Self-dual property of the Potts model in one dimension

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Abstract

A duality relation is derived for the Potts model in one dimension from a graphical consideration. It is shown that the partition function is self-dual with the nearest-neighbor interaction and the external field appearing as dual parameters. Zeroes of the partition function are analyzed. Particularly, we show that the duality relation implies a circle theorem in the complex temperature plane for the one-dimensional Ising model.

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I. INTRODUCTION

It is now well-known that the Potts model [1,2] possesses a duality relation in two dimensions. In fact, it is using this duality relation that Potts [1] first determined the transition temperature for the square lattice. The purpose of this note is to point out a self-dual relation of the Potts model in one dimension, a curious and somewhat surprising result which appears to have escaped heretofore attention. For the Ising model this duality implies a new circle theorem.

Consider \( N \) Potts spins placed on a ring interacting with nearest-neighbor interactions \( K \) and an external field \( L \) for one specific spin state. Number the sites by \( i = 1, 2, \cdots, N \) and denote the spin state at site \( i \) by \( \sigma_i = 1, 2, \cdots, q \). The partition function is

\[
Z_N(q; K, L) = \sum_{\sigma_i = 1}^{q} \prod_{i=1}^{N} T(\sigma_i, \sigma_{i+1})
\]

with \( \sigma_{N+1} = \sigma_1 \) and

\[
T(\sigma_i, \sigma_{i+1}) = \exp[K\delta_{Kr}(\sigma_i, \sigma_{i+1})] \exp[L\delta_{Kr}(\sigma_i, 1)],
\]

where \( \delta_{Kr} \) is the Kronecker delta function. In ensuing discussions we shall make use of the fact that, due to symmetry, the partition function (1) is the same if the external field is applied to any spin state, namely, if \( \delta_{Kr}(\sigma_i, 1) \) in (2) is replaced by \( \delta_{Kr}(\sigma_i, \alpha) \) for any \( \alpha = 2, 3, \cdots, q \).

Regarding \( T(\sigma, \sigma') \) as the elements of a \( q \times q \) matrix \( T \), then the partition function (1) can be evaluated by the standard technique of the transfer matrix. The characteristic equation for \( T \) is

\[
\det \left| \begin{array}{cccc}
e^{K+L} - \lambda & e^L & \cdots & e^L \\
1 & e^K - \lambda & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & e^K - \lambda \\
\end{array} \right| = 0.
\]

This equation can be factorized leading to eigenvalues \( e^K - 1 \), which is \((q-2)\)-fold degenerate, and \( \lambda_{\pm} \), which are the two roots of the quadratic equation

\[
\lambda^2 - (e^{K+L} + e^K + q - 2)\lambda + e^L(e^K - 1)(e^K + q - 1) = 0.
\]

This gives rise to the following explicit expression for the partition function,

\[
Z_N(q; K, L) = \lambda_+^N + \lambda_-^N + (q - 2)(e^K - 1)^N.
\]

Explicitly, we have

\[
\lambda_{\pm} = \frac{1}{2} \left[ e^{K+L} + e^K + q - 2 \pm \left( (e^{K+L} - e^K - q + 2)^2 + 4(q - 1)e^L \right)^{1/2} \right].
\]

Note that terms involving branch cuts are cancelled in expanding (4), and as a consequence \( Z_N(q; K, L) \) is indeed a polynomial in \( e^K \) and \( e^L \).
II. THE DUALITY RELATION

Our main result is the self-dual relation for the partition function \((\text{5})\)

\[
[(e^K - 1)(e^L - 1)]^{-N/2} Z_N(q; K, L) = [(e^{K^*} - 1)(e^{L^*} - 1)]^{-N/2} Z_N(q; K^*, L^*),
\]

(7)

where the dual variables \(K^*\) and \(L^*\) are related to \(L\) and \(K\), respectively, by

\[
(e^{K^*} - 1)(e^L - 1) = q, \quad (e^{L^*} - 1)(e^K - 1) = q.
\]

(8)

The validity of (7) can be explicitly verified using (5) for \(q = 2\) and for specific values of \(N = 2, 3, \cdots\). But its validity for general \(q\) and \(N\) is not very obvious by looking at the solution.

