Inverse dynamic and spectral problems for the one-dimensional Dirac system on a finite tree.

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Abstract. We consider inverse dynamic and spectral problems for the one-dimensional Dirac system on a finite tree. Our aim will be to recover the topology of a tree (lengths and connectivity of edges) as well as matrix potentials on each edge. As inverse data we use the Weyl-Titchmarsh matrix function or the dynamic response operator.

1. Introduction

Let $\Omega$ be a finite connected compact graph without cycles (a tree). The graph consists of edges $E = \{e_1, \ldots, e_N\}$ connected at the vertices $V = \{v_1, \ldots, v_{N+1}\}$. Every edge $e_j \in E$ is identified with an interval $(0, l_j)$ of the real line. The edges are connected at the vertices $v_j$ which can be considered as equivalence classes of the edge end points, we write $e \sim v$ if the vertex $v$ is a boundary of the edge $e$. The boundary $\Gamma = \{v_1, \ldots, v_m\}$ of $\Omega$ is a set of vertices having multiplicity one (the exterior nodes). In what follows we assume that one boundary node (say $v_m$) is clamped, i.e. zero Dirichlet boundary condition is imposed at $v_m$, and everywhere below we will be dealing with the reduced boundary $\Gamma = \Gamma \setminus \{v_m\}$. Since the graph under consideration is a tree, for every $a, b \in \Omega$, $a \neq b$, there exists the unique path $\pi[a, b]$ connecting these points.

For simplicity of the formulation of the balance conditions at the internal vertexes, we introduce the special parametrization of $\Omega$: we assume that at any internal vertex all the edges connected at it have this vertex as start point or as end point. We assume that clamped vertex $v_m$ is the start point of the edge $e_m$, which fix the parametrization.

Let $J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, at each edge $e_i$ we are given with a real matrix-valued potential $V_i = \begin{pmatrix} p_i & q_i \\ q_i & -p_i \end{pmatrix}$, $p_i, q_i \in C^1(e_i)$. The space of real vector

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valued square integrable functions on the graph \( \Omega \) is denoted by \( L^2(\Omega) := \bigoplus_{i=1}^{N} L^2(e_i, \mathbb{R}^2) \). For the element \( U \in L^2(\Omega) \) we write
\[
U := \begin{pmatrix} u_1^1 \\ u_2^1 \end{pmatrix} = \left\{ \begin{pmatrix} u_1^1 \\ u_2^1 \end{pmatrix} \right\}_{i=1}^{N}, \quad u_1^1, u_2^1 \in L^2(e_i).
\]

The continuity condition at the internal vertexes reads:
\[
u_i^1(v) = u_j^1(v), \quad e_i \sim v, \quad e_j \sim v, \quad v \in V \setminus \tilde{\Gamma}. \quad (1.1)
\]

The second condition (force balance) at the internal vertex \( v \) is introduced as
\[
\sum_{i \mid e_i \sim v} u_i^2(v) = 0, \quad v \in V \setminus \tilde{\Gamma}. \quad (1.2)
\]

We put \( \Psi := \begin{pmatrix} \psi_1^1 \\ \psi_2^1 \end{pmatrix} \in L^2(\Omega), \psi_1^1, \psi_2^1 \in H^1(e_i) \) and introduce the operator
\[
\mathcal{L} \Psi := \left\{ J \frac{d}{dx} \begin{pmatrix} \psi_1^1 \\ \psi_2^1 \end{pmatrix} + V_i \begin{pmatrix} \psi_1^1 \\ \psi_2^1 \end{pmatrix} \right\}, \quad x \in e_i
\]
with the domain
\[
D(\mathcal{L}) = \left\{ \Psi \in L^2(\Omega) \mid \psi_1^1, \psi_2^1 \in H^1(e_i), \; i = 1, \ldots, N, \right. \left. \Psi \text{ satisfies } (1.1), (1.2), \quad \psi^1(v) = 0, \; v \in \Gamma \right\}
\]

