Is $\Upsilon(3S)$ a pure $S$–wave?

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Abstract

Assuming the QCD multipole expansion is applicable to hadronic transitions of $\Upsilon(3S)$ into lower level bottomonia, we consider the possibility that $\Upsilon(3S)$ has a $D$–wave component. This assumption leads to a natural explanation of the $\pi\pi$ spectrum in $\Upsilon(3S) \to \Upsilon(1S) \pi\pi$. Consequences of this assumption on other hadronic and radiative transitions of $\Upsilon(3S)$ are also discussed in the same context.

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1. Introduction

It has been suggested recently by two of us (S.C. and P.K.) that the $\pi\pi$ spectrum in $\Upsilon(3S) \to \Upsilon(1S)\pi\pi$ can be explained by including a $D$–wave amplitude for the dipion system [1]. The most general amplitude for a spin–1 particle decaying into another spin–1 particle with the emission of two pions is given by

$$M = A_0 \epsilon^\mu \epsilon'^\nu \left[ (q^2 + B E_1 E_2 + C m^2_\pi) g_{\mu \nu} + D (p_\mu p'_\nu + p_\nu p'_\mu) \right],$$

in the lowest order in pion momenta expansion. Here, $\epsilon$ and $\epsilon'$ are the polarization vectors of the initial and the final $\Upsilon$’s, $p$ and $p'$ are the 4-momenta of two pions, $q^2 = (p + p')^2 \equiv m^2_{\pi\pi} = s_\pi$, and $E_1$ and $E_2$ are the energies of each pion in the rest frame of the initial $\Upsilon$. Two sets of parameters give the best fit to the $m_{\pi\pi}$ distribution in $\Upsilon(3S) \to \Upsilon(1S)\pi\pi$ [1]. Various angular distributions of the decay products in $e^+ e^- \to \Upsilon(3S) \to \Upsilon(1S)\pi\pi$ are predicted in Ref. [1], and these need to be verified by future experiments.

However, the reason for $D \neq 0$ in [1] was not clearly discussed in Ref. [1]. Two possibilities were briefly mentioned: either a $D$–wave admixture in $\Upsilon(3S)$ or, a breakdown of QCD multipole expansion for hadronic transitions of $\Upsilon(3S)$. It is our purpose to explore the first possibility in detail. Since QCD multipole expansion enables us to understand hadronic transitions between heavy quarkonia other than $\Upsilon(3S)$, it is desirable to try to understand the amplitude [1] in the same framework. If this is possible, then other hadronic transitions of $\Upsilon(3S)$ can be studied in the same
context. We note that theoretical predictions on hadronic transitions of $\Upsilon(3S)$ in the literature are not reliable, since they do not correctly describe $\Upsilon(3S) \rightarrow \Upsilon(1S)\pi\pi$. If our predictions are in serious contradiction with the experiments, then we may have to conclude that the QCD multipole expansion breaks down in case of $\Upsilon(3S)$.

This work is organized as follows. In Section 2, the amplitude (1) is interpreted in the framework of QCD multipole expansion. It is found that results in Ref. [1] can be readily obtained, once $\Upsilon(3S)$ is assumed to be a mixture of $S$- and $D$-waves with a mixing angle $\phi$:

$$|\Upsilon(3S)\rangle = \cos \phi |3S\rangle + \sin \phi |D\rangle.$$ (2)

Consequences of this assumption on other decays of $\Upsilon(3S)$ are then explored in detail. First of all, it turns out that the current upper limit on $B(\Upsilon(3S) \rightarrow \Upsilon(1S) + \eta)$ selects $P2$ from the two sets of parameters of Ref. [1]. In Section 3, various radiative transitions of $\Upsilon(3S)$ are considered. There, a tight constraint on the $D$-wave mixing arises from electric dipole radiative transitions $\Upsilon(3S) \rightarrow \chi_{bJ}(2P) + \gamma$. In the presence of a $D$-wave component in $\Upsilon(3S)$, some new and interesting radiative decays appear. It can affect the decay rate of $\Upsilon(3S) \rightarrow \eta_b + \gamma$, and allows the following cascade transitions:

$$\Upsilon(3S) \rightarrow 1D \rightarrow h_b(1P) \rightarrow \eta_b.$$ (3)

Besides these decays, $\Upsilon(3S) \rightarrow h_b(1P) + \pi^0$ and $\Upsilon(3S) \rightarrow h_b(1P) \pi\pi$ are also interesting, and the $D$-wave contributions to these processes are considered in Section 4.
For these decays, we adopt the approach proposed by Voloshin \cite{2}, which correctly predicts the ratio of the charmonium $^1P_1$ state decaying into $J/\psi + \pi^0$ and $J/\psi + \pi\pi$ \cite{3}. All of these decays reach a spin–singlet $P$–wave state, $h_b(1P)$, that is hard to produce in the $e^+e^-$ annihilation. $h_b(1P)$ can be a source of a spin–singlet $S$–wave state ($\eta_b$) through electric dipole radiative transition, $h_b(1P) \rightarrow \eta_b + \gamma$. Finally, our results are summarized in Section 5.

