Pattern-avoiding permutations and Brownian excursion, part II: fixed points

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Abstract Permutations that avoid given patterns are among the most classical objects in combinatorics and have strong connections to many fields of mathematics, computer science and biology. In this paper we study fixed points of both 123- and 231-avoiding permutations. We find an exact description for a scaling limit of the empirical distribution of fixed points in terms of Brownian excursion. This builds on the connections between pattern-avoiding permutations and Brownian excursion developed in Hoffman et al. (Pattern-avoiding permutations and Brownian excursion, Part 1: Shapes and fluctuations, to appear Random Structures and Algorithms. arXiv:1406.5156, 2016) and strengthens the recent results of Elizalde (Electron J Comb 18(2):17, 2011) and Miner and Pak (Adv Appl Math 55:86–130, 2014) on fixed points of pattern-avoiding permutations.

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1 Introduction

In this paper we study the asymptotic behavior of the fixed points of pattern-avoiding permutations. The study of random pattern-avoiding permutations has drawn considerable attention in the recent literature. A body of work has developed around studying geometric properties of the graph of the permutation. A surprising result was that Brownian excursion began to appear, in various guises, in descriptions of the limiting objects, see for example the recent work of Janson [12], Madras and Liu [19], Madras and Pehlivan [20] and Miner and Pak [22]. In Part I of this series [11], we gave a strong pathwise connection between the graph of a pattern-avoiding permutation and Brownian excursion that explains the large scale behavior of the graph. This result, however, does not immediately yield information about local properties, such as fixed points of the permutation. The fixed points of pattern-avoiding permutations have drawn special attention in the literature, see for example [6–9,22]. In this paper we show that the fixed points of pattern-avoiding permutations are related to Brownian excursion. This result is surprising because, although Brownian excursion is related to the bulk behavior of the graph of a pattern-avoiding permutation [11], the property of being a fixed point is a very local property.

Our main result is the following theorem, which as far as we know is the first to give a connection between the asymptotic distribution of fixed points of pattern-avoiding permutations and Brownian excursion. Recall that if $\pi \in S_k$ and $\tau \in S_n$, we say that $\tau$ avoids $\pi$, or is $\pi$-avoiding, if $\tau$ does not contain $\pi$.

**Theorem 1.1** Let $(\varrho_t, 0 \leq t \leq 1)$ be a standard Brownian excursion.

(a) Let $\sigma_n$ be a uniformly random $231$-avoiding permutation of $[n]$. Then

$$\lim_{n \to \infty} \frac{1}{n^{1/4}} \sum_{i=1}^{n} \frac{\delta_{i/n}}{\sqrt{n}} 1_{\{\sigma_n(i) = i\}} = d \frac{1}{2^{7/4} \pi^{1/2}} \varrho_t^{-3/2} dt,$$

where the convergence is with respect to weak convergence of finite measures on $[0, 1]$ and $\delta_a$ is the point mass at $a$.

(b) Let $\rho_n$ be a uniformly random $123$-avoiding permutation of $[n]$ and let $A$ and $B$ be independent Bernoulli $(1/4)$ random variables, also jointly independent of $(\varrho_t, 0 \leq t \leq 1)$. Then

$$\lim_{n \to \infty} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \delta_{i-rac{n}{\sqrt{2n}}} 1_{\{\rho_n(i) = i\}} = d A \delta_{\varrho_t(1/2)/2} + B \delta_{\varrho_t(1/2)/2},$$

where the convergence is with respect to weak convergence of finite measures on $\mathbb{R}$.

This result builds on the large scale connection between pattern-avoiding permutations and Brownian excursion developed in [11], where it was used to show that the
bulk of a pattern-avoiding permutation can be asymptotically described by Brownian excursion. Part (b) of Theorem 1.1 has a nice interpretation. A 123-avoiding permutation can have at most two fixed points, one above \( n/2 \) and one below it. Part (b) of the theorem says that, asymptotically, these fixed points occur independently. Moreover, conditionally given that both fixed points exist they are reflections of each other across \( n/2 \) and the fluctuation of their distance from \( n/2 \) is given by the midpoint of a Brownian excursion. We emphasize that the limiting measure has the additional randomness of \((A, B)\) that is not part of the Brownian excursion. In the proof we will see that this is a consequence of the fact that having \( \sigma(i) = i \) is in a sense a local property of the permutation. In Part (a) of Theorem 1.1, such extra randomness is present at the discrete level, but does not appear in the limit for reasons related to the Law of Large Numbers.

The appearance of Brownian excursion in Theorem 1.1 will be explained by particular bijections between pattern-avoiding permutations and Dyck paths. The bijection we use for 231-avoiding permutations was first used in Part I [11] and is better suited to extracting probabilistic information than more classical bijections, while the bijection we use for 123-avoiding permutations is a classical bijection from [1].

The fixed points of random permutations have been well studied in both probability and combinatorics. We will not survey the field here, but for the sake of comparison we state the classical result of Montmort and Bernoulli [5] on the distribution of the number of fixed points in a uniformly chosen random permutation in language similar to ours.

**Theorem 1.2** ([5]) Let \( \pi_n \) be a uniformly random permutation of \([n]\) and let \( N \) be a Poisson random measure on \([0, 1]\) with intensity equal to Lebesgue measure. Then

\[
\lim_{n \to \infty} \sum_{i=1}^{n} \delta_{i/n} 1_{[\pi_n(i) = i]} \rightharpoonup N,
\]

where the convergence is with respect to weak convergence of finite measures on \([0, 1]\).

Montmort [5] shows that the number of fixed points converges to a Poisson random variable, but the extension to convergence of the empirical distribution of fixed points to a Poisson random measure is straight-forward, see e.g [3, Theorem 11] for a strong version of this result based on Stein’s method. Comparing Theorems 1.1 and 1.2, we see that 231-avoiding permutations have many more fixed points than uniformly random permutations and these fixed points are more likely to appear near 1 or \(n\), while 123-avoiding permutations have fewer fixed points than uniformly random permutations and they are more closely concentrated around \(n/2\).

The previous strongest results on the fixed points of pattern-avoiding permutations were established in [7,22], which we summarize in the following theorem.

**Theorem 1.3** For a permutation \( \pi \), let \( \text{fp}(\pi) \) be the number of fixed points of \( \pi \).

(a) (Theorem 6.4 [22]) Let \( \sigma_n \) be a uniformly random 231-avoiding permutation of \([n]\). Then
\[ \lim_{n \to \infty} n^{-1/4} \mathbb{E}(\text{fp}(\sigma_n)) = \frac{\Gamma\left(\frac{1}{4}\right)}{2\sqrt{\pi}}. \]

(b) (Proposition 5.3 [7], Theorems 6.3 [22]) Let \( \rho_n \) be a uniformly random 123-avoiding permutation of \([n]\). Then

\[ \lim_{n \to \infty} \mathbb{E}(\text{fp}(\rho_n)) = \frac{1}{2} \]

and for every \( \epsilon > 0 \)

\[ \lim_{n \to \infty} \mathbb{P}\left( \rho_n(i) = i \text{ for some } i \text{ such that } |i - \frac{n}{2}| > \epsilon n \right) = 0. \]

We remark that there is a small mistake in [22, Theorem 6.4], where the limit in Part (a) is given as \(2\Gamma\left(\frac{1}{4}\right)/\sqrt{\pi}\), but it is easily seen from the proof of [22, Theorem 6.4] that the value we give here is correct. From this we see that our results in Theorem 1.1 are the first to give detailed information about the asymptotic distribution of fixed points of pattern-avoiding permutations.

1.1 Connections with invariance principles

In this section we give more detail on the relationship between Theorem 1.1 and the results of Part I [11]. As in [11], our results here are derived from bijections between Dyck paths and pattern-avoiding permutations. Throughout the paper we use the following definition of a Dyck path.

**Definition 1.4** A Dyck path \( \gamma \) is a sequence \( \{\gamma(x)\}_{x=0}^{2n} \) that satisfy the following conditions:

- \( \gamma(0) = \gamma(2n) = 0 \)
- \( \gamma(x) \geq 0 \) for all \( x \in \{0, 1, \ldots, 2n\} \) and
- \( |\gamma(x + 1) - \gamma(x)| = 1 \) for all \( x \in \{0, 1, \ldots, 2n - 1\} \).

We often want to consider the function generated by a Dyck path through linear interpolation. Throughout this paper we often use the same notation to denote a sequence and the continuous function generated by extending it through linear interpolation.

Brownian excursion is the process \((e_t)_{0 \leq t \leq 1}\) which is Brownian motion conditioned to be 0 at 0 and 1 and positive in the interior [23]. It is well known that the scaling limit of Dyck paths are Brownian excursion [13] and that Dyck Paths of length \(2n\) are in bijection with 321-avoiding and 231-avoiding permutations [15, 18].

**Fixed points of 123-avoiding permutations**

The convergence developed in [11] is strongly suggestive of the general form of the limit distribution of fixed points of 123-avoiding permutations. To see this, we rephrase [11, Theorem 1.2] in terms of 123-avoiding permutations.
Theorem 1.5 [11, Theorem 1.2] Let $\rho_n$ be a uniformly random $123$-avoiding permutation of $[n]$. Then there exists a (random) partition of $[n]$ into $S^+$ and $S^-$ such that if $F_n^\pm : [0, n + 1] \to \mathbb{R}$ is the linear interpolation of the points

$$\{(i, \rho_n(i)) : i \in S^\pm\} \cup \{(0, n)\} \cup \{(n + 1, 0)\}$$

Then

$$\left(\frac{F_n^+(nt) - n(1 - t)}{\sqrt{2n}}, \frac{F_n^-(nt) - n(1 - t)}{\sqrt{2n}}\right)_{t \in [0, 1]} \Rightarrow (\Theta_t, -\Theta_t)_{t \in [0, 1]}.$$

Theorem 1.5 shows that $\rho_n(i) = n - i + O(\sqrt{n})$, where the fluctuation is described by Brownian excursion. Consequently, if $\rho_n(i) = i$ then $i = n - i + O(\sqrt{n})$, so that $i = (n/2) + O(\sqrt{n})$. Thus if $\rho_n$ has any fixed points then they then they are within $O(\sqrt{n})$ of $n/2$. This already gives an improvement over Part (b) of Theorem 1.3 in terms of the location of the fixed points. To establish Part (b) of Theorem 1.1, we must carefully examine the local structure of $\rho_n$ and this is what leads to the independent Bernoulli random variables appearing in the theorem. This will be done using the bijection with Dyck paths introduced in Sect. 2.

Although Theorem 1.5 strongly suggests the general form of Theorem 1.1 Part (b), our proof of Theorem 1.1 does not depend on Theorem 1.5. We also remark that [11, Theorem 1.2] is stated for $321$-avoiding permutations, however, this theorem shows that the fixed points of $321$-avoiding permutations are concentrated near 1 and $n$ (see also [22, Theorem 6.1]) and are not well-described by Brownian excursion.

**Fixed points of $231$-avoiding permutations**

The situation for $231$-avoiding permutations is quite different from the case of $123$-avoiding permutations. In order to explain the connection with the invariance principle from [11], we make use of the bijection we define below in (17). The details of this bijection are not needed for the present discussion, but will be used later in the paper. The following result is the invariance principle we obtain from [11], see also Fig. 1.

**Theorem 1.6** [11, Theorem 1.3] Let $\Gamma^n$ be a uniformly chosen Dyck path of length $2n$ and let $\sigma_{\Gamma^n}$ be the image of $\Gamma^n$ under the bijection (17), so that $\sigma_{\Gamma^n}$ is a uniformly random $231$-avoiding permutation. For any $\epsilon > 0$ there exists a sequence of sets $SE\Gamma^n$ such that

![Fig. 1](https://via.placeholder.com/150)

$\Gamma^n(2nt)/\sqrt{2n}$ along with $-F_n$
\[ P\left( |SE_{\Gamma^n}| > n - n^{75+\epsilon} \right) \rightarrow 1 \]

and
\[
\left( \frac{\Gamma^n(2nt)}{\sqrt{2n}}, \frac{F_n(nt) - nt}{\sqrt{2n}} \right)_{t \in [0,1]} \Rightarrow (e_t, -e_t)_{t \in [0,1]},
\]
where \( F_n \) is the linear interpolation of the points
\[ \{(i, \sigma/Gamma_1^n(i)) : i \in SE_{\Gamma^n}\} \cup \{(0, 0)\} \cup \{(n + 1, n + 1)\} \]

This theorem shows that the bulk of the points in a uniformly random 231-avoiding permutation closely follow a Brownian excursion. However, most of the fixed points of \( \sigma/Gamma_1^n \) are in the set \((\lfloor n \rfloor \setminus SE_{\Gamma^n})\) of exceptional points that Theorem 1.6 does not provide much information about. Nonetheless, we can still describe the asymptotic distribution of fixed point in terms of the limiting excursion of the Dyck path \( \Gamma^n \). Our description of these points will allow us to greatly generalize the results in [22] about distribution of a random 231-avoiding permutation close to the diagonal. We count the number of fixed points in an interval by
\[ \theta_{[an, bn]}(\sigma) = |\{i \in [an, bn] : \sigma(i) = i\}| \]
where \( 0 \leq a < b \leq 1 \). Miner and Pak proved that the expected number of fixed points in the interval \([1, n]\) is of order \( n^{1/4} \) [22]. The next theorem shows that a typical 231-avoiding permutation has on the order of \( n^{1/4} \) fixed points. Moreover it allows us to calculate the distribution of
\[ \frac{1}{n^{1/4}} \theta_{[an, bn]}(\sigma_{\Gamma^n}). \]

We now state a version of Theorem 1.1 Part (a) that gives joint convergence of the Dyck path and the fixed points of the associated 231-avoiding permutation.

**Theorem 1.7** Let \( \Gamma^n \) be a uniformly random Dyck path of length \( 2n \) and let \( (\Gamma^n(t), 0 \leq t \leq 2n) \) be its linear interpolation. We then have the joint convergence
\[
\left( \frac{\Gamma^n(2ns)}{\sqrt{2n}}, \frac{\theta_{[0,tn]}(\sigma_{\Gamma^n})}{n^{1/4}} \right)_{(s,t) \in [0,1]^2} \xrightarrow{dist} \left( e_s, \int_0^t 1 \div 2^{7/4} \pi^{1/2} \omega_u^{3/2} du \right)_{(s,t) \in [0,1]^2}
\]
in distribution on \( D([0, 1], \mathbb{R}) \times D([0, 1], \mathbb{R}) \), where \( (e_t, 0 \leq t \leq 1) \) is Brownian excursion and \( D([0, 1], \mathbb{R}) \) is the space of right continuous functions with left limits equipped with the Skorokhod topology.

**Remark** We may replace \( D([0, 1], \mathbb{R}) \) with \( C([0, 1], \mathbb{R}) \) in the preceding theorem if we replace \( \theta_{[0,sn]}(\sigma_{\Gamma^n}) \) with the linear interpolation of the points \( (\theta_{[0,i]}(\sigma_{\Gamma^n}), i = 0, \ldots, n) \).
Remark Theorem 1.7 does more than just tell us the distribution of fixed points on an interval. It relates (with high probability) the number of fixed points of $\sigma_{\Gamma^n}$ on an interval $[an, bn]$ with the shape of $\Gamma^n$ on the same interval. Roughly speaking if we know a Brownian excursion that approximates a scaled Dyck path then with high probability we can quite closely determine the number of fixed points for the corresponding 231-avoiding permutation. Also if we know the location of fixed points for a 231-avoiding permutation then we can use that to find a Brownian excursion that does a good job approximating the corresponding scaled Dyck path.

Remark Our methods are quite different from [22], but we do use the result of [22] about the expected number of fixed points in the proof. Our methods could be used to prove this as well, but the resulting proof is no simpler than that in [22]. The methods in [22] can also be adapted to find the asymptotics of $\mathbb{E}(\theta_{[an, bn]}(\sigma_{\Gamma^n}))$ for any interval $[an, bn]$, but we will get these asymptotics essentially for free from our proof of Theorem 1.7.

“Almost fixed points”

Perhaps the most interesting result in [22] is a phase transition it shows in

$$\mathbb{P} \left( \sigma(i) = \left[ i - \left( \frac{(i(n - i))^\alpha}{n} \right) \right] \right)$$

that occurs at $\alpha = 3/8$. In particular they show that

$$\mathbb{P} \left( \sigma(i) = \left[ i - \left( \frac{(i(n - i))^\alpha}{n} \right) \right] \right) \sim \begin{cases} Cn^{-3/4} & \text{if } \alpha \in (0, 3/8) \\ Cn^{-3/2-2\alpha} & \text{if } \alpha \in [3/8, .5) \end{cases} \quad (1)$$

This result is particularly intriguing because it is not clear what is driving the phase transition. Miner and Pak say that their results on 231-avoiding permutations “are extremely unusual, and have yet to be explained even on a qualitative level” [22]. In this paper we use a generalization of Theorem 1.7 to give an explanation of these results.

First we show that the difference is not due to the number of “almost” fixed points on a typical path. To make this precise we define

$$\theta_{[an, bn]}^{K, \alpha}(\sigma) = \left\{ i : \sigma(i) = i - \left[ K \left( \frac{(i(n - i))^\alpha}{n} \right) \right] \right\} \cap [an, bn].$$

Then we follow the proof of Theorem 1.7 very closely to show

Corollary 1.8 Let $\Gamma^n$ be a uniformly random Dyck path of length $2n$ and let $(\Gamma^n(t), 0 \leq t \leq 2n)$ be its linear interpolation. Fix any $0 < a < b < 1$. We then have the joint convergence
in distribution on $D([a, b], \mathbb{R}) \times D([a, b], \mathbb{R})$, where $(\varepsilon_t, 0 \leq t \leq 1)$ is Brownian excursion and $D([a, b], \mathbb{R})$ is the space of right continuous functions with left limits equipped with the Skorokhod topology.

