ON EXTRA ZEROS OF P-ADIC L-FUNCTIONS: THE CRYSTALLINE CASE

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Abstract. We formulate a conjecture about extra zeros of p-adic L-functions at near central points which generalizes the conjecture formulated in [Ben2]. We prove that this conjecture is compatible with Perrin-Riou’s theory of p-adic L-functions. Namely, using Nekovář’s machinery of Selmer complexes we prove that our $\mathcal{L}$-invariant appears as an additional factor in the Bloch-Kato type formula for special values of Perrin-Riou’s module of L-functions.

Nous avons toutefois supposé pour simplifier que les opérateurs $1 - \varphi$ et $1 - p^{-1} \varphi^{-1}$ sont inversibles laissant les autres cas, pourtant extrêmement intéressant pour plus tard.

Introduction to Chapter III of [PR2]

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0.1. **Extra zeros.** Let $M$ be a pure motive over $\mathbb{Q}$. Assume that the complex $L$-function $L(M, s)$ of $M$ extends to a meromorphic function on the whole complex plane $\mathbb{C}$. Fix an odd prime $p$. It is expected that one can construct $p$-adic analogs of $L(M, s)$ interpolating $p$-adically algebraic parts of its special values. This program was realised and the corresponding $p$-adic $L$-functions were constructed in many cases, but the general theory remains conjectural. In [PR2], Perrin-Riou formulated precise conjectures about the existence and arithmetic properties of $p$-adic $L$-functions in the case then the $p$-adic realisation $V$ of $M$ is crystalline at $p$. Let $D_{\text{cris}}(V)$ denote the filtered Dieudonné module associated to $V$ by the theory of Fontaine. Let $D$ be a subspace of $D_{\text{cris}}(V)$ of dimension $d_+(V) = \dim_{\mathbb{Q}_p} V_{c=1}$ stable under the action of $\varphi$. One says that $D$ is regular if one can associate to $D$ a $p$-adic analog of the six-term exact sequence of Fontaine and Perrin-Riou (see [PR2] for exact definition). Fix a lattice $T$ of $V$ stable under the action of the Galois group and a lattice $N$ of a regular module $D$. Perrin-Riou conjectured that one can associate to this data a $p$-adic $L$-function $L_p(T, N, s)$ satisfying some explicit interpolation property. Let $r$ denote the order of vanishing of $L(M, s)$ at $s = 0$ and let $L^*(M, 0) = \lim_{s \to 0} s^{-r} L(M, s)$. Then at $s = 0$ the interpolation property writes

$$\lim_{s \to 0} \frac{L_p(T, N, s)}{s^r} = \mathcal{E}(V, D) R_{V, D}(\omega_{V, N}) \frac{L^*(M, 0)}{R_{M, \infty}(\omega_M)}.$$ 

Here $R_{M, \infty}(\omega_M)$ (resp. $R_{V, D}(\omega_{V, N})$) is the determinant of the Beilinson (resp. the $p$-adic regulator) computed in some compatible bases $\omega_M$ and $\omega_{V, N}$ and $\mathcal{E}(V, D)$ is an Euler-like factor given by

$$\mathcal{E}(V, D) = \det(1 - p^{-1} \varphi^{-1} | D) \det(1 - \varphi | D_{\text{cris}}(V)/D).$$

If either $D_{\text{cris}} = p^{-1}$ or $(D_{\text{cris}}(V)/D)^{\varphi=1} \neq 0$ we have $\mathcal{E}(V, D) = 0$ and the order of vanishing of $L_p(N, T, s)$ should be $> r$. In this case we say that $L(T, N, s)$ has an extra zero at $s = 0$. The same phenomenon occurs in the case then $V$ is semistable and non-crystalline at $p$. An archotypical example is provided by elliptic curves having split multiplicative reduction [MTT]. Assume that $0$ is a critical point for $L(M, s)$ and that $H^0(M) = H^0(M^*(1)) = 0$. In [Ben2] using the theory of $(\varphi, \Gamma)$-modules we associated to each regular $D$ an invariant $\mathcal{L}(V, D) \in \mathbb{Q}_p$ generalising both Greenberg’s $\mathcal{L}$-invariant [G] and Fontaine-Mazur’s $\mathcal{L}$-invariant [M]. This allows to formulate a quite general conjecture about the behavior of $p$-adic $L$-functions at extra zeros in the spirit of [G]. To the best of our knowledge this conjecture is actually proved in the following cases:

1) Kubota-Leopoldt $p$-adic $L$-functions [FG], [GK]. Here the $\mathcal{L}$-invariant can be interpreted in terms of Gross $p$-adic regulator [Gs].

2) Modular forms of even weight [K], [GS], [S]. Here the $\mathcal{L}$-invariant coincides with Fontaine-Mazur’s $\mathcal{L}(f)$.

3) Modular forms of odd weight [Ben3]. The associated $p$-adic representation $V$ is either crystalline or potentially crystalline at $p$ and we do need the theory of $(\varphi, \Gamma)$-modules to define the $\mathcal{L}$-invariant.

4) Symmetric square of an elliptic curve having either split multiplicative reduction [Ro] or a good ordinary reduction (Dasgupta, work in progress). Here $V$ is ordinary and the $\mathcal{L}$-invariant reduces to Greenberg’s construction [G].

5) Symmetric powers of CM-modular forms [HL].

0.2. **Extra zero conjecture.** In this paper we generalise the conjecture from [Ben2] to the non critical point case. Assume that $V$ is crystalline at $p$. Then the weight argument shows that $\mathcal{E}(V, D)$ can vanish only if $wt(M) = 0$ or $-2$. In particular, we expect that the interpolation


factor does not vanish at \( s = 0 \) if \( wt(M) = -1 \) i.e. that the \( p \)-adic \( L \)-function can not have an extra zero at the central point in the good reduction case. To fix ideas assume that \( wt(M) \leq -2 \) and that \( M \) has no subquotients isomorphic to \( \mathbb{Q}(1) \). Then \( D \) is regular if and only if the associated \( p \)-adic regulator map

\[
r_{V,D} : H^1_{\text{cris}}(V) \to D_{\text{cris}}(V)/(\text{Fil}^0 D_{\text{cris}}(V) + D)
\]

is an isomorphism. The semisimplicity of \( \varphi : D_{\text{cris}}(V) \to D_{\text{cris}}(V) \) (which conjecturally always holds) allows to decompose \( D \) into a direct sum

\[
D = D_{-1} \oplus D_{\varphi=p^{-1}}.
\]

Under some mild assumptions (see 3.1.2 and 4.1.2 below) we associate to \( D \) an \( \mathcal{L} \)-invariant \( \mathcal{L}(V,D) \) which is a direct generalization of the main construction of [Ben2]. The Beilinson-Deligne conjecture predicts that \( V(t) \) does not vanish at \( s = 0 \) and that \( L(M^*(1), s) \) has a zero of order \( r = \dim_{\mathbb{Q}} H^1_f(V) \) at \( s = 0 \). We propose the following conjecture:

**Extra zero conjecture.** Let \( D \) be a regular subspace of \( D_{\text{cris}}(V) \) and let \( e = \dim_{\mathbb{Q}}(D_{\varphi=p^{-1}}) \). Then

1) The \( p \)-adic \( L \)-function \( L_p(T, N, s) \) has a zero of order \( e \) at \( s = 0 \) and

\[
\frac{L_p^*(T, N, 0)}{R_{V,D}(\omega_{V,N})} = -\mathcal{L}(V,D) \mathcal{E}^+(V,D) \frac{L(M,0)}{R_{M,\infty}(\omega_M)}.
\]

2) Let \( D^\perp \) denote the orthogonal complement to \( D \) under the canonical duality \( D_{\text{cris}}(V) \times D_{\text{cris}}(V^*(1)) \to \mathbb{Q}_p \). The \( p \)-adic \( L \)-function \( L_p(T^*(1), N^\perp, s) \) has a zero of order \( e + r \) where \( r = \dim_{\mathbb{Q}} H^1_f(V) \) at \( s = 0 \) and

\[
\frac{L_p^*(T^*(1), N^\perp, 0)}{R_{V^*,(1),D^\perp}(\omega_{V^*,(1)},N^\perp)} = \mathcal{L}(V,D) \mathcal{E}^+(V^*(1), D^\perp) \frac{L^*(M^*(1),0)}{R_{M^*,(1,\infty}(\omega_{M^*,(1)})}.
\]

In the both cases

\[
\mathcal{E}^+(V,D) = \mathcal{E}^+(V^*(1), D^\perp) = \det(1 - p^{-1} \varphi^{-1} \mid D_{-1}) \det(1 - p^{-1} \varphi^{-1} \mid D_{\text{cris}}(V^*(1))).
\]

**Remarks.** 1) \( \mathcal{E}^+(V,D) \) is obtained from \( \mathcal{E}(V,D) \) by excluding zero factors. It can be also written in the form

\[
\mathcal{E}^+(V,D) = E_p^*(V,1) \det_{\mathbb{Q}_p} \left( \frac{1 - p^{-1} \varphi^{-1}}{1 - \varphi} \mid D_{-1} \right)
\]

where \( E_p(V,t) = \det(1-\varphi t \mid D_{\text{cris}}(V)) \) is the Euler factor at \( p \) and \( E_p^*(V,t) = E_p(V,t) \left( 1 - \frac{t}{p} \right)^{-e} \).

2) Assume that \( H^1_f(V) = 0 \). Since \( H^1_f(V^*(1)) \) should also vanish by the weight reason, our conjecture in this cases reduces to the conjecture 2.3.2 from [Ben2].

3) The regularity of \( D \) supposes that the localisation \( H^1_f(V) \to H^1_f(\mathbb{Q}_p, V) \) is injective. Jannsen’s conjecture (precised by Bloch and Kato) says that the \( p \)-adic realisation map \( H^1_f(M) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p \to H^1_f(\mathbb{Q}_p, V) \) is injective. The last condition is not really essential and can be suppressed.
$\mathbb{Q}_p \to H^1_f(V)$ is an isomorphism. The composition $H^1_f(M) \to H^1_f(\mathbb{Q}_p, V)$ of these two maps is essentially the syntomic regulator. Its injectivity seems to be a difficult open problem.

0.3. Selmer complexes and Perrin-Riou’s theory. In the last part of the paper we show that our extra zero conjecture is compatible with the Main Conjecture of Iwasawa theory as formulated in [PR2]. The main technical tool here is the descent theory for Selmer complexes [Ne2]. We hope that the approach to Perrin-Riou’s theory based on the formalism of Selmer complexes can be of independent interest.

For a profinite group $G$ and a continuous $G$-module $X$ we denote by $C_c^\bullet(G, X)$ the standard complex of continuous cochains. Let $S$ be a finite set of primes containing $p$. Denote by $G_S$ the Galois group of the maximal algebraic extension of $\mathbb{Q}$ unramified outside $S \cup \{ \infty \}$. Set $R\Gamma_S(X) = C_c^\bullet(G_S, X)$ and $R\Gamma(\mathbb{Q}_p, X) = C_c^\bullet(G_\mathbb{Q}_p, X)$, where $G_\mathbb{Q}_p$ is the absolute Galois group of $\mathbb{Q}_p$. Let $\Gamma$ be the Galois group of the cyclotomic $p$-extension $\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}$, $\Gamma_1 = \text{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}(\zeta_p))$ and $\Delta = \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$. Let $\Lambda(\Gamma) = \mathbb{Z}_p[[\Gamma]]$ denote the Iwasawa algebra of $\Gamma$. Each $\Lambda(\Gamma)$-module $X$ decomposes into the direct sum of its isotypical components $X = \bigoplus_{\eta \in \Delta} X^{(\eta)}$ and we denote by $X^{(\eta)}$ the component which corresponds to the trivial character $\eta_0$. Set $\Lambda = \Lambda(\Gamma)^{(\eta_0)}$. Let $\mathcal{H}$ denote the algebra of power series with coefficients in $\mathbb{Q}_p$ which converge on the open unit disk. We will denote again by $\mathcal{H}$ the associated large Iwasawa algebra $\mathcal{H}(\Gamma_1)$. In this paper we consider only the trivial character component of the module of $p$-adic $L$-functions because it is sufficient for applications to trivial zeros, but in the general case the construction is exactly the same. We keep notation and assumptions of section 0.2. Assume that the weak Leopoldt conjecture holds for $(V, \eta_0)$ and $(V^*(1), \eta_0)$. We consider global and local Iwasawa cohomology $R\Gamma_{Iw,S}(T) = R\Gamma_S(\Lambda(\Gamma) \otimes \mathbb{Z}_p, T)^\dagger$ and $R\Gamma_{Iw}(\mathbb{Q}_p, T) = R\Gamma(\mathbb{Q}_p, (\Lambda(\Gamma) \otimes \mathbb{Z}_p) T)^\dagger$ where $\dagger$ is the canonical involution on $\Lambda(\Gamma)$. Let $D$ be a regular submodule of $D_{\text{cris}}(V)$. For each non archimedean place $v$ we define a local condition at $v$ in the sense of [Ne2] as follows. If $v \neq p$ we use the unramified local condition which is defined by

$$R\Gamma_{Iw,f}(\mathbb{Q}_v, N, T) = R\Gamma_{Iw,f}^{(\eta_0)}(\mathbb{Q}_v, T) = \left[T^I_v \otimes \Lambda^i \xrightarrow{f_v} T^I_v \otimes \Lambda^i\right]$$

where $I_v$ is the inertia subgroup at $v$ and $f_v$ is the geometric Frobenius. If $v = p$ we define

$$R\Gamma_{Iw,f}^{(\eta_0)}(\mathbb{Q}_v, N, T) = (N \otimes \Lambda)[-1].$$

The derived version of the large exponential map $\text{Exp}_{V,h}$, $h \gg 0$ (see [PR1]) gives a morphism

$$R\Gamma_{Iw,f}^{(\eta_0)}(\mathbb{Q}_v, N, T) \to R\Gamma_{Iw}^{(\eta_0)}(\mathbb{Q}_p, T) \otimes \mathcal{H}.$$

Therefore we have a diagram

$$R\Gamma_{Iw,S}^{(\eta_0)}(T) \otimes_\Lambda \mathcal{H} \xrightarrow{\bigoplus_{v \in S} R\Gamma_{Iw,f}^{(\eta_0)}(\mathbb{Q}_v, T) \otimes_\Lambda \mathcal{H}} \left(\bigoplus_{v \in S} R\Gamma_{Iw,f}^{(\eta_0)}(\mathbb{Q}_v, N, T)\right) \otimes_\Lambda \mathcal{H}.$$

Let $R\Gamma_{Iw,h}^{(\eta_0)}(D, V)$ denote the Selmer complex associated to this data. By definition it sits in the distinguished triangle

$$R\Gamma_{Iw,S}^{(\eta_0)}(D, V) \to \left(R\Gamma_{Iw,S}^{(\eta_0)}(V) \oplus \left(\bigoplus_{v \in S} R\Gamma_{Iw,f}^{(\eta_0)}(\mathbb{Q}_v, D, V)\right)\right) \otimes \mathcal{H} \to \left(\bigoplus_{v \in S} R\Gamma_{Iw}^{(\eta_0)}(\mathbb{Q}_v, V)\right) \otimes \mathcal{H}. \quad (0.1)$$
where \( \chi \).

**Theorem 2.** Assume that \( \mathcal{L}(V, D) \neq 0 \). Then

i) The cohomology \( R^i \Gamma_{Iw,h}^{(\eta_0)}(D, V) \) are \( \mathcal{H} \)-torsion modules for all \( i \).

ii) \( R^i \Gamma_{Iw,h}^{(\eta_0)}(D, V) = 0 \) for \( i \neq 2, 3 \) and

\[
R^3 \Gamma_{Iw,h}^{(\eta_0)}(D, V) \simeq (H^0(\mathbb{Q}((p^\infty)), V^*(1))^*)^{(\eta_0)} \otimes_A \mathcal{H}.
\]

iii) The complex \( R^i \Gamma_{Iw,h}^{(\eta_0)}(D, V) \) is semisimple i.e. for each \( i \) the natural map

\[
R^i \Gamma_{Iw,h}^{(\eta_0)}(D, V)^\Gamma \to R^i \Gamma_{Iw,h}^{(\eta_0)}(D, V)_{\Gamma}
\]

is an isomorphism.

Assume that \( \mathcal{L}(V, D) \neq 0 \). Let \( \mathcal{K} \) be the field of fractions of \( \mathcal{H} \). Then Theorem 1 together with (0.1) define an injective map

\[
i_{V,Iw,h} : \Delta_{Iw,h}(N, T) \to \mathcal{K}
\]

and the module of \( p \)-adic \( L \)-functions is defined as

\[
L_{Iw,h}^{(\eta_0)}(N, T) = i_{V,Iw,h}(\Delta_{Iw,h}(N, T)) \subset \mathcal{K}.
\]

Let \( \gamma_1 \) be a fixed generator of \( \Gamma_1 \). Choose a generator \( f(\gamma_1 - 1) \) of the free \( A \)-module \( L_{Iw,h}^{(\eta_0)}(N, T) \) and define a meromorphic \( p \)-adic function

\[
L_{Iw,h}(T, N, s) = f(\chi(\gamma_1)^s - 1),
\]

where \( \chi : \Gamma \to \mathbb{Z}_p^* \) is the cyclotomic character.

**Theorem 2.** Assume that \( \mathcal{L}(V, D) \neq 0 \). Then

1) The \( p \)-adic \( L \)-function \( L_{Iw,h}(T, N, s) \) has a zero of order \( e = \dim_{\mathbb{Q}_p}(D^{e=p^{-1}}) \) at \( s = 0 \).

2) One has

\[
\frac{L_{Iw,h}^*(T, N, 0)}{R_{V,D}(\omega_{T,N})} \sim_p \Gamma(h)^{d_+(V)} \mathcal{L}(V, D) \mathcal{E}^+(V, D) \frac{\# \mathcal{U}(T^*(1)) \text{ Tam}_{\omega_M}^0(T)}{\# H_{\mathbb{S}}^0(V/T) \# H_{\mathbb{S}}^0(V^*(1)/T^*(1))},
\]

where \( \mathcal{U}(T^*(1)) \) is the Tate-Shafarevich group of Bloch-Kato [BK] and \( \text{ Tam}_{\omega_M}^0(T) \) is the product of local Tamagawa numbers of \( T \).

**Remarks.** 1) Using the compatibility of Perrin-Riou’s theory with the functional equation we obtain analogous results for the \( L_p(T^*(1), N^\perp, s) \) (see section 5.2.9).

2) If \( D_{\text{cris}}(V)^{\varphi=1} = D_{\text{cris}}(V)^{\varphi=p^{-1}} = 0 \) the phenomenon of extra zeros does not appear, \( \mathcal{L}(V, D) = 1 \) and Theorem 2 was proved in [PR2], Theorem 3.6.5. We remark that even in this case our proof is different. We compare the leading term of \( L_{Iw,h}^*(T, N, s) \) with the trivialisation
$i_{\omega, M, p} : \Delta_{EP}(T) \to \mathbb{Q}_p$ of the Euler-Poincaré line $\Delta_{EP}(T)$ (see [F3]) and show that in compatible bases one has

$$\frac{L_{Iw, h}^i(T, N, 0)}{R_{V, D}(\omega_{V, N})} \sim_p \Gamma(h)^{d_+(V)} \mathcal{L}(V, D) \mathcal{E}^+(V, D) i_{\omega, M, p}(\Delta_{EP}(T))$$

(see Theorem 5.2.5). Now Theorem 2 follows from the well known computation of $i_{\omega, M, p}(\Delta_{EP}(T))$ in terms of the Tate-Shafarevich group and Tamagawa numbers ([FP], Chapitre II).

3) Let $E/\mathbb{Q}$ be an elliptic curve having good reduction at $p$. Consider the $p$-adic representation $V = \text{Sym}^2(T_p(E)) \otimes \mathbb{Q}_p$, where $T_p(E)$ is the $p$-adic Tate module of $E$. It is easy to see that $D = D_{\text{cris}}(V)^{e=p-1}$ is one dimensional. In this case some versions of Theorem 2 were proved in [PR3] and [D] with an ad hoc definition of the $\mathcal{L}$-invariant. Remark that $p$-adic $L$-functions attached to the symmetric square of a newform were constructed by Dabrowski and Delbourgo [DD].

4) Another approach to Iwasawa theory in the non-ordinary case was developped by Pottharst in [Pt1], [Pt2]. Pottharst uses the formalism of Selmer complexes but works with local conditions coming from submodules of the $(\varphi, \Gamma)$-module associated to $V$ rather then with the large exponential map. This approach has many advantages, in particular it allows to develop an interesting theory for representations which are not necessarily crystalline. Nevertheless it seems that the large exponential map is crucial for the study of extra zeros at least in the good reduction case.

5) The Main conjecture of Iwasawa theory [PR2], [C2] says that the analytic $p$-adic $L$-function $L_p(N, T, s)$ multiplied by a simple explicit $\Gamma$-factor depending on $h$ can be written in the form $L_p(N, T, s) = f(\chi(\gamma_1)^s - 1)$ for an appropriate generator $f(\gamma_1 - 1)$ of $L_{Iw, h}^i(N, T)$. Therefore the main conjecture implies Bloch-Kato style formulas for special values of $L_p(N, T, s)$. We remark that the Bloch-Kato conjecture predicts that

$$\frac{L^*(M, 0)}{RM, \infty(\omega_M)} \sim_p \# \mathcal{I}(T^*(1)) \frac{\text{Tam}_{\omega, M}(T)}{\#H^0_S(V/T) \#H^0_S(V^*(1)/T^*(1))}$$

and therefore Theorem 2 implies the compatibility of our extra zero conjecture with the Main conjecture. Note that this also follows directly from (0.2) if we use the formalism of Fontaine and Perrin-Riou [F3] to formulate Bloch-Kato conjectures.

0.4. The organisation of the paper is as follows. In §1 we review the theory of $(\varphi, \Gamma)$-modules which is the main technical tool in our definition of the $\mathcal{L}$-invariant. We also give the derived version of computation of Galois cohomology in terms of $(\varphi, \Gamma)$-modules. This follows easily from the results of Herr [H1] and Liu [Li] and the proofs are placed in Appendix. Similar results can be found in [Pt1], [Pt2]. In §2 we recall preliminaries on the Bloch-Kato exponential map and review the construction of the large exponential map of Perrin-Riou given by Berger [Ber3] using again the basic language of derived categories. The $L$-invariant is constructed is section 3.1. In section 3.2 we relate this construction to the derivative of the large exponential map. This result plays a key role in the proof of Theorem 2. The extra zero conjecture is formulated in §4. In §5 we interpret Perrin-Riou’s theory in terms of Selmer complexes and prove Theorems 1 and 2.

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§1. Preliminaries

1.1. \((\varphi, \Gamma)\)-modules.

1.1.1. The Robba ring (see [Ber1],[C3]). In this section \(K\) is a finite unramified extension of \(\mathbb{Q}_p\) with residue field \(k_K\), \(O_K\) its ring of integers, and \(\sigma\) the absolute Frobenius of \(K\). Let \(\overline{K}\) an algebraic closure of \(K\), \(G_K = \text{Gal}(\overline{K}/K)\) and \(C\) the completion of \(\overline{K}\). Let \(v_p : C \to \mathbb{R} \cup \{\infty\}\) denote the \(p\)-adic valuation normalized so that \(v_p(p) = 1\) and set \(|x|_p = \left(\frac{1}{p}\right)^{v_p(x)}\). Write \(B\) for the \(p\)-adic annulus \(B(\mu^r, 1) = \{x \in C \mid r \leq |x| < 1\}\). As usually, \(\mu^p_n\) denotes the group of \(p^n\)-th roots of unity. Fix a system of primitive roots of unity \(\zeta_n = (\zeta_{p^n})_{n \geq 0}\), \(\zeta_{p^n} \in \mu_{p^n}\) such that \(\zeta_{p^n}^p = \zeta_{p^{n-1}}\) for all \(n\). Set \(K_n = K(\zeta_{p^n})\), \(K_\infty = \bigcup_{n=0}^\infty K_n\), \(H_K = \text{Gal}(\overline{K}/K_\infty)\), \(\Gamma = \text{Gal}(K_\infty/K)\) and denote by \(\chi : \Gamma \to \mathbb{Z}_p^*\) the cyclotomic character.

Set \(\mathfrak{E}^+ = \lim_{x \to x^p} O_C/pO_C = \{x = (x_0, x_1, \ldots, x_n, \ldots) \mid x_i^p = x_i \ \forall i \in \mathbb{N}\}\). Let \(\hat{x}_n \in O_C\) be a lifting of \(x_n\). Then for all \(m \geq 0\) the sequence \(\hat{x}_m^p\) converges to \(x^{(m)} = \lim_{n \to \infty} \hat{x}_n^p \in O_C\) which does not depend on the choice of liftings. The ring \(\mathfrak{E}^+\) equipped with the valuation \(v_E(x) = v_p(x^{(0)})\) is a complete local ring of characteristic \(p\) with residue field \(\overline{k}_K\). Moreover it is integrally closed in its field of fractions \(\mathfrak{E} = \text{Fr}(\mathfrak{E}^+)\).

