ANALOGUES OF THE CENTRAL POINT THEOREM FOR FAMILIES WITH $d$-INTERSECTION PROPERTY IN $\mathbb{R}^d$

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Abstract. In this paper we consider families of compact convex sets in $\mathbb{R}^d$ such that any subfamily of size at most $d$ has a nonempty intersection. We prove some analogues of the central point theorem and Tverberg’s theorem for such families.

1. Introduction

Let us start with some definitions.

Definition 1.1. A family of sets $\mathcal{F}$ has property $\Pi_k$ if for any nonempty $\mathcal{G} \subseteq \mathcal{F}$ such that $|\mathcal{G}| \leq k$ the intersection $\bigcap \mathcal{G}$ is not empty.

Helly’s theorem [6] states that a finite family of convex sets (or any family of convex compact sets) with $\Pi_{d+1}$ property in $\mathbb{R}^d$ has a common point. In the review [3] Helly’s theorem and a lot of its generalizations are considered in detail.

In this paper we mostly consider the families with $\Pi_d$ property in $\mathbb{R}^d$, the “almost” Helly property. The typical example of a family with $\Pi_d$ property is any family of hyperplanes in general position. It can be easily seen that such a family need not have a common point, and even need not have a bounded piercing number (compare [9], where some bounds on piercing number are given for families of particular sets).

An important consequence of Helly’s theorem is the central point theorem [4, 15, 16] for measures. Here we discuss its discrete version for finite point sets instead of measures.

Theorem (The discrete central point theorem). For a finite set $X \subset \mathbb{R}^d$ there exists a point $x \in \mathbb{R}^d$ such that any half-space $H \ni x$ contains at least

$$r = \left\lceil \frac{|X|}{d+1} \right\rceil$$

points of $X$. Here $|X|$ denotes the cardinality of $X$.

In [10] a “dual” analogue of the central point theorem was established for the families of hyperplanes. Here it is proved for every family with $\Pi_d$ property, and in a stronger form.

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Theorem 1.1. Let a finite family $\mathcal{F}$ of convex closed sets in $\mathbb{R}^d$ have property $\Pi_d$. Then there exists a point $x \in \mathbb{R}^d$ such that any unbounded continuous curve that passes through $x$ intersects at least

$$r = \left\lceil \frac{|\mathcal{F}|}{d+1} \right\rceil$$

sets in $\mathcal{F}$.

Similar to what is done in [10] it is natural to generalize this theorem in the spirit of Tverberg’s theorem [17].

**Definition 1.2.** Consider a family $\mathcal{G}$ of $d+1$ convex compact sets in $\mathbb{R}^d$ with $\Pi_d$ property. The family $\mathcal{G}$ either has a common point, or (by the Alexander duality [5], Theorem 3.44) the complement $\mathbb{R}^d \setminus \bigcup \mathcal{G}$ consists of two connected components: $X$ and $Y$, where $X$ is bounded and $Y$ is unbounded. For a point $x \in X$ we say that $\mathcal{G}$ surrounds $x$.

**Conjecture 1.1.** Let a finite family $\mathcal{F}$ of convex compact sets in $\mathbb{R}^d$ have property $\Pi_d$. Then there exists a point $x \in \mathbb{R}^d$ and

$$r = \left\lceil \frac{|\mathcal{F}|}{d+1} \right\rceil$$

pairwise disjoint nonempty subfamilies $\mathcal{F}_1, \ldots, \mathcal{F}_r \subseteq \mathcal{F}$ such that the following condition holds for any $i = 1, \ldots, r$:

1) either some member of $\mathcal{F}_i$ contains $x$;
2) or the family $\mathcal{F}_i$ surrounds $x$.

It is easy to see that Conjecture 1.1 would imply Theorem 1.1 because any unbounded continuous curve through $x$ must intersect some element of every $\mathcal{F}_i$. This conjecture remains open, but in order to deduce Theorem 1.1 it is sufficient (see Section 5) to prove Conjecture 1.2 for large enough set of values $r$, as it is done in the following theorem.

