CANCELLATION OF PROJECTIVE MODULES IN POLYNOMIAL RINGS OF PRIME CHARACTERISTIC

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ABSTRACT. Let \( A \) be a commutative Noetherian ring of characteristic \( p > 0 \), such that \( \dim(A) = d \). Let \( P \) be a projective \( A[T_1, ..., T_n] \)-module of rank \( d \). We show that \( P \) is cancellative if and only if \( P/ < T_1, ..., T_n > P \) is cancellative. We deduce some applications. In one of the interesting consequences, we show that the Bass-Quillen conjecture has an affirmative answer in dimension three, when \( 2 \) is invertible.

1. INTRODUCTION

All rings in this article are assumed to be commutative Noetherian with \( 1(\neq 0) \) and all projective modules are finitely generated with constant rank function. Let \( A \) be a ring of finite (Krull) dimension \( d \). Let \( P \) be a projective \( A \)-module of rank \( n \). \( P \) is said to be cancellative if \( P \oplus A^k \cong Q \oplus A^k \) for some integer \( k \geq 1 \) and some projective \( A \)-module \( Q \), implies that \( P \cong Q \). By a classical result of Bass [1] it is well known that if \( n > d \) then \( P \) is cancellative. Therefore, the study of the cancellation property becomes interesting whenever the equality \( n = d \) holds. Let \( \text{Um}(P) \) be the set of all elements \( p \in P \) such that there exists a \( A \)-linear surjection \( \phi_p : P \rightarrow A \) with the property \( \phi_p(p) = 1 \). It is well documented in the literature that the study of the cancellation problem (whenever \( n = d \) holds) turns out to be the study of the orbit space \( \text{Um}(P \oplus A)/\text{Aut}(P \oplus A) \).

Let \( P \) be a projective \( A[T] \)-module of rank \( d + 1 \). Plumstead [14, Theorem 1] used some sheaf patching techniques successfully to prove that \( P \) is cancellative and settled the conjectures of Eisenbud-Evans [8] on polynomial rings in affirmative. We would also like to mention that the third conjecture of Eisenbud-Evans have an affirmative answer due to the works of Sathaye [20] and Mohan Kumar [10]. In this article our primary focus is on Plumstead’s philosophy of using sheaf patching techniques to obtain the cancellation result on the polynomial ring \( A[T] \). Inspired by their work we prove the following (please see Theorem 2.9 for the proof):

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Theorem 1.1. Let $A$ be a commutative Noetherian ring of characteristic $p > 0$. Let $\dim(A) = d$. Let $Q$ be a projective $A[T_1, \ldots, T_n]$-module of rank $d$. $Q$ is cancellative if and only if $Q/ < T_1, \ldots, T_n > Q$ is cancellative.

In fact, in section 2 we prove a stronger but rather technical version of the above result [please see Proposition 2.5]. For this reason, in the introduction purpose we will stick to the above mention result only. Theorem 1.1 was proved by Rao [17] (when $n = 1$) and Wiemers [29], with an extra assumption $\frac{1}{a^d} \in A$ (with no restriction on $\text{char}(A)$). In the literature of projective modules their results turned out to be very crucial. For instance, in the same paper, Rao solved the Bass-Quillen conjecture for rings of dimension 3 whose residue characteristic is $\neq 2, 3$. In general, the conjecture is open till date.

While dealing with the cancellation problem, the hypothesis "$\frac{1}{a^d} \in A$", appeared in the literature due to the seminal technique of factorial rows, introduced by Swan-Tower [25] and Suslin [23]. One of the main goals in this article is to investigate how far one can extend the above mentioned results due to Rao and Wiemers, without the hypothesis "$\frac{1}{a^d} \in A$".

In the remaining part of the introduction we will discuss briefly the idea of the proof of Theorem 1.1. Before that, we need to recall the steps of Plumstead’s proof of [14, Theorem 1]. For this purpose we shall assume that $P$ is a projective $A[T]$-module of rank $d + 1$. Plumstead used the following fiber product diagram:

$$
\begin{array}{ccc}
A[T] & \longrightarrow & A_s[T] \\
\downarrow & & \downarrow \\
A_{1+<s> A}[T] & \longrightarrow & A_{s(1+<s> A)}[T]
\end{array}
$$

where $s \in A$ is a non-zero divisor, and the maps are canonical localization maps. Note that using standard arguments one can get a non-zero divisor $s \in A$ such that $P_s$ is a free $A_s[T]$-module. One of the crucial steps of Plumstead’s proof is to show that $P_{1+<s> A}$ is cancellative in the second corner. As, since the above fiber product diagram respects the polynomial structure in the fourth corner, one can use Quillen’s splitting lemma [15, Theorem 1] and the universal property of the fiber product, to obtain the required cancellation in $A[T]$. It was Plumstead’s ingenious vision, to use generalized dimension function efficiently on the ring $A_{1+<s> A}[T]$, to establish that $P_{1+<s> A}[T]$ is cancellative. Unfortunately, their arguments will not work whenever the rank of the projective module $P$ is $d$. We still do not know whether one can give a proof of Theorem 1.1 using arguments involving generalized dimension function.
Instead of working with the same fiber product diagram, as above, we use a slightly different fiber product diagram, as follows:

\[
\begin{array}{c}
A[T] \\
\downarrow \\
(A[T])_{1+<s>A[T]}
\end{array} \longrightarrow \begin{array}{c}
A_s[T] \\
\downarrow \\
(A[T])_{s(1+<s>A[T])}
\end{array}
\]

The benefits of the above fiber product diagram is that one can obtain the cancellation in the second corner using some basic elementary arguments (as the height of Jacobson radical of \((A[T])_{1+<s>A[T]}\) is \(\geq 1\)). But as this diagram does not respect the polynomial structure in the fourth corner, the patching part is crucial here. We circumvent this issue with the assumption that the base ring has characteristic \(p > 0\).

