THE CHARACTERISTIC CYCLES AND SEMI-CANONICAL BASES ON TYPE A QUIVER VARIETY

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Abstract. In this article we study a conjecture of Geiss-Leclerc-Schröer, which is an analogue of a classical conjecture of Lusztig in the Weyl group case. It concerns the relation between canonical basis and semi-canonical basis through the characteristic cycles. We formulate an approach to this conjecture and prove it for type $A_2$ quiver. In general type A case, we reduce the conjecture to show that certain nearby cycles have vanishing Euler characteristic.

Contents

1. Introduction
2. Characteristic cycles
   2.1. Micro-support and characteristic cycle
   2.2. Constructible functions, Lagrangian cycles and characteristic cycles
3. Canonical Bases and Semi-canonical Bases
   3.1. Representation of quiver algebras and preprojective algebras
   3.2. Convolution products and canonical bases
4. Quiver of type A
   4.1. Constructible functions
   4.2. Characteristic cycles
   4.3. Compatibility with convolutions
   4.4. Inductions
   4.5. Further reduction of (7)
5. Quiver of type $A_2$
   5.1. Vanishing cycle
   5.2. Euler characteristic of Milnor fiber
   5.3. Application
6. Appendix
   6.1. Regularity of stratification
   6.2. A vanishing cycle calculation
References

1. Introduction

In [5], the authors studied the semi-canonical basis for the enveloping algebra $U(n)$ which was constructed by Lusztig in [10], here $n$ is the maximal nilpotent subalgebra of some symmetric Kac-Moody Lie algebra over $\mathbb{C}$. They raised the question about the relation between semi-canonical basis, canonical basis and the singular support (cf. [5], 1.5), referring to a conjecture made by Lusztig for the Weyl group algebra(cf. [9], 4.17).

In this paper, we consider the conjecture by Geiss-Leclere-Schröer mentioned above. We consider the quiver $(I,Q)$ of type $A$ with orientation $\Omega : i \rightarrow i+1$. The variety of quiver representations $E_{V,\Omega}$ in an $I$-graded vector space $V$ admits a stratification by an action of a reductive group $G_V$. For each orbit $S$, we can associate an

Key words and phrases. Quiver, characteristic cycles, canonical basis, semi-canonical basis, vanishing cycles, Milnor fiber.
pervasive sheaf $\text{IC}(\mathcal{S}, \mathbb{C})$. They give rise to a basis for $U(n)$ which is the canonical basis and we refer to as $g_S$ the corresponding element. Also by considering the union of conormal bundles over the orbits on $E_{V,\Omega}$

$$\Lambda_V := \bigcup_S T_S^* E_{V,\Omega},$$

whose irreducible components are naturally indexed by orbits, Lusztig constructed the semi-canonical basis for $U(n)$ and to which we refer as $\phi_S$. Let

$$g_S = \sum_{S'} m_{S',S} \phi_{S'}.$$

On the other hand, Kashiwara and Shapira have constructed a characteristic cycle $CC(F)$ for a constructible sheaf $F$ on a manifold (cf. [6]). And we have the following formula

$$CC(\text{IC}(\mathcal{S}, \mathbb{C})) = [T_S^* E_{V,\Omega}] + \sum_{S' \subseteq S} n_{S',S}[T_{S'}^* E_{V,\Omega}], \quad n_{S',S} \geq 0.$$

Furthermore, they constructed a morphism

$$E_u : L(T^* E_{V,\Omega}) \rightarrow M(E_{V,\Omega})$$

where $L(T^* E_{V,\Omega})$ denotes the group of Lagrangian cycles and $M(E_{V,\Omega})$ the space of constructible functions on $E_{V,\Omega}$. The relation between $g_S$ and $\text{IC}(\mathcal{S}, \mathbb{C})$ is as follows

$$E_u(CC(\text{IC}(\mathcal{S}, \mathbb{C}))) = (-1)^{\dim S} g_S.$$

The following two conjectures are equivalent.

**Conjecture 1.1.** $E_u([T_S^* E_{V,\Omega}]) = (-1)^{\dim S} \phi_S$.

**Conjecture 1.2.** $m_{S',S} = (-1)^{\dim S'-\dim S} n_{S',S}$.

In this paper we develop a strategy to approach this problem. First let us formulate the dual statement. Let $M(E_{V,\Omega})^{GV}$ be the space of $G_V$-invariant constructible functions on $E_{V,\Omega}$. Then Lusztig [8] showed that there is an algebra isomorphism

$$U(n) \cong \mathcal{M}_{\Omega} := \bigoplus_{V \in \mathcal{V}} M(E_{V,\Omega})^{GV},$$

where $\mathcal{V}$ is the set of isomorphism classes of $I$-graded vector spaces and the product on $\mathcal{M}_{\Omega}$ is given by convolution. So we can view the canonical and semi-canonical bases as elements in $\mathcal{M}_{\Omega}$. Let $M(\Lambda_V)^{GV}$ be the space of $G_V$-invariant constructible functions on $\Lambda_V$. The pullback along $E_{V,\Omega} \leftarrow \Lambda_V$ induces an algebra homomorphism

$$\Psi : \mathcal{M}_{\Pi} := \bigoplus_{V \in \mathcal{V}} M(\Lambda_V)^{GV} \rightarrow \mathcal{M}_{\Omega}$$

where the product on $\mathcal{M}_{\Pi}$ is also given by convolution. Lusztig [8] showed that this induces an isomorphism $\Psi_0$ on a subalgebra $\mathcal{M}_{\Pi}$ of $\mathcal{M}_{\Pi}$. Let $\mathcal{M}_{\Pi}(V) := \mathcal{M}_{\Pi} \cap M(\Lambda_V)^{GV}$. We have a diagram

$$\begin{array}{ccc}
\mathcal{M}_{\Pi}(V) & \xrightarrow{\cong} & M(\Lambda_V)^{GV} \\
\downarrow_{\Psi_0} & & \downarrow_{\Psi} \\
M(E_{V,\Omega})^{GV} & & \\
\end{array}$$

Lusztig [10] showed that there exists a basis $\{\tilde{\phi}_S\}$ of $\mathcal{M}_{\Pi}(V)$ parametrized by $G_V$-orbits $S$ in $E_{V,\Omega}$ satisfying

$$\tilde{\phi}_S(x, y) = \begin{cases} 
0 & \text{if } (x, y) \in \mathcal{O}_{S'} \text{ and } S' \neq S \\
1 & \text{if } (x, y) \in \mathcal{O}_S 
\end{cases}$$
where $\mathcal{O}_S$ is some open dense subset of $T^*_x E_{V,\Omega}$. By definition, $\phi_S = \Psi_0^{-1}(\phi)$. We define the dual semi-canonical basis to be $\rho_S(\phi) := \Psi_0^{-1}(\phi)$. Let $K_{G_V}(E_{V,\Omega})$ be the Grothendieck group of $G_V$-equivariant perverse sheaves on $E_{V,\Omega}$. The local Euler characteristic gives an isomorphism

$$\chi : K_{G_V}(E_{V,\Omega}) \otimes \mathbb{Z} \mathbb{C} \cong M(E_{V,\Omega})^{G_V}, \quad \mathcal{F} \mapsto \phi_{\mathcal{F}}$$

where $\phi_{\mathcal{F}}(x) = \chi(\mathcal{F}_x)$. Define $\chi_{S}^{\text{mic}}(\phi_{\mathcal{F}}) := m_S(C(C(\mathcal{F})))$, the multiplicity of $[T^*_x E_{V,\Omega}]$ in $CC(\mathcal{F})$. Then Conjecture 1.2 is equivalent to the following dual statement.

**Conjecture 1.3.** $(-1)^{\dim S} \chi_{S}^{\text{mic}} = \rho_S$.

In order to approach this conjecture, we define a section of $\Psi_0$

$$\eta_V : M(E_{V,\Omega})^{G_V} \longrightarrow M(\Lambda V)^{G_V}$$

by $\eta_V(\phi)(x, y) := \chi(\Phi f_y[-1](\mathcal{F}))$, where $f_y : E_{V,\Omega} \rightarrow \mathbb{C}$ is the linear functional defined by $y$. This map has been introduced in [3] in a more general setting. The link with characteristic cycles is as follows.

**Proposition 1.4.** (cf. Proposition [1.2]) For $\mathcal{F} \in D_{G_V}(E_{V,\Omega})$ and $(x, y) \in (T^*_x E_{V,\Omega})_{\text{reg}}$,

$$\eta_V(\phi_{\mathcal{F}})(x, y) = (-1)^{\dim \Lambda V - \dim \tilde{S}} m_S(C(C(\mathcal{F}))).$$

where $\tilde{S}$ is the dual orbit of $S$.

As a consequence, Conjecture 1.3 is equivalent to

$$\Psi_0^{-1}(\phi)|_{\mathcal{O}_S} = (-1)^{\dim \Lambda V - \dim \tilde{S} - \dim S} \eta_V(\phi)|_{\mathcal{O}_S}$$

for all $\phi \in M(E_{V,\Omega})^{G_V}$. Indeed, it is possible to show

$$\dim \Lambda V - \dim \tilde{S} - \dim S \equiv 0 \mod 2$$

from the fact that $IC(\tilde{S}, \mathbb{C})^\vee = IC(S, \mathbb{C})$, where $(\cdot)^\vee$ is the Fourier-Sato transform. We will not include the argument here, since it is not our main focus. Now we can state our main result.

**Theorem 1.5.** For type $A_2$ quiver, $\Psi_0^{-1} = \eta_V$.

Conjecture 1.3 for type $A_2$ quiver follows from this theorem and (2). We shall point out that Conjecture 1.3 in this case also follows from the known results $(-1)^{\dim S} \chi_{S}^{\text{mic}} = g^*_S \ recipient[12]$ and $\rho_S = g^*_S \ recipient[5]$, where $g^*_S$ is the dual canonical basis. Nevertheless, the purpose of this paper is to develop a strategy for studying Conjecture 1.3 in all cases. We plan to apply our strategy to study this conjecture for some special orbits in the future.

The paper is organized as follows. In section 2 we review the notion of characteristic cycles and in section 3 we review the classical work of Lusztig on canonical basis and semi-canonical basis. Both sections contain no new results and we mainly follow Lusztig’s notations. In §4.1 we introduce the map $\eta_V$ and show its image are constructible functions. The ideas are from [3]. In §4.2 we show Proposition 1.6. Our main tool is the stratified Morse theory (cf. [14]). Note that in order to apply the results of [14], some Whitney type regularity condition is required and is verified in the appendix. In §4.3 we show that the equality $\eta_V = \Psi_0^{-1}$ is equivalent to the compatibility of $\eta_V$ with convolution (cf. Proposition 4.12). In §4.4 and §4.5 we reduce it further to a problem of vanishing cycle calculation.

**Conjecture 1.6.** (Cf. Conjecture [4.2]) $\chi(\Phi h_{\Psi_0}[-1](\mathcal{F}))(1, x_0) = 1$.

Finally, in §5 we show that the above conjecture is true for type $A_2$ quiver. We prove this result by showing the relevant nearby cycle have Euler characteristic 0. We should remark that even in this case, the singular locus of $h_{\Psi_0}$ can be very complicated, and it could involve singular irreducible components of various dimensions. We show that the relevant nearby cycle have Euler characteristic 0 by constructing certain fiber of the Milnor fiber over some compact space and showing the fibers all have Euler characteristic 0. Then the result follows from the Leray spectral sequence.

**Acknowledgement.** The project was discussed when both authors were in Max Planck Institute for Mathematics of Bonn and started when both were in Yau Mathematical Sciences Center of Tsinghua University. They
would like to thank both institute for their excellent working environment. This second author is supported by Tsinghua University Initiative Scientific Research Program No. 2019Z07L02016.

2. Characteristic cycles

In this section we review some generalities on characteristic cycles, our main reference is [6]. Nothing is new in this section.

2.1. Micro-support and characteristic cycle. To introduce the micro-support of a $\mathbb{C}^\times$-sheaf on a manifold, we follow [9] section 8.6 to give a definition using vanishing cycles.

Consider $X$ to be a complex manifold with a holomorphic function $f: X \to \mathbb{C}$, moreover, we assume that $Y = f^{-1}(0)$ is non-singular. Also, let $p: \mathbb{C} \to \mathbb{C}$ be the function $p(z) = \exp(2\pi \sqrt{-1}z)$, considered as the universal covering map of $\mathbb{C}^\times$. Finally, let $\overline{p}: \overline{X} \to X$ be the pullback of $p$ along $f$.

Definition 2.1. Let $\mathcal{F} \in D^b(X)$. We define

$$R\Psi_f(\mathcal{F}) = i^*R\overline{p}_*\overline{p}^*(\mathcal{F})$$

where $i: Y \to X$, and call it the nearby-cycle functor.

We also need to consider the vanishing cycles, which is

Definition 2.2. Let $R\Phi_f(\mathcal{F}) \in D^b(\mathcal{F})$ be the unique element such that we have the following distinguished triangles

$$i^*(\mathcal{F}) \to R\Psi_f(\mathcal{F}) \to R\Phi_f(\mathcal{F}) \xrightarrow{+1}.$$ 

Now we can define the micro-support of a constructible sheaves

Definition 2.3. Let $D^b_c(X)$ be the full subcategory of $D^b(X)$ consisting elements with bounded constructible cohomology sheaves. Let $p \in T^*X$ and $\mathcal{F} \in D^b_c(X)$, then we define a subset $SS(\mathcal{F}) \subseteq T^*(X)$ by the following

1: $p \notin SS(\mathcal{F})$.

2: There exists some open neighborhood $U$ of $p$ such that for any $x \in X$ and any holomorphic function $f: W \to \mathbb{C}$ defined in a neighborhood $W \subseteq X$ of $x$ with $f(x) = 0$ and $df(x) \in U$, one gets $R\Phi_f(\mathcal{F})_x = 0$.

Remark: Note that such a definition works well for varieties over other field. More precisely, Beilinson [2] has constructed micro-support for arbitrary base field, and Saito [13] constructed characteristic for varieties over finite field.

Finally, we want to assign a multiplicity to each irreducible component of $SS(\mathcal{F})$ in a functorial way. We do not give the exact definition but just list some of the properties shared by characteristic cycles.

Proposition 2.4. Let $X$ and $Y$ be complex manifolds, and $\mathcal{F} \in D^b_c(X), \mathcal{G} \in D^b_c(Y)$. We have

1: $CC(\mathcal{F} \boxtimes \mathcal{G}) = CC(\mathcal{F}) \boxtimes CC(\mathcal{G})$.

2: $CC(D_X(\mathcal{F})) = CC(\mathcal{F})$, where $D_X$ is the Verdier dual.

3: Let $F' \to F \to F'' \xrightarrow{+1}$ be a distinguished triangle in $D^b_c(X)$, then $CC(\mathcal{F}) = CC(\mathcal{F}') + CC(\mathcal{F}'')$.

4: Assume that $\mathcal{F}$ is a local system, then we have

$$CC(\mathcal{F}) = (-1)^{\dim(X)} \text{rank}(\mathcal{F})[T^*_XX].$$

5: We have $\text{supp}(CC(\mathcal{F})) = SS(\mathcal{F})$.

6: (Milnor type formula) Let $x \in U \subseteq X$ be an open subset and $f: U \to \mathbb{C}$ be holomorphic. Assume that the section $C_f = (y, df(y))$ intersects $SS(F)$ transversally, then we have

$$-\chi(R\Phi_f(\mathcal{F}|_U)_x) = (CC(\mathcal{F}), C_f)_{T^*U,x}$$

7: Let $\mathcal{F}$ be perverse, then $CC(\mathcal{F}) \geq 0$. 

4 TAIWANG DENG AND BIN XU
Proof. (1) is (9.4.1) in [6], and (2) is proved in Proposition 9.4.4 in loc.cit. and (3) is proved in Proposition 9.4.5 in loc.cit. Note that in (2) our formula differs from that of [6] by an antipodal pullback since we are working with complex varieties. For (4), we refer to lemma 4.11 of [13], and for (5) and (6), see Theorem 4.9 and Proposition 4.14 [13]. Again, we note that the characteristic cycle in [6] differs from ours by a sign since we require 

\[ CC(F) \geq 0 \]

for perverse sheaf \( F \), following [13]. Finally, (7) follows from Proposition 5.14 of [13]. □

2.2. Constructible functions, Lagrangian cycles and characteristic cycles. We introduce the following set of constructible functions on complex varieties.

Definition 2.5. A function \( \phi : X \to \mathbb{C} \) is constructible if \( f(X) \) is finite and \( f^{-1}(c) \) is a constructible subset of \( X \) in Zarisky for any \( c \in \mathbb{C} \). The set of constructible functions on \( X \) is denoted by \( M(X) \).

Remark: Our definition of constructible functions is more restrictive than that of [6].

Notation 2.6. Let \( K(X) \) be the Grothendieck group of \( D^b_c(X) \), i.e, the full subcategory of \( D^b(X) \) consisting of elements with bounded constructible \( \mathbb{C} \)-coefficients cohomology sheaves.

