Fekete-Szegő Inequalities for Certain Subclasses of Bi-Univalent Functions

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Abstract. In this paper, we consider the subclasses $\mathcal{H}_{\sigma}^{a,b;c}(\varphi)$ and $\mathcal{B}_{\sigma}^{a,b;c}(\varphi,\lambda)$ of bi-univalent functions associated with the Hohlov operator. These subclasses are defined in terms of subordination. The Fekete-Szegő inequalities are determined and some previous results can be obtained using these findings.

1. Introduction

Let $A$ be the class of functions $f(z)$ of the following Taylor-Maclaurin series form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are normalized and analytic in the open unit disk $D := \{ z : z \in C, |z|<1 \}$. 

The convolution or Hadamard product of two analytic functions $f, h \in A$ is the analytic functions denoted by $f \ast h$ and it is defined as:

$$(f \ast h)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n ,$$

where $f$ is given by (1) and $h = z + \sum_{n=2}^{\infty} b_n z^n$.

Given the complex parameters of $a, b$ and $c$ where $(c \neq 0, -1, -2, -3,...)$ the Gaussian hypergeometric function $\, _2F_1(a,b,c;z)$ is defined as:
\[
_{2}F_{1}(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_{n}(b)_{n}}{(c)_{n}n!} z^{n}
\]

\[
= 1 + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} z^{n-1} \quad (z \in D)
\]  

(2)

where \((\alpha)_{n}\) is symbol of Pochhammer, given by

\[
(\alpha)_{n} = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \begin{cases} 
1 & \text{if } n = 0 \\
\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1) & \text{if } n = 1, 2, 3,\ldots
\end{cases}
\]

The familiar Hohlov [1,2] convolution operator \(I_{a,b,c}\) which is a special case of the Dziok-Srivastava operator [3,4] is introduced as follows:

\[
I_{a,b,c}f(z) = z_{2}F_{1}(a, b, c; z) \ast f(z) = z + \sum_{n=2}^{\infty} \phi_{n}a_{n}z^{n} \quad (z \in D)
\]  

(3)

where \(\phi_{n} = \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}\).

Hohlov discussed some interesting geometrical properties shown by operator \(I_{a,b,c}\) and the three-parameter family of operators \(I_{a,b,c}\) contain, as its special cases, most of the known linear integral or differential operators. In particular, if \(b = 1\) in (2), then \(I_{a,b,c}\) reduces to the Carlson-Shaffer operator, a special case of Hohlov operator. Meanwhile, Hohlov operator \(I_{a,b,c}\) is a generalization of the Ruscheweyh derivative operator and Bernadi-Libera-Livingston operator.

Let \(S\) be the subclass of \(A\) which has univalent functions in \(D\). For each \(f \in S\), there is an inverse function \(f^{-1}\) in the neighborhood. According to the Koebe one-quarter theorem [5], \(f^{-1}\) which can be defined in some neighborhood of the origin contain a disk radius of \(\frac{1}{4}\), which \(|w| < \frac{1}{4}\). In some cases, \(f^{-1}\) can be extended to whole \(D\). Obviously, \(f^{-1}\) is also a univalent function. For that, every univalent function \(f\) on \(D\) with inverse \(f^{-1}\) is defined as:

\[
f^{-1}(f(z)) = z, \quad z \in D
\]

and

\[
f(f^{-1}(w)) = w, |w| < r_0(f), r_0(f) \geq \frac{1}{4}
\]
where \( f^{-1}(w) = w - a_1 w^2 + (2a_2^2 - a_1) w^3 - (5a_3^3 - 5a_2 a_1 + a_4) w^4 + \ldots \)

A function \( f \in A \) is said to be bi-univalent in \( D \) if both \( f \) and \( f^{-1} \) are univalent in \( D \). Let us denote the class of bi-univalent functions in \( D \) as \( \sigma \) given in expansion (1). Some examples of functions in the class \( \sigma \) are as follows:

\[
\frac{z}{1-z}, \quad -\log(1-z) \quad \text{and} \quad \frac{1}{z} \log\left(\frac{1+z}{1-z}\right).
\]

For brief history and interesting examples in the class \( \sigma \) (see [6]). Lewin [7] was the first mathematician to study bi-univalent class functions in 1967 and found \( |a_2| < 1.51 \). Subsequently, in 1979, Brannan and Clunie [8] conjectured that the bound \( \sqrt{2} \) for modulus of initial coefficient \( |a_2| \) and Brannan and Taha [9] theorized that \( |a_2| \leq 1 \) in 1986. Due to that, it is interesting to discuss on so-called bi-univalent functions problem.