By \( \mathbf{S} \) we denote the spectral problem on the graph:
\[
J \Psi_x + V \Psi = \lambda \Psi \quad x \in e_i, \quad (1.3)
\]
\[
\Psi \text{ satisfies } (1.1), (1.2) \text{ at } v \in V \setminus \tilde{\Gamma} \quad (1.4)
\]
\[
\psi^1(v) = 0 \quad \text{for } v \in \Gamma, \quad (1.5)
\]

We introduce the Titchmarsh-Weyl (TW) matrix-function as an analog to Dirichlet-to-Neumann map \([3, 2, 5]\) in the following way: for \( \lambda \notin \mathbb{R} \) and \( \xi \in \mathbb{R}^{m-1} \) we consider the problem \((1.3), (1.4)\) with the nonhomogeneous boundary condition:
\[
\psi^1(v_i) = \xi_i, \quad i = 1, \ldots, m - 1. \quad (1.6)
\]

The TW matrix function connects the values of the solution \( \Psi(\cdot, \lambda) \) to \((1.3), (1.4), (1.6)\) in the first and second channels at the boundary:
\[
\psi^2(\cdot, \lambda)|_{\Gamma} = \mathbf{M}(\lambda) \psi^1(\cdot, \lambda)|_{\Gamma}, \quad (1.7)
\]
\[
(\psi^2(v_1, \lambda), \ldots, \psi^2(v_{m-1}, \lambda))^T = \mathbf{M}(\lambda) (\xi_1, \ldots, \xi_{m-1})^T.
\]

The inverse problem for the problem \( \mathbf{S} \) is to recover the tree \( \Omega \), i.e. connectivity of edges and their lengths, and parameters \( p_i, q_i \) on edges \( e_i \) from \( \mathbf{M}(\lambda) \).

Along with the spectral, we consider the dynamic inverse problem. We introduce the outer space \( \mathcal{F}_T := L^2([0, T], \mathbb{R}^{m-1}) \), the space of controls
acting on the reduced boundary of \( \Omega \). The forward problem is described by the Dirac system on the each edge of the tree:

\[
iU_t(x,t) + JU_x(x,t) + V(x)U(x,t) = 0 \quad x \in e_i, \ t \geq 0,
\]

(1.8)

conditions at internal vertexes:

\[
U(v,t) \text{ satisfies (1.1), (1.2) for all } t \geq 0, \ v \in V \setminus \tilde{\Gamma},
\]

(1.9)

Dirichlet boundary conditions:

\[
U^1|_{\Gamma} = F, \text{ on } \Gamma \times [0,T],
\]

(1.10)

where \( F = (f^1(t), \ldots, f^{m-1}(t))^T \in \mathcal{F} \), and (1.10) means that

\[
(u^1(v_1,t), \ldots, u^1(v_{m-1},t))^T = (f^1(t), \ldots, f^{m-1}(t))^T.
\]

By \( D \) we denote the dynamic problem on \( \Omega \), described by system (1.8), compatibility conditions (1.9) at all internal vertices for all \( t > 0 \), Dirichlet boundary condition (1.10) and zero initial condition \( U(\cdot,0) = 0 \). The solution to this problem is denoted by \( U^F \). We introduce the response operator for the problem \( D \) by

\[
R^T\{F\}(t) := u^2(\cdot, t)|_{\Gamma}, \ t \in [0,T].
\]

(1.11)

In other words, \( R^T \) connects values of the solution \( U^F \) to the problem \( D \) in the first and the second channels at the boundary:

\[
\left( R^T (f^1(t), \ldots, f^{m-1}(t))^T \right) (t) = (u^2(v_1,t), \ldots, u^2(v_{m-1},t))^T.
\]

The operator \( R^T \) has a form of convolution:

\[
(R^TF)(t) = (R \ast F)(t) = \int_0^t R(t-s)F(s) \, ds,
\]

where \( R(t) = \{R_{ij}\}_{i,j=1}^{m-1} \) is a response matrix. The entries \( R_{ij}(t) \) are defined in the following way: let \( U_i \) be a solution to the boundary value problem (1.8), (1.9), \( U_i(\cdot,0) = 0 \) with special boundary condition (1.10) where \( F = (0, \ldots, \delta(t), \ldots, 0)^T \) with only nonzero element at \( i \)-th place. Then

\[
R_{ij}(t) = u^2_i(v_j,t).
\]

(1.12)

The inverse problem for the problem \( D \) is to recover the tree (connectivity of the edges and their lengths) and the matrix potential on edges from the response operator \( R^T(t), \ t > 0 \).

