Lax pairs for the equations describing compatible nonlocal Poisson brackets of hydrodynamic type, and integrable reductions of the Lamé equations

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1 Introduction. Basic definitions

In the present work, the nonlinear equations for the general nonsingular pairs of compatible nonlocal Poisson brackets of hydrodynamic type are derived and the integrability of these equations by the method of inverse scattering problem is proved. For these equations, the Lax pairs with a spectral parameter are presented. Moreover, we demonstrate the integrability of the equations for some especially important partial classes of compatible nonlocal Poisson brackets of hydrodynamic type, in particular, for the most important case when one of the compatible Poisson brackets is local and also for the case when one of the compatible Poisson brackets is generated by a metric of constant Riemannian curvature. The case when one of the compatible Poisson brackets of hydrodynamic type is local and nondegenerate was studied in detail in our previous paper [1], where the corresponding equations were derived and the integrability of these equations was announced. This case is very important, since, as was shown in [1], any solution of these equations generates an integrable bi-Hamiltonian hierarchy of hydrodynamic type by explicit formulae. Moreover, these equations describe an important class of integrable reductions of the classical Lamé equations. Accordingly, in the case when one of the compatible Poisson brackets is generated by a metric of constant Riemannian curvature, the corresponding equations describe integrable reductions of the equations for orthogonal curvilinear coordinate systems in spaces of constant curvature.

1.1 Local Poisson brackets of hydrodynamic type

An arbitrary local homogeneous first-order Poisson bracket, that is, a Poisson bracket of the form

\[ \{I, J\} = \int \frac{\delta I}{\delta u^i(x)} \left( g^{ij}(u(x)) \frac{d}{dx} + b^{ij}_k(u(x)) u^k_x \right) \frac{\delta J}{\delta u^i(x)} dx, \]

(1.1)

where \(u^1, ..., u^N\) are local coordinates on a certain given smooth \(N\)-dimensional manifold \(M\), is called a local Poisson bracket of hydrodynamic type or a Dubrovin–Novikov bracket [2]. Here \(u^i(x), 1 \leq i \leq N\), are functions (fields) of single independent variable \(x\), the

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coefficients $g^{ij}(u)$ and $b_{jk}^{ij}(u)$ of bracket (1.1) are smooth functions of local coordinates, $I[u]$ and $J[u]$ are arbitrary functionals on the space of fields $u^i(x)$, $1 \leq i \leq N$. A local bracket (1.1) is called nondegenerate if $\det(g^{ij}(u)) \neq 0$.

**Theorem 1.1 (Dubrovin, Novikov [2])** If $\det(g^{ij}(u)) \neq 0$, then bracket (1.1) is a Poisson bracket, that is, it is skew-symmetric and satisfies the Jacobi identity, if and only if

1. $g^{ij}(u)$ is an arbitrary flat pseudo-Riemannian contravariant metric (a metric of zero Riemannian curvature),

2. $b_{jk}^{ij}(u) = -g^{is}(u)\Gamma_{sk}^{ij}(u)$, where $\Gamma_{sk}^{ij}(u)$ is the Riemannian connection generated by the contravariant metric $g^{ij}(u)$ (the Levi–Civita connection).

Consequently, for any local nondegenerate Poisson bracket of hydrodynamic type, there always exist local coordinates $v^1, ..., v^N$ (flat coordinates of the metric $g^{ij}(u)$) in which all the coefficients of the bracket are constant:

$$\tilde{g}^{ij}(v) = \eta^{ij} = \text{const}, \quad \tilde{\Gamma}_{sk}^{ij}(v) = 0, \quad \tilde{b}_{jk}^{ij}(v) = 0,$$

that is, the bracket has the form

$$\{I, J\} = \int \frac{\delta I}{\delta v^i(x)} \eta^{ij} \frac{d}{dx} \frac{\delta J}{\delta v^j(x)} dx,$$  \hspace{1cm} (1.2)

where $(\eta^{ij})$ is a nondegenerate symmetric constant matrix:

$$\eta^{ij} = \eta^{ji}, \quad \eta^{ij} = \text{const}, \quad \det(\eta^{ij}) \neq 0.$$

### 1.2 Nonlocal Poisson brackets of hydrodynamic type

Nonlocal Poisson brackets of hydrodynamic type (the Mokhov–Ferapontov brackets) were introduced and studied in the work of the present author and Ferapontov [3]. They have the following form:

$$\{I, J\} = \int \frac{\delta I}{\delta u^i(x)} \left( g^{ij}(u(x)) \frac{d}{dx} + b_{jk}^{ij}(u(x)) u_k^x + K u^2_j \left( \frac{d}{dx} u^i_x \right)^{-1} \right) \frac{\delta J}{\delta u^j(x)} dx,$$  \hspace{1cm} (1.3)

where $K$ is an arbitrary constant. A bracket of form (1.3) is called nondegenerate if $\det(g^{ij}(u)) \neq 0$.