In the following, the absolute decay rate or its lower/upper bound is derived for each decay process. It depends on the mixing angle $\phi$ and quarkonium matrix elements of operators, $r$ and $r^2$, where $\vec{r}$ is the spatial separation of $b$ and $\bar{b}$. The matrix element of $r^2$ between $\Upsilon(3S)$ and $\Upsilon(1S)$ can be directly extracted from the spectrum and the absolute decay rate of $\Upsilon(3S) \rightarrow \Upsilon(1S) \pi\pi$, and gives information that is independent of specific potential models. Absolute decay rates of $\Upsilon(3S) \rightarrow \chi_b J(2P) + \gamma$ give useful information on the matrix element of $r$ between $\Upsilon(3S)$ and $\chi_b(2P)$ and the mixing angle $\phi$. Other unknown quarkonium matrix elements will be fixed by the results from potential model calculations. We use $m_b = 4.8$ GeV in this work. This induces some uncertainty less than $\sim 20\%$ in the numerical estimates of $1/m_b^2$. Finally, some of our results in Sections 3 and 4 show explicit dependence on $G_8$, the Green’s function of the color octet $b\bar{b}$ states (defined in \cite{7}.) These results should be regarded as being order–of–magnitude estimates because of the approximation we will make about $G_8$. 

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2. Hadronic transitions of $\Upsilon(3S)$ into $\Upsilon(1S)$

Let us begin with the $\pi\pi$ spectrum in $\Upsilon(3S) \rightarrow \Upsilon(1S)\pi\pi$. In QCD multipole expansion, this process occurs through $E1 - E1$ multipole interaction, where the $E1$ interaction Hamiltonian of quarkonium with a gluon is

$$H_{\text{int}}(E1) = -\frac{1}{2} g \xi^a r_i E^a_i(0). \quad (4)$$

Here, $g \equiv (4\pi\alpha_s)^{1/2}$ is the $SU(3)_c$ gauge coupling constant, $\vec{r}$ is the relative position of the quark ($b$) and the antiquark ($\bar{b}$), and $\xi^a \equiv t^a_b - t^a_{\bar{b}}$ is the difference of the $SU(3)_c$ color generators that act on the quark and the antiquark, respectively. When acting between colorless states, $\xi^a$ satisfies

$$\langle \text{color singlet} | \xi^a \xi^b | \text{color singlet} \rangle = \frac{2}{3} \delta^{ab}. \quad (5)$$

From (4), the amplitude for $i \rightarrow f\pi\pi$ is given by

$$\mathcal{M}(i \rightarrow f\pi\pi) = \frac{2}{3} \langle f | r_i G_8 r_j | i \rangle \langle \pi\pi | \pi\alpha_s E^a_i E^a_j | 0 \rangle, \quad (6)$$

where $G_8$ is the Green’s function for the color octet $Q\bar{Q}$ states:

$$G_8(E) = \sum_k \frac{|k\rangle \langle k|}{E_k - E}. \quad (7)$$

Here $k$ runs over color octet $Q\bar{Q}$ states only. $E$ and $E_k$ are the energies of the initial and the intermediate states. $G_8$ is unknown due to our ignorance of quark confinement in QCD, and will be treated as a constant. Then, a lower bound on $G_8$ can be derived

$$|G_8|^2 > 18 \text{ GeV}^{-2},$$
using a sum rule on $\langle nS \mid r^2 \mid 1S \rangle$ and the decay rate of $\Upsilon(2S) \to \Upsilon(1S) \, \pi\pi$ \cite{3}, $G_8$ can be determined if the absolute decay rate of $h_c(1P) \to J/\psi + \pi^0$ is known experimentally \cite{3}.

Now we show that the $D$–term in (3) naturally arises from the $|\Delta L| = 2$ transition, $^3D_1 \to ^3S_1 \, \pi\pi$. We assume that the $D$–wave mixes with the initial quarkonium $\Upsilon(3S)$, since the final quarkonium $\Upsilon(1S)$ is the lowest level bottomonium and it is hard to imagine that it would contain any contamination of a $D$–wave. According to potential model calculations \cite{3}, there are two $D$–wave levels below $BB$ threshold, with $m(1D) = 10.16$ GeV and $m(2D) = 10.44$ GeV, respectively. Since $\Upsilon(2D)$ is closer to $\Upsilon(3S)$ than $\Upsilon(1D)$, one might guess the $D$–wave in (3) to be $\Upsilon(2D)$. However, as discussed below (23), a larger value of $\langle 1S \mid r^2 \mid D \rangle$ is desirable for mixing. Thus, $\Upsilon(1D)$ may be preferred because it has no node in the radial wave function. In this work, we do not address questions regarding the origin of the $S$– and $D$–wave mixing and which of the two $D$–wave levels enters in (3). The discussion in this section is independent of such issues. In the next section on radiative transitions of $\Upsilon(3S)$, we consider both $1D$ and $2D$ mixing.

The angular part of the matrix elements between quarkonia can be easily performed, and we get \cite{7}

$$
\langle (^3S_1)_j \mid r_k \, G_8 \, r_l \mid (^3S_1)_i \rangle = \frac{1}{3} \, I_{S,S'} \, \delta_{ij} \, \delta_{kl}, \quad (8)
$$

$$
\langle (^3S_1)_j \mid r_k \, G_8 \, r_l \mid (^3D_1)_i \rangle = \frac{\sqrt{2}}{10} \, I_{S,D} \, (\delta_{ik} \, \delta_{jl} + \delta_{il} \, \delta_{jk} - \frac{2}{3} \, \delta_{ij} \, \delta_{kl}), \quad (9)
$$
where
\[ I_{i,f} \equiv \langle f | r_k G_8 r_k | i \rangle = \int_0^{\infty} R_f r_k G_8 r_k R_i r^2 \, dr. \]  
(10)

