Adaptive Reduced Rank Regression

Qiong Wu  
College of William & Mary

Felix Ming Fai Wong  
Independent Researcher

Zhenming Liu  
College of William & Mary

Yanhua Li  
Worcester Polytechnic Institute

Varun Kanade  
University of Oxford and  
The Alan Turing Institute

Abstract

Low rank regression has proven to be useful in a wide range of forecasting problems. However, in settings with a low signal-to-noise ratio, it is known to suffer from severe overfitting. This paper studies the reduced rank regression problem and presents algorithms with provable generalization guarantees. We use adaptive hard rank-thresholding in two different parts of the data analysis pipeline. First, we consider a low rank projection of the data to eliminate the components that are most likely to be noisy. Second, we perform a standard multivariate linear regression estimator on the data obtained in the first step, and subsequently consider a low-rank projection of the obtained regression matrix. Both thresholding is performed in a data-driven manner and is required to prevent severe overfitting as our lower bounds show. Experimental results show that our approach either outperforms or is competitive with existing baselines.

1 Introduction

This paper studies inference algorithms for the linear model $y = Mx + \epsilon$, where $x \in \mathbb{R}^{d_1}$, $y \in \mathbb{R}^{d_2}$, $M \in \mathbb{R}^{d_2 \times d_1}$, and $\epsilon \in \mathbb{R}^{d_2}$ is a zero mean noise. We focus on the high dimensional setting, in which the number of observations $n$ is significantly smaller than the number of learnable parameters. While this problem arises often in different areas, such as identification of biomarkers [43], understanding risks associated with various diseases [14, 3], and image recognitions [41, 32, 12], we are specifically motivated by its application in forecasting equity returns in financial markets [10, 30, 36, 25, 5]. We aim to forecast the next day/week returns $y_t$ (it is a vector) of all the equities in a specific universe on day $t$, using the linear model $y_t \sim Mx_t$, where $x_t$ denotes a large collection of features that could be relevant to the return.

Regularization by controlling the ranks. Regularization is needed to produce non-trivial forecasts. A major regularization technique is the low rank regularization, which builds estimators whose ranks are much lower than $\min\{d_1, d_2\}$. There are two major ways to implement the low rank regularizer:

(i) Regularizing the nuclear norm of $M$: The nuclear norm $\|M\|_*$ of $M$ is the sum of singular values of $M$. This approach adds the cost term $\lambda \|M\|_*$ and it has been extensively studied in the literature (see e.g., [27, 22]). Theoretical results for such models are established, however, the algorithm for solving the nuclear-norm regularized approach is usually computationally burdensome and unscalable because it needs to solve a sequence of singular value decomposition problems for large matrices.

(ii) Imposing hard low rank constraints on $M$: A hard low rank constraint is imposed over $M$ (rank$(M) \leq k$ for a suitable $k$) as shown in Fig. 1A. This approach is called reduced rank regression (RRR) [6, 9, 16, 24, 26, 35]. While the hard rank constraint is not convex, an efficient and scalable algorithm exists to solve the problem [39].

Current landscape. Nuclear-norm based regularization methods are not computationally scalable. While algorithms for reduced rank regression are highly scalable, these models appear to experience severe overfitting problems, even when $k$ (the rank constraint) is set to be small as shown in Table 1.
It is often straightforward to identify a large collection of features. We establish the optimality of our algorithm by comparing its performance against the best algorithm’s performance and the optimal one from the benchmark family is small for every single M. A key challenge is to properly define the benchmark algorithms. These algorithms need to be sufficiently powerful (e.g., have oracle access to information that is unavailable to realistic algorithms), but not unrealistically strong, so that performing competitive analysis becomes feasible.

2 Preliminary

Our model. We consider the model $y = Mx + \epsilon$, where $x \in \mathbb{R}^{d_1}$ is a multivariate Gaussian, $y \in \mathbb{R}^{d_2}$, $M \in \mathbb{R}^{d_2 \times d_1}$, and $\epsilon \in \mathbb{R}^{d_2}$. We assume a PAC learning framework, i.e., we observe a sequence $\{(x_i, y_i)\}_{i \leq n}$ of independent samples and our goal is to find an $M$ that minimizes the out-of-sample error $\mathbb{E}[\|Mx - (Mx + \epsilon)\|^2]$. Assumptions. We make four assumptions based on the structure of datasets we see in practice.

(A) Reduced rank regression:  
$$\min \frac{1}{n}\|Y - XM^\top\|^2_F \quad \text{s.t. } \text{rank}(M) \leq k.$$  
(B) Reduced rank ridge regression:  
$$\min \frac{1}{n}\|Y - XM^\top\|^2_F + \lambda\|M\|^2_F \quad \text{s.t. } \text{rank}(M) \leq k.$$  

Figure 1: Existing techniques for imposing hard rank constraints on $M$

Various heuristics, such as [26] (see Fig.1B) have been proposed to alleviate the overfitting problem, but it remains unclear whether these methods have provable generalization errors.

Table 1: Train and test error of the reduced rank regression model in a synthetic data. $k$ represents the rank constraint (see App.F.1 for more details). The gap between train and test errors persists even when $k = 1$.

| $k$ | Training error | Test error |
|----|----------------|------------|
| 1  | 0.2008         | 10.9501    |
| 2  | 0.0846         | 15.2094    |
| 3  | 0.0699         | 19.3182    |
| 4  | 0.0623         | 20.0598    |
| 5  | 0.0557         |            |

Our research questions and contributions. This work examines algorithmic and statistical aspects of reduced rank regressions and answers three research questions.

Research Q1. Why is the reduced rank regression model (in Fig.7A) ineffective? We start with explaining the root cause of RRR’s overfitting problem. Using a hard rank constraint to control out-of-sample error, one needs to implicitly estimate the precision matrix of the features $x$. Optimizing the mean squared error (MSE) with only the rank constraint on $M$ corresponds to the use of the empirical precision matrix as the estimate. However, the empirical precision matrix is known to be inaccurate in high-dimensional settings; such inaccuracy leads to RRR’s overfitting problem.

Research Q2. How can we fix the overfitting problem? We develop a new statistical technique for RRR to circumvent the overfitting problem. Our new model possesses three salient properties: (i) Compute efficiency: The algorithm is highly scalable. (ii) Adaptive model selection: Our algorithm adapts the model to signal quality. When the signal-to-noise ratio in the data is low, our algorithm will automatically choose an estimate $\hat{M}$ with low rank to reduce the variance error. When the ratio is high, it chooses an $M$ with higher rank to better extract signals. The rank of $M$ does not need to match that of the ground truth. (iii) Leveraging spectral properties of $x$: Correctness of our algorithm critically relies on an assumption of the spectral properties of $x$. We assume that the eigenvalues of the covariance matrix follow a power law distribution. Extensive studies confirm this assumption [1, 28, 38]. To the best of our knowledge, our algorithm/model is the first mathematically rigorous solution that leverages the power law structure of the features’ covariance matrices.

Research Q3. How can we prove the optimality of our algorithm? Most theoretical results on high dimensional statistics establish optimality by using the so-called minimax bound (see e.g., [7, 40, 20, 4, 15]): One first computes MSE of an estimator, which is often parametrized by unobservable properties of the ground-truth matrix $M$ (such as its rank or nuclear norm) and then construct a worst case $M$ to justify the optimality of the MSE. This approach is unsuitable for our settings for two reasons: (i) We often do not know the true rank or nuclear norm of $M$. When we use conservative estimate of $M$’s hidden/latent parameters (e.g., assuming that $M$’s rank is $\min\{d_1, d_2\}$), existing techniques often give us trivial results. (ii) Minimax type bounds only guarantee the worst case optimality for a family of matrices (e.g., all $M$ of rank $k$) but they do not tell whether we can find better estimates for a specific dataset $M$.

We establish the optimality of our algorithm by comparing its performance against the best algorithm of a “reasonable” family of benchmark algorithms. We shall argue that the gap between our algorithm’s performance and the optimal one from the benchmark family is small for **every single $M$**. A key challenge is to properly define the benchmark algorithms. These algorithms need to be sufficiently powerful (e.g., have oracle access to information that is unavailable to realizable algorithms), but not unrealistically strong, so that performing competitive analysis becomes feasible.
Let \( Y \in \mathbb{R}^{n \times d_1} \) and \( X \in \mathbb{R}^{n \times d_2} \) be data matrices with their \( i \)-th rows representing the \( i \)-th observation. For matrix \( A \), we denote its singular value decomposition as \( A = U \Sigma V^\top \) and \( A_r := U_r \Sigma_r V_r^\top \) is the rank \( r \) approximation obtained by keeping the top \( r \) singular values and the corresponding singular vectors. When the context is clear, we drop the superscript \( A \) and use \( U, \Sigma \), and \( V \) (\( U_r, \Sigma_r \), and \( V_r \)) instead. We use MATLAB notation when we refer to a specific row or column, i.e., \( V_{i,:} \) is the first row of \( V \) and \( V_{1:3} \) is the first column. \( \|A\|_F \), \( \|A\|_1 \), and \( \|A\|_2 \) are Frobenius, spectral, and nuclear norms of \( A \). In general, we use boldface to denote data matrices and regular fonts to denote other matrices. Let \( C^* = \mathbb{E}[\|x\|] \) and \( C = \frac{1}{2}X^\top X \) be the empirical estimate of \( C^* \). Let \( C^* = V^* A^\top V^* \) be the eigen-decomposition of the matrix \( C^* \), and let \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq 0 \) be the diagonal entries of \( \Lambda^* \). Let \( \{U_1, U_2, \ldots, U_r\} \) be an arbitrary set of column vectors. Let \( \text{Span}(\{U_1, U_2, \ldots, U_r\}) \) be the subspace spanned by \( \{U_1, U_2, \ldots, U_r\} \). An event happens with high probability means that it happens with probability \( \geq 1 - n^{-\beta} \). 3 is an arbitrarily chosen large constant and is not optimized.

### 3 The original RRR algorithm and its overfitting problem

#### The algorithm

We next explain algorithms for RRR. Let us recall the intuitions for solving the standard regression problem with uni-variate response. We aim to find \( \min_{M} \|Y - X M \|^2 \), where \( Y \in \mathbb{R}^{n \times 1} \), \( X \in \mathbb{R}^{n \times d_1} \), and \( M \in \mathbb{R}^{1 \times d_2} \). This corresponds to a “degenerated instance” of RRR, in which \( d_1 = 1 \) and \( r = 1 \). The solution is to project \( Y \) onto \( X \)'s column space. The RRR generalizes this simple geometric intuitions in two ways: (i) \( d_2 > 1 \): We need to project a total number of \( d_2 \) column vectors of \( Y \in \mathbb{R}^{n \times d_2} \) onto a subspace from \( X \); (ii) \( \text{rank}(M) \leq r \): Because \( \text{rank}(M) \leq r \), we have \( \text{rank}(X M^\top) \leq r \). We can only project columns of \( Y \) onto a subspace of rank at most \( r \) from column space of \( X \).

In other words, RRR aims to find a subspace \( S \) of rank at most \( r \) from \( X \)'s column space so that when we project columns of \( Y \) onto \( S \), the total variations of the projected vectors are maximized. This can be exactly solved by Principal Component Analysis. We now execute this idea.
The direction problems of eigenvectors.

The scale problem of the eigenvalues.

We use PCA to solve Eq. 2. Furthermore, we can safely discard them. Therefore, by using the coordinate system defined by columns of \( C \)'s left singular vectors, we express the column vectors of \( Y \) and those of \( U_{X}N^{T} \) into independent features. When orthogonal features are fed to a linear regression solver, they will be treated as being mutually independent. The PCA procedure leverages this property to compute the total variations of \( Y \) explained by a chosen subspace.

In other words, operation (i) aims to find a linear map \( \Pi(x) \) that transforms \( x \) into independent features (see Fig. 3(c)). Our crucial observation is that \( \Pi(\cdot) \) found by the standard RRR model is of poor quality and thus it causes the severe overfitting problem. Specifically, because \( x \) is a multivariate Gaussian, there is only one way to construct \( \Pi \) up to a unitary transform: i.e., \( \Pi(x) = (\Lambda^{*})^{-\frac{1}{2}}(V^{*})^{T}x \) (recalling that \( C^{*} = V^{*}\Lambda^{*}(V^{*})^{T} \)) so that \( z = \Pi(x) \) becomes a standard multivariate Gaussian. On the other hand, when the RRR model uses \( U_{X} \) as the basis, it is implicitly using the empirical covariance matrix \( C \) as \( C^{*} \)'s estimate and set \( \Pi(x) = (\Lambda^{*})^{-\frac{1}{2}}(V^{*})^{T}x \). When there is insufficient data, using \( C \) causes two major problems, which lead to severe overfitting.

(i) The scale problem of the eigenvalues. The empirical covariance estimates over-estimates large eigenvalues and underestimates small eigenvalues (e.g., \( \lambda_{1}(C) > \lambda_{1}(C^{*}) \) and \( \lambda_{d_{i}}(C) \ll \lambda_{d_{i}}(C^{*}) \)). The underestimated eigenvalues cause more problems. This is because the precision matrix \( (X^{T}X)^{-1} \) is used to estimate the linear coefficients \( M \), and therefore the impact of under-estimation errors is explained by the difference between \( 1/\lambda_{i}(C) \) and \( 1/\lambda_{i}(C^{*}) \) (for \( i \) close to \( d_{2} \)).