Let \(\Lambda = W(\mathfrak{E})\) be the ring of Witt vectors with coefficients in \(\mathfrak{E}\). Denote by \([\ ] : \mathfrak{E} \to W(\mathfrak{E})\) the Teichmüller lift. Any \(u = (u_0, u_1, \ldots) \in \Lambda\) can be written in the form

\[ u = \sum_{n=0}^\infty [u^p^n]p^n.\]

Set \(\pi = [e] - 1\), \(A^+_K = O_K[[\pi]]\) and denote by \(A_K\) the \(p\)-adic completion of \(A^+_K[1/\pi]\). Let \(B = \Lambda[1/p], B_K = A_K[1/p]\) and let \(B\) denote the completion of the maximal unramified extension of \(B_K\) in \(B\). Set \(A = B \cap \Lambda, A^+ = W(\mathfrak{E}^+), A^+ = \Lambda^+ \cap A\) and \(B^+ = A^+[1/p]\). All these rings are endowed with natural actions of the Galois group \(G_K\) and Frobenius \(\varphi\).

Set \(A_K = A^{H_K}\) and \(B_K = A_K[1/p]\). Remark that \(\Gamma\) and \(\varphi\) act on \(B_K\) by

\[ \tau(\pi) = (1 + \pi)^{\chi(\tau)} - 1, \quad \tau \in \Gamma \]

\[ \varphi(\pi) = (1 + \pi)^p - 1. \]

For any \(r > 0\) define

\[ \mathfrak{B}^{\downarrow, r} = \left\{ x \in \mathfrak{B} \mid \lim_{k \to +\infty} \left( v_E(x_k) + \frac{pr}{p-1} k \right) = +\infty \right\}. \]

Set \(\mathfrak{B}^{\downarrow, r} = B \cap \mathfrak{B}^{\downarrow, r}, \mathfrak{B}^{\downarrow, r}_K = B_K \cap \mathfrak{B}^{\downarrow, r}\), \(\mathfrak{B}^{\downarrow} = \bigcup_{r > 0} \mathfrak{B}^{\downarrow, r}\) \(\mathfrak{A}^{\downarrow} = A \cap \mathfrak{B}^{\downarrow}\) and \(\mathfrak{B}^{\downarrow}_K = \bigcup_{r > 0} \mathfrak{B}^{\downarrow, r}_K\).

It can be shown that for any \(r \geq p - 1\)

\[ \mathfrak{B}^{\downarrow, r}_K = \left\{ f(\pi) = \sum_{k \in \mathbb{Z}} a_k \pi^k \mid a_k \in K\text{ and } f \text{ is holomorphic and bounded on } B(r, 1) \right\}. \]

Define

\[ \mathfrak{B}^{\downarrow, r}_{\text{rig, } K} = \left\{ f(\pi) = \sum_{k \in \mathbb{Z}} a_k \pi^k \mid a_k \in K\text{ and } f \text{ is holomorphic on } B(r, 1) \right\}. \]
Set $R(K) = \bigcup_{r \geq p-1} B_{rig,K}^r$ and $R^+(K) = R(K) \cap K[[\pi]]$. It is not difficult to check that these rings are stable under $\Gamma$ and $\varphi$. To simplify notations we will write $R = R(Q_p)$ and $R^+ = R^+(Q_p)$. As usual, we set
\[ t = \log(1 + \pi) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\pi^n}{n} \in R \]
Note that $\varphi(t) = pt$ and $\tau(t) = \chi(\gamma)t$, $\tau \in \Gamma$.

1.1.2. $(\varphi, \Gamma)$-modules (see [F2], [CC1]). Let $A$ be either $B_K^\dagger$ or $R(K)$. A $(\varphi, \Gamma)$-module over $A$ is a finitely generated free $A$-module $D$ equipped with semilinear actions of $\varphi$ and $\Gamma$ commuting to each other and such that the induced linear map $\varphi : A \otimes_\varphi D \to D$ is an isomorphism. Such a module is said to be etale if it admits a $A_K^\dagger$-lattice $N$ stable under $\varphi$ and $\Gamma$ and such that $\varphi : A_K^\dagger \otimes_\varphi N \to N$ is an isomorphism. The functor $D \mapsto R(K) \otimes_{B_K^\dagger} D$ induces an equivalence between the category of etale $(\varphi, \Gamma)$-modules over $B_K^\dagger$ and the category of $(\varphi, \Gamma)$-modules over $R(K)$ which are of slope 0 in the sense of Kedlaya’s theory ([Ke] and [C5], Corollary 1.5). Then Fontaine’s classification of $p$-adic representations [F2] together with the main result of [CC1] lead to the following statement.

Proposition 1.1.3. i) The functor
\[ D^\dagger : V \mapsto D^\dagger(V) = (B^\dagger \otimes_{Q_p} V)^{H_K} \]
establishes an equivalence between the category of $p$-adic representations of $G_K$ and the category of etale $(\varphi, \Gamma)$-modules over $B_K^\dagger$.

ii) The functor $D_{rig}^\dagger(V) = R(K) \otimes_{B_K^\dagger} D^\dagger(V)$ gives an equivalence between the category of $p$-adic representations of $G_K$ and the category of $(\varphi, \Gamma)$-modules over $R(K)$ of slope 0.

Proof. see [C4], Proposition 1.7.

1.1.4. Cohomology of $(\varphi, \Gamma)$-modules (see [H1], [H2], [Li]). Fix a generator $\gamma$ of $\Gamma$. If $D$ is a $(\varphi, \Gamma)$-module over $A$, we denote by $C_{\varphi, \gamma}(D)$ the complex
\[ C_{\varphi, \gamma}(D) : 0 \xrightarrow{f} D \xrightarrow{g} D \oplus D \xrightarrow{g} D \to 0 \]
where $f(x) = ((\varphi - 1)x, (\gamma - 1)x)$ and $g(y, z) = (\gamma - 1)y - (\varphi - 1)z$. Set $H^i(D) = H^i(C_{\varphi, \gamma}(D))$. A short exact sequence of $(\varphi, \Gamma)$-modules
\[ 0 \to D' \to D \to D'' \to 0 \]
gives rise to an exact cohomology sequence:
\[ 0 \to H^0(D') \to H^0(D) \to H^0(D'') \to H^1(D') \to \cdots \to H^2(D'') \to 0. \]

Proposition 1.1.5. Let $V$ be a $p$-adic representation of $G_K$. Then the complexes $R\Gamma(K, V)$, $C_{\varphi, \gamma}(D^\dagger(V))$ and $C_{\varphi, \gamma}(D_{rig}^\dagger(V))$ are isomorphic in the derived category of $Q_p$-vector spaces $D(Q_p)$.

Proof. This is a derived version of Herr’s computation of Galois cohomology [H1]. The proof is given in the Appendix, Propositions A.3 and Corollary A.4.
1.1.6. Iwasawa cohomology. Recall that $\Lambda$ denotes the Iwasawa algebra of $\Gamma$, $\Delta = \text{Gal}(K_1/K)$ and $\Lambda(\Gamma) = \mathbb{Z}_p[\Delta] \otimes_{\mathbb{Z}_p} \Lambda$. Let $\iota: \Lambda(\Gamma) \rightarrow \Lambda(\Gamma)$ denote the involution defined by $\iota(g) = g^{-1}$, $g \in \Gamma$. If $T$ is a $\mathbb{Z}_p$-adic representation of $G_K$, then the induced module $\text{Ind}_{K}^{K_\infty}(T)$ is isomorphic to $(\Lambda(\Gamma) \otimes \mathbb{Z}_p) \otimes T$ and we set

$$R\Gamma_{\text{Iw}}(K,T) = R\Gamma(K,\text{Ind}_{K_\infty}^{K_\infty}(T)).$$

Write $H^i_{\text{Iw}}(K,T)$ for the Iwasawa cohomology

$$H^i_{\text{Iw}}(K,T) = \lim_{\text{cor} K_n/K_{n-1}} H^i(K_n,T).$$

Recall that there are canonical and functorial isomorphisms

$$R^i\Gamma_{\text{Iw}}(K,T) \simeq H^i_{\text{Iw}}(K,T), \quad i \geq 0,$$

$$R\Gamma_{\text{Iw}}(K,T) \otimes_{\Lambda(\Gamma)} \mathbb{Z}_p[G_n] \simeq R\Gamma(K_n,T)$$

(see [Ne2], Proposition 8.4.22). The interpretation of the Iwasawa cohomology in terms of $(\varphi, \Gamma)$-modules was found by Fontaine (unpublished but see [CC2]). We give here the derived version of this result. Let $\psi: B \rightarrow B$ be the operator defined by the formula $\psi(x) = \frac{1}{p} \varphi^{-1} (\text{Tr}_{B/\varphi(B)}(x))$. We see immediately that $\psi \circ \varphi = \text{id}$. Moreover $\psi$ commutes with the action of $G_K$ and $\psi(A^/) = A^\dagger$. Consider the complexes

$$C_{1w,\psi}(T) : D(T) \xrightarrow{\psi-1} D(T),$$

$$C^{\dagger}_{1w,\psi}(T) : D^\dagger(T) \xrightarrow{\psi-1} D^\dagger(T).$$

Proposition 1.1.7. i) The complexes $R\Gamma_{\text{Iw}}(K,T)$, $C_{1w,\psi}(T)$ and $C^\dagger_{1w,\psi}(T)$ are naturally isomorphic in the derived category $D(\Lambda(\Gamma))$ of $\Lambda(\Gamma)$-modules.

Proof. See Proposition A.7 and Corollary A.8.

1.1.8. $(\varphi, \Gamma)$-modules of rank 1. Recall the computation of the cohomology of $(\varphi, \Gamma)$-modules of rank 1 following Colmez [C4]. As in op. cit., we consider the case $K = \mathbb{Q}_p$ and put $\mathcal{R} = B^\dagger_{\text{rig}, \mathbb{Q}_p}$ and $\mathcal{R}^+ = B^+_{\text{rig}, \mathbb{Q}_p}$. The differential operator $\partial = (1 + \tau) \frac{d}{d\tau}$ acts on $\mathcal{R}$ and $\mathcal{R}^+$. If $\delta: \mathbb{Q}_p^* \rightarrow \mathbb{Q}_p^*$ is a continuous character, we write $\mathcal{R}(\delta)$ for the $(\varphi, \Gamma)$-module $\mathcal{R}e_\delta$ defined by $\varphi(e_\delta) = \delta(p)e_\delta$ and $\gamma(e_\delta) = \delta(\chi(\tau)) e_\delta$. Let $x$ denote the character induced by the natural inclusion of $\mathbb{Q}_p$ in $L$ and $|x|$ the character defined by $|x| = p^{-\nu_p(x)}$.

Proposition 1.1.9. Let $\delta: \mathbb{Q}_p^* \rightarrow \mathbb{Q}_p^*$ be a continuous character. Then:

i) $$H^0(\mathcal{R}(\delta)) = \begin{cases} \mathbb{Q}_p t^m & \text{if } \delta = x^{-m}, k \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

ii) $$\dim_{\mathbb{Q}_p}(H^1(\mathcal{R}(\delta))) = \begin{cases} 2 & \text{if either } \delta(x) = x^{-m}, m \geq 0 \text{ or } \delta(x) = |x| x^m, k \geq 1, \\ 1 & \text{otherwise.} \end{cases}$$
iii) Assume that $\delta(x) = x^{-m}$, $m \geq 0$. The classes $\text{cl}(t^m, 0) e_\delta$ and $\text{cl}(0, t^m) e_\delta$ form a basis of $H^1(\mathcal{R}(x^{-m}))$.

iv) Assume that $\delta(x) = |x|x^m$, $m \geq 1$. Then $H^1(\mathcal{R}(|x|^{x^m}))$, $m \geq 1$ is generated by $\text{cl}(\alpha_m)$ and $\text{cl}(\beta_m)$ where

$$
\alpha_m = \frac{(-1)^{m-1}}{(m-1)!} \vartheta^{m-1} \left( \frac{1}{\pi} + \frac{1}{2} a \right) e_\delta, \quad (1 - \varphi) a = (1 - \chi(\gamma) \gamma) \left( \frac{1}{\pi} + \frac{1}{2} \right),
$$

$$
\beta_m = \frac{(-1)^{m-1}}{(m-1)!} \vartheta^{m-1} \left( b, \frac{1}{\pi} \right) e_\delta, \quad (1 - \varphi) \left( \frac{1}{\pi} \right) = (1 - \chi(\gamma) \gamma) b.
$$

Proof. See [C4], sections 2.3-2.5.

1.2. Crystalline representations.

1.2.1. The rings $B_{\text{cris}}$ and $B_{\text{dR}}$ (see [F1], [F4]). Let $\theta_0 : A^+ \to O_C$ be the map given by the formula

$$
\theta_0 \left( \sum_{n=0}^{\infty} [u_n] p^n \right) = \sum_{n=0}^{\infty} u_n^{(0)} p^n.
$$

It can be shown that $\theta_0$ is a surjective ring homomorphism and that $\ker(\theta_0)$ is the principal ideal generated by $\omega = \sum_{i=0}^{p-1} [e]^i/p$. By linearity, $\theta_0$ can be extended to a map $\theta : \tilde{B}^+ \to C$. The ring $B_{\text{dR}}^+$ is defined to be the completion of $\tilde{B}^+$ for the $\ker(\theta)$-adic topology:

$$
B_{\text{dR}}^+ = \lim_n \tilde{B}^+ / \ker(\theta)^n.
$$

This is a complete discrete valuation ring with residue field $C$ equipped with a natural action of $G_K$. Moreover, there exists a canonical embedding $\tilde{K} \subset B_{\text{dR}}^+$. The series $t = \sum_{n=0}^{\infty} (-1)^{n-1} \pi^n / n$ converges in the topology of $B_{\text{dR}}^+$ and it is easy to see that $t$ generates the maximal ideal of $B_{\text{dR}}^+$. The Galois group acts on $t$ by the formula $g(t) = \chi(g) t$. Let $B_{\text{dR}} = B_{\text{dR}}^+ \{ t^{-1} \}$ be the field of fractions of $B_{\text{dR}}^+$. This is a complete discrete valuation field equipped with a $G_K$-action and an exhaustive separated decreasing filtration $\text{Fil}^i B_{\text{dR}} = t^i B_{\text{dR}}^+$. As $G_K$-module, $\text{Fil}^i B_{\text{dR}} / \text{Fil}^{i+1} B_{\text{dR}} \simeq C(i)$ and $B_{\text{dR}}^{G_K} = K$.

Consider the $PD$-envelope of $A^+$ with a respect to the map $\theta_0$

$$
A_{\text{PD}} = A^+ \left[ \omega^2, \omega^3, \frac{\omega}{3!}, \ldots, \frac{\omega^n}{n!}, \ldots \right]
$$

and denote by $A_{\text{cris}}^+$ its $p$-adic completion. Let $B_{\text{cris}}^+ = A_{\text{cris}}^{\mathbb{Z}_p} \otimes \mathbb{Q}_p$ and $B_{\text{cris}} = B_{\text{cris}}^{\mathbb{Z}_p} \{ t^{-1} \}$. Then $B_{\text{cris}}$ is a subring of $B_{\text{dR}}$ endowed with the induced filtration and Galois action. Moreover, it is equipped with a continuous Frobenius $\varphi$, extending the map $\varphi : A^+ \to A^+$. One has $\varphi(t) = pt$.

1.2.2. Crystalline representations (see [F5], [Ber1], [Ber2]). Let $L$ be a finite extension of $\mathbb{Q}_p$. Denote by $K$ its maximal unramified subextension. A filtered Dieudonné module over $L$ is a finite dimensional $K$-vector space $M$ equipped with the following structures:
• a $\sigma$-semilinear bijective map $\varphi : M \to M$;
• an exhaustive decreasing filtration $(\text{Fil}_i M_L)$ on the $L$-vector space $M_L = L \otimes_K M$.

A $K$-linear map $f : M \to M'$ is said to be a morphism of filtered modules if

1. $f(\varphi(d)) = \varphi(f(d))$, for all $d \in M$;
2. $f(\text{Fil}_i M_L) \subseteq \text{Fil}_i M'_L$, for all $i \in \mathbb{Z}$.

The category $\text{MF}^\varphi_L$ of filtered Dieudonné modules is additive, has kernels and cokernels but is not abelian. Denote by $1$ the vector space $K_0$ with the natural action of $\sigma$ and the filtration given by

$$\text{Fil}^i 1 = \begin{cases} 
K, & \text{if } i \leq 0, \\
0, & \text{if } i > 0.
\end{cases}$$

Then $1$ is a unit object of $\text{MF}^\varphi_L$ i.e. $M \otimes 1 \simeq 1 \otimes M \simeq M$ for any $M$.

If $M$ is a one dimensional Dieudonné module and $d$ is a basis vector of $M$, then $\varphi(d) = \alpha v$ for some $\alpha \in K$. Set $t_N(M) = v_p(\alpha)$ and denote by $t_H(M)$ the unique filtration jump of $M$. If $M$ is of an arbitrary finite dimension $d$, set $t_N(M) = t_N(\wedge^d M)$ and $t_H(M) = t_H(\wedge^d M)$. A Dieudonné module $M$ is said to be weakly admissible if $t_H(M) = t_N(M)$ and if $t_H(M') \leq t_N(M')$ for any $\varphi$-submodule $M' \subseteq M$ equipped with the induced filtration. Weakly admissible modules form a subcategory of $\text{MF}_L$ which we denote by $\text{MF}^{s,f}_L$.

If $V$ is a $p$-adic representation of $G_L$, define $D_{\text{dR}}(V) = (B_{\text{dR}} \otimes V)^{G_L}$. Then $D_{\text{dR}}(V)$ is a $L$-vector space equipped with the decreasing filtration $\text{Fil}^i D_{\text{dR}}(V) = (\text{Fil}^i B_{\text{dR}} \otimes V)^{G_L}$. One has $\dim_L D_{\text{dR}}(V) \leq \dim_{\mathbb{Q}_p}(V)$ and $V$ is said to be de Rham if $\dim_L D_{\text{dR}}(V) = \dim_{\mathbb{Q}_p}(V)$. Analogously one defines $D_{\text{cris}}(V) = (B_{\text{cris}} \otimes V)^{G_L}$. Then $D_{\text{cris}}(V)$ is a filtered Dieudonné module over $L$ of dimension $\dim_K D_{\text{cris}}(V) \leq \dim_{\mathbb{Q}_p}(V)$ and $V$ is said to be crystalline if the equality holds here. In particular, for crystalline representations one has $D_{\text{dR}}(V) = D_{\text{cris}}(V) \otimes_K L$. By the theorem of Colmez-Fontaine [CF], the functor $D_{\text{cris}}$ establishes an equivalence between the category of crystalline representations of $G_L$ and $\text{MF}^{s,f}_L$. Its quasi-inverse $V_{\text{cris}}$ is given by $V_{\text{cris}}(D) = \text{Fil}^0(D \otimes_{K_0} B_{\text{cris}})^{\varphi=1}$.

An important result of Berger ([Ber 1], Theorem 0.2) says that $D_{\text{cris}}(V)$ can be recovered from the $(\varphi, 1)$-module $D_{\text{rig}}^+(V)$. The situation is particularly simple if if $L/\mathbb{Q}_p$ is unramified. In this case set $D^+(V) = (V \otimes_{\mathbb{Q}_p} B^+)^{H_K}$ and $D_{\text{rig}}^+(V) = \mathcal{R}^+(K) \otimes_{B_K^+} D^+(V)$. Then

$$D_{\text{cris}}(V) = \left(D_{\text{rig}}^+(V) \left[\frac{1}{t}\right]^r\right)$$

(see [Ber2], Proposition 3.4).

\section{The exponential map}

\subsection{The Bloch-Kato exponential map ([BK], [Ne1], [FP]).}

\subsection{Cohomology of Dieudonné modules.} Let $L$ be a finite extension of $\mathbb{Q}_p$. Recall that we denote by $\text{MF}^\varphi_L$ the category of filtered Dieudonné modules over $L$. If $M$ is an object of $\text{MF}^\varphi_L$, define

$$H^i(L, M) = \text{Ext}^i_{\text{MF}^\varphi_L}(1, M), \quad i = 0, 1.$$
is weakly admissible too and we can write $H^i(L, M) = \operatorname{Ext}^i_{M_{L^f}}(1, M)$.

2.1.2. The exponential map. Let $\operatorname{Rep}_{\text{cris}}(G_K)$ denote the category of crystalline representations of $G_K$. For any object $V$ of $\operatorname{Rep}_{\text{cris}}(G_K)$ define

$$H^i_f(K, V) = \operatorname{Ext}^i_{\operatorname{Rep}_{\text{cris}}(G_K)}(\mathbb{Q}_p(0), V).$$

An easy computation shows that

$$H^i_f(K, V) = \begin{cases} H^0(K, V), & \text{if } i = 0, \\ \ker (H^1(K, V) \rightarrow H^1(K, V \otimes B_{\text{cris}})), & \text{if } i = 1, \\ 0, & \text{if } i \geq 2. \end{cases}$$

Let $t_V(K) = D_{\text{dR}}(V)/\text{Fil}^0 D_{\text{dR}}(V)$ denote the tangent space of $V$. The rings $B_{\text{dR}}$ and $B_{\text{cris}}$ are related to each other via the fundamental exact sequence

$$0 \rightarrow \mathbb{Q}_p \rightarrow B_{\text{cris}} \xrightarrow{f} B_{\text{dR}}/\text{Fil}^0 B_{\text{dR}} \oplus B_{\text{cris}} \rightarrow 0$$

where $f(x) = (x \mod \text{Fil}^0 B_{\text{dR}}), (1 - \varphi) x$ (see [BK], §4). Tensoring this sequence with $V$ and taking cohomology one obtains an exact sequence

$$0 \rightarrow H^0(K, V) \rightarrow D_{\text{cris}}(V) \rightarrow t_V(K) \oplus D_{\text{cris}}(V) \rightarrow H^1_f(K, V) \rightarrow 0.$$

The last map of this sequence gives rise to the Bloch-Kato exponential map

$$\exp_{V,K} : t_V(K) \oplus D_{\text{cris}}(V) \rightarrow H^1(K, V).$$

Following [F3] set

$$\mathbf{R}\Gamma_f(K, V) = C^*(D_{\text{cris}}(V)) = \left[ D_{\text{cris}}(V) \xrightarrow{f} t_V(K) \oplus D_{\text{cris}}(V) \right].$$

From the classification of crystalline representations in terms of Dieudonné modules it follows that the functor $V_{\text{cris}}$ induces natural isomorphisms

$$r^{i}_{V,p} : \mathbf{R}\Gamma_f(K, V) \rightarrow H^i_f(K, V), \quad i = 0, 1.$$

The composite homomorphism

$$t_K(V) \oplus D_{\text{cris}}(V) \rightarrow \mathbf{R}\Gamma_f(K, V) \xrightarrow{r^{i}_{V,p}} H^i_f(K, V)$$

coincides with the Bloch-Kato exponential map $\exp_{V,K}$ ([Ne1], Proposition 1.21).

2.1.3. The map $\mathbf{R}\Gamma_f(K, V) \rightarrow \mathbf{R}\Gamma(K, V)$. Let $g : B^* \rightarrow C^*$ be a morphism of complexes. We denote by $\operatorname{Tot}^n(g)$ the complex $\operatorname{Tot}^n(g) = C^{n-1} \oplus B^n$ with differentials $d^n : \operatorname{Tot}^n(g) \rightarrow \operatorname{Tot}^{n+1}(g)$ defined by the formula $d^n(c, b) = (-1)^n g^n(b) + d^{n-1}(c), d^n(b))$. It is well known that if $0 \rightarrow A^* \xrightarrow{f} B^* \xrightarrow{g} C^* \rightarrow 0$ is an exact sequence of complexes, then $f$ induces a quasi
isomorphism $A^\bullet \sim \text{Tot}^\bullet(g)$. In particular, tensoring the fundamental exact sequence with $V$, we obtain an exact sequence of complexes

$$0 \rightarrow \text{R} \Gamma(K,V) \rightarrow C^\bullet_c(G_K,V \otimes B_{\text{cris}}) \xrightarrow{f} C^\bullet_c(G_K,(V \otimes (B_{\text{dR}}/\text{Fil}^0 B_{\text{dR}})) \oplus (V \otimes B_{\text{cris}})) \rightarrow 0$$

which gives a quasi isomorphism $\text{R} \Gamma(K,V) \sim \text{Tot}^\bullet(f)$. Since $\text{R} \Gamma_f(K,V)$ coincides tautologically with the complex

$$C^0_c(G_K,V \otimes B_{\text{cris}}) \xrightarrow{f} C^0_c(G_K,(V \otimes (B_{\text{dR}}/\text{Fil}^0 B_{\text{dR}})) \oplus (V \otimes B_{\text{cris}}))$$

we obtain a diagram

$$\xymatrix{ \text{R} \Gamma(K,V) & \text{Tot}^\bullet(f) \ar[ll]_{\sim} \ar[dl] \text{R} \Gamma_f(K,V) }$$

which defines a morphism $\text{R} \Gamma_f(K,V) \rightarrow \text{R} \Gamma(K,V)$ in $\mathcal{D}(\mathbb{Q}_p)$ (see [BF], Proposition 1.17). Remark that the induced homomorphisms $\text{R}^i \Gamma_f(K,V) \rightarrow H^i_f(K,V)$ ($i = 0,1$) coincide with the composition of $r^\dagger_{V,p}$ with natural embeddings $H^i_f(K,V) \rightarrow H^i(K,V)$.

