ON THE DECOMPOSITION MATRICES OF THE QUANTIZED SCHUR ALGEBRA

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Abstract. We prove the decomposition conjecture for the Schur algebra stated in [LT]. We also give a new approach to the Lusztig conjecture via canonical bases of the Hall algebra.

0. Introduction and general notations.

0.1. The aim of this paper is to give a proof of the decomposition conjecture for the quantized Schur algebra [LT, Conjecture 5.2] which generalizes the theorem of Ariki (see [A]) on the decomposition numbers of the Hecke algebra of type $A$. More precisely, let $\bigwedge^\infty$ be the level 1 Fock space of type $A$ and let $B^\pm$ be the bases of $\bigwedge^\infty$ introduced in [LT]. The decomposition conjecture links the decomposition matrices of the quantized Schur algebra and the basis $B^+$. Our proof consists in two steps: first we express $B^\pm$ in terms of some Kazhdan-Lusztig polynomials. Then we note that a simple module of the quantized Schur algebra can be pulled-back to a simple module of the Lusztig integral form of the quantized enveloping algebra of $\mathfrak{sl}_k$ (denoted by $U(\mathfrak{sl}_k)$). Thus, the Lusztig conjecture for the dimension of the simple $U(\mathfrak{sl}_k)$-modules at roots of unity identifies the entries of the decomposition matrices with some Kazhdan-Lusztig polynomials. It suffices to observe that these polynomials are precisely the ones which appear in $B^+$.

Let $U^-_n$ be the Hall algebra of nilpotent representations of the cyclic quiver. Set $\varepsilon = \exp(2i\pi/n')$. Put $n = n'$ if $n$ odd, $n = n'/2$ else, i.e. $n$ is the order of $\varepsilon^2$. Let $U_\varepsilon(\mathfrak{sl}_k)$ be the specialization at $v = \varepsilon$ of $U(\mathfrak{sl}_k)$. We give a new approach to the proof of the Lusztig conjecture on the character of the simple modules of $U_\varepsilon(\mathfrak{sl}_k)$ in terms of the canonical basis of $U^-_n$. Recall that this conjecture (proved by Kashiwara-Tanisaki and Kazhdan-Lusztig) gives the multiplicity of the Weyl module of $U_\varepsilon(\mathfrak{sl}_k)$ with highest weight $\mu$, say $W_\mu$, in the simple $U_\varepsilon(\mathfrak{sl}_k)$-module with highest weight $\lambda$, say $V_\lambda$, i.e.

\[ [V_\lambda : W_\mu] = \sum_y (-1)^l(yx) P_{yx}(1), \]

where $x \in \hat{S}_k$ is minimal such that $\nu = \lambda \cdot x^{-1}$ satisfies

\[ \nu_i < \nu_{i+1}, \quad \forall i = 1, 2, ..., k - 1, \quad \nu_1 - \nu_k \geq 1 - k - n. \]

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and $\mu = \lambda \cdot x^{-1}y$. We proceed as follows. First we prove that $\bigwedge^\infty$ is a cyclic $U_n^-$-module generated by the vacuum vector $|\emptyset\rangle$. Then we define a basis $B'$ of $U_n^-$ using intersection cohomology. We construct a basis $B$ of $\bigwedge^\infty$ via the action of $B'$ on the vacuum vector. We prove that $B$ and $B^+$ are fixed by the same semi-linear involution (see Theorem 6.3). At last, we prove that the equality $B = B^+$ is a $q$-analogue of the Lusztig conjecture (see Subsection 11.4). The reader should be warned that we endow the Hall algebra with the product opposite to the usual one (used in [G1] or [L1-4]).

The plan of the paper is the following. In Sections 1-4 we recall the definitions and the main properties of the basic objects. In Sections 5-6 we construct an action of $U_n^-$ on the Fock space $\bigwedge^\infty$. Proposition 6.1 is new. In Section 7 we introduce the convolution algebra on pairs of affine flags. This algebra is a geometric analogue of the affine Schur algebra (Proposition 7.4) and is related to $U_n^-$ in Proposition 7.6. In Sections 8-9 we give a representation of $U_n^-$ on the finite wedges space, $\bigwedge^l$, via the coproduct of $U_n^-$. This action is related to the convolution algebra on affine flags by Lemma 8.3. In Section 10 we interpret the action of $U_n^-$ on $\bigwedge^\infty$ as a “limit” of $\bigwedge^l$ when $l$ goes to infinity. Using the results of Sections 7-9 we prove that the elements of $B$ are fixed by the Leclerc-Thibon involution (Theorem 6.3). In Section 11 we prove the Decomposition Conjecture. Let us observe that the proof only uses the results of Sections 8 and 9. In Section 12 we reinterpret the Lusztig conjecture. We use in an essential way the construction of the representation of $U_n^-$ on $\bigwedge^\infty$ given in Section 6.

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0.2. We now fix a few general notations. Set $S = \mathbb{C}[v]$, $A = \mathbb{C}[v, v^{-1}]$. Let $F$ be a field with $q^2$ elements and let $\bar{F}$ be the algebraic closure of $F$. Fix a set $I$. For any $i \in I$ and $r \in \mathbb{N}^I$, let $\bar{F}_r[i]$ be the $I$-graded $\bar{F}$-vector space with a single $r$-dimensional component, in degree $i$. Let $e_i \in \mathbb{N}^{(I)}$ be the dimension of $\bar{F}_r[i]$. For any $d \in \mathbb{N}^{(I)}$ set $|d| = \sum_{i \in I} d_i$. If $i \in \mathbb{Z}$ let $\bar{i}$ be the class of $i$ in $\mathbb{Z}/n\mathbb{Z}$. Given a positive integer $l$ let $\Pi(l)$ be the set of all the partitions of $l$ and let $\Pi_l$ be the set of partitions with at most $l$ parts. Put $\Pi = \cup_l \Pi(l)$. The set $\Pi$ is endowed with the usual order. If $\lambda \in \Pi$ let $\lambda'$ be the dual partition. For an irreducible algebraic variety $X$ we denote by $\mathcal{H}_i^l(\text{IC}_X)$ the $i$-th cohomology sheaf of the intersection complex of $X$. Then, for any stratum $Y \subset X$, let $\dim \mathcal{H}_Y^l(\text{IC}_X)$ be the dimension
of the stalk of $\mathcal{H}^i(IX)$ at a point of $Y$. For any set $X$ with the action of a group $G$ let $C_G(X)$ be the set of $G$-invariant functions $X \to \mathbb{C}$ supported on a finite number of orbits. For any subset $X$ of an algebraic variety let $\bar{X}$ denote its Zariski closure.

1. The Hecke algebra.

1.1. Fix $n \in \mathbb{N}^\times$ and set

$$A^n_l = \{i \in \mathbb{Z}^l | 1 - n \leq i_1 \leq i_2 \leq \cdots \leq i_l \leq 0\}.$$  

Let $\mathfrak{S}_l$ be the symmetric group and let $\hat{\mathfrak{S}}_l = \mathfrak{S}_l \ltimes \mathbb{Z}^l$ be the extended affine Weyl group. Let $\hat{S}_l \subset \hat{\mathfrak{S}}_l$ be the set of simple affine reflexions and put $S_l = \hat{S}_l \cap \mathfrak{S}_l$. As usual, the simple affine reflexions are denoted by $s_0, s_1, ..., s_{l-1}$ in such a way that $S_l = \{s_1, s_2, ..., s_{l-1}\}$. Let $\pi \in \hat{\mathfrak{S}}_l$ be the zero length element such that $s_{i-1} = \pi^{-1}s_i\pi$. The group $\hat{\mathfrak{S}}_l$ acts on $\mathbb{Z}^l$ on the right in such a way that

\[
\begin{align*}
(i)\lambda &= i + n\lambda & \text{if } \lambda \in \mathbb{Z}^l \\
(ii)s_j &= (i_1, i_2, ..., i_{j+1}, i_j, ..., i_l) & \text{if } j \neq 0 \\
(iii)s_0 &= (i_l - n, i_2, ..., i_{l-1}, i_1 + n).
\end{align*}
\]

The alcove $A^n_l$ is a fundamental domain for this action. If $i \in A^n_l$ let $\mathfrak{S}_l \subset \hat{\mathfrak{S}}_l$ be its isotropy group, $S_l = \hat{S}_l \cap \mathfrak{S}_l$, and let $S^l$ be the set of minimal length representatives of the cosets in $\mathfrak{S}_l \setminus \hat{\mathfrak{S}}_l$. For any $x \in \hat{\mathfrak{S}}_l$, let $x_1 \in \mathfrak{S}_l$ and $x^1 \in S^l$ be such that $x = x_1x^1$. Let $\omega \in \hat{\mathfrak{S}}_l$ be the longest element. Set $\rho = (0, -1, -2, ..., 1 - l) \in \mathbb{Z}^l$ and put

$$\lambda \cdot x = (\lambda + \rho)x - \rho, \quad x \in \hat{\mathfrak{S}}_l, \quad \forall \lambda \in \mathbb{Z}^l.$$  

1.2. The Hecke algebra of type $GL_l$, say $H_l$, is the unital associative $\mathbb{A}$-algebra generated by $T_{i}^{\pm 1}, i = 1, 2, ..., l - 1$ modulo the following relations

\[
(T_i - 1)(T_i - v^{-2}) = 0, \quad (T_i + 1)T_i - T_i^{i+1}T_i = 0, \quad |i - j| > 1 \Rightarrow T_iT_j = T_jT_i.
\]

The affine Hecke algebra of type $GL_l$, say $H_l$, is the unital associative $\mathbb{A}$-algebra generated by $T_{i}^{\pm 1}, X_{i}^{\pm 1}, i = 1, 2, ..., l - 1, j = 1, 2, ..., l$ modulo the relations (a) and

\[
X_iX_j = X_jX_i, \quad X_i, X_j = X_iX_j.
\]

For all $x \in \mathfrak{S}_l \ltimes \mathbb{Z}^l$ let $l(x)$ be the length of $x$ and let $\tilde{T}_x$ be the normalized element $\tilde{T}_x = v^{l(x)}T_x$. The algebra $H_l$ is isomorphic to the Hecke algebra of the extended affine Weyl group $\hat{\mathfrak{S}}_l \ltimes \mathbb{Z}^l$ via the Bernstein isomorphism which maps $\tilde{T}_\lambda$ to $X^{\lambda} = X^\lambda_1X^\lambda_2 \cdots X^\lambda_l$ if $\lambda \in \mathbb{Z}^l$ is dominant, i.e. if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$. The semilinear involution $- : H_l \to H_l$ is such that $\tilde{T}_x = \tilde{T}_{x^{-1}}$ for all $x$. For all $x$ put $\tilde{T}_x = v^{l(x)}T_x$. If $t \in \mathbb{C}^\times$ let $\tilde{H}_l|_t$ be the specialization of $\tilde{H}_l$ at $v = t$.  

2. The quantum group.

Put $I = \{1, 2, \ldots, n - 1\}$ (resp. $I = \{0, 1, \ldots, n - 1\}$) and let $a_{ij}$ be the entries of the Cartan matrix of type $A_{n-1}$ (resp. $A_{n-1}^{(1)}$). The quantized enveloping algebra of $\mathfrak{sl}_n$ (resp. $\widehat{\mathfrak{sl}}_n$) is the unital associative $\mathbb{C}(v)$-algebra generated by $e_i, f_i, k_i^{\pm 1}, i \in I$, modulo the Kac-Moody type relations

\[ k_i k_i^{-1} = 1 = k_j^{-1} k_i, \quad k_i k_j = k_j k_i, \]

\[ k_i e_j = v^{a_{ij}} e_j k_i, \quad k_i f_j = v^{-a_{ij}} f_j k_i, \quad [e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{v - v^{-1}}, \]

\[ \sum_{k=0}^{1-a_{ij}} (-1)^k e_k^{(k)} e_j e_i^{(1-a_{ij} - k)} = \sum_{k=0}^{1-a_{ij}} (-1)^k f_k^{(k)} f_j f_i^{(1-a_{ij} - k)} = 0 \quad \text{if} \quad i \neq j, \]

where

\[ [k] = \frac{v^k - v^{-k}}{v - v^{-1}}, \quad [k]! = [k][k-1] \cdots [1], \quad e_i^{(k)} = \frac{e_i^k}{[k]!}, \quad f_i^{(k)} = \frac{f_i^k}{[k]!}. \]

We denote by $U(\mathfrak{sl}_n)$ (resp. $U(\widehat{\mathfrak{sl}}_n)$) the Lusztig integral form, i.e. the $\mathbb{A}$-subalgebra generated by the divided powers $e_i^{(k)}, f_i^{(k)}$, and by $k_i^{\pm 1}$. If $n = \infty$ the algebra $U(\mathfrak{sl}_\infty)$ is well defined. The algebras above are Hopf algebras. The coproduct is

\[ \Delta e_i = e_i \otimes k_i + 1 \otimes e_i, \quad \Delta f_i = f_i \otimes 1 + k_i^{-1} \otimes f_i, \quad \Delta k_i = k_i \otimes k_i. \]

Let $U^{-}(\mathfrak{sl}_n) \subset U(\mathfrak{sl}_n)$ and $U^{-}(\mathfrak{sl}_\infty) \subset U(\mathfrak{sl}_\infty)$ be the subalgebras generated by the elements $f_i^{(k)}$.

3. The Hall algebra.

In this section we recall some of the results of [L1-4] and [G1].

