THE MODULI SPACE OF 5 POINTS ON $\mathbb{P}^1$ AND K3 SURFACES

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ABSTRACT. We show that the moduli space of ordered 5 points on $\mathbb{P}^1$ is isomorphic to an arithmetic quotient of a complex ball by using the theory of periods of K3 surfaces. We also discuss a relation between our uniformization and the one given by Shimura [S], Terada [Te], Deligne-Mostow [DM].

Dedicated to Professor Yukihiko Namikawa on his 60th birthday

1. INTRODUCTION

The purpose of this note is to show that the moduli space of 5 ordered points on $\mathbb{P}^1$ is isomorphic to an arithmetic quotient of a 2-dimensional complex ball by using the theory of periods of K3 surfaces (Theorem 6.10). This was announced in [K2], Remark 6. The main idea is to associate a K3 surface with an automorphism of order 5 to a set of 5 ordered points on $\mathbb{P}^1$ (see 3). The period domain of such K3 surfaces is a 10-dimensional bounded symmetric domain of type IV. We remark that a non-zero holomorphic 2-form on the K3 surface is an eigen-vector of the automorphism, which implies that the period domain of the pairs of these K3 surfaces and the automorphism of order 5 is a 2-dimensional complex ball associated to a hermitian form of the signature $(1;2)$ defined over $\mathbb{Z}$. Here we use several fundamental results of Nikulin [N1], [N2], [N3] on automorphisms of K3 surfaces and the lattice theory.

Note that this moduli space is isomorphic to the moduli space of nodal del Pezzo surfaces of degree 4. For the moduli space of del Pezzo surfaces of degree 1, 2 or 3, the similar description holds. See [K2], Remark 5, [K1], [DGK], respectively.

On the other hand, Shimura [S], Terada [Te], Deligne-Mostow [DM] gave a complex ball uniformization by using the periods of the curve $C$ which is the 5-fold cyclic covering of $\mathbb{P}^1$ branched along 5 points. We shall discuss a relation between their uniformization and ours in 7. In fact, the above K3 surface has an isotrivial pencil whose general member is the unique smooth curve $D$ of genus 2 admitting an automorphism of order 5 (see Lemma 3.4). We show that the above K3 surface is birational to the quotient of $C$ by a diagonal action of $\mathbb{Z}/5\mathbb{Z}$ in 7.

In this paper, a lattice means a $\mathbb{Z}$-valued non-degenerate symmetric bilinear form on a free $\mathbb{Z}$-module of finite rank. We denote by $U$ or $V$ the even lattice defined by the matrix

\[
\begin{pmatrix}
0 & 1 & 2 & 1 \\
1 & 0 & 1 & 2
\end{pmatrix},
\]

respectively and by $A_m$, $D_n$ or $E_1$ the even negative definite lattice defined by the Dynkin matrix of type $A_m$, $D_n$ or $E_1$ respectively. If $L$ is a lattice and $m$ is an integer, we denote by $L(\mathfrak{m})$ the lattice over the same $\mathbb{Z}$-module with the symmetric bilinear form multiplied by $m$. We also denote by $L^m$ the orthogonal direct sum of $m$ copies of $L$, by $L^*$ the dual of $L$ and by $A_L$ the finite abelian group $L/L^*$.

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2. QUARTIC DEL PEZZO SURFACES

2.1. Five points on \( \mathbb{P}^1 \). Consider the diagonal action of \( \text{PGL} (2) \) on \( (\mathbb{P}^1)^5 \). In this case, the semistable points and stable points in the sense of [Mu] coincide and the geometric quotient \( \mathbb{P}^5_1 \) is smooth and compact. The stable points are \( f_{p_1; \cdots ; p_5} \) no three of which coincide. It is known that \( \mathbb{P}^5_1 \) is isomorphic to the quintic del Pezzo surface \( D_5 \), that is, a smooth surface obtained by blowing up four points \( f_{q_1; \cdots ; q_4} \) in general position on \( \mathbb{P}^2 \) (e.g. Dolgachev [D], Example 11.5). The quintic del Pezzo surface \( D_5 \) contains 10 lines corresponding to the 4 exceptional curves over \( q_1; \cdots ; q_4 \) and the proper transforms of 6 lines through two points from \( f_{q_1; \cdots ; q_4} \). These ten lines correspond to the locus consisting of \( p_1; \cdots ; p_5 \) with \( p_i = p_j \) for some \( i \neq j \). The group of automorphisms of \( D_5 \) is isomorphic to the Weyl group \( W ( \mathfrak{A}_4 ) \) \( \cong \text{S}_5 \) which is induced from the natural action of \( \text{S}_5 \) on \( (\mathbb{P}^1)^5 \).

2.2. Quartic Del Pezzo surfaces. Let \( S \) be a smooth quartic Del Pezzo surface. It is known that \( S \) is a complete intersection of two quadrics in \( \mathbb{P}^4 \). Consider the pencil of quadrics whose base locus is \( S \). Its discriminant is a union of distinct five points of \( \mathbb{P}^1 \). Conversely any distinct five points \( (1 : 1) \) on \( \mathbb{P}^1 \), the intersection of quadrics

\[
(3.1) \quad x_0^5 = f_5 (x_1, x_2) = \sum_{i=1}^{5} x_i^2 = 0
\]

is a smooth quartic Del Pezzo surface. Thus the moduli space of smooth quartic Del Pezzo surfaces is isomorphic to \( ((\mathbb{P}^1)^5 \setminus \Delta)/\text{PGL} (2)) = \mathbb{S}_5 \) where \( \Delta \) is the locus consisting of points \( (x_1, \cdots , x_5) \) with \( x_i = x_j \) for some \( i \neq j \). If five points are not distinct, but stable, the equation (2.1) defines a quartic Del Pezzo surface with a node. Thus \( \mathbb{P}^5_1 \) is the coarse moduli space of nodal quartic Del Pezzo surfaces.

3. K3 SURFACES ASSOCIATED TO FIVE POINTS ON \( \mathbb{P}^1 \)

3.1. A plane quintic curve. Let \( f_{p_1; \cdots ; p_5} \) be an ordered stable point in \( (\mathbb{P}^1)^5 \). It defines a homogeneous polynomial \( f_5 (x_1, x_2) \) of degree 5. Let \( C \) be the plane quintic curve defined by

\[
(3.1) \quad x_0^5 = f_5 (x_1, x_2) = \sum_{i=1}^{5} x_i^2 = 0
\]

The projective transformation

\[
(3.2) \quad g : (x_0 : x_1 : x_2) \mapsto (x_0 : x_1 : x_2)
\]

acts on \( C \) as an automorphism of \( C \) of order 5 where \( \zeta \) is a primitive 5-th root of unity. Let \( E_0, L_1 \) \((1 : 1 : 5)\) be lines defined by

\[
E_0 : x_0 = 0; \quad L_1 : x_1 = x_2;
\]

Note that all \( L_1 \) are members of the pencil of lines through \((1 : 0 : 0)\) and \( L_1 \) meets \( C \) at \((0 : 1 : 1)\) with multiplicity 5.
3.2. K3 surfaces. Let \( X \) be the minimal resolution of the double cover of \( \mathbb{P}^2 \) branched along the sextic curve \( E_0 + C \). Then \( X \) is a K3 surface. We denote by \( g \) the covering transformation. The projective transformation \( g \) in (3.2) induces an automorphism \( A \) of \( X \) of order 5. We denote by the same symbol \( E_0 \) the inverse image of \( E_0 \).

