HIGHER INTEGRABILITY OF WEAK SOLUTION OF A NONLINEAR PROBLEM ARISING IN THE ELECTRORHEOLOGICAL FLUIDS

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ABSTRACT. In this paper, we study the Dirichlet problem arising in the electrorheological fluids

\[ \begin{cases} -\text{div} \, a(x, Du) = k(u^{\gamma-1} - u^{\beta-1}) & x \in \Omega, \\ u = 0 & x \in \partial \Omega, \end{cases} \]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) and \( \text{div} \, a(x, Du) \) is a \( p(x) \)-Laplace type operator with \( 1 < \beta < \gamma < \inf_{x \in \Omega} p(x) \), \( p(x) \in (1, 2] \). By establish a reversed Hölder inequality, we show that for any suitable \( \gamma, \beta \), the weak solution of previous equation has bounded \( p(x) \) energy satisfies \( |Du|^{p(x)} \in L^\delta_{\text{loc}} \) with some \( \delta > 1 \).

1. Introduction. In this paper, we consider the following nonlinear boundary value problem

\[ \begin{cases} -\text{div} \, a(x, Du) = f(x, u) & x \in \Omega, \\ u = 0 & x \in \partial \Omega, \end{cases} \] (1.1)

where \( \Omega \subseteq \mathbb{R}^n \), \( n \geq 3 \) is a bounded domain with smooth boundary, \( p(x) > 1 \) and \( p(x) \in C(\bar{\Omega}) \). Let us make some remarks concerning problems with \( p(x) \)-growth in general. On one hand, the \( p(x) \)-growth problem present the borderline case between standard \( p \)-growth [1] and so called \( (p, q) \) growth conditions introduced in [13], therefore involving delicate perturbation arguments to treat the variable growth situation. On the other hand, the study of variational problems involving \( p(x) \)-growth conditions is a consequence of their applications. Such as image restoration considered in [2] and electrorheological fluids, at this point, we must mention about electrorheological fluids. These are special fluids characterized by their ability to change in a dramatic way their mechanical properties when in presence of an external electromagnetic field. According to the model proposed by Rajagopal and Růžička [10], the system can be looked as the delicate interaction

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between the electromagnetic fields and the moving fluids:

\[
\begin{aligned}
\text{div } E &= 0 \quad \text{curl } E = 0 \quad \text{in } \Omega, \\
u_t - \text{div } a(x, \varepsilon(u), E) + \text{div } (u \otimes u) + D\phi = f & \quad \text{in } \Omega, \\
\text{div } u &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]  

(1.2)

where \(E(x)\) is the electromagnetic field, \(u : \Omega(\subset R^3) \rightarrow R^3\) is the velocity of field, \(\varepsilon(u)\) is the symmetric part of the gradient, \(a(x, \varepsilon(u), E)\) is the extra stress tensor and \(\phi\) is the pressure. The main point in the previous system as just mentioned, is that the monotone vector field \(a : R^9 \rightarrow R^9\) depends in a nonlinear way by \(\varepsilon(u)\):

\[
a(x, \varepsilon(u), E) \approx \mu (1 + |\varepsilon(u)|^2)^\frac{p(\varepsilon(u))^2 - 2}{2} \varepsilon(u) + \text{terms with similar growth}.
\]

For the equation like (1.1) above, Rajagopal and Ružička established an existence theory in [12], which is particularly satisfying in the steady case in [11] read as

\[
\begin{aligned}
-\text{div } a(x, \varepsilon(u)) + \text{div } (u \otimes u) + D\phi &= f(x), \\
\text{div } u &= 0.
\end{aligned}
\]  

(1.3)

Note the connection between (1.1) and (1.3), our paper can be regard as a generalize for (1.3). Notice that the existence and multiplicity of solution for problems with \(p(x)-\text{growth}\) have been established in [14]. Based on the result in [14], then we proved higher integrability of weak solutions for equation like (1.1). For more details we consider the problem of the type

\[
\begin{aligned}
-\text{div } a(x, Du) &= k(u^{\gamma-1} - u^{\beta-1}) \quad x \in \Omega, \\
u &= 0 \quad x \in \partial \Omega, \\
u &\geq 0 \quad x \in \Omega,
\end{aligned}
\]

(1.4)

where \(1 < \beta < \gamma < \inf_{x \in \Omega} p(x)\), and \(k > k_0\) for a positive constant \(k_0\) defined in [14]. Here, \(\Omega \subset R^n\) is a bounded domain, the continuous vector field \(a : \Omega \times R^n \rightarrow R^n\) is assumed to be \(C^1\)-regular in the gradient variable \(z \in R^n\) satisfy the following structure conditions

\[
|a(x, \xi)| \leq L (1 + |\xi|^{p(x)-1}),
\]

(1.5)

\[
a(x, \xi) \cdot \xi \geq \nu |\xi|^{p(x)},
\]

(1.6)

for all \(x \in \Omega\) and \(\xi \in R^n\) with \(L \geq 1, \nu \in (0, 1]\). Throughout this paper, we assume that \(2 \geq p(x) > 1, p(x) \in C^{0, \alpha}(\Omega)\) with \(\alpha \in (0, 1]\). Furthermore,

\[
1 < \gamma_1 \leq p(x) \leq \gamma_2 \leq 2, \quad \text{and} \quad |p(x) - p(y)| \leq \omega(|x - y|),
\]

(1.7)

for all \(x, y \in \Omega\), where \(\gamma_1 := \min_{z \in \Omega} p(x)\) and \(\gamma_2 := \max_{z \in \Omega} p(x) + \frac{1}{\log \omega} \omega : R^+ \rightarrow R^+\) is the modulus of continuity of \(p(x)\), which satisfies

\[
\begin{aligned}
\omega(6R_0) &\leq \frac{(\theta - 1)\gamma_1}{2}, \\
\omega(R) \log \frac{1}{R} &\leq L, \quad \text{for all } R < 1,
\end{aligned}
\]

(1.8)

where \(R_0 < 1, \theta > 1\) will be specified in later.

In what follows, we shall repeat write \(p(\cdot)\) instead of \(p(x)\) if there is no danger of confusion and our aim is to prove such a higher integrability for solutions of degenerate elliptic equation with \(p(x)-\text{growth}\). The result read as follows
Theorem 1.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then for any suitable $\gamma, \beta$, there exist a constant $\delta > 1$ such that for any solution $u \in W_{0}^{1,p(\cdot)}(\Omega)$ of (1.4) with $p(x) \in (1,2]$ under the assumption (1.5)–(1.8) there holds
$$|Du|^{p(x)} \in L^\delta_{\text{loc}}(\Omega).$$
Moreover, there exist $R_0 = R_0(n, \gamma_1, \gamma_2, \omega(\cdot), ||Du|^{p(\cdot)(1-(a-1)/p_1)}||)$ such that for any ball $B_p$, with $B_{3p} \subset \subset \Omega$ and $p \leq R_0$ there holds
$$\int_{B_p} |Du|^{\delta p(x)} dx \leq c_1 \left( \int_{B_{3p}} |Du|^{p(x)} \left( 1 - \frac{n-1}{p(x)} \right) dx \right)^{\delta / \left( 1 - \frac{n-1}{p(x)} \right)} + c_2 \left( \int_{B_{3p}} |u|^{p(x)} dx + 1 \right), \quad (1.9)$$
where $c_1 = c_1(n)$, $c_2 = c_2(n, \nu, \gamma_1, \gamma_2, L, k)$, and $p_1, a$ will be specified in later.
Indeed, (1.9) is a proper reverse Hölder inequality, i.e., the constants $c_1, c_2$ are independent of the energy of the weak solution.