The connection between spectral and the dynamic inverse data is known \([3, 2, 5]\) and was used for solving inverse spectral and dynamic problems. Let \( F \in \mathcal{F} \cap (C^\infty_0(0, \infty))^{m-1} \) and

\[
\tilde{F}(k) := \int_0^\infty F(t) e^{ikt} \, dt
\]

be its Fourier transform. The systems (1.3) and (1.8) are clearly connected: going formally in (1.3) over to the Fourier transform, we obtain (1.3) with \( \lambda = k \). It is not difficult to check that the response matrix function \( R(t) \) and
TW matrix function $M(\lambda)$ (Nevanlinna type matrix function) are connected by the same transform:

$$M(k) = \int_0^\infty R(t)e^{ikt}dt$$

(1.13)

where this equality is understood in a weak sense. We use this relationship between dynamic and spectral data solve the inverse problem from either $M(k)$ or $R(t)$, $t \geq 0$.

We will use the Boundary Control method [7], first applied to the problems on trees in [8, 10] and its modification, so called leaf-peeling method introduced in [3] and developed in [5, 2, 6]. This method, as it follows from its name is connected with the controllability property of the dynamical system under the consideration. The general principal [7] says that better controllability of dynamical system leads to better identifiability. Introduce the control operator: $W^T : F_T^T \hookrightarrow L_2(\Omega)$ acting by the rule:

$$W^T F := U^F(\cdot, T).$$

For the wave equation on a tree [8, 10] the corresponding control operator is boundedly invertible for certain values of time. For the two-velocity system [5, 4] the corresponding operator is not invertible, but at least there is some "local" controllability. But for the Dirac system there are no even "local" controllability, the latter causes the consequences for the inverse problem. To overcome this difficulty, we will use some ideas from [9]. In [2] the authors developed purely dynamic version of the leaf-peeling method for the inverse problem for the wave equation with potential on a finite tree, which allows one to solve the inverse problem using $R^T$ for some finite $T$. We are planning to return to this (optimal in time) setting for a Dirac system on a tree elsewhere.

The next section is devoted to the solution of IP. On analyzing the reflection of a wave propagating from a boundary from an inner vertex, we obtain the length of boundary edge. On the next step, using the method from [9], we find $p_i, q_i$, for boundary edges $e_i$ (i.e. $e_i \sim v_i, i = 1, \ldots, m - 1$). Then we determine sheaf – a star-shaped subgraph of $\Omega$, consisting of boundary edges $e_1, \ldots, e_m$ and only one non-boundary edge. And on the last step we consider the new tree: $\Omega \setminus \bigcup_{i=1}^{m_0} e_i$ and recalculate the Weyl-Titchmarsh matrix $\tilde{M}(\lambda)$ for this reduced tree.

2. Inverse problem.

2.1. Reflection from the inner vertex. Let $U^\delta$ be a solution to the special boundary value problem for Dirac system on a tree: on the edges $e_i, i = 1, \ldots, N$, $U^\delta$ satisfies (1.8), at internal vertexes the continuity and force balance conditions (1.9) hold, $U^\delta(\cdot, T) = 0$, and on the boundary we prescribe the special condition