3.1. Fix a finite field $\mathbb{F}$ with $q^2$ elements as in the introduction. Let $\Gamma = (I, J)$ be an oriented graph : $I$ is the set of vertices and $J$ is the set of arrows. Given an arrow $j \in J$ let $j_1$ and $j_2$ be respectively the input vertex and the output vertex. Fix $d \in \mathbb{N}(I)$ and let $V$ be an $I$-graded $\mathbb{F}$-vector space of dimension $d$. Let $E_{\mathbb{V}} \subseteq \bigoplus_{j \in J} \text{Hom}(V_{j_1}, V_{j_2})$ be the subset of nilpotent representations of $\Gamma$ on $V$. In this paper we will suppose that $\Gamma$ is one of the following two graphs :

(a) $\Gamma = \Gamma_n$ is the cyclic quiver of type $A_n^{(1)}$, i.e. $I = \mathbb{Z}/n\mathbb{Z}$ and $J = \{ \bar{i} \mapsto \bar{i} + 1 | \bar{i} \in \mathbb{Z}/n\mathbb{Z} \}$,

(b) $\Gamma = \Gamma_\infty$ is the infinite quiver of type $A_\infty$, i.e. $I = \mathbb{Z}$ and $J = \{ i \mapsto i + 1 | i \in \mathbb{Z} \}$.

3.2. Set $A_d = C_{G_V}(E_{\mathbb{V}})$ where $G_V = \prod_{i \in I} GL(V_i)$. Given $a, b \in \mathbb{N}(I)$ such that $d = a + b$, fix $I$-graded $\mathbb{F}$-vector spaces $U, W$ of dimensions $a, b$. Let consider the diagram

\[ E_U \times E_W \xleftarrow{p_1} E \xrightarrow{p_2} F \xrightarrow{p_3} E_V, \]

where
(c) $E$ is the set of triples $(x, \phi, \psi)$ such that $x \in E_V$,

$$0 \to U \xrightarrow{\phi} V \xrightarrow{\psi} W \to 0$$

is an exact sequence of $I$-graded vector spaces and $\phi(U)$ is stable by $x$.

(d) $F$ is the set of pairs $(x, U')$ where $x \in E_V$ and $U' \subset V$ is a $x$-stable $I$-graded subspace of dimension $a$.

Given $f \in \mathbb{C}_{G_U}(E_U)$ and $g \in \mathbb{C}_{G_W}(E_W)$ set

$$f \circ g = q^{-m(b, a)}(p_b) h \in \mathbb{C}_{G_V}(E_V),$$

where $h \in \mathbb{C}(F)$ is the function such that $p^*_b h = p^*_b (fg)$ and $m(b, a) = \sum_{j \in J} b_j a_j + \sum_{i \in I} b_i a_i$. Then $(A, o)$, where $A = \bigoplus_d A_d$, is an associative algebra.

3.3. Given $a, b \in \mathbb{N}(J)$ such that $d = a + b$, fix a $I$-graded $\mathbb{F}$-vector space $U \subset V$ of dimension $a$. Let consider the diagram

$$E_U \times E_{V/U} \xleftarrow{p} E \xrightarrow{i} E_V.$$

Here $E \subset E_V$ is the subset of representations preserving $U$, the map $i$ is the inclusion and $p$ is the obvious projection. Set

$$\Delta_{a, b} : A_d \to A_a \otimes A_b, \quad f \mapsto q^{-n(b, a)} p_i^* f,$$

where $n(b, a) = \sum_{j \in J} b_j a_j - \sum_{i \in I} b_i a_i$.

3.4. Recall that $\Gamma = \Gamma_n$ or $\Gamma_\infty$. The classification of the isomorphism classes of nilpotent representations of $\Gamma$ does not depend on the ground field $\mathbb{F}$. It is proved in [R] that the structural constants of $A$ in the basis formed by the characteristic functions of the $G_V$-orbits in $E_V$ are the value at $v = q$ of universal polynomials in $A$. Thus $A$ can be viewed as the specialization at $v = q$ of a $\hat{A}$-algebra, called the generic Hall algebra. Let $U^-_n$ (resp. $U^-_\infty$) be the generic Hall algebra if $\Gamma = \Gamma_n$ (resp. $\Gamma = \Gamma_\infty$). It is known that $U^-_\infty$ is isomorphic to $U^- (\mathfrak{sl}_\infty)$ and that $U^- (\mathfrak{sl}_n)$ embeds in $U^-_n$ (see [G1]). Let $A^0$ be the $\hat{A}$-linear span of elements $k_d$ with $d \in \mathbb{Z}^{(J)}$ such that

$$k_0 = 1 \quad \text{and} \quad k_a k_b = k_{a+b}, \quad \forall a, b.$$

For simplicity we will write $k_i = k_{e_i}$ for all $i \in I$. Set $\hat{A} = A \otimes_{\hat{A}} A^0$ and put

$$(f \otimes k_a) \circ (g \otimes k_b) = v^{-a} (f \circ g) \otimes k_{a+b}, \quad \forall g \in A_d \quad \forall f \in A,$$

where $a \cdot d = -n(a, d) - n(d, a)$. Consider the map $\Delta : \hat{A} \to \hat{A} \otimes_{\hat{A}} \hat{A}$ such that

$$\Delta(f \otimes k_c) = \sum_{d = a+b} \Delta_{a, b} (f) (k_{b+c} \otimes k_c), \quad \forall f \in A_d.$$

Then $(\hat{A}, o, \Delta)$ is a $\hat{A}$-bialgebra (it is due to Lusztig for the composition algebra, the general case is due to Green). Put $U^-_n = \hat{A}$ if $\Gamma = \Gamma_n$ and $U^-_\infty = \hat{A}$ if $\Gamma = \Gamma_\infty$. 


3.5. Given a $G_V$-orbit $O \subset E_V$ let $f_O \in A$ be the $v^{\dim O}$ times the characteristic function of $O$. For any $G_V$-orbit $O \subset E_V$ set
\[ b_O = \sum_{i,O'} v^{-i+\dim O'-\dim O'} \dim \mathcal{H}^i_{O'}(IC_O) f_{O'} . \]
The elements $b_O$ form a basis of $A$. If $d \in \mathbb{N}^{(I)}$ let $f_d \in A$ be the characteristic function of the zero representation of $\Gamma$ in a $d$-dimensional space. The following result is proved in Section 13.

**Proposition.** The algebra $A$ is generated by the $f_d$, $d \in \mathbb{N}^{(I)}$. \hfill \square

3.6. Given two integers $i \leq j$, let $\bar{F}[i,j]$ be the unique indecomposable representation of $\Gamma_\infty$ (resp. $\Gamma_n$) with dimension $\sum_{k=i}^j \epsilon_k$ (resp. $\sum_{k=i}^j \epsilon_k$). For any partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ let $\bar{F}[\lambda]$ be the representation of $\Gamma$ such that
\[ \bar{F}[\lambda] = \bigoplus_{k \geq 1} \bar{F}[1-k, \lambda_k - k] . \]
Let $O_\lambda$ be the orbit of $\bar{F}[\lambda]$ and put $d_\lambda = \dim O_\lambda$.

4. The Fock space.

In this section we recall the construction of the quantized Fock space, due to [H], as it is re-interpreted in [MM].

4.1. Let $T(\lambda)$ be the tableau of shape $\lambda$ whose box with coordinates $(x,y)$ is filled with $y-x$. For instance if $\lambda = (432)$ we get
\[
\begin{array}{ccc}
-2 & -1 \\
-1 & 0 & 1 \\
0 & 1 & 2 & 3
\end{array}
\]
Let $\Lambda^\infty$ be a $A$-module with basis $\{ |\lambda\rangle | \lambda \in \Pi \}$. If $i \in \mathbb{Z}$, a removable $i$-box of $T(\lambda)$ is a box with the color $i$ which can be removed in such a way that the new tableau still comes from a partition. Similarly, an indent $i$-box corresponds to a box with the color $i$ which can be added to $T(\lambda)$. Given $i \in \mathbb{Z}/n\mathbb{Z}$, $i \in \bar{i}$, and a partition $\lambda$ put
\[ n_i(\lambda) = \sharp \{ \text{indent } i\text{-box of } T(\lambda) \} - \sharp \{ \text{removable } i\text{-box of } T(\lambda) \} , \]
and $n_i(\lambda) = \sum_{i \in \bar{i}} n_i(\lambda)$, $n_i^-(\lambda) = \sum_{j < i \& j \in \bar{i}} n_j(\lambda)$, $n_i^+(\lambda) = \sum_{j > i \& j \in \bar{i}} n_j(\lambda)$.

4.2. The algebra $U(\mathfrak{sl}_\infty)$ acts on $\Lambda^\infty$ by
\[
k_i(|\lambda\rangle) = v^{n_i(\lambda)} |\lambda\rangle , \quad e_i(|\lambda\rangle) = |\nu\rangle , \quad f_i(|\lambda\rangle) = |\mu\rangle ,
\]
where the partitions $\mu, \nu$ are such that $T(\mu) - T(\lambda)$ and $T(\lambda) - T(\nu)$ are a box with color $i$. It is known that $\Lambda^\infty$ is the simple module with highest weight $\Lambda_0$ (the fundamental weight) and that the canonical basis of $\Lambda^\infty$ is $\{ |\lambda\rangle | \lambda \in \Pi \}$. The weight multiplicities in $\Lambda^\infty$ are 0 or 1, i.e. $\Lambda_0$ is a minuscule weight.
4.3. The algebra $\mathbf{U}(\hat{\mathfrak{sl}}_n)$ acts on $\Lambda^\infty$ by
\[ k_i(|\lambda|) = v^{n_i(\lambda)}|\lambda|, \quad e_i(|\lambda|) = \sum_{i \in I} v^{-n_i(\lambda)}e_i(|\lambda|), \quad f_i(|\lambda|) = \sum_{i \in I} v^{n_i(\lambda)}f_i(|\lambda|). \]

5. The representation of $\mathbf{U}_\infty^-$ on $\Lambda^\infty$.

The algebras $\mathbf{U}_\infty^-$ and $\mathbf{U}^-(\hat{\mathfrak{sl}}_\infty)$ are isomorphic. Thus $\Lambda^\infty$ may be viewed as the quotient of $\mathbf{U}_\infty^-$ by a left ideal $\mathbf{I}$. Let us describe $\mathbf{I}$. Let $\Gamma_\infty$ be the quiver $\Gamma_\infty$ with the opposite orientation. For any $\mathbb{Z}$-graded $\mathbb{F}$-vector space $V$ let $\Lambda_V$ be the variety of pairs $(x, \bar{x})$ of commuting representations respectively of $\Gamma_\infty$ and $\bar{\Gamma}_\infty$ on $V$. The variety $\Lambda_V$ is reducible. For any $G_V$-orbit $O \subseteq E_V$ set
\[ \Lambda_O = \{(x, \bar{x}) \in \Lambda_V \mid x \in O\}. \]

According to [N] the orbit $O$ is stable if there exists a triple
\[ (x, \bar{x}, i) \in \mathcal{O}_O \times \text{Hom}(\mathbb{F}[0], V) \]
such that $i$ is homogeneous of degree $0$ and that for any graded subspace $W \subseteq V$,
\[ (a) \quad (x(W), \bar{x}(W) \subseteq W \text{ and } \text{Im } i \subseteq W) \Rightarrow W = V \]
(since the Hall algebra is endowed with the product opposite to the usual one, we use the stability condition opposite to the one in [N]).

**Proposition.** The ideal $\mathbf{I}$ is linearly spanned by the elements $\mathbf{b}_O$ such that $O \neq O_\lambda$ for all $\lambda$. Moreover the map $\mathbf{U}_\infty^-/\mathbf{I} \rightarrow \Lambda^\infty$, $\mathbf{b}_O + \mathbf{I} \mapsto |\lambda|$, is an isomorphism of $\mathbf{U}_\infty^-$-modules.

**Proof.** From [N, Theorem 11.7 and Proposition 3.5], $\mathbf{I}$ is linearly generated by the elements $\mathbf{b}_O$ such that $O$ is unstable. Let us show that for any $\lambda \in \Pi$ the orbit $O_\lambda$ is stable. A dimension counting then shows that the orbits $O_\lambda$ are precisely all the stable orbits.

Recall that $\mathbb{F}[i, j]$ is the representation $x$ of $\Gamma_\infty$ on the graded space $\bigoplus_{k=1}^j \mathbb{F} v_k$, where $v_k$ is a non-zero vector of degree $k$, such that $x(v_k) = v_{k+1}$ if $k < j$ and $x(v_j) = 0$. Fix $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r) \in \Pi$. Fix non-zero vectors $v_{k, s} \in \mathbb{F}[1 - k, \lambda_k - k]$ with degree $s$. The representation $\mathbb{F}[\lambda]$ is given by the endomorphism $x$ such that for all $k$,
\[ x(v_{k, s}) = v_{k, s+1} \quad \text{if} \quad s \in [1 - k, \lambda_k - k), \quad \text{and} \quad x(v_{k, \lambda_k - k}) = 0. \]

Let us exhibit a pair $(i, \bar{x})$ satisfying $(a)$. Fix a graded homomorphism $i \in \text{Hom}(\mathbb{F}[0], \mathbb{F}[\lambda])$ such that $v_{1, 0} \in \text{Im } i$. Consider the degree $-1$ linear operator $\bar{x}$ on $\mathbb{F}[\lambda]$ such that $\bar{x}(v_{k, s}) = v_{k+1, s-1}$ if $k \neq r$ and $s \leq \lambda_{k+1} - k$, $\bar{x}(v_{k, s}) = 0$ else.

