Zhang $L^2$-Regularity for the solutions of Backward Doubly Stochastic Differential Equations under globally Lipschitz continuous assumptions

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August 31, 2018

Abstract

We prove an $L^2$-regularity result for the solutions of Forward Backward Doubly Stochastic Differential Equations (F-BDSDEs in short) under globally Lipschitz continuous assumptions on the coefficients. Therefore, we extend the well known regularity results established by Zhang (2004) for Forward Backward Stochastic Differential Equations (F-BSDEs in short) to the doubly stochastic framework. To this end, we prove (by Malliavin calculus) a representation result for the martingale component of the solution of the F-BDSDE under the assumption that the coefficients are continuous in time and continuously differentiable in space with bounded partial derivatives. As an (important) application of our $L^2$-regularity result, we derive the rate of convergence in time for the (Euler time discretization based) numerical scheme for F-BDSDEs proposed by Bachouch et al.(2016) under only globally Lipschitz continuous assumptions.

Keywords: Forward Backward Doubly Stochastic Differential Equations; $L^2$-regularity, Malliavin calculus; representation result; numerical scheme; rate of convergence

MSC2010: Primary 60H10, Secondary 65C30

1 Introduction

Stochastic partial differential equations (SPDEs in short) appear in many applications, like Zakai equations in non linear filtering, stochastic control with partial

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observations and genetic populations. The SPDE of our interest is of the following form

$$u_t(x) = \Phi(x) + \int_t^T [Lu_s(x) + f(s, x, u_s(x), (\nabla u_s\sigma)(x))]ds$$

$$+ \int_t^T h(s, x, u_s(x), (\nabla u_s\sigma)(x))\tilde{dB}_s,$$

(1)

where, \(T > 0\) is fixed, \(u_t(x) = u(t, x)\) is a predictable random field, \(f\) and \(h\) are non-linear deterministic coefficients, \(Lu = (Lu_1, \cdots, Lu_k)\) is a second order differential operator and \(\sigma\) is the diffusion coefficient. The differential term with \(\tilde{dB}_t\) refers to the backward stochastic integral with respect to an \(l\)-dimensional Brownian motion on \((\Omega, \mathcal{F}, P, (B_t)_{t \geq 0})\). F-BDSDEs have been introduced to give a Feynman-Kac representation for the classical solution of the stochastic semilinear PDE (1), see the seminal work of [PP94]. The BDSDE \((Y^{t,x}, Z^{t,x})\) of our interest is of the following form

$$Y^{t,x}_s = \Phi(X^{t,x}_T) + \int_s^T f(r, X^{t,x}_r, Y^{t,x}_r, Z^{t,x}_r)dr$$

$$+ \int_s^T h(r, X^{t,x}_r, Y^{t,x}_r, Z^{t,x}_r)\tilde{dB}_r - \int_s^T Z^{t,x}_r dW_r,$$

(2)

where \((X^{t,x}_s)_{t \leq s \leq T}\) is a \(d\)-dimensional diffusion process starting from \(x\) at time \(t\) driven by the finite \(d\)-dimensional Brownian motion \((W_t)_{0 \leq t \leq T}\) (independent from \(B\)) with infinitesimal generator \(L\). Under some regularity assumptions on the coefficients \(b, \sigma, \Phi, f\) and \(h\), the authors in [PP94] proved that \(u_t(x) = Y^{t,x}_t\) and \(\nabla u_t\sigma(x) = Z^{t,x}_t\), \(\forall(t, x) \in [0, T] \times \mathbb{R}^d\) (see [PP94] Theorem 3.1 for details). Many generalizations studying more general nonlinear SPDEs have been made by different approaches of the notion of weak solutions, that is, Sobolev’s solutions (see [K99, BM01, MS02]) and stochastic viscosity solutions (see [LS98, BuM01, LS02]). Essentially, SPDEs have been numerically resolved by an analytic approach, that is, based on time-space discretization of the equations. The discretization is achieved by different methods such as finite difference, finite element and spectral Galerkin methods [GN95, G99, W05, GK10, JK10]. More precisely, the Euler finite-difference scheme was studied in [GN95], [G99] and [GK10]. Its convergence was proved in [GN95] and the order of convergence was determined in [G99]. Very interesting results are presented in [GK10] when they studied a symmetric finite difference scheme for a class of linear SPDEs driven by an infinite dimensional Brownian motion. The authors proved that the approximation error is proportional to \(\bar{h}^2\) where \(\bar{h}\) is the discretization step in space. They even proved (using the Richardson acceleration method) that if the SPDE is non degenerate and the coefficients are \(m\)-times continuously differentiable in the state variable (of dimension \(d\)) with \(m > 1 + \frac{d}{2}\), then the error is proportional to \(\bar{h}^4\). Finite element based schemes for parabolic SPDEs were
studied in [W05] in the one-dimensional case, with a study of the rate of convergence for the Forward and Backward Euler and the Crank-Nicholson schemes. The obtained rate of convergence is similar to the rate of the finite difference schemes. The spectral Galerkin approximation was investigated in [JK10]. The method is based on Taylor expansions derived from the solution of the SPDE, under sufficient regularity conditions. This approach was also used in [LMR97] to approximate the solution of the Zakai equation.

Only recently some works took an active interest in the simulation and approximation of (2). This interest was motivated by results and advances in the approximation and simulation of the standard F-BSDEs during the last fifteen years. Indeed, when $h \equiv 0$, SPDE (1) becomes a deterministic PDE and we deal with a standard F-BSDE. The numerical resolution of F-BSDEs has already been studied in the literature by Bally [B97], Zhang [Z04], Bouchard and Touzi [BT04], Gobet, Lemor and Warin [GLW06] and Bouchard and Elie [BE08] among others. Zhang [Z04] suggested a discrete-time approximation, by step processes, for a class of decoupled F-BSDEs with possible path-dependent terminal values. He established an $L^2$-type regularity result for the F-BSDE’s solution. Then he proved the convergence of his numerical scheme and he derived the rate of convergence in time. Bouchard and Touzi [BT04] proposed a similar numerical scheme for decoupled F-BSDEs. They computed the conditional expectations involved in their numerical scheme using the kernel regression estimation and used the Malliavin approach and the Monte Carlo method for their computation. Gobet, Lemor and Warin in [GLW06] suggested an explicit (time discretization based) numerical scheme. They also proposed an empirical regression scheme to approximate the nested conditional expectations arising from the time discretization of the standard F-BSDE. The latter method, also known as regression Monte-Carlo method or least-squares Monte-Carlo method, is popular and known to perform well for high-dimensional problems.

In the stochastic PDEs’ case, that is $h \neq 0$, Aman [Ama13] and Aboura [Abo11] considered the particular case when $h$ does not depend on the control variable $z$. Aman [Ama13] proposed a numerical scheme following Bouchard and Touzi [BT04] and obtained a convergence of order $\hat{h}$ of the square of the $L^2$-error ($\hat{h}$ is the time discretization step). Aboura [Abo11] studied the same numerical scheme under the same kind of assumptions, but following Gobet et al. [GLW05]. He obtained a convergence of order $\hat{h}$ in time and used the regression Monte Carlo method to implement his scheme, as in [GLW05]. Also, when $h$ doesn’t depend on $z$, a first order scheme was proposed in [BCMZ16] using the two sided Ito-Taylor expansion when the forward process is a drifted Brownian motion. Under the assumption that the coefficients are 3 times continuously differentiable with all partial derivatives bounded, they obtained a rate of convergence of order $\hat{h}^2$ for the component $y$ and of order $\hat{h}$ for the component $z$.

In the general case, that is, $h$ depends on the variable $z$, the authors in [BCZ11]
studied the time discretization error for the time discretization based approximation scheme for F-BDSDEs when the forward process is simply a drifted Brownian motion. They derived a rate of convergence of order $\hat{h}$ under the assumption that the coefficients are continuously differentiable in space with all partial derivatives uniformly bounded. In [BBMM16], the authors extended the approach of Bouchard-Touzi-Zhang to F-BDSDEs. They gave an upper bound for the time discretization error under globally Lipschitz continuous assumptions on the coefficients. However, they derived the rate of convergence of this scheme under rather strong assumptions, namely, all the coefficients are 2 times continuously differentiable with all partial derivatives bounded. Finally, they deduced a numerical scheme for the weak solution of the semilinear SPDE (1) and gave the rate of convergence in time for the latter numerical scheme. The problem of approximation of nested conditional expectations arising from the time discretization of F-BDSDEs was recently resolved in [BGM16] using the regression Monte-Carlo method. The resolution was done conditionally to the paths of the Brownian motion $B$, in the spirit of SPDE (1), under globally Lipschitz continuous assumptions on the coefficients.