$$u^1(v_1, t) = \delta(t), \; u^1(v_2, t) = 0, \ldots, u^1(v_{m-1}, t) = 0.$$
We denote by \( l_1 \) the length of the boundary edge \( e_1 = [v_1, v'] \), which is identified with the interval \([0, l_1]\). We assume that this edge is connected at the inner vertex \( v' \) with other \( n - 1 \) edges \( e_2, \ldots, e_n \) which we identify with the intervals \([l_1, l_1 + l_i]\), where \( l_i \) is a length of \( e_i, i = 2, \ldots, n \). When \( t < l_1 \) (i.e. the wave generated at \( v_1 \) does not reach the inner vertex \( v' \)), the solution to the above problem is zero on all edges except \( e_1 \). And on this edge it is given by \([9]\):

\[
U_1^\delta(x, t) = \left( \frac{\delta(t-x)}{i\delta(t-x)} \right) + \Gamma(x, t), \quad x \in e_1, \ t < l_1,
\]

where \( \Gamma(x, t) \) is a smooth function in the region \( \{0 < x < t\} \). At \( t = l_1 \) the wave reaches the inner vertex \( v' \), on the time interval \( l_1 < t < l_1 + L \) where \( L = \min_{i=2,\ldots,n}\{l_i\} \), the solution on the edges \( e_2, \ldots, e_n \) has a form

\[
U_i^\delta(x, t) = \left( \frac{\alpha \delta(t+x-l_1-l_i)}{-i\alpha \delta(t+x-l_1-l_i)} \right) + \Gamma_i(x, t), \quad x \in e_i, \ l_1 < t < l_1 + L.
\]

and on the first edge, \( x \in e_1 \):

\[
U_1^\delta(x, t) = \left( \frac{\delta(t-x)}{i\delta(t-x)} \right) + \left( \frac{\gamma \delta(t+x-2l_1)}{-i\gamma \delta(t+x-2l_1)} \right) + \Gamma'(x, t), \ l_1 < t < l_1 + L.
\]

In the above representations \( \Gamma_i, \Gamma', i = 2, \ldots, n \) are smooth functions and the constants \( \alpha, \gamma \) are subjected to determination.

We use the first continuity condition \([1.1]\) at the vertex \( v' \) to get the relation

\[1 + \gamma = \alpha,\]

and use the force balance condition \([1.2]\) at \( v' \), which yields

\[1 - \gamma - (n - 1)\alpha = 0.\]

Two above equalities lead to the following formulas

\[\gamma = \frac{2 - n}{n}, \quad \alpha = \frac{2}{n}.\]

Bearing in mind the definition of a response matrix \([1.12]\), we see that its component \( R_{11}(t) \) has a form:

\[R_{11}(t) = u_2^\delta(v_1, t) = i\delta(t) + i\frac{n-2}{n} \delta(t-2l_1) + \Gamma'_2(0, t), \quad t \in [0, 2l_1 + L) \] (2.1)

with some smooth \( \Gamma'_2(0, t) \). Thus by knowing the response matrix entry \( R_{11}(t) \), one can determine the length \( l_1 \) of the edge \( e_1 \) (it is contained in the argument of the second singular term) and the number of edges, \( e_1 \) connected with. The representation \([2.1]\) implies that from the diagonal elements \( R_{ii}, i = 1, \ldots, m - 1 \) of the response matrix one can extract the lengths \( l_i \) of the boundary edges \( e_i, i = 1, \ldots, m - 1 \).
2.2. Inverse problem on a half-line. Here we show following [9] that diagonal elements of the response matrix determine not only the lengths $l_i$ of boundary edges $e_i$, but also matrix potentials $V_i$ on $e_i$, $i = 1, \ldots, m - 1$.

