GENERALIZED BERNSTEIN OPERATORS DEFINED BY INCREASING NODES

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Abstract. We study certain generalizations of the classical Bernstein operators, defined via increasing sequences of nodes. Such operators are required to fix two functions, \( f_0 \) and \( f_1 \), such that \( f_0 > 0 \) and \( f_1/f_0 \) is increasing on an interval \([a,b]\). A characterization regarding when this can be done is presented. From it we obtain, under rather general circumstances, the following necessary condition for existence: if nodes are non-decreasing, then \((f_1/f_0)' > 0\) on \((a,b)\), while if nodes are strictly increasing, then \((f_1/f_0)' > 0\) on \([a,b]\).

1. Introduction

Let \( \mathbb{P}_n = \mathbb{P}_n[a,b] \) denote the space of polynomials of degree bounded by \( n \), over the interval \([a,b]\). In recent years there has been a continued interest in finding generalizations or modifications of the classical Bernstein operators \( B_n : C[a,b] \rightarrow \mathbb{P}_n[a,b] \), defined by

\[
B_n f(x) = \sum_{k=0}^{n} f\left(a + \frac{k}{n}(b-a)\right) \binom{n}{k} \frac{(x-a)^k (b-x)^{n-k}}{(b-a)^n},
\]

to more general spaces of functions, but still reproducing a two-dimensional subspace, say \( \text{Span}\{f_0, f_1\} \), with \( f_0 > 0 \) and \( f_1/f_0 \) injective. Within the realm of polynomial spaces, one asks when the exact reproduction of functions, other than the affine ones, is possible. Also, similar questions have been asked about related positive operators, (cf. for instance [1]).

Sometimes fixing a subspace different from the affine functions is achieved by modifying the Bernstein bases (consider, for instance, the nowadays called King’s operators, after [9]). Within the line of research followed here (cf. [12], [3], [4], [5], [10], [6], [11], [2], [7]) fixing \( f_0 \) and \( f_1 \) is achieved, when possible, by modifying the location of the nodes \( t_{n,k} \) (instead of having \( t_{n,k} = a + \frac{k}{n}(b-a) \) as in (1)). A motivation for this approach is that it allows us to keep the Bernstein bases unchanged, a desirable feature given their several optimality properties, cf. for instance [8]. Multiplying by \(-1\) if needed, we may assume that \( f_1/f_0 \) is strictly increasing.

The situation regarding the existence of generalized Bernstein operators, defined by strictly increasing sequences of nodes, is well understood in the context of extended Chebyshev spaces, cf. [4]: one considers a two-dimensional extended Chebyshev space \( U_1 \), for which a generalized

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Bernstein operator fixing it can always be defined with increasing nodes (since they are the endpoints of the interval), and inductively, via the interlacing property of nodes (cf. [4, Theorem 6]) this definition is extended to $U_1 \subset U_2 \subset \cdots \subset U_n$, where each $U_k$ is a $k + 1$-dimensional extended Chebyshev space.

But this framework is insufficient even for spaces of polynomials, since for instance, it cannot handle the case where we have $U_1 := \text{Span}\{1, x^3\}$ over $[a, b]$, with $a < 0 < b$ (cf. Example 2.5). It is thus natural to try to go beyond chains of extended Chebyshev spaces. Now, a salient difference between various definitions of generalized Bernstein operators appearing in the literature, is whether one should require the sequence of nodes to be strictly increasing (as in [11]), or not (as in [4, 5]). Of course, having strictly increasing nodes leads to better properties from the point of view of shape preservation, but existence will be obtained in fewer cases.

We shed light on this issue by characterizing, in terms of the spaces $U_n$ and $D_{f_0}U_n := \left\{ \frac{d}{dx} \left( f/f_0 \right) : f \in U_n \right\}$, when the sequence of nodes is non-decreasing, and when it is strictly increasing, cf. Theorem 3.3 below for full details. This Theorem improves on [3, Theorem 1] (the main result of [3]) and generalises [7, Theorem 3.2], which deals exclusively with the polynomial case $P_n[a, b]$. From Theorem 3.3 the following necessary condition is obtained: if both spaces $U_n$ and $D_{f_0}U_n := \left\{ \frac{d}{dx} \left( f/f_0 \right) : f \in U_n \right\}$ have positive Bernstein bases (a hypothesis weaker than being extended Chebyshev spaces) then the existence of a generalized Bernstein operator having non-decreasing nodes entails that $(f_1/f_0)' > 0$ on $(a, b)$, while if nodes are strictly increasing, then $(f_1/f_0)' > 0$ on $[a, b]$, cf. Corollary 3.4.

