On the Density of a Graph and its Blowup

Asaf Shapira ∗ Raphael Yuster†

Abstract

The theorem of Chung, Graham, and Wilson on quasi-random graphs asserts that of all graphs with edge density $p$, the random graph $G(n, p)$ contains the smallest density of copies of $K_{t,t}$, the complete bipartite graph of size $2t$. Since $K_{t,t}$ is a $t$-blowup of an edge, the following intriguing open question arises: Is it true that of all graphs with triangle density $p^3$, the random graph $G(n, p)$ contains the smallest density of $K_{t,t,t}$, which is the $t$-blowup of a triangle?

Our main result gives an indication that the answer to the above question is positive by showing that for some blowup, the answer must be positive. More formally we prove that if $G$ has triangle density $p^3$, then there is some $2 \leq t \leq T(p)$ for which the density of $K_{t,t,t}$ in $G$ is at least $p^{(3+o(1))t^2}$, which (up to the $o(1)$ term) equals the density of $K_{t,t,t}$ in $G(n, p)$. We also consider the analogous question on skewed blowups, showing that somewhat surprisingly, the behavior there is different. We also raise several conjectures related to these problems and discuss some applications to other areas.

1 Introduction

One of the main family of problems studied in extremal graph theory is how does the lack/number of copies of one graph $H$ in a graph $G$ affects the lack/number of copies of another graph $H'$ in $G$. Perhaps the most well known problems of this type are Ramsey’s Theorem and Turán’s Theorem. Our investigation here is concerned with the relation between the densities of certain fixed graphs in a given graph. Some well known results of this type are Goodman’s Theorem [11] and the Chung-Graham-Wilson Theorem [7]. Some recent results of this type have been obtained recently by Razborov [16] and an abstract investigation of problems of this type was taken recently by Lovász and Szegedy [14]. In this paper we introduce an extremal problem of this type, which is related to some of these well studied problems, and to problems in other areas such as quasi-random graphs and Communication Complexity.

Let us start with some standard notation. Given a graph $H$ on $h$ vertices $v_1, \ldots, v_h$ and a sequence of $h$ positive integers $a_1, \ldots, a_h$ we denote by $B = H(a_1, \ldots, a_h)$ the $(a_1, \ldots, a_h)$-blowup of

∗School of Mathematics and College of Computing, Georgia Institute of Technology, Atlanta GA, 30332. E-mail: asafico@math.gatech.edu

†Department of Mathematics, University of Haifa, Haifa 31905, Israel. E-mail: raphy@math.haifa.ac.il
Let $H$ obtained by replacing vertex $v_i$ of $H$ with an independent set $I_i$ of $a_i$ vertices, and by replacing every edge $(v_i, v_j)$ of $H$ with a complete bipartite graph connecting the independent sets $I_i$ and $I_j$. For brevity, we will call $B = H(b_1, \ldots, b)$ the $b$-blowup of $H$, that is, the blowup in which all vertices are replaced with an independent set of size $b$. For a fixed graph $H$ and a graph $G$ we denote by $c_H(G)$ the number of copies of $H$ in $G$, or more formally the number of injective mappings from $V(H)$ to $V(G)$ which map edges of $H$ to edges of $G$. For various reasons, it is usually more convenient to count homomorphisms from $H$ to $G$ rather than count copies of $H$ in $G$. Let us then denote this quantity by $\text{Hom}_H(G)$, that is, the number of (not necessarily injective) mappings from $V(H)$ to $V(G)$ which map edges of $H$ to edges of $G$ (allowing two endpoints of an edge to be mapped to the same vertex of $G$). We now let $d_H(G) = \frac{\text{Hom}_H(G)}{n^h}$ denote the $H$-density of $G$ (or the density of $H$ in $G$). Note that $0 \leq d_H(G) \leq 1$ and we can think of $d_H(G)$ as the probability that a random map $\phi : V(H) \rightarrow V(G)$ is a homomorphism. We will also say that a graph on $n$ vertices has edge density $p$ if it has $p\binom{n}{2}$ edges. Finally, let us say that a graph $G$ on $n$ vertices has asymptotic $H$-density $\gamma$ if $d_H(G) = \gamma + o(1)$ where (as usual) the $o(1)$ term represents a quantity that goes to $0$ when $n$ goes to infinity. For brevity, for the rest of the paper whenever we refer to the $H$-density of a graph we will always refer to the asymptotic $H$-density of $G$.

The main motivation of our investigation here comes from the theory of quasi-random graphs. The fundamental theorem of quasi-random graphs, the Chung-Graham-Wilson Theorem \cite{Chung} (CGW-Theorem for short), asserts \cite{CGW} that of all graphs with edge density $p$, the random graph $G(n, p)$ contains the smallest asymptotic density of copies of $C_4$, the cycle of length $4$. Let $K_{a,b}$ denote the complete bipartite graph on sets of vertices of sizes $a$ and $b$ and note that $K_{a,b}$ is the $(a, b)$-blowup of an edge and that $C_4$ is just $K_{2,2}$. So the CGW-Theorem states that of all graphs with edge-density $p$, the random graph has the smallest density of the $2$-blowup of an edge. Actually, essentially the same argument as in \cite{Chung} shows that for every $a, b$, of all graphs with edge density $p$, the random graph contains the smallest density of $K_{a,b}$ (if $a = 1$ then any regular graph with edge density $p$ also has this property).

The question we raise in this paper can thus be thought of as an extension of the CGW-Theorem from blowups of an edge, to blowups of arbitrary graphs. Let us state it explicitly.

**Problem 1** Let $H$ be a fixed graph and set $B = H(a_1, \ldots, a_h)$. Assuming the $d_H(G) = \gamma$, how small can $d_B(G)$ be? Furthermore, is it true that of all graphs with $d_H(G) = \gamma$, the (appropriate) random graph $G(n, p)$ has the smallest $d_B(G)$?

As is well-known, the CGW-Theorem further states that if a graph has edge density $p$ and its $C_4$-density is the same as that of $G(n, p)$ then $G$ must be quasi-random, that is, behave like $G(n, p)$ in some well defined way (see the excellent survey on quasi-random graphs by Krivelevich and Sudakov \cite{Krivelevich} for the precise definitions). Again, this result on $C_4$ ($\equiv K_{2,2}$) can be extended to any $K_{a,b}$
(assuming $K_{a,b}$ is not a star). We can thus ask the following question that again tries to generalize the result of [7] from blowups of an edge to blowups of other graphs $H$:

**Problem 2** Let $H$ be a fixed graph, set $B = H(a_1, \ldots, a_h)$ and suppose $d_H(G)$ and $d_B(G)$ equal those of $G(n, p)$. Must $G$ be quasi-random?

