SEMIGROUP-CONTROLLED ASYMPTOTIC DIMENSION

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ABSTRACT. We introduce the idea of semigroup-controlled asymptotic dimension. This notion generalizes the asymptotic dimension and the asymptotic Assouad-Nagata dimension in the large scale. There are also semigroup controlled dimensions for the small scale and the global scale. Many basic properties of the asymptotic dimension theory are satisfied by a semigroup-controlled asymptotic dimension. We study how these new dimensions could help in the understanding of coarse embeddings and uniform embeddings. In particular we have introduced uncountable many invariants under quasi-isometries and uncountable many bi-Lipschitz invariants. Hurewicz type theorems are generalized and some applications to geometric group theory are shown.

CONTENTS

1. Introduction and preliminaries 1
2. Control semigroups 3
3. Semigroup-controlled dimensions: basic properties 5
4. Large scale and small scale dimensions 7
5. Non equivalent semigroup-controlled dimensions 10
6. Maps between metric spaces and dimension 14
7. Hurewicz type theorems 17
References 19

1. INTRODUCTION AND PRELIMINARIES

The asymptotic dimension was introduced by Gromov in [8] as one important coarse invariant in the study of geometric group theory. The idea behind its definition is to analyze a metric space as a large scale object.

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An analogous concept to the asymptotic dimension but for small scale objects would be the uniform dimension introduced by Isbell in [10]. These two dimensions suggest the idea of seeing a metric space as a global object and then we would get a global definition of dimension. In [4] (see also [12]) it was given such definition and it was analyzed some properties of zero dimensional spaces.

The following notions associated to a cover \( U = \{U_s\}_{s \in S} \) of a metric space \((X, d_X)\) are standard concepts in the theory of metric spaces. They are used in many equivalent definitions of asymptotic dimension.

Let \( U = \{U_s\}_{s \in S} \) be a cover of a metric space \((X, d_X)\), not necessarily open. Associated to this cover there is a family a natural family of functions \( \{f_s\}_{s \in S} \) with \( f_s : X \rightarrow \mathbb{R}^+ \) defined by:

\[
f_s(x) := d_X(x, X \setminus U_s).
\]

With such functions we can define:

- **Local Lebesgue number** \( L_x(U) \) of \( U \) at \( x \in X \): \( L_x(U) := \sup \{f_s(x) | S \in S\} \).
- **Global Lebesgue number** \( L(U) := \inf \{L_x(U) | x \in X\} \).
- **Local s-multiplicity** \( s - m(U) \) of \( U \) at \( x \in X \) is defined as the number of elements of \( U \) that intersect \( B(x, s) \).
- **Global s-multiplicity** \( s - m(U) := \sup \{s - m(U) | x \in X\} \). If \( s = 0 \) then the 0-multiplicity will be called multiplicity of \( U \) and it will be noted by \( m(U) \).

Given a family of subsets \( U \) of a metric space \((X, d_X)\) it is said that \( U \) is \( C \)-bounded with \( C > 0 \) if \( \text{diam}(U) \leq C \) for every \( U \in U \). If \( U \) is \( C \)-bounded for some \( C > 0 \) it is said that \( U \) is uniformly bounded and if \( d_X(U, V) > s \) for every \( U, V \in U \), it is said also that \( U \) is \( s \)-disjoint with \( s > 0 \).

A definition of asymptotic dimension could be the following:

**Definition 1.1.** We will say that a metric space \((X, d_X)\) has asymptotic dimension at most \( n \) (notation: \( \text{asdim}X \leq n \)) if there is an \( s_0 \) such that for every \( s \geq s_0 \) there exists an uniformly cover(colored cover) \( U = \bigcup_{i=1}^{n+1} U_i \) so that each \( U_i \) is \( s \)-disjoint.

From this definition we can deduce that there exist a function \( f : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) with \( \lim_{x \rightarrow \infty} f(x) = \infty \) such that each \( s \)-disjoint \( n + 1 \) colored cover is \( f(s) \)-bounded. If we restricted the range of functions \( f \) allowed to the linear ones(or asymptotically linear ones) we would get the notion of asymptotic Assouad-Nagata dimension, also called linear controlled asymptotic dimension or asymptotic dimension with Higson property. Many results have appeared recently associated to this dimension, see [3], [4], [6], [7] and [9], for example. Here is the definition:

**Definition 1.2.** We will say that a metric space \((X, d_X)\) has asymptotic Assouad-Nagata dimension at most \( n \) (notation: \( \text{asdim}X \leq n \)) if there is a \( s_0 > 0 \) and a \( C > 0 \) such that for every \( s \geq s_0 \) there exist a \( C \)-\( s \)-bounded cover(colored cover) \( U = \bigcup_{i=1}^{n+1} U_i \) so that each \( U_i \) is \( s \)-disjoint.
If we just change the condition of $s \geq s_0$ by $s \leq s_0$ we will get the notion of capacity dimension introduced by Buyalo in [5]. The asymptotic Assouad-Nagata dimension and the capacity dimension are just the large scale and the small scale versions of the Assouad-Nagata dimension introduced by Assouad in [1]. The Assouad-Nagata dimension is a bi-Lipschitz invariant. Lang and Schlichenmaier proved in [11] that the Assouad-Nagata dimension is in fact a quasi-isometry invariant. Many other interesting properties of the Assouad-Nagata dimension appeared in that work.

The relations among the capacity dimension, the asymptotic Assouad-Nagata dimension and the Assouad-Nagata dimension were showed in [3].

In this paper we generalize all these notions using the concept of semigroup-controlled asymptotic dimension, semigroup-controlled small scale dimension and semigroup-controlled global dimension. The idea is to modify the range of functions allowed to control the size of the colored covers in definition 1.2. These functions will have a semigroup structure. In section 2 we will study some basic properties of this kind of semigroups. In the next section we will define the notion of semigroup-controlled dimension and we will show that many basic properties of asymptotic dimension (see [2]) are satisfied for these new dimensions. Section 4 is dedicated to relate the large scale, small scale and global dimensions following the same ideas of [3]. In section 5 we will prove that this generalization is not trivial i.e. we have introduced uncountable many different dimensions. The last two sections are dedicated to important topics. In section 6 we will study the types of maps between metric spaces for which these new dimensions are invariant. In particular we will prove that all these dimensions are quasi-isometric invariants in the large scale theory or bi-Lipschitz invariants in the global case. Last section generalizes the results of [9] about the Hurewicz type theorem. Some applications to geometric group theory are obtained as corollaries.

2. Control semigroups

In this work we will consider properties $\mathcal{P}(s)$ depending on positive real numbers $s \in \mathbb{R}^+$. We will say that $\mathcal{P}(s)$ is satisfied in a neighborhood of $\infty$ (respectively in a neighborhood of 0) if there is a $s_0$ such that $\mathcal{P}(s)$ is true for every $s \geq s_0$ (resp. $s \leq s_0$).

Properties that are satisfied in a neighborhood of $\infty$ will be called large scale properties or asymptotic properties, properties that are satisfied in a neighborhood of 0 will be called small scale properties. If a property $\mathcal{P}(s)$ is satisfied in a neighborhood of $\infty$ and in a neighborhood of 0 we will say that $\mathcal{P}(s)$ is a global property.

In this work we will usually give proofs and statements for the large scale case. The small scale case and the global case can be usually done using a dual reasoning and in many cases they will be left to the reader.

