Comments on Backreaction

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Abstract. We respond to the criticisms of a recent paper of Buchert et al.
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Buchert et al have now published a revised version [1] of their original paper [2] criticizing our work. In the revised version, the tone of their criticisms has been improved and all assertions that we made mathematical errors in our work have been removed. However, the essential content of their paper remains unchanged. Therefore, we shall hereby revise our rebuttal by simply adding some remarks to our original rebuttal to note changes and address new points. In addition, elsewhere [3] we will give a simple, heuristic discussion of why backreaction is negligible in cosmology. Except for the remarks identified below as “[NOTE ADDED: . . .],” the remainder of this paper is unchanged from our original posting.

In a recent paper, Buchert et al [2] have criticized nearly all aspects of our analysis of backreaction produced by small scale inhomogeneities in cosmology as presented in [4–7]. Most of these criticisms concern points that are irrelevant to our main arguments and/or are based upon misinterpretations of our work. Nevertheless, in order to avoid the appearance that we are ignoring or evading potentially valid issues, we will give a brief response in the Appendix to all of the scientific/mathematical points raised in [2]. In short, we stand firmly behind all of the assertions, arguments, and conclusions presented in all of our previous papers [4–7].

Nevertheless, we believe that the discussion of [2] has the potential to create some confusion about the nature of our analysis of backreaction and the conclusions that can be drawn from it. Therefore, we feel that it would be appropriate for us to take this opportunity to explain, in an informal way, what we mean by “backreaction” and how the main tools of our analysis fit in with prior efforts, particularly the work of Isaacson [8, 9].

The context for the phenomenon of backreaction by small scale inhomogeneities concerns a situation where the actual spacetime metric $g_{ab}$ has large curvature fluctuations on small scales but, nevertheless, $g_{ab}$ can be well approximated by a metric $g_{ab}^{(0)}$ that does not have large curvature fluctuations. Although our analysis is valid in a much more general context, the main situation we have in mind is where $g_{ab}$ is the actual metric of the universe and $g_{ab}^{(0)}$ is a metric with FLRW symmetry. For simplicity, we will restrict our discussion below to the case where $g_{ab}^{(0)}$ has FLRW symmetry.

The issue at hand is whether the small scale inhomogeneities of $g_{ab}$ can contribute nontrivially to the dynamics of $g_{ab}^{(0)}$. A priori, this is possible even though $\gamma_{ab} \equiv g_{ab} - g_{ab}^{(0)}$ is assumed to be small: Einstein’s equation for $g_{ab}$ contains derivatives of $\gamma_{ab}$, which need not be small even when $\gamma_{ab}$ is small. Consequently, the Einstein tensor, $G_{ab}$, of $g_{ab}$ need not be close to the Einstein tensor, $G_{ab}^{(0)}$, of $g_{ab}^{(0)}$. Thus, although $g_{ab}$ is assumed to be an exact solution of Einstein’s equation with some stress-energy source $T_{ab}$, it is possible that $g_{ab}^{(0)}$ may not be close to a solution to Einstein’s equation with a suitably averaged stress-energy source $T_{ab}^{(0)}$. If this occurs, we say that there is a substantial backreaction effect of the small scale inhomogeneities on the effective dynamics of $g_{ab}^{(0)}$.

There are two cases where our analysis does not apply (and was never intended or claimed to apply). The first case is where the actual metric $g_{ab}$ is not close to any FLRW metric $g_{ab}^{(0)}$. A good example of this relevant to cosmology is obtained by taking $g_{ab}$ to
be an LTB model \[10–14\] with significant voids/overdensities on scales comparable to the Hubble radius. A simpler (but much less physically motivated) example discussed in \[15\] is the case where the universe consists of two disconnected FLRW parts at different stages of expansion/contraction. In such cases, since \(g_{ab}\) differs substantially from any single FLRW metric, there is little point in even attempting to find a “best fit” FLRW metric \(g_{ab}^{(0)}\) by averaging or by any other scheme. If one insists on finding a “best fit” \(g_{ab}^{(0)}\), the dynamics of \(g_{ab}^{(0)}\) would be expected to differ significantly from standard FLRW dynamics. However, we would not describe this as “backreaction.”

