Estimates of functions with vanishing periodizations *

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Abstract
We prove that if a function \( f \in L^p(\mathbb{R}^d) \) has vanishing periodizations then \( \|f\|'_p \lesssim \|f\|_p \), provided \( 1 \leq p < \frac{2d}{d+2} \) and dimension \( d \geq 3 \).

1 Introduction
Let \( f \in L^1(\mathbb{R}^d) \). Define a family of its periodizations with respect to a rotated integer lattice:

\[
g_\rho(x) = \sum_{\nu \in \mathbb{Z}^d} f(\rho(x - \nu)) \tag{1}
\]

for all rotations \( \rho \in SO(d) \). We have a trivial estimate \( \|g_\rho\|_1 \leq \|f\|_1 \) and \( \hat{g}_\rho(m) = \hat{f}(\rho m) \) where \( m = (m_1, \ldots, m_d) \in \mathbb{Z}^d \). The author has shown recently that \( g_\rho \) is in \( L^2([0,1]^d \times SO(d)) \) if and only if \( f \in L^2(\mathbb{R}^d) \), provided the dimension \( d \geq 5 \). The requirement \( f \in L^1(\mathbb{R}^d) \) can be replaced by \( f \in L^p(\mathbb{R}^d) \) for a certain range of \( p \), see for details ([6]), ([7]).

The main object of our research will be functions \( f \) whose periodizations \( g_\rho \) identically vanish for a.e. rotations \( \rho \in SO(d) \). It is equivalent to the statement that \( \hat{f} \) vanishes on all spheres of radius \( |m| = (m_1^2 + \ldots + m_d^2)^{\frac{1}{2}} \) where \( m \in \mathbb{Z}^d \). Such functions are closely related to the Steinhaus tiling set

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problem ([4], [5]): does there exists a (measurable) set \( E \subset \mathbb{R}^d \) such that every rotation and translation of \( E \) contains exactly one integer lattice point? M. Kolountzakis ([4]) showed that if \( f \in L^1 \) and \( |x|^\alpha f(x) \in L^1 \) for a certain \( \alpha > 0 \) and \( f \) has constant periodizations then \( \hat{f} \in L^1 \) when dimension \( d = 2 \). M. Kolountzakis and T. Wolff ([5], Theorem 1) proved that if periodizations of a function from \( L^1(\mathbb{R}^d) \) are constants then the function is continuous and, in fact, bounded, provided that the dimension \( d \) is at least three. We will generalize the last result for functions \( f \) in \( L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d) \):

**Theorem 1** Let \( d \geq 3 \) and \( f \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d) \), \( 1 \leq p < \frac{2d}{d+2} \), has identically vanishing periodizations then \( f \in L^{p'}(\mathbb{R}^d) \):

\[
\|f\|_{p'} \leq C\|f\|_p
\]

where \( C \) depends only on \( d \) and \( p \).

The main reason why the dimension \( d \geq 3 \) comes from the famous Lagrange theorem saying that every positive integer can be represented as sums of four squares and actually from the fact that every integer of form \( 8k+1 \) can be written as sums of three squares. Since relatively few integers can be represented as sums of two squares, we will show in Section 3 that the result of M. Kolountzakis and T. Wolff doesn’t hold if \( d = 2 \) and that is why there is no theorem for \( d = 2 \). Another reason why the dimension \( d \geq 3 \) is because we consider the family of periodizations with respect to the \( SO(d) \) group of rotations. It leads to estimates involving the decay of spherical harmonics. The rate of decay for \( d = 2 \) is not fast enough although it is almost fast enough. That is why for \( d = 2 \) the range of \( p \) in the theorem becomes empty: \( 1 \leq p < 1 \).

**Remark 1** There is no essential difference between the case of identically vanishing periodizations and the case of \( g_\rho \) being trigonometric polynomials of uniformly bounded degrees for all \( \rho \in SO(d) \).

**Corollary 1** If \( p \leq r \leq p' \) then under the conditions of Theorem 1

\[
\|f\|_r \leq C\|f\|_p
\]

where \( C \) depends only on \( d \) and \( p \).

We will show in Section 3 that this range of \( r \) is sharp.

We will use the notation \( x \lesssim y \) meaning \( x \leq Cy \), and \( x \sim y \) meaning that \( x \lesssim y \) and \( y \lesssim x \) for some constant \( C > 0 \) independent from \( x \) and \( y \).
2 Proof of the theorem

Define the following functions $h, h_1, h_2 : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{C}$

\[ h(y, t) = \int \hat{f}(\xi)e^{i2\pi y \cdot \xi}d\sigma_t(\xi) \]  (2)

\[ = \int_{\mathbb{R}^d} f(x)\hat{d}\sigma_t(y - x)dx \]  (3)

\[ = \int_{\mathbb{R}^d} f(y - x)\hat{d}\sigma_t(x)dx, \]  (4)

\[ h_1(y, t) = \int_{|x| \leq 1} f(y - x)\hat{d}\sigma_t(x)dx, \]  (5)

\[ h_2(y, t) = \int_{|x| > 1} f(y - x)\hat{d}\sigma_t(x)dx \]  (6)

where $d\sigma_t$ is the Lebesgue surface measure on a sphere of radius $t$. Clearly, $h = h_1 + h_2$. To proceed further we will need certain technical estimates associated with $h_1$ and $h_2$ proven in two lemmas below. The proof of the theorem itself starts after Remark 2 to Lemma 2. The Fourier transforms in these two lemmas below are taken with respect to variable $t$, except in the second part of the proof of Lemma 2. $L^p'$ norms are taken over variable $y$. We will apply some technique from M. Kolountzakis and T. Wolff ([5]) and O. Kovrijkine ([6], [7]).