Definition 2.7. Let \( L(X) \) be the free abelian group generated by complex Lagrangian sub-varieties of \( X \).

Remark: Naturally we have \( CC(F) \in L(T^*X) \) for any \( F \in K(X) \).

Definition 2.8. We have group morphisms

\[
\chi : K(X) \to M(X), \quad F \mapsto \chi(F), \quad \chi(F)(x) = \chi(F_x)
\]

and

\[
CC : K(X) \to L(T^*X), \quad F \mapsto CC(F)
\]

Theorem 2.9. The morphisms \( \chi \) and \( CC \) are isomorphisms.

Proof. Cf. [6] theorem 9.7.1, 9.7.10]. Note that regardless of the modification we made on the relevant objects, the proof is exactly the same. □

Following [6], we define an Euler morphism \( Eu \) from \( L(T^*X) \) to \( M(X) \) as follows

Definition 2.10. Let \( x \in X, U \subseteq X \) a neighborhood of \( x \) and \( \phi : U \to \mathbb{R} \) satisfying \( \phi(x) = 0, d\phi(x) = 0 \) and the Hessian of \( \phi \) at \( x \) is positive definite. Let \( \lambda \in L(T^*X) \), then we put

\[
Eu(\lambda)(x) = \sharp(C_\phi \cap \lambda)_x
\]

where \( C_\phi = \{ (y, d\phi(y)) | y \in U \} \).

Remark: In loc.cit, it is shown to be well defined(cf. (9.7.26)).

We are ready to state the following

Theorem 2.11. (cf. [6] Theorem 9.7.11) The diagram:

\[
\begin{array}{ccc}
K(X) & & M(X) \\
\downarrow CC & & \downarrow \chi \\
L(T^*X) & \xrightarrow{Eu} & M(X)
\end{array}
\]

is commutative, and the arrows are isomorphic.

3. Canonical Bases and Semi-canonical Bases

In this section we recall the classical construction of canonical basis and semi-canonical basis due to Lusztig. We only state the relevant facts in case of simply laced type quivers, we refer to [10], [8] for a detailed discussion.
3.1. **Representation of quiver algebras and preprojective algebras.** Let $Q = (I, H, s, e)$ be a finite quiver without loops, where

- (1): $I$ is the finite set of vertices;
- (2): $H$ is the finite set of edges with direction;
- (3): $s(\text{resp. } e) : H \to I$ sends an arrow to its starting point (resp. end points);
- (4): There is an involution $H \to H$ satisfying $e(\overline{h}) = s(h), s(\overline{h}) = e(h)$.

Let $\Omega \subseteq H$ be an orientation, i.e., $\Omega \cup \overline{\Omega} = H, \Omega \cap \overline{\Omega} = \emptyset$. For $i \in I$, set

$$r_i = \sum_{h \in \Omega : s(h) = i} \overline{h}h - \sum_{\alpha \in \Omega : e(h) = i} h\overline{h}$$

**Notation 3.1.** We denote by $(Q, \Omega)$ by the sub-quiver generated by $\Omega$.

**Definition 3.2.** Let $H(Q, \Omega) = \mathbb{C}\Omega$ be the quiver algebra generated by $\Omega$ and

$$\Pi(Q) = \mathbb{C}H/J$$

where $J$ is the ideal generated by the elements $r_i$. We call $\Pi(Q)$ the preprojective algebra associated to $Q$.

**Notation 3.3.** Let $V = \bigoplus_{i \in I} V_i$ be a $I$-graded vector spaces. Let

$$|V| := (\dim(V_i))_{i \in I}$$

be its dimension vector.

**Definition 3.4.** Consider the following variety

$$E_{V, \Omega} = \{(x_h)_{h \in \Omega} : x_h \in \text{Hom}(V_{s(h)}, V_{e(h)})\}$$

viewed as the representation variety of $H(Q, \Omega)$ with underlying space $V$.

**Definition 3.5.** A representation of the preprojective algebra $\Pi(Q)$ on $V$ is to give $(x_h)_{h \in H}$ subjecting to the relation

$$\sum_{\alpha \in \Omega : s(h) = i} x_\alpha x_h - \sum_{h \in \Omega : e(h) = i} x_h x_\overline{h} = 0.$$ 

Let $p = h_1h_2 \cdots h_t$ be a path in $H$, set

$$x_p = x_{h_1}x_{h_2} \cdots x_{h_t}$$

We say that the representation is nilpotent if $x_p = 0$ for any path $p$ of length greater than some $N \in \mathbb{N}$. Let $\Lambda_V$ be the set of nilpotent representations on $V$.

**Remark:** Note that if $V^i$ is an $I$-graded vector space with $|V^i| = (\delta_{i,j})_{j \in I}$ (here $\delta$ is the Kronecker symbol), then $\Lambda_V$ consists of one single point and we denote by $Z_i$ the corresponding representation. We also note that the nilpotency condition is equivalent to requiring the representation admits a composition series consists of only simple modules isomorphic to $Z_i$ for $i \in I$.

We recall some basic results concerning the algebra $\Pi(Q)$.

**Proposition 3.6.** (cf. [5] Proposition 3.1) The following are equivalent

(a) The algebra $\Pi(Q)$ is finite dimensional.
(b) Every finite dimensional representation of $\Pi(Q)$ is nilpotent.
(c) $(Q, \Omega)$ is a Dynkin quiver.
3.2. Convolution products and canonical bases. We recall the construction of canonical bases through convolution products.

**Definition 3.7.** Let $X$ be a complex variety and $f : X \to \mathbb{C}$ be a constructible function. We define
\[
\int_{x \in X} f(x) = \sum_{c \in \mathbb{C}} c \chi(f^{-1}(c))
\]
where $\chi$ is the Euler characteristic with compact support.

**Notation 3.8.** Let $G V = \prod_{i \in I} GL(V_i)$ be the automorphism group of $V$, which actions on $E V, V$ by conjugation.

**Definition 3.9.** Let $M(E V, \Omega)^{G V}$ be the set of $G V$-invariant constructible functions on $E V$. Similarly one can define $M(\Lambda V)^{G V}$.

**Definition 3.10.** Let $V, V', V''$ be $I$-graded vector spaces such that $|V| = |V'| + |V''|$ Then we have a bilinear map
\[
* : M(E V', \Omega)^{G V''} \times M(E V'', \Omega)^{G V''} \to M(E V, \Omega)^{G V}
\]
by
\[
(\phi' \ast \phi'')(x) = \int_{y \subseteq x} \phi'(y) \phi''(x/y), \quad x \in E V
\]
where $y$ runs through all the sub-representations of $x$ such that the underlying vector space is isomorphic to $V'$.

**Definition 3.11.** Let
\[
\Omega = \bigoplus_{V \in V} M(E V, \Omega)^{G V}, \quad \Pi = \bigoplus_{V \in V} M(\Lambda V)^{G V}.
\]
where $V$ is the set of isomorphism classes of $I$-graded vector spaces.

**Proposition 3.12.** The vector spaces $\Omega$ and $\Pi$ with the convolution product $*$ are unital associative algebras.

**Proof.** We refer to [5] section 5.4 and [8] section 10.19. \hfill \Box

**Definition 3.13.** Let $M(1) = \Omega$ and $M(\Pi) = \Pi$ be the sub-algebra of $\Omega$ and $\Pi$ respectively.

**Proposition 3.14.** [7, Proposition 9.8] If $Q$ is simply-laced, then $\Omega = \Pi$.

**Notation 3.15.** Let $g$ be a symmetric Kac-Moody algebra and $n$ be the maximal nilpotent Lie subalgebra. Let $Q$ be the associated quiver. Also, let $U(n)$ be the enveloping algebra of $n$.

**Theorem 3.16.** We have isomorphisms of algebras
\[
\Psi : U(n) \to M(1), \quad \Phi : U(n) \to M(\Pi)
\]
with
\[
\Psi(e_i) = 1_{S_i}, \quad \Phi(e_i) = 1_{Z_i},
\]
where $e_i, i \in I$ is a set of Chevalley generators for $U(n)$.

**Proof.** For $\Psi$, we refer to [8] Proposition 10.20, and for $\Phi$, we refer to [10]. \hfill \Box

We give another description of the map $\Psi$ in terms of the quantum enveloping algebras, which is also due to Lusztig. We briefly recall the construction.
Notation 3.17. For each $V$, Lusztig defined a subset of perverse sheaves $P_O(V)$. Let $K_v(\Omega, V)$ be the $\mathbb{Z}[v^\pm]$-modules generated by elements of $P_O(V)$. Moreover, he defined the following convolution products

\[ * : K_v(\Omega, V') \times K_v(\Omega, V'') \to K_v(\Omega, V) \]

for $I$-graded vector spaces $V', V''$ such that $|V| = |V'| + |V''|$. Finally, let

\[ K_v(\Omega) = \bigoplus_{V \in \mathcal{V}} K_v(\Omega, V) \]

be the resulted unital associative algebra.

Theorem 3.18. We have isomorphism of algebras

\[ \Psi_v : U_v(n) \to K_v(\Omega) \otimes_{\mathbb{Z}[v^\pm]} \mathbb{Q}(v), \quad \Psi(E_i) = 1_{S_i} \]

where $E_i, i \in I$ is a set of Chevalley generators for the quantized algebras $U_v(n)$ and $1_{S_i}$ is the constant sheaf on the variety corresponding to the one dimensional representation $S_i$.

Proof. Cf. [8], §10.17.

Remark: By letting $v = 1$, we recover the previous map $\Psi$ by identifying $E_i$ to $e_i$ (cf. [8], §10.20).

Definition 3.19. Let $U_{v,Z}(n) = \Psi_v^{-1}(K_v(\Omega))$

Let $\text{Irr}(\Lambda_V)$ be the set of irreducible components of $\Lambda_V$.

Definition 3.20. Following Lusztig, we define for each graded vector space $V$ a $\mathbb{C}$-basis

\[ \{ \phi_Z | Z \in \text{Irr}(\Lambda_V) \} \]

of $\mathcal{M}_\Omega$. The function $\phi_Z$ is uniquely characterized by the fact that it is equal to 1 on a dense open subset of $Z$ and equal to 0 on a dense open subset of any other irreducible component $Z'$ of $\Lambda_V$ (cf. [10], Lemma 2.5).

4. Quiver of Type A

Let $Q = (I, H, s, e)$ be a quiver of type $A$. Let $I = \{1, 2, \cdots, r\}$ and $\Omega$ be the orientation $i \to i + 1$. Let $V = \bigoplus_{i \in I} V_i$ be an $I$-graded vector space.

\[ E_{V,\Omega} = \bigoplus_{1 \leq i < r} \text{Hom}(V_i, V_{i+1}), \quad G_V = \prod_{1 \leq i \leq r} GL(V_i) \]

Let $D_{G_V}(E_{V,\Omega})$ be the $G_V$-equivariant derived category of constructible complexes on $E_{V,\Omega}$ and $K_{G_V}(E_{V,\Omega})$ be the corresponding Grothendieck group. Then the local Euler characteristic gives an isomorphism

\[ \chi : K_{G_V}(E_{V,\Omega}) \otimes_\mathbb{Z} \mathbb{C} \cong M(E_{V,\Omega})^{G_V}, \quad \mathcal{F} \leftrightarrow \phi_\mathcal{F} \]

where $\phi_\mathcal{F}(x) = \chi(\mathcal{F}_x)$. Let $\bar{\Omega}$ be the opposite orientation, and

\[ E_{V,\bar{\Omega}} = \bigoplus_{1 \leq i < r} \text{Hom}(V_{i+1}, V_i) \]

Let

\[ E_V = E_{V,\Omega} \oplus E_{V,\bar{\Omega}} \hookrightarrow \text{End}(V), \quad G_V \hookrightarrow GL(V) \]

We define a $GL(V)$-invariant nondegenerate bilinear form on $\text{End}(V)$ by the trace

\[ \langle , \rangle : \text{End}(V) \times \text{End}(V) \to \mathbb{C}, \quad \langle x, y \rangle = \text{tr}(xy) \]

It induces a $G_V$-invariant nondegenerate bilinear form on $\mathfrak{g}_V := \text{Lie}(G_V)$, and a $G_V$-invariant nondegenerate pairing

\[ \langle , \rangle : E_{V,\Omega} \times E_{V,\Omega} \to \mathbb{C} \]

Under this pairing, we can identify

\[ E_{V,\Omega} \cong E_{V,\Omega}^* \text{ and } T^*E_{V,\Omega} \cong E_V \cong T^*E_{V,\Omega}. \]
On $\text{End}(V)$, we have the Lie bracket $[x, y] := xy - yx$ and
$$\Lambda_V = \{(x, y) \in E_V \mid [x, y] = 0\}.$$We decompose $\Lambda_V$ into irreducible components,
$$\Lambda_V \cong \bigcup_S T^*_S E_{V, \Omega} \cong \bigcup_C T^*_C E_{V, \Omega},$$which are closures of conormal bundles over orbits on $E_{V, \Omega}$ and $E_{V, \bar{\Omega}}$ respectively. For any orbit $S \subseteq E_{V, \Omega}$, we define the dual orbit $\hat{S} \subseteq E_{V, \bar{\Omega}}$ by the condition that
$$T^*_S E_{V, \Omega} \cong T^*_S E_{V, \bar{\Omega}}.$$We also define
$$(T^*_S E_{V, \Omega})_{\text{reg}} := T^*_S E_{V, \Omega} \setminus \cup_{S \neq S'} T^*_S E_{V, \Omega}.$$Then it is easy to see that
$$(T^*_S E_{V, \Omega})_{\text{reg}} \subseteq S \times \hat{S}.$$4.1. Constructible functions. We will define a map
$$\eta_V : M(E_{V, \Omega})^G \to M(\Lambda_V)^G,$$which has been introduced in [3] in a more general setting.
**Definition 4.1.** For any $(x, y) \in \Lambda_V$, $\eta_V(\phi_F)(x, y) = \chi(R\Phi f_y[-1](F)_x)$, where $f_y : E_{V, \Omega} \to \mathbb{C}$ is defined by $f_y(z) = \langle z, y \rangle$.
Next we show the image of $\eta_V$ lies in $M(\Lambda_V)^G$.

**Proposition 4.2.** For $F \in D_{G_V}(E_{V, \Omega})$, $\eta_V(\phi_F) \in M(\Lambda_V)^G$.

To prove this, we will give another description of $\eta_V$ following [3]. Let $S \subseteq E_{V, \Omega}$ be any orbit and $\hat{S} \subseteq E_{V, \bar{\Omega}}$ be its dual. We would like to define $\eta_V(\phi_F)$ on each $T^*_S E_{V, \Omega}$ as follows.

$$\begin{align*}
E_{V, \Omega} \times \hat{S} &\longrightarrow E_{V, \Omega} \times E_{V, \Omega} \\
\pi &\downarrow \quad \quad \quad \downarrow (,)
E_{V, \Omega} &\quad \quad \quad \quad \mathbb{C}
\end{align*}$$

Note
$$T^*_S E_{V, \bar{\Omega}} \subseteq \langle , \rangle^{-1}(0).$$
Denote the restriction of $\langle , \rangle$ to $E_{V, \Omega} \times \hat{S}$ by $f_S$.

**Lemma 4.3.** For $F \in D_{G_V}(E_{V, \Omega})$ and $(x, y) \in T^*_S E_{V, \Omega}$,
$$(R\Phi f_S[-1](\pi^* F))_{(x,y)} \cong (R\Phi f_y[-1](F))_x.$$**Proof.** Let $Z_{G_V}(y)$ be the stabilizer of $y$ in $G_V$. We have an isomorphism
$$G_V \times Z_{G_V}(y) E_{V, \Omega} \cong E_{V, \Omega} \times \hat{S}, \quad (g, z) \mapsto (gz, gy).$$The inclusion
$$E_{V, \Omega} \to G_V \times Z_{G_V}(y) E_{V, \Omega}, \quad z \mapsto (1, z)$$gives a section of $\pi$
$$i : E_{V, \Omega} \to E_{V, \Omega} \times \hat{S}, \quad z \mapsto (z, y).$$The pullback along the inclusion induces an equivalence of categories
$$D_{G_V}(G_V \times Z_{G_V}(y) E_{V, \Omega}) \cong D_{Z_{G_V}(y)}(E_{V, \Omega}).$$
Since $\langle \cdot, \cdot \rangle$ is $G_V$-invariant, for any $G \in D_{G_V}(E_V, \Omega) \cong D_{G_V}(G \times Z_{G_V}(y) E_V, \Omega)$, we get
\[ R\Phi_f(y)(i^*G) \cong i^*R\Phi_{(y)}(G). \]

Let $G = \pi^*F$, then $i^*G = F$. So
\[ R\Phi_f(F) \cong i^*R\Phi_{fS}(\pi^*F). \]

In particular,
\[ R\Phi_f(F)_x \cong i^*(R\Phi_{fS}(\pi^*F))_x \cong R\Phi_{fS}(\pi^*F)(x,y). \]

\[ \square \]

**Remark 4.4.** In the lemma, we have used the following general fact. Suppose $H$ is a closed subgroup of $G$ and $X$ is an $H$-space. Then the inclusion
\[ i : X \hookrightarrow G \times_H X, \quad x \mapsto (1, x) \]
induces an equivalence of categories
\[ D_G(G \times_H X) \cong D_H(X), \quad G \mapsto i^*G \]
Let $f : G \times_H X \to \mathbb{C}$ be a $G$-invariant continuous function. Then we have base change
\[ R\Phi_{fG}(i^*G) \cong i^*R\Phi_f(G), \]
for any $G \in D_G(G \times_H X)$.