In the remaining sections we deduce some interesting consequences of Theorem 1.1. We request to see Sections 3, 5, 6, 7 for further details.

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\section{Main theorem}

The purpose of this section is to give the proof of the main results in this paper. As a preparation we require the following lemmas.

\begin{lemma}
Let \(A\) be a ring and \(J \subset \text{Jac}(A)\) be an ideal. Let \(n \geq 2\) be an integer. Let \(v \in \text{Um}_n(A)\). Let ‘bar’ denote going modulo \(J\). If \(\overline{v}\) is completable to a matrix in \(E_n(A/J)\) (respectively \(\text{SL}_n(A/J)\)) then \(v\) is elementarily completable to a matrix in \(E_n(A)\) (respectively \(\text{SL}_n(A)\)).
\end{lemma}

Proof. Let \(\epsilon \in E_n(A/J)\) (respectively \(\text{SL}_n(A/J)\)) be such that \(v\epsilon = e_1\). Since the canonical map \(E_n(A) \rightarrow E_n(A/J)\) (respectively \(\text{SL}_n(A) \rightarrow \text{SL}_n(A/J)\)) is surjective there exists \(\alpha_1 \in E_n(A)\) (respectively \(\text{SL}_n(A)\)) such that \(\overline{\alpha_1} = \epsilon\). Therefore, we get \(v\alpha_1 = (1+j_1, j_2, \ldots, j_n)\), where \(j_i \in J\) for \(i = 1, \ldots, n\). Since \(j_1 \in J \subset \text{Jac}(A)\), the element \(1+j_1\) is invertible in the ring \(A\). Hence we can find \(\alpha_2 \in E_n(A)\) such that \((1+j_1, j_2, \ldots, j_n)\alpha_2 = e_1\). Let us define \(\alpha = \alpha_1\alpha_2 \in E_n(A)\) (respectively \(\text{SL}_n(A)\)). Then we get \(v\alpha = e_1\). This completes the proof. \qed
The next lemma is very special in characteristic $p > 0$ set-up. Before that, let us recall a matrix is called a nilpotent matrix if some positive power of the matrix is the zero matrix.

**Lemma 2.2.** Let $A$ be a ring of characteristic $p > 0$. Let $\eta \in SL_n(A)$ be such that $\eta = \prod_{i=1}^k (I_n + \eta_i)$, where $\eta_i \in M_n(A)$ are nilpotent matrices. Moreover, assume that $e_1\eta = e_1$. Then there exists $\eta(X) \in GL_n(A[X])$ such that

1. $e_1\eta(X) = e_1$;
2. $\eta(0) = I_n$ and $\eta(1) = \eta$.

Proof. Note that finite product of nilpotent matrices is again a nilpotent matrix. Since $\text{char}(A) = p > 0$, sum of finitely many nilpotent matrices in $A$ is also a nilpotent matrix in $A$. Therefore, $\eta = \prod_{i=1}^k (I_n + \eta_i) = I_n + \eta'$, for some nilpotent matrix $\eta' \in M_n(A)$. $e_1\eta = e_1$ implies that $e_1\eta' = (0, \ldots, 0)$. Let $\eta'^k = 0$. We define $\eta(X) := (I_n + X\eta') \in M_n(A[X])$. Then note that $\eta(X)e_1 = e_1$ and $\eta(0) = I_n$ and $\eta(1) = \eta$. Therefore, the only remaining part is to show that $\eta(X) \in GL_n(A[X])$. This follows from the fact that $\eta(X)^p + k = I_n$. This completes the proof. \hfill $\Box$

Let us recall the definition of homotopy matrices.

**Definition 2.3.** Let $A$ be a ring. A matrix $\alpha \in GL_n(A)$ is said to be homotopic to the identity matrix (or simply homotopy matrix) if there exists $\alpha(T) \in GL_n(A[T])$ such that $\alpha(0) = I_n$ and $\alpha(1) = \alpha$. Let $H_n(A) = \{\alpha \in GL_n(A) : \alpha \text{ is homotopic to the identity matrix}\}$. Then it is easy to check that $H_n(A)$ is a normal subgroup of $GL_n(A)$, containing $E_n(A)$.

**Notation.** Let $A$ be a ring and $u, v \in Um_n(A)$. We write $u \sim_{E_n(A)} v$ if and only if there exists an elementary matrix $e \in E_n(A)$ such that $ue = v$.

The next result is a consequence of Lemma 2.2. It shows that in rings with positive characteristic, stably elementary matrices are homotopy matrices. Moreover, if the ring is regular, which is essentially of finite type over a field, then stably elementary matrices are actually elementary matrix. Before that recall for two square matrices $A$ and $B$ of order $m$ and $n$ respectively, $A \perp B := \begin{pmatrix} A & 0_{m,n} \\ 0_{n,m} & B \end{pmatrix}$ is also a square matrix of order $m + n$.

**Theorem 2.4.** Let $A$ be a ring of characteristic $p > 0$. Let $n \geq 3$. Let $\eta \in SL_n(A) \cap E_{n+1}(A)$. Then $\eta \in H_n(A)$. Moreover, if $A$ is regular such that it is essentially of finite type over a field, then $\eta \in E_n(A)$.