Lemma 1 Let $q : \mathbb{R} \to \mathbb{R}$ be a Schwartz function supported in $[\frac{1}{4}, 2]$, let $f \in L^p(\mathbb{R}^d)$ where $1 \leq p \leq 2$ and let $b \in [0, 1)$. Define $H_{1,N} : \mathbb{R}^d \times \mathbb{R} \to \mathbb{C}$

\[ H_{1,N}(y, t) = \frac{1}{\sqrt{t + b}}h_1(y, \sqrt{t + b})q(\frac{\sqrt{t + b}}{N}). \]

Then

\[ \sum \sum \Vert \hat{H}_{1,2}(y, \nu)\Vert_{p'} \leq C\|f\|_p \] (7)

where $C$ depends only on $q$ and $d$.

Proof of Lemma 1:

It will be enough to show that

\[ \sum \nu \neq 0 \Vert \hat{H}_{1,N}(y, \nu)\Vert_{p'} \leq \frac{C\|f\|_p}{N}. \] (8)
We have
\[ \hat{H}_{1,N}(y,\nu) \leq C |\nu|^{k} \int |\partial_{t}^{k}H_{1,N}(y,t)| dt \] (9)
for \( \nu \neq 0 \). Applying Minkowski’s inequality to (9) we have
\[ \| \hat{H}_{1,N}(y,\nu) \|_{p'} \leq C |\nu|^{k} \int \|\partial_{t}^{k}H_{1,N}(y,t)\|_{L^{p'}(dy)} dt. \] (10)

We need to estimate the integrand on the right side of (10). To do so we will first estimate the \( L^{p'} \) norm of derivatives of \( h_{1}(y,t) \) when \( t \geq 1 \):
\[ \|\partial_{t}^{k}h_{1}(y,t)\|_{p'} \lesssim t^{d-1} \|f\|_{p} \] (11)
with an implicit constant depending only on \( k \) and \( d \). In order to obtain (11), rewrite the definition of \( h_{1} \) (5) in the following way:
\[ h_{1}(y,t) = \int_{|x| \leq 1} f(y-x)\hat{\sigma}_{t}(x)dx \]
\[ = t^{d-1} \int_{\mathbb{R}^{d}} f(y-x) \cdot \chi_{\{|x| \leq 1\}} \int_{|\xi| = 1} e^{-i2\pi tx \cdot \xi}d\sigma(\xi)dx, \]
differentiate the last equality \( k \) times and apply Young’s inequality.

We can easily prove by induction that
\[ \frac{d^{k}}{dt^{k}} \left( \frac{h_{1}(\sqrt{t} + b)}{\sqrt{t} + b} \right) = \sum_{i=0}^{k} C_{i,k} \frac{h_{1}^{(i)}(\sqrt{t} + b)}{\left(\sqrt{t} + b\right)^{2k+1-i}}. \] (12)

Combining (12) and (11) we obtain for \( t \sim N^{2} \)
\[ \|\partial_{t}^{k} \left( \frac{h_{1}(y,\sqrt{t} + b)}{\sqrt{t} + b} \right)\|_{p'} \leq CN^{d-k-2} \|f\|_{p} \] (13)
with \( C \) depending only on \( k \) and \( d \).

Since \( q(\frac{\sqrt{t} + b}{N}) = q(\sqrt{t'} + b') = \tilde{q}(t') \) with \( t' = \frac{t}{N^{2}} \) and \( b' = \frac{b}{N} \) and \( \tilde{q}(t') \) is a Schwartz function supported in \( t' \sim 1 \), we have
\[ \left| \frac{d^{k}}{dt^{k}} q(\frac{\sqrt{t} + b}{N}) \right| = N^{-2k} \left| \frac{d^{k}}{dt^{k}} \tilde{q}(t') \right| \leq CN^{-2k} \] (14)
with C depending only on k and q.

\[ q\left(\sqrt{t+b}\right) \text{ is supported in } t \sim N^2 \text{ hence we obtain from (13) and (14) that} \]

\[
\left\| \frac{\partial^k}{\partial t^k} H_{1,N}(y,t) \right\|_{p'} = \left\| \frac{d^k}{dt^k} \left( \frac{h_1(y,\sqrt{t+b})}{\sqrt{t+b}} q(\sqrt{t+b}/N) \right) \right\|_{p'} \leq C N^{d-2-k} \|f\|_p
\]

with C depending only on k, d and q. Since \( H_{1,N}(y,t) \) is also supported in \( t \sim N^2 \) we have

\[
\int \left\| \frac{\partial^k}{\partial t^k} H_{1,N}(y,t) \right\|_{L^{p'}(dy)} dt \leq C N^{d-k} \|f\|_p.
\]

Substituting the above estimate to (10) we obtain

\[
\left\| \hat{H}_{1,N}(y,\nu) \right\|_{p'} \leq \frac{C N^{d-k} \|f\|_p}{|\nu|^k} \quad (16)
\]

for every \( \nu \neq 0 \).

Summing (16) over all \( \nu \neq 0 \) and putting \( k = d+1 \) we get our desired result

\[
\sum_{\nu \neq 0} \| \hat{H}_{1,N}(y,\nu) \|_{p'} \leq C \|f\|_p
\]

where C depends only on q and d. Sum over dyadic \( N \) to obtain the statement of the lemma.

The next lemma will be proven in the spirit of the Stein-Tomas restriction theorem ([1], p.104).

