Seiches in a Closed Basin of Various Geometric Shape

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Abstract. In this paper, we will observe wave's profile when the resonance occur in a closed basin of various geometric shape, which are rectangular, triangular, and semi-parabolic. The governing equation is shallow water equation. We solve the governing equation analytically to obtain the value of natural resonant period of the basin, which causes resonance phenomena. The analytical results were compared to the experimental results to see how good our model is. The equation will also be solved numerically using finite volume on a staggered grid method. Then, the numerical scheme will be validated against the analytical solutions we derived before.

1. Introduction

Seiches are long-period standing oscillations in an enclosed basin or in a locally isolated part of a basin. The example of seiches in a closed basin is an oscillations occur in a glass of water when the glass is disturbed, or the natural oscillations occur in a lake, pool, etc. Seiches occur at the natural resonant period of the basin which depends on the basin's geometry and depth ([1]). Physically, seiches are similar to the vibration of guitar string or elastic membrane. When the natural resonant period of the basin matches with the dominant period of external forces, such as wind, air pressure, or seismic vibrations, further resonance occurs. Resonance can result in possible damage of the land around the basin, even can cause tsunami at the worst scenario ([1]).

Resonance phenomena and the natural resonant period in a closed basin of various geometric shape have been investigated by several researchers using experimental approach, such as by Rabinovich [1] and Wilson [2]. The results of the experimental researches have been used by engineers in the related field. Wilson [2] has also determined the natural resonant period for closed rectangular basin analytically. However, there is no reference that has derived the natural resonant period for closed triangular and semi-parabolic basin. In this paper, we will formulate a mathematical model for wave's profile in a closed basin based on linear shallow water equation to investigate the resonance phenomena over various geometric of topography. Then, the model will be solved analytically using separation of variables method, to get the value of natural resonant period and confirm the experiment results. Further, we derive numerical scheme to be implemented to the model to simulate the profile of the wave when resonance occur.

There are six sections in this paper. In the first section, we will give a brief introduction about seiches and resonance phenomena. Then, the mathematical model will be explained in the
second section, followed by the analytical solution of natural resonant period of the basin that will be calculated in the third section. For the next section, we provide a description of the numerical methods we used, which is finite volume on a staggered grid method. In the fifth section, the numerical results are presented and validated against the analytical solutions. By the end of the paper, we will give a brief conclusion to summarize our results in the sixth section.

2. Model Formulation

![Figure 1. Illustration of a wave pass through a free water domain](image)

We consider a wave’s profile in a free water domain as described in Figure 1. Wave elevation calculated from undisturbed condition, denoted by $\eta(x, t)$, while $u(x, t)$ is horizontal velocity. In this model, we assumed that $u(x, t)$ is uniform along $z-axis$ and there is no vertical velocity. As shown in Figure 1, the total depth of basin denoted by $h = d + \eta$, where $d(x)$ is the depth of basin computed from undisturbed condition. Here, because the value of $\eta$ is much smaller compared to the value of $h(x)$, then we can assume that $h(x) = d(x)$. This wave’s profile can be illustrated using governing equation which called as Linear Shallow Water Equation written in

\begin{align}
\eta_t + (hu)_x &= 0, \\
u_t + g\eta_x &= 0,
\end{align}

with $g$ is the gravitational acceleration of 9.81 $m/s^2$. In this paper, we will investigate the wave’s profile in three different geometric shape of closed basin, which are:

(i) Rectangular basin with $h(x) = h_0$.
(ii) Triangular basin with $h(x) = h_0(1 - \frac{2x}{L})$.
(iii) Semi-parabolic basin with $h(x) = h_0(1 - \frac{4x^2}{L^2})$,

in which $h_0$ is the maximum water depth ($h(x)$) and $L$ is the length of the basin.

