Convergence of the High-Accuracy Algorithm for Solving the Dirichlet Problem of the Modified Helmholtz Equation

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Received 19 September 2019; Accepted 26 December 2019; Published 20 January 2020

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In this paper, we derive the convergence for the high-accuracy algorithm in solving the Dirichlet problem of the modified Helmholtz equation. By the boundary element method, we transform the system to be a boundary integral equation. The high-accuracy algorithm using the specific quadrature rule is developed to deal with weakly singular integrals. The convergence of the algorithm is proved based on Anselone’s collective compact theory. Moreover, an asymptotic error expansion shows that the algorithm is of order $O(h^3_0)$. The numerical examples support the theoretical analysis.

1. Introduction

Modified Helmholtz equation arises in many important fields of science and engineering, for instance, wave propagation and scattering [1], structural vibration [2], implicit marching schemes for the heat equation [3], the Navier–Stokes equations [4], and so on. With the rapid development of computer power, many numerical methods [5–11] can be used to solve the modified Helmholtz equation. Among all methods for numerically solving the boundary value problem, the boundary integral equation approach is one of the most fundamental treatments. By the fundamental solutions, it can transform the boundary problem into an integral equation defined on the boundary. In the literature, several popular numerical methods for solving the boundary integral equations reformulated from the modified Helmholtz equation are used. For instance, Steinbach and Tchoualag [12] used the Galerkin method with one-periodic B-spline as basis functions and a spectral collocation method to solving the modified Helmholtz equation with the mixed boundary value problem; Kropinski and Quaife [13] used the fast multipole-accelerated integral equation methods to solving the modified Helmholtz equation with linear boundary conditions, and so on. Motivation of this work is to present an accuracy algorithm for solving the following modified Helmholtz problem:

$$\begin{cases} \Delta u(x) - \alpha^2 u(x) = 0, & \in \Omega, \\
 u(x) = g(x), & \in \Gamma, \end{cases}$$

where $\alpha$ is a real and positive constant and called the acoustic wave number. $\Omega \subset \mathbb{R}^2$ is a bounded domain with the boundary $\Gamma = \bigcup_{m=1}^{d} \Gamma_m$, $d > 1$ which is a closed curve. $g(x)$ is a given function.

Based on the boundary element method, the function $u(y)$ satisfies

$$u(y) = \int_\Gamma K^*(y,x)v(x)dx, \quad y \in \Omega,$$

where $x = (x_1, x_2), \ y = (y_1, y_2)$, and $v(x)$ is the density function, and

$$K^*(y,x) = -\frac{1}{2\pi}K_0(\alpha|x - y|),$$

is the fundamental solution of the modified Helmholtz equation. $K_0$ is the modified Bessel function [14]:

$$K_0(z) = -\ln z + \ln 2 - \gamma, \quad z \rightarrow 0,$$
where \( y \) is Euler’s constant.

Since \( u(y) \) is a continuous function in \( \Omega \cup \Gamma \), we have
\[
g(y) = \int_{\Gamma} K^*(y,x)v(x)ds_x, \quad y \in \Gamma. \tag{5}
\]

The aforementioned equation is weakly singular boundary integral equation of first kind, whose solution exists and is unique as long as \( C_\Gamma \neq 1 \) [15], where \( C_\Gamma \) is the logarithmic capacity. As soon as \( v(x) \) is solved from (5), the function \( u(y) (y \in \Omega) \) can be calculated by (2). Galerkin methods and collocation methods have been used to solve (5) based on the projective theory. However, there exist the following disadvantages: (a) the discrete matrix is full and each element has to calculate the weakly singular integral for collocation methods or the double weakly singular integral for Galerkin methods; (b) the order of accuracy is lower [16].

In this paper, we present a high-accuracy algorithm to solve (5). By specific quadrature rule, the high-accuracy algorithm is constructed to discrete the boundary integral equation, and a linear system is obtained. The calculation of the discrete matrix becomes very simple and straightforward without any singular integrals. We prove the convergence of the linear system by estimating eigenvalues of the discrete matrix and Anselone’s collective compact theory. Moreover, an asymptotic expansion of errors is obtained. Finally, numerical results verify the theoretical analysis.