Thus for all $K$ and $\alpha$ the distribution of the number of “almost” fixed points is asymptotically the same as the distribution of the number of fixed points and we do not see the same phase transition that Miner and Pak observed.

But there is no inconsistency between our results and [22] in the regime $K > 0$ and $\alpha \in [3/8, .5)$. This is because a small number of permutations drive the probability that Miner and Pak calculate in (1). This is missed by our convergence in distribution. This small number of permutations are the ones $\sigma_\gamma$ whose corresponding Dyck paths $\gamma$ have height $\gamma(i) = [K(i(n-i)/n)^\alpha]$ for some $i \in [2an, 2bn]$. As the density of these permutations becomes vanishingly small as $n \to \infty$, these permutations do not affect the limiting distribution of $n^{-1/4}\theta_{[an, bn]}(\sigma)$ that we calculate. But these are the permutations that dominate the probabilities that Miner and Pak calculate.

2 321-avoiding permutations

If $\rho$ is a 123-avoiding permutation of length $n$ and we define $\tau(k) = n + 1 - \rho(k)$ it is clear by symmetry that $\tau$ is a 321-avoiding permutation of length $n$. We now describe a bijection (which is often known as the Billey-Jockusch-Stanley or BJS bijection) from Dyck paths of length $2n$ to 321-avoiding permutations of length $n$ [2]. By the above relation, this will allow us to prove results about 123-avoiding permutations. Fix a Dyck path $\gamma: \{0, 1, \ldots, 2n\} \to \mathbb{N}$ of length $2n$. Given $\gamma$ define the following. Let $m$ be the number of runs of increases (or decreases) in $\gamma$. Let $a_i$ be the number of increases in the $i$th run of increases in $\gamma$. Let $A = \sum_{i=1}^{m-1} A_i$ and $\bar{A} = \{1, 2, \ldots, n\}\backslash(1 + A)$. Similarly we define $d_i$ and $D_i = \sum_{i=1}^{j} d_i$ based on the length of the descents. Then define $D = \cup_{i=1}^{m-1} \{D_i\}$ and $\bar{D} = \{1, 2, \ldots, n\}\backslash D$. We also set $A_0 = D_0 = 0$. Let $\tau_\gamma$ be the corresponding 321-avoiding permutation with the BJS bijection. This is defined by $\tau_\gamma(D_i) = 1 + A_i$ on $D$ and $\tau_\gamma$ increasing on $\bar{D}$ with $\tau_\gamma(\bar{D}) = \bar{A}$.

We note that

$$A_i - D_i = \gamma(A_i + D_i).$$

(2)

We will make use of the following property of this bijection whose proof we leave to the reader (see Fig. 2).

**Lemma 2.1** For any $\gamma \in \text{Dyck}^{2n}$ and $j \in \{1, 2, \ldots, n\}$

$$\tau_\gamma(j) > j \quad \text{if} \ j \in D$$

and

$$\tau_\gamma(j) \leq j \quad \text{if} \ j \notin D$$
We now undertake a more detailed analysis to show that for most $321$-avoiding permutations if $i, j$ are such that $D_{i-1} < j < D_i$ then

$$\tau_\gamma(j) \approx j - A_i - D_i$$

and if $j = D_i$ then

$$\tau_\gamma(j) \approx j + A_i - D_i.$$ 

Our first step is the following lemma.

**Lemma 2.2** Fix a Dyck path $\gamma \in \text{Dyck}^{2n}$ and $j \notin \mathcal{D}$. There exists $i$ such that $D_{i-1} < j < D_i$. Then for $k \in \{1, \ldots, m-1\}$.

(a) If $A_k - (k - 1) > D_i - i$ then $\tau_\gamma(j) < A_k$.

(b) If $A_k - (k - 1) < D_{i-1} - (i - 1)$ then $\tau_\gamma(j) > A_k + 1$.

**Proof** Let $x = \max(\bar{\mathcal{A}} \cap \{1, 2, \ldots, A_k\})$. In the first case we note that

$$|\bar{\mathcal{A}} \cap \{1, 2, \ldots, A_k\}| = A_k - (k - 1) > D_i - i = |\{1, 2, \ldots, D_i\} \cap \bar{\mathcal{D}}|.$$ 

Thus

$$\tau_\gamma^{-1}(x) > D_i > j.$$ 

As $\tau_\gamma$ is monotone on the complement of $\mathcal{D}$ we get that

$$A_k \geq x = \tau_\gamma(\tau_\gamma^{-1}(x)) > \tau_\gamma(j).$$ 

In the second case

$$|\bar{\mathcal{A}} \cap \{1, 2, \ldots, A_k\}| = A_k - (k - 1) < D_{i-1} - (i - 1) = |\{1, 2, \ldots, D_{i-1}\} \cap \bar{\mathcal{D}}|.$$
Thus $\tau^{-1}_\gamma(x) < D_{i-1} < j$ and as $\tau_\gamma$ is monotone on the complement of $\mathcal{D}$

$$x = \tau_\gamma(\tau^{-1}_\gamma(x)) < \tau_\gamma(j).$$

As $\tau_\gamma(j) > x$ and $\tau_\gamma(j) \in \bar{A}$ we get that

$$\tau_\gamma(j) > 1 + A_k.$$

$\square$

**Definition 2.3** We say that a Dyck path $\gamma \in \text{Dyck}^{2n}$ with associated sequences $A_i$ and $D_i$ satisfies the Petrov conditions if

(a) $\max_{x \in \{0, 1, \ldots, 2n\}} \gamma(x) < .4n^6$

(b) $|\gamma(x) - \gamma(y)| < .5n^4$ for all $x, y$ with $|x - y| < 2n^6$

(c) $|A_i - A_j - 2(i - j)| < .1|i - j|^6$ for all $i, j$ with $|i - j| \geq n^3$ and

(d) $|D_i - D_j - 2(i - j)| < .1|i - j|^6$ for all $i, j$ with $|i - j| \geq n^3$

**Lemma 2.4** With high probability the Petrov conditions are satisfied. The probability that they are not satisfied is decaying exponentially in $n^c$ for some $c > 0$.

*Proof* These results are standard Petrov style moderate deviation results [24]. The general type of conditioning argument we need appears in [21, 25]. The specific statements we need can be found in the appendix of [11]. $\square$

From these conditions we can derive many other moderate deviation results. We now list the ones that we will need. The proofs of these lemmas are contained in [11].

**Lemma 2.5** If a Dyck path $\gamma \in \text{Dyck}^{2n}$ with associated sequences $A_i$ and $D_i$ satisfies the Petrov conditions then $A_i - D_i < n^4$ for all $i < n^6$ and for all $i > n - n^6$. Also, $|A_i - A_{i-1}|, |D_i - D_{i-1}| < n^{18}$ for all $i$. Finally every consecutive sequence of length at least $n^3$ has at least one element of $\mathcal{D}$ and at least one element of $\bar{\mathcal{D}}$.

**Lemma 2.6** For any Dyck path $\gamma \in \text{Dyck}^{2n}$ that satisfies the Petrov conditions and any $j = D_i \in \mathcal{D}$

$$|\tau_\gamma(j) - j - \gamma(2j)| < 10n^4.$$  

Also for any such $\gamma$, $j$ and $i$ with $D_{i-1} < j < D_i$

$$|\tau_\gamma(j) - j + \gamma(2j)| < 10n^4.$$  

3 Fixed points of 123-avoiding permutations

In this section we use our analysis of 321-avoiding permutations from Sect. 2 to study the fixed points of a random 123-avoiding permutation. A permutation with three distinct fixed points has the pattern 123. Thus a 123-avoiding permutation can have at
most 2 fixed points. At most one of them can be in the interval $[1, n/2]$ and at most one of them in the interval $(n/2, n]$. Elizalde showed that as $n \to \infty$ the expected number of fixed points in a random 123-avoiding permutation is converging to 1/2 [7]. Miner and Pak refined this by showing that the number of fixed points outside of the interval $[(1-\epsilon)n/2, (1+\epsilon)n/2]$ is converging to 0 [22]. In this section we give an asymptotic description of the distribution of fixed points in terms of Brownian excursion.

We start with our main combinatorial lemma.

**Lemma 3.1** Let $\rho$ be a 123-avoiding permutation of length $n$, let $\tau$ be defined by $\tau(k) = n + 1 - \rho(k)$ and let $\gamma \in \text{Dyck}^{2n}$ be the image of $\tau$ under the BJS bijection. There is a local minimum of $\gamma$ at $n$ if and only if there exists $k$ such that $\rho(k) = k \leq n/2$. If there exists a fixed point at some $k \leq n/2$ then the fixed point $k$ satisfies

$$k = \frac{n - \gamma(n)}{2}. \tag{3}$$

**Proof** It is clear by symmetry that $\rho$ is 123-avoiding if and only if $\tau$ is 321-avoiding. We note that there is a fixed point $k = \rho(k)$ if and only if there is a $k$ such that $(k, \tau(k))$ is on the anti-diagonal of the graph of $\tau$, i.e. $\tau(k) = n + 1 - k$.

If $k \leq n/2$ and $\rho(k) = k$ then we have $\tau(k) = n + 1 - k > k$ and $(k, \tau(k))$ lies above the diagonal and on the upper sequence. Thus we must have that $k = D_i = D_i(\gamma)$ for some $i$ and

$$\tau(k) = 1 + A_i = n + 1 - D_i.$$ 

Rearranging we get that $A_i + D_i = n$. This implies that

$$(A_i + D_i, A_i - D_i) = (n, A_i - D_i)$$

is a local minimum on the graph of $\gamma$.

Similarly if $(n, n - 2j)$ is a local minimum of $\gamma$ then there exists $k$ such that $A_k + D_k = n$. Then

$$\tau(D_k) = 1 + A_k = 1 + n - D_k$$

and $(D_k, \tau(D_k))$ lies on the anti-diagonal. As $D_k \leq A_k = n - D_k$ we get $D_k \leq n/2$. Solving $A_i + D_i = n$ and $A_i - D_i = \gamma(n)$ for $D_i$ we get that

$$D_i = \frac{n - \gamma(n)}{2}$$

which is the location of the fixed point. $\square$

We now translate Lemma 3.1 into a statement about the distribution of fixed points. To perform this analysis we define several random variables on the set of 123-avoiding permutations.

- For any $x \in \mathbb{R} \cup \infty$ let $\delta_x$ be the point mass at $x$.
- $\tilde{A}_n(\rho) = \# \{ i \in [1, n/2] : \rho(i) = i \}$
\[ \tilde{B}_n(\rho) = \# \{ i \in (n/2, n] : \rho(i) = i \} \]

\[ \tilde{X}_n(\rho) \text{ to be the fixed point in } [1, n/2] \text{ if it exists and } \infty \text{ if there are none.} \]

\[ \tilde{Y}_n(\rho) \text{ to be the fixed point in } (n/2, n] \text{ if it exists and } \infty \text{ if there are none.} \]

From these two we define the random measures

\[ \tilde{X}_n(\rho) = \tilde{A}_n(\rho) \delta_{\hat{X}_n - n/2} = \sum_{i=1}^{[n/2]} \delta_{i - \frac{n}{2\sqrt{2n}}} \mathbf{1}_{\rho(i) = i} \]

and

\[ \tilde{Y}_n(\rho) = \tilde{B}_n(\rho) \delta_{\hat{Y}_n - n/2} = \sum_{i=[n/2]+1}^{n} \delta_{i - \frac{n}{2\sqrt{2n}}} \mathbf{1}_{\rho(i) = i}. \]

So \( \tilde{X}_n(\rho) + \tilde{Y}_n(\rho) \) encodes the number and location of the fixed points and is appropriately scaled. For the above random variables we often drop the \( \rho \) when we are referring to uniformly random 123-avoiding permutation.

Now we identify the limit of \( \tilde{X}_n + \tilde{Y}_n \) which is the main result of this section.

**Theorem 3.2** Let \( A \) and \( B \) be independent Bernoulli(1/4) random variables and let \( X \) be a random variable that is independent of \( A \) and \( B \) and distributed like \( \frac{1}{2} \mathcal{E}_{1/2} \), half the height of a Brownian excursion at 1/2. Then

\[ \tilde{X}_n + \tilde{Y}_n \overset{\text{dist}}{\longrightarrow} A \delta_X + B \delta_X. \]

Note that this is an equivalent formulation of Theorem 1.1 Part (b). The first step in the proof is the following lemma (see Fig. 3).

**Lemma 3.3**

\[ \tilde{X}_n \overset{\text{dist}}{\longrightarrow} A \delta_X \quad \text{and} \quad \tilde{Y}_n \overset{\text{dist}}{\longrightarrow} B \delta_X. \]

**Proof** Let \( \rho \), \( \tau \) and \( \gamma \) be as in Lemma 3.1. If \( \rho \) is a 123-avoiding permutation then \( \tau \) is 321-avoiding. By Lemma 3.1 we have that if \( D_i \leq n/2 \) is a fixed point for \( \rho \) then \( (n, A_i - D_i) \) is a local minimum for \( \gamma \). The set of Dyck paths of length \( 2n \) that have a local minimum at the point \((n, h)\) is in 1-1 correspondence with the set of Dyck paths of length \( 2n - 2 \) that go through \((n - 1, h + 1)\). If \( n \) is even we get

\[ \mathbb{P}(\tilde{A}_n = 1) = \frac{C_{n-1} - C_{n-1}}{C_n} \to \frac{1}{4}. \]

If \( n \) is odd we get

\[ \mathbb{P}(\tilde{A}_n = 1) = \frac{C_{n-1} - C_{n-1}C_{n-1} / 2}{C_n} \to \frac{1}{4}. \]
Fig. 3 The pictures above show two Dyck paths $\gamma, \gamma' \in \text{Dyck}^{2n}$ and corresponding $321$-avoiding permutations $\tau_\gamma, \tau_{\gamma'}$. The path $\gamma$ has a local minimum at $(n, \gamma(n))$. This corresponds with the point $\left(\frac{n-\gamma(n)}{2}, \frac{n+\gamma(n)}{2} + 1\right)$ on the anti-diagonal in $\tau_\gamma$. This becomes a fixed point for $\rho_\gamma = n + 1 - \tau_\gamma$. The other path $\gamma'$ does not have a local minimum at $n$ and correspondingly $\tau_{\gamma'}$ has no point on the anti-diagonal with $x$-coordinate less than or equal to $n/2$ and no fixed point of $\rho$ less than or equal to $n/2$.

Because of (3) in Lemma 3.1 we get that

$$\frac{\hat{X}_n - n/2}{\sqrt{2n}} \quad \text{given } (\hat{A} = 1) \overset{\text{dist}}{=} \frac{-\Gamma^n(n)}{2\sqrt{2n}} \quad \text{given } (n, \Gamma^n(n)) \text{ is a local minimum},$$

which is equal in distribution (if $n$ is even) to

$$\frac{-\Gamma^{n-1}(n - 1) + 1}{2\sqrt{2n}}$$

where $\Gamma^{n-1}$ is a uniformly chosen Dyck path of length $2n - 2$. This last quantity is converging in distribution to $-X = -e_{1/2}/2$. The case when $n$ is odd is virtually identical. This proves the first claim in the lemma.

The second follows by symmetry as the permutation defined by

$$\bar{\rho}(k) = n + 1 - \rho(n + 1 - k)$$

is also $123$-avoiding. The fixed points of $\bar{\rho}$ are $n + 1$ minus the fixed points of $\rho$. Thus the convergence of $\hat{Y}_n \to B\delta_X$ follows by symmetry. □
Our next goal is to prove the following lemma.

**Lemma 3.4**

\[(\tilde{A}_n, \tilde{B}_n) \xrightarrow{\text{dist}} (A, B).\]

To prove that \(\tilde{A}_n\) and \(\tilde{B}_n\) are asymptotically independent consider \(\gamma \in \text{Dyck}^{2n}\) and its image under the BJS bijection \(\rho_\gamma\). As we saw in Lemma 3.1 \(\tilde{A}_n = 1\) is the event

\[\gamma(n - 1) - \gamma(n) = \gamma(n + 1) - \gamma(n) = 1\]

which is determined by the increments of \(\gamma\) in the region \([n - 1, n + 1]\). We will now show that \(\tilde{B}_n\) is essentially determined by \(\gamma\) in

\([0, 2n]\backslash[n - 1, n + n^{41}]\).

As the increments of a Dyck path in \([n - 1, n + 1]\) are roughly independent of the values of \(\gamma\) in \([0, 2n]\backslash[n - 1, n + n^{41}]\) we will get that \(\tilde{A}_n\) is asymptotically independent of \(\tilde{B}_n\).

To make this formal we define an equivalence relation on \(\text{Dyck}^{2n}\).

**Definition 3.5** For \(\gamma, \gamma' \in \text{Dyck}^{2n}\) we write \(\gamma \sim \gamma'\) if

- \(\gamma(m) = \gamma'(m)\) for all \(m \notin [n, n + l - 2]\) where \(l = \lfloor n^{41} \rfloor\) and
- \(\gamma\) and \(\gamma'\) have the same number of local maximums and local minimums in the interval \([n - 2, n + l]\).