(ii) The direction problems of eigenvectors. Vector \( V^{*} \) controls whether the coordinates in \( \Pi(x) \) will continue to be independent. We observe that a large fraction of \( V^{*} \)'s columns deviates significantly from \( C^{*} \). Fig. 3(a) plots the pairwise angles between the eigenvectors of \( C \) and those of \( C^{*} \). We can see that only the first 20% eigenvectors of \( C \) are sufficiently close to those of \( C^{*} \). The rest provides limited information about the ground-truth directions (i.e., they are "garbage directions").

4 Adaptive Reduced Rank Regression

As shown in Fig. 4, our new model/algorithm (ARRR) consists of two components.

C1. Linear mapping through precision matrix estimation (Sec. 4.1). We construct a \( \tilde{\Pi} \) such that the \( \tilde{\Pi}(x) \) becomes standard Gaussian. We first produce estimates of \( V^{*} \) and \( \Lambda^{*} \) and call them \( \hat{V} \)
and $\hat{\Lambda}$. Then we set $\hat{\Pi}(x) = \hat{V}(\hat{\Lambda})^{-\frac{1}{2}}x$. We call this procedure as **Precision-Matrix-Estimate** because building $\hat{V}$ and $\hat{\Lambda}$ is equivalent to building a precision matrix estimate.

Simple information theoretic argument can show that it is impossible to construct good estimates of all the column vectors of $V^*$. Since keeping the “garbage directions” will adversarially impact the performance of the PCA-based denoiser, we will **cut out** small eigenvectors that we cannot estimate (i.e., our $\hat{\Pi}$ is not full rank). We make sure: 1. The transformed vectors are independent. Our $\hat{\Pi}(x)$ contains $k_1 \leq d_1$ variables that are independent standard Gaussian. 2. We do not cut out too many signals. When a direction is cut out, signals associated with that direction will be permanently destroyed. Thus, we need to ensure only a negligible fraction of signals are destroyed.

**C2. Solving Orthogonal Reduced Rank Regressions (Sec 4.2).** Once we obtain a reliable linear map $\hat{\Pi}$, we will reduce the original RRR problem into a simpler one $y = Nz + \epsilon$, where $z = \hat{\Pi}(x)$ and it is a standard Gaussian. We refer to this simplified problem as Orthogonal Reduced Rank Regression (OrthogonalRR). Let $Z \in \mathbb{R}^{n \times d_1}$ be the matrix stacking together all $z$’s ($Z_{i,:}$ is the $i$-th observation). Note that $\frac{1}{n}Z^T Y = \frac{1}{n}Z^T N z^T + \frac{1}{n}Z^T \epsilon \approx N^T + \frac{1}{n}Z^T E$. Thus, the problem is similar to a matrix denoiser problem [4, 7, 15, 20, 40]: We need to estimate $N$ but we only observe a noisy matrix $N^T + \frac{1}{n}Z^T E$. We develop a PCA-based method to solve it.

**Benchmark algorithms (Sec. 4.3).** We define a family of benchmark algorithms (hereafter BA) to assess the performance of our algorithm. BA needs to be powerful to serve as a performance upper bound but reasonable so that competitive analysis is feasible. The BA family behaves as follows:

1. **Oracle access to $C$:** We assume that the BA has oracle access to $C$ so that it will reduce the original RRR to an OrthogonalRR problem $y = Nz + \epsilon$, where $z$ is a standard Gaussian.

2. **Oracle access to destroyed signals.** Recall that $k_1$ is $\hat{\Pi}$’s rank. The BA also knows $N_{\cdot, k_1+1:d_1}$. Intuitively, these signals are along the directions that are truncated. The BA’s behavior is tied to the behavior of our algorithm. When our algorithm cuts out more signals ($k_1$ becomes small), the BA knows more about $N$ so this characterizes signal loss due to using low rank approximation of $\Pi$.

3. **Use PCA to solve OrthogonalRR.** Let $N = [N_+, N_-]$, where $N_+ \in \mathbb{R}^{d_2 \times k_1}$ and $N_- \in \mathbb{R}^{d_2 \times (d_1-k_1)}$. Because $N_-$ is known to the BA algorithm, it will further reduce our problem to $y_+ = N_+ z_+ + \epsilon$ and solve the problem by using PCA-based techniques.

### 4.1 Precision matrix estimation (Precision-Matrix-Estimate in Fig. 2)

**Algorithm overview.** Our algorithm constructs $\hat{\Pi}$ by producing an estimate of the covariance/precision matrix. We find the eigenvectors’ threshold by a large gap between two consecutive eigenvalues and cutting out all the eigenvectors that are behind the gap. Recall that $C = VAV^T$ and $C_{k_1} = V_{k_1} \Lambda_{k_1} V_{k_1}^T$ is a rank-$k_1$ approximation of $C$. Similarly, $C_{k_1}^* = V_{k_1}^* \Lambda_{k_1}^* (V_{k_1}^*)^T$ is a rank-$k_1$ approximation of $C^*$.

**Proposition 4.1.** Let $\epsilon$ and $\delta$ be two tunable parameters such that $\epsilon = \omega(\log^3 n/\sqrt{k_1})$ and $\delta^2 = \omega(\epsilon)$. Assume that $\Lambda_{k_1}^* = O(i^ {-\rho})$ follows a power law distribution. Consider running the algorithm Precision-Matrix-Estimate, with high probability, we have

(i) Leading eigenvectors/values are close: $\exists$ a unitary matrix $W$ and a constant $c_1$ such that

$$
\|V_{k_1} (\Lambda_{k_1})^{-\frac{1}{2}} - V_{k_1}^* (\Lambda_{k_1}^*)^{-\frac{1}{2}} W\| \leq c_1 \frac{\epsilon}{\delta^2}.
$$

(ii) Gap implies tail bound: $\sum_{i \geq k_1} \lambda_i^* \leq c_1 \delta^{\frac{\rho-1}{\rho+1}}$.

**Interpretation.** Our algorithm aims to find a $k_1$ and set $\hat{\Pi} = V_{k_1} \Lambda_{k_1}^{\frac{1}{2}}$. We observe that (i) if $\Lambda_i(C) - \Lambda_{i+1}(C)$ is large, the first $i$ eigenvectors/eigenvalues of $C$ are close to those of $C^*$. It essentially comes from Davis-Kahan. (ii) Total signal loss is proportional to $\sum_{i \geq k_1} \lambda_i^*$.

Let $\Pi_{k_1} = V_{k_1} (\Lambda_{k_1}^*)^{\frac{1}{2}}$. We face a tradeoff: $\lambda_i^*(C) - \lambda_{i+1}(C)$ needs to be large so that $\Pi_{k_1} \approx \Pi_{k_1}$ and $\sum_{i \geq k_1} \lambda_i^*$ to be small so that $\Pi_{k_1} \approx \Pi$. Our major technical contribution is to show that when $\lambda_{k_1}^* \approx \lambda_{k_1+1}^*$ is small, the tail sum $\sum_{i \geq k_1} \lambda_i^*$ is also small (i.e., small gap implies small tail). See Fig. 3(b). This enables us to simultaneously argue $\Pi_{k_1} \approx \Pi_{k_1}$ and $\Pi_{k_1} \approx \Pi$.

**Analysis outline.** Our analysis consists of the four steps. Steps 1 to 3 argue that $\hat{\Pi}_{k_1}$ and $\Pi_{k_1}$ are close. Step 4 argues that $\Pi_{k_1}$ and $\Pi$ are close. See App. for the full analysis.
Step 1. Dimension-free Chernoff bound for matrices. We first use a dimension-free Chernoff bound from \cite{29} to give a bound on $\|C^* - C\|$:

$$\Pr[\|C^* - C\| \geq \epsilon] \leq (2n^2) \exp(-n\epsilon^2/(\log^4 n)) + n^{-10}. \quad (4)$$

The exponent 10 is chosen arbitrarily and is not optimized.

Step 2. Davis-Kahan bound. We next show that the first few eigenvectors of $C$ are close to those of $C^*$ by using the Davis-Kahan theorem \cite{11}.

Lemma 4.2. Let $\mathcal{P}^* = V_{k_1}(V_{k_1}^*)^\top$ and $\mathcal{P} = V_{k_1}V_{k_1}^\top$. Let $\epsilon = \omega((\log^3 n)/\sqrt{n})$ and $\delta^3 = \omega(\epsilon)$. When $\|C^* - C\| \leq \epsilon$, we have $\|\mathcal{P}^* - \mathcal{P}\| \leq 2\epsilon^2$.

Step 3. Commuting the unitary matrix. Lemma 4.2 shows that there exists a unitary matrix $W$ such that $\|V_{k_1}W - V_{k_1}^*\|$ are close to 0 and $\Lambda_{k_1}$ and $\Lambda_{k_1}^*$ are close. This gives us that $V_{k_1}W\Lambda_{k_1}^{-1/2}$ and $V_{k_1}^*(\Lambda_{k_1}^*)^{-1/2}$ are close, whereas we need that $V_{k_1}W\Lambda_{k_1}^{-1/2}W$ and $V_{k_1}^*(\Lambda_{k_1}^*)^{-1/2}$ are close. The unitary matrix $W$ is not in the right place. This is a standard technical obstacle for analyzing PCA based techniques \cite{12,13,23}, which is addressed by the following lemma:

Lemma 4.3. Let $U_1, U_2$ be $n \times d$ matrices such that $U_1^\top U_1 = U_2^\top U_2 = 1$. Let $S_1, S_2$ be diagonal matrices with strictly positive entries, and let $W \in \mathbb{R}^{d \times d}$ be a unitary matrix. Then,

$$\|U_1S_1^{-1}W - U_2S_2^{-1}\| \leq \frac{\|U_1S_1W - U_2S_2\|}{\min\{\{S_1\}_{ii}\} \cdot \min\{\{S_2\}_{ii}\}} + \frac{\|U_1S_1W - U_2S_2\|}{\min\{\{S_2\}_{ii}\}}.$$

The results from Step 1 to Step 3 suffice to prove the first part of Proposition 4.1.

Step 4. Gap implies tail bound. We finally prove that a local gap (information about two consecutive eigenvalues) suffices to bound the mass of the entire tail.

Lemma 4.4. Let $\{\lambda_i\}_{i \leq n}$ be a sequence such that $\sum_{i \leq n} \lambda_i = 1$ and $\lambda_i = O(n^{-\rho})$ for some $\rho \geq 2$. Define $\delta_i = \lambda_i - \lambda_{i+1}$ for $i \geq 1$. Let $d_0$ be a sufficiently large number. Let $d$ be any number such that $d \geq d_0$. Let $\tau$ be any parameter such that $\tau > 1$. There exists an $i^*$ such that (1) Small local gap: $\delta_{i^*} = \Theta(\sqrt{d/(\log n)})$, and (2) Small tail sum: $\sum_{i \geq d} \lambda_i \leq c_1/2d^{-\tau}$.

By setting $\tau = \rho - 1$ in Lemma 4.4 we may prove part ii of Proposition 4.1.

4.2 The orthogonal reduced rank problem (MOMENT-ESTIMATE in Fig. 2)

We now examine the orthogonal reduced rank problem: $y = N\epsilon + e$, where $N \in \mathbb{R}^{d_2 \times k_1}$, $\epsilon \sim N(0, I_{d_2})$, and $y \in \mathbb{R}^{d_2}$ is the noise. Wlog, we assume $\sigma_e \geq 1$.

Moment-based thresholding algorithm. We focus on comparing our algorithm against the so-called moment-based thresholding (MBT) family of algorithms. An MBT algorithm $\mathcal{A}$ is associated with a thresholding function $f_{\mathcal{A}}(n, d_2, k_1, \sigma_e)$ and produces an estimate as follows: Step 1. Let $\tilde{N}^\top = \frac{1}{\sigma_e}Z^\top Y$. Step 2. Let $r = \min_{r} \{\lambda_{r+1}(\tilde{N}) < f_{\mathcal{A}}(n, d_2, k_1, \sigma_e)\}$ and output $\tilde{N}_r$, the rank-$r$ approximation of $\tilde{N}$ (i.e., all eigenvectors with eigenvalues $\geq f_{\mathcal{A}}(n, d_2, k_1, \sigma_e)$ are kept).

Why comparing against MBT family? We use MBT family as the baseline for two important reasons: (i) Most algorithms rely only on $Z^\top Y$. $Z^\top Y$ is the sufficient statistics for the joint distribution $(y, z)$. Most known estimators only use the sufficient statistics information. (ii) Hard thresholding is effective to find the estimator. Other soft-thresholding methods exist \cite{15} but they often do not significantly improve asymptotic performance. We use simple rules to keep the analysis intuitive.

Our algorithm and the thresholding function. Our algorithm is an MBT algorithm with $f_{\mathcal{A}}(n, d_2, k_1, \sigma_e) = \frac{d_2}{\sqrt{n}}(\sqrt{k_1} + \sqrt{d_2})$. Our first result shows that the MSE will strictly decrease as we increase the rank of our estimate until we hit $k_2$ (the dimension we keep). See also App. D.2.