As we’ll see in this paper, while the result of [7] give a positive answer to Problem 1 for all blowups of an edge, the answer to Problem 1 is negative for some blowups of other graphs, and we conjecture that the answer is positive for other blowups. As for Problem 2, we currently don’t have an indication if it has a positive answer for any blowup of any graph other than blowups of an edge. Hence, we will focus our attention on Problem 1. As we will see shortly, Problem 1 seems challenging even for the first non-trivial case of $H$ being the triangle (denoted $K_3$), so we will mainly consider this special case. To simplify the notation, let us denote by $K_{a,b,c}$ the $(a, b, c)$-blowup of $K_3$. So $K_{2,2,2}$ is the 2-blowup of the triangle and the question we are interested in is the following: Suppose the density of triangles in $G$ is $\gamma$. How small can the density of $B = K_{a,b,c}$ be in $G$? Let us denote by $f_B(\gamma)$ the infimum of this quantity over all graphs with triangle density $\gamma$. So Problem 1 can be restated as asking for a bound for the function $f_B(\gamma)$.

A simple upper bound for $f_B(\gamma)$ can be obtained by considering the number of triangles and copies of $K_{a,b,c}$ in the random graph $G(n, \gamma^{1/3})$. In the other direction, a simple lower bound can be obtained from the Erdős-Simonovits Theorem [8] regarding the number of copies of complete 3-partite hypergraphs in dense 3-uniform hypergraphs. These two bounds give the following:

**Proposition 1.1** Let $B = K_{a,b,c}$. Then we have the following bounds

$$\gamma^{abc} \leq f_B(\gamma) \leq \gamma^{(ab+bc+ac)/3}.$$

So Problem 1 can be formulated as asking for a characterization of the blowups $B$ for which we have $f_B(\gamma) = \gamma^{(ab+bc+ac)/3}$. Our main results in this paper, discussed in the next subsections, show that in some cases the lower bound in the above proposition is (essentially) tight, while it seems that for other cases the upper bound gives the correct answer.

We conclude with noting that one can naturally consider the following variant of Problem 1. Let $H$ be a fixed graph and let $B'$ be any subgraph of $H(a_1, \ldots, a_h)$. How small can $f_{B'}(G)$ be if $f_H(G) = \gamma$? We note that while Problem 1 for the case of $H$ being an edge is well understood (via the CGW-Theorem), the above variant of Problem 1 is open even when $H$ is an edge. This is the long standing conjecture of Sidorenko [18] and Simonovits [19] that states that for every bipartite graph $B$, the random graph $G(n, p)$ has the smallest $B$-density over all graphs with edge density $p$. We thus focus our attention on Problem 1.
1.1 Balanced blowups and the main results

When considering the case $B = K_{2,2,2}$, the bounds given by Proposition 1.1 become $\gamma^8 \leq f_B(\gamma) \leq \gamma^4$, and when $B = K_{t,t,t}$ the above bounds become $\gamma^t \leq f_B(\gamma) \leq \gamma^t$. The question we are interested in is, therefore, whether $f_B(\gamma) = \gamma^t$. Currently, we cannot obtain any (real) improvement over Proposition 1.1 for the case $B = K_{2,2,2}$, but we are able to show that some fixed blowup must have density asymptotically close to $\gamma^t$, namely, as the density expected in the random graph. More formally, our main result is the following.

**Theorem 1** For every $0 < \gamma, \delta < 1$ there are $N = N(\gamma, \delta)$ and $T = T(\gamma, \delta)$ such that if $G$ is a graph on $n \geq N$ vertices and its triangle density is $\gamma$, then there is some $2 \leq t \leq T$ for which the $K_{t,t,t}$-density of $G$ is at least $\gamma^{(1+\delta)t^2}$.

As we have previously mentioned, we focus our attention on blowups of $K_3$, although the proof of Theorem 1 extends to blowups of larger complete graphs. Denote by $K^t_k$ the $t$-blowup of $K_k$. The precise same conclusion of Theorem 1 holds if $\gamma$ is the density of $K_k$, $T = T(k, \gamma, \delta)$, and $N = N(k, \gamma, \delta)$. As the proof of Theorem 1 is already quite involved, we only prove it for triangles, as stated.

Let us state a related result that was recently obtained by Alon [1].

**Theorem 2 (Alon [1])** Set $B = K_{t,t,t}$. Then we have

$$f_B(\gamma) \geq \gamma^{t^2}/\gamma^t.$$  

Alon’s result implies that for any $t \geq 1/\gamma^2$ we can improve upon the lower bound of Proposition 1.1. Note that Alon’s result does not imply that for large enough $t$, the density of $B = K_{t,t,t}$ gets closer to the density of $B$ in $G(n, p)$. Alon’s argument is based on an idea used by Nikiforov [15] to tackle an Erdős-Stone [9] type question. In Section 4 we observe that a slightly weaker bound can be obtained directly from Nikiforov’s result.

Let us finally mention another unexpected motivation for studying $f_B(\gamma)$. As it turns out, in the case $t = 2$ (i.e., when $B = K_{2,2,2}$), the question of bounding $f_B(\gamma)$, was also considered recently (and independently) due to a different motivation. Barak and Raz observed that improving the lower bound of $B = K_{2,2,2}$ from $f_B(\gamma) \geq \gamma^8$ to $f_B(\gamma) \geq \gamma^{8-c}$ for some $c > 0$ would have certain non-trivial applications in communication complexity.

1.2 Skewed blowups

We now turn our discussion to small skewed-blowups, which somewhat surprisingly, seem to behave quite differently from the symmetric blowups considered in the previous subsection. Proposition 1.1 implies that when $B = K_{1,1,2}$ we have $\gamma^2 \leq f_B(\gamma) \leq \gamma^{5/3}$. In another independent recent
investigation, motivated by an attempt to improve the bounds in the well-known Triangle Removal Lemma (see Theorem 9), Trevisan (see [21] page 239) observed that the $\gamma^2$ lower bound for the case $B = K_{1,1,2}$ can be (slightly) improved:

**Theorem 3** Set $B = K_{1,1,2}$. Then we have the following bound

$$f_B(\gamma) \geq \omega(\gamma^2).$$

We note that since the proof of Theorem 3 applies the so called triangle removal-lemma (see Theorem 9), which, in turn, applies Semerédi’s Regularity Lemma, the $\omega(\gamma^2)$ bound in Theorem 3 “just barely” beats the simple $\gamma^2$ bound of Proposition 1.1. The bound which the proof gives is roughly of order $\log^* (1/\gamma^2)$, and Tao [21] asked if it is possible to improve this bound to something like $\log \log (1/\gamma^2)$. While we can not rule out such a bound, we can still rule out a polynomially better bound by improving the upper bound of Proposition 1.1.

**Theorem 4** Set $B = K_{1,1,2}$. Then we have the following bound

$$f_B(\gamma) \leq \gamma^2 - o(1)$$

where the $o(1)$ term goes to 0 with $\gamma$.

Observe that Theorem 4 implies that the random graph does not minimize the density of $K_{1,1,2}$ over all graphs with a given triangle density. So we see that the answer to Problem 1 is negative for this case. Also, note that Theorems 3 and 4 together determine the correct exponent of $f_B(\gamma)$ for $B = K_{1,1,2}$. The problem of determining the correct order of the $o(1)$ terms remains open and seems challenging.