**Corollary 4.5.** $\eta^0(\phi_F)|_{T^*_S(E_V, \Omega)} = \chi(R\Phi_{fS}[-1](\pi^*F))|_{T^*_S(E_V, \Omega)}$.

In particular, $\eta^0(\phi_F)$ is constructible on $T^*_S(E_V, \Omega)$. Since
\[ \Lambda_V = \bigsqcup_S T^*_S(E_V, \Omega), \]
we see $\eta^0(\phi_F) \in M(\Lambda_V)^{G_V}$.

4.2. **Characteristic cycles.** For $F \in D_{G_V}(E_V, \Omega)$, let $m_S(CC(F))$ be the multiplicity of $[T^*_S(E_V, \Omega)]$ in $CC(F)$. Let $f$ be the restriction of $\langle \cdot, \cdot \rangle$ to $S \times \hat{S}$. The goal of this subsection is to prove the following proposition.

**Proposition 4.6.** For $F \in D_{G_V}(E_V, \Omega)$ and $(x, y) \in (T^*_S(E_V, \Omega))_{\text{reg}}$,
\[ \eta^0(\phi_F)(x, y) = (-1)^{\text{dim} \Lambda_V - \text{dim} \hat{S}} m_S(CC(F)). \]

The proof will occupy the whole section. Recall $\eta^0(\phi_F)(x, y) = \chi(R\Phi_{fS}[-1](F)_x)$. By [14 Lemma 1.3.2],
\[ (R\Phi_{fS}[-1](F)_x) \cong (R\Gamma_{re(f_y) > 0}(F)_x) \]
In terms of stratified Morse theory, the right hand side is called the local Morse data, denoted by LMD($L, re(f_y), x$). We have the following splitting formula for the local Morse data.

**Theorem 4.7.** For $F \in D_{G_V}(E_V, \Omega)$ and $(x, y) \in (T^*_S(E_V, \Omega))_{\text{reg}}$,
\[ \text{LMD}(F, re(f_y), x) \cong \text{TMD}(F, re(f_y), x) \otimes \hat{S} \text{NMD}(F, re(f_y), x) \]
with
\[ \text{TMD}(F, re(f_y), x) := (R\Gamma_{re(f_y) > 0}(1_S))_x \]
the tangential Morse data, and
\[ \text{NMD}(F, re(f_y), x) := (R\Gamma_{re(f_y) > 0}(F|_{NS}))_x \]
the normal Morse data with respect to a normal slice $NS \subseteq E_V, \Omega$ to $S$ at $x$.

**Proof.** It follows from [14 Theorem 5.3.3], which has some regularity condition on the stratification. We will verify this condition for our case in the appendix. \[ \square \]

As a direct consequence, we have

**Corollary 4.8.** $\chi(\text{LMD}(F, re(f_y), x)) = \chi(\text{TMD}(F, re(f_y), x)) \cdot \chi(\text{NMD}(F, re(f_y), x))$. 
It is the normal Morse data that relates to the characteristic cycle, namely
\((-1)^{\dim_S} \chi(\text{NMD}(\mathcal{F}, re(f_y), x)) = m_S(\text{CC}(\mathcal{F})).\)
This differs from [14] (5.21) by a sign \((-1)^{\dim_S}\), which makes \(m_S(\text{CC}(\mathcal{F}))\) positive whenever \(\mathcal{F}\) is perverse. By [14] Lemma 1.3.2,
\[
\begin{align*}
\text{TMD}(\mathcal{F}, re(f_y), x) & \cong R\Phi_f[-1](1_S)_x, \\
\text{NMD}(\mathcal{F}, re(f_y), x) & \cong R\Phi_f[-1](\mathcal{F}|_{NS})_x.
\end{align*}
\]
So it remains to determine \(\chi(\text{TMD}(\mathcal{F}, re(f_y), x))\). Instead of computing it directly, we shall apply the splitting formula to the other vanishing cycle \(R\Phi_f[-1](\pi^*\mathcal{F})_{(x,y)}\) in (4). By [14] Lemma 1.3.2 again
\[
(R\Phi_f[-1](\pi^*\mathcal{F}))_{(x,y)} \cong (R\Gamma_{re(f_S)\geq 0}(\pi^*\mathcal{F}))_{(x,y)} = \text{LMD}(\pi^*\mathcal{F}, re(f_S), (x,y)).
\]
The stratification of \(E_{V, \Omega}\) by \(G_{V}\)-orbits induces a stratification of \(E_{V, \Omega} \times \widehat{S}\), which satisfies the same condition on regularity. Note \((x,y) \in S \times \widehat{S}\).

**Lemma 4.9.** \(d(f_S)|_{(x,y)} = (\pi^* df_y)|_{(x,y)}\).

*Proof.* For \((v, w) \in T_x(E_{V, \Omega}) \oplus T_y(\widehat{S}),\) let us choose curves \(x(t), y(t)\) on \(E_{V, \Omega}\) and \(\widehat{S}\) respectively such that \(x(0) = x, x'(0) = v\) and \(y(0) = y, y'(0) = w\).

We compute the image of \((v, 0)\) and \((0, w)\) separately under \(d(f_S)_{(x,y)}:\)
\[
\begin{align*}
(v, 0) & \mapsto \frac{d(x(t), y)}{t}|_{t=0} = (v, y), \\
(0, w) & \mapsto \frac{d(x(y(t))}{t}|_{t=0} = (x, w) = 0
\end{align*}
\]
where the last equality follows from (3). This finishes the proof.

Since \((x, y) \in (T^*_SE_{V, \Omega})_{\text{reg}},\) then \(d(f_S)|_{(x,y)} \in T^*_S(E_{V, \Omega} \times \widehat{S})_{\text{reg}}\). So we can apply the splitting formula again.

**Theorem 4.10.** For \(\mathcal{F} \in D_{G_{V}}(E_{V, \Omega})\) and \((x, y) \in (T^*_S E_{V, \Omega})_{\text{reg}},\)
\[
(6) \quad \text{LMD}(\pi^*\mathcal{F}, re(f_S), (x,y)) \cong \text{TMD}(\pi^*\mathcal{F}, re(f_S), (x,y)) \otimes C \text{NMD}(\pi^*\mathcal{F}, re(f_S), (x,y)).
\]
where
\[
\begin{align*}
\text{TMD}(\pi^*\mathcal{F}, re(f_S), (x,y)) & = (R\Gamma_{re(f)\geq 0}(1_{S \times \widehat{S}}))_{(x,y)}, \\
\text{NMD}(\pi^*\mathcal{F}, re(f_S), (x,y)) & = (R\Gamma_{re(f|_{NS \times \{y\}})\geq 0}(\pi^*\mathcal{F}|_{NS \times \{y\}}))_{(x,y)}.
\end{align*}
\]

*Proof.* It follows from [14] Theorem 5.3.3.

By [14] Lemma 1.3.2 again,
\[
\begin{align*}
\text{TMD}(\pi^*\mathcal{F}, re(f_S), (x,y)) & \cong R\Phi_f[-1](1_{S \times \widehat{S}})_{(x,y)}, \\
\text{NMD}(\pi^*\mathcal{F}, re(f_S), (x,y)) & \cong R\Phi_{f|_{NS \times \{y\}}[-1]}(\pi^*\mathcal{F}|_{NS \times \{y\}})_{(x,y)}.
\end{align*}
\]
By the natural isomorphism \(NS \cong NS \times \{y\},\) we have
\[
R\Phi_{f|_{NS \times \{y\}}}((\pi^*\mathcal{F}|_{NS \times \{y\}})_{(x,y)} \cong R\Phi_{f|_{NS}}(\mathcal{F}|_{NS})_{x}.
\]
Hence,
\[
\text{NMD}(\pi^*\mathcal{F}, re(f_S), (x,y)) \cong \text{NMD}(\mathcal{F}, re(f_y), x).
\]
So it suffices to compute \(\text{TMD}(\pi^*\mathcal{F}, re(f_S), (x,y))\), equivalently \(R\Phi_f[-1](1_{S \times \widehat{S}})_{(x,y)}\).

**Proposition 4.11.** \(R\Phi_f[-1](1_{S \times \widehat{S}})_{(x,y)} = C[\dim\Lambda_V - \dim\widehat{S} - \dim S].\)

This proposition is a special case of [3] Theorem 6.7.5. For the convenience of the reader, we will reproduce its proof in the appendix.
4.3. Compatibility with convolutions. Starting from this subsection, we will investigate when \( \eta_V = \Psi_0^{-1} \).

**Proposition 4.12.** The following statements are equivalent.

1. \( \eta_V = \Psi_0^{-1} \);
2. \( \text{Im} \eta_V \subseteq \mathcal{M}_\Pi \);
3. For any decomposition of \( I \)-graded vector spaces \( V = V^1 \oplus V^2 \),

\[
\eta_{V^1}(\phi_1) \ast \eta_{V^2}(\phi_2) = \eta_V(\phi_1 \ast \phi_2)
\]

for any \( \phi_1 \in M(E_{V^1,\Omega})^{G_{V^1}} \) and \( \phi_2 \in M(E_{V^2,\Omega})^{G_{V^2}} \).

**Proof.** It is clear that (1) implies (2). Since \( \Psi \circ \eta_V = \text{id} \), then (2) implies (1) and (3). It follows from [Proposition 7.3](#) that (3) implies (2).

We begin by recalling the definitions of the two convolutions in (7). Consider the following diagram

\[
\begin{array}{ccc}
E'_{V^1,V^2,\Omega} & \xrightarrow{p_2} & E''_{V^1,V^2,\Omega} \\
\downarrow p_1 & & \downarrow p_3 \\
E_{V^1,\Omega} \times E_{V^2,\Omega} & & E_{V,\Omega}
\end{array}
\]

where

\[
E'_{V^1,V^2,\Omega} := \{(x, \text{Fil}) | x \in E_{V,\Omega}, \text{Fil} : 0 = W^0 \subseteq W^1 \subseteq W^2 = V \text{ \text{x-stable with } } |W^k/W^{k-1}| = |V^k| \text{ for } k = 1, 2\},
\]

\[
E''_{V^1,V^2,\Omega} := \{(x, \text{Fil}, \varphi_1, \varphi_2) | (x, \text{Fil}) \in E'_{V^1,V^2,\Omega} \text{ and } \varphi_k : V^k \hookrightarrow W^k/W^{k-1} \text{ for } k = 1, 2\},
\]

and

\[
p_3 : E''_{V^1,V^2,\Omega} \rightarrow E_{V,\Omega}, \quad (x, \text{Fil}, \varphi_1, \varphi_2) \mapsto x
\]

is proper;

\[
p_2 : E'_{V^1,V^2,\Omega} \rightarrow E''_{V^1,V^2,\Omega}, \quad (x, \text{Fil}, \varphi_1, \varphi_2) \mapsto (x, \text{Fil})
\]

is a principal \( G_{V^1} \times G_{V^2} \)-bundle;

\[
p_1 : E'_{V^1,V^2,\Omega} \rightarrow E_{V^1,\Omega} \times E_{V^2,\Omega}, \quad (x, \text{Fil}, \varphi_1, \varphi_2) \mapsto (\varphi_1^{-1}x\varphi_1, \varphi_2^{-1}x\varphi_2)
\]

is smooth, where we denote the induced morphisms on \( W^k/W^{k-1} \) still by \( x \). To see the properties of \( p_1, p_2, p_3 \) more easily, we will give another description of the diagram.

We fix a filtration \( \text{Fil} : 0 = W^0 \subseteq W^1 \subseteq W^2 = V \), where \( \bar{W}^1 = V^1 \). Let \( \bar{\varphi}_1 : V^1 \rightarrow \bar{W}^1/W^0 \) be the identity and \( \bar{\varphi}_2 : V^2 \rightarrow \bar{W}^2/W^1 \) be the composition of \( V^2 \rightarrow V^1 \). Let

\[
E^0_{V^1,V^2,\Omega} := \{x \in E_{V,\Omega} | x \text{ stabilizes } \text{Fil} \} \hookrightarrow E'_{V^1,V^2,\Omega}, \quad x \mapsto (x, \text{Fil}, \bar{\varphi}_1, \bar{\varphi}_2)
\]

It admits an action by

\[
G^0_{V^1,V^2} := \{g \in G_V | g \text{ stabilizes } \text{Fil} \}
\]

a parabolic subgroup of \( G_V \). It has a Levi component \( G_{V^1} \times G_{V^2} \) and the unipotent radical is

\[
G^+_{V^1,V^2} := \{g \in G^0_{V^1,V^2} | \tilde{\varphi}_k^{-1}g\tilde{\varphi}_k = \text{id} \text{ for } k = 1, 2\}.
\]

The following lemma is immediate.

**Lemma 4.13.** We have \( G_V \)-equivariant isomorphisms

\[
G_V \times_{G^0_{V^1,V^2}} E^0_{V^1,V^2,\Omega} \cong E''_{V^1,V^2,\Omega}, \quad (g, x) \mapsto (gx, g\text{Fil})
\]

\[
G_V \times_{G^+_{V^1,V^2}} E^0_{V^1,V^2,\Omega} \cong E'_{V^1,V^2,\Omega}, \quad (g, x) \mapsto (gx, g\text{Fil}, g\bar{\varphi}_1, g\bar{\varphi}_2).
\]
By this lemma, we can rewrite diagram (5) as

$$G_V \times_{G_{v_1,v_2}} E_{v_1,v_2,\Omega}^{\geq 0} \xrightarrow{p'_2} G_V \times_{G_{v_1,v_2}} E_{v_1,v_2,\Omega}^{\geq 0}$$

\[\xrightarrow{p_3} \]

\[E_{v_1,\Omega} \times E_{v_2,\Omega} \]

\[\xrightarrow{\phi} E_{v,\Omega} \]

where

$$p'_3 : G_V \times_{G_{v_1,v_2}} E_{v_1,v_2,\Omega}^{\geq 0} \to E_{v,\Omega}, \quad (g,x) \mapsto gx$$

$$p'_2 : G_V \times_{G_{v_1,v_2}} E_{v_1,v_2,\Omega}^{\geq 0} \to G_V \times_{G_{v_1,v_2}} E_{v_1,v_2,\Omega}^{\geq 0}, \quad (g,x) \mapsto (g,x)$$

$$p'_1 : G_V \times_{G_{v_1,v_2}} E_{v_1,v_2,\Omega}^{\geq 0} \to E_{v_1,v_2,\Omega} \xleftarrow{p_1} E_{v_1,v_2,\Omega} \times E_{v_2,\Omega}$$

For $F_1 \in D_{G_{v_1}}(E_{v_1,\Omega}), F_2 \in D_{G_{v_2}}(E_{v_2,\Omega})$, we define

$$F_1 * F_2 := p_3 F''$$

where

$$p_3^* F'' \cong p_1^*(F_1 \boxtimes F_2).$$

For $\phi_1 \in M(E_{v_1,\Omega})^{G_{v_1}}, \phi_2 \in M(E_{v_1,\Omega})^{G_{v_1}}$, we define $\phi_1 * \phi_2 \in M(E_{v,\Omega})^{G_{v}}$ by

$$(\phi_1 * \phi_2)(x) = \int_{p_3^{-1}(x)} \phi''(x, \text{Fil})$$

where

$$\phi''(x, \text{Fil}) = \phi_1(x_1)\phi_2(x_2)$$

with

$$x_1 = \varphi_1^{-1}x \varphi_1, x_2 = \varphi_2^{-1}x \varphi_2$$

for any choice of isomorphisms $\varphi_k : V^k \to W^k/W^{k-1}$.