Lemma 4.5. Let $\tilde{N}^\top = \frac{1}{\sigma_e}Z^\top Y$. Let $\tilde{N}_r$ be a rank-$i$ approximation of $\tilde{N}$. Let $k_2$ be the number determined by MOMENT-ESTIMATE in Fig. 2. We have

$$\|N\|_F > \|\tilde{N}_1 - N\|_F > \|\tilde{N}_2 - N\|_F > \cdots > \|\tilde{N}_{k_2} - N\|_F. \quad (5)$$

Optimality of our algorithm. We show that our algorithm uses asymptotically the same amount of samples to achieve the same MSE, compared to the optimal MBT algorithm. First, we define the optimal MBT algorithm. Second, we provide a lower bound on the optimal algorithm’s thresholding function. Third, we prove an optimality gap between our algorithm and the optimal one.

Step 1. Optimal MBT algorithm. Let the cost of an MBT algorithm $\mathcal{A}$ for a specific $N$ with sample size $n$ be $\text{Cost}(\mathcal{A}, N, n, \sigma_e) = E[\|\hat{N}(\mathcal{A}) - N\|_F^2]$. Because $z$ is a standard Gaussian,
We have (see App. D.4) for a full analysis.

\[
\Theta(1)
\]

No surprise.

We define a new family of baselines. Theorem 4.10. Tracking error.

\[
\delta
\]

\[
\text{variance error increases. When}
\]

Because

Step 2. Lower bound. Below is our main lower bound result. See also App. D.3.

Lemma 4.6. Let \( A^* \) be the optimal MBT algo. We have \( f_{A^*}(n, d_2, k_1, \sigma) \leq \frac{\delta}{\sqrt{n}}(\sqrt{F_1} + \sqrt{F_2}) \).

Step 3. Optimal thresholding implies optimal sampling. Because \( f_{A^*}(n, d_2, k_1, \sigma) = \Theta(f_{A}(n, d, k, \sigma)) \), there exists a sufficiently large constant \( c \) such that \( f_{A^*}(n, d_2, k_1, \sigma) > 3f_{A}(cn, d_2, k_1, \sigma) \).

Intuitively, this means when the sample size is increased by a constant factor, our threshold value shrinks to the optimal threshold. Consequently, we are expected to extract more signals with \( cn \) samples.

Definition 4.7. Let \( A_1 \) and \( A_2 \) be two MBT algorithms. \( A_1 \) is \((c, f)\)-approximation to \( A_2 \) if for any matrix \( N \) and any sufficiently large \( n \), \( \text{Cost}(A_1, N, cn, \sigma) + f\|N\|_{F}^{2} \leq \text{Cost}(A_2, N, n, \sigma, \sigma) \), where \( c \) and \( f \) can be a function of \( n \). In addition, when \( A_1 \) is \((c, f)\)-approximation to \( A^* \), we say \( A_1 \) is \((c, f)\)-close to the optimal algorithm.

We have (see App. D.4) for a full analysis.

Proposition 4.8. Let \( \max\{k_1, d_2\} = o(n/\log^2 n) \). Our algorithm for solving OrthogonalRR is \((\Theta(1), f^2)\)-close to the optimal MBT, where \( f = \Theta\left(\frac{\max\{\sqrt{\kappa_1}, \log^2 n\}}{\sqrt{n}}\right) \).

4.3 Optimality of Adaptive Reduced Rank Regression

This section “puts everything together” and shows that our algorithm is close to optimal. We have two major tasks/messages: (i) Definition of baseline algorithms. Since \( x \) is not standard Gaussian, we define a new family of baselines. (ii) No surprise. Our final error consists of the sum of two terms. One comes from estimation error of \( \hat{\Pi} \) and the other comes from tracking error of solving an OrthogonalRR problem.

General Reduced Rank Regression Family (GRRRF). An algorithm is in a general reduced rank regression family if it takes two steps to solve an RRR problem: Step 1. First, it builds a rank-\( k_1 \) linear map \( \hat{\Pi} \) to perform the transformation \( \hat{z} = \hat{\Pi}(x) \), where \( \hat{z} \in \mathbb{R}^{k_1} \). Step 2. Then, it solves the reduced rank regression problem \( \hat{y} = N\hat{z} + \epsilon \), where \( N = M\hat{\Pi}^{-1} \) and \( \hat{\Pi}^{-1} \) is the (pseudo)-inverse of \( \Pi \). Note that in this family, we do not make any requirement on \( \hat{\Pi} \), i.e., the algorithm can have its own rule to determine \( k_1 \) or not attempt to track the leading eigenvectors of \( C^* \).

Oracle benchmark family. An Oracle Benchmark Algorithm is tied to the rank \( k_1 \) of \( \hat{\Pi} \). Recall from Sec. [4] that a rank-\( k_1 \) oracle benchmark algorithm (rank-\( k_1 \) OBA) needs to have oracle access to \( C^* \), to the signals along the small destroyed directions, and will reduce the original problem to a OrthogonalRR problem. See App. [E] and Fig. [6] for a precise definition.

Tracking error. Let \( A \) be an algorithm from GRRRF. Let \( \hat{M}(A) \) be the output of \( A \). The cost of \( A \) with sample size \( n \) is \( \text{Cost}(A, M, n, \sigma) = E[\|\hat{M}(A)x - Mx\|] \). The cost is expressed in terms of \( \ell_2 \)-norm instead of the square of \( \ell_2 \)-norm. Both definitions are meaningful. We use \( \ell_2 \)-norm to simplify the analysis. Techniques developed here can be used to produce a bound for the squared cost.

Definition 4.9. An algorithm \( A \) in GRRRF is \((c, f)\)-close to the optimal algorithm in the oracle benchmark family if for a sufficiently large \( n \): 1. The rank of \( \hat{\Pi} \) is \( k_1 \). 2. For the optimal rank-\( k_1 \) oracle benchmark family of algorithm, we have

\[
\text{Cost}(A, M, cn, \sigma) + fE\|Mx\| \leq \text{Cost}(A, M, n, \sigma).
\]

Theorem 4.10. Assume Al to A4 hold, let \( n/\log^2 n > d_2 \). Let \( \epsilon = \omega(\log n/\sqrt{n}) \) and \( \delta^3 = \omega(\epsilon) \) be tunable parameters and \( k_1 \) be the rank of \( \hat{\Pi} \) determined by our algorithm. There exists a constant \( c_0 \) such that our AARR is \((c_0, f)\)-close to the optimal algorithm, where \( f = O\left(\frac{\kappa_1}{n} + \frac{\delta^2}{\sigma^2} + \delta^{\frac{\sqrt{n}}{\log^2 n}}\right) \).

We make two remarks. (i) The first two terms \( \frac{\kappa_1}{n} + \frac{\delta^2}{\sigma^2} \) come from the error \( \hat{\Pi} - \Pi \) and the last term comes from the error of MOMENT-ESTIMATOR so there is no surprise. (ii) We see a variance-bias trade off: When \( \delta \) shrinks, \( k_1 \) becomes larger and \( \hat{\Pi} \) becomes less accurate (compared to \( \Pi k_1 \)) so variance error increases. When \( \delta \) grows, \( \sum_{i \geq k_1} \lambda_i^2 \) grows so bias error increases.
5 Experiments

Synthetic data. We produce synthetic data using the linear model $Y = XM^T + E$. Eigenvalues of $X$’s covariance matrix follow a power law distribution. To generate $M$, we randomly sample entries from $\{-1, 0, 1\}$ and run SVD to construct a low rank approximation of it. $E$’s entries are i.i.d. $N(0, \sigma^2_{\text{noise}})$ with $\sigma_{\text{noise}} = \eta \cdot \text{std(vec}(XM^T))$ and $\eta$ is a noise multiplier parameter. We set $n = 500$, $d_1 = 500$, $d_2 = 100$, rank$(M) = 10$, and $\rho = 2$. Our experiments are robust against other parameters. We compare our algorithm with ridge regression, reduced rank ridge regression (“Reduced ridge”) [26], LASSO, nuclear norm regularized regression, and standard reduced rank regression (“RRR”). Nuclear norm regularized regression is optimized for estimating low rank matrices subject to linear constraints, while the nuclear norm is both slow and significantly worse than other methods because the part $\sigma_{\text{min}}(C^*)$ is small in our data. Fig. 4c shows the rank of $M$ from the regression methods explicitly regularized by rank. Only ARRR and Reduced ridge can effectively adjust $M$’s rank to signal quality.

Real dataset. To examine our forecasting methods in predicting equity returns, we use a stock market dataset from an emerging market. The dataset consists of approximately 3600 stocks between 2011 and 2018. We focus on predicting the next 5-day returns. For each asset in the universe, we compute its past 1-day, past 5-days and past 10-days returns as part of the feature. We use a standard approach to translate forecasts into positions [18, 31, 2, 42]. We examine two different universes in this market: (i) Universe 1 is equivalent to S&P 500 and consists of 983 stocks, and (ii) Full universe consists of all stocks except for illiquid ones. See Appendix F.2 for the details.

Results. Nuclear norm regularized regression is too slow for this dataset so we cannot include it. Table 2 reports the forecasting power and portfolio return for out-of-sample periods in two universes. We observe that (i) The data has low signal-to-noise ratio. The out-of-sample $R^2$ values of all the methods are close to 0. (ii) ARRR has the highest forecasting power. Note that reduced rank ridge has considerably worse performance, suggesting that our model appears to be more robust in financial datasets. (iii) Small in-sample and out-of-sample gap. Our method has the worst in-sample $R^2$ and has the smallest gap between in-sample and out-of-sample performance (see column MSE$_{in-out}$), suggesting that our model is better at avoiding spurious signals.

6 Conclusion

This paper studies the high-dimensional regression problem $y = X\epsilon + \epsilon$ with a low signal-to-noise ratio which is known to suffer from severe overfitting. First, we analyze reduced rank regression and its overfitting problem. Second, we propose adaptive reduced rank regression (ARRR) with better
generalization guarantees. Our ARRR leverages the spectral properties of $x$ and can adapt the model to signal quality. Third, we prove the optimality of our algorithm. Additionally, our approach either outperforms or is competitive with existing baselines in the synthetic experiments and achieves the best performance in real dataset (predicting equity returns).
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A Notation table

In this section, we list the notation in our paper.

- $(y, x)$: A pair of observations, in which $x \in \mathbb{R}^{d_1}$ is the feature and $y \in \mathbb{R}^{d_2}$ is the response.
- $M$: Parameter matrix in $\mathbb{R}^{d_2 \times d_1}$.
- $\epsilon \in \mathbb{R}^{d_2}$: A zero mean noise.
- $\sigma_\epsilon$: Standard deviation of the noise.
- $X \in \mathbb{R}^{n \times d_1}$ and $Y \in \mathbb{R}^{n \times d_2}$: Data matrices with their $i$-th rows representing the $i$-th observation.
- $n$: Total number of samples.
- $V_{i,:}$: The $i$-th row of $V$.
- $V_{i,:}$: The $i$-th column $V$.
- $C^*$: Covariance matrix of $x$.
- $\hat{C}$: Empirical estimate of $C^*$.
- $\hat{V}$: An estimate of $V^*$.
- $\hat{A}$: An estimate of $\Lambda^*$, where $C^* = V^*\Lambda^*(V^*)^\top$.
- $\lambda_i$ (or $\lambda_i(C)$): the $i$-th largest eigenvalue of $C$.
- $C_{k_1}$: Recall that $C = V\Lambda V^\top$ and $C_{k_1} = V_{k_1}\Lambda_{k_1}V_{k_1}^\top$ is a rank-$k_1$ approximation of $C$.
- $\lambda_i^*$: The $i$-th largest eigenvalue of $C^*$.
- $C_{k_1}^*$: Recall that $C^* = V^*\Lambda^*(V^*)^\top$ and $C_{k_1}^* = V_{k_1}^*\Lambda_{k_1}^*(V_{k_1}^*)^\top$ is a rank-$k_1$ approximation of $C$.
- $\Theta$: The precision matrix of $x$.
- $\Pi(\cdot)$: The linear map and $\Pi(x) = (\Lambda^*)^{-\frac{1}{2}}(V^*)^\top x$
- $\hat{\Pi}$: An estimate of $\Pi$. $\hat{\Pi}(x) = \hat{V}(\hat{\Lambda})^{-\frac{1}{2}} x$.
- $\Pi^{-1}$: The (pseudo)-inverse of $\Pi$.
- $N$: $M\Pi^{-1}$.
- $\|A\|_F$: Frobenius norm of $A$.
- $\|A\|$: Spectral norm of $A$.
- $\|A\|_*$: Nuclear norms of $A$.
- $k_1$: The rank of our $\hat{\Pi}$.
- $\hat{N}$: An estimate of $N$.
- $k_2 = \min_{k_2}(\sigma_{k_2} \geq \frac{4\sigma_\epsilon}{\sqrt{n}}(\sqrt{k_1} + \sqrt{d_2}))$ where $\sigma_i$ be the $i$-th singular value of $\frac{1}{n}Z^\top Y$.
- $\ell$: $\max\{k_1, n/\log^2 n\}$.
- $z$: $\Pi(x)$: $\Pi(x)$ transforms $x$ into independent features.
- $Z \in \mathbb{R}^{n \times d_1}$: The matrix stacking together all $z$'s where $Z_{i,:}$ is the $i$-th observation.
- $\text{Span}\{\{U_{1}, U_{2}, \ldots, U_{\ell}\}\}$: The subspace spanned by $\{U_{1}, U_{2}, \ldots, U_{\ell}\}$.
- $A^*$: The optimal MBT algorithm.
- $F_1 \in \mathbb{R}^{d_2 \times g}$: Column space of $F_1$ is the same as $\text{Span}\{\{U_{i}\} \}_{i \in \mathcal{G}}$.
- $F_2 \in \mathbb{R}^{d_2 \times (k_2 - g)}$: Columns form an orthonormal basis for the subspace $\text{Span}\{\{U_{i}\} \}_{i \leq k_2} - \text{Span}\{\{U_{i}\} \}_{i \in \mathcal{G}}$.
- $F_3 \in \mathbb{R}^{d_2 \times (d_2 - k_2)}$: Columns form an orthonormal basis of the subspace that’s orthogonal to $\text{Span}\{\{U_{i}\} \}_{i \leq k_2}$.
- ARRR: Adaptive Reduced Rank Regression.
- RRR: Reduced Rank Regression.
- OrthogonalRR: Orthogonal Reduced Rank Regression.
- Reduced ridge/ Reduced rank ridge: Reduced rank ridge regression.
- BA: Benchmark algorithms.
- OBA: Oracle benchmark family.
- GRRRF: General Reduced Rank Regression Family.
• MBT: Moment-based thresholding.
• PCA: Principal components analysis.
• SVD: Singular-value decomposition.
• $r_{i,t}$: Return of the $i$-th asset for time interval $t$. 
B Exiting building blocks

