Boundary Terms in Generalized Geometry and Doubled Field Theory

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Abstract

We propose a boundary action to complement the recently developed duality manifest actions in string and M–theory using generalized geometry. This boundary action combines the Gibbons–Hawking term with boundary pieces that were previously neglected in the construction of these actions. The combination may be written in terms of the metric of generalized geometry. The result is to produce an action that is duality invariant including boundary terms.

1 Introduction

Dualities have always played a central role in string theory. T–duality is a hidden symmetry from the spacetime point of view and a nonperturbative symmetry from the world–sheet point of view [1]. Already during the early days of string theory there were attempts to reformulate the theory to make T–duality a manifest symmetry [2, 3, 4, 5, 6, 7]. This was done by doubling the dimension of the space where T–duality acts so that T–duality was linearly realized on this space. The metric on this doubled space turns out to be the same as that of the generalized geometry introduced by Hitchin [8]. A useful and natural by product is the incorporation of the NS–NS two form potential into the generalized metric so that one only needs the generalized metric on the doubled space and the dilaton. The dilaton is not doubled but is shifted from the usual string frame dilaton; this is related to how the dilaton is required to shift under T-duality. (In doubled field theory the Ramond Ramond sector must be incorporated separately see [9, 10] for recent work in this direction.)

Recently the doubled approach has had something of a rebirth [11, 12]. Its quantum properties have been investigated in [13, 14, 15, 16]. It has also been extended to M–theory in [17, 18, 19, 20, 21] where the duality group is related to the U–duality groups of string theory. (For a review of U-duality see [22].) The relationship between the doubled...
formalism in string theory and the M-theory extended geometry is explored in detail in [23]. The type II string has been developed in [24]. All these recent approaches may be viewed as a small part of the larger programme to encode the symmetries of M-theory into $E_{11}$. For early work in this direction see [25] and more recent work directly related to these ideas see [26].

The use of the generalized metric on the doubled (or, in the case of M–theory, extended) space allows one to construct a duality manifest action for supergravity in terms of this generalized metric. Importantly, even though the duality was derived using a flat toroidal space, it turns out no such requirement is necessary for this reformulation of the spacetime action. The duality frame is chosen by simply specifying how one dimensionally reduces from the whole extended space to physical spacetime. This is discussed in detail in [19, 20, 27].

The duality manifest actions of supergravity will be the topic of this paper. These actions are quadratic in derivatives whereas the Einstein-Hilbert action is of course second order in derivatives. Thus, the equivalence of the duality manifest actions to the usual action was demonstrated only after an integration by parts which turns the Einstein-Hilbert into an action quadratic in first derivatives and then one of course neglects boundary terms. A natural question given generalized geometry defined by a metric is how does one construct a connection and curvature. The attempts to construct these ideas in generalized geometry are described in [28, 29, 30, 10].

This equivalence (and subsequent neglect of boundary terms) was described for the doubled string in [27], the extended heterotic string in [31] and in M–theory in [19, 20, 21].

There are two possibilities. The boundary action could break the duality manifest form or it can be reformulated in terms of the generalized metric and the boundary action itself becomes reformulated in a duality invariant way. Both possibilities would be allowed logically.

Famously, gravity already contains a boundary action, the York– Gibbons–Hawking term [32, 33] (usually just called Gibbons–Hawking). The necessity for the Gibbons–Hawking term in the action has its origins in deriving the equations of motion from the Einstein-Hilbert subject to appropriate boundary conditions. This term is also required for completeness of transition amplitudes in quantum gravity [34]. The definition of the complete gravitational action then has the rather useful by product of allowing a simple calculation of blackhole thermodynamics. The action when evaluated on shell in a Euclideanized background becomes equivalent to the free energy of the system. Thermodynamic quantities such as energy and entropy are then found from the free energy just by taking derivatives. For the Schwarzschild black hole the bulk action vanishes on shell and all the thermodynamic information is contained in the the Gibbons-Hawking boundary term.

