DISCRETIZATIONS OF A FRACTIONAL ORDER LOGISTIC EQUATION ARISING FROM A SIMPLE SI-TERORIST MODEL

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Abstract

The simplest model of terrorist growth model consists of two subpopulations, namely the susceptible subpopulation (S) and the militant or infected and infectious subpopulation (I). The model is governed by a coupled of differential equation reflecting the growth of the susceptible and infected subpopulations. Assuming a constant human population, the system can be reduced to a logistic differential equation. In this paper a fractional order delayed logistic equation is discussed and the discretization in the form of piecewise constant argument is used to find the solution. We use the first and the second order discretization method in the numerical scheme and investigate the effect of the fractional order in the growth of the underlying population modelled by the equation. We found that in general the discretization method can mimic the behavior of the original logistic equation for some parameters. However, destabilizing effect may occur depending on the combination of the values of related parameters, such as the fractional order, the intrinsic growth rate, and the piecewise constant argument parameter.

Keywords: SI-terrorist model, logistic differential equation, fractional order, piecewise constant argument

I. Introduction

Nowadays the word terrorism is often heard and becomes a popular word in the world. It is connected with a militant group in human population. Terrorism is "the use of force or threat to weaken morals, intimidate and conquer".
Mathematically, spreading ideology can be viewed analogously as a process of infection or disease transmission in epidemiology. For this reason, some authors in literatures have adopted the same process in modeling the spread of believe and ideology as the process of disease transmission. For examples the model in [XIII] and [XVII] can be viewed as a modified SI model usually found in modeling a contagious disease. The very early model of $SI$ is constructed by Kermack and McKendrick in 1927, and republished in reputable journal “Bulletin of Mathematical Biology” [XL, XLI, XLII]. In the case of constant population, the $SI$ model can be simplified to just one equation as will be explained in the following.

Broadly speaking, in modeling terrorism or militant growth, we will look at the problem to find the number of infectives (those people who are affected by the belief or ideology under consideration) and the number of susceptibles (those people who are prone to ideology transmission as a receiver) change over time periods other than a single day. Let us use the function $I(t)$ to denote the number of infectives at time $t$ and $S(t)$ to denote the number of susceptibles at time $t$. Then the daily transmission rate is $\frac{dI(t)}{N}$, where $N = I(t) + S(t)$ is total population size. We assume that $N$ constant during the course of the epidemic so we can set $\frac{d}{N} = \beta$ constant. By assuming a short period of time, the instantaneous rate of change is given by $\frac{dI(t)}{dt} = \beta I(t)S(t)$, where $\beta I(t)S(t) = \beta I(t)(N - I(t))$. The last equation is the well-known logistic differential equation.

The logistic equation has been applied in many areas of applications, especially in population modeling [II, III, IV, V, VI, XV, XVI]. It suits to describe the natural growth of many populations, including human, wild and captive animals, and other biological creatures. Some complexities might occur in the population growth process, such as losing or retaining memory to the previous event. However, many authors argued that those equations in the above literatures do not take into account several important factors, such as the dependence on historical property by ignoring the status of the earlier stages of the respective variables [I]. Hence, it cannot predict exactly the physical problem under consideration. The author in [I] argued in order to extend the limitations of this model, we should consider the fractional derivative rather than the local derivative so we obtain the equation $\frac{d^\alpha I(t)}{dt^\alpha} = \beta I(t)S(t) = \beta I(t)(N - I(t))$. Hence in this situation the growth is usually modelled by the used of fractional order derivative instead of the usual derivative in the differential equation. In this paper we will explore the solution of the fractional order logistic differential equation by looking at its piecewise constant argument counterpart.

II. Materials and Methods

Many mathematical models have been developed to accommodate the phenomenon that represented as fractional order derivative. Some examples are [XX] and [XXXIX] for general population models and [XXX, XXXIV, XXXVI] for logistic growth population models. Some of most common methods for finding the...
solution of the fractional order differential equations found in literatures are analytical method and exact solution [IX, XXXIX], Legendre polynomial method [XXXVII], numerical methods [XLIX, XII].

The author in [L] devises a numerical method for solving constant or time-varying delay nonlinear fractional-order differential equations. The author assumes an arbitrary positive real number order of derivative with the Caputo definition of the differential operator. The solution is approximated by the general Adams-Bashforth-Moulton method combined with the linear interpolation method. The author also presents the detailed of error analysis for the method and compare the result with the exact analytical solution. It is shown that the proposed numerical method is effective in approximating the solution. Recently the author in [XXXI] presents an approximate solution for a non-linear fractional logistic differential equation (FLDE) by using the operational matrices of Bernstein polynomials (BPs). The author has used the Riemann-Liouville sense for both the fractional derivative and the fractional integration. The author reduces the problem of FLDE to the problem of non-linear system of algebraic equations. The algebraic equation is then solved by using Newton iteration method. Numerical examples show that the method gives results with an excellent agreement with the exact solution. In this paper we will explore the solution of the fractional order logistic differential equation by looking at its piecewise constant argument counterpart.