**Comment 1.2** Both Theorems 3 and 4 were also obtained independently by N. Alon [11].

If we consider $B = K_{1,2,2}$, then Proposition 1.1 gives $\gamma^4 \leq f_B(\gamma) \leq \gamma^{8/3}$. Essentially the same proof as that of Theorem 3 and the same construction used for the proof of Theorem 4 give the following improved bounds (we omit the proofs).

**Theorem 5** Set $B = K_{1,2,2}$. Then we have the following bounds

$$\omega(\gamma^4) \leq f_B(\gamma) \leq \gamma^{3-o(1)}$$

in which the $o(1)$ term goes to 0 with $\gamma$, while the $\omega(1)$ term goes to $\infty$.

Note that as opposed to the case of $B = K_{1,1,2}$ in which our bounds determined the correct exponent of $f_B(\gamma)$, in the case of $B = K_{1,2,2}$ we only know that the correct exponent of $f_B(\gamma)$ is between 3 and 4. Also, $B = K_{1,2,2}$ is another example of a blowup of $K_3$ for which the answer to Problem 1 is negative.
1.3 Organization

The rest of this paper is organized as follows. In section 2 we focus on large blowups and prove our main result, Theorem 1. Our first main tool for the proof of Theorem 1 is the quantitative version of the Erdős-Stone theorem, the so called Bollobás-Erdős-Simonovits theorem [4, 5], regarding the size of blowups of $K_r$ in graphs whose density is larger than the Turán density of $K_r$. Our second main tool is a functional variant of Szemerédi’s regularity lemma [20] due to Alon et al. [2]. In section 3 we consider small skewed blowups and prove Theorems 4 and 3. The proof of these results apply the so called triangle-removal lemma of Rusza-Szemerédi as well as the Rusza-Szemerédi graphs. The final section contains some concluding remarks.

2 The Density of Large Symmetric Blowups

2.1 Background on the Regularity Lemma

We start with the basic notions of regularity, some of the basic applications of regular partitions and state the regularity lemmas that we use in the proof of Theorem 1. See [12] for a comprehensive survey on the Regularity Lemma. We start with some basic definitions. For every pair of nonempty disjoint vertex sets $A$ and $B$ of a graph $G$, we define $e(A, B)$ to be the number of edges of $G$ between $A$ and $B$. The edge density of the pair is defined by $d(A, B) = e(A, B)/|A||B|$. 

Definition 2.1 ($\gamma$-regular pair) A pair $(A, B)$ is $\gamma$-regular, if for any two subsets $A' \subseteq A$ and $B' \subseteq B$, satisfying $|A'| \geq \gamma|A|$ and $|B'| \geq \gamma|B|$, the inequality $|d(A', B') - d(A, B)| \leq \gamma$ holds.

Let $G$ be a graph obtained by taking a copy of $K_3$, replacing every vertex with a sufficiently large independent set, and every edge with a random bipartite graph. The following well known lemma shows that if the bipartite graphs are “sufficiently” regular, then $G$ contains the same number of triangles as the random graphs does. For brevity, let us say that three vertex sets $A, B, C$ are $\epsilon$-regular if the three pairs $(A, B), (B, C)$ and $(A, C)$ are all $\epsilon$-regular. Several versions of this lemma were previously proved in papers using the Regularity Lemma. See e.g. Lemma 4.2 in [10].

Lemma 2.2 For every $\zeta$ there is an $\epsilon = \epsilon(\zeta)$ satisfying the following. Let $A, B, C$ be pairwise disjoint independent sets of vertices of size $m$ each that are $\epsilon$-regular and satisfy $d(A, B) = \alpha_1$, $d(A, C) = \alpha_2$ and $d(A, C) = \alpha_3$. Then $(A, B, C)$ contain at most $(\alpha_1\alpha_2\alpha_3 + \zeta)m^3$ triangles.

The following lemma also follows from Lemma 4.2 in [10].

Lemma 2.3 For every $t$ and $\zeta$ there is an $\epsilon = \epsilon(t, \zeta)$ such that if $G$ is a $3t$-partite graph on disjoint vertex sets $A_1, \ldots, A_t, B_1, \ldots, B_t, C_1, \ldots, C_t$ of size $m$, and these sets satisfy:

- $(A_i, B_j, C_k)$ are $\epsilon$-regular for every $1 \leq i, j, k \leq t$.
• For every $1 \leq i, j, k \leq t$ we have $d(A_i, B_j) \geq \alpha_1$, $d(A_i, C_k) \geq \alpha_2$ and $d(B_j, C_k) \geq \alpha_3$.

Then $G$ contains at least $(\alpha_1 \alpha_2 \alpha_3 - \zeta)^t m^{3t}$ copies of $K_{t,t,t}$ each having precisely one vertex from each partite set.

The following lemma is an easy consequence of Lemma 2.3 obtained by taking $t$ multiple copies of each partite set.

Lemma 2.4 For every $t$ and $\zeta$ there is an $\epsilon = \epsilon_{2.4}(t, \zeta)$ such that if $G$ is a 3-partite graph on disjoint vertex sets $A, B, C$ of size $m$ and these sets satisfy:

- $(A, B, C)$ is $\epsilon$-regular.
- $d(A, B) \geq \alpha_1$, $d(A, C) \geq \alpha_2$ and $d(B, C) \geq \alpha_3$.

Then $G$ contains at least $(\alpha_1 \alpha_2 \alpha_3 - \zeta)^t m^{3t}$ distinct homomorphisms of $K_{t,t,t}$.

A partition $A = \{V_i \mid 1 \leq i \leq k\}$ of the vertex set of a graph is called an equipartition if $|V_i|$ and $|V_j|$ differ by no more than 1 for all $1 \leq i < j \leq k$ (so in particular each $V_i$ has one of two possible sizes). The order of an equipartition denotes the number of partition classes ($k$ above). A refinement of an equipartition $A$ is an equipartition of the form $B = \{V_{i,j} \mid 1 \leq i \leq k, 1 \leq j \leq \ell\}$ such that $V_{i,j}$ is a subset of $V_i$ for every $1 \leq i \leq k$ and $1 \leq j \leq \ell$.

Definition 2.5 ($\gamma$-regular equipartition) An equipartition $B = \{V_i \mid 1 \leq i \leq k\}$ of the vertex set of a graph is called $\gamma$-regular if all but at most $\gamma k^2$ of the pairs $(V_i, V_j)$ are $\gamma$-regular.

The Regularity Lemma of Szemerédi can be formulated as follows.

Lemma 2.6 ([20]) For every $m$ and $\gamma > 0$ there exists $T = T_{2.6}(m, \gamma)$ with the following property: If $G$ is a graph with $n \geq T$ vertices, and $A$ is an equipartition of the vertex set of $G$ of order at most $m$, then there exists a refinement $B$ of $A$ of order $k$, where $m \leq k \leq T$ and $B$ is $\gamma$-regular.

Our main tool in the proof of Theorem 1 is Lemma 2.8 below, proved in [2]. This lemma can be considered a strengthening of Lemma 2.6 as it guarantees the existence of an equipartition and a refinement of this equipartition that poses stronger properties compared to those of the standard $\gamma$-regular equipartition. This stronger notion is defined below.

