Complex hyperbolic representations of $SL_2(\mathbb{R})$

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Abstract

We construct a two-parameter family of irreducible representations of $SL_2(\mathbb{R})$ on the infinite-dimensional complex hyperbolic space. To this end, we introduce the notion of horospherical combination of two representations. Our family then appears as horospherical combinations of two known one-parameter families.

Introduction

0.1 Context

Representations of one semi-simple Lie group into another have no mysteries: by the Karpelevich–Mostow theorem (see [15] or for a proof in the hyperbolic case see [2]), they are all “standard” in the sense that they correspond to totally geodesic or trivial embedding of the corresponding symmetric spaces of the simple factors.

The situation changes dramatically when allowing infinite-dimensional hyperbolic spaces. The irreducible representations of $\text{Isom}(H^1_\mathbb{R})$ into $\text{Isom}(H^\infty_\mathbb{R})$ have been classified by Monod & Py in [13] and it turns out that they form an exotic one-parameter deformation family. The same holds for self-representations of

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Isom(ℋ₀ₓ) (see Monod & Py [14]). Over C, an analogous exotic family has been constructed by Monod in [12] although a complete classification is not yet known.

The present paper focuses on the special case where real and complex hyperbolic spaces meet, namely the Lie group $PSL_2(\mathbb{R})$. Indeed, this group can be viewed as the connected component of either Isom(ℋ₀ₓ) or Isom(ℋ₁ₓ). This might seem innocuous since the spaces $ℋ₀ₓ$ and $ℋ₁ₓ$ are isometric (after rescaling in our conventions). However, the real and complex viewpoint do give us two completely different families of irreducible representations on the infinite-dimensional complex hyperbolic space $ℋ₂ₓ$:

On the one hand from the representations $Isom(ℋ₀ₓ) \to Isom(ℋ₂ₓ)$ studied in [8] and [13], we obtain a family of representations into $Isom(ℋ₂ₓ)$ by complexification. Although they are irreducible (in the complex sense), they preserve a totally real subspace and hence have vanishing Cartan invariant.

On the other hand, the representations $Isom(ℋ₁ₓ) \to Isom(ℋ₂ₓ)$ constructed in [12] have a non-trivial Cartan invariant, which actually parametrizes that family.

The main result of this paper is an interpolation between these two families of representations of the group $PSL_2(\mathbb{R}) \cong Isom(ℋ₀ₓ)_o \cong Isom(ℋ₁ₓ)$. We thus obtain a two-parameter family of representations of $PSL_2(\mathbb{R})$ into $Isom(ℋ₂ₓ)$ which, to the author’s knowledge, have not been described before.

0.2 Statements

Given an irreducible representation $ρ$ of $Isom(ℋ₁ₓ)$ (with $1 \leq n \leq \infty$) into $Isom(ℋ₂ₓ)$, in [12] Monod has introduced two invariants (with a different notation than here). The first invariant, $ℓ(ρ)$ is a length invariant already considered for the real case in [13]. The definition of $ℓ(ρ)$ hinges on the fact that for any hyperbolic isometry $g \in Isom(ℋ₁ₓ)$ with displacement length $ℓ(γ)$, the isometry $ρ(g)$ will also be hyperbolic and its displacement length $ℓ(ρ(g))$ is proved to be asymptotically proportional to $ℓ(γ)$ as $ℓ(γ) \to \infty$, with a proportionality factor not depending on $g$. This factor is by definition $ℓ(ρ)$.

Whilst this first invariant makes sense over the reals as well, the second invariant considered by Monod in [12] is specific to the complex case. Recall that the second cohomology of $Isom(ℋ₁ₓ)$ is generated by the Kähler class, which can be geometrically realized by the Cartan invariant of triples of points in $ℋ₁ₓ$. Since this class generates the cohomology, the pull-back through $ρ$ of the Cartan invariant in $ℋ₁ₓ$ must be some multiple of the Cartan invariant in $ℋ₂ₓ$. For $n = 1$, this multiple times $π/2$ is by definition the second invariant, the Cartan argument of $ρ$ denoted $Arg(ρ)$.

In this language, the first family of representations $PSL_2(\mathbb{R}) \to Isom(ℋ₂ₓ)$ from [13] is parametrized by $0 < ℓ(ρ) < 2$ and satisfies $Arg(ρ) = 0$. The second family, from [12], is parametrized by $0 < ℓ(ρ) < 1$ but satisfies $Arg(ρ) = π/2ℓ(ρ)$. We caution the reader that there is a scaling convention to be taken into account since $ℓ$ depends on the metric chosen on $ℋ₀ₓ$ respectively $ℋ₁ₓ$.

We now have the necessary language to describe our two-parameter family:

**Theorem 0.2.1.** For every $0 < t < 1$ and $0 \leq r \leq \frac{πt}{2}$, there exists an irreducible representation $ρᵣₜ : PSL_2(\mathbb{R}) \to Isom(ℋ₂ₓ)$ such that $Arg(ρᵣₜ) = r$ and $ℓ(ρᵣₜ) = t$.

Moreover, this representation is unique up to conjugation.
Our construction is a form of interpolation between the families constructed in [12] and [13]. In fact, we introduce an operation that we call the horospherical combination of two given representations $\rho_1, \rho_2$ that have the same length invariant $\ell(\rho_1) = \ell(\rho_2)$. We denote the resulting representation by $\rho_u \wedge \rho_2$, with $0 \leq u \leq 1$.

As the name is intended to suggest, this operation resembles a convex combination, but only of the “horospherical part” of the representations. For this to make sense, we need to assume that the length invariants coincide. As for the Cartan argument of $\rho_u \wedge \rho_2$, it satisfies

$$\operatorname{Arg}(\rho \wedge \tau) = (1 - u)\operatorname{Arg}(\rho) + u\operatorname{Arg}(\tau).$$

1 Preliminaries

The results described in this section are well known, they are presented here in order to make explicit all the conventions and definitions that will be used. In the subsections 1.1 and 1.2, following [5] and [7], the different hyperbolic spaces are defined and some results about their groups of isometries that will be useful are presented.

In section 1.3, following the ideas of [13], general properties of some complex hyperbolic representations of $\text{SL}_2(\mathbb{R})$ are studied and the notation used in the rest of the article is defined.

1.1 The hyperbolic spaces and their isometries

Following Burger, Iozzi & Monod [3], let $H$ be a separable Hilbert space over $F = \mathbb{R}, \mathbb{C}$, with $\dim F(H) \in \mathbb{N}_{\geq 2} \cup \{\infty\}$, endowed with a non-degenerate form $Q$, linear in the first argument and antilinear in the second.

Define

$$\iota(Q) = \sup\{\dim_F(W) \mid W \leq H \text{ and } Q|_{W \times W} = 0\},$$

and

$$\iota_+ Q = \sup\{\dim_F(W) \mid W \leq H \text{ and } Q|_{W \times W} \text{ is positive definite}\}.$$

Suppose that $\iota(Q) = \iota_+(Q) = 1$. A form with these properties will be called a form of signature $(1, n)$, where $n = \dim_F(H) - 1$.

For every $v \in H$, denote $[v] = Fv$. If $\dim_F(H) = m - 1$, where $m \geq 3$ if $F = \mathbb{R}$ and $m \geq 2$ if $F = \mathbb{C}$, define

$$H^m_F = \{[v] \mid B(v, v) > 0\}.$$

The space $H^m_F$ is equipped with a metric given by the formula

$$\cosh(d([v], [w])) = \frac{|B(v, w)|}{B(v, v)^{1/2}B(w, w)^{1/2}}.$$

A $\pm$-orthogonal decomposition of $H$ is a $B$-orthogonal decomposition

$$H = W_+ \oplus W_-,$$

with $B|_{W_+ \times W_+}$ positive/negative definite. Given a $\pm$-orthogonal decomposition of $H$, define the sesquilinear form $B_\pm$ as $B_\pm|_{W_+ \times W_+} = B$, $B_\pm|_{W_- \times W_-} = -B$ and
$B(W_+, W_-) = 0$. A form of signature $(1, \infty)$ on $H$ is called **strongly non-degenerate** if for every (any) $\pm$-orthogonal decomposition, the space $(H, B_{\pm})$ is a Hilbert space (see Lemma 2.4 of [5]).

The metric space $(H^m_F, d)$ is complete if, and only if, $B$ is strongly non-degenerate (see Proposition 3.3 in [5]). From now on the space $H^m_F$ will be always considered associated to a strongly non-degenerate sesquilinear form and it will be called the **$m$-dimensional $F$-hyperbolic space** (see Proposition 3.7 of [5]). For further reading on these spaces see [5], [7] for the infinite-dimensional case, and [10] for the finite-dimensional complex one. From now on $H$ will denote a separable Hilbert space over $F$ provided with $B$, a strongly non-degenerate sesquilinear form of signature $(1, m)$.

If $F = \mathbb{C}$, let $K = \mathbb{R}, \mathbb{C}$ and if $F = \mathbb{R}$ define $K = \mathbb{R}$. Denote $\pi$ the projectivization map of $H \to \mathbf{P}(H)$. A $K$-**hyperbolic subspace** of $H^m_F$ is the image under $\pi$ of a closed $K$-vector subspace $L$ of $H$ such that $B|_{L \times L}$ is non-degenerate of signature $(1, m')$. The restriction of $B$ to $L$ is strongly non-degenerate (see Proposition 2.8 of [5]), therefore $\pi(L)$ is a (complete) hyperbolic space. In the complex finite-dimensional case, this is a characterization for totally geodesic subspaces (see 3.1.11 of [10]).

For every finite set of points $X$ of $H^\infty_F$ there is $W \subset H$, a finite-dimensional space over $F$, that contains representatives of each of the elements of $X$. The restriction of $B$ to $W$ is a non-degenerate form of signature $(1, n)$, therefore $\pi(W)$ is isometric to a finite-dimensional $F$-hyperbolic space. This shows that many statements about finite sets of points in $H^\infty_F$ can be reduced to a finite-dimensional question. For example, the space $H^\infty_F$ is a geodesically complete CAT(-1) space because this is true for every finite dimensional $H^m_F$ (see Proposition II.10.10 of [3]).

Every geodesic ray in $H^\infty_F$ lies inside a finite dimensional $F$-hyperbolic space. It is not surprising that $\partial H^\infty_F$, the visual boundary of $H^\infty_F$, is in a natural bijection with the set of isotropic vectors of $H$, because this is true at a finite-dimensional level (see Proposition 3.5.3 in [7]).

Observe that the space

$$\{ [v] \in \mathbf{P}(H) \mid B(v, v) \geq 0 \}$$

can be provided with the subspace topology of the projective space (with the quotient topology) associated to $H$. The hyperbolic spaces are Gromov hyperbolic, therefore $H^m_F \cup \partial H^m_F$ has a natural topology. In this case both topologies are the same and coincide in $H^m_F$ with the metric topology (see Proposition 3.5.3 of [7]).

Given a CAT(0) space $X$ and $\sigma$ a geodesic ray representing $\xi \in \partial X$. For every $y \in X$, the limit

$$b_{\xi, \sigma(0)}(y) = \lim_{t \to \infty} d(y, \sigma(t)) - t$$

eexists and defines a continuous and convex function (see Lemma II.8.18 of [3]),

$$X \xrightarrow{b_{\xi, \sigma(0)}} \mathbb{R}$$

called the **Busemann function associated to $\xi$ and normalized in $\sigma(0)$**.

If $\sigma$ and $\tau$ are two asymptotic geodesic rays, there exists a constant $C$ such that $b_{\xi, \sigma(0)} - b_{\xi, \tau(0)} = C$ (see Corollary II.8.20 of [3]). Thus, for every $\xi \in \partial X$, all
the Busemann functions associated to $\xi$ have the same level (resp. sublevel) sets. This subsets are called horospheres (resp. horoballs) centered at $\xi$.

For $H^m_F$ there is an explicit description of the Busemann functions. If $x \in H^m_F$ it can be shown at a finite-dimensional level that every geodesic ray $\sigma$ issuing from $x$ admits a lift to $H$ of the shape $t \mapsto \cosh(t)\tilde{x} + \sinh(t)u$, where $\tilde{x}$ is a lift of $x$, $B(\tilde{x}, \tilde{x}) = 1 = -B(u, u)$ and $B(\tilde{x}, u) = 0$. If $y \in H^m_F$ and $\tilde{y}$ is a lift of $y$ such that $B(\tilde{y}, \tilde{y}) = 1$, then

$$b_{\xi,\sigma(0)}(y) = \lim_{t \to \infty} d(y, \sigma(t)) - t = \lim_{t \to \infty} \arccosh(|B(\tilde{y}, \cosh(t)\tilde{x} + \sinh(t)u)|) - t = \ln(|B(\tilde{y}, \tilde{x} + u)|).$$

Observe that $\xi$ is represented by the isotropic vector $\tilde{x} + u$ (see Proposition 3.5.3 of [7]). Let $g$ be an isometry of a metric space $X$. Let $d_g : X \to \mathbb{R}$ be the function given by $d_g(x) = d(gx, x)$. The displacement of $g$ is defined as

$$\ell(g) = \inf_{x \in X} \{d_g(x)\}.$$  

The following proposition is well known, here just a sketch of the proof is presented.

**Proposition 1.1.1.** For every isometry $g$ of a complete $\text{CAT}(-1)$ space $X$, there is a trichotomy:

1. The map $g$ is of elliptic type: $g$ fixes a point in $X$.

2. The map $g$ is of hyperbolic type: $g$ preserves a geodesic and it does not fix any point in $X$.

3. The map $g$ is of parabolic type: $g$ fixes a unique point $\xi \in \partial X$, it does not fix any point in $X$, it leaves invariant all the horospheres centered at $\xi$ and $\ell(g) = 0$.

**Proof.** Suppose $\ell(g)$ is achieved. If $\ell(g) = 0$, then 1. holds. If $\ell(g) > 0$, then $g$ preserves a geodesic (see Theorem II.6.8 of [3]). The geodesic is unique and there are no fixed points by $g$ in $X$ because the projection onto a non-empty convex and closed set is a strict contraction (see II.2.12 of [3]). Thus 2. holds.

Suppose $\ell(g)$ is not achieved. If $g$ fixes two points in $\partial X$, then it preserves the geodesic connecting them (see Proposition 4.4.4 of [4]). Therefore $\ell(g)$ is achieved in this geodesic, but this is a contradiction. Thus $g$ fixes at most one point at infinity.

