Perturbed Gauged WZNW Models

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Abstract

We discuss a new type of unitary perturbations around conformal theories inspired by the \(\sigma\)-model perturbation of the nonunitary WZNW model. We show that the nonunitary level \(k\) WZNW model perturbed by its sigma model term goes to the unitary level \(-k\) WZNW model. When plugged into the gauged WZNW model the given perturbation results in the perturbed gauged WZNW model which no longer describes a coset construction. We consider the BRST invariant generalization of the sigma model perturbation around the gauged WZNW model. In this way we obtain perturbed coset constructions. In the case of the \(SU_{m-2}(2) \times SU_1(2)/SU_{m-1}(2)\) coset, the BRST invariant sigma model perturbation is identical to Zamolodchikov’s \(\Phi_{(3,1)}\) perturbation of the minimal conformal series. The existence of general geometry flows is clarified.

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1 Introduction

There has recently been much interest attracted to gauged Wess-Zumino-Novikov-Witten (GWZNW) models [1]-[5] which turn out to give rise to exact (to all orders in $\alpha'$) nontrivial string solutions [6]-[11]. Such an outstanding situation occurs because of the fact that GWZNW models provide a proper Lagrangian formulation to algebraic coset constructions [15]-[16] which are exactly solvable conformal field theories. On the other hand there are examples of many other exact conformal field theories which are distinct from cosets. One possible route away from coset constructions may be through some relevant perturbations of gauged WZNW models. Therefore, there may exist some room for possibly new exact string backgrounds by perturbations of GWZNW models.

All string solutions are believed to be points in the coordinate space of string field theory [17], which is supposed to be independent of any particular choice of coordinates in analogy with general relativity. Such a string field theory formulation, though still being developed, apparently necessitates this coordinate space to be a connected multitude. Obviously conformal models alone cannot form any connected multitude of string solutions. Therefore, the whole configuration space of string field theory is going to be much more complex than the space of conformal models.

Some interesting properties of this configuration space have been analyzed in [18]-[19]. The main tools which appear to be very effective in elaborating the features of this highly nontrivial object are the methods of perturbed conformal field theory [20]-[23]. From the obtained results, the coordinate space of string field theory comes along to have the structure of a group-like manifold whose local properties are determined by the algebra of all integrable relevant deformations around conformal (Lagrangian) theories. Unfortunately the knowledge of the algebra of all relevant integrable perturbations around conformal Lagrangian (unitary) models is still incomplete. Accomplishing these tasks may turn out to need considerable effort.

In these notes we would like to consider some special relevant unitary deformations around (gauged) WZNW models some of which have a clear space-time interpretation. Some of the results were presented in [24]-[26]. In the present paper we will describe those
results in more details and will discuss a new type of relevant perturbations of the gauged WZNW model.

The perturbations we are going to exploit are generated by the sigma model term of the nonunitary WZNW model \[24\]. In spite of the nonunitarity of the underlying WZNW model, the perturbing operator provides one with a unitary Virasoro representation, as will be shown in section 2. The nonunitary WZNW model perturbed by this operator will be demonstrated to go to the unitary WZNW model. The theory, in which the nonunitary WZNW model emerges as an intrinsic element, is the GWZNW model. In section 3 we will apply our perturbation to the GWZNW model. In section 4 we will define a BRST invariant perturbing operator which may act on the coset projection of the GWZNW model. In this way we will obtain perturbed coset constructions. We will exhibit that in the case of the \( SU_{m-2}(2) \times SU_1(2)/SU_{m-1}(2) \) GWZNW model, our BRST invariant perturbation reproduces the known deformation of the minimal conformal models by the \( \Phi_{(3,1)} \) operator. In section 5 we will summarize our results and comment on them.

2 The sigma model perturbation of the nonunitary WZNW model

Our starting point is the level \( k \) WZNW model \[27]-[29]\ described by the action

\[
S_{WZNW}(g;k) = -\frac{k}{4\pi} \int [\text{Tr} |g^{-1}dg|^2 + \frac{i}{3} d^{-1}\text{Tr}(g^{-1}dg)^3],
\]

(2.1)

where \( g \) is the matrix field taking its values on the Lie group \( G \). The theory possesses the affine symmetry \( \hat{G} \times \hat{G} \) which entails an infinite number of conserved currents \[7,29\]. The latter can be derived from the basic currents \( J \) and \( \bar{J} \),

\[
J = J^a t^a = -\frac{k}{2} g^{-1} \partial g,
\]

(2.2)

\[
\bar{J} = \bar{J}^a t^a = -\frac{k}{2} \bar{\partial} g \cdot g^{-1},
\]

satisfying the equations of motion

\[
\bar{\partial} J = 0, \quad \partial J = 0.
\]

(2.3)
In eqs. (2.2) $t^a$ are the generators of the Lie algebra $\mathcal{G}$ associated with the Lie group $G$,

$$[t^a, t^b] = f^{abc} t^c,$$  \hspace{1cm} (2.4)

with $f^{abc}$ the structure constants.

