Accuracy analysis of automodel solutions for Lévy flight-based transport: from resonance radiative transfer to a simple general model

A B Kukushkin\textsuperscript{1,2} and P A Sdvizhenskii\textsuperscript{1}

\textsuperscript{1} National Research Centre ‘Kurchatov Institute’, Moscow, 123182, Russian Federation
\textsuperscript{2} National Research Nuclear University MEPhI, Moscow, 115409, Russian Federation

E-mail: kukushkin_ab@nrcki.ru

Abstract. The results of accuracy analysis of automodel solutions for Lévy flight-based transport on a uniform background are presented. These approximate solutions have been obtained for Green’s function of the following equations: the non-stationary Biberman-Holstein equation for three-dimensional (3D) radiative transfer in plasma and gases, for various (Doppler, Lorentz, Voigt and Holtsmark) spectral line shapes, and the 1D transport equation with a simple long-tailed step-length probability distribution function with various power-law exponents. The results suggest the possibility of substantial extension of the developed method of automodel solution to other fields far beyond physics.

1. Introduction

The analysis of the Green’s function of the non-stationary Biberman-Holstein equation for radiative transfer in plasma and gases has shown [1] that there is an approximate automodel solution based on three scaling laws: for the propagation front (i.e. relevant-to-superdiffusion average displacement of perturbation’s carrier from an instant point source) and asymptotic solutions far beyond and far in advance of the propagation front. All these scaling laws are determined essentially by the long-free-path carriers (named, by Mandelbrot [2], Lévy flights, cf. page IX in Ref. [3]). The validity of the suggested automodel solution was proved by its comparison with analytical solutions by Veklenko [4] in the 3D case of the Biberman–Holstein equation of the resonance radiation transfer for various (Doppler, Lorentz, Voigt and Holtsmark) spectral line shapes with complete redistribution over frequency (within spectral line width) in the elementary act of the resonance scattering (i.e. absorption and subsequent emission) of the photon by an atom/ion. The approach [1] was extended in [5] on the wide class of non-stationary superdiffusive transport on a uniform background with a simple long-tailed step-length probability distribution function (PDF) with various power-law exponents.

In this paper, we present the results of accuracy analysis of automodel solutions for Lévy flight-based transport, including the resonance radiative transfer (section 2) and a simple general model (section 3). Possible applications of the method [1, 5] to other problems, where the role of the processes with dominating role of Lévy flights is identified and widely used, is discussed in section 4.

2. Self-similarity accuracy estimations for the Biberman-Holstein equation

The Biberman–Holstein equation for radiative transfer in a uniform medium of two-level atoms/ions is obtained from a system of equations for spatial density of excited atoms, \( F(\mathbf{r}, t) \), and spectral intensity...
of resonance radiation. This system is reduced to a single equation for \(F(r, t)\), which appears to be an integral equation, non-reducible to a differential diffusion-type equation:

\[
\frac{\partial F(r, t)}{\partial t} = \frac{1}{\tau} \int G(|\mathbf{r} - \mathbf{r}'|) F(r', t) dV' - \left(\frac{1}{\tau} + \sigma\right) F(r, t) + q(r, t).
\]

(1)

where \(\tau\) is the lifetime of the excited atomic state with respect to spontaneous radiative decay; \(\sigma\) is the rate of the collisional quenching of excitation; \(q\) is the source of excited atoms, different from populating by the absorption of resonant photons (e.g., collisional excitation). Hereafter we use the dimensionless time coordinate, assuming the normalization of time by \(\tau\). The kernel \(G\) is determined by the (normalized) emission spectral line shape \(\varepsilon_\omega\) and the absorption coefficient \(k_\omega\). In homogeneous media, \(G\) depends on the distance between the points of emission and the absorption of the photon:

\[
G(r) = -\frac{1}{4\pi r^2} \frac{dT(r)}{dr}, \quad T(r) = \int_0^\infty \varepsilon_\omega \exp(-k_\omega r) d\omega.
\]

(2)