Let \(S\) denote the set of all equivalence classes for \(\sim\).

We now define a good set of equivalence classes. Then we show that almost all the Dyck paths are in their union, which we call \(G_n\).

**Definition 3.6** Define \(G_n\) to be the set of all equivalence classes \(s \in S\) such that

(a) some element \(\gamma \in s\) satisfies the Petrov conditions (defined in Definition 2.3) and

(b) \(\gamma(n - 2) > n^{45}\) for some (all) \(\gamma \in s\).

Also define

\[G_n = \bigcup_{s \in G_n} s\]  \hspace{1cm} (6)

**Lemma 3.7** \(\mathbb{P}(G_n) \to 1\) as \(n \to \infty\).

**Proof** The probability that the Petrov conditions are not satisfied is decaying exponentially in \(n^c\) for some \(c > 0\) by Lemma 2.4. The second condition in the definition of \(G_n\) is satisfied with probability going to 1 since the convergence of rescaled Dyck paths to Brownian excursion, see [13], shows that \(\gamma(n - 2) \approx n^{-5}\) on all but a set of vanishing probability. \(\square\)
Lemma 3.8

\[
\lim_{n \to \infty} \max_{s \in G_n} \left| \mathbb{E}(\tilde{A}_n \mid \gamma \in s) - \frac{1}{4} \right| \to 0.
\]

Proof This is a straightforward but tedious calculation. Fix \( a, b \in \{\pm 1\} \) and \( h, h', j \in \mathbb{N} \). Let \( s \) be an equivalence class such that, for all \( \gamma \in s \),

\[
\gamma(n - 2) = h + a, \quad \text{and} \quad \gamma(n - 1) = h, \quad \text{and} \quad \gamma(n + l - 1) = h' + b \quad \text{and} \quad \gamma(n + l) = h'
\]

and there are \( j \) peaks in the interval \([n - 2, n + l]\). We break \( s \) up into four sets based on whether \( a \) and \( b \) are positive or negative.

Consider the case when \( a = b = 1 \). The paths \( \gamma \in s \) that have a local minimum at \( n \) have a decrease between \( n - 1 \) and \( n \) followed by an increase between \( n \) and \( n + 1 \). Using the notation of Proposition 9 of [17] the number of such walks is

\[
\#T_{ud}^{j,\geq}(l, h - 1, h' + 1) = \left( \frac{l + (h' + 1) - (h - 1)}{2} - 1 \right) \left( \frac{l - (h' + 1) + (h - 1)}{2} - 1 \right) \overset{\text{def}}{=} X_1 \cdot Y_1.
\]

Similarly the number of walks that start with an increase followed by a decrease is

\[
\#T_{dd}^{j,\geq}(l, h + 1, h' + 1) = \left( \frac{l + (h' + 1) - (h + 1)}{2} - 1 \right) \left( \frac{l - (h' + 1) + (h + 1)}{2} - 1 \right) \overset{\text{def}}{=} X_2 \cdot Y_2,
\]

the number of walks that start with two decreases is

\[
\#T_{dd}^{j,\geq}(l, h - 1, h' + 1) = \left( \frac{l + (h' + 1) - (h - 1)}{2} - 1 \right) \left( \frac{l - (h' + 1) + (h - 1)}{2} - 1 \right) \overset{\text{def}}{=} X_1 \cdot Y_3,
\]

and the number of walks that start with two increases is

\[
\#T_{ud}^{j,\geq}(l, h + 1, h' + 1) = \left( \frac{l + (h' + 1) - (h + 1)}{2} - 1 \right) \left( \frac{l - (h' + 1) + (h + 1)}{2} - 1 \right) \overset{\text{def}}{=} X_3 \cdot Y_2.
\]

(Note that in [17] these sets on the left are counted by the difference of two products of binomial coefficients. In \( G_n \) the walks are sufficiently high at \( n - 2 \), at least \( n^{45} \), that they can never go negative between \( n \) and \( n + n^{41} \). Thus the second term in the difference is zero.)

Let \( i_0 \) be the last peak before \( n - 2 \), that is \( i_0 = \max\{i : A_i + D_{i-1} < n - 2\} \). Then \( i_0 + j \) is the last peak at or before \( n + l \), i.e. \( j = \max\{k : A_{i_0 + k} + D_{i_0 + k - 1} \leq n + l\} \). Then if some element of \( s \) satisfies the Petrov conditions by Lemma 2.5 then we can bound the distance between the last peak before \( n - 2 \) and the last peak at or before \( n + l \) by

\[
l + 2 - 2n^{18} \leq A_{i_0 + j} + D_{i_0 + j - 1} - (A_{i_0} + D_{i_0 - 1}) \leq l + 2 + 2n^{18}.
\]
As some element of \( s \) satisfies the third and fourth Petrov conditions then
\[
|A_{i_0+j} - A_{i_0} - 2j|, |D_{i_0+j-1} - D_{i_0-1} - 2j| \leq l^6 \leq n^{25}
\]
and we have that
\[
l + 2 - 2n^{18} - 2n^{25} \leq 4j \leq l + 2 + 2n^{18} + 2n^{25}
\]
and \( l = j(4 + o(1)) \). If some element of \( s \) satisfies the second Petrov condition then \(|h' - h| \leq 5n^{-4} = o(l)\), as \( l \approx n^{41} \). Thus the top term of each binomial coefficient is of order \((l/2)(1 + o(1))\) and we have that the binomial coefficients are all of the form
\[
\binom{l/2 + o(1)}{j(1 + o(1))} = \binom{2j(1 + o(1))}{j(1 + o(1))}.
\]
Adding one to the top term increases the binomial coefficient by a factor of \( 2 + o(1) \). Changing the bottom term by one multiplies the binomial coefficient by a factor of \( 1 + o(1) \). Thus \( X_1/X_2 = 2(1 + o(1)) \) and \( X_2/X_3 = 1 + o(1) \). Similarly we have that \( Y_1/Y_3 = 1 + o(1) \) and \( Y_2/Y_1 = 2(1 + o(1)) \). Thus the cardinalities of the four sets are comparable and the conditional probability of a local minimum at \( n \) is asymptotically \( 1/4 \). The other three cases \( (a = -b = 1, -a = b = 1 \) and \( a = b = -1 \) are virtually identical. We leave the details to the reader. Combining these four calculations proves the result. \( \Box \)

**Lemma 3.9** For any \( \gamma, \gamma' \in G_n \) with \( \gamma \sim \gamma' \) we have
\[
\tilde{B}_n(\rho_\gamma) = \tilde{B}_n(\rho_{\gamma'}) \quad \text{and} \quad \hat{Y}_n(\rho_\gamma) = \hat{Y}_n(\rho_{\gamma'})
\]

**Proof** Remember the definitions of \( \mathcal{D}, \mathcal{A} \) and \( \bar{A} \) from the start of Sect. 2. First we claim that
\[
\mathcal{D}_\gamma \oplus \mathcal{D}_{\gamma'} \subset [1, n/2] \quad \text{and} \quad \bar{A}_\gamma \oplus \bar{A}_{\gamma'} \subset [1 + n/2, n], \quad (7)
\]

where \( \oplus \) is the symmetric difference operator. To see this note that both of these sets are defined by the points which are a local minimum for one Dyck path but not the other. Based on the definition of the equivalence relation the local minima of \( \gamma \) and \( \gamma' \) can only differ in the interval \( I = (n - 2, n + n^{41}) \). By the second condition in the definition of \( \mathcal{G}_n \) each of the local minima in the interval \( I \) is preceded by at least \( n/2 + n^{45}/2 - 2 > 1 + n/2 \) up-steps. This proves the second claim in (7). Also by the second condition in the definition of \( \mathcal{G}_n \) each of the local minima in the interval \( I \) is preceded by at most \( n/2 - n^{45}/2 + n^{41} < n/2 \) down-steps. This proves the first claim in (7). Also by the previous argument and the second condition in the equivalence relation
\[
|\mathcal{D}_\gamma \cap [1, n/2]| = |\mathcal{D}_{\gamma'} \cap [1, n/2]|. \quad (8)
\]
If $\tilde{B}_n(\rho(\gamma)) = 1$ then there exists $j > n/2$ which is a fixed point of $\rho_\gamma$ and it lies on the anti-diagonal of $\tau_\gamma$. As $j > n/2$ we get

$$\tau_\gamma(j) = n + 1 - j \leq j \quad \text{and} \quad \tau_\gamma(j) = n + 1 - j < 1 + n/2. \quad (9)$$

So by the first part of (9) we have $(j, \tau_\gamma(j))$ lies on the lower sequence for $\tau_\gamma$. Thus $j \notin D_\gamma$. By the first part of (7) and the fact that $j > n/2$ we also have $j \notin D_{\gamma'}$. Thus $\tau_\gamma(j) \in \tilde{A}_\gamma$ and $\tau_{\gamma'}(j) \in \tilde{A}_{\gamma'}$. By (8) and the fact that $D_\gamma$ and $D_{\gamma'}$ are equal after $n/2$ there exists $k$ such that

$$D_k = D_k' < j < D_{k+1} = D_{k+1}'.$$

So $\tau_\gamma(j)$ is the $(j - k)$’th element of $\tilde{A}_\gamma$ and $\tau_{\gamma'}(j)$ is the $(j - k)$’th element of $\tilde{A}_{\gamma'}$. By the second half of (7) and the second part of (9) we know that

$$\tau_\gamma(j) \in \tilde{A}_\gamma | [1,1+n/2) = \tilde{A}_{\gamma'} | [1,1+n/2).$$

Thus $\tau_\gamma(j)$ must be equal to $\tau_{\gamma'}(j)$ as they are both the $(j - k)$’th term in the same set. Thus $(j, \tau_\gamma(j))$ and $(j, \tau_{\gamma'}(j))$ lie on the anti-diagonal and $j = \rho_{\gamma'}(j)$ is a fixed point of $\rho_{\gamma'}$. As the roles of $\gamma$ and $\gamma'$ are symmetric this establishes the claim of the lemma. \hfill $\square$

**Proof of Lemma 3.4** From (4) and (5) we have that $\tilde{A}_n \to A$ and by symmetry we have that $\tilde{B}_n \to B$. Thus we just need to show that $\mathbb{E}(\tilde{A}_n \tilde{B}_n) \to \frac{1}{16}$.

$$\left| \mathbb{E}(\tilde{A}_n \tilde{B}_n) - \frac{1}{16} \right| = \left| \mathbb{E}(\tilde{B}_n) \mathbb{E}(\tilde{A}_n \tilde{B}_n = 1) - \frac{1}{16} \right|$$

$$\leq \left| \mathbb{E}(\tilde{B}_n) - \frac{1}{4} \right| + \left| \mathbb{E}(\tilde{A}_n \tilde{B}_n = 1) - \frac{1}{4} \right|$$

$$\leq \left| \mathbb{E}(\tilde{B}_n) - \frac{1}{4} \right| + \left| \mathbb{E}(\tilde{A}_n \tilde{B}_n = 1) - \mathbb{E}(\tilde{A}_n) \mathbb{E}(\tilde{B}_n = 1) \right| + \left| \mathbb{E}(\tilde{A}_n) - \frac{1}{4} \right|$$

$$\leq \left| \mathbb{E}(\tilde{B}_n) - \frac{1}{4} \right| + \left| \mathbb{E}(\tilde{A}_n | \tilde{B}_n = 1) \cap G_n) - \mathbb{E}(\tilde{A}_n) \mathbb{E}(\tilde{B}_n = 1) \right| + \left| \mathbb{E}(\tilde{A}_n) - \frac{1}{4} \right|$$

$$\leq \left| \mathbb{E}(\tilde{B}_n) - \frac{1}{4} \right| + \max_{s \in G_n} \left| \mathbb{E}(\tilde{A}_n \gamma \in s) - \mathbb{E}(\tilde{A}_n | \gamma \in s) \right| + \mathbb{P}(G_n^C)$$

$$+ \left| \mathbb{E}(\tilde{A}_n) - \frac{1}{4} \right|. \quad (10)$$

The last inequality is valid because of Lemma 3.9. The first and last terms on the right hand side of (10) go to zero by Lemma 3.4. The second term goes to zero by Lemmas 3.4 and 3.8 and the third term goes to zero by Lemma 3.7. \hfill $\square$

In contrast to Lemma 3.1 the event $\{\tilde{B}_n = 1\}$ and the location $\hat{Y}_n$ of the fixed point after $n/2$ is more complicated to describe.
Lemma 3.10  For all \( n \) sufficiently large and all \( \gamma \in G_n \) with \( \tilde{B}_n = 1 \)

\[
|\hat{Y}_n - n/2 - \gamma(n)/2| \leq 100n^4
\]

Proof Suppose \( \gamma \in G_n \). By the definition of \( G_n \) there exists \( \gamma' \) such that \( \gamma \sim \gamma' \) and \( \gamma' \) satisfies the Petrov conditions. By Lemma 3.9 it causes no loss of generality to assume that \( \gamma \) satisfies the Petrov conditions.

Restricted to the lower sequence \( \bar{D} \) we have that \( \tau_\gamma(j) = j \) is an increasing sequence as each component is increasing. We will show that if \( j \in \bar{D} \) and

\[
j < \frac{n + \gamma(n) - 100n^4}{2}
\]

then \( \tau_\gamma(j) + j < n \). Similarly we will show that if \( j \in \bar{D} \) and

\[
j > \frac{n + \gamma(n) + 100n^4}{2}
\]

then \( \tau_\gamma(j) + j > n + 1 \). Then for any \( j \) with \( (j, \tau_\gamma(j)) \) on the anti-diagonal and the lower sequence we must have \( j + \tau_\gamma(j) = n + 1 \) and thus

\[
j \in \left( \frac{n + \gamma(n) - 100n^4}{2}, \frac{n + \gamma(n) + 100n^4}{2} \right).
\]

Let \( j \) be the smallest value in \( \bar{D} \) not satisfying (11). Then by Lemma 2.5

\[
j < \frac{n + \gamma(n) - 98n^4}{2}
\]

Since \( \gamma \) satisfies the Petrov conditions by Lemma 2.6 we have

\[
\tau_\gamma(j) - j + \gamma(2j) \leq 10n^4.
\]

Then manipulating this we get

\[
\tau_\gamma(j) + j \leq 2j + 10n^4 - \gamma(2j)
\leq n + \gamma(n) - 98n^4 + 10n^4 - \gamma(n) + (\gamma(n) - \gamma(2j))
\leq n - 88n^4 + n^4
< n.
\]

The inequality in (13) is true because by Petrov condition (a) \( n \) and \( 2j \) are within \( n^6 \) and thus by Petrov condition (b) \( \gamma(n) \) and \( \gamma(2j) \) are within \( n^4 \).

Let \( j \) be the largest value in \( \bar{D} \) not satisfying (12). Then by Lemma 2.5

\[
j > \frac{n + \gamma(n) + 98n^4}{2}
\]
Since $\gamma$ satisfies the Petrov conditions by Lemma 2.6

$$\tau_\gamma(j) - j + \gamma(2j) \geq -10n^4$$

Then manipulating and making the same estimates we get

$$\tau_\gamma(j) + j \geq 2j - 10n^4 - \gamma(2j)$$

$$\geq n + \gamma(n) + 98n^4 - 10n^4 - \gamma(n) + (\gamma(n) - \gamma(2j))$$

$$\geq n + 88n^4 + (\gamma(n) - \gamma(2j))$$

$$\geq n + 80n^4.$$  

The last line follows in the same way as the first computation. \[\square\]

**Lemma 3.11** Conditional on $\tilde{A}_n = \tilde{B}_n = 1$ we have that

$$\hat{Y}_n \xrightarrow{\text{dist}} \delta_{X}.$$  

**Proof** Since $X$ is continuous using Lemma 3.4 it suffices to show that for any $0 < a < b$

$$\Pr(\tilde{A}_n = \tilde{B}_n = 1 \text{ and } \hat{Y}_n \in (a\sqrt{2n}, b\sqrt{2n})) \to \frac{1}{16}\Pr(X \in (a, b)).$$  \hspace{1cm} (14)

By Lemma 3.3

$$\Pr(\tilde{B}_n = 1 \text{ and } \hat{Y}_n \in (a\sqrt{2n}, b\sqrt{2n})) \to \frac{1}{4}\Pr(X \in (a, b)).$$  \hspace{1cm} (15)

By Lemma 3.9 for any $s \in G_n$ the event on the left hand side of (15) either happens for all $\gamma \in s$ or for no $\gamma \in s$. Thus Lemma 3.8 implies that for each $s \in G_n$

$$\Pr(\gamma \in s \text{ and } \tilde{A}_n = \tilde{B}_n = 1 \text{ and } \hat{Y}_n \in (a\sqrt{2n}, b\sqrt{2n}))$$

$$= \frac{1}{4}\Pr(\{\gamma \in s\} \cap \tilde{B}_n = 1 \text{ and } \hat{Y}_n \in (a\sqrt{2n}, b\sqrt{2n})) (1 + \Delta(s))$$  \hspace{1cm} (16)

where $\sup_{s \in G_n} \Delta(s) = o(1)$. Summing over all $s \in S$ and using Lemma 3.7 to see that $\Pr(G_n)$ is close to 1 completes the proof. \[\square\]

**Lemma 3.12** Conditional on $\tilde{A}_n = \tilde{B}_n = 1$ we have that

$$\tilde{X}_n + \tilde{Y}_n \xrightarrow{\text{dist}} \delta_{-X} + \delta_{X}.$$  

**Proof of Lemma 3.12** By Lemma 3.1 for any $\gamma \in G_n$ with $\tilde{A}(\gamma) = 1$ we have that

$$\hat{X}(\gamma) = \frac{n - \gamma(n)}{2}.$$  

\[\square\]
Similarly by Lemma 3.10 we have that for any $\gamma \in G_n$ with $\tilde{B}_n(\gamma) = 1$

$$\left| -\frac{n + \gamma(n)}{2} + \hat{Y}(\gamma) \right| \leq 100n^4.$$

Combining these two lines gives us

$$\left| \hat{X} - \frac{n}{\sqrt{2n}} + \frac{\hat{Y} - n}{\sqrt{2n}} \right| = \left| -\frac{\gamma(n)}{2\sqrt{2n}} + \frac{\hat{Y} - \frac{\gamma(n)}{2}}{\sqrt{2n}} + \frac{\gamma(n)}{2\sqrt{2n}} \right| \leq 100n^4 \leq 100n^{-1}.$$

Combining this with Lemma 3.11 establish the lemma. \qed

**Proof of Theorem 3.2** Because of Lemma 3.4 it is sufficient to show that

- conditional on $\tilde{A}_n = \tilde{B}_n = 1$ then $\tilde{A}_n \hat{X}_n + \tilde{B}_n \hat{Y}_n \rightarrow \delta_X + \delta_X$,
- conditional on $\tilde{A}_n = 1 - \tilde{B}_n = 1$ then $\tilde{A}_n \hat{X}_n + \tilde{B}_n \hat{Y}_n \rightarrow \delta_X$ and
- conditional on $1 - \tilde{A}_n = \tilde{B}_n = 1$ then $\tilde{A}_n \hat{X}_n + \tilde{B}_n \hat{Y}_n \rightarrow \delta_X$.

