Minimum Connected Dominating set for Certain Circulant Networks

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Abstract

A Minimum Connected Dominating Set is a minimum set of connected nodes such that every other node in the network is one hop connected with a node in this set. In general, the problem is proved to be NP-hard. In this paper we find a Minimum Connected Dominating Set for certain Circulant Networks.

Keywords: Circulant Network, Dominating Set, Connected Dominating set, Maximum Leaf Spanning Tree.

1. Introduction

A graph is a collection of nodes interconnected by edges. Thus ‘networks’ are nothing but graphs in which nodes represent processors or processes and edges represent communication links between them. The domination problem is a fundamental problem in graph theory. Given a graph $G$, a dominating set of the graph is a set of nodes such that every node in $G$ is either in the set or has a direct neighbouring node in the set. This problem, along with its variations, such as the connected dominating set or the $k$-dominating set, play significant role in wireless sensor networks [1], ad hoc sensor networks [2], peer-to-peer networks, Interconnection Networks [3] etc. A connected dominating set (CDS) of a graph $G$ is a dominating set whose induced graph is connected. A connected dominating set serves as a virtual backbone of a network which can help with routing. Any vertex outside the virtual backbone can send message or signal to another vertex through the virtual backbone. A virtual backbone supports shortest path routing [4], fault-tolerant routing [4], multi-casting [4], radio broadcasting [4], clustering [5] and so on. Furthermore, a virtual backbone of a wireless network may reduce communication overhead, increase bandwidth efficiency, and decrease energy consumption [6].

A spanning tree of a connected graph $G$ is defined as a maximal set of edges of $G$ that contains no cycle, or as a minimal set of edges that connect all vertices. The Maximum Leaf Spanning Tree (MLST) problem is to find a spanning tree of a graph with as many leaves as possible. A leaf in a graph $G$ of a spanning tree is a vertex in $G$ of degree one in the spanning tree. Given a graph $G$, the Maximum Leaf Spanning Tree problem is to find the maximum number of vertices of degree one in a spanning tree over all spanning trees of $G$.

Nomenclature

| $G$ | Graph |
| --- | --- |
| $G(n;\pm\{1,2,3\ldots j\})$ | Circulant Network |
| $S$ | Dominating Set |
| $R$ | Connected Dominating Set |
| $N[v]$ | Closed neighbourhood of $v$ |

Greek symbols

| $\gamma(G)$ | Minimum Dominating set (MDS) |
| $\gamma'(G)$ | Minimum Connected Dominating set (MCDS) |
| $\tau_l(G)$ | Maximum Leaf Spanning Tree (MLST) |

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Some broadcasting problems in network design ask to minimize the number of broadcasting nodes, which must be connected to a single root. This translates the problem into finding a spanning tree with many leaves and few internal nodes. Ongoing research on this topic is motivated by the fact that variants of this problem occur frequently in real life applications [7,8,9]. The Maximum Leaf Spanning Tree problem can be found in the area of communication networks and circuit layouts [7]. In communication networks where the vertices correspond to terminals, the problem on message routing is to design a tree-like layout in the network where “leaf terminals” may have lighter workloads than “intermediate terminals” of degree at least two. Hence, in this case, the solution of MLST problem provides a reasonable layout [9].

Minimum Connected Dominating Set and Maximum Leaf Spanning Tree problem are known to be NP-hard [10] and are equivalent [17]. Many literature references discussed the MCDS and MLST problems by approximation algorithms [11,13,14,16,17]. The problems are discussed for some generalized trapezoid graphs [18], grid graphs [19], Hypercube and Star networks [20]. Even though there are numerous results and discussions on MCDS and MLST problems, most of them deal with only approximate results.

In this paper we exhibit a Minimum Connected Dominating Set for the circulant network $G(n, \pm\{1,2,3,\ldots,j\})$, thereby solving the Maximum Leaf Spanning Tree problems also.

