A LINEAR CONDITION FOR NON-VERY GENERIC DISCRIMINANTAL ARRANGEMENTS

SIMONA SETTEPANELLA AND SO YAMAGATA

Abstract. The discriminantal arrangement is the space of configurations of \( n \) hyperplanes in generic position in a \( k \) dimensional space (see [15]). Differently from the case \( k = 1 \) in which it corresponds to the well known braid arrangement, the discriminantal arrangement in the case \( k > 1 \) has a combinatorics which depends from the choice of the original \( n \) hyperplanes. It is known that this combinatorics is constant in an open Zarisky set \( \mathbb{Z} \), but to assess weather or not \( n \) fixed hyperplanes in generic position belongs to \( \mathbb{Z} \) proved to be a nontrivial problem. Even to simply provide examples of configurations not in \( \mathbb{Z} \) is still a difficult task. In this paper, moving from a recent result in [23], we define a weak linear independency condition among sets of vectors which, if imposed, allows to build configurations of hyperplanes not in \( \mathbb{Z} \). We provide 3 examples.

1. Introduction

In 1989 Manin and Schechtman ([15]) introduced the discriminantal arrangement \( \mathcal{B}(n, k, \mathcal{A}^0) \) which hyperplanes consist of the non-generic parallel translates of the generic arrangement \( \mathcal{A}^0 \) of \( n \) hyperplanes in \( \mathbb{C}^k \). \( \mathcal{B}(n, k, \mathcal{A}^0) \) is a generalization of the braid arrangement ([18]) with which \( \mathcal{B}(n, 1) = \mathcal{B}(n, 1, \mathcal{A}^0) \) coincides. These arrangements, which have several beautiful relations with diverse problems (see, for instance Kapranov-Voevodsky [11], [9], [10], the vanishing of cohomology of bundles on toric varieties [19], the representations of higher braid groups [12]) proved to be not only very interesting, but also quite tricky. Indeed, differently from the braid arrangement \( \mathcal{B}(n, 1) \), their combinatorics depends on the original arrangement \( \mathcal{A}^0 \) if \( k > 1 \). This generated few misunderstanding over the years.

Manin and Schechtman introduced the discriminantal arrangement in order to model higher Bruhat orders. To do so, they only used the combinatorics of \( \mathcal{B}(n, k, \mathcal{A}^0) \) when \( \mathcal{A}^0 \) varies in an open Zarisky set \( \mathbb{Z} \). In 1991 Ziegler showed (see Theorem 4.1 in [25]) that their description was not complete and that a slightly different construction was needed. In particular in [8] Felsner and Ziegler showed that, in the representable case, what was missed in Manin and Schechtman construction was the part coming from the case \( \mathcal{A}^0 \not\in \mathbb{Z} \).

As pointed out by Falk in 1994 ([21]), the construction in [15] led to the misunderstanding that the intersection lattice of \( \mathcal{B}(n, k, \mathcal{A}^0) \) was independent of the arrangement \( \mathcal{A}^0 \) (see, for instance, [17] [18], or [13]). In [7] Falk shows that this was not the case, providing an example of a combinatorics of \( \mathcal{B}(6, 3, \mathcal{A}^0) \) different from the one presented by Manin and Schechtman. Nevertheless Falk too was not aware that such an example already existed in [4].

Actually, Crapo already introduced the discriminantal arrangement in 1985 with the name of geometry of circuits (see [4]). In a more combinatorial setting, he defined it as the matroid \( M(n, k, C) \) of circuits of the configuration \( C \) of \( n \) generic points in \( \mathbb{R}^k \). The circuits of the matroid \( M(n, k, C) \) are the hyperplanes of \( \mathcal{B}(n, k, \mathcal{A}^0) \), when \( \mathcal{A}^0 \) is the arrangement of the hyperplanes in \( \mathbb{R}^k \) orthogonal to the vectors joining the origin with the \( n \) points in \( C \) (for further development see [5]). In this paper, Crapo provides also an example of an arrangement \( \mathcal{A}^0 \) of 6 lines in the real plane for which the combinatorics of \( \mathcal{B}(6, 2, \mathcal{A}^0) \) in rank 3 is different from the one described in [15].

In 1997 Bayer and Brandt (see [3]) cast a light on this difference naming very generic the arrangements \( \mathcal{A}^0 \in \mathbb{Z} \) (the one simply called generic in [15]) and non very generic the others. They conjectured a description of the intersection lattice of \( \mathcal{B}(n, k, \mathcal{A}^0), \mathcal{A}^0 \in \mathbb{Z} \), subsequently proved by Athanasiadis in 1999 in [2] (as far as it is known to the authors, this is the first paper on the literature on discriminantal arrangements in which there is a reference to the Crapo’s paper [4]). It is worthy to notice that Bayer and Brandt mentioned in [3] that, in their opinion, a candidate for a very generic

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arrangement could have been the cyclic arrangement. In Subsection 2.2 by means of the Theorem 4.1 in [24], we show that this is not the case since there are cyclic arrangements which are non-very generic. Moreover even if Athanasiadis proved that the combinatorics of $B(n, k, \mathcal{A})$, $\mathcal{A} \in \mathbb{Z}$ can be described by opportune\textsuperscript{1}ly defined sets of subsets of $\{1, \ldots, n\}$, the contrary is not true in general. In Section 4 by means of our main result, we provide two examples of non-very generic arrangements $\mathcal{A}^0$ for which the combinatorics of $B(n, k, \mathcal{A}^0)$ satisfies Bayer-Brandt-Athanasiadis numerical properties (see Examples 4.2 and 4.3). We point out that to provide such examples is a nontrivial result since, until very recently, 1985 Crapo’s and 1997 Falk’s examples were, essentially, the only known examples of non-very generic arrangements.