**Proposition 4.14.** For $F_1 \in D_{G_{v_1}}(E_{v_1,\Omega}), F_2 \in D_{G_{v_2}}(E_{v_2,\Omega})$,

$$\phi_{F_1} * \phi_{F_2} = \phi_{F_1 * F_2}.$$

**Proof.** We have

$$\phi_{F_1 * F_2}(x) = \chi(H^*(p_3^{-1}(x), F'')) = \int_{p_3^{-1}(x)} \chi(F''),$$

where for the second equality we refer to [1] Proposition 24.16. It is easy to see that $\chi(F'') = \phi''$. \qed

Next consider the following diagram

$$\begin{array}{c}
\Lambda_{v_1,v_2}' \xrightarrow{q_2} \Lambda_{v_1,v_2}'' \\
q_1 \downarrow \quad \quad \quad \downarrow q_3 \\
\Lambda_{v_1} \times \Lambda_{v_2} \quad \quad \quad \Lambda_V
\end{array}$$

where

$$\Lambda_{v_1,v_2}'' := \{(x,y,\text{Fil})| (x,y) \in \Lambda_V, \text{Fil} : 0 = W^0 \subseteq W^1 \subseteq W^2 = V (x,y)-stable \text{ with } |W^k/W^{k-1}| = |V^k| \text{ for } k = 1,2\}$$

$$\Lambda_{v_1,v_2}''' := \{(x,y,\varphi_1,\varphi_2) | (x,y,\text{Fil}) \in \Lambda_{v_1,v_2}'' \text{ and } \varphi_k : V^k \sim W^k/W^{k-1} \text{ for } k = 1,2\},$$

and

$$q_3 : \Lambda_{v_1,v_2}''' \to \Lambda_V, \quad (x,y,\text{Fil}) \mapsto (x,y).$$
is proper;

\[ q_2 : \Lambda_{V,1}^{\prime} \rightarrow \Lambda_{V,1}^{\prime}\prime, \quad (x, y, \text{Fil}, \varphi_1, \varphi_2) \mapsto (x, y, \text{Fil}) \]

is a principal \(G_{V,1} \times G_{V,2}\)-bundle;

\[ q_1 : \Lambda_{V,1}^{\prime} \rightarrow \Lambda_{V,1}^{\prime}, \quad (x, y, \text{Fil}, \varphi_1, \varphi_2) \mapsto ((\varphi_1^{-1} x \varphi_1, \varphi_1^{-1} y \varphi_1), (\varphi_2^{-1} x \varphi_2, \varphi_2^{-1} y \varphi_2)) \]

where we denote the induced morphisms on \(W^k/W^{k-1}\) still by \(x, y\).

For \(\phi_1 \in M(\Lambda_{V,1}^{G_{V,1}}), \phi_2 \in M(\Lambda_{V,2}^{G_{V,2}})\), we define \(\phi_1 \ast \phi_2 \in M(\Lambda_{V}^{G_{V}})\) by

\[ (\phi_1 \ast \phi_2)(x, y) = \int_{q_3^{-1}(x, y)} \phi''(x, y, \text{Fil}) \]

where

\[ \phi''(x, y, \text{Fil}) = \phi_1((x_1, y_1))\phi_2((x_2, y_2)) \]

with

\[ x_1 = \varphi_1^{-1} x \varphi_1, \quad y_1 = \varphi_1^{-1} y \varphi_1; \quad x_2 = \varphi_2^{-1} x \varphi_2, \quad y_2 = \varphi_2^{-1} y \varphi_2 \]

for any choice of isomorphisms \(\varphi_k : V^k \rightarrow W^k/W^{k-1}\).

One can easily extend this convolution to constructible functions on \(E_{V,1}, E_{V,2}\) by considering the diagram

\[ \begin{array}{ccc}
E_{V,1}' & \xrightarrow{q_2} & E_{V,1}' \\
\downarrow q_1 & & \downarrow q_3 \\
\Lambda_{V,1}^{\prime} & \xrightarrow{\phi_1} & \Lambda_{V,1}^{\prime}\prime \\
\downarrow \Lambda_{V,1} & & \downarrow \Lambda_{V} \\
E_{V,1} \times E_{V,2} & & E_{V} \\
\end{array} \]

where

\[ E_{V,1}' := \{(x, y, \text{Fil}) \mid (x, y) \in E_{V,1}, \text{Fil} : 0 = W_0 \subseteq W_1 \subseteq W_2 = V (x, y)\text{-stable with } |W^k/W^{k-1}| = |V^k| \text{ for } k = 1, 2\}, \]

\[ E_{V,1}' := \{(x, y, \text{Fil}, \varphi_1, \varphi_2) \mid (x, y, \text{Fil}) \in E_{V,1}' \text{ and } \varphi_k : V^k \rightarrow W^k/W^{k-1} \text{ for } k = 1, 2\}. \]

Note both the top and right squares are Cartesian, but the left one is not. The following lemma is immediate from the definition.

**Lemma 4.15.** For \(\phi_1 \in M(E_{V,1}^{G_{V,1}}), \phi_2 \in M(E_{V,2}^{G_{V,2}})\),

\[ (\phi_1 \ast \phi_2)|_{\Lambda_{V}} = \phi_1|_{\Lambda_{V,1}} \ast \phi_2|_{\Lambda_{V,2}}. \]

For \(\mathcal{F}_1 \in D_{G_{V,1}}(E_{V,1}), \mathcal{F}_2 \in D_{G_{V,2}}(E_{V,2})\), we can also define convolution

\[ \mathcal{F}_1 \ast \mathcal{F}_2 := q_3^{\prime} \mathcal{F}'' \]

where

\[ q_3^{\prime} \mathcal{F}'' \cong q_3^\prime (\mathcal{F}_1 \boxtimes \mathcal{F}_2). \]

One can also show easily that

\[ \phi_{\mathcal{F}_1} \ast \phi_{\mathcal{F}_2} = \phi_{\mathcal{F}_1 + \mathcal{F}_2}. \]

Now we attempt to establish \([7]\) directly. It suffices to show that for any \(\mathcal{F}_1 \in D_{G_{V,1}}(E_{V,1}, \Omega), \mathcal{F}_2 \in D_{G_{V,2}}(E_{V,2}, \Omega),\)

\[ \eta_{V,1}(\phi_{\mathcal{F}_1}) \ast \eta_{V,2}(\phi_{\mathcal{F}_2}) = \eta_{V}(\phi_{\mathcal{F}_1 + \mathcal{F}_2}). \]

We first expand the right hand side. For \((x, y) \in \Lambda_{V},\)

\[ \eta_{V}(\phi_{\mathcal{F}_1 + \mathcal{F}_2})(x, y) = \chi(R\Phi_{\mathcal{F}_2}[-1](\mathcal{F}_1 \ast \mathcal{F}_2)_x) = \chi(R\Phi_{\mathcal{F}_2}[-1](\mathcal{F}_1 \ast \mathcal{F}_2)_x). \]
By proper base change, 
\[ R\Phi_{f_3}(p_3!F'') \cong p_3!(R\Phi_{f_3\circ p_3}F''). \]
Hence,
\[ \chi(R\Phi_{f_3}(p_3!F'')) = \chi(H^*(p_3^{-1}(x), R\Phi_{f_3\circ p_3}F'')) = \int_{p_3^{-1}(x)} \chi(R\Phi_{f_3\circ p_3}F'') \]
where for the second equality we refer to [11] Proposition 24.16. By smooth base change, 
\[ p_2^*(R\Phi_{f_3\circ p_3}F'') \cong R\Phi_{f_3\circ p_3\circ p_2}(p_2^*F''). \]
We can also express the left hand side as an integration.
\[ (\eta_{V,1}^*(\phi_{F_1}) \ast \eta_{V,2}^*(\phi_{F_2}))(x, y) = \int_{q_3^{-1}(x, y)} \eta_{V,1}^*(\phi_{F_1})(x_1, y_1) \cdot \eta_{V,2}^*(\phi_{F_2})(x_2, y_2) \]
Comparing the two integrals, we see
\[ q_3^{-1}(x, y) \hookrightarrow p_3^{-1}(x), \quad (x, y, \text{Fil}) \mapsto (x, \text{Fil}). \]
Then both integrals are equal if
\begin{enumerate}
  \item \( \chi(R\Phi_{f_3\circ p_3}F'') = 0 \) over \( p_3^{-1}(x)\setminus q_3^{-1}(x, y) \).
  \item \( \chi(R\Phi_{f_3\circ p_3}[-1]F'')(x, \text{Fil}) = \eta_{V,1}^*(\phi_{F_1})(x_1, y_1) \cdot \eta_{V,2}^*(\phi_{F_2})(x_2, y_2) \) for \( (x, y, \text{Fil}) \in q_3^{-1}(x, y) \).
\end{enumerate}
In (2), we can rewrite the right hand side as
\[ \eta_{V,1}^*(\phi_{F_1})(x_1, y_1) \cdot \eta_{V,2}^*(\phi_{F_2})(x_2, y_2) = \chi(R\Phi_{f_3\circ p_3}[-1]F_1)_{x_1} \chi(R\Phi_{f_3\circ p_3}[-1]F_2)_{x_2}. \]

**Theorem 4.16 (Sebastiani-Thom).**
\[ R\Phi_{f_3\circ p_3}(F_1) \cong R\Phi_{f_3\circ p_3}\circ p_2(F_1 \otimes F_2), \]
where
\[ f_{y_1} \oplus f_{y_2} : E\,_{V,1,\Omega} \times E\,_{V,2,\Omega} \to \mathbb{C}, \quad (x_1', x_2') \mapsto f_{y_1}(x_1') + f_{y_2}(x_2'). \]
By smooth base change,
\[ p_2^*R\Phi_{f_3\circ p_3}(F_1 \otimes F_2) \cong R\Phi_{(f_{y_1} \oplus f_{y_2}) \circ p_1}(p_2^*(F_1 \otimes F_2)). \]
So (2) is equivalent to
\[ \chi(R\Phi_{(f_{y_1} \oplus f_{y_2}) \circ p_1}(p_2^*(F_1 \otimes F_2))(x, y, \text{Fil})) = \chi(R\Phi_{f_3\circ p_3\circ p_2}(p_2^*F'')(x, y, \text{Fil})). \]
Note \( p_2^*(F_1 \otimes F_2) \cong p_2^*F'' \), but
\[ (f_{y_1} \oplus f_{y_2}) \circ p_1 \neq f_y \circ p_3 \circ p_2. \]
So we cannot conclude the equality directly. This is the main reason that we have to approach [7] in a roundabout way.

4.4. **Inductions.** We fix an \( I \)-graded isomorphism \( V = V^1 \oplus V^2 \oplus \cdots \oplus V^n \) and a filtration
\[ \text{Fil} : 0 = W^0 \subset W^1 \subset \cdots \subset W^n = V \]
where
\[ W^k := V^1 \oplus \cdots \oplus V^k \]
Let \( \bar{\varphi}_k : V^k \hookrightarrow W^k \to W^k/W^{k-1} \). The goal is to calculate
\[ \phi_1 \ast \cdots \ast \phi_n \]
for \( \phi_k \in M(E_{V^k,\Omega})^{G_{V^k}}. \) We will define
\[ \text{Ind}_{V^1, \ldots, V^n} : \otimes_{k=1}^n M(E_{V^k,\Omega})^{G_{V^k}} \to M(E_{V,\Omega})^{G_V} \]
as follows. Consider the diagram

\[
\begin{array}{ccc}
E'_{V^1, \ldots, V^n, \Omega} & \xrightarrow{p_2} & E''_{V^1, \ldots, V^n, \Omega} \\
p_1 \downarrow & & \downarrow p_3 \\
E_{V^1, \Omega} \times \cdots \times E_{V^n, \Omega} & & E_{V, \Omega}
\end{array}
\]

where

\[
E'_{V^1, \ldots, V^n, \Omega} := \{(x, \text{Fil})| x \in E_{V, \Omega}, \text{Fil} : 0 = W^0 \subset \cdots \subset W^n = V x\text{-stable with } |W^k/W^{k-1}| = |V^k| \text{ for } k = 1, \ldots, n\}
\]

\[
E''_{V^1, \ldots, V^n, \Omega} := \{(x, \text{Fil}, \{\varphi_k\}_{k=1}^n)| (x, \text{Fil}) \in E''_{V^1, \ldots, V^n, \Omega} \text{ and } \varphi_k : V^k \to W^k/W^{k-1} \text{ for } k = 1, \ldots, n\},
\]

and

\[
p_3 : E''_{V^1, \ldots, V^n, \Omega} \to E_{V, \Omega}, \quad (x, \text{Fil}) \mapsto x
\]

is proper;

\[
p_2 : E'_{V^1, \ldots, V^n, \Omega} \to E''_{V^1, \ldots, V^n, \Omega}, \quad (x, \text{Fil}, \{\varphi_k\}_{k=1}^n) \mapsto (x, \text{Fil}), \quad p_1 : E'_{V^1, \ldots, V^n, \Omega} \to E_{V^1, \ldots, V^n, \Omega}, \quad (x, \text{Fil}, \{\varphi_k\}_{k=1}^n) \mapsto \{\varphi_k^{-1}x\varphi_k\}_{k=1}^n
\]

is smooth, where we denote the induced morphisms on \(W^k/W^{k-1}\) still by \(x\). To see the properties of \(p_1, p_2, p_3\) more easily, we will give another description of the diagram. Let

\[
E_{V^1, \ldots, V^n, \Omega}^{>0} := \{x \in E_{V, \Omega}| x \text{ stabilizes } \text{Fil}\} \to E'_{V^1, \ldots, V^n, \Omega}, \quad x \mapsto (x, \text{Fil}, \{\varphi_k\}_{k=1}^n)
\]

It admits an action by

\[
G_{V^1, \ldots, V^n}^{>0} := \{g \in G_V| g \text{ stabilizes } \text{Fil}\}
\]

a parabolic subgroup of \(G_V\). It has a Levi component \(G_{V^1, \ldots, V^n}^{+}\) and the unipotent radical is

\[
G_{V^1, \ldots, V^n}^{+} := \{g \in G_{V^1, \ldots, V^n}^{>0}| \varphi_k^{-1}g\varphi_k = id \text{ for } k = 1, \ldots, n\}.
\]

**Lemma 4.17.**

\[
G_V \times_{G_{V^1, \ldots, V^n}^{>0}} E_{V^1, \ldots, V^n, \Omega}^{>0} \cong E'_{V^1, \ldots, V^n, \Omega}, \quad (g, x) \mapsto (gx, g\text{Fil})
\]

\[
G_V \times_{G_{V^1, \ldots, V^n}^{+}} E_{V^1, \ldots, V^n, \Omega}^{>0} \cong E'_{V^1, \ldots, V^n, \Omega}, \quad (g, x) \mapsto (gx, g\text{Fil}, \{g\varphi_k\}_{k=1}^n)
\]

\[
E_{V^1, \Omega} \times \cdots \times E_{V^n, \Omega} \quad \xrightarrow{p_2'} \quad G_V \times_{G_{V^1, \ldots, V^n}^{>0}} E_{V^1, \ldots, V^n, \Omega}^{>0}
\]

\[
E_{V^1, \Omega} \times \cdots \times E_{V^n, \Omega} \quad \xrightarrow{p_1} \quad E_{V^1, \ldots, V^n, \Omega}
\]

where

\[
p_2' : G_V \times_{G_{V^1, \ldots, V^n}^{>0}} E_{V^1, \ldots, V^n, \Omega}^{>0} \to E_{V^1, \ldots, V^n, \Omega}, \quad (g, x) \mapsto gx,
\]

\[
p_2' : G_V \times_{G_{V^1, \ldots, V^n}^{+}} E_{V^1, \ldots, V^n, \Omega}^{>0} \to G_V \times_{G_{V^1, \ldots, V^n}^{>0}} E_{V^1, \ldots, V^n, \Omega}^{>0}, \quad (g, x) \mapsto (g, x),
\]

\[
p_1' : G_V \times_{G_{V^1, \ldots, V^n}^{+}} E_{V^1, \ldots, V^n, \Omega}^{>0} \to E_{V^1, \ldots, V^n, \Omega}^{>0}, \quad \phi \mapsto \phi''(x, \text{Fil})
\]

For \(\phi_k \in M(E_{V^k, \Omega})^{G_{V^k}}(k = 1, \ldots, n)\), we define

\[
\text{Ind}_{V^1, \ldots, V^n}(\phi_1 \otimes \cdots \otimes \phi_n)(x) = \int_{p_3^{-1}(x)} \phi''(x, \text{Fil})
\]
where
\[ \phi''(x, \text{Fil}) = \phi_1(x_1) \cdots \phi_n(x_n) \]
with
\[ x_k = \varphi_k^{-1} x \varphi_k \]
for any choice of isomorphisms \( \varphi_k : V^k \to W^k / W^{k-1} \). For \( F_k \in DG_{Vk}(E_{Vk}, \Omega)(k = 1, \cdots, n) \), we define
\[ \text{Ind}_{V^1, \cdots, V^n}(F_k \boxtimes \cdots \boxtimes F_n) := p_3 F'' \]
where
\[ p_2^* F'' \cong p_1^*(F_k \boxtimes \cdots \boxtimes F_n). \]

**Proposition 4.18.** For \( F_k \in DG_{V^k}(E_{V^k}, \Omega)(k = 1, \cdots, n) \),
\[ \text{Ind}_{V^1, \cdots, V^n}(\phi F_1 \otimes \cdots \otimes \phi F_n) = \phi \text{Ind}_{V^1, \cdots, V^n}(F_1 \otimes \cdots \otimes F_n). \]

**Proof.** We have
\[ \phi \text{Ind}_{V^1, \cdots, V^n}(F_1 \otimes \cdots \otimes F_n)(x) = \chi(H^*(p_3^{-1}(x), F'')) = \int_{p_3^{-1}(x)} \chi(F'') \]
where for the second equality we refer to \([\text{I}]\) Proposition 24.16. It is easy to see that \( \chi(F'') = \phi'' \).