For every $n \in \mathbb{N}$ define,

$$H_n = \{x \in X \mid d_g(x) \leq \ell(g) + \frac{1}{n}\}.$$

The sets $H_n$ are non-empty, closed and convex. The family $\{H_n\}_{n \in \mathbb{N}}$ is such that

$$I = \bigcap_{n \in \mathbb{N}} H_n = \emptyset.$$

This implies that for every $y \in X$, $d(y, H_n) \to \infty$ as $n \to \infty$. Therefore

$$J = \bigcap_{n \in \mathbb{N}} \partial H_n \neq \emptyset.$$
In fact, \( PO \) and \( RH \). Observe that the diagonal matrix acts trivially on \((\text{see Theorem 2.2.3 of [7]}\)), thus if \( J \) contains more than one point, then the geodesic connecting any two elements of the intersection is contained in \( I \), which is a contradiction. The sets \( H_n \) are \( g \)-invariant, thus the only element of \( J \) is fixed by \( g \).

Consider a geodesic segment \( \sigma \) that represents \( \xi \). There exists \( c(g) \in R \) such that for every \( x \in X \),

\[
b_{\xi,\sigma(0)}(gx) - b_{\xi,\sigma(0)}(x) = b_{\xi,g^{-1}\sigma(0)}(x) - b_{\xi,\sigma(0)}(x) = c(g).
\]

By the triangle inequality \( |c(g)| \leq \ell(g) \), but \( \ell(g) = 0 \) (see Proposition 3.1 of [16]), therefore \( b_{\xi,\sigma(0)} \) is \( g \)-invariant. This completes the proof for 3.

The following proposition can be deduced from the arguments in [3].

**Proposition 1.1.2.** If \( x \in H_F^m \), \( y_1,y_2 \in \partial H_F^m \) and if \( s \in O_F(B) \) is such that \( sy_1 = y_2 \) and \( sy_2 = y_1 \), then the following hold:

1. The action of \( O_F(B) \) on \( H_F^m \) is transitive.
2. The action of \( O_F(B) \) is transitive on metric spheres centered at \( x \).
3. The action of \( O_F(B) \) is double transitive on \( \partial H_F^m \).
4. If \( m < \infty \) and \( F = R \), 1., 2., and 3. hold for \( SO(1,m) \).
5. \( O_F(B) = O_F(B)_{y_1} \sqcup (O_F(B)_{y_1} \cdot s \cdot O_F(B)_{y_1}) \).

The group \( O_F(B) \) is denoted by \( U(1,m) \) (resp. \( O(1,m) \)) if \( F = C \) (resp. \( F = R \)). For every \( G \leq O_F(B) \), denote \( PG \) the natural image under projectivization. In fact, \( PO(1,m) = \text{Isom}(H_R^m) \) and \( PU(1,m) \) is an index 2 subgroup of \( \text{Isom}(H_R^m) \) (see Theorem 2.2.3 of [17]). Moreover

\[
\text{Isom}(H_C^m)_{o} = PU(1,m)
\]

and

\[
\text{Isom}(H_R^m)_{o} = PSO(1,m).
\]

Observe that the diagonal matrix act trivially on \( H_F^m \), therefore if \( m < \infty \), then \( PSU(1,m) = PU(1,m) \).

The topology of these groups will be the quotient topology of the projectivization map. This topologies coincide, for \( m < \infty \), with the topology of uniform convergence on compact sets.

Suppose \( \xi \in \partial H_F^m \) and \( G < \text{Isom}(H_F^m)_{\xi} \). Let \( b_{\xi,\sigma(0)} \) be a Busemann function centered at \( \xi \) and normalized in \( \sigma(0) \), for some geodesic ray \( \sigma \). The geodesic ray \( \sigma \) admits a lift

\[
\tilde{\sigma}(t) = cosh(t)\tilde{x} + sinh(t)u,
\]

with \( u, \tilde{x} \in H \) such that \( B(\tilde{x}, \tilde{x}) = 1 = -B(u, u) \) and \( B(u, \tilde{x}) = 0 \). For every \( g \in G \), there exists \( c(g) \in R \) such that for every \( y \in H_F^m \),

\[
b_{\xi,\sigma(0)}(y) = b_{\xi,\sigma(0)}(gy) + c(g).
\]
The map \( c : G \to \mathbb{R} \), called the Busemann character associated to \( \xi \), is a continuous homomorphism and does not depend on the choice of \( \sigma \).

Observe that if \( \tilde{y} \) is a normalized lift of \( y \), then

\[
c(g) = \ln \left( \frac{|B(\tilde{y}, \tilde{y} + u)|}{|B(\tilde{y}, \tilde{y} + u)|} \right),
\]

where \( \tilde{y} \) is any linear representative of the isometry \( g \). Thus, if

\[
\tilde{g}(\tilde{x} + u) = \theta(\tilde{g})(\tilde{x} + u),
\]

with \( \theta(\tilde{g}) \in \mathbb{C} \setminus \{0\} \), then \( c(g) = \ln(|\theta(\tilde{g})|) \). Therefore the map \( g \mapsto |\theta(\tilde{g})| \in \mathbb{R}_{>0} \) is a continuous homomorphism.

**Proposition 1.1.3.** If \( G < POP(B)_{\xi} \) and \( c : G \to \mathbb{R} \) is the Busemann character associated to \( \xi \), then

1. \( \ker(c) = \{ T \in G \mid T \text{ is elliptic or parabolic} \} \).
2. For every \( T \in G, \ell(T) = |c(g)| \).

**Proof.** 1. Suppose \( \xi \) is represented by the isotropic element \( y_1 \). Let \( T \in G \) and let \( \tilde{T} \) be a linear representative of \( G \). If \( T \) is hyperbolic, \( \tilde{T} \) leaves invariant two isotropic lines with respective representatives \( y_1 \) and \( y_2 \). Suppose that \( B(y_1, y_2) = 1 \). Thus, if \( \tilde{T}(y_1) = \theta_1 y_1, \) then \( \theta_1 \theta_2 = 1 \).

The point \( x \) represented by \( \frac{1}{\sqrt{2}}(y_1 + y_2) \) belongs to the geodesic connecting \( y_1 \) and \( y_2 \) because \( 2d(T(x), x) = d(T^2(x), x) \). Observe that \( d(T(x), x) = |\ln(|\theta_1|)| \). This implies that \( |\theta_1| \neq 1 \), and as it was noticed before, \( c(g) = \ln(|\theta_1|) \). Therefore \( T \notin \ker(c) \).

If \( T \) is parabolic, by Proposition 1.1.1 \( c(T) = 0 \). If \( T \) is elliptic then \( T \) fixes pointwise every geodesic ray representing \( \xi \) that starts on a \( T \)-fixed point in \( H^\mathbb{R} \).

Therefore \( T \) fixes pointwise the entire geodesic containing any of these geodesic rays. Using the arguments for the hyperbolic case it is possible to conclude that \( c(g) = 0 \).

The point 2. follows from the arguments of 1 and Proposition 1.1.1. \( \square \)

Let \( \{e_1, e_2\} \) be the canonical base of \( \mathbb{C}^2 \) and define the basis \( \{\xi_1, \xi_2\} \), where \( \xi_1 = \frac{1}{\sqrt{2}}(e_1 + e_2) \) and \( \xi_2 = \frac{1}{\sqrt{2}}(e_1 - e_2) \). Observe that \( B(\xi_1, \xi_1) = 0 = B(\xi_2, \xi_2) \) and \( B(\xi_1, \xi_2) = 1 \).

In the basis \( \{\xi_1, \xi_2\} \) every element of \( U(1,1)_{\xi_1} \) has the form

\[
\begin{pmatrix}
\lambda & z \\
0 & \gamma
\end{pmatrix}
\]

with \( \lambda \bar{\gamma} = 1 \) and \( Re(\gamma \bar{z}) = 0 \). Thus for every \( T \in \text{Isom}_B(H^\mathbb{R} | \xi_1) \) there exists

\[
g(\lambda, b) = \begin{pmatrix}
\lambda & ib \\
0 & \lambda^{-1}
\end{pmatrix} \in SU(1,1)
\]

with \( \lambda > 0 \) and \( b \in \mathbb{R} \) such that \( \pi(g(\lambda, b)) = T \), where \( \pi \) is the projectivization map. Define the the group

\[
P = \{ g(\lambda, b) \}_{\lambda > 0, b \in \mathbb{R}} \leq SU(1,1).\]
Observe that for \( g(\lambda, b), g(\gamma, b) \in P \),

\[
g(\lambda, b) \cdot g(\gamma, d) = g(\lambda \gamma, \gamma^{-1} b + \lambda d)
\]

and that \( \pi_P : P \to \text{Isom}_o(H^1_C) \) is an isomorphism. A transformation \( g(\lambda, b) \) is parabolic if, and only if, \( \lambda = 1 \) and \( b \neq 0 \). All the nontrivial maps \( g(\lambda, 0) \) are hyperbolic.

If \( s \in SU(1, 1) \) is defined by \( s(\xi_1) = i\xi_2 \) and \( s(\xi_2) = i\xi_1 \), then by Proposition 1.1.2

\[
\text{Isom}(H^1_C) \circ \pi(P) = \pi(P) \sqcup \pi(P) \pi(s) \pi(P).
\]

Every element of \( SU(1, 1) \) has the form

\[
M(\alpha, \beta) = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix},
\]

where \( |\alpha|^2 - |\beta|^2 = 1 \). The map \( SU(1, 1) \xrightarrow{\psi} SL_2(\mathbb{R}) \) given by

\[
\psi(M(\alpha, \beta)) = \begin{pmatrix} Re(\alpha) + Im(\beta) & Re(\beta) + Im(\alpha) \\ Re(\beta) - Im(\alpha) & Re(\alpha) - Im(\beta) \end{pmatrix}
\]

is an isomorphism. Let \( T \in SL_2(\mathbb{R}) \) be

\[
\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}
\]

and define the map \( SU(1, 1) \xrightarrow{\Psi} SL_2(R) \) as \( \Psi(A) = T^{-1}\psi(A)T \). The map \( \Phi \) is such that

\[
\psi(g(\lambda, b)) = \begin{pmatrix} \lambda & b \\ 0 & \lambda^{-1} \end{pmatrix}
\]

and

\[
\psi(s) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

The group \( SU(1, 1) \) admits a simple description in terms of generators and the relations between them. The following theorem is a well known fact and it will play a crucial role in the proof of the main theorem of this article. A proof of it can be found in p. 209 of \[11\].

**Theorem 1.1.4.** Let \( F \) be the free group generated by the family \( \{u(r)\}_{r \in \mathbb{R} \setminus 0} \) and an element \( w \). For \( r \neq 0 \), denote

\[
s(r) = wu(r^{-1})wu(r)wu(r^{-1}).
\]

Consider the relations

1. \( u \) is an additive homomorphism.
2. \( s \) is a multiplicative homomorphism.
3. \( w^2 = s(-1) \)
4. \( s(a)u(b)s(a^{-1}) = u(ba^2), \) for every \( a, b \neq 0 \).
If $G$ is the quotient of $F$ under these relations then $G$ is isomorphic to $SU(1,1)$.

Let $SU(1,1) \xrightarrow{\phi} GL_3(\mathbb{R})$ be the map defined by

$$
\phi(M(\alpha, \beta)) = \begin{pmatrix}
-\frac{1}{2}(\beta^2 + \beta^2 - \alpha^2 - \alpha^2) & \frac{1}{2}(\alpha^2 - \alpha^2 - \alpha^2 + \alpha^2) & i(\alpha \beta - \alpha \beta) \\
-\frac{1}{2}(\beta^2 + \beta^2 - \alpha^2 + \alpha^2) & \frac{1}{2}(\beta^2 + \beta^2 + \alpha^2) & \frac{i}{\alpha \beta + \alpha \beta} \\
-\sqrt{2} & 2 & 0
\end{pmatrix}
$$

and let

$$
T = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & 0 & \sqrt{2} \\
1 & -1 & 0 \\
1 & 1 & 0
\end{pmatrix}.
$$

Define the map $SU(1,1) \xrightarrow{\Phi} SO(1,2)$ given by

$$
\Phi(M(\alpha, \beta)) = T^{-1}\phi(M(\alpha, \beta))T.
$$

The map $\Phi$ is a homomorphism and $\ker(\Phi) = \{Id, -Id\}$. With an appropriate choice of a basis $\{\xi'_1, \xi'_2, u\}$ of $\mathbb{R}^3$, where

$$
B(\xi'_i, \xi'_i) = 0 = B(\xi_i, u)
$$

and

$$
B(\xi'_1, \xi'_2) = 1 = -B(u, u),
$$

the map $\Phi$ is such that,

$$
\Phi(s) = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & -1
\end{pmatrix},
$$

$$
\Phi(g(1, b)) = \begin{pmatrix}
1 & b^2 & -\sqrt{b} \\
0 & 1 & 0 \\
0 & -\sqrt{b} & 1
\end{pmatrix}
$$

and

$$
\Phi(g(\lambda, 0)) = \begin{pmatrix}
\lambda^2 & 0 & 0 \\
0 & \lambda^{-2} & 0 \\
0 & 0 & 1
\end{pmatrix}.
$$

Every elliptic transformation is contained in a compact subgroup, therefore its image under $\Phi$ is elliptic too (see Proposition II.2.7 of [3]). Thus, by Proposition 1.1.2 and the previous argument, $\Phi$ preserves the type.

Consider the following commutative diagram,

$$
\begin{array}{ccc}
SU(1,1) & \xrightarrow{\Phi} & SO(1,2) \\
\downarrow & & \downarrow \\
\text{Isom}(H^1_C)_o & \xrightarrow{\overline{\Phi}} & \text{Isom}(H^2_R)_o,
\end{array}
$$

where the vertical arrows are the projectivization maps and $\overline{\Phi}$ is the induced isomorphism.
Lemma 1.1.5. For every \( g \in \text{Isom}_o(H^1_C) \),
\[
2\ell(g) = \ell(\Phi(g)).
\]

Proof. The map \( \Phi \) preserves the type (elliptic, parabolic and hyperbolic), therefore it is enough to prove the claim for hyperbolic elements. Up to conjugation, every hyperbolic element in \( \text{Isom}(H^1_C)_o \) has a representative \( g(\lambda, 0) \) for some \( \lambda > 0 \). For these elements the claim follows from the arguments in the proof of Proposition 1.1.3. \( \square \)

1.2 Isometric representations

In this subsection the generalities of representations into groups of isometries of hyperbolic spaces and the tools used for the statements of the last section are discussed. Most of the results and definitions of this subsection are in [12]. Here they will be presented in particular for the case that this article deals with.

Given a topological group \( G \), a homomorphism \( \rho : G \rightarrow \text{Isom}(H^\infty_F) \) is called a representation. If for every \( x \in H^\infty_F \), the map \( g \mapsto \rho(g)x \) is continuous, then the representation is called continuous. In this work every representation will be considered continuous.

The representation \( \rho \) is called non-elementary if it does not fix any point in \( H^\infty_F \cup \partial H^\infty_F \) and it does not permute two points in \( \partial H^\infty_F \). The representation \( \rho \) is called irreducible if it is non-elementary and \( H^\infty_F \) does not admit a proper \( G \)-invariant \( F \)-hyperbolic subspace.