The Piecewise Constant Argument Differential Equations

The piecewise constant argument differential equations are proposed by several authors, such as in [XXI, XXII, XXXV, XXXVIII]. Since then it is applied in numerous areas of economy, biology and physics, because it is much more suitable than delay differential equations in treating dynamic phenomena [XXIII], it can be used to approximate delay differential equations [XXVII], and it is able to describe an impulsive delay properly [XLIV] (see [XIX, XLIII, XLV] for the traditional delay theory). Recently the author in [XI] has studied the dynamics of a harvested logistic differential equation model by considering time delay and piecewise constant argument. The author discussed both discrete and piecewise constant delays in the logistic equation and found the positive solutions. Other works of the application of the piecewise constant argument dedicated to logistic equations are [XIV, XVIII, XXVIII, XLVI, LI].

The Piecewise Constant Argument Logistic Differential

The classical logistic equation goes back to the work of Verhulst who developed the equation to describe the population growth in a limited environment in 1838 [XXIX]. The equation was reappeared and used by Pearl and Reed to study the growth of the U.S. population [XXXIII] and by MacLean and Willard Turner to study the growth of the Canada population [XXV]. The Classical model of logistic equation is given by

\[ Dx(t) = \frac{dx(t)}{dt} = rx(t)\left(1 - \frac{x(t)}{K}\right) \]  

(1)
with \( x(t) \) denotes the number of population at time \( t \) and \( r \) is the intrinsic growth rate of the population. The population is limited above by a maximum number \( K \), which is called the carrying capacity. Equation (1) can be normalized such that the maximum number is 1 in the form

\[
Dx(t) = \frac{dx(t)}{dt} = \rho x(t)(1-x(t))
\]

for suitable value of \( \rho \). Figure 1 shows the typical graph of the logistic equation for \( \rho = 1 \) with three different initial values: \( x_0 = 0.25 \), \( x_0 = 0.025 \), and \( x_0 = 0.0025 \). All the initial values grow approaching the carrying capacity eventually.

Many works have been done to modify the equation for certain condition to accommodate some biological complexities to make the model more realistic, such as varying \( K \) [XXIV], meta-population [II, III, IV], intra-specific competition intensity [V, VI], etc. In this paper we consider a fractional derivative in the equation in the form

\[
D^\alpha x(t) = \rho x(t)(1-x(t))
\]

for \( t > 0 \) and initial condition \( x(0) = x_0 \). The order of the derivative is denoted by \( \alpha \in \mathbb{R} \). In relation to our problem in the introduction, equation (3) represent the growth of militant individual in an SI-terrorism model.

To study the equation, we look at the piecewise constant argument counterpart by discretizing the equation as in [VII, XLVIII]. The piecewise constant arguments problem originally appeared in Cooke and Wiener [XXI, XXII]. In [VII, XLVIII] equation (3) can be discretized as

\[
D^\alpha x(t) = \rho x \left( \left\lfloor \frac{t}{\varrho} \right\rfloor \varrho \left( 1 - x \left( \left\lfloor \frac{t}{\varrho} \right\rfloor \varrho \right) \right) \right)
\]

with \( t > 0 \) and initial condition \( x(0) = x_0 \). Here \( \lfloor \cdot \rfloor \) is the greatest-integer function. To proceed further we define the basic definitions and properties of fractional-order differentiation and integration in the sense of Caputo [XXVI, XLVII].

Fig. 1: The graphs of logistic equation (2) for \( \rho = 1 \) with three different initial values.
Definition 1: The fractional integral of order $\alpha$ of the function $f(t), \ t > 0$ is defined by

$$I^\alpha f(t) = \int_0^t (t-s)^{\alpha-1} f(s) ds.$$

Definition 2: The fractional derivative of order $\alpha$ of the function $f(t), \ t > 0$ is defined by

$$D^\alpha f(t) = I^{n-\alpha} D^n f(t) = \int_0^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)} f^{(n)}(s) ds, \quad \text{with} \quad D = \frac{d}{dt} \quad \text{and} \quad n-1 < \alpha, n \in \mathbb{N}^*.$$

The authors in [XLVIII, LI] proceed as the following steps:

- First, let $t \in [0, r)$, then $\frac{t}{r} \in (0, 1)$. So, we get

$$D^\alpha x(t) = \rho x(t)\left(1-x(t)\right) = D^\alpha x(t) = \rho x_0\left(1-x_0\right) \text{ with } t \in [0, r). \quad (5)$$