2. Circulant Network

The circulant network is a natural generalization of double loop network, which was first considered by Wong and Coppersmith [21]. Circulant graphs have been used for decades in the design of computer and telecommunication networks due to their optimal fault-tolerance and routing capabilities [22]. It is also used in VLSI design and distributed computation. Theoretical properties of circulant graphs have been studied extensively and surveyed by Bermond et al. [23]. Every circulant graph is a vertex transitive graph and a Cayley graph [3]. Most of the earlier research concentrated on using the circulant graphs to build interconnection networks for distributed and parallel systems [22,23]. Classes of graphs that are circulant graphs include the complete graphs, complete bipartite graphs, Paley graphs of prime order, prism graphs, möbius ladder graph, tetrahedral graph and torus grid graphs.

Definition 1 [24]: A circulant graph denoted by $G(n, \pm\{1,2,\ldots,j\})$, $1 < j \leq \lfloor n/2 \rfloor$, $n \geq 3$ is defined as a graph consisting of the vertex set $V = \{0,1,2,3,\ldots,n-1\}$ and the edge set $E = \{(i, j) \mid j-i \equiv s(\text{mod } n), s \in \{1,2,\ldots,j\}\}$. See Figure 1.

3. Main Results

The following problems have been considered in the literature and are NP-complete [10].

Problem 1: Given a connected graph $G(V,L)$ and a vertex set $S \subseteq V$, $S$ is a dominating set if each vertex in $G$ is either in $S$ or has at least one neighbour in $S$. A dominating set with minimum cardinality is called Minimum Dominating Set. Minimum Domination Problem is to determine $\gamma(G)$ defined as $\gamma(G) = \min |S|$ where the minimum is taken over all dominating sets of $G$. 

![Figure 1. Circulant Network $\Gamma_4(17, \pm(1,2))$](image)
**Problem 2**: $R$ is a connected dominating set if $R$ is a dominating set and the induced subgraph $G[R]$ is connected. A connected dominating set with minimum cardinality is called a minimum connected dominating set. Minimum Connected domination Problem is to determine $\gamma_c(G)$ defined as $\gamma_c(G) = \min |R|$ where the minimum is taken over all connected dominating sets of $G$.

**Problem 3**: Given an undirected graph $G(V, E)$, $|V| = m$, $|E| = n$, find a spanning tree $T$ of $G$ with a maximum number of leaves $\tau_L(G)$ among all spanning trees of $G$.

The following result is easy to observe.

**Lemma 1**: Let $G$ be an $r$-regular graph on $n$ vertices. Then $\gamma(G) \geq \lceil n/r + 1 \rceil$

**Domination Algorithm** $G(n \pm \{1, 2, \ldots, j\})$

**Input**: Circulant Graph $G(n, \pm\{1, 2, \ldots, j\})$

**Algorithm**: Label the vertices of graph $G(n, \pm\{1, 2, \ldots, j\})$ as $0, 1, 2, \ldots, n-1$ in the clockwise sense. Select the vertices $S = \{0, 2j, 1, 2(2j + 1), 3(2j + 1), \ldots, (k-1)(2j + 1)\}$ where $k = \lceil n/2j + 1 \rceil$.

**Output**: Minimum Dominating Set $S$ for $G(n, \pm\{1, 2, \ldots, j\})$. See Figure 2.

**Proof of correctness**: Clearly $N[k_3(2j + 1)] \cap N[k_2(2j + 1)] = \emptyset$, $k_3 \leq k_2 \leq \lceil n/2j + 1 \rceil$ and $S$ dominates all the vertices in $G$.

**Theorem 1**: Let $G$ be the circulant graph $G(n, \pm\{1, 2, \ldots, j\})$. Then $\gamma(G) = \lceil n/2j + 1 \rceil$

![Figure 2. Minimum Dominating Set $S$ for $G(17, \pm\{1, 2\})$](image)

**Lemma 2**: Let $G$ be a graph and $V = \{v_1, v_2, v_3, \ldots, v_m\}$ be an ordered set of vertices in $G$. Let $R = \{u_1, u_2, u_3, \ldots, u_k\} \subseteq V$ be a dominating set with $u_1 < u_2 < u_3 < \ldots < u_k$ satisfying the following conditions:

i) $N[u_i] \cap N[u_{i+1}] = \{u_i\}$, and $N[u_i] \cap N[u_j] = \emptyset$, $i < j$, $i \neq i + 1, i + 2$.

ii) $N[u_{i+1}] \subseteq N[u_i] \cup N[u_{i+2}]$.

