KP web-solitons from wave patterns: an inverse problem

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Abstract. Nonlinear interactions among small amplitude, long wavelength, obliquely propagating waves on the surface of shallow water often generate web-like patterns. In this article, we discuss how line-soliton solutions of the Kadomtsev-Petviashvili (KP) equation can approximate such web-pattern in shallow water wave. We describe an “inverse problem” which maps a certain set of measurable data from the solitary waves in the given pattern to the parameters required to construct an exact KP soliton that describes the non-stationary dynamics of the pattern. We illustrate the inverse problem using explicit examples of shallow water wave pattern.

1. Introduction

Web-like wave patterns as shown in Figure 1 below formed on the surface of shallow water are often observed at a beach with nearly flat bottom. These patterns are non-stationary in general, and are formed due to the interaction of solitary waves coming onto the shore. On a smaller scale, such wave patterns can be artificially created in table-top experiments [13] (see also movies available at the website http://www.needs-conferences.net/2009/), as well as in more accurate water tank experiments in the laboratory [15, 19]. The purpose of this note is to provide a mathematical model describing the dynamics of such wave patterns based on the Kadomtsev-Petviashvili (KP) equation [10]

\[(4u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0, \tag{1.1}\]

where \(u = u(x,y,t)\) represents the (normalized) wave amplitude. The KP equation describes the propagation of small amplitude, long wavelength, uni-directional waves with small transverse variation (i.e., quasi-two-dimensional waves). An important family of solitary wave solutions of the KP equation are the line-soliton solutions which are globally non-singular, and do not decay along distinct lines in

Figure 1. Wave patterns on a beach. Photographs by M. J. Ablowitz and D. E. Baldwin [2]
the $xy$-plane. Such solutions consist of arbitrary number of asymptotic line solitons for $|y| \gg 0$ and complex web-like interaction patterns of line solitons in the finite region of the $xy$-plane. Because of such interaction patterns, these solutions are sometimes called the web-solitons. Several simple yet exact web-soliton solutions including the Y-shape soliton and several others have been experimentally demonstrated in recent laboratory wave tank experiments by Harry Yeh’s group at Oregon State University [15, 19]. The general line-soliton solutions have been studied extensively by the authors who gave a complete classification of these solutions using geometric and combinatorial techniques [5, 6, 12].

In this article, we consider simple examples of shallow water wave patterns such as in Figure 1 which are assumed to be formed under the same physical conditions where the KP equation is valid. We then formulate the inverse problem which determines the information required to construct the exact line-soliton solution of the KP equation, simply referred to as the KP line-solitons, from the wave pattern data consisting of the amplitude and slopes of the solitary waves, and the locations of those waves on the $xy$-plane for fixed $t$.

Resonant interaction among solitary waves plays an important role in the formation of surface wave patterns that can be approximated by the KP line-solitons. Resonant interactions for the KP solitons in the form of a Y-shape wave-form were first investigated by Miles [16]. It turns out that such Y-shape wave-form is an exact solution of the KP equation, and is referred to as Y-soliton. Subsequently, more general type of resonant and partially-resonant line-soliton solutions have been discovered (see e.g., [4, 11, 3]). In this paper, we demonstrate using explicit examples of shallow water wave patterns that the resonant line-soliton solutions of the KP equation can be used to approximate such patterns and their dynamics fairly well.

Recent interest in the study of the KP line-soliton solutions has generated a wealth of data in the form of photographs and video recordings of wave patterns on flat beaches (see e.g., [18, 1, 2]). There are several works in recent literature on the modeling of shallow water wave patterns by the KP line-solitons (see e.g., [1, 17, 18]). Most of these works are qualitative studies and compares a pattern with a KP soliton at a fixed instant of time $t = t_0$. In contrast, our algorithm makes explicit use of the measurements from a given wave pattern to obtain the analytical form of the KP line-soliton solution which then describes the non-stationary dynamics of the pattern for both $t \geq t_0$ and $t \leq t_0$. It should be noted that the analysis presented here is only a leading order approximation since it is limited to the comparison between the wave patterns and the exact KP theory. In other words, we do not consider higher order corrections to the KP equation due to large amplitude, finite angle (departure from quasi-two-dimensionality), uneven bottom, or any such physical perturbations.