Proof. Since $\eta \in E_{n+1}(A)$, $\eta$ satisfies the hypothesis of Lemma 2.2. Therefore, using the argument given in Lemma 2.2 there exists a nilpotent matrix $\eta' \in M_{n+1}(A)$ such
that \(1 \perp \eta = I_{n+1} + \eta'\). We define \(\eta(X) := I_{n+1} + X\eta'\). It is proved in Lemma 2.2 that \(\eta(X) \in \text{GL}_{n+1}(A[X])\). Note that since \(e_1(1 \perp \eta) = e_1\) and \((1 \perp \eta)e_1^T = e_1^T\) we get \(e_1\eta' = (0, \ldots, 0)\) and \(\eta'e_1^T = (0, \ldots, 0)^T\). This implies that \(\eta' \in M_n(A)\). Therefore, \(\eta \in H_n(A)\).

Moreover, if we assume that \(A\) is regular and essentially of finite type over a field, then by [28] Theorem 3.3], we get \(\eta(X) \in E_n(A[X])\). Evaluating \(\eta(X)\) at \(X = 1\) we get \(\eta \in E_n(A)\). This completes the prove. \(\square\)

Now we are ready to give the proof of the main result in this paper. The following proposition asserts a sufficient condition for the cancellation problem (of free modules) in the polynomial rings of characteristic \(p > 0\).

**Proposition 2.5.** Let \(A\) be a ring of characteristic \(p > 0\). Let \(v(T_1, \ldots, T_n) \in \text{Um}_{d+1}(A[T_1, \ldots, T_n])\).

Suppose that there exists a non-zero divisor \(s \in A\) such that the followings hold:

1. \(v(T) \sim_{E_{d+1}(A[sT])} v(0)\);
2. There exists \(\beta \in \text{SL}_{d+1}(A)\) such that \(v(0, \ldots, 0)\beta = e_1\);
3. \(v(T_1, \ldots, T_n)\) is completable to an elementary matrix in \((A/ < s > A)[T]\), where 'bar' denote going modulo \(< s > A[T]\).

Then \(v(T_1, \ldots, T_n)\) is completable to an invertible matrix in \(A[T_1, \ldots, T_n]\).

Proof: For \(d < 2\) the result is trivial. Therefore, we may assume that \(d \geq 2\). Let \(T = (T_1, \ldots, T_n)\). Note that if \(s\) is a unit then \(A_s = A\), hence from hypothesis (1) we can find \(\theta \in E_{d+1}(A[T])\) such that \(v(T)\theta = v(0)\). Since \(v(0)\) is completable to an invertible matrix the result follows. Therefore, we may assume that \(s\) is not a unit. We give the proof in the following claims.

**Claim - 1.** Let \(R = (A[T])_{1+<s>\bar{A}[T]}\). We claim that there exists \(\alpha'_1 \in E_{d+1}(R)\) such that \(v(T)\alpha'_1 = v(0)\).

Proof of claim - 1. Let \(B = A/ < s > A\). By the hypothesis (3), we can find \(\tau \in E_{d+1}(B[T])\) such that \(v(T)\tau = \tau_1\). Note that \(R/ < s > R = B[T]\). Since \(v(T)\) is completable to an elementary matrix in \(B[T]\), the unimodular row \(v(0)\) is completable to an elementary matrix in \(B \subset B[T]\). Therefore, we get \(\frac{v(T)}{v(0)} \sim_{E_{d+1}(R/sR)} \frac{\tau_1}{E_{d+1}(R/sR)}\). As the ideal \(< s > R \subset \text{Jac}(R)\), by Lemma 2.1 we have \(v(T) \sim_{E_{d+1}(R)} e_1 \sim_{E_{d+1}(R)} v(0)\). Hence there exists \(\alpha'_1 \in E_{d+1}(R)\), such that \(v(T)\alpha'_1 = v(0)\). This completes the proof of Claim - 1.

By hypothesis (1), there exists \(\alpha_2(T)' \in E_{d+1}(A[sT])\) such that \(v(T)\alpha_2(T)' = v(0)\). Moreover, replacing \(\alpha_2(T)'\) by \(\alpha_2(T)'(\alpha_2(0)'^{-1}\) we may assume that \(\alpha_2(0)' = 1_{d+1}\).

Observe that we can always find \(f(T) \in A[T]\) such that if we define \(t(T) := 1 + sf(T) \in A[T]\) then \(\alpha'_1 \in E_{d+1}(A[T]_t(T))\) and \(v(T)\alpha'_1 = v(0)\) in the ring \(A[T]_t(T)\). Let us define \(\alpha_1 := \alpha'_1(\beta_t(T))\) and \(\alpha_2 := \alpha_2(T)'(\beta)_s\). Then we get \(v(T)\alpha_i = e_1\) for \(i = 1, 2\).
Let $\eta = (\alpha_2)^{-1}(\alpha_1)s$. Since $d \geq 2$, it follows from Suslin’s normality theorem that $E_{d+1}(A[T]_{st(T)})$ is a normal subgroup of $\text{SL}_{d+1}(A[T]_{st(T)})$. Therefore, we get

$$\eta = (\beta)^{-1}(\alpha(T)_{t(T)})^{-1}(\alpha_1)s(\beta)_{st(T)} \in E_{d+1}(A[T]_{st(T)})$$

Notice that we also have $e_1\eta = e_1(\alpha_2)^{-1}(\alpha_1)s = v(T)(\alpha_1)s = e_1$.

**Claim - 2.** We claim that there exist $\eta_1 \in E_{d+1}(A[T]_{l(T)})$ and $\eta_2 \in E_{d+1}(A_s[T])$ such that $\eta = (\eta_2)_{l(T)}(\eta_1)s$ and $e_1\eta_i = e_1$ for $i = 1, 2$.

