Fate of a Thin-Shell Wormhole Powered by Morris-Thorne Wormhole

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Asymmetric thin-shell wormholes from two traversable Morris-Thorne wormhole spacetimes, with identical shape but different redshift functions, are constructed. Energy density of the thin-shell wormhole derives its power from a Morris-Thorne wormhole which is already exotic. By choice, the weak energy condition for the thin-shell wormhole is satisfied. A linear barotropic equation of state is assumed to hold after the radial perturbations. The fate of our thin-shell wormhole, after the perturbation, is striking: the asymmetric thin-shell wormhole is destined either to collapse to the original Morris-Thorne wormhole or expand indefinitely along with the radius of the throat. In case it collapses to the original wormhole, the result is an asymmetric Morris-Thorne wormhole. Although this asymmetry does not reflect into the embedding diagram of the wormhole, passing across the throat, the wormhole adventurer feels a different redshift function.

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I. INTRODUCTION

The project of minimizing the indispensable exotic matter necessary in constructing wormholes was accomplished successfully under the title of thin-shell wormholes (TSWs) [1]. The idea was to confine the entire notorious negative energy density to a narrow band, ideally at the scales of Planck length. The reason in doing this is to earn legitimacy to such a negative energy concept within the domain of classical physics. The cut and paste procedure may cover anything to anything provided the consistency requirement of junction conditions are met. The expanding literature on TSWs is the best evidence to support the proposal of Visser. Furthermore, the idea is not restricted by spherical symmetry above but equally well finds rooms of application in cylindrical symmetry, albeit with more strict regulations.

Besides symmetric TSWs, which has been employed in the past [2], recently we have promoted the idea of asymmetric thin-shell wormholes (ATSWs) which hosts non-isotropic and inhomogeneous spacetimes [3, 4]. Thus, when Alice crosses the throat into the other universe, she encounters a truly different world.

On the other hand, there exists in the wormhole literature spacetimes which represent wormholes in essence [5]. As explained in detail by Morris and Thorne [6], asymptotic flatness, flare-out conditions and other conditions must all be satisfied for such wormhole spacetimes in order to have a safe wormhole travel across them. Expectedly, some of these conditions such as asymptotic flatness must be taken in a more restrictive sense when one deals with cylindrical symmetries instead of spherical. However, conditions like asymptotic flatness in radial infinity when one concerns the throat’s neighborhood only [7], or flare-out condition when it comes to TSWs [8], could be disregarded unrestrictedly.

Our aim in this paper is to construct a TSW from two different patches of Morris-Thorne traversable wormholes (MTWs). The latter is characterized by two basic functions and a key parameter; these are respectively the redshift function $\Phi(r)$, the shape function $b(r)$, and the minimum radius $r_0$ measuring the throat. We glue two such MTWs with different $\Phi(r)$ functions to construct a viable ATSW. Our ATSW will naturally draw its energy from MTW, which will be exotic. However, degree of freedom available at our disposal allows us to exploit the choice of throat radius of our ATSW such that the weak energy condition (WEC), i.e. $\sigma \geq 0$ and $\sigma + p \geq 0$, holds under some certain conditions, where $\sigma$ and $p$ refer to the energy density and the tangential pressure on the ATSW’s throat, respectively. This takes care of exotic energy problem in TSWs, whereas another serious problem concerns about the stability. To work out the latter, we introduce a barotropic fluid equation of state (EoS) after the perturbation and investigate the consequences. Based on such a barotropic fluid, we conclude some results in the following sections.

The organization of the paper is as follows. In section II we give a general description of ATSW construction from MTW. Stability analysis of our ATSW is considered in section III. The paper is completed with conclusion in part IV.