**Lemma 2** Let \( q : \mathbb{R} \to \mathbb{R} \) be a Schwartz function supported in \([1/2, 2]\), let \( f \in L^p(\mathbb{R}^d) \) where \( 1 \leq p < \frac{2d}{d+2} \) and let \( b \in [0,1) \). Define \( H_{2,N} : \mathbb{R}^d \times \mathbb{R} \to \mathbb{C} \)

\[
H_{2,N}(y,t) = \frac{1}{\sqrt{t+b}} h_2(y,\sqrt{t+b}) q(\sqrt{t+b}/N).
\]

Then we have

\[
\sum_{\nu \neq 0} \| \sum_{l \geq 0} \hat{H}_{2,2^l}(y,\nu) \|_{p'} \leq C \|f\|_p \quad (17)
\]

with C depending only on p, q and d.
Proof of Lemma 2:

We have

\[ \hat{H}_{2,N}(y,\nu) = \int H_{2,N}(y,t)e^{-i2\pi\nu t}dt \]

\[ = 2e^{i2\pi\nu b} \int Nq(t)h_2(y,tN)e^{-i2\pi\nu(Nt)^2}dt \]

\[ = 2e^{i2\pi\nu b} \int Nq(t)e^{-i2\pi\nu(Nt)^2} \int_{|x|>1} f(y-x)d\sigma_{Nt}(x)dxdt \]

\[ = 2e^{i2\pi\nu b} \int_{|x|>1} f(y-x) \int Nq(t)e^{-i2\pi\nu(Nt)^2}(Nt)^{d-1}d\sigma(Ntx)dtdx \]

(18)

where

\[ D_{N,\nu}(x) = 2e^{i2\pi\nu b} \chi_{\{|x|>1\}} \int Nq(t)e^{-i2\pi\nu(Nt)^2}(Nt)^{d-1}d\sigma(Ntx)dtdx. \]  

(19)

Denote by

\[ K_\nu(x) = \sum_{l\geq0} D_{2^l,\nu}(x). \]  

(20)

We need to estimate

\[ \| \sum_{l\geq0} \hat{H}_{2,2^l}(y,\nu) \|_{p'} = \| K_\nu * f \|_{p'}. \]

If \( p' = \infty \) or \( p' = 2 \) we have

\[ \| K_\nu * f \|_{\infty} \leq \| K_\nu \|_{\infty} \| f \|_1 \]

\[ \| K_\nu * f \|_2 \leq \| K_\nu \|_{\infty} \| f \|_2. \]

First we will show that

\[ \| K_\nu \|_{\infty} \leq \| \sum_{l\geq0} |D_{2^l,\nu}(x)| \|_{\infty} \]

\[ \leq C|\nu|^{-\frac{d}{2}}. \]  

(21)

To do so we need to estimate \( D_{N,\nu} \).

We will use a well-known fact that \( \hat{\sigma}(x) = \text{Re}(B(|x|)) \) with \( B(r) = a(r)e^{i2\pi r} \) and \( a(r) \) satisfying estimates

\[ |a^k(r)| \leq \frac{C}{r^{\frac{d}{2}+k}} \]  

(22)
with \( C \) depending only on \( k \) and \( d \). Now we will estimate the integral in (19) with \( B(|x|) \) instead of \( d\sigma(x) \)

\[
\int Nq(t)e^{-i2\pi\nu(Nt)^2}(Nt)^{d-1}a(N|x|t)e^{i2\pi N|x|t}dt
\]

\[
= \frac{N^{\frac{d+1}{2}}}{|x|} \int q(t)e^{-i2\pi\nu(Nt)^2}t^{d-1}a(N|x|t)(N|x|)^{\frac{d+1}{2}}e^{i2\pi N|x|t}dt
\]

\[
= \frac{N^{\frac{d+1}{2}}}{|x|}e^{i2\pi \frac{|x|}{2\nu}} \int q(t)a(N|x|t)(N|x|)^{\frac{d+1}{2}}t^{d-1}e^{-i2\pi\nu N^2(t-\frac{|x|}{2\nu})^2}dt
\]

\[
= \frac{N^{\frac{d+1}{2}}}{|x|}e^{i2\pi \frac{|x|}{2\nu}} \int \phi(t, |x|)e^{-i2\pi\nu N^2(t-\frac{|x|}{2\nu})^2}dt \tag{23}
\]

where \( \phi(t, |x|) = q(t)a(N|x|t)(N|x|)^{\frac{d+1}{2}}t^{d-1} \) is a Schwartz function with respect to variable \( t \) supported in \([\frac{N}{2}, 2] \) which is bounded, together with each derivative uniformly in \( t \), \( |x| \geq 1 \) and \( N \) because of (22). Note that we used here the fact that \( N|x| \geq 1 \). We can say even more. Let \( |x| = c \cdot r \) where \( c \geq 2 \) and \( r \geq \frac{1}{2} \). Then all partial derivatives of \( \phi(t, c \cdot r) \) with respect to \( t \) and \( r \) are also bounded uniformly in \( t \), \( r \), \( c \) and \( N \). Hence \( \phi(t, c \cdot t) \) is a Schwartz function supported in \([\frac{N}{2}, 2] \) which is bounded, together with each derivative uniformly in \( t \), \( c \) and \( N \). We will use this fact later to estimate \( \Delta_v \).

Fix some \( |x| \geq 1 \). In the calculations below we will write just \( \phi(t) \) instead of \( \phi(t, |x|) \) for simplicity. From the method of stationary phase ([3], Theorem 7.7.3) it follows that if \( k \geq 1 \) then

\[
| \int \phi(t)e^{-i2\pi\nu N^2(t-\frac{|x|}{2\nu})^2}dt - \sum_{j=0}^{k-1}c_j(\nu N^2)^{-j-\frac{1}{2}}\phi^{(2j)}(\frac{|x|}{2\nu N}) | \leq c_k(\nu N^2)^{-k-\frac{1}{2}} \tag{24}
\]

where \( c_j \) are some constants.

