Abstract. Dynamically typed object-oriented languages enable programmers to write elegant, reusable and extensible programs. However, with the current methodology for program verification, the absence of static type information creates significant overhead. Our proposal is two-fold: First, we propose a layer of abstraction hiding the complexity of dynamic typing when provided with sufficient type information. Since this essentially creates the illusion of verifying a statically-typed program, the effort required is equivalent to the statically-typed case. Second, we show how the required type information can be efficiently derived for all type-safe programs by integrating a type inference algorithm into Hoare logic, yielding a semi-automatic procedure allowing the user to focus on those typing problems really requiring his attention. While applying type inference to dynamically typed programs is a well-established method by now, our approach complements conventional soft typing systems by offering formal proof as a third option besides modifying the program (static typing) and accepting the presence of runtime type errors (dynamic typing).

1 Introduction

Dynamically typed programming languages refrain from restricting their programs to ensure operations are only applied to suitable operands. While this allows experienced programmers to write more elegant, concise and reusable code, it has the obvious drawback that type errors may occur at runtime.

Recently, object-oriented dynamically typed languages like Python, Ruby and JavaScript are gaining popularity also on the server-side (Ruby on Rails, node.js) and are used even for business- [27] and safety-critical [3] applications.

Unfortunately, despite the growing need for correctness guarantees, the lack of type information causes a large overhead in formal methods like Hoare logic and severely decreases the effectiveness of automatic reasoning engines compared to the statically-typed setting (see Section 2.1).
There are two ways to deal with this problem:

1) **Annotation**: Most contemporary approaches to verifying dynamically typed programs ask the user to manually supply the needed type information in loop invariants and method contracts [12,23,29,21]. For larger programs, this induces significant overhead. We argue that manually supplying type information for all variables is not only tedious, but also often unnecessary, as most of this information could have been inferred automatically.

2) **Translation**: Obviously, translating the dynamically typed program into an equivalent statically typed version\(^1\) and then using a Hoare logic for statically-typed programs (like [2,5]) for verification is also possible. In such a translation process, type inference algorithms like [17,11] are usually of significant help. Note, however, that gradual typing [26,4] it not useful in this context, as such Hoare logics require the entire program to be well-typed prior to verification. Additionally, this approach removes any benefit of dynamic typing since it is equivalent to verifying a statically typed language with type inference.

We propose to get the best of both worlds by integrating an automatic type safety verifier with Hoare logic into a semi-automatic procedure and using the derived type information to reduce overhead and enable effective automated reasoning about dynamically typed programs just like with statically typed ones. In the context of soft typing [6], our approach can also be understood as offering proofs of type safety as a third option besides rewriting the program (static typing) and runtime-checks (dynamic typing).

Concretely, in this paper we describe two components:

1) A layer of abstraction that, given suitable type information, abstracts from the complexities of dynamic typing and hence reduces the verification of dynamically typed programs to that of statically typed ones. This also works with partial type information on a per-expression basis (see Section 2.1).

2) A construction for complementing a Hoare logic with an automatic type safety verifier, yielding a semi-automatic procedure for deriving type information with the following properties (see Section 2.4):

- Automation – only typing problems beyond the reach of the automatic verifier require manual intervention.
- Completeness relative to the Hoare logic – if the Hoare logic is complete, then type information can be derived for all typesafe programs (see Section 5.2).
- Bidirectional exchange of results – automatically derived type information can be used in Hoare logic proofs and vice versa, proof results are used by the automatic verifier to increase precision.

Together, these two components form a novel verification system that makes the effort additionally required to verify a dynamically typed program proportional to the total complexity of hard typing problems in this program. Unlike

\(^{1}\) After this translation, the static type system should be able to ensure the absence of type errors, unlike in the embeddings discussed in [15]. Finding such an equivalent version is undecidable in general and hence requires manual effort (see Section 2.2)
in annotation-based-approaches, programs with only trivial typing problems require no additional effort and unlike in translation-based-approaches, all typesafe programs can be verified.

This paper constitutes our first step towards connecting the (relative) complete Hoare logics [2,5] and advanced reasoning engines developed for statically-typed object-oriented languages with the advancing automatic type safety verifiers for dynamically typed languages [21,29,9,17]. In this extended version, proofs for all theorems and lemmas can be found in Appendix E.

Notation $p$ is a sequence $p_1, ..., p_n$ where $n$ is obvious from context or does not matter, $\{p\}$ the smallest set containing all its elements and $a \cdot b$ sequence concatenation. $\triangleq$ means “is defined as” and $\mathbb{N}_n \triangleq \{0, ..., n\}$, $\mathbb{N}_n^1 \triangleq \{1, ..., n\}$.

2 Overview / Motivation

We will first discuss how correctness proofs can be simplified using sufficient type information and then how this information can be derived.

2.1 Static- vs. Dynamically-typed Hoare Logic

Apart from the additional need to establish type safety, there are other differences between Hoare logic for dynamically typed- and statically typed languages ($HL_d$ and $HL_s$). The latter (like [2,5]), usually share a type system between programming- and assertion language: the assertion $x > 8$ denotes the set of states where the value of a numeric program variable $x$ is larger than 8. In $HL_d$ (like [12]) however, as types are not statically known, all variables are of type $\mathbb{O}$ (object). The assertion $x > 8$ is hence meaningless as $>$ is not defined for type $\mathbb{O}$. In this setting, a similar set of states can be denoted by the assertion\footnote{the precise meaning of $\mathbb{N}(x, i)$ will be explained in Subsection 4.2.} $\exists i. \mathbb{N}(x, i) \land i > 8$ which can be automatically derived from $x > 8$ given sufficient type information (the fact that the object referenced by $x$ always represents a number).

Furthermore, $HL_s$ usually include side-effect-free (pure) program expressions ($e$) into the assertion language, allowing efficient reasoning using proof rules like

$$\{q[u := e]\}u := e\{q\}$$

Here, $q[u := e]$ denotes the substitution of all occurrences of a variable $u$ by $e$ in the assertion $q$. This rule allows directly deducing weakest preconditions over assignments like $\{x + 5 > 8\}x := x + 5\{x > 8\}$ (1) by letting the expression $e$ traverse the boundary between program and logic. In $HL_d$, this is not possible since program expressions could have side-effects. While a subset of side-effect-free methods can be defined, identifying such pure expressions requires type information. Without it, establishing a property equivalent to (1) requires $\geq 6$ rule applications.
This observation is given significance by the fact that usually most expressions only involve immutable data types like numbers and strings. Regarding them as side-effecting operations on general object-structures not only complicates proofs, but also significantly decreases the effectiveness of automated reasoning engines. For instance, assertions can often be efficiently established by SMT solvers over Presburger arithmetic while no similar decision procedure exists for arbitrary operations over general object-structures.

Section 4 will show how type information can be used to counter these problems and create the illusion of proving a statically typed program.

2.2 Providing Type Information

Sufficient type information for dynamically typed programs is uncomputable in general (Section 5.1). However, a number of good approximations exist [17,11] that we will refer to as automatic type safety verifiers.

It is known that many dynamically typed programs only occasionally diverge from what would also be possible in static typing disciplines and consequently, that the output of such algorithms is usually sufficient for typing most of their subexpressions [17, Section 5][11, Section 6].

If the entire program can be typed by a sound automatic verifier, then HLs could be applied. However, the whole point of dynamic typing is the possibility to go beyond the limits of such automatic procedures (type systems). Approaches to verifying these languages thus must also be able to operate under less ideal circumstances. The following example will illustrate this point.

2.3 The Evaluator Example

Figure 1 depicts a dynamically typed program evaluating arithmetic expressions. While crafted to provide a hard typing problem, its use of ad-hoc data structures is not uncommon in Ruby, Python or Javascript.

The class \texttt{Evaluator} has two methods \texttt{parse()} and \texttt{calc()}. The former parses a string and stores the resulting parse tree in the instance variable \texttt{@tree}, while the latter evaluates a given parse tree (defaulting to \texttt{@tree}) over a given environment (a mapping from variable names (strings) to integers).

The example is hard to type because the parse trees are represented as ad-hoc constructions of nested lists. Numeric constants \texttt{VALUE, VAR} and \texttt{OP} in the first element distinguish value-, variable- and operation nodes. The types of the remaining list elements depend on these node types: the second element is numeric (the value) for value-nodes, a string (the variable name to be looked up in the environment) for var-nodes and numeric (representing the operation to be performed) for op-nodes. Only op-nodes use nesting: further list elements are sub-parse-trees that are to be recursively evaluated to operands.

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3 Advanced dynamic features like mixins, traits, method update and dynamic class hierarchies increase the complexity of type inference. However, in this paper we aim to study the problem of dynamic typing in isolation and leave them as future work.
class Evaluator {
    method parse(str) { ... }

    method calc(env, tree = @tree) {
        if tree[0] = VALUE then tree[1]
        elseif tree[0] = VAR then env[tree[1]]
        elseif tree[0] = OP then
            if tree[1] = ADD then calc(env, tree[2]) + calc(env, tree[3])
            elseif ...
            else nil
        fi
        else nil
    fi
}
}
new Evaluator().parse(input).calc(ENV)

Fig. 1. Relevant part of the evaluator example source code

Typing this example requires deducing precise types for heterogeneous lists from propositions (like $\text{tree}[0] = \text{VALUE}$) about their first element. To the best of our knowledge there is no automatic procedure able to establish such implications. Also note that the typing problem can be made even harder: allowing an arbitrary number of operands in op-nodes, returning strings instead of null, etc. This example will be used to demonstrate our technique.

2.4 Semi-Automation

In the concept depicted in Figure 2, the correctness proof is split into two “layers” (see Section 6.3). While the user (supported by a theorem prover) derives his proof in the higher layer, the lower layer contains type information and is created and modified solely by the automatic type safety verifier. For this purpose, the typings ($ty$) derived for the program $\pi$ by the verifier are translated into proofs (see Section 6.2). While the information contained in this lower layer proof is already useful for supporting the user’s higher-layer proof (see
Section 4), the user may at any time decide to refine it by deriving more precise type information in the higher layer. This information is filtered to make it interpretable for the verifier and then supplied as trusted assumptions to refine the lower-layer type information (see Sections 6.1, 6.3).

Note that deriving type information and using the layer of abstraction is not a strict 2-step process. The latter requires the former only on a per-expression basis, allowing an interleaving of steps. Concretely, the layer of abstraction applies to all expressions proven type-safe (see Section 4). Expressions with open typing problems may be included at any time by proving them type-safe. This interleaving is possible as our refinements are monotonic (see Section 6.3).

3 Setting

3.1 Model Languages: Dyn and Stat

To explain our methodology in a setting facilitating formal proof we introduce a pair of minimalistic programming languages that differ only in the fact that one is dynamically typed (dyn) while the other uses a static type system with type inference (stat). Like their real-world siblings, the two are imperative, class-based object-oriented languages including inheritance, method renaming, dynamic dispatch and constructors. However, they do not support advanced dynamic features like a dynamic class hierarchy, method update or eval().

Syntax of dyn:

```
Class ≡ class S
Meth ≡ class C \{ meth \}
Expr ≡ e
Stmt ≡ S
```

Basic data types in stat:

```
true, false ∈ Cnst, \( \Delta : \text{Int} \rightarrow \text{Int} \in \text{Op} \)
```

Fig. 3. Syntax of dyn and stat

Syntax: The syntax of both dyn and stat is depicted in Figure 3. In dyn, method bodies consist of statements (S) which in contrast to expressions (e) can contain sequential composition. Expressions are composed of null, the only
constant, local- and instance variables (prefixed with @), the self-reference this, operators for object identity and dynamic type checks, method- and constructor calls, assignments, conditionals and while loops. Note that equality (=) is desugared to a (class-specific) method call, while object identity (==) is a built-in operation yielding true iff the two expressions refer to the same object (We stipulate null == null yields true).

Each class except the predefined class object must specify a parent class whose methods are inherited. The inheritance relation must be acyclic. Every class thus transitively inherits from object. Inherited methods may be overwritten or renamed (using rename). Like in actual dynamically typed languages, inheritance is mere code reuse and can be removed using an automatic expansion step [22]. Furthermore, we will assume this step to be completed and not concern ourselves any further with inheritance or renaming.

