Regularization of Legendre Function Series for Charged Particles
Improved Nearside-Farside Subamplitudes

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(Dated: August 20, 2002)

A simple regularization procedure is proposed for the Legendre function series of improved nearside-farside subamplitudes for charged particles elastic scattering. The procedure is the extension of the usual one which defines the partial wave series for the scattering amplitude in the presence of a long range Coulomb term in the potential, and it provides the same convergence rate.

PACS numbers: 24.10.Ht, 25.70.Bc, 03.65.Sq

The nearside-farside (NF) method proposed by Fuller is an effective tool to separate the full elastic scattering amplitude \( f(\theta) \), where \( \theta \) is the scattering angle, into simpler subamplitudes. The Fuller NF subamplitudes are usually more slowly varying and less structured then \( f(\theta) \). This allows one to explain the complicated patterns appearing in some cross sections, as interference effects between simpler nearside (N) and farside (F) subamplitudes. These subamplitudes can often be interpreted as contributions from simple scattering mechanisms allowing a physical understanding of the scattering process.

Sometimes, particularly when applied to scattering of \( \alpha \) particles and light heavy-ions at intermediate and high energies, the Fuller NF subamplitudes are biased by the presence of unphysical contributions, making the NF subamplitudes more structured then desired. Recently an improved NF method has been proposed to further extend the effectiveness of the original Fuller technique. The improved NF method is based on a modified Yennie, Ravenall, and Wilson (YRW) resummation identity, which holds for Legendre polynomial series (LPS). The increased effectiveness descends from using resummation parameters with values reducing the unphysical contributions to the Fuller NF subamplitudes.

The Legendre function series (LFS) for the improved NF subamplitudes are, however, not convergent in the usual sense. A resummation technique named in the following extended YRW (EYRW) resummation, was used in Refs. to overcome convergence series. At forward angles, the rate of convergence of the EYRW series is not satisfactory in the presence of a long range Coulomb term in the potential. For \( \alpha \) particles, light and heavy ions scattering this fact is disturbing, because it compels one to use more partial waves than necessary in standard optical potentials calculations and in the usual Fuller NF method. Here we present a regularization procedure that, if applied to LFS of improved NF subamplitudes, makes these series as rapidly convergent as those of more conventional approaches.

The starting point for the improved NF method is the quantum mechanical partial wave series (PWS) of the elastic scattering amplitude

\[
f(\theta) = \sum_{l=0}^{\infty} a_l P_l(\cos \theta),
\]

where \( x = \cos \theta \), \( P_l(x) \) is the Legendre polynomial of degree \( l \), \( x \neq 1 \), and \( a_l \) is given in terms of the scattering matrix element \( S_l \) by

\[
a_l = \frac{1}{2ik}(2l + 1)S_l,
\]

where \( k \) is the wavenumber.

To obtain the improved NF subamplitudes, one substitutes the usual factor \( S_l - 1 \) with \( S_l \) on the r.h.s. of the dropped term ensured the convergence of the for scattering by short range potentials, for which \( S_l \to 1 \) exponentially for \( l \to \infty \). In this case, having omitted a term \( \propto \delta(1 - x) \), where \( \delta \) indicates the Dirac distribution (e.g. see [1], p. 52), the sum in (1) is defined only in a distributional sense. In the presence of a long range Coulomb term in the potential the dropped 1 is not relevant for convergence. With or without the 1, the sum in (1) is convergent only in a distributional sense. In and references therein, one can find more or less recent discussions on the convergence of the Coulombic PWS, and of the different techniques solving the problem.

The improved NF subamplitudes are obtained by using for \( f(\theta) \), in place of (1), its resummed form

\[
f(\theta) = \left( \prod_{i=0}^{r} \frac{1}{1 + \beta_i x} \right) \sum_{n=0}^{\infty} \alpha_n^{(r)} P_n(x),
\]