This leads to the motivation of this paper. For the numerical resolution of F-BDSDEs with coefficient $h$ depending on $z$, the regression Monte-Carlo scheme studied in [BGM16] converges under Lipschitz continuous assumptions on the coefficients, while the rate of convergence in time is proved in [BBMM16] under the assumption that the coefficients are 2 times continuously differentiable with all partial derivatives bounded. A natural problem of interest is to derive the same rate of convergence in time (obtained in [BBMM16]) under only Lipschitz continuous assumptions on the coefficients. This enables the approximation and simulation of F-BDSDEs under only Lipschitz continuous conditions on the coefficients, which in turn enables the numerical approximation of weak solutions of SPDE (1) (via F-BDSDEs) under rather mild conditions.

To this end, we proceed as follows. First, we study the Malliavin derivative of the solution $(Y, Z)$ of the F-BDSDE and we prove a representation result for the martingale component $Z$ of the solution under the assumption that all the coefficients are continuous in time and continuously differentiable in space with all partial derivatives uniformly bounded. Afterwards, we use this representation result to prove an $L^2$-regularity result for the solution of the F-BDSDE under globally Lipschitz continuous assumptions, extending the well known $L^2$- regularity results for F-BSDEs (proved by Zhang in [Z04]) to the doubly stochastic framework. Then, our $L^2$-regularity result is used to derive the rate of convergence in time of the numerical scheme for F-BDSDEs proposed in [BBMM16] under globally Lipschitz continuous assumptions.

The paper is organized as follows. In section 2, we make a recall about F-BDSDES and introduce the notations and the assumptions needed in our work. In
section 3, we prove a representation result for the martingale component $Z$ of the solution. Then, we prove our main result which is an $L^2$-regularity result for the solution of the F-BDSDE. In section 4, we apply our $L^2$-regularity result to derive the rate of convergence in time of the numerical scheme for F-BDSDEs proposed in [BBMM16].

Usual notations. If $x$ is in an Euclidean space $E$, $|x|$ denotes its norm. If $A$ is a matrix, $|A|$ stands for its Hilbert-Schmidt norm.

2 Notations, preliminaries on Forward Backward Doubly Stochastic Differential Equations and assumptions

We assume that $W$ and $B$ are two independent Brownian motions defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where we define the sigma-fields $\mathcal{F}^W_{t,s} := \sigma\{W_r - W_t, t \leq r \leq s\}$, $\mathcal{F}^B_{s,T} := \sigma\{B_r - B_s, s \leq r \leq T\}$, $\mathcal{F}^W := \mathcal{F}^W_{0,T}$, $\mathcal{F}^B := \mathcal{F}^B_{0,T}$, $\mathcal{F} := \mathcal{F}^W \lor \mathcal{F}^B$, all completed with the $\mathbb{P}$-null sets. The solution of (2) is measurable at each $s$ with respect to $\mathcal{F}^t_s$, where $((\mathcal{F}^t_s)_{t \leq s \leq T})$ is the collection of sigma-fields defined as follows. For fixed $t \in [0, T]$ and for all $s \in [t, T]$

$$\mathcal{F}^t_s := \mathcal{F}^W_{t,s} \lor \mathcal{F}^B_{s,T}.$$ 

We denote $\mathcal{F}^0_s$ by $\mathcal{F}_s$ for simplicity.

We also need to introduce the following spaces:

- $C^l_b(\mathbb{R}^p, \mathbb{R}^q)$ denotes the set of all functions $\phi : \mathbb{R}^p \rightarrow \mathbb{R}^q$ such that they are $l$-times continuously differentiable with all partial derivatives uniformly bounded. We denote $C^l_b$ when the context is clear.
- $C^{k,l}_b([0, T] \times \mathbb{R}^p, \mathbb{R}^q)$ denotes the set of all functions $\phi : [0, T] \times \mathbb{R}^p \rightarrow \mathbb{R}^q$ such that they are $k$-times continuously differentiable in time and $l$-times continuously differentiable in space with all partial derivatives uniformly bounded. We denote $C^{k,l}_b$ when the context is clear.
- $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^k)$ denotes the set of $\mathcal{F}_T$-measurable square integrable random variables with values in $\mathbb{R}^k$.

For any $m \in \mathbb{N}$ and $t \in [0, T]$, the following notations are standard:

- $\mathbb{H}^2_m([t, T])$ denotes the set of $(classes of dP \times dt)$ a.e equal) $\mathbb{R}^m$-valued jointly measurable processes $\{\psi_u; u \in [t, T]\}$ satisfying:
  (i) $||\psi||^2_{\mathbb{H}^2_m([t, T])} := E[\int_t^T |\psi_u|^2 du] < \infty$,
  (ii) $\psi_u$ is $\mathcal{F}_u$-measurable, for a.e. $u \in [t, T]$.
- $\mathcal{S}^2_m([t, T])$ denotes similarly the set of $\mathbb{R}^m$-valued continuous processes satisfying:
  (i) $||\psi||^2_{\mathcal{S}^2_m([t, T])} := E[\sup_{t \leq u \leq T} |\psi_u|^2] < \infty$,
  (ii) $\psi_u$ is $\mathcal{F}_u$-measurable, for any $u \in [t, T]$.
For any random variable \( v \) where \( \hat{E} (v) \) is then a Sobolev space.

\[ \mathcal{M}^{2}_{k \times d}([t, T], \mathbb{D}^{1.2}) \]

\[ L^{2}([t, T], \mathbb{D}^{1.2}) \]

\( (X^{t,x}_{s})_{t \leq s \leq T} \) be the unique strong solution of the following stochastic differential equation:

\( dX^{s,x}_{s} = b(X^{s,x}_{s})ds + \sigma(X^{t,x}_{s})dW_{s}, \quad s \in [t, T], \quad X^{t,x}_{t} = x, \quad 0 \leq s \leq t \)
where \( b \) and \( \sigma \) are two functions on \( \mathbb{R}^d \) with values respectively in \( \mathbb{R}^d \) and \( \mathbb{R}^{d \times d} \). We consider the following BDSDE: For all \( t \leq s \leq T \),

\[
\begin{align*}
  dY^t_{s,x} &= -f(s,X^t_{s,x},Y^t_{s,x},Z^t_{s,x})ds - h(s,X^t_{s,x},Y^t_{s,x},Z^t_{s,x})dB_s + Z^t_{s,x}dW_s, \\
  Y^t_{T,x} &= \Phi(X^t_{T,x}),
\end{align*}
\]

where \( f \) and \( \Phi \) are two functions respectively on \([0,T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \) and \( \mathbb{R}^d \) with values in \( \mathbb{R}^k \) and \( h \) is a function on \([0,T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \) with values in \( \mathbb{R}^{k \times l} \). We will omit the dependence of the process \( X \) on the initial condition if it starts at time \( t = 0 \).

The following assumptions will be needed in our work.

**Assumption (H1)** There exists a non-negative constant \( K \) such that

\[ |b(x) - b(x')| + |\sigma(x) - \sigma(x')| \leq K|x - x'|, \forall x, x' \in \mathbb{R}^d. \]

**Assumption (H2)** There exist two constants \( K \geq 0 \) and \( 0 \leq \alpha < 1 \) such that for any \((t_1, x_1, y_1, z_1), (t_2, x_2, y_2, z_2) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d}, \)

\[ \begin{align*}
  (i) |f(t_1, x_1, y_1, z_1) - f(t_2, x_2, y_2, z_2)| &\leq K\left(\sqrt{|t_1 - t_2| + |x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|}\right), \\
  (ii) |h(t_1, x_1, y_1, z_1) - h(t_2, x_2, y_2, z_2)| &\leq K\left(|t_1 - t_2| + |x_1 - x_2|^2 + |y_1 - y_2|^2 + \alpha^2|z_1 - z_2|^2\right), \\
  (iii) |\Phi(x_1) - \Phi(x_2)| &\leq K|x_1 - x_2|, \\
  (iv) \sup_{0 \leq t \leq T}(|f(t, 0, 0, 0)| + |h(t, 0, 0, 0)|) &\leq K.
\end{align*} \]

**Remark 1** Pardoux and Peng [PP94, Theorem 1.1] proved that under assumptions (H1) and (H2), there exists a unique solution \((Y, Z)\) in \( S^2_k([t,T]) \times \mathbb{H}^2_{k \times d}([t,T]) \) to the F-BDSDE \((3)-(4)\).

