_4(rL)}s_X(rL). \]
Now, applying Lemma 4.17 to the cone $\mathcal{C}$, we may pick $C_4(rL)$ with the property that 

$$C_4(rL) \geq \tilde{C}_4 \| rL \|$$

for some constant $\tilde{C}_4$ (depending only on the basis $\{e_i\}$) and for all $r$ and all $L \in \mathcal{C}$. Using this in (4.21), we get

$$C(rL) \leq \frac{C^{d+1}_1}{\| rL \|^{d+1}(d+1)!} \left( \frac{\text{vol}(rL)}{C_4 \| rL \|} + 2^{d-1}C_3 \| rL \|^{d-1} \right) + \frac{4}{C_4 \| rL \|} s_X(rL)$$

$$= \frac{C^{d+1}_1}{(d+1)!\| L \|^{d+1}r^{d+1}} \left( \frac{r^d \text{vol}(L)}{rA_1} + 2^{d-1}C_3 r^{d-1} \| L \|^{d-1} \right) + \frac{4}{r^2 C_4} s_X(L)$$

$$= \frac{1}{r^2} \left( \frac{C^{d+1}_1}{(d+1)!\| L \|^{d+1}} \left( \frac{\text{vol}(L)}{C_4 A_1} + 2^{d-1}C_3 A_2^{d-1} \right) + \frac{4}{C_4 A_1} \right) \quad (4.22)$$

Now, using the fact that $\text{vol}(L)$ is a continuous function of $L$ [LM09], we may find a constant $A_3$ (depending only on the compact set $\kappa$) such that $\text{vol}(L) \leq A_3$ for all $L$ in $\kappa$. Using this together with the bounds $A_1 \leq \| L \| \leq A_2$ in (4.22), we get

$$C(rL) \leq \frac{1}{r^2} \left( \frac{C^{d+1}_1}{(d+1)!\| L \|^{d+1}} \left( A_3 \frac{2^{d+1}}{C_4 A_1} + 2^{d-1}C_3 A_2^{d-1} \right) + \frac{4}{C_4 A_1} \right) \quad \text{.}$$

So setting

$$C'(D) = \left( \frac{C^{d+1}_1}{(d+1)!\| L \|^{d+1}} \left( A_3 \frac{2^{d+1}}{C_4 A_1} + 2^{d-1}C_3 A_2^{d-1} \right) + \frac{4}{C_4 A_1} \right),$$

and using it in (4.20), we have

$$| s_X(rL) - s_X(rL') | \leq \frac{1}{r^2} C'(D) \| rL - rL' \|. $$

Using the scaling property of $s_X$ for $\mathbb{Q}$-divisors (Theorem 3.2), this in turn implies,

$$| s_X(L) - s_X(L') | \leq C'(D) \| L - L' \| \quad (4.23)$$

for any two $\mathbb{Q}$-divisors $L, L'$ in $\kappa$ such that $L' - L$ is big and $\| L - L' \| \leq \frac{\| L \|}{2}$. Note that $C'(D)$ only depends on the set $\kappa$ and hence only on $D$.

To complete the proof of Theorem 4.2, we need to remove the assumption that $L' - L$ is big from inequality (4.23). For any $L$ in $\mathcal{C}$, let $\lambda_i(L)$ denote the $i$th-coordinate of $L$ with respect to the basis $\{e_i\}$ (for $1 \leq i \leq \rho$). Now, since $D$ is contained in the interior of $\kappa$, by Lemma 4.19 there exists a positive $r(D)$ satisfying the following three conditions:

(a) $r(D) < \frac{A_1}{2}$.
(b) The closed ball $B_{r(D)} := \{ D' \mid \| D' - D \| \leq r(D) \}$ is contained in the interior of $\kappa$. 