### 2.1.4. Exponential map for $(\varphi, \Gamma)$-modules.

In this subsection we define an analog of the exponential map for crystalline $(\varphi, \Gamma)$-modules. See [Na] for a more general setting. Let $K/\mathbb{Q}_p$ be an unramified extension. If $D$ is a $(\varphi, \Gamma)$-module over $\mathcal{R}(K)$ define

$$\mathcal{D}_{\text{cris}}(D) = (D[1/t])^\Gamma.$$  

It can be shown that $\mathcal{D}_{\text{cris}}(D)$ is a finite dimensional $K$-vector space equipped with a natural decreasing filtration $\text{Fil}^d \mathcal{D}_{\text{cris}}(D)$ and a semilinear action of $\varphi$. One says that $D$ is crystalline if

$$\dim_K(\mathcal{D}_{\text{cris}}(D)) = \text{rg}(D).$$  

From [Ber4], Théorème A it follows that the functor $D \mapsto \mathcal{D}_{\text{cris}}(D)$ is an equivalence between the category of crystalline $(\varphi, \Gamma)$-modules and $\text{MF}_{K}^\varphi$. Remark that if $V$ is a $p$-adic representation of $G_K$ then $D_{\text{cris}}(V) = \mathcal{D}_{\text{cris}}(D_{\text{rig}}^\dagger(V))$ and $V$ is crystalline if and only if $D_{\text{rig}}(V)$ is.

Let $D$ be a $(\varphi, \Gamma)$-module. To any cocycle $\alpha = (a,b) \in Z^1(C_{\varphi, \Gamma}(D))$ one can associate the extension

$$0 \rightarrow D \rightarrow D_{\alpha} \rightarrow \mathcal{R}(K) \rightarrow 0$$

defined by

$$D_{\alpha} = D \oplus \mathcal{R}(K)e, \quad (\varphi - 1)e = a, \quad (\gamma - 1)e = b.$$  

As usual, this gives rise to an isomorphism $H^1(\mathcal{D}) \simeq \text{Ext}^1_{\mathcal{R}}(\mathcal{R}(K),D)$. We say that $\text{cl}(\alpha)$ is crystalline if $\dim_K(\mathcal{D}_{\text{cris}}(D_{\alpha})) = \dim_K(\mathcal{D}_{\text{cris}}(D)) + 1$ and define

$$H^1_f(\mathcal{D}) = \{\text{cl}(\alpha) \in H^1(\mathcal{D}) \mid \text{cl}(\alpha) \text{ is crystalline} \}$$

(see [Ben2], section 1.4.1). If $D$ is crystalline (or more generally potentially semistable) one has a natural isomorphism

$$H^1(\mathcal{D}_{\text{cris}}(D)) \rightarrow H^1_f(\mathcal{D}).$$
Set \( t_D = D_{\text{cris}}(D)/\text{Fil}^0D_{\text{cris}}(D) \) and denote by \( \exp_D : t_D \oplus D_{\text{cris}}(D) \to H^1(D) \) the composition of this isomorphism with the projection \( t_D \oplus D_{\text{cris}}(D) \to H^1(K, D_{\text{cris}}(D)) \) and the embedding \( H^1_f(D) \hookrightarrow H^1(D) \).

Assume that \( K = \mathbb{Q}_p \). To simplify notation we will write \( D_m \) for \( R([x]x^m) \) and \( e_m \) for its canonical basis. Then \( D_{\text{cris}}(D_m) \) is the one dimensional \( \mathbb{Q}_p \)-vector space generated by \( t^{-m}e_m \).

As in [Ben2], we normalise the basis \( (\text{cl}(\alpha_m), \text{cl}(\beta_m)) \) of \( H^1(D_m) \) putting \( \alpha_m^* = (1 - 1/p) \text{cl}(\alpha_m) \) and \( \beta_m^* = (1 - 1/p) \log(\chi(\gamma)) \text{cl}(\beta_m) \).

**Proposition 2.1.5.** i) \( H^1_f(D_m) \) is the one-dimensional \( \mathbb{Q}_p \)-vector space generated by \( \alpha_m^* \).

ii) The exponential map

\[
\exp_{D_m} : t_{D_m} \to H^1(D_m)
\]

sends \( t^{-m}w_m \) to \( -\alpha_m^* \).

**Proof.** This is a reformulation of [Ben2], Proposition 1.5.8 ii).

### 2.2. The large exponential map.

#### 2.2.1. Notation.

In this section \( p \) is an odd prime number, \( K \) is a finite unramified extension of \( \mathbb{Q}_p \) and \( \sigma \) the absolute Frobenius acting on \( K \). Recall that \( K_\infty = K(\zeta_p^*) \) and \( K_\infty = \bigcup_{n=1}^{\infty} K_n \).

We set \( \Gamma = \text{Gal}(K_\infty/K) \), \( \Gamma_n = \text{Gal}(K_\infty/K_n) \) and \( \Delta = \text{Gal}(K_1/K) \). Let \( \Lambda = \mathbb{Z}_p[\Gamma_1] \) and \( \Lambda(\Gamma) = \mathbb{Z}_p[\Delta] \otimes_{\mathbb{Z}_p} \Lambda \). We will consider the following operators acting on the ring \( K[[X]] \) of formal power series with coefficients in \( K \):

- The ring homomorphism \( \sigma : K[[X]] \to K[[X]] \) defined by \( \sigma \left( \sum_{i=0}^\infty a_iX^i \right) = \sum_{i=0}^\infty \sigma(a_i)X^i \);

- The ring homomorphism \( \varphi : K[[X]] \to K[[X]] \) defined by

\[
\varphi \left( \sum_{i=0}^\infty a_iX^i \right) = \sum_{i=0}^\infty \sigma(a_i)\varphi(X)^i, \quad \varphi(X) = (1 + X)^p - 1.
\]

- The differential operator \( \partial = (1 + X) \frac{d}{dX} \). One has \( \partial \circ \varphi = p\varphi \circ \partial \).

- The operator \( \psi : K[[X]] \to K[[X]] \) defined by \( \psi(f(X)) = \left( \frac{1}{p} \varphi^{-1} \right) \left( \sum_{\zeta^p=1} f((1 + X)\zeta - 1) \right) \).

It is easy to see that \( \psi \) is a left inverse to \( \varphi \), i.e. that \( \psi \circ \varphi = \text{id} \).

- An action of \( \Gamma \) given by \( \gamma \left( \sum_{i=0}^\infty a_iX^i \right) = \sum_{i=0}^\infty a_i\gamma(X)^i, \quad \gamma(X) = (1 + X)^{\chi(\gamma)} - 1 \).

Remark that these formulas are compatible with the definitions from sections 1.1.1 and 1.1.6.

Fix a generator \( \gamma_1 \in \Gamma_1 \) and define

\[
\mathcal{H} = \{ f(\gamma_1 - 1) \mid f \in \mathbb{Q}_p[[X]] \text{ is holomorphic on } B(0, 1) \}, \quad \mathcal{H}(\Gamma) = \mathbb{Z}_p[\Delta] \otimes_{\mathbb{Z}_p} \mathcal{H}.
\]

#### 2.2.2. The map \( \Xi_{V,n}^\epsilon \).

It is well known that \( \mathbb{Z}_p[[X]]^{\psi=0} \) is a free \( \Lambda \)-module generated by \( (1 + X) \) and the operator \( \partial \) is bijective on \( \mathbb{Z}_p[[X]]^{\psi=0} \). If \( V \) is a crystalline representation of \( G_K \) put \( D(V) = D_{\text{cris}}(V) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[X]]^{\psi=0} \).

Let \( \Xi_{V,n}^\epsilon : D(V)_{\Gamma_n}[-1] \to \mathbf{R}\Gamma_f(K_1, V) \) be the map defined by

\[
\Xi_{V,n}^\epsilon(\alpha) = \begin{cases} \left( p^{-n}(\sum_{k=1}^n (\sigma \otimes \varphi)^{-k} \alpha(\zeta_p^k - 1), -\alpha(0)) \right) & \text{if } n \geq 1, \\ \text{Tr}_{K_1/K} (\Xi_{V,1}^\epsilon(\alpha)) & \text{if } n = 0. \end{cases}
\]
An easy computation shows that \( \Xi_{V,0}^\epsilon : D_{\text{cris}}(V)[-1] \to R\Gamma_f(K, V) \) is given by the formula
\[
\Xi_{V,0}^\epsilon(a) = \frac{1}{p}(-\varphi^{-1}(a), -(p-1)a).
\]
In particular, it is homotopic to the map \( a \mapsto -(0, (1-p^{-1}\varphi^{-1})a) \). Write
\[
\Xi_{V,n}^\epsilon : D(V) \to R\Gamma(K_n, V) = \frac{t_V(K_n) \otimes D_{\text{cris}}(V)}{D_{\text{cris}}(V)/V^{G_K}}
\]
denote the homomorphism induced by \( \Xi_{V,n}^\epsilon \). Then
\[
\Xi_{V,0}^\epsilon(a) = -(0, (1-p^{-1}\varphi^{-1})a) \pmod{D_{\text{cris}}(V)/V^{G_K}}.
\]
If \( D_{\text{cris}}(V)^\varphi = 0 \) the operator \( 1-\varphi \) is invertible on \( D_{\text{cris}}(V) \) and we can write
\[
\Xi_{V,0}^\epsilon(a) = \left( \frac{1-p^{-1}\varphi^{-1}}{1-\varphi} a, 0 \right) \pmod{D_{\text{cris}}(V)/V^{G_K}}. \tag{2.1}
\]
For any \( i \in \mathbb{Z} \) let \( \Delta_i : D(V) \to \frac{D_{\text{cris}}(V)}{(1-p^i\varphi)D_{\text{cris}}(V)} \otimes \mathbb{Q}_p(i) \) be the map given by
\[
\Delta_i (\alpha(X)) = \partial^i \alpha(0) \otimes \varepsilon^{\otimes i} \pmod{(1-p^i\varphi)D_{\text{cris}}(V)}.
\]
Set \( \Delta = \bigoplus_{i \in \mathbb{Z}} \Delta_i \). If \( \alpha \in D(V)^{\Delta = 0} \), then by [PR1], Proposition 2.2.1 there exists \( F \in D_{\text{cris}}(V) \otimes \mathbb{Q}_p \mathbb{Q}_p[[X]] \) which converges on the open unit disk and such that \( (1-\varphi)F = \alpha \). A short computation shows that
\[
\Xi_{V,n}^\epsilon(\alpha) = p^{-n}((\sigma \otimes \varphi)^{-n}(F)(\zeta_p^n - 1), 0) \pmod{D_{\text{cris}}(V)/V^{G_K}}, \quad \text{if } n \geq 1
\]
(see [BB], Lemme 4.9).

2.2.3. Construction of the large exponential map. As \( \mathbb{Z}_p[[X]][1/p] \) is a principal ideal domain and \( \mathcal{H} \) is \( \mathbb{Z}_p[[X]][1/p] \)-torsion free, \( \mathcal{H} \) is flat. Thus
\[
C^+_{1w,\psi}(V) \otimes_{\Lambda_{\mathbb{Q}_p}} \mathcal{H}(\Gamma) = C^+_{1w,\psi}(V) \otimes_{\Lambda_{\mathbb{Q}_p}} \mathcal{H}(\Gamma) = \left[ \mathcal{H}(\Gamma) \otimes_{\Lambda_{\mathbb{Q}_p}} D^+_{\text{rig}}(V) \xrightarrow{\psi^{-1}} \mathcal{H}(\Gamma) \otimes_{\Lambda_{\mathbb{Q}_p}} D^+_{\text{rig}}(V) \right].
\]
By proposition 1.1.7 on has an isomorphism in \( D(\mathcal{H}(\Gamma)) \)
\[
R\Gamma_{1w}(K, V) \otimes_{\Lambda_{\mathbb{Q}_p}} \mathcal{H}(\Gamma) \cong C^+_{1w,\psi}(V) \otimes_{\Lambda_{\mathbb{Q}_p}} \mathcal{H}(\Gamma).
\]
The action of \( \mathcal{H}(\Gamma) \) on \( D^+_{\text{rig}}(V)^{\psi = 1} \) induces an injection \( \mathcal{H}(\Gamma) \otimes_{\Lambda_{\mathbb{Q}_p}} D^+_{\text{rig}}(V)^{\psi = 1} \hookrightarrow D^+_{\text{rig}}(V)^{\psi = 1} \). Composing this map with the canonical isomorphism \( H^1_{\text{lw}}(K, V) \cong D^+_{\text{rig}}(V)^{\psi = 1} \) we obtain a map \( \mathcal{H}(\Gamma) \otimes_{\Lambda_{\mathbb{Q}_p}} H^1_{\text{lw}}(K, V) \hookrightarrow D^+_{\text{rig}}(V)^{\psi = 1} \). For any \( k \in \mathbb{Z} \) set \( \nabla_k = t \partial - k \). An easy induction shows that \( \nabla_{k-1} \circ \nabla_{k-2} \circ \cdots \circ \nabla_0 = t^k \partial^k \).

Fix \( h \geq 1 \) such that \( \text{Fil}^{-h}D_{\text{cris}}(V) = D_{\text{cris}}(V) \) and \( V(-h)^{G_K} = 0 \). For any \( \alpha \in D(V)^{\Delta = 0} \) define
\[
\Omega_{V,h}^\epsilon(\alpha) = (-1)^{h-1} \frac{\log (\chi(\gamma_i))}{p} \nabla_{h-1} \circ \nabla_{h-2} \circ \cdots \circ \nabla_0(F(\pi)),
\]
where \( F \in \mathcal{H}(V) \) is such that \( (1-\varphi)F = \alpha \). It is easy to see that \( \Omega_{V,h}^\epsilon(\alpha) \in D^+_{\text{rig}}(V)^{\psi = 1} \). In [Ber3] Berger shows that \( \Omega_{V,h}^\epsilon(\alpha) \in \mathcal{H}(\Gamma) \otimes_{\Lambda_{\mathbb{Q}_p}} D^+_{\text{rig}}(V)^{\psi = 1} \) and therefore gives rise to a map
\[
\text{Exp}_{V,h} : D(V)^{\Delta = 0}[-1] \to R\Gamma_{1w}(K, V) \otimes_{\Lambda_{\mathbb{Q}_p}} \mathcal{H}(\Gamma)
\]
Let
\[
\text{Exp}_{V,h} : D(V)^{\Delta = 0} \to \mathcal{H}(\Gamma) \otimes_{\Lambda_{\mathbb{Q}_p}} H^1_{\text{lw}}(K, V)
\]
denote the map induced by \( \text{Exp}_{V,h} \) in degree 1. The following theorem is a reformulation of the construction of the large exponential map given by Berger in [Ber3].
Theorem 2.2.4. Let
\[ \text{Exp}_{V,h,n}^\varepsilon : D(V)_{I_n}^{\Delta=0}[-1] \to R\Gamma_{Iw}(K,V) \otimes_{\Lambda_{\text{gr}}} Q_p[G_n]. \]
denote the map induced by Exp_{V,h}. Then for any \( n \geq 0 \) the following diagram in \( D(Q_p[G_n]) \) is commutative:

\[
\begin{array}{ccc}
D(V)_{I_n}^{\Delta=0}[-1] & \xrightarrow{\text{Exp}_{V,h,n}^\varepsilon} & R\Gamma_{Iw}(K,V) \otimes_{\Lambda_{\text{gr}}} Q_p[G_n] \\
E_{V,n} & \downarrow & \downarrow \cong \\
R\Gamma_f(K_n,V) & \xrightarrow{(h-1)!} & R\Gamma(K_n,V).
\end{array}
\]

In particular, Exp_{V,h} coincides with the large exponential map of Perrin-Riou.

Proof. Passing to cohomology in the previous diagram one obtains the diagram

\[
\begin{array}{ccc}
D(V)^{\Delta=0} & \xrightarrow{\text{Exp}_{V,h}^\varepsilon} & H(\Gamma) \otimes_{\Lambda_{\text{gr}}} H^1_{Iw}(K,V) \\
E_{V,n} & \downarrow & \downarrow \text{pr}_{V,n} \\
D_{dR/K_n}(V) \oplus D_{\text{cris}}(V) & \xrightarrow{(h-1)! \text{exp}_{V,K_n}} & H^1(K_n,V)
\end{array}
\]

which is exactly the definition of the large exponential map. Its commutativity is proved in [Ber3], Theorem II.13. Now the theorem is an immediate consequence of the following remark.

Let \( D \) be a free \( A \)-module and let \( f_1, f_2 : D[-1] \to K^* \) be two maps from \( D[-1] \) to a complex of \( A \)-modules such that the induced maps \( h_1(f_1) \) and \( h_2(f_2) : D \to H^1(K^*) \) coincide. Then \( f_1 \) and \( f_2 \) are homotopic.

Remark. The large exponential map was first constructed in [PR1]. See [C1] and [Ben1] for alternative constructions and [PR4], [Na1] and [Ri] for generalizations.

§3. The \( \mathcal{L} \)-invariant

3.1. Definition of the \( \mathcal{L} \)-invariant.

3.1.1. Preliminaries. Let \( S \) be a finite set of primes of \( \mathbb{Q} \) containing \( p \) and \( G_S \) the Galois group of the maximal algebraic extension of \( \mathbb{Q} \) unramified outside \( S \cup \{\infty\} \). For each place \( v \) we denote by \( G_v \) the decomposition at \( v \) group and by \( I_v \) and \( \text{Frob}_v \) the inertia subgroup and Frobenius automorphism respectively. Let \( V \) be a pseudo-geometric \( p \)-adic representation of \( G_S \). This means that the restriction of \( V \) on the decomposition group at \( p \) is a de Rham representation. Following Greenberg, for any \( v \notin \{p, \infty\} \) set

\[ R\Gamma_f(Q_v,V) = \left[ V^{I_v} \xrightarrow{1-f_v^{I_v}} V^{I_v} \right], \]

where the terms are placed in degrees 0 and 1 (see [F3], [BF]). Note that there is a natural quasi-isomorphism \( R\Gamma_f(Q_v,V) \simeq C^*_c(G_v/I_v, V^{I_v}) \). Note that \( R^0\Gamma(Q_v,V) = H^0(Q_v,V) \) and \( R^1\Gamma_f(Q_v,V) = H^1_f(Q_v,V) \) where

\[ H^1_f(Q_v,V) = \ker(H^1(Q_v,V) \to H^1(Q_v^w, V)). \]
For \( v = p \) the complex \( R\Gamma_f(Q_v, V) \) was defined in \( \S 2 \). To simplify notation write \( H^i_S(V) = H^i(G_S, V) \) for the continuous Galois cohomology of \( G_S \) with coefficients in \( V \). The Bloch-Kato’s Selmer group of \( V \) is defined as

\[
H_1^f(V) = \ker \left( H_1^S(V) \to \bigoplus_{v \in S} H_1^f(Q_v, V) \right).
\]

We also set

\[
H_1^{f, (p)}(V) = \ker \left( H_1^S(V) \to \bigoplus_{v \in S - \{p\}} H_1^f(Q_v, V) \right).
\]

From the Poitou-Tate exact sequence one obtains the following exact sequence relating these groups (see for example [PR2], Lemme 3.3.6)

\[
0 \to H_1^f(V) \to H_1^{f, (p)}(V) \to H_1^f(Q_p, V) \to H_1^f(V^*(1)) \to 0.
\]

We also have the following formula relating dimensions of Selmer groups (see [FP], II, 2.2.2)

\[
\dim_{\mathbb{Q}_p} H_1^f(V) - \dim_{\mathbb{Q}_p} H_1^f(V^*(1)) - \dim_{\mathbb{Q}_p} H_0^S(V) + \dim_{\mathbb{Q}_p} H_0^S(V^*(1)) = \dim_{\mathbb{Q}_p} t_V(Q_p) - \dim_{\mathbb{Q}_p} H^0(\mathbb{R}, V).
\]

Set \( d_{\pm}(V) = \dim_{\mathbb{Q}_p}(V^{c=\pm 1}) \), where \( c \) denotes the complex conjugation.

### 3.1.2. Basic assumptions.

Assume that \( V \) satisfies the following conditions

**C1)** \( H_1^f(V^*(1)) = 0 \).

**C2)** \( H_0^S(V) = H_0^S(V^*(1)) = 0 \).

**C3)** \( V \) is crystalline at \( p \) and \( \varphi : D_{\text{cris}}(V) \to D_{\text{cris}}(V) \) is semisimple at 1 and \( p^{-1} \).

**C4)** \( D_{\text{cris}}(V)^{q=1} = 0 \).

**C5)** The localisation map

\[
\text{loc}_p : H_1^f(V) \to H_1^f(Q_p, V)
\]

is injective.

These conditions appear naturally in the following situation. Let \( X \) be a proper smooth variety over \( \mathbb{Q} \). Let \( H_p^i(X) \) denote the \( p \)-adic etale cohomology of \( X \). Consider the Galois representations \( V = H_p^i(X)(m) \). By Poincaré duality together with the hard Lefschetz theorem we have

\[
H_p^i(X)^* \simeq H_p^i(X)(i)
\]

and thus \( V^*(1) \simeq V(i + 1 - 2m) \). The Beilinson conjecture (in the formulation of Bloch and Kato) predict that

\[
H_1^f(V^*(1)) = 0 \quad \text{if} \quad w \leq -2.
\]

This corresponds to the hope that there are no nontrivial extensions of \( \mathbb{Q}(0) \) by motives of weight \( \geq 0 \). If \( X \) has a good reduction at \( p \), then \( V \) is crystalline [Fa] and the semisimplicity of \( \varphi \) is a well known (and difficult) conjecture. By a result of Katz and Messing [KM] \( D_{\text{cris}}(V)^{q=1} \neq 0 \).
can occur only if $i = 2m$. Therefore up to eventually replace $V$ by $V^*(1)$ the conditions C1, C3-4) conjecturally hold with except the weight $-1$ case $i = 2m - 1$.

The condition $D_{cris}(V)^{\varphi=1} = 0$ imples that the exponential map $t^V\langle \mathbb{Q}_p \rangle \rightarrow H^1_f(\mathbb{Q}_p, V)$ is an isomorphism and we denote by $\log_V$ its inverse. The composition of the localisation map $\text{loc}_p$ with the Bloch-Kato logarithm

$$r_V : H^1_f(\mathbb{Q}_p, V) \rightarrow t^V\langle \mathbb{Q}_p \rangle$$

coincides conjecturally with the $p$-adic (syntonic) regulator. We remark that if $H^0(\mathbb{Q}_v, V) = 0$ for all $v \neq p$ (and therefore $H^1_f(\mathbb{Q}_v, V) = 0$ for all $v \neq p$) then $\text{loc}_p$ is injective for all $m \neq i/2, i/2 + 1$ by a result of Jannsen ([Ja], Lemma 4 and Theorem 3).

If $H^0_S(V) \neq 0$, then $V$ contains a trivial subextension $V_0 = \mathbb{Q}_p(0)^k$. For $\mathbb{Q}_p(0)$ our theory describes the behavior of the Kubota-Leopoldt $p$-adic $L$-function and is well known. Therefore we can assume that $H^0_S(V) = 0$. Applying the same argument to $V^*(1)$ we can also assume that $H^0_S(V^*(1)) = 0$.

From our assumptions we obtain an exact sequence

$$0 \rightarrow H^1_f(\mathbb{Q}_p, V) \rightarrow H^1_f(\mathbb{Q}_p, V) \rightarrow H^1_f(\mathbb{Q}_p, V) \rightarrow 0. \quad (3.1)$$

Moreover

$$\dim_{\mathbb{Q}_p} H^1_f(\mathbb{Q}_p, V) = \dim_{\mathbb{Q}_p} t^V\langle \mathbb{Q}_p \rangle - d_+(V),$$
$$\dim_{\mathbb{Q}_p} H^1_f(\mathbb{Q}_p, V) = d_-(V) + \dim_{\mathbb{Q}_p} H^0(\mathbb{Q}_p, V^*(1)). \quad (3.2)$$

### 3.1.3. Regular submodules

In the remainder of this section we assume that $V$ satisfies C1-5).

**Definition** (Perrin-Riou). 1) A $\varphi$-submodule $D$ of $D_{cris}(V)$ is regular if $D \cap \text{Fil}^0 D_{cris}(V) = 0$ and the map

$$r_{V,D} : H^1_f(\mathbb{Q}_p, V) \rightarrow D_{cris}(V)/(\text{Fil}^0 D_{cris}(V) + D)$$

induced by $r_V$ is an isomorphism.

2) Dually, a $(\varphi, N)$-submodule $D$ of $D_{cris}(V^*(1))$ is regular if $D + \text{Fil}^0 D_{cris}(V^*(1)) = D_{cris}(V^*(1))$ and the map

$$D \cap \text{Fil}^0 D_{cris}(V^*(1)) \rightarrow H^1_f(V)^*$$

induced by the dual map $r^*_{V,D} : \text{Fil}^0 D_{cris}(V^*(1)) \rightarrow H^1_f(V)^*$ is an isomorphism.

