Abstract

Las Vergnas’s active orders are a collection of partial orders on the bases of a matroid which are derived from the classical notion of matroid activity. In this paper, we construct a generalization of Las Vergnas’s external order which is defined on the independence complex of a matroid. We show that this poset is a refinement of the geometric lattice of flats of the matroid, and has the structure of a supersolvable join-distributive lattice. We uniquely characterize the lattices which are isomorphic to the external order of a matroid, and we explore a correspondence between matroid and antimatroid minors which arises from the poset construction.

1 Introduction

The classical notion of matroid activity plays an important role in understanding fundamental properties of a matroid, including the $h$-vector of its independence complex and the matroid Tutte polynomial. In 2001, Michel Las Vergnas introduced another structure derived from matroid activity, a collection of partial orders on the bases of a matroid which he called the active orders [15]. These orders elegantly connect matroid activity to a system of basis exchange operations, and are closely related to the broken circuit complex and the Orlik-Solomon algebra of a matroid.

In [1] and [2], the combinatorial structure of these active orders arises in relation to the initial ideal of certain projective varieties derived from affine linear spaces. In the theory of zonotopal algebra (see for instance [12], [3] and [16]), the active orders connect with a class of combinatorial objects called forward exchange matroids, where the bases associated with a forward exchange matroid satisfy axioms which are equivalent to their forming an order ideal in the external order.

The primary purpose of the present work is to define a generalization of Las Vergnas’s external order which extends the order to the independent sets of a matroid. If $M$ is an

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ordered matroid, we define for each independent set $I$ a set of externally passive elements, $\text{EP}_M(I)$, using a general definition given in [15]. The external order can be generalized to independent sets by the following.

**Definition.** If $I$ and $J$ are independent sets of the ordered matroid $M$, then we define the **generalized external order** $\leq_{\text{ext}}$ by

$$I \leq_{\text{ext}} J \iff \text{EP}_M(I) \supseteq \text{EP}_M(J)$$

By [15], Proposition 3.1, this is equivalent to Las Vergnas’s ordering in the case where $I$ and $J$ are two bases. For a variety of technical reasons, throughout this exposition we will instead work with the *reverse* of this ordering:

$$I \leq^*_{\text{ext}} J \iff \text{EP}_M(I) \subseteq \text{EP}_M(J)$$

Whenever we refer to the “external order” in this work, we will be referring to this reversed order unless otherwise noted. We use distinct notation for these two orders to reduce ambiguity, particularly because there are other contexts in which Las Vergnas’s original ordering convention fits more naturally with existing literature.

By associating each independent set with its corresponding set of externally passive elements, we define a set system $\mathcal{F}_{\text{ext}} := \{\text{EP}(I) : I \in \mathcal{I}(M)\}$, and show:

**Theorem 1.** If $M$ is an ordered matroid, then the set system $\mathcal{F}_{\text{ext}}$ of externally passive sets of $M$ is an antimatroid.

An **antimatroid** is a special class of greedoid which appears particularly in connection with convexity theory. Specifically, associated with any antimatroid is a **convex closure operator**, a closure operator on the ground set which combinatorially abstracts the operation of taking a convex hull, in the same way that a matroid closure operator abstracts the operation of taking a linear span. The convex closure operator on an ordered matroid derived from $\mathcal{F}_{\text{ext}}$ in particular bears a strong similarity to the convex closure operator for oriented matroids, which were first explored by Las Vergnas in [14].

In a 1985 survey paper [10], American mathematicians Paul Edelman and Robert Jamison noted:

> The authors have previously referred to these objects by the cacophonous name of ‘antimatroids’. We hope there is time to rectify this and that Gresham’s Law does not apply to mathematical nomenclature.

In the intervening 30 years, the name nevertheless appears to have become ensconced in the mathematical literature. However, in light of our Theorem 1 and other structural results of antimatroids, the name is perhaps not so poorly chosen, as the generalized external order provides an explicit connection between antimatroids and their combinatorial namesake.

The characterization of $\mathcal{F}_{\text{ext}}$ as an antimatroid further allows us to connect the external order with the large existing literature on lattice theory. The feasible sets of an antimatroid have a highly structured inclusion ordering called a **join-distributive lattice**, which is thus

\[^1\text{In fact, join-distributive lattices are essentially equivalent to antimatroids via a construction similar to that of Birkhoff’s representation theorem.}\]
inherited by the generalized external order. Moreover, the poset is in fact a refinement of the geometric lattice of flats associated with the matroid $M$, obtained by suitably combining copies of Las Vergnas’s original external order for the different flats of $M$.

Figure 1 compares Las Vergnas’s external order with the generalized order for the linear matroid represented by the columns of the matrix

$$X = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

where the numbers 1 through 4 indicate the column number, labeled from left to right.

Las Vergnas’s original construction required the inclusion of an additional zero element (the ‘∗’ in Figure 1) in order to form a proper lattice structure. In the generalized order, bases whose meet in the original order would have been the extra zero element instead are joined at an independent set of lower rank.

The fact that the external order comes from an antimatroid allows us to describe features of the lattice structure combinatorially. In addition, using results of Gordon and McMahon [11] for general greedoids, we are able to further derive the following explicit partition of the boolean lattice.

**Proposition 4.23.** If $M$ is an ordered matroid with ground set $E$, then the intervals

$$[I, I \cup EA(I)]$$

for $I$ independent

form a partition of the boolean lattice $2^E$.

This partition bears a resemblance to the well-known partition of Crapo, described in [6], and in fact it can be shown that this partition is a proper refinement of Crapo’s.

Another main purpose of this exposition is to discuss the way in which the external orders fit into the context of antimatroids and join-distributive lattices. To refine our understanding, we characterize a proper subclass of the join-distributive lattices which we call matroidal join-distributive lattices.
**Definition.** Given a lattice $L$ and an element $x \in L$, let $r_c(x)$ denote the number of elements in $L$ which cover $x$. A join-distributive lattice $L$ is called **matroidal** if $r_c$ is decreasing in $L$, and it satisfies the semimodular inequality

$$r_c(x \land y) + r_c(x \lor y) \leq r_c(x) + r_c(y)$$

For an element $x$ of a join-distributive lattice $L$, one can associate a set $I(x)$ called the **independent set** corresponding with $x$. If $L$ is the external order lattice of an ordered matroid $M$, then the $I$ operator recovers the matroid independent sets of $M$. Even for an arbitrary join-distributive lattice, the collection of independent sets is closed under taking subsets, and thus forms a simplicial complex. This join-distributive independence complex in fact provides an alternate characterization of matroidal join-distributive lattices.

**Theorem 2.** A join-distributive lattice $L$ is matroidal if and only if its independent sets are those of a matroid.

In particular, this shows that the external order of an ordered matroid is a matroidal join-distributive lattice. This result goes a long way towards understanding where the external order sits among all join-distributive lattices, but surprisingly, there are matroidal join-distributive lattices which are not an external order. If we denote the class of join-distributive lattices by JD, the class of matroidal join-distributive lattices by MJD, and the class of lattices derived from the external order by EO, then

$$\text{EO} \subsetneq \text{MJD} \subsetneq \text{JD}$$

Figure 3 in Section 5.2 gives an example of a lattice in MJD but not EO, and Figure 2 in Section 2.3 gives an example of a lattice in JD but not MJD.

A further refinement is necessary to precisely classify the lattices isomorphic to an external order, and that refinement comes from the notion of **edge lexicographic** or **EL-shellability**. A graded poset $P$ is EL-shellable if its Hasse diagram admits a labeling of its edges by integers which satisfies certain lexicographic comparability conditions on unrefinable chains. EL-shellability of a graded poset implies shellability of its order complex, and the notion has been widely studied for different classes of posets.

The external order is EL-shellable, and in fact it satisfies a stronger property called $S_n$ **EL-shellability**. We study how $S_n$ EL-shellability relates to antimatroids, and we show that

**Theorem 3.** A finite lattice $L$ is isomorphic to the external order $\leq^*_\text{ext}$ of an ordered matroid if and only if it is join-distributive, matroidal, and $S_n$ EL-shellable.

McNamara introduced $S_n$ EL-shellability in [17] as a way to characterize the **supersolvable** lattices of Stanley [19], and in particular, he proved that the two properties are equivalent. This implies that one may replace “$S_n$ EL-shellable” with “supersolvable” in the above classification of the external order.

The remainder of the document is structured as follows. Section 2 gives a brief overview of necessary background material in the areas of matroid theory, antimatroid theory, and the theory of join-distributive lattices. Section 3 develops additional technical results relating
feasible and independent sets of join-distributive lattices. Section 4 constructs the generalized external order and explores its structure and connections with greedoid theory. Section 5 then characterizes matroidal join-distributive lattices and relates them to $S_n$ EL-shellability, and Section 6 relates the deletion and contraction operations of matroids and antimatroids.

2 Background

We will be studying the relations between several objects in the areas of lattice theory and discrete geometry, for which significant theory has been developed. We provide a brief review of relevant background here, and refer the reader to standard sources for additional details.

For general matroid notions, Oxley [18] is comprehensive, and for concepts related to matroid activity, Björner [6] gives a concise overview. For the topics of greedoids and antimatroids, our primary references are Björner and Ziegler’s survey [7], as well as the book [13] by Korte, Lovász and Schrader. General lattice theory is developed in detail in Stanley [20], Chapter 3, and the literature on join-distributive lattices is discussed in some detail in the introduction of Czédli [8].

2.1 Matroids

To begin, we define matroids, a combinatorial object which generalizes both the concept of linear independence of vectors in a vector space, and the concept of cycle-freeness of edge sets in a graph. The basic object of interest is the set system.

Definition 2.1. If $E$ is a finite set, a set system is a pair $(E, F)$ where $F$ is a nonempty collection of subsets of $E$. We will sometimes refer to $F$ as a set system when we don’t need to emphasize the ground set.

A common notation in the study of finite set systems is to use a string of lower-case characters or numbers to refer to a small finite set. For instance, if $a, b \in E$ are elements of a ground set, then the string $ab$ denotes the set $\{a, b\}$. If $A \subseteq E$, then $A \cup ab$ denotes the set $A \cup \{a, b\}$. In practice this notation enhances rather than confounds communication, so we will adopt it in the present work when the meaning is clear from the context.

We can now define matroids in terms of their collections of “independent sets” as follows.

Definition 2.2. A set system $M = (E, I)$ is called a matroid if

- If $I \in I$ and $J \subseteq I$, then $J \in I$; and
- For $I, J \in I$, if $|I| > |J|$, then there is an element $x \in I$ such that $J \cup x \in I$.

A set in $I$ is called an independent set of the matroid $M$.

The first property above is called the hereditary property for a set system, and the second is called the matroid independence exchange axiom.

The independence axioms for matroids are one of many different equivalent definitions of matroids frequently called “cryptomorphisms”. Among the classical cryptomorphisms are axiom systems for bases, circuits, rank functions, closure operators, and the greedy algorithm.
A fluent understanding of the definitions of these concepts and the relations between them will be helpful in the remainder of this work, and is explored in detail in [18] Chapter 1.

A pair of constructions which will be used frequently are the basic circuit and basic bond.

**Definition 2.3.** Let $M = (E, \mathcal{I})$ be a matroid, and let $B$ be a basis of $M$. For $x \notin B$, define $c_i(M, B, x)$ the basic circuit of $x$ in $B$ to be the unique circuit contained in $B \cup x$.

Dually, for $b \in B$ define $b_0(M, B, b)$ the basic cocircuit or basic bond of $b$ in $B$ to be the unique cocircuit contained in $(E \setminus B) \cup b$.

A classical characterization of the basic circuit and basic bond is given by the following lemma.

**Lemma 2.4.** Let $M$ be a matroid with a basis $B$, and let $b \in B$ and $x \notin B$. Then the following are equivalent:

- $b \in c_i(B, x)$
- $x \in b_0(B, b)$
- $B \setminus b \cup x$ is a basis of $M$

For notational convenience, we extend the definition of basic circuits and basic cocircuits in the following way.

**Definition 2.5.** Let $M$ be a matroid, let $I \in \mathcal{I}(M)$, and denote $F = \text{span}(I)$. For $x \in F \setminus I$, define

$$c_i(I, x) = c_i(M|_F)(I, x)$$

and for $y \in I$, define

$$b_0(I, y) = b_0(M|_F)(I, y)$$

For elements outside of $F$, neither of these expressions are defined.

A concept of fundamental importance in the remainder of this work is the notion of matroid activity.

**Definition 2.6.** An ordered matroid is a matroid $M = (E, \mathcal{I})$ along with a total order $\leq$ on the ground set $E$. We will frequently refer to $M$ as an ordered matroid without specifying the order when no ambiguity arises.

**Definition 2.7.** Let $M = (E, \mathcal{I})$ be an ordered matroid, and let $B$ be a basis of $M$. For $x \in E \setminus B$, we call $x$ externally active with respect to $B$ if $x$ is the minimum element of the basic circuit $c_i(B, x)$, and externally passive otherwise. For $b \in B$, we call $b$ internally active with respect to $B$ if $b$ is the minimum element of the basic cocircuit $b_0(B, b)$, and internally passive otherwise.

We denote the sets of externally active and externally passive elements with respect to a basis $B$ by $\text{EA}_M(B)$ and $\text{EP}_M(B)$, and the sets of internally active and internally passive elements by $\text{IA}_M(B)$ and $\text{IP}_M(B)$.
Note in particular that the internal and external activities are dual notions. If $M^*$ is the dual matroid of $M$, then $EA_M(B) = IA_{M^*}(E \setminus B)$, and similarly for the other sets.