The automodel solution of equation (1) with a point instant source \(q(r, t) = \delta(\mathbf{r} - \mathbf{r}_0)\delta(t - t_0)\) (i.e., Green’s function) was suggested in [5], which in the 3D case and for arbitrary space-time coordinates of the instant point source takes the form:

\[
F_{\text{auto}}(\mathbf{r}, t; \mathbf{r}_0, t_0) = (t - t_0) \cdot G \left( |\mathbf{r} - \mathbf{r}_0| \cdot g \left( \frac{r_0(t - t_0)}{|\mathbf{r} - \mathbf{r}_0|} \right) \right).
\]

(3)

where \(g\) is a function of a single variable, and its asymptotic behavior is known: \(g(s) = 1, s << 1; g(s) \propto s, s >> 1\). The relation between \(g\) and the exact solution of equation (1), \(f_{\text{exact}}\), is described by the following equations:

\[
Q_G(\rho, t) \equiv \frac{1}{\rho} \hat{G}^{-1} \left( \frac{f_{\text{exact}}(\rho, t)}{t} \right),
\]

(4)

where \(\hat{G}^{-1}\) is the function reciprocal to the \(G\) function, \(\rho \equiv k_0 |\mathbf{r} - \mathbf{r}_0|\), \(k_0\) is the absorption coefficient for photons with frequency \(\omega_0\), corresponding to the line shape center,

\[
Q_{G_1}(\rho, \rho) \equiv Q_{G_1}(s, \rho) = g(s),
\]

(5)

\[
Q_{G_2}(\rho(t), t) \equiv Q_{G_2}(s, t) = g(s),
\]

(6)

where the functions \(\tau(\rho, s)\) and \(\rho(\tau, s)\) are determined by the relation

\[
s = \rho(\tau) / \rho.
\]

(7)

To prove the automodel solution one has to show weak dependence (independence) of \(Q_{W_1}\) and \(Q_{W_2}\) functions on, respectively, space coordinate and time. The results of the validation of the automodel solution and the reconstruction of function \(g\) from comparison of function (3) with computations of the Green’s function [4] for the Lorentz, Doppler, Voigt, and Holtsmark line shapes are shown in figure 1.
Figure 1. Function $Q_{G2}$ for $\rho > \rho_{\text{min}}, t > t_{\text{min}}$ for different line shapes: (a) dispersive (Lorentz), (c) – Doppler, (e) Voigt ($a = \sqrt{\frac{2}{3}} \Delta \omega_{\text{Lorentz}}/\Delta \omega_{\text{Doppler}} = 1$), (g) Holtsmark. Projection of the ratio $Q_{G2}(s, t)/Q_{G2}(s, t^*)$ onto the $(Q_{G2}, t)$ plane for different line shapes: (b) dispersive (Lorentz), $t^* = 50$, (d) Doppler, $t^* = 1500$, (f) Voigt, $a = 1$, $t^* = 80$, (h) Holtsmark, $t^* = 160$. The dependence of the error, which is given by the deviation from unity, on the $t_{\text{min}}$ value is shown in the right-hand-side figures. The range of variable $s$ is limited by the conditions $\rho < \rho_{\text{max}}$.

It is seen from figure 1 that the function (3) is indeed an automodel Green’s function of equation (1) with an accuracy defined by the coincidence of the curves $Q_{G2}(s, t)$, for various values of the time $t$. Maximum deviation of $Q_{G1}$ and $Q_{G2}$ functions from a function of $s$ only is as follows. For $\{\rho > 30, t > 40\}$ we have 0.2% for the Lorentz line shape, and 4%, 3%, and 8% for, respectively, Doppler, Voigt ($a = 1$), and Holtsmark line shapes. The maximum deviation amounts to 1% for the last-named three line
shapes if we, for \( \rho > 30 \), restrict the time to \( t > 950, 280, 700 \), respectively. The largest deviations take place at \( s \approx 1 \).