(c) For any two \( \mathbb{Q} \)-divisors \( L \) and \( L' \) in \( B_{r(D)} \), setting \( \lambda = \max_i \{ \frac{\lambda_i(L)}{\lambda_i(L')} \} \), we have

\[
A_1 < \lambda \| L' \| < A_2
\]

and

\[
\| \lambda L' - L \| < \frac{A_1}{2}.
\]

Fix such an \( r(D) \). For any two \( \mathbb{Q} \)-divisors \( L \) and \( L' \) in \( B_{r(D)} \) such that \( L' \) is not a multiple of \( L \), setting \( \lambda = \max_i \{ \frac{\lambda_i(L)}{\lambda_i(L')} \} \), then \( \lambda L' - L \) is ample (hence, big). Indeed, recall that \( e_i \)'s are an ample basis for \( N_1^1(X) \) and the \( j \)-th coordinate of \( \lambda L' - L \) is

\[
\lambda \lambda_j(L') - \lambda_j(L) = \lambda_j(L') \left( \lambda - \frac{\lambda_j(L)}{\lambda_j(L')} \right) \geq 0
\]

The right hand side is non-negative since \( \lambda \) is the maximum of \( \lambda_i(L)/\lambda_i(L') \). Now, if \( \lambda = \lambda_j(L)/\lambda_j(L') \) for all \( j \), then \( L' \) is a multiple of \( L \). Therefore, if \( L' \) is not a multiple of \( L \), one of the coefficients of \( \lambda L' - L \) is strictly positive, which implies \( \lambda L' - L \) is ample.

Furthermore, \( \lambda L' \in \kappa \) and \( \| \lambda L' - L \| < \frac{\| L \|}{2} \) (these are ensured by condition (c) on \( r(D) \)). Hence, using (4.23) we have

\[
|s_X(\lambda L') - s_X(L)| \leq C'(D)\| \lambda L' - L \|
\]

for any two ample \( \mathbb{Q} \)-divisors \( L \) and \( L' \) contained in \( B_{r(D)} \).

Pick a positive constant \( A_4 \) (depending only on \( D \) and \( r(D) \)) such that we \( \lambda_i(L) \geq A_4 \) for any \( L \) in \( B_{r(D)} \) and all \( i \). This is possible because \( \kappa \), hence the closed ball \( B_{r(D)} \) is contained in the interior of the cone \( \mathcal{C} \). Since for some \( i \), we have \( \lambda = \frac{\lambda_i(L)}{\lambda_i(L')} \), we have

\[
|\lambda - 1| \leq \frac{|\lambda_i - \lambda_i'|}{\lambda_i'} \leq \frac{\| L - L' \|}{A_4}
\]

Similarly, we have

\[
\left| \frac{1}{\lambda} - 1 \right| \leq \frac{\| L - L' \|}{A_4}
\]

To conclude the argument, we note that

\[
|s_X(L) - s_X(L')| \leq |s_X(L) - s_X(\lambda L')| + |s_X(\lambda L') - s_X(L)|
\leq C'(D)\| L - \lambda L' \| + \frac{1}{\lambda} - 1|s_X(L')|
\leq C'(D)\| L - L' \| + C'(D)|1 - \lambda|\| L' \| + \frac{1}{\lambda} - 1|s_X(L')|
\leq C'(D)\| L - L' \| + C'(D)\frac{A_2}{A_4}\| L - L' \| + \frac{1}{A_4}\| L - L' \|.
\]

Lastly, if \( L' \) were a multiple of \( L \), then only the last term in the above inequality suffices. Thus, we see that for our choice of \( r(D) \), choosing \( C(D) = C'(D) + C'(D)\frac{A_2}{A_4} + \frac{1}{A_4} \) works for the inequaity (4.1), hence proving Theorem 4.2. \( \square \)
5. Extending the $F$-signature function to the Nef Cone.

In this section, we will prove that the $F$-signature function, originally defined in Section 3 only on the ample cone (Definition 2.2) extends continuously to the non-zero classes in the nef cone.

**Theorem 5.1.** Suppose that $X$ is a globally $F$-regular projective variety of dimension $d$. Then the $F$-signature function $s_X$ extends continuously to all non-zero classes of the Nef cone $\text{Nef}_R(X)$. Moreover, if $D$ is a nef Cartier divisor which is not big, then $s_X(D) = 0$.