Historically, the most important property of the notions of matroid activity is that they generate an important algebraic invariant of matroids called the Tutte polynomial.

**Proposition 2.8.** Given an ordered matroid $M$, the Tutte polynomial of $M$ is given by

$$T_M(x, y) = \sum_{B \in \mathcal{B}(M)} x^{\lvert IA(B) \rvert} y^{\lvert EA(B) \rvert}$$

and is independent of the ordering of $M$.

The Tutte polynomial is what is called the universal Tutte-Grothendieck invariant for the class of all matroids, and in particular it encodes a breadth of combinatorial data corresponding to a matroid.

### 2.2 Antimatroids

Greedoids are a generalization of matroids which capture the structure necessary for the matroid greedy algorithm to apply. The generalization gives rise to a rich hierarchy of subclasses, including matroids, which are outlined in exquisite detail in [7], Figure 8.5.

**Definition 2.9.** A set system $G = (E, F)$ is called a greedoid if

- For every non-empty $X \in F$, there is an $x \in X$ such that $X \setminus x \in F$; and
- For $X, Y \in F$, if $\lvert X \rvert > \lvert Y \rvert$, then there is an element $x \in X$ such that $Y \cup x \in F$.

A set in $F$ is called a feasible set of the greedoid $E$.

The first property above is a weakening of the matroid hereditary property called accessibility, and the second property above is exactly the matroid independence exchange axiom, which we sometimes will call the greedoid exchange axiom for clarity.

For our discussion, the most important subclass of greedoids aside from matroids is the antimatroids, defined by:

**Definition 2.10.** A set system $(E, F)$ is called an antimatroid if

- $F$ is a greedoid; and
- if $X \subseteq Y$ are sets in $F$ and $a \in E \setminus Y$ with $X \cup a \in F$, then $Y \cup a \in F$.

The second property in this definition is called the interval property without upper bounds. Antimatroids as set system of feasible sets can be formulated in a variety of equivalent manners, and we will state for reference several of these which will also be useful.

**Proposition 2.11** ([7], Proposition 8.2.7). If $F$ is a set system, then the following conditions are equivalent.

- $F$ is an antimatroid;
• $\mathcal{F}$ is accessible, and closed under taking unions; and

• $\emptyset \in \mathcal{F}$, and $\mathcal{F}$ satisfies the exchange axiom that if $X, Y$ are sets in $\mathcal{F}$ such that $X \nsubseteq Y$, then there is an element $x \in X \setminus Y$ such that $Y \cup x \in \mathcal{F}$.

Before moving on to some of the essential characteristics of these objects, we offer a remark concerning the name “antimatroid”.

### 2.2.1 Independent Sets, Circuits and Cocircuits

As with matroids, the theory of antimatroids admits a number of cryptomorphic definitions, which include a theory of rooted circuits and a dual theory of rooted cocircuits. For more details, see [7] Section 8.7.C as well as [13] Section 3.3.

**Definition 2.12.** If $(E, \mathcal{F})$ is a set system and $A \subseteq E$, define the trace $\mathcal{F}: A$ by

$$\mathcal{F}: A \coloneqq \{X \cap A : X \in \mathcal{F}\}$$

If $\mathcal{F}$ is a greedoid, then $A \subseteq E$ is called **free** or **independent** if $\mathcal{F}: A = 2^A$. If $A$ is not independent, it is called **dependent**.

**Definition 2.13.** If $(E, \mathcal{F})$ is a set system and $A \in \mathcal{F}$, then the **feasible extensions** of $A$ are the elements of

$$\Gamma(A) \coloneqq \{x \in E \setminus A : A \cup x \in \mathcal{F}\}$$

The following lemma relates freeness to feasible extensions, and follows directly from Lemma 3.1 of [13].

**Lemma 2.14.** If $(E, \mathcal{F})$ is an antimatroid, then $X \subseteq E$ is independent if and only if it is equal to the feasible extensions $\Gamma(A)$ of some feasible set $A \in \mathcal{F}$.

Of particular note is that the collection of independent sets of an antimatroid is closed under taking subsets, and thus forms a simplicial complex as a set system. We will discuss more properties of independent sets and their relationship with feasible sets of an antimatroid in Section 3.

The cryptomorphisms of rooted circuits and rooted cocircuits are presented in terms of rooted sets:

**Definition 2.15.** If $A$ is a set and $a \in A$, then the pair $(A, a)$ is called a **rooted set** with **root** $a$. In this case, we may equivalently refer to $A$ as a rooted set if the root is clear from context.

Now we can define the circuits of an antimatroid.

**Definition 2.16.** A **circuit** of an antimatroid $(E, \mathcal{F})$ is a minimal dependent subset of $E$.

In particular, the following holds for circuits of an antimatroid.

**Proposition 2.17** ([7]). If $(E, \mathcal{F})$ is an antimatroid and $C \subseteq E$, then there is a unique element $a \in C$ such that $\mathcal{F}: C = 2^C \setminus \{\{a\}\}$. We call the rooted set $(C, a)$ a **rooted circuit** of $\mathcal{F}$. 8
Let \( \mathcal{C}(\mathcal{F}) \) denote the collection of rooted circuits of an antimatroid \( \mathcal{F} \). Rooted circuits give a cryptomorphism for antimatroids due to the following fundamental result.

**Proposition 2.18** ([7], Proposition 8.7.11). Let \( (E, \mathcal{F}) \) be an antimatroid and \( A \subseteq E \). Then \( A \) is feasible if and only if \( C \cap A \neq \{a\} \) for every rooted circuit \( (C, a) \).

That is, an antimatroid is fully determined by its collection of rooted circuits. Further, we can axiomatize the rooted families which give rise to an antimatroid.

**Proposition 2.19** ([7], Theorem 8.7.12). Let \( \mathcal{C} \) be a family of rooted subsets of a finite set \( E \). Then \( \mathcal{C} \) is the family of rooted circuits of an antimatroid if and only if the following two axioms are satisfied:

*(CI1)* If \( (C_1, a) \in \mathcal{C} \), then there is no rooted set \( (C_2, a) \in \mathcal{C} \) with \( C_2 \subsetneq C_1 \).

*(CI2)* If \( (C_1, a_1), (C_2, a_2) \in \mathcal{C} \) and \( a_1 \in C_2 \setminus a_2 \), then there is a rooted set \( (C_3, a_2) \in \mathcal{C} \) with \( C_3 \subseteq C_1 \cup C_2 \setminus a_1 \).

Bj"orners and Ziegler noted that these axioms bear a curious resemblance to the circuit axioms for matroids, and we will see in Section 4 that this resemblance is not superficial.

A second cryptomorphism for antimatroids is their rooted cocircuits, which form a certain type of dual to their rooted circuits.

**Definition 2.20.** If \( (E, \mathcal{F}) \) is an antimatroid and \( F \in \mathcal{F} \), then an element \( a \in F \) is called an **endpoint** of \( F \) if \( F \setminus a \in \mathcal{F} \). If \( F \in \mathcal{F} \) has a single endpoint \( a \), then we call \( F \) a **cocircuit**, and we call the rooted set \( (F, a) \) a **rooted cocircuit** of \( \mathcal{F} \). Equivalently, \( (F, a) \) is a rooted cocircuit iff \( F \in \mathcal{F} \) is minimal containing \( a \). We denote by \( \mathcal{C}^*(\mathcal{F}) \) the collection of rooted cocircuits of an antimatroid \( \mathcal{F} \).

In many places in the literature, antimatroid cocircuits are also called **paths**, but we use the name cocircuit to emphasize their duality with antimatroid circuits. The descriptive power of these rooted sets is exemplified by the following lemma.

**Lemma 2.21** ([13], Lemma 3.12). If \( (E, \mathcal{F}) \) is an antimatroid and \( A \subseteq E \), then \( A \) is feasible if and only if it is a union of cocircuits. If \( A \) has \( k \) endpoints \( \{a_1, \ldots, a_k\} \), then \( A \) is a union of \( k \) cocircuits \( \{A_1, \ldots, A_k\} \), where the root of each \( A_i \) is \( a_i \).

In particular, this shows that the cocircuits of an antimatroid also uniquely determine the feasible sets. As with circuits, there is also an axiomatic characterization of the set systems which form the collection of rooted cocircuits of an antimatroid.

**Proposition 2.22.** Let \( \mathcal{C}^* \subseteq \{(D, a) : D \subseteq E, a \in D\} \) be a family of rooted subsets of a finite set \( E \). Then \( \mathcal{C}^* \) is the family of rooted cocircuits of an antimatroid \( (E, \mathcal{F}) \) if an only if the following two axioms are satisfied:

*(CC1)* If \( (D_1, a) \in \mathcal{C}^* \), then there is no rooted set \( (D_2, a) \in \mathcal{C}^* \) with \( D_2 \subsetneq D_1 \).

*(CC2)* If \( (D_1, a_1) \in \mathcal{C}^* \) and \( a_2 \in D_1 \setminus a_1 \), then there is a rooted set \( (D_2, a_2) \in \mathcal{C}^* \) with \( D_2 \subseteq D_1 \setminus a_1 \).
Since rooted circuits and rooted cocircuits suffice to specify an antimatroid, when convenient we will sometimes denote an antimatroid using these rooted set systems, as a pair $(E, C)$ or $(E, C^*)$.

Finally, we describe the duality which relates the circuits and cocircuits of an antimatroid.

**Definition 2.23.** If $E$ is a finite set and $\mathcal{U}$ is a collection of subsets of $E$, then $\mathcal{U}$ is called a clutter if no set in $\mathcal{U}$ is contained in another. If $\mathcal{U}$ is a clutter, then the blocker of $\mathcal{U}$, denoted $B(\mathcal{U})$ is the collection of minimal subsets

$$B(\mathcal{U}) := \min \{ V \subseteq E : V \cap U \text{ is nonempty for each } U \in \mathcal{U} \}$$

A basic result of blockers is that the operation of taking blockers is an involution on clutters.

**Lemma 2.24.** For any clutter $\mathcal{U}$, the blocker $\mathcal{V} = B(\mathcal{U})$ is a clutter, and $B(B(\mathcal{V})) = \mathcal{U}$.

In particular, this involution provides the essential connection between antimatroid circuits and cocircuits.

**Definition 2.25.** If $\mathcal{A}$ is a collection of rooted subsets of a ground set $E$ and $x \in E$, let $\mathcal{A}_x$ denote the collection of sets $\{ A \setminus x : (A, x) \in \mathcal{A} \}$.

**Proposition 2.26.** Let $(E, \mathcal{F})$ be an antimatroid with circuits and cocircuits $\mathcal{C}$ and $\mathcal{C}^*$ respectively. Then for each $x \in E$, we have that $\mathcal{C}_x$ and $\mathcal{C}_x^*$ are clutters, and $\mathcal{C}_x^*$ is the blocker of $\mathcal{C}_x$ and vice versa.

### 2.2.2 Minors

Finally, we will recall two notions of minors which may be defined respectively for greedoids and for antimatroids. First, we give the standard definitions of deletion and contraction for general greedoids.

**Definition 2.27.** If $G = (E, \mathcal{F})$ is a greedoid and $A \subseteq E$, then the greedoid deletion $G \setminus A$ is the set system $(E \setminus A, \mathcal{F} \setminus A)$, where

$$\mathcal{F} \setminus A = \{ F \subseteq E \setminus A : F \in \mathcal{F} \}$$

The greedoid contraction $G / A$ is the set system $(E \setminus A, F / A)$ where

$$F / A = \{ F \subseteq E \setminus A : F \cup A \in \mathcal{F} \}$$

A greedoid deletion $G \setminus A$ is always a greedoid, while in general a greedoid contraction $G / A$ is a greedoid only when $A$ is feasible, as otherwise $\emptyset$ is not included in the resulting set system.

A greedoid minor is a deletion of a contraction of a greedoid. Aside from the limitation that the contracting set is feasible, greedoid minors behave like matroid minors in that the deletion and contraction operations commute with themselves and each other.

We provide these definitions for arbitrary greedoids primarily for background and context. For antimatroids in particular, there is an alternate formulation of minors based on rooted circuits which will be central to the discussion in Section 6.
Definition 2.28. If \( \mathcal{A} = (E, \mathcal{C}) \) is an antimatroid with rooted circuits \( \mathcal{C} \) and \( A \subseteq E \), then the antimatroid deletion \( \mathcal{A} \setminus A \) is the pair \( (E \setminus A, \mathcal{C} \setminus A) \) where

\[
\mathcal{C} \setminus A = \{ (C, x) : (C, x) \in \mathcal{C}, C \cap S = \emptyset \}
\]

The antimatroid contraction \( \mathcal{A} / A \) is the pair \( (E \setminus A, \mathcal{C} / A) \) where

\[
\mathcal{C} / A = \min \{ (C \setminus S, x) : (C, x) \in \mathcal{C}, x \notin S \}
\]

and where \( \min \mathcal{R} \) for a collection \( \mathcal{R} \) of rooted sets denotes the subcollection of those which are (non-strictly) minimal under inclusion as non-rooted sets.

In particular, these deletion and contraction operations produce antimatroids, and also behave like matroid minors.

