Effective coherence in centralizer extensions of locally quasi-convex hyperbolic groups

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Abstract

We prove that every group $G$ discriminated by a locally quasi-convex torsion-free hyperbolic group $\Gamma$ is effectively coherent: that is, presentations for finitely generated subgroups can be computed from the subgroup generators. We study $G$ via its embedding into an iterated centralizer extension of $\Gamma$, and prove that this embedding can be computed. We also give algorithms to enumerate all finitely generated groups discriminated by $\Gamma$, and to decide whether a given group is discriminated by $\Gamma$.

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1 Introduction

A group $G$ is called coherent if every finitely generated subgroup is isomorphic to a finitely presented group. When $G$ is known to be coherent, an algorithmic question immediately arises: can such a presentation be computed from the subgroup
generators? We say that \(G\) is \textit{effectively coherent} if there is an algorithm that, given a finite set \(X \subset G\), produces a presentation for the subgroup \(\langle X \rangle\).

The study of algorithmic problems in combinatorial group theory has produced a great number of algorithms which, starting with a presentation for a group \(H\) (and usually some additional knowledge), compute various properties of \(H\). The ability to compute presentations for subgroups allows us to use these algorithms in the study of finitely generated subgroups.

Our main interest is in groups \textit{discriminated} by another group \(\Gamma\). A group \(G\) is discriminated by \(\Gamma\) (or is \textit{fully residually} \(\Gamma\)) if for every finite set \(\{g_1, \ldots, g_n\}\) of non-trivial elements of \(G\) there exists a homomorphism \(\phi : G \to \Gamma\) such that \(\phi(g_i)\) is non-trivial for \(i = 1, \ldots, n\). When \(\Gamma\) is a free group and \(G\) is finitely generated, \(G\) is called a \textit{limit group}. Limit groups figured prominently in the work of Kharlampovich-Miasnikov \[12\] and Sela \[25\] on the solution to Tarski’s problems on the elementary theory of free groups. Many equivalent characterizations of limit groups are known, and these characterizations extend beyond the case when \(\Gamma\) is free (see \[13\] for a summary).

In particular, when \(\Gamma\) is hyperbolic these characterizations hold. But not all hyperbolic groups are coherent (this follows from the well-known construction in \[24\]), and if \(\Gamma\) is not coherent then there are non-coherent groups discriminated by \(\Gamma\) (\(\Gamma\) itself is an example). We will consider the case when \(\Gamma\) is hyperbolic, locally quasi-convex, and torsion-free. In this case, we prove that every finitely generated group \(G\) which is discriminated by \(\Gamma\) is effectively coherent.

The group \(G\) is known to embed into a group obtained from \(\Gamma\) by iterated centralizer extensions \[15\], and a partial result regarding the computation of this embedding, for the general case when \(\Gamma\) is hyperbolic and torsion-free, was given in \[14\]. We complete this result, giving an algorithm to compute the embedding provided \(\Gamma\) is also locally quasi-convex.

For limit groups, this embedding was computed in \[11\]. Effective coherence for limit groups we proved in \[6\], as a corollary of an algorithm which enumerates all limit groups. We prove these results for \(\Gamma\) in the other order: as a corollary of effective coherence for groups discriminated by \(\Gamma\), we obtain an algorithm enumerating all (finitely generated) groups discriminated by \(\Gamma\), and an algorithm that recognizes whether or not a group is discriminated by \(\Gamma\). Our principal results are summarized below.

\textbf{Theorem.} Let \(\Gamma\) be a hyperbolic group which is torsion-free and locally quasi-convex. Then there exist algorithms to solve each of the following problems.

(i) Given a group \(G\) known to be discriminated by \(\Gamma\), compute a sequence of centralizer extensions of \(\Gamma\),

\[\Gamma = G_0 < G_1 < \ldots < G_n,\]

and an embedding of \(G\) into \(G_n\) (Theorem \[21\]).

(ii) Given a group \(G\) known to be discriminated by \(\Gamma\) and a finite subset \(X \subset G\), compute a presentation for the subgroup generated by \(X\) (Theorem \[23\]).
(iii) Given a group $G$ and a solution to the word problem in $G$, determine whether or not $G$ is discriminated by $\Gamma$ (Theorem 21).

(iv) Enumerate, through presentations, all finitely generated groups discriminated by $\Gamma$, without repeating isomorphic groups (Theorem 24).

We approach effective coherence in $G$ by first proving effective coherence in any iterated centralizer extension $G_n$ of $\Gamma$ (Theorem 20). We view $G_n$ as the fundamental group of a graph of groups $\mathbb{A}$ with two vertices, one with vertex group $G_n - 1$ and the other with an abelian vertex group, and one edge with group $\mathbb{Z}$. A graph-folding algorithm, along with a structure called an $\mathbb{A}$-graph, was developed in [10] to find the induced decomposition of, and hence a presentation for, any finitely presented subgroup of the fundamental group of $\mathbb{A}$. While certain algorithmic properties are required of $\mathbb{A}$ (it should be ‘benign’, see Definition 3), when the only edge group is $\mathbb{Z}$ these properties reduce to the decidability of the following problem in vertex groups.

**Problem 1.** The power coset membership problem for a group $G$ asks to decide, given two elements $x, g \in G$ and a finitely generated subgroup $H$ of $G$, whether or not there exists a non-zero integer $m$ such that $g^m \in xH$.

Decidability of the power coset membership problem implies decidability of several other important algorithmic problems. On input $H = 1$ and $g = 1$ the answer is ‘Yes’ if and only if $x = 1$, so the word problem is decidable in $G$. On input $g = 1$, the answer is ‘Yes’ if and only if $x \in H$, so the membership problem is decidable in $G$.

On input $x = 1$, the answer is ‘Yes’ if and only if $H \cap \langle g \rangle$ is non-trivial, and we call this the power membership problem. Note that $m = 0$ is specifically excluded in the problem specification so that this (non-trivial) problem arises when $x = 1$. If the intersection is non-trivial, a generator for $H \cap \langle g \rangle$ may be produced by finding the smallest $m$ such that $g^m \in H$. Finally, we can decide if the intersection $xH \cap \langle g \rangle$ is non-empty, since this occurs if and only if either the answer to the power coset membership problem is ‘Yes’ or $x \in H$.

To solve the power coset membership problem in $G_n - 1$, we view $G_n - 1$ itself as the fundamental group of a two-vertex graph of groups $\mathbb{A}'$. We construct a ‘folded $\mathbb{A}'$-graph’ $B^{(x)}$ representing the coset $xH$: doing so requires that $\mathbb{A}'$ be benign, and we argue inductively. To decide if $p^m \in xH$, we develop in §2 an algorithm $\text{ReadPower}$ that determines whether or not a power of $p$ can be ‘read’ in $B^{(x)}$ (Theorem 13).

While reading a fixed power of $p$ in $xH$ is straightforward, ensuring algorithm termination while trying to read an arbitrary power is difficult. We provide some general, though somewhat involved, conditions under which $\text{ReadPower}$ terminates (Property 12), giving a solution to the power coset membership problem for certain graphs of groups (Theorem 14). As an example, power coset membership is decidable whenever all edge groups are finite and, necessarily, the problem is decidable in vertex groups (Corollary 15). The proof that $\mathbb{A}'$ satisfies these conditions is given in §3, and we rely on the local quasi-convexity of $\Gamma$ and the resulting toral relatively hyperbolic locally (relatively) quasi-convex structure of $G_{n-1}$. 

3
2 Subgroups of a fundamental group of a graph of groups

When a group $G$ is presented as the fundamental group of a graph of groups, every subgroup $H$ inherits from $G$ a graph of groups decomposition, which yields a presentation of $H$. To compute this presentation, and to solve the problem of membership in $H$, [10] developed a graph-folding algorithm similar to the folding procedure used by Stallings to study subgroups of free groups. We apply this technique to study the power coset membership problem in fundamental groups of certain graphs of groups.

2.1 $\mathbb{A}$-graphs and subgroup graphs

We recall from [10] the notion of an $\mathbb{A}$-graph and some important properties of $\mathbb{A}$-graphs. All of the results of §2.1 are from [10], to which we refer the reader for proofs.

A graph consists of a set $V$ called vertices, a set $E$ called edges, an involution $-^1 : E \rightarrow E$ that has no fixed points, and two functions $o : E \rightarrow V$ and $t : E \rightarrow V$ that satisfy $o(e) = t(e^{-1})$ for all $e \in E$.

A graph of groups $A$ consists of a graph $A$ together with, for each vertex $v \in V$ a group $A_v$, for each edge $e \in E$ a group $A_e$, and for each edge group $A_e$ two monomorphisms $\alpha_e : A_e \rightarrow A_{o(e)}$ and $\omega_e : A_e \rightarrow A_{t(e)}$. For inverse edges $e^{-1}$ we insist that $A_{e^{-1}} = A_e$, $\alpha_{e^{-1}} = \omega_e$.

An $\mathbb{A}$-path $p = < p_0, e_1, p_1, \ldots, e_n, p_n >$ from vertex $v_0$ to $v_n$ consists of an underlying edge-path $(e_1, \ldots, e_n)$ from $v_0$ to $v_n$ and a choice of vertex group elements $p_i \in A_{t(e_i)}$ for $i = 1, \ldots, n$ and $p_0 \in A_{v_0}$. The length of $p$ is the number $n$ of edges.

An elementary reduction replaces a subpath of the form

$$< a, e, \omega_e(c), e^{-1}, b >,$$

where $c \in A_e$, with the subpath $< a\alpha_e(c) b >$. Elementary reductions and their inverses generate an equivalence relation on $\mathbb{A}$-paths, with the equivalence class of $p$ denoted $\overline{p}$. A path is reduced if no elementary reduction is applicable.

The set of equivalence classes of $\mathbb{A}$-loops based at a given vertex $v_0$ forms a group under concatenation and is called the fundamental group of the graph of groups $\mathbb{A}$ and is denoted $\pi_1(\mathbb{A}, v_0)$.

$\mathbb{A}$-graphs

Let $\mathbb{A}$ be a graph of groups with underlying graph $A$ and base vertex $v_0$. An $\mathbb{A}$-graph $B$ consists of a graph $B$ together with the following additional data:

(i) a graph morphism $[\cdot] : B \rightarrow A$;

(ii) for each vertex $u$ of $B$ a subgroup $B_u \leq A_{[u]}$;
(iii) for each edge \( f \) of \( B \) two group elements \( f_\alpha \in A_{\alpha(f)} \) and \( f_\omega \in A_{\omega(f)} \) such that \((f^{-1})_\alpha = (f_\omega)^{-1}\). 

The \( A \)-vertex \([u]\) is called the type of a vertex \( u \in B \), the \( A \)-edge \([f]\) is the type the edge \( f \) of \( B \), and we say that \( f \) has the label \((f_\alpha, [f], f_\omega)\). 

The \( A \)-graph \( B \) defines a graph of groups \( B \) as follows. Vertex groups are the groups \( B_u \) above, for each edge \( f \) the edge group \( B_f \) is the subgroup of \( A_{\alpha(f)} \) defined by 

\[
B_f = \alpha_f^{-1}(f\alpha^{-1}B_{\alpha(f)}f_\alpha) \cap \omega_f^{-1}(f\omega B_{\omega(f)}f_\omega^{-1}),
\]

and the monomorphism \( \alpha_f : B_f \to B_{\alpha(f)} \) is defined by 

\[
\alpha_f(g) = f_\alpha \alpha([f])(g) f_\alpha^{-1}.
\]

Since \( \omega_f = \alpha_{f^{-1}} \), we have that \( \omega_f : B_f \to B_{\omega(f)} \) is given by 

\[
\omega_f(g) = f_\omega^{-1} \omega([f])(g) f_\omega.
\]

Henceforth we will not distinguish between the \( A \)-graph \( B \) and its associated graph of groups \( B \), referring to both as \( B \) and saying that \('B is an \( A \)-graph'.