The important observation which has been made in [29] is that the spectrum of the WZNW model contains states which correspond to the primary fields of the underlying affine symmetry. By definition, $\phi_i$ is an affine primary field, if it has the following operator product expansion (OPE) with the affine current $J_{[29]}$

$$J^a(w) \phi_i(z, \bar{z}) = \frac{t^a_i}{w - z} \phi_i(z, \bar{z}) + \text{reg.},$$  \hspace{1cm} (2.5)

where the matrices $t^a_i$ correspond to the left representation of $\phi_i(z, \bar{z})$. In the WZNW model, any affine primary field is Virasoro primary and its conformal dimensions are given by [29]

$$\Delta_i = \frac{c_i}{c_V + k}, \quad \bar{\Delta}_i = \frac{\bar{c}_i}{c_V + k},$$  \hspace{1cm} (2.6)

where $c_i = t^a_i t^a_i$, $\bar{c}_i = \bar{t}^a_i \bar{t}^a_i$ and $c_V$ is defined according to

$$f^{acd} f^{bcd} = c_V \delta^{ab}.$$  \hspace{1cm} (2.7)

The point to be made is that there are Virasoro primary fields in the spectrum of the WZNW model which are not affine primary fields, but their descendants. Our examples of such fields will be restricted to one particular composite state.

Let us consider the following composite field

$$O =: J^a \bar{J}^\bar{a} \phi^{a\bar{a}} :$$  \hspace{1cm} (2.8)

which is defined as a normal ordered product of the affine currents $J^a$, $\bar{J}^\bar{a}$ with the spin (1,1) affine-Virasoro primary field in the adjoint representation of $G \times G$. The product of the three operators in eq. (2.8) can be properly defined according to

$$O(z, \bar{z}) = \oint \frac{dw}{2\pi i} \oint \frac{d\bar{w}}{2\pi i} \frac{J^a(w) \cdot \bar{J}^\bar{a}(\bar{w}) \cdot \phi^{a\bar{a}}(z, \bar{z})}{|z - w|^2},$$  \hspace{1cm} (2.9)

where in the numerator the product is understood as an OPE. It is easy to see that the given product does not contain singular terms.
From the definition it follows that the operator $O$ is an affine descendant of the affine-Virasoro primary field $\phi$. Indeed, $O$ can be presented in the form

$$O(0) = J^a_{-1} \bar{J}^{\bar{a}}_{-1} \phi^{a\bar{a}}(0),$$

(2.10)

where

$$J^a_{-1} = \oint \frac{dw}{2\pi i} w^{-1} J^a(w), \quad \bar{J}^{\bar{a}}_{-1} = \oint \frac{d\bar{w}}{2\pi i} \bar{w}^{-1} \bar{J}^{\bar{a}}(\bar{w}).$$

(2.11)

At the same time, the operator $O$ continues to be a Virasoro primary operator. Indeed, one can check that the state $O(0)|0\rangle$ is a highest weight vector of the Virasoro algebra, with $|0\rangle$ the $SL(2, C)$ invariant vacuum. That is,

$$L_0 O(0)|0\rangle = \Delta O(0)|0\rangle,$$

(2.12)

$$L_{m>0} O(0)|0\rangle = 0.$$  

Here the generators $L_n$ are given by the contour integrals

$$L_n = \oint \frac{dw}{2\pi i} w^{n+1} T(w),$$

(2.13)

where $T(w)$ is holomorphic component of the Sugawara stress tensor of the conformal WZNW model,

$$T(z) = \frac{J^a(z) J^a(z)}{k + c_V}.$$

(2.14)

In eqs. (2.12), $\Delta$ is the conformal dimension of the operator $O$. We find

$$\Delta = \bar{\Delta} = 1 + \frac{c_V}{k + c_V}.$$  

(2.15)

Here $\bar{\Delta}$ is the conformal dimension of $O$ associated with antiholomorphic conformal transformations.

Let us turn to the large $k$ limit. Then the operator $O$ becomes a quasimarginal operator with anomalous dimensions

$$\Delta = \bar{\Delta} = 1 + \frac{c_V}{k} + O(k^{-2}).$$

(2.16)

When $k$ is negative, the given operator $O$ is relevant; whereas for positive $k$ it is an irrelevant operator. Relevant quasimarginal operators are of a great interest because they can be used as perturbing operators around given conformal theories.

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The point to be made is that the WZNW model with negative $k$ is a nonunitary theory, that is there are states with negative norms in the spectrum and the Hamiltonian is unbounded below. At the same time the theory is not completely meaningless. We will show that there are states which have positive norms with respect to the $SL(2, C)$ vacuum. Thus, there exists a subsector in which the theory can be properly defined by means of a certain truncation of the fusion algebra. Note that $|k|$ is thought of being integer and, hence, such a truncation must exist.

In the large negative $k$ limit, the operator $O$ appears to be suitable for performing a perturbation around the nonunitary WZNW model. The significant point to be made is that despite the nonunitarity of the conformal model to be perturbed, the perturbation operator $O$ corresponds to a unitary highest weight vector of the Virasoro algebra. Indeed, as we have shown above, $O$ has positive conformal dimensions. In turn, the Virasoro central charge of the nonunitary WZNW model in the large negative level limit is bigger than one, 

$$c_{WZNW}(k) = \frac{k \dim G}{k + c_V} = \dim G + O(1/k) > 1.$$  

(2.17)

In fact, the operator $O$ lies in the unitary range of the Kac determinant and, hence, it provides a unitary representation of the Virasoro algebra.

