SYMBOLIC SUBSTITUTION SYSTEMS BEYOND ABELIAN GROUPS

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Abstract. Symbolic substitution systems are an important source of aperiodic Delone sets in abelian locally compact groups. In this article, we consider a large class of non-abelian nilpotent Lie groups with dilation structures, which we refer to as rationally homogeneous Lie groups with rational spectrum (RAHOGRASPs). We show, by explicit construction, that every RAHOGRASP admits a lattice with a primitive symbolic substitution system and hence contains a weakly aperiodic linearly repetitive Delone set. These are the first examples of weakly aperiodic linearly repetitive Delone sets in non-abelian Lie groups. Building on our previous work, we establish unique ergodicity of the corresponding Delone dynamical systems. Our construction applies in particular to all two-step nilpotent Lie groups defined over $\mathbb{Q}$ such as the Heisenberg group. In this case, the Delone sets in question are in fact strongly aperiodic, i.e. the underlying action of the corresponding Delone dynamical system is free.

1. Introduction

1.1. Motivation. The theory of aperiodic order is an important source of examples of dynamical systems over locally compact abelian groups: If $\Lambda$ is an aperiodic Delone set in a locally compact abelian group $A$ and $\Omega_\Lambda$ denotes its orbit closure in the space of all such Delone sets (with respect to the Chabauty–Fell topology), then $A \ltimes \Omega_\Lambda$ is a topological dynamical system. In principle, such systems can be studied for arbitrary Delone sets $\Lambda$, but in practice they have been studied mostly for three (overlapping) classes of Delone sets:

- Aperiodic Meyer sets, i.e. Delone sets arising from cut-and-project constructions;
- Aperiodic Delone sets arising from symbolic substitutions inside a lattice in $A$;
- Aperiodic Delone sets arising from geometric substitutions.

The present article continues a general program initiated in [BH18, BHP18] and developed further in [BH21, Fis19, BHS19, BHP20, BHP21a, BHP21b, BP20, BH20, Mac20, CHT20, Hru20, Mac21a, Mac21b] which aims to expand the theory of aperiodic order to non-abelian locally compact groups and to study the resulting dynamical systems. So far the focus in this program has been on extending the cut-and-project construction. Thanks to the work cited above there is now a far-reaching theory of Meyer sets in non-abelian groups; the dynamical properties of the resulting systems can sometimes be surprisingly different from the abelian case. The present article is a first attempt to extend the theory of abelian substitution dynamical systems [Que87, Sol97, Sol98, AP98, Dur00, Fog02, DL06, BG13, KL13, BO14] to certain non-abelian groups, focusing on the symbolic setting.

1.2. General framework. We will develop a general framework to study and construct symbolic substitution systems, leading to aperiodic Delone sets in a large class of nilpotent (non-abelian) Lie groups. Already in dimension less than ten, our construction applies to hundreds of families of Lie groups and is both very explicit and very flexible; this flexibility will allow us to construct aperiodic Delone sets with peculiar properties, most notably linear repetitivity. To the best of our knowledge, these are the first examples of aperiodic linearly repetitive Delone sets.
sets in non-abelian Lie groups. Before we get into the details of our construction, let us roughly outline the results that can be expected from our framework; precise definitions will be given later. In order to define a symbolic substitution we need two types of input, namely geometric and combinatorial data.

**Geometric data:** Our ambient space will always be a 1-connected Lie group $G$ together with a left-invariant metric $d$ inducing the topology on $G$. Moreover we need a lattice $\Gamma \triangleleft G$ and a corresponding fundamental domain $V \subseteq G$ as well as a one-parameter group $(D_\lambda)_{\lambda > 0}$ of dilations of $(G, d)$ such that $D_{\lambda_0}(\Gamma) \subseteq \Gamma$ for some $\lambda_0 > 1$. We then refer to $D = (G, d, (D_\lambda)_{\lambda > 0}, \Gamma, V)$ as a dilation datum.

The group $G$ underlying a dilation datum must be nilpotent (in fact, a so-called homogeneous group) and admit a rational structure. Conversely, every 2-step nilpotent group with a rational structure (such as the Heisenberg group) gives rise to a dilation datum. In this article we introduce the class of rationally homogeneous Lie groups with rational spectrum (RAHOGRASPs), which contains all of these 2-step examples, but also many higher step groups. We then show that groups in this class admit particularly nice dilation data called homogeneous dilation data (see Definition 4.21 and Corollary 4.22). For dimension 7, the smallest dimension for which there is a large zoo of nilpotent Lie groups including infinite families, we provide a census of indecomposable RAHOGRASPs in Theorem 3.13 and Theorem 3.14 below, leading to hundreds of explicit examples in dimension less than than ten.

**Combinatorial data:** On top of our dilation datum $D$ we also need to choose a finite alphabet $A$, a sufficiently large stretch factor $\lambda_0 \gg 1$ (see Section 2.2) such that $D_{\lambda_0}(\Gamma) \subseteq \Gamma$ and a substitution rule $S_0 : A \rightarrow A^{D_{\lambda_0}(V)\cap \Gamma}$.

We then refer to $S = (A, \lambda_0, S_0)$ as a substitution datum over $D$.

In applications one usually wants to impose further conditions on the substitution datum. Two important such conditions, to be discussed below, are primitivity and non-periodicity, and in all of our examples these can be arranged.

**Theorem 1.1.** For every RAHOGRASP $G$ of dimension at least 2 and any finite alphabet $A$ with $|A| \geq 2$ there exists a dilation datum $D$ over $G$ and a primitive and non-periodic substitution datum over $D$ with alphabet $A$.

Our proof of Theorem 1.1 is actually constructive: In the proof of Corollary 4.22 we construct for every RAHOGRASP of dimension $\geq 2$ a homogeneous dilation datum in the sense of Definition 4.21. In the proof of Proposition 6.2 we then construct a primitive non-periodic substitution datum over an arbitrary homogeneous dilation datum. The proof shows that there is actually a huge freedom in choosing such a substitution datum.

**The construction:** With every substitution datum $S = (A, \lambda_0, S_0)$ over a dilation datum $D$ we associate a substitution map $S : A^\Gamma \rightarrow A^\Gamma$ and a canonical $S$-invariant subshift $\Omega(S) \subseteq A^\Gamma$ called the associated substitution system. While the construction of $S$ is more complicated than in the abelian case, the resulting map can still be characterized as the unique $D_{\lambda_0}$-equivariant extension of $S_0$ which is “locally defined” (cf. Proposition 2.7). Once a substitution map has been constructed, the definition of the associated substitution system is just as in the abelian case, i.e. $\Omega(S)$ consists of all $\omega \in A^\Gamma$ whose patches are $S$-legal (see Notation 2.8 below).
1.3. **Main results.** The following is our main theorem (see Proposition 7.1, Theorem 7.2, Theorem 7.4 and Theorem 8.1 respectively):

**Theorem 1.2** (Non-abelian primitive symbolic substitution systems). Let $S$ be a substitution datum over a dilation datum $D$ with associated substitution map $S: \mathcal{A}^\Gamma \to \mathcal{A}^\Gamma$ and substitution system $\Omega(S)$. Then the following hold:

(a) The subspace $\Omega(S) \subseteq \mathcal{A}^\Gamma$ is an $S$-invariant subshift, i.e. compact, $S$-invariant and $\Gamma$-invariant.

(b) Some power of $S$ has a fixpoint in $\Omega(S)$; in particular $\Omega(S)$ is non-empty.

If $S$ is primitive, then

(c) the action $\Gamma \curvearrowright \Omega(S)$ is minimal, i.e. every $\Gamma$-orbit in $\Omega(S)$ is dense.

(d) every element in $\omega \in \Omega(S)$ is linearly repetitive.

If $S$ is non-periodic, then

(e) $\Omega(S)$ is weakly aperiodic, i.e. there is an $\omega \in \Omega(S)$ with trivial $\Gamma$-stabilizer. In fact, every fixpoint of a power of $S$ has trivial $\Gamma$-stabilizer.

Here, linear repetitivity of an element $\omega \in \mathcal{A}^\Gamma$ means that every pattern in $\omega$ of sufficiently large size $R$ is contained in every ball of radius $C \cdot R$ in $G$ for some universal constant $C$ independent of $R$, where “size” is measured with respect to the invariant metric $d$ on $G$ (see Section 7.2 below for details).

**Comparison to the Euclidean case:** Parts (a) - (d) of Theorem 1.2 are classical in the case where the underlying metric group $(G,d)$ is the Euclidean space, cf. for instance the work by Solomyak [Sol97, Sol98], Anderson and Putnam [AP98], Durand [Dur00], Fogg [Fog02], Damanik and Lenz [DL06], Baake and Grimm [BG13] and references therein. On the contrary, our notion of non-periodic substitutions seems to be new even in the Euclidean case of symbolic substitutions. It is an adaption of similar notions in the geometric setting [Sol97, Sol98, AP98], where injectivity of the substitution is needed to get aperiodic tilings. In the one-dimensional situation, weak aperiodicity of a subshift is characterized by the existence of proximal pairs [BO14], cf. also the discussions in higher dimensions in [BG13, KL13].

In the Euclidean case, one can deduce two additional properties of $\Omega(S)$ from the properties listed in Theorem 1.2:

- If $S$ is primitive, then $\Omega(S)$ is a uniquely ergodic $\Gamma$-space [Dur00, DL01, Len02, LP03, DL06].
- If $S$ is primitive and non-periodic, then $\Omega(S)$ is strongly aperiodic in the sense that every $\omega \in \Omega(S)$ has trivial $\Gamma$-stabilizer, [BO14, BG13].

Both of these conclusions follow by abstract arguments: If $(G,d)$ is Euclidean and $\omega \in \mathcal{A}^\Gamma$ is linearly repetitive, then the orbit closure $\Omega_\omega$ is uniquely ergodic by [LP03, DL06]. Similarly, if $(G,d)$ is Euclidean and $\omega \in \mathcal{A}^\Gamma$ has trivial $\Gamma$-stabilizer and minimal orbit closure $\Omega_\omega$, then this orbit closure is strongly aperiodic.

If $G$ is non-abelian and hence $(G,d)$ is a non-Euclidean geometry, then there does not seem to be an abstract argument which would guarantee the unique ergodicity or strong aperiodicity of orbit closures. However, both unique ergodicity and strong aperiodicity can be established under additional assumptions. By the following theorem (cf. Theorem 9.3), unique ergodicity holds in all of our explicit examples:

**Theorem 1.3 (Unique ergodicity).** Let $S$ be a primitive substitution datum over a homogeneous dilation datum. Then the associated substitution system $\Omega(S)$ is uniquely ergodic.
The theorem holds more generally if $S$ is primitive and the underlying metric group $(G,d)$ of $D$ has exact polynomial growth (see the discussion in Subsection 3.4). In fact, by the main theorem of [BHP20, Theorem 1.4] the orbit closure of a linearly repetitive pattern is always uniquely ergodic in a group of exact polynomial growth. In this generality, the proof of unique ergodicity is substantially more involved than in the abelian case. In fact, most of the companion paper [BHP20] is devoted to this proof.

While unique ergodicity holds at least in all the examples considered here, the question of strong aperiodicity is more subtle, and we confine ourselves with a sample result (see Section 8.2 for details). Concerning the following theorem we remark that every 1-connected 2-step nilpotent Lie group with a rational structure admits a canonical homogeneous structure.

**Theorem 1.4** (Strong aperiodicity). Let $D$ be a homogeneous dilation datum whose underlying homogeneous Lie group is 2-step nilpotent and endowed with its canonical homogeneous structure. Then for every non-periodic primitive substitution datum $S$ over $D$ the associated substitution system $\Omega(S)$ is strongly aperiodic, i.e. every $\omega \in \Omega(S)$ has trivial $\Gamma$-stabilizer.

It is likely that strong aperiodicity can also be established for certain (maybe even all) higher step examples by an inductive argument similar to ours, but since the set-up gets quite technical if one increases the step-size, we will not pursue this here.

**Delone sets:** Our symbolic results can be used to establish corresponding results for Delone sets in the usual way. Given a Delone subset $\Lambda$ of a Lie group $G$ we denote by $\Omega_\Lambda$ the $G$-orbit closure of $\Lambda$ with respect to the Chabauty-Fell topology (see Subsection 9.2); from Theorems 1.1, 1.2, 1.3 and 1.4 we can then deduce the following corollary by standard methods (see Section 9.2).

**Corollary 1.5** (Linearly repetitive Delone sets). Every RAHOGRA$SP$ $G$ contains a linearly repetitive Delone set $\Lambda$ with trivial $G$-stabilizer such that $\Omega_\Lambda$ is minimal and uniquely ergodic. If $G$ is 2-step nilpotent and endowed with its canonical homogeneous structure, then we can moreover arrange for $\Omega_\Lambda$ to be strongly aperiodic.

**Related work and open problems:** Although there is a large literature concerning dynamical systems over abelian groups which satisfy some form of linear repetitivity, very few examples seem to be known in the non-abelian case. In fact, the only examples that we are aware of are those constructed recently by Pérez [Pér20] in the context of Schreier graphs associated with certain actions of spinal groups. We thus believe that the present article provides the first construction of aperiodic linearly repetitive Delone sets in non-abelian nilpotent Lie groups and thus (via the Voronoi construction) of aperiodic self-similar tilings in such groups. On the contrary, periodic self-similar tilings of RAHOGRA$SP$s have been a subject of study ever since Strichartz’ highly influential article [Str92]; see in particular the work of Gelbrich [Gel94] for a very explicit construction of periodic self-similar tilings in the Heisenberg group.

Concerning general Delone sets in nilpotent Lie groups, not a lot seems to be known yet. Machado [Mac20] has established that such Delone sets satisfy the Meyer condition if and only if they are relatively dense subsets of approximate lattices. The linearly repetitive Delone sets constructed in the current article are actually Meyer, but for the trivial reason that they are even relatively dense in – and hence bounded displacement (BD) equivalent and bilipschitz (BL) equivalent to – a lattice. On the contrary, the work of Dymarz, Kelly, Li and Lukyanenko [DKLL18] shows that there are many different BD and BL equivalence classes of Delone sets in nilpotent Lie groups, which in general cannot be represented by lattices. We currently do not know whether BL equivalence classes, which do not contain a lattice, can admit linearly repetitive Delone sets.
Our examples are also special from a dynamical point of view, since the horizontal factors (in the sense of [BH21]) of their hull dynamical systems are homogeneous. Nevertheless, it might be interesting to study their central diffraction in the sense of [BH21] and compare it to the central diffraction of the ambient lattices.

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2. Examples of dilation data and substitution data

In this section we will explain our framework in more details and discuss various concrete examples.

2.1. The geometry: Dilation data. The precise definition of a dilation group and a dilation datum is as follows:

Definition 2.1. A dilation group is a triple $(G, d, (D_\lambda)_{\lambda > 0})$, where

(D1) $G$ is a connected locally compact second countable group, called the underlying group, and $d$ is a left-invariant metric on $G$ which induces the given topology on $G$;

(D2) $(D_\lambda)_{\lambda > 0}$ is a one-parameter group of automorphisms of $G$ (i.e. there is a group homomorphism $D : (\mathbb{R}^+ , \cdot) \to \text{Aut}(G)$ such that $D(\lambda) = D_\lambda$), called the underlying dilation family, such that

$$d(D_\lambda(g), D_\lambda(h)) = \lambda \cdot d(g, h) \quad \text{for all } \lambda > 0, g, h \in G.$$  

If $(G, d, (D_\lambda)_{\lambda > 0})$ is a dilation group, then a subgroup $\Gamma < G$ will be called invariant if $D_\lambda(\Gamma) \subseteq \Gamma$ for some $\lambda > 1$. We recall that a subgroup $\Gamma < G$ is called a uniform lattice if it is discrete and cocompact. In this case there exists a bounded Borel subset $V \subseteq G$ which intersects each $\Gamma$-orbit (for the left-multiplication action of $\Gamma$ on $G$) in a single point and contains an open identity neighbourhood. We refer to such a set $V$ as a convenient fundamental domain for $\Gamma$.

Definition 2.2. A dilation datum $D = (G, d, (D_\lambda)_{\lambda > 0}, \Gamma, V)$ consists of an underlying dilation group $(G, d, (D_\lambda)_{\lambda > 0})$, an invariant uniform lattice $\Gamma < G$ and a convenient fundamental domain $V$ for $\Gamma$.

The following notion will be needed later (see Remark 2.10 below for an explanation): Given a dilation datum $D$ as above, we say that a real number $\lambda_0$ is sufficiently large relative to $V$ if $\lambda_0 > 1 + \frac{r_+ - r_-}{r_+}$, where $V$ contains an open ball or radius $r_-$ and is contained in an open ball of radius $r_+$ around the identity. Dilation data exist both in the abelian and the non-abelian world:
Example 2.3. Let $G = (\mathbb{R}^n, +)$ with Euclidean metric $d$ and let $D_\lambda \in \text{Aut}(G)$ be given by

$$D_\lambda(x_1, \ldots, x_n) := (\lambda^{r_1}x_1, \ldots, \lambda^{r_n}x_n)$$

for some parameters $r_1, \ldots, r_n > 0$. Then $(G, d, (D_\lambda)_{\lambda > 0})$ is a dilation group and if $r_1, \ldots, r_n$ are integers, then $\Gamma := \mathbb{Z}^n$ is an invariant lattice and $V := [-\frac{1}{2}, \frac{1}{2})^n$ is a convenient fundamental domain. In this case, every $\lambda_0 \geq 3$ is sufficiently large with respect to $V$.

Example 2.4. Let $G$ be the 3-dimensional Heisenberg group, which we will think of as $\mathbb{R}^3$ with multiplication given by

$$(x, y, z) \ast (x', y', z') = (x + x', y + y', z + z' + \frac{1}{2}(xy' - yx')).$$

We then have a dilation datum $D_\mathbb{H} := (\mathbb{H}_3(\mathbb{R}), d, (D_\lambda)_{\lambda > 0}, \Gamma, V)$, where

- the homogeneous metric $d$ on $G$ is the unique left-invariant metric such that
  $$d(((x, y, z), (0, 0, 0)) = ((x^2 + y^2)^2 + z^2)^{1/4},$$
- the dilation family $(D_\lambda)_{\lambda > 0}$ is defined by $D_\lambda(x, y, z) := (\lambda x, \lambda y, \lambda^2 z)$,
- the invariant lattice is $\Gamma := \{(x, y, z) \in G \mid x, y, z \in 2\mathbb{Z}\}$,
- and $V := [-1, 1]^3$ is a convenient fundamental domain for $\Gamma$.

Note that a more obvious choice of invariant lattice would be given by $\Gamma' := 2\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, but the above choice is technically convenient for us. We note for later use that $B_1 \subseteq V \subseteq B_{1.5}$ (since for $(x, y, z) \in V$ we have $d((x, y, z), (0, 0, 0)) < \sqrt{5} < 1.5$), and hence any $\lambda_0 \geq 3$ is sufficiently large with respect to $V$.

While all of the results in the present article are new and interesting already in the case of the Heisenberg group, this example does not fully capture the generality of our current setting. The following is a more general example, taken from our census of indecomposable 7-dimensional RAHOSRPs:

Example 2.5 (An infinite family of 3-step nilpotent homogeneous dilation data). For every $\mu \in \mathbb{R} \setminus \{0, 1\}$ there is a 1-connected 3-step nilpotent Lie group $G_\mu$ (corresponding to the Lie algebra $(147E)_\mu$ from the table [Gon98]) which is isomorphic to $\mathbb{R}^7$ with multiplication given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} \ast \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ x_4 + y_4 + \frac{\mu}{4}(x_1y_2 - x_2y_1) \\ x_5 + y_5 + \frac{\mu}{2}(x_2y_3 - x_3y_2) \\ x_6 + y_6 - \frac{\mu}{2}(x_1y_3 - x_3y_1) \\ x_7 + y_7 + \frac{\mu}{2}(x_2y_6 - x_6y_2) + \frac{1 - \mu}{2}(x_3y_4 - x_4y_3) \end{bmatrix}.$$\(^1\)

It turns out that $(G_\mu)_{\mu \in \mathbb{R}}$ is an infinite family; in fact we have $G_{\mu_1} \cong G_{\mu_2}$ if and only if

$$\frac{(1 - \mu_1 + \mu_1^2)^3}{\mu_1^2(1 - \mu_1)^2} = \frac{(1 - \mu_2 + \mu_2^2)^3}{\mu_2^2(1 - \mu_2)^2};$$

A homogeneous dilation structure on $G_\mu$ is given by

$$D_\lambda(x_1, \ldots, x_7) = (\lambda x_1, \lambda x_2, \lambda x_3, \lambda^2 x_4, \lambda^2 x_5, \lambda^2 x_6, \lambda^3 x_7).$$

\(^1\)The factor $\frac{1}{2}$ can be replaced by any non-zero real number without changing the isomorphism type of the group; our normalization follows a convention explained in more details below.
Moreover, there exists \( \varepsilon_0 \in (0, \frac{1}{2}) \) (depending on \( \mu \)) such that for all \( \varepsilon \leq \varepsilon_0 \) the Euclidean ball
\[
A_\varepsilon := \{ x \in G_\mu : x_1^2 + \cdots + x_n^2 \leq \varepsilon^2 \}
\]
of radius \( \varepsilon \) is convex in the sense that \( D_t(x) * D_{1-t}(y) \in A_\varepsilon \) holds for all \( x, y \in A_\varepsilon \) and \( t \in [0,1] \) (see [HS90, Theorem B]). For any such \( \varepsilon \) there then exists a unique left-invariant metric \( d_\varepsilon \) on \( G_\mu \) such that
\[
d_\varepsilon(x,0) = \inf \{ t > 0 : D_{1/t}(x) \in A_\varepsilon \},
\]
and the triple \( (G_\mu, d_\varepsilon, (D_\lambda)_{\lambda > 0}) \) is a homogeneous dilation group. It turns out that this homogeneous dilation group admits a homogeneous dilation datum provided \( \mu \in \mathbb{Q} \). If this is the case and \( \mu = p/q \) for some \( p \in \mathbb{Z} \) and \( q \in \mathbb{N} \), then an example of an invariant lattice in \( G_\mu \) is given by
\[
\Gamma_{\mu} := \mathbb{Z} \times (2q\mathbb{Z}) \times (2q\mathbb{Z}) \times \mathbb{Z}^4,
\]
and we obtain a dilation datum \( D_\mu := (G_{\mu}, d_\varepsilon, (D_\lambda)_{\lambda > 0}, \Gamma_{\mu}, V) \) by choosing
\[
V := [-\frac{1}{2}, \frac{1}{2}] \times [-q, q] \times [-\frac{1}{2}, \frac{1}{2}].
\]
One checks easily that \( B_{r_-} \subseteq V \subseteq B_{r_+} \) with \( r_\pm = \frac{1}{ \sqrt{2\varepsilon} } \) and \( r_+ = \sqrt{2q+1} \), and hence any \( \lambda_0 \geq 1 + 2(q + 1) \cdot \varepsilon^{-\frac{1}{2}} \) is sufficiently large relative to \( V \).

### 2.2. The combinatorics: From substitution data to substitution maps.

In this subsection we first provide a precise definition of substitution data. We then explain how such a datum gives rise to a substitution map and associated substitution system. Finally, we provide explicit examples of substitution data and discuss their additional properties. We will assume throughout that a dilation datum \( D := (G, d, (D_\lambda)_{\lambda > 0}, \Gamma, V) \) has been fixed.

**Definition 2.6.** A substitution datum \( (\mathcal{A}, \lambda_0, S_0) \) over \( D \) consists of

(S1) a finite set \( \mathcal{A} \) called the underlying alphabet;

(S2) a sufficiently large real number \( \lambda_0 \) relative to \( V \), called the underlying stretch factor, such that \( D_{\lambda_0}(\Gamma) \subseteq \Gamma \);

(S3) a map \( S_0 : \mathcal{A} \to \mathcal{A}^{D_{\lambda_0}(V) \cap \Gamma} \) called the underlying substitution rule.

From now on let \( S = (\mathcal{A}, \lambda_0, S_0) \) be a substitution datum and \( D := D_{\lambda_0} \). Given a (finite) subset \( M \subseteq \Gamma \), we refer to an element \( P \in \mathcal{A}^M \) as a (finite) patch with support \( \text{supp}(P) := M \). Given \( a \in \mathcal{A} \) we denote by \( P_a \) the patch with \( \text{supp}(P_a) = \{ e \} \) and \( P_a(e) = a \). We denote by \( \mathcal{A}_\Gamma^* \) the \( \Gamma \)-space of all (possibly infinite) patches. The following result will be established in Proposition 5.6 and Proposition 5.7:

**Proposition 2.7** (Associated substitution map). There is a unique map \( S : \mathcal{A}_\Gamma^* \to \mathcal{A}_\Gamma^* \) with the following properties:

(E1) \( S(P_a) = S_0(a) \) for all \( a \in \mathcal{A} \).

(E2) For all \( \gamma \in \Gamma \) and \( P \in \mathcal{A}_\Gamma^* \) we have
\[
\text{supp}(S(\gamma P)) = D(\gamma) \cdot \text{supp}(S(P)) \quad \text{and} \quad S(\gamma P) = D(\gamma) S(P).
\]

(E3) If \( (M_i)_{i \in I} \) is a family of subsets of \( \Gamma \) and \( M = \bigcup M_i \), then for all \( P \in \mathcal{A}^M \) we have
\[
\text{supp}(S(P)) = \bigcup_{i \in I} \text{supp}(S(P|_{M_i})) \quad \text{and} \quad S(P)(\gamma) = S(P|_{M_i})(\gamma) \quad \text{for all} \ \gamma \in \text{supp}(S(P|_{M_i})).
\]

Moreover, this map preserves the subset \( \mathcal{A}_\Gamma \subseteq \mathcal{A}_\Gamma^* \), and the restriction \( S : \mathcal{A}_\Gamma \to \mathcal{A}_\Gamma \) is continuous with respect to the product topology.