We consider the inverse problem for the Dirac system on a half-line, which is set up in the following way:

$$\begin{cases}
    iu_t + J u_x + V u = 0, & x > 0, \quad 0 < t < 2T \\
    u_t|_{t=0} = 0, & x \geq 0 \\
    u_1|_{x=0} = f, & 0 \leq t \leq 2T,
\end{cases} \quad (2.2)$$

where $V = \begin{pmatrix} p & q \\ q & -p \end{pmatrix}$ is a matrix potential, $p = p(x)$ and $q = q(x)$ being real-valued $C^1$-smooth functions. We associate a response operator to the above system, acting in $L^2(0,2T; \mathbb{C})$ by the rule

$$RF := u^f_2(t) |_{0 \leq t \leq 2T}.$$ 

This operator has a form of convolution: $RF = if + r \ast f$, where $r|_{0 \leq t \leq 2T}$ is a response function. The response function $r(t)$ for $t \in (0,2T)$ is determined by the values of the potential $V(x)$ for $x \in (0, T)$ only, therefore, the relevant dynamic setup of the inverse problem is: for a fixed $T > 0$, given $r|_{0 \leq t \leq 2T}$ to recover $V|_{0 \leq x \leq T}$. We assume that all functions of time $t \geq 0$ are extended to $t < 0$ by zero. Also, for a $z \in \mathbb{C}, \bar{z} := \text{Re } z - i \text{Im } z$ is its conjugate.

A $\mathbb{C}$-valued function $r|_{0 \leq t \leq 2T}$ determines an operator $C^T$ acting in $L^2(0,2T; \mathbb{C}^2)$ by the rule

$$(C^T a)(t) = 2a(t) + \int_0^T c^T(t,s)a(s) \, ds, \quad 0 \leq t \leq T, \quad (2.3)$$

where $a = \begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix}$, and the the element of matrix kernel $c^T$ are

$$c_{11}(t,s) = -i \left[ r(t-s) - r(s-t) \right], \quad c_{12}(t,s) = -i \bar{r}(2T-t-s), \quad c_{21}(t,s) = i r(2T-t-s), \quad c_{22}(t,s) = i \left[ \bar{r}(t-s) - r(s-t) \right].$$

In [9] it is proved

**Theorem 1.** The function $r \in C^1([0,2T]; \mathbb{C})$ is the response function of a system (2.2) with a $C^1$-smooth real zero trace potential $V$ if and only if operator $C^T$ is a positive definite isomorphism.

Below we describe a procedure of recovering a potential $V$ from a response function $r$:

1. Given a response function $r(t)$, $0 \leq t \leq 2T$ of the system (2.2), determine the operator $C^T$ and the matrix-kernel $c^T$ by (2.3).
2. For $0 < \xi < T$ solve the family of the linear integral equations

$$\frac{1}{2} c^T(t,s) + 2k^\xi(t,s) + \int_{T-\xi}^T k^\xi(t,\eta)c^T(\eta,s) \, d\eta = 0, \quad T-\xi \leq s, t \leq T, \quad (2.4)$$
which determines the matrix-valued function $k^\xi$, $0 < \xi \leq T$ via $c$. The solvability is guaranteed by the positive-definiteness of $C^T$. By standard integral equations theory arguments, the solution $k^\xi$ is of the same smoothness as $c^T$, i.e., is $C^1$-smooth outside the diagonal $t = s$.

3. Define the matrix $w(x, x)$ by

$$w(x, x) = -2\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} k^x(T - x, T - x), \quad x \in (0, T),$$

(2.5)

3. Define the matrix $w(x, x)$ by

$$w(x, x) = -2\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} k^x(T - x, T - x), \quad x \in (0, T),$$

(2.5)

Take its first column $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ and recover the entries of matrix potential by:

$$p(x) = \text{Im} w_1(x, x) + \text{Re} w_2(x, x), \quad x \in (0, T),$$

$$q(x) = -\text{Re} w_1(x, x) + \text{Im} w_2(x, x), \quad x \in (0, T).$$

We use the method described above to recover the potential $V_i$ on each boundary edge $e_i$, $i = 1, \ldots, m - 1$. Where for a fixed boundary vertex $v_i$, we consider the response $R_{ii}^{T_i}$ with $T_i = l_i$ and $l_i$ is a length of $e_i$, were recovered from $R_{ii}(t)$ as explained in the previous subsection.