This paper is organized as follows: In Section 2, the singularity of integral kernels and solutions are discussed; in Section 3, the high-accuracy algorithm is shown; in Section 4, the convergence analysis of the algorithm is given; in Section 5, an asymptotic expansion of the errors is obtained; and in Section 6, numerical examples are shown.

2. Singularity of Integral Kernels and Solutions

Let \( \Gamma = \bigcup_{m=1}^{d+1} \Gamma_m, (d > 1) \) be closed polygons with \( C_{\Gamma_{11}} \) and \( \Gamma_m (m = 1, \ldots, d) \) be a piecewise smooth curve. Define the boundary integral operators on \( \Gamma_m \):
\[
(K_{qm}v_m)(y) = \int_{\Gamma_m} K_{qm}(y,x)v_m(x)ds_x, \quad y \in \Gamma, (m, q = 1, \ldots, d), \tag{6}
\]
where \( K_{qm}(y,x) = K^*(y,x) \).

Then, (5) can be converted to a matrix operator equation:
\[
Kv = G, \tag{7}
\]
where \( K = [K_{qm}]_{q=1}^{d+1}, v = (v_1(x), v_2(x), \ldots, v_d(x))^T \), and \( G = (g_1(y), g_2(y), \ldots, g_d(y))^T \). Assume that \( \Gamma_m \) be described by the parameter mapping: \( x_m(s) = (x_m(s), x_m(s)): [0, 1] \rightarrow \Gamma_m, \) with \( |x_m'(s)| = [ (x_m'(s))^2 + (x_m'(s))^2 ]^{1/2} > 0, m = 1, \ldots, d \). Using the \( \sin^2 \) transformation [17],
\[
\psi_p(t) = \frac{\theta_p(t)}{\theta_p(1)}: [0, 1] \rightarrow [0, 1], \quad p \in N, \tag{8}
\]
where \( \theta_p(t) = \int_t^1 \sin(p\nu)\rho d\nu \). The operators in (6) are equivalent to the following integral operators on [0,1]:
\[
(K_{qm}\omega_m)(t) = \int_0^t K_{qm}(t,\tau)\omega_m(\tau)d\tau, \quad t \in [0,1], \tag{9}
\]
where \( K_{qm}(t,\tau) = -(1/2\pi)K_0(\alpha|x_q(t)-x_m(\tau)|), \) \( x_q(t) = (x_q(\psi_q(t)), x_m(\psi_q(t)) \), \( \omega_m(\tau) = \omega_m(\psi_q(\tau))|x_m'(\psi_q(\tau))| \psi_q'' \), and \( |x_q(t)-x_m(\tau)| = [ (x_q(t)-x_m(\tau))^2 + (x_q(t)-x_m(\tau))^2 ]^{1/2} \).

Hence, (7) is equivalent to
\[
K\omega = G, \tag{10}
\]
where \( \omega = (\omega_1, \omega_2, \ldots, \omega_d)^T \). By (4), it is clear that the operator \( K \) is a weakly singular integral operator, which is decomposed into the sum of the integral operators \( A \) and \( B \), where one carries the main singularity characteristic of \( A \) and the other is a compact operator with a smooth kernel. Define
\[
(A_{mm}\omega_m)(t) = \int_0^t a_{mm}(t,\tau)\omega_m(\tau)d\tau, \quad t \in [0,1], \tag{11}
\]
and
\[
(B_{qm}\omega_m)(t) = \int_0^t b_{qm}(t,\tau)\omega_m(\tau)d\tau, \quad t \in [0,1], \tag{12}
\]
where \( a_{mm}(t,\tau) = (1/2\pi)|\ln(2e^{-(1/2)}\sin(\pi(t-\tau)))| \) and
\[
b_{qm}(t,\tau) = \left\{ \begin{array}{ll} k_{qm}(t,\tau) - a_{mm}(t,\tau), & \text{for } q = m, \\ k_{qm}(t,\tau), & \text{for } q \neq m, \end{array} \right. \tag{13}
\]
where \( k_{mm}(t,\tau) \rightarrow (1/2\pi)|\ln([x'_m(\psi_p(\tau))\psi_p'']/2ne^{-(1/2)}| + \ln(\alpha/2) + \gamma) \) as \( t \rightarrow \tau \).