Since this result contradicts some statements made in [11, Section 7.2], in an effort to clarify these issues we have emphasized concrete examples and explicit computations throughout the paper.

To sum up, the difference between the cases where $f_1'$ vanishes at some point inside $(a, b)$, and where $f_1' > 0$ on $(a, b)$, turns out to be very important from the point of view of the ordering of the nodes, and hence, of shape preservation and of the existence of generalized Bernstein operators.

2. Definitions and motivating examples.

**Definition 2.1.** Let $U_n$ be an $n + 1$ dimensional subspace of $C^n([a, b], K)$, where $K = \mathbb{R}$ or $K = \mathbb{C}$. A Bernstein basis $\{p_{n,k} : k = 0, \ldots, n\}$ of $U_n$ is a basis with the property that each $p_{n,k}$ has a zero of order $k$ at $a$, and a zero of order $n - k$ at $b$. The function $p_{n,k}$ might have additional zeros inside $(a, b)$; this is not excluded by the preceding definition. A Bernstein basis is non-negative if for all $k = 0, \ldots, n$, $p_{n,k} \geq 0$ on $[a, b]$, and positive if $p_{n,k} > 0$ on $(a, b)$. Finally, a non-negative Bernstein basis is normalized if $\sum_{k=0}^{n} p_{n,k} \equiv 1$.

It is easy to check that non-negative Bernstein bases are unique up to multiplication by a positive scalar, and that normalized Bernstein bases are unique.
Definition 2.2. If \( U_n \) has a non-negative Bernstein basis \( \{ p_{n,k} : k = 0, \ldots, n \} \), we define a generalized Bernstein operator \( B_n : C[a,b] \rightarrow U_n \) by setting

\[
B_n (f) = \sum_{k=0}^{n} f(t_{n,k}) \alpha_{n,k} p_{n,k},
\]

where the nodes \( t_{n,0}, \ldots, t_{n,n} \) belong to the interval \([a,b]\), and the weights \( \alpha_{n,0}, \ldots, \alpha_{n,n} \) are positive.

Non-negativity of the functions \( p_{n,k} \) and positivity of the weights \( \alpha_{n,0}, \ldots, \alpha_{n,n} \) are required so that the resulting operator is positive, a natural property from the viewpoint of shape preservation. Strict positivity of the weights entails that all the basis functions are used in the definition of the operator, something useful if we want families of operators to converge to the identity. Finally, the nodes must belong to \([a,b]\); otherwise, the operator will not be well defined on \( C[a,b] \). But no requirement is made about the ordering of the nodes, and in particular, we do not ask that they be strictly increasing, i.e., that \( t_{n,0} < t_{n,1} < \cdots < t_{n,n} \). When we only have \( t_{n,0} \leq t_{n,1} \leq \cdots \leq t_{n,n} \) we say that the sequence of nodes is increasing, or equivalently, non-decreasing.

The problem of existence, as studied in [4] and [5], arises when we choose two functions \( f_0, f_1 \in U_n \), such that \( f_0 > 0 \), \( f_1/f_0 \) is strictly increasing, and we require that

\[
B_n (f_0) = f_0 \quad \text{and} \quad B_n (f_1) = f_1.
\]

If these equalities can be satisfied, they uniquely determine the location of the nodes and the values of the coefficients, cf. [4, Lemma 5]; in other words, there is at most one Bernstein operator \( B_n \) of the form (2) satisfying (3). We will consistently use the following notation. Assume that \( p_{n,k}, k = 0, \ldots, n, \) is a Bernstein basis of the space \( U_n \). Given \( f_0, f_1 \in U_n \), there exist coefficients \( \beta_{n,0}, \ldots, \beta_{n,n} \) and \( \gamma_{n,0}, \ldots, \gamma_{n,n} \) such that

\[
f_0 (x) = \sum_{k=0}^{n} \beta_{n,k} p_{n,k} (x) \quad \text{and} \quad f_1 (x) = \sum_{k=0}^{n} \gamma_{n,k} p_{n,k} (x).
\]

