Elementary Particles as Solutions of a 4-Dimensional Source Equation

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The author discusses particular solutions of a second order equation designated by source equation. This equation is special because the metric of the space where it is written is influenced by the solution, rendering the equation recursive. The recursion mechanism is established via a first order equation which bears some resemblance to Dirac equation. In this paper the author limits the discussion to solutions with constant norm but makes use of 4-dimensional hypercomplex numbers in matrix representation, a concept that is formally introduced in a section devoted to that aspect. The particular solutions that are found exhibit symmetries that can be assigned to spin, electric and color charges of elementary particles, leaving mass as a free parameter. Massless particles can also be assigned to special solutions of the source equation, with the cases of photons, gluons and gravitons clearly identified, together with another massless particle which does not seem to be related to anything detected experimentally. Another section deals with particle dynamics under fields, showing that both gravitational and electrodynamics can be modelled by geodesics of the spaces whose metric tensors result from the recursion mechanism. Finally the author suggests two lines of future work, one deriving fields from densities and currents of masses and charges and the other one aimed at determining particles’ masses.

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I. BACKGROUND

One can conceive of a very large computer with the sole task of solving a small number of simultaneous differential equations, with the only itch that the equations are evolutionary, meaning that in the course of time their form evolves as a result of previously found solutions. Assuming that the problem could be appropriately formulated, this would create an evolutionary system of equations whose complexity would surely outgrow the computing power of any machine. In a science-fiction pass one could also assume the computer to be itself a product of the equations, such that its computing power would grow in parallel to the equations’ complexity. The author proposes that the Universe as a whole is such a system, whose evolution can be understood up to our ability to understand the basic equations. We will deal with evolutionary equations by writing non-linear equations in 4-dimensional space; the solutions to these will be seen as evolutionary when 3-dimensional space is considered. To this effect we will focus on one particular form of the basic equations as such equations are not known to us in general, although we can discuss some of their characteristics. This particular form is designated by source equation in Ref. [1] because it is the source of dynamics and dynamic space, an essential concept introduced in the reference cited earlier. We believe the source equation to be a condensation of the basic equations in a similar way to which Klein-Gordon and Helmholtz equations are condensations of Dirac and Maxwell equations [2, 3]. The reader may also find useful a look at Ref. [4] as this is a precursor of the present paper, establishing most of the assumptions used here, including setting the problem in a very specific 4-dimensional space, with the justification for such an approach.

This discussion is about a mathematical problem, i.e. without any preexisting physical assumptions: All the variables in the equations are pure numbers, unless clearly stated otherwise, so the question of dimensional homogeneity is never an issue. This facet of the issues at hand is detailed in Ref. [1] where it was shown that dimensions and physical interpretation can be introduced at a later stage, simultaneously with the definition of the physical constants \( c, G, \hbar \) and \( \epsilon \).

The general form of the source equation is

\[
g^{\mu\nu} \nabla_\mu \psi = - \psi, \tag{1}
\]

where \( g^{\mu\nu} \) is a space metric, \( \psi \) is a function whose mathematical characteristics will be discussed in the following section, and where \( \nabla_\mu \) designates the covariant derivative with respect to the coordinate \( x^\mu \). We use Einstein’s summation convention, with Greek indices and superscripts taking values between 0 and 3 and Roman ones running from 1 to 3. It is presumed that the basic equations of the dynamics will provide a means of determining \( g^{\mu\nu} \) from \( \psi \), in such a way that the successive equation-solution sets resulting from Eq. (1) become coupled in a recursive manner, i.e. the solutions become self sustained. This process then links the space metric to the dynamics, defining what we have designated as dynamic space, a space which contains and is generated by the dynamics. The space metric will be considered separable in two components: an “inertia” reflecting the influence of the particular point on the dynamic space whose coordinates are under consideration, and a “field” reflecting the influence of all other points.
It is important to emphasize that dynamic space excludes anything that does not participate in the dynamics under study. If we take the viewpoint of Physics for one moment, at the Universe’s scale everything participates in the grand dynamics and is contained in the Universe’s dynamic space. When we decide to study some particular dynamics isolated from the rest of the Universe, this dynamics defines a space for itself, which excludes everything else. Influences from other dynamic spaces can be accounted for via the field component of the metric. Consider the Earth’s orbit around the Sun: We have the choice of using the dynamic space defined by the two bodies, or the dynamic space of the Earth alone, influenced then by a field component due to the Sun. These two approaches are equivalent if within the dynamic space concept we take universal space as a superposition of dynamic spaces, within which we can choose the components for the need of a particular study.

In connection with dynamic space within the above view of superposed spaces, we define also an observer space, which is by definition obtained through a coordinate change that removes the inertia component from the metric of the dynamic space under consideration by the observer. Observer space can be then seen as the result of a continuum of superposed dynamic spaces where each point can be mapped to a point of one particular dynamic space via inserting an inertia component in the dynamics equation of that space (as discussed in Ref. [1]). In Physics, imagine one galaxy as dynamic space: When the galaxy’s dynamics is studied in isolation its dynamic space extends to infinity. With the inertia component related to mass density, and with this density tending to zero with distance, the conversion to observer space confines the galaxy to a limited region of observer space. A similar link between dynamic and observer spaces can be applied to planetary systems, to the atom, and eventually to particles.

This paper is the first in a series which will examine some mechanisms of dynamic space definition in a gradual process which has the potential to lead to universal space, starting here with “elementary” particles. Next section deals with the mathematical characteristics of the function \( \psi \) as the solution of Helmholtz equation, also known as the “standing wave” equation, itself identified as contained in the source equation. We examine first 2- and 3- dimensional settings, with the ultimate intent to obtain a 4-dimensional formalism adequate for expressing the source equation solutions and metric. Once such solutions are found, their physical interpretation with and without fields follows to confirm their adequacy by leading to key features of gravitation and electromagnetism. Particles involved with chromodynamics will only be introduced, without the necessary dynamic details.

II. 4-DIMENSIONAL COMPLEX NUMBERS

A complex number can be represented in the exponential form by

\[
\psi = me^{i\theta}
\]

or equivalently in real-imaginary form as

\[
\psi = m(\cos \theta + i \sin \theta),
\]

with \( m \) a positive number called the norm or modulus of the complex number. Either form is well suited to represent harmonic functions of one variable relative to another by taking just the real part. In the second form the variable \( \psi \) as a harmonic function would be dependent on the variable \( \theta \) by the relation \( \psi = m \cos \theta \), with the first form representing solutions

\[
\psi = me^{\pm iux}
\]

of Helmholtz equation in one dimension

\[
\frac{d^2}{dx^2} \psi = -v^2 \psi,
\]

with \( \theta = ux \).

A complex number is also appropriate for representing 2-dimensional rotations since such rotations are harmonic relations between two orthogonal directions of space. In this interpretation a variation of coordinates in one of the directions is represented by the real part, while the coefficient of the imaginary part deals with the orthogonal direction. If the two coordinates are associated as a complex number

\[
x = x^1 + ix^2,
\]

the rotated coordinates are given by

\[
x' = \psi x,
\]

which can be expressed in complex form as,

\[
x' = x^1 \cos \theta - x^2 \sin \theta + i \left( x^1 \sin \theta + x^2 \cos \theta \right).
\]

Complex numbers with norm unity belong to a group usually designated by \( U(1) \), isomorphic to the group of 2-dimensional rotations \( SO(2) \), whereby rotations in opposite directions are represented by plus and minus signs added to the exponent of their exponential form as shown in Eq. (4), and a \( 2\pi \) rotation in either direction is equivalent to no rotation at all, i. e. the null rotation, with the exponential replaced by unity, and is commonly referred as a “full” rotation.