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II. ATSWS FROM WORMHOLE SPACETIMES

In this section we begin by setting a general 4–dimensional wormhole spacetime metric in spherical coordinates

$$ds^2 = -A(\ell)\, dt^2 + B(\ell)\, d\ell^2 + C(\ell)\, d\Omega^2,$$

in which the metric functions $A(\ell)$, $B(\ell)$ and $C(\ell)$ are all positive functions of radial coordinate $\ell$, whereas $d\Omega^2$ is traditionally a 2–sphere line element defined by $d\Omega^2 = d\theta^2 + \sin^2 \theta \, d\varphi^2$. On this spacetime we define a time-dependent timelike hypersurface expressed by

$$\mathcal{H}(\ell, \tau) := \ell - a(\tau) = 0,$$

in which $\tau$ is the proper time measured by the observer defined through the constraint $-A(\ell)\, dt^2 + B(\ell)\, d\ell^2 = -d\tau^2$ on the hypersurface. Therefore, the induced metric on the hypersurface $\mathcal{H}$

$$ds_{\mathcal{H}}^2 = -d\tau^2 + C(a)\, d\Omega^2$$

admits

$$\ell^2 = \frac{1 + B\dot{a}^2}{A},$$

where an overdot stands for a total derivative with respect to the proper time $\tau$. Then, the spacelike normal 4–vector components to the hypersurface are denoted by

$$n_\mu = \left( g^{\alpha\beta} \frac{\partial \mathcal{H}}{\partial x^\alpha} \right)^{-1/2} \frac{\partial \mathcal{H}}{\partial x^\mu},$$

which can be used to attain the components of the $3 \times 3$ curvature tensor on $\mathcal{H}$ as follows

$$K_{ab} = -n_\mu \left( \frac{\partial x^\mu}{\partial \xi^a} + \Gamma^\mu_{\alpha\beta} \frac{\partial x^\alpha}{\partial \xi^a} \frac{\partial x^\beta}{\partial \xi^b} \right).$$

Herein, the Christoffel symbols $\Gamma^\mu_{\alpha\beta}$ are the ones compatible with the metric of the bulk spacetime $g_{\alpha\beta}$, while $x^\alpha = \{t, \ell, \theta, \varphi\}$ and $\xi^a = \{\tau, \theta, \varphi\}$ stand for the coordinates of the bulk spacetime and the hypersurface, respectively. Using Eqs. (3)-(6) one acquires the explicit form of the mixed extrinsic curvature tensor given by

$$K^a_b = diag(K^\tau_b, K^\ell_b, K^\varphi_b) = \sqrt{\frac{B}{1 + B\dot{a}^2}} diag \left[ \tilde{a} + \frac{A'}{2AB}, \frac{\dot{\theta}^2}{2}, \frac{1 + B\dot{a}^2}{2B} \left( \frac{A'}{A} + \frac{B'}{B} \right), \frac{1 + B\dot{a}^2}{2B} \left( \frac{C'}{C} \right), \frac{1 + B\dot{a}^2}{2B} \left( \frac{C'}{C} \right) \right].$$

with its trace

$$K \equiv K^a_a = \sqrt{\frac{B}{1 + B\dot{a}^2}} \left[ \tilde{a} + \frac{1}{2B} \left( \frac{A'}{A} + 2\frac{C'}{C} \right) + \frac{\dot{\varphi}^2}{2} \left( \frac{A'}{A} + \frac{B'}{B} + 2\frac{C'}{C} \right) \right].$$

In the last two equations a prime represents a total derivative with respect to the metric functions’ argument $\ell$.

Having calculated all these, the following is how to construct a TSW in Visser’s sense [1]. Assume we have two (not necessarily the same) spacetimes of the general metric brought in Eq. (1). From both spacetimes, the inner part of the hypersurface $\mathcal{H}$ gets cut out and the outer parts are glued together at $\mathcal{H}$, which now becomes the two spacetimes’ 2 + 1–dimensional boundary; the whole structure is called a TSW and the hypersurface $\mathcal{H}$ becomes the throat of the wormhole. We note that the hypersurface is selected such that the cut radius $a$ is greater than any possible horizon existed in the two spacetimes. If the metric functions $A(\ell)$ and $B(\ell)$ of the two spacetimes are exactly the same, the result is a symmetric TSW, while if they are not (under certain circumstances), an ATSWS [3].