Proposition 2.29 ([9], Propositions 12 and 14). If \( \mathcal{A} = (E, \mathcal{F}) \) is an antimatroid and \( A \subseteq E \), then \( \mathcal{A} / A \) and \( \mathcal{A} \setminus A \) are antimatroids. If \( A, B \subseteq E \) are disjoint, then

- \( (\mathcal{A} \setminus A) \setminus B = (\mathcal{A} \setminus B) \setminus A \)
- \( (\mathcal{A} \setminus A) / B = (\mathcal{A} / B) \setminus A \)
- \( (\mathcal{A} / A) / B = (\mathcal{A} / B) / A \)

An antimatroid minor may then be defined as a deletion of a contraction of an antimatroid. Although not immediately obvious from the circuit definition, these operations may also be characterized in the following way in terms of antimatroid feasible sets.

Proposition 2.30. If \( (E, \mathcal{F}) \) is an antimatroid and \( A \subseteq E \), then

- \( \mathcal{F} \setminus A \) is given by the trace \( \mathcal{F} : (E \setminus A) \)
- \( \mathcal{F} / A \) is given by the greedoid deletion \( \mathcal{F} / A = \{ F \in \mathcal{F} : F \cap A = \emptyset \} \)

Antimatroid deletion by a set \( A \) can in general be thought of as collapsing the edges of the antimatroid Hasse diagram whose labels for the natural edge labeling (see Definition 3.2) are elements of \( A \).

2.3 Join-distributive Lattices

Finally, we review background on the class of posets called join-distributive lattices, which fundamentally connect antimatroids with lattice theory. Beyond standard notions of lattice theory, we require the following definitions, which follow the exposition of [8].

Definition 2.31. A lattice \( L \) is called semimodular or upper semimodular if for all \( x, y \in L \), if \( x \gg x \wedge y \), then \( x \vee y \gg y \).

Definition 2.32. A lattice \( L \) is called meet semidistributive if it satisfies the meet semidistributive law, that for all \( x, y \in L \) and for any \( z \in L \), if \( x \wedge z = y \wedge z \), then the common value of these meets is \( (x \vee y) \wedge z \).
Definition 2.33. Given a lattice $L$, an element $x \in L$ is called meet-irreducible if it is covered by exactly one element of $L$, and is called join-irreducible if it covers exactly one element of $L$. We denote the set of meet-irreducibles of $L$ by $\text{MI}(L)$, and the set of join-irreducibles of $L$ by $\text{JI}(L)$.

Definition 2.34. Given a lattice $L$ and an element $x \in L$, an irredundant meet decomposition of $x$ is a representation $x = \bigwedge Y$ with $Y \subseteq \text{MI}(L)$ such that $x \neq \bigwedge Y'$ for any proper subset $Y'$ of $Y$. The lattice $L$ is said to have unique meet-irreducible decompositions if each $x \in L$ has a unique irredundant meet-decomposition.

Definition 2.35. If $x \in L$ is a member of a locally finite lattice, let $j(x)$ denote the join of all elements covering $x$.

Using this terminology, we can define join-distributive lattices and give several equivalent formulations which will be variously useful for our discussion.

Definition 2.36. A finite lattice is called join distributive if it is semimodular and meet-semidistributive.

Proposition 2.37 ([8], Proposition 2.1). For a finite lattice $L$, the following are equivalent.

1. $L$ is join-distributive
2. $L$ has unique meet-irreducible decompositions
3. For each $x \in L$, the interval $[x, j(x)]$ is a boolean lattice
4. The length of each maximal chain in $L$ is equal to $|\text{MI}(L)|$.

The most important property of join-distributive lattices for our purposes is a remarkable correspondence with antimatroids, very similar to the correspondence of Birkhoff’s representation theorem for finite distributive lattices.

Definition 2.38 ([7]). Given a finite join-distributive lattice $L$, let $\mathcal{F}(L)$ denote the set system which is the image of the map $T : L \to 2^{\text{MI}(L)}$ given by

$$T : x \mapsto \{ y \in \text{MI}(L) : y \nleq x \}$$

Proposition 2.39 ([7], Theorem 8.7.6). $T$ is a poset isomorphism from $L$ to $\mathcal{F}(L)$ ordered by inclusion, and joins in $L$ correspond to unions in $\mathcal{F}(L)$. $\mathcal{F}(L)$ is an antimatroid with ground set $\text{MI}(L)$, and the poset $\mathcal{F}$ of feasible sets of any antimatroid, ordered by inclusion, forms a join-distributive lattice.

Figure 2 demonstrates the application of this map to produce an antimatroid from a join-distributive lattice.

The primary consequence of this correspondence is that join-distributive lattices are essentially equivalent to antimatroids: $T$ gives a one-to-one correspondence between join-distributive lattices and antimatroids $\mathcal{F}$ which have no loops, or equivalently, for which the ground set $E$ is covered by the feasible sets of $\mathcal{F}$.
Figure 2: The $T$ map applied to a join-distributive lattice with labeled meet irreducibles

Explicitly, if $\mathcal{F}$ is an antimatroid with ground set $E = \bigcup_{F \in \mathcal{F}} F$, let $L(\mathcal{F})$ denote the join-distributive lattice formed by the feasible sets of $\mathcal{F}$ under set inclusion. Then the elements of $E$ are in bijection with the meet irreducibles of $L(\mathcal{F})$ by the map $x \mapsto S_x$, where $S_x \in \mathcal{F}$ is the unique meet irreducible in $L(\mathcal{F})$ covered by $(S_x \cup x) \in \mathcal{F}$. This bijection of ground sets induces a canonical isomorphism between $\mathcal{F}$ and $T(L(\mathcal{F}))$.

In general, we will allow for antimatroids with loops. This introduces a slight ambiguity in the equivalence between antimatroids and join-distributive lattices, as an antimatroid with loops has the same feasible sets and associated join-distributive lattice as a corresponding antimatroid with loops removed. This should not cause confusion in practice, however, so we will often refer to general antimatroids and join-distributive lattices interchangeably, keeping this subtlety in mind.

### 3 Feasible and Independent Sets of Join-distributive Lattices

Before moving to the main new results of this work, we will develop some additional theory in the realm of antimatroids and join-distributive lattices which will be useful later. Our aim is to explore the robust connections between the independent sets and the feasible sets of an antimatroid, so we will work in the equivalent context of join-distributive lattices, which provide a more symmetric way to represent these set systems.

To begin, we give some notation to describe covering relations and independent sets in join-distributive lattices.

**Definition 3.1.** For a poset $P$, let $\text{Cov}(P) \subseteq P \times P$ denote the covering pairs $(x, y)$, with $x \prec y$.

**Definition 3.2.** Let $L$ be a join-distributive lattice. Recall from Definition 2.38 the map $T : L \to 2^{\text{MI}(L)}$ which maps $L$ to its associated antimatroid, and let $e : \text{Cov}(L) \to \text{MI}(L)$ denote the natural edge labeling, given by $e : (x, y) \mapsto T(y) \setminus T(x)$. Such set differences are singletons, hence the map is well-defined into $\text{MI}(L)$.

**Definition 3.3.** If $x \in L$ is an element of a join-distributive lattice, let $I(x)$ denote the set of elements

$$I(x) = \{ e(x, y) : y \in L, (x, y) \in \text{Cov}(L) \}$$
and let \( J(x) \) denote the set of elements

\[
J(x) = \{ e(w, x) : w \in L, (w, x) \in \text{Cov}(L) \}
\]

\( I(x) \) is the **independent set** associated to \( x \), and is equal to the independent set of feasible extensions of \( T(x) \) in the antimatroid corresponding to \( L \). We adopt the following additional notation.

**Definition 3.4.** If \( L \) is a join-distributive lattice,

- Let \( \mathcal{F}(L) = (\text{MI}(L), \{ T(x) : x \in L \}) \) denote the (loopless) antimatroid associated with \( L \)
- Let \( \mathcal{I}(L) = \{ I(x) : x \in L \} \) denote collection of independent sets of \( L \)
- Let \( \mathcal{C}(L) \) denote the collection of rooted circuits of \( \mathcal{F}(L) \), which we interchangeably refer to as the rooted circuits of \( L \)

Notice that \( I(x) \) is disjoint from \( T(x) \), and \( J(x) \) is a subset of \( T(x) \). The meet-irreducible elements \( x \in \text{MI}(L) \) are characterized by the condition \(|I(x)| = 1\), in which case \( I(x) = \{x\} \). The join-irreducible elements \( y \in \text{JI}(L) \) are characterized by the condition \(|J(x)| = 1\), and in particular correspond with the rooted cocircuits of \( \mathcal{F}(L) \).

Of particular importance is the following:

**Lemma 3.5.** For \( x, y \in L \) elements of a join-distributive lattice, \( T(x) \) has empty intersection with \( I(y) \) if and only if \( x \leq y \).

**Proof.** If \( x \leq y \), then \( T(x) \subseteq T(y) \). If \( a \in I(y) \cap T(x) \), then \( a \) is a member of both \( I(y) \) and \( T(y) \), contradicting disjointness.

Otherwise, \( x \lor y > y \). In particular, there is a covering element \( y_a \) for some \( a \in I(y) \) such that \( T(y_a) = T(y) \cup a \), and \( y_a \leq x \lor y \). Thus \( a \in T(y_a) \subseteq T(x \lor y) = T(x) \cup T(y) \), so because \( a \notin T(y) \) we conclude that \( a \in T(x) \cap I(y) \). \( \square \)

**Corollary 3.6.** The map \( I : L \to \mathcal{I}(L) \) is one-to-one.

**Proof.** If \( x, y \in L \) satisfy \( I(x) = I(y) \), then \( T(x) \cap I(y) = T(y) \cap I(x) = \emptyset \), so \( x \leq y \) and \( y \leq x \).

In particular, an element of a join-distributive lattice is uniquely identified with its corresponding independent set. In fact, this property characterizes the antimatroids among all greedoids.

**Proposition 3.7.** A greedoid \((E, \mathcal{F})\) is an antimatroid if and only if the feasible extension operator \( \Gamma : A \mapsto \{ x \in E \setminus A : A \cup x \in \mathcal{F} \} \) is one-to-one.

**Proof.** The forward direction is just restating Corollary 3.6 in the context of antimatroids. So suppose that \( \mathcal{F} \) is a greedoid and the map \( \Gamma \) is one-to-one.

To see that \( \mathcal{F} \) is an antimatroid, we prove that it satisfies the interval property without upper bounds. As a base case, suppose \( A, B \in \mathcal{F} \) with \( B = A \cup x \) for some \( x \notin A \). If \( A \cup y \in \mathcal{F} \) for some \( y \notin B \), we want to show that \( B \cup y \in \mathcal{F} \) as well.
Suppose this is not the case, so that $B \cup y = A \cup xy \notin \mathcal{F}$. Then we will show that $A \cup x$ and $A \cup y$ are mapped to the same set under $\Gamma$. To this end, suppose that $z \in \Gamma(A \cup x)$ for some $z$, so that $A \cup xz \in \mathcal{F}$.

Then in particular, $|A \cup y| < |A \cup xz|$, so by the greedoid exchange axiom we know there is an element $w \in (A \cup xz) \setminus (A \cup y) = xz$ such that $A \cup yw \in \mathcal{F}$. However, by assumption we know that $A \cup xy \notin \mathcal{F}$, so we must have $w = z$. Then $A \cup yz \in \mathcal{F}$, so $z \in \Gamma(A \cup y)$.

This implies that $\Gamma(A \cup x) \subseteq \Gamma(A \cup y)$. A symmetric argument proves the reverse inclusion, so we see that $\Gamma$ maps the two sets to the same independent set, a contradiction. We conclude that in this context, $B \cup y = A \cup xy \in \mathcal{F}$.

In general if $A, B \in \mathcal{F}$ with $A \subseteq B$, then by repeatedly applying the greedoid exchange axiom, there is a sequence of covering sets $A_i \in \mathcal{F}$ with

$$A = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_k = B$$

where $A_{i+1} = A_i \cup x_i$ for some $x_i \in E$. The interval property without upper bounds follows by inducting on the length of this chain using the previous base case.

As mentioned previously, the independent sets of an antimatroid are closed under taking subsets, and so form a simplicial complex. In terms of the lattice structure of $L$, we get a stronger fact, that the inclusion order on the complex embeds in $L$ in the following way. If $A \in I(L)$, let $x_A$ denote the corresponding lattice element $I^{-1}(A)$.

**Lemma 3.8.** If $J$ is an independent set of a join-distributive lattice $L$, and $I \subseteq J$, then $I$ is independent, and $x_I \geq x_J$.

**Proof.** If $I \subsetneq J$, there is a lattice element $x_{J'} > x_I$ such that $I \subseteq J'$. This follows because if $a \in J \setminus I$, then by definition of independent sets, there is a covering element $x_{J'} > x_I$ such that $T(x_{J'}) \setminus T(x_I) = \{a\}$. In particular, because $L$ is join-distributive, the interval $[x_I, j(x_I)]$ is boolean, and so $j(x_{J'}) \geq j(x_I)$. Noting that for any $x \in L$ the relation $I(x) = T(j(x)) \setminus T(x)$ holds, we have

$$J' = T(j(x_{J'})) \setminus T(x_{J'}) \supseteq T(j(x_I)) \setminus (T(x_I) \cup a) = J \setminus a \supseteq I$$

Since $L$ is of finite length, repeated applications of the above must terminate, producing a saturated chain whose greatest element is $x_K$ for an independent set $K$ satisfying $K \supseteq I$ but not $K \supset I$. Hence $I = K$ is independent, and $x_I \geq x_J$. \qed

We now state and prove some additional lemmas concerning independent sets of join-distributive lattices which will be useful in later sections.

