HOMOTOPY DIMENSION OF ORBITS OF MORSE FUNCTIONS ON SURFACES

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Abstract. Let $M$ be a compact surface, $P$ be either the real line $\mathbb{R}$ or the circle $S^1$, and $f : M \to P$ be a $C^\infty$ Morse map. The identity component $D_{id}(M)$ of the group of diffeomorphisms of $M$ acts on the space $C^\infty(M, P)$ by the following formula: $h \cdot f = f \circ h^{-1}$ for $h \in D_{id}(M)$ and $f \in C^\infty(M, P)$. Let $O(f)$ be the orbit of $f$ with respect to this action and $n$ be the total number of critical points of $f$. In this note we show that $O(f)$ is homotopy equivalent to a certain covering space of the $n$-th configuration space of the interior $\text{Int}M$. This in particular implies that the (co-)homology of $O(f)$ vanish in dimensions greater than $2n - 1$, and the fundamental group $\pi_1 O(f)$ is a subgroup of the $n$-th braid group $B_n(M)$.

Keywords: Morse function, orbits, classifying spaces, homotopy dimension, geometric dimension

AMS Classification 2000: 14F35, 46T10

1. Introduction

Let $M$ be a compact surface, $P$ be either the real line $\mathbb{R}$ or the circle $S^1$. Then the group $D(M)$ of $C^\infty$ diffeomorphisms of $M$ acts on the space $C^\infty(M, P)$ by the following formula:

\begin{equation}
(1.1) \quad h \cdot f = f \circ h^{-1}
\end{equation}

for $h \in D(M)$ and $f \in C^\infty(M, P)$.

We say that a smooth ($C^\infty$) map $f : M \to P$ is Morse if

(i) critical points of $f$ are non-degenerate and belong to the interior of $M$;

(ii) $f$ is constant on every connected component of $\partial M$.

Let $f \in C^\infty(M, P)$, $\Sigma_f$ be the set of critical points of $f$, and $D(f, \Sigma_f)$ be the subgroup of $D(M)$ consisting of diffeomorphisms $h$ such that $h(\Sigma_f) = (\Sigma_f)$.

Then we can define the stabilizers $S(f)$ and $S(f, \Sigma_f)$, and orbits $O(f)$ and $O(f, \Sigma_f)$ with respect to the actions of the groups $D(M)$.
and $\mathcal{D}(f, \Sigma_f)$. Thus

$$S(f) = \{ h \in \mathcal{D}(M) : f \circ h^{-1} = f \}, \quad \mathcal{O}(f) = \{ f \circ h^{-1} : h \in \mathcal{D}(M) \},$$

$$S(f, \Sigma_f) = S(f) \cap \mathcal{D}(f, \Sigma_f).$$

We endow the spaces $\mathcal{D}(M)$ and $C^\infty(M, P)$ with the corresponding $C^\infty$ Whitney topologies. They induce certain topologies on the stabilizers and orbits.

Let $\mathcal{D}_{id}(M)$ and $\mathcal{D}_{id}(f, \Sigma_f)$ be the identity path components of the groups $\mathcal{D}(M)$ and $\mathcal{D}(f, \Sigma_f)$, $S_{id}(f)$ and $S_{id}(f, \Sigma_f)$ be the identity path components of the corresponding stabilizers, and $\mathcal{O}_f(f)$ and $\mathcal{O}_f(f, \Sigma_f)$ be the path-components of $f$ in the corresponding orbits with respect to the induced topologies.