2. Preliminaries.

2.1. Some notes and definition. We recall in what follows some definitions and basic properties of the generalized Lebesgue-Sobolev spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$. Let $P(\Omega)$ is the set consist with Lebesgue measurable function $p : \Omega \rightarrow [1, \infty]$, where $\Omega \subset \mathbb{R}^n (n \geq 2)$ nonempty. For all measurable function $u$, define
$$\rho_{p(x)}(u) = \int_{\Omega, \Omega_{\infty}} |u|^{p(x)} dx + \sup_{x \in \Omega_{\infty}} |u(x)|,$$
where $\Omega_{\infty} = \{ x \in \Omega : p(x) = \infty \}$.
Variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ consist of $u$ that satisfies the property: $\exists t_0 > 0$ s.t. $\rho_{p(x)}(t_0 u) < \infty$. $\forall u \in L^{p(\cdot)}(\Omega)$, define
$$\|u\|_{p(\cdot)} = \inf \left\{ t > 0 : \rho_{p(x)} \left( \frac{u}{t} \right) \leq 1 \right\},$$
then $L^{p(\cdot)}(\Omega)$ endowed with the norm above is a Banach space. $\forall p \in P(\Omega)$, define its dual exponent :
$$p'(x) = \begin{cases} \infty, & x \in \Omega_1 = \{ x \in \Omega : p(x) = 1 \}, \\
1, & x \in \Omega_{\infty}, \\
p(x) \frac{(p(x) - 1)}{p(x)}, & \text{other,} \end{cases} \quad (2.1)$$
the generalized Sobolev space $(W^{k,p(\cdot)}(\Omega), k \in \mathbb{N}, \| \cdot \|_{k,p})$, is defined as the set
$$W^{k,p(\cdot)}(\Omega) = \{ f \in L^{p(\cdot)}(\Omega); \partial^\alpha f \in L^{p(\cdot)}(\Omega), \text{for all } \alpha \leq k \},$$
which is endowed with the norm
$$\| f \|_{k,p(\cdot)} = \sum_{\alpha \leq k} \| \partial^\alpha f \|_{p(\cdot)}. \quad (2.2)$$
The space $W^{0,p(\cdot)}(\Omega)$ is defined as the completion of $C^\infty_0(\Omega)$ in the norm (2.2). Using standard arguments one can derive from the properties of the space $L^{p(\cdot)}(\Omega)$ that $W^{k,p(\cdot)}(\Omega)$ and $W^{0,k,p(\cdot)}(\Omega)$ are separable, reflexive Banach spaces.
Let us now discuss the embedding properties of the generalized Lebesgue space. Firstly, we know that
\[ L^{p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega), \tag{2.3} \]
if and only if
\[ q(x) < p(x) \quad \text{a.e. in } \Omega. \]

We denote by \( L^{p'}(\Omega) \) the conjugate of \( L^{p(x)}(\Omega) \), where \( 1/p(x) + 1/p'(x) = 1 \). Moreover for all \( p \in P(\Omega), u \in L^{p(x)}(\Omega), v \in L^{p(x)}(\Omega) \) we have
\[ \int_{\Omega} |u(x)v(x)| \, dx \leq 2 \|u(x)\|_{p(x)} \|v(x)\|_{p'(x)}. \]

We note that if \( q \in P(\Omega) \) and \( q(x) < p^*(x) \) and for all \( x \in \Omega, \) then the embedding \( W^{1,p(x)} \hookrightarrow L^{q(x)}(\Omega) \) is compact and continuous, where \( p^*(x) = np(x)/(n - p(x)) \) if \( p(x) < n \) or \( p^*(x) = +\infty \) if \( p(x) = n \). For more details one can see Zhou 5, [3, 4, 5, 8, 9].

2.2. Definition of the weak solution. We say that \( u \in W^{1,p(x)}_0 \) with \( p(x) \geq 1 \) is a weak solution of problem (1.4), if \( u \geq 0 \) a.e in \( \Omega \) and
\[ \int_{\Omega} a(x,Du) \cdot D\varphi \, dx - k \int_{\Omega} (u^\gamma - u^{\beta - 1}) \cdot \varphi \, dx = 0, \tag{2.4} \]
for all \( \varphi \in W^{1,p(x)}_0(\Omega) \).

In the proof of Theorem 1.1, we will use Giaquinta’s lemma in a version formulated by Giaquinta [6] or Giusti [7]:

**Theorem 2.1.** Let \( \Omega \subset \mathbb{R}^n, 0 < m < 1, \) and \( f \in L^1_{\text{loc}}(\Omega), g \in L^p_{\text{loc}}(\Omega) \) for some \( \sigma > 1 \) be two nonnegative functions such that for any ball \( B_\rho \) with \( B_{3\rho} \subset \subset \Omega \) there holds
\[ \int_{B_\rho} f \, dx \leq b_1 \left( \int_{B_{3\rho}} f^m \, dx \right)^{\frac{1}{m}} + b_2 \int_{B_{3\rho}} g \, dx + k \int_{B_{3\rho}} f \, dx, \]
where \( b_1, b_2 > 1 \) and \( 0 < k \leq k_0 = k_0(m,n) \). Then there exists a constant \( r_0 = r_0(n,m,b_1) > 1 \) such that
\[ f \in L^r_{\text{loc}} \quad \text{for all } 1 < r < \min\{r_0, \sigma\}. \]
Moreover for all balls \( B_\rho \) with \( B_{3\rho} \subset \subset \Omega \) there holds
\[ \int_{B_\rho} f^r \, dx \leq t \left[ \left( \int_{B_{3\rho}} f \, dx \right)^r + \mu b_2 \int_{B_{3\rho}} g^r \, dx \right], \]
where \( t = t(n) > 0 \) and \( \mu = \mu(m,n) > 0 \).