**Theorem 1.2.** Conjecture 1.1 holds when $r$ is a prime power.

It is also possible to give a generalization in the spirit of Tverberg’s transversal conjecture [18], see [2, 8, 10, 19, 21] for proofs of some particular cases of Tverberg’s transversal conjecture and similar results.

**Definition 1.3.** Consider a family $\mathcal{G}$ of $d - m + 1$ convex compact sets in $\mathbb{R}^d$ with $\Pi_{d-m}$ property and an $m$-flat $L$. We say that $\mathcal{G}$ surrounds $L$ if $\pi(\mathcal{G})$ surrounds the point $\pi(L)$, where $\pi$ is the projection along $L$.

**Conjecture 1.2.** Suppose that each of $m + 1$ families $\mathcal{F}_i$ ($i = 0, \ldots, m$) of convex compact sets in $\mathbb{R}^d$ have property $\Pi_{d-m}$. Then there exists an $m$-flat $L$ and, for every $i = 0, \ldots, m$

$$r_i = \left\lceil \frac{|\mathcal{F}_i|}{d-m+1} \right\rceil$$

pairwise disjoint nonempty subfamilies $\mathcal{F}_{i1}, \ldots, \mathcal{F}_{ir_i} \subseteq \mathcal{F}_i$ such that for any pair $i = 0, \ldots, m$, $j = 1, \ldots, r_i$ the following condition holds:

1) either some member of $\mathcal{F}_{ij}$ intersects $L$;
2) or the family $F_{ij}$ surrounds $L$.

**Theorem 1.3.** Conjecture [1.2] is true when $r_i$ are powers of the same prime $p$, and
1) either $p = 2$;
2) or $d - m$ is even;
3) or $m = 0$.

The case $m = 0$ is inserted here to make a unified statement with Theorem [1.2]. Actually, in this theorem the sets need not be convex, it is sufficient that all their projections to $d - m$-flats are convex.

The restriction that $r_i$ are prime powers is essential in the proof of Theorem [1.3], since the action of a $p$-torus on the configuration space is required, see Section 4.

## 2. Facts from topology

We consider topological spaces with continuous (left) action of a finite group $G$ and continuous maps between such spaces that commute with the action of $G$. We call them $G$-spaces and $G$-maps. We mostly consider groups $G = (\mathbb{Z}_p)^k$ for prime $p$ here ($p$-tori).

For basic facts about (equivariant) topology and vector bundles the reader is referred to the books [7, 11, 14]. The cohomology is taken with coefficients in $\mathbb{Z}_p$ ($p$ is the same as in the definition of $G$), in notations we omit the coefficients. Let us start from some standard definitions.

**Definition 2.1.** Denote $EG$ the classifying $G$-space, which can be thought of as an infinite join $EG = G \ast \cdots \ast G \ast \cdots$ with diagonal left $G$-action. Denote $BG = EG/G$. For any $G$-space $X$ denote $X_G = (X \times EG)/G$, and put (equivariant cohomology in the sense of Borel) $H^*_G(X) = H^*(X_G)$. It is easy to verify that for a free $G$-space $X$ the space $X_G$ is homotopy equivalent to $X/G$.

Consider the algebra of $G$-equivariant cohomology of the point $A_G = H^*_G(pt) = H^*(BG)$. For a group $G = (\mathbb{Z}_p)^k$ the algebra $A_G = H^*_G(\mathbb{Z}_p)$ has the following structure (see [7]). In the case $p$ odd it has $2k$ multiplicative generators $v_i, u_i$ with dimensions $\dim v_i = 1$ and $\dim u_i = 2$ and relations

$$v_i^2 = 0, \quad \beta v_i = u_i.$$

We denote $\beta(x)$ the Bockstein homomorphism.

