Improved almost Morawetz estimates for the cubic nonlinear Schrödinger equation

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Abstract: We prove global well-posedness for the cubic, defocusing, nonlinear Schrödinger equation on $\mathbb{R}^2$ with data $u_0 \in H^s(\mathbb{R}^2)$, $s > 1/4$. We accomplish this by improving the almost Morawetz estimates in [9].

1 Introduction

The cubic, defocusing, nonlinear Schrödinger equation on $\mathbb{R}^2$,

$$iu_t + \Delta u = |u|^2u,$$
$$u(0, x) = u_0(x) \in H^s(\mathbb{R}^2),$$

has been the subject of a great deal of research in recent years. It was proved in [4] that for any $s > 0$, (1.1) has a local solution on some interval $[0, T]$, $T(\|u_0\|_{H^s}) > 0$. Moreover, for a solution to fail to extend to a global solution, but instead exist only on a maximal interval $[0, T_*)$,

$$\lim_{t \to T_*} \|u(t)\|_{H^s(\mathbb{R}^2)} = \infty.$$ (1.2)

The first progress to proving the existence of a global solution was proved in [3].

Theorem 1.1 (1.1) has a global solution for $u_0 \in H^1(\mathbb{R}^2)$.

Sketch of Proof: (1.1) has the conserved quantities

$$M(u(t)) = \int |u(t, x)|^2 dx = M(u(0)).$$ (1.3)

$$E(u(t)) = \frac{1}{2} \int |\nabla u(t, x)|^2 dx + \frac{1}{4} \int |u(t, x)|^4 dx = E(u(0)).$$ (1.4)
Combining this fact with the Sobolev embedding theorem implies
$E(u(0)) \lesssim \|u_0\|_{H^1(R^2)}^2$. Since (1.4) is positive definite, this implies
$\|u(t)\|_{H^1}^2 \leq C(\|u_0\|_2)\|u_0\|_{H^1}^2$
for all time. Thus there exists a solution for all time. \(\Box\)

The reader will notice there is a gap between the regularity necessary to
prove local well-posedness \((s > 0), [4]\) and the regularity needed in Theorem
1.1 to prove a global solution, \([3]\). Many have undertaken to close this gap.
The first progress was made in \([2]\).

**Theorem 1.2** If \(u_0 \in H^s(R^2), s > 3/5\), then (1.1) has a global solution of
the form

\[
\begin{align*}
u(t,x) &= e^{it\Delta}u_0 + w(t,x), \\
w(t,x) &\in H^1(R^2). \\
\end{align*}
\]  

In this case the method of proof was the Fourier truncation method. Take
\(\phi(\xi) \in C_0^\infty,\)

\[
\phi(\xi) = \begin{cases}
1, & |\xi| \leq 1; \\
0, & |\xi| > 2.
\end{cases}
\]

Then split the initial data into low frequency and high frequency components.

\(\hat{u}_0(\xi) = \phi(\frac{\xi}{N})\hat{u}_0(\xi) + (1 - \phi(\frac{\xi}{N}))\hat{u}_0(\xi) = \hat{u}_l(\xi) + \hat{u}_h(\xi).\)

Since \(\|u_l\|_{H^1} \lesssim N^{1-s}\|u_0\|_{H^s}\), the equation

\[
\begin{align*}
iu_t + \Delta u &= |u|^2v, \\
v(0,x) &= u_l,
\end{align*}
\]  

has a global solution with

\(E(v(t,x)) \lesssim N^{2-2s}\|u_0\|_{H^s(R^2)}.\)

Also, if \(s > 3/5\), the equation

\[
\begin{align*}
iw_t + \Delta w &= |v + w|^2(v + w) - |v|^2v, \\
w(0,x) &= u_h,
\end{align*}
\]  

has a solution on \([0, T]\) of the form

\(e^{it\Delta}u_h + q(t,x),\)
\( q(t, x) \in H^1(\mathbb{R}^2) \quad \forall t. \)

This approach was modified in [6] to produce the I-method. The I-operator,

\[
I_N : H^s(\mathbb{R}^2) \rightarrow H^1(\mathbb{R}^2),
\]

is the smooth, radial Fourier multiplier

\[
\hat{I}_N f(\xi) = m_N(\xi) \hat{f}(\xi),
\]

\[
m_N(\xi) = \begin{cases} 
1, & \text{for } |\xi| \leq N; \\
\left(\frac{\xi}{N}\right)^{s-1}, & \text{when } |\xi| > 2N.
\end{cases}
\]

From this point on, we will understand that \( I \) refers to the I - operator \( I_N \).

\[
\|I f\|_{H^1(\mathbb{R}^3)} \lesssim N^{1-s} \|f\|_{H^s(\mathbb{R}^3)},
\]

\[
\|f\|_{H^s(\mathbb{R}^2)} \lesssim \|I f\|_{H^1(\mathbb{R}^2)}.
\]

Therefore, if

\[
E(Iu(t)) = \frac{1}{2} \int |\nabla Iu(t, x)|^2 dx + \frac{1}{4} \int |Iu(t, x)|^4 dx,
\]

was a conserved quantity then the existence of a global solution would follow for any \( s > 0 \). This is not the case, however. Instead, it was proved in [6] that

**Lemma 1.3** If \( E(Iu(t)) \leq 1 \), then there exists \( \delta > 0 \) such that

\[
\sup_{t \in [0, \delta]} |E(Iu(t)) - E(Iu(0))| \leq O\left(\frac{1}{N^{3/2}}\right).
\]

This implies global well-posedness for \( u_0 \in H^s(\mathbb{R}^2), s > 4/7 \). Subsequent papers (see [8], [5], [9]) have decreased the necessary regularity to

**Theorem 1.4** (1.1) has a global solution for \( u_0 \in H^s(\mathbb{R}^2), s > 1/3 \).

This was proved by combining the I-method, a modified energy functional, and almost Morawetz estimates. The method will be described in more detail in the subsequent sections. In addition, the almost Morawetz estimates will be improved, thus improving Theorem 1.4 to