2.3. Choice of some global constants. In what follows, we will repeat use some constants: \( a, \theta \). To begin with, we choosing a positive number \( \theta > 1 \) which satisfies the following condition
(i): \( \theta \in (1, 1 + \frac{2}{3m}) \cap (1, \gamma_1), \)
(ii): \( \theta \leq \frac{\gamma_1}{\gamma_2 - \gamma}. \)

From now on \( \theta \) will be a constant only depends on \( \gamma_1, \gamma_2, n \). Next, we put a restriction on \( a > 1 \):
\[ a \in I_1 \cap I_2 \cap I_5 \cap I_4 \cap I_5 \cap I_6 \cap I_7, \tag{2.5} \]
where
\[
I_1 = \left(1, \frac{\gamma_1 + 1}{2}\right) \quad I_2 = (1, 1 + \gamma_1 - \theta(\gamma_2 - 1)),
\]
\[
I_3 = \left(1, \frac{\gamma_1}{\gamma_2} + 1\right) \quad I_4 = \left(1, \frac{3n + 2 - 3n\theta}{3n + 2} - \gamma_2 + 1\right),
\]
\[
I_5 = (1, 1 + \gamma_1 - \theta) \quad I_6 = \left(1, \frac{\gamma_1 + \sqrt{(\gamma_1 - 1)^2 + 4\theta}}{2}\right),
\]
\[
I_7 = \left(1, 1 + \frac{\gamma_1}{\gamma_2}, \frac{1}{10n}\right).
\]
Furthermore, Let \(a\) satisfies
\[
\frac{1}{(2 - a)^2} \leq \frac{\gamma_1 + 1 - a}{\theta}.
\]
From (2.6), one can see that the valid set of \(a\) satisfies previous inequality is \(a \in (1, B)\), where \(1 < B < 2\) dependent on \(\gamma_1\) and \(\theta\). Obviously, \(a\) depends only on \(n, \gamma_1, \gamma_2\). Indeed, since \(I_6\) we should restrict \(\theta < \gamma_1\), and from (ii) one can find that \(I_2\) is valid.

Having fixed the constants \(\theta, a\) we choose an open ball \(B_{\rho}\) such that \(B_{3\rho} \subset \subset \Omega\) and \(\rho \leq R_0\).

For \(x \in \Omega\) we define
\[
q_i \equiv q_i(a, \theta) = \frac{p_i \left(1 - \frac{a - 1}{p_i}\right)}{\theta}\quad (i = 1, 2), \quad q(x) \equiv q(x, a, \theta) = \frac{p(x) \left(1 - \frac{a - 1}{p_i}\right)}{\theta},
\]
where
\[
p_1 \equiv \inf_{x \in B_{3\rho}} p(x), \quad p_2 \equiv \sup_{x \in B_{3\rho}} p(x),
\]
from the definition of \(q_i\) and \(I_5\) one can see that \(q_i \geq 1\).

2.4. **A equivalent form of equation (1.4).** For the weak solution \(u\) of (1.4), we can find a function \(v \in \mathbb{R}^n\) such that
\[
v \cdot Du = 0, \quad \text{and} \quad \frac{1}{C} \leq |v| \leq C,
\]
with \(C \geq 1\) is a positive constant.

Indeed, without loss of generality, we set \(n = 2\) and \(Du\) has the form
\[
Du = \left(\frac{g_1(x_1, x_2), g_2(x_1, x_2)}{f_1(x_1, x_2), f_2(x_1, x_2)}\right),
\]
with
\[
f_i(x_1, x_2) = a_i^1 x^{b_i^1} + a_i^2 x^{b_i^2} \quad g_i(x_1, x_2) = A_i^1 x^{B_i^1} + A_i^2 x^{B_i^2},
\]
where \(b_j^i, B_j^i \geq 0, \quad (i, j = 1, 2)\). Then we just take \(v = (v_1, v_2) = (f_1 g_2 h, -f_2 g_1 h)\), where multiply the factor \(h(x_1, x_2) > 0\) such that the exponent of \(x_i\) for \(v_j\) more than 1. Since \(\Omega\) is bounded, we can get the previous claim. Moreover, we can choose suitable \(h\) such that \(1/c \leq \text{div} \ v \leq c\) for a positive constant \(c\), for \(n > 2\) and \(Du\) has the form different with above, we have the same argument.
From previous argument, for the weak solution of (1.4), then we have an equivalence form

\[
\begin{cases}
-\text{div } a(x, Du) = k \cdot \frac{1}{\text{div } v} \text{div } [v(u^{\gamma-1} - u^{\beta-1})] & x \in \Omega, \\
u = 0 & x \in \partial \Omega, \\
u \geq 0 & x \in \Omega,
\end{cases}
\]

It is obvious for the weak solution \( u \) of (1.4) satisfy (2.8), one may notice that (1.4) is different with (2.8) with other general \( u \), i.e., (2.8) may have other solution which do not satisfy (1.4), and in follows paper we just omit this case without consideration.

By (2.4) we have \( u \in W^{1,p}_0(\Omega) \) as a weak solution of problem (1.4), if \( u \geq 0 \) a.e in \( \Omega \)

\[
\int_\Omega a(x, Du) \cdot D\varphi dx - k \cdot \frac{1}{\text{div } v} \text{div } [v(u^{\gamma-1} - u^{\beta-1})] \cdot \varphi dx = 0,
\]

far all \( \varphi \in W^{1,p}_0(\Omega) \).

3. Higher integrability of weak solution. Let \( \eta \in C^\infty_0(B_{2\rho}) \) be a cut off function for \( B_{2\rho} \), i.e.

\[
\eta = 1, \quad \text{in } B_{\rho}, \quad 0 \leq \eta \leq 1 \quad \text{in } B_{2\rho},
\]

\[
\left| \frac{\partial \eta}{\partial x_i} \right| \leq \frac{c}{\rho}, \quad \left| \frac{\partial^2 \eta}{\partial x_i \partial x_j} \right| \leq \frac{c}{\rho^2} \quad \text{in } B_{2\rho} \quad (i, j = 1, \ldots, n),
\]

where \( c \) is a positive constant independent of \( \rho \). Then the function

\[
\varphi := \eta(u - (u)_{2\rho}),
\]

is a truncated function, where \( (u)_{2\rho} := \int_{B_{2\rho}} u dx \) denotes the mean value of \( u \) on \( B_{2\rho} \). Then we have

\[
|D\varphi| \leq |Du| + c\frac{1}{\rho}|u - (u)_{2\rho}| \quad \text{on } B_{2\rho}, \quad \text{spt } \varphi \subset B_{2\rho}.
\]

Now let us begin to prove the main result of this paper

**Proof of Theorem 1.1**. Let \( \lambda > 0 \), define

\[
E_\lambda \equiv \{ x \in \mathbb{R}^n : M(x) \leq \lambda \},
\]

where

\[
M(x) \equiv \left[ M \left( |D\varphi|^{\frac{\mu(\gamma)}{2(2-a)}} \right)(x) \right]^{2-a} \equiv \sup_{r > 0} \left( \int_{B_r(x)} |D\varphi|^{\frac{\mu(\gamma)}{2(2-a)}} dy \right)^{2-a},
\]

for \( x \in \mathbb{R}^n \). Since \( \text{spt } \varphi \subset B_{2\rho} \) we find that for \( x \in \mathbb{R}^n \setminus B_{3\rho} \), there holds

\[
M(x) = \left( \sup_{r > \rho} \int_{B_r(x)} |D\varphi|^{\frac{\mu(\gamma)}{2(2-a)}} dy \right)^{2-a} \leq c(n) \left( \int_{B_{2\rho}(x)} |D\varphi|^{\frac{\mu(\gamma)}{2(2-a)}} dy \right)^{2-a}.
\]
Then, we deduce that
\[ \int_{B_{2\rho}} |D\varphi|^{\frac{p(y)}{2(\gamma_2 - a)}} dy \]
\[ \leq c(\gamma_2) \left[ \int_{B_{2\rho}} |Du|^{\frac{p(y)}{2(\gamma_2 - a)}} dy + \int_{B_{2\rho}} \frac{|u - (u)_{2\rho}|}{\rho} D\varphi^{\frac{p(y)}{2(\gamma_2 - a)}} dy \right] \]
\[ \leq c(\gamma_2) \left[ \left( \int_{B_{2\rho}} |Du|^{q(y)} dy \right)^{\frac{1}{q_1(2 - a)}} + \int_{B_{2\rho}} \frac{|u - (u)_{2\rho}|}{\rho} D\varphi^{\frac{p(y)}{2(\gamma_2 - a)}} dy + 1 \right]. \quad (3.3) \]

Indeed, observing that
\[ \frac{p(y)}{\gamma_2(2 - a)} \leq \frac{p_2}{\gamma_2(2 - a)} \leq \frac{1}{2 - a}, \]
by (3.6) we infer that
\[ \int_{B_{2\rho}} \frac{|u - (u)_{2\rho}|}{\rho} D\varphi^{\frac{p(y)}{2(\gamma_2 - a)}} dy \leq 1 \]
for \( y \in B_{2\rho} \), from previous inequality we obtain (3.3).