The next proposition shows that for non-elementary representations the concept of irreducibility behaves in a very different way than in the linear setting. The proposition was proved for the real case in Proposition 4.3 of [5]. The proof for the complex case works exactly in the same way.

Proposition 1.2.1. If \( G \rho \rightarrow \text{Isom}(H^\infty_F) \) is non-elementary, there exists a unique \( G \)-invariant \( F \)-hyperbolic subspace \( L \) such that for every \( G \)-invariant \( F \)-hyperbolic space \( N \), \( L \subset N \).

The space \( L \) will be called the irreducible part of \( \rho \). The following result is similar to Proposition 5.1 of [12] or to Proposition 2.1 of [13]. The proof for this case can be mimicked from those of any of the aforementioned propositions.

Proposition 1.2.2. A non-elementary representation \( \text{Isom}(H^1_C)_o \xrightarrow{\rho} \text{Isom}(H^\infty_F) \) preserves the type (elliptic, parabolic and hyperbolic) and \( \rho(P) \) fixes a unique point \( \xi \in \partial H^\infty_F \).

In [12] the author developed a Gelfand-Naimark-Segal type of construction for actions by isometries on complex hyperbolic spaces. This construction is the key idea behind the main result of this paper.

Given \( x, y, z \in H^\infty_C \), the Cartan argument of \( (x, y, z) \) is defined as
\[
\text{Cart}(x, y, z) = \text{Arg}(B(\tilde{x}, \tilde{y}), B(\tilde{y}, \tilde{z}), B(\tilde{z}, \tilde{x})),
\]
where \( \tilde{x}, \tilde{y} \) and \( \tilde{z} \) are any lifts of \( x, y \) and \( z \). This definition can be extended for triples of distinct points in \( \partial H^\infty_C \).
The map
\[ H^m_C \times H^m_C \times H^m_C \xrightarrow{\text{Cart}} \mathbb{R} \]
is an alternating 2-cocycle and its image is contained in \( (-\frac{\pi}{2}, \frac{\pi}{2}) \). In the complex hyperbolic space this invariant for three points plays a very important role. For further reading see [4], [10] and [12].

A set \( X \subset H^m_C \) is contained in a real hyperbolic subspace if, and only if, for every \( x, y, z \in X \), \( \text{Cart}(x, y, z) = 0 \) (see Lemma 2.1 in [4]). A set \( X \subset H^m_F \) is called total if there is not a proper and closed \( F \)-vector space that contains the lifts of \( X \).

Following [12], a pair \((\alpha, \beta)\) is called a \textit{G-invariant kernel of hyperbolic type} defined on a topological group \( G \), if
\[
\alpha : G^3 \rightarrow \left( -\frac{\pi}{2}, \frac{\pi}{2} \right)
\]
is a continuous \( G \)-invariant (with respect to the diagonal action) alternating 2-cocycle,
\[
\beta : G \rightarrow \mathbb{R}_{>0}
\]
is a continuous function, symmetric with respect to the inversion of the group, such that \( \beta(e) = 1 \) and such that the map
\[
(g, k) \mapsto \beta(g)\beta(k) - e^{-i\alpha(g, k, e)}\beta(gk^{-1})
\]
is a kernel of positive type. See chapter II C of [1] for the definition and some properties of the kernels of positive type.

The next result is Theorem 1.11 of [12].

**Theorem 1.2.3.** The pair \((\alpha, \beta)\) is a \textit{G-invariant kernel of hyperbolic type} if, and only if, there exist, up to a conjugation by an isometry of \( H^m_C \), a unique representation \( G \xrightarrow{\rho} \text{Isom}(H^m_C)_o \) and \( p \in H^m_C \) such that the orbit of \( p \) is total and
\[
\beta(g) = \cosh(d(\rho(g)p, p))
\]
and
\[
\alpha(g_1, g_2, g_3) = \text{Cart}(\rho(g_1)p, \rho(g_2)p, \rho(g_3)p)
\]
Moreover \( \beta \) and \( \alpha \) are continuous if, and only if, \( \rho \) is orbitally continuous.

Recall from the proof of the previous theorem that for every \( g \in G \), \( \rho(g) \) is the isometry induced by a linear map \( T_g \) preserving a sesquilinear form of signature \((1, \infty)\) defined on a Hilbert space. These linear maps are such that for every \( g, l \in G \),
\[
T_gT_l = e^{i\alpha(gl, g, e)}T_{gl}.
\]

**Proposition 1.2.4.** Let \( G \xrightarrow{\rho} \text{Isom}(H^m_C)_o \) be a representation and suppose \( x \in H^m_C \) is a point with total orbit. If there exists \( \omega \), an alternating \( G \)-invariant 1-cochain, such that \( d\omega = \alpha \), where \( \alpha \) is the 2-cocycle associated to \( x \), then \( \rho \) admits a lift to a representation \( G \xrightarrow{\hat{\rho}} U(1, m) \).
Proof. Let $T_g$ the map defined in the proof of Theorem 1.2.3 Define $T'_g = e^{-i\omega(g,e)}T_g$. Observe that on one side

\[ T'_gT'_h = e^{-i\omega(g,e)}e^{-i\omega(h,e)}T'_gT'_h = e^{-i\omega(g,e)}e^{-i\omega(h,e)}\alpha(g,h,e)T'_gT'_h, \]

and that on the other side,

\[ \alpha(g,h,e) = \omega(g,e) - \omega(g,h,e) + \omega(h,e). \]

Therefore the map $g \mapsto T'_g$ is a homomorphism. \[ \square \]

**Proposition 1.2.5.** Let $G \overset{\rho}{\rightarrow} \text{Isom}(H^m_C)$, be a representation and let $x \in H^m_C$. If $\rho$ fixes a point $y \in \partial H^m_C$, then there exists $\omega$, an alternating $G$-invariant $1$-cochain such that $\partial\omega = \alpha$, where $\alpha$ is the $2$-cocycle associated to $x$.

Proof. The continuity arguments will not be discussed, but it will be clear from the arguments used that they can be deduced. Define $\omega(g,l) = \text{Cart}(y, \rho(g)x, \rho(l)x)$. Choose $\tilde{x}$ a lift of $x$ and $\tilde{y}$ a lift of $y$ such that $B(\tilde{x}, \tilde{y}) > 0$. Let $\tilde{\rho}$ be a linear lift of $\rho$ such that for every $g \in G$, $\tilde{\rho}(g)(\tilde{y}) = \theta_g\tilde{y}$, with $\theta_g > 0$. Thus,

\[ \text{Cart}(y, \rho(l)x, \rho(k)x) = \text{Cart}(\rho(l^{-1})y, x, \rho(l^{-1})\rho(k)x) = \text{Arg}(B(\theta_{l^{-1}}\tilde{y}, \tilde{x})B(\tilde{x}, \rho(l^{-1}k)\tilde{x})B(\rho(l^{-1}k)\tilde{x}, \theta_{l^{-1}}\tilde{y})) = \text{Arg}(B(\tilde{x}, \tilde{\rho}(l^{-1}k)\tilde{x})B(\tilde{x}, \tilde{\rho}(k^{-1}l)\tilde{y})) = \text{Arg}(B(\tilde{\rho}(l)\tilde{x}, \tilde{\rho}(k)\tilde{x})). \]

For every $x_1, x_2, x_3 \in H^m_C$,

\[ |\text{Cart}(x, y, z)| < \pi/2 \]

and for $y \in \partial H^m_C$,

\[ |\text{Cart}(y, x_1, x_2)| \leq \pi/2. \]

Therefore,

\[ \text{Arg}(B(\tilde{\rho}(l)\tilde{x}, \tilde{\rho}(k)\tilde{x})) - \text{Arg}(B(\tilde{\rho}(g)\tilde{x}, \tilde{\rho}(k)\tilde{x})) + \text{Arg}(B(\tilde{\rho}(g)\tilde{x}, \tilde{\rho}(l)\tilde{x})) = \text{Arg}(B(\tilde{\rho}(l)\tilde{x}, \tilde{\rho}(k)\tilde{x})B(\tilde{\rho}(k)\tilde{x}, \tilde{\rho}(g)\tilde{x})B(\tilde{\rho}(g)\tilde{x}, \tilde{\rho}(l)\tilde{x})). \]

In other words,

\[ \partial\omega(g, l, k) = \text{Cart}(y, \rho(l)x, \rho(k)x) - \text{Cart}(y, \rho(g)x, \rho(k)x) + \text{Cart}(y, \rho(g)x, \rho(l)x) = \alpha(g, l, k). \]

\[ \square \]

The following corollary is a consequence of propositions 1.2.4 and 1.2.5.

**Corollary 1.2.6.** Let $G \overset{\rho}{\rightarrow} \text{Isom}(H^m_C)$, be a representation and suppose $x \in H^m_C$ has a total orbit. If $\rho$ fixes a point at infinity, then $\rho$ admits an orbitally continuous lift $\tilde{\rho}$ to $U(1, m)$. 

12
1.3 Non-elementary representations of Isom($H^1_C$)$_o$

The technique of studying non-elementary representations through their restrictions to stabilizers of points at infinity can be tracked back to [5] and [13]. In particular the results of this subsection follow the ideas of the latter.

In this subsection the notation and general properties of non-elementary representations of Isom($H^1_C$)$_o$ into a the group of holomorphic isometries of a infinite-dimensional complex hyperbolic space will be discussed.

Fix $\rho$ an irreducible representation of Isom($H^1_C$)$_o$ into the group of holomorphic isometries of $H^1_C$. With an abuse of notation, it will be supposed often that $\rho$ is defined on SU(1, 1). By Proposition 1.2.2, $\rho(P)$ fixes a unique point $y_1 \in \partial H^1_C$. The subgroup $\{g(\lambda, 0)\}_{\lambda > 0}$ is abelian and again by Proposition 1.2.2 there is a unique $y_2 \in \partial H^1_C$ such that $y_1$ and $y_2$ are the extremes of the axis shared by every element $\rho(g(\lambda, 0))$.

Without lost of generality, fix a representative $\eta_1$ (resp. $\eta_2$) of $y_1$ (resp. $y_2$) such that $B(\eta_1, \eta_2) = 1$. By Corollary 1.2.5 $\rho|_P$ admits a lift into an homomorphism $P \rightarrow U(B)$. With a small abuse of notation, the name $\rho$ will be kept for this lift. Up to a multiplication by a continuous homomorphism $P \rightarrow S^1$, it is possible to suppose that $B(\rho(\lambda, b), \eta_1, \eta_2) > 0$, for every $\lambda > 0$ and $b \in \mathbb{R}$.

The following proposition is inspired by Proposition 2.3 and the arguments in p.8 of [13].

**Proposition 1.3.1.** There exists a continuous isomorphism $\chi : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ such that for every $g = g(\lambda, b) \in P$, $\rho(g) = \rho(\lambda, b)$ can be represented by the transformation

$$
\begin{pmatrix}
\chi(\lambda) & -\chi(\lambda)|c(\lambda, b)|^2 + i\Delta(\lambda, b) & -\chi(\lambda)\langle \pi(\lambda, b)(\cdot), c(\lambda, b) \rangle \\
0 & \chi(\lambda)^{-1} & 0 \\
0 & c(\lambda, b) & \pi(\lambda, d)
\end{pmatrix},
$$

with respect to the decomposition $\mathbb{C}\eta_1 \oplus \mathbb{C}\eta_2 \oplus (\eta_1^+ \cap \eta_2^+)$, where $\langle \cdot, \cdot \rangle$ is the restriction of $B$ to $\eta_1^+ \cap \eta_2^+$, $|c(\lambda, b)|^2 = \langle c(\lambda, b), c(\lambda, b) \rangle$ and where for every $\lambda > 0$ and $b, d \in \mathbb{R}$,

1. $\Delta(\lambda, b) \in \mathbb{R}$ and the map $(\lambda, b) \mapsto \Delta(\lambda, b)$ is continuous.
2. $c(\lambda, b) \in \eta_1^+ \cap \eta_2^+$ and the map $g(\lambda, b) \mapsto c(\lambda, b)$ is continuous.
3. $\pi(\lambda, b)$ is a unitary map of $\eta_1^+ \cap \eta_2^+$ and the map $g(\lambda, b) \mapsto \pi(\lambda, b)$ is a strongly continuous unitary representation.
4. $c(1, b + d) = c(1, b) + \pi(1, b)c(1, d)$.
5. $\Delta(\lambda, 0) = 0$ and $c(\lambda, 0) = 0$.
6. $\chi(\lambda)\pi(\lambda, 0)c(1, b) = c(1, \lambda^2 b)$.
7. $\chi(\lambda)^2\Delta(1, b) = \Delta(1, \lambda^2 b)$.
8. $-\Delta(1, b) = \Delta(1, -b)$.
9. $\text{Im}(\langle c(1, d), c(1, b) \rangle) = \Delta(1, d - b) - \Delta(1, d) + \Delta(1, b)$.
10. \( \text{Re}((c(1,d), c(1,b))) = -|c(1,-b+d)|^2 + |c(1,b)|^2 + |c(1,d)|^2. \)

**Proof.** Let \( c \) be the Busemann character associated to \( \eta_1 \). By Proposition 1.1.3 and the comments before it, for every \( g \in P \), \( \rho(g) \eta_1 = \theta(g) \eta_1 \), where
\[ c(\rho(g)) = \ln(|\theta(g)|). \]

Observe that \( g(\lambda, b) = g(\lambda, 0)g(1, \lambda^{-1}b) \), therefore
\[ c(\rho(\lambda, b)) = c(\rho(\lambda, 0)). \]

Thus, the map \( \lambda \mapsto e^{c(\rho(\lambda, 0))} = \chi(\lambda) \) is a non-trivial continuous isomorphism \( \mathbb{R}_{>0} \to \mathbb{R}_{>0} \) (see propositions 1.1.3 and 1.2.2).

The points 1., 2., 3. and 4. are consequences of: the map \( \rho \) is a homomorphism to \( U(B) \) which is orbitally continuous, \( \rho(P) \) preserves the line generated by \( \eta_1 \) and the comments before Proposition 1.1.2.

For 5. observe that by construction \( \Delta(\lambda, 0) = 0 \) and \( c(\lambda, 0) = 0 \) because \( \rho(\lambda, 0) \) is hyperbolic and \( \eta_1 \) and \( \eta_2 \) are representatives of the extremes of the axis preserved by it.