The gluonic matrix element, \( \langle \pi \pi | \pi \alpha_s E_i^a E_f^a | 0 \rangle \), can be calculated by considering \( \langle \pi \pi | \alpha_s G_{\mu \rho} G_{\nu \sigma} | 0 \rangle \) and the QCD scale anomaly. Detailed procedures are discussed in Refs. [8], [2] and [7]. The result can be summarized as

\[ \langle \pi^+ \pi^- | \alpha_s G_{\mu \rho} G_{\nu \sigma} | 0 \rangle \]  
(11)

\[ = A (q^2 + m^2) + B (g_{\mu \nu} \tau_{\rho \sigma} - g_{\mu \rho} \tau_{\nu \sigma} + g_{\rho \sigma} \tau_{\mu \nu} - g_{\rho \nu} \tau_{\mu \sigma}), \]

where

\[ \tau_{\mu \nu} = p_\mu p'_\nu + p_\nu p'_\mu = \frac{1}{2} (q_{\mu} q_{\nu} - r_{\mu} r_{\nu}), \]  
(12)

\[ A = \frac{1}{3} \frac{\alpha_s^2}{\beta} = -\frac{1}{3} \frac{2\pi}{9}, \]  
(13)

\[ B = \frac{1}{2} \alpha_s \rho_G = \frac{1}{2\pi} \lambda, \]  
(14)

and \( q = p + p', \quad r = p - p' \).

The \( A \)–term receives contribution from the QCD scale anomaly [4], while the \( B \)–term arises from the gluonic contribution to the energy momentum tensor of QCD [8]. The parameter \( \lambda \) can be determined from the \( \pi \pi \) spectrum in \( \Upsilon(2S) \rightarrow \Upsilon(1S) \pi \pi \) [4] : \( \lambda = 1.6 \sim 1.9 \). This is consistent with what we obtain below from the \( \pi \pi \) spectrum in \( \Upsilon(3S) \rightarrow \Upsilon(1S) \pi \pi \).

Using the information given above, one can calculate the \( S \)– and \( D \)–wave con-
tributions to $\Upsilon(3S) \to \Upsilon(1S)\pi\pi$:

$$
\mathcal{M}(3S \to 1S \pi\pi) = \frac{4\pi^2}{81} I_{3S,1S} \hat{e} \cdot \hat{e}'
\times \left[ q^2 + m_{\pi}^2 - \frac{9\lambda}{4\pi^2} \left( (q^0)^2 - (r^0)^2 + q^2 + 2m_{\pi}^2 \right) \right],
$$

$$
\mathcal{M}(kD \to 1S \pi\pi) = \sqrt{2} \frac{\lambda}{30} I_{kD,1S} \hat{e}_k \cdot \hat{e}'_l \left[ \{q_k q_l - r_k r_l\} - \frac{1}{3} \delta_{kl} \{\vec{q}^2 - \vec{r}^2\} \right],
$$

$$
\mathcal{M}(\Upsilon(3S) \to \Upsilon(1S)\pi\pi) = \mathcal{M}(3S \to 1S \pi\pi) \cos \phi + \mathcal{M}(kD \to 1S \pi\pi) \sin \phi.
$$

Note that the structure of the $D$–term in (1) comes from the first curly bracket in (16), as mentioned at the beginning of this section.

We fit the $\pi\pi$ spectrum in $\Upsilon(3S) \to \Upsilon(1S)\pi\pi$ using the above amplitude with three free parameters, $I_{kD,1S} \sin \phi$, $I_{3S,1S} \cos \phi$ and $\lambda$. The best fit is given by two sets of solutions (see Fig. 1):

$$
\frac{I_{kD,1S}}{I_{3S,1S}} \tan \phi = \pm(2.4 \pm 0.5),
$$

$$
\lambda = (2.0 \pm 0.1),
$$

with $\chi^2/d.o.f. = 11.2/7$ (equivalent to 13.2 % C.L.). These correspond to two best fits (called P1 and P2) obtained in Ref. [4] using amplitude (4). More specifically, one can express (17) in the form of (1) using (15) and (16), and find the value of $D$.

It turns out that the upper and the lower signs in (18) correspond to the parameter set P2 and P1 (See table 1.) in Ref. [4], respectively. They can be distinguished by measuring various angular distributions of the final decay products as suggested in Ref. [4]. Also, as discussed below in detail, the decay rate for $\Upsilon(3S) \to \Upsilon(1S) + \eta$ can
resolve this twofold ambiguity, and the parameter set $P2$ is preferred. The value of $\lambda = (2.0 \pm 0.1)$ obtained here is consistent with the $\lambda$ extracted from the $\pi \pi$ spectrum in $\Upsilon(2S) \to \Upsilon(1S) \pi \pi$ \cite{4}.

From the absolute decay rate of $\Upsilon(3S) \to \Upsilon(1S) \pi \pi$, we obtain the absolute values of $I_{1D,1S} \sin \phi$ and $I_{3S,1S} \cos \phi$:

$$|I_{3S,1S} \cos \phi| = |\langle 1S | r G_8 r | 3S \rangle| \cos \phi \approx 0.78 \text{ GeV}^{-3}$$

(19)

$$|I_{kD,1S} \sin \phi| = |\langle 1S | r G_8 r | kD \rangle| \sin \phi \approx 1.92 \text{ GeV}^{-3}$$

(20)

$$\tan \phi = 2.46 \frac{|\langle 1S | r G_8 r | 3S \rangle|}{|\langle 1S | r G_8 r | kD \rangle|}.$$  \hspace{1cm} (21)

Up to now, we didn’t care about $G_8$. If we assume that $G_8$ is a constant, (21) becomes

$$\tan \phi = 2.46 \frac{|\langle 1S | r^2 | 3S \rangle|}{|\langle 1S | r^2 | kD \rangle|}.$$ 

(22)

Using the values of quarkonia matrix elements quoted in Ref. \cite{7},

$$|\langle 1S | r^2 | 3S \rangle| = 0.3 \text{ GeV}^{-2},$$

$$|\langle 1S | r^2 | kD \rangle| = 1.65 \text{ GeV}^{-2},$$

(23)

we get $\phi \approx \pm 24^\circ$. However, as discussed in Ref. \cite{7}, the matrix element $|\langle 1S | r^2 | 3S \rangle|$ can be much smaller than the above number, since the $3S$ state has two nodes which may lead to almost complete cancellation. Furthermore, the accuracy of the wave functions determined in potential models is about 10%. Therefore, the actual mixing angle $\phi$ may be much smaller than 24°. Also, because of our approximation on $G_8$,}
this kind of determination of $\phi$ is less reliable than that obtained in the next section from radiative decays, $\Upsilon(3S) \rightarrow \chi_{bJ}(2P) + \gamma$. In fact, too large a mixing angle ($\phi$) may result in severe discrepancy between the theoretical predictions and the experiments for these radiative decays.