Another important property of the operator $O$ is that it obeys the following fusion rule

$$O \cdot O = [O] + [I] + \ldots,$$  

(2.18)

where the square brackets denote the contributions of $O$ and identity operator $I$ and the corresponding descendants of $O$ and $I$; whereas dots stand for operators with conformal dimensions greater than one. The fusion given by eq. (2.18) is easy to prove in the large $k$ limit. The proof is as follows. The limit of large level corresponds to the classical limit of the conformal WZNW model. In this limit, the affine-Virasoro primary field $\phi^{a\bar{a}}$ can be naturally identified with the classical composite field

$$\phi^{a\bar{a}} = \text{Tr}(g^{-1}t^at^{\bar{a}}).$$  

(2.19)

*In fact, there are many other operators in the spectrum of the nonunitary WZNW model which correspond to unitary highest weight states of the Virasoro algebra.*
So the operator $O$ goes to the classical function

$$O = -\frac{k^2}{4} \text{Tr}(\partial g \bar{g} g^{-1}), \quad (2.20)$$

which is the sigma model term. Clearly the operator $O$ preserves explicitly the global $G \times G$ symmetry of the conformal WZNW model. Therefore, on the right hand side of eq. (2.18) there must appear only primaries and their descendants which are scalars under the given symmetry. Apparently these primaries are to be built up of the field $g$ and derivatives. Then, it is not difficult to convince oneself that there are no other relevant primary operators but $O$ and $I$. Otherwise, the renormalizability of the corresponding sigma model would be broken.

All in all, the properties of the operator $O$ are sufficient to give a unitary renormalizable perturbation of the nonunitary conformal WZNW model. From now on by the perturbed nonunitary WZNW model we will understand the following theory

$$S_{PWZNW}(\epsilon) = S_{WZNW}(g, k) - \epsilon \int \text{d}^2 z \, O(z, \bar{z}), \quad (2.21)$$

where $\epsilon$ is a small parameter measuring a deviation from the conformal model $S_{WZNW}(g, k)$.

We go on to compute the renormalization beta function associated to the coupling $\epsilon$. Away of criticality, where $\epsilon \neq 0$, the beta function is defined according to [20]-[23],[30]

$$\beta = [2 - (\Delta + \bar{\Delta})] \epsilon - \pi C \epsilon^2 + O(\epsilon^3), \quad (2.22)$$

where $(\Delta, \bar{\Delta})$ are given by eq. (2.16). The constant $C$ is taken here to be the coefficient of the three point function

$$\langle O(z_1, \bar{z}_1)O(z_2, \bar{z}_2)O(z_3, \bar{z}_3) \rangle = C \|O\|^2 \prod_{i<j} \frac{1}{|z_{ij}|^{2\Delta}}, \quad (2.23)$$

where $\|O\|^2 = \langle O(1)O(0) \rangle$. Let us emphasize that the two and three point functions are calculated with respect to the $SL(2, C)$ vacuum.\footnote{It is necessary to point out that there are two distinct directions in which the perturbation can be performed: $\epsilon < 0$ and $\epsilon > 0$. In what follows we will discuss the perturbation(s) in the (massless) direction ($\epsilon > 0$).}
One can easily solve equation (2.22) to find critical points of the beta function. There are two solutions

\[ \epsilon_1 = 0, \quad \epsilon_2 = -\frac{2c_V}{\pi C k}. \] (2.24)

The first one obviously corresponds to the unperturbed WZNW model; whereas the second solution signifies a new (infrared) conformal point in the vicinity of the ultraviolet fixed point, \( \epsilon_1 = 0 \). In order to understand the meaning of the second critical point, one has to compute the coefficient \( C \) in eq. (2.23).

Computation of coefficients of three point functions of primary fields is one of the most difficult technical problems in conformal field theory. However, in the case under consideration, we need only calculate it to leading order in \( 1/k \). This can be done relatively easily. First of all, by using the definition of \( O \) given by eq. (2.9) we can present the three point function in eq. (2.23) in the form

\[ \langle O(z_1, \bar{z}_1)O(z_2, \bar{z}_2)O(z_3, \bar{z}_3) \rangle = \langle \Pi_{i<j}^3 \int \frac{dw_i}{2\pi i} \int \frac{d\bar{w}_i}{2\pi i} \frac{1}{|w_i - z_i|^2} J^a_i(w_i) \bar{J}^{\bar{a}}_i(\bar{w}_i) \phi^{a\bar{a}}(z_i, \bar{z}_i) \rangle. \] (2.25)

The right hand side of the last equation can be simplified with the Ward identities to the following compact expression

\[ \langle O(z_1, \bar{z}_1)O(z_2, \bar{z}_2)O(z_3, \bar{z}_3) \rangle = k^2 \frac{f^{abc} f^{\bar{a}\bar{b}\bar{c}}}{4} C_{\phi\phi\phi}^{a\bar{b}c} \frac{1}{\Pi_{i<j}^3 |z_{ij}|^{2\Delta}}. \] (2.26)

where \( C_{\phi\phi\phi} \) is the coefficient of the three point function

\[ \langle \phi^{a\bar{a}}(z_1, \bar{z}_1)\phi^{b\bar{b}}(z_2, \bar{z}_2)\phi^{c\bar{c}}(z_3, \bar{z}_3) \rangle = M C_{\phi\phi\phi}^{a\bar{b}c} \frac{1}{\Pi_{i<j}^3 |z_{ij}|^{2\Delta}}. \] (2.27)

The factor \( M \) in eqs. (2.26), (2.27) is the matter of the \( \phi \)-field normalization, \( \langle \phi^{a\bar{a}}(1)\phi^{b\bar{b}}(0) \rangle = M \delta^{ab}\delta^{\bar{a}\bar{b}} \). It is necessary to emphasize that formula (2.26) is only given to leading order in \( 1/k \).