**Theorem 1.5** (1.1) has a global solution for \( u_0 \in H^s(\mathbb{R}^2), s > 1/4 \).
In §2 the modified energy functional of [8] will be recalled, as well as a modified local well-posedness theorem. In §3, the Morawetz inequality for \( u(t, x) \) will be proved (originally proved in [5]),

\[
\| u \|_{L^4_t([0,T] \times \mathbb{R}^2)}^4 \lesssim T^{1/3} \| u_0 \|_{L^2(\mathbb{R}^2)}^3 \| u \|_{L^{\infty}_t([0,T], H^1(\mathbb{R}^2))}^4 + T^{1/3} \| u_0 \|_{L^2(\mathbb{R}^2)}^4. \tag{1.12}
\]

In §4, the known almost-Morawetz estimate in [9] for \( Iu(t, x) \) will be improved. Finally, in §5, this improvement will be used to prove Theorem 1.5.

\section{Modified Energy Functional}

In this section the known results concerning the modified energy functional will merely be stated. All of these results have been proved before (see [8] and [9]). If \( u(t, x) \) solves (1.1), then \( Iu(t, x) \) solves

\[
i Iu_t + \Delta Iu = I(|u|^2 u). \tag{2.1}
\]

If the nonlinearity was of the form \( |Iu|^2 Iu \), then \( E(Iu(t)) \) would be conserved. However, since \( |Iu|^2 Iu \neq I(|u|^2 u) \),

\[
\partial_t E(Iu(t)) = 2Re \int (Iu_t(t, x))I(|u(t, x)|^2 \overline{u(t, x)}) - |Iu(t, x)|^2 Iu(t, x)dx. \tag{2.2}
\]

The change in energy decreases as \( N \to \infty \).

\begin{theorem}
If \( E(Iu(0)) \leq 1 \), then there exists \( \delta > 0 \) such that

\[
|E(Iu(t)) - E(Iu(0))| \leq O\left( \frac{1}{N^{3/2}} \right), \tag{2.3}
\]

for \( t \in [0, \delta] \).
\end{theorem}

\textit{Proof:} See [6].

In [8], the authors proved the existence of a modified energy functional \( \tilde{E}(u(t)) \) satisfying the properties:

1. \( \tilde{E}(u(t)) \) has a slower variation than \( E(Iu(t)) \).

2. \( \tilde{E}(u(t)) \) is close to \( E(Iu(t)) \) in the sense that \( E(Iu(t)) \) can be controlled by \( \tilde{E}(u(t)) \).
Proposition 2.2 There exists a modified energy functional $\tilde{E}$ satisfying the fixed time estimate,

$$|\tilde{E}(u(t)) - E(Iu(t))| \lesssim \frac{1}{\theta N^{2-}}\|Iu(t)\|^4_{H^1(R^2)}. \quad (2.4)$$

Proof: See §4 of [8].

Proposition 2.3 $\tilde{E}(u(t))$ has the energy increment for a time interval $J$,

$$|\sup_{t \in J} \tilde{E}(u(t)) - \tilde{E}(u(a))| \lesssim \left(\frac{\theta^{1/2}}{N^{3/2-2}} + \frac{1}{N^{2-}} + \frac{1}{\theta N^{3-}}\right)\|Iu\|^4_{X^{1,1/2+}(J \times R^2)}. \quad (2.5)$$

Proof: See §7 and §8 of [8].

The $X^{1,1/2+}$ norm will not be defined in this paper, because it will not be needed.

Proposition 2.4 Assume that

$$\sup_{t \in J} E(Iu(t)) \leq 2, \quad (2.6)$$

and for some $\epsilon > 0$,

$$\|Iu\|_{L^4_t \cdot (J \times R^2)} \leq \epsilon. \quad (2.7)$$

then

$$|\sup_{t \in J} \tilde{E}(u(t)) - \tilde{E}(u(a))| \lesssim \frac{1}{N^{2-}} + \frac{\theta^{1/2}}{N^{3/2-2}} + \frac{1}{\theta N^{3-}}. \quad (2.8)$$

In particular, taking $\theta = \frac{1}{N}$ implies

$$\sup_{t, t' \in J} |\tilde{E}(u(t)) - E(u(t'))| \lesssim \frac{1}{N^{2-}}. \quad (2.9)$$

Proof: See §4 of [9].

Theorem 2.5 Let

$$\|\langle \nabla \rangle Iu_0\|_{L^2(R^2)} = 1$$

and
\[\int_{J_k} \int |Iu(t,x)|^4 dx dt < \mu_0, \quad (2.10)\]

for some \(\mu_0 > 0\) sufficiently small. Then (1.1) is locally well-posed on \([0,T]\) and

\[Z_I(J_k,u) = \sup_{(q,r) \text{ admissible}} \|\langle \nabla \rangle Iu(t,x)\|_{L_t^q L_x^r(J_k \times \mathbb{R}^2)} \leq C. \quad (2.11)\]

\((q,r)\) is an admissible pair if

\[\frac{2}{q} = 2\left(\frac{1}{2} - \frac{1}{r}\right)\]

and \(q > 2\).

Proof: See §3 of [5].

### 3 Morawetz inequalities

In this section we will recall the proof of the following Morawetz inequality from [5]. This recollection will be useful for the arguments given in the next section.