Next definition describes the type of functions that are going to control the dimension.
Definition 2.1. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing continuous function

a. We will say that it is a large scale (or asymptotic) dim-control function if $f(x) \geq x$ in a neighborhood of $\infty$ and $f(\infty) = \infty$ i.e. $\lim_{x \to \infty} f(x) = \infty$.

b. If we require that $f(x) \geq x$ in a neighborhood of 0 and $f(0) = 0$ we will say that $f$ is a small scale dim-control function.

c. If a function is a small scale dim-control function and a large scale dim-control function then we will say that it is a global dim-control function.

If a large scale dim-control function $f$ is equal to some linear function in a neighborhood of $\infty$ we will say that it is a large scale (or asymptotically) linear dim-control function. Analogously it can be defined the notion of small scale linear dim-control function and global linear dim-control function.

Now we define the notion of large scale (or asymptotic) control semigroup. This notion is the main concept of this paper.

Definition 2.2. Let $\mathcal{S}$ be a set of asymptotic dim-control functions. We will say that $\mathcal{S}$ is a large scale (or asymptotic) control semigroup if the following properties are satisfied:

(1) Every asymptotically linear dim-control function is in $\mathcal{S}$ (linear condition).

(2) For every pair of functions $g_1, g_2 \in \mathcal{S}$ we have $g_1 \circ g_2 \in \mathcal{S}$ (semigroup condition).

We define the notion of small scale and global control semigroup for a set of small scale (resp. global) dim-control function analogously. We will note it by $\xi$ (resp. $\bar{S}$)

Remark 2.3. In this work we will refer to asymptotic control semigroup just as control semigroups unless it were necessary to remark the large scale condition.

Here are the main examples of control semigroups.

Example 2.4. The set of all asymptotic dim-control functions is a control semigroup. It will be called the uniform control semigroup and we will note it by $U$ (u and $\bar{U}$ for the small scale and the global case respectively)

The set of all asymptotically linear dim-control functions is a control semigroup. It will be called the Nagata (or the linear) control semigroup and it will be noted by $N$ (n and $\bar{N}$ for the small scale and large scale case respectively).

Remark 2.5. Let $K$ be a set of asymptotic dim-control functions. From the theory of semigroups we know that such set generates a control semigroup $L(K)$ i.e. the intersection of all control semigroups $\mathcal{S}$ so that $K \subset \mathcal{S}$. It will be called the control semigroup generated by $K$. Note that $\mathcal{N} = L(\{x\})$. We have also obtained many examples of control semigroups using this procedure.
In the set of all control semigroups we can define a partial order.

**Definition 2.6.** Given two control semigroups $S_1$ and $S_2$, we will say that $S_2$ is finer than $S_1$ (notation: $S_1 \preceq S_2$) if for every asymptotic dim-control function $f \in S_2$ there is a dim-control function $g \in S_1$ such that $f \leq g$ in a neighborhood of $\infty$. If $S_1 \preceq S_2$ and $S_2 \preceq S_1$ we will say that both control semigroups are equivalent and we will note it by $S_2 \approx S_1$.

The following result is easy to check and it is left as an exercise.

**Proposition 2.7.** Let $S$ be a control semigroup then:

1. $U \preceq S \preceq N$.
2. $L(K \cup S) \preceq S$ with $K$ is any set of asymptotic dim-control functions.

**Remark 2.8.** As a consequence of the second statement in the previous proposition we obtain that $L(S_2 \cup S_1) \preceq S_2$ and $L(S_2 \cup S_1) \preceq S_1$ for every pair of control semigroups. Then the set of all control semigroups with the relation $\preceq$ is a directed set.

Now we show an example of two different equivalent control semigroups. In the following section we could see how this example allows us to give another definition of asymptotic Assouad-Nagata dimension:

**Example 2.9.** Let $\{C_i\}_{i=1}^n$ be a finite set of constants with $C_i \geq 1$. For each finite set of those constants we can define an asymptotic dim-control function $f_{\{C_i\}}$ as a continuous piecewise linear function built with linear functions of slope $C_i$ such that $f_{\{C_i\}}(\infty) = \infty$. It can be easily checked that the set of all functions of the form $f_{\{C_i\}}$ is a control semigroup. We will note it by $\mathcal{PL}$. Clearly we have $\mathcal{PL} \preceq \mathcal{N}$. Take a $f_{\{C_i\}} \in \mathcal{PL}$. Let $C$ be the maximum of the $C_i$ then the linear function $f(x) = Cx$ satisfies $f(x) \geq f_{\{C_i\}}(x)$ for all $x$ in a neighborhood of $\infty$. So we have proved that $\mathcal{N} \preceq \mathcal{L}$ and both semigroups are equivalent.

The following is an easy consequence of the semigroup theory and it will be used later.

**Proposition 2.10.** Let $K$ be a set of dim-control functions. If $g$ is a dim-control function $g$ of $L(K)$ then there exist a finite sequence of dim-control functions $\{f_i\}_{i=1}^n$ such that:

$$g = f_1 \circ f_2 \circ ... \circ f_n$$

where each $f_i$ belongs to $\mathcal{N} \cup K$.

3. **Semigroup-controlled dimensions: basic properties**

Now, using the notion of control semigroup we can give the definition of semigroup-controlled asymptotic dimension. It generalizes the notions of Assouad-Nagata asymptotic dimension and asymptotic dimension.

**Definition 3.1.** Let $S$ be a control semigroup. We will say that a metric space $(X, d)$ has $S$-controlled asymptotic dimension at most $n$ (notation:
asdim\(_S\)X ≤ n) if there is an \(f \in S\) such that for every \(s\) in some neighborhood of ∞ there exist a cover (colored cover) \(U = \bigcup_{i=1}^{n+1} U_i\) so that each \(U_i\) is \(s\)-disjoint and \(f(s)\)-bounded.

A metric space is said to have \(S\)-controlled asymptotic dimension \(n\) if it has \(S\)-controlled asymptotic dimension at most \(n\) and for every \(k < n\) it does not happen that \(\text{asdim}_S\)X ≤ \(k\).

**Remark 3.2.**

a. The function \(f \in S\) for which \(\text{asdim}_S\)X ≤ \(n\) will be called \((n, S)\)-dimensional control function of \(X\). Such notion will become very important in the last section.

b. The small scale \(ξ\)-controlled dimension will be noted by \(\text{smdim}_ξ\)X.

c. Special remark is needed for the definition of global dimension. The strictly analogous definition to the Assouad-Nagata dimension would be the following:

\((X, d)\) has \(S\)-controlled global dimension at most \(n\) (notation: \(\text{dim}_S\)X ≤ \(n\)) if there is a \(f \in S\) such that for every \(s\) there is a cover (colored cover) \(U = \bigcup_{i=1}^{n+1} U_i\) so that each \(U_i\) is \(s\)-disjoint and \(f(s)\)-bounded.

We will say that two control semigroups \(S_2\) and \(S_1\) are \(\text{dim-equivalent}\) (notation: \(\text{asdim}_{S_2} \equiv \text{asdim}_{S_1}\)) if for every metric space \((X, d)\) we have \(\text{asdim}_{S_2}\)X = \(\text{asdim}_{S_1}\)X.

Next proposition and corollaries justify our definition of \(\leq\).

**Proposition 3.3.** Let \(S_1\) and \(S_2\) be two control semigroups. If \(S_1 \leq S_2\) then for every metric space \(X\) we have \(\text{asdim}_{S_1}\)X ≤ \(\text{asdim}_{S_2}\)X.