Case (i) Assume that the equation \( f_5 = 0 \) has no multiple roots. In this case there are 5 \((2)\)-curves, denoted by \( E_1 (1 \ i \ 5) \), obtained as exceptional curves of the minimal resolution of singularities of \( A \) corresponding to the intersection of \( C \) and \( E_0 \). The inverse image of \( E_1 \) is the union of two smooth rational curves \( F_1; G_1 \) such that \( F_1 \) is tangent to \( G_1 \) at one point. Let \( p; q \) be the inverse image of \((1:0:0)\). We may assume that all \( F_1 \) (resp. \( G_1 \)) are through \( p \) (resp. \( q \)). Obviously \( A \) preserves each curve \( E_1; F_2; G_3 \) \((0 \ i \ 5; 1 \ j \ 5)\) and preserves each \( E_1 \) and \( F_1 \) = \( G_1 \).

Case (ii) If \( f_5 = 0 \) has a multiple root, then the double cover has a rational double point of type \( D_7 \). Hence \( E_1 \) contains 7 smooth rational curves \( E_1 ^0 \) \((1 \ i \ 7)\) whose dual graph is of type \( D_7 \). We assume that \( E_1 ^0 \) meets \( E_0 \) and \( E_1 ^{0} \); \( E_2 ^{0} \); \( E_3 ^{0} \); \( E_4 ^{0} \); \( E_5 ^{0} \); \( E_6 ^{0} \); \( E_7 ^{0} \); \( E_8 ^{0} \); \( E_9 ^{0} \) = 1. If \( f_1 \) is a multiple root, then \( F_1 \) and \( G_1 \) are disjoint and each of them contains one component of \( D_7 \), for example, \( F_1 \) meets \( E_0 ^0 \) and \( G_1 \) meets \( E_9 ^0 \).

3.3. A pencil of curves of genus two. The pencil of lines on \( \mathbb{P}^2 \) through \((1:0:0)\) gives a pencil of curves of genus two on \( X \). Each member of this pencil is invariant under the action of the automorphism \( A \) of order 5. Hence a general member is a smooth curve of genus two with an automorphism of order five. Such a curve is unique up to isomorphism and is given by

\[
y^2 = x(x^5 + 1)
\]

(see Bolza [Bo]). If \( f_1 \) is a simple root of the equation \( f_5 = 0 \), then the line \( L_1 \) is a singular member of this pencil consisting of three smooth rational curves \( E_1 ^0 + F_1 ^0 + G_1 ^0 \). We call this singular member a singular member of type I. If \( f_1 \) is a multiple root of \( f_5 = 0 \), then the line \( L_1 \) is a singular member consisting of nine smooth rational curves \( E_1 ^0 ; \cdots ; E_9 ^0 ; F_1 ^0 ; G_1 ^0 \). We call this a singular member of type II. The two points \( p; q \) are the base points of the pencil. After blowing up at \( p; q \) we have a base point free pencil of curves of genus two. The singular fibers of such pencils are completely classified by Namikawa and Ueno [NU]. The type I (resp. type II) corresponds to [IX-2] (resp. [IX-4] ) in [NU]. We now conclude:

3.4. Lemma. The pencil of lines on \( \mathbb{P}^2 \) through \((1:0:0)\) gives a pencil of curves of genus two on \( X \). A general member is a smooth curve of genus two with an automorphism of order five. In case that \( f_5 = 0 \) has no multiple roots, it has five singular members of type I. In case that \( f_5 = 0 \) has a multiple root \( \varphi \) (resp. two multiple roots), it has three singular members of type I and one singular member of type II (resp. one of type I and two of type II).

3.5. A 5-fold cyclic cover of \( \mathbb{P}^1 \). The following is due to I. Dolgachev. The above K3 surface has an automorphism of order 5 by construction. This implies that \( X \) is obtained as a 5-fold cyclic cover of a rational surface. Let \( \Delta \) be a divisor of \( \mathbb{P}^1 \) defined by

\[
\Delta = 4(1, \ldots, 1) + 2m_1 + m_2 + 3m_3
\]

where \( 1, \ldots, 1 \) are the fibers of the first projection from \( \mathbb{P}^1 \) branched along \( \ldots \). Then
taking the normalization and resolving the singularities, and blowing down the proper transforms of $l_1; \ldots; l_m_1;m_2;m_3$ which are exceptional curves of the first kind, we have a $K3$ surface $Y$. Locally the singularities over the intersection points of $l_i$ and $m_1; m_2$ are given by $z^5 = x^4y$ and those over the intersection points of $l_i$ and $m_3$ are given by $z^5 = x^4y^3$. Note that the ruling of the first projection from $P_1 \to \mathbb{P}^1$ gives a pencil of curves of genus 2 on $Y$. On the other hand, consider the involution of $P_1 \to \mathbb{P}^1$ which changes $m_1$ and $m_2$, and fixes $m_3$. Let $m_4$ be the another fixed fiber of this involution. Then this involution induces an involution of $Y$ which fixes the inverse image $C$ of $m_4$. The last curve $C$ corresponds to the plane quintic curve given in (3.1).

4. PICARD AND TRANSCENDENTAL LATTICES

In this section we shall study the Picard lattice and the transcendental lattice of $K3$ surfaces $X$ given in (3.2). We denote by $S_X$ the Picard lattice of $X$ and by $T_X$ the transcendental lattice of $X$.

4.1. The Picard lattice.

4.2. Lemma. Assume that $f_5 = 0$ has no multiple roots. Let $S$ be the sublattice generated by $E_0$ and components of the singular members of the pencil in Lemma 3.4. Then rank $(S) = 10$ and $det (S) = 5^3$. Moreover if $X$ is generic in the sense of moduli, then the Picard lattice $S_X = S$.