**Theorem 1.2 ([3])** If $\det(g^{ij}(u)) \neq 0$, then bracket (1.3) is a Poisson bracket, that is, it is skew-symmetric and satisfies the Jacobi identity, if and only if

1. $g^{ij}(u)$ is an arbitrary pseudo-Riemannian contravariant metric of constant Riemannian curvature $K$, 

2. $b_{jk}^{ij}(u) = -g^{is}(u)\Gamma_{sk}^{ij}(u)$, where $\Gamma_{sk}^{ij}(u)$ is the Riemannian connection generated by the contravariant metric $g^{ij}(u)$ (the Levi–Civita connection).
(2) $b^{ij}_k(u) = -g^{is}(u)\Gamma^{jk}_{sk}(u)$, where $\Gamma^{jk}_{sk}(u)$ is the Riemannian connection generated by the contravariant metric $g^{ij}(u)$ (the Levi-Civita connection).

In [4] Ferapontov introduced and studied more general nonlocal Poisson brackets of hydrodynamic type (the Ferapontov brackets), namely, the Poisson brackets of the form

$$\{I, J\} = \int \frac{\delta I}{\delta u^i(x)} \left( g^{ij}(u(x)) \frac{d}{dx} + b^{ij}_k(u(x)) u^k_x \right) + \sum_{\alpha=1}^{L} \varepsilon_\alpha (w^{\alpha})^k_i(u(x)) u^k_x \left( \frac{d}{dx} \right)^{-1} (w^{\alpha})^l_j(u(x)) u^l_x \frac{\delta J}{\delta \omega^j(x)} dx,$$

where $\varepsilon_\alpha = \pm 1$, $\alpha = 1, \ldots, L$.

**Theorem 1.3 ([4])** Bracket (1.4) is a Poisson bracket, that is, it is skew-symmetric and satisfies the Jacobi identity, if and only if

1. $b^{ij}_k(u) = -g^{is}(u)\Gamma^{jk}_{sk}(u)$, where $\Gamma^{jk}_{sk}(u)$ is the Riemannian connection generated by the contravariant metric $g^{ij}(u)$ (the Levi-Civita connection),
2. the metric $g^{ij}(u)$ and the set of the affinors $(w^{\alpha})^i_j(u)$ satisfies relations

$$g_{ik}(u)(w^{\alpha})^k_j(u) = g_{jk}(u)(w^{\alpha})^k_i(u), \quad \alpha = 1, \ldots, L,$$

(1.5)

$$\nabla_k(w^{\alpha})^i_j(u) = \nabla_j(w^{\alpha})^i_k(u), \quad \alpha = 1, \ldots, L,$$

(1.6)

$$R^{ij}_{kl}(u) = \sum_{\alpha=1}^{L} \varepsilon_\alpha \left( (w^{\alpha})^l_j(u)(w^{\alpha})^k_i(u) - (w^{\alpha})^k_j(u)(w^{\alpha})^l_i(u) \right).$$

(1.7)

In addition, the family of the affinors $w^{\alpha}(u)$ is commutative: $[w^{\alpha}, w^{\beta}] = 0$.

Let us write out all the relations on the coefficients of the nonlocal Poisson bracket (1.4) in a convenient form for further repeated use.

**Lemma 1.1** Bracket (1.4) is a Poisson bracket if and only if its coefficients satisfy the relations

$$g^{ij} = g^{ji},$$

(1.8)

$$\frac{\partial g^{ij}}{\partial u^k} = b^{ij}_k + b^{ji}_k,$$

(1.9)

$$g^{is}b^{jk}_s = g^{js}b^{ik}_s,$$

(1.10)

$$g^{is}(w^{\alpha})^j_s = g^{js}(w^{\alpha})^i_s,$$

(1.11)

$$(w^{\alpha})^l_j(u)(w^{\beta})^s_i = (w^{\beta})^s_j(u)(w^{\alpha})^l_i,$$

(1.12)
Definition 1.1 (Magri [5]) Two Poisson brackets, that is, the property of a system to have two compatible Hamiltonian representations. This approach demonstrated that integrability is closely related to the bi-Hamiltonian property.

1.3 Compatible Poisson brackets

In [6], Magri proposed a bi-Hamiltonian approach to the integration of nonlinear systems. This approach demonstrated that integrability is closely related to the bi-Hamiltonian property, that is, the property of a system to have two compatible Hamiltonian representations.

