Analysis of polarity
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Abstract

We develop a differential theory for the polarity transform parallel to that of the Legendre transform, which is applicable when the functions studied are “geometric convex”, namely convex, non-negative and vanish at the origin. This analysis may be used to solve a family of first order equations reminiscent of Hamilton–Jacobi and conservation law equations, as well as some second order Monge–Ampère type equations. A special case of the latter, that we refer to as the homogeneous polar Monge–Ampère equation, gives rise to a canonical method of interpolating between convex functions.

1 Introduction

The Legendre transform $L$, introduced by Mandelbrojt and Fenchel, is a classical operation mapping functions on $\mathbb{R}^n$ to convex lower-semi-continuous functions. It has numerous applications in many areas of mathematics, in physics and in economics. Restricted to convex lower-semi-continuous functions, it is an involution and on twice differentiable convex functions satisfies

$$\nabla f^* = (\nabla f)^{-1}, \quad \text{and} \quad \nabla^2 f^*_x = (\nabla^2 f^*)^{-1}|_{\nabla f(x)},$$

where we denote $f^* := Lf$. These properties lead to the classical fact that $L$ can be used to solve various first- and second-order equations, in particular equations of conservation laws, Hamilton–Jacobi equations, and Monge–Ampère equations.

Our main focus in this article is another duality transform $P$, called polarity. Our main goal here is to develop a differential theory for $P$. We introduce the notion of a polar subdifferential for a function, and analyze its properties. The analysis turns out to be more delicate than the corresponding analysis for $L$, due to the more non-linear nature of this transform. We further identify a wide class of convex functions for which second order analysis can be developed. As applications of this analysis we are led to introduce certain PDEs that are natural analogues of the classical Hamilton–Jacobi, conservations law and Monge-Ampère equations. These can solved by the polarity transform. They provide new processes for interpolation between convex functions.

Due to the ubiquitous role of the Legendre transform, the results here naturally raise the possibility of deriving many other parallel constructions and applications for polarity. The differential analysis of polarity we initiate here can be seen as a first step in this direction. Further generalizations, applications, and interpretation in terms of affine differential geometry will be considered elsewhere.

This article is organized as follows. After deriving some basic identities for polars of non-negative functions in Section 2, Section 3 is concerned with the basic sub-differential theory
for polarity. Here the polar subgradient is defined and some of its basic properties are studied. Section 4 computes the Hessian under polarity. In Section 5 we compute the first- and second-variation formulas for families of polars. Sections 6–7 derive the canonical Hamilton–Jacobi and Monge–Ampère type equations associated to polarity. In Section 8 we derive some explicit formulas for these solutions.

2 Polars of nonnegative functions

Recently it was shown [1] that the Legendre–Mandelbrojt–Fenchel transform [2, 3]

\[ f^*(y) \equiv (\mathcal{L}f)(y) = \sup_{x \in \mathbb{R}^n} (\langle x, y \rangle - f(x)), \]

and polarity [6, §15]

\[ f^0(y) \equiv (\mathcal{P}f)(y) = \inf \{ c \geq 0 : \langle x, y \rangle \leq 1 + cf(x), \forall x \in \mathbb{R}^n \}, \]

are essentially the only order reversing involutions on the class

\[ \text{Cvx}_0(\mathbb{R}^n) := \{ f \text{ convex and lower semi-continuous on } \mathbb{R}^n, f \geq 0, f(0) = 0 \}, \]

referred to as the class of “geometric convex functions.” We denote by Cvx(\mathbb{R}^n) the set of lower semi-continuous convex functions \( f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{ +\infty \}. \) Note that functions in Cvx_0(\mathbb{R}^n) are always proper and closed in the terminology of Rockafellar [5]. The domain of a function in Cvx(\mathbb{R}^n) is defined to be the (convex) set on which it attains finite values, and is denoted dom(\( f \)). Let us remark that the notation in the present article clashes somewhat with that in [1].

The epigraph of a function is defined as the set

\[ \text{epi } f = \{ (x, y) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq y \}, \]

Note that a function \( f \) belongs to Cvx_0(\mathbb{R}^n) if and only if the epigraph is a closed convex set containing \( \{0\} \times \mathbb{R}^+ \) and contained in the half-space \( \mathbb{R}^n \times \mathbb{R}^+ \).

**Lemma 2.1.** Let \( f \) be a non-negative function. (i) Then \( f^0 \in \text{Cvx}_0(\mathbb{R}^n) \) and

\[ f^0(y) = \begin{cases} \sup_{x \in f^{-1}(0)} \frac{\langle x, y \rangle - 1}{f(x)}, & \text{for } 0 \neq y \in \{ x : f(x) = 0 \}^\circ, \\ 0, & y = 0, \\ +\infty, & \text{otherwise}. \end{cases} \]

(ii) The double polar of \( f \) is the convex envelope,

\[ f^{**} = f^{00} = \sup \{ g \in \text{Cvx}_0(\mathbb{R}^n) : g \leq f \} \leq f. \]

(iii) The epigraph of \( f^0 \) is the reflection with respect to \( \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1} \) of the polar of the epigraph of \( f \).

**Proof.** If \( f \equiv 0 \) then both [3] and [1] give \( f^0 = 1_{\{0\}} \). So assume that \( f \neq 0 \).

(i) First observe that by [3], \( f^0(0) = 0 \). If \( f \in \text{Cvx}_0(\mathbb{R}^n) \), the first line of [1] already implies
\( f^0(0) = 0 \) since \( f \) is unbounded. For general \( f \geq 0 \),
\[
\sup_{x \notin f^{-1}(0)} \left( \frac{\langle x, y \rangle - 1}{f(x)} \right) = \inf \{ c : \langle x, y \rangle \leq 1 + cf(x) \ \forall x \notin f^{-1}(0) \}.
\]
If \( 0 \neq y \in (f^{-1}(0))^o \) then for all \( x \in f^{-1}(0) \) we have that \( \langle x, y \rangle \leq 1 \) and
\[
\sup_{x \notin f^{-1}(0)} \left( \frac{\langle x, y \rangle - 1}{f(x)} \right) = \inf \{ c : \langle x, y \rangle \leq 1 + cf(x) \ \forall x \in \mathbb{R}^n \}
= \inf \{ c \geq 0 : \langle x, y \rangle \leq 1 + cf(x) \ \forall x \in \mathbb{R}^n \},
\]
since for some \( x \) with \( \langle x, y \rangle > 1 \) we have \( f(x) > 0 \) (as \( f \neq 0 \)). If \( y \notin (f^{-1}(0))^o \), then there exists some \( x \) with \( f(x) = 0 \) and with \( \langle x, y \rangle > 1 \), thus by (4), \( f^0(y) = +\infty \), in agreement with (3). Finally, to see that \( f^0 \in \text{Cvx}_0(\mathbb{R}^n) \), it only remains to show that it is convex. If \( f \) is unbounded, then as already noted the first line of (4) already implies that \( f^0(0) = 0 \), and then \( f^o \) is a supremum of linear functionals and \( 1^o_{(f^{-1}(0))^o} \), hence convex. Finally, if \( f \) is bounded then \( f^0 = 1^o_{\{0\}} \).

(ii) By the classical properties of the Legendre transform \([5]\) it suffices to show the first equality in (5). First, if \( f \in \text{Cvx}_0(\mathbb{R}^n) \), \( f^{**} = f^{oo} = \text{cl} f = f \), where \( \text{epi} \text{cl} f = \text{epi} f \) \([5]\). Observe now that \( f^{oo} \leq f \), indeed:
\[
f^{oo}(x) = \inf \{ c^* \geq 0 : \langle x, z \rangle \leq 1 + c^* f^0(z), \ \forall z \}
\leq \inf \{ c^* \geq 0 : \langle x, z \rangle \leq 1 + c^* \frac{\langle x, z \rangle - 1}{f(x)}, \ \forall z \} = f(x).
\]
Since clearly \( \mathcal{P} \) is order reversing, we see that \( \text{cl} f \leq f \) implies \( (\text{cl} f)^o \geq f^o \) and \( \text{cl} f = (\text{cl} f)^{oo} \leq f^{oo} \leq f \). However, \( f^{oo} \) is closed by (i) and thus must equal \( \text{cl} f \).