**Proof.** Given \(f \in S_2\) take \(g \in S_1\) such that \(g(s) \geq f(s)\) in a neighborhood of ∞, then any \(f(s)\)-bounded family \(U\) of subsets of \(X\) is \(g(s)\)-bounded with \(s\) in a neighborhood of ∞.

□

**Corollary 3.4.** Let \(S\) be a control semigroup. For every metric space \(X\) we have \(\text{asdim}_U\)X ≤ \(\text{asdim}_S\)X ≤ \(\text{asdim}_N\)X.

**Corollary 3.5.** If two control semigroups are equivalent then they are \(\text{dim-equivalent}\), i.e. they define the same asymptotic dimension.

**Example 3.6.** Note that \(\text{dim}_U\)X and \(\text{asdim}_U\)X are the uniform dimension defined in [4] and the asymptotic dimension of [8] respectively. We have also that \(\text{asdim}_N\)X, \(\text{smdim}_ξ\)X and \(\text{dim}_S\)X are the asymptotic dimension with Higson property (see for example [13]), the capacity dimension (see [5]) and the Assouad Nagata dimension (introduced in [1]) respectively.

Note that applying 3.5 to the example 2.9 we get another way of defining the Assouad-Nagata asymptotic dimension.

The following proposition covers some basic properties of a semigroup-controlled asymptotic dimension. The proofs are highly similar to that ones for Assouad Nagata dimension and asymptotic Assouad Nagata dimension.
We quote between brackets the works where the analogous proofs can be founded.

**Proposition 3.7.** Let \((X,d_X)\) and \((Y,d_Y)\) be metric spaces and \(\mathcal{S}\) a control semigroup. Then it is satisfied:

1. \(\text{asdim}_\mathcal{S} A \leq \text{asdim}_\mathcal{S} X\) for every \(A \subset X\). (This is trivial)
2. \(\text{asdim}_\mathcal{S} X \times Y \leq \text{asdim}_\mathcal{S} X + \text{asdim}_\mathcal{S} Y\) ([9] or [11])
3. If \(X = A \cup B\) then \(\text{asdim}_\mathcal{S} X = \max\{\text{asdim}_\mathcal{S} A, \text{asdim}_\mathcal{S} B\}\) ([9] or [11]).
4. The following conditions are equivalent: ([2])
   a. \(\text{asdim}_\mathcal{S} X \leq n\).
   b. There is a \(f \in \mathcal{S}\) such that for every \(s\) in some neighborhood of \(\infty\) there exist a cover \(U\) with \(s - m(U) \leq n + 1\) and \(f(s)\)-bounded.
   c. There is a \(g \in \mathcal{S}\) such that for every \(s\) in some neighborhood of \(\infty\) there exist a cover \(U\) with \(m(U) \leq n + 1\), \(L(U) \geq s\) and \(g(s)\)-bounded.
   d. There is a \(h \in \mathcal{S}\) such that for every \(\epsilon\) in some neighborhood of 0 there is a map \(\epsilon\text{-Lipschitz} p : X \to K^n\) with \(K^n\) a \(n\)-dimensional simplicial complex such that the family \(p^{-1}(st_v)\) is \(h(1/\epsilon)\)-bounded.

**Remark 3.8.** Note that in the proof of the third property it is used the semigroup condition. For the second and fourth properties it is necessary that given \(g_1, g_2 \in \mathcal{S}\) there exists a \(g_3 \in \mathcal{S}\) such that \(g_1 + g_2 \leq g_3\) in a neighborhood of \(\infty\). Using the semigroup condition and the linear condition define \(g_3 = 2 \cdot g_1 \circ g_2\) if \(g_1(x) > x\) and \(g_2(x) > x\) in a neighborhood of \(\infty\).

4. **Large scale and small scale dimensions**

The aim of this section is to study how the large scale, small scale and the global dimensions are related. Many of these results are based on [3].

Given a global control semigroup we can see it just as a large scale(or small scale) control semigroup. This is the idea behind next definition.

**Definition 4.1.** Let \(\tilde{\mathcal{S}}\) be a global control semigroup we will define the large scale (resp. small scale) truncated semigroup of \(\tilde{\mathcal{S}}\) as the semigroup of all functions \(g\) for which there exist a \(\dim\)-control function \(f \in \tilde{\mathcal{S}}\) with \(g(x) = f(x)\) in a neighborhood of \(\infty\) (resp. in a neighborhood of 0). We will note it by: \(\text{Trunc}_\ast(\tilde{\mathcal{S}})\) (resp. \(\text{Trunc}_{\ast\ast}(\tilde{\mathcal{S}})\)).

Now we present some kind of inverse operation of truncation. Given a small scale control semigroup \(\xi\) and a large scale control semigroup \(\mathcal{S}\) we want to create a global control semigroup \(\tilde{\mathcal{S}}\) associated to those ones.

**Definition 4.2.** We define the linked set of \(\xi\) and \(\mathcal{S}\) as the set of all continuous increasing functions \(g\) for which there exist a small scale \(\dim\)-control function \(g_1\) with \(g_1 \in \xi\) and a large scale \(\dim\)-control function \(g_2\) with \(g_2 \in \mathcal{S}\) such that \(g(x) = g_1(x)\) in a neighborhood of 0 and \(g(x) = g_2(x)\) in a neighborhood of \(\infty\). It will be noted by \(\text{Link}(\xi, \mathcal{S})\).
Clearly $\text{Trunc}^{\ast\ast}(S)$ and $\text{Trunc}_{\ast\ast}(S)$ are small scale and large scale control semigroups respectively. Next proposition shows we have the same property for linked sets.

**Proposition 4.3.** Let $\text{Link}(\xi, S)$ be a linked set then it is a global control semigroup.

*Proof.* The linear condition is trivial. Now let $f, g$ be two dim-control functions of $\text{Link}(\xi, S)$ and let $(f_1, f_2), (g_1, g_2)$ its small scale and large scale associated functions. we have that $f(g(x)) = f_1(g_1(x))$ in a neighborhood of 0 and $f(g(x)) = f_2(g_2(x))$ in a neighborhood of $\infty$ then as $f_1 \circ g_1 \in \xi$ and $f_2 \circ g_2 \in S$ the semigroup condition is satisfied. \hfill $\Box$

The relationship between truncation and linking is given in the following result:

**Proposition 4.4.** Let $S$ be a global control semigroup then:

$$\text{Link}(\text{Trunc}_{\ast\ast}(S), \text{Trunc}^{\ast\ast}(S)) \approx S.$$ Conversely $\text{Trunc}^{\ast\ast}(\text{Link}(\xi, S)) \approx S$ and $\text{Trunc}_{\ast\ast}(\text{Link}(\xi, S)) \approx \xi.$$

*Proof.* Let $\mathcal{P}$ be the global control semigroup $\text{Link}(\text{Trunc}_{\ast\ast}(S), \text{Trunc}^{\ast\ast}(S))$. Clearly by $S \subset \mathcal{P}$ we have $\mathcal{P} \leq S$.