2.3. Recovery of the boundary sheaves. At this point we assume that we already know the lengths $l_i$ of boundary edges $e_i$, $i = 1, \ldots, m - 1$. We will need the reduced response function $R(t) = \{R_{ij}(t)\}_{i,j=1}^{m-1}$, (1.11). If our inverse data is a TW function $M(\lambda)$, we can pass to $R(t)$ taking the inverse Fourier transform of (1.13).

First of all we identify the edges connected at the same vertex. Two boundary edges, $e_i$ and $e_j$, $1 \leq i, j \leq m - 1$ have a common vertex if and only if

$$R_{ij}(t) = \begin{cases} 0, & \text{for } t < l_i + l_j, \\ \neq 0, & \text{for } t > l_i + l_j. \end{cases}$$

This relation allows us to divide the boundary edges into groups, such that edges from one group have a common vertex. We call these groups presheaves. More exactly, we introduce the following

**Definition 1.** We consider a subgraph of $\Omega$ which is a star graph consisting of all edges incident to an internal vertex $v$. This star graph is called a presheaf if it contains at least one boundary edge of $\Omega$. A presheaf is called a sheaf if all but one its edges are the boundary edges of $\Omega$.

The sheaves are especially important to our identification algorithm. To extract them we denote the found pre-sheaves by $P_1, \ldots, P_L$, and define the distance $d(P_k, P_m)$ between two pre-sheaves in the following way: we take boundary edges $e_i \in P_k$ and $e_j \in P_m$ and then put

$$d(P_k, P_m) = \max\{t > 0 : R_{ij}(t - l_i - l_j) = 0\}.$$ 

Clearly this definition does not depend on the particular choice of $e_i \in P_k$ and $e_j \in P_m$ and gives the distance between the internal vertices of the
pre-sheaves $P_k$ and $P_m$. Then we consider

$$\max_{k,m \in 1,\ldots,N, k \neq m} d(P_k, P_m).$$

It is not difficult to see that two pre-sheaves on which this maximum is attained (we denote them by $P$ and $P'$) are sheaves. Indeed, since $\Omega$ is a tree, there is only one path between $P$ and $P'$. If we assume the existence of an “extra” internal edge in $P$ or $P'$, this leads to contradiction, since there would necessarily exist sheaves with a distance between them greater than $d(P, P')$.

### 2.4. Leaf peeling method.

Let the sheaf $P$, found on the previous step, consist of the boundary vertices $v_1, \ldots, v_{m_0}$ from $\Gamma$, the corresponding boundary edges $e_1, \ldots, e_{m_0}$ and an internal edge $e_{m'_0}$. We assume that we already recovered lengths $l_i$ and potentials $V_i$, $i = 1, \ldots, m'_0$, i.e. on boundary edges of $P$. We identify each edge $e_{m'_0}, e_i, i = 1, \ldots, m_0$, with the interval $[0, l_i]$ and the vertex $v_{m'_0}$, the internal vertex of the sheaf, — with the set of common endpoints $x = 0$. At this point it is convenient to renumber the edge $e_{m'_0}$ as $e_0$ and the vertex $v_{m'_0}$ as $v_0$.

By $\widetilde{M}(\lambda)$ we denote the reduced TW matrix function associated with the new graph $\tilde{\Omega} = \Omega \setminus \bigcup_{i=1}^{m_0} \{e_i\}$ with boundary points $v_0 \cup \Gamma \setminus \bigcup_{i=0}^{m_0} v_i$.