**Proposition 3.1** If \(u(t,x)\) solves (1.1) then

\[\|u(t,x)\|_{L_{t,x}^4([0,T] \times \mathbb{R}^2)}^4 \lesssim T^{1/3} \|u_0\|_{L^2(\mathbb{R}^2)}^3 \|u(t,x)\|_{L_{t,x}^\infty([0,T],H^1(\mathbb{R}^2))} + T^{1/3} \|u_0\|_{L^2(\mathbb{R}^2)}^4. \quad (3.1)\]

Proof: Suppose that \(v(t,z)\) solves the partial differential equation

\[iv_t + \Delta_z v = F. \quad (3.2)\]

Then define the quantities

\[T_{0j}(t,z) = 2Im(u(t,z)\partial_j u(t,z)), \quad (3.3)\]

\[L_{jk}(t,z) = -\partial_j \partial_k (|u|^2) + 4Re(\overline{\partial_j u(t,z)} \partial_k u(t,z)). \quad (3.4)\]

These quantities obey the relation,
\[ \partial_t T_{0j} + \partial_k L_{jk} = 2(F(t, z)\partial_j u(t, z) - u(t, z)\partial_j F(t, z)) + F(t, z)\partial_j u(t, z) - u(t, z)\partial_j F(t, z). \] (3.5)

Let \( v(t, z) \) be a tensor product of solutions to (1.1) on \( \mathbb{R}^2 \times \mathbb{R}^2 \),
\[ (u_1 \otimes u_2)(t, z) = u_1(t, x)u_2(t, y) = v(t, z), \] (3.6)
\[ iv_t + \Delta v = i\partial_k(u_1(t, x)u_2(t, y) + iu_1(t, x)\partial_k(u_2(t, y)) + (\Delta_x u_1(t, x))u_2(t, y) + u_1(t, x)(\Delta_y u_2(t, y)) = |u_1(t, x)|^2u_1(t, x)u_2(t, y) + |u_2(t, y)|^2u_1(t, x)u_2(t, y). \] (3.7)

Define the Morawetz action,
\[ M_a^{\otimes 2}(t) = 2\int_{\mathbb{R}^2 \times \mathbb{R}^2} \nabla a(z) \cdot \text{Im}(\overline{v(t, z)}\nabla v(t, z))dz, \] (3.8)
\[ \partial_t M_a^{\otimes 2}(t) = 2\int \partial_j a(z)\partial_t T_{0j}(t, z)dz, \] (3.9)
following the convention that repeated indices are summed.
\[ \partial_t M_a^{\otimes 2}(t) = 2\int \partial_j \partial_k(|v|^2)\partial_j a(z) \] (3.10)
\[ -8\int \partial_k \text{Re}(\overline{\partial_j v(t, z)}\partial_k v(t, z))\partial_j a(z)dz \] (3.11)
\[ +4\int \{F(t, z)\partial_j v(t, z) - v(t, z)\partial_j F(t, z) + F(t, z)\partial_j v(t, z) - v(t, z)\partial_j F(t, z)\} \partial_j a(z). \] (3.12)

Let \( v(t, z) = u(t, x)u(t, y) \), where \( u \) solves (1.1). Take the term (3.10) first.
\[ 2\int \partial_j \partial_k(|v(t, z)|^2)\partial_j a(z) = -2\int |v(t, z)|^2(\Delta a(z))dz. \]

Now let \( a(z) = a(x, y) = f(|x - y|), \) where \( f \) is a smooth, convex function.
Let
\[ f(x) = \begin{cases} \frac{1}{2M}x^2(1 - \log \frac{x}{M}), & \text{if } |x| < \frac{M}{\sqrt{e}}; \\ 100x, & \text{if } |x| > M. \end{cases} \] (3.13)
For $|x - y| < \frac{M}{\sqrt{e}}$, 
\[ \Delta a(x, y) = \frac{2}{M} \log \left( \frac{M}{|x - y|} \right) \Rightarrow -\Delta \Delta a(x, y) = \frac{2}{M} \delta_{x=y}, \]
and for $|x - y| > M$, 
\[ -\Delta \Delta a(x, y) = O \left( \frac{1}{|x - y|^3} \right) = O \left( \frac{1}{M^3} \right). \]

\[
\int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}^2} (-\Delta \Delta a(x, y)) |u(t, x)|^2 |u(t, y)|^2 dxdydt = \frac{2}{M} \int_0^T \int_{\mathbb{R}^2} |u(t, x)|^4 dx dt 
+ O \left( \frac{1}{M^3} \right) \int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}^2} |u(t, x)|^2 |u(t, y)|^2 dxdydt.
\]
Since $M$ will be large, $|\nabla a(z)|$ is uniformly bounded on $\mathbb{R}^2 \times \mathbb{R}^2$, and 
\[ |M_a^{\otimes^2}(t)| = 2 \left| \int_{\mathbb{R}^2 \times \mathbb{R}^2} \nabla a(z) \cdot \text{Im}(\nabla v(t, z)) \nabla v(t, z) dz \right| \lesssim \|u_1(t, x)\|_{L^\infty_t([0,T],L^2(\mathbb{R}^2))}^2 \|u_2(t, y)\|_{L^\infty_t([0,T],\dot{H}^1(\mathbb{R}^2))} \|u_2(t, y)\|_{L^\infty_t([0,T],L^2(\mathbb{R}^2))} 
+ \|u_2(t, y)\|_{L^\infty_t([0,T],L^2(\mathbb{R}^2))}^2 \|u_1(t, x)\|_{L^\infty_t([0,T],\dot{H}^1(\mathbb{R}^2))} \|u_1(t, x)\|_{L^\infty_t([0,T],L^2(\mathbb{R}^2))} \|u_1(t, x)\|_{L^\infty_t([0,T],L^2(\mathbb{R}^2))}. \]
This implies, 
\[
\frac{2}{M} \int_0^T \int_{\mathbb{R}^2} |u(t, x)|^4 dx dt + O \left( \frac{1}{M^3} \right) \int_0^T \int_{\mathbb{R}^2 \times \mathbb{R}^2} |u(t, x)|^2 |u(t, y)|^2 dxdydt 
+ (3.11) + (3.12) \lesssim \|u_0\|_{L^2(\mathbb{R}^2)}^3 \|u\|_{L^\infty_t(\dot{H}^1([0,T] \times \mathbb{R}^3))}^3.
\]
The proof will be complete once we prove (3.11) and (3.12) are positive.