Finally, in order to better understand the difficulty behind this problem, we remark that even with the Bayer-Brandt-Athanasiadis description, the enumerative problem of funding the characteristic polynomial of the discriminantal arrangement in the very generic case is nontrivial (see, for instance, [16]).

Recently, following a result in [14], which completely describes the rank 2 combinatorics of $B(n, k, \mathcal{A}^0)$ for any $\mathcal{A}^0$, the authors tried to better understand the combinatorics of the discriminantal arrangement (see [21], [22], [23]) and its connection with special configurations of points in the space (see [6], [20]). In particular in [23] the authors generalize the dependency condition given in [14] providing a sufficient condition for the existence, in rank $r > 2$, of non-very generic intersections, i.e. intersections which do not appear in $B(n, k, \mathcal{A}^0), \mathcal{A}^0 \in \mathbb{Z}$.

In this paper, moving from the construction in [23], we introduced the notion of weak linear independency among sets of vectors proving that if there are vectors in the hyperplanes of $\mathcal{A}^0$ which form weakly linearly independent sets, then $\mathcal{A}^0$ is non-very generic. This result allowed to build several examples of non-very generic arrangements in higher dimension (see also [24]).

The content of the paper is the following. In Section 2 we recall the definition of discriminantal arrangement and, following [23], the definition of simple intersection, $K_r$-translated and $K_r$-configuration. In Section 3 we introduce the notion of $K_r$-vector sets, weak linear independency and we prove our main result, Theorem 3.6. In the last section we provide three examples of non-very generic arrangements obtained by imposing the condition stated in Theorem 3.6.

2. Preliminaries

2.1. Discriminantal arrangement. Let $\mathcal{A}^0 = \{ H_1^0, \ldots, H_n^0 \}$ be a central arrangement in $\mathbb{C}^k, k < n$ such that any $m$ hyperplanes intersect in codimension $m$ at any point except for the origin for any $m \leq k$. We will call such an arrangement a central generic\textsuperscript{1} arrangement. The space $\mathfrak{S}[\mathcal{A}^0]$ (or simply $\mathfrak{S}$ when dependence on $\mathcal{A}^0$ is clear or not essential) will denote the space of parallel translates of $\mathcal{A}^0$, that is the space of the arrangements $\mathcal{A} = \{ H_1^0, \ldots, H_n^0 \}$, $t = (x_1, \ldots, x_n) \in \mathbb{C}^n$, $H_i^0 = H_0^0 + a_i x_i$, $a_i$ a vector normal to $H_i^0$. There is a natural identification of $\mathfrak{S}$ with the $n$-dimensional affine space $\mathbb{C}^n$ such that the arrangement $\mathcal{A}^0$ corresponds to the origin. In particular, an ordering of hyperplanes in $\mathcal{A}^0$ determines the coordinate system in $\mathfrak{S}$ (see [14]).

The closed subset of $\mathfrak{S}$ formed by the translates of $\mathcal{A}^0$ which fail to form a generic arrangement is an union of hyperplanes $D_L \subset \mathfrak{S}$ (see [15]). Each hyperplane $D_L$ corresponds to a subset $L = \{ i_1, \ldots, i_{k+1} \} \subset \{ 1, \ldots, n \}$ and it consists of $n$-tuples of translates of hyperplanes $H_{i_1}^0, \ldots, H_{i_{k+1}}^0$ in which translates of $H_{i_0}^0, \ldots, H_{i_{k+1}}^0$ fail to form a generic arrangement. The arrangement $B(n, k, \mathcal{A})$ of hyperplanes $D_L$ is called discriminantal arrangement and has been introduced by Manin and Schechtman in [15].

It is well known (see, among others [4], [15]) that there exists an open Zarisky set $\mathcal{Z}$ in the space of (central) generic arrangements of $n$ hyperplanes in $\mathbb{C}^k$, such that the intersection lattice of the discriminantal arrangement $B(n, k, \mathcal{A})$ is independent from the choice of the arrangement $\mathcal{A} \in \mathcal{Z}$. Accordingly to Bayer and Brandt in [3] we will call the arrangements $\mathcal{A} \in \mathcal{Z}$ very generic and non-very generic the others.