The point here is to emphasize that actions have applications beyond simply encoding the equations of motion and naive quantization. In gravity, they encode the thermodynamics of solutions.

This paper will combine boundary terms arising from two sources the boundary terms, required to equate the reformulated duality manifest actions with the Einstein-Hilbert action and the Gibbons-Hawking boundary term. The combination will then be written
in terms of the generalized metric. The equivalence to the usual form will come from restricting the boundary on the doubled (or extended space). These restrictions on the boundary will be consistent with the dimensional reduction conditions required to show equivalence of the bulk actions.

Other recent interesting work on doubled field theory is given in [35, 36, 37, 38, 39, 40].

2 Doubled field theory and generalized geometry

The usual bosonic part of the low energy effective action for NS-NS sector of the conventional string is as follows:

$$S = \int_M \sqrt{g}e^{-2\phi} \left( R[g] + 4(\partial\phi)^2 - \frac{1}{12} H^2 \right),$$

where $H_{ijk} = 3\partial_i b_{jk}$ is a field strength for the Kalb–Ramond field.

To make the $O(d, d)$ symmetry manifest, the space is doubled by including so called winding-mode coordinates which then allow the action to be written in terms of a generalized metric $H_{MN}$ on the doubled space. We denote the doubled space by $M^*$ with coordinates $X^M = (\tilde{x}_i, x^j)$, where $M = 1, \ldots, 2d$. The action in terms of the generalized metric on the doubled space is given by:

$$S = \int_{M^*} e^{-2d} \left( \frac{1}{8} H^{MN} \partial_M H^{KL} \partial_N H_{KL} - \frac{1}{2} H^{KL} \partial_L H^{MN} \partial_N H_{KM} 
- 2\partial_M d \partial_N H^{MN} + 4 H^{MN} \partial_M d \partial_N d \right).$$

The doubled dilaton $d$ is written in terms of the usual dilaton $\phi$ as follows

$$e^{-2d} = \sqrt{g}e^{-2\phi}.$$

The generalized metric, $H_{MN}$ is a quadratic form on this doubled space with coordinates $X^M$ and is constructed from the usual metric $g_{ij}$ and two form $b_{ij}$ on the nondoubled space as below:

$$H_{MN} = \begin{bmatrix} g^{ij} & b^i_k \\ -b^j_i & g_{kl} + b^a_k b_{la} \end{bmatrix}, \quad X^M = \begin{bmatrix} \tilde{x}_m \\ x^m \end{bmatrix}.$$

$H^{MN}$ denotes the inverse of $H_{MN}$.

The theory is now also equipped with a constraint on the fields that has its origin in the level matching condition. This is given by,

$$\partial_{\tilde{x}_i} \partial_{x_i} = 0.$$

This may be written in the doubled $O(d, d)$ invariant form as:

$$\eta^{AB} \partial_A \partial_B = 0,$$
where $\eta_{AB}$ is the $O(d, d)$ metric:

$$\eta_{AB} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$  \hfill (7)

Different duality frames are chosen by different choices of solution of this condition with the most obvious choices being (though by no means the only ones):

$$\partial \tilde{x} = 0 \quad \text{or its dual} \quad \partial x = 0.$$  \hfill (8)

Taking the first choice so that all fields are taken to be independent on the winding coordinates $\tilde{x}$, it was shown [12] that (2) reduces to (1) up to boundary terms. For us it will be important to keep the previously neglected boundary terms. Imposing this constraint but keeping boundary terms we obtain the resulting action:

$$S = \int_M \sqrt{g} e^{-2\phi} \left( R[g] + 2(\partial \phi)^2 - \frac{1}{12} H^2 \right) - \int_M \partial_m \left[ e^{-2\phi} \sqrt{gg^{mn}} g^{nb} \partial_n g_{bc} - e^{-2\phi} \sqrt{gg^{mn}} g^{nb} \partial_c g_{nb} \right].$$  \hfill (9)