Then, (10) becomes
\[
(A + B)\omega = G, \tag{14}
\]
where \( A = \text{diag}(A_{11}, A_{22}, \ldots, A_{dd}), B = K - A = [B_{qm}]_{q=1}^{d+1}, \) and \( G = (g_1(t), g_2(t), \ldots, g_d(t))^T \). \( A_{mm}(m = 1, \ldots, d) \) is an isometry operator from \( H^r[0,1] \) to \( H^{r+1}[0,1] \) for any real number \( r \). \( A \) is also an isometry operator from \( (H^r[0,1])^d \) to \( (H^{r+1}[0,1])^d \). Hence, \( A \) is invertible, and (14) is equivalent to
\[
(E + A^{-1}B)\omega = A^{-1}G. \tag{15}
\]

Since \( \psi_p(t) \in C^\infty[0,1] \) increases monotonously on \([0,1] \) with \( \psi_p(0) = 0 \) and \( \psi_p(1) = 1 \), the solutions of (14) are equivalent to those of (7) [17].

Now, let us study the solution singularity for (7). Denote the corner points \( Q_1, \ldots, Q_d \) of the closed polygonal boundary \( \Gamma \) and \( Q_1 = Q_d \). Denote \( \chi_m \in [-1, 1] \) by \( \chi_1 = \chi_d = -1 \) for the endpoints \( Q_1 \) and \( Q_d \) or \( (1 - \chi_m)\pi = \alpha\chi_m\pi\chi_{m-1} \chi_{m+1} \) for the middle corner \( Q_m \). Based on the potential theory [15, 18], near the corner or endpoint \( Q_m \) the solution \( \psi_m \) has a singularity \( O(|s - \chi_m|^{\beta_m}/(|s - \chi_m| + 1/2)), \) where \( \beta_m = -\chi_m/(1 + |\chi_m|) \in (-1/2), \) and \( s = \chi_m \) at \( Q_m \) is the arc parameter. Since the singularity of \( \psi_m \) results from the singularities of the potential \( u(y) \), when \( u(y) \) is nonsingular in the exterior region, the singularity of \( \psi_m \) becomes \( O(1/(s - \chi_m)^{(1/2)(1 + |\chi_m|)}) \), and when it is nonsingular in the interior, the singularity of \( \psi_m \) becomes \( O(1/(s - \chi_m)^{(1/2)(1 + |\chi_m|)}) \). Although \( \psi_m(s) \) has a singularity, \( \omega_m(s) \) has no singularity.
by (8). The kernel \( b_{nm} \) of \( B_{nm} \) has a singularity at corners, but \( \sin^2(\pi s)b_{nm} \) has no singularity.

3. The High-Accuracy Algorithm

**Lemma 1 (see [19]).** Assume that \( g(x) \) and \( h(x) \in C^{2m}[a,b] \). \( G(x) = \ln|x - t|g(x) + h(x) \) is a periodic function with a period of \( b - a \) and at least 2 \( m \) differentiable in \( L = [0, \infty) \). There is the following quadrature formula:

\[
Q_n(G) = \frac{h}{2} \sum_{j=1}^{n} G(x_j) + h(t)h + \ln\left(\frac{h}{2\pi}\right)G(t)h, \quad (16)
\]

and the error

\[
E_n(G) = \int_{a}^{b} G(x)dx - Q_n(G) = 2 \sum_{j=1}^{m} \frac{\zeta(2j)}{(2j)!} G^{(2j)}(t)h^{2j+1} + O(h^{2m}),
\]

where \( h \) is the mesh width and \( \zeta(t) \) is the derivative of the Riemann zeta function.

Let \( h_m = 1/n_m \) be mesh widths and \( t_j = \tau_j = (j - 1/2)h_m(j = 1, \ldots, n_m) \) be nodes.