2.2. Simple intersections. According to [23] we call an element $X$ in the intersection lattice of the discriminantal arrangement $B(n, k, \mathcal{A})$ a simple intersection if

$$X = \bigcap_{i=1}^r D_{L_i}, |L_i| = k + 1, \text{ and } \bigcap_{i \in I} D_{L_i} \neq D_{S_i}, \text{ for any } I \subset [r], |I| \geq 2.$$

\textsuperscript{1}Here we use the word generic to stress that $\mathcal{A}^0$ admits a translated which is a generic arrangement.
We call multiplicity of the simple intersection \( X \) the number \( r \) of hyperplanes intersecting in \( X \). In \cite{23} authors proved that the following Proposition holds.

**Proposition 2.1.** If the intersection lattice of the discriminantal arrangement \( \mathcal{B}(n, k, \mathcal{A}) \) contains a simple intersection of rank strictly less than its multiplicity, then \( \mathcal{A} \) is non-very generic.

It is nontrivial to assess whether or not an arrangement is very generic. For example, in \cite{3} Bayer and Brandt guessed that the cyclic arrangement could have been a good candidate to build very generic arrangements. We can now show that this is not true in general. For instance, the cyclic arrangement \( \mathcal{A}^0 \in \mathbb{R}^3 \) with hyperplanes normal to the vectors \( \alpha_1 = (1, t_1 t_2), (t_1 t_2, t_1 t_3, t_2 t_4, t_5 t_6) = (1, -1, a, -a, b, -b), a, b \neq 1, a \neq b \), is generic but non-very generic. Indeed the vectors \( \alpha_1 \times \alpha_2, \alpha_3 \times \alpha_4 \) and \( \alpha_5 \times \alpha_6 \) are linearly dependent and by Theorem 4.1 in \cite{21} this is equivalent to the intersection \( X = D_{(1,2,3,4)} \cap D_{(1,2,5,6)} \cap D_{(3,4,5,6)} \) be a simple intersection of multiplicity 3 in rank 2. By Proposition 2.1 we get that \( \mathcal{A}^0 \) is non-very generic.

### 2.3. \( K_T \)-translated and \( K_T \)-configurations.

Fixed a set \( \mathbb{T} = \{ L_1, \ldots, L_r \} \) of subsets \( L_i \subset [n], |L_i| = k + 1 \), for any arrangement \( \mathcal{A} = \{ H_1, \ldots, H_n \} \) translated of \( \mathcal{A}^0 \) we will denote by \( P_t = \bigcap_{p \in L_t} H_p \) and \( H_{i,j} = \bigcap_{p \in L_i \cap L_j} H_p \). Notice that \( P_t \) is a point if and only if \( \mathcal{A} \in D_{L_t} \), it is empty otherwise. Following \cite{23} we will call the set \( \mathbb{T} \) an \( r \)-set if the conditions

\[
\bigcup_{i=1}^{r} L_i = \bigcup_{i \in \mathbb{I}} L_i \quad \text{and} \quad L_i \cap L_j \neq \emptyset
\]

are satisfied for any subset \( I \subset [r], |I| = r - 1 \) and any two indices \( 1 \leq i < j \leq r \).

Given an \( r \)-set \( \mathbb{T} \) authors in \cite{23} defined:

**\( K_T \)-translated.** A translated \( \mathcal{A} = \{ H_1, \ldots, H_n \} \) of \( \mathcal{A}^0 \) will be called \( K_T \) or \( K_T \)-translated if each point \( P_t = \bigcap_{p \in L_t} H_p \neq \emptyset \) is the intersection of exactly the \( k + 1 \) hyperplanes indexed in \( L_t \) for any \( L_t \in \mathbb{T} \).

**\( K_T \)-configuration \( K_T(\mathcal{A}) \).** Given a \( K_T \)-translated \( \mathcal{A} \), the complete graph having the points \( P_t \) as vertices and the vectors \( P_t \) as edges will be called \( K_T \)-configuration and denoted by \( K_T(\mathcal{A}) \).

### 3. An algebraic condition for non-very genericity

The discriminantal arrangement \( \mathcal{B}(n, k, \mathcal{A}) \) is not essential arrangement of center \( D_{[n]} = \bigcap_{L \subset [n], |L| = k+1} D_L \approx \mathbb{C}^k \). The center is formed by all translated \( \mathcal{A}t \) of \( \mathcal{A}^0 \) which are central arrangements. If we consider its essentialization \( \text{ess}(\mathcal{B}(n, k, \mathcal{A})) \) in \( \mathbb{C}^{n-k} = \mathbb{S}/D_{[n]} \), an element \( \mathcal{A}t \in \text{ess}(\mathcal{B}(n, k, \mathcal{A})) \) will corresponds uniquely to a translation \( t \in \mathbb{C}^n/C \approx \mathbb{C}^{n-k} \), \( C = \{ t \in \mathbb{C}^n | \mathcal{A}t \text{ is central} \} \). The following proposition arises naturally.

**Proposition 3.1.** Let \( \mathcal{A}^0 \) be a generic central arrangement of \( n \) hyperplanes in \( \mathbb{C}^k \). Translations \( \mathcal{A}^0, \ldots, \mathcal{A}^0 \) of \( \mathcal{A}^0 \) are linearly independent vectors in \( \mathbb{S}/D_{[n]} \approx \mathbb{C}^{n-k} \) if and only if \( t_1, \ldots, t_d \) are linearly independent vectors in \( \mathbb{C}^n/C \).