According to Lanczos [9] and Israel [10], such a structure must follow two certain conditions. The first condition, which was implicitly thought of in the explanations above, states that the metric of the throat must be continuous across the shell / throat i.e., $[g_{ij}^{(2)} - g_{ij}^{(1)}] = 0$. This implies that the metric function $A(\ell)$, $B(\ell)$ and $C(\ell)$ should satisfy

$$\left\{ -A_1(a)\, \ell^2 + B_1(a)\, \dot{\ell}^2 = -A_2(a)\, \ell^2 + B_2(a)\, \dot{\ell}^2 = -1 \right\}$$

$$C_1(a) = C_2(a) = C(a)$$

(9)
on the throat, where the indices 1 and 2 distinguish the bulk spacetimes.

Meanwhile, the second condition, imposes a discontinuity on the curvature tensor $K^a_b$ passing across the throat, and relates it to the energy-momentum tensor $S^a_b ( = \text{diag} (-\sigma, p, p))$ of the matter that unavoidably must exist on the throat. This condition is mathematically expressed as $(8\pi G = 1)$

$$\left[ K^a_b \right] - \delta^a_b \left[ K \right] = - S^a_b, \quad (10)$$

where the square brackets impart a jump in the included entity i.e. $[\Upsilon] = \Upsilon_2 - \Upsilon_1$. The latter equation, together with Eqs. (7) and (8) amount to two equations for the energy density $\sigma$ and the angular pressure $p$ on the throat as follow

$$\sigma = - \frac{C'}{C} \sum_{i=1}^{2} \sqrt{1 + B_i \dot{a}^2} \quad (11)$$

and

$$p = \sum_{i=1}^{2} \left\{ \sqrt{\frac{B_i}{1 + B_i \dot{a}^2}} \left[ \dot{a} + \frac{(A_i B_i C_i)'}{2A_i B_i C_i} \dot{a}^2 + \frac{(C_i')^2}{2B_i A_i C_i} \right] \right\}, \quad (12)$$

By manipulating Eq. (11) one is able to extract a dynamic mechanical equation for the throat such as

$$\dot{a}^2 + V (a) = 0, \quad (13)$$

where the potential

$$V (a) = \frac{1}{2} \left( \frac{1}{B_1} + \frac{1}{B_2} \right) - \left[ \frac{C'}{2C \sigma} \left( \frac{1}{B_1} - \frac{1}{B_2} \right) \right]^2 - \left( \frac{C \sigma}{2C'} \right)^2 \quad (14)$$

against the radius $a$ determines the stability of the throat under a radial perturbation. Note that in case of a perfect similarity between the two bulk spacetimes ($A_1 (a) = A_2 (a)$ and $B_1 (a) = B_2 (a)$), the equations above would simplify to a great deal.

Before we proceed, let us be more specific about the metric of the two spacetimes. The metric of the MTW [6] is traditionally expressed in the form

$$ds^2_{MT} = - e^{2\Phi (r)} dt^2 + \frac{dr^2}{1 - \frac{b (r)}{r}} + r^2 d\Omega^2, \quad (15)$$

in which $\Phi (r)$ and $b (r)$ are known as the redshift and the shape functions, respectively. For the spacetime to represent a proper spatial wormhole geometry and not to have any horizons or singularities, these two functions obey the following conditions; throughout the spacetime $1 - b (r)/r \geq 0$, and $\Phi (r)$ is finite everywhere. Therefore, the wormhole possesses a throat at $r_0 = b(r_0)$ as the minimum radius.

One example of such functions which fits to the criterion above for the shape function is

$$b (r) = \frac{r_0^2}{r}, \quad (16)$$

with which, alongside the new radial coordinate

$$\ell = \pm \left( r^2 - r_0^2 \right)^{1/2}, \quad -\infty < \ell < \infty \quad (17)$$

the MTW metric transforms to

$$ds^2_{MT} = - e^{2\Phi (\ell)} d\ell^2 + \ell^2 + (\ell^2 + r_0^2) d\Omega^2. \quad (18)$$