Definition 2.7 ($\mathcal{E}$-regular equipartition) For a function $\mathcal{E}(r) : \mathbb{N} \mapsto (0, 1)$, a pair of equipartitions $A = \{V_i \mid 1 \leq i \leq k\}$ and its refinement $B = \{V_{i,j} \mid 1 \leq i \leq k, 1 \leq j \leq \ell\}$, where $V_{i,j} \subset V_i$ for all $i, j$, are said to be $\mathcal{E}$-regular if

1. All but at most $\mathcal{E}(0)k^2$ of the pairs $(V_i, V_j)$ are $\mathcal{E}(0)$-regular.
2. For all \(1 \leq i < i' \leq k\), for all \(1 \leq j, j' \leq \ell\) but at most \(\mathcal{E}(k)\ell^2\) of them, the pair \((V_{i,j}, V_{i',j'})\) is \(\mathcal{E}(k)\)-regular.

3. All \(1 \leq i < i' \leq k\) but at most \(\mathcal{E}(0)k^2\) of them are such that for all \(1 \leq j, j' \leq \ell\) but at most \(\mathcal{E}(0)\ell^2\) of them \(|d(V_i, V_{i'}) - d(V_{i,j}, V_{i',j'})| < \mathcal{E}(0)\) holds.

It will be very important for what follows to observe that in Definition 2.7, we may use an arbitrary function rather than a fixed \(\gamma\) as in Definition 2.5 (such functions will be denoted by \(\mathcal{E}\) throughout the paper). The following is one of the main results of [2].

**Lemma 2.8 ([2])** For any integer \(m\) and function \(\mathcal{E}(r) : \mathbb{N} \mapsto (0, 1)\) there is \(S = S_{2,8}(m, \mathcal{E})\) such that any graph on at least \(S\) vertices has an \(\mathcal{E}\)-regular equipartition \(A, B\) where \(|A| = k \geq m\) and \(|B| = kt \leq S\).

### 2.2 Main Idea and Main Obstacle

Let us describe the main intuition behind the proof of Theorem 1 and where its naive implementation fails. Recall that Lemma 2.2 says that an \(\epsilon\)-regular triple contains the “correct” number of triangles we expect to find in a “truly” random graph with the same density. So given a graph with triangle density \(\gamma\), we can apply the Regularity Lemma with (say) \(\epsilon = \gamma\). Suppose we get a partition into \(k\) sets, for (say) some \(k \leq \ell = T_{2,6}(\gamma, 1/\gamma^2)\). So the situation now is that the number of triangles spanned by any triple \(V_i, V_j, V_k\) is more or less determined by the densities between the sets. Since \(G\) has triangle density \(\gamma\), we get (by averaging) that there must be some triple \(V_i, V_j, V_k\) whose triangle density is also close to being at least \(\gamma\). Suppose the densities between \(V_i, V_j, V_k\) are \(\alpha_1, \alpha_2\) and \(\alpha_3\). Since the number of triangles between \(V_i, V_j, V_k\) is determined by the densities connecting them, we get that \(\alpha_1\alpha_2\alpha_3\) is close to \(\gamma\). Now, if \(\epsilon\) is small enough, then we can also apply Lemma 2.3 on \(V_i, V_j, V_k\) in order to infer that they contain close to \((n/k)^{3\ell}\alpha_1^2\alpha_2^2\alpha_3^2\) copies of \(K_{t,t,t}\). Hence, by the above consideration, this number is close to \((n/k)^{3\ell}\gamma^{1/3}\). Now, since for large enough \(t \geq t(k)\) we have \((n/k)^{3\ell}\gamma^{1/3} = n^{3\ell}\gamma(1+o(1))^{1/3}\), we can choose a large enough \(t = t(k)\) to get the desired result. Since \(k\) is bounded by a function of \(\gamma\) so is \(t\).

The reason why the above argument fails is that in order to apply Lemma 2.3 with a given \(t\), the value of \(\epsilon\) in the \(\epsilon\)-regular partition needs to depend on \(t\). So we arrive at a circular situation in which \(\epsilon\) needs to be small enough in terms of \(t\) (to allow us to apply Lemma 2.3), and \(t\) needs to be large enough in terms of \(\epsilon\) (to allow us to infer that \((n/k)^{3\ell}\gamma^{1/3} = n^{3\ell}\gamma(1+o(1))^{1/3}\)).

We overcome the above problem by applying Lemma 2.8 which more or less allows us to find a partition which is \(f(k)\)-regular where \(k\) is the number of partition classes. However, this is an over simplification of this result (as can be seen from Definition 2.7), and our proof requires several other ingredients that enable us to apply Lemma 2.8. Most notably, we need to use a classic result of Bollobás, Erdős and Simonovits [4, 5] and adjust it to our setting.
2.3 Some preliminary lemmas

We now turn to discuss two simple (yet crucial) lemmas that will be later used in the proof of Theorem 1. Let us recall that Turán’s theorem asserts that every graph with edge density larger than \(1 - \frac{1}{r-1}\) contains a copy of \(K_r\), the complete graph on \(r\) vertices. The Erdős-Stone theorem strengthens this result by asserting that if the edge density is larger than \(1 - \frac{1}{r-1}\), then the graph actually contains a blowup of \(K_r\). More precisely, there is a function \(f(n, c, r)\) such that every \(n\)-vertex graph with edge density \(1 - \frac{1}{r-1} + \beta\) contains an \(f(n, \beta, r)\)-blowup of \(K_r\). The determination of the growth rate of \(f(n, \beta, r)\) received a lot of attention until Bollobás, Erdős and Simonovits [4, 5] determined that for fixed \(\beta\) and \(r\) we have \(f(n, \beta, r) = \Theta(\log n)\). See [15] for a short proof of this result and for related results and references. As it turns out, the bound \(\Theta(\log n)\) will be crucial for our proof (a bound like \(\log 1 - \varepsilon n\) would not be useful for us). Let us state an equivalent formulation of this result for the particular choice of \(r = 3\) and \(\beta = 1/24\).

**Theorem 6 (Bollobás-Erdős-Simonovits [4, 5])** There is an absolute constant \(c\), such that every graph on at least \(c^t\) vertices and edge density at least \(13/24\) contains a copy of \(K_{t, t, t}\).

As a 3-partite graph with edge density at least \(7/8\) between any two parts has overall density greater than \(13/24\) we have:

**Corollary 1** There is an absolute constant \(C\), such that every 3-partite graph with parts of equal size \(C^t\) and edge density at least \(7/8\) between any two parts, contains a copy of \(K_{t, t, t}\).

We will need the following lemma guaranteeing many copies of a large blowup of \(K_3\).

**Lemma 2.9** If \(G\) is a 3-partite graph on vertex sets \(X, Y\) and \(Z\) of equal size \(m\), and the three densities \(d(X, Y), d(X, Z)\) and \(d(Y, Z)\) are all at least \(15/16\), then \(G\) contains at least \(\lfloor m/3^t/C^3t^2 \rfloor\) copies of \(K_{t, t, t}\).