It is easy to see that if $D$ is a regular submodule of $D_{cris}(V)$, then

$$D^\perp = \text{Hom}(D_{cris}(V)/D, D_{cris}(\mathbb{Q}_p(1)))$$

is a regular submodule of $D_{cris}(V^*(1))$. From (3.2) we also obtain that

$$\dim D = d_+(V), \quad \dim D^\perp = d_-(V) = d_+(V^*(1)).$$

Let $D \subset D_{cris}(V)$ be a regular subspace. As in [Ben2] we use the semisimplicity of $\varphi$ to decompose $D$ into the direct sum

$$D = D_{-1} \oplus D^\varphi = p^{-1}.\$$

which gives a four step filtration

$$\{0\} \subset D_{-1} \subset D \subset D_{cris}(V).$$
Let $\mathcal{D}$ and $\mathcal{D}_{-1}$ denote the $(\varphi, \Gamma)$-submodules associated to $D$ and $D_{-1}$ by Berger’s theory, thus

$$D = \mathcal{D}_{\text{cris}}(\mathcal{D}), \quad D_{-1} = \mathcal{D}_{\text{cris}}(\mathcal{D}_{-1}).$$

Set $W = \text{gr}_0 D_{\text{rig}}^+(V)$. Thus we have two tautological exact sequences

$$0 \to D \to D_{\text{rig}}^+(V) \to D' \to 0,$$

$$0 \to D_{-1} \to D \to W \to 0.$$

Note the following properties of cohomology of these modules:

a) The natural maps $H^1(D_{-1}) \to H^1(D)$ and $H^1(D) \to H^1(D_{\text{rig}}^+(V)) = H^1(Q_p, V)$ are injective. This follows from the observation that $\mathcal{D}_{\text{cris}}(\mathcal{D}')_{\varphi=1} = 0$ by C4. Since $H^0(D') = \text{Fil}^0 \mathcal{D}_{\text{cris}}(\mathcal{D}')_{\varphi=1}$ ([Ben2], Proposition 1.4.4) we have $H^0(D') = 0$. The same argument works for $W$.

b) $H^1_f(D_{-1}) = H^1(D_{-1})$. In particular the exponential map $\exp_{D_{-1}} : D_{-1} \to H^1(D_{-1})$ is an isomorphism. This follows from the computation of dimensions of $H^1(D_{-1})$ and $H^1_f(D_{-1})$.

Namely, since $D_{\varphi=1} = D_{\varphi=p^{-1}} = \{0\}$ the Euler-Poincaré characteristic formula [Li] together with Poincaré duality give

$$\dim_{Q_p} H^1(D_{-1}) = \dim_{Q_p} H^0(D_{-1}) - \dim_{Q_p} H^0(D^*_1(\chi)) = \dim_{Q_p} (D_{-1}).$$

On the other hand since

$$\text{Fil}^0 D_{-1} = D_{-1} \cap \text{Fil}^0 \mathcal{D}_{\text{cris}}(V) = \{0\}$$

one has $\dim_{Q_p} H^1_f(D_{-1}) = \dim_{Q_p} (D_{-1})$ by [Ben2], Corollary 1.4.5.

c) The exponential map $\exp_D : D \to H^1_f(D)$ is an isomorphism. This follows from $\text{Fil}^0 D = \{0\}$ and $D_{\varphi=1} = \{0\}$.

The regularity of $D$ is equivalent to the decomposition

$$H^1_f(Q_p, V) = H^1_f(V) \oplus H^1_f(D). \quad (3.3)$$

Since $\text{loc}_p$ is injective by C5, the localisation map $H^1_f(V) \to H^1(Q_p, V)$ is also injective. Let

$$\kappa_D : H^1_f(V) \to \frac{H^1(Q_p, V)}{H^1_f(D)}$$

denote the composition of this map with the canonical projection.

**Lemma 3.1.4.** i) One has

$$H^1_f(Q_p, V) \cap H^1(D) = H^1_f(D).$$

ii) $\kappa_D$ is an isomorphism.

**Proof.** i) Since $H^0(D') = 0$ we have a commutative diagram with exact rows and injective columns

$$
\begin{array}{cccccc}
0 & \to & H^1_f(D) & \to & H^1_f(Q_p, V) & \to & H^1_f(D') \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H^1(D) & \to & H^1(Q_p, V) & \to & H^1(D').
\end{array}
$$
This gives i).

ii) Since $H^1_f(D) \subset H^1_f(\mathbb{Q}_p, V)$ one has $\ker(\kappa_D) \subset H^1_f(\mathbb{Q}_p, V)$. One the other hand (3.3) shows that $\kappa_D$ is injective on $H^1_f(V)$. Thus $\ker(\kappa_D) = \{0\}$. On the other hand, because $\dim_{\mathbb{Q}_p} H^1_f(D) = \dim_{\mathbb{Q}_p}(D)$ we have

$$\dim_{\mathbb{Q}_p} \left( \frac{H^1(\mathbb{Q}_p, V)}{H^1_f(D)} \right) = d_-(V) + \dim_{\mathbb{Q}_p} H^0(\mathbb{Q}_p, V^*(1)).$$

Comparing this with (3.2) we obtain that $\kappa_D$ is an isomorphism.

3.1.5. The main construction. Set $e = \dim_{\mathbb{Q}_p}(D^{\varphi=p^{-1}})$. The $(\varphi, \Gamma)$-module $W$ satisfies

$$\text{Fil}^0 \mathcal{D}_{\text{cris}}(W) = 0, \quad \mathcal{D}_{\text{cris}}(W)^{\varphi=p^{-1}} = \mathcal{D}_{\text{cris}}(W).$$

(Recall that $\mathcal{D}_{\text{cris}}(W) = D^{\varphi=p^{-1}}$.) The cohomology of such modules was studied in detail in [Ben2], Proposition 1.5.9 and section 1.5.10. Namely, $H^0(W) = 0$, $\dim_{\mathbb{Q}_p} H^1(W) = 2e$ and $\dim_{\mathbb{Q}_p}(W) = e$. There exists a canonical decomposition

$$H^1(W) = H^1_f(W) \oplus H^1_c(W)$$

of $H^1(W)$ into the direct sum of $H^1_f(W)$ and some canonical space $H^1_c(W)$. Moreover there exist canonical isomorphisms

$$i_{D,f} : \mathcal{D}_{\text{cris}}(W) \simeq H^1_f(W), \quad i_{D,c} : \mathcal{D}_{\text{cris}}(W) \simeq H^1_c(W).$$

These isomorphisms can be described explicitly. By Proposition 1.5.9 of [Ben2]

$$W \simeq \bigoplus_{i=1}^e D_{m_i},$$

where $D_{m_i} = \mathcal{D}(\mathbb{Q}_p[x|x^{m_i}]), m_i \geq 1$. By Proposition 2.1.5 $H^1_f(D_{m_i})$ is generated by $\alpha_{m_i}^*$ and $H^1_c(D_{m_i})$ is the subspace generated by $\beta_{m_i}^*$ (see also Proposition 1.1.9). Then

$$i_{D_{m_i},f}(x) = x\alpha_{m_i}^*, \quad i_{D_{m_i},c}(x) = x\beta_{m_i}^*.$$

Since $H^0(W) = 0$ and $H^2(D_{-1}) = 0$ we have exact sequences

$$0 \to H^1(D_{-1}) \to H^1(D) \to H^1(W) \to 0,$$

$$0 \to H^1_f(D_{-1}) \to H^1_f(D) \to H^1_f(W) \to 0.$$

Since $H^1_f(D_{-1}) = H^1(D)$ we obtain that

$$\frac{H^1(D)}{H^1_f(D)} \simeq \frac{H^1(W)}{H^1_f(W)}.$$

Let $H^1(D, V)$ denote the inverse image of $H^1(D)/H^1_f(D)$ by $\kappa_D$. Then $\kappa_D$ induces an isomorphism

$$H^1(D, V) \simeq \frac{H^1(D)}{H^1_f(D)}.$$
By Lemma 3.1.4 the localisation map $H^1(D, V) \to H^1(W)$ is well defined and injective. Hence, we have a diagram

$$
\begin{array}{ccc}
\mathcal{D}_{\text{cris}}(W) & \xrightarrow{i_{D,f}} & H^1(W) \\
\downarrow \rho_{D,f} & & \downarrow \rho_{D,f} \\
H^1(D, V) & \xrightarrow{i_D} & H^1(W) \\
\downarrow \rho_{D,c} & & \downarrow \rho_{D,c} \\
\mathcal{D}_{\text{cris}}(W) & \xrightarrow{i_{D,c}} & H^1_c(W),
\end{array}
$$

where $\rho_{D,f}$ and $\rho_{D,c}$ are defined as the unique maps making this diagram commute. From Lemma 3.1.4 iii) it follows that $\rho_{D,c}$ is an isomorphism. The following definition generalise (in the crystalline case) the main construction of [Ben2] where we assumed in addition that $H^1_f(V) = 0$.

**Definition.** The determinant

$$\mathcal{L}(V, D) = \det \left( \rho_{D,f} \circ \rho_{D,c}^{-1} \mid \mathcal{D}_{\text{cris}}(W) \right)$$

will be called the $\mathcal{L}$-invariant associated to $V$ and $D$.

### 3.2. $\mathcal{L}$-invariant and the large exponential map.

#### 3.2.1. Derivation of the large exponential map.

In this section we interpret $\mathcal{L}(V, D)$ in terms of the derivative of the large exponential map. This interpretation is crucial for the proof of the main theorem of this paper. Recall that $H^1(Q_p, \mathcal{H}(\Gamma) \otimes Q_p V) = \mathcal{H}(\Gamma) \otimes \Lambda(\Gamma) H^1_{Iw}(Q_p, V)$ injects into $D^\dagger_{\text{rig}}(V)$. Set

$$
F_0 H^1(Q_p, \mathcal{H}(\Gamma) \otimes Q_p V) = D \cap H^1(Q_p, \mathcal{H}(\Gamma) \otimes Q_p V),
$$

$$
F_{-1} H^1(Q_p, \mathcal{H}(\Gamma) \otimes Q_p V) = D_{-1} \cap H^1(Q_p, \mathcal{H}(\Gamma) \otimes Q_p V).
$$

As in section 2.2 we fix a generator $\gamma \in \Gamma$. The following result is a straightforward generalisation of [Ben3], Proposition 2.2.2. For the convenience of the reader we give here the proof which is the same as in op. cit. modulo obvious modifications.

**Proposition 3.2.2.** Let $D$ be an admissible subspace of $D_{\text{cris}}(V)$. For any $a \in D^{p-1}$ let $\alpha \in D(V)$ be such that $\alpha(0) = a$. Then

i) There exists a unique $\beta \in F_0 H^1(Q_p, \mathcal{H}(\Gamma) \otimes V)$ such that

$$(\gamma - 1) \beta = \exp^\varepsilon_{V,h}(\alpha).$$

ii) The composition map

$$\delta_{D,h} : D^{p-1} \to F_0 H^1(Q_p, \mathcal{H}(\Gamma) \otimes V) \to H^1(W)$$

$$\delta_{D,h}(a) = \beta \pmod{H^1(D_{-1})}$$

is given explicitly by the following formula:

$$\delta_{D,h}(\alpha) = -(h - 1)! \left(1 - \frac{1}{p}\right)^{-1} (\log \chi(\gamma))^{-1} i_{D,c}(\alpha).$$
Proof. Since $D_{\text{cris}}(V)^{\varphi=1} = 0$, the operator $1 - \varphi$ is invertible on $D_{\text{cris}}(V)$ and we have a diagram

$$
\begin{array}{ccc}
D(V)^{\Delta=0} & \xrightarrow{\text{Exp}_{t,h}} & H^1(\mathbb{Q}_p, \mathcal{H}(\Gamma) \otimes V) \\
\downarrow{\Xi_{V,0}} & & \downarrow{\text{pr}_V} \\
D_{\text{cris}}(V) & \xrightarrow{(h-1)! \exp_{\nu}} & H^1(\mathbb{Q}_p, V).
\end{array}
$$

where $\Xi_{V,0}(\alpha) = \frac{1 - p^{-1} \varphi^{-1}}{1 - \varphi} \alpha(0)$ (see (2.1)). If $\alpha \in D^{\varphi=p-1} \otimes \mathbb{Z}_p[[X]]^{\psi=0}$, then $\Xi_{V,0}(\alpha) = 0$ and $\text{pr}_V \left( \text{Exp}_{t,h}(\alpha) \right) = 0$. On the other hand, as $V^{G_K} = 0$ the map $\left( \mathcal{H}(\Gamma) \otimes_{\Lambda_{\mathbb{Q}_p}} H^1_{\text{cris}}(\mathbb{Q}_p, V) \right)_1 \rightarrow H^1(\mathbb{Q}_p, V)$ is injective. Thus there exists a unique $\beta \in \mathcal{H}(\Gamma) \otimes_{\Lambda} H^1_{\text{cris}}(\mathbb{Q}_p, T)$ such that $\text{Exp}_{t,h}(\alpha) = (\gamma - 1) \beta$. Now take $a \in D^{\varphi=p-1}$ and set

$$
f = a \otimes \ell \left( \frac{(1 + X)^{\chi(\gamma)} - 1}{X} \right),
$$

where $\ell(g) = \frac{1}{p} \log \left( \frac{g^p}{\varphi(g)} \right)$. An easy computation shows that

$$
\sum_{\zeta \in \mathbb{Z}_p^1} \ell \left( \frac{(\chi(\gamma)(1 + X)^{\chi(\gamma)} - 1)}{\zeta(1 + X) - 1} \right) = 0.
$$

Thus $f \in D^{\varphi=p-1} \otimes \mathbb{Z}_p[[X]]^{\psi=0}$. Write $\alpha$ in the form $\alpha = (1 - \varphi)(1 - \gamma)(a \otimes \log(X))$. Then

$$
\Omega_{V,h}(\alpha) = (-1)^{h-1} \frac{\log \chi(\gamma_1)}{p} t^h \partial^h((\gamma - 1)(a \log(\pi))) = \frac{\log \chi(\gamma_1)}{p} (\gamma - 1) \beta
$$

where

$$
\beta = (-1)^{h-1} t^h \partial^h(a \log(\pi)) = (-1)^{h-1} t^h \partial^h \left( \frac{1 + \pi}{\pi} \right).
$$

This implies immediately that $\beta \in \mathcal{D}$. On the other hand $D^{\varphi=p-1} = D_{\text{cris}}(W) = (W[1/t])^\Gamma$ and we will write $\tilde{a}$ for the image of $a$ in $W[1/t]$. By [Ben2], sections 1.5.8-1.5.10 one has $W \simeq \bigoplus_{i=1}^s D_{m_i}$ where $D_{m_i} = \mathfrak{B}([x]x^m)$ and we denote by $e_m$ the canonical base of $D_{m_i}$. Then without loss of generality we may assume that $\tilde{a} = t^{-m_i}e_{m_i}$ for some $i$. Let $\tilde{\beta}$ be the image of $\beta$ in $W^{\psi=1}$ and let $h_0 : W^{\psi=1} \rightarrow H^1(W)$ be the canonical map furnished by Proposition 1.1.7. Recall that $h_0(\tilde{\beta}) = \text{cl}(e_m)\tilde{\beta}$ where $(1 - \gamma)c = (1 - \varphi)\tilde{\beta}$. Then $\tilde{\beta} = (-1)^{h-1} t^{h-m_i} \partial^h \log(\pi)$. By Lemma 1.5.1 of [CC1] there exists a unique $b_0 \in B_{\mathbb{Q}_p}^{1,\psi=0}$ such that $(\gamma - 1)b_0 = \ell(\pi)$. This implies that

$$(1 - \gamma) \left( t^{h-m_i} \partial^h b_0 e_{m_i} \right) = (1 - \varphi) \left( t^{h-m_i} \partial^h \log(\pi) e_{m_i} \right) = (-1)^{h-1} (1 - \varphi) \tilde{\beta}.$$

Thus $c = (-1)^{h-1} t^{h-m_i} \partial^h b_0 e_{m_i}$ and $\text{res}(ct^{m_i-1}dt) = (-1)^{h-1} \text{res}(t^{h-1} \partial^h b_0 dt) e_{m_i} = 0$. Next from the congruence $\tilde{\beta} \equiv (h-1)! t^{-m_i}e_{m_i} \pmod{\mathbb{Q}_p[[\pi]] e_{m_i}}$, it follows that $\text{res}(\tilde{\beta} t^{m_i-1}dt) = (h-1)! e_{m_i}$. Therefore by [Ben2], Corollary 1.5.6 we have

$$
\text{cl}(c,\tilde{\beta}) = (h-1)! \text{cl}(\beta_m) = (h-1)! \frac{p}{\log \chi(\gamma_1)} i_{W,c}(a).
$$
On the other hand
\[
\alpha(0) = a \otimes \ell \left( \frac{(1 + X)^{\chi(\gamma)} - 1}{X} \right) \big|_{X=0} = a \left( 1 - \frac{1}{p} \right) \log(\chi(\gamma)).
\]
These formulas imply that
\[
\delta_{D,h}(\alpha) = (h - 1)! \left( 1 - \frac{1}{p} \right)^{-1} (\log \chi(\gamma))^{-1} i_{\delta,h}(\alpha).
\]
and the proposition is proved.

3.2.3. Interpretation of the $L$-invariant. From the definition of $H^1(D, V)$ and Lemma 3.1.4 we immediately obtain that
\[
\frac{H^1(\mathbb{Q}_p, V)}{H_{f,\{p\}}^1(V) + H^1(D_{-1})} \cong \frac{H^1(D)}{H^1(D, V) + H^1(D_{-1})} \cong \frac{H^1(W)}{H^1(D, V)}.
\]
Thus, the map $\delta_{D,h}$ constructed in Proposition 3.2.2 induces a map
\[
D_{\varphi=p^{-1}}^{\varphi=p^{-1}} \rightarrow H^1_1(\mathbb{Q}_p, V)
\]
which we will denote again by $\delta_{D,h}$. On the other hand, we have isomorphisms
\[
D_{\varphi=p^{-1}}^{\varphi=p^{-1}} \cong \frac{H^1_1(\mathbb{Q}_p, V)}{H^1_1(D_{-1})} \cong \frac{H^1_1(\mathbb{Q}_p, V)}{H^1_1(D_{-1})} \cong \frac{H^1_1(\mathbb{Q}_p, V)}{H^1_1(D_{-1})}.
\]

**Proposition 3.2.4.** Let $\lambda_D : D_{\varphi=p^{-1}}^{\varphi=p^{-1}} \rightarrow D_{\varphi=p^{-1}}^{\varphi=p^{-1}}$ denote the homomorphism making the diagram

\[
\begin{array}{ccc}
D_{\varphi=p^{-1}}^{\varphi=p^{-1}} & \xrightarrow{\lambda_D} & D_{\varphi=p^{-1}}^{\varphi=p^{-1}} \\
\delta_{D,h} \downarrow & & \downarrow (h-1)! \exp_V \\
H^1_1(\mathbb{Q}_p, V) & \xrightarrow{\sim} & H_{f,\{p\}}^1(V) + H^1_1(D_{-1})
\end{array}
\]

commute. Then
\[
\det \left( \lambda_D \mid D_{\varphi=p^{-1}}^{\varphi=p^{-1}} \right) = (\log \chi(\gamma))^{-e} \left( 1 - \frac{1}{p} \right)^{-e} \mathcal{L}(V, D).
\]

**Proof.** The proposition follows from Proposition 3.2.2 and the following elementary fact. Let $U = U_1 \oplus U_2$ be the decomposition of a vector space $U$ of dimension $2e$ into the direct sum of two subspaces of dimension $e$. Let $X \subset U$ be a subspace of dimension $e$ such that $X \cap U_1 = \{0\}$. Consider the diagrams

\[
\begin{array}{ccc}
X & \xrightarrow{p_1} & U_1 \\
\downarrow p_2 & & \downarrow f \\
U_2 & \xleftarrow{i_1} & U_1
\end{array}
\quad
\begin{array}{ccc}
U/X & \xleftarrow{i_2} & U_1 \\
\downarrow v_2 & & \downarrow g \\
U_2 & \xrightarrow{g} & U_2
\end{array}
\]

where $p_k$ and $i_k$ are induced by natural projections and inclusions. Then $f = -g$. Applying this remark to $U = H^1(W), X = H^1(D, V), U_1 = H^1_{f_1}(W), U_2 = H^1_{c_2}(W)$ and taking determinants we obtain the proposition.
§4. Special values of \( p \)-adic \( L \)-functions

4.1. The Bloch-Kato conjecture.

4.1.1. The Euler-Poincaré line (see [F3], [FP],[BF]). Let \( V \) be a \( p \)-adic pseudo-geometric representation of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \). Thus \( V \) is a finite-dimensional \( \mathbb{Q}_p \)-vector space equipped with a continuous action of the Galois group \( G_S \) for a suitable finite set of places \( S \) containing \( p \). Write \( R\Gamma_S(V) = C^\bullet_c(G_S, V) \) and define

\[
R\Gamma_{S,c}(V) = \text{cone} \left( R\Gamma_S(V) \to \bigoplus_{v \in S \cup \{\infty\}} R\Gamma(\mathbb{Q}_v, V) \right) [-1].
\]

Fix a \( \mathbb{Z}_p \)-lattice \( T \) of \( V \) stable under the action of \( G_S \) and set \( \Delta_S(V) = \det_{\mathbb{Q}_p}^{-1} R\Gamma_{S,c}(V) \) and \( \Delta_S(T) = \det_{\mathbb{Z}_p}^{-1} R\Gamma_{S,c}(T) \). Then \( \Delta_S(T) \) is a \( \mathbb{Z}_p \)-lattice of the one-dimensional \( \mathbb{Q}_p \)-vector space \( \Delta_S(V) \) which does not depend on the choice of \( T \). Therefore it defines a \( p \)-adic norm on \( \Delta_S(V) \) which we denote by \( \| \cdot \|_S \). Moreover, \( \langle \Delta_S(V), \| \cdot \|_S \rangle \) does not depend on the choice of \( S \). More precisely, if \( \Sigma \) is a finite set of places which contains \( S \), then there exists a natural isomorphism \( \Delta_S(V) \to \Delta_{\Sigma}(V) \) such that \( \| \cdot \|_S = \| \cdot \|_\Sigma \). This allows to define the Euler-Poincaré line \( \Delta_{EP}(V) \) as \( (\Delta_S(V), \| \cdot \|_S) \) where \( S \) is sufficiently large. Recall that for any finite place \( v \in S \) we defined

\[
R\Gamma_f(\mathbb{Q}_v, V) = \begin{cases} [V_i \xrightarrow{1-f} V_i] & \text{if } v \neq p \\ D_{\text{cris}}(V) \xrightarrow{(\text{pr}, 1-v)} t_V(\mathbb{Q}_p) \oplus D_{\text{cris}}(V) & \text{if } v = p. \end{cases}
\]

At \( v = \infty \) we set \( R\Gamma_f(\mathbb{R}, V) = [V^+ \to 0] \), where the first term is placed in degree 0. Thus \( R\Gamma_f(\mathbb{R}, V) \xrightarrow{\sim} R\Gamma(\mathbb{R}, V) \). For any \( v \) we have a canonical morphism \( \text{loc}_p : R\Gamma_f(\mathbb{Q}_v, V) \to R\Gamma(\mathbb{Q}_v, V) \) which can be viewed as a local condition in the sense of [Ne2]. Consider the diagram

\[
\begin{array}{ccc}
R\Gamma_S(V) & \xrightarrow{\oplus} & \bigoplus_{v \in S \cup \{\infty\}} R\Gamma(\mathbb{Q}_v, V) \\
\downarrow & & \downarrow \\
\bigoplus_{v \in S \cup \{\infty\}} R\Gamma_f(\mathbb{Q}_v, V)
\end{array}
\]

and define

\[
R\Gamma_f(V) = \text{cone} \left( R\Gamma_S(V) \oplus \bigoplus_{v \in S \cup \{\infty\}} R\Gamma_f(\mathbb{Q}_v, V) \right) \to \bigoplus_{v \in S \cup \{\infty\}} R\Gamma(\mathbb{Q}_v, V) [-1].
\]

Thus, we have a distinguished triangle

\[
R\Gamma_f(V) \to R\Gamma_S(V) \oplus \left( \bigoplus_{v \in S \cup \{\infty\}} R\Gamma_f(\mathbb{Q}_v, V) \right) \to \bigoplus_{v \in S \cup \{\infty\}} R\Gamma(\mathbb{Q}_v, V). \tag{4.1}
\]

Set

\[
\Delta_f(V) = \det_{\mathbb{Q}_p}^{-1} R\Gamma_f(V) \otimes \det_{\mathbb{Q}_p}^{-1} t_V(\mathbb{Q}_p) \otimes \det_{\mathbb{Q}_p} V^+.
\]