**Lemma 3.2** Let $f$ be a convex function. Then 
\[ \partial_j \partial_k a(z), \]
gives a positive definite matrix for all $z \in \mathbb{R}^2 \times \mathbb{R}^2$ if $a(z) = f(|x - y|)$.  

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Proof:

\[ \partial_j \partial_k f(|x-y|) = f''(|x-y|) \frac{(x-y)_j(x-y)_k}{|x-y|^2} + f'(|x-y|) \frac{(x-y)_j(x-y)_k}{|x-y|^2} \delta_{jk} - \frac{(x-y)_j(x-y)_k}{|x-y|^2}. \]

Take the inner product defined by this matrix.

\[ \langle z_j z_k | f''(|x-y|) \frac{(x-y)_j(x-y)_k}{|x-y|^2} \rangle = \frac{f''(|x-y|)}{|x-y|^2} (z \cdot (x-y))^2. \]

\[ |\langle z_j z_k | f'(|x-y|) \frac{(x-y)_j(x-y)_k}{|x-y|^2} \rangle| \leq \frac{|f'(|x-y|)|}{|x-y|} |z|^2, \]

\[ \langle z_j z_k | f'(|x-y|) \delta_{jk} \rangle = \frac{f'(|x-y|)}{|x-y|} |z|^2. \]

This proves the lemma. □

In particular, after integrating by parts, \( \text{(3.11)} \) ≥ 0.

To evaluate \( \text{(3.12)} \), without loss of generality take \( j = 1 \).

\[ \overline{F(t,z)} \partial_1 v(t,z) - v(t,z) \partial_1 F(t,z) \]

\[ = |u(t,y)|^2 \overline{u(t,x)u(t,y) \partial_1 (u(t,x)u(t,y))} - \overline{u(t,x)u(t,y) \partial_1 (|u(t,y)|^2 u(t,y)u(t,x))} \]

\[ + |u(t,x)|^2 \overline{u(t,x)u(t,y) \partial_1 (u(t,x)u(t,y))} - \overline{u(t,x)u(t,y) \partial_1 (|u(t,x)|^2 u(t,y)u(t,x))}. \]

\[ |u(t,y)|^2 \overline{u(t,x)u(t,y) \partial_1 (u(t,x)u(t,y))} - \overline{u(t,x)u(t,y) \partial_1 (|u(t,y)|^2 u(t,y)u(t,x))} = 0. \]

\[ |u(t,x)|^2 \overline{u(t,x)u(t,y) \partial_1 (u(t,x)u(t,y))} - \overline{u(t,x)u(t,y) \partial_1 (|u(t,x)|^2 u(t,y)u(t,x))} \]

\[ = - |u(t,x)|^2 \overline{u(t,x)u(t,y) \partial_1 (u(t,x)u(t,y))} - |u(t,x)|^2 \overline{(u(t,x)u(t,y)) \partial_1 (u(t,x)u(t,y))}. \]
\[
= -\frac{1}{2} \partial_1 (|u(t,x)|^4 |u(t,y)|^2).
\]

Similarly,
\[
F \partial_1 u - u \partial_1 F = -\frac{1}{2} \partial_1 (|u(t,x)|^4 |u(t,y)|^2). \tag{3.15}
\]

Make a similar calculation \(j = 2, 3, 4\), although when \(j = 3\) or 4 switch \(x\) and \(y\) in (3.15). Therefore, (3.12) is a sum of terms of the form
\[
- \int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \partial_j (|u(t,x)|^4 |u(t,y)|^2) a_j(z) dxdydt,
\]
when \(j = 1, 2\) and
\[
- \int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \partial_j (|u(t,y)|^4 |u(t,x)|^2) a_j(z) dxdydt,
\]
when \(j = 3, 4\). Integrating by parts and noticing
\[
a_{jj}(z) = f''(|x-y|) \frac{(x-y)^2}{|x-y|^2} + \frac{f'(|x-y|)}{|x-y|} (1 - \frac{(x-y)^2}{|x-y|^2}) \geq 0
\]
proves (3.12) \(\geq 0\). Combining terms,
\[
\frac{2}{M} \int_0^T \int_{\mathbb{R}^2} |u(t,x)|^4 dxdt \lesssim \sup_{[0,T]} \|u(t,x)\|_{L^2}^2 \|u(t,x)\|_{\dot{H}^1} + O\left( \frac{T}{M^3} \right) \sup_{[0,T]} \|u(t,x)\|_{L^4}^4.
\]

Choosing \(M = T^{1/3}\) proves the proposition. \(\Box\)

4 Almost Morawetz Inequalities

In this section, the almost Morawetz estimate in [5, 9] will be improved. For \(u_0\) with regularity below \(s = 1\), if \(u(t,x)\) solves (1.1) then \(Iu(t,x)\) solves
\[
iIu(t,x) + \Delta Iu(t,x) = I(|u(t,x)|^2 u(t,x)). \tag{4.1}
\]

**Proposition 4.1** Define the quantity
\[
Z_I([0,T]) = \sup_{(q,r) \text{ admissible}} \|\langle D \rangle Iu\|_{L_t^q L_x^r([0,T] \times \mathbb{R}^2)}.
\tag{4.2}
\]
\[ \|Iu(t,x)\|_{L^4_t([0,T] \times \mathbb{R}^2)}^4 \lesssim T^{1/3} \|u_0\|_{L^2(\mathbb{R}^2)}^2 \|Iu(t,x)\|_{L^\infty_T(\mathbb{R}^2)} + T^{1/3} \|u_0\|_{L^2(\mathbb{R}^2)}^4 + T^{1/3} \sum_k \frac{Z_1(J_k)^6}{N^2}, \]  

(4.3)

where \( J_k \) is a partition of \([0,T]\). 