Proof. First note that the dimension of $P_5 \otimes \mathbb{R}$ is 2. On the other hand, $X$ has an automorphism of order 5 induced from $g$ given in (3.1) which acts non trivially on $H^0(X; \omega_X^2)$. Nowhere vanishing holomorphic 2-forms are eigenvectors of $g$. We can see that the dimension of the period domain is $(\dim H^0(X; \omega_X^2) = 5)$ (N2, Theorem 3.1. Also see the following section 5). Hence the local Torelli theorem implies that rank $(S_X) = 10$ for generic $X$. Let $S_0$ be the sublattice of $S_X$ generated by $E_i; F_i; (1 \leq i \leq 5)$. Then a direct calculation shows that rank $(S_0) = 10$ and $det (S_0) = 5^3$. The first assertion now follows from the relations:

$$5E_0 = \sum_{i=1}^5 (F_i \cdot 2E_0);$$
$$X = G_1 + F_1 = 2E_0 + \sum_{j \neq 0, i} E_j;$$

Note that $S = S = (\mathbb{Z}/5\mathbb{Z})^3$. Now assume that rank $(S_X) = 10$. If $S_X \notin S$, then $S_X \notin S$ and hence there exists an algebraic cycle $C$ not contained in $S$ and satisfying

$$5C = X^5 \sum_{i=0}^5 a_i E_i + \sum_{i=1}^5 b_i F_i; a_i, b_i \in \mathbb{Z};$$

By using the relations

$$h5C; E_i \equiv 0 \pmod{5}; h5C; F_i \equiv 0 \pmod{5};$$

we can easily show that

$$a_i \equiv 0 \pmod{5}; b_i \equiv 0 \pmod{5};$$

This is a contradiction.
4.3. **Discriminant quadratic forms.** Let $L$ be an even lattice. We denote by $L^*$ the dual of $L$ and put $A_L = L = L^*$. Let

$$q_L : A_L \to \mathbb{Z}$$

be the discriminant quadratic form defined by

$$q_L(x \mod L) = hx; x \mod 2\mathbb{Z}$$

and

$$b_L : A_L \to \mathbb{Z}$$

the discriminant bilinear form defined by

$$b_L(x \mod L; y \mod L) = hx; y \mod Z$$

Let $S$ be as in Lemma 4.2. Then $A_S$ is generated by

$$E_1 + 2F_1 + 3F_2 + 4E_2 = 5$$

with $q_S(\cdot) = q_S(\cdot) = 4 = 5$, and $b_S(\cdot; \cdot) = b_S(\cdot; \cdot) = 3 = 5$.

4.4. **The transcendental lattice.** Let $T$ be the orthogonal complement of $S$ in $H^2(X; \mathbb{Z})$. For generic $X$, $T$ is isomorphic to the transcendental lattice $T_X$ of $X$ which consists of transcendental cycles, that is, cycles not perpendicular to holomorphic 2-forms on $X$.

4.5. **Lemma.** Assume that $\ell_5$ has no multiple roots. Then

$$S' \ V \ A_4 \ A_4; \ T' \ U \ V \ A_4 \ A_4$$

where $V$ or $U$ is the lattice defined by the matrix

$$\begin{array}{cc}
2 & 1 \\
1 & 2
\end{array}, \begin{array}{cc}
0 & 1 \\
1 & 0
\end{array}$$

respectively.

**Proof.** We can see that $q_S$ and the discriminant quadratic form of $V \ A_4 \ A_4$ coincide. Also note that $q_S = q_S$ (Nikulin [N1], Corollary 1.6.2). Now the assertion follows from Nikulin [N1], Theorem 1.14.2.

4.6. **Lemma.** Let $S_1$ be the sublattice generated by $E_0$ and components of the singular members of the pencil in Lemma 3.4 where $i = 1$ or $2$ is the number of multiple roots of $\ell_5 = 0$. Let $T_1$ be the orthogonal complement of $S_1$ in $H^2(X; \mathbb{Z})$. Then

$$S_1' \ V \ E_8 \ A_4; \ T_1' \ U \ V \ A_4;$$

$$S_2' \ V \ E_8 \ E_8; \ T_2' \ U \ V.$$
4.7. The Kähler cone. Let $S_X$ be the Picard lattice of $X$. Denote by $P(\mathfrak{X})^+$ the connected component of the set $f \times 2 S_X \setminus R : hx; xi > 0 \mathfrak{g}$ which contains an ample class. Let $(\mathfrak{X})$ be the set of effective classes $\mathfrak{r}$ with $\mathfrak{r}^2 = 2$. Let

$$C(\mathfrak{X}) = f \times 2 P(\mathfrak{X})^+ : hx; ri > 0; r 2(\mathfrak{X}) \mathfrak{g}$$

which is called the Kähler cone of $X$. It is known that $C(\mathfrak{X}) \setminus S_X$ consists of ample classes. Let $W(\mathfrak{X})$ be the subgroup of $O(S_X)$ generated by reflections defined by

$$s_x : x ! x + hw; ri; r 2(\mathfrak{X});$$

Note that the action of $W(\mathfrak{X})$ on $S_X$ can be extended to $H^2(\mathfrak{X}; Z)$ acting trivially on $T_X$ because $r 2 S_X = T_X^2$. The Kähler cone $C(\mathfrak{X})$ is a fundamental domain of the action of $W(\mathfrak{X})$ on $P(\mathfrak{X})^+$. 

5. Automorphisms

We use the same notation as in §3, 4. In this section we study the covering involution of $X$ over $\mathbb{P}^2$ and the automorphism of $X$ of order 5.

5.1. The automorphism of order 2.

5.2. Lemma. Let $= \ldots$. Then the invariant sublattice $M = H^2(\mathfrak{X}; Z)^< >$ is generated by $E_1(0 1 5)$.

Proof. Note that $M$ is a 2-elementary lattice, that is, its discriminant group $A_M = M = M$ is a finite 2-elementary abelian group. Let $r$ be the rank of $M$ and let $l$ be the number of minimal generator of $A_M = (\mathbb{Z} = 2 \mathbb{Z})^l$. The set of fixed points of $\mathfrak{C}$ is the union of $C$ and $E_0$. It follows from Nikulin [N3], Theorem 4.2.2 that $(\mathbb{Z} r 1) = 2 = g(\mathfrak{C}) = 6$ and the number of components of fixed points set of other than $C$ is $(r 1) = 2 = 1$. Hence $r = 6; l = 4$. On the other hand we can easily see that $f E_1 : 0 i 5 \mathfrak{g}$ generates a sublattice of $M$ with rank 6 and discriminant $2^4$. Now the assertion follows.

5.3. Lemma. Let $N$ be the orthogonal complement of $M = H^2(\mathfrak{X}; Z)^< >$ in $S$. Then $N$ is generated by the classes of $F_1 F_2 G_1 G_2 i 5$ and contains no $(2)$-vectors.