It is easy to see that \( R\Gamma_f(V) \) and \( \Delta_f(V) \) do not depend on the choice of \( S \). Consider the distinguished triangle

\[
R\Gamma_{S,c}(V) \to R\Gamma_f(V) \to \bigoplus_{v \in S \cup \{\infty\}} R\Gamma_f(\mathbb{Q}_v, V).
\]
Since $\det_{Q_p} R \Gamma_f(Q_p, V) \simeq \det_{Q_p}^{-1} t_V(Q_p)$ and $\det_{Q_p} R \Gamma_f(R, V) = \det_{Q_p} V^+$ tautologically, we obtain canonical isomorphisms

$$\Delta_f(V) \simeq \det_{Q_p}^{-1} R \Gamma_{S,c}(V) \simeq \Delta_{EP}(V).$$

The cohomology of $R \Gamma_f(V)$ is as follows:

$$R^0 \Gamma_f(V) = H^0_S(V), \quad R^1 \Gamma_f(V) = H^1_f(V), \quad R^2 \Gamma_f(V) \simeq H^1_f(V^*(1))^*, \quad R^3 \Gamma_f(V) = \text{coker} \left( H^2_S(V) \to \bigoplus_{v \in S} H^2(Q_v, V) \right) \simeq H^0_S(V^*(1))^*.$$  

These groups seat in the following exact sequence:

$$0 \to R^1 \Gamma(V) \to H^1_S(V) \to \bigoplus_{v \in S} H^1_f(Q_v, V) \to R^2 \Gamma_f(V) \to \text{coker} \left( H^2_S(V) \to \bigoplus_{v \in S} H^2(Q_v, V) \right) \to R^3 \Gamma_f(V) \to 0.$$  

The $L$-function of $V$ is defined as the Euler product

$$L(V, s) = \prod_v E_v(V, (Nv)^{-s})^{-1}$$

where

$$E_v(V, t) = \begin{cases} 
\det (1 - f_v t | V^f_v), & \text{if } v \neq p \\
\det (1 - \varphi t | D_{\text{cris}}(V)), & \text{if } v = p.
\end{cases}$$

4.1.2. Canonical trivialisations. In this paper we treat motives in the formal sense and assume all conjectures about the category of mixed motives $\mathcal{M}$ over $\mathbb{Q}$ which are necessary to state the Bloch-Kato conjecture (see [F3], [FP]). Let $M$ be a pure motive over $\mathbb{Q}$ and let $M_B$ and $M_{dR}$ denote its Betti and de Rham realisations respectively. Fix an odd prime $p$ and denote by $V = M_p$ the $p$-adic realisation of $M$. Then one has comparison isomorphisms

$$M_B \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} M_{dR} \otimes_{\mathbb{Q}} \mathbb{C},$$

$$M_B \otimes_{\mathbb{Q}} \mathbb{Q}_p \xrightarrow{\sim} V.$$  

which induce trivialisations

$$\Omega_{M}^{(H, \infty)} : \det_{Q} M_B \otimes \det_{Q}^{-1} M_{dR} \to \mathbb{C},$$

$$\Omega_{M}^{(\ell, p)} : \det_{Q_p} V \otimes \det_{Q}^{-1} M_B \to \mathbb{Q}_p.$$  

The complex conjugation acts on $M_B$ and $V$ and decomposes the last isomorphism into $\pm$ parts which we denote again by $\Omega_{M}^{(\ell, p)}$ to simplify notation

$$\Omega_{M}^{(\ell, p)} : \det_{Q_p} V^\pm \otimes \det_{Q}^{-1} M_B^\pm \to \mathbb{Q}_p.$$  

The restriction of $V$ on the decomposition group at $p$ is a de Rham representation and $D_{dR}(V) \simeq M_{dR} \otimes_{\mathbb{Q}} \mathbb{Q}_p$. The comparison isomorphism

$$V \otimes B_{dR} \xrightarrow{\sim} D_{dR}(V) \otimes B_{dR}.$$
induces a map
\[ \tilde{\Omega}_M^{(H,p)} : \det_{Q_p} V \otimes \det_{Q_p}^{-1} D_{dR}(V) \to B_{dR}. \]

It is not difficult to see that there exists a finite extension \( L \) of \( \hat{\mathbb{Q}}_p \) such that \( \text{Im}(\tilde{\Omega}_M^{(H,p)}) \subset L_{\text{tor}}(V) \) and we define
\[ \Omega_M^{(H,p)} : \det_{Q_p} V \otimes \det_{Q_p}^{-1} D_{dR}(V) \to L \]
by \( \Omega_M^{(H,p)}(\tilde{\Omega}_M^{(H,p)}) \). We remark that if \( V \) is crystalline at \( p \) then one can take \( L = \hat{\mathbb{Q}}_p \) (see [PR2], Appendice C.2).

Assume that the groups \( H^i(M) = \text{Ext}^i_{\mathcal{M},\mathcal{M}}(\mathbb{Q}(0), M) \) are well defined and vanish for \( i \neq 0,1 \). It should be possible to define a \( Q \)-subspace \( \tilde{H}_f^1(M) \) of \( H^1(M) \) consisting of "integral" classes of extensions which is expected to be finite dimensional. It is convenient to set \( H_0^1(M) = H^0(M) \).

In this paper \( M \) will always denote a motive satisfying the following conditions

**M1)** \( M \) is pure of weight \( w \leq -2 \).

**M2)** The \( p \)-adic realisation \( V \) of \( M \) is crystalline at \( p \).

**M3)** \( M \) has no subquotients isomorphic to \( \mathbb{Q}(1) \).

These conditions imply that \( H^0(M) = H^0(M^+(1)) = 0 \) and \( H^1(M^+(1)) = 0 \) by the weight argument. and by (4.10) the representation \( V \) should satisfy the conditions \( \textbf{C1,2,4}) \) of section 3.1.2. In particular, from (4.2) it follows that
\[ \det_{Q_p} R\Gamma_f(V) \sim \det_{Q_p}^{-1} H_f^1(V). \]  
(4.11)

The semisimplicity of \( \varphi \) is a well known conjecture which is actually known for abelian varieties. Finally \( \textbf{C5}) \) should follow from the injectivity of the syntomic regulator.

The comparision isomorphism (4.3) induces an injective map
\[ \alpha_M : M^+_{\mathbb{B}} \otimes_{\mathbb{Q}} \mathbb{R} \to t_M(\mathbb{R}) \]
and the six-term exact sequence of Fontaine and Perrin-Riou ([F3], section 6.10) degenerates into an isomorphism (the regulator map)
\[ r_{M,\infty} : H_f^1(M) \otimes_{\mathbb{Q}} \mathbb{R} \sim \text{coker}(\alpha_M). \]

The maps \( \alpha_M \) and \( r_{M,\infty} \) define a map
\[ R_{M,\infty} : \det_{\mathbb{Q}}^{-1} t_M(\mathbb{Q}) \otimes \det_{\mathbb{Q}} M^+_{\mathbb{B}} \otimes \det_{\mathbb{Q}} H_f^1(M) \to \mathbb{R} \]

Fix bases \( \omega_f \in \det_{\mathbb{Q}} H_f^1(M), \omega_{t_M} \in \det_{\mathbb{Q}} t_M(\mathbb{Q}) \) and \( \omega_{M_{\mathbb{B}}}^+ \in \det_{\mathbb{Q}} M^+_{\mathbb{B}} \). Set \( \omega_M = (\omega_f, \omega_{t_M}, \omega_{M_{\mathbb{B}}}^+) \) and define
\[ R_{M,\infty}(\omega_M) = R_{M,\infty}(\omega_{t_M}^{-1} \otimes \omega_{M_{\mathbb{B}}}^+ \otimes \omega_f). \]
Using (4.11) and the isomorphisms (4.10) define

$$i_{\omega, p} : \Delta_{EP}(V) \cong \det_{\mathbb{Q}_p}t_V(\mathbb{Q}_p) \otimes \det_{\mathbb{Q}_p}V^+ \otimes \det_{\mathbb{Q}_p}H^1_f(V) \to \mathbb{Q}_p$$

by $x = i_{\omega, p}(x) (\omega_{t_M}^{-1} \otimes \omega_M^+ \otimes \omega_f)$.

Consider now the case of the dual motive $M^*(1)$. Again one has $\Delta_{EP}(V^*(1)) \simeq \Delta_f(V^*(1))$ where

$$\Delta_f(V^*(1)) \cong \det_{\mathbb{Q}_p}t_{V^*(1)}(\mathbb{Q}_p) \otimes \det_{\mathbb{Q}_p}V^*(1) \otimes \det_{\mathbb{Q}_p}H^1_f(V).$$

The map $\alpha_{M^*(1)} : M^*(1) \rightarrow t_{M^*(1)}(\mathbb{R})$ is surjective and it is related to $\alpha_M$ by the canonical duality $\ker(\alpha_M) \times \ker(\alpha_{M^*(1)}) \to \mathbb{R}$ (see [F3], section 5.4). The six-term exact sequence degenerates into an isomorphism

$$r_{M^*(1), \infty} : H^1_f(M)^* \otimes_{\mathbb{Q}} \mathbb{R} \simeq \ker(\alpha_{M^*(1)}).$$

This allows to define a map

$$R_{M^*(1), \infty} : \det_{\mathbb{Q}_p}t_{M^*(1)}(\mathbb{Q}) \otimes \det_{\mathbb{Q}_p}M^*(1)^* \otimes \det_{\mathbb{Q}_p}H^1_f(M) \to \mathbb{R}.$$

Fix bases $\omega_{t_{M^*(1)}} \in \det_{\mathbb{Q}_p}t_{M^*(1)}(\mathbb{Q})$ and $\omega_{M^*(1)}^+ \in \det_{\mathbb{Q}_p}M^*(1)^*$. Set $\omega_{M^*(1)} = (\omega_{t_{M^*(1)}}, \omega_{M^*(1)}^+, \omega_f)$ and $R_{M^*(1), \infty}(\omega_{M^*(1)}) = R_{M^*(1), \infty}(\omega_{t_{M^*(1)}}^{-1} \otimes \omega_{M^*(1)}^+ \otimes \omega_f)$. Then again this data defines a trivialisation

$$i_{\omega_{M^*(1)}, p} : \Delta_{EP}(V^*(1))) \to \mathbb{Q}_p.$$ (4.13)

It is conjectured that the $L$-functions $L(V, s)$ and $L(V^*(1), s)$ are well defined complex functions have meromorphic continuation to the whole $\mathbb{C}$ and satisfy some explicit functional equation ([FP] chapitre III). One expects that they do not depend on the choice of the prime $p$ and we will denote them by $L(M, s)$ and $L(M^*(1), s)$ respectively. The conjectures about special values of these functions state as follows.

**Conjecture (Beilinson-Deligne).** The $L$-function $L(M, s)$ does not vanish at $s = 0$ and

$$\frac{L(V, 0)}{R_{M, \infty}(\omega_M)} \in \mathbb{Q}^*.$$ 

The $L$-function $L(M^*(1), s)$ has a zero of order $r = \dim_{\mathbb{Q}_p} H^1_f(M)$ at $s = 0$. Let $L(M^*(1), 0) = \lim_{s \to 0} s^{-r} L(M^*(1), s)$. Then

$$\frac{L(M^*(1), 0)}{R_{M^*(1), \infty}(\omega_{M^*(1)})} \in \mathbb{Q}^*.$$ 

**Conjecture (Bloch-Kato).** Let $T$ be a $\mathbb{Z}_p$-lattice of $V$ stable under the action of $G_S$. Then

$$i_{\omega, p}(\Delta_{EP}(T)) = \frac{L(M, 0)}{R_{M, \infty}(\omega_M)} \mathbb{Z}_p,$$

$$i_{\omega_{M^*(1)}, p}(\Delta_{EP}(T^*(1))) = \frac{L(M^*(1), 0)}{R_{M^*(1), \infty}(\omega_{M^*(1)})} \mathbb{Z}_p.$$
4.1.3. Compatibility with functional equation. The compatibility of the Bloch-Kato conjecture with the functional equation follows from the conjecture \( C_{EP}(V) \) of Fontaine and Perrin-Riou about local Tamagawa numbers ([FP], chapitre III, section 4.5.4). More precisely, define

\[
\Gamma^*(V) = \prod_{i \in \mathbb{Z}} \Gamma^*(-i)^{h_i(V)}
\]

where \( h_i(V) = \dim_{\mathbb{Q}_p} (\text{gr}_i(D_{dR}(V))) \) and

\[
\Gamma(i) = \begin{cases} 
(i-1)! & \text{if } i > 0 \\
(-1)^i & \text{if } i \leq 0.
\end{cases}
\]

The exact sequence

\[
0 \rightarrow t_{V^*}(1)(\mathbb{Q}_p)^* \rightarrow D_{dR}(V) \rightarrow t_V(\mathbb{Q}_p) \rightarrow 0
\]

allows to consider \( \omega_{M_{dR}} = \omega_{t_M} \otimes \omega_{M_{dR}}^{-1} \in \det_{\mathbb{Q}_p}D_{dR}(V) \). Choose bases \( \omega_T^+ \in \det_{\mathbb{Z}_p}T^+ \) and \( \omega_T^- \in \det_{\mathbb{Z}_p}T^- \) and set \( \omega_T = \omega_T^+ \otimes \omega_T^- \in \det_{\mathbb{Z}_p}T \) and \( \omega_{T^*}(1) = (\omega_T^-)^* \in \det_{\mathbb{Z}_p}T^*(1)^+ \). Then the conjecture \( C_{EP}(V) \) implies that

\[
\frac{i_{\omega_{M_{dR}}(1,1)}(\Delta_{EP}(T^*(1)))}{\Omega_{M^*}(\omega_T^+,\omega_{M_{dR}})(\omega_{T^*}(1),\omega_{M_{dR}})} = \frac{\Gamma^*(V) \cdot \Omega_{M,\text{cris}}(H^1,\omega_{M_{dR}})}{\Omega_{M,\text{cris}}(H^1,\omega_{T^*}(1),\omega_{M_{dR}})}
\]

(see [PR2], Appendice C). We remark that for crystalline representations \( C_{EP}(V) \) is proved in [BB08].

4.2. \( p \)-adic \( L \)-functions.

4.2.1. \( p \)-adic Beilinson’s conjecture. We keep previous notation and conventions. Let \( M \) be a motive which satisfies the conditions (M1-3) of section 4.1.2 and let \( V \) denote the \( p \)-adic realisation of \( M \). We fix bases \( \omega_{M_B}^+ \in \det_{\mathbb{Q}_p}M_B^+ \), \( \omega_{t_M} \in \det_{\mathbb{Q}_p}t_M(\mathbb{Q}) \) and \( \omega_f \in \det_{\mathbb{Q}_p}H^1_f(M) \). We also fix a lattice \( T \) in \( V \) stable under the action of \( G_S \) and a base \( \omega_T^+ \in \det_{\mathbb{Z}_p}T^+ \). To simplify notation we will assume that the choices of \( \omega_{M_B}^+ \) and \( \omega_T^+ \) are compatible, namely that

\[
\Omega_{M,\text{cris}}(\omega_T^+,\omega_{M_B}^+) = 1.
\]

Let \( D \) be a regular subspace of \( D_{\text{cris}}(V) \). We fix a \( \mathbb{Z}_p \)-lattice \( N \) of \( D \) and a basis \( \omega_N \in \det_{\mathbb{Z}_p}N \). By the analogy with the archimedean case we can consider the \( p \)-adic regulator as a map \( r_{V,D} : H^1_f(V) \rightarrow \text{coker}(\alpha_{V,D}) \) where

\[
\alpha_{V,D} : D \rightarrow t_V(\mathbb{Q}_p)
\]

is the natural projection. Set \( \omega_{V,N} = (\omega_{t_M},\omega_N,\omega_f) \) and denote by \( R_{V,D}(\omega_{V,N}) \) the determinant of \( r_{V,D} \) computed in the bases \( \omega_f \) and \( \omega_{t_M} \otimes \omega_N^{-1} \). Namely, \( R_{V,D}(\omega_{V,N}) \) is the image of \( \omega_{t_M}^{-1} \otimes \omega_N \otimes \omega_f \) under the induced isomorphism

\[
R_{V,D} : \det_{\mathbb{Q}_p}t_V(\mathbb{Q}_p) \otimes \det_{\mathbb{Q}_p}D \otimes \det_{\mathbb{Q}_p}H^1_f(V) \rightarrow \mathbb{Q}_p.
\]

Now, consider the projection

\[
\alpha_{V^*}(1,D^+) : D^+ \rightarrow t_{V^*}(1)(\mathbb{Q}_p).
\]
A standard argument from the linear algebra shows that $\alpha_{V^{\ast}(1),D^\perp}$ is surjective and is related to $\alpha_{V,D}$ by the canonical duality $\text{coker}(\alpha_{V,D}) \times \ker(\alpha_{V^{\ast}(1),D^\perp}) \to \mathbb{Q}_p$. This defines isomorphisms

$$\det_{\mathbb{Q}_p}^{-1} t_{V^{\ast}(1)}(\mathbb{Q}_p) \otimes \det_{\mathbb{Q}_p} D^\perp \simeq \det_{\mathbb{Q}_p} (\ker(\alpha_{V^{\ast}(1),D^\perp}) \cong \det_{\mathbb{Q}_p}^{-1}(\text{coker}(\alpha_{V,D}))$$

and composing this map with the determinant of $r_{V,D}$ we have again a trivialisation

$$R_{V^{\ast}(1),D^\perp} : \det_{\mathbb{Q}_p}^{-1} t_{V^{\ast}(1)}(\mathbb{Q}_p) \otimes \det_{\mathbb{Q}_p} D^\perp \otimes \det_{\mathbb{Q}_p} H^1_f(V) \to \mathbb{Q}_p.$$  

Choose a lattice $N^\perp \subset D^\perp$, fix bases $\omega_{t_{M^\ast}(1)}$ and $\omega_{N^\perp}$ of $\det_{\mathbb{Q}_p} t_{V^{\ast}(1)}(\mathbb{Q}_p)$ and $\det_{\mathbb{Q}_p} N^\perp$ respectively and set $\omega_{V,N^\perp} = (\omega_{t_{M^\ast}(1)}, \omega_{N^\perp}, \omega_f)$.

Perrin-Riou conjectured [PR2] that there exists an analytic $p$-adic $L$-function $L_p(T, \omega_N, s)$ which interpolates special values of the complex $L$-function $L(M,s)$. In particular one expects that if $p^{-1}$ is not an eigenvalue of $\varphi$ acting on $D$ then $L_p(T, \omega_N, s)$ does not vanish at $s = 0$ and

$$\frac{L_p(T, N, 0)}{R_{V,D}(\omega_{V,N})} = \mathcal{E}(V,D) \frac{L(M,0)}{R_{M,\infty}(\omega_M)}.$$

where

$$\mathcal{E}(V,D) = \det(1 - p^{-1} \varphi^{-1} | D) \det(1 - p^{-1} \varphi^{-1} | D^\perp) = \det(1 - p^{-1} \varphi^{-1} | D) \det(1 - \varphi | D_{\text{cris}}(V)/D).$$

Dually it is conjectured that there exists a $p$-adic $L$-function $L_p(T^\ast(1), \omega_{N^\perp}, s)$ which interpolates special values of $L(M^\ast(1), s)$. One expects that if $1$ is not an eigenvalue of $\varphi$ acting on the quotient $D_{\text{cris}}(V^\ast(1))/D^\perp$ then $L_p(T^\ast(1), \omega_{N^\perp}, s)$ has a zero of order $r = \dim_{\mathbb{Q}} H^1_f(M)$ at $s = 0$ and

$$\frac{L_p^*(T^\ast(1), N^\perp, 0)}{R_{V^{\ast}(1),D^\perp}(\omega_{V^{\ast}(1),N^\perp})} = \mathcal{E}^*(V^{\ast}(1), D^\perp) \frac{L^*(M^\ast(1),0)}{R_{M^\ast(1),\infty}(\omega_M)}.$$  

These properties of $p$-adic $L$-functions can be viewed as $p$-adic analogues of Beilinson’s conjectures and we refer the reader to [PR2], chapitre 4 and [C2], section 2.8 for more detail. Note that from the definition it is clear that $\mathcal{E}(V,D) = \mathcal{E}(V^\ast(1), D^\perp)$. One can also write $\mathcal{E}(V,D)$ in the form

$$\mathcal{E}(V,D) = E_p(V,1) \det \left( \frac{1 - p^{-1} \varphi^{-1}}{1 - \varphi} | D \right).$$

### 4.2.2. Trivial zero conjecture.

Assume now that $D^{\varphi=p^{-1}} \neq 0$. Since $M$ is crystalline at $p$, this can occur only if $M$ is of weight $-2$. Set

$$e = \dim_{\mathbb{Q}_p} D^{\varphi=p^{-1}} = \dim_{\mathbb{Q}_p} (D^\perp + D_{\text{cris}}(V^\ast(1))^{\varphi=1}/D^\perp).$$

Assume that the $p$-adic realisation $V$ of $M$ satisfies the conditions (C1-5) of section 3.1.2. Decompose $D$ into the direct sum $D = D_{-1} \oplus D^{\varphi=p^{-1}}$ and define

$$\mathcal{E}^+(V,D) = \mathcal{E}^+(V^\ast(1), D^\perp) = \det(1 - p^{-1} \varphi^{-1} | D_{-1}) \det(1 - p^{-1} \varphi^{-1} | D^\perp).$$

(4.15)
We propose the following conjecture about the behavior of $p$-adic $L$-functions at $s = 0$.

**Trivial zero conjecture.** Let $D$ be a regular subspace of $D_{\text{cris}}(V)$. Then

1) The $p$-adic $L$-function $L_p(T, N, s)$ has a zero of order $e$ at $s = 0$ and
$$\frac{L_p^*(T, N, 0)}{R_{V, D}(\omega_{V,N})} = -\mathcal{L}(V,D) \mathcal{E}^+(V,D) \frac{L(M,0)}{R_{M,\infty}(\omega_M)}.$$ 

2) The $p$-adic $L$-function $L_p(T^*(1), N^\perp, s)$ has a zero of order $e + r$ where $r = \dim_{\mathbb{Q}} H^1_f(M)$ at $s = 0$ and
$$\frac{L_p^*(T^*(1), N^\perp, 0)}{R_{V^*(1), D^\perp}(\omega_{V^*,N^\perp})} = \mathcal{L}(V,D) \mathcal{E}^+(V^*(1), D^\perp) \frac{L^*(M^*(1),0)}{R_{M^*(1), \infty}(\omega_{M^*(1)})}.$$ 

**Remarks.** 1) If $H^1_f(M) = 0$ the $p$-adic regulator vanishes and we recover the conjecture formulated in [Ben2], section 2.3.2.

2) The regulators $R_{M,\infty}(\omega_M)$ and $R_{V, D}(\omega_{V,N^\perp})$ are well defined up to the sign and in order to obtain equalities in the formulation of our conjecture one should make the same choice of signs in the definitions of $R_{M,\infty}(\omega_M)$ and $R_{V, D}(\omega_{V,N^\perp})$. See [PR2], section 4.2 for more detail.

3) Our conjecture is compatible with the expected functional equation for $p$-adic $L$-functions. See section 2.5 of [PR2] and section 5.2.7 below.

§5. **The module of $p$-adic $L$-functions**

5.1. The Selmer complex.

5.1.1. **Iwasawa cohomology.** Let $\Gamma$ denote the Galois group of $\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}$ and $\Gamma_n = \text{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}(\zeta_{p^n}))$. Set $\Lambda = \mathbb{Z}_p[[\Gamma]]$ and $\Lambda(\Gamma) = \mathbb{Z}_p[\Delta] \otimes_{\mathbb{Z}_p} \Lambda$. For any character $\eta \in X(\Delta)$ put
$$e_\eta = \frac{1}{|\Delta|} \sum_{g \in \Delta} \eta^{-1}(g) g.$$ 

Then $\Lambda(\Gamma) = \bigoplus_{\eta \in X(\Delta)} \Lambda(\Gamma)^{(\eta)}$ where $\Lambda(\Gamma)^{(\eta)} = \Lambda e_\eta$ and for any $\Lambda(\Gamma)$-module $M$ one has a canonical decomposition
$$M \simeq \bigoplus_{\eta \in X(\Delta)} M^{(\eta)}, \quad M^{(\eta)} = e_\eta(M).$$

We write $\eta_0$ for the trivial character of $\Delta$ and identify $\Lambda$ with $\Lambda(\Gamma) e_{\eta_0}$.

Let $V$ be a $p$-adic pseudo-geometric representation unramified outside $S$. Set $d(V) = \dim(V)$ and $d_{\pm}(V) = \dim(V^{e=\pm 1})$. Fix a $\mathbb{Z}_p$-lattice $T$ of $V$ stable under the action of $G_S$. Let $\iota : \Lambda(\Gamma) \to \Lambda(\Gamma)$ denote the canonical involution $g \mapsto g^{-1}$. Recall that the induced module $\text{Ind}_{\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}}(T)$ is isomorphic to $(\Lambda(\Gamma) \otimes_{\mathbb{Z}_p} T)^\iota$ ([Ne2], section 8.1). Define

$$H^i_{\text{Iw},S}(T) = H^i_S((\Lambda(\Gamma) \otimes_{\mathbb{Z}_p} T)^\iota),$$ 

$$H^i_{\text{Iw}}(\mathbb{Q}_v, T) = H^i(\mathbb{Q}_v, (\Lambda(\Gamma) \otimes_{\mathbb{Z}_p} T)^\iota) \quad \text{for any finite place $v$.}$$

From Shapiro’s lemma it follows immediately that

$$H^i_{\text{Iw},S}(T) = \lim_{\text{cores}} H^i_S(\mathbb{Q}(\zeta_{p^n}), T), \quad H^i_{\text{Iw}}(\mathbb{Q}_p, T) = \lim_{\text{cores}} H^i(\mathbb{Q}_p(\zeta_{p^n}), T).$$
Set $H^1_{Iw,S}(V) = H^1_{Iw,S}(T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $H^1_{Iw}(Q_v, V) = H^1_{Iw}(Q_v, T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. In [PR2] Perrin-Riou proved the following results about the structure of these modules.