We prove Theorem 5.1 in two parts, depending on whether or not $L$ is big. First, we have the following comparison of the $F$-signature function with the volume function:

**Lemma 5.2.** Let $X$ be a globally $F$-regular projective variety of dimension $d$. Fix a norm $\| \|$ on the Néron-Severi space of $X$. Let $C_1$ be a constant such that for any non-zero effective divisor $L$, we have (see Definition 4.5 for the notation),

$$I_e(mL) = H^0(mL)$$

for all $m > \frac{C_1 p^e}{\|L\|}$.

The existence of such a constant is guaranteed by Theorem 4.8. Then, for any ample Cartier divisor $D$ on $X$, we have

$$s_X(D) \leq \frac{C_1^{d+1} \text{vol}(D)}{\|D\|^{d+1}(d + 1)!}.$$  \hspace{1cm} (5.1)

Note that the right-hand side has the same order of decay as the $F$-signature function, decaying in the order of $1/\|D\|$ as the norm of divisor $\|D\| \to \infty$.

**Proof.** Using Lemma 4.6 to calculate the $F$-signature $s_X(D)$, we have

$$s_X(D) = \lim_{e \to \infty} \frac{1}{p^{(d+1)}} \sum_{m=0}^{\infty} \dim_k \frac{H^0(mD)}{I_e(mD)}$$

$$= \lim_{e \to \infty} \frac{1}{p^{(d+1)}} \sum_{m=0}^{C_1 p^e} \dim_k \frac{H^0(mD)}{I_e(mD)} \quad \text{(By Theorem 4.8)}$$

$$\leq \lim_{e \to \infty} \frac{1}{p^{(d+1)}} \sum_{m=0}^{C_1 p^e} \dim_k H^0(mD)$$

$$\leq \lim_{e \to \infty} \frac{\text{vol}(D)}{p^{(d+1)}} (\frac{C_1 p^e}{\|D\|})^{d+1} \quad \text{(using the Hilbert-polynomial of $D$)}$$

$$= \frac{C_1^{d+1} \text{vol}(D)}{\|D\|^{d+1}(d + 1)!}$$

$\square$

**Proof of Theorem 5.1.** First suppose that $D$ is a non-zero nef divisor that is not big. Then, for any sequence $\{L_t\}_t$ of ample $\mathbb{Q}$-divisors approaching $D$, choose a positive integer $r_t$ for...
each $t \geq 1$ such that $r_t L_t$ is integral Cartier. Then, we see that using Lemma 5.2,

$$s_X(L_t) = r_t s_X(r_t L_t) \leq \frac{C_{d+1}^d \text{vol}(r_t L_t)}{\|r_t L_t\|^{d+1}(d+1)!} = \frac{C_{d+1}^d \text{vol}(L_t)}{\|L_t\|^{d+1}(d+1)!}.$$ 

Since $\|D\| \neq 0$, we have that $\|L_t\|$ approaches a non-zero number (namely, $\|D\|$) and $\text{vol}(L_t)$ approaches 0 as $t \to \infty$ (since $D$ is not big), this shows that $s_X(L_t) \to 0$ as $t \to \infty$. As the sequence $\{L_t\}$ chosen was arbitrary, this shows that the $F$-signature function $s_X$ extends continuously by zero to all non-zero nef divisors $L$ that are not big.

Now suppose that $D$ is a big and nef divisor. Following the proof of Theorem 4.2, to prove that $s_X$ is locally Lipschitz for ample divisors around $D$, by Lemma 4.18, we may a pick a suitable norm depending on $D$. Since the big cone $\text{Big}(X)$ is an open subset of $N^1_{\mathbb{R}}(X)$, given $D \in \text{Big}(X)$, we may pick a basis $e_1, \ldots, e_\rho$ for $N^1_{\mathbb{R}}(X)$ such that each $e_i$ is the class of a big invertible sheaf and such that $D$ is contained in the interior of the cone generated by the $e_i$’s (equivalently, $D = \sum a_i e_i$ with each $a_i > 0$). Let $C = \{a_i e_i | a_i \geq 0\}$ denote the closed cone generated by the $e_i$’s and $\|\|$ denote the sup-norm with respect to the basis $\{e_i\}$. Then, arguing verbatim as in the final step of the proof of Theorem 4.2 and applying the argument to all ample $\mathbb{Q}$-divisors $L, L'$ contained in $C$, we get positive numbers $r(D)$ and $C(D)$ such that