**Lemma 3.9.** If $x \leq y$ in a join-distributive lattice $L$, then $I(x) \subseteq I(y) \cup T(y)$.

**Proof.** Suppose that $a \in I(x)$, and $a \notin T(y)$. Then there is an element $x_a > x$ such that $T(x_a) = T(x) \cup a$, and by the antimatroid interval property without upper bounds, there must be an element $y_a \in L$ such that $T(y_a) = T(y) \cup a$, and so we have $y_a > y$. We conclude that $a \in I(y)$. \qed

**Lemma 3.10.** If $I, J$ are independent sets of a join-distributive lattice $L$, then if $x_I \wedge x_J = x_K$ for $K$ independent, then $K \subseteq I \cup J$.  

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Proof. Let $a \in K$, and suppose that $a \notin I \cup J$. Since $x_K \leq x_I, x_J$, we know that $a \in I \cup T(x_I)$ and $a \in J \cup T(x_J)$. Thus since $a$ is in neither $I$ nor $J$, we can conclude that $a \in T(x_I) \cap T(x_J)$.

However, since $a \in K$, there exists $K'$ independent such that $T(x_K') = T(x_K) \cup a$. Since $T(x_K) \subseteq T(I) \cap T(J)$ and $a \in T(I) \cap T(J)$, we have that $T(x_K') \subseteq T(I) \cap T(J)$. We see now that $x_K < x_K' \leq x_I, x_J$, and this contradicts the claim that $x_K$ is the meet of $x_I$ and $x_J$. \qed

If $A \subseteq M(L)$, let $x_A$ denote the meet of all elements $x_I$ for $I \subseteq A$ independent. The element $x_A \in L$ is equal to $x_K$ for some independent set $K$, and by induction on Lemma 3.10, we have that $K \subseteq A$. Let $I(A)$ denote this independent set, and note that if $A$ is itself independent, then $I(A) = A$ by Lemma 3.8.

Lemma 3.11. If $A, B \subseteq M(L)$, then $x_A \lor x_B \leq x_{A \cap B}$, and $x_A \land x_B \leq x_{A \cup B}$.

Proof. For the first inequality, let $I$ be independent with $I \subseteq A \cap B$, and note that $I \subseteq A$ and $I \subseteq B$, so $x_A \leq x_I$ and $x_B \leq x_I$. In particular, $x_A \lor x_B \leq x_I$, so since this holds for arbitrary $I \subseteq A \cap B$, it is also true for the meet of all such elements, hence $x_A \lor x_B \leq x_{A \cap B}$.

For the second inequality, let $I$ be independent with $I \subseteq A \cup B$, and let $I_1 = I \cap A$, and $I_2 = I \cap B$. By Lemma 3.8 both $I_1$ and $I_2$ are independent, and they satisfy $x_{I_1}, x_{I_2} \geq x_I$. Thus $x_{I_1} \land x_{I_2} \geq x_I$, and in fact we will see that $x_{I_1} \land x_{I_2} = x_I$.

If $K$ is independent with $x_K = x_{I_1} \land x_{I_2}$, then by Lemma 3.10, we have that $K \subseteq I_1 \cup I_2 = I$. For $a \in I$, suppose without loss of generality that $a \in I_1$. In particular, $a \notin T(x_K)$, and this implies $a \notin T(x_K)$ because $x_{I_1} \geq x_K$. But by Lemma 3.9, since $x_K \geq x_1$, we have that $I \subseteq K \cup T(x_K)$, and so we conclude that $a \in K$. Since $a \in I$ was arbitrary, we thus have $I \subseteq K$, so the two sets are equal.

Finally, note that since $x_{I_1} \land x_{I_2} = x_I$ and $I_1 \subseteq A$, $I_2 \subseteq B$, we have that $x_A \land x_B \leq x_I$. Since $I$ was chosen arbitrarily in $A \cup B$, we conclude $x_A \land x_B \leq x_{A \cup B}$. \qed

4 Extending Las Vergnas’s External Order

In [15], Michel Las Vergnas defined partial orderings on the bases of an ordered matroid which are derived from the notion of matroid activity. His external order, defined in terms of matroid external activity, is the starting point for the remainder of this work.

Definition 4.1 ([15]). Let $M$ be an ordered matroid. Then Las Vergnas’s external order on the set of bases of $M$ is defined by:

$$B_1 \leq_{\text{ext}} B_2 \text{ iff } \text{EP}(B_1) \supseteq \text{EP}(B_2)$$

The poset obtained by this definition depends on the ordering associated with $M$, but has some suggestive properties, summarized in the following.

Proposition 4.2. Let $M = (E, I)$ be an ordered matroid, and let $P = (B(M), \leq_{\text{ext}})$ be the external order on the bases of $M$. Let $L$ denote the poset $P$ with an additional minimal element $0$ added to the ground set. Then

- $P$ is a graded poset, graded by $|\text{EP}(B)|$
• Two bases $B_1$ and $B_2$ satisfy a covering relation $B_1 \preceq B_2$ in $P$ iff $B_2 = B_1 \setminus b \cup a$, where $b \in B_1$, and $a$ is the maximal element of $bo(B_1, b)$ externally active with respect to $B_1$. In this case, $EP(B_2) = EP(B_1) \cup b$

• $L$ is a lattice with combinatorially defined meet and join operators

A dual order, the **internal order**, can be derived from the external order on the dual ordered matroid $M^*$, and has analogous properties.

4.1 The Generalized External Order

In the same paper, Las Vergnas defined a generalized notion of matroid activity which will be the key to generalizing the external order.

**Definition 4.3.** Let $M = (E, \mathcal{I})$ be an ordered matroid, and let $A \subseteq E$. Then we say that $x \in E$ is **M-active** with respect to $A$ if there is a circuit $C$ of $M$ with $x \in C \subseteq A \cup x$ such that $x$ is the smallest element of $C$. We denote the set of such $M$-active elements by $Act_M(A)$, and define

1. $EA_M(A) := Act_M(A) \setminus A$
2. $EP_M(A) := (E \setminus A) \setminus EA_M(A)$
3. $IA_M(A) := Act_{M^*}(E \setminus A) \cap A$
4. $IP_M(A) := A \setminus IA_M(A)$

In particular, the above definition reduces to the classical definition of matroid activity when $A$ is chosen to be a basis of $M$.

One of the primary properties of external activity that allows the construction of the external lattice on bases is the fact that the map

$$B \mapsto EP(B)$$

is one-to-one. This characteristic fails spectacularly for the generalized definition of external activity. However, when we restrict our attention to independent sets, the situation is better.

**Lemma 4.4.** Let $M = (E, \mathcal{I})$ be an ordered matroid, and let $I \in \mathcal{I}$. Then if $F$ is the flat spanned by $I$, we have

$$Act_M(I) = Act_{M|_{F}}(I)$$

**Proof.** Suppose that $x \in Act_{M|_{F}}(I)$. Then $x \in F$, and there is a circuit $C$ of $M|_{F}$ such that $x \in C \subseteq I \cup x$ and $x$ is the smallest element of $C$. However, the circuits of $M|_{F}$ are just the circuits of $M$ which are contained in $F$, so in particular we have that $C$ is also a circuit of $M$, which shows that $x \in Act_M(I)$.

Now suppose that $x \in Act_M(I)$. Then there is a circuit $C$ of $M$ such that $x \in C \subseteq I \cup x$ and $x$ is the smallest element of $C$. In particular, we have that $C \setminus x$ is an independent subset of $F$.  

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If $x \notin F$, then we would have $x \notin \text{span}(C \setminus x)$, which would imply that $C = (C \setminus x) \cup x$ is independent, a contradiction. Thus it must be the case that $x \in F$. This means that $C \subseteq I \cup x \subseteq F$, so this implies that $C$ is also a circuit of $M|_F$. Since $C$ still satisfies the conditions required by the definition of activity in $M|_F$, we conclude that $x \in \text{Act}_{M|_F}(I)$. \qed

In particular, we have the following.

**Corollary 4.5.** If $M = (E, I)$ is an ordered matroid and $I \in I$ with $F = \text{span}(I)$, then

$$\text{EP}_M(I) = \text{EP}_{M|_F}(I) \cup (E \setminus F)$$

and in particular

$$F = \text{span}(E \setminus \text{EP}_M(I))$$

*Proof.* The first equality follows directly from the above lemma, noting that $\text{Act}_M(I) = \text{Act}_{M|_F}(I) \subseteq F$. The second equality follows because

$$I \subseteq E \setminus \text{EP}_M(I) \subseteq F$$

\qed

**Corollary 4.6.** If $M$ is an ordered matroid, then the map $\text{EP}_M : I \to 2^E$ is one-to-one.

*Proof.* From previous theory we know that $\text{EP}$ is one-to-one when restricted to the bases of a matroid. Now let $I, J$ be distinct independent sets of $M$, with $F_I = \text{span}(I)$ and $F_J = \text{span}(J)$. If $F_I \neq F_J$, then by the above lemma,

$$\text{span}(E \setminus \text{EP}_M(I)) = F_I \neq F_J = \text{span}(E \setminus \text{EP}_M(J))$$

Thus in this case the two passive sets cannot be equal.

If $F_I = F_J$, call this common spanning flat $F$. Then $I$ and $J$ are distinct bases of the restriction matroid $M|_F$. This gives that $\text{EP}_{M|_F}(I) \neq \text{EP}_{M|_F}(J)$, so

$$\text{EP}_M(I) = \text{EP}_{M|_F}(I) \cup (E \setminus F) \neq \text{EP}_{M|_F}(J) \cup (E \setminus F) = \text{EP}_M(J)$$

because the unions with $(E \setminus F)$ are disjoint unions. \qed

With this result in mind, we extend Las Vergnas’s external order to the independent sets of an ordered matroid.

**Definition 4.7.** Let $M$ be an ordered matroid. Then the external order on the independent sets of $M$ is defined by:

$$I_1 \leq_{\text{ext}} I_2 \iff \text{EP}(I_1) \supseteq \text{EP}(I_2)$$

In particular, because $\text{EP}$ restricted to the bases of $M$ is the same as the classical definition used by Las Vergnas, the original external order on the bases of $M$ appears as a subposet of this generalization. As noted in the introduction, for technical convenience we will work with the reverse of this order,

$$I_1 \leq^* I_2 \iff \text{EP}(I_1) \subseteq \text{EP}(I_2)$$
Whenever we refer to the external order, we will be referring to the reversed order \( \leq^*_{\text{ext}} \) unless otherwise noted.

To understand the properties of the generalized external order, we will relate the notion of matroid external activity to an analogous notion for antimatroids, as follows. We first note that the rooted circuits of an antimatroid can be thought of as minimal obstructions to extending feasible sets.

**Lemma 4.8.** Let \( (E, \mathcal{F}) \) be an antimatroid with associated join-distributive lattice \( L \), let \( x \in L \), and let \( a \in E \setminus T(x) \). Then \( a \in I(x) \) if and only if each rooted circuit \( (C, a) \) of \( \mathcal{F} \) has nonempty intersection with \( T(x) \).

*Proof.* If \( a \in I(x) \), then \( T(x) \cup a \) is a feasible set. If a rooted circuit \( (C, a) \) is disjoint from \( T(x) \), then the intersection of \( C \) with \( T(x) \cup a \) is equal to the singleton set \( \{a\} \). However, this violates the definition from Proposition 2.17 of the root of a rooted circuit.

On the other hand, if each rooted circuit \( (C, a) \) has nonempty intersection with \( T(x) \), then the intersection of \( C \) with \( T(x) \cup a \) is not equal to the singleton set \( \{a\} \), and so by Proposition 2.18, we have that \( T(x) \cup a \in \mathcal{F} \), so \( a \in I(x) \).

A consequence of this fact is that the rooted circuits of an antimatroid allow us to recover the feasible set associated to a given independent set without reference to any other global structure of the antimatroid.

**Lemma 4.9.** Let \( (E, \mathcal{F}) \) be an antimatroid with associated join-distributive lattice \( L \), and let \( x \in L \). Then

\[
T(x) = \{ a \in E \setminus I(x) : C \notin I(x) \cup a \text{ for any } (C, a) \in \mathcal{C}(\mathcal{F}) \}
\]

*Proof.* Let \( T_0(x) \) denote the set in the right side of the equality, and let \( a \) be an arbitrary element in \( E \setminus I(x) \). If \( a \notin T_0(x) \), then there is a rooted circuit \( (C, a) \in \mathcal{C} \) such that \( C \subseteq I(x) \cup a \). But then \( C \setminus a \subseteq I(x) \), so \( C \cap T(x) \) is either \( \{a\} \) if \( a \in T(x) \) or empty if \( a \notin T(x) \). By Proposition 2.18, since \( T(x) \in \mathcal{F} \), we see that \( C \cap T(x) \neq \{a\} \), so we conclude that in this case, \( a \notin T(x) \). Thus \( T(x) \subseteq T_0(x) \).

Now suppose that \( a \in T_0(x) \). If \( I(x) \cup a \) is independent, say \( I(y) = I(x) \cup a \), then by Lemma 3.8 we know that \( x \geq y \), so by Lemma 3.9, \( I(y) \subseteq I(x) \cup T(x) \), and thus \( a \in T(x) \).

