EXISTENCE OF SOLUTION FOR A CLASS OF ELLIPTIC EQUATION WITH DISCONTINUOUS NONLINEARITY AND ASYMPTOTICALLY LINEAR

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Abstract. This paper concerns the existence of a nontrivial solution for the following problem

\begin{align*}
(P) \quad \left\{ \begin{array}{l}
-\Delta u + V(x)u \in \partial_s F(x,u) \text{ a.e. in } \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N),
\end{array} \right.
\end{align*}

where \( F(x,t) = \int_0^t f(x,s) \, ds \), \( f \) is a discontinuous function and asymptotically linear at infinity, \( \lambda = 0 \) is in a spectral gap of \( -\Delta + V \), and \( \partial_t F \) denotes the generalized gradient of \( F \) with respect to variable \( t \). Here, by employing Variational Methods for Locally Lipschitz Functionals, we establish the existence of solution when \( f \) is periodic and non periodic.

1. Introduction

In this paper we study the existence of nontrivial solution for the following class of elliptic problems

\begin{align*}
(P) \quad \left\{ \begin{array}{l}
-\Delta u + V(x)u \in \partial_s F(x,u) \text{ a.e. in } \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N),
\end{array} \right.
\end{align*}

where \( F(x,t) = \int_0^t f(x,s) \, ds \), \( f \) is a discontinuous function and asymptotically linear at infinity, \( \lambda = 0 \) is in a spectral gap of \( -\Delta + V \) and \( \partial_t F \) denotes the generalized gradient of \( F \) with respect to variable \( t \).

The problem \( (P) \) for the case where \( f \) is a continuous function becomes

\begin{align*}
(P_1) \quad \left\{ \begin{array}{l}
-\Delta u + V(x)u = f(x,u) \text{ a.e. in } \mathbb{R}^N, \\
u \in H^1(\mathbb{R}^N),
\end{array} \right.
\end{align*}

and it has been studied for some authors. In [20], Li and Szulkin have improved the generalized link theorem obtained in Kryszewski and Szulkin [17] to establish the existence of solution for \( (P_1) \) with \( f \) being asymptotically linear at infinity and asymptotic to a \( \mathbb{Z}^N \)-periodic function. Motivated by results found in [20], some authors studied the problem \( (P_1) \) with the same conditions on \( V \) and supposing other conditions on \( f \), but \( f \) still being asymptotically linear, see for example, Ding and Lee [12], Chen and Dawei [11], Tang [30], Lin and Tang [23], Wu and Qin [35] and their references.

The main motivation of the present paper comes from the study found in Li and Szulkin [20] and Alves and Patricio [2]. In [2], the authors have studied the existence of nontrivial solution for problem \( (P) \) for a class of superlinear problem where the nonlinearity is a discontinuous function and \( \lambda = 0 \) is in a spectral gap of \( -\Delta + V \). In that paper, it was proved a generalized link theorem for Locally Lipschitz functionals that improves the generalized link theorem found

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in [17], and after that, the authors used their link theorem to prove the existence of nontrivial solution for \((P)\).

Hereafter, we assume the following conditions on \(f\) and \(V\):

\[ (H1) \quad V : \mathbb{R}^N \to \mathbb{R} \text{ is continuous, } \mathbb{Z}^N\text{-periodic and } \quad 0 \not\in \sigma(-\Delta + V). \]

In the sequel, \((\mu_-, \mu_1)\) is the spectral gap containing 0 and \(\mu_0 := \min\{-\mu_1, \mu_1\}\).

\[ (H2) \quad \lim_{t \to 0} \frac{f(x,t)}{t} = 0 \quad \text{uniformly in } \quad x \in \mathbb{R}^N, \]

and for any \(0 < a < b < +\infty\), there is \(M = M(a,b) > 0\) such that
\[ |f(x,t)| \leq M, \quad \forall x \in \mathbb{R}^N \quad \text{and} \quad |t| \in [a,b]. \]

\[ (H3) \quad f(x,t) = V_\infty(x) + f_\infty(x,t) \quad \text{where} \quad f_\infty(x,t) \text{ is a measurable function defined on } \mathbb{R}^N \times \mathbb{R}, \]
\[ \lim_{t \to \infty} \frac{f_\infty(x,t)}{t} = 0 \quad \text{uniformly with respect to } \quad x \quad \text{as} \quad |t| \to +\infty, \quad V_\infty \text{ is } \mathbb{Z}^N\text{-periodic and} \]
\[ V_\infty(x) \geq \mu, \quad \forall x \in \mathbb{R}^N, \text{ for some } \mu > \mu_1. \]

Moreover, the functions
\[ \underline{f}(x,t) = \lim_{r \downarrow 0} \text{ess inf} \{f(x,s); |s-t| < r\} \]
and
\[ \overline{f}(x,t) = \lim_{r \downarrow 0} \text{ess sup} \{f(x,s); |s-t| < r\}. \]

are \(N\)-measurable functions, see Chang [6–8] for more details.

\[ (H4) \quad 0 \leq F(x,t) \leq \frac{1}{2} \rho t \quad \text{for all } \rho \in \partial_t F(x,t) \text{ and } t \in \mathbb{R}, \text{ where} \]
\[ F(x,t) = \int_0^t f(x,s)ds. \]

\[ (H5) \quad \text{There exists } \delta \in (0, \mu_0) \text{ such that if } \quad \frac{\rho}{t} \geq \mu_0 - \delta \quad \text{and} \quad \rho \in \partial_t F(x,t), \text{ then } \frac{1}{2} \rho t - F(x,t) \geq \delta. \]

A nonlinearity \(f\) that satisfies the conditions above is the following: Fixed \(b > 0\), let us consider the function
\[ f(x,t) = \begin{cases} \mu t - \mu \arctan(t), & \text{if } |t| \leq b \\ \mu t + \mu (\gamma - 1) \arctan(t), & \text{if } |t| > b \end{cases} \]

where \(\mu > \mu_1\) and \(0 < \gamma \) is such that \(\gamma < \frac{\mu_0}{\mu}\).

Our main result is the following:

**Theorem 1.1.** (The periodic case) Assume \((H1) - (H5)\) and that \(f\) is \(\mathbb{Z}^N\)-periodic. Then, the problem \((P)\) has a nontrivial solution.

The Theorem 1.1 complements the study made in [20] and [2] in the following sense: It complements [20], because in that paper the nonlinearity is continuous, while in the present paper the nonlinearity is discontinuous. Moreover, since in our paper the functional is not \(C^1\), it was necessary to prove a version for Locally Lipschitz Functionals of the linking theorem developed found [20], see Sections 4 and 5. Related to the [2], we are working with a nonlinearity that is asymptotically linear at infinity, while in that paper the nonlinearity is superlinear.

In order to study the non periodic case, that is, the case where \(f\) is not necessarily a periodic function, we will assume that there is \(h \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})\) with \(h_t \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})\) such that
(H6) \( h \) is \( \mathbb{Z}^N \)-periodic and
\[
0 < h(x, t) t < t^2 h_t(x, t) \leq V_{\infty}(x) t^2,
\]
whenever \( t \neq 0 \),
where \( h_t \) denotes the derived from function \( h \) with respect to \( t \) and \( V_{\infty} \) was given in (H3).

(H7) \( F(x, t) \geq H(x, t) \) for all \( t \in \mathbb{R} \) and
\[
|f(x, t) - h(x, t)| \leq a(x)|t|, \ \forall \ t \in \mathbb{R},
\]
\[
|\rho - h(x, t)| \leq a(x)|t|, \ \forall \ t \in \mathbb{R} \text{ and } \forall \ \rho \in \partial_t F(x, t).
\]

where \( a(x) > 0 \) for all \( x \in \mathbb{R}^N \), \( a \in L^\infty(\mathbb{R}^N) \), \( a(x) \to 0 \) as \( |x| \to +\infty \) and
\[
H(x, t) = \int_0^t h(x, s)ds.
\]

Next, we show an example of a function \( f \) that satisfies the assumptions (H2) – (H7). Fixed \( b > 0 \), let us consider the function
\[
f(x, t) = \begin{cases}
h(x, t), & \text{if } |t| \leq b \\
h(x, t) + \mu \gamma a(x) \arctan(t), & \text{if } |t| > b,
\end{cases}
\]
where \( \mu > \mu_1 \) and
(a) \( h(x, t) = \mu \left[ t - \arctan(t) \right] \);
(b) \( a(x) = e^{-|x|^2} \);
(c) \( 0 < \gamma \) is such that \( \|a\|_\infty \gamma < \frac{\mu_0}{\mu} \).

Our main result involving the non periodic case is the following:

**Theorem 1.2.** (The non periodic case) Assume (H1) – (H7). Then, the problem \( (P) \) has a nontrivial solution.

The Theorem 1.2 also complements the study made in [20], because we are considering that \( f \) can be a discontinuous function.

The plan of the paper is as follows. In Section 2, we recall some definitions and basic results on the critical point theory of Locally Lipschitz functionals. In Section 3, we study a deformation lemma for Locally Lipschitz functionals. In Section 4, we prove a linking theorem for Locally Lipschitz Functionals. Finally, in Sections 5 and 6, we employ the linking theorem to prove Theorems 1.1 and 1.2.

Before concluding this section, still in the context of asymptotically linear problems, we would like to cite the papers [1, 3, 9–11, 15, 18, 19, 21, 22, 24, 25, 28, 29, 31–33, 36]

**Notation:** From now on, otherwise mentioned, we use the following notations:
- \( \| \|_X \) denotes the norm of the space \( X \).
- \( X^* \) denotes the dual topological space of \( X \) and \( \| \|_* \) denotes the norm in \( X^* \).
- \( B_r(u) \) is an open ball centered at \( u \in X \) with radius \( r > 0 \).
- \( \| \|_p \) denotes the usual norm of the Lebesgue space \( L^p(\mathbb{R}^N) \), for \( p \in [1, +\infty] \).
- \( l : X \to \mathbb{R} \) denotes a continuous linear functional.
- \( I_d : X \to X \) denotes the identity application.
- \( C_i \) denote (possibly different) any positive constants, whose values are not relevant.
2. Preliminary results

In this section we recall some facts involving nonsmooth analysis and the energy functional associated with problem \((P)\).

2.1. Basic results from nonsmooth analysis. In this subsection, for the reader’s convenience, we recall some definitions and basic results on the critical point theory of Locally Lipschitz Functionals as developed by Chang [6], Clarke [4,5], and Grossinho and Tersian [26].