Given \( \mathcal{A}^0, \ldots, \mathcal{A}^0 \) \( K_T \)-translated of \( \mathcal{A}^0 \), we will say that the \( K_T \)-configurations \( K_T(\mathcal{A}^i) \) are independent if \( \mathcal{A}^i, i = 1, \ldots, d \) are.

### 3.1. \( K_T \)-vector sets.

Let \( \mathcal{A}^0, t = (x_1, \ldots, x_n) \) be a \( K_T \)-translated of \( \mathcal{A}^0 \) and \( P^i_t \) denote the intersection \( \bigcap_{p \in L_t} H_p^i \). Then to the \( K_T \)-configuration \( K_T(\mathcal{A}^i) \) corresponds a unique family \( \{ v^i_{t,j} \} \) of vectors such that \( P^i_t + v^i_{t,j} = P^j_t \). Notice that two different \( K_T \)-configurations can define the same family \( \{ v^i_{t,j} \} \). With the above notations, we provide the following definition.

**Definition 3.2.** Let \( \mathcal{A}^0 \) be a central generic arrangement, \( \mathbb{T} \) an \( r \)-set, \( \mathcal{A}^i a K_T \)-translated of \( \mathcal{A}^0 \) and \( i_0 \in [r] \) a fixed index. We call \( K_T \)-vector set the set of vectors \( \{ v^i_{t,j} \}_{t \neq i_0} \) which satisfies \( P^i_{i_0} + v^i_{t,j} = P^j_t \) for any \( j \in [r], j \neq i_0 \).\footnote{It is unique in the quotient space \( \mathbb{S}/D_{[n]} \approx \mathbb{C}^{n-k} \)}
Remark 3.4. Let \( K_T \) be the \( K_T \)-configuration of the arrangement \( A \). Then for any \( c \in \mathbb{C} \), the \( K_T \)-configuration \( K_T(A^c) \) is an "expansion" by \( c \) of \( K_T(A) \), that is \( v_{i,j}^c = cv_{i,j} \). This is consequence of the fact that for any \( i \in [r] \) the vector \( OP_{i,j}^c \) joining the origin with the points \( P_i \) satisfies \( OP_{i,j}^c = cOP_{i,j} \) by definition of translation. Hence \( P_i^cP_j^c = cP_iP_j \), i.e.

\[
v_{i,j}^c = cv_{i,j}.
\]

Analogously we have that, if \( t_1, t_2 \in \mathbb{C}^n \) are two translations then

\[
v_{i,j}^{t_1} + v_{i,j}^{t_2} = v_{i,j}^{t_1 + t_2}.
\]

We can now prove the main lemma of this section.

Lemma 3.5. Let \( A \) be a central generic arrangement of \( n \) hyperplanes in \( \mathbb{C}^k \) and \( T = \{ L_1, \ldots, L_r \} \) be an \( r \)-set such that \( |n| = \sum_{i=1}^r |L_i| \). The \( K_T \)-translated arrangements \( A^{t_1}, \ldots, A^{t_r} \) of \( A \) are linearly independent if and only if their associated \( K_T \)-vector sets \( \{v_{i,j}^{t_h}\}_{h=1}^d \) are weakly linearly independent.
Proof. By definition, $\mathcal{A}^0, \ldots, \mathcal{A}^v$ are linearly independent if and only if the translations $t_1, \ldots, t_d$ are linearly independent vectors in $\mathbb{C}^0/C$. Let’s consider a linear combination $\sum_{h=1}^{d} a_h t_h$ of the vectors $t_h$ and the translated arrangements $\mathcal{A}^{a_h}$. By Remark 3.4 we have that the $K_T$-vector sets associated to $\mathcal{A}^{a_h}$ satisfy the equalities:

$$\{v_{i_{h,j}}^{d}a_h\}_{i \neq i_0} = \sum_{h=1}^{d} \{v_{i_{h,j}}^{d}a_h\}_{i \neq i_0} = \sum_{h=1}^{d} a_h \{v_{i_{h,j}}^{d}\}_{i \neq i_0}.$$ 

Hence $\sum_{h=1}^{d} a_h (v_{i_{h,j}}^{d}a_h)_{i \neq i_0} = 0$ if and only if $\sum_{h=1}^{d} a_h (v_{i_{h,j}}^{d}a_h)_{i \neq i_0} = 0$ for any $j$, that is $D_{i=1}^{d} a_h = 0$. This is equivalent to $\mathcal{A}^{\sum_{h=1}^{d} a_h}$ be a central arrangement of center $P^{\sum_{h=1}^{d} a_h}$, i.e. $\sum_{h=1}^{d} a_h t_h \in C$ and the statement follows from Proposition 3.1.