$$|s_X(L) - s_X(L')| \leq C(D)\|L - L'\|$$

for any two ample $\mathbb{Q}$-divisors $L$ and $L'$ contained in a ball of radius $r(D)$ around $D$. This proves that $s_X$ is uniformly continuous in a neighbourhood of $D$, which gives us a unique continuous extension of $s_X$ to $D$. \qed

The $F$-signature function of the blow-up of $\mathbb{P}^2$ at a point provides an instructive example of the behavior of the function on the boundary. For a formula for general Hirzebruch surfaces, see [HS17].

**Example 5.3.** Let $X = \text{Bl}_x(\mathbb{P}^2)$ be the blow-up of $\mathbb{P}^2$ at $x = [0 : 0 : 1]$. Let $H$ denote the pull-back of a line in $\mathbb{P}^2$ passing through $x$ and $E$ be the exceptional divisor for the blow-up. Then $H$ and $E$ form a basis for the Néron-Severi space and the nef cone of $X$ is given by the divisors $aH - bE$ such that $0 \leq b \leq a$. For $L = aH - bE$, we can compute the $F$-signature of $L$ using the formula described in [VK12], and it is given by

$$s_X(L) = \begin{cases} \frac{a-b}{ab} \frac{2b(a-b)}{a} + \frac{(3b-a)(2a-3b)}{6b(a-b)^2} + \frac{(2a-3b)^2}{2a(a-b)^2}, & \text{if } b \leq a \leq \frac{3}{2}b \\ \frac{1}{a} - \frac{b^2+(a-2b)^2+(a-3b)(a-2b)+(a-3b)^2}{6a(a-b)^2}, & \text{if } \frac{3}{2}b \leq a \leq 3b \\ \frac{1}{a} - \frac{b^2+(a-2b)^2+(a-3b)(a-2b)+(a-3b)^2}{6a(a-b)^2}, & \text{if } 2b \leq a \leq 3b \\ \frac{1}{a} - \frac{b^2+(a-2b)^2+(a-3b)(a-2b)+(a-3b)^2}{6a(a-b)^2}, & \text{if } 3b \leq a \end{cases}$$

Note that along the line $a = b$, which corresponds to a nef but not big boundary face, the $F$-signature extends to the zero function (as proved in Theorem 5.1). On the other hand, along $b = 0$, which is the big and nef boundary face, letting $b \to 0$ yields $s_X(L) = \frac{1}{a} - \frac{a^2+a^2+a^2}{6a^2} = \frac{1}{2a}$. It turns out that this corresponds to the $F$-signature of a pair $(\mathbb{P}^2, m)$ with respect to the divisor $aL$ on $\mathbb{P}^3$ (see [BST11, Theorem 4.20] for the definition of the $F$-signature of pairs).

**Remark 5.4.** Theorem 5.1 gives us a unique extension of the $F$-signature function to the non-zero classes in the nef cone of $X$. Further, we also know that for nef divisors that are not big, the extension is 0. Thus, it is natural to ask what the extension to a big and nef
divisor is. In forthcoming work, we explore this question and provide some answers in terms of $F$-signature of pairs, as indicated by Example 5.3.

6. LOCAL UPPER BOUNDS FOR THE $F$-SIGNATURE FUNCTION

In this section, we prove effective local upper bounds for the $F$-signature function (Definition 2.2).