Definition 1.1 (Magri [6]) Two Poisson brackets \( \{ \cdot , \cdot \}_1 \) and \( \{ \cdot , \cdot \}_2 \) are called compatible if an arbitrary linear combination of these Poisson brackets

\[
\{ \cdot , \cdot \} = \lambda_1 \{ \cdot , \cdot \}_1 + \lambda_2 \{ \cdot , \cdot \}_2,
\]

where \( \lambda_1 \) and \( \lambda_2 \) are arbitrary constants, is also a Poisson bracket. In this case, we shall also say that the brackets \( \{ \cdot , \cdot \}_1 \) and \( \{ \cdot , \cdot \}_2 \) form a pencil of Poisson brackets.

As was shown by Magri in [6], compatible Poisson brackets generate integrable hierarchies of systems of differential equations. In particular, for a system, the bi-Hamiltonian property generates recurrent relations for the conservation laws of this system.

1.4 Compatible pseudo-Riemannian metrics

Two pseudo-Riemannian contravariant metrics \( g^{ij}_1(u) \) and \( g^{ij}_2(u) \) are called compatible if for any linear combination of these metrics \( g^{ij}(u) = \lambda_1 g^{ij}_1(u) + \lambda_2 g^{ij}_2(u) \), where \( \lambda_1 \) and \( \lambda_2 \) are arbitrary constants for which \( \det(g^{ij}(u)) \neq 0 \), the coefficients of the corresponding Levi-Civita connections and the components of the corresponding tensors of Riemannian curvature are related by the same linear formula: \( \Gamma^k_i(u) = \lambda_1 \Gamma^k_i^{(1)}(u) + \lambda_2 \Gamma^k_i^{(2)}(u) \) and \( R^k_{ijl}(u) = \lambda_1 R^k_{ijl}^{(1)}(u) + \lambda_2 R^k_{ijl}^{(2)}(u) \) (in this case, we shall say also that the metrics \( g^{ij}_1(u) \) and \( g^{ij}_2(u) \) form a pencil of metrics) [6], [7]. Flat pencils of metrics, that is nothing but compatible nondegenerate local Poisson brackets of hydrodynamic type (compatible Dubrovin–Novikov brackets [8]), were introduced in [6]. Two pseudo-Riemannian contravariant metrics \( g^{ij}_1(u) \) and \( g^{ij}_2(u) \) of constant Riemannian curvature \( K_1 \) and \( K_2 \) respectively are called compatible if any linear combination of these metrics \( g^{ij}(u) = \lambda_1 g^{ij}_1(u) + \lambda_2 g^{ij}_2(u) \), where \( \lambda_1 \) and \( \lambda_2 \) are arbitrary constants for which \( \det(g^{ij}(u)) \neq 0 \), is a metric of constant Riemannian curvature \( \lambda_1 K_1 + \lambda_2 K_2 \) and the coefficients of the corresponding Levi-Civita connections are related by the same linear formula: \( \Gamma^k_i^{(j)}(u) = \lambda_1 \Gamma^k_i^{(1,j)}(u) + \lambda_2 \Gamma^k_i^{(2,j)}(u) \) [6], [7]. In this case, we shall also say that the metrics \( g^{ij}_1(u) \) and \( g^{ij}_2(u) \) form a pencil of metrics of constant Riemannian curvature [6], [7]. It is obvious that all these definitions are mutually consistent, so that if compatible metrics are metrics of constant Riemannian curvature, then they form a
Consider two arbitrary nonlocal Poisson brackets of hydrodynamic type. A pair of pseudo-Riemannian metrics \( g_1^{ij}(u) \) and \( g_2^{ij}(u) \) is called *nonsingular* if the eigenvalues of this pair of metrics, that is, the roots of the equation 

\[
\det(g_1^{ij}(u) - \lambda g_2^{ij}(u)) = 0
\]

are distinct (a pencil of metrics which is formed by a nonsingular pair of metrics is also called *nonsingular*). In \([6]\), it was proved that if the Poisson brackets (2.1) and (2.2) are compatible, then the metrics \( g_1^{ij}(u) \) and \( g_2^{ij}(u) \) are compatible if and only if the eigenvalues of this pair of metrics, that is, the roots of the equation 

\[
\det(u_1^{ij} g_1^{ij}(u(x)) - \lambda g_2^{ij}(u(x))) = 0
\]

are distinct (a pencil of metrics which is formed by a nonsingular pair of metrics is also called *nonsingular*). In \([6]\), it was proved that an arbitrary nonsingular pair of metrics is compatible if and only if there exist local coordinates \( u = (u^1, ..., u^N) \) such that both the metrics are diagonal in these coordinates and have the following special form (one can consider that one of the metrics, here \( g_2^{ij}(u) \), is an arbitrary diagonal metric): 