Now let $g$ be a dim-control function in $\mathcal{P}$. There exist two dim-control functions $g_1, g_2 \in S$ such that $g(x) = g_1(x)$ in a neighborhood of 0 and $g(x) = g_2(x)$ in a neighborhood of $\infty$. Let $x_1 \leq x_2$ be two positive numbers such that $g(x) = g_1(x)$ if $x \leq x_1$ and $g(x) = g_2(x)$ if $x \geq x_2$. Let $M$ be the maximum of $g$ in $[x_1, x_2]$ and let $m$ be the minimum of $g_1 \circ g_2$ in the same interval. Suppose $m < M$ then there exist a $C > 1$ such that $Cm \geq M$. Define the function $g'(x) = C(g_1(g_2(x)))$. We have $g'(x) \geq g(x)$ for every $x$ and then $S \leq \mathcal{P}$.

The converse is obvious. \hfill $\Box$

Next definition is the key to connect large scale, small scale and global dimensions, see [3].

**Definition 4.5.** Let $S$ be a global control semigroup and let $X$ be any metric space. We will say that $(X, d)$ has $S$-microscopic controlled dimension at most $n$ and we will note by $m - \dim_S X \leq n$ if the metric space $(X, d'_1 = \min(d, 1))$ has $S$-controlled dimension at most $n$. In a similar way we will say that a metric space $(X, d)$ has $S$-macroscopic controlled dimension at most $n$ $(M - \dim_S X \leq n)$ if $\dim_S(X, d''_n) \leq n$ with $d''_n = \max(1, d)$.

**Lemma 4.6.** Let $(X, d)$ be a metric space and let $S$ be a global control semigroup. If for every $c > 0$ we define the metrics $d'_c = \min(c, d)$ and $d''_c = \max(c, d)$ then $m - \dim_S X = \dim_S(X, d'_c)$ and $M - \dim_S X = \dim_S(X, d''_c)$.

The proof of this result is given in [3]. In such proof the authors used that the Assouad-Nagata dimension is invariant under Lipschitz functions. We give another proof without using this fact.
Proof. We just do the proof for the microscopic case. The macroscopic case is similar. Suppose without loss of generality that $c < 1$. Let $s$ be any positive number. If $s \geq c$ then pick the cover $U = X$. Assume that $s < c$ then there is a cover $U$ with $s$-Lebesgue number in $(X, d'_1)$, multiplicity at most $m - \dim S X + 1$ and $f(s)$-bounded for some dim-control function $f \in S$. But as $d'_1(x, y) = d'_c(x, y) = d(x, y)$ if $d(x, y) \leq c$ then such cover satisfy $L(U) \geq s$ in $(X, d'_c)$ and it is $f(s)$-bounded. We have proved $\dim S(X, d'_c) \leq m - \dim S X$. The remaining case is similar. \hfill \Box

**Corollary 4.7.** Let $S_1$ be a global control semigroup. If $X$ is a bounded metric space then $m - \dim S_1 X = \dim S_1 X$. If $X$ is a discrete metric space then $M - \dim S_1 X = \dim S_1 X$.

Next lemma shows how the microscopic dimension of a global control semigroup is greater or equal than the semigroup-controlled asymptotic dimension associated to the large scale truncated semigroup.

**Lemma 4.8.** Let $X$ be a metric space and let $S$ be a global control semigroup then the following properties are equivalent:

1. $M - \dim S X \leq n$
2. There is a function $f \in S$ such that for all $s$ in a neighborhood of $\infty$ there is a colored cover $U = \bigcup_{i=1}^{n+1} U_i$ with each $U_i$ $s$-disjoint and $f(s)$-bounded.

The proof is almost equal to the proof of Lemma 2.7. of [3]. It will be left to the reader.

Using the same reasoning we can get the analogous Lemma for the microscopic case:

**Lemma 4.9.** Let $X$ be a metric space and $\tilde{S}$ a global control semigroup then the following properties are equivalent:

1. $m - \dim \tilde{S} X \leq n$
2. There is a function $f \in \tilde{S}$ such that for all $s$ in a neighborhood of $0$ there is a colored cover $U = \bigcup_{i=1}^{n+1} U_i$ with each $U_i$ $s$-disjoint and $f(s)$-bounded.

Next lemma could be considered some kind of converse of the previous ones.

**Lemma 4.10.** Let $S$ be a global control semigroup. For every metric space $X$ we have $\dim S X \leq n$ if and only if $m - \dim S X \leq n$ and $M - \dim S X \leq n$.

Proof. The necessary condition is obvious by lemma 4.8. Let us prove the sufficient condition. Suppose $m - \dim S X \leq n$ and $M - \dim S X \leq n$. Without loss of generality we can assume that the $f$ associated to the bounds of the microscopic covers and the macroscopic covers is the same, if not take the composition. Let $s$ be a positive real number. We want to find a dim-control function $g \in S$ and a colored covering $U$ $s$-disjoint and $g(s)$-bounded. It is clear that if $s > 1$ or $f(s) < 1$ the result is obvious. Assume that $s \leq 1$
and $f(s) \geq 1$. Pick $s_0 = f^{-1}(1)$ and define the function $g(x) = f(2f(x))$. So if $s_0 \leq s \leq 1$ take a colored covering $U$ of $(X, d''_X)$ so that it is $2f(s)$-disjoint and $f(2(f(s)))$-bounded.

Combining all the results of this section we get that the global dimension can be obtained just studying the dimension in a neighborhood of 0 and in a neighborhood of $\infty$.

**Theorem 4.11.** Let $\hat{S}$ be a global control semigroup. For every metric space $X$ we have that $smdim_{\text{Trunc}^\ast}(\hat{S})X = m - dim_{\hat{S}}X$ and $asdim_{\text{Trunc}^{\ast\ast}}(\hat{S})X = M - dim_{\hat{S}}X$.

In the other hand given $\xi$ a small scale control semigroup and $S$ a large scale control semigroup then:

$$\dim_{\text{Link}}(\xi, S)X = \max\{smdim_\xi X, asdim_\xi S\}. $$

**Proof.** By the two previous lemmas we get $M - dim_\hat{S}X \geq asdim_{\text{Trunc}^{\ast\ast}}(\hat{S})X$. Suppose $asdim_{\text{Trunc}^{\ast\ast}}(\hat{S})X \leq n$. Then for every $s \geq s_0$ there is a cover $U^s = \bigcup_{i=1}^{n+1} U^s_i$ that is $s$-disjoint and $f(s)$-bounded with $f \in \text{Trunc}^{\ast\ast}(\hat{S})$. That means that there is a $f_1 \in \hat{S}$ such that $f(x) = f_1(x)$ if $x \geq x_1$ for some $x_1$. Let $s'_0$ be the maximum of $x_1$ and $s_0$. We have that for every $s \geq s'_0$ there is a cover $U^s = \bigcup_{i=1}^{n+1} U^s_i$ $s$-disjoint and $f_1(s)$-bounded. Applying 4.8 we get the result. The microscopic case is analogous.

For the second statement just note:

$$\dim_{\text{Link}}(\xi, S)X = \max\{m - \dim_{\text{Link}}(\xi, S)X, M - \dim_{\text{Link}}(\xi, S)X\} = \max\{smdim_{\text{Trunc}^\ast}(\text{Link}(\xi, S))X, asdim_{\text{Trunc}^{\ast\ast}}(\text{Link}(\xi, S))X\}$$

So the result follows from the second statement of 4.4 and 3.3.

5. **Non equivalent semigroup-controlled dimensions**

Let $\overline{ASDIM}$ (respectively $\overline{SMDIM}$) be the quotient set of all large scale (resp. small scale) control semigroups with the equivalence relation $\equiv$ defined in section 3. In this section we will estimate the cardinality of $\overline{ASDIM}$ and $\overline{SMDIM}$.