Note that \( q_1(2 - a) \geq 1 \), applying Poincaré's inequality and Hölder's inequality we obtain
\[ \int_{B_{2\rho}} \frac{|u - (u)_{2\rho}|}{\rho} D\varphi^{\frac{p(y)}{2(\gamma_2 - a)}} dy \leq 1 \]
\[ \leq 1 + \left( \int_{B_{2\rho}} |Du|^{q_1} dy \right)^{\frac{1}{q_1(2 - a)}} \leq c \left[ 1 + \left( \int_{B_{2\rho}} |Du|^{q(y)} dy \right)^{\frac{1}{q_1(2 - a)}} \right]. \quad (3.4) \]

Joining (3.3) with (3.4), we finally obtain
\[ \int_{B_{2\rho}} |D\varphi|^{\frac{p(y)}{2(\gamma_2 - a)}} dy \leq c(\gamma_2) \left[ 1 + \left( \int_{B_{2\rho}} |Du|^{q(y)} dy \right)^{\frac{1}{q_1(2 - a)}} \right]. \quad (3.5) \]

From (3.5) and taking into account (3.2), it follows that for \( x \in R^n \setminus B_{3\rho} \)
\[ M(x) \leq c \left[ 1 + \left( \int_{B_{2\rho}} |Du|^{q(y)} dy \right)^{\frac{1}{q_1(2 - a)}} \right] \equiv \lambda_1, \quad (3.6) \]
where \( c = c(n, \gamma_2) \). By (3.6), we infer that
\[ R^n \setminus B^{3\rho} \subset E_\lambda \quad \text{for } \lambda \geq \lambda_1. \]

Recall back the Kirszbraun extension theorem, for all \( \lambda \geq \lambda_1 \), there exist a Lipschitz continuous function \( \varphi_\lambda : \ R^n \longrightarrow R \) with the following properties:
\[ \left\{ \begin{array}{rcl}
\varphi_\lambda & = & \varphi \\
\|D\varphi_\lambda\|_\infty & \leq & c(n)\lambda.
\end{array} \right. \quad (3.7) \]
Since \(spt\varphi \subset B_{2\rho}\) we have \(spt\varphi \subset B_{3\rho}\). Now we choose \(\varphi_\lambda\) as a test function in (2.9), then we obtain
\[
\int_{B_{3\rho}} a(x, Du) \cdot D\varphi_\lambda \leq c \int_{B_{3\rho}} k \frac{1}{\text{div} v} \text{div}[v(u^{\gamma-1} - u^{\beta-1})]\varphi_\lambda \cdot \text{sign}(A)dx,
\]
with \(A = |v(u^{\gamma-1} - u^{\beta-1})|\varphi_\lambda\) and \(\beta < \gamma\). From above we arrive at
\[
\int_{E_\lambda \cap B_{3\rho}} a(x, Du) \cdot D\varphi_\lambda - c\text{sign}(A)(u^{\gamma-1} - u^{\beta-1})v \cdot D\varphi dx
\]
\[\leq -c \int_{B_{3\rho} \setminus E_\lambda} a(x, Du) \cdot D\varphi_\lambda - k\text{sign}(A)(u^{\gamma-1} - u^{\beta-1})v \cdot D\varphi_\lambda dx.
\]
Note that \(u \geq 0\), taking into account (1.5) and (3.7), it follows that
\[
\int_{E_\lambda \cap B_{2\rho}} a(x, Du) \cdot D\varphi - k\text{sign}(B)(u^{\gamma-1} - u^{\beta-1})v \cdot D\varphi dx
\]
\[\leq c(n)\lambda \int_{B_{3\rho} \setminus E_\lambda} L \left(1 + |Du|^{p(-1)}\right) + k(u^{\gamma-1} + u^{\beta-1})dx,
\]
with \(B = v \cdot D\varphi_\lambda\). Multiplying both side by \(\lambda^{-a}\) and integrating on \((\lambda_1, \infty)\) we infer that
\[
\int_{\lambda_1}^{\infty} \lambda^{-a} \int_{E_\lambda \cap B_{2\rho}} a(x, Du) \cdot D\varphi - k\text{sign}(B)(u^{\gamma-1} - u^{\beta-1})v \cdot D\varphi dx \lambda d\lambda
\]
\[\leq c \int_{\lambda_1}^{\infty} \lambda^{1-a} \int_{B_{3\rho} \setminus E_\lambda} L \left(1 + |Du|^{p(-1)}\right) + k(u^{\gamma-1} + u^{\beta-1})dx \lambda^{1-a} d\lambda,
\]
with \(c = c(n)\).

Set \(m(x) = \max\{\lambda_1, M(x)\}\), then by Fubini theorem, we arrive at
\[
\int_{B_{2\rho}} a(x, Du) \cdot D\varphi - k\text{sign}(B)(u^{\gamma-1} - u^{\beta-1})v \cdot D\varphi dx \int_{m_0}^{\infty} \lambda^{-a} d\lambda
\]
\[\leq c(n)(L + k) \int_{B_{3\rho}} |Du|^{p(-1)} + u^{\gamma-1} + u^{\beta-1} + 1 \int_{\lambda_1}^{M} \lambda^{1-a} d\lambda.
\]
From above, it follows that
\[
\frac{1}{a-1} \int_{B_{2\rho}} a(x, Du) \cdot D\varphi - k\text{sign}(B)(u^{\gamma-1} - u^{\beta-1})v \cdot D\varphi \frac{dx}{m(x)^{a-1}}
\]
\[\leq \frac{c(n)(L + k)}{2-a} \int_{B_{3\rho}} |Du|^{p(-1)} + u^{\gamma-1} + u^{\beta-1} + 1 \right) M(x)^{2-a}dx. \tag{3.8}
\]