From [EPQ97], [PP94] and [K84], the following standard estimates for the solution of the F-BDSDE \((3)-(4)\) hold and we remind the following theorem.

**Theorem 1** Under assumptions (H1) and (H2), there exists a positive constant \( C \) such that

\[ E\left[\sup_{t \leq s \leq T}|X^t_{s,x}|^2\right] \leq C(1 + |x|^2), \tag{5}\]

\[ E\left[\sup_{t \leq s \leq T}|Y^t_{s,x}|^2 + \int_t^T |Z^t_{s,x}|^2 ds\right] \leq C(1 + |x|^2). \tag{6}\]
3 Representation result and Zhang $L^2$-regularity

The aim of this section is to prove an $L^2$-regularity result for the solution of the F-BDSDE (3)-(4) under globally Lipschitz continuous assumptions on the coefficients. To this end, we prove a representation and a path regularity results for the martingale component $Z$ of the solution under the assumption that the coefficients $b, \sigma$ and $\Phi$ are in $C^1_b$ and $f$ and $h$ are in $C^{0,1}_b$. These results enable us to derive a rate of convergence in time for the numerical scheme for F-BDSDEs studied in [BBMM16] (see subsections 4.1 and 4.2 for details) under only globally Lipschitz continuous assumptions on the coefficients $b, \sigma, f, h$ and $\Phi$, in the spirit of the results of Zhang [Z04].

Let us stress that the representation and the path regularity results for the component $Z$ are proved in [BBMM16] under the assumption that all the coefficients are 2 times continuously differentiable with all partial derivatives uniformly bounded. Here, the proofs are given under weaker assumptions compared to [BBMM16].

3.1 Malliavin calculus for the solutions of forward SDEs

In this subsection, we recall some results on the differentiability in the Malliavin sense of the forward process $X^{t,x}$. Under the assumption that $b$ and $\sigma$ are in $C^1_b$, Nualart [N06] stated that $X^{t,x}_s \in D^{1,2}$ for any $s \in [t,T]$ and for $l \leq k$, the derivative $D^l_rX^{t,x}_s$ is given by:

(i) $D^l_rX^{t,x}_s = 0$, for $s < r \leq T$,

(ii) For any $t < r \leq T$, a version of $\{D^l_rX^{t,x}_s, r \leq s \leq T\}$ is the unique solution of the following linear SDE

$$D^l_rX^{t,x}_s = \sigma^l(X^{t,x}_r) + \int_r^s \nabla b(X^{t,x}_u)D^l_rX^{t,x}_u du + \sum_{i=1}^d \int_r^s \nabla \sigma_i(X^{t,x}_u)D^l_rX^{t,x}_u dW^i_u,$$  

(7)

where $(\sigma^i)_{i=1,...,d}$ denotes the i-th column of the matrix $\sigma$.

The following inequalities will be useful later. From [N06], we know that for any $0 \leq r \leq s \leq T$, there exists a non-negative constant $C$ such that

$$E\left[ \sup_{0 \leq u \leq T} \left| D_s X_u \right|^2 \right] \leq C(1 + |x|^2),$$  

(8)

$$E\left[ \sup_{s \vee r \leq u \leq T} \left| D_s X_u - D_r X_u \right|^2 \right] \leq C|s - r|(1 + |x|^2).$$  

(9)

3.2 Malliavin calculus for the solutions of F-BDSDEs

In this subsection, we study the differentiability in the Malliavin sense of the solution of the F-BDSDE (3)-(4). First, we recall the following result from Pardoux and Peng [PP92] about the Malliavin derivative of the classical Itô integral.
Lemma 1 ([PP92]) Let $U \in \mathbb{H}^2_2([t,T])$ and $I_i(U) = \int_t^T U_r dW^i_r, i = 1,\ldots,d$. Then, for each $\theta \in [0,T]$ we have $U_\theta \in \mathbb{D}^{1,2}$ if and only if $I_i(U) \in \mathbb{D}^{1,2}, i = 1,\ldots,d$ and for all $\theta \in [0,T]$, we have

$$D_\theta I_i(U) = \int_\theta^T D_\theta U_r dW^i_r + U_\theta, \theta > t,$$

$$D_\theta I_i(U) = \int_t^\theta D_\theta U_r dW^i_r, \theta \leq t.$$

We also recall the following lemma from [BBMM16] which shows that a backward Itô integral is differentiable in the Malliavin sense if and only if its integrand is so. More precisely, since the Malliavin derivative is with respect to the Brownian motion $W$, we have

Lemma 2 ([BBMM16]) Let $U \in \mathbb{H}^2_2([t,T])$ and $I_i(U) = \int_t^T U_r dW^i_r, i = 1,\ldots,l$. Then for each $\theta \in [0,T]$ we have $U_\theta \in \mathbb{D}^{1,2}$ if and only if $I_i(U) \in \mathbb{D}^{1,2}, i = 1,\ldots,l$ and for all $\theta \in [0,T]$, we have

$$D_\theta I_i(U) = \int_\theta^T D_\theta U_r dW^i_r, \theta > t,$$

$$D_\theta I_i(U) = \int_t^\theta D_\theta U_r dW^i_r, \theta \leq t.$$

The following result will be needed to prove Proposition 2. It can be proved using the same arguments as in the classical BSDEs’ setting (see [EPQ97]).

Proposition 1 Let $(\phi^1, f^1, h^1)$ and $(\phi^2, f^2, h^2)$ be two standard parameters of the BDSDE (6) and $(Y^1, Z^1)$ and $(Y^2, Z^2)$ the associated solutions. Let assumptions (H1) and (H2) hold. For $s \in [t,T]$, set $\delta Y_s := Y^1_s - Y^2_s$, $\delta_2 f_s := f^1(s, X_s, Y^2_s, Z^2_s) - f^2(s, X_s, Y^2_s, Z^2_s)$ and $\delta_2 h_s := h^1(s, X_s, Y^2_s, Z^2_s) - h^2(s, X_s, Y^2_s, Z^2_s)$. Then, we have

$$||\delta Y||^2_{\mathbb{H}^2_2([t,T])} + ||\delta Z||^2_{\mathbb{H}^2_{2 \times 2}([t,T])} \leq C E[|\delta Y_T|^2 + \int_t^T |\delta_2 f_s|^2 ds + \int_t^T |\delta_2 h_s|^2 ds], \quad (10)$$

where $C$ is a positive constant depending only on $K$, $T$ and $\alpha$.

In the next proposition, we prove that the Malliavin derivative of the solution of the BDSDE (6) is a solution of a linear BDSDE (see [PP92] for the standard BSDEs’ case). The same proposition is proved in [BBMM16] under the assumption that all the coefficients are 2 times continuously differentiable with all partial derivatives uniformly bounded. We give here the proof under weaker assumptions.

Proposition 2 Assume that (H1) and (H2) hold and that the coefficients $b, \sigma$ and $\Phi$ are in $C^1_b$ and $f$ and $h$ are in $C^{0,1}_b$. For any $t \in [0,T]$ and $x \in \mathbb{R}^d$, let \{(Y_s, Z_s), t \leq s \leq T\} denote the unique solution of the following BDSDE

$$Y_s = \Phi(X_s^{t,x}) + \int_s^T f(r, X_s^{t,x}, Y_r, Z_r) dr + \int_s^T h(r, X_s^{t,x}, Y_r, Z_r) dB_r.$$
We define recursively the sequence \( (\theta_s) \) and let us show that 
\[
\theta(s) \leq m,
\]
\( \leq \theta \leq T \).