The first of these is the statement of Lemma 3.12. Combining Lemma 3.12 with Lemma 3.3 shows that the second and third statements are true. This completes the proof. \qed

### 4 A Bijection between Dyck Paths and 231-avoiding permutations

The total number of Dyck paths from 0 to $2n$ is given by $C_n$, the $n$th Catalan number. The number of 231-avoiding permutations in $S_n$ is also given by the $n$th Catalan number. Hence there is a bijection between the two sets. We now define a particular bijection that uses geometric properties of the path. This bijection can be viewed as a rotation of the bijection used in [16] and later [9]. For the sake of completeness we include a sketch of the proof that it is a bijection here. For our purposes the most important geometric aspect of a Dyck path is an excursion.

**Definition 4.1** An excursion in a Dyck Path starting at $x$ with height $h$ and length $l$ is a path interval $\gamma([x, x + l])$ such that

(a) $\gamma(x) = \gamma(x + l) = h - 1$
(b) $\gamma(x + 1) = \gamma(x + l - 1) = h$ and
(c) $l = \min\{j \geq 1 : \gamma(x + j) = h - 1\}$.

Note that there are $n$ excursions in a Dyck Path of length $2n$ as there is one excursion that begins with every up-step. Based on this correspondence we say the $i$th excursion, $Exc(i)$ is the one that begins with the $i$th up-step.

**Definition 4.2** For a Dyck path $\gamma$, define the following:

- $Exc(i) :=$ the $i$th excursion.
- $v_i :=$ the position after the $i$th up-step, or $1 +$ the start of $Exc(i)$.
- $h_i := \gamma(v_i) =$ the height of the path after the start of $Exc(i)$.
- $l_i :=$ the length of the same excursion.
Figure 4 illustrates these definitions for a particular γ.

For a path γ ∈ Dyck_{2n} we define pointwise the map \( σ_γ : \mathbb{Z} \to \mathbb{Z} \) by

\[
σ(i) = i + l_i/2 - h_i. \tag{17}
\]

**Theorem 4.3** For γ ∈ Dyck_{2n} let \( σ = σ_γ \) be defined as above. Then \( σ \in S_n(231) \). Moreover, \( γ \mapsto σ_γ \) is a bijection from Dyck_{2n} \( → S_n(231) \).

**Proof** A detailed version of this proof can be found in [11]. We include a sketch of the arguments here.

For any Dyck path \( γ \) and any \( i < j \) if

\[
Exc(j) ⊂ Exc(i) \quad \text{then} \quad σ_γ(j) < σ_γ(i) \quad \tag{18}
\]

and if

\[
Exc(j) \cap Exc(i) = \emptyset \quad \text{then} \quad σ_γ(i) < σ_γ(j). \quad \tag{19}
\]

If \( σ \notin S_n(231) \), then there exists \( i < j < k \) such that \( σ(k) < σ(i) < σ(j) \). Note that \( σ_γ(k) < σ_γ(i) \) implies the \( k \)th up-step occurs before the end of the \( i \)th excursion. Therefore the \( j \)th up-step also occurs before the end of the \( i \)th excursion which implies \( σ_γ(j) < σ_γ(i) \) by 18, and \( σ_γ \) must be 231-avoiding.

To see that \( γ \mapsto σ_γ \) is injective, note that a Dyck path, \( γ \), is uniquely specified by specifying its peaks. The peaks of \( γ \) uniquely determine the left minima of \( σ_γ \) (\( \{(i, σ_γ(i)), σ_γ(i) < σ_γ(j) \text{ for all } j > i\} \)). Since a 231-avoiding is uniquely determined by its left minima, \( γ \mapsto σ_γ \) is injective. \( \square \)

**5 Fixed points for 231-avoiding permutations**

For a 231-avoiding permutation \( σ ∈ S_n(231) \), let \( θ_I(σ) \) denote the number of fixed points of \( σ \) contained in the subset \( I \subset [n] \). Based on our bijection from Sect. 4, for a \( γ \in \text{Dyck}_{2n} \) and \( σ = σ_γ \) and \( σ(i) = i \) precisely when \( l_i/2 = h_i \).
Theorem 5.1  Fix $0 < a < b < 1$ and $\epsilon > 0$. Let $\Gamma^n$ be chosen uniformly at random from $\text{Dyck}^{2n}$. Then

$$\lim_{n \to \infty} \mathbb{P} \left( \left| \frac{1}{n^{1/4}} \theta_{[a_n, b_n]}(\sigma_{\Gamma^n}) - \frac{1}{2 \pi^{1/2}} \int_a^b \left( \frac{n^{1/2}}{\Gamma^n(2nt)} \right)^{3/2} \, dt \right| > \epsilon \right) = 0.$$ 

Using this theorem and a result from [22] we are able to prove our main result.

Proof of Theorem 1.7  Since $((2n)^{-1/2} \Gamma^n(2nt), 0 \leq t \leq 1) \to_d (\varphi_t, 0 \leq t \leq 1)$ in $C([0, 1], \mathbb{R})$ by [13], where $\varphi_t$ denotes a standard Brownian excursion from 0 to 1, Theorem 5.1 implies that for every fixed $0 < a < b < 1$, we have

$$\frac{1}{n^{1/4}} \theta_{[a_n, b_n]}(\sigma_{\Gamma^n}) \xrightarrow{d} \frac{1}{2^{7/4} \pi^{1/2}} \int_a^b \varphi_t^{-3/2} \, dt.$$  (20)

Our first step is to extend this convergence to $a = 0$ and $b = 1$. For any $0 \leq \alpha \leq \beta \leq 1$ we define the random variables

$$F_{\alpha, \beta} = \frac{1}{2^{7/4} \pi^{1/2}} \int_\alpha^\beta \varphi_t^{-3/2} \, dt.$$ 

In [4] the density function for the height of a Brownian excursion at time $t \in [0, 1]$ is determined to be

$$\mathbb{P}(\varphi_t \in dh) = \frac{2^{1/2} h^2}{(\pi t^3 (1 - t)^3)^{1/2}} \exp \left( - \frac{h^2}{2t(1 - t)} \right) dh$$

We can compute $\mathbb{E}[F_{0, 1}]$ by taking the expectation inside the integral and get

$$\mathbb{E}[F_{0, 1}] = \frac{1}{2^{7/4} \pi^{1/2}} \int_0^1 \mathbb{E} \left[ \varphi_t^{-3/2} \right] \, dt = \frac{\text{Gamma}(1/4)}{2\pi^{1/2}}.$$ 

In [22, Theorem 6.4] it is shown that

$$\frac{\mathbb{E} \theta_{[1, n]}}{n^{1/4}} \to \frac{\text{Gamma}(1/4)}{2\pi^{1/2}}.$$  (21)

(In [22] the result is stated incorrectly and is off by a factor of four.) Consequently, we have $n^{-1/4} \mathbb{E} \theta_{[1, n]} \to \mathbb{E} F_{0, 1}$. By the monotone convergence theorem we have both $F_{\delta, 1 - \delta} \to F_{0, 1}$ almost surely as $\delta \downarrow 0$ and $\mathbb{E} F_{\delta, 1 - \delta} \to \mathbb{E} F_{0, 1}$ as $\delta \downarrow 0$. Furthermore, by (20) and the version of Fatou’s Lemma for convergence in distribution (e.g. [14, Lemma 4.11]) we have for $0 < a < b < 1$,

$$\mathbb{E} F_{a, b} \leq \liminf_{n \to \infty} \frac{1}{n^{1/4}} \mathbb{E} \theta_{[a_n, b_n]}.$$  (22)
Thus, using the fact that \( \theta_{[\delta n, (1-\delta)n]} \leq \theta_{[1,n]} \), we have

\[
\lim_{\delta \downarrow 0} \lim_{n \to \infty} \sup_{\frac{1}{n^{1/4}}} \left| \frac{1}{n^{1/4}} \theta_{[1,n]} - \frac{1}{n^{1/4}} \theta_{[\delta n, (1-\delta)n]} \right| \leq \lim_{\delta \downarrow 0} \left( \mathbb{E}F_{0,1} - \mathbb{E}F_{\delta,1-\delta} \right) = 0.
\]

By [14, Theorem 4.28] this implies that \( n^{-1/4} \theta_{[1,n]}(\sigma^{\Gamma_n}) \to_d F_{0,1} \).

Turning to the general case, by [14, Lemma 4.11] the convergences \( n^{-1/4} \theta_{[1,n]}(\sigma^{\Gamma_n}) \to_d F_{0,1} \) and \( n^{-1/4} \mathbb{E} \theta_{[1,n]} \to \mathbb{E}F_{0,1} \) imply that \( (n^{-1/4} \theta_{[1,n]}(\sigma^{\Gamma_n}))_{n \geq 1} \) is uniformly integrable. Since for \( 0 \leq a \leq b \leq 1 \) we have \( \theta_{[a,n,b]} \leq \theta_{[1,n]} \), this implies that for fixed \( 0 \leq a \leq b \leq 1 \) the sequence \( (n^{-1/4} \theta_{[a,n,b]}(\sigma^{\Gamma_n}))_{n \geq 1} \) is also uniformly integrable. Combining with Eq. (20), this implies that if \( 0 < a < b < 1 \) then \( n^{-1/4} \mathbb{E} \theta_{[a,n,b]} \to \mathbb{E}F_{a,b} \). Fixing \( 0 < b < 1 \), observe that for every \( b < c < 1 \), we have

\[
n^{-1/4} \mathbb{E} \theta_{[1,b]} \leq n^{-1/4} \mathbb{E} \left( \theta_{[1,b]} + \theta_{[b,n]} - \theta_{[b,n,c]} \right) = n^{-1/4} \mathbb{E} \left( \theta_{[1,b]} - \theta_{[b,n,c]} \right) \to \mathbb{E}F_{0,1} - \mathbb{E}F_{b,c}.
\]

Letting \( c \) tend to 1 and using monotone convergence, we find that

\[
\lim_{n \to \infty} n^{-1/4} \mathbb{E} \theta_{[1,b]} \leq \mathbb{E}F_{0,b}.
\]

Combining this with Eq. (22) we have

\[
\lim_{\delta \downarrow 0} \lim_{n \to \infty} \sup_{\frac{1}{n^{1/4}}} \left| \frac{1}{n^{1/4}} \theta_{[1,b]} - \frac{1}{n^{1/4}} \theta_{[\delta n, b]} \right| \leq \lim_{\delta \downarrow 0} \left( \mathbb{E}F_{0,b} - \mathbb{E}F_{\delta,b} \right) = 0. \tag{23}
\]

Another use of [14, Theorem 4.28] shows that \( n^{-1/4} \theta_{[1,b]}(\sigma^{\Gamma_n}) \to_d F_{0,b}, \) for every \( 0 \leq b \leq 1 \). That is, we have shown that for every \( t \in [0,1] \), we have

\[
\frac{1}{n^{1/4}} \theta_{[1,tn]}(\sigma^{\Gamma_n}) \to_d \frac{1}{\sqrt{2\pi}} \int_0^t e^{-u^2/2} \, du. \tag{24}
\]

Since, as established above, \( (n^{-1/4} \theta_{[1,tn]}(\sigma^{\Gamma_n}))_{n \geq 1} \) is uniformly integrable, we also have

\[
\lim_{n \to \infty} n^{-1/4} \mathbb{E} \theta_{[0,tn]} = \mathbb{E}F_{0,t},
\]

for every \( t \in [0,1] \). Since \( t \mapsto n^{-1/4} \mathbb{E} \theta_{[0,tn]} \) is nondecreasing and \( t \mapsto \mathbb{E}F_{0,t} \) is continuous, this convergence happens uniformly on \([0,1]\).

We now extend the one-dimensional convergence of Eq. (24) to functional convergence. Since \((2n)^{-1/2} \Gamma_n(2nt), 0 \leq t \leq 1 \to_d (\xi_t, 0 \leq t \leq 1) \) in \( C([0,1], \mathbb{R}) \), using the Skorokhod representation [14, Theorem 4.30] we may take a version of
$(\Gamma^n)_{n \geq 1}$ such that $((2n)^{-1/2} \Gamma^n(2nt), 0 \leq t \leq 1) \to (\varphi_t, 0 \leq t \leq 1)$ almost surely in $C([0, 1], \mathbb{R})$. It follows from Theorem 5.1 that for fixed $0 < a < b < 1$ we have
\[
\frac{1}{n^{1/4}} \theta_{[an, bn]}(\sigma \Gamma^n) \xrightarrow{p} \frac{1}{2^{7/4} \pi^{1/2}} \int_a^b e^{-t^2/2} dt.
\]
Moreover, note that Eq. (23) and Markov’s inequality imply that for every $\epsilon > 0$ we have
\[
\lim_{\delta \downarrow 0} \limsup_{n \to \infty} \mathbb{P}\left(\left| \frac{1}{n^{1/4}} \theta_{[1, bn]} - \frac{1}{n^{1/4}} \theta_{[bn, bn]} \right| > \epsilon \right) = 0,
\]
and, since $F_{a,b} \to F_{0,b}$ almost surely as $a \downarrow 0$, it follows that
\[
\frac{1}{n^{1/4}} \theta_{[1, bn]}(\sigma \Gamma^n) \xrightarrow{p} \frac{1}{2^{7/4} \pi^{1/2}} \int_0^b e^{-t^2/2} dt.
\]
From any subsequence $\{(n^{-1/4} \theta_{[1, t n_k]}, 0 \leq t \leq 1)\}_{k \geq 1}$ of $\{(n^{-1/4} \theta_{[1, t n]}, 0 \leq t \leq 1)\}_{n \geq 1}$ we may use a diagonal argument to extract a further subsequence $\{(n^{-1/4} \theta_{[1, t n_k]}, 0 \leq t \leq 1)\}_{j \geq 1}$ such that
\[
\mathbb{P}\left(\lim_{j \to \infty} n^{-1/4} \theta_{[1, b n_k]}(\Gamma^{n_k}) - \frac{1}{2^{7/4} \pi^{1/2}} \int_0^b e^{-t^2/2} dt \text{ for all } b \in \mathbb{Q}\right) = 1.
\]
Since $t \mapsto n^{-1/4} \theta_{[1, t n_k]}$ is almost surely nondecreasing and $t \mapsto F_{0,t}$ is almost surely continuous, it follows that
\[
\lim_{j \to \infty} \sup_{b \in [0,1]} \left| n^{-1/4} \theta_{[1, b n_k]}(\Gamma^{n_k}) - \frac{1}{2^{7/4} \pi^{1/2}} \int_0^b e^{-t^2/2} dt \right| = 0
\]
on the set
\[
\left(\lim_{j \to \infty} n^{-1/4} \theta_{[1, b n_k]}(\Gamma^{n_k}) - \frac{1}{2^{7/4} \pi^{1/2}} \int_0^b e^{-t^2/2} dt \text{ for all } b \in \mathbb{Q}\right),
\]
and thus
\[
\mathbb{P}\left(\lim_{j \to \infty} \sup_{b \in [0,1]} \left| n^{-1/4} \theta_{[1, b n_k]}(\Gamma^{n_k}) - \frac{1}{2^{7/4} \pi^{1/2}} \int_0^b e^{-t^2/2} dt \right| = 0\right) = 1.
\]
Consequently, since convergence in probability is metrizable, we have
\[
\left(n^{-1/4} \theta_{[1, t n]}(\Gamma^n), 0 \leq t \leq 1\right) \xrightarrow{p} \left(\frac{1}{2^{7/4} \pi^{1/2}} \int_0^t e^{-u^2/2} du, 0 \leq t \leq 1\right),
\]
in the Skorokhod space $D([0, 1], \mathbb{R})$, which completes the proof. \hfill \Box

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Now we set up the notation necessary to prove Theorem 5.1. We break the interval \([an, bn]\) up into subintervals of size about \(n^{0.9}\). In each of these intervals we will estimate the expected number of fixed points using the height of Dyck path at the start of the interval. Then we will bound the variance to show that with high probability the number of fixed points is close to the expected value.