In order to compute the coefficient on the right hand side of eq. (2.26), one can use the following useful observation. It turns out that in the large \( |k| \) limit the operator

\[ K^a = :\phi^{a\bar{a}}\bar{J}^{\bar{a}} := \text{Tr}(-\frac{k}{2} g^{-1} \bar{\partial} g t^a) \] (2.28)
acquires the canonical dimension (0,1). Therefore, in the quasiclassical limit $K^a$ must behave as a current. In particular, the OPE of $K^a$ with $\phi^{b\bar{b}}$ has to be as follows:

$$K^a(w, \bar{w})\phi^{b\bar{b}}(z, \bar{z}) = \frac{f_{abc}}{\bar{w} - \bar{z}} \phi^{c\bar{b}}(z, \bar{z}) + \text{reg.} \quad (2.29)$$

Note that one can check the last formula using the classical Poisson brackets of the WZNW model \([32]\). This equation gives rise to the useful relation

$$C^{\alpha \bar{a}}_{\phi \phi \phi} C^{\beta \bar{b}}_{\phi \phi \phi} f^{\alpha \beta \bar{d}} = f^{abc} \delta^{\bar{b} \bar{c}} \delta^{\bar{a} \bar{d}} + \mathcal{O}(1/k). \quad (2.30)$$

Taking this identity in eq. (2.26) we obtain the following expression

$$\langle O(z_1, \bar{z}_1)O(z_2, \bar{z}_2)O(z_3, \bar{z}_3) \rangle = M \ c_V \ (k \dim G/2)^2 \ \Pi_{i<j}^{3} \frac{1}{|z_{ij}|^{2\Delta}}. \quad (2.31)$$

To get the constant $C$ from the last formula, one has to normalize the two point function to one. The norm of the operator $O$ is given by

$$||O||^2 = \langle O(1)O(0) \rangle = M \ (k \dim G/2)^2. \quad (2.32)$$

Finally we find

$$C = \frac{M \ c_V (k \dim G/2)^2}{||O||^2} = c_V + \mathcal{O}(1/k). \quad (2.33)$$

With the given expression for $C$ the second solution in (2.24) comes out as follows

$$\epsilon_2 = -\frac{2}{\pi k}. \quad (2.34)$$

It is important that the value of $\epsilon_2$ does not depend on the normalization constant $M$. Substituting this solution in eq. (2.21) we come to the interesting result

$$S_{PWZNW}(\epsilon_2) = S_{WZNW}(g, -k). \quad (2.35)$$

In other words, the WZNW model with negative level $k$ perturbed by the operator $O$ arrives at the conformal WZNW model with positive level $l = -k = |k|$. Thus, the

\[\text{From the dimension one could argue that}
K^a(w, \bar{w})\phi^{b\bar{b}}(z, \bar{z}) = \frac{\lambda f_{abc}}{\bar{w} - \bar{z}} \phi^{c\bar{b}}(z, \bar{z}) + \text{reg.}
\]

By comparing the last formula with the classical Poisson bracket $\{K^a(w, t), \phi^{b\bar{b}}(z, t)\}$, we find $\lambda = 1 + \mathcal{O}(1/k)$.
two conformal points of the WZNW model discovered by Witten \[7\], turn out to be the ultraviolet and infrared fixed points of the one renormalization group flow. It might be interesting to understand whether there is a sort of duality symmetry between these two conformal systems. In ref. \[31\] it has been observed that the nonunitary WZNW model can come into being by duality transformations in the space of Thirring models with positive kinetic energy.

Once we know exactly the critical theory corresponding to the perturbative conformal point given by eq. (2.34), we can compute the exact Virasoro central charge at the point $\epsilon_2$. We find

$$c(\epsilon_2) = c(\epsilon_1) - \frac{2k c_V \dim G}{c_V^2 - k^2}.$$  

(2.36)

Apparently when $k < -c_V$, the difference $\Delta c = c(\epsilon_2) - c(\epsilon_1)$ is less than zero in full agreement with the $c$-theorem \[20\]. We might expect this result since the perturbation was done by the operator $O$ with positive norm.

The point to be made is that the Virasoro central charge at the infrared conformal point $\epsilon_2$ may also be estimated by perturbation theory (see for example, the Cardy-Ludwig formula \[23\]). By comparing the perturbative result

$$\Delta c = -\frac{y^3}{C^2} ||O||^2 = -\frac{[2 - (\Delta + \bar{\Delta})^3]||O||^2}{c_V^2}$$  

(2.37)

with the exact expression in (2.36), we can fix the normalization constant $M$. We find

$$M = \frac{1}{\dim G}.$$  

(2.38)

The factor $||O||^2$ in eq. (2.37) comes from the Zamolodchikov metric in the formula for the $c$-function. One can use the sigma model representation of the operator $O$ and free field Green functions to verify the formula for the norm.

It is instructive to compute anomalous dimensions of the operator $O$ at the second conformal point. We can use the perturbative formula due to Redlich \[33\]. To given order in $1/k$ the formula yields

$$\Delta(\epsilon_2) = \bar{\Delta}(\epsilon_2) = 1 - c_V/k$$  

(2.39)

in full agreement with the exact result

$$\Delta(\epsilon_2) = \bar{\Delta}(\epsilon_2) = 1 + \frac{c_V}{c_V - k}.$$  

(2.40)
where \( k \) is negative.