(iii) The statement holds for \( f \in \text{Cvx}_0(\mathbb{R}^n) \) essentially from the definition \([3]\) \([5]\) p. 137]. In general, by (i) \( f^o \in \text{Cvx}_0(\mathbb{R}^n) \), and so by (ii) \( f^{oo} = f^o \), implying the desired statement. \( \square \)

The previous lemma recovers well-known properties of polars of non-convex sets. Let \( K \) be a set in \( \mathbb{R}^n \), and let \( K^o \) denote its polar, given by
\[
K^o = \{ y \in \mathbb{R}^n : \langle x, y \rangle \leq 1, \ \forall x \in K \}.
\]
For a closed convex set \( K \) let \( 1^c_K \) denote the convex indicator function, equal to 0 on \( K \) and \(+\infty \) elsewhere. Then \( \mathcal{P} 1^c_K = 1^o_K \).

One more useful fact is that for \( f \in \text{Cvx}_0(\mathbb{R}^n) \) we have that
\[
(6) \quad \text{dom}(f^0) = \{ x : f(x) = 0 \}^o \quad \text{and} \quad \text{dom}(f) = \{ y : f^o(y) = 0 \}^o.
\]
Indeed, a closed convex set \( K \) satisfies that \( K = \text{dom}(f) \) if and only if \( f \geq 1^c_K \) and \( f \geq 1^c_{K'} \), for any closed \( K' \supseteq K \). Similarly, \( \{ f = 0 \} = T^f \) if and only if \( f \leq 1^c_T \) and \( f \leq 1^c_{T'} \), for any \( T \subseteq T'. \) Since polarity on \( \text{Cvx}_0(\mathbb{R}^n) \) is an involution which changes order and replaces \( 1^c_K \) by \( 1^c_{K^o} \), the claim follows.

Next we briefly discuss the composition of \( \mathcal{P} \) and \( \mathcal{L} \). It is not hard to check that the two transformations commute, and thus the composition is an involution on \( \text{Cvx}_0(\mathbb{R}^n) \) which is order preserving. We list two of its properties which shall be useful in the sequel.
Lemma 2.2. Let \( f \in \text{Cvx}_0(\mathbb{R}^n) \) and \( x \) with \( f(x) \neq 0, +\infty \). If \( f \) is differentiable at the origin, we have that

\[ f(x)f^{\circ \circ}(x/f(x)) = 1. \tag{7} \]

Moreover, the above conclusion holds whenever \( f|_{[0,x(1+\delta)]} \) is not linear for any \( \delta > 0 \).

Proof. We will prove this lemma using mainly the order-preservation property of \( \mathcal{CP} \), together with our knowledge on how it acts on simple functions. Indeed, by the properties above it is enough to consider functions in \( \text{Cvx}_0(\mathbb{R}^+) \). Clearly \( f \leq \max(1_{[0,x]}^{c}, l_{f(x)}) \), where \( l_c \) denotes the function \( l_c(t) = ct \). Since \((1_{[0,x]}^{c})^{\circ \circ} = l_{1/x} \) and \((l_{c})^{\circ \circ} = 1_{[0,1/c]}^{c} \) we get that

\[ f^{\circ \circ} \leq \max(1_{[0,x/f(x)]}^{c}, l_{1/x}), \]

so that \( f^{\circ \circ}(x/f(x)) \leq 1/f(x) \). Next, we use the assumption that \( f \) has a supporting functional at \( x \) which is not the linear function \( l_{x/f(x)} \). Denote this support function by \( f^{(x)} \). Then \( f^{\circ \circ} \geq h^{\circ \circ} \) which is easily computed to be \(\max(0, \frac{1}{w}(t - \frac{x-w}{f(x)}) \). In particular we have \( f^{\circ \circ}(x/f(x)) \geq h^{\circ \circ}(x/f(x)) = 1/f(x) \). The proof of the lemma is complete. \( \square \)

Remark 2.3. In the case not covered in Lemma 2.2, the product in (7) can still be computed. Indeed, a function \( f \in \text{Cvx}_0(\mathbb{R}^n) \) is linear on some interval \( [0,y] \), if and only if the mapping \( x \mapsto x/f(x) \) is not injective. Assume that \( f(ty) = ct \) for \( 0 \leq t \leq 1 \), and that \( f \) is not linear on any extension of \([0,y] \). Then \( f^{\circ \circ}|_{[0,y]} \) is supported on \([0, y/f(y)] \), and the value it attains on \( y/f(y) = ty/f(ty) \) is \( 1/f(y) \). Thus for any \( 0 < t \leq 1 \),

\[ f(ty)f^{\circ \circ}(ty/f(ty)) = t. \]

The second property of \( f^{\circ \circ} \) which we shall need is a geometric description, which will help us investigate how properties such as smoothness and strict convexity are affected by this transformation \( f \mapsto f^{\circ \circ} \). Define the mapping \( F: \mathbb{R}^n \times (0, \infty) \to \mathbb{R}^n \times (0, \infty) \) by

\[ F(x_1, \ldots, x_n, t) = (x_1/t, \ldots, x_n/y, 1/t). \]

The following was shown in [1]:

Lemma 2.4. Let \( f \in \text{Cvx}_0(\mathbb{R}^n) \), then

\[ \text{int}(\text{epi} \ (f^{\circ \circ})) = F(\text{int}(\text{epi} \ (f))). \]

Remark 2.5. The mapping \( F \) is “fractional linear” (sometimes called “projective linear”), and in particular maps segments to segments and subdomains of affine \( k \)-dimensional subspaces to subdomains of affine \( k \)-dimensional subspaces. In particular, if \( f \in \text{Cvx}_0(\mathbb{R}^n) \) is strictly convex outside of \( \{f = 0\} \) then \( f^{\circ \circ} \) is strictly convex outside \( \{f^{\circ \circ} = 0\} \), and if \( f \) is differentiable in \( \text{dom}(f) \setminus \{f = 0\} \) then \( f^{\circ \circ} \) is differentiable in \( \text{dom}(f^{\circ \circ}) \setminus \{f^{\circ \circ} = 0\} \).

Note, however, that even for \( f \) which is everywhere differentiable and strictly smooth, the function \( f^{\circ \circ} \) may have a non-zero set \( \{f^{\circ \circ} = 0\} \) and may fail to be differentiable at the boundary of this set.

The following definition will be of use to us in the sequel.
Definition 2.6. Let $f \in \text{Cvx}_0(\mathbb{R}^n)$.

(i) The ray-linearity zone of $f$ is the set of $x \in \mathbb{R}^n \setminus \{0\}$ such that $f|_{[0,x]}$ is linear.

(ii) $f$ is nonlinear at infinity if for every ray $\mathbb{R}^+ x$ the one dimensional function $f(tx)$ defined on $t \in \mathbb{R}^+$ has domain $\mathbb{R}^+$ and is not between $h(t) = at$ and $h(t) - b$ for any $a \geq 0$ and $b > 0$.

Lemma 2.7. A function $f \in \text{Cvx}_0(\mathbb{R}^n)$ is nonlinear at infinity if and only if the linearity zone of $f^\circ$ is empty.