Then, \((Y, Z) \in B^2([t, T], \mathbb{D}^{1,2})\) and \( \{D_0 Y_s, D_0 Z_s; t \leq s, \theta \leq T\} \) is given by

(i) \( D_0 Y_s = 0, D_0 Z_s = 0 \) for all \( t \leq s < \theta \leq T \),

(ii) for any fixed \( \theta \in [t, T] \), \( \theta \leq s \leq T \) and \( 1 \leq i \leq d \), a version of \( (D_i^t Y_s, D_i^t Z_s) \) is the unique solution of the following BDSDE

\[
D_i^t Y_s = \nabla \Phi(X_i^{t,x}) D_i^t X_i^{t,x} + \int_s^T \left( \nabla_y f(r, X_i^{t,x}, Y_r, Z_r) D_i^t Y_r + \sum_{j=1}^d \nabla_{z,j} f(r, X_i^{t,x}, Y_r, Z_r) D_i^t Z_j^r \right) dr \\
+ \int_s^T \left( \nabla_y h(r, X_i^{t,x}, Y_r, Z_r) D_i^t Y_r + \sum_{j=1}^d \nabla_{z,j} h(r, X_i^{t,x}, Y_r, Z_r) D_i^t Z_j^r \right) \, dB_r^i \\
+ \sum_{n=1}^l \int_s^T \sum_{j=1}^d \left( \nabla_{z,j} h(r, X_i^{t,x}, Y_r, Z_r) D_i^t Z_j^r \right) dB_r^i - \int_s^T \sum_{j=1}^d D_i^t Z_j^r \, dW_r^j, \tag{11}
\]

where \((z^j)_{1 \leq j \leq d}\) denotes the \( j \)-th column of the matrix \( z \), \((h^n)_{1 \leq n \leq l}\) denotes the \( n \)-th column of the matrix \( h \) and \( B = (B^1, \ldots, B^l) \).

**Proof.** To simplify the notations, we restrict ourselves to the case \( k = d = l = 1 \).

\((D_0 Y, D_0 Z)\) is well defined and from inequalities [8] and [8] we deduce that for each \( \theta \leq T \)

\[
E[\sup_{t \leq s \leq T} |D_0 Y_s|^2] + E[\int_t^T |D_0 Z_s|^2 ds] \leq C(1 + |x|^2).
\]

We define recursively the sequence \( (Y^m, Z^m) \) as follows. First we set \((Y^0, Z^0) = (0, 0)\). Then, given \((Y^{m-1}, Z^{m-1})\), we define \((Y^m, Z^m)\) as the unique solution in \( S_k^2([t, T]) \times H_{k \times d}^2([t, T]) \) of

\[
Y^m_s = \Phi(X_i^{t,x}) + \int_s^T f(r, X_i^{t,x}, Y_r^{m-1}, Z_r^{m-1}) dr + \int_s^T h(r, X_i^{t,x}, Y_r^{m-1}, Z_r^{m-1}) \, dB_r^i - \int_s^T Z_r^{m-1} \, dW_r.
\]

We recursively show that \((Y^m, Z^m) \in B^2([t, T], \mathbb{D}^{1,2})\). Suppose that

\((Y^m, Z^m) \in B^2([t, T], \mathbb{D}^{1,2})\)

and let us show that

\((Y^{m+1}, Z^{m+1}) \in B^2([t, T], \mathbb{D}^{1,2})\).
Set $\Sigma^m_r := (X^{l,x}_r, Y^m_r, Z^m_r)$. From the induction assumption, we have

$$
\Phi(X_T) + \int_s^T f(r, \Sigma^m_r)dr \in \mathbb{D}^{1,2}.
$$

We have $h(r, \Sigma^m_r) \in \mathbb{D}^{1,2}$ for all $r \in [t, T]$. From Lemma 2, we have

$$
\int_t^T h(r, \Sigma^m_r)dB_r \in \mathbb{D}^{1,2}.
$$

Then

$$ Y^{m+1}_s = E[\Phi(X^{l,x}_T)] + \int_s^T f(r, \Sigma^m_r)dr + \int_s^T h(r, \Sigma^m_r)dB_r | \mathcal{F}^W_{t,s} \cap \mathcal{F}^B_{t,T}] \in \mathbb{D}^{1,2}.
$$

Hence

$$
\int_t^T Z^{m+1}_r dW_r = \Phi(X^{l,x}_T) + \int_t^T f(r, \Sigma^m_r)dr + \int_t^T h(r, \Sigma^m_r)dB_r - Y^{m+1}_t \in \mathbb{D}^{1,2}.
$$

It follows from Lemma 1 that $Z^{m+1} \in \mathcal{M}_F^2([t, T], \mathbb{D}^{1,2})$ and we have for $t \leq s \leq \theta$, $D_\theta Y^{m+1}_s = D_\theta Z^{m+1}_s = 0$, while for $\theta \leq s \leq T$, we have

$$
D_\theta Y^{m+1}_s = \nabla \Phi(X^{l,x}_T)D_\theta X^{l,x}_T
+ \int_s^T \left( \nabla_x f(r, \Sigma^m_r)D_\theta X_r + \nabla_y f(r, \Sigma^m_r)D_\theta Y^m_r + \nabla_z f(r, \Sigma^m_r)D_\theta Z^m_r \right)dr
+ \int_s^T \left( \nabla_x h(r, \Sigma^m_r)D_\theta X_r + \nabla_y h(r, \Sigma^m_r)D_\theta Y^m_r + \nabla_z h(r, \Sigma^m_r)D_\theta Z^m_r + \nabla_z Z^m_r \right)d\tilde{B}_r
- \int_s^T D_\theta Z^{m+1}_r dW_r. \tag{12}
$$

From inequality 8, we deduce that for each $\theta \leq T$

$$
E[\sup_{t \leq s \leq T}|D_\theta Y^{m+1}_s|^2] + \int_t^T E[|D_\theta Z^{m+1}_s|^2]ds \leq C(1 + |x|^2).
$$

It is known that inequality 8 holds for $(Y^{m+1}, Z^{m+1})$ and we deduce that

$$
\|Y^{m+1}\|_{1,2} + \|Z^{m+1}\|_{1,2} < \infty,
$$

which shows that $(Y^{m+1}, Z^{m+1}) \in \mathcal{B}^2([t, T], \mathbb{D}^{1,2})$. Using the contraction mapping argument as in EPQ97, we deduce that $(Y^{m+1}, Z^{m+1})$ converges to $(Y, Z)$ in $S^2([t, T]) \times \mathbb{H}^2([t, T])$. We will show that $(D_\theta Y^m, D_\theta Z^m)$ converges to $(Y^0, Z^0)$ in $L^2(\Omega \times [t, T] \times [t, T], dP \otimes dt \otimes dt)$, where $Y^0_s = Z^0_s = 0$ for all $t \leq s \leq \theta$ and $(Y^0_s, Z^0_s, \theta \leq s \leq T)$ is the solution of the following BDSDE

$$
Y^\theta_s = \nabla \Phi(X^{l,x}_T)D_\theta X^{l,x}_T.
$$
Therefore, we obtain

\[ D_\theta Y_s^{m+1} - Y_s^\theta = \int_s^T \left( (\nabla_x f(r, \Sigma_r) - \nabla_x f(r, \Sigma_r)) D_\theta X_t^{l,x} + \nabla_y f(r, \Sigma_r) Y_r^\theta + \nabla_z f(r, \Sigma_r) Z_r^\theta \right) dr \]

From equations 12 and 13 we have

\[ E[ \sup_{\theta \leq s \leq T} |D_\theta Y_s^{m+1} - Y_s^\theta|^2 ] + E[ \int_s^T |D_\theta Z_r^{m+1} - Z_r^\theta|^2 dr ] \]

\[ \leq CE \left[ \int_s^T \left| (\nabla_x f(r, \Sigma_r^m) - \nabla_x f(r, \Sigma_r)) D_\theta X_t^{l,x} + \nabla_y f(r, \Sigma_r^m) Y_r^\theta - \nabla_y f(r, \Sigma_r) Y_r^\theta + \nabla_z f(r, \Sigma_r^m) Z_r^\theta - \nabla_z f(r, \Sigma_r) Z_r^\theta \right|^2 dr \right] \]

Therefore, we obtain

\[ E \left[ \int_t^T \int_t^T |D_\theta Y_{st}^{m+1} - Y_{st}^\theta|^2 dsd\theta + \int_t^T \int_t^T |D_\theta Z_{st}^{m+1} - Z_{st}^\theta|^2 dsd\theta \right] \]

\[ \leq CE \left[ \int_t^T \int_t^T |\delta_{r,\theta}^m|^2 drd\theta + \int_t^T \int_t^T |p_{r,\theta}^m|^2 drd\theta \right], \quad (14) \]
and
\[
\rho_{r,\theta}^m = (\nabla_x h(r, \Sigma^m_x) - \nabla_x h(r, \Sigma_r)) D_\theta X_{\theta}^{t,x} + \nabla_y h(r, \Sigma^m_y) Y_{\theta}^y - \nabla_y h(r, \Sigma_r) Y_{\theta}^y + \nabla_z h(r, \Sigma^m_z) Z_{\theta}^z - \nabla_z h(r, \Sigma_r) Z_{\theta}^z.
\]