In other words, \( S \) is the unique \( D \)-equivariant extension of \( S_0 \) which is “locally defined”. The restriction \( S : \mathcal{A}_\Gamma \to \mathcal{A}_\Gamma \) is called the substitution map associated with \( S \).
Notation 2.8 (Substitution map and substitution system). We say that a finite patch $P$ is $S$-legal if $P$ is in the same $\Gamma$-orbit as some restriction of $S^n(P_a)$ for some $n \in \mathbb{N}$ and $a \in A$. Similarly, an element of $A^\Gamma$ is called $S$-legal if each of its finite subpatches is legal. We then define the associated substitution system by

$$\Omega(S) := \{ \omega \in A^\Gamma : \omega \text{ is } S\text{-legal} \}$$

Moreover, given $\omega \in \Omega(S)$ we denote by $\Omega_\omega$ the orbit closure of $\omega$ in $A^\Gamma$. It is easy to see that $\Omega_\omega \subseteq \Omega(S)$ for every $\omega \in \Omega(S)$.

We now consider a first example of a non-abelian substitution datum:

Example 2.9. Let $D_H := (\mathbb{H}_3(\mathbb{R}), d, (D_\lambda)_{\lambda > 0}, \Gamma, V)$ be the dilation datum over the 3-dimensional Heisenberg group from Example 2.4. We are going to define a substitution datum over $D_H$ with underlying alphabet $A := \{a, b\}$ and stretch factor $\lambda_0 := 3$; note that this stretch factor is sufficiently large by Example 2.4. We then have

$$D_{\lambda_0}(V) \cap \Gamma = \{-2, 0, 2\} \times \{-2, 0, 2\} \times \{-8, -6, -4, -2, 2, 4, 6, 8\},$$

and if we define a substitution rule $S_0 : A \to A^{D_{\lambda_0}(V) \cap \Gamma}$ as in Figure 1, then $(A, 3, S_0)$ defines a substitution datum over $D_H$.

![Figure 1](image-url)

Figure 1. An example of a substitution rule in the Heisenberg group with $\lambda_0 = 3$.

Remark 2.10 (Support growth). Our assumption that the stretch factor $\lambda_0$ is sufficiently large relative to $V$ guarantees that supports of patches grow arbitrarily large under iterated application of $S$. It is thus similar to the well-known requirement for one-dimensional symbolic substitutions that the length of the words should tend to infinity. More precisely, if $P$ is any finite patch whose support contains the identity, then the supports of the iterates $S^n(P)$ exhaust all of $\Gamma$ (see Theorem 5.13 below). If $P$ is an arbitrary finite patch (not necessarily containing the identity), then the supports of these patches will still become arbitrary large, but if $P$ is sufficiently far away from the identity, then they will be further and further away from $e$. 
Figure 2. The pattern $S^2(P_a)$ for the “same” substitution in the Heisenberg group and Euclidean space.

Example 2.11 (Euclidean vs. non-euclidean support growth). In Example 2.9 we have seen a substitution datum $S = (A, 3, S_0)$ over the Heisenberg dilation datum $\mathcal{D}_H$; the same substitution

Figure 3. Maximal and minimal $z$-values in the support of $S^4(P_a)$ for the “same” substitution in the Heisenberg group and Euclidean space.
datum can also be considered over the Euclidean dilation datum $D_{\mathbb{R}^3} := (\mathbb{R}^3, d, (D_\lambda)_{\lambda > 0}, \Gamma_{\mathbb{R}}, V)$, where $(\mathbb{R}^3, d, (D_\lambda)_{\lambda > 0})$ is as in Example 2.3 and where we choose $\Gamma_{\mathbb{R}} = (2\mathbb{Z})^3$ and $V := [-1, 1]^3$ as in the Heisenberg case.

Due to the difference in geometry between Euclidean space and the Heisenberg group, the “same” substitution datum $S$ gives rise to two very different substitution maps over these two dilation data, and it is interesting to compare how patches grow under iterated applications of the substitution map $S$ in both cases. The pattern $S^2(P_a)$ is sketched in Figure 2 for both groups. Plotting higher iterations is difficult (for $N = 3$ the plot already consists of more than half a million points), hence only the maximal and minimal $z$-values in the support of $S^4(P_a)$ are plotted in Figure 3.

In the Euclidean case, the support of $S^\omega(P_a)$ equals the cube $[-\lambda_0^n, \lambda_0^n]^2 \times [-\lambda_0^n, \lambda_0^n] \cap \Gamma$. In the case of the Heisenberg group the projection of $S^\omega(P_a)$ onto the third coordinate $(x = 0, y = 0)$ is still given by $[-\lambda_0^{2n}, \lambda_0^{2n}] \cap 2\mathbb{Z}$, but for $|x| > 0, |y| > 0$ the fibers over $x$ and $y$ get more and more shifted, where the precise shift depends on the $x$ and $y$ coordinate.

**Remark 2.12 (Comparison of fixpoints).** Let $D$ be either $D_{\mathbb{H}}$ or $D_{\mathbb{R}^3}$, let $S$ be the substitution datum from Example 2.9 and define $a^\infty(\gamma) := a$ for all $\gamma \in \Gamma$. From the fact that $S_0(a)(e) = a$ one deduces easily that in both cases the limit $a := \lim_{n \to \infty} S^n(a^\infty)$ exists and defines a fixpoint for $S$ in the corresponding substitution system $\Omega(S)$. Again, due to the differences in geometry, the two fixpoints in the Heisenberg and the Euclidean case have a very different structure.

Since there is a huge freedom in choosing a substitution datum $S$ over a given dilation datum $D$, it is natural to ask whether one can choose substitution data with additional properties. Two important such properties for us in the sequel are primitivity and non-periodicity.

The notion of a primitive substitution datum is well-known in the abelian setting, and the definition carries over verbatim: Given two patches $P$ and $Q$ we say that $P$ occurs in $Q$ and write $P \prec Q$ if $P$ lies in the same $\Gamma$-orbit as some restriction of $Q$.

**Definition 2.13.** A substitution datum $(A, \lambda_0, S_0)$ is called primitive, if there exists $L \in \mathbb{N}$ such that for all $a, b \in A$ we have $P_a \prec S^L(P_b)$.

As mentioned earlier, primitivity of a substitution datum can be used to ensure minimality (and even linear repetitivity) of the associated substitution system. Similarly we would like to provide a condition on the substitution datum which ensures aperiodicity (at least in some weak form, compare Theorem 1.2 (e)). It is well-known that for symbolic substitutions in abelian groups the existence of proximal pairs leads to weakly aperiodic subshifts [BO14, BG13]. On the other hand, injectivity of substitution in the geometric setting of abelian groups leads to weakly aperiodic tilings [Sol97, Sol98, AP98]. Inspired by the latter, we introduce the following notion:

**Definition 2.14.** A substitution datum $(A, \lambda_0, S_0)$ is called non-periodic if $S_0$ is injective and

$$ \left( \gamma^{-1}S(P_a) \right)|_{\gamma^{-1}D(V) \cap D(V)} \neq S(P_b)|_{\gamma^{-1}D(V) \cap D(V)} $$

for all $\gamma \in (D(V) \cap \Gamma) \setminus \{e\}$ and $a, b \in A$.

One can check that the substitution datum in Example 2.9 above is both primitive (with $L = 1$) and non-periodic. Similarly, one can find examples of non-periodic and primitive substitution data for the 7-dimensional dilation data in Example 2.5, see Section 6 for explicit constructions. In fact, such substitution data exist for all RAHOGRASPs, cf. Theorem 1.1.
3. Homogeneous dilation groups

In this section we are going to survey some well-known results concerning so-called \textit{gradable Lie groups}, following [FR16, LD17]. This is motivated by the fact that, as we will see, 1-connected graduable Lie groups are precisely the groups which underly dilation data (see Subsection 3.3). A 1-connected graduable Lie group together with a choice of grading of its Lie algebra will be called a \textit{homogeneous Lie groups}, and these groups carry a canonical dilation structure that we will discuss in some detail. Important examples of graduable Lie groups are given by 2-step nilpotent Lie groups, and we will use these examples to illustrate the general theory throughout.

3.1. Nilpotent and graduable Lie algebras. Let \( \mathbb{K} \) be a field and let \((g_\mathbb{K}, [\cdot, \cdot])\) be a finite-dimensional Lie algebra over \( \mathbb{K} \), i.e. \( g_\mathbb{K} \) is a finite-dimensional \( \mathbb{K} \)-vector space and \([\cdot, \cdot] : g_\mathbb{K} \times g_\mathbb{K} \to g_\mathbb{K} \) is a bilinear, antisymmetric map satisfying the Jacobi identity

\[
[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.
\]

We recall that the lower central series of \((g_\mathbb{K}, [\cdot, \cdot])\) is defined by

\[
L^0(g_\mathbb{K}) := g_\mathbb{K} \quad \text{and} \quad L^{k+1}(g_\mathbb{K}) := [g_\mathbb{K}, L^k(g_\mathbb{K})], \quad k \geq 0.
\]

It then follows by induction that for all \( k \in \mathbb{N} \) the subspace \( L^k(g_\mathbb{K}) \) is an ideal in \( g_\mathbb{K} \) (i.e. invariant under all inner automorphisms) and in fact even characteristic in \( g_\mathbb{K} \) (i.e. invariant under all Lie algebra automorphisms of \( g_\mathbb{K} \)).

\textbf{Definition 3.1.} We say that \( g_\mathbb{K} \) is at most \( s \)-step nilpotent if \( L^s(g_\mathbb{K}) = \{0\} \) and \( s \)-step nilpotent if moreover \( s = 0 \) or \( L^{s-1}(g) \neq \{0\} \). If \( g \) is at most 1-step nilpotent, then it is called abelian.

\textbf{Example 3.2.} The real Heisenberg Lie algebra is the three-dimensional real Lie algebra with basis \((X, Y, Z)\) and Lie bracket determined by \([X, Y] = Z\) and \([X, Z] = [Y, Z] = 0\). It is 2-step nilpotent, and in fact the only non-abelian 3-dimensional nilpotent Lie algebra.

\textbf{Remark 3.3} (Tabulations of nilpotent Lie algebras). While nilpotent Lie algebras cannot be classified, there are tabulations of nilpotent Lie algebras in low dimensions. See in particular [Gon98] for a tabulation of all nilpotent (real) Lie algebras of dimension \( \leq 7 \). In these tables, the Lie algebras are given by a basis \( X_1, \ldots, X_d \) (with \( d \leq 7 \)) together with commutator relations of the form

\[
[X_i, X_j] = \sum_{k=1}^n \alpha_{ij}^k X_k;
\]

by convention, all commutators between basic elements which are not listed are understood to be 0. It clearly suffices to tabulate the nilpotent real Lie algebras which are \textit{indecomposable}, i.e. not direct sums of lower-dimensional Lie algebras. In dimension 7 there are (up to isomorphism) 9 uncountable families of such Lie algebras which depend on a real parameter \( \lambda \) and moreover 140 such Lie algebras which are isolated (i.e. not part of a family).

\textbf{Example 3.4} (An infinite family). For every \( \mu \in \mathbb{R} \) the real Lie algebra \( g_\mu \) labelled \((147E)_\mu \) in [Gon98] is given by generators \( X_1, \ldots, X_7 \) and non-trivial relations

\[
[X_1, X_2] = X_4, \quad [X_1, X_3] = -X_6, \quad [X_1, X_5] = -X_7, \quad [X_2, X_3] = X_5
\]

\[
[X_2, X_6] = \mu X_7, \quad [X_3, X_4] = (1 - \mu)X_7.
\]

It is 3-step nilpotent and in fact isomorphic to the Lie algebra of the group \( G_\mu \) from Example 2.5.
Definition 3.5. A positive $K$-grading on a Lie algebra $(\mathfrak{g}_K, [\cdot, \cdot])$ is a $K$-vector space decomposition

\begin{equation}
\mathfrak{g}_K = \bigoplus_{\alpha \in (0, \infty)} \mathfrak{g}_\alpha \text{ such that } [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta} \text{ for all } \alpha, \beta > 0.
\end{equation}

If $X \in \mathfrak{g}_\alpha$, then we say that $X$ is homogeneous of degree $\alpha$ with respect to the given grading. We say that a positive $K$-grading is compatible with a basis $(X_1, \ldots, X_d)$ if $X_1, \ldots, X_d$ are homogeneous. A grading of $\mathfrak{g}_K$ is called a stratification of length $s$ if it is of the form

\[ \mathfrak{g}_K = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_s \]

with $\mathfrak{g}_j \neq \{0\}$ for $j \in \{1, \ldots, s\}$ and $[\mathfrak{g}_1, \mathfrak{g}_j] = \mathfrak{g}_{j+1}$ for $j \in \{1, \ldots, s-1\}$. We say that $\mathfrak{g}_K$ is $K$-gradable (resp. $K$-stratifiable) if it admits a positive $K$-grading (resp. a $K$-stratification).

Remark 3.6 (Stratifiable vs. gradable vs. nilpotent). Since $\mathfrak{g}_K$ is finite-dimensional, for any $K$-grading on $\mathfrak{g}_K$ as in (1) the set $I := \{\alpha > 0 : \mathfrak{g}_\alpha \neq \{0\}\}$ is finite. We refer to $I \subseteq \mathbb{R}_{>0}$ as the spectrum of the $K$-grading. If we set

\[ \alpha_{\min} := \min I, \quad \alpha_{\max} := \max I \quad \text{and} \quad s := \lceil \alpha_{\max}/\alpha_{\min} \rceil + 1, \]

then every $s$-fold commutator is trivial, and hence $\mathfrak{g}$ is at most $s$-step nilpotent. In particular, every $K$-gradable Lie algebra is nilpotent; the converse is not true (see e.g. [LD17, Ex. 1.8]). Also, a $K$-gradable Lie algebra may admit many different $K$-gradings. On the contrary, a $K$-stratification, if it exists, is unique up to automorphisms (see [LD17, Rem. 1.3]).

Example 3.7 (2-step nilpotent implies stratifiable). Assume that $\mathfrak{g}_K$ is 2-step nilpotent. Set $\mathfrak{g}_2 := L^1(\mathfrak{g}_K)$ and let $\mathfrak{g}_1$ be an arbitrary vector space complement of $\mathfrak{g}_2$ in $\mathfrak{g}_K$ so that

\begin{equation}
\mathfrak{g}_K = \mathfrak{g}_1 \oplus \mathfrak{g}_2.
\end{equation}

Since $\mathfrak{g}_K$ is 2-step nilpotent, we have $[X, Y] = 0$ for all $X \in \mathfrak{g}_K$ and $Y \in \mathfrak{g}_2$, which implies that $[\mathfrak{g}_1, \mathfrak{g}_1] = [\mathfrak{g}_K, \mathfrak{g}_K] = \mathfrak{g}_2$. This shows that (2) is a stratification of $\mathfrak{g}_K$, and hence $\mathfrak{g}_K$ is stratifiable. By the previous remark, it thus admits a unique stratification up to automorphisms.

We will be mostly interested in real and rational Lie algebras, i.e. Lie algebras over $\mathbb{R}$ and $\mathbb{Q}$.

Remark 3.8 (Real Lie algebras and rational forms). If $\mathfrak{g}$ is a $d$-dimensional real Lie algebra, then a basis $X_1, \ldots, X_d$ of $\mathfrak{g}$ is called rational (resp. integral) if for $i, j \in \{1, \ldots, d\}$ we have

\[ [X_i, X_j] = \sum_{k=1}^{d} \alpha_{ij}^k X_k \quad \text{with} \quad \alpha_{ij}^k \in \mathbb{Q} \text{ (resp. } \alpha_{ij}^k \in \mathbb{Z}) \]

In this case, the $\mathbb{Q}$-span $\mathfrak{g}_Q$ of $X_1, \ldots, X_d$ is a rational Lie algebra such that $\mathfrak{g}_Q \otimes Q \mathbb{R} = \mathfrak{g}$. We refer to such a rational Lie algebra as a rational form of $\mathfrak{g}$, and say that $\mathfrak{g}$ is defined over $\mathbb{Q}$ if such a form exists. Note that every rational form of a Lie algebra $\mathfrak{g}$ admits an integral basis. Indeed, starting from an arbitrary rational basis we can multiply by the largest common denominator of the $\alpha_{ij}^k$.

Remark 3.9 (Real gradings as automorphisms). Assume that $\mathfrak{g}$ is a real Lie algebra. With every positive $\mathbb{R}$-grading on $\mathfrak{g}$ we can associate an automorphism $A \in \text{Aut}(\mathfrak{g})$ by demanding that $A(X) = \alpha \cdot X$ if $X$ is homogeneous of degree $\alpha$. Note that the grading can be recovered from the associated automorphism as its eigenspace decomposition, and hence there is a bijection between positive $\mathbb{R}$-gradings on $\mathfrak{g}$ and $\mathbb{R}$-diagonalizable automorphisms of $\mathfrak{g}$ with positive spectrum. In view of this bijection, we sometimes also refer to the pair $(\mathfrak{g}, A)$ as a positively graded real...
Lie algebra. Note that, by construction, the spectrum of $A$ coincides with the spectrum of the underlying grading.

**Example 3.10.** Assume that $\mathfrak{g}_K$ is 2-step nilpotent equipped with its canonical stratification $\mathfrak{g}_K = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ (see Example 3.7). If we pick a basis of $\mathfrak{g}_1$ and extend it to a basis of $\mathfrak{g}_2$ by a basis of $\mathfrak{g}_2$, then the automorphism $A$ takes the form diag$(1, \ldots, 1, 2, \ldots, 2)$. For example, if we consider the Heisenberg algebra with standard basis $(X, Y, Z)$ as in Example 3.2, then $A = \text{diag}(1, 1, 2)$.

**Remark 3.11 (Rational gradings on real Lie algebras).** Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{g}_\mathbb{Q}$ be a rational form of $\mathfrak{g}$. Then every positive $\mathbb{Q}$-grading (respectively stratification) of $\mathfrak{g}_\mathbb{Q}$ induces a positive $\mathbb{R}$-grading (respectively stratification) on $\mathfrak{g}$ by tensoring each of the summands in (1) with $\mathbb{R}$ over $\mathbb{Q}$. We refer to such an $\mathbb{R}$-grading of $\mathfrak{g}$ as a rational grading (respectively rational stratification), and we say that $\mathfrak{g}$ is rationally graduable (respectively rationally stratifiable) if it admits a rational grading (respectively stratification). By Remark 3.9, $\mathfrak{g}$ is rationally graduable if and only if it admits a diagonalizable automorphism with positive rational spectrum.

We observe that if $\mathfrak{g}$ is a $k$-step nilpotent Lie algebra, then every rational form of $\mathfrak{g}$ is also $k$-step nilpotent. In particular, if $\mathfrak{g}$ is 2-step nilpotent and defined over $\mathbb{Q}$, then any rational form admits a stratification by Example 3.7, hence $\mathfrak{g}$ is rationally stratifiable.

By definition, a rationally graduable Lie algebra is both graduable and defined over $\mathbb{Q}$. We believe that the converse is not true, but we could not find a counterexample in dimension $\leq 7$.

It is an interesting problem to decide whether a given nilpotent Lie algebra $\mathfrak{g}$ is rationally graduable; we are not aware of any easy algorithm to decide this. On the other hand, if we fix a rational basis $(X_1, \ldots, X_4)$ of $\mathfrak{g}$, then finding all rational gradings which are compatible with the given basis is an easy problem in linear algebra. We illustrate this in an example:

**Example 3.12.** Let $\mathfrak{g}$ be the Lie algebra $(257G)$ from [Gon98] with generators $(X_1, \ldots, X_7)$ and non-trivial relations

$$[X_1, X_2] = X_3, \ [X_1, X_3] = X_6, \ [X_1, X_5] = X_7, \ [X_2, X_4] = X_7, \ [X_4, X_5] = X_6.$$ 

There then exists a graduation on $\mathfrak{g}$ compatible with $(X_1, \ldots, X_7)$ such that $X_j$ has degree $\alpha_j$ if and only if the system of equations

$$\alpha_1 + \alpha_2 = \alpha_3, \ \alpha_1 + \alpha_3 = \alpha_6, \ \alpha_1 + \alpha_5 = \alpha_7, \ \alpha_2 + \alpha_4 = \alpha_7, \ \alpha_4 + \alpha_5 = \alpha_6.$$ 

has a solution in positive real numbers. The real solutions $\alpha = (\alpha_1, \ldots, \alpha_7)$ of this system are given by

$$\alpha = s \cdot (2, -3, -1, 1, -2, 1, 0) + t \cdot (-2, 4, 2, -3, 3, 0, 1) \ \ (s, t \in \mathbb{R}),$$

and these are positive if and only if $s > t > 3/4s$. For example, choosing $s = 5$, $t = 4$ we obtain a graduation with

$$\alpha = (2, 1, 3, 3, 2, 5, 4).$$

Note that we were not only able to find a compatible graduation, but even a compatible graduation with rational spectrum (given by $\{1, 2, 3, 4, 5\}$). This is not a coincidence: if a system of linear (in-)equalities with rational coefficients has a real solution, then it has a rational solution. In summary, we have seen that $(257G)$ is rationally graduable, and that the grading can be chosen to be compatible with the basis from [Gon98].

To get a census of the examples to which our theory will apply, it is instructive to apply the algorithm underlying the previous example to all indecomposable nilpotent real Lie algebras of dimension 7. Let us first consider those 140 Lie algebras, which are not part of a one parameter
family. It turns out that all of them are defined over $\mathbb{Q}$ - in fact, the bases given in [Gon98] are always rational. Applying our algorithm to these 140 bases we obtain, using computer assistance:

**Theorem 3.13** (Classification of compatible rational gradings, I). Let $\mathfrak{g}$ be an indecomposable nilpotent real Lie algebra of dimension 7 which is not part of a one parameter family and let $\mathcal{B} := (X_1, \ldots, X_7)$ be the rational basis of $\mathfrak{g}$ given in [Gon98]. Then $\mathfrak{g}$ admits a rational grading which is compatible with $\mathcal{B}$ unless $\mathfrak{g}$ is either $(13457G)$, $(13457I)$, $(12457B)$, $(12457G)$, $(12457J)$, $(12457K)$, $(12357B)$, $(123457E)$, $(123457F)$, $(123457H)$, $(12457J)_1$, $(12457N)_1$, $(12457N)_2$, $(12357B)_1$ or $(123457H)_1$. In particular, at least 126 of the 140 examples are rationally graduable.

**Proof.** For each of the Lie algebras to which the theorem applies, an explicit grading, which is compatible with $\mathcal{B}$, is given in the first table in Appendix A. These gradings have been found by solving systems of linear equations as in Example 3.12, using computer assistance. However, no computer assistance is needed to check that they indeed have the desired property. □

We emphasize that we are not making any statement concerning the existence of rational gradings on the remaining 14 examples; these cannot be treated by our method, but may still admit a grading. The methods from [HKMT21] may be helpful to decide this.

We also reiterate that our notation follows [Gon98]. Some older sources also list a Lie algebra denoted $(12457M)$, which is not contained in [Gon98], since it is contained in the family $(12457N)$ as the special case $\lambda = 0$, and thus is not covered by Theorem 3.13.

Besides the examples covered by Theorem 3.13 there also exist 9 one parameter families $(\mathfrak{g}_\mu)$ of indecomposable nilpotent real Lie algebra of dimension 7. In this case, the basis $(X_\mu^1, \ldots, X_\mu^7)$ given in [Gon98] is rational if and only if $\mu$ is rational, and we have the following result:

**Theorem 3.14** (Classification of compatible rational gradings, II). Let $(\mathfrak{g}_\mu)$ be a one-parameter family of indecomposable nilpotent real Lie algebras of dimension 7, let $\mu \in \mathbb{Q}$ and let $\mathcal{B} := (X_\mu^1, \ldots, X_\mu^7)$ be the rational basis of $\mathfrak{g}_\mu$ given in [Gon98]. Then $\mathfrak{g}_\mu$ admits a rational grading which is compatible with $(X_\mu^1, \ldots, X_\mu^7)$ unless $\mathfrak{g}$ is $(12457N)$ or $(12457N)_2$. In particular, for rational parameters $\mu$ at least 7 of the 9 families are rationally graduable.

**Proof.** Again we apply the method of Example 3.12 and use computer assistance to find the desired gradings. Note that the resulting linear systems do not depend on $\mu$ (except if a coefficient turns out to be 0), and hence all families can be checked by a finite number of computations. Explicit gradings for the above 7 families can be found in the second table in Appendix A. □

The conclusion of our census is that in small dimensions the overwhelming majority of nilpotent real Lie algebras which are defined over $\mathbb{Q}$ are rationally graduable and thus the class of rationally graduable Lie algebras is indeed very rich.

### 3.2. Homogeneous groups and their associated dilation groups

Throughout this article, the term “Lie group” will be used as abbreviation for “finite-dimensional real Lie groups”.

**Remark 3.15** (Nilpotent Lie groups). Recall that if $\mathfrak{g}$ is a finite-dimensional real Lie algebra, then there exists a 1-connected (i.e. connected and simply-connected) Lie group $G(\mathfrak{g})$ with Lie algebra $\mathfrak{g}$. Moreover, this Lie group is unique up to Lie group isomorphism (i.e. up to a smooth group isomorphism with smooth inverse), and every other connected Lie group with Lie algebra $\mathfrak{g}$ is a quotient of $G(\mathfrak{g})$ by a discrete central subgroup. Moreover, the abstract group underlying $G(\mathfrak{g})$ is abelian, nilpotent or (at most) $k$-step nilpotent for some $k \in \mathbb{N}$ if and only if $\mathfrak{g}$ has...
the corresponding property. We then refer to \( G(\mathfrak{g}) \) as an abelian, nilpotent or (at most) \( k \)-step nilpotent Lie group accordingly.