In addition, Brannan and Taha have also introduced several certain subclasses of the bi-univalent function class \( \sigma \) which are similar to the commonly used subclasses, starlike and convex functions of order \( \alpha \) \((0 < \alpha \leq 1)\). For each function classes of bi-starlike and bi-convex functions of order \( \alpha \), non-sharp estimates on the initial coefficients are found. In fact, in the Srivastava’s et al [6] work, they were basically investigated the diversified subclasses of the bi-univalent function class \( \sigma \) in recent times. Currently, many researchers explore on bounds for various subclasses of bi-univalent functions [6,10,11,12,13,14,15]. The study of bi-univalent functions is the state-of-the-art topic in the study of geometric function theory. Some researchers like [6,15,16] introduce new subclasses of the bi-univalent function class \( \sigma \) and obtained non-sharp bounds on the initial coefficient \( |a_2| \text{ and } |a_3| \). Accordingly, the coefficient estimate problem for each Taylor-Maclaurin series \( |a_n| \) \((n \in N \setminus \{1,2\} : N = \{1,2,3,\ldots\}\)

is still an open problem. Hence, the need to find coefficient values should be emphasized.

Given two functions \( f, g \in A \). The analytic function \( f \) is said to be subordinate to the analytic function \( g \) in \( D \) as follows:

\[ f(z) \prec g(z) \]

if there is a Schwarz function \( w(z) \), analytic in \( D \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) for all \( z \in D \) satisfying the following conditions:

\[ f(z) = g(w(z)) \text{ for all } z \in D. \]

Furthermore, if the function \( g \) is univalent in \( D \), then we have the following equivalence [17]:

\[ f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(D) \subset g(D). \]

Let \( \varphi(z) \) is an analytic function with positive real part on \( D \), where \( \varphi(0) = 1, \varphi'(0) > 0 \) and \( \varphi \) maps the unit disk \( D \) on starlike region with respect to 1 which is symmetric to the real axis. Ma and Minda [18] unified various subclasses of starlike and convex functions. \( \varphi \) has a series expansion of the form
\[ \varphi(z) = 1 + B_1 z + B_2 z^2 \ldots \]  

where \( B_1, B_2, B_3, \ldots \) are real and \( B_1 > 0 \). Let \( S^*(\varphi) \) be the Ma-Minda class of starlike function in \( f(z) \in A \) for which

\[
\frac{zf'(z)}{f(z)} < \varphi(z), \quad (z \in D)
\]

and \( C(\varphi) \) be the class of Ma-Minda convex function in \( f(z) \in A \) for which

\[
1 + \frac{zf''(z)}{f'(z)} < \varphi(z), \quad (z \in D).
\]

Ma and Minda had also obtained the Feketo-Szegö inequality for functions in the class \( C(\varphi) \). Given \( f \in C(\varphi) \) if and only if \( zf'(z) \in S^*(\varphi) \), Feketo-Szegö inequality for functions in the Ma-Minda starlike class can be obtained. Feketo-Szegö is the inequality for coefficients of univalent analytic functions discovered by Feketo and Szegö in 1933, which is related to the Bieberbach conjecture. Feketo-Szegö is widely used since that year. With the known value of \( |a_2| \) and \( |a_3| \), it is natural to find the relationship between \( a_3 \) and \( a_2^2 \) for the class \( C(\varphi) \). The Feketo-Szegö inequality states if (1) is a univalent analytic function on \( D \), then

\[
|a_3 - \lambda a_2^2| \leq \begin{cases} 3 - 4\lambda & \text{if } \lambda \leq 0 \\ 1 + 2e^{(\frac{2\lambda}{1-\lambda})} & \text{if } 0 \leq \lambda < 1 \\ 4\lambda - 3 & \text{if } \lambda \geq 1 \end{cases}
\]

This classical result of Feketo and Szegö specifies the maximum value of \( |a_3 - \lambda a_2^2| \), as non-linear functional of the real parameter \( \lambda \), for the class of normalized univalent functions (1).