Observing that
\[
\int_{B_{2\rho}} a(x, Du) \cdot D\varphi \frac{dx}{m(x)^{a-1}}
\]
\[= \int_{B_{2\rho}} (\eta a(x, Du) \cdot Du + a(x, Du) \cdot (u - (u)_{2\rho}) \otimes D\eta) \frac{dx}{m(x)^{a-1}}
\]
\[\geq \nu \int_{B_{2\rho}} |Du|^{p(-1)} \frac{dx}{m(x)^{a-1}} - L\lambda_1^{1-a} \int_{B_{2\rho}} (1 + |Du|^{p(-1)}) |u - (u)_{2\rho}| \rho \frac{dx}{\rho}. \tag{3.9}
\]
Since \(m^{-1} \leq M^{-1}\), then we obtain that
\[
|D\varphi| m(x)^{1-a} \leq |D\varphi| M(x)^{1-a} \leq |D\varphi(x)| \cdot |D\varphi(x)|^{\frac{p(x)(1-a)}{2}} \leq 1 + |D\varphi(x)|.
\]
Therefore, we deduce that
\[
k \int_{B_{2r}} \text{sign}(B) \left( |u^{\gamma-1} - u^{\beta-1}| v \cdot D\varphi \right) \frac{dx}{m(x)^{a-1}} \leq c \int_{B_{2r}} (u^{\gamma-1} + u^{\beta-1})(1 + |D\varphi|)dx,
\] (3.10)
where \( c = c(k) \). From inequality (3.10), take into account (3.8), (3.9), then we have
\[
\nu \int_{B_{\rho}} \frac{|Du|^{p(x)}}{(x)_{m(x)}^{a-1}} dx \leq \frac{c(n)(a-1)}{(2-a)} (L+k) \int_{B_{\rho}} |Du|^{p(x)-1} + u^{\gamma-1} + u^{\beta-1} + 1 M(x)^{2-a} dx
\]
\[+ c(k) \int_{B_{2\rho}} (u^{\gamma-1} + u^{\beta-1}) (1 + |D\varphi|) dx
\]
\[+ L\lambda_1^{1-a} \int_{B_{2\rho}} (1 + |Du|^{p(x)-1}) \frac{|u - (u)_{2\rho}|}{\rho} dx.
\] (3.11)
Set
\[J := \left\{ x \in B_\rho : M(x) \leq |Du(x)|^{\frac{p(x)}{p_1}}, M(x) \geq \lambda_1 \right\}.
\]
Note that for \( x \in J \), we have
\[|Du(x)|^{-p(x)} \leq M(x)^{-p_1} = m(x)^{-p_1},
\]
and for \( x \in B_\rho \setminus J \) we have
\[|Du(x)|^{p(x)} \leq M(x)^{p_1} \quad \text{or} \quad |Du(x)| = |D\varphi(x)| \leq M(x) \leq \lambda_1.
\]
Observing that
\[
\int_{B_{\rho}} |Du(x)|^{p(x)} \left(1 - \frac{a-1}{p_1}\right) dx
\]
\[= \int_{J} |Du(x)|^{p(x)} \left(1 - \frac{a-1}{p_1}\right) dx + \int_{B_\rho \setminus J} |Du(x)|^{p(x)} \left(1 - \frac{a-1}{p_1}\right) dx
\]
\[\leq \int_{J} |Du(x)|^{p(x)} \left(1 - \frac{a-1}{p_1}\right) dx + \int_{B_\rho \setminus J} M(x)^{p_1 - (a-1)} + \lambda_1^{p_1 - (a-1)} dx,
\] (3.12)
Inserting (3.12) into (3.11), then we obtain
\[
\int_{B_{\rho}} |Du(x)|^{p(x)} \left(1 - \frac{a-1}{p_1}\right) dx
\]
\[\leq c \int_{B_{3\rho}} (Du|^{p(x)-1} + u^{\gamma-1} + u^{\beta-1} + 1) M(x)^{2-a} dx
\]
\[+ c_1 \lambda_1^{1-a} \int_{B_{2\rho}} (1 + |Du|^{p(x)-1}) \frac{|u - (u)_{2\rho}|}{\rho} dx
\]
\[+ c_2 \int_{B_{2\rho}} (u^{\gamma-1} + u^{\beta-1}) (1 + |D\varphi|) dx + \int_{B_{\rho}} M(x)^{p_1 - (a-1)} dx + \int_{B_{2\rho}} \lambda_1^{p_1 - (a-1)} dx
\]
\[=: H_1 + H_2 + H_3 + H_4 + H_5,
\] (3.13)
where \( c = c(n, \nu, \gamma_1, \gamma_2, L, k), c_1 = c_1(L, \nu), c_2 = c_2(k, \nu) \).
Since \( \varphi := \eta(u - (u)_{2\rho}) \), then
\[|D\varphi| \leq |Du| + \frac{c}{\rho} |u - (u)_{2\rho}| \quad \text{on} \quad B_{2\rho},
\] (3.14)
By virtue of (3.14), we have
\[ \int_{B_{2\rho}} |D\varphi|^p(x)^{(1-\frac{a-1}{p_1})} \, dx \]
\[ \leq c(\gamma_2) \left( \int_{B_{2\rho}} |Du(x)|^p(x)^{(1-\frac{a-1}{p_1})} \, dx + \int_{B_{2\rho}} \frac{|u-(u)_{2\rho}|}{\rho}^p(x)^{(1-\frac{a-1}{p_1})} \, dx \right). \quad (3.15) \]

In what follows, we propose to prove
\[ p_2 \left( 1 - \frac{a-1}{p_1} \right) \leq \frac{nq_1}{n-q_1}. \quad (3.16) \]

Indeed, by \( I_1, (1.8)_1 \), and the definition of \( p_1, p_2 \), we have
\[ p_1 \left( 1 - \frac{a-1}{p_1} \right) + \frac{np_1}{p_2} \geq p_1 \left( 1 - \frac{a-1}{p_1} \right) + n - \frac{n(p_2 - p_1)}{p_2} \geq p_1 \left( \frac{1}{2} + \frac{1}{2\gamma_1} \right) + n - \frac{n(2\gamma_1)}{p_2} \geq 1 + n - \frac{n(\theta - 1)}{2} \geq n\theta. \]

From above, it follows that
\[ p_2 \left( n\theta - p_1 \left( 1 - \frac{a-1}{p_1} \right) \right) \leq np_1, \]
which implies
\[ p_2 \left( 1 - \frac{a-1}{p_1} \right) \leq \frac{n(p_1 + 1 - a)}{n\theta - p_1 \left( 1 - \frac{a-1}{p_1} \right)}, \]
thus, we have (3.16).