**Remark 3.16 (Exponential coordinates).** For every Lie group \( G \) with Lie algebra \( \mathfrak{g} \) there is an exponential function \( \exp : \mathfrak{g} \to G \) which restricts to a diffeomorphism between a neighbourhood of 0 in \( \mathfrak{g} \) and a neighbourhood of the neutral element \( e_G \) in \( G \). If \( \mathfrak{g} \) is nilpotent and \( G \) is connected, then \( \exp \) is a covering map, and if \( G \) is moreover simply-connected, then \( \exp \) is a global diffeomorphism. Assume this from now on and denote by \( \log : G \to \mathfrak{g} \) its inverse. We then define a multiplication * on \( \mathfrak{g} \) by

\[
X \ast Y := \log(\exp(X) \exp(Y)),
\]

and observe that the Lie group \( G \) is isomorphic to the Lie group \((\mathfrak{g}, \ast)\). From now on, given a finite-dimensional nilpotent Lie algebra \( \mathfrak{g} \), we reserve the symbol \( G(\mathfrak{g}) \) to denote the specific model of the 1-connected Lie group with Lie algebra \( \mathfrak{g} \) given by \( G(\mathfrak{g}) = (\mathfrak{g}, \ast) \). We note that the maps \( \exp : \mathfrak{g} \to G(\mathfrak{g}) \) and \( \log : G(\mathfrak{g}) \to \mathfrak{g} \) are just the identity maps of the underlying sets; nevertheless we sometimes write \( \exp(X) \) for an element \( X \in \mathfrak{g} \) to indicate that \( X \) is now considered as a group element.

The multiplication * on \( G(\mathfrak{g}) \) is called the **Baker-Campbell-Hausdorff (BCH) multiplication**, and if we abbreviate \( \text{ad}(X)(Y) := [X, Y] \), then it is given by the (universal) Baker-Campbell-Hausdorff series

\[
X \ast Y = X + \sum_{k,m \geq 0 \atop p_1 + q_1 > 0} \frac{(-1)^k}{(k + 1)(q_1 + \cdots + q_k + 1)} \text{ad}(X)^p \text{ad}(Y)^q \cdots \text{ad}(X)^p \text{ad}(Y)^q \text{ad}(X)^m \frac{1}{p_1! q_1! \cdots p_k! q_k! m!} (Y).
\]

Note that if \( \mathfrak{g} \) is \( s \)-step nilpotent, then \( \text{ad}(X)^p = \text{ad}(Y)^q = 0 \) for all \( p, q > s \) and \( X, Y \in \mathfrak{g} \), and thus the sum defining the BCH multiplication on a nilpotent Lie algebra is always finite. Note that the first few terms of the Baker-Campbell-Hausdorff series are given by

\[
X \ast Y = X + Y + \frac{1}{2} [X, Y] + \frac{1}{12} [X, [X, Y]] + \frac{1}{12} [Y, [Y, X]] - \frac{1}{24} [Y, [X, [X, Y]]] - \ldots
\]

In particular, if \( \mathfrak{g} \) is abelian, then \( X \ast Y = X + Y \) and if \( \mathfrak{g} \) is 2-step nilpotent, then \( X \ast Y = X + Y + \frac{1}{2} [X, Y] \).

**Remark 3.17 (Nilpotent Lie groups in coordinates).** If we choose a basis of a nilpotent Lie algebra \( \mathfrak{g} \) and use it to identify \( \mathfrak{g} \) with \( \mathbb{R}^n \) for some \( n \geq 0 \), then the coordinates of \( X \ast Y \) are polynomial in the entries of \( X \) and \( Y \). Indeed, this follows from bilinearity of the Lie bracket and the fact that the Baker-Campbell-Hausdorff multiplication is polynomial in these Lie brackets. Up to isomorphism, we may thus think of a 1-connected nilpotent Lie group as \( \mathbb{R}^n \) together with some exotic polynomial multiplication law, which can be expressed in terms of the Lie bracket.

**Example 3.18 (Heisenberg group).** Let \( \mathfrak{g} \) be the 3-dimensional real Heisenberg algebra with basis \((X, Y, Z)\) as in Example 3.2. If we identify \( xX + yY + zZ \in \mathfrak{g} \) with \((x, y, z) \in \mathbb{R}^3\) then the group \( G(\mathfrak{g}) \) becomes identified with \( \mathbb{R}^3 \) with multiplication given by

\[
(x, y, z) \ast (x', y', z') = (x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y)).
\]

This explains our convention to include the factor \( \frac{1}{2} \) in the definition of the Heisenberg group.

**Example 3.19 (A 3-step example).** If we denote by \( \mathfrak{g}_\mu \) the Lie algebra from Example 3.4 then we may identify \( \mathfrak{g}_\mu \) with \( \mathbb{R}^7 \) using the basis \((X_1, \ldots, X_7)\). Under this identification, the BCH multiplication corresponds precisely to the multiplication on \( \mathbb{R}^7 \) defined in Example 2.5. This shows that the group \( G_\mu \) is indeed isomorphic to \( G(\mathfrak{g}_\mu) \).
We now come to the main definition of this section.

**Definition 3.20.** A 1-connected nilpotent Lie group \( G \) is called (rationally) graduable if its Lie algebra \( g \) is (rationally) graduable. In this case, we refer to a (rational) positive \( \mathbb{R} \)-grading on \( g \) as a (rational) homogeneous structure on \( G \). A (rationally) graduable group together with a (rational) homogeneous structure is called a (rationally) homogeneous group.

The terminology varies quite a lot in the literature: The term graduable groups is used in [LD17], whereas [Sie86] uses the term contractible group. The term homogeneous group is used e.g. in [FS82, FR16], whereas [LD17] uses the term graded groups. We emphasize that, in our terminology, a (rationally) graduable group may admit many different (rationally) homogeneous structures. Note that Theorem 3.13 and Theorem 3.14 provide plenty of examples of nilpotent homogeneous groups.

**Example 3.21 (2-step nilpotent Lie groups).** If \( G \) is a 1-connected 2-step nilpotent Lie group defined over \( \mathbb{Q} \), then by Example 3.7 it is rationally graduable, since there even exists a rational stratification on \( g \). Since stratifications (if they exist) are unique up to automorphisms [LD17, Rem. 1.3], we conclude that up to automorphisms every 1-connected 2-step nilpotent Lie group which is defined over \( \mathbb{Q} \) admits a unique canonical homogeneous structure up to automorphisms in the sense of the following definition.

**Definition 3.22.** Let \( G \) be a 1-connected 2-step nilpotent Lie group defined over \( \mathbb{Q} \). Any homogeneous structure on \( G \) which arises from a stratification of a rational form of \( G \) is called a canonical homogeneous structure.

**Remark 3.23 (Parametrizing homogeneous structures).** We recall from Remark 3.9 that a positive grading on a Lie algebra \( g \) corresponds to a certain automorphism \( A \) of \( g \). We may thus equivalently consider a homogeneous group as a pair \((G, A)\), where \( G \) is a graduable group with Lie algebra \( g \) and \( A \in \text{Aut}(g) \) is the associated automorphism of the underlying grading.

From now on \((G, A)\) will denote a homogeneous group. For every \( \lambda > 0 \) we define a map

\[
D_\lambda : G \to G, \quad g \mapsto \exp_G(\exp_{\text{Aut}(G)}(\log \lambda \cdot A)(\log_G(g))),
\]

where \( \exp_{\text{Aut}(G)} \) denotes the exponential map of the Lie group \( \text{Aut}(G) \). Since \( A \) has strictly positive eigenvalues, the following properties of the family \((D_\lambda)_{\lambda > 0}\) are immediate from the definition:

**Proposition 3.24.** The map \( D : \mathbb{R}^+ \to \text{Aut}(G), \lambda \mapsto D_\lambda \) is a group homomorphisms. Moreover, for every \( g \in G \) we have

\[
\lim_{\lambda \to 0} D_\lambda(g) = e, \quad D_1(g) = g \quad \text{and} \quad \lim_{\lambda \to \infty} D_\lambda(g) = \infty. \quad \square
\]

**Definition 3.25.** The one-parameter group \((D_\lambda)_{\lambda > 0}\) of automorphisms is called the associated dilation family of the homogeneous group \((G, A)\).

With every homogeneous group \((G, A)\) one can also associate a canonical bi-Lipschitz class of left-invariant, proper, continuous metrics inducing the given topology on \( G \) which are compatible with the dilation family. The following definition and results concerning homogeneous (quasi-)norms can be found in [FR16, Sec. 3.1.6], hence we omit the proofs.

**Definition 3.26.** A continuous non-negative function \( |\cdot| : G \to \mathbb{R}_{\geq 0} \) is called a homogeneous quasinorm if

\[
\text{(H1)} \quad |g| = 0 \text{ if and only if } g = e.
\]
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(H2) \(|g^{-1}| = |g| \) for all \(g \in G\).

(H3) \(|D_\lambda(g)| = \lambda|g|\).

It is called a homogeneous norm if moreover

(H4) \(|gh| \leq |g| + |h|\).

If \(|\cdot|\) is a homogeneous (quasi-)norm then the associated homogeneous (quasi-)metric is the map

\[ d_{\|\cdot\|} : G \times G \to \mathbb{R}_{\geq 0}, \quad (g, h) \mapsto |g^{-1}h|. \]

Moreover, if \(R > 0\) and \(g \in G\) then the open \(R\)-ball around \(g\) with respect to \(|\cdot|\) is given by

\[ B_{\|\cdot\|}(g, R) = \{ h \in G : d_{\|\cdot\|}(g, h) < R \}. \]

As the name suggests, the homogeneous metric associated with a homogeneous norm is a metric. If \(|\cdot|\) is merely a homogeneous quasinorm, then the associated quasimetric may fail to satisfy the triangle inequality; however there will always exist a constant \(C > 0\) such that

\[ d_{|\cdot|}(x, y) \leq C(d_{|\cdot|}(x, z) + d_{|\cdot|}(z, y)) \quad (x, y, z \in G). \]

In either case, the associated (quasi-)metric is invariant under the action of \(G\) on itself by left-multiplication. We have the following result due to W. Hebisch and A. Sikora [HS90].

**Proposition 3.27** (Hebisch-Sikora). Let \((G, A)\) be a homogeneous group.

(a) There exists a homogeneous norm \(\|\cdot\|\) on \(G\).

(b) If \(|\cdot|\) is any other homogeneous quasinorm on \(G\), then there exist \(C \geq 1\) such that

\[ C^{-1}|g| \leq \|g\| \leq C|g| \quad \text{for all } g \in G. \]

(c) If \(|\cdot|\) is any homogeneous quasinorm on \(G\), then the open balls with respect to \(|\cdot|\) are precompact and generate the topology of \(G\).

In particular, all homogeneous metrics on \(G\) are left-invariant, continuous, proper metrics and bi-Lipschitz to each other.

**Example 3.28** (Explicit quasinorms). While it is not always easy to write down an explicit homogeneous norm, it is always easy to write down an explicit homogeneous quasinorm: Let \((X_1, \ldots, X_n)\) be an eigenbasis of \(g\) with respect to \(A\) with corresponding eigenvalues \((\mu_1, \ldots, \mu_n)\). We then obtain a family of homogeneous quasi-norms on \(G\) by setting

\[ \left| \exp \left( \sum_{i=1}^{n} \alpha_i X_i \right) \right|_p := \left( \sum_{i=1}^{n} |\alpha_i|^{p/\mu_i} \right)^{1/p} \quad (1 \leq p < \infty) \quad \text{and} \quad \left| \exp \left( \sum_{i=1}^{n} \alpha_i X_i \right) \right|_{\infty} := \max_{i=1,\ldots,n} |\alpha_i|^{1/\mu_i}. \]

Note that if \(|\cdot|\) is a homogeneous quasinorm on a homogeneous group \((G, A)\), then it follows from Property (H3) that the associated quasimetric \(d := d_{|\cdot|}\) satisfies

\[ d(D_\lambda(g), D_\lambda(h)) = \lambda d(g, h). \]

Together with Proposition 3.24 this implies:

**Corollary 3.29.** If \((G, A)\) is a homogeneous group with associated dilation family \((D_\lambda)_{\lambda > 0}\) and \(d\) is a homogeneous metric on \(G\), then \((G, d, (D_\lambda)_{\lambda > 0})\) is a dilation group.

**Definition 3.30.** We refer to the dilation group \((G, d, (D_\lambda)_{\lambda > 0})\) from Corollary 3.29 as a homogeneous dilation group associated with the homogeneous group \((G, A)\).
Example 3.31 (Heisenberg group). We consider the Heisenberg Lie algebra from Example 3.2. Its canonical stratification is given by $g_1 = \mathbb{R} X \oplus \mathbb{R} Y$ and $g_2 = \mathbb{R} Z$ and the corresponding homogeneous dilation structure is precisely the one from Example 2.4.

Example 3.32 (3-step examples). The family $g_\mu$ from Example 3.4 also admits a stratification given by $g_1 = \mathbb{R} X_1 \oplus \mathbb{R} X_2 \oplus \mathbb{R} X_3$, $g_2 = \mathbb{R} X_4 \oplus \mathbb{R} X_5 \oplus \mathbb{R} X_6$ and $g_3 = \mathbb{R} X_7$. The corresponding homogeneous dilation structure on $G_\mu = G(g_\mu)$ is precisely the one from Example 2.5.

3.3. Siebert’s theorem. We can now state the classification of those groups which underly dilation groups, which is essentially due to Siebert [Sie86]:

**Theorem 3.33** (Siebert). For a connected lcsc group $G$ the following are equivalent.

(a) $G$ is the underlying group of a dilation group.

(b) There exists a group homomorphism $D : \mathbb{R}^+ \to \text{Aut}(G)$, $\lambda \mapsto D_\lambda$ such that for every $g \in G$ we have
$$\lim_{\lambda \to 0} D_\lambda(g) = e, \quad D_1(g) = g \quad \text{and} \quad \lim_{\lambda \to \infty} D_\lambda(g) = \infty.$$ (c) There exists an automorphism $D \in \text{Aut}(G)$ such that $D^{-k}(g) \to e$ for all $g \in G$.

(d) $G$ is a simply-connected nilpotent Lie group whose Lie algebra $g$ admits a positive grading.

In particular, the underlying group of a dilation group is always a nilpotent Lie group.

**Proof.** The non-trivial implication is (c) $\implies$ (d), and this is contained in [Sie86, Cor. 2.4]. Now assume (d) and fix a homogeneous structure on $G$. If $(D_\lambda)_{\lambda > 0}$ denotes the associated dilation family, then (b) holds by Proposition 3.24 and (a) holds by Corollary 3.29. Finally, both (a) and (b) imply (c) by setting $D := D_\lambda$ for some $\lambda > 1$. □

Note that while the underlying group of a dilation group always admits a homogeneous structure by Siebert’s theorem, this does not mean that every dilation group is homogeneous. Indeed, every real $(n \times n)$-matrix with positive spectrum defines a dilation family on the abelian Lie group $(\mathbb{R}^n, +)$, but it is homogeneous if and only if the matrix is diagonalizable. It is mostly for technical convenience that we restrict ourselves to the homogeneous (or “diagonalizable”) case in the present article.

**Remark 3.34.** If $G$ is the underlying group of a dilation group and $D$ is as in Condition (iii) of Theorem 3.33, then there exists a metric $d$ on $G$ such that $d(D(x), D(y)) = \lambda d(x, y)$ for some $\lambda > 1$ and all $x, y \in G$, i.e. $D$ is a metric dilation with respect to $d$. Conversely, if $(X, d)$ is any homogeneous metric space which admits a metric dilation, then $X$ is the underlying space of a connected lcsc group $G$ and $D$ is an automorphism of $G$ by [CKLD+21, Theorem D]. This automorphism then automatically satisfies Condition (iii) of Theorem 3.33, hence $G$ is the underlying group of a dilation group. In other words, if we merely want to work with homogeneous metric spaces which admit a metric dilation (as is necessary to define substitution systems), then we have no other choice than to work with dilation groups.

3.4. Exact polynomial growth for homogeneous dilation groups. In this subsection we are going to establish that homogeneous groups and their lattices have exact polynomial growth in the sense of the following definition:

**Definition 3.35.** Let $G$ be a locally compact group and let $d$ be a left-invariant metric on $G$ inducing the given topology on $G$. We say that $(G, d)$ has exact polynomial growth if there exist constants $C > 0$ and $\kappa \geq 0$ such that
$$\lim_{r \to \infty} \frac{m_G(B_r)}{C r^\kappa} = 1,$$
where $B_r := \{ g \in G : d(g,e) < r \}$.

We will prove the following statement which is an immediate consequence of Lemma 3.39 and Lemma 3.40 given below.

**Proposition 3.36** (Exact polynomial growth). Let $(G,A)$ be a homogeneous group and $d$ be a homogeneous metric on $G$. Then $(G,d)$ has exact polynomial growth. Moreover, if $\Gamma < G$ is a lattice, then $(\Gamma, d_{\Gamma \times \Gamma})$ also has exact polynomial growth.

As explained in the companion paper [BHP20], exact polynomial growth can be used to deduce unique ergodicity from linear repetitivity. Towards the proof of Proposition 3.36 we start with some general remarks concerning balls in dilation groups.

**Remark 3.37** (Balls and dilations). Note that if $(G,d,(D_\lambda)_{\lambda > 0})$ is a dilation group, then for $x \in G$ and $\lambda, r > 0$ we have

$$D_\lambda(B(x,r)) = D_\lambda(xB(e,r)) = D_\lambda(x)B(e,\lambda r) = B(D_\lambda(x),\lambda r).$$

Combining left-invariance of $d$ with the triangle inequality we also obtain

$$d(gh,x) \leq d(gh,g) + d(g,x) \leq d(h,e) + d(g,x) \quad (g,h,x \in G),$$

and hence the corresponding balls satisfy

$$B(x,r)B(e,s) \subseteq B(x,r+s).$$

In particular (4) and (5) hold for balls with respect to homogeneous metrics on homogeneous groups. In fact, the former even holds for balls with respect to homogeneous quasimetrics (but the latter does not).

In the case of homogeneous groups we can use these general formulas to estimate Haar volumes of balls.

**Remark 3.38** (Volume growth). If $|\cdot|$ is a homogeneous quasinorm on a homogeneous group $(G,A)$, then we may deduce from (4) that for all $x \in G$ and $r > 0$ we have

$$B^{|\cdot|}(x,r) = x \cdot B^{|\cdot|}(e,r) = x \cdot D_r(B^{|\cdot|}), \quad B^{|\cdot|} := B^{|\cdot|}(e,1).$$

Now set $\kappa := \text{tr}(A)$ and $C := m_G(B^{|\cdot|})$. If $m_G$ is any choice of left-Haar measure on $G$ (which is then also a right-Haar measure, since $G$ is nilpotent, and hence unimodular), then by [FR16, p. 100, Equality (3.6)] we have $m_G(D_\lambda(S)) = \lambda^\kappa \cdot m_G(S)$ for any Borel subset $S \subseteq G$. Using left-invariance of $m_G$ we then obtain

$$m_G(B^{|\cdot|}(x,r)) = C \cdot r^\kappa$$

In particular, homogeneous groups have polynomial volume growth. The growth exponent $\kappa$ is sometimes called the homogeneous dimension of $(G,A)$.

**Lemma 3.39** (Polynomial volume growth). If $|\cdot|$ is a homogeneous quasinorm on a homogeneous group $(G,A)$, then there exist $C > 0$, $\kappa \geq 1$ such that for all $x,y \in G$ and $r > 0$ we have

$$m_G(B^{|\cdot|}(x,r)) = C \cdot r^\kappa \quad \text{and} \quad \lim_{s \to \infty} \frac{m_G(B^{|\cdot|}(x,r+s))}{m_G(B^{|\cdot|}(y,s))} = 1.$$

**Proof.** The first statement is contained in Remark 3.38, and since

$$\lim_{s \to \infty} \frac{C \cdot (r+s)^\kappa}{C \cdot r^\kappa} = 1,$$

it implies the second statement. \qed
This statement carries over to lattices, at least for homogeneous norms (rather than semi-norms):

**Lemma 3.40 (Lattice point counting).** Let \((G,A)\) be a homogeneous group with homogeneous norm \(\|\cdot\|\) and let \(\Gamma\) be a lattice in \(G\). Let \(C\) and \(\kappa\) as in Lemma 3.39 and set \(B_r^\Gamma := B^\parallel(e,r)\cap\Gamma\). Then there exists \(r_0 > 0\) and a function \(\vartheta : (r_0, \infty) \to [0, \infty)\) with \(\lim_{r \to \infty} \vartheta(r) = 0\) such that

\[
\frac{C}{\text{covol}(\Gamma)} \cdot r^\kappa - \vartheta(r) \cdot r^\kappa \leq |B_r^\Gamma| \leq \frac{C}{\text{covol}(\Gamma)} \cdot r^\kappa + \vartheta(r) \cdot r^\kappa \quad \text{for all } r > r_0
\]

where \(\text{covol}(\Gamma)\) denotes the covolume of \(\Gamma\). In particular,

\[
\lim_{r \to \infty} \frac{|B_{r+s}^\Gamma|}{|B_r^\Gamma|} = 1.
\]

*Proof.* Since \(G\) is nilpotent, the lattice \(\Gamma\) is uniform and thus we may fix a relatively compact fundamental domain \(V\) containing \(e\) as an inner point. By definition we then have \(\text{covol}(\Gamma) = m_G(V)\). Given \(r > 0\) we now abbreviate \(B_r := B^\parallel(e,r)\) and \((B_r)\Gamma := B_r^\Gamma V\). Since \(V\) is relatively compact, there exists \(r_0\) such that \(V \subseteq B_{r_0}\), and since \(V\) is a fundamental domain we have

\[
(B_r)\Gamma = \bigcup_{\gamma \in B_r^\Gamma} \gamma V, \quad \text{hence} \quad |B_r^\Gamma| = \frac{m_G((B_r)\Gamma)}{m_G(V)} = \frac{m_G((B_r)\Gamma)}{\text{covol}(\Gamma)}.
\]

It follows from (5) that \((B_r)\Gamma \subseteq B_r V \subseteq B_r B_{r_0} \subseteq B_{r+r_0}\), and similarly one obtains \(B_{r-r_0} \subseteq (B_r)\Gamma\) for all \(r \geq r_0\). We thus deduce from (6) and Lemma 3.39 that

\[
\frac{C(r - r_0)^\kappa}{\text{covol}(\Gamma)} \leq |B_r^\Gamma| \leq \frac{C(r + r_0)^\kappa}{\text{covol}(\Gamma)} \quad \text{for all } r > r_0.
\]

If we now define \(\vartheta : (r_0, \infty) \to [0, \infty)\) by

\[
\vartheta(r) := \frac{C}{\text{covol}(\Gamma)} \cdot \left(\frac{(r + r_0)^\kappa - (r - r_0)^\kappa}{r^\kappa}\right).
\]

then \(\vartheta(r) = O(1/r)\) and the first statement follows. For the second statement we then observe that

\[
\limsup_{r \to \infty} \frac{|B_{r+s}^\Gamma|}{|B_r^\Gamma|} \leq \limsup_{r \to \infty} \frac{C}{m_G(V)} \frac{(r + s)^\kappa + \vartheta(r)(r + s)^\kappa}{(r)^\kappa - \vartheta(r)(r)^\kappa} = 1 \leq \liminf_{r \to \infty} \frac{|B_{r+s} \cap \Gamma|}{|B_r \cap \Gamma|}.
\]

This finishes the proof. \(\square\)

### 4. Homogeneous dilation data

We have seen in the previous section that homogeneous groups give rise to dilation groups. In order to obtain dilation data for such dilation groups, we need to find suitable lattices in homogeneous groups. We will see that such lattices exist in rationally homogeneous groups with rational spectrum; this will provide plenty of examples of dilation data via Theorem 3.13 and Theorem 3.14. Moreover, we will associate a canonical dilation datum with every 2-step nilpotent Lie algebra defined over \(\mathbb{Q}\).
4.1. Adapted lattices. Throughout this section, \( g \) denotes a nilpotent real Lie algebra of dimension \( d < \infty \) which is defined over \( \mathbb{Q} \), and \( g_\mathbb{Q} \) denotes a rational form of \( g \). We recall that \( G(g) := (g,\ast) \) is a model for the 1-connected Lie group with Lie algebra \( g \), where \( \ast \) denotes Baker-Campbell-Hausdorff multiplication on \( g \).

Remark 4.1 (Lattices in nilpotent Lie groups). If \( B = (X_1, \ldots, X_d) \) is a basis of \( g_\mathbb{Q} \), then \( \Gamma_B := \langle \exp(X_1), \ldots, \exp(X_d) \rangle \) is a uniform lattice, i.e. a discrete cocompact subgroup of \( G(g) \). Different choices of bases for \( g_\mathbb{Q} \) lead to commensurable lattices, hence \( g_\mathbb{Q} \) uniquely determines a commensurability class of lattices in \( G(g) \). By [Rag72, Thm. 2.12] and the subsequent remark, every lattice in \( G(g) \) lies in a unique such commensurability class, hence determines a rational form of \( g \). In particular, \( G(g) \) admits a lattice if and only if \( g \) is defined over \( \mathbb{Q} \).

While the commensurability class of \( \Gamma_B \) is independent of the choice of basis \( B \), its fine structure depends very much on \( B \). By choosing a special kind of basis, we can simplify the structure of \( \Gamma_B \) considerably.

Definition 4.2. A basis \( B = (X_1, \ldots, X_d) \) of \( g_\mathbb{Q} \) is called adapted if it is integral, i.e.

\[
[X_i, X_j] = \sum_{k=1}^{n} \alpha_{ij}^k X_k \quad \text{with} \quad \alpha_{ij}^k \in \mathbb{Z},
\]

and all of the structure constants \( \alpha_{ij}^k \) are multiples of the finitely many integers which appear as denominators in the Baker-Campbell-Hausdorff product for \( g \). In this case, the lattice \( \Gamma_B \) is called an adapted lattice.