Let \((X, ||\cdot||_X)\) be a real Banach space. A functional \(I : X \to \mathbb{R}\) is locally Lipschitz, \(I \in \text{Lip}_{loc}(X, \mathbb{R})\) for short, if given \(u \in X\) there is an open neighborhood \(V := V_u \subset X\) of \(u\), and a constant \(K = K_u > 0\) such that

\[|I(v_2) - I(v_1)| \leq K||v_1 - v_2||_X, \quad v_i \in V, \quad i = 1, 2.\]

The generalized directional derivative of \(I\) at \(u\) in the direction of \(v \in X\) is defined by

\[I^\circ(u; v) := \lim_{h \to 0, \delta h \to 0} \frac{1}{\delta} (I(u + h + \delta v) - I(u + h)).\]

The generalized gradient of \(I\) at \(u\) is the set

\[\partial I(u) = \{\xi \in X^* ; I^\circ(u; v) \geq \langle \xi, v \rangle ; \forall v \in X\}.\]

Moreover, we denote by \(\lambda_I(u)\) the following real number

\[\lambda_I(u) := \min\{|||\xi||_* : \xi \in \partial I(u)\}\].

We recall that \(u \in X\) is a critical point of \(I\) if \(0 \in \partial I(u)\), or equivalently, when \(\lambda_I(u) = 0\).

Lemma 2.1. If \(I\) is continuously differentiable to Fréchet in an open neighborhood of \(u \in X\), we have \(\partial I(u) = \{I'(u)\}\).

Lemma 2.2. If \(Q \in C^1(X, \mathbb{R})\) and \(\Psi \in \text{Lip}_{loc}(X, \mathbb{R})\), then for each \(u \in X\)

\[\partial(Q + \Psi)(u) = Q'(u) + \partial \Psi(u).\]

2.2. Preliminaries results involving the energy functional. By assumptions \((H2)\) and \((H3)\), the energy functional associated with problem \((P)\) is given by

\[I(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2)dx - \int_{\mathbb{R}^N} F(x, u)dx, \quad u \in H^1(\mathbb{R}^N),\]

where \(F(x, t) = \int_0^t f(x, s)ds\).

By standard argument, the functional \(Q : H^1(\mathbb{R}^N) \to \mathbb{R}\) given by

\[Q(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)|u|^2)dx\]

belongs to \(C^1(H^1(\mathbb{R}^N), \mathbb{R})\) and

\[Q'(u)v = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)dx, \quad \forall \; u, v \in H^1(\mathbb{R}^N).\]

However, the functional \(\Psi : H^1(\mathbb{R}^N) \to \mathbb{R}\) given by

\[(2.1) \quad \Psi(u) = \int_{\mathbb{R}^N} F(x, u)dx\]

is only Locally Lipschitz, because the function \(f\) is not continuous in whole \(\mathbb{R}\). Hence, the functional \(I\) is also Locally Lipschitz, from where it follows that we cannot use the variational
methods for $C^1$ functionals in order to get critical points for $I$, and so, we must use variational methods for Locally Lipschitz. Have this in mind, by Lemma 2.2,
\[ \partial I(u) = Q'(u) - \partial \Psi(u), \quad \forall u \in H^1(\mathbb{R}^N). \]

By $(H1)$, it is well known that $H^1(\mathbb{R}^N) = E^+ \oplus E^-$ is an orthogonal decomposition and there is an equivalent norm $|| \cdot ||_{H^1(\mathbb{R}^N)}$ with
\[ ||u||^2 = ||u^+||^2 - ||u^-||^2, \quad \forall u = u^+ + u^- \in E^+ \oplus E^-. \]

Recalling that $(\mu_1, \mu_1)$ is the spectral gap of $-\Delta + V$ containing 0 and $\mu_0 = \min\{ -\mu_1, \mu_1 \}$, by Stuart [27],
\[ \|u^+\|^2 \geq \mu_1 \|u^+\|^2_2, \quad \forall u^+ \in E^+ \quad \text{and} \quad \|u^-\|^2 \geq -\mu_1 \|u^-\|^2_2, \quad \forall u^- \in E^-.
\]

Therefore,
\[ \|u\|^2 \geq \mu_0 \|u\|^2_2, \quad \forall u \in H^1(\mathbb{R}^N). \]

Moreover, it is possible to prove that
\[ \langle Q'(u), v \rangle = \langle u^+, v \rangle - \langle u^-, v \rangle, \quad \forall u, v \in H^1(\mathbb{R}^N), \]

where $(.,.)$ denotes the usual inner product in $H^1(\mathbb{R}^N)$.

Using the notations above, we can rewrite $I$ of the form
\[ I(u) = \frac{1}{2} \|u^+\|^2 - \frac{1}{2} \|u^-\|^2 - \Psi(u), \quad \forall u = u^+ + u^- \in E^+ \oplus E^-.
\]

In order to apply variational methods for Locally Lipschitz, we claim that $\Psi : L^2(\mathbb{R}^N) \to \mathbb{R}$ is well defined, because by $(H2) - (H3)$,
\[ |f(x, t)| \leq C|t|, \quad \forall t \in \mathbb{R}, \quad \forall x \in \mathbb{R}^N, \]

and so,
\[ |F(x, t)| \leq C|x|^2, \quad \forall t \in \mathbb{R}, \quad \forall x \in \mathbb{R}^N.
\]

Consequently,
\[ \partial \Psi(u) \subset \partial_t F(x, u) = \left[ \overline{f(x, u(x))}, \overline{f(x, u(x))} \right] \text{ a.e in } \mathbb{R}^N,
\]

where
\[ \underline{f(x, t)} = \lim_{r \downarrow 0} \text{ess inf} \{ f(x, s); |s - t| < r \}
\]

and
\[ \overline{f(x, t)} = \lim_{r \downarrow 0} \text{ess sup} \{ f(x, s); |s - t| < r \}.
\]

The inclusion above means that given $\xi \in \partial \Psi(u) \subset (L^2(\mathbb{R}^N))^* \approx L^2(\mathbb{R}^N)$, there is $\tilde{\xi} \in L^2(\mathbb{R}^N)$ such that
\[ \cdot \langle \xi, v \rangle = \int_{\mathbb{R}^N} \tilde{\xi} v, \quad \forall v \in L^2(\mathbb{R}^N), \]

\[ \cdot \xi(x) \in \partial_t F(x, u(x)) = \left[ \underline{f(x, u(x))}, \overline{f(x, u(x))} \right] \text{ a.e in } \mathbb{R}^N.
\]

The following proposition is very important to establish the existence of a critical point for the functional $I$.

**Proposition 2.3.** (See [2]). If $(u_n) \subset H^1(\mathbb{R}^N)$ is such that $u_n \rightharpoonup u_0$ in $H^1(\mathbb{R}^N)$ and $\rho_n \in \partial \Psi(u_n)$ satisfies $\rho_n \rightharpoonup \rho_0$ in $(H^1(\mathbb{R}^N))^*$, then $\rho_0 \in \partial \Psi(u_0)$.
3. A special deformation lemma

From now on, $X$ is a Hilbert space with $X = Y \oplus Z$, where $Y$ is a separable closed subspace of $X$ and $Z = Y^\perp$. If $u \in X$, $u^+$ and $u^-$ denote the orthogonal projections from $X$ in $Z$ and in $Y$, respectively. In $X$ let us define the norm

$$||| \cdot ||| : X \to \mathbb{R}$$

$$u \mapsto |||u||| = \max \left\{ ||u^+||, \sum_{k=1}^{\infty} \frac{1}{2^k} \|u^-(e_k)\| \right\},$$

where $(e_k)$ is a total orthonormal sequence in $Y$. The topology on $X$ generated by $||| \cdot |||$ will be denoted by $\tau$ and all topological notions related to it will include this symbol. From [17], if $(u_n) \subset X$ is a bounded sequence, then

$$u_n \overset{\tau}{\rightharpoonup} u \text{ in } X \iff u_n^- \rightharpoonup u^- \text{ and } u_n^+ \rightharpoonup u^+ \text{ in } X.$$ 

Let $I : X \to \mathbb{R}$ be a Locally Lipschitz functional, $I \in \text{Lip}_{loc}(X, \mathbb{R})$ for short, of the form

$$I(u) = \frac{1}{2} ||u^+||^2 - \frac{1}{2} ||u^-||^2 - \Psi(u),$$

where $\Psi \in \text{Lip}_{loc}(X, \mathbb{R})$ is bounded from below, $I$ is $\tau$-upper semicontinuous, and $|| \cdot ||$ is an equivalent norm to $|| \cdot ||_X$. Moreover, we suppose that

$$\partial I(u) = Q'(u) - \partial \Psi(u),$$

where $Q \in C^1(X, \mathbb{R})$ and

$$\langle Q'(u), v \rangle = (u^+, v) - (u^-, v), \quad u, v \in X,$$

with $(\cdot, \cdot)$ being the inner product of $X$.

From now on, we will say that a functional $I : X \to \mathbb{R}$ verifies the condition $(H)$ when:

If $(u_n) \subset I^{-1}(\{\alpha, \beta\})$ is such that $u_n \overset{\tau}{\rightharpoonup} u_0$ in $X$, then there exists $M > 0$ such that $\partial \Psi(u_n) \subset B_M(0) \subset X^*$, $\forall n \in \mathbb{N}$. In addition, if $\xi_n \in \partial I(u_n)$ with $\xi_n \overset{\tau}{\rightharpoonup} \xi_0$ in $X^*$, we have $\xi_0 \in \partial I(u_0)$.

**Lemma 3.1.** If $A \subset X$ and $\varepsilon > 0$ is such that

$$(1 + ||u||)\lambda_I(u) \geq \varepsilon, \quad \forall u \in A,$$

then, for each $u \in A$, there exists $\chi_u \in X$ with $||\chi_u|| = 1$ satisfying

$$\langle l, \chi_u \rangle \geq \frac{\varepsilon}{2}, \quad \forall l \in (1 + ||u||)\partial I(u).$$

**Proof.** Given $u \in A$ and $l \in \partial I(u)$, we have

$$0 < \varepsilon \leq (1 + ||u||)\lambda_I(u) \leq (1 + ||u||)||l||_*,$$

that is,

$$B_2^e(0) \cap (1 + ||u||)\partial I(u) = \emptyset.$$ 

Since $B_2^e(0)$ and $(1 + ||u||)\partial I(u)$ are closed sets, convex, not empty and disjoint, by Hahn-Banach Theorem there are $\psi_u \in X^{**}\setminus\{0\}$ and $R > 0$ satisfying

$$\langle \psi_u, l \rangle \geq R \geq \langle \psi_u, w \rangle, \quad \forall l \in (1 + ||u||)\partial I(u), \forall w \in B_2(0).$$

By reflexivity of $X$, there exists $v_u \in X$ such that

$$\langle \psi_u, l \rangle = \langle l, v_u \rangle, \quad \forall l \in X^*.$$

So,

$$\langle l, v_u \rangle \geq \langle w, v_u \rangle, \quad \forall l \in (1 + ||u||)\partial I(u), \forall w \in B_2(0).$$
or yet,
\begin{equation}
\langle l, \chi_u \rangle \geq \langle w, \chi_u \rangle, \quad \forall \ l \in (1 + ||u||)\partial I(u), \forall \ w \in \overline{B}_2(0),
\end{equation}
where \( \chi_u = \frac{n}{||u||} \).