Another observation is that the perturbation of the nonunitary conformal WZNW model by the operator \( O \) provides some insight into the nonconformal WZNW model with arbitrary coupling constant in front of its sigma model term. Indeed, in the large \(|k|\) limit we can make use of the quasiclassical formula (2.20) to alter the perturbed theory to the form

\[
S_{PWZNW}(\epsilon) = \frac{1}{4\lambda} \int d^2 x \ Tr(\partial_\mu g \partial^\mu g^{-1}) + k \Gamma, \quad (2.41)
\]

where \( \Gamma \) is the Wess-Zumino term; whereas the sigma model coupling \( \lambda \) is related to the perturbation parameter \( \epsilon \) by the formula

\[
\frac{1}{\lambda} = \frac{k}{4\pi} + \frac{\epsilon k^2}{4}. \quad (2.42)
\]

The constant \( \epsilon \) runs from 0 to \( \epsilon_2 = -(2/\pi k) \). Correspondingly the coupling \( \lambda \) changes from \(-\infty\) to \(+\infty\) except the small interval \([- (4\pi/k), (4\pi/k)]\) which goes to zero as \(|k| \to \infty\). We find it rather amazing that inside the perturbation interval there lies the point \( \epsilon_{WZ} = -(1/\pi k) \) at which the sigma model term drops out and we are left with the pure Wess-Zumino term.

\[
S_{PWZNW}(\epsilon_{WZ}) = k \Gamma. \quad (2.43)
\]

Some time ago classical two dimensional models described by the pure Wess-Zumino term attracted some attention [34]. Now we have exhibited that such peculiar theories can be properly understood at the quantum level as the massive perturbation of the nonunitary WZNW model. Our conjecture is that the pure Wess-Zumino term theory separates the nonunitary phase from the unitary phase of the WZNW model given by eq. (2.41). That is, at this point, all states with negative norms have to turn into null vectors, perhaps due to appearance of the additional symmetry. We are going to explore this point elsewhere.
3  The sigma model perturbation of the gauged WZNW model

In this section we would like to consider one application of the sigma model perturbation of the nonunitary WZNW model. It is the GWZNW model in which the nonunitary WZNW theory emerges as an intrinsic component (see e.g. ref.[5]). Indeed, at the classical level the GWZNW model can be described as a combination of usual conformal WZNW models

\[ S_{GWZNW} = S_{WZNW}(hg\tilde{h}; k) + S_{WZNW}(h\tilde{h}; -k). \]  (3.44)

Here the matrix field \( g \) takes its values on the Lie group \( G \); whereas \( h, \tilde{h} \) take values on the subgroup \( H \) of \( G \). Apparently one of these two WZNW models has to be nonunitary.

The usual action of the GWZNW model in terms of the group element \( g \) and the vector nonabelian fields \( A_z \) and \( \bar{A}_\bar{z} \) is obtained from eq. (3.44) upon using the following definition

\[ A_z = h^{-1} \partial h, \quad \bar{A}_{\bar{z}} = \overline{\partial}\tilde{h}\tilde{h}^{-1}. \]  (3.45)

Correspondingly the gauge symmetry in the variables \( g, h, \tilde{h} \) is given by

\[ g \rightarrow \Omega g\Omega^{-1}, \quad h \rightarrow h\Omega^{-1}, \quad \tilde{h} \rightarrow \Omega\tilde{h}, \]  (3.46)

where \( \Omega \) is the parameter of the gauge group \( H \).

At the quantum level, the GWZNW model is described by the action

\[ S_{QGWZNW} = S_{WZNW}(hg\tilde{h}; k) + S_{WZNW}(h\tilde{h}; -k - 2c_V(H)) + S_{Gh}(b, c, \bar{b}, \bar{c}). \]  (3.47)

Compared to the classical expression, the quantum action has the second (nonunitary) WZNW model of the product \( h\tilde{h} \) with the level shifted by twice the eigenvalue of the quadratic Casimir operator in the adjoint representation of the subalgebra \( \mathcal{H} \). In addition, the QGWZNW model has the ghost-like contribution

\[ S_{Gh} = \text{Tr} \int d^2 z \ (b\partial c + \bar{b}\partial \bar{c}). \]  (3.48)

The modifications to the quantum action have a very elegant explanation [35]. All these changes in the quantum theory work to convert the second class constraints of the classical
GWZNW model into first class constraints. A systematic investigation of the conversion was first discussed in [36] and then extensively studied in [37]. The theory also has a nilpotent BRST operator [5].