\[
\begin{align*}
\frac{\partial g_2^{ij}(u)}{\partial u^i} &= g_1^{ij}(u)\delta^j_1 \\
\frac{\partial g_2^{ij}(u)}{\partial u^j} &= g_1^{ij}(u)\delta^i_1
\end{align*}
\]

and 

\[
\begin{align*}
\frac{\partial g_1^{ij}(u)}{\partial u^i} &= g_2^{ij}(u)\delta^j_1 \\
\frac{\partial g_1^{ij}(u)}{\partial u^j} &= g_2^{ij}(u)\delta^i_1
\end{align*}
\]

In \([6]\) it was proved that if the Poisson brackets (2.1) and (2.2) are compatible, then the metrics \( g_1^{ij}(u) \) and \( g_2^{ij}(u) \) of the brackets are also compatible. Moreover, it was also proved in \([6]\) that if 1) the pair of metrics \( g_1^{ij}(u) \) and \( g_2^{ij}(u) \) is nonsingular, 2) both the metrics \( g_1^{ij}(u) \), \( g_2^{ij}(u) \) and the affinors \( (w_1^{ij})_k^l(u) \), \( (w_2^{ij})_k^l(u) \) can be simultaneously diagonalized in a domain of local coordinates, then the Poisson brackets are compatible if and only if the metrics are compatible. Here we prove in some sense the converse theorem, which is very important for our method of integrating the equations for the general nonsingular pairs of arbitrary compatible nonlocal Poisson brackets of hydrodynamic type.

**Theorem 2.1** If the pair of metrics \( g_1^{ij}(u) \) and \( g_2^{ij}(u) \) is nonsingular, then the Poisson brackets \( \{I, J\}_1 \) and \( \{I, J\}_2 \) are compatible if and only if the metrics are compatible and
both the metrics \( g^{ij}_1(u), g^{ij}_2(u) \) and the affinors \((w^1)_{ij}^s(u), (w^2)_{ij}^s(u)\) can be simultaneously diagonalized in a domain of local coordinates.

Proof. It is sufficient to prove here that if the pair of metrics is nonsingular and the Poisson brackets are compatible, then both the metrics \( g^{ij}_1(u), g^{ij}_2(u) \) and the affinors \((w^1)_{ij}^s(u), (w^2)_{ij}^s(u)\) can be simultaneously diagonalized in a domain of local coordinates. All the rest was proved in [6]. First of all, it was proved that in this case the metrics are diagonal and have the following special form in these coordinates:

\[ g^{ij}_1(u) \text{ and } g^{ij}_2(u) \text{ are diagonal and have the compatible.} \]

Recall that if the pair of metrics is nonsingular, then the metrics are compatible if and only if there exist local coordinates such that the metrics are diagonal and have the following special form in these coordinates: \( g^{ij}_1(u) = g^i(u) \delta^j \) and \( g^{ij}_2(u) = f^i(u) g^j(u) \delta^j, \)

where \( f^i(u^i), 1 \leq i \leq N, \) are functions of single variable \( u^i. \) The functions \( f^i(u^i) \) are the eigenvalues of the pair of metrics \( g^{ij}_1(u) \) and \( g^{ij}_2(u) \) so that they are distinct by assumption of the theorem even in the case if they are constants. It follows from the compatibility of the Poisson brackets \( \{I, J\}_1 \) and \( \{I, J\}_2 \) (it is necessary to consider relation (1.11) for the pencil \( \{I, J\}_1 + \lambda \{I, J\}_2 \) that

\[ g^{is}_1(w^2)_s = g^{is}_1(w^2)_s, \]

\[ g^{is}_2(w^2)_s = g^{is}_2(w^2)_s. \]

Besides, from relation (1.11) for the Poisson brackets \( \{I, J\}_1 \) and \( \{I, J\}_2 \) we have

\[ g^{is}_1(w^1)_s = g^{is}_1(w^1)_s, \]

\[ g^{is}_2(w^1)_s = g^{is}_2(w^1)_s. \]

So from (2.3) and (2.6) in our special local coordinates we get

\[ g^i(w^2)_j = g^i(w^2)_j, \]

\[ f^i(u^j) g^i(w^2)_j = f^i(u^j) g^i(w^2)_j. \]

Thus

\[ (w^2)_j^i = g^i(w^2)_j = \frac{f^i(u^j) g^i}{f^j(u^j)}(w^2)_j^i, \]

that is,

\[ \left( 1 - \frac{f^i(u^j)}{f^j(u^j)} \right) (w^2)_j^i = 0. \]