3. Analytical Solution

In this section, we will solve the model to determine the fundamental natural resonant period, which is the period that can cause a resonance, of each type of closed basin using Separation of Variables Method. In that case, we consider a harmonic wave with certain value of $\omega$, which is the wave’s angular frequency, written as

\begin{align}
\eta(x, t) &= F(x)e^{-i\omega t}, \\
u(x, t) &= G(x)e^{-i\omega t},
\end{align}

in which $F(x)$ and $G(x)$ are functions of $x$. Then, we got the partial derivatives for $\eta$ and $u$ read as
\[ \eta_x = F_x e^{-i\omega t}, \quad (5) \]
\[ \eta_t = -i\omega F(x)e^{-i\omega t}, \quad (6) \]
\[ u_x = G_x e^{-i\omega t}, \quad (7) \]
\[ u_t = -i\omega G(x)e^{-i\omega t}. \quad (8) \]

Then we substitute equation (5) and (8) to equation (2) to get the relation between function \( G \) and \( F \) as written below.

\[ G = \frac{-i g}{\omega} F_x. \quad (9) \]

Next, we substitute equation (6), (7), and (9) to equation (1), so that we get second order differential equation

\[ \omega^2 F + gF_x h_x + gF_{xx} h = 0. \quad (10) \]

For the next steps, we will use equation (10) as the base differential equation that will be solved for each type of closed basin, depends on the basin’s depth \( h(x) \).

3.1. Rectangular Basin

![Closed Rectangular basin](image)

For closed basin with rectangular type of bottom, the depth is uniform along basin’s domain, written in mathematical form as \( h(x) = h_0 \), so that we have \( h_x = 0 \). Consequently, equation (10) will become

\[ F_{xx} + \frac{\omega^2}{gh_0} F = 0. \quad (11) \]

The boundary conditions for this type of basin, following Figure 2, can be written as

\[ (\text{Boundary Condition}) \quad u(0,t) = 0 \quad \rightarrow \quad G(0) = 0, \]

and \( u(L,t) = 0 \quad \rightarrow \quad G(L) = 0. \)

Applying the relation in equation (9), our boundary conditions will become

\[ F_x(0) = 0 \quad \text{and} \quad F_x(L) = 0. \quad (12) \]

Solution for equation (11) is easily calculated using Characteristic Method, and that is

\[ F(x) = C_1 e^{\frac{\omega}{\sqrt{gh_0}} x} + C_2 e^{-\frac{\omega}{\sqrt{gh_0}} x}, \quad (13) \]

in which \( C_1 \) and \( C_2 \) are certain constants. The first derivative of \( F(x) \) is

\[ F_x = i \frac{\omega}{\sqrt{gh_0}} C_1 e^{\frac{\omega}{\sqrt{gh_0}} x} - i \frac{\omega}{\sqrt{gh_0}} C_2 e^{-\frac{\omega}{\sqrt{gh_0}} x}. \quad (14) \]
Then, we substitute the first boundary condition from (12) to (14) so that we got
\[ F_x(0) = i \frac{\omega}{\sqrt{gh_0}} (C_1 - C_2) = 0, \]
which will cause \( C_1 = C_2 \). On the other hand, from the second boundary condition (12) and applying the new condition \( C_1 = C_2 \), we got equation below.
\[ F_x(L) = i \frac{\omega}{\sqrt{gh_0}} C_1 (e^{i \frac{\omega}{\sqrt{gh_0}} L} - e^{-i \frac{\omega}{\sqrt{gh_0}} L}) = 0. \]
Equation (15) will be satisfied when \( C_1 = 0 \) or \( e^{i \frac{\omega}{\sqrt{gh_0}} L} - e^{-i \frac{\omega}{\sqrt{gh_0}} L} = 0 \). If \( C_1 = 0 \), then solution of \( F(x) \) is the trivial solution that is avoided in this case. Therefore, the second option must be satisfied, which equals to \( \frac{\omega}{\sqrt{gh_0}} L = m\pi \), where \( m = 0, 1, 2, \ldots \). Substituting \( \omega = \frac{2\pi}{T} \), with \( T \) is wave’s period, then we got
\[ \frac{2\pi}{T} L \sqrt{gh_0} = m\pi. \]
From equation (16), we can determine the value of fundamental natural resonant period \((m = 1)\), which is
\[ T_1 = \frac{2L}{\sqrt{gh_0}}. \]