Over the years, while many spacetimes have been studied in the TSW framework, the MT spacetime has amazingly been ignored; perhaps because the idea of constructing a TSW out of a traversable wormhole is bizarre enough not to draw attentions. Nonetheless, we dare to use the metric in Eq. (15) to construct a TSW, though, not even a symmetric one but an ATSW by entailing different redshift functions on the two sides, i.e. $\Phi_1 (\ell) \neq \Phi_2 (\ell)$. Therefore,
the energy density $\sigma$, the angular pressure $p$, and the potential $V (a)$ of the throat versus its radial coordinate amount to ($\ell = a$)

$$\sigma = \frac{4a\sqrt{1 + \dot{a}^2}}{a^2 + r_0^2},$$ (19)

$$p = \sqrt{1 + \dot{a}^2} \left( \frac{2\dot{a}}{1 + \dot{a}^2} + \Phi_1' + \Phi_2' + \frac{2a}{a^2 + r_0^2} \right),$$ (20)

and

$$V (a) = 1 - \left[ \frac{(a^2 + r_0^2)}{4a} \sigma \right]^2,$$ (21)

respectively. Let us add that herein $a$ is the radius of the throat of the ATSW while $r_0$ stands for the radius of the original MTW. Hence, it holds that $a \geq 0$.

In the case of a static ATSW (not necessarily at equilibrium) one sets $a = a_0$ and $\dot{a} = \ddot{a} = 0$ which in turn imply

$$\sigma_0 = -\frac{4a_0}{(a_0^2 + r_0^2)},$$ (22)

and

$$p_0 = \Phi_1' (a_0) + \Phi_2' (a_0) + \frac{2a_0}{a_0^2 + r_0^2}.$$ (23)

As we mentioned above, generally $a_0 \geq 0$ but for the specific case when $a_0 = 0$ the two throats (throat of the original MTW and the throat of the ATSW) coincide; It is like cutting two bulk wormhole metrics at their throats and rejoining them at ATSW’s throat. Under these circumstances, $\sigma_0 = 0$, which implies that there is no need of matter at the throat. On the other hand, $p_0 = \Phi_1' (a_0) + \Phi_2' (a_0)$, which is generally nonzero and could potentially be positive. This entails that, under static condition, the weak and so the null energy conditions are satisfied at the thin-shell’s throat. Furthermore, it is worth mentioning that setting $a_0 = 0$ actually turns the ATSW to an asymmetric wormhole. A specific choice of $\Phi_1 = \Phi_2 = \Phi$ turns the resultant ATSW into a good old TSW. For the latter case one finds $p_0 = 2\Phi' (0)$.

### III. STABILITY ANALYSIS OF MT POWERED ATSWS

With reference to Eq. (21), in this section we would like to further analyze the potential of such a structure by employing a linear barotropic EoS, i.e.

$$p = p_0 + \omega (\sigma - \sigma_0),$$ (24)

wherein $\sigma_0$ and $p_0$ are given in Eqs. (22) and (23), and $\omega$ is the propagation speed of sound through the matter on the throat; accordingly, $\omega \in (0, 1)$. This form of EoS guarantees that when $a \to a_0$ we get $p \to p_0$.

Besides, applying the energy conservation on the throat, $\nabla_j S^{ij} = 0$, when $i = \tau$, one is able to recover the equation

$$\sigma' + \frac{2a}{a^2 + r_0^2} (\sigma + p) = 0$$ (25)

for the energy density and the tangential pressure on the throat. Using the above equation together with the EoS in Eq. (24), we arrive at an expression for the energy density

$$\sigma = \frac{1}{\omega + 1} \left[ (\sigma_0 + p_0) \left( \frac{a_0^2 + r_0^2}{a^2 + r_0^2} \right)^{\omega + 1} + \omega \sigma_0 - p_0 \right],$$ (26)

which with Eqs. (22) and (23), can be used to rewrite the potential at Eq. (21) as

$$V (a) = 1 - \left\{ \frac{(a^2 + r_0^2)}{4a (\omega + 1)} \left[ \left( \Psi - \frac{2a_0}{a_0^2 + r_0^2} \right) \left( \frac{a_0^2 + r_0^2}{a^2 + r_0^2} \right)^{\omega + 1} - 1 \right] - \frac{4a_0}{a_0^2 + r_0^2} (\omega + 1) \right\}^2,$$ (27)
FIG. 1: $V''(a_C)$ versus $\Psi$ for various values of $\omega$ within its permitted range. The figure accent that $V''(a_C)$ is negative definite, saying that $a_C$ is the maximum point of the potential $V(a)$.