The following elementary fact regarding bases will be used throughout (cf. [4, Lemma 5]): If there exists a generalized Bernstein operator \( B_n \) of the form (2), fixing \( f_0 \) and \( f_1 \), then it must be the case that for each \( k = 0, \ldots, n, \)

\[
\beta_{n,k} = f_0 (t_{n,k}) \alpha_{n,k} \quad \text{and} \quad \gamma_{n,k} = f_1 (t_{n,k}) \alpha_{n,k}.
\]

Note that since by hypothesis \( f_0 > 0 \) and \( \alpha_{n,k} > 0 \), if \( B_n \) exists then \( \beta_{n,k} > 0 \). Now, using the injectivity of \( f_1/f_0 \), the nodes are uniquely determined by

\[
t_{n,k} := \left( \frac{f_1}{f_0} \right)^{-1} \left( \frac{\gamma_{n,k}}{\beta_{n,k}} \right),
\]

and the weights, by

\[
\alpha_{n,k} := \frac{\beta_{n,k}}{f_0 (t_{n,k})}.
\]
Finally, when the Bernstein basis is non-negative and normalized, and \( f_0 = 1 \), we have that
\[
1 = \alpha_{n,k} = \beta_{n,k} \quad \text{and} \quad t_{n,k} = f_1^{-1}(\gamma_{n,k}).
\]

Suppose that instead of fixing \( 1 \) and \( x \) over \([a, b]\), we want a generalized Bernstein operator that reproduces \( f_0 = 1 \) and some strictly increasing function other than \( x \). Possibly the simplest choice is to fix \( f_1(x) = x^3 \), since \( x^2 \) is not increasing over arbitrary intervals. Already in this case we observe a wide range of behavior, depending on the values of \( a \) and \( b \).

Recall that \( \mathbb{P}_n[a, b] \) denotes the space of polynomials on \([a, b]\), of degree bounded by \( n \). In this case the Bernstein bases are given by
\[
p_{n,k}(x) = \binom{n}{k}(x-a)^k(b-x)^{n-k}/(b-a)^n.
\]

**Example 2.3.** Consider \( \mathbb{E}_1 = \text{Span}\{1, x^3\} \) over \([-1, 1]\). Then \( \{p_{1,0}(x) := (1-x^3)/2, p_{1,1}(x) := (1+x^3)/2\} \) is the unique normalized Bernstein basis for \( \mathbb{E}_1 \). Define, as in [1] Formula (21),
\[
B_1 f := f(a)p_{1,0} + f(b)p_{1,1}.
\]
Then it is clear that \( B_1 1 = p_{1,0} + p_{1,1} = 1 \) and \( B_1 x = -p_{1,0} + p_{1,1} = x^3 \). Note that \( \text{Span}\{1, x^3\} \) is not an extended Chebyshev space, a notion defined next.

**Definition 2.4.** An extended Chebyshev space \( U_n \) of dimension \( n+1 \) over the interval \([a, b]\) is an \( n+1 \) dimensional subspace of \( C^n([a, b]) \) such that each \( f \in U_n \) has at most \( n \) zeros in \([a, b]\), counting multiplicities, unless \( f \) vanishes identically.

Extended Chebyshev spaces of dimension \( n+1 \) generalize the space of polynomials of degree at most \( n \) by retaining the bound on the number of zeros. It is well known that extended Chebyshev spaces always have positive Bernstein bases.

**Example 2.5.** Let \( \mathbb{E}_2 = \text{Span}\{1, x, x^3\} \) over \([-1, 1]\). In this case it is impossible to obtain a non-negative Bernstein basis for \( \mathbb{E}_2 \), whence the corresponding generalized Bernstein operator cannot be defined. To see why this is true, note that if \( p_{2,1}(x) = a + bx + cx^3 \) has one zero at \(-1\) and another at \( 1 \), then \( a-b-c = a+b+c = 0 \), so \( a = 0 \) and \( b = -c \), with \( b \neq 0 \). But for any such \( b \), \( p_{2,1}(x) = b(x-x^3) \) crosses the \( y \)-axis at \( 0 \). Note that Bernstein bases do exist for \( \mathbb{E}_2 \): One such basis is given by \( \{p_{2,0}(x) = 2-3x+x^3, p_{2,1}(x) = x-x^3, p_{2,2}(x) = 2+3x-x^3\} \).