From the definition of \((\delta_{r,\theta}^m)_{t \leq r, \theta \leq T}\), we have
\[
E \left[ \int_t^T \int_t^T |\delta_{r,\theta}^m|^2 dr d\theta \right] \leq C \int_t^T (A_m(\theta, t, T) + B_m(\theta, t, T)) d\theta,
\]
where
\[
A_m(\theta, t, T) = E \left[ \int_t^T |(\nabla_x f(r, \Sigma^m_r) - \nabla_x f(r, \Sigma_r)) D_\theta X_{\theta}^{t,x}|^2 dr \right],
\]
\[
B_m(\theta, t, T) = E \left[ \int_t^T |(\nabla_y f(r, \Sigma_r) - \nabla_y f(r, \Sigma^m_r)) Y_{\theta}^y|^2 dr \right] + E \left[ \int_t^T |(\nabla_z f(r, \Sigma_r) - \nabla_z f(r, \Sigma^m_r)) Z_{\theta}^z|^2 dr \right].
\]

Moreover, since \(\nabla_x f\) is bounded and continuous with respect to \((x, y, z)\), it follows by the dominated convergence theorem and inequality \([5]\) that
\[
\lim_{m \to \infty} \int_t^T A_m(\theta, t, T) d\theta = 0.
\]

Furthermore, since \(\nabla_y f\) and \(\nabla_z f\) are bounded and continuous with respect to \((x, y, z)\), it follows also by the dominated convergence theorem and inequality \([6]\) that
\[
\lim_{m \to \infty} \int_t^T B_m(\theta, t, T) d\theta = 0.
\]

From the definition of \((\rho_{r,\theta}^m)_{s \leq r, \theta \leq T}\), we have
\[
E \left[ \int_t^T \int_t^T |\rho_{r,\theta}^m|^2 dr d\theta \right] \leq C \int_t^T (A'_{m}(\theta, t, T) + B'_{m}(\theta, t, T)) d\theta,
\]
with
\[
A'_m(\theta, t, T) = E \left[ \int_t^T |(\nabla_x h(r, \Sigma^m_x) - \nabla_x h(r, \Sigma_r)) D_\theta X_{\theta}^{t,x}|^2 dr \right],
\]
\[
B'_m(\theta, t, T) = E \left[ \int_t^T |(\nabla_y h(r, \Sigma_r) - \nabla_y h(r, \Sigma^m_r)) Y_{\theta}^y|^2 dr \right] + E \left[ \int_t^T |(\nabla_z h(r, \Sigma_r) - \nabla_z h(r, \Sigma^m_r)) Z_{\theta}^z|^2 dr \right].
\]

Similarly as shown above, since \(\nabla_y h\) and \(\nabla_z h\) are bounded and continuous with respect to \((x, y, z)\) we can show that
\[
\lim_{m \to \infty} \int_t^T A'_m(\theta, t, T) d\theta = \lim_{m \to \infty} \int_t^T B'_m(\theta, t, T) d\theta = 0.
\]
Using 15, 16 and 17 in the estimate 14 we deduce that
\[
\lim_{m \to \infty} E \left[ \int_t^T \int_t^T |D_\theta Y^m_s| ds d\theta \right] = 0.
\]
It follows that \((Y^m, Z^m)\) converges to \((Y, Z)\) in \(L^2([t,T], \mathbb{D}^{1,2})\) and a version of \((D_\theta Y, D_\theta Z)\) is given by \((Y^\theta, Z^\theta)\), which is the desired result. □

3.3 Representation and path regularity results for the martingale component of the solution of the F-BDSDE

In this subsection, we prove a representation result for the martingale component \(Z\) (that implies a path regularity result) which will be useful to prove the \(L^2\)-regularity of the solution of the F-BDSDE.

**Proposition 3** Let assumptions (H1) and (H2) hold and assume that the coefficients \(b, \sigma\) and \(\Phi\) are in \(C^1_b\) and \(f\) and \(h\) are in \(C^{0,1}_b\). Then, \(\{D^i_s Y^i_{t,x}, t \leq s \leq T\}\) is a version of \(\{(Z^i_{t,x})_i, t \leq s \leq T\}\), where \((Z^i_{t,x})_i\) denotes the \(i\)-th component of the matrix \(Z^i_{t,x}\).

**Proof.** To simplify the notations, we restrict ourselves to the case \(k = d = 1\).

Notice that for \(t \leq s\), we have
\[
Y^i_{t,x} = Y^i_{t,x} - \int_t^s f(r, \Sigma^i_{t,r}) dr - \int_t^s h(r, \Sigma^i_{t,r}) dB_r + \int_t^s Z^i_{t,r} dW_r,
\]
where \(\Sigma^i_{t,r} := (X^i_{t,r}, Y^i_{t,r}, Z^i_{t,r})\).

It follows from Lemma 1 and Lemma 2 that, for \(t < \theta \leq s\)
\[
D_\theta Y^i_{t,x} = Z^i_{\theta,x} + \int_\theta^s D_\theta Z^i_{t,r} dW_r
\]
\[
- \int_\theta^s \left( \nabla_x f(r, \Sigma^i_{t,r}) D_\theta X^i_{t,r} + \nabla_y f(r, \Sigma^i_{t,r}) D_\theta Y^i_{t,x} + \nabla_z f(r, \Sigma^i_{t,r}) D_\theta Z^i_{t,x} \right) dr
\]
\[
- \int_\theta^s \left( \nabla_x h(r, \Sigma^i_{t,r}) D_\theta X^i_{t,r} + \nabla_y h(r, \Sigma^i_{t,r}) D_\theta Y^i_{t,x} + \nabla_z h(r, \Sigma^i_{t,r}) D_\theta Z^i_{t,x} \right) dB_r.
\]
The result follows by taking \(\theta = s\). □

**Corollary 1** Let assumptions (H1) and (H2) hold and assume that the coefficients \(b, \sigma\) and \(\Phi\) are in \(C^1_b\) and \(f\) and \(h\) are in \(C^{0,1}_b\). Then, for any \(0 \leq t \leq s \leq T\) and \(x \in \mathbb{R}^d\),
\[
Z^i_{t,x} = \nabla Y^i_{t,x} [\nabla X^i_{t,x}]^{-1} \sigma(X^i_{t,x}).
\]
In particular, \(Z^i_{t,x}\) has continuous paths.
Proof. Recall that the matrix \( \nabla X_s = \left( \frac{\partial X_t^{ij}}{\partial x} \right)_{1 \leq i, j \leq d} \) solves the SDE
\[
\nabla X_s^{i,x} = I_d + \int_t^s \nabla b(X_u^{i,x}) \nabla X_u^{i,x} \, du + \int_t^s \nabla \sigma(X_u^{i,x}) \nabla X_u^{i,x} \, dW_u. \tag{19}
\]
From the uniqueness of the solution of the SDE (7) satisfied by \( D_\theta X_s^{i,x} \), it follows that
\[
D_\theta X_s^{i,x} = \nabla X_s^{i,x} [\nabla X_s^{i,x}]^{-1} \sigma(X_s^{i,x}). \tag{20}
\]
Now, consider the equation
\[
\nabla Y_s^{t,x} = \nabla \Phi(X_T^{t,x}) \nabla X_T^{t,x} + \int_s^T \left( \nabla_x f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \nabla X_r^{t,x} + \nabla_y f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \nabla Y_r^{t,x} \right) \, dr \\
+ \int_s^T \left( \nabla_x^{\sigma} h(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \nabla X_r^{t,x} + \nabla_y^{\sigma} h(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \nabla Y_r^{t,x} \right) \, dW_r - \int_s^T \nabla Z_r^{t,x} \, dB_r. \tag{21}
\]
Denote by \((\nabla X_s^{t,x}, \nabla Y_s^{t,x}, \nabla Z_s^{t,x})\) the solution of the F-BDSDE (19)–(21). From the uniqueness of the solution of BDSDE (11) and the formula (20), we deduce that
\[
D_\theta Y_s^{t,x} = \nabla Y_s^{t,x} [\nabla X_s^{t,x}]^{-1} \sigma(X_s^{t,x}). \tag{22}
\]
Thus
\[
D_\theta Y_s^{t,x} = \nabla Y_s^{t,x} [\nabla X_s^{t,x}]^{-1} \sigma(X_s^{t,x}).
\]
By Proposition 3 the representation (18) follows. The continuity of \( Z_s^{t,x} \) follows from that of \( D_\theta Y_s^{t,x} \), which follows from that of \( \nabla Y_s^{t,x}, \nabla X_s^{t,x} \) and \( X_s^{t,x} \).