**Proof:** Let \(C\) be the constant of Corollary 1. If \(m < C^t\) there is nothing to prove (as \(\lfloor m/3^t/C^3t^2 \rfloor = 0\)) so let us assume that \(m \geq C^t\). We first claim that at least 1/2 of the graphs spanned by three sets of vertices \(X' \subseteq X, Y' \subseteq Y, Z' \subseteq Z\), where \(|X'| = |Y'| = |Z'| = C^t\), have edge density at least 7/8. Indeed, suppose we randomly pick the sets \(X', Y'\) and \(Z'\). The expected density of non-edges between \((X', Y'), (X', Z')\) and \((Y', Z')\) is 1/16, so by Markov’s inequality, with probability at least 1/2 this density is at most 1/8.

By Corollary 1, every graph of size at least \(C^t\), whose edge density is at least 7/8, contains a copy of \(K_{t, t, t}\). So by the above consideration, at least half of the \(\binom{m}{C^t}^3\) choices of \(A', B', C'\) contain a \(K_{t, t, t}\). Since each such \(K_{t, t, t}\) is counted \(\binom{m-t}{C^t-t}^3\) times, we have that the number of distinct copies of \(K_{t, t, t}\) in \(G\) is at least

\[
\frac{1}{2} \binom{m}{C^t}^3 / \binom{m-t}{C^t-t}^3 \geq m^{3t}/C^{3t^2}.
\]

9
The proof of Theorem 1 we give in the next subsection only covers the case of \( \gamma \ll \delta \). As the following lemma shows, we can then “lift” this result to arbitrary \( 0 < \gamma, \delta < 1 \).

**Lemma 2.10** If Theorem 1 holds for every \( \delta > 0 \) and every small enough \( \gamma < \gamma_0(\delta) \), then it also holds for every \( 0 < \gamma, \delta < 1 \).

**Proof:** Assume to the contrary that there exist a \( \delta > 0 \), a \( \gamma \geq \gamma_0(\delta) \) and arbitrarily large graphs with triangle-density \( \gamma \) in which the \( K_{t,t,t} \)-density is smaller than \( \gamma(1+\delta)t^2 \) for every \( 2 \leq t \leq T(\gamma_0^2(\delta), \delta) \). Let \( G \) be one such graph on at least \( N(\gamma_0^2(\delta), \delta) \) vertices. For an integer \( k \) let \( G^{\otimes k} \) be the \( k \)th tensor product of \( G \), that is, the graph whose vertices are sequences of \( k \) (not necessarily distinct) vertices of \( G \), and where vertex \( v = (v_1, \ldots, v_k) \) is connected to vertex \( u = (u_1, \ldots, u_k) \) if and only if \( v_i \) is connected to \( u_i \) for every \( 1 \leq i \leq k \). The key observation is that for every graph \( H \), if the \( H \)-density of \( G \) is \( \gamma \) then the \( H \)-density of \( G^{\otimes k} \) is \( \gamma^k \). Let then \( k \) be the smallest integer satisfying \( \gamma^k < \gamma_0(\delta) \) and note that in this case we have \( \gamma_0^2(\delta) \leq \gamma^k < \gamma_0(\delta) \). We thus get that \( G^{\otimes k} \) is a graph on at least \( N(\gamma_0^2(\delta), \delta) \geq N(\gamma^k, \delta) \) vertices, with triangle density \( \gamma^k \) and for all \( 2 \leq t \leq T(\gamma^k, \delta) \leq T(\gamma_0^2(\delta), \delta) \) its \( K_{t,t,t} \) density is smaller than \( \gamma^k(1+\delta)t^2 \), which contradicts the assumption of the lemma. \( \blacksquare \)

### 2.4 Proof of Theorem 1

We prove the theorem for every \( 0 < \delta < 1 \) and for every \( 0 < \gamma < 1 \) which is small enough so that

\[
\gamma < \left( \frac{1}{128C^3} \right)^{2/\delta} \tag{1}
\]

where \( C \) is the absolute constant from Lemma 2.9. By Lemma 2.10 this will establish the theorem for all \( 0 < \delta, \gamma < 1 \).

For a given positive integer \( r \), let \( t = t(r, \delta, \gamma) \) be a large enough integer such that

\[
\frac{1}{r^{3t}} \left( \frac{\gamma}{64} \right)^{t^2} \geq 2C^{3t^2} \gamma(1+\delta)t^2 \tag{2}
\]

holds. Since we assume that \( \gamma \) and \( \delta \) satisfy (1), it is enough to make sure that \( t \) satisfies

\[
\frac{1}{r^{3t}} \geq \gamma^{4\delta t^2},
\]

hence we can take

\[
t(r, \delta, \gamma) = \max\{2, \frac{6 \log r}{\delta \log \frac{1}{\gamma}} \}. \tag{3}
\]
We now define a function $\mathcal{E}(r)$ as follows:

\[
\mathcal{E}(r) = \begin{cases} 
\min\{\frac{1}{16}, \gamma/30, \frac{2.2}{\gamma}(t(0), \gamma/4)\}, & r = 0 \\
\min\{\frac{1}{16}, \frac{2.4}{\gamma}(t(r, \delta, \gamma), \gamma/64), \frac{2.3}{\gamma}(t(r, \delta, \gamma), \gamma/64)\}, & r \geq 1 .
\end{cases}
\]  

(4)

Given $\gamma$ and $\delta$ let $\mathcal{E}(r)$ be the function defined above. Set $m = 30/\gamma$ and let $S = S_{2.8}(m, \mathcal{E})$ be the constant from Lemma 2.8. Given a graph $G$ on $n \geq S$ vertices and parameters $\gamma$ and $\delta$, we apply Lemma 2.8 with $m = 30/\gamma$ and with the function $\mathcal{E}(r)$ defined above. The lemma returns an $\mathcal{E}$-regular partition consisting of an equipartition $A = \{V_1 | 1 \leq i \leq k\}$ and a refinement $B = \{V_{i,j} | 1 \leq i \leq k, 1 \leq j \leq \ell\}$, where $k\ell \leq S(m, \mathcal{E})$ and $k \geq m$. Note that $S$ depends only on $\delta$ and $\gamma$.

We now remove from $G$ any edge whose endpoints belong to the same set $V_i$. We thus remove at most $n^2/(2m) < \frac{1}{30}n^2$ edges. We also remove any edge connecting pairs $(V_i, V_j)$ that are not $\mathcal{E}(0)$-regular. The first property of an $\mathcal{E}$-regular partition guarantees that we thus remove at most $\mathcal{E}(0)n^2 < \frac{1}{30}n^2$ edges. We also remove any edge connecting a pair $(V_i, V_j)$ for which there are more than $\mathcal{E}(0)\ell^2$ pairs $i', j'$ which do not satisfy $|d(V_i, V_j) - d(V_{i', j'})| < \mathcal{E}(0)$. By the third property of an $\mathcal{E}$-partition we infer that we thus remove at most $\frac{1}{30}n^2$ edges. All together we have removed less than $\frac{1}{30}n^2$ edges and so we have destroyed at most $\frac{1}{2}n^3$ triangles in $G$ (recall that we are counting homomorphisms so each triangle is counted 6 times). And so the new graph we obtain has triangle density at least $\gamma/2$. Let us call this new graph $G'$.