**Semantics:** Both dyn and stat programs consist of a main statement $S$ and sets of classes $C$, methods $M$ and variables $V = V_L \cup V_I$ where $V_L$ and $V_I$ are the sets of local- and instance variables respectively. While each class $C \in C$ has a subset of method declarations $M_C \subseteq M$ and instance variables $V_C \subseteq V_I$, every method $C.m \in M$ has a subset of local variables $V_{C,m} \subseteq V_L$ used in its method body $S_{C,m}$. $V_S = \{ \text{this, r} \} \subseteq V_L$ is a set of special variables. While this references the current object and is not allowed to be assigned to in programs, r holds the result of the last evaluated expression and cannot be used in programs.

**Dyn**'s value domain is the set of all objects $D_d = D \emptyset$ and its type system is the lattice of union types represented as sets of class names $\{ C_1, ..., C_n \} \in T_d = 2^C$ with the subset-ordering $\subseteq$ (see Figure 4). The null value is contained in every such type. **Stat** on the contrary distinguishes basic data types $T_s = \{ \emptyset, \text{null} \}$ and its value domain $D_s \cong \bigcup_T D_T$ includes objects, numbers, booleans, strings, lists and finite maps. We omit definitions of states, state update etc. as they are standard. To keep track of instance-class relationships we use class references and for every class $C$ introduce a distinct object $\rho_C$ as well as a special instance variable @c such that $o.@c = \rho_C$ iff $o$ is an instance of class $C$. Using @c in programs is not permitted.

Comparing Dyn with Stat: Dyn is a pure object-oriented language (objects are the only values) while stat has basic data types. However, both provide the same constants and pure (i.e. side-effect-free) operations on them. Dyn desugars
them to constructor and method calls (see Figure 3), while \textit{stat} (like usual in statically typed languages) provides them build-in ($c_e$ and $op(c_e)$ in Figure 3).

Also, \textit{stat} expressions are pure. Side-effects are only allowed in statements, which must only have pure subexpressions. This is not a restriction, as every \textit{dyn}-expression can be transformed into a sequence of \textit{stat} statements by recursively (and in the order of evaluation) replacing subexpressions $e$ by fresh local variables $u$ and prepending the assignment $u := e$.

Every \textit{stat} program is also a \textit{dyn} program that evaluates to (an object-oriented version of) the same result. The only reason that the opposite direction does not hold is the language restriction imposed by \textit{stat}'s static type system.

\textbf{Type Errors:} Contrary to \textit{stat}, which rejects programs deemed unsafe at compile time, \textit{dyn} allows every syntactically correct program to be executed and raises type errors at runtime when

- a method call is not supported by its receiver (in this arity) or
- a condition of a conditional or while loop is not boolean

While "message not understood"-errors are fundamentally linked to type-checking in class-based OO-languages, dynamically typed languages often allow conditions to be of arbitrary type. Nevertheless, the second error condition models a common error class where a built-in operation supports a fixed set of types.

Many dynamically typed languages raise type errors when accessing variables prior to assignment. We will leave this as future work and consider all local (instance) variables to be initialized to \textit{null} prior to method executions (on instantiation). Also, type errors are often treated as exceptions, allowing interception and handling. For simplicity, we will consider them as fatal.

\subsection*{3.2 Hoare Logic}

The presentation of \textit{dyn} and \textit{stat}'s program logics closely follows [2,1]. We start by introducing the assertion language (Figure 5). Essentially, it is weak second order logic, extended with the same constants $c_e$, operations $op(T)$ and types used in \textit{stat}. It will be used to reason about both \textit{dyn} and \textit{stat}, however.

Assertions contain typed logical expressions ($l$). Such expressions consist of typed logical variables, local/instance program variables $u/l.x$ (of type $O$ in \textit{dyn} and of some type $T \in T_s$ in \textit{stat} / same, with $l$ being of type $O$) including \textit{this}, typed constants and typed operations. Contrary to program expressions, logical expressions can access instance variables of objects other than \textit{this}.

Logical expressions may only occur as parts of well-typed equations. Following [5], undefined operations like dereferencing a \textit{null} value or accessing a sequence with an index out of bounds ($l[n]$ with $n \geq |l|$) yield a \textit{null} value and equality is non-strict with respect to such values (\textit{null = null} is \textit{true}) to ensure a two-valued logic. Assertions are boolean combinations of such equations allowing quantification over finite sequences of elements of basic types.

We also introduce the following abbreviation for making reasoning about runtime types more convenient:
Asrt ⊨ p,q ::= [l] ∈ T, T ∈ T
[l] ∈ {C₁, ..., Cₙ} ⊨ l ≠ null → [l]@c = ρC₁ ∨ ... ∨ l.@c = ρCₙ

The reader may convince himself/herself that the following implications hold:

\[ J L K ∈ T \land J L K ∈ T \rightarrow J L K ∈ T_1 \sqcap T_2 \]
\[ J L K ∈ T \lor J L K ∈ T \rightarrow J L K ∈ T_1 \sqcup T_2 \]
\[ l₁ = l₂ → \exists T, [l₁] ∈ T \land [l₂] ∈ T \]

Selected differences between the Hoare-style axiomatic semantics for dyn and stat are contrasted in Figure 6. While the semantics for stat are standard⁴, the rules for dyn were modeled after [12]. Omitted rules are listed in Appendix B. In Hoare triples \{p\}S\{q\}, the special variable r is only allowed in the postcondition q and denotes the return value of S. The rules will be analyzed in the next section.

4 Layer of Abstraction

Let us compare the proof rules given in Figure 6. Obviously, the dyn rules are more complicated than their stat counterparts. Analyzing their differences, one can identify three core reasons why reasoning about dynamically typed programs is more complex than reasoning about statically typed ones.

1. Type safety: In Figure 6, the parts ensuring type safety are marked. Such type safety preconditions are unnecessary in statically typed languages.

2. Mapping objects to values: Hoare logic for dynamically typed languages often uses predicates to map between program objects and logical values. For instance, the COND rule has to use the predicate \( B() \) to establish a correspondence between the program expressions e and the logical expression b of type \( B \).

3. Side-effecting expressions: In the stat-rules ASGN and COND, pure program expressions \( e_ε \) and \( b_ε \) are directly used in logical assertions. Here, the clever design choice of a shared type system pays off. Unfortunately, dynamic typing forces us to relinquish this benefit, as the types of expressions are not statically known and impure expressions are ill-suited for logical reasoning. Observe also how dyn’s METH rule models the evaluation order using a sequence of intermediate predicates \( p_i \), which would not be necessary for pure expressions. However, since dyn treats operations as method calls, the METH rule needs to be applied even for pure operations like +, <, ∧, etc, making properties of assignments and conditionals even more tedious to derive.

The following sections will explain how the layer of abstraction mitigates these issues.

4.1 Type Safety Preconditions

Like already mentioned, the fact that type errors are runtime events in dynamically typed languages gives rise to the following notion of correctness:

⁴ they closely follow other Hoare logics for statically typed languages [2,5,1]
where $m$ is a method and $\mathbf{true}$ a predicate and

$\exists \mathbf{v} : \mathcal{T}^* : \mathbf{v}' = 1 \land p[v/v'[0]]$, $\forall \mathbf{v} : \mathcal{T} \mathbf{p} \triangleq \exists \mathbf{v} : \mathcal{T} \mathbf{p}$

Fig. 5. Syntax of the assertion language

Hoare logic rules for

### dyn

RULE: Assignment (ASGN)

\[
\begin{align*}
\{p\} e[v := r] & \quad \{p\} u := v[q] \\
\{p\} & \quad \{p[u := e] \} u := e\{p\}
\end{align*}
\]

RULE: Conditional (COND) (type-safe) partial correctness

\[
\begin{align*}
\{p\} e[r \land \text{bool_test}] & \quad \{r \land b\} S_1(q) \\
\{r \land \neg b\} & \quad S_2(q) \\
\{p\} & \quad \{p \land b\} S_1(q) \quad \{p \land \neg b\} S_2(q) \\
\{p\} \text{ if } e \text{ then } S_1 \text{ else } S_2 \text{ if } q
\end{align*}
\]

where $b$ is a predicate and $\text{bool_test} \triangleq [r] \in \{\text{bool}\} \land \mathbb{B}(r, b)$

RULE: Method Call (METH)

\[
\begin{align*}
\{p_i\} e_i(p_{i+1}[u_i := r]) & \quad \text{for } i \in \mathbb{N}_n \\
\{p_{i+1}\} u_0.m(u_1, \ldots, u_n) & \quad \{q\}
\end{align*}
\]

\[
\begin{align*}
\{p_i\} e_0.m(e_1, \ldots, e_n) & \quad \{q\}
\end{align*}
\]

where $u_i \in \mathcal{V}_L$ fresh, $u_i \notin \text{var}(e_j) \cup \text{change}(e_j)$ for all $i, j \in \mathbb{N}_n$.

RULE: Recursion (REC) (type-safe) partial correctness

\[
A \vdash \{p\} S(q), \\
A \vdash \{p_i\} \begin{array}{l}
\text{begin local this, } \overline{u_i} := \overline{v_i}, \overline{v_i} ; S_i \end{array} \{q_i\}, i \in \mathbb{N}_n
\]

\[
\begin{array}{l}
p_i \rightarrow [v_i] \in \{C_i\}, i \in \mathbb{N}_n
\end{array}
\]

\[
\{p\} S(q)
\]

where method $m_i(\overline{u_i})S_i \in \mathcal{M}_{C_i}$, $A \equiv \{p_1\} v_1.m_1(\overline{v_1})q_1, \ldots, \{p_n\} v_n.m_n(\overline{v_n})q_n$.

Fig. 6. Comparison of dynamically typed and statically typed Hoare logic rules
Definition 1 (Type-safe partial correctness). \( \{ p \} S \{ q \} \) holds in the sense of type-safe partial correctness (written \( \models_{\tau} \{ p \} S \{ q \} \)) if every non-diverging, non-failing computation starting in a state satisfying \( p \) does not abort with a type error, but ends in a state satisfying \( q \).

The preconditions particular to proof rules for type-safe partial correctness are called type safety preconditions. Being orthogonal to preventing divergence (\( t \)) and failures (\( f \)) (calling a method on a null value) as well as ensuring the post-condition (\( p \)), these correctness notions may be freely combined. Total correctness would hence be denoted \( \tau tfp \). In the proof rules given in Figure 6 (and Appendix B), \( \tau \)-preconditions are marked.

In \( HL_s \), type-safety preconditions are unnecessary. Regarding such preconditions, correctness proofs in statically-typed languages resemble those in dynamically typed languages for type-unsafe correctness notions. Omitting these preconditions hence is a first step in proving dynamically typed programs like statically-typed ones. This can be achieved by treating type safety issues separately from other correctness issues.

Definition 2 (Decomposition). The following rule is added to the proof system for a type-safe notion of correctness (\( \tau X \)):

\[
\begin{align*}
\vdash_X \{ p \} S \{ q \} \\
\vdash_{\tau p} \{ p \} S \{ \text{true} \}
\end{align*}
\]

where \( \vdash_X \) refers to the corresponding type-unsafe variant of the proof system while \( \vdash_{\tau p} \) always refers to the proof system for type-safe partial correctness.

Correctness of the decomposition rule follows directly from the semantic definition for type-safe partial correctness. Intuitively, it states that whenever \( \models_{\tau p} \{ p \} S \{ \text{true} \} \) and the precondition \( p \) have been established for some statement \( S \), we can omit type safety preconditions when reasoning about \( S \), although our program is dynamically-typed.

4.2 Mapping Objects to Values

Mapping predicates are a further peculiarity of \( HL_d \). However, when the types of all used variables are known, those predicates can be generated automatically. We will now provide a “virtual” variable \( \hat{u} \) of the corresponding base type for each object variable \( u \) that can be safely mapped.