Theorem 6.1. Let $X$ be a globally $F$-regular projective variety. Let $d = \dim X$. Fix a basis $e_1, \ldots, e_\rho$ for the Néron-Severi space $N_1^X$ such that each $e_i$ corresponds to the class of an ample and globally generated invertible sheaf. Let $C$ denote the simplicial cone generated by the $e_i$’s, that is, $C = \{ \sum a_i e_i | a_i \in \mathbb{R}_{\geq 0} \}$. Let $\| \|$ denote the sup-norm on $N_1^X$ with respect to the $e_i$’s. Then, for any non-zero class $L$ in $C$, we have

$$s_X(L) \leq \frac{(d^2 + 2d)d^{d+1}\text{vol}(L)}{\left\| L \right\|^{d+1}(d+1)!}. \quad (6.1)$$

Lemma 6.2. Suppose $L$ is a globally generated ample divisor and $H$ any nef divisor on $X$. Then, for all $e \geq 1$, we have:

(a) $I_e(mL) = H^0(mL)$ for $m > (d^2 + d)p^e$,

(b) $I_e(m(nL + H)) = H^0(m(nL + H))$ for all $m > \frac{(d^2 + 2d)p^e}{n}$.

Proof. Let $S$ be the section ring of $X$ with respect to $L$. And for any $j \geq 0$, let $M^j$ be the $S$-module $\bigoplus_{t \geq 0} \mathcal{O}_X(jH + tL)$.

(a) First, we claim that $S$ is generated as a graded ring by homogeneous elements of degree at most $d$. This follows from Mumford’s Theorem [Laz04, Theorem 1.8.5], if we show that the trivial bundle $\mathcal{O}_X$ is $d$-regular with respect to $L$. Since $X$ is globally $F$-regular and $L$ is ample, by Theorem 2.13, we have that

$$H^i(X, \mathcal{O}_X((d-i)L)) = 0 \quad \text{for all } i > 0.$$  

This implies that $\mathcal{O}_X$ is $d$-regular with respect to $L$ and hence that $S$ is generated by elements of degree at most $d$. 

Figure 1. The $F$-signature function of the blow up of $\mathbb{P}^2$ at a point.
Since the section ring $S$ is generated by elements of degree $\leq d$, the homogeneous maximal ideal $m = S_{>0}$ is generated in degrees $\leq d$. By [HS06, Proposition 8.3.8], there exist elements $x_0, \ldots, x_d$ (not necessarily homogeneous), such that all terms of each $x_i$ have degree at most $d$, and the integral closure $(x_0, \ldots, x_d)$ is equal to the maximal ideal $m$. Now, by using the Briançon-Skoda theorem in the strongly $F$-regular ring $S$ [HH90, Theorem 5.4], we have

$$m^{(d+1)p^e} = (x_0^{p^e}, \ldots, x_d^{p^e})^{d+1} \subseteq (x_0^{p^e}, \ldots, x_d^{p^e}).$$

Therefore, if $m \geq d(d+1)p^e$, for any element $x \in S_m = H^0(mL)$, by the pigeon-hole principle, we have $x \in m^{(d+1)p^e}$, and consequently, $x \in (x_0^{p^e}, \ldots, x_d^{p^e})$. Hence, the map $O_X \to F^p_\ast O_X(mL)$ sending $1 \mapsto F^p_\ast x$ cannot split.

(b) Similarly as in part (a), we claim that for any $j \geq 0$, $M^j$ is generated over $S$ by elements of degree at most $d$. For this, again by Mumford’s theorem, it is enough to show that $O_X(jH)$ is $d$-regular with respect to $L$. Since $H$ is nef, by Theorem 2.13 we again have:

$$H^i(X, O_X(jH + (d - i)L)) = 0 \text{ for all } i > 0.$$ 

Suppose $f \in H^0(m(nL + H) \setminus I_e(m(nL + H)))$ and $m > \frac{(d^2 + 2d)p^e}{\|L\|}$, then we may write $f = \sum r_i f_i$ for $r_i \in S$ and $f_i \in M^m$ with degree of $f_i$ at most $d$. Then the degree of each $r_i$ is at least $(d^2 + d)p^e$. Now, since by assumption, the map $O_X \to F^p_\ast O_X(m(nL + H))$ sending $1$ to $F^p_\ast f$ splits, we must have that for some $i$, $r_i \in H^0(kL) \setminus I_e(kL)$ for a suitable $k > (d^2 + d)p^e$, contradicting part (a) of the lemma. Hence, we must have $I_e(m(nL + H)) = H^0(m(nL + H))$ for all $m > \frac{(d^2 + 2d)p^e}{\|L\|}$. This completes the proof of the lemma.