In order to generalize complex numbers we designate by imaginary unit any traceless matrix \( u \) that verifies the condition

\[
u^2 = -I,
\]
then an imaginary matrix is defined as the product of a real number by an imaginary unit; the following relation is always verified $\dddot{8}$:

$$e^{\theta a} = \frac{e^{\theta a} + e^{-\theta a}}{2} + \frac{e^{\theta a} - e^{-\theta a}}{2} = \cos \theta + u \sin \theta.$$  (10)

In a similar formal manner, we can define also traceless matrices $h$ designated Hermitian units such that $h^2 = I$. For such matrices we have

$$e^{\theta h} = \frac{e^{\theta h} + e^{-\theta h}}{2} + \frac{e^{\theta h} - e^{-\theta h}}{2} = \cos \theta + h \sin \theta.$$  (11)

Rotations in 3 dimensions are associated with hypercomplex numbers called quaternions (or 3D complex numbers) seen as an extension of complex numbers to 3-dimensional space $\dddot{9}$. We will define them through the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$  (12)

complemented with $I$ as the identity matrix. They verify the well-known unitary and anti-commutator relations

$$\sigma^i \sigma^j + \sigma^j \sigma^i = 2 \delta^{ij} I,$$  (13)

with $\delta^{ij}$ being the Kronecker delta, making them Hermitian units as well as imaginary units when multiplied by $i$.

When multiplied by $i$ they do form a quaternion representational basis as defined by Hamilton $\dddot{10}$ if the quaternion product is simultaneously defined by:

$$i \sigma^j \otimes i \sigma^k = \begin{cases} \sigma^j \sigma^k, & j \neq k; \\ -I, & j = k. \end{cases}$$  (14)

A unitary $2 \times 2$ matrix representing unitary 3D complex numbers has the form

$$\psi = e^{ia_j \sigma^j},$$  (15)

with the $a_j$'s real number components of a vector $\vec{a}$ with length $\theta = (\delta^{ij} a_i a_j)^{1/2}$. From Eq. (14), Eq. (15) is equivalent to

$$\psi = I \cos \theta + i \frac{a_j \sigma^j}{\theta} \sin \theta,$$  (16)

which is a unitary quaternion, with the identity matrix $I$ as the 0th order of its matrix representation. It is associated with a rotation of an angle $2 \theta$ about the direction of unit vector $\vec{a}/\theta$, as we shall see below.

Defining the Hermitian coordinate matrix of a point $(x^1, x^2, x^3)$ in 3D via the Pauli matricial basis

$$x = i \sum x^j \sigma^j = \begin{pmatrix} ix^3 \\ -x^2 + ix^1 \\ -ix^3 \end{pmatrix},$$  (17)

we can see that the coordinates remain on a sphere of squared radius $\det x = \delta_{ij} x^j x^i$ when transformed by the matricial similarity transformation

$$x' = \psi x \psi^{-1} = \psi x \psi^\dagger,$$  (18)

where $\psi^\dagger$ is the Hermitian conjugate of $\psi$; notice that the second equality results from $\psi$ being a unitary matrix. The transformation above performs a generic rotation in 3D containing the earlier 2D rotations with double their angle as we can see by taking a vector $\vec{a}$ along $\sigma^i$ (only $a_1$ is non-zero) and using Eqs. (10) and (11) in the similarity Eq. (18). Coordinate $x^3$ is then unchanged, and the other coordinates transform as follows:

$$x'^2 = x^2 \cos(2\theta) - x^3 \sin(2\theta),$$
$$x'^3 = x^2 \sin(2\theta) + x^3 \cos(2\theta).$$  (19)

The same relation holds true when a circular permutation of the indices is performed. This result means that a $4\pi$ rotation angle is induced or two full rotations, for one cycle of the exponent $\dddot{11}$. The group of $2 \times 2$ unitary matrices defined by Eq. (13) is designated by $U(2)$, with $SU(2)$ being the sub-group of those matrices whose determinant is unity $\dddot{12}$ and the group of real orthogonal matrices of dimension 3, representing 3D rotations, is designated by $SO(3)$. It is said that $SU(2)$ is a double cover or two-fold covering of $SO(3)$ in view of the angle doubling referred above $\dddot{13}$.

Generalization of complex numbers to 3 dimensions implies that we consider also non-unitary matrices; here the situation is more complicated than in 2 dimensions because the entities we are defining are $2 \times 2$ matrices and so must also be whatever generalizes the modulus of a standard complex. Eq. (15) can be generalized with the insertion of a traceless Hermitian matrix before the exponential; this factor is designated as modulus of the 3D complex, which we will represent by $|\psi|$: $\dddot{14}$

$$|\psi| = |\psi| e^{ia_j \sigma^j}.$$  (20)

The modulus relates to the norm $||\psi||$ through

$$|\psi|^2 = \psi^\dagger \psi = |\psi|^2 |\psi| = ||\psi||^2.$$  (21)

We will impose a further restriction on the modulus by taking the norm as a positive real number:

$$|\psi|^2 |\psi| = m^2.$$  (22)

Now we are ready to verify that the 3D complexes defined through Eq. (20) can be used as solutions of a 3-dimensional Helmholtz equation. Consider the following equation with $\psi$ taken as a standard complex number function of coordinates $x^j$ and $\vec{v}$ a vector of components $v_j$ and norm $v$:

$$\delta^{jk} \partial_{jk} \psi = -v^2 \psi;$$  (23)

usually the solution to this equation is written

$$\psi = |\psi| e^{i x^i v_j x^j}.$$  (24)
where we have ignored an arbitrary phase factor and

\[ v^2 = \delta^{jk} v_j v_k. \tag{25} \]

This solution represents a plane standing wave with \( \psi \) seen as defining the orientation of the wave. We call standing wave to a standing harmonic 3-dimensional pattern. Imagine that \( \psi \) represents light intensity: The space will then be seen as alternating between bright and dark along the wave direction; by defining planes normal to that direction spaced at half the wavelength we could split 3-dimensional space into alternating bright and dark zones. Using instead \( \psi \) as a 3D complex we can write different solutions for the same equation \( \psi \):

\[ \psi = |\psi|e^{\pm iv_j\sigma^j x^j}. \tag{26} \]

We must verify that Eq. \( \psi \) together with Relations \( \Delta \) and \( \nabla \) is a solution of Eq. \( \Delta \). The partial derivatives of \( \psi \) are of the form

\[ \partial_j \psi = \pm |\psi| iv_j \sigma_j \exp(\pm iv_k \sigma^k x^k) = \pm iv_j \sigma_j \psi \tag{27} \]

where we have lowered the index of \( \sigma_j \) in order to avoid summation over \( j \). The second derivative in \( x^j \) of \( \psi \) produces a term \( \sigma_j \sigma_j \), and considering Relations \( \Delta \), we get

\[ \partial_{jj} \psi = -(v_j)^2 \psi. \tag{28} \]

When the three second derivatives are added we obtain Condition \( \nabla \), and thus Eq. \( \Delta \) is verified.