The operators $x$ and $\bar{x}$ commute since
\[ x\bar{x}(v_{r, s}) = 0 = \bar{x}x(v_{r, s}) \quad \forall s, \]
\[ x\bar{x}(v_{k, s}) = 0 = \bar{x}x(v_{k, s}) \quad \forall s \geq \lambda_{k+1} - k, \]
\[ x\bar{x}(v_{k, s}) = v_{k+1, s} = \bar{x}x(v_{k, s}) \quad \forall s < \lambda_{k+1} - k, \quad \forall k \neq r. \]
Now if $W \subseteq V$ is such that $x(W) \subseteq W$ and $\text{Im} \ i \subseteq W$ then $\mathbb{F}[0, \lambda_1 - 1] \subseteq W$; namely $v_{1,0} \in W$ and thus $v_{1,s} = x^s(v_{1,0}) \in W$ for all $s$. By definition of $\bar{x}$ we have for all $t < r$

$$\bar{x}^t(\mathbb{F}[0, \lambda_1 - 1]) = \mathbb{F}[-t, \lambda_{1+t} - 1].$$

The dimension $d_\lambda$ of $\mathbb{F}[\lambda]$ is such that $d_{\lambda,i}$ is the multiplicity of the color $i$ in the tableau $T(\lambda)$ (see Section 4). Thus the linear isomorphism $b_{O_\lambda} \mapsto |\lambda|$ preserves the weights. Moreover it preserves the canonical base up to a permutation of its elements. Since $\Lambda_0$ is minuscule there is at most one vector of a given weight in the canonical basis. Hence the canonical bases are fully identified. 

6. The representation of $U^-_n$ on $\wedge^\infty$.

6.1. Fix $d \in \mathbb{N}(\mathbb{Z})$ and let $\bar{V}$ be a $\mathbb{Z}$-graded $\mathbb{F}$-vector space of dimension $d$. Let $\bar{d} \in \mathbb{N}/n\mathbb{Z}$ be the multi-index such that $d_i = \sum_{\bar{j} \in i} d_{\bar{j}}$ for all $i \in \mathbb{Z}/n\mathbb{Z}$, and let $\bar{V}$ be the $\bar{d}$-dimensional $\mathbb{Z}/n\mathbb{Z}$-graded vector space such that $\bar{V}_i = \bigoplus_{i \in \bar{j}} V_j$. The vector space $\bar{V}$ is filtered by the subspaces

$$\bar{V}_{\geq i} = \bigoplus_{j \geq i} V_j, \quad \forall i \in \mathbb{Z}. $$

The associated graded is naturally identified with the $\mathbb{Z}$-graded space $V$. Set

$$E_{\bar{V},V} = \{ x \in E_{\bar{V}} \mid x(\bar{V}_{\geq i}) \subseteq \bar{V}_{\geq i+1}, \forall i \}. $$

The map $p : E_{\bar{V},V} \to E_V$ associate to a representation of $\Gamma_n$ in $\bar{V}$ the corresponding graded representation of $\Gamma_\infty$ in $V$. Let $j : E_{\bar{V},V} \hookrightarrow E_{\bar{V}}$ be the closed embedding. Let consider the map $\gamma_d : U^-_{n,d} \to U^-_{\infty,d}$ such that

$$\gamma_{\bar{d}|_{V}=q^{-1}} : C_{\bar{G}_\lambda}(E_{\bar{V}}) \to C_{\bar{G}_\lambda}(E_V), \quad f \mapsto q^{-h(\bar{d})}p_{\bar{d}j}^*f,$$

where $h(\bar{d}) = \sum_{i < j \& \bar{i}=j} d_{i}(d_{j+1}-d_{j})$. Put $k(b,a) = \sum_{i > j \& \bar{i}=j} b_{i}(2a_{j} - a_{j-1} - a_{j+1})$. The following is proved in Section 13.

**Proposition.** Fix $\alpha, \beta \in \mathbb{N}/n\mathbb{Z}$ and $d \in \mathbb{N}(\mathbb{Z})$ such that $\bar{d} = \alpha + \beta$. Then,

$$\sum_{a+b=d \atop \bar{d} = \alpha, \beta} v^{-k(b,a)} \gamma_{\alpha}(f) \circ \gamma_{\beta}(g) = \gamma_{\alpha}(f \circ g) \quad \forall f \in U^-_{n,\alpha}, \forall g \in U^-_{n,\beta},$$

\[\square\]

**Remark.** With the notations in Section 3.5 we have $\gamma_{\bar{d}}(f_{\bar{i}}) = v^{h(\bar{d})} f_{\bar{i}}$. Observe that $f_{\bar{d}}$ is the product of the divided powers $f_{i}^{(d_{i})}$’s ordered from $i = -\infty$ to $\infty$.

6.2. For all $\lambda \in \Pi$ and all $x \in U^-_n$ put

$$(a) \quad x(|\lambda|) = \sum_{d} \gamma_{\bar{d}}(x)k_{\bar{d}}(|\lambda|) \quad \text{where} \quad \bar{d}' = \sum_{j < i, \bar{i}=j} d_{j} \epsilon_{i}. $$

**Corollary.** Formula $(a)$ extends the Hayashi action of $U^-_{n}(\bar{\mathfrak{sl}}_n)$ on $\wedge^\infty$ to a representation of $U^-_{n}$. 


Lemma. The compatibility with the product in $U^-_n$ follows from Proposition 6.1. Formula (a) implies that

$$f_t(\lambda) = \sum_{i \in \bar{t}} \sum_{\mu} v^{\varphi(\epsilon_i, \delta_{\lambda})}|\mu|,$$

where $\mu$ is a partition such that $T(\mu) - T(\lambda)$ is a box with color $i$ and

$$g(\epsilon_i, \delta_{\lambda}) = - \sum_{i<j} (2d_{\lambda,j} - d_{\lambda,j-1} - d_{\lambda,j+1}) + \alpha_i,$$

where $\alpha_i = 1$ if $i < 0$ and $\bar{i} = 0$, and $\alpha_i = 0$ else. We have already observed that $d_{\lambda,i}$ is the multiplicity of the color $i$ in $T(\lambda)$. Thus, $n_j(\lambda) = -2d_{\lambda,j} + d_{\lambda,j-1} + d_{\lambda,j+1} + \delta_{j0}$, and

$$g(\epsilon_i, \delta_{\lambda}) = \sum_{i<j} n_j(\lambda) = n_i(\lambda).$$

6.3. For any $\lambda \in \Pi$ set $b_\lambda = b_{O_\lambda}(0)$ where $O_\lambda$ is the isomorphism class of representations of $\Gamma_n$ defined in Subsection 3.6. Put $B = \{b_\lambda | \lambda \in \Pi\}$. Leclerc and Thibon have introduced in [LT] a semi-linear involution on $\wedge^\infty$.

Theorem. $B$ is a basis of $\wedge^\infty$ whose elements are fixed by the Leclerc-Thibon involution.

The theorem is proved in Subsection 10.1. We first introduce some more material.

6.4. Let $r : E_V \to E_{\bar{V}}$ be such that

$$r(x)|_{V_i} = \bigoplus_{i \in I} x|_{V_i} \quad \forall x \in E_V.$$

Fix a $G_V$-orbit $O \subset E_V$ such that $O \cap E_{\bar{V},V} \neq \emptyset$. If $x \in p(O \cap E_{\bar{V},V})$ then

$$(b) \quad \sharp(p^{-1}(x) \cap O) \in \begin{cases} q^{2N} & \text{if } r(x) \in O \\ (q^2 - 1)N & \text{else.} \end{cases}$$

Indeed, fix $y \in p^{-1}(x) \cap O$. It suffices to consider the case where $y$ is indecomposable. Then, fix a basis of homogeneous vectors $\{v_i | i \in [1,r]\}$ of $\bar{V}$ such that

$$(c) \quad y(v_k) = v_{k+1} \quad \forall k = 1, 2, ..., r - 1.$$
where $O' \subseteq E_V$ is the unique $G_V$-orbit such that $r(O') \subseteq O$. \hfill $\Box$

7. Flag varieties.

7.1. Fix a positive integer $l$. Set $\mathbb{L} = \mathbb{F}((z))$ and $G = GL_1(\mathbb{L})$. A lattice in $\mathbb{L}^l$ is a free $\mathbb{F}[[z]]$-submodule of rank $l$. Let $Y$ be the set of sequences of lattices $L = (L_i)_{i \in \mathbb{Z}}$ such that

$$L_i \subseteq L_{i+1} \text{ and } L_{i+n} = z^{-1} L_i.$$ 

The group $G$ acts on $Y$ in the obvious way. Let $M$ be the set of all $\mathbb{Z} \times \mathbb{Z}$-matrices with non-negative entries, say $m = (m_{ij})_{i,j \in \mathbb{Z}}$, such that $m_{i+n,j+n} = m_{ij}$. Set

$$M^l = \{m \in M \mid \sum_{i \in \mathbb{Z}} \sum_{j=1}^n m_{ij} = l\}.$$ 

The set $M^l$ parametrizes the orbits of the diagonal action of $G$ in $Y \times Y$ to $m$ corresponds the set $Y_m$ of the pairs $(L', L)$ such that

$$m_{ij} = \dim_{\mathbb{F}} \left( \frac{L_{i+1} \cap L'_{j+1}}{(L_i \cap L'_{j+1}) + (L_{i+1} \cap L'_j)} \right).$$

For all $L \in Y$ let $Y_{m,l}$ be the fiber over $L$ of the first projection $Y_m \to Y$. If $Y_{m,L} \neq \emptyset$ then $Y_{m,L}$ is the set of $\mathbb{F}$-points of an algebraic variety whose dimension, denoted by $y(m)$, is independent of $L$. Let $1_m \in G(\mathbb{F}(Y \times Y))$ be $q^{-y(m)}$ times the characteristic function of $Y_m$. The convolution product, denoted $\ast$, endows $G(\mathbb{F}(Y \times Y))$ with the structure of an associative algebra.

7.2. Let $X$ be the set of sequences of lattices $L = (L_i)_{i \in \mathbb{Z}}$ such that

$$L_i \subseteq L_{i+1}, \quad L_{i+l} = z^{-1} L_i \quad \text{and} \quad \dim_{\mathbb{F}}(L_{i+1}/L_i) = 1.$$ 

The group $G$ acts on $X$ in the obvious way. The orbits of the diagonal action of $G$ in $Y \times X$ are labelled by functions $i : \mathbb{Z} \to \mathbb{Z}$ such that $i(k+l) = i(k) + n$ for all $k$. Let $X_1$ be the orbit of the pair $(L_1, L_0)$ such that

$$L_{i,1} = \prod_{i(j) \leq i} \mathbb{F} e_j \quad \text{and} \quad L_{0,i} = \prod_{j \leq i} \mathbb{F} e_j.$$ 

Here $(e_1, e_2, ..., e_l)$ is a fixed $\mathbb{L}$-basis of $\mathbb{L}^l$ and $e_{i+lk} = z^{-k} e_i$ for all $k \in \mathbb{Z}$. A periodic function $i$ as above is identified with the $l$-uple $(i(1), i(2), ..., i(l)) \in \mathbb{Z}^l$. If $L \in Y$ let $X_{1,L}$ be the fiber over $L$ of the projection $X_1 \to Y$. If $X_{1,L} \neq \emptyset$, then $X_{1,L}$ is the set of $\mathbb{F}$-points of an algebraic variety of dimension $l(\omega_l)$. Let $1_l \in G(\mathbb{F}(Y \times X))$ be $q^{-l(\omega_l)}$ times the characteristic function of $X_1$. The space $G(\mathbb{F}(Y \times X))$ is a left $G(\mathbb{F}(Y \times X))$-module and a right $G(\mathbb{F}(X \times X))$-module.

7.3. For all $x \in \widetilde{G}_l$ let $X_x \subseteq X \times X$ be the $G$-orbit of the pair $(x(L_0), L_0)$. There is an algebras isomorphism $\widetilde{H}_{lq^{-1}} \to G(\mathbb{F}(X \times X))$ which maps $T_x$ to the characteristic function of $X_x$ (see [IM]). Put $P = X_1^{-1} T_1^{-1} T_2^{-1} \cdots T_{l-1}^{-1}$. 


Lemma. The right action of $\widehat{H}_l$ on $C_G(Y \times X)$ is such that if $i \in A^n_l$, $x \in S^1$, and $s \in \hat{S}_l$, then $(1_i)P = \otimes 1_{(i)x}$ and

$$
(1_{(i)x})^\hat{T}_s = \begin{cases} 
  v^{-1}1_{(i)x} & \text{if } xs \notin S^1 \text{ (then } xs > x), \\
  1_{(i)x} & \text{if } xs > x \text{ and } xs \in S^1,
  \\
  1_{(i)x} + (v^{-1} - v)1_{(i)x} & \text{if } xs < x \text{ (then } xs \in S^1).
\end{cases}
$$

Proof. To simplify the notations fix $l = 2$. Fix $i \in A^n_l$ and $x \in S^1$. Set $(i, j) = (i)x$. Then

$$
x_{s1} \notin S^1 \iff (\exists t \in S_l \text{ such that } xs_1 = tx) \iff (i, j)s_1 = (i, j).
$$

Moreover,

$$
xs_1 > x \text{ and } xs_1 \in S^1 \iff i \leq j \text{ and } (i, j)s_1 \neq (i, j).
$$

The formulas in the proposition gives

$$
(1_{(i,j)})^\hat{T}_1 = \begin{cases} 
  v^{-2}1_{(i,j)} & \text{if } i = j, \\
  v^{-1}1_{(j,i)} & \text{if } i < j, \\
  v^{-1}1_{(j,i)} + (v^{-2} - 1)1_{(i,j)} & \text{if } i > j.
\end{cases}
$$

These are precisely the formulas in [VV, Section 5] taking into account the different normalizations for the Hecke algebra and the factor $q^{-l(\omega_i)}$. The result follows from [VV, Proposition 6].