In the case $p = 2$ the algebra $A_G$ is the algebra of polynomials of $k$ one-dimensional generators $v_i$.

We are going to find the equivariant cohomology of a $G$-space $X$ using the following spectral sequence (see [7, 13]).

**Theorem 2.1.** The natural fiber bundle $\pi_{X_G} : X_G \to BG$ with fiber $X$ gives the spectral sequence with $E_2$-term

$$E^{x,y}_2 = H^x(BG, \mathcal{H}^y(X)),$$

that converges to the graded module, associated with the filtration of $H^*_G(X)$. 
The system of coefficients $H^\nu(X)$ is obtained from the cohomology $H^\nu(X)$ by the action of $G = \pi_1(BG)$. The differentials of this spectral sequence are homomorphisms of $H^*(BG)$-modules.

This theorem implies that if the space $X$ is $n-1$-connected, than the natural map $A^m_G \to H^m_G(X)$ is injective in dimensions $m \leq n$.

Any representation of $G$ can be considered as a vector bundle over the point $pt$, and it has corresponding characteristic classes in $H^*(G)$. We need the following lemma, that follows from the results of [7], Chapter III §1.

**Lemma 2.1.** Let $G = (Z_p)^k$, and let $I[G]$ be the subspace of the group algebra $\mathbb{R}[G]$, consisting of elements

$$\sum_{g \in G} a_g g, \quad \sum_{g \in G} a_g = 0.$$  

Then the Euler class $e(I[G]) \neq 0 \in A_G$ and is not a divisor of zero in $A_G$.

Note that in this lemma the fact that $G = (Z_p)^k$ is essential.

We also need the following fact on the Grassmann variety from [2, 8, 21]. Consider the canonical bundle over the Grassmann variety $\gamma \to G^{d-m}_d$. In the case $p = 2$ we consider the variety of non-oriented $d-m$-subspaces, and for odd $p$ we consider the variety of oriented subspaces.

**Lemma 2.2.** For the Euler class $e(\gamma)$ modulo $p$ the following holds

$$e(\gamma)^m \neq 0 \in H^{m(d-m)}(G^{d-m}_d, Z_p),$$  

if either $p = 2$, or $d - m$ is even, or $m = 0$. In the latter case we put $e(\gamma)^0 = 1 \in H^0(G^{d-m}_d, Z_p)$ by definition.

3. Topology of Tverberg’s theorem

In Tverberg’s theorem and its topological generalizations (see [11, 20] for example) it is important to consider the configuration space of $r$-tuples of points $x_1, \ldots, x_r \in \Delta^N$ with pairwise disjoint supports. Here $\Delta^N$ is a simplex of dimension $N$. Let us make some definitions, following the book [12].

**Definition 3.1.** Let $K$ be a simplicial complex. Denote $K^r_\Delta$ the subset of the $r$-fold product $K^r$, consisting of the $r$-tuples $(x_1, \ldots, x_r)$ such that every pair $x_i, x_j$ ($i \neq j$) has disjoint supports in $K$. We call $K^r_\Delta$ the $r$-fold deleted product of $K$.

**Definition 3.2.** Let $K$ be a simplicial complex. Denote $K^r_\Delta^*$ the subset of the $r$-fold join $K^r$, consisting of convex combinations $w_1 x_1 \oplus \cdots \oplus w_r x_r$ such that every pair $x_i, x_j$ ($i \neq j$) with weights $w_i, w_j > 0$ has disjoint supports in $K$. We call $K^r_\Delta^*$ the $r$-fold deleted join of $K$.

Note that the deleted join is a simplicial complex again, while the deleted product has no natural simplicial complex structure, although it has some cellular complex structure.
The $r$-fold deleted product of the simplex $\Delta^{(r-1)(d+1)}$ is the natural configuration space in Tverberg’s theorem, but sometimes it is simpler to use the deleted join because of the following fact. Denote $[r]$ the set $\{1, \ldots, r\}$ with the discrete topology, the following lemma is well-known, see [12] for example.