As $G'$ has triangle density at least $\gamma/2$, we get (by averaging) that there must be three sets $(V_i, V_j, V_k)$ that contain at least $\frac{1}{2}\gamma(n/k)^3$ triangles with one vertex in each of the sets $V_i, V_j, V_k$ (we are using $k$ as both an index and as the number of parts in the partition, but there is no confusion). For what follows, let us set $\alpha_1 = d(V_i, V_j)$, $\alpha_2 = d(V_i, V_k)$ and $\alpha_3 = d(V_j, V_k)$. Because we have removed edges between non-$\mathcal{E}(0)$-regular pairs, we get that $(V_i, V_j, V_k)$ must be $\mathcal{E}(0)$-regular. Letting $\Delta$ denote the number of triangles spanned by $(V_i, V_j, V_k)$ we see that as $\mathcal{E}(0) \leq 2.2(\gamma/4)$, we can apply Lemma 2.2 on $(V_i, V_j, V_k)$ to conclude that

\[
\frac{1}{2}\gamma \left(\frac{n}{k}\right)^3 \leq \Delta \leq (\alpha_1 \alpha_2 \alpha_3 + \frac{1}{4}\gamma) \left(\frac{n}{k}\right)^3 ,
\]

implying that

\[
\alpha_1 \alpha_2 \alpha_3 \geq \frac{1}{4}\gamma .
\]  

(5)

Let us say that a $3s$-tuple (where $s$ is any positive integer) $1 \leq i_1 < \cdots < i_s \leq \ell$, $1 \leq j_1 < \cdots < j_s \leq \ell$, $1 \leq k_1, \cdots, k_s \leq \ell$ is good if it satisfies the following four properties:

1. For every $i_a, j_b, k_c$ we have that $(V_{i_a}, V_{j_b}, V_{k_c})$ are $\mathcal{E}(k)$-regular.
2. For every $i_a, j_b$ we have $d(V_{i_a}, V_{j_b}) \geq \alpha_1 - \mathcal{E}(0) \geq \alpha_1 - \frac{1}{s}\gamma \geq \frac{1}{2}\alpha_1$. 

3. For every $i_a, k_c$ we have $d(V_{i_a}, V_{k_c}) \geq \alpha_2 - \mathcal{E}(0) \geq \frac{1}{2} \gamma \geq \frac{1}{8} \alpha_2$.

4. For every $j_b, k_c$ we have $d(V_{j_b}, V_{k_c}) \geq \alpha_3 - \mathcal{E}(0) \geq \frac{1}{2} \gamma \geq \frac{1}{8} \alpha_3$.

Suppose $i_1, \ldots, i_t$, $j_1, \ldots, j_t$, $k_1, \ldots, k_t$ is a good 3-tuple. Then the definition of $\mathcal{E}$ (via the function $\mathcal{E}(r)$ from Lemma 2.3) and the first property of a good 3-tuple, guarantee that we can apply Lemma 2.4 on this 3-tuple, to conclude that it has at least $K$ number of homomorphisms of $E$. Then definition of $3$-tuple, guarantee that we can apply Lemma 2.4 on this 3-tuple, to conclude that it has at least

$$\left(\frac{n}{k}\right)^{3t} \left(\frac{1}{8} \alpha_1 \alpha_2 \alpha_3 - \frac{1}{64} \gamma\right)^2 \geq \left(\frac{n}{k}\right)^{3t} \left(\frac{\gamma}{64}\right)^2$$

copies of $K_{t,t,t}$, where we have also used (5). Our choice of $t = t(k, \delta, \gamma)$ in (3) guarantees (via (2)) that the number of copies of $K_{t,t,t}$ in a good 3-tuple is at least

$$\left(\frac{n}{k}\right)^{3t} \left(\frac{\gamma}{64}\right)^2 \geq 2C^{3t^2} \left(\frac{n}{\ell}\right)^{3t} \gamma^{(1+\delta)t^2}.$$  \hfill (6)

But how can we be certain that a good 3-tuple exists? And if they do, how many are there? We first consider the case $\ell \geq C^t$. Let us now recall that $\mathcal{E}(r) \leq \frac{1}{16}$ for every $r \geq 0$ and so the second and third properties of a $\mathcal{E}$-regular partition guarantee that at least $\frac{1}{16} \ell^2$ of the choices $1 \leq i', j' \leq \ell$ are such that $(V_i, V_j)$ is $\mathcal{E}(k)$-regular and satisfies $|d(V_i, V_j) - d(V_{i'}, V_{j'})| \leq \mathcal{E}(0)$. The same holds with respect to the other two pairs $(V_j, V_k)$ and $(V_i, V_k)$. Therefore, by Lemma 2.9 the sets $V_i, V_j, V_k$ contain at least $\left[\frac{\ell^3}{C^{3t}}\right] \geq 0.5 \frac{\ell^3}{C^{3t}}$ choices of good 3-tuples. Hence, combining this with (6) we infer that the number of copies of $K_{t,t,t}$ spanned by $(V_i, V_j, V_k)$ is at least

$$\frac{\ell^3}{2C^{3t^2}} \cdot 2C^{3t^2} \left(\frac{n}{\ell}\right)^{3t} \gamma^{(1+\delta)t^2} = n^{3t} \gamma^{(1+\delta)t^2},$$

implying that the density of $K_{t,t,t}$ in $G'$ (and so also in $G$) is at least $\gamma^{(1+\delta)t^2}$.

We now consider the case $\ell < C^t$. Assume that in this case we can find just one good 3-tuple. Then definition of $\mathcal{E}$ (via the function $\mathcal{E}(r)$ from Lemma 2.3) and the first property of a good 3-tuple, guarantee that we can apply Lemma 2.4 on this 3-tuple, to conclude that it has at least

$$\left(\frac{n}{k}\right)^{3t} \left(\frac{1}{8} \alpha_1 \alpha_2 \alpha_3 - \frac{1}{64} \gamma\right)^2 \geq \left(\frac{n}{k}\right)^{3t} \left(\frac{\gamma}{64}\right)^2$$

distinct homomorphisms of $K_{t,t,t}$. Our choice of $t = t(k, \delta, \gamma)$ in (3) guarantees (via (2)) that the number of homomorphisms of $K_{t,t,t}$ in a good 3-tuple is at least

$$\left(\frac{n}{k}\right)^{3t} \left(\frac{\gamma}{64}\right)^2 \geq 2C^{3t^2} \left(\frac{n}{\ell}\right)^{3t} \gamma^{(1+\delta)t^2} \geq n^{3t} \gamma^{(1+\delta)t^2},$$

implying that the density of $K_{t,t,t}$ in $G'$ (and so also in $G$) is at least $\gamma^{(1+\delta)t^2}$. To see that a single good 3-tuple $i_1, j_1, k_1$ exists, consider picking $i_1, j_1$ and $k_1$ randomly and uniformly from $[\ell]$. Since
we infer that with positive probability $i_1, j_1$ and $k_1$ will satisfy the four properties of a good 3-tuple, so a good 3-tuple exists.