7.4. Fix $i \in A^n_l$. Let $H_i \subseteq \widehat{H}_l$ be the parabolic subalgebra associated to $S^1$. Set $e_1 = \sum_{x \in e^i} T_x$ and $\pi_1 = \sum_{x \in e^i} v^{-2l(x)}$. Thus $e_1^2 = \pi_1 e_1$ and $e_1 = v^{2l(\omega_i)} e_1$. Set

$$
T_{n,l} = \bigoplus_{i \in A^n_l} e_i \widehat{H}_l.
$$

The affine $q$-Schur algebra $\hat{S}_{n,l}$, introduced in [G2], is the endomorphism ring of the right $\widehat{H}_l$-module $T_{n,l}$. If $j \in A^n_l$ set

$$
M_{ij} = \{ m \in M^l \mid Y_m \cap (G(L_i) \times G(L_j)) \neq \emptyset \}.
$$

A matrix $m \in M_{ij}$ is identified with the class in $S^1 \setminus \hat{S}_l/S_j$ of the elements $x$ such that $(L_{(i)x}, L_j) \in Y_m$. Set $T_m = \sum_{x \in m} T_x$. Let $\widehat{H}_{ij} \subseteq \widehat{H}_l$ be the $A$-linear span of the elements $T_m$ with $m \in M_{ij}$. The $A$-linear homomorphism

$$
\bigoplus_{i,j \in A^n_l} \widehat{H}_{ij} \to \hat{S}_{n,l}
$$
which maps \( T_m, m \in M_{lj} \), to the endomorphism such that \( e_k \mapsto \delta_{k,j} T_m \in e_l \hat{H}_l \), is invertible. The product in the affine \( q \)-Schur algebra, denoted \( \cdot \), is
\[
T_m \cdot T_n = \delta_{k,j} \pi_j^{-1} T_m T_n \quad \forall m \in M_{lj} \quad \forall n \in M_{kl}.
\]

If \( t \in \mathbb{C}^\times \) let \( \hat{S}_{n,l|t} \) and \( T_{n,l|t} \) be the specializations of \( \hat{S}_{n,l} \) and \( T_{n,l} \) at \( v = t \).

**Proposition.** (a) The map \( \Phi: \hat{S}_{n,l|q^{-1}} \to \mathbb{C}_G(Y \times Y), T_m \mapsto q^{y(m)} 1_m \), is an isomorphism of algebras.

(b) There is a unique isomorphism of \( \hat{S}_{n,l|q^{-1}} \times \hat{H}_{l|q^{-1}} \)-modules, still denoted \( \Phi \), from \( T_{n,l|q^{-1}} \) to \( \mathbb{C}_G(Y \times X) \) such that \( e_1 \mapsto q^{l(\omega_i)} 1_i \) for all \( i \in A_l^n \).

**Proof.** The map \( \Phi \) is a linear isomorphism. For all \( x, y, z \in \hat{S}_l \) let \( B_{xy}^z(v) \in A \) be such that
\[
T_x T_y = \sum_z B_{xy}^z(v) T_z.
\]

If \( (L'', L) \in X_z \) then,
\[
B_{xy}^z(q^{-1}) = \sharp\{ L' \in X \mid (L'', L') \in X_x \quad \& \quad (L', L) \in X_y \}.
\]

Fix \( m \in M_{lk}, n \in M_{lj}, \) and \( o \in M_{jk} \). Let \( A_{no}^m \in \mathbb{N} \) be such that
\[
1_n \ast 1_o = \sum_m q^{-y(n)-y(o)+y(m)} A_{no}^m 1_m.
\]

Then for any \( z \in m, \)
\[
A_{no}^m = h_{j}^{-1} \sum_{z \in n} B_{zy}^z(q^{-1}),
\]

where \( h_j = \pi_j|_{v=q^{-1}} \) is the cardinal of the fiber of the projection \( X \to G(L_j) \). Claim (a) follows from the identity
\[
T_n \cdot T_o = \pi_j^{-1} \sum z \sum_{z \in n} B_{xy}^z(v) T_z = \sum_m A_{no}^m T_m \quad \text{mod} \ (v - q^{-1}).
\]

Let \( \Phi \) be the unique isomorphism of right \( \hat{H}_l \)-modules \( T_{n,l|q^{-1}} \to \mathbb{C}_G(Y \times X) \) such that \( \Phi(e_1) = q^{l(\omega_i)} 1_i \) for all \( i \in A_l^n \). Let us prove that \( \Phi \) commutes to the action of \( \hat{S}_{n,l} \). We must prove that for all \( i, j \in A_l^n \) and all \( m \in M_{lj} \) then
\[
q^{y(m)} 1_m \ast 1_j = q^{-l(\omega_i) - l(\omega_j)} h_i^{-1} \Phi(e_1 T_m) = q^{l(\omega_i) - l(\omega_j)} h_i^{-1} (1_i) T_m.
\]

Put
\[
1_m \ast 1_j = \sum_{k \in \mathbb{Z}^l} q^{-y(m) - l(\omega_j) + l(\omega_k)} A_{mj}^k 1_k, \quad (1_i) T_m = \sum_{k \in \mathbb{Z}^l} q^{-l(\omega_i) + l(\omega_k)} A_{mj}^k 1_k.
\]

Fix \( z \in \hat{S}_l \) and \( (L'', L) \in X_z \) whose projection in \( Y \times X \) is in \( X_k \). Then
\[
A_{mj}^k = \sharp\{ L' \in Y \mid (L'', L') \in Y_m, \quad (L', L) \in X_j \} = h_j^{-1} \sum_{y \in m, z \in \varphi_j} B_{zy}^z(q^{-1}),
\]
\[ A_{sk}^n = \sum_{y \in \mathbf{m}} \sharp \{ L' \in X \mid (L'', L') \in X_i, \ (L', L) \in X_y \} = \sum_{y \in \mathbf{m}} B_{xy}^z(q^{-1}). \]

Claim (b) follows from the equality
\[ \pi_i^{-1} \sum_{y \in \mathbf{m}} T_x T_y = \pi_j^{-1} \sum_{y \in \mathbf{m}} T_y T_x. \]

7.5. The set \( M^+ = \{ \mathbf{m} \in M \mid i > j \Rightarrow m_{ij} = 0 \} \) parametrizes the isomorphism classes of nilpotent representations of the quiver \( \Gamma_n : O_{\mathbf{m}} \) is the class of \( \bigoplus_{i=1}^n \bigoplus_{j \geq i} \mathbb{F}[i, j]^{m_{ij}} \). Let \( \bar{\sim} \) be the unique semilinear involution on \( U_n^+ \) fixing the elements \( \mathbf{b}_{O_{\mathbf{m}}} \).

**Proposition.** The involution \( \bar{\sim} \) on \( U_n^+ \) is a ring homomorphism and \( \bar{\mathbf{f}}_\alpha = \mathbf{f}_\alpha \) for all \( \alpha \in \mathbb{N}^{(\mathbb{Z}/n\mathbb{Z})} \).

**Proof.** The second claim is obvious since \( \mathbf{f}_\alpha \) is the characteristic function of a single point. We now prove the first claim. For any algebraic variety \( X \) over \( \bar{\mathbb{F}} \) let \( \mathcal{D}(X) \) be the bounded derived category of complexes of \( \mathbb{Q}_l \)-sheaves on \( X \) (see [L2], [L3]). If \( G \) is a connected algebraic group acting on \( X \) let \( \mathcal{D}^G_\mathcal{D}(X) \) be the full subcategory whose objects are sums of shifted simple \( G \)-equivariant objects in \( \mathcal{D}(X) \). Lusztig has defined in [L2, Section 3.1] a convolution product
\[ * : \mathcal{D}^G_\mathcal{D}(E_U) \times \mathcal{D}^G_\mathcal{D}(E_W) \to \mathcal{D}^G_\mathcal{D}(E_V) \]
such that \( \mathcal{F} \star \mathcal{G} = (p_3)_! \mathcal{H} \) where \( \mathcal{H} \) satisfies \( p_2^* \mathcal{H} \simeq p_1^* (\mathcal{F} \otimes \mathcal{G}) \). Let \( D \) be the Verdier duality. Since \( p_1 \) and \( p_2 \) are smooth with connected fibers and since \( p_3 \) is proper we get \( D(\mathcal{F} \star \mathcal{G}) = (D\mathcal{F}) \star (D\mathcal{G}) \mid [2d_1 - 2d_2] \] where \( d_1 \) and \( d_2 \) are the dimensions of the fibers of \( p_1 \) and \( p_2 \). Let \( \alpha, \beta \) be the dimension of \( U, W \). We know that \( d_2 = \sum_{i} \alpha_i^2 + \sum_{l} \beta_l^2 \) and \( d_1 = d_2 + m(\beta, \alpha) \). Thus \( D\mathcal{F} \star \mathcal{G} = (D\mathcal{F}) \star (D\mathcal{G}) \mid [2m(\beta, \alpha)] \). Finally observe that the elements \( \mathbf{b}_{O_{\mathbf{m}}} \) are the Frobenius traces of the simple perverse sheaves on the \( E_V \) since the varieties \( \bar{\mathbf{O}}_{\mathbf{m}} \) are pure (see [L1, Corollary 11.6]).

7.6. If \( L', L \in Y \) are such that \( L' \subseteq L \) then \( L/L' \) may be viewed as a nilpotent representation of \( \Gamma_n \) of dimension \( \alpha \) where \( \alpha_i = \dim_{\bar{\mathbb{F}}}(L_i/L'_i) \) (see [L2, Section 11], [GV]). Then, set
\[ a(L', L) = \sum_{i=1}^n \dim_{\bar{\mathbb{F}}}(L_i/L'_i)(\dim_{\bar{\mathbb{F}}}(L_{i+1}/L'_i) - \dim_{\bar{\mathbb{F}}}(L_i/L'_i)). \]

Let \( \Theta : U_n^+ \to \tilde{S}_{n, 1} \) be the \( \mathbb{A} \)-linear map such that
\[ \Phi \circ \Theta(f)(L', L) = q^{-a(L', L)} f(L/L') \quad \text{if} \quad L' \subseteq L, \quad 0 \quad \text{else}. \]

If \( i \in A_n^l \) and \( \mathbf{m} \in M^+ \) let \( \mathbf{m}^l \subseteq \cup_j \mathbf{m}_j \) be the matrix with the \((i, j)\)-th entry equal to
\[ \delta_{ij}(q^{-1}(j + 1) - \sum_{k \leq j} m_{kj}) + (1 - \delta_{ij}) m_{i+1,j}. \]
Let \( \phi \) be the semilinear involution on \( \hat{S}_{n,I} \) such that \( \phi(u) = u^{-2t(\omega_i)\vec{u}} \) for all \( u \in \hat{H}_I \).

**Proposition.** The map \( \Theta : U_n \to \hat{S}_{n,I} \) is an algebra homomorphism. Moreover if \( u \in U_n^- \) and \( m \in M^+ \) we have

\[
\phi \circ \Theta(u) = \Theta(\vec{u}) \quad \text{and} \quad \Phi \circ \Theta(f_{O_m}) = \sum_{i \mid m' \in M} 1_{m'},
\]

**Proof.** The first claim is immediate from the formula

\[
a(L'', L) - a(L', L) - a(L'', L') = -m(L/L', L'/L'')
\]

and from the definition of the product in \( U_n^- \) and \( \mathbb{C}_G(Y \times Y) \). We know that \( f_\alpha = f_\alpha \) for all \( \alpha \in \mathbb{N}^{(Z/2Z)} \). Similarly, \( \phi \circ \Theta(f_\alpha) = \Theta(f_\alpha) \) since for any flag \( L' \) the \( L \)'s such that \( L' \subseteq L \) and \( f_\alpha(L/L') \neq 0 \) are the rational points of a smooth variety. Hence, the second claim results from the first claim, Proposition 3.5, and the fact that \( \phi \) is a ring homomorphism. Now let us first prove that

\[
\text{(c)} \quad (L', L) \in Y_{m'} \iff \left( L'/L' \in O_m \text{ and } \dim_{\mathbb{F}}(L_i'/L_{i-1}) = n^{-1}(i) \right).
\]

By definition \( (L', L) \in Y_{m'} \) if and only if

\[
\delta_{ij}(n^{-1}(j + 1) - \sum_{k \leq j} m_{kj}) + (1 - \delta_{ij})m_{i+1,j} = \dim_{\mathbb{F}} \left( \frac{L_i \cap L_{i+1}'}{(L_i \cap L_{i+1})} \right).
\]

Thus it suffices to prove that if \( \dim_{\mathbb{F}}(L_i'/L_i') = n^{-1}(i + 1) \) and \( L' \subseteq L \) then

\[
\text{(d)} \quad \dim_{\mathbb{F}} \left( \frac{L_i \cap L_{i+1}'}{(L_i \cap L_{i+1})} \right)
\]

is the multiplicity of \( \mathbb{F}[i, j] \) in \( L'/L' \) for all \( i \leq j \),

\[
\text{(e)} \quad \text{if } x_i : L_i/L_i' \to L_{i+1}/L_{i+1}' \text{ is the map induced by the inclusion } L_i \subseteq L_{i+1}, \text{ then }
\]

\[
\dim_{\mathbb{F}} \ker(x_i) = \dim(L_{i+1}/L_i \cap L_{i+1}').
\]

Claim (e) is immediate since

\[
\ker(x_i) = (L_i \cap L_{i+1}')/L_i', \quad L_{i+1}'/(L_i \cap L_{i+1}') \simeq \frac{L_{i+1}'/L_i'}{(L_i \cap L_{i+1}').}
\]

and since \( n^{-1}(i + 1) = \dim(L_i'/L_i) \). Part (d) is due to the fact that \( \mathbb{F}[i, j] \) is a direct summand of \( L'/L' \) if and only if there is a vector \( w \in L_{j+1}' \setminus L_j' \) such that \( w \in L_i \setminus L_{i-1} \). The second claim follows from (c) and the formula

\[
(m \in M^+ \text{ and } m'^1 \in M) \Rightarrow \dim_{\mathbb{F}} O_m + a(m') = y(m'^1),
\]

which is left to the reader. \( \square \)

**Remark.** For any \( m \in M^1 \) set

\[
c_m = \sum_{i, n} v^{-i+y(m)} \dim H^{I}_{Y_{n,L}}(IC_{\gamma_{m,L}}) T_n,
\]
where \( L \) is such that \( Y_{m,L} \neq \emptyset \). The elements \( c_m \) form a \( \mathbb{A} \)-basis of \( \hat{S}_{n,l} \) and \( \varphi(c_m) = c_m \). Proposition 7.6 implies that for any \( m \in M^+ \) we have

\[
\Theta(b_{O_m}) = \sum_{i|m| \in M} c_{mi}.
\]

We will not use this.