3.4 Zhang \( L^2 \)-Regularity result under globally Lipschitz continuous assumptions

In this subsection, we prove the Zhang \( L^2 \)-regularity result for the solution of the F-BDSDE (3)–(11) under globally Lipschitz continuous assumptions on the coefficients. Thus, we extend the results of Zhang [Z04] on F-BSDEs to the doubly stochastic framework. The following lemma gives estimates and stability results (after a perturbation on the coefficients) for the solution of a F-BDSDE. Its proof is omitted since it is based on techniques which are classical in BSDEs’ theory.

Lemma 3 Assume that assumptions (H1) and (H2) hold. Let \((X, Y, Z)\) denote the solution of the F-BDSDE (3)–(11). Then we have the following:
(i) \( L^p \) estimates: For all \( p \geq 2 \), there exists a constant \( C_p \) depending only on \( T, K, \alpha \) and \( p \) such that
\[
E^{\sup_{0 \leq s \leq T} |Y_s|^p + \left( \int_0^T |Z_s|^2 \, ds \right)^{\frac{p}{2}}} \leq C_p E^{\Phi(X_T)^p} + \int_0^T |f(s, X_s, 0, 0)|^p \, ds
\]
there exists a constant $C$ solution of the F-BDSDE (3)-(4), for Lemma 4 assume that assumptions (H1) and (H2) hold. Then we have

$$
E\left[|Y_s - Y_t|^p\right] \leq C_p \left\{ E\left[|\Phi(X_T)|^p + \sup_{0 \leq s \leq T} |f(s, X_s, 0, 0)|^p \right]
+ \sup_{0 \leq s \leq T} |h(s, X_s, 0, 0)|^p ds \right\}^{\frac{p}{2}}. \tag{24}
$$

(ii) Stability result: Let $(X^\epsilon, Y^\epsilon, Z^\epsilon)$ denote the solution of the perturbed F-BDSDE (3)-(4) with coefficients replaced by $b^\epsilon, \sigma^\epsilon, f^\epsilon, h^\epsilon$ and $\Phi^\epsilon$ and initial condition replaced by $x^\epsilon$. Assume that $b^\epsilon, \sigma^\epsilon, f^\epsilon, h^\epsilon$ and $\Phi^\epsilon$ satisfy assumptions (H1) and (H2), that $\lim_{\epsilon \to 0} x^\epsilon = x$ and that for fixed $(x,y,z)$ in $\mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times d}$,

$$
\lim_{\epsilon \to 0} b^\epsilon(x) - b(x) + |\sigma^\epsilon(x) - \sigma(x)|^2 = 0, \\
\lim_{\epsilon \to 0} |\Phi^\epsilon(x) - \Phi(x)|^2 + \int_0^T |h^\epsilon(s, x, y, z) - h(s, x, y, z)|^2 ds \\
+ \int_0^T |f^\epsilon(s, x, y, z) - f(s, x, y, z)|^2 ds = 0.
$$

Then we have

$$
\lim_{\epsilon \to 0} E\left[ \sup_{0 \leq s \leq T} |X^\epsilon_s - X_s|^2 + \sup_{0 \leq s \leq T} |Y^\epsilon_s - Y_s|^2 + \int_0^T |Z^\epsilon_s - Z_s|^2 ds \right] = 0. \tag{25}
$$

The next lemma provides $L^p$ estimates for the martingale component $Z$ of the solution of the F-BDSDE (3)-(4) for $p \geq 2$. It gives also estimates for the continuous component $Y$.

**Lemma 4** Assume that assumptions (H1) and (H2) hold. Then for all $p \geq 2$, there exists a constant $C_p > 0$ depending on $T, K, \alpha$ and $p$ such that

$$
\left( E[|Z^\epsilon_s|^p]\right)^{\frac{1}{p}} \leq C_p (1 + |x|) \text{ a.e. s } \in [t, T]. \tag{26}
$$

In addition, there exists a positive constant $C$ independent from $\hat{h}$ the time step of a given uniform time-grid $\pi := \{0 = t_0 < \ldots < t_N = T\}$ such that

$$
\max_{0 \leq n \leq N-1} \sup_{t_n \leq s \leq t_{n+1}} E\left[ |Y^\epsilon_{s,t} - Y^\epsilon_{t_n,t_n}|^2 + |Y^\epsilon_{s,t} - Y^\epsilon_{t_{n+1},t_{n+1}}|^2 \right] \leq C \hat{h} (1 + |x|^2). \tag{27}
$$

**Proof.** First, we consider the case when $b, \sigma$ and $\Phi$ are in $C^1_b$ and $f$ and $h$ are in $C^{0,1}_b$ and satisfying assumptions (H1) and (H2). Let $(\nabla X^\epsilon, \nabla Y^\epsilon, \nabla Z^\epsilon)$ be the solution of the F-BDSDE (19)-(21).
Since $\nabla X^{t,x}$ is the solution of the SDE (19), $[\nabla X^{t,x}]^{-1}$ is also the solution of an SDE and we have the following estimate

$$E\left[ \sup_{0 \leq t \leq T} |[\nabla X^{t,x}]^{-1}|^p \right] \leq C_p. \quad (28)$$

On the other hand, $\nabla Y^{t,x}$ is the solution of the linear BDSDE (21). Using estimate (23), we get

$$E\left[ \sup_{0 \leq t \leq T} |\nabla Y^{t,x}|^p \right] \leq C_p. \quad (29)$$

Now, recall the representation result (18)

$$Z^{t,x}_s = \nabla Y^{t,x}_s [\nabla X^{t,x}_s]^{-1} \sigma(X^{t,x}_s), \quad P - a.s., \text{ for all } s \in [t, T].$$

Using Hölder’s inequality, we get

$$\left( E|Z^{t,x}_s|^p \right)^{\frac{1}{p}} \leq \left( E|\nabla Y^{t,x}_s|^3 \right)^{\frac{1}{3p}} \left( E|[\nabla X^{t,x}_s]^{-1}|^3 \right)^{\frac{1}{3p}} \left( E|\sigma(X^{t,x}_s)|^3 \right)^{\frac{1}{3p}} \leq C_p(1 + |x|), \quad \forall s \in [t, T].$$

Now the aim is to generalize the previous estimate to the globally Lipschitz continuous coefficients’ case. So let $b, \sigma, \Phi, f$ and $h$ be coefficients satisfying the assumptions (H1) and (H2) and let $b^k, \sigma^k, \Phi^k, f^k$ and $h^k$ be smooth mollifiers of these coefficients (take $b^k, \sigma^k, \Phi^k$ in $C^1_b$ and $f^k$ and $h^k$ in $C^0_1$). Denoting $Z^{t,x,k}$ the solution of the F-BDSDE associated to the smooth coefficients, we deduce from (30) that

$$\lim_{k \to +\infty} E\left[ \int_t^T |Z^{k,t,x}_s - Z^{t,x}_s|^2 \, ds \right] = 0. \quad (31)$$

We deduce that for a.e. $s \in [t, T]$, there exist a subsequence of $(Z^{k,t,x})_k$ such that

$$\lim_{k \to +\infty} Z^{k,t,x}_s = Z^{t,x}_s \quad \text{in probability.}$$

By the Fatou’s Lemma, we get $E|Z^{t,x}_s|^p \leq C_p(1 + |x|)$. Inserting the latter inequality in (24), we get the estimate (27). \hfill \square

Now we are in position to prove our main result which is the $L^2$-regularity of the solutions of F-BDSDEs. First, we need to define the step process $\bar{Z}$.

