A METRIZABLE SEMITOPOLOGICAL SEMILATTICE
WITH NON-CLOSED PARTIAL ORDER

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Abstract. We construct a metrizable semitopological semilattice $X$ whose partial order $P = \{(x, y) \in X \times X : xy = x\}$ is a non-closed dense subset of $X \times X$. As a by-product we find necessary and sufficient conditions for the existence of a (metrizable) Hausdorff topology on a set, act, semigroup or semilattice, having a prescribed countable family of convergent sequences.

1. Introduction

In this paper we shall construct an example of a metrizable semitopological semilattice with non-closed partial order.

A semilattice is a commutative semigroup $X$ whose any element $x \in X$ is an idempotent in the sense that $xx = x$. A typical example of a semilattice is any partially ordered set $X$ in which any finite non-empty set $F \subset X$ has the greatest lower bound $\inf(F)$. In this case the binary operation $X \times X \to X$, $(xy) \mapsto \inf\{x, y\}$, turns $X$ into a semilattice.

Each semilattice $X$ carries a partial order $\leq$ defined by $x \leq y$ iff $xy = x$. For this partial order we have $xy = \inf\{x, y\}$.

A (semi)topological semilattice is a semilattice $X$ endowed with a topology such that the binary operation $X \times X \to X$, $xy \mapsto xy$, is (separately) continuous.

The continuity of the semilattice operation in a Hausdorff topological semilattice implies the following well-known fact, see [5, VI-1.14].

Proposition 1.1. For any Hausdorff topological semilattice $X$ the partial order $P = \{(x, y) \in X \times X : xy = x\}$ is a closed subset of $X \times X$.

It is natural to ask whether this proposition remains true for Hausdorff semitopological semigroups. The following example answers this question in negative.

Example 1.2. There exists a metrizable countable semitopological semilattice $X$ whose partial order is dense and non-closed in $X \times X$.

This example will be constructed in Section 6 after some preliminary work, made in Sections 2–5. In Section 2 we establish necessary and sufficient conditions on a set $X$ and function $\ell : \text{dom}(\ell) \to X$ defined on a subset $\text{dom}(\ell) \subset X^\omega$ ensuring that $X$ admits a (metrizable) Hausdorff topology in which every sequence $s \in \text{dom}(\ell)$ converges to the point $\ell(s)$. In Section 3 we study the analogous problem for acts, i.e. sets endowed with monoids of self-maps and in Section 4–5 we apply the obtained results about acts to constructing topologies with prescribed convergent sequences on semigroups and semilattices. More information on the closedness of the partial order in semitopological semilattices can be found in [1, §7].
Observe that the indiscrete topology \{\emptyset, X\} on X is \ell-admissible. So, the family of \ell-admissible topologies is not empty. This family has the largest element. This is the topology \(\tau\) consisting of all subsets \(U \subset X\) such that for any sequence \(s = (s_n)_{n \in \omega} \in \ell(\ell)\) with \((s_n) \in U\) the set \(\{n \in \omega : s_n \notin U\}\) is finite. The topology \(\tau\) will be referred to as the largest \(\ell\)-admissible topology on \(X\).

In this section we discuss the following problem.

**Problem 2.1.** Under which conditions the largest \(\ell\)-admissible topology \(\tau_\ell\) on \(X\) is Hausdorff?

Below we define two necessary conditions of the Hausdorffness of the topology \(\tau_\ell\).

The function \(\ell : (\ell(\ell)) \to X\) is defined to be

- \(T_1\)-separating if for any sequence \(s \in \text{dom}(\ell)\) any point \(x \in X\) with \(x \neq \ell(s)\) the set \(\{n \in \omega : s_n = x\}\) is finite;
- \(T_2\)-separating if \(\ell\) is \(T_1\)-separating and for any sequences \(s, t \in \text{dom}(\ell)\) with \(\ell(s) \neq \ell(t)\) there exists a finite set \(F \subset \omega\) such that \(s_n \neq t_m\) for any \(n, m \in \omega \setminus F\).

We say that a topology \(\tau\) on a set \(T\) satisfies the separation axiom \(T_1\) if each finite subset of \(T\) is \(\tau\)-closed in \(T\). In this case we say that \((T, \tau)\) is a \(T_1\)-space.

**Lemma 2.2.** The function \(\ell\) is \(T_1\)-separating if and only if the topology \(\tau_\ell\) satisfies the separation axiom \(T_1\).

**Proof.** Assume that \(\ell\) is \(T_1\)-separating and take any finite set \(F \subset X\). To show that the set \(U := X \setminus F\) belongs to the topology \(\tau_\ell\), it suffices to check that for every \(s \in \text{dom}(\ell)\) with \(\ell(s) \in U\) the set \(\{n \in \omega : s(n) \notin U\}\) is finite. By the \(T_1\)-separating property of \(\ell\), for every \(x \in F\) the set \(\Omega_x = \{n \in \omega : s_n = x\}\) is finite and so is the set \(\{n \in \omega : s_n \notin U\}\).

Now assuming that each finite subset \(F \subset X\) is closed in the topology \(\tau_\ell\), we shall prove that the function \(\ell\) is \(T_1\)-separating. Given any point \(x \in X \setminus \{\ell(s)\}\), observe that \(\ell(s) \in X \setminus \{x\}\) implies that the set \(\{n \in \omega : s_n \notin X \setminus \{x\}\}\) is finite, which means that \(\ell\) is \(T_1\)-separating.