To establish (7) for general \(q\) and \(N\), we construct a lattice \(L\) of \(N + 1\) sites as shown in Fig. 1, by introducing an extra (ghost) spin interacting with all \(N\) sites with an equal interaction \(L\). Note that the lattice \(L\) is planar and self-dual. Namely, if one places dual spins, one in each face of \(L\) including the face exterior to \(L\), and connects dual spins with edges crossing each of the edges of \(L\), one arrives at a lattice which is precisely \(L\).

Let \(Z_L(q; K, L)\) be the partition function of the Potts model on \(L\). Following the remark after (2), one has the identity

\[
Z_L(q; K, L) = q Z_N(q; K, L).
\]

(9)

It is then sufficient to show that (7) holds for \(Z_L(q; K, L)\).

Generally, the Potts model possesses a duality relation for any planar lattice, or graph, with arbitrary edge-dependent interactions \(K_{ij}\). Let \(Z(K_{ij})\) be the partition function of the Potts model on a planar graph of \(M\) sites with edge-dependent interactions \(K_{ij}\), and \(Z^{(D)}(K^*_{ij})\) the partition function of the dual model. Then, the duality relation given in \(3\) can be written as

\[
\frac{q^{-M/2} Z(K_{ij})}{\prod_{\text{edges}} \sqrt{e^{K_{ij}} - 1}} = \frac{q^{-M_D/2} Z^{(D)}(K^*_{ij})}{\prod_{\text{edges}} \sqrt{e^{K^*_{ij}} - 1}},
\]

(10)

where \(M_D\) is the number of sites of the dual graph and

\[
(e^{K_{ij}} - 1)(e^{K^*_{ij}} - 1) = q.
\]

(11)

Now the lattice \(L\) is self-dual, namely, \(M = M_D = N + 1\) and \(Z = Z^{(D)} = Z_L\). The application of (10) to \(L\) now leads to an expression which is precisely (7) with \(Z_L\) in place of \(Z_N\). The duality relation (7) now follows after introducing (9).

III. PARTITION FUNCTION ZEROES

The zeroes of the partition function can be computed from

\[
\lambda^N_+ + \lambda^N_- + (q - 2)(e^K - 1)^N = 0,
\]

(12)
at least numerically, for any given $N$. The analysis is particularly simple if the three terms in (12) coalesce into two. For the $q = 2$ Ising model, for example, one finds zeroes located at

$$
\lambda_+ = e^{i(2n+1)\pi/N} \lambda_-, \quad n = 0, 1, 2, \cdots, N - 1.
$$

(13)

This leads to the Yang-Lee circle $\mathcal{F}$ in the complex $e^L$ plane for $K \geq 0$. For the zero-field Potts model, $L = 0$, one has $\lambda_- = e^K - 1$ and finds from (12) the zeroes at

$$
e^K + q - 1 = (1 - q)^{1/N}(e^K - 1), \quad n = 0, 1, 2, \cdots, N - 1.
$$

(14)

In the limit of $N \to \infty$, the partition function (4) is dominated by the eigenvalue with the largest magnitude. It follows that the zeroes lie continuously on the loci

$$
|\lambda_+| = |\lambda_-|, \quad \text{in } |\lambda_+| \geq |e^K - 1|
$$

$$
|\lambda_\pm| = |e^K - 1|, \quad \text{in } |\lambda_\pm| \geq |\lambda_\mp|,
$$

(15)

generalizing a previous result [7] for real $q$ and $L = 0$. Here, the loci (14) and (15) apply to real and complex $K, L, q$. Particularly, for $L = e^K = 0$, $Z_N(q; K, L)$ gives the ground state entropy of the antiferromagnetic Potts model and the chromatic polynomial (in $q$) of a ring. In this case we have $\lambda_+ = q - 1, \lambda_- = e^K - 1 = -1$, and we find the zeroes on the circle

$$
|q - 1| = 1
$$

(16)
in the complex $q$ plane. This result has previously been reported by Shrock and Tsai [8].