The assumption that $\bigcup_{r=1}^{r'} L_r = [n]$ in Lemma 3.5 is equivalent to consider a subset $\mathcal{A}^0 \subset \mathcal{A}^0$ which only contains the hyperplanes indexed in the $\bigcup_{r=1}^{r'} L_r \subset [n]$ in the more general case. Indeed if a (central) generic arrangement $\mathcal{A}^0$ contains a subarrangement $\mathcal{A}^0$ which is non-very generic then $\mathcal{A}^0$ is obviously non-very generic. Analogously, if there is a restriction arrangement $\mathcal{A}^r = [H \cap Y_\pi \mid H \in \mathcal{A}^0 \setminus \mathcal{A}_r], Y_\pi = \bigcap_{H \in \mathcal{A}_r} H$ of $\mathcal{A}^0$ which is non-very generic, then $\mathcal{A}^0$ is non-very generic. The following main theorem of this Section follows.

Theorem 3.6. Let $\mathcal{A}^0$ be a central generic arrangement of $n$ hyperplanes in $\mathbb{C}^0$. If there exists an $r$-set $T = \{L_1, \ldots, L_r\}$ with $\bigcup_{i=1}^{r} L_i = m$ and rank $\cap_{p \in \bigcup_{r=1}^{r'} L_r} H_p = y$, which admits $m - y - k - r'$ weakly linearly independent $K_T$-vector sets for some $r' < r$, then $\mathcal{A}^0$ is non-very generic.

Proof. Let’s consider the subarrangement $\mathcal{A}'$ of $\mathcal{A}^0$ given by the hyperplanes indexed in the $\bigcup_{r=1}^{r'} L_r$ and its essentialization, i.e. the restriction arrangement $\mathcal{A}'^r, Y = \cap_{p \in \bigcup_{r=1}^{r'} L_r} H_p$. If $y = \text{rank } Y$ then the arrangement $\mathcal{A}'^r$ is a central essential arrangement in $\mathbb{C}^{m-y}, m = \bigcup_{r=1}^{r'} L_r$. By Lemma 3.5 if $\mathcal{A}'^1, \ldots, \mathcal{A}'^{m-r'}$ are $K_T$-translated of $\mathcal{A}'^r$ associated to the $m - y - k - r'$ independent $K_T$-vector sets, then $\mathcal{A}^0, \ldots, \mathcal{A}^{m-r'}$ are linearly independent vectors in $\mathbb{S}[\mathcal{A}']/D_{m} \simeq \mathbb{C}^{m-r-k}$. That is $\mathcal{A}'^1, \ldots, \mathcal{A}'^{m-r'}$ span a subspace of dimension $m - y - k - r'$. On the other hand, by construction, $\mathcal{A}^0$ is $K_T$-translated, i.e. $\mathcal{A}^0 \in \text{ess}(X)$, $X = \cap_{i=1}^{r} D_{L_i}$ for any $j = 1, \ldots, m - y - k - r'$, that is the space spanned by $\mathcal{A}^0, \ldots, \mathcal{A}^{m-r'}$ is included in $\text{ess}(X)$. This implies that the simple intersection $\text{ess}(X)$ has dimension $d \geq m - y - k - r > m - y - k - r$ that is its codimension is smaller than $r$, i.e. rank $\text{ess}(X) < r$ and hence rank $X < r$. This implies that $\mathcal{A}'$ is non-very generic and hence $\mathcal{A}^0$ is non-very generic. 

Theorem 3.6 allows to build non-very generic arrangements simply imposing linear conditions on vectors $v_{i,j} \in H_{i,j}^0$. This linearity is a non trivial achievement since the conditions to check the (non) very genericity are Plücker-type conditions. We point out that while Theorem 3.6 provides a quite useful tool to build non-very generic arrangements, we are still far away from being able to check whether a given arrangement is very generic or not.

In the next section we will provide non trivial examples of how to build non-very generic arrangements by means of Theorem 3.6.

4. Examples of non-very generic arrangements

In this section we present few examples to illustrate how to use the Theorem 3.6 to construct non-very generic arrangements. To construct the numerical examples we used the software CoCoA-5.2.4 (see [1]).

Example 4.1 (B(12, 8, $\mathcal{A}^0$) with an intersection of multiplicity 4 in rank 3). Let $L_1 = [12] \setminus \{10, 11, 12\}, L_2 = [12] \setminus \{7, 8, 9\}, L_3 = [12] \setminus \{4, 5, 6\}$ and $L_4 = [12] \setminus \{1, 2, 3\}$ be subsets of $[12]$ of $k + 1 = 9$ indices. It is an easy computation that the set $\mathcal{T} = \{L_1, L_2, L_3, L_4\}$ is a 4-set. Let’s consider a central generic arrangement $\mathcal{A}^0$ of 12 hyperplanes in $\mathbb{C}^8$. In this case $m = n = 12, y = 0$ and $m - k - r = 12 - 8 - 4 = 0$, hence, by Theorem 3.6 in order for $\mathcal{A}^0$ to be non-very generic it is enough the existence of just one $K_T$-vector set $v_{1,2,1,3,1,4}$, that is the vectors $v_{3,4} = v_{1,2} - v_{1,3} \in \cap_{p \in \bigcup_{r=1}^{r'} L_r} H_p$ and $v_{2,4} = v_{1,4} - v_{1,2} \in \cap_{p \in \bigcup_{r=1}^{r'} L_r} H_p$ have to belong to $H_{12}^0$ (see Figure 2). Notice that since $v_{3,4} = v_{2,4} - v_{2,3} \in$
Let’s see a numerical example. Let us consider hyperplanes of equation $H_i^0 : \alpha_i \cdot x = 0$, with $\alpha_i$, $i = 1, \ldots, 11$ assigned as following:

$$
\begin{align*}
\alpha_1 &= (0, 0, 1, 1, 0, -1, 1), \\
\alpha_2 &= (0, 0, 0, 1, 1, 1, -1), \\
\alpha_3 &= (0, 0, 1, 0, 0, 0, 1), \\
\alpha_4 &= (0, 1, 0, 1, 1, 0, 1), \\
\alpha_5 &= (0, 2, 0, -1, -1, 0, 1), \\
\alpha_6 &= (0, -1, 2, 0, -1, -1, 1), \\
\alpha_7 &= (1, 0, 0, 1, 0, -1, 1), \\
\alpha_8 &= (-1, 0, 0, 1, 1, -1, 0), \\
\alpha_9 &= (-4, 0, 0, 0, 0, 1, 1), \\
\alpha_{10} &= (1, 1, 1, -1, 1, 1, 0), \\
\alpha_{11} &= (1, 1, 1, 2, 2, 2, 3, 0).
\end{align*}
$$

(3)

In this case, we have the $K_T$-vector set

$$
\{v_{1,2}, v_{1,3}, v_{1,4}\} = \{(1, 0, 0, 0, 0, 0, 0, 0, 0), (0, 1, 0, 0, 0, 0, 0, 0, 0), (0, 0, -1, 0, 0, 0, 0, 0)\}.
$$

The other vectors are obtained by means of relations $v_{2,3} = v_{1,3} - v_{1,2}, v_{2,4} = v_{1,4} - v_{1,2}, v_{3,4} = v_{1,4} - v_{1,3}$, that is

$$
\begin{align*}
v_{2,3} &= (-1, 1, 0, 0, 0, 0, 0, 0, 0), \\
v_{2,4} &= (-1, 0, 1, 0, 0, 0, 0, 0, 0), \\
v_{3,4} &= (0, -1, 1, 0, 0, 0, 0, 0, 0)
\end{align*}
$$

(4)

and, finally, we get $\alpha_{12} = (-2, -2, -2, 3, 4, -5, 6, 7)$ by imposing the condition that $\alpha_{12}$ has to be orthogonal to $v_{2,3}$ and $v_{2,4}$.

\textbf{Example 4.2} ($B(16, 11, \mathcal{A}^0)$ with an intersection of multiplicity 4 in rank 3). Let $L_1 = [16] \setminus \{13, 14, 15, 16\}, L_2 = [16] \setminus \{9, 10, 11, 12\}, L_3 = [16] \setminus \{5, 6, 7, 8\}$ and $L_4 = [16] \setminus \{1, 2, 3, 4\}$ be subsets of [16] of $k + 1 = 12$ indices. The set $\mathcal{T} = \{L_1, L_2, L_3, L_4\}$ is a 4-set. Let’s consider a central generic arrangement $\mathcal{A}^0$ of 16 hyperplanes in $\mathbb{C}^{11}$. In this case $m = n = 16, y = 0$ and $m - k - r = 16 - 11 = 4 = 1$, hence, by Theorem 3.6 in order for $\mathcal{A}^0$ to be non-very generic we need two weakly linearly independent $K_T$-vector sets $\{v_{1,2}^1, v_{1,3}^1, v_{1,4}^1\}$ and $\{v_{2,3}^2, v_{2,4}^2, v_{3,4}^2\}$ that is the vectors $v_{2,3}^1 \in \bigcap_{p \in L_1 \cap L_2 \setminus \{16\}} H_p^0$ and $v_{2,4}^1 \in \bigcap_{p \in L_2 \cap L_4 \setminus \{16\}} H_p^0, k = 1, 2$, have to belong to $H_{16}^0$. Notice that since $v_{3,4}^2 = v_{2,4}^2 - v_{2,3}^2 \in \bigcap_{p \in L_2 \cap L_4 \setminus \{16\}} H_p^0$, if $v_{2,3}^2, v_{2,4}^2 \in H_{16}^0$ then $v_{3,4}^2 \in H_{16}^0$. That is all hyperplanes in $\mathcal{A}^0$ can be chosen freely, but $H_{16}^0$ which has to contain the vectors $v_{2,3}^2, v_{2,4}^2, k = 1, 2$. 

\footnote{Here and in the rest of this section, freely means that we only impose the condition that $\mathcal{A}^0$ is a central generic arrangement. In particular this condition is always taken as given and imposed even if not written.} 

\footnote{The graphic representation in this case can be simply obtained replacing the number 12 with 16 in Figure 1.}
Let’s see a numerical example. Let us consider hyperplanes of equation \( H^0_i : \alpha_i \cdot x = 0 \), with \( \alpha_i , i = 1, \ldots , 15 \) assigned as following.