Note that if \( (Y_1, \ldots, Y_d) \) is an arbitrary basis of \( g_\mathbb{Q} \) then there always exist integers \( n_1, \ldots, n_d \) such that the elements \( X_j := n_j Y_j \) form an adapted basis of \( g_\mathbb{Q} \). In particular, every rational form of \( g \) contains an adapted lattice (unique up to commensurability).

Remark 4.3 (Structure of adapted lattices). If \( B \) is an adapted basis of \( g_\mathbb{Q} \) and \( g_\mathbb{Z} \) denotes the \( \mathbb{Z} \)-span of \( B \), then by the very definition of an adapted basis we have \( X_i \ast X_j \in g_\mathbb{Z} \) for all \( i, j \in \{1, \ldots, d\} \). This implies that \( \Gamma_B = \exp(g_\mathbb{Z}) \), i.e. every element of \( \log(\Gamma_B) \) can be written as an integral linear combination of \( X_1, \ldots, X_d \).

We now consider a homogeneous dilation group \((G, d, (D_\lambda)_{\lambda > 0})\) whose underlying homogeneous group \((G, A)\) is rationally homogeneous. This means that we can find a \( \mathbb{Q} \)-form \( g_\mathbb{Q} \) of the Lie algebra \( g \) of \( G \) together with a positive \( \mathbb{Q} \)-grading

\[
g_\mathbb{Q} = \bigoplus_{\alpha \in \text{spec}(A)} g_{\mathbb{Q}, \alpha},
\]

which induces the given grading on \( g \).

Lemma 4.4. \( g_\mathbb{Q} \) admits an adapted basis of eigenvectors for \( A \).

Proof. Combining bases of the eigenspaces \( g_{\mathbb{Q}, \alpha} \) we obtain an eigenbasis \((Y_1, \ldots, Y_d)\) for \( A \). Now replace each \( Y_j \) by a suitable integer multiple. \( \square \)

Definition 4.5. If \( B \) is an adapted basis of eigenvectors for \( A \), then we refer to the corresponding lattice \( \Gamma_B \) as an adapted lattice for the rationally homogeneous group \((G, A)\).

Proposition 4.6 (Invariance of adapted lattices). Assume that \( \text{spec}(A) \subseteq \mathbb{Q} \). Then for every adapted lattice \( \Gamma < G \) for \((G, A)\) there exists \( \lambda_0 \in \mathbb{N}_{>1} \) such that \( D_{\lambda_0}(\Gamma) \subseteq \Gamma \).
Proof. Let $\Gamma = \Gamma_B$, where $B = (X_1, \ldots, X_d)$ is an adapted eigenbasis. For every $j \in \{1, \ldots, d\}$, the element $X_j$ is an eigenvector of $A$ with some rational eigenvalue $\mu_j$. Let $\lambda_0$ be the smallest common denominator of the $\mu_j$ and choose $p_j \in \mathbb{N}$ so that $\mu_j = p_j/\lambda_0$. Then, by definition, $D_{\lambda_0}(\exp(X_j)) = \exp(X_j)^{p_j}$. Since the $\exp(X_j)$ generate $\Gamma$ we deduce that $D_{\lambda_0}(\Gamma) \subseteq \Gamma$. \hfill $\Box$

Proposition 4.6 holds more general for the lattice generated by any $\mathbb{Q}$-basis of $\mathfrak{g}_\mathbb{Q}$; the advantage of adapted lattices will become apparent only in the next subsection when dealing with product coordinates. The following example shows that the assumption $\text{spec}(A) \subseteq \mathbb{Q}$ in Proposition 4.6 is necessary:

Example 4.7. Consider the rationally homogeneous group $(G, A)$ with $G = (\mathbb{R}^2, +)$ with $A(x, y) = (x, \pi y)$ and rational structure given by $\mathbb{Q}^2 \subseteq \mathbb{R}^2$. The standard basis vectors $e_1$ and $e_2$ then form a rational basis of eigenvectors of $A$, but the corresponding lattice $\Gamma = \mathbb{Z}^2$ is not invariant under any $D_{\lambda}$, since $D_{\lambda}(x, y) = (\lambda x, \lambda^2 y)$.

In the sequel we refer to a rationally homogeneous group $(G, A)$ as a RAHOGRASP if its spectrum is contained in $\mathbb{Q}$. We can then summarize the results of this subsection as follows:

Corollary 4.8. Let $(G, A)$ be a RAHOGRASP with associated dilation family $(D_{\lambda})_{\lambda > 0}$ and let $d$ be a homogeneous metric on $G$. Then $(G, d, (D_{\lambda})_{\lambda > 0})$ is a homogeneous dilation group and there exist an adapted lattice $\Gamma < G$ and $\lambda_0 > 1$ such that $D_{\lambda_0}(\Gamma) \subseteq \Gamma$. \hfill $\Box$

Note that all of the examples of invariant lattices given in the introduction (i.e. Example 2.3, Example 2.4 and Example 2.5) are adapted lattices in RAHOGRASPs.

4.2. Product splittings with respect to an adapted basis. For the remainder of this section $(G, A)$ denotes a RAHOGRASP and $\Gamma < G$ denotes an adapted lattice.

Remark 4.9 (Cocycles and central extensions). Recall that if $H$ is a Lie group and $m \in \mathbb{N}$, then a smooth map $\beta : H \times H \to \mathbb{R}^m$ is called a cocycle, provided

$$\beta(g_2, g_3) + \beta(g_1, g_2 g_3) = \beta(g_1 g_2, g_3) + \beta(g_1, g_2),$$

$g_1, g_2, g_3 \in H$.

It is called normalized provided $\beta(e, e) = 0$. We observe that for every normalized cocycle we have the identities

$$\beta(g, g^{-1}) = \beta(g^{-1}, g) \quad \text{and} \quad \beta(g, e) = \beta(e, g) = 0 \quad \text{for all } g \in H.$$  

Indeed, these follows from (7) by setting $g_1 := g_3 := g^{-1}, g_2 := g$, respectively $g_1 := g, g_2 := g_3 := e$. Given a normalized cocycle $\beta : H \times H \to \mathbb{R}^m$ there is a unique Lie group structure on $H \times \mathbb{R}^m$ such that for all $g, h \in H$ and $x, y \in \mathbb{R}^m$ we have

$$(g, x)(h, y) := (gh, x + y + \beta(g, h)) \quad \text{and} \quad (g, x)^{-1} := (g^{-1}, -x - \beta(g, g^{-1})),$$

and we denote the corresponding Lie group by $H \times_\beta \mathbb{R}^m$.

We are going to show:

Proposition 4.10 (Splittings of adapted lattices in RAHOGRASP). Let $\Gamma$ be an adapted lattice in a RAHOGRASP $(G, A)$ with spectral radius $\rho(A)$.

(a) There exist $m \in \mathbb{N}$, a rationally homogeneous group $(G_H, A_H)$ with rational spectrum and a smooth function $\beta : G_H \times G_H \to \mathbb{R}^m$ satisfying (7) and (8) such that $G \cong G_H \times_\beta \mathbb{R}^m$.

(b) There exists an adapted lattice $\Gamma_H < G_H$ with $\beta(\Gamma_H, \Gamma_H) \subseteq \mathbb{Z}^m$ such that, under the isomorphism from (a), $\Gamma \cong \Gamma_H \times_\beta \mathbb{Z}^m$. 

(c) If \((D_{\lambda})_{\lambda>0}\) and \((D_{\lambda}^H)_{\lambda>0}\) denote the dilation families of \((G,A)\) and \((G_H, A_H)\) respectively, then under the isomorphism from (a) we have

\[D_{\lambda}(g_H, v) = (D_{\lambda}^H(g_H), \lambda^0(A)v), \quad g_H \in G_H, v \in \mathbb{R}^m.\]

(d) There exists \(\lambda_0 > 1\) such that \(D_{\lambda_0}(\Gamma) \subseteq \Gamma\), i.e. \(D_{\lambda_0}^H(\Gamma_H) \subseteq \Gamma_H\) and \(\lambda_0 := \lambda_0^0(A) \in \mathbb{Z} \).

Example 4.11. The Heisenberg group \(\mathbb{H}_3(\mathbb{R})\) admits the splitting \(\mathbb{H}_3(\mathbb{R}) = \mathbb{R}^2 \times_{\beta} \mathbb{R}\), where \(\beta((x_1, x_2), (y_1, y_2)) := \frac{1}{2}(x_1y_2 - x_2y_1)\). Since \(\beta((2\mathbb{Z})^2 \times (2\mathbb{Z})^2) \subseteq 2\mathbb{Z}\) we obtain a corresponding splitting \(2\mathbb{Z}^2 \times_{\beta} 2\mathbb{Z}\) for the lattice from Example 2.4. (By choosing different coordinates, we could replace the second factor by \(\mathbb{Z}\).)

Similarly, the group \(G_{\mu}\) from Example 2.5 admits a splitting of the form \(G_H \times_{\beta} \mathbb{R}^m\), where \(G_H := \{x \in G_{\mu} : x_7 = 0\}\) is 2-step nilpotent, and the corresponding lattice then splits as \(\Gamma_{\mu} = \Gamma_H \times_{\beta} \mathbb{Z}\), where \(\Gamma_H := \{\gamma \in \Gamma_{\mu} : x_7 = 0\}\).

Remark 4.12 (Non-trivial splittings). We call a splitting of the form \(G \cong G_H \times_{\beta} \mathbb{R}^m\) as in Proposition 4.10 (a) non-trivial, if \(G_H \neq \{e\}\). If \(G\) is non-abelian, then every splitting is necessarily non-trivial. If \(G \cong \mathbb{R}^n\) is abelian, then we can always find a non-trivial splitting provided \(\text{dim } G \geq 2\). Some of our results below depend on the assumption that the splitting is non-trivial, but these results will be either trivial or well-known in the one-dimensional case.

For the proof of Proposition 4.10 we fix an adapted basis \((X_1, \ldots, X_d)\) of \(\Gamma\). Each \(X_j\) is an eigenvector of \(A\), and we will arrange the order of the basis so that the corresponding eigenvalues satisfy \(\mu_1 \leq \cdots \leq \mu_d\) and hence the spectral radius of \(A\) is given by \(\rho(A) := \mu_d\).

Remark 4.13 (Splitting of the Lie algebra). We denote by \(g_V\) the \(A\)-eigenspace of \(g\) with eigenvalue \(\mu_2 = \rho(A)\). For every \(j \in \{1, \ldots, d\}\) and \(Y \in g_V\) we then have

\[ [X_j, Y] \in g_{\rho(A) + \mu_j} = \{0\}, \quad \text{and hence } [X_j, Y] = 0. \]

This shows that \(g_V\) is a central ideal of \(g\); in particular, \(g_V\) is abelian and we may form the quotient Lie algebra \(g_H := g/g_V\). Given \(i \in \{1, \ldots, d\}\) we denote by \(\overline{X}_i\) the image of \(X_i\) in \(g_H\). If \(j_o := \min\{j \in \{1, \ldots, d\} : \mu_j = \rho(A)\}\), then \(\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_{j_o-1}\) is a basis of \(g_H\) and \(X_{j_o}, \ldots, X_d\) is a basis of \(g_V\). We refer to \(g_V\) and \(g_H\) as the vertical, respectively horizontal part of \(g\). We then have a short exact sequence of Lie algebras given by

\[ 0 \to g_V \to g \to g_H \to 0, \]

and there is a unique linear splitting \(s : g_H \to g_H\) of the underlying exact sequence of vector spaces which maps \(\overline{X}_j\) to \(X_j\) for every \(j < j_o\). Using this splitting we may thus identify \(g\) with \(g_H \oplus g_V\) via the linear isomorphism

\[ \iota_B : g \to g_H \oplus g_V, \quad X_j \mapsto \begin{cases} (\overline{X}_j, 0), & j < j_o, \\ (0, X_j), & j \geq j_o. \end{cases} \]

The Lie bracket on \(g\) then induces via the isomorphism \(\iota_B\) a Lie bracket on \(g_H \oplus g_V\), which is of the form

\[ [(X_H, X_V), (Y_H, Y_V)] = ([X_H, Y_H]H, \omega(X_H, Y_H)), \quad \omega([X_H, Y_H]H, Z_H) = \omega([X_H, Z_H]H, Y_H) - \omega([Y_H, Z_H]H, X_H). \]

In the sequel we will denote by \(g_H \oplus_{\omega} g_V\) the Lie algebra with underlying vector space \(g_H \oplus g_V\) and Lie bracket given by (10).
Remark 4.14 (Splitting of the group). We use the Lie algebra isomorphism $\iota_B : \mathfrak{g} \to \mathfrak{g}_H \oplus \omega \mathfrak{g}_V$ to identify $G \cong G(\mathfrak{g}) \cong G(\mathfrak{g}_H \oplus \omega \mathfrak{g}_V)$. If we set $G_H := G(\mathfrak{g}_H)$ and $G_V := G(\mathfrak{g}_V)$, then $G_V$ is a closed normal central subgroup of $G$ and we obtain a short exact sequence of groups

$$G_V \to G \to G_H, \quad \text{where} \quad p : G \to G_H, \quad p \left( \exp \left( \sum_{i=1}^d \alpha_i X_i \right) \right) = \exp \left( \sum_{i=1}^{j_0-1} \alpha_i \overline{X}_i \right)$$

denotes the canonical projection. By definition we have $G_H = \{ \exp(\sum_{i<j_0} \alpha_i \overline{X}_i) : \alpha_i \in \mathbb{R} \}$ and hence we can define a section of $p$ by

$$\sigma : G_H \to G, \quad \exp \left( \sum_{i<j_0} \alpha_i \overline{X}_i \right) \mapsto \exp \left( \sum_{i<j_0} \alpha_i X_i \right).$$

This section then yields an identification of $G$ with $G_H \times G_V$, and the induced product on $G_H \times G_V$ is given by

$$(g_H, g_V)(h_H, h_V) = (g_H h_H, g_V h_V \beta(g_H, h_H)), \quad \text{where} \quad \beta(g_H, h_H) := \sigma(g_H h_H)^{-1} \sigma(g_H) \sigma(h_H) \in G_V.$$

If we now identify $G_V$ with the additive group $(\mathbb{R}^m, +)$, where $m := \dim \mathfrak{g}_V = d - j_0 + 1$ via the Lie group isomorphism

$$\mathbb{R}^m \to G_V, \quad x = (x_1, \ldots, x_m)^\top \mapsto \sum_{j=1}^{m} x_j X_{j_0-1+j},$$

then $\beta : G_H \times G_H \to \mathbb{R}^m$ is a normalized cocycle and $G \cong G_H \times_\beta \mathbb{R}^m$.

At this point we have established Part (a) of Proposition 4.10. Part (b) now follows from the fact that $\Gamma$ is adapted:

Remark 4.15 (Splitting of the lattice). Let us define $\Gamma_V := \Gamma \cap G_V$ and $\Gamma_H := p(\Gamma)$. It then follows from the adaptedness of $\Gamma$ that

$$\Gamma_V = \exp(\mathbb{Z} X_{j_0} + \cdots + \mathbb{Z} X_d) \quad \text{and} \quad \Gamma_H = \exp(\mathbb{Z} \overline{X}_1 + \cdots + \mathbb{Z} \overline{X}_{j_0-1})$$

In particular, we have a short exact sequence

$$\Gamma_V \leftarrow \Gamma \xrightarrow{p|_{\Gamma}} \Gamma_H.$$

Moreover, the section $\sigma : G_H \to G$ restricts to a section $\sigma_\Gamma : \Gamma_H \to \Gamma$ of $p|_{\Gamma}$, and consequently $\beta$ restricts to a cocycle $\beta_\Gamma : \Gamma_H \times \Gamma_H \to \Gamma_Z$. Under our identification of $G$ with $G_H \times_\beta \mathbb{R}^m$ the lattice $\Gamma$ thus becomes identified with the lattice

$$\Gamma_H \times_{\beta_\Gamma} \mathbb{Z}^m < G_H \times_\beta \mathbb{R}^m.$$

At this point we have proved Part (b) of Proposition 4.10.

Remark 4.16 (Splitting of the dilation family). By definition the dilation family $(D_\lambda)_{\lambda > 0}$ of $(G, A)$ is given by

$$D_\lambda \left( \exp \left( \sum_{i=1}^d \alpha_i X_i \right) \right) = \exp \left( \sum_{i=1}^d \alpha_i \lambda m X_i \right).$$
If we identify $G$ with $G_H \times_\beta G_V$ then we obtain a corresponding dilation family on $G_H \times_\beta G_V$ given by
\[
D_\lambda \left( \exp \left( \sum_{i=1}^{j_0-1} \alpha_i X_i \right), \exp \left( \sum_{i=j_0}^{d} \alpha_i X_i \right) \right) = \left( \exp \left( \sum_{i=1}^{d} \alpha_i \lambda^{\mu_i} X_i \right), \exp \left( \lambda^{\rho(A)} \cdot \sum_{i=j_0}^{d} \alpha_i X_i \right) \right).
\]
We thus have $D_\lambda = (D^H_\lambda, D^V_\lambda)$, where $D^H_\lambda$ and $D^V_\lambda$ are dilations of $G_H$ and $G_V$ respectively. More precisely, $(D^H_\lambda)$ is the dilation family associated with the induced homogeneous structure on $G_H$, and if we identify $G_V$ with $(\mathbb{R}^m, +)$ as above, then $D^V_\lambda(x) = \lambda^{\rho(A)} x$.

This proves Part (c) of the proposition, and Part (d) was already established in Corollary 4.8, hence the proposition is proved. We close this subsection by spelling out Proposition 4.10 in the special case of a 2-step nilpotent RAHOGRASP:

**Remark 4.17** (The 2-step nilpotent case). If $(G, A)$ is a 2-step nilpotent RAHOGRASP and $A$ corresponds to the canonical stratification of $g$ (cf. Example 3.7 and Example 3.21), then the situation is considerably simpler: In this case we have $g = \mathfrak{g}_1 \oplus [\mathfrak{g}, \mathfrak{g}]$, and if $m = \dim [\mathfrak{g}, \mathfrak{g}]$, then $[\mathfrak{g}, \mathfrak{g}] \cong \mathbb{R}^m$ is a central ideal in $\mathfrak{g}$ and $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \cong \mathbb{R}^{d-m}$ is an abelian Lie algebra cf. Example 3.7. Together with Remark 3.9, we conclude that the spectrum of $A$ equals $\{1, 2\}$ and so $\rho(A) = 2$.

**Corollary 4.18** (Splittings of adapted lattices in the 2-step case). Let $(G, A)$ be a 2-step nilpotent Lie group with its canonical homogeneous structure and let $\Gamma$ be an adapted lattice in $(G, A)$.

(a) There exist $m \in \mathbb{N}$ and a smooth function $\beta : \mathbb{R}^{d-m} \times \mathbb{R}^{d-m} \to \mathbb{R}^m$ satisfying (7) and (8) such that $G \cong \mathbb{R}^{d-m} \times_\beta \mathbb{R}^m$.

(b) There exists an adapted lattice $\Gamma_H < \mathbb{R}^{d-m}$ with $\beta(\Gamma_H, \Gamma_H) \subseteq \mathbb{Z}^m$ such that, under the isomorphism from (a), $\Gamma \cong \Gamma_H \times_\beta \mathbb{Z}^m$.

(c) Under the isomorphism from (a) we have
\[
D_\lambda(g_H, v) = (\lambda g_H, \lambda^2 v), \quad g_H \in \mathbb{R}^{d-m}, v \in \mathbb{R}^m.
\]

(d) There exists $\lambda_0 > 1$ such that $D_{\lambda_0}(\Gamma) \subseteq \Gamma$, i.e. $\lambda_0 \Gamma_H \subseteq \Gamma_H$ and $\lambda_V := \lambda_0^2 \in \mathbb{Z}$. \qed

### 4.3. Fundamental domains and homogeneous dilation data

We recall that $(G, A)$ denotes a RAHOGRASP and $\Gamma < G$ denotes an adapted lattice. In view of Proposition 4.10 we will assume without loss of generality that, with notation as in the proposition, we have
\[
\Gamma = \Gamma_H \times_\beta \Gamma_0 < G = G_H \times_\beta \mathbb{R}^m,
\]
where $\Gamma_0 < \mathbb{R}^m$ is a cocompact lattice. We could even choose our coordinates such that $\Gamma_0 = \mathbb{Z}^m$, but in examples it will often be convenient to allow $\Gamma_0$ to be of the form $\Gamma_0 = M\mathbb{Z}^m$ for some invertible matrix $M$. Throughout this section we are going to assume that the splitting is non-trivial, i.e. $G_H \neq \{e\}$; then $G_H$ is automatically non-compact and thus $\Gamma_H$ is an infinite group. Given subsets $A \subseteq G_H$ and $B \subseteq \mathbb{R}^m$ we will denote by $A \times_\beta B$ the corresponding subset of $G$. We also recall that a Borel fundamental domain for $\Gamma$ is called *convenient* if it contains an open identity neighbourhood.

**Lemma 4.19** (Fundamental boxes). If $V_H$ is a (convenient) fundamental domain for $\Gamma_H$ in $G_H$ and $V_0$ is a (convenient) fundamental domain for $\Gamma_0$ in $\mathbb{R}^m$, then $V := V_H \times_\beta V_0$ is a (convenient) fundamental domain for $\Gamma$ in $G$. 


Proof. Let $g = (g_H, x) \in G$. There then exists $\gamma H \in \Gamma H$ such that $v_H := \gamma H g_H \in V_H$. Let $y \in \mathbb{R}^m$ such that $(\gamma H, 0)(g_H, x) = (v_H, y)$. Then there exists $\gamma V \in \Gamma_0$ such that $y + \gamma V \in V_0$, and hence if we set $\gamma := (\gamma H, \gamma V) = (e, \gamma V)(\gamma H, 0)$, then

$$
\gamma g = (e, \gamma V)(\gamma H, 0)(g_H, x) = (e, \gamma V)(v_H, y) = (v_H, y + \gamma V) \in V_H \times V_0.
$$

Conversely let $v = (v_H, v_0) \in V$ and $\gamma = (\gamma H, \gamma V)$ in $\Gamma$ with $\gamma v = (\gamma H v_H, \gamma V + v_0 + \beta(\gamma H, v_H)) \in V$, i.e. $\gamma H v_H \in V_H$ and $\gamma V + v_0 + \beta(\gamma H, v_H) \in V_0$. Since $V_H$ is a fundamental domain for $\Gamma_H$ and $\{v_H, v_H\} \subseteq V_H$ we deduce that $\gamma H = e$. This in turn implies by (8) that $\{v_0, \gamma V + v_0\} \in V_0$, and since $V_0$ is a fundamental domain for $\Gamma_0$ we deduce that $\gamma V = 0$. Thus $\gamma$ is trivial, and hence $V$ is a fundamental domain. \qed

For the following lemma we recall that the dilation $D_{\lambda_0}$ acts on $\mathbb{R}^m$ by multiplication by some constant factor $\lambda_V$ (which depends on $\lambda_0$).

**Lemma 4.20.** There exists a convenient fundamental domain $V_0$ for $\Gamma_0$ with the following property: If $\lambda_0 > 1$ has been chosen such that $D_{\lambda_0}(\Gamma) \subseteq \Gamma$ and $\lambda_V > 1$, then

$$
\bigcap_{x \in F_V} x + F_V \neq \emptyset, \quad \text{where } F_V := \lambda_V V_0 \cap \Gamma_0 \subseteq \Gamma_0.
$$

**Proof.** We may assume without loss of generality that $\Gamma_0 = \mathbb{Z}^m$; by Proposition 4.10 we then have $\lambda_V \in \mathbb{Z}$. We claim that we can simply choose $V_0 := [-1/2, 1/2]^m$. Indeed, we have

$$
F_V = \begin{cases} 
\{ -j, -j + 1, \ldots, j - 2, j - 1 \}^m, & \lambda_V = 2j \text{ even}, \\
\{ -j, -j + 2, \ldots, j - 2, j \}^m, & \lambda_V = 2j + 1 \text{ odd},
\end{cases}
$$

and thus the intersection in (11) always contains the vector $(-1, \ldots, -1)^T$ (if $\lambda_V$ is even) or $(0, \ldots, 0)^T$ (if $\lambda_V$ is odd). \qed

**Definition 4.21.** A dilation datum $D = (G, d, (D_\lambda)_{\lambda>0}, \Gamma, V)$ is called homogeneous if

- $(G, d, (D_\lambda)_{\lambda>0})$ is a homogeneous dilation group and $\Gamma < G$ is an adapted lattice;
- there exists a non-trivial splitting $G = G_H \times_\beta \mathbb{R}^m$ (compatible with the dilation structure) and corresponding splitting $\Gamma = \Gamma_H \times_\beta \Gamma_0$ such that $V = V_H \times_\beta V_0$, where $V_H$ and $V_0$ are convenient fundamental domains for $\Gamma_H$ and $\Gamma_0$ respectively;
- $V_0$ satisfies the intersection property from Lemma 4.20.

In this situation, $V$ is called a fundamental box for the given splitting.

**Corollary 4.22.** Every RAHOGRA SP of dimension at least 2 admits a homogeneous dilation datum.

**Proof.** This follows from combining Corollary 4.8, Proposition 4.10, Remark 4.12, Lemma 4.19 and Lemma 4.20. \qed

Note that the abelian RAHOGRA SP $(\mathbb{R}^1, +)$ is the only example which has to be excluded from Corollary 4.22, since it does not admit a non-trivial splitting. However, all of the results which we are going to establish for substitutions over homogeneous dilation data are anyway known in the one-dimensional case [Que87, BG13].
5. Substitution maps from substitution data

In the previous two sections we have constructed plenty of examples of dilation data. We now turn to the combinatorial side of our construction and discuss substitution data and associated substitution maps. We will see in the next section that interesting substitution data exist in the setting of homogeneous dilation data.