Proof. Since $F_1 = G_1$, the classes of $F_1 G_1$ are contained in $N$. A direct calculation shows that their intersection matrix $(f F_1 G_1 F_2 G_2 i)_{i,j} 4$ is

$$\begin{bmatrix}
0 & 8 & 2 & 2 & 1 \\
8 & 2 & 8 & 2 & C \\
2 & 8 & 2 & 2 & A \\
2 & 2 & 8 & 2 & 8
\end{bmatrix}$$

whose discriminant is $2^4 \mathfrak{A}$. On the other hand, $N$ is the orthogonal complement of $M$ in $S$, and $M$ (resp. $S$) has the discriminant $2^4$ (resp. $5^4$). Hence the discriminant of $N$ is $2^4 \mathfrak{A}$. Therefore the first assertion follows. It follows from the above intersection matrix that $N$ contains no $(2)$-vectors.
5.4. Lemma. Let \( r \) be a \((2)\)-vector in \( H^2(X; \mathbb{Z}) \). Assume that \( r \not\in M \) in \( H^2(X; \mathbb{Z}) \). Then \( h_r ; x \in \mathbb{G} \).

**Proof.** Assume that \( h_r ; x \in \mathbb{G} \). Then \( r \) is represented by a divisor. By Riemann-Roch theorem, we may assume that \( r \) is effective. By assumption \( (r) = r \). On the other hand the automorphism preserves effective divisors, which is a contradiction.

5.5. Lemma. Let \( P (\mathcal{M})^+ \) be the connected component of the set

\[ \text{fx 2 } M \quad R : h_r ; x < 0 \mathfrak{g} \]

which contains the class of \( C \) where \( C \) is the fixed curve of \( M \) of genus 6. Put

\[ C (\mathcal{M}) = \text{fx 2 } P (\mathcal{M})^+ : h_r ; E_i > 0 ; i = 0 ; 1 ; \ldots ; 5 \mathfrak{g} \]

Let \( W (\mathcal{M}) \) be the subgroup generated by reflections associated with \((2)\)-vectors in \( M \). Then \( C (\mathcal{M}) \) is a fundamental domain of the action of \( W (\mathcal{M}) \) on \( P (\mathcal{M})^+ \) and \( O (\mathcal{M}) = \{ 1 \} \). Let \( W (\mathcal{M}) \) be the symmetry group of degree 5 which is the automorphism group of \( C (\mathcal{M}) \).

**Proof.** First consider the dual graph of \( E_0 ; \ldots ; E_5 \). Note that any maximal extended Dynkin diagram in this dual graph is \( \Gamma_4 \) with the maximal rank 4 \( (\text{rank } \mathcal{M}) = 2 \). It follows from Vinberg [V], Theorem 2.6 that the group \( W (\mathcal{M}) \) is of finite index in \( O (\mathcal{M}) \) the orthogonal group of \( M \). The assertion now follows from Vinberg [V], Lemma 2.4.

5.6. Lemma. Let \( W (\mathcal{M}) \) be the subgroup of \( O (\mathcal{M}) \) generated by all reflections associated with negative norm vectors in \( M \). Then \( C (\mathcal{M}) \) is a fundamental domain of the action of \( W (\mathcal{M}) \) on \( P (\mathcal{M})^+ \). Moreover \( W (\mathcal{M}) = W (\mathcal{M}) \). \( \mathfrak{S} \)

**Proof.** First note that \( W (\mathcal{M}) \). Let \( r = E_i \) \( E_i \) \( i < j \) \( 5 \) which is a \((4)\)-vector in \( M \). Since \( h_r ; E_i ; M i = 2 \mathfrak{Z} \), the reflection defined by

\[ s_x : x \mapsto x + h_r ; xi = 2 \]

is contained in \( O (\mathcal{M}) \). These reflections generate \( S_5 \) acting on \( C (\mathcal{M}) \) as the automorphism group of \( C (\mathcal{M}) \). Now Lemma 5.5 implies that \( O (\mathcal{M}) = \{ 1 \} \). Let \( W (\mathcal{M}) \).

5.7. Lemma. Let \( C (X) \) be the Kähler cone of \( X \). Then

\[ C (\mathcal{M}) = C (X) \setminus P (\mathcal{M})^+ : \]

**Proof.** Since the class of \( C \) is contained in the closure of \( C (X) \), \( C (X) \setminus P (\mathcal{M})^+ \), and hence it suffices to see that any face of \( C (X) \) does not cut \( C (\mathcal{M}) \) along proper interior points of \( C (\mathcal{M}) \). Let \( r \) be the class of an effective cycle with \( r^2 = 2 \). If \( r \not\in M \), Lemma 5.5 implies the assertion. Now assume \( (r) \in r \). Then \( r = (r + (r)) \not\in F (r) = 2 + (r + (r)) = 2 \). By Hodge index theorem, \( (r + (r)) \not\in S \). Since \( r^2 = 2 \), this implies that \( (r + (r)) \not\in 0 \) or \( 1 \). If \( (r + (r)) = 2 \), again by Hodge index theorem, \( h_r ; r + (r) = 2h_r ; r + (r) = 2 \). Hence the \((4)\)-vector \( r + (r) \) defines a reflection in \( W (\mathcal{M}) \). It follows from Lemma 5.6 that \( h_r ; x + (r) ; x > 0 \) for any \( x \in 2 \mathfrak{C} (\mathcal{M}) \). Since \( r \) acts trivially on \( M \), \( h_r ; x > 0 \) for any \( x \in 2 \mathfrak{C} (\mathcal{M}) \). Thus we have proved the assertion.
5.8. An isometry of order five. Let \( \sigma \) be the automorphism of \( X \) of order 5 induced by the automorphism given in 3.2. In the following Lemma 5.9 we shall show that \( \tilde{j}T \) is conjugate to the isometry \( \sigma \) defined as follows:

Let \( e_1f \) be a basis of \( U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) satisfying \( e_1^2 = f^2 = 0; \) \( e_1f = 1 \). Let \( x;y \) be a basis of \( V = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \) satisfying \( x^2 = y^2 = 2; \) \( x;y = 1 \), and let \( e_1;e_2;e_3;e_4 \) be a basis of \( A_4 \) so that \( e_1^2 = 2; e_2; e_3; e_4 = 1 \) and other \( e_i \) are orthogonal.

Let \( \sigma \) be an isometry of \( U \) \( V \) defined by

\[
(5.1) \quad \sigma(e) = f; \quad \sigma(f) = e; \quad \sigma(x) = f; \quad \sigma(y) = 3f + x + y.
\]

Also let \( \theta \) be an isometry of \( A_4 \) defined by

\[
(5.2) \quad \theta(e_1) = e_2; \quad \theta(e_2) = e_3; \quad \theta(e_3) = e_4; \quad \theta(e_4) = (e_1 + e_2 + e_3 + e_4).
\]

Combining \( \sigma \) and \( \theta \), we define an isometry of \( T = U + V + A_4 + A_4 \). By definition, is of order 5 and has no non-zero fixed vectors in \( T \). Moreover the action of the discriminant group \( T = T \) is trivial. Hence can be extended to an isometry (we use the same symbol) of \( H^2(X;\mathbb{Z}) \) acting trivially on \( S \) (Nikulin [N1], Corollary 1.5.2).