By considering the Caputo definition of the differential and integral operator, the solution of (4), is then given by

$$x(t) = x_0 + \int_0^t \rho x_0(1-x_0) \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds$$

$$= x_0 + \rho x_0(1-x_0) \frac{t^\alpha}{\Gamma(1+\alpha)}. \quad (6)$$

- Next, let $t \in [r, 2r)$, then $\frac{t}{r} \in [1, 2)$. So, we get

$$D^\alpha x(t) = \rho x_0\left(1-x_0\right) \text{ with } t \in [r, 2r). \quad (7)$$

Again, by considering the Caputo definition of the differential and integral operator, the solution of (4), is then given by

$$x(t) = x(r) + \int_r^t \rho x_0(1-x_0) \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} ds$$

$$= x(r) + \rho x_0(1-x_0) \frac{(t-r)^{\alpha}}{\Gamma(1+\alpha)}.$$ 

with $t \in [r, 2r)$.

- By repeating the process with $t \in [nr, (n+1)r)$ we get

$$x_{n+1}(t) = x_n(nr) + \rho x_n(nr)(1-x_n(nr)) \frac{(t-nr)^n}{\Gamma(1+n)}.$$ 

By assuming $t \to (n+1)r$ then we obtain the discretization of equation (4) given by

$$x_{n+1}((n+1)r) = x_n(nr) + \rho x_n(nr)(1-x_n(nr)) \frac{(r)^{\alpha}}{\Gamma(1+\alpha)}$$

which is equivalent to

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\[ x_{n+1} = x_n + \rho x_n (1-x_n) \frac{r^{\alpha}}{\Gamma(1+\alpha)}. \] (10)

Now, instead of (4) we consider the corresponding equation of (3) is the piecewise constant arguments

\[ D^\alpha x(t) = \rho x \left( \left\lfloor \frac{t}{r} \right\rfloor r \right) \left( 1 - x \left( \left\lfloor \frac{t-r}{r} \right\rfloor r \right) \right). \] (11)

The authors in [46] show that in this case we obtain a second order (11), i.e.

\[ x_{n+1} = x_n + \rho x_n (1-x_{n-1}) \frac{r^{\alpha}}{\Gamma(1+\alpha)}. \] (12)

Repeating the process we will have third order,

\[ x_{n+1} = x_n + \rho x_n (1-x_{n-2}) \frac{r^{\alpha}}{\Gamma(1+\alpha)}, \] (13)

fourth order,

\[ x_{n+1} = x_n + \rho x_n (1-x_{n-3}) \frac{r^{\alpha}}{\Gamma(1+\alpha)}, \] (14)

and so on. In the next section we will explore the numerical examples of the discretization of the first- and second- of the piecewise constant argument fractional differential equations.

III. Results and Discussions

In this section we explore the discretization in equations (10) and (12) to see the behavior with the changes of the parameters.

The First-Order Discretization

It is evident that the first order discretization (10) mimics the behavior of the logistic growth in some circumstances. The trajectory goes to the stable value (the carrying capacity) as expected. Figure 2 shows the plots of equation (10) for \( \rho = 1, r = 1, \alpha_1 = \alpha_2 = \alpha_3 = 1 \) with three different initial values: \( x_0 = 0.25, x_0 = 0.025 \), and \( x_0 = 0.0025 \). Unlike the original logistic equation in Figure 1, here different initial values are trapped to the same trajectory immediately. This implies that a number of different initial values which are close to each other do not affect the eventual trajectory of the solution. Meanwhile, Figure 3 shows the fact that a higher fractional order yields in a slower growth of the function. The example in Figure 3 assumes a low value of \( r \), i.e. \( r = 1 \). However, when the value of \( r \) increases, e.g. \( r = 2 \) in Figure...
4, then the situation reversed, i.e., a higher fractional order yields in a faster growth of
the function. Beside the effect on the increasing growth velocity, a higher fractional
order also tends to destabilize the system in approaching the carrying capacity – i.e. it
results in a higher amplitude of fluctuation (green color graph in Figure 3). The
destabilizing effect of a higher fractional order in Figure 3 assumes the parameter
values $\rho = 1$ and $r = 2$. However, if the condition is reversed, i.e. $\rho = 2$, $r = 1$, then
a higher fractional order stabilize the system (green color graph in Figure 5).