First, a subset of “pure” (i.e. immutable) classes \( C_{\tau} \subseteq C \) along with a function \( \Psi \) mapping classes from \( C_{\tau} \) to corresponding base types \( T \in T_s \) of the assertion language must be defined. For \textbf{dyn}, this mapping is

\[
\Psi(\text{num}) = \mathbb{N}, \Psi(\text{bool}) = \mathbb{B}, \Psi(\text{list}) = \mathbb{L}, ...
\]

The mapping can be extended to union types \( T \in T_d \) by defining

\[
\Psi(\{\}) = \text{Null}, \Psi(\{C\}) = \Psi(C) \text{ for } C \in C_{\tau} \text{ and } \Psi(T) = \emptyset \text{ otherwise.}
\]
Translating Assertions:

For each type \( T \in \mathcal{T}_s \), there is usually already a mapping predicate \( T(o, v) : \mathcal{O} \times \mathcal{D}_T \rightarrow \mathcal{B} \) for mapping objects to values as well as a safety predicate \( \text{safe}_T(o) : \mathcal{O} \rightarrow \mathcal{B} \) defining under what condition this mapping is safe. For \( \mathbb{N} \) these are\(^5\)

\[
N(o, n) \triangleq \text{safe}_N(o) \rightarrow (o.@pred = \text{null} \rightarrow n = 0 \land \\
\text{o.@pred} \neq \text{null} \rightarrow N(o.@pred, n - 1)) \quad \text{and} \\
\text{safe}_N(o) \triangleq o \neq \text{null} \land \llbracket o \rrbracket \in \{\text{num}\}
\]

We then introduce a new assertion language \( \text{Asrt}_T \) allowing the use of automatically mapped virtual variables \( \hat{x} \). Its semantics is defined in terms of a mapping \( Y : \text{Asrt}_T \rightarrow \text{Asrt} \) to the old assertion language.

**Definition 3 (Automatic Variable Mapping).**

Let \( x_1, \ldots, x_n \) be a sequence of variables that can be safely mapped to types \( T_1, \ldots, T_n \) and for which \( \hat{x}_i \) occur free in \( p \) for \( i \in \mathbb{N}_n^1 \). Also, let \( v_{\hat{x}_1}, \ldots, v_{\hat{x}_n} \) be a corresponding sequence of logical variables of types \( T_1, \ldots, T_n \). Then,

\[
Y(p) \triangleq \exists v_{\hat{x}_1} : T_1, \ldots, v_{\hat{x}_n} : T_n. \ Y_S(p) \land Y_M(p) \\
Y_S(p) \triangleq \{x_1, \ldots, \hat{x}_n := v_{\hat{x}_1}, \ldots, v_{\hat{x}_n} \}, \ Y_M(p) \triangleq T_1(x_1, v_{\hat{x}_1}) \land \ldots \land T_n(x_n, v_{\hat{x}_n})
\]

The precise definition of which variables can be “safely mapped” depends on the type information available. For the verifier that will be discussed in Section 6.1, the \( x_i \) may be local variables \( u \) or instance variables of the current object this.@x. Note that \( \text{Asrt}_T \) conservatively extends \( \text{Asrt} \), as any assertion \( p \in \text{Asrt} \) is mapped to itself. We hence assume \( Y \) to be implicitly applied to all assertions, enabling the pervasive use of automatic object mapping. For instance, assuming that \( \text{safe}_U(u) \) could be established in the lower layer, the \( Y \)-assertion \( u < 5 \) can be used instead of the equivalent \( \exists v_U : \mathbb{N}, v_U < 5 \land \mathbb{N}(u, v_U) \). To formally show that the automatic object mapping allows us to trivially map \( \text{stat} \) assertions into \( \text{dyn} \) assertions, we need a mapping \( \Theta \) between their states.

**Translating States:** \( \Theta(\sigma_x) \triangleq \sigma_d \) where \( \sigma_d \) is derived from \( \sigma_x \) by introducing for every base type \( T \in \mathcal{T}_s \setminus \{\mathcal{O}, \text{Null}\} \) a (possibly infinite) set of objects \( \{o, | v \in \mathcal{D}_T \land T(o, v)\} \) and substituting every variable \( x \) of base type \( T \), holding the value \( v \in \mathcal{D}_T \) by a variable \( x \) of type \( T \), referencing the object \( o_v \). Furthermore, for each base type \( T \in \mathcal{T}_s \setminus \{\mathcal{O}, \text{Null}\} \), we identify any two objects \( o_1, o_2 \) iff \( T(o_1, v_1) \), \( T(o_2, v_2) \) and \( v_1 = v_2 \). We lift this equivalence to \( \text{dyn} \) states in the natural way.

**Translating Assertions:** \( \Theta(p) \triangleq p[x_1, \ldots, x_n := \hat{x}_1, \ldots, \hat{x}_n] \) where \( x_i \) are all variables that can be safely mapped and occur free in \( p \).

**Theorem 1.** For all assertions \( p \) and \( \text{stat} \) states \( \sigma : \sigma \models p \iff \Theta(\sigma) \models \Theta(p) \).

The automatic mapping requires safety predicates to be pre-established in the lower layer, which requires both type information and tracking of null values.

### 4.3 Pure Expressions

\( HL_o \) allow highly effective reasoning by including (syntactically identified) pure program expressions into their logical assertions. In this section, we will show

\(^5\) Expressing them using quantification over sequences instead of recursion is possible, but less readable.
that assuming the availability of type information in the lower layer, this concept is also applicable to dynamically-typed languages.

To define a pure subset of dyn expressions, one complements the set of “pure” classes $C_e$ with a set of “pure” (i.e. side-effect-free) methods $M_e \subseteq M$ and extends the function $\Psi$ to also map method- and constructor calls to corresponding logical expressions. Such an expression $l \in LExp$ of type $T$ with free variables $v_0,...,v_n$ of types $T_0,...,T_n$ can be interpreted as a function $f_l : T_0 \times \ldots \times T_n \mapsto T$. We hence denote its type as $LExp(T_0 \times \ldots \times T_n \mapsto T)$. The extension of the mapping $\Psi$ can then be written as follows:

For every pure operation $m$ of arity $n$: $\Psi : (T_0.m(T_1,...,T_n) \mapsto T) \mapsto LExp(T_0 \times \ldots \times T_n \mapsto T)$

For every pure constructor new $C$ of arity $n$: $\Psi : (\Psi(C).init(T_1,...,T_n) \mapsto T) \mapsto LExp(T_1 \times \ldots \times T_n \mapsto T)$

For the type $\mathbb{N}$ these are $\Psi(\mathbb{N}.init(\text{Null})) = 0$, $\Psi(\mathbb{N}.init(\mathbb{N})) = v_1 + 1$, $\Psi(\mathbb{N}.add(\mathbb{N})) = v_0 + v_1$, $\Psi(\mathbb{N}.succ()) = v_0 + 1$.

It is then possible to define a predicate $\text{pure}(e)$ automatically identifying pure expressions given type information for all variables used. $\Psi$ can be extended to map such pure program expressions to typed logical expressions. We denote the type of a pure expression by $\tau(e)$. Then, after establishing that $\{p | \hat{\Psi} = \Psi(T_0.m(T_1,...,T_n) \mapsto T)]u_0.m(u_1,...,u_n)\{p\}$

with $T_i = \tau(u_i)$ for all $i \in \mathbb{N}_n$ holds for all methods in $M_e$, the following axiom can be established by induction over the structure of $e$

AXIOM: PURE EXPR: $\{p | \hat{\Psi} = \Psi(e)\}|e[p\}$ where $\text{pure}(e)$

Combining the axiom with dyn-specific proof rules yields simplified rules for pure expressions that closely resemble those for stat. For instance:

AXIOM: PURE ASGN $\{p | \hat{\Psi} = \Psi(e)\}|x := e[p\}$

RULE: PURE COND $\{p \land \Psi(e)\}S_1\{q\}$ $\{p \land \neg\Psi(e)\}S_2\{q\}$ where $\text{pure}(e)$ and $\tau(e) = B$.

Definitions for $\text{pure}(e)$, $\Psi : Expr_d \mapsto LExp$ and $\tau(e)$ as well as omitted rules and soundness proofs can be found in Appendix A. Finally, we are able to state the main theorem of this section: in combination with decomposition and automatic object mapping, above rules allow verification just like in statically typed languages. This follows from the fact that stat proofs closely resemble dyn proofs using these techniques.

Translating Programs: Since $\text{stat} \subseteq \text{dyn}$, we simply have $\Theta(S) = S$.

Translating Proofs: $\Theta(\phi) = \varphi$ is defined inductively over the structure of the proof $\phi$ in Hoare logic for stat. Applications of the rules ASGN,COND,LOOP and METH need to be substituted for applications of PURE ASGN, PURE COND, PURE LOOP and PURE METH + PURE ASGN respectively. Note that this is always possible as stat expressions are pure and well-typed pure
assignments preserve safety predicates. Applications of all other rules can be preserved, as they are identical for \textit{dyn} and \textit{stat}.

\textbf{Theorem 2.} For every \textit{stat} program $S$ and every correctness proof $\phi$ of a property $\{p\}S\{q\}$ in Hoare logic for a particular correctness notion of \textit{stat} programs, $\Theta(\phi)$ is a valid proof of the property $\{\Theta(p)\}S\{\Theta(q)\}$ in Hoare logic for a corresponding type-unsafe correctness notion of \textit{dyn} programs.

Furthermore, since types for \textit{stat} programs can be inferred, their type-safety proofs can be constructed automatically (see Section 6.2). Applying the decomposition rule (Definition 2) then yields a proof for type-safe correctness. It follows that for statically typable programs, deriving a proof in \textit{dyn} (using the layer of abstraction) does not require any more effort than deriving it in \textit{stat}. The remainder of this paper will discuss how the layer of abstraction can be applied to arbitrary dynamically typed programs by deriving the necessary type information. In Section 7, we will demonstrate this point by proving the evaluator example correct.

5 Deriving Type Information

5.1 Type Information and Type Safety

A program $\pi$ is called \textit{type-safe} if no execution of $\pi$ can result in a type error. Type safety is the problem of deciding whether a given program is type-safe. Since type errors can be regarded as a form of output, type safety is a nontrivial semantic property and hence undecidable for Turing complete languages by Rice’s theorem.

A type $T$ is an element of a complete lattice $(T, \subseteq) = (2^C, \subseteq)$. A typing $\text{ty}$ of a program $\pi$ is an arbitrary data structure giving rise to a mapping $\text{ty}(S) : \text{Stmt} \rightarrow T$ from sub-statements $S$ of $\pi$ to types. It is important to stress that a sub-statement occurring multiple times in $\pi$ is treated as multiple different statements. One can think of statements as represented by their parse tree nodes.

A typing $\text{ty}$ for a program $\pi$ is called \textit{sound} iff in every execution of $\pi$, whenever a sub-statement $S$ is evaluated to a value $v$, then $v$ is of a type $T \subseteq \text{ty}(S)$. A typing $\text{ty}$ is at least as \textit{precise} as another typing $\text{ty}'$, written $\text{ty} \subseteq \text{ty}'$, iff for all statements $S$ it holds that $\text{ty}(S) \subseteq \text{ty}'(S)$.

For a program $\pi$, the \textit{least precise type-safe typing} $\text{ty}_\perp^\pi$ is a typing where for every method call $e_0.m(e_1, ..., e_n)$, $\text{ty}_\perp^\pi(e_0) = \{C \in C \mid C$ supports method $m$ of arity $n\}$ and for every conditional or while loop with condition $e$, $\text{ty}_\perp^\pi(e) = \{\text{bool}\}$ and for all other sub-statements $S$, $\text{ty}_\perp^\pi(S) = \top$. By definition, a program $\pi$ is type-safe iff it has a sound\textsuperscript{6} typing $\text{ty}$ that is precise enough to establish type safety ($\text{ty} \subseteq \text{ty}_\perp^\pi$).

Type safety verifiers (type inference algorithms) derive a typing for a given program by over-approximating its behavior. A verifier is \textit{sound} iff the typings it derives are.

\textsuperscript{6} if a method call, conditional or while loop is unreachable, sound typings may assign the type $\bot$ to its receiver / condition.
Note that given a typing \( ty \) for a program \( \pi \), it is straightforward to decide \( ty \sqsubseteq ty^* \). In fact, this would even be possible if \( ty \), like \( ty^* \), would assign \( \top \) to all non-receiver, non-condition subexpressions. However, deciding soundness usually requires more information. For this reason, sound type safety verifiers usually \( a \) assign types to all subexpressions and \( b \) provide a set of inference rules (commonly called a “type system”) allowing to check safety of their derived typings using this additional type information. A soundness proof for these rules with respect to the semantics of the programming language is a crucial part of proving such algorithms sound.