5.9. Lemma. The isometry \( \tilde{j}T \) is conjugate to \( \sigma \).

Proof. By the surjectivity of the period map of \( K \) 3 surfaces, there exists a 3 surface \( X \) whose transcendental lattice \( T_X \) is isomorphic to \( T \). Moreover we may assume that \( \tilde{j}X \) is an eigenvector of \( \sigma \) under the isomorphism \( T_X \cong T \). Since \( \tilde{j}X \) acts trivially on \( S_X \), there exists an automorphism \( \tilde{j}X \) of \( X \) with \( ( \tilde{j}X ) = (P_5) \).

Since \( S_X \cong S \), there exist 16 (2)-classes in \( S_X \) whose dual graph coincides with that of \( E_1, \) \( E_2; G_k (1 \leq j; k \leq 5) \) on \( X \) in 3.2. We denote by \( E_1^0; F_1^0; G_k^0 \) these divisors corresponding to \( E_1; F_1; G_k \). We shall show that if necessary by changing them by \( w \) \( (E_1^0); w \) \( (F_1^0); w \) \( (G_k^0) \) for a suitable \( w \) \( 2; \) \( 2; \) \( \mathbb{Z} \), all \( E_1^0; F_1^0; G_k^0 \) are smooth rational curves. Consider the divisors \( D = 2E_1^0 + E_1^0 + E_2^0 + E_3^0 + E_4^0 \). Obviously \( D \) is a singular fiber of type \( I_0 \) and \( E_1^0 (0 \leq i \leq 4) \) are components of singular fibers. Thus we may assume that \( E_1^0, \) \( E_1^0 \) are smooth rational curves. Next consider the divisor \( D = 2E_1^0 + E_1^0 + E_2^0 + E_3^0 + E_4^0 \). By replacing \( D \) by \( w \) \( (D) \) \( w \) \( 2; \) \( \mathbb{Z} \), we may assume that \( D \) is a singular fiber of type \( I_0 \) and all \( E_1^0 \) are smooth rational curves.

Next we shall show that the incidence relation of \( E_1^0; F_1^0; G_k^0 \) is the same as that of \( E_1; F_1; G_k \). Obviously \( E_1^0 \) is a pointwisely fixed by \( \tilde{j}X \). Recall that \( \tilde{j}X \) acts on \( H^0(\mathbb{Z}; \chi; \gamma) \) non trivially. By considering the action of \( \tilde{j}X \) on the tangent space of \( E_1^0 \), \( E_1^0 \) acts on \( E_0^0 \) non trivially. Now consider the elliptic fibration defined by the linear system \( 2E_1^0 + E_1^0 + E_2^0 + E_3^0 + E_4^0 \). By replacing \( D \) by \( w \) \( (D) \) \( w \) \( 2; \) \( \mathbb{Z} \), we may assume that \( D \) is an elliptic fibration. Thus we may assume that all \( E_1^0 \) are smooth rational curves. Since \( jF_1^0 + G_1^0 \) \( j \) \( 2E_0^0 + E_1^0 + E_2^0 + E_3^0 + E_4^0 \), all \( F_1^0; G_1^0 \) are also smooth rational curves.

Next we shall show that the incidence relation of \( E_1^0; F_1^0; G_k^0 \) is the same as that of \( E_1; F_1; G_k \). Obviously \( E_0^0 \) is pointwisely fixed by \( \tilde{j}X \). Recall that \( \tilde{j}X \) acts on \( H^0(\mathbb{Z}; \chi; \gamma) \) non trivially. By considering the action of \( \tilde{j}X \) on the tangent space of \( E_0^0 \), \( \tilde{j}X \) acts on \( E_0^0 \) non trivially. Now consider the elliptic fibration defined by the linear system \( 2E_0^0 + E_1^0 + E_2^0 + E_3^0 + E_4^0 \). By replacing \( D \) by \( w \) \( (D) \) \( w \) \( 2; \) \( \mathbb{Z} \), we may assume that all \( E_1^0 \) are smooth rational curves. Since \( \tilde{j}F_0^0 + G_0^0 \) \( \tilde{j} \) \( 2E_0^0 + E_1^0 + E_2^0 + E_3^0 + E_4^0 \), all \( F_0^0; G_0^0 \) coincide with that of \( E_1; F_1; G_k \).
Finally define the isometry $^0$ of order 2 of $S_Y$ by $^0(\mathcal{E}) = \mathcal{G}_1 (1, 1, 5)$ and $^0(\mathcal{E}) = \mathcal{E}_1$. Then $^0$ can be extended to an isometry of $H^2(\mathcal{M}, \mathbb{Z})$ acting on $T_{X^0}$ as $1$. By definition of $^0$, it preserves $\mathcal{M}$, and hence preserves the Kähler cone $\mathcal{K}$ (Lemma 5.6). By the Torelli theorem, there exists an automorphism $^0$ with $(^0) = 0$. It follows from Nikulin [N3], Theorem 4.2.2 that the set of fixed points of $^0$ is the disjoint union of $E_0$ and a smooth curve of genus 6. By taking the quotient of $X^0$ by $^0$, we have the same configuration as in 3.2. Thus $X^0$ can be deformed to $X$ smoothly and hence $^0$ is conjugate to $^0$.

5.10. Lemma. Let $\mathbb{E} = 0$. Let $K$ be the sublattice generated by $^i(e) (0, 1, 4)$. Then $K$ contains a vector with positive norm.

Proof. First note that $e$, $(e)$, $^2(e)$ are linearly independent isotropic vectors. Since the signature of $T$ is $(2; 8)$, we may assume that $x = t^3$; $(e)e = 0$: Then $e$ is a desired one.

5.11. Lemma. Let $\mathbb{E} = 2$. Let $R$ be the lattice generated by $^i(e) (0, 1, 4)$. Assume that $R$ is negative definite. Then $R$ is isometric to the rank lattice $A_4$.

Proof. Put $m_1 = \mathbb{E}^2; ^i(e)i = 1$. Also obviously $m_1 = m_4, m_2 = m_3$ and $^i(1) = 0$. Then

$$2 = x^2 = \mathbb{E}^2; ^i(e)i = 2m_1, 2m_2.$$

Hence $\langle m_1, m_2 \rangle = (1; 0)$ or $(0; 1)$. Therefore $\mathbb{E}^i(\mathbb{E}) : 0, 1, 3z$ is a basis of the root lattice $A_4$.