\( r = 0, 1, 2, \ldots \), where

\[
\alpha_n^{(i)} = \beta_i \frac{n}{2n - 1} \alpha_{n-1}^{(i-1)} + \alpha_n^{(i-1)} + \beta_i \frac{n + 1}{2n + 3} \alpha_{n+1}^{(i-1)},
\]

with \( \beta_n = 0 \), \( \alpha_n^{(0)} = a_n \), and \( \alpha_n^{(-1)} = 0 \). The resummed form is an exact mathematical identity deriving from the recurrence property of the Legendre polynomials. It holds for real or complex values of the resummation parameters \( \beta_i \) (\( i \neq 0 \)), restricted only by the condition...
1 + βi x ≠ 0, for −1 ≤ x < 1. The integer index r is the order of the resummation, and r = 0 means no resummation of the original PWS. In (1) we changed the index of the sum (l) from l to n to remark that the index of the resummed Legendre polynomial series (LPS) in (1) has not, for r ≠ 0, the physical meaning of orbital quantum number, differently from the index of the original PWS (l). Similarly the terms αn(0) have not the physical meaning of partial wave amplitudes. The usual YRW resummed form (1) for f(θ) is obtained by setting βi = −1 (i ≠ 0) in (3).

We note that for pure Coulomb scattering, for which

\[ P_n(x) = Q_n^{-}(x) + Q_n^{+}(x), \]

with Q_n(x) the Legendre function of the second kind of degree n. By inserting (6) into (3), f(θ) is separated into the sum of two subamplitudes

\[ f(θ) = f_{\{β\}}^{-}(θ) + f_{\{β\}}^{+}(θ), \]

with

\[ f_{\{β\}}^{±}(θ) = \left( \prod_{i=0}^{r} \frac{1}{1 + β_i x} \right) \sum_{n=0}^{∞} α_n^{(r)} Q_n^{±}(x). \]

In (10), with the subscript \{β\} we indicate that the N \( f_{\{β\}}^{-}(θ) \) and F \( f_{\{β\}}^{+}(θ) \) subamplitudes depend, differently from f(θ), on the resummation order r and parameters βi. This occurs because the resummed form of series (LFS) of linear combination of first and second kind Legendre functions, of integer degree, is different from \( f_{\{β\}}^{-}(θ) \). In fact, let us indicate with

\[ F(θ) = \sum_{n=0}^{∞} d_n L_n(x) \]

a LFS in \( L_n(x) = p P_n(x) + q Q_n(x) \), with p and q independent of n. Owing to the property \( nQ_{n-1}(x) → 1 \) as \( n \rightarrow 0 \), the resummed form of \( F(x) \), of order s and parameters γi, is

\[ F(θ) = \left( \prod_{i=0}^{s} \frac{1}{1 + γ_i x} \right) \sum_{n=0}^{∞} δ_n^{(s)} L_n(x) \]

+ \( q \sum_{i=0}^{s} γ_i δ_n^{(s)} \frac{1}{1 + γ_j x} \).

Equation (12) is an exact mathematical identity extending the validity of (4) to more general LFS, and it reduces to (3) for LPS (q = 0). The conditions of validity of (12), and the recurrence relation for the resummed coefficients, are the same as those for (4), after substituting r, β, α, and a with s, γ, δ, and d, respectively.

Because the \( Q_n^{±}(x) \) used to split \( P_n(x) \) in (10) are a particular case of the more general \( L_n(x) \) (with p = 1/2, and q = ±i/π), the presence of the last term in (12) is responsible for the dependence of \( f_{\{β\}}^{±}(θ) \) on r and βi.

The last term on the r.h.s. of (12) gives a contribution if the splitting (4) is inserted in (4). This contribution is absent if the splitting is inserted in (3).

In Refs. [4, 5] it was observed that unphysical contributions, when appearing in the Fuller NF subamplitudes (r = 0 in (10)), decrease by increasing r in (4) (the values r = 1, and 2 were tested), if the \( β_i \) are selected to make null the coefficients α0(1), α1(1), . . . , αr−1. The resummed LFS (α0(1), and α0(2) for the cases tested). In this way one drops the contributions to the NF resummed subamplitudes from low n values for which the splitting (4), though exact by construction, is not expected to be physically meaningful.

The \( α_n^{(r)} \) in (3) and (10) go asymptotically to a constant for short range potentials, or are Coulombic in the presence of a Coulomb term in the potential. Because of this the corresponding LFS are not convergent in the usual sense. In Refs. [4, 5] the convergence was forced, and accelerated, by applying to the improved LFS a final (EYRW) resummation (12) of order s ≥ 1, with \( d_n = α_0^{(r)}, γ_i = −1, \) and i ≠ 0. The final EYRW resummation ensures the numerical convergence of the LFS, with a convergence rate increasing with s. The increased rate of convergence costs, however, the cancellation of significant digits (see [8] for details), and numerically the procedure may results not convenient or even impossible, using arithmetic with a fixed digit number.