For the Ising model the duality relation (7) provides some further implications on the partition function zeroes. To conform with usual notations, we consider $N$ Ising spins on a ring with nearest-neighbor interactions $K$ and an external magnetic field $L \geq 0$. The Ising partition function is

$$
Z_{\text{Ising}}(K_1, L_1) = \sum_{\sigma_i = \pm 1} \prod_{i=1}^N \left[ e^{K\sigma_i\sigma_{i+1}} e^{L\sigma_i} \right].
$$

(17)

Using the identity $\sigma_i\sigma_{i+1} = 2 \delta_{K1}(\sigma_i, \sigma_{i+1}) - 1$ and $\sigma_i = 2 \delta_{Kr}(\sigma_i, 1) - 1$, we relate $Z_{\text{Ising}}$ to the 2-state Potts partition function via

$$
Z_{\text{Ising}}(K_1, L_1) = e^{-N(L_1+K_1)} Z_N(2; 2K_1, 2L_1).
$$

(18)

Thus, after substituting (18) into (7) and some re-arrangement, we obtain the following self-dual relation for the Ising model on $L$,

$$
\frac{Z_{\text{Ising}}(K_1, L_1)}{(\sinh 2K_1 \sinh 2L_1)^{N/4}} = \frac{Z_{\text{Ising}}(K^*_1, L^*_1)}{(\sinh 2K^*_1 \sinh 2L^*_1)^{N/4}},
$$

(19)

where the dual variables are

$$
e^{-2K^*_1} = \tanh L_1, \quad e^{-2L^*_1} = \tanh K_1.
$$

(20)

We note the resemblance of (19) with the duality relation of the Ising model formulated by Syozi [6].
From the Yang-Lee circle theorem [5] we know that, for $K \geq 0$, zeroes of $Z_{\text{Ising}}(K_1, L_1)$ lie on the unit circle in the complex field $e^{-2L_1}$ plane. Now, since we have $K_1^* \geq 0$, the duality relation (19) implies that zeroes of $Z_{\text{Ising}}(K_1, L_1)$ lie also on the unit circle in the complex temperature $\tanh K_1$ plane, a result which holds for any $L_1 \geq 0$ and which is not readily recognized otherwise. A consequence of this is that zeroes of $Z_{\text{Ising}}(K_1, L_1)$ lie on the pure imaginary axis in the $e^{-2K_1}$ plane, generalizing a result previously known only for $L_1 = 0$ [7]. Our circle theorem contrasts with the square lattice result, valid only in the thermodynamic limit and in zero field, which states that in the complex $\tanh K_1$ plane zeroes of the partition function $Z_{\text{Sq Ising}}(K_1)$ lie on two circles of radius $\sqrt{2}$ and centered at $\pm 1$ [9].

Note added: After the completion of this manuscript, it has been pointed out to me that results similar to those described here have been reported previously [10]. The duality relation (7), which is derived in this manuscript from a graphical consideration, was obtained in [10] algebraically by considering the transfer matrix. I would like to thank A. Glumac for calling my attention to [10].

IV. ACKNOWLEDGEMENT

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FIGURE CAPTION

Fig. 1. The lattice $\mathcal{L}$ with $N + 1$ sites.
[1] R. B. Potts, Proc. Camb. Phil. Soc. 48, 106 (1952).
[2] F. Y. Wu, Rev. Mod. Phys. 54, 235 (1982).
[3] F. Y. Wu and Y. K. Wang, J. Math. Phys. 17, 439 (1976).
[4] The characteristic equation (3) is factorized by subtracting the second row from the third and each of the subsequent rows, followed by adding the third and all subsequent columns to the second column.
[5] C. N. Yang and T. D. Lee, Phys. Rev. 87, 404 (1952); T. D. Lee and C. N. Yang, Phys. Rev. 87, 410 (1952).
[6] I. Syozi, in Phase Transitions and Critical Phenomena, Vol. 1, Edited by C. Domb and M. S. Green (Academic Press, New York, 1972).
[7] V. Matveev and R. Shrock, Phys. Lett. A204, 353 (1995).
[8] R. Shrock and S.-H. Tsai, Phys. Rev. E55, 5165 (1997).
[9] M. E. Fisher, in Lecture Notes in theoretical Physics, edited by W. E. Brittin (University of Colorado Press, Boulder, 1965), Vol. 7c.
[10] A. Glumac and K. Uzelac, J. Phys. A 27, 7709 (1994).
Fig. 1