(1) Since \( k_{qm}(t, \tau) = b_{qm}(t, \tau) \) are the smooth functions on \( [0,1] \) for \( q \neq m \), by the trapezoidal or the midpoint rule [20], we construct the Nyström's approximate operator \( B_{qm}^h \) of the integral operator \( B_{qm} \) defined by

\[
(B_{qm}^h \omega_m)(t) = h_m \sum_{j=1}^{n_m} b_{qm}(t, t_j) \omega_m(t_j), \quad t \in [0,1], (q, m = 1, \ldots, d).
\]

(18)

We have the following error bounds [20]:

(a) For \( \Gamma_q \cap \Gamma_m = \emptyset \),

\[
(B_{qm}^h \omega_m)(t) - (B_{qm} \omega_m)(t) = O(h_m^{2l}), \quad l \in N, \quad (19)
\]

(b) For \( \Gamma_q \cap \Gamma_m \neq \emptyset \in \{Q_m\} \),

\[
(B_{qm}^h \omega_m)(t) - (B_{qm}^h \omega_m)(t) = O(h_m^l), \quad (20)
\]

where [17]

\[
\lambda = \begin{cases} \min((p+1)(\beta_m+1), p+1), & \text{odd}, \\
\min((p+1)(\beta_m+1), 2p+2), & \text{even}.
\end{cases} \quad (21)
\]

(2) Since \( k_{mn}(t, \tau) = a_{mn}(t, \tau) + b_{mn}(t, \tau) \) have the double logarithmic singularities on \([0,1]\), by Lemma 1, we get the following approximations \( A_{mn}^h \) of the integral operator \( A_{mn} \) defined by

\[
(A_{mn}^h \omega_m)(t) = \frac{h_m}{2\pi} \sum_{j=1}^{n_m} \ln|2e^{-1/2} \sin \pi(t - \tau_j)| \omega_m(\tau_j)
\]

\[
+ \frac{h_m}{2\pi} \ln\left|2\pi e^{-1/2} \frac{h_m}{2\pi}\right| \omega_m(t), \quad t \in [t_j],
\]

and the error bounds

\[
(A_{mn} \omega_m)(t) - (A_{mn}^h \omega_m)(t) = \frac{\ln|2e^{-1/2} \sin \pi(t - \tau_j)| \omega_m(\tau_j)}{2\pi} \frac{2\pi e^{-1/2}}{h_m} \omega_m(t) + O(h_m^{2l}).
\]

(22)

The approximations \( B_{mn}^h \) of the integral operator \( B_{mn} \) are defined by

\[
(B_{mn}^h \omega_m)(t) = h_m \sum_{j=1}^{n_m} b_{mn}(t_j, \tau_j) \omega_m(\tau_j)
\]

\[
+ \frac{h_m}{2\pi} \ln\left|2\pi e^{-1/2} \frac{h_m}{2\pi}\right| \omega_m(t), \quad t \in [t_j].
\]

(23)

Setting \( t = t_i \) \( (i = 1, \ldots, n_m) \), we obtain the discrete equations of (14):

\[
(A + B^h) \omega^h = C^h,
\]

(25)

where \( A^h = \text{diag}(A_{11}^h, \ldots, A_{dd}^h), \quad A_{mn} = [a_{nm}(t_j, \tau_j)]_{j=1}^{n_m} \),

\[
B^h = [B_{qm}^h]_{q,m=1}^{d}, \quad B_{qm}^h = [b_{qm}(t_j, \tau_j)]_{j=1}^{n_q}, \quad C^h = (g_1(t_1), \ldots, g_1(t_n), \ldots, g_d(t_1), \ldots, g_d(t_n))^T, \quad \omega^h = [\omega_{mq}]_{m=1}^{d}.
\]

\[
\omega_{nm} = (\omega_m(t_1), \ldots, \omega_m(t_n))^T,
\]

\[
a_{nm}(t_i, \tau_j) = \begin{cases} h_m \ln|2e^{-1/2} \sin \pi(t_i - \tau_j)|, & \text{as } i \neq j, \\
\frac{h_m \ln|2e^{-1/2} \sin \pi(t_i - \tau_j)|}{2\pi}, & \text{as } i = j.
\end{cases}
\]

(26)