The second case to which our analysis does not apply is where the actual metric \(g_{ab}\) is close to an FLRW metric \(g_{ab}^{(0)}\), but one applies an averaging or other scheme to \(g_{ab}\) to construct an “effective” FLRW metric \(g_{ab}^{(0)}\) intended to reproduce averages of observables calculated in the actual metric. In general, this approach will give rise to a metric \(g_{ab}^{(0)}\) that differs significantly from \(g_{ab}^{(0)}\). This will typically happen if one tries to match the geodesics of \(g_{ab}\) in a naive way to corresponding geodesics in an FLRW model, since—although \(g_{ab}\) is close to \(g_{ab}^{(0)}\)—the geodesics of \(g_{ab}\) are not close to the geodesics of \(g_{ab}^{(0)}\). We provided an illustration of this in our “ball bearing” example in \[7\], where we constructed a smooth metric \(q_{ab}\) on a 2-sphere that was everywhere close to the round sphere metric \(q_{ab}^{(0)}\), but the geodesics and curvature of \(q_{ab}\) are not close to those of \(q_{ab}^{(0)}\). As we explained in \[7\], observers living in “Sphereland” and doing experiments with taut ropes (geodesics) would find extremely large deviations from any round sphere model on small scales and also would find some significant deviations on large scales. If they used their geodesic and curvature data to determine a “best fit” round metric \(q_{ab}^{(0)}\), they could easily make a substantial error, depending upon exactly what “averaging scheme” they used. Returning to the cosmological case, we noted in section 3 of \[7\] that the Buchert approach uses timelike geodesics in an essential way and the Clarkson and Umeh approach uses null geodesics in an essential way, so \textit{a priori} both approaches may produce an FLRW metric \(g_{ab}^{(0)}\) that differs significantly from \(g_{ab}^{(0)}\). The dynamics of \(g_{ab}^{(0)}\) may then differ significantly from that of \(g_{ab}^{(0)}\). If this occurs, we shall refer to the effect as pseudo-backreaction, since it has the appearance of being a dynamical effect produced by small scale inhomogeneities but is actually merely an artifact produced by a poor choice of representative FLRW metric \(g_{ab}^{(0)}\). Our analysis does not apply to pseudo-backreaction.

If, as we claim, naive averaging schemes making use of geodesics and curvature are prone to producing errors in the choice of representative FLRW model \(g_{ab}^{(0)}\), how should one go about determining \(g_{ab}^{(0)}\)? As we discussed in section 4.3 of \[7\], this should be done in the manner that is normally already done in practice by cosmologists (see, e.g., \[16\]): Consider FLRW metrics (with appropriate free parameters) and their perturbations (with appropriate free parameters). Calculate observable quantities within the models and find the model parameters that provide the best fit to all of the data. If one does not find an acceptable fit, this shows that something is wrong or incomplete in the theoretical model. If one does find an acceptable fit, this determines the parameters of the model, and one thereby obtains a representative FLRW metric \(g_{ab}^{(0)}\). Of course,
this does not “prove” that the theoretical model is correct or that $g_{ab}^{(0)}$ is close to the actual metric of the universe, $g_{ab}$. As in all other areas of science, even if one has a simple model that provides an extremely good fit to a wide variety of disparate data, one always should be open to the possibility that there are alternative explanations.

None of what has been said above addresses the issue of whether, in our actual universe, there might be significant backreaction effects on the large scale dynamics produced by small scale inhomogeneities. As discussed above, this is possible a priori because even if $\gamma_{ab} = g_{ab} - g_{ab}^{(0)}$ is small, its contribution to Einstein’s equation need not be small. How can one compute these backreaction effects?