To each \( B \)-path \( q = < q_0, f_1, q_1, \ldots, f_m, q_m > \) from \( u_0 \) to \( u_1 \) we associate an \( A \)-path \( \mu(q) \) from \([u_0]\) to \([u_1]\) defined by 

\[
\mu(q) = < q_0(f_1)\alpha, [f_1], (f_1)\omega q_1(f_2)\alpha, [f_2], \ldots, f_m, (f_m)\omega q_m > .
\]

Note that if path \( p \) ends at vertex \( u \) and \( q \) begins at \( u \), then for the concatenated path \( pq \) we have 

\[
\mu(pq) = \mu(p)\mu(q).
\]

For two equivalent \( B \)-paths \( q \) and \( q' \), their corresponding \( A \)-paths \( \mu(q) \) and \( \mu(q') \) are equivalent. Let 

\[
\overline{L}(B, u_0, u_1) = \{ \overline{\mu(q)} | q \text{ is a reduced } B \text{-path from } u_0 \text{ to } u_1 \},
\]

where \( \overline{\mu(q)} \) denotes the \( A \)-equivalence class of \( \mu(q) \). If \( u_0 \) is a vertex of \( B \) such that \([u_0] = v_0\), then \( \mu \) induces a homomorphism \( \nu : \pi_1(B, u_0) \to \pi_1(A, v_0) \) whose image is precisely \( \overline{L}(B, u_0, u_0) \).

We are interested in the case when the homomorphism \( \nu \) is injective, which occurs when \( B \) is folded. An \( A \)-graph \( B \) is said to be not folded if at least one of the following conditions holds.

(I) There exists a vertex \( u \) and two distinct edges \( f_1 \) and \( f_2 \) with \( u = t(f_1) = o(f_2) \) and \([f_1] = [f_2]^{-1} = e\), such that 

\[
(f_1)\omega b(f_2)\alpha = \omega_e(e)
\]

for some \( b \in B_u \) and \( e \in A_e \).

(II) There is an edge \( f \) such that 

\[
\alpha_e^{-1}(f\alpha^{-1}B_{\alpha(f)}f_\alpha) \neq \omega_e^{-1}(f\omega B_{\omega(f)}f_\omega^{-1}),
\]

where \( e = [f] \).
Otherwise, \( B \) is folded. Note in (I) that an edge \( f \) and its inverse \( f^{-1} \) are distinct edges.

**Lemma 2.** Suppose \( B \) is folded. Then for every reduced \( B \)-path \( q \) the \( A \)-path \( \mu(q) \) is reduced, hence the homomorphism \( \nu: \pi_1(B, u_0) \to \pi_1(A, v_0) \) is injective.

Starting with an \( A \)-graph \( B' \) that is not folded, there are six folding moves \( F1-F6 \) and three auxiliary moves \( A0-A2 \) that may be applied to eliminate instances of (I) and (II) while preserving the image of \( \pi_1(B, u_0) \) in \( \pi_1(A, v_0) \).

The folding algorithm (Proposition 5.4 of [10]) consists of performing a sequence of folding moves (in any order) until a folded graph is obtained. To carry out the folding moves effectively, and to ensure the existence of a terminating sequence, the following condition is sufficient.

**Definition 3.** A finite connected graph of finitely generated groups \( A \) is said to be benign if all of the following conditions are satisfied.

1. For each vertex \( v \) and edge \( e \) with \( o(e) = v \) there is an algorithm that, given a finite set \( X \subset A_v \) and an element \( a \in A_v \), decides whether or not \( \langle X \rangle \cap \alpha_{e}(A_e) \) is empty and if non-empty produces an element of this intersection.
2. Every edge group is Noetherian (all subgroups are finitely generated).
3. Every edge group has decidable uniform membership problem. That is, there is an algorithm that, given a finite set \( X \subset A_e \) and an element \( a \in A_e \), decides whether or not \( a \in \langle X \rangle \).
4. For each vertex \( v \) and edge \( e \) with \( o(e) = v \) there is an algorithm that, given a finite set \( X \subset A_v \) computes a generating set for \( \langle X \rangle \cap \alpha_{e}(A_e) \).

The main theorem of [10] produces, from a set of subgroup generators, a folded \( A \)-graph representing the subgroup and a presentation for the subgroup.

**Theorem 4.** Let \( A \) be a benign graph of groups with base vertex \( v_0 \).

(i) There is an algorithm that, given a finite subset \( X \subset \pi_1(A, v_0) \), constructs a folded \( A \)-graph \( B \) with base vertex \( u_0 \) such that \( L(B, u_0, u_0) = \langle X \rangle \). Each vertex group in \( B \) is described by a generating set of elements of vertex groups of \( A \).

(ii) If each vertex group of \( A \) is effectively coherent then \( \pi_1(A, v_0) \) is effectively coherent.

### 2.2 Cosets of a subgroup

In order to solve the power coset membership problem in \( \pi_1(A, v_0) \), we will construct a folded graph \( B^{(x)} \) similar to the graph \( B \) constructed in Theorem 4. While in \( B \) elements of the subgroup \( H = \langle X \rangle \) are represented by loops at a vertex \( u_0 \), elements of \( xH \) are represented by paths in \( B^{(x)} \) from a distinguished vertex \( u_x \) to \( u_0 \). The graph \( B^{(x)} \) is not strictly necessary to solve power coset membership:
to decide if \( p^n \in xH \), we may instead check if \( x^{-1}p^n \) is represented by a loop in \( B \). However, this requires that the path \( x^{-1}p^n \) be reduced, and since the reduced path depends on \( n \), this introduces complications. The use of \( B(x) \) provides a more elegant solution.

**Theorem 5.** Let \( G = \pi_1(A, v_0) \), where \( A \) is a benign graph of groups. There is an algorithm that, given a finitely generated subgroup \( H = \langle h_1, \ldots, h_m \rangle \leq G \) and \( x \in G \), produces either

1. an element \( y \in G \) with path length 0 such that \( yH = xH \), if such \( y \) exists, and otherwise
2. a folded \( A \)-graph \( B(x) \) with distinguished vertices \( u_x \) and \( u_0 \) such that

\[
L(B(x), u_x, u_0) = xH.
\]

The analogous result for right cosets holds.

**Proof.** Construct an \( A \)-graph \( B_0(x) \) as follows. Begin with a base vertex \( u_0 \). We may assume that each generator \( h \) of \( H \) is given as reduced \( A \)-path

\[
p_h =< h_0, e_1, h_1, \ldots, e_k, h_k >
\]

from \( v_0 \) to \( v_0 \) (Property (1) of the definition of benign implies that we may compute a reduced representative for any non-reduced path). If \( k > 0 \), attach at \( u_0 \) a loop with \( k \) edges having labels

\[
(h_0, e_1, 1), (h_1, e_2, 1), \ldots, (h_{k-2}, e_{k-1}, 1), (h_{k-1}, e_k, h_k).
\]

These labels define the types of the vertices along this path, i.e. the first has type \( o(e_1) = v_0 \), the second has type \( o(e_2) \), and so on. Each vertex group along this path, except for \( B_{u_0} \), is set to be trivial. Set \( B_{u_0} \) to be

\[
B_{u_0} = \{ h \mid p_h = 0 \} \leq A_{v_0}.
\]

Denote this \( A \)-graph by \( B_0(x) \) and note that it is precisely the \( A \)-graph \( B_0 \) defined in Definition 5.1 of [10].

Now let \( x \) be represented by the reduced \( A \)-path

\[
p_x =< x_0, e_1(x), x_1, \ldots, e_l(x), x_l >
\]

from \( v_0 \) to \( v_0 \). Construct a 'line' of edges \( f_1^{(x)}(x), f_2^{(x)}(x), \ldots, f_l^{(x)}(x) \), with \( t(f_i^{(x)}(x)) = o(f_{i+1}^{(x)}(x)) \) for \( i = 1, \ldots, l-1 \), having labels

\[
(x_0, e_1^{(x)}(x), 1), (x_1, e_2^{(x)}(x), 1), \ldots, (x_{l-2}, e_{l-1}^{(x)}(x), 1), (x_{l-1}, e_l^{(x)}(x), x_l),
\]

with vertex types assigned according these labels as above. Attach this line to \( B_0(x) \) by setting \( t(f_1^{(x)}(x)) = u_0 \). Denote \( o(f_1^{(x)}(x)) = u_x \), the initial vertex of the first edge, and note that it is of type \( v_0 \). Each vertex group along the path is set to be trivial (except \( B_{u_0}, \) which has been assigned above). We have now constructed \( B_0(x) \).
Lemma 6. Let $B_{0}^{(x)}$ be as above. Then

$$L(B_{0}^{(x)}, u_0, u_0) = H \quad \text{and} \quad L(B_{0}^{(x)}, u_x, u_0) = xH.$$  

Proof. Since the vertex groups along the line $f_{1}^{(x)}, \ldots, f_{l}^{(x)}$ are all trivial, a reduced path from $u_0$ to $u_0$ cannot contain any of these edges. Hence the first statement is immediate (cf. Lemma 5.3 of [10]).

For the second statement, let $p$ be any reduced path from $u_x$ to $u_0$. Since all the vertex groups along the $f_{1}^{(x)}, \ldots, f_{l}^{(x)}$ branch are trivial, $p$ can be written as a concatenation of paths $p = qp'$ where

$$q = < 1, f_{1}^{(x)}, 1, f_{2}^{(x)}, \ldots, 1, f_{l}^{(x)}, 1 >$$

and $p'$ is a reduced path from $u_0$ to $u_0$. Clearly $\mu(q) = x$, and $\mu(p') = h \in H$ by the first statement, so $\mu(p) = xh \in xH$, hence $L(B_{0}^{(x)}, u_x, u_0) \subset xH$. Conversely, each $xh$ can be represented by a path $qp'$ as above and hence $xH \subset L(B_{0}^{(x)}, u_x, u_0)$. \hfill \Box

We now apply folding moves to $B_{0}^{(x)}$, in any order, producing a sequence of $A$-graphs

$$B_{0}^{(x)}, B_{1}^{(x)}, \ldots$$

We adopt the convention that the images in every $B_{i}^{(x)}$ of edges $f$ and vertices $u$ from $B_{0}^{(x)}$ continue to be denoted by the same letters $f$, $u$. As well, $\mu$ will denote the map from $B_{i}^{(x)}$-paths to $A$-paths, for all values of $i$.

If at any point in the sequence the next move will identify $u_x$ with $u_0$, we stop and do not perform this move (the algorithm will output $y$ in this case, as we will see below). We assume then that $u_x$ and $u_0$ are distinct in every $B_{i}^{(x)}$. We also make two small changes to the folding moves, regarding the use of auxiliary move $A0$. Recall that to apply $A0$ at a vertex $B_u$ we take an element $g \in A_{[u]}$ and replace

(i) $B_u$ by $gB_u g^{-1}$,

(ii) $f_o$ by $gf_o$ for edges $f$ with $o(f) = u$, and

(iii) $f_o$ by $f_og$ for edges $f$ with $t(f) = u$.