**Lemma 1.** If $\Sigma_f$ is discrete set, e.g. when $f$ is Morse, then $S_{id}(f, \Sigma_f) = S_{id}(f)$.

*Proof.* Since $S(f, \Sigma_f) \subset S(f)$, we have that $S_{id}(f, \Sigma_f) \subset S_{id}(f)$. Conversely, let $h_t : M \to M$ be an isotopy such that $h_0 = id_M$ and $h_t \in S(f)$ for all $t \in I$, i.e. $f \circ h_t = f$. We have to show that $h_t \in S(f, \Sigma_f)$ for all $t \in I$. Notice that $d(f \circ h_t) = h_t^*df = df$, whence $h_t(\Sigma_f) = \Sigma_f$. Since $\Sigma_f$ is discrete and $h_0 = id_M$ fixes $\Sigma_f$, we see that so does every $h_t$, i.e. $h_t \in S(f, \Sigma_f)$. $\square$

Let $f : M \to P$ be a Morse map. Denote by $c_i$, $(i = 0, 1, 2)$, the total numbers of critical points of $f$ of index $i$ and let $n = c_0 + c_1 + c_2$ be the total number of critical points of $f$.

Notice that for every Morse map $f$ its orbits $\mathcal{O}(f)$ and $\mathcal{O}(f, \Sigma_f)$ are Fréchet submanifolds of $C^\infty(M, P)$ of finite codimension, see [4, 5]. Therefore, e.g. [3], these orbits have the homotopy types of CW-complexes. But in general these complexes may have infinite dimensions.

Let $X$ be a topological space which is homotopy equivalent to some CW-complex. Then a *homotopy dimension* h.d. $X$ of $X$ is the minimal dimension of a CW-complex homotopy equivalent to $X$. In particular h.d. $X$ can be equal to $\infty$. It is also evident that if h.d. $X < \infty$, then (co-)homology of $X$ vanish in dimensions greater that h.d. $X$.

If $\pi$ is a finitely presented group $\pi$, then the geometric dimension of $\pi$, denoted g.d. $\pi$, is the homotopy dimension of its Eilenberg-MacLane space $K(\pi, 1)$:

$$\text{g.d. } \pi := \text{h.d. } K(\pi, 1).$$

In [2] Theorems 1.3, 1.5, 1.9] the author described the homotopy types of $S_{id}(f)$, $\mathcal{O}_f(f)$, and $\mathcal{O}_f(f, \Sigma_f)$. It follows from these results
that
\[ \text{h.d. } \mathcal{S}_{id}(f), \quad \text{h.d. } \mathcal{O}_f(f, \Sigma_f) \leq 1. \]

In fact, \( \mathcal{S}_{id}(f) \) is contractible provided either \( f \) has at least one critical point of index 1, i.e., \( c_1 \geq 1 \) or \( M \) is non-orientable. Otherwise \( \mathcal{S}_{id}(f) \simeq S^1 \).

Also, \( \mathcal{O}_f(f) \simeq S^1 \) for Morse mappings \( T^2 \to S^1 \) and \( K^2 \to S^1 \) without critical points, and \( \mathcal{O}_f(f) \) is contractible in all other cases, where \( K \) stands for the Klein bottle.

For \( \mathcal{O}_f(f) \) the description is not so complete. But if \( f \) is generic, i.e., it takes distinct values at distinct critical points, then
\[ \text{h.d. } \mathcal{O}_f(f) \leq \max\{c_0 + c_2 + 1, c_1 + 2\} < \infty. \]

Actually, in this case \( \mathcal{O}_f(f) \) is either contractible or homotopy equivalent to \( T^k \) or to \( \mathbb{R}P^3 \times T^k \) for some \( k \geq 0 \), where \( T^k \) is a \( k \)-dimensional torus.

Thus the upper bound for \( \text{h.d. } \mathcal{O}_f(f) \) (at least in generic case) depends only on the number of critical points of \( f \) at each index.

In this note we will show that \( \text{h.d. } \mathcal{O}_f(f) \leq 2n - 1 \) for arbitrary Morse mapping \( f : M \to P \) having exactly \( n \geq 1 \) critical points. Notice that if \( n = 0 \), then \( f \) is generic, and in fact \( \text{h.d. } \mathcal{O}_f(f) \leq 1 \), see [2, Table 1.10].