The paper is organized as follows: In Section 2, we provide some basic background for the one line-soliton and the resonant Y-soliton solutions of the KP equation. Then in Section 3 we describe an algorithmic method to construct an exact KP line-soliton solution from a given wave pattern, and illustrate our algorithm using explicit examples of actual shallow water wave patterns. Finally, we conclude with some remarks in Section 4.

2. The KP line-soliton solutions
This section contains background information on the KP line-soliton solutions and brief descriptions of some particular solutions (see [5, 6, 12] for the details).

The solutions of the KP equation (1.1) are given in terms of the $\tau$-function $\tau(x, y, t)$ as

$$u(x, y, t) = 2(\ln \tau)_{xx}. \quad (2.1)$$

For line-solitons the $\tau$-function consists of (i) $M$ distinct real parameters ordered as $k_1 < k_2 < \cdots < k_M$, and (ii) an $N \times M$ real matrix $A$ of full rank with $N < M$. The $\tau$-function is explicitly given by

$$\tau(x, y, t) = \sum_I \Delta_I(A) E_I(x, y, t), \quad (2.2)$$

where $\Delta_I(A)$ is a determinant of order $N$. The matrix $E_I(x, y, t)$ is an $N \times M$ matrix whose $(i, j)$-th element is $e^{i j k_I}$ for $j = 1, 2, \ldots, M$ and $i = 1, 2, \ldots, N$. The $\Delta_I(A)$ are the determinants of order $N$ formed from the matrix $A$ and its first $N - 1$ consecutive powers. The $E_I(x, y, t)$ are determined by the boundary conditions at the $x$-axis.

The $\tau$-function is related to the solution of the KP equation by the formula

$$\tau(x, y, t) = \sum_{\alpha \in \mathbb{Z}^M} c_\alpha(x, y, t) e^{i \sum_{k=1}^M \alpha_k k k}.$$ 

For line-solitons, the coefficients $c_\alpha(x, y, t)$ are given by

$$c_\alpha(x, y, t) = \sum_{I} \Delta_I(A) E_I(x, y, t) e^{i \sum_{k=1}^M \alpha_k k k},$$

where $\Delta_I(A)$ is the determinant of order $N$ formed from the matrix $A$ and its first $N - 1$ consecutive powers.
where the sum is over all (ordered) \( N \)-element subsets \( I \) of the set \( \{1, \ldots, M\} \). The coefficient \( \Delta_I(A) \) is the \( N \times N \) minor of the matrix \( A \) with the column set \( I \), and \( E_I(x, y, t) := K_I \exp \Theta_I(x, y, t) \) where

\[
K_I = \prod_{j>m} (k_i - k_m), \quad \Theta_I(x, y, t) = \sum_{m=1}^{N} (k_i x + k_m^2 y - k_m^3 t).
\]  

(2.3)

The soliton solution \( u(x, y, t) \) is regular if and only if \( \Delta_I(A) \geq 0 \) for all the \( k \)-element subsets \( I \) [14]. In this case, the matrix \( A \) is called a totally non-negative matrix.

**Remark 2.1** In [7], it was shown that the solution (2.1) can be also expressed using the \( \tau \)-function in the Grammian form, i.e. \( u = 2(\ln \tau_h)_{xx} \) with

\[
\tau_h(x, y, t) = \det(I + CF), \quad F_{ij} = \int f_{ij} g_j dx
\]

where \( C \) is an \( N \times (M-N) \) constant matrix, and \( F \) is an \( (M-N) \times N \) matrix for \( f_i = e^{\Phi(p)} \) and \( g_j = e^{-\Phi(q)} \) with \( \Phi(k) = k x + k^2 y - k^3 t \). Those parameters \( q_1, q_2, \ldots, q_N, p_1, p_2, \ldots, p_{M-N} \), and the coefficient matrix \( C \) are uniquely determined by \( k_j \)'s and the matrix \( A \) in (2.2).

In [5, 6], we showed that the general soliton solution given by equations (2.1) and (2.2) consists of \( N \) and \( (M-N) \) asymptotic line-solitons as \( y \gg 0 \) and \( y \ll 0 \), respectively. Each asymptotic soliton is uniquely parametrized by a pair of distinct \( k \)-parameters \( \{k_i, k_j\} \) for \( i < j \), and we let \([i, j]\) denote the index pair for this soliton (see section 2.1 for further explanation). The index pair \([i, j]\) is uniquely characterized by a map \( \pi \) such that \( \pi(i) = j \) if \([i, j]\) labels an asymptotic soliton for \( y \gg 0 \), and \( \pi(j) = i \) if \([i, j]\) labels an asymptotic soliton for \( y \ll 0 \). The map \( \pi \) turns out to be fixed-point free permutation of the index set \( \{1, \ldots, M\} \) known as derangement which is conveniently represented by a chord diagram as shown in examples below. Thus the soliton solution generated by the \( \tau \)-function (2.2) is represented by the chord diagram associated to the derangement \( \pi \).