If \( I(x) \cup a \) is not independent, it contains a rooted circuit \( (C, b) \in \mathcal{C} \). Since any subset of \( I(x) \) is independent and thus not a circuit, we must have that \( a \in C \). However, \( a \) cannot be the root of \( C \) because in this case \( C \subseteq I(x) \cup a \) violates the fact that \( a \in T_0(x) \). However, if \( b \neq a \) then \( b \in I(x) \), so by Lemma 4.8 we have that \( C \cap T(x) \) is nonempty. Since all elements of \( C \) aside from \( a \) are in \( I(x) \) which is disjoint from \( T(x) \), we conclude that \( a \in T(x) \). Thus \( T_0(x) \subseteq T(x) \) as well.

In light of this lemma, it makes sense to define the external activity in an antimatroid as follows.

**Definition 4.10.** Let \( (E, \mathcal{F}) \) be an antimatroid with rooted circuits \( C \), and let \( I \) be an independent set. Then for \( a \in E \setminus I \), we say that \( a \) is **externally active** with respect to \( I \) if there exists a rooted circuit \( (C, a) \in \mathcal{C} \) such that \( C \subseteq I \cup a \). Otherwise we say that \( a \) is **externally passive**.
We denote the active elements of $\mathcal{F}$ by $\text{EA}_F(I)$, and the passive elements by $\text{EP}_F(I)$, where the subscripts may be omitted if there is no risk of ambiguity. If $L$ is a join-distributive lattice, then $\text{EA}_L(x)$ and $\text{EP}_L(x)$ denote the active and passive elements of $I(x)$ in the associated antimatroid $F(L)$.

In particular, for $x \in L$ a join-distributive lattice, Lemma 4.9 shows that $T(x)$ is the set of externally passive elements of $I(x)$.

We can now connect the external order with the theory of antimatroids.

**Proposition 4.11.** If $M$ is an ordered matroid, then the collection of rooted sets

$$C = C_{\text{ext}}(M) \coloneqq \{(C, \min(C)) : C \in C(M)\}$$

satisfies the axioms of rooted antimatroid circuits.

**Proof.** For axiom (CI1), note that if $(C_1, a)$ and $(C_2, a)$ are in $C$, then $C_1$ and $C_2$ are circuits of $M$, and thus $C_1$ is not a proper subset of $C_2$ by properties of matroid circuits.

For axiom (CI2), suppose $(C_1, a_1), (C_2, a_2) \in C$ with $a_1 \in C_2 \setminus a_2$. By definition of $C$ we know that $a_1 = \min(C_1)$ and $a_2 = \min(C_2)$, so in particular we know that $a_1 > a_2$, and $a_2 \notin C_1$.

Note that matroid circuits satisfy the following strong elimination axiom: If $C_1, C_2$ are circuits with $a_1 \in C_1 \cap C_2$ and $a_2 \in C_2 \setminus C_1$, then there is a circuit $C_3 \subseteq (C_1 \cup C_2) \setminus a_1$ which contains $a_2$.

Applying this elimination axiom to our present circuits, we obtain a matroid circuit $C_3 \subseteq (C_1 \cup C_2) \setminus a_1$ with $a_2 \in C_3$. $a_2$ is minimal in $C_1 \cup C_2$, so this implies that $a_2 = \min(C_3)$, and $(C_3, a_2) \in C$. Thus $C$ satisfies axiom (CI2) as well. \hfill $\square$

Proposition 4.11 allows us to conclude the following structural characterization of the generalized external order.

**Definition 4.12.** If $M$ is an ordered matroid, let

$$\mathcal{F}_{\text{ext}} = \mathcal{F}_{\text{ext}}(M) \coloneqq \{\text{EP}_M(I) : I \in \mathcal{I}(M)\}$$

**Theorem 1.** If $M$ is an ordered matroid, then $\mathcal{F}_{\text{ext}}(M)$ is the collection of feasible sets of the antimatroid with rooted circuits $C_{\text{ext}}(M)$.

**Proof.** Denote $M = (E, \mathcal{I})$. By Proposition 4.11, $C_{\text{ext}}(M)$ forms the rooted circuits of an antimatroid $(E, \mathcal{F})$. Let $L$ be the associated join-distributive lattice. By definition of antimatroid circuits as minimal dependent sets, we have that $\mathcal{I}(L) = \mathcal{I}$ so that the sets $I(x), x \in L$ are in correspondence with the matroid independent sets of $M$.

By Lemma 4.9, any element $x \in L$ has

$$T(x) = \text{EP}_L(x) = \{a \in E \setminus I(x) : C \notin I(x) \cup a \text{ for any } (C, a) \in C_{\text{ext}}(M)\}$$

In particular, we can see that $\text{EP}_L(x) = \text{EP}_M(I(x))$ for each $x \in L$, and so the feasible set of $\mathcal{F}$ associated with each independent set $I(x)$ is given by the set of (matroid) externally passive elements of $I(x)$. Thus the feasible sets of $\mathcal{F}$ are exactly the sets in $\mathcal{F}_{\text{ext}}(M)$, as we wished to show. \hfill $\square$
A further consequence of this argument is that the independent set associated with each feasible set \( EP_M(I) \) in \( F_{\text{ext}}(M) \) is in fact \( I \). Following from this correspondence with antimatroids, we may apply Proposition 2.39 to obtain the following.

**Corollary 4.13.** If \( M = (E, \mathcal{I}) \) is an ordered matroid, then the external order \( \leq_{\text{ext}} \) on \( \mathcal{I} \) is a join-distributive lattice. Meet-irreducible sets in the lattice correspond with the non-loops of \( E \), and joins correspond to taking unions of externally passive sets.

### 4.2 Combinatorial Structure

Using the antimatroid structure of the generalized external order, we are able to prove a variety of properties of the poset, many of which generalize the properties enjoyed by the classical order on matroid bases. In the following, \( M = (E, \mathcal{I}) \) denotes an ordered matroid.

**Lemma 4.14.** The following basic properties hold for independent sets and externally passive sets in \( M \).

1. If \( I, J \in \mathcal{I} \), then \( I \leq_{\text{ext}} J \) if and only if \( EP(I) \cap J = \emptyset \)
2. If \( I, J \in \mathcal{I} \) and \( J \supseteq I \), then \( J \leq_{\text{ext}} I \)
3. If \( I, J \in \mathcal{I} \), then \( I \wedge J \subseteq I \cup J \)

**Proof.** The three parts are restatements of Lemmas 3.5, 3.8 and 3.10 respectively in the context of the generalized external order. \( \square \)

**Lemma 4.15.** If \( M = (E, \mathcal{I}) \) is an ordered matroid, \( I \in \mathcal{I} \), and \( a \in E \setminus EP(I) \), the set \( EP(I) \cup a \) is the set of externally passive elements of some independent set iff \( a \in I \).

**Proof.** Let \( L \) be the join-distributive lattice associated with the antimatroid \( F_{\text{ext}}(M) \), and let \( x \in L \) be the element with \( I(x) = I \). Then \( I(x) \) is the set of feasible extensions of \( T(x) = EP(I) \), so \( EP(I) \cup a \) is feasible in \( F_{\text{ext}}(M) \) iff \( a \in I(x) = I \). The result follows because the feasible sets are exactly the sets of externally passive elements. \( \square \)

We now characterize the covering relations in the external order.

**Definition 4.16.** For an ordered matroid \( M \), if \( I \) is independent and \( a \in I \), define the **active chain** of \( a \) in \( I \) to be the set

\[
ch(I, a) = EA_M(I) \cap bo(I, a)
\]

**Proposition 4.17.** Let \( M \) be an ordered matroid, and let \( I \in \mathcal{I}(M) \). Then for each \( a \in I \), define the independent set \( J_a \) by

- If \( ch(I, a) \) is nonempty, \( J_a = I \setminus a \cup \max(ch(I, a)) \).
- If \( ch(I, a) \) is empty, \( J_a = I \setminus a \).

For each \( a \in I \), we have \( EP(J_a) = EP(I) \cup a \), and thus the sets \( J_a \) are the independent sets covering \( I \) in the external order.
Proof. Let \( a \in I \), and denote \( F = \text{span}(I) \), \( I_0 = I \setminus a \), and \( F_0 = \text{span}(I_0) \).

From Lemma 4.15 we know that there exists an independent set \( J \) such that \( \text{EP}(J) = \text{EP}(I) \cup a \). Since \( E \setminus F \subseteq \text{EP}(I) \) and \( \text{EP}(J) \cap J = \emptyset \), we have that \( J \subseteq F \).

Using the antimatroid interval property without upper bounds, with the fact that independent sets are the sets of antimatroid feasible extensions, we know that \( I_0 \subseteq J \). Thus since \( J \) is independent and contained in \( F \), either \( J = I_0 \), or \( J = I_0 \cup b \) for some \( b \in \text{bo}(I, a) \).

In the latter case, since \( b \in J \), \( b \notin \text{EP}(J) = \text{EP}(I) \cup a \), so this implies that \( b \) is an element of the active chain \( \text{ch}(I, a) \).

If \( \text{ch}(I, a) \) is empty, then we must be in the first case above, so \( J = I \setminus a = J_a \) as desired.

If \( \text{ch}(I, a) \) is nonempty, let \( c \) be its maximal element, which in particular is in \( F \setminus F_0 \), and is not in \( \text{EP}(I) \cup a \). On one hand, suppose that \( J = I_0 \). Then \( F \setminus \text{span}(J) = F \setminus F_0 \subseteq \text{EP}(J) \), so \( c \in \text{EP}(J) \), and this implies that \( \text{EP}(J) \neq \text{EP}(I) \cup a \), a contradiction.

On the other hand, suppose that \( J = I_0 \cup c' \) for some \( c' \in \text{ch}(I, a) \), \( c' < c \). Then because \( c \notin F_0 \), we must have \( \text{ci}(J', c) \notin I_0 \cup c \), so \( c' \in \text{ci}(J', c) \). This implies that \( c \) is externally passive since \( c' < c \), so again \( \text{EP}(J) \neq \text{EP}(I) \cup a \).

Since there is only one remaining possibility for \( J \), we conclude that \( J = I \setminus a \cup c = J_a \). \( \square \)

The downward covering relations are somewhat more complicated to describe in general, but a particular covering always exists.

**Lemma 4.18.** Let \( M \) be an ordered matroid. If \( I \) is independent and \( x = \min(\text{EP}(I)) \), then there is an independent set \( J \) such that \( \text{EP}(J) = \text{EP}(I) \setminus x \).

**Proof.** If \( x \notin \text{span}(I) \), then let \( J = I \cup x \). Then the active chain \( \text{ch}(J, x) \) is empty, so from Proposition 4.17, \( \text{EP}(J) = \text{EP}(I) \setminus x \).

If \( x \in \text{span}(I) \), then let \( y = \min(\text{ci}(I, x)) \), and let \( J = I \setminus y \cup x \). Then \( \text{ci}(J, y) = \text{ci}(I, x) \), so since \( y < x \), we have that \( y \) is externally active with respect to \( J \), and in particular is contained in the active chain \( \text{ch}(J, x) \).

In fact, we can show that \( y = \max(\text{ch}(J, x)) \). If this were not the case, then there is an element \( z > y \) with \( z \in \text{EA}(J) \cap \text{bo}(J, x) \). Then \( z \in \text{bo}(J, x) \), which means that \( x \in \text{ci}(J, z) \) and \( y \in \text{ci}(I, z) \). Since \( z \in \text{EA}(J) \), we have \( z < x \), and since \( z > y \) we have that \( z \in \text{EP}(I) \). This contradicts the assumption that \( x \) was minimal in \( \text{EP}(I) \).

We conclude that \( y = \max(\text{ch}(J, x)) \), so again by Proposition 4.17, we have that \( \text{EP}(J) = \text{EP}(I) \setminus x \). \( \square \)

**Corollary 4.19.** If \( M = (E, \mathcal{I}) \) is an ordered matroid and \( I, J \in \mathcal{I} \) satisfy \( I \leq^*_\text{ext} J \), then \( I \) is lexicographically greater than or equal to \( J \), where prefixes are considered small.

**Proof.** This follows because \( \text{ch}(I, x) \) consists only of elements smaller than \( x \), so any covering relation corresponds with either a replacement of an element with a smaller one, or with removal of an element entirely. \( \square \)

We can give explicit combinatorial formulations for the meet and join of independent sets in the external order.

**Lemma 4.20.** If \( A \subseteq E \), then the lex maximal basis \( B \) of \( M \setminus A \) satisfies \( \text{EP}(B) \subseteq A \). If \( I >^*_\text{ext} B \) for some independent set \( I \), then \( \text{EP}(I) \setminus A \) is nonempty.
Proof. Suppose \( x \in \text{EP}(B) \setminus A \). Then the element \( y = \min(\text{ci}(B, x)) \) is an element of \( B \), and the basis \( B' = B \setminus y \cup x \) gives a basis in \( M \setminus A \) which is lex greater than \( B \), a contradiction. Thus \( \text{EP}(B) \subseteq A \).

If \( I >^\ast \text{ext} B \), then there is an independent set \( J \leq^\ast \text{ext} I \) which covers \( B \), so that \( \text{EP}(J) = \text{EP}(B) \cup x \) for some \( x \in E \). However, such a \( J \) exists exactly when \( x \in B \), so since \( B \subseteq E \setminus A \), we have \( x \notin A \). Thus \( \text{EP}(I) \setminus A \) is nonempty. 