Move $A0$ is only used in folding moves $F1$ and $F2$.

In $F1$, two distinct non-loop edges $f_1$ and $f_2$ are folded together. At the beginning of this move, $A0$ must be applied at one of the two vertices $t(f_1)$ or $t(f_2)$ (we may choose which) in order to obtain identical edge labels on $f_1$ and $f_2$. Since no folding move in the sequence (2) identifies $u_x$ with $u_0$, at least one of the vertices $t(f_1)$ or $t(f_2)$ is neither $u_x$ nor $u_0$. We choose to apply $A0$ at this vertex.

In $F2$, a non-loop edge $f_2$ is folded onto a loop edge $f_1$. Auxiliary move $A0$ must be applied at $t(f_1)$, with an element $g \in A_{[t(f_1)]}$, in order to equalize the edge labels on $f_1$ and $f_2$. If $t(f_1)$ is equal to $u_x$ or $u_0$, we will, at the conclusion of the folding move, apply $A0$ with the element $g^{-1}$ at the vertex $t(f_1)$ (which now
coincides with \( t(f_2) \). Note that in the description of F2 in [10], this is done only when \( t(f_1) = u_0 \).

It follows from [10] Prop. 4.15 that \( L(B_i^{(x)}, u_0, u_0) = H \) for all \( i \). We claim that
\[
L(B_i^{(x)}, u_x, u_0) = xH
\]
for every \( i \). From the description of the folding moves, and the considerations regarding A0 above, it follows that for every path \( p \) in \( B_0^{(x)} \) from \( u_x \) to \( u_0 \) there exists, in each \( B_i^{(x)} \), a path \( p' \) from \( u_x \) to \( u_0 \) such that \( \mu(p') = \mu(p) \). Hence \( L(B_i^{(x)}, u_x, u_0) \supseteq xH \). In particular, for the path \( p' \) which represents \( x \) in \( B_0^{(x)} \), there exists a path \( q' \) in \( B_i^{(x)} \) from \( u_x \) to \( u_0 \) such that \( \mu(q') = x \). For the opposite inclusion, let \( p \) be any path from \( u_x \) to \( u_0 \) in \( B_i^{(x)} \). Since \( (q')^{-1}p \) is a loop based at \( u_0 \), we have \( \mu((q')^{-1}p) = h \in H \) and so
\[
\mu(p) = \mu(q'(q')^{-1}p) = \mu(q') \mu((q')^{-1}p) = xh,
\]
proving the claim.

Now if at no point in the folding sequence is there a move that would identify \( u_x \) with \( u_0 \), the folding algorithm terminates producing the folded graph \( B^{(x)} \) as required (see the proof of Theorem 5.8 of [10]). Assume then that for the graph \( B_i^{(x)} \) there is an applicable folding move which would identify \( u_x \) with \( u_0 \).

This move must be of type F1 or F2, involving edges \( f_1^{(x)} \) and an edge \( f \) with \( o(f) = t(f_1^{(x)}) \) and \( t(f) = u_0 \). Two possibilities are shown in Figure 2.1 (the third possibility has \( f_1^{(x)} \) as the loop edge). Let \( e = e_1^{(x)} \), \( u = o(f) \), and let \( f_1^{(x)} \) have label \( (a_1, e, b_1) \) and \( f \) have label \( (a_2, e^{-1}, b_2) \). Since \( B_i^{(x)} \) is not folded, there exists \( b \in B_u \) and \( c \in A_x \) such that
\[
b_1ba_2 = \omega_e(c).
\]
Then
\[
p = <1, f_1^{(x)}, b, f, 1>
\]
is a path in \( B_i^{(x)} \) from \( u_x \) to \( u_0 \) hence \( \mu(p) = xh \) for some \( h \in H \). But
\[
\mu(p) = <a_1, e, b_1ba_2, e^{-1}, b_2> = <a_1, e, a_\epsilon(c), e^{-1}, b_2>
\]
which is equivalent to the length 0 path \( y = a_1\omega_e(c)b_2 \in A_{u_0} \). Hence \( xH = yH \), and the algorithm returns the element \( y \).

\[ \square \]

### 2.3 Semi-canonical forms for paths in certain \( \mathbb{A} \)-graphs

Let \( \mathbb{B} \) be a folded \( \mathbb{A} \)-graph and let

\[ p = (p_0, e_1, p_1, e_1, \ldots, p_{n-1}, e_n, p_n) \]

be an \( \mathbb{A} \)-path. If there exists a \( \mathbb{B} \)-path \( q \) such that \( \mu(q) = \overline{p} \), then there will (usually) be infinitely many equivalent such paths. We will define a semi-canonical form for these paths, which will depend on the choice of representative \( p \) for the class \( \overline{p} \). While semi-canonical forms are not required to solve the problem of determining the existence of \( q \), they are essential in solving the more difficult problem of whether or not there exists \( q \) and \( m \neq 0 \) such that \( \mu(q) = \overline{p^m} \) (discussed in the next section).

We place the following restriction on \( \mathbb{B} \): assume that every non-trivial edge group \( B_f \) of \( \mathbb{B} \) has finite index in the corresponding edge group \( A_c \) of \( \mathbb{A} \). This is a simple generalization of the case in which all edge groups of \( \mathbb{A} \) are cyclic groups.

For each non-trivial \( B_f \), fix a set \( R_f \) of left coset representatives of \( B_f \) in \( A_c \), using \( R_f = \{1\} \) in the case when \( B_f = A_f \).

Our definition of semi-canonical forms corresponds to the following procedure for searching for a path \( q = (q_0, f, \ldots) \) such that \( \mu(q) = \overline{p} \). We start searching at a vertex \( u' \) such that \( [u'] = o(e_1) \). We locate an edge \( f \) of type \( e_1 \) incident on \( u' \).

Since the map \( \mu \) will multiply \( q_0 \) on the right by \( f_o \), and the target element \( p_0 \) may be replaced by \( p_0\alpha e_1(c) \) for any \( c \in A_{e_1} \), the element \( p_0\alpha e_1(c)f^{-1}_{\alpha} \) must be in the vertex group \( B_{u'} \) for some \( c \in A_{e_1} \) for \( q \) to exist. Having selected \( q_0 = p_0\alpha e_1(c)f^{-1}_{\alpha} \), we proceed to the next vertex group, but with \( p_1 \) replaced by \( \omega e_1(c^{-1})p_1 \). Though there may be infinitely many choices for the ‘adjustment’ \( c \), we will show that only finitely many ‘canonical’ choices need to be considered. Along the next edge we may have another adjustment \( c' \), and so produce a sequence of adjustments, each depending on the previous.

#### Canonical adjustments

Let \( f \) be an edge of \( \mathbb{B} \) and denote \([f] = e \) and \( u = o(f) \). For an element \( c \in B_f \), we define the left adjustment corresponding to \( c \) as

\[ l_c = \alpha c f^{-1}_{\alpha} \quad (3) \]

and the right adjustment corresponding to \( c \) as

\[ r_c = f^{-1}_{\omega}\omega (c^{-1}) \quad (4) \]

Note that while \( l_c \) and \( r_c \) depend on both \( c \) and \( f \), we will assume that the edge groups of \( \mathbb{A} \) are pairwise disjoint, making \( f \) uniquely determined by \( c \).

For an element \( a \in A_{\alpha(e)} \) define a set \( C(f, a) \subseteq A_c \) called the canonical adjustment set as follows. If there is no element \( c \in A_c \) such that \( ac \in B_{u} \), then \( C(f, a) \) is empty. If such an element \( c \) does exist, then
Let $\alpha$ be a reduced $A$-path and let $\sigma$ be an adjustment sequence associated with the pair $(p, \mathcal{F})$. Let $c_i = A_{e_i}$ for $i = 1, \ldots, n$ that satisfies the following properties:

(i) $c_1 \in C(f_1, p_0)$,
(ii) $c_i \in C(f_i, r_{c_{i-1}}, p_i)$ for $i = 2, \ldots, n - 1$, and

(iii) otherwise $B_f$ has finite index in $A_e$ and we set

$$C(f, a) = \{ c \in R_f \mid al_c \in B_u \}.$$
Note that the possible values for $c_i$ depend on $c_{i-1}$. To every adjustment sequence $\sigma$ we associate a $B$-path

$$Q(\sigma) = \langle p_0 l_{c_1}, f_1, r_{c_1} p_1 l_{c_2}, f_2, \ldots, f_n, r_{c_n} p_n \rangle.$$  

If the path $F$ has length 0, we associate with $F$ an empty adjustment sequence $\sigma = \emptyset$ and define $Q(\sigma) = p_0$. The path $Q(\sigma)$ is in fact a $B$-path since the elements $p_0 l_{c_1}, r_{c_1} p_1 l_{c_2}, \ldots, r_{c_n} p_n$ lie in the appropriate $B$-vertex groups, by construction (see Lemma 7).

**Lemma 8.** For every adjustment sequence $\sigma$ associated with $(p, F)$, the path $Q(\sigma)$ is a $B$-path satisfying

$$\mu(Q(\sigma)) = \bar{p}.$$

**Proof.** We have

$$\mu(Q(\sigma)) = \langle p_0 l_{c_1}(f_1)\omega_1, (f_1)\omega r_{c_1} p_1 l_{c_2}(f_2)\alpha, \ldots, e_n, (f_n)\omega r_{c_n} p_n \rangle = \langle p_0 \alpha e_1(c_1), e_1, \omega e_1(c_1^{-1}) p_1 \alpha e_2(c_2), \ldots, e_n, \omega e_n(c_n^{-1}) p_n \rangle = \bar{p}. $$

\[ \square \]

**Semi-canonical paths**

Now let $\Phi$ be the set of all edge paths $F = (f_1, \ldots, f_n)$ with $o(f_1) = u'$, $t(f_n) = u''$, and $[f_i] = e_i$ for $i = 1, \ldots, n$. We define the set of **semi-canonical paths** $SC(p, u', u'')$ as

$$SC(p, u', u'') = \bigcup_{F \in \Phi} \{Q(\sigma) \mid \sigma \text{ is an adjustment sequence for } (p, F)\}.$$ 

The key properties of semi-canonical paths (Theorem 9) are that every path mapping onto $\bar{p}$ is equivalent to a semi-canonical path and that the number of semi-canonical paths is finite.

**Theorem 9.** Let $\mathcal{A}$ be a graph of groups and let $\mathcal{B}$ be a folded $\mathcal{A}$-graph such that every non-trivial edge group of $\mathcal{B}$ has finite index in its corresponding edge group of $\mathcal{A}$. Let

$$p = \langle p_0, e_1, p_1, \ldots, e_n, p_n \rangle$$

be an $\mathcal{A}$-path and let $u'$, $u''$ be vertices of $\mathcal{B}$ with $[u'] = o(e_1)$ and $[u''] = t(e_n)$. Then every $\mathcal{B}$-path $q$ from $u'$ to $u''$ that satisfies

$$\mu(q) = \bar{p} \tag{5}$$

is equivalent to a path in the finite set $SC(p, u', u'')$. Consequently, the set $SC(p, u', u'')$ is non-empty if and only if there exists a path $q$ from $u'$ to $u''$ satisfying (5).
Proof. The fact that $\mathcal{SC}(p, u', u'')$ is finite follows from the fact that every set $C(f, a)$ is finite and there are finitely many edge paths of length $n$ in $B$, hence there are finitely many adjustment sequences.