These troubles can be avoided by investigating the properties of the resummation identity (12) with \( d_n \) equal to the pure Coulomb \( α_n^{(r)} \) given by (3). In this case we explicitly know the l.h.s. of (12) for the relevant p and q values. In fact, if p = 1 and q = 0 it is the Rutherford scattering amplitude \( f_R(θ) \), while for \( p = 1/2 \) and \( q = ±i/π \) one obtains the Fuller-Rutherford NF subamplitudes \( f_{FR}^{±}(θ) \) (Eqs. 14 a, b). Because (12) is exact
it holds for arbitrary \( \gamma_i \), and therefore also for \( \gamma_i = \beta_i \),
with \( \beta_i \) obtained by applying the improved resummation
method to the exact \( S_i \). With this choice the pure Coulomb resummed coefficients \( \alpha_n^{C(r)} \) asymptotically
approach \( \alpha_n^{(r)} \) as rapidly as the pure Coulomb \( S \)-matrix elements, \( S_i^C \), approach \( S_i \) in the usual optical potential
calculations.

With the change of notation \( f^{(0)} = f, f^{(1)} = f^{(1)}(\beta) \),
\( f_R^{(0)} = f_R, f_R^{(1)} = f_R^{(1)}(\beta) \), \( L_n^{(0)} = P_n \), and \( L_n^{(1)} = Q_n^{(1)} \),
by subtracting from \( f \), or \( f_R \), the corresponding resummed forms \( f^{(1)} \), applied to pure Coulomb scattering
\( (s = r, \gamma_i = \beta_i, \alpha_n^{(r)} = \alpha_n^{C(r)} \text{ and } q = 0, \pm 1) \), one obtains the final result

\[
 f^{(m)}(\theta) = \left( \prod_{i=0}^{r} \frac{1}{1 + \beta_i x_i} \right) \sum_{n=0}^{\infty} \left[ \alpha_n^{(r)} - \alpha_n^{C(r)} \right] L_n^{(m)}(x)
 + f_R^{(m)}(\theta) + m \int_{\infty}^{1} \beta_i \alpha_0^{C(i-1)} \prod_{j=0}^{i} \frac{1}{1 + \beta_j x} \, dx.
\]

with \( m = 0 \) for the full amplitude and \( m = \mp 1 \) for the NF subamplitudes. For \( r = 0 \) and \( m = 0, \) or \( m = \mp 1 \), Eq. (13) is the usual regularization procedure defining
the r.h.s of \( f \), or \( f_R \), in the presence of a long range Coulomb term in the potential. This procedure is based on
adding the explicit expression of \( f_R(\theta) \), or \( f_R^{(1)}(\theta) \), and subtracting its formal PWS, or LFS, for the full amplitude
(Ref. \( \) p. 428), or the Fuller NF subamplitudes \( \) \( \). For \( r \geq 1 \), Eq. (13) is the generalization of this
regularization procedure to resummed forms of the full amplitude, or NF subamplitudes. The sum appearing in
this term is as rapidly convergent as the usual sum with \( r = 0 \).

Before showing the effectiveness of our regularization procedure in a physically interesting case, we show the
difficulties met by the EYRW technique \( \) to ensure, and speed up, the convergence of improved, or not, LFS
for pure Coulomb scattering. In this case \( a_{\infty} \equiv a_{0}^{C(1)} \) and
the LFS on the r.h.s. of (13) is identically null, with arbitrary choice of \( \beta_i \). For \( r = 0 \), Eq. (13) trivially states
that the scattering amplitude \( (m = 0) \) is the Rutherford amplitude, and the NF subamplitudes \( (m = \mp 1) \) are the
usual Fuller-Rutherford ones. For \( r > 0 \), by choosing \( \beta_i \) accordingly with the improved resummation method,
Eq. (13) gives the explicit expression of the improved NF subamplitudes \( (m = \mp 1) \) in term of the usual Fuller-
Rutherford ones, and of simple functions depending on \( \beta_i \) and \( a_{0}^{C(i-1)} \). For simplicity we will name \( \) explicit expression for pure Coulomb improved NF subamplitudes.