**Proposition 4.21.** The minimum element of the external order is the lex maximal basis of \( M \), and the maximum element of the external order is the empty set. If \( I, J \in \mathcal{I} \), then meets and joins in the external order are described by

- \( I \wedge J \) is the lex maximal basis of \( M \setminus (\text{EP}(I) \cap \text{EP}(J)) \)
- \( I \vee J \) is the lex maximal basis of \( M \setminus (\text{EP}(I) \cup \text{EP}(J)) \)

**Proof.** The proof is by repeated application of Lemma 4.20. The lex maximal basis \( B \) of \( M = M \setminus \emptyset \) has \( \text{EP}(B) \subseteq \emptyset \), so \( B \) is the minimum element in the external order. Likewise, \( \text{EP}(\emptyset) \) is the ground set of \( M \) minus any loops (which are never externally passive), so \( \emptyset \) is the maximum element.

To characterize meets, let \( K \) be the lex maximal basis of \( M \setminus (\text{EP}(I) \cap \text{EP}(J)) \). Then \( \text{EP}(K) \subseteq \text{EP}(I) \cap \text{EP}(J) \), so we have that \( K \leq^\ast \text{ext} I \wedge J \). Further, if \( K' \geq^\ast \text{ext} K \), then \( \text{EP}(K') \) contains an element outside of \( \text{EP}(I) \cap \text{EP}(J) \), which shows that \( K' \) is not less than one of \( I \) or \( J \). Since \( K \leq^\ast \text{ext} I, J \) and no larger independent set is, we conclude that \( K = I \wedge J \).

To characterize joins, let \( K \) be the lex maximal basis of \( M \setminus (\text{EP}(I) \cup \text{EP}(J)) \), so that \( \text{EP}(K) \subseteq \text{EP}(I) \cup \text{EP}(J) \). By properties of antimatroids, \( \text{EP}(I \vee J) = \text{EP}(I) \cup \text{EP}(J) \), so in particular, we have \( K \leq^\ast \text{ext} I \vee J \). If this relation is not equality however, we note that \( \text{EP}(I \vee J) \) contains an element outside of \( \text{EP}(I) \cup \text{EP}(J) \), which is a contradiction. Thus we must have equality, so \( K = I \vee J \).

From this we also conclude

**Corollary 4.22.** \( I \) is the lex maximal basis of \( M \setminus \text{EP}(I) \) for any independent set \( I \).

As a further consequence, we obtain the following partition of the boolean lattice into boolean subintervals.

**Proposition 4.23.** If \( M \) is an ordered matroid with ground set \( E \), then the intervals

\[ [I, I \cup \text{EA}(I)] \text{ for } I \text{ independent} \]

form a partition of the boolean lattice \( 2^E \).

This partition resembles the classic partition of Crapo (see for instance [6]), and in fact, it can be shown that this partition is a refinement of Crapo’s. Gordon and McMahon [11] mention that the existence of such a partition is implied by their Theorem 2.5 applied to matroid independent sets, and this explicit form can be proved by first generalizing the idea of their Proposition 2.6 to external activity for arbitrary independent sets. Interestingly, an
independent proof is obtained by instead applying Theorem 2.5 to the antimatroid $\mathcal{F}_{\text{ext}}(M)$. This gives the interval partition

$$[\text{EP}(I), E \setminus I]$$

for $I$ independent

and the desired interval partition is obtained from this by taking set complements. The details of these proofs are omitted.

Finally, we note that the external order is a refinement of the geometric lattice of flats of the associated matroid.

**Proposition 4.24.** The natural map from the external order $\leq^*_{\text{ext}}$ on $M$ to the geometric lattice of flats of $M$ given by $I \mapsto \text{span}(I)$ is surjective and monotone decreasing. In particular, the external order on $M$ is a refinement of the geometric lattice of flats of $M$.

**Proof.** Suppose $I$ and $J$ are independent with $I \leq^*_{\text{ext}} J$. In particular, EP($I$) contains all elements outside of span($I$), and by Lemma 4.14, we also have EP($I$) $\cap$ $J$ $=$ $\emptyset$. Thus $J \subseteq \text{span}(I)$, so we conclude $\text{span}(J) \subseteq \text{span}(I)$. $\square$

Note in particular that the classical ordering convention $\leq_{\text{ext}}$ which is consistent with Las Vergnas’s original definition then gives an order preserving surjection onto the geometric lattice of flats of a matroid. This is a significant reason why in some contexts the classical order convention, rather than the reverse, may be more convenient.

5 Lattice Theory of the Extended Order

With the external order identified as a join-distributive lattice, a natural question which arises is to classify the lattices this construction produces. To do so, we will need to incorporate two main ideas.

First, we will define the subclass of matroidal join-distributive lattices which characterizes the join-distributive lattices whose independent are those of a matroid. Second, we will identify a property, $S_n$ EL-shellability, which ensures a certain order consistency condition for the roots of circuits.

We will see in Theorem 3 that these two lattice-theoretic properties, which are satisfied by the external order, are in fact enough to characterize the lattices isomorphic to the external order of an ordered matroid.

5.1 Matroidal Join-distributive Lattices

The most apparent connection between the external order and the underlying ordered matroid is in the equality of the matroid and antimatroid independent sets. We now define the class of matroidal join-distributive lattices to further explore this connection.

**Definition 5.1.** If $L$ is a join-distributive lattice, define the covering rank function $r_c$ of $L$ by

$$r_c : x \mapsto |I(x)|$$

counting the number of elements in $L$ which cover $x$. 

Definition 5.2. We call a join-distributive lattice $L$ matroidal if the covering rank function $r_c$ is decreasing, and satisfies the semimodular inequality

$$r_c(x \land y) + r_c(x \lor y) \leq r_c(x) + r_c(y)$$

Proposition 5.3. If $L$ is a matroidal join-distributive lattice, then $\mathcal{I}(L)$ is the collection of independent sets of a matroid with ground set $\text{MI}(L)$.

Proof. For notational convenience, let $\mathcal{I} = \mathcal{I}(L)$ and let $E = \text{MI}(L)$. We will show that the function $r : 2^E \to \mathbb{Z}_{\geq 0}$ defined by

$$r(A) = \max \{|I| : I \in \mathcal{I}, I \subseteq A\}$$

is a matroid rank function on $2^E$ whose independent sets are $\mathcal{I}$.

Both the fact that $0 \leq r(A) \leq |A|$ for any subset $A$ and that $r(A) \leq r(B)$ for subsets $A \subseteq B \subseteq E$ are clear from the definition of $r$. Thus all that remains is to prove the semimodular inequality

$$r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$$

for any subsets $A, B \subseteq E$.

Recall that for arbitrary $A \subseteq E$, we denote by $x_A$ the meet of the elements

$$\mathcal{I}_A = \{x_I : I \subseteq A \text{ is independent}\}$$

In general, $x_A$ is equal to a minimal element $x_I$ with $I \subseteq A$ independent, and since $r_c$ is decreasing in $L$, covering rank is maximized in $\mathcal{I}_A$ by $x_I$. This means that $I$ is a maximal size independent subset of $A$, so we conclude that $r(A) = r_c(x_A)$.

Now for $A, B \subseteq E$, by Lemma 3.11 we know $x_A \land x_B \leq x_{A \cup B}$ and $x_A \lor x_B \leq x_{A \cap B}$. Thus with the semimodular inequality for $r_c$ and because $r_c$ is a decreasing function, we have

$$r(A \cup B) + r(A \cap B) = r_c(x_{A \cup B}) + r_c(x_{A \cap B})$$
$$\leq r_c(x_A \land x_B) + r_c(x_A \lor x_B)$$
$$\leq r_c(x_A) + r_c(x_B)$$
$$= r(A) + r(B)$$

Thus $r$ satisfies the semimodular inequality.

Finally, note that if $A$ is independent, then $r(A) = |A|$, and if $A$ is not independent, then the independent subsets of $A$ are proper, so $r(A) < |A|$. Thus the sets $A \in \mathcal{I}$ are exactly the subsets of $E$ for which $r(A) = |A|$, and so $\mathcal{I}$ is the set of independent sets of the matroid with rank function $r$. \qed

With a little more work, we can also prove the converse of this statement: a join-distributive lattice whose independent sets form a matroid is itself matroidal. To this end, a few additional lemmas will be useful.

Definition 5.4. Let $L$ be a join distributive lattice whose independent sets are the independent sets of a matroid. Then for $x \in L$, let $F_x$ denote the matroid flat $\text{cl}(T(x)^c)$. 

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Lemma 5.5. If $L$ is a join-distributive lattice whose independent sets are the independent sets of a matroid $M$, then for any $x \in L$, the independent set $I(x)$ is a basis of $F_x$. In particular, $r_c(x) = |I(x)| = r(F_x)$.

Proof. Since $I(x) \subseteq T(x)^c$, we have $I(x) \subseteq F_x$ for any $x$, so suppose there is an $x \in L$ such that $I(x)$ doesn’t span $F_x$. In particular, by properties of matroids there is an element $a \in T(x)^c \setminus I(x)$ such that $I(x) \cup a$ is independent in $M$, and since $\mathcal{I}(L) = \mathcal{I}(M)$, there is an element $y \in L$ with $I(y) = I(x) \cup a$. By Lemma 3.8, we have $y < x$, and by Lemma 3.9, this means that $I(y) \subseteq I(x) \cup T(x)$. However, this is a contradiction since $a \in T(x)^c \setminus I(x)$. □

Lemma 5.6. Let $L$ be a join-distributive lattice whose independent sets are the independent sets of a matroid $M$. If $x, y \in L$ satisfy $I(x) \supseteq I(y)$, then the elements of $T(y) \setminus T(x)$ lie outside of $F_y$.

Proof. If $x = y$ this is vacuously true, so suppose $x \neq y$. By Lemma 3.8, we have $x < y$, so there is a sequence of elements $x = z_0 < z_1 < \cdots < z_k = y$ with edge labels $a_i = e(z_{i-1}, z_i)$. In particular, $T(y) \setminus T(x) = \{a_1, \ldots, a_k\}$.

For each $i$, $a_i \in I(z_{i-1})$. If $a_i$ were in $I(y)$ for some $i$, then we would have $a_i \in T(z_i) \subseteq T(y)$, so in particular this contradicts disjointness of $T(y)$ and $I(y)$. By induction using Lemma 3.9, we see that $I(z_i) \supseteq I(y)$ for each $i$. Thus the sets $I(y) \cup a_i \subseteq I(z_{i-1})$ are independent, and $a_i \notin \text{cl}(I(y))$ for each $i$. The conclusion follows from Lemma 5.5. □

Lemma 5.7. Let $L$ be a join-distributive lattice whose independent sets are the independent sets of a matroid $M$. If $x, y \in L$, then

- $F_{x \lor y} \subseteq F_x \cap F_y$
- $F_{x \land y} = \text{cl}(F_x \cup F_y)$

Proof. For the first relation, note that $T(x \lor y) = T(x) \cup T(y)$, so

$$F_{x \lor y} = \text{cl}((T(x) \cup T(y))^c) = \text{cl}((T(x)^c \cap T(y))^c) \subseteq \text{cl}(T(x)^c) \cap \text{cl}(T(y)^c) = F_x \cap F_y$$

For the second, begin by noticing that $T(x \land y) \subseteq T(x) \cap T(y)$, so

$$F_{x \land y} = \text{cl}(T(x \land y)^c) \supseteq \text{cl}((T(x) \cap T(y))^c) = \text{cl}(T(x)^c \cup T(y)^c) = \text{cl}(F_x \cup F_y)$$

Let $G_{x \land y} := \text{cl}(F_x \cup F_y)$, and suppose the containment $F_{x \land y} \supseteq G_{x \land y}$ is proper. Then since $I(x \land y)$ is a basis for $F_{x \land y}$, we have $I(x \land y) \setminus G_{x \land y}$ is nonempty, containing an element $a$. Then there exists $z \in L$ with $I(z) = I(x \land y) \setminus a$, and by Lemma 3.8, we have $z > x \land y$.

Since $a$ lies outside of $G_{x \land y}$, we know that $I(z) = I(x \land y) \setminus a$ has span $F_z \supseteq G_{x \land y}$, so in particular $F_z$ contains both $F_x$ and $F_y$. By Lemma 5.6, since $I(x \land y) \supseteq I(z)$, we know that $T(z) \setminus T(x \land y)$ contains only elements outside of $F_z$. However, since $F_x, F_y \subseteq F_z$, we have

$$T(z) \setminus T(x \land y) \subseteq F_z^c \subseteq F_x^c \cap F_y^c \subseteq T(x) \cap T(y)$$

Noting that $T(x \land y) \subseteq T(x) \cap T(y)$, we further conclude that $T(z) \subseteq T(x) \cap T(y)$, and thus $z \leq x \land y$. This contradicts $z > x \land y$, so we see that the inclusion $F_{x \land y} \supseteq G_{x \land y}$ must be equality as desired. □
Finally, we can prove the converse to Proposition 5.3.

**Proposition 5.8.** Let $L$ be a join-distributive lattice. If $\mathcal{I}(L)$ is the collection of independent sets of a matroid, then $L$ is matroidal.