1) $H^1_{Iw,S}(V) = 0$ and $H^1_{Iw}(Q_v, T) = 0$ if $i \neq 1, 2$;

2) If $v \neq p$, then for each $\eta \in X(\Delta)$ the $\eta$-component $H^1_{Iw}(Q_v, T)^{(\eta)}$ is a finitely generated torsion $\Lambda$-module. In particular, $H^1_{Iw}(Q_v, T) \simeq H^1(Q_{ur}/Q_v, (\Lambda(\Gamma) \otimes_{\mathbb{Z}_p} T^*)^\vee)$.

3) If $v = p$, then $H^2_{Iw}(Q_p, T)^{(\eta)}$ are finitely generated torsion $\Lambda$-modules. Moreover, for each $\eta \in X(\Delta)$

$$\text{rg}_\Lambda \left( H^1_{Iw}(Q_p, T)^{(\eta)} \right) = d, \quad H^1_{Iw}(Q_p, T)^{(\eta)} \simeq H^0(Q_p(\zeta_2), T)^{(\eta)}.$$ 

Remark that by local duality $H^1_{Iw}(Q_p, T) \simeq H^0(Q_p(\zeta_2), V^*(1)/T^*(1))$.

4) If the weak Leopoldt conjecture holds for the pair $(V, \eta)$ i.e. if $H^2_S(Q(\zeta_2, \nu), V/T)^{(\eta)} = 0$ then $H^2_{Iw,S}(T)^{(\eta)}$ is $\Lambda$-torsion and

$$\text{rank}_\Lambda \left( H^1_{Iw,S}(T)^{(\eta)} \right) = \begin{cases} \text{d}^-(V), & \text{if } \eta(c) = 1 \\ \text{d}^+(V), & \text{if } \eta(c) = -1. \end{cases}$$

Passing to the projective limit in the Poitou-Tate exact sequence one obtains an exact sequence

$$0 \rightarrow H^2_S(Q(\zeta_2, \nu), V^*(1)/T^*(1)) \rightarrow H^1_{Iw,S}(T) \rightarrow \bigoplus_{v \in S} H^1_{Iw}(Q_v, T) \rightarrow H^1_S(Q(\zeta_2, \nu), V^*(1)/T^*(1)) \rightarrow 0. \quad (5.1)$$

Define

$$\text{RG}_{Iw,S}(T) = C^*_\Lambda(G_S, (\Lambda(\Gamma) \otimes_{\mathbb{Z}_p} T)^\vee),$$

$$\text{RG}_{Iw}(Q_v, T) = C^*_\Lambda(G_v, (\Lambda(\Gamma) \otimes_{\mathbb{Z}_p} T)^\vee),$$

$$\text{RG}_S(Q(\zeta_2, \nu), V^*(1)/T^*(1)) = C^*_\Lambda(G_S, \text{Hom}_{\mathbb{Z}_p}(\Lambda(\Gamma), V^*(1)/T^*(1))).$$

Then the sequence (5.1) is induced by the distinguished triangle

$$\text{RG}_{Iw,S}(T) \rightarrow \bigoplus_{v \in S} \text{RG}_{Iw}(Q_v, T) \rightarrow (\text{RG}_S(Q(\zeta_2, \nu), V^*(1)/T^*(1))^\vee \leftarrow$$

$$\leftarrow (5.1) \leftarrow 2$$

([Ne2], Theorem 8.5.6). Finally, we have usual descent formulas

$$\text{RG}_{Iw,S}(T) \otimes_{\mathbb{Z}_p} F^1_T \simeq \text{RG}_S(T), \quad \text{RG}_{Iw}(Q_v, T) \otimes_{\mathbb{Z}_p} F^1_T \simeq \text{RG}(Q_v, T)$$

([Ne2], Proposition 8.4.21).

5.1.2. The complex $\text{RG}^{(\eta_0)}_{Iw,h}(D, V)$. For the remainder of this chapter we assume that $V$ satisfies the conditions C1-5) of section 3.1.2 and that the weak Leopoldt conjecture holds for $(V, \eta_0)$ and $(V^*(1), \eta_0)$. We remark that these assumptions are not independent. Namely, by [PR2], Proposition B.5 C4 and C5) imply the weak Leopoldt conjecture for $(V^*(1), \eta_0)$. From the same result it follows that the vanishing of $H^1_f(V^*(1))$ implies the weak Leopoldt conjecture
for \((V, \eta_0)\) if in addition we assume that \(H^0(\mathbb{Q}_p, V^*(1)) = 0\).

To simplify notations we write \(\mathcal{H}\) for \(\mathcal{H}(\Gamma_1)\). Fix a regular subspace \(D\) of \(\text{D}_{\text{cris}}(V)\) and a \(\mathbb{Z}_p\)-lattice \(N\) of \(D\). Set \(\mathcal{D}_p(N,T)^{(\eta_0)} = N \otimes_{\mathbb{Z}_p} \Lambda\), \(\text{R} \Gamma_{\text{lw},f}^{(\eta_0)}(\mathbb{Q}_p, N, T) = \mathcal{D}_p(N,T)^{(\eta_0)}[-1]\) and \(\text{R} \Gamma_{\text{lw},f}^{(\eta_0)}(\mathbb{Q}_p, D, V) = \text{R} \Gamma_{\text{lw},f}^{(\eta_0)}(\mathbb{Q}_p, N, T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\). Consider the map

\[
\text{Exp}_{V,h} : \text{R} \Gamma_{\text{lw},f}^{(\eta_0)}(\mathbb{Q}_p, T) \otimes_{\Lambda} \mathcal{H} \to \text{R} \Gamma_{\text{lw}}^{(\eta_0)}(\mathbb{Q}_p, T) \otimes_{\Lambda} \mathcal{H}
\]

which will be viewed as a local condition at \(p\). If \(v \neq p\) the inertia group \(I_v\) acts trivially on \(\Lambda\) set

\[
\text{R} \Gamma_{\text{lw},f}^{(\eta_0)}(\mathbb{Q}_v, N, T) = \left[ T^I \otimes \Lambda' \xrightarrow{1-f_v} T^I \otimes \Lambda' \right]
\]

where the first term is placed in degree 0. We have a commutative diagram

\[
\begin{array}{ccc}
\text{R} \Gamma_{\text{lw},f}^{(\eta_0)}(T) \otimes_{\Lambda} \mathcal{H} & \cong & \bigoplus_{\nu \in S} \text{R} \Gamma_{\text{lw},f}^{(\eta_0)}(\mathbb{Q}_v, T) \otimes_{\Lambda} \mathcal{H} \\
\downarrow & & \downarrow \\
\left( \bigoplus_{\nu \in S} \text{R} \Gamma_{\text{lw},f}^{(\eta_0)}(\mathbb{Q}_v, N, T) \right) \otimes_{\Lambda} \mathcal{H} & \cong & \bigoplus_{\nu \in S} \text{R} \Gamma_{\text{lw},f}^{(\eta_0)}(\mathbb{Q}_v, T) \otimes_{\Lambda} \mathcal{H}
\end{array}
\]

Consider the associated Selmer complex

\[
\text{R} \Gamma_{\text{lw},h}^{(\eta_0)}(D, V) = \text{cone} \left[ \left( \text{R} \Gamma_{\text{lw},f}^{(\eta_0)}(T) \oplus \bigoplus_{\nu \in S} \text{R} \Gamma_{\text{lw},f}^{(\eta_0)}(\mathbb{Q}_v, N, T) \right) \otimes_{\Lambda} \mathcal{H} \to \bigoplus_{\nu \in S} \text{R} \Gamma_{\text{lw},f}^{(\eta_0)}(\mathbb{Q}_v, T) \otimes_{\Lambda} \mathcal{H} \right] [-1]
\]

It is easy to see that it does not depend on the choice of \(S\). Our main result about this complex is the following theorem.

**Theorem 5.1.3.** Assume that \(V\) satisfies the conditions C1-5) and that the weak Leopoldt conjecture holds for \((V, \eta_0)\) and \((V^*(1), \eta_0)\). Let \(D\) be a regular subspace of \(\text{D}_{\text{cris}}(V)\). Assume that \(\mathcal{L}(V,D) \neq 0\). Then

i) \(\text{R}^i \Gamma_{\text{lw},h}^{(\eta_0)}(D, V)\) are \(\mathcal{H}\)-torsion modules for all \(i\).

ii) \(\text{R}^i \Gamma_{\text{lw},h}^{(\eta_0)}(D, V) = 0\) for \(i \neq 2, 3\) and

\[
\text{R}^3 \Gamma_{\text{lw},h}^{(\eta_0)}(D, V) \simeq \left( H^0(\mathbb{Q}(\mathbb{G}_{m,\text{cris}}), V^*(1))^*(\eta_0) \right) \otimes_{\Lambda} \mathcal{H}.
\]

iii) The complex \(\text{R} \Gamma_{\text{lw},h}^{(\eta_0)}(D, V)\) is semisimple i.e. for each \(i\) the natural map

\[
\text{R}^i \Gamma_{\text{lw},h}^{(\eta_0)}(D, V) \Gamma \to \text{R}^i \Gamma_{\text{lw},h}^{(\eta_0)}(D, V) \Gamma
\]

is an isomorphism.

**5.1.4. Proof of Theorem 5.1.3.** We leave the proof of the following lemma as an easy exercise.
Lemma 5.1.4.1. Let $A$ and $B$ be two submodules of a finitely generated free $\mathcal{H}$-module $M$. Assume that the natural maps $A_{\Gamma} \to M_{\Gamma}$ and $B_{\Gamma} \to M_{\Gamma}$ are both injective. Then $A_{\Gamma} \cap B_{\Gamma} = \{0\}$ implies that $A \cap B = \{0\}$.

5.1.4.2. Since $H_{Iw}^0(S(V))$ and $H_{Iw}^0(Q_v, V)$ are zero, we have $R^1\Gamma^0_{Iw,h}(D, V) = 0$. Next, by definition $R^1\Gamma^0_{Iw,h}(D, V) = \ker(f)$ where

$$f : \left( H_{Iw,S}^1(T)^{(\eta_0)} \oplus D_p(N, T)^{(\eta_0)} \oplus \bigoplus_{v \in S \setminus \{p\}} H_{Iw,f}(Q_v, T)^{(\eta_0)} \right) \otimes \mathcal{H} \to \bigoplus_{v \in S} H_{Iw}^1(Q_v, T)^{(\eta_0)} \otimes \mathcal{H}$$

is the map induced by (5.2). If $v \in S - \{p\}$ one has

$$H_{Iw,f}(Q_v, T)^{(\eta_0)} = H_{Iw,1}(Q_v, T)^{(\eta_0)} = H^1(Q_v, (\Lambda \otimes T^f_v)^e).$$

Thus

$$R^1\Gamma^0_{Iw,h}(D, V) = \left( H_{Iw,S}^1(T)^{(\eta_0)} \otimes_{\mathcal{H}} \mathcal{H} \right) \cap \left( \text{Exp}_{V,h}^e \left( D_p(D, T)^{(\eta_0)} \right) \otimes_{\mathcal{H}} \mathcal{H} \right)$$

in $H_{Iw}^1(Q_v, T)^{(\eta_0)} \otimes_{\mathcal{H}} \mathcal{H}$. Put

$$A = \text{Exp}_{V,h}^e(D_{\Gamma} \otimes \mathcal{H}) \oplus X^{-1} \text{Exp}_{V,h}^e(D^{\psi=p^{-1}} \otimes \mathcal{H}) \subset H_{Iw}^1(Q_v, T)^{(\eta_0)} \otimes_{\mathcal{H}} \mathcal{H}.$$

By Theorem 2.2.4 and Proposition 3.2.2 $A_{\Gamma}$ injects into $H^1(Q_v, V)$. The $\mathcal{H}$-module $M = \left( H_{Iw}^1(Q_v, T)^{(\eta_0)} \otimes_{\mathcal{H}} \mathcal{H} \right)$ is free and $A \hookrightarrow M$. Since $T^{\mathcal{C}_{Q_p}} = 0$ one has $M_{\Gamma} = H_{Iw}^1(Q_v, V)_{\Gamma} \subset H^1(Q_v, V)$ and we obtain that $A_{\Gamma}$ injects into $M_{\Gamma}$.

Set $B = \left( H_{Iw,S}^1(T)^{(\eta_0)} \otimes_{\mathcal{H}} \mathcal{H} \right)$. The weak Leopoldt conjecture for $(V^* (1), \eta_0)$ together with the fact that $H_{Iw}^1(Q_v, T)$ are $\Lambda$-torsion for $v \in S - \{p\}$ imply that $B \hookrightarrow M$. Since the image of $H_{Iw}^1(Q_v, V)_p$ in $H^1(Q_v, V)$ is contained in $H_1^1(Q_v, V)$, the image of $H_{Iw,S}^1(S(V))_{\Gamma}$ in $H_1^1(S(V))$ is in fact contained in $H_{f,\{p\}}^1(V)$. From C5 it follows that $H_{f,\{p\}}^1(V)$ injects into $H^1(Q_v, V)$ and we have

$$H_{Iw,S}^1(V)_{\Gamma} = H_{Iw,S}^1(V)_{\Gamma} \hookrightarrow H_{f,\{p\}}^1(V) \hookrightarrow H^1(Q_v, V).$$

Thus $B_{\Gamma} \subset M_{\Gamma}$. We shall prove that $R^1\Gamma^0_{Iw,h}(D, V) = 0$. By Lemma 5.1.4.1 it suffices to show that $A_{\Gamma} \cap B_{\Gamma} = \{0\}$. Now we claim that $A_{\Gamma} \cap H^1_{f,\{p\}}(V) = \{0\}$. First note that by Lemma 3.1.4

$$H_{f,\{p\}}^1(V) \hookrightarrow \frac{H^1(Q_v, V)}{H^1(D_{\Gamma})}.$$

On the other hand, from Theorem 2.2.4 it follows that

$$\text{Exp}_{V,h}^e(D_{-1} \otimes \mathcal{H})_{\Gamma} = \text{exp}_{\mathcal{C}_{Q_p}}(D_{-1}) \subset H^1(D_{-1}).$$

Now Proposition 3.2.2 implies that the image of $A_{\Gamma}$ in \(H^1(Q_v, V)_{\Gamma}/H^1(D_{-1})\) coincides with $H^1_{\epsilon}(W)$. But $\mathcal{L}(V, D) \neq 0$ if and only if $H^1_B(V) \cap H^1_{\epsilon}(W) = 0$ where $H^1_B(V)$ denotes the inverse image of $H^1(W)$ in $H^1_{f,\{p\}}(V)$ (see Lemma 3.1.4 iii)). This proves the claim and implies that $R^1\Gamma^0_{Iw,h}(D, V) = 0$. 
5.1.4.3. We shall show that $\mathbf{R}^2\Gamma^{(\eta_0)}_{Iw,h}(D,V)$ is $\mathcal{H}$-torsion. By definition, we have an exact sequence

$$0 \to \text{coker}(f) \to \mathbf{R}^2\Gamma^{(\eta_0)}_{Iw,h}(D,V) \to \bigoplus_{\Lambda_{q,p}} \mathcal{H} \to 0,$$

where

$$\bigoplus_{\Lambda_{q,p}} \mathcal{H} = \ker \left( H_{Iw,S}(V) \to \bigoplus_{v \in S} H_{Iw}(\mathbb{Q}_v,V) \right).$$

It follows from the weak Leopoldt conjecture that $\bigoplus_{\Lambda_{q,p}} \mathcal{H}$ is $\Lambda_{q,p}$-torsion. On the other hand, as $\mathcal{H}$ is a Bezout ring $[La]$, the formulas

$$\text{rank}_A H^1_{Iw,S}(T)^{(\eta_0)} = d_-(V), \quad \text{rank}_A H^1_{Iw,q_p,T} = d(V), \quad \text{rank}_A D_p(N,T) = d_+(V)$$

together with the fact that $\mathbf{R}^1\Gamma^{(\eta_0)}_{Iw,h}(D,V) = 0$ imply that $\text{coker}(f)$ is $\mathcal{H}$-torsion. We have therefore proved that $\mathbf{R}^2\Gamma_{Iw,h}(D,V)$ is $\mathcal{H}$-torsion. Finally, the Poitou-Tate exact sequence gives that

$$\mathbf{R}^3\Gamma_{Iw,h}(D,V) = (H^0(\mathbb{Q}(\zeta_{p^\infty}), V^*(1))^*)^{(\eta_0)} \otimes_{\Lambda_{q,p}} \mathcal{H}$$

is also $\mathcal{H}$-torsion.

5.1.4.4. Now we prove the semisimplicity of $\mathbf{R}^1\Gamma_{Iw,h}(D,V)$. First write $H^1_{Iw,S}(V)^{(\eta_0)} \simeq \bigoplus_{\Lambda_{q,p}} \mathcal{H}$. Since $H^1_{Iw,S}(V)^{(\eta_0)}$ is also $\mathcal{H}$-torsion. We have

\[
\begin{array}{cccccc}
0 & \longrightarrow & H^1_{Iw,S}(V)^{(\eta_0)} & \longrightarrow & H^1_{f,(p)}(V) & \longrightarrow & H^0(\mathbb{Q}_p,V^*(1))^*) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^1_{Iw}(\mathbb{Q}_p,V)^{(\eta_0)} & \longrightarrow & H^1(\mathbb{Q}_p,V) & \longrightarrow & H^0(\mathbb{Q}_p,V^*(1))^*) & \longrightarrow & 0
\end{array}
\]

with obviously exact upper line the bottom line is also exact. This implies immediately that the natural map

$$\frac{H^1_{Iw}(\mathbb{Q}_p,V)^{(\eta_0)}}{H^1_{Iw,S}(V)^{(\eta_0)} + H^1(D_{-1})} \to \frac{H^1(\mathbb{Q}_p,V)}{H^1_{f,(p)}(V) + H^1(D_{-1})}$$

is an isomorphism.

Consider the exact sequence

$$0 \to \left( H^1_{Iw,S}(T)^{(\eta_0)} \oplus D_p(N,T)^{(\eta_0)} \right) \otimes \mathcal{H} \to H^1_{Iw}(\mathbb{Q}_p,T)^{(\eta_0)} \otimes \mathcal{H} \to \text{coker}(f) \to 0.$$  

Recall that $\text{Exp}_{V,h,0} : \mathcal{D} \to H^1_{Iw}(\mathbb{Q}_p,V)_{\Gamma}$ denotes the homomorphism induced by the large exponential map. Applying the snake lemma, and taking into account that $\text{Im}(\text{Exp}_{V,h,0}) = \text{exp}_{V,Q_p}(D_{-1}) = H^1(D_{-1})$ and $\ker(\text{Exp}_{V,h,0}) = D_{\varphi = p^{-1}}$ (see for example [BB], Propositions 4.17 and 4.18 or the proof of Proposition 3.3.2) we obtain

$$\text{coker}(f)_{\Gamma_1} = \ker \left( H^1_{Iw,S}(V)^{(\eta_0)} \oplus D \stackrel{\text{Exp}_{V,h,0}}{\longrightarrow} H^1(\mathbb{Q}_p,V) \right) = D_{\varphi = p^{-1}}, \quad \text{(by the regularity of } D),$$

$$\text{coker}(f)_{\Gamma_1} = \frac{H^1_{Iw}(\mathbb{Q}_p,V)^{(\eta_0)}}{H^1_{Iw,S}(V)^{(\eta_0)} + H^1(D_{-1})} = \frac{H^1(\mathbb{Q}_p,V)}{H^1_{f,(p)}(V) + H^1(D_{-1})}.$$
Thus one has a commutative diagram

\[ \begin{array}{ccc}
\text{coker}(f)_{\Gamma_1} & \longrightarrow & D^{\varphi=p^{-1}} \\
\downarrow & & \downarrow \delta_{D,h} \\
\text{coker}(f)_{\Gamma_1} & \longrightarrow & H^1(Q_p,V) \\
& & \uparrow \\
& & H^1_{f,\{p\}}(V) + H^1(D_{-1}) \\
\end{array} \tag{5.5} \]

where horizontal arrows are isomorphisms, the left vertical arrow is the natural projection and the right vertical row is the map defined in section 3.2.3. From Proposition 3.2.4 it follows that coker\((f)_{\Gamma_1} \rightarrow \text{coker}(f)_{\Gamma_1} \) is an isomorphism if and only if \( \mathcal{L}(V,D) \neq 0 \).

On the other hand, the arguments [PR2], section 3.3.4 show that \( \mathbf{W}^2_{Iw,S}(V)_{\Gamma} = \mathbf{W}^3_{Iw,S}(V)_{\Gamma} = 0 \). Remark that Perrin-Riou assumes that \( D_{\text{cris}}(V)^{\varphi=1} = D_{\text{cris}}(V)^{\varphi=p^{-1}} = 0 \), but her proof works in our case without modifications and we repeat it for the commodity of the reader. Consider the commutative diagram (where we write \( \mathbf{W}^2_{Iw}(V) \) instead \( \mathbf{W}^2_{Iw,S}(V) \) and \( H^1(Q(\zeta_p^{\infty}),V^*(1))_{\Gamma} \) instead \( H^1_S(Q(\zeta_p^{\infty}),V^*(1))_{\Gamma} \) to abbreviate notation)

\[ \begin{array}{ccccccc}
0 & \longrightarrow & H^1_{Iw}(V)_{\Gamma} & \longrightarrow & \oplus_{v \in S} H^1_{Iw}(Q_v,V)_{\Gamma} & \longrightarrow & H^1(Q(\zeta_p^{\infty}),V^*(1))_{\Gamma} & \longrightarrow & \mathbf{W}^2_{Iw}(V)_{\Gamma} & \longrightarrow & 0 \\
0 & \longrightarrow & H^1_{f,\{p\}}(V) & \longrightarrow & \oplus_{v \in S \setminus \{p\}} H^1_{Iw}(Q_v,V) \oplus H^1(Q_p,V) & \longrightarrow & H^1(V^*(1))^* & \longrightarrow & 0 \\
H^0(Q_p,V^*(1))^* & \longrightarrow & H^0(Q_p,V^*(1))^* & \longrightarrow & 0 \\
0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\end{array} \]

The top row of this diagram is obtained by taking coinvariants in the Poitou-Tate exact sequence. Thus it is exact. The middle row is obtained from the exact sequence

\[ 0 \rightarrow H^1_S(V^*(1)) \rightarrow H^1(Q_p,V^*(1)) \oplus \bigoplus_{v \in S \setminus \{p\}} \frac{H^1(Q_v,V^*(1))}{H^1_f(Q_v,V^*(1))} \rightarrow H^1_{f,\{p\}}(V)^* \rightarrow 0 \]

by taking duals. Here we use the condition \( H^1_f(V^*(1)) = 0 \). The exactness of the left and middle columns follows from the diagram (5.4). The isomorphism from the right column comes from the exact sequence

\[ 0 \rightarrow H^1(\Gamma, H^0_S(Q(\zeta_p^{\infty}),V^*(1))) \rightarrow H^1_S(V^*(1)) \rightarrow H^1_S(Q(\zeta_p^{\infty}),V^*(1))_{\Gamma} \rightarrow 0 \]

together with the remark that \( H^1(\Gamma, H^0_S(Q(\zeta_p^{\infty}),V^*(1))) = 0 \) because \( H^0(\Gamma, H^0_S(Q(\zeta_p^{\infty}),V^*(1))) = H^0_S(Q,V^*(1)) = 0 \) by C2. Now an easy diagram search shows that \( \mathbf{W}^2_{Iw,S}(V)_{\Gamma} = 0 \). Finally,
from \( \dim_{\mathbb{Q}_p} \mathcal{W}_{Iw,S}(V)^{\Gamma} \leq \dim_{\mathbb{Q}_p} \mathcal{W}_{Iw,S}(V)^{\Gamma} \) it follows that \( \mathcal{W}_{Iw,S}(V)^{\Gamma} = 0 \). Therefore, applying the snake lemma to (5.3) we obtain a commutative diagram

\[
\begin{array}{ccc}
\text{coker}(f)^{\Gamma_1} & \longrightarrow & R^2\Gamma_{Iw,h}^{(\eta_0)}(D, V)^{\Gamma_1} \\
\downarrow & & \downarrow \\
\text{coker}(f)^{\Gamma_1} & \longrightarrow & R^2\Gamma_{Iw,h}^{(\eta_0)}(D, V)^{\Gamma_1},
\end{array}
\]

in which the horizontal arrows are isomorphisms and the vertical arrows are natural projections. This proves that \( R\Gamma_{Iw,h}^{(\eta_0)}(D, V) \) is semisimple in degree 2. Remark that the semisimplicity in degree 3 is obvious because by ii) \( R^3\Gamma_{Iw,h}^{(\eta_0)}(D, V)^{\Gamma_1} = R^3\Gamma_{Iw,h}^{(\eta_0)}(D, V)^{\Gamma_1} = 0 \). This completes the proof of Theorem 5.1.3.