Label the intervals \(I_k = [a_k, b_k]\) for \(k \in [0, \ldots, K - 1]\), where \(K = \lfloor (b - a)n^{0.1} \rfloor\) and

\[
a_k = \lfloor an + (k/K)(bn - an) \rfloor \quad \text{and} \quad b_k = a_{k+1}.
\]

Denote a sequence of heights \(\alpha = \{\alpha^n_k\}_{k=0}^{K-1}\) and define

\[
\Omega^n(\alpha) := \bigcap_{k=0}^{K-1} \left\{ \gamma \in \text{Dyck}^{2n} | \gamma (v_{a_k}) = \alpha^n_k \right\}
\]

where \(v_{a_k}\) is the number of steps in \(\gamma\) up to and including the \(a_k\)th up-step. Note that \(\Omega^n(\alpha) \cap \Omega^n(\alpha') = \emptyset\) if \(\alpha \neq \alpha'\). Let \(\mathcal{A}\) denote the collection of all \(\alpha\).

**Definition 5.2 (A Proper Subset of Dyck\(^{2n}\))** We say a sequence of heights \(\alpha = \{\alpha^n_k\}\) is proper if the following are satisfied for all \(k = 0, \ldots, K\):

- \(n^{0.499} < \alpha^n_k < n^{0.501}\) and
- \(|\alpha^n_k - \alpha^n_{k+1}| < n^{0.451}\).

We say \(\Omega^n(\alpha)\) is proper if \(\alpha\) is proper.

**Definition 5.3** Recalling Definition 4.2, we define the random variables for a random path \(\Gamma_n \in \text{Dyck}^{2n}\):

- \(V_i^n :=\) number of steps up to and including the \(i\)th up-step.
- \(H_i^n := \Gamma^n(V_i^n)\).
- \(L_i^n :=\) the length of the \(i\)th excursion.

Let \(\mathcal{B}_n\) denote the collection of proper \(\alpha \in \mathcal{A}\). Most \(\Gamma^n \in \text{Dyck}^{2n}\) will be in some proper \(\Omega^n(\alpha)\).

**Lemma 5.4** For \(n\) sufficiently large, and \(\Gamma^n\) be chosen uniformly at random from \(\text{Dyck}^{2n}\),

\[
\mathbb{P}\left( \Gamma^n \in \bigcup_{\alpha \in \mathcal{B}_n} \Omega^n(\alpha) \right) = 1 - o(1).
\]

Moreover

\[
\mathbb{P}\left( \bigcap_{i \in [an, bn]} \left\{ n^{0.49} < H_i^n < n^{0.51} \right\} \bigg| \Omega^n(\alpha) \right) > 1 - e^{-n^{0.0001}}
\]

for all proper \(\Omega^n(\alpha)\).
Proof The first statement follows from Lemmas 6.10 and 6.11. The second statement follows by applying Lemma 6.12 to the intervals $I_k$ for $0 \leq k < K$. □

For a fixed sequence of heights $\alpha$, let $\hat{k}(x) = \sup_k \{2a_k - \alpha_k^i \leq x\}$. We define the following function $\rho_\alpha : [2an, 2bn] \rightarrow [0, n]$

$$\rho_\alpha(x) := \alpha_k^n(x).$$

For $\gamma \in \Omega^n(\alpha)$, $v_{a_k} = 2a_k - \alpha_k^n$ and $\gamma(v_{a_k}) = \rho_\alpha(v_{a_k}) = \alpha_k^n$. For most $\gamma \in \Omega^n(\alpha)$, $\gamma$ will be close to $\rho_\alpha$.

**Lemma 5.5** Fix $0 < a < b < 1$, and $\epsilon > 0$. For all $n$ sufficiently large,

$$\max_{\alpha \in \mathcal{B}_n} \left\{ \mathbb{P} \left( \sup_{t \in [a,b]} \left| \left( \frac{n^{1/2}}{\rho_\alpha(2nt)} \right)^{3/2} - \left( \frac{n^{1/2}}{\Gamma^n(2nt)} \right)^{3/2} \right| > n^{-0.01} \left| \Omega^n(\alpha) \right| \right) \right\} < e^{-n^{0.001}}.$$

**Proof** Let $\hat{k} = \hat{k}(2nt)$. By definition $v_{a_k} < 2nt < v_{a_{k+1}}$ so

$$|2nt - v_{a_k}| < |2a_k - 2d_k + \alpha_k^n - \alpha_k^{n+1}| < 2n^{0.9} + n^{0.451} < 3n^{0.9}.$$

Using Lemma 6.12 we obtain deviation bounds corresponding to all $i \in (a_k, a_{k+1})$ for $0 \leq k < K$. In particular we have for $t < 3$,

$$\mathbb{P}(|\Gamma^n(v_{a_k} + tn^{0.9}) - \Gamma^n(v_{a_k})| > n^{0.46} |\Omega^n(\alpha)|) < e^{-n^{0.001}}.$$ 

By Lemma 5.4, we may also conclude that $\Gamma^n(2nt) > n^{0.49} - 1$ with probability $1 - e^{-n^{0.0001}}$, so with probability at least $1 - 2e^{-n^{0.0001}}$,

$$\left| \left( \frac{n^{1/2}}{\rho_\alpha(2nt)} \right)^{3/2} - \left( \frac{n^{1/2}}{\Gamma^n(2nt)} \right)^{3/2} \right| < \left| \left( \frac{n^{1/2}}{\Gamma^n(2nt)(\rho_\alpha(2nt)/\Gamma^n(2nt))} \right)^{3/2} - \left( \frac{n^{1/2}}{\Gamma^n(2nt)} \right)^{3/2} \right| \leq n^{0.015}n^{-0.03} < n^{-0.01}. $$

**Lemma 5.6** Fix $0 < a < b < 1$. For all $n$ sufficiently large,

$$\max_{\alpha \in \mathcal{B}_n} \left| n^{-1/4} \mathbb{E}_\theta(a_n, b_n) \left| \Omega^n(\alpha) \right| - 1 \right| < n^{-0.001}.$$
Lemma 5.7 Fix $0 < a < b < 1$. For all $n$ sufficiently large,
\[
\max_{\alpha \in B_n} \Var[\theta_{[\beta n, \beta n]}] ^ n(\alpha) < n^{0.48}.
\]

Because these bounds are uniform over all proper $\Omega^n(\alpha)$ we will drop the $\alpha$ where no confusion should arise. We delay the proofs of these two lemmas until after the proof of Theorem 5.1 as they are long and somewhat technical.

Proof of Theorem 5.1 Let $\Gamma^n$ chosen uniformly from Dyck$^{2n}$. Since $(2n)^{-1/2} \Gamma^n(2nt), 0 \leq t \leq 1 \to_d (\epsilon_t, 0 \leq t \leq 1)$ in $C([0, 1], \mathbb{R})$, we may take a version of $(\Gamma^n)_{n \geq 1}$ such that $(2n)^{-1/2} \Gamma^n(2nt), 0 \leq t \leq 1) \to_d (\epsilon_t, 0 \leq t \leq 1)$ almost surely in $C([0, 1], \mathbb{R})$.

For sufficiently large $n$, by Lemma 5.7 and Chebyshev’s inequality
\[
\mathbb{P}
\left[
\theta_{[\beta n, \beta n]}(\Gamma^n) - \mathbb{E}
\left[
\theta_{[\beta n, \beta n]}|\Gamma^n \in \Omega^n
\right]
\right]
\geq \mathbb{E}
\left[
\theta_{[\beta n, \beta n]}|\Gamma^n \in \Omega^n
\right)
\leq \mathbb{P}
\left[
\Gamma^n \notin \cup_{\alpha \in B_n} \Omega^n(\alpha)
\right]
\leq \mathbb{P}
\left[
\Gamma^n \notin \cup_{\alpha \in B_n} \Omega^n(\alpha)
\right] + n^{-0.01} \mathbb{P}
\left[
\Gamma^n \in \cup_{\alpha \in B_n} \Omega^n(\alpha)
\right] = o(1).
\]

Using a similar decomposition in each case, Lemma 5.6 implies that for sufficiently large $n$
\[
\mathbb{P}
\left[
\frac{\sum_{\alpha} \mathbb{E}
\left[
\theta_{[\beta n, \beta n]}|\Gamma^n \in \Omega^n(\alpha)
\right] \mathbb{1}_{(\Gamma^n \in \Omega^n(\alpha))}
\right]
\leq \mathbb{E}
\right[
\theta_{[\beta n, \beta n]}|\Gamma^n \in \Omega^n(\alpha)
\right] - 1
\leq n^{-0.001}
\right]) = o(1),
\]

and Lemma 5.5 implies that for sufficiently large $n$
\[
\mathbb{P}
\left[
\frac{1}{\sum_{\alpha} \mathbb{E}
\left[
\theta_{[\beta n, \beta n]}|\Gamma^n \in \Omega^n(\alpha)
\right] \mathbb{1}_{(\Gamma^n \in \Omega^n(\alpha))}
\right]
\leq \mathbb{E}
\left[
\theta_{[\beta n, \beta n]}|\Gamma^n \in \Omega^n(\alpha)
\right] - 1
\leq n^{-0.001}
\right]) = o(1).
\]

Since $(2n)^{-1/2} \Gamma^n(2nt), 0 \leq t \leq 1) \to_d (\epsilon_t, 0 \leq t \leq 1)$ almost surely
\[
\frac{1}{\sum_{\alpha} \mathbb{E}
\left[
\theta_{[\beta n, \beta n]}|\Gamma^n \in \Omega^n(\alpha)
\right] \mathbb{1}_{(\Gamma^n \in \Omega^n(\alpha))}
\right]
\leq \mathbb{E}
\left[
\theta_{[\beta n, \beta n]}|\Gamma^n \in \Omega^n(\alpha)
\right] - 1
\leq n^{-0.001}
\right]) = o(1).
\]
and therefore
\[
\sum_{\alpha} \frac{1}{2\pi^{1/2}} \int_{a}^{b} \left( \frac{n^{1/2}}{\rho(2nt)} \right)^{3/2} dt \mathbf{1}_{(\Gamma_n \in \Omega^n(\alpha))} \xrightarrow{p} \frac{1}{2^{3/4}\pi^{1/2}} \int_{a}^{b} e^{-3/2t} dt.
\]

By Eq. (26) and the fact that if two sequences converge in probability so does their product (see e.g. [14, Corollary 4.5]), we have
\[
n^{-1/4} \sum_{\alpha} E \left[ \theta_{|an, bn|} \middle| \Gamma_n \in \Omega^n(\alpha) \right] \mathbf{1}_{(\Gamma_n \in \Omega^n(\alpha))} \xrightarrow{p} \frac{1}{2^{3/4}\pi^{1/2}} \int_{a}^{b} e^{-3/2t} dt.
\]

Thus it follows from Eq. (25) that
\[
n^{-1/4} \theta_{|an, bn|}(\sigma_{\Gamma^n}) \xrightarrow{p} \frac{1}{2^{7/4}\pi^{1/2}} \int_{a}^{b} e^{-3/2t} dt,
\]
completing the proof.

5.1 Proof of Lemma 5.6

For \(i \in [an, bn]\) we have that \(\theta_i := \theta_i(\sigma_{\Gamma^n})\) is a 0-1 valued random variable where
\[
\mathbb{P}(\theta_i = 1) = \mathbb{P}(L_{i/n} = H_{i/n}).
\]

Let \(I_k = I_k^{\text{int}} \cup I_k^{\text{out}}\) where \(I_k^{\text{out}}\) consists of the \(2n^{0.6}\) values both directly after \(a_k\) or directly before \(a_k + 1\) and \(I_k^{\text{int}}\) is the rest of \(I_k\).

Lemma 5.8 Fix \(0 < a < b < 1\). For all proper \(\Omega^n\) and for each \(k\), and \(i \in I_k^{\text{int}}\).
\[
E[\theta_i | \Omega^n] = \frac{1}{2\pi^{1/2} (\alpha_k^n)^{3/2}} (1 + \Delta),
\]
where \(\Delta = \Delta(i, k, \Omega^n) = o(n^{-0.01})\) uniformly in \(i, k\) and proper \(\Omega^n\).

Proof For each \(k\) the and each \(i\) in \(I_k^{\text{int}}\) the conditions for Lemma 6.8 are satisfied since \(\Omega^n\) is proper. Therefore
\[
E[\theta_i | \Omega^n] = \frac{1}{2\pi^{1/2} (\alpha_k^n)^{3/2}} (1 + \Delta)
\]
as desired.

For \(I_k^{\text{out}}\) we look at \(E[\theta_{|I_k^{\text{out}}|} | \Omega^n]\) as a whole rather than computing \(E[\theta_i | \Omega^n]\) for each individual \(i\).
Lemma 5.9 Fix $0 < a < b < 1$ and proper $\Omega^n$. For all $\gamma \in \Omega^n$

$$\sum_k \mathbb{E} \left[ \theta_{I_{k_{\text{out}}}} | \Omega^n \right] \leq 6n^{0.21}.\$$

Proof For each $k$, $I_{k_{\text{out}}}$ consists of two intervals of length $2n^{0.6}$. As $\Omega^n$ is proper, Lemma 5.4 implies that $h_i > n^{0.49}$ for every $i \in [an, bn]$ with probability $e^{-n^{0.0001}}$. Given this lower bound for each $h_i$, Lemma 6.13 implies that there are at most $5n^{0.11}$ fixed points in $I_{k_{\text{out}}}$. Then $\mathbb{E} \left[ \theta_{I_{k_{\text{out}}}} | \Omega^n \right] \leq 5n^{0.11} + 4n^{0.6}e^{-n^{0.0001}}$ for each $0 \leq k < K < n^{0.1}$. Adding them up proves the lemma.

Lemma 5.10 For fixed $0 < a < b < 1$ and proper $\Omega^n$,

$$\mathbb{E} \left[ \theta_{[an, bn]} | \Omega^n \right] = (1 + \Delta) \sum_{j=\lfloor an \rfloor}^{\lfloor bn \rfloor} \frac{1}{2\pi^{1/2} \rho \left( V^n_j \right)^{3/2}},$$

where $\Delta = o(n^{-0.01})$ uniformly in proper $\Omega^n$.

Proof By linearity of expectation:

$$\mathbb{E} \left[ \theta_{[an, bn]} | \Omega^n \right] = \sum_{k=0}^{K-1} \sum_{i' \in I_{k_{\text{int}}}} \mathbb{E} \left[ \theta_{i'} | \Omega^n \right] = \sum_{k=0}^{K-1} \sum_{i=0}^{\lfloor |I_k| \rfloor - 1} \mathbb{E} \left[ \theta_{a_k+i} | \Omega^n \right]$$

For each $k$, and $a_k + i \in I_{k_{\text{int}}}$, we can apply Lemma 5.8 to conclude

$$\mathbb{E} \left[ \theta_{a_k+i} | \Omega^n \right] = \frac{1}{2\pi^{1/2} \left( \alpha^n_k \right)^{3/2}} \left( 1 + \Delta \left( i, k, \Omega^n \right) \right)$$

where $\Delta(i, k, \Omega^n) = o(n^{-0.01})$ uniformly in $i$, $k$ and proper $\Omega^n$.

By Lemma 5.9, $\mathbb{E} \left[ \sum_k \theta_{I_{k_{\text{out}}}} | \Omega^n \right] < n^{0.22}$. On the other hand $\alpha^n_k < n^{0.51}$ implies

$$\sum_k \mathbb{E} \left[ \theta_{I_{k_{\text{int}}}} | \Omega^n \right] \geq \sum_k \sum_{i \in I_{k_{\text{int}}}} \frac{1}{2\pi^{1/2} \left( \alpha^n_k \right)^{3/2}} \left( 1 + \Delta(i, k, \Omega^n) \right) > \sum_k \sum_{i \in I_{k_{\text{int}}}} n^{-0.765} > n^{0.23}$$
so the contribution from \( \sum_k \mathbb{E}[\theta_{i\text{out}}^k | \Omega^n] \) is dominated by \( \sum_k \mathbb{E}[\theta_{i\text{int}}^k | \Omega^n] \). Then by Lemmas 5.8 and 5.9

\[
\mathbb{E}[\theta_{[a_n, b_n]} | \Omega^n] = \sum_k \mathbb{E}[\theta_{i\text{out}}^k | \Omega^n] + \sum_k \mathbb{E}[\theta_{i\text{int}}^k | \Omega^n] \\
= 5n^{0.21} + \sum_k \frac{|I_{i\text{int}}^k|}{2\pi^{1/2}(\alpha_k^n)^{3/2}} (1 + \Delta (k, \Omega^n)) \\
= (1 + \Delta (\Omega^n)) \sum_k \frac{|I_{i\text{int}}^k|}{2\pi^{1/2}(\alpha_k^n)^{3/2}}
\]

where \( \Delta(k, \Omega^n) = o(n^{-0.01}) \) uniformly in \( k \) and proper \( \Omega^n \) and \( \Delta(\Omega^n) = o(n^{-0.01}) \) uniformly in \( \Omega^n \).