The issue of backreaction was addressed in 1964 by Brill and Hartle \cite{17} in the context of trying to find “geon solutions” to Einstein’s equation, consisting of a ball of gravitational radiation that is held together by its self-gravitation. In this case, the small scale inhomogeneities consist of gravitational radiation, whose backreaction substantially alters the background metric. The Brill-Hartle approach was then significantly generalized by Isaacson \cite{8,9}. The Isaacson work is nicely summarized in subsections 35.13–15 of Misner, Thorne, and Wheeler \cite{18}.

However, although the approximations made in Isaacson’s work are well motivated physically, some serious difficulties arise if one tries to give a mathematically precise justification of them. The most serious difficulties involve the justification of equations satisfied by $\gamma_{ab}$ (denoted as $h_{\mu\nu}$ in \cite{18}), such as eq. (35.59a) of \cite{18}. The nature of these difficulties were elucidated at the end of section III of our first paper \cite{4}.

In the late 1980’s, one of us (R.M.W.) was very troubled by these difficulties and felt that the best way to resolve them would be to reformulate the Isaacson approximation scheme in terms of one-parameter families of metrics $g_{ab}(\lambda)$, similar to what is done to justify ordinary perturbation theory (see section 7.5 of \cite{19}). A reformulation of the Isaacson scheme in terms of one-parameter families would allow one to say with much more clarity what approximations are being made and would enable one to rigorously derive the equations satisfied by the various quantities at each order of approximation. However, to describe the Isaacson scheme in this way, one would need to consider one parameter families where $\gamma_{ab}(\lambda) \to 0$ as $\lambda \to 0$ but spacetime derivatives of $\gamma_{ab}(\lambda)$ do not go to zero, in order that that they may continue to make nontrivial contributions to the second order Einstein tensor in the limit, thereby providing nontrivial backreaction. It is therefore far from obvious how to describe the limiting behavior as $\lambda \to 0$ in a mathematically precise way. G. Burnett was a graduate student in the Chicago relativity group and R.M.W. posed to Burnett the problem of reformulating the Isaacson approximation scheme using one-parameter families. Burnett solved this problem brilliantly in \cite{20}, using the notion of weak limits to give a mathematically precise characterization of the one-parameter families $g_{ab}(\lambda)$. A discussion of the relationship between Burnett’s reformulation and the original Isaacson scheme can be found in Burnett’s paper and our previous papers (particularly sections I and II of \cite{4}).

Our contribution to the analysis of backreaction was to recognize that the
methods used in the Isaacson approach—as reformulated by Burnett—could be directly imported to describe backreaction in cosmology. The small scale inhomogeneities of primary interest now are not gravitational waves oscillating in space and time but are the perturbations associated with density inhomogeneities varying mainly in space. Nevertheless, the scheme for calculating the “Isaacson average” of the second order Einstein tensor of these perturbations does not depend on any wavelike character of the perturbation and works just as well for treating the backreaction effects of Newtonian-like cosmological perturbations as it does for treating the backreaction effects of gravitational waves.

Our general analysis presented in [4] allowed for the presence of gravitational radiation. It thereby allowed for backreaction produced by gravitational radiation as well as by matter inhomogeneities. Our main results are contained in two theorems. These theorems state that if the matter stress-energy tensor satisfies the weak energy condition, then the effective stress-energy tensor describing the leading order backreaction effects of the small scale inhomogeneities must (i) be traceless and (ii) satisfy the weak energy condition. In essence, our theorems say that significant backreaction effects can be produced only by gravitational radiation and not by matter inhomogeneities.

At sub-leading order, matter inhomogeneities do produce nontrivial backreaction effects, as we analyzed in [5]. In particular, in Appendix B of [5] we computed the backreaction effects of matter inhomogeneities on the expansion rate of the universe, under the assumption that these inhomogeneities are Newtonian-like in nature. A lengthy calculation revealed that backreaction effectively modifies the matter stress-energy by adding in the effects of kinetic motion and the Newtonian potential energy and stresses. In particular, it “renormalizes” the proper mass density to an “ADM mass density.”