It is natural to then combine the total derivative term in the above with the Gibbons–Hawking term (modified by dilaton). This boundary term introduced by Gibbons and Hawking in [33] is given by:

$$S_{GH} = 2 \oint_{\partial M} \sqrt{e} e^{-2\phi} K = 2 \oint_{\partial M} \sqrt{h} e^{-2\phi} h^{ab} (\partial_a n_b - \Gamma^m_{ab} n_m)$$

$$= 2 \oint_{\partial M} \sqrt{h} e^{-2\phi} h^{ab} \partial_a n_b - \oint_{\partial M} \sqrt{h} e^{-2\phi} h^{mn} (2\partial_m h_{ab} - \partial_n h_{ab}) n_m$$  \hfill (10)

where $K = \nabla_i n^i$ is the trace of the second fundamental form for the induced metric on the boundary, $n_a$ is normal on the boundary and $h_{ab}$ is metric on the boundary.

Comparing (9) and (10) (and with the replacement of $g$ by $h$) one obtains:

$$\int_M \sqrt{g} e^{-2\phi} \left( R[g] + 4(\partial \phi)^2 - \frac{1}{12} H^2 \right) + S_{GH} = S + \oint_{\partial M} \sqrt{e} e^{-2\phi} (2h^{ab} n_a n_b - n^c h^{ab} \partial_c h_{ac}).$$  \hfill (11)

We now wish to write the boundary term on the right hand side of (11) in $O(d, d)$–covariant form by recasting it in terms of the generalized metric. This produces,

$$S_{tot} = S + \oint_{\partial M^*} e^{-2d} \left[ 2 \mathcal{H}^{AB} \partial_A N_B + N_A \partial_B \mathcal{H}^{AB} \right].$$  \hfill (12)

The normal $N_A$ is now the unit normal to the boundary in the doubled space. The expression (12) is $O(d, d)$ covariant and should be true in any duality frame.

In order for this term to match the boundary term in (11) (after a duality frame is chosen to give the usual bulk action) we require that the possible normal vector for the boundary in the doubled space be restricted to the form:

$$N_A = \begin{bmatrix} 0 \\ n_a \end{bmatrix}, \quad N^A = \begin{bmatrix} -b^j n_i \\ n^a \end{bmatrix}.$$  \hfill (13)
This normal is such that the normalization condition doesn’t imply any constraints to the dynamical fields $g_{ij}$ and $b_{ij}$:

$$N^A N^B \mathcal{H}_{AB} = 1 \implies n_a n^a = 1.$$  \hspace{1cm} (14)

The fact that the normal is only allowed components along the $x^i$ directions is due to the fact that we chose the particular duality frame where fields are independent of $\tilde{x}_i$. A direct consequence of this is that there could be no boundary located in $\tilde{x}_i$ as this would break $\tilde{x}_i$ translation invariance. Of course, if we chose the T-dual frame where fields are independent of $x^i$ then we would have to choose the opposite condition on the boundary normal. A natural conjecture is that the general restriction on the boundary normal follows from the constraint which has its origins in the level matching condition so that in general we require that:

$$N^A \eta_{AB} N^B = 0.$$  \hspace{1cm} (15)

There is also a doubled geometry for the heterotic string \[31\]. Now there are also new coordinates denoted by $y_i$ that are dual to the gauge fields. A similar story follows with an action written in terms of some metric on the extended space the usual low energy action for the heterotic string following from reducing the theory.