Proof. Put $g := f^\circ$. A function $g$ in $\text{Cvx}_0(\mathbb{R}^n)$ is linear on a segment $[0, w]$ if and only if it is above some linear function $l(x) = \langle x, u \rangle + (u \not\perp \text{w})$ and below the function which in $+\infty$ everywhere and equal to $\langle x, u \rangle$ on the segment $[0, w]$. This function may be written as $\sup(l, 1_{[0, w]})$. By taking polars of these conditions, we get that this happens if and only if $g^\circ$ is below $l^\circ(x)$ and above $\inf(l^\circ, 1_{[0, w]^\circ})$. Note that $l$ is simply the norm associated with a certain body (a halfspace in fact) so that $l^\circ$ is simply the norm associated with the polar of this body, which is the segment $[0, u]$. This norm is infinity outside $\mathbb{R}^+ u$ and equals to $\langle x, \frac{u}{|u|^2} \rangle$ on this ray. As for $\inf(l^\circ, 1_{[0, w]^\circ})$, it is the convexified minimum of $l^\circ$ and of the indicator of a halfspace, so it is 0 on the halfspace and linear outside, with slope the same as $l^\circ$ was. By Hahn Banach theorem, this is equivalent to the fact that $g^\circ(tw)$ restricted to $t \in \mathbb{R}^+$ is below the linear function $h(t) = t\langle w, \frac{u}{|u|^2} \rangle$ and above $h(t) - b$ for some $b > 0$. \hfill \Box

3 Polar subdifferential map

A central feature of the Legendre transform is that it is related to a gradient mapping. Namely, when $f \in C^1(\mathbb{R}^n)$ is strictly convex, $\nabla f : \mathbb{R}^n \to (\mathbb{R}^n)^* \cong \mathbb{R}^n$ is injective, and

$$f^\ast(y) := \sup_{x \in \mathbb{R}^n} [\langle x, y \rangle - f(x)]$$

can be computed explicitly from the function and its gradient map,

$$f^\ast(y) = \langle (\nabla f)^{-1}(y), y \rangle - f(\nabla f)^{-1}(y)).$$

More generally, for any proper closed convex function $f$, one uses the subdifferential map $\partial f(x) = \{ y : f(z) \geq f(x) + \langle z - x, y \rangle \ \forall z \}$, and the inverse of the subdifferential map detects the points where the supremum is attained, so that $\partial f(x) = \{ y : f^\ast(y) + f(x) = \langle x, y \rangle \}$. Moreover, $\partial f = (\partial f^\ast)^{-1}$, i.e., $y \in \partial f(x)$ if and only if $x \in \partial f^\ast(y)$. The above facts motivate the following definition for the polar subdifferential map.

Definition 3.1. For a function $f : \mathbb{R}^n \to \mathbb{R}^+ \cup \{\infty\}$ define the polar subdifferential map $\partial^\circ f$ at a point $x \in \text{dom}(f)$ by

$$\partial^\circ f(x) := \{ y \in \text{dom}(f^\circ) : f^\circ(y)f(x) = \langle x, y \rangle - 1 \}.$$ 

We say that $y \in \partial^\circ f(x)$ is a polar subgradient of $f$ at $x$. The domain of $\partial^\circ f$, $\text{dom}(\partial^\circ f)$, is defined as the set of $x$ with $\partial^\circ f(x) \neq \emptyset$.

Note that $\partial^\circ f(x)$ is a convex set. Indeed, $f^\circ((1 - \lambda)y_1 + \lambda y_2) \leq (1 - \lambda)f^\circ(y_1) + \lambda f^\circ(y_2)$ hence if $y_1, y_2 \in \partial^\circ f(x)$ then $f^\circ((1 - \lambda)y_1 + \lambda y_2)f(x) \leq \langle x, (1 - \lambda)y_1 + \lambda y_2 \rangle - 1$, and as the opposite inequality is always true by definition of $f^\circ$, we have equality on $[y_1, y_2]$. Also note that $\partial^\circ f(x)$ is closed relatively to $\text{dom}(f^\circ)$.
Remark 3.2. One could roughly restate the definition above in words as follows: “\( y \) is a polar sub-gradient of \( f \) at \( x \) if the supremum in the definition of \( f^\circ(y) \) is attained at \( x \).” The case for which this second definition does differ from the one above is when \( f(x) = 0 \). Let us shortly discuss this case: First note that \( \partial f(0) = \emptyset \). Consider some \( x \neq 0 \) with \( f(x) = 0 \). In such a case \( \partial f(x) = \{ y \in \text{dom}(f) : \langle x, y \rangle = 1 \} \). If the function \( f \in \text{Cvx}_0(\mathbb{R}^n) \) has zero set \( K \), that is, \( f|_K = 0 \) and \( f|_{\mathbb{R}^n \setminus K} \neq 0 \), then \( \text{dom}(f^\circ) = K^\circ \) so that the above definition becomes, for \( x \in K \),

\[
\partial f(x) := \{ y \in K^\circ : \langle x, y \rangle = 1, f^\circ(y) \neq \infty \}.
\]

For \( x \) in the relative interior of \( K \) this is again an empty set, and for \( x \) on the boundary of \( K \) the polar subdifferential is the set of supporting functionals to \( K \) at \( x \) (which are in the boundary of \( K^\circ \)) and which belong to \( \text{dom}(f^\circ) \). For example we may consider a function \( f^\circ \) with \( \text{dom}(f) = \text{int}(K^\circ) \), so that \( f \) itself is 0 on \( K \) and for all \( x \) in the boundary of \( K \), the polar-subdifferential is empty.

Note that if \( f \in \text{Cvx}_0(\mathbb{R}^n) \) then by (53), \( (\partial f)^{-1} = \partial f^\circ \), i.e., \( x \in \partial f^\circ(y) \) if and only if \( y \in \partial f(x) \). This means that one may write

\[
\partial f^\circ(y) = \{ x \in \text{dom}(f) : y \in \partial f(x) \},
\]

that is, the polar subdifferentials of \( f^\circ \) at a point \( y \) with \( f^\circ(y) \neq 0 \) are precisely the points for which the supremum in the definition of \( f^\circ \) is attained. This allows us to examine many examples for which \( \partial f^\circ \) is empty, for example when \( f \) is a norm then clearly in the definition of \( f^\circ \) (which is the dual norm) the supremum is never attained.

Note that unlike the usual subdifferential, \( \partial^2 f(x) \) can be empty even when \( f \) is smooth and convex at \( x \). This and other properties of \( \partial^2 \) will follow from the following basic lemmas. Below we say that “\( g|[0,x+] \) in not linear” when there is no interval \([0,t x]\) with \( t > 1 \) on which \( g \) is linear.

Lemma 3.3. Let \( f \in \text{Cvx}_0(\mathbb{R}^n) \). Then for each \( x \in \text{int}(\text{dom}(f)) \) with \( f(x) \neq 0 \),

\[
\partial^2 f(x) = \{ y : f^\circ(y), f^\circ(y/f^\circ(y)) \neq 0, +\infty, f^\circ|[0,y+) \text{ not linear and } y/f^\circ(y) \in \partial f(x) \} = \{ z/f^\circ(z) : z \in \partial f(x), f^\circ|[0,z+) \text{ not linear and } f^\circ(z), f^\circ(z/f^\circ(z)) \neq 0, +\infty \}.
\]

If, in addition, \( f \) is not linear on \([0,x+]\) and \( f(x), f^\circ(x/f^\circ(x)) \neq 0, +\infty \) we have that

\[
\partial^2 f(x) = (\partial f^\circ)^{-1}(x/f^\circ(x)) := \{ y : x/f^\circ(x) \in \partial f^\circ(y) \}.
\]

Note that the condition \( x \in \text{int}(\text{dom}(f)) \) is important, for instance check the example \( f(x) = \inf \left( 1_{[0,1]}, \max(x/2, 1_{[0,2]}) \right) \) at \( x = 2 \). Here \( \hat{f} \) denotes the closed convex envelope of \( f \). In this example \( \partial^2 f(2) = [1/2, 1] \) whereas the expression in the right hand side of the first equation in the lemma gives \((1/2, 1]\).