B.1 Existing lemmas we use

Lemma B.1. Let \( A \) and \( B \) be \( n \times n \) positive semidefinite matrices with the same rank of \( d \). Let \( X \) and \( Y \) be of full column rank such that \( XX^\top = A \) and \( YY^\top = B \). Let \( \delta \) be the smallest non-zero eigenvalue of \( B \). Then there exists a unitary matrix \( W \in \mathbb{R}^{d \times d} \) such that
\[
\| XW - Y \| \leq \frac{\| A - B \| (\sqrt{\| A \|} + \sqrt{\| B \|})}{\delta}.
\]

B.2 Algorithms for the original RRR and negative examples

Fig. 5 presents the algorithm for the original RRR. We next give an example that coordinates in

\[
\text{REDUCED-RANK-REGRESSION}(X, Y, k)
\]

1. \( \triangleright \) solve \( \| Y - XM^\top \|_F^2 \)
2. \( \triangleright \) s.t. \( \text{rank}(M) \leq k \)
3. \( [U, \Sigma, V] = \text{svd}(X) \)
4. \( N^\top = U^\top Y \)
5. \( [U^N, \Sigma^N, V^N] = \text{svd}(N) \)
6. \( N_k \leftarrow U_k^N \Sigma_k^N (V_k^N)^\top \)
7. \( \text{return } M_k = N_k \Sigma^{-1} V^\top \)

Figure 5: Algorithm for the reduced rank regression.

C Analysis of our algorithm for precision matrix estimation

This section proves Proposition 4.1.

Our proof consists of four major steps. Step 1 to 3 argue that \( k_{i_1} \) and \( k_{i_2} \) are close. Step 4 argues that \( \sqrt{k_1} \) and \( \sqrt{k_2} \) are close.

Step 1. Dimension-free Chernoff bound for matrices. We first give a bound on \( \| C^* - C \| \), which characterizes the tail probability by using only the first and second moments of random vectors. This is the key device enabling us to meaningfully recover signals even when \( n \ll d_1 \).

**Lemma C.1.** Recall that \( C^* = \mathbb{E}(xx^\top) \) and \( C = \frac{1}{n}X^\top X \). We have
\[
\Pr[\| C^* - C \| \geq \epsilon] \leq (2n^2) \exp(-n \epsilon^2/(\log^4 n)) + n^{-10}.
\]

The exponent 10 is chosen arbitrarily and is not optimized.

**Proof of Lemma C.1.** We use the following specific form of Chernoff bound (29)

**Lemma C.2.** Let \( z_1, z_2, \ldots, z_n \) be i.i.d. random vectors such that \( \| z_i \| \leq \alpha \) a.s. and \( \| \mathbb{E}[z_i z_i^\top] \| \leq \beta \). Then for any \( \epsilon > 0 \),
\[
\Pr \left[ \left\| \frac{1}{n} \sum_{i=1}^{n} z_i z_i^\top - \mathbb{E}[z_i z_i^\top] \right\| \geq \epsilon \right] \leq (2n^2) \exp \left( - \frac{n \epsilon^2}{16 \beta \alpha^2 + 8 \alpha^2 \epsilon} \right)
\]

We aim to use Lemma C.2 to show Lemma C.1. But the \( \ell_2 \)-norm of \( z_i \)'s are unbounded so we need to use a simple coupling technique to circumvent the problem. Specifically, let \( c_0 \) be a suitable constant and define
\[
\tilde{z}_i = \begin{cases} 
  z_i & \text{if } |x_i| \leq c_0 \log^2 n \\
  0 & \text{otherwise}.
\end{cases}
\]

By using a standard Chernoff bound, we have
\[
\Pr[\exists i : \tilde{z}_i \neq z_i] \leq \frac{1}{n^{10}}.
\]

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Let us write $\hat{C} = \frac{1}{n} \sum_{i \leq n} \hat{z}_i \hat{z}_i^\top$. We shall set $\alpha = c_0 \log^2 n$ and $\beta = \Theta(1)$ in Lemma 2. One can see that
\begin{equation}
\Pr[\|C^* - C\| \geq \epsilon] \leq \Pr \left[ (\|\hat{C} - C\| \geq \epsilon) \lor (\hat{C} \neq C) \right] \leq 2 n^2 \exp \left( - \frac{nc^2}{\log^2 n} \right) + \frac{1}{n^{10}}.
\end{equation}

\hfill \Box

**Step 2. Davis-Kahan bound.** The above analysis gives us that $\|C^* - C\| \leq \epsilon$. We next show that the first a few eigenvectors of $C$ are close to those of $C^*$.

**Lemma C.3.** Let $P^* = V_{k_1} (V_{k_1}^*)^\top$ and $P = V_{k_1} V_{k_1}^\top$. Let $\epsilon = \omega(\log^3 n/\sqrt{n})$ and $\delta = \omega(\epsilon)$. When $\|C^* - C\| \leq \epsilon$,
\begin{equation}
\|P^* - P\| \leq \frac{2 \epsilon}{\delta}.
\end{equation}

**Proof.** Recall that $\lambda_1^*, \lambda_2^*, \ldots, \lambda_{d_1}^*$ are the eigenvalues of $C^*$. Let also $\lambda_1, \lambda_2, \ldots, \lambda_{d_1}$ be the eigenvalues of $C$. Define
\begin{equation}
S_1 = [\lambda_{k_1} - \delta/10, \infty] \quad \text{and} \quad S_2 = [0, \lambda_{k_1+1} + \delta/10].
\end{equation}
The constant 10 is chosen in an arbitrary manner. Because $\|P^* - P\| \leq \epsilon$, we know that $S_1$ contains $\lambda_1^*, \ldots, \lambda_k^*$ and that $S_2$ contains $\lambda_{k+1}^*, \ldots, \lambda_{n}^*$ [21]. Using the Davis-Kahan Theorem [11], we get
\begin{equation}
\|P^* - P\| \leq \frac{\|C^* - C\|}{0.8\delta} \leq \frac{2 \epsilon}{\delta}.
\end{equation}

\hfill \Box

**Step 3. Commuting the unitary matrix.** Lemma C.3 shows that there exists a unitary matrix $W$ such that $\|V_{k_1} W - V_{k_1}^*\|$ are close to 0 and $\lambda_{k_1}^*$ and $\lambda_{k_1}$ are close. This gives us that $V_{k_1} W \lambda_{k_1}^{-1/2}$ and $V_{k_1}^* (\lambda_{k_1}^*)^{-1/2}$ are close, whereas we need that $V_{k_1} \lambda_{k_1}^{-1/2} W$ and $V_{k_1}^* (\lambda_{k_1}^*)^{-1/2}$ are close. The unitary matrix $W$ is not in the right place. This is a standard technical obstacle for analyzing PCA based techniques [37] [13] [23], which is addressed by the following lemma:

**Lemma C.4.** Let $U_1, U_2$ be $n \times d$ matrices such that $U_1^\top U_1 = U_2^\top U_2 = I$. Let $S_1, S_2$ be diagonal matrices with strictly positive entries, and let $W \in \mathbb{R}^{d \times d}$ be a unitary matrix. Then,
\begin{equation}
\|U_1 S_1^{-1} W - U_2 S_2^{-1}\| \leq \frac{\|U_1 S_1 W - U_2 S_2\|}{\min \{S_1\}_{ii} \cdot \min \{S_2\}_{ii}} + \frac{\|U_1 U_1^\top - U_2 U_2^\top\|}{\min \{S_1\}_{ii}}.
\end{equation}

**Proof.** Observe that,
\begin{equation}
U_1 S_1^{-1} W - U_2 S_2^{-1} = U_1 S_1^{-1} W (S_2 U_2^\top - W^\top S_1 U_1^\top) U_2 S_2^{-1} + U_1 U_1^\top U_2 S_2^{-1} - U_2 U_2^\top U_2 S_2^{-1}.
\end{equation}
The result then follows by taking spectral norms of both sides, the triangle inequality and the sub-multiplicativity of the spectral norm.

\hfill \Box

Results from Step 1 to Step 3 suffice to prove the first part of Proposition 4.1.

**Proof of part (i) in Proposition 4.7** First, we use Lemma C.3 and Lemma B.1 (adopted from [37]) to get that
\begin{equation}
\|V_{k_1}^* (\lambda_{k_1}^*)^{-1/2} W - V_{k_1} (\lambda_{k_1})^{-1/2}\| \leq \frac{c_0 \epsilon}{\delta^2}.
\end{equation}

Next, observe that $\lambda_{k_1}, \lambda_{k_1}^* = \Omega(\delta)$. By applying Lemma C.4 with $U_1 = V_{k_1}^*$ and $S_1 = (\lambda_{k_1}^*)^{-1/2}$, $U_2 = V_{k_1}$ and $S_2 = (\lambda_{k_1})^{-1/2}$, we obtain
\begin{equation}
\|V_{k_1}^* \lambda_{k_1}^{-1/2} - V_{k_1} \lambda_{k_1}^*^{-1/2}\| \leq \frac{\|V_{k_1}^* \Lambda_{k_1}^* \lambda_{k_1}^{-1/2} W - V_{k_1} \Lambda_{k_1} \lambda_{k_1}^{-1/2}\|}{\delta} + \frac{\|P^* - P\|}{\delta} \leq \frac{c_1 \epsilon}{\delta^3}.
\end{equation}

\hfill \Box
Step 4. Gap implies tail bound. Our thresholding rule ensures that $\lambda^*_k - \lambda^*_{k+1} = \Omega(\delta)$. Our next goal is to prove that a local gap (information about two consecutive eigenvalues) suffices to bound the mass of the entire tail.

Lemma C.5. Let $\{\lambda_i\}_{i \leq n}$ be a sequence such that $\sum_{i \leq n} \lambda_i = 1$ and $\lambda_i = O(n^{-\rho})$ for some $\rho \geq 2$. Define $\delta_i = \lambda_i - \lambda_{i+1}$ for $i \geq 1$. Let $d_0$ be a sufficiently large number. Let $d$ be any number such that $d \geq d_0$. Let $\tau$ be any parameter such that $\tau > \rho - 1$. There exists an $i^*$ such that $1$ (Small local gap). $\delta_{i^*} = \Theta(\frac{1}{\sqrt{n}d^{\rho-1}})$, and $2$ (Small tail sum). $\sum_{i \geq i^*} \lambda_i \leq c_1/2d^{-\tau}$.

We can use Lemma C.5 to prove part ii of Proposition 4.1 in a straightforward manner.

Proof of Lemma C.5. Because $\lambda_i = O(i^{-\rho})$, there exists a constant $c_0$ such that $\lambda_i \leq c_0/i^\rho$. We let $\text{utail}(\ell) = \sum_{i \geq \ell} c_0/i^\rho = c_1/\ell^\rho - 1$.

Next, let us define

$$i_1 = \arg \max_{i_1} \left\{ \sum_{i \leq i_1} \lambda_i \leq 1 - \text{utail}(d^{\frac{\tau}{\rho-1}}) \right\} \quad \text{and} \quad i_2 = \arg \max_{i_2} \left\{ \sum_{i \leq i_2} \lambda_i \leq 1 - 0.5 \times \text{utail}(d^{\frac{\tau}{\rho-1}}) \right\}$$

We can check that when $i_1 = d^{\frac{\tau}{\rho-1}}$, we have $\sum_{i \leq i_1} \lambda_i \geq 1 - \text{utail}(d^{\frac{\tau}{\rho-1}})$. Therefore, there exists a constant $c_3$ such that $i_1 \leq i_2 \leq c_3 d^{\frac{\tau}{\rho-1}}$.

Next, note that $\sum_{i \leq i_2 + 1} \lambda_i \geq 1 - \frac{c_4 d^{-\tau}}{2}$ and $\sum_{i \leq i_1} \lambda_i \leq 1 - c_2 d^{\tau}$. This implies that

$$\sum_{i_1 + 1 \leq i \leq i_2 + 1} \lambda_i \geq \frac{c_1 d^{-\tau}}{2}.$$ (19)

By using an averaging argument, there exists an $i_3 \in [i_1 + 1, i_2 + 1]$ such that

$$\lambda_{i_3} \geq \frac{c_1 d^{-\tau}}{2(i_2 - i_1)} \geq c_4 d^{-\frac{\tau}{\rho-1}}$$

for some constant $c_4$.