Consequently, since all the functions \( f^i(u^i) \) are distinct, we get

\[ (w^2)_j^i = 0 \quad \text{for } i \neq j. \]

Similarly, from (2.4) and (2.7) we have

\[ (w^1)_j^i = 0 \quad \text{for } i \neq j. \]

Thus theorem 2.1 is proved.
3 Equations for nonsingular pairs of compatible nonlocal Poisson brackets of hydrodynamic type

Consider an arbitrary nonsingular pair of compatible nonlocal Poisson brackets of hydrodynamic type (2.1) and (2.2), that is, we assume that the Poisson brackets are compatible and the pair of metrics \( g^{ij}_{2}(u) \) and \( g^{ij}_{1}(u) \) is nonsingular.

**Theorem 3.1** General nonsingular pairs of compatible nonlocal Poisson brackets of hydrodynamic type are described by the following consistent integrable nonlinear systems:

\[
\frac{\partial H_{2,i}^{\alpha}}{\partial u^i} = \beta_{ij} H_{2,j}^{\alpha}, \quad i \neq j, \tag{3.1}
\]

\[
\frac{\partial \beta_{ij}}{\partial u^k} = \beta_{ik} \beta_{kj}, \quad i \neq j, \quad i \neq k, \quad j \neq k, \tag{3.2}
\]

\[
\epsilon_{2}^{i} \frac{\partial \beta_{ij}}{\partial u^i} + \epsilon_{2}^{j} \frac{\partial \beta_{ij}}{\partial u^j} + \sum_{s \neq i, s \neq j} \epsilon_{2}^{s} \beta_{si} \beta_{sj} + \sum_{\alpha=1}^{L_{2}} \varepsilon_{2,\alpha} H_{1,i}^{\alpha} H_{2,j}^{\alpha} = 0, \quad i \neq j. \tag{3.3}
\]

\[
\epsilon_{2}^{i} f^i(u^i) \frac{\partial \beta_{ij}}{\partial u^i} + \frac{1}{2} \epsilon_{2}^{i} (f^i)' \beta_{ij} + \epsilon_{2}^{j} f^j(u^j) \frac{\partial \beta_{ij}}{\partial u^j} + \frac{1}{2} \epsilon_{2}^{j} (f^j)' \beta_{ij} + \sum_{s \neq i, s \neq j} \epsilon_{2}^{s} f^s(u^s) \beta_{si} \beta_{sj} + \sum_{\alpha=1}^{L_{1}} \varepsilon_{1,\alpha} H_{1,i}^{\alpha} H_{1,j}^{\alpha} = 0, \quad i \neq j, \tag{3.4}
\]

where \( f^i(u^i), i = 1, \ldots, N, \) are arbitrary given functions of one variable.

According to theorem 2.1 there exist local coordinates such that in these coordinates we have

\[
g^{ij}_{2}(u) = g^{i}(u) \delta^{ij}, \quad g^{ij}_{1}(u) = f^i(u^i) g^{i}(u) \delta^{ij}, \tag{3.6}
\]

\[
(w_{1}^{a})_{j}^{i}(u) = (w_{2}^{a})_{j}^{i}(u) \delta^{ij}, \quad (w_{2}^{a})_{j}^{i}(u) = (w_{1}^{a})_{j}^{i}(u) \delta^{ij}. \tag{3.7}
\]

Moreover, according to the same theorem 2.1 any pair of Poisson brackets of this form is compatible. Thus it is sufficient to consider the conditions that the brackets \( \{I, J\}_{1} \) and \( \{I, J\}_{2} \) of this special form are Poisson brackets. If the metric \( g^{ij}(u) \) and the affinors \( (w^{a})_{j}^{i}(u) \) of bracket (1.4) are diagonal, then the conditions (1.8)–(1.14) take an especially simple form. Consider the conditions for the bracket \( \{I, J\}_{2} \) in the special local coordinates.

Introduce the standard classical notation

\[
g^{i}(u) = \frac{e_{2}^{i}(H_{1}(u))^{2}}{(H_{1}(u))^{2}}, \quad ds^{2} = \sum_{i=1}^{N} e_{2}^{i}(H_{1}(u))^{2}(du^{i})^{2}, \tag{3.8}
\]

\[
\beta_{ik}(u) = \frac{1}{H_{1}(u)} \frac{\partial H_{k}}{\partial u^i}, \quad i \neq k, \tag{3.9}
\]
where $H_i(u)$ are the Lamé coefficients and $\beta_{ik}(u)$ are the rotation coefficients, $\epsilon_2^i = \pm 1$, $i = 1, ..., N$. Although, in our case, all the functions are, generally speaking, complex, we shall use formulae, which are convenient for using also in the purely real case.