Some algorithms (e.g. context-sensitive ones) even assign multiple types to each statement (one for each context). While \( ty(S) \) in this case yields the union of all types assigned to \( S \), the soundness proof may differentiate these types.

As type safety verifiers differ, so do their typings. We associate with each verifier \( V_X \) a kind of typing capturing its respective format and restrictions. Between such kinds, it is possible to translate in both directions. However, as the precision achievable with a verifier \( V_X \) varies, so does the precision expressible using \( V_X \)-typings. For instance, while it is usually possible to translate path-insensitive typings into path-sensitive ones by assigning the same types to each path, the reverse direction entails merging paths and thus a loss of precision.

### 5.2 Type Safety Proofs are Type Information

A type safety proof for a statement \( S \) is a proof of the property \( \{p\}S\{true\} \) for some precondition \( p \) in Hoare logic for type-safe partial correctness. When run from a state satisfying \( p \), it ensures type-safety of \( S \) by establishing all type-safety preconditions.

Such proofs constitute a kind of typing as their assertions contain type information that is by definition sufficient to establish type safety. Soundness of these typings can be validated using the proof rules of Hoare logic. Before discussing how to extract type information from a Hoare logic proof, one should state that this information needs to be compatible with the type safety verifier to be useful for our purpose. We hence define typing assertions \( TAsrt \subset Asrt \) as a subset of the assertion language modeling the capabilities of this verifier.

For instance, the verifier \( V_{Ex} \) that will be presented in Section 6.1 is based on a flow-sensitive, path-sensitive data flow analysis. As usual, only local variables of the current method and instance variables of the current object are tracked flow-sensitively. The remainder of the heap is abstracted into a finite number of type variables \([C.@x]\) – one for each instance variable @x of each class C.

Logically, \( V_{Ex} \) establishes a global typing invariant of the form

\[
I_{Ex}(ty) \triangleq \forall o. \left( \bigwedge_{C \in \mathcal{C}} \left( [o] \in \{C\} \rightarrow \bigwedge_{@x \in \mathcal{V}_C} [o.@x] \in ty([C.@x]) \right) \right)
\]

for a \( V_{Ex} \)-typing \( ty \), stating the fact that the types assigned to the type variables \([C.@x]\) in \( ty \) are over-approximating the actual types of those instance variables. Also, automatic verifiers provide for each program location the return type of the
For every assertion $p$, Lemma 1.

For precision, type information can always be supplied for type-safe programs.

Practically, those can be regarded as a conjunction of typing literals (see below). Additionally, path sensitivity allows differentiating different paths leading to a program location and hence requires expressing alternatives, leading us to a disjunctive normal form of typing literals. Hence, only the literals allowed in typing assertions are verifier-specific$^7$. For $\mathcal{V}_{\text{Ex}}$ we define

$$\text{TAsrt} \triangleright \tau ::= \mu \mid \tau \lor \tau \mid \tau \land \tau, \quad \text{TLit}_{\text{Ex}} \triangleright \mu ::= \neg \mu \mid [[u]] \in T \mid [[\text{this}@x]] \in T$$

We will now define how to extract type information from Hoare logic proofs. In such a proof, each postcondition may contain flow-sensitive type information about variables as well as the return value $r$ of the previous expression. Given an assertion $p$, one extracts this information by first converting $p$ into disjunctive normal form, treating typing literals, equations and quantifiers as literals and then applying a projection $pr_X : \text{Asrt} \mapsto \text{TAsrt}_X$ that preserves $\land, \lor, \mu$ while mapping all literals $\not\in \text{TLit}_X$ to true (i.e. $[[\text{this}] \in T]$). Every assertion $p$ thus implies $pr_X(p)$. Note that depending on the structure of $p$, there might be a significant loss of precision. This is unproblematic, however, as supplying type information is in the user’s interest. The following theorems show that sufficiently precise type information can always be supplied for type-safe programs.

**Lemma 1.** For every assertion $p$ and every $\mathcal{V}_X$-typing assertion $\tau$ such that $p \rightarrow \tau$, there exists an equivalent assertion $p' \leftrightarrow p$ such that $pr_X(p') = \tau$.

A $\mathcal{V}_X$-typing assertion $\tau$ is most precise for an assertion $p$ iff $p \rightarrow \tau$ and for all $\mathcal{V}_X$-typing assertions $\tau'$, $p \rightarrow \tau'$ implies $\tau \rightarrow \tau'$.

**Theorem 3.** For every verifier $\mathcal{V}_X$, each type safety proof $\psi$ has an equivalent proof $\psi'$, such that for every assertion $p'$ in $\psi'$, $pr_X(p')$ is most precise for $p'$.

Furthermore, one can define a projection $pr_X^X$ further projecting $\mathcal{V}_X$ typing assertions to summary types for the variable $x$ such that for all assertions $p$, all variables $x$ and all verifiers $\mathcal{V}_X$ we have $p \rightarrow [[x]] \in pr_X^X(p)$. For $\mathcal{V}_{\text{Ex}}$:

$$pr_{\text{Ex}}^X([x]) \in T, \quad pr_{\text{Ex}}^X([x']) \in T \quad \text{with } x' \neq x,$$

$$pr_{\text{Ex}}^X(\neg \mu) \triangleq T \setminus pr_{\text{Ex}}^X(\mu),$$

$$pr_{\text{Ex}}^X(\tau \land \tau') \triangleq pr_{\text{Ex}}^X(\tau) \cap pr_{\text{Ex}}^X(\tau'), \quad pr_{\text{Ex}}^X(\tau \lor \tau') \triangleq pr_{\text{Ex}}^X(\tau) \cup pr_{\text{Ex}}^X(\tau')$$

We extend $pr_X^X$ to assertions by defining $pr_X^X = pr_X^X \circ pr_X$. Using it, every type safety proof $\psi$ gives rise to a $\mathcal{V}_X$-typing $ty_\psi$ assigning every sub-statement $S$ the type $pr_X^X(q_1 \land \ldots \land q_k)$ where $q_i$ are the postconditions of all Hoare triples of the form $\{p_i\}S\{q_i\}$ in $\psi$.

**Theorem 4 (Completeness relative to Hoare logic).** Given completeness of the Hoare logic, for every type-safe program $\pi$ there exists a type safety proof $\psi$ such that $ty_\psi$ is sound and precise enough to establish type safety: $ty_\psi \sqsubseteq ty^\dagger_\pi$.

It follows that no (sound) typing can be more precise than a type safety proof. It is hence possible to translate from all other kinds of typings into them without incurring any loss of precision.

$^7$ Adding the literals $u = \text{null}$ and this.$@x = \text{null}$ allows tracking null values.
6 Semi-Automation

As type safety is undecidable and full automation hence only achievable at the expense of completeness, we instead aim at semi-automation by integrating a suitable automatic type safety verifier into Hoare logic. Such verifiers have to satisfy the following requirements:

- Soundness - it safely over-approximates the actual program behavior.
- Monotonicity - increasing precision cannot create type errors.
- Refinements - provides an interface to supply trusted assumptions for increasing precision (Section 6.1). These must be treated
  - Flow sensitively - assumptions should have an associated program location and affect only data flows from that location onward.
  - Path sensitively - assumptions should be able to use disjunctions ($\lor$) to express alternatives. The verifier should treat these alternatives like different paths reaching the associated program location.
- Termination - terminates on all inputs (programs).\footnote{Potentially non-terminating analyses like \cite{16} must be performed iteratively by first generating a base result and then refining it towards higher precision (similar to \cite{28}); thus, they can be interrupted any time and yield the most precise result reached.} 

Note that flow- and path sensitivity are required only for refinements\footnote{Flow- or path-insensitive assumptions would increase the annotation-burden.}, not for the verifier itself. Also, the chosen Hoare logic must be powerful enough to express the verifier’s reasoning. While we consider these requirements to be modest and can hardly imagine an analysis not amendable using this approach, proving this is difficult. In this paper we will therefore concentrate on automatic verifiers based on flow-analysis which are known to contain quite powerful analyses \cite{18,32,28}.

6.1 An Exemplary Automatic Type Safety Verifier

In this section we will introduce an exemplary automatic type safety verifier $\mathcal{V}_{\text{Ex}}$ to in the following complement our abstract discussion with concrete examples using $\mathcal{V}_{\text{Ex}}$-typings.

To also shed some light on the minimum requirements above, we chose a minimalistic one exactly fulfilling these criteria. $\mathcal{V}_{\text{Ex}}$ is based on a sound, flow-sensitive data flow analysis and resembles the work of Palsberg et al. \cite{22}. As required, we allow specifying a set of path-sensitive trusted assumptions. However, the analysis does not introduce path-sensitivity by itself. **Flow Sensitivity** Intra-procedurally, local variables as well as instance variables of the current object are tracked flow-sensitively. As usual, this is realized by converting all statements to static single-assignment form (SSA) with respect to these variables prior to analysis. A $\mathcal{V}_{\text{Ex}}$-typing $ty$ hence assigns one type $ty([x], L)$ per program location $L$ to each such variable $x$.

**Path Sensitivity** To realize intra-procedural path-sensitivity, for each path $j \in \text{path}(S^4)$ from the start of a method to each program location $S^4$ in the
method, the previous sub-statement $S$ and each flow-sensitively tracked variable $x$ are assigned distinct types $ty([S], j)$ and $ty([x], S^i, j)$ respectively. Program locations are denoted by $\uparrow S$ (or $S^\downarrow$) for the beginning (end) of a sub-statement $S$ of $\pi$.

**Null Pointers** Although [22] is a pure type analysis, we use the same algorithm to also perform null-pointer analysis. This is realized by defining the value $null$ as the only instance of a class $Null$ and furthermore explicitly inserting the class $Null$ into all types whose expression may evaluate to $null$ instead of implicitly allowing $null$ to be element of every type’s domain. We hence define

\[
J_x^K \in \{Null, C_1, \ldots, C_n\} \equiv x = null \lor J_x^K \in \{C_1, \ldots, C_n\}
\]

The interested reader may find a more detailed account of algorithm $\mathcal{V}_{Ex}$ in Appendix C.

**Typings to Logic** The function $\Xi_X$ maps the type information contained in a flow-sensitive, path-sensitive $\mathcal{V}_X$-typing $ty$ for each program location $L$ into a typing assertion. $\mathcal{V}_{Ex}$-typings $ty$ are flow-sensitive, as $ty([x], S^\downarrow)$, the type of the variable $x$ at program location $L$ takes strong updates into account. Hence, for $\mathcal{V}_{Ex}$-typings, the function $\Xi_{Ex}$ can be defined as:

\[
\Xi_{Ex}(ty, S^\downarrow) \equiv \exists j \in \text{path}(S^\downarrow) \land \forall x \in \mathcal{V}_{M(S^\downarrow)}, [x] \in ty([x], S, j)
\]

where $S$ is a sub-statement of $\pi$, $M(L)$ denotes the method a program location $L$ belongs to and $\mathcal{V}_{C.m} \equiv \text{var}(S_{C.m}) \cup \text{change}(S_{C.m})$.

**Definition 4 (Refinement of Typings).** Let $ty$ be a $\mathcal{V}_{Ex}$-typing derived for a program $\pi$. Then a conjunctive refinement step of $ty$ using the trusted assumption $\tau$ at program location $L$ is a quadruple $(ty, \tau, L, ty')$ with the $\mathcal{V}_{Ex}$-typing $ty'$ being derived for a program $\pi'$ resulting from $\pi$ by inserting the Statement $R_\tau$ just before $L$ and $R_\tau$ being defined inductively as:

\[
\begin{align*}
R_{[x]\in T} & \equiv x := x \cap T^{10} \\
R_{\mu \land \mu'} & \equiv R_{\mu}; R_{\mu'} \\
R_{\nu \lor \nu'} & \equiv \text{if... then} R_{\nu} \text{ else } R_{\nu'} \text{ end}^{11}
\end{align*}
\]

\(^{10}\) the type filter $x \cap T$ is defined in Appendix C

\(^{11}\) the condition does not matter as $\mathcal{V}_{Ex}$ will treat conditionals non-deterministically anyway.
Theorem 5. For all conjunctive refinements $ty \xrightarrow{\tau,L} ty'$, $ty' \sqsubseteq ty$ holds.

