ANYONS ON A TORUS

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Abstract

We prove the equivalence between anyon quantum mechanics on a torus and Chern-Simons gauge theory. It is also shown that the Hamiltonian and total momenta commute among themselves only in the physical Hilbert space.

INTRODUCTION

A few years ago Einarsson\cite{1} gave braid group analysis of quantum mechanics of $q$ anyons (with the statistics phase $\theta_s$) on a torus. He showed that Schrödinger wavefunctions must have $M$-components, and that $q$, $M$, and $\theta_s$ must satisfy

$$e^{2iq\theta_s} = 1 = e^{2iM\theta_s}. \tag{1}$$

In particular, for $\theta_s = \pi/N$ ($N$: an integer), $q$ and $M$ must be multiples of $N$.

On the other hand it is known\cite{2} that anyon quantum mechanics on a plane is equivalent to Chern-Simons gauge theory coupled to non-relativistic matter fields. Does the equivalence remain valid on a torus? If it does, how does the constraint (1) result from Chern-Simons gauge theory?

In this paper we show\cite{3,4} that the equivalence is exact and everything follows from Chern-Simons gauge theory.

CHERN-SIMONS THEORY

The Lagrangian is given by

$$\mathcal{L} = \frac{\kappa}{4\pi} \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho$$

$$+ i\psi^\dagger D_0 \psi - \frac{1}{2m} (D_k \psi)^\dagger (D_k \psi), \tag{2}$$

where $D_0 = \partial_0 + ia_0$ and $D_k = \partial_k - ia_k$. The Chern-Simons coefficient $\kappa$ is related to $\theta_s$ by $\theta_s = \pi/\kappa$. To be definite, $\psi(x)$ is taken to be a fermion field.

On a torus ($0<x_j<L_j$, $j=1,2$) there are two non-integrable phases of Wilson line integrals along non-contractible loops:

$$\exp \left( i \int_{C_j} dx \cdot a \right) \to W_j = e^{i\theta_j}. \tag{3}$$

They form a conjugate pair:\cite{5}

$$[\theta_1, \theta_2] = \frac{2\pi i}{\kappa}. \tag{4}$$

Fields are not single-valued in general:

$$a_\mu[T_j x] = a_\mu[x] + \partial_\mu \beta_j(x)$$

$$\psi[T_j x] = e^{-i\beta_j(x)} \psi[x] \tag{5}$$
where $T_1 x = (x_1 + L_1, x_2)$ etc. Field operators must be smooth on the covering space, and therefore $\psi[T_1 T_2 x] = \psi[T_2 T_1 x]$, from which the flux quantization condition follows:

$$\Phi = -\int dx \ f_{12} = 2\pi m \ (m: \text{integer}). \quad (6)$$

By solving Chern-Simons field equations $(\kappa/4\pi)\varepsilon^{\mu\nu\rho} f_{\mu\nu} = j^\rho$, $a_\mu(x)$ can be expressed in terms of $\theta_j(t)$ and $\psi(x)$. The resulting Hamiltonian is

$$H = \frac{1}{2m} \int dx \ (D_k \psi)\dagger(D_k \psi),$$

$$a^j(x) = \frac{\theta_j(t)}{L_j} - \frac{\Phi}{2L_j L_2} e^{ik x_k} + \int dy \ e^{ik \partial_y G(x-y)} \left( \frac{2\pi}{\kappa} \psi\dagger \psi(y) + \frac{\Phi}{L_1 L_2} \right) \quad (7)$$

where $G(r)$ is the periodic Green’s function on a torus satisfying $\Delta G(r) = \delta(r) - (1/L_1 L_2)$. Furthermore one has to impose a constraint on physical states,

$$Q + \frac{\kappa}{2\pi} \Phi \approx 0 \quad (Q = \int dx \ \psi\dagger \psi) \quad (8)$$

as (8), despite being a part of the original Chern-Simons field equations, does not follow from the Hamiltonian in (7).