Presently, some results obtained in the previous literature, each of which deals with \( |a_3 - \lambda a_2^2| \), for various classes of functions are defined in terms of subordination [20,21,22,23]. For some history of Feketo-Szegö problem for class of starlike, convex and close-to-convex functions, refer to work produced by by Srivastava et al. [22]. Besides that, some authors [24,25] have studied the Feketo-Szegö inequalities for certain subclasses of bi-univalent associated with operators. Therefore, the main focus of this paper is to introduce new subclasses of \( \mathcal{H}_{\sigma}^{a,b,c}(\varphi) \) and \( \mathcal{B}_{\sigma}^{a,b,c}(\varphi,\lambda) \) of bi-univalent functions associated with Hohlov operator based on concept of Ma-Minda. Subordination is used to define subclasses. Finally, the Feketo-Szegö inequalities are determined and by using these findings, some previous results can be obtained.

Firstly, we introduce two subclasses of bi-univalent functions using the Hohlov operator.

**Definition 1.** Let \( \varphi \) is given in (4), a function \( f \in \sigma \) is said to be in the class \( \mathcal{H}_{\sigma}^{a,b,c}(\varphi) \) if the following conditions are satisfied:

\[
\left[ I_{a,b,c} f(z) \right]^\prime < \varphi(z)
\]

and
Proof of Theorem 1

Now, we begin by concerning the Fekete-Szegő inequality.

Lemma 1. Let \( f \) be of the form (1) be in the class \( \mathcal{H}_\sigma^{a,b,c}(\varphi) \) and \( \mu \in \mathbb{R} \). Then

\[
|a_3 - \mu a_2^2| \leq \left\{ \begin{array}{ll}
\frac{B_1^3 |(1 - \mu)|}{|3 \phi_3 B_1^2 + 4 \phi_2^2(B_1 - B_2)|} & \text{for } |1 - \mu| \geq 1 + \frac{4 \phi_2^2(B_1 - B_2)}{3 \phi_3 B_1^2} \\
\frac{B_1}{3 \phi_3} & \text{for } |1 - \mu| \leq 1 + \frac{4 \phi_2^2(B_1 - B_2)}{3 \phi_3 B_1^2} \end{array} \right.
\]

(6)

Proof: Let \( f \) be in \( \mathcal{H}_\sigma^{a,b,c}(\varphi) \) and \( g = f^{-1} \). Then, there exist analytic functions \( u \) and \( v \) in...
$D$ with $u(0) = v(0) = 0$ and $|u(z)| < 1, |v(z)| < 1$ for all $z \in D$. From Definition 1, it satisfies

$$[I_{a,b,c}f(z)]' = \varphi(u(z))$$

and

$$[I_{a,b,c}g(w)]' = \varphi(v(w)). \tag{7}$$

Let $p$ and $q$ are analytic functions in $D$ with $p(0) = q(0) = 1$ and have positive real parts in $D$. Define these two functions as:

$$p(z) := \frac{1+u(z)}{1-u(z)} = 1 + p_1 z + p_2 z^2 + \ldots$$

and

$$q(z) := \frac{1+v(z)}{1-v(z)} = 1 + q_1 z + q_2 z^2 + \ldots. \tag{8}$$

It can be derived that

$$u(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2} \left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2 + \ldots\right] \tag{9}$$

and

$$v(z) = \frac{q(z) - 1}{q(z) + 1} = \frac{1}{2} \left[q_1 z + \left(q_2 - \frac{q_1^2}{2}\right) z^2 + \ldots\right].$$