Let $\Omega$ be the set of all the countable ordinals($\Omega_0$) union the first uncountable ordinal. This set is uncountable and it has a natural well order. We will prove that there exist two order preserving maps $i_L : \Omega \rightarrow \overline{ASDIM}$ and $i_S : \Omega \rightarrow \overline{SMDIM}$. As usual we will prove the results for the large scale case. The small scale case will be left to the reader.

As a first step we will estimate the cardinality of the set of all large scale (resp. small scale) control semigroups modulo the equivalence relation $\sim$. They will be noted by $\overline{ASDIM}$ and $\overline{SMDIM}$.

The next lemmas are necessary to show that there exist at least countable many semigroups.
Lemma 5.1. Let \( g : [a_i, a_{i+1}] \to [g(a_i), g(a_{i+1})] \) be an increasing continuous function defined in some interval and let \( f(a_i), f(a_{i+1}) \) be any pair of points that satisfies \( f(a_i) \geq g(a_i), f(a_{i+1}) \geq g(a_{i+1}) \) then there is a continuous increasing function \( f \) defined in the same interval such that \( f \geq g \) with \( f(a_i) = f(a_i) \) and \( f(a_{i+1}) = f(a_{i+1}) \).

Proof. Define the function \( f_{aux}(x) = g(x) + f(a_i) - g(a_i) \). This is an increasing continuous function in the interval. If \( f_{aux}(a_{i+1}) = g(a_{i+1}) \) then \( f_{aux} = f \). Otherwise let \( z \) be the greatest point in \([a_i, a_{i+1}]\) such that the segment \( (a_{i+1}, f_{aux}(a_{i+1})) \) intersects the graph of \( g \). Such point exists by continuity. Then define:

\[
\tilde{f}(x) = \begin{cases} 
  f_{aux}(x) & \text{if } x \in [a_i, z] \\
  \frac{f(a_{i+1}) - f_{aux}(a_{i+1})}{a_{i+1} - z}(x - z) + f_{aux}(z) & \text{if } x \in (z, a_{i+1}]
\end{cases}
\]

It is clear that this function satisfies the requirements of the lemma. \( \square \)

Lemma 5.2. Let \( f \) be a large scale dim-control function such that it is strictly bigger than the identity in a neighborhood of \( \infty \). Then there exist a large scale dim-control function \( g \) such that for every \( n \in \mathbb{N} \) \( g(x) > f^n(x) \) in some neighborhood of \( \infty \).

Proof. As the function is strictly bigger than the identity we have that there is an increasing sequence \( x_n \to \infty \) such that if \( x \geq x_n \) then \( f^n(x) > f^i(x) \) for all \( i \leq n - 1 \). Using 5.1 define in each interval \([x_n, x_{n+1}]\) a function \( g_n \) so that \( g_n(x_n) = f^n(x_n), g_n(x_{n+1}) = f^{n+1}(x_{n+1}) \) and \( g_n(x) \geq f^n(x) \). Paste all these functions and we get \( g \) the required function.

Using a similar reasoning we can give the following lemma:

Lemma 5.3. Let \( f \) be a small scale dim-control function such that it is strictly bigger than the identity in a neighborhood of \( 0 \). Then there exist a small scale dim-control function \( g \) such that for every \( n \in \mathbb{N} \) \( g(x) > f^n(x) \) in some neighborhood of \( 0 \).

Next lemma shows that there are at least countable many non equivalent large scale control semigroups.

Lemma 5.4. There exist a sequence \( \{S_i\}_{i \in \mathbb{N}} \) of large scale control semigroups with \( S_i \prec S_{i-1} \) and \( S_i = L(\{f_i\} \cup S_{i-1}) \) where \( f_i \) is an asymptotic dim-control function such that for every \( g \in S_{i-1} \) \( f_i(x) > g(x) \) in a neighborhood of \( \infty \).

Proof. Take \( S_1 = \mathcal{N} \). Let \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) be the dim-control function defined by:

\[
f(x) = \begin{cases} 
  x^2 & \text{if } x \in [1, \infty) \\
  x & \text{otherwise}
\end{cases}
\]

It is clear that for every asymptotically linear dim-control function \( g \) there is a point \( x_0 \in \mathbb{R}_+ \) such that \( f(x) > g(x) \) if \( x \geq x_0 \) then we have \( S_2 = L(\{f\} \cup \mathcal{N}) \prec \mathcal{N} = S_\infty \).
Suppose we have constructed a sequence of control semigroups with $\mathcal{S}_n \prec ... \prec \mathcal{S}_1$ so that for each control semigroup $\mathcal{S}_i$ there is an asymptotic dim-control function $f_i$ that $\mathcal{S}_i = L(\{f_i\} \cup \mathcal{S}_{i-1})$ and for every dim-control function $g$ of $\mathcal{S}_{i-1}$ there exists a $x_0 \in \mathbb{R}_+$ such that $f_i(x) > g(x)$ if $x > x_0$. Now apply lemma 5.2 to $f_n$ in order to get a dim-control function $f_{n+1}$ so that for every $j \in \mathbb{N}$, $f_{n+1}(x) > f_j(x)$ if $x > x_0$ for some $x_0$. We claim that $f_{n+1}$ satisfies the same property for all $g \in \mathcal{S}_n$. Let $g$ be a dim-control function in $\mathcal{S}_n$. Using 2.10 we have that:

$$g = f_1 \circ f_2 \circ ... \circ f_p$$

For some functions $f_i$ in $\{f_n\} \cup \mathcal{S}_{n-1}$. For every $f_i$ there exist an $x_i$ and a $j_i$ such that $f_n^{j_i}(x) > f_i(x)$ if $x \geq x_i$. Let $x'_0$ be the maximum of all $x_i$ then we have that $g(x) < f_n^{j_i}(x)$ for some $j$ if $x \geq x'_0$ and by the method we have built $f_{n+1}$ we have $f_{n+1}(x) > g(x)$ in a neighborhood of $\infty$.

Note that for getting $\mathcal{S}_2$ we just need an asymptotic dim-control function $f$ that were strictly greater than any asymptotically linear dim-control function in a neighborhood of $\infty$.

Doing a dual reasoning we can get:

Lemma 5.5. There exist a sequence $\{\xi_i\}_{i \in \mathbb{N}}$ small scale control semigroups with $\xi_i \prec \xi_{i-1}$ and $\xi_i = L(\{f_i\} \cup \xi_{i-1})$ where $f_i$ is a dim-control function such that for every $g \in \xi_{i-1}$ $f_i(x) > g(x)$ in a neighborhood of 0.

We have proved that the sets $\text{SMDIM}$ and $\text{ASDIM}$ are at least countable. The lemmas of above suggest the following definition:

Definition 5.6. We will say that a large scale (small scale) control semigroup $\mathcal{S}$ (resp. $\xi$) is mono-bounded at $\infty$ (respectively at 0) if there exist an asymptotic dim-control function $f$ such that for every $g \in \mathcal{S}$ (resp $g \in \xi$) we have $f(x) > g(x)$ in a neighborhood of $\infty$ (resp. in a neighborhood of 0). The function $f$ will be called the bound function of $\mathcal{S}$ (of $\xi$) at $\infty$ (resp. at 0).

Using the sequence generated in 5.4 we can build the control semigroup generated by such sequence. Such semigroup will be mono-bounded and then we can begin again a similar process as in 5.4. This is the idea of the next two lemmas.