where $\Psi = \Phi_1' + \Phi_2'$ is a constant. Since $r_0$ is merely a scale factor, without loss of generality we can set it equal to 1. Having this done, out of $V(a)$, taking the first derivative with respect to $a$ and setting $a = a_0$, we observe that $V'(a_0)$ has an extremum at a critical radius given by

$$a_C = \frac{(\sqrt{27} + \Psi^2 + \sqrt{27})^{2/3} - \Psi^{2/3}}{\sqrt{3} \left[ \Psi (\sqrt{27} + \Psi^2 + \sqrt{27}) \right]^{1/3}}$$

which is independent of $\omega$. Our numerical calculations (Fig. 1) confirms that the second derivative of the potential at the critical radius $a_C$, i.e. $V''(a_C)$ is negative definite for admissible domain of $\omega$ and arbitrary values of $\Psi$. Therefore, $a_C$ represents an unstable equilibrium radius for the TSW. On the other hand, if one rewrites Eq. (28) for $\Psi$ to obtain

$$\Psi = \frac{2}{a_C (a_C^2 + 1)}$$

then it could be brought this back to $V'(a_0)$ to see that for any $a_0 < a_C$ the numerical value of $V'(a_0)$ is positive, while for any $a_0 > a_C$ this value is negative. Accordingly, we state that, if the two MTW spacetimes are glued together at a radius $a_0 < a_C$, the TSW collapses to the original wormhole ($V(a) \to -\infty$ as $a \to 0$), while, if $a_0$ is selected such that $a_0 > a_C$, the TSW evaporates ($V(a) \to -\infty$ as $a \to \infty$). Note that, if $\Phi_1$ and $\Phi_2$ were selected such that $\Psi = 0$, no equilibrium point would exist and $V'(a_0)$ would always be positive, stating that the TSW would again collapse to the original wormhole. In either case of collapsing to the original wormhole, since there is no restrictions on the redshift functions $\Phi_1$ and $\Phi_2$ (except for their finiteness throughout the corresponding spacetimes), the resultant wormhole will be an asymmetric wormhole, in general. Stated otherwise, $b(r)$ function is common to both MTV whereas the redshift functions will differ.

IV. CONCLUSION

The first prototype example of a traversable wormhole was introduced in 1988 by Morris and Thorne [6], where they investigated the physical details of an observer for a safe passage through a wormhole structure. Exotic matter and inherent instability constitute the perilous items that threaten such a passage. Employing such MTWs, we construct TSWs, both symmetric and asymmetric at equal ease. The energy density remains still negative, unless we adjust the radius of both items to make it zero. The latter case renders satisfaction of the WEC possible, leaving the pressure components dependant on the redshift functions. Radial perturbation of the throat gives rise to an energy-momentum tensor which is identified as the EoS for a barotropic fluid with a critical radius $a_C$. Steep rise towards/away from
the critical radius indicates a collapsing/ever expanding TSW; either it collapses to the original MTW or attains a negative infinite potential in the equation $\dot{a}^2 + V(a) = 0$. Such a behavior determines the unstable fate of a TSW established on the MTW. It is worthwhile to note that, since the constructed TSW is asymmetric in general, in case it collapses to the original MTW, the resultant construction will be an asymmetric MTW i.e. same shape function but different redshift functions. Although this asymmetry does not reflect into the embedding diagram of the wormhole [6, 11] (since the shape function $b(r)$ which determines the geometrical shape of the embedding), due to different redshift functions considered here, it emanates in the traveler’s perception of time, once they pass the throat. Let’s add that asymmetric wormhole in different framework has been considered in literature [12]. It remains to be seen, whether the discussed ends for other TSWs powered by the MTW is generic or not. Our choice of the linear barotropic EoS is the simplest one; will there be different results otherwise? For example, what happens if a variable EoS [13] or a Chaplygin gas [14,15] are employed?

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