For each \( k, |I_k| = (1 + O(n^{-0.3}))|I_{i\text{int}}^k| \) by the definitions of \( I_k \) and \( I_{i\text{int}}^k \). Then the above expression becomes

\[
\mathbb{E}[\theta_{[a_n, b_n]} | \Omega^n] = (1 + \Delta'(\Omega^n)) \sum_k \frac{|I_k|}{2\pi^{1/2}(\alpha_k^n)^{3/2}} \\
= (1 + \Delta'(\Omega^n)) \sum_k \sum_{j \in I_k} \frac{1}{2\pi^{1/2}(\alpha_k^n)^{3/2}}
\]

with \( \Delta'(\Omega^n) = o(n^{-0.01}) \) uniformly in all proper \( \Omega^n \). For \( j \in I_k, \rho(V_{n^j}) = \alpha_k^n \), finishing the proof.

Proof of Lemma 5.6 By Lemma 5.10 we can write the conditional expectation of \( \theta_{[a_n, b_n]} \) as

\[
\mathbb{E}[\theta_{[a_n, b_n]} | \Omega^n] = (1 + \Delta) \sum_{i=\lfloor a_n \rfloor}^{\lfloor b_n \rfloor} \frac{1}{2\pi^{1/2}\rho \left( V_{n^j} \right)^{3/2}}
\]

where \( \Delta = o(n^{-0.01}) \) uniformly in all proper \( \Omega^n \). Converting the sum into an integral we have

\[
\mathbb{E}[\theta_{[a_n, b_n]} | \Omega^n] = (1 + \Delta) \int_{\lfloor a_n \rfloor}^{\lfloor b_n \rfloor} \frac{1}{2\pi^{1/2}\rho \left( V_{n^j} \right)^{3/2}} du.
\]

The change of variables \( nt = u \) gives

\[
\mathbb{E}[\theta_{[a_n, b_n]} | \Omega^n] = (1 + \Delta) \int_a^b \frac{1}{2\pi^{1/2}\rho \left( V_{n^j} \right)^{3/2}} n dt.
\]
Since $\Omega^n$ is proper, $|2nt - V^n_{\lfloor nt \rfloor}| < n^{0.51}$. Therefore either $\hat{k}(V^n_{\lfloor nt \rfloor}) = \hat{k}(2nt)$ or $\hat{k}(2nt) - 1$. In either case by properness of $\Omega^n$, $|\rho(V^n_{\lfloor nt \rfloor}) - \rho(2nt)| < n^{0.451}$ and $\rho(2nt) > n^{0.499}$ for $t \in [a, b]$ so

$$\frac{1}{\rho(V^n_{\lfloor nt \rfloor})^{3/2}} = \left(1 + o\left(n^{-0.001}\right)\right) \frac{1}{\rho(2nt)^{3/2}}.$$

Scaling by $n^{1/4}$ completes the proof. \qed

### 5.2 Proof of Lemma 5.7

Now that we have the conditional expectation $\mathbb{E}[\theta_{\lfloor an, bn \rfloor} | \Omega^n]$, we will bound the conditional variance, $\text{Var}[\theta_{\lfloor an, bn \rfloor} | \Omega^n]$.

Our basic variance equation is

$$\text{Var} \left[ \theta_{\lfloor an, bn \rfloor} | \Omega^n \right] = \sum_{i,j} \left( \mathbb{E} \left[ \theta_i \theta_j | \Omega^n \right] - \mathbb{E} \left[ \theta_i | \Omega^n \right] \mathbb{E} \left[ \theta_j | \Omega^n \right] \right).$$

The key to bounding the conditional variance for a proper $\Omega^n$ is understanding $\mathbb{E}[\theta_i \theta_j | \Omega^n]$ for various ranges of $i$ and $j$. We cover $[an, bn]^2$ with $\cup_{i=1}^5 B_i$ where each $B_i$ is defined as follows:

- $B_1 = \bigcup_k \bigcup_{k'} I_{k}^{\text{out}} \times I_{k'}^{\text{out}},$
- $B_2 = \bigcup_k \bigcup_{k'} \{ I_{k}^{\text{int}} \times I_{k'}^{\text{out}} \} \bigcup \{ I_{k}^{\text{int}} \times I_{k'}^{\text{int}} \},$
- $B_3 = \bigcup_k \bigcup_{k' \neq k} I_{k}^{\text{int}} \times I_{k'}^{\text{int}},$
- $B_4 = \bigcup_k I_{k}^{\text{int}} \times \{ j \in I_{k}^{\text{int}} \text{s.t. } |j - i| \leq 2n^{0.6} \},$
- $B_5 = \bigcup_k I_{k}^{\text{int}} \times \{ j \in I_{k}^{\text{int}} \text{s.t. } |j - i| > 2n^{0.6} \}.$

Consider the property

$$P = \bigcap_{i \in [an, bn]} \left\{ n^{0.49} < H^n_i < n^{0.51} \right\}.$$

For each $B_i$ we will show that

$$\sum_{(i,j) \in B_i} \mathbb{E} \left[ \theta_i \theta_j 1_P | \Omega^n \right] = \mathbb{E} \left[ \theta_i 1_P | \Omega^n \right] \mathbb{E} \left[ \theta_j 1_P | \Omega^n \right] = o \left( n^{0.48} \right).$$

Hence the total variance, $\text{Var}[\theta_{\lfloor an, bn \rfloor} 1_P | \Omega^n]$ is $o(n^{0.48})$.

The following lemma allows us extend this bound to $\text{Var}[\theta_{\lfloor an, bn \rfloor} | \Omega^n]$.

**Lemma 5.11** Fix $0 < a < b < 1$. For $n$ sufficiently large and proper $\Omega^n$

$$\sum_{(i,j) \in [an, bn]^2} \mathbb{E} \left[ \theta_i \theta_j | \Omega^n \right] \leq \sum_{(i,j) \in [an, bn]^2} \mathbb{E} \left[ \theta_i \theta_j 1_P | \Omega^n \right] + n^5 \exp \left( -n^{0.001} \right).$$
Proof Since $\theta_i \theta_j 1_{PC} < 1_{PC}$, by Lemma 5.4 we have

$$\mathbb{E} [\theta_i \theta_j 1_{PC} | \Omega^n] \leq \mathbb{E} [1_{PC} | \Omega^n] \leq n^3 \exp \left( -n^{0.0001} \right).$$

Noting that $\theta_i \theta_j = \theta_i \theta_j 1_P + \theta_i \theta_j 1_{PC}$ and taking expectation gives

$$\mathbb{E} [\theta_i \theta_j | \Omega^n] \leq \mathbb{E} [\theta_i \theta_j 1_P | \Omega^n] + \mathbb{E} [1_{PC} | \Omega^n] \leq \mathbb{E} [\theta_i \theta_j 1_P | \Omega^n] + n^3 \exp \left( -n^{0.0001} \right).$$

Summing over $i$ and $j$ completes the proof. 

For the following series of lemmas we will assume the following standard hypotheses.

- Fix $0 < a < b < 1$.
- $\Omega^n \subset \text{Dyck}^{2n}$ is proper.
- Let $n$ be large enough such that $n^2 e^{-n^{0.0001}} < o(1)$.

Lemma 5.12 Assuming the standard hypotheses,

$$\sum_{(i,j) \in B_2} \mathbb{E} [\theta_i \theta_j 1_P | \Omega^n] < n^{0.47}.$$

Proof For fixed $k'$ we may use Lemma 6.13 to show $\sum_{j \in I_{k'}^{out}} \mathbb{E} [\theta_j | \ast] < 4n^{0.6-0.49}$ no matter the conditions given by $\ast$. In particular we have

$$\sum_{j \in I_{k'}^{out}} \mathbb{E} [\theta_i \theta_j 1_P | \Omega^n] \leq \sum_{j \in I_{k'}^{out}} \mathbb{E} [\theta_j | \Omega^n, \theta_i 1_P = 1] \mathbb{E} [\theta_i 1_P | \Omega^n]
\leq \mathbb{E} [\theta_i 1_P | \Omega^n] 8n^{0.6-0.49}.$$

Then

$$\sum_{k} \sum_{k'} \sum_{i \in I_{k'}^{out}} \mathbb{E} [\theta_i \theta_j 1_P | \Omega^n] \leq \sum_{k} \sum_{k'} \sum_{j \in I_{k'}^{out}} \mathbb{E} [\theta_i 1_P | \Omega^n] 4n^{0.6-0.49}
\leq \sum_{k} \sum_{k'} 8n^{0.22}
\leq n^{0.42}.$$

Lemma 5.13 Assuming the standard hypotheses,

$$\sum_{(i,j) \in B_2} \mathbb{E} [\theta_i \theta_j 1_P | \Omega^n] < 6n^{0.47}.$$
Proof This follows the proof of Lemma 5.12 closely. By Lemma 6.13

\[
\sum_{k} \sum_{k'} \sum_{i \in I^i_k} \sum_{j \in I^j_{k'}} \mathbb{E}\left[\theta_i \theta_j 1_p | \Omega^n\right] \leq \sum_{k} \sum_{k'} \sum_{i \in I^i_k} \mathbb{E}[\theta_i 1_p | \Omega^n] 2n^{0.6-0.49}.
\]

For each \(k\) and each \(i \in I^i_k\), Lemma 6.8 and the properness of \(\Omega^n\) imply that

\[
\mathbb{E}[\theta_i 1_p | \Omega^n] < (n^{0.495})^{3/2} < n^{0.74}.
\]

Then

\[
\sum_{k} \sum_{k'} \sum_{i \in I^i_k} 3n^{0.11} \mathbb{E}[\theta_i 1_p | \Omega^n] \leq 3n^{0.1} n^{0.9} n^{0.11} n^{0.74} < 3n^{0.47}.
\]

Changing the roles of \(i\) and \(j\) and doubling the upper bounded completes the proof. \(\Box\)

Lemma 5.14 Assuming the standard hypotheses,

\[
\sum_{(i,j) \in B_3} \mathbb{E}\left[\theta_i \theta_j 1_p | \Omega^n\right] - \mathbb{E}\left[\theta_i 1_p | \Omega^n\right] \mathbb{E}\left[\theta_j 1_p | \Omega^n\right] = 0.
\]

Proof The flavor of this proof is somewhat different from the previous lemmas. Without loss of generality we may assume that \(k < k'\).

If \(\theta_i 1_p = 1\), then the corresponding \(i\)th excursion will end before the \(a_{k'}\)th excursion begins as

\[
L_i^n/2 = H_i^n < n^{0.51} < 2n^{0.6}
\]

and \(a_{k'} > i + |I^i_k'|\). Therefore, for \(i \in I_k\) and \(j \in I_{k'}\),

\[
\mathbb{E}\left[\theta_i \theta_j 1_p | \Omega^n\right] = \mathbb{E}\left[\theta_i 1_p | \Omega^n\right] \mathbb{E}\left[\theta_j 1_p | \Omega^n\right]
\]

and

\[
\sum_{(i,j) \in B_3} \left(\mathbb{E}\left[\theta_i \theta_j 1_p | \Omega^n\right] - \mathbb{E}\left[\theta_i 1_p | \Omega^n\right] \mathbb{E}\left[\theta_j 1_p | \Omega^n\right]\right) = 0.
\]

\(\Box\)

Lemma 5.15 Assuming the standard hypotheses,

\[
\sum_{(i,j) \in B_4} \mathbb{E}[\theta_i \theta_j 1_p | \Omega^n] < n^{0.47}.
\]

Proof By Lemma 6.13

\[
\sum_{|i-j| < 2n^{0.6}} \mathbb{E}[\theta_i \theta_j 1_p | \Omega^n] \leq 5n^{0.6-0.49} \mathbb{E}[\theta_i 1_p | \Omega^n].
\]
For each \( k \), \( |I_k^{int}| \leq n^{0.9} \) so
\[
\sum_{k} \sum_{i \in I_k^{int}} 5n^{0.11} n^{-0.73} \leq 5n^{0.1+0.9+0.11-0.735} < n^{0.47}.
\]
\( \square \)

The last possibility is the one which requires the most care.

**Lemma 5.16** Assuming the standard hypotheses,
\[
\sum_{(i,j) \in B_3} \mathbb{E}[\theta_i \theta_j 1_p | \Omega^n] < n^{0.47}.
\]

**Proof** We proceed in a manner similar to Lemmas 6.6 and 6.7. For \( (i, j) \in B_3 \), with \( i \in I_k^{int} \),
\[
\mathbb{E}[\theta_i \theta_j 1_p | \Omega^n] = \sum_{n^{0.49} < h < n^{0.51}} \sum_{n^{0.49} < h' < n^{0.51}} G(i, j, h, h', a_k, a_{k+1}, \alpha_k^n, \alpha_{k+1}^n).
\]

where
\[
G(i, j, h, h', a_k, a_{k+1}, \alpha_k^n, \alpha_{k+1}^n) = C_{h-1} C_{h'-1} \mathbb{E}[2i-h,h|\mathcal{E}_{v_k}] \mathbb{E}[2j-h',h'|\mathcal{E}_{v_{k+1}}]^{-1}.
\]

For fixed \( i, j \) and \( k \) there are two cases to consider for values \( h \) and \( h' \). One where we can use Lemma 6.2 for each each section of the path, and one where we bound \( G \) by \( e^{-n^{0.001}} \) using Lemma 6.1.

Define the set of pairs of heights \( D_{i,j,k} \) such that for \( (h, h') \in D_{i,j,k} \), Lemma 6.2 is valid for each of the path sections. For \( (h, h') \notin D_{i,j,k} \) the contribution to \( \mathbb{E}[\theta_i \theta_j 1_p | \Omega^n] \) is bounded by \( e^{-n^{0.001}} \). Otherwise
\[
\mathbb{E}[\theta_i \theta_j 1_p | \Omega^n] \leq \sum_{(h, h') \in D_{i,j,k}} \sqrt{n^{0.9}} \frac{1}{(j-i)(n^{0.9} - j)h^3 h'^3} F(i, j, h, h', a_k^n, a_{k+1}^n)
\]

where
\[
F(i, j, h, h', a_k^n, a_{k+1}^n) = \exp \left( -\frac{(h - a_k^n)^2}{4i} - \frac{(h' - h)^2}{4(j - i)} - \frac{(a_{k+1}^n - h')^2}{4(n^{0.9} - j)} + \frac{(a_{k+1}^n - a_k^n)^2}{4n^{0.9}} \right).
\]
For \( h \) and \( h' \in (n^{0.49}, n^{0.51}) \) we may replace \( \frac{1}{(hh')^{3/2}} \) with \( n^{-1.47} \). By Lemma 6.15 there exists a large constant \( C \) such that

\[
\sum_{h, h'} F(i, j, h, h', \alpha_k^n, \alpha_{k+1}^n) < C \sqrt{i (j - i) (n^{0.9} - j) / n^{0.9}}
\]

uniformly over all choices of \((i, j) \in B_5\) and \( \alpha_k^n \) and \( \alpha_{k+1}^n \) from a proper sequence of heights.

This gives

\[
\sum_{(i, j) \in B_5} \mathbb{E} [\theta_i \theta_j 1_p | \Omega^n] \leq \sum_{(i, j) \in B_5} C n^{-1.47} \leq C n^{1.9} n^{-1.47} < n^{0.47}.
\]

\[\square\]

**Proof of Lemma 5.7** Since \( \mathbb{E}[\theta_i 1_p | \Omega^n] \geq 0 \), Lemmas 5.12, 5.13, 5.14, 5.15, and 5.16 combine to show that

\[
\text{Var} \left[ \theta_{[an, bm]} 1_p | \Omega^n \right] = \sum_{k=1}^{5} \sum_{(i, j) \in B_k} \left( \mathbb{E} [\theta_i \theta_j 1_p | \Omega^n] - \mathbb{E} [\theta_i 1_p | \Omega^n] \mathbb{E} [\theta_j 1_p | \Omega^n] \right)
\]

\[
= \sum_{k=1}^{5} \left( \sum_{(i, j) \in B_k} \mathbb{E} [\theta_i \theta_j 1_p | \Omega^n] \right) - \sum_{k=3}^{5} \mathbb{E} [\theta_i 1_p | \Omega^n] \mathbb{E} [\theta_j 1_p | \Omega^n]
\]

\[
< n^{0.47} + 6n^{0.47} + n^{0.47} + n^{0.47} = 9n^{0.47}.
\]

Using Lemma 5.11 and the fact that

\[
\mathbb{E}[\theta_i | \Omega^n] \geq \mathbb{E}[\theta_i 1_p | \Omega^n] \geq 0,
\]

we see that

\[
\text{Var} \left[ \theta_{[an, bm]} | \Omega^n \right] = \sum_{(i, j) \in [an, bm]^2} \left( \mathbb{E} [\theta_i \theta_j | \Omega^n] - \mathbb{E} [\theta_i | \Omega^n] \mathbb{E} [\theta_j | \Omega^n] \right)
\]

\[
\leq n^5 \exp \left( -n^{0.0001} \right) + \sum_{(i, j) \in [an, bm]^2} \left( \mathbb{E} [\theta_i 1_p | \Omega^n] - \mathbb{E} [\theta_i 1_p | \Omega^n] \mathbb{E} [\theta_j 1_p | \Omega^n] \right)
\]

\[\square\]
$$= n^5 \exp\left(-n^{0.0001}\right) + \text{Var}\left[\theta_{an, bn} \mathbf{1}_p | \Omega^m\right]$$

$$< n^{0.48},$$

which completes the proof.