3.2. Triangular Basin

For triangular type, the basin’s depth is \( h(x) = h_0(1 - 2x/L) \), so that the first derivative of \( h(x) \) is \( h_x = -\frac{2h_0}{L} \), then equation (10) becomes
\[ \omega^2 F - \frac{2h_0}{L} F_x + gh_0(1 - \frac{2x}{L}) F_{xx} = 0. \]
Solution for the second order differential equation (18) is written as
\[ F(x) = C_3 J_0(\sqrt{\omega^2 L\alpha(x)}) + C_4 Y_0(\sqrt{\omega^2 L\alpha(x)}), \]
where \( \alpha(x) = \sqrt{\frac{L-2x}{gh_0}} \), while \( C_3 \) and \( C_4 \) are certain constants. \( J_k(s) \) is Bessel Function of the first kind and \( Y_k(s) \) is Bessel Function of the second kind, with \( s \) is independent variable and \( k \) is the order of bessel function. Then we differentiate (19) so that we got
\[ F_x = \frac{C_3 J_1(\sqrt{\omega^2 L\alpha(x)})\sqrt{\omega^2 L}}{\alpha(x)gh_0} + \frac{C_4 Y_1(\sqrt{\omega^2 L\alpha(x)})\sqrt{\omega^2 L}}{\alpha(x)gh_0}. \]
Because it is a closed basin, then we have hard wall boundary condition at the both side of basin. In mathematical form, it can be written as

\[ F_x(-L/2) = 0 \quad \text{and} \quad F_x(L/2) = 0. \]  

(21)

In order to find the fundamental natural resonant period, we need to define another wave’s condition, which is

\[ F(0) = 0. \]  

(22)

Afterwards, we substituting the boundary condition to solution (20). Considering that

\[ \lim_{x \to L/2} \frac{J_1(\sqrt{\omega^2 L \alpha(x)})}{\sqrt{\omega^2 L \alpha(x)}} = 0 \quad \text{and} \quad \lim_{x \to L/2} \frac{C_4 Y_1(\sqrt{\omega^2 L \alpha(x)})}{\alpha(x) g h_0} = \infty, \]

then from the second boundary condition, we established that \( C_4 = 0 \) and \( C_3 \neq 0 \). Then, we substitute the condition (22) to equation (19). Here, we applying \( C_4 = 0 \) to the equation, so that we got

\[ F(0) = C_3 J_0(\sqrt{\omega^2 L \alpha(0)}) = 0. \]

Because \( C_3 \neq 0 \), then \( J_0(\sqrt{\omega^2 L \alpha(0)}) = 0 \) must be satisfied. The first root of Bessel Function \( J_0(s) \) mentioned in Wolfram Language is \( s = 2.4048 \). So that the term inside the bessel function’s bracket must follow

\[ \sqrt{\omega^2 L \alpha(0)} = 2.4048, \]

and it will be satisfied when the fundamental natural resonant period is

\[ T_1 = 1.3064 \frac{2L}{\sqrt{gh_0}}. \]

### 3.3. Semi-parabolic Basin

![Figure 4. Closed Semi-parabolic basin](image)

Next, we will calculate the fundamental natural resonant period for closed semi-parabolic basin which depth is \( h(x) = h_0(1 - \frac{4x^2}{L^2}) \) for domain \([-L/2, L/2]\). The first derivative of \( h(x) \) is

\[ h_x = \frac{-8h_0 x}{L^2}. \]

Therefore, equation (10) becomes

\[ \omega^2 F - g \frac{8h_0 x}{L^2} F_x + gh_0(1 - \frac{4x^2}{L^2}) F_{xx} = 0, \]

(23)

and the solution can be written as

\[ F(x) = C_5 P_{\nu}(\frac{2x}{L}) + C_6 Q_{\nu}(\frac{2x}{L}), \]

(24)

in which \( C_5 \) and \( C_6 \) are certain constants, while we define \( \nu = -\frac{1}{2} \sqrt{\frac{gh_0}{\sqrt{gh_0^2 + \omega^2 L^2}}} \). \( P_k(s) \) is first kind legendre function and \( Q_k(s) \) is second kind legendre function, while \( k \) is the order of legendre function and \( s \) is independent variable. The frist derivative of \( F(x) \) is
\[ F_x = \beta(x)(\nu + 1)(2C_5[P_{\nu+1}\frac{2x}{L} - \frac{2}{L}xP_{\nu}\frac{2x}{L}]) + 2C_6[Q_{\nu+1}\frac{2x}{L} - \frac{2}{L}xQ_{\nu}\frac{2x}{L}]), \]

(25)

where \( \beta(x) = \frac{1}{L(-1 + \frac{4x^2}{L^2})} \). The boundary conditions for this type of basin are

\[ F_x(-L/2) = 0 \quad \text{and} \quad F_x(L/2) = 0. \]

(26)

By substituting boundary conditions (26) to equation (25) we got

\[ F_x(-L/2) = \beta(-L/2)(\nu + 1)(2C_5[P_{\nu+1}(-1) + P_{\nu}(-1)] + 2C_6[Q_{\nu+1}(-1) + Q_{\nu}(-1))] = 0, \]