Proof. First, \( y \in \partial^2 f(x) \) satisfies \( f^\circ(y) \neq 0 \) otherwise by Remark 3.2 \( x \) would be at the boundary of the domain of \( f \), contrary to the assumption. Thus, in equation (69) one may divide by \( f^\circ(y) \) so that the condition is equivalent to \( \langle x, y/f^\circ(y) \rangle = \frac{1}{f^\circ(y)} \) defining a supporting hyperplane for \( f \) at \( x \). In particular, \( y/f^\circ(y) \in \partial f(x) \). This in turn implies (by Legendre theory) that \( f(x) = \langle x, y/f^\circ(y) \rangle - f^\circ(\frac{y}{f^\circ(y)}) \) and comparing this with the definition of \( \partial^2 f(x) \) we get that \( f^\circ(\frac{y}{f^\circ(y)}) = \frac{1}{f^\circ(y)} \) so that in particular \( f^\circ(\frac{y}{f^\circ(y)}) \neq 0, +\infty \). Finally, were it the case
that \( f^\circ \) was linear on some interval \([0, yt]\), for some \( t > 1 \) then we’d have that

\[
f^\circ(ty) = tf^\circ(y) = \frac{\langle x, ty \rangle - t}{f(x)} < \frac{\langle x, ty \rangle - 1}{f(x)} \leq f^\circ(ty),
\]
a contradiction.

On the other hand, suppose that \( f^\circ(y), f^*(\frac{y}{f^\circ(y)}) \neq 0, +\infty, f^\circ|_{[0, y+]} \) not linear and \( y/f^\circ(y) \in \partial f(x) \). Then \( \langle \cdot, y/f^\circ(y) \rangle - f^*(\frac{\cdot}{f^\circ(y)}) \) is a supporting hyperplane for \( f \) at \( x \), and in particular

\[
f(x) = \langle x, y/f^\circ(y) \rangle - f^*(\frac{x}{f^\circ(y)}).
\]
By Lemma 3.3 applied to \( f^\circ \) (which can be applied since \( f^\circ \) is not linear on \([0, y+]\)) we have that \( f^*(\frac{y}{f^\circ(y)}) = \frac{1}{f^\circ(y)} \) which concludes the proof of second inclusion and thus completes the first equality.

For the second equality, note that by the above argument, letting \( z = y/f^\circ(y) \) we have that \( y = z/f^*(z) \). Thus, if \( y \in \partial^o f(x) \) then it can be written as \( z/f^*(z) \) for some \( z \) satisfying the conditions. Indeed, one only should notice that linearity of \( f^\circ \) on \([0, w]\) is equivalent to linearity of \( f^* = (f^\circ)^* \) on \([0, w/f^\circ(w)]\), by the properties of the transform \( \mathcal{P} \mathcal{L} \) discussed in Section 2. On the other hand, given \( y = z/f^*(z) \) in \( \text{dom}(f^\circ) \) such that \( f^\circ(y) \neq 0 \) and \( f^\circ \) is not linear on \([0, y+]\), we see again, that \( y/f^\circ(y) = z \), otherwise \( f^\circ(y/f^\circ(y))f^*(z) \neq 1 \) which implies that \( f^* \) is linear on \([0, z+]\), a contradiction to the assumptions. This completes the proof of the first assertion.

For the second, simply use that \( y \in \partial^o f(x) \) if and only if \( x \in \partial^o f^\circ(y) \) together with the first assertion.

An easy consequence is the following:

**Corollary 3.4.** Let \( f \in \text{Cvx}_0(\mathbb{R}^n) \) and \( x \in \text{int}(\text{dom}(f)) \) with \( f(x) \neq 0 \), and assume \( f \) is not linear on \([0, x+]\). Then

\[
(10) \quad \partial^o f(x) = \partial f^*(x/f(x)),
\]
and

\[
(11) \quad \partial^o f^*(x/f(x)) = \partial f(x).
\]

**Proof.** By Lemma 3.3 for \( f \) satisfying these conditions, \( y \in \partial^o f(x) \) is equivalent to \( x/f(x) \in \partial f^\circ(y) \), which in turn, by Legendre theory, is equivalent to \( y \in \partial f^*\circ(x/f(x)) \). The second equation follows after noticing that \( f \) is linear on \([0, w]\) if and only if \( f^\circ \) is linear on \([0, w/f^\circ(w)]\), and then applying the first equality to \( f^* \) at \( x/f(x) \), noticing that under non-linearity of \( f \) on \([0, x+]\) we have \((x/f(x))/f^*(x/f(x)) = x \) by (7).

\[
(12) \quad y = \frac{\nabla f(x)}{f^*(\nabla f(x))}, \quad \text{and} \quad f^\circ(y) = \frac{1}{f^*(\nabla f(x))} = \frac{1}{\langle x, \nabla f(x) \rangle - f(x)}.
\]

Conversely, if \( f \in \text{Cvx}_0(\mathbb{R}^n) \), \( x \in \text{int}(\text{dom}(f)) \) with \( f(x) \neq 0 \), satisfy \( \partial^o f(x) = \{y\} \) then \( \partial f(x) = \{y/f^\circ(y)\} \). In particular, \( f \) is differentiable at \( x \), and \( y = \nabla f(x)/f^*(\nabla f(x)) \).

**Proof.** The first part follows directly from Lemma 3.3 Indeed, from differentiability there exists only one \( y \in \partial f(x) \) so that the set \( \partial^o f(x) \) can include at most one element, and in
case it includes an element, this element must be \( y/f^*(y) \). If indeed \( \partial^0 f(x) \) includes one element, \( y \), then, still from Lemma 3.3, \( y/f^*(y) = \nabla f(x) \) and \( f^\circ \) is not linear on \([0, y+]) \) so that \( f^\circ(y) = 1/f^*(y/f^*(y)) = 1/f^*(\nabla f(x)) \). This completes the proof of the first part.

Suppose now that \( \partial^0 f(x) = \{y\} \) for some \( x \in \text{int}(\text{dom}(f)) \) with \( f(x) \neq 0 \). By Lemma 3.3 we have that \( y \in \text{dom}(f^\circ) \), \( f^\circ(y) \neq 0 \), \( f^*(y/f^*(y)) \neq 0 \) and \( f^\circ[0,y+] \) is not linear. Letting \( z := y/f^*(y) \) we have that \( z \in \partial f(x) \). By (7) (and the remark following it) we thus have that \( f^* \) is not linear on \([0, z+] \) and \( f^*(y) = \frac{1}{f^*(z)} \). In particular, \( y = \frac{z}{f^*(z)} \).

Assume that there was another element \( z' \in \partial f(x) \). From convexity of the set \( \partial f(x) \) we can clearly assume that \( z' \) is as close as we wish to \( z \).

We claim that for \( z' \) close enough to \( z \) we have that \( f^* \) is not linear on \([0, z'+] \). If indeed this is true, we can make sure by continuity of \( f^* \) on its domain that \( y = z/f^*(z) \) and \( y' := z'/f^*(z') \) are close and thus also \( f^*(y') \neq 0 \), and by Lemma 2.2 also \( f^\circ(y') \neq +\infty \) so that \( y' \in \partial^0 f(x) \).

We thus must have that \( y' = y \) but this means that \( f^* \) is linear on \([0, z] \) and \( z' = tz \) for some \( t < 1 \) which is not the case we are considering.

The remaining case is that we cannot find \( z' \in \partial f(x) \) close to \( z \) such that \( f^* \) is not linear on \([0, z'+] \). This means that \( f^* \) is linear on \([0, z] \) and the only other \( z' \in \partial f(x) \) are \( z' = tz \) for \( t < 1 \). Since \( z' \in \partial f(x) \) if and only if \( x \in \partial f^\circ(z) \) this linearity implies that \( \partial f(x) = [0, z] \). This already implies that \( f(x) = 0 \) (since we have for any \( 0 \leq t \leq 1 \) in particular that \( f(0) \geq f(x) - \langle tz, x \rangle \)) which contradicts the assumption on \( x \).