Let us now consider \( \mathbb{E}_2 = \text{Span}\{1, x, x^3\} \) over \([-1, 2]\). In this case it is impossible to obtain a Bernstein basis for \( \mathbb{E}_2 \), even allowing for changes of signs. Suppose there is such a basis, and let us try to compute \( p_{2,2} \). Note that the coefficient of \( x^3 \) cannot be zero, since \( p_{2,2}(x) \) has degree at least two (hence three); dividing by the said coefficient, we may assume that \( p_{2,2}(x) = a+bx+x^3 = (x+1)^2(x+c) = c+(1+2c)x+(2+c)x^2+x^3 \). Equating coefficients we see that \( c = -2 \). Hence \( p_{2,2}(2) = 0 \), which is a contradiction.

**Example 2.6.** Next we consider \( \mathbb{P}_3[a, b] = \text{Span}\{1, x, x^2, x^3\} \), with the standard Bernstein bases over \([a, b] = [-1, 1]\) and over \([a, b] = [-1, 2]\). In the first case a generalized Bernstein operator fixing \( f_0 = 1 \) and \( f_1(x) = x^3 \) exists, but the sequence of nodes fails to be increasing. In fact, this must be the case, by Corollary 3.1 below, since \( f'_1(0) = 0 \). More explicitly, solving for the coefficients \( \gamma_{3,k} \) of \( x^3 \) we have \( \gamma_{3,0} = -1, \gamma_{3,1} = 1, \gamma_{3,2} = -1, \) and \( \gamma_{3,3} = 1 \), so in this particular instance the coordinates and the nodes take the same values, oscillating between \(-1\) and \( 1 \). Note that \( B_3 \) is just the projection from \( C[a, b] \) onto \( \text{Span}\{1, x^3\} \). This can be seen
by observing that for \( k \geq 0 \), \( B_3 x^{2k} = 1 \) and \( B_3 x^{2k+1} = x^3 \). Alternatively, given \( f \in C[a,b] \), if we simplify the expression for \( B_3 f(x) \), we find that \( B_3 f(x) \in \text{Span}\{1, x^3\} \).

When \([a, b] = [-1, 2]\), a generalized Bernstein operator fixing \( 1 \) and \( x^3 \) does not exist on \( \mathbb{P}_3[a, b] \), since solving for the coefficients \( \gamma_{3,k} \) of \( x^3 \) we find that \( \gamma_{3,0} = -1 \), \( \gamma_{3,1} = 2 \), \( \gamma_{3,2} = -4 \), \( \gamma_{3,3} = 8 \), so the node \( t_{3,2} = (-4)^{1/3} \) falls outside \([-1, 2]\).

However, a generalized Bernstein operator fixing \( 1 \) and \( x^3 \) does exist on \( \mathbb{P}_4[-1, 2] \), for now the coefficients \( \gamma_{4,k} \) of \( x^3 \) are \( \gamma_{4,0} = -1 \), \( \gamma_{4,1} = 5/4 \), \( \gamma_{4,2} = -1 \), \( \gamma_{4,3} = -1 \) and \( \gamma_{4,4} = 8 \), so all the nodes fall inside \([-1, 2]\).

Thus, not only the cases where \( f_1 \) is strictly increasing and where \( f'_1 > 0 \) on \((a,b)\) are different, but also the (relative) location of the possible zeros of \( f'_1 \) is relevant; a more extreme instance of this phenomenon can be found in [7] Theorem 5.2.

Next we present some counterexamples. The results are analogous to the instances seen so far, but spaces and bases are chosen to specifically address some claims made in [11] pages 121-122, where it is stated “Our purpose here is not to develop a comprehensive theory on Bernstein-type operators, but to convince the reader via a few relevant examples, that there do exist similar operators in more general situations.” While Bernstein-type operators can be defined in more general situations, they do not exist in some of the relevant examples presented there. The following spaces are considered in [11] page 123: Let \( a < 0 < b \), and let \( n \geq 4 \). Consider the sequence \( \mathbb{E}_1 \subset \mathbb{E}_2 \subset \mathbb{P}_3 \subset \cdots \subset \mathbb{P}_{n-1} \subset \mathbb{E}_n \), where \( \mathbb{E}_1 := \text{Span}\{1, x^3\} \), \( \mathbb{E}_2 := \text{Span}\{1, x, x^3\} \), and for \( n \geq 4 \), \( \mathbb{E}_n := \text{Span}\{1, x, \ldots, x^{n-1}, x^{n+2}\} \). The domain of definition of these functions is taken to be the interval \([a,b]\). In [11] Definitions 3.1 and 3.2 “Bernstein-like operators” are required to have strictly increasing sequences of nodes.