\[\square\]

**Lemma 6.3.** Fix a basis $e_1, \ldots, e_p$ of $N^1(X)$ such that each $e_i$ corresponds to the class of an ample and globally generated invertible sheaf. Let $\mathcal{C}$ denote the simplicial cone generated by the $e_i$’s, that is, $\mathcal{C} = \{ \sum a_i e_i \mid a_i \in \mathbb{R}_{\geq 0} \}$. Let $\| \|$ denote the sup-norm on $N^1(X)$ with respect to the $e_i$’s. Then, for any invertible sheaf $\mathcal{L}$ such that its class $L$ in the Néron-Severi space satisfies $L \in \mathcal{C}$ and $\|L\| \geq d$ (where $d$ is the dimension of $X$), we have

(a) $\mathcal{L}$ is ample and globally generated.

(b) Further,

$$I_e(mL) = H^0(mL) \text{ for all } m > \frac{(d^2 + 2d)p^e}{\|L\|}.$$ 

**Proof.** (a) Ampleness of $\mathcal{L}$ follows from the assumption that $L$ lies in $\mathcal{C}$ and $L$ is non-zero since $\|L\| \neq 0$. It remains to show global generation of $\mathcal{L}$. For this, we note that since $\|L\| \geq d$, there is some $i$ such that we may decompose the divisor $L$ as $L = dL_i + H$ where $H$ is some nef Cartier divisor and $L_i$ is a Cartier divisor corresponding to the class $e_i$. This follows from the assumption that $e_i$’s are integral, ample and globally generated and the fact that the sup-norm is achieved by some coordinate of $L$. Hence, applying Theorem 2.13, we have

$$H^p(X, O_X(L - p L_i)) = 0 \text{ for all } p > 0.$$ 

Therefore, $\mathcal{L}$ is 0-regular with respect to the globally generated ample divisor $L_i$. Hence, $\mathcal{L}$ is globally generated itself.
(b) Since $\|L\| \geq d$, for some $0 \leq i \leq \rho$, we may write $L = \|L\| e_i + H$ for some integral and nef class $H$. Now, applying Part (b) of Lemma 6.2, we get

$$I_e(mL) = H^0(mL) \text{ for all } m > \frac{(d^2 + 2d)p^e}{\|L\|}.$$

□

Proof of Theorem 6.1: By Theorem 4.2, the $F$-signature function is continuous, hence we may prove Theorem 6.1 only when $L$ is an ample $\mathbb{Q}$-divisor. Further, since both sides of (6.1) scale inverse-linearly, we may assume that $L$ is a Cartier divisor and $\|L\| \geq d$. Then, applying Part (b) of Lemma 6.3, the Theorem follows from Lemma 5.2 by using $\frac{d^2 + 2d}{\|L\|}$ instead of $\frac{C_1}{\|L\|}$.

□

Corollary 6.4. Let $X$ be a globally $F$-regular variety, and $L$ be an ample $\mathbb{R}$-divisor on $X$. Then, for any compact neighborhood $B$ of $L$ such that $B \subset \text{Amp}_\mathbb{R}(X)$, there exists a constant $C > 0$ such that for classes $D \in B$, we have

$$s_X(L) \geq C \text{vol}(L).$$

Proof. This follows from the continuity of the $F$-signature and the volume functions (Corollary 4.3) and the fact that the $F$-signature function is positive for all ample $\mathbb{Q}$-divisors (Remark 2.11).