**Lemma 2.3.** If the topology \(\tau_\ell\) is Hausdorff, then the function \(\ell\) is \(T_2\)-separating.

**Proof.** By Lemma 2.2 the function \(\ell\) is \(T_1\)-separating. To prove that \(\ell\) is \(T_2\)-separating, take two sequences \(s, t \in \text{dom}(\ell)\) with \(\ell(s) \neq \ell(t)\). By the Hausdorff property of the topology \(\tau_\ell\), there are disjoint open sets \(U, V \subset \ell(\ell)\) such that \(\ell(s) \in U\) and \(\ell(t) \in V\). By the definition of the topology \(\tau_\ell\), the sets \(F := \{n \in \omega : s_n \notin U\}\) and \(E := \{m \in \omega : t_m \notin V\}\) are finite. Then \(s_n \neq t_m\) for any \(n, m \in \omega \setminus (F \cup E)\).

Now we shall prove that the largest \(\ell\)-admissible topology \(\tau_\ell\) is Hausdorff if the function \(\ell\) is \(T_2\)-separating and \(\text{dom}(\ell)\) is at most countable.

For a sequence \(s \in \text{dom}(\ell)\) and a subset \(I \subset \omega\) let \(s[I] := \{s_n : n \in I\}\) and \(s[I]^* := s[I] \cup \{\ell(s)\}\).

**Lemma 2.4.** Assume that the function \(\ell : \text{dom}(\ell) \to X\) is \(T_2\)-separating. Let \(s \in \text{dom}(\ell)\) and \(I \subset \omega\).

1. the set \(s[I]^*\) is closed in \((X, \tau_\ell)\);
2. each point \(x \in s[I]^* \setminus \{\ell(s)\}\) is isolated in \(s[I]^*\);
3. the subspace \(s[I]^*\) of \((X, \tau_\ell)\) is compact and Hausdorff.

**Proof.** 1. The inclusion \(X \setminus s[I]^* \in \tau_\ell\) will follow as soon as we show that for any sequence \(t \in \text{dom}(\ell)\) with \(\ell(t) \notin s[I]^*\) the set \(\{n \in \omega : t_n \in s[I]^*\}\) is finite. By the \(T_2\)-separating property of \(\ell\), there exists a finite set \(\Omega \subset \omega\) such that \(s_n \neq t_m\) for any \(n, m \in \omega \setminus \Omega\). Consider the finite set \(E = \{\ell(s)\} \cup \{s_n : n \in \Omega\} \setminus \{\ell(t)\}\). By the \(T_1\)-separating property of \(\ell\), the set \(\Lambda = \Omega \cup \{n \in \omega : t_n \in E\}\) is finite. Then the set \(\{n \in \omega : t_n \in s[I]^*\}\) is closed in \((X, \tau_\ell)\).

2. Given any point \(x \in s[I]^* \setminus \{\ell(s)\}\), observe that \(s[I]^* \setminus \{x\} = s[J]^*\) where \(J = I \setminus s^{-1}(x)\).

By Lemma 2.4(1), the subspace \(s[J]^*\) is closed in \((X, \tau_\ell)\) and then the singleton \(\{x\} = s[I]^* \setminus \{\ell(s)\}\) is open in \(s[I]^*\).

3. The compactness of \(s[I]^*\) follows from the fact that each neighborhood \(U \in \tau_\ell\) of \(\ell(s)\) contains all but finitely many points of the set \(s[I]\). To see that \(s[I]^*\) is Hausdorff, take any two
distinct points \( x, y \in s[I]^\ast \). One of these points is distinct from the limit point \( \ell(s) \) of the sequence \( s \) and we lose no generality assuming that \( x \neq \ell(s) \). By Lemmas \( \ref{lem:closure} \) and \( \ref{lem:diagonal} \), the singleton \( U_x = \{ x \} \) closed-and-open in \( s[I]^\ast \) and so is its complement \( U_y = s[I]^\ast \setminus U_x \). Then \( U_x, U_y \) are disjoint neighborhoods of the points \( x, y \) in \( s[I]^\ast \), witnessing that the subspace \( s[I]^\ast \) of \( (X, \tau_k) \) is Hausdorff.

**Corollary 2.5.** If the function \( \ell : \text{dom}(\ell) \to X \) is \( T_2 \)-separating, then for every finite subset \( S \subset \text{dom}(\lambda) \) the subspace \( S[\omega]^\ast = \bigcup_{s \in S} s[\omega]^\ast \) of \( (X, \tau_\ell) \) is compact, Hausdorff, and closed in \( (X, \tau_k) \).

This corollary follows from Lemma \( \ref{lem:closure} \) and the following known fact.

**Lemma 2.6.** The union \( A \cup B \) of two closed Hausdorff subspaces of a topological space \( T \) is Hausdorff.

**Proof.** We lose no generality assuming that \( T = A \cup B \). The Hausdorff property of the space \( T = A \cup B \) will follow as soon as we check that its diagonal \( \Delta_T = \{(x, x) : x \in T\} \) is closed in \( T \times T \). For this observe that \( \Delta_T = \Delta_A \cup \Delta_B \). Since the space \( A \) is Hausdorff and closed in \( T \), its diagonal \( \Delta_A \) is closed in \( A \times A \) and in \( T \times T \). By analogy, the diagonal \( \Delta_B \) is closed in \( B \times B \) and in \( T \times T \). Then the union \( \Delta_T = \Delta_A \cup \Delta_B \) is closed in \( T \times T \) and the space \( T \) is Hausdorff. \( \Box \)

We say that the topology \( \tau \) of a topological space \( X \) is generated by a family \( K \) of subspaces of \( X \) if a set \( U \subset X \) is open in \( X \) if any only if for every \( K \in K \) the intersection \( U \cap K \) is open in the subspace topology of \( K \). A topology \( \tau \) on a set \( X \) is called a \( k_\omega \)-topology if it is generated by a countable family of compact subsets of the topological space \( (X, \tau) \).