8. The tensor representation of \( \hat{U}_n^- \).

8.1. Let \( \mathbb{A}^{(Z)} \) be the \( \mathbb{A} \)-linear span of vectors \( x_i, i \in \mathbb{Z} \). Let \( e_{ij} \in M \) be the matrix with 1 at the spot \((k,l)\) if \((k,l) \in (i,j) + \mathbb{Z}(n,n)\) and 0 elsewhere.

Lemma. \( \hat{U}_n^- \) acts on \( \mathbb{A}^{(Z)} \) in such a way that for all \( m \in M \) and all \( \alpha \in \mathbb{N}^{Z/nZ}, \)

\[
f_{O_m}(x_i) = \sum_{j \geq i} \delta_{m,e_{ij}} x_{j+1} \quad \text{and} \quad k_{\alpha}(x_i) = v^{-\alpha} x_i.
\]

Proof. It is the action obtained by taking \( l = 1 \) in the geometric construction of Section 7 via the isomorphism \( T_{n,1} \xrightarrow{\sim} \hat{A}^{(Z)}, 1 \mapsto x_i \) (if \( l = 1 \) then \( X \) and \( Y \) are zero dimensional).

8.2. Put \( \mathcal{O}^l = (\mathbb{A}^{(Z)})^{\otimes l} \). For any sequence \( i = (i_1,i_2,\ldots,i_l) \in \mathbb{Z}^l \) set \( \otimes x_i = x_{i_1} \otimes \cdots \otimes x_{i_l} \). On one hand \( \mathcal{O}^l \) is a left \( \hat{U}_n^- \)-module via the coproduct \( \Delta \). On the other hand \( \hat{H}_l \) acts on \( \mathcal{O}^l \) as follows for all \( k = 1,2,\ldots,l-1 \) and \( j = 1,2,\ldots,l \):

(a) \( (\otimes x_i)T_k = \begin{cases} v^{-2} x_i & \text{if } i_k = i_{k+1} \\ v^{-1} x_{(i)s_k} & \text{if } -n < i_k < i_{k+1} \leq 0 \\ v^{-1} x_{(i)s_k} + (v^{-2} - 1) x_i & \text{if } -n < i_{k+1} < i_k \leq 0, \end{cases} \)

(b) \( (\otimes x_i)X_j^{-1} = \otimes x_{(i)\epsilon_j} \).

Lemma. The representations of \( \hat{U}_n^- \) and \( \hat{H}_l \) on \( \mathcal{O}^l \) commute.

Proof. Since the coproduct is coassociative (see [G1, Theorem 1(ii)]), we are reduced to the case \( l = 2 \). By definition

\[
\Delta(f_\alpha) = \sum_{\beta+\gamma = \alpha} v^{n(\gamma,\beta)} f_\beta k_\gamma \otimes f_\gamma.
\]

Thus \( f_\alpha \) acts on \( \mathcal{O}^2 \) as

\[
\begin{align*}
f_x \otimes 1 + k_x \otimes f_x & \quad \text{if } \alpha = \epsilon_i \\
f_x \otimes f_j + f_x \otimes f_j & \quad \text{if } \alpha = \epsilon_i + \epsilon_j \quad \text{and} \quad i \neq j \\
f_x \otimes f_x & \quad \text{if } \alpha = 2\epsilon_i \\
0 & \quad \text{else.}
\end{align*}
\]
The commutation results from a direct computation. □

8.3. Lemma. The map $\otimes x_i \mapsto v^{l(\omega)} e_1$, for all $\mathbf{i} \in A_l^n$, extends uniquely to an isomorphism of $U_n^- \times \hat{H}_l$-bimodules $\otimes v^{-l} \sim T_{n,l}$.

Proof. The map above extends uniquely to an isomorphism of $\hat{H}_l$-modules. Let us prove that this isomorphism commutes to $U_n^-$. For any $\mathbf{i} \in \mathbb{Z}^l$ Lemma 8.1 and formula (c) give

$$f_\alpha(\otimes x_i) = \sum_n v^{c(1,1+n)} \otimes x_{i+n},$$

where $n = (n_1, n_2, ..., n_l) \in \{0, 1\}^l$ describes the set of all sequences such that $\alpha = \sum_{s=1}^l n_s \epsilon_{i_s}$ and

$$c(i, i + n) = - \sum_{1 \leq s < t \leq l} n_t(1 - n_s)n(\epsilon_{i_t}, \epsilon_{i_s}).$$

By Proposition 7.4, after the specialisation $v = q^{-1}$ we have $q^{-l(\omega)}f_\alpha(e_1) = (\Phi \circ \Theta)(f_\alpha) \ast 1_l$. The R.H.S. is the convolution product of $1_l$ and a function supported by the set of all pairs $(L', L)$ such that for all $i$ we have

$$(d) \quad L'_i \subseteq L_i \subseteq L'_{i+1}, \quad \dim_F(L_i/L'_i) = \alpha_i, \quad \text{and} \quad \dim_F(L_i/L_{i-1}) = \# i^{-1}(i).$$

By definition of the convolution product $(\Phi \circ \Theta)(f_\alpha) \ast 1_l$ (simply denoted by $f_\alpha(1_l)$) is a linear combination of the $1_l$’s such that it exists $L \in Y$ such that $(L, L_\emptyset) \in X_1$ and $(L_j, L)$ satisfies (d). Suppose first that $\mathbf{i} \in A_l^n$. Then $X_1 \cap (Y \times \{L_\emptyset\}) = \{(L_i, L_\emptyset)\}$. Thus $f_\alpha(1_l)$ is a linear combination of the $1_l$’s such that

$$(e) \quad i \leq j \leq i + 1 \quad \text{and} \quad \alpha_i = \#(i^{-1}(i) \cap j^{-1}(i + 1)).$$

More precisely we get

$$f_\alpha(1_l) = \sum_n q^{-a(i+n, l)-l(\omega)+l(\omega_i+1)} 1_{i+n},$$

where the $n$’s are as above and $a(i+n, l) = \sum_i \alpha_i(\# i^{-1}(i + 1) - \alpha_{i+1})$. A simple computation using (e) gives

$$a(i+n, l) = \sum_{s, t=1}^l n_t(1 - n_s)\delta_{i_t+1, i_s}.$$

Moreover for any sequence $j$ we have $l(\omega_l) = \dim(P_j/(B \cap P_j))$ where $P_j, B \subseteq G$ are the isotropy subgroup of $L_j$ and $L_\emptyset$. Thus we obtain

$$l(\omega_{i+n}) - l(\omega_i) = \sum_{1 \leq t < s \leq l} n_t(1 - n_s)\delta_{i_t, i_s+1}(1 - \delta_{i_t, 0}) +$$

$$+ \sum_{1 \leq t < s \leq l} n_s(1 - n_t)\delta_{i_t, i_s-1} - \sum_{1 \leq t < s \leq l} n_s(1 - n_t)\delta_{i_t, i_s}.$$
To conclude it suffices to compute the image of $\otimes x_{i+n}$ in $\mathbb{C}_G(Y \times X)$. Using the identity

$$X_k^{-1} = \tilde{T}_{k-1}^{-1} \cdots \tilde{T}_1^{-1} P \tilde{T}_{l-1} \cdots \tilde{T}_k$$

and Lemmas 7.3 and 8.2, we get that $\otimes x_{i+n}$ is mapped to

$$q^{-d(i,i+n)}(i+n) \rightarrow 1 \begin{cases} 1 \leq s, t \leq l | i_s = 0, i_t = 1 - n, n_s = 1, n_t = 0 \end{cases}.$$ 

Then, the equality results from an easy computation. The general case (i.e. $i \notin A^0_l$) follows since the isomorphism we consider commutes to $	ilde{\mathbf{H}}_l$.

8.4. Let $\psi$ be the semilinear involution on $T_{n,t} \simeq \otimes^l$ such that $\psi(e_t) = \tilde{e}_t$ for all $t \in \tilde{\mathbf{H}}_l$. Proposition 7.6 and the definition of the involutions $\phi$ and $\psi$ imply the following lemma (see Subsection 7.6).

**Lemma.** For all $u \in U^r_n$ and all $t \in \otimes^l$ we have $\psi(ut) = \tilde{u}\psi(t)$. \hfill $\square$

9. The action of $U^r_n$ on wedges.

9.1. Set $\Omega^l = \sum_i \text{Im} (T_i + 1) \subset \otimes^l$. We have

$$\otimes^l / \Omega^l \simeq \bigoplus_{i \in A^+} e_i \tilde{H}_l e^-,$$

where $e^- = \sum_{x \in \Omega^l} (-v)^{l(x)} \tilde{T}_x$. For any $i \in \mathbb{Z}^l$ let $\Delta x_i$ be the image of $\otimes x_i$ in $\otimes^l / \Omega^l$. Set

$$P^{++} = \{ i \in \mathbb{Z}^l | i_1 > i_2 > ... > i_l \}.$$

The $\Delta x_i$’s such that $i \in P^{++}$ form a basis of $\otimes^l / \Omega^l$ (see [KMS, Proposition 1.3]). For any $\lambda \in \Pi_l$ set $\{ \lambda \} = \Delta x_l$ if $i = \lambda + \rho$, where $\rho$ is as in Section 1.1. Let $\wedge^l \subset \otimes^l / \Omega^l$ be the linear span of the vectors $\{ \lambda \}$.

9.2. The representation of $U^r_n$ on $\otimes^l$ descends to $\otimes^l / \Omega^l$ (use Lemma 8.2). For all $\lambda \in \Pi_l$ set $b_{\lambda} = b_{O_{\lambda}}(\emptyset)$ and put $B_l = \{ b_{\lambda} | \lambda \in \Pi_l \}$. Let consider the involution $\psi$ on $\otimes^l / \Omega^l$ such that

$$\psi(e_{i} h e^-) = v^{2t(\omega)} \tilde{e}_i h e^- \quad \forall h \in \tilde{\mathbf{H}}_l.$$ 

**Proposition.** $B_l$ is a basis of $\wedge^l$ whose elements are fixed by $\psi$.

**Proof.** Lemma 8.1, the definition of $\Delta$, and the normal ordering rule [KMS, (43) and (45)] imply that for any $\lambda \in \Pi_l$ and any orbit $O \subset O_{\lambda} \setminus O_{\lambda}$ we have

$$f_{O_{\lambda}}(\emptyset) \in v^2|\lambda \rangle + \bigoplus_{\mu < \lambda} A_{\lambda} |\mu \rangle \quad \text{and} \quad f_{O}(\emptyset) \in \bigoplus_{\mu < \lambda} A_{\mu} |\mu \rangle.$$ 

Thus, $B_l$ is a basis. Now Lemma 8.4 implies that the action of $b_{O_{\lambda}}$ on $\wedge^l$ commutes to $\psi$. Since $\psi(\emptyset) = |\emptyset \rangle$ we get $\psi(b_{\lambda}) = b_{\lambda}$ for all $\lambda$. \hfill $\square$

9.3. Let $P^+_l \subset \mathbb{Z}^l$ be the subset of integral dominant weights. Let $\mathfrak{S}^{l,l}$ be the set of minimal length representatives of the cosets in $\mathfrak{S}_l \setminus \mathfrak{S}_l$. Thus $\mathfrak{S}^{l,l} = \mathfrak{S}^l \cap \mathfrak{S}^l$. 


where \( \mathcal{S}^l \) is the set of minimal length representatives of the cosets in \( \hat{\mathcal{S}}_l / \mathcal{S}_l \). For any \( x \in \hat{\mathcal{S}}_l \) let \( \tilde{x} \) be the smallest element in the double coset \( \mathcal{S}_l x \mathcal{S}_l \). Set
\[
\mathcal{S}(i, l) = \{ x \in \mathcal{S}^l | S_i x \cap x S_i = \emptyset \}.
\]

**Lemma.** Fix \( i \in A_1^n \). Then,
\[
\begin{align*}
(a) & \quad e_1 \tilde{T}_x e^- \neq 0 \Rightarrow x \in \mathcal{S}_l \mathcal{S}(i, l) \mathcal{S}_l, \\
(b) & \quad x \in \mathcal{S}^l \ \& \ (i)x \in P_{i}^{++} \Rightarrow e_1 \tilde{T}_x e^- = v^{-(\omega_i)} \wedge x(i)x, \\
(c) & \quad (i)x \in P_{i}^{++} \iff x \in \mathcal{S}_l \mathcal{S}(i, l) \omega.
\end{align*}
\]