**Lemma 3.1.** The deleted join of the simplex $(\Delta^N)^r_\Delta = [r]^*N + 1$ is $N - 1$-connected.

If $r$ is a prime power $r = p^k$, then the group $G = (\mathbb{Z}_p)^k$ can be somehow identified with $[r]$, so a $G$-action on $K^r_\Delta$ and $K^r_\Delta$ by permuting $[r]$ arises. In this case Theorem [2.1] and the above lemma imply that the natural map $A_G^l \to H^l_G((\Delta^N)^r_\Delta)$ is injective in dimensions $l \leq N$. We need the similar fact for deleted products.

**Lemma 3.2.** Let $r = p^k$, $G = (\mathbb{Z}_p)^k$, and let $K$ be a simplicial complex. If the natural map $A^l_G \to H^l_G(K^r_\Delta)$ is injective for $l \leq N$, then the similar map $A^l_G \to H^l_G(K^r_\Delta)$ is injective for $l \leq N - r + 1$.

*Proof. Define the map $\alpha : K^r \to G$ as follows. Let $\alpha$ map a convex combination $w_1x_1 \oplus w_rx_r \in K^r$ to $(w_1, \ldots, w_r) \in \mathbb{R}^r$, the latter space is identified with $G$, if we identify the set $[r]$ with $G$. This map is $G$-equivariant.

Consider the natural orthogonal projection $\pi : \mathbb{R}[G] \to I[G]$ (the latter $G$-representation is defined in Lemma [2.1] and the natural inclusion $\iota : K^r_\Delta \to K^r$. The map $\beta = \pi \circ \alpha \circ \iota : K^r_\Delta \to I[G]$ is $G$-equivariant, and it can be easily checked that

$$K^r_\Delta = \{y \in K^r_\Delta : \beta(y) = 0\}.$$

Now assume the contrary: the image of some nonzero $\xi \in A^l_G$ is zero in $H^l_G(K^r_\Delta)$ and $l \leq N - r + 1$. We denote the classes in $A_G$ and their natural images in the equivariant cohomology of $G$-spaces by the same letters if it does not lead to confusion. Denote $e(I[G]) = e \in A^{-1}_G$ for brevity. The Euler class of a vector bundle is zero outside the zero set of a section of the bundle, so $e = 0 \in H^{-1}_G(K^r_\Delta \setminus K^r_\Delta)$, and by the standard property of the cohomology product

$$e\xi = 0 \in H^{l+r-1}_G((K^r_\Delta \setminus K^r_\Delta) \cup K^r_\Delta) = H^{l+r-1}_G(K^r_\Delta).$$

By Lemma [2.1] $e\xi \neq 0 \in A^{l+r-1}_G$, and we come to contradiction with the injectivity condition in the statement of this lemma. \qed

**4. Proof of Theorem 1.3**

It is sufficient to prove Theorem [1.3] since Theorem [1.2] is its particular case. The reasoning is essentially the same as in [10], compare also [8].

For any $m$-flat $L$ denote the unique $d - m$-subspace in $\mathbb{R}^d$, orthogonal to $L$, by $L^\perp$. It is easy to see that $L$ is determined uniquely by $L^\perp$ and the point $L \cap L^\perp$. So the variety of all $m$-flats is the total space of the canonical bundle $\gamma^{d-m}_d$ over the Grassmann variety $G^{d-m}_d$.