Our results are fully in accord with the “back of the envelope” estimates of backreaction given in [15]. Our results on the sub-leading backreaction effects of Newtonian-like matter perturbations are fully in accord with the analysis of [21].

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Appendix A. Response to Specific Criticisms

In this Appendix, we will provide brief responses to the specific criticisms of [2]. However, we will not address the following two categories of criticism: (1) Criticism or discussion of any results not presented in [4–7]; (2) criticisms that are of no scientific
importance. The second category includes the subcategories of (a) criticisms that are unrelated to the main issues at hand and (b) numerous insinuations and innuendos‡ that suggest that we were sloppy in our assertions and arguments and/or overlooked various points. [NOTE ADDED: The tone has been improved in [1].] Our silence on these categories of criticism should not be interpreted as indicating agreement. The section numbers below correspond to the numbering of [2].

Section 2: This section criticizes our ball bearing example (mentioned in the body of the paper above) where we constructed a smooth metric $q_{ab}$ on a 2-sphere that was everywhere close to the round sphere metric $q^{(0)}_{ab}$, but the geodesics and curvature of $q_{ab}$ are not close to those of $q^{(0)}_{ab}$. The purpose of this example was simply to illustrate how two metrics could be close without their geodesics and curvature being close. Therefore, logically, it does not make sense to criticize the example unless we made an error in our claim that the metrics are close or our claim that their geodesics and curvature are not close. Neither claim appears to be challenged in section 2. Thus, as a matter of principle, we do not understand what is being criticized.

Section 3: This section points out that the Buchert formalism violates the conclusions of our theorems on the effective stress-energy tensor. The reasons why our results do not apply to the Buchert formalism were already explained in the body of the paper above. The remainder of this section mainly reviews our approach and raises numerous mathematical questions concerning the validity of our work, referring to Appendix B for details. We will give our response to these points under “Appendix B” below, except for the following comment: In contrast to what is said in subsection 3.7, backreaction in our formalism does not suddenly “turn on” at $\lambda = 0$; the one-parameter family of metrics $g_{ab}(\lambda)$ converges uniformly on compact sets to $g^{(0)}_{ab}$, so the dynamics of $g^{(0)}_{ab}$ accurately describes the (large scale) dynamics of $g_{ab}(\lambda)$ for sufficiently small $\lambda$. Indeed, for $\lambda > 0$, backreaction terms are present, and are described to leading order by the second order Einstein tensor; for $\lambda = 0$, they are accounted for by the effective stress-energy tensor $t^{(0)}_{ab}$, which is the weak limit as $\lambda \to 0$ of the second order Einstein tensor.

Section 4: Subsection 4.1 makes the point that whenever one has a one-parameter family of metrics $g_{ab}(\lambda)$, one could apply a $\lambda$-dependent diffeomorphism $\psi(\lambda)$ to generate a new one-parameter family of metrics $\psi^*(\lambda)g_{ab}(\lambda)$. If $g_{ab}(\lambda)$ approaches a limit as $\lambda \to 0$, then a $\psi(\lambda)$ may be chosen so that $\psi^*(\lambda)g_{ab}(\lambda)$ approaches a different limit§ or no limit at all. In this sense, the limit of any one-parameter family of metrics (or any other tensor fields) is always “coordinate dependent.” The situation with regard to our approach is no different from ordinary perturbation theory (see section 7.5 of [19]) where one could

‡ As a small but typical example, footnote 5 of [2] states that “Green and Wald obviously use a fixed $\lambda$-independent Riemannian metric for computing norms...” The word “obviously” suggests that we omitted to say this. In fact, we stated that we were using a fixed, $\lambda$-independent Riemannian metric for computing norms very clearly and explicitly in our papers. [NOTE ADDED: The word “obviously” has been deleted from footnote 5 in [2].]