5.1. Patches. Let Γ be a countable group and let A be a finite alphabet.

Notation 5.1. Given a subset $M \subseteq \Gamma$, we consider $A^M$ as a compact topological space with the product topology. An element $P \in A^M$ is called a patch with support $M$, and we set $\text{supp}(P) := M$. A patch $P$ is called finite if $\text{supp}(P)$ is finite. We then denote the space of all patches, respectively all finite patches by

$$A^*_\Gamma := \bigcup_{M \subseteq \Gamma} A^M \quad \text{and} \quad A^*_\Gamma := \bigcup_{M \subseteq \Gamma \text{ finite}} A^M \subseteq A^*_\Gamma.$$  

The group $\Gamma$ acts on $A^*_\Gamma$ by

$$\text{supp}(\gamma.P) := \gamma.\text{supp}(P) \quad \text{and} \quad \gamma.P(x) := P(\gamma^{-1}x), \quad \gamma \in \Gamma, \ P \in A^*_\Gamma,$$

and we denote by $\sim$ the corresponding orbit relation on $A^*_\Gamma$, i.e. $P \sim Q$ if there exists $\gamma \in \Gamma$ with $\gamma.P = Q$. If $M \subseteq \Gamma$, or more generally $M \subseteq G$, and $P \in A^*_\Gamma$, then we denote by $P|_M := P|_{\text{supp}(P) \cap M}$ the restriction of $P$ to $\text{supp}(P) \cap M$.

Definition 5.2. Given two patches $P$ and $Q$ we say that $P$ occurs in $Q$ and write $P \prec Q$ if there exists a subset $M \subseteq \Gamma$ such that $P \sim Q|_M$.

Note that $\prec$ defines a partial order on $A^*_\Gamma$.

5.2. Extending substitution rules. Throughout this subsection we fix a dilation datum $D = (G,d, (D_\lambda)_{\lambda > 0}, \Gamma, V)$ and a substitution datum $(A, \lambda_0, S_0)$ over $D$. As before, we then set $D := D_{\lambda_0}$ and $\lambda_V := \lambda_0^{d(A)} > 1$.

Remark 5.3 (Basic properties). Since $\Gamma$ is a uniform lattice in $G$, there exist $R \geq r > 0$ such that $\text{dist}(g,\Gamma) \leq R$ for all $g \in G$ and $d(\gamma_1, \gamma_2) \geq r$ for all distinct elements $\gamma_1, \gamma_2 \in \Gamma$; we then say that $\Gamma$ is $r$-uniformly discrete and $R$-relatively dense, or a $(r, R)$-Delone set for short. Conversely, every $(r, R)$-Delone subgroup of $G$ is a uniform lattice.

If $\Gamma < G$ is an $(r, R)$-Delone subgroup for some $R > r > 0$, then $D^n(\Gamma)$ is again a uniform lattice, since it is $(\lambda_0^n r, \lambda_0^n R)$-Delone. The latter implies in particular that

$$D(\Gamma) \subseteq \Gamma. \tag{12}$$

Indeed, otherwise $\Gamma$ would be $(\lambda_0^n r)$-discrete for all $n \in \mathbb{N}$, contradicting relative denseness.

The fact that $V$ is a convenient fundamental domain for $\Gamma$ implies that

$$G = \bigcup_{\gamma \in \Gamma} \gamma V. \tag{13}$$

It also implies that there exist $r_+ > r_- > 0$ such that

$$B(e, r_-) \subseteq V \subseteq B(e, r_+). \tag{14}$$

In this situation we call $r_-$ an inner radius and $r_+$ an outer radius for $V$. We say that $\lambda_0$ is sufficiently large relative to $V$ if there exist an inner radius $r_-$ and an outer radius $r_+$ such that

$$\lambda_0 > 1 + \frac{r_+}{r_-} \tag{15}$$
For many of our results below we will have to assume that this is the case.

In the sequel we will often use the following consequences of (13): Firstly, for every $\gamma \in \Gamma$ we have $\gamma V \cap \Gamma = \{\gamma\}$, and hence
\begin{equation}
MV \cap \Gamma = M \quad \text{for every } M \subseteq \Gamma.
\end{equation}

Secondly, since $D$ is an automorphism we have
\begin{equation}
G = D^n(G) = \bigcup_{\gamma \in \Gamma} D^n(\gamma)D^n(V).
\end{equation}

In particular, $D^n(V)$ is a convenient fundamental domain for $D^n(\Gamma)$. Finally, for all $M \subseteq \Gamma$ we have
\begin{equation}
D(MV) \cap \Gamma = \bigcup_{\eta \in M} (D(\eta)D(V)) \cap \Gamma = \bigcup_{\eta \in M} D(\eta)(D(V) \cap \Gamma).
\end{equation}

Here the last equality follows from (12) and the fact that $(\gamma A) \cap \Gamma = \gamma(A \cap \Gamma)$ for every $\gamma \in \Gamma$ and $A \subseteq G$.

As mentioned in the introduction, every substitution rule gives rise to a substitution map on the level of patches:

**Construction 5.4** (Extending the substitution rule). Let $P \in \mathcal{A}_\Gamma^{**}$ be a patch and let $M := \text{supp}(P)$. We then define a patch $S(P)$ as follows:

(a) We choose the support of $S(P)$ to be $\text{supp}(S(P)) := D(MV) \cap \Gamma$.

(b) Now let $\gamma \in D(MV) \cap \Gamma$. By (18) there then exists a unique $\eta \in M$ such that $\gamma \in D(\eta)(D(V) \cap \Gamma)$.

(c) If $\eta$ is as in (b), then $P(\eta) \in A$ and $D(\eta)^{-1}\gamma \in D(V) \cap \Gamma$, hence we may define $S(P)(\gamma) := S_0(P(\eta))(D(\eta)^{-1}\gamma)$.

**Definition 5.5.** The map $S : \mathcal{A}_\Gamma^{**} \to \mathcal{A}_\Gamma^{**}$ constructed in Construction 5.4 is called the substitution map associated with the substitution datum $(A, \lambda_0, S_0)$.

**Proposition 5.6.** The substitution map $S : \mathcal{A}_\Gamma^{**} \to \mathcal{A}_\Gamma^{**}$ is the unique map which satisfies Axioms (E1)–(E3) from Proposition 2.7. Moreover, we have
\begin{equation}
S^n(\gamma P) = D^n(\gamma)S^n(P) \quad \text{for all } P \in \mathcal{A}_\Gamma^{**} \text{ and each } n \in \mathbb{N}.
\end{equation}

**Proof.** To prove uniqueness of $S$, let $P \in \mathcal{A}_\Gamma^{**}$. If $\text{supp}(P) = \{e\}$, then $S(P)$ is uniquely determined by (E1). If $|\text{supp}(P)| = 1$, then $S(P)$ is uniquely determined by (E2) and the previous case, since $\text{supp}(P)$ is a translate of $\{e\}$. Finally, in the general case $S(P)$ is uniquely determined by Axiom (E3) and the previous cases, since we can decompose $\text{supp}(P)$ into a union of singletons. This shows that there exists at most one function $S$ satisfying (E1)–(E3), and it remains to show that the map $S$ from Construction 5.4 satisfies the Axioms (E1)–(E3) and (19). Since (E1) holds by definition and (19) follows by applying (E2) $n$ times, we are left with showing (E2) and (E3).

(E2) Let $\gamma \in \Gamma$ and $P \in \mathcal{A}_\Gamma^{**}$. Set $M := \text{supp}(P)$ and note that $\text{supp}(\gamma P) = \gamma M$. Since $D$ is an automorphism and $\Gamma$ is $D$-invariant we have
\begin{align*}
\text{supp}(S(\gamma P)) &= D(\gamma MV) \cap \Gamma = D(\gamma)(D(MV) \cap \Gamma) \\
&= D(\gamma)(D(MV) \cap \Gamma) = D(\gamma) \cdot \text{supp}(S(P)),
\end{align*}

which shows the first part of (E2). Now let \( \zeta \in \text{supp}(S(\gamma P)) = D(\gamma)D(MV) \cap \Gamma \) and set \( \zeta' := D(\gamma^{-1})\zeta \); then \( \zeta' \in \text{supp}(P) \) and we need to show that
\[
S(\gamma P)(\zeta) = D(\gamma)S(P)(\zeta) = S(P)(\zeta').
\]
Let \( \eta' \in M \) be the unique element such that \( \zeta' \in D(\eta')(D(V) \cap \Gamma) \) and set \( \eta := \gamma \eta' \in \gamma M \) so that \( \zeta \in D(\eta)(D(V) \cap \Gamma) \). By construction we then have
\[
S(\gamma P)(\zeta) = S_0(\gamma P(\eta))(D(\eta)^{-1}\zeta) \quad \text{and} \quad S(P)(\zeta') = S_0(P(\eta'))(D(\eta')^{-1}\zeta').
\]
Since \( \gamma P(\eta) = P(\gamma^{-1}\eta) = P(\eta') \) and \( D(\eta)^{-1}\zeta = D(\gamma \eta')^{-1}D(\gamma)\zeta' = D(\eta')^{-1}\zeta' \) this implies (20) and finishes the proof of (E2).

(E3) Let \( P \in A^*_M^\Gamma \) and \( M := \text{supp}(P) \), and suppose that \( (M_i)_{i \in I} \) is a family of subsets of \( M \) such that \( M = \bigcup_i M_i \). By (18) we have
\[
\text{supp}(S(P)) = D(MV) \cap \Gamma = \bigcup_{\eta \in M} D(\eta)(D(V) \cap \Gamma) = \bigcup_{i \in I} \bigcup_{\eta \in M_i} D(\eta)(D(V) \cap \Gamma)
\]
\[
= \bigcup_{i \in I} \bigcup_{\eta \in M_i} D(M_i V) \cap \Gamma = \bigcup_{i \in I} \text{supp}(S(P|M_i)),
\]
which establishes the first part of (E3). Finally, let \( \gamma \in \text{supp}(S(P|M_i)) \) and \( \eta \in M_i \subseteq M \) such that \( \gamma \in D(\eta)(D(V) \cap \Gamma) \). Then by construction we have \( S(P)(\gamma) = S_0(P(\eta))(D(\eta)^{-1}\gamma) = S(P|M_i)(\gamma) \), which shows (E3) and finishes the proof.

Note that, by construction, the substitution map \( S \) restricts to maps (also denoted by \( S \))
\[
S : A^\Gamma \to A^\Gamma \quad \text{and} \quad S : A^*_M^\Gamma \to A^*_M^\Gamma.
\]
If we equip \( A^\Gamma \) with the product topology, then it is a compact space and we observe:

**Proposition 5.7** (Continuity). The map \( S : A^\Gamma \to A^\Gamma \) is continuous.

**Proof.** Let \( M \subseteq \Gamma \) be finite and pick an arbitrary patch \( P \in A^M \). Define the open set \( U := U_P := \{ \omega \in A^\Gamma : \omega|M = P \} \). By definition of the product topology on \( A^\Gamma \) it suffices to show that \( S^{-1}(U) \) is open in \( A^\Gamma \). Recall that \( \Gamma = \bigcup_{\eta \in \Gamma} (D(\eta)D(V) \cap \Gamma) \) and therefore, there are \( m \in \mathbb{N} \) and pairwise distinct \( \eta_j \in \Gamma, 1 \leq j \leq m \), such that
\[
M \subseteq \bigcup_{j=1}^m (D(\eta_j)D(V) \cap \Gamma).
\]
We define \( M' := \{ \eta_j : 1 \leq j \leq m \} \). For each \( \omega \in S^{-1}(U) \) we define the set \( W_\omega := \{ \omega' \in A^\Gamma : \omega'|M' = \omega|M' \} \). Then each \( W_\omega \) is an open neighborhood of \( \omega \) and we have
\[
\bigcup_{\omega \in S^{-1}(U)} W_\omega \supseteq S^{-1}(U),
\]
and the above union is open as a union of open sets. Thus, in order to show that \( S^{-1}(U) \) is open, it suffices to prove that \( S(W_\omega) \subseteq U \) for all \( \omega \in S^{-1}(U) \). To this end, pick \( \omega \in S^{-1}(U) \) and let \( \omega' \in W_\omega \), i.e. \( \omega(\eta_j) = \omega'(\eta_j) \) for \( 1 \leq j \leq m \). This in turn gives \( S_0(\omega(\eta_j)) = S_0(\omega'(\eta_j)) \) for all \( 1 \leq j \leq m \). Now let \( \gamma \in \bigcup_{j=1}^m D(\eta_j)D(V) \cap \Gamma \) such that by the aforementioned disjointness property there is a unique \( 1 \leq i \leq m \) such that \( \gamma \in D(\eta_i)D(V) \cap \Gamma \). By definition of the map \( S \) (see the properties of the Construction 5.4) we obtain
\[
S(\omega)(\gamma) = S_0(\omega(\eta_i))(D(\eta_i)^{-1}\gamma) = S_0(\omega'(\eta_i))(D(\eta_i)^{-1}\gamma) = S(\omega')(\gamma).
\]
Since $M \subseteq \bigcup_{j=1}^m D(\eta_j)D(V) \cap \Gamma$ by the very definition of the $\eta_j$, the equality $S(\omega)(\gamma) = S(\omega')(\gamma)$ holds in particular for all $\gamma \in M$. Consequently, $S(\omega') \in U$. Hence $S(W_\omega) \subseteq U$ and since $\omega \in S^{-1}(U)$ was chosen arbitrarily, the proof of the proposition is finished. \hfill \Box

5.3. Support growth. We keep the notation of the previous subsection. In particular, $(A, \lambda_0, S_0)$ denotes a substitution datum over a fixed dilation datum $D = (G, d, (D_\lambda)_{\lambda \geq 0}, \Gamma, V)$ and $D := D_\lambda_0$.

**Notation 5.8.** Given $g \in G$ and $r > 0$ we denote by $B(g, r)$ the open ball of radius $r$ around $g$ with respect to $d$. Given a bounded subset $B \subseteq G$ we define

$$B_{+r} := \bigcup_{x \in B} B(x, r) \quad \text{and} \quad B_{-r} := \{x \in B : B(x, r) \subseteq B\}.$$ 

Note that for all $g \in G$ and $r', r > 0$ we have $B(g, r')_{+r} \subseteq B(g, r' + r)$, and if $r' > r$, then we also have $B(g, r')_{-r} \supseteq B(g, r' - r)$, hence the notation.

We consider the following problem:

**Problem 5.9 (Support growth problem).** Let $P \in A_1^*$ be a finite patch. What is the support of $S^n(P)$? In what sense does it grow as $n \to \infty$?

The answer to this question will be roughly as follows: If the stretch factor $\lambda_0$ was close to 1, then the support of $S^n(P)$ would not have to grow at all. However, since we are assuming that $\lambda_0$ is actually sufficiently large relative to $V$ (as part of the definition of a substitution datum), the sets $\text{supp}(S^n(P))$ will actually grow arbitrarily large in cardinality as $n \to \infty$. If the initial set $M$ contains the identity element $e \in G$, then they even cover larger and larger balls in $\Gamma$, but if the initial set $M$ is far away from $e$, then they will run further and further away from $e$. To make this precise, we introduce the following concept.

**Definition 5.10.** The inner $\Gamma$-approximation of a bounded subset $B \subseteq G$ is defined as

$$B_{\Gamma} := (B \cap \Gamma)V.$$ 

In other words, $B_{\Gamma}$ is obtained from $B$ by first passing to the finite sets of lattice points inside $B$ and then thickening the resulting finite set by $V$. It is thus a finite union of translates of $V$. For the following proposition we recall from (14) the notion of an outer radius for $V$.

**Proposition 5.11 (Size of inner approximation).** If $r_+$ denotes an outer radius for $V$, then

$$B_{-r_+} \subseteq B_{\Gamma} \subseteq B_{+r_+}.$$ 

**Proof.** Every $x \in G$ is contained in a set of the form $\gamma V$ for a unique $\gamma \in \Gamma$. If $v := \gamma^{-1}x \in V$, then $d(x, \gamma) = d(\gamma v, \gamma) = d(v, e) < r_+$ and thus $\gamma \in B(x, r_+)$. Now if $x \in B_{-r_+}$, then $B(x, r_+) \subseteq B$ and hence $\gamma \in B \cap \Gamma$. This implies $x \in B_{\Gamma}$ and shows the first inclusion. The second inclusion is immediate from $V \subseteq B(e, r_+)$.

For example, if $B$ is a large open ball around the identity, then $B_{\Gamma}$ is very close to $B$. At the other extreme, if $B$ contains no lattice points, then $B_{\Gamma}$ is empty. We can now describe how the support of a patch transforms under iterations of $S$:

**Proposition 5.12 (Support formula).** If $P \in A_1^*$ with $\text{supp}(P) = M \neq \emptyset$, then

$$\text{supp}(S^n(P)) = V(n, M) \cap \Gamma,$$

where $V(0, M) := MV$ and $V(n, M) := D(V(n-1, M)_{\Gamma})$ for all $n \in \mathbb{N}$. 
Proof. If \( n = 1 \), then \( \text{supp}(S(P)) = D(MV) \cap \Gamma \) holds by definition. Then \( MV \cap \Gamma = M \) (since \( e \in V \)) implies \( D(MV) = D((MV \cap \Gamma)V) = V(1, M) \) proving the induction base. By induction hypothesis suppose \( \text{supp}(S^n(P)) = V(n, M) \cap \Gamma \). Since \( S^{n+1}(P) = S(S^n(P)) \), we conclude

\[
\text{supp}(S^{n+1}(P)) = D((V(n, M) \cap \Gamma)V) \cap \Gamma = V(n + 1, M) \cap \Gamma,
\]

proving the desired statement.

By the proposition, the support growth problem amounts to study the growth of the sets \( V(n, M) \). This growth can be described under the assumption that \( \lambda_0 \) is sufficiently large relative to \( V \) in the sense of Remark 5.3.

**Theorem 5.13** (Support growth theorem). Let \( M \subseteq \Gamma \) be a non-empty finite subset. If \( \lambda_0 \) is sufficiently large relative to \( V \), then the following hold.

(a) If \( e \in M \), then for every bounded set \( B \subseteq G \), there exists \( n_0 \in \mathbb{N} \) such that \( B \subseteq V(n_0, M) \).

(b) For all \( k \in \mathbb{N} \), the inclusions

\[
B(e, \lambda_0^{k+1}C_-) \subseteq V(k + 1) \subseteq B(e, \lambda_0^{k+1}C_+)
\]

holds with \( C_- := (\lambda_0 - (1 + \frac{r}{r-\varepsilon}))\frac{r}{\lambda_0} \) and \( C_+ := \lambda_0 r_-(1 + \frac{\lambda_0}{\lambda_0 - 1}) \).

(c) If \( \text{dist}(M, \{e\}) \geq \frac{2\lambda_0r_+}{\lambda_0 - 1} \), then \( \text{dist}(V(n, M), \{e\}) \to \infty \) as \( n \to \infty \).

In particular, \( |V(n, M) \cap \Gamma| \to \infty \) as \( n \to \infty \).

**Remark 5.14.** It is worth mentioning that for specific groups the assumption that \( \lambda_0 \) is sufficiently large relative to \( V \) can be relaxed. For instance, we observed in the Heisenberg group \( G = \mathbb{H}_3(\mathbb{R}) \) that \( \lambda_0 \) is sufficiently large relative to \( V \) for the lattice \( \Gamma = 2\mathbb{Z}^3 \) with \( V = [-1, 1]^3 \) if \( \lambda_0 \geq 3 \), cf. Example 2.9. However, one can show for \( \lambda_0 = 2 \) the statements of Theorem 5.13 (a) and (b) are still valid for different constants \( C_- \) and \( C_+ \). This follows by straightforward computation, cf. also [Bec21]. On the other hand, the definition of \( \lambda_0 \) being sufficiently large relative to \( V \) is universal in the realm of RAHOGRASPs.

The case where \( M = \{e\} \) will play a crucial role and so we abbreviate \( V(n) := V(n, \{e\}) \). The key observation is that the sets \( V(n) \) exhaust \( G \) as soon as they manage to exhaust a sufficiently large ball.

**Lemma 5.15.** Let \( r > r_+ \) and \( \lambda_0 > \frac{r}{r_r} \). If \( B(e, r) \subseteq V(n_0) \) for some \( n_0 \in \mathbb{N} \), then every bounded subset of \( G \) is contained in \( V(n) \) for some \( n \in \mathbb{N} \). Moreover, the inclusions

\[
B(e, c_- \lambda_0^{k+1}) \subseteq V(n_0 + k) \subseteq B(e, c_+ \lambda_0^{n_0+1})
\]

hold, where \( c_- := \lambda_0(r - r_+) - r > 0 \) and \( c_+ := r(1 + \frac{\lambda_0}{\lambda_0 - 1}) > 0 \).

**Proof.** Let \( \varepsilon := \frac{\lambda_0(r - r_+)}{r} - 1 > 0 \), then \( \lambda_0(r - r_+) = (1 + \varepsilon)r \). We prove for all \( k \in \mathbb{N} \) that

\[
B(e, \lambda_0^{k+1}\varepsilon r + r) \subseteq V(n_0 + k).
\]

For \( k = 1 \), Proposition 5.11 and Proposition 5.12 imply

\[
B(e, r) \subseteq V(n_0) \implies B(e, r - r_+) \subseteq V(n_0) \implies B(e, \lambda_0(r - r_+)) \subseteq V(n_0 + 1).
\]

Since \( \lambda_0(r - r_+) = (1 + \varepsilon)r = \lambda_0^{-1}\varepsilon r + r \), the induction base case is proven. Then we see by the induction hypothesis that

\[
B(e, \lambda_0^{k+1}\varepsilon r + r) \subseteq V(n_0 + k) \implies B(e, \lambda_0^{k+1}\varepsilon r + r - r_+) \subseteq V(n_0 + k) \implies B(e, \lambda_0^k\varepsilon r + \lambda_0 r - \lambda_0 r_+) \subseteq V(n_0 + k + 1).
\]
while $\lambda_0 r - \lambda_0 r_+ = (1 + \varepsilon) r \geq r$. This proves the first inclusion since $c_- = \varepsilon r$. Moreover, we conclude that every bounded subset of $G$ is eventually contained in $V(n)$.

Next we prove for all $k \in \mathbb{N}$ that

$$V(k) \subseteq B(e, \lambda_0^k r(1 + S_{k-1}))$$

where $S_k := \sum_{n=0}^{k} \frac{1}{\lambda_0^n}$ for $k \geq 0$. For $k = 1$, we have

$$V(0) \subseteq B(e, r_+) \subseteq B(e, r) \implies V(0) \Gamma \subseteq B(e, r + r_+) \implies V(1) \subseteq B(e, \lambda_0(r + r_+)).$$

This proves the induction base case as $\lambda_0(r + r_+) = \lambda_0 r (1 + \frac{r_+}{r})$ and $\frac{r_+}{r} < 1 = S_0$. Then the induction hypothesis leads to

$$V(k) \subseteq B(e, \lambda_0^k r(1 + S_{k-1})) \implies V(k) \Gamma \subseteq B(e, \lambda_0^k r(1 + S_{k-1}) + r_+) \implies V(k+1) \subseteq B(e, \lambda_0^{k+1} r(1 + S_{k-1}) + \lambda_0 r_+)).$$

Since

$$\lambda_0^{k+1} r(1 + S_{k-1}) + \lambda_0 r_+ = \lambda_0^{k+1} r (1 + S_{k-1} + \frac{r_+}{\lambda_0}) < \lambda_0^{k+1} r (1 + S_{k-1} + \frac{1}{\lambda_0}) = \lambda_0^{k+1} r (1 + S_k),$$

the inclusion $V(k) \subseteq B(e, \lambda_0^k r(1 + S_{k-1}))$ is proven for all $k \in \mathbb{N}$.

The previous considerations lead to the desired inclusion $V(n_0 + k) \subseteq B(e, c_+ \lambda_0^{n_0+k})$ as $r > 0$, $\lim_{k \to \infty} S_k = \frac{\lambda_0}{\lambda_0 - 1}$ (geometric series) and $S_k \leq S_{k+1}$ for all $k \in \mathbb{N}$.

**Proof of Theorem 5.13 for $M = \{e\}$.** Assume that $M = \{e\}$ and define $r := \lambda_0 r_-$. Then $V(0) \Gamma = \{e\} V = V \supseteq B(e, r_-) \implies V(1) = D(V(0) \Gamma) \supseteq D(B(e, r_-)) = B(e, \lambda_0 r_-) = B(e, r)$.

Since $\lambda_0 > 1 + \frac{r_+}{r_-}$ holds by assumption, we conclude $r = \lambda_0 r_- > r_- + r_+ > r_+$. Similarly, multiplying both sides of $\lambda_0 > 1 + \frac{r_+}{r_-}$ with $\lambda_0 r_-$ and rearranging terms yields

$$(\lambda_0 r_- - r_+) \lambda_0 > \lambda_0 r_- \implies (r - r_+) \lambda_0 > r \implies \lambda_0 > \frac{r}{r - r_+}.$$

Part (a) and (b) of the Theorem then follows by applying Lemma 5.15 with $n_0 := 1$.