□

Remark. In the spirit of Lemma 5.2, this raises the natural question:

Question 6.5. Let $X$ be a globally $F$-regular projective variety and $\|\|$ be a fixed norm on $N^1(X)$. Then, does there exist a constant $C > 0$ (depending only on $X$) such that, for all ample $\mathbb{Q}$-divisors $L$?

References

[AL03] I. M. ABERBACH and G. J. LEUSCHKE: The $F$-signature and strong $F$-regularity, Math. Res. Lett. 10 (2003), no. 1, 51–56. MR1960123 (2004b:13003)

[BST11] M. BLICKLE, K. SCHWEDE, and K. TUCKER: $F$-signature of pairs and the asymptotic behavior of Frobenius splittings, arXiv:1107.1082.

[Bou02] S. BOUCKSOM: On the volume of a line bundle, Internat. J. Math. 13 (2002), no. 10, 1043–1063. 1945706

[BFJ09] S. BOUCKSOM, C. FAVRE, and M. JONSSON: Differentiability of volumes of divisors and a problem of Teissier, J. Algebraic Geom. 18 (2009), no. 2, 279–308. 2475816

[CRST18] J. CARVAJAL-ROJAS, K. SCHWEDE, and K. TUCKER: Fundamental groups of $F$-regular singularities via $F$-signature, Ann. Sci. Éc. Norm. Supér. (4) 51 (2018), no. 4, 993–1016.

[CR22] J. A. CARVAJAL-ROJAS: Finite torsors over strongly $F$-regular singularities, Épizod ets Geom. Algébrique 6 (2022), Art. 1, 30. 4391081

[ELM+05] L. EIN, R. LAZARSFELD, M. MUSTAŢĂ, M. NAKAMAYE, and M. POPA: Asymptotic invariants of line bundles, Pure Appl. Math. Q. 1 (2005), no. 2, Special Issue: In memory of Armand Borel. Part 1, 379–403. 2194730

[Fol99] G. B. FOLLAND: Real analysis, second ed., Pure and Applied Mathematics (New York), John Wiley & Sons, Inc., New York, 1999, Modern techniques and their applications, A Wiley-Interscience Publication. 1681462
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[GLP+15] Y. Gongyo, Z. Li, Z. Patakfalvi, K. Schwede, H. Tanaka, and R. Zong: On rational connectedness of globally F-regular threefolds, Adv. Math. 280 (2015), 47–78.

[GOST15] Y. Gongyo, S. Okawa, A. Sannai, and S. Takagi: Characterization of varieties of fano type via singularities of cox rings, J. Algebraic Geom. 24 (2015), no. 1, 159–182.

[GT19] Y. Gongyo and S. Takagi: Kollár’s injectivity theorem for globally F-regular varieties, Eur. J. Math. 5 (2019), no. 3, 872–880.

[HM06] C. D. Hacon and J. McKernan: Boundedness of pluricanonical maps of varieties of general type, Invent. Math. 166 (2006), no. 1, 1–25.

[HX15] C. D. Hacon and C. Xu: On the three dimensional minimal model program in positive characteristic, J. Amer. Math. Soc. 28 (2015), no. 3, 711–744.

[Har77] R. Hartshorne: Algebraic geometry, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52.

[Har15] R. Hartshorne: Algebraic geometry, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52.

[Har77] R. Hartshorne: Algebraic geometry, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52.

[HK95] R. Hartshorne and M. Mustaţă: Vanishing, singularities and effective bounds via prime characteristic local algebra, Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., vol. 62, Amer. Math. Soc., Providence, RI, 1997, pp. 289–325.

[HL02] J. Huneke and G. J. Leuschke: Tight closure and strong F-regularity, Mem. Amer. Math. Soc. 154 (2001), no. 732, vi+110 pp.

[HM10] J. Huneke and C. Miller: Low Krull dimension and the de Jong conjecture, Duke Math. J. 150 (2009), no. 3, 311–340.

[HS17] D. Hirose and T. Sawada: Korff F-signatures of Hirzebruch surfaces, arXiv:1701.01905 (2017).

[HH97] M. Hochster and C. Huneke: Tight closure and strong F-regularity, Mem. Amer. Math. Soc. 114 (1995), no. 545, x+250 pp.