§ The fact that different limits can be obtained in this manner for spacetimes representing a body that “shrinks to zero size” plays a central role in the Gralla-Wald derivation of self-force [22][23].
start with a one-parameter family \( g_{ab}(\lambda, x) \) that is jointly smooth in \( \lambda \) and \( x \), and then apply a diffeomorphism \( \psi(\lambda) \) that is not smooth in \( \lambda \) at \( \lambda = 0 \) (as in the examples given in subsection 4.1) so that \( \psi^*(\lambda)g_{ab}(\lambda) \) approaches a different limit (or no limit) as \( \lambda \to 0 \). Obviously, the fact that one can perform such a construction is completely irrelevant to the validity of ordinary perturbation theory, and it is similarly irrelevant to the validity of our results.

A much less trivial issue—which does not appear to have been considered in [2]—concerns the case where \( \psi^*(\lambda) \) goes to the identity as \( \lambda \to 0 \) (so that \( \text{w-lim}_{\lambda \to 0} \psi^*g_{ab} = g_{ab}^{(0)} \) but derivatives of \( \psi^*(\lambda) \) do not go to zero. Such transformations are of no relevance if \( \psi^*(\lambda)g_{ab}(\lambda) \) fails to satisfy our conditions (i)–(iv), just as non-jointly-smooth diffeomorphisms are of no relevance to ordinary perturbation theory. However, if \( \psi^*(\lambda)g_{ab}(\lambda) \) continues to satisfy our conditions (i)–(iv), it gives rise to a gauge dependence of \( \mu_{abcdef} \equiv \text{w-lim}_{\lambda \to 0} \nabla_a \gamma_{bc} \nabla_d \gamma_{ef} \) analogous to the gauge dependence of quantities appearing in ordinary perturbation theory. Nevertheless, as shown by Burnett [20], the effective stress-energy \( t_{ab} \) describing backreaction is gauge invariant. This corresponds to the statement that the Isaacson average of the second order Einstein tensor becomes gauge invariant in the short-wavelength limit. Thus, our backreaction results are fully “coordinate invariant” in the desired sense.

In subsection 4.2, our example given in [6] of a family of Gowdy metrics exhibiting nontrivial backreaction is criticized on the grounds that it is “ultra-local” and also on the grounds that we could have chosen a different family\([\|]\) that would have given no backreaction. Our purpose in giving the Gowdy family was to provide an explicit example of a one-parameter family of metrics \( g_{ab}(\lambda) \) that satisfies our assumptions (i)–(iv) with nonvanishing backreaction, \( t_{ab} \neq 0 \). Logically, it can be invalidated only by showing that our choice of \( g_{ab}(\lambda) \) violates at least one of our assumptions (i)–(iv) or that \( t_{ab} = 0 \). This is not claimed/shown in subsection 4.2.

**Section 5:** This section is largely a defense against our previous criticisms of the Buchert approach. We stand by our previous criticisms. Subsection 5.4 discusses the issue of how to obtain the “smoothed” metric \( g_{ab}^{(0)} \) from \( g_{ab} \). As already discussed above in the body of the paper, it is our view that using geodesics and/or curvature of \( g_{ab} \) to directly construct \( g_{ab}^{(0)} \) is a bad way to proceed, since it is prone to making significant errors because the geodesics and curvature of the actual metric \( g_{ab} \) differ significantly from \( g_{ab}^{(0)} \). Rather, one should determine \( g_{ab}^{(0)} \) by building a model for \( g_{ab}^{(0)} \) and \( \gamma_{ab} \) and fitting all available data to the model.