We can also consider the similar decay $\Upsilon(3S) \rightarrow \Upsilon(2S) \pi\pi$. The best fit for $\phi = 0^\circ$ (no $D$–wave mixing) yields $\chi^2/d.o.f. = 1.7/7$. If the $D$–wave mixing is allowed, we get the best fit with $\chi^2/d.o.f. = 1.4/6$. Therefore, it is difficult to tell which of the two is a better fit, and our assumption on the $D$–wave mixing cannot be tested clearly in this decay mode. Improved measurements of the $\pi\pi$ spectrum in $\Upsilon(3S) \rightarrow \Upsilon(2S) \pi\pi$ are welcome.

The message of this work can be put in the following way: a small mixture of a $D$–wave component in $\Upsilon(3S)$ can explain the $\pi\pi$ spectrum in $\Upsilon(3S) \rightarrow \Upsilon(1S) \pi\pi$ in the framework of QCD multipole expansion, if $|\langle1S \mid r \ G_8 \ r \mid 3S\rangle|$ is much more suppressed compared to $|\langle1S \mid r \ G_8 \ r \mid kD\rangle|$.

One can also consider another transition, $\Upsilon(3S) \rightarrow \Upsilon(1S) + \eta$ (or $\pi^0$), which occurs through interference of $E1$ and $M2$ interaction. The $M2$ interaction Hamiltonian is given by [4]

$$H_{int}(M2) = -\frac{1}{4m} g \ S_j \ \xi^a \ r_i D_i H_j^a(0),$$

(24)

where $m$ is the mass of the quark, $\vec{S}$ is the total spin of the quark and the antiquark, and $D_i$ is the spatial component of the covariant derivative.
From (4) and (24), we can derive the amplitude for this transition and calculate the gluonic matrix element using the $U_A(1)$ anomaly in QCD [4] :

$$M(\Upsilon(3S) \to \Upsilon(1S) + \eta) = \left( \partial_k \langle \eta | \pi \alpha_s E^a_k H^a_j | 0 \rangle \right) m_Q^{-1} \epsilon_{ijk} \epsilon_i^j \epsilon_j^k \left[ I_{3S,1S} \cos \phi - \frac{1}{\sqrt{2}} I_{kD,1S} \sin \phi \right]$$

(25)

where $I_{i,f}$ is defined in (10). Using the results (19) and (20), we predict

$$\Gamma(\Upsilon(3S) \to \Upsilon(1S) + \eta) = \begin{cases} 
58 \text{ eV (for P2)}, \\
870 \text{ eV (for P1)},
\end{cases}$$

or 0.2% or 3.6% in the branching ratio. Current upper limit on this decay mode is 0.22% [10], which prefers the first set (P2) :

$$B(\Upsilon(3S) \to \Upsilon(1S) + \eta) = 0.2\% \text{ (for P2)},$$

(26)

which is close to the current upper limit. Therefore, twofold ambiguity encountered in Ref. [1] is lifted in the present work, and the parameter set P2 is preferred. Observation of $\Upsilon(3S) \to \Upsilon(1S) + \eta$ at the anticipated branching ratio would constitute one of the cleanest tests of our assumption : applicability of QCD multipole expansion to $\Upsilon(3S)$, and a small admixture of $D$–wave component in $\Upsilon(3S)$.

### 3. Radiative transitions of $\Upsilon(3S)$

In order to further check the $D$–wave mixing in $\Upsilon(3S)$, we consider electric dipole radiative transitions, $\Upsilon(3S) \to \chi_bJ(2P) + \gamma$ and $\Upsilon(3S) \to \chi_bJ(1P) + \gamma$. In this section
and the following one, we assume the $D$–wave in $\Upsilon(3S)$ can be either $1D$ or $2D$ state and consider both possibilities on the same footing. The transition rate of these decays is given by

$$\Gamma(\Upsilon(3S) \to \chi_bJ(nP) + \gamma) = \frac{4}{27} \alpha Q_b^2 \omega^3 (2J_f + 1) |\langle nP_J | r | \Upsilon(3S) \rangle|^2,$$  \hspace{1cm} (27)

where

$$\langle nP_0 | r | \Upsilon(3S) \rangle = \langle nP | r | 3S \rangle \cos \phi + \sqrt{2} \langle nP | r | kD \rangle \sin \phi,$$

$$\langle nP_1 | r | \Upsilon(3S) \rangle = \langle nP | r | 3S \rangle \cos \phi - \frac{1}{\sqrt{2}} \langle nP | r | kD \rangle \sin \phi,$$  \hspace{1cm} (28)

$$\langle nP_0 | r | \Upsilon(3S) \rangle = \langle nP | r | 3S \rangle \cos \phi + \frac{1}{5\sqrt{2}} \langle nP | r | kD \rangle \sin \phi.$$