Obviously, (25) is a system of linear equations with \( n(= \sum_{m=1}^{d} n_m) \) unknowns. Once \( \omega^h \) is solved by (25), the solution \( u(y) (y \in \Omega) \) can be computed by
\[ u^h(y) = -\frac{1}{2\pi} \sum_{m=1}^{d} h_m \sum_{i=1}^{n_m} K_0 \left[ d(y - x(t_i)) \right] \omega^h_m(t_i). \] (27)

4. Convergence of the Algorithm

From (11) and (26), we have

\[ A^h_{nm} = \text{circulate} \left( \frac{h_m}{2\pi} \ln \left( \frac{2e^{-1/2}h_m}{\pi} \right), \ldots, \frac{h_m}{2\pi} \ln 2e^{-1/2} \sin(\pi(n_m - 1)h_m) \right). \] (28)

Lemma 2 (see [21]). (1) There exists a positive \( c_1 > 0 \) so that the eigenvalues \( \lambda_\beta (\beta = 1, \ldots, n_m) \) of \( A^h_{nm} \) satisfy \( c_1 \lambda_\beta > 1/(2\pi n_m) \); (2) \( A_{nm} \) is invertible, and \( (A^h_{nm})^{-1} \) is uniformly bounded.

Based on Lemma 2, we immediately get the following corollary:

**Corollary 1.** \( A^h \) is invertible, and \( (A^h)^{-1} \) is uniformly bounded.

From Corollary 1, we know (25) is equivalent to

\[ (L^h + (A^h)^{-1}B^h)\omega^h = (A^h)^{-1}G^h, \] (29)

where \( L^h \) denotes the unit matrix.

Now we give the following definitions to discuss the approximate convergence in (29).

Define a subspace \( C_0[0,1] = \{ v(t) \in C[0,1] : v(t)/\sin^2(\pi t) \in C[0,1] \} \) of the space \( C[0,1] \) with a norm \( \|v\|_1 = \max_{t \in [0,1]} \|v(t)/\sin^2(\pi t)\|_1 \). Let \( S^h_n \subset C_0[0,1] \) be a piecewise linear function subspace with base points \( (t_j)_{j=1}^{n_m} \) and \( e_j(t) = \delta_{jt} \). Define a prolongation operator \( P^h_n : R^{h_n} \rightarrow S^{h_n} \) satisfying

\[ P^h_n \omega = \sum_{j=1}^{n_m} \omega_j e_j(t), \quad \forall \omega = (\omega_1, \ldots, \omega_{n_m}) \in R^{h_n}. \] (30)

Define a restricted operator \( R^h_n : C_0[0,1] \rightarrow R^{h_n} \) satisfying

\[ R^h_n v_n = (v_n(t_1), \ldots, v_n(t_{n_m})) \in R^{h_n}, \quad \forall v_n \in C_0[0,1]. \] (31)

Replacing \( (A^h)^{-1}B^h = P^h(A^h)^{-1}R^h B^h, \omega^h = P^h \omega_h, \) and \( G = P^h G^h, \) we construct an operator equation

\[ (L^h + (A^h)^{-1}B^h)\omega^h = (A^h)^{-1}G^h, \] (32)

where \( P^h = \text{diag}(P_1, \ldots, P_d), R^h = \text{diag}(R_1, \ldots, R_d) \). Obviously, if \( \omega^h \) is the solution of (32), then \( R^h \omega^h \) must be the solution of (29); conversely, if \( \omega^h \) is the solution of (29), then \( P^h \omega^h \) must be the solution of (32).

In order to prove the convergence of the algorithm, we give the following lemma.

**Lemma 3** (see [21]). The operator sequence \( \{P^h(A^h_{nm}^{-1}) R^h A_{nm} : C^2[0,1] \rightarrow C[0,1]\} \) is uniformly bounded and convergent to the embedding operator \( I \).