Proof of claim - 2. Since $\eta \in E_{d+1}(A[T]_{st(T)})$, it can be written as $\eta = \prod_{i=1}^{k} (I_{d+1} + e_{i,j_i}(\lambda_i))$, where $e_{i,j_i}(\lambda_i) \in M_{d+1}(A[T]_{st(T)})$, such that the only non-zero entry is $\lambda_i$ at the position $(i, j_i)$ ($i \neq j_i$). Note that $e_{i,j_i}(\lambda_i)^2 = 0$. Implies that $e_{i,j_i}(\lambda_i)$ are nilpotent matrices (* for all $l = 1, ..., k$). Therefore, using Lemma 2.2, there exists $\eta(X) \in \text{GL}_{d+1}((A[T]_{st(T)})[X])$ such that $\eta(0) = I_{d+1}$ and $\eta(1) = \eta$ and $e_1\eta(X) = e_1$.

Using Quillen’s splitting lemma [15] Theorem 1, paragraph 2, for $g = (s)^N$ with large $N$, we get

$$\eta(X) = (\eta(X)\eta(gX)^{-1})_{l(T)}(\eta(gX))s,$$

with $\eta(X)\eta(gX)^{-1} \in \text{GL}_{d+1}((A_s[T])[X])$ and $\eta(gX) \in \text{GL}_{d+1}((A[T]_{l(T)}))[X])$. Since $e_1\eta(X) = e_1$, this gives us $e_1\eta(gX) = e_1$ and $e_1\eta(X)\eta(gX)^{-1} = e_1$. Let us define $\eta_1 := \eta(g)$ and $\eta_2 := \eta(1)\eta(g)^{-1}$. Then note that $\eta(gX) \in \text{GL}_{d+1}(A[T]_{l(T)}[X])$ and $\eta(0) = I_{d+1}$ and $\eta(1, g) = \eta_1$. Similarly, $\eta(X)\eta(gX)^{-1} \in \text{GL}_{d+1}((A_s[T])[X])$ and $\eta(0)\eta(0)^{-1} = I_{d+1}$ and $\eta(1)\eta(g)^{-1} = \eta_2$. Therefore, we have $\eta = (\eta_2)_{l(T)}(\eta_1)s$. Since $e_1\eta_1(gX) = e_1$ and $e_1\eta(X)\eta(gX)^{-1} = e_1$, this implies that, $e_1\eta_i = e_1$, for $i = 1, 2$. This completes the proof of Claim - 2.

Let $\sigma_t := \alpha_1\eta_1^{-1} \in \text{GL}_{d+1}(A[T]_{l(T)})$ and $\sigma_s := \alpha_2\eta_2 \in \text{GL}_{d+1}(A[T]_s)$. In the remaining part of the proof we will patch $\sigma_t$ and $\sigma_s$ to produce a matrix $\alpha \in \text{GL}_{d+1}(A[T])$ such that $v(T)\alpha = e_1$.

Notice that $(\sigma_t)_s = (\sigma_s)_{l(T)}$ (as $\eta = (\alpha_2)^{-1}(\alpha_1)s = (\eta_2)_{l(T)}(\eta_1)s$ ). Therefore, using [11] Proposition 2.2, page no 211 there exists a unique $\alpha \in \text{GL}_{d+1}(A[T])$ such that $(\alpha)_s = \sigma_s$ and $(\alpha)_{l(T)} = \sigma_t$. Observe that $v(T)\sigma_s = v(T)\alpha_2\eta_2 = e_1\eta_2 = e_1$ and $v(T)\sigma_t = v(T)\alpha_1\eta_1^{-1} = e_1\eta_1^{-1} = e_1$. Since $s > A[T] + t(T) > A[T] = A[T]$, no maximal ideal $m \subset A[T]$ can contain both $s$ and $t(T)$ simultaneously. Therefore, we either have $(\alpha)_m = (\alpha)_m = (\alpha)_m$ or $(\alpha)_m = (\alpha)_m = (\alpha)_m$. Hence a local checking ensures that $v(T)\alpha = e_1$. This completes the proof.

Now we move towards the result mention in abstract of the paper. Before that, we recall a lemma, for a proof please see [13] Chapter 6, page no 78, Lemma 6.1.2.

**Lemma 2.6.** Let $A$ be a ring. Let $I \subset A[T]$ be an ideal. Let $l(I) = \{a \in A : \text{there is } f(T) = aT^n + a_{n-1}T^{n-1} + ... + a_0 \in I \text{ with } a_i \in A\}$. Then $l(I)$ is an ideal in $A$ and $ht(l(I)) \geq ht(I)$.
Theorem 2.7. Let $A$ be a ring of characteristic $p > 0$ such that $\dim(A) = d$. Let $v(T_1, ..., T_n) \in \text{Um}_{d+1}(A[T_1, ..., T_n])$. The unimodular row $v(T_1, ..., T_n)$ is completable to an invertible matrix in $A[T_1, ..., T_n]$ if and only if $v(0, ..., 0)$ is completable to an invertible matrix in $A$.

Proof. Let $T = (T_1, ..., T_n)$. Note that if $v(T)$ is completable to an invertible matrix in $A[T]$, then evaluating at $T = 0$, we get $v(0)$ is completable to an invertible matrix in $A$.

Conversely, assume that $v(0)$ is completable to an invertible matrix in $A$. Since any unimodular row of length two is completable, without loss of generality we may assume that $d \geq 2$. Note that it enough to show that all hypotheses of Proposition 2.5 are satisfying. The remaining part of the proof is devoted to show this.

Let $J_v = \{ s \in A : v(T) \sim_{E_{d+1}(A[T])} \{ v(0) \} \}$. Then it was proved in [11, Chapter VI, page no 211, Theorem 2.3] that $J_v$ is an ideal in $A$. Let $v(T) = (v_1(T), ..., v_{d+1}(T))$. Using prime avoidance lemma on $< v_1(T), ..., v_{d+1}(T) > A[T]$ we may assume that (after a suitable alteration) $v_1(T)$ is a non-zero divisor in $A[T]$. Note that $\{ (s < v_1(T) > A[T]) = < s > A$, where $s$ is the leading coefficient of $v_1(T)$. Then by Lemma 2.6 $s$ is a non-zero divisor in $A$. It follows from [16, Corollary 2.5] that $s \in J_v$.