The measured rates of $\Upsilon(3S) \to \chi_bJ(2P) + \gamma$ are available \cite{11} for $J = 0, 1, 2$. Analysis of these decay rates yield a solution for the two variables $\langle 2P|r|3S \rangle \cos \phi$ and $\langle 2P|r|kD \rangle \sin \phi$:

$$\langle 2P | r | 3S \rangle \cos \phi = + (2.66 \pm 0.16) \text{ GeV}^{-1},$$

$$\langle 2P | r | kD \rangle \sin \phi = - (0.14 \pm 0.18) \text{ GeV}^{-1}. \hspace{1cm} (29)$$

To determine $\phi$ from (29), we use the potential model calculations of $\langle f | r | i \rangle$ given in \cite{6}:

$$\langle 1P | r | 3S \rangle = -0.023 \text{ GeV}^{-1},$$

$$\langle 1P | r | 1D \rangle = -2.0 \text{ GeV}^{-1},$$

$$\langle 1P | r | 2D \rangle = -0.26 \text{ GeV}^{-1}.$$
\begin{align*}
\langle 2P \mid r \mid 3S \rangle &= +2.7 \text{ GeV}^{-1}, \\
\langle 2P \mid r \mid 1D \rangle &= +1.9 \text{ GeV}^{-1}, \\
\langle 2P \mid r \mid 2D \rangle &= -2.7 \text{ GeV}^{-1}.
\end{align*}

Multiplicative relativistic correction factors to the matrix elements involving the \( S \)-wave have been calculated in Ref. \[12\]. The corrections depend on the state of \( \chi_{bJ}(nP) \) into which the \( \Upsilon(3S) \) decays:

\begin{align*}
\langle 2P_J \mid r \mid 3S \rangle : & \quad J = 2 \quad 1.02, \\
& \quad J = 1 \quad 1.00, \\
& \quad J = 0 \quad 1.95. \\
\langle 1P_J \mid r \mid 3S \rangle : & \quad J = 2 \quad 2.3, \\
& \quad J = 1 \quad 1.2, \\
& \quad J = 0 \quad 1.9.
\end{align*}

From the second equation of (29) and (30), we get a value for the \( D \)-wave mixing angle \( \phi \) as deduced from \( \Upsilon(3S) \) decaying into \( \chi_{bJ}(2P) \):

\[ \phi_{1D} = -4^\circ \pm 6^\circ \]
\[ \phi_{2D} = +3^\circ \pm 4^\circ \]  (32)

The error in the angles is an estimate only. Moreover, the angles are seen to be consistent with zero. \( \phi \) may also be estimated from the first equation of (29) and (30).
Eqs. (30) and (31). $-12^\circ < \phi < +12^\circ$ obtained this way is not very useful because the allowed range is large.

A determination of the mixing angle is also possible by making use of experimental bounds on various combinations of branching ratios for $\Upsilon(3S)$ decaying into $\chi_{bJ}(1P)$. Form Ref. [11] we have the experimentally measured

\[
B(\chi_{b2}(1P) \to \Upsilon(1S)\gamma) = 0.22 \pm 0.04
\]
\[
B(\chi_{b1}(1P) \to \Upsilon(1S)\gamma) = 0.35 \pm 0.08
\]
\[
B(\chi_{b0}(1P) \to \Upsilon(1S)\gamma) < 0.06.
\]

$B(\Upsilon(3S) \to \chi_{bJ}(1P)\gamma)$ is sensitive to the $D$–wave mixing angle and can be calculated from Eq. (27), (30) and (31) knowing the total decay width. These branching ratios lead to a bound on $\phi$ because one has to satisfy the following experimental relation [13] (individual branching ratios are not available at this time):

\[
F(\phi) = \sum_{J=1,2} B(\Upsilon(3S) \to \chi_{bJ}(1P)\gamma) \cdot B(\chi_{bJ}(1P) \to \Upsilon(1S)\gamma)
\]
\[
= (1.2 ^{+0.4}_{-0.3} \pm 0.09) \times 10^{-3}.
\]

In Figs. 2 (a) and (b) we show $F(\phi)$ for mixing with $|1D\rangle$ and $|2D\rangle$ states respectively. The allowed region of $\phi$ is larger in the case of $2D$ mixing if one demands the consistency between the $\chi_{bJ}(nP)$ decays. We like to emphasize that the present experimental data is consistent with the assumption of a $D$–wave mixing. Based on our analysis, mixing with the $2D$ state is seen to be more plausible.
An estimate of the mixing angle (32) may now be used to predict the branching ratio of $\chi_{b0}$ decaying into $\Upsilon(1S)$ using [13]

$$\sum_{J=0,1,2} B(\Upsilon(3S) \to \chi_{bJ}(1P)\gamma) \cdot B(\chi_{bJ}(1P) \to \Upsilon(1S)\gamma) = (1.7 \pm 0.4 \pm 0.6) \times 10^{-3}.$$  \hspace{1cm} (35)

We finally write down expected branching ratios of $\Upsilon(3S) \to \chi_{bJ}(1P) + \gamma$ for $J = 0, 1, 2$ assuming $|1D\rangle$ and $|2D\rangle$ mixing using the central values of Eq. (32):

$$B(\Upsilon(3S) \to \chi_{b0}(1P) + \gamma) = 1.1\% \hspace{0.5cm} (1D), \hspace{1cm} 1.8\% \hspace{0.5cm} (2D),$$

$$B(\Upsilon(3S) \to \chi_{b1}(1P) + \gamma) = 2.0\% \hspace{0.5cm} (1D), \hspace{1cm} 0.2\% \hspace{0.5cm} (2D),$$

$$B(\Upsilon(3S) \to \chi_{b2}(1P) + \gamma) = 0.3\% \hspace{0.5cm} (1D), \hspace{1cm} 1.0\% \hspace{0.5cm} (2D),$$  \hspace{1cm} (36)

It is clear that a better determination of branching ratios of these radiative decays, $\Upsilon(3S) \to \chi_{bJ}(nP) + \gamma$ with $n = 1, 2$ can resolve $2D$ mixing from $1D$ mixing, or vice versa.