Now, applying the Hölder’s inequality, Sobolev-Poincaré’s inequality, we obtain
\[ \int_{B_{2\rho}} \frac{|u-(u)_{2\rho}|^{p_2(1-\frac{a-1}{p_1})}}{\rho} \, dx \leq 1 + \left( \int_{B_{2\rho}} \frac{|u-(u)_{2\rho}|^{\frac{np_1}{p_2}}}{\rho} \right)^{\frac{np_1}{p_2}} \int_{B_{2\rho}} |Du(x)|^{q_1} \, dx \]
\[ \leq 1 + c \left( \int_{B_{2\rho}} |Du|^{q_1} \right)^{\left( 1-\frac{a-1}{p_1} \right) \frac{p_2}{p_1}}. \quad (3.17) \]

Furthermore, observing that
\[ p_2 \left( 1 - \frac{a-1}{p_1} \right) = \frac{\theta}{p_1} p_2, \]
then from (3.17) we find that
\[ \int_{B_{2\rho}} \frac{|u-(u)_{2\rho}|^{p_2(1-\frac{a-1}{p_1})}}{\rho} \, dx \]
\[ \leq c \left[ 1 + \left( \int_{B_{2\rho}} |Du|^{q_1} \, dx \right)^{\theta} \left( \int_{B_{2\rho}} |Du|^{q_1} \, dx \right)^{\frac{(p_2 - p_1)\theta}{p_1}} \right] \]
\[ \leq c(n, L, \gamma_1, \gamma_2) \left[ 1 + \left( \int_{B_{2\rho}} |Du|^{q_1} \, dx \right)^{\theta} \right]. \quad (3.18) \]
By (3.18), and taking into account (3.15), we finally obtain

$$\int_{B_{2r}} |D\varphi|^{p(x)} \left(1 - \frac{a-1}{p_1}\right) dx \leq c \left(1 + \int_{B_{2r}} |Du|^{p(x)} \left(1 - \frac{a-1}{p_1}\right) dx + \left(\int_{B_{2r}} |Du|^{q(x)} dx\right)^{\theta}\right),$$

(3.19)

where $c = c(n, \gamma_1, \gamma_2, L)$.

**Estimation of $H_1$.** Make use of $I_3$, we can see that

$$\frac{p(\cdot)}{p(\cdot) - 1} \left(1 - \frac{a-1}{p_1}\right) > 1.$$

By the aid of Young’s inequality with exponents

$$\frac{p(x)}{p(x) - 1} \left(1 - \frac{a-1}{p_1}\right) \quad \text{and} \quad \frac{p(x)}{p(x) - 1} \left(1 + a\right) = \frac{p(x)}{p(x) - 1} \left(1 - \frac{a-1}{p_1}\right) + \frac{p(x)}{p(x) - 1} \left(1 + a\right) = \frac{p(x)}{p(x) - 1} \left(1 - \frac{a-1}{p_1}\right) + 1,$$

then we obtain that

$$H_1 \leq c \int_{B_{2r}} |Du|^{p(x)} \left(1 - \frac{a-1}{p_1}\right) dx + |u|^{\frac{a-1}{p_1}} p(x) \left(1 - \frac{a-1}{p_1}\right)$$

$$+ |u|^{\frac{a-1}{p_1}} p(x) \left(1 - \frac{a-1}{p_1}\right) + M(x) \int_{B_{2r}} |Du|^{p(x)} \left(1 - \frac{a-1}{p_1}\right) dx + 1 dx,$$

(3.20)

where $c = c(n, \gamma_1, \gamma_2, L, k, \nu)$.

Observing that by $I_7$ we have

$$\frac{p(x)}{p(x) - 1} \left(1 + a\right) \leq \gamma_2 \left(1 - \frac{a-1}{p_1}\right),$$

and

$$\gamma_2 \left(1 - \frac{a-1}{p_1}\right) (2 - a) \geq p_1 - (a - 1) \geq 1,$$

then by virtue of Hardy-Littlewood maximal theorem and make use of (3.19), and note that the factor $2 - a \in (0, 1)$, then we have

$$\int_{B_{2r}} M(x) \frac{p(x)}{p(x) - 1} \left(1 - \frac{a-1}{p_1}\right) dx$$

$$\leq c \left(1 + \int_{B_{2r}} M(x)^{\gamma_2} \left(1 - \frac{a-1}{p_1}\right) dx\right) \leq c \left(1 + \int_{B_{2r}} |D\varphi|^{p(x)} \left(1 - \frac{a-1}{p_1}\right) dx\right)$$

$$\leq c \left(1 + \int_{B_{2r}} |Du|^{p(x)} \left(1 - \frac{a-1}{p_1}\right) dx + \left(\int_{B_{2r}} |Du|^{q(x)} dx\right)^{\theta}\right),$$

(3.21)

where $c = c(n, \gamma_1, \gamma_2, L)$. Inserting (3.21) into (3.20), we finally obtain

$$H_1 \leq c \left[\int_{B_{2r}} |Du|^{p(x)} \left(1 - \frac{a-1}{p_1}\right) dx + |u|^{\frac{a-1}{p_1}} p(x) \left(1 - \frac{a-1}{p_1}\right) + |u|^{\frac{a-1}{p_1}} p(x) \left(1 - \frac{a-1}{p_1}\right) + 1 dx\right]$$

$$+ c \left(\int_{B_{2r}} |Du|^{q(x)} dx\right)^{\theta},$$

where $c = c(n, \gamma_1, \gamma_2, L, k, \nu)$.
**Estimation of $H_2$.** Note that $q_1 = p_1 \left(1 - \frac{a-1}{p_1} \right) / \theta$, by $I_2$ and (ii), it is obvious to deduce that $\frac{q_1}{p_2 - 1} > 1$, then we infer that

$$H_2 = c \lambda_1^{1-a} \int_{B_{2\rho}} \left(1 + |Du|^{p(x)-1} \right) \frac{|u - (u)_{2\rho}|}{\rho} \, dx$$

$$\leq c \lambda_1^{1-a} \int_{B_{2\rho}} (1 + |Du|)^{p_2-1} \frac{|u - (u)_{2\rho}|}{\rho} \, dx$$

$$\leq c \lambda_1^{1-a} \left( \int_{B_{2\rho}} (1 + |Du|)^{q_1} \, dx \right)^{\frac{q_1 - p_2 + 1}{q_1}} \cdot \left( \int_{B_{2\rho}} \frac{|u - (u)_{2\rho}|}{\rho} \right)^{\frac{nq_1}{n - p_2 + 1}} \, dx. \tag{3.22}$$

At this point, by $I_4$ we have

$$\frac{p_2}{q_1} = \frac{p_2}{p_1} \left(1 - \frac{2-1}{p_1} \right) \leq \frac{2}{3n} \leq 1 + \frac{1}{n}.$$ 

By the aid of inequality above, then we obtain that

$$\frac{nq_1}{n - p_2 + 1} = \frac{nq_1}{nq_1 - np_2 + n} = \frac{nq_1}{(n + 1)q_1 - np_2 + n - q_1} \leq \frac{nq_1}{n - q_1}.$$ 

From above, once again using Hölder’s inequality and Sobolev-Poincaré’s inequality we infer that

$$\left( \int_{B_{2\rho}} \frac{|u - (u)_{2\rho}|}{\rho} \right)^{q_1 - p_2 + 1} \leq c \left( \int_{B_{2\rho}} \frac{|u - (u)_{2\rho}|}{\rho} \right)^{\frac{nq_1}{n - q_1}} \, dx \leq c(n, \gamma_1, \gamma_2) \left( \int_{B_{2\rho}} (1 + |Du|)^{q_1} \, dx \right)^{\frac{1}{q_1}}. \tag{3.23}$$