Now consider some $\alpha \in G^{d-m}_d$ and a point $b \in \alpha$, denote the orthogonal projection onto $\alpha$ by $\pi_\alpha$. For any $X \in \bigcup_{i=0}^m \mathcal{F}_i$ denote $\phi(b, X)$ the closest to $b$ point in $\pi_\alpha(X)$. This point depends continuously on the pair $(\alpha, b)$.
Fix some \(i = 0, \ldots, m\) and denote a linear map \(\psi_i : K_i = \Delta^{[F_i]+1} \to \alpha_i\), determined so that it maps the vertices of the simplex to the points \(\phi(b, X) - b\) for \(X \in F_i\), and is piecewise linear. Denote \(\xi : (K_i)_{\Delta_i}^r \to \alpha^{r_i}\) the corresponding map of the deleted products. Let the group \(G_i = (Z_p)^{k_i}\), where \(r_i = p^{k_i}\) act on the deleted product \(L_i = (K_i)_{\Delta_i}^r\) and on \(\alpha^{r_i}\) by permutations, we denote \(\alpha^{r_i} = \alpha_i[G_i]\) to indicate this action, the map \(\xi_i\) is \(G_i\)-equivariant.

In the sequel we denote \(\gamma_{d-m}^d = \gamma\) for brevity. Summing up all the maps we obtain a map

\[
\xi : L_0 \times \cdots \times L_m \to \alpha[G_0] \oplus \cdots \oplus \alpha[G_m].
\]

The map \(\xi\) also depends on the pair \((\alpha, b)\) continuously, so actually it gives a section \(\xi\) of the vector bundle

\[
U = \alpha[G_0] \oplus \cdots \oplus \alpha[G_m] \to \gamma \times L_0 \times \cdots \times L_m.
\]

Here \(\alpha\) can be treated as the pullback of the vector bundle \(\gamma \to G_{d-m}^d\) by the map \(\pi : \gamma \to G_{d-m}^d\), so \(\alpha\) is a vector bundle over \(\gamma\).

To prove the theorem we have to find such \(\alpha \in G_{d-m}^d, b \in \alpha, (y_0, \ldots, y_m) \in L_0 \times \cdots \times L_m\) that \(\xi(\alpha, b, y_0, \ldots, y_m) = 0\).

If we take the bundle of large enough balls \(B(\gamma)\) in \(\gamma\), the section \(\xi\) obviously has no zeros on \(\partial B(\gamma) \times L_0 \times \cdots \times L_m\). To guarantee the zeros for the section \(\xi\), we have to find the relative Euler class (see [10] for properties of the relative Euler class)

\[
e(\xi) \in H^{(d-m)(r_0 + \cdots + r_m)}_{G_0 \times \cdots \times G_m}(B(\gamma) \times L_0 \times \cdots \times L_m, \partial B(\gamma) \times L_0 \times \cdots \times L_m).
\]

Denote for brevity \(G = G_0 \times \cdots \times G_m\).

Let us decompose the bundle \(U\) and its section \(\xi\) in the following way. Any \(\alpha[G_i]\) can be split \(\alpha[G_i] = \alpha \otimes R[G] = \alpha \otimes R \oplus \alpha \otimes I[G_i] = \alpha \oplus \alpha \otimes I[G_i]\). So the \(\xi\) splits into section \(\eta\) of the bundle \(V = \alpha^{m+1}\) and \(\zeta\) of the bundle \(W = \alpha \oplus \bigoplus_{i=0}^{m} I[G_i]\), and \(U = V \oplus W\).

The section \(\eta\) has no zeroes on \(\partial B(\gamma) \times L_0 \times \cdots \times L_m\) and, in fact, for large enough balls in \(B(\gamma)\) the homotopy \(\eta_t = (1 - t)\eta + t(-b, \ldots, -b)\) connects it to the section \((-b, \ldots, -b)\) so that \(\eta_t\) has no zeroes on \(\partial B(\gamma) \times L_0 \times \cdots \times L_m\) for all \(t \in [0,1]\). The section \(\eta\) does not depend on \(L_0 \times \cdots \times L_m\) and it can be easily seen that (see [10], the proof of Theorem 6)