The spin–flip radiative transition $\Upsilon(3S) \to \eta_b + \gamma$ is also affected by the $D$–wave component in $\Upsilon(3S)$. The decay rate is given by

$$\Gamma(\Upsilon(3S) \to \eta_b + \gamma) = \frac{2}{3} \alpha Q_b^2 \frac{\omega^3}{m_b^2} |F|^2, \quad \text{ (37)}$$

where

$$F = \sqrt{2} \langle 1S | \ j_0(\omega r/2) \ | 3S \rangle \cos \phi + \langle 1S | j_2(\omega r/2) \ | kD \rangle \sin \phi, \quad \text{ (38)}$$

where $j_n(x)$ is the $n$–th spherical Bessel function. In the long wavelength limit
(\omega r \to 0)$, we can approximate:

$$j_0(x) = 1 - \frac{x^2}{6},$$

$$j_2(x) = \frac{x^2}{15},$$

so that

$$F = \frac{1}{G_8} \left[ -\frac{\sqrt{2}}{24} I_{3S,1S} \cos \phi + \frac{1}{60} I_{kD,1S} \sin \phi \right],$$

(39)

assuming $G_8$ is a constant parameter, as in Ref. [5]. Therefore, one can again use (19) and (20) to evaluate $F$, and calculate the decay rate from (37):

$$\Gamma(\Upsilon(3S) \to \eta_b + \gamma) = \begin{cases} 4 \left(18/G_8^2(\text{GeV}^{-2})\right) \text{eV} & \text{(for P2)}, \\ 75 \left(18/G_8^2(\text{GeV}^{-2})\right) \text{eV} & \text{(for P1)}, \end{cases}$$

(40)

Note that (i) there is no $\phi$–dependence left over, once we use the results (19) and (20), and (ii) this result is independent of which of the $D$-wave actually mixes. If the parameter set P1 were the correct one and $G_8$ not too large, this could open up a new option for the discovery of $\eta_b$ in the $e^+e^-$ annihilation bypassing the intermediate stage involving $h_b(1P)$. Unfortunately, the upper limit on $B(\Upsilon(3S) \to \Upsilon(1S) + \eta)$ prefers the parameter set P2, for which $B(\Upsilon(3S) \to \eta_b + \gamma) < 1.6 \times 10^{-4}$. Therefore, this channel may compete with other possibilities discussed below in the search of $\eta_b$, only if $G_8^2$ is not too large.

The $D$–wave component in $\Upsilon(3S)$ can generate other interesting radiative transitions:

$$\Upsilon(3S) \xrightarrow{\gamma} 1^1D_2 \xrightarrow{\gamma} h_b(1P) \xrightarrow{\gamma} \eta_b.$$
The first chain is energetically allowed only for 1D mixing. These decay rates can be readily obtained from the results of Ref. [14]. Omitting all details, the final results are given below:

\[
\Gamma(\Upsilon(3S) \to 1^1D_2 + \gamma) = \frac{16}{3} \omega^3 \frac{\alpha Q_b^2}{(2m_b)^2} \sin^2 \phi
\]

\[
\approx 3.8 \left( \frac{\omega (\text{MeV})}{200} \right)^3 \left( \frac{\sin \phi}{0.1} \right)^2 \text{eV,}
\]

\[
\Gamma(1^1D_2 \to h_b(1P) + \gamma) = \frac{8}{15} \alpha Q_b^2 \omega^3 |\langle 1P | r | 1D \rangle|^2 \approx 30 \text{ keV,}
\]

\[
\Gamma(h_b(1P) \to \eta_b + \gamma) \approx 40 \text{ keV.}
\]

This corresponds to the production of \( \approx 80 \) \( h_b(1P) \)'s in decays of \( 10^6 \) \( \Upsilon(3S) \)'s for \( \sin \phi = \pm 0.1 \), or \( \phi = \pm 6^\circ \), and if \( B(1^1D_2 \to h_b(1P) + \gamma) = 50\% \). These are sensitive to the mixing angle \( \phi \), and may be useless for producing \( h_b(1P) \) and \( \eta_b \), if \( \sin \phi < 0.1 \).

4. Hadronic transitions of \( \Upsilon(3S) \) into \( h_b(1P) \)

Finally, let us consider \( \Upsilon(3S) \to h_b(1P) + X \) with \( X = \pi^0 \) or \( \pi\pi \). This may serve as a source of the spin–singlet \( P \)–wave state, \( h_b(1P) \), if its branching ratio is appreciable. In QCD multipole expansion, the above transitions are generated by the interference between \( E1 \) and \( M1 \) interactions, where the \( M1 \) interaction Hamiltonian is [2]

\[
H_{\text{int}}(M1) = \frac{1}{2m} g \xi^a \Delta_i H_i^a(0),
\]

with \( \Delta \) being the difference of the spin operators for the quark and the antiquark.
The amplitude for $\Upsilon(3S) \to h_b(1P) + X$ is

$$M(\Upsilon(3S) \to h_b(1P) + X) = -\frac{2}{3m} \hat{\epsilon}_i \hat{\epsilon}'_j \langle X | \pi \alpha_s E_i^a H_k^a | 0 \rangle$$

$$\times \langle (h_b(1P))_j | r_l G_8 \Delta_k + \Delta_k G_8 r_l | (\Upsilon(3S))_i \rangle,$$  \hspace{1cm} (45)

where $\hat{\epsilon}$ and $\hat{\epsilon}'$ are the polarization vectors of $\Upsilon(3S)$ and $h_b(1P)$, respectively.