Since \( \phi \) is supported in \([\frac{N}{2}, 2] \) we conclude from (24) that

\[
| \int \phi(t)e^{-i2\pi\nu N^2(t-\frac{|x|}{2\nu})^2}dt | \leq \begin{cases} C(\nu N^2)^{-\frac{1}{2}} & \text{if } N \in \left[ \frac{|x|}{4\nu}, \frac{|x|}{\nu} \right] \\ C_k(\nu N^2)^{-k-\frac{1}{2}} & \text{if } N \notin \left[ \frac{|x|}{4\nu}, \frac{|x|}{\nu} \right] \end{cases} \tag{25}
\]

Replacing in (19) \( \Delta_v(x) \) with \( \frac{B(|x|)+B(|x|)}{2} \) it follows from (25) that

\[
|D_{N,\nu}(x)| \leq \frac{N^{\frac{d+1}{2}}}{|x|} \begin{cases} C(\nu N^2)^{-\frac{1}{2}} & \text{if } N \in \left[ \frac{|x|}{4\nu}, \frac{|x|}{\nu} \right] \\ C_k(\nu N^2)^{-k-\frac{1}{2}} & \text{if } N \notin \left[ \frac{|x|}{4\nu}, \frac{|x|}{\nu} \right] \end{cases} \tag{26}
\]

The number of dyadic \( N \in \left[ \frac{|x|}{4\nu}, \frac{|x|}{\nu} \right] \) is at most 3. Therefore choosing \( k \geq \frac{d+1}{4} \)
and summing (26) over all dyadic $N$ we have
\[ |K_\nu(x)| \leq \sum_{l \geq 0} |D_{2^l, \nu}(x)| \leq C|\nu|^{-\frac{d}{q}} \]
with $C$ depending only on $d$ and $q$. Thus we proved (21).

Now we will show that
\[ \| \hat{K}_\nu \|_\infty \leq \| \sum_{l \geq 0} \hat{D}_{2^l, \nu}(y) \|_\infty \leq C. \]  
(27)

Since supp $\phi \in [\frac{1}{4}, 2]$ we can re-write (24) for a stronger version of the method of stationary phase ([3], Theorems 7.6.4, 7.6.5, 7.7.3)
\[ \left| \int \phi(t)e^{-i2\pi N^2(t-\frac{y}{2\nu N})^2} dt \right| \leq \sum_{j=0}^{k-1} c_j(\nu N^2)^{-j-\frac{1}{2}} \phi^{(2j)}(\frac{|x|}{2\nu N}) \leq \frac{C_k(\nu N^2)^{-k-\frac{1}{2}}}{\max(1, \frac{|x|}{8\nu N^2})^k} \]
where $c_j$ are some constants. Therefore, if $\nu > 0$,
\[ D_{N, \nu}(x) = \chi_{\{|x|>1\}} \frac{N^{d+1}}{|x|^{d+1}} e^{i2\pi \frac{|x|^2}{\nu N}} \sum_{j=0}^{k-1} c_j(\nu N^2)^{-j-\frac{1}{2}} \phi^{(2j)}(\frac{|x|}{2\nu N}) + \phi_k(x) \]  
(28)
where $|\phi_k(x)| \leq \chi_{\{|x|>1\}} \frac{N^{d+1}}{|x|^{d+1}} \frac{C_k(\nu N^2)^{-k-\frac{1}{2}}}{\max(1, \frac{|x|}{8\nu N^2})^k}$. If $\nu < 0$ then just replace $\phi^{(2j)}(\frac{|x|}{2\nu N})$ with $\overline{\phi^{(2j)}(\frac{|x|}{2\nu N})}$. We further assume that $\nu > 0$. Choosing $k \geq \frac{d+1}{2}$ we have
\[ \| \hat{\phi}_k \|_\infty \leq \| \phi_k \|_1 \]
\[ = \int_{|x| \leq 8\nu N} |\phi_k| dx + \int_{|x| > 8\nu N} |\phi_k| dx \]
\[ \leq \frac{C}{N} \]
(29)
where $C$ depends only on $d$ and $q$. We can ignore $\chi_{\{|x|>1\}}$ in front of the sum in (28) because if $\frac{1}{8\nu N} \in [\frac{1}{4}, 2]$, then $|x| > \nu N \geq 1$. We will consider only the zero term in the sum. The other terms can be treated similarly. The Fourier transform of
\[ \frac{N^{d+1}}{|x|^{d+1}} e^{i2\pi \frac{|x|^2}{\nu N}} (\nu N^2)^{-\frac{1}{2}} \phi(\frac{|x|}{2\nu N}) \]

at point $y$ is equal to
\[ N^{\frac{d+1}{2}} (2\nu N)^{\frac{d+1}{2}} (\nu N^2)^{-\frac{1}{2}} \int_{\mathbb{R}^d} \psi(|x|) e^{i2\pi N^2 |x|^2} e^{-i2\pi \nu N^2 x \cdot y} dx = C(\nu N^2)^{\frac{d}{2}} e^{-i2\pi \nu |y|^2} \int_{\mathbb{R}^d} \psi(|x|) e^{i2\pi \nu N^2 |x|^2} e^{-\frac{\|x-y\|^2}{\nu N^2}} dx \]  
(30)
where \( \psi(t) = \phi(t, 2\nu Nt) t^{-\frac{d-1}{2}} \) is a Schwartz function supported in \([\frac{1}{2}, 2]\) whose derivatives and the function itself are bounded uniformly in \(t, \nu\) and \(N\) (see remark after (23)). The same is true about partial derivatives of \( \psi(|x|) \).

Applying the stationary phase method for \( \mathbb{R}^d \) (33, Theorem 7.7.3) we get

\[
| \int_{\mathbb{R}^d} \psi(|x|) e^{i2\pi \nu N^2 |x - \frac{y}{N}|^2} dx | \leq \begin{cases} C & \text{if } N \in [\frac{|y|}{2}, 2|y|] \\ C_k(\nu N^2)^{-k} & \text{if } N \notin [\frac{|y|}{2}, 2|y|] \end{cases}
\]

(31)

Therefore the absolute value of (30) can be bounded from above by:

\[
\leq \begin{cases} C & \text{if } N \in [\frac{|y|}{2}, 2|y|] \\ C_k(\nu N^2)^{-k} & \text{if } N \notin [\frac{|y|}{2}, 2|y|] \end{cases}
\]

(32)

Similar inequalities hold for Fourier transforms for the rest of the terms in the sum in (28). The number of dyadic \( N \in [\frac{|y|}{2}, 2|y|] \) is bounded by 3. Using (29), choosing \( k \geq 1 \) in (32) and summing over all dyadic \( N \) we get

\[
\sum_{l \geq 0} |\hat{D}_{2^l, \nu}(y)| \leq C
\]

(33)

with \( C \) depending only on \( d \) and \( q \), provided \( \nu \neq 0 \). Thus we proved (27).