With either complex or 3D complex solutions for \( \psi \) the general solution of Eq. \( \Delta \) is obtained through a superposition of positive and negative exponential solutions with different phase factors. The types of solutions, though, are different: Both equations represent standing plane waves but Eq. \( \psi \) is a associated to spinning of the points along the wave direction at twice the wave frequency; we will now call this a spin 1/2 wave. General solutions with 3D complex numbers are obtained by superposition of spin 1/2 waves; the norm of the 3D complex solution is interpreted as the wave amplitude, while the modulus contains information about its polarization. Spin 1/2 waves are especially indicated to describe axially guided waves, while no-spin waves, described by complex numbers, are better suited to describe unguided waves.

The question remains about the possibility of defining hypercomplex numbers representing solutions of Helmholtz equation in 4 dimensions. It is impossible to add a 4th imaginary unit on a par with the 3 quaternion units we used and end up with a division algebra \( \mathbb{R} \); this does not mean, however, that we cannot define 4-dimensional complex numbers. We will start with a discussion about 4-dimensional rotations, since in 2 and 3 dimensions complex numbers have been associated to rotations and we are led to expect a similar relationship in 4D.

Rotations in n-dimensional space are isomorphic to group \( SO(n) \) and determined by \( n(n - 1)/2 \) parameters.

\[ 8 \ 3 \ 11, \] usually chosen as rotation angles; this means that in 4 dimensions we have to select different angles to characterize all possible rotations. There are 6 very special 4D rotations which preserve two of the coordinates while rotating the other two. For instance the orthogonal 4D rotations which preserve two of the coordinates along the wave direction; by defining planes normal to that direction spaced at half the wavelength we could split 3-dimensional space into alternating bright and dark zones. Using instead \( \psi \) as a 3D complex we can write different solutions for the same equation \( \psi \):

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \theta & -\sin \theta & 0 \\
0 & \sin \theta & \cos \theta & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \tag{29}
\]

\[
\begin{pmatrix}
\cos \theta & 0 & 0 & -\sin \theta \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\sin \theta & 0 & 0 & \cos \theta
\end{pmatrix} \tag{30}
\]

When right multiplied by the column vector of the coordinates these special matrices rotate two coordinates by an angle \( \theta \) while preserving the other two coordinates.

Group \( SO(4) \) is doubly covered by group \( SU(2) \times SU(2) \) in a similar way to the double covering of \( SO(3) \) by \( SU(2) \); this suggests that we can replace Pauli \( \sigma \) matrices by Dirac matrices built from them through matrix direct products. There are 15 of such matrices excluding the identity which can be generated by the formulas

\[ \lambda^j = I_2 \otimes \sigma^j, \]

\[ \rho^j = \sigma^j \otimes I_2, \tag{31} \]

where \( I_2 \) is the \( 2 \times 2 \) identity matrix and \( A \otimes B \) is the matrix direct product. Explicitly this set of Dirac matrices is given by

\[
\lambda^1 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \lambda^2 = \begin{pmatrix}
0 & -i & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & i \\
0 & 0 & -i & 0
\end{pmatrix},
\]

\[
\lambda^3 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \rho^1 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix},
\]

\[
\rho^2 = \begin{pmatrix}
0 & 0 & -i & 0 \\
0 & 0 & 0 & i \\
i & 0 & 0 & 0 \\
0 & i & 0 & 0
\end{pmatrix}, \quad \rho^3 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0
\end{pmatrix}.
\]

The complete set of 16 Dirac matrices is generated by

\[ E^{\mu \nu} = \rho^\mu \lambda^\nu, \tag{32} \]

where \( \lambda^0 = \rho^0 = I \).

The set of 16 Dirac matrices describes 15-dimensional space in the same way as the set of 4 Pauli matrices describes 3-dimensional space. The source equation and the 4D Helmholtz equation don’t require this high dimensionality because they can be associated to 4D rotations and hence to group \( SU(2) \times SU(2) \), which can be represented
by Dirac matrices if appropriate relations are introduced between its elements, effectively reducing the dimensionality from 15 to 4. The problem we will be facing is more complex, however because the source equation is obtainable from a first order equation in much the same way as Klein-Gordon equation can be obtained from Dirac equation; the process is the same that was used in Ref. 1. From the cited reference we recover the first order equation, here in a slightly modified form

\[ i s^\mu \nabla_\mu \psi = \psi. \]  

(33)

where \( s^\mu \) are 4 \( \times \) 4 Hermitian matrices defined by

\[ s^0 = \frac{\lvert \psi \rvert}{\lvert \psi \rvert^2}, \quad s^j = \frac{\alpha^j}{\lvert \psi \rvert}; \]  

(34)

where we define 4 \( \alpha^\mu \) matrices by

\[ \alpha^0 = E^{10}, \quad \alpha^j = E^{3j}, \]  

(35)

which verify the relation

\[ \alpha^\mu \alpha^\nu + \alpha^\nu \alpha^\mu = 2\delta^{\mu\nu}. \]  

(36)

Relation (21) is used for the construction of \( s^\mu \). This is a more general procedure than was used in Ref. 1.

Applying the operator \( is^\mu \nabla_\mu \) to both members, and assuming the condition

\[ \nabla_\sigma s^\mu = 0, \]  

(37)

the space metric of Eq. 11 generated by the \( s^\mu \) matrices is:

\[ g^{\mu\nu} = s^\mu s^\nu. \]  

(38)

The condition imposed by Eq. 37 automatically verifies Riemann’s condition \( \nabla_\sigma g^{\mu\nu} = 0 \) although the converse is not necessarily true.

There is an apparent inconsistency in Eq. 38 because \( s^\mu s^\nu \neq s^\nu s^\mu \) and thus the metric is defined as a non-symmetric tensor of 4 \( \times \) 4 matrix elements. However, in many circumstances, as will be the case in this section, the metric will be anti-symmetric and cancellation between corresponding terms will allow a diagonal metric to be used.

Consider here expressions of the type \( g^{\mu\nu} \nabla_{\mu\nu} \), which imply an addition over all the \( \mu\nu \) combinations; since the terms \( \mu\nu \) and \( \nu\mu \) are symmetric and cancel each other, the expression is equivalent to what would be obtained with a diagonal metric. In index raising or lowering operations it is not indifferent whether the metric is symmetric or anti-symmetric; we will always consider the metric to be defined by Eq. 38 followed by elimination of anti-symmetric elements, making this elimination an integral part of dynamic space construction.

In order to verify Eq. 34 and considering relations 21 we must try solutions of the type

\[ \psi = \psi_0 e^{p_\mu \alpha^\mu x^\mu}, \]  

(39)

where we have used \( \psi_0 \) to denote the modulus and \( x^0 \) dependence of \( \psi \). Solutions of the form given above can be inserted into Eq. 36 resulting in

\[ \frac{i \lvert \psi \rvert \nabla_0 \psi}{\lvert \psi \rvert^2} - \sum a^\mu_j \psi = \psi, \]  

(40)

leaving only the \( x^0 \) dependence to be contended with.