**VACUUM**

The field theory defined by (7) and (8) with commutation relations for $\theta_j$ and $\psi(x)$ is invariant under large gauge transformations:

$$\theta_j \rightarrow \theta_j + 2\pi n_j,$$

$$\psi(x) \rightarrow e^{2\pi i(n_1 x_1/L_1 + n_2 x_2/L_2)} \psi(x) \quad (9)$$

where $(n_1, n_2) = (1,0)$ and $(0,1)$. The associated unitary operators are given by

$$U_j = \exp \left\{ ie^{ik \kappa} \theta_k \right\} \quad -2\pi i \int dx \frac{x_j}{L_j} \psi\dagger \psi(x) \right\} . \quad (10)$$

$U_j$’s and $W_j$’s satisfy

$$U_1 U_2 = U_2 U_1 e^{-2\pi i \kappa},$$

$$W_1 W_2 = W_2 W_1 e^{-2\pi i \kappa} . \quad (11)$$

Two gauge transformations $U_1$ and $U_2$ do not commute with each other in general.\(^5\)\(^6\) Consistent quantum theory is possible only if the coefficient $\kappa$ is a rational number.\(^7\) Two cases are important, an integer $\kappa$ in the anyon superconductivity and an inverse integer $\kappa$ in the fractional quantum Hall effect.

Let us concentrate on the integer $\kappa$=$N$ case, in which $U_1$ and $U_2$ commute. As a consequence of (11) there are $N$ degenerate vacua. Choosing $U_j |0_a\rangle = e^{i \alpha_j} |0_a\rangle$, one finds

$$W_1 |0_a\rangle = e^{-i \alpha_2 / N} |0_a\rangle ,$$

$$W_2 |0_a\rangle = e^{+i \alpha_1 / N} |0_{a-1}\rangle . \quad (12)$$

**WAVEFUNCTIONS**

A $q$-particle Schrödinger wavefunction in quantum mechanics is a matrix element of $q$ field operators $\psi(x)$ between the vacuum and corresponding $q$-particle state $|\Psi_q\rangle$. One elaboration is necessary on a torus. It is given by

$$\phi_a^q(t; x_1, \cdots, x_q) = \langle 0_a | \Omega \psi(1) \cdots \psi(q) | \Psi_q \rangle ,$$

$$\Omega = \exp \left\{ -i \sum_{p=1}^q \left( \frac{x_1^p}{L_1} \theta_1 + \frac{x_2^p}{L_2} \theta_2 \right) \right\} \quad (13)$$

A couple of things should be noted. There are $N$ degenerate vacua so that the wavefunction must have $N$-components: $\phi_a^q \ (a = 1, \cdots, N)$. Secondly, the operator $\Omega$ is necessary in the definition of $\phi_a^q$ to make it invariant under large gauge transformation (9).
Recalling \( \theta_s = \pi / \kappa \), we see that the constraint (1) is satisfied. We have just shown that \( M = N \), and (6) and (8) imply that \( q = mN \).

**BRAID GROUP**

The wavefunction \( \phi^f \) in (13) satisfies the braid group algebra on a torus. There are three sets of operations on a torus: (a) \( \sigma_j \): the (counterclockwise) interchange of the \( j \)th and \((j + 1)\)th particles, (b) \( \tau_j \): the loop transport of the \( j \)th particle in the \( x_1 \)-direction, (c) \( \rho_j \): the corresponding transport in the \( x_2 \)-direction. These three, \( \sigma_j \), \( \tau_j \), and \( \rho_j \) satisfy the braid group algebra, from which Einarsson derived the aforementioned constraint.\(^1\)

Action of these operators on \( \phi^f \) is simple:

\[
\begin{align*}
\sigma_j &: \ x_j \leftrightarrow x_{j+1} \\
\tau_j &: \ x_j \rightarrow T_1 x_j \\
\rho_j &: \ x_j \rightarrow T_2 x_j
\end{align*}
\]

(14)

In particular, since \( \psi \) is a fermion,

\[
\sigma_j \phi^f = -\phi^f .
\]

(15)

\( \phi^f \) is the wavefunction in the fermion representation.

As a consequence of (14), the braid group algebra is trivially satisfied. However, \( \phi^f \) is transformed quite nontrivially under the action of \( \tau_j \) and \( \rho_j \):

\[
\begin{align*}
(\tau_j \phi^f)_a &= \exp \left[ -i \beta_1(x^j) + i \pi N \sum_p \frac{x_p^a}{L_2} + 2 \pi i a \right] \phi^f_a \\
(\rho_j \phi^f)_a &= \exp \left[ -i \beta_2(x^j) - i \pi N \sum_p \frac{x_p^a}{L_1} \right] \phi^f_{a-1}
\end{align*}
\]

(16)

Notice that \( \phi^f \) is a regular function of \( \{x^p\} \), without any singularity. Under \( \tau_j \) and \( \rho_j \), \( \phi^f \) picks up \( \{x^p\} \) dependent phases, natural in gauge theory. In Einarsson’s analysis it was implicitly assumed that phases must be constant, which demands multi-valued wavefunctions.