6.2 Translating Typings into Proofs

Definition 5 (Typing Proof). A typing proof $\psi$ for a $\forall X$-typing $ty$ of a statement $S$ is a minimal\footnote{all Hoare triples in $\psi$ must contribute to establishing the conclusion.} proof of the property $\{p\}S\{true\}$ for some precondition $p$ in Hoare logic for partial correctness that for every sub-expression $e$ of $S$ contains a Hoare triple $\{p,e\}e\{q_e\}$ with $pr_X^e(q_e) \sqsubseteq ty(e)$.

Technically, $\psi$ only establishes soundness of the typing $ty$ (by being a Hoare logic proof and $ty(\psi) \sqsubseteq ty$). However, when $ty(\psi) \sqsubseteq ty(\psi)'$, $\psi$ can be turned into a type safety proof by changing the proof system to Hoare logic for type-safe partial correctness and trivially establishing the type safety preconditions. Hence, typing proofs are well-suited as intermediate steps towards type safety proofs.

Recall that $\forall X$-typings can be checked for soundness using a $\forall X$-specific inference system. It is hence possible to extend $\Xi_X$ to mechanically derive a typing proof $\psi = \Xi_X(\pi,ty)$ for a sound $\forall X$-typing $ty$ for a program $\pi$ by translating the rules of this inference system into Hoare logic and establishing $I_X(ty)$ as a global invariant. In such a proof, each assertion $p$ at program location $L$ is exactly the typing assertion $\Xi_X(\pi,L)$. We hence write $\Xi_X(\psi,L) \triangleq \Xi_X(\pi,L)$.

The interested reader may find an exemplary translation for $\forall Ex$-typings in Appendix D. Note that $\Xi_X$ allows using automatically derived type information in Hoare logic proofs even in theorem proving environments trusting only propositions that they verified a proof for.

6.3 Two-Layered Proofs

A two-layered proof is a Hoare logic proof for a type-safe notion of correctness of a dynamically-typed program, in which every assertion has the form $\tau \land p$ for a typing assertion $\tau$ and an assertion $p$. While $p$ is user-editable, $\tau$ is meant to be created and modified solely by an automated type safety verifier. We refer to $\tau$ as the “lower layer” and $p$ as the “higher layer” of the proof/assertion.

Theorem 6 (Two-Layered Proof Construction). Given a typing proof $\phi_l$ and a proof $\phi_h$ for the same program $\pi$, it is always possible to construct a two-layered-proof $\phi$ with $\phi_l$ as lower and $\phi_h$ as higher layer.

Starting from a typing proof $\Xi_X(\pi,ty)$ in the lower layer and only $true$ in the higher layer, proofs in the higher layer are supported by type information from the lower layer (Section 4). The type information may also be refined:

Definition 6 (Refinement of Typing Proofs). Let $\psi = \Xi_X(\pi,ty)$ be a typing proof generated by a typing ty of a program $\pi$. Then each conjunctive refinement step $ty \xrightarrow{\tau,L} ty'$ gives rise to a conjunctive proof refinement step $\psi \xrightarrow{\tau,L} \psi'$ with $\psi' = \Xi_X(\pi,ty')$. 

12 all Hoare triples in $\psi$ must contribute to establishing the conclusion.
Let $\psi_l = \Xi_X(\pi, ty)$ be the lower layer of a two-layered proof $\psi$. Then whenever a typing literal appears within the higher layer $p$ of an assertion at program location $L$ in $\psi$, the lower-layer proof $\psi_l$ is substituted by the result $\psi'_l$ of the conjunctive proof refinement step $\psi_l^{prX(p),L} \rightarrow \psi'_l$. In such refinements $\Xi_X(\psi'_l, L') \rightarrow \Xi_X(\psi_l, L')$ holds for all $L'$ due to Theorem 5. Higher layer proof steps depending on lower layer information hence remain valid.

7 Verifying the Evaluator Example

To demonstrate how the techniques developed enable the convenient verification of dynamically typed programs despite hard typing problems, we will proof the evaluator example both type-safe and correct. Figure 8 shows all annotations necessary to prove that $\text{calc}()$ derives a given term’s value.

**Type safety:** The given invariant enables deriving the assertions on lines 2-4 and hence a proper typing of the remaining program. As a property of the ad-hoc data structure it must be established in the (omitted) method $\text{parse}()$. With the types of $\text{env} (S \rightarrow N)$ and $r (N)$ known, their mapping can be automated. The complex ad-hoc data structure $\text{tree}$ is given the (imprecise) type $N \rightarrow O$ and its elements hence need manual mapping. These mappings are encapsulated in predicates ($\text{valuetree2}$, $\text{vartree3}$, $\text{optree5}$) and furthermore ignored.

**Correctness (marked):** The lower layer information allows identifying numerous pure expressions ($\text{tree[1]}$, $\text{env[tree[1]]}$, $\text{tree[1]} \Rightarrow \text{ADD}$, etc.). Establishing the specified property for the first two branches then only requires applying $\text{PURE EXPR}$. The conditional in line 5 can be handled by $\text{PURE COND}$. Since all arguments to the recursive method calls in that line are also pure, 

\[
\{ \text{optree2}(\hat{\text{tree}}, \hat{\text{env}}) \land \hat{\text{tree[1]}} = \hat{\text{ADD}} \} \quad \text{calc}(\text{env, tree[2]}) + \text{calc}(\text{env, tree[3]})
\]

\[
\{ \text{treeval3}(\hat{\text{tree}}, \hat{\text{env}}, r) \}
\]

can be derived automatically. Note that all implications can be handled by SMT solvers with theories for Presburger arithmetic and lists for which effective decision procedures are known and do not require reasoning about graph-like object structures.

8 Related Work

There are several threads of related work regarding dynamically typed programs. In each, we can only discuss those works most closely related to ours.

**Type Safety:** Cartwright [6] pioneered a strand of work called “soft typing”, applying automated type safety verifiers to dynamically typed languages with the aim of improving performance. Another line of work is “gradual typing” [26,4], letting the user decide which parts of the program should be statically checked for type errors, while dynamically typing the remaining program.  

\footnote{Again, all recursive predicates can instead be expressed using quantification over sequences, at the expense of readability}

\footnote{The two may also be combined [24]}
\[
\text{def } \text{valuete}2(t, n) \triangleq N(t[0], \text{VALUE}) \land N(t[1], n), \text{valuete}1(t) \triangleq \exists n. \text{valuete}2(t, n)
\]

\[
\text{def } \text{var}3(t, e, x) \triangleq N(t[0], \text{VAR}) \land S(t[1], x) \land e[x] \neq \text{null}
\]

\[
\text{def } \text{opt}5(t, e, l, r) \triangleq N(t[0], \text{OP}) \land N(t[1], \text{op}) \land \text{L}(t[2], l) \land \text{parsete}2(l, e) \land \text{L}(t[3], r) \land \text{parsete}2(r, e)
\]

\[
\text{def } \text{opt}2(t, e) \triangleq \exists \text{op}, l, r. \text{opt}5(t, e, l, r)
\]

\[
\text{def } \text{parsete}2(t, e) \triangleq \text{valuete}1(t) \lor \text{var}2(t, e) \lor \text{opt}2(t, e)
\]

\[
\text{def } \text{treeval}3(\text{tree}, \text{env}, n) \triangleq
\]

\[
\text{valuete}2(\text{tree}, n) \lor
\]

\[
\exists x. \text{var}3(\text{tree}, \text{env}, x) \land \text{env}[x] = n \lor
\]

\[
\exists \text{op}, l, r, n_l, n_r. \text{opt}5(\text{tree}, \text{env}, \text{op}, l, r) \land \text{treeval}3(l, \text{env}, n_l) \land \text{treeval}3(r, \text{env}, n_r) \land \text{op} = \text{ADD} \rightarrow n = n_l + n_r \land ...
\]

\[
\text{inv } \forall t, e. \text{parsete}2(t, e) \rightarrow
\]

\[
t[0] = \text{VALUE} \rightarrow \text{valuete}1(t) \land t[0] = \text{FIN} \rightarrow \text{var}2(t, e) \land t[0] = \text{OP} \rightarrow \text{opt}2(t, e)
\]

\[
\{ \text{parsete}2(\text{tree}, \text{en})\}
\]

\[
1 \text{ method calc(env, tree = @tree) } \{ \\
2 \quad \text{if tree}[0] = \text{VALUE then } \{ \text{valuete}1(\text{tree}) \} \text{ tree}[1] \\
3 \quad \text{elseif tree}[0] = \text{VAR then } \{ \text{var}2(\text{tree}, \text{en}) \} \text{ env}[\text{tree}[1]] \\
4 \quad \text{elseif tree}[0] = \text{OP then } \{ \text{opt}2(\text{tree}, \text{en}) \} \\
5 \quad \text{if tree}[1] = \text{ADD then calc(env, tree}[2]) + \text{calc(env, tree}[3]) \\
6 \quad \text{elseif ...} \\
7 \quad \text{else nil} \\
8 \quad \text{fi} \\
9 \quad \text{else nil} \\
10 \quad \text{fi} \\
11 \} \\
\{ \text{treeval}3(\text{tree}, \text{en}, \text{r})\}
\]

\textbf{Fig. 8.} Correctness proof for the evaluator example
Both soft- and gradual typing require rewriting of program parts in exchange for type safety guarantees. On the contrary, our approach is able to provide such guarantees also for parts that are not statically typable. Also, the user is free to omit type safety from the specification (dynamic typing) and may still rewrite the program to allow automatic checking (static typing), both on a per-expression basis (gradual). With respect to correctness, our approach hence subsumes both soft and gradual typing. However, it does not (yet) increase performance.

Others [13,20], have extended such abstraction-based verifiers to handle many idioms common in dynamically typed languages.

Correctness: To the best of our knowledge, [12] currently is the only\textsuperscript{15} axiomatic semantics for a type-safe notion of correctness of a dynamically typed language. Like discussed in Section 2.1 it uses type safety preconditions, considers all variables to be of object type and does not use pure expressions and would thus benefit from our approach.

Nguyen et al. [21] proposed an automatic contract verifier for untyped higher-order functional languages based on symbolic execution inserting run-time checks for contracts it cannot statically guarantee. Since they use a mechanism similar to widening to enforce termination, their approach also combines abstraction-based and symbolic reasoning.

Drawing on their work on the verification of untyped higher-order functional programs [9], Chugh et al. [8,7] provide a dependent-type system for an untyped functional “core calculus” $\lambda_{JS}$ JavaScript programs can be translated into. No soundness is demonstrated for their system.

Swamy et al. [29] semi-automatically reason about a wide range of JavaScript idioms by translating into the dependently-typed functional language $F^\ast$ and using its SMT-based reasoning engine. They also noticed that the type information generated by an abstraction-based type safety verifier (GateKeeper in their case) are useful to improve the effectiveness of automatic reasoning engines. However, they did not feed the symbolically derived proof results back into GateKeeper and did not use the type information to ease the annotation burden for their users. Since their main focus lies on a novel encoding of Dijkstra's predicate transformer semantics allowing $F^\ast$'s dependent type inference to effectively reason about imperative programs in a style similar to Hoare logic, we consider the approaches to be largely complementary.

In general, all fully automatic approaches [6,21,13,9] are necessarily incomplete. They can however be used as automatic type safety verifiers. Furthermore, all purely symbolic approaches [9,8,29,12,23] require all type information to be manually specified in method contracts and loop invariants.

Both the idea and the term “Layer of abstraction” are inspired by the work of Gardner et al. [12] on reasoning about JavaScript. However, their work abstracts from the peculiarities of the JavaScript variable store, while ours abstracts from the complexity of dynamic typing and is applicable to virtually any dynamically typed language.

\textsuperscript{15} [23] only treat partial correctness. Also, they restrict the programming language to allow a form of (type-unsafe) pure expressions.
The decomposition rule used to establish the layer of abstraction is inspired by similar constructions in [1].

Some tools for verification of statically typed imperative programs [10] allow using a “pure” subset of the programming language (that is side-effect-free and guaranteed to terminate) within assertions. The ability of our layer of abstraction to allow the use of well-typed “pure” program expressions in assertions can be seen as an extension of this idea to dynamically typed programs.