By Hahn-Banach Theorem
\[ 1 = ||\chi_u|| = \max \{ \langle w, \chi_u \rangle : w \in X^* \text{ and } ||w||_* \leq 1 \}, \]
then,
\[ \max \left\{ \left( \frac{\varepsilon}{2} w, \chi_u \right) : w \in X^* \text{ and } ||w||_* \leq 1 \right\} = \frac{\varepsilon}{2}. \]
Consequently,
\begin{equation}
\max \left\{ \langle \tilde{w}, \chi_u \rangle : \tilde{w} \in \overline{B}_2(0) \right\} = \frac{\varepsilon}{2}.
\end{equation}

For \( l \in \partial(1 + ||u||)I(u), \) (3.3) and (3.4) combine to give
\[ \langle l, \chi_u \rangle \geq \sup_{w \in \overline{B}_2(0)} \langle w, \chi_u \rangle = \frac{\varepsilon}{2}, \]
that is,
\[ \langle l, \chi_u \rangle \geq \frac{\varepsilon}{2}, \ \forall \ l \in (1 + ||u||)\partial I(u), \]
finishing the proof. \( \square \)

**Theorem 3.2.** Assume the condition (H) and let \( \alpha < \beta \) and \( \varepsilon > 0 \) satisfying
\[ \lambda_I(u)(1 + ||u||) \geq \varepsilon, \ \forall \ u \in I^{-1}([\alpha, \beta]). \]

Then, for each \( u_0 \in I^{-1}([\alpha, \beta]) \), there exists \( \eta_0 > 0 \) such that
\[ \langle \xi, \chi_{u_0} \rangle > \frac{\varepsilon}{3}, \ \forall \ \xi \in (1 + ||u||)\partial I(u), \ u \in B_{\eta_0}(u_0) \cap I^{-1}([\alpha, \beta]), \]
where \( B_{\eta_0}(u_0) = \{ u \in X ; ||u - u_0|| < \eta_0 \} \) with \( \chi_{u_0} \) given in Lemma 3.1.

**Proof.** Arguing by contradiction, assume that there exist \( (u_n) \subseteq I^{-1}([\alpha, \beta]) \) with \( u_n \rightharpoonup u_0 \) in \( X \) and \( \xi_n \in (1 + ||u_n||)\partial I(u_n) \) such that
\[ \langle \xi_n, \chi_{u_0} \rangle \leq \frac{\varepsilon}{3}, \ \forall \ n \in \mathbb{N}. \]
Note that
\[ \xi_n = (1 + ||u_n||)(Q'(u_n) - \rho_n), \]
with \( \rho_n \in \partial \Psi(u_n) \). It follows from (H) that, going to a subsequence if necessary, there is \( \rho_0 \in X^* \) such that \( \rho_n \rightharpoonup \rho_0 \) in \( X^* \). In addition, \( \rho_0 \in \partial \Psi(u_0) \). On the other hand, using the lower limitation of \( \Psi, \) (3.2), the limit \( u_n \rightharpoonup u_0 \) in \( X \), the fact that \( I(u_n) \geq \alpha \ \forall \ n \in \mathbb{N}, \) and (3.1), we can assume that \( u_n \rightharpoonup u_0 \) in \( X \). Since \( \xi_0 := (1 + ||u_0||)(Q'(u_0) - \rho_0) \in (1 + ||u_0||)\partial I(u_0), \) by Lemma 3.1,
\[ \frac{\varepsilon}{3} < \frac{\varepsilon}{2} \leq (1 + ||u_0||) \langle Q'(u_0) - \rho_0, \chi_{u_0} \rangle. \]

Hence, without loss of generality, we can suppose that
\[ \langle Q'(u_n) - \rho_n, \chi_{u_0} \rangle > 0, \ \forall \ n \in \mathbb{N}. \]
So,
\[ \frac{\varepsilon}{3} < (1 + ||u_0||) \langle Q'(u_0) - \rho_0, \chi_{u_0} \rangle \leq \liminf_{n \to +\infty} (1 + ||u_n||) \langle Q'(u_n) - \rho_n, \chi_{u_0} \rangle = \liminf_{n \to +\infty} \langle \xi_n, \chi_{u_0} \rangle \leq \frac{\varepsilon}{3}, \]
which is absurd. □

**Lemma 3.3.** Under the assumptions of Theorem 3.2, there exist a \( \tau \)-open neighborhood \( V \) of \( I^{\beta} = \{ u \in X : I(u) \leq \beta \} \) and a vector field \( P : V \rightarrow X \) satisfying:

(P1) \( P \) is locally Lipschitz continuous and \( \tau \)-locally Lipschitz continuous,
(P2) each point \( u \in V \) has a \( \tau \)-neighborhood \( V_u \) such that \( P(V_u) \) is contained in a finite-dimensional subspace of \( X \),
(P3) for \( u \in V \), \( ||P(u)|| \leq 1 + 2||u|| \) and \( \langle \xi, P(u) \rangle \geq 0 \), \( \forall \xi \in (1 + ||u||)\partial I(u) \);
(P4) for \( u \in I^{-1}([\alpha, \beta]) \)

\[ \langle \xi, P(u) \rangle \geq \frac{\varepsilon}{3}, \forall \xi \in (1 + ||u||)\partial I(u). \]

**Proof.** For each \( u_0 \in I^{-1}([\alpha, \beta]) \), the Theorem 3.2 guarantees the existence of \( \eta_0 > 0 \) satisfying

\[ \langle \xi, \chi_{u_0} \rangle \geq \frac{\varepsilon}{3}, \forall \xi \in (1 + ||u||)\partial I(u), \quad u \in B_{\eta_0}(u_0) \cap I^{-1}([\alpha, \beta]). \]

**Claim 3.4.** For each \( u \in I^{-1}([\alpha, \beta]) \) there exists a \( \tau \)-neighborhood \( N_u \) of \( u \) such that

\[ ||u|| \leq 2||v||, \forall v \in N_u. \]

If \( u = 0 \), then (3.6) is immediate. Now, for \( u \in I^{-1}([\alpha, \beta]) \) \( \setminus \{ 0 \} \), suppose that (3.6) is not true. Then, there exists

\[ (v_j) \subset B_{\frac{1}{j}}(u) = \left\{ v \in X : ||v - u|| \leq \frac{1}{j} \right\} \]

with

\[ 2||v_j|| \leq ||u||, \forall j \in \mathbb{N}. \]

As \( v_j \xrightarrow{\tau} u \) in \( X \), by (3.1), we derive that \( v_j \rightharpoonup u \) in \( X \). Consequently,

\[ 0 < ||u|| \leq \liminf_j ||v_j|| \leq \frac{||u||}{2}, \]

which is absurd.

Consider \( U_u = B_{\eta_0}(u) \cap I^{-1}([\alpha, \beta]) \cap N_u \), which is still a \( \tau \)-neighborhood of \( u \in I^{-1}([\alpha, \beta]) \).

Since \( I \) is \( \tau \)-upper semicontinuous, then

\[ U_0 = I^{-1}((\alpha, \beta)) \]

is \( \tau \)-open in \( X \). Therefore, the family

\[ \mathcal{N} := \{ U_u \}_{u \in I^{-1}([\alpha, \beta])} \cup U_0 \]

is a \( \tau \)-open covering for \( I^{-1}((\alpha, \beta)) \).

In addition \((I^\beta, \tau)\) is a metric space, then there exists a \( \tau \)-locally finite \( \tau \)-open covering \( \mathcal{V} = \{ \mathcal{V}_i : i \in \mathcal{I} \} \) of \( I^\beta \) (see [13]) more fine than \( \mathcal{N} \). Next, we define the \( \tau \)-open neighborhood of \( I^\beta \) by

\[ V = \bigcup_{i \in \mathcal{I}} \mathcal{V}_i \]

and set \( \{ \gamma_i : i \in \mathcal{I} \} \) as being a \( \tau \)-Lipschitz continuous partition of unity subordinated to \( \mathcal{M} \).

Employing the notations above, we set the vector field \( P : V \rightarrow X \) by

[\[ P(u) = \sum_{i \in \mathcal{I}} \gamma_i(u) w_i, \]

where:

- If \( \mathcal{V}_i \subseteq U_{u_i} \), we choose \( w_i = \chi_{u_i} (1 + ||u||) \) (where \( \chi_{u_i} \in X \) is given in Lemma 3.1).
EXISTENCE OF SOLUTION FOR A CLASS OF ELLIPTIC EQUATION ...

- If $\mathcal{V}_i \subseteq U_0$, we choose $w_i = 0$.

The items (P1), (P2) and (P4) follow in a similar way as done in [2].

(P3) Note that, if $u \in U_0$, by Claim 3.4,

$$||w_i|| = ||\chi_u(1 + ||u_i||)|| = (1 + ||u||) \leq (1 + 2||u||).$$

Therefore, given $u \in V$,

$$||P(u)|| \leq \sum_{i \in J} \gamma_i(u)||w_i|| \leq (1 + 2||u||) \sum_{i \in J} \gamma_i(u) = (1 + 2||u||).$$

Let $u \in V$, then $u \in \mathcal{V}_i$ for some $i \in J$.

- If $\mathcal{V}_i \subset U_0$, then

$$(\xi, P(u)) = 0, \forall \xi \in (1 + ||u||)\partial I(u).$$

- If $\mathcal{V}_i \subset U_{u_i}$, for some $i \in J_0$, by (3.5), for $\xi \in (1 + ||u||)\partial I(u)$, we have

$$(\xi, P(u)) = \sum_{i \in J} \gamma_i(u) (\xi, \chi_{u_i}(1 + ||u_i||))$$

$$= \sum_{i \in J} (1 + ||u_i||)\gamma_i(u) (\xi, \chi_{u_i})$$

$$\geq \frac{\varepsilon}{3} \sum_{i \in J} \gamma_i(u) = \frac{\varepsilon}{3} > 0.$$
(b) \( g \) is \( \tau \)-locally finite-dimensional, i.e., for each \( (t,u) \in [0,1] \times U \) there is a neighbourhood \( \mathcal{N} \) of \( (t,u) \) in the product topology of \([0,1] \times (X,\tau)\) such that \( g(\mathcal{N} \cap ([0,1] \times U)) \) is contained in a finite-dimensional subspace of \( X \).

