Spectral densities of superconducting qubits with environmental resonances.

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(Dated: May 15, 2008)

PACS numbers: 74.50.+r, 85.25.Dq, 03.67.Lx

In this paper we derive the environmental spectral density for flux [1], phase [2] and charge [3] qubits when each of them is coupled to an environment with a resonance. From the spectral density we obtain the characteristic spontaneous emission (relaxation) lifetimes \( T_1 \) for each of these qubits, and show that there is a substantial enhancement of \( T_1 \) beyond the resonant frequency of the environment. The circuits considered are shown in Fig. 1, 2 and 3.

The flux qubit shown in Fig. 1 is measured by a dc-SQUID. Hence to study decoherence and relaxation time scales, one has to consider the noise that is transferred from the qubit to the dc-SQUID.

The classical equation of motion for the dc-SQUID is

\[
\ddot{\phi} + \frac{2\pi}{\Phi_0} L_c \phi \sin \phi - \frac{2\pi}{\Phi_0} I + \int_0^t dt' Y(t-t') \dot{\phi}(t') = 0 \tag{1}
\]

where \( \phi \) is the gauge invariant phase across the Josephson junction of the outer dc-SQUID loop, \( I_{c0} \) is the critical current of its junction, \( \Phi_0 = h/2e \) is the flux quantum, and the total induced current in the outer dc-SQUID is

\[
I = \frac{4}{L_{dc}} \langle \delta \phi_0 \sigma_z \rangle + \frac{4}{L_{dc}} \langle \phi_m \rangle + \left( \frac{2\pi}{\Phi_0} \right)^2 J_1 \langle \phi_p \rangle. \tag{2}
\]

Here \( \phi_p \) and \( \phi_m \) are the sum and the difference of the gauge invariant phases across the junctions of the inner SQUID, \( L_{dc} \) is the self-inductance of the inner SQUID, and \( J_1 \) is the bilinear coupling between \( \phi_m \) and \( \phi_p \) at the potential energy minimum. The term \( \delta \phi_0 = \pi M_q I_{cir}/\Phi_0 \), where \( I_{cir} \) is the circulating current of the localized states of the qubit (described in terms of Pauli matrix \( \sigma_z \)), and \( M_q \) is the mutual inductance between the qubit and the outer dc-SQUID. The last term in Eq. (1) is the dissipation term due to effective admittance \( Y(\omega) \) felt by the outer dc-SQUID.

For the charge qubit shown in Fig. 2 the classical equation of motion for the charge \( Q \) is

\[
V_g(\omega) = -\frac{\omega^2 I_f(\omega)}{1-\omega^2 I_f(\omega) C_J} + \frac{1}{C_g} + i\omega Z(\omega) \tag{3}
\]

Here, \( V_g \) is the gate voltage, \( C_g \) is the gate capacitance, \( I_f \) and \( C_J \) are the Josephson inductance and capacitance respectively, \( Z(\omega) \) is the effective impedance seen by the charge qubit due to a transmission line resonator (cavity), \( \Omega \) is the resonant frequency of the resonator, and \( Q \) is the charge across \( C_g \).

For the phase qubit shown in Fig. 3 the classical equation of motion is

\[
C_0 \ddot{\gamma} + \frac{2\pi}{\Phi_0} I_{c0} \sin \gamma - \frac{2\pi}{\Phi_0} I + \int_0^t dt' Y(t-t') \dot{\gamma}(t') = 0 \tag{4}
\]

where \( I_{c0} \) is the critical current of Josephson junction \( J \) in Fig. 2, \( I \) is the bias current, and \( \Phi_0 = h/2e \) is the flux quantum. The last term is the dissipation term due to \( Y(\omega) \) which is the effective admittance as seen by the dc-SQUID.

The equations of motion described in Eqs. (1), (3), and (4), can be all approximatelly described by the effective spin-boson Hamiltonian

\[
\tilde{H} = \frac{\hbar \omega_{01}}{2} \sigma_z + \sum_b \hbar \omega_b b_b^\dagger b_b + H_{SB}, \tag{5}
\]

where \( \sigma_z \) is the Pauli matrix, \( b_b^\dagger \) and \( b_b \) are the boson creation and destruction operators, \( \omega_b \) is the boson frequency, and \( H_{SB} \) is the spin-boson interaction Hamiltonian.
The Ohmic resistance of the circuit is modelled by $C$ capacitance for charge qubits.

In this case, the environmental spectral density is $J$ and $\gamma$ for the phase qubit. The role of the resonant frequency, where $\omega_i$ represents a two-level approximation for the qubit (system) described by states $|0\rangle$ and $|1\rangle$ with energy difference $\hbar \omega_{01}$. The second term corresponds to the isolation network (bath) represented by a bath of bosons, where $b_k$ and $b_k^\dagger$ are the annihilation and creation operator of the $k$-th bath mode with frequency $\omega_k$. The third term is the system-bath (SB) Hamiltonian which corresponds to the coupling between the environment and the qubit.