**Combining Static Analysis with Program Logics:** There has been a considerable amount of work on integrating algorithmic decision procedures (mostly model checking) and deductive methods for program verification (See [31] for pointers). Due to the deep connection between data flow analysis and model checking [25], many of these techniques can be considered as related.

Note that our conjunctive refinement differs from abstraction refinement since it is not the abstraction that is refined, but the analysis result.

Also, translations from typings (i.e. from type systems for information-flow properties) to program logics are commonly used in the Proof-Carrying Code Community [14] to avoid the need for property-specific proof-checkers. Although PCC is a completely different application area, their aim was also to integrate results derived by different inference systems into one common representation – and incidentally they also chose a program logic as their “lingua franca”.

A closely related proposal also integrating symbolic with abstraction-based reasoning is MIXY [19], a framework for mixing symbolic execution with type checking. In their system, the user divides his/her program into s-blocks and t-blocks. While s-blocks are analysed using symbolic execution, type analysis is applied to t-blocks. The results of both analyses are bidirectionally exchanged using so-called MIX-rules: Type analysis results are translated into a matching start environment for symbolic execution and types ensured by (exhaustive!) symbolic execution can be used for type analysis. Also, the aim is related: What Phang et. al called “balancing precision vs. efficiency” is the same as “combining automation with completeness”, although Phang et al. do not prove their system complete. Our approach could most likely be integrated into their framework as “Hoare-Logic blocks” \{h \in h\} with typing \(\Gamma \vdash \{h \in h\} : \tau\) for which a Hoare-triple \(\{p\} e \{q\}\) must be derived where \(p \triangleq \Xi_X(\Gamma)\) and \(pr_X(q_e) = \tau\) for some verifier \(V_X\).

**9 Conclusion & Future Work**

The approach presented allows verifying dynamically typed programs just like statically typed ones, requiring manual assistance only on hard typing problems. If a program is statically-typable, there is no difference. Otherwise, the user may freely choose whether going beyond the limits of the type system is worth the verification effort. While the stated requirements for automated verifiers allow conveniently using the technique, more powerful verifiers can be expected to significantly increase the degree of automation. Being gradually applicable like gradual typing and automated like soft typing, the approach allows deriving type
safety guarantees also for code parts that cannot be statically typed. Should it turn out to be practically usable, it would suggest dynamic typing as a serious alternative to static typing for verifiable languages that have all necessary infrastructure readily available. Until then, there are several useful ways to extended this work:

**Completeness:** Currently, there is no (relative) complete Hoare logic for a dynamically typed language. While the approach is also applicable to incomplete program logics, no completeness guarantee can be provided in this case.

**Other Program Logics:** [8,29] are both based on refinement types. Recently, it has been shown [30] how to extend such systems to provide (relative) completeness like Hoare logic. It would be interesting to investigate if our approach is also applicable to such program logics.

**Other Program Analyses:** The formalization of semi-automation suggests that it could be generalizable to arbitrary data flow analyses.

**Performance:** The derived type information could be used to omit run-time checks and generate more efficient binaries.

**Features:** The current concept excludes optional variables and type errors as exceptions. Also, closures and advanced dynamic features like method update, dynamic type hierarchies and eval() should be studied.

**Implementation:** An implementation would allow evaluating the practical usefulness of the concept.

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A Appendix: Identifying and Translating Pure Expressions

Identifying pure expressions \( (\text{pure} : T_s \times Expr_d \mapsto \mathbb{B}) \)
\[ pure(e) \triangleq \tau(e) \] defined,
\[ \tau(e) = \inf\{ \mathbb{T} \mid pure(T, e) \} \]
\[ pure(\text{Null}, \text{null}) \triangleq \text{true}, \quad \forall e \in \text{Expr}_d. \ pure(\text{Null}, e) \rightarrow pure(\text{Null}, e) \]
\[ pure(T, u) \triangleq [u] \in T \land \Psi(T) = \mathbb{T} \land safe_T(u) \] (includes the case of this)
\[ pure(T, \text{@x}) \triangleq [\text{this.}@x] \in T \land \Psi(T) = \mathbb{T} \land safe_T(@x) \]
\[ pure(B, e \text{ is}_A C) \triangleq pure(\text{Null}, e) \]
\[ pure(B, e_1 == e_2) \triangleq \exists T \in T_s. \ pure(T, e_1) \land pure(T, e_2) \]
\[ pure(T, e_0.m(e_1, ..., e_n)) \triangleq \exists T_0, ..., T_n, \ pure(T_1, e_i) \text{ for } i \in \mathbb{N}_n \land \Psi(T_0.m(T_1, ..., T_n) \rightarrow T) \text{ defined} \]
\[ pure(T, u := e) \triangleq \text{false} \]
\[ pure(T, @v := e) \triangleq \text{false} \]
\[ pure(T, \text{while } e \text{ do } S \text{ od}) \triangleq \text{false} \]
\[ pure(T, \text{if } e \text{ then } S_1 \text{ else } S_2 \text{ fi}) \triangleq pure(B, e) \land S_i \equiv e_i \land pure(T, e_i) \text{ for } i \in \{1, 2\} \]

**Translation of pure expressions into logical expressions** \( (\Psi : T_s \times \text{Expr}_d \mapsto LExp) \)
\[ \Psi(e) \triangleq \Psi(e)(e), \quad \Psi(x) \triangleq x \]
\[ \Psi_{\text{Null}}(\text{null}) \triangleq \text{null}, \quad (\Psi_{\text{Null}}(e) = p) \rightarrow (\Psi_0(e) = p) \]
\[ \Psi_T(e_0.m(e_1, ..., e_n)) \triangleq l[v_0, ..., v_n := \Psi_T(e_0), ..., \Psi_T(e_n)] \]
where \( pure(T_1, e_i) \text{ for } i \in \mathbb{N}_n \land \Psi(T_0.m(T_1, ..., T_n) \rightarrow T) = l \).
\[ \Psi_T(\text{new } C(e_1, ..., e_n)) \triangleq l[v_1, ..., v_n := \Psi_T(e_1), ..., \Psi_T(e_n)] \]
where \( pure(T_1, e_i) \text{ for } i \in \mathbb{N}_n \land \Psi(\Psi(\{C\}).\text{init}(T_1, ..., T_n) \rightarrow T) = l \).
\[ \Psi_B(e_1 == e_2) \triangleq \Psi_T(e_1) = \Psi_T(e_2) \text{ where } pure(T, e_i) \text{ for } i \in \{1, 2\}. \]
\[ \Psi_B(e \text{ is}_A C) \triangleq [\Psi_0(e)] \in \{C\} \]
\[ \Psi_T(\text{if } e \text{ then } e_1 \text{ else } e_2 \text{ fi}) \triangleq \text{if } \Psi_B(e) \text{ then } \Psi_T(e_1) \text{ else } \Psi_T(e_2) \text{ fi} \]

**Proof Rules for pure expressions**

**RULE: PURE LOOP** (**strong** (**type-safe** partial correctness))
\[
\frac{\{p \land \Psi(e)\}S\{p\}}{\{p\} \text{ while } e \text{ do } S \text{ od } \{p \land \neg\Psi(e) \land r = \text{null}\}} \quad \text{where } pure(e), \ \tau(e) = \mathbb{B}
\]

**RULE: PURE METH**
\[
\frac{\{p\}u_0.m(u_1, ..., u_n)\{q\}}{\{p[|u_0, ..., u_n := \Psi(e_0), ..., \Psi(e_n)|]e_0.m(e_1, ..., e_n)\{q\}} \quad \text{where } u_i \in V_L \text{ fresh and } pure(e_i) \text{ for all } i \in \mathbb{N}_n.\]
Soundness

Proof. The Axiom PURE EXPR can be established by induction over the structure of the pure expression $e$, using the guarantees provided by $\text{pure}(e)$. In the cases for variables, $\text{pure}(x)$ implies $\text{safe}_T(x)$ for some type $T$. In the case for method calls, we assume

$$\{p[r := \Psi(T_0.m(T_1, ..., T_n) \rightarrow T)]u_0.m(u_1, ..., u_n)\{p\}$$

with $T_i = \tau(u_i)$ for all $i \in \mathbb{N}_n$ to be established for all methods in $\mathcal{M}_e$ (which is precisely the meaning of “correspondence between methods and operations with respect to the mapping $\Psi$” in Section 4.3).

The rules PURE ASGN, PURE COND, PURE LOOP and PURE METH can then be derived by combining the axiom PURE EXPR with the dyn rules for ASGN, COND, LOOP and METH respectively.

\[\square\]

B Appendix: Axiomatic Semantics for dyn

AXIOM: CONST

$$\{p[r := \text{null}]\}\text{null\{p\}}$$

AXIOM: VAR

$$\{p[r := u]\}u\{p\}$$

Note: includes the case of $u \equiv \text{this}$.

AXIOM: IVAR

$$\{p[r := \text{this}@v]\}@v\{p\}$$

RULE: ASGN (both normal and instance variables)

$$\frac{\{p\}e\{q[u := r]\}}{\{p\}u := e\{q\}} \text{ where } u \in \mathcal{V}$$

RULE: SEQ

$$\frac{\{p\}S_1(r)}{\{p\}S_1; S_2\{q\}} \frac{\{r\}S_2\{q\}}{}$$

RULE: Conditional (COND) (strong type-safe partial correctness)

$$\frac{\{p\}e\{r \land \text{bool\_test}\}}{\{r \land b\}S_1\{q\}} \frac{\{r \land \neg b\}S_2\{q\}}{}$$

$$\{p\} \text{ if } e \text{ then } S_1 \text{ else } S_2 \text{ fi } \{q\}$$
where $b$ is a predicate and $bool_{\text{test}} \equiv (r \neq \text{null}) \land (r \in \{\text{bool}\}) \land \exists (r, b)$

RULE: LOOP (strong type-safe partial correctness)

$$
\frac{\{ p \} \{ p \land r \land bool_{\text{test}} \} \{ p \land r \land b \} S \{ p \}}{\{ p \} \text{ while } e \text{ do } S \text{ od } \{ p \land \neg b \land r = \text{null} \}}
$$

where $b$ is a predicate and $bool_{\text{test}}$ is defined as in COND.

RULE: CONS

$$
p \rightarrow p_i, \{ p_i \} \{ q_i \}, q_i \rightarrow q
$$

RULE: PASGN

$$\{ p[u := t] \} u := t \{ q \}
$$

where $\{ u \} \subseteq V_L$ and $\{ t \} \subseteq V_L \cup \{ \text{null} \}$.

RULE: BLCK

$$\frac{\{ p \} \{ u := t \} \{ \text{null} \} ; S \{ q \}}{\{ p \} \begin{array}{l}
\text{begin local } \{ v \} := t ; S \{ q \} \\
\text{end}
\end{array}}
$$

where $\{ u \} \subseteq V_L$, $\{ t \} \subseteq V_L \cup \{ \text{null} \}$, $\{ v \} = V_L \setminus (\{ u \} \cup V_S)$ and $|\text{null}| = |v|$.

RULE: METH

$$\frac{\{ p_i \} e_i \{ p_{i+1}[u_i := r] \} \text{ for } i \in \mathbb{N}_n \backslash 1 \ \{ p_n \} u_0, m(u_1, \ldots, u_n) \{ q \}}{\{ p \} e_0, m(e_1, \ldots, e_n) \{ q \}}
$$

where $u_i$ fresh, $u_i \in V_L, u_i \notin \text{var}(e_j) \cup \text{change}(e_j)$ for all $i, j \in \mathbb{N}_n$.

RULE: REC (strong typesafe partial correctness)

$$\frac{A \vdash \{ p \} S \{ q \}, \begin{array}{l}
A \vdash \{ p_i \} \text{begin local this, } u_i := v_i, \overrightarrow{c} ; S_i \text{ end} \{ q_i \}
\end{array}}{\{ p \} \text{begin local this, } u_i := v_i, \overrightarrow{c} \{ q_i \}, i \in \mathbb{N}_n \backslash 1}
$$

where method $m_i(u \{ v \})\{ S_i \} \in M_C$, and $A \equiv \{ p_1 \} v_1, m_1(u \{ v \})\{ q_1 \}, \ldots, \{ p_n \} v_n, m_n(u \{ v \})\{ q_n \}$.