5.12. Lemma. Let $R = A_4$ be a sublattice of $T$. Assume that $R$ is invariant under the action of $^0$. Then the orthogonal complement $R^0$ of $R$ in $T$ is isomorphic to $U \vee A_4$.

Proof. Let $T^0$ be the orthogonal complement of $R$ in $T$. Then $T = R \vee T^0$ or $T$ contains $R \vee T^0$ as a sublattice of index 5. We shall show that the second case does not occur. Assume that $\{T: R \vee T^0\} = 5$. Then $A_T = \langle T \rangle = T^0 = \langle \mathcal{E} \rangle$ because $jA_X \in \{T: R \vee T^0\} = \langle \mathcal{E} \rangle$. Let $^0$ be an isometry of $L$ so that $^0jT^0 = jT^0$ and $^0j(T^0) = 1$. The existence of such $^0$ follows from [N1], Corollary 1.5.2. It follows from the surjectivity of the period map of $K$ surfaces that there exists a $K$ 3 surface $Y$ whose transcendental lattice isomorphic to $T^0$ and whose period is an eigen-vector of $^0$ under a suitable marking. Since $^0$ acts trivially on the Picard lattice $(T^0) = 0$, $0$ is induced from an automorphism $^0$ of $Y$. It follows from Vorontsov's theorem [Vo] that the number of minimal generator of $A_T$ is at most rank $(T^0) = 5 = 2$ where $^0$ is the Euler function. This contradicts the fact $A_T = \langle \mathcal{E} \rangle$. Thus we have proved that $T = R \vee T^0$. Since $\mathcal{E} = \mathcal{E} \vee A_4$, the assertion now follows from Nikulin [N1], Theorem 1.14.2.

5.13. Discriminant locus. Let $\mathbb{E} = 2$. Let $R$ be the sublattice generated by $^i(e) (0, 1, 4)$. Assume that $R$ is negative definite. Then $R = A_4$ and the orthogonal complement of $R$ in $T$ is isomorphic to $T^0 = U \vee A_4$ (Lemmas 5.11, 5.12). Let $^0$ be an isometry of $L$ so that $^0jT^0 = jT^0$ and $^0j(T^0) = 1$. Then there exists an $K$ 3 surface $Y$ and an automorphism $^0$ such that the transcendental lattice of $Y$ is isomorphic to $T^0$, the period of $Y$ is an eigen-vector of $^0$ and $^0$ acts trivially on the Picard lattice of $Y$. By the same argument as in the proof of Lemma 5.9, we can see that $Y$ is corresponding to the case that $f_5 = 0$ has a multiple root in 3.2.
6. A COMPLEX BALL UNIFORMIZATION

6.1. Hermitian form. Let $\mathbb{P}^{n} = \mathbb{C}$. We consider $T$ as a free $\mathbb{Z}[x]$-module by $(a + b)x = ax + b \langle x \rangle$.

Let

$$h(\langle x \rangle, y) = \sum_{i=0}^{5} X^i h_{x}^i (y) i,$$

Then $h(\langle x \rangle, y)$ is a hermitian form on $\mathbb{Z}[x]$-module. With respect to a $\mathbb{Z}[x]$-basis $e_i$ of $A_4$, the hermitian matrix of $h$ is given by (6.1)

$$h = \begin{pmatrix}
0 & 5 & 1 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{pmatrix}.$$

Let

$$' : \mathbb{P}^{n} \rightarrow \mathbb{P}^{n}$$

be a linear map defined by

$$' (x) = \begin{pmatrix}
X^3 \\
(i + 1)^{i} (x) = 5 \\
i = 0
\end{pmatrix}.$$

Note that $' (\langle 1 \rangle x) = ' (\langle x \rangle (x)) = 4 (x) 2 \mathbb{P}^{n}$. Hence $'$ induces an isomorphism

$$' (\langle 1 \rangle )' A_4 = T = T.$$

6.2. Reflections. Let $a \in A_4$ with $h(\langle a \rangle, a) = 1$. Then the map

$$R_a : \mathbb{P}^{n} \rightarrow \mathbb{P}^{n}$$

is an automorphism. This automorphism $R_+^a$ has order 5 and $R_a$ has order 10 both of which fix the orthogonal complement of $a$. They are called reflections. Consider a decomposition

$$T = U \bigoplus A_4 \bigoplus A_4.$$

If $a = e_i$ of the last component $A_4$ as in \[\text{[5.2]}\], we can easily see that

$$R_a = s_{e_i} s_{e_i} s_{e_i} s_{e_i}$$

where $s_{e_i}$ is a reflection in $O(T)$ associated with $(2)$-vector $e_i$ defined by

$$s_{e_i} : x \mapsto x + h(x; e_i e_i).$$

In other words,

$$R_{e_i} = 1_U \bigoplus 1_V \bigoplus 1_{A_4} \bigoplus 4,$$

Since $s_{e_i}$ acts trivially on $A_4$, $R_+^a$ acts trivially on $A_4$ and $R_a$ acts on $A_4$ as a reflection associated with $(e_1 + 2e_2 + 3e_3 + 4e_4) = 5 2 A_4$:
6.3. **The period domain and arithmetic subgroups.** We use the same notation as in 5.8. Let
\[
\mathbb{T} = \mathbb{T}_1 \oplus \mathbb{T}_2 \oplus \mathbb{T}_3 \oplus \mathbb{T}_4
\]
be the decomposition of \( T \)-eigenspaces where \( T \) is a primitive 5-th root of unity (see Nikulin [N2], Theorem 3.1). An easy calculation shows that
\[
e = e_1 + (4 + 1)e_2 + (2)e_3 + e_4
\]
is an eigenvector of \( T \) with the eigenvalue \( h; i = 5 \). On the other hand,
\[
e = e_1 + (4 + 1)e_2 + (2)(e + f + y)(e + f + x)
\]
is an eigenvector of \( T \) with the eigenvalue \( h; i = 5(2 + 3) \). Thus if \( e \in \mathbb{P} \mathbb{T}^5 \), the hermitian form \( h!; i=5 \) on \( T \) is of signature \((1; 2)\) and is given by
\[
(6.3)
\begin{pmatrix}
0 & p & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]
For other , the hermitian form is negative definite. Now we take \( e \in \mathbb{P} \mathbb{T}^5 \) and define
\[
B = \{z \in \mathbb{T} : hz; zi > 0\};
\]
Then \( B \) is a 2-dimensional complex ball. For a \((2)\)-vector \( \alpha \) in \( T \), we define
\[
H_\alpha = \frac{\alpha}{\alpha^2} B; \quad H = \bigcup_{\alpha \in T} H_\alpha
\]
where \( \alpha \) runs over \((2)\)-vectors in \( T \). Let
\[
(6.5)
\begin{pmatrix}
f & 0 & 0 & 0 & 0 & 0 \\
0 & f & 2 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]
6.4. **Remark.** The hermitian form \( h \) in (6.1) coincides with the one of Shimura [S], Yamazaki and Yoshida [YY]. This and the isomorphism (6.2) imply that our groups \( 0 \) coincide with \( (1; 2) \) in Yamazaki and Yoshida [YY].