Lemma 5.7. Let $\{\mathcal{S}_i\}_{i \in \mathbb{N}}$ be a sequence of large scale control semigroups mono-bounded at $\infty$ such that $\mathcal{S}_i \prec \mathcal{S}_{i-1}$ and $\mathcal{S}_i = L(\{f_i\} \cup \mathcal{S}_{i-1})$ where $f_i$ is a bound function of $\mathcal{S}_{i-1}$ then the large scale control semigroup given by $L(\bigcup_{i=1}^{\infty} \mathcal{S}_i)$ is mono-bounded at $\infty$.

Proof. The reasoning is similar to the previous ones.

Firstly we note that $\mathcal{S} := L(\bigcup_{i=1}^{\infty} \mathcal{S}_i) = \bigcup_{i=1}^{\infty} \mathcal{S}_i$.

We will prove that $\mathcal{S}$ is mono-bounded at $\infty$. We have that there is a sequence of points $\{a_i\}_{i \in \mathbb{N}}$ with $a_i \to \infty$ and $a_{i+1} > a_i$ such that $f_{i+1}(x) > ...
function $f(x)$ for all $x \geq a_{i+1}$. Using 5.1 we can build a function $f$ such that $f(a_i) = 2f_i(a_i)$ and $f(x) \geq f_i(x)$ in $[a_i, a_{i+1}]$. Let $g$ be a dim-control function of $S_i$ by hypothesis there is a $x_0$ such that $f_{i+1}(x) > g(x)$ for every $x \geq x_0$ and as we have $f(x) > f_{i+1}(x)$ for every $x \geq a_{i+2}$ then $S$ is mono-bounded at $\infty$ and $f$ is a bound function.

Here is the lemma for the small scale:

**Lemma 5.8.** Let $\{\xi_i\}_{i \in \mathbb{N}}$ be a sequence of small scale control semigroups mono-bounded at 0 such that $\xi_i \prec \xi_{i-1}$ and $\xi_i = L(\{f_i\} \cup \xi_{i-1})$ where $f_i$ is the bound function of $\xi_{i-1}$ then the small scale control semigroup given by $L(\bigcup_{i=1}^{\infty} \xi_i)$ is mono-bounded at 0.

Now as $\bigcup_{i=1}^{\infty} S_i$ is mono-bounded we can apply again 5.4 and generate another strictly decreasing sequence of control semigroups and such semigroups will be mono-bounded. Applying repeatedly the arguments of 5.7 and 5.4 (or 5.8 and 5.5 in the small scale case) we get:

**Corollary 5.9.** There are two injective maps $i_L : \Omega \to ASDIM$, $i_s : \Omega \to SMDIM$ such that $i_L(\beta) \prec i_L(\alpha)$ ($i_s(\beta) \prec i_s(\alpha)$) for every pair of ordinals with $\alpha < \beta$.

**Proof.** By the previous reasoning it is obvious that there exist an injective map $i_0^L : \Omega_0 \to ASDIM \setminus \{U\}$ that it is order preserving. Moreover we have $U \prec i_0^L(\alpha)$ for every $\alpha \in \Omega_0$. Define:

$$i_L(\alpha) = \begin{cases} i_0^L(\alpha) \text{ if } \alpha \in \Omega_0 \\ U \text{ Otherwise} \end{cases}$$

The small scale case is analogous. 

Now we will prove that the control semigroups given by the map of 5.9 are not dim-equivalent.

**Lemma 5.10.** Let $S$ be a mono-bounded control semigroup and let $f$ be a bound function at $\infty$ of $S$. Then there exist a metric space $X$ such that $0 = asdim_{L(f) \cup S} X < asdim_{S} X$.

**Proof.** Clearly $f$ is strictly greater than the identity. Let $\{a_i\}_{i \in \mathbb{N}} \subset \mathbb{R}_+$ be the sequence defined by $a_0 = 0$ and $a_i = f(n_i)$ with $n_0 = 0$ and $n_i$ being the smallest natural number such that $n_i > n_{i-1}$ and $a_{i-1} + n_i < f(n_i)$ for every $i \in \mathbb{N}$. Define the sequence of sets $\{C_i\}_{i=1}^{\infty}$ as:

$$C_i := \{a_i\} \cup \{a_i\} \cup \{a_{i-1} + m \cdot n_i\}_{m \in \mathbb{N}} \cap [a_{i-1}, a_i] \}.$$

Take the set $X := \bigcup_{i=1}^{\infty} C_i$ we claim that this set satisfies the required properties.

Note that $U_0^{n_k} := \bigcup_{i=1}^{k} C_i$ is $n_k$-connected and $diam(U_0^{n_k}) = f(n_k)$. Also note that if $s \in \{n_k, n_{k+1}\}$ then the $s$-connected components of $X \setminus U_0^{n_k}$ have cardinality at most two and $d(X \setminus U_0^{n_k}, U_0^{n_k}) > n_{k+1}$. Hence $asdim_{L(f) \cup S} X \leq 0.$
Now if asdim_§X ≤ 0 with g as (0, §)-dimensional control function take x_0
such that f(x) > g(x) if x ≥ x_0. Then for every n_k ≥ x_0 we have that every
n_k-disjoint cover contains the n_k-connected set U_{n_k}^p but as diam(U_{n_k}^p) =
f(n_k) > g(n_k) we get a contradiction. □

For the small scale the lemma is similar. The proof will follow the steps
of the previous lemma but we will construct X as a convergent sequence to
0 nor to ∞. The details are left to the reader.

Lemma 5.11. Let ξ be a mono-bounded small scale control semigroup and
let f be a bound function at 0 of ξ. Then there exist a metric space X such
that 0 = smdim_L(∪f_1: S) X < smdim_S X.

As a trivial corollary of the previous results we get the following theorem
that gives us an estimation of the cardinalities of ASDIM and SMDIM.

Theorem 5.12. There exist two injective maps:
i_L : Ω → ASDIM
i_s : Ω → SMDIM
such that each i_L(α) (resp. i_s(α)) is mono-bounded at ∞(resp. at 0) if
α ∈ Ω_0. Moreover we have that i_L(β) ≺ i_L(α) (i_s(β) ≺ i_s(α)) for every pair
of ordinals such that α < β.

6. Maps between metric spaces and dimension

The aim of this section is to study when a semigroup-controlled dimension
of a metric space is invariant under modifications of the metric. The type
of modifications allowed is described in the next definition.

Definition 6.1. Consider a set Q of positive real continuous increasing
functions {ρ_λ}.

1. If ρ_λ(∞) = ∞ for every ρ_λ ∈ Q we will say that Q is a large scale
metric perturbation set or just metric perturbation set.
2. If ρ_λ(0) = 0 for every ρ_λ ∈ Q we will say that it is a small scale
metric perturbation set.
3. If Q is a large scale and a small scale metric perturbation set we will
say that it is a global perturbation set.

Using this definition we can define the notion of coarse embedding in the
following way:

Definition 6.2. Let Q be a metric perturbation set. Let X, Y be two metric
spaces and let f : X → Y be a function. We will say that f is a Q-coarse
embedding if there are two functions ρ_− and ρ_+ with ρ_− , ρ_+ ∈ Q such that:
ρ_−(d_X(x, y)) ≤ d_Y(f(x), f(y)) ≤ ρ_+(d_X(x, y)) for every pair of points
x, y ∈ X.
In particular if a function f satisfies the previous inequality we can say that
it is a (ρ_−, ρ_+)-coarse embedding. Functions ρ_− and ρ_+ are called dilatation
and contraction respectively.