**Proof.** Suppose that $x \leq y$ in $L$, so that $T(x) \subseteq T(y)$. Then in particular, $F_x = \text{cl}(T(x)^c) \supseteq \text{cl}(T(y)^c) = F_y$, so

$$r_c(x) = r(F_x) \geq r(F_y) = r_c(y)$$

and thus $r_c$ is decreasing. To prove that $r_c$ satisfies the semimodular inequality, we appeal to the corresponding inequality for matroid rank functions. Using Lemmas 5.5 and 5.7, we have

$$r_c(x \land y) + r_c(x \lor y) = r(F_x \land y) + r(F_x \lor y)$$

$$\leq r(\text{cl}(F_x \cup F_y)) + r(F_x \cap F_y)$$

$$= r(F_x \cup F_y) + r(F_x \cap F_y)$$

$$\leq r(F_x) + r(F_y)$$

$$= r_c(x) + r_c(y)$$

Gathering the above results, we have proven the following.

**Theorem 2.** A join-distributive lattice $L$ is matroidal if and only if $\mathcal{I}(L)$ is the collection of independent sets of a matroid.

It is clear from this result that the generalized external order for an ordered matroid $M$ gives a matroidal join-distributive lattice. A natural question to address, then, is whether all matroidal join-distributive lattices arise as the external order for some ordering of their underlying matroid. In fact, this question can be answered in the negative, as the following counterexample demonstrates.

**Example.** Consider the antimatroid on ground set $E = \{a, b, c, d\}$ whose feasible sets are $\mathcal{F} = \{\emptyset, d, c, bd, cd, ac, abd, bcd, acd, abc\}$. The Hasse diagram for the corresponding join-distributive lattice appears in Figure 3.

In particular, the collection of independent sets of this antimatroid is the uniform matroid $U_4^2$ of rank 2 on 4 elements. Suppose this were the external order with respect to some total ordering $<$ on $E$. In this case, we observe that

- $a$ is active with respect to $I = bc$, so $a$ is smallest in the basic circuit $\text{ci}(bc, a) = abc$

- $b$ is active with respect to $I = ad$, so $b$ is smallest in the basic circuit $\text{ci}(ad, b) = abd$

But this implies that both $a < b$ and $b < a$, a contradiction. Thus this matroidal join-distributive lattice cannot come from a total ordering on the ground set $E$. 

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5.2 The External Order and $S_n$ EL-labelings

To bridge the gap between matroidal join-distributive lattices and the external order, we will need one more key notion, a combinatorial construction on a graded poset called an $S_n$ EL-labeling, or snelling.

Definition 5.9. If $P$ is a finite poset, then a map $\lambda : \text{Cov}(P) \rightarrow \mathbb{Z}$ on the covering pairs of $P$ is called an edge labeling of $P$.

If $m$ is an unrefinable chain $x_0 \preceq x_1 \preceq \cdots \preceq x_k$ in $P$, then the sequence

$$\lambda(m) = (\lambda(x_0, x_1), \lambda(x_1, x_2), \ldots, \lambda(x_{k-1}, x_k))$$

is called the label sequence of $m$, and an unrefinable chain $m$ is called increasing if $\lambda(m)$ is increasing.

Definition 5.10. If $P$ is a finite graded poset, then an edge labeling $\lambda$ is called an edge lexicographic or EL-labeling if

- Any interval $[x, y] \subseteq P$ has a unique increasing maximal chain $m_0$
- Any other maximal chain in $[x, y]$ has edge labels which are lex greater than the edge labels of $m_0$

The existence of an EL-labeling on a poset $P$ in particular implies that the order complex of $P$ is shellable, and this is the application for which the notion was introduced by Björner in [5]. In particular, a poset which admits an EL-labeling is called EL-shellable.

EL-labelings are not sufficiently rigid to capture the combinatorial property we are trying to isolate, but the following strengthening, first introduced by McNamara in [17], couples well with the set system structure of antimatroids.

Definition 5.11. An EL-labeling on a finite graded poset $P$ is called an $S_n$ EL-labeling or snelling if the label sequence $\lambda(m)$ of any maximal chain in $P$ is additionally a permutation of the integers 1 to $n$. A poset which admits an $S_n$ EL-labeling is called $S_n$ EL-shellable.
We proceed to relate $S_n$ EL-labelings of join-distributive lattices to the following useful property for antimatroid circuits.

**Definition 5.12.** If $(E, F)$ is an antimatroid with rooted circuits $C$, we say that $F$ is **confluent** if there is an ordering $\leq$ on the elements of $E$ such that the root of any rooted circuit $C \in C$ is given by $x = \max_{\leq}(C)$. We call such an ordering a **confluent ordering** for $F$. Similarly, a join-distributive lattice is called confluent if its corresponding antimatroid is confluent.

This definition captures the essential structure that distinguishes the external order from other matroidal join-distributive lattices. A useful consequence of confluence is that comparable feasible sets in a confluent antimatroid have lex comparable independent sets in the following sense.

**Lemma 5.13.** In a confluent join-distributive lattice $L$, if $x, y \in L$ satisfy $x \leq y$, then $I(x) \leq I(y)$ in lex ordering, where prefixes of a word $S$ are considered larger than $S$.

**Proof.** If $x < y$, then $T(y) = T(x) \cup a$ for some $a \in E = MI(L)$, and in particular $a \in I(x)$. By Lemma 3.9, $I(x) \setminus a \subseteq I(y)$. Since $I(y)$ is the set of elements in $E \setminus T(y)$ which are not the root of a circuit disjoint from $T(y)$, any new elements in $I(y) \setminus I(x)$ are elements $b$ which are the root of a circuit $(C, b)$ with $a \in C$. Since the ordering on $E$ is confluent, the root $b$ is maximal in $C$, so $b > a$.

This shows that $I(y)$ consists of the elements in $I(x) \setminus a$ plus a (potentially empty) set of elements $S$ all of which are larger than $a$. The ordering $I(x) < I(y)$ follows, and the general fact for $y$ not covering $x$ follows by induction on the length of a maximal chain between $x$ and $y$. \[\square\]

The main structural result of this section is Proposition 5.15, which is similar to the work of Armstrong in [4] characterizing supersolvable matroids. In fact, our result can be derived from Armstrong’s Theorem 2.13, which lists several conditions which are equivalent to $S_n$ EL-shellability of a join-distributive lattice. Our result in particular shows that the condition “$(E, F)$ is a confluent antimatroid” is also equivalent to the conditions listed in Armstrong’s theorem.

We provide an independent proof of Proposition 5.15 for the reader’s convenience. The proof has the particular advantage of more directly relating $S_n$ EL-labelings with the natural labelings of antimatroids without needing to pass through the theory of supersolvable lattices.

We begin by proving the following lemma.

**Lemma 5.14.** Let $L$ be a join-distributive lattice. Then any $S_n$ EL-labeling of $L$ is equivalent to the natural edge labeling of $L$ for some ordering of its labels.

**Proof.** Let $\epsilon : \text{Cov}(L) \to [n]$ be an $S_n$ EL-labeling of $L$, and let $e : \text{Cov}(L) \to MI(L)$ denote the natural edge labeling of $L$. First we prove that for any diamond of elements $x, y, x', y' \in L$ as below, we have that $\epsilon(x, x') = \epsilon(y, y')$.
To see this, suppose that $m$ is a maximal chain of $L$ which includes the covering relations $x \prec x' \prec y'$, and let $m'$ be the maximal chain of $L$ which is identical to $m$ except that it replaces the covering relations $x \prec x' \prec y'$ with the relations $x \prec y \prec y'$. Then the edge labels of $m$ and $m'$ form permutations of $[n]$, and the edge labels below $x$ and above $y'$ in each chain are identical.

In particular, since both are permutations, the sets of labels $\{\epsilon(x, x'), \epsilon(x', y')\}$ and $\{\epsilon(x, y), \epsilon(y, y')\}$ are the same, say $\{a, b\}$ with $a < b$. Since $\epsilon$ is an $S_n$ EL-labeling, exactly one chain in the interval $[x, y']$ is in increasing order, which means that $\epsilon$ gives one of the two labelings:

In either case, $\epsilon(x, x') = \epsilon(y, y')$, as we wished to show.

Now let $x, x' \in L$ be a covering pair, $x \prec x'$, let $y \in \text{MI}(L)$ be the edge label $\epsilon(x, x')$, and let $y'$ be the unique element covering $y$ in $L$. We will show that in this case, $\epsilon(x, x') = \epsilon(y, y')$.

To see this, note that $x \preceq y$, and let $m$ be a maximal chain between $x$ and $y$, given by $x = z_0 \prec z_1 \prec \cdots \prec z_k = y$. If $k = 0$, then $x = y$ and the desired relation holds trivially. Otherwise, by the interval property without upper bounds, there exist elements $z_i' > z_i$ with $\epsilon(z_i, z_i') = y$, and we observe a parallel chain $m'$ given by $x' = z_0' < z_1' < \cdots < z_k' = y'$. Then each pair of coverings $z_i < z_i'$ and $z_{i+1} < z_{i+1}'$ form a diamond of elements as in the previous argument, and so $\epsilon(z_i, z_i') = \epsilon(z_{i+1}, z_{i+1}')$ for each $i$. This shows that $\epsilon(x, x') = \epsilon(y, y')$.

Finally, let $m$ now denote the unique increasing maximal chain of $L$ in the labeling $\epsilon$, given by $0 = x_0 \prec x_1 \prec \cdots \prec x_n = 1$. In particular, since the labels of $m$ are an increasing permutation of $[n]$, we have that $\epsilon(x_{i-1}, x_i) = i$ for each $i$. Then each covering in this chain corresponds with the meet irreducible $y_i = e(x_{i-1}, x_i)$, which is covered by a unique element $y_i'$. By the above argument, $\epsilon(y_i, y_i') = \epsilon(x_i, x_i') = i$ as well.

In particular, this implies that for any covering relation $x \prec x'$ in $L$, the label $\epsilon(x, x')$ is given by the label $\epsilon(y_i, y_i') = i$, where $\epsilon(x, x') = y_i$. Thus $\epsilon(x, x') = \varphi(\epsilon(x, x'))$ for the bijection $\varphi : \text{MI}(L) \rightarrow [n]$ given by $y_i \mapsto i$, and we see that $\epsilon$ is equivalent to $e$ under the ordering induced by $\varphi$.

Applying this lemma, we can demonstrate the equivalence of confluence and $S_n$ EL-shellability for join-distributive lattices. We will prove in two parts the following:
Proof. Fix a confluent ordering of $E = \text{MI}(L)$, and as usual, let $e : \text{Cov}(L) \to E$ denote the natural edge labeling of $L$. The fact that the sequence of labels of any maximal chain gives a permutation of $E$ is clear from the fact that the union of the edge labels of a maximal chain is equal to $E = T(1)$.

Thus it is sufficient to show that every interval $[x, y]$ has a unique increasing maximal chain. Further, since the edge labels of any maximal chain in $[x, y]$ are a permutation of $T(y) \setminus T(x)$ and determine the chain uniquely, it is enough to prove that there is a chain whose edge labels are the increasing sequence of the elements of $T(y) \setminus T(x)$.

For this, we proceed by induction on the size of $T(y) \setminus T(x)$. If $x = y$, then the empty chain is sufficient, so suppose that $x < y$, and let $a = \min(T(y) \setminus T(x))$.

For any $z \in [x, y]$, we have that $I(z)$ is lex greater than or equal to $I(x)$ in the sense of Lemma 5.13. Further, if we denote $J = I(x) \setminus T(y)$, then we have $J \subseteq I(z)$ by the antimatroid interval property without upper bounds.

Thus the smallest element of lexicographic divergence between $I(x)$ and $I(z)$ must be an element $b$ of $I(x) \cap T(y)$ which is contained in $I(x)$ but not in $I(z)$. In particular we have $b \in T(y) \setminus T(x)$.

Since $a$ is smallest in $T(y) \setminus T(x)$, if $a \notin I(x)$, then the smallest element of divergence between $I(x)$ and $I(z)$ is larger than $a$, so $a \notin I(z)$.

However, this holds for any $z \in [x, y]$, so if it were the case that $a \notin I(x)$, then we would conclude that there are no edges in $[x, y]$ labeled by $a$, which would imply that $a \notin T(y)$, a contradiction. Thus we must have $a \in I(x)$.

In particular, this means that there is an element $x'$ covering $x$ such that $T(x') = T(x) \cup a$, and by induction, there is a unique increasing chain in the interval $[x', y]$, whose labels are the increasing permutation of the elements in $T(y) \setminus (T(x) \cup a)$. Appending this chain to the covering relation $x \prec x'$ gives an increasing chain in $[x, y]$, and completes the proof. 

Lemma 5.17. If $L$ is a non-confluent join-distributive lattice, then $L$ is not $S_n$ EL-shellable.

Proof. Let $(E, F)$ be the associated antimatroid of $L$, and suppose that $L$ is non-confluent. Then for any ordering of $E$, there is a rooted circuit $C$ whose root is not maximal in $C$.

Suppose that nevertheless, $L$ is $S_n$ EL-shellable. By Lemma 5.14, an $S_n$ EL-labeling corresponds with the natural labeling $e : \text{Cov} \to E$ for some ordering of $E$. With respect to that ordering, there is a rooted circuit $(C, a)$ of $F$ such that $a \neq \text{max}(C)$.

Let $b = \text{max}(C)$. By Proposition 2.26, the stem $C \setminus a$ of $C$ is in the blocker for the clutter of stems

$$C_a^* = \{ D \setminus a : (D, a) \text{ an antimatroid cocircuit of } F \}$$

In particular, since a blocker consists of the minimal sets intersecting each set in a clutter, we have that $(C \setminus a) \setminus b$ is not in the blocker of $C_a^*$, and so some antimatroid cocircuit $(D, a)$ must include $b$ in its stem $D \setminus a$.