On the other hand, we have $\lambda_d \leq \frac{c_0}{d^{\rho}}$. Therefore, there exists an $i^* \in [i_3, d]$ such that $\delta_{i^*} = \Theta(d^{-\frac{\tau}{\rho-1}})$. Finally, one can check that

$$\sum_{i \geq i^*} \lambda_i \leq \sum_{i \geq i_2} \lambda_i \leq \frac{c_4}{2d^{\tau}}.$$ (21)

This completes our analysis.

By setting $\tau = \rho - 1$ in Lemma C.5 we may prove part ii of Proposition 4.1.

D Missing proofs for Orthogonal Reduced Rank Regression

D.1 Building blocks

We first describe a few self-contained building blocks.

D.1.1 $\frac{1}{n}Z^T Z$ is close to $I$

Recall that $Z \in \mathbb{R}^{n \times k_1}$ so that each entry is a standard Gaussian. We can intuitively see that $\frac{1}{n}Z^T Z$ is close to $I$. The lemma below quantifies the difference.

Lemma D.1. Let $Z \in \mathbb{R}^{n \times k_1}$, where $k_1 < n$. Let each entry of $Z$ be an independent Gaussian. We have

$$\| \frac{1}{n}Z^T Z - I \| \leq \max \left\{ 10 \log^2 n, 4 \sqrt{\frac{k_1}{n}} \right\}$$ (22)

Proof of Lemma D.1. We rely on the Lemma 35:
Lemma D.2. Let $S \in \mathbb{R}^{n \times k}$ ($n > k$) be a random matrix so that each $S_{i,j}$ is an independent standard Gaussian random variable. Let $\sigma_{\max}(S)$ be the maximum singular value of $S$ and $\sigma_{\min}(S)$ be the minimum singular value of it. We have

$$\Pr[\sqrt{n} - \sqrt{k} - t \leq \sigma_{\min}(S) \leq \sigma_{\max}(S) \leq \sqrt{n} + \sqrt{k} + t] \geq 1 - 2 \times \exp(-t^2/2).$$ (23)

Here, we may set $t = \max \left\{ \frac{\sqrt{k}}{10}, \log^2 n \right\}$. Let us start with considering the case $\frac{\sqrt{k}}{10} > \log^2 n$. We have

$$\sigma_{\min}(Z^T Z) \geq n - 2.2 \sqrt{nk} + 1.21 k_1 \geq n - 2.2 \sqrt{nk}. \quad (24)$$

and

$$\sigma_{\max}(Z^T Z) \leq n + 2.2 \sqrt{nk} + 1.21 k_1 \leq n + 4 \sqrt{nk}. \quad (25)$$

The case $\frac{\sqrt{k}}{10} \leq \log^2 n$ can be analyzed in a similar fashion so that we can get

$$\|\frac{1}{n} Z^T Z - I\| \leq \max \left\{ \frac{10 \log^2 n}{\sqrt{n}}, \frac{4 \sqrt{k_1}}{n} \right\}.$$ (26)

\[ \square \]

D.1.2 Properties of $\|Z^T E\|$  

The noise term $\frac{1}{n} Z^T E$ plays a major role in determining $k_2$. Recall that $Z = U^Z \Sigma^Z (V^Z)^T$ be SVD of $Z$. The lemma below characterizes the behavior of $\|Z^T E\|$.

Lemma D.3. Using our notations for OrthogonalRR, we have

1. $\|\frac{1}{n} Z^T E\| \leq \frac{1}{\sqrt{n}} (\sqrt{k_1} + \sqrt{d_2})$.

2. Let $B = V^Z (U^Z)^T E / \sigma_e$. There exists a $\Delta$ with $\|\Delta\| \leq \frac{20 \max\{\log^2 n, k_1\}}{\sqrt{n}}$ such that

$$\frac{1}{n} Z^T E = \frac{\sigma_e}{\sqrt{n}} (B + \Delta).$$ (27)

Proof of Lemma D.3. Let $t = \max \left\{ \frac{10 \log^2 n}{\sqrt{n}}, \frac{4 \sqrt{k_1}}{n} \right\}$. By Lemma D.1 with high probability $\|\frac{1}{n} Z^T Z - I\| \leq t$. This implies that the eigenvalues of $Z^T Z$ are all within the range $n(1 \pm t)$. Note that for $0 < \eta < 1/3$, if $\xi \in [1 - \eta, 1 + \eta]$, then $\sqrt{\xi} \in [1 - 2\eta, 1 + 2\eta]$. This implies that the singular values of $Z$ are within the range $\sqrt{n}(1 \pm 2t)$. Let $\Sigma^Z / \sqrt{n} = I + \Delta^Z$, where $\|\Delta^Z\| \leq 2t$. We have

$$V^Z \left( \frac{\Sigma^Z}{\sqrt{n}} \right)^T \frac{E}{\sqrt{n}} = V^Z (I + \Delta^Z)(U^Z)^T \frac{E}{\sqrt{n}} = V^Z (U^Z)^T \frac{E}{\sqrt{n}} + V^Z \Delta^Z (U^Z)^T \frac{E}{\sqrt{n}}.$$ (28)

As the columns $V^Z$ are also unit vectors, $V^Z (U^Z)^T E / \sqrt{n}$ is a matrix with i.i.d. Gaussian entries with standard deviation $\sigma_e / \sqrt{n}$.  

Let $B = V^Z (U^Z)^T E / \sigma_e$ and $\hat{B} = (U^Z)^T B / \sigma_e$. Then, combining (27) and (28), we have

$$\frac{1}{n} Z^T E = \frac{\sigma_e}{\sqrt{n}} (B + V^Z \Delta^Z \hat{B}).$$ (29)

The entries in $B (\hat{B})$ are all i.i.d Gaussian. By Marchenko-Pastar’s law (and the finite sample version of it [33]), we have with high probability, $\|\hat{B}\|, \|B\| = \sqrt{k_1} + \sqrt{d_2} + o(\sqrt{k_1} + \sqrt{d_2})$. Therefore, with high probability:

$$\left\| \frac{1}{n} Z^T E \right\| \leq \frac{1.1 \sigma_e}{\sqrt{n}} (\sqrt{k_1} + \sqrt{d_2}).$$

Using the fact that $\|\Delta^Z\| \leq 2t$ and (29), we also prove the second part of the lemma. \[ \square \]
D.2 Proof of Lemma 4.5

Proof. Without loss of generality, assume that \( k_1 \leq d_2 \). We shall show that

\[
\| \hat{N}_{k_2 - 1} - N \|_F > \| \hat{N}_{k_2} - N \|_F.
\]

We treat the remaining inequalities in a similar fashion. Let \( \hat{N} = U \Sigma V^\top \) be the SVD of \( N \) and \( N^\top = V \Sigma^\top U^\top \). Treat the \((k_2 - 1)\)th and \( k_2 \)th rows of the matrices \( \hat{N}^\top \) and \( N^\top \) as vectors in \( \mathbb{R}^{d_2} \). Then we express these vectors in terms of the orthonormal basis \( U^N \). In a thin SVD, \( U^N \) only has \( k_1 \) columns. Let \( (U^N)^\perp \) be any orthonormal basis for the subspace that’s orthogonal to the space spanned by the columns of \( U^N \). The columns of \( U^N \) and \((U^N)^\perp \) span \( \mathbb{R}^{d_2} \). Recall that \( U_{i:k}^N \) is the \( i \)-th column of \( U^N \). For any \( \ell \leq k_2 \), we have \( \| \hat{N}_{k_2}^\top - N^\top U_{i:k}^N \|_F^2 = \sum_{i=1}^{k_1} \| (\hat{N}_{k_2}^\top - N^\top) U_{i,k}^N \|_2^2 + \| N^\top (U^N)^\perp \|_F^2 \). Therefore,

\[
\| \hat{N}_{k_2 - 1}^\top - N^\top \|_F^2 - \| \hat{N}_{k_2}^\top - N^\top \|_F^2 = \| N^\top U_{i:k_2}^N \|_F^2 - \| (\hat{N}_{k_2}^\top - N^\top) U_{i:k_2}^N \|_F^2.
\]

Note that

\[
\| (\hat{N}_{k_2}^\top - N^\top) U_{i:k_2}^N \|_F = \| (\hat{N}_{k_2}^\top - N^\top) U_{i:k_2}^N \|_2 \leq \frac{1}{n} \| Z^\top e U_{i:k_2}^N \| \leq \frac{1.1 \sigma_1}{\sqrt{n}} (\sqrt{k_1} + \sqrt{d_2}).
\]

On the other hand, our thresholding rule gives \( \| N^\top U_{i:k_2}^N \|_F \geq \frac{4 \sigma_1}{\sqrt{n}} (\sqrt{k_1} + \sqrt{d_2}) \).

We use a triangle inequality to complete the proof. \( \square \)

D.3 Lower bounds

This section shows that our thresholding function is optimal (i.e., prove Lemma 4.6). Our goal is to

construct \( M \) so that if we are more aggressive in choosing a larger \( k_2 \), out-of-sample error increases. Our lower bound instance is a strong one because it contains a linear number of signals that are right below the threshold. In other words, having a massive amount of “almost strong enough” signals will not help. We remark that rank-

\( k_2 \) matrices are trivial lower bounds: if an algorithm recovers \( k_2 \) dimensional subspace for the signal, the remaining subspace consists of only noises so including more subspace will only harm. Our lower bound highlights an impossible result for a more interesting setting in which \( n \) is insufficiently large to recover the true rank of \( N \). In this case, any subspace in \( \hat{N}_{k_2}^\top \) (the subspace that is orthogonal to \( \hat{N}_{k_2} \)) almost certainly contains some signals. Our lower bound asserts that the noise is consistently larger for most of the remaining subspace.

Lemma D.4. For any \( k_1, d_2, \) and \( \sigma, \) there exists a matrix \( N \in \mathbb{R}^{d_2 \times k_1} \) such that it has a total number of \( c_0 k_1 \) non-zero singular values, each of which is \( \theta = \frac{c_0 \sigma}{\sqrt{n}} (\sqrt{k_1} + \sqrt{d_1}) \) for some suitably small \( c_0 \). Any estimator \( \hat{N} \) for \( \ell > 0 \) (defined above) will give worse than trivial out-of-sample performance:

\[
\| \hat{N}_{\ell} - N \|_F^2 > \| N \|_F^2.
\]

We make a few remarks. First, \( \mathbb{E} \| A z \|_2^2 = \| A \|_F^2 \) when the covariance of \( z \) is \( I \) and \( z \) is sub-Gaussian. Therefore, \( \| \hat{N}_{\ell} - N \|_F^2 > \| N \|_F^2 \) implies that \( \hat{N}_{\ell} \) produces a forecast that’s worse than trivial (trivial forecast is 0). Second, the threshold our algorithm uses is \( \frac{4 \sigma_1}{\sqrt{n}} \sqrt{k_1} + \sqrt{d_1} \). In this lower bound example, our algorithm will decide to produce trivial forecast. On the other hand, when the threshold is set to be \( \frac{c_0 \sigma}{\sqrt{n}} \sqrt{k_1} + \sqrt{d_1} \) for some suitably small constant \( c_2 \), we will choose a \( k_2 > 0 \), leading to worse-than-trivial forecast. Finally, we can also see that Lemma D.4 implies Lemma 4.6.

Proof of Lemma D.4. Without loss of generality, assume that \( k_1 \leq d_2 \). Let \( \theta = \frac{c_0 \sigma_1}{\sqrt{n}} (\sqrt{k_1} + \sqrt{d_1}) \) for some sufficient small constant \( c_0 \). Our matrix \( M \) is defined as follows:

\[
M_{i,j} = 
\begin{cases} 
\theta & \text{if } i = j \text{ and } i \leq c_0 k_1, \\
0 & \text{otherwise.}
\end{cases}
\]

In other words, all the off-diagonal entries are 0 and only the first \( c_0 k_1 \) diagonal entries are non-zero (set to be \( \theta \)).

We shall show that for any \( \ell \),

\[
\| \hat{N}_{\ell} - N \|_F \geq \| N \|_F.
\]

It suffices to analyze the case \( \ell \leq c_0 k_1 \). The case \( \ell > c_0 k_1 \) can be generalized in a straightforward manner.
We use a similar technique developed for the upper bound analysis. But before proceeding we notice that because of \( N \)’s structure, for any unit vector \( v \in \mathbb{R}^{d_2} \), we have
\[
\|N^T v\| \leq \theta \|I_{d_2 \times d_2} v\| \leq \theta.
\]  

Let \( \hat{N} = U^N \Sigma^N (V^N)^\top \). Because we have \( \|\hat{N}_\ell^\top - N^\top\|_F^2 = \sum_{i=1}^\ell \|\hat{N}_{\ell,i}^\top - N_{\ell,i}^\top\|^2 + \|N^\top (U^N)_{\ell}^\top\|_2^2 \), we only need show a term-wise inequality:
\[
\|\|\hat{N}_\ell^\top - N^\top\|U^N_{\ell,i}\|^2 > \|N^\top U^N_{\ell,i}\|_2^2.
\]  

This is equivalent to showing \( \frac{1}{n}Z^\top E U^N_{\ell,i} > \|N^\top U^N_{\ell,i}\|_2 \). We use triangle inequality and finite sample version of Marchenko-Pastur’s law \[33\]. Let \( F = V^Z (U^Z)^\top E/\sigma_\epsilon \) and \( \hat{F} = (U^Z)^\top E/\sigma_\epsilon \) and we have
\[
\frac{1}{n}Z^\top E = \frac{\sigma_\epsilon F}{\sqrt{n}} + \frac{\sigma_\epsilon V^Z \Delta^2 \hat{F}}{\sqrt{n}}.
\]  

The first \( \ell \) singular values of \( F \) are all \( \geq \sqrt{d_2 + c_2 \sqrt{k_1}} \) so long as \( \ell \leq c_0 k_1 \) for sufficiently small \( c_0 \). The singular values of the second term \( (\sigma_\epsilon V^Z \Delta^2 \hat{F}) \) are \( o(\sqrt{k_1} + \sqrt{d_2}) \). So with high probability
\[
\sigma_i \left( \frac{\sigma_\epsilon F}{\sqrt{n}} + \frac{\sigma_\epsilon V^Z \Delta^2 \hat{F}}{\sqrt{n}} \right) \geq \frac{\sigma_\epsilon}{n} (c_4 \sqrt{k_1} + \sqrt{d_2})
\]  

for \( i \leq \ell \). Together with \[35\], we see that \( \frac{1}{n}Z^\top E U^N_{\ell,i} > \|N^\top U^N_{\ell,i}\|_2 \) for a suitably small \( c_1 \).