**Proof.** Suppose that \( x \in \mathcal{S}^l, s_i \in S_l, \) and \( s \in S_l, \) are such that \( s_i x = xs \). Then
\[
v^{-1} e_1 \tilde{T}_x e^- = e_1 \tilde{T}_s \tilde{T}_x e^- = e_1 \tilde{T}_s \tilde{T}_x e^- = -ve_1 \tilde{T}_x e^-.
\]
Thus \( e_1 \tilde{T}_x e^- = 0 \). Any \( x \in \hat{\mathcal{S}}_l \) decomposes in \( x = x_i \tilde{x} x_l \) where \( x_i \in \mathcal{S}_i, \tilde{x} \in \mathcal{S}^l, \)
\( x_l \in \mathcal{S}_l, \) and \( l(x) = l(x_i) + l(\tilde{x}) + l(x_l) \). In particular
\[
e_1 \tilde{T}_x e^- = v^{-(x_i)} (-v)^{l(\tilde{x})} e_1 \tilde{T}_x e^-.
\]
Claim (a) follows. Let us prove claim (b). Recall that if \( \lambda \) is dominant then \( \tilde{T}_\lambda^{-1} \) is mapped to \( X^\lambda = X_{\lambda}^{\lambda_1} X_{\lambda}^{\lambda_2} \cdots X_{\lambda}^{\lambda_l} \) by the Bernstein isomorphism. Then (8.2.b) implies that
\[
(\otimes_{\mathcal{S}_i}) \tilde{T}_\lambda = \otimes_{x(i)} \lambda \ \forall \lambda \in P_{i}^{++} \ \forall i \in \mathcal{Z}^l,
\]
Moreover (8.2.a) implies that
\[
(\otimes_{\mathcal{S}_i}) \tilde{T}_x = \otimes_{x(i)} x \ \forall x \in \mathcal{S}_l \cap \mathcal{S}_l \ \forall i \in A_1^n.
\]
Fix \( x \in \hat{\mathcal{S}}_l \). Then \( x \) decomposes uniquely as \( x = y \lambda \) where \( y \in \mathcal{S}_l \) and \( \lambda \in \mathcal{Z}^l \). If
\[
(i)x = (i)y + n \lambda \in P_{i}^{++} \text{ then } \lambda \in P_{i}^{+}.
\]
Since \( s \lambda > \lambda \) for all \( s \in S_l \) if \( \lambda \) is dominant, we get \( \tilde{T}_x = \tilde{T}_y \tilde{T}_\lambda \). Suppose moreover that \( x \in \mathcal{S}^l \). Then for any \( s \in S_l \) we have
\[
sy \lambda > y \lambda.
\]
Since \( l(z \lambda) = l(z) + l(\lambda) \) for any \( z \in \mathcal{S}_l \) (\( \lambda \) is dominant), we obtain that
\( y \in \mathcal{S}_l \cap \mathcal{S}_l \). Hence, Section 8.2 implies that
\[
e_1 \tilde{T}_x e^- = e_1 \tilde{T}_y \tilde{T}_\lambda e^- = v^{-(\omega_i)} \wedge x(i)x.
\]
Finally, claim (c) follows from
\[
\begin{align*}
(x \in \mathcal{S}^l \ \& \ (i)x \in P_{i}^{++}) \iff (s_i x > x > xs \ \forall s \in S_l \ \forall s_i \in S_l) \\
\iff x \in \mathcal{S}(i, l) \omega.
\end{align*}
\]
\[\square\]

**Proposition.** If \( i \in A_1^n \) and \((i)x \in P_{i}^{++} \) then \( \psi(\otimes_{x(i)} x) = (-1)^{(\lambda)(\omega)} v^{(\omega)(\omega)} \wedge x(i)x \omega \).

**Proof.** First recall that if \( \lambda \in P_{i}^{+} \) and \( \lambda^* = -\omega(\lambda) \), then \( \tilde{T}_\lambda = \tilde{T}_\omega \tilde{T}_\lambda \tilde{T}_\omega \) (indeed,
since \( \lambda, \lambda^* \) are dominant weights we have \( \tilde{T}_\omega \tilde{T}_\lambda = \tilde{T}_{-\lambda} \tilde{T}_\omega = \tilde{T}_{-\lambda} \tilde{T}_\omega \). In particular,
\[ \tilde{T}_{-\lambda} = \tilde{T}_\omega \tilde{T}_\lambda \tilde{T}_\omega^{-1}. \] Fix \( i \in A^+_n \) and \( x \in \mathfrak{S}^i \) such that \((i)x \in P^+_l \). As above fix \( x = y\lambda \) with \( y \in \mathfrak{S}_l \cap \mathfrak{S}^i \) and \( \lambda \in P^+_l \). Using Lemma 9.3.b we get

\[
\psi(\wedge x_{(i)x}) = \psi((\otimes x_i) \tilde{T}_y \tilde{T}_\lambda e^-) = v^{l(\omega_i)} \psi(e_i \tilde{T}_y \tilde{T}_\lambda e^-) = v^{l(\omega_i)+2l(\omega)} e_i \tilde{T}_y^{-1} \tilde{T}_{-\lambda}^{-1} \tilde{T}_\lambda e^- =
\]

\[
v^{l(\omega_i)+2l(\omega)} e_i \tilde{T}_y^{-1} \tilde{T}_\lambda \tilde{T}_{-\lambda}^{-1} e^- = (-1)^{l(\omega)} v^{2l(\omega)-l(\omega^i)} e_i \tilde{T}_y \tilde{T}_{-\lambda}^{-1} e^-.
\]

For all \( s \in \mathfrak{S}_l \), \( sy > y \) implies that \( sy \omega < y \omega \). Thus

\[
\psi(\wedge x_{(i)x}) = (-1)^{l(\omega)} v^{l(\omega^i)} (\otimes x_{(i)y\omega}) \tilde{T}_{-\lambda}^{-1} e^- = (-1)^{l(\omega)} v^{l(\omega^i)} \wedge x_{(i)y\omega}.
\]

\[ \square \]

9.4. Fix \( x \in \mathfrak{S}^i \). The element

\[ D_{x\omega} = \sum_{y \in \mathfrak{S}^i, y \leq x} (-v)^{l(y) - l(x)} P_{y\omega, x\omega} \tilde{T}_{y\omega} e^- \]

is fixed by the involution on \( \tilde{\mathfrak{H}}_l e^- \) such that \( h e^- \mapsto v^{2l(\omega)} h e^- \) and

\[ D_{x\omega} - \tilde{T}_{x\omega} e^- \in \bigoplus_{y \in \mathfrak{S}^l} v^{-1} \mathfrak{S} \tilde{T}_{y\omega} e^- \]

(see [D1]). If \( i \in A^+_n \) and \( x \in \mathfrak{S}(i, l) \) set \( b^+_{(i)x\omega} = v^{l(\omega^i)} e_i D_{x\omega} \). Lemma 9.3 gives

\[ b^+_{(i)x\omega} = \sum_{y \in \mathfrak{S}^i, y \leq x} (-v)^{l(y) - l(x)} v^{-l(y)} P_{y\omega, x\omega} \wedge x_{(i)y\omega} \]

Similarly fix \( i \in A^+_n \) and \( x \in \mathfrak{S}^i \). Then

\[ C'_{\omega^i x} = v^{l(x) + l(\omega^i)} \sum_{y \in \mathfrak{S}^i, z \in \mathfrak{S}_l} \sum_{y \leq x} P_{zy, \omega x} T_{zy}. \]

If \( y \in \mathfrak{S}^i \) and \( y \leq x \) then \( P_{zy, \omega x} = P_{zy, \omega x} \) for all \( z \in \mathfrak{S}_l \) (see [D1, page 491]). Thus

\[ C'_{\omega^i x} = v^{l(x) + l(\omega^i)} e_i \sum_{y \in \mathfrak{S}^i, y \leq x} P_{zy, \omega x} T_y. \]

If \( x \in \mathfrak{S}(i, L) \) set \( b^+_{(i)x\omega} = (-v)^{l(\omega)} C'_{\omega^i x} e^- \). Then Lemma 2.2 gives

\[ b^+_{(i)x\omega} = \sum_{y, z} v^{l(x) - l(yz)} (-v)^{l(z)} P_{yz, \omega x} \wedge x_{(i)y\omega} = \sum_{y \in \mathfrak{S}(i, L)} v^{l(x) - l(y)} Q_{yz, \omega x} \wedge x_{(i)y\omega}, \]

where the first sum is over all couples \((y, z) \in \mathfrak{S}(i, L) \times \mathfrak{S}_l \) such that \( yz \leq x \) and \( Q_{yz, \omega x} = \sum_z (-1)^{l(z)} P_{yz, \omega x} \) is a parabolic Kazhdan-Lusztig polynomial. Observe that \( b^\pm_{(i)x\omega} \) is completely characterized by the following properties:

\[ \psi(b^\pm_{(i)x\omega}) = b^\pm_{(i)x\omega}, \quad b^+_{(i)x\omega} \wedge x_{(i)x\omega} \in \bigoplus_{y \in \mathfrak{S}(i, L)} v^{-1} \mathfrak{S} \wedge x_{(i)y\omega}, \]
and \( b^+_{(i)x_\omega} \wedge x_{(i)x_\omega} \in \bigoplus_{y \in \mathfrak{S}(\Pi_i)} vS \wedge x_{(i)y_\omega} \).

In particular \( \{b^-_i \mid i \in P^{++}_l\} \) and \( \{b^+_i \mid i \in P^{++}_l\} \) are bases of \( \mathfrak{S}_\Omega^l \). For all \( \lambda \in \Pi_l \) set \( b^+_\lambda = b^+_i \) if \( i = \lambda + \rho \). Put \( B^+_l = \{b^+_\lambda \mid \lambda \in \Pi_l\} \).

Remark. If \( i \in A^n_l \) and \( x, y \in \mathfrak{S}_l \) are such that if \( (i)x, (i)y \in P^{++}_l \) then \( y \leq x \Rightarrow (i)x - (i)y \) is a positive root.

9.5. The space \( \wedge^l \) is endowed with four bases: \( B^+_l = \{b^+_\lambda \mid \lambda \in \Pi_l\} \), \( B_l = \{b_\lambda \mid \lambda \in \Pi_l\} \), and \( \{\langle \lambda \rangle \mid \lambda \in \Pi_l\} \). Moreover, \( B^+_l \) are characterized by

\[
\psi(b^+_\lambda) = b^+_\lambda, \quad b^-_\lambda - \langle \lambda \rangle \in \bigoplus_{\mu < \lambda} v^{-1}S|\mu\rangle \quad \text{and} \quad b^+_\lambda - \langle \lambda \rangle \in \bigoplus_{\mu < \lambda} vS|\mu\rangle
\]

(in particular \( \psi(\wedge^l) = \wedge^l \)). Recall that if \( x \in \mathfrak{S}_l \) and \( \lambda \in \mathbb{Z}^l \), then \( \lambda \cdot x = (\lambda + \rho)x - \rho \) (see 1.1). Section 9.4 implies the following theorem.

Theorem. (a) If \( \lambda \in \Pi_l \) and \( x \) is minimal such that \( i = \lambda \cdot x^{-1} + \rho \in A^n_l \), then

\[
b^-_\lambda = \sum_y (-v)^{l(y) - l(x)}v^{-l(\mu)}P_{y_\omega}|\lambda \cdot x^{-1}y\rangle,
\]

where the sum is over all the \( y \) such that \( y \leq x \) and \( \lambda \cdot x^{-1}y \in \Pi_l \).

(b) For all \( \lambda \in \Pi_l \) the coordinates of \( b^+_\lambda \) in the wedges are some parabolic Kazhdan-Lusztig polynomials (w.r.t. the parabolic subgroup \( \mathfrak{S}_l \subset \mathfrak{S}_l \)). \( \square \)

Now, suppose that \( l \leq n \). Let consider \( i, j \in A^n_l \), \( n \in M^+ \) and \( m = n^j \in M_{ji} \cap M^+ \).

If \( x \in \mathfrak{S}_l \) is such that \( (i)x \in P^{++}_l \) then Section 7 gives

\[
f_{O_n}(\wedge^x_{(i)x}) = v^{l(\omega_j) + y(m)}T_{m}e^- \in e_j\hat{H}_le^-.
\]

In particular if \( i = (\rho)\omega \in A^n_l \) and \( x = \omega \) we get

\[
f_{O_n}(\wedge^x_{(\rho)\omega}) = v^{l(\omega_j)}e_j\hat{T}_me^-,
\]

where \( m = \mathfrak{m} \) is the smallest element. Let suppose that \( f_{O_n}(\wedge^x_{\mathfrak{m}}) \neq 0 \). Let \( \mathfrak{S}_l \) be the set of minimal length representatives of the cosets in \( \mathfrak{S}_l/\mathfrak{S}_l \). Then Lemma 9.3 implies that \( m = yt \) with \( t \in \mathfrak{S}_l \) and \( y \in \mathfrak{S}_l \cap \mathfrak{S}_l^{-} \) such that \( S_{jy} \cap yS_l = \emptyset \). Then,

\[
f_{O_n}(\wedge^x_{\mathfrak{m}}) = v^{l(\omega_j)}e_j\hat{T}_ye^- = v^{l(\omega_j)}e_j\hat{T}_y\hat{T}_w\hat{T}_we^- = (-v)^{l(t)}\wedge x_{(j)y_\omega} \in \bigoplus_{\lambda} \mathfrak{S}|\lambda\rangle.
\]

As a consequence if \( l \leq n \) then \( B^+_l = B_l \).

Conjecture. The bases \( B_l \) and \( B^+_l \) coincide for all \( l \). \( \square \)

10. Proof of Theorem 6.3.
10.1. Let $\bigotimes^\infty$ be the free $\mathbb{A}$-module linearly generated by the semi-infinite monomials
\[ \bigotimes x_i = x_{i_1} \otimes x_{i_2} \otimes x_{i_3} \otimes \cdots \]
where $i = (i_1, i_2, \ldots)$ is a sequence of integers such that $i_k = 1 - k$ for $k \gg 1$.