The points 6. and 7. are consequences of the identity
\[ g(\lambda, 0)g(1, b)g(\lambda^{-1}, 0) = g(1, \lambda^2 b). \]

For 8. observe that form the identity \( g(1, b)g(1, -b) = g(1, 0), \)
\[ \Delta(0) = \Delta(b) + \Delta(-b) - \text{Im}(\langle \pi(1, b)c(1, -b), c(1, b) \rangle) \]
and by 4.,
\[ \text{Im}(\langle \pi(1, b)c(1 - b), c(b) \rangle) = -\text{Im}(\langle c(1, b), c(b) \rangle) = 0. \]

Therefore \( -\Delta(1, b) = \Delta(1, -b). \)

The points 9. and 10. can be deduced from 4. and 8. and the fact that \( \rho \) is a homomorphism. Indeed, observe that
\[ |c(1, -b + d)|^2 = |c(1, b)|^2 + |c(1, d)|^2 + 2\text{Re}(\langle c(1, -b), \pi(1, -b)c(1, d) \rangle) \]
\[ = |c(1, b)|^2 + |c(1, d)|^2 - 2\text{Re}(\langle c(1, b), c(1, d) \rangle) \]
and that
\[ \Delta(1, d - b) = \Delta(1, d) - \Delta(1, b) - \text{Im}(\langle \pi(1, -b)c(1, d), c(1, -b) \rangle) \]
\[ = \Delta(1, d) - \Delta(1, b) + \text{Im}(\langle c(1, d), c(1, b) \rangle). \]

\[ \Box \]

By 4., if \( \pi \) and \( c \) are restricted to \( \{g(1,b)\}_{b \in \mathbb{R}} \), then \( c \) is an affine cocycle associated to the representation \( \pi \). In the rest of the text, for an irreducible representation, the notation of the previous proposition is fixed. The conventions \( \Delta(1, b) = \Delta(b), c(1, b) = c(b) \) and \( \pi(1, b) = \pi(b) \) will be used from now on.

There will be often an abuse of notation: \( g(\lambda, b) \) (resp. \( \rho(\lambda, b) \)) will denote either the isometry of \( \mathbb{H}_C^n \) (resp. \( \mathbb{H}_C^n \)) or a representative in \( SU(1, 1, \mathbb{R}) \) (resp. \( U(1, \mathbb{R}) \)). It will be clear at all times what use of the notation is being made. Also the symbols \( \xi_i \) (resp. \( \eta_i \)) will be used either for the points in \( \partial \mathbb{H}_C^n \) (resp. \( \partial \mathbb{H}_C^n \)) or for their representatives. Again, this will not generate confusion.
Lemma 1.3.2. If $\chi(\lambda) = \lambda^t$, then $0 < t \leq 2$.

Proof. If $b \in \mathbb{R} \setminus \{0\}$, the transformation $g(1, b)$ is parabolic, therefore $c(b) \neq 0$ or $\Delta(b) \neq 0$. The maps $b \mapsto c(b)$ and $b \mapsto \Delta(b)$ are continuous and such that $c(0) = 0$ and $\Delta(0) = 0$. For every $\lambda > 0$, $\lambda^t |c(b)| = |c(\lambda^2t)|$ and $\lambda^{2t} \Delta(b) = \Delta(\lambda^2b)$, thus $t > 0$.

Observe that $c(2b) = c(b) + \pi(b)c(b)$, therefore $2^t |c(b)| \leq 2|c(b)|$. Thus, if $c(b) \neq 0$ for some (every) $b$, then $t \leq 2$. If this is not the case, by Proposition 1.3.1, the map $b \mapsto \Delta(b)$ is a (non-trivial) homomorphism. Therefore

$$2^t \Delta(b) = \Delta(2b) = 2\Delta(b)$$

and $t = 1$.

Define

$$K(b) = - \frac{|c(b)|^2}{2} + i\Delta(b).$$

Observe that

$$K(b) = \frac{B(\eta_2, \rho(1, b)\eta_1)}{|B(\eta_2, \rho(1, b)\eta_1)|} B(\rho(1, b)\eta_2, \eta_2),$$

therefore $\text{Arg}(K(b))$ does not depend on $\eta_1$ and $\eta_2$, the representatives of the extremes of the axis preserved by the maps $\rho(\lambda, 0)$, if the normalization condition $B(\eta_1, \eta_2) = 1$ is imposed.

The following lemma is an immediate consequence of Proposition 1.3.1.

Lemma 1.3.3. Given a non-elementary representation, for every $\lambda > 0$ and every $b \in \mathbb{R}$, the following hold.

1. $K(\lambda b) = \lambda^t K(b)$.
2. $K(-b) = \overline{K(b)}$.
3. $K(b + d) = K(b) + K(d) + \langle c(d), c(-b) \rangle$.

Define the displacement of $\rho$ as $t$ and denote it by $\ell(\rho)$. The following lemma is similar to Theorem B in [13]. The proof there works for this particular case (see propositions 1.1.2 and 1.1.3).

Lemma 1.3.4. For every $g \in \text{Isom}(\mathbb{H}_C^t)$, $\ell(\rho(g)) = \ell(\rho)\ell(g)$.

In [13], among other things, the authors classified the irreducible representations $\text{Isom}(\mathbb{H}_R^t) \xrightarrow{\ell} \text{Isom}(\mathbb{H}_C^t)$. They showed that for every $0 < t < 1$ there exists a unique, up to a conjugation, irreducible representation $\rho_t$ such that for every $g \in \text{Isom}(\mathbb{H}_R^t)$, $\ell(\rho) = t$. For $t = 1$ they showed that there is not an irreducible representation $\rho$ such that $\ell(\rho) = 1$.

Every representation $\text{Isom}(\mathbb{H}_R^t) \xrightarrow{\ell} \text{Isom}(\mathbb{H}_C^t)$ is in fact a linear representation into $O(1, \infty)$. By Corollary 5.2 of [102], $\rho_t$ restricted to $\text{Isom}(\mathbb{H}_R^t)_{\rho}$ remains non-elementary, thus it has an irreducible part. With a small abuse of notation, keep the notation $\rho_t$ for the irreducible representation. There is a natural embedding $O(1, \infty) < U(1, \infty)$ through complexification. In Proposition 5.10 of
Then \(\Delta(\ell)\) be the homomorphism of Lemma 1.1.5 and recall that if \(g\) is irreducible and such that \(\ell\) is not an irreducible representation Isom\((H)\), then \(\ell(\Phi(g)) = 2\ell(g)\). Therefore for every \(t \in (0, 1)\) and every \(g \in Isom(H)\),

\[
\ell(\rho_t^C \circ \Phi(g)) = t\ell(\Phi(g)) = 2t\ell(g).
\]

This shows that for every \(q \in (0, 2)\) there exists an irreducible representation

\[
Isom(H) \to Isom(H^C)
\]

such that \(\ell(\rho) = q\).

**Lemma 1.3.5.** If \(\rho\) is the complexification of an irreducible representation

\[
Isom(H) \to Isom(H^C)
\]

then \(\Delta(b) = 0\), for every \(b \in R\).

**Proof.** Observe that Arg\((K(b))\) does not depend on the choice of representatives \(\eta_1, \eta_2\) of the extremes of the axis preserved by the isometries \(\rho(\lambda, 0)\) as long as \(B(\eta_1, \eta_2) = 1\) (see the definition before Lemma 1.3.3). Therefore if \(\eta_1\) and \(\eta_2\) are chosen in the totally real subspace that contains the representatives of the real hyperbolic subspace of \(H^C\) preserved by \(\rho\), it is clear that \(K(b) \in R\).

The following proposition follows some of the ideas of 13 and shows that there is not an irreducible representation \(Isom(H) \to Isom(H^C)\) such that \(\ell(\rho) = 2\).

**Proposition 1.3.6.** If \(\rho\) is only supposed non-elementary and such that \(\ell(\rho) = 2\), then \(b \mapsto c(b)\) is a non-trivial linear map and \(\Delta(b) = 0\), for every \(b \in R\).

**Proof.** The decomposition of an isometry \(\rho(g)\) and the properties of the maps in Proposition 1.3.1 are still valid if \(\rho\) is only supposed non-elementary. Observe that

\[
2|c(b)| = |c(2b)| = |c(b) + \pi(b)c(b)| \leq |c(b)| + |\pi(b)c(b)| \leq 2|c(b)|.
\]

Therefore \(\pi(b)c(b) = c(b)\), for every \(b \in R\). Observe that for every \(b, d \in R\),

\[
\pi(b)c(d) + c(b) = c(b + d) = \pi(b + d)c(b + d) = \pi(b)c(d) + \pi(d)c(b) = \pi(b)c(d) + \pi(d)c(b).
\]

Thus, for every \(b, d \in R\), \(\pi(d)c(b) = c(b)\), or in other words, \(b \mapsto c(b)\) is a linear map. This implies that for every \(b, d \in R\), \(Im(\langle c(b), c(d)\rangle) = 0\), thus by Proposition 1.3.1, the map \(b \mapsto \Delta(b)\) is linear, but \(\Delta(2) = 4\Delta(1)\). Therefore \(c(b) \neq 0\) and \(\Delta(b) = 0\), for every \(b \neq 0\).
Lemma 1.3.7. If $\ell(\rho) \neq 1,2$, the family $\{c(b)\}_{b \in \mathbb{R} \setminus 0}$ is $\mathbb{C}$-linearly independent.

Proof. Suppose $\sum_{i} a_i c(b_i) = 0$ with $b_i \neq 0$. Without lost of generality, suppose that $b_1 > b_i$ for every $i \neq 1$. For every $d \in \mathbb{R}$,

$$
0 = Re\left(\sum_{i} a_i \langle c(b_i), c(d) \rangle\right) = \sum_{i} Re(a_i) Re(\langle c(b_i), c(d) \rangle) - Im(a_i) Im(\langle c(b_i), c(d) \rangle)
$$

and

$$
0 = Im\left(\sum_{i} a_i \langle c(b_i), c(d) \rangle\right) = \sum_{i} Re(a_i) Im(\langle c(b_i), c(d) \rangle) + Im(a_i) Re(\langle c(b_i), c(d) \rangle).
$$

Consider an interval $(b_1, b_1 + r)$ such that $0 \not\in (b_1, b_1 + r)$ and consider $d \in (b_1, b_1 + r)$. By Proposition 1.3.1 there are constants $C_0, C_1, D_0, D_1$ such that

$$
C_0 d^t + \sum_{i} Re(\overline{K(1)} a_i)(d - b_i)^t = C_1
$$

and

$$
D_0 d^t + \sum_{i} Im(\overline{K(1)} a_i)(d - b_i)^t = D_1.
$$

Thus, there exist constants $E_0, E_1$ such that for every $d \in (b_1, b_1 + r)$,

$$
E_0 d^t + \sum_{i} \overline{K(1)} a_i (d - b_i)^t = E_1.
$$

After differentiating twice the previous equality with respect to $d$ in the interval $(b_1, b_1 + r)$, it follows that

$$
t(t - 1) E_0 d^{t-2} + t(t - 1) \sum_{i} \overline{K(1)} a_i ((d - b_i)^{t-2} = 0.
$$

If $d \to b_1^+$, then $(d - b_i)^{t-2}$ is unbounded, but for every $i \neq 1$, $(d - b_i)^{t-2}$ is bounded. Therefore $a_1 = 0$. Repeating the same argument, it is possible to show that for every $i$, $a_i = 0$. \(\square\)

Let $\sigma$ be the isometry of $\mathbf{H}^1_{\mathbb{C}}$ represented, in the basis $\{\xi_1, \xi_2\}$, by

$$
s = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.
$$

The next proposition and the corollary after it follow the arguments of Proposition 2.4 of [13].

Proposition 1.3.8. The isometry $\rho(\sigma)$ can be represented in the decomposition $\mathbb{C} \eta_1 \oplus \mathbb{C} \eta_2 \oplus (\eta_1^\perp \cap \eta_2^\perp)$, by

$$
\begin{pmatrix} 0 & \nu^{-1} & 0 \\ \nu & 0 & 0 \\ 0 & 0 & A \end{pmatrix},
$$

for some $\nu > 0$ and some unitary map $A$ such that, for every $b \in \mathbb{R}$,

$$
Ac(b) = \nu K(b)c(1, -1/b).
$$
Proof. Observe that \( g(\lambda, 0)s = s g(\lambda^{-1}, 0) \), therefore \( \rho(\sigma) \) preserves the set \( \{\eta_1, \eta_2\} \). If \( \rho(\sigma) \) fixes it pointwise, then \( \eta_1 \) would be a \( \text{Isom}(\mathbb{H}_C)_{\sigma} \)-fixed point (see Proposition \([1.1.2]\)) which is a contradiction. Thus \( \rho(\sigma) \) admits a linear representative

\[
\begin{pmatrix}
0 & \nu^{-1} & 0 \\
\nu & 0 & 0 \\
0 & 0 & A
\end{pmatrix},
\]

with \( \nu > 0 \) and \( A \) a unitary map of \( \eta_1^+ \cap \eta_2^+ \).

Observe that as elements of \( SU(1, 1) \),

\[
s \cdot g(1, b) \cdot s \cdot g(1, b/|b|^2) \cdot s = \begin{pmatrix} b^{-1} & -i \\ 0 & b \end{pmatrix},
\]

thus,

\[
\rho(\sigma) \rho(1, b) \rho(\sigma) \rho(1, b/|b|^2) \rho(\sigma) = \rho(1/|b|, -b/b)).
\]

Notice that using the canonical linear representatives of the isometries in the previous identity, on one side,

\[
\rho(\sigma) \rho(1, b) \rho(\sigma) \rho(1, b/|b|^2) \rho(\sigma)(\eta_2) = \rho(\sigma) \rho(1, b)(\eta_2) = \rho(\sigma) \left( K(b) \eta_1 + \eta_2 + c(b) \right) = \nu^{-1} \eta_1 + \nu K(b) \eta_2 + A c(b),
\]

and on the other side, by Proposition \([1.3.1]\) and Lemma \([1.3.3]\)

\[
\rho(1/|b|, -b/b)(\eta_2) = \rho(1/|b|, 0) \rho(1, -b)(\eta_2) = \rho(1/|b|, 0) \left( K(-b) \eta_1 + \eta_2 + c(-b) \right) = |b|^{-1} K(b) \eta_2 + \eta_2 + \pi(1/|b|, 0)c(-b) = K(-b/|b|) \eta_1 + |b|^t \eta_2 + |b|^t c(-1/b).
\]

Thus, there exist a unitary complex number \( \theta \) such that

\[
\theta \left( \nu^{-1} \eta_1 + \nu K(b) \eta_2 + A c(b) \right) = K(-b/|b|) \eta_1 + |b|^t \eta_2 + |b|^t c(-1/b).
\]

Observe that \( \theta \nu K(b) = |b|^t \); therefore, by Lemma \([1.3.3]\) \( \nu K(b/|b|) = \theta^{-1} \). This implies that

\[
Ac(b) = \nu K(b/|b|) |b|^t c(-1/b) = \nu K(b) c(-1/b).
\]

\[
\square
\]

**Corollary 1.3.9.** The representation \( \rho \) is determined by its restriction to \( P \).

Proof. The identity \( g(\lambda, b) = g(1, \lambda b) g(\lambda, 0) \) implies that \( c(\lambda, b) = \lambda^{-1} c(\lambda b) \). Let \( W \) be the closed complex vector space generated by \( \{c(b)\}_{b \in \mathbb{R}} \) in \( \eta_1^+ \cap \eta_2^+ \). By Proposition \([1.3.1]\) and Proposition \([1.3.8]\) \( C \eta_1 \oplus C \eta_2 \oplus W \) is a closed and invariant complex subspace of signature \((1, \infty)\). Therefore, as \( \rho \) is irreducible, \( W = \eta_1^+ \cap \eta_2^+ \).