We also remark that \( N \) does not need to be a diagonal matrix. Any matrix \( N \) with \( c_0 k_1 \) copies of singular values \( \theta \) can serve as lower bound instances. This is because unitary transformations of the noise matrix will not rescale its singular values.

### D.4 Proof of Proposition 4.8

**Design of the coupling.** Let \( Z^{(0)} \), \( Y^{(0)} \), and \( E^{(0)} \) be the data for \( \mathcal{A}^* \) and \( Z^{(1)} \), \( Y^{(1)} \), and \( E^{(1)} \) be the data for our algorithm.

Let \( \hat{N}^{(0)} = \frac{1}{n_0} (Z^{(0)})^\top (Y^{(0)}) \) and \( \hat{N}^{(1)} = \frac{1}{n_1} (Z^{(1)})^\top Y^{(1)} \). Let \( \lambda^* = f_{\mathcal{A}^*}(n, d_1, k_2, \sigma_\epsilon) \) be the threshold used by the optimal algorithm. Let \( \lambda \) be the threshold used by our algorithm. Finally, let \( k_2^* \) be the rank of \( \hat{N}(\mathcal{A}^*) \) and \( k_2 \) be the rank of \( \tilde{N}(\mathcal{A}) \). Let \( n_0 = n_1 \) be the number of samples for \( \mathcal{A}^* \) and \( n_1 = cn \) be the number of samples for \( \mathcal{A} \).

Let \( U^{(0)}_{:,1}, U^{(0)}_{:,2}, \ldots U^{(0)}_{:,k_2^*} \in \mathbb{R}^{d_2} \) be the first \( k_2^* \) left singular vectors of \( \hat{N}^{(0)} \). Let \( U^{(1)}_{:,1}, \ldots, U^{(1)}_{:,k_2} \) be the first \( k_2 \) singular vectors of \( \hat{N}^{(1)} \).

Let the SVD of \( Z^{(0)} \) and \( Z^{(1)} \) be
\[
Z^{(0)} = U^{Z^{(0)}} \Sigma^{Z^{(0)}} (V^{Z^{(0)}})^\top \quad \text{and} \quad Z^{(1)} = U^{Z^{(1)}} \Sigma^{Z^{(1)}} (V^{Z^{(1)}})^\top
\]  

Note that \( U^{Z^{(i)}} \), \( \Sigma^{Z^{(i)}} \), and \( V^{Z^{(i)}} \) \( (i \in \{0, 1\}) \) are independent \[8\]. We may use the independence property and Lemma D.5 to have
\[
V^{Z^{(i)}} (U^{Z^{(0)}})^\top E^{(0)} = V^{Z^{(i)}} (U^{Z^{(1)}})^\top E^{(1)}
\]  

**Lemma D.5.** By using the coupling procedure defined in Eq. 40 there exists a \( B, \Delta^{(0)}, \) and \( \Delta^{(1)} \in \mathbb{R}^{k_2 \times d_2} \) such that
\[
\frac{1}{n_0} (Z^{(0)})^\top E^{(0)} = \frac{\sigma_\epsilon}{\sqrt{n_0}} (B + \Delta^{(0)})
\]
\[
\frac{1}{n_1} (Z^{(1)})^\top E^{(1)} = \frac{\sigma_\epsilon}{\sqrt{n_1}} (B + \Delta^{(1)}),
\]  

where (i) each entry in \( B \) is an independent standard Gaussian, and (ii) \( \|\Delta^{(i)}\| \leq 20 \max (\log^2 n_2, k_2) \). Lemma D.5 also implies that
\[
\frac{1}{n_1} (Z^{(1)})^\top E^{(1)} = \frac{1}{\sqrt{n_1 n_0}} (Z^{(0)})^\top E^{(0)} + \frac{\sigma_\epsilon}{\sqrt{n_1}} (\Delta^{(1)} - \Delta^{(0)}).
\]

**Good and bad sets of \( \{U^{(0)}_{:,i}\}_{i \leq k_2^*} \).** We categorize the vectors \( \{U^{(0)}_{:,i}\}_{i \leq k_2^*} \) into good and bad sets:
(i) **Good** $U_{i:i}^{(0)}$ set $\mathcal{G}$: It intuitively means when we keep this direction we are better off. Specifically $U_{i:i}^{(0)} \in \mathcal{G}$ if and only if
\[
\|(\hat{N}^{(0)} - N)^T U_{i:i}^{(0)}\| \leq \|N^T U_{i:i}^{(0)}\|. \tag{43}
\]

(ii) **Bad** $U_{i:i}^{(0)} \in \mathcal{B}$: It means that when we remove this direction, we are better off. $U_{i:i}^{(0)} \in \mathcal{B}$ if and only if
\[
\|(\hat{N}^{(0)} - N)^T U_{i:i}^{(0)}\| > \|N^T U_{i:i}^{(0)}\|. \tag{44}
\]

Let $g$ be the number of elements in $\mathcal{G}$ and $b$ be the number of elements in $\mathcal{B}$.

**Lemma D.6.** Let $F \in \mathbb{R}^{d_2 \times g}$ such that $F_{i,:}$ is the $i$-th vector in $\mathcal{G}$. We have
\[
\|\hat{N} (A^*) - N\|^2_F \leq \|N^T F\|^2_F. \tag{45}
\]

**Proof.** We have
\[
\|\hat{N} (A^*) - N\|^2_F = \sum_{i \in \mathcal{G}} \|(\hat{N} (A^*) - N)^T U_{i:i}^{(0)}\|^2_F 
\leq \sum_{i \in \mathcal{G}} \|N^T U_{i:i}^{(0)}\|^2_F \quad \text{(definition of $\mathcal{G}$)}
\leq \|N^T F\|^2_F.
\]

\[\square\]

**Lemma D.7.** Using the above coupling rule, for any $i \in \mathcal{G}$, we have $U_{i:i}^{(0)} \in \text{Span}\{U_{i,k}^{(1)}\}_{i \leq k_2}$.

**Proof.** We shall prove that $(\hat{N}^{(1)})^T U_{i:i}^{(0)} \geq \lambda$ for all $i \in \mathcal{G}$. First, for any $i \in \mathcal{G}$,
\[
\|N^T U_{i:i}^{(0)}\| \geq \|\hat{N} (A^*) - N\|^T U_{i:i}^{(0)} \geq \|\hat{N}^T (A^*) U_{i:i}^{(0)} - \|N^T U_{i:i}^{(0)}\|.
\]

This implies that $2\|N^T U_{i:i}^{(0)}\| \geq \|\hat{N} (A^*) - N\|^T U_{i:i}^{(0)} \geq \|N^T U_{i:i}^{(0)}\|$.

Now we apply triangle inequality on $\|N^T U_{i:i}^{(0)}\| \geq \|\hat{N} (A^*) - N\|^T U_{i:i}^{(0)}$ again. Using the fact that $\hat{N} (A^*) U_{i:i}^{(0)} = \hat{N}^{(0)} U_{i:i}^{(0)}$ for $i \in \mathcal{G}$, we have
\[
\left\| \left( \frac{1}{n_0} (Z(0))^T E(0) + \left( \frac{1}{n_0} (Z(0))^T Z(0) - I \right) N^T \right) U_{i:i}^{(0)} \right\| \leq \|N^T U_{i:i}^{(0)}\| \tag{47}
\]

Therefore,
\[
\left\| \frac{1}{n_0} (Z(0))^T E(0) U_{i:i}^{(0)} \right\| \leq \|N^T U_{i:i}^{(0)}\| + \|\frac{1}{n_0} (Z(0))^T Z(0) - I\| N^T \| \|N^T U_{i:i}^{(0)}\| \tag{48}
\]

Now we can compute $\|(\hat{N}^{(1)})^T U_{i:i}^{(0)}\|$ for $i \in \mathcal{G}$:

\[
\|(\hat{N}^{(1)})^T U_{i:i}^{(0)}\|
= \left\| \frac{1}{n_1} (Z(1))^T Z(1) N^T U_{i:i}^{(0)} + \frac{1}{n_1} (Z(1))^T E(1) U_{i:i}^{(0)} \right\|
= \left\| N^T U_{i:i}^{(0)} + (I - \frac{1}{n_1} (Z(1))^T Z(1)) N^T U_{i:i}^{(0)} + \frac{1}{n_1} (Z(1))^T E(1) U_{i:i}^{(0)} \right\|
\geq \|N^T U_{i:i}^{(0)}\| - \left\| (I - \frac{1}{n_1} (Z(1))^T Z(1)) N^T U_{i:i}^{(0)} \right\| - \|\frac{1}{n_1} (Z(1))^T E(1) U_{i:i}^{(0)}\|
\geq \|N^T U_{i:i}^{(0)}\| - \left\| (I - \frac{1}{n_1} (Z(1))^T Z(1)) N^T U_{i:i}^{(0)} \right\| - \frac{\|\hat{N}^{(1)} - \Delta(0)\|}{\sqrt{n_0 n_1}} \tag{Using Eq. [42]}
\geq \lambda^* \frac{3}{3} \quad \text{(Using Lemma D.7, Lemma D.5 and Eq. [48])}
\geq \lambda
\]

\[\square\]
Now we are ready to prove Proposition 4.8. We need to utilize the following fact:

\[
(\hat{N}(0) - N)^\top = \frac{1}{n_0}(Z(0)^\top Z(0) N^\top + \frac{1}{n_0} Z(0)^\top E(0) - N^\top).
\]

Therefore,

\[
\frac{1}{n_0}(Z(0)^\top E(0) = (\hat{N}(0) - N)^\top + \frac{1}{n_0}(Z(0)^\top Z(0) - I)N^\top.
\]

Let us define three matrices that correspond to an orthonormal basis of \( R^{d_2} \):

1. \( F_1 \in \mathbb{R}^{d_2 \times k_2} \): The column space of \( F_1 \) is the same as \( \text{Span}(\{U_{:,i}\}_{i \in G}) \).

2. \( F_2 \in \mathbb{R}^{d_2 \times (k_2-g)} \): Its columns form an orthonormal basis for the subspace \( \text{Span}(\{U_{:,i}\}_{i \leq k_2}) - \text{Span}(\{U_{:,i}\}_{i \in G}) \). In other words, columns of \( F_2 \) form a basis for the subspace \( \text{Span}(\{U_{:,i}\}_{i \leq k_2}) \) that’s not covered by \( \text{Span}(\{U_{:,i}\}_{i \in G}) \).

3. \( F_3 \in \mathbb{R}^{d_2 \times (d_2-k_2)} \): Its columns form an orthonormal basis of the subspace that is orthogonal to \( \text{Span}(\{U_{:,i}\}_{i \leq k_2}) \).

We have

\[
\|\hat{N}(A) - N\|_F^2 = \sum_{i \leq 3} \|((\hat{N}(A) - N)^\top F_i\|^2.
\]

We first analyze \( \|((\hat{N}(A) - N)^\top F_1\|^2 \):

\[
\begin{align*}
\|((\hat{N}(A) - N)^\top F_1\|^2 & = \|((\hat{N}(1) - N)^\top F_1\|^2 \\
& = \|\frac{1}{n_1}(Z(1)^\top Z(1) - I)N^\top + \frac{1}{n_1} Z(1)^\top E(1) - N^\top)F_1\|^2 \\
& \leq 2\|\frac{1}{n_1}(Z(1)^\top Z(1) - I)N^\top\|^2 + 2\left\|\frac{1}{\sqrt{n_0 n_1}}(Z(0)^\top E(0) + \frac{\sigma_1}{n_0 n_1} (\Delta(1) - \Delta(0)))\right\|^2_F \\
& \leq 2\|\frac{1}{n_1}(Z(1)^\top Z(1) - I)N^\top\|^2 + 4\left\|\frac{n_0}{n_1} \left[ (\hat{N}(0) - N)^\top + \frac{1}{n_0}(Z(0)^\top Z(0) - I)N^\top\right]\right\|^2_F \\
& + \frac{\sigma_2^2}{n_1} \left(\|\Delta(0)\|_F^2 + \|\Delta(1)\|_F^2\right) \quad \text{(Use Eq. [50])} \\
& \leq \left(\Theta\left(\frac{\max\{\sqrt{k_1, \log^2 n}\}}{\sqrt{n}}\right)\right)^2 \|N\|_F^2 + 4\frac{n_0}{n_1} \|\hat{N}(0) - N\|_F^2.
\end{align*}
\]

On the other hand, we have

\[
\|((\hat{N}(A) - N)^\top[F_2, F_3]\| \leq \| N^\top[F_2, F_3]\|_F^2 = \|\hat{N}(A^*) - N\|_F^2 - \|((\hat{N}(A^*) - N)^\top F_1\|_F^2.
\]

Therefore,

\[
\|\hat{N}(A) - N\|_F^2 = \sum_{i \leq 3} \|((\hat{N}(A) - N)^\top F_i\|^2 \leq \|\hat{N}(A^*) - N\|_F^2 + \Theta\left(\frac{\max\{\sqrt{k_1, \log^2 n}\}}{\sqrt{n}}\right)\|N\|_F^2.
\]

\section{Optimality of ARRR}

This section first describes the oracle-benchmark family, then proves Theorem 4.10.