The angular part of the quarkonium matrix elements can be performed as before, and we get

$$\langle (1P)_j | r_l G_8 \Delta_k | (3S)_i \rangle = \frac{1}{\sqrt{3}} \langle R_P | r G_8 | R_S \rangle \delta_{ik} \delta_{jl}, \hspace{1cm} (46)$$

$$\langle (1P)_j | r_l G_8 \Delta_k | (3D)_i \rangle = \frac{\sqrt{3}}{5 \sqrt{2}} \langle R_P | r G_8 | R_D \rangle$$

$$\times \left( \delta_{ij} \delta_{kl} + \delta_{il} \delta_{jk} - \frac{2}{3} \delta_{ik} \delta_{jl} \right).$$  \hspace{1cm} (47)

Now, consider the case $X = \pi^0$, for which the matrix element of the gluonic operators are determined by $U_A(1)$ anomaly and the mass difference between $u$ and $d$ quarks [2, 4]:

$$\langle \pi^0 | \pi \alpha_s E_i^a H_k^a | 0 \rangle = \delta_{ik} \frac{\pi^2}{3 \sqrt{2}} \left( \frac{m_u - m_d}{m_u + m_d} \right) f_\pi m_\pi^2 \equiv A_0 \delta_{ik}. \hspace{1cm} (48)$$

Using the pion decay constant $f_\pi = 132$ MeV and $(m_u - m_d)/(m_u + m_d) = 0.3$, we get $A_0 = 1.7 \times 10^{-3}$ GeV$^3$. The amplitude for $\Upsilon(3S) \to h_b(1P) + \pi^0$ becomes

$$M(\Upsilon(3S) \to h_b(1P) + \pi^0) = A_0 I_\pi \hat{\epsilon} \hat{\epsilon}',$$  \hspace{1cm} (49)

where

$$I_\pi = -\frac{4 \sqrt{3}}{9} \frac{G_8}{m_b} \left[ \langle 1P | r | 3S \rangle \cos \phi + \sqrt{2} \langle 1P | r | 1D \rangle \sin \phi \right],$$  \hspace{1cm} (50)

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and the decay rate is

\[ \Gamma(\Upsilon(3S) \rightarrow h_b(1P) + \pi^0) = \frac{1}{2\pi} \left( A_0 \, I_{\pi} \right)^2 |\vec{p}_{\pi}|. \] (51)

The expression in the bracket of (50) is the same as the first one in (28). Therefore, \( \Upsilon(3S) \rightarrow \chi_{b0}(1P) + \gamma \) and \( \Upsilon(3S) \rightarrow h_b(1P) + \pi^0 \) are related with each other. The ratio of the decay rates of these two seemingly different decays is independent of quarkonium matrix elements of \( r \) or the mixing angle \( \phi \), and determines \( G_8 \). From (27) and (51), we find

\[ \frac{\Gamma(\Upsilon(3S) \rightarrow h_b(1P) + \pi^0)}{\Gamma(\Upsilon(3S) \rightarrow \chi_{b0}(1P) + \gamma)} = \frac{27}{8\pi\alpha Q^2_b} \left( \frac{4\sqrt{3}}{9} \frac{G_8 A_0}{m_{b\omega}} \right)^2 \approx 8.7 \times 10^{-3} \left( \frac{G_8^2 \text{ (GeV}^{-2})}{18} \right). \] (52)

The absolute decay rate of \( \Upsilon(3S) \rightarrow h_b(1P) + \pi^0 \) is

\[ \Gamma(\Upsilon(3S) \rightarrow h_b(1P) + \pi^0) = \left( \frac{G_8^2 \text{ (GeV}^{-2})}{18} \right) \times \left\{ \begin{array}{l} 1 \text{ eV} \quad \text{(for 1D)} \\ 0.3 \text{ eV} \quad \text{(for 2D)} \end{array} \right\} \] (53)

(For numerical estimates, we have used the values of \( \langle 1P|r|3S \rangle \) and \( \langle 1P|r|1D \rangle \) quoted in (30) with relativistic correction factors (31) and central values of (32).) This amounts to the branching ratio greater than \( 4 \times 10^{-5} \) for 1D mixing, and \( 1.2 \times 10^{-5} \) for 2D mixing. Therefore, this decay may be the best for reaching \( h_b(1P) \), and subsequently \( \eta_b \) through \( h_b(1P) \rightarrow \eta_b + \gamma \).

A similar decay, \( \Upsilon(3S) \rightarrow h_b(1P)\pi\pi \), does not receive any contribution from the trace of the energy–momentum tensor in QCD, and is not enhanced over \( \Upsilon(3S) \rightarrow \)
\[ h_b(1P) + \pi^0. \]

From the general expression (11)–(14), we get

\[
\langle \pi^+ \pi^- | \pi \alpha_\pi E^\alpha_i H^\alpha_j | 0 \rangle = \frac{1}{2} \lambda \epsilon_{ijk} (E_1 p_{2k} + E_2 p_{1k}),
\]

(54)

where \( p^\mu = (E_1, \vec{p}_1) \), \( p'^\mu = (E_2, \vec{p}_2) \) are the four–momenta of the pions. Then, the decay rate for \( \Upsilon(3S) \to h_b(1P)\pi^+\pi^- \) is [2]

\[
\Gamma(\Upsilon(3S) \to h_b(1P)\pi^+\pi^-) = \frac{\lambda^2}{48\pi^3} \varphi \frac{\Delta^7}{70} |I_{2\pi}|^2,
\]

(55)

where \( \Delta = m(\Upsilon(3S)) - m(h_b(1P)) \), \( \varphi = 0.22 \) is the suppression factor of the phase space integral due to the pion mass [4], and

\[
I_{2\pi} = -\frac{4\sqrt{3}}{9} \frac{G_s}{m_b} \left[ \langle 1P | r | 3S \rangle \cos \phi - \frac{1}{\sqrt{2}} \langle 1P | r | 1D \rangle \sin \phi \right].
\]

(56)

Again, the expression in the square bracket is the same as the second equation of (28), and this decay is related to \( \Upsilon(3S) \to \chi_b(1P) + \gamma \) in the same way as in (52).