Finally, note that since $k \leq S$ we see that $k$ is upper bounded by some function of $\gamma$ and $\delta$. As $t = t(k, \delta, \gamma)$ is chosen in (3) we see that $2 \leq t \leq T(\gamma, \delta)$ for some function $T(\gamma, \delta)$ and so the proof is complete.

3 The Density of Small Skewed Blowups

In this section we focus our attention on small skewed blowups of $K_3$. We start with that proof of Theorem 4 in which we will apply the following well known result of Ruzsa and Szemerédi [17]. For completeness, we include the short proof.

Theorem 7 ([17]) Suppose $S \subseteq [n]$ is a set of integers containing no 3-term arithmetic progression. Then there is a graph $G = (V, E)$ with $|V| = 6n$ and $|E| = 3n|S|$, whose edges can be (uniquely) partitioned into $n|S|$ edge disjoint triangles. Furthermore, $G$ contains no other triangles.

Proof: We define a 3-partite graph $G$ on vertex sets $A$, $B$ and $C$, of sizes $n$, $2n$ and $3n$ respectively, where we think of the vertices of $A$, $B$ and $C$ as representing the sets of integers $[n]$, $[2n]$ and $[3n]$. For every $1 \leq i \leq n$ and $s \in S$ we put a triangle $T_{i,s}$ in $G$ containing the vertices $i \in A$, $i + s \in B$ and $i + 2s \in C$. It is easy to see that the above $n|S|$ triangles are edge disjoint, because every edge determines $i$ and $s$. To see that $G$ does not contain any more triangles, let us observe that $G$ can only contain a triangle with one vertex in each set. If the vertices of this triangle are $a \in A$, $b \in B$ and $c \in C$, then we must have $b = a + s_1$ for some $s_1 \in S$, $c = b + s_2 = a + s_1 + s_2$ for some $s_2 \in S$, and $a = c - 2s_3 = a + s_1 + s_2 - 2s_3$ for some $s_3$. This means that $s_1, s_2, s_3 \in S$ form an arithmetic progression, but because $S$ is free of 3-term arithmetic progressions it must be the case that $s_1 = s_2 = s_3$ implying that this triangle is one of the triangles $T_{i,s}$ defined above.

For the proof of Theorem 4 we will need to combine Theorem 7 with the following well known result of Behrend [3].

Theorem 8 (Behrend [3]) For every $n$, there exists $S \subseteq [n]$ of size $n/8^{\sqrt{\log n}} = n^{1-o(1)}$ containing no 3-term arithmetic progression.

Proof of Theorem 4: Let $m$ be any integer and let $S$ be a 3AP-free subset of $[m]$ of size $m/8^{\sqrt{\log m}}$ as guaranteed by Theorem 8. Let $G'$ be the graph of Theorem 7 when using $[m]$ and the above set $S$. Finally, let $G$ be an $n/6m$ blowup of $G'$, that is, the graph obtained by replacing every vertex $v$ of $G'$ with an independent set $I_v$ of size $n/6m$, and replacing every edge $(u,v)$ of $G'$ with a complete
bipartite graph connecting \( I_u \) and \( I_v \). Observe that \( G \) has \( n \) vertices, and that each triangle in \( G' \) gives rise to \((n/6m)^3\) triangles in \( G \). Hence, the number of ways to map a triangle into \( G \) is

\[
6m|S| \left( \frac{n}{6m} \right)^3 = \frac{n^3}{6^2m^8\sqrt{\log m}}
\]

(recall that there are six ways to map a labeled triangle into a triangle of \( G \)). The crucial observation is that because all the triangles in \( G' \) are edge disjoint, the only copies of \( K_{1,1,2} \) in \( G \) are those that are formed by picking 4 vertices from the sets \( I_a, I_b \) and \( I_c \) for which \( a, b \) and \( c \) formed a triangle in \( G' \). This means that the number of ways to map a \( K_{1,1,2} \) into \( G \) is

\[
12m|S| \left( \frac{n}{6m} \right)^2 \left( \frac{|m|}{2} \right) \leq \frac{n^4}{6^3m^28\sqrt{\log m}}.
\]

Now setting

\[
\gamma = \frac{1}{6^2m^8\sqrt{\log m}}
\]

we see that the density of triangles in \( G \) is \( \gamma \), while the density of \( K_{1,1,2} \) in \( G \) is at most

\[
\frac{1}{6^3m^28\sqrt{\log m}} = \gamma^2 2^c \sqrt{\log 1/\gamma} = \gamma^2 - o(1),
\]

for some absolute constant \( c \), thus completing the proof.

For completeness, we reproduce the short proof of Theorem 3. We will need the so called “triangle removal lemma” of [17]:

**Theorem 9 ([17])** If \( G \) is an \( n \) vertex graph from which one should remove at least \( \varepsilon n^2 \) edges in order to destroy all triangles, then \( G \) contains at least \( f(\varepsilon)n^3 \) triangles.

**Proof of Theorem 3**: Suppose \( G \) has \( \gamma n^3 \) triangles. Then by Theorem 9 we know that \( G \) contains a set of edges \( F \) of size at most \( f(\gamma)n^2 \), the removal of which makes \( G \) triangle-free, where \( f(\gamma) = o(1) \). For each edge \( e \in E(G) \) let \( t(e) \) be the number of triangles in \( G \) containing \( e \) as one of their edges. Observe that a copy of \( K_{1,1,2} \) is obtained by taking two triangles sharing an edge. Also, as the removal of edges in \( F \) makes \( G \) triangle-free, every triangle in \( G \) has an edge of \( F \) as one if its edges. Therefore, by Cauchy-Schwartz we have that the number of copies of \( K_{1,1,2} \) in \( G \) is

\[
\sum_{e \in F} t(e)^2 \geq \frac{1}{|F|} \left( \sum_{e \in F} t(e) \right)^2 \geq \frac{1}{|F|} \gamma^2 n^6 \geq \frac{1}{f(\gamma)} \gamma^2 n^4,
\]

implying the desired result with \( 1/f(\gamma) \) being the \( \omega(1) \) term in the statement of the theorem.
4 Concluding Remarks and Open Problems

• Our main result given in Theorem 1 states that in any graph $G$ with $K_3$-density $\gamma$, there is some $t$ for which the $K_{t,t,t}$-density in $G$ is almost as large as the $K_{t,t,t}$-density in a random graph with the same triangle density. This motivates us to raise the following conjecture, stating that the upper bound in Proposition 1.1 gives the correct order of $f_B(\gamma)$ and hence that the answer to Problem 1 is positive for balanced blow-ups of $K_3$.