\[
e(\eta) = u(\gamma)e(\gamma)^m \times 1 \in H^{(d-m)(m+1)}(B(\gamma), \partial B(\gamma)) \times H^{0}_{G}(L_0 \times \cdots \times L_m) \subset \subset H^{(d-m)(m+1)}(B(\gamma) \times L_0 \times \cdots \times L_m, \partial B(\gamma) \times L_0 \times \cdots \times L_m),
\]

where \(u(\gamma)\) is the Thom’s class of \(\gamma\), \(e(\gamma)\) (the same as \(e(\alpha)\)) is its Euler class, and the last inclusion is the Kunneth formula. Lemma 2.1 shows that \(u(\gamma)e(\gamma)^m \neq 0\) (compare [10], the proof of Theorem 6).

Now we consider the class \(e(\zeta) \in H^{(d-m)(r_0 + \cdots + r_m - m - 1)}_{G}(B(\gamma) \times L_0 \times \cdots \times L_m)\). Taking some fixed \(b \in B(\gamma)\) and considering the inclusion

\[
t_b : L_0 \times \cdots \times L_m = \{b\} \times L_0 \times \cdots \times L_m \to B(\gamma) \times L_0 \times \cdots \times L_m
\]
and the induced bundle \( \iota^*_b(W) = \bigoplus_{i=0}^{m} (I[G_i])^{d-m} \), we obtain

\[
\iota^*_b(e(\zeta)) = e(I[G_0])^{d-m} \times e(I[G_1])^{d-m} \times \cdots \times e(I[G_m])^{d-m} \in H^*_G(L_0 \times \cdots \times L_m) = \\
= H^*_G(L_0) \times \cdots \times H^*_G(L_m),
\]

the last equality being the K"unneth formula. By Lemmas 2.1 and 3.2, for any \( i = 0, \ldots, m \)
the Euler class \( e(I[G])^{d-m} \neq 0 \in H^*_G(L_i) \) and, by the K"unneth formula, \( \iota^*_b(e(\zeta)) = a \neq 0 \). From one more K"unneth formula for the product \( B(\gamma) \times L_0 \times \cdots \times L_m \) it follows that

\[
e(\zeta) = 1 \times a + \sum_j b_j \times c_j,
\]

where \( b_j \in H^*(B(\gamma)), c_j \in H^*_G(L_0 \times \cdots \times L_m) \), and \( \dim b_j > 0 \) for all \( j \). So

\[
e(\xi) = u(\gamma)e(\gamma)^m \times a + \sum_j u(\gamma)e(\gamma)^m b_j \times c_j,
\]

and \( e(\xi) \neq 0 \) by the K"unneth formula.

Now we can consider some zero of \( \xi \). Let us have some subspace and a point \( \alpha \ni b \),
and \( (y_0, \ldots, y_m) \in L_0 \times \cdots \times L_m \) such that \( \xi(\alpha, b, y_0, \ldots, y_m) = 0 \). Every point \( y_i \in L_i \) is actually an \( r_i \)-tuple of points \( y_{i_1}, \ldots, y_{ir_i} \in K_i = \Delta^{I_i} \) with pairwise disjoint supports. We identify the vertices of \( K_i \) with \( F_i \) and write

\[
y_{ij} = \sum w(i, j, X)X.
\]

Denote \( F_{ij} = \{ X \in F_i : w(i, j, X) > 0 \} \), each \( X \) is assigned to no more than one of \( F_{ij} \), because \( y_{ij} \) have pairwise disjoint supports. The condition \( \xi = 0 \) implies that for any \( i = 0, \ldots, m \), \( j = 1, \ldots, r_i \) the point \( b \) is a convex combination of its projections

\[
b = \sum_{X \in F_{ij}} w(i, j, X)\phi(b, X).
\]

If \( b \) coincides with one of \( \phi(b, X) \), then \( L \) (the \( m \)-flat, perpendicular to \( \alpha \) and passing through \( b \)) intersects the corresponding \( X \). If \( b \) lies in the interior of the convex hull of some \( d - m + 1 \) points of \( \phi(b, X) \), we change \( F_{ij} \) so that it contains only those \( d - m + 1 \) corresponding sets \( X \) and note, that \( \{ \pi_\alpha(X) \} \} X \in F_{ij} \) surround \( X \) by Lemma 4.1 (see below), and therefore \( F_{ij} \) surrounds \( L \).