We will define 4D complex numbers to be 4 \( \times \) 4 matrices, solutions of Eq. 38, of the form

\[ \psi = \lvert \psi \rvert e^{p_\mu u^\mu x^\mu}, \]  

(41)

where \( u^j = i \alpha^j \) and \( u^0 \) is some imaginary unit independent of the other 3. If we were only concerned with the simplified source equation above any Hermitian matrix with a trace of 4m could be used as modulus; this does not occur when we consider also Eq. 34. For simplicity we start by examining the situations where \( p_0 = m, p_j = 0 \); equation 35 is simplified to

\[ is^0 \nabla_0 \psi = \psi. \]  

(42)

Considering the definition of \( s^0 \) in Eq. 34 and noting that the covariant derivative of \( \psi \) is equal to its partial derivative

\[ i \lvert \psi \rvert m u^0 \psi = m^2 \psi, \]  

(43)

where we have made \( \lvert \psi \rvert = m \) and multiplied both members by \( m^2 \) to remove the denominator. This equation is verified if

\[ \lvert \psi \rvert = im u^0. \]  

(44)

We are now in position to remove the simplifying condition inserted above and write the general solution

\[ \psi = im u^0 e^{s_p_\mu u^\mu x^\mu}, \]  

(45)

where \( s \) is a sign factor with the values \( \pm 1 \) and necessarily Condition 12 is verified.

The extra imaginary unit \( u^0 \) needed to build a 4-dimensional complex number is built with recourse to matrices \( E^{2j} = \rho^2 \lambda^j \). Any combination \( i q_j \rho^2 \lambda^j / n_q \), with \( q_j \) equal to 0 or 1 and \( (n_q)^2 \) the number of non-zero \( q_j \) produces a suitable imaginary unit. Such combinations bear strong links with octonions 11. Assuming justifiable special rules on matrix multiplication that will ensure non-associativity as required by the algebra, we can
define an octonion basis as follows
\begin{align*}
e_1 &= i\rho^2\lambda^1, \\
e_2 &= i\rho^2\lambda^2, \\
e_3 &= i\rho^2\lambda^3, \\
e_4 &= \frac{i\rho^2 (\lambda^1 + \lambda^2)}{\sqrt{2}}, \\
e_5 &= \frac{i\rho^2 (\lambda^2 + \lambda^3)}{\sqrt{2}}, \\
e_6 &= \frac{i\rho^2 (\lambda^1 + \lambda^2 + \lambda^3)}{\sqrt{3}}, \\
e_7 &= \frac{i\rho^2 (\lambda^1 + \lambda^3)}{\sqrt{2}}.
\end{align*} (46)

The multiplication table for octonions can be inferred from Fig. 1 where the arrows indicate directions of forward cycling, i.e., directions where sign is maintained; for instance \(e_1e_2 = e_4\) or \(e_4e_5 = e_3\). In the reverse direction the sign of the result is negative.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{Fano Plane.png}
\caption{Fano plane: the arrows indicate the directions of forward cycling; there are unseen arrows closing the cycles, for instance between \(e_4\) and \(e_5\) or \(e_3\) and \(e_5\).}
\end{figure}

Using the multiplication table and the basis definition of Eq. (46) we can establish the non-associative multiplication rule for octonions based on the \(\alpha\) matrices:

1. remove all normalization factors,
2. perform the matrix addition of the two elements,
3. replace any occurrence of \(2\lambda^j\) by zero,
4. renormalize to unit determinant according to the number of matrices in the sum,
5. change sign if multiplying in the reverse direction as defined by the Fano plane,
6. any element squared follows matrix multiplication rules and equals \(-1\).

The justification for this rules is solely found on the fact that the elements of Eqs. (46) equipped with such product produce an octonion basis. As an example of use of the rules, we can check that \(e_6e_4 = -e_3\):

1. remove normalization factors,
2. perform addition,
3. replace factors of 2 by zero,
4. no renormalization needed,
5. change sign for reverse direction \(\rightarrow -e_3\),
6. not applicable.

Under the assumptions above a 4D complex number has the form
\[
\psi = \frac{p_0 q_0 \rho^2}{n_q} \exp \left( \frac{i p_0 q_0 \rho^2 \lambda^j x^0}{n_q} + i \alpha^k x^k \right). \tag{47}
\]

These numbers cannot be associated to transformations of 4-space because they involve all 15 dimensions. Presumably the 15 coordinates are not independent and the actual space dimension is lower than 15 but this is a subject which needs profound research which is beyond the scope of the present work. Some insight into the sort of transformations induced by the \(i\alpha^j\) imaginary units can be gained by considering the following matrix
\[
\begin{pmatrix}
x^0 + i x^3 & -x^2 - ix^1 & 0 & 0 \\
x^2 - ix^1 & x^0 - ix^3 & 0 & 0 \\
0 & 0 & -x^4 - ix^7 & x^6 + i x^5 \\
0 & 0 & -x^6 + i x^5 & -x^0 + ix^7
\end{pmatrix}. \tag{48}
\]

Considering all the particular situation \(\psi = \exp(i\alpha^j / 2)\) and applying a similarity transformation coordinates \(x^0, x^1, x^4\) and \(x^5\) are preserved while the other coordinates are transformed as
\[
\begin{align*}
x^2 &= x^2 \cos \theta + x^3 \sin \theta, \\
x^3 &= -x^2 \sin \theta + x^3 \cos \theta, \\
x^6 &= x^6 \cos \theta - x^7 \sin \theta, \\
x^7 &= x^6 \sin \theta + x^7 \cos \theta.
\end{align*} \tag{49}
\]

If the other \(i\alpha^j\) units were tested we would find the sets \((x^1, x^2, x^3)\) and \((x^5, x^6, x^7)\) to undergo rotations of opposite sign. We are thus led to assume that coordinates \(x^0\) to \(x^3\) and \(x^4\) to \(x^7\) are interrelated, reducing the space dimension to 11; this is not an unexpected dimension and is coincident with what is used by superstring theory \[\|\). Transformations involving any of the \(n^0\) units are more
complex, involving all the elements of the coordinate matrix.

Considering \( p_1 = 0 \) in Eq. (14), the unitary hypercomplex numbers represented through the imaginary units of Eq. (10) have the form

\[
q_j q^2 \rho^j \exp \left( \frac{i \theta q_j q^2 \rho^j}{2 n_q} \right).
\]

(50)

where \( \theta = 2 p \rho x^0 \) is an arbitrary real number. If the upper left quarter of the coordinate matrix represents 4-space, as in (15), these numbers represent 2 coordinate preserving transformations in the cases of \( e_1, e_2, e_3, \) and 1 coordinate preserving transformations in the cases of \( e_4, e_5, e_7 \) and no coordinate preserving transformations in the case of \( e_6 \). These numbers can be classified according to the signs of \( p \) and \( q_j \), and according to the number of non-zero \( q_j \). For each sign of \( p \) there is one hypercomplex number for \( (n_q)^2 \) equal to 3 and 3 hypercomplex numbers for \( (n_q)^2 \) either 1 or 2.

There is another unexplored possibility for the 0th imaginary unit which consist of making \( u^0 = i \alpha^0 \); this generates yet a different sort of transformation analogous to a global harmonic pulsation of the whole 4-space. Consider \( \psi = \exp(i \alpha^0 \theta/2) \) and apply a similarity transformation to the coordinate matrix; all the elements in the upper left quarter, those representing 4-space, will appear multiplied by \( \cos \theta \).

These generalized complex numbers will be used in the present and following sections to write solutions of the first order and source equations. We may have used some discretion in choosing the mathematical entities we defined as 4D complex numbers; we were guided in this choice by need to make a connection with elementary particles later on.