**Proof:** Split the nonlinearity 

\[ F = I(|u(t,x)|^2 u(t,x))Iu(t,y) + Iu(t,x)I(|u(t,y)|^2 u(t,y)) = N_g + N_b, \]

\[ N_g = |Iu(t,x)|^2 Iu(t,x)Iu(t,y) + |Iu(t,y)|^2 Iu(t,x)Iu(t,y), \]

\[ N_b = F - N_g. \]  

(4.4)

After taking a tensor product of solutions \( v(t,z) = Iu(t,x)Iu(t,y) \), repeat the procedure from §3 to obtain

\[
-2 \int_0^T |v(t,z)|^2 (\Delta \Delta a(z)) dz \\
+ 8 \int \text{Re}(\bar{\partial}_j v(t,z) \partial_k v(t,z)) dz \\
+ 4 \int (\bar{F}(t,z) \partial_j v(t,z) - \bar{v}(t,z) \partial_j F(t,z)) \\
+ F(t,z) \partial_j v(t,z) - v(t,z) \partial_j F(t,z) \partial_j a(z) dz \\
= M_a(T) - M_a(0) \]  

(4.5)

Once again, the second term \( 8 \int \text{Re}(\partial_j v(t,z) \partial_k v(t,z)) dz \) is strictly positive and can be discarded, as well as the parts of the third term with \( N_g \) in place of \( F \). Therefore

\[
\int_0^T \int |Iu(t,x)|^4 dx dt \lesssim T^{1/3} \|u_0\|_{L^2(\mathbb{R}^2)}^3 \|Iu\|_{L^\infty_T([0,T],H^1(\mathbb{R}^2))} + T^{1/3} \|u_0\|_{L^4(\mathbb{R}^2)}^4 + T^{1/3} \int_0^T \int (\overline{N_b \partial_j v(t,z)} - \overline{v(t,z) \partial_j N_b} + N_b \overline{\partial_j v(t,z)} - v(t,z) \overline{\partial_j N_b}) \partial_j a(z) dz \\
+ T^{1/3} \int_0^T \int (\overline{N_b \partial_j v(t,z)} - \overline{v(t,z) \partial_j N_b} + N_b \overline{\partial_j v(t,z)} - v(t,z) \overline{\partial_j N_b}) \partial_j a(z) dz \\
+ T^{1/3} \int_0^T \int (N_b \partial_j v(t,z) - v(t,z) \partial_j N_b + \overline{N_b \partial_j v(t,z)} - v(t,z) \overline{\partial_j N_b}) \partial_j a(z) dz \\
= (4.6)

To handle 

\[
T^{1/3} \int_0^T \int (\overline{N_b \partial_j v(t,z)} - \overline{v(t,z) \partial_j N_b} + N_b \overline{\partial_j v(t,z)} - v(t,z) \overline{\partial_j N_b}) \partial_j a(z) dz \\
+ T^{1/3} \int_0^T \int (N_b \partial_j v(t,z) - v(t,z) \partial_j N_b + \overline{N_b \partial_j v(t,z)} - v(t,z) \overline{\partial_j N_b}) \partial_j a(z) dz \\
+ T^{1/3} \int_0^T \int (\overline{N_b \partial_j v(t,z)} - \overline{v(t,z) \partial_j N_b} + N_b \overline{\partial_j v(t,z)} - v(t,z) \overline{\partial_j N_b}) \partial_j a(z) dz \\
= (4.6)

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it suffices to handle terms of the form
\[ \int_{J_k} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \nabla a \cdot (N_b \nabla v(t, x)) dz dt, \quad (4.7) \]
as well as terms of the form
\[ \int_{J_k} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \nabla a \cdot (\nabla N_b) v(t, z) dz dt. \quad (4.8) \]
Integrating by parts in \( x \), (4.8) is a sum of terms of the form (4.7), along with terms of the form
\[ \int_{J_k} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \Delta a(z) N_b(t, z) v(t, z) dz dt. \quad (4.9) \]
(4.7) will be tackled first.

\[ \int_{J_k} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \nabla a \cdot (N_b \nabla v(t, z)) dz dt \]
is a sum of terms of the form
\[ \int_{J_k} \int_{\mathbb{R}^2} N_b(t, z) |Iu(t, y)||\nabla Iu(t, x)| dx dy dt. \quad (4.10) \]
\[ N_b = Iu(t, x)[I(|u(t, y)|^2 u(t, y)) - |Iu(t, y)|^2 Iu(t, y)] + Iu(t, y)[I(|u(t, x)|^2 u(t, x)) - |Iu(t, x)|^2 Iu(t, x)]. \]
This implies
\[ (4.10) \lesssim \|I(|u(t, x)|^2 u(t, x)) - |Iu(t, x)|^2 Iu(t, x)\|_{L^1_t L^2_x(J_k \times \mathbb{R}^2)} Z_I(J_k)^3. \quad (4.11) \]
The quantity
\[ \|I(|u(t, x)|^2 u(t, x)) - |Iu(t, x)|^2 Iu(t, x)\|_{L^1_t L^2_x} \quad (4.12) \]
can be estimated by making a Littlewood-Paley partition of \( u(t, x) \). Define a quantity \( F(t, \xi) \)
\[ F(t, \xi) = \int_{\xi_1 + \xi_2 + \xi_3 = \xi} [m(\xi_1 + \xi_2 + \xi_3) - m(\xi_1)m(\xi_2)m(\xi_3)] \times \hat{u}(t, \xi_1)\hat{u}(t, \xi_2)\hat{u}(t, \xi_3) d\xi_1 d\xi_2 \]
\[
\int_{\xi_1 + \xi_2 + \xi_3 = \xi} \frac{[m(\xi) - m(\xi_1)m(\xi_2)m(\xi_3)]}{m(\xi_1)m(\xi_2)m(\xi_3)} \hat{u}(t, \xi_1) \hat{u}(t, \xi_2) \hat{u}(t, \xi_3).
\]

Suppose \(\hat{u}(t, \xi_i)\) is supported on the frequency region \(|\xi_i| \sim N_i\), and without loss of generality suppose \(N_1 \geq N_2 \geq N_3\). Consider four regions separately.