**Proposition 4.19.** For \( \phi_k \in M(E_{V^k}, \Omega)^{G_{V^k}}(k = 1, \cdots, n) \),
\[ \phi_1 \ast \cdots \ast \phi_n = \text{Ind}_{V^1, \cdots, V^n}(\phi_1 \otimes \cdots \otimes \phi_n) \]

**Proof.** We will prove it by induction on \( n \). When \( n = 2 \), there is nothing to show. Suppose \( n > 2 \). By induction assumption,
\[ \phi_1 \ast \cdots \ast \phi_n = (\phi_1 \ast \phi_2) \ast \cdots \ast \phi_n = \text{Ind}_{V^1, V^2}(\phi_1 \otimes \phi_2) \ast \cdots \ast \phi_n \]
\[ = \text{Ind}_{V^1, \cdots, V^n}(\text{Ind}_{V^1, V^2}(\phi_1 \otimes \phi_2) \otimes \cdots \otimes \phi_n) \]

Consider the following diagram

\[
\begin{array}{ccc}
E'_{V^1, V^n, \Omega} & \xrightarrow{F''} & E''_{V^1, V^n, \Omega} \\
\downarrow & & \downarrow \\
E'_{W^1, V^n, \Omega} & \xrightarrow{F''} & E''_{W^1, V^n, \Omega} \\
\end{array}
\]

where
\[
E'_{W^2, \cdots, V^n, \Omega} = \left\{ (x, \text{Fil}, \{ \varphi_k \}_{k=3}^n, \psi_2) | (x, \text{Fil}) \in E'_{V^1, V^n, \Omega}, \varphi_k : V^k \xrightarrow{\sim} W^k / W^{k-1} (k \geq 3), \psi_2 : W^2 \xrightarrow{\sim} W^2 \right\}
\]
and
\[
\tilde{E}'_{V^1, \cdots, V^n, \Omega} = \left\{ (x, \text{Fil}, \{ \varphi_k \}_{k=1}^n, \psi_2) | (x, \text{Fil}, \{ \varphi_k \}_{k=1}^n) \in E'_{V^1, \cdots, V^n, \Omega}, \psi_2 : \tilde{W}^2 \xrightarrow{\sim} W^2 \right\}
\]
\[
\tilde{E}'_{V^1, \cdots, V^n, \Omega} \to E'_{V^1, V^2, \Omega} \times \prod_{k=3}^n E_{V^k, \Omega}, \quad \text{Fil}, \{ \varphi_k \}_{k=1}^n, \psi_2 \mapsto (x', \text{Fil}', \{ \varphi_k' \}_{k=1}^n, \psi_2) \mapsto ((x', \text{Fil}', \{ \varphi_k' \}_{k=1}^n, \{ \varphi_k^{-1} x \varphi_k \}_{k=3}^n) \]
where
\[ x' = \psi_2^{-1}x\psi_2, \quad \text{Fil}' : 0 = \psi_2^{-1}(W^1) \subseteq \psi_2^{-1}(W^2) = \bar{W}^2 \]
and
\[ \varphi_1' = \psi_2^{-1}\varphi_1, \quad \varphi_2' = \psi_2^{-1}\varphi_2. \]
In particular,
\[
\tilde{E}'_{V'_2, \ldots, V'_n, \Omega} = (E''_{V'_1, V'_2, V'_2} \times \prod_{k=3}^{n} E_{V'_k, \Omega}) \times (E_{W'_2, V'_2} \times \bigwedge_{k=3}^{n} E_{V'_k, V'_k, \Omega})
\]
and
\[
\tilde{E}'_{V'_1, \ldots, V'_n, \Omega} = (E''_{V'_1, V'_2, V'_2} \times \prod_{k=3}^{n} E_{V'_k, \Omega}) \times (E_{W'_2, \ldots, V'_n, \Omega} \times E_{V'_1, \ldots, V'_n, \Omega})
\]
Let \( \phi_k = \phi_{\mathcal{F}_k} \). It suffices to show
\[
\text{Ind}_{V'_1, \ldots, V'_n} (\mathcal{F}_1 \times \cdots \times \mathcal{F}_n) = \text{Ind}_{W'_2, \ldots, V'_n} (\text{Ind}_{V'_1, \ldots, V'_n} (\mathcal{F}_1 \times \mathcal{F}_2) \times \cdots \times \mathcal{F}_n).
\]
This can be seen easily by tracing the diagram.

We will also define
\[
\text{Ind}_{V'_1, \ldots, V'_n} : \otimes_{k=1}^{n} M(\Lambda_{V'_k})^{G_{V'_k}} \to M(\Lambda_V)^{G_V}
\]
as follows. Consider the diagram
\[
\begin{array}{ccc}
\Lambda'_{V'_1, \ldots, V'_n} & \xrightarrow{q_2} & \Lambda''_{V'_1, \ldots, V'_n} \\
q_1 \downarrow & & \downarrow q_3 \\
\Lambda_{V'_1} \times \cdots \times \Lambda_{V'_n} & & \Lambda_V
\end{array}
\]
where
\[
\Lambda''_{V'_1, \ldots, V'_n} = \left\{ (x, y, \text{Fil}) | (x, y) \in \Lambda_V, \text{Fil} : 0 = W^0 \subseteq \cdots \subseteq W^n = V (x, y)-stable \ | W^k/W^{k-1} | = | V^k | (1 \leq k \leq n) \right\}
\]
\[
\Lambda'_{V'_1, \ldots, V'_n} = \left\{ (x, y, \text{Fil}, (\varphi_k)_{k=1}^{n}) | (x, y, \text{Fil}) \in \Lambda''_{V'_1, \ldots, V'_n} \text{ and } \varphi_k : V^k \cong W^k/W^{k-1} (1 \leq k \leq n) \right\}
\]
and
\[
q_3 : \Lambda''_{V'_1, \ldots, V'_n} \to \Lambda_V, \quad (x, y, \text{Fil}) \mapsto (x, y)
\]
is proper;
\[
q_2 : \Lambda'_{V'_1, \ldots, V'_n} \to \Lambda''_{V'_1, \ldots, V'_n}, \quad (x, y, \text{Fil}, (\varphi_k)_{k=1}^{n}) \mapsto (x, y, \text{Fil})
\]
is a principal \( G_{V'_1} \times \cdots \times G_{V'_n} \)-bundle;
\[
q_1 : \Lambda'_{V'_1, \ldots, V'_n} \to \Lambda_{V'_1} \times \cdots \times \Lambda_{V'_n}, \quad (x, y, \text{Fil}, (\varphi_k)_{k=1}^{n}) \mapsto \left\{ (\varphi_k^{-1} x \varphi_k, \varphi_k^{-1} y \varphi_k) \right\}_{k=1}^{n}
\]
where we denote the induced morphisms on \( W^k/W^{k-1} \) still by \( x, y \).

For \( \phi_k \in \text{M}(\Lambda_{V'_k})^{G_{V'_k}} \), we define
\[
\text{Ind}_{V'_1, \ldots, V'_n} (\phi_1 \otimes \cdots \otimes \phi_k)(x, y) = \int_{q_3^{-1}(x, y)} \phi''(x, y, \text{Fil})
\]
where
\[
\phi''(x, y, \text{Fil}) = \phi_1((x_1, y_1)) \cdots \phi_n((x_n, y_n))
\]
with
\[
x_k = \varphi_k^{-1} x \varphi_k, \quad y_k = \varphi_k^{-1} y \varphi_k
\]
for any choice of isomorphisms \( \varphi_k : V^k \to W^k/W^{k-1} \).
One can easily extend this induction constructible functions on $E_{V^k}$ by considering the diagram

\[ \begin{array}{ccc}
E'_{V^1, \ldots, V^n} & \xrightarrow{q_2} & E''_{V^1, \ldots, V^n} \\
\Lambda'_{V^1, \ldots, V^n} & \xrightarrow{q_1} & \Lambda''_{V^1, \ldots, V^n} \\
\Lambda_{V^1 \times \cdots \times V^n} & \xrightarrow{q_3} & \Lambda_V
\end{array} \]

where

\[
E''_{V^1, \ldots, V^n} := \left\{ (x, y, \text{Fil}) | (x, y) \in E_{V^1, \Omega}, \text{Fil} : 0 = W^0 \subseteq W^1 \subseteq W^2 = V(x, y)\text{-stable } |W^k/W^{k-1}| = |V^k| (1 \leq k \leq n) \right\}
\]

\[
E'_{V^1, \ldots, V^n} := \left\{ (x, y, \{\varphi_k\}_{k=1}^n) | (x, y, \text{Fil}) \in E''_{V^1, \ldots, V^n} \text{ and } \varphi_k : V^k \cong W^k/W^{k-1} (1 \leq k \leq n) \right\}.
\]

Note both the top and right squares are Cartesian, but the left one is not. The following lemma follows immediately from the definition.

**Lemma 4.20.** For $\phi_k \in M(E_{V^k}, \Omega)^{G_{V^k}} (1 \leq k \leq n),$

\[
\text{Ind}_{V^1, \ldots, V^n}(\phi_1 \otimes \cdots \otimes \phi_n)|_{\Lambda_V} = \text{Ind}_{V^1, \ldots, V^n}(\phi_1|_{\Lambda_{V^1}} \otimes \cdots \otimes \phi_n|_{\Lambda_{V^n}}).
\]

For $\mathcal{F}_k \in D_{G_{V^k}}(E_{V^k}) (1 \leq k \leq n),$ we can also define induction

\[
\text{Ind}_{V^1, \ldots, V^n}(\mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_n) := q'_2 \ast \mathcal{F}''
\]

where

\[
q'_2 \ast \mathcal{F}'' \cong q'_1 \ast (\mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_n).
\]

One can also show easily that

\[
\text{Ind}_{V^1, \ldots, V^n}(\phi_{\mathcal{F}_1} \otimes \cdots \otimes \phi_{\mathcal{F}_n}) = \phi_{\text{Ind}_{V^1, \ldots, V^n}(\mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_n)}.
\]

**Proposition 4.21.** For $\phi_k \in M(\Lambda_{V^k})^{G_{V^k}} (k = 1, \ldots, n),$

\[
\phi_1 * \cdots * \phi_n = \text{Ind}_{V^1, \ldots, V^n}(\phi_1 \otimes \cdots \otimes \phi_n).
\]

**Proof.** It suffices to show for $\phi_k \in M(E_{V^k})^{G_{V^k}} (1 \leq k \leq n),$

\[
\phi_1 * \cdots * \phi_n = \text{Ind}_{V^1, \ldots, V^n}(\phi_1 \otimes \cdots \otimes \phi_n).
\]

The proof is similar to that of Proposition 4.19.

\[\square\]

4.5. **Further reduction of \([7]\).** Let $|V| = (d_i)_{i \in I}$ and $d = \sum_{i \in I} d_i.$ Let

\[
S_{|V|} := \{ a \in I^{1, \ldots, d} | |a^{-1}(i)| = d_i \}
\]

For any $a \in S_{|V|},$ we fix an $I$-graded isomorphism $V = V^1 \oplus \cdots \oplus V^d$ such that

\[
|V^k| = (\delta_0(k), i)_{i \in I}
\]

Note $E_{V^k, \Omega} = \Lambda_{V^k} = E_{V^k} = \{0\}.$ The following statement is a special case of \([7]\).

**Conjecture 4.22.**

\[
\eta_{V^1}(1_1) * \cdots * \eta_{V^d}(1_d) = \eta_V(1_1 * \cdots * 1_d).
\]

**Remark 4.23.** $\eta_{V^k}(1_k) = 1_{\Lambda_{V^k}}.$
Indeed, we have

**Lemma 4.24.** Conjecture 4.22 is equivalent to (7).

**Proof.** We only need to show that (7) follows from Conjecture 4.22. By Proposition 7.3, it suffices to show (7) for

\[ \phi_I = 1_{I_1} \cdots 1_{I_d} \in M(E_{V^I} G_{VI}) \]
\[ \phi_{II} = 1_{II_1} \cdots 1_{II_d} \in M(E_{V^{II}} G_{V^{III}}) \]

associated with \( a_I \in S_{V^I} \) and \( a_{II} \in S_{V^{II}} \) respectively. By Conjecture 4.22,

\[ \eta_{V^I}(\phi_I) \ast \eta_{V^{II}}(\phi_{II}) = (\eta_{V^I}(1_{I_1}) \cdots \eta_{V^{II}}(1_{II_1})) \ast (\eta_{V^I}(1_{I_1}) \cdots \eta_{V^{II}}(1_{II_1})) \]
\[ = \eta_V(1_{I_1} \cdots 1_{I_d} 1_{II_1} \cdots 1_{II_d}) \]
\[ = \eta_V(\phi_I \ast \phi_{II}) \]

\( \square \)

Now we will describe our approach to Conjecture 4.22. By Proposition 4.19 and Proposition 4.21, it suffices to show

\[ \text{Ind}_{V^1 \cdots V^d}(\eta_{V^1}(1_1) \otimes \cdots \otimes \eta_{V^d}(1_d)) = \eta_V(\text{Ind}_{V^1 \cdots V^d}(1_1 \cdots 1_d)). \]

Note \( 1_k = \phi_{1_k} \), where \( 1_k \in D_{G_{V^k}}(E_{V^k} G_{V^k}). \) So it is the same as

\[ \text{Ind}_{V^1 \cdots V^d}(\eta_{V^1}(\phi_{1_1}) \otimes \cdots \otimes \eta_{V^d}(\phi_{1_d})) = \eta_V(\text{Ind}_{V^1 \cdots V^d}(1_1 \cdots 1_d)). \]

We first expand the right hand side. For \((x, y) \in \Lambda_V, \)

\[ \eta_V(f_{\text{Ind}_{V^1 \cdots V^d}(1_1 \cdots 1_d)})(x, y) = \chi(R\Phi_{f_y}[-1](\mathbb{1}_1 \otimes \cdots \otimes \mathbb{1}_d), x) = \chi(R\Phi_{f_y}[-1](p_3^d, F'))(x) \]

By proper base change,

\[ R\Phi_{f_y}(p_3^d, F') \cong p_3(R\Phi_{f_y} p_{3^d} F') \]

Hence,

\[ \chi(R\Phi_{f_y}(p_3^d, F'))(x) = \chi(H^*(p_3^{-1}(x), R\Phi_{f_y} p_{3^d} F')) = \int_{p_3^{-1}(x)} \chi(R\Phi_{f_y} p_{3^d} F'). \]

By smooth base change,

\[ p_3^*(R\Phi_{f_y} p_{3^d} F') \cong R\Phi_{f_y} p_{3^d} p_2(p_2^d F') \cong R\Phi_{f_y} p_{3^d} p_2(1). \]

We can also express the left hand side as an integration,

\[ \text{Ind}_{V^1 \cdots V^d}(\eta_{V^1}(\phi_{1_1}) \otimes \cdots \otimes \eta_{V^d}(\phi_{1_d}))(x, y) = \int_{q_3^{-1}(x, y)} \eta_{V^1}(\phi_{1_1})(x_1, y_1) \cdots \eta_{V^d}(\phi_{1_d})(x_d, y_d) = \int_{q_3^{-1}(x, y)} 1, \]

due to the fact that \((x_k, y_k) = 0 \) for \( 1 \leq k \leq d. \) Comparing the two integrals, we see

\[ q_3^{-1}(x, y) \mapsto p_3^{-1}(x), \quad (x, y, \text{Fil}) \mapsto (x, \text{Fil}). \]

If we want to prove that the two integrals are equal, it suffices to show

\[ R\Phi_{f_y} p_{3^d} p_2(1)(x, \text{Fil}, \{\varphi_k\})_{k=1}^d = 0 \quad \text{for } (x, \text{Fil}) \in p_3^{-1}(x) \setminus q_3^{-1}(x, y), \]
\[ (14) \]
\[ \chi(R\Phi_{f_y} p_{3^d} p_2([-1])(x, \text{Fil}, \{\varphi_k\})_{k=1}^d) = 1 \quad \text{for } (x, \text{Fil}) \in q_3^{-1}(x, y), \]
\[ (15) \]
To show these, we adopt the diagram

\[
\begin{array}{c}
G_V \times E_{V_1,\ldots,V_d}^{\geq 0} \\
\downarrow \rho'_0 \\
G_V \times G_{V_1,\ldots,V_d}^+ \times E_{V_1,\ldots,V_d}^{\geq 0} \\
\downarrow \rho'_1 \\
E_{V_1,\Omega} \times \cdots \times E_{V_d,\Omega}
\end{array} \xrightarrow{p'_2} \begin{array}{c}
G_V \times E_{V_1,\ldots,V_d}^{\geq 0} \\
\downarrow \rho'_3 \\
E_{V_1,\Omega} \times \cdots \times E_{V_d,\Omega}
\end{array}
\]

Let

\[
G_V \times E_{V_1,\ldots,V_d,\Omega}^{\geq 0} \ni (g_0, x_0) \mapsto (x, \text{Fil}, \{\varphi_k\}_{k=1}^d)
\]

and

\[
h_y : G_V \times E_{V_1,\ldots,V_d,\Omega}^{\geq 0} \to \mathbb{C}, \quad (g, x') \mapsto \langle gx', y \rangle
\]

be the pullback of \( f_y \circ p_3 \circ p_2 \) along \( \rho'_0 \). Then by smooth base change,

\[
R\Phi_{f_y \circ p_3 \circ p_2}(1)_{(x, \text{Fil}, \{\varphi_k\}_{k=1}^d)}(g_0, x_0) \cong R\Phi_{h_y}(1)_{(g_0, x_0)}.
\]

If \( R\Phi_{h_y}(1)_{(g_0, x_0)} \neq 0 \), then \( h \) is singular at \((g_0, x_0)\). So we compute

\[
\langle g_0, y \rangle = \langle v, g_0^{-1}y \rangle,
\]

this is also equivalent to require that \( g_0^{-1}y \) stabilizes \( \Fil \), which is the same to say \( y \) stabilizes \( g_0\Fil = \Fil \), i.e., \((x, \Fil) \in q_3^{-1}(x, y)\). So we have shown \([15]\).