6.5. **Proposition.** (1) is generated by reflections \( R_a \) with \( h(a; a) = 1 \) and \( 0 \) is generated by \( R_a^2 \) with \( h(a; a) = 1 \). The quotient \( = 0 \) is isomorphic to \( O(3; \mathbb{F}_5) \)' \( \mathbb{Z} = 22 \) \( S_5 \).

(2) \( H = 0 \) consists of 10 smooth rational curves forming 10 lines on the quintic del Pezzo surface.

**Proof.** The assertions follow from the above Remark 6.4 and Propositions 4.2, 4.3, 4.4 in Yamazaki and Yoshida [YY].
6.6. **Discriminant quadratic forms and discriminant locus.** Let
\[ q_T : \mathbb{A}_T \to \mathbb{P}^1 \]
be the discriminant quadratic form of \( T \). The discriminant group \( \mathbb{A}_T \) consists of the following 125 vectors:

- **Type (0):** \( q_T(\cdot) = 0; \# = 1; \)
- **Type (0):** \( \mathcal{E} 0; q_T(\cdot) = 0; \# = 24; \)
- **Type (2=5):** \( q_T(\cdot) = 2=5; \# = 30; \)
- **Type (2=5):** \( q_T(\cdot) = 2=5; \# = 30; \)
- **Type (4=5):** \( q_T(\cdot) = 4=5; \# = 20; \)
- **Type (4=5):** \( q_T(\cdot) = 4=5; \# = 20. \)

Let \( \mathbb{A}_3 \) be a component of \( T \) with a basis \( e_1, e_2, e_3, e_4 \) as in \( \S 5.6. \) Then \( (e_1 + 2e_2 + 3e_3 + 4e_4) = 5 = (e_1^2 - 2e_2 + 3^2(e_3) + 4^3(e_1)) \) \( \pmod T \) is a vector in \( \mathbb{A}_T \) with norm \( 4=5. \) It follows from Proposition \( 6.5 \) that \( 0 \) acts transitively on the set of \( (4=5) \)-vectors in \( \mathbb{A}_T \). Hence for each \( 2 \mathbb{A}_T \) with \( q_T(\cdot) = 4=5 \) there exists a vector \( 2 \mathbb{T} \) with \( r^2 = 2 \) satisfying \( r^2 = (x + 2(x) + 3^2(x) + 4^3(x)) = 5 \) \( \pmod T \). Moreover, defines
\[ H = \begin{bmatrix} x \\ r \end{bmatrix} \]
where \( r \) moves over the set
\[ f x 2 \mathbb{T} : r^2 = 2; \]
Thus the set
\[ f 2 \mathbb{A}_T : q_T(\cdot) = 4=5 \]
bijectively corresponds to the set of components of \( H = 0 \). Let
\[ \sim = f 2 \mathcal{O} (L) : \sim = \sim ; \]

6.7. **Lemma.** The restriction map \( \sim ! \) is surjective.

**Proof.** We use the same notation as in \( \S 5.2 \). The symmetry group \( S_5 \) of degree 5 naturally acts on the set \( \mathcal{E} 1; \cdots; \mathcal{E}_5 \) as permutations. This action can be extended to the one on \( S \). Together with the action of \( S \), the natural map
\[ \mathcal{O} (S) ! \mathcal{O} (\mathcal{E}_5) = f 1 \mathcal{E}_5 \]
is surjective. Let \( g 2 \). Then the above implies that there exists an isometry \( g^0 \) in \( \mathcal{O} (S) \) whose action on \( \mathbb{A}_5 = \mathbb{A}_T \) coincides with the one of \( g \) on \( \mathbb{A}_T \). Then it follows from Nikulin [N1], Proposition 1.6.1 that the isometry \( (g^0; g) \) of \( S \) \( \to \) \( T \) can be extended to an isometry in \( \sim \) which is the desired one.

6.8. **Period map.** We shall define an \( S_5 \)-equivariant map
\[ \mathbb{P} : \mathbb{P}_1 ! \mathbb{P}^1 \]
called the period map. Denote by \( (\mathbb{P}^5)^0 \) the locus of distinct five ordered points on \( \mathbb{P}^1 \). Let \( f_{\mathbb{P}_1}; \cdots; f_{\mathbb{P}_5} 2 (\mathbb{P}^5)^0 \). Let \( X \) be the corresponding \( \mathbb{R} 3 \) surface with the automorphism \( \mathcal{O} (S) \) of order 5 as in \( \S 5.2 \) The order of \( f_{\mathbb{P}_1}; \cdots; f_{\mathbb{P}_5} \) defines an order of smooth rational curves
\[ E_i; \quad (0 \ i \ 5) \quad F_j; G_j; \quad (1 \ j \ 5) \]
modulo the action of the covering involution \( \). It follows from Lemma \( \S 5.9 \) that there exists an isometry
\[ : L ! \mathbb{H}^2 (X ; \mathbb{Z}) \]
satisfying \( p \). Now we define
\[
p(\mathfrak{X} ; \mathfrak{Y}) = (\mathcal{C} \backslash \{p\})
\]

6.9. Lemma. \( p(\mathfrak{X} ; \mathfrak{Y}) \) and \( B \cap H \).

Proof. If not, there exists a vector \( x \) with \( x^2 = 2 \) which is represented by an effective divisor on \( X \) as in the proof of Lemma 5.4. Obviously \( x + r(\mathfrak{X}) + \cdots + r(\mathfrak{Y}) = 0 \). On the other hand \( x + r(\mathfrak{X}) + \cdots + r(\mathfrak{Y}) \) is non-zero effective because is an automorphism. Thus we have a contradiction.

Thue we have a holomorphic map
\[
p : \mathbb{P}^5_1 ! \quad (B \cap H) = 0:
\]
The group \( S_5 \) naturally acts on \( \mathbb{P}^5_1 \) which induces an action on \( S \) as permutations of \( E_1, \ldots, E_5 \). On the other hand, \( S_5 = \mathfrak{g} \) naturally acts on \( B = 0 \). Under the natural isomorphism \( O(\mathfrak{g}) = O(\mathfrak{f}) = \mathfrak{g} 1 \mathfrak{g} \), \( p \) is equivariant under these actions of \( S_5 \).