Next we briefly describe the one line-soliton and the Y-solitons which form the building blocks of a web-soliton solution of the KP equation.

### 2.1. One line-soliton solution

Asymptotic analysis of the \( \tau \)-function in (2.2) reveals that the solution \( u(x, y, t) \) is exponentially vanishing in regions of the \( xy \)-plane where a single exponential \( E_I \) is dominant over all other exponentials in (2.2), while it is localized along certain lines denoted by an index pair \([i, j]\) where a pair of dominant exponentials \( E_I, E_J \) are in balance. In the neighborhood of this line \([i, j]\), the \( \tau \)-function can be approximated by

\[
\tau \approx \Delta_I(A) E_I + \Delta_J(A) E_J,
\]

where \( I = I_0 \cup \{i\} \) and \( J = I_0 \cup \{j\} \) with \( |I_0| = N - 1 \). The solution \( u(x, y, t) \) along this line approximates a single line-soliton solution,

\[
u(x, y, t) \approx A_{[i, j]} \sech \Theta_{[i, j]}(x, y, t),
\]

and will be referred to as the \([i, j]\)-soliton. Here the phase \( \Theta_{[i, j]} \) is given by

\[
\Theta_{[i, j]} := \frac{1}{2}(\Theta_I - \Theta_J) = \frac{1}{2}(k_i - k_j)(x + \tan \Psi_{[i, j]} y - C_{[i, j]} t + x_{[i, j]}),
\]

\[
x_{[i, j]} = \frac{1}{k_i - k_j} \ln \frac{\Delta_I(A) K_I}{\Delta_J(A) K_J},
\]

(2.4)

and \( K_I, \Theta_I \) are as in (2.3). The soliton amplitude \( A_{[i, j]} \), slope \( \Psi_{[i, j]} \) and velocity in the positive \( x \)-direction \( C_{[i, j]} \), are defined respectively by

\[
A_{[i, j]} = \frac{1}{2}(k_i - k_j)^2, \quad \tan \Psi_{[i, j]} = k_i + k_j, \quad C_{[i, j]} = k_i^2 + k_j^2 + k_i k_j > 0.
\]

(2.5)
A contour plot of this line-soliton solution is shown in Figure 2. Note that the angle \( \Psi_{[i,j]} \) is measured counterclockwise from the y-axis, and \(-\frac{\pi}{2} < \Psi_{[i,j]} < \frac{\pi}{2}\).

The line \([i,j]\) forms the boundary between the regions of dominant exponentials \(E_I\) and \(E_J\). The parameters \(k_i\) and \(k_j\) are exchanged during the transition from the region of one dominant exponential to the other by crossing the \([i,j]\)-soliton. Hence the line \([i,j]\) can be viewed as representing a permutation of the index set \([i,j]\). The chord diagram in Figure 2 depicting the exchange \(k_i \leftrightarrow k_j\) represents the permutation \(\pi = (i \ j \ j \ i)\) of the index set \([i,j]\).

![Figure 2. Contour plot and the chord diagram representation of the \([i,j]\)-soliton](image)

### 2.2. Y-soliton solutions

The Y-soliton is formed due to the resonant interaction of three line solitons labeled \([i,j]\), \([j,l]\) and \([i,l]\) with \(i < j < l\). If we represent each line soliton as a traveling wave \(u = \Phi(K_{a,b} \cdot r - \Omega_{a,b} t)\) with \(r = (x,y)\), and \(K_{a,b} = \frac{1}{2}(k_b - k_a, k^2_b - k^2_a)\), \(\Omega_{a,b} = \frac{1}{2}(k^3_b - k^3_a)\) being the wave-vector and the frequency of the wave, then the soliton triplet satisfy the resonant conditions

\[
K_{[i,j]} + K_{[j,l]} = K_{[i,l]}, \quad \Omega_{[i,j]} + \Omega_{[j,l]} = \Omega_{[i,l]}.
\]

Near the trivalent vertex, the \(\tau\)-function has the form

\[
\tau \approx \Delta_I(A)E_I + \Delta_J(A)E_J + \Delta_L(A)E_L.
\]

There are two cases for the index sets \([I,J,L]\) leading to the contour plots in Figures 3.