In the case where \( \partial^0 f \) is a single point \( \{y\} \) we say that \( f \) is polar differentiable at \( x \) and denote the polar gradient by

\[
\nabla^0 f(x) = y.
\]

Some further consequences of Lemmas 3.3 and 3.5 are the following. First, as already remarked in Remark 3.2 above, \( \partial^0 f(x) = \emptyset \) if \( x \in \text{int} f^{-1}(0) \). Second, if \( f \) is differentiable at \( x \neq 0 \) in the boundary of \( f^{-1}(0) \) (and hence \( \nabla f(x) = 0 \)) we also have \( \partial^0 f(x) = \emptyset \). Indeed, by the same remark, were the equation in the definition to hold, we would need \( y \) to belong to the boundary of \((f^{-1}(0))^\circ\), and \( \langle x, y \rangle = 1 \). However, for such point we have that \( f^\circ(y) = +\infty \) since by definition

\[
f^\circ(y) \geq \frac{\langle x(1+\varepsilon), y \rangle - 1}{f(x(1+\varepsilon))} = \frac{\varepsilon \langle x, y \rangle}{o(\varepsilon)} \rightarrow +\infty.
\]

Finally, there is a close connection between smoothness of \( f \) and differentiability of \( f^\circ \), similar to the one holding for Legendre transform.

Our main concern in sections below will be that if a function is both strongly convex and twice continuously differentiable, then so is \( f^\circ \). Most of this claim is proved in Section 4 where we derive a precise formula for the Hessian of \( f^\circ \). We shall need, however, a simpler claim regarding differentiability. To this end, we introduce the following class of functions:

**Definition 3.6.** Denote by \( S_1(\mathbb{R}^n) \) the class of \( f \in \text{Cvx}_0(\mathbb{R}^n) \) which attain only finite values, are continuously differentiable on \( \mathbb{R}^n \setminus \{0\} \) and are strictly convex (that is, their graph does not contain any segment).

Note that these functions vanish only at the origin.

**Proposition 3.7.** Assume \( f \in S_1(\mathbb{R}^n) \). Then
1. \( \text{dom}(f^\circ) = \mathbb{R}^n \), \( f^\circ \) vanishes only at the origin.
2. \( f \) is polar differentiable at any point \( x \neq 0 \).
3. \( f^\circ \) is differentiable at any point \( y \neq 0 \) such that \( f^\circ[0,y] \) is not linear.
4. \( f^\circ \) is strictly convex at any point \( y \) such that \( f^\circ[0,y] \) is not linear.
Proof. Clearly \( \text{dom}(f^\circ) = \{f = 0\}^\circ = \mathbb{R}^n \) and that \( \{f^\circ = \} = \text{dom}(f)^\circ = \{0\} \).

If \( f \) is strictly convex and differentiable at \( x \in \text{int}(\text{dom}(f)) \), then \( f \) is polar differentiable at \( x \). Indeed, by strict convexity of \( f \) it cannot linear on \([0, x] \), which means \( f^* (\nabla f(x)) \neq 0 \). Thus by Lemma 3.3 the set \( \partial^o f(x) \) is non empty, whereas by Lemma 4.3 it consists of at most one point.

To show differentiability of \( f^\circ \), we consider the intermediate function \( f^{\circ\circ} \). By Lemma 2.4

\[
\text{int}(\text{epi}(f^{\circ \circ})) = F(\text{int}(\text{epi}(f))).
\]

and in particular (as \( F \) is continuous) there can be no segments on the graph of \( f^{\circ \circ} \) outside the set \( \{(x, 0): f^{\circ \circ}(x) = 0\} \). This means that \( f^{\circ \circ} \) is strictly convex at any point \( x \) with \( f^{\circ \circ}(x) \neq 0 \).

Therefore \( f^\circ = (f^{\circ \circ})^* \) is differentiable at any point \( y \) such that \( y = \nabla f^{\circ \circ}(x) \) for some \( x \) with \( f^{\circ \circ}(x) \neq 0 \). This amounts precisely to \( f^\circ \) not being linear on \([0, y] \).

To get strict convexity of \( f^\circ \) we use that \( f^{\circ \circ} \) is differentiable at any point with \( f^{\circ \circ}(x) \neq 0 \). Indeed, this follows from the remark after Lemma 2.4 as any supporting \((n - 1)\) dimensional region of \( f \) is mapped via \( F \) to a supporting \((n - 1)\) dimensional region of \( f^{\circ \circ} \) and vice versa. This means there is precisely one subgradient to \( f^{\circ \circ} \) at any such point, and by Legendre theory there are no two points \( y_1 \neq y_2 \) such that \( Pf|_{[0, y_i]} \) is non-linear, in which \( f^\circ \) has the same gradient. That is, outside the “ray-linearity zone” of \( f^\circ \), it is strictly convex. \( \square \)

4 Second order differentiability

In this section we explain the relation between the Hessian of \( f \) and the Hessian of \( f^\circ \). We shall mainly work in the following class of functions.

**Definition 4.1.** Denote by \( S_2(\mathbb{R}^n) \subset S_1(\mathbb{R}^n) \) the class of twice continuously differentiable in \( \mathbb{R}^n \setminus \{0\} \) such that \( \nabla^2 f(x) > 0 \) for all \( x \neq 0 \).

By Proposition 2.4 such functions are polar differentiable in \( x \neq 0 \). In the following proposition we derive a precise formula for the Hessian of \( f^\circ \) in terms of \( \nabla^2 f \), at those points for which one can be sure \( f^\circ \) is twice differentiable. \( \square \)

**Proposition 4.2.** Assume \( f \in S_2(\mathbb{R}^n) \). Let \( x \in \mathbb{R}^n \setminus \{0\} \) and \( y = \nabla^o f(x) \). If \( f^o|_{[0, y]} \) is not linear then \( f^\circ \) is twice differentiable at \( y \), \( \nabla^2 f^\circ(y) > 0 \) and we have

\[
(f(x)f^\circ(y))^2 (\nabla^2 f^\circ)(y) = (f(x)I - x\nabla f(x))(\nabla^2 f(x))^{-1} (f^\circ(y)I - (\nabla f^\circ(y)y)^T),
\]

and

\[
(\nabla^2 f^\circ(y))^{-1} = f(x)f^\circ(y)(I - xy)^T \nabla^2 f(x) (I - xy).
\]

In particular,

\[
det(\nabla^2 f(x)) \det(\nabla^2 f^\circ(y)) = \frac{1}{(f(x)f^\circ(y))^{n+2}}.
\]

\[\text{1}\) After presenting the results from this article in the Cortona Convex Geometry Conference in June 2011, we were informed by X.-N. Ma that (13) was also obtained independently by H.-Y. Jian, X.-J. Wang.

\[\text{2}\) We regard vectors \( x \in \text{dom}(f) \) as a column vector, \( y \in \text{dom}(f^\circ) \) as row vectors, and the various differentials accordingly. For example, the differential of \( f^\circ \), which is a function of \( y \), is a matrix that is to be multiplied with vectors \( v \in T_y(\mathbb{R}^n)^* \) from the right. When taking the differential of a function \( G: X \to Y \) where points of \( X \) are considered as column vectors and points of \( Y \) as row vectors, we let \( DG(x) \) act on \( w \in T_x \mathbb{R}^n \) by \((DGw)^T\), and similarly if \( H: Y \to X \) (e.g., the kind of map \( \nabla^o f^\circ \) is) we let \( DH \) act on a vector \( v \) by \((vDH)^T\).
\textbf{Proof.} By equation (13), the domain of \( f^\circ \) is \( \mathbb{R}^n \) and it vanishes only at 0. By Proposition 3.7, the function \( f^\circ \) is differentiable at \( y \). Thus \( \nabla f^\circ(y) = x \). By Lemma 3.5:

\begin{equation}
(16) \quad \nabla f^\circ(y) = \frac{x}{f(x)}, \quad \text{and} \quad \nabla f(x) = \frac{y}{f^\circ(y)} .
\end{equation}

The second equation implies \( x = \nabla f^\circ(y) \) is differentiable with respect to \( y \) and then the first that \( f^\circ \) is twice differentiable.