Observe that,

\[
|c(1)|^2 = \langle Ac(1), Ac(1) \rangle = \nu^2 |K(1)|^2 |c(1)|^2.
\]

This and Proposition \([1.3.8]\) show that \( \rho(\sigma) \) is determined by \( \rho|_P \). \[
\square
\]


2 A new family of representations

In this section a family of irreducible representations of Isom($\mathbb{H}_C^1$)$_o$ is built, that to the best of the author’s knowledge were not known before. The way this is done is using a binary product for irreducible representations that is developed in Subsection 2.2.

In Subsection 2.1 it is defined a complete invariant for irreducible representations. This invariant allows to assure that this new representations are not equivalent to any of those described in [12] and [13] (see the comments before Lemma 1.3.5 and before Proposition 2.1.4). In particular all the irreducible representations $\rho$ such that $\ell(\rho) = 1$ are classified.

2.1 Invariants for representations of Isom($\mathbb{H}_C^1$)$_o$

In this subsection the theory of kernels of complex hyperbolic type developed in [12] is used to find a complete invariant for irreducible representations of Isom($\mathbb{H}_C^1$)$_o$.

Lemma 2.1.1. For every $y \in \mathbb{H}_C^1$,

$$\lim_{b \to \infty} \text{Cart}(g(1, b)y, g(1, -b)y, y) = -\frac{\pi}{2}.$$  

Proof. If $y$ is represented by $w = a\xi_1 + \xi_2$, then $\text{Re}(a) > 0$ and

$$\begin{align*}
\text{Cart}(g(1, b)y, g(1, -b)y, y) &= \text{Arg}\left( B(g(1, 2b)w, w)B(g(1, -b)w, w)^2 \right) = \\
&= \text{Arg}\left( 2\text{Re}(a)(4\text{Re}(a)^2 - b^2) + 8\text{Re}(a)b^2 + i((4\text{Re}(a)^2 - b^2)2b - 8\text{Re}(a)^2b) \right). \\
\end{align*}$$

Therefore,

$$\lim_{b \to \infty} \text{Cart}(g(1, b)y, g(1, -b)y, y) = -\frac{\pi}{2}. \quad \square$$

Suppose $\rho$ is defined on $SU(1, 1)$. Let $y \in \mathbb{H}_C^1$ and let $K$ be the maximal compact subgroup of $SU(1, 1)$ that fixes $y$. Denote $x \in \mathbb{H}_C^\infty$ the point fixed by $\rho(K)$ (see Proposition 5.8 and Remark 5.9 in [12]). Then there exists $0 \leq s$ such that for every $g_1, g_2 \in SU(1, 1)$,

$$s\text{Cart}(g_1y, g_2y, y) = \text{Cart}(\rho(g_1)x, \rho(g_2)x, x).$$

This is a consequence of the fact that the action of $SU(1, 1)$ on $\mathbb{H}_C^1$ is doubly transitive (see Remark 2.5 in [12]).

Observe that $s \leq 1$ because there exist $g_1, g_2 \in SU(1, 1)$ such that

$$|\text{Cart}(g_1y, g_2y, y)|$$

is as close as desired to $\pi/2$ and

$$|\text{Cart}(\rho(g_1)x, \rho(g_2)x, x)| < \pi/2.$$
Lemma 2.1.2. If \( x \in H_C^2 \), then
\[
\lim_{b \to \infty} \text{Cart}(\rho(b)x, \rho(-b)x, x) = \text{Arg}(K(-1)).
\]
Moreover, if \( K \leq SU(1, 1) \), the stabilizer of \( x \), is a maximal compact subgroup, \( y \in H_C^1 \) is the point fixed by \( K \) and \( 0 < s \leq 1 \) is such that
\[
scart(g(1, b)y, g(1, -b)y, y) = \text{Cart}(\rho(b)x, \rho(-b)x, x),
\]
then \( \frac{\pi}{2} = \text{Arg}(K(1)) \).

Proof. If \( \tilde{x} = a\eta_1 + \eta_2 + u \) is a representative of \( x \), then
\[
\text{Cart}(\rho(1, b)x, \rho(1, -b)x, x) = \text{Arg}\left(B(\rho(1, 2b)\tilde{x}, \tilde{x})B(\rho(1, -b)\tilde{x}, \tilde{x})^2\right)
\]
\[
= \text{Arg}\left(\begin{pmatrix} 2Re(a) + K(2b) + \langle u, c(2b) \rangle + \langle c(2b) + \pi(2b)u, u \rangle \
2Re(a) + K(-b) + \langle u, c(b) \rangle + \langle c(-b) + \pi(-b)u, u \rangle \end{pmatrix}\right).
\]
There exist constants \( C_1, C_2 > 0 \) such that for every \( b > 0 \),
\[
|\langle u, c(b) \rangle + \langle c(-b) + \pi(-b)u, u \rangle| \leq C_1 b^\frac{1}{2} + C_2.
\]
Therefore,
\[
\lim_{b \to \infty} \text{Cart}(\rho(b)x, \rho(-b)x, x) = \lim_{b \to \infty} \text{Arg}(K(2b)K(-b)^2) = \text{Arg}(K(-1)).
\]
The second claim is immediate from Lemma 2.1.1.

Observe that the previous lemma, Lemma 2.1.1 and the fact that the Cartan argument is left-invariant imply that neither \( \text{Arg}(K(1)) \) nor \( s \) depend on the choice of the point \( x \in H_C^2 \) fixed by a maximal compact subgroup of \( SU(1, 1) \). The previous lemma shows also that \( \Delta(1) \geq 0 \).

In view of the previous observations define \( \text{Arg}(\rho) \), the angular invariant of \( \rho \), as \( \text{Arg}(K(1)) \). This invariant can be defined for non non-elementary representations not necessarily irreducible. With this normalization, for a non-elementary representation \( \rho \), \( 0 \leq \text{Arg}(\rho) \leq \frac{\pi}{2} \).

Proposition 2.1.3. If \( \rho \) is non-elementary and \( \text{Arg}(\rho) = \pi/2 \), then \( \rho \) preserves a copy of \( H_C^1 \).

Proof. Observe that if \( \text{Re}(K(1)) = 0 \), then for every \( b \in R \), \( c(b) = 0 \).

The previous proposition, which is trivial in this context, is contained in the much more general Theorem 1.1 of [9].

In [12], the author showed that if \( G \) is a topological group and \((\beta, \alpha)\) is a \( G \)-invariant kernel of hyperbolic type, then for every \( 0 < t < 1 \), \((\beta^t, t\alpha)\) is a \( G \)-invariant kernel of hyperbolic type (see Theorem 1.12 of the aforementioned article). The author also showed, in particular, that if \((\beta, \alpha)\) is a kernel of hyperbolic type associated to the tautological action of isomorphism \( H_C^2 \), then for every \( 0 < t < 1 \), \((\beta^t, t\alpha)\) induces (see Theorem 1.2.3) a non-elementary representation \( \text{Isom}(H_C^2) \to \text{Isom}(H_C^2) \), such that \( \ell(\rho) = t \) (see Theorem 1.15 and Lemma 2.2 of [12]). The following proposition is a direct consequence of Theorem 1.2.3 and Lemma 2.1.2.
Proposition 2.1.4. Let \( x \in \mathbf{H}^1_{\mathbb{C}} \) and let \( 0 < t < 1 \). If \( \rho \) is the irreducible part of the non-elementary representation associated to the kernel \((\beta^t, \alpha^t)\), where \((\beta, \alpha)\) is the kernel of hyperbolic type associated to \( x \) and the tautological action of \( \text{Isom}(\mathbf{H}^1_{\mathbb{C}}) \) on \( \mathbf{H}^1_{\mathbb{C}} \), then
\[
\text{Arg}(\rho) = \frac{t\pi}{2}.
\]

Lemma 2.1.5. If \( x \in \mathbf{H}^\infty_{\mathbb{C}} \) is represented by \( \frac{1}{\sqrt{2}}(\eta_1 + \eta_2) \), then the function of hyperbolic type associated to \( x \) can be reconstructed from \( K(1) \) and \( \ell(\rho) \).

Proof. The transformation \( \rho(\sigma) \) is determined by the restriction of \( \rho \) to the subgroup \( P \) (see Proposition 1.3.8). The claim is that the restriction of \( \rho \) to \( P \) is entirely determined by \( K(1) \) and the parameter \( t \).

For every \( b, d \in \mathbb{R} \) and for every \( \lambda, \gamma > 0 \),
\[
\text{Cart}(\rho(\lambda, b)x, \rho(\gamma, d)x, x) = \text{Arg}
\begin{align*}
B\left(\rho(\lambda^{-1}, \gamma^{-1}b - \lambda^{-1}d)(\eta_1 + \eta_2), \eta_1 + \eta_2 \right) & \times \\
B\left(\rho(\lambda^{-1}, -b)(\eta_1 + \eta_2), \eta_1 + \eta_2 \right) = \\
\text{Arg}
\begin{align*}
B\left(\rho(1, \lambda^{-1}b - \lambda^{-2}d)(\eta_1 + \eta_2), \rho(\lambda^{-1}, 0)(\eta_1 + \eta_2) \right) & \times \\
B\left(\rho(1, \gamma^{-1}d)(\eta_1 + \eta_2), \rho(\gamma^{-1}, 0)(\eta_1 + \eta_2) \right) & \times \\
B\left(\rho(1, -\lambda b)(\eta_1 + \eta_2), \rho(\lambda, 0)(\eta_1 + \eta_2) \right)
\end{align*}
\end{align*}
\]

Observe that, by Lemma 1.3.3, the last term can be recovered knowing the values of \( K(1) \) and \( t \). For the same reasons,
\[
\text{coshd}(\rho(\lambda, b)x, x) = \text{Arg}
\begin{align*}
\frac{1}{2} |B\left(1 + K(\lambda^{-1}b)\eta_1 + \eta_2, \lambda^{-t}\eta_1 + \lambda^{t}\eta_2 \right)|
\end{align*}
\]
can be also recovered from \( K(1) \) and \( t \). Therefore the claim follows from Theorem 1.2.3 and the fact that the \( P \)-orbit of \( \frac{1}{\sqrt{2}}(\eta_1 + \eta_2) \) is total (see Proposition 1.1.2). \( \square \)

Theorem 2.1.6. Let \( \rho_1 \) and \( \rho_2 \) be two irreducible representations of \( \text{Isom}(\mathbf{H}^1_{\mathbb{C}}) \) into \( \text{Isom}(\mathbf{H}^\infty_{\mathbb{C}}) \) such that \( \ell(\rho_1) = \ell(\rho_2) \). Then \( \rho_1 \) and \( \rho_2 \) are equivalent if, and only if, \( \text{Arg}(\rho_1) = \text{Arg}(\rho_2) \).

Proof. Suppose that \( \rho_1(P) \) and \( \rho_2(P) \) share the same fixed point in \( \partial \mathbf{H}^\infty_{\mathbb{C}} \) and that the families \( \{\rho_1(\lambda, 0)\}_{\lambda > 0} \) and \( \{\rho_2(\lambda, 0)\}_{\lambda > 0} \) preserve the same axis.

If \( \rho_1 \) and \( \rho_2 \) are equivalent, their restrictions to the group \( P \) are equivalent. Therefore there exists \( T \) an isometry of \( \mathbf{H}^\infty_{\mathbb{C}} \) such that \( T\rho_1|_P T^{-1} = \rho_2|_P \). Notice that \( T(\eta_i) = \eta_i \), where \( \eta_i \) for \( i = 1, 2 \) are the extremes of the axis preserved by the isometries \( \rho_i(\lambda, 0) \). If
\[
K_i(1) = \frac{-|c_i(1)|^2}{2} + i\Delta_i(1) = \frac{Q(\eta_2, \rho_i(1, 1)\eta_1)}{|Q(\eta_2, \rho_i(1, 1)\eta_1)|^2} B(\rho_i(1, 1)\eta_2, \eta_2),
\]
then it is clear that \( \text{Arg}(K_1(1)) = \text{Arg}(K_2(1)) \) because \( \text{Arg}(K_i(1)) \) does not depend on the choice of the representatives of \( \eta_i \), as long as the condition \( B(\eta_1, \eta_2) = 1 \) is fulfilled.
Suppose \( \text{Arg}(K_1(1)) = \text{Arg}(K_2(1)) \). After conjugating \( \rho_1 \) by an isometry \( \rho_1(\gamma, 0) \) if needed, it is possible to suppose that \( K_1(1) = K_2(1) \). Let \( x \in H^\infty_C \) be the point represented by
\[
\frac{1}{\sqrt{2}}(\eta_1 + \eta_2).
\]
Consider the respective kernels of hyperbolic type associated to \( x \). By Lemma 2.1.5 the representations \( \rho_1|_P \) and \( \rho_2|_P \) can be supposed identical, therefore by Proposition 1.3.8 \( \rho_1 \) and \( \rho_2 \) are equivalent.

Observe that the assuming that \( \rho_1 \) and \( \rho_2 \) have the same distinguished points at infinity is not restrictive. Indeed, if \( \rho \) is an irreducible representation, by Lemma 2.1.2 \( \text{Arg}(K(1)) \) is invariant under conjugations and given any \( \eta_1, \eta_2 \in \partial H^\infty_C \), up to a conjugation, it is possible to suppose that \( \eta_1 \) is the unique fixed point of \( \rho(P) \) and that \( \eta_2 \) is the other extreme of the axis preserved by the family \( \{\rho(\lambda, 0)\}_{\lambda > 0} \).

### 2.2 Extending certain representations of a parabolic subgroup

In this section a procedure to produce from two irreducible representations with the same displacement a third one, which in general will not be equivalent to any of the previous two, is described. With this method a new family of non-equivalent representations will be constructed and in particular, for irreducible representations with displacement 1 this family will be exhaustive.

The definition of this binary operation relies on the fact that with certain conditions, the representations of \( P \) into \( \text{Isom}(H^\infty_C)_o \) can be extended to representations of \( SU(1, 1) \). This is done using Proposition 1.3.8 as a definition for the image under the representation of the map \( s \) (defined before the aforementioned proposition).