Without loss of generality we may assume that $s$ is not a unit in $A$. Let $B = A/ < s > A$ and ‘bar’ denote going modulo $< s > A$. Since $s$ is a non-zero divisor we have $\dim(B) \leq d - 1$. Therefore using [14] and [24, Theorem 2.6] we can find $\tau \in E_{d+1}(B[T])$ such that $v(T)\tau = \tau_1$. Hence the result follows from Proposition 2.5.

The next theorem is a “projective” version of the above Theorem. Before that we recall the definition of Quillen’s ideal.

Definition 2.8. Let $A$ be a ring $P$ be a projective $A[T_1, ..., T_n] -$ module. Let $T = (T_1, ..., T_n)$. We write $J_P = \{ s \in A : P_s \cong \frac{P}{P} \otimes^L A[T] \}$ That is, $J_P$ is the Quillen ideal with respect to the projective module $P$. It is proved in [15, Theorem 1] that $J_P$ is an ideal of $A$. An easy deduction from Quillen-Suslin theorem ([15], [24]) establish the fact that $\text{ht}(J_P) \geq 1$ (see [7, Remark 2.9]).

Theorem 2.9. Let $A$ be a ring of characteristic $p > 0$ such that $\dim(A) = d$. Let $P$ be a projective $A[T_1, ..., T_n] -$ module of rank $d$. $P$ is cancellative if and only if $P/ < T_1, ..., T_n > P$ is cancellative.

Proof. Let $T = (T_1, ..., T_n)$. Note that if $P$ is cancellative then so is $P/TP$. Therefore, without loss of generality we may assume that $P/TP$ is cancellative. Let $(a(T), p) \in \text{Um}(A[T] \oplus P)$. To show $P$ is cancellative it is enough to show that there exists $\tau \in \text{Aut}(A[T] \oplus P)$ such that $\tau(a(T), p) = (1, 0)$. The remaining part of the proof is devoted to show this.

Let $S$ be the set of all non-zero divisors in $A$. Then note that $P \otimes S^{-1}A[T]$ is a free module. Therefore, there exists a non-zero divisor $s_1 \in A$ such that $P_{s_1}$ is a free module.
Note that if $s_1$ is a unit then this will imply that $P$ is free. In this case the proof follows from Theorem 2.7. Therefore, without loss of generality we may assume that $s_1$ is not a unit.

Since the Quillen’s ideal $J_P$ has height $\geq 1$, there exists a non-zero divisor $s_2 \in A$ such that $P/TP \otimes A_{s_2}[T] \cong P \otimes A_{s_2}[T]$. Without loss of generality we may assume that $s_2$ is not a unit. Let $s = s_1s_2$. Let where ‘tilde’ denote going modulo $< T > A[T]$. Then note that we have the followings:

1. $P_s$ is free;
2. $\tilde{P} \otimes A_s[T] \cong P \otimes A_s[T]$;
3. $s$ is a non-zero divisor but not a unit in $A$.

Therefore, in the ring $A_s[T]$, using the argument used in [Proposition 2.5 Case - 2], we can find $\alpha_2(T)' \in E(A_s[T] \oplus P_s) = E_{d+1}(A_s[T])$ such that $\alpha_2(T)'(a(T), p) = (a(0), \overline{p})$ and $\alpha_2(0)' = I_{d+1}$.

Let $B = A/ < s > A$ and $R = (A[T])_{1+<s>A[T]}$. Then $\dim(B) \leq d - 1$. Let ‘bar’ denote going modulo $s$. Note that in the polynomial ring $B[T]$, using [12, Theorem 2.6] we can find $\overline{\alpha_1}' \in E(B[T] \oplus \overline{P})$ such that $\overline{\alpha_1}'(\overline{a(T)}, \overline{p}) = \overline{(a(0), p)}$. Since the canonical map $E(R \oplus (P \otimes R)) \rightarrow E(((R/ < s > R) \oplus (P \otimes R/ < s > R)) = E(B[T] \oplus \overline{P})$ is surjective and $< s > R \subset \text{Jac}(R)$, we can find $\alpha_1' \in E(R \oplus (P \otimes R))$ such that $\alpha_1'(a(T), p) = (a(0), \overline{p})$. Now following the same patching arguments given in Proposition 2.5 the result follows.

3. Application 1: Bass-Quillen conjecture in dimension 3 and characteristic 3

Let $A$ be a regular local ring. The famous Bass-Quillen conjecture asks whether projective $A[T]$-modules are free. In this section we give an affirmative answer of the Bass-Quillen conjecture in $3$-dimensional regular local ring $A$, in which $2$ is a unit.

**Theorem 3.1.** Let $A$ be a local ring of dimension 3 such that char$(A) = p > 2$ and $\frac{1}{2} \in A$. Then $Um_n(A[T]) = e_1SL_n(A[T])$ for all $n \geq 2$.

Proof. By standard stability argument it is enough to show that $Um_n(A[T]) = e_1SL_n(A[T])$ for $n = 3, 4$. If $n = 4$ then the result follows using Theorem 2.7. If $n = 3$ then this is done in [18, Theorem 3.1].

**Theorem 3.2.** Let $A$ be a regular local ring of dimension 3 and $\frac{1}{2} \in A$. Let $P$ be a projective $A[T]$-module. Then $P$ is free.