(5)
\[
\begin{align*}
\alpha_1 &= (0, 0, 1, 0, 0, 1, 0, 0, 0, 1, -1), \quad \alpha_2 = (0, 0, -1, 0, 0, 1, 1, 1, -1, 0), \quad \alpha_3 = (0, 0, 2, 0, 0, 1, 1, 0, 1, 1, 0), \\
\alpha_4 &= (0, 0, 1, 0, 0, 1, 0, 0, 0, 1), \quad \alpha_5 = (0, -1, 0, 0, 1, 1, 1, -1, 0), \quad \alpha_6 = (0, 1, 0, 0, 2, 0, 0, -1, -1, 0, 1), \\
\alpha_7 &= (0, 2, 0, 0, -1, 0, -1, 0, 0, 1, 1), \quad \alpha_8 = (0, -1, 0, 2, 0, 1, 1, 1, 0), \quad \alpha_9 = (1, 0, 0, -3, 0, 0, -1, -1, 1, 1), \\
\alpha_{10} &= (2, 0, 0, 5, 0, 0, 1, -1, -1, 1, 1), \quad \alpha_{11} = (3, 0, 0, 1, 0, 0, 1, -1, 2, 0, 1), \quad \alpha_{12} = (1, 0, 0, 5, 0, 0, 1, 0, 1, 1, 0), \\
\alpha_{13} &= (1, 1, 1, -3, -3, -3, -1, -3, 2, -2, -1), \quad \alpha_{14} = (1, 1, 1, 0, 0, -2, -1, -8, 1, 1), \quad \alpha_{15} = (0, 0, 0, -5, -5, -5, 1, 2, -3, -4, 7).
\end{align*}
\]

In this case, we have the \( K_7 \)-vector sets
\[
\begin{align*}
\{ v^1_{1,2}, v^1_{1,3}, v^1_{1,4} \} &= \{(1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), (0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), (0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)\} , \\
\{ v^2_{1,2}, v^2_{1,3}, v^2_{1,4} \} &= \{(0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0)\} .
\end{align*}
\]

The other vectors are obtained by means of relations \( v^k_{2,3} = v^i_{1,3} - v^i_{1,2}, v^k_{2,4} = v^i_{1,4} - v^i_{1,2}, v^k_{3,4} = v^i_{1,4} - v^i_{1,3}, k = 1, 2, \) that is

(6)
\[
\begin{align*}
v^1_{2,3} &= (-1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), \quad v^1_{2,4} = (-1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), \quad v^1_{3,4} = (0, -1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0) , \\
v^2_{2,3} &= (0, 0, 0, -1, 1, 0, 0, 0, 0, 0, 0, 0, 0), \quad v^2_{2,4} = (0, 0, 0, -1, 1, 0, 0, 0, 0, 0, 0, 0, 0), \quad v^2_{3,4} = (0, 0, 0, 0, -1, 1, 0, 0, 0, 0, 0, 0, 0, 0) .
\end{align*}
\]

and, finally, we get \( \alpha_{16} = (1, 1, 1, -2, -2, -2, 5, 6, 7, 8, 9) \) by imposing the conditions that \( \alpha_{16} \) has to be orthogonal to \( v^k_{2,3} \) and \( v^k_{2,4} \), \( k = 1, 2 \).

Example 4.3 (\( \mathcal{B}(10, 3, \mathcal{A}^0) \) with an intersection of multiplicity 5 in rank 4). Let \( L_1 = \{1, 2, 3, 4\}, L_2 = \{1, 5, 6, 7\}, L_3 = \{2, 5, 8, 9\}, L_4 = \{3, 6, 8, 10\} \) and \( L_5 = \{4, 7, 9, 10\} \) be subsets of \( [10] \) of \( k + 1 = 4 \) indices. The set \( \mathcal{T} = \{L_1, L_2, L_3, L_4, L_5\} \) is a 5-set. Let’s consider a central generic arrangement \( \mathcal{A}^0 \) of 10 hyperplanes in \( \subset^3 \). In this case \( m = n = 10, y = 0 \) and \( m - k - r = 10 - 3 - 5 = 2 \), hence, by Theorem 5.6 in order for \( \mathcal{A}^0 \) to be non-very generic we need three weakly linearly independent \( K_7 \)-vector sets \( \{ v^1_{1,2}, v^1_{1,3}, v^1_{1,4}, v^1_{1,5} \}, \{ v^2_{1,2}, v^2_{1,3}, v^2_{1,4}, v^2_{1,5} \} \) and \( \{ v^3_{1,2}, v^3_{1,3}, v^3_{1,4}, v^3_{1,5} \} \) that is the vectors \( v^k_{1,5} \), \( k = 1, 2, 3 \), have to belong to \( H^0_9 \) (see Figure 3). Notice that since in this case hyperplanes are planes, then the three vectors \( v^k_{i,j}, k = 1, 2, 3 \) will be linearly dependent for any choice of indices \( i, j \), \( i \neq j \). This additional condition forces that at most 8 hyperplanes in \( \mathcal{A}^0 \) can be chosen freely, while both \( H^0_9 \) and \( H^0_{10} \) have to contain the dependent vectors \( v^k_{1,5} \) and \( v^k_{2,5} \), \( k = 1, 2, 3 \), respectively.