Given \( R > r > 0 \) and \( z \in Z \setminus \{0\} \), we set

\[
\mathcal{M} = \{ u = y + tz ; \|u\| \leq R, t \geq 0 \text{ and } y \in Y \}
\]

\[
\mathcal{M}_0 = \{ u = y + tz ; y \in Y, \|u\| = R \text{ and } t \geq 0 \text{ or } \|u\| \leq R \text{ and } t = 0 \}
\]

\[
S = \{ u \in Z ; \|u\| = r \}
\]

and

\[
\Gamma := \{ \gamma \in C([0,1] \times \mathcal{M}, X) : \gamma \text{ is admissible, } \gamma(0,u) = u \\
\text{and } I(\gamma(s,u)) \leq \max\{ I(u), -1 \}, \forall s \in [0,1] \}.
\]

**Theorem 4.1. (Linking theorem).** Let \( X = Y \oplus Z \) be a separable Hilbert space with \( Y \) orthogonal to \( Z \). Suppose that:

(i) \( I \in \text{Lip}_{loc}(X,\mathbb{R}) \) is \( \tau \)-upper semicontinuous with

\[
I(u) = \frac{1}{2} \|u^+\|^2 - \frac{1}{2} \|u^-\|^2 - \Psi(u),
\]

where \( \Psi \in \text{Lip}_{loc}(X,\mathbb{R}) \) is bounded from below.

(ii) There exists \( z_0 \in Z \setminus \{0\} \), \( \delta > 0 \) and \( R > r > 0 \) such that \( I|_{S} \geq \delta \) and \( I|_{\mathcal{M}_0} \leq 0 \).

If \( I \) satisfies the condition \((H)\), then there exists a sequence \((C)_c\) for \( I \), where

\[
c = \inf_{\gamma \in \Gamma} \sup_{u \in \mathcal{M}} I(\gamma(1,u)).
\]

In addition, \( c \geq \delta \).

**Proof.** The inequality \( c \geq \delta \) follows as in [20]. Suppose by contradiction that there is \( \varepsilon > 0 \) such that

\[
(1 + \|u\|) \|\lambda_I(u)\| \geq \varepsilon, \forall u \in I^{-1}([c - \varepsilon, c + \varepsilon]).
\]

As \( I \) verifies the condition \((H)\), we can employ the Lemma 3.5 with \( \alpha = c - \varepsilon \) and \( \beta = c + \varepsilon \).

Since

\[
c = \inf_{\gamma \in \Gamma} \sup_{u \in \mathcal{M}} I(\gamma(1,u)),
\]

there is \( \tilde{\gamma} \in \Gamma \) such that \( \tilde{\gamma}([1] \times \mathcal{M}) \subset I^{c+\varepsilon} \). Thereby, by Lemma 3.5, there exists \( T > 0 \) such that

\[
I(\eta(T, \tilde{\gamma}(1,u))) \leq c - \varepsilon, \forall u \in I^{c+\varepsilon}.
\]

Moreover, according to Lemma 3.5, \( \eta : [0,T] \times I^{c+\varepsilon} \to X \) is an admissible homotopy.

Now, let us consider the homotopy \( \gamma : [0,1] \times \mathcal{M} \to X \) defined as follows:

\[
\gamma(s,u) = \left\{ \begin{array}{ll}
\tilde{\gamma}(2s,u), & 0 \leq s \leq \frac{1}{2} \\
\eta(T(2s - 1), \tilde{\gamma}(1,u)), & \frac{1}{2} \leq s \leq 1.
\end{array} \right.
\]

Then \( \gamma \in \Gamma \), and by (4.1),

\[
\sup_{u \in \mathcal{M}} I(\gamma(1,u)) \leq c - \varepsilon,
\]

which contradicts the definition of \( c \).

**Corollary 4.2.** Under the hypotheses of Theorem 4.1 and assuming that

\[
c = \sup_{u \in \mathcal{M}} I(u),
\]

there is \( v \in \mathcal{M} \) such that \( I(v) = c \) and \( 0 \in \partial I(v) \).
Proof. Seeking for a contradiction, we suppose that $\mathcal{M} \cap K_c = \emptyset$ where

$$K_c = \{ u \in X : I(u) = c \text{ and } 0 \in \partial I(u) \}.$$

**Claim 4.3.** There exists $\varepsilon > 0$ such that

$$\lambda_I(u) \geq \varepsilon, \forall u \in I^{-1}([c - \varepsilon, c + \varepsilon]) \cap \mathcal{M}.$$

In fact, otherwise there is $(u_n) \subset \mathcal{M}$ such that

$$\lambda_I(u_n) \to 0 \text{ and } I(u_n) \to c.$$

As $||u_n|| \leq R$ and $X$ is a Hilbert space, going to a subsequence if necessary, there is $u \in X$ such that

$$u_n \to u \text{ in } X.$$

Hence, $||u|| \leq \liminf_{n \to +\infty} ||u_n|| \leq R$.

On the other hand, using the fact that $u_n = u^-_n + t_n u^+_0$, it follows that $(t_n) \subset [0, +\infty)$ and $(u^-_n) \subset X^-$ are bounded sequences. Thus, going to a subsequence if necessary, there are $u^- \in X^-$ and $t_0 \in [0, +\infty)$ such that

(4.2) $$u^-_n \to u^- \text{ and } t_n \to t_0,$$

then $u_n \to u^- + t_0 u^+_0$, and so, $u = u^- + t_0 u^+_0$ and $||u|| \leq R$, that is, $u \in \mathcal{M}$. In addition, using (4.2), we deduce that $c \leq I(u)$, because by Fatou’s lemma

$$c = \limsup_{n \to +\infty} I(u_n)$$

$$= \limsup_{n \to +\infty} \left(\frac{t_n^2}{2}||u^+_0||^2 - \frac{1}{2}||u^-_n||^2 - \Psi(u_n)\right)$$

$$= \frac{t_0^2}{2}||u^+_0||^2 - \frac{1}{2} \liminf_n ||u^-_n||^2 - \liminf_n \Psi(u_n)$$

$$\leq \frac{t_0^2}{2}||u^+_0||^2 - \frac{1}{2} ||u^-||^2 - \Psi(u) = I(u).$$

Finally, since $I(u) \leq \sup_{\mathcal{M}} I = c$, we can conclude that $I(u) = c$.

Now we are ready to show that $0 \in \partial I(u)$. First of all note that $u_n \rightharpoonup u$ in $X$, because $(u_n) \subset X$ is bounded,

$$u^+_n \to u^+ \text{ and } u^-_n \to u^- \text{ in } X.$$

In what follows, let us set $w_n \in \partial I(u_n)$ with $\lambda_I(u_n) = ||w_n||$, and $\rho_n \in \partial \Psi(u_n)$ satisfying

$$\langle w_n, \varphi \rangle = \langle Q'(u_n), \varphi \rangle - \langle \rho_n, \varphi \rangle, \forall \varphi \in X \text{ and } \forall n \in \mathbb{N}.$$

Since $\lambda_I(u_n) \to 0$ and $u_n \rightharpoonup u$ in $X$,

$$\langle \rho_n, \varphi \rangle \to \langle Q'(u), \varphi \rangle, \forall \varphi \in X,$$

that is, $\rho_n \rightharpoonup^\ast Q'(u)$ in $X^\ast$. Therefore, $Q'(u) \in \Psi(u)$ (condition (H)), that is, $Q'(u) = \rho$ for some $\rho \in \Psi(u)$, then $0 \in \partial I(u)$. Therefore, $u \in K_c \cap \mathcal{M}$, which is absurd because we are supposing that $K_c \cap \mathcal{M} = \emptyset$. This proves the Claim 4.3.

By hypothesis, $I(u) \leq c$ for all $u \in \mathcal{M}$. Thus, Claim 4.3 together with Lemma 3.5 yield there is $T > 0$ such that

(4.3) $$I(\eta(T, u)) \leq c - \varepsilon, \forall u \in \mathcal{M}.$$
Now, let us consider the homotopy $\gamma: [0, 1] \times \mathcal{M} \to X$ defined by
\[
\gamma(s, u) = \begin{cases} 
  u, & 0 \leq s \leq \frac{1}{2} \\
  \eta(T(2s - 1), u), & \frac{1}{2} \leq s \leq 1.
\end{cases}
\]
Analogous to what was done in Theorem 4.1, we have $\gamma \in \Gamma$. In addition, (4.3) ensures that $\gamma(\{1\} \times \mathcal{M}) \subset I^{c-\varepsilon}$, that is,
\[
\sup_{u \in \mathcal{M}} I(\gamma(1, u)) \leq c - \varepsilon,
\]
which contradicts the definition of $c$. □

5. Proof of Theorem 1.1

In order to prove Theorem 1.1, we would like point out that the same arguments explored in [2] guarantee that the energy function $I$ associated with problem $(P)$, see (2.4), satisfies the condition $(H)$.

In what follows, our goal is to show that functional $I$ verifies the link geometry of the

Theorem 4.1 with $X = H^1(\mathbb{R}^N)$, $Z = E^+$ and $Y = E^-$.

Lemma 5.1. Suppose $(H2) - (H3)$. Then, there are $\beta > 0$ and $r > 0$ such that
\[
\inf_{u \in S} I(u) \geq \beta.
\]

Proof. From (2.2), for $0 < \varepsilon < \frac{\mu_1}{2}$, the Sobolev continuous embedding together with (2.6) leads to
\[
I(u) = \frac{1}{2}||u||^2 - \int_{\mathbb{R}^N} F(x, u) \, dx
\geq \frac{1}{2}||u||^2 - \frac{\varepsilon}{\mu_1}||u||^2 - \tilde{C}_{\varepsilon}||u||^p
= \left( \frac{\mu_1 - 2\varepsilon}{\mu_1} \right) \frac{||u||^2}{2} - \tilde{C}_{\varepsilon}||u||^p.
\]
From this, there are $\beta, r > 0$ of a such way that
\[
I(u) \geq \beta > 0, \text{ for } ||u|| = r.
\]

\hfill □

Lemma 5.2. Suppose $(H2) - (H3)$. If $z_0 \in E^+ \setminus \{0\}$ is such that
\[
(5.1) \quad \tau^2||z_0||^2 - ||y||^2 - \int_{\mathbb{R}^N} V_\infty(x)(\tau z_0 + y)^2 \, dx < 0, \quad \forall \tau > 0 \quad \text{and} \quad y \in X^-,
\]
then
\[
\sup_{u \in \mathcal{M}_0} I(u) \leq 0.
\]

Proof. Let $E = E^+ \oplus \mathbb{R}z_0 \equiv E^- \oplus \mathbb{R}^+z_0$. If $u \in E$, then
\[
I(u) = \frac{\tau^2}{2}||z_0||^2 - \frac{||y||^2}{2} - \int_{\mathbb{R}^N} F(x, y + \tau z_0) \, dx.
\]