If none of the above alternatives holds, then \( b \) lies in the relative interior of the convex hull of some \( n < d - m + 1 \) points \( \phi(b, \phi(b, X)), \phi(b, X), X_1, \ldots, X_n \in F_{ij} \). Denote the half-spaces

\[
H_X = \{ x \in \mathbb{R}^d : (x, \phi(b, X) - b) \geq (\phi(b, x), \phi(b, X) - b) \}.
\]

Note that \( X \subseteq H_X \) (since \( \phi \) is the projection) and the half-spaces \( H_{X_1}, \ldots, H_{X_n} \) have empty intersection. So some \( n < d - m + 1 \) sets of \( F_{ij} \) have an empty intersection, that contradicts the \( \Pi_{d-m} \) property.

Now we only have to prove the lemma.
Lemma 4.1. Let a family $G = \{G_1, \ldots, G_{d+1}\}$ of convex compact sets in $\mathbb{R}^d$ have property $\Pi_d$. Let a point $b \in \mathbb{R}^d$ be such that $b$ lies in the interior of the convex hull of $g_1, \ldots, g_{d+1}$, where $g_i$ is the closest to $b$ point in $G_i$. Then $G$ surrounds $b$.

Proof. Again, denote the half-spaces

$$H_i = \{ x \in \mathbb{R}^d : (x, g_i - b) \geq (g_i, g_i - b) \}$$

and note that $G_i \subseteq H_i$. Clearly, $\bigcap_{i=1}^{d+1} H_i = \emptyset$.

For any $i = 1, \ldots, d+1$ the nonempty intersection $\bigcap_{j \neq i} G_j$ is contained in $\bigcap_{j \neq i} H_i$, take one point $x_i \in \bigcap_{j \neq i} G_j$. The simplex $\Delta = \text{conv}_{i=1}^{d+1} \{x_i\}$ contains $\mathbb{R}^d \setminus \bigcup_{i=1}^{d+1} H_i \ni b$ (compare [9], Lemma 1), and every its facet $\partial_i \Delta = \text{conv}_{j \neq i} \{x_i\}$ is contained in the corresponding $G_i$.

Thus $b \notin \bigcup_{i=1}^{d+1} G_i$ and is separated from infinity by $\bigcup_{i=1}^{d+1} G_i \supseteq \partial \Delta$, so $G$ surrounds $b$ by definition. \qed

5. Proof of Theorem 1.1

In this theorem we can assume that $F$ consists of compact sets. Indeed, for a large enough ball $B$ the family $\{X \cap B\}_{X \in F}$ consists of compact sets and has property $\Pi_d$.

As it was already noted, this theorem follows from Theorem 1.2 directly when $r$ is a prime power. Consider some other $r$. Obviously, it is sufficient to prove the theorem in the case $N = |F| = (d+1)(r-1) + 1$.

By the Dirichlet theorem on arithmetic progressions, we can find a positive integer $k$ such that $R = k(r-1) + 1$ is a prime. Now take the family $F'$ of size $kN$ by simply repeating each set in $F$ exactly $k$ times. Note that

$$kN = k(d+1)(r-1) + k = (d+1)(R-1) + k \geq (d+1)(R-1) + 1,$$

so we can apply the case of the theorem, that is already proved, to $F'$ to get some point $x$.

Every unbounded closed curve $C \ni x$ intersects at least $R = k(r-1) + 1$ sets of $F'$. Each set of $F$ is counted no more that $k$ times, then we conclude that $C$ intersects at least $r$ sets of $F$.

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