We are now coming back to estimation of $H_2$ in (3.22). Recalling the definition of $\lambda_1$, choose a suitable $R_0 < 1$, then for $2\rho < R_0$, there holds

$$|B_\rho|^{-(p_2 - p_1)} \leq |B_\rho|^{-\omega(2\rho)}.$$ 

Observing that from (1.8)

$$|B_r|^{-(p_2 - p_1)} = \left(\frac{2}{\alpha_n}\right)^{n(p_2 - p_1)} e^{-n(p_2 - p_1) \log 2r}$$

$$\leq \left(\frac{2}{\alpha_n}\right)^{n\omega(2r)} e^{n\omega(2r) \log \frac{r}{\delta r}} \leq c(n, L),$$

for $2r \leq 6\rho \leq 6R_0 \leq 1$, which implies that

$$\left( \int_{B_{2\rho}} |Du|^{p(x)} \, dx \right)^{p_2 - p_1} \leq c(n, L).$$
By the previous inequality we obtain that

\[
\left( \int_{B_{2r}} |Du|^{q(x)} dx \right)^{\frac{p_2 + 1 - a}{q_1}} \leq \left( \int_{B_{2r}} |Du|^{q(x)} dx \right)^{\frac{p_2 + 1 - a}{q_1}} \left( \int_{B_{2r}} |Du|^{q(x)} dx \right)^{\frac{p_1 + 1 - a}{q_1}}
\]

\[
\leq c(n, L) \left( \int_{B_{2r}} |Du|^{q(x)} dx \right)^{\frac{1 - a}{q_1}} . (3.24)
\]

Furthermore, by the definition of \( \lambda_1 \) one can see that

\[
\lambda_1^{1 - a} \leq 1, \quad \lambda_1^{1 - a} \leq \left( \int_{B_{2r}} |Du|^{q(x)} dx \right)^{\frac{1 - a}{q_1}} . (3.25)
\]

Thus, by inequality (3.23)–(3.25) we can obtain

\[
H_2 \leq c \lambda_1^{1 - a} \left[ 1 + \left( \int_{B_{2r}} |Du|^{q_1} dx \right)^{\frac{p_2}{q_1}} \right] \leq c \lambda_1^{1 - a} \left[ 1 + \left( \int_{B_{2r}} |Du|^{q(x)} dx \right)^{\frac{p_2}{q_1}} \right]
\]

\[
\leq c \left[ 1 + \left( \int_{B_{2r}} |Du|^{q(x)} dx \right)^{\frac{p_2 + 1 - a}{q_1}} \right] \leq c \left[ 1 + \left( \int_{B_{2r}} |Du|^{q(x)} dx \right)^{\theta} \right],
\]

where \( c = c(n, L, \gamma_1, \gamma_2, \nu) \).

Estimation of \( H_3 \). Make use of Young’s inequality with the exponents \( \frac{1}{2 - a} \) and \( \frac{1}{1 - (2 - a)^2} \), we obtain that

\[
H_3 \leq c \left( \int_{B_{2r}} |D\varphi|^{\frac{1}{(2 - a)^2}} dx + \int_{B_{2r}} (|u|^{\gamma - 1} + |u|^{\beta - 1})^{\frac{1}{1 - (2 - a)^2}} + 1 dx \right),
\]

where \( c = c(n, \gamma_1, \gamma_2, k, \nu) \).

Since (2.6) we have \( 1/(2 - a)^2 \leq q_1 \leq q(\cdot) \), similar with (3.3) and (3.4), we finally obtain

\[
H_3 \leq c \left[ 1 + \left( \int_{B_{2r}} |Du|^{q(x)} dx \right) \right] + c \left( \int_{B_{2r}} (|u|^{\gamma - 1} + |u|^{\beta - 1})^{\frac{1}{1 - (2 - a)^2}} dx \right.
\]

\[
\leq c \left[ 1 + \left( \int_{B_{2r}} |Du|^{q(x)} dx \right) \right] + c \left( \int_{B_{2r}} (|u|^{\gamma - 1} + |u|^{\beta - 1})^{\frac{1}{1 - (2 - a)^2}} dx \right.
\]

where \( c = c(n, \gamma_1, \gamma_2, \nu, k) \).

Estimation of \( H_4 \). Observing that

\[
\gamma_2 \left( 1 - \frac{a - 1}{p_1} \right) \geq p_1 - (a - 1).
\]
From above, and and taking into account (3.21), it follows that

\[
H_4 \leq c \left( 1 + \int_{B_p} M(x)^{\gamma_2 \left( 1 - \frac{a-1}{\nu} \right)} \, dx \right) \\
\leq c \left[ 1 + \int_{B_p} |Du|^{p(x) \left( 1 - \frac{a-1}{\nu} \right)} \, dx + \left( \int_{B_p} |Du|^{q(x)} \, dx \right)^{\theta} \right],
\]

where \( c = c(n, \gamma_1, \gamma_2, L) \).

**Estimation of** \( H_5 \). By the definition of \( \lambda_1 \) and note that \( (p_1 + 1 - a)/q_1 = \theta \), then there holds

\[
H_5 \leq c \left[ 1 + \left( \int_{B_{2p}} |Du|^{q(x)} \, dx \right)^{\theta} \right],
\]

where \( c = c(n, L, \gamma_2, \nu) \).

Taking into account the bounds \( H_1 \text{--} H_5 \) with (3.13), we can finally obtain the following reverse Hölder’s type inequality

\[
\int_{B_p} |Du|^{p(x) \left( 1 - \frac{a-1}{p_1} \right)} \, dx \leq c \left( \int_{B_{3p}} |Du|^{q(x)} \, dx \right)^{\theta} + c \int_{B_{3p}} |Du|^{p(x) \left( 1 - \frac{a-1}{p_1} \right)} \, dx \\
+ c \int_{B_{3p}} |u|^{p(x) \left( \frac{\gamma_2(\gamma_2-1)}{p_1} \right) \left( 1 - \frac{a-1}{p_1} \right)} + |u|^{p(x) \left( \frac{\gamma_2(\gamma_2-1)}{p_1} \right) \left( 1 - \frac{a-1}{p_1} \right)} \, dx + 1 \, dx,
\]

where \( c = c(n, \gamma_1, \gamma_2, L, k, \nu) \). Now, if

\[
\frac{p(x)}{p(x) - 1} \left( 1 - \frac{a-1}{p_1} \right) \geq \frac{p(x)}{p(x) - \gamma_2(2-a)^2}.
\]

From (3.26), then we get

\[
\int_{B_p} |Du|^{p(x) \left( 1 - \frac{a-1}{p_1} \right)} \, dx \leq c_1 \left( \int_{B_{3p}} |Du|^{q(x)} \, dx \right)^{\theta} + c_1 \int_{B_{3p}} |Du|^{p(x) \left( 1 - \frac{a-1}{p_1} \right)} \, dx \\
+ c_2 \int_{B_{3p}} |u|^{p(x) \left( \frac{\gamma_2(\gamma_2-1)}{p_1} \right) \left( 1 - \frac{a-1}{p_1} \right)} + |u|^{p(x) \left( \frac{\gamma_2(\gamma_2-1)}{p_1} \right) \left( 1 - \frac{a-1}{p_1} \right)} \, dx + 1 \, dx.
\]