We are now left with \([15]\). Assume \((x, \Fil) \in q_3^{-1}(x, y)\), i.e., \( y_0 := g_0^{-1}y \in E_{V_1,\ldots,V_d,\Omega}^{\geq 0} \). By applying \( g_0 \), we get

\[
\begin{array}{c}
(g, x') \quad G_V \times E_{V_1,\ldots,V_d,\Omega}^{\geq 0} \\
\downarrow h_{y_0} \\
(g_0g, x') \quad G_V \times E_{V_1,\ldots,V_d,\Omega}^{\geq 0}
\end{array} \xrightarrow{h_y} \mathbb{C}
\]

where \( R\Phi_{h_y}(1)_{(g_0, x_0)} \cong R\Phi_{h_{y_0}}(1)_{(1, x_0)} \). So we have reduced it to the following statement.

**Conjecture 4.25.** \( \chi(R\Phi_{h_{y_0}}[-1](1))_{(1, x_0)} = 1 \).

In the next section, we will prove this for type \( A_2 \) quiver.

## 5. Quiver of Type \( A_2 \)

Let \( V = V_1 \oplus V_2 \) be a graded vector space and \( \Omega \) be the orientation \( 1 \rightarrow 2 \). Let \( d_i = \dim V_i \) and \( d_1 + d_2 = d \).

\[
E_{V,\Omega} = \text{Hom}(V_1, V_2), \quad E_{V,\Omega} = \text{Hom}(V_2, V_1), \quad G_V = GL(V_1) \times GL(V_2).
\]

For \((x, y) \in E_{V,\Omega} \times E_{V,\Omega}\) and \( g = (g_1, g_2) \in G_V \), we have the group action

\[
g \cdot x = g_2 x g_1^{-1}, \quad g \cdot y = g_1 y g_2^{-1}
\]

We also have the Lie bracket

\[
[x, y] = (-yx, xy) \in \text{End}(V_1) \times \text{End}(V_2)
\]
and $G_V$-invariant nondegenerate pairing

$$\langle x, y \rangle = \text{tr}(xy).$$

We fix an $I$-graded isomorphism $V = V^1 \oplus V^2 \oplus \cdots \oplus V^d$ such that $\dim V^k = 1$. Then

$$V_1 = V^{t_1} \oplus \cdots \oplus V^{t_{d_1}}, \quad V_2 = V^{s_1} \oplus \cdots \oplus V^{s_{d_2}},$$

where $t_1 < \cdots < t_{d_1}$ and $s_1 < \cdots < s_{d_2}$. Select all indexes

$$1 = \mu_1 < \mu_2 < \cdots < \mu_e = d_1 + 1, \quad 1 = \nu_1 < \nu_2 < \cdots < \nu_f = d_2 + 1,$

such that

$$t_{\mu_i - 1} + 1 < t_{\mu_i} = t_{\mu_i + 1} - 1, \quad \text{for } i < e;$$

$$s_{\nu_i - 1} + 1 < s_{\nu_i} = s_{\nu_i + 1} - 1, \quad \text{for } i < f.$$

Then the list $\{1, 2, \cdots, d\}$ would correspond to either of these cases below:

1. $s_{\nu_1}, \cdots, s_{\nu_2 - 1}; t_{\mu_1}, \cdots, t_{\mu_2 - 1}; s_{\nu_2} \cdots \cdots ; s_{\nu_f - 1}, \cdots, s_{\nu_f - 1}; t_{\mu_e - 1} \cdots t_{\mu_e - 1}, \quad e = f$,

2. $s_{\nu_1}, \cdots, s_{\nu_2 - 1}; t_{\mu_1}, \cdots, t_{\mu_2 - 1}; s_{\nu_2} \cdots \cdots ; t_{\mu_e - 1} \cdots t_{\mu_e - 1}; s_{\nu_f - 1}, \cdots, s_{\nu_f - 1}, \quad e = f - 1$,

3. $t_{\mu_1}, \cdots, t_{\mu_2 - 1}; s_{\nu_1}, \cdots, s_{\nu_2 - 1}; t_{\mu_2} \cdots \cdots ; t_{\mu_e - 1} \cdots t_{\mu_e - 1}; s_{\nu_f - 1}, \cdots, s_{\nu_f - 1}, \quad e = f$,

4. $t_{\mu_1}, \cdots, t_{\mu_2 - 1}; s_{\nu_1}, \cdots, s_{\nu_2 - 1}; t_{\mu_2} \cdots \cdots ; t_{\mu_e - 1} \cdots t_{\mu_e - 1}, \quad e = f + 1$.

We fix a filtration

$$\mathbf{F} : 0 = \bar{W}^0 \subset \bar{W}^1 \subset \cdots \subset \bar{W}^d = V$$

where

$$\bar{W}^k := V^1 \oplus \cdots \oplus V^k.$$

Let

$$E_{V^1, \cdots, V^d, \Omega}^{\geq 0} := \{ x \in E_{V, \Omega} | x \text{ stabilizes } \mathbf{F} \}.$$

It admits an action by

$$G_{V^1, \cdots, V^d, \Omega}^{\geq 0} := \{ g \in G_V | g \text{ stabilizes } \mathbf{F} \}$$

a Borel subgroup of $G_V$. It has a Levi component $G_{V^1} \times \cdots \times G_{V^d}$, which is a maximal torus, and the unipotent radical is

$$G_{V^1, \cdots, V^d}^+ := \{ g \in G_{V^1, \cdots, V^d} | \bar{g}^{-1} g \bar{g} = id \text{ for } k = 1, \cdots, d \},$$

where $\bar{g} : V^k \hookrightarrow W^k \hookrightarrow W^k/W^{k-1}$. Similarly we can define $E_{V^1, \cdots, V^d, \Omega}^{\geq 0}$.

We also fix basis vectors $v_k$ for $V^k$, then they give a basis for each $V_i$. Under these basis, we have

$$E_{V, \Omega} = \text{Hom}(V_1, V_2) \cong \text{Mat}_{d_2 \times d_1}(\mathbb{C}),$$

$$E_{V^1, \Omega} = \text{Hom}(V_2, V_1) \cong \text{Mat}_{d_1 \times d_2}(\mathbb{C}),$$

$$G_V = \text{GL}(V_1) \times \text{GL}(V_2) \cong \text{GL}(d_1, \mathbb{C}) \times \text{GL}(d_2, \mathbb{C}).$$

Let $B_i$ be the Borel subgroup of $\text{GL}(d_i, \mathbb{C})$, consisting of upper triangular matrices with unipotent radical $U_i$, then

$$G_{V^1, \cdots, V^d}^{\geq 0} \cong B_1 \times B_2, \quad G_{V^1, \cdots, V^d}^+ \cong U_1 \times U_2.$$

To describe the elements of $E_{V^1, \cdots, V^d, \Omega}^{\geq 0}$ in terms of matrices, we should turn a $d_2 \times d_1$ matrix into a block matrix by requiring the first row in $k$-th row block is row $\nu_k$ and the the first column in $k$-th column block is column $\mu_k$. There are $f - 1$ row blocks and $e - 1$ column blocks. Depending on the previous four cases, we will
get the following shape of elements $x = \{X_{i,j}\} \in E_{V_1, \ldots, V_d, \Omega}^{\geq 0}$. We will use * and 0 to indicate the blocks. In case (1),

$$
\begin{pmatrix}
* & * & \cdots & * & * \\
0 & * & \cdots & * & * \\
\vdots & \vdots & \ddots & \vdots & * \\
0 & 0 & \cdots & * & * \\
0 & 0 & \cdots & 0 & *
\end{pmatrix}.
$$

In case (2),

$$
\begin{pmatrix}
* & * & \cdots & * \\
0 & * & \cdots & * \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & * \\
0 & 0 & \cdots & 0
\end{pmatrix}.
$$

In these two cases, if $X_{i,j} \neq 0$, then

$$
\nu_k \leq i < \nu_{k+1} \text{ for } k < f \Rightarrow j \geq \mu_k.
$$

In case (3),

$$
\begin{pmatrix}
0 & * & \cdots & * \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & * \\
0 & 0 & \cdots & 0
\end{pmatrix}.
$$

In case (4),

$$
\begin{pmatrix}
0 & * & \cdots & * \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & * \\
0 & 0 & \cdots & 0
\end{pmatrix}.
$$

In these two cases, if $X_{i,j} \neq 0$, then

$$
\nu_k \leq i < \nu_{k+1} \text{ for } k < f \Rightarrow j \geq \mu_{k+1}.
$$

Similarly, for $y = \{Y_{i,j}\} \in E_{V_1, \ldots, V_d, \bar{\Omega}}^{\geq 0}$. In case (1) and (2), if $Y_{i,j} \neq 0$, then

$$
\mu_k \leq i < \mu_{k+1} \text{ for } k < e \Rightarrow j \geq \nu_{k+1}.
$$

In case (3) and (4), if $Y_{i,j} \neq 0$, then

$$
\mu_k \leq i < \mu_{k+1} \text{ for } k < e \Rightarrow j \geq \nu_k.
$$

5.1. **Vanishing cycle.** For $y_0 \in E_{V_1, \ldots, V_d, \Omega}^{\geq 0}$, we want to compute the vanishing cycle of

$$
\hat{h}_{y_0} : G_V \times E_{V_1, \ldots, V_d, \Omega}^{\geq 0} \to \mathbb{C}, (g, x) \mapsto \langle gx, y_0 \rangle
$$

at $(1, x_0)$, where $(x_0, y_0) \in \Lambda_V$. First we would like to show that it suffices to consider those $y_0$ in nice shape. Suppose $y'_0 = g' \cdot y_0$ for $g' \in G_{V_1, \ldots, V_d}^{\geq 0}$, then

\[
\begin{array}{ccc}
(g, x) & \xrightarrow{G_V \times E_{V_1, \ldots, V_d, \Omega}^{\geq 0}} & \mathbb{C} \\
\downarrow & & \downarrow h_{y_0} \\
(g' g' g^{-1}, g' \cdot x) & \xrightarrow{G_V \times E_{V_1, \ldots, V_d, \Omega}^{\geq 0}} & \mathbb{C} \\
\downarrow & & \downarrow h_{y_0'}
\end{array}
\]
Let \( x'_0 = g' \cdot x_0 \), then it is the same to consider the vanishing cycle of \( h_{y_0} \) at \((1, x'_0)\). So we can change \( y_0 \) by the action of \( G_{V^1, \ldots, V^d}^{\geq 0} = B_1 \times B_2 \). Note the action of \( B_1 \) on \( y_0 \) is by row operations and the action of \( B_2 \) on \( y_0 \) is by column operations. For each nonzero column \( s \) of \( y_0 \), let \( \alpha_s \) be the first nonzero entry from the bottom. By the action of \( B_1 \), we can make all entries above \( \alpha_s \) be zero. Then by the action of \( B_2 \), we can make all entries on the right of \( \alpha_s \) be zero. If we do this process from the first column to the last column, then we can make each row and column of \( y_0 \) contain at most one nonzero entry, which can be further normalized to be one. From now on, we will assume \( y_0 \) satisfies this property. Let us index the nonzero entries in \( y_0 \) by a set \( A \) and \( \alpha \in A \) corresponds to the entry \((i_\alpha, j_\alpha)\). Let \( I = \{i_\alpha \mid \alpha \in A\} \), \( J = \{j_\alpha \mid \alpha \in A\} \). Our choice of \( y_0 \) has the following consequence on \( x_0 \).

**Lemma 5.1.** \( X_{ij} = 0 \) for \( j \in I \) or \( i \in J \) at \( x_0 \)

**Proof.** Since \([x_0, y_0] = 0\), then \( x_0 y_0 = y_0 x_0 = 0 \). The result follows immediately from our assumption on \( y_0 \). \( \square \)

Let \( \check{U}_i \) the unipotent radical of the opposite Borel subgroup \( \check{B}_i \). Then the map

\[
(U_1 \times \check{U}_2) \times (B_1 \times B_2) \rightarrow GL(d_1, \mathbb{C}) \times GL(d_2, \mathbb{C}), \quad (u, b) \mapsto ub
\]

is smooth. By smooth base change, it suffices to consider the vanishing cycle of the pullback

\[
\tilde{h}_{y_0} : (U_1 \times \check{U}_2) \times (B_1 \times B_2) \times E_{V^1, \ldots, V^d, \Omega}^{\geq 0} \rightarrow \mathbb{C}
\]

One can also consider the composition of the projection

\[
pr : (U_1 \times \check{U}_2) \times (B_1 \times B_2) \times E_{V^1, \ldots, V^d, \Omega}^{\geq 0} \rightarrow (U_1 \times \check{U}_2) \times E_{V^1, \ldots, V^d, \Omega}^{\geq 0}
\]

with the restriction of \( h_{y_0} \) to \((U_1 \times \check{U}_2) \times E_{V^1, \ldots, V^d, \Omega}^{\geq 0}\), and we denote it by \( \bar{h}_{y_0} \). Then we have a commutative diagram

\[
\begin{array}{ccc}
(U_1 \times \check{U}_2) \times (B_1 \times B_2) \times E_{V^1, \ldots, V^d, \Omega}^{\geq 0} & \rightarrow & \mathbb{C} \\
| & & | \\
(U_1 \times \check{U}_2) \times (B_1 \times B_2) & \downarrow & \downarrow \\
(u, b, x) & \Rightarrow & (u, b, x) \\
\end{array}
\]

So it is the same to consider \( \tilde{h}_{y_0} \). Finally, since \( \tilde{h}_{y_0} \) factors through \( pr \), by smooth base change it suffices to consider the restriction of \( h_{y_0} \) to \((U_1 \times \check{U}_2) \times E_{V^1, \ldots, V^d, \Omega}^{\geq 0}\), and we denote it still by \( \tilde{h}_{y_0} \). So we have shown

**Lemma 5.2.** \( R\Phi_{h_{y_0}}(1)_{(1,x_0)} \cong R\Phi_{\tilde{h}_{y_0}}(1)_{(1,x_0)} \).

Let us denote the entries of \( g^{-1}_1 \) by \( M_{ij} \) and that of \( g_2 \) by \( N_{ij} \). We want to calculate \( \tilde{h}_{y_0} \) explicitly,

\[
\tilde{h}_{y_0}(g_1, g_2, x) = \text{tr}(x g^{-1}_1 y_0 g_2)
\]

\[
= \sum_{i, j, s > i} X_{ij} Y_{js} N_{si} + \sum_{i, j, r < j} X_{ij} M_{jr} Y_{rs} + \sum_{i, j, s > i} X_{ij} M_{jr} Y_{rs} N_{si}
\]

\[
= \sum_{\alpha \in A, i} X_{i_\alpha} N_{j_\alpha i} + \sum_{\alpha' \in A, j} X_{j_{\alpha'} i_\alpha} M_{j_{\alpha'} i_\alpha} + \sum_{\alpha'' \in A, i, j} X_{ij} M_{j_{\alpha''} i_\alpha} N_{j_{\alpha''} i_\alpha}.
\]

There are three terms in the summation. We first consider \( X_{ij} \) appearing in both of the first two terms. They are necessarily of the form \( X_{j_{\alpha'} i_\alpha} \). Let us define

\[
T = \{(\alpha', \alpha) \in A^2 | X_{j_{\alpha'} i_\alpha} \neq 0\}.
\]

We have an inclusion

\[
\pi : T \rightarrow J \times I, \quad (\alpha', \alpha) \mapsto (j_{\alpha'}, i_\alpha).
\]

**Lemma 5.3.** For any \((\alpha', \alpha) \in T\), \( X_{j_{\alpha'} i_\alpha} \) appears in both of the first two terms.
Then we need to make some change of variables. For any $T$ to see this is well-defined, we impose a partial order on $T$ which only appear in (16). Note in the latter case, $X$ which only appear in (16) and (21), and deal with it remains to quadratic variables (called (21)) set $M_{i\alpha}^{\prime} = N_{j\alpha}j_{\alpha} + M_{i\alpha}i_{\alpha} + \sum_{\alpha'' \in A_{j\alpha}j_{\alpha} > j_{\alpha}^{\prime}}^{i_{\alpha}'' < i_{\alpha}} M_{iai_{\alpha}''} N_{iai_{\alpha}''j_{\alpha}^{'}}. \right)