From (51) and (55), we get

\[
\frac{\Gamma(\Upsilon(3S) \to h_b(1P)\pi^+\pi^-)}{\Gamma(\Upsilon(3S) \to h_b(1P) + \pi^0)} \approx \left\{ \begin{array}{ll}
0.2 & \text{(for 1D)} \\
0.03 & \text{(for 2D)}
\end{array} \right.
\]

(57)

where we have used \( \lambda = 2 \) and Eqs. (51), (54) and (52). This corresponds to \( \sim 10^{-5} \) (for the 1D mixing) and \( \sim 4 \times 10^{-7} \) (for the 2D mixing) in the branching ratio of \( \Upsilon(3S) \to h_b(1P)\pi\pi \). The current upper limit is about two orders of magnitude above our prediction [10]. Our analysis shows that one has to look for \( \Upsilon(3S) \to h_b(1P) + \pi^0 \) \( (10^{-5} \sim 10^{-4} \text{ in the branching ratio}) \) rather than \( \Upsilon(3S) \to h_b(1P)\pi\pi \) to identify the spin–singlet \( P \)–wave bottomonium state, \( h_b(1P) \), even if \( \Upsilon(3S) \) has an admixture of a \( D \)–wave.
5. Conclusion

In conclusion, we find that the $\pi\pi$ spectrum in $\Upsilon(3S) \rightarrow \Upsilon(1S) \pi\pi$ can be explained in a natural way in the framework of QCD multipole expansion by assuming the physical $\Upsilon(3S)$ state is an admixture of the $S-$ and $D-$waves. The $\pi\pi$ spectrum determines the mixing angle $\phi$ to be less than $\sim 24^\circ$. ( Decay rates for $\Upsilon(3S) \rightarrow \chi_{bJ}(2P) + \gamma$ give much tighter and more reliable value for $\phi$ to be $-(4^\circ \pm 6^\circ)$ for $1D$ mixing, and $+(3^\circ \pm 4^\circ)$ for $2D$ mixing.) Effects of this $D-$wave component on other decays of $\Upsilon(3S)$ are discussed in detail. Twofold ambiguity encountered in Ref. [1] is resolved by the upper limit on $B(\Upsilon(3S) \rightarrow \Upsilon(1S) + \eta)$, and the parameter set P2 is preferred for which

$$B(\Upsilon(3S) \rightarrow \Upsilon(1S) + \eta) = 0.2\%.$$  

This is one of the cleanest tests of our assumptions on the $D-$wave mixing in $\Upsilon(3S)$ and applicability of QCD multipole expansion to hadronic transitions of $\Upsilon(3S)$. Consistency of our approach can be cross–checked by measuring the polar angle distribution ( with respect to the $e^+e^-$ beam direction ) of $\Upsilon(1S)$ in $\Upsilon(3S) \rightarrow \Upsilon(1S) \pi\pi$ as discussed in Ref. [1]. Further tests of our assumptions will be possible by checking our predictions [30] by improving measurements of $B(\Upsilon(3S) \rightarrow \chi_{bJ}(nP) + \gamma)$. More informations on $\phi$ and/or $G_8$ can be extracted by checking [10], [11], [12], [52] and (53). The most promising place to look for $h_b(1P)$ may be $\Upsilon(3S) \rightarrow h_b(1P) + \pi^0$, but the branching ratio is rather small, (53). (If $\sin \phi \approx 0.1$ and $1D$ mixing is the correct
one, the cascades (3) may be useful.) Because of the small branching ratio of decay modes involving $h_b(1P)$ in the final state, it may be worth while to look at the direct transition $\Upsilon(3S) \rightarrow \eta_b + \gamma$ for $\eta_b$ search.

In this work, we have derived several new results on the branching ratios of hadronic and radiative transitions of $\Upsilon(3S)$. If any of our predictions is in serious contradiction with the experiments, then we may conclude that QCD multipole expansion breaks down for hadronic transitions of $\Upsilon(3S)$ to lower quarkonia because of the large radius of $\Upsilon(3S)$ and its being close to the $B\bar{B}$ threshold.

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Figure Caption

**Fig. 1** The best fit to the $\pi\pi$ spectrum in $\Upsilon(3S) \to \Upsilon(1S) \pi\pi$ using the amplitude \(^{(17)}\). The results of this fit is given in \(^{(18)}\), for which $\chi^2/d.o.f. = 11.2/7$.

**Fig. 2** Plots of $F(\phi)$ of Eq. \(^{(34)}\) assuming the mixing angle ($\phi$) to be a parameter. Fig. 2 (a) assumes a mixing with $|1D\rangle$, while Fig. 2 (b) assumes a mixing with $|2D\rangle$. The shaded region is allowed by Eqs. \(^{(32)}\) and \(^{(34)}\). These may be used to obtain bounds on $\phi$.

Table 1.

Two sets of parameters giving the best $\chi^2$ fit to the $\pi\pi$ spectrum from Ref. \(^{(1)}\).

| Parameters | Fit 1 (P1)   | Fit 2 (P2)   |
|------------|-------------|-------------|
| A          | 101.60 ± 103.7 | 366.34 ± 61.94 |
| B          | −5.80 ± 8.29  | −3.12 ± 2.67  |
| C          | 20.53 ± 84.09 | 4.33 ± 24.86  |
| D          | 3.73 ± 4.35   | −1.03 ± 0.30  |

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