(c) Let $r := \text{dist}(M, \{e\}) \geq \frac{2 \lambda_0 r_+}{\lambda_0 - 1}$. We show by induction that for all $n \in \mathbb{N}$ we have

$$\text{dist}(V(n, M), \{e\}) \geq r + (n-1) \lambda_0 r_+.$$

Indeed, the assumption on $r$ amounts to $\lambda_0(r - 2r_+) \geq r$. For $n = 1$ this yields

$$
\text{dist}(M V, \{e\}) \geq r - r_+ \implies \text{dist}(V(0), \{e\}) \geq r - 2r_+ \implies \text{dist}(V(1), \{e\}) \geq r.
\$$

For $n \geq 2$ we then see by induction that

$$
\text{dist}(V(n), \{e\}) \geq r + (n-1) \lambda_0 r_+ \implies \text{dist}(V(n, M), \{e\}) \geq (r - 2r_+) + r_+ + (n-1) \lambda_0 r_+ \implies \text{dist}(V(n + 1), \{e\}) \geq r + \lambda_0 r_+ + (n-1) \lambda_0^2 r_+,
\$$

and since the latter is bounded below by $r + n \lambda_0 r_+$, the theorem follows.

The following lemma collects various properties of the sets $V(n, M)$:

**Lemma 5.16.** Let $\gamma \in \Gamma$, $M, M' \subseteq \Gamma$ be finite non-empty sets and $n, m \in \mathbb{N}$. Then the following hold:

(a) $V(n, \gamma M) = D^n(\gamma)V(n, M)$,

(b) If $M \subseteq M'$, then $V(n, M) \subseteq V(n, M')$,

(c) $V(n, V(m, M) \cap \Gamma) = V(n + m, M)$,
(d) $V(n - 1) \cap \Gamma \subseteq V(n) \cap \Gamma$.

**Proof.** Each of the statements follows by a simple induction argument. Here are the details.

(a) For $n = 1$, the identities

$$V(1, \gamma M) = D((\gamma M \cap \Gamma)V) = D(\gamma)D((M \cap \gamma^{-1}\Gamma)V) = D(\gamma)D((M \cap \Gamma)V)$$

follow as $D$ is an automorphism and $\gamma \in \Gamma$. For general $n \in \mathbb{N}$, we obtain

$$V(n + 1, \gamma M) = D((V(n, \gamma M) \cap \Gamma)V) = D(D^n(\gamma)V(n, M) \cap \Gamma)V)$$

$$= D^{n+1}(\gamma)D((V(n, M) \cap D^n(\gamma)^{-1}\Gamma)V) = D^{n+1}(\gamma)D((V(n, M) \cap \Gamma)V)$$

using the induction hypothesis, that $D$ is an automorphism and $D^n(\Gamma) \subseteq \Gamma$.

(b) The equalities $V(1, M) = D(MV) \subseteq D(M'V) = V(1, M')$ prove the case $n = 1$. Then the induction hypothesis leads to

$$V(n + 1, M) = D((V(n, M) \cap \Gamma)V) \subseteq D((V(n, M') \cap \Gamma)V) = V(n + 1, M').$$

(c) We first prove the statement for $m = 1$ by an induction over $n \in \mathbb{N}$. For $n = 1$, the equality

$$V(1, V(1, M) \cap \Gamma) = D((V(1, M) \cap \Gamma)V) = V(2, M)$$

follows by definition. Then the induction hypothesis $V(n, V(1, M)) \cap \Gamma = V(n+1, M) \cap \Gamma$ implies

$$V(n + 1, V(1, M) \cap \Gamma) = D((V(n, V(1, M) \cap \Gamma)V) = D((V(n + 1, M) \cap \Gamma)V) = V(n + 2, M).$$

Having the statement for $m = 1$, we conclude the desired identity for any $m \in \mathbb{N}$ by induction. Specifically, the induction step follows from

$$V(n + m + 1, M) = V(n + m, V(1, M) \cap \Gamma) = V(n, V(m, V(1, M) \cap \Gamma) \cap \Gamma) = V(n, V(m + 1, M) \cap \Gamma).$$

(d) For $n = 1$, we get $V(1) \cap \Gamma = D(V \cap \Gamma) \cap \Gamma \supseteq \{e\} = V \cap \Gamma$. If it holds for $n \geq 1$, then

$$V(n + 1) \cap \Gamma = D((V(n) \cap \Gamma)V) \cap \Gamma \supseteq D((V(n - 1) \cap \Gamma)V) \cap \Gamma = V(n) \cap \Gamma$$

proves the desired result. \qed

6. Existence of Good Substitution Data

The purpose of this section is to prove Theorem 1.1, i.e. to establish the existence of primitive substitution data over arbitrary dilation data and – in the case of homogeneous dilation data – even of non-periodic primitive substitution data.

6.1. Primitive Substitution Data. We fix a dilation datum $D := (G, d, (D_\lambda)_{\lambda > 0}, \Gamma, V)$. The goal of this subsection is to show that there exists a primitive substitution datum $(A, \lambda_0, S_0)$ over $D$. Recall that a substitution datum $(A, \lambda_0, S_0)$ is primitive if there exists $L \in \mathbb{N}$ such that for all $a, b \in A$ we have $P_a \prec S^L(P_b)$.

**Construction 6.1.** Since $D$ is a dilation datum, there is a $\lambda > 1$ such that $D_\lambda(\Gamma) \subseteq \Gamma$. Since $V$ is an identity neighborhood, there is a positive radius $r_- > 0$ such that $B(e, r_-) \subseteq V$. Thus,

$$B(e, \lambda^N r_-) \subseteq D^N_\lambda (B(e, r_-)) \subseteq D^N_\lambda (V)$$

follows and so $D^N_\lambda (V)$ contains arbitrary large balls. Since $\Gamma$ is relatively dense and $\lambda > 1$, there is an $N \in \mathbb{N}$ such that $|D^N_\lambda (V) \cap \Gamma| \geq 2$. Let $\lambda_0 := \lambda^N$, $D := D_{\lambda_0}$ and $A := \{a, b\}$ be an alphabet with two different letters. Since $|D(V) \cap \Gamma| \geq 2$, there are two disjoint non-empty sets $\Xi_a, \Xi_b \subseteq \Gamma$ satisfying $D(V) \cap \Gamma = \Xi_a \cup \Xi_b$. Define a substitution rule $S_0 : A \to A^{D(V) \cap \Gamma}$
by \( S_0(c)(\Xi_o) = \{ a \} \) and \( S_0(c)(\Xi_o) = \{ b \} \). Then \((A, \lambda_0, S_0)\) is clearly primitive with \( L = 1 \). It is worth pointing out that the same construction would work for any finite alphabet.

6.2. Non-periodic primitive substitution data. We have seen in the previous subsection that any given dilation datum admits a primitive substitution datum. We do not know whether every dilation datum admits a non-periodic primitive substitution datum, but as we will see in the present subsection, this is indeed the case in all of our examples. Specifically, we will show the following proposition and thereby complete the proof of Theorem 1.1, see Proposition 6.6 and the preceding considerations.

**Proposition 6.2.** If \( A \) is a finite alphabet with \( |A| \geq 2 \) and \( D \) is a homogeneous dilation datum, then there exists a primitive and non-periodic substitution datum \( S \) over \( D \) with alphabet \( A \).

Throughout this subsection we fix a homogeneous dilation datum \( D = (G, d, (D_\lambda)_{\lambda > 0}, \Gamma, V) \) and an alphabet \( A \) with \( |A| \geq 2 \). There then exists a splitting \( G = G_H \times_\beta \mathbb{R}^m \), compatible with the dilation structure, and corresponding splittings \( \Gamma = \Gamma_H \times_\beta \Gamma_0 \) and \( V = V_H \times_\beta V_0 \) of the lattice and fundamental domain such that \( V_0 \) satisfies the intersection condition of Lemma 4.20, and we fix all these data as well. We remind the reader that in the definition of homogeneous dilation datum, we suppose that \( G_H \neq \{ e \} \) and so \( G_H \) is a non-compact group.

**Lemma 6.3 (Choice of stretch factor).** There exists \( \lambda_0 > 1 \) which is sufficiently large relative to \( V \) and such that \( D := D_{\lambda_0} \) satisfies

\[
(21) \quad D(\Gamma) \subseteq \Gamma, \quad D(g_H, v) = (D^H(g_H), \lambda_V v) \quad \text{for some } \lambda_V > 1, \quad |F_H| \geq 3 \quad \text{and} \quad |F_V| \geq |A| + 1,
\]

where

\[
(22) \quad F_H := D^H(V_H) \cap \Gamma_H \subseteq \Gamma_H \quad \text{and} \quad F_V := \lambda_V V_0 \cap \Gamma_0 \subseteq \Gamma_0.
\]

**Proof.** By definition of a dilation datum we can find \( \lambda_0 \) satisfying the first property of (21). Replacing \( \lambda_0 \) by \( n\lambda_0 \) for some large \( n \in \mathbb{N} \) we can ensure that \( \lambda_0 \) is sufficiently large relative to \( V \) and satisfies the second condition of (21). It thus remains to show only that by enlarging \( \lambda_0 \) even further if necessary we can ensure the final two conditions. For this let \( r_- > 0 \) such that \( B(e, r_-) \subseteq V \). Note that if \( x \in B(e, \lambda_0^N r_-) \), then \( d(D^{-N}x, e) = d(D^{-N}x, D^{-N}e) \leq \lambda_0^{-N} d(x, e) \leq r_- \) and hence \( D^{-N}x \in B(e, r_-) \); this shows that

\[
B(e, \lambda_0^N r_-) \subseteq D^N(B(e, r_-)) \subseteq D^N(V),
\]

i.e. \( D^N(V) \) contains arbitrary large balls as \( N \to \infty \). In particular, if \( B_H \subseteq G_H \) and \( B_V \subseteq \mathbb{R}^m \) are any two bounded sets, then \( B_H \times B_V \subseteq D^N(V) \) for all sufficiently large \( N \). Now since \( \Gamma_H \) is relatively dense in \( G_H \) (which is non-compact) and \( \Gamma_0 \) is relatively dense in \( \mathbb{R}^m \) we deduce that for any \( p, q \in \mathbb{N} \) (in particular for \( p := 3 \) and \( q := |A| + 1 \)) we can find \( N \in \mathbb{N} \) such that

\[
|D^N(V) \cap (\Gamma_H \times_\beta \{ 0 \})| \geq p \quad \text{and} \quad |D^N(V) \cap (\{ e_H \} \times_\beta \Gamma_0)| \geq q.
\]

We may thus replace \( \lambda_0 \) by \( \lambda_0^N \), and the proposition follows.

From now on we fix a stretch factor \( \lambda_0 > 1 \) satisfying (21) and abbreviate \( D := D_{\lambda_0} \). In view of the splitting of our dilation \( D \) we then have

\[
(23) \quad D(V) \cap \Gamma = F_H \times_\beta F_V,
\]

where \( F_H \) and \( F_V \) are as in (22). Since \( |F_H| \geq 3 \) we may choose two distinct points \( \gamma_1 \neq \gamma_2 \) in \( F_H \setminus \{ e \} \). Since \( V \) satisfies the intersection condition of Lemma 4.20 we may moreover choose \( x_1 \in \bigcap_{x \in F_V} x + F_V \). We observe that in particular \( x_1 \in F_V \), since \( 0 \in F_V \). Since
\(|F_V| \geq |A| + 1 \geq 2\) we may then choose \(x_2 \in F_V \setminus \{x_1\}\). From now on we assume that 
\(x_1, x_2, \gamma_1, \gamma_2\) are chosen in this way. We also fix a letter \(a \in A\). Since \(|F_V| \geq |A| + 1\) we may 
now choose for every \(c \in A \setminus \{a\}\) an element \(x_c \in F_V \setminus \{x_1, x_2\}\) such that \(x_c \neq x_d\) if \(c \neq d\). We 
fix all of these choices and define a subset  
\[\Xi := \Xi_a \cup \Xi_o \cup \{(\gamma_2, x_c) : c \in A \setminus \{a\}\} \cup \{(\gamma_2, x_2)\} \subseteq F_H \times \beta F_V,\]
where  
\[\Xi_a := (\{e\} \times \beta F_V) \cup (\{\gamma_1\} \times \beta (F_V \setminus \{x_1\})) \quad \text{and} \quad \Xi_o := \{(\gamma, x_1) : \gamma \in F_H \setminus \{e\}\}.\]
In view of (23) we then have \(\Xi \subseteq D(V) \cap \Gamma\), hence for every substitution rule \(S_0\) we need in 
particular to specify the values of \(S_0(d)\) for every \(d \in A\).

**Definition 6.4.** Let \(A\) be a finite alphabet with \(|A| \geq 2\) and \(a \in A\). We call \(S_0 : A \to A^{D(V) \cap \Gamma}\) 
a good substitution rule if the following conditions are satisfied:

- For every \(c \in A\) we have \(S_0(c)(\Xi_a) = \{a\}\) and \(S_0(c)(\Xi_o) \in A \setminus \{a\}\).
- For every \(b \in A\) and \(c \in A \setminus \{a\}\), we have \(S_0(b)(\gamma_2, x_c) = c\).
- For every \(b \in A\) we have \(S_0(b)(\gamma_2, x_2) = b\).

**Remark 6.5.** Clearly for every dilation datum, alphabet and stretch factor \(\lambda_0\) satisfying (21) 
there are plenty of choices for good substitution rules, since we are only specifying very few of the 
values of \(S_0\). Note in particular that (in the case where \(|A| \geq 3\) we can choose \(S_0(c)(\Xi_o)\) 
completely arbitrary as long as it does not coincide with the letter \(a\). The last condition is used 
only to ensure that \((A, \lambda_0, S_0)\) primitive and can be relaxed if we know by other reasons that 
the substitution is primitive.

Proposition 6.2 is now a consequence of the following observation:

**Proposition 6.6.** If \(S_0\) is a good substitution rule, then the associated substitution \(S\) is primitive 
and non-periodic.

**Proof.** Firstly, we claim that \((A, \lambda_0, S_0)\) is primitive with with exponent \(L = 1\), i.e. that for 
\(c, d \in A\) we have \(P_c \prec S^L(P_d)\). Indeed this follows since for each \(d \in A\), 
\[S_0(d)(\gamma_2, x_c) = c, \quad \text{for } c \in A \setminus \{a\}, \text{ and } S_0(d)(e, 0) = a\]
hold where we used that \((e, 0) \in \{e\} \times \beta F_V \subseteq \Xi_a\).

Secondly, we claim that \(S_0\) is injective. This follows directly from \(S_0(d)(\gamma_2, x_2) = d\) for all 
\(d \in A\).

Thirdly, let \(\eta = (g, x) \in (D(V) \cap \Gamma) \setminus \{(e, 0)\}\) and \(c, d \in A\). It remains to be shown that 
\[\exists \eta' \in \eta^{-1}D(V) \cap D(V) \cap \Gamma : \quad (\eta^{-1}S_0(c)) (\eta') \neq S_0(d)(\eta').\]
Assume first that \(g = e\) and so \(x \neq 0\). Since \(D(V) \cap \Gamma = F_H \times \beta F_V\), we then have \(x \in F_V \setminus \{0\}\). 
Since \((\gamma_1, x_1) \in D(V) \cap \Gamma\) we have  
\[\eta' := (\gamma_1, x_1 - x) = \eta^{-1}(\gamma_1, x_1) \in \eta^{-1}(D(V) \cap \Gamma) = \eta^{-1}D(V) \cap \eta^{-1}\Gamma = \eta^{-1}D(V) \cap \Gamma.\]
On the other hand, \(x_1 \in x + F_V\) and thus \(x_1 - x \in F_V\) and thus \(\eta' \in F_H \times \beta F_V = D(V) \cap \Gamma\). This 
shows that \(\eta' \in \eta^{-1}D(V) \cap D(V) \cap \Gamma\). Furthermore, since \(x \neq 0\) we have \(\eta' \in \{\gamma_1\} \times \beta (F_V \setminus \{x_1\}) \subseteq \Xi_o\), while \((\gamma_1, x_1) \in \Xi_a\). We deduce that 
\[\eta^{-1}S_0(c))(\eta') = S_0(c)(\eta') = S_0(c)(\gamma_1, x_1) \in A \setminus \{a\} \neq a = S_0(d)(\eta').\]
This finishes the case $g = e$ and we may thus assume that $g \neq e$. We then set $\eta' := (e, x_1 - x)$. Since $x_1 \in x + F_V$ we then have $x_1 - x \in F_V$ and thus $\eta' \in F_H \times_{\beta} F_V = D(V) \cap \Gamma$. Moreover, $\eta' = (g, x_1) \in \Xi \subseteq D(V) \cap \Gamma$ holds and hence $\eta' \in \eta^{-1}D(V)$. Finally, since $\eta' \in \Xi_a$ and $(g, x_1) \in \Xi_o$ we have

$$(\eta^{-1} S_0(c))(\eta') = S_0(c)(\eta \eta') = S_0(c)((g, x_1)) \in A \setminus \{a\} \neq a = S_0(d)(\eta'),$$

finishing the proof. \hfill \Box

**Figure 4.** The substitution rule $S_0$ is plotted together with the sets $\Xi_a$ and $\Xi_o$. This shows that $S_0$ is a good substitution rule.

**Example 6.7.** We will show that the substitution datum defined in Example 2.9 is a good substitution rule for an alphabet with two letters for the Heisenberg group $H$. We use the splitting defined in Example 4.11 where $G_H = \mathbb{R}^2 \times \beta \mathbb{R}$, $\Gamma = 2\mathbb{Z}^3$ and $\Gamma_H = 2\mathbb{Z}^2$. We have seen that $\lambda_0 := 3$ is sufficiently large relative to $V$, and for this choice of $\lambda_0$ we have

$$D(V) \cap \Gamma = \{-2, 0, 2\} \times \{-2, 0, 2\} \times \{-8, -6, -4, -2, 0, 2, 4, 6, 8\},$$

hence in the above notation we have

$$F_H = \{-2, 0, 2\}^2 \quad \text{and} \quad F_V = \{-8, -6, -4, -2, 0, 2, 4, 6, 8\}.$$ 

Since $|F_H| \geq 3$, $|F_V| \geq 2$ and $\bigcap_{x \in F_V} x + F_V = \{0\}$ we see that $\lambda_0$ does indeed satisfy all the conditions of (21). We now choose $\gamma_1 := (0, 2)^\top$ and $\gamma_2 := (0, -2)^\top$ in $F_H \setminus \{e\}$. Define $x_1 := 0 \in \bigcap_{x \in F_V} x + F_V$ and $x_2 := 4, x_b := 8 \in F_V \setminus \{x_1\}$. With these conventions fixed, it is straightforward to check that the map $S_0 : A \to A^{D(V)\cap \Gamma}$ defined in Example 2.9 is a good substitution rule, cf. Figure 4. Thus, the substitution datum $(A, 3, S_0)$ over $D_\mathbb{H} := (\mathbb{H}_3(\mathbb{R}), d, (D_\lambda)_{\lambda > 0}, \Gamma, V)$ is primitive and non-periodic by Proposition 6.6.

In a similar way one can show that with the same choice for $F_H, F_V, x_1, x_2, x_b, \gamma_1$ and $\gamma_2$, the substitution rule $(A, 3, S_0)$ over $D_\mathbb{R} := (\mathbb{R}^3, d, (D_\lambda)_{\lambda > 0}, \Gamma, V)$ is also primitive and non-periodic.
7. Substitution dynamical systems

Throughout this section $S = (A, \lambda_0, S_0)$ denotes a substitution datum over a dilation datum $D := (G, d, (D_\lambda)_{\lambda \geq 0}, \Gamma, V)$ and $S$ denotes the associated substitution map.

7.1. Existence of legal fixpoints. Recall from the introduction that a finite patch $P$ is called $S$-legal if there is an $n \in \mathbb{N}$ and a letter $a \in A$ such that $P < S^n(P_a)$, where $P_a$ denotes the patch with $\supp(P_a) = \{e\}$ and $P_a(e) = a \in A$. Also recall that $\Omega(S) = \{\omega \in A^\Gamma : \omega \text{ is } S\text{-legal}\}$, where $\omega \in A^\Gamma$ is called $S$-legal if each finite subpatch of $\omega$ is $S$-legal. The following proposition then holds for purely formal reasons:

**Proposition 7.1.** The set $\Omega(S)$ is compact, metrizable, $S$-invariant and $\Gamma$-invariant.

**Proof.** By the definition of the product topology any patch of a limit point $\omega$ of a sequence $(\omega_n) \subseteq \Omega(S)$ is a patch of $\omega_n$ for $n$ large enough. Hence $\Omega(S)$ is closed. Thus, it is compact and metrizable as a closed subset of a compact metrizable space $A^\Gamma$. If $\omega \in \Omega(S)$ is $S$-legal then also any patch of $S(\omega)$ is $S$-legal by definition and so $\Omega(S)$ is $S$-invariant. Finally, if $\omega \in \Omega(S)$ then also $\gamma \omega \in \Omega(S)$ holds for all $\gamma \in \Gamma$ since every patch of $\omega$ is a patch of $\gamma \omega$ and vice versa. □

In fact, the proposition even holds without the assumption that the stretch factor $\lambda_0$ is sufficiently large relative to $V$, but without this assumption we cannot guarantee that $\Omega(S)$ is non-empty. Recall that this notion was crucial to control the growth of patches under iterated applications of the substitution map $S$ in Theorem 5.13; it is therefore also crucial for the proof of the following theorem. We will show that there always exists a $S^k$-fixpoint in $\Omega(S)$ for some $k \in \mathbb{N}$, i.e. there is an $\omega \in \Omega(S)$ satisfying $S^k(\omega) = \omega$.

**Theorem 7.2.** The set $\Omega(S)$ is non-empty. In fact, $\Omega(S)$ contains a $S^k$-fixpoint for some $k \in \mathbb{N}$.

The following lemma is the main step in the proof and already shows that $\Omega(S)$ is non-empty.

**Lemma 7.3.** Let $\omega \in A^\Gamma$. Then every limit point of $(S^n(\omega))_{n \in \mathbb{N}}$ is $S$-legal.

**Proof.** Let $\omega \in A^\Gamma$ and suppose $S^{n_k}(\omega) \to \rho$ for $k \to \infty$ and $n_k \to \infty$. Consider a patch $P$ of $\rho$ with support $M$, namely $\rho|_M = P$. By Theorem 5.13, there is a $k_0 \in \mathbb{N}$ such that $V(n_k)$ contains $M$ for all $k \geq k_0$. Furthermore, the convergence $S^{n_k}(\omega) \to \rho$ in the product topology yields that there is a $k_1 \geq k_0$ such that $S^{n_k}(\omega)|_M = \rho|_M$ for $k \geq k_1$. Thus,

$$P = \rho|_M = S^{n_k}(\omega)|_M = S^{n_k}(\omega)|_{V(n_k)} = S^{n_k}(\omega)|_{\{e\}}$$

follows for $k \geq k_1 \geq k_0$. Hence, the patch $P$ occurs in $S^{n_k}(P_a)$ for the letter $a := \omega(e) \in A$, namely $P$ is $S$-legal. □

**Proof of Theorem 7.2.** Let $\omega_0 \in A^\Gamma$. Since the alphabet is finite, there are $k_0, k \in \mathbb{N}$ such that $S^{k_0}(\omega_0)|_{\{e\}} = S^{k_0+k}(\omega_0)|_{\{e\}}$. Define $\omega_1 := S^{k_0}(\omega_0)$ and $\omega_{n+1} := S^k(\omega_n)$. We will show that $(\omega_n)$ is convergent to some $\omega \in \Omega(S)$ and that $S^k(\omega) = \omega$.

By construction, we have $\omega_1(e) = \omega_n(e)$ for all $n \in \mathbb{N}$. Since $S^{mk}(\omega_n) = \omega_{n+m}$ holds for all $n, m \in \mathbb{N}$, the latter leads to

$$\omega_{n+1}|_{V(mk)} = S^{mk}(\omega_1)|_{V(mk)} = S^{mk}(P_{\omega_1(e)}) = S^{mk}(P_{\omega_n(e)}) = S^{mk}(\omega_n)|_{V(mk)} = \omega_{n+m}|_{V(mk)}$$

for all $n, m \in \mathbb{N}$.

Let $M \subseteq \Gamma$ be finite. By assumption, $\lambda_0$ is sufficiently large relative to $V$ and so there exists an $m_0 \in \mathbb{N}$ such that $M \subseteq V(mk)$ for $m \geq m_0$ by Theorem 5.13. Hence, the previous considerations lead to

$$\omega_{m_0+1}|_M = \omega_{n+m_0}|_M, \quad n \in \mathbb{N},$$

and the previous considerations lead to

$$\omega_{m_0+1}|_M = \omega_{n+m_0}|_M, \quad n \in \mathbb{N},$$
follows proving that \((ωₙ)_{n ∈ ℤ^+}\) is convergent in the product topology \(A^Γ\). Let \(ω := \lim_{n→∞} ω_n\). Due to Lemma 7.3, \(ω ∈ Ω(S)\) holds. Furthermore the continuity of \(S : A^Γ → A^Γ\) (Proposition 5.7) implies that

\[S^k(ω) = \lim_{n→∞} S^k(ω_n) = \lim_{n→∞} ω_{n+1} = ω.\]

Thus, \(ω\) is an \(S\)-legal fixpoint of some power of \(S\). \(\square\)

### 7.2. Consequences of primitivity

In this subsection we are going to show, see Theorem 7.2, Proposition 7.6, Proposition 7.5 and Lemma 7.8 proven below:.

**Theorem 7.4** (Linear repetitivity and minimality). If \(S\) is primitive, then \(Ω(S)\) is minimal and every \(ω ∈ Ω(S)\) is linearly repetitive.

We recall that an element \(ω ∈ A^Γ\) is called repetitive if, for each finite subpatch \(P\) of \(ω\), there is a radius \(r > 0\) such that \(P ⊆ ω|_{B(x,r)}\) for all \(x ∈ Γ\). It is called linearly repetitive if there exists a constant \(C > 0\) such that for all \(r ≥ 1\), \(x, y ∈ Γ\), we have \(ω|_{B(x,r)} ⊆ ω|_{B(y,Cr)}\).