Analogous definitions are given for the small scale and global theory. Note that a small scale embedding and a global embedding are always injective, but a coarse embedding is not necessarily injective.

**Remark 6.3.**  
(1) Let $Q$ be the set of all positive continuous increasing functions $\rho_\lambda$ with $\rho_\lambda(\infty) = \infty$, a large scale $Q$-coarse embedding is called a coarse embedding.  
(2) If $Q$ is the set of all linear functions then a $Q$-coarse embedding is called a quasi-isometric embedding. See [8].  
(3) If $Q$ is the set of all positive continuous increasing functions $\rho_\lambda$ with $\rho_\lambda(0) = 0$ then a small scale $Q$-embedding is called uniform embedding [10]. Also it is known as a uniformly continuous embedding.

The following proposition gives a sufficient condition for a semigroups controlled dimension to be invariant under a metric perturbation set.

**Proposition 6.4.** Let $Q$ be a metric perturbation set. Let $S_1, S_2$ be two control semigroups. Suppose that given any dim-control function $f$ with $f \in S_2$ and given any two functions $\rho_-, \rho_+$ with $\rho_-, \rho_+ \in Q$ there exist a dim-control function $g \in S_1$ that satisfies $\rho_- \circ f \circ \rho_+ \leq g$ in a neighborhood of $\infty$. Then if $F : X \to Y$ is a $Q$-embedding between metric spaces $X, Y$ we have that $\text{asdim}_{S_1} X \leq \text{asdim}_{S_2} Y$.

**Proof.** Suppose $\text{asdim}_{S_2} Y \leq n$. Let $t > 0$ be a positive number in a neighborhood of $\infty$. Take a cover $U$ in $Y$ of multiplicity at most $n + 1$, with Lebesgue number at least $\rho_+(t)$ and uniformly bounded by $f(\rho_+(t))$ with $f \in S_2$. Let $V$ be the covering given by $V = F^{-1}(U)$. Note that the multiplicity of $V$ is $n + 1$. Also we have that if $d_X(x, y) \leq t$ then $d_Y(F(x), F(y)) \leq \rho_+(t)$ and the Lebesgue number of $V$ is at least $t$. Now if we have two points $x, y$ that are in the same set $V \in V$ we have:

$$d(x, y) \leq \rho_-^{-1}(d(F(x), F(y)) \leq \rho_+^{-1}(f(\rho_+(t)))$$

By hypothesis there is a $g \in S_1$ such that $\rho_-^{-1} \circ f \circ \rho_+ \leq g$. Hence $V$ is $g(t)$-bounded and we have proved $\text{asdim}_{S_1} X \leq \text{asdim}_{S_2} Y$. □

It is clear that the proposition of above also works for small scale and global embeddings.

Now given a control semigroup $S$ we would like to find a metric perturbation set $Q$ for which the associated semigroup-controlled asymptotic dimension $\text{asdim}_S$ is invariant under any $Q$-embedding, i.e. if $f : X \to Y$ is a $Q$-embedding then $\text{asdim}_S X \leq \text{asdim}_S Y$. This is the idea behind next definition.

**Definition 6.5.** Let $S$ be a control semigroup. We define the metric perturbation set $\Sigma(S)$ generated by $S$ as the set of all continuous increasing functions $\rho_\lambda$ with and $\lim_{x \to \infty} \rho_\lambda(x) = \infty$ for which there exist a dim-control function $g \in S$ such that $g \geq \rho_\lambda$ in a neighborhood of $\infty$. 
Remark 6.6. Note that the semigroup condition allow us to show that if \( f : X \to Y \) is a \( \Sigma(S) \)-coarse embedding and \( g : Y \to Z \) is a \( \Sigma(S) \)-coarse embedding then \( g \circ f : X \to Z \) is a \( \Sigma(S) \)-coarse embedding.

Using 6.4 it is easy to check that \( \text{asdim}_S \) is invariant under a \( \Sigma(S) \)-embedding.

**Corollary 6.7.** Let \( S \) be any control semigroup. Let \( X, Y \) be any two metric spaces. If there exist a \( \Sigma(S) \)-coarse embedding \( f : X \to Y \) then \( \text{asdim}_S X \leq \text{asdim}_S Y \).

**Proof.** Just note that the conditions of 6.4 are satisfied. \( \square \)

Clearly the analogous corollary is true for global scale and small scale dimensions.

Remark 6.8. Using 6.7 and the results of 5.12 we have found uncountable many invariants under quasi-isometries or uncountable many invariants under bi-Lipschitz equivalences for the global case. Note also that if a metric perturbation set \( Q \) is mono-bounded by \( f \) then using a reasoning similar as in 5.12 but taking \( S_1 = L(\{f\}) \) in 5.4 and applying again 6.7 we get that there exist uncountable many \( Q \)-invariants.

For the zero dimensional case we have the following nice theorem. The proof appears in [4].

**Theorem 6.9.** Let \( \bar{S} \) be a global control semigroup. For every metric space \((X,d_X)\) with \( \dim \bar{S} X \leq 0 \) there exist an ultrametric \( d_u \) in \( X \) such that the identity \( \text{id} : (X,d_X) \to (X,d_u) \) is a \( \Sigma(\bar{S}) \)-embedding.

Following the ideas of [4] we have also that every ultrametric space can be embedded via a bi-Lipschitz function in \( L_\omega \). The construction of \( L_\omega \) is the following:

Let \( S \) be a countable set and fix an element \( s_0 \in S \). The set \( L_\omega \) is the subset of all sequences \( \bar{x} = \{x_n\}_{n \in \mathbb{Z}} \) with \( x_n \in S \). We will say that a sequence is in \( L_\omega \) if there exist a \( k \) such that \( x_n = s_0 \) for every \( n < k \). The distance between two elements of \( L_\omega \) is given by \( d(\bar{x}, \bar{y}) = 3^{-m} \) if \( m \in \mathbb{Z} \) is the minimum index such that \( x_m \neq y_m \). We have:

**Corollary 6.10.** Let \( \bar{S} \) be a global control semigroup. For every metric space \((X,d_X)\) with \( \dim \bar{S} X \leq 0 \) there exist a \( \Sigma(\bar{S}) \)-embedding \( f : X \to L_\omega \).

The idea of a \( \Sigma(S) \)-coarse embedding suggests us an analogous definition of a \( \Sigma(S) \)-coarse map or global \( \Sigma(\bar{S}) \)-map. That definitions would be the generalization of coarse maps(or asymptotically Lipschitz maps) and uniform maps. This notion will be very important in the next section.

**Definition 6.11.** Let \( S \) be a control semigroup. We will say that a function \( f \) between metric spaces \( f : X \to Y \) is a \( \Sigma(S) \)-coarse map if there is a \( \rho \in \Sigma(S) \) such that:

\[
d_Y(f(x), f(y)) \leq \rho(d(x, y)) \quad \text{for every} \quad x, y \in X.
\]
7. Hurewicz type theorems

The aim of this section is to generalize the results of [9]. In order to get such generalization we need to modify the notion of an \(m\)-dimensional control function of a function \(f: X \to Y\) between metric spaces. That is is the idea behind the next definition.