3. Self-similarity accuracy estimations for general model (simple PDF, 1D case)

We consider the 1D transport on a uniform background, described by the equation for spatial density \( f(x, t) \) of an excitation of the background medium, which may evolve due to the exchange of excitation between various points of the medium via emission and absorption of the carriers (here the retardation caused by the finite velocity of carriers is neglected; the derivation of this equation for a possible mechanism of interaction between the medium and the carriers of medium’s perturbation is given in Appendix in [5]):

\[
\frac{\partial f(x, t)}{\partial t} = \frac{1}{\tau} \int_{-\infty}^{\infty} W(|x - x'|) f(x', t) \, dx' - \left( \frac{1}{\tau} + \sigma \right) f(x, t) + q(x, t) \tag{8}
\]

where \( W(x) \) is a step-length PDF (i.e. the probability that the carrier, emitted at some point, is absorbed at a distance \( x \) from that point), \( 1/\tau \) is the absolute value of the emission rate (i.e. \( \tau \) is the average waiting time between the absorption and reemission of the carrier), \( q \) is the source function, which is the rate of production of excitation by an external source (i.e. a source which differs from the excitation of the medium due to absorption described by the \( W \) function), and \( \sigma \) is the rate of the quenching of excitation.

The uniformity of the background assumes that, first, the \( W \) is a function of only one variable — the distance between the points of emission and absorption — and, second, \( \tau \) and \( \sigma \) are the constants. The latter makes the role of quenching simply described by the time exponent \( \exp(-\sigma \tau) \), therefore in what follows we omit this process. Hereafter we use the dimensionless time and space coordinate, assuming the normalization of time by \( \tau \) and using a dimensionless PDF. We will seek for the Green’s function, taking, respectively, the source function as a point instant source, \( q(x, t) = \delta(x) \delta(t) \).

We take the PDF in the following simple form which possesses a long tail and the infinite value of the mean square displacement:

\[
W(\rho) = \frac{\gamma}{2(1+\rho)^{\gamma+1}}, \quad 0 < \gamma < 2, \quad \rho = |x - x'|, \quad \int_{-\infty}^{\infty} W(|x - x'|) \, dx' = 1. \tag{9}
\]

It was shown in [5] that one can derive automodel Green’s function in the form

\[
f(x, t) = t W \left( \frac{\rho_{fr}(t)}{\rho} \right), \quad \rho = |x|, \tag{10}
\]

or, for the PDF (9),

\[
f(x, t) = t \frac{\gamma}{2 \left[ 1 + \rho g \left( \frac{\rho_{fr}(t)}{\rho} \right) \right]^{\gamma+1}}, \quad \rho = |x|, \tag{11}
\]

where \( \rho_{fr}(t) = (t + 1)^{1/\gamma} - 1 \). The procedure of the reconstruction of function \( g \) is quite similar to that of equations (4)–(7):

\[
Q_{W}(x, t) = \frac{1}{\rho} \tilde{W}^{-1} \left( \frac{f_{exact}(x, t)}{t} \right) = \frac{1}{\rho} \left[ \frac{\gamma}{2 f_{exact}(x, t)} \right]^{\gamma+1} - 1, \quad \rho = |x|, \tag{12}
\]

where \( \tilde{W}^{-1} \) is the function reciprocal to \( W \) function.

\[
Q_{W}(\rho, t(\rho, s)) = Q_{W1}(s, \rho) = g(s), \tag{13}
\]

\[
Q_{W}(\rho(t, s), t) = Q_{W2}(s, t) = g(s), \tag{14}
\]

where the functions \( t(\rho, s) \) and \( \rho(t, s) \) are determined by the relation

\[
s = ((t + 1)^{1/\gamma} - 1) / \rho \tag{15}
\]

(the propagation front \( \rho_{fr}(t) = \rho(t, s = 1) \)).
The $Q_{W2}$ function for various values of time coordinates is shown in figure 2 for three values of the exponent $\gamma$.