We now proceed by induction on the length $n$ of $p$. Suppose that the path $q$ exists. We may assume that $q$ is reduced. When $n = 0$, $q$ must be precisely the length 0 path $q = < p_0 >$, and $u' = u''$. This path is the unique element of $\mathcal{SC}(p, u', u'')$.

Now assume the statement holds for paths of length less than $n$ and suppose that there exists a path $q = < q_0, f_1, q_1, \ldots, f_n, q_n >$ from $u'$ to $u''$ such that $\mu(q) = \overline{p}$. Denote

$$\text{Tail}(q) = < q_1, f_2, q_3, \ldots, f_n, q_n > .$$

Then there exists $c_1 \in A_{e_1}$ such that $q_0(f_1)_\alpha = p_0\alpha_{e_1}(c_1)$ and $\text{Tail}(q)$ satisfies

$$\mu(\text{Tail}(q)) = r_{c_1}\text{Tail}(p).$$

Let $f = f_1$. We will show that we may assume $c_1 \in C(f, p_0)$ by replacing $q$ with an equivalent path $\tilde{q}$. We consider the three cases in the definition of $C(f, p_0)$.

In the first case, when $A_{e_1} = 1$, we have $c_1 = 1$ which is the only element in $C(f, p_0)$. In the second case, when $B_f = 1$, the element $c_1$ is unique and is the only element in $C(f, p_0)$. In the third case, there exist $c_1' \in R_f$ and $b \in B_{u'}$ such that $c_1 = c_1'b$. Since $b^{-1} \in B_f$, we may replace $q$ by the equivalent path

$$\tilde{q} = < q_0\alpha_f(b^{-1}), f, \omega_f(b)\text{Tail}(q) > .$$

Since the first element $q_0\alpha_f(b^{-1})$ of $\tilde{q}$ satisfies

$$(q_0\alpha_f(b^{-1})) f_\alpha = q_0f_\alpha\alpha_{e_1}(b^{-1}) = p_0\alpha_{e_1}(c_1b^{-1}) = p_0\alpha_{e_1}(c_1'),$$

we may assume from the beginning that $c_1 \in C(f, p_0)$.

Since $\text{Tail}(q)$ has length $n - 1$ it is equivalent, by induction, to a semi-canonical path. That is, there exists an adjustment sequence $\sigma' = (c_2, \ldots, c_n)$ associated with the path $r_{c_1}\text{Tail}(p)$ and the edge path $(f_2, \ldots, f_n)$ such that $\text{Tail}(q)$ is equivalent to $Q(\sigma')$. If $n = 1$ then $r_{c_1}p_1 \in B_{u''}$ and if $n \geq 2$ then $c_2 \in C(f_2, r_{c_1}p_1)$, hence the sequence $\sigma = (c_1, \ldots, c_n)$ is an adjustment sequence associated with $p$ and $(f_1, \ldots, f_n)$. Then

$$q = < q_0, f_1, Q(\sigma') > = < p_0c_1, f_1, r_{c_1}p_1c_2, f_2, \ldots, r_{c_n}p_n > = Q(\sigma)$$

hence $q$ is equivalent to the semi-canonical path $Q(\sigma)$.

\[\square\]

Remark 10. If $p'$ is an $A$-path with $\overline{p'} = \overline{p}$, the set $\mathcal{SC}(p', u', u'')$ need not coincide with $\mathcal{SC}(p, u', u'')$, hence the adjective ‘semi-canonical’. This arises from the fact that whenever $B_f = 1$, the unique element of $C(f_i, r_{e_{i-1}}p_i)$ depends on $p_i$.  

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2.4 Reading powers in an $A$-graph

When $A$ is a benign graph of groups and $B$ is a folded $A$-graph, there is an algorithm that, given an $A$-path $p$, decides whether or not there exists a $B$-path $q$ such that $\mu(q) = \overline{p}$ (see Claim 5.14 of [10]). We require a stronger version of this result (the algorithm $\text{ReadPower}$ on page 16), which will decide whether or not there exists $q$ such that $\mu(q) = \overline{p}^m$ for some $m > 0$, under certain conditions on $A$ and $B$.

We use Theorem [9] to restrict $q$ to semi-canonical forms, but consequently we must insist that non-trivial edge groups of $B$ have finite index in the corresponding $A$-edge groups. We assume that $B$ is already folded, so rather than insist that $A$ be benign (the requirement for folding), we specify a set of properties (Property 12) for the pair $(A, B)$ that are sufficient for our algorithm to run and terminate. To simplify the description of these properties, we begin by defining the following property.

**Property 11.** Let $G$ be a group, and let $C$ and $C'$ be subgroups of $G$.

(i) We say that $G$ satisfies Property 11 with respect to $C$ if for every finitely generated subgroup $H \leq G$ and every $x, g \in G$, if $g^{-1}x \not\in C$ and $H \cap C = H \cap C' = 1$

then there exist finitely many pairs $(c, c') \in C \times C'$ such that $cgc' \in H$.

(ii) We say that $G$ satisfies Property 11 with respect to $(C, C')$ if for every finitely generated subgroup $H \leq G$ and every $g \in G$, if $H \cap C = H \cap C' = 1$,

then there exist finitely many pairs $(c, c') \in C \times C'$ such that $cgc' \in H$.

We now state the conditions we will need in order to use $\text{ReadPower}$. By an edge cycle we mean a finite edge path $(f_1, f_2, \ldots, f_m)$ with $t(f_m) = o(f_1)$. Let $f_{m+1}$ denote $f_1$.

**Property 12.** Let $A$ be a graph of groups and let $B$ be a folded $A$-graph. We say that the pair $(A, B)$ satisfies Property 12 if all of the following conditions hold.

(i) Every vertex group of $A$ has decidable membership and power membership problem.

(ii) There is an algorithm that, given a vertex $v$ of $A$, an edge $f$ of $B$ with $[o(f)] = v$, and an element $x \in A_v$, decides whether or not the intersection $xB_{o(f)} \cap \alpha[f](A_{[f]})f^{-1}$ is empty.
(iii) For every edge $f$ of $B$, $B_f$ is trivial or has finite index in $A[f]$.

(iv) For every edge cycle $f_1, \ldots, f_m$ of $B$ in which $B_{f_i} = 1$ for all $i$, there exists $i \in \{1, \ldots, m\}$ such that

1. if $e_{i+1} \neq e_i^{-1}$, then $A_{t(f_i)}$ satisfies Property 11 with respect to the pair of subgroups
   \[ \left( \omega_{e_i}(A_{e_i})^{(f_i)} \omega, \alpha_{e_{i+1}}(A_{e_{i+1}})^{(f_{i+1})} \omega^{-1} \right), \]

2. if $e_{i+1} = e_i^{-1}$, then $A_{t(f_i)}$ satisfies Property 11 with respect to
   \[ \omega_{e_i}(A_{e_i})^{(f_i)} \omega, \]

where $e_i = [f_i]$ and $e_{i+1} = [f_{i+1}]$.

Let $p = < p_0, e_1, p_1, \ldots, e_n, p_n >$ be a reduced $A$-path which is a loop (i.e. $o(e_1) = t(e_n)$). We say that $p$ is cyclically reduced if $p^2$ is also reduced. For $i \in \{0, \ldots, n-1\}$, define

\[ \hat{p}_0 = \hat{p}_n = p_0, \]
\[ \hat{p}_i = p_i, \text{ for } i \neq 0. \]

Indices of $\hat{p}_i$ are taken modulo $n$. Note that for any $m > 0$,

\[ p^m = < p_0, e_1, \hat{p}_1, e_2, \hat{p}_2, \ldots, e_n, \hat{p}_n, e_1, \hat{p}_1, \ldots, e_n, p_n >. \]

The following algorithm ReadPower is a modified breadth-first search on the graph $B$. It uses a data structure $T$ called the search tree which is a rooted, ordered tree. Each vertex $\tau$ of $T$ is labelled by a pair $(u, i)$ where $u$ is a vertex of $B$ and $i$ is the distance modulo $n$ from the root to $\tau$, and is marked as either ‘explored’ or ‘unexplored’. Though $i$ is determined by the depth of $\tau$ in $T$, having explicitly recorded will simplify our description and proof of the algorithm. The root is labelled by $(u', 0)$. Each edge from a vertex labelled by $(u, i)$ to a vertex labelled by $(u', i+1)$ is labelled by a pair $(f, c)$ where $f$ is an edge of $B$ from $u$ to $u'$ and $c \in A[f]$. The element $c$ will be a canonical adjustment which is ‘pushed forward’ to the next vertex group.

**Theorem 13.** There is an algorithm ReadPower that, given

- a graph of groups $A$ and a folded $A$-graph $B$ satisfying the conditions of Property [12]
- a cyclically reduced $A$-loop $p$,
- vertices $u', u''$ of $B$,

decides whether or not there exists a $B$-path $q$ from $u'$ to $u''$ and an integer $m > 0$ such that

\[ \mu(q) = p^m \]

and if so, produces $q$ and the minimum such $m$. 

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Algorithm 1: ReadPower

Input: Graph of groups $A$ and a folded $A$-graph $B$ satisfying Property 12, a reduced $A$-loop $p$, and vertices $u', u''$ of $B$.

Output: A pair $(q, m)$ where $q$ is a $B$-path from $u'$ to $u''$ and $m$ is a positive integer $m$ such that $\mu(q) = p^m$, with $m$ minimal over all such pairs. If no pair exists, the word 'No'.

1. $T :=$ single root vertex labelled by $(u', 0)$ and marked 'unexplored';
2. if $[u'] \neq o(e_1)$ or $[u''] \neq t(e_n)$ then
   3. return 'No';
4. while $T$ has unexplored vertices do
   5. $\tau :=$ unexplored vertex having minimum distance $d$ from the root and, of all unexplored vertices at distance $d$, is left-most;
   6. $(u, i) :=$ label of $\tau$;
   7. $e := e_i + 1$;
   8. if $\tau$ is the root then
      9. if $\text{Length}(p) = 0$ then
         10. if $B_{u'} \cap (p_0) \neq 1$ then
             11. return $(p^m_0, m)$ with $m > 0$ minimal such that $p^m_0 \in B_{u'}$;
         12. else
             13. return 'No';
      14. else
         15. $a := p_0$;
   16. else
      17. $\tau' :=$ Parent($\tau$);
      18. $(f', c') :=$ label of edge $\tau' \rightarrow \tau$;
      19. $a := r_{c'} \bar{p}_i$;
      20. $F := \{ f \in \text{Edges}(B) \mid [f] = e, o(f) = u \}$;
      21. for each $f \in F$ do
         22. if $\exists c \in A_e$ such that $a_l e_c B_{u'}$ then
            23. for each $c \in C(f, a)$ do
               24. if $i = n - 1$ and $t(f) = u''$ and $r_{c} p_n \in B_{u''}$ then
                  25. $m := (d + 1)/n$;
                  26. Let $(f_1, c_1), \ldots, (f_d, c_d)$ be the edge labels of the path from the root to $\tau$;
                  27. $\sigma := (c_1, \ldots, c_d, c)$;
                  28. return $(Q(\sigma), m)$;
               29. else
                  30. if no vertex $(t(f), i + 1)$ in $T$ has incoming edge $(f, c)$ then
                     31. Insert a child $\tau''$ of $\tau$ with label $(t(f), i + 1)$;
                     32. Label the edge $\tau \rightarrow \tau''$ by $(f, c)$;
                     33. Mark $\tau''$ as unexplored;
                  34. Mark $\tau$ as explored;
               35. return 'No';
Proof. Algorithm effectiveness. First, we check that all steps of the algorithm may in fact be carried out. The only non-obvious steps are those on lines 10, 11, 22 and 24.