Conjecture 1 Let $t \geq 2$ and set $B = K_{t,t,t}$. Then

$$f_B(\gamma) = \gamma^2.$$  

We remind the reader of our remark in Section 1 that any (polynomial) improvement over the lower bound of Proposition 1.1 would have interesting applications in theoretical computer science.

• Theorems 4 and the upper bound of 5 show that when considering the skewed blowups $B = K_{1,1,2}$ or $B = K_{1,2,2}$ the random graph does not minimize the density of $B$. This motivates us to raise the following conjecture.

Conjecture 2 If $a, b, c$ are not all equal, then there is $c > 0$ such that

$$f_B(\gamma) \leq \gamma^{\frac{1}{3}(ab+bc+ac)+c}.$$  

The construction we used in order to prove Theorems 4 and 5 can be used to verify Conjecture 2 for other skewed blowups. For example, for every $B = K_{1,1,t}$ it establishes that $f_B \leq \gamma^{t-o(1)}$ which matches the lower bound in Proposition 1.1 up to the $o(1)$ term. It may be possible to modify this construction in order to establish Conjecture 2.

• As we have mentioned in Section 1, Alon [1] has recently shown that if the triangle density of $G$ is $\gamma$ then its $K_{t,t,t}$-density is at least $\gamma^{O(t^2/\gamma^3)}$. We now show that a slightly weaker bound can be derived directly from a recent result of Nikiforov [15].

Theorem 10 If a graph has triangle density $\gamma$, then its $K_{t,t,t}$-density is at least $2^{-O(t^2/\gamma^4)}$.

Proof (sketch): By a result of Nikiforov [15], a graph with triangle density $\gamma$ has a $K_{t,t,t}$ with $t = \gamma^3 \log n$. Or in other words, every graph on at least $2^{t^2/\gamma^4}$ vertices, whose triangle density is $\gamma$, has a copy of $K_{t,t,t}$. As in the proof of Lemma 2.2, if a graph has triangle density $\gamma$, then most subsets of vertices of size $2^{t^2/\gamma^4}$ have (roughly) the same density, so they contain
a $K_{t,t,t}$. We thus get that $G$ has $\frac{n}{2}(\binom{n}{3})$ sets which contain a $K_{t,t,t}$ and since each $K_{t,t,t}$ is counted $\binom{n-3t}{3}$ times we get that $G$ has $n^3/2O(t^3)$ distinct copies of $K_{t,t,t}$.

We note that although this is not stated explicitly in [15], Nikiforov’s arguments actually shows that a graph of triangle density $\gamma$ has a $K_{t,t,t}$ with $t = \gamma^2 \log_1/\gamma n$ and so the argument above can actually give Alon’s result.

- Observe that in a random graph $G(n, \gamma^{1/3})$, whose triangle density is $\gamma$, we expect to find a $K_{t,t,t}$ with $t = c \log_1/\gamma n$ for some absolute constant $c$. It seems very interesting to try and improve Nikiforov’s result [15] mentioned above by showing the following:

**Conjecture 3** There is an absolute constant $c > 0$, such that if a graph $G$ has triangle density $\gamma$, then $G$ has a $K_{t,t,t}$ of size $t = c \log_1/\gamma n$.

Besides being an interesting problem on its own, we note that such an improved bound, together with the argument we gave in the proof of Theorem[10], would imply that if the triangle density of a graph is $\gamma$, then its $K_{t,t,t}$ density is at least $\gamma^{O(t^2)}$, which would come close to establishing Conjecture[1]

Acknowledgement: We would like to thank Noga Alon, Guy Kindler and Benny Sudakov for helpful discussions related to this paper.

References

[1] N. Alon, private communication, 2008.

[2] N. Alon, E. Fischer, M. Krivelevich and M. Szegedy, Efficient testing of large graphs, Proc. of 40th FOCS, New York, NY, IEEE (1999), 656–666. Also: Combinatorica 20 (2000), 451-476.

[3] F. A. Behrend, On sets of integers which contain no three terms in arithmetic progression, Proc. National Academy ofSciences USA 32 (1946), 331-332.

[4] B. Bollobás and P. Erdős, On the structure of edge graphs, Bull. London Math. Soc. 5 (1973), 317-321.

[5] B. Bollobás, P. Erdős and M. Simonovits, On the structure of edge graphs II, J. London Math. Soc. (2) 12 (1976), 219-224.

[6] S. A. Burr and P. Erdős, On the magnitude of generalized Ramsey numbers for graphs. In Infinite and Finite Sets I, Vol. 10 of Colloquia Mathematica Soc. Janos Bolyai, North-Holland, Amster- dam/London, 1975, pp. 214-240.
[7] F. R. K. Chung, R. L. Graham and R. M. Wilson, Quasi-random graphs, Combinatorica 9 (1989), 345-362.

[8] P. Erdős, On some new inequalities concerning extremal properties of graphs, Theory of Graphs (Proc. Colloq., Tihany, 1966), Academic Press, New York, 1968, 77–81.

[9] P. Erdős and A. H. Stone, On the structure of linear graphs, Bull. Amer. Math. Soc. 52 (1946), 1089-1091.

[10] E. Fischer, The difficulty of testing for isomorphism against a graph that is given in advance, SIAM J. on Computing 34 (2005), 1147-1158.

[11] A. W. Goodman, On sets of acquaintances and strangers at any party, Amer. Math. Monthly 66 (1959), 778-783.

[12] J. Komlós and M. Simonovits, Szemerédi’s Regularity Lemma and its applications in graph theory. In: Combinatorics, Paul Erdős is Eighty, Vol II (D. Miklós, V. T. Sós, T. Szönyi eds.), János Bolyai Math. Soc., Budapest (1996), 295–352.

[13] M. Krivelevich and B. Sudakov, Pseudo-random graphs, in: More Sets, Graphs and Numbers, Bolyai Society Mathematical Studies 15, Springer, 2006, 199-262.

[14] L. Lovász and M. Szegedy, Finitely forcible graphons, arXiv:0901.0929v1.

[15] V. Nikiforov, Graphs with many $r$-cliques have large complete $r$-partite subgraphs, Bull. London Math. Soc. 40 (2008), 23-25

[16] A. Razborov, On the minimal density of triangles in graphs, Combinatorics, Probability and Computing, 17 (2008), 603-618.

[17] I. Ruzsa and E. Szemerédi, Triple systems with no six points carrying three triangles, in Combinatorics (Keszthely, 1976), Coll. Math. Soc. J. Bolyai 18, Volume II, 939-945.

[18] A. F. Sidorenko, A correlation inequality for bipartite graphs, Graphs Combin. 9 (1993), 201-204.

[19] M. Simonovits, Extremal graph problems, degenerate extremal problems and super-saturated graphs, in: Progress in graph theory (J. A. Bondy ed.), Academic, New York, 1984, 419-437.

[20] E. Szemerédi, Regular partitions of graphs, In: Proc. Colloque Inter. CNRS (J. C. Bermond, J. C. Fourrier, M. Las Vergnas and D. Sotteau, eds.), 1978, 399–401.

[21] T. Tao, Structure and Randomness: pages from year one of a mathematical blog, AMS 2008.