Let’s see a numerical example. Let us consider hyperplanes of equation \( H^0_i : \alpha_i \cdot x = 0 \), with \( \alpha_i , i = 1, \ldots , 8 \) assigned as following.

(7)
\[
\begin{align*}
\alpha_1 &= (0, 10, 3), \quad \alpha_2 = (20, 0, -9), \quad \alpha_3 = (2, -3, 0), \quad \alpha_4 = (3, 1, 0), \\
\alpha_5 &= (0, 0, 1), \quad \alpha_6 = (1, -1, 1), \quad \alpha_7 = (1, 2, 2), \quad \alpha_8 = (4, -1, -3).
\end{align*}
\]

In this case, we have the \( K_7 \)-vector sets
\[
\begin{align*}
\{ v^1_{1,2}, v^1_{1,3}, v^1_{1,4}, v^1_{1,5} \} &= \{(1, -3, 10), (\frac{21}{2}, 2, 10), (\frac{9}{2}, 3, \frac{25}{2}), (\frac{77}{9}, \frac{77}{3}, -\frac{125}{9})\}, \\
\{ v^2_{1,2}, v^2_{1,3}, v^2_{1,4}, v^2_{1,5} \} &= \{(-2, 6, -20), (-9, -47, -20), (-3, -2, -27), (-\frac{2}{3}, 2, -\frac{50}{3})\}, \\
\{ v^3_{1,2}, v^3_{1,3}, v^3_{1,4}, v^3_{1,5} \} &= \{(-3, 3, -10), (-\frac{9}{2}, -\frac{239}{80}, -10), (-\frac{1467}{1040}, -\frac{489}{520}, -\frac{1651}{1040}), (-\frac{4}{3}, 4, -\frac{71}{6})\}.
\end{align*}
\]
The other vectors are obtained by means of relations \(v_{ij}^k = v_{ij}^l - v_{ij}^k\), where \(2 \leq i < l \leq 5\), \(k = 1, 2, 3\), that is

\[
\begin{align*}
\text{Example 4.3:} & \quad v_{2,3} = \left(\frac{7}{2}, \frac{27}{2}, 0\right), v_{1,4} = \left(\frac{7}{2}, 6, \frac{5}{2}\right), v_{2,5} = \left(-\frac{86}{9}, \frac{86}{9}, -\frac{215}{9}\right), \\
v_{3,4} = \left(0, \frac{15}{2}, \frac{5}{2}\right), v_{3,5} = \left(-\frac{235}{18}, \frac{91}{6}, \frac{215}{9}\right), v_{4,5} = \left(-\frac{235}{18}, \frac{68}{3}, \frac{475}{18}\right), \\
v_{2,3} = (-7, -53, 0), v_{2,4} = (-1, -8, -7), v_{2,5} = \left(\frac{4}{3}, -4, \frac{10}{3}\right) \\
\end{align*}
\]

Finally, we get \(\alpha_9 = (314, -40, -197)\) and \(\alpha_{10} = (139, 30, -43)\) by imposing the conditions that \(\alpha_9\) and \(\alpha_{10}\) have to be orthogonal to \(v_{3,5}\) and \(v_{4,5}\), \(k = 1, 2, 3\).

**Remark 4.4.** Notice that Example 4.3 is slightly different from other examples for two reasons. Firstly, it uses a different combinatorics. In the Examples 4.1 and 4.2 the 4-sets \(T = \{L_1, L_2, L_3, L_4\}\) are of the form \(L_i = [n] \setminus K_i\) with \(K_i\)’s which satisfy the properties \(\bigcup_{i=1}^4 K_i = [n]\) and \(K_i \cap K_j = \emptyset\) while in the Example 4.3 they are not. Secondly, in the Examples 4.1 and 4.2 in order to obtain non-very generic arrangement we could choose all hyperplanes freely but one, while in the Example 4.3 two hyperplanes had to be fixed as a result of the need of three weakly independent \(K_i\)-vector sets in two dimensional hyperplanes. Indeed this dependency condition gives rise to 27 independent equations of the form

\[
v_{j,i}^3 = \alpha v_{i,j}^l + \beta v_{i,j}^l
\]

which fix the entries of the vectors \(v_{i,j}^k\), \(i = 3, 4, 5\) uniquely for any choice of three dependent vectors \(v_{i,j}^k\), \(k = 1, 2, 3\). Hence the vectors \(v_{3,5}^5\) and \(v_{4,5}^5\), \(k = 1, 2, 3\) are determined and so are the two hyperplanes \(H_0^5\) and \(H_0^5\).

**Remark 4.5.** Notice that both Example 4.1 and Example 4.2 satisfy Athanasiadis condition while the first one fails when the set \(I\) has maximal cardinality. This essentially shows how the problem to describe the \(r\)-sets \(T\) that can give rise to (simple) non-very generic intersections is non trivial.

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5Notice that this is a generalization of the combinatorics used in [14].
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(1) Department of Economics and Statistics, Turin University, Turin, Italy.

Email address: simona.settepanella@unito.it

(2) Department of Mathematics, Hokkaido University, Sapporo, Japan.

Email address: so.yamagata@math.sci.hokudai.ac.jp