**Theorem 2.** Let \( f : M \to P \) be a Morse map and \( n \) be the total number of critical points of \( f \). Assume that \( n \geq 1 \). Denote by \( \mathcal{F}_n(\text{Int}M) \) the configuration space of \( n \) points of the interior \( \text{Int}M \) of \( M \). Then \( \mathcal{O}_f(f) \) is homotopy equivalent to a certain covering space \( \mathcal{F}(f) \) of \( \mathcal{F}_n(\text{Int}M) \).

**Corollary 3.** \( \text{h.d. } \mathcal{O}_f(f) \leq 2n - 1 \), whence (co-)homology of \( \mathcal{O}_f(f) \) vanish in dimensions \( \geq 2n \).

**Proof.** Since \( \mathcal{F}_n(\text{Int}M) \) and its connected covering spaces are open manifolds of dimension \( 2n \), they are homotopy equivalent to CW-complexes of dimensions not greater than \( 2n - 1 \). \( \square \)

For simplicity denote \( \pi = \pi_1 \mathcal{O}_f(f) \). Since the covering map \( \mathcal{F}(f) \to \mathcal{F}_n(\text{Int}M) \) yields a monomorphisms of fundamental groups, we obtain the following:

**Corollary 4.** The fundamental group \( \pi \) of \( \mathcal{O}_f(f) \) is a subgroup of the \( n \)-th braid group \( \mathcal{B}_n(M) = \pi_1(\mathcal{F}_n(\text{Int}M)) \) of \( M \).

**Corollary 5.** Suppose that \( M \) is aspherical, i.e., \( M \neq S^2, \mathbb{R}P^2 \). Then \( \mathcal{O}_f(f) \) is aspherical as well, i.e., \( K(\pi, 1) \)-space, whence g.d. \( \pi \leq 2n - 1 \).

**Proof.** Actually the asphericity of \( \mathcal{O}_f(f) \) for the case \( M \neq S^2, \mathbb{R}P^2 \) is proved in [2, Theorems 1.5, 1.9].
But it can be shown by another arguments. It is well known and can easily be deduced from [1] that for an aspherical surface \( M \) every of its configuration spaces \( \mathcal{F}_n(\text{Int} M) \) and thus every covering space of \( \mathcal{F}_n(\text{Int} M) \) are aspherical as well. Hence so is \( \mathcal{F}(f) \) and thus \( \mathcal{O}_f(f) \) itself. □

A presentation for \( \pi \) will be given in another paper.

2. ORBITS OF THE ACTIONS OF \( \mathcal{D}_{\text{id}}(M) \) AND \( \mathcal{D}_{\text{id}}(f, \Sigma_f) \)

**Proposition 6.** Let \( f : M \to P \) be a Morse map and

\[
(2.1) \quad p : \mathcal{D}(M) \mapsto \mathcal{O}(f), \quad p(h) = f \circ h^{-1}
\]

be the natural projection. Then \( \mathcal{O}_f(f) \) is the orbit of \( f \) with respect to \( \mathcal{D}_{\text{id}}(M) \) and \( \mathcal{O}_f(f, \Sigma_f) \) is the orbit of \( f \) with respect to \( \mathcal{D}_{\text{id}}(f, \Sigma_f) \). In other words,

\[
p(\mathcal{D}_{\text{id}}(M)) = \mathcal{O}_f(f) \quad \text{and} \quad p(\mathcal{D}_{\text{id}}(f, \Sigma_f)) = \mathcal{O}_f(f, \Sigma_f).
\]

**Proof.** The proof is based on the following general statement. Let \( G \) be a topological group transitively acting on a topological space \( O \) and \( f \in O \). Denote by \( G_e \) the path-component of the unit \( e \) in \( G \) and let \( O_f \) be the path-component of \( f \) in \( O \).

**Lemma 7.** Suppose that the mapping \( p : G \to O \) defined by

\[
p(\gamma) = \gamma \cdot f, \quad \forall \gamma \in G
\]

satisfies a covering path axiom (in particular, this holds when \( p \) is a locally trivial fibration). Then \( O_f \) is the orbit of \( f \) with respect to the induced action of \( G_e \) on \( O \), i.e., \( p(G_e) = O_f \).