![Figure 3. Contour plots and chord diagrams of the Y-solitons. Resonant soliton triplet \([i,j],[j,l]\) and \([i,l]\). The horizontal line in each chord diagram gives the values of the \(k\)-parameters \(k_i < k_j < 0 < k_l\).](image)
Due to the ordering \( k \in \mathbb{N} \) of the slopes of the solitons \( k_i, k_j \) and \( k_l \) appear below it to form a \( \Lambda \)-shape as shown in Figure 3(a). Following the transitions of the dominant exponentials clockwise in Figure 3(a), one recovers the permutation \( \pi = \left( \begin{array}{ccc} i & j & l \\ i & j & l \end{array} \right) \) given by the three line solitons \( [i, j], [j, l], [i, l] \). The chord diagram corresponding to this permutation is shown below the contour plot.

(b) The index sets are \( I = I_0 \cup \{i\}, J = J_0 \cup \{j\} \) and \( L = I_0 \cup \{l\} \) where \( I_0 \) is the common set of indices with \( |I_0| = N - 1 \). In this case the \([i, l]\)-soliton is above the trivalent vertex while the solitons \([i, j]\) and \([j, l]\) appear above it to form an \( \Lambda \)-shape as shown in Figure 3(a). Following the transitions of the dominant exponentials clockwise in Figure 3(a), one recovers the permutation \( \pi = \left( \begin{array}{ccc} i & j & l \\ i & j & l \end{array} \right) \) given by the three line solitons \([i, j], [j, l], [i, l]\). The chord diagram corresponding to this permutation is shown below the contour plot.

Due to the ordering \( k_i < k_j < k_l \), the \([i, l]\)-soliton has the largest amplitude \( A_{[i, l]} = \frac{1}{2}(k_l^2 - k_i^2) \), and the \([j, l]\)-soliton has the largest slope \( \tan \Psi_{[j, l]} = k_j + k_i \), while the \([i, j]\)-soliton has the smallest slope. Note that all these information can be gathered directly from the chord diagrams presented in Figure 3, which is one of the main reasons we use the chord diagrams in our analysis. One should also note that the \( k \)-parameters in the Y-soliton are uniquely determined only from the slopes of the three solitons. Recall that the slopes of the solitons \([i, j], [j, l], [i, l]\) are respectively given by

\[
\tan \Psi_{[i, j]} = k_i + k_j, \quad \tan \Psi_{[j, l]} = k_j + k_i, \quad \tan \Psi_{[i, l]} = k_i + k_l.
\]

Then one can compute the values of the \( k \)-parameters as

\[
k_i = \frac{1}{2} \left( \tan \Psi_{[i, j]} + \tan \Psi_{[i, l]} - \tan \Psi_{[j, l]} \right),
\]

\[
k_j = \tan \Psi_{[i, j]} - k_i, \quad k_l = \tan \Psi_{[i, l]} - k_i
\]

**Remark 2.2** The KP equation also admits solutions where two line solitons interact to form an X-vertex. In this case, the interaction is non-resonant and the wave pattern is stationary. An important feature of X-soliton solution is the existence of a constant phase shift which appears as a shift of the crest-line of each line-soliton at the interaction region, and is determined by the amplitudes and slopes of the solitary waves forming X-shape pattern [6]. In a real physical situation, the phase shift is typically at most twice the soliton wavelength, and large phase shifts several times the soliton wavelength are highly improbable (see [8] for more details). A long stem observed in a real physical pattern does not correspond to an X-soliton, rather it corresponds an entirely different non-stationary, resonant solution of the KP equation. For example, the stationary wave pattern in Figure 1(a) shows an X-type interaction, while the wave pattern in Figure 1(b) is not of X-type. The long stem appearing in the middle of Figure 1(b) is an intermediate solitary wave which interact resonantly with other two solitons at the trivalent vertices.