Using (21) and (27) and interpolating between \( p = 1 \) and \( p = 2 \), we obtain

\[
\| K_\nu \ast f \|_{p'} \leq C \| \nu |^{-\alpha_p} \| f \|_p
\]

(34)

where \( \alpha_p = \frac{d-2}{p} \). \( \alpha_p > 1 \) if \( p < \frac{2d}{d+2} \). Summing (34) over all \( \nu \neq 0 \), we get the desired inequality

\[
\sum_{\nu \neq 0} \| \sum_{l \geq 0} \hat{H}_{2^l, \nu}(y, \nu) \|_{p'} \leq C \| f \|_p.
\]

\( \square \)

**Remark 2** It is clear from the proof that we have the same inequality if the summation over \( l \geq 0 \) is replaced by summation over any subset of nonnegative integers.

Now we are in a position to proceed with the proof of the theorem. Let \( q : \mathbb{R} \to \mathbb{R} \) be a fixed nonnegative Schwartz function supported in \([\frac{1}{2}, 2]\) such that

\[
q(t) + q(t/2) = 1
\]

when \( t \in [1, 2] \). It follows that

\[
\sum_{l \geq 0} q(\frac{t}{2^l}) = 1
\]

(35)
when $t \geq 1$. Define

$$q_0(t) = 1 - \sum_{l \geq 0} q(t)$$

for $t \geq 0$. It is clear that $q_0(|x|)$ is a Schwartz function supported in $|x| \leq 1$. Let $\psi(t) = q_0(t) + q(t)$ then

$$\psi_k(t) = \psi(t) = q_0(t) + \sum_{l \geq 0} q(t)$$

and $\psi(|x|)$ is a Schwartz function supported in $|x| \leq 2$ such that $\psi(|x|) = 1$ if $|x| \leq 1$. Therefore

$$\int \hat{f}(x)e^{2\pi x \cdot y}\psi\left(\frac{|x|}{2^k}\right)dx = (\hat{f} \hat{\psi}_k)(y)$$

converges to $f$ in $L^p$ as $k \to \infty$. To prove that $f \in L^p'$ and $\|f\|_{p'} \lesssim \|f\|_p$ it will be enough to show that

$$\|f \hat{\psi}_k\|_{p'} \leq C\|f\|_p$$

since the claim will follow by an application of Fatou’s lemma to a subsequence of $f \hat{\psi}_k$ converging a.e. to $f$.

We have

$$\begin{align*}
(f \hat{\psi}_k)(y) &= (f \hat{q}_0)(y) + \sum_{l \geq 0} \int \hat{f}(x)e^{2\pi x \cdot y}q\left(\frac{|x|}{2^l}\right)dx \\
&= (f \hat{q}_0)(y) + \sum_{l \geq 0} \int q\left(\frac{t}{2^l}\right)\int \hat{f}(\xi)e^{i2\pi y \cdot \xi}d\sigma_1(\xi)dt \\
&= (f \hat{q}_0)(y) + \sum_{l \geq 0} \int q\left(\frac{t}{2^l}\right)h(y,t)dt. \quad (36)
\end{align*}$$

Applying Young’s inequality we estimate the first term:

$$\|f \hat{q}_0\|_{p'} \lesssim \|f\|_p$$

for $1 \leq p \leq 2$. Now we have to estimate the sum over $l$.

It is a well-known fact from Number Theory proven by Lagrange that every positive integer can be represented as sums of four squares ([2], p.25), moreover there exists an infinite arithmetic progression of positive integers,
e.g., \(8n + 1\), which can be represented as sums of three squares ([2], p. 38).

We will use only the latter fact. Therefore, rescaling we can assume that \(\hat{f}\) vanishes on all spheres of radius \(\sqrt{n + b}\) where \(n\) is a nonnegative integer and \(0 < b < 1\) is a fixed number. Therefore \(h(y, \sqrt{n + b}) = 0\) for all \(y \in \mathbb{R}^d\).

Making a change of variables and keeping in mind that \(q\) is supported in \([\frac{1}{2}, 2]\) we re-write every term in the sum in the following way:

\[
\int_0^{\infty} q\left(\frac{t}{N}\right) h(y, t) dt = \int \frac{1}{2\sqrt{t + b}} q\left(\frac{\sqrt{t + b}}{N}\right) h(y, \sqrt{t + b}) dt.
\]

An application of Poisson’s summation formula gives us

\[
0 = \sum_n \frac{1}{\sqrt{n + b}} q\left(\frac{\sqrt{n + b}}{N}\right) h(y, \sqrt{n + b})
\]

\[
= \sum_{\nu} \left( \frac{1}{\sqrt{\nu + b}} q\left(\frac{\sqrt{\nu + b}}{N}\right) h(y, \sqrt{\nu + b}) \right)^{\wedge} (\nu)
\]

\[
= \int \frac{1}{\sqrt{\nu + b}} q\left(\frac{\sqrt{\nu + b}}{N}\right) h(y, \sqrt{\nu + b}) dt + \sum_{\nu \neq 0} \hat{H}_{1,N}(y, \nu) + \sum_{\nu \neq 0} \hat{H}_{2,N}(y, \nu)
\]

where

\[
H_{i,N}(y, t) = \frac{1}{\sqrt{t + b}} q\left(\frac{\sqrt{t + b}}{N}\right) h_i(y, \sqrt{t + b}), \quad i = 1, 2.
\]