**Proof.** Evidently, \( p(G_e) \subset O_f \). Conversely, let \( g \in O_f \). Then there exists a path \( \omega : I \to O_f \) between \( f \) and \( g \), i.e., \( \omega(0) = f \) and \( \omega(1) = g \). Since \( p \) satisfies the covering path axiom, \( \omega \) lifts to the path \( \tilde{\omega} : I \to G \) such that \( \tilde{\omega}(0) = e \) and \( \omega = p \circ \tilde{\omega} \). Then \( g = \omega(1) = p \circ \tilde{\omega}(1) \in p(G_e) \).

Thus \( p(G_e) = O_f \). □

It remains to note that the mapping \( (2.1) \) is a locally trivial fibration, see e.g. [1, 5], and \( \mathcal{D}(M) \) (resp. \( \mathcal{D}(f, \Sigma_f) \)) transitively acts on the orbit \( \mathcal{O}_f(f) \) (resp. \( \mathcal{O}_f(f, \Sigma_f) \)). Therefore the conditions of Lemma 7 are satisfied. □
3. Proof of Theorem \[2\]

Let \( \mathcal{F}_n(\text{Int}M) \) be the configuration space of \( n \) points of the interior \( \text{Int}M \) of \( M \). Thus

\[
\mathcal{F}_n(\text{Int}M) = \mathcal{P}_n(\text{Int}M)/\mathbb{S}_n,
\]

where

\[
\mathcal{P}_n(\text{Int}M) = \{(x_1, \ldots, x_n) \mid x_i \in \text{Int}M \text{ and } x_i \neq x_j \text{ for } i \neq j\}
\]
is called the pure \( n \)-th configuration space of \( \text{Int}M \), and \( \mathbb{S}_n \) is the symmetric group of \( n \) symbols freely acting on \( \mathcal{P}_n(\text{Int}M) \) by permutations of coordinates.

We can regard \( \mathcal{F}_n(\text{Int}M) \) as the space of \( n \)-tuples of mutually distinct points of \( \text{Int}M \).

Denote by \( \Sigma f = \{x_1, \ldots, x_n\} \) the set of critical points of \( f \). Then for every \( g \in \mathcal{O}_f(f) \) the set \( \Sigma_g \) of its critical points is a point in \( \mathcal{F}_n(\text{Int}M) \).

Hence the correspondence \( g \mapsto \Sigma_g \) is a well-defined mapping

\[
k : \mathcal{O}_f(f) \rightarrow \mathcal{F}_n(\text{Int}M), \quad k(g) = \Sigma_g.
\]

Lemma 8. (i) The mapping \( k \) is a locally trivial fibration. The connected component of the fiber containing \( f \) is homeomorphic to \( \mathcal{O}_f(f, \Sigma f) \).

(ii) Let \( k_i : \pi_i(\mathcal{O}_f(f), f) \rightarrow \pi_i(\mathcal{F}_n(\text{Int}M), \Sigma f), \ (i \geq 1) \), be the corresponding homomorphism of homotopy groups induced by \( k \). Then \( k_1 \) is a monomorphism and all other \( k_i \) for \( i \geq 2 \) are isomorphisms.

Assuming that Lemma 8 is proved we will now complete our theorem. Let \( \mathcal{F}(f) \) be the covering space of \( \mathcal{F}_n(\text{Int}M) \) corresponding to the subgroup

\[
\pi_1\mathcal{O}_f(f) \approx k_1(\pi_1\mathcal{O}_f(f)) \subset \pi_1\mathcal{F}_n(\text{Int}M).
\]

Then \( k \) lifts to the mapping \( \hat{k} : \mathcal{O}_f(f) \rightarrow \mathcal{F}(f) \) which induces isomorphism of all homotopy groups. Since \( \mathcal{O}_f(f) \) and \( \mathcal{F}(f) \) are connected, we obtain from (2) that \( \hat{k} \) is a desired homotopy equivalence. Theorem 2 is proved modulo Lemma 8.