The source equation (11) can become extremely intricate; this is to be expected, since it is applicable to all the problems of dynamics. The exposition of this section will be made clearer by a preliminary exploration of the 4D Helmholtz equation’s solutions using the 4D complexes introduced above. Consider Eq. (23) with \( j \) and \( k \) indices replaced by \( \mu \) and \( \nu \) in order to make it 4-dimensional

\[
\delta^{\mu \nu} \partial_\mu \psi = -m^2 \psi.
\]

(51)

Just as in the 3D case, this equation accepts complex solutions of the type \( \psi = ||\psi|| \exp(\pm ip_\mu x^\mu) \), with the condition

\[
\delta^{\mu \nu} p_\mu p_\nu = m^2.
\]

(52)

We are interested in exploring 4D complex solutions of the form given by Eq. (11); no matter which choice is made for \( u^0 \), it will always be possible to write a solution in the form of Eq. (45) with Condition (52) verified. This solution represents a spin 1/2 wave in the direction of the unit 4-vector \( \hat{p} \) whose components are \( p_\mu/m; m \) is also the wave frequency along the direction \( \hat{p} \); in 4D we will associate the designation Compton frequency to the wave frequency. The 4D wave is made up of two components: One 3-space component generated by the 3 elements \( p_j \), with frequency \( \Sigma (p_j)^2 \), corresponding to the De Broglie frequency, and one component in the direction \( x^0 \), with frequency \( p_0 \), corresponding to a transformation that does not preserve \( x^0 \). When \( u^0 = i \alpha^0 \) the \( x^0 \) wave induces a pulsation of 4-space, as referred.

In this section we want to explore the simplest possible forms of the source equation, together with its solutions. The form of the equation is governed by the metric which includes an inertial component and a field component; we will assume here the latter to be the identity matrix, so we will limit ourselves to identifying the recursion mechanism which determines the inertia component.

If we consider 4D complexes defined by Eq. (41), a subsequent application of Eqs. (23) and (53) generates an anti-symmetric metric given by the following relations

\[
g^{00} = ||\psi||^2 = \frac{1}{||\psi||^2},
\]

\[
g^{0j} = \frac{\psi^j}{||\psi||^2},
\]

\[
g^{j0} = \frac{\alpha^j}{||\psi||^2},
\]

\[
g^{jj} = \frac{1}{||\psi||^2},
\]

\[
g^{jk} = -\frac{\alpha^j \alpha^k}{||\psi||^2},
\]

(53)

If \( ||\psi|| \) anti-commutes with \( \alpha^j \) this is a situation where the metric is anti-symmetric and can be replaced by a diagonal metric; making the substitution \( ||\psi||^2 \rightarrow m^2 \) one gets

\[
g^{\mu \nu} = \frac{1}{m^2} \delta^{\mu \nu},
\]

(54)

as the metric to be used in the source equation.

Leaving aside the anti-symmetry conditions, having assumed the norm to be constant, and noting that a constant metric zeroes the Christoffel symbols allowing the covariant derivatives to be replaced by partial ones, the source equation (11) becomes the 4D Helmholtz equation (54). Apart from a phase factor, this equation allows solutions of the type given by Eq. (11) with Condition (52) met and with diagonal modulus. If we want not only the source equation but also the first order equation (38) to be verified, then the modulus is given by Eq. (14).

III. A PHYSICAL EVALUATION OF THE MASSIVE PARTICLE SOLUTIONS

What is specially interesting about 4-dimensional standing waves in comparison to their 3D counterparts is that there are different sorts, according to the imaginary unit \( u^0 \) that is chosen. Looking at the octonion
basis elements \((e_1, e_2, e_3)\) identify transformations where 2 of the 3-space coordinates are preserved, elements \((e_4, e_5, e_2)\) refer to transformations preserving only one 3-space coordinate and element \(e_6\) 4D combined transformations involving all 3-space coordinates. For the 7 possible \(u^0\) choices that form an octonion basis the concept of octonion units can be extended to the waves they generate, suggesting that octonion products performed among 4D waves may describe particles involved in chromodynamics. The different possible choices for \(u^0\), including \(i\omega^0\), point to the interpretation that Eq. (45) represents 8 particles and their respective anti-particles, with this solution describing the known elementary particle dynamics. While the equations have so far been written as pure numbers, dimensions and physical interpretation are possible through the use of dimensional factors introduced in Ref. [1].

Returning to Eq. (16), assume that \(p_1 = p_2 = 0\) so that the particle moves along the \(x^3\) direction. We have \(m^2 = (p_0)^2 + (p_3)^2\) and the first thing we notice is that the frequency \((p_0)\) of the \(x^0\) wave is reduced as \((p_3)\) increases, i.e. the transformation involving \(x^0\) with 1, 2 or 3 spatial coordinates, is transferred to rotation around \(x^3\). Secondly we notice that the De Broglie frequency is \(p_3\) with spin frequency of twice that value; we refer to this fact saying that the particle has spin 1/2; actually the spin can also be \(-1/2\) when the sign factor \(s\) is negative. Rewriting Eq. (16) for a stationary particle, defined by \(p_j = 0\), whose \(u^0\) is chosen among the octonion basis elements

\[
\psi = \frac{m u_j p^2 \lambda_j}{n_q} \exp\left(\frac{\text{sin} q_j p^2 \lambda_j}{n_q}\right),
\]

we identify the particle’s mass as \(m\), connecting it with the norm of the particle’s wave function; for a stationary particle the mass coincides with the frequency of the \(x^0\) wave.

The particle’s electric charge can be defined as \(\sum q_j / 3\) and comes from the transformation involving the 0th coordinate through one octonion unit. For this reason, when \((n_q)^2\) equals 1 or 3, we will choose negative rather than positive \(q_j\), thereby allowing the electric charge to be \(-1\) when \(n_q^2 = 3\). Such a particle can be then associated with an electron.

The electric charge is \(+2/3\) when \(n_q^2 = 2\) and \(-1/3\) when \(n_q^2 = 1\), corresponding to the up and down quarks, respectively [2, 10]. Color charge is associated with the various \(q_j \neq 0\) possibilities and we can see that electric charge appears as a spherically symmetric color charge. Anti-particles are in turn represented by an equation identical to Eq. (16) where the sign of the \(q_j\)’s is reversed. The interpretation made above shows that Eq. (55) has all the necessary features to represent all leptons and quarks with the exception of neutrinos, with the particle’s mass being the only free parameter. It is of course desirable to remove all free parameters, deriving mass from some characteristics of the source equation’s solutions, but we will not address the subject in this paper.

We are going to associate neutrinos with the \(u^0 = i\omega^0\) situation; as for the other particles, the neutrinos’ masses appear as free parameters.

### IV. SOLUTIONS WITH ZERO NORM - THE QUESTION OF MASSLESS PARTICLES

It is important to investigate solutions of Eq. (51) when \(m = 0\) noting that the equation can be rearranged as

\[
\delta^k_j \nabla_k \psi = -\nabla_0 \psi. \tag{56}
\]

Whenever

\[
\nabla_0 \psi = \omega^2 \psi \tag{57}
\]

and covariant derivatives can be replaced by partial ones, the equation above becomes the 3D Helmholtz equation \([23]\), for which there are plane wave solutions of the type given by Eq. (24), i.e. no-spin waves. Being in 4D, integration of Eq. (57) has to be made with \(\psi\) as a \(4 \times 4\) matrix; after integration we get

\[
\psi = \psi_{1,2,3} e^{\pm i \omega x^a}, \tag{58}
\]

where \(\hbar\) is an Hermitian unit and \(\psi_{1,2,3}\) is a \(4 \times 4\) matrix function of the 3 spatial coordinates. The general form of a plane wave solution for Eq. (56) can be written

\[
\psi = Ce^{\pm i \omega x^a} e^{i \rho p_j x^j}; \tag{59}
\]

with \(C\) an Hermitian matrix integration constant.