(27)

and

\[ F_x(L/2) = \beta(L/2)(\nu + 1)(2C_5[P_{\nu+1}(1) - P_{\nu}(1)] + 2C_6[Q_{\nu+1}(1) - Q_{\nu}(1))] = 0. \]

(28)

Based on the form of legendre function itself, for \( \nu \in \{0, 1, 2, \ldots\} \), the value of legendre function of first and second kind for \( s = 1 \), respectively, is \( P_{\nu}(1) = 1 \) and \( Q_{\nu}(1) = \infty \). Now, substituting those values to equation (28), we discovered that the equation will be satisfied if \( C_6 = 0 \) and \( C_5 \neq 0 \). Consequently, from equation (27) we got

\[ 2C_5[P_{\nu+1}(-1) + P_{\nu}(-1)] = 0. \]

(29)

Technically, equation (29) will always be satisfied, because for \( \nu \in \{0, 1, 2, 3, 4, \ldots\} \), we will get \( P_{\nu+1}(-1) = -P_{\nu}(-1) \). However, we need to choose the minimum value of \( \nu \neq 0 \) in order to find the value of fundamental natural resonant period. In that case, we got

\[ \nu = \frac{1}{2} \sqrt{\frac{gh_0}{gh_0} - \sqrt{\frac{L^2}{gh_0} + gh_0}} = 1, \]

(30)

which will be satisfied when

\[ T_1 = 1.1107 \frac{2L}{\sqrt{gh_0}}. \]

(31)

Finally, all the three analytical solutions are compared to the solutions calculated using experimental approach defined by Rabinovich [1]. Table 1 presents the comparison between the analytical and experimental fundamental natural resonant period measured by \( Error = \frac{|Analytical T_1 - Experimental T_1|}{Experimental T_1} \).

| Type of Basin | \( h(x) \) | \( T_1 \) (Analytical Solution) | \( T_1 \) (Experimental) | Error |
|---------------|-------------|-------------------------------|--------------------------|-------|
| Rectangular   | \( h_0 \)   | \( \frac{2L}{\sqrt{gh_0}} \) | \( \frac{2L}{\sqrt{gh_0}} \) | 0.00000 |
| Triangular    | \( h_0(1 - \frac{2x}{L}) \) | \( 1.3064 \frac{2L}{\sqrt{gh_0}} \) | \( 1.3050 \frac{2L}{\sqrt{gh_0}} \) | 0.00107 |
| Semi-parabolic| \( h_0(1 - \frac{4x^2}{L^2}) \) | \( 1.1107 \frac{2L}{\sqrt{gh_0}} \) | \( 1.1100 \frac{2L}{\sqrt{gh_0}} \) | 0.00063 |

Table 1. Analytical solution compared to experimental solution
From Table 1 we can see that the errors were small enough to say that our model actually fit the experimental approach well. It also means that the analytical solutions we have derived are indeed the fundamental natural resonant period that can cause a resonance in the actual harbor or basin.

4. Numerical Method
In this section, we will explain the numerical scheme that we used to simulate the wave’s profile in a closed basin with rectangular, triangular, and semi-parabolic type of bottom. We implement the finite volume on a staggered grid method as described in [3], [4] and [5]. Using the scheme, we will confirm whether our analytical solutions from previous section produce a resonance phenomena or not. Here, we consider equations (1) and (2) as our mathematical model for wave’s profile in domain \([0, L]\). Using staggered grid method, we discretize the domain into \(0 = x_{1/2}, x_1, x_3/2, ..., x_{N_x}, x_{N_x+1/2} = L\) as shown in Figure 5.

![Staggered Grid Method Illustration](image)

Figure 5. Illustration of staggered grid method

Mass conservation (1) is approximated at a cell centered at \(x_i\). This will be used to compute the values of water elevation \(\eta\) at every full grid points \(x_i\), with \(i = 1, 2, 3, ..., N_x\). At the same time, momentum conservation (2) is approximated at a cell centered at \(x_{i+1/2}\) that will be used to compute the values of wave’s velocity \(u\) at every staggered grid points \(x_{i+1/2}\), with \(i = 0, 1, 2, ..., N_x\). Then, using finite volume on a staggered grid method, equation (1) and (2) will be approximated by the numerical equations below.

\[
\eta_i^{n+1} - \eta_i^n + \frac{(hu)_i^{n+1/2} - (hu)_{i-1/2}^n}{\Delta x} = 0 \tag{32}
\]

\[
u_{i+1/2}^{n+1} - u_{i+1/2}^n + \frac{g\eta_{i+1}^{n+1} - \eta_{i+1}^n}{\Delta x} = 0 \tag{33}
\]

In the term of stability condition, using Von Neumann stability analysis, we got the Courant-Friedrichs-Lewy condition for (32) and (33) which is \(\sqrt{gh_0 \frac{\Delta t}{\Delta x}} \leq 1\). Using numerical equation (32) and (33), we will simulate the wave’s profile in the closed rectangular, triangular, and semi-parabolic basin.