Now in [11] Example 7.1, the existence of “Bernstein-like operators” \( B_n : C[a,b] \to \mathbb{E}_n \) for \( a < 0 < b \), fixing \( 1 \) and \( x^3 \), is asserted. We show here, by explicit computation, that when \( n = 4 \) and \([a,b] = [-1,2]\), a generalized Bernstein operator fixing \( 1 \) and \( x^3 \) does not exist. When \([a,b] = [-1,1]\), such an operator exists, but one must give up the condition of increasing nodes.

**Example 2.7.** First we take \([a,b] = [-1,1]\). In this case a generalized Bernstein operator \( B_4 : C[a,b] \to \mathbb{E}_4 \) fixing \( 1 \) and \( x^3 \) does exist, but it is not defined via an increasing sequence of nodes. The normalized Bernstein basis on \( \mathbb{E}_4 \) can simply be found by writing arbitrary linear combinations of the functions \( \{1, x, x^2, x^3, x^6\} \), and imposing the conditions of having precisely 4 zeros, \( k \) of them at \(-1\) and the other \( 4-k \) at \( 1 \). Multiplying by \(-1 \) if needed, these basis functions can be assumed to be non-negative at 0 (in fact, we shall see that they are positive inside \((-1,1)\)). Imposing the additional condition that they add up to 1, we find
the unique normalized Bernstein basis \( \{ p_{4,0}, \ldots, p_{4,4} \} \) on \([-1, 1]\), where

\[
\begin{align*}
p_{4,0}(x) &= \frac{5}{56} - \frac{9}{28} x + \frac{45}{112} x^2 - \frac{5}{28} x^3 + \frac{1}{112} x^6, \\
p_{4,1}(x) &= \frac{2}{7} - \frac{3}{7} x - \frac{3}{14} x^2 + \frac{3}{7} x^3 - \frac{1}{14} x^6, \\
p_{4,2}(x) &= \frac{1}{4} - \frac{3}{8} x^2 + \frac{1}{8} x^6, \\
p_{4,3}(x) &= \frac{2}{7} - \frac{3}{7} x - \frac{3}{14} x^2 - \frac{3}{7} x^3 - \frac{1}{14} x^6, \\
p_{4,4}(x) &= \frac{5}{56} + \frac{9}{28} x + \frac{45}{112} x^2 + \frac{5}{28} x^3 + \frac{1}{112} x^6.
\end{align*}
\]

These functions are positive at zero, add up to 1, and have the correct number of zeros at the endpoints. To see that they form a positive basis, since \( p_{4,k}(0) > 0 \) for \( k = 0, \ldots, 4 \), it suffices to show that they have no additional zeros inside \((-1, 1)\). But this is easily checked, for we already know the location of four zeros of each \( p_{4,k} \). Using the division algorithm, we factor all the corresponding linear terms \((x + 1)\) and \((x - 1)\), and are left in each case with a second degree polynomial having no real roots.

Once we have found the normalized Bernstein bases, we use the condition (3) to determine nodes: Equating coefficients in \( x^3 = \sum_{k=0}^{4} \gamma_{4,k} p_{4,k}(x) \) and solving for \( \gamma_{4,k} \) we find that \( \gamma_{4,0} = -1, \gamma_{4,1} = 3/4, \gamma_{4,2} = 0, \gamma_{4,3} = -3/4 \) and \( \gamma_{4,4} = 1 \). Since the nodes are the cube roots of these coordinates, it follows that \( t_{4,0} < t_{4,3} < t_{4,2} < t_{4,1} < t_{4,4} \), and we see that the nodes do not form an increasing sequence in \( k \).