Line 10 is an instance of the power membership problem in $A[u']$, hence is decidable. For line 11, we check, using decidability of the membership problem in $A[u']$, if each of $p, p', p^3, \ldots$ is in $B_u'$ and return the first success.

On line 22 deciding the existence of $c$ is equivalent to deciding whether or not the intersection $(f_\alpha a^{-1}) B_u \cap \alpha_c (A_e)^{f_{\alpha}^{-1}}$, in the vertex group $A[u]$, is empty. This is decidable by Property 12(ii).

On line 23, we compute the set $C(f, a)$. According to the definition of $C(f, a)$, this amounts to either searching for the unique element $c \in A_e$ such that $a_l c \in B_u$ or checking the condition $a_l c \in B_u$ for finitely many $c$. Line 24 also involves a membership problem in a vertex group of $A$, and this problem is decidable by Property 12(i).

Algorithm termination. Next, we ensure that the algorithm terminates. Every iteration of the while loop on line 4 marks one vertex of $T$ as explored, so it suffices to show that the size of $T$ is bounded. Since $F$ and $C(f, a)$ are always finite sets, every vertex in $T$ has finitely many children, so it suffices to show that the depth of $T$ is bounded.

Observe that an edge of $T$ labelled by $(f, c)$ having terminal vertex labelled by $(t(f), i)$ occurs at most once in $T$: once such an edge exists, the conditional on line 30 prevents it from being created a second time. The possible values for $f$ and $i$ are clearly finite. If $B_f \neq 1$, then the set $C(f, a)$ is, regardless of $a$, a subset of the fixed, finite set of coset representatives $R_f$. It follows that there are finite many edges in $T$ labelled by $(f, c)$ such that $B_f \neq 1$. When $B_f = 1$, the set $C(f, a)$ consists of a single element, but this element may (in principle) be any element of $A[f]$. We will argue that there are only finitely many possible values for $c$ in this case.

Assume that the depth of $T$ is unbounded. Then $T$ contains an infinite path

$$\tau_0 \rightarrow \tau_1 \rightarrow \tau_2 \rightarrow \ldots, \quad (7)$$

and we denote by $(f_j, c_j)$ the label of the edge from $\tau_j$ to $\tau_{j+1}$. By the above observation, there exists $M > 0$ such that $B_{f_j} = 1$ for all $j > M$. Let $E$ be the number of edges in $B$. For all $j > M$, the sequence of edges

$$f_j, f_{j+1}, \ldots, f_{j+E}$$

must contain a subsequence which is an edge cycle in $B$, and so must contain a sequential pair of edges $f, f'$ satisfying the statements in Property 12(iv). Since the path (7) is infinite, such a pair must occur infinitely often. Hence there exists a sequential pair of edges $f, f'$ satisfying the statements in Property 12(iv), an integer $i \in \{0, \ldots, n - 1\}$, and an infinite subset $J \subset \mathbb{N} > M$ such that for all $j \in J$, $f_{j-1} = f, f_j = f'$, and $\tau_j$ is labelled by $(u, i)$, where $u = t(f)$. 

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Let \( e = [f] \), \( e' = [f'] \), and denote \( C = \omega_e(A_e)^f \) and \( C' = \alpha_{e'}(A_{e'})^{(f')}_{\alpha^{-1}} \). Since \( B \) is folded, the edge group \( B_f \) is defined by

\[
B_f = \omega_e^{-1} \left( B_u^{f_{\alpha^{-1}}} \cap \omega_e(A_e) \right).
\]

But \( B_f \) is trivial, hence \( B_u^{f_{\alpha^{-1}}} \cap \omega_e(A_e) = 1 \) and so

\[
B_u \cap \omega_e(A_e)^{f_{\alpha^{-1}}} = 1 = B_u \cap C.
\]

Since \( B_f \) is also equal to an inverse image under \( \alpha_{e'} \), we similarly obtain

\[
B_u \cap \alpha_{e'}(A_{e'})^{(f')}_{\alpha^{-1}} = 1 = B_u \cap C'.
\]

Consider first the case when \( e' \neq e^{-1} \). We know that \( A_{[u]} \) satisfies Property \( \Box \) with respect to \( (C, C') \). By construction, \( r_{c_{j-1}}(\hat{p}_il_{c_j}) \in B_u \) for all \( j \in J \), and we may rewrite this element as

\[
(f_{\omega^{-1}}^{-1} \omega_e(c_{j-1})) \hat{p}_i \left( \alpha_{e'}(c_j)(f'_{\alpha^{-1}})^{-1} \right) = \omega_e(c_{j-1})^{f^{-1}} \left( f_{\omega^{-1}}^{-1} \hat{p}_i(f'_{\alpha^{-1}})^{-1} \right) \alpha_{e'}(c_j)(f'_{\alpha^{-1}})^{-1}.
\]

Setting \( H = B_u \) and \( g = f_{\omega^{-1}}^{-1} \hat{p}_i(f'_{\alpha^{-1}})^{-1} \in A_{[u]} \), since \( (8) \) and \( (9) \) hold, Property \( \Box \) states that there are finitely many pairs \((c_{j-1}, c_j)\) for which \( r_{c_{j-1}}(\hat{p}_il_{c_j}) \in B_u \). Hence there exists \( c \) such that \( c_{j-1} = c \) for infinitely many \( j \in J \). This contradicts the fact that \( T \) contains at most one edge labelled by \((f, c)\) with terminal vertex labelled by \((t(f), i)\).

If \( e' = e^{-1} \), we have \( \alpha_{e'} = \omega_e \) and we may rewrite the element \( r_{c_{j-1}}(\hat{p}_il_{c_j}) \) as

\[
\omega_e(c_{j-1})^{f_{\omega^{-1}}} \left( f_{\omega^{-1}}^{-1} \hat{p}_i(f'_{\alpha^{-1}})^{-1} \right) \left( \alpha_{e'}(c_j)(f'_{\alpha^{-1}})^{-1} \right).
\]

Set \( H = B_u \), \( g = \left( f_{\omega^{-1}}^{-1} \hat{p}_i(f'_{\alpha^{-1}})^{-1} \right) \), and \( x = f_{\omega^{-1}}^{-1}(f'_{\alpha^{-1}})^{-1} \). Since \( gx^{-1} = \hat{p}_il_{c_j} \), if \( gx^{-1} \in C \) then \( \hat{p}_i \in \omega_e(A_e) \) which contradicts the fact that \( p \) is cyclically reduced. Hence \( gx^{-1} \notin C \), so Property \( \Box \) states that there are finitely many pairs \((c_{j-1}, c_j)\) such that \( r_{c_{j-1}}(\hat{p}_il_{c_j}) \in B_u \) and we obtain a contradiction as above.

**Algorithm correctness.** Finally, we prove that the algorithm is correct. The cases when \( p \) has length zero, and when \([u'] \neq o(e_1) \) or \([u''] \neq t(e_n) \) are clearly correct.

First, suppose the algorithm returns the pair \((Q(\sigma), m)\) on line \( 28 \). It follows immediately from lines \( 23 \) and \( 24 \) that \((c_1, \ldots, c_d, c)\) is an adjustment sequence for \( p^m \), hence the desired path \( q = Q(\sigma) \) exists. We will argue the minimality of \( m \) below.

Now suppose that the algorithm reaches line \( 35 \) returning ‘No’. Assume, for contradiction, that there exists a \( B \)-path \( q \) from \( u' \) to \( u'' \) and an integer \( m > 0 \) such that \( \overline{\mu(q)} = p^m \), and assume that \( m \) is minimal. Then the set \( SC(p^m, u', u'') \) is non-empty, so there exists an edge sequence \( F = (f_1, \ldots, f_{nm}) \) and an adjustment sequence \( \sigma = (c_1, \ldots, c_{nm}) \) associated with \((p^m, F)\). Since the algorithm searches exhaustively for adjustment sequences, it will discover this sequence, recording it in \( T \) during lines \( 31 \) and \( 32 \) and reaching the return statement on line \( 28 \) unless
the conditional on line 30 fails. Assume then that such a failure first occurs at position \( j \) in the adjustment sequence: that is,

\[
i = j \pmod{n},
\]

and the edge \( \tau' \to \tau \) has label \((f_j, c_j)\), but the required child \( \tau'' \) of \( \tau \) is not created because \( T \) already contains a vertex with label

\[
(t(f_{j+1}), j+1)
\]

having incoming edge with label

\[
(f_{j+1}, c_{j+1}).
\]

Let \( \tau_{k+1} \) be this pre-existing vertex, let \( \tau_0, \tau_1, \ldots, \tau_{k+1} \) be the vertices along the unique path from the root \( \tau_0 \) to \( \tau_{k+1} \), and let \((f'_0, c'_0), \ldots, (f'_k, c'_k)\) be the sequence of edge labels along this path. Note that \( \tau_k \) is at distance \( k \) from the root, \( \tau \) is at distance \( j \) from the root, and that \( k = j \pmod{n} \).

Since \( \tau_k \) has children (and is not equal to \( \tau \)), it must be marked ‘explored’. Since vertices of \( T \) are processed in order of distance from the root then left-to-right, either \( k < j \) or \( k = j \) and \( \tau_k \) appears to the left of \( \tau \). Consider the sequence of elements

\[
\sigma'' = (c'_1, c'_2, \ldots, c'_k, c_{j+1}, c_{j+2}, \ldots, c_{mn})
\]

and the edge path

\[
\mathcal{F}'' = (f'_1, f'_2, \ldots, f'_k, f_{j+1}, f_{j+2}, \ldots, f_{mn}).
\]

Since

\[
(f'_{k+1}, c'_{k+1}) = (f_{j+1}, c_{j+1}),
\]

it follows that \( \sigma'' \) is an adjustment sequence associated with either \((p^{m'}, \mathcal{F}'')\), in the case \( k = j \), or with \((p^{m'}, \mathcal{F}'')\) where

\[
m' = \frac{j - k}{n},
\]

in the case \( k < j \). In the latter case, the path \( Q(\sigma'') \) is in \( \mathcal{SC}(p^{m'}, u', u'') \), contradicting the minimality of \( m \). In the former case, we may replace the original edge path \( \mathcal{F} \) and adjustment sequence \( \sigma \) by \( \mathcal{F}'' \) and \( \sigma'' \). Since \( \tau_k \) was to the left of \( \tau \) in \( T \) we may, after finitely many such replacements, reduce to the case \( k < j \) and obtain the contradiction above.

The above argument also demonstrates the minimality of the value \( m \) returned on line 28. If \( m \) is not minimal, there is a semi-canonical path \( q \) for \( p^{m'} \), with \( m' \) minimal, that the algorithm does not find (it searches breadth-first, and returns the first success). The failure cannot be due to the case \( k < j \), since this implies that \( m' \) is not minimal. The failure must occur in the \( k = j \) case, but we may then repeat the argument with a semi-canonical path \( q' \) passing through the vertex \( \tau_k \) to the left of \( \tau \). After finitely many steps we again reduce to the case \( k < j \) and obtain a contradiction.