**Lemma 2.7.** If the function \( \ell : \text{dom}(\ell) \to X \) is \( T_2 \)-separating and \( \text{dom}(\ell) \) is at most countable, then the topology \( \tau_k \) is Hausdorff and normal. Moreover, \( \tau_k \) is a \( k_\omega \)-topology, generated by the countable family \( K = \{ s[\omega]^\ast : s \in \text{dom}(\ell) \} \) of compact sets in \( (X, \tau_\ell) \).

**Proof.** The definition of the topology \( \tau_k \) ensures that it is generated by the countable family \( K \). Now we show that the topology \( \tau_k \) is Hausdorff and normal. By Lemma \( \ref{lem:closure} \), the topology \( \tau_k \) satisfies the separation axiom \( T_1 \). Now it suffices to check that this topology is normal. From now on, we consider \( X \) as a topological space endowed with the topology \( \tau_k \).

Given two disjoint closed sets \( A, B \subset X \) we should find two disjoint open sets \( V, W \subset X \) such that \( A \subset V \) and \( B \subset W \).

Let \( \text{dom}(\ell) = \{s_n\}_{n \in \omega} \) be an enumeration of the countable set \( \text{dom}(\ell) \). By Corollary \( \ref{lem:closure} \), for every \( n \in \omega \) the subspace \( K_n := \bigcup_{i \leq n} s_i[\omega]^\ast \) of \( (X, \tau_k) \) is compact, Hausdorff, and closed in \( (X, \tau_k) \).

Let \( A_0 := A \cap K_0 \) and \( B_0 := B \cap K_0 \). By induction we shall construct sequences \( (A_n)_{n \in \omega}, (B_n)_{n \in \omega}, (V_n)_{n \in \omega}, (W_n)_{n \in \omega} \) of subsets in \( X \) such that for every \( n \in \omega \) the following conditions are satisfied:

1. the sets \( A_n, B_n \) are disjoint and closed in \( K_n \);
2. \( A_n \subset V_n \subset K_n \) and \( B_n \subset W_n \subset K_n \);
3. the sets \( V_n, W_n \) are open in \( K_n \) and \( V_n \cap W_n = \emptyset \);
4. \( A_{n+1} = V_n \cup (K_{n+1} \cap A) \) and \( B_{n+1} = W_n \cup (K_{n+1} \cap B) \);
5. \( A_n \cap B_n = \emptyset \).

Assume that for some \( n \in \omega \) disjoint closed sets \( A_n, B_n \subset K_n \) with \( A_n \cap A \subset A_n \) and \( K_n \cap B \subset B_n \) have been constructed. By the normality of the compact Hausdorff space \( K_n \), there are open sets \( V_n, W_n \subset K_n \) satisfying the conditions (2),(3). Define the sets \( A_{n+1}, B_{n+1} \) by the formula (4) and observe that

\[ A_{n+1} \cap B_{n+1} = (A_n \cup B_n) \cap K_n \subset (V_n \cap W_n) \subset A \cap B = \emptyset. \]

After completing the inductive construction, observe that \( V := \bigcup_{n \in \omega} V_n \) and \( W = \bigcup_{n \in \omega} W_n \) are disjoint open sets in \( X \) such that \( A \subset V \) and \( B \subset W \).

The following theorem is the main result of this section.

**Theorem 2.8.** For a set \( X \) and function \( \ell : \text{dom}(\ell) \to X \) defined on a countable subset \( \text{dom}(\ell) \subset X^\omega \) the following conditions are equivalent:
(1) \(X\) admits a metrizable topology \(\tau\) in which every sequence \(s \in \text{dom}(\ell)\) converges to the point \(\ell(s)\).

(2) \(X\) admits a Hausdorff topology \(\tau\) in which every sequence \(s \in \text{dom}(\ell)\) converges to the point \(\ell(s)\).

(3) The following two properties are satisfied:

(3a) for any \(s \in \text{dom}(\ell)\) and \(x \in X\) with \(s \neq \ell(s)\) the set \(\{n \in \omega : s_n = x\}\) is finite;

(3b) for any sequences \(s, t \in \text{dom}(\ell)\) with \(\ell(s) \neq \ell(t)\) there exists a finite set \(F \subset \omega\) such that \(s_n \neq t_m\) for any \(n, m \in \omega \setminus F\).

Proof. The implications \((1) \Rightarrow (2) \Rightarrow (3)\) are trivial.

To prove that \((3) \Rightarrow (1)\), assume the the condition \((3)\) is satisfied. Then the function \(\ell\) is \(T_2\)-separating and by Lemma 2.7, the largest \(\ell\)-admissible topology \(\tau_\ell\) is Hausdorff and normal. Consider the countable subset \(D = \bigcup_{s \in \text{dom}(\ell)} s[\omega]^*\) of \(X\) and observe that \(X\setminus D\) is a closed-and-open discrete subspace of the topological space \(X\) endowed with the topology \(\tau_\ell\). Being countable and Tychonoff, the closed-and-open discrete subspace \(D\) of \(X\) is zero-dimensional. Then for any distinct points \(x, y \in D\) we can choose a closed-and-open subset \(U_{x,y} \subset X\) such that \(x \in U_{x,y}\) and \(y \notin U_{x,y}\).