First we recalculate entries $\widetilde{M}_{0i}(\lambda)$, $i = 0, m_0 + 1, \ldots, m - 1$. Let us fix $v_1$, the boundary point of the sheaf $P$. Let $\Psi$ be a solution to the problem (1.3)-(1.4) with the boundary conditions given by

$$\psi^1(v_1) = 1, \quad \psi^1(v_2) = 0, \quad j = 2, \ldots, m.$$ 

We point out that on the boundary edge $e_1$ the function $\Psi$ solves the Cauchy problem

$$\begin{cases} J\Psi_x + V\Psi = \lambda\Psi, & x \in e_1, \\ \psi^1(v_1) = 1, \quad \psi^2(v_1) = M_{11}(\lambda). \end{cases} \quad (2.6)$$

On other boundary edges of $P$, the function $\Psi$ solves the problems

$$\begin{cases} J\Psi_x + V\Psi = \lambda\Psi, & x \in e_i, \\ \psi^1(v_i) = 0, \quad \psi^2(v_i) = M_i(\lambda), & i = 2, \ldots, m_0 \end{cases} \quad (2.7)$$

Since we know potentials $V_i$ on the edges $e_1, \ldots, e_{m_0}$, we can solve the Cauchy problems (2.6) and (2.7), and use the conditions (1.1), (1.2) at the internal vertex $v_0$ to recover $\psi^1_0(v_0, \lambda), \psi^2_0(v_0, \lambda)$ — the value of the solution $\Psi$ at $v_0$, i.e. at the “new” boundary point of the new tree $\tilde{\Omega}$. Then we obtain:

$$\widetilde{M}_{00}(\lambda) = \frac{\psi^2_0(v_0, \lambda)}{\psi^1_0(v_0, \lambda)},$$

$$\widetilde{M}_{0i}(\lambda) = \frac{M_i(\lambda)}{\psi^1_0(v_0, \lambda)}, \quad i = m_0 + 1, \ldots, m - 1.$$ 

To find $\widetilde{M}_{i0}(\lambda), i = m_0 + 1, \ldots, m - 1$ we fix boundary point $v_i, i \notin \{1, \ldots, m_0, m\}$ and consider the solution $\Psi$ to (1.3)-(1.4) with the boundary
conditions given by
\[ \psi^1(v_i) = 1, \quad \psi^1(v_j) = 0, \quad j \neq i. \]
The function \( \Psi \) solves the Cauchy problems on the edges \( e_1, \ldots, e_{m_0} \):
\[
\begin{align*}
J \Psi_x + V \Psi &= \lambda \Psi, \quad x \in e_j \\
\psi^1(v_j) &= 0, \quad \psi^2(v_j) = M_{ij}(\lambda), \quad j = 1, \ldots, m_0.
\end{align*}
\] (2.8)

Since we know the potential on the boundary edges of \( P \), we can solve Cauchy problems (2.8) and use conditions at the internal vertex \( v_0 \) to recover \( \psi^1_0(v_0, \lambda) \), \( \psi^2_0(v_0, \lambda) \) – the value of solution at the “new” boundary point \( v_0 \) of reduced tree \( \tilde{\Omega} \).

On the other hand, on new tree \( \tilde{\Omega} \) the function \( U \) solves the problem (1.3)-(1.4) with the boundary conditions
\[ \psi^1(v_i) = 1, \quad \psi^1(v_0) = \psi^1_0(v_0, \lambda), \quad \psi^1(v_j) = 0, \quad j = m_0 + 1, \ldots, m, \]
\[ v_j \neq v_i, \quad v_j \neq v_0. \]

Thus for the entries of \( \tilde{M}(\lambda) \) the following relations hold:
\[
\begin{align*}
\tilde{M}_{i0}(\lambda) &= \psi^2_0(v_0, \lambda) - \psi^1_0(v_0, \lambda) \tilde{M}_{00}(\lambda), \\
\tilde{M}_{ij}(\lambda) &= M_{ij}(\lambda) - \psi^1_0(v_0, \lambda) \tilde{M}_{0j}(\lambda).
\end{align*}
\]

To recover all elements of the reduced matrix \( \tilde{M}(\lambda) \) we need to repeat this procedure for all \( i, j = m_0 + 1, \ldots, m - 1. \)

Thus using the described procedure we can recalculate the truncated TW matrix \( M(\lambda) \) for the new ‘peeled’ tree \( \tilde{\Omega} \). Repeating the procedure sufficient number of times, we step by step recover the tree and the matrix potential.

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