RULE: CNSTR

$$\frac{\{ p \} \text{new } C.\text{init}(\overrightarrow{c}) \{ q \}}{\{ p \} \text{new } C(\overrightarrow{c}) \{ q \}}
$$

AXIOM: NEW

$$\frac{\{ p[r := \text{new } C] \} \text{new } C\{ p \}}{}
B.1 Auxiliary Rules

RULE: DISJ
\[ \{p\}S\{q\} \quad \{r\}S\{q\} \]
\[ \frac{}{\{p \lor r\}S\{q\}} \]

RULE: CONJ
\[ \{p_1\}S\{q_1\} \quad \{p_2\}S\{q_2\} \]
\[ \frac{}{\{p_1 \land p_2\}S\{q_1 \land q_2\}} \]

RULE: ∃-INT
\[ \{p\}S\{q\} \]
\[ \frac{}{\{\exists x.p\}S\{q\}} \]
where \( x \notin \text{var}(M) \cup \text{var}(S) \cup \text{free}(q) \).

RULE: INV
\[ \{r\}S\{q\} \]
\[ \frac{}{\{p \land r\}S\{p \land q\}} \]
where \( \text{free}(p) \cap (\text{change}(M) \cup \text{change}(S)) = \emptyset \) and \( p \) does not contain quantification over objects.

RULE: SUBST
\[ \{p\}S\{q\} \]
\[ \frac{}{\{p[\overrightarrow{z} := \overrightarrow{t}]\}S\{q[\overrightarrow{z} := \overrightarrow{t}]\}} \]
where \( \text{var}(\overrightarrow{z}) \cap (\text{var}(M) \cup \text{var}(S)) = \text{var}(\overrightarrow{t}) \cap (\text{change}(M) \cup \text{change}(S)) = \emptyset \).

C Appendix: Automatic Type Safety Verifier \( V_{Ex} \)

To allow for intra-procedural flow-sensitivity, all statements \( S \) are converted to static single-assignment form (SSA) for local- as well as instance variables of the current object. This necessitates that each occurrence of such a variable \( x \) having some number of assignments, say \( n \), is replaced by one of its \( k \geq n \) “versions” \( x_1, \ldots, x_k \) such that each version has exactly one assignment dominating all its occurrences (except \( \phi \)-occurrences\(^{16}\)). We maintain a mapping \( \nu(x, L) \) from variables \( x \in \mathcal{V} \) and program locations \( L \) of \( \pi \) to the version \( x_i \) whose assignment dominates this location.

Next, each sub-statement \( S \) of the program \( \pi \) is given a type variable \( \llbracket S \rrbracket \). For each method \( C.m \), additional type variables \( \llbracket P_{i,m}^{C,m} \rrbracket \) for each of its parameters, \( \llbracket R^{C,m} \rrbracket \) for its return value and \( \llbracket V_{u,i}^{C,m} \rrbracket \) for each version \( u_i \) of each local variable \( u \) are added. Instance variables \( @v \in \mathcal{V}_C \) are given both a global type variable

\(^{16}\) for \( k - n \) versions, so-called \( \phi \)-assignments are inserted at control-flow joins. The occurrences in such assignments need not be dominated.
and type variables $\llbracket V_{C, \text{@v}_i} \rrbracket$ for each version $\text{@v}_i$ in methods $C.m$ using them.

The constraint generation then proceeds as follows: for every method declaration method $m(u_1, ..., u_n)\{S_{C,m}\}$ of class $C$ we generate the following constraints:

$$
\begin{align*}
[ P_{i}^{C,m} ] & \sqsubseteq [ u_i ], & [ S_{C,m} ] & \subseteq [ R_{C,m} ], & [ V_{C, \text{@v}_i} ] & \subseteq [ V_{C, \text{@v}_i} ] \\
\text{for all } i \in \mathbb{N}_1^n & \quad & \text{for all } \text{@v} \text{ used in } C.m
\end{align*}
$$

Additionally, we traverse the parse tree of the body $S_{C,m}$ of the method $C.m$ applying the rules given in Figure 9.

In the body of a method $C.m$:

- $null \Rightarrow [null] = \{\}$
- $u_i \Rightarrow [u_i] = [V_{u_i, \text{@v}_i}]$
- $\text{@v}_i \Rightarrow [\text{@v}_i] = [V_{C.m, \text{@v}_i}]$
- $e.m(e_1, ..., e_n) \Rightarrow [e] \subseteq [m(e_1, ..., e_n) \Rightarrow [e.m(...)]]$
- $\text{this} \Rightarrow [\text{this}] = [C]$
- $S; e \Rightarrow [S; e] = [e]$
- if $e$ then $S_1$ else $S_2$ fi \Rightarrow $[e] \sqsubseteq [\text{if} ... \text{fi}]$
- while $e$ do $S$ od \Rightarrow $[e] \sqsubseteq [\text{while} ... \text{od}] = \{\}$
- new $C'(e_1, ..., e_n) \Rightarrow [\text{new} C'(\ldots)]$
- $\phi(e_1, ..., e_n) \Rightarrow [\phi(e_1, ..., e_n)]$
- $e \sqcap T \Rightarrow [e \sqcap T]$

**Fig. 9.** $\mathcal{V}_{E}$ typing rules for $\text{dyn}$

The resulting system of set inclusion constraints can be solved by propagating the type information from constructor calls forwards along the data flow and whenever a type $C$ reaches a type variable of the form $m(T_1, ..., T_n) \rightarrow R$ (generated by a method call), the method $C.m$ of arity $n$ is looked up and the following connection constraints are added

$$
[ T_i ] \sqsubseteq [ P_{i}^{C,m} ] \quad \text{for all } i \in \mathbb{N}_1^n, \quad [ R_{C,m} ] \sqsubseteq [ R ]
$$

Upon reaching a fixpoint, the analysis provides a solution $ty$ (a typing) mapping each type variable (and thus every sub-statement, variable, parameter and return value) to a union type in $T$. To mask the initial conversion to SSA we define $ty(x) \equiv \sqcup i, ty(x_i)$ for all variables $x$ tracked flow-sensitively.

A typing $ty$ is called consistent iff for every constraint $[S_1] \sqsubseteq [S_2]$ it holds that $ty([S_1]) \sqsubseteq ty([S_2])$. An important property of typing rules is for consistency to imply soundness. Note that the constraints marked in Figure 9 serve to ensure sufficient precision ($ty \sqsubseteq ty')$ rather than soundness. When omitting these constraints, the algorithm outputs a sound typing even when its precision is insufficient to establish type safety.

By intra-procedural path-sensitivity we mean that the algorithm maintains a set of alternatives $\text{path}(L)$ for each program location $L$ and for each $j \in \text{path}(L)$ separately derives the types of flow-sensitively tracked variables $ty(v(x, L), j)$ as well as results of the previous sub-statement $ty([S], j)$ with $L = S^j$. In $\mathcal{V}_{E}$,
path sensitivity is only introduced by specially marked if\textsubscript{PS}-conditionals duplicating the number of paths leading to them. Note that eliminating equivalent alternatives is important to keep the analysis terminating in the presence of loops\textsuperscript{17}. However, as these markings are only used in the definition of trusted assumptions below, usually $|\text{path}(L)| = 1$ for all $L$.

**Logic to Typings** As listed in the requirements in Section 2.4, verifiers $\mathcal{V}_X$ need an interface for supplying trusted assumptions. Abstractly, one defines a refinement relation $ty \overset{\tau}{\rightarrow} ty'$ between $\mathcal{V}_X$-typings. Here, $ty'$ refines $ty$ by taking the trusted assumption $\tau$ at program location $L$ into account. For $\mathcal{V}_Ex$ we first extend dyn expressions by introducing a type filter operation $e := T \cap e$ generating the (monotone) typing constraint $T \cap [e] \subseteq [t \cap e]$. By inserting type filters, it is possible to refine a typing assertion $\tau \equiv \Xi(ty, L)$ for a program location $L$ to $\tau \wedge \tau'$ for some assumption $\tau'$:

**Definition 7 (Refinement of Constraint Systems).** Let $G$ be a constraint system generated by applying $\mathcal{V}_Ex$’s constraint generation to a program $\pi$. A conjunctive refinement step of $G$ using the trusted assumption $\tau$ at program location $L$ is a quadruple $(G, \tau, L, G')$, written $G \overset{\tau}{\rightarrow}^{\mathcal{V}_Ex} G'$ with $G'$ being the constraint system generated by applying $\mathcal{V}_Ex$’s constraint generation to the program $\pi$ with the marked conditional $S_{TA}(\tau)$ inserted just before program location $L$. For the definition of $S_{TA}(\tau)$, we assume without loss of generality $\tau \equiv \nu_1 \lor \ldots \lor \nu_n$ to be in disjunctive normal form with all conjunctions $\nu_i$ mentioning each variable in at most one typing literal:

\[
S_{TA}(\nu \lor \tau) \overset{\Delta}{=} \text{ifPS} \ldots \text{then } S_{TA}(\nu) \text{ else } S_{TA}(\tau) \text{ fi},
\]
\[
S_{TA}([x] \in T \land \nu) \overset{\Delta}{=} S_{TA}([x] \in T); S_{TA}(\nu), \quad S_{TA}([x] \in T) \overset{\Delta}{=} x := T \cap x
\]

Note that the condition does not matter as $\mathcal{V}_Ex$ regards conditionals as nondeterministic choice. In essence, if $\tau$ has $n$ disjuncts $\nu_j$ then all paths reaching the marked conditional are split into $n$ paths and in each of them, the types $[x_i]$ of all variables $x_i \in \text{free}($\text{pr}_{x_i}$(\nu_j))$ are refined to $[x_i] \cap \text{pr}_{x_i}($\text{pr}_{x_i}$(\nu_j))$ for program locations dominated by $L$.

**D Appendix: Translation from Typings to Typing Proofs**

We will now show how to translate a given $\mathcal{V}_Ex$-typing $ty$ into a typing proof $\psi$ with $ty_{\psi} = ty$. Note that in contrast to type safety proofs this is possible for every sound typing $ty$.

Recall from Section 6.2 that for a $\mathcal{V}_Ex$-typing $ty$, the global typing invariant $I_{Ex}(ty)$ states that the types $ty([C \cdot \$_{x}]$) assigned to instance variables $C \cdot \$_{x}$ in $ty$ safely over-approximate the actual types of these variables for all runs of the program. Establishing $I_{Ex}(ty)$ as an invariant of all method bodies and the main statement is thus an important step in constructing a typing proof. Fortunately, $I_{Ex}(ty)$ can be shown to be invariant under most proof rules in our logic. The

\[\text{\textsuperscript{17}} [x] \in \{C_1\} \lor [x] \in \{C_2\} \text{ is not equivalent to } [x] \in \{C_1, C_2\}! \text{ Otherwise, no assumption could type if b then x := } \text{"foo" else x := 21 end; x + x.} \]
only complicated cases are the rules for assignment to instance variables and object creation. For these cases, we introduce new rules that explicitly preserve the global typing invariant:

**RULE: θ-IASGN**

\[
\frac{\{I_{Ex}(ty) \land \tau\} \cdot \{I_{Ex}(ty) \land \tau'[\text{this.}@x := r]\}}{\{I_{Ex}(ty) \land \tau\} \cdot \exists x \in V_I \{I_{Ex}(ty) \land \tau'\}}
\]

where \(\tau \rightarrow [\text{this}] \in \{C\}\), \(\tau' \rightarrow [r] \in T\), \(T \in 2^C\), \(T \subseteq ty([V_C.\alpha_x])\).

**AXIOM: θ-NEW**

\[
\{I_{Ex}(ty) \land \tau := \text{new}_C\} \cdot \text{new}_C\{I_{Ex}(ty) \land \tau\}
\]

Formally deriving soundness of these rules requires intricate details of the substitutions involved and rather lengthy proofs. Intuitively, assignment to instance variables preserves the global typing invariant if the assignment is compatible with the typing \((T \subseteq ty([V_C.\alpha_x])))\) and object creation does so because it initializes all instance variables to \text{null} and thus the newly created object satisfies the global typing invariant. For a detailed treatment of the substitutions, the interested reader is referred to [5,1].