Let \(\tau\) be the topology on \(D\), generated by the countable subbase \(\{U_{x,y} : x, y \in D, x \neq y\}\). It is clear that the topology \(\tau\) is second-countable, Hausdorff, zero-dimensional and hence regular.

By Urysohn Metrization Theorem [4 4.2.9], the topological space \(D = (D, \tau)\) is metrizable. Then the topology of the topological sum on \((X\setminus D) \oplus D\) is also metrizable. Since \(\tau \subset \tau_\ell\), the topology \(\tau\) is \(\ell\)-admissible, which means that each sequence \(s \in \text{dom}(\ell)\) converges to the point \(\ell(s)\).

The countability of the domain \(\text{dom}(\ell)\) in Theorems 2.8 is essential as shown by the following example.

Example 2.9. There exists a \(T_2\)-separating function \(\ell : \text{dom}(\ell) \to \{0, 1\} \subset \omega\) defined on a subset \(\text{dom}(\ell) \subset [\omega]^{\omega}\) of cardinality \(|\text{dom}(\ell)| = \omega_1\) such that the largest \(\ell\)-admissible topology \(\tau_\ell\) is not Hausdorff.

Proof. To construct such function \(\ell\), take any Hausdorff \((\omega_1, \omega_1)\)-gap on \(\omega\), which is a pair \(((A_i)_{i \in \omega_1}, (B_i)_{i \in \omega_1})\) of families of infinite subsets of \(\omega\) satisfying the following two conditions:

(H1) for any \(i < j < \omega_1\) we have \(A_i \subset^* A_j\) and \(B_i \subset^* B_j\);

(H2) \(A_i \cap B_j\) is finite for any \(i, j \in \omega_1\);

(H3) for any set \(C \subset \omega\) one of the sets \(\{i \in \omega_1 : A_i \subset^* C\}\) or \(\{i \in \omega_1 : B_i \subset^* \omega \setminus C\}\) is at most countable.

Here the notation \(A \subset^* B\) means that the complement \(A \setminus B\) is finite.

It is well-known [3 Ch.20] that Hausdorff \((\omega_1, \omega_1)\)-gaps do exist in ZFC.

For every \(i \in \omega_1\) choose any bijective functions \(\alpha_i : \omega \to A_i\) and \(\beta_i : \omega \to B_i\), and put \(\text{dom}(\ell) = \{\alpha_i, \beta_i : i \in \omega_1\}\). Let \(\ell : \text{dom}(\ell) \to \{0, 1\} \subset \omega\) be the function such that \(\ell^{-1}(0) = \{\alpha_i\}_{i \in \omega_1}\) and \(\ell^{-1}(1) = \{\beta_i\}_{i \in \omega_1}\). The injectivity of the functions \(\alpha_i, \beta_i\) and the condition (H2) ensure that the function \(\ell\) is \(T_2\)-separating. Assuming that the \(\ell\)-admissible topology on \(\omega\) is Hausdorff, we could find two disjoint open sets \(U_0, U_1 \in \tau_\ell\) such that \(0 \in U_0\) and \(1 \in U_1\). By condition (H3), there exists \(i \in \omega_1\) such that \(A_i \not\subset^* U_0\) or \(B_i \not\subset^* U_1\). In the first case the set \(\{n \in \omega : \alpha_i(n) \notin U_0\}\) is infinite, which contradicts \(U_0 \in \tau_\ell\). In the second case the set \(\{n \in \omega : \beta_i(n) \notin U_1\}\) is infinite, which contradicts \(U_1 \in \tau_\ell\).

3. Convergent sequences in topological acts

For a set \(X\) denote by \(X^X\) the set of all self-maps \(X \to X\). The set \(X^X\) endowed with the operation of composition is a monoid whose unit is the identity map \(\text{id}_X\) of \(X\).

An act is a pair \((X, A)\) consisting of a set \(X\) and a submonoid \(A \subset X^X\). Elements of the set \(A\) are called the shifts of the act \((X, A)\).

A topology \(\tau\) on the underlying set \(X\) of an act \((X, A)\) is called an shift-continuous if each shift \(\alpha \in A\) is a continuous self-map of the topological space \((X, \tau)\).

Theorem 3.1. For an act \((X, A)\) with countable set \(A\) of shifts and a function \(\ell : \text{dom}(\ell) \to X\) defined on a countable subset \(\text{dom}(\ell) \subset X^\omega\) the following conditions are equivalent:

\[\begin{align*}
\begin{array}{ll}
(1) & X \text{ admits a metrizable topology } \tau \text{ in which every sequence } s \in \text{dom}(\ell) \text{ converges to the point } \ell(s). \\
(2) & X \text{ admits a Hausdorff topology } \tau \text{ in which every sequence } s \in \text{dom}(\ell) \text{ converges to the point } \ell(s). \\
(3) & \text{The following two properties are satisfied:} \\
(3a) & \text{for any } s \in \text{dom}(\ell) \text{ and } x \in X \text{ with } s \neq \ell(s) \text{ the set } \{n \in \omega : s_n = x\} \text{ is finite;} \\
(3b) & \text{for any sequences } s, t \in \text{dom}(\ell) \text{ with } \ell(s) \neq \ell(t) \text{ there exists a finite set } F \subset \omega \text{ such that } s_n \neq t_m \text{ for any } n, m \in \omega \setminus F. \\
\end{array}
\end{align*}\]
(1) $X$ admits a shift-continuous metrizable topology $\tau$ in which every sequence $s \in \operatorname{dom}(\ell)$ converges to the point $\ell(s)$;

(2) $X$ admits a shift-continuous Hausdorff topology $\tau$ in which every sequence $s \in \operatorname{dom}(\ell)$ converges to the point $\ell(s)$;

(3) the following two properties hold:

(3a) for any $s \in \operatorname{dom}(\ell)$, $\alpha \in \mathcal{A}$ and $x \in X$ with $x \neq \alpha \circ \ell(s)$ the set $\{n \in \omega : \alpha \circ s_n = x\}$ is finite;

(3b) for any sequences $s, t \in \operatorname{dom}(\ell)$ and any shifts $\alpha, \beta \in \mathcal{A}$ with $\alpha \circ \ell(s) \neq \beta \circ \ell(t)$ there exists a finite set $F \subset \omega$ such that $\alpha \circ s_n \neq \beta \circ t_m$ for any $n, m \in \omega \setminus F$.