\subsection{Benchmark algorithms}

We first describe the intuition for defining our benchmark algorithms.

**Intuition.** A rank-\(k_1\) Oracle Benchmark Algorithm (rank-\(k_1\) OBA) has oracle access to \(C^*\) and it reduces the original RRR problem to an OrthogonalRRR problem, where \(y = N z + \epsilon\), \(z\) is a standard Gaussian, and \(N = MV^*(A^*)^\dagger\). In addition, the OBA has oracle access to the signals along the directions that are removed by a GRRRF algorithm, and along the “excessively small” directions. Specifically, we let \(\ell = \min\{k_1, n/\log^2 n\}\). We assume that an OBA knows \(N, \ell, k_1, d_2\). Thus, the
OBA will solve an OrthogonalRR problem to find an estimated $N_{1:t}$. Note that (i) When $k_1$ gets close to $n$, MBT algorithms fail to work. Thus, we give our OBA “free access” to $N$’s columns $(n^2/\log^2 n$ to $n)$ to address the limitation of the MBT algorithm so that the tracking error is still well defined for large $k_1$. (ii) For our ARR, we have $k_1 = o(n^{1/2})$ (see Proposition 4.1), so we always have $k_1 = \ell$ in our analysis.

**Definition.** We now formally define the rank-$k_1$ oracle benchmark family of algorithms (rank-$k_1$ OBF). Let $\ell = \min\{k_1, n/\log^2 n\}$. Let $Z \in \mathbb{R}^{n \times d_1}$ be the orthonormal features seen by the optimal algorithm and then “split” $Z$ into left and right halves $Z = [Z_+, Z_-]$, where $Z_+ = \mathbb{R}^{n \times \ell}$ consists of the leftmost $\ell$ column vectors of $Z$, and $Z_- \in \mathbb{R}^{n \times (d_1 - \ell)}$ consists of the remaining column vectors. Similarly, let $N = [N_+, N_-]$, where $N_+ \in \mathbb{R}^{d_2 \times \ell}$ and $N_- \in \mathbb{R}^{d_2 \times (d_1 - \ell)}$. Next, let $Y_+ = Y - Z \cdot N_+^\top$ (so that $Y_+ = Z_+ N_+^\top + E$) and $(\hat{N})^\top = (Z_+)^\top Y$. We may similarly define $y_+$, $y_-$, and $z_+ = [z_+, z_-]$ at the instance level. An algorithm is in rank-$k_1$ OBF if it has oracle access to $C^*$ and $N_-$ and finds $\hat{N}_+$ by running an MBT algorithm on $\hat{N}$. A rank-$k_1$ OBF is optimal if and only if it runs an optimal MBT algorithm on $Y_+^\top Z_+$ to find $\hat{N}_+$. See also Fig. 6.

**Oracle-Benchmark-Algorithm** $(X, Y, k_1)$

1. $\triangleright$ the algorithm has oracle access to $C^*$.
2. Let $Z = X(V)^\top (\Lambda^*)^{-\frac{1}{2}}$.
3. The regression problem becomes $y = Nz + \epsilon$.
4. $\triangleright$ the algorithm has oracle access to $N_\ell d_2$, where $\ell = \min\{k_1, n/\log^2 n\}$.
5. $\triangleright$ Let $Z = [Z_+, Z_-]$, where $Z_+ \in \mathbb{R}^{n \times \ell}$ and $Z_- \in \mathbb{R}^{n \times (d_1 - \ell)}$.
6. $\triangleright$ Let $N = [N_+, N_-]$, where $N_+ \in \mathbb{R}^{d_2 \times \ell}$ and $N_- \in \mathbb{R}^{d_2 \times (d_1 - \ell)}$.
7. Let $E = N_\ell d_2$ be the output of $A^*$ when it solves the OrthogonalRR problem $Y_+^\top Z_+ + E + 1$.
8. $\triangleright$ Find $\hat{N}_+$ by using optimal MBT on $[Y_+, Z_+]$.
9. $\triangleright$ Return $\hat{M}$ by using $\hat{N}$ and $C^*$.  

Figure 6: The optimal algorithm in the oracle benchmark family (OBF). Any algorithm in OBF has oracle access to $C^*$ and a part of $N$.

**E.2 Proof of Theorem 4.10**

**Design of coupling.** We continue to couple $\hat{M}(A)$ with $\hat{M}^*(A)$.

Data seen by $A^*$. Let $n_0$ be the number of samples seen by $A^*$. Let $X^{(0)}$, $Y^{(0)}$, and $E^{(0)}$ be the data associated with the process of $A^*$. Let $Z^{(0)} = X^{(0)} \Pi^\top$. Furthermore, let $Z^{(0)} = [Z_+^{(0)}, \Pi^{(1)}]$, where $Z_+^{(0)} \in \mathbb{R}^{n \times k_1}$ and $Z_-^{(0)} \in \mathbb{R}^{n \times (d_1 - k_1)}$. Here we know that $k_1 = \min\{k_1, n/\log^2 n\}$. Let $(\hat{Y}^{(0)})^\top = Y^{(0)} - Z^{(0)} N_-^\top = Z_+^{(0)} (N_+^\top) + E^{(0)}$. Let $(\hat{N})^\top = \frac{1}{n_0} (Z_+^{(0)})^\top Y_+^{(0)}$. Let $\hat{N}_+ (\Lambda^*) \in \mathbb{R}^{d_2 \times k_1}$ be the output of $A^*$ when it solves the OrthogonalRR problem $Y_+^{(0)} Z_+^{(0)} + E^{(1)}$.

Data seen by $A$. Let $n_1 = c n_0$ be the number of samples seen by $A$. Let $X^{(1)}$, $Y^{(1)}$, and $E^{(1)}$ be associated with the process of $A$. Let $\hat{Z}_+^{(1)} = X \Pi^\top$ and $Z^{(1)} = [\hat{Z}_+^{(1)}, \Pi^{(1)}]$, where $\hat{Z}_+^{(1)}$ are the last $d_1 - k_1$ columns of $X \Pi^\top$. Let $\hat{N}_+ (\Lambda^*) \in \mathbb{R}^{d_2 \times k_1}$ be the output of $A$ when it solves the OrthogonalRR sub-problem (i.e., finding a low rank approximation of $(\hat{Z}_+^{(1)})^\top Y_+^{(1)}$).

Note that we have

$$
(\hat{N})^\top = \frac{1}{n_0} (Z_+^{(0)})^\top Z_+^{(0)} N_-^\top + \frac{1}{n_0} Z_+^{(0)} E^{(0)}
$$

$$(\hat{N})^\top = \frac{1}{n_1} (Z_+^{(1)})^\top [Z_+^{(1)} N_+^\top + Z_-^{(1)} N_-^\top + E^{(1)}]$$

We couple $Z^{(0)}$ and $E^{(0)}$ with $Z^{(1)}$ and $E^{(1)}$ in a similar manner as we did for the analysis for OrthogonalRR, i.e., we let

$$
\frac{1}{n_0} (Z_+^{(0)})^\top E^{(0)} = \frac{\sigma}{\sqrt{n_0}} (B + \Delta^{(0)})
$$

$$
\frac{1}{n_1} (Z_+^{(1)})^\top E^{(1)} = \frac{\sigma}{\sqrt{n_1}} (B + \Delta^{(1)}),
$$

where $\|\Delta^{(i)}\| \leq \frac{20 \max\{\log^2 n_1, k_1\}}{\sqrt{n_0}}$.
Let $\Delta_+ = \min_W \text{unitary } \hat{Z}^{(1)} - Z^{(1)}_+ W$. By Proposition 4.1 we have $\|\Delta_+\| \leq \frac{c}{n_1^2} \sqrt{n_1}$ for some constant $c_1$. We now have

$$\begin{align*}
(N^{(1)})^T & = \frac{1}{n_1} (Z^{(1)}_+ W + \Delta_+)^T \left[ Z^{(1)}_+ N^{(1)}_+ + Z^{(1)}_- N^{(1)}_- + E^{(1)} \right] \\
& = \frac{1}{n_1} W^T (Z^{(1)}_+)^T Z^{(1)}_+ N^{(1)}_+ + \frac{1}{n_1} W^T (Z^{(1)}_-)^T Z^{(1)}_- N^{(1)}_+ \\
& \quad + \frac{1}{n_1} W (Z^{(1)}_+)^T E^{(1)} + \frac{1}{n_1} \Delta_+^T (Z^{(1)}_+ N^{(1)}_+ + Z^{(1)}_- N^{(1)}_- + E^{(1)}).
\end{align*}$$

Let $\hat{\Delta} = (Z^{(1)}_+)^T (Z^{(1)}_+ N^{(1)}_+)$. We mimic the analysis for OrthogonalRR to categorize $\{U^{(0)}_{i:i} \}_{1 \leq k_2}$ into good and bad sets:

(i) Good $U^{(0)}_{i:i}$ set $G$: $U^{(0)}_{i:i} \in G$ if and only if

$$\| (\hat{N}^{(0)} - N)^T U^{(0)}_{i:i} \| \leq \| N^T U^{(0)}_{i:i} \|.$$

(ii) Bad $U^{(0)}_{i:i}$ set $B$: $U^{(0)}_{i:i} \in B$ if and only if

$$\| (\hat{N}^{(0)} - N)^T U^{(0)}_{i:i} \| > \| N^T U^{(0)}_{i:i} \|.$$

We may use Fact E.1 to generalize Lemma D.6

**Lemma E.1.** For any $U^{(0)}_{i:i} \in G$, we have $U^{(0)}_{i:i} \in \text{Span} (\{U^{(1)}_{i:i} \}_{1 \leq k_2})$.

We then may use Lemma E.1 and techniques developed in Sec. D.6 (i.e., decompose $R^{d_2}$ into $F_1$, $F_2$, and $F_3$) to show that

$$\| \hat{N} (A) - N \|^2_F \leq \| \hat{N} (A^*) - N \|^2_F + \Theta \left( \frac{k_1}{n_1} + \frac{\epsilon}{\delta^3} \right) \| N \|^2_F.$$

Finally, we have that

$$\begin{align*}
E & | \hat{M} (A)x - Mx | \\
& = E | \hat{M} (A) \hat{T}^{-1} \hat{T} (x) - Mx | \\
& = E | \hat{N} (A) \hat{z} - Nz | \\
& = E | (\hat{N} (A) z_+ - N_+ z_+ + \hat{N} (A) (\hat{z}_+ - z_+)) - N_- z_- | \\
& \leq \| \hat{N} (A) - N_+ \|_F + \| \hat{N} (A) \| E \| \hat{z}_+ - z_+ \| + \Theta \left( \sum_{i \geq k_1} \lambda_i^2 \right) \| N \| \\
& \leq \| \hat{N} (A^*) - N_+ \|_F + \Theta \left( \frac{k_1}{n_1} + \frac{\epsilon}{\delta^3} \right) \| N \|_F + \| \hat{N} \| E \| \hat{z}_+ - z_+ \| + \Theta \left( \sum_{i \geq k_1} \lambda_i^2 \right) \| N \| \\
& \leq E | \hat{M} (A^*) x - Mx | + \Theta \left( \frac{k_1}{n_1} + \frac{\epsilon}{\delta^3} \right) E \| Mx \| + \Theta \left( \frac{\delta^{\frac{d_1-1}{2}}}{\delta^{\frac{d_1}{4}}} \right) E \| Mx \|.
\end{align*}$$

This completes the proof of the Theorem.

**F  Full experimental results**

In this section, we describe our experimental results obtained from both synthetic and real datasets.
F.1 Synthetic data

We evaluate our algorithm on synthetic data to gain further insights into its theoretical properties. We produce synthetic data using the linear model \( Y = X M^\top + E \). \( X \) comes from a matrix of i.i.d. standard Gaussian random variables, whose covariance matrix’s eigenvalues follow a power law distribution. To generate \( M \), we first randomly sample entries from \( \{-1, 0, 1\} \), and then we run SVD to construct a low rank approximation of it. Finally, \( E \)'s entries are i.i.d. \( \mathcal{N}(0, \sigma^2_{\text{noise}}) \) with \( \sigma_{\text{noise}} = \eta \cdot \text{std} (\text{vec}(X M^\top)) \) and \( \eta \) is a noise multiplier parameter. We fix \( n = 500 \), \( d_2 = 100 \), and \( \rho = 2 \). Setting the parameters in other ways results in similar behavior.