A general line-soliton solution of the KP equation can be regarded as a collection of one-solitons and Y-solitons which fit together to form a web-structure with X- and Y-vertices in the interaction region. A combinatorial description of this solution is obtained by gluing the local chord diagrams corresponding to those components to form the complete chord diagram for the derangement \( \pi \) associated with the general soliton solution. We will implement this technique of “gluing” of chords in the next section in order to reconstruct a KP soliton solution from a real wave pattern.

### 3. The Inverse problem

In this section we discuss our main result which is the problem of constructing an exact soliton solution of the KP equation from a given wave pattern. As examples, we consider the surface wave patterns on shallow water observed on flat ocean beaches. The construction procedure consists of the following steps:
(i) Set the $xy$-coordinates, so that all solitary waves in the pattern are traveling almost in the positive $x$-direction (towards the shore).

(ii) Using (2.5) and (2.6), find the $k$-parameters by measuring the amplitudes and slopes of the solitary waves in the pattern.

(iii) Define the derangement $\pi$ and its chord-diagram from the $k$-parameters obtained in the step (ii).

(iv) Determine the form of the matrix $A$ from the derangement $\pi$ using combinatorial techniques [14].

(v) Using (2.4), find the matrix elements of $A$ from the minors $\Delta_i(A)$ by measuring the locations of the solitary waves in the pattern.

For a simple illustration of our method, in this note, we consider wave patterns with only two trivalent vertices consisting of two Y-type (one Y-shape and other $\lambda$-shape) waves which have a common intermediate solitary wave. Such patterns are observed fairly regularly in flat ocean beaches. We distinguish the Y-type waves by assigning a white trivalent vertex for the $\lambda$-shape wave and a black trivalent vertex for the Y-shape wave. There are 5 distinct cases for such wave pattern as shown in Figure 4. The resonant vertices in each of these patterns are labeled A (upper vertex) and B (lower vertex). The solitary waves incident at the vertex A are numbered as 1, 2 and 3, while the solitary waves numbered as 3, 4 and 5 are incident at the vertex B. Thus each pattern consists of two Y-type waves glued together by the common solitary wave 3 joining vertices A and B. We first identify each solitary wave with an $[i, j]$-soliton for some parameters $k_i < k_j$ to be determined. If we denote the parameters $\{k_a, k_b, k_c\}$ at vertex A, and $\{k_{a'}, k_{b'}, k_{c'}\}$ at vertex B, then each one-soliton can be parametrized as follows:

Soliton 1 Soliton 2 Soliton 3 Soliton 4 Soliton 5
\{k_a, k_b\} \{k_{a'}, k_{b'}\} \{k_b, k_c\} \{k_{b'}, k_{c'}\} \{k_{a'}, k_{c'}\}

Note that there are two parametrizations (with respect to the vertices A and B) for the intermediate soliton 3. Let $\tan \Psi_i, i = 1, 2, \ldots, 5$ denote the slope of each one-soliton. In particular, the slope of soliton 3 is given as $\tan \Psi_3 = k_b + k_c = k_{b'} + k_{c'}$. Then the $k$-values can be solved uniquely from the slope measurements at the vertices A and B for each Y-soliton as given by (2.6), i.e.

\[
\begin{align*}
&\begin{cases}
  k_a = \frac{1}{2}(\tan \Psi_1 + \tan \Psi_2 - \tan \Psi_3), \\
  k_b = \tan \Psi_1 - k_a, \quad k_c = \tan \Psi_2 - k_a,
\end{cases} \\
&\begin{cases}
  k_{a'} = \frac{1}{2}(\tan \Psi_4 + \tan \Psi_5 - \tan \Psi_3), \\
  k_{b'} = \tan \Psi_4 - k_{a'}, \quad k_{c'} = \tan \Psi_5 - k_{a'}
\end{cases}
\end{align*}
\]

(3.1)

Without loss of generality, the two sets of $k$-parameters for soliton 3 can be ordered as $k_b < k_c$ and $k_{b'} < k_{c'}$. If the given wave pattern is an exact KP soliton, then one should have $k_b = k_{b'}$ and $k_c = k_{c'}$. But from equations (3.1) we have

\[k_b - k_{b'} = -(k_c - k_{c'}) = \frac{1}{2}[(\tan \Psi_1 - \tan \Psi_2) - (\tan \Psi_4 - \tan \Psi_5)],\]
which in general, has a non-zero value due to the deviation from the exact KP theory which is only a leading order approximation to a real wave pattern. To approximate the pattern by a KP soliton, we take the average of the two sets of k-values obtained from the measurements at the vertices A and B. That is, we define the k-parameters of soliton 3 as