Proof. The implications $(1) \Rightarrow (2) \Rightarrow (3)$ are trivial. To prove that $(3) \Rightarrow (1)$, assume that the condition $(3)$ is satisfied. Consider the set $\operatorname{dom}(\mathcal{A}^\ell) = \{\alpha \circ s : \alpha \in \mathcal{A}, s \in \operatorname{dom}(\ell)\} \subset X^\omega$ and the function $\mathcal{A}^\ell : \operatorname{dom}(\mathcal{A}^\ell) \rightarrow X$ defined by the formula $\mathcal{A}^\ell(\alpha \circ s) = \alpha(\ell(s))$. The conditions (3a) and (3b) ensure that the function $\mathcal{A}^\ell$ is well-defined and is $T_2$-separating. By Lemma 2.7, the largest $\mathcal{A}^\ell$-admissible topology $\tau_{\mathcal{A}^\ell}$ on $X$ is Hausdorff and normal. Since the monoid $\mathcal{A}$ contains the identity map of $X$, the topology $\tau_{\mathcal{A}^\ell}$ is $\ell$-admissible, which implies that each sequence $s \in \operatorname{dom}(\ell)$ converges to $\ell(s)$ in the topological space $(X, \tau_{\mathcal{A}^\ell})$.

We claim that the topology $\tau_{\mathcal{A}^\ell}$ is shift-continuous. Given any shift $\alpha \in \mathcal{A}$ and open set $U \in \tau_{\mathcal{A}^\ell}$, we need to check that $\alpha^{-1}(U) \in \tau_{\mathcal{A}^\ell}$. The latter inclusion holds if any only if for any sequence $s \in \operatorname{dom}(\mathcal{A}^\ell)$ with $\mathcal{A}^\ell(s) \in \alpha^{-1}(U)$ the set $\{n \in \omega : s_n \notin \alpha^{-1}(U)\} = \{n \in \omega : \alpha \circ s_n \notin U\}$ is finite. Since $\mathcal{A}$ is a monoid, the sequence $\alpha \circ s$ belongs to $\operatorname{dom}(\mathcal{A}^\ell)$. Since $U \in \tau_{\mathcal{A}^\ell}$ and $\mathcal{A}^\ell(\alpha \circ s) = \mathcal{A}^\ell(s) \in U$, the set $\{n \in \omega : \alpha \circ s_n \notin U\} = \{n \in \omega : s_n \notin \alpha^{-1}(U)\}$ is finite and we are done.

The countability of the sets $\operatorname{dom}(\ell)$ and $\mathcal{A}$ imply the countability of the sets $\operatorname{dom}(\mathcal{A}^\ell)$ and $D = \{s[\omega]^* : s \in \operatorname{dom}(\mathcal{A}^\ell)\}$. Observe that for every $\alpha \in \mathcal{A}$ we have $\alpha[D] \subset D$.

By the definition of the topology $\tau_{\mathcal{A}^\ell}$, the set $D$ is open-and-closed in $(X, \tau_{\mathcal{A}^\ell})$ and the complement $X \setminus D$ is discrete. Being Tychonoff, the countable subspace $D$ of $(X, \tau_{\mathcal{A}^\ell})$ is zero-dimensional. This allows us to choose a countable family $B \subset \mathcal{A}^\ell$ of open-and-closed sets that separate points of the countable set $D$ in the sense that for any distinct points $x, y \in D$ there exists a set $B \in B$ such that $x \notin B$ and $y \notin B$. Since $\mathcal{A}^\ell$ is an act topology on $(X, \mathcal{A})$, for every $\alpha \in \mathcal{A}$ and $B \in B$ the set $\alpha^{-1}(B)$ is closed-and-open. Then the topology $\tau_D$ on $D$ generated by the subbase $\{D \cap \alpha^{-1}(B), D \setminus \alpha^{-1}(B) : \alpha \in \mathcal{A}, B \in B\}$ is second-countable, Hausdorff and zero-dimensional.

By the Urysohn metrization Theorem [4 4.2.9], the topological space $D_{\tau} = (D, \tau_D)$ is metrizable. Then the topology $\tau$ of topological sum $D_{\tau} \oplus (X \setminus D)$ of $D_{\tau}$ and the discrete topological space $X \setminus D$ is metrizable. By the definition of the topology $\tau_D$, for every $\alpha \in \mathcal{A}$ the restriction $\alpha[D]$ is a continuous self-map of the topological space $D_{\tau}$. Since $X \setminus D$ is a closed-and-open discrete subspace of $(X, \tau)$, the continuity of $\alpha[D]$ implies that $\alpha$ is a continuous self-map of the metrizable topological space $(X, \tau)$. This means that the topology $\tau$ is shift-continuous. Since $\tau \subset \tau_{\mathcal{A}^\ell}$, the metrizable topology is $\ell$-admissible. 