**Proof of Corollary 1.8** We follow the proof of Theorem 1.7 with very minor changes. In particular we use Corollaries 6.9 and 6.14 in place of Lemmas 6.8 and 6.13. Everything else follows in exactly the same manner when \(\alpha < .48\). For \(\alpha \in [.48, .5]\) we follow the proof of Theorem 1.7 changing the exponents to \(.5 \pm \delta\). We leave the details to the reader.

The following appendix contains various technical lemmas that have been used throughout the paper. The statements of the lemmas are similar to results found elsewhere, but modified for use in this paper.

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**Appendix A: Technical lemmas**

We begin with a useful Lemma that will help count non-negative lattice paths between points. Let \(A_n\) denote the set of points \((i, m) \in \mathbb{Z}^2\) such that \(0 < n^{0.6} < i < 2n^{0.9}\) and \(|m| < i^{0.55}\).

**Lemma 6.1** For \((i, m) \in A_n\).

$$\binom{2i - m}{i} = \frac{(1 + \Delta(i, m)) 4^i}{2^m \sqrt{\pi i}} e^{-\frac{m^2}{\pi i}}$$

where \(\Delta(i, m) = o(n^{-0.1})\) uniformly in \(i\) and \(m\) in \(A_n\).

For \(i > n^{0.6}\) and \(|m| > i^{0.55}\)

$$\binom{2i - m}{i} \leq \frac{4^i}{2^m e^{-n^{0.05}}}.$$

**Proof** This first equality follows from IX.1 on page 615 of Flajolet and Sedgewick [10]. For the second equality we let \(m = i^{0.55} + r\) or \(m = -i^{0.55} - r\) for some \(r > 0\). Using the first equality,

$$\binom{2i - m}{i} = \binom{2i - i^{0.55}}{i} \prod_{k=0}^{r-1} \frac{i - i^{0.55} - k}{2i - i^{0.55} - k} \leq \frac{4^i}{2^{i^{0.55} + r}} e^{-i^{0.1}/4} \prod_{k=0}^{r-1} \frac{2i - 2i^{0.55} - 2k}{2i - i^{0.55} - k} \leq \frac{4^i}{2^m e^{-n^{0.05}}}.$$

A similar computation holds for \(m = -i^{0.55} - r\). \(\square\)
Consider a lattice path starting at \((v_0, h_0)\). Recall Definition 4.2. We may extend those definitions to general lattice paths with a slight modification. The definitions \(v_i\) and \(h_i\) remain the same, the position and the height after the \(i\)th up-step from the start of the path. For \(l_i\) we do not necessarily have an excursion. If the path never returns below \(h_i\) at some time later than \(v_i\) then we say that \(l_i = \infty\).

**Lemma 6.2** Suppose \((i, m) \in A_n\) and \(h = h_0 + m\). The number of lattice paths starting from \((v_0, h_0)\) ending with an \(i\)th up-step to \((v_i, h_0 + m)\) is given by

\[
\binom{2(i-1)-(m-1)}{i-1} = \frac{4^i}{2^{m+1} \sqrt{\pi i}} \exp \left(-\frac{m^2}{4i}\right) (1 + \Delta(i, m))
\]

where \(\Delta(i, m)\) is as defined in Lemma 6.1.

**Proof** Let \(i\) and \(d\) denote the number of up and down steps respectively in a lattice path starting at \(v_0\) ending with the \(i\)th up-step at position \(v_i\). We denote the total number of steps by \(v_i = v_0 + i + d\). The change in height for that path is given by \(h - h_0 = m = i - d\) so \(v_i = v_0 + 2i - m\). The second to last position of the path is \((v_0 + 2i - m - 1, h_0 + m - 1)\) since the last step is assumed to be an up-step. The total number of lattice paths from \((v_0, h_0)\) to \((v_0 + 2i - m - 1, h_0 + m - 1)\) is counted by Lemma 6.1, giving the equation found in Lemma 6.2. \(\square\)

Now that we can accurately count the number of lattice paths from one point to another we can count the number of non-negative paths between two points. For a pair of points \((v_0, h_0)\) and \((v_i, h_i)\) we let \(\mathcal{E}^{v_i, h_i}_{v_0, h_0}\) denote the set of non-negative lattice paths ending with an up-step between the two points.

**Lemma 6.3** For \((i, m) \in A_n\), let \(h_0 > n^{0.499}\), \(v_i = v_0 + 2i - m\), and \(h_i = h_0 + m\). Then

\[
|\mathcal{E}^{v_i, h_i}_{v_0, h_0}| = \frac{4^i}{2^{m+1} \sqrt{\pi i}} \exp \left(-\frac{m^2}{4i}\right) (1 + \Delta'(i, m))
\]

where \(\Delta'(i, m) = o(n^{-0.1})\) uniformly in \(i, m \in A_n\).

**Proof** We count using standard ballot counting arguments. It is enough to consider only \(h_0 < i - m\) in the following approximations.

\[
|\mathcal{E}^{v_0 + 2i - m, h_0 + m}_{v_0, h_0}| = \binom{2i - m - 1}{i - 1} - \binom{2i - m - 1}{i + h_0}.
\]

For \(h_0 > n^{0.499}\), if \(h_i > h_0\) then for \(m > 0\)

\[
\binom{2i - m - 1}{i + h_0} / \binom{2i - m - 1}{i - 1} \leq \prod_{r=0}^{h_0} \left(\frac{i - m - r}{i + r}\right) \leq \prod_{r=0}^{h_0} \left(1 - \frac{r}{4i}\right) \leq \exp\left(-\frac{(h_0)^2}{100i}\right).
\]
If \( h_i < h_0 \) then \( h_0 - |m| > h_0/2 \) giving

\[
\left( \frac{2i + |m| - 1}{i + h_0} \right) / \left( \frac{2i + |m| - 1}{i - 1} \right) \leq \prod_{r=0}^{\lfloor m/2 \rfloor} \left( \frac{i + |m| - r}{i + r} \right) \prod_{r=\lfloor m/2 \rfloor + 1}^{h_0} \left( \frac{i + |m| - r}{i + r} \right) \\
\leq \prod_{r=\lfloor m/2 \rfloor + 1}^{h_0} \left( 1 - \frac{2r - |m|}{i + r} \right) \\
\leq \prod_{s=0}^{h_0/2} \left( 1 - \frac{s}{4i} \right) \\
\leq \exp \left( -(h_0)^2/100i \right).
\]

Moreover \((h_0)^2/100i > n^{0.07}\), so by Lemma 6.2

\[
\left| \mathcal{E}_{v_0, h_0}^{v_i, h_i} \right| = \left( \frac{2i - m - 1}{i - 1} \right) \left( 1 + O \left( \exp \left( -n^{0.06} \right) \right) \right) \\
= \frac{4^i}{2^{m+1} \sqrt{\pi i}} \exp \left( -\frac{m^2}{4i} \right) \left( 1 + \Delta' (i, m) \right).
\]

\( \square \)

For paths chosen uniformly from \( \mathcal{E}_{v_0, h_0}^{v_j, h_j} \) for \( 0 < i < j \) we would like to know for various values of \( i \) and \( h \) how many of these path go through the point \((v_i, h)\) after the \( i \)th up-step. Given \( \Gamma^n \in \mathcal{E}_{v_0, h_0}^{v_j, h_j} \) chosen uniformly at random what is the probability that \( H_i^n = h \)?

**Lemma 6.4** Fix \( v_0, h_0, v_j \) and \( h_j \). Let \( X^n \) be chosen uniformly from \( \mathcal{E}_{v_0, h_0}^{v_j, h_j} \). For \( h > 0 \) and \( 0 < i < j \),

\[
\mathbb{P}(H_i^n = h) = \left| \mathcal{E}_{v_0, h_0}^{v_0 + 2i - (h - h_0), h} \right| \left| \mathcal{E}_{v_0, h_0}^{v_j, h_j} \right| \left| \mathcal{E}_{v_0, h_0}^{v_0 + 2i - (h - h_0), h} \right|^{-1}.
\]

**Proof** Any path in \( \mathcal{E}_{v_0, h_0}^{v_j, h_j} \) can be decomposed uniquely into a concatenation of two paths, one in \( \mathcal{E}_{v_0, h_0}^{v_i, h_i} \) and the other in \( \mathcal{E}_{v_0, h_0}^{v_j, h_j} \) for some appropriate values of \( v_i \) and \( h_i \) that satisfy \( v_i = v_0 + 2i - (h_1 - h_0) \). If \( h_i = h \), then \( v_i = v_0 + 2i - (h - h_0) \). The set \( \{ X^n \in \mathcal{E}_{v_0, h_0}^{v_j, h_j} \mid H_i = h \} \) is in bijection with \( \mathcal{E}_{v_0, h_0}^{v_0, h_0} \times \mathcal{E}_{v_i, h_i}^{v_j, h_j} \). Then

\[
\mathbb{P}(H_i^n = h) = \left| \left\{ \Gamma^n \in \mathcal{E}_{v_0, h_0}^{v_j, h_j} \mid H_i^n = h \right\} \right| \left| \mathcal{E}_{v_0, h_0}^{v_j, h_j} \right|^{-1} \\
= \left| \mathcal{E}_{v_0, h_0}^{v_0 + 2i - (h - h_0), h} \right| \left| \mathcal{E}_{v_0, h_0}^{v_j, h_j} \right| \left| \mathcal{E}_{v_0, h_0}^{v_0 + 2i - (h - h_0), h} \right|^{-1}
\]
as desired (see Fig. 5). \( \square \)
Fix $v_0, h_0, v_j$, with $h_j$ such that $v_j - v_0 = 2j - (h_j - h_0)$ as above. Also fix $i$, $h > 0$ and $i < j - h$. For $X^n$ chosen uniformly from $\mathcal{E}^{v_j,h_j}_{v_0,h_0}$,

$$\mathbb{P}(L_i^n/2 = h | H^n_i = h) = C_{h-1} \left| \mathcal{E}^{v_j,h_j}_{v_0+2i-(h-h_0)+2h-1,h-1} \right|^{-1} \left| \mathcal{E}^{v_j,h_j}_{v_0+2i-(h-h_0),h} \right|^{-1}. $$

**Proof** Let $X^n \in \mathcal{E}^{v_j,h_j}_{v_0,h_0}$ also satisfy $H^n_i = h$. From Lemma 6.4 there are precisely

$$\left| \mathcal{E}^{v_j,h_j}_{v_0+2i-(h-h_0),h} \right|$$

such paths. Each of these paths that satisfies $L_i^n/2 = h$ has a unique decomposition into three parts:

- A path $X_1^n \in \mathcal{E}^{v_j,h_j}_{v_0+2i-(h-h_0),h}$,
- an excursion $X_2^n$ from $(v_0 + 2i - (h - h_0), h)$ to $(v_0 + 2i - (h - h_0) + 2h - 2, h)$, staying above $h - 1$,
- a single step from $(v_0 + 2i - (h - h_0) + 2h - 2, h)$ to $(v_0 + 2i - (h - h_0) + 2h - 1, h - 1)$,
- and a path $X_3^n \in \mathcal{E}^{v_j,h_j}_{v_0+2i-(h-h_0)+2h-1,h-1}$.

The choice of $X_1^n$, $X_2^n$, and $X_3^n$ uniquely determines $X^n$. There are

$$\left| \mathcal{E}^{v_j,h_j}_{v_0+2i-(h-h_0),h} \right|, C_{h-1}, \text{ and } \left| \mathcal{E}^{v_j,h_j}_{v_0+2i-(h-h_0)+2h-1,h-1} \right|$$

such choices for $X_1^n$, $X_2^n$, and $X_3^n$ respectively. Therefore

$$\mathbb{P}(L_i^n/2 = h | H_i^n = h) = C_{h-1} \left| \mathcal{E}^{v_j,h_j}_{v_0+2i-(h-h_0)+2h-1,h-1} \right|^{-1} \left| \mathcal{E}^{v_j,h_j}_{v_0+2i-(h-h_0),h} \right|^{-1}. $$

$\square$
Lemma 6.6 For $X^n \in \mathcal{E}_{v_0,h_0}^{v_j,h_j}$ chosen uniformly at random and $0 < i < j$,
\[
\mathbb{P}(L^n_i / 2 = H^n_j) = \sum_{h_{0+i} \leq h = \max(0,h_{j-(j-i)})} C_h \frac{P_{v_0,h_0}^{v_j,h_j}}{P_{v_0+2i-(h-h_0),h}^{v_0+2i-(h-h_0)+2h-1,h-1} \cdot P_{v_0,h_0}^{v_j,h_j}}.
\]

Proof If $L^n_i / 2 = H^n_j$ then there is some $h \in \mathbb{N}$ such that $\{H^n_j = h\} \cap \{L^n_i / 2 = h\}$ occurs. Therefore
\[
\{L^n_i / 2 = H^n_j\} = \bigcup_h \{\{H^n_j = h\} \cap \{L^n_i / 2 = h\}\}.
\]

Luckily the event $\{H^n_j = h \cap L^n_i / 2 = h\}$ is disjoint from $\{H^n_j = h' \cap L^n_i / 2 = h'\}$ for $h \neq h'$. Then
\[
\mathbb{P}\left(\bigcup_h \{H^n_j = h\} \cap \{L^n_i / 2 = h\}\right) = \sum_h \mathbb{P}\left(\{H^n_j = h\} \cap \{L^n_i / 2 = h\}\right).
\]

If $h \notin (\max(0, H^n_j - (j-i), h_0 + i))$ then $\mathbb{P}(H^n_j = h) = 0$. Otherwise we have
\[
\mathbb{P}\left(\{H^n_j = h\} \cap \{L^n_i / 2 = h\}\right) = \mathbb{P}\left(L^n_i / 2 = h \mid H^n_j = h\right) \mathbb{P}(H^n_j = h).
\]
Combining Lemmas 6.4 and 6.5 provides the result. □

Lemma 6.7 Let $0 < i < j \leq 2n^{0.9}$ with $i > 2n^{0.6}$ and $j - i > 2n^{0.6}$, and let $h_0 \in (n^{0.499}, n^{0.501})$. Define $m$ and $m_j$ such that $h_j = h_0 + m_j$ and $h = h_0 + m$ where $|m_j| < \min(j^{0.55}, n^{0.451})$. Let $m_{\max} = \min(i^{0.55}, m_j + (j-i)^{0.55})$ and $m_{\min} = \max(-i^{0.55}, m_j - (j-i)^{0.55})$. For $m_{\min} < m < m_{\max}$
\[
\mathbb{P}(L^n_i / 2 = h \mid H^n_j = h) \mathbb{P}(H^n_j = h) = \frac{\sqrt{j}}{4\pi \sqrt{i(j-i)(h_0)^3}} \exp\left(-\frac{(jm - im_j)^2}{4ij(j-i)}\right) (1 + \Delta)
\]
where $\Delta = o(n^{-0.001})$ uniformly in $i, j, m, m_j, h_0$ that satisfy the above conditions.

For $m < m_{\min}$ or $m > m_{\max}$,
\[
\mathbb{P}(L^n_i / 2 = h \mid H^n_j = h) \mathbb{P}(H^n_j = h) < \exp\left(-n^{0.001}\right).
\]

Proof The summand in Lemma 6.6 is given by
\[
\mathbb{P}(L^n_i / 2 = h \mid H^n_j = h) \mathbb{P}(H^n_j = h) = C_h \frac{P_{v_0,h_0}^{v_j,h_j}}{P_{v_0+2i-m+2h-1,h-1}^{v_0+2i-m,h} \cdot P_{v_0,h_0}^{v_j,h_j}}.
\]
By Lemma 6.3 we can make the following substitutions:

\[ C_{h-1} = \frac{4^{h-1}}{\pi^{1/2} h^{3/2}} (1 + \Delta_1(h)), \]

\[ \mathcal{E}_{v_0+h_0}^{v_0+2i-m,h} = \binom{2(i-1) - (m-1)}{i-1} \]

\[ = \frac{4^i}{2^{m+1} \pi^{1/2} i^{1/2}} e^{-m^2/4i} (1 + \Delta_2(i-1, m-1)), \]

\[ \mathcal{E}_{v_0+h_0}^{v_j,h_j} = \binom{2(j-1) - (m_j-1)}{j-1} \]

\[ = \frac{4^j}{2^{m_j+1} \pi^{1/2} j^{1/2}} e^{-m_j^2/4j} (1 + \Delta_2(j-1, m_j-1)). \]

where both \( \Delta_1 \) and \( \Delta_2 \) are bounded uniformly by \( n^{-0.01} \) over all parameters satisfying the conditions of the lemma. Furthermore,

\[ \mathcal{E}_{v_0+2i-m+2h-1,h-1}^{v_j,h_j} = \binom{2(j-i-h) - (m_j-m)}{j-i-h} \]

\[ = \frac{4^{j-i-h}}{2^{m_j-m} \pi^{1/2} (j-i-h)^{1/2}} e^{-(m_j-m)^2/4(j-i-h)} (1 + \Delta_2(j-i-h \quad 0 \quad m_j-m)), \]

\[ = \frac{4^{j-i-h}}{2^{m_j-m} \pi^{1/2} (j-i)^{1/2}} e^{-(m_j-m)^2/4(j-i)} (1 + \Delta_3(j-i-h \quad 0 \quad m_j-m)) \]

where \( \Delta_3 \) is also uniformly bounded by \( n^{-0.01} \) over all parameters satisfying the conditions of the lemma. Combining these equations together proves the first statement of Lemma 6.7. For the second statement we use the second approximation in Lemma 6.1 to bound \( \mathbb{P}(H^n_i = h_0 + m) \) using the formula in Lemma 6.4. \( \square \)

Let’s consider the special case where \( j \approx n^{0.9} \).