![Graphs showing the $Q_{W2}$ function for different $\gamma$ values.](image)

**Figure 2.** Function $Q_{W2}$ for $\rho > \rho_{\text{min}}$, $t > t_{\text{min}}$ for different $\gamma$ values: (a) $\gamma = 0.5$; (c) $\gamma = 1$; (e) $\gamma = 1.5$. Projection of the relation $Q_{W2}(s, t)/Q_{W2}(s, t^*)$ in the plane \{ $Q_{W2}$, $t$ \} for different $\gamma$ values: (b) $\gamma = 0.5$ for $\rho > 100$, $t > 30$, $t^* = 150$; (d) $\gamma = 1$ for $\rho > 30$, $t > 30$, $t^* = 2500$; (f) $\gamma = 1.5$ for $\rho > 30$, $t > 100$, $t^* = 7000$. The dependence of the error, which is given by the deviation from unity, on the $t_{\text{min}}$ value is shown in the right-hand-side figures. The range of variable $s$ is limited by the conditions $\rho < \rho_{\text{max}}$.

It is seen from figure 2 that for given values of $\gamma$ the function (11) is indeed an automodel solution of equation (8) with the PDF of equation (9) with an accuracy defined by the coincidence of the curves $Q_{W2}(s, t)$, for various values of the time $t$. According to exact asymptotic solutions, used for constructing
solution (10), one may expect high accuracy of the respective solution (11) for large enough values of dimensionless time and distance from the source. Indeed, we have the following numbers for the maximum deviation of $Q_{W1}$ and $Q_{W2}$ functions from a function of $s$ only. For instance, for $\gamma = 1$ this deviation amounts to $3\%$ for \{\( \rho > 30, t > 30 \)\} and $1\%$ for \{\( \rho > 30, t > 270 \)\} and \{\( \rho > 100, t > 30 \)\}. The accuracy for a weaker tail appears to be worse: for $\gamma = 1.5$ and $\rho > 30$ the deviation amounts to $12\%, 7\%$ and $1\%$ for, respectively, $t > 30, t > 100$, and $t > 2400$. For a stronger tail, $\gamma = 0.5$, the propagation front moves substantially faster, and high accuracy is achieved at larger distances: $5\%$ for $\rho > 100, t > 30$ and $1\%$ for $\rho > 100, t > 100$. The largest deviations take place at $s \sim 1$.

4. Conclusions

The results of Sec. 2 and 3 shows that the identification of main features of nonlocal (superdiffusive) mechanism of transport of resonance radiation may be fruitful for extending the method of approximate automodel solution to other problems. There is an example of physics model which gives precisely the power-law PDF of Eq. (9); see Eq. (1) in [6] for photon-assisted transport of minority carriers in semiconductors (photo-excited holes in n-type InP [7]). The simple long-tailed step-length probability distribution function (9) with various power-law exponents may serve a bridge between physics and other fields. The general features of superdiffusion based on Lévy flights are recognized and applied in many fields (see, e.g., [1], [8]). Derivation of scaling laws, and especially of approximate automodel solutions, may be of practical interest.

Acknowledgments

This work is partially supported by the Russian Foundation for Basic Research (project RFBR-15-07-07850-a).

References

[1] Kukushkin A B, Sdvizhenskii P A 2014 Proc. 41st EPS Conference on Plasma Physics (Berlin) P4.133 http://ocs.ciemat.es/EPS2014PAP/pdf/P4.133.pdf
[2] Mandelbrot B B 1982 The Fractal Geometry of Nature (New York: Freeman)
[3] Shlesinger M, Zaslavsky G M and Frisch U. (ed) 1995 Lévy Flights and Related Topics in Physics (New York: Springer)
[4] Veklenko B A 1959 Soviet Phys. JETP 9 138
[5] Kukushkin A B and Sdvizhenskii P A 2016 J. Phys. A: Math. Theor. 49 255002
[6] Subashiev A V, Semyonov O, Chen Z, Luryi S 2014 Phys. Lett. A 378 pp 266–269
[7] Luryi S, Semyonov O, Subashiev A V, Chen Z 2012 Phys. Rev. B 86 201201(R)
[8] Klafter J, Sokolov I M 2005 Anomalous diffusion spreads its wings Physics world 18 p 29