Claim 5.3. There exists $R > 0$ such that
\[
I(u) \leq 0 \quad \text{for} \quad ||u|| = R.
\]
Indeed, suppose that there are \((y_n) \subset E^-\) and \((t_n) \subset [0, +\infty)\) such that \(||y_n + t_nz|| \to +\infty\) and \(I(y_n + t_nz_0) > 0\) for all \(n \in \mathbb{N}\). So, for \(u_n = y_n + t_nz\),

\[
0 < \frac{I(u_n)}{||u_n||^2} = \frac{1}{2}\frac{t_n^2}{||u_n||^2||z_0||^2} - \frac{1}{2}\frac{||y_n||^2}{||u_n||^2} - \int_{\mathbb{R}^N} \frac{F(x, u_n)}{||u_n||^2} dx,
\]

that is,

\[
(5.2) \quad \frac{I(u_n)}{||u_n||^2} = \frac{1}{2}\tau_n^2||z_0||^2 - \frac{1}{2}||v_n||^2 - \int_{\mathbb{R}^N} \frac{F(x, u_n)}{||u_n||^2} dx
\]

where

\[
\tau_n = \frac{t_n}{||u_n||} \quad \text{and} \quad v_n = \frac{y_n}{||u_n||}.
\]

Note that

\[
\tau_n^2||z_0||^2 + ||v_n||^2 = \frac{||t_nz_0||^2 + ||y_n||^2}{||u_n||^2} = \frac{||u_n||^2}{||u_n||^2} = 1.
\]

Hence, going to a subsequence if necessary, there are \(\tau \geq 0\) and \(v \in X^-\) such that

\[
\tau_n \to \tau \quad \text{in} \quad \mathbb{R} \quad \text{and} \quad v_n \to v \quad \text{in} \quad H^1(\mathbb{R}^N).
\]

Setting

\[
\overline{v} = \tau z_0 + v \quad \text{and} \quad \overline{v}_n = \tau_n z_0 + v_n,
\]

we obtain

\[
\overline{v}_n = \frac{u_n}{||u_n||} \quad \text{and} \quad \overline{v}_n \to \overline{v} \quad \text{in} \quad H^1(\mathbb{R}^N).
\]

For each \(R > 0\), (5.2) combined with (H3) gives

\[
0 \leq \frac{I(u_n)}{||u_n||^2} = \frac{1}{2}\tau_n^2||z_0||^2 - \frac{1}{2}||v_n||^2 - \int_{\mathbb{R}^N} \frac{F(x, u_n)}{||u_n||^2} dx
\]

\[
\leq \frac{1}{2}\tau_n^2||z_0||^2 - \frac{1}{2}||v_n||^2 - \frac{1}{2} \int_{B_R} V_{\infty}(x)u_n^2 dx - \int_{B_R} \frac{F_{\infty}(x, u_n)}{||u_n||^2} dx
\]

\[
= \frac{1}{2}\tau_n^2||z_0||^2 - \frac{1}{2}||v_n||^2 - \frac{1}{2} \int_{B_R} V_{\infty}(x)\overline{v}_n^2 dx - \int_{B_R} \frac{F_{\infty}(x, u_n)}{||u_n||^2} dx.
\]

In what follows, we will show that

\[
\lim_{n \to +\infty} \int_{B_R} \frac{F_{\infty}(x, u_n)}{||u_n||^2} dx = 0.
\]

Since

\[
\int_{B_R} \frac{F_{\infty}(x, u_n)}{||u_n||^2} dx = \int_{B_R} \frac{F_{\infty}(x, u_n)}{u_n^2} \frac{u_n^2}{||u_n||^2} dx = \int_{B_R} \frac{F_{\infty}(x, u_n)}{u_n^2} \overline{v}_n^2 dx
\]

it is enough to show that

\[
\lim_{n \to +\infty} \int_{B_R} \frac{F_{\infty}(x, u_n)}{u_n^2} \overline{v}_n^2 dx = 0.
\]

In the proof this limit, the condition (H3) applies an essential rule, because it ensures that

\[
\lim_{|t| \to +\infty} \frac{F_{\infty}(x, t)}{t^2} = 0
\]

and

\[
|F_{\infty}(x, t)| \leq \frac{c_0}{2} |t|^2, \quad \forall t \in \mathbb{R} \quad \text{and} \quad \forall x \in \mathbb{R}^N,
\]

for some \(c_0 > 0\). Next, we will analyze the cases \(\tau = 0\) and \(\tau \neq 0\).
• If $\overline{\varphi} = 0$, then

\begin{equation}
(5.3) \quad \left| \int_{B_R} \frac{F_\infty(x, u_n)}{u_n^2} \overline{\varphi}_n^2 \, dx \right| \leq \frac{c_0}{2} \int_{B_R} |\overline{\varphi}_n|^2 \, dx.
\end{equation}

Using the fact that $\overline{\varphi}_n \to 0$ in $L^2(B_R)$, we derive

$$\lim_{n \to +\infty} \int_{B_R} \frac{F_\infty(x, u_n)}{u_n^2} \overline{\varphi}_n^2 \, dx = 0.$$  

• If $\overline{\varphi} \neq 0$, as $\overline{\varphi}_n(x) \to \overline{\varphi}(x)$ a.e in $B_R$ and $||u_n|| \to +\infty$, we must have $u_n(x) = \overline{\varphi}_n(x)||u_n|| \to +\infty$ a.e. in $B_R(0)$, and so,

$$\lim_{n \to +\infty} \frac{F_\infty(x, u_n)}{u_n^2} \overline{\varphi}_n = 0, \text{ a.e in }B_R.$$

As

$$\left| \frac{F_\infty(x, u_n)}{u_n^2} \overline{\varphi}_n \right| \leq \frac{c_0}{2} |\overline{\varphi}_n|^2,$$

the Lebesgue’s dominated convergence theorem yields

$$\lim_{n \to +\infty} \int_{B_R} \frac{F_\infty(x, u_n)}{u_n^2} \overline{\varphi}_n^2 \, dx = 0,$$

or equivalently,

$$\lim_{n \to +\infty} \int_{B_R} \frac{F_\infty(x, u_n)}{||u_n||^2} \, dx = 0.$$  

Therefore,

$$0 \leq \limsup_{n \to +\infty} \left( \frac{1}{2} \tau^2 ||z_0||^2 - \frac{1}{2} ||v_n||^2 - \frac{1}{2} \int_{B_R} V_\infty(x)v_n^2 \, dx - \int_{B_R} \frac{F_\infty(x, u_n)}{||u_n||^2} \, dx \right),$$

$$\leq \frac{1}{2} \tau^2 ||z_0||^2 - \frac{1}{2} \liminf_{n \to +\infty} ||v_n||^2 - \frac{1}{2} \int_{B_R} V_\infty(x)v^2 \, dx,$$

$$\leq \frac{1}{2} \tau^2 ||z_0||^2 - \frac{1}{2} ||v||^2 - \frac{1}{2} \int_{B_R} V_\infty(x)(\tau z_0 + v)^2 \, dx.$$  

Taking the limit of $R \to +\infty$ and using the fact that $V_\infty(\tau z_0 + y)^2 \in L^1(\mathbb{R}^N)$, we find the inequality below

$$0 \leq \frac{1}{2} \tau^2 ||z_0||^2 - \frac{1}{2} ||v||^2 - \frac{1}{2} \int_{\mathbb{R}^N} V_\infty(x)(\tau z_0 + v)^2 \, dx,$$

which contradicts (5.1). This finishes the proof of Claim 5.3, and the lemma is proved. \(\square\)

Before continuing our study, we would like to point out that the condition (5.1) is not empty, because by [20, Remark 3.6] there exists $z_0 \in X^+$ with $||z_0|| = 1$, such that

$$t^2 ||z_0||^2 - ||y||^2 - \mu||y + tz_0||_2^2 < 0, \quad \forall t \in \mathbb{R} \quad \text{and} \quad \forall y \in E^-.$$  

Hence, by (H3),

$$\tau^2 ||u_0^+||^2 - ||y||^2 - \int_{\mathbb{R}^N} V_\infty(x)(\tau u_0^+ + y)^2 \, dx < 0, \quad \forall t \in \mathbb{R} \quad \text{and} \quad \forall y \in E^-.$$  

**Lemma 5.4.** All sequences $(C)_c$ for the functional $I$ are bounded.
Proof. Let \((u_n) \subset H^1(\mathbb{R}^n)\) be a sequence \((C)_c\) for the functional \(I\), that is,

\[ I(u_n) \to c \quad \text{and} \quad (1 + ||u_n||)\lambda_I(u_n) \to 0 \quad \text{as} \quad n \to +\infty. \]

Next, we set \(w_n \in \partial I(u_n)\) with \(\lambda_I(u_n) = ||w_n||_r\) and \(\rho_n \in \partial \Psi(u_n)\) such that

\[ w_n = Q'(u_n) - \rho_n. \]

Suppose by contradiction that for some subsequence, still denoted by \((u_n)\),

\[ ||u_n|| \to +\infty \quad \text{as} \quad n \to +\infty. \]

Setting the sequence \(v_n = \frac{w_n}{||w_n||}\), then \((v_n)\) is either:

1. *(Vanishing)*: For each \(r > 0\)

\[ \lim_{n \to +\infty} \sup_{y \in \mathbb{R}^N} \int_{B_r(y)} |v_n|^2 \, dx = 0, \]

or

2. *(Non-vanishing)*: There exist \(r, \eta > 0\) and a sequence \((z_n) \subset \mathbb{Z}^N\) such that

\[ \limsup_{n \to +\infty} \int_{B_r(z_n)} |v_n|^2 \, dx \geq \eta. \]

Suppose first that \((v_n)\) is non-vanishing. Given \(\varphi \in C_0^\infty(\mathbb{R}^N)\) and \(\varphi_n(x) = \varphi(x - z_n)\), we obtain that

\[ o_n(1) = \frac{1}{||u_n||} \langle w_n, \varphi_n \rangle = \langle v_n^+, v_n^- \rangle - \frac{1}{||u_n||} \int_{\mathbb{R}^N} \rho_n \varphi_n \, dx \]

\[ = \langle v_n^+, v_n^- \rangle - \frac{1}{||u_n||} \int_{\mathbb{R}^N} V_\infty(x) u_n \varphi_n \, dx - \frac{1}{||u_n||} \int_{\mathbb{R}^N} \rho_n^\infty \varphi_n \, dx \]

\[ = \langle v_n^+, v_n^- \rangle - \int_{\mathbb{R}^N} V_\infty(x) v_n \varphi_n \, dx - \int_{\mathbb{R}^N} \rho_n^\infty \frac{v_n}{u_n} \varphi_n \, dx, \]

where \(\rho_n^\infty \in \partial_t F_\infty(x, u_n)\), that is,

\[ \langle v_n^+, v_n^- \rangle - \int_{\mathbb{R}^N} V_\infty(x) v_n \varphi_n \, dx - \int_{\mathbb{R}^N} \rho_n^\infty u_n v_n \varphi_n \, dx \to 0. \]

Let

\[ \tilde{v}_n(x) = v_n(x + z_n) \quad \text{and} \quad \tilde{u}_n(x) = u_n(x + z_n). \]

Knowing \(||\tilde{v}_n|| = ||v_n|| = 1\), then, going to a subsequence if necessary, there exists \(\tilde{v} \in H^1(\mathbb{R}^N)\) such that

\[ \tilde{v}_n \to \tilde{v} \quad \text{in} \quad H^1(\mathbb{R}^N) \]

and so, \(\tilde{v}_n \to \tilde{v}\) in \(L_{loc}^2(\mathbb{R}^N)\). Since

\[ \int_{B_r(0)} |\tilde{v}|^2 \, dx = \limsup_{n \to +\infty} \int_{B_r(0)} |\tilde{v}_n|^2 \, dx = \limsup_{n \to +\infty} \int_{B_r(z_n)} |v_n|^2 \, dx \geq \eta, \]

it follows that \(\tilde{v} \neq 0\).