$$\tilde{k}_b = \frac{1}{2}(k_b + k_b'), \quad \tilde{k}_c = \frac{1}{2}(k_c + k_c'),$$

which preserves the slope of soliton 3 since $$\tan \Psi_3 = \tilde{k}_b + \tilde{k}_c$$. Then we have the following re-parametrization of the line solitons:

$$\{k_a, \tilde{k}_b\} \quad \{k_a, \tilde{k}_c\} \quad \{\tilde{k}_b, \tilde{k}_c\} \quad \{k_a', \tilde{k}_b\} \quad \{k_a', \tilde{k}_c\}$$

In what follows, we provide a simple, step-by-step algorithm to reconstruct the exact KP soliton solution from a given wave pattern with two trivalent resonant vertices.

3.1. Step-by-step construction of the KP line-soliton solution

For specificity, we use a real example of the wave pattern in Figure 5 which corresponds to the first pattern in Figure 4.

![Figure 5](image)

**Figure 5.** A real example of the wave pattern corresponding to the first pattern in Figure 4. Photograph of the wave pattern taken by D. E. Baldwin [2]

**Step 1: The k-parameters** We first set the xy-coordinates, so that the wave is propagating along almost x-direction, and choose the origin as the vertex A in Figure 5. Then we trace the wave crests as a graph, and measure the angles of all the line solitons with respect to the positive y-axis. For this example, we estimate the angles for the line solitons to be

$$\Psi_1 = -15^\circ, \quad \Psi_2 = 3^\circ, \quad \Psi_3 = 25^\circ, \quad \Psi_4 = -13^\circ, \quad \Psi_5 = 5^\circ.$$  

Next from (3.1), one obtains $$k_a = -0.341, k_b = 0.073$$ and $$k_c = 0.393$$ at the vertex A, and $$k_a' = -0.305, k_b' = 0.074$$ and $$k_c' = 0.392$$ at the vertex B. After averaging to get $$\tilde{k}_b, \tilde{k}_c$$ and then arranging the k-parameters in increasing order gives

$$(k_a, k_b', \tilde{k}_b, \tilde{k}_c) = (k_1, k_2, k_3, k_4) = (-0.341, -0.305, 0.0735, 0.3925).$$

Note that the slope data at the resonant vertices is sufficient to obtain the k-parameters; the amplitude data for the solitons, which is difficult to measure from a photograph, is not necessary in this case. The amplitude of each one-soliton can be calculated from (2.5) once the k-values have been found.

From this set of k-parameters, the procedure to find the derangement $$\pi$$ for the data may be illustrated as a process of gluing the chord diagrams of the two Y-solitons with vertices A and B through the common chord as illustrated in Figure 6. At the vertex A, one has a X-soliton of the type formed by line solitons...
labeled 1, 2, and 3, and the corresponding chord diagram is indicated by the top diagram in the middle of Figure 6. The dots in this diagram denote the \( k \)-parameters \( k_a < k_b < k_c \). Similarly, the bottom chord diagram corresponds to the Y-soliton at the vertex B formed by solitons 3, 4, and 5 with the dots marking the \( k \)-parameters \( k_d < k_f < k_e \). We then superpose the top and bottom diagrams and erase the common chord for the intermediate soliton 3 and recover the complete chord diagram of the asymptotic solitons shown by the rightmost diagram in Figure 6. In this diagram the \( k \)-parameters are marked (by the 4 dots) as \( k_a = k_1, k_d = k_2 \) and the average values \( \bar{k}_b = k_3, \bar{k}_c = k_4 \) with \( k_1 < k_2 < k_3 < k_4 \). This diagram corresponds to the derangement \( \pi = (4312) \) in the one-line notation of permutation. The top chords in this diagram are associated to the asymptotic solitons 2 and 4 for \( y \gg 0 \) which are identified as \([1, 4]-\) and \([2, 3]-\) soliton. Similarly, the bottom chords in this diagram are associated to the \([1, 3]-\) and \([2, 4]-\) asymptotic line solitons for \( y \ll 0 \). This is summarized in Figure 6.