**Definition 7.1.** Let \(S_1\) and \(S_2\) be two control semigroups. We will say that a function \(D^{(S_1, S_2)}: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+\) is a \((S_1, S_2)\)-function if for every \(\rho_1 \in S_1\) and for every \(\delta \in S_2\) there is a function \(\rho_2 \in S_1\) such that \(D^{(S_1, S_2)}(\rho_1(x), \delta(x)) \leq \rho_2(x)\) in a neighborhood of \(\infty\).

The definitions for the small scale case and the global case are obtained as usual.

Now the following definition generalize the definition 4.4. of [9]. We recall the following definition of the cited work: Given a function \(f: X \to Y\) between metric spaces a subset \(A \subset X\) is said to be \((r, R)\)-bounded if for any \(x, y \in A\) with \(d_X(x, y) \leq r\) then \(d_Y(f(x), f(y)) \leq R\).

**Definition 7.2.** Let \(S_1, S_2\) be any control semigroups. Given a function \(f: X \to Y\) between metric spaces and given \(m \geq 0\), an \(m\)-dimensional \((S_1, S_2)\)-control function of \(f\) is a \((S_1, S_2)\)-function \(D_f^{(S_1, S_2)}: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+\) such that for all \(r_X > 0\) and \(R_Y > 0\) in a neighborhood of \(\infty\) any \((\infty, R_Y)\)-bounded subset \(A\) of \(X\) can be expressed as the union of \(m + 1\) sets whose \(r_X\)-components are \(D_f^{(S_1, S_2)}(r_X, R_Y)\)-bounded.

The generalization of 4.6. of [9] is similar:

**Definition 7.3.** Let \(S_1, S_2\) be any control semigroups. Given a function \(f: X \to Y\) between metric spaces and given \(k \geq m + 1 \geq 1\), an \((m, k)\)-dimensional \((S_1, S_2)\)-control function of \(f\) is a \((S_1, S_2)\)-function \(D_f^{(S_1, S_2)}: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+\) such that for all \(r_X > 0\) and \(R_Y > 0\) in a neighborhood of \(\infty\) any \((\infty, R_Y)\)-bounded subset \(A\) of \(X\) can be expressed as the union of \(k\) sets \(\{A_i\}_{i=1}^k\) whose \(r_X\)-components are \(D_f^{(S_1, S_2)}(r_X, R_Y)\)-bounded so that any \(x \in A\) belongs to at least \(k - m\) elements of \(\{A_i\}_{i=1}^k\).

Definitions for the small scale and global case are obvious.

With these generalizations the analogous theorem (for the large scale case) of 4.9. in [9] would be the following.

**Proposition 7.4.** Let \(S_1, S_2\) be any control semigroups and let \(k = m + n + 1\) where \(m, n \geq 0\). Suppose \(f: X \to Y\) is a \(\Sigma(S_2)\)-coarse function of metric spaces and \(\mathop{asdim}_{S_2} Y \leq n\). If there is an \((m, k)\)-dimensional \((S_1, S_2)\)-control function \(D_f^{(S_1, S_2)}\) of \(f\) then:

\[
\mathop{asdim}_{S_1} X \leq m + n
\]

The proof is almost the same as in [9]. Note that in the proof the semigroup condition is used strongly. For small scale case and the global case there are analogous theorems. We just give the global one:
Proposition 7.5. Let \( \tilde{S}_1, \tilde{S}_2 \) be any global control semigroups and let \( k = m + n + 1 \) where \( m, n \geq 0 \). Suppose \( f : X \to Y \) is a global \( \Sigma(\tilde{S}_2) \)-function of metric spaces and \( \dim_{\tilde{S}_2} Y \leq n \). If there is an \((m, k)\)-dimensional \((\tilde{S}_1, \tilde{S}_2)\)-control function \( D_{f}^{(\tilde{S}_1, \tilde{S}_2)} \) of \( f \) then:

\[
\dim_{\tilde{S}_1} X \leq m + n
\]

Now note that as a trivial consequence of proposition 4.7 of [9] we have:

Proposition 7.6. Let \( f : X \to Y \) be a function of metric spaces and \( m \geq 0 \). Suppose that there exist an \( m \)-dimensional \((S_1, S_2)\) control function of \( f \) with \( S_1, S_2 \) two control semigroups. Then there exist an \((m, k)\)-dimensional \((S_1, S_2)\)-control function of \( f \) for every \( k > m + 1 \)

We can define the dimension of a function between metric spaces in the following way:

Definition 7.7. Given \( S_1, S_2 \) two control semigroups and given a function \( f : X \to Y \) of metric spaces we define the \((S_1, S_2)\)-asymptotic dimension \( \text{asdim}_{S_1, S_2}(f) \) of \( f \) as the minimum of \( m \) for which there is an \( m \)-dimensional \((S_1, S_2)\)-control function.

Finally we have as a corollary of 7.4

Theorem 7.8. If \( f : X \to Y \) is a \( \Sigma(S_2) \)-coarse function of metric spaces then:

\[
\text{asdim}_{S_1} X \leq \text{asdim}_{(S_1, S_2)}(f) + \text{asdim}_{S_2} Y.
\]

Here is the global version:

Theorem 7.9. If \( f : X \to Y \) is a global \( \Sigma(S_2) \)-function of metric spaces then:

\[
\dim_{\tilde{S}_1} X \leq \dim_{(\tilde{S}_1, \tilde{S}_2)}(f) + \dim_{\tilde{S}_2} Y.
\]

The following lemma appears in [9] (Proposition 8.4.) and it will be useful to apply the Hurewicz theorem to exact sequences of finitely generated groups.

Lemma 7.10. If \( 1 \to K \to G \to H \to 1 \) is an exact sequence and \( G \) is a finitely generated group, then there are word metrics \( d_G \) on \( G \) and \( d_H \) on \( H \) such that \( f : (G, d_G) \to (H, d_H) \) is 1-Lipschitz and for any \( m \)-dimensional control function \( D_K \) on \( K \) the function:

\[
D_f(r_G, R_H) := D_K(r_G + 2R_H) + 2R_H
\]

is an \( m \)-dimensional control function of \( f \).

Then if \( D_K \in S \) for some control semigroup \( S \) we will have

Corollary 7.11. If \( 1 \to K \to G \to H \to 1 \) is an exact sequence of groups so that \( G \) is finitely generated then

\[
\text{asdim}_S(G, d_G) \leq \text{asdim}_S(K, d_K|_K) + \text{asdim}_S(H, d_H).
\]
for any word metrics $d_G$ on $G$ and $d_H$ on $H$.

Finally, we will show the analogous version of the Hurewicz theorem for groups acting on spaces with finite semigroup-controlled asymptotic dimension.

Let $G$ be a group acting by isometries on a metric space $X$ and let $R > 0$. Given $x_0 \in X$ the $R$-stabilizer of $x_0$ is defined by $W_R(x_0) = \{ \gamma \in G : d(\gamma x_0, x_0) \leq R \}$.

The following theorem is the analogous to Theorem 8.9 in [9]. The proof is completely similar.

**Theorem 7.12.** Let $S_1$ and $S_2$ be two control semigroups and let $G$ be a finitely generated group acting by isometries on a metric space $X$ such that $\text{asdim}_{S_2}X < +\infty$. Fix a point $x_0 \in X$. If there exists a $(S_1, S_2)$-function $D(S_1, S_2) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $R > 0$ the function $g : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $g(r) = D(r, R)$ is a $(k, S_1)$-dimensional control function of $W_R(x_0)$, then:

\[
\text{asdim}_{S_1}G \leq k + \text{asdim}_{S_2}X.
\]

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