**Example 2.8.** Let us now take \([a, b] = [-1, 2]\). In this case a generalized Bernstein operator \( B_4 : C[a, b] \to \mathbb{E}_4 \) fixing 1 and \( x^3 \), cannot be defined. Using the same steps as in the preceding example, we find the following Bernstein basis functions:

\[
\begin{align*}
p_{4,0}(x) &= \frac{640}{2673} - \frac{128}{297} x + \frac{80}{297} x^2 - \frac{160}{2673} x^3 + \frac{1}{2673} x^6, \\
p_{4,1}(x) &= \frac{5776}{13365} - \frac{152}{1485} x - \frac{532}{1485} x^2 + \frac{2318}{13365} x^3 - \frac{38}{13365} x^6, \\
p_{4,2}(x) &= \frac{98}{405} + \frac{14}{45} x - \frac{7}{405} x^2 - \frac{56}{810} x^3 + \frac{7}{810} x^6, \\
p_{4,3}(x) &= \frac{16}{243} + \frac{27}{54} x + \frac{2}{243} x^2 - \frac{2}{243} x^3 - \frac{2}{243} x^6, \\
p_{4,4}(x) &= \frac{5}{243} + \frac{2}{27} x + \frac{5}{243} x^2 + \frac{10}{243} x^3 + \frac{1}{486} x^6.
\end{align*}
\]

Positivity of these functions on \((-1, 2)\) is obtained by noticing, first, that \( p_{4,k}(0) = 1 \), and second, that after factoring the linear terms \((x + 1)\) and \((x - 2)\), in each case we are left with a second degree polynomial having no real roots. Adding up we see that the basis is normalized, so it is enough to compute the coordinates \( \gamma_{4,k} \) of \( x^3 \). Doing so, we find that \( \gamma_{4,2} = -16/7 < -1 \), so the node \( t_{4,2} \notin [-1, 2] \).
3. Characterizing when nodes increase for general spaces.

The following technical results, used to prove Theorem 3.3, come from [5]. Proposition 3.1 appears in [5] Proposition 3], while Lemma 3.2 is a less general version of [5] Lemma 6.

**Proposition 3.1.** Assume that \( U_n \subset C^n([a, b], \mathbb{K}) \) has a Bernstein basis \( p_{n,k} \), \( k = 0, \ldots, n \). Let \( f_0 \in U_n \) be strictly positive and suppose that \( D_{f_0} U_n := \left\{ \frac{d}{dx} \left( \frac{f}{f_0} \right) : f \in U_n \right\} \) has a Bernstein basis \( q_{n-1,k} \), \( k = 0, \ldots, n-1 \). Set \( c_0 := 0, q_{n-1,-1} := 0, d_n := 0, \) and \( q_{n-1,n} := 0 \). For \( k = 1, \ldots, n \), define the non-zero numbers

\[
(10) \quad c_k := \frac{1}{f_0(a)} \lim_{x \downarrow a} \frac{d}{dx} p_{n,k}(x) q_{n-1,k-1}(x) = \frac{1}{f_0(a)} p_{n,k}^{(k)}(a) q_{n-1,k-1}^{(k-1)}(a)
\]

and for \( k = 0, \ldots, n-1 \), the non-zero numbers

\[
(11) \quad d_k := \frac{1}{f_0(b)} \lim_{x \uparrow b} \frac{d}{dx} p_{n,k}(x) q_{n-1,k}(x) = \frac{1}{f_0(b)} p_{n,k}^{(n-k)}(b) q_{n-1,k}^{(n-k-1)}(b).
\]

Then for every \( k = 0, \ldots, n \),

\[
(12) \quad \frac{d}{dx} \frac{p_{n,k}(x)}{f_0(x)} = c_k q_{n-1,k-1}(x) + d_k q_{n-1,k}(x).
\]

**Lemma 3.2.** Let \( p_{n,k} \), \( k = 0, \ldots, n \), be a non-negative Bernstein basis of \( U_n \subset C^n([a, b], \mathbb{K}) \). Then there exists a \( \delta > 0 \) such that \( p_{n,k}'(x) < 0 \) for all \( x \in [b - \delta, b] \) and all \( k = 0, \ldots, n-1 \), while \( p_{n,k}'(x) > 0 \) for all \( x \in [a, a + \delta] \) and all \( k = 1, \ldots, n \). Thus, the numbers \( c_k \) defined in (10) for \( k = 1, \ldots, n \) are positive, and the numbers \( d_k \) defined in (11) for \( k = 0, \ldots, n-1 \) are negative.