Applying Lemma 1 and Lemma 2 with Remark 2 we bound the sum:

\[
\| \sum_{l \geq 0}^{k} \int_0^{\infty} q\left(\frac{t}{2^l}\right) h(y, t) dt \|_{p'} \leq \sum_{l \geq 0} \sum_{\nu \neq 0} \| \hat{H}_{1,2^l}(y, \nu) \|_{p'} + \sum_{l \geq 0} \| \sum_{\nu \neq 0} \hat{H}_{2,2^l}(y, \nu) \|_{p'}
\]

\[
\leq C \| f \|_{p}.
\]

Combining (36), (37) and the last inequality we obtain the desired result

\[
\| f * \widehat{\psi}_k \|_{p'} \leq C \| f \|_{p}
\]

from which the statement of the theorem follows. \(\square\)

Remark 3 We say that a function \(f \in L^p\) has vanishing periodizations if there exists a sequence of Schwartz functions \(f_k\) with vanishing periodizations converging to \(f\) in \(L^p\). It follows from Theorem 1 that \(f \in L^{p'}\) and \(f_k\) converge to \(f\) in \(L^p\) if dimension \(d \geq 3\) and \(1 \leq p < \frac{3d}{d+2}\).
3 Counterexamples and open questions

Theorem 1 does not say what happens when $d = 1$ and $d = 2$.

$d = 1$ is not an interesting case. We can easily construct examples of functions $f$ with vanishing periodizations such that their $L^p$ norms are not bounded by their $L^q$ norms for any given pair of $p \neq q$.

When $d = 2$ Theorem 1 does not hold. More precisely, Lemma 3 below shows that if $1 \leq p < 2$ then the following inequality does not hold for functions with vanishing periodizations:

$$\|f\|_{p'} \lesssim \|f\|_p.$$  

In this lemma we will deal with a sequence of functions $f_n$ such that $\hat{f}_n$ vanish on all circles of radius $\sqrt{l^2 + k^2}$. Denote by $X_2$ the Banach space of functions from $L^1(\mathbb{R}^2)$ whose Fourier transforms vanish on all circles of radius $\sqrt{l^2 + k^2}$

$$X_2 = \{ f \in L^1(\mathbb{R}^2) : \hat{f}(r) = 0 \text{ if } |r| = \sqrt{l^2 + k^2}, (k,l) \in \mathbb{Z}^2 \}.$$  

The next lemma crucially depends on the following fact from the Number Theory ([2], p.22):

The number of integers in $[n,2n]$ which can be represented as sums of two squares is $n\epsilon_n$ where $\epsilon_n \lesssim \frac{1}{\sqrt{\ln n}} \to 0$ as $n \to \infty$.

We only use the fact that $\lim \epsilon_n = 0$.

**Lemma 3** Let $1 \leq p < 2$ and $d = 2$ then there exists a sequence of Schwartz functions $f_n \in X_2$ such that

$$\lim_{n \to \infty} \frac{\|f_n\|_{p'}}{\|f_n\|_p} = \infty.$$  

**Proof of Lemma 3:** Let $a_1 < a_2 < a_3 < ...$ be the enumeration of numbers $a_m = \sqrt{l^2 + k^2}$ in ascending order. Denote $\delta_m = a_{m+1} - a_m$. As we already said the number of $a_m$ in $[\sqrt{n}, 2\sqrt{n}]$ is $n\epsilon_n$. Let $a_m_0$ and $a_m_1$ be correspondingly the smallest and the largest such $a_m$. Then

$$\sum_{m=m_0}^{m_1-1} \delta_m = a_{m_1} - a_{m_0} \sim \sqrt{n}.$$  

Let

$$\delta = \frac{C}{\sqrt{n}\epsilon_n}$$  

(38)
with small enough constant $C > 0$ so that if

$$M = \{m, m_0 \leq m < m_1 : \delta_m \geq \delta\}$$

then

$$\sqrt{n} \lesssim \sum_{m \in M} \delta_m$$

since $m_1 - m_0 \sim n\epsilon_n$. Choose coordinate axes $x$ and $y$. We will construct $\hat{f}_n$ supported in $\bigcup_{m \in M} R_m$ where $R_m$ is a largest possible rectangle inscribed between circles of radius $a_m$ and $a_{m+1}$ with sides parallel to the coordinate axes. Then $R_m$ is of size $\sim \delta_m \times \sqrt{\delta_m a_m} \gtrsim \delta_m \times \sqrt{n} \gtrsim \delta_m \times 1$. We will split each rectangle $R_m$ further into smaller rectangles $r$ of the same size $\sim \delta \times 1$. The number of these rectangles $r$ is

$$N = \sum_{m \in M} \left\lfloor \frac{\delta_m}{\delta} \right\rfloor \sim \sum_{m \in M} \frac{\delta_m}{\delta} \sim \frac{\sqrt{n}}{\sqrt{\epsilon_n}} = n\epsilon_n$$

(39)

since $\delta_m \geq \delta$ for $m \in M$. Enumerate these rectangles $r_k, k = 1, \ldots, N$. Let $r_k$ be centered at $(\lambda_k, 0)$ It is clear that $|\lambda_k - \lambda_l| \geq \delta$ for $k \neq l$. Let $\phi$ be a nonnegative Schwartz function on $\mathbb{R}$ supported in $[-\frac{1}{2}, \frac{1}{2}]$. We have that $\hat{\phi}(x) \geq C > 0$ when $x$ is small enough. Define $\hat{f}_n$ as the following sum:

$$\hat{f}_n(x, y) = \sum_{k=1}^{N} \phi\left(\frac{x - \lambda_k}{\delta}\right)\hat{\phi}(y).$$

(40)