Repetitivity can also be characterized dynamically: Given an element \(ω ∈ A^Γ\) we denote by \(Ω_ω := \{γω : γ ∈ Γ\}\) the orbit closure of \(ω\). By definition of the product topology we have \(ω’ ∈ Ω_ω\) if and only if \(ω'|F < ω\) for every finite subset \(F ⊆ Γ\), i.e. all finite patches of \(ω’\) appears in \(ω\). A subset \(Ω ⊆ A^Γ\) is called minimal if every orbit is dense. This means that \(Ω = Ω_ω\) for all \(ω ∈ Ω\), i.e. all elements of \(Ω\) have exactly the same finite patches up to translations.

**Proposition 7.5.** The element \(ω ∈ A^Γ\) is repetitive if and only if \(Ω_ω\) is minimal. In this case every element of \(Ω_ω\) is repetitive. Similarly, if \(ω\) is linearly repetitive, then every element of \(Ω_ω\) is linearly repetitive.

**Proof.** The first statement is [BG13, Prop. 4.3] - note that abelianness of \(Γ\) is assumed, but never used in the proof. The second statement follows from the first, since all elements of \(Ω_ω\) have orbit closure \(Ω_ω\) by minimality of the latter. As for the third statement, if \(ω\) is linearly repetitive, then all elements in the orbit closure have the same finite patches as \(ω\) by minimality, and since the notion of linear repetitivity only depends on properties of finite patches, the proposition follows. \(\square\)

In the present situation this applies because of Theorem 7.2 and the following observation:

**Proposition 7.6.** If \(S\) is primitive and \(ω ∈ Ω(S)\) is an \(S^k\)-fixpoint for some \(k ∈ ℤ^+\), then \(Ω(S) = Ω_ω\).

Note that the proposition amounts to showing that every \(S\)-legal patch appears in \(ω\). This is a consequence of the following lemma:

**Lemma 7.7.** If \(S\) is primitive, then for every \(S\)-legal patch \(P\) there exists \(N_P ∈ ℤ^+\) such that \(P < S^n(P_a)\) for all \(n ≥ N_P\) and \(a ∈ A\).

**Proof.** Let \(P\) be an \(S\)-legal patch, namely there is an \(n_P ∈ ℤ^+\) and a letter \(b ∈ A\) such that \(P < S^{n_P}(P_b)\). We note that if \(Q\) is any patch such that \(P_b < Q\), then \(P < S^{n_P}(Q)\) follows. Since \((A, A₀, S₀)\) is primitive with exponent \(L\), we conclude that \(P < S^{n_P+L}(P_a)\) for all \(a ∈ A\). Set \(N_P := L + n_P\). If \(n ≥ N_P\) and \(a ∈ A\) then

\[P < S^n(P_b) < S^n(S^{n_P}(P_a)) = S^n(P_a)\]

follows since \(n − n_P ≥ L\) and so \(P_b < S^n(P_a)\). \(\square\)
Proof of Proposition 7.6. Let $P$ be an $S$-legal patch, then Lemma 7.7 and Proposition 5.12 imply that there is an $m \in \mathbb{N}$ such that
\[ P \prec S^{mk}(P_{\omega(e)}) \prec S^{mk}(\omega)|_{V(mk)} = \omega|_{V(mk)}. \]
Thus every $S$-legal patch appears in $\omega$, and the proposition follows. \qed

At this point we have reduced the proof of Theorem 7.4 to the following lemma:

**Lemma 7.8.** If $S$ is primitive and $\omega \in \Omega(S)$ is an $S^k$-fixpoint for some $k \in \mathbb{N}$, then $\omega$ is linearly repetitive.

**Proof.** We first show that $\omega$ is repetitive. For this let $P$ be a finite subpatch of $\omega$. Using Lemma 7.7 we find $m \in \mathbb{N}$ such that $P \prec S^{mk}(P_a)$ for all $a \in A$. Since $D^{mk}(\Gamma) \subseteq G$ is relatively dense there is an $r > 0$ such that for each $x \in \Gamma$ there is an $\eta_x \in \Gamma$ such that $D^{mk}(\eta_x)V(mk) \subseteq B(x, r)$. By Lemma 5.16 we have $D^{mk}(\eta_x)V(mk) = V(mk, \{\eta_x\})$ and we conclude with Proposition 5.12 that for all $x \in \Gamma$ we have
\[ P \prec S^{mk}(P_{\omega(\eta_x)}) = S^{mk}(\omega)|_{V(mk, \{\eta_x\})} = \omega|_{D^{mk}(\eta_x)V(mk)} \prec \omega|_{B(x,r)}. \]
Since $r$ is independent of $x$, this shows that $\omega$ is indeed repetitive.

We now upgrade this statement to linear repetitivity. Let $r_{x, z}$ denote an outer, respectively inner radius for $V$. Define the constants $C_+ := \lambda_0 r_{x, z} (1 + \frac{\lambda_0}{\lambda_0 - 1})$ and $C_- := (\lambda_0 - 1) (1 + \frac{r_{x, z}}{\lambda_0})$ as in Theorem 5.13. Since $\lambda_0$ is sufficiently large relative to $V$, we have $C_- > 0$. Since $\lambda_0 > 1$, there exists an $i \in \mathbb{N}$ such that
\[ \lambda_0^i \geq 2 \frac{r_+}{C_-}. \]
Due to Theorem 5.13 (b), there is an $r(i) > 0$ such that $V(i) \subseteq B(e, r(i))$.

Recall that we write $P \sim Q$ provided $P$ and $Q$ are in the same $\Gamma$-orbit. The set
\[ \{ P \in A^M : M = B(e, r) \cap \Gamma, P \sim \omega|_{B(x,r)} \text{ for some } x \in \Gamma, 0 < r \leq \max\{r_+, r(i)\} \} \]
is finite, since $A$ is finite and $\Gamma$ is uniformly discrete. Then the repetitivity of $\omega$ yields the existence of a constant $C_1 \geq 1$ such that
\[ \omega|_{B(x,r)} \prec \omega|_{B(y,C_1)}, \quad x, y \in \Gamma, 0 < r \leq \max\{r_+, r(i)\}. \]
Due to Theorem 5.13 (a), there is a $j \in \mathbb{N}$ such that $B(e, C_1) \subseteq V(j)$. Define
\[ C := \max\{C_1, \lambda_0^i C_+ \frac{r_+}{r_-} + \lambda_0 \} \geq 1. \]
If $1 \leq r \leq r_+$, we conclude
\[ \omega|_{B(x,r)} \prec \omega|_{B(y,C_1)} \prec \omega|_{B(y,C_1)}, \quad x, y \in \Gamma. \]
Let $r > r_+$. Since $\lambda_0 > 1$, we have
\[ \bigcup_{n \in \mathbb{N}} [\lambda_0^{kn-r_+} r_+, \lambda_0^{kn} r_+] = [r_+, \infty), \]
and so there is a unique $n_0 \in \mathbb{N}$ such that
\[ \lambda_0^{kn_0-r_+} r_+ \leq r < \lambda_0^{kn_0} r_+. \]
Let $x, y \in \Gamma$. Since $G = D^{kn_0}(\Gamma)D^{kn_0}(V)$, there are $\eta_x, \eta_y \in \Gamma$ and $v_x, v_y \in V$ such that
\[ x = D^{kn_0}(\eta_x)D^{kn_0}(v_x) \quad \text{and} \quad y = D^{kn_0}(\eta_y)D^{kn_0}(v_y). \]
Since \( V \subseteq B(e, r_+) \) and \( D^{kna}(B(e, s)) = B(e, \lambda_0^{kna} s) \), we conclude
\[
x \in B\left(D^{kna}(\eta_x), \lambda_0^{kna} r_+ \right) \quad \text{and} \quad D^{kna}(\eta_y) \in B(y, \lambda_0^{kna} r_+).
\]

Due to (25) and \( B(z, r)B(e, s) \subseteq B(z, r + s) \), we first observe
\[
B(x, r) \subseteq B\left(D^{kna}(\eta_x), \lambda_0^{kna} r_+ \right)B(e, r) \subseteq B\left(D^{kna}(\eta_x), r + \lambda_0^{kna} r_+ \right) \subseteq B\left(D^{kna}(\eta_x), \lambda_0^{kna} 2r_+ \right).
\]

By the choice of \( i \in \mathbb{N} \), we derive
\[
\lambda_0^{kna} 2r_+ = \lambda_0^{kna} 2\frac{T}{C} C_\pm \leq \lambda_0^{kna+i} C_\pm.
\]

The inclusion \( B(e, \lambda_0^{kna+i} C_\pm) \subseteq V(kn_0 + i) \) holds by Theorem 5.13 (b). Hence, the previous considerations and Lemma 5.16 (a) and (c) lead to
\[
B(x, r) \subseteq D^{kna}(\eta_x)B(e, \lambda_0^{kna+i} C_\pm) \subseteq D^{kna}(\eta_x)V(kn_0 + i) = V(kn_0, \eta_x(V(i) \cap \Gamma)).
\]

Thus, \( S^{kna}(\omega) = \omega \) and Proposition 5.12 yield
\[
\omega|_{B(x, r)} < S^{kna}(\omega)|_{V(kn_0, \eta_x(V(i) \cap \Gamma))} = S^{kna}(\omega|_{\eta_x(V(i) \cap \Gamma)}).
\]

Hence, we conclude from \( V(i) \subseteq B(e, r(i)), (24) \) and \( B(e, C_1) \subseteq V(j) \) that
\[
\omega|_{\eta_x(V(i) \cap \Gamma)} < \omega|_{B(\eta_x, r(i) \cap \Gamma)} < \omega|_{B(\eta_y, C_1 \cap \Gamma)} < \omega|_{\eta_y(V(j) \cap \Gamma)}.
\]

If we combine this with the previous consideration, Proposition 5.12 implies
\[
\omega|_{B(x, r)} < S^{kna}(\omega|_{\eta_x(V(i) \cap \Gamma)}) < S^{kna}(\omega|_{\eta_y(V(j) \cap \Gamma)} = S^{kna}(\omega)|_{V(kn_0, \eta_y(V(j) \cap \Gamma))}.
\]

Focusing again on the support, Lemma 5.16 (a) and (c) lead to
\[
V(kn_0, \eta_y(V(j) \cap \Gamma)) = D^{kna}(\eta_y)V(kn_0, V(j) \cap \Gamma) = D^{kna}(\eta_y)V(kn_0 + j).
\]

Then Theorem 5.13 (b) and \( D^{kna}(\eta_y) \in B(y, \lambda_0^{kna} r_+) \) yield
\[
V(kn_0, \eta_y(V(j) \cap \Gamma)) \subseteq D^{kna}(\eta_y)B(e, \lambda_0^{kna+i} C_\pm) \subseteq B(y, \lambda_0^{kna}(\lambda_0^{j} C_\pm + r_+)),
\]

using again \( B(z, r)B(e, s) \subseteq B(z, r + s) \). Then (25) and the choice of \( C \) give us the estimate
\[
\lambda_0^{kna}(\lambda_0^{j} C_\pm + r_+) = \left( \frac{\lambda_0^{j} C_\pm + r_+}{r_+} \right) \lambda_0^{kna-1} r_+ \leq C r,
\]

implying \( V(kn_0, \eta_y(V(j) \cap \Gamma)) \subseteq B(y, Cr) \). Hence, we finally conclude
\[
\omega|_{B(x, r)} < S^{kna}(\omega)|_{V(kn_0, \eta_y(V(j) \cap \Gamma))} < \omega|_{B(y, Cr)},
\]

using \( S^{kna}(\omega) = \omega \). This proves that \( \omega \) is linearly repetitive. \( \square \)

**Proof of Theorem 7.4.** This follows from Theorem 7.2, Proposition 7.6, Proposition 7.5 and Lemma 7.8 proven below. \( \square \)

8. APERIODICITY OF SYMBOLIC SUBSTITUTION SYSTEMS

As in the previous section \( S = (\mathcal{A}, \lambda_0, S_0) \) denotes a substitution datum over a dilation datum \( \mathcal{D} = (G, d, (D_{\lambda})_{\lambda > 0}, \Gamma, V) \) and \( S \) denotes the associated substitution map.
8.1. Weak aperiodicity. We recall from Definition 2.14 that the substitution datum $S$ is non-periodic if $S_0$ is injective and
\[(\gamma^{-1}S(P_a)|_{\gamma^{-1}D(V)\cap D(V)} \neq S(P_b)|_{\gamma^{-1}D(V)\cap D(V)}) \quad \text{for all } \gamma \in (D(V) \cap \Gamma) \setminus \{e\} \text{ and } a, b \in A.
\]
We are going to show that this condition implies that $\Omega(S)$ is weakly aperiodic. In fact we can establish a slightly stronger statement. For this we recall that by Theorem 7.2 some power of $S$ is guaranteed to have a fixpoint in $\Omega(S)$.

**Theorem 8.1.** If $S$ is non-periodic, then $\Omega(S)$ is weakly aperiodic. More precisely, if $\omega \in \Omega(S)$ is an $S^k$-fixpoint for some power $k \in \mathbb{N}$, then $\omega$ has trivial $\Gamma$-stabilizer.

Our proof of Theorem 8.1 builds on two key observations. We first show in Lemma 8.2 below that injectivity of $S_0$ implies that $S^n : \mathcal{A}^\Gamma \to \mathcal{A}^\Gamma$ is injective for all $n \in \mathbb{N}$, and that this injectivity is witnessed by very specific elements of $(S)$. We then use this result to show that the set of potential $S^n$-fixpoints in $\Gamma$ decreases with $n$ (see Proposition 8.3). We recall from Section 5.3 that for finite $M \subseteq \Gamma$ and $n \in \mathbb{N}$ we denote
\[V(0, M) := MV \quad \text{and} \quad V(n, M) := D(V(n - 1, M)_{\Gamma}).\]
Moreover, we use the abbreviation $V(n) := V(n, \{e\})$ for all $n \in \mathbb{N}_0$.

**Lemma 8.2.** If the substitution $S_0 : A \to A^{D(V)\cap \Gamma}$ is injective, then $S^n : \mathcal{A}^\Gamma \to \mathcal{A}^\Gamma$ is injective for all $n \in \mathbb{N}$. More precisely, if for $\eta \in \Gamma$ we have $\omega_1(\eta) \neq \omega_2(\eta)$, then
\[S^n(\omega_1)|_{V(n, \{\eta\}) \cap \Gamma} \neq S^n(\omega_2)|_{V(n, \{\eta\}) \cap \Gamma} \quad \text{for all } n \in \mathbb{N}.
\]
**Proof.** We first show that $S$ is injective, thus settling the case $n = 1$. So fix some distinct $\omega_1, \omega_2 \in A$ along with $\eta \in \Gamma$ such that $\omega_1(\eta) \neq \omega_2(\eta)$. By injectivity of $S_0$ there must be some $\gamma \in D(V) \cap \Gamma$ such that $S_0(\omega_1(\eta))(\gamma) \neq S_0(\omega_2(\eta))(\gamma)$. Set $\gamma := D(\eta)\gamma$. The definition of the map $S$ now gives $S(\omega_1)(\gamma) = S_0(\omega_1(\eta))(\gamma) \neq S_0(\omega_2(\eta))(\gamma) = S(\omega_2)(\gamma)$. This shows that $S$ is injective and noting that $\gamma \in D(\eta)D(V) = V(1, \{\eta\})$ we find $S(\omega_1)|_{V(1, \{\eta\}) \cap \Gamma} \neq S(\omega_2)|_{V(1, \{\eta\}) \cap \Gamma}$. This settles the case $n = 1$.

Proceeding by induction, suppose that the assertion of the lemma is true for some $n \geq 1$. Then $S^{n+1} = S \circ S^n$ is clearly injective as a composition of two injective maps. Further fix some distinct $\omega_1, \omega_2 \in A$ along with $\eta \in \Gamma$ such that $\omega_1(\eta) \neq \omega_2(\eta)$. By the induction hypothesis we find some $\gamma \in V(n, \{\eta\}) \cap \Gamma$ such that $S^n(\omega_1)(\eta)|_{V(n, \{\eta\}) \cap \Gamma} \neq S^n(\omega_2)(\eta)|_{V(n, \{\eta\}) \cap \Gamma}$. By injectivity of $S_0$ we find some $\gamma \in D(V) \cap \Gamma$ such that $S_0(\omega_1)(\eta)|_{V(n, \{\eta\}) \cap \Gamma} \neq S_0(\omega_2)(\eta)|_{V(n, \{\eta\}) \cap \Gamma}$. Setting $\gamma := D(\eta)\gamma$ we obtain in the same manner as in the proof for $n = 1$ that $S^{n+1}(\omega_1)(\gamma) \neq S^{n+1}(\omega_2)(\gamma)$. The observation that $\gamma \in D((V(n, \{\eta\}) \cap \Gamma)D(V) = V(n + 1, \{\eta\})$ now finishes the proof.

We now turn to the second key statement.

**Proposition 8.3.** Suppose that the substitution datum $(A, \lambda_0, S_0)$ is non-periodic. Then for each $n \in \mathbb{N}$, every $\gamma \in \Gamma \setminus D^n(\Gamma)$ and all $\omega_1, \omega_2 \in \mathcal{A}^\Gamma$, we have
\[\gamma^{-1}S^n(\omega_1)|_{V(n, \{\eta\}) \cap \Gamma} \neq S^n(\omega_2)|_{V(n, \{\eta\}) \cap \Gamma}.
\]

The proof of the proposition will be based on induction. The case $n = 1$ is taken care of by the following lemma.

**Lemma 8.4.** Suppose that the substitution datum $(A, \lambda_0, S_0)$ is non-periodic. Then for each $\gamma \in \Gamma \setminus D(\Gamma)$ and all $\omega_1, \omega_2 \in \mathcal{A}^\Gamma$, we have
\[\gamma^{-1}S(\omega_1)|_{D(V) \cap \Gamma} \neq S(\omega_2)|_{D(V) \cap \Gamma}.
\]
Proof. Fix $\gamma \in \Gamma \setminus D(\Gamma)$ and $\omega_1, \omega_2 \in A^\Gamma$. By equality (17), the union $G = \bigsqcup_{\eta \in \Gamma} D(\eta V)$ consists of pairwise disjoint sets and so there is a unique $\eta \in \Gamma$ satisfying $\gamma \in D(\eta V) \cap \Gamma = D(\eta)D(V) \cap \Gamma$. Moreover we have $\gamma \neq D(\eta)$ by assumption. Since $D(\eta) \in \Gamma$, we conclude that $D(\eta)^{-1} \gamma \in (D(V) \cap \Gamma) \setminus \{e\}$. Since the substitution datum is non-periodic, we infer

$$
(D(\eta)^{-1} \gamma)^{-1} S(P_{\omega_1(\eta)})(g) \neq S(P_{\omega_2(e)})(g)
$$

for some $g \in (D(\eta)^{-1} \gamma)^{-1} D(V) \cap D(V) \cap \Gamma$. By definition of $S : A^\Gamma \to A^\Gamma$, we get

$$(D(\eta)^{-1} \gamma)^{-1} S(P_{\omega_1(\eta)})(g) = S_0(\omega_1(\eta)) (D(\eta)^{-1} \gamma g) = S(\omega_1)(\gamma g) = \gamma^{-1} S(\omega_1)(g)$$

and, since $g \in D(V) \cap \Gamma$, also $S(P_{\omega_2(e)})(g) = S_0(\omega_2(e))(g) = S(\omega_2)(g)$. Hence,

$$\gamma^{-1} S(\omega_1)(g) \neq S(\omega_2)(g)$$

follows. This finishes the proof. \hfill \Box

With the lemma at our disposal we can now prove Proposition 8.3.

Proof of Proposition 8.3. We proceed by induction on $n \in \mathbb{N}$. Since $V(1) = D(V)$, the case $n = 1$ has already been proven in Lemma 8.4. So assume that $n \geq 1$ and suppose that the statement of the proposition is true for $n$. Let $\gamma \in \Gamma \setminus D^{n+1}(\Gamma)$. We will distinguish two cases, namely (a) $\gamma \in \Gamma \setminus D^n(\Gamma)$ and (b) $\gamma \in D^n(\Gamma) \setminus D^{n+1}(\Gamma)$.

(a) Let $\gamma \in \Gamma \setminus D^n(\Gamma)$ and $\omega_1, \omega_2 \in A^\Gamma$. Then the induction hypothesis leads to

$$\gamma^{-1} S^{n+1}(\omega_1)|_{V(n) \cap \Gamma} = \gamma^{-1} S^n(S(\omega_1))|_{V(n) \cap \Gamma} \neq S^n(S(\omega_2))|_{V(n) \cap \Gamma} = S^{n+1}(\omega_2)|_{V(n) \cap \Gamma}.$$

Since $V(n) \cap \Gamma \subseteq V(n+1) \cap \Gamma$ holds by Lemma 5.16 (d), the desired result follows.

(b) Since $\gamma \in D^n(\Gamma) \setminus D^{n+1}(\Gamma)$, there is a $\tilde{\gamma} \in \Gamma$ such that $\gamma = D^n(\tilde{\gamma})$ but $\tilde{\gamma} \notin D(\Gamma)$. Now the Lemma 8.4 implies that

$$\tilde{\gamma}^{-1} S(\omega_1)(\eta) \neq S(\omega_2)(\eta)$$

holds for some $\eta \in D(V) \cap \Gamma$. In combination with the injectivity of the substitution $S_0 : A \to A^{D(V) \cap \Gamma}$, Lemma 8.2 asserts that

$$S^n(\tilde{\gamma}^{-1} S(\omega_1))|_{V(n,\{\eta\}) \cap \Gamma} \neq S^n(S(\omega_2))|_{V(n,\{\eta\}) \cap \Gamma}. $$

Since $V(n,\{\eta\}) \subseteq V(n, D(V) \cap \Gamma)$ and $D(V) = V(1)$, Lemma 5.16 (b) and (c) lead to $V(n,\{\eta\}) \cap \Gamma \subseteq V(n+1) \cap \Gamma$, which in turn yields

$$S^n(\tilde{\gamma}^{-1} S(\omega_1))|_{V(n+1) \cap \Gamma} \neq S^n(S(\omega_2))|_{V(n+1) \cap \Gamma}.$$ 

Invoking Proposition 5.6, we conclude

$$\gamma^{-1} S^{n+1}(\omega_1) = D^n(\tilde{\gamma}^{-1}) S^n(S(\omega_1)) = S^n(\tilde{\gamma}^{-1} S(\omega_1)).$$

Since also $S^{n+1}(\omega_2) = S^n(S(\omega_2))$ holds, the desired result follows. \hfill \Box

We are now ready to prove Theorem 8.1.

Proof of Theorem 8.1. Assume that $\omega \in \Omega(S)$ is such that $S^k(\omega) = \omega$ for some $k \in \mathbb{N}$. Moreover, fix $h \in \Gamma \setminus \{e\}$. By uniform discreteness of $\Gamma$ we observe that there is some $n \in \mathbb{N}$ such that $h \in \Gamma \setminus D^{nk}(\Gamma)$ since otherwise one would find a sequence $(\gamma_n)$ of non-trivial elements in $\Gamma$ such that $d(\gamma_n, e) = \lambda_n^{-nk} d(h,e) \xrightarrow{n} 0$. With the assumption that $S^k(\omega) = \omega$ it now follows from Proposition 8.3 applied to $\omega_1 = \omega_2 = \omega$ that

$$h^{-1} \omega|_{V(nk) \cap \Gamma} = h^{-1} S^{nk}(\omega)|_{V(nk) \cap \Gamma} \neq S^{nk}(\omega)|_{V(nk) \cap \Gamma} = \omega|_{V(nk) \cap \Gamma}. $$


This yields $h^{-1}\omega \neq \omega$ and since $h \in \Gamma \setminus \{e\}$ was chosen arbitrarily the proof is finished.

8.2. Strong aperiodicity in the 2-step case. We have seen in the previous subsection that the substitution system associated with an arbitrary non-periodic substitution datum is weakly aperiodic. Under certain additional assumptions this conclusion can actually be strengthened. We recall that $\Omega(S)$ is called strongly aperiodic if the $\Gamma$-action is free, i.e. if every $\omega \in \Omega(S)$ has trivial $\Gamma$-stabilizer.

It is easy to see that every minimal dynamical system over an abelian group is strongly aperiodic if and only if it is weakly aperiodic, see Lemma 8.5 below. However, this implication is not known to hold beyond the abelian case. In the case of nilpotent groups one can hope to be able to reduce to the abelian case by an inductive procedure. We were able to carry this out in the 2-step nilpotent case; we believe that our strategy also applies more generally, but the corresponding inductive arguments get very technical. Thus, for the remainder of this subsection we are going to assume that $G$ is a 2-step nilpotent Lie group endowed with its canonical homogeneous structure (cf. Definition 3.22) and that $D$ is a homogeneous dilation datum over $G$. We also assume that $S$ is a primitive non-periodic substitution datum over $D$. Our goal is to establish Theorem 1.4, which claims that $\Omega(S)$ is strongly aperiodic.

From now on we adopt the notation of Corollary 4.18: We assume without loss of generality that

$$\Gamma = \Gamma_H \times_\beta \mathbb{Z}^m < G = G_H \times_\beta \mathbb{R}^m,$$

where $G_H \cong (\mathbb{R}^{d-m}, +)$ is an abelian Lie group, $\beta : G_H \times G_H \to \mathbb{R}^m$ is a smooth cocycle and $\Gamma_H < G_H$ is a lattice, and we are going to assume that $V = V_H \times_\beta V_0$ is a fundamental box, see Lemma 4.19. We then pick a primitive and non-periodic substitution datum $S = (A, \lambda_0, S_0)$ over $D$; our goal is to show that the corresponding substitution system $\Gamma \acts \Omega(S)$ is strongly aperiodic.