In Fig. 1 the thick curves show the ratio to the Rutherford cross section, \( \sigma_{R}(\theta) \), of the exact pure Coulomb improved F cross sections, of order \( r = 0,1 \), and \( 2 \)
\( (r = 0 \text{ meaning the original Fuller method}) \). In the same figure the thin curves show the F cross sections
obtained by forcing, and accelerating, the convergence of (10) with an additional EYRW resummation of order
\( s = 4 \), and fixing the maximum number of the summed partial waves to \( l_{\text{max}} = 100 \) and 1000. The results
were obtained with \( \eta = 10 \), which is a typical value of the Sommerfeld parameter for heavy-ion scattering. For
this \( \eta \) value the improved resummation parameters are \( \beta_1 = 0.9802 + 0.1980 i \) (for \( r = 1 \)), \( \beta_2 = 1.0072 + 0.1166 i \) and \( \beta_2 = 0.7804 + 0.6052 i \) (for \( r = 2 \)). Figure 1 shows that the final EYRW resummation \( \) ensures the convergence of the LFS \( \) \( \), but the convergence rate is
low. For \( \theta \lesssim 5^\circ \) a numerically satisfactory result is not obtained even with \( l_{\text{max}} = 1000 \). By fixing \( l_{\text{max}} \) and the final resummation order, the angle at with the truncated LFS disagrees with the exact result increases with the improved resummation order.

Figure 1 also shows that the improved resummation method reduces, particularly at forward angles, the
unphysical F contribution present in the original Fuller NF method. However it does not suppress it, and is ineffective
at \( \theta \approx 180^\circ \). This is an insurmountable difficulty connected with the NF splitting \( \), mathematically con-
tinuing (at \( \theta = 180^\circ \)) the N subamplitude into a F one, or vice versa. This also in absence of physically meaningful
subamplitudes justifying this continuation. In these situations the only practical suggestion we can give is to
not take seriously the NF subamplitudes at \( \theta \approx 180^\circ \), if in a neighbourhood of this angle the cross section and the
LIP of the full amplitude have a non oscillatory behavior, suggesting the dominance of a single side (positive LIP
for F and negative for N) contribution. We remember that in \( \) the LIP (local impact parameter) is defined as the
derivative of the argument of the scattering amplitude with respect to the scattering angle, named LAM (local angular momentum) by Fuller, divided by the wavenumber \( k \).
As a second example of the effectiveness of our regularization procedure, we consider the first order improved F cross section and LIP of the phenomenological optical potential WS2, used to fit the $^{16}\text{O} + ^{16}\text{O}$ elastic cross section at $E_{\text{lab}} = 145 \text{ MeV}$. The improved resummation parameter is in this case $\beta_1 = -0.9997 - 0.0798i$ [3]. The upper panel of Fig. 2 shows, for $\theta < 30^\circ$ and $l_{\text{max}} = 150$, the F LIP calculated using our regularization procedure (thick curve) and different order (thin lines) EYRW resummations [3]. The lower panel shows the corresponding F cross sections. Symmetrization effects were ignored.

Note that 150 partial waves are more than really necessary to obtain reliable scattering amplitudes using our regularization procedure. Using an EYRW resummation of order 1 (thin dotted curves), this partial wave number is not sufficient to obtain a satisfactory result. By increasing the EYRW resummation order it decreases the angular width of the region where the thin curves differ from the corresponding thick ones. However, for $\theta \lesssim 5^\circ$, the 150 partial waves used are not enough, even using a fourth order final EYRW resummation.

These results show, in practical examples, that EYRW resummed LFS for asymptotically Coulombic $S_l$ are convergent, with a convergence rate increasing by increasing the resummation order. Compared with the extension here given of the usual regularization procedure for asymptotically Coulombic $S_l$ the EYRW resummation technique effectiveness is, however, computationally poor. The regularization procedure here described can be easily extended to make rapidly convergent the LFS in (3) and (10) for scattering by short range potentials. In these cases, however, also an additional first order EYRW resummation makes the LFS convergent with the same rapidity, and there is no practical advantage in using a different procedure.

Acknowledgments

The author is indebted to J. N. L. Connor for stimulating and helpful discussions.