\(N_1 << N\): In this case the multipliers \(m(\xi) = 1\), so

\[
\frac{[m(\xi) - m(\xi_1)m(\xi_2)m(\xi_3)]}{m(\xi_1)m(\xi_2)m(\xi_3)} = 0.
\]

\(N_2 << N \lesssim N_1\): By the fundamental theorem of calculus,

\[
m(\xi_1 + \xi_2 + \xi_3) - m(\xi_1) \lesssim |\xi_2 + \xi_3| \nabla m(\xi_1).
\]

\[
\frac{\nabla m(\xi_1)}{m(\xi_1)} \lesssim \frac{1}{|\xi_1|}.
\]

\[
\frac{|\xi_2 + \xi_3||\xi|}{|\xi_1|^2} \int_{\xi_1} \hat{u}_1(t, \xi_1) \hat{u}_2(t, \xi_2) \hat{u}_3(t, \xi_3) \|_{L^1_x L^2_t}\]

\[
\lesssim \frac{1}{N^2} \|\nabla u_1\|_{L^2_x L^2_t} \|\nabla u_2\|_{L^2_x L^2_t} \|u_3\|_{L^2_x L^2_t} \lesssim \frac{N_1^{-1} N_2^{-1} N_3^{-1}}{N^2} Z^3.
\]

\(N_3 << N \lesssim N_2 \leq N_1\): In this case, make the trivial multiplier estimate,

\[
\frac{|m(\xi_1 + \xi_2 + \xi_3) - m(\xi_1)m(\xi_2)|}{m(\xi_1)m(\xi_2)} \leq \frac{m(\xi_1 + \xi_2 + \xi_3)}{m(\xi_1)m(\xi_2)} + 1
\]

\[
\lesssim \frac{m(\xi_1 + \xi_2 + \xi_3)}{m(\xi_1)m(\xi_2)},
\]

since \(m(\xi_1 + \xi_2 + \xi_3) \sim m(\xi_1)\) and \(m(\xi_2) \leq 1\),

\[
1 + \frac{m(\xi_1 + \xi_2 + \xi_3)}{m(\xi_1)m(\xi_2)} \lesssim \frac{1}{m(\xi_2)}.
\]

\[
\frac{1}{m(\xi_2)|\xi_2||\xi_1|} \lesssim \frac{1}{m(N)|\xi_1|} \lesssim \frac{1}{N^2}.
\]

This uses the fact that \(m(\xi)\xi\) is monotone increasing for any \(s > 0\) and \(m(N)N = N\). Therefore,
\[
\left\| \int_{\xi_1 + \xi_2 + \xi_3 = \xi} \frac{m(\xi_1 + \xi_2 + \xi_3)}{m(\xi_1) m(\xi_2)} \widehat{Iu}(t, \xi_1) \widehat{Iu}(t, \xi_2) \widehat{Iu}(t, \xi_3) d\xi_1 d\xi_2 \right\|_{L^1_t L^2_x} \lesssim \frac{1}{N_3^3 Z_I^3}.
\]

Finally, consider the region
\[
N \lesssim N_3 \leq N_2 \leq N_1: \text{ Doing the same analysis,}
\]
\[
\left\| \int_{\xi_1 + \xi_2 + \xi_3 = \xi} \frac{m(\xi_1 + \xi_2 + \xi_3)}{m(\xi_1) m(\xi_2)|\xi_1||\xi_2||\xi_3|} \frac{1}{m(\xi_1) m(\xi_2) m(\xi_3)} \lesssim \frac{1}{N_3^3 Z_I^3}.
\]
This is because \( m(\xi_1 + \xi_2 + \xi_3) \sim m(\xi_1) \), and \( m(\xi_2)|\xi_2| \gtrsim m(N) N, m(\xi_3)|\xi_3| \gtrsim m(N) N \).
\[
\left\| \int_{\xi_1 + \xi_2 + \xi_3 = \xi} \frac{m(\xi_1 + \xi_2 + \xi_3)}{m(\xi_1) m(\xi_2)|\xi_1||\xi_2||\xi_3|} \nabla \widehat{Iu}(t, \xi_1) \nabla \widehat{Iu}(t, \xi_2) \nabla \widehat{Iu}(t, \xi_3) d\xi \right\|_{L^1_t L^2_x} \lesssim \frac{N_1^1 N_2^1 N_3^1}{N^3 Z_I^3}.
\]
This proves the proposition for terms of the form (4.7).

Turning to (4.9),
\[
(4.9) \lesssim \int_J \int |I(\{u(t, x)\}^2 u(t, x)) - |Iu(t, x)|^2 (Iu(t, x))|Iu(t, x)| \Delta a(|x-y|)|Iu(t, y)|^2 dx dy dt.
\]
On \(|x - y| < \frac{M}{\sqrt{\varepsilon}}\),
\[
\Delta a(x, y) = \frac{2}{M} \log \left( \frac{M}{|x - y|} \right),
\]
and for large \(|x - y|\),
\[
\Delta a(x, y) = O\left( \frac{1}{|x - y|} \right).
\]
Therefore, for $|x - y| > 1$, $\Delta a(x, y)$ is uniformly bounded. This bound is uniform for $M \geq 1$.

\[
\int_0^T \int_{|x-y|>1} |I(|u(t,x)|^2u(t,x)) - |u(t,x)|^2|u(t,x)| \times |u(t,x)|||u(t,y)||^2 \Delta a(x, y) dx dy dt \\
\leq \sup_x \int_{|x-y|>1} \Delta a(x, y)|Iu(t,y)||^2 dy \\
\times \int_0^T \int_{|x-y|>1} |I(|u(t,x)|^2u(t,x)) - |u(t,x)|^2|u(t,x)||u(t,x)| dx dt.
\]

\[
\int_{|x-y|>1} \Delta a(x, y)|Iu(t,y)||^2 \lesssim ||Iu(t,y)||^2_{L^2}.
\]

\[
\int_0^T \int_{|x-y|>1} |I(|u(t,x)|^2u(t,x)) - |u(t,x)|^2|u(t,x)| \times |u(t,x)|||u(t,y)||^2 \Delta a(x, y) dx dy dt \\
\lesssim \|u_0\|^2_3 \|I(|u(t,x)|^2u(t,x)) - |u(t,x)|^2|u(t,x)||L^1_t L^2_x(J_k \times \mathbb{R}^2).
\]