Let $\pi := \{0 = t_0 < \ldots < t_N = T\}$ be a uniform time-grid with time step $\bar{h}$. We define $\bar{Z}$ by

$$\begin{align*}
\bar{Z}_t &= \frac{1}{\bar{h}} E_{t_n} \left[ \int_{t_n}^{t_{n+1}} Z_s \, ds \right], \quad \text{for all } t \in [t_n, t_{n+1}), \text{ for all } n \in \{0, \ldots, N - 1\}, \\
\bar{Z}_{t_N} &= 0.
\end{align*} \quad (32)$$

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**Theorem 2** \((L^2\text{-regularity})\) Under assumptions \((H1)\) and \((H2)\), we have

\[
\max_{0 \leq n \leq N-1} \sup_{0 \leq t_n \leq s \leq s_{n+1}} E \left| Y_s - Y_{t_n} \right|^2 + \left| Y_s - Y_{t_{n+1}} \right|^2 \\
+ \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} E \left[ \left| Z_s - \bar{Z}_{t_n} \right|^2 + \left| Z_s - \bar{Z}_{t_{n+1}} \right|^2 \right] ds \leq C\hat{h}(1 + |x|^2). \tag{33}
\]

**Proof.** Using the estimate \((27)\), one obtains

\[
\max_{0 \leq n \leq N-1} \sup_{0 \leq t_n \leq s \leq s_{n+1}} E \left| Y_s - Y_{t_n} \right|^2 + \left| Y_s - Y_{t_{n+1}} \right|^2 \\
+ \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} E \left[ \left| Z_s - \bar{Z}_{t_n} \right|^2 + \left| Z_s - \bar{Z}_{t_{n+1}} \right|^2 \right] ds \\
\leq C\hat{h}(1 + |x|^2) + C \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} E \left[ \left| Z_s - \bar{Z}_{t_n} \right|^2 \right] ds + C \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} E \left[ \left| Z_s - \bar{Z}_{t_{n+1}} \right|^2 \right] ds. \tag{34}
\]

Let \(b^k, \sigma^k, \Phi^k, f^k\) and \(h^k\) be smooth mollifiers of \(b, \sigma, \Phi, f\) and \(h\) (we take \(b^k, \sigma^k\) and \(\Phi^k\) in \(C^1_0\) and \(f^k\) and \(h^k\) in \(C^0_0\)). We denote by \((X^k, Y^k, Z^k)\) the solution of the F-BDSDE associated to the smooth coefficients.

First, we deal with the term \(\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} E \left[ \left| Z_s - \bar{Z}_{t_n} \right|^2 \right] ds\). Since the conditional expectation minimizes the conditional mean square error, we have

\[
\sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} E \left| Z_s - \bar{Z}_{t_n} \right|^2 ds \leq \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} E \left| Z_s - Z^k_s \right|^2 ds
\]

\[
\leq 2 \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} E \left| Z_s - Z^k_s \right|^2 ds + 2 \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} E \left| Z^k_s - Z^k_{t_n} \right|^2 ds,
\]

\[
= 2 \int_0^T \left| Z_s - Z^k_s \right|^2 ds + 2 \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} E \left| Z^k_s - Z^k_{t_n} \right|^2 ds.
\]

By the stability result \((25)\), we have

\[
\lim_{k \to +\infty} \int_0^T \left| Z^k_s - Z_s \right|^2 ds = 0. \tag{35}
\]

Now, using the representation result \((18)\) for \(Z^k\), we have

\[
Z^k_s - Z^k_{s'} = \nabla Y^k_s \nabla X^k_s^{-1} \sigma^k(X^k_s) - \nabla Y^k_{s'} \nabla X^k_{s'}^{-1} \sigma^k(X^k_{s'}, s, s' \in [t_n, t_{n+1}]). \tag{36}
\]

Then, by inserting \(\nabla Y^k_s \nabla X^k_s^{-1} \sigma^k(X^k_s)\) and \(\nabla Y^k_{s'} \nabla X^k_{s'}^{-1} \sigma^k(X^k_{s'})\), we obtain

\[
\left| Z^k_s - Z^k_{s'} \right|^2 \leq 3 \left| \nabla Y^k_s - \nabla Y^k_{s'} \right|^2 \left| \nabla X^k_s^{-1} \right|^2 \left| \sigma^k(X^k_s) \right|^2
\]

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Then, inserting $Z^k$ here we also used also the same kind of estimation as (24) but for $\nabla X_k$ of $\nabla X_k^{-1}$ we get

$$|Z_s^k - Z_{tn}^k|^2 \leq C\left\{ |\nabla Y_s^k| [\nabla Y_{tn}^k]^{-1} \cdot |\nabla X_s^k|^{-1} |\sigma^k(X_s^k)|^2 + 3|\nabla Y_s^k| [\nabla X_s^k]^{-1} |\sigma^k(X_s^k)|^2 + 3|\nabla Y_{tn}^k| [\nabla X_{tn}^k]^{-1} |\sigma^k(X_{tn}^k)|^2 \right\}.$$

We conclude by using Hölder’s inequality and the estimate (27) that

$$\sum_{n=0}^{N-1} E\left[ \int_{tn}^{tn+1} |Z_s^k - Z_{tn}^k|^2 ds \right] \leq \hat{C}h(1 + |x|^2), \quad (37)$$

here we also used also the same kind of estimation as (24) but for $[\nabla X_s^k]^{-1}$ as it is a solution of an SDE.

Now, it reminds to handle the error term

$$\sum_{n=0}^{N-1} \int_{tn}^{tn+1} E[|Z_s^k - \bar{Z}_{tn+1}|^2] ds = \sum_{n=0}^{N-2} \int_{tn}^{tn+1} E[|Z_s^k - \bar{Z}_{tn+1}|^2] ds + \int_{tn}^{\infty} E[|Z_s^k|^2] ds.$$

Define

$$\bar{Z}_n^k = \frac{1}{h} E_t \left[ \int_{tn}^{tn+1} Z_s^k ds \right], \quad \text{for all } t \in [tn, tn+1), \quad \text{for all } n \in \{0, \ldots, N-1\}, \quad (38)$$

Then, inserting $Z_s^k, Z_{tn+1}^k$ and $\bar{Z}_{tn+1}^k$, we get

$$\sum_{n=0}^{N-1} \int_{tn}^{tn+1} E[|Z_s^k - \bar{Z}_{tn+1}|^2] ds \leq C \sum_{n=0}^{N-2} \int_{tn}^{tn+1} E[|Z_s^k - Z_{tn+1}^k|^2] ds$$

$$+ C \sum_{n=0}^{N-2} \int_{tn}^{tn+1} E[|Z_s^k - Z_{tn+1}^k|^2] ds + C \sum_{n=0}^{N-2} \int_{tn}^{tn+1} E[|Z_{tn+1}^k - \bar{Z}_{tn+1}^k|^2] ds$$

$$+ C \sum_{n=0}^{N-2} \int_{tn}^{tn+1} E[|\bar{Z}_{tn+1}^k - \bar{Z}_{tn+1}^k|^2] ds.$$

Note that

$$\sum_{n=0}^{N-2} \int_{tn}^{tn+1} E[|Z_s^k - Z_{tn+1}^k|^2] ds \leq \sum_{n=0}^{N-1} \int_{tn}^{tn+1} E[|Z_s^k - Z_{tn+1}^k|^2] ds = \int_0^T |Z_s^k - Z_{tn+1}^k|^2 ds,$$

which tends to zero when $k$ tends to infinity, again by the stability result

$$\sum_{n=0}^{N-2} \int_{tn}^{tn+1} E[|Z_{tn+1}^k|^2] ds$$

The term $\sum_{n=0}^{N-2} \int_{tn}^{tn+1} E[|Z_s^k - Z_{tn+1}^k|^2] ds$ is bounded by $\sum_{n=0}^{N-1} \int_{tn}^{tn+1} E[|Z_s^k - Z_{tn+1}^k|^2] ds$.
which is handled exactly like \( \sum_{n=0}^{N-2} \int_{t_n}^{t_{n+1}} E[|Z_k^s - Z_{t_n}^k|^2]ds \) using the representation result 18 for \( Z^k \) (take \( s' = t_{n+1} \) in (36)). We get

\[
\sum_{n=0}^{N-1} E \left[ \int_{t_n}^{t_{n+1}} |Z_k^s - Z_{t_n+1}^k|^2 ds \right] \leq C\hat{h}(1 + |x|^2).
\]

We deal with the term \( \sum_{n=0}^{N-2} \int_{t_n}^{t_{n+1}} E[|Z_{t_{n+1}}^k - \bar{Z}_{t_{n+1}}^k|^2]ds \) as follows.