Proof. Note that if char$(A) \neq 3$, then this done in [17, Theorem 2.10]. Therefore, it is enough to proof the theorem for char$(A) = 3$. Since $A$ is regular local ring $K_1(A[T]) \cong$
\textbf{4. Application 2: On van der Kallen group}

Let \( A \) be a ring of dimension \( d \geq 2 \). Then from \([26]\), it follows that, the \( SL_{d+1}(A[T]) \)-orbit space of \((d+1)\)-unimodular rows in \( A[T] \) has a group structure. In this section we prove some results on the van der Kallen group.

\textbf{Theorem 4.1.} Let \( A \) be a ring of dimension \( d \geq 2 \) and \( \text{char}(A) = p > 0 \). Then the evaluation map (evaluating at \( T = 0 \)) \( \Gamma : \frac{Um_{d+1}(A[T])}{SL_{d+1}(A[T])} \rightarrow \frac{Um_{d+1}(A)}{SL_{d+1}(A)} \) is an isomorphism.

Proof. Note that since the canonical set-theocratic map \( Um_{d+1}(A[T]) \rightarrow Um_{d+1}(A) \) is surjective, \( \Gamma \) is also surjective. Therefore, only remaining is to show that \( \Gamma \) is injective. Let \([-\cdot]\) denote the class of \((d+1)\)-unimodular \( A[T] \) vector in the group \( \frac{Um_{d+1}(A[T])}{SL_{d+1}(A[T])} \). Let \( v(T) \in Um_{d+1}(A[T]) \) such that \( \Gamma([v(T)]) = 0 \) in the group \( \frac{Um_{d+1}(A)}{SL_{d+1}(A)} \). This implies that, the unimodular row \( v(0) \) is completable. Therefore, using Theorem 2.7 we get \( v(T) \) is completable. This completes the proof. \( \Box \)

The following theorem is an improvement of \([17, \text{Corollary 2.5}]\), in characteristic \( p > 0 \).

\textbf{Theorem 4.2.} Let \( A \) be a ring of dimension \( d \) of characteristic \( p > 0 \). Let \( v(T) \in Um_{d+1}(A[T]) \). Then there exists \( \alpha(T) \in SL_{d+1}(A[T]) \) such that \( v(T)\alpha(T) = v(0) \).

Proof. For \( d = 1 \), the theorem holds trivially. Therefore, without loss of generality we may assume that \( d \geq 2 \). Let \( i : \frac{Um_{d+1}(A)}{SL_{d+1}(A)} \rightarrow \frac{Um_{d+1}(A[T])}{SL_{d+1}(A[T])} \) be the canonical homomorphism induced by the natural embedding \( Um_{d+1}(A) \hookrightarrow Um_{d+1}(A[T]) \). Then note that \( \Gamma \circ i \equiv id \). By Theorem 4.1 \( \Gamma \) is an isomorphism. Therefore, we get \( i \circ \Gamma \equiv id \). Let \([-\cdot]\) denote the class of \((d+1)\)-unimodular \( A[T] \) vectors, in the group \( \frac{Um_{d+1}(A[T])}{SL_{d+1}(A[T])} \). Then note that \([v(T)] = i \circ \Gamma([v(T)]) = i([v(0)]) = [v(0)] \). This completes the proof. \( \Box \)

\textbf{5. Application 3: Some cancellation results}

In this section we will derive some cancellation results, which are direct consequence of Theorem 2.7.

\textbf{Theorem 5.1.} Let \( A \) ring of dimension \( d \) such that one of the following conditions hold:

1. \( A \) is a semi-local ring of characteristic \( p > 0 \);
2. \( F \) be a field which is algebraic over a finite field. \( A \) is an affine \( F \)-algebra.

Let \( v(T_1, \ldots, T_n) \in Um_{d+1}(A[T_1, \ldots, T_n]) \) be a unimodular row of length \( d + 1 \). Then there exists \( \alpha \in SL_{d+1}(A[T_1, \ldots, T_n]) \) such that \( v(T_1, \ldots, T_n)\alpha = e_1 \).
Proof. Let \( T = (T_1, \ldots, T_n) \). Note that if \( d < 2 \) then the result follows trivially. Therefore, we may assume that \( d \geq 2 \). In view of Theorem 2.7 it is enough to show that \( v(0) \) is completable. If \( A \) is a semi-local ring of characteristic \( p > 0 \) the result is direct consequence of Theorem 2.7. If \( A \) is an affine \( F \)-algebra then this result follows from [27] Corollary 17.3.

Over \( \mathbb{F}_p \), we pose the following question.

**Question 5.2.** Let \( A \) be an affine algebra over \( \mathbb{F}_p \) of dimension \( d \geq 2 \). Let \( v(T_1, \ldots, T_n) \in \text{Um}_{d+1}(A[T_1, \ldots, T_n]) \) be an unimodular row of length \( d + 1 \). Then does there exist \( \epsilon \in E_{d+1}(A[T_1, \ldots, T_n]) \) such that \( v(T_1, \ldots, T_n) \epsilon = e_1 \)?

An affirmative answer of the above question will give us the following cancellation result, which improves [17, Corollary 2.5] over \( \mathbb{F}_p \).

**Theorem 5.3.** Let \( A \) be an affine algebra over \( \mathbb{F}_p \) of dimension \( d \geq 4 \) and \( \frac{1}{(d-1)!} \in A \). Let \( v(T_1, \ldots, T_n) \in \text{Um}_d(A[T_1, \ldots, T_n]) \). Moreover, assume that Question 5.2 has an affirmative answer. Then there exists \( \alpha \in \text{SL}_d(A[T_1, \ldots, T_n]) \) such that \( v(T_1, \ldots, T_n) \alpha = e_1 \).