**Theorem 3.3.** Assume that both \( U_n \subset C^n([a, b], \mathbb{K}) \) and \( D_{f_0} U_n := \left\{ \frac{d}{dx} \left( \frac{f}{f_0} \right) : f \in U_n \right\} \) possess non-negative Bernstein basis \( p_{n,k} \), for \( k = 0, \ldots, n \), and \( q_{n-1,k} \), for \( k = 0, \ldots, n-1 \), respectively. Suppose \( f_0, f_1 \in U_n \) are such that \( f_0 > 0 \), its coordinates \( \beta_{n,k} \) satisfy \( \beta_{n,k} > 0 \), and \( f_1/f_0 \) is strictly increasing on \([a, b]\). Then the following statements are equivalent:

a) There exists a generalized Bernstein operator \( B_n : C[a, b] \rightarrow U_n \) defined by a sequence of non-decreasing (resp. strictly increasing) nodes, and fixing both \( f_0 \) and \( f_1 \).

b) For \( k = 0, \ldots, n \), the numbers \( \gamma_{n,k}^{\beta} \) are non-decreasing (resp. strictly increasing).

c) The coefficients \( w_k \), defined by

\[
(13) \quad \frac{d}{dx} \frac{f_1}{f_0} = \sum_{k=0}^{n-1} w_k q_{n-1,k}
\]

for \( k = 0, \ldots, n-1 \), are non-negative (resp. strictly positive).

**Proof.** We begin with some technical preliminaries, under the assumption that for \( k = 0, \ldots, n \), the coordinates \( \beta_{n,k} \) of \( f_0 \) are strictly positive. Let \( k_0 \in \{0, \ldots, n-1\} \). Since
\[ p_{n,k}, k = 0, \ldots, n, \text{ is a basis, there exists numbers } \delta_1, \ldots, \delta_n \text{ such that} \]

\[ \psi_{k_0} := f_1 - \frac{\gamma_{n,k_0}}{\beta_{n,k_0}} f_0 = \sum_{k=0}^{n} \delta_k p_{n,k}. \]

From (4) we get

\[ \delta_k = \gamma_{n,k} - \frac{\gamma_{n,k_0}}{\beta_{n,k_0}} \beta_{n,k} \]

for \( k = 0, \ldots, n. \) Setting \( k = k_0 \) we obtain

\[ \delta_{k_0} = \gamma_{n,k_0} - \frac{\gamma_{n,k_0}}{\beta_{n,k_0}} \beta_{n,k_0} = 0. \]

Let us write

\[ \frac{f_1}{f_0} - \frac{\gamma_{n,k_0}}{\beta_{n,k_0}} = \frac{\psi_{k_0}}{f_0} = \sum_{k=1}^{n} \delta_k \frac{p_{n,k}}{f_0}. \]

Differentiating we get

\[ \frac{d}{dx} \frac{f_1}{f_0} = \sum_{k=0}^{n} \delta_k \frac{d}{dx} \left( \frac{p_{n,k}}{f_0} \right). \]

Proposition 3.1 together with Lemma 3.2 show that

\[ \frac{d}{dx} \frac{f_1}{f_0} = \sum_{k=0}^{n} \delta_k \left[ c_k q_{n-1} - 1 + d_k q_{n-1} \right], \]

where \( c_0 = 0 \) and \( c_k > 0 \) for \( k = 0, \ldots, n, \) while \( d_k < 0 \) for \( k = 0, \ldots, n-1 \) and \( d_n = 0. \) Thus,

\[ \frac{d}{dx} \frac{f_1}{f_0} = \delta_0 d_0 q_{n-1,0} + \sum_{k=1}^{n-1} \delta_k \left[ c_k q_{n-1,k-1} + d_k q_{n-1,k} \right] + c_n \delta_n q_{n-1,n-1} \]

\[ = \sum_{k=1}^{n} \delta_k c_k q_{n-1,k-1} + \sum_{k=0}^{n-1} \delta_k d_k q_{n-1,k} = \sum_{k=0}^{n-1} \left( \delta_{k+1} c_{k+1} + \delta_k d_k \right) q_{n-1,k}. \]

Using (13) we conclude that

\[ c_{k+1} \delta_{k+1} = w_k - \delta_k d_k \]

for \( k = 0, \ldots, n-1. \) Inserting \( k = k_0 \) in (16), from (15) we get

\[ c_{k_0+1} \delta_{k_0+1} = w_{k_0} - \delta_{k_0} d_{k_0} = w_{k_0} \]

whenever \( k_0 \in \{0, \ldots, n-1\} \). Now the result is easily obtained from this equality. We mention only the non-decreasing case, since the strictly increasing one is handled in an identical manner.