Physical interpretation of Eq. (59) reveals a plane wave restricted to 3D space and evanescent in the 0th dimension, if the positive sign is rejected. The phenomenon of evanescent waves is well known from total reflection and waveguides, where both exponent signs are also present but the positive sign is ignored for lack of physical significance. Here we interpret the solutions given by Eq. (59) as representing massless particles, of which we distinguish three different types according to the Hermitian unit \(\hbar\) that is present: An Hermitian unit obtained from the octonion basis element \(e_6\) by product with \(i\) carries electric charge in the evanescent wave and will be associated to photons; an Hermitian unit obtained from one of the other octonion basis elements by product with \(i\) carries color charge in the evanescent wave and will be associated to gluons; finally the Hermitian unit \(\alpha^0\) carries no charge and will be discussed later in this section.

Some words are needed here about the reason why we chose no-spin wave solutions for Eq. (56) while spin 1/2 solutions exist. We could argue that spin 1/2 solutions cannot coexist in the ground state due to the Pauli exclusion principle, but we can do better than that: A no-spin wave defines a direction in 4-space but not a worldline; this is true both for dynamic or observer space. On the contrary, the observer space appearance of a spin 1/2 wave is similar to a 4-dimensional waveguide and it defines a worldline coinciding with the waveguide axis; this
idea was first suggested in Ref. [4]. The number of cycles along any waveguide length is actually a measure of time interval in the waveguide’s (or particle’s) own clock. We can also assume all clocks in all waveguides to be synchronized from a common “zero time” in the origin of the Universe. When two waveguides meet in the same 3D position and their time measurement coincides they have to interact; Ref. [14] includes the discussion of a collision problem in 4-dimensional optics, which is an illustration of this fact. The same does not happen with no-spin waves which never actually meet in one position because they are not associated to lines and this matches the behavior of massless particles. The interpretation of time as the count of cycles along the waveguide is entirely compatible with the use of $dt$ as line element that we make in the next section.

A superposition of an even number of $\pm 1/2$ spin waves has many characteristics of a no-spin wave; consider for instance the superposition of 2 equal modulus equal frequency $\pm 1/2$ spin waves with different phase, both spinning around the $x^1$ direction:

$$\psi = |\psi| \left[ e^{i\alpha x^1} + e^{i\alpha(\omega x^1 + \rho)} \right]; \quad (60)$$

it is sufficient to make $\rho = \pi$ for the equation above to degenerate into a no-spin wave given by $\psi = |\psi| \cos(\omega x^1)I$; naturally the same could be done if the two waves were counter-spinning or with any even number of waves co- or counter-spinning. The spin of any wave will thus be expressed by the algebraic sum of the superposed waves spins.

All solutions verifying Eq. (59) have complex dependence on the spatial coordinates, not quaternionic-like solutions. As we said before, complex numbers describe no-spin waves, whereby these solutions all exhibit no spin. Only combinations of an even number of particles with spin $1/2$ can then lead to integral spin waves, behaving like no-spin ones. We shall examine this case next.

We know from experiment that photons are associated to the annihilation of an electron and a positron; since we have established the wavefunctions of the intervening particles it is elucidating to investigate how the annihilation process can be understood in terms of those wavefunctions. The wavefunctions of an electron and a positron are given by

$$\psi_- = i e_6 e^{-sme e x^0} + i p_j \alpha^j x^j,$$
$$\psi_+ = i e_6 e^{sme e x^0} + i p_j \alpha^j x^j; \quad (61)$$

where $e_6$ is the octonion basis element corresponding to electron and positron and the exponent sign designates each particle’s electric charge; the sign of $s$ stands for the two possible spin orientations. By adding an $i e_6 e x^0$ term within both exponents to represent the coupling between spins and multiplying the two wave functions we get

$$\psi_- \otimes \psi_+ \Rightarrow m^2 e^{-sme e x^0(1-i)} \otimes e^{sme e x^0(1+i)} \Rightarrow m^2 e^{2sm e e x^0} \left( 2 e^{i\rho_j \alpha^j x^j} \right) \Rightarrow me^{sm e e x^0} \left( 2 e^{i\rho_j \alpha^j x^j} \right) \otimes me^{sm e e x^0} \left( 2 e^{i\rho_j \alpha^j x^j} \right) \Rightarrow (62)$$

In the first stage of the process (62), the two particles are assumed to interact by means of the $i x^0$ terms in the exponents; in the second stage (63) an intermediate particle is formed, which absorbs the result from the annihilation in the $p_j$; the superposition of the two waves is indicated by $2\ast$. Finally the third stage (64) produces 2 identical photons of spin 1 and frequency equal to the mass of one particle. The final result is a solution of Eq. (65), representing a superposition of 3D spin waves, evanescent in the $x^0$ direction. As before only a negative evanescent exponent is acceptable and since $s$ can have both signs the sense of the mutual orbiting must change according to the spin sign.

The annihilation of two neutrinos, if at all possible, leads to a similar expression, where $i e_6$ is replaced by $\alpha^0$; the resulting particles are

$$\psi = me^{m^0 x^0} \left( 2 e^{i\rho_j \alpha^j x^j} \right) \Rightarrow (63)$$

This would be a massless particle not involved in any of the four known fundamental interactions and we do not dare proposing an interpretation for it, although it might be related to dark matter in the Universe.

Returning to Eq. (62), further annihilation is possible through the interaction of two $\psi_- \psi_+$ pairs with opposing evanescent exponent signs $s$. We rejected the positive sign for its lack of physical significance, however two pairs with opposite signs produce pure 3-dimensional waves, not accompanied by any $x^0$ evanescence; these waves are the superposition of 4 wavefunctions and thus have spin 2. Four identical particles with mass $m$ can be annihilated producing 4 resulting particles represented by

$$\psi = m \left( 4 e^{i\rho_j \alpha^j x^j} \right). \quad (64)$$

We believe these are good candidates for gravitons which may have been detected experimentally long ago by Allais’ measurements of Foucault’s pendulum anomalies [15] and already discussed by the author [16].

V. SOLUTIONS WITH FIELDS

In sections [15] and [16] we discussed field free solutions of the source equation in situations which reduced that equation to the 4D Helmholtz equation; those solutions where found to have characteristics which suggested their physical interpretation as massive and massless particles. However the real test of this interpretation relies
on the ability of the particles thus defined to react properly to applied fields, accurately predicting the dynamics of those particles. Fields are assumed to alter the particle’s dynamic space through the $s^\mu$ matrices defined in Eq. (33). The exact way in which fields are created is not yet clearly understood but we believe it is connected to the flux of massless particles; we think that the flux of massless particles determines the field gradient, the net value of the flux being linked to the norm of the field gradient and the flux direction to its direction. Things get more complicated when massless particles are originated in moving frames because an extra rotational factor intervenes. The whole process will eventually be governed by equations similar to Maxwell’s equations albeit generalized to encompass the three field types: gravitational, electromagnetic and color. In the following exposition fields are presented axiomatically and the dynamics they induce are then seen to represent dynamics under the influence of gravity and electromagnetism; a loose association is made with the massless particles identified in the previous section but no attempt is made to propose a formal link between particle flux and field gradient, which we intend to do in a future paper.