For a fixed $x$ take the region $|x - y| \leq 1$,

\[
\int_{|x-y|\leq 1} \Delta a(x, y)|Iu(t,y)||^2 dy \leq \|Iu(t,y)||^2_{L^2} \left( \frac{1}{M} \log \left( \frac{M}{|x-y|} \right) \right)_{L^1(|x-y|\leq 1)}.
\]

Since $\|Iu(t,y)||^4_{L^4(\mathbb{R}^2)} \leq \|(\nabla)^{1/2}Iu||^2_{L^2(\mathbb{R}^2)}$, therefore,

\[
\sup_x \int \Delta a(x, y)|Iu(t,y)||^2 dy \leq C. \tag{4.14}
\]

\[
\int_{J_k} \int \left|I(|u(t,x)|^2u(t,x)) - |u(t,x)|^2|u(t,x)||Iu(t,x)||\Delta a(x, y)||u(t,y)||^2 dx dy dt \\
\lesssim \|I(|u(t,x)|^2u(t,x)) - |u(t,x)|^2|u(t,x)||L^1_t L^2_x(J_k \times \mathbb{R}^2) Z_I(J_k)^3 \lesssim \frac{1}{N^2} Z_I(J_k)^6.
\]

This completes the proof of the proposition. \(\square\)
5 Proof of Theorem 1.5

Fix a time interval \([0, T_0]\). We wish to show that (1.1) has a solution on that time interval. If \(u(t, x)\) is a solution on \([0, T]\) then
\[
\frac{1}{\lambda} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right)
\]
is a solution on \([0, \lambda^2 T]\). Let \(u_0, \lambda\) denote the rescaled solution at \(t = 0\), and let \(u_{\lambda}(t)\) be the rescaled solution.

\[
\|u_{0, \lambda}\|_{H^s(\mathbb{R}^2)} = \lambda^{-s}\|u_0\|_{H^s(\mathbb{R}^2)}.
\]

(5.1)

Choose \(\lambda = C_0\|u_0\|_{H^s(\mathbb{R}^2)}\) \(N^{(1-s)/s}\) so that

\[
E(Iu_{0, \lambda}) = \frac{2}{5}.
\]

We now wish to prove \(E(Iu_{\lambda}(t)) \leq 1\) on \([0, \lambda^2 T_0]\).

Next, define a subset of \([0, \lambda^2 T_0]\),

\[
F_T = \{t : \tilde{E}(u_{\lambda}(t)) \leq \frac{3}{4}\}.
\]

(5.2)

By the fixed time estimate (2.4), \(|\tilde{E}(u(t)) - E_{\lambda}(Iu(0))| \lesssim \frac{1}{\theta N^2}\), assume \(\tilde{E}(u_{\lambda}(0)) \leq \frac{1}{2}\) since \(E(Iu_{\lambda}(0)) \leq \frac{2}{5}\), therefore, \(0 \in F_T\). Furthermore, \(F_T\) is closed in \([0, \lambda^2 T_0]\) by the dominated convergence theorem. It remains therefore to show \(F_T\) is open in \([0, \lambda^2 T_0]\). If \(\tilde{E}(u_{\lambda}(t)) \leq \frac{3}{4}\) on \([0, T']\), then for some \(\delta > 0\), \(\tilde{E}(u_{\lambda}(t)) \leq \frac{3}{4}\) on \([0, T' + \delta]\), which in turn implies \(E(Iu_{\lambda}(t)) \leq 1\) on \([0, T' + \delta]\).

Because \(E(Iu_{\lambda}(t)) \leq 1\) on \([0, T' + \delta]\), by the Sobolev embedding theorem \(|\|Iu_{\lambda}(t, x)\|_{L^4_t([0, T'+\delta] \times \mathbb{R}^2)}| is finite. Next apply the local well-posedness theorem 2.5 If \(|\|Iu_{\lambda}\|_{L^4_t,J_k \times \mathbb{R}^2} \leq \epsilon\) and \(|\|\nabla\rangle Iu_{0, \lambda}\|_{L^2(\mathbb{R}^2)} \leq 1\), then

\[
Z(J_k, u_{\lambda}) \leq C.
\]

(5.3)

The interval \([0, T' + \delta]\) can be partitioned into

\[
\frac{\|Iu_{\lambda}\|_{L^4_t,J_k \times \mathbb{R}^2}}{\epsilon^4}
\]

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Proof of Theorem 1.5: Let

\[ \|Iu_{\lambda}\|_{L^4_t, x([0, T + \delta])}^4 \leq C' \lambda^{2/3} T_0^{1/3} (\|u_0\|_{L^2}^1 + \|u_0\|_{L^2}^1) + \frac{1}{N^{2-}} \sum_k Z_k(J_k, u_\lambda)^6 \]

\[ \leq C' \lambda^{2/3} T_0^{1/3} (\|u_0\|_{L^2}^1 + \|u_0\|_{L^2}^1) \sup_{[0, T + \delta]} E(Iu_{\lambda}(t)) + \frac{C_6}{e^{4N^{2-}} \|Iu_{\lambda}\|_{L^4_t, x}^4}. \]

(5.4)

\[ \lambda^{2/3} N^{-2} \sim N^{\frac{2-8s}{3s}}, \]

so for \( s > 1/4 \), choosing \( N \) sufficiently large,

\[ \frac{C'C_6}{e^{4}} \lambda^{2/3} T_0^{1/3} \leq \frac{1}{2}. \]