By the definition of \( \bar{Z}_{t_{n+1}}^k \), Jensen’s inequality and Cauchy-Schwarz’s inequality, we have for all \( n = 0, \ldots, N - 2 \)

\[
\int_{t_n}^{t_{n+1}} E[|Z_{t_{n+1}}^k - \bar{Z}_{t_{n+1}}^k|^2]ds = \hat{h}E[|Z_{t_{n+1}}^k - \bar{Z}_{t_{n+1}}^k|^2]
\]

\[
= \frac{1}{\hat{h}} E \left[ \left( \int_{t_n}^{t_{n+1}} (Z_{t_{n+1}}^k - Z_s^k)ds \right)^2 \right]
\]

\[
\leq \frac{1}{\hat{h}} E \left[ \left( \int_{t_n}^{t_{n+1}} (Z_{t_{n+1}}^k - Z_s^k)ds \right)^2 \right]
\]

\[
\leq \int_{t_n}^{t_{n+1}} E|Z_{t_{n+1}}^k - Z_s^k|^2 ds.
\]

Thus,

\[
\sum_{n=0}^{N-2} \int_{t_n}^{t_{n+1}} E[|Z_{t_{n+1}}^k - \bar{Z}_{t_{n+1}}^k|^2]ds \leq \sum_{n=0}^{N-2} \int_{t_n}^{t_{n+1}} E|Z_{t_{n+1}}^k - Z_s^k|^2 ds
\]

\[
= \sum_{n=1}^{N-1} \int_{t_n}^{t_{n+1}} E|Z_{t_n}^k - Z_s^k|^2 ds
\]

\[
\leq \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} E|Z_{t_n}^k - Z_s^k|^2 ds
\]

\[
\leq C\hat{h}(1 + |x|^2),
\]

by (37).

Finally, we deal with \( \sum_{n=0}^{N-2} \int_{t_n}^{t_{n+1}} E[|\bar{Z}_{t_{n+1}}^k - \bar{Z}_{t_{n+1}}|^2]ds \) as follows. By the definitions of \( \bar{Z}_{t_{n+1}}^k \) and \( \bar{Z}_{t_{n+1}} \), Jensen’s inequality and Cauchy-Schwarz’s inequality, we have for all \( n = 0, \ldots, N - 2 \)

\[
\int_{t_n}^{t_{n+1}} E[|\bar{Z}_{t_{n+1}}^k - \bar{Z}_{t_{n+1}}|^2]ds = \hat{h}E[|\bar{Z}_{t_{n+1}}^k - \bar{Z}_{t_{n+1}}|^2]
\]

\[
= \frac{1}{\hat{h}} E \left[ \left( \int_{t_n}^{t_{n+1}} (Z_s^k - Z_s^k)ds \right)^2 \right]
\]

\[
= \frac{1}{\hat{h}} E \left[ \left( \int_{t_n}^{t_{n+1}} 0 ds \right)^2 \right]
\]

\[
= 0.
\]
\[
\begin{align*}
&\leq \frac{1}{h} E\left[ \int_{t_{n+1}}^{t_{n+2}} (Z^k_k - Z_s) ds \right]^2 \\
&\leq \int_{t_{n+1}}^{t_{n+2}} E|Z^k_k - Z_s|^2 ds.
\end{align*}
\]

Hence
\[
\sum_{n=0}^{N-2} \int_{t_n}^{t_{n+1}} E[|Z^k_{t_{n+1}} - Z_{t_{n+1}}|^2] ds \leq \sum_{n=0}^{N-2} \int_{t_{n+1}}^{t_{n+2}} E|Z^k_k - Z_s|^2 ds \\
= \sum_{n=1}^{N-1} \int_{t_n}^{t_{n+1}} E|Z^k_k - Z_s|^2 ds \\
\leq \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} E|Z^k_k - Z_s|^2 ds \\
= \int_0^T |Z^k_k - Z_s|^2 ds,
\]

which tends to zero when \( k \) goes to infinity by (35). To conclude the proof, observe that by (26), \( \int_{t_{N-1}}^{t_N} E|Z_s|^2 ds \leq C \hat{h}(1 + |x|^2) \).

\[\square\]

4 Application: Rate of convergence in time for a numerical scheme for F-BDSDEs under globally Lipschitz continuous conditions

In this section, we give the main application of our \( L^2 \)-regularity result stated in Theorem 2. This application will be in Corollary 2 where we derive, under globally Lipschitz continuous conditions, a rate of convergence in time for the numerical scheme for the F-BDSDE (3)-(4) studied in BBMM16.

4.1 Numerical scheme for F-BDSDEs

We recall from BBMM16 the following discretized version of (3)-(4). Let

\[\pi : t_0 = 0 < t_1 < \ldots < t_N = T,\]

be a partition of the time interval \([0,T]\). For simplicity, we take an equidistant partition of \([0,T]\) i.e. \( \hat{h} = \frac{T}{N} \) and \( t_n = n \hat{h}, 0 \leq n \leq N \). In the sequel, the notations \( \Delta W_n = W_{t_{n+1}} - W_{t_n} \) and \( \Delta B_n = B_{t_{n+1}} - B_{t_n} \), for \( n = 0, \ldots, N - 1 \) will be used.

The forward component \( X \) is approximated by the classical forward Euler scheme:

\[
\begin{cases}
X^N_{t_0} = x, \\
X^N_{t_{n+1}} = X^N_{t_n} + \hat{h}b(X^N_{t_n}) + \sigma(X^N_{t_n}) \Delta W_n, \text{ for } n = 0, \ldots, N - 1.
\end{cases}
\]
The solution \((Y, Z)\) of (4) is approximated by \((Y^N, Z^N)\) defined by
\[
Y^N_{t_N} = \Phi(X^N_T) \quad \text{and} \quad Z^N_{t_N} = 0,
\]
and for \(n = N-1, \ldots, 0\), we set
\[
Y^N_{t_n} = E_{t_n} \left[ Y^N_{t_{n+1}} + h(t_{n+1}, \Theta^N_{n+1}) \Delta B_n + \hat{h}f(t_n, \Theta^N_n) \right],
\]
\[
\hat{h} Z^N_{t_n} = E_{t_n} \left[ Y^N_{t_{n+1}} \Delta W^\top_n + h(t_{n+1}, \Theta^N_{n+1}) \Delta B_n \Delta W^\top_n \right],
\]
where
\[
\Theta^N_n := (X^N_{t_n}, Y^N_{t_n}, Z^N_{t_n}), \quad \text{for all} \quad n = 0, \ldots, N.
\]

\(\top\) denotes the transpose operator and \(E_{t_n}\) denotes the conditional expectation w.r.t. the \(\sigma\)-algebra \(F_{t_n}\).

We also recall the continuous approximation of the solution of BDSDE (4). For \(n = 0, \ldots, N-1\)
\[
Y^N_t := Y^N_{t_{n+1}} + \int_t^{t_{n+1}} f(t_n, \Theta^N_n) ds + \int_t^{t_{n+1}} h(t_{n+1}, \Theta^N_{n+1}) dB_s - \int_t^{t_{n+1}} Z^N_s dW_s, \quad t_n \leq t < t_{n+1}.
\]

4.2 Rate of convergence for the Euler time discretization based numerical scheme for F-BDSDEs

In order to derive the rate of convergence in time of the numerical scheme (39)-(40), the authors in [BBMM16] proved the \(L^2\)-regularity for the martingale integrand \(Z\) under strong assumptions on the coefficients. Indeed, they assume that the coefficients \(b, \sigma\) and \(\Phi\) are in \(C^2_b\) and \(f\) and \(h\) are in \(C^2\). Our \(L^2\)-regularity result stated in Theorem 2 requires the coefficients to be only globally Lipschitz continuous but enables us to derive the same rate of convergence in time derived in [BBMM16]. This is an important improvement for that numerical scheme.

Let us recall the following upper bound result (Theorem 3.1 in [BBMM16]) for the time discretization error.

**Theorem 3 ([BBMM16])** Define the time discretization error by
\[
\text{Error}_N(Y, Z) := \sup_{0 \leq s \leq T} E[|Y_s - Y^N_s|^2] + \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} E[|Z_s - Z^N_s|^2] ds,
\]
where \(Y^N\) and \(Z^N\) are given by (41). Under assumptions (H1) and (H2) we have
\[
\text{Error}_N(Y, Z) \leq C\hat{h}(1 + |x|^2) + C \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} E[|Z_s - \hat{Z}_{t_n}|^2] ds
\]

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\[
+ C \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} E[|Z_s - \tilde{Z}_{tn+1}|^2]ds + C \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} E[|Y_s - Y_{tn}|^2]ds \\
+ C \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} E[|Y_s - Y_{tn+1}|^2]ds.
\]

(43)

The rate of convergence in time of our scheme under globally Lipschitz continuous assumptions is derived in the next corollary.

**Corollary 2** Under Assumptions (H1) and (H2), we have

\[
\text{Error}_N(Y, Z) \leq \hat{C} \hat{h}(1 + |x|^2).
\]

(44)

**Proof.** The result follows by using the estimate (33) in the upper bound estimate (43).

\[
\square
\]

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