Fix \( \rho_1 \) and \( \rho_2 \) two irreducible representations of \( \text{Isom}(H^\infty_C)_o \) into \( \text{Isom}(H^\infty_C)_o \) such that \( \ell(\rho_1) = \ell(\rho_2) = t \). With the conventions of the previous section, suppose without lost of generality, that \( \rho_i \) share the two distinguished points \( \eta_i \in \partial H^\infty_C \).

That is, for every \( g(\lambda, b) \), \( \rho_i(\lambda, b)(\eta_1) = \eta_1 \) and for every \( g(\lambda, 0) \), \( \rho_i(\lambda, 0)(\eta_2) = \eta_2 \).

In a matrix representation with respect to the decomposition
\[
C\eta_1 \oplus C\eta_2 \oplus (\eta_1^\perp \cap \eta_2^\perp),
\]
by Proposition 1.3.8 \( \rho_i(\lambda, b) \) has the shape,
\[
\begin{pmatrix}
\lambda |c_i(\lambda, b)|^2/2 + i\Delta_i(\lambda, b) & -\lambda^{-t}(\pi_i(\lambda, b)(\cdot), c_i(\lambda, b)) \\
0 & 0 \\
0 & c_i(\lambda, b) & \pi_i(\lambda, d)
\end{pmatrix},
\]
and the isometry \( \rho_i(\sigma) \) has the representation
\[
\begin{pmatrix}
0 & \nu_i^{-1} & 0 \\
\nu_i & 0 & 0 \\
0 & 0 & A_i
\end{pmatrix},
\]
where \( \nu_i > 0 \) and, by Proposition 1.3.8
\[
A_i c_i(b) = \nu_i K_i(b) c_i(1, -1/b).
\]
Define a model for the hyperbolic space in the following way. Consider $\mathbb{C}^2 = \mathbb{C}^{\eta_1} \oplus \mathbb{C}^{\eta_2}$ and consider the Hilbert space $L = H_1 \oplus H_2$, where $H_i = \eta_i^+ \cap \eta_i^-$. Define the form $Q$ in $\mathbb{C}^{\eta_1} \oplus \mathbb{C}^{\eta_2} \oplus L$ which is $\mathbb{C}$ linear in the first entry, antilinear in the second and that is given by

1. $Q|_{H_i} = \langle \cdot, \cdot \rangle_i$.
2. $Q(H_1, H_2) = 0$.
3. $Q(\eta_i, H_j) = 0$, for $i, j = 1, 2$.
4. $Q(\eta_i, \eta_i) = 0$, for $i = 1, 2$.
5. $Q(\eta_1, \eta_2) = 1$.

This defines a strongly non-degenerate form of signature $(1, \infty)$ in $\mathbb{C}^{\eta_1} \oplus \mathbb{C}^{\eta_2} \oplus L$.

Define $c(b) = c_1(b) \oplus c_2(b)$ and $\pi(\lambda, b) = \pi_1(\lambda, b) \oplus \pi_2(\lambda, b)$, for every $b \in \mathbb{R}$ and $\lambda > 0$. Observe that $\pi$ is a unitary representation of the group $P$ on $L$.

Define $\rho(\lambda, 0)$ as the isometry represented by

$$
\begin{pmatrix}
\lambda^t & 0 & 0 \\
0 & \lambda^{-t} & 0 \\
0 & 0 & \pi(\lambda, 0)
\end{pmatrix}
$$

and $\rho(1, b)$ as the isometry represented by

$$
\begin{pmatrix}
1 & -|c(b)|^2/2 + i\Delta(b) & -\langle \pi(b)(\cdot), c(b) \rangle \\
0 & 1 & 0 \\
0 & c(b) & \pi(b)
\end{pmatrix},
$$

where $\Delta(b) = \Delta_1(b) + \Delta_2(b)$. Observe that the transformations $\rho(\lambda, 0)$ and $\rho(1, b)$ are isometries of $H_2^{\infty}$, the hyperbolic space associated to $\mathbb{C}^{\eta_1} \oplus \mathbb{C}^{\eta_2} \oplus L$ and $Q$. Denote $K(b) = -|c(b)|^2/2 + i\Delta(b)$ and $K_i(b) = -|c_i(b)|^2/2 + i\Delta_i(b)$.

The next proposition is a consequence of Proposition 1.3.1 and Lemma 1.3.3.

**Proposition 2.2.1.** If $c$, $\pi$, $K$ and $K_i$ are defined as above, then for every $\lambda > 0$ and $b, d \in \mathbb{R}$, the following properties hold.

1. $Im(\langle c(b), c(d) \rangle) = \Delta(b - d) - \Delta(b) + \Delta(d)$.
2. $Re(\langle c(b), c(d) \rangle) = -\frac{|c(b - d)|^2}{2} + \frac{|c(b)|^2}{2} + \frac{|c(d)|^2}{2}$.
3. $K(b) = K_1(b) + K_2(b)$.
4. $K(\lambda b) = \lambda^t K(b)$.
5. $K(-b) = K(b)$.
6. $K(b + d) = K(b) + K(d) + \langle c(d), c(-b) \rangle$.
7. $\pi(\lambda, 0)c(b) = \lambda^{-t}c(\lambda^2b)$.
Lemma 2.2.3. For every $b \neq 0$, $K(b) \neq 0$.

Proof. Suppose $K(1) = 0$. By Lemma 2.2.2, $\Delta_i(1) \geq 0$, therefore $\Delta_i(1) = 0$. The isometries $\rho_i(1, 1)$ are parabolic, thus $c_i(1) \neq 0$, which is a contradiction.

Observe that $g(\lambda, 0)g(1, b) = g(\gamma, 0)g(1, d)$ if, and only if, $\lambda = \gamma$ and $b = d$, and that $g(\lambda, 0)g(1, b) = g(1, \lambda^2b)g(\lambda, 0)$. It will be shown that with the formulas for $\rho(\lambda, 0)$ and $\rho(1, b)$ it is possible to define an homomorphism on $P$.

Lemma 2.2.4. For every $\lambda, \gamma > 0$ and $b, d \in \mathbb{R}$, the following identities hold.

1. $K(\lambda^{-1}\gamma^{-1}b + d) = K(d) + \gamma^{-t}K(\lambda^{-1}b) + \langle \pi(\gamma, 0)c(\gamma^{-1}d), c(-\lambda^{-1}b) \rangle$.
2. $\pi(\lambda^{-1}b)\pi(\gamma, 0)c(\gamma^{-1}d) + \gamma^{-t}c(\lambda^{-1}b) = \pi(\gamma, 0)c(\lambda^{-1}\gamma^{-2}b + \gamma^{-1}d)$.

Proof. By Proposition 2.2.1

$$K(\lambda^{-1}\gamma^{-1}b + d) = K(d) + K(\lambda^{-1}b) + \langle c(d), c(-\lambda^{-1}b) \rangle = K(d) + K(\gamma^{-1}b) + \langle \gamma^{-t}c(d), \gamma^{-t}c(-\lambda^{-1}b) \rangle = K(d) + K(\gamma^{-1}b) + \langle \pi(\gamma^{-1}d), c(\gamma^{-1}b) \rangle = K(d) + \gamma^{-t}K(\lambda^{-1}b) + \langle \pi(\gamma, 0)c(\gamma^{-1}d), c(-\lambda^{-1}b) \rangle$$

and

$$\pi(\lambda^{-1}b)\pi(\gamma, 0)c(\gamma^{-1}d) + \gamma^{-t}c(\lambda^{-1}b) = \gamma^{-t}\pi(\lambda^{-1}b)c(\gamma^{-1}d) + \gamma^{-t}c(\lambda^{-1}b) = \gamma^{-t}c(\lambda^{-1}b + \gamma d) = \pi(\gamma, 0)c(\lambda^{-1}\gamma^{-2}b + \gamma^{-1}d).$$

Lemma 2.2.4. For every $\gamma, \lambda > 0$, $b, d \in \mathbb{R}$ and $u \in \eta_1^+ \cap \eta_2^+$,

$$\gamma^t\langle u, c(-\gamma^{-1}d) \rangle + \langle u, \pi(-\gamma^{-1}d)\pi(\gamma^{-1}, 0)c(-\lambda^{-1}b) \rangle = \gamma^t\langle u, c(1, -\lambda^{-1}\gamma^{-2}b - \gamma^{-1}d) \rangle.$$

Proof. By Proposition 2.2.1

$$\gamma^t\langle u, c(-\gamma^{-1}d) \rangle + \langle u, \pi(-\gamma^{-1}d)\pi(\gamma^{-1}, 0)c(-\lambda^{-1}b) \rangle = \gamma^t\langle u, c(-\gamma^{-1}d) \rangle + \gamma^t\langle u, \pi(-\gamma^{-1}d)c(-\lambda^{-1}\gamma^{-2}b) \rangle = \gamma^t\langle u, c(1, -\lambda^{-1}\gamma^{-2}b - \gamma^{-1}d) \rangle.$$

Proposition 2.2.5. The map $g(\lambda, b) \mapsto \rho(\lambda, 0)\rho(1, \lambda^{-1}b)$ is a homomorphism and for every $x \in H_\mathbb{C}^+$, the map $g(\lambda, b) \mapsto \rho(\lambda, 0)\rho(1, \lambda^{-1}b)x$ is continuous.

Proof. Observe that

$$g(\lambda, b)g(\gamma, d) = g(\lambda, 0)g(1, \lambda^{-1}b)g(\gamma, 0)g(1, \gamma^{-1}d) = g(\lambda, 0)g(\gamma, 0)g(1, \lambda^{-1}\gamma^{-2}b)g(1, \gamma^{-1}d) = g(\lambda\gamma, 0)g(1, \lambda^{-1}\gamma^{-2}b + \gamma^{-1}d).$$
Therefore the first claim of the proposition is that
\[
\rho(\lambda, 0)\rho(1, \lambda^{-1}b)\rho(\gamma, 0)\rho(1, \gamma^{-1}d) = \rho(\lambda\gamma, 0)\rho(1, \lambda^{-1}\gamma^{-2}b + \gamma^{-1}d).
\]
It is clear that \(\lambda \mapsto \rho(\lambda, 0)\) is a homomorphism, thus to show the previous claim is equivalent to show that
\[
\rho(1, \lambda^{-1}b)\rho(\gamma, 0)\rho(1, \gamma^{-1}d) = \rho(\gamma, 0)\rho(1, \lambda^{-1}\gamma^{-2}b + \gamma^{-1}d).
\]
This will be done by comparing the columns of the matrix representations of both sides of the identities with respect to the decomposition \(C\eta_1 \oplus C\eta_2 \oplus L\).

It is clear from the definition of \(\rho\) that for \(\eta_1\),
\[
\rho(1, \lambda^{-1}b)\rho(\gamma, 0)\rho(1, \gamma^{-1}d)\eta_1 = \gamma^t\eta_1 = \rho(\gamma, 0)\rho(1, \lambda^{-1}\gamma^{-2}b + \gamma^{-1}d)\eta_1.
\]

By Proposition 2.2.1 and Lemma 2.2.3 for \(\eta_2\),
\[
\rho(1, \lambda^{-1}b)\rho(\gamma, 0)\rho(1, \gamma^{-1}d)\eta_2 = \rho(1, \lambda^{-1}b)\rho(\gamma, 0)\left(K(\gamma^{-1}d)\eta_1 + \eta_2 + c(\gamma^{-1}d)\right) = \rho(1, \lambda^{-1}b)\left(\gamma^tK(\gamma^{-1}d)\eta_1 + \gamma^{-t}\eta_2 + \pi(\gamma, 0)c(\gamma^{-1}d)\right)
\]
\[
= \left(K(d) + \gamma^{-t}K(\lambda^{-1}b) + \langle \pi(\gamma, 0)c(\gamma^{-1}d), c(\lambda^{-1}b)\rangle\right)\eta_1 + \gamma^{-t}\eta_2 + \pi(\gamma, 0)c(\gamma^{-1}d)\eta_2 = \rho(\gamma, 0)\left(K(\lambda^{-1}\gamma^{-2}b + \gamma^{-1}d)\eta_1 + \eta_2 + c(\lambda^{-1}\gamma^{-2}b + \gamma^{-1}d)\right).
\]

And last, by Proposition 2.2.1 and Lemma 2.2.4 for \(u \in \eta_1^+ \cap \eta_2^+\),
\[
\rho(1, \lambda^{-1}b)\rho(\gamma, 0)\rho(1, \gamma^{-1}d)u = \rho(1, \lambda^{-1}b)\rho(\gamma, 0)(\langle u, c(-\gamma^{-1}d)\rangle\eta_1 + \gamma^t\eta_1) = \rho(1, \lambda^{-1}b)(\gamma^t\langle u, c(-\gamma^{-1}d)\rangle\eta_1 + \pi(\gamma, 0)\pi(\gamma^{-1}d)u)
\]
\[
= \left(\gamma^t\langle u, c(-\gamma^{-1}d)\rangle + \langle \pi(\gamma, 0)c(\gamma^{-1}d), c(\lambda^{-1}b)\rangle\right)\eta_1 + \pi(\gamma, 0)\pi(\gamma^{-1}d)u = \gamma^t\langle u, c(\lambda^{-1}\gamma^{-2}b + \gamma^{-1}d)\rangle\eta_1 + \pi(\gamma, 0)\pi(\gamma^{-1}d)u.
\]

Therefore the map
\[
g(\lambda, b) \mapsto \rho(\lambda, 0)\rho(1, \lambda^{-1}b) \in \text{Isom}(H^\infty_C)
\]
is a homomorphism.