The affine Hecke algebra of type $\mathfrak{g}l_\infty$ acts on $\bigotimes^\infty$ via formulas (8.b) and (8.c). Set $\Omega^\infty = \sum_i \text{Im} (T_i + 1) \subset \bigotimes^\infty$. As above $\wedge x_i$ is the class of $\otimes x_i$ in $\bigotimes^\infty/\Omega^\infty$. The formulas in Section 8 and [KMS, Lemma 2.2] imply that for all $\alpha \in \mathbb{N}/\mathbb{Z}$ we have

\[ \forall i \quad \exists l \in \mathbb{N}_\times \quad \text{such that} \quad f_\alpha(\wedge x_i) = f_\alpha(x_{i_1} \wedge \cdots \wedge x_{i_l} \wedge x_{i_{l+1}} \wedge x_{i_{l+2}} \wedge \cdots) \]

Thus the action of $U^-_n$ on $\wedge^l$ induces an action on $\wedge^\infty$.

**Lemma.** The map $|\lambda\rangle \mapsto \wedge x_i$, where $i_k = 1 + \lambda_k - k$, gives an embedding of the representation of $U^-_n$ on $\wedge^\infty$ given in Section 6 into $\bigotimes^\infty/\Omega^\infty$.

**Proof.** The proof goes by a direct computation. First observe that $\wedge x_i$ and $|\lambda\rangle$ have the same weight for any $\lambda \in \Pi$ if $i$ is the sequence such that $i_k = 1 + \lambda_k - k$. Fix $\lambda \in \Pi$ and $\alpha \in \mathbb{N}/\mathbb{Z}$. Fix $i$ as above. Formula 6.2.a gives

\[ f_\alpha(\wedge x_i) = f_\alpha(|\lambda\rangle) = \sum_{\text{a s.t. } \bar{a} = \alpha} \gamma_a (f_\alpha) \prod_{j < \ell} k_{a,j}^j (|\lambda\rangle). \]

Moreover Remark 6.1 implies that $\gamma_a(f_\alpha)$ is $v^{(a)}$ times the product of the $f_i^{(a)}$'s ordered from $i = -\infty$ to $\infty$. Using the formulas in Section 4 we first observe that $f_i^{(2)}$ acts by zero on the Fock space for any $i$. The elements $|\lambda\rangle$ and $\wedge x_i$ have the same weight with respect to $k_i$. Thus we get

\[ f_\alpha(\wedge x_i) = \sum_n e^{c(i,i+n)} \wedge x_{i+n}, \]

where $n = (n_1, n_2, \ldots) \in \{0,1\}^\times$ describes the set of all sequences such that $\alpha = \sum_{a \geq 1} n_a \epsilon_{i_a}$ and

\[ e(i,i+n) = \sum_{i_k > i_l} n_l \delta_{i_l,i_k} - \sum_{i_k > 1 + i_l} n_l \delta_{i_l+1,i_k} - \]
\[ - \sum_{i_k < i_l} n_l n_k \delta_{i_l,i_k} + \sum_{1 + i_k < i_l} n_l n_k \delta_{i_l+1,i_k}. \]

If $\wedge x_{i+n} \neq 0$ then $e(i,i+n) = \sum_{i_k > i_l} n_l (1 - n_k) (\delta_{i_l,i_k} - \delta_{i_l+1,i_k})$. On the other hand the formula in Section 8.3 gives

\[ f_\alpha(\wedge x_i) = \sum_n e^{c(1,i+n)} \wedge x_{i+n}, \]

where $n$ describes the same set and

\[ c(i,i+n) = - \sum_{1 \leq k < l} n_l (1 - n_k) n(\epsilon_{i_l}, \epsilon_{i_k}). \]
We are through (recall that \( i \) is decreasing). \( \square \)

Theorem 6.3 follows from Proposition 9.2 and Lemma 10.1.

10.2. The involution \( \psi \) on \( \bigwedge^i \) induces the semilinear involution \( \psi \) on \( \bigwedge^\infty \) such that,

\[
\forall i, \ l \geq \sum_k (i_k - 1 + k) \Rightarrow \psi(\wedge x_i) = \psi(x_{i_1} \wedge \cdots \wedge x_{i_l}) \wedge x_{i_{l+1}} \wedge x_{i_{l+2}} \wedge \ldots
\]

Proposition 9.3 implies that \( \psi \) coincides with the involution on \( \bigwedge^\infty \) used in [LT]. In [LT] Leclerc and Thibon have defined two bases \( B^\pm = \{ b^\pm_\lambda | \lambda \in \Pi \} \) in \( \bigwedge^\infty \) such that for all \( \lambda \)

\[
\psi(b^\pm_\lambda) = b^\pm_\lambda, \quad b^\pm_\lambda - |\lambda| \in \bigoplus_{\mu < \lambda} v^{-1} S(\mu) \quad \text{and} \quad b^+_\lambda - |\lambda| \in \bigoplus_{\mu < \lambda} v S(\mu).
\]

Thus, Conjecture 9.5 is equivalent to

**Conjecture.** The bases \( B \) and \( B^+ \) coincide. \( \square \)

Set \( b_\lambda = \sum_\mu d_{\mu \lambda} |\mu| \) and \( b^+_\lambda = \sum_{\mu} c^+_{\mu \lambda} |\mu| \). Conjecture 10.2 is precisely \( d_{\lambda \mu} = c^+_{\lambda \mu} \).

11. Proof of the Decomposition Conjecture.

Let \( \varepsilon \) be a \( n \)-th root of unity. The quantized Schur algebra \( S_{n,I} \) is the subalgebra of \( \hat{\mathcal{S}}_{n,I} \) spanned by the elements \( T_m \) with \( m \in \mathcal{S}_l \setminus \mathcal{S}_l / \mathcal{S}_j \) (see Subsection 7.4). Fix \( l \leq k \). Let consider the subalgebra

\[ S_l = S_{k,l} \cap \bigoplus_{\lambda, \mu \in \Pi(l)} \mathcal{H}_{i_\lambda i_\mu} \]

where \( i_\lambda = (1 - k)^{\lambda_1} (2 - k)^{\lambda_2} \ldots 0^{\lambda_k} \) for any \( \lambda = (\lambda_1 \geq \lambda_2 \geq \ldots) \in \Pi(l) \). We want to compute the decomposition matrices of the simple \( S_l \)-modules under the specialization \( v \mapsto \varepsilon \). The algebra \( S_{k,l} \) is Morita equivalent to \( S_l \). For any \( i \in \mathbb{C}^* \) let \( S_{l,I} \) be the specialization of \( S_l \) at \( v = i \). The simple modules of \( S_{l,I} \) are parametrized by \( \Pi(l) \). For any \( k \) let \( \mathcal{U}(gl_k) \) be the Lusztig integral form of the quantized enveloping algebra of \( gl_k \) and let \( \mathcal{U}_\varepsilon(gl_k) \) be the specialization at \( v = \varepsilon \). The set \( \Pi_k \) is identified with the set of dominant weights of \( gl_k \) with non-negative components. If \( \lambda \in \Pi_k \), let \( V_\Lambda \) and \( W_\Lambda \) be respectively the simple and the Weyl \( \mathcal{U}_\varepsilon(gl_k) \)-module with highest weight \( \lambda \). There exists a surjective map \( \pi : \mathcal{U}_\varepsilon(gl_k) \twoheadrightarrow S_{k,l|\varepsilon} \) (see [D2]). If \( \lambda \in \Pi(l) \) let \( L_\Lambda, M_\Lambda \) be the simple and the Specht \( S_{k,l|\varepsilon} \)-modules such that

\[ \pi^*[L_\Lambda] = [V_\Lambda] \quad \text{and} \quad \pi^*[M_\Lambda] = [W_\Lambda] \]

in the Grothendieck ring.

**Theorem.** The specialization at \( v = 1 \) of the matrix \((e^+_{\lambda \mu})_{\lambda, \mu \in \Pi(l)}\), \( \lambda, \mu \in \Pi(l) \), is the decomposition matrix of the Specht modules of \( S_l \).

**Proof.** The Lusztig conjecture (proved by Kashiwara-Tanisaki and Kazhdan-Lusztig) gives the multiplicity of \( W_\mu \) in \( V_\lambda \). More precisely we have

\[ [V_\lambda : W_\mu] = \sum_y (-1)^y P_{yx}(1), \]
where $x \in \hat{\mathfrak{g}}_l$ is minimal such that $\nu = \lambda \cdot x^{-1}$ satisfies
\[
\nu_i < \nu_{i+1} \quad \forall i = 1, 2, \ldots, k - 1, \quad \nu_1 - \nu_k \geq 1 - k - n,
\]
and $\mu = \lambda \cdot x^{-1} y$. According to Theorem 9.5.a, the Lusztig Conjecture is equivalent to
\[
(a) \quad [L_{\lambda'}] = \sum_{\mu} e_{\lambda\mu}^-(1) [M_{\mu'}], \quad \forall \lambda \in \Pi_k.
\]
Recall that $(e_{\lambda\mu}^+)_{\lambda\mu} = (e_{\lambda'\mu'}^-)_{\lambda\mu}^{-1}$ (see [LT, Section 4]). Thus
\[
(a) \iff [M_{\lambda}] = \sum_{\mu} e_{\lambda\mu}^+(1) [L_{\mu}].
\]
\[\square\]

12. The Lusztig conjecture.

12.1. Let $F$ be the variety of partial flags in $\mathbb{C}^l$ of the type
\[
\{0\} \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_k = \mathbb{C}^l.
\]
The linear group $GL_l$ acts diagonally on $F \times F$. Let $Z \subseteq T^*F \times T^*F$ be the Steinberg variety ($Z$ is a reducible variety whose irreducible components are the closure of the conormal bundles to the $GL_l$-orbits in $F \times F$). The group $G = GL_l \times \mathbb{C}^\times$ acts naturally on $Z$ : the linear group acts diagonally and $z \in \mathbb{C}^\times$ acts by multiplication by $z^{-2}$ along the fibers. The complexified Grothendieck group of equivariant coherent sheaves on $Z$, denoted by $K_{k,l}$, is endowed with an associative convolution product (see [GV], [V2]) denoted by $\ast$. For any $z \in \mathbb{C}^\times$, a parametrization of the simple modules of the specialized algebra $K_{k,l}|_{z=\bar{z}}$ is given in [GV] (see in [V2] the remark after Theorem 4 for the case of roots of unity): the simple modules are labelled by orbits of pairs $(s, x) \in GL_l \times \mathfrak{gl}_l$ where $s$ is semi-simple, $x^k = 0$, and $ss^{-1} = z^{-2}x$. As usual the $GL_l$-orbit of $x$ is labelled by the partition $\lambda \in \Pi(l)$ such that $\lambda_i$ is the length of the $i$-th Jordan block of $x$. Then $\lambda' \in \Pi(l) \cap \Pi_k$. The orbits of the pairs $(s, x)$ such that the spectrum of $s$ is in $z^{2\mathbb{Z}}$ are labelled by isomorphism class of nilpotent representations of $\Gamma_\infty$ if $z$ is generic and of $\Gamma_n$ if $z = \varepsilon$ (recall that $\varepsilon^2$ is a primitive $n$-th root of unity). Let $\Omega_{k,l}$ and $\Omega_{k,l}^\infty$ be the corresponding sets of isomorphism classes of representations of $\Gamma_n$ and $\Gamma_\infty$. If $O \in \Omega_{k,l}^\infty$ (resp. $O \in \Omega_{k,l}$) let $L_O^\infty$ (resp. $L_O$) be the simple $K_{k,l}$-module labelled by $O$. Similarly let $M_O^\infty$ and $M_O$ be the standard modules labelled by $O$ (see [V2]). Let $[M]$ be the class of the module $M$ in the complexified Grothendieck ring. Let $\hat{\mathfrak{g}}_l$ and $\hat{\mathfrak{g}}_l^\infty$ be the linear span of the elements $[L_O]$ and $[L_O^\infty]$ where $O \in \Omega_{k,k}$ or $O \in \Omega_{k,k}^\infty$ and $k \geq 1$. The restricted dual $\hat{\mathfrak{g}}_l^\ast$ (resp. $\hat{\mathfrak{g}}_l^\ast^\infty$) is spanned by the linear forms $l_O$ (resp. $l_O^\infty$) such that
\[
l_O([L_{O'}]) = \delta_{O,O'} \quad \text{and} \quad l_O^\infty([L_{O'}^\infty]) = \delta_{O,O'}.
\]

12.2. The quantized enveloping algebra of $\hat{\mathfrak{g}}_k$ is generated by elements $e_{i,s}$, $f_{i,s}$, $h_{j,t}$ and $k_j^{\pm 1}$ ($0 < i < k$, $0 < j \leq k$, $s \in \mathbb{Z}$, $t \in \mathbb{Z}^\times$) which satisfy the relations...
The spaces formula $\text{GV}$, Theorem 6.6 implies that there are two linear isomorphisms $f$. To avoid confusions let $\text{basis of the spaces}$ $\text{First observe that the classes of the standard modules}$ $\Psi_{k,l} : U(\widehat{\mathfrak{gl}}_k) \otimes_k \mathbb{C}(v) \to K_{k,l} \otimes_k \mathbb{C}(v)$. It is proved in $[S]$ that $\Psi_{k,l}$ restricts to a surjective homomorphism $U(\widehat{\mathfrak{gl}}_k) \to K_{k,l}$. Observe that the restriction of a simple $U_z(\widehat{\mathfrak{gl}}_k)$-module to $U_z(\widehat{\mathfrak{sl}}_k)$ is simple. Thus $\Psi_{k,l} L_O$ for $O \in \Omega_{k,l}$ (resp. $\Psi_{k,l} L_O^{-\infty}$ for $O \in \Omega_{k,l}^{-\infty}$), may be viewed as a simple $U_z(\widehat{\mathfrak{sl}}_k)$-module when $z = \varepsilon$ (resp. $z$ generic).