\( \square \)
By combining the ReadPower algorithm with \(\mathbb{A}\)-graph folding, we may solve the power coset membership problem in certain graphs of groups.

**Theorem 14.** Let \(G\) be the fundamental group of a benign graph of groups \(\mathbb{A}\) in which every vertex group has decidable power coset membership problem, and such that for every folded \(\mathbb{A}\)-graph \(\mathbb{B}\), the pair \((\mathbb{A}, \mathbb{B})\) satisfies Property 12. Then the power coset membership problem is decidable in \(G\).

*Proof.* Let \(v_0\) be the base vertex of \(\mathbb{A}\), so that \(G = \pi_1(\mathbb{A}, v_0)\). Assume we are given as input to the power coset membership problem elements \(x, p, h_1, \ldots, h_n \in G\), expressed as \(\mathbb{A}\)-loops based at \(v_0\). The decidability of the membership problem in vertex groups allows us to reduce paths, so we may assume that these paths are reduced. Let \(H = \langle h_1, \ldots, h_n \rangle\) and \(p = <p_0, e_1, p_1, e_2, \ldots, e_n, p_n>\).

Suppose that \(p\) is not cyclically reduced, that is, \(p^2\) is not a reduced path. Then \(e_1 = e_n^{-1}\) and \(p_np_0 = \omega_{e_n}(c)\) for some \(c \in A_{e_n}\). Consider the conjugate

\[p^{p_0} = <1, e_1, p_1, \ldots, e_n, p_np_0 > \sim 1, e_1, p_1, \ldots, p_{n-1} \omega_{e_n}(c), e_n, 1>.

Since \((p^{p_0})^m \in x_{v_0}H^{p_0}\) if and only if \(p^m \in xH\), we may assume from the beginning that \(p_0 = p_n = 1\).

Let \(v_1 = t(e_1)\). Under the natural isomorphism \(\pi_1(\mathbb{A}, v_0) \simeq \pi_1(\mathbb{A}, v_1)\), the image of \(p\) is the reduced path \(p' = <p_1, e_2, \ldots, e_{n-1}, p_{n-1}>\) and we consider the same problem with \(v_1\) as the base vertex instead of \(v_0\) (replacing \(x\) and \(H\) by their images under this isomorphism). If \(p'\) is not cyclically reduced, we may repeat this procedure, reducing the length of \(p\) each time, until \(p\) is cyclically reduced. Hence we may assume from the beginning that \(p\) is cyclically reduced, and we continue to denote the base vertex by \(v_0\).

Since \(\mathbb{A}\) is benign, we may apply Theorem 3 with input \(H\) and \(x\). Suppose the algorithm reaches the second case, returning the coset graph \(\mathbb{B}^{(x)}\).

The power coset membership problem has a positive answer if and only if there exists a path \(q\) in \(\mathbb{B}^{(x)}\) from \(u_s\) to \(u_0\) such that \(\mu(q) = \tilde{p}^n\) or \(\mu(q) = (\tilde{p}^{-1})^n\) for some \(n > 0\). We can decide this using Theorem 13 (i.e. the ReadPower algorithm), since the pair \((\mathbb{A}, \mathbb{B}^{(x)})\) satisfies Property 12 by assumption.

Now suppose the algorithm of Theorem 5 returns a path \(y\) of length 0 such that \(yH = xH\). Construct the folded \(\mathbb{A}\)-graph \(\mathbb{B}\) described in Theorem 4 and let \(u_0\) be the base vertex of \(\mathbb{B}\) and \(A_{u_0}\) the associated vertex group.

Consider first the case when \(p\) has length 0. Since \(p^x\) and \(y\) are both in the base vertex group \(A_{v_0}\), \(p^x \in yH\) if and only if \(p^n \in y(H \cap A_{v_0})\). But \(H \cap A_{v_0} = B_{u_0}\) (since \(\mu\) preserves path length), hence the problem reduces to an instance of power coset membership in the vertex group \(A_{v_0}\), which is decidable.

Now consider the case when \(p\) has non-zero length. We must determine whether or not there exists a \(B\)-path \(q\) from \(u_0\) to \(u_0\) such that \(\tilde{\mu}(q) = y^{-1}p^\alpha\) or \(\tilde{\mu}(q) = y^{-1}(p^{-1})^\alpha\), for some \(n > 0\). We make a slight modification to the ReadPower algorithm, replacing the statement ‘\(a := p_0\)’ on line 15 by ‘\(a := y^{-1}p_0\)’ (and similarly for \(p^{-1}\)). The modified algorithm clearly solves the problem. \(\square\)

Note that the preconditions for Theorem 14, in particular Property 12[iv], may be difficult to establish. In the next section, we will use quasi-convexity properties
to establish this for groups obtained by iterated centralizer extensions, but let us mention here the simple case when edge groups are finite.

**Corollary 15.** Let $G$ be the fundamental group of a graph of groups in which all edge groups are finite and all vertex groups have decidable power coset membership problem. Then the power coset membership problem is decidable in $G$.

**Proof.** Property \[12\] is immediate. Since edge groups are finite, the various algorithmic problems involving edge groups reduce to finitely many word problems. Finite groups are Noetherian, so the graph of groups is benign. 

3 Effective coherence, embedding, enumeration

We prove in this section that any finitely generated group $G$ that is discriminated by a locally quasi-convex torsion-free hyperbolic group $\Gamma$ is effectively coherent. We approach this via the fact that $G$ embeds into a group $G_n$ obtained from $\Gamma$ by iterated centralizer extensions. A centralizer extension corresponds to a graph of groups, so we will prove that this graph of groups is benign and hence the folding algorithm may be used to compute the induced decomposition of, and hence a presentation for, a subgroup. Since the only edge group is $\mathbb{Z}$, being ‘benign’ reduces to the power coset membership problem being decidable in vertex groups. We prove this by applying the results from \[2\] inductively. As applications, we compute the embedding of $G$ into $G_n$, and provide enumeration and recognition algorithms for groups discriminated by $\Gamma$.

3.1 Iterated centralizer extensions

Let $\Gamma$ be any group. An extension of a centralizer of $\Gamma$ is a group $G_1$ presented by

$$G_1 = \langle \Gamma, t_1, \ldots, t_r \mid [C(g), t_i] = [t_i, t_j] = 1, 1 \leq i, j \leq r \rangle$$

where $g \in G$ and $C(g)$ is the centralizer of $g$ in $\Gamma$. Note that $G_1$ is isomorphic to the amalgamated product $\Gamma \ast_{C(g)} (C(g) \times \mathbb{Z})$, which we may regard as the fundamental group of a graph of groups with two vertices.

The operation of forming an extension of a centralizer may be iterated to form a chain

$$\Gamma = G_0 < G_1 < \ldots < G_n$$

(10)

where $G_i$ is an extension of a centralizer of $G_{i-1}$. We say that $G_n$ is obtained from $\Gamma$ by iterated extensions of centralizers.

Suppose now that $\Gamma$ is torsion-free and that for every $g \in G$, its centralizer $C(g)$ is cyclic and malnormal (in particular, $G$ is CSA). It follows (see \[20\]) that all centralizers in $G_n$ are free abelian groups and that the chain \[10\] may be arranged so that each $G_i$ is obtained from $G_{i-1}$ by extension of a cyclic centralizer, i.e.

$$G_i = G_{i-1} \ast_{\langle g \rangle} \mathbb{Z}^{r_i+1}$$

(11)

We prove here that in the presence of Property \[11\] the decidability of the power coset membership problem propagates up the chain \[10\].
Theorem 16. Let \( G \) be a group, \( g \in G \), and let \( G' = G \ast_{\langle g \rangle} (\langle g \rangle \times \mathbb{Z}^r) \). Assume that

(i) \( G \) has decidable power coset membership problem, and
(ii) \( G \) satisfies Property \([11]\) with respect to every conjugate of \( \langle g \rangle \).

Then \( G' \) has decidable power coset membership problem and its corresponding graph of groups is benign.

Proof. We regard \( G' \) as the fundamental group of a graph of groups \( \mathbb{A} \) having two vertices \( v_0 \) and \( v_1 \) with vertex groups \( G \) and \( \langle g \rangle \times \mathbb{Z}^r \) (respectively) and edge group \( \langle g \rangle \). We will apply Corollary \([11]\) to \( \mathbb{A} \).

Power coset membership in \( G \) is decidable by assumption, and in the abelian vertex group reduces to deciding whether or not a linear system of equations over \( \mathbb{Z} \) has a solution in \( \mathbb{Z} \), which is a decidable problem. Since the only edge group of \( \mathbb{A} \) is cyclic, it follows immediately that \( \mathbb{A} \) is benign.

Let \( B \) be a folded \( \mathbb{A} \)-graph. Properties \([12] (i) \) and \([12] (ii) \) follow from decidability of the power coset membership problem in vertex groups, since the only edge group is cyclic, and Property \([12] (iii) \) holds since every non-trivial subgroup of a cyclic group has finite index.

For Property \([12] (iv) \), consider any edge cycle \( f_1, \ldots, f_m \) in \( B \) with \( B f_i = 1 \) for \( i = 1, \ldots, m \). Every edge \( f \) of \( B \) has satisfies \( |o(f)| = v_0 \) and \( |t(f)| = v_1 \), or vice-versa, so the cycle must be of length at least two. Hence there exists \( i \) such that \( t(f_i) = v_0 = o(f_{i+1}) \), and \( f_i = f_{i+1}^{-1} \). Since \( A_{v_0} = G \) satisfies Property \([11]\) with respect to conjugates of the edge group \( \langle g \rangle \), it follows that \([iv]\) holds.

In the next section we will choose the base group \( \Gamma \) so that we are able to prove that Property \([11]\) holds at each step of the chain.

3.2 Locally quasi-convex hyperbolic groups \( \Gamma \)

Let \( G \) be a group generated by a finite set \( X \). Let \( \mathcal{X} = \text{Cay}(G, X) \) be the Cayley graph of \( G \) with respect to \( X \), and \( d_X \) the associated metric. For a path \( \gamma \) in \( \mathcal{X} \) we denote by \( |p| \) the length of \( p \) and by \( o(\gamma) \) and \( t(\gamma) \) the initial and terminal vertices of \( \gamma \), respectively. A path \( p \) is a \((\lambda, c)\)-quasi-geodesic if for every subpath \( q \) of \( p \),

\[
|q| \leq \lambda d_X(o(q), t(q)) + c.
\]

A subgroup \( R \subseteq G \) is called quasi-convex if there exists a constant \( \epsilon \geq 0 \) ('quasi-convexity constant') such that the following holds: for every pair of elements \( r_1, r_2 \in R \) and every geodesic \( \gamma \) with \( o(\gamma) = r_1 \) and \( t(\gamma) = r_2 \), every vertex of \( \gamma \) is within distance \( \epsilon \) of a vertex belonging to \( R \). Though quasi-convexity of a subgroup may depend on the choice of generating set \( X \), when \( G \) is hyperbolic it is independent of the choice of finite generating set \([2]\) Prop. 10.4.1).

If every finitely generated subgroup of \( G \) is quasi-convex then \( G \) is called locally quasi-convex. Essential to our purposes is the fact that locally quasi-convex hyperbolic groups are effectively cohererent (Proposition 6.1 of \([7]\)). Examples of locally quasi-convex groups include free groups, (most) surface groups, and a variety of small-cancellation, Coxeter, and one-relator groups recently studied by Martínez-Pedroza, McCammond, and Wise \([18], [19], [16]\).
We will use a torsion-free locally quasi-convex hyperbolic group $\Gamma$ as the base of a chain of centralizer extensions, as in [10]. In order to obtain Property [11] at each step $G_k$ of the chain, we will need to use relative hyperbolicity and relative quasi-convexity properties.