Destabilizing effect is more apparent in combination of higher parameters, such
as $\rho = 1.25$ and $r = 2$. In this case chaotic behavior occurs for a fractional order
$\alpha = 1.125$ (Figure 6). The destabilization effect is only moderate for a moderate
fractional order $\alpha = 1$ (Figure 7) and becomes a stabilization effect for lower
fractional orders $\alpha = 0.5$ and $\alpha = 0.25$ (Figures 8 and 9). But interestingly, whenever
we fixed $\rho = 1.25$ and $r = 1$ chaotic behavior occurs for a small fractional order
(Figure 11) which then is stabilizes by a higher fractional order (Figure 10). The
authors in [X] have provided a more prudent scheme so that the difference equation
performs better in capturing the dynamics of the underlying fractional differential
equation. However, current method is able to discretize a second-order difference
equation. The method also succeed in discretizing a system of the autonomous
differential equations that generating chaotic behavior [XXXII]. Furthermore, the
authors in [VIII] have applied the method to “Predict, Evaluate, Correct, Evaluate” a
chaotic biological system and consider a Delayed Feedback Control into the system.

**Fig.2:** The plots of the logistic equation (10) for $\rho = 1$, $r = 1$, $\alpha_1 = \alpha_2 = \alpha_3 = 1$
with three different initial values: $x_0 = 0.25$ (blue), $x_0 = 0.025$ (red), and $x_0 = 0.0025$ (green).

**Fig.3:** The plots of the logistic equation (10) for $\rho = 1$, $r = 1$, $x_0 = 0.025$ with
three different fractional order $\alpha_1 = 0.5$ (blue), $\alpha_2 = 1$ (red), and $\alpha_3 = 1.5$ (green).
Fig. 4: The plots of the logistic equation (10) for $\rho = 1$, $r = 2$, $x_0 = 0.025$ with three different fractional order $\alpha_1 = 0.5$ (blue), $\alpha_2 = 1$ (red), and $\alpha_3 = 1.5$ (green).

Fig. 5: The plots of the logistic equation (10) for $\rho = 2$, $r = 1$, $x_0 = 0.025$ with three different fractional order $\alpha_1 = 0.5$ (blue), $\alpha_2 = 1$ (red), and $\alpha_3 = 1.5$ (green).

Fig. 6: The plot of the logistic equation (10) for $\rho = 1.25$, $r = 2$, $x_0 = 0.025$ with the fractional order $\alpha = 1.125$.

Fig. 7: The plot of the logistic equation (10) for $\rho = 1.25$, $r = 2$, $x_0 = 0.025$ with the fractional order $\alpha = 1$.

Fig. 8: The plot of the logistic equation (10) for $\rho = 1.25$, $r = 2$, $x_0 = 0.025$ with the fractional order $\alpha = 0.5$.

Fig. 9: The plot of the logistic equation (10) for $\rho = 1.25$, $r = 2$, $x_0 = 0.025$ with the fractional order $\alpha = 0.25$. 

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A richer structure is revealed for the second discretization in equations (12). It is worth to explore in a much more detail but here we will not attempt to uncover the detail structure, instead we illustrate them visually in Figure 12 to Figure 15. The most important one is that the discretization can also mimic the behavior of the logistic growth for some parameters as described in the caption of Figure 12. Moreover, Figure 15 shows that an envelope occurs and behaves like the logistic growth equation.

The second-order discretization

Fig. 10: The plot of the logistic equation (10) for $\rho = 2.5$, $r = 1$, $x_0 = 0.025$ with the fractional order $\alpha = 1.5$.

Fig. 11: The plot of the logistic equation (10) for $\rho = 2.5$, $r = 1$, $x_0 = 0.025$ with the fractional order $\alpha = 0.25$.

Fig. 12: The plots of the logistic equation (12) for $\rho = 1$, $r = 1$, $x_0 = 0.025$ with the fractional order $\alpha = 4$, $\alpha = 4.5$ and $\alpha = 5$.

Fig. 13: The plots of the logistic equation (12) for $\rho = 2.5$, $r = 1$, $x_0 = 0.025$ with the fractional order $\alpha = 0.95$, $\alpha = 1$ and $\alpha = 1.1$. 
IV. Conclusion

In this paper we discuss the numerical solution of a fractional order delayed logistic equation in the form of piecewise constant argument. The equation arised from a simple SI-terrorist mathematical model. We use the first and the second order discretization method in the numerical scheme and investigate the effect of the fractional order in the growth of the underlying population modelled by the equation. We show that in general the discretization method can mimic the behavior of the original logistic equation for some parameters. However, destabilizing effect may occur depending on the combination of the values of related parameters, such as the fractional order, the intrinsic growth rate, and the piecewise constant argument parameter. This motivates further investigation to search for alternative method that able to capture the dynamics of the underlying fractional differential equation.

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Fig. 14: The plot of the logistic equation (12) for $\rho = 2.5$, $r = 1$, $x_0 = 0.025$ with the fractional order $\alpha = 0.05$, $\alpha = 0.1$, and $\alpha = 0.15$.

Fig. 15: The plot of the logistic equation (12) for $\rho = 2.5$, $r = 1$, $x_0 = 0.025$ with the fractional order $\alpha = 0.005$, $\alpha = 0.01$, and $\alpha = 0.015$. 
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