Recall that there is an algebra homomorphism $ev : U(\widehat{\mathfrak{sl}}_k) \to U(\mathfrak{gl}_k)$ such that

$$ev(f_0) = (-1)^k v^{k-1} \{e_{k-1}, \{e_{k-2}, \{e_2, e_1\} \} \} k^{-1} k_{k-1}$$

$$ev(f_i) = f_i, \quad ev(e_i) = e_i, \quad i = 1, 2, \ldots, k - 1,$$

where $\{x, y\} = xy - yx - v^{-1}yx$. If $\lambda \in \Pi_k$ let $V_\lambda$ (resp. $V_\lambda^{-\infty}$) be the simple $U_z(\mathfrak{gl}_k)$-modules with highest weight $\lambda$ where $z = \varepsilon$ (resp. $z$ generic). The Drinfeld polynomials of $L_O$ and $L_O^{-\infty}$ are computed in $[V^2]$. If $\lambda \in \Pi(k)$, then $\Psi_{k,k} L_O$ and $\Psi_{k,k} L_O^{-\infty}$ are the pull-backs of the modules $V_\lambda$ and $V_\lambda^{-\infty}$ by the evaluation map $ev$ (see [CP, Proposition 12.2.13]). Let $R_n$ and $R_\infty$ be the linear span of the classes $[V_\lambda]$ and $[V_\lambda^{-\infty}]$ for all $\lambda$ and all $k$. The restricted dual spaces $R_n^*$ and $R_\infty^*$ are spanned by the linear forms $l_\lambda$ and $l_\lambda^{-\infty}$ such that

$$l_\lambda([V_\mu]) = \delta_{\lambda\mu} \quad \text{and} \quad l_\lambda^{-\infty}([V_\mu^{-\infty}]) = \delta_{\lambda\mu}.$$

The element $[V_\lambda^{-\infty}]$ may be viewed as the class in $R_n$ of the Weyl module $W_\lambda$ with highest weight $\lambda$. Let $s_\lambda \in R_n^*$ be such that

$$s_\lambda([W_\mu]) = \delta_{\lambda\mu}.$$

12.3. In this subsection $U_n$, $U_\infty$ and $\land^{-\infty}$ stand for their specializations at $v = 1$.

**Theorem.** The linear isomorphism $R_n^* \cong \land^{-\infty}$ such that $s_\lambda \mapsto [\lambda]$ maps $l_\lambda$ to $b_\lambda$.

**Proof.** First observe that the classes of the standard modules $[M_O]$ and $[M_O^{-\infty}]$ form a basis of the spaces $\hat{R}_n$ and $\hat{R}_\infty$. Let $m_O$ and $m_O^{-\infty}$ be the elements of the dual basis. To avoid confusions let $f_O^\infty$, $b_O^\infty$, denote the generators of $U_\infty$. The multiplicity formula $[GV, Theorem 6.6]$ implies that there are two linear isomorphisms

$$\iota_n : U_n^{-} \to \hat{R}_n^* \quad \text{and} \quad \iota_\infty : U_\infty^{-} \to \hat{R}_\infty^*$$

such that

$$(a) \quad \iota_n(f_O) = m_O, \quad \iota_n(b_O) = l_O, \quad \iota_\infty(f_O^\infty) = m_O^{-\infty}, \quad \iota_\infty(b_O^\infty) = l_O^{-\infty}.$$

The spaces $R_n^*$ and $R_\infty^*$ are identified with $\land^{-\infty}$ via the maps

$$s_\lambda \mapsto [\lambda] \quad \text{and} \quad l_\lambda^{-\infty} \mapsto [\lambda].$$
We obtain the following commutative square
\[
\begin{array}{ccc}
\Lambda^\infty & \overset{\cong}{\to} & \mathbb{R}_n^* \\
U_n^- & \overset{\cong}{\to} & \mathbb{R}_\infty^* \\
\end{array}
\]
where the horizontal arrows are the dual of the specialization maps and the vertical arrows are the dual of the evaluation maps. By definition the upper arrow maps \( s_\lambda \) to \( l_\lambda^\infty \) and both elements are identified with the vacuum vector \( |\lambda\rangle \). The right vertical arrow is such that
\[
b_O^\infty = l_O^\infty \mapsto l_\lambda^\infty \text{ if } O = O_\lambda, \quad b_O^\infty \mapsto 0 \text{ else.}
\]
By Proposition 5 it is the quotient map
\[
U_n^- \to \Lambda^\infty, \quad u \mapsto u(|\emptyset\rangle).
\]
Suppose first that the lower horizontal arrow is the map \( \gamma \) introduced in Section 6. Then the left vertical arrow is the quotient map
\[
U_n^- \to \Lambda^\infty, \quad u \mapsto u(|\emptyset\rangle).
\]
Hence \((a)\) implies that the left vertical arrow maps \( l_{O_\lambda} \) to \( b_\lambda \). Since this arrow is the transpose of the evaluation map we get \( l_\lambda = b_\lambda \) and we are through. By Subsection 6.4, to prove that the map \( \hat{\mathbb{R}}_n^* \to \hat{\mathbb{R}}_\infty^* \) is \( \gamma \) we are reduced to prove that if \( r(O') \subseteq O \) then \( [M_O^\infty] \) specializes to \( [M_O] \). This is obvious by the localization theorem in equivariant K-theory.

\[12.4.\] Theorem 12.3 implies that \( V_\Lambda^\infty = \sum_{\mu} d_{\lambda'\mu'}(1) [V_\mu] \). According to Section 11 the Lusztig Conjecture can be written as
\[
[W_\lambda] = \sum_{\mu} e_{\lambda'\mu'}^+(1) [V_\mu], \quad \forall \lambda,
\]
which is precisely Conjecture 10.2.

\[13. \text{Proof of Proposition 6.1.}\]

\[13.1.\] Fix \( \Gamma = \Gamma_n \text{ or } \Gamma_\infty \). Let \( S_d \) be the set of finite sequences \( d = (d^1, d^2, \ldots, d^l) \) of elements in \( \mathbb{N}^{(l)} \) such that \( \sum_k d^k = d \). Fix a \( I \)-graded vector space \( V \) of dimension \( d \). For each \( d \in S_d \) let \( F_d \) be the set of flags of \( V \) of type \( d \), i.e. \( F_d \) is the set of filtrations \( F = (\{0\} = F^0 \subseteq F^1 \subseteq \cdots \subseteq F^l = V) \) such that \( F^k \) is \( I \)-graded and has dimension \( d^1 + d^2 + \cdots + d^k \). Given \( x \in E_V \) we say that a flag \( F \in F_d \) is \( x \)-stable if \( x(F^k) \subseteq F^{k-1} \) for all \( k \). Let \( \hat{F}_d \) be the variety of all pairs \( (x, F) \) such that \( x \in E_V \) and \( F \in F_d \) is \( x \)-stable. The group \( G_V \) acts on \( \hat{F}_d \) in the obvious way. Let \( \pi_d : \hat{F}_d \to E_V \) be the first projection. The map \( \pi_d \) commutes to \( G_V \). Thus the function \( f_d = \pi_d!(1) \) belongs to \( \mathbb{C}_{G_V}(E_V) \).

\[\text{Lemma.} \ (a) \ The \ space \ \mathbb{C}_{G_V}(E_V) \ is \ linearly \ spanned \ by \ the \ elements \ f_d \ with \ d \in S_d.\]
(b) For any \(a, b \in \mathbb{N}^{(I)}\) and any \(a \in S_a, b \in S_b\), we have \(f_a \circ f_b = q^{-m(b,a)} f_{ab}\) where \(ab \in S_{a+b}\) is the sequence \(a\) followed by the sequence \(b\).

**Proof.** Claim (b) is proved as in [L2, Lemma 3.2.b]. Let us prove claim (a). If a flag \(F\) is \(x\)-stable then \(F^k \subseteq \text{Ker}(x^k)\). Thus if \(d \in S_d\) is such that
\[
d^1 + d^2 + \cdots + d^k = \dim \text{Ker}(x^k) \quad \forall k = 1, 2, 3, ...
\]
then \(\pi_d^{-1}(x)\) is reduced to the single flag
\[
\{0\} \subseteq \text{Ker}(x) \subseteq \text{Ker}(x^2) \subseteq \cdots \subseteq V.
\]
In particular \(f_d(x) = 1\). Moreover, in this case \(f_d\) is supported on the \(G_V\)-orbits of the \(y\)'s such that
\[
\dim \text{Ker}(x^k) \leq \dim \text{Ker}(y^k) \quad \forall k = 1, 2, 3, ...
\]
i.e. \(y \in \overline{G_V \cdot x}\). We are through. \(\square\)

**Remark.** It is easy to see that for any \(d \in \mathbb{N}^{\mathbb{Z}/n\mathbb{Z}}\) and \(d = (d)\) we have \(f_d = f_d\). Thus Proposition 3.5 is a consequence of (a) and (b).

13.2. We fix a \(\mathbb{Z}\)-graded vector space \(V\) of dimension \(d\). Let \(\tilde{V}\) be the associated \(\mathbb{Z}/n\mathbb{Z}\)-graded vector space, of dimension \(\bar{d}\). The space \(\tilde{V}\) is endowed with the \(\mathbb{Z}\)-filtration whose associated graded is identified with \(V\). Fix \(d \in S_d\). We have the following commutative diagram
\[
\tilde{F}_{\bar{d}} \xrightarrow{\pi_{\bar{d}}} E_{\tilde{V}} \xrightarrow{p} E_V,
\]
where \(\tilde{F}_{\bar{d},d} = \pi_{\bar{d}}^{-1}(E_{\tilde{V},V})\) and the vertical arrows are the embeddings. We have clearly
\[
(p_j^* f_d) = (p_{\pi_d,d})!(1).
\]
Let \(S_{d,d} \subset S_d\) be the set of sequences \(d\) such that \(\sum_{i \in I} d^i_i = \bar{d}_i^i\) for any \(k\) and \(i\). If \(d \in S_{d,d}\) let \(\tilde{F}_{\bar{d},d} \subset \tilde{F}_{\bar{d},d}\) be the set of pairs \((x, F)\) such that the associated graded of \(F^k\) with respect to the filtration induced by the \(\mathbb{Z}\)-filtration on \(\tilde{V}\) has dimension \(d^1 + d^2 + \cdots + d^k\). The sets \(\tilde{F}_{\bar{d},d}\) form a partition of \(\tilde{F}_{\bar{d},d}\). We have a commutative square
\[
\tilde{F}_{\bar{d},d} \xrightarrow{p_{\pi_d,d}} E_V \xrightarrow{\pi_d} \tilde{F}_{\bar{d}},
\]
where the left vertical arrow is the inclusion and \(\tau\) maps the pair \((x, F)\) to the associated graded. Thus
\[
(p_{\pi_{\bar{d},d}})! (1) = \sum_{d \in S_{d,d}} (\pi_d \tau)! (1).
\]
Lemma. The map $\tau$ is a vector bundle of rank

$$r(d) = \sum_{k \geq 1} \sum_{i \geq j} d_k^i d_{i+1}^j + \sum_{k < 1} \sum_{i > j} d_k^i d_j^i.$$ 

Proof. The proof goes as the proof of [L2, Lemma 4.4]. More precisely fix $(x, F) \in \bar{F}_d$ and compute the fiber $\tau^{-1}(x, F)$. Giving a $\mathbb{Z}/n\mathbb{Z}$-graded subspace $\bar{F}^k \in V$ of dimension $d^1 + d^2 + \cdots + d^k$ whose associated $\mathbb{Z}$-graded is $F^k$ is the same as giving a map

$$z^k = \oplus z^k_{ij} \in \bigoplus_{i \geq j} \text{Hom}(F^k_j, V/F^k_i).$$

Then $\bar{F}^k \subset \bar{F}^{k+1}$ if and only if $z^{k+1} = z^k : F^k \to V/F^{k+1}$. On the other hand, giving $\bar{x} \in E_{V,V}$ such that $p(\bar{x}) = x$ is the same as giving a map

$$y = \oplus y_{i+1,j} \in \bigoplus_{i > j} \text{Hom}(V_j, V_{i+1}).$$

Then $\bar{F}$ is $\bar{x}$-stable if and only if

$$z^k_{i+1,j+1} \circ x_j - x_i \circ z^k_{ij} - y_{i+1,j} = 0 : F^k_j \to V_{i+1}/F^k_{i+1}.$$ 

The Lemma results from a direct computation. \qed

The Lemma and (e), (d), give

$$\gamma_d(f_a) = \sum_d \in \mathcal{S}_{\alpha,a} q^{2r(d) - h(d)} f_d.$$ 

Fix $\alpha, \beta \in \mathbb{N}^{\mathbb{Z}/n\mathbb{Z}}$, $a \in \mathcal{S}_\alpha$, and $b \in \mathcal{S}_\beta$. Using Lemma 13.1.b we get

$$\gamma_d(f_a \circ f_b) = \sum_{a,b} q^{m(b,a) - m(\beta,a) + 2r(ab) - h(d)} f_a \circ f_b,$$

where the sum is over all $(a, b) \in \mathcal{S}_{\alpha,a} \times \mathcal{S}_{b,b}$ and all $(a, b)$ such that $\bar{a} = \alpha$, $\bar{b} = \beta$, and $d = a + b$. We are thus reduced to prove the following identity

$$(e) \quad m(b,a) - m(\beta,a) + 2r(ab) - 2r(b) - 2r(a) + h(a) + h(b) - h(d) = k(b,a).$$

Set

$$l_+(b,a) = \sum_{i \geq j} (b_i a_j + b_j a_{i+1}) \quad \text{and} \quad l_-(b,a) = \sum_{i < j} (b_i a_j + b_j a_{i+1}).$$

Then (e) follows from the following equalities which are easy to prove:

$$m(b,a) - m(\beta,a) = -l_+(b,a) - l_-(b,a),$$

$$h(a) + h(b) - h(d) = k(b,a) - l_+(b,a) + l_-(b,a),$$

$$r(ab) - r(a) - r(b) = l_+(b,a).$$
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