**Lemma 6.8** For \( X^n \) chosen uniformly from \( \mathcal{E}_{v_0,h_0}^{v_j,h_j} \) with \( i, j, h_0, \) and \( h_j \) satisfying

- \( j = n^{0.9}(1 + \Delta') \), \( \Delta' \leq n^{-0.1} \) uniformly.
- \( n^{0.499} < h_0 < n^{0.501} \).
- \( i \in (2n^{0.6}, n^{0.9} - 2n^{0.6}), \)
- \( h_j = h_0 + m_j \) where \( |m_j| \leq n^{0.451} \),

then

\[ \mathbb{P}(L^n_i/2 = H^n_i) = \frac{1}{2\pi^{1/2} (h_0)^{3/2}} (1 + \Delta), \]

where \( \Delta = \Delta(i, h_0, h_j) = o(n^{-0.001}) \) uniformly in \( i, h_0, h_j \) in the ranges above.
Proof. Let $m_{\text{min}} = \max(-i^{0.55}, m_j - (j - i)^{0.55})$ and $m_{\text{max}} = \min(i^{0.55}, m_j + (j - i)^{0.55})$ and consider the inequality which follows from Lemma 6.6.

\[
\sum_{m = m_{\text{min}}}^{m_{\text{max}}} \mathbb{P}(L_i^n / 2 = h_0 + m | H_i^n = h_0 + m) \mathbb{P}(H_i^n = h_0 + m) \leq \mathbb{P}(L_i^n / 2 = H_i^n) \leq ne^{-n^{0.001}} + \sum_{m = m_{\text{min}}}^{m_{\text{max}}} \mathbb{P}(L_i^n / 2 = h_0 + m | H_i^n = h_0 + m) \mathbb{P}(H_i^n = h_0 + m).
\]  

(27)

Lemma 6.7 gives

\[
\mathbb{P}(L_i^n / 2 = H_i^n) = \left(1 + o\left(n^{-0.01}\right)\right) \sum_{m = m_{\text{min}}}^{m_{\text{max}}} \frac{\sqrt{j}}{4\pi \sqrt{i (j - i) (h_0)^3}} \exp \left(- \frac{(jm - im_j)^2}{4ij (j - i)}\right) \left(1 + o\left(n^{-0.01}\right)\right).
\]

\[
= \left(1 + o\left(n^{-0.01}\right)\right) \int_{m_{\text{min}}}^{m_{\text{max}}} \frac{1}{4\pi \sqrt{i (1 - i/j) (h_0)^3}} \exp \left(- \frac{(m - im_j/j)^2}{4i (1 - i/j)}\right) \left(1 + o\left(n^{-0.01}\right)\right) \, dm.
\]

By our definition $(m_{\text{min}} - \frac{i}{j}m_j) < -n^{0.01}$ and $(m_{\text{max}} - \frac{i}{j}m_j) > n^{0.01}$. Therefore the integral above is computed in the standard way, with

\[
\int_{-t}^{t} \exp \left(- \frac{(m - m_0)^2}{4c}\right) \, dm = 2c^{1/2} \pi^{1/2} + \delta(t).
\]

where $\delta(t)$ is an error function with exponential decay.

\[
\mathbb{P}(L_i^n / 2 = H_i^n) = \frac{1}{2\pi^{1/2} (h_0)^{3/2}} \left(1 + o\left(n^{-0.001}\right)\right).
\]

\[\square\]

Corollary 6.9 For any $k \in \mathbb{R}$ and $\alpha \in (0, 0.48)$ let $\Gamma^n$ chosen uniformly from $\mathcal{E}_{v_j, h_j}^{v_j, h_j}$ with $i, j, h_0$, and $h_j$ satisfying

- $j = n^{0.9}(1 + \Delta'), \Delta' < n^{-0.1}$ uniformly.
- $n^{0.499} < h_0 < n^{0.501}$.
- $i \in (2n^{0.6}, n^{0.9} - 2n^{0.6})$.
- $h_j = h_0 + m_j$ where $|m_j| \leq n^{0.451}$.
Lemma 6.12
For sufficiently large \( n \), for every modifications to the parameters. \( \square \)

This follows by a similar argument to Corollaries 6.19 and 6.20, with slight

Proof
The proof goes exactly as in Lemma 6.8 with \( L_i^n = H_i^n \) replaced by \( L_i^n = H_i^n - k(i(n-1)/n)^\alpha \). The order of \( k(i(n-1)/n)^\alpha \) is less than \( n^{0.49} \) so it will not affect the approximation. \( \square \)

Lemma 6.10
Fix \( 0 < a < b < 1 \) and let \( a_k = \lfloor an + nk/K \rfloor \) where \( K = \lfloor (b-a)n^{0.1} \rfloor \)
For \( \Gamma^n \in \text{Dyck}^{2n} \) chosen uniformly at random,

\[
\mathbb{P}\left( \bigcap_{k=0}^K \{n^{0.499} < \Gamma^n(V^n_{a_k}) < n^{0.501}\} \right) > 1 - o(1).
\]

Proof
This is an immediate consequence of Corollary 6.19 along with the convergence of Dyck paths to Brownian excursion. \( \square \)

Lemma 6.11
Fix \( 0 < a < b < 1 \). For any \( n \) large enough and \( \Gamma^n \in \text{Dyck}^{2n} \),

\[
\mathbb{P}\left( \bigcup_{k=0}^{K-1} \left\{ \left| \Gamma^n(V^n_{a_k}) - \Gamma^n(V^n_{a_{k+1}}) \right| > n^{0.451} \right\} \bigcap \left\{ n^{0.499} < \Gamma^n(V^n_{a_k}) < n^{0.501} \right\} \right) < e^{-n^{0.001}}.
\]

Proof
This follows by a similar argument to Corollaries 6.19 and 6.20, with slight modifications to the parameters. \( \square \)

Lemma 6.12
For sufficiently large \( n \), for every \( \frac{1}{2}n^{0.9} < j \leq 2n^{0.9} \), and \( h_0, h_j \) both bounded between \( n^{0.499} \) and \( n^{0.501} \) with \( |h_0 - h_j| < n^{0.451} \) we have that if \( X^n \in \mathcal{E}^{V_j,h_j}_{v_0,h_0} \) is chosen uniformly at random then

\[
\mathbb{P}\left( \sup_{t \in [0,1]} \left| X^n(v_0 + tv_j) - h_0 \right| > 2n^{0.452} \right) < e^{-n^{0.001}}.
\]

Proof
First note that the maximum fluctuation of \( |X^n(v_0 + tv_j) - h_0| \) is within 1 of the maximum fluctuation of \( |H_i^n - h_0| \) for \( i \leq j \). Suppose \( i > j/2 \), \( H_i^n = h \), and \( |h - h_0| > n^{0.452} \). Lemma 6.3 gives

\[
|\mathcal{E}^{V_j,h}_{v_0,h_0}| \leq \frac{4^i}{2^{h-h_0+1}} e^{-n^{0.904}/4j},
\]

for sufficiently large \( n \), independent of \( j, h_0 \), and \( h_j \) satisfying the hypotheses of the lemma. There are \( 2(j-i) - (h_j - h) \) steps remaining to go from \( (v_i, h) \) to \( (v_j, h_j) \). So
Again by Lemma 6.3,
\[
|E_{v_j, h_j}| \geq \frac{4^j}{2^{h_j-h}} \exp \left( -\frac{(h_j-h_0)^2}{4^j} \right),
\]
for sufficiently large \( n \), independent of \( j, h_0, \) and \( h_j \) satisfying the hypotheses of the lemma. Then we can conclude that
\[
P \left( H^n_i = h \right) \leq 4 \sqrt{\pi j} \exp \left( -\frac{n^{0.904}}{4^j} + \frac{(n^{0.451})^2}{4^j} \right) \leq 8 \sqrt{\pi n^{0.9}} e^{-n^{0.004}/8}.
\]
A similar bound can be used for \( i < j/2 \) with a little more work. Note that if \( |h-h_0| > n^{0.452} \) and \( |h_j-h_0| < n^{0.451} \) then \( |h-h_j| \geq \frac{1}{2} n^{0.452} \). The same argument now applies. Using the union bound and summing over the possible values of \( h \) and \( i \) now gives the result. \( \square \)

**Lemma 6.13** Let \( I \subset [an, bn] \) denote an interval of length at most \( n^\alpha \). For \( \gamma \in \text{Dyck}^{2n} \), if \( h_i > n^{0.49} \) for all \( i \in I \), then
\[
\theta_I \leq 1 + n^{\alpha-0.49}.
\]

**Proof** If the \( j \)th excursion is contained in \( i \)th excursion, then both \( h_j > h_i \) and \( l_j/2 < l_i/2 \). If \( \theta_i = 1 \) then for \( i < j < l_i/2, \theta_j = 0 \).

For an interval \( I \) with \( |I| < n^{0.49} \) suppose at least one fixed point exists. Let \( i^* \) denote the first excursion that satisfies \( \theta_i = 1 \). Since \( i^* \) corresponds to a fixed point,
\[
l_{i^*} = h_i > n^{0.49}.
\]
Therefore, for all \( j \in I \) such that \( j > i^* \) the \( j \)th excursion is contained in the \( i^* \)th excursion and \( \theta_j = 0 \). Therefore either \( \theta_I \) is either 0 or 1. For an interval of size greater than or equal to \( n^\alpha \), it can be covered by \( n^{\alpha-0.49} \) intervals of size \( n^{0.49} \) each of which has at most one fixed point so the total number of fixed points will be bounded by \( n^{\alpha-0.49} \). \( \square \)

**Corollary 6.14** Let \( I \subset [an, bn] \) denote an interval of length at most \( n^\alpha \). For \( \gamma \in \text{Dyck}^{2n} \), if \( h_i > n^{0.49} \) for all \( i \in I \), then
\[
\theta_I^{K, \alpha} \leq 2n^{\alpha-0.49}.
\]

**Proof** The function \( f(i) = (i(n-i)/n)^\alpha \) can change by at most 1 over any interval of length \( n^{0.49} \). This implies that over any interval \( I' \) of length \( n^{0.49} \) has \( \theta_{I'}^{K, \alpha} \leq 2 \). Then the result follows as in Lemma 6.13. \( \square \)
Lemma 6.15 There exists constant $C > 0$ such that for every $n$ large enough and $i, j, w$ that satisfy the following:

(a) $2n^{0.6} < i < j < n^{0.9} - 2n^{0.6}$,
(b) $|i - j| > 2n^{0.6}$,
(c) and $w < n^{0.451}$

$$\Psi(i, j, w, n) = \sum_{m'=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \exp \left( -\frac{1}{4} \left( \frac{m^2}{i} + \frac{(m - m')^2}{j - i} + \frac{(w - m')^2}{n^{0.9} - j} - \frac{w^2}{n^{0.9}} \right) \right)$$

$$< C \sqrt{\frac{i(j - i)(n^{0.9} - j)}{n^{0.9}}}.$$  

Proof Our goal will be to convert this double sum into a recognizable form.

$$\Psi(i, j, w, n) \leq \sum_{m'=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \exp \left( -\frac{1}{4} \left( \frac{m^2}{i} + \frac{(m - m')^2}{j - i} + \frac{(w - m')^2}{n^{0.9} - j} - \frac{w^2}{n^{0.9}} \right) \right)$$

$$\leq \sum_{m'=-\infty}^{\infty} \exp \left( -\frac{1}{4} \left( \frac{(w - m')^2}{n^{0.9} - j} - \frac{w^2}{n^{0.9}} \right) \right) G(i, j, m')$$

where

$$G(i, j, m') = \sum_{m=-\infty}^{\infty} \exp \left( -\frac{1}{4} \left( \frac{m^2}{i} + \frac{(m - m')^2}{j - i} \right) \right).$$

With some algebra we see

$$G(i, j, m') = \exp \left( -\frac{1}{4} \frac{m'^2}{j} \right) \sum_{m=-\infty}^{\infty} \exp \left( -\frac{1}{4} \frac{j(m - m'i/j)^2}{i(j - i)} \right)$$

$$\leq C_1 \exp \left( -\frac{1}{4} \frac{m'^2}{j} \right) \int_{-\infty}^{\infty} \exp \left( -\frac{1}{4} \frac{j}{i(j - i)}(t - m'i/j)^2 \right) dt$$

$$\leq C_1 \exp \left( -\frac{1}{4} \frac{m'^2}{j} \right) \sqrt{\frac{i(j - i)}{j}}$$

for some positive constant $C_1$ that does not depend on $i, j$ or $m'$. Inserting this into the upper bound for $\Psi(i, j, w, n)$ gives

$$\Psi(i, j, w, n) \leq C_1 \sqrt{\frac{i(j - i)}{j}} \sum_{m'=-\infty}^{\infty} \exp \left( -\frac{1}{4} \left( \frac{m^2}{j} + \frac{(m' - w)^2}{n^{0.9} - j} - \frac{w^2}{n^{0.9}} \right) \right)$$

$$\leq C_1 \sqrt{\frac{i(j - i)}{j}} \sum_{m'=-\infty}^{\infty} \exp \left( -\frac{1}{4} \left( \frac{n^{0.9}(m' - wj/n^{0.9})^2}{j(n^{0.9} - j)} \right) \right).$$
\[ \leq C \sqrt{\frac{i(j-i)(n^{0.9} - j)}{n^{0.9}}} \]

where \( C > 0 \) and does not depend on \( i, j, w, \) and \( n \).

The next two results are special cases of [24, Theorem III.12, Theorem III.15] respectively (see also [21, Lemma A1, Lemma A2]).

**Lemma 6.16** Let \( X_1, X_2, \ldots \) be i.i.d with \( \mathbb{E}X_1 = 0 \) and let \( S_n = X_1 + \cdots + X_n \). Suppose that \( \sigma^2 = \mathbb{E}(X_1^2) < \infty \). For all \( x \) and \( n \) we have

\[ \mathbb{P}\left( \max_{1 \leq k \leq n} S_k \geq x \right) \leq 2 \mathbb{P}\left( S_n \geq x - \sqrt{2n\sigma^2} \right). \]

**Lemma 6.17** Let \( X_1, X_2, \ldots \) be i.i.d with \( \mathbb{E}X_1 = 0 \) and let \( S_n = X_1 + \cdots + X_n \). Suppose that there exists a \( g > 0 \) such that \( \mathbb{E}(e^{t|X_1|}) < \infty \). Then there exist constants \( g, T > 0 \), independent of \( n \), such that

\[ \mathbb{P}(S_n \geq x) \leq \begin{cases} 
\exp\left(-\frac{x^2}{2gn}\right) & \text{if } 0 \leq x \leq ngT \\
\exp\left(-\frac{Tx}{2}\right) & \text{if } x \geq ngT.
\end{cases} \]

These lemmas lead immediately to the following corollary.

**Corollary 6.18** Maintaining the hypotheses of Lemma 6.17, fix \( \epsilon, c > 0 \) and \( 0 < \alpha < 2\beta \) and let \( \nu = \min(\beta, 2\beta - \alpha) \). There exist constants \( A, B > 0 \) such that

\[ \mathbb{P}\left( \max_{1 \leq i \leq n} \max_{1 \leq j \leq n} |S_j - S_i| \geq \epsilon n^\beta \right) \leq A \exp\left(-Bn^\nu\right). \]

Let \( S = (S_m, m \geq 0) \) be a simple symmetric random walk on \( \mathbb{Z} \) with \( S_0 = 0 \). Define \( V_0 = 0 \) and for \( m \geq 1 \) let \( V_m = \inf\{k > V_{m-1} : S_k - S_{k-1} = 1\} \). Let \( \eta(S) = \inf\{k : S_k = -1\} \). Observe that \( (V_m - V_{m-1})_{m \geq 1} \) is an i.i.d sequence of geometric random variables with parameter \( 1/2 \).

**Corollary 6.19** Fix \( \epsilon > 0 \) and \( 1/2 < \alpha \leq 1 \). There exist constants \( A, B > 0 \) such that

\[ \mathbb{P}\left( \max_{1 \leq i \leq n} |V_i - 2i| \geq \epsilon n^\alpha \right) \leq A \exp\left(-Bn^{2\alpha-1}\right). \]

**Corollary 6.20** Let \( \Gamma^n \) be a uniformly random Dyck path of length \( 2n \). For any \( \delta > 0 \), there exist constants \( A, B, \nu > 0 \) such that for all \( n \geq 1 \)

\[ \mathbb{P}\left( \max_{1 \leq i \leq 2n} \Gamma^n_i \geq 0.4n^{0.5+\delta} \right) \leq A \exp\left(-Bn^\nu\right). \]
and
\[
\mathbb{P}\left( \max_{1 \leq i \leq 2n} \max_{|i-j| \leq 2n^{0.25+\delta}} |\Gamma^n_j - \Gamma^n_i| \geq 0.5n^{0.25+\delta} \right) \leq A \exp\left(-Bn^{\nu}\right)
\]

Proof Noting that
\[
\Gamma^n \stackrel{d}{=} (S_k, 0 \leq k \leq 2n) \text{ given } \eta(S) = 2n + 1,
\]
the first claim is an immediate consequence of Lemmas 6.16 and 6.17 combined with the fact that \( \mathbb{P}(\eta(S) = 2n + 1) \sim cn^{-3/2} \) for some \( c > 0 \). The second claim follows similarly from Corollary 6.18.

\[\square\]

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