In the considered “diagonal” case, the conditions that $\{I, J\}_2$ is a Poisson bracket are equivalent to the Gauss-Codazzi equations for submanifolds with flat normal bundle and holonomic net of curvature lines (see [9], [10]). Following [9], [10] introduce the functions equivalent to the Gauss-Codazzi equations for submanifolds with flat normal bundle and holonomic net of curvature lines (see [9], [10]). Following [9], [10] introduce the functions $H_{a,i}^0(u)$, $1 \leq i \leq N$, $1 \leq \alpha \leq L_2$, such that

$$
(u_2^a)'(u) = \frac{H_{2,i}^0(u)}{H_i(u)},
$$

(3.10)

Then the conditions (1.8)–(1.14) for the bracket $\{I, J\}_2$ take the form (see [11])

$$
\frac{\partial H_{a,i}^0}{\partial u^i} = \beta_{ij} H_{2,i}^0, \quad i \neq j,
$$

(3.11)

$$
\frac{\partial \beta_{ij}}{\partial u^k} = \beta_{ik} \beta_{kj}, \quad i \neq j, \quad i \neq k, \quad j \neq k,
$$

(3.12)

$$
e_2^i \frac{\partial \beta_{ij}}{\partial u^i} + e_2^j \frac{\partial \beta_{ij}}{\partial u^j} + \sum_{s \neq i, s \neq j} e_2^s \beta_{si} \beta_{sj} + \sum_{\alpha = 1}^{L_2} \epsilon_2^\alpha H_{2,i}^\alpha H_{2,j}^\alpha = 0, \quad i \neq j.
$$

(3.13)

For the metric $g_{ij}^0(u) = f^i(u') g^j(u) \delta^{ij}$, the Lamé coefficients and the rotation coefficients have the form

$$
\tilde{H}_i(u) = \frac{H_i(u)}{\sqrt{\epsilon_1^i f^i(u')}}, \quad f^i(u') g^i(u) = \frac{\epsilon_1^i \epsilon_2^i}{(H_i(u))^2}, \quad \epsilon_1^i = \pm 1,
$$

(3.14)

$$
\tilde{\beta}_{ik}(u) = \frac{1}{H_i(u)} \frac{\partial \tilde{H}_k}{\partial u^i} = \frac{\sqrt{\epsilon_1^i f^i(u')}}{\sqrt{\epsilon_1^k f^k(u')}} \left( \frac{1}{H_i(u)} \frac{\partial H_k}{\partial u^i} \right) = \frac{\sqrt{\epsilon_1^i f^i(u')}}{\sqrt{\epsilon_1^k f^k(u')}} \beta_{ik}(u), \quad i \neq k.
$$

(3.15)

Besides, introduce the functions $H_{a,i}^\alpha(u)$, $1 \leq i \leq N$, $1 \leq \alpha \leq L_1$, such that

$$
(u_2^a)'(u) = \frac{\tilde{H}_{1,i}^\alpha(u)}{H_i(u)} = \sqrt{\epsilon_1^i f^i(u')} \frac{H_{a,i}^\alpha(u)}{H_i(u)} = \frac{H_{1,i}^\alpha(u)}{H_i(u)}.
$$

(3.16)

Accordingly, equations (3.12) are automatically satisfied also for the rotation coefficients $\tilde{\beta}_{ik}(u)$, equations (3.11) become equations (3.5), and equations (3.13) for $\beta_{ik}(u)$ give equations (3.4).

Note that it is easy to show that equations (3.3), (3.4) for nonsingular pairs of metrics (that is, all the functions $f^i(u')$ must be distinct also in the case if they are constants)
are equivalent to the following equations (in particular, it is more convenient to use these equations for checking the consistency of system (3.1)–(3.5)):

\[
\frac{\partial \beta_{ij}}{\partial u^i} = \frac{1}{2} \frac{(f^i(u'))'}{f^j(w) - f^i(w')} \beta_{ij} + \frac{e_2^j e_2^j}{2} \frac{(f^j(u'))'}{f^j(w) - f^i(w')} \beta_{ji} - \sum_{s \neq i, s \neq j} e_2^i e_2^s (f^j(u')) (f^j(u'))' \beta_{si} \beta_{sj} + \sum_{\beta = 1}^{L_1} \frac{e_2^i e_{1,\beta}}{f^j(w) - f^i(w')} H_{1,i}^\beta H_{1,j}^\beta - \\
\sum_{\alpha = 1}^{L_2} \frac{e_2^i e_{2,\alpha} f^j(u')}{(f^j(w') - f^i(w'))} H_{2,i}^\alpha H_{2,j}^\alpha, \quad i \neq j.
\]