For the second claim of the proposition it is enough to show that for every \(x \in H^\infty_C\), the map \(g(\lambda, b) \mapsto \rho(\lambda, 0)\rho(1, \lambda^{-1}b)x\) is continuous around the identity in \(P\). Suppose \(g_i = g(\lambda_i, b_i) \to Id\) in \(P\), then
\[
\frac{1}{4}|B(g(\lambda_i, b_i)(\eta_1 + \eta_2), \eta_1 + \eta_2)|^2 = \frac{1}{4}((\lambda_i + \lambda_i^{-1})^2 + b_i^2) \to 1,
\]

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or equivalently,

\[(\lambda_i - \lambda_i^{-1})^2 + b_i^2 \to 0.\]

Therefore \(\lambda_i \to 1\) and \(b_i \to 0\). If \(x = \alpha \eta_1 + \beta \eta_2 + u\) is such that \(Q(x, x) = 1\), then

\[
\rho(\lambda_i, 0)\rho(1, \lambda_i^{-1}b_i)x = \\
\rho(\lambda_i, 0)(\alpha + \beta K(\lambda_i^{-1}b_i) + \langle u, c(-\lambda_i^{-1}b_i) \rangle)\eta_1 + \\
\rho(\lambda_i, 0)(\beta \eta_2 + \beta c(\lambda_i^{-1}b_i) + \pi(\lambda_i^{-1}b_i)u) = \\
\lambda_i^t(\alpha + \beta K(\lambda_i^{-1}b_i) + \langle u, c(-\lambda_i^{-1}b_i) \rangle)\eta_1 + \lambda_i^{-1} \beta \eta_2 + \\
\pi(\lambda_i, 0)(\beta c(\lambda_i^{-1}b_i) + \pi(\lambda_i^{-1}b_i)u) = \\
(\lambda_i^t \alpha + \beta K(\lambda_i^{-1}b_i) + \langle u, \pi(\lambda_i^{-1}, 0)c(-\lambda_i b_i) \rangle)\eta_1 + \lambda_i^{-1} \beta \eta_2 + \\
\lambda_i^{-1} \beta c(\lambda_i b_i) + \pi(\lambda_i, 0)\pi(\lambda_i^{-1}b_i)u.
\]

Therefore, since

\[g(\lambda, b) \mapsto \pi(\lambda, b) = \pi_1(\lambda, b) \oplus \pi_2(\lambda, b)\]

is orbitally continuous,

\[
\lim_{i \to \infty} |B(\rho(\lambda_i, 0)\rho(1, \lambda_i^{-1}b_i)x, x)| = \\
\lim_{i \to \infty} |\overline{\beta}(\lambda_i^t \alpha + \beta K(\lambda_i^{-1}b_i) + \langle u, \pi(\lambda_i^{-1}, 0)c(-\lambda_i b_i) \rangle) + \overline{\lambda_i^{-1}} \beta + \\
\langle \lambda_i^{-1} \beta c(\lambda_i b_i) + \pi(\lambda_i, 0)\pi(\lambda_i^{-1}b_i)u, u \rangle| = \\
|\overline{\beta} \alpha + \overline{\alpha} \beta + \langle u, u \rangle| = 1.
\]

Now it is possible to define the representation \(P \overset{\rho}{\to} \text{Isom}(H_C^\infty)\), given by

\[\rho(\lambda, b) = \rho(\lambda, 0)\rho(1, \lambda^{-1}b).\]

The next results are devoted to prove that \(\rho\) can be extended to a homomorphism defined on \(SU(1, 1)\).

**Lemma 2.2.6.** The only point fixed in \(H_C^\infty \cup \partial H_C^\infty\) by \(\rho\) is \(\eta_1\).

**Proof.** The isometries \(\rho(\lambda, 0)\) are hyperbolic by construction, therefore the only other candidate to be fixed by \(\rho\) is \(\eta_2\), but \(\rho(1, b)\) does not fix it because \(K(1) \neq 0\) (see Lemma 2.2.2).

Observe that

\[K(b) = \frac{Q(\eta_2, \rho(1, b)\eta_1)}{|Q(\eta_2, \rho(1, b)\eta_1)|^2}Q(\rho(1, b)\eta_2, \eta_2)\]

is also true for the representation \(\rho\). After a conjugation by an isometry \(\rho(\gamma, 0)\) if necessary, assume that \(|K(1)| = 1\). Notice that this conjugation does not change the argument of \(K(1)\).

The following is the uniqueness part of the GNS construction (see Theorem C.1.4 of [1]).

**Lemma 2.2.7.** Let \(X\) be a set and let \(H\) be a Hilbert space. Suppose \(f\) and \(g\) are two functions \(X \to H\) such that their images are total in \(H\). If for every \(x, y \in X\), \(\langle f(x), f(y) \rangle = \langle g(x), g(y) \rangle\), then there exists \(A\), a unitary map of \(H\), such that \(Af(x) = g(x)\).
Proposition 2.2.8. The map

\[ Ac(b) = K(b)c(-1/b) \]

defines a unitary map in \( L' = \langle \langle c(b) \rangle \rangle_{b \in \mathbb{R}} \) such that \( A^2 = 1d \).

Proof. Due to Lemma 2.2.7, it is enough to show that \( \langle Ac(b), Ac(d) \rangle = \langle c(b), c(d) \rangle \).
Suppose \( b \neq d \). By Proposition 2.2.1, on one side,

\[
\langle A(c(b)), A(c(d)) \rangle = K(b)K(d)\langle c(-1/b), c(-1/d) \rangle = K(b)K(d)i\left( \Delta(-1/b + 1/d) - \Delta(-1/b) + \Delta(-1/d) \right) =
|b|^t|d|^t K\left( \frac{b}{|b|} \right) K\left( \frac{-d}{|d|} \right) i\left( \frac{|b-d|^t(b-b)[b|b|]}{|d||b|} + \frac{b}{|b|} - \frac{d}{|d|} \right) \Delta(1) =
K\left( \frac{b}{|b|} \right) K\left( \frac{-d}{|d|} \right) i\left( \frac{|b-d|^t(b-b)[b|b|]}{|d||b|} + \frac{d^t}{|d|} - \frac{|b^t|}{|b|} \right) \Delta(1).
\]

On the other side,

\[
\langle c(b), c(d) \rangle =
-|c(b)|^2 + \frac{|c(b)|^2}{2} + \frac{|c(d)|^2}{2} + i\left( \Delta(b - d) - \Delta(b) + \Delta(d) \right) =
( -|b - d|^t + |b|^t + |d|^t ) \Delta(1) =
\]

There are three cases to analyse: 1) \( b > 0 > d \), 2) \( b > d > 0 \) and 3) \( b < d < 0 \).

1) If \( b > 0 > d \),

\[
K\left( \frac{b}{|b|} \right) K\left( \frac{-d}{|d|} \right) = K(1)^2
\]

and

\[
\frac{|b-d|^t(b-b)[b|b|]}{|d||b|} + \frac{|d|^t}{|d|} - \frac{|b|^t}{|b|} = -|b - d|^t + |b|^t + |d|^t.
\]

Therefore

\[
\langle Ac(b), Ac(d) \rangle = K(1)^2( -|b - d|^t + |b|^t + |d|^t ) \left( \frac{\Delta(1)}{2} + i\Delta(1) \right) =
( -|b - d|^t + |b|^t + |d|^t ) K(1)^2(-K(1)) =
-(-|b - d|^t + |b|^t + |d|^t) K(1)
\]

and

\[
\langle c(b), c(d) \rangle =
( -|b - d|^t + |b|^t + |d|^t ) \left( \frac{\Delta(1)}{2} - i\Delta(1) \right) =
-(-|b - d|^t + |b|^t + |d|^t) K(1).
\]

2) If \( b > d > 0 \),

\[
K\left( \frac{b}{|b|} \right) K\left( \frac{-d}{|d|} \right) = 1
\]

and

\[
\frac{|b-d|^t(b-b)[b|b|]}{|d||b|} - \frac{|b|^t}{|b|} + \frac{|d|^t}{|d|} = |b - d|^t - |b|^t + |d|^t.
\]
Thus

\[ \langle A(c(b)), A(c(d)) \rangle = ( - |d - b|^t + |d|^t + |b|^t \frac{|c(1)|^2}{2} + (|b - d|^t + |d|^t - |b|^t) \Delta(1) \]

and

\[ \langle c(b), c(d) \rangle = ( - |d - b|^t + |d|^t + |b|^t \frac{|c(1)|^2}{2} + i(|b - d|^t - |b|^t + |d|^t) \Delta(1). \]

3) If \( b < d < 0 \),

\[ K \left( \frac{b}{|b|} \right) K \left( \frac{-d}{|d|} \right) = 1 \]

and

\[ \frac{|b - d|^t(b - d)(|b|d)}{|b|d} + \frac{|d|^t b - |b|^t d}{|b|^t d} = -|b - d|^t - |d|^t + |b|^t. \]

Therefore

\[ \langle A(c(b)), A(c(d)) \rangle = ( - |d - b|^t + |d|^t + |b|^t \frac{|c(1)|^2}{2} + i(|b - d|^t - |b|^t + |d|^t) \Delta(1) \]

and

\[ \langle c(b), c(d) \rangle = ( - |b - d|^t + |b|^t + |d|^t \frac{|c(1)|^2}{2} + i(|b - d|^t + |b|^t - |d|^t) \Delta(1). \]

The case \( b = d \) is an immediate consequence of Proposition 2.2.1.

By Lemma 2.2.7, \( A \) induces a unitary map on \( L \). And last, observe that

\[ A^2(c(b)) = K(b)K(-1/b)c(b) \]

\[ = |b|^t K \left( \frac{b}{|b|} \right) |b| K \left( \frac{-b}{|b|} \right) c(b) \]

\[ = c(b). \]

Consider now \( \mathbf{H}_C^\infty \) as the hyperbolic space associated to \( \mathbf{C} \eta_1 \oplus \mathbf{C} \eta_2 \oplus L' \) and the restriction of the form \( Q \) defined in the beginning of this section. Denote by \( \tilde{\sigma} \in \text{Isom} (\mathbf{H}_C^\infty) \) the order two transformation represented by

\[
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & A
\end{pmatrix}
\]

The claim is that the representation \( \rho \) can be extended to a representation of \( SU(1,1) \) using \( \tilde{\sigma} \). That is to say, if \( g(1,b) \), with \( b \in \mathbf{R} \), \( g(\lambda,0) \), with \( \lambda > 0 \), and

\[ s = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \]

are understood as elements of \( SU(1,1) \), then the map defined by :

1. \( T(g(\lambda,0)) = \rho(\lambda,0) \),
2. \( T(-g(\lambda,0)) = \rho(\lambda,0) \),
3. $T(g(1, b)) = \rho(1, b)$,
4. $T(s) = \tilde{\sigma}$,

where $\rho(\lambda, 0)$, $\rho(1, b)$ and $\tilde{\sigma}$ are interpreted as elements of $\text{Isom}(H^\infty_C)$, is a homomorphism. 

In order to prove that $T$ is a homomorphism it is enough to show that $T$ is coherent with the relations of Theorem 1.1.4, that is to show that, for $\lambda > 0$ and $b \in \mathbb{R}$,

1. $\rho(\lambda, 0) = \tilde{\sigma} \rho(1, \lambda^{-1}) \tilde{\sigma} \rho(1, \lambda) \tilde{\sigma} \rho(1, \lambda^{-1}).$
2. $\rho(\lambda, 0) = \tilde{\sigma} \rho(1, -\lambda^{-1}) \tilde{\sigma} \rho(1, -\lambda) \tilde{\sigma} \rho(1, -\lambda^{-1}).$
3. $\lambda \mapsto \rho(\lambda, 0)$ is a homomorphism.
4. $b \mapsto \rho(1, b)$ is a homomorphism.
5. $\rho(\lambda, 0) \rho(1, b) \rho(\lambda^{-1}, 0) = \rho(1, \lambda^2 b)$.
6. $\tilde{\sigma}^2 = \text{Id}.$

Observe that $T$ is coherent with the points from 3., 4. and 5. because $\rho$ is a homomorphism defined on $P$ (see Proposition 2.2.5). By Proposition 2.2.8, point 6. holds, therefore the only two families of relations left to be verified are that for every $b > 0$,

$\rho(b, 0) = \sigma \rho(1, b^{-1}) \sigma \rho(1, b) \sigma \rho(1, b^{-1}) = \sigma \rho(1, -b^{-1}) \sigma \rho(1, -b) \sigma \rho(1, -b^{-1}).$

**Lemma 2.2.9.** For $\epsilon = \pm 1$ and for every $b > 0$,

1. $1 + K(\epsilon b) K(\epsilon b^{-1}) + \langle Ac(\epsilon b), c(-\epsilon b^{-1}) \rangle = 0.$
2. $K(\epsilon b) c(\epsilon b^{-1}) + \pi(\epsilon b^{-1}) Ac(\epsilon b) = 0.$

**Proof.** Indeed,

\[
1 + K(\epsilon b) K(\epsilon b^{-1}) + \langle Ac(\epsilon b), c(-\epsilon b^{-1}) \rangle = 1 + K(\epsilon)^2 + K(\epsilon) \langle c(-\epsilon b^{-1}), c(-\epsilon b^{-1}) \rangle = 1 + K(\epsilon)^2 + b^{-t} K(\epsilon) |c(-1)|^2 = K(\epsilon) \left( K(-\epsilon) + K(\epsilon) + |c(-1)|^2 \right) = 0.
\]

and

\[
K(\epsilon b) c(\epsilon b^{-1}) + \pi(\epsilon b^{-1}) Ac(\epsilon b) = K(\epsilon b) \left( c(\epsilon b^{-1}) + \pi(\epsilon b^{-1}) c(-\epsilon b^{-1}) \right) = 0.
\]

**Lemma 2.2.10.** If $b > 0$ and $\epsilon = \pm 1$, then

$\rho(b, 0) = \tilde{\sigma} \rho(1, \epsilon b^{-1}) \tilde{\sigma} \rho(1, \epsilon b) \tilde{\sigma} \rho(1, \epsilon b^{-1}).$
Proof. The procedure will be to compare the columns of the canonical matrix representation of both sides of the identities with respect to the decomposition $C\eta_1 \oplus C\eta_2 \oplus L$. In fact, it will be shown that

$$\tilde{\sigma}\rho(b,0)\rho(1,-eb^{-1})\tilde{\sigma} = \rho(1,eb^{-1})\tilde{\sigma}\rho(1,eb).$$

With a small abuse of notation, keep the notation above for the canonical linear and that

$$\rho(1,eb^{-1})\tilde{\sigma}\rho(1,eb)(\eta_1) = K(\epsilon)(\eta_1) + \eta_2 + c(\epsilon)$$

and on the other side,

$$\rho(1,eb^{-1})\tilde{\sigma}\rho(1,eb)(\eta_1) = K(\epsilon)(\eta_1) + \eta_2 + c(\epsilon)$$

Observe that

$$K(\epsilon)(\eta_1) + \eta_2 + c(\epsilon) = b^{-\epsilon}\eta_1 + K(\epsilon)\eta_2 + K(\epsilon)c(\epsilon)$$

Therefore as linear transformations, what has to be shown is that

$$\tilde{\sigma}\rho(b,0)\rho(1,-eb^{-1})\tilde{\sigma} = K(\epsilon)\rho(1,eb^{-1})\tilde{\sigma}\rho(1,eb).$$