Theorem 3.1 can be compared with the following result proved in [2 3.4].

Theorem 3.2 (Banakh, Protasov, Sipacheva). Let $\kappa$ be an infinite cardinal, $(X, \mathcal{A})$ be an act, $x \in X$, and $(x_i)_{i \in \kappa}$ be a transfinite sequence of points in $X$. Assume that there exists a (not necessarily bijective) enumeration $\mathcal{A} = \{\alpha_i\}_{i \in \kappa}$ of the set $\mathcal{A}$ such that for each ordinal $m \in \kappa$ and ordinals $i, j, k < \kappa$ the following conditions are satisfied:

(1) if $\alpha_i(x) \neq \alpha_j(x)$, then $\alpha_i(x_m) \neq \alpha_j(x_m)$;

(2) if $\alpha_i(x) \neq \alpha_j(x_k)$, then $\alpha_i(x_m) \neq \alpha_j(x_k)$.

Then $X$ admits a shift-continuous hereditarily normal topology $\tau$ in which the transfinite sequence $(x_i)_{i \in \kappa}$ converges to the point $x$ in the sense that for every neighborhood $O_x \in \tau$ of $x$ there exists $n \in \kappa$ such that $x_i \in O_x$ for all $i \geq n$ in $\kappa$. 

4. Convergent sequences in semigroups

Let $X$ be a semigroup and $X^1$ be the semigroup $X$ with attached unit. A topology $\tau$ on $X$ is called *shift-continuous* if for every $a, b \in X^1$ the two-sided shift

$$X \to X, \ x \mapsto axb,$$

is a continuous self-map of the topological space $(X, \tau)$.

Each semigroup $X$ has the structure of an act $(X, \mathcal{A})$ endowed with the family of shifts $\mathcal{A} = \{s_{a,b}: a, b \in X^1\}$. Applying Theorem 3.1 to this act, we obtain the following theorem, which is a main result of this section.

**Theorem 4.1.** For a countable semilattice $X$ and a function $\ell: \text{dom}(\ell) \to X$ defined on a countable subset $\text{dom}(\ell) \subset X^\omega$, the following conditions are equivalent:

1. The semigroup $X$ admits a shift-continuous metrizable topology $\tau$ in which every sequence $s \in \text{dom}(\ell)$ converges to the point $\ell(s)$;
2. The semigroup $X$ admits a shift-continuous Hausdorff topology $\tau$ in which every sequence $s \in \text{dom}(\ell)$ converges to the point $\ell(s)$;
3. The following two properties hold:
   a. for any $s \in \text{dom}(\ell)$, $a, b \in X^1$ and $x \in X$ with $x \neq a \cdot \ell(s) \cdot b$ the set $\{n \in \omega: a \cdot s_n \cdot b = x\}$ is finite;
   b. for any $s, t \in \text{dom}(\ell)$ and $a, b, c, d \in X^1$ with $a \cdot \ell(s) \cdot b \neq c \cdot \ell(t) \cdot d$ there exists a finite subset $F \subset \omega$ such that $a \cdot s_n \cdot b \neq c \cdot t_m \cdot d$ for all $n, m \in \omega \setminus F$.

For commutative semigroups, Theorem 4.1 has a bit simpler form.

**Theorem 4.2.** For a countable commutative semigroup $X$ and a function $\ell: \text{dom}(\ell) \to X$ defined on a countable subset $\text{dom}(\ell) \subset X^\omega$, the following conditions are equivalent:

1. The semigroup $X$ admits a shift-continuous metrizable topology $\tau$ in which every sequence $s \in \text{dom}(\ell)$ converges to the point $\ell(s)$;
2. The semigroup $X$ admits a shift-continuous Hausdorff topology $\tau$ in which every sequence $s \in \text{dom}(\ell)$ converges to the point $\ell(s)$;
3. The following two properties hold:
   a. for any $s \in \text{dom}(\ell)$, $a \in X^1$ and $x \in X$ with $x \neq a \cdot \ell(s)$ the set $\{n \in \omega: a \cdot s_n = x\}$ is finite;
   b. for any $s, t \in \text{dom}(\ell)$ and $a, b \in X^1$ with $a \cdot \ell(s) \neq b \cdot \ell(t)$ there exists a finite subset $F \subset \omega$ such that $a \cdot s_n \neq b \cdot t_m$ for any $n, m \in \omega \setminus F$.

5. Convergent sequences in semilattices

Applying Theorem 4.2 to semilattices we obtain the following characterization.

**Theorem 5.1.** For a countable semilattice $X$ and a function $\ell: \text{dom}(\ell) \to X$ defined on a countable subset $\text{dom}(\ell) \subset X^\omega$, the following conditions are equivalent:

1. The semilattice $X$ admits a shift-continuous metrizable topology $\tau$ in which every sequence $s \in \text{dom}(\ell)$ converges to the point $\ell(s)$;
2. The semilattice $X$ admits a shift-continuous Hausdorff topology $\tau$ in which every sequence $s \in \text{dom}(\ell)$ converges to the point $\ell(s)$;
3. The following two conditions hold:
   a. for any $s \in \text{dom}(\ell)$, $a \in X$ and $x \in X$ with $x \neq a \cdot \ell(s)$ the set $\{n \in \omega: a \cdot s_n = x\}$ is finite;
   b. for any $s, t \in \text{dom}(\ell)$ and $a, b \in X$ with $a \cdot \ell(s) \neq b \cdot \ell(t)$ there exists a finite subset $F \subset \omega$ such that $a \cdot s_n \neq b \cdot t_m$ for any $n, m \in \omega \setminus F$.