The point to be made is that the product $h\tilde{h}$ is a gauge invariant quantity. So is any function of $h\tilde{h}$. Moreover, the gauge symmetry allows us to impose the following gauge condition

$$\tilde{h} = 1,$$

(3.49)

which does not lead to any additional propagating Faddeev-Popov ghosts. Therefore, any function of $h$ has to respect the gauge symmetry. Let us consider the following one

$$O_H =: J^a\bar{J}^{\bar{a}}\phi^{a\bar{a}}:,$$

(3.50)

where all the three operators on the right hand side are defined in terms of the nonunitary level $(-k - 2c_V(H))$ WZNW model on $H$:

$$\phi^{a\bar{a}} = \text{Tr}(h^{-1}t^a h^{\bar{a}}),$$

$$J^a = \frac{1}{2}\eta^{ab}\text{Tr}[(k + 2c_V(H))h^{-1}\partial h^a],$$

(3.51)

$$\bar{J}^{\bar{a}} = \frac{1}{2}\eta^{\bar{a}\bar{b}}\text{Tr}[(k + 2c_V(H))\bar{h}\partial \bar{h}^{-1}t^a].$$

Normal ordering is understood in accordance with eqs. (2.9), (2.10). The operator $\int d^2z O_H(z, \bar{z})$ respects the gauge symmetry (3.46) in the sense that being substituted into any correlation functions it will not spoil the Ward identity attributed to the gauge symmetry given by (3.46). Namely,

$$\langle \partial \bar{J}^{\text{tot}}(\bar{z}) \ldots \int d^2w O_H(w, \bar{w}) \ldots \rangle - \langle \bar{\partial} J^{\text{tot}}(z) \ldots \int d^2w O_H(w, \bar{w}) \ldots \rangle = 0,$$

(3.52)

where

$$J^{\text{tot}, a} = \frac{1}{2}\text{Tr}[(k + 2c_V(H))h^{-1}\partial h^a] - \frac{1}{2}\text{Tr}(kg^{-1}\partial g^a) - f^{abc}b^b_2 c^c$$

(3.53)

$$\bar{J}^{\text{tot}, \bar{a}} = \frac{1}{2}\text{Tr}[(k + 2c_V(H))\bar{h}\partial \bar{h}^{-1}t^a] - \frac{1}{2}\text{Tr}(k\partial g^{-1}t^a) - f^{abc}\bar{b}^b_2 c^c.$$
The given operator $O_H$ is a good physical operator as it gives rise to a unitary Virasoro representation in the GWZNW model. This operator possesses all the merits of the operator $O$ considered in the previous section. Therefore, we can make use of $O_H$ to perturb the GWZNW model. Henceforth we will call perturbed GWZNW model the following theory

$$S_{PGWZNW}(\epsilon) = S_{QGWZNW} - \epsilon \int d^2z \, O_H(z, \bar{z}). \tag{3.54}$$

Due to the striking analogy between $O_H$ and $O$ from the previous section, we can conclude that the PGWZNW model given by eq. (3.54) has to have a second (infrared) conformal point. The value of the parameter $\epsilon$ at this point can be deduced from eq. (2.34). To leading order in $1/k$ we find

$$\epsilon_2 = \frac{2}{\pi k}. \tag{3.55}$$

The quantity $\epsilon_2$ occurs with a positive sign because $k$ - the level of the ungauged WZNW model - is a positive integer in the case under consideration.

The exact conformal theory corresponding to the perturbative fixed point can be recognized as follows

$$S_{PGWZNW}(\epsilon_2) = S_{WZNW}(g; k) + S_{WZNW}(h; k + 2c_V(H)) + S_{Gh}. \tag{3.56}$$

Correspondingly the Virasoro central charge at the infrared critical point is given by

$$c(\epsilon_2) = c(G/H) + c_{WZNW}(k + 2c_V(H)) - c_{WZNW}(-k - 2c_V(H))$$

$$= c(G/H) - \frac{2(k + 2c_V(H))c_V(H) \dim H}{k^2 + 4kc_V(H) + 3c_V(H)^2}. \tag{3.57}$$

Here $c(G/H)$ is the Virasoro central charge of the $G/H$ coset construction; whereas $c_{WZNW}(l)$ denotes the Virasoro central charge of the level $l$ WZNW model on the subgroup $H$. It is clear from eq. (3.57) that $c(\epsilon_2) < c(G/H)$ as it should be according to Zamolodchikov’s c-theorem.

Both relation (3.56) and (3.57) indicate that the second (infrared) critical point no longer corresponds to a gauged WZNW theory. It is not surprising because the perturbation by the operator $O_H$ breaks the BRST invariance of the GWZNW model. This may
have some interesting consequences. First of all, we want to point out that all ghost free states of the GWZNW model after perturbation by the operator $O_H$ have to go to unitary states at the infrared conformal point. Indeed, it follows from eq. (3.56) that at the perturbative conformal point any ghost free state belongs to the direct product of Fock spaces of the unitary WZNW models. Since the perturbing operator does not involve the ghosts, all ghost free states of the GWZNW model go to ghost free states of the perturbed theory. It was observed in ref. [3] that ghost free states of the GWZNW model may be representatives of the cohomology classes forming the physical subspace $\text{Ker} Q/\text{Im} Q$ (with $Q$ the BRST operator) of the GWZNW model. If it is the case (at least for some particular models), then it may be very interesting to investigate what is happening to the fusion algebra of these ghost free representatives under the perturbation described by eq. (3.54). Apparently such unitary primary states have to continue being unitary vectors at the perturbative conformal point. By unitary states we mean states which have positive definite inner product with themselves. Of particular interest are GWZNW models with a finite number of cohomology classes such as $SU_k(2) \times SU_1(2)/SU_{k+1}(2)$ models. It might be important to understand the mechanism of deformation of fusions of these models in the course of the $O_H$ perturbation and to see how (an infinite number of) new primaries will enter deformed fusion rules at the infrared conformal point.