**Lemma 4** (see [22]). Let the integral operator \( M_{nm} \in L(C[0,1], C[0,1]) \) exist, and let the integral operator \( L_{nm} \in L(C[0,1], C[0,1]) \) with the kernel \( l_m(t,s) \) of \( L_{nm} \) satisfy \( M_{nm}^{-1}l_m(s,t) = l_m(t,s) \), where \( L(C[0,1], C[0,1]) \) is a linear bounded operator space, and \( l_m \) is the kernel of \( L_{nm} \). Assume that \( l_m(t,s) \) and \( (\partial/\partial t)l_m(t,s) \) are continuous on \([0,1]^2\). Let \( s_m = (i-1/2)h_m (i = 1, \ldots, n_m) \) be nodes. Then, the Nyström approximation of \( T_{nm} \),

\[ T_{nm}^h \omega_m = h_m \sum_{i=1}^{n_m} i_m(t,s_m) \omega_m(s_m), \quad \forall \omega_m \in [0,1], \] (33)

is collectively compact convergent to \( T_{nm} \),

\[ M_{nm}^{-1}l_m \rightarrow M_{nm}^{-1}l_m \in L(C^0[0,1], C^1[0,1]). \] (34)

Based on Lemma 4, we can immediately obtain the following corollary.

**Corollary 2.** Let \( \Gamma = \bigcup_{m=1}^{d} \Gamma_m \) with \( C_\Gamma \neq 1 \), and let \( |q - m| \neq 1 \) or \( d - 1 \). Then, under (8), the Nyström approximation \( B_{qm}^h \) of \( B_{qm} \) has

\[ P_{q} A_{qm}^{-1} R_{qm} B_{qm}^h \xrightarrow{cc} A_{qm}^{-1} B_{qm}, \quad \text{in } C[0,1] \rightarrow C[0,1], \] (35)

by the trapezoidal or the midpoint rule, where \( \xrightarrow{cc} \) denotes the collectively compact convergence.

**Lemma 5.** Let \( \Gamma = \bigcup_{m=1}^{d} \Gamma_m \) with \( C_\Gamma \neq 1 \), and let \( |q - m| = 1 \) or \( d - 1 \). Then, we have

\[ P_{q} A_{qm}^{-1} R_{qm} B_{qm}^h \xrightarrow{cc} A_{qm}^{-1} B_{qm}, \quad \text{in } L(C^0[0,1], C^1[0,1]), \] (36)

where the kernel of \( B_{qm} \) is \( \psi_{q} B_{qm} \).

**Proof:** The kernel of \( B_{qm} \) has logarithmic singularity at the corner \( Q_{m+1} = \Gamma_{q} \cap \Gamma_{m+1} \) with an inner angle \( \theta_{m+1} \) (replacing \( m+1 = 1 \) when \( m = d \)). Without loss of generality, suppose that \( Q_{m+1} = (0,0) \), then by (4), the kernel of \( B_{qm} \) can be expressed by

\[ b_{qm}(t,s) = \frac{1}{2\pi} \ln \left[ a_0(t) + a_1(s)^2 - 2a_0(t)a_1(s) \cos \theta_{m+1} \right]^2 \] (37)

where \( a_0(t) = |x_q(\psi_q(t))| \) and \( a_1(s) = |x_m(\psi_q(s))| \). We assume that \( a_0(0) = a_1(0) = 0 \), namely, \( B_{qm} \) has the logarithmic singularity at points \( (0,0) \) and are continuously differentiable in \([0,1]^2\). Consider
arbitrarily take a sequence \( \{Z_h\}_{h \in H} \) in space \( \Theta \), where \( Z_h = (Z_{h1}, Z_{h2}, \ldots, Z_{hd})^T \) with \( \|Z_h/(\sin^2(\pi t))\|_{\infty} \leq 1 \), \( m = 1, \ldots, d \). We firstly consider the first component of \( P^h (A^h)^{-1} R^h B^h Z_h \):

\[
\sum_{m=1}^{d} P_j (A_{11})^{-1} R_1 B_{1m}^h Z_{bm}. \tag{44}
\]

For \( \Gamma_m = \Gamma_1 \) or \( \Gamma_m \cap \Gamma_1 = \emptyset \), by Corollary 2, \( P_j (A_{11})^{-1} R_1 B_{1m}^h \xrightarrow{c.c} A_{11} B_{1m} \) in \( C[0,1] \), and there exists a convergent subsequence in \( \{P_j (A_{11})^{-1} R_1 B_{1m}^h Z_{bm}\}_m \). For \( \Gamma_m \cap \Gamma_1 \neq \emptyset \), then

\[
\begin{align*}
&\left\|P_j (A_{11})^{-1} R_1 B_{1m}^h Z_{bm}\right\|_{0,0} \\
\leq &\left\|P_j (A_{11})^{-1} \right\|_{1,0} \left\|A_{11}^{-1} B_{1m}^h \right\|_{2,0} \left\|Z_{bm}\right\|_{\infty}.
\end{align*}
\tag{45}
\]