The $k$-th term in (40) is supported in $r_k$. Therefore, $\hat{f}_n$ is a Schwartz function supported in $\bigcup_{m \in M} R_m$. Hence $\hat{f}_n$ vanishes on all circles of radius $a_l$. Taking the inverse Fourier transform of (40), we get

$$f_n(\xi, \eta) = \delta \hat{\phi}(\xi \delta)\hat{\phi}(\eta) \sum_{k=1}^{N} e^{i\lambda_k \xi}.$$
Assume that $p' < \infty$. Then
\[
\int |f_n(\xi, \eta)|^{p'} d\xi d\eta \geq \| \tilde{\phi} \|_{p'}^{p'} \int_{|\xi| \leq \frac{100}{\sqrt{n}}} |\tilde{\phi}(\xi \delta)|^{p'} \left| \sum_{k=1}^{N} e^{i\lambda_k \xi} \right|^{p'} d\xi
\]
\[
\gtrsim \delta^{p'} N^{p'} \frac{1}{\sqrt{n}}
\]
\[
\sim (\sqrt{n})^{p' - 1}.
\]
To obtain the second inequality we used that
\[
|\sum_{k=1}^{N} e^{i\lambda_k \xi}| \geq |\sum_{k=1}^{N} \cos (\lambda_k \xi)| \gtrsim N
\]
since $|\lambda_k \xi| \leq \frac{1}{50}$. We used (38) and (39) to obtain the last estimate. Therefore
\[
\|f_n\|_{p'} \gtrsim (\sqrt{n})^{\frac{1}{p'}}.
\]
(42)

If $p' = \infty$ we can obtain in a similar way that
\[
\|f_n\|_{\infty} \geq |f_n(0)| \gtrsim \sqrt{n}.
\]
(43)

Now we will estimate the $L^p$ norm from above. Denote
\[
g(x) = \sum_{k=1}^{N} e^{i\lambda_k \xi}.
\]
Since $|\frac{\lambda_k - \lambda_l}{\delta}| \geq \frac{\delta}{\delta} = 1$ for $k \neq l$ we have
\[
\int_I |g|^2 \sim N
\]
for any interval $I$ of length $4\pi$ (see ([8], Theorem 9.1)). Therefore,
\[
\int_I |g|^p \leq |I|^{1 - \frac{p}{2}} \left( \int_I |g|^2 \right)^{\frac{p}{2}}
\]
\[
\lesssim N^{\frac{p}{2}}
\]
(44)

for any interval $I$ of length $4\pi$. Since $\tilde{\phi}$ is a Schwartz function, we have that
\[
|\tilde{\phi}(x)| \lesssim \frac{1}{1 + x^2}.
\]
Therefore
\[
\int |f_n(\xi, \eta)|^p d\xi d\eta = \|\hat{f}_n\|_p^p - 1 \int |\hat{\phi}(\xi)|^p \cdot |\sum_{k=1}^N e^{i\xi_k^2/2} \hat{\phi}^p| d\xi
\]
\[
= C\delta^{p-1} \sum_{l=-\infty}^{\infty} \int_{\pi}^{(l+1)\pi} |\hat{\phi}(\xi)|^p \cdot |g(\xi)|^p d\xi
\]
\[
\lesssim \delta^{p-1} \sum_{l=-\infty}^{\infty} \frac{1}{(1+l^2)^p} N^p
\]
\[
\lesssim \sqrt{n}\epsilon_n^{1/2}. \tag{45}
\]

We used (38) and (39) to obtain the last estimate. Therefore
\[
\|f_n\|_p \lesssim (\sqrt{n})^{\frac{1}{p+1}} \epsilon_n^p.
\]

Dividing (42) by (45) we obtain the desired result
\[
\frac{\|f_n\|_{p'}}{\|f_n\|_p} \geq \frac{(\sqrt{n})^{\frac{1}{p'}}}{(\sqrt{n})^{\frac{1}{p+1}} \epsilon_n^{\frac{2-p}{2p}}}
\]
\[
= \frac{1}{\epsilon_n^{\frac{2-p}{2p}}} \to \infty
\]
as \(n \to \infty\) since \(p < 2\).

**Corollary 2** There exists a function \(f \in X_2\) such that
\[
\|f\|_{L^\infty(D(0,1))} = \infty.
\]

It follows immediately from the lemma and (43) that if \(p = 1\) then
\[
\sup_{f \in X_2} \frac{\|f\|_{L^\infty(D(0,1))}}{\|f\|_1} = \infty.
\]

We claim that there exists a function \(f \in X_2\) such that \(\|f\|_{L^\infty(D(0,1))} = \infty\). Suppose towards a contradiction that this is not true. Then the restriction operator
\[
T : f \to f|_{D(0,1)}
\]
maps \(X_2\) to \(L^\infty(D(0,1))\). Note that if \(f_n \to f\) in \(L^1\) and \(f_n \to g\) in \(L^\infty(D(0,1))\), then \(f = g\) a.e. on \(D(0,1)\). An application of the Closed Graph Theorem shows that \(T\) is a bounded operator acting from \(X_2\) to \(L^\infty(D(0,1))\).
This contradicts to the Corollary 2. Thus we proved our claim. □

Obviously, this function $f$ is not continuous. Therefore, it can serve as a counterexample to the theorem of M. Kolountzakis and T. Wolff ([5], Theorem 1) mentioned in Introduction when $d = 2$.

**Remark 4** However, it is not known whether the following inequality holds for $f \in X_2$:

$$\|f\|_r \lesssim \|f\|_p$$

where $1 \leq p < 2$ and $p < r < p'$.

Now we will show that the range of $r$ in Corollary 1 is sharp. We need to check two cases: $r > p'$ and $r < p$. In the former case the argument will be similar to the one in the previous lemma. Therefore we will give only a sketch of the proof. We will deal with a sequence of functions $f_n$ such that $\hat{f}_n$ vanish on all circles of radius $\sqrt{m_1^2 + \ldots + m_d^2}$. Denote by $X_d$ the Banach space of functions from $L^1(\mathbb{R}^d)$ whose Fourier transforms vanish on all circles of radius $m_1^2 + \ldots + m_d^2$.

$$X_d = \{f \in L^1(\mathbb{R}^d) : \hat{f}(r) = 0 \text{ if } |r| = \sqrt{m_1^2 + \ldots + m_d^2}, (m_1, \ldots, m_d) \in \mathbb{Z}^d \}.$$  

We will construct a sequence of Schwartz functions $f_n$ with Fourier transforms supported outside of spheres of radius $\sqrt{m}$. Therefore these functions automatically belong to $X_d$.