![Figure 6. Chord gluing algorithm for the wave pattern in Figure 5](image)

**Step 2: Form of the A-matrix** The totally non negative matrix \( A \) in the \( \tau \)-function (2.2) can be derived from the derangement \( \pi \) obtained in Step 1 by employing combinatorial methods developed in a recent paper by Kodama and Williams [14], we refer the interested readers to that article and omit the details in this short note. Alternatively, the interested reader may also consult the work of Chakravarty and Kodama [5, 6] where in particular, the \( A \)-matrices for those patterns in Figure 4 were explicitly given in reduced row echelon form for the corresponding derangements \( \pi \). For the current example with \( \pi = (4312) \), the \( A \)-matrix is given by

\[
A = \begin{pmatrix}
1 & 0 & -b & -c \\
0 & 1 & a & 0
\end{pmatrix}
\]

with three real parameters \( \{a, b, c\} \). Note that \( A \) is a totally nonnegative matrix, i.e., \( \Delta_{i,j}(A) \geq 0 \) for all \( 1 \leq i < j \leq 4 \), if all \( a, b, c > 0 \). Then the resulting exact solution of KP is non-singular for all \( x, y \) and \( t \).

**Step 3: Evaluation of the A-matrix elements** The last step is to explicitly find the entries of the \( A \)-matrix. These are related to the locations of certain line-solitons in the given wave pattern as explained below. Figure 7 shows the asymptotic solitons labeled by their index pairs \([i, j]\) corresponding to the wave pattern of our example. The index set \( I = \{i, j\} \) indicates each region in the \( xy \)-plane where the exponential \( E_j \) is dominant. The corresponding minor \( \Delta_i(A) \) of the matrix \( A \) with column set \( I \) is given in terms of the parameters \( a, b \) and \( c \). The line corresponding to the \([i, j]\)-soliton is defined by \( \Theta_{[i,j]} = 0 \). The expression (2.4) for \( \Theta_{[i,j]} \) contains the constant \( x_{i,j} \) which involves the ratios of \( \Delta_i(A) \) and \( \Delta_j(A) \) where \( I \) and \( J \) label the adjacent regions separated by the line \([i, j]\).

We pick a point \((x_0, y_0)\) on the line corresponding to the \([1, 4]\)-soliton in Figure 7. Setting the time \( t = 0 \), we have the equation of this line,

\[
x + (k_1 + k_4)y + \frac{1}{k_1 - k_4} \ln \frac{\Delta_{1,2}(A)(k_2 - k_1)}{\Delta_{2,4}(A)(k_4 - k_2)} = 0.
\]
Δ₁,₂ = 1,   Δ₂,₃ = b
Δ₂,₄ = c,       Δ₃,₄ = ac

Figure 7. Dominant exponentials and the minors of the matrix A for the wave pattern in Figure 5

Using the facts that Δ₁,₂(A) = 1 and the point (x₀, y₀) is on the line, one obtains

Δ₂,₄(A) = c = \frac{k₂ - k₁}{k₄ - k₂} \exp \left( - \left( k₄ - k₁ \right)x₀ - \left( k₄² - k₂² \right)y₀ \right).

Similarly, selecting a point (x₀, y₀) on line corresponding to the [1, 3]-soliton, yields

Δ₂,₃(A) = b = \frac{k₂ - k₁}{k₃ - k₂} \exp \left( - \left( k₃ - k₁ \right)x₀ - \left( k₃² - k₂² \right)y₀ \right),

and finally, by choosing a point on the [2, 3]-line, one can get a = Δ₃,₄(A)/Δ₂,₄(A). In this way, we find the parameters a, b, c (hence the A-matrix).

For this example, we chose the origin as the vertex A common to lines corresponding to the [1, 3]- and [1, 4]-solitons, and estimate the location of the vertex B as (50, -107.2) which lie on the [2, 3]- and [2, 4]-soliton lines. Taking (x₀, y₀) as those points, we obtain the values a = 0.110 × 10⁻¹¹, b = 0.095, c = 0.052 which gives the exact A-matrix in Step 2. From the A-matrix and the k-parameters found in Step 1, one can construct the τ-function and the soliton solution using (2.1). Plots of the exact solution, u = 2(ln τ)ₓₓ, are shown in Figure 8. The middle plot at t = 0 corresponds to the photograph of the wave pattern in Figure 5, the left and right plots are for t < 0 and t > 0, respectively. Note that the wave pattern is non-stationary, and the stem (soliton 3) is an intermediate soliton rather than a stationary phase shift.