Given $g, h \in G$ we denote by $[g, h] := ghg^{-1}h^{-1}$ the commutator of $g$ and $h$, and we write $[G, G]$ for the commutator subgroup of $G$; recall that, because of the special structure of $G$, we have

$$[G, G] = \{e_H\} \times_\beta \mathbb{R}^m \subseteq Z(G),$$

where $e_H$ denotes the neutral element of $G_H$, cf. Corollary 4.18. Moreover, this corollary asserts that the dilations $D_\lambda$ split as

$$D_\lambda : G_H \times_\beta \mathbb{R}^m \to G_H \times_\beta \mathbb{R}^m, \quad D_\lambda(g_H, v) = (D_\lambda^H(g_H), \lambda^2 v),$$

and hence $D_\lambda^H(\Gamma_H) \subseteq \Gamma_H$ and $\lambda_0^2 \in \mathbb{Z}$, where $(D_\lambda^H)_{\lambda > 0}$ is a dilation family on $G_H$.

Now let $\rho \in \Omega(S)$. We have to show that the stabilizer of $\rho$ in $\Gamma$ is trivial. As a first step we are going to show that no non-trivial central element of $\Gamma$ can fix $\rho$. Indeed, this follows immediately from the weak aperiodicity and minimality of $\Omega(S)$ by the following classical argument:

**Lemma 8.5.** Let $H$ be an lcsc group with center $Z$ and suppose that $H$ acts minimally by homeomorphisms on a compact metrizable space $\Omega$. If there is an $\omega \in \Omega$ such that its $H$-stabilizer intersects trivially with $Z$, then for all $\rho \in \Omega$ and each $z \in Z$ we have

$$z\rho = \rho \implies z = e.$$

In particular, if $H$ is abelian and there is some $\omega \in \Omega$ with trivial $H$-stabilizer, then the action $H \acts \Omega$ is strongly aperiodic, i.e. every element in $\Omega$ has trivial $H$-stabilizer.
Lemma 8.6. Denote by $Z$ the center of $G$ and write $D := D_{\lambda_0}$ as before. Then for all $\gamma \in \Gamma \setminus Z$, there exists some $N_0 \in \mathbb{N}$ such that

$$\gamma D^N(\{e_H\} \times V_0) \cap D^N(\Gamma) = \emptyset \quad \text{for all} \quad N \geq N_0,$$

where $e_H$ denotes the unit element in $G_H$.

Proof. Fix $\gamma = (\gamma_H, \gamma_0) \in \Gamma \setminus Z$. Since $\gamma \notin Z$ we then have $\gamma_H \neq e_H$. By uniform discreteness of $\Gamma_H$ and since $\lambda_0 > 1$ there must be some $N_0 \in \mathbb{N}$ such that $\gamma_H \notin (D_{\lambda_0/H})^N(\Gamma_H)$ for all $N \geq N_0$. Now fix an arbitrary $N \geq N_0$. Using that $\beta(e_H, t) = 0$ for all $t \in \mathbb{R}^m$ we compute

$$\gamma D^N(\{e_H\} \times V_0) = (\gamma_H, \gamma_0)(\{e_H\} \times (\lambda_0^2)^NV_0) = (\gamma_H, \gamma_0 + \lambda_0^2V_0).$$

Since $D^N(\Gamma) = (D_{\lambda_0/H})^N(\Gamma_H) \times \lambda_0^2N\mathbb{Z}^m$ and $\gamma_H \notin (D_{\lambda_0/H})^N(\Gamma_H)$ for $N \geq N_0$, the above calculation shows that $\gamma D^N(\{e_H\} \times V_0) \cap D^N(\Gamma) = \emptyset$ for all $N \geq N_0$. \hfill $\square$

We are now in position to prove Theorem 1.4.

Proof of Theorem 1.4. Since $S$ is primitive, $\Omega(S)$ is minimal by Theorem 7.4. By Theorem 7.2 there exist $\omega \in \Omega(S)$ and $k \in \mathbb{N}$ such that $S^k(\omega) = \omega$ and we fix such a pair $(\omega, k)$ once and for all. Recall from Theorem 8.1 that $\omega$ has trivial $\Gamma$-stabilizer.

Now let $\rho \in \Omega(S)$ and $\gamma \in \Gamma$ with $\gamma \cdot \rho = \rho$. We have to show that $\gamma = e$. If $\gamma \in Z$, then $\gamma = e$ by Lemma 8.5 and minimality of $\Omega(S)$; we will thus assume for contradiction that $\gamma \in \Gamma \setminus Z$. By Lemma 8.6 we may then choose $k_0 \in \mathbb{N}$ such that for $N := k_0$, we have

$$S^N(\omega) = \omega \quad \text{and} \quad \gamma D^N(\{e_H\} \times V_0) \cap D^N(\Gamma) = \emptyset.$$
If we combine this formula with the fact that $\gamma \rho = \rho$ and $S^N(g, \omega) = D^N(g)S^N(\omega)$ and use continuity of the $\Gamma$-action, then we obtain
\[
S^N(\eta_n^{-1}\omega) = D^N(\eta_n^{-1})S^N(\omega) = \eta_0\eta_n^{-1}D^N(\eta_n^{-1})\omega = \eta_0\eta_n^{-1}\omega \xrightarrow{n \to \infty} \eta_0\rho
\]
By definition of the (product) topology induced from $A^\Gamma$, there is an $n_0 \in \mathbb{N}$ such that
\[
(\gamma S^N(\tilde{\eta}^{-1}_n\omega)) |_{V(\eta_0) \cap \Gamma} = (S^N(\eta_n^{-1}\omega)) |_{V(\eta_0) \cap \Gamma}, \quad n \geq n_0.
\]
If we set $\omega_1 := \tilde{\eta}_0^{-1}\omega$ and $\omega_2 := \tilde{\eta}_n^{-1}\omega$ and $\eta := (\gamma S)^{-1}$, then this becomes
\[
(\eta^{-1}S^N(\omega_1)) |_{V(\eta_0) \cap \Gamma} = (S^N(\omega_2)) |_{V(\eta_0) \cap \Gamma}.
\]
Now recall that $N \in \mathbb{N}$ was chosen such that $\gamma D^N(\{e_H\} \times V_0) \cap D^N(\Gamma) = \emptyset$. Since $z_0 \in D^N(\{e_H\} \times V_0)$ we thus deduce that $\gamma S \not\in D^N(\Gamma)$ and hence $\gamma^{-1} \in \Gamma \setminus D^N(\Gamma)$. This implies $\eta \in \Gamma \setminus D^N(\Gamma)$ contradicting Proposition 8.3 since $(A, \lambda_0, S_0)$ is non-periodic.

9. Unique ergodicity and associated Delone dynamical systems

In this section, $D = (G, d, (D_\lambda)_{\lambda>0}, \Gamma, V)$ denotes a homogeneous dilation datum and $S = (A, \lambda_0, S_0)$ denotes a substitution datum over $D$ with associated substitution map $S$ and substitution system $\Omega(S)$. We recall from Theorem 7.4 that $\Omega(S) = \Omega_\omega$ for every $\omega \in \Omega(S)$ provided that $S$ is primitive.

9.1. Unique ergodicity of substitution systems.

Remark 9.1 (Measures and weighted Delone sets). For $H \in \{\Gamma, G\}$ we denote by $\mathcal{R}(H)$ the space of all Radon measures on $H$. Then $H$ acts on $\mathcal{R}(H)$ via push-forward of measures and the weak-*$*$-topology with respect to $C_c(H)$ defines a compact metrizable $H$-invariant topology on $\mathcal{R}(H)$ (see e.g. [BHP20]). Given $\mu \in \mathcal{R}(H)$ we denote by $\Omega^H_\mu$ the orbit closure of $\mu$ with respect to this topology, which is then a topological dynamical system over $H$, i.e. a compact metrizable $H$-space. Following [BHP20] a measure $\Lambda \in \mathcal{R}(H)$ is called a weighted Delone set provided its support is a Delone set.

For any injective map $\iota : A \to (0, \infty)$ and $H \in \{\Gamma, G\}$ we define $\Gamma$-equivariant injections
\[
(28) \quad \iota_{\ast,H} : A^\Gamma \to \mathcal{R}(H), \quad \omega \mapsto \Lambda_\omega := \sum_{\gamma \in \Gamma} \iota(\omega(\gamma)) \cdot \delta_{\gamma}.
\]
Since $\Gamma$ is a Delone set in $H$, each element $\Lambda_\omega$ is a weighted Delone set in $H$.

Proposition 9.2. If $\iota : A \to (0, \infty)$ is injective and $\omega \in A^\Gamma$, then the map $\iota_{\ast,H}$ restricts to a $\Gamma$-equivariant homeomorphism $\Omega_\omega \to \Omega^\Gamma_{\Lambda_\omega}$. In particular, the dynamical systems $\Gamma \curvearrowright \Omega_\omega$ and $\Gamma \curvearrowright \Omega^\Gamma_{\Lambda_\omega}$ are topologically isomorphic.

Proof. See [BHP20, Lemma 5.1 and Proposition 5.2].

In other words, the weighted Delone dynamical system $\Gamma \curvearrowright \Omega^\Gamma_{\Lambda_\omega}$ is just a different model for the symbolic system $\Gamma \curvearrowright \Omega_\omega$. The system $G \curvearrowright \Omega^G_{\Lambda_\omega}$ is just the induced $G$-system, i.e.
\[
G \curvearrowright \Omega^G_{\Lambda_\omega} \cong (G \times \Omega^\Gamma_{\Lambda_\omega})/\Gamma,
\]
where $\Gamma$ acts diagonally on $G \times \Omega^H_{\Lambda}$ and the $G$-action on the quotient is induced by the action on the first factor. We can now establish the following main result, which immediately implies Theorem 1.3.

**Theorem 9.3** (Unique ergodicity). If $S$ is primitive, then for all $\omega \in \Omega(S)$ the dynamical systems $\Gamma \curvearrowright \Omega(S) = \Omega_\omega$, $\Gamma \curvearrowright \Omega^H_{\Lambda_\omega}$ and $G \curvearrowright \Omega^G_{\Lambda_\omega}$ are uniquely ergodic and minimal.

**Proof.** In view of Proposition 9.2 it suffices to show unique ergodicity and minimality of $H \curvearrowright \Omega^H_{\Lambda_\omega}$ for $H \in \{\Gamma, G\}$. For this we need two observations: Firstly, $\omega$ is linearly repetitive by Theorem 7.4, which in the terminology of [BHP20] means that $\Lambda_\omega$ is symbolically linearly repetitive with respect to $d|_{\Gamma \times \Gamma}$. Secondly, let $d_H := d$ if $H = G$ and $d_H := d|_{\Gamma \times \Gamma}$ if $H = \Gamma$; then $H$ has exact polynomial growth with respect to $d_H$ by Proposition 3.36. In view of these observations the theorem now follows from [BHP20, Theorem 5.7(b)]. \qed

In [BHP20] we introduced a notion of *tempered repetitivity* with respect to strong Følner sequences. The relevance of this notion to the proof of Theorem 9.3 is as follows: from the fact that $(G,d)$ and $(\Gamma, d|_{\Gamma \times \Gamma})$ have exact polynomial growth one can deduce that balls in these groups form strong Følner sequences. The fact that $\omega$ is linearly repetitive then implies that $\Lambda_\omega$ (seen as an element of either $R(\Gamma)$ or $R(G)$) is tempered repetitive with respect to these sequences of balls. Unique ergodicity is then a consequence of this tempered repetitivity by [BHP20, Theorem 1.3 and Theorem 1.4].

### 9.2. Associated Delone dynamical systems

The results of this article can also be formulated in the language of Delone sets. We refer the reader to [BHP18] for background on Delone sets in groups. In the sequel it suffices to know that the space of closed subsets of a locally compact second countable group $H$ carries a natural $H$-invariant compact metrizable $H$-invariant topology called the *Chabauty-Fell topology*. Given a uniformly discrete set $\Lambda \subseteq H$ we denote by $\Omega^H_{\Lambda}$ the orbit closure of $\Lambda$ in this topology, which is a topological dynamical system over $H$.

For the remainder of this subsection, $(G,A)$ denotes a RAHGRASP of dimension $\geq 2$ and let $A = \{a, b\}$ be an alphabet with two letters. By Theorem 1.1 we may then choose a dilation datum $D$ over $G$ and primitive and non-periodic substitution datum $S$ over $D$ with alphabet $A$. We fix these choices and denote by $\Omega(S)$ the associated substitution system. We also pick an embedding $\iota : A \rightarrow (0, \infty)$ and some $\omega \in \Omega(S)$ which is fixed under some power of $S$ (cf. Theorem 7.2); for concreteness’ sake let us pick $\iota(a) = 1$ and $\iota(b) = 2$.

By the results of the previous section, $\omega$ corresponds to a weighted Delone set $\Lambda_\omega$ (given by $\Lambda_\omega = \sum_{\gamma \in \Gamma} \iota(\omega(\gamma)) \cdot \delta_\gamma \in R(\Gamma) \subseteq R(G)$). However, we can avoid the language of measures and simply identify $\omega$ with the uniformly discrete set $\Lambda'_\omega \subseteq \Gamma \subseteq G$ given by

$$\Lambda'_\omega := \{\gamma \in \Gamma : \omega(\gamma) = a\}.$$

**Lemma 9.4** (Delone sets vs. measures). For $H \in \{\Gamma, G\}$ the Delone dynamical system $H \curvearrowright \Omega^H_{\Lambda'_\omega}$ is isomorphic to the measure dynamical system $H \curvearrowright \Omega^H_{\Lambda_\omega}$.

**Proof.** By [BHP21a, Proposition 3.2], the Delone dynamical system $H \curvearrowright \Omega^H_{\Lambda'_\omega}$ is isomorphic to the measure dynamical system $H \curvearrowright \Omega^H_{\Lambda_\omega}$, where $\Lambda'_\omega = \sum_{\lambda \in \Lambda'_\omega} \delta_\lambda$ is the associated Dirac comb. On the other hand, the map

$$\Omega^H_{\Lambda_\omega} \rightarrow R(H), \quad \mu \mapsto \sum_{\{x \in H : \mu(x) = 1\}} \delta_x,$$
is continuous, $H$-equivariant and maps $\Lambda_\omega$ to $\Lambda''_\omega$. It thus maps the compact orbit closure $\Omega^H_{\Lambda_\omega}$ homeomorphically onto its image, which is the orbit closure $\Omega^H_{\Lambda''_\omega}$ of $\Lambda''_\omega$. Combining these identifications yields the lemma.

We note in particular that $\Omega^H_{\Lambda''_\omega}$ has no $H$-fixpoints, since $\Omega^H_{\Lambda_\omega}$ is minimal, and hence $\emptyset \not\in \Omega^H_{\Lambda''_\omega}$.

By [BH18, Prop. 4.4] this implies that $\Lambda'_\omega$ is relatively dense, i.e. a Delone set in $H$. This now implies Corollary 1.5 from the introduction:

**Proof of Corollary 1.5.** We consider the Delone set $\Lambda := \Lambda'_\omega$. Since $\omega$ is linearly repetitive, $\Lambda$ is a linearly repetitive Delone set in the sense of [BHP20].

By Lemma 9.4 and Proposition 9.2 the system $\Gamma \acts \Omega(\Gamma_{\Lambda_\omega})$ is isomorphic to $\Gamma \acts \Omega(\Gamma_{\Lambda''_\omega})$ or, equivalently, $\Gamma \acts (G\times \Omega(S)) / \Gamma$. We thus deduce from Theorem 9.3 that $\Omega^G_A$ is minimal and uniquely ergodic.

Under the isomorphism $\Omega^G_A \cong (G\times \Omega(S)) / \Gamma$, the basepoint $\Lambda$ is mapped to the class of $(e,\omega)$. If $g \in G$ stabilizes this class, then the first factor tells us that $g \in \Gamma$, and the second factor tells us that $g \in \Gamma$ stabilizes $\omega \in \Omega(S)$. With Theorem 8.1 we thus deduce that the stabilizer of $\Lambda$ is trivial.

If $G$ is 2-step nilpotent and endowed with its canonical homogeneous structure, then the action of $\Gamma$ on $\Omega(S)$ is free by Theorem 1.4, and hence the induced action of $G$ on $(G\times \Omega(S)) / \Gamma$ is free. This shows that $\Omega^G_A$ is strongly aperiodic and finishes the proof.

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In this appendix we record the data used in the proofs of Theorem 3.13 and Theorem 3.14. We provide this data in the following format: The first column in the following tables refers to the classification in [Gon98]. For each of the labels in this column, the corresponding table in [Gon98] provides an indecomposable nilpotent real Lie algebra of dimension 7 (in the case of the first table) or a one-parameter family of such Lie algebras (in the case of the second table) together with a rational basis $x_1, \ldots, x_7$. The following columns list positive integers $\alpha_1, \ldots, \alpha_7$. One can check that the corresponding Lie algebra admits a unique rational grading such that the generator $x_i$ has degree $\alpha_i$. In particular, the simply-connected Lie groups corresponding to the Lie algebras listed in these tables all admit the structure of a RAHOGASP. The degrees $\alpha_1, \ldots, \alpha_7$ have been computed – using computer assistance – by the method from Example 3.12. Due to the limitations of this method, we do not know which of the remaining Lie algebras from [Gon98] give rise to RAHOGASPs.

| Lie algebra | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\alpha_4$ | $\alpha_5$ | $\alpha_6$ | $\alpha_7$ | Lie algebra | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\alpha_4$ | $\alpha_5$ | $\alpha_6$ | $\alpha_7$ |
|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|------------|
| 37A        | 1          | 1          | 1          | 1          | 2          | 2          | 2          | 37B        | 1          | 1          | 1          | 1          | 2          | 2          | 2          |
| 37C        | 1          | 1          | 1          | 1          | 2          | 2          | 2          | 37D        | 1          | 1          | 1          | 1          | 2          | 2          | 2          |
| 357A       | 1          | 1          | 2          | 1          | 3          | 2          | 2          | 357B       | 1          | 1          | 2          | 1          | 3          | 3          | 2          |
| 357C       | 1          | 1          | 2          | 2          | 3          | 3          | 3          | 27A        | 1          | 1          | 1          | 1          | 2          | 2          | 2          |
| 27B        | 1          | 1          | 1          | 1          | 1          | 2          | 2          | 257A       | 1          | 1          | 2          | 2          | 1          | 3          | 2          |
| 257B       | 1          | 1          | 2          | 1          | 1          | 3          | 2          | 257C       | 1          | 1          | 2          | 2          | 1          | 3          | 2          |
| 257D       | 1          | 1          | 2          | 2          | 2          | 3          | 3          | 257E       | 1          | 1          | 2          | 1          | 2          | 3          | 3          |
| 257F       | 1          | 1          | 2          | 1          | 1          | 3          | 2          | 257G       | 2          | 1          | 3          | 3          | 2          | 5          | 4          |
| 257H       | 1          | 1          | 2          | 2          | 1          | 3          | 3          | 257I       | 1          | 1          | 2          | 2          | 2          | 3          | 3          |
| 257J       | 1          | 1          | 2          | 2          | 2          | 3          | 3          | 257K       | 1          | 1          | 2          | 1          | 2          | 3          | 3          |
| 257L       | 1          | 1          | 2          | 2          | 1          | 3          | 3          | 247A       | 1          | 1          | 1          | 2          | 2          | 3          | 3          |
| 247B       | 1          | 1          | 1          | 2          | 2          | 3          | 3          | 247C       | 1          | 1          | 1          | 2          | 2          | 3          | 3          |
| 247D       | 1          | 1          | 1          | 2          | 2          | 2          | 3          | 247E       | 1          | 1          | 1          | 2          | 2          | 3          | 3          |
| 247F       | 1          | 1          | 1          | 2          | 2          | 3          | 3          | 247G       | 1          | 1          | 1          | 2          | 2          | 3          | 3          |
| 247H       | 1          | 1          | 1          | 2          | 2          | 3          | 3          | 247I       | 1          | 1          | 1          | 2          | 2          | 3          | 3          |
| 247J       | 1          | 1          | 1          | 2          | 2          | 3          | 3          | 247K       | 1          | 1          | 1          | 2          | 2          | 3          | 3          |
| 247L       | 1          | 1          | 2          | 2          | 3          | 3          | 4          | 247M       | 1          | 1          | 2          | 2          | 3          | 3          | 5          |
| 247N       | 1          | 1          | 1          | 2          | 2          | 3          | 3          | 247O       | 1          | 1          | 2          | 2          | 3          | 3          | 5          |
| 247P       | 1          | 1          | 1          | 2          | 2          | 2          | 3          | 247Q       | 1          | 1          | 2          | 2          | 3          | 3          | 4          |
| 247R       | 1          | 2          | 2          | 3          | 3          | 4          | 5          | 247A       | 1          | 1          | 2          | 3          | 1          | 4          | 2          |
| 2457B      | 1          | 2          | 3          | 4          | 1          | 5          | 2          | 2457C      | 1          | 1          | 2          | 3          | 3          | 4          | 4          |
| 2457D      | 1          | 2          | 3          | 4          | 3          | 5          | 4          | 2457E      | 1          | 1          | 2          | 3          | 2          | 3          | 4          |
| 2457F      | 1          | 2          | 3          | 4          | 1          | 5          | 2          | 2457G      | 1          | 1          | 2          | 3          | 2          | 3          | 4          |
| 2457H      | 1          | 2          | 3          | 4          | 3          | 5          | 4          | 2457I      | 1          | 2          | 3          | 4          | 1          | 5          | 3          |
| 2457J      | 1          | 2          | 3          | 4          | 3          | 5          | 5          | 2457K      | 2          | 3          | 5          | 7          | 6          | 8          | 9          |
| 2457L      | 1          | 1          | 2          | 3          | 3          | 4          | 4          | 2457M      | 1          | 1          | 2          | 3          | 3          | 4          | 4          |
| 2357A      | 1          | 1          | 2          | 2          | 3          | 3          | 4          | 2357B      | 1          | 1          | 2          | 2          | 3          | 3          | 4          |
| 2357C      | 1          | 1          | 2          | 2          | 3          | 3          | 4          | 2357D      | 1          | 1          | 2          | 2          | 3          | 3          | 4          |
\begin{center}
\begin{tabular}{|c|c|c|c|c|c|c|c|c|}
\hline
23457A & 1 & 1 & 2 & 3 & 4 & 5 & 3 & \\
23457C & 1 & 1 & 2 & 3 & 4 & 5 & 5 & \\
23457E & 1 & 2 & 3 & 4 & 5 & 6 & 5 & \\
23457G & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\
157 & 1 & 1 & 2 & 2 & 1 & 2 & 3 & \\
147B & 1 & 1 & 1 & 2 & 2 & 2 & 3 & \\
147F & 1 & 1 & 2 & 2 & 2 & 3 & 3 & \\
1457B & 1 & 2 & 3 & 4 & 1 & 4 & 5 & \\
137B & 1 & 2 & 1 & 2 & 3 & 3 & 4 & \\
137D & 1 & 2 & 1 & 2 & 3 & 3 & 4 & \\
1357B & 1 & 1 & 2 & 2 & 3 & 2 & 4 & \\
1357D & 1 & 1 & 2 & 4 & 3 & 3 & 4 & \\
1357F & 2 & 1 & 3 & 2 & 4 & 3 & 5 & \\
1357H & 1 & 1 & 2 & 2 & 3 & 3 & 4 & \\
1357J & 4 & 2 & 6 & 3 & 8 & 7 & 10 & \\
1357O & 1 & 1 & 2 & 2 & 3 & 3 & 4 & \\
1357Q & 1 & 1 & 2 & 2 & 3 & 3 & 4 & \\
13457A & 1 & 1 & 2 & 3 & 4 & 4 & 5 & \\
13457C & 1 & 1 & 2 & 3 & 4 & 4 & 5 & \\
13457E & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\
13457F & 1 & 1 & 2 & 3 & 3 & 4 & 4 & \\
12457A & 1 & 1 & 2 & 3 & 4 & 5 & 5 & \\
12457D & 2 & 1 & 3 & 5 & 6 & 7 & 8 & \\
12457F & 1 & 2 & 3 & 4 & 3 & 5 & 7 & \\
12457I & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\
12357A & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\
123457A & 1 & 1 & 2 & 3 & 4 & 5 & 6 & \\
123457C & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\
37B & 1 & 1 & 1 & 2 & 2 & 2 & 2 & \\
257J & 1 & 1 & 2 & 2 & 2 & 3 & 3 & \\
247F & 1 & 1 & 1 & 2 & 2 & 3 & 3 & \\
247P & 1 & 1 & 1 & 2 & 2 & 2 & 3 & \\
2457L & 1 & 1 & 2 & 3 & 3 & 4 & 4 & \\
147A & 1 & 1 & 2 & 2 & 2 & 3 & 3 & \\
137B & 1 & 1 & 2 & 2 & 3 & 3 & 4 & \\
1357P & 1 & 1 & 2 & 2 & 3 & 3 & 4 & \\
12457L & 1 & 1 & 2 & 3 & 3 & 4 & 5 & \\
\hline
\end{tabular}
\end{center}

\begin{center}
\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
Lie algebra & \(\alpha_1\) & \(\alpha_2\) & \(\alpha_3\) & \(\alpha_4\) & \(\alpha_5\) & \(\alpha_6\) & \(\alpha_7\) & Lie algebra & \(\alpha_1\) & \(\alpha_2\) & \(\alpha_3\) & \(\alpha_4\) & \(\alpha_5\) & \(\alpha_6\) & \(\alpha_7\) \\
\hline
147E & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 13457D & 1 & 1 & 2 & 3 & 3 & 4 & 4 \\
1357N & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 1357M & 1 & 1 & 2 & 3 & 3 & 4 & 4 \\
123457I & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 1357S & 1 & 1 & 2 & 3 & 3 & 4 & 4 \\
1357QRS & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 147E & 1 & 1 & 2 & 2 & 3 & 3 & 4 \\
\hline
\end{tabular}
\end{center}
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