4 The BRST invariant sigma model perturbation of the GWZNW model

We have described in the previous section the operator $O_H$ which is suitable for performing relevant unitary deformations of GWZNW models. However, this operator $O_H$ as it was defined in eq. (3.50) has no proper action on the physical subspace $\text{Ker} Q/\text{Im} Q$ of the GWZNW model. Yet it might be interesting to have perturbations genuinely defined on the space $\text{Ker} Q/\text{Im} Q$. The aim of the present section is to exhibit an operator which provides relevant perturbations on the given space.

Let us start with the nilpotent BRST operator $Q$ of the given GWZNW model. The
nilpotent BRST operator $Q$ is defined as follows \[5\]

\[
Q = \oint \frac{dz}{2\pi i} [c^a (\tilde{J}^a + J^a) : (z) - \frac{1}{2} f^{abc} : c^a b^b c^c : (z)],
\]

(4.58)

where we have used the following notations

\[
J = \frac{(k + 2cV(H))}{2} h^{-1} \partial h,
\]

(4.59)

\[
\tilde{J} = -\frac{k}{2} g^{-1} \partial g|_H.
\]

Here the current $\tilde{J}$ is a projection of the $G$-valued current on the subalgebra $H$ of $G$. Note that our notations are slightly different from those in [5].

It is instructive to verify that the operator $O_H$ in eq. (3.50) is not annihilated by $Q$. However it would be the case if the current $J$ in the definition of $O_H$ had vanishing central charge. The last observation gives a hint at the way one has to alter $O_H$ to end up with a BRST invariant operator.

Another crucial point is that the currents $J$ and $\tilde{J}$ form the affine algebras with central elements of the opposite signs. As a matter of fact, there must exist a linear combination of $J$ and $\tilde{J}$ which acting on $\phi$ has to give rise to an affine primary vector with respect to the affine current $J + \tilde{J}$ from $Q$. Indeed, one can check that the following relations

\[
(J^a_{n \geq 0} + \tilde{J}^a_{n \geq 0}) \cdot (J^b_{-1} + \frac{(k + 4cV(H))}{k} \tilde{J}^b_{-1}) \phi^b|_0 = 0
\]

(4.60)

hold. Thus, $(J^b_{-1} + \frac{(k + 4cV(H))}{k} \tilde{J}^b_{-1}) \phi^b|_0$ is an affine primary scalar.

After that, it is easy to prove that the following operator

\[
O_{BRST} =: [J^a + \frac{(k + 4cV(H))}{k} \tilde{J}^a][J^\bar{a} + \frac{(k + 4cV(H))}{k} \tilde{J}^\bar{a}] \phi^{a\bar{a}} : ,
\]

(4.61)

where $\phi^{a\bar{a}}$ is as in eq. (3.51), is BRST invariant. That is,

\[
[Q, O_{BRST}(z, \bar{z})] = \oint \frac{dw}{2\pi i} : c^a (\tilde{J}^a + J^a) : (w) O_{BRST}(z, \bar{z}) = 0,
\]

(4.62)

where the contour surrounds the point $z$. Obviously $O_{BRST}$ has essentially the same algebraic structure as $O$ and $O_H$.  

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The point to be made is that in the classical limit $|k| \to \infty$ the operator $O_{BRST}$ can be presented in the form

$$O_{BRST} =: J'^a \bar{J}^{\bar{a}} :,$$

where

$$J' = -\frac{k}{2} (gh^{-1})^{-1} \partial (gh^{-1}),$$

$$\bar{J}' = \bar{\partial} \bar{J} + \bar{J} = -\frac{k}{2} (\bar{\partial} g g^{-1} |_{H} - \bar{\partial} h h^{-1}).$$

The last formulas might be useful in the course of Lagrangian interpretation of the operator $O_{BRST}$. It shows that $O_{BRST}$ to certain extent can be understood as a Thirring-like interaction between the currents $J'$ and $\bar{J}'$. It is necessary to point out that in the classical theory, the current $\bar{J}'$ coincides with the classical gauge constraint. So that at the classical level, $O_{BRST}$ goes to zero.

Also it is obvious that $O_{BRST}$ and $O_H$ share the same conformal dimensions, so that in the large $k$ limit, the given operator $O_{BRST}$ has to behave as a relevant quasimarginal operator. It still provides a unitary conformal representation since $O_{BRST}$ does not contain the ghost-like fields. Furthermore, $O_{BRST}$ has to continue satisfying the fusion algebra

$$O_{BRST} \cdot O_{BRST} = [O_{BRST}] + [I] + ...,$$

(4.63)

where the dots represents contributions of operators with irrelevant conformal dimensions (larger than one).

Indeed, in order to prove eq. (4.63) in the large $k$ limit, we can again turn to the symmetry and renormalizability arguments as we did it in the case of the operator $O$ as well as $O_H$.

All in all, the operator $O_{BRST}$ appears to be an appropriate relevant quasimarginal unitary operator for carrying out perturbations from the physical subspace Ker $Q$/Im $Q$ of the GWZNW model.