**Lemma 4** Let $1 < p \leq 2$ and $r > p'$ then there exists a sequence of Schwartz functions $f_n \in X$ such that

$$\lim_{n \to \infty} \frac{\|f_n\|_r}{\|f_n\|_p} = \infty.$$  

**Proof of Lemma 4:** A maximal rectangle inscribed between spheres of radius $\sqrt{n}$ and $\sqrt{n+1}$ has dimensions $\sim \frac{1}{\sqrt{n}} \times 1 \times 1 \times \ldots \times 1$. Let $r_k$ be parallel identical rectangles inscribed between spheres of radius $\sqrt{n+k}$ and $\sqrt{n+k+1}$, where $k = 0, 1, \ldots, n-1$, with dimensions $\sim \frac{1}{\sqrt{n}} \times 1 \times 1 \times \ldots \times 1$ and centered at $(\lambda_k, 0, 0, \ldots, 0)$. It is clear that $\lambda_{k+1} - \lambda_k \sim \frac{1}{\sqrt{n}}$. Let $\phi$ be a nonnegative Schwartz function on $\mathbb{R}$ supported in $[-\frac{1}{100}, \frac{1}{100}]$. We have that $\hat{\phi}(x) \geq C > 0$ when $x$ is small enough. Define $\hat{f}_n$ as the following sum:

$$\hat{f}_n(x_1, x_2, \ldots, x_d) = \sum_{k=0}^{n-1} \phi((x_1 - \lambda_k)\sqrt{n}) \prod_{l=2}^{d} \phi(x_l).$$ (46)
The $k$-th term in (46) is supported in $r_k$. Therefore, $\hat{f}_n$ is a Schwartz function vanishing on all spheres of radius $\sqrt{m}$. Taking the inverse Fourier transform of (46), we get

$$f_n(y_1, y_2, \ldots, y_d) = \prod_{l=2}^{d} \phi(y_l) \frac{1}{\sqrt{n}} \tilde{\phi}(\frac{y_1}{\sqrt{n}}) \sum_{k=0}^{n-1} e^{i \lambda_k y_1}. \quad (47)$$

Arguments analogous to those in Lemma 3 show that

$$\|f_n\|_r \gtrsim (\sqrt{n})^{\frac{1}{p}}$$

and

$$\|f_n\|_p \lesssim (\sqrt{n})^{\frac{1}{p}}.$$ 

Therefore

$$\frac{\|f_n\|_r}{\|f_n\|_p} \gtrsim (\sqrt{n})^{\frac{1}{p} - \frac{1}{q}} \to \infty$$

as $n \to \infty$ since $r > p'$. □

The case when $r < p$ is very simple. Let

$$\hat{f}(x) = \phi(\frac{x - x_0}{\epsilon})$$

where $\phi$ is a Schwartz function supported in $B^d(0, 1)$ so that $\hat{f}$ is supported in a small ball $B^d(x_0, \epsilon)$ placed between two fixed spheres of radius $\sqrt{n}$ and $\sqrt{n} + 1$. Then $f(y) = e^d \phi(\epsilon y)$ and

$$\frac{\|f\|_r}{\|f\|_p} \sim \frac{\epsilon \frac{d}{p}}{\epsilon \frac{d}{p'}} \to \infty$$

as $\epsilon \to 0$ since $r < p$. Note that we didn't put any restriction on $p$ here.

Now we will show that Theorem 1 does not hold if $p > 2$. More precisely, let $p > 2$ and $r \neq p$ then the following inequality is not true for functions with vanishing periodizations:

$$\|f\|_r \lesssim \|f\|_p.$$ 

We just considered the case when $r < p$ therefore we need to consider only the case $r > p$. The argument is almost the same as in Lemma 4. We can construct a sequence of Schwartz functions $f_n$ with Fourier transforms vanishing on all spheres of radius $\sqrt{m}$ and such that $\|f_n\|_r \gtrsim (\sqrt{n})^{\frac{1}{p}}$ and $\|f_n\|_p \leq \|f_n\|_p \lesssim (\sqrt{n})^{\frac{1}{p'}}$. Therefore

$$\frac{\|f_n\|_r}{\|f_n\|_p} \gtrsim (\sqrt{n})^{\frac{1}{p} - \frac{1}{q}} \to \infty.$$
Remark 5 Since Theorem 1 trivially holds for $p = 2$ it is natural to expect that it should hold for $1 \leq p \leq 2$. It is unknown whether the Theorem 1 holds for $\frac{2d}{d+2} \leq p < 2$.

Another interesting question is whether the following is true:

$$\| \hat{f} \|_p \lesssim \| f \|_p$$

for some range of $p < 2$ if $f$ has vanishing periodizations. It would then follow that

$$\| \hat{f} \|_r \lesssim \| f \|_p$$

for $p \leq r \leq p'$. All we know from Theorem 1 is that (49) holds when $2 \leq r \leq p'$, $1 \leq p < \frac{2d}{d+2}$ and $d \geq 3$ since $\| f \|_2 \lesssim \| f \|_p$.

Our final open question is whether the following inequalities are true for functions with not necessarily vanishing periodizations $g_\rho$:

$$\| f \|_{p'} \lesssim \| f \|_p + \| g \|_{p'}$$

and

$$\| g \|_{p'} \lesssim \| f \|_p + \| f \|_{p'}$$

for some range of $p \leq \frac{2d}{d+1}$ where

$$\| g \|_{p'} = \left( \int_{\rho \in SO(d)} \| g_\rho \|_{p'}^p \, d\rho \right)^{\frac{1}{p}}.$$

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