The remaining terms are

\[ \sum_{\alpha \in A, j \notin J} X_{i\alpha} N_{ja}^i \]
\[ \sum_{\alpha' \in A, j \notin J} X_{j\alpha}^i M_{ji}^{i\alpha'} \]
\[ \sum_{\alpha'' \in A, i \notin J} X_{ij} M_{ji}^{i\alpha''} N_{ja}^i \]
\[ \sum_{\alpha'' \in A, i \notin J} X_{ij} M_{ji}^{i\alpha''} N_{ja}^i \]
\[ \sum_{\alpha'' \in A, i \notin J} X_{ij} M_{ji}^{i\alpha''} N_{ja}^i. \]

The goal is to separate the variables so that the function can be viewed as a quadratic form in some variables (called quadratic variables) with coefficients in different variables (called coefficient variables). First we set $M_{ji} (j \notin I, i \in I), N_{ji} (j \in J, i \notin J)$ and $X_{ij} (i \notin J, j \notin J)$ or $(i \notin J, j \in I)$ quadratic variables; set $N_{ji} (j \notin J, i \notin J), X_{ij} (i \notin J, j \notin J)$ coefficient variables. Note the variables $M_{ji} (i \notin I), N_{ji} (j \notin J)$ will never appear, so we can set them arbitrary. (We will set them as coefficient variables if not specified.) It remains to deal with

\[ M_{ji} \quad (j \in I, i \in I) \]

which only appear in (16) and (21), and

\[ X_{ij} \quad (i \in J, j \in I) \]

which only appear in (16). Note in the latter case, $X_{ij} \neq 0$ only when $(i, j) \in \pi(T)$. To achieve our goal, we need to make some change of variables. For any $(\alpha', \alpha) \in T$, \[ M_{iai_{\alpha}^{\prime}} = N_{jai_{\alpha}^{\prime}} + M_{iai_{\alpha}^{\prime}} + \sum_{\alpha'' \in A_{jai_{\alpha}^{\prime} > j_{\alpha}^{'}}}^{i_{\alpha}'' < i_{\alpha}} M_{iai_{\alpha}''} N_{iai_{\alpha}''j_{\alpha}^{'}}. \right)\]

To see this is well-defined, we impose a partial order on $T$ such that

\[ (\alpha', \alpha) >_T (\alpha'', \alpha) \quad \text{if} \quad j_{\alpha'} < j_{\alpha''} \]

Then

\[ M_{iai_{\alpha}} = M_{iai_{\alpha}^{\prime}} + \sum_{(\alpha'', \alpha) \in T \quad (\alpha'', \alpha) >_T (\alpha', \alpha)} U_{jai_{\alpha}''}^{\alpha} M_{iai_{\alpha}''} + U_{jai_{\alpha}''}^{\alpha} \]

\[ \text{Proof.} \quad \text{By the shape of } x \text{ and } y_0, \text{ we know if } X_{jai_{\alpha}^{\prime}} \neq 0, \text{ then } j_{\alpha'} > j_{\alpha''} \text{ and } i_{\alpha'} > i_{\alpha''}. \text{ The rest is clear.} \]
where \( \mathcal{U}_{j^{(i)},a^{(i)}}^{\alpha} \) are polynomials in \( N_{ij} \) (\( i, j \in J \)), and \( \mathcal{U}_{j^{(i)},a^{(i)}}^{\alpha} \) are polynomials in \( N_{ij} \) (\( i, j \in J \)) and \( M_{i^{(i)},a^{(i)}} \) for any \( (\alpha'', \alpha) \notin T \). Set \( \mathcal{U}_{j^{(i)},a^{(i)}}^{\alpha} = 1 \). After this change of variables, (16) becomes

\[
(23) \quad \sum_{(\alpha', \alpha) \in T} X_{j^{(i)},a^{(i)}} M'_{i^{(i)},a^{(i)}}.
\]

We can also split (21) into two parts:

\[
(21)a : \sum_{(\alpha'', \alpha) \notin T, i \notin J} X_{i^{(i)},a^{(i)}} M_{i^{(i)},a^{(i)}} N_{j^{(i)},a^{(i)}}
\]
and

\[
(21)b : \sum_{(\alpha'', \alpha) \in T, i \notin J} X_{i^{(i)},a^{(i)}} M_{i^{(i)},a^{(i)}} N_{j^{(i)},a^{(i)}}
\]

Substitute (22) into (21)b, we get

\[
\sum_{(\alpha'', \alpha) \in T, i \notin J} X_{i^{(i)},a^{(i)}} M_{i^{(i)},a^{(i)}} N_{j^{(i)},a^{(i)}} = \sum_{(\alpha'', \alpha) \notin T, i \notin J} X_{i^{(i)},a^{(i)}} M_{i^{(i)},a^{(i)}} N_{j^{(i)},a^{(i)}} + \sum_{(\alpha'', \alpha) \in T} \mathcal{U}_{j^{(i)},a^{(i)}}^{\alpha} M'_{i^{(i)},a^{(i)}} + \mathcal{U}_{j^{(i)},a^{(i)}}^{\alpha} N_{j^{(i)},a^{(i)}}
\]

Combined with (23), we get

\[
\sum_{(\alpha', \alpha) \in T} \left( X_{j^{(i)},a^{(i)}} + \sum_{(\alpha'', \alpha) \notin T, i \notin J} X_{i^{(i)},a^{(i)}} \mathcal{U}_{j^{(i)},a^{(i)}}^{\alpha} M'_{i^{(i)},a^{(i)}} + \sum_{(\alpha'', \alpha) \in T} X_{i^{(i)},a^{(i)}} \mathcal{U}_{j^{(i)},a^{(i)}}^{\alpha} N_{j^{(i)},a^{(i)}} \right)
\]

For \( (\alpha', \alpha) \in T \), let

\[
(24) \quad X'_{j^{(i)},a^{(i)}} = X_{j^{(i)},a^{(i)}} + \sum_{(\alpha'', \alpha) \notin T, i \notin J} X_{i^{(i)},a^{(i)}} \mathcal{U}_{j^{(i)},a^{(i)}}^{\alpha} N_{j^{(i)},a^{(i)}}.
\]

Substitute \( X'_{j^{(i)},a^{(i)}} \) into the previous expression, we get

\[
(25) \quad \sum_{(\alpha', \alpha) \in T} X'_{j^{(i)},a^{(i)}} M'_{i^{(i)},a^{(i)}} + \sum_{(\alpha'', \alpha) \notin T, i \notin J} X_{i^{(i)},a^{(i)}} \mathcal{U}_{j^{(i)},a^{(i)}}^{\alpha} N_{j^{(i)},a^{(i)}}.
\]

In sum, after the substitutions by \( M'_{i^{(i)},a^{(i)}} \) and \( X'_{j^{(i)},a^{(i)}} \) for all \( (\alpha', \alpha) \in T \), we see

\[
(16) + (21)b = (25),
\]

Combined with (21)a, we rewrite them as

\[
(16)' : \sum_{(\alpha', \alpha) \in T} X'_{j^{(i)},a^{(i)}} M'_{i^{(i)},a^{(i)}}
\]
and

\[
(21)' : \sum_{i \notin J, (\alpha'', \alpha') \notin T} X_{i^{(i)},a^{(i)}} M_{i^{(i)},a^{(i)}} N_{j^{(i)},a^{(i)}} + \sum_{i \notin J, (\alpha'', \alpha') \in T} X_{i^{(i)},a^{(i)}} \mathcal{U}_{j^{(i)},a^{(i)}}^{\alpha} N_{j^{(i)},a^{(i)}}
\]

so

\[
\text{tr}(xyg_1^{-1}y_0g_2) = (16)' + (17) + (18) + (19) + (20) + (21)' .
\]

We will set \( M'_{i^{(i)},a^{(i)}} \) and \( X'_{j^{(i)},a^{(i)}} \) for \( (\alpha', \alpha) \in T \) as quadratic variables. We will also set the variables \( M_{i^{(i)},a^{(i)}} \) for \( (\alpha', \alpha) \notin T \) as coefficient variables.
5.2. Euler characteristic of Milnor fiber. We want to compute
\[ \chi(R\Psi_{h_{y_{0}}}[-1](1)(1,x_{0})) = 1 - \chi(R\Psi_{h_{y_{0}}}(1)(1,x_{0})). \]
The idea is to relate \( \chi(R\Psi_{h_{y_{0}}}(1)(1,x_{0})) \) with the Euler characteristic of the Milnor fiber for \( \tilde{h}_{y_{0}} \) at \( (1,x_{0}) \). We will recall the definition of the Milnor fiber below.

Let \( f \) be an analytic function germ at the origin of \( \mathbb{C}^{n+1} \) with \( f(0) = 0 \). Let
\[ B_{\epsilon} := \{ z \in \mathbb{C}^{n+1} | |z_{0}|^2 + \cdots |z_{n}|^2 < \epsilon \} \]
and \( S_{\epsilon}^{2n+1} = \partial B_{\epsilon} \).

**Theorem 5.4 (Milnor [11]).**
\[ \varphi_{\epsilon} : S_{\epsilon}^{2n+1}\setminus f^{-1}(0) \longrightarrow \mathbb{S}^{1}, \; z \mapsto f(z)/|f(z)| \]
is a smooth locally trivial fibration for \( \epsilon \) sufficiently small.

**Definition 5.5.** For any \( \theta \in \mathbb{S}^{1} \) and \( \epsilon \) sufficiently small as in the above theorem, \( \varphi_{\epsilon}^{-1}(\theta) \) is called the Milnor fiber of \( f \) at the origin.

To compare with the nearby cycle, we consider another description of the Milnor fiber. For \( 0 < \delta \ll \epsilon \), let
\[ D_{\delta}^{s} = \{ t \in \mathbb{C} | 0 < |t| < \delta \}. \]

**Theorem 5.6 (Lê [15]).**
\[ \psi : B_{\epsilon} \cap f^{-1}(D_{\delta}^{s}) \longrightarrow D_{\delta}^{s} \]
is a smooth locally trivial fibration for \( 0 < \delta \ll \epsilon \) both sufficiently small.

**Proposition 5.7.** For sufficiently small \( \delta, \epsilon \) as in the above theorem and any \( a \in D_{\delta}^{s} \), \( \psi^{-1}(a) \) is diffeomorphic to the Milnor fiber of \( f \) at the origin.

**Proof.** Cf. [4, Proposition 1.4]. \( \square \)

As a consequence, we can also define the Milnor fiber to be \( \psi^{-1}(a) \). By [14, Lemma 1.1.1],

(26) \[ \chi(R\Psi_{f}(1)) \cong \chi(\psi^{-1}(a)). \]

Now let us assume \( f(z) \) is a homogeneous polynomial of degree \( N \). Following [4], we call \( f^{-1}(1) \) the global Milnor fiber of \( f \) at the origin.

**Proposition 5.8.** \( f^{-1}(1) \) is diffeomorphic to the Milnor fiber of \( f \) at the origin.

**Proof.** We can construct a homeomorphism \( g \) such that the following diagram commutes
\[
\begin{array}{ccc}
S_{\epsilon}^{2n+1}\setminus f^{-1}(0) & \xrightarrow{g} & f^{-1}(\mathbb{S}^{1}) \\
\varphi & \swarrow & f \\
\mathbb{S}^{1} & \xleftarrow{f} & \mathbb{S}^{1}
\end{array}
\]

Here
\[ g : z \mapsto |f(z)|^{-\frac{1}{N}} \cdot z. \]

Note for any \( z \in f^{-1}(\mathbb{S}^{1}) \), there exists unique \( t \in \mathbb{R}_{+} \) such that \( t^{\frac{1}{N}} \cdot z \in S_{\epsilon}^{2n+1} \). One can check that this gives the inverse. \( \square \)

**Remark 5.9.** Since the above diagram holds for all \( \epsilon \), then in the case of homogeneous polynomials the Milnor fiber at the origin is homeomorphic to \( \varphi_{\epsilon}^{-1}(\theta) \) for any \( \epsilon \).
5.3. Application. Let \( f = \tilde{h}_{y_0}(\langle g_1, g_2, x \rangle) \), whose variables are denoted by \((X_{ij}, M_{ij}, N_{ij})\). After change of variables in Section 5.1 we denote the set of new variables by \( z = (X'_{ij}, M'_{ij}, N'_{ij}) \). Then we want to compute the Euler characteristic of the Milnor fiber of \( f(z) \) at the point \( z^0 = (x'_{ij}, 0, 0) \). We choose a small ball around this point

\[
B_\varepsilon := \left\{ (X'_{ij}, M'_{ij}, N'_{ij}) \mid \sum_{ij} |X'_{ij} - x'_{ij}|^2 + \sum_{ij} |M'_{ij}|^2 + \sum_{ij} |N'_{ij}|^2 \leq \varepsilon \right\}
\]

such that

\[
\varphi_\varepsilon = f/|f| : S_\varepsilon \setminus f^{-1}(0) \to \mathbb{S}^1
\]
is a smooth fibration as in Theorem 5.4. Let \( V \) be the subset of coefficient variables and \( W \) be the subset of quadratic variables. Let

\[
\tilde{B}_\varepsilon^V := \left\{ (X'_{ij}, M'_{ij}, N'_{ij})_V \mid \sum_{ij} |X'_{ij} - x'_{ij}|^2 + \sum_{ij} |M'_{ij}|^2 + \sum_{ij} |N'_{ij}|^2 \leq \varepsilon \right\}
\]

which is the projection of \( \tilde{B}_\varepsilon \) onto the coefficient variables. It is a ball around the projection \( z_V^0 \) of \( z^0 \). For any \( z_V \in \tilde{B}_\varepsilon^V \), we have

\[
|z_V - z_V^0|^2 \leq \varepsilon.
\]

Let

\[
f_{z_V}(z_W) = f(z_V, z_W) \quad \text{for} \quad z_W = (X'_{ij}, M'_{ij}, N'_{ij})_W.
\]

It is a quadratic form. Let \( z_W^0 \) be the projection of \( z^0 \) to the quadratic variables. Note \( X'_{ij} \in W \) if and only if \( i \in J \) or \( j \in I \). By Lemma 5.1 and the formula (24) we have \( z_W^0 = 0 \). Let

\[
\varphi_{z_V, \varepsilon - |z_V - z_V^0|^2} = f_{z_V}/|f_{z_V}| : S_{\varepsilon - |z_V - z_V^0|^2}^{2|W|-1} \setminus f_{z_V}^{-1}(0) \to \mathbb{S}^1
\]

So we have a diagram

\[
\begin{array}{ccc}
S_{\varepsilon - |z_V - z_V^0|^2}^{2|W|-1} \setminus f_{z_V}^{-1}(0) & \to & \mathbb{S}^1 \\
\downarrow & & \downarrow \\
S_\varepsilon \setminus f^{-1}(0) & \to & \mathbb{S}^1 \\
\pi_V & & \\
\tilde{B}_\varepsilon^V & & \\
\end{array}
\]

which gives a fibration of the Milnor fiber \( \varphi_\varepsilon^{-1}(\theta) \) for some \( \theta \in \mathbb{S}^1 \) over a closed subset \( C_\varepsilon^V \) of \( \tilde{B}_\varepsilon^V \). In view of Remark 5.9 the fiber \( \varphi_\varepsilon^{-1}_{z_V, \varepsilon - |z_V - z_V^0|^2}(\theta) \) is homeomorphic to the Milnor fiber of \( f_{z_V} \) at the origin. By (26),

\[
(27) \quad \chi((\varphi_\varepsilon^{-1}_{z_V, \varepsilon - |z_V - z_V^0|^2}(\theta))) = \chi(R\Psi f_{z_V}(1)_0).
\]

Next we would like to compute the Euler characteristic of \( \varphi_\varepsilon^{-1}_{z_V, \varepsilon - |z_V - z_V^0|^2}(\theta) \) through the nearby cycle. To do so, we need the following lemma.