Furthermore, from (8) and making use (4), it follows that

$$\varphi(u(z)) = \varphi\left[\frac{p(z) - 1}{p(z) + 1}\right] = 1 + \frac{1}{2} B_1 p_1 z + \left[\frac{1}{2} B_1 \left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4} B_2 p_1^2\right] z^2 + \ldots \tag{10}$$

and

$$\varphi(v(z)) = \varphi\left[\frac{q(w) - 1}{q(w) + 1}\right] = 1 + \frac{1}{2} B_1 q_1 w + \left[\frac{1}{2} B_1 \left(q_2 - \frac{q_1^2}{2}\right) + \frac{1}{4} B_2 q_1^2\right] w^2 + \ldots. \tag{11}$$

On the other hand,

$$[I_{a,b,c}f(z)]' = 1 + 2\varphi_2 a_2 z + 3\varphi_3 a_3 z^2 + \ldots \tag{11}$$

and

$$[I_{a,b,c}g(w)]' = 1 - 2\varphi_2 a_2 w + 3\varphi_3 (2a_2^2 - a_3) w^2 - \ldots \tag{12}$$

Now, by comparing the coefficients in (9)-(12), we obtain
\[ 2 \varnothing_z a_z = \frac{1}{2} B_1 p_1 \]  
(13)

\[ 3 \varnothing_3 a_3 = \frac{1}{2} B_1 \left( p_2 - \frac{p_1^2}{2} \right) + \frac{1}{4} B_2 p_1^2 \]  
(14)

\[ -2 \varnothing_z a_z = \frac{1}{2} B_1 q_1 \]  
(15)

and

\[ 3 \varnothing_3 (2a_z^2 - a_3) = \frac{1}{2} B_1 \left( q_2 - \frac{q_1^2}{2} \right) + \frac{1}{4} B_2 q_1^2 . \]  
(16)

From (13) and (15), there is

\[ p_1 = -q_1 \]  
(17)

and

\[ \frac{32 \varnothing_z^2 a_z^2}{B_1^2} = p_1^2 + q_1^2 . \]  
(18)

From (14), (16) and (18), it can be derived that

\[ a_z^2 = \frac{B_1^2 (p_2 + q_2)}{4 \varnothing_3 B_1^2 + 4 \varnothing_3^2 (B_1 - B_2)} . \]  
(19)

By subtracting (16) from (14) and making use (17) with further computations, we have

\[ a_3 = a_z^2 + \frac{1}{12 \varnothing_3} B_1 (p_2 - q_2) . \]  
(20)

Now, applying (19) and (20), we get

\[ |a_3 - \mu a_z^2| = \frac{B_1}{4} \left| \left( H(\mu) + \frac{1}{3 \varnothing_3} \right) p_2 + \left( H(\mu) - \frac{1}{3 \varnothing_3} \right) q_2 \right| \]

where

\[ H(\mu) = \frac{B_1^2 (1 - \mu)}{3 \varnothing_3 B_1^2 + 4 \varnothing_3^2 (B_1 - B_2)} . \]

Since all \( B_j \) are real, \( B_1 > 0, |p_2| \leq 2, |q_2| \leq 2 \) and from Lemma 1, we conclude that

\[ |a_3 - \mu a_z^2| = \begin{cases} 
B_1 |H(\mu)| & \text{for } |H(\mu)| \geq \frac{1}{3 \varnothing_3} \\
\frac{B_1}{3 \varnothing_3} & \text{for } |H(\mu)| \leq \frac{1}{3 \varnothing_3} 
\end{cases} \]

which completes the proof.

**Theorem 2.** Let \( f \) be given by (1) be in the class \( \mathfrak{B}_b^{a,b,c}(\varphi, \lambda), \lambda \geq 1 \) and \( \mu \in \mathbb{R} \). Then
Applying the similar method as in the earlier proof, we get

\[ |a_3 - \mu a_3^2| \leq \frac{B_1^3 (1 - \mu)}{|(1 + 2\lambda) \varnothing_3 B_1^2 + (1 + \lambda)^2 \varnothing_2^2 (B_1 - B_2)|} \quad \text{for } |1 - \mu| \geq \frac{1 + (1 + \lambda)^2 \varnothing_2^2 (B_1 - B_2)}{(1 + 2\lambda) \varnothing_3 B_1^2}. \]