First we prove that a) and b) are equivalent. It follows from (4) that \( \frac{f_1(a)}{f_0(a)} = \frac{\gamma_{n,a}}{\beta_{n,a}} \) and \( \frac{f_1(b)}{f_0(b)} = \frac{\gamma_{n,b}}{\beta_{n,b}}. \) By (5), the nodes are non-decreasing (and hence they belong to \([a, b]\)) if and only if so are the numbers \( \frac{\gamma_{n,a}}{\beta_{n,a}} \).
To finish, we revisit Example 2.7 under the light of the preceding results.

Example 3.6. The condition \( \left( \frac{f_1}{f_0} \right)' > 0 \) on \([a, b]\) is not sufficient to ensure non-decreasing nodes, as [4, Example 4.1] shows: consider \( \mathbb{P}_3[0, 1] \) with the standard Bernstein basis, let \( f_0 = 1 \), and let \( f_1(x) = 3x/8 - x^2/2 + x^3/3 \). Then \( f_1'(x) := (x-1/2)^2 + 1/8 = 3p_{2,0}(x)/8 - p_{2,1}(x)/8 + 3p_{2,2}(x)/8 \), so by Theorem 3.3 no generalized Bernstein operator \( B_3 : C([a, b], \mathbb{K}) \to \mathbb{P}_3[a, b] \), fixing 1 and \( f_1 \), can be defined via a non-decreasing sequence of nodes.

Example 3.5. The condition \( \left( \frac{f_1}{f_0} \right)' > 0 \) on \([a, b]\) is not sufficient to ensure non-decreasing nodes, as [4, Example 4.1] shows: consider \( \mathbb{P}_3[0, 1] \) with the standard Bernstein basis, let \( f_0 = 1 \), and let \( f_1(x) = 3x/8 - x^2/2 + x^3/3 \). Then \( f_1'(x) := (x-1/2)^2 + 1/8 = 3p_{2,0}(x)/8 - p_{2,1}(x)/8 + 3p_{2,2}(x)/8 \), so by Theorem 3.3 no generalized Bernstein operator \( B_3 : C([a, b], \mathbb{K}) \to \mathbb{P}_3[a, b] \), fixing 1 and \( f_1 \), can be defined via a non-decreasing sequence of nodes.

As before, we can find a positive Bernstein basis of

\[
D_{f_0}E_4 = \text{Span}\{1, x, x^2, x^3\}
\]

over \([a, b] = [-1, 1]\), by writing arbitrary linear combinations of the functions \( \{1, x, x^2, x^3\} \), and imposing the conditions of having precisely 3 zeros, \( k \) of them at \(-1\) and the other \( 3 - k \) at 1. Multiplying by \(-1\) if needed, these basis functions can be assumed to be non-negative.
at 0 (we shall see that they are actually positive inside $(−1,1)$). In this way we find the following Bernstein basis $\{p_{3,0}, \ldots, p_{3,3}\}$ on $[-1,1]$:

\[
\begin{align*}
p_{3,0}(x) &= 1 - \frac{5}{2}x + \frac{5}{3}x^2 - \frac{1}{6}x^5, \\
p_{3,1}(x) &= 1 - \frac{1}{2}x - x^2 + \frac{1}{2}x^5, \\
p_{3,2}(x) &= 1 + \frac{1}{2}x - x^2 - \frac{1}{2}x^5, \\
p_{3,3}(x) &= 1 + \frac{5}{2}x + \frac{5}{3}x^2 + \frac{1}{6}x^5.
\end{align*}
\]

These functions have the correct number of zeros at the endpoints. To see that they form a positive basis, since $p_{3,k}(0) = 1 > 0$ for $k = 0, \ldots, 3$, it suffices to show that they have no additional zeros inside $(−1,1)$, which follows by factoring all the corresponding linear terms $(x+1)$ and $(x-1)$. In each case we are left with a second degree polynomial having no real roots. So we are within the realm of Theorem 3.3 or Corollary 3.4. Since the derivative of $f_1(x) = x^3$ vanishes at 0, we conclude that no generalized Bernstein operator $B_3 : C([-1,1], \mathbb{K}) \to \mathbb{E}_4[-1,1]$, fixing 1 and $f_1$, can be defined via a non-decreasing sequence of nodes.

Observe however that this result is less informative than Example 2.7, since it does not tell us whether a generalized Bernstein operator can be defined, by dropping the requirement that nodes be non-decreasing.

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