It is known that the quotient \( B = 0 \) is compact (see Shimura [S]). We remark that cusps of \( B \) correspond to totally isotropic sublattices of \( T \) invariant under \( \mathfrak{g} \). Hence the compactness also follows from Lemma 5.10.

6.10. Main theorem. The period map \( p \) can be extended to an \( S_5 \)-equivariant isomorphism
\[
p : \mathbb{P}^5_1 ! \quad B = 0:
\]

Proof. Let \( M \) be the space of all 5 stable points on \( \mathbb{P}^1 \) and \( M \cap 0 \) the space of all distinct 5 points on \( \mathbb{P}^1 \). We can easily see that \( M \cap \mathbb{P}^1 \) is locally contained in a divisor with normal crossing. By construction, \( p \) is locally liftable to \( B \). It now follows from a theorem of Borel ([Borel]) that \( p \) can be extended to a holomorphic map from \( M \) to \( B = 0 \) which induces a holomorphic map \( p \) from \( \mathbb{P}^5_1 \) to \( B = 0 \). Next we shall show the injectivity of the period map over \( (B \cap H) = 0 \). Let \( C \cap C^0 \) be two plane quintic curves as in [3.1]. Let \( (\mathfrak{X} ; \mathfrak{Y}) \) (resp. \( (\mathfrak{X}^0 ; \mathfrak{Y}) \)) be the associated marked \( K \) 3 surfaces with automorphisms \( \mathfrak{Y} \) (resp. \( \mathfrak{Y}^0 \)). Assume that the periods of \( (\mathfrak{X} ; \mathfrak{Y}) \) and \( (\mathfrak{X}^0 ; \mathfrak{Y}) \) coincide in \( B = 0 \). Then there exists an isometry
\[
\mathfrak{Y} : H^2(\mathfrak{X}^0 ; Z) ! H^2(\mathfrak{X} ; Z)
\]

preserving the periods and satisfying \( \mathfrak{Y}(\mathfrak{g}) = \mathfrak{Y}^0 \) and \( \mathfrak{Y}^0(\mathfrak{g}) = \mathfrak{Y}^0 \) (Lemma 6.7). It follows from Lemma 5.5 that \( \mathfrak{Y} \) preserves the Kähler cones. The Torelli theorem for \( K \) surfaces implies that there exists an isomorphism \( \mathfrak{f} : X ! X^0 \) with \( \mathfrak{f} = \mathfrak{Y} \). Then \( \mathfrak{f} \) induces an isomorphism between the corresponding plane quintic curves \( C \) and \( C^0 \). Thus we have proved the injectivity of the period map.

Since both \( \mathbb{P}^5_1 \) and \( B = 0 \) are compact, \( p \) is surjective. Recall that both \( \mathbb{P}^5_1 \cap (\mathbb{P}^5_1)^0 \) and \( H = 0 \) consist of 10 smooth rational curves. The surjectivity of \( p \) implies that no components of \( \mathbb{P}^5_1 \cap (\mathbb{P}^5_1)^0 \) contract to a point. Now the Zariski main theorem implies that \( p \) is isomorphic. By construction, \( p \) is \( S_5 \)-equivariant over the Zariski open set \( (\mathbb{P}^5_1)^0 \). Hence \( p \) is \( S_5 \)-equivariant isomorphism between \( \mathbb{P}^5_1 \) and \( B = 0 \).
7. Shimura-Terada-Deligne-Mostow’s Reflection Groups

The plane quintic curve $C$ defined by (3.1) appeared in the papers of Shimura [S], Terada [Te], Deligne-Mostow [DM], and the moduli space of these curves has a complex ball uniformization. As we remarked, the hermitian form (6.1) coincides with those of Shimura [S], Terada [Te], Deligne-Mostow [DM] (see Remark 6.4). This implies

7.1. Theorem. The arithmetic subgroup is the one appeared in Deligne-Mostow’s list [DM]:

$$\frac{2}{5}; \frac{3}{5}; \frac{2}{5}; \frac{3}{5}; \frac{2}{5};$$

A geometric meaning of this theorem is as follows. Recall that $X$ has an isotrivial pencil of curves of genus two whose general member is the smooth curve $D$ of genus two with an automorphism of order 5 given by the equation (3.3). The $X$ is given by

$$s^2 = x_0 (\kappa_0^5 + f_5 (x_1; x_2))$$

where $x_1=x_2$ is the parameter of this pencil. On the other hand, we consider $C$ as a base change $C! P^1$ given by

$$(v; x_1; x_2) : (x_1; x_2):$$

Then over $C$, $v^5 = f_5 (x_1; x_2)$ and hence the pencil is given by

$$s^2 = x_0 (\kappa_0^5 + v^5)$$

which is nothing but the equation of the curve $D$. Thus the $K_3$ surface $X$ is birational to the quotient of $C$ by an diagonal action of $Z=\mathbb{Z}$. This correspondence gives a relation between the Hodge structures of $C$ and $X$.

7.2. Problem. Let $i$ be a positive rational number ($0 \leq i \leq d+1$ or $i=1$) satisfying $\rho = 2$. Set

$$F_{gh}(\kappa_2; \ldots; x_{d+1}) = \prod (\kappa_1)^i (\kappa_1 x_1)^{-i} d\kappa_1$$

where $g; h \neq 1; 0; 1; x_2; \ldots; x_{d+1}; i; g$. Then $F_{gh}$ is a multivalued function on

$$M = f (x_1) 2 (P^1)^{d+3} jx_1 \in 1 \; ; 0; 1; x_1 \in x_3 (i \in j)q:$$

These functions generate a $(d+1)$-dimensional vector space which is invariant under monodromy. Let $\{ i \}$ be the image of $\{ i \}$ in PGL $(d+1; \mathbb{C})$ under the monodromy action. In Deligne-Mostow [DM] and Mostow [Mo], they gave a sufficient condition for which $\{ i \}$ is a lattice in the projective unitary group $P U (d; 1)$, that is, $\{ i \}$ is discrete and of finite covolume, and gave a list of such $\{ i \}$ (see [Th] for the correction of their list).

Denote $D = \frac{D}{D}$ where $D$ is the common denominator. As remarked in Theorem 7.1 in the case $D = 5$, $\{ i \}$ is related to $K_3$ surfaces. In case of $D = 3; 4$ or $6$, $\{ i \}$ is also related to $K_3$ surfaces (see [K2], [K3], [DGK]). In these cases, the corresponding $K_3$ surfaces have an isotrivial elliptic fibration whose general fiber is an elliptic curve with an automorphism of order 4 or 6.

For the remaining arithmetic subgroups $\{ i \}$ with $D > 6$ in the Deligne-Mostow’s list, are they related to $K_3$ surfaces?
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