**Claim 5.5.** For \(\rho_n^\infty \in \partial_t F_\infty(x, u_n)\),

\[ \lim_{n \to +\infty} \int_{\mathbb{R}^N} \frac{\rho_n^\infty}{u_n} v_n \varphi_n \, dx = 0. \]
The claim follows from the limits below
\[
\lim_{n \to +\infty} \int_{\mathbb{R}^N} \frac{|f_{\infty}(x, u_n)|}{|u_n|} |v_n \varphi_n| \, dx = 0 \quad \text{and} \quad \lim_{n \to +\infty} \int_{\mathbb{R}^N} \frac{|f_{\infty}(x, u_n)|}{|u_n|} |v_n \varphi_n| \, dx = 0,
\]
because
\[
|\rho_n^\infty| \leq |f_{\infty}(x, u(x))| + |f_{\infty}(x, u(x))|,
\]
leads to
\[
\frac{|\rho_n^\infty|}{|u_n|} |v_n \varphi_n| \leq \frac{|f_{\infty}(x, u(x))|}{|u_n|} |v_n \varphi_n| + \frac{|f_{\infty}(x, u(x))|}{|u_n|} |v_n \varphi_n|.
\]
By (H2),
\[
\lim_{|t| \to +\infty} \frac{f_{\infty}(x, t)}{t} = 0 \quad \text{and} \quad \lim_{|t| \to +\infty} \frac{f_{\infty}(x, t)}{t} = 0 \quad \text{uniformly in} \quad \mathbb{R}^N.
\]
and by (H3),
\[
|f_{\infty}(x, t)| \leq c_0 |t| \quad \text{and} \quad |f_{\infty}(x, t)| \leq c_0 |t|, \quad \forall t \in \mathbb{R} \quad \text{and} \quad \forall x \in \mathbb{R}^N,
\]
for some $c_0 > 0$.

Notice
\[
\int_{\mathbb{R}^N} \frac{|f_{\infty}(x, u(x))|}{|u_n(x)|} |v_n(x) \varphi_n(x)| \, dx = \int_{\mathbb{R}^N} \frac{|f_{\infty}(x + z_n, \tilde{u}_n(x))|}{|\tilde{u}_n(x)|} |\tilde{v}_n(x) \varphi(x)| \, dx
\]
and
\[
|\frac{f_{\infty}(x + z_n, \tilde{u}_n(x))}{\tilde{u}_n(x)}| |\tilde{v}_n(x) \varphi(x)| \leq c_0 |\tilde{v}_n(x)| \varphi(x)|.
\]
Furthermore, fixing $\omega_n(x) = |\tilde{v}_n(x)| \varphi(x)|$ and $\omega(x) = |\tilde{v}(x)| \varphi(x)|$, we have
\[
\sup_n \|\omega_n\|_{L^1(\mathbb{R}^N)} < \infty, \quad \int_{\mathbb{R}^N} \omega_n(x) \, dx \to \int_{\mathbb{R}^N} \omega(x) \, dx
\]
and $\omega_n(x) \to \omega(x)$ a.e in $\mathbb{R}^N$. So,
\[
\omega_n \to \omega \quad \text{in} \quad L^1(\mathbb{R}^N).
\]
Going to a subsequence if necessary, there exists $h \in L^1(\mathbb{R}^N)$ such that
\[
|\tilde{v}_n(x)| \varphi(x)| \leq h(x) \quad \text{a.e in} \quad \mathbb{R}^N, \quad \forall n \in \mathbb{N}
\]
and
\[
\left| \frac{f_{\infty}(x + z_n, \tilde{u}_n(x))}{\tilde{u}_n(x)} \right| |\tilde{v}_n(x) \varphi(x)| \leq c_0 h(x), \quad \text{a.e in} \quad \mathbb{R}^N, \quad \forall n \in \mathbb{N}.
\]
Now, let us consider the sets
\[
A_0 = \{x \in \mathbb{R}^N : \tilde{v}(x) = 0\} \quad \text{and} \quad A = \{x \in \mathbb{R}^N : \tilde{v}(x) \neq 0\}.
\]
Thereby
\[
\left| \frac{f_{\infty}(x + z_n, \tilde{u}_n(x))}{\tilde{u}_n(x)} \right| |\tilde{v}_n(x) \varphi(x)| \, dx \leq c_0 |\tilde{v}_n(x) \varphi(x)| \, dx \to 0 \quad \text{a.e in} \quad A_0.
\]
By Lebesgue’s dominated convergence theorem,
\[
\lim_{n \to +\infty} \int_{A_0} \left| \frac{f_{\infty}(x + z_n, \tilde{u}_n(x))}{\tilde{u}_n(x)} \right| |\tilde{v}_n(x) \varphi(x)| \, dx = 0.
\]
Using the fact that $\tilde{u}_n(x) = \tilde{v}_n(x)||u_n|| \to +\infty$ in $A$, by (5.6),
\[
\lim_{n \to +\infty} \left| \frac{f_{\infty}(x + z_n, \tilde{u}_n(x))}{\tilde{u}_n(x)} \right| |\tilde{v}_n(x) \varphi(x)| = 0. \quad \text{a.e in} \quad A.
Again, by Lebesgue’s dominated convergence theorem,
\[ \lim_{n \to +\infty} \int_A \left| \frac{f_\infty(x + z_n, \bar{u}_n(x))}{\bar{u}_n(x)} \right| |\bar{v}_n(x)| \varphi(x) \, dx = 0. \]
Thus,
\[ \int_{\mathbb{R}^N} \left| \frac{f_\infty(x + z_n, \bar{u}_n(x))}{\bar{u}_n(x)} \right| |\bar{v}_n(x)| \varphi(x) \, dx = 0. \]
Analogously
\[ \lim_{n \to +\infty} \int_{\mathbb{R}^N} \left| \frac{f_\infty(x + z_n, \bar{u}_n(x))}{\bar{u}_n(x)} \right| |\bar{v}_n(x)| \varphi(x) \, dx = 0, \]
proving the claim.

By Claim 5.5,
\[ \lim_{n \to +\infty} \left( \int_{\mathbb{R}^N} V_\infty(x)v_n \varphi_n \, dx - \int_{\mathbb{R}^N} \frac{\rho_n}{u_n} v_n \varphi_n \, dx \right) = \int_{\mathbb{R}^N} V_\infty(x)\tilde{v}\varphi \, dx. \]
Then, from (5.5),
\[ \langle \tilde{v}^+ - \tilde{v}^- , \varphi \rangle - \int_{\mathbb{R}^N} V_\infty(x)\tilde{v}\varphi \, dx = 0, \forall \varphi \in C_0^\infty(\mathbb{R}^N), \]
that is, \( \tilde{v} \in H^1(\mathbb{R}^N) \setminus \{0\} \) verifies
\[ -\Delta \tilde{v} + (V - V_\infty)\tilde{v} = 0, \text{ in } \mathbb{R}^N. \]
However, since \((V - V_\infty)\) is periodic, the spectrum of \(-\Delta + V - V_\infty\) is absolutely continuous, then it has no eigenvalues [see [16], Theorem 4.59]. This shows that \( (v_n) \) cannot be non-vanishing.

Suppose that \( (v_n) \) is vanishing. As in (5.5),
\[ ||v_n^+||^2 - \int_{\mathbb{R}^N} \rho \frac{v_n}{u_n} v_n^+ \, dx = \langle v_n^+ - v_n^- , v_n^+ \rangle - \int_{\mathbb{R}^N} \rho \frac{v_n}{u_n} v_n^+ \, dx \to 0 \]
and
\[ -||v_n^-||^2 - \int_{\mathbb{R}^N} \rho \frac{v_n}{u_n} v_n^- \, dx = \langle v_n^+ - v_n^- , v_n^- \rangle - \int_{\mathbb{R}^N} \rho \frac{v_n}{u_n} v_n^- \, dx \to 0. \]
Since \( ||v_n|| = 1 \), (5.8) and (5.9) lead to
\[ 1 - \int_{\mathbb{R}^N} \rho \frac{v_n}{u_n} (v_n^+ - v_n^-) \, dx \to 0, \]
that is,
\[ \lim_{n \to +\infty} \left( \int_{\mathbb{R}^N} \rho \frac{v_n}{u_n} (v_n^+ - v_n^-) \, dx \right) = 1. \]

By the definition of \( \mu_0, \mu_1 \) and \( \mu_{-1} \), if \( u \in H^1(\mathbb{R}^N) \),
\[ ||u^+||^2 \geq \mu_1 ||u^+||_2^2, \forall u^+ \in E^+ \text{ and } ||u^-||^2 \geq -\mu_{-1} ||u^-||_2^2, \forall u^- \in E^- . \]
Hence,
\[ ||u||^2 \geq \mu_0 ||u||_2^2, \forall u \in H^1(\mathbb{R}^N). \]
In what follows, let us consider the set
\[ \Omega_n = \left\{ x \in \mathbb{R}^N : \frac{\rho_n(x)}{u_n(x)} \leq \mu_0 - \delta \right\}, \]
where \( \delta > 0 \) was given in (H5).
By Hölder inequality, (5.11) and the orthogonality of \(v^+_n\) and \(v^-_n\) in \(L^2(\mathbb{R}^N)\), it follows that
\[
\int_{\Omega_n} \frac{\rho_n}{u_n} v_n (v^+_n - v^-_n) \, dx \leq (\mu_0 - \delta) \int_{\mathbb{R}^N} |v^+_n| \, |v^-_n| \, dx
\]
\[
= (\mu_0 - \delta) ||v^+_n||_2 \, ||v^-_n||_2
\]
\[
= (\mu_0 - \delta) ||v_n||_2^2 \leq \frac{(\mu_0 - \delta)}{\mu_0} < 1.
\]