Differentiating the second identity of (16) gives

\[ \nabla \nabla f^\circ(y)(\nabla^2 f)(x) = \frac{1}{f^\circ(y)^2} \left[f^\circ(y)I - \nabla f^\circ(y) \right], \]

where we denoted the differential of the map \( x(y) = \nabla f^\circ(y) \) by \( \nabla \nabla f^\circ(y) \). Recall that for \( \langle w, z \rangle \neq 1 \) one has

\[ (I - zw^T)^{-1} = I + \frac{zw^T}{1 - \langle w, z \rangle}, \]

so that as \( \nabla f^\circ(y)/f^\circ(y) = x \) and \( \langle x, y \rangle \neq 1 \) (that is implied by \( f(x) \neq 0 \)), \( f^\circ(y)I - \nabla f^\circ(y) y \) is invertible. As \( f \) is strongly convex it follows that \( \nabla \nabla f^\circ(y) \) is positive-definite. As \( \nabla^\circ f \circ \nabla^\circ f = Id \) the inverse function theorem implies that \( \nabla^\circ f \) is differentiable at \( y \) and that \( \nabla \nabla f^\circ(y)(\nabla \nabla f(x))^T = Id \). Thus, \( y = \nabla f^\circ(x) \) is differentiable in \( x \), and the first identity of (16) gives

\[ \nabla \nabla f(x) = \frac{1}{f^\circ(x)^2} (\nabla^2 f^\circ(y))^{-1} [f(x)I - x \nabla f(x)]. \]

which, after simplification, proves (13). Similarly, \( f(x)I - x \nabla f(x) \) is invertible. We re-write, using Lemma 3.5, (13) as

\[ f(x)f^\circ(y)(\nabla^2 f^\circ)(y) = \left(I - \frac{x}{f(x)} \frac{y}{f^\circ(y)} \right) (\nabla^2 f(x))^{-1} \left(I - \left(\frac{x}{f(x)} \frac{y}{f^\circ(y)} \right)^T \right). \]

We invert as above, using that \( \langle x, y \rangle - f(x)f^\circ(y) = 1 \), to get

\[ (I - \frac{x}{f(x)} \frac{y}{f^\circ(y)})^{-1} = I - xy, \]

since

Thus, (14) follows. Finally, \( \det(I - xy) = 1 - \langle x, y \rangle = -f(x)f^\circ(y) \). Thus, (15) follows by taking determinants in (14).

One may readily derive similar formulas relating the Hessians of \( f \) at \( x \) and \( f^\circ \) at \( x/f(x) \) under appropriate regularity assumptions, for example:

\[ \nabla^2 f^\circ \left( \frac{x}{f(x)} \right) = (\nabla^2 f^\circ)^{-1}(y) = f(x)f^\circ(y)(I - xy)^T \nabla^2 f(x)(I - xy), \]

where \( \nabla^\circ f(x) = y \). We omit the calculations.

5 Variation of polarity

In this section we consider one-parameter families \( \{f_t(x)\}_{t \in \mathbb{R}} \) of convex functions.
5.1 First variation

The well-known first variation formula for the Legendre transform is:

**Proposition 5.1.** Let \( u(t,x) \in C^2(\mathbb{R} \times \mathbb{R}^n) \) with \( u_t(\cdot) = u(t,\cdot) \in \mathcal{S}_2 \) for each \( t \). Denote by \( w(t,y) \) the Legendre transforms of \( u_t(x) = u(t,x) \) in the space variable, that is, \( w(t,y) = \sup_{x \in \mathbb{R}^n} \left[ (y,x) - u(t,x) \right] \). Then,

\[
\frac{\partial w}{\partial t}(t,y) = -\frac{\partial u}{\partial t}(t,(\nabla u_t)^{-1}(y)).
\]

For the proof, take a variation in \( t \) of \( w(t,y) \), with \( x = x(t,y) = (\nabla_x u)^{-1}(y) \),

\[
\frac{\partial w}{\partial t} \bigg|_y = \frac{dw}{dt} \bigg|_y = \sum_{j=1}^n \frac{\partial x_j}{\partial t} - \frac{\partial u}{\partial t} \bigg|_x - \sum_{j=1}^n \frac{\partial u}{\partial x_j} \frac{\partial x_j}{\partial t} = -\frac{\partial u}{\partial t} \bigg|_x,
\]

since \( \nabla u(t,x) = y \).

The corresponding result for polarity is the following.\(^3\)

**Proposition 5.2.** (First variation of polarity) Let \( u(t,x) \in C^2(\mathbb{R} \times (\mathbb{R}^n \setminus \{0\})) \) with \( u_t(\cdot) = u(t,\cdot) \in \mathcal{S}_2 \) for each \( t \). Denote by \( w_t = w(t,\cdot) = u_t^\circ \) the fiberwise polar. Then for any \( t \), and any \( y \) such that \( u_t \) is not linear on \( [0,y] \),

\[
\left( \frac{\partial}{\partial t} \log w \right) (t,y) = -\left( \frac{\partial}{\partial t} \log u \right) (t,\nabla^o w(t,y)).
\]

**Proof.** Let \( y \in \text{Im}\, \partial^o u_t \). Proposition \( \text{5.7} \) implies that for every \( t \) and every \( y \) such that \( w_t \) is not linear on \( [0,y] \), \( w_t \) is polar differentiable at \( y \). Denote \( x(t,y) = \nabla^o w(t,y) \). Since \( u_t \) is differentiable, by Lemma \( \text{3.5} \) \( y = \nabla^o u_t(x) \), and

\[
w(t,y)^{-1} = \langle x(t,y), \nabla u(t,x(t,y)) \rangle - u(t,x(t,y)).
\]

Differentiating with respect to \( t \) gives

\[
\frac{1}{w^2(t,y)} \frac{\partial}{\partial t} w(t,y) = -\frac{\partial u}{\partial t}(t,x(t,y)) + \left\langle x(t,y), \left( \nabla \frac{\partial}{\partial t} u \right)(t,x(t,y)) \right\rangle
\]

\[
+ \left\langle x(t,y), \nabla^2 u(t,x(t,y)) \frac{\partial x}{\partial t}(t,y) \right\rangle
\]

(where two terms have cancelled). By Lemma \( \text{3.3} \) \( x(t,y) = (\nabla u_t)^{-1}(y/w(t,y)) \), therefore

\[
\frac{\partial}{\partial t} x(t,y) = \left( \frac{\partial}{\partial t} (\nabla u_t)^{-1} \right) \left( \frac{y}{w(t,y)} \right) + (\nabla_y (\nabla u_t)^{-1})(y/w(t,y)) \frac{\partial}{\partial t} \left( \frac{y}{w(t,y)} \right).
\]

By the chain rule, \((\nabla_y (\nabla u_t)^{-1})(y/w(t,y)) = (\nabla^2 u_t)^{-1}(x(t,y)) \). Thus the last term in (18) equals

\[
\left\langle x(t,y), \nabla^2 u(t,x(t,y)) \left( \frac{\partial}{\partial t} (\nabla u_t)^{-1} \right) \left( \frac{y}{w(t,y)} \right) \right\rangle + \left\langle x(t,y), \frac{\partial}{\partial t} \left( \frac{y}{w(t,y)} \right) \right\rangle.
\]

\(^3\)We shall use \( \nabla \) and \( \nabla^o \) to denote differentiation and polar differentiation with respect to the space variables.
Plugging everything back into the original equation yields

$$\begin{align*}
- \frac{1}{w^2(t, y)} \frac{\partial}{\partial t} w(t, y) &= \left\langle x(t, y), (\nabla \frac{\partial u}{\partial t})(t, x(t, y)) \right\rangle \\
&\quad + \left\langle x(t, y), (\nabla^2 u(t, x(t, y)))(\frac{\partial}{\partial t}(\nabla u_t)^{-1})(\frac{y}{w(t, y)}) \right\rangle \\
&\quad - \left\langle x(t, y), \frac{y}{w^2(t, y)} \frac{\partial w}{\partial t}(t, y) \right\rangle - \frac{\partial u}{\partial t}(t, x(t, y)),
\end{align*}$$

or,

$$\begin{align*}
\frac{\langle x(t, y), y \rangle - 1}{w^2(t, y)} \frac{\partial}{\partial t} w(t, y) &= \left\langle x(t, y), (\nabla \frac{\partial u}{\partial t})(t, x(t, y)) \right\rangle - \frac{\partial u}{\partial t}(t, x(t, y)) \\
&\quad + \left\langle x(t, y), \nabla^2 u(t, x(t, y)) \left( \frac{\partial}{\partial t}(\nabla u_t)^{-1} \right) \left( \frac{y}{w(t, y)} \right) \right\rangle.
\end{align*}$$