By Lemma 5, there exists a convergent subsequence in \( \{P_j (A_{11})^{-1} R_1 B_{1m}^h Z_{bm}\}_m \). Based on the above two cases, it is proved that there exists an infinite subsequence \( H^{(1)} \subset H \) such that the first component converge. Similarly, it can be concluded that there exists an infinite subsequence \( H^{(d-1)} \subset \cdots \subset H^{(1)} \subset H \) such that \( P^h (A^{(d)})^{-1} R^h B^h Z_h \) converge. Therefore, \( P^h (A^{(d)})^{-1} R^h B^h \) is collectively compact convergent sequence, and \( P^h (A^{(d)})^{-1} R^h B^h \xrightarrow{P} A^h B \), where \( \xrightarrow{P} \) shows the point convergence. We complete the proof. \( \square \)

5. The Error of the Algorithm

In this section, we derive the asymptotic expansion of the errors. We first provide the main result.

**Theorem 2.** Let \( \Gamma = \cup_{m=1}^{d} \Gamma_m \) with \( C_r \neq 1 \) and \( \Gamma_m (m = 1, \ldots, d) \) is the smooth curve. The operator sequence \( \{P^h (A^h)^{-1} R^h B^h\}_h \) is collectively convergent to \( A^{-1} B \) in \( V = (C_0[0,1])^d \),

\[
P^h (A^h)^{-1} R^h B^h \xrightarrow{c.c} A^{-1} B. \tag{43}
\]

**Proof.** Let \( \Theta = \{v : \|v\| \leq 1, v \in V\} \) be a unit ball. \( H = \{h^1, h^2, \ldots \} \) is the grid step sequence, where \( h^n = \{h_1^{(n)} , h_2^{(n)}, \ldots, h_d^{(n)} \} \) with \( n \to \infty, \max_{1 \leq m \leq d} h_m^{(n)} \to 0 \). We
6. Numerical Examples

In this section, in order to verify the errors in the previous sections, we carry out some numerical examples for the modified Helmholtz equation with Dirichlet boundary value problems by the high-accuracy algorithm.

**Example 1.** We consider the modified Helmholtz equation with $\alpha^2 = 8$ and $\alpha^2 = 18$ on a plate domain $\Omega$ with the boundary $\Gamma = \bigcup_{m=1}^{r} \Gamma_m$ and $\Gamma_1 = \{(x_1, x_2) = (t, 0) : 0 \leq t \leq 1\}$ and $\Gamma_2 = \{(x_1, x_2) = (0.5 \cos \pi t + 0.5, 0.5 \sin \pi t) : 0 \leq t \leq 1\}$. We consider the Dirichlet boundary conditions corresponding to the exact solution $e^{-\sqrt{2} \alpha/2} e^{i(x_1x_2)}$.

We calculate the errors of $u(x)$ at the points $(0.5, 0.3)$ and $(0.5, 0.35)$ by $\psi_4(t)$. The errors $e_{1,2} = |u^h(x) - u(x)|$ are given in Table 1. Average errors of 100 points are given in Table 2. We let ratio denote $(e_{1,2} |_{n_1=2^n})/(e_{1,2} |_{n_1=2^{n+1}})$, $(i = 1, 2)$. From the numerical results, we can see that the accuracy order of $u^h(x)$ is $O(h_0^\alpha)$ for the high-accuracy algorithm.

**Example 2.** We consider the modified Helmholtz equation with $\alpha^2 = 8$ and $\alpha^2 = 18$ on a plate domain $\Omega$ with the boundary $\Gamma = \bigcup_{m=1}^{r} \Gamma_m$ and $\Gamma_1 = \{(x_1, x_2) = (t, 0) : 0 \leq t \leq 1\}$ and $\Gamma_2 = \{(x_1, x_2) = (0.5 \cos \pi t + 0.5, 0.5 \sin \pi t) : 0 \leq t \leq 1\}$. We consider the Dirichlet boundary conditions corresponding to the exact solution $e^{-\sqrt{2} \alpha/2} e^{i(x_1x_2)}$.