**Proof.** The implications $(1) \Rightarrow (2) \Rightarrow (3)$ are trivial. The implication $(3) \Rightarrow (1)$ will follow from Theorem 4.2 as soon as we check that the condition (3) of Theorem 5.1 implies condition (3) of Theorem 4.2. So, assume that condition (3) of Theorem 5.1 is satisfied.

To check the condition (3a) of Theorem 4.2 take any sequence $s \in \text{dom}(\ell)$ and element $a \in X^1$ and $x \in X$ with $x \neq a \cdot \ell(s)$. If $a \in X$, then the set $\{n \in \omega: a \cdot s_n = x\}$ is finite by the
condition (3a) of Theorem 5.1. So, we assume that \( a \) is an external unit for \( X \). In this case \( \{ n \in \omega : a \cdot s_n = x \} = \{ n \in \omega : s_n = x \} \subset \{ n \in \omega : x \cdot s_n = x \cdot x = x \} \). Assuming that the set \( \{ n \in \omega : s_n = x \} \) is infinite, we conclude that the set \( \{ n \in \omega : x \cdot s_n = x \} \) is infinite, which implies that \( x \cdot \ell(s) = x \). Since \( \ell(s) \cdot \ell(s) = \ell(s) = a \cdot \ell(s) \neq x = \ell(s) \cdot x \), the set

\[
\{ n \in \omega : \ell(s) \cdot s_n = x \} = \{ n \in \omega : \ell(s) \cdot s_n = \ell(s) \cdot x \} \supset \{ n \in \omega : s_n = x \}
\]

is finite, which contradicts our assumption.

Next, we check the condition (3b) of Theorem 5.1. Given any sequences \( s, t \in \text{dom}(\ell) \) and elements \( a, b \in X^1 \) with \( a \cdot \ell(s) \neq b \cdot \ell(t) \), we need to find a finite set \( F \subset \omega \) such that \( a \cdot s_n \neq b \cdot t_m \) for any \( n, m \in \omega \setminus F \). The condition (3a) of Theorem 5.1 ensures that \( a \) or \( b \) does not belong to \( X \). We lose no generality assuming that \( a \notin X \) and hence \( a \) is the external unit to \( X \). In this case the inequality \( a \cdot \ell(s) \neq b \cdot \ell(t) \) transforms into the inequality \( \ell(s) \neq b \cdot \ell(t) \). We claim that there exists an element \( c \in X \) such that \( c \cdot \ell(s) \neq b \cdot \ell(t) \). If \( \ell(s) \neq \ell(s) \cdot b \cdot \ell(t) \), then put \( c = \ell(s) \). If \( \ell(s) = \ell(s) \cdot b \cdot \ell(t) \), then put \( c = b \cdot \ell(t) \) and conclude that \( c \cdot \ell(s) = \ell(s) \neq b \cdot \ell(t) = cb \cdot \ell(t) \).

In both cases we get \( c \cdot \ell(s) \neq cb \cdot \ell(t) \). By condition (3b) of Theorem 5.1, there exists a finite set \( F \subset \omega \) such that \( c \cdot s_n \neq cb \cdot t_m \) and hence \( s_n \neq b \cdot t_m \) for any \( n, m \in \omega \setminus F \).

A topology on a semilattice \( X \) is called Lawson if it has a base consisting of open subsemilattices.

**Example 5.2.** There exists a countable semilattice \( X \) and a function \( \ell : \{ s, t \} \rightarrow X \) defined on a subset \( \{ s, t \} \subset X^\omega \) such that

1. The semilattice \( X \) admits a shift-continuous metrizable topology \( \tau \) in which the sequence \( s \) converges to \( \ell(s) \) and the sequence \( t \) converges to \( \ell(t) \).
2. The semilattice \( X \) admits no Lawson Hausdorff topology \( \tau \) in which the sequence \( s \) converges to \( \ell(s) \) and the sequence \( t \) converges to \( \ell(t) \).

**Proof.** By [3, 2.21], there exists a compact metrizable topological semilattice \( K \) containing two points \( x, y \in K \) such that for any neighborhoods \( O_x, O_y \subset K \) of the points \( x, y \) there exists a finite subset \( F \subset O_x \) such that \( \inf F \in O_y \). Fix countable neighborhood bases \( \{ V_n \}_{n \in \omega} \) and \( \{ W_n \}_{n \in \omega} \) at the points \( x, y \), respectively. For every \( n \in \omega \) choose a finite subset \( F_n \subset \bigcap_{k \leq n} V_k \) such that \( \inf F_n \in \bigcap_{k \leq n} W_k \). Let \( s \in X^\omega \) be a sequence such that \( s_n = \inf F_n \) for every \( n \in \omega \) and \( t \in X^\omega \) be a sequence such that \( F_n = \{ t_k : \sum_{i \leq n} |F_i| < k \leq \sum_{i \leq n} |F_i| \} \) for every \( n \in \omega \). Let \( \ell(s) = y \) and \( \ell(t) = x \).

Let \( X \) be the countable semilattice generated by the countable set \( \{ x, y \} \cup \{ \omega \} \cup \{ t |\omega | \} \). The metrizable topology on \( X \) inherited from \( K \) satisfies that the condition (1) is satisfied.