**Lemma 5.10.** rank \( \text{Hessian}(f_{z_V})_0 \) is even.

**Proof.** We can divide the set \( W \) of variables into two classes:

\[
W_1 = \{ M'_{ia} \mid (\alpha', \alpha) \in T \} \cup \{ M'_{ji} \mid j \notin I, i \in I \} \cup \{ X'_{ij} \mid i \notin J, j \in I \}
\]

and

\[
W_2 = \{ X'_{ia} \mid (\alpha', \alpha) \in T \} \cup \{ N'_{ji} \mid j \notin J, i \notin J \} \cup \{ X'_{ij} \mid i \in J, j \notin I \}
\]

such that the Hessians of the restrictions of \( f_{z_V} \) to variables in \( W_1 \) (resp. \( W_2 \)) are both zero at 0. Then the Hessian must be in the form

\[
\text{Hessian}(f_{z_V}) = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}.
\]

So its rank is even. \( \square \)
Corollary 5.11. \( \chi(R\Psi_{f_{0,1}}(1)_0) = 0. \)

Proof. Since \( f_{z_V} \) is a quadratic form, we can change the coordinates such that

\[
f_{z_V}(u) = \sum_{i=1}^{r} u_i^2
\]

where \( r = \text{rank Hessian}(f_{z_V}) \) is even by the previous lemma. By Sebastiani-Thom theorem,

\[
R\Phi_{f_{z_V}}[-1](1)_0 = \mathbb{C}[-r].
\]

Hence \( \chi(R\Phi_{f_{z_V}}[-1](1)_0) = (-1)^r = 1. \) It follows \( \chi(R\Phi_{f_{z_V}}(1)_0) = 0. \)

\[\square\]

Proposition 5.12. \( \chi(R\Psi_{f}(1)_0) = 0. \)

Proof. We have \( \chi(R\Psi_{f}(z_V)\theta) = \chi(\varphi^{-1}(\theta)), \) where \( \theta \in S^1. \) The latter admits a fibration over \( C^V \subseteq B^V \) with fibers \( \varphi_{z_V,\epsilon,-|z_V-x'|-\epsilon}^{-1}(\theta). \) By the Leray spectral sequence

\[
H^p(C^V, (R^q\pi_V)_*1) \Rightarrow H^n(\varphi_{z_V}^{-1}(\theta)),
\]

we have

\[
\chi(\varphi_{z_V}^{-1}(\theta)) = \sum_{p,q} (-1)^{p+q} \dim H^p(C^V, (R^q\pi_V)_*1) = \sum_q (-1)^q \chi(H^*(C^V, (R^q\pi_V)_*1))
\]

\[
= \sum_q (-1)^q \int_{C^V} \chi((R^q\pi_V)_*1) = \int_{C^V} \chi((R\pi_V)_*1).
\]

By Corollary 5.11 and [27],

\[
\chi((R\pi_V)_*1)_{z_V} = \chi(\pi_V^{-1}(z_V)) = \chi(\varphi_{z_V,\epsilon,-|z_V-x'|-\epsilon}^{-1}(\theta)) = 0.
\]

Hence, \( \chi(\varphi_{z_V}^{-1}(\theta)) = 0. \)

\[\square\]

This completes the proof of Conjecture 4.25 for type \( A_2 \) quiver.

6. Appendix

6.1. Regularity of stratification. Let \( X \) be a closed subset of a smooth real manifold \( M \) of dimension \( m. \) A smooth stratification of \( X \) is a filtration

\[
X_0 \subseteq \cdots \subseteq X_n \subseteq X
\]

by closed subsets such that \( X^i := X_i \setminus X_{i-1} \) is a smooth \( i \)-dim submanifold of \( M. \) For \( j > i, \) we say \( X^j \) is \( w \)-regular over \( X^i \) at \( x \in X^i \cap X^j \) if there exists a neighborhood \( U \) of \( x \) in \( M \) and constant \( C \) such that in suitable local coordinates

\[
d(T_{x'}X^i, T_{x''}X^j) < C\|x'' - x'\|
\]

for all \( x' \in U \cap X^i, x'' \in U \cap X^j. \) Here we have chosen a norm \( \| \cdot \| \) on \( \mathbb{R}^m \) and identify \( T_{x'}X^i, T_{x''}X^j \) with subspaces of \( \mathbb{R}^m. \) We define the distance between any two subspaces \( V, W \) of \( \mathbb{R}^m \) to be

\[
d(V, W) := \sup_{v \in V, \|v\| = 1} d(v, W)
\]

In our setting, we will take \( M \) to be a complex variety and \( X_1 \) to be semialgebraic subsets of \( M. \) By [14] Remark 4.1.9, we have

\[
\text{w-regular} \Rightarrow \text{Whitney a-regular, b-regular and d-regular}
\]

So in order to apply [14] Theorem 5.3.3, it suffices to show w-regularity.

Proposition 6.1. The stratification of \( E_{V,\Omega} \) by \( G_V \)-orbits is w-regular.
Proof. Let $S_i, S_j$ be $G_V$-orbits such that $\tilde{S}_j \supseteq S_i$. For $x \in S_i$, we choose a small neighborhood $U$ of $x$ in $E_{\Omega} \cap S_i$ such that $\overline{U \cap S_i} \subseteq S_i$ is compact. For any $x' \in U \cap S_i$, let $T_xS_i = [g_V, x']$. We fix a norm on $E_{\Omega}$. Let $N(x')$ be the subspace of $g_V$ orthogonal to the kernel of

$$g_V \to E_{\Omega}, \quad h \mapsto [h, x']. $$

Let

$$C(x') := \sup\{\|h\| : h \in N(x') \text{ and } \|h, x'\| = 1\}$$

It is bounded by some positive constant $C$ on $U \cap S_i$. For $x'' \in U \cap S_j, x' \in U \cap S_i$, we can find $h \in g_V$ with $\|h\| \leq C$ such that

$$d(T_xS_i, T_{x''}S_j) = d([h, x'], [h, x'']) \leq d([h, x'], [h, x''])$$

for some positive constant $C'$ independent of $h, x', x''$.

$\square$

6.2. A vanishing cycle calculation. We will prove Proposition 6.4 following that of [3, Theorem 6.7.5]. Let us recall the statement.

**Proposition 6.2.** $R\Phi_f[-1](\mathbb{1}_{S \times \tilde{S}})(x, y) = \mathbb{C}[\dim \Lambda \dim \tilde{S} - \dim S]$, where $f$ is the restriction of $(\cdot, \cdot)$ to $S \times \tilde{S}$.

First we need to make some preparations.

**Lemma 6.3.** The function $f$ is singular over $T^*_S(E_{\Omega})_{\text{reg}}$, i.e., $df|_{T^*_S(E_{\Omega})_{\text{reg}}} = 0$.

*Proof.* For any $(x, y) \in T^*_S(E_{\Omega})_{\text{reg}}$ and $u \in T_xS$, we have $df(x, y)(u) = \langle u, y \rangle = 0$. Similarly, we have $df(x, y)(v) = \langle x, v \rangle = 0$ for any $v \in T_y\tilde{S}$. This finishes the proof.

*Fix* $(x, y) \in T^*_S(E_{\Omega})_{\text{reg}}$ and let $N \subseteq S \times \tilde{S}$ be a normal slice to $T^*_S(E_{\Omega})_{\text{reg}}$ at $(x, y)$. In particular, we require $N \cap T^*_S(E_{\Omega})_{\text{reg}} = (x, y)$. The key step is to show

**Proposition 6.4.** The Hessian of $f$ at $(x, y)$ has rank $\dim S + \dim \tilde{S} - n = \dim N$. Moreover, the Hessian of $f|_N$ at $(x, y)$ is non-degenerate.

We can pull back $f$ to the Lie algebras $g_V \times g_V$ of $G_V \times G_V$ near a neighborhood of $(x, y)$ as follows

$$F(h_1, h_2) = \langle \exp(h_1)x, \exp(h_2)y \rangle : g_V \times g_V \to \mathbb{C}.$$

It is easy to see that the rank of Hessian of $f$ at $(x, y)$ is the same as that of $F$ at $(0, 0)$.

**Lemma 6.5.** In a small neighborhood of $0$ in $g_V$, one can express

$$\exp(h)x = x + [h, x] + \frac{1}{2}[h, [h, x]] + \cdots + \frac{1}{n!}[h, [h, \ldots, [h, x] \ldots]] + \cdots$$

$$\exp(h)y = y + [h, y] + \frac{1}{2}[h, [h, y]] + \cdots + \frac{1}{n!}[h, [h, \ldots, [h, y] \ldots]] + \cdots$$

*Proof.* For any $h \in g_V$ we can find real number $\delta > 0$, which only depends on the norm of $h$, such that the vector-valued function

$$G(t) := \exp(th)x : [-\delta, \delta] \to V,$$

can be expressed as

$$G(t) = G(0) + G'(0)t + \frac{1}{2}G''(0)t^2 + \cdots + \frac{1}{n!}G^{(n)}(0)t^n + \cdots$$

Since

$$G^{(n)}(t) = \exp(th)[h, [h, \ldots, [h, x] \ldots]]$$

then

$$\exp(th)x = x + [th, x] + \frac{1}{2}[th, [th, x]] + \cdots + \frac{1}{n!}[th, [th, \ldots, [th, x] \ldots]]t^n + \cdots$$

$$= x + [th, x] + \frac{1}{2}[th, [th, x]] + \cdots + \frac{1}{n!}[th, [th, \ldots, [th, x] \ldots]] + \cdots$$
This proves the first equality. The second equality can be proved in the same way.

Let us write \( Z_1(h) = \exp(h)x - (x + [h, x]) \) and \( Z_2(h) = \exp(h)y - (y + [h, y]) \). Then

\[
F(h_1, h_2) = (x + [h_1, x], y + [h_2, y]) + (x + [h_1, x], Z_2(h_2)) + \langle Z_1(h_1), y + [h_2, y] \rangle + \langle Z_1(h_1), Z_2(h_2) \rangle
\]

The degree 2 terms in the above expression can only come from \( \langle [h_1, x], [h_2, y] \rangle, \langle x, Z_2(h_2) \rangle \) and \( \langle Z_1(h_1), y \rangle \).

Therefore,

\[
\text{Hessian}(F)(0,0) = \begin{pmatrix}
\text{Hessian}(\langle Z_1(h_1), y \rangle)(0,0)
& B \\
B^T & \text{Hessian}(\langle x, Z_2(h_2) \rangle)(0,0)
\end{pmatrix}
\]

where

\[
B = \left( \frac{\partial^2}{\partial h_1 \partial h_2} \langle [h_1, x], [h_2, y] \rangle \right)(0,0)
\]

It is not hard to see that \( B \) corresponds to the bilinear form

\[\langle [h_1, x], [h_2, y] \rangle : g_V \times g_V \to \mathbb{C}\]

after we identify \( T_{(0,0)}(g_V \times g_V) \) with \( g_V \times g_V \). Since

\[\langle [h_1, x], [h_2, y] \rangle = \langle [y, [h_1, x]], h_2 \rangle,
\]

then the rank of the above bilinear form is \( \dim [y, [g_V, x]] \). So we have shown

**Lemma 6.6.** \( \text{rank Hessian}(F)(0,0) \geq \dim [y, [g_V, x]] \).

Next, we would like to show

**Lemma 6.7.** \( \dim [y, [g_V, x]] = \dim S + \dim \widehat{S} - \dim \Lambda_V \).

**Proof.** It is easy to see that \( [g_V, x] = T_\alpha S \) and \( \text{Ker}[y, \cdot]|_{E_V;\Omega} = T^*_\alpha E_V;\Omega \). Since \( (x, y) \) is regular, \( T^*_\alpha E_V;\Omega \cap S \times \{ y \} \) contains an open neighborhood of \( (x, y) \) in \( T^*_\alpha E_V;\Omega \). Hence \( T^*_\alpha E_V;\Omega \subseteq T_\alpha S \). So \( \dim [y, [g_V, x]] = \dim T_\alpha S - \dim T^*_\alpha E_V;\Omega = \dim T_\alpha S - (\dim \Lambda_V - \dim T_y \widehat{S}) = \dim S + \dim \widehat{S} - \dim \Lambda_V \). \( \square \)

**Corollary 6.8.** \( T_{(x,y)}(T^*_\alpha(E_V;\Omega)) \subseteq \{(u, v) \in T_x S \times T_y \widehat{S} | [u, y] + [x, v] = 0 \} \).

**Proof.** For any \( (u, v) \in T_{(x,y)}(\Lambda_V) \), one can choose smooth \( \alpha : [0,1] \to V, \beta : [0,1] \to E_V;\Omega \) such that

\[
\alpha(0) = x, \beta(0) = y, d\alpha(1) = u, d\beta(1) = v
\]

and \( (\alpha(t), \beta(t)) \in \Lambda_V \). Then \( \alpha(t), \beta(t) = 0 \). Differentiate it at \( t = 0 \):

\[
0 = \lim_{t \to 0} \frac{1}{t} \left( [\alpha(t), \beta(t)] - [x, y] \right) = \lim_{t \to 0} \frac{1}{t} \left( [\alpha(t) - x, \beta(t)] + [x, \beta(t) - y] \right)
\]

\[
= \left[ \lim_{t \to 0} \frac{\alpha(t) - x}{t}, \lim_{t \to 0} \frac{\beta(t) - y}{t} \right] = [u, y] + [x, v]
\]

It follows

\[
T_{(x,y)}(\Lambda_V) \subseteq \{(u, v) \in T^*_\alpha(E_V;\Omega) | [u, y] + [x, v] = 0 \}.
\]

Since \( (x, y) \) is regular, \( T_{(x,y)}(\Lambda_V) = T_{(x,y)}(T^*_\alpha(E_V;\Omega)) \subseteq T_x S \times T_y \widehat{S} \). So

\[
T_{(x,y)}(\Lambda_V) \subseteq \{(u, v) \in T_x S \times T_y \widehat{S} | [u, y] + [x, v] = 0 \}
\]

and it is enough to show the dimension of the right hand side is equal to \( \dim \Lambda_V \). Now let us consider

\[
\varphi(u, v) = [u, y] + [x, v] : T_x S \times T_y \widehat{S} \to g_V
\]

Note \( \text{Ker} \varphi = \{(u, v) \in T_x S \times T_y \widehat{S} | [u, y] + [x, v] = 0 \} \). The image of \( \varphi \) is \( \{[g_V, x], y] + [x, [g_V, y]] = [[g_V, x], y] \} \). By the previous lemma, \( \dim \text{Im} \varphi = \dim S + \dim \widehat{S} - \dim \Lambda_V \). Hence \( \dim \text{Ker} \varphi = \dim \Lambda_V \). This finishes the proof. \( \square \)
Next we would like to compute the Hessian of \( f \) at \((x, y)\) in a different way. Let us choose local coordinates for a neighborhood \( U \) of \((x, y)\) in \( S \times \hat{S} \) such that
\[
U \cap T^*_S(E_{V, \Omega})_{\text{reg}} = \{ \xi = (\xi_i)_{i=1}^m \in U \mid \xi_{n+1} = \cdots = \xi_m = 0 \}
\]
and
\[
U \cap N = \{ \xi = (\xi_i)_{i=1}^m \in U \mid \xi_1 = \cdots = \xi_n = 0 \}.
\]
By taking \( U \) sufficiently small, we can assume \( f|_U \) has analytic expansion
\[
f = \sum_{i_1, \ldots, i_l} c_{i_1, \ldots, i_l} \xi_{i_1}^{m_{i_1}} \cdots \xi_{i_l}^{m_{i_l}}.
\]
Since \( f \) is singular over \( T^*_S(E_{V, \Omega})_{\text{reg}} \), the above expression can not have terms \( \xi_i \xi_j \) with \( i \leq n, j > n \). Note \( f|_{T^*_S(E_{V, \Omega})_{\text{reg}}} = 0 \). So
\[
\text{Hessian}(f|_U)_0 = \begin{pmatrix} 0 & 0 \\ 0 & \text{Hessian}(f|_N)_0 \end{pmatrix}.
\]
It follows \( \text{rank Hessian}(f|_U)_0 = \text{rank Hessian}(f|_N)_0 \leq \dim N \). Combining Lemma 6.6 and Lemma 6.7 we have proved Proposition 6.4. Now we can prove Proposition 6.2.

**Proof.** Let us choose local coordinates for a neighborhood \( U \) of \((x, y)\) in \( S \times \hat{S} \) such that
\[
U \cap T^*_S(E_{V, \Omega})_{\text{reg}} = \{ \xi = (\xi_i)_{i=1}^m \in U \mid \xi_{n+1} = \cdots = \xi_m = 0 \}.
\]
Let
\[
N = \{ \xi = (\xi_i)_{i=1}^m \in U \mid \xi_1 = \cdots = \xi_n = 0 \},
\]
which is a normal slice to \( U \cap T^*_S(E_{V, \Omega}) \) at \((x, y)\). Since \( f \) vanishes and is singular on \( T^*_S(E_{V, \Omega})_{\text{reg}} \), we can assume
\[
f(\xi) = \sum_{i,j > n} \alpha_{i,j}(\xi) \xi_i \xi_j
\]
By Proposition 6.4, the Hessian of \( f|_N \) is non-degenerate. So we can make a change of coordinates
\[
\xi'_k = \sum_{k,l} b_{k,l}(\xi) \xi_l
\]
following the Graham-Schmidt process such that
\[
\left( b_{k,l}(\xi) \right)_{k,l} = \begin{pmatrix} I_n & 0 \\ 0 & B(\xi) \end{pmatrix}
\]
where \( B(\xi) \) is upper-triangular with constant function 1 on the diagonal, and
\[
f(\xi') = \sum_{i > n} \beta_i(\xi') \xi'^2_i.
\]
for \( \beta_i(\xi') \) nonzero on a small neighborhood \( W \subseteq U \). By choosing a branch of square roots, we can make a further change of coordinates by
\[
\xi''_i = \begin{cases} 
\xi'_i & \text{if } i \leq n \\
\sqrt{\beta_i(\xi')} \xi'_i & \text{if } i > n
\end{cases}
\]
Then
\[
f(\xi'') = \sum_{i > n} \xi''^2_i
\]
It follows from Sebastiani-Thom theorem that
\[
(R\Phi_f[-1]|_{(1_U)})(x,y) = \mathbb{C}[-\dim N].
\]
This finishes the proof. \(\square\)
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