For $\eta_2$, observe that,

$$\tilde{\sigma}\rho(b,0)\rho(1,-eb^{-1})\tilde{\sigma}(\eta_2) = b^{\epsilon}\eta_2$$

and that

$$\rho(1,eb^{-1})\tilde{\sigma}\rho(1,eb)(\eta_2) =$$

$$\rho(1,eb^{-1})\tilde{\sigma}\left(K(\epsilon)\eta_1 + \eta_2 + c(\epsilon)\right) =$$

$$\rho(1,eb^{-1})\left(b^{\epsilon}K(\epsilon)\eta_1 + b^{-\epsilon}\eta_2 + b^{-\epsilon}c(\epsilon)\right) =$$

$$(1 + K(\epsilon))K(\epsilon)\eta_1 + \langle Ac(\epsilon),c(\epsilon)\rangle \eta_1 + K(\epsilon)\eta_2 + K(\epsilon)c(\epsilon) + \pi(\epsilon)c(\epsilon)Ac(\epsilon).$$

Therefore, by Lemma 2.2.9,

$$\tilde{\sigma}\rho(b,0)\rho(1,-eb^{-1})\tilde{\sigma}(\eta_2) = K(\epsilon)\rho(1,eb^{-1})\tilde{\sigma}\rho(1,eb)(\eta_2).$$

And last, for every $d \in \mathbb{R} \setminus \{0\},$

$$\tilde{\sigma}\rho(b,0)\rho(1,-eb^{-1})\tilde{\sigma}(d) =$$

$$K(d)\tilde{\sigma}\rho(b,0)\rho(1,-eb^{-1})c(d^{-1}) =$$

$$K(d)\tilde{\sigma}\rho(b,0)\left(\langle c(-d^{-1}),c(\epsilon)\rangle \eta_1 + \pi(-eb^{-1})c(-d^{-1})\right) =$$

$$K(d)\left(b^{\epsilon}\langle c(-d^{-1}),c(\epsilon)\rangle \eta_1 + \pi(b,0)\pi(-eb^{-1})c(-d^{-1})\right) =$$

$$K(d)\left(b^{\epsilon}\langle c(-d^{-1}),c(\epsilon)\rangle \eta_2 + A\pi(b,0)\pi(-eb^{-1})c(-d^{-1})\right).$$
On the other hand,

\[\rho(1, eb^{-1}) \bar{\sigma} \rho(1, eb)(c(d)) = \]
\[\rho(1, eb^{-1}) \bar{\sigma} \left( \langle c(d), c(-eb) \rangle \eta_1 + \pi(eb)c(d) \right) = \]
\[\rho(1, eb^{-1}) \left( \langle c(d), c(-eb) \rangle \eta_2 + A \pi(eb)c(d) \right) = \]
\[\langle c(d), c(-eb) \rangle K(eb^{-1}) + \langle A \pi(eb)c(d), c(-eb^{-1}) \rangle \eta_1 + \]
\[\langle c(d), c(-eb) \rangle \eta_2 + \langle c(d), c(-eb) \rangle c(eb^{-1}) + \pi(eb^{-1})A \pi(eb)c(d).\]

Therefore the claim is that

\[K(-e) \left( \langle c(d), c(-eb) \rangle \eta_2 + \langle c(d), c(-eb) \rangle c(eb^{-1}) + \pi(eb^{-1})A \pi(eb)c(d) \right) = \]
\[K(d) \left( b^t \langle c(-d^{-1}), c(eb^{-1}) \rangle \eta_2 + A \pi(b, 0) \pi(-eb^{-1})c(-d^{-1}) \right).\]

Observe that

\[K(e)K(d)b^t \langle c(-d^{-1}), c(eb^{-1}) \rangle = \]
\[\langle K(d)c(-d^{-1}), K(-eb)c(b^{-1}) \rangle = \]
\[\langle Ac(d), Ac(-eb) \rangle = \langle c(d), c(-eb) \rangle.\]

Therefore

\[K(-e) \langle c(d), c(-eb) \rangle = K(d)b^t \langle c(-d^{-1}), c(eb^{-1}) \rangle.\]

The only identity remaining to show is that

\[K(d)A \pi(b, 0) \pi(-eb^{-1})c(-d^{-1}) = \]
\[K(-e) \left( \langle c(d), c(-eb) \rangle c(eb^{-1}) + \pi(eb^{-1})A \pi(eb)c(d) \right).\]

Suppose \(0 \neq d \neq eb\). Notice that

\[\pi(eb^{-1})A \pi(eb)c(d) = \]
\[\pi(eb^{-1})A \left( c(eb + d) - c(eb) \right) = \]
\[\pi(eb^{-1}) \left( K(eb + d) c(-eb + d)^{-1} - K(eb)c(-eb^{-1}) \right) = \]
\[K(eb + d) \left( c \left( \frac{d}{eb + d} \right) - c(eb^{-1}) \right) + K(eb)c(eb^{-1}) = \]
\[\left( K(eb) + K(d) + \langle c(d), c(-eb) \rangle \right) \left( c \left( \frac{d}{eb + d} \right) - c(eb^{-1}) \right) + K(eb)c(eb^{-1}) = \]
\[K(eb + d) c \left( \frac{d}{eb + d} \right) - \left( K(d) + \langle c(d), c(-eb) \rangle \right) c(eb^{-1}).\]

Therefore if \(R = K(-e)\pi(eb^{-1})A \pi(eb)c(d),\)

\[K(-e) \left( \langle c(d), c(-eb) \rangle c(eb^{-1}) + \pi(eb^{-1})A \pi(eb)c(d) \right) = \]
\[K(-e) \langle c(d), c(-eb) \rangle c(eb^{-1}) + R = \]
\[K(-e) \left( K(eb + d) c \left( \frac{d}{eb + d} \right) - K(d) c(eb^{-1}) \right).\]

On the other hand

\[A \pi(b, 0) \pi(-eb^{-1})c(-d^{-1}) = \]
\[A \pi(b, 0) \left( c \left( -\frac{eb}{eb(d) - c(eb^{-1})} \right) - c(-eb^{-1}) \right) = \]
\[b^{-t} A \left( c \left( -\frac{eb}{eb(d) - c(eb^{-1})} \right) - c(-eb) \right) = \]
\[b^{-t} \left( K \left( -\frac{eb}{eb(d) - c(eb^{-1})} \right) c \left( \frac{d}{eb(d)} \right) - K(-eb)c(eb^{-1}) \right) = \]
\[K \left( -\frac{eb}{eb(d) - c(eb^{-1})} \right) c \left( \frac{d}{eb(d)} \right) - K(-eb)c(eb^{-1}).\]
Therefore, what is left is to show that
\[
K(d) \left( K\left(-\frac{\epsilon(d)b}{d}\right)c\left(\frac{d}{\epsilon(d)b}\right) - K\left(-\epsilon\right)c\left(e^{-1}\right) \right) = \\
K\left(-\epsilon\right) \left( K\left(e + d\right)c\left(\frac{d}{\epsilon(e+b-d)}\right) - K\left(d\right)c\left(e^{-1}\right) \right),
\]
which is equivalent to show that
\[
K(d)K\left(-\frac{\epsilon d - b}{d}\right) = K\left(-\epsilon\right)K\left(e + d\right).
\]
This can be easily proved considering all the different cases.

Define
\[
f_{eb}(d) = K(d)A\pi(b,0)\pi(-e^{-1})c(-d^{-1})
\]
and
\[
g_{eb}(d) = K\left(-\epsilon\right) \left( \langle c(d), c(-eb)\rangle c(e^{-1}) + \pi(e^{-1})A\pi(e)b(c(d)) \right).
\]

Observe that for a given value \(b_0 \in \mathbb{R} \setminus \{0\}\), the functions \(f_{b_0}(d)\) and \(g_{b_0}(d)\) are continuous on \(d\) and such that \(f_{b_0}(0) = 0 = g_{b_0}(0)\). It has been shown that for every \(0 \neq d \neq eb_0\), \(f_{b_0}(d) = g_{b_0}(d)\), therefore by continuity \(f_{b_0} = g_{b_0}\).

This concludes the proof for the equalities,
\[
\rho(b,0) = \tilde{\sigma}\rho(1,b^{-1})\tilde{\sigma}\rho(1,b)\tilde{\sigma}\rho(1,b^{-1}) = \tilde{\sigma}\rho(1,-b^{-1})\tilde{\sigma}\rho(1,-b)\tilde{\sigma}\rho(1,-b^{-1}).
\]

The previous lemma completes the argument that shows that
\[
SU(1,1) \xrightarrow{T} \text{Isom}(H_C^\infty)
\]
is a homomorphism. The next theorem deals with the continuity and fixed point properties of \(T\).

**Theorem 2.2.11.** The map \(T\) induces an irreducible (orbitally continuous) representation \(\rho\) of \(\text{Isom}(H^1_C)\) into \(\text{Isom}(H^\infty_C)\) with \(\ell(\rho) = t\) and
\[
\text{Arg}(\rho) = \text{Arg}(K_1(1) + K_2(1)).
\]

**Proof.** Observe that \(T\) does not have fixed points in \(H^\infty_C \cup \partial H^\infty_C\) because \(\sigma\) does not fix \(\eta_1\) (see Lemma 2.2.6). If \(T\) preserves a geodesic, then it permutes the two limits of it, but this is a contradiction because every homomorphism \(SU(1,1) \to \mathbb{Z}_2\) is constant.

Let \(SU(1,1) \xrightarrow{T} \text{Isom}(H^1_C)\) be the projectivization map. The group \(\pi(P)\) is closed in \(\text{Isom}(H^1_C)\) and, by Proposition 1.1.2, there is a decomposition
\[
\text{Isom}(H^1_C) = \pi(PsP) \cup \pi(P).
\]
Therefore \(\pi(PsP)\) is an open neighborhood of \(Id \in \text{Isom}(H^1_C)\). Thus, it is enough to show that, if \((g_j)\) is a sequence in \(\pi(PsP)\) such that \((g_j) \to \pi(s)\), then for every \(x \in H^\infty_C\),
\[
\rho(g_j)x \to T(s)x = \tilde{\sigma}x.
\]
Observe that every element of $PsP$ can be written as

$$g(\lambda, b)sg(1, d) = \begin{pmatrix} -b & i(\lambda - bd) \\ i\lambda^{-1} & -\lambda^{-1}d \end{pmatrix}. $$

If $g_j = \pi(g(\lambda_j, b_j)sg(1, d_j))$, then $b_j \to 0$, $\lambda_j \to 1$ and $d_j \to 0$. Therefore, for every $x \in H_\infty^\infty$, $\rho(\lambda_j, b_j)x \to x$ and $\rho(1, d_j)x \to x$, hence with a triangle inequality argument it is possible to conclude that $g_jx \to \sigma x$.

The irreducible part of $\rho$ has to contain the axis (and its limits) preserved by the maps $\rho(\lambda, 0)$, therefore $\rho$ is irreducible by construction.

2.3 A new family of representations

With the results of the previous subsection a continuum of non-equivalent representations will be constructed.

Given an irreducible representation $\rho$ denote $K(1) = K(1)_\rho$. If $p, q \in \mathbb{R} > 0$ and $\rho, \tau : \text{Isom}(H^\infty_\infty) \to \text{Isom}(H^\infty)$ are two irreducible representations such that $\ell(\rho) = \ell(\tau) = t$, let $\rho_\rho$ and $\tau_\tau$ be two irreducible representations, equivalent to $\rho$ and $\tau$ respectively, such that $|K(1)_{\rho_\rho}| = p$ and $|K(1)_{\tau_\tau}| = q$. Observe that with the procedure describe in Theorem 2.2.11 it is possible to obtain an irreducible representation $\omega$ such that $\ell(\omega) = t$ and

$$\text{Arg}(\omega) = \text{Arg} \left( \frac{pK_{\rho}(1)}{|K_{\rho}(1)|} + \frac{qK_{\tau}(1)}{|K_{\tau}(1)|} \right).$$

Therefore for every

$$s \in [\min\{\text{Arg}(\rho), \text{Arg}(\tau)\}, \max\{\text{Arg}(\rho), \text{Arg}(\tau)\}]$$

there is an irreducible representation $\phi$ such that $\ell(\phi) = t$ and $\text{Arg}(\phi) = s$.

Given $u \in [0, 1]$, denote $\rho \wedge_{u} \tau$ the irreducible representation such that $\ell(\rho \wedge_{u} \tau) = t$ and

$$\text{Arg}(\rho \wedge_{u} \tau) = (1 - u)\text{Arg}(\rho) + u\text{Arg}(\tau).$$

This representation will be called a horospherical combination of $\rho$ and $\tau$.

The representation $\rho \wedge_{u} \tau$ is well defined in the following sense. If $\ell(\rho) = \ell(\tau)$ and $\rho'$ and $\tau'$ are equivalent to $\rho$ and $\tau$ respectively, then $\rho \wedge_{u} \tau$ is equivalent to $\rho' \wedge_{u} \tau'$ (see Theorem 2.1.6).

Although in the definition of the horospherical combination, for simplicity, the representations were supposed acting on the same hyperbolic space, nothing prevents to define the horospherical combination of two irreducible representations with one possibly having finite-dimensional target. This could be the case, by Mostow-Karpelevich theorem or in particular by Lemma 1.3.7 if $t = 1$.

Using the families constructed in [12] and [13] and the horospherical combination a new family of non-equivalent representations is built.
Recall that for every $0 < t < 2$ on one hand, up to a conjugation, there exists a unique irreducible representation
\[ \text{Isom}(H^1_C)_o \xrightarrow{\rho_t} \text{Isom}(H^\infty_C)_o \]
such that $\ell(\rho_t) = t$ and that preserves a real hyperbolic space. These representations are such that $\text{Arg}(\rho_t) = 0$ (see Lemma 1.3.5, the comments before it and Theorem 2.1.6). On the other hand, for every $0 < t < 1$ there exists an irreducible representation
\[ \text{Isom}(H^1_C)_o \xrightarrow{\tau_t} \text{Isom}(H^\infty_C)_o \]
such that $\text{Arg}(\tau_t) = \frac{t\pi}{2}$ and $\ell(\tau_t) = t$ (see Proposition 2.1.4 and the comments before it).

**Theorem 2.3.1.** If $0 < t < 1$ and $r \in [0, t\pi/2]$ or if $t = 1$ and $r \in [0, \pi/2)$, there exists a unique, up to a conjugation, irreducible representation $\rho_{t,r}$ such that $\text{Arg}(\rho_{t,r}) = r$ and $\ell(\rho_{t,r}) = t$.

**Proof.** For $t < 1$, consider the family of irreducible representations $\rho_t \land \tau_t$.

For $t = 1$, let $id$ be the identity map $\text{Isom}(H^1_C)_o \rightarrow \text{Isom}(H^\infty_C)_o$. Observe that for every $u \in [0, 1)$, by construction the representation $\rho_t \land id$ is irreducible and the target is an infinite-dimensional complex hyperbolic space. \hfill \Box

By Lemma 2.1.2 the representations listed in the previous theorem are representatives of all the irreducible representations of $\text{Isom}(H^1_C)_o$ into $\text{Isom}(H^\infty_C)_o$ with displacement 1.

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