It remains to prove that \( X \) admits no Lawson Hausdorff topology \( \tau \) in which the sequence \( s \) converges to \( \ell(s) = y \) and the sequence \( t \) converges to \( \ell(t) = x \). To derive a contradiction, assume that such topology \( \tau \) exists. Then the points \( x, y \) have disjoint open neighborhoods \( O_x, O_y \in \tau \) such that \( O_x \) is a subsemilattice of \( X \). Since the sequences \( s \) and \( t \) converge to \( y \) and \( x \), respectively, there exists \( n \in \omega \) such that \( s_k \in O_x \) and \( t_k \in O_x \) for all \( k \geq n \). Then \( F_n \subset \{ t_k \}_{k \geq n} \subset O_x \) and hence \( \inf F_n \in O_x \) (as \( O_x \) is a subsemilattice of \( X \)). On the other hand, \( \inf F_n = s_n \in O_y \). Then \( \inf F_n \in O_x \cap O_y \), which contradicts the choice of the neighborhoods \( O_x \) and \( O_y \).

\[ \square \]

6. The Example

In this section we shall apply Theorem 5.1 to construct an example of a metrizable semitopological semilattice with dense non-closed partial order.

Consider the semilattice \( \{ 0, 1, 2 \} \) endowed with the operation of taking minimum. In the semilattice \( \{ 0, 1, 2 \}^\omega \) consider the countable subsemilattice \( X \) consisting of functions \( f : \omega \rightarrow \{ 0, 1, 2 \} \) having non-empty finite support \( \text{supp}(f) := f^{-1}(\{ 0, 1 \}) \).

It is easy to see that the partial order on \( \{ 0, 1, 2 \} \) induced by the semilattice operation (of minimum) coincides with the usual linear order on \( \{ 0, 1, 2 \} \). Then the semilattice operation (of coordinatewise minimum) on \( X \subset \{ 0, 1, 2 \}^\omega \) induces the natural partial order on \( X \).
For every $n \in \omega$ consider the functions $0_n, 1_n \in X$ defined by
\[
0_n(i) = \begin{cases} 
0 & \text{if } i = n \\
2 & \text{otherwise}
\end{cases} \quad \text{and} \quad 1_n(i) = \begin{cases} 
1 & \text{if } i = n \\
2 & \text{otherwise}.
\end{cases}
\]
It is clear that $0_n \leq 1_n$ and hence $0_n \leq 1_n$ for all $n \in \omega$.

**Theorem 6.1.** The semilattice $X$ admits a metrizable shift-continuous topology $\tau$ such that the set $\{(0_n, 1_n) : n \in \mathbb{N}\}$ is dense in the square $X \times X$ of the semitopological semilattice $(X, \tau)$. Since
\[
\{(0_n, 1_n)\}_{n \in \omega} \subseteq P := \{(x, y) \in X \times X : xy = x\} \neq X \times X
\]
the partial order $P$ of $X$ is dense and non-closed in $X \times X$.

**Proof.** Write the countable set $X \times X$ as $X \times X = \{(x_k, y_k) : k \in \omega\}$. For every $k, n \in \omega$ consider the elements $z_{k,n} = 0_{2^k3^n}$ and $u_{k,n} = 1_{2^k3^n}$ of the set $X$. These elements form sequences $\tilde{z}_k = (z_{k,n})_{n \in \omega}$ and $\tilde{u}_k = (u_{k,n})_{n \in \omega}$, which are elements of the set $X^\omega$. Let $\text{dom}(\ell) = \{\tilde{z}_k, \tilde{u}_k : k \in \omega\}$ and $\ell : \text{dom}(\ell) \to X$ be the function defined by $\ell(\tilde{z}_k) = x_k$ and $\ell(\tilde{u}_k) = y_k$ for $k \in \omega$.

We claim that for the function $\ell$ the condition (3b) of Theorem 5.1 is satisfied. In fact, the condition (3a) is satisfied in the stronger form: for any $s \in \text{dom}(\ell)$ and $a, b \in X$, the set $\{n \in \omega : a \cdot s_n = b\} \subseteq \{n \in \omega : 2k \in \omega \text{ with } 2^k3^n \in \text{supp}(b)\}$ is finite.

Now we check the condition (3b). Fix elements $a, b \in X$ and sequences $s, t \in \text{dom}(\lambda)$ such that $a \cdot t(s) = b \cdot \ell(t)$. It is easy to see that $a \cdot s_n \neq b \cdot t_m$ for any $n, m \in \omega \setminus F$ where $F = \{n \in \omega : \exists k \in \omega \text{ such that } 2^k3^n \in \text{supp}(a) \cup \text{supp}(b)\}$.

By Theorem 5.1 the semilattice $X$ admits a metrizable shift-continuous topology $\tau$ in which every sequence $s \in \text{dom}(\ell)$ converges to $\ell(s)$. In particular, for every $k \in \omega$ the sequence $(0_{2^k3^n})_{n \in \omega} = \tilde{z}_k$ converges to $x_k = \ell(\tilde{z}_k)$ and the sequence $(1_{2^k3^n})_{n \in \omega} = \tilde{u}_k$ converges to $y_k = \ell(\tilde{u}_k)$. Consequently, the set $\{(0_{2^k3^n}, 1_{2^k3^n}) : k \in \omega\} \subset \{(0_m, 1_m) : m \in \omega\}$ is dense in $X \times X = \{(x_k, y_k) : k \in \omega\}$.

Since
\[
\{(0_n, 1_n)\}_{n \in \omega} \subseteq P := \{(x, y) \in X \times X : xy = x\} \neq X \times X,
\]
the partial order $P$ is a dense non-closed subset of $X \times X$. \qed

**Problem 6.2.** Does the semilattice $X$ admit a Lawson Hausdorff shift-continuous topology such that the partial order $P = \{(x, y) \in X \times X : xy = x\}$ is not closed (and dense) in $X \times X$?

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