Therefore, the remainder can be absorbed into the left hand side and

\[ \|Iu_{\lambda}\|_{L^4_t, x([0, T + \delta])}^4 \leq 2C' \lambda^{2/3} T_0^{1/3} (\|u_0\|_{L^2}^1 + \|u_0\|_{L^2}^1). \]

Partitioning \([0, T + \delta]\) into \( 2C' \lambda^{2/3} T_0^{1/3} (\|u_0\|_{L^2}^1 + \|u_0\|_{L^2}^1) \) pieces,

\[ \sup_{t \in [0, T + \delta]} |\tilde{E}(u_{\lambda}(t))| \leq \frac{1}{2} + \frac{2C' \lambda^{2/3} T_0^{1/3} (\|u_0\|_{L^2}^1 + \|u_0\|_{L^2}^1)}{e^{4N^{2-}}} \]

(5.5)

Taking \( N(T_0, \|u_0\|_2) \) sufficiently large, this implies \( F_T = [0, \lambda^2 T_0] \).

Proof of Theorem 1.5: Let

\[ N = \left( \frac{20C'C_0^{2/3} T_0^{1/3} (m_0^4 + m_0^3) 3s/(8s-2)+}{e^{4}} \right)^{3s/(8s-2)} = C(m_0) T_0^{\frac{s}{8s-2}+}. \]

This implies

\[ \sup_{[0, \lambda^2 T_0]} \tilde{E}(u_{\lambda}(t)) \leq \frac{3}{5}, \]

which in turn implies \( E(Iu_{\lambda}(t)) \leq 1 \) on \([0, \lambda^2 T_0]\). Splitting the solution \( u_{\lambda}(t) = P_{\leq N} u_{\lambda}(t) + P_{> N} u_{\lambda}(t), \)

\[ \|P_{\leq N} u_{\lambda}\|_{H^s(\mathbb{R}^2)} \leq \|I_N u_{\lambda}\|_{H^1(\mathbb{R}^2)} \leq E(Iu_{\lambda}(t)), \]

\[ \|P_{> N} u_{\lambda}\|_{H^s(\mathbb{R}^2)} \leq N^{s-1} \|I_N u_{\lambda}\|_{H^1(\mathbb{R}^2)} \leq N^{s-1} E(Iu_{\lambda}(t)), \]

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which proves that
\[ \|u_\lambda(t)\|_{H^s(\mathbb{R}^2)} \leq 2. \]

Finally, \( \lambda = C_0N^{\frac{1-s}{2}} = C(m_0)C_0(\|u_0\|_{H^s})T_0^{\frac{1-s}{2-s}} \), so rescaling back,
\[
\sup_{t \in [0,T]} \|u(t)\|_{H^s(\mathbb{R}^2)} \leq C(m_0)C_0(\|u_0\|_{H^s})T^{\frac{s(1-s)}{2-s}}. \tag{5.6}
\]

This proves the theorem. \( \square \)
References

[1] J. Bourgain. Refinements of Strichartz’ inequality and applications to 2D-NLS with critical nonlinearity. *International Mathematical Research Notices*, 5:253 – 283, 1998.

[2] J. Bourgain. *Global Solutions of Nonlinear Schrödinger Equations*. American Mathematical Society Colloquium Publications, 1999.

[3] T. Cazenave and F. Weissler. The Cauchy problem for the nonlinear Schrödinger Equation in $H^1$. *Manuscripta Mathematica*, 61:477 – 494, 1988.

[4] T. Cazenave and F. Weissler. The Cauchy problem for the nonlinear Schrödinger Equation in $H^s$. *Nonlinear Analysis*, 14:807 – 836, 1990.

[5] J. Colliander, M. Grillakis, and N. Tzirakis. Improved interaction Morawetz inequalities for the cubic nonlinear Schrödinger equation on $\mathbb{R}^2$. *Int. Math. Res. Not. IMRN*, (23):90 – 119, 2007.

[6] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Almost conservation laws and global rough solutions to a nonlinear Schrödinger equation. *Mathematical Research Letters*, 9:659 – 682, 2002.

[7] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Global existence and scattering for rough solutions of a nonlinear Schrödinger equation on $\mathbb{R}^3$. *Communications on Pure and Applied Mathematics*, 21:987 – 1014, 2004.

[8] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Resonant decompositions and the I-method for cubic nonlinear Schrödinger equation on $\mathbb{R}^2$. *Discrete and Continuous Dynamical Systems A*, 21:665 – 686, 2007.

[9] J. Colliander and T. Roy. Bootstrapped Morawetz estimates and resonant decomposition for low regularity global solutions of cubic NLS on $\mathbb{R}^2$. *preprint, arXiv:0811.1803*.

[10] D. De Silva, N. Pavlović, G. Staffilani, and N. Tzirakis. Global well-posedness for the $L^2$-critical nonlinear Schrödinger equation in higher dimensions. *to appear, Communications on Pure and Applied Analysis*.

[11] M. Keel and T. Tao. Endpoint Strichartz estimates. *American Journal of Mathematics*, 120:955 – 980, 1998.
[12] C. Sogge. *Fourier Integrals in Classical Analysis*. Cambridge University Press, 1993.

[13] E. Stein. *Harmonic Analysis: Real Variable Methods, Orthogonality, and Oscillatory Integrals*. Princeton University Press, 1993.

[14] T. Tao. *Nonlinear Dispersive Equations: Local and Global Analysis*. American Mathematical Society, 2006.

[15] M. Taylor. *Pseudodifferential Operators and Nonlinear PDE*. Birkhauser, 1991.

[16] M. Taylor. *Partial Differential Equations*. Springer Verlag Inc., 1996.

[17] Y. Tsutsumi. $L^2$ solutions for nonlinear Schrödinger equation and nonlinear groups. *Funktional Ekvacioj*, 30:115 – 125, 1987.