**Proof.** For \( f \in \mathcal{S}_a^{b;c}(\varphi, \lambda) \) and \( g = f^{-1} \), there are analytic functions \( u \) and \( v \) in \( D \) with \( u(0) = v(0) = 0 \) and \( |u(z)| < 1 \) for all \( z \in D \). From definition 2, it follows that

\[
(1 - \lambda) \frac{I_{a,b,c}(z)}{z} + \lambda I_{a,b,c}(z)' = \varphi(u(z))
\]

(21)

and

\[
(1 - \lambda) \frac{I_{a,b,c'}(w)}{w} + \lambda I_{a,b,c'}(w)' = \varphi(v(w)).
\]

(22)

Since

\[
(1 - \lambda) \frac{I_{a,b,c}(z)}{z} + \lambda I_{a,b,c}(z)' = 1 + (1 + \lambda) \varnothing_2 a_2 z + (1 + 2\lambda) \varnothing_2 a_3 z^2 + \ldots
\]

and

\[
(1 - \lambda) \frac{I_{a,b,c}(w)}{w} + \lambda I_{a,b,c}(w)' = 1 - (1 + \lambda) \varnothing_2 a_2 w + (1 + 2\lambda) \varnothing_3 (2a_2^2 - a_3) w^2 + \ldots
\]

Therefore, from (9) and (10) together with (21) and (22), it follows that

\[
(1 + \lambda) \varnothing_2 a_2 = \frac{1}{2} B_1 p_1
\]

(23)

\[
(1 + 2\lambda) \varnothing_3 a_3 = \frac{1}{2} B_1 \left( p_2 - \frac{p_1^2}{2} \right) + \frac{1}{4} B_2 p_1^2
\]

(24)

\[-(1 + \lambda) \varnothing_2 a_2 = \frac{1}{2} B_1 q_1
\]

(25)

and

\[
(1 + 2\lambda) \varnothing_3 (2a_2^2 - a_3) = \frac{1}{2} B_1 \left( q_2 - \frac{q_1^2}{2} \right) + \frac{1}{4} B_2 q_1^2.
\]

(26)

Computation from (23) - (26) yield

\[
a_2^2 = \frac{B_1^3 (p_2 + q_2)}{4[B_1^2 (1 + 2\lambda) \varnothing_3 + (B_1 - B_2)(1 + \lambda)^2 \varnothing_2^2]}
\]

(27)

Applying the similar method as in the earlier proof, we get
\[ a_3 = a_2^2 + \frac{1}{4(1+2\lambda)} B_1 (p_2 - q_2). \]  

(28)

Furthermore, making use (27) and (28), we obtain

\[ |a_3 - \mu a_2^2| = \frac{B_1}{4} \left| \left( H(\mu) + \frac{1}{(1 + 2\lambda)\psi_3} \right) p_2 + \left( H(\mu) - \frac{1}{(1 + 2\lambda)\psi_3} \right) q_2 \right| \]

where

\[ H(\mu) = \frac{B_1^2 (1 - \mu)}{(1 + 2\lambda)\psi_3 B_1^2 + (1 + \lambda)^2 \psi_2 (B_1 - B_2)}. \]

Therefore, since all \( B_j \) are real, \( B_1 > 0, |p_2| \leq 2, |q_2| \leq 2 \) and applying Lemma 1, we have

\[ |a_3 - \mu a_2^2| \leq \begin{cases} 
B_1 |H(\mu)| & \text{for } |H(\mu)| \geq \frac{1}{(1 + 2\lambda)\psi_3} \\
B_1 \frac{1}{(1 + 2\lambda)\psi_3} & \text{for } |H(\mu)| \leq \frac{1}{(1 + 2\lambda)\psi_3}
\end{cases} \]

and proves the result.

**Remark:** For \( a = c \) and \( b = 1 \), Theorem 1 reduces to the first theorem in [28]. Also, the result from Theorem 2 will be a result of Theorem 2.7 in [20].

**Acknowledgments**

This research is supported by Malaysian Ministry of Higher Education grant FRGS/1/2016/STG06/UiTM/02/1 and Universiti Teknologi Mara Cawangan Sabah.

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