(3.17)

4 Lax pair for the general nonsingular pair of compatible nonlocal Poisson brackets of hydrodynamic type

The Lax pair with a spectral parameter for the system (3.1)–(3.5) can be derived from the linear problem for the system (3.1)–(3.3) describing all submanifolds with flat normal bundle and holonomic net of curvature lines. The equations (3.1)–(3.3) are the conditions of consistency for the following linear system:

\[
\frac{\partial \varphi_i}{\partial u^k} = \frac{\sqrt{e_2^i}}{\sqrt{e_2^k}} \beta_{ik} \varphi_k, \quad i \neq k,
\]

(4.1)

\[
\frac{\partial \varphi_i}{\partial u^i} = -\sum_{k \neq i} \frac{\sqrt{e_2^i}}{\sqrt{e_2^k}} \beta_{ki} \varphi_k + \sum_{\alpha = 1}^{L_2} \frac{\sqrt{e_2^i}}{\sqrt{e_2^k}} H_{2,i}^\alpha \psi^\alpha,
\]

(4.2)

\[
\frac{\partial \psi^\alpha}{\partial u^i} = -\frac{\sqrt{e_2^i}}{\sqrt{e_2^k}} H_{2,i}^\alpha \varphi_i.
\]

(4.3)

The condition that the bracket \( \{ I, J \}_1 + \lambda \{ I, J \}_2 \) is a Poisson bracket for any \( \lambda \) is equivalent to the system (3.1)–(3.3) corresponding to the metric \( (\lambda + f^i(u')) g^i(u) \delta^{ij} \) and the affinors \( (w^\beta_1)^i_j(u) \), \( 1 \leq \beta \leq L_1 \), and \( \sqrt{\lambda (w^\alpha_2)^i_j(u)} \), \( 1 \leq \alpha \leq L_2 \). In this case, the linear problem (4.1)–(4.3) becomes the Lax pair with the spectral parameter \( \lambda \) for the general nonsingular pair of arbitrary compatible nonlocal Poisson brackets of hydrodynamic type:

\[
\frac{\partial \varphi_i}{\partial u^k} = \frac{\sqrt{e_2^i (\lambda + f^i)}}{\sqrt{e_2^k (\lambda + f^k)}} \beta_{ik} \varphi_k, \quad i \neq k,
\]

(4.4)

\[
\frac{\partial \varphi_i}{\partial u^i} = -\sum_{k \neq i} \frac{\sqrt{e_2^i (\lambda + f^k)}}{\sqrt{e_2^k (\lambda + f^i)}} \beta_{ki} \varphi_k + \sum_{\alpha = 1}^{L_2} \frac{\sqrt{e_2^i (\lambda + f^k)}}{\sqrt{e_2^k (\lambda + f^i)}} H_{2,i}^\alpha \psi^\alpha,
\]

(4.2)

\[
\frac{\partial \psi^\alpha}{\partial u^i} = -\frac{\sqrt{e_2^i (\lambda + f^i)}}{\sqrt{e_2^k (\lambda + f^k)}} H_{2,i}^\alpha \varphi_i.
\]

(4.3)
The Lax pair with the spectral parameter proves the integrability of the system (3.1)–(3.5) for the general nonsingular pair of compatible nonlocal Poisson brackets of hydrodynamic type.

If \( H_{1,i}^\alpha(u) = 0 \), \( 1 \leq i \leq N \), \( 1 \leq \alpha \leq L_1 \), and \( H_{2,i}^\alpha(u) = 0 \), \( 1 \leq i \leq N \), \( 1 \leq \alpha \leq L_2 \), then the system (3.1)–(3.5) describes all compatible local Poisson brackets of hydrodynamic type (compatible Dubrovin–Novikov brackets or flat pencils of metrics). This system was derived and integrated by the method of inverse scattering problem in [11], [12] with using the Zakharov method of differential reductions [13]. The Lax pair for this system was demonstrated by Ferapontov in [14] (here \( \varepsilon_i^2 = 1 \), \( 1 \leq i \leq N \)):

\[
\frac{\partial \varphi_i}{\partial u^j} = \frac{\lambda + f_i}{\lambda + f_j} \beta_{ij} \varphi_j, \quad i \neq j, \tag{4.10}
\]

\[
\frac{\partial \varphi_i}{\partial u^i} = -\sum_{k \neq i} \beta_{ki} \varphi_k. \tag{4.11}
\]

The condition of consistency for the linear system (4.10), (4.11) defines the Lamé equations.