This combined with (5.10) provides
\[
\text{(5.12)} \quad \liminf_{n \to +\infty} \int_{\mathbb{R}^N \setminus \Omega_n} \frac{\rho_n}{u_n} v_n (v^+_n - v^-_n) \, dx > 0.
\]

**Claim 5.6.**

\[
\lim_{n \to +\infty} ||\mathbb{R}^N \setminus \Omega_n|| = +\infty.
\]

Suppose that
\[
\limsup_{n \to +\infty} ||\mathbb{R}^N \setminus \Omega_n|| < \infty.
\]

Fixed \(p \in (2, 2^*)\), the limit (5.12) combines with Hölder inequality to give
\[
0 < \liminf_{n} \int_{\mathbb{R}^N \setminus \Omega_n} \frac{\rho_n}{u_n} v_n (v^+_n - v^-_n) \, dx
\]
\[
\leq c_0 \liminf_{n \to +\infty} \int_{\mathbb{R}^N \setminus \Omega_n} |v_n| |v^+_n - v^-_n| \, dx
\]
\[
\leq c_0 \liminf_{n \to +\infty} \left[ (||\mathbb{R}^N \setminus \Omega_n||) \frac{p - 2}{p} \left( \int_{\mathbb{R}^N \setminus \Omega_n} |v_n|^\frac{p}{2} |v^+_n - v^-_n|^\frac{2}{p} \, dx \right) \right]
\]
\[
\leq c_0 \liminf_{n \to +\infty} \left[ (||\mathbb{R}^N \setminus \Omega_n||) \frac{p - 2}{p} \left( \int_{\mathbb{R}^N \setminus \Omega_n} |v_n|^p \, dx \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}^N \setminus \Omega_n} |v^+_n - v^-_n|^p \, dx \right)^{\frac{1}{p}} \right]
\]
\[
\leq \tilde{c}_0 \liminf_{n \to +\infty} \left[ (||\mathbb{R}^N \setminus \Omega_n||) \frac{p - 2}{p} \left( \int_{\mathbb{R}^N \setminus \Omega_n} |v_n|^p \, dx \right)^{\frac{1}{p}} \right].
\]

As we are supposing that \((v_n)\) is vanishing, \(v_n \to 0\) in \(L^s(\mathbb{R}^N)\) for \(s \in (2, 2^*)\), and so,
\[
0 < \tilde{c}_0 \liminf_{n \to +\infty} \left[ (||\mathbb{R}^N \setminus \Omega_n||) \frac{p - 2}{p} \left( \int_{\mathbb{R}^N \setminus \Omega_n} |v_n|^p \, dx \right)^{\frac{1}{p}} \right] \to 0,
\]
which is absurd, and the Claim 5.6 is proved. Accordingly to (H4), (H5) and Claim 5.6,
\[
\int_{\mathbb{R}^N} \left( \frac{1}{2} \rho_n u_n - F(x, u_n) \right) \, dx \geq \int_{\mathbb{R}^N \setminus \Omega_n} \left( \frac{1}{2} \rho_n u_n - F(x, u_n) \right) \, dx
\]
\[
\geq \int_{\mathbb{R}^N \setminus \Omega_n} \delta \, dx \to +\infty,
\]
that is,
\[
\int_{\mathbb{R}^N} \left( \frac{1}{2} \rho_n u_n - F(x, u_n) \right) \, dx \to +\infty.
\]
On the other hand, since \( (u_n) \) is a sequence \((C)_c\) and \( \langle w_n, u_n \rangle \to 0 \), we find
\[
\int_{\mathbb{R}^N} \left( \frac{1}{2} \rho_n u_n - F(x, u_n) \right) \, dx = I(u_n) - \frac{1}{2} \langle w_n, u_n \rangle \to c,
\]
which is a contradiction. This completes the proof. \( \square \)

Now, we are ready to conclude the proof of Theorem 1.1.

**Proof of Theorem 1.1:**

By Fatou’s lemma the functional \( I \) is \( \tau \)-upper semicontinuous, see [34, Lemma 6.15]. Therefore, by Lemmas 5.1 and 5.2, the functional \( I \) satisfies the hypotheses of Theorem 4.1, and so, by Lemma 5.4 there exists a bounded \((C)_c\) sequence for the functional \( I \), denoted by \( (u_n) \subset H^1(\mathbb{R}^N) \), i.e.,
\[
I(u_n) \to c, \quad (1 + ||u_n||) \lambda_1(u_n) \to 0 \quad \text{and} \quad ||u_n|| \leq K, \quad \forall \ n \in \mathbb{N},
\]
for some \( K > 0 \). By [34, Lemma 1.21] and (2.6), there exists \( \delta_1 > 0 \) such that
\[
\liminf_{n \to +\infty} \sup_{y \in \mathbb{R}^N} \int_{B(y,1)} |u_n|^2 \, dx \geq \delta_1.
\]
In addition, there is \( (z_n) \subset \mathbb{Z}^N \) such that
\[
\int_{B(z_n, 1 + \sqrt{N})} |u_n|^2 \, dx \geq \frac{\delta_1}{4}, \quad n \geq n_0.
\]
Setting \( v_n(x) = u_n(x + z_n) \), we compute
\[
(5.13) \quad \int_{B(0, 1 + \sqrt{N})} |v_n(x)|^2 \, dx = \int_{B(z_n, 1 + \sqrt{N})} |u_n(x)|^2 \, dx \geq \frac{\delta_1}{4}, \quad n \geq n_0.
\]
Moreover, a simple computation also shows \( (v_n) \subset H^1(\mathbb{R}^N) \) is a \((PS)_c\) sequence for \( I \) (see [2] for details) and \( ||v_n|| = ||u_n|| \). Since \( (u_n) \) is bounded, going to a subsequence if necessary, \( v_n \rightharpoonup v \) in \( H^1(\mathbb{R}^N) \) and by (5.13) \( v \neq 0 \). Now, by using the Proposition 2.3, we can argue as in [2] to conclude that
\[-\Delta v(x) + V(x)v(x) \in \partial_u F(x, u) \quad \text{a.e in} \quad \mathbb{R}^N,
\]
showing the desired result.

### 6. The non periodic case

In this section, we will prove the Theorem 1.2. Similarly to Section 3, the \((H2)\) and \((H3)\) ensure that the energy functional associated with problem \((P)\) defined by
\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 \, dx - \int_{\mathbb{R}^N} F(x, u) \, dx, \quad u \in H^1(\mathbb{R}^N),
\]
is well defined. Furthermore, since the lemmas showed in Section 5 do not depend on the periodicity of function \( f \), but only of its growth, all of them are also true in this section, and we have the same link geometry. Consequently, there is a bounded sequence \( (u_n) \subset H^1(\mathbb{R}^N) \) such that
\[
(1 + ||u_n||) \lambda_1(u_n) \to 0 \quad \text{and} \quad I(u_n) \to c,
\]
where
\[
c = \inf_{\gamma \in \Gamma} \sup_{u \in \mathcal{M}} I(\gamma(1, u)).
\]
Next, we are going to recall some facts involving the periodic problem
\[
\begin{aligned}
\begin{cases}
-\Delta u + V(x)u &= h(x, u), \quad \text{in } \mathbb{R}^N, \ N \geq 3 \\
 u &\in H^1(\mathbb{R}^N),
\end{cases}
\end{aligned}
\tag{A}
\]
where \( h : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \) is the continuous function that satisfies the assumption \((H6)\).

First of all, we would like point out that the energy functional associated with problem \((A)\), given by
\[
J(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx - \int_{\mathbb{R}^N} H(x, u)dx, \ u \in H^1(\mathbb{R}^N),
\]
is well defined and \( J \in C^1(\mathbb{R}^N, \mathbb{R}) \), where \( H(x, t) = \int_0^t h(x, s)ds \).

It is proved in [20] that the functional \( J \) has a nontrivial critical \( u_0 \in H^1(\mathbb{R}^N) \setminus \{0\} \) that satisfies:
\[
\tau^2||u_0^+||^2 - ||y||^2 - \int_{\mathbb{R}^N} V_{\infty}(x)(\tau u_0^+ + y)^2dx < 0, \ \forall \tau \in \mathbb{R} \ \text{and} \ y \in X^-,
\]
\[
J(u_0) = \min\{J(u) : u \neq 0 \ \text{and} \ J'(u) = 0\},
\]
and
\[
\sup_{u \in \mathcal{M}} J(u) = J(u_0) > 0.
\]

In particular, if we choose \( z_0 = u_0^+ \) in the definition of \( \mathcal{M} \), see Section 4, the Lemma 5.2 still holds with \( z_0 = u_0^+ \).

Using the above information, we are ready to conclude the proof of Theorem 1.2.

\textbf{Proof of Theorem 1.2:}

From above commentaries, there is a bounded sequence \( (u_n) \subset H^1(\mathbb{R}^N) \) satisfying
\[
(1 + ||u_n||)\lambda_J(u_n) \to 0 \ \text{and} \ J(u_n) \to c
\]
where
\[
c = \inf_{\gamma \in \Gamma} \sup_{u \in \mathcal{M}} J(\gamma(1, u)).
\]

Therefore, there are \( w_n \in \partial I(u_n) \) with \( \lambda_I(u_n) = ||w_n||_* \) and \( \rho_n \in \partial \Psi(u_n) \) such that
\[
\langle w_n, \varphi \rangle = \langle Q'(u_n), \varphi \rangle - \langle \rho_n, \varphi \rangle, \ \forall \varphi \in X, \ \forall n \in \mathbb{N}.
\]

Since \( H^1(\mathbb{R}^N) \) is reflexive, going to a subsequence if necessary, there is \( u \in H^1(\mathbb{R}^N) \) such that
\[
u_n \rightharpoonup u \ \text{in} \ H^1(\mathbb{R}^N).
\]

If \( u \neq 0 \), then the Theorem 1.2 is proved. If \( u = 0 \), we have
\[
u_n \rightarrow 0 \ \text{in} \ H^1(\mathbb{R}^N).
\]

From \((H7)\), \( G(x, t) \geq H(x, t) \) for all \( t \in \mathbb{R} \), and so,
\[
0 < c := \inf_{\gamma \in \Gamma} \sup_{u \in \mathcal{M}} I(\gamma(1, u)) \leq \sup_{u \in \mathcal{M}} I(u) = \sup_{u \in \mathcal{M}} J(u) = J(u_0)
\]
that is,
\[
c \leq J(u_0).
\]