Then $R$ is a minimum connected dominating set of $G$.

**Proof**:

By condition (i), $R$ is a connected dominating set. 

By condition (ii), the closed neighborhood of $u_{i+1}$ is contained in $N[u_i] \cup N[u_{i+2}]$. Thus $u_{i+1}$ will not dominate any vertex in $V \backslash N[u_i] \cap N[u_{i+2}]$. Hence we need at least $k$ elements to dominate $V$. Thus $R$ is minimum.

**Connected Domination Algorithm** $G(n, \pm\{1, 2, \ldots, j\})$

**Input**: Circulant Graph $G(n, \pm\{1, 2, \ldots, j\})$

**Algorithm**: Label the vertices of $G(n, \pm\{1, 2, \ldots, j\})$ as $0, 1, 2, 3, \ldots, k$ where $k = (n - (j + 2))/j$.

**Output**: Minimum Connected Dominating Set $R$ for $(n, \pm\{1, 2, \ldots, j\})$. See Figure 3.

![Figure 3. Minimum Connected Dominating Set $R$ for $G(17, \pm\{1, 2\})$](image)
**Proof of Correctness:** $R$ satisfies the conditions of Lemma 2. Hence $R$ is a Minimum Connected Dominating Set of the circulant graph $G(\eta, \pm\{1,2, \ldots, j\})$.

The proofs of the following theorems are easy consequences of Lemma 2 and the Connected Domination Algorithm $G(\eta, \pm\{1,2, \ldots, j\})$.

**Theorem 2:** Let $G$ be circulant graph $G(\eta, \pm\{1,2, \ldots, j\})$. Then $\gamma_c(G) = 1 + [\eta - (2j + 1)]/j$.

**Theorem 3:** Let $G$ be circulant graph $G(\eta, \pm\{1,2, \ldots, j\})$. Then $\tau_t(G) = \eta - \gamma_c(G)$.

Next we consider another class of circulant network, namely $G(\eta, \pm\{1,3\})$.

**Connected Domination Algorithm $G(\eta, \pm\{1,3\})$**

**Input:** Circulant Graph $G(\eta, \pm\{1,3\})$

**Algorithm:** Label the vertices of $G(\eta, \pm\{1,3\})$ as $0,1,2, \ldots, n-1$ in the clockwise sense. Select the labelled vertices $R = \{0,3,6, \ldots, (i + 1) \cdot 3\}$ if $\eta \equiv 1 \mod 5$ and $R = \{0,3,6, \ldots, i \cdot 3\}$ otherwise, where $i = \lfloor \eta / 5 \rfloor$.

**Output:** Minimum Connected Dominating Set $R$ for $G(\eta, \pm\{1,3\})$.

**Proof of Correctness:** $R$ satisfies the conditions of Lemma 2. Hence $R$ is a Minimum Connected Dominating Set of the circulant graph $G(\eta, \pm\{1,3\})$.

The proof of the following theorem is an easy consequence of Connected Domination Algorithm $G(\eta, \pm\{1,3\})$.

**Theorem 4:** Let $G$ be $G(\eta, \pm\{1,3\})$. Then $\gamma_c(G) = i + 2$, if $\eta \equiv 1 \mod 5$, $\gamma_c(G) = i + 1$ otherwise. where $i = \lfloor \eta / 5 \rfloor$.

4. Conclusion

In this paper we calculated the exact value for Minimum Dominating Set and Minimum Connected Dominating Set, thereby solving Maximum Leaf Spanning Tree problem for certain Circulant Networks.

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