Figure 8. Evolution of the KP soliton constructed from the wave pattern in Figure 5.

One should note that the τ-function for the solution obtained here contains five exponential terms, i.e.

\[ \tau = \sum_{1 \leq i < j \leq 4} \Delta_{i,j}(A)E_{i,j} \quad \text{with} \quad \Delta_{i,j}(A) \neq 0 \quad \text{except} \quad \Delta_{1,4}(A) = 0, \]

while the τ-function for an X-soliton has only four exponential terms. The main difference is that the X-soliton solutions form a stationary pattern whereas the solution corresponding to our example above
is non-stationary. The Grammian form (used by some authors) for the solution in Figure 8 is obtained from Remark 2.1 if one chooses \( \{k_1, k_2, k_3, k_4\} = \{q_1, q_2, p_1, p_2\} \) and \( C = D_1^{-1}JD_2 \) where

\[
J = \begin{pmatrix}
-b & -c \\
-a & 0
\end{pmatrix}, \quad D_1 = \begin{pmatrix}
q_1 - q_2 & 0 \\
0 & q_2 - q_1
\end{pmatrix}, \\
D_2 = \begin{pmatrix}
(p_1 - q_1)(p_1 - q_2) & 0 \\
0 & (p_2 - q_1)(p_2 - q_2)
\end{pmatrix}.
\]

### 3.2. Example: Wave patterns in Lake Peipsi, Estonia

To further illustrate our algorithm we present another example where we consider a succession of photographs of an evolving wave pattern and compare the dynamics with the time evolution of the exact KP line-soliton solution reconstructed by the algorithm described in Section 3.1.

The top panel in Figure 9 shows a snapshot of a wave pattern observed on a beach of Lake Peipsi, Estonia. This photograph is part of a video footage showing the dynamics of the pattern over time. We apply our step-by-step algorithm to this snapshot chosen to be at \( t = 0 \) to construct the exact soliton solution \( u(x, y, 0) \), then compare \( u(x, y, t) \) for \( t \neq 0 \) with other snapshots taken from the video footage.

The middle picture of the top panel in Figure 9 shows the graph traced from the wave crests in the pattern. This corresponds to the fourth pattern in Figure 4. From the graph, the angles of the line-solitons are estimated to be \( \Psi_1 = 14^\circ, \Psi_2 = 37^\circ, \Psi_3 = 0^\circ, \Psi_4 = -42^\circ, \Psi_5 = -20^\circ \). Following Step 1 in Section 3.1, we obtain the \( k \)-parameters as \( \{k_1, k_2, k_3, k_4\} = \{-0.632, -0.268, 0.268, 0.517\} \). Then we have the derangement \( \pi = (2413) \) as shown in the top right panel of Figure 9. Here solitons 2 and 5 correspond to the \([3,4]\)- and \([1,3]\)-solitons respectively, for \( y < 0 \), whereas solitons 1 and 4 correspond to \([2,4]\)- and \([1,2]\)-solitons for \( y > 0 \). Finally, we construct the exact soliton solution following Steps 2 and 3 in Section 3.1. The bottom two panels of Figure 9 compares the time evolutions of the exact soliton solution and the actual wave pattern. The last snapshot in the middle panel corresponds to \( t = 0 \) from which the exact soliton solution was constructed, the previous snapshots in this panel correspond to \( t < 0 \). Also note that the leftmost snapshot corresponds to the wave pattern of the third case in Figure 4 indicating that the evolving solution can coincide with more than one configuration shown in Figure 4.

![Figure 9. Waves in Lake Peipsi, Estonia. Snapshots from a video courtesy Ira Didenkulova.](image-url)
4. Conclusion
In this note we provided an algorithm to construct an exact solution of the KP equation approximating a given non-stationary pattern of small amplitude, long wavelength and primarily unidirectional shallow water waves. In this “inverse problem” one can directly read off the parameters for the exact soliton solution by measuring the angles and locations of the solitary waves in the given pattern with respect to a fixed reference frame. We have illustrated our method by applying it to photographs and videos of real wave patterns and have shown that the constructed exact solutions and their time evolution are in good agreement with the dynamics of these non-stationary patterns. Our method is only a leading order approximation to the pattern as we do not consider higher order corrections to the KP equation to take into account effects such as large amplitude, obliqueness of interaction, non constant water depth at the beach etc.

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