(19)

Since $$(\nabla u_t)^{-1}(\nabla u_t(x)) = x$$

$$\left( \frac{\partial}{\partial t}(\nabla u_t)^{-1} \right)(\nabla u_t(x)) + \langle \nabla(\nabla u_t)^{-1}(\nabla u_t(x)), \frac{\partial}{\partial t}(\nabla u_t) \rangle = 0$$

or,

$$\left( \frac{\partial}{\partial t}(\nabla u_t)^{-1} \right)(\nabla u_t(x)) = -\nabla^2 u_t(x)^{-1} \frac{\partial}{\partial t} \nabla u_t(x).$$

Using that $$\nabla u_t(x(t, y)) = \frac{y}{w(t, y)},$$ the first and third term on the right hand side of (19) cancel. The result now follows since $$\langle x(t, y), y \rangle - 1 = u(t, x(t, y))w(t, y).$$

One may readily combine Propositions 5.1 and 5.2 to get a similar formula for the first variation of $$f^\circ \circ.$$ Under the appropriate regularity condition it reads

$$\frac{\partial w}{\partial t}(t, x, u_t(x)) = \frac{1}{u_t(x)u_t^*(\nabla u_t(x))} \frac{\partial u}{\partial t}(t, x),$$

where $$w(t, y) = u_t^*(y).$$

### 5.2 Second variation

We recall the well-known formula for the second variation of the Legendre transform. Its proof follows immediately upon differentiating (17) (see, e.g., [6, p. 87]).

**Proposition 5.3.** Let $$u(t, x) \in C^2(\mathbb{R} \times \mathbb{R}^n)$$ with $$u_t(\cdot) = u(t, \cdot) \in S_2$$ for each $$t.$$ Let $$w(t, y) = u_t^*(y).$$ Then

$$\frac{\partial^2 w}{\partial t^2}(t, y) = - \frac{\partial^2 u}{\partial t^2}(t, (\nabla u_t)^{-1}(y)) - \left\langle \nabla \frac{\partial u}{\partial t}(t, (\nabla u_t)^{-1}(y)), \nabla \frac{\partial w}{\partial t}(t, y) \right\rangle,$$

or equivalently

$$\frac{\partial^2 w}{\partial t^2}(t, y) = - \frac{\partial^2 u}{\partial t^2} + \left\langle \nabla \frac{\partial u}{\partial t}(t, (\nabla^2 u_t)^{-1} \nabla \frac{\partial u}{\partial t}) \right\rangle,$$

where the right hand side is evaluated at $$t, (\nabla u_t)^{-1}(y)).$$
For polarity we have:

**Theorem 5.4.** (Second variation of polarity) Let $u(t, x) \in C^2(\mathbb{R} \times (\mathbb{R}^n \setminus \{0\}))$ with $u_t(\cdot) = u(t, \cdot) \in S_2$ for each $t$. Denote by $w_t = w(t, \cdot) = u_t^\circ$ the fiberwise polar. Then for every $t$ and $y$ such that $w_t$ is not linear on $[0, y]$ we have

$$
(22) \quad \left( \frac{\partial^2}{\partial t^2} \log w \right)(t, y) = -\left( \frac{\partial^2}{\partial t^2} \log u \right) + u \left\langle \nabla \frac{\partial}{\partial t}(\log u), (\nabla^2 u)^{-1}\nabla \frac{\partial}{\partial t}(\log u) \right\rangle,
$$

where the right hand side is evaluated at $(t, \nabla^\circ w_t(y))$. Equivalently,

$$
(23) \quad \frac{\bar{w}}{w}(t, y) = -\frac{1}{w} \det \left( \begin{array}{cc} -u^2(\tilde{u}/u) & u\nabla(\tilde{u}/u) \\ u\nabla(\tilde{u}/u)^T & \nabla^2 u \end{array} \right) \bigg|_{(t, \nabla^\circ w(t)(y))}.
$$

**Proof.** We differentiate the first variation formula (Proposition 5.2) proving (22). Equation (23) follows from this and Proposition 5.2.

We omit the computations.

**Remark 5.5.** Note that the last term can be expressed more symmetrically as follows:

$$
u \left\langle \nabla \frac{\partial}{\partial t}(\log u), (\nabla^2 u)^{-1}\nabla \frac{\partial}{\partial t}(\log u) \right\rangle = uw \left\langle \nabla \frac{\partial}{\partial t}(\log u), \nabla \frac{\partial}{\partial t}(\log w)(I - (\nabla u)^T \cdot \nabla w)^{-1} \right\rangle.
$$

We omit the computations.

### 6 First order equations

The first order analysis enables us to linearize a family of first order PDEs, analogous to the linearization of the Hamilton–Jacobi equation by the Legendre transform. Define the operation
This is shown to be a sort of geometric inf-convolution in §8 where a precise formula is derived.

**Theorem 6.1.** Let \( g \in \text{Cvx}_0(\mathbb{R}^n) \cap C^2(\mathbb{R}^n \setminus \{0\}) \). Let \( f \in S_2 \) and non-linear at infinity. Then the equation

\[
\frac{1}{u} \frac{\partial u}{\partial t} + u^* (\nabla u) g \left( \frac{\nabla u}{u^* (\nabla u)} \right) = 0, \quad u(0, x) = f(x),
\]

admits a unique non-linear at infinity solution \( u(t) \in S_2 \) given by

\[
u(t, x) = f \boxplus \frac{1}{t} g^\circ.
\]

In particular, there exists a solution for all time \( t \geq 0 \).

A similar result holds for the Dirichlet problem with convex data.

**Proof.** Note that by Lemma 2.7 the function \( f^\circ \) has an empty linearity zone, and therefore so does \( f^\circ + tg \) for any \( t \geq 0 \). We may thus apply Proposition 5.2, which implies that the function \( (f^\circ + tg)^\circ \) satisfies our original equation. 

As an application, an equation reminiscent of Burgers’ equation,

\[
\frac{\partial u}{\partial t}(t, x) + \| \nabla u(t, x) \| u(t, x) = 0, \quad u(0, x) = f(x),
\]

can be solved for all \( t \geq 0 \), with

\[
u(t) \equiv u(t, \cdot) = f \boxplus \frac{1}{t} \| \cdot \|^* = (f^\circ + t\| \cdot \|)^\circ,
\]

where the polarity operation is performed with respect to the variable \( x \) only. Here \( \| \cdot \|^* \) denotes the norm dual to \( \| \cdot \| \). If \( f \) is a norm then so is \( u(t) \) for each \( t \).

Similarly, a solution of

\[
\frac{\partial}{\partial t} \log u(t, x) + \frac{1}{2u^* (\nabla u)} |\nabla u|^2 = 0, \quad u(0, x) = f(x),
\]

is given by

\[
u(t) \equiv u(t, \cdot) = f \boxplus \frac{1}{2t} \| \cdot \|^2 = (f^\circ + \frac{t}{2} | \cdot |^2)^\circ.
\]

We remark that analogously there are PDEs of first and second order linearized by the transform \( P \circ L \), and we omit the calculations for brevity.

**7 The polar Monge–Ampère equation**

To put our results in this section in perspective, it is good to keep in mind the classical result that the partial Legendre transform linearizes the homogeneous real Monge–Ampère (HRMA), written schematically as \( \det \nabla^2 f = 0 \) [7]. This is contained in Proposition 5.3 if \( u(t) \in \text{SCvx} \cap C^\infty \) for each \( t \) then \( \det \nabla^2 u = \det \nabla^2 u(\ddot{u} - |\nabla u|^2 \nabla^2 u)^{-1} = 0 \) if and only if \( \ddot{u}(t)^* = 0 \), where \( u(t)^* \) denotes the Legendre transform of \( u(t, \cdot) \) in the \( x \) variables.
The following is a consequence of Theorem 5.4. We denote $|X|_g^2 := g(X, X)$ for any semi-Riemannian metric $g$.