We are going to assume that fields leave the particle wavefunction $\psi$ virtually unaltered, so that we can use the wavefunctions derived previously; this means that we can make $|\psi| = imu^0$ and $|\psi| = m$. It will frequently be convenient to write the modulus in the equivalent form $|\psi| = im \exp(\pm i u^0 \pi/2)$; as a first generalization step the $s^\mu$ matrices are modified from Eq. (33) by introducing $G_{\mu\nu}$, scalars, and $A_{\mu\nu}$ matrices:

$$
G_{\mu\nu} = \frac{iG_{\mu\nu} e^{\pm i u^0 A_{\mu}\pi/2}}{m},
$$

Eq. (33) maintains its validity and by applying to both members the operator $is^\mu \nabla_\mu$ we get the source equation with $g^{\mu\nu}$ defined by Eq. (33) as before.

In a general situation the metric defined by Eq. (33) can be expanded to

$$
g^{00} = -\frac{1}{m} \sum_{\mu} (G_{\mu})^2 e^{\pm i u^0 A_{\mu}\pi},
$$

$$
g^{0j} = \frac{1}{m^2} G_{\mu} A^\mu e^{\pm i u^0 A_{\mu}\pi/2} G_{j} A^j,
$$

$$
g^{ij} = \frac{1}{m^2} G_{\mu} A^\mu e^{\pm i u^0 A_{\mu}\pi/2},
$$

$$
g^{ik} = \frac{1}{m^2} G_{\mu} A^\mu G_{k} A^k.
$$

This metric will now have to be examined for various possible forms of $G_{\mu}$ and $A_{\mu}$ and the respective dynamics related to known interactions. Simultaneously a connection will be made to the massless particles identified in the previous section in order to associate them with mediators of the various interactions.

The simplest case to examine is of course the one where all the $A_{\mu}$ are zero and the $G_{\mu}$ coefficients are all equal, $G_{\mu} = G/4$; inserting into Eq. (33), the metric becomes anti-symmetric and we can reduce it to the diagonal metric

$$
g^{\mu\nu} = \left(\frac{G}{m}\right)^2 \delta^{\mu\nu}.
$$

This metric defines a space whose geodesics reproduce the dynamics of a particle with mass $m$ in a gravitational field $G$, as long as the field sources are stationary. This problem was analyzed in Ref. [1] and will be reproduced below for completeness.

Inserting Eq. (67) into Eq. (11), with $\psi$ given by Eq. (11) and assuming $G$ to be a slowly varying function so that the Christoffel symbols can be neglected

$$
\left(\frac{G}{m}\right)^2 \delta^{\mu\nu} p_{\mu} p_{\nu} = 1;
$$

then replacing $p_{\mu} = g_{\mu\nu} \dot{x}^\nu$, with $\dot{x}^\mu = dx^\mu/dt$ the derivative of the coordinate with respect to the geodesic line element, here designated by $dt$,

$$
\left(\frac{m}{G}\right)^2 \delta^{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 1.
$$

Here we remind the reader that although we have chosen the letter $t$ for the line element, this is in no way associated with time at this stage; the procedure whereby this association can be made later on was detailed in Ref. [1].

The equation above is called a dynamic space equation because the inertia element $m$ is present in the metric; it can be converted into an observer space equation by the coordinate change $X^\mu = m \dot{x}^\mu$. We can then write

$$
\dot{X}^0 G^2 = 1.
$$

The procedure to derive the 4 geodesic equations from Eq. (70) can be found in many textbooks, for instance [17]. We define a constant Lagrangian with the value 1/2 and insert into the second member of Eq. (70) which becomes $2L$. Following that, we derive the Euler-Lagrange equations in a standard way. If $G$ is independent from coordinate $X^0$ there is conservation of the corresponding conjugate momentum

$$
\dot{X}^0 G^2 = \frac{1}{\gamma},
$$

with $\gamma$ an arbitrary constant. For the other 3 coordinates we can write

$$
\frac{d}{dt} \left( \frac{\dot{X}^i}{G^2} \right) = -\delta_{\mu\nu} \dot{X}^\mu \dot{X}^\nu \partial_\nu G + \frac{G^3}{G} \partial_\nu G,
$$

using (70).

The reader is now referred to Ref. [1] in order to verify that in the particular case of a single body of mass $M$ as origin of the field introduced in Eq. (35), one must
make $G = \exp(-M/r)$, with $r$ the distance between the two masses, thereby giving a clue that $\ln G$ is the Newtonian gravitational field. In the cited reference it was also demonstrated that predictions of general relativity are verified to the first order approximation by the equations above.

The link of gravitational interaction introduced in Eq. (54) with the graviton equation (53) can be made by noting that both $G$ and graviton’s modulus are products of the identity matrix by a scalar; it is thus arguable that the gravitational field can be mediated by such particles.

Next we want to investigate the adequacy of function $\psi$ to represent the electrodynamics of a charged particle. We start by taking $G$ as unity and

$$A_\mu = e_6 A_\mu, \quad (73)$$

with $A_\mu$ scalar; we will also assume that $u^0 = \pm e_6$, effectively limiting the analysis to the electron/positron family members. As a result the exponentials $\exp(\pm u^0 A_\mu/2)$ in Eq. (53) can be replaced by $\exp(\pm A_\mu/2)$. It is also convenient to define

$$V_\mu = e^{\pm A_\mu} \pi/2. \quad (74)$$

The metric elements become

$$g^{00} = \frac{\delta^{\mu\nu} V_\mu V_\nu}{m^2},$$

$$g^{0j} = \frac{V_\mu \delta_\mu^j}{m^2},$$

$$g^{j0} = \frac{V_\mu \delta^j_\mu}{m^2},$$

$$g^{jk} = \frac{\alpha \delta^k_\mu}{m^2}. \quad (75)$$

This is no longer an anti-symmetric metric such as we have found in all previous cases because $g^{0j}$ and $g^{j0}$ do not cancel each other completely. After cancellation of the anti-symmetric parts the metric becomes

$$g^{\mu\nu} = \frac{1}{m^2} \begin{pmatrix} \delta^{\mu\nu} & V_\Lambda & V_1 & V_2 & V_3 \\ V_1 & 1 & \cdot & \cdot & \cdot \\ V_2 & \cdot & 1 & \cdot & \cdot \\ V_3 & \cdot & \cdot & 1 & \cdot \\ V_\Lambda & \cdot & \cdot & \cdot & 1 \end{pmatrix}. \quad (76)$$

The lower subscript metric $g_{\mu\nu}$ is obtained, as usual, by calculating the inverse of $g^{\mu\nu}$

$$g^{\mu\nu} = \frac{1}{m^2} \begin{pmatrix} 1 & -V_1 & -V_2 & -V_3 \\ -V_1 & (V_\Lambda)^2 + (V_1)^2 & V_1 V_2 & V_1 V_3 \\ -V_2 & V_1 V_2 & (V_\Lambda)^2 + (V_2)^2 & V_2 V_3 \\ -V_3 & V_1 V_3 & V_2 V_3 & (V_\Lambda)^2 + (V_3)^2 \end{pmatrix}. \quad (77)$$

We don’t develop here the details of electrodynamics generated by the metric above; the procedure is similar to what was used for gravity and can be found in Ref. [1], where both electric and magnetic field dynamics were examined.