In particular, $D$ is feasible and corresponds with a join-irreducible element of $L$ where the single feasible set covered by $D$ is $D \setminus a$. If $x \in L$ satisfies $T(x) = D$, then any chain $m$
given by \( 0 = z_0 \preceq z_1 \preceq \cdots \preceq z_k = x \) has edge labels which are a permutation of the elements of \( D \).

Further, since the only feasible set covered by \( D \) is \( D \setminus a \), we have that \( e(z_{k-1}, z_k) = a \). This implies that \( a \) comes after \( b \) in the sequence of edge labels of \( m \), and so \( m \) is not an increasing chain. This contradicts the fact that in an \( S_n \) EL-labeling, any interval must have a unique increasing maximal chain. We conclude that no \( S_n \) EL-labeling exists, and so a join-distributive lattice which is non-confluent is not \( S_n \) EL-shellable.

Finally, Proposition 5.15 allows us to classify the matroidal join-distributive lattices which are the external order for a matroid. Specifically, it is immediate that a matroidal join-distributive lattice \( L \) is the external order of an ordered matroid iff it is confluent, in which case the underlying matroid may be ordered by the reverse of any confluent ordering of \( L \). Thus we immediately conclude

**Corollary 5.18.** A matroidal join-distributive lattice \( L \) with corresponding matroid \( M \) is the external order for some ordering of \( M \) if and only if \( L \) is \( S_n \) EL-shellable.

Aggregating our results to this point, we can now state a complete characterization of lattices corresponding with the external order of an ordered matroid.

**Theorem 3.** A finite lattice \( L \) is isomorphic to the external order \( \leq^{*}_{\text{ext}} \) of an ordered matroid if and only if it is join-distributive, matroidal, and \( S_n \) EL-shellable.

### 6 Deletion and Contraction

We continue by exploring a correspondence between the deletion and contraction operations of matroids and antimatroids which is introduced by the external order construction. In the following, let \((E, \mathcal{F})\) denote an antimatroid, and unless otherwise noted, for \( A \subseteq E \) let \( \mathcal{F} \setminus A \) and \( \mathcal{F} / A \) denote antimatroid deletion and contraction, as defined in Section 2.2.2.

**Definition 6.1.** We call an element \( a \in E \) an **extending element** of \( \mathcal{F} \) if \( a \) is the root of any circuit of \( \mathcal{F} \) which contains it. We say that \( A \subseteq E \) is an **extending set** of \( \mathcal{F} \) if there is an ordering \( A = \{a_1, \ldots, a_k\} \) such that \( a_i \) is an extending element of \( \mathcal{F} \setminus \{a_1, \ldots, a_{i-1}\} \) for each \( i \).

It is not hard to show that an antimatroid \((E, \mathcal{F})\) is confluent (cf. Section 5.2) if and only if \( E \) is an extending set. The following lemma relates antimatroid deletion with the standard greedoid deletion and contraction operations.

**Lemma 6.2.** If \( A \in \mathcal{F} \) is a feasible set, then the antimatroid deletion \( \mathcal{F} \setminus A \) is equal to the greedoid contraction \( \mathcal{F} / A \). If \( A \) is an extending set of \( \mathcal{F} \), then the antimatroid deletion \( \mathcal{F} \setminus A \) is equal to the greedoid deletion \( \mathcal{F} \setminus A \).

The first part of this lemma is discussed in [9], Section 4, but we will prove both parts here for completeness.
Proposition 6.3. Suppose that $F$ is matroidal with associated matroid $M$. Then for $A \subseteq E$, the antimatroid deletion $F \backslash A$ is matroidal with associated matroid $M \backslash A$. If $F = F_{\text{ext}}(M)$ for an ordered matroid $M$, then $F \backslash A = F_{\text{ext}}(M \backslash A)$, where the order on $M \backslash A$ is induced by the order on $M$.

Proof. Recall that the circuits of an antimatroid are the minimal non-independent sets, so an antimatroid is matroidal with associated matroid $M$ iff its circuits are the circuits of $M$.

Now let $C$ denote the collection of rooted circuits of $F$. Then the circuits of $F \backslash a$ are given by

$$C \backslash a = \{C \in C : C \cap \{a\} = \emptyset\}$$

Forgetting the roots, these are exactly the circuits of $M \backslash a$, so we conclude that $F \backslash a$ is matroidal with associated matroid $M \backslash a$.

Remembering the roots, if $M$ is ordered then $F = F_{\text{ext}}(M)$ iff every circuit $C$ of $F$ has root $x = \min(C)$. This property is preserved by restricting to a subset of the circuits, so we see that if $F = F_{\text{ext}}(M)$, then $F \backslash a = F_{\text{ext}}(M \backslash a)$.

Antimatroid contractions do not behave as nicely as deletions with respect to matroidal structure — in many cases, contraction does not even preserve the property of being matroidal! However, for certain contraction sets the situation is still favorable.
Proposition 6.4. Suppose that $F$ is matroidal with associated matroid $M$.

- For $A$ feasible, the antimatroid contraction $F / A$ is matroidal with associated matroid $M' = M / A$.

- For $A$ extending, the antimatroid contraction $F / A$ is matroidal with associated matroid $M' = M \setminus A$.

For either case, if $F = F_{\text{ext}}(M)$ for an ordered matroid $M$, then $F / A = F_{\text{ext}}(M')$, where the order on $M'$ is induced by the order on $M$.

Proof. As in Lemma 6.2, it is sufficient to prove these cases when $A = \{a\}$ is a singleton set because of commutativity properties of minors.

If $A = \{a\}$ is a feasible set, then $A \cap C \neq \{a\}$ for any rooted circuit $C$, and so $a$ is never the root of a circuit of $F$. In particular, this means that

$$C(F / a) = \min \{(C \setminus a, x) : (C, x) \in C(F)\}$$

The circuits of $M / a$ are exactly the underlying sets of the rooted circuits of $F / a$, so we conclude that $F / a$ is matroidal with associated matroid $M / a$. If $M$ is ordered and $F = F_{\text{ext}}(M)$, then any rooted circuit $(C', x)$ of $F / a$ corresponds with a rooted circuit $(C, x)$ of $F$, where $C' = C \setminus a$. Since $F = F_{\text{ext}}(M)$, we have $x = \min(C)$, and since $x \neq a$, we have also that $x = \min(C')$, so the root of each circuit of $F / a$ is the minimal element of the circuit. This implies that $F / a = F_{\text{ext}}(M / a)$ for the induced order on $M / a$.

If $A = \{a\}$ for $a$ an extending element of $F$, then $a$ is the root of any circuit containing it. In particular this means that

$$C(F / a) = \min \{(C \setminus a, x) : (C, x) \in C(F), x \neq a\}$$

$$= \{(C, x) : (C, x) \in C(F), a \notin C\} = C(F \setminus a)$$

Thus in this case, $F / a = F \setminus a$, and the result follows from Proposition 6.3. \qed

Although antimatroid contraction doesn’t preserve matroid structure for arbitrary contraction sets, if $F$ is the external order for an ordered matroid, the resulting set system is related nicely to the external orders for the corresponding matroid deletion and contraction. We start with two lemmas, one due to Dietrich, and the other a short technical lemma on matroid deletions.

Lemma 6.5 ([9], Lemma 13). If $(C, x) \in C(F)$ and $A \subseteq E$ with $x \notin A$, then there exists a rooted circuit $(C', x) \in C(F / A)$ with $C' \subseteq C \setminus A$.

Lemma 6.6. Let $M$ be a matroid on ground set $E$, and let $A \subseteq E$. If $C \in C(M)$, then for each $x \in C \setminus A$, there exists $C' \in C(M / A)$ with $C' \subseteq C$ and $x \in C'$.

Proof. We induct on the size of $A$. If $A = \emptyset$, then the lemma holds trivially. Now suppose that $|A| \geq 1$, and let $a \in A$. We will apply a result from [18] Exercise 3.1.3, which states that

- If $a \in C$, then either $a$ is a loop or $C \setminus a$ is a circuit of $M / a$
• If \( a \notin C \), then \( C \) is a union of circuits of \( M / a \)

Let \( C \in \mathcal{C} \), assume without loss of generality that \( C \setminus A \) is nonempty, and let \( x \in C \setminus A \).

Suppose first that \( a \in C \). If \( a \) were a loop, this would imply \( C = \{a\} \), which contradicts our assumption that \( C \setminus A \) is nonempty. By the above, we now have that \( C \setminus a \) is a circuit of \( M / a \). In particular, \( x \in (C \setminus a) \setminus (A \setminus a) \), so by induction there exists a circuit \( C' \) of \( M / A = (M / a) / (A \setminus a) \) such that \( C' \subseteq C \setminus a \subseteq C \) and \( x \in C' \). Thus the lemma holds.

Now suppose that \( a \notin C \). Then \( C \) is a union of circuits of \( M / a \), so in particular there is a circuit \( C' \in \mathcal{C}(M / a) \) with \( C' \subseteq C \) and \( x \in C' \). Inductively there exists a circuit \( C'' \) of \( M / A = (M / a) / (A \setminus a) \) such that \( x \in C'' \) and \( C'' \subseteq C' \subseteq C \). This completes the proof.

Using these lemmas, we prove the following.

**Proposition 6.7.** Let \( M \) be an ordered matroid with ground set \( E \), and suppose \( \mathcal{F} = \mathcal{F}_{\text{ext}}(M) \) is the external order for \( M \). Then for \( A \subseteq E \), we have

\[
\mathcal{F}_{\text{ext}}(M / A) \subseteq \mathcal{F} / A \subseteq \mathcal{F}_{\text{ext}}(M \setminus A)
\]

**Proof.** We begin with the left inclusion. Suppose that \( F \subseteq E \setminus A \) is not feasible in \( \mathcal{F} / A \), so that there exists a rooted circuit \((C, x)\) of \( \mathcal{F} / A \) such that \( F \cap C = x \). Then in particular, \( C = C_0 \setminus A \) for a rooted circuit \((C_0, x) \in \mathcal{C}(\mathcal{F})\) with \( x \notin A \).

Since \( \mathcal{F} = \mathcal{F}_{\text{ext}}(M) \), the set \( C_0 \) is a circuit of \( M \), and \( x = \min(C_0) \). By Lemma 6.6, there exists a circuit \( C' \in \mathcal{C}(M / A) \) with \( C' \subseteq C_0 \setminus A = C \) and \( x \in C' \). Since \( x = \min(C_0) \), we also have \( x = \min(C') \), so \((C', x)\) is a rooted circuit of \( \mathcal{F}_{\text{ext}}(M / A) \). In particular we see that \( C' \cap F = \{x\} \), so we conclude that \( F \) is also not feasible in \( \mathcal{F}_{\text{ext}}(M / A) \).

For the right inclusion, suppose that \( F \subseteq E \setminus A \) is not feasible in \( \mathcal{F}_{\text{ext}}(M \setminus A) \), so that there exists a rooted circuit \((C, x)\) of \( \mathcal{F}_{\text{ext}}(M \setminus A) \) with \( C \) disjoint from \( A \) and \( F \cap C = x \).

Then \((C, x) \in \mathcal{C}(\mathcal{F})\), and by Lemma 6.5, there is a circuit \((C', x) \in \mathcal{C}(\mathcal{F} / A)\) with \( C' \subseteq C \setminus A = C \). In particular, \( F \cap C' = x \), so we conclude that \( F \) is also not feasible in \( \mathcal{F} / A \).

### 7 Further Work

We conclude by discussing some potentially fruitful directions for future study.

One common thread which was encountered when investigating the external order is the incongruity between matroids and antimatroids in the area of duality. Matroids admit a classical involutive duality operator \( M \mapsto M^* \), where \( M^* \) is defined on the same ground set as \( M \) and the bases of \( M^* \) are the complements of the bases of \( M \). On the other hand, in [9], Dietrich proves that for antimatroids, no involutive duality operator exists which satisfies certain desirable properties with respect to antimatroid deletion and contraction.

Given that the external order provides a correspondence between ordered matroids and a subclass of antimatroids, it would be interesting if matroid duality could be lifted to at least a subclass of antimatroids.

**Question 1.** Is there a natural notion of duality for matroidal antimatroids which corresponds with matroid duality of independence complexes?
A positive answer to this question would likely provide the basis for a generalized notion of internal and external matroid activity derived from antimatroid circuit roots rather than a total order on the ground set, and one might study whether such a generalization exhibits the standard behaviors and properties of matroid activity. If such a generalization exists, it would be interesting if it could be related to the decision trees of [11].

Another notion which arises from the external order is the idea that we can isolate certain distinguished local exchange moves between the independent sets of a matroid, which correspond with the covering relations of the external order (cf. Proposition 4.17). Due to the rooted circuit structure of antimatroids, and in particular by Lemma 4.8, similar exchange structure can be isolated for the independence complexes of arbitrary antimatroids.

**Question 2.** Are there other natural classes of simplicial complexes that can be represented as the independence complexes of suitable antimatroids? What exchange structure emerges from such representations, and what does this reveal about the structure of these complexes?

The independence complexes of antimatroids seem to be a rather broad class. One place to start could be to determine whether there exist simplicial complexes which cannot be realized as an antimatroid independence complex.

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