**Corollary 5.1.5.** The exponential map induces an isomorphism of \( D^{\varphi = p^{-1}} \) onto \( \text{coker}(f)^{\Gamma_1} \cong R^2\Gamma_{Iw,h}^{(\eta_0)}(D, V)^{\Gamma_1} \) and the diagram

\[
\begin{array}{ccc}
D^{\varphi = p^{-1}} & \sim & R^2\Gamma_{Iw,h}^{(\eta_0)}(D, V)^{\Gamma_1} \\
\downarrow \lambda_D & & \downarrow \\
D^{\varphi = p^{-1}} & \stackrel{(h-1)!\exp_V}{\longrightarrow} & R^2\Gamma_{Iw,h}^{(\eta_0)}(D, V)^{\Gamma_1},
\end{array}
\]

in which the map \( \lambda_D \) is defined in Proposition 3.2.4, commutes.

### 5.2. The module of \( p \)-adic \( L \)-functions.

#### 5.2.1. The canonical trivialisation.** We conserve the notation and conventions of section 4.2. Let \( D \) be an admissible subspace of \( \mathcal{D}_{\text{vis}}(V) \) and assume that \( \mathcal{L}(V, D) \neq 0 \). We review Perrin-Riou’s definition of the module of \( p \)-adic \( L \)-functions using the formalism of Selmer complexes. Set

\[
\Delta_{Iw,h}(D, V) = \text{det}_{\Lambda_F}^{-1} \left( \Gamma_{Iw,S}^{(\eta_0)}(V) \bigoplus_{v \in S} \Gamma_{Iw,f}^{(\eta_0)}(\mathbb{Q}_v, D, V) \right) \otimes \text{det}_{\Lambda_F} \left( \bigoplus_{v \in S} \Gamma_{Iw}^{(\eta_0)}(\mathbb{Q}_v, V) \right).
\]

The exact triangle

\[
R\Gamma_{Iw,S}^{(\eta_0)}(D, V) \to \left( \Gamma_{Iw,S}^{(\eta_0)}(V) \bigoplus_{v \in S} \Gamma_{Iw,f}^{(\eta_0)}(\mathbb{Q}_v, D, V) \right) \otimes \mathcal{H} \to \left( \bigoplus_{v \in S} \Gamma_{Iw}^{(\eta_0)}(\mathbb{Q}_v, V) \right) \otimes \mathcal{H}
\]

gives an isomorphism \( \Delta_{Iw,h}(D, V) \otimes_{\Lambda_F} \mathcal{H} \cong \text{det}_{\mathcal{H}}^{-1} R\Gamma_{Iw,S}^{(\eta_0)}(D, V) \). Let \( \mathcal{K} \) denote the field of fractions of \( \mathcal{H} \). By Theorem 5.1.3, all \( R\Gamma_{Iw,S}^{(\eta_0)}(D, V) \) are \( \mathcal{H} \)-torsion and we have a canonical map.

\[
\text{det}_{\mathcal{H}}^{-1} R\Gamma_{Iw,S}^{(\eta_0)}(D, V) \cong \bigoplus_{i \in \{2, 3\}} \text{det}_{\mathcal{H}}^{-i+1} R^i\Gamma_{Iw,S}^{(\eta_0)}(D, V) \to \mathcal{K}.
\]

The composition of these maps gives a trivialization \( i_{V,Iw,h} : \Delta_{Iw,h}(D, V) \to \mathcal{K} \).
5.2.2. Local conditions. In this section we compare local conditions coming from Perrin-Riou’s theory to the Bloch-Kato’s one. Set $R\Gamma_f(\mathbb{Q}_p, D, V) = D[-1]$ and define

$$S = \text{cone} \left( \frac{1 - p^{-1}\varphi^{-1}}{1 - \varphi} : R\Gamma_f(\mathbb{Q}_p, D, V) \to R\Gamma_f(\mathbb{Q}_p, V) \right) [-1].$$

(5.6)

Thus, explicitly

$$S = [D \oplus D_{\text{cris}}(V) \to D_{\text{cris}}(V) \oplus t_V(\mathbb{Q}_p)] [-1] \simeq [D \oplus D_{\text{cris}}(V) \to D_{\text{cris}}(V) \oplus D] [-1],$$

where the unique non-trivial map is given by

$$(x, y) \mapsto \left((1 - \varphi)y, \left(\frac{1 - p^{-1}\varphi^{-1}}{1 - \varphi}x + y\right) \mod \text{Fil}^0 D_{\text{cris}}(V)\right).$$

Thus $H^1(S) = D^{\varphi = p^{-1}}$ and $H^2(S) = \frac{t_V(\mathbb{Q}_p)}{(1 - p^{-1}\varphi^{-1})D} \simeq D_{\text{cris}}(V) \mod \text{Fil}^0 D_{\text{cris}}(V) + D^{-1}$. From the semi-simplicity of $\frac{1 - p^{-1}\varphi^{-1}}{1 - \varphi}$ it follows that the natural projection $H^1(S) \oplus H^1_f(V) \to H^2(S)$ is an isomorphism and we have a canonical trivialization

$$\alpha_S : \text{det}_{\mathbb{Q}_p} S \otimes \text{det}_{\mathbb{Q}_p} R\Gamma_f(V) \simeq \text{det}_{\mathbb{Q}_p}^{-1} H^1(S) \otimes \text{det}_{\mathbb{Q}_p} H^2(S) \otimes \text{det}_{\mathbb{Q}_p}^{-1} H^1_f(V) \simeq \mathbb{Q}_p.$$

(5.7)

Hence the distinguished triangle

$$S \to R\Gamma_f(\mathbb{Q}_p, D, V) \to R\Gamma_f(\mathbb{Q}_p, V) \to S[1]$$

induces isomorphisms

$$\beta_S : \text{det}_{\mathbb{Q}_p} t_V(\mathbb{Q}_p) \otimes \text{det}_{\mathbb{Q}_p} R\Gamma_f(V) \simeq \text{det}_{\mathbb{Q}_p}^{-1} R\Gamma_f(\mathbb{Q}_p, V) \otimes \text{det}_{\mathbb{Q}_p} R\Gamma_f(V)$$

$$\simeq \text{det}_{\mathbb{Q}_p}^{-1} R\Gamma_f(\mathbb{Q}_p, D, V) \otimes \text{det}_{\mathbb{Q}_p} S \otimes \text{det}_{\mathbb{Q}_p} R\Gamma_f(V)$$

$$\simeq \text{det}_{\mathbb{Q}_p} D \otimes \text{det}_{\mathbb{Q}_p} S \otimes \text{det}_{\mathbb{Q}_p} R\Gamma_f(V)$$

(5.8)

and

$$\vartheta_S : \text{det}_{\mathbb{Q}_p} t_V(\mathbb{Q}_p) \otimes \text{det}_{\mathbb{Q}_p} R\Gamma_f(V) \simeq \text{det}_{\mathbb{Q}_p} D \otimes \text{det}_{\mathbb{Q}_p} S \otimes \text{det}_{\mathbb{Q}_p} R\Gamma_f(V) \simeq \text{det}_{\mathbb{Q}_p} D. \quad (5.9)$$

Fix bases $\omega_V \in \text{det}_{\mathbb{Q}_p} t_V(\mathbb{Q}_p)$, $\omega_D \in \text{det}_{\mathbb{Q}_p} D$ and $\omega_f \in \text{det}_{\mathbb{Q}_p} H^1_f(V)$. Let $R_{V,D}(\omega_{V,D})$ denote the determinant of the regulator map

$$r_{V,D} : H^1_f(V) \to D_{\text{cris}}(V)/(\text{Fil}^0 D_{\text{cris}}(V) + D)$$

with respect to $\omega_f$ and $\omega_V \otimes \omega_D^{-1}$.

**Lemma 5.2.3.** i) Let $f : W \to W$ be a semi-simple endomorphism of a finitely dimensional $k$-vector space $W$. The canonical projection $\ker(f) \to \text{coker}(f)$ is an isomorphism and the tautological exact sequence

$$0 \to \ker(f) \to W \xrightarrow{f} W \to \text{coker}(f) \to 0$$
induces an isomorphism
\[ \det^* f : \det_k(W) \to \det_k(W) \otimes \det_k(\ker(f)) \otimes \det_k^{-1}(\text{coker}(f)) \to \det_k(W). \]
Then \( \det^* f(x) = \det(f | \ker(f)). \)

ii) The map \( \vartheta_S \) sends \( \omega_v \otimes \omega_f^{-1} \) onto
\[ \det^* \left( \frac{1 - p^{-1} \varphi^{-1}}{1 - \varphi} | D \right)^{-1} E_p(V,1)^{-1} R_{V,D}(\omega_V,D)^{-1} \omega_D \]

**Proof.** The proof is straightforward and is omitted here.

### 5.2.4. Definition of the module of \( p \)-adic \( L \)-functions.

In this subsection we interpret Perrin-Riou’s construction of the module of \( p \)-adic \( L \)-functions in terms of \([Ne2]\). Fix a \( \mathbb{Z}_p \)-lattice \( N \) of \( D \) and set
\[ \Delta_{1w,h}(N,T) = \det_A^{-1} \left( R\Gamma_{1w,S}(T) \oplus \left( \oplus_{v \in S} R\Gamma_{1w,f}(Q_v, N, T) \right) \right) \otimes \det_A \left( \oplus_{v \in S} R\Gamma_{1w}(Q_v, T) \right). \]
The module of \( p \)-adic \( L \)-functions associated to \( (N, T) \) is defined as
\[ L_{1w,h}^{(n_0)}(N, T) = i_{V,1w,h}(\Delta_{1w,h}(N, T)) \subset \mathcal{K}. \]
Fix a generator \( f(\gamma_1 - 1) \) of \( L_{1w,h}^{(n_0)}(N, T) \) and define a meromorphic \( p \)-adic function
\[ L_{1w,h}(T, N, s) = f(\chi(\gamma)^s - 1). \]

Let now \( V \) be the \( p \)-adic realisation of a pure motive \( M \) over \( \mathbb{Q} \) which satisfies the conditions \( \text{M1-3)} \) of section 4.1.2. As we saw in section 4.1.2 on expects that \( V \) satisfies \( \text{C1-5)} \). We fix bases \( \omega_f \in \det_Q H_f^1(M), \omega_{\text{tr}} \in \det_Q t_M(\mathbb{Q}) \) and use the same notation for their images in \( \det_Q H_f^1(V) \) and \( \det_Q t_V(Q_p) \) respectively. Choose bases \( \omega_{M_f}^{\pm} \in \det_Q M_f^\pm \) and \( \omega_{T}^{\pm} \in \det_Q T^\pm \) and define the \( p \)-adic period \( \Omega_M^{(e,p)}(\omega_T^+, \omega_{M_f}^+) \in \mathbb{Q}_p \) by \( \omega_T^+ = \Omega_M^{(e,p)}(\omega_T^+, \omega_{M_f}^+) \) using the comparision isomorphism (4.4) and (4.7). Let \( \omega_N \) be a generator of \( \det_{\mathbb{Z}_p} N \).

**Theorem 5.2.5.** Assume that \( V \) satisfies \( \text{C1-5)} \) and that the weak Leopoldt conjecture holds for \( (V, \eta_0) \) and \( (V^*, 1, \eta_0) \). Let \( D \) be an admissible subspace of \( \mathcal{D}_{\text{cris}}(V) \). Assume that \( \mathcal{L}(V, D) \neq 0 \). Then
i) \( L_{1w,h}(T, N, s) \) is a meromorphic \( p \)-adic function which has a zero at \( s = 0 \) of order \( e = \dim_{\mathbb{Q}_p}(D^{e-1}'). \)
ii) Let \( L_{1w,h}^{*}(T, N, 0) = \lim_{s \to 0} s^{-e} L_{1w,h}(T, N, s) \) be the special value of \( L_{1w,h}(T, N, s) \) at \( s = 0 \). Then
\[ \frac{L_{1w,h}^{*}(T, N, 0)}{R_{V,D}(\omega_V,N)} \sim_p \Gamma(h)^{d_+(V)} \mathcal{L}(V, D) \mathcal{E}^+(V, D) \frac{i_{\omega_{M,f}}(\Delta_{EF}(T))}{\Omega_M^{(e,p)}(\omega_T^+, \omega_{M_f}^+)}, \]
where \( i_{\omega_{M,f}} \) and \( \mathcal{E}^+(V, D) \) are defined by (4.12) and (4.16) respectively and \( \Gamma(h) = (h - 1)! \).

### 5.2.6. Proof of Theorem 5.2.5.

5.2.6.1. First recall the formalism of Iwasawa descent which will be used in the proof. The
result we need is proved in [BG]. This is a particular case of Nekovár’s descent theory [Ne2]. Let $C^\bullet$ be a perfect complex of $\mathcal{H}$-modules and let $C_0^\bullet = C^\bullet \otimes^L \mathbb{Q}_p$. We have a natural distinguished triangle

$$C^\bullet \xrightarrow{\sim} C^\bullet \rightarrow C_0^\bullet,$$

where $X = \gamma_1 - 1$. In each degree this triangle gives a short exact sequence

$$0 \rightarrow H^n(C^\bullet)_{\Gamma_1} \rightarrow H^n(C_0^\bullet) \rightarrow H^{n+1}(C^\bullet)_{\Gamma_1} \rightarrow 0.$$

One says that $C^\bullet$ is semisimple if the natural map

$$H^n(C^\bullet)_{\Gamma_1} \rightarrow H^n(C^\bullet) \rightarrow H^n(C_0^\bullet)$$

is an isomorphism in all degrees. If $C^\bullet$ is semisimple, there exists a natural trivialisation of $\det_{\mathbb{Q}_p} C_0^\bullet$, namely

$$\vartheta : \det_{\mathbb{Q}_p} C_0^\bullet \simeq \otimes_{n \in \mathbb{Z}} \left( \det_{\mathbb{Q}_p}^{(-1)^n} H^n(C_0) \simeq \otimes_{n \in \mathbb{Z}} \left( \det_{\mathbb{Q}_p}^{(-1)^n} H^n(C^\bullet)_{\Gamma_1} \otimes \det_{\mathbb{Q}_p}^{(-1)^n} H^{n+1}(C^\bullet)_{\Gamma_1} \right) \right) \simeq \mathbb{Q}_p$$

where the last map is induced by (5.10). We now suppose that $C \otimes_{\mathcal{H}} \mathcal{K}$ is acyclic and write $i_\infty : \det_{\mathcal{H}} C^\bullet \rightarrow \mathcal{K}$ for the associated morphism in $\mathcal{P}(\mathcal{K})$. Then $i_\infty(\det_{\mathcal{H}} C^\bullet) = f\mathcal{H}$, where $f \in \mathcal{K}$. Let $r$ be the unique integer such that $X^{-r} f$ is a unit of the localization $\mathcal{H}_0$ of $\mathcal{H}$ with respect to the principal ideal $X\mathcal{H}$.

**Lemma 5.2.6.2.** Assume that $C^\bullet$ is semisimple. Then $r = \sum_{n \in \mathbb{Z}} (-1)^{n+1} \dim_{\mathbb{Q}_p} H^n(C^\bullet)_{\Gamma_1}$ and there exists a commutative diagram

$$\begin{array}{ccc}
\det_{\mathcal{H}} C^\bullet & \xrightarrow{X^{-r} i_\infty} & \mathcal{H}_0 \\
\otimes L_{\mathbb{Q}_p} & \downarrow & \\
\det_{\mathbb{Q}_p} C_0^\bullet & \xrightarrow{\vartheta} & \mathbb{Q}_p
\end{array}$$

in which the right vertical arrow is the augmentation map.

**Proof.** See [BG], Lemma 8.1. Remark that Burns and Greither consider complexes over $\Lambda \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ but since $\mathcal{H}$ is a Bézout ring, all their arguments work in our case and are omitted here.

**5.2.6.3.** By Theorem 5.1.3 the complex $\Gamma_{1w,h}^{(m_0)}(D, V)$ is semisimple and the first assertion follows from Lemma 5.2.6.2 together with Corollary 5.1.5.

**5.2.6.4.** Now we can prove Theorem 5.2.5. Define

$$\mathbf{R}\Gamma_f(\mathbb{Q}_v, N, T) = \mathbf{R}\Gamma_{1w,f}^{(m_0)}(\mathbb{Q}_v, N, T) \otimes^L \mathbb{Z}_p, \quad \mathbf{R}\Gamma_f(\mathbb{Q}_v, D, V) = \mathbf{R}\Gamma_f(\mathbb{Q}_v, N, T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Remark that for $v = p$ this definition coincides with the definition given in 5.2.2. Applying $\otimes^L_{\mathcal{H}} \mathbb{Q}_p$ to the map $\mathbf{R}\Gamma_{1w,f}^{(m_0)}(\mathbb{Q}_v, D, V) \rightarrow \mathbf{R}\Gamma_{1w}^{(m_0)}(\mathbb{Q}_v, T) \otimes^L_{\mathcal{H}} \mathcal{H}$ we obtain a morphism

$$\mathbf{R}\Gamma_f(\mathbb{Q}_v, D, V) \rightarrow \mathbf{R}\Gamma(\mathbb{Q}_v, V).$$
If \( v \neq p \), then \( \Gamma_f(Q_v, D, V) = \Gamma_f(Q_v, V) \) and this morphism coincides with the natural map \( \Gamma_f(Q_v, V) \to \Gamma(Q_v, V) \). If \( v = p \), then \( \Gamma_f(Q_v, D, V) = D[-1] \) and by Theorem 2.2.4 it coincides with the composition

\[
\begin{array}{ccc}
D & \xrightarrow{\text{cris}} & D_{\text{cris}}(V) \\
\downarrow & & \downarrow \\
& \xrightarrow{(h-1)! \exp_V} & H^1(Q_p, V).
\end{array}
\]

Let \( \Gamma_f(D, V) \) denote the Selmer complex associated to the diagram

\[
\begin{array}{ccc}
\Gamma_f(D, V) & \to & \bigoplus_{v \in S} \Gamma(Q_v, V) \\
\downarrow & & \downarrow \\
& \to & \bigoplus_{v \in S} \Gamma_f(Q_v, D, V)
\end{array}
\]

Then we have a distinguished triangle

\[
\begin{array}{c}
\Gamma_f(D, V) \\
\xrightarrow{n} \Gamma(S) \\
\xrightarrow{\bigoplus_{v \in S} \Gamma_f(Q_v, D, V)} \\
\to \bigoplus_{v \in S} \Gamma(Q_v, V)
\end{array}
\]  

(5.11)

which induces isomorphisms

\[
\det_{Q_r}^{-1} \Gamma_f(D, V) \otimes D \xrightarrow{\sim} \det_{Q_r}^{-1} \Gamma_f(D, V),
\]

\[
\xi_{D, h} : \Delta_{EP}(V) \otimes D \otimes \det_{Q_r}^{-1} \nabla^+ \xrightarrow{\sim} \det_{Q_r}^{-1} \Gamma_f(D, V).
\]

Next, \( \Gamma_f(D, V) = \Gamma_{1w, h}^{(\eta_0)}(D, V) \otimes_{\mathcal{H}} \mathbb{Q}_p \) and for any \( i \) one has an exact sequence

\[
0 \to \Gamma_f^{(\eta_0)}(D, V)_{\Gamma_1} \to \Gamma_f(D, V) \to \Gamma_f^{(\eta_0)}(D, V)_{\Gamma_1} \to 0.
\]

From Theorem 5.1.3 it follows that

\[
\Gamma_f(D, V) = \begin{cases} 
\mathbb{R}^2 \Gamma_{1w, h}^{(\eta_0)}(D, V)_{\Gamma_1} & \text{if } i = 1 \\
\mathbb{R}^2 \Gamma_{1w, h}^{(\eta_0)}(D, V)_{\Gamma_1} & \text{if } i = 2 \\
0 & \text{if } i \neq 1, 2.
\end{cases}
\]

Therefore, the isomorphism \( \mathbb{R}^2 \Gamma_{1w, h}(D, V)_{\Gamma_1} \to \mathbb{R}^2 \Gamma_{1w, h}(D, V)_{\Gamma_1} \) induces a canonical trivialization

\[
\vartheta_{D, h} : \det_{\mathbb{Q}_p} \Gamma_f(D, V) \xrightarrow{\sim} \mathbb{Q}_p.
\]

By Lemma 5.2.6.2 we have a commutative diagram

\[
\begin{array}{ccc}
\det_{\mathcal{H}}^{-1} \Gamma_{1w, h}^{(\eta_0)}(D, V) & \xrightarrow{X^{-\epsilon_{V, 1w, h}}} & \mathcal{H}_0 \\
\downarrow \otimes_{\mathcal{H}} \mathbb{Q}_p & & \downarrow \\
\det_{\mathbb{Q}_p}^{-1} \Gamma_f(D, V) & \xrightarrow{\vartheta_{D, h}^{-1}} & \mathbb{Q}_p
\end{array}
\]
Since
\[ \Delta_{Iw,h}(N,T) \otimes_{\Lambda}^{L} \mathbb{Z}_p \simeq \Delta_{EP}(T) \otimes_{\mathbb{Z}_p} \omega_N \otimes_{\mathbb{Z}_p} (\omega^+_T)^{-1} \]
it implies that
\[ \vartheta_{D,h}^{-1} \circ \xi_{D,h}(\Delta_{EP}(T) \otimes_{\mathbb{Z}_p} \omega_N \otimes_{\mathbb{Z}_p} (\omega^+_T)^{-1}) = \log(\chi(\gamma))^{-e} L_{Iw,h}^{*}(T, N, 0) \mathbb{Z}_p. \quad (5.12) \]
Consider the diagram
\[
\begin{array}{c}
\begin{array}{c}
\mathbf{R} \Gamma_f(V) \longrightarrow \mathbf{R} \Gamma_S(V) \oplus \bigoplus_{v \in S \cup \{\infty\}} \mathbf{R} \Gamma_f(Q_v, V) \longrightarrow \bigoplus_{v \in S} \mathbf{R} \Gamma(Q_v, V) \\
\mathbf{R} \Gamma_{f,h}(D, V) \longrightarrow \mathbf{R} \Gamma_S(V) \oplus \bigoplus_{v \in S} \mathbf{R} \Gamma_f(Q_v, D, V) \longrightarrow \bigoplus_{v \in S} \mathbf{R} \Gamma(Q_v, V)
\end{array}
\end{array}
\]
where \( L = \text{cone} (\mathbf{R} \Gamma_{f,h}(D, V) \to \mathbf{R} \Gamma_f(V)) \to [-1] \) and the upper and middle rows coincide with (4.1) and (5.11) up to the following modification: the map \( \text{loc}_p : \mathbf{R} \Gamma_f(Q_p, V) \to \mathbf{R} \Gamma(Q_p, V) \) is replaced by \( \Gamma(h) \text{loc}_p \). Hence \( S \) is isomorphic to \( L \) in the derived category \( D^p(Q_p) \) and we have an exact triangle
\[ S \to \mathbf{R} \Gamma_{f,h}(D, V) \to \mathbf{R} \Gamma_f(V) \to S[1]. \]
An easy diagram search shows that \( H^1(S) \simeq \mathbf{R}^1 \Gamma_{f,h}(D, V) \) coincides with \( \text{id} : D^{\varphi=p^{-1}} \to D^{\varphi=p^{-1}} \) and that
\[ 0 \to H^1_f(V) \to H^2(S) \to \mathbf{R}^2 \Gamma_{f,h}(D, V) \to 0 \]
coincides with
\[ 0 \to H^1_f(V) \to \frac{D_{\text{cris}}(V)}{\text{Fil}^0 D_{\text{cris}}(V) + D_{-1}} \xrightarrow{\Gamma(h) \exp_p} \frac{H^1(Q_p, V)}{H^1_{f,\{p\}}(V) + H^1(D_{-1})} \to 0. \]
Therefore, we have a commutative diagram
\[
\begin{array}{c}
\begin{array}{c}
\text{det}_{Q_p} S \otimes \text{det}_{Q_p} \mathbf{R} \Gamma_f(V) \xrightarrow{\alpha} \text{det}_{Q_p} \mathbf{R} \Gamma_{f,h}(D, V) \\
\vartheta_S \downarrow \quad \downarrow \vartheta_{D,h} \\
Q_p \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qua
Next, (5.12) and (5.13) give

$$\Gamma(h)^{d_+(V)} \frac{i_{\omega_{M,p}}(\Delta_{EP}(T))}{\Omega_M^{(\ell,t,p)}(\omega_T^+, \omega_B^+)}.$$  \hspace{1cm} (5.15)

From Lemma 5.2.3 it follows that the composition of left vertical maps \( \vartheta_S = (\text{id} \otimes \beta_S) \) sends \( \Delta_{EP}(T) \otimes (\omega_T \otimes (\omega_T^+)^{-1} \otimes \omega_f) \) onto

$$\det^* \left( \frac{1 - p^{-1} \varphi - 1}{1 - \varphi} \left| D \right. \right)^{-1} E_p(V,1)^{-1} R_{V,D}(\omega_{V,N}) \Delta_{EP}(T) \otimes (\omega_N \otimes (\omega_T^+)^{-1})$$  \hspace{1cm} (5.16)

Next, (5.12) and (5.13) give

$$\vartheta_D^{-1} \circ (\xi_{D,h} \otimes \kappa)(\Delta_{EP}(T) \otimes \omega_N \otimes (\omega_T^+)^{-1}) = \left( 1 - \frac{1}{p} \right)^e \mathcal{L}(V,D)^{-1} L^*_{i_{w,h}}(T,N,0) \mathbb{Z}_p. \hspace{1cm} (5.17)$$

Putting together (5.15), (5.16) and (5.17) we obtain that

$$\frac{L^*_{i_{w,h}}(T,N,0)}{R_{V,D}(\omega_{V,N})} \sim_p \Gamma(h)^{d_+(V)} \mathcal{L}(V,D) E_p^*(V,1) \det_{\mathbb{Q}_p} \left( \frac{1 - p^{-1} \varphi - 1}{1 - \varphi} \left| D \right. \right) \frac{i_{\omega_{M,p}}(\Delta_{EP}(T))}{\Omega_M^{(\ell,t,p)}(\omega_T^+, \omega_B^+)}$$

and the theorem is proved.