5. Result and Discussion
For this section, we will present the simulation’s results of wave’s profile in each type of closed basin using the numerical scheme we have formulated before. By observing the simulation’s results, we will find out whether our numerical scheme fits the analytical solutions or not. In this simulation, we use numerical domain \([0, 20]\) m with gravitational acceleration \(g = 9.81\ m/s^2\), and \(h_0 = 10\ m\). The boundary conditions for each type of basin are the same boundary conditions that have been mentioned in analytical solution section. While the initial conditions for this simulation are \(\eta(x, 0) = 0\ m\) and \(u(x, 0) = 0\ m\), which are used for all type of closed basin. Figure 6 presents the simulation’s result for each type of basin.
Figure 6. Simulation of resonance in closed (top) rectangular, (middle) triangular, and (bottom) semi-parabolic basin. Horizontal axis represents the time steps or iterations, while the vertical axis represents wave elevation calculated at $x = L$ m.
Figure 7. Simulation of wave profile in closed (top) rectangular, (middle) triangular, and (bottom) semi-parabolic basin when resonance did not occur. Horizontal axis represents the time steps or iterations, while the vertical axis represents wave elevation calculated at $x = L$ m.
The results in Figure 6 show the behaviour of wave’s elevation or wave’s amplitude in all three different basins, which are rectangular, triangular, and semi-parabolic, along the observation time which is \( T = 300 \) time steps. The simulated wave has the analytical solutions we have calculated before as its period. From the results alone, we can see that the elevation or the amplitude of the wave has increased as time goes by. By the escalation pattern as described in Figure 6, we can say that if the observation time \( T \) becomes bigger, then most likely the value of the wave’s amplitude will also becomes bigger. This condition is a sign of a resonance phenomena. It means, our numerical scheme fit the analytical solutions nicely. Yet, in order to validate our numerical scheme, we also will simulate the wave’s profile in all types of basin when the incoming wave period were not the same as the analytical solutions we have derived.

In the simulations illustrated in Figure 7, we have chose different values of wave period, which are not the same value as the analytical solutions we have derived. In this case, the wave periods were slightly less than the one in the previous simulations. For rectangular basin, we have used \( T_1 = 0.98 \times \frac{2L}{\sqrt{gh_0}} \) and \( T_1 = 1.2864 \times \frac{2L}{\sqrt{gh_0}} \) for triangular basin. As for semi-parabolic basin, we have chose \( T_1 = 1.0807 \times \frac{2L}{\sqrt{gh_0}} \) as the incoming wave period. We can see in Figure 7 that the profile of the wave amplitude in all basins were not always increased. Sometimes the amplitude would be decreased and sometimes it would rose again. This is not the characteristic of resonance phenomena.

From the two simulations we did, the first one, which is when we use the analytical solutions or the fundamental natural resonant period as the incoming wave period, the numerical results show that in all three basins, resonance occur. On the other hand, in the second simulation, which is when we use different value from the analytical solutions as the incoming wave period, we did not found any sign of resonance phenomena. These numerical results fit the analytical solutions well, which imply that resonance phenomena will occur when the incoming wave period match the natural resonant period, and will not occur when the period is different.

6. Conclusion

A mathematical model of wave’s profile in a closed basin was formulated successfully to investigate a resonance phenomena in the basin. The model was solved analytically using separation of variables method to get the value of fundamental natural resonant period for closed rectangular, triangular, and semi-parabolic basin. Those results were compared to the experimental data and produce a very small error from the calculation. It means, the model we have formulated was fit the actual resonance phenomena well. Further, numerical scheme was derived using finite volume on a staggered grid method and was validated against the analytical solutions. The simulation results show that our numerical scheme fitted the analytical solutions nicely, which also means that it fitted the experimental data or the actual resonance phenomena.

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