At the charge degeneracy point for the charge qubit (gate charge $N_y = 0$), at the flux degeneracy point (external flux $\Phi_{ext} = \pi \Phi_0$) for the flux qubit, and the suitable flux bias condition for the phase qubit (external flux $\Phi_a = L_1 \phi_0$) $H_{SB}$ reduces to,

$$H_{SB} = \frac{1}{2} \sigma_x \hbar (1|v|0) \sum_k \lambda_k (b_k^\dagger + b_k)$$

where $v = \phi$ for the flux qubit, $Q$ for the charge qubit, and $\gamma$ for the phase qubit. The spectral density of the bath modes $J(\omega) = \hbar \sum_k \lambda_k^2 \delta (\omega - \omega_k)$ has dimensions of energy and can be written as $J(\omega) = \omega \text{Re} Y(\omega)/(\Phi_0/2\pi)^2$ for flux and phase qubits and $J(\omega) = 2\hbar \omega^2/\hbar \text{Re} \mathcal{Z}(\omega)$ for charge qubits.

For the flux qubit circuit shown in Fig. 1, the shunt capacitance $C_s$ is used to control the environment, while the Ohmic resistance of the circuit is modelled by $R$. In this case, the environmental spectral density is

$$J_1(\omega) = \frac{\alpha_1 \omega}{(1 - \omega^2/\Omega_1^2)^2 + 4\omega^2 \Gamma_1^2/\Omega_1^4},$$

when $\omega_m \gg \max(\omega_p, \omega_0)$, and when the dc-SQUID is far away from the switching point to be modelled by an ideal inductance $L_J$. Here, $\Omega_1 = 1/\sqrt{L_J C_s} = \sqrt{4\pi I_c/(C_s \Phi_0)}$ is the plasma frequency of the inner SQUID and plays the role of the resonant frequency, where $I_c$ is the critical current for each of two Josephson junctions. Also, $\Gamma_1 = 1/(C_s R)$ plays the role of the resonance width, and $\alpha_1 = 2(eI_p I_b M_q)^2/(C_s^2 \hbar^2 R \Omega_1^2)$ reflects the low frequency behavior. The coupling between the flux qubit and the outer dc-SQUID occurs emerges from the interaction of the persistent current $I_p$ of the qubit and the bias current $I_b$ of the dc-SQUID via their mutual inductance $M_q$.

The spectral density for the charge qubit shown in Fig. 2 is obtained by solving for the normal modes of the resonator and transmission lines, including an input impedance $R$ at each end of the resonator. It is given by

$$J_3(\omega) = \frac{e^2 \Omega_2}{\ell c} \frac{\Gamma_2}{(\omega - \Omega_2)^2 + (\Gamma_2/2)^2}$$

were $\Omega_2$ is the resonator frequency, $\ell$ is resonator length, $c$ is the capacitance per unit length of the transmission line, $C_q$ is the gate capacitance, and $C_J$ is the junction capacitance. The quantity $\Gamma_2 = \Omega_2/Q$ where $Q$ is the quality factor of the cavity.

For negligible $Y_{int}(\omega)$, the spectral density of the phase qubit shown in Fig. 3 is

$$J_3(\omega) = \frac{\Phi_0}{2\pi} \left( \frac{\alpha_{3\omega}}{(1 - \omega^2/\Omega_3^2)^2 + 4\omega^2 \Gamma_3^2/\Omega_3^4} \right),$$

where $\alpha_3 = L^2/(L + L^2)R \approx (L/L_0)^2/R$ is the leading order term in the low frequency ohmic regime, $\Omega_3 = \sqrt{(L + L_0)/(LL_0C)} \approx 1/\sqrt{LC}$ is essentially the resonance frequency, and $\Gamma_3 = 1/(2CR)$ plays the role of resonance width. Here, we used $L_1 \gg L$ corresponding to the relevant experimental regime.

Once the spectral functions of the environments are known, the relaxation rates are derived following standard methods [4]. At finite temperatures, the relaxation time

$$\frac{\hbar}{T_{1,i}} = \beta_i J_i(\omega) \coth \frac{\hbar \omega}{2k_B T}$$

is directly related to the environmental spectral density. Here, the index $i = 1, 2, 3$ labels the results for flux, charge and phase qubits, respectively. The parameter $\beta_i$ takes values $\beta_1 = 1$ for the flux qubit of Fig. 1, $\beta_2 = C_q^2/(C_q + C_J)^2$ for the charge qubit of Fig. 2 and $\beta_3 = 1/[(\Phi_0/2\pi)^2 C_0 \omega_{01}]$ for the phase qubit of Fig. 3.

An inspection of Eq. 10 and the corresponding spectral functions $J_i(\omega)$ shows that suitably detuning the qubit to higher frequencies beyond the environmental resonance can enhance the characteristic spontaneous emission lifetime (relaxation time) $T_{1,i}$ of the qubit by a couple of orders of magnitude as discussed in Ref. [4].

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