[HM90] M. Hochster and C. Huneke: Tight closure, invariant theory, and the Briançon-Skoda theorem, J. Amer. Math. Soc. 3 (1990), no. 1, 31–116.

[HLS01] C. Huneke, A. Leuschke, and C. Miller: Strong F-regularity and Gorenstein Noetherian rings, Adv. Math. 155 (2000), no. 2, 223–250.

[HS06] C. Huneke and I. Swanson: Integral closure of ideals, rings, and modules, London Mathematical Society Lecture Note Series, vol. 336, Cambridge University Press, Cambridge, 2006.

[Kaw21] T. Kawakami: Bogomolov-Sommese type vanishing for globally F-regular threefolds, Math. Z. 299 (2021), no. 3-4, 1821–1835.

[Kle66] S. L. Kleiman: Toward a numerical theory of ampleness, Ann. Math. (2) 84 (1966), 293–344.

[K06] A. Küronya: Asymptotic cohomological functions on projective varieties, Amer. J. Math. 128 (2006), no. 6, 1475–1519.

[Laz04] R. Lazarsfeld: Positivity in algebraic geometry. I, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics, vol. 48, Springer-Verlag, Berlin, 2004, Classical setting: line bundles and linear series.

[LM09] R. Lazarsfeld and M. Mustaţă: Convex bodies associated to linear series, Ann. Sci. Éc. Norm. Supér. (4) 42 (2009), no. 5, 783–835.

[LLX20] C. Li, Y. Liu, and C. Xu: A guided tour to normalized volume, Geometric analysis—in honor of Gang Tian’s 60th birthday, Progr. Math., vol. 333, Birkhäuser/Springer, Cham, 2020.

[MPST19] L. Ma, T. Polstra, K. Schwede, and K. Tucker: F-signature under birational morphisms, Forum Math. Sigma 7 (2019), Paper No. e11, 20.

[Mar22] I. Martín: The number of torsion divisors in a strongly F-regular ring is bounded by the reciprocal of F-signature, Comm. Algebra 50 (2022), no. 4, 1595–1605.

[Muk22] A. Mukhopadhyay: Frobenius-poincaré function and hilbert-kunz multiplicity, 2022.

[Pol22] T. Polstra: A theorem about maximal Cohen-Macaulay modules, Int. Math. Res. Not. IMRN (2022), no. 3, 2086–2094.

[SS10] K. Schwede and K. E. Smith: Globally F-regular and log Fano varieties, Adv. Math. 224 (2010), no. 3, 863–894.

[Smi97] K. E. Smith: Vanishing, singularities and effective bounds via prime characteristic local algebra, Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., vol. 62, Amer. Math. Soc., Providence, RI, 1997, pp. 289–325.

[Smi00] K. E. Smith: Globally F-regular varieties: applications to vanishing theorems for quotients of Fano varieties, Michigan Math. J. 48 (2000), 553–572.
[SVdB97] K. E. Smith and M. Van den Bergh: Simplicity of rings of differential operators in prime characteristic, Proc. London Math. Soc. (3) 75 (1997), no. 1, 32–62. MR1444312 (98d:16039)

[Sta] T. Stacks Project Authors: Stacks Project.

[Tak06] S. Takayama: Pluricanonical systems on algebraic varieties of general type, Invent. Math. 165 (2006), no. 3, 551–587. 2242627

[Tay19] G. Taylor: Inversion of adjunction for f-signature, 2019.

[Tri18] V. Trivedi: Hilbert-Kunz density function and Hilbert-Kunz multiplicity, Trans. Amer. Math. Soc. 370 (2018), no. 12, 8403–8428. 3864381

[Tuc12] K. Tucker: F-signature exists, To appear in Inventiones Mathematicae, arXiv:1103.4173.

[VK12] M. R. Von Korff: The F-Signature of Toric Varieties, ProQuest LLC, Ann Arbor, MI, 2012, Thesis (Ph.D.)–University of Michigan. 3093997

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