**SINGULAR TRANSFORMATION**

Einarsson’s wavefunction, \( \phi^E \), is related to \( \phi^f \) by a singular gauge transformation.\(^3,8\)

\[
\begin{align*}
\phi^E_a &= \Omega_{\mathrm{sing}} \phi^f_a \\
\Omega_{\mathrm{sing}} &= \prod_{j<k} \left[ \frac{\vartheta_1(w_{jk})}{\vartheta_1(w_{jk})} \right]^{\frac{1}{\pi}} \cdot e^{i \pi x_1^j x_2^k / L_1 L_2}
\end{align*}
\]

(17)

where \( \vartheta_1(w) \) is Jacobi’s theta function, \( x_1^j = x_1 - x_1^k \), \( w = (x_1 + i x_2) / L_1 \), etc.

It is straightforward to see that

\[
\sigma_j \phi^E = -e^{-i \pi/N} \phi^E .
\]

(18)

The action of \( \tau_j \) and \( \rho_j \) is somewhat simplified. The result generalizes Einarsson’s to arbitrary particle configurations. All topological information is contained in \( \Omega_{\mathrm{sing}} \).

**SCHRÖDINGER EQUATION**

The Schrödinger equation for \( \phi^f \)

\[
\begin{align*}
i \frac{\partial}{\partial t} \phi^f(t; x_1, \ldots, x_q) &= \hat{H} \phi^f \\
&= (q!)^{-1/2} \langle 0_a | \Omega \psi(1) \cdots \psi(q) \rangle H |\Psi_q\rangle
\end{align*}
\]

(19)

is obtained by permuting \( H \) (defined in(7)) to the left of \( \Omega \) and \( \psi(j) \)’s. \( \hat{H} \) is given by

\[
\begin{align*}
\hat{H} &= -\frac{1}{2m} \sum_j \left( \nabla^{(j)} - i \mathbf{A}^{(j)} \right)^2 \\
\mathbf{A}^{(j)k} &= \frac{2 \pi}{N} \sum_{p \neq j} \left( \frac{x_p^j - x_p^k}{2 L_1 L_2} + \nabla^{(j)} G(x^j - x^k) \right)
\end{align*}
\]

(20)
The equation for $\phi^E$ is obtained by inserting (17) into (20). The result is very simple:

$$i\frac{\partial}{\partial t} \phi^E_a = -\frac{1}{2m} \sum_j (\nabla^{(j)})^2 \phi^E_a .$$

(21)

It is a “free” equation. The anyon interaction is hidden in the boundary condition (18).

**TRANSLATION INVARIANCE**

The total momentum operator in the second quantized theory is given by

$$P^k = -i \int d\mathbf{x} \, \psi^\dagger D_k \psi .$$

(22)

The corresponding operator in quantum mechanics is found to be

$$\hat{P}^k = -i \sum_j \nabla^{(j)}_k ,$$

(23)

where $\hat{P}^k \phi^E_a = (q!)^{-1/2} (0_a| \cdots P^k |\Psi_q).$

$P^k$ and $H$ in (7) form an algebra:

$$[P^j, P^k] = i \epsilon^{jk} \frac{2\pi}{\kappa L_1 L_2} Q \left( Q + \frac{\kappa}{2\pi} \Phi \right) ,$$

$$[P^j, H] = i \epsilon^{jk} \frac{2\pi}{\kappa L_1 L_2} J^k \left( Q + \frac{\kappa}{2\pi} \Phi \right) .$$

(24)

where $J^k = \int d\mathbf{x} j^k$ and $= P^k / m$ in the nonrelativistic theory. They do not commute among themselves as operators, but do commute in the physical Hilbert space defined by the constraint (8).

Hence the translation invariance is maintained in the Hilbert space. Obviously the corresponding operators in quantum mechanics, $\hat{P}^j$ and $H$, commute with each other.\(^\text{10}\)

Another important set of commutators are

$$[W_j, P^k] = \epsilon^{jk} \frac{\pi}{\kappa L_1 L_2} \{ Q, W_j \} ,$$

$$[W_j, H] = \epsilon^{jk} \frac{\pi}{\kappa L_1 L_2} \{ J^k, W_j \} .$$

(25)

It turns out that the relations (24) and (25) remain valid even for relativistic theory with Dirac fields. They are universal relations in Chern-Simons theory.

**SUMMARY**

Chern-Simons gauge theory was born many years ago. It is simple, beautiful, and rich. It is important in the fractional quantum Hall effect and superconductivity. It is the most powerful and fruitful way of describing anyon physics, embodied with a unique algebraic structure. Much is hidden to be discovered in future.

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