**Section 6:** It is difficult for us to determine exactly what is being criticized in this section, but we take this opportunity to make two clarifying remarks related to the discussion:

1. Our dictionary given in [5] that translates a Newtonian model into a general relativistic spacetime will yield a spacetime metric that solves Einstein’s equation to

\[ \text{Note that our choice of one-parameter family with nonvanishing backreaction gives rise to a } g_{ab}^{(0)} \text{ that provides a much better approximation to } g_{ab}(\lambda_0) \text{ than the alternative families suggested in subsection 4.2.} \]
very high accuracy. The distinction between having a quantity nearly solve an equation at all times versus having this quantity provide a good approximation to a solution at all times was explained in the second-to-last paragraph of the introduction of [5]. (2) If $g_{ab}$ is close to $g_{ab}^{(0)}$, then by flux conservation, the average apparent luminosity of sources (including multiple images) in the metric $g_{ab}$ must closely match that of $g_{ab}^{(0)}$. This is a simple fact, not an “argument against backreaction.” [NOTE ADDED: In [1] some text has been added to try to rebut this simple fact. Here the authors confuse average apparent luminosity (which closely matches that of the background FLRW metric, by conservation of flux) with average luminosity distance (which is a nonlinear function of apparent luminosity and thus can differ significantly from an FLRW model). The fact that nonlinear functions of average apparent luminosity can be significantly affected by inhomogeneities was explained clearly in section 4.3 of [7].]

Appendix B: Appendix B contains the core of the arguments against our results. It is referred to many times in the body of [2] to justify their criticisms. Words such as “error” appear frequently in Appendix B when describing of our work. [NOTE ADDED: In [1] these words have been removed.]

The authors of [2] appear to assume that we wished to consider metrics $g_{ab}(\lambda)$ of low regularity in spacetime. This is not the case. Indeed, until we read [2], it never occurred to us to consider spacetime metrics that are not $C^\infty$ in spacetime at each fixed $\lambda$, and there is no reason to consider metrics that are not smooth in spacetime. [NOTE ADDED: In [1], the authors assert that “one cannot safely carry out computations by declaring that second derivatives of the perturbing tensor $\gamma$ may be unboundedly large, and then treat them as ordinary smooth functions.” This suggests that some of the misunderstandings of the authors may originate from a confusion between unboundedness of second spacetime derivatives in the limit $\lambda \to 0$ (which is allowed in our formalism) with unboundedness of these derivatives in spacetime at fixed $\lambda$ (which we had never contemplated allowing).] For smooth spacetime metrics, all of our manipulations are trivially justified. For example, in their eq. (B.9) one can simply integrate by parts a second time to take all the derivatives off of $\gamma_{ab}$, and then use our assumption (ii) (eq. (7) of [2]) to see that the limit vanishes.

Nevertheless, the discussion of Appendix B raises an issue of some mathematical interest: If one did wish to consider metrics of low spacetime regularity, precisely what regularity would be needed to make our analysis valid? Indeed, since $\nabla_a \gamma_{bc}$ need not have a pointwise limit as $\lambda \to 0$ and $\nabla_a \nabla_b \gamma_{cd}$ may become unboundedly large as $\lambda \to 0$, it would be surprising if our results actually required smoothness of $\gamma_{ab}(\lambda)$ for all fixed $\lambda > 0$.

To analyze the regularity needed, we note that our condition (ii) (namely, $|\gamma_{ab}(\lambda, x)| < \lambda C_1(x)$) makes sense only if $\gamma_{ab}$ is a tensor field that is defined (almost) everywhere. Condition (ii) then immediately further implies that $\gamma_{ab}$ is locally $L^1$ and thus defines a distribution. In particular, its (weak) second derivatives are