Consider now the situation where $\partial_\mu V_\nu = 0$ except for $\partial_1 V_2 = q B_3$; for simplicity we make $V_0 = 1$, $V_1 = V_3 = 0$ and $V_2 = x^1 q B_3$; $B_3$ represents the magnetic field along $x^3$ and $q = \pm 1$. The Lagrangian is now defined by

$$\frac{2L}{m^2} = \delta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - 2q x^1 B_3 \dot{x}^0 \dot{x}^2 + \left(q x^1 B_3 \dot{x}^2\right)^2. \quad (77)$$

Some straightforward calculations lead to the conjugate momenta

$$\frac{p_0}{m^2} = \dot{x}^0 - q x^1 B_3 \dot{x}^2, \quad (78)$$

$$\frac{p_1}{m^2} = \dot{x}^1, \quad (79)$$

$$\frac{p_2}{m^2} = \dot{x}^2 - q x^1 B_3 \dot{x}^0 + \left(q x^1 B_3\right)^2 \dot{x}^2, \quad (80)$$

$$\frac{p_3}{m^2} = \dot{x}^3. \quad (81)$$

Since the Lagrangian is independent from $x^3$, Eq. (77) gives

$$\dot{x}^3 = \text{constant}; \quad (82)$$

then, independence of the Lagrangian from $x^0$ and Eq. (77) give

$$\dot{x}^0 = \frac{1}{m^2} q x^1 B_3 \dot{x}^2, \quad (83)$$

with $\gamma$ a constant greater than 1, unity applying to a stationary particle.

Inserting Eq. (83) into Eq. (80) and deriving the second member with respect to the line element $d\gamma$ and equating to zero because the Lagrangian is independent from $x^2$

$$\dot{x}^2 = \frac{q B_3^2}{m^2} \dot{x}^1. \quad (84)$$

Finally, from Eq. (77), deriving the Lagrangian with respect to $x^1$ and making the needed substitutions we get

$$\dot{x}^1 = -\frac{q B_3}{m^2} \dot{x}^2. \quad (85)$$

Equations (77) and (85) represent the Lorentz force exerted by the magnetic field $B_3$ over a particle of electric charge $q$ and moving with velocity $\dot{x}^j$. The frequency $q B_3/m\gamma$ is the angular frequency of the movement.

In the limit $\gamma = 1$ and $\dot{x}^j = 0$ the particle approach breaks down because this approach is similar to the geometrical or ray approach of optics and cannot be used when any features of the worldline are of the same order of magnitude as the particle’s Compton wavelength. If we take $q = -1$, for the electron case, the angular frequency becomes $\omega = -B_3/m\gamma$ and one possible solution for Eqs. (77) and (85) is

$$x^1 = r \cos (\omega t),$$

$$x^2 = r \sin (\omega t). \quad (86)$$
The use of $t$ as an independent variable is here linked to a specific worldline in dynamic space; $t$ can always be expressed in terms of the coordinates by inversion of the equations above $t = \arctan(x^2/x^3)/\omega$. Inserting Eqs. \([86]\) into Eqs. \([78]\) to \([81]\) we get the conjugate momenta

\[
\begin{align*}
p_0 &= \frac{m}{\gamma}, \\
p_1 &= \frac{mB_3r \sin(\omega t)}{\gamma}, \\
p_2 &= 0, \\
p_3 &= \text{arbitrary constant}.
\end{align*}
\]  

Choosing zero for $p_3$ the particle’s wavefunction can be written

\[
\psi = \exp \left( \frac{m}{\gamma} u^0 x^0 + \frac{mB_3r \sin(\omega t)}{\gamma} i\alpha^1 x^1 \right) \\
= \exp \left[ \frac{m}{\gamma} u^0 x^0 + i\alpha^1 m^2 \omega r^2 \sin^2(\omega t) \right]; 
\]  

which is now an expression not involving $\dot{x}^3$ and not breaking down when $\gamma \to 1$. In the limit there is spin precession with the cyclotron frequency $\omega = -B_3/m$; for a positron the frequency would show a positive sign.

Similarly to the relation of gravitons with the gravitational field, photons can be seen as related to the EM field by their modulus when comparing Eq. \([18]\) with Eq. \([62]\). In both cases there is an exponent which can interact with the exponent in the wavefunction modulus; photons can be considered as mediators of this field. Naturally relations like Eq. \([23]\) can be established for the other octonion basis elements, originating the color fields and chromodynamics; this is a subject for a forthcoming paper.

We cannot end this analysis without touching upon the situation where the $G_\mu$ differ from each other; this must surely be related to gravity, since we have seen how gravitational field is modelled by equal coefficients. We presume at this point that different $G_\mu$’s must be associated with mass currents in a similar way to what relates the $V_\mu$’s to electrical currents. This would mean that different $G_\mu$ coefficients model situations of non-stationary gravitational sources. As we said at the beginning of this section, it is believed that a set of coupled differential equations links the components of each field and that these equations are a generalization of Maxwell’s equations; we will investigate this possibility also in a forthcoming paper.

\section{VI. CONCLUSION AND FURTHER WORK}

Massive and non-massive wave-particles have been found as solutions of a 4-dimensional source equation. Those solutions were identified as 4D standing waves and may be related to the probability waves of quantum mechanics by a switch from the pseudo-Euclidean space used in this work to the more traditional hyperbolic spacetime. The character of their dynamical symmetry has been found to follow that of elementary particles, leaving mass as the only free parameter. Electric and color charge result directly from rotational symmetry properties in 4D of standing waves, and massive particles were shown to verify the laws of dynamics following geodesics of a suitably defined 4-space. The physical origin of spin has been identified without appeal to Relativity principles, and found to result merely from Helmholtz equation requirements in 4D. The modifications to the source equation needed to express the complete dynamics were loosely tied to the character of massless particles, this being an area of further development as detailed below.

Two complementary lines of future work are envisaged, following paths only suggested in the present work. The first line will try to complete the set of recursive equations, deriving fields from particle densities and currents in dynamic and observer space, and in particular deriving chromodynamics. As a mere working hypothesis we believe that a composite field can be defined as

\[
\phi_\mu = G_\mu \alpha_\mu e^{A_\nu}, \quad \text{(summation is suppressed)}
\]  

where $G_\mu$ represents the contribution from gravitational sources both stationary and non-stationary and $A_\mu$ represents both electromagnetic and color fields, as discussed in the previous section. The composite field will then be related to the flux of massless particles by a composite field tensor defined by

\[
F_{\mu\nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu,
\]  

which is expected to verify relations similar to Maxwell’s equations, namely

\[
\partial_\mu F_{\mu\nu} + \partial_\nu F_{\mu\nu} + \partial_\nu F_{\nu\lambda} = 0, \\
\partial^{\mu} F_{\mu\nu} = J_\nu.
\]  

These are untested hypothesis and must be seen only as a working program.

The second line of work will be a search for variable norm solutions of the source equation. Preliminary work has already shown that allowance for a variable norm and the consequent consideration of Christoffel symbols can send the source equation into some modified forms of Emden-Fowler equation \([15]\) which can determine mass, thus eliminating the only remaining free parameter.

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