**Theorem 7.1.** Let $u_0, u_T \in \mathcal{S}_2$ and non-linear at infinity. The Dirichlet problem

$$
(1/u) + |\nabla (\dot{u}/u)|^{2}_{(\nabla^2 u)^{-1}} = 0, \quad u(0, \cdot) = u_0, \quad u(T, \cdot) = u_T,
$$

admits a unique non-linear at infinity solution $u(t) \in \mathcal{S}_2$ given by

$$
u(t, x) = \left(1 - \frac{t}{T}\right) u_0 + \frac{t}{T} u_1 \right)^{\circ} (x) = \left(\frac{T u_0 - \nabla u_1}{T - t}ight)(x).
$$

We call (26) the homogeneous polar Monge–Ampère equation. Similarly, the solution to the Cauchy problem for the homogeneous polar Monge–Ampère equation follows by combining Proposition 5.2 and Theorem 5.4.

**Theorem 7.2.** Let $u_0 \in \mathcal{S}_2$ and non-linear at infinity, and let $\dot{u}_0 \in C^2(\mathbb{R}^n)$ satisfy $\dot{u}_0(0) = 0$. The Cauchy problem

$$
(1/u) + |\nabla (\dot{u}/u)|^{2}_{(\nabla^2 u)^{-1}} = 0, \quad u(0, \cdot) = u_0, \quad \dot{u}(0, \cdot) = \dot{u}_0,
$$

admits a unique non-linear at infinity solution $u(t) \in \mathcal{S}_2$ given by

$$
u(t, x) = \left(u_0 \cdot (1 - tv)\right)^{\circ} (x), \quad t \in [0, T),
$$

where $v(0) = 0$ and

$$v(y) = \frac{\dot{u}_0(\nabla^{\circ} u_0(y))}{u_0(\nabla^{\circ} u_0(y))}, \quad y \neq 0,$

and where $t \in [0, T)$ with $T = T(u_0, \dot{u}_0)$ the supremum over all $t > 0$ such that the function $u_0 \cdot (1 - tv) \in \mathcal{S}_2$ and is nonlinear at infinity.

8 Geometric inf-convolution

In this section we derive an explicit formula for the the solutions to the PDEs presented in the preceding sections. We refer to

$$f \square g = (f^o + g^o)^o.$$

as geometric inf-convolution of $f$ and $g$. The next lemma justifies this name. It gives a formula for such as expression, reminiscent of the formula for inf-convolution [5, p. 33]

$$f \square g(x) = \inf_{y+z=x} (f(y) + g(z)) = (f^* + g^*)^*.$$

**Lemma 8.1.** For $f, g \in \text{Cvx}_0(\mathbb{R}^n)$ that vanish only at the origin, $f \square g \in \text{Cvx}_0(\mathbb{R}^n)$ and

$$(f \square g)(x) = \inf \left\{ (f(y)^{-1} + g(z)^{-1})^{-1} : \frac{x-y}{f(y)} = \frac{z-x}{g(z)} \right\}.$$
check that \( h \) is geometric convex.

\[
(f^o + g^o)(x) = \sup_{y,z \in \mathbb{R}^n} \left( \frac{(x, y) - 1}{f(y)} + \frac{(x, z) - 1}{g(z)} \right)
\]

\[= \sup_{y,z \in \mathbb{R}^n} \left( \frac{(x, g(z)y + f(y)z) - 1}{f(y)g(z)(f(y) + g(z))} \right)\]

\[= \sup_{w \in \mathbb{R}^n} \left( \inf_{y,z \in \mathbb{R}^n : w = g(z)y + f(y)z} \left( \frac{(f(y) - 1)(g(z) - 1)}{f(y) + g(z)} \right) \right)
\]

Letting

\[h(w) = \inf\{(f(y) - 1)(g(z) - 1) : y, z \in \mathbb{R}^n, \ w = \frac{g(z)y + f(y)z}{f(y) + g(z)}\},\]

we see that the last expression is \( h^o(x) \). Then rearrange \( w = \frac{g(z)y + f(y)z}{f(y) + g(z)} \) as \( w - y = \frac{w - y}{g(z)} \) or \((w - y)g(z) = (z - w)f(y)\).

It remains to verify that the resulting function is geometric convex. Denote

\[K_\varphi = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^+ : \varphi(x/y) \leq 1\}.
\]

Then \( \mathbb{K}_f \boxtimes g = \mathbb{K}_f + \mathbb{K}_g \).

Write

\[K_f + K_g = \{(z, y) : x = z_1 + z_2, y = y_1 + y_2, y_1 f(z_1/y_1) \leq 1, y_2 g(z_2/y_2) \leq 1\}
\]

\[= \{(x, y, z) : x = \frac{x_1y_1 + x_2y_2}{y_1 + y_2}, y = y_1 + y_2, y_1 f(x_1) \leq 1, y_2 g(x_2) \leq 1\}.
\]

Thus, \( K_h = K_f + K_g \), and

\[h(x) = \|(x, 1)\|_{K_h} = \inf\{1/y : (x, y, z) \in K_h\} = \inf\{1/y : (x, y) \in K_f + K_g\}.
\]

Therefore

\[(f \boxtimes g)(x) = \inf\{\frac{1}{y_1 + y_2} : x_1y_1 + x_2y_2, y_1 f(x_1) \leq 1, y_2 g(x_2) \leq 1\}.
\]

In the strictly convex case, the boundary of \( K_h \) is a subset of the closure of the Minkowski sum of the boundaries of the sets \( K_f \) and \( K_g \), which means that we can without loss of generality assume in the infimum above \( y_1 = 1/f(x_1) \) and \( y_2 = 1/f(x_2) \). We end up with

\[(f \boxtimes g)(x) = \inf\{\frac{1}{f(x_1) - 1 + f(x_2) - 1} : x = \frac{x_1(f(x_1) - 1) + x_2(f(x_2) - 1)}{(f(x_1) - 1 + (f(x_2) - 1)}\},
\]

Rearranging, the result follows.

Next we present a formula for the polar gradient of the function \( f + g \) at a point \( x \).

Lemma 8.2. Let \( f, g \in \text{Cvx}_0 \mathbb{R}^n \) with \( \text{dom}(f) = \text{dom}(g) = \mathbb{R}^n \). Assume both are polar differe-
Using Lemma 8.1 we have that
\[ \nabla^o(f + g)(x) = \left( \frac{g^o(\nabla^o g(x))}{f^o(\nabla^o f(x)) + g^o(\nabla^o g(x))} \right) \nabla^o f(x) + \left( \frac{f^o(\nabla^o f(x))}{f^o(\nabla^o f(x)) + g^o(\nabla^o g(x))} \right) \nabla^o g(x). \]

Note that \( \nabla^o(tf)(x) = \nabla^o(f)(x) \) by the definition of the polar gradient. Thus we get the formula
\[ \nabla^o(f + tg) = \left( \frac{g^o(\nabla^o g(x))}{tf^o(\nabla^o f(x)) + g^o(\nabla^o g(x))} \right) \nabla^o f(x) + \left( \frac{tf^o(\nabla^o f(x))}{tf^o(\nabla^o f(x)) + g^o(\nabla^o g(x))} \right) \nabla^o g(x). \]

Proof. As \( f \) and \( g \) are polar differentiable at \( x \), by Lemma 3.5 they are both differentiable at \( x \), hence so is \( f + g \). Denote \( y_1 = \nabla^o f(x) \), \( y_2 = \nabla^o g(x) \) and
\[ z = y_1 \frac{g^o(y_2)}{f^o(y_1) + g^o(y_2)} + y_2 \frac{f^o(y_1)}{f^o(y_1) + g^o(y_2)}. \]
We shall show that \( z \in \partial^o(f + g)(x) \), and by Lemma 3.5 once again, get that \( z = \