5.2.7. Special values of \( L^*_{i_{w,h}}(T,N,s) \). Let \( \tilde{H}^1_f(T) \) denote the image of \( H^1_f(T) \) in \( H^1_f(V) \) and let \( \omega_{T,f} \) be a base of \( \det_{\mathbb{Z}_p} \tilde{H}^1_f(T) \). Let \( R_{V,D}(\omega_{T,N}) \) denote the determinant of \( r_{V,D} \) computed in the bases \( \omega_{T,m}, \omega_N \) and \( \omega_{T,f} \).

Corollary 5.2.8. Under the assumptions of Theorem 5.2.5 one has

$$\frac{L^*_{i_{w,h}}(T,N,0)}{R_{V,D}(\omega_{T,N})} \sim_p \Gamma(h)^{d_+(V)} \mathcal{L}(V,D) E^+(V,D) \frac{\# \mathcal{W}(T^*(1)) \text{ Tam}^0_{\omega_M}(T)}{\# H^0_S(V/T) \# H^0_S(V^*(1)/T^*(1))},$$

where \( \mathcal{W}(T^*(1)) \) is the Tate-Shafarevich group of Bloch-Kato [BK] and \( \text{ Tam}^0_{\omega_M}(T) \) is the product of local Tamagawa numbers of \( T \) taken over all primes and computed with respect to a fixed base \( \omega_{T,m} \) of \( \det_{\mathbb{Q}_M}(\mathbb{Q}) \).

Proof. The computation of the trivialisation of the Euler-Poincaré line (see for example [FP], chapitre II, Théorème 5.6.3) together with the definition of \( i_{\omega_{M,p}} \) by (4.12) give

$$i_{\omega_{M,p}}(\Delta_{EP}(T)) = \frac{\# \mathcal{W}(T^*(1)) \text{ Tam}^0_{\omega_M}(T)}{\# H^0_S(V/T) \# H^0_S(V^*(1)/T^*(1))} \frac{\Omega_M^{(\ell,t,p)}(\omega_T^+, \omega_B^+)}{\omega_f : \omega_{T,f}}.$$

Since \( R_{V,D}(\omega_{T,N}) = R_{V,D}(\omega_{V,N}) \left[ \omega_f : \omega_{T,f} \right] \) the corollary follows from Theorem 5.2.5.

5.2.9. The functional equation. Recall that we set \( h_i(V) = \dim_{\mathbb{Q}_p}(\text{ gr } D_{\text{dr}}(V)) \) and \( m = \sum_i h_i(V) \). Since \( V \) is crystalline, \( \det_{\mathbb{Q}_p}(V) \) is a one dimensional crystalline representation and \( \det_{\mathbb{Z}_p}(T) = T_0(m) \) where \( T_0 \) is an unramified \( G_{\mathbb{Q}_p} \)-module of rank 1 over \( \mathbb{Z}_p \). The module \( (T_0 \otimes \)
where \( r = (t^{-1} \otimes \varepsilon)^{\otimes m} \) is a \( \mathbb{Z}_p \)-lattice in \( \det_{\mathbb{Q}_p}(D_{\text{cris}}(V)) = D_{\text{cris}}(V_0(m)) \) which depends only on \( T \) and which we denote by \( D_{\text{cris}}(T_0(m)) \).

Let \( D^\perp \) be the dual regular module. The exact sequence
\[
0 \to D \to D_{\text{cris}}(V) \to (D^\perp)^* \to 0
\]
gives an isomorphism
\[
\det_{\mathbb{Q}_p} D \otimes \det_{\mathbb{Q}_p}^{-1} D^\perp \simeq \det_{\mathbb{Q}_p} D_{\text{cris}}(V)
\]
and we fix a lattice \( N^\perp \subset D^\perp \) such that
\[
\det_{\mathbb{Z}_p} N \otimes \det_{\mathbb{Z}_p}^{-1} N^\perp \simeq D_{\text{cris}}(T_0(m)).
\]
Set \( \Gamma_{V,h}(s) = \prod_{j \geq -h} (j + s)^{\dim \text{Fil}^j D_{\text{an}}(V)}. \) The conjecture \( \delta_{\mathbb{Z}_p}(V) \) of [PR1] proved in [BB] implies that for \( h \gg 0 \)
\[
L_{Iw,h}(T^*(1), N^\perp, -s) \sim_{\Lambda^*} \Gamma_{V,h}(s) \prod_{-h < j < h} (j + s)^{d_+ (V^*(1))} L_{Iw,h}(T, N, s)
\]
(see [PR2], Théorème 2.5.2). This can seen as the algebraic counterpart of the functional equation for \( p \)-adic \( L \)-functions. An elementary computation (see [BB], Lemme 4.7) shows that
\[
\Gamma_{V,h}(s) \prod_{-h < j < h} (j + s)^{d_+ (V^*(1))} = \Gamma^*(V) \Gamma(h)^{d_+(V^*(1)) - d_+(V)^s r} + o(s^r).
\]
where \( r = \dim_{\mathbb{Q}_p} t_V(\mathbb{Q}_p) - d_+(V) = \dim_{\mathbb{Q}_p} H^1_J(V) \) and \( \Gamma^*(V) \) is defined by (4.14). Therefore \( L_{Iw,h}(T^*(1), N^\perp, s) \) has a zero of order \( \dim_{\mathbb{Q}_p} H^1_J(V) + e \) at \( s = 0 \). Moreover one has
\[
\frac{L_{Iw,h}^*(T^*(1), N^\perp, 0)}{\Gamma(h)^{d_+(V^*(1))}} \sim_p \frac{\Gamma^*(V)}{\Gamma(h)^{d_+(V)}} L_{Iw,h}(T, N, 0).
\]

From the definition of \( R_{V^*(1), D^\perp} \) (see section 4.2.1) one obtains easily that \( R_{V^*(1), D^\perp} (\omega_{V^*(1), N^\perp}) = \Omega_{M}^{(H,p)}(\omega_T, \omega_{M_{\text{IR}}})^{-1} R_{V,D}(\omega_V, N) \) where \( \Omega_{M}^{(H,p)} \) denotes the period map defined (4.8) and (4.9) and \( \omega_{M_{\text{IR}}} = \omega_{M}^{\perp} \otimes \omega_{E_{M_{\text{IR}}}}^{-1} \). Taking into account (4.15) we obtain that
\[
\frac{L_{Iw,h}^*(T^*(1), N^\perp, 0)}{R_{V^*(1), D^\perp} (\omega_V, N^\perp)} \sim_p \frac{\Gamma(h)^{d_+(V^*(1))} \mathcal{L}(V, D) \mathcal{E}^+(V^*(1), D^\perp)}{\Gamma(h)^{d_+(V)}} \frac{i_{\omega_{M_{\text{IR}}}}(\Delta_{\text{EP}}(T^*(1)))}{\Omega_{M}^{(H,p)}(\omega_T, \omega_{M_{\text{IR}}})}
\]
which is the analog of Theorem 5.2.5 for \( L_{Iw,h}(T^*(1), N^\perp, s) \).

**Appendix. Galois cohomology of \( p \)-adic representations**

**A.1.** Let \( K \) be a finite extension of \( \mathbb{Q}_p \) and \( T \) a \( p \)-adic representation of \( G_K \). Fix a topological generator \( \gamma \) of \( \Gamma \). Let \( D(T) = (T \otimes_{\mathbb{Z}_p} A)^{H_K} \) be the \((\varphi, \Gamma)\)-module associated to \( T \) by Fontaine’s theory [F2]. Consider the complex
\[
C_{\varphi, \gamma}(D(T)) = \left[D(T) \xrightarrow{f} D(T) \oplus D(T) \xrightarrow{g} D(T)\right]
\]
where the modules are placed in degrees 0, 1 and 2 and the maps \( f \) and \( g \) are given by
\[
f(x) = ((\varphi - 1)x, (\gamma - 1)x), \quad g(y, z) = (\gamma - 1)y - (\varphi - 1)z.
\]
Proposition A.2. There are canonical and functorial isomorphisms
\[ h^i : H^i(C_{\varphi,\gamma}(D(T))) \xrightarrow{\sim} H^i(K, T) \]
which can be described explicitly by the following formulas:
i) If \( i = 0 \), then \( h^0 \) coincides with the natural isomorphism
\[ D(T)_{\varphi=1, \gamma=1} = H^0(K, T \otimes_{\mathbb{Z}_p} A_{\varphi=1}) = H^0(K, T). \]
ii) Let \( \alpha, \beta \in D(T) \) be such that \((\gamma - 1)\alpha = (1 - \varphi)\beta\). Then \( h^1 \) sends \( \text{cl}(\alpha, \beta) \) to the class of the 2-cocycle
\[ \mu_1(g) = (g - 1)x + \frac{g - 1}{\gamma - 1}\beta, \]
where \( x \in D(T) \otimes_{A_K} A \) is a solution of the equation \((1 - \varphi)x = \alpha\).
iii) Let \( \hat{\gamma} \in G_K \) be a lifting of \( g \in \Gamma \) and let \( x \) be a solution of \((\varphi - 1)x = \alpha\). Then \( h^2 \) sends \( \alpha \) to the class of the 2-cocycle
\[ \mu_2(g_1, g_2) = \hat{\gamma}^{k_1}(h_1 - 1) \frac{\hat{\gamma}^{k_2} - 1}{\gamma - 1}x \]
where \( g_i = \hat{\gamma}^{k_i}h_i, \ h_i \in H_K \).

Proof. The isomorphisms \( h^i \) were constructed in [H1], Theorem 2.1. Remark that i) follows directly from this construction (see [H1], p.573) and that ii) is proved in [Ben1], Proposition 1.3.2 and [CC2], Proposition I.4.1. The proof of iii) follows along exactly the same lines. Namely, it is enough to prove this formula modulo \( p^n \) for each \( n \). Let \( \alpha \in D(T)/p^nD(T) \). By Proposition 2.4 of [H1] there exists \( r \geq 0 \) and \( y \in D(T)/p^nD(T) \) such that \((\varphi - 1)\alpha = (\gamma - 1)^r\beta\). Let
\[ N_x = (D(T)/p^nD(T)) \oplus (\oplus_{i=1}^r(A_K/p^nA_K) t_i), \]
where \( \varphi(t_i) = t_i + (\gamma - 1)^{r-i}(\alpha) \) and \( \gamma(t_i) = t_i + t_{i-1} \). Then \( N_x \) is a \((\varphi, \Gamma)\)-module and we have a short exact sequence
\[ 0 \rightarrow D \rightarrow N_x \rightarrow X \rightarrow 0 \]
where \( X = N_x/M \simeq \oplus_{i=1}^r A_K/p^nA_K t_i \). An easy diagram search shows that the connecting homomorphism \( \delta_D : H^1(C_{\varphi,\gamma}(D(X))) \rightarrow H^2(C_{\varphi,\gamma}(D(T))) \) sends \( \text{cl}(0, t_r) \) to \( -\text{cl}(\alpha) \). The functor \( V(D) = (D \otimes_{A_K} A)^{\varphi=1} \) is a quasi-inverse to \( D \). Thus one has an exact sequence of Galois modules
\[ 0 \rightarrow T/p^nT \rightarrow T_x \rightarrow V(X) \rightarrow 0 \]
where \( T_x = V(N_x) \). From the definition of \( x \) it follows immediately that \( t_r - x \in T_x \). By ii), \( h^1(\text{cl}(0, t_r)) \) can be represented by the cocycle \( c(g) = \frac{g - 1}{\gamma - 1}t_r \) and we fix its lifting \( \hat{c} : G_K \rightarrow N_x \) putting \( \hat{c}(g) = \frac{g - 1}{\gamma - 1}(t_r - x) \). As \( g_1\hat{c}(g_2) - \hat{c}(g_1g_2) + \hat{c}(g_1) = -\mu_2(g_1, g_2) \), the connecting map \( \delta^1_1 : H^1(K, V(X)) \rightarrow H^2(K, T/p^nT) \) sends \( \text{cl}(c) \) to \( -\text{cl}(\mu_2) \) and iii) follows from the commutativity of the diagram
\[ \begin{array}{ccc}
H^1(C_{\varphi,\gamma}(X)) & \xrightarrow{\delta_1^1} & H^2(C_{\varphi,\gamma}(T/p^nT)) \\
\downarrow h^1 & & \downarrow h^2 \\
H^1(K, V(X)) & \xrightarrow{\delta_2^1} & H^2(K, T/p^nT).
\end{array} \]
**Proposition A.3.** The complexes $R\Gamma(K, T)$ and $C_{\varphi, \gamma}(T)$ are isomorphic in $D(\mathbb{Z}_p)$.

**Proof.** The proof is standard (see for example [BF], proof of Proposition 1.17). The exact sequence

$$0 \to T \to D(T) \otimes_{A_K} A \xrightarrow{\varphi^{-1}} D(T) \otimes_{A_K} A \to 0$$

gives rise to an exact sequence of complexes

$$0 \to C_c^*(G_K, T) \to C_c^*(G_K, D(T) \otimes_{A_K} A) \xrightarrow{\varphi^{-1}} C_c^*(G_K, D(T) \otimes_{A_K} A) \to 0$$

Thus $R\Gamma(K, T)$ is quasi-isomorphic to the total complex

$$K^*(T) = \text{Tot}^* \left( C_c^*(G_K, D(T) \otimes_{A_K} A) \xrightarrow{\varphi^{-1}} C_c^*(G_K, D(T) \otimes_{A_K} A) \right).$$

On the other hand $C_{\varphi, \gamma}(T) = \text{Tot}^* \left( A^*(T) \xrightarrow{\varphi^{-1}} A^*(T) \right)$, where $A^*(T) = [D(T) \xrightarrow{\gamma^{-1}} D(T)]$. Consider the following commutative diagram of complexes

\[
\begin{array}{ccc}
D(T) & \xrightarrow{\gamma^{-1}} & D(T) \\
\downarrow \beta_0 & & \downarrow \beta_1 \\
C^0(G_K, D(T) \otimes_{A_K} A) & \to & C^1(G_K, D(T) \otimes_{A_K} A) \\
\downarrow & & \downarrow \\
\cdots & & \cdots
\end{array}
\]

in which $\beta_0(x) = x$ viewed as a constant function on $G_K$ and $\beta_1(x)$ denotes the map $G_K \to D(T) \otimes_{A_K} A$ defined by $(\beta_1(x))(g) = \frac{g^{-1}}{\gamma - 1} x$. This diagram induces a map $\text{Tot}^* \left( A^*(T) \xrightarrow{\varphi^{-1}} A^*(T) \right) \to K^*(T)$ and we obtain a diagram

$$C_{\varphi, \gamma}(T) \to K^*(T) \leftarrow R\Gamma(K, T)$$

where the right map is a quasi-isomorphism. Then for each $i$ one has a map

$$H^i(C_{\varphi, \gamma}(T)) \to H^i(K^*(T)) \simeq H^i(K, T)$$

and an easy diagram search shows that it coincides with $h^i$. The proposition is proved.

**Corollary A.4.** Let $V$ be a $p$-adic representation of $G_K$. Then the complexes $R\Gamma(K, V)$, $C_{\varphi, \gamma}(D(V^\dagger))$ and $C_{\varphi, \gamma}(D_{\text{rig}}(V))$ are isomorphic in $D(\mathbb{Q}_p)$.

**Proof.** This follows from Theorem 1.1 of [Li] together with Proposition A.2.

**A.5.** Recall that $K_\infty/K$ denotes the cyclotomic extension obtained by adjoining all $p^n$-th roots of unity. Let $\Gamma = \text{Gal}(K_\infty/K)$ and let $A(\Gamma) = \mathbb{Z}_p[[\Gamma]]$ denote the Iwasawa algebra of $\Gamma$. For any $\mathbb{Z}_p$-adic representation $T$ of $G_K$ the induced representation $\text{Ind}_{K_\infty/K} T$ is isomorphic to $(T \otimes_{\mathbb{Z}_p} A(\Gamma))^\dagger$ and we set $R\Gamma_{\text{Iw}}(K, T) = C_c^*(G_K, \text{Ind}_{K_\infty/K} T)$.

Consider the complex

$$C_{\text{Iw}, \psi}(T) = \left[ D(T) \xrightarrow{\psi^{-1}} D(T) \right]$$

in which the first term is placed in degree 1.
Proposition A.6. There are canonical and functorial isomorphisms
\[ h^1_{\text{lw}} : H^i(C_{1w, \psi}(T)) \to H^i_{\text{lw}}(K, T) \]
which can be described explicitly by the following formulas:
i) Let \( \alpha \in D(T)^{\psi=1} \). Then \( (\varphi-1) \alpha \in D(T)^{\psi=0} \) and for any \( n \) there exists a unique \( \beta_n \in D(T) \) such that \( (\gamma_n - 1) \beta_n = (\varphi-1) \alpha \). The map \( h^1_{\text{lw}} \) sends \( \cl(\alpha) \) to \( (h^1_n(\cl(\beta_n, \alpha)))_{n \in \mathbb{N}} \in H^1_{\text{lw}}(K_n, T) \).
ii) If \( \alpha \in D(T) \), then \( h^2_{\text{lw}}(\cl(\alpha)) = -(h^2_n(\varphi(\alpha)))_{n \in \mathbb{N}} \).

Proof. The proposition follows from Theorem II.1.3 and Remark II.3.2 of [CC2] together with Proposition A.2.

Proposition A.7. The complexes \( R\Gamma_{\text{lw}}(K, T) \) and \( C_{1w, \psi}(T) \) are isomorphic in the derived category \( D(\Lambda(\Gamma)) \).

Proof. We repeat the arguments used in the proof of Proposition A.1.2 with some modifications. For any \( n \geq 1 \) there is an exact sequence
\[ 0 \to \Ind_{K_n/K}T \to (D(T) \otimes_{\mathbb{Z}} \mathbb{Z}_p[G_n]^t) \otimes_{\Lambda_K} A \xrightarrow{\varphi-1} (D(T) \otimes_{\mathbb{Z}} \mathbb{Z}_p[G_n]^t) \otimes_{\Lambda_K} A \to 0. \]
Set \( D(\Ind_{K_\infty/K}T) = D(T) \otimes_{\mathbb{Z}} \Lambda(\Gamma)^t \) and
\[ D(\Ind_{K_\infty/K}(T))^\wedge_{\Lambda_K} A = \varprojlim_n (D(T) \otimes_{\mathbb{Z}} \mathbb{Z}_p[G_n]^t) \otimes_{\Lambda_K} A. \]
As \( \Ind_{K_n/K}T \) are compact, taking projective limit one obtains an exact sequence
\[ 0 \to \Ind_{K_\infty/K}T \to D(\Ind_{K_\infty/K}(T))^\wedge_{\Lambda_K} A \xrightarrow{\varphi-1} D(\Ind_{K_\infty/K}(T))^\wedge_{\Lambda_K} A \to 0. \]
Thus \( R_{1w}(K, T) \) is quasi-isomorphic to
\[ K_{1w}^\bullet(T) = \text{Tot}^\bullet \left( C_c^\bullet(G_K, D(\Ind_{K_\infty/K}T)^\wedge_{\Lambda_K} A) \xrightarrow{\varphi-1} C_c^\bullet(G_K, D(\Ind_{K_\infty/K}T)^\wedge_{\Lambda_K} A) \right). \]
We construct a quasi-isomorphism \( f_* : C_{1w, \psi}(T) \to K_{1w}^\bullet(T) \). Any \( x \in D(T) \) can be written in the form \( x = (1 - \varphi \psi) x + \varphi \psi x \) where \( \psi(1 - \varphi \psi) x = 0 \). Then for each \( n \geq 0 \) the equation \( (\gamma_n - 1) y_n = (\varphi \psi - 1) x \) has a unique solution \( y_n \in D(T)^{\psi=0} \) ([CC2], Proposition I.5.1). In particular, \( y_n = \frac{\gamma_{n+1} - 1}{\gamma_n - 1} y_{n+1} \) and we have a compatible system of elements
\[ Y_n = \sum_{k=0}^{[G_n]} \gamma^k \otimes \gamma^k(y_n) \in D(T) \otimes_{\mathbb{Z}} \mathbb{Z}_p[G_n]^t. \]
Put \( Y = (Y_n)_{n \geq 0} \in D(\Ind_{K_\infty/K}T) \). Then
\[ (\gamma_n - 1) Y_n = (\gamma - 1) Y \pmod{D(\Ind_{K_n/K}T)}. \]
Let \( \eta_x \in C_c^1(G_K, D(\Ind_{K_\infty/K}T)^\wedge_{\Lambda_K} A) \) be the map defined by \( \eta_x(g) = \frac{g - 1}{\gamma - 1} (1 \otimes x) \). Define \( f_1 : D(T) \to K_{1w}^1(T) = C_c^0(G_K, D(\Ind_{K_\infty/K}T)^\wedge_{\Lambda_K} A) \oplus C_c^1(G_K, D(\Ind_{K_\infty/K}T)^\wedge_{\Lambda_K} A) \) by \( f_1(x) = (Y, \eta_x) \) and \( f_2 : D(T) \to C_c^1(G_K, D(\Ind_{K_\infty/K}T)^\wedge_{\Lambda_K} A) \subset K_{1w}^2(T) \) by \( f_2(z) = -\eta_{\varphi(z)} \). It is easy to check that \( f_* \) is a morphism of complexes. This gives a diagram
\[ C_{1w, \psi}(T) \to K_{1w}^\bullet(T) \xleftarrow{R\Gamma_{1w}(K, T)} \]
in which the right map is a quasi-isomorphism. Using Proposition A.1.4 it is not difficult to check that for each \( i \) the induced map
\[ H^i(C_{1w, \psi}(T)) \to H^i(K_{1w}^\bullet(T)) \simeq H^i_{1w}(K, T) \]
coincides with \( h^i_{1w} \). The proposition is proved.
Corollary A.8. The complexes $\mathcal{R} \Gamma_{IW}(K,T)$ and $C_{\psi}(T)$ are isomorphic in $\mathcal{D}(\Lambda(\Gamma))$.

Proof. One has $D^{\dagger}(T)^{\psi=1} = D(T)^{\psi=1}$ ([C1], Proposition 3.3.2) and $D^{\dagger}(T)/(\psi-1) = D(T)/(\psi-1)$ ([L], Lemma 3.6). This shows that the inclusion $C_{\psi}(T) \rightarrow D(T)^{\psi=1}$ is a quasi-isomorphism.

Remark A.9. These results can be slightly improved. Namely, set $r_n = (p-1)p^{n-1}$. The method used in the proof of Proposition III.2.1 [C2] allows to show that $\psi(D^{\dagger}r_n(T)) \subset D^{\dagger}r_n-1(T)$ for $n \gg 0$. Moreover, for any $a \in D^{\dagger}r_n(T)$ the solutions of the equation $(\psi - 1)x = a$ are in $D^{\dagger}r_n(T)$. Thus $C_{\psi}^{\dagger}r_n(T) = \left[D^{\dagger}r_n(T) \xrightarrow{\psi-1} D^{\dagger}r_n(T)\right]$, $n \gg 0$ is a well-defined complex which is quasi-isomorphic to $C_{\psi}(T)$. Further, as $\varphi(A^{\dagger}r/p) = A^{\dagger}r$ we can consider the complex

$$C_{\varphi,\gamma}^{\dagger}r_n(T) = \left[D^{\dagger}r_n-1(T) \xrightarrow{f} D^{\dagger}r_n(T) \oplus D^{\dagger}r_n-1(T) \xrightarrow{g} D^{\dagger}r_n(T)\right], \quad n \gg 0$$

in which $f$ and $g$ are defined by the same formulas as before. Then the inclusion $C_{\varphi,\gamma}^{\dagger}r_n(T) \rightarrow C_{\varphi,\gamma}(T)$ is a quasi-isomorphism.

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