Proof. Note that in view of Proposition 2.5 it is enough to show that all hypotheses of Proposition 2.5 satisfy. Let \( T = (T_1, \ldots, T_n) \). Let \( v(T) \in \text{Um}_d(A[T]) \). We can find a non-zero divisor \( s \in A \) such that \( v(T) \sim_{E_d(A_s[T])} v(0) \). By [9, Theorem 1.6] \( v(0) \) is completable. Let ‘bar’ denote going modulo \( < s > A \). Let \( B = A/ < s > A \). Then \( B \) is an affine \( \mathbb{F}_p \)-algebra of dimension \( d - 1 \) and \( v(T) \in \text{Um}_d(B[T]) \). Therefore, by our hypothesis \( v(T) \) is completable to an elementary matrix in \( B[T] \). This completes the proof. \( \square \)

6. **Application 4: Monic Inversion Principles and Euler Class Groups**

From this section onwards, we will shift the theme of the paper from the cancellation problem of projective modules to the splitting problem of projective modules and efficient generation of ideals. Unlike in the previous sections, we will not give the proofs of the theorems in this section. Most of the proofs in this section will follow from the original works, one just need to use Theorem 4.2 in place of Rao’s result [17, Corollary 2.5]. We will only comment on them. Unless otherwise stated, in this section we will denote \( A \) as a commutative Noetherian ring of dimension \( d \), such that \( A \) contains an infinite field of characteristic \( p > 0 \).

**Remark 6.1.** Let \( \dim(A) = d \geq 2 \). Let \( k \subset A \) be an infinite field. Using Theorem 4.2 one can remove the hypothesis “\((n - 1)! \) is invertible” from [3, Proposition 3.1], when \( \text{char}(k) = p > 0 \). Therefore, one can remove the hypothesis \((n - 1)! \) is invertible (which is written in the form of the ring contains the field of rationals) from [3, Section 4]. In brief, we have the following theorem:
Theorem 6.2. Let $\dim(A) = d \geq 2$. Let $P$ be a projective $A$–module of rank $d$. Let $\chi : \wedge^d P \cong L$ be a $L$–orientation of $P$. Then the Euler class $e(P, \chi)$, of the pair $(P, \chi)$, defined in [3, Section 4], is well defined. Moreover, $P$ has a unimodular element if and only if $e(P, \chi) = 0$.

Now we move towards the polynomial ring $A[T]$. Recall that the ring $A(T)$ is obtained from $A[T]$ by inverting all monic polynomials in $T$. Here we remark that in [5, Theorem 3.10], Das actually proved the following theorem:

**Theorem 6.3 (Das).** Let $R$ be a commutative Noetherian ring of dimension $d \geq 3$, such that

1. $R$ contains an infinite field;
2. $\frac{1}{m} \in R$.

Let $I \subset R[T]$ be an ideal of height $d$ such that $I = \langle f_1, \ldots, f_d \rangle + T^2 I$. Assume that there exists $g_i \in IR(T)$ such that $IR(T) = \langle g_1, \ldots, g_d \rangle$ where $f_i - g_i \in IR(T)^2$. Then there exists $F_i \in I$ such that $I = \langle F_1, \ldots, F_d \rangle$, where $f_i - F_i \in I^2 T$.

The only place where Das required the hypothesis $\frac{1}{m} \in R$, is that he needs to use Rao’s result [17, Corollary 2.5]. Therefore, in characteristic $p > 0$, one can omit the hypothesis (2) from Theorem 6.3 by using Theorem 4.2. Therefore, we have the following:

**Theorem 6.4.** Let $\dim(A) = d \geq 3$. Let $I \subset A[T]$ be an ideal of height $d$ such that $I = \langle f_1, \ldots, f_d \rangle + T^2 I$. Assume that there exists $g_i \in IA(T)$ such that $IA(T) = \langle g_1, \ldots, g_d \rangle$ where $f_i - g_i \in IA(T)^2$. Then there exists $F_i \in I$ such that $I = \langle F_1, \ldots, F_d \rangle$, where $f_i - F_i \in I^2 T$.

Let $\dim(A) = d \geq 3$. Using the same arguments given in [3, Section 4] one can define the $d$–th Euler class group $E^d(A[T])$ and the $d$–th weak Euler class group $E^d_0(A[T])$, and prove the following theorem:

**Theorem 6.5.** Let $\dim(A) = d \geq 3$. Let $P$ be a projective $A[T]$–module (with trivial determinant) of rank $d$. Let $\chi : \wedge^d P \cong A[T]$ be an orientation of $P$. Then the Euler class $e(P, \chi)$, of the pair $(P, \chi)$, defined in [3, Section 4], is well defined. Moreover, $P$ has a unimodular element if and only if $e(P, \chi) = 0$.

Mimicking the proofs of [5, Theorem 5.4 and Proposition 5.8] the following theorems follow:

**Theorem 6.6.** Let $\dim(A) = d \geq 3$. Then the following sequence of groups is exact.

$$0 \to E^d(A) \to E^d(A[T]) \to \prod_m E^d(A_m[T]),$$

where the direct product runs over all maximal ideals $m$ of $A$ such that $\text{ht}(m) = d$. 
**Theorem 6.7.** Let \( \dim(A) = d \geq 3 \). Then the canonical map \( \Gamma : E^d(A[T]) \to E^d(A(T)) \) is injective in the following cases:

1. \( \text{ht}(J) \geq 1 \), where \( J \) denotes the Jacobson radical of \( A \);
2. \( A \) is an affine domain over an algebraically closed field of characteristic \( p > 0 \).

7. **APPLICATION 5: A splitting criterion of projective modules of even rank**

Let \( R \) be a ring and \( Q \) be a projective \( R \)-module. Recall that a \( R \)-linear map \( \alpha : Q \to R \) is called a generic section of \( Q \) if \( \text{rank}(Q) = \text{ht}(\alpha(Q)) \). In this section we prove a splitting criterion of projective modules (of even rank) over some polynomial rings, in terms of its generic section.