We calculate the errors of $u(x)$ at the points $(0.6, 0.6)$ and $(0.6, 0.5)$ by $\psi_4(t)$. The errors $e_{1,2} = |u^h(x) - u(x)|$ are given in Table 3. Average errors of 100 points are given in Table 4. We let ratio denote $(e_{1,2} |_{n_1=2^n})/(e_{1,2} |_{n_1=2^{n+1}})$, $(i = 1, 2)$. From the numerical results, we can see that...

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### Table 1: The simulation results for Example 1.

| $(n_1, n_2)$ | $\alpha^2 = 8$ | Ratio | $\alpha^2 = 18$ | Ratio |
|--------------|----------------|-------|----------------|-------|
| (8, 8)       | 1.3001e-3      | —     | 1.5000e-3      | —     |
| (16, 16)     | 1.5737e-4      | 8.2162| 1.7826e-4      | 8.5450|
| (32, 32)     | 1.9508e-5      | 8.0668| 2.2899e-5      | 7.7844|
| (64, 64)     | 2.4256e-6      | 8.0428| 2.8596e-6      | 8.0078|
| (128, 128)   | 2.9535e-7      | 8.2125| 3.5122e-7      | 8.1420|
| (256, 256)   | 3.2949e-8      | 8.9638| 4.0697e-8      | 8.6301|

### Table 2: The simulation results for Example 1.

| $(n_1, n_2)$ | $\alpha^2 = 8$ | Ratio | $\alpha^2 = 18$ | Ratio |
|--------------|----------------|-------|----------------|-------|
| (8, 8)       | 1.4512e-4      | —     | 1.4195e-4      | —     |
| (16, 16)     | 1.7060e-5      | 8.5065| 1.6596e-5      | 8.5534|
| (32, 32)     | 2.1358e-6      | 7.9874| 2.0716e-6      | 8.0113|
| (64, 64)     | 2.6607e-7      | 8.0271| 2.5935e-7      | 7.9876|
| (128, 128)   | 3.2564e-8      | 8.1709| 3.2011e-8      | 8.1019|
| (256, 256)   | 3.7158e-9      | 8.7636| 3.7756e-9      | 8.4786|

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The errors $\eta$ at the points $(0.5, 0.3)$ and $(0.5, 0.35)$ by $\psi_4(t)$. The errors $e_{1,2} = |u^h(x) - u(x)|$ are given in Table 1. Average errors of 100 points are given in Table 2. We let ratio denote $(e_{1,2} |_{n_1=2^n})/(e_{1,2} |_{n_1=2^{n+1}})$, $(i = 1, 2)$. From the numerical results, we can see that the log ratio = 3, which shows that the accuracy order of $u^h(x)$ is $O(h_0^\alpha)$ for the high-accuracy algorithm.
to solve the three-dimensional axisymmetric boundary in-

(3) this algorithm will be used of the present algorithm, and other existing numerical

Acknowledgments

he authors declare that they have no conflicts of interest.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This study was supported by the National Natural Science Foundation of China (11661005; 11301070) and the Program of Chengdu Normal University (YJRC2018-1; CS18ZDZ02).

7. Conclusions

In the paper, we introduce a high-accuracy algorithm using the specific quadrature rule to solve the modified Helmholtz equation with Dirichlet boundary value problems. A few concluding remarks can be made: (1) evaluation on entries of discrete matrices is very simple and straightforward without any singular integrals by the algorithm. Hence, the algorithm is appropriate to solve singularity problems; (2) the numerical results show that the algorithm retains the optimal convergence order \(O(h^6)\). These are remarkable advantages of the present algorithm, and other existing numerical methods, such as Galerkin methods and collocation methods, did not possess; and (3) this algorithm will be used to solve the three-dimensional axisymmetric boundary integral equations in the future.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

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Acknowledgments

This study was supported by the National Natural Science Foundation of China (11661005; 11301070) and the Program of Chengdu Normal University (YJRC2018-1; CS18ZDZ02).

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