We go on to define the BRST invariant perturbed GWZNW model

$$S_{BRST}(\epsilon) = S_{QGWZNW} - \epsilon \int d^2z \ O_{BRST}.$$  

(4.64)
We want to point out that in accordance with eq. (4.61), the given perturbation around
the GWZNW model can be understood as a linear combination of two perturbations
by the sigma model term and by a Thirring-like current-current interaction. The latter
introduces novel features in computations with $O_{BRST}$. We can no longer rely on results
obtained for the operators $O$ and $O_H$ from the previous sections. This is mainly because of
the fact that the linear combination $J + \frac{k+4c_V(H)}{k} \tilde{J}$ in the large $k$ limit behaves as an affine
current with central charge $3c_V(H)$. Therefore, the central term of the underlying affine
algebra no longer dictates the leading order in $1/k$ and, hence, more intricate computations
are required. Most of the difficulties reside in the three point function of the spin (1,1)
operator $\phi^{aa}$ from the nonunitary WZNW model. To date, three point functions of all
affine primaries have been computed only in the case of the $SU(2)$ WZNW model [38].
Fortunately the identity (2.30) allows us to overcome most of the technical obstacles.

Let us consider one very interesting GWZNW model with $G = SU_{m-2}(2) \times SU_1(2)$ and
$H = SU_{m-1}(2)$. The physical subspace $\text{Ker} Q/\text{Im} Q$ of this theory properly describes
the Fock space of the minimal conformal model with the Virasoro central charge [35]
\begin{equation}
c = 1 - \frac{6}{m(m+1)}, \quad m = 3, 4, \ldots
\end{equation}

It is straightforward to compute the conformal dimensions of the corresponding operator $O_{BRST}$. We find
\begin{equation}
\Delta = \tilde{\Delta} = 1 - \frac{c_V(SU(2))}{m - 1 + c_V(SU(2))} = 1 - \frac{2}{m + 1}.
\end{equation}

There is one primary operator from $\text{Ker} Q/\text{Im} Q$ with the given conformal dimensions.
This is the $\Phi_{(3,1)}$ operator. Since $O_{BRST}$ belongs to the same physical subspace, we have
to identify $O_{BRST}$ with $\Phi_{(3,1)}$ (up to $Q$-exact terms). Note that the fusion algebra of
$\Phi_{(3,1)}$ agrees with eq. (4.63). This way we learn that our perturbation of the GWZNW
model coincides with the perturbation of the minimal conformal series first discovered by
Zamolodchikov [20], [30]. It is rather amazing that the $\Phi_{(3,1)}$ perturbation, in fact, stems
from the relevant perturbation around the nonunitary WZNW model [24]. With this in
mind, formula (4.61) exhibits a microscopical structure of the $\Phi_{(3,1)}$ operator. Surprisingly
the value of the perturbation parameter $\epsilon$ at the infrared conformal point in the theory
in eq. (4.64) is still given by formula (3.55). On the other hand for the perturbative Virasoro central charge formula (2.37) gives rise to the expression

\[ \Delta c = -\frac{2c_V(H) \dim^2 H}{m^3} M, \]  

(4.67)

where we have used the following result

\[ ||O_{BRST}||^2 = M \dim^2 H/4. \]  

(4.68)

Here the normalization constant \( M \) is given by eq. (2.38). In order to derive the last formula, one has to make use of the definition of \( O_{BRST} \) in eq. (4.61) with normal ordering as in eq. (2.9). In the case under consideration, one finds

\[ \Delta c = -\frac{12}{m^3} + O(m^{-4}). \]  

(4.69)

The last formula agrees with the exact result for the given coset. It is interesting that the normalization constant \( M \) is universal for all cosets and is fixed in the nonunitary WZNW model. Apparently, the infrared conformal point obtained by perturbation of the minimal conformal model, guarantees the existence of the second conformal point at the same value of \( \epsilon \) for all \( G/SU(2) \) GWZNW models with arbitrary \( G \). By computing the Virasoro central charge at the infrared conformal point, we establish the possibility of flow between the following coset constructions

\[ \tilde{G}_1 \times SU_{k-1}(2) \rightarrow SU_k(2) \]

\[ \rightarrow \tilde{G}_1 \times SU_{k-2}(2). \]  

(4.70)

By using our perturbation on the GWZNW model, we can easily prove the conjecture about the flow between symmetric \( G_k \times G_l/G_{k+l} \) and \( G_{k-l} \times G_l/G_k \) cosets (see e.g. [39]). Indeed, for arbitrary \( G \) we find (in the large \( k \) limit)

\[ \Delta c = -\frac{2c_V(G) \dim G l^2}{k^3} + O(1/k^4). \]  

(4.71)

One can verify that the given difference agrees with the exact result for the coset constructions under consideration. Note that our BRST invariant perturbation coincides with the perturbation operator of the \( G_k \times G_l/G_{k+l} \) cosets considered in [39]. Therefore, it is suggestive that \( S \)-matrices for perturbed gauged WZNW models can be related to the restricted \( R \)-matrices of the Toda field theories [39].

Because coset constructions describe certain geometries it makes sense to introduce a notion of geometry flow along the trajectory of the renormalization group.
5 Conclusion

We have started with the relevant perturbation around the nonunitary WZNW model and proceeded to define proper deformations of gauged WZNW models as well as general coset constructions. We found that our perturbations being generated by the operators agree with the Zamolodchikov $c$-theorem. We have established that the BRST invariant perturbing operator in the case of the $SU_k(2) \times SU_1(2)/SU_{k+1}(2)$ GWZNW model coincides with the $\Phi_{(3,1)}$ operator of the minimal conformal series. At the same time, our perturbation operator seems to be equally suitable for performing relevant hermitian perturbations around general GWZNW models. In this way one can discover renormalization group flows between different conformal theories describing different geometries of the target space.

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