On the other hand, if

\[
\frac{p(x)}{p(x) - 1} \left( 1 - \frac{a-1}{p_1} \right) \leq \frac{p(x)}{p(x) - \gamma_2(2-a)^2},
\]

we infer that

\[
\int_{B_p} |Du|^{p(x) \left( 1 - \frac{a-1}{p_1} \right)} \, dx \leq c_1 \left( \int_{B_{3p}} |Du|^{q(x)} \, dx \right)^{\theta} + c_1 \int_{B_{3p}} |Du|^{p(x) \left( 1 - \frac{a-1}{p_1} \right)} \, dx \\
+ c_2 \int_{B_{3p}} (u^{\gamma-1} + u^{\beta-1})^{\frac{1}{1-(2-a)^2}} + 1 \, dx,
\]

where \( c_1 = c_1(n, \nu, L, k, \gamma_1, \gamma_2) \), \( c_2 = c_2(n, \nu, L, k, \gamma_1, \gamma_2) \geq 1 \).
Now, by using of Theorem 2.1, set
\[ f(x) = |Du|^{p(x)\left(1 - \frac{a-1}{p_1}\right)}, \]
\[ g(x) = |u|^{\frac{p(x)(\gamma - 1)}{p(x) - 1} \left(1 - \frac{a-1}{p_1}\right)} + |u|^{p(x)(\beta - 1)\left(1 - \frac{a-1}{p_1}\right)} + 1 \text{ or } (u^{\gamma - 1} + u^{\beta - 1}) \frac{1}{1-(x-a)^2} + 1. \]
By Theorem 2.1, we only need to choose \( \gamma, \beta \) such that
\[ g(x) \in L_{loc}^\sigma(\Omega), \quad \sigma > 1. \]
Since \( u \in W^{1,p(\cdot)}(\Omega) \) and
\[
\begin{cases}
\frac{p(x)}{p(x) - 1} \left(1 - \frac{a-1}{p_1}\right) \leq \frac{\gamma_2}{\gamma_1 - 1} \left(1 - \frac{a-1}{\gamma_2}\right) := A_1, \\
\frac{1}{1 - (2-a)^2} := A_2,
\end{cases}
\]
then if \( \gamma, \beta \) are close to 1 enough, there holds
\[
\frac{p(x)}{(\gamma - 1)A_1} > 1, \quad \frac{p(x)}{(\gamma - 1)A_2} > 1.
\]
We choose \( \gamma \) such that
\[
\frac{p(x)}{(\gamma - 1)A_1} \geq \frac{\gamma_1}{(\gamma - 1)A_1} > 1, \quad \frac{p(x)}{(\gamma - 1)A_2} \geq \frac{\gamma_1}{(\gamma - 1)A_2} > 1,
\]
then we can find a \( \sigma > 1 \), there holds
\[ g(x) \in L_{loc}^\sigma. \]
By the aid of Theorem 2.1, we obtain a higher integrability exponent \( r_0 = r_0(n, \gamma_1, \gamma_2, L, k, \nu) \) and for a suitable \( a \) in (2.5) we have
\[ |Du|^{p(\cdot)\left(1 - \frac{a-1}{p_1}\right)} \in L_{loc}^r \quad \text{for all } r \in (1, \min\{r_0, \sigma\}), \]
and
\[
\int_{B_r} |Du|^{p(x)\left(1 - \frac{a-1}{p_1}\right)} dx 
\leq c \left[ \left( \int_{B_{3r}} |Du|^{p(\cdot)\left(1 - \frac{a-1}{p_1}\right)} dx \right)^r + bc2 \left( u^{(\gamma - 1)\alpha} + u^{(\beta - 1)\alpha} + 1 \right)^\sigma \right], \tag{3.29}
\]
where \( c = c(n), b = (n, \gamma_2), \) and \( \alpha = p(x) \left(1 - \frac{a-1}{p_1}\right) / (p(x) - 1) \) or \( \frac{1}{1-(2-a)^2} \).

At last, we fixed the exponent \( r \in (1, \min\{r_0, \sigma\}) \), for \( r \) does not depend on \( a \) in (2.5), then we choose \( a \) such that
\[ \left(1 - \frac{a-1}{p_1}\right) \geq \left(1 - \frac{a-1}{\gamma_1}\right) > \frac{1}{r}. \]
Set \( \delta = \left(1 - \frac{a-1}{\gamma_1}\right) r \) then there holds
\[ p(x) \left(1 - \frac{a-1}{p_1}\right) r \geq p(x)\delta > p(x). \]
Observing that
\[ \frac{\gamma - 1}{p(\cdot) - 1} p(\cdot) \left(1 - \frac{a-1}{p_1}\right) < p(x), \]
by imbedding property (2.3) we have

$$\int_{B_{\rho}} |Du|^p(x) \delta \, dx \leq c_1 \left( \int_{B_{3\rho}} |Du|^{p(x)} \left( 1 - \frac{2}{p_1} \right) \right)^r + c_2 \int_{B_{3\rho}} |u|^{p(x)} + 1 \, dx,$$

where $c_1 = c_1(n)$, $c_2 = c_2(n, \gamma_1, \gamma_2, L, k, \nu)$. For $\alpha = \frac{1}{1 - (2-a)^2}$ we have the same conclusion, then we have completed the proof of Theorem 1.1.

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REFERENCES

[1] S. Chen and Z. Tan, Optimal partial regularity of second order parabolic systems under controllable growth condition, J. Funct. Anal., 66 (2014), 4908–4937.
[2] Y. Chen, S. Levine and M. Rao, Variable exponent, linear growth functionals in image restoration, SIAM J. Appl. Math., 66 (2006), 1383–1406.
[3] L. Diening, P. Harjulehto, P. Hästö and M. Růžička, Lebesgue and Sobolev Spaces with Variable Exponents, Lecture Notes in Mathematics, 2017, Springer-Verlag, Berlin Heidelberg, 2011.
[4] D. Edmunds and J. Rákosník, Sobolev embeddings with variable exponent, Stud. Math., 143 (2000), 267–293.
[5] D. Edmunds and J. Rákosník, Sobolev embeddings with variable exponent II, Math. Nachr., 246/247 (2002), 53–67.
[6] M. Giaquinta, Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems, Princeton Univ. Press, Princeton, NJ, 1983.
[7] E. Giusti, Direct Methods in the Calculus of Variations, World Scientific, Singapore, 2003.
[8] H. Hudzik, The problems of separability, duality, reflexivity and of comparison for generalized Orlicz-Sobolev spaces $W^{k,p}(\Omega)$, Comment. Math. Prace Mat., 21 (1980), 315–324.
[9] O. Kováčik and J. Rákosník, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, Czechoslovak Math. J., 116 (1991), 592–618.
[10] K. R. Rajagopal and M. Ruzička, Mathematical modelling of electrorheological fluids, Continuum. Mech. Thermodyn., 13 (2001), 59–78.
[11] M. Růžička, A note on steady flow of fluids with shear dependent viscosity, Nonlin. Anal. Theory, Meth. Appl., 30 (1997), 3029–3039.
[12] M. Růžička, Electrorheological fluids: Modeling and Mathematical Theory, Lecture Notes in Math., 1748, Springer-Verlag, Berlin, 2000.
[13] P. Marcellini, Regularity and existence of solutions of elliptic equations with p,q-growth conditions, J. Differential Equations, 90 (1991), 1–30.
[14] M. Mihăilescu and V. Rădulescu, A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids, Proc. R. Soc., 462 (2006), 2625–2641.

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