Next, we are going to prove that \( c = J(u_0) \).
Claim 6.1. 

\[ \rho_n - \tilde{\Psi}'(u_n) \to 0 \text{ and } \Psi(u_n) - \tilde{\Psi}(u_n) \to 0, \]

where

\[ \tilde{\Psi}(u) = \int_{\mathbb{R}^N} F(x,u)dx \text{ and } \tilde{\Psi}(u) = \int_{\mathbb{R}^N} H(x,u)dx. \]

Let \( \varphi \in H^1(\mathbb{R}^N) \) with \( \|\varphi\| \leq 1 \). By (H7), Hölder inequality and using the inequality \( \|u\|^2 \geq \mu_0\|u\|^2 \) \( \forall u \in H^1(\mathbb{R}^N) \), we obtain

\[ \left| \left\langle \rho_n - \tilde{\Psi}'(u_n), \varphi \right\rangle \right| = \left| \int_{\mathbb{R}^N} [\rho_n - h(x,u_n)]\varphi dx \right| \]
\[ \leq \int_{\mathbb{R}^N} a(x)|u_n| |\varphi|dx \]
\[ \leq \left( \int_{\mathbb{R}^N} |a(x)|^2|u_n|^2dx \right)^{\frac{1}{2}} ||\varphi||_2 \]
\[ \leq \left( \frac{1}{\mu_0} \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |a(x)|^2|u_n|^2dx \right)^{\frac{1}{2}} ||\varphi||. \]

Using the fact that \( a(x) \to 0 \) whenever \( |x| \to +\infty \), given \( \varepsilon > 0 \) there is \( R_0 > 0 \) such that \( |a(x)| \leq \varepsilon \) for \( |x| > R_0 \). By compact embedding \( H^1(B_{R_0}) \hookrightarrow L^2(B_{R_0}) \) we have \( u_n \to 0 \) in \( L^2(B_{R_0}) \), thereby there is \( n_0 \in \mathbb{N} \) such that \( \|u_n\|_{L^2(B_{R_0})} \leq \varepsilon \), \( \forall n \geq n_0 \), and so,

\[ \int_{\mathbb{R}^N} |a(x)|^2|u_n|^2dx = \int_{B_{R_0}} |a(x)|^2|u_n|^2dx + \int_{B_{R_0}^c} |a(x)|^2|u_n|^2dx \]
\[ \leq ||a||^2_{\infty} \int_{B_{R_0}} |u_n|^2dx + \varepsilon^2 \int_{B_{R_0}^c} |u_n|^2dx \]
\[ \leq \varepsilon^2(||a||^2_{\infty} + K^2), \]

where \( \|u_n\| \leq K \) for all \( n \in \mathbb{N} \). As \( \varepsilon \) is arbitrary,

\[ \rho_n - \tilde{\Psi}'(u_n) \to 0 \quad \text{in} \quad (H^1(\mathbb{R}^N))^*. \]

A similar argument guarantees that

\[ \Psi(u_n) - \tilde{\Psi}(u_n) \to 0 \quad \text{in} \quad \mathbb{R}. \]

Then, by Claim 6.1,

\[ J'(u_n) \to 0 \quad \text{and} \quad J(u_n) \to c, \quad \text{as} \quad n \to +\infty. \]

As \( (u_n) \) is bounded, it follows that \( (u_n) \) is a \((C)_{c}\)-sequence for the functional \( J \), and so, arguing as in the proof of Theorem 1.1, there is a nontrivial critical point \( u_1 \neq 0 \) of \( J \), with \( J(u_1) \leq c \). On the other hand, by (6.1), we must have \( J(u_0) \leq J(u_1) \), from where it follows that

\[ c \leq \sup_{u \in \mathcal{M}} I(u) \leq J(u_0) \leq J(u_1) \leq c, \]

that is,

\[ \sup_{u \in \mathcal{M}} I(u) = c. \]

Now, as Corollary 4.2 still holds when \( f \) is non periodic, we can conclude that there is \( v \in H^1(\mathbb{R}^N) \) such that \( 0 \in \partial I(v) \) and \( I(v) = c > 0 \). This finishes the proof of Theorem 1.2
References

[1] S. Alama and Y. Y. Li, *On Multibump bound states for certain semilinear elliptic equations*, Indiana J. Math. 41 (1992) 983-1026.

[2] C.O Alves and G. F. Patricio, *Existence of solution for a class of indefinite variational problems with discontinuous nonlinearity*, arXiv:2012.03641v1[math.AP] 1, 2, 5, 9, 12, 19.

[3] D. G. Costa and H. Tehrani, *On a class of asymptotically linear elliptic problems in $\mathbb{R}^N$*, J. Differential Equations 173 (2001), 470–494.

[4] F. H. Clark, *Generalized gradients and applications*, Trans. Amer. Math. Soc. 205 (1975), 247-262.

[5] F. H. Clark., *Optimization and Nonsmooth Analysis*, Wiley, New York 1983.

[6] K. C. Chang, *Variational methods for nondifferentiable functionals and their applications to partial differential equations*, J. Math. Anal. Appl. 80, 102-129 (1981).

[7] K. C. Chang, *On the multiple solutions of the elliptic differential equations with discontinuous nonlinear terms*, Sci. Sin. 21, 139-158 (1978).

[8] K. C. Chang, *The obstacle problem and partial differential equations with discontinuous nonlinearities*, Commun. Pure Appl. Math. 33, 117-146 (1980).

[9] M. Clapp and L.A. Maia, *A positive bound state for an asymptotically linear or superlinear Schrödinger equation*, J. Differential Equations, 260 (2016), pp. 3173-3192.

[10] X. Chang, *Ground state solutions of asymptotically linear fractional Schrödinger equations*, J. Math. Phys. 54, 061504 (2013).

[11] S. Chen and Z. Dawei, *Existence of nontrivial solutions for asymptotically linear elliptic equations with vanishing at infinity*, J. Differential Equations 231, 501–512 (2006) Zbl pre05115328 MR 2287894.

[12] Y. Ding and C. Lee, *Multiple solutions of Schrödinger equations with indefinite linear part and super or asymptotically linear terms*, J. Differential Equations 222, (2006), 137–163.

[13] R. Engelking, *General Topology*, Monografie Matematyczne, tom 60, PWN-Polish Scientific Publishers, Warszawa 1977.

[14] N. Fukagai, M. Ito and K. Narukawa, *Positive solutions of quasilinear elliptic equations with critical Orlicz-Sobolev nonlinearity on $\mathbb{R}^N$*, Funkcial. Ekvac. 49 (2006), 235-267.

[15] L. Jeanjean and K. Tanaka, *A positive solution for an asymptotically linear elliptic problem on $\mathbb{R}^N$, autonomous at infinity*, J. Differential Equations 245, 201–222 (2008). doi:10.1016/j.jde.2008.01.006.

[16] P. Kuchment, *Floquet Theory for Partial Differential Equations*, Birkhäuser, Basel, 1993.

[17] W. Kryszewski and A. Szulkin, *Generalized linking theorem with an application to semilinear Schrödinger equations*, Adv. Differential Equations (1998), 441-472.

[18] R. Lehrer and L.A. Maia, *Positive solutions of asymptotically linear equations via Pohozaev manifold J. Funct. Anal.*, 266 (2014), pp. 213-246.

[19] G. Li and H. S. Zhou, *The existence of a positive solution to asymptotically linear scalar field equations*, Proc. Royal Soc. Edinburgh 130A (2000) 81-105.

[20] G. B. Li and A. Szulkin, *An asymptotically periodic Schrödinger equation with indefinite linear part*, Commun. Contemp. Math, 2002, 4: 763-776.

[21] C.Y. Liu, Z.P. Wang, Z. P. and H.S. Zhou, *Asymptotically linear Schrödinger equation with potential vanishing at infinity*, J. Differential Equations 245, 201–222 (2008). doi:10.1016/j.jde.2008.01.006.

[22] Z.L. Liu, J.B. Su and T. Weth, *Compactness results for Schrödinger equations with asymptotically linear terms*, J. Differential Equations 231, 501–512 (2006) Zbl pre05115328 MR 2287894.

[23] X. Y. Lin and X. H. Tang, *An asymptotically periodic and asymptotically linear Schrödinger equation with indefinite linear part*, Comput. Math. Appl., 70 (2015), 726-736.

[24] L.A. Maia, J.C. Oliveira Junior and R. Ruviaro, *A non-periodic and asymptotically linear indefinite Schrödinger equation on $\mathbb{R}^N$, Comm. Partial Differential Equations 24 (1999), 1731–1758.

[25] D.D. Qin and X.H. Tang., *Asymptotically linear Schrödinger equation with zero on the boundary of the spectrum*, Electron. J. Differ. Equ. 213, 1–15 (2015).

[26] G. Rosário and S. A. Tersian, *An Introduction to Minimax Theorems and Their Applications to Differential Equations*, Springer-Science+Business Media, B.V., (2001).

[27] C. A. Stuart, *Bifurcation into spectral gaps*, Bull. Belg. Math. Soc., Supplement (1995), 59.

[28] C. A. Stuart and H. S. Zhou, *Applying the mountain pass theorem to an asymptotically linear elliptic equation on $\mathbb{R}^N$, Comm. Partial Differential Equations 24 (1999), 1731–1758.

[29] H. Tehrani, *A note on asymptotically linear elliptic problems in $\mathbb{R}^N$, J. Math. Anal. Appl. 271 (2002), 546–554.*
[30] X. H. Tang, *Non-Nehari manifold method for asymptotically linear Schrödinger equation*, J. Aust. Math. Soc., 98 (2015), 104-116.  
[31] F. A. Van Heerden, *Multiple solutions for a Schrödinger type equation with an asymptotically linear term*, Nonlinear Anal. 55 (2003) 739–758.  
[32] F. A. Van Heerden, Z.-Q. Wang, *Schrödinger type equations with asymptotically linear nonlinearities*, Differential Integral Equations 16, (2003), 257–280.  
[33] F. A. Van Heerden; *Homoclinic solutions for a semilinear elliptic equation with an asymptotically linear nonlinearity*, Calc. Var. Partial Differential Equations, 20 (2004), 431-455.  
[34] M. Willem, *Minimax Theorems*, Birkhäuser, Boston, 1996.  
[35] Q. F. Wu and D. D. Qin, *Ground and bound states of periodic Schrödinger equations with super or asymptotically linear terms*, Electronic Journal of Differential Equations, 25 (2018), 1-26.  
[36] H.S. Zhou and H.B. Zhu, *Asymptotically linear elliptic problem on $\mathbb{R}^N$*. Quart. J. Math. 59, 523–541 (2008).doi:10.1093/qmath/ham047.  

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