**Lemma 7.1.** Let \( d \geq 3 \) be an even integer. Let \( A \) be a ring of dimension \( d \) such that one of the following conditions hold:

1. \( A \) is a semi local ring of characteristic \( p > 0 \);
2. Let \( F \) be a field which is algebraic over a finite field. \( A \) is an affine \( F \)-algebra.

Let \( I \subset A[T] \) be an ideal of height \( d \) such that \( \mu(I/I^2) = \mu(I) = d \), where \( \mu(-) \) is the minimal number of generators of \(-\). Then any set of generators of \( I = \langle f_1, \ldots, f_d \rangle + I^2 \) lifts to a set of generators of \( I \).

Proof. Note that since the canonical map \( E^d(A[T]) \to E^d_0(A[T]) \) is surjective it is enough to show that any local orientation \( I = \langle f_1, \ldots, f_d \rangle + I^2 \) has a lift. Let \( \omega_1 \) be the local orientation of \( I \) induced by the set of generators \( \{f_1, \ldots, f_d\} \).

Let \( I = \langle g_1, \ldots, g_d \rangle \). Let \( \omega_2 \) be the local orientation of \( I \) induced by the set of generators \( \{g_1, \ldots, g_d\} \). Then note that \( (I, \omega_2) = 0 \) in the \( d \)-th Euler class group \( E^d(A[T]) \). Let ‘bar’ denote going modulo \( I \). As a \( A[T]/I \)-module two sets of generators of \( I/I^2 \) must differ by some invertible matrix \( \alpha \in \text{GL}_d(A[T]/I) \) that is \((\overline{f}_1, \ldots, \overline{f}_d) = (\overline{g}_1, \ldots, \overline{g}_d)\). Let \( \det(\alpha) = \overline{\pi} \in (A[T]/I)^* \) then we have \((I, \overline{\pi}^{-1}\omega_2) = (I, \omega_1)\). We get \( b \in A[T] \) such that \( ab - 1 \in I \). Then note that \( (b, g_1, g_2, \ldots, g_d) \in \text{Um}_{d+1}(A[T]) \). By Theorem 5.1 the unimodular row \((b, g_1, g_2, \ldots, g_d)\) is completable to an invertible matrix.

We can follow the arguments as in [21] Proposition, page 956, second proof to conclude that there is a matrix \( \sigma \in M_d(A[T]) \) with determinant \( a^{d-1} \) modulo \( I \) such that, if \((g_1, \ldots, g_d)\sigma = (F_1, \ldots, F_d)\), then \( I = \langle F_1, \ldots, F_d \rangle \). Let \( \omega'_1 \) be the local orientation of \( I \) induced by the set of generators \( \{F_1, \ldots, F_d\} \). Then note that we have the followings \( 0 = (I, \omega'_1) = (I, \overline{\pi}^{d-1}\omega_2) \) in \( E^d(A[T]) \).

Only remaining is to show that \( f_i - F_i \in I^2 \). To show this note that it is enough to show that \( (I, \omega_1) = 0 \) in \( E^d(A[T]) \).
As \( d \) is even we have \((I, \omega_1) = (I, \omega_1^{-1}) = (I, \omega_2)\) (by [6, Proposition 4.9]) \((I, \omega'_1) = 0\). This completes the proof. \(\square\)

**Theorem 7.2.** Let \( d \geq 3 \) be an even integer. Let \( A \) be a ring of dimension \( d \) containing an infinite field such that one of the following conditions hold:

1. \( A \) is a semi local ring of characteristic \( p > 0 \);
2. Let \( F \) be a field which is algebraic over a finite field. \( A \) is an affine \( F \)-algebra.

Let \( P \) be a projective \( A[T] \)-module with trivial determinant of rank \( d \). Let \( I \subset A[T] \) be an ideal of height \( d \) such that there is a surjection \( \phi : P \to I \). Then \( P \) has a unimodular element if and only if \( \mu(I) = d \).

Proof. Suppose that \( P \) has a unimodular element. Fix a trivialization \( \chi : \wedge^d P \cong A[T] \).

Since \( A \) containing an infinite field, we may assume that \( \text{ht}(I(0)) \geq d \). As \( P \) has a unimodular element, so does \( P \otimes A(T) \). Let \((P, \chi, \phi)\) induce the set of generators of \( I = \langle f_1, \ldots, f_d \rangle + I^2 \).

Since \( P \otimes A(T) \) has a unimodular element using Theorem [6,2] there exists \( g_i \in A(T) \) such that \( I A(T) = \langle g_1, \ldots, g_d \rangle \), with \( f_i - g_i \in I^2 A(T) \). This will induce a set of generators of \( I(0) = \langle a_1, \ldots, a_d \rangle \) such that \( f_i(0) - a_i \in I(0)^2 \).

Combining these we can find a set of generators of \( I = \langle F_1, \ldots, F_d \rangle + I^2 T \) such that \( F_i - f_i \in I^2 \) [please see [2, Remark 3.9] for details]. Notice that \( F_i - g_i \in I A(T)^2 \). Therefore, using Theorem [6,4] we get \( \mu(I) = d \).

Conversely, assume that \( \mu(I) = d \). Let \( \chi \) be a trivialization \( P \). Let \((I, \omega)\) be a Euler cycle induced by the triplet \((P, \chi, \phi)\). Since \( \mu(I) = d \) using Lemma [2,1] we get \((I, \omega) = 0 \) in \( E^d(A[T]) \). That is, \((I A(T), \omega \otimes A(T)) = 0 \) in \( E^d(A(T)) \). Using Theorem [6,2] we get \( P \otimes A(T) \) has a unimodular element. Therefore, it follows from [4, Theorem 3.4] that \( P \) has a unimodular element. This completes the proof. \(\square\)

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