¶ [NOTE ADDED: What we are calling the “weak derivative” here corresponds to what the authors
automatically well defined as a distribution via
\[
\nabla_a \nabla_b \gamma_{cd}[f^{abcd}] \equiv \gamma_{cd}[\nabla_b \nabla_a f^{abcd}] = \int_M \gamma_{cd} \nabla_b \nabla_a f^{abcd}
\]
(A.1)
for any test \((C^\infty_0)\) tensor field \(f^{abcd}\). Similarly, condition (iii) (namely, \(|\nabla_a \gamma_{bc}(\lambda, x)| < C_2(x)\)) makes sense only if \(\gamma_{ab}\) is differentiable (almost) everywhere. Condition (iii) then immediately further implies that \(\nabla_a \gamma_{bc}\) is locally \(L^2\). It follows immediately from the above together with the smoothness \(g^{(0)}_{ab}\) that the Einstein tensor of \(g_{ab}(\lambda, x) = g^{(0)}_{ab}(x) + \gamma_{ab}(\lambda, x)\) is well defined as a distribution on spacetime for all (sufficiently small) \(\lambda\). Thus, for all \(\lambda > 0\), it makes sense to require that Einstein’s equation holds weakly. Our results then follow by imposing Einstein’s equation weakly at \(\lambda > 0\) and taking the weak limit as \(\lambda \to 0\). The linear terms in the second derivatives of \(\gamma_{ab}\)—the main focus of the critical remarks of Appendix B—are easily seen to make no contribution in this limit on account of (A.1) and condition (ii).

Thus, our results continue to hold under the weakest regularity conditions under which our assumptions make any sense. This, of course, is a side-point since we were, in any case, considering only metrics that are smooth in spacetime at each fixed \(\lambda\)—in which case all of our manipulations are much more trivially justified. We conclude that the criticisms of Appendix B are completely without merit. Similarly, all of the statements sprinkled throughout the body of [2] that rely on Appendix B for justification are completely without merit.

Appendix C: Appendix C purports to show that something is wrong with our second example given in [6]. (Our first example was criticized in subsection 4.2—see the discussion above.) For our second example, we simply wrote down a one-parameter family of metrics \(g_{ab}(\lambda)\) that satisfied all of our assumptions except Einstein’s equation. We then declared it to solve Einstein’s equation with matter source \(T_{ab}(\lambda) = G_{ab}(\lambda)/8\pi\) (“Synge’s method” [24]). As one might expect, the resulting \(T_{ab}(\lambda)\) does not satisfy the weak energy condition. We then calculated the effective stress-energy of backreaction, \(t_{ab}\), and showed that it failed to satisfy the conclusions of our two theorems (tracelessness and the weak energy condition). This example is useful because it shows that, for the validity of our theorems, we cannot dispense with the hypothesis of the weak energy condition on \(T_{ab}(\lambda)\).

The authors of [2] correctly say that we obtain our matter stress-energy \(T_{ab}(\lambda)\) by Synge’s method, which they call “case (i).” They then say that another approach would be to obtain a different \(T_{ab}(\lambda)\) from a conformally invariant matter Lagrangian, which they call “case (ii).” Of course, case (ii) has nothing to do with our work. In case (ii), the matter stress-energy tensor would be conformally invariant, and their eq. (C.18) of [1] call the “distributional derivative.”

[NOTE ADDED: Despite the improved tone of Appendix B, the fundamental misunderstandings displayed in [3] remain present in [1]. For example, the first sentence of the paragraph following the paragraph containing eq. (B.20) in [1] asserts that “w-lim \(\nabla_a \nabla_b \gamma_{cd}\) = 0 is not a consequence of the GW hypotheses (ii), (iii), (iv) but a strong a priori assumption . . . .” This assertion is false, since w-lim \(\nabla_a \nabla_b \gamma_{cd}\) = 0 is a trivial consequence of (A.1) and condition (ii).]
would hold. They then say, “Let us look more closely at Green and Wald’s choice (i): using (C.18) . . . ” They then proceed to derive contradictions. But eq. (C.18) applies only to case (ii), not to case (i). \[NOTE ADDED: In [11], the reference to eq. (C.18) has been removed. Nevertheless, the association \textit{(made below eq. (C.22))} of the weak limit of $T_{ab}(g(1))$ (which, obviously, is equal to $T_{ab}(g(1))$) with $T_{ab}(0)$ makes no sense to us.\]

Since it is logically impossible for a metric and resulting stress-energy tensor obtained by Synge’s method to fail to be a solution to Einstein’s equation, it is logically impossible for our example to be wrong unless we made a computational error in calculating $t_{ab}$. We do not believe that we made any such error, and no evidence of such an error is presented in Appendix C.

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