If $G$ is torsion-free and hyperbolic relative to a collection $\mathcal{H} = \{H_1, \ldots, H_m\}$ of abelian subgroups then $G$ is called toral relatively hyperbolic. Subgroups of $G$ that are conjugate into subgroups in $\mathcal{H}$ are called parabolic. Denote $\hat{\mathcal{H}} = \cup_{s=1}^m H_s \setminus \{1\}$ let $\hat{\mathcal{X}} = \text{Cay}(G, X \cup \hat{\mathcal{H}})$. We remove from $X$ all parabolic elements.

We will employ aspects of the geometry of $\hat{\mathcal{X}}$ developed in [23]. For a path $p$ in $\hat{\mathcal{X}}$, a maximal subpath consisting of edges from $H_i$ is called an $H_i$-component. Every vertex that does not lie in the interior of some $H_i$-component is called a phase vertex. Two $H_i$-components $p_1$ and $p_2$ are connected if there is an edge from a vertex of $p_1$ to a vertex of $p_2$ labelled by an element of $H_i$. An $H_i$-component is isolated if it is not connected to any other $H_i$-component.

A subgroup $R$ of $G$ is called relatively quasi-convex if there exists a constant $\epsilon \geq 0$ such that the following condition holds: for every two elements $r_1, r_2 \in R$, every geodesic path $\gamma$ in $\hat{\mathcal{X}}$ with $o(\gamma) = r_1$ and $t(\gamma) = r_2$, and every vertex $v$ of $\gamma$, there exists a vertex $w \in R$ such that

$$d_{\hat{\mathcal{X}}}(v, w) \leq \epsilon.$$

Note that this distance is in $\mathcal{X}$, not $\hat{\mathcal{X}}$. As with quasi-convexity, relative quasi-convexity (in a relatively hyperbolic group) does not depend on the finite generating set $X$ ([23] Prop. 4.10). If every finitely generated subset of $G$ is relatively quasi-convex then $G$ is called locally relatively quasi-convex.

We wish to apply Theorem [16] inductively to a chain starting from $\Gamma$, as in [10]. The base case requires decidability of the power coset membership problem in $\Gamma$.

**Lemma 17.** Every torsion-free locally quasi-convex hyperbolic group $\Gamma$ has decidable power coset membership problem.

**Proof.** Let $H \leq \Gamma$ be finitely generated, and let $x, g \in \Gamma$. Let $\delta$ be a hyperbolicity constant for $\Gamma$, and compute a quasi-convexity constant $\epsilon$ for $H$ (using [9] Prop. 4).

Suppose that $g^m \in xH$ for some $m > 0$. In the Cayley graph of $\Gamma$, the path labelled by $g^m$ is a $(\lambda, c)$-quasi-geodesic, for some $\lambda$ and $c$ depending on $\delta$ and $|g|$ but not on $m$ ([21] Lem. 1.11). Consequently, the path $q$ labelled by $x^{-1}g^m$ is a $(\lambda, c')$-quasi-geodesic, where $c' = c + (\lambda + 1)|x|$. Let $p$ be a geodesic path from 1 to $x^{-1}g^m$. There exists a constant $K = K(\delta, \lambda, c')$ such that every vertex of $q$ is within distance $K$ of $p$ ([21] Lem. 1.9). Since $H$ is $\epsilon$-quasi-convex, every vertex of $p$ lies within $\epsilon$ of $H$. Hence every vertex of $q$ lies within distance $K + \epsilon$ of $H$. In particular, there exist elements $y_0, y_1, \ldots, y_m$ in the ball $B$ of radius $K + \epsilon$ centered at 1 such that

$$x^{-1}g^iy_i \in H,$$

for $i = 0, \ldots, m$ (see Figure [3.1]).

Let $R = |B|$. We claim that there exists $0 < n \leq R$ such that $x^{-1}g^n \in H$. If $m \leq R$, the claim holds, so assume $m > R$. Then there exist $0 \leq i < j < m$ such that $y_i = y_j$.  

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Let $h_1 = x^{-1}g^iy_i$, $h_2 = (y_i^{-1}g^{-1}x)x^{-1}g^jy_j$, and $h_3 = (y_j^{-1}g^{-j}x)x^{-1}g^m$, which are all elements of $H$. Then

$$h_1h_3 = h_1y_i^{-1}g^{m-j} = h_1(y_i^{-1}g^{j-i}y_i)(y_i^{-1}g^{m-j})g^{-(j-i)}$$

so $x^{-1}g^{m-(j-i)} \in H$. If $m - (j-i) \leq R$, the claim holds, and otherwise we repeat the above argument with $m' = m - (j-i)$ in place of $m$ (note that $m' > 0$). This proves the claim.

If $g^m \in xH$ for some $m < 0$, a similar argument shows that $x^{-1}g^n \in H$ for some $0 > n \geq -R$. We conclude that if there exists $m \neq 0$ such that $g^m \in xH$, then there exists $n \neq 0$ in the interval $[-R,R]$ such that $g^n \in xH$. Then to solve the power coset membership problem, it suffices to check if any of the elements $x^{-1}g^i$, for $i \in [-R,R] \setminus \{0\}$, is in $H$. Membership in $H$ may be determined using the algorithm in [7] Prop. 6.1.

To prove Property [11] we need the fact that $G_n$ is toral relatively hyperbolic and locally relatively quasi-convex. We apply a combination theorem to the amalgamated product structure to prove this fact.

**Lemma 18.** Let $G_n$ be obtained from a locally quasi-convex torsion-free hyperbolic group $\Gamma$ by iterated centralizer extensions. Then $G_n$ is toral relatively hyperbolic, and every finitely generated subgroup of $G_n$ is relatively quasi-convex and toral relatively hyperbolic.

**Proof.** It follows from results of [20] that we may arrange for $G_n$ to be obtained from $\Gamma$ via a chain

$$\Gamma = G_0 < G_1 < \ldots < G_n$$

in which $G_i$ is obtained from $G_{i-1}$ by an HNN-extension with one stable letter,

$$G_i = \langle G_{i-1},t_i \mid [C(g_i),t_i] = 1 \rangle,$$

where $g_i \in G_{i-1}$ and $C(g_i)$ denotes the centralizer of $g_i$ in $G_{i-1}$. Further, each $C(g_i)$ is a maximal abelian subgroup of $G_{i-1}$ and is free abelian of finite rank.
We proceed by induction on \( n \). The base case holds since finitely generated subgroups of \( \Gamma \) are quasi-convex, hence hyperbolic [2] Prop. 10.4.2.

Assume that the statements hold for \( G_{n-1} \). In particular, \( G_{n-1} \) is hyperbolic relative to a collection \( \mathcal{H} = \{H_1, \ldots, H_m\} \) of abelian subgroups which necessarily includes, up to conjugation, all non-cyclic maximal abelian subgroups. We may assume that \( C(g_i) \) is included in the set \( \mathcal{H} \). Indeed, since \( C(g_i) \) is maximal abelian it is either conjugate to some \( H_i \), and we may replace \( H_i \) by \( C(g_i) \) in \( \mathcal{H} \), or is cyclic. If it is cyclic, then it is hyperbolically embedded in \( G_{n-1} \), i.e. \( G_{n-1} \) is hyperbolic relative to \( \mathcal{H} \cup \{C(g_i)\} \) [22] Cor. 1.7.

Since \( C(g_i) \in \mathcal{H} \) and \( G_i \) is the HNN-extension \( G_{i-1}*_{C(g_i)} = C(g_i) \), it follows that \( G_i \) is hyperbolic relative to

\[
\mathcal{H}' = (\mathcal{H} \setminus \{C(g_i)\}) \cup \{\langle C(g_i), t_i \rangle\},
\]

and that every finitely generated subgroup of \( G_n \) is relatively quasi-convex ([1] Cor. 1.7 and Thm. 3.1). In particular, \( G_n \) is toral relatively hyperbolic. If \( H \leq G_n \) is finitely-generated, then relative quasi-convexity of \( H \) implies that \( H \) is (toral) relatively hyperbolic (by [8] Thm. 9.1 or [17] Thm. 1.8).

We may now use relative quasi-convexity to show that Property 11 holds at every step of the chain.

\textbf{Lemma 19.} Let \( G \) be a locally relatively quasi-convex toral relatively hyperbolic group. For every \( c \in G \) such that \( C(c) = \langle c \rangle \), \( G \) satisfies Property 11 with respect to \( \langle c \rangle \).

\textbf{Proof.} Let \( g, x \in G \) with \( gx^{-1} \notin \langle c \rangle \) and let \( H \leq G \) be finitely generated such that \( H \cap \langle c \rangle = H \cap \langle c^x \rangle = 1 \). Since \( \langle c \rangle \simeq \langle c^x \rangle \simeq \mathbb{Z} \), we must prove that there exist finitely many pairs \( (n, l) \in \mathbb{Z}^2 \) such that

\[
e^n g(c^x)^l \in H.
\]

(12)

First, suppose that there are two such pairs \( (n, l) \) and \( (n, k) \) with \( k \neq l \). Then

\[
(c^x)^{-l} g^{-1} c^n g(c^x)^k = (c^x)^{k-l} \in H,
\]

contradicting \( H \cap \langle c^x \rangle = 1 \). If there exist two such pairs \( (n, k) \) and \( (m, k) \) with \( n \neq m \), then

\[
e^n g(c^x)^k (c^x)^{-k} g^{-1} c^{-m} = c^{n-m} \in H
\]

contradicting \( H \cap \langle c \rangle = 1 \). So we assume, for contradiction, that for each \( i \in \mathbb{N} \) we have a pair \( (n_i, k_i) \) with

\[
e^{n_i} g(c^x)^{k_i} \in H
\]

(13)

and such that \( n_i \neq n_j \) and \( k_i \neq k_j \) for all \( i \neq j \).

Let \( G \) be hyperbolic relative to \( \mathcal{H} = \{H_1, \ldots, H_m\} \). We may assume that \( \langle c \rangle = H_1 \) (reasoning as in the proof of Lemma 18). An edge of \( \mathcal{X} \) labelled by \( c^n \) will be denoted \( e^{(n)} \), where \( e^{(0)} \) denotes an empty edge. Note that every vertex of \( \mathcal{X} \) has an outgoing edge \( e^{(n)} \), for all \( n \).
Proof. First we show that every \( \langle c \rangle \)-component is isolated. There are at most two values of \( i \) for which either \( n_i' = 0 \) or \( k_i' = 0 \), so we may ignore these cases and assume that \( n_i' \neq 0 \) and \( k_i' \neq 0 \). Then the \( \langle c \rangle \)-components of \( p_i \) are the edges \( e^{(n_i')} \) and \( e^{(k_i)} \) and the \( \langle c \rangle \)-edges appearing in \( p_2 \) and \( p_3 \).