The Lax pair (4.12), (4.13) can be easily derived from the classical linear problem (4.10), (4.11) for the Lamé equations (see [17], [18]).

The Lax pair (4.8), (4.9) is generalized also to the case of arbitrary nonsingular pencils of metrics of constant Riemannian curvature (see examples in [14]). This Lax pair can be also easily derived from the corresponding linear problem for the system describing all the orthogonal curvilinear coordinate systems in \( N \)-dimensional spaces of constant curvature \( K_2 \):

\[
\frac{\partial \varphi_i}{\partial u} = \frac{\sqrt{\varepsilon_i^2}}{\sqrt{\varepsilon_j^2}} \beta_{ij} \varphi_j, \quad i \neq j, \tag{4.12}
\]
\[
\frac{\partial \phi_i}{\partial u^i} = \sum_{k \neq i} \frac{\sqrt{\varepsilon^k}}{\sqrt{\varepsilon^i}} \beta_{ki} \varphi_k + \frac{\sqrt{K_2}}{\sqrt{\varepsilon^i}} H_i \psi, \quad (4.13)
\]
\[
\frac{\partial \psi}{\partial u^i} = -\frac{\sqrt{K_2}}{\sqrt{\varepsilon^i}} H_i \varphi_i. \quad (4.14)
\]

The condition of consistency for the linear system (4.12)–(4.14) gives the equations for all orthogonal curvilinear coordinate systems in \(N\)-dimensional spaces of constant Riemannian curvature \(K_2\).

The corresponding Lax pair with a spectral parameter for nonsingular pencils of metrics of constant Riemannian curvature has the form (see [17], [18]):

\[
\frac{\partial \phi_i}{\partial u^i} = \sum_{k \neq i} \frac{\sqrt{\varepsilon^k}}{\sqrt{\varepsilon^i}} \beta_{ki} \varphi_k + \frac{\sqrt{K_2 + K_1}}{\sqrt{\varepsilon^i}} \beta_{i1} \varphi_1, \quad (4.15)
\]
\[
\frac{\partial \phi_i}{\partial u^i} = -\sum_{k \neq i} \frac{\sqrt{\varepsilon^k}}{\sqrt{\varepsilon^i}} \beta_{ki} \varphi_k + \frac{\sqrt{K_2 + K_1}}{\sqrt{\varepsilon^i}} \beta_{i1} \varphi_1, \quad (4.16)
\]
\[
\frac{\partial \psi}{\partial u^i} = \frac{\sqrt{K_2 + K_1}}{\sqrt{\varepsilon^i}} H_i \varphi_i. \quad (4.17)
\]

where \(\lambda\) is a spectral parameter. The condition of consistency for the linear system (4.13)–(4.17) is equivalent to the equations for nonsingular pencils of metrics of constant Riemannian curvature.

If \(H_{\alpha_2,i}(u) = 0, 1 \leq i \leq N, 1 \leq \alpha \leq L_1\), then the corresponding integrable systems (3.1)–(3.5) describe compatible pairs of Poisson brackets of hydrodynamic type one of which is local. These systems always give integrable reductions of the classical Lamé equations. The corresponding Lax pairs with a spectral parameter have the form:

\[
\frac{\partial \phi_i}{\partial u^k} = \frac{\sqrt{\varepsilon^k}}{\sqrt{\varepsilon^2}} \beta_{ik} \varphi_k, \quad i \neq k, \quad (4.18)
\]
\[
\frac{\partial \phi_i}{\partial u^i} = -\sum_{k \neq i} \frac{\sqrt{\varepsilon^k}}{\sqrt{\varepsilon^2}} \beta_{ki} \varphi_k + \sum_{\beta=1}^{L_1} \frac{\sqrt{\varepsilon_1,\beta}}{\sqrt{\varepsilon^2}} H^{\beta}_{1,i} \lambda^\beta, \quad (4.19)
\]
\[
\frac{\partial \lambda^\beta}{\partial u^i} = -\frac{\sqrt{\varepsilon_1,\beta}}{\sqrt{\varepsilon^2}} H^{\beta}_{1,i} \varphi_i. \quad (4.20)
\]

Similarly, we can construct the Lax pair with a spectral parameter for any special partial case of the equations for compatible nonlocal Poisson brackets of hydrodynamic type, for example, we get integrable reductions of the equations for orthogonal curvilinear coordinate systems in spaces of constant Riemannian curvature \(K\) (in this case \(H_{\alpha_2,i}(u) = \sqrt{K} H_{2,i}(u)\), \(L_2 = 1, \alpha = 1\)).
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