The function \(\Xi\) defined below will construct typing proofs from proof steps like \(\phi_1 \rightarrow \cdots \rightarrow \phi_n\) (RULE)

where \(X\) is a conclusion of the form \(A \vdash \{p\} S\{q\}\), RULE is the name of the proof rule applied, and the \(\phi_i\) for \(i \in \mathbb{N}_n\) are subproofs establishing the premises. For reasoning about recursive method calls, the REC rule needs a set of assumptions about the methods in \(\pi\).

**Definition 8.** For a \(\text{dyn}\) program \(\pi\) with classes \(C\) each with a set of methods \(M_C\) and a typing \(ty\) for \(\pi\), the set \(A_{\pi,ty}\) of method call assumptions is

\[
A_{\pi,ty} \overset{\Delta}{=} \{(\bar{p}_{C,m})V_0.m(v_1, \ldots, v_n)\{\bar{q}_{C,m}\} | \exists C \in C, m \in M_C\}
\]

where \(\bar{p}_{C,m} \overset{\Delta}{=} I(ty) \land \{V_0\} \in \{C\} \land \bigwedge_{i=1}^n\{v_i\} \in ty([P_{C,m}^i]) \land \bigwedge_{i=1}^k\{v_0.\alpha_{v_i}\} \in ty(V_{C.\alpha_{v_i}})\) and \(\bar{q}_{C,m} \overset{\Delta}{=} I(ty) \land \{r\} \in ty([R_{C,m}])\), method \(m\) is of arity \(n\) and \(C\) has \(k\) instance variables.

We are now ready to state the definition of \(\Xi:\)

**Definition 9. Translation \(\Xi\) for Programs**

Given a \(\text{dyn}\) program \(\pi\) with a main statement \(S\) and a set of methods \(M\) and a \(V_{Ex}\)-typing \(ty\) for \(\pi\), the function \(\Xi(\pi, ty)\) yields a typing proof for \(ty\). It is defined as follows:

\[
\Xi(\pi, ty) \overset{\Delta}{=} \Xi(A_{\pi,ty} \vdash \{true\} S\{true\}, ty)
\]

\[
\Xi(A_{\pi,ty} \vdash b_{C,m}, ty) \text{ for } C.m(\bar{u})\{S_{C,m}\} \in \mathcal{M} \overset{(REC)}{\vdash \{true\} S\{true\}}
\]

with \(b_{C,m} \overset{\Delta}{=} \{\bar{p}_{C,m}\}\) begin local this, \(\bar{u} := \bar{V}; S_{C,m} end\{\bar{q}_{C,m}\}\) where \(\bar{p}_{C,m}\) and \(\bar{q}_{C,m}\) are given in Definition 8.
Definition 10. Translation \( \Xi \) for Statements and Expressions

For a given \( \mathcal{V}_{\Xi \Theta} \)-typing \( \text{ty} \) of a dyn statement \( S \) and a Hoare logic statement \( X \) of the form

\[
X \equiv A \vdash \{ I(\text{ty}) \land \Xi(\text{ty}, \downarrow S) \} S \{ I(\text{ty}) \land \Xi(\text{ty}, S^1) \}
\]

with \( A \) a set of assumptions, \( \Xi(X, \text{ty}) \) yields a typing proof for \( \text{ty} \) with preconditions \( \Xi(\text{ty}, \downarrow S) \) and under the assumptions \( A \).

\( \Xi \) is defined inductively over the structure of \( S \) and in essence models the reasoning of the verifier \( \mathcal{V}_{\Xi \Theta} \) as an equivalent combination of Hoare rule applications. The Hoare triples of above form are then assembled into a full typing proof.

if \( S \equiv \text{null} \), \( \Xi(\text{ty}, \downarrow S) \equiv \Xi(\text{ty}, S^1)[r := \text{null}] \) then

\[
\Xi(X, ty) = \frac{A \vdash \{ I(\text{ty}) \land \Xi(\text{ty}, \downarrow S) \} \text{null}\{ I(\text{ty}) \land \Xi(\text{ty}, S^1) \}}{(\text{CONST})^S}
\]

if \( S \equiv u \), \( \Xi(\text{ty}, \downarrow S) \equiv \Xi(\text{ty}, S^1)[r := u] \) then

\[
\Xi(X, ty) = \frac{A \vdash \{ I(\text{ty}) \land \Xi(\text{ty}, \downarrow S) \} u\{ I(\text{ty}) \land \Xi(\text{ty}, S^1) \}}{(\text{VAR})}
\]

if \( S \equiv \text{@x}, \Xi(\text{ty}, \downarrow S) \equiv \Xi(\text{ty}, S^1)[\text{this}@x := r] \) then

\[
\Xi(X, ty) = \frac{A \vdash \{ I(\text{ty}) \land \Xi(\text{ty}, \downarrow S) \} \text{@x}\{ I(\text{ty}) \land \Xi(\text{ty}, S^1) \}}{(\text{IVAR})}
\]

if \( S \equiv u := e \), \( \Xi(\text{ty}, \downarrow S) \rightarrow \Xi(\text{ty}, e^1) \), \( \Xi(\text{ty}, e^1) \rightarrow \Xi(\text{ty}, S^1)[u := r] \) then

\[
\Xi(X, ty) = \frac{A \vdash \{ I(\text{ty}) \land \Xi(\text{ty}, \downarrow S) \} e\{ I(\text{ty}) \land \Xi(\text{ty}, S^1)[u := r] \}}{(\text{CONS})}
\]

\[
\Xi(X, ty) = \frac{A \vdash \{ I(\text{ty}) \land \Xi(\text{ty}, \downarrow S) \} e\{ I(\text{ty}) \land \Xi(\text{ty}, S^1)[u := r] \}}{(\text{ASGN})}
\]

if \( S \equiv \text{@x} := e \), \( \Xi(\text{ty}, \downarrow S) \rightarrow \Xi(\text{ty}, e^1) \), \( \Xi(\text{ty}, e^1) \rightarrow \Xi(\text{ty}, S^1)[\text{this}@x := r] \) then

\[
\Xi(X, ty) = \frac{A \vdash \{ I(\text{ty}) \land \Xi(\text{ty}, \downarrow S) \} \text{@x} := e\{ I(\text{ty}) \land \Xi(\text{ty}, S^1) \}}{(\text{CONS})}
\]

\[
\Xi(X, ty) = \frac{A \vdash \{ I(\text{ty}) \land \Xi(\text{ty}, \downarrow S) \} \text{@x} := e\{ I(\text{ty}) \land \Xi(\text{ty}, S^1) \}}{(\text{CONS})}
\]

if \( S \equiv \text{if} \text{then} S_1 \text{else} S_2 \text{ end}, \Xi(\text{ty}, e^1) \rightarrow [r] \in \{ \text{bool} \}, \Xi(\text{ty}, \downarrow S) \rightarrow \Xi(\text{ty}, e^1), \Xi(\text{ty}, e^1) \rightarrow \Xi(\text{ty}, S^1), \Xi(\text{ty}, e^1) \rightarrow \Xi(\text{ty}, S^2), \Xi(\text{ty}, S^2) \rightarrow \Xi(\text{ty}, S^1) \) then

\[
\Xi(X, ty) =
\]
Lemma 2. For every dyn program π and every consistent Vect typing ty of π, \( \Xi(\pi, ty) \) can be constructed and is a valid typing proof for ty in π.

Proof. By induction over the structure of the program π comparing the application conditions of Hoare logic rules, the typing rules for Vect and the preconditions for the respective cases in the translation \( \Xi \).

Note that this implies soundness of \( \text{Vect} \).

E Appendix: Omitted Proofs

Proof for Theorem 1

Proof. By definition of \( \Theta(\sigma) \), for all variables x of a base type T in \( \sigma \), \( \Theta(\sigma) \models \text{safe}_T(x) \) holds and x can hence be safely mapped. Under the assumption that for all such variables x it holds that \( \llbracket x \rrbracket(\Theta(\sigma)) = \llbracket x \rrbracket(\sigma) \), the following lemma can be established by induction over the structure of the assertion language: \( \llbracket \cdot \rrbracket(\sigma) = \llbracket \Theta(l) \rrbracket(\Theta(\sigma)) \) for all logical expressions l and stat states \( \sigma \). As the assumption is guaranteed by the mapping predicates introduced by \( \mathcal{T}_M \), the desired result can then be established by induction over the structure of the assertion language. 

Proof for Theorem 2

Proof. By induction over the structure of the proof \( \phi \), using Theorem 1 and the fact that the application conditions for the pure expression rules are satisfied when \( S \) is a statically typed program and all assertions where translated using \( \Theta \).
Proof for Lemma 1

Proof. $p' \equiv p \land \tau$ has the described properties.

Proof for Theorem 3

Proof. Follows from Lemma ?? and the definition of the most precise typing assertion.

Proof for Theorem 4

Proof. Follows from completeness of the Hoare logic, Theorem 3 and the fact that type safety proofs must establish the absence of type errors.

Proof for Lemma 2

Proof. By comparing the application conditions for cases of $\Xi$ with the typing rules given in Figure 9 and the side conditions of applied Hoare-logic rules. For instance, in the case for assignment, consistency of the typing implies $ty([e]) = ty([u := e])$ and $ty([e]) \subseteq ty([u])$ and that $u$ is the only variable whose (flow-sensitive) type may have changed between $e \downarrow$ and $S \downarrow$ ($r$ stays the same).

With the definition of $\Xi([ty],L)$ we conclude that $\Xi([ty],L)$ if $u := r \land [u] \in t$ for some type $t$. \hfill \Box$

Proof of Theorem 5

Proof. Let $ty \xrightarrow{\tau L} ty'$ be a conjunctive refinement step and $x \in \mathcal{V}_{C.m}$. Then $ty'([x],L) = ty([x],L) \cap \Omega_X^L(\tau) \subseteq ty([x],L)$. This difference is induced by the constraints generated from $\mathcal{R}_\tau$. Since all other constraints are identical between $\pi$ and $\pi'$ and all constraints are monotonic, $ty'([S],L) \subseteq ty([S],L)$ for all sub-statements $S$ of $\pi$ and consequently $ty' \subseteq ty$ follows by induction over the constraint system. \hfill \Box$

Proof for Theorem 6

For our proof we need the following definition:

Definition 11 (Fusion of Hoare Logic Proofs). Let $\phi$ be a Hoare logic proof for $\{p\}S\{q\}$ in some notion of correctness $X$ and $\varphi$ be a typing proof for $\{\tau\}S\{\tau'\}$. Then, the fusion $\phi + \varphi$ is a two-layered proof for $\{p \land \tau\}S\{q \land \tau'\}$ in the sense of $X$-correctness.

WLOG, we assume $\phi$ and $\varphi$ to be minimal (all Hoare triples contribute to the proof’s conclusion). They hence have a tree-like structure. Their fusion can then be constructed by recursion over this structure.

Induction basis = Fusing axioms. All axioms in our Hoare logic (Appendix B), are invariant under conjunction: if $\{p\}S\{q\}$ and $\{\tau\}S\{\tau'\}$ can be derived using this axiom, then $\{p \land \tau\}S\{q \land \tau'\}$ can also.

Induction step = Fusing rules. All rules in our Hoare logic have the following properties
– Invariant under fusion: If
\[
\frac{\{p_1\} S_1 \{q_1\}, \ldots, \{p_n\} S_n \{q_n\}}{(X)}
\]
and
\[
\frac{\{\tau_1\} S_1 \{\tau_1'\}, \ldots, \{\tau_n\} S_n \{\tau_n'\}}{(X)}
\]
are valid rule applications, then
\[
\frac{\{p_1 \wedge \tau_1\} S_1 \{q_1 \wedge \tau_1'\}, \ldots, \{p_n \wedge \tau_n\} S_n \{q_n \wedge \tau_n'\}}{(X)}
\]
is also.

– They are either syntax-directed (and hence must appear in both proofs) or have a neutral application \(\{p\} S\{q\}\) (X) that can be inserted into a proof to make its structure match the other one (having an application of rule \(X\)).

For most rules, this is obvious. For applications of CONJ and DISJ, one needs to fuse the proof with both premises. To see that the properties hold for the SUBST rule, consider that all variables occurring in typing assertions are being read in some method of the program (otherwise, typing them is useless). Hence, the side-condition of the SUBST rule does not allow them to be substituted for and all applications of this rule hence are neutral for all typing assertions.

Both proofs can hence be made structurally equivalent by inserting neutral rule applications and then fused using the invariance property. □