Proof of Lemma 8. (i) Recall, [1], that the following evaluation map

\[
e : \mathcal{D}_{id}(M) \rightarrow \mathcal{F}_n(\text{Int}M), \quad e(h) = h(\Sigma f)
\]
is a locally trivial principal fibration with fiber

\[
\hat{\mathcal{D}}(f) = \mathcal{D}_{id}(M) \cap \mathcal{D}(f, \Sigma f).
\]

Let \( p : \mathcal{D}_{id}(M) \rightarrow \mathcal{O}_f(f) \) be the projection defined by \( p(h) = f \circ h^{-1} \).

Then the set of critical points of the function \( f \circ h^{-1} \in \mathcal{O}_f(f) \) is \( h(\Sigma f) \).

Therefore \( e \) coincides with the following composition:

\[
e = k \circ p : \mathcal{D}_{id}(M) \xrightarrow{p} \mathcal{O}_f(f) \xrightarrow{k} \mathcal{F}_n(\text{Int}M).
\]
Since \( e \) and (by Proposition 6) the mapping \( p \) are principal locally trivial fibrations, we obtain that \( k \) is also a locally trivial fibration with fiber \( \hat{O}(f) \) being the orbit of \( f \) with respect to the group \( \hat{D}(f) \).

It is easy to see that the identity component of the group \( \hat{D}(f) \) coincides with \( D_{id}(f, \Sigma f) \), whence by Proposition 6 the connected component of \( \hat{O}(f) \) containing \( f \) is \( O_f(f, \Sigma f) \).

(ii) As noted above since \( n \geq 1 \), it follows from [2, Theorems 1.5(i), 1.9] that \( O_f(f, \Sigma f) \) is contractible. Then from the exact sequence of homotopy groups of the fibration \( k \) we obtain that for \( i \geq 2 \) every \( k_i \) is an isomorphism, and \( k_1 \) is a monomorphism. Lemma 8 is proved. □

**Remark 9.** In general the covering map \( \mathcal{F}(f) \to \mathcal{F}_n(\text{Int}M) \) is not regular, i.e., \( \pi_1 O_f(f) \approx \pi_1 \mathcal{F}(f) \) is not a normal subgroup of \( \mathcal{B}_n(M) = \pi_1 \mathcal{F}_n(\text{Int}M) \).

**Remark 10.** Theorem 2 does not answer the question whether \( O_f(f) \) has the homotopy type of a finite CW-complex. Indeed, since \( M \) is compact, it follows from [3,4] that \( \mathcal{B}_n(M) \) can be regarded as an open cellular (i.e. consisting of full cells) subset of a finite CW-complex \( \prod_n M/S_n \). Therefore if the covering map \( \mathcal{F}(f) \to \mathcal{F}_n(\text{Int}M) \) is an infinite sheet covering, i.e., \( \pi_1 O_f(f) \) has an infinite index in \( \mathcal{B}_n(M) \), then we obtain a priori an infinite cellular subdivision of \( \mathcal{F}(f) \). On the other hand, as noted above, for a generic Morse map \( f : M \to P \) a finiteness of the homotopy type of \( O_f(f) \) follows from [2].

**References**

[1] E. Fadell, L. Neuwirth, *Configuration spaces*, Math. Scand. 10 (1962) 111-118.
[2] S. Maksymenko, *Homotopy types of stabilizers and orbits of Morse functions on surfaces*, Annals of Global Analysis and Geometry, 29 no. 3, (2006) 241-285, [http://xxx.lanl.gov/math.GT/0310067](http://xxx.lanl.gov/math.GT/0310067)
[3] R. S. Palais, *Homotopy theory of infinite dimensional manifolds*, Topology, 5 (1966) 1-16.
[4] V. Poenaru, *Un théorème des fonctions implicites pour les espaces d’applications \( C^\infty \)*, Publ. Math. Inst. Hautes Étud. Sci., 38 (1970) 93-124.
[5] F. Segre, *Un théorème de fonction implicites sur certains espaces de Fréchet et quelques applications*, Ann. scient. éc. norm. sup., 4-e serie, 5 (1972) 599-660.

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