![Figure 7: Reconstruction error \( \| M - \hat{M} \|_F \) at different \((k_1, k_2)\) for varying noise levels. The dashed lines correspond to the true rank of \( M \) and the 'x' marks the \((k_1, k_2)\) with the lowest error.](image)

In the Figs. 7, 8 and 9 we consider rank \( M = 40 \) and \( d_1 = 100 \) (similar results under other settings). We sweep the \((k_1, k_2)\) space and measure the reconstruction error on \( M \) due to our algorithm in Fig. 7. We observe that when the noise level is low (\( \eta = 0.2 \)), the optimal \( k_1 \) can be much larger than rank \( M \), while the optimal \( k_2 \) is rank \( M \). This is consistent with our theoretical results. However, as the noise level increases, the optimal \((k_1, k_2)\) shrinks to values smaller than rank \( M \), which demonstrates the ability of our algorithm to adapt \( M \)'s rank to signal-to-noise ratios. Figs. 8 and 9 show how our algorithm performs in terms of out-of-sample MSE and correlation on \( X_{\text{out}} \) and \( Y_{\text{out}} \), which are generated in the same way as \( X \) and \( Y \) under fixed \( M \). The two metrics (commonly used in real applications) are qualitatively similar to reconstruction error (which is usually unobservable).

![Figure 8: Out-of-sample MSE: \( \| Y_{\text{out}} - X_{\text{out}} \hat{M}^\top \|_F \).](image)

Table 3: In-sample and out-of-sample error on \( Y \) and \( Y_{\text{out}} \) for RRR (\( d_1 = 500 \) and \( \eta = 0.2 \)). A Repeat of Table 1

| \( k = 1 \) | \( k = 2 \) | \( k = 3 \) | \( k = 4 \) | \( k = 5 \) |
|---|---|---|---|---|
| Training error | 0.2008 | 0.0846 | 0.0699 | 0.0623 | 0.0557 |
| Test error | 9.0845 | 10.9501 | 15.2094 | 19.3182 | 20.0598 |

We also examine the in-sample and out-of-sample MSE in reduced rank regression (RRR) when \( d_1 = 500 \) and rank \( M = 10 \). Table 3 shows that even \( k \) in Fig. 3 is very small, there is a huge gap between training and test error, which indicates severe overfitting problem.

Then, we compare our algorithm with other regularized regression methods (\( d_1 = 500 \)), including ridge regression (“Ridge”), reduced rank ridge regression [26] (“Reduced ridge”), LASSO (“Lasso”), nuclear norm regularized regression (“Nuclear”), and reduced rank regression (“RRR”). For each regression method, we sweep its regularization parameter(s) and report the best performance. To generate \( M \), we randomly sample entries from \( \{-1, 0, 1\} \), and we run SVD to construct a low
Figure 9: Out-of-sample correlation.

Figure 10: Algorithm comparison with varying noise multipliers $\eta$ and fixed rank $(M) = 10$. Error bars report 1 standard deviation from 10 runs.

The algorithm performs significantly better than other commonly used regularization strategies (ridge regression, LASSO, and nuclear norm regularization), and has similar performance as reduced rank ridge regression. Such observations echo our theoretical analysis well. Fig. 10c shows the rank of $\hat{M}$ from the regression methods. Both our algorithm and Reduced ridge are able to adapt (reduce) the rank when the noise level is high, while Nuclear consistently attempts to recover the true rank of $M$, which sometimes leads to poor performance.

F.2 Real dataset

This section examines the performances of our forecasting methods for predicting equity returns. We use an emerging stock market. The data consists of approximately 3600 stocks between 2011 and 2018. We use the open (9:30 am) to open price to compute returns at daily frequency. The returns only cover regular trading days. We examine two different universes in this market: (i) Universe 1 is equivalent to the S&P 500 and consists of 983 stocks, and (ii) Full universe consists of all stocks except for illiquid ones.

Responses. We focus on predicting the next 5-day returns for different universes. In this section, we take Table 6 as an example. The universe includes 800 equities. Therefore $d_2 = 800$. The next-5-day forecast is produced for every trading day. Let $r_{i,t}$ be the return of the $i$-th asset for time interval $t$. Let $r_t = (r_{i,t} : i \leq d_2)$.

Features. For each asset in the universe, we compute its past 1-days, past-5 days and past 10-days returns as part of the feature. Therefore, the total number of features is $d_1 = 2400$. Let $x_t$ be the feature vector at time $t$. We remark that each individual feature has limited forecasting power (which is implicitly implied from LASSO).

Model, training, and hyper-parameters. Our regression model is

$$r_{t+1} = Mx_t + \epsilon. \quad (60)$$

We use three year data for training, one year for validation and one year for testing. The model is re-trained every test year. For example, the training period is May 1 2011 to May 1, 2014 and the validation period is from June 1, 2014 to June 1, 2015. We use the validation set to determine the hyperparameters and build the model, and then we use the trained model to forecast returns of equity.
in the same universe from July 1, 2015 to July 1, 2016. Then the model is retrained by using data in the second training period (May 1, 2012 to May 1, 2015). This workflow repeats. To avoid the look-ahead issue, we set the gap between the training and validation set as one month, which is the same as the gap between the validation and testing set.

**Optimization.** We use standard approach to translate forecasts into positions \([18, 31, 2, 42]\). Roughly speaking, the position is proportional to the product of forecasts and a function of average dollar volume. We allow short-selling. We do not consider transaction cost and market impact. We do not hold overnight positions so the exposure to the market is minimum.

**Results.** Our data span May 1, 2011 through July 30, 2018. We report the forecasting power and portfolio return for out-of-sample periods. Each training period consists of approximately 720 observations (one observation is a \((x_t, r_t)\) pair). Each forecasting period consists of approximately 240 observations. Table 5 and 6 report the full results of the two universes. See Table 4 for detailed descriptions of data columns.

**Obs 1.** High signal to noise. For all the methods we examined, the out-of-sample \(R^2\) are all close to 0.

**Obs 2.** ARRR has the highest forecasting power \((R^2)\). Among all models, our method has the highest out-of-sample \(R^2\). An interesting property here is that Reduced ridge has considerably worse performance, suggesting that our model appears to be more robust in financial datasets.

**Obs 3.** In-sample and out-of-sample gap. Our method has the worst in-sample \(R^2\) but achieves the best out-of-sample \(R^2\) and the smallest gap between in-sample and out-of-sample performance (in \(R^2\)). MSE shows a similar pattern. This suggests that our model is better at avoiding spurious signals, and thus achieves our major design goal.

**Distribution of the eigenvalues and gaps.** Fig. 11(a) shows the distribution of the eigenvalues of the feature covariance matrix from training periods in Universe 1. Their consistent behaviors are evidence of time series stationarity. It indeed exhibits decay and long-tail behaviors, and thus satisfies our assumption about power law decay (see Sec. 2). The tails are in general small, so it is necessary to cut out them to achieve improved performance (confirmed in experiments in Sec. 5). Fig. 11(b) shows the gap between two consecutive eigenvalues of the covariance matrix in log scale. They all are non-monotonic, confirming the need to develop a theory using only the asymptotic decay assumptions (asymptotic decays allow non-monotonic gaps).

**Characteristics of the portfolio.** We next describe the characteristics of the portfolio constructed by using our model’s forecast. See Table 5, 6 for the breakdown of returns. Note that we also report weighted correlation. The weights are determined by the historical dollar volume of the asset. This statistics is useful because the positions taken by the optimizer are sensitive to the historical dollar volume. Fig. 12(a) shows the distribution of the standard deviation of our returns at daily granularity from the forecast in Table 6. This measures the “aggressiveness” of our forecasts. When the forecasts have high standard deviation, they push the optimizer to take larger positions. We note that while the

Figure 11: Statistics from eigenvalues of the feature covariance matrix and the x-axis is the training duration.
standard deviations are time-varying, our forecasts rarely bet more than 60bps. This is an evidence that the forecast picks up the market structure from training. Fig. 12(b) plots the distribution of returns from our portfolio at daily granularity. We see that this is a “statistical betting”: a considerable fraction of our betting loses money. However, if we bet frequently enough, our portfolio is still able to maintain a high Sharpe. We remark that the performance may reduce when we consider the transaction costs.

Table 4: Data column names and their explanations

| Column name | Explanation                                                                 |
|-------------|-----------------------------------------------------------------------------|
| Test start date | The start date of the testing period                                    |
| $N$         | The universe size                                                          |
| $\text{MSE}_{\text{out}}$ | Normalized out-of-sample mean squared error (normalized by standard deviation of the responses) |
| $\text{MSE}_{\text{in}}$ | Normalized in-sample mean squared error                                    |
| $\text{MSE}_{\text{in}}-\text{out}$ | $\text{MSE}_{\text{in}} - \text{MSE}_{\text{out}}$ |
| $R^2_{\text{out}}$ | Out-of-sample $R^2$                                                        |
| $R^2_{\text{in}}$ | In-sample $R^2$                                                            |
| $R^2_{\text{out, w}}$ | Out-of-sample weighted $R^2$ (weighted by a function of dollar volume). |
| $R^2_{\text{in, w}}$ | In-sample weighted $R^2$                                                    |
| Sharpe       | The Sharpe ratio                                                           |
| $t$-statistic | The $t$-statistic by using Newey-West estimator                           |

Table 5: Adaptive reduced rank regression on full universe

| Test year      | N   | $\text{MSE}_{\text{out}}$ | $\text{MSE}_{\text{in}}$ | $R^2_{\text{out}}$ | $R^2_{\text{in}}$ | $R^2_{\text{out, w}}$ | $R^2_{\text{in, w}}$ | $t$-statistic | Sharpe |
|----------------|-----|--------------------------|--------------------------|-------------------|-------------------|-----------------------|---------------------|--------------|--------|
| 20150701       | 2331| 1.023                    | 0.9667                   | 1.802             | 330.247           | 1.2012                | 330.596             | 0.8867       | 1.3445  |
| 20160701       | 2491| 1.0205                   | 0.937                    | 37.4936           | 997.3532          | 20.0247               | 868.7226            | 6.6677       | 3.2371  |
| 20170701       | 2617| 0.99                     | 0.93                     | 7.4185            | 695.0658          | 11.4901               | 736.9868            | 38.7093      | 0.9553  |
| all, glued     | 2838| 1.0056                   | 0.905                    | 18.5795           | 689.1728          | 12.1668               | 699.375            | 15.4134      | 1.6239  |

Figure 12: Statistics from our predicted returns on Universe 1. The x-axis is the testing duration.
Table 6: Adaptive reduced rank regression on *Universe I*

| Test year | N   | MSE \(_{\text{out}}\) | MSE \(_{\text{in}}\) | \(R^2_{\text{out}}\) | \(R^2_{\text{in}}\) | \(R^2_{\text{out, w}}\) | \(R^2_{\text{in, w}}\) | \(t\)-statistic | Sharpe |
|-----------|-----|-----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------|---------|
| 20150701  | 800 | 1.0339                | 0.9728               | 99.2344              | 142.5158             | 142.8957             | 157.8843            | 3.6759         | 4.2709  |
| 20160701  | 800 | 0.9971                | 0.9638               | 30.1577              | 178.912              | 35.018               | 206.934             | 4.3541         | 3.2096  |
| 20170701  | 800 | 1.0123                | 0.9889               | 2.2629               | 150.4449             | 0.0266               | 134.4185            | 14.945         | 0.0517  |
| all, glued | 983 | 0.9935                | 1.014                | 46.3162              | 158.3772             | 70.4223              | 164.1192            | 8.3268         | 2.435   |

Table 7: Summary of results for equity return forecasts

| Model          | MSE \(_{\text{out}}\) | MSE \(_{\text{in}}\) | MSE \(_{\text{in} - \text{out}}\) | \(R^2_{\text{out}}\) | \(R^2_{\text{in}}\) | \(R^2_{\text{in} - \text{out}}\) | \(t\)-statistic | Sharp | \(t\)-statistic |
|----------------|-----------------------|----------------------|------------------------|----------------------|----------------------|------------------------|----------------|-------|----------------|
| ARRR, N= 983   | 0.9935                | 1.014                | 0.0205                 | 46.3761              | 158.2564             | 2.4350                 | 8.3268         |       |                 |
| Lasso          | 1.1158                | 0.3953               | -0.7205                | 6.6049               | 7147.0116            | 2.1462                 | 0.0601         |       |                 |
| Ridge          | 1.2158                | 0.1667               | -1.0491                | 9.8596               | 8511.9076            | 0.6603                 | -0.0497        |       |                 |
| Reduced ridge  | 1.0900                | 0.8687               | -0.2213                | 13.0321              | 1555.5136            | 0.3065                 | -0.3275        |       |                 |
| ARRR, N= 2838  | 1.0856                | 0.9050               | -0.1056                | 18.3761              | 689.0625             | 1.0239                 | 15.4134        |       |                 |
| Lasso          | 1.0625                | 0.5286               | -0.5339                | 1.1236               | 6029.5225            | 0.5954                 | 0.0179         |       |                 |
| Ridge          | 1.0289                | 0.6741               | -0.3548                | 0.2116               | 5342.1481            | 0.5739                 | 0.0670         |       |                 |
| Reduced ridge  | 1.9722                | 0.7373               | -1.2349                | 1.0816               | 2416.7056            | 1.5482                 | 0.0619         |       |                 |