Assume for contradiction that \( p_1 \) has a non-isolated \( \langle c \rangle \)-component. The component \( e^{(n_i)} \) is not connected to a \( \langle c \rangle \)-component of \( p_2 \), as this would imply that either \( p_2 \) is not a geodesic or that \( p_2 \) begins with a \( \langle c \rangle \)-edge. Similarly, \( e^{(k_i)} \) is neither connected to a \( \langle c \rangle \)-component of \( p_2 \) nor to a \( \langle c \rangle \)-component of \( p_3 \). If \( e^{(n_i)} \) and \( e^{(k_i)} \) are connected, it implies that \( p_2 \) represents an element of \( \langle c \rangle \), hence \( gx^{-1} \in \langle c \rangle \), which is false.

The only remaining possibility is that \( p_2 \) has a \( \langle c \rangle \)-component which is connected to either \( e^{(n_i')} \) or to a \( \langle c \rangle \)-component of \( p_2 \). We claim that this occurs for only finitely many \( i \). Assume otherwise. Then for infinitely many \( i \), there exists \( 1 \leq a_i \leq |p_2| \) and \( 2 \leq b_i \leq |p_2| + 1 \) such that the \( a_i \)-th vertex of \( p_2 \) is connected via a \( \langle c \rangle \)-edge to the \( b_i \)-th vertex of \( p_3 \). Since \( p_2 \) and \( p_3 \) are finite, there exist \( i \neq j \) such that \( a_i = a_j \) and \( b_i = b_j \). It follows that

\[
g_1c_i^{k_i'}g_2 \in \langle c \rangle, \quad g_1c_j^{k_j'}g_2 \in \langle c \rangle
\] (14)

where \( g_1 \in G \) is the element represented by the length \( |p_2| - a_i + 1 \) terminal segment of \( p_2 \) and \( g_2 \in G \) is the length \( b_i - 1 \) initial segment of \( p_3 \). Taking the difference of the elements appearing in (14), we conclude that

\[
(c_i^{k_i'} - c_j^{k_j'})g_2 \in \langle c \rangle.
\]

Since \( G \) is toral relatively hyperbolic, it satisfies the CSA property (i.e. maximal abelian subgroups are malnormal, see [15] or [5]). The group \( \langle c \rangle \) is a centralizer,
and CSA groups are commutation-transitive, hence \( \langle c \rangle \) is a maximal abelian sub-
group. Since \( k_i \neq k_j \), the conjugator \( g_2 \) must lie in \( \langle c \rangle \). But \( g_2 \) is represented by
an initial segment of \( p_3 \), but \( p_3 \) is geodesic and does not begin with a \( \langle c \rangle \)-edge, so
we must have \( g_2 = 1 \). Hence \( b_i = 1 \), which is a contradiction.

Now we check that for every \( H_s \in \mathcal{H}, s \neq 1 \), every \( H_s \)-component in \( p_i \) is
isolated for infinitely many \( i \). Assume otherwise. The \( H_s \)-components may only occur in \( p_2 \) and \( p_3 \).
Since each is a geodesic, no two \( H_s \)-components of \( p_2 \) are connected, and similarly for \( p_3 \). Hence an \( H_s \)-component of \( p_2 \) is connected to an
\( H_s \)-component of \( p_3 \), for infinitely many \( i \). As in the case for \( \langle c \rangle \)-components, we
conclude there exist \( i \neq j \), \( g_1 \in G \) corresponding to a terminal segment of \( p_2 \), and
\( g_2 \in G \) corresponding to an initial segment of \( p_3 \) such that

\[ g_1 e^{k_i'} g_2 \in H_s, \quad g_1 e^{k_j'} g_2 \in H_s. \]

We conclude that \( \langle e^{k_i'} e^{k_j'} \rangle g_2 \in H_s \). This contradicts the fact that the maximal
abelian subgroups \( \langle c \rangle g_2 \) and \( H_s \) must intersect trivially in a CSA group.

We return to the proof of Lemma 19. The length of \( p_i \) is bounded by

\[ \gamma = 2 + |x| + 2|g|, \]

so \( p_i \) is, trivially, a \((1, \gamma)\)-quasi-geodesic. Let \( q_i \) be a geodesic in \( \hat{X} \) from 1 to \( e^{n_i} g x^{-1} e^{k_i} x \). Since \( p_i \) and \( q_i \) are both quasi-geodesic paths with the same endpoints
and \( p_i \) is a path without backtracking, there exists a constant \( \sigma = \sigma(\gamma) \) such that
every \( (\text{phase}) \) vertex of \( p_i \) is within \( d_{\hat{X}} \)-distance \( \sigma \) of a vertex of \( q_i \) [71] Prop.
3.15. In particular, for the endpoint \( u_i \) of \( e^{(n_i)} \), there exists a vertex \( v_i \) in \( q_i \) at
\( d_{\hat{X}} \)-distance at most \( \sigma \) from \( u_i \). Let \( \epsilon \) be a quasi-convexity constant for \( H \). Since
\( q_i \) is a geodesic in \( \hat{X} \) between elements of \( H \), \( v_i \) is at \( d_{\hat{X}} \)-distance at most \( \epsilon \) from
a vertex belonging to \( H \). Hence for infinitely many \( i \), \( u_i \) is connected in \( \hat{X} \) to a vertex of \( H \) by a path \( r_i \) of \( d_{\hat{X}} \)-length at most \( \sigma + \epsilon \), as shown in Figure 3.2.

Since the ball of radius \( \sigma + \epsilon \) is finite in \( \hat{X} \), there exist \( i \neq j \) such that \( r_i \) and \( r_j \) are labelled by the same element \( g' \in G \). We conclude that

\[ e^{n_i} g' \in H, \quad e^{n_j} g' \in H \]

hence \( e^{n_i - n_j} \in H \). Since \( n_i' \neq n_j' \), this contradicts the fact that \( H \cap \langle c \rangle = 1 \), so the
assumption that there are infinitely many pairs \((n_i, k_i)\) satisfying [13] is false. \( \square \)
Finally, we prove that \( G_n \) is effectively coherent. This is the central theorem of this paper.

**Theorem 20.** There is an algorithm that, given

- a finitely presented group \( \Gamma \) known to be hyperbolic, torsion-free, and locally quasi-convex,
- a chain of centralizer extensions \( \Gamma = G_0 < G_1 < \ldots < G_n \) and,
- a finite subset \( X \subset G_n \),

produces a presentation for the subgroup \( \langle X \rangle \) generated by \( X \).

**Proof.** The chain of centralizer extensions may be rewritten so that \( G_i \) is an extension of a cyclic centralizer of \( G_{i-1} \), for \( i = 1, \ldots, n \), as in (11). We proceed by induction on \( n \). The base group \( \Gamma \) is effectively coherent.

Assume that \( G_{n-1} \) is effectively coherent and we are given \( X = \{h_1, \ldots, h_m\} \leq G_n \). We have

\[
G_n = G_{n-1} \ast \langle g_n \rangle \mathbb{Z}^{r_n+1},
\]

where \( C_{G_{n-1}}(g_n) = \langle g_n \rangle \). It follows by induction from Theorem 16, using Lemma 17, Lemma 18, and Lemma 19, that the graph of groups corresponding to the decomposition (15) is benign. Then by [10] Theorem 5.8, \( G_n \) is effectively coherent.

\[\square\]

### 3.3 Groups discriminated by \( \Gamma \)

When a group \( G \) is discriminated by \( \Gamma \), it embeds into a group \( G_n \) obtained from \( \Gamma \) by centralizer extensions. In [13], it was shown that one may compute a finite collection of homomorphisms from \( G \) to groups obtained from \( \Gamma \) by centralizer extensions, at least one of which must be injective. However, an algorithm to identify an injective homomorphism from this collection could not be given. Using effective coherence in \( G_n \), we may isolate which of these homomorphisms is an embedding.

**Theorem 21.** Let \( \Gamma \) be a torsion-free locally quasi-convex hyperbolic group. There is an algorithm that, given a finitely presented group \( G \):

(i) if \( G \) is discriminated by \( \Gamma \), computes a sequence centralizer extensions

\[
\Gamma = G_0 < G_1 < \ldots < G_n
\]

and an embedding \( G \hookrightarrow G_n \);

(ii) if \( G \) is not discriminated by \( \Gamma \), runs forever.

If in addition a solution to the word problem in \( G \) is given, the algorithm terminates when \( G \) is not discriminated by \( \Gamma \), reporting this fact.
Proof. From Theorem 3.17 of [14], we may effectively construct finitely many groups \( H_1, \ldots, H_m \), each given as sequence centralizer extensions of \( \Gamma \), and homomorphisms \( \phi_i : G \to H_i \) such that if \( G \) is discriminated by \( \Gamma \), at least one \( \phi_i \) is injective.

For each \( i \), construct a presentation for \( \phi_i(G) \) using Theorem 20. By Lemma 18 each \( \phi_i(G) \) is a toral relatively hyperbolic group. If \( G \) is discriminated by \( \Gamma \), then \( G \) is also toral relatively hyperbolic since it is isomorphic to some \( \phi_i(G) \). We apply the algorithm given in Theorem 0.2 of [3], which terminates when \( G \) is toral relatively hyperbolic and runs forever when \( G \) is not.

If this algorithm terminates, we check for each \( i = 1, \ldots, m \) whether or not \( G \) and \( \phi_i(G) \) are isomorphic, using the solution to the isomorphism problem for toral relatively hyperbolic groups given in [4]. If \( G \not\simeq \phi_i(G) \), then \( \phi_i \) is not injective. If \( G \simeq \phi_i(G) \), then \( G \) is discriminated by \( \Gamma \) (since it is isomorphic to a subgroup of \( H_i \), which is discriminated by \( \Gamma \)), and hence is Hopfian (26). Consequently, \( \phi_i \) is injective.

If a solution to the word problem in \( G \) is given, we may run in parallel the following algorithm. Enumerate non-trivial elements \( g \in G \) and check whether or not \( \phi_i(g) \) is trivial for each \( i \). If \( G \) is not discriminated by \( \Gamma \), one of the \( \phi_i \) must fail to be injective and the algorithm will eventually find \( g \) and \( i \) such that \( \phi_i(g) = 1 \).

Note that this theorem provides a recognition algorithm for groups \( G \) discriminated by \( \Gamma \), provided the word problem is decidable in \( G \). Since we may compute an embedding of \( G \) into an iterated centralizer extension \( G_n \) of \( \Gamma \), algorithmic problems regarding \( G \) reduce to the same problem in a subgroup of \( G_n \). In particular, we have the following.

**Theorem 22.** Every group discriminated by a torsion-free locally quasi-convex hyperbolic group is effectively coherent.

In the case when \( \Gamma \) is a free group, the above result was obtained in [6]. Our result provides an alternate proof.

**Corollary 23.** Limit groups are effectively coherent.

We may also give an algorithm that enumerates all groups discriminated by \( \Gamma \).

**Theorem 24.** There is an algorithm that, given a presentation of a group \( \Gamma \) that is locally quasi-convex, torsion-free, and hyperbolic, enumerates without repetition all finitely generated groups discriminated by \( \Gamma \).

**Proof.** Every finitely generated group discriminated by \( \Gamma \) is isomorphic to a subgroup of a group obtained from \( \Gamma \) by a finite chain of centralizer extensions. Enumerate all groups \( G_n \) obtained from \( \Gamma \) by a finite chain of centralizer extensions and all finite subsets \( X_i \subset G_n \). Since each \( G_n \) is effectively coherent, we can compute a presentation for \( \langle X_i \rangle \). Since every \( \langle X_i \rangle \) is toral relatively hyperbolic, we may use a solution to the isomorphism problem [4] to eliminate repetitions. \( \square \)
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