The second order derivatives appear only in the Ricci scalar term term of the effective action. Hence, the additional pieces in the generalized metric for the heterotic string do not affect the form of the boundary term and it remains the same as before ie. equation (12). The normal $N_A$ will be restricted by the requirement of reproducing the usual action once the conditions,

$$\partial_{\tilde{x}} = 0, \quad \partial_{y} = 0$$  \hspace{1cm} (16)

are chosen so that now similarly to before

$$N_A = \begin{bmatrix} 0 \\ n_a \\ 0 \end{bmatrix}.$$  \hspace{1cm} (17)

### 3 M–theory

The M–theory version of the doubled formalism introduces coordinates for all possible wrapped branes. As such the duality groups are more complicated and the generalized metric becomes much more complicated. This will not effect the boundary issues we are discussing and so we will illustrate the boundary action for the case where the duality group is $SL(5)$ and we have only four dimensions involved in the duality transformation. In four dimensions only the membranes can wrap. The dual coordinates may be labelled by $y_{ab}$ (in four dimensions there are six such directions leading to a total generalized space of ten dimensions). The coordinates will be in the 10 of $SL(5)$ and the generalized metric will transform linearly under $SL(5)$ transformations. The coordinates of generalized space are denoted by:

$$X^M = \begin{bmatrix} x^a \\ y_{ab} \end{bmatrix}.$$  \hspace{1cm} (18)
(Note, the difference in notation as compared with [11]. This is so as to be consistent in each case with the original literature). A metric on this space can be constructed which will combine the ordinary metric and $C$ field, [19] as follows.

\[
M_{MN} = \begin{bmatrix}
g_{ab} + \frac{1}{2} C^c_{a} e^{f} C_{bf} & \frac{1}{\sqrt{2}} C_{a}^{kl} \\
\frac{1}{\sqrt{2}} C_{b}^{mn} & g_{mn,kl}
\end{bmatrix};
\] (19)

where $g^{kl,rs} = \frac{1}{2}(g^{kr}g^{ls} - g^{ks}g^{lr})$. The action in terms of this metric is given by:

\[
V = \sqrt{g} \left[ \frac{1}{12} M^{MN}(\partial_{M}M_{KL})(\partial_{N}M^{KL}) - \frac{1}{2} M^{MN}(\partial_{N}M^{KL})(\partial_{L}M_{MK}) + \frac{1}{12} M^{MN}(M^{KL}\partial_{M}M_{KL})(M^{RS}\partial_{N}M_{RS}) + \frac{1}{4} M^{MN}M^{PQ}(M^{RS}\partial_{P}M_{RS})(\partial_{M}M_{NQ}) \right],
\] (20)

where $\partial_{M} = (\frac{\partial}{\partial x^a}, \frac{\partial}{\partial y^{ab}})$ and $M$ is the generalized metric.

As before, if one evaluates this action under the assumption that the fields are independent of the dual coordinates, i.e., $\partial_{y^{ab}} = 0$ then one recovers the usual Einstein-Hilbert action with the kinetic term for the $C$ field.

The necessary boundary term is similar to the string theory cases. It again has the form of (12) but without the dilaton and the normal is again restricted to obey:

\[
N_{M} = \begin{bmatrix} n_{m} \\ 0 \end{bmatrix}, \quad N^{M} = \begin{bmatrix} \eta^{m} \\ -\frac{1}{\sqrt{2}} C^{n}_{rs} n_{n} \end{bmatrix}.
\] (21)

This restriction of the normal is where one has chosen the fields to be independent of the $y$ coordinates. The general duality invariant restriction on the normal is not known since this is tied up with the physical section condition. Despite some obvious conjectures the physical section condition for the M-theory extended geometry is not known.

### 4 Discussion

The motivation for including these boundary terms is not just to provide a more thorough treatment of the duality reformulated actions but importantly to include the Gibbons-Hawking boundary term in the reformulated actions. The point then is to allow the thermodynamic properties to be calculated from these duality manifest actions through evaluating the action on shell. The hope would then be to use this to derive duality manifest thermodynamic properties of supergravity solutions.

The other application of these surface terms is in constructing globally well-defined solutions. These boundary terms provide important constraints on the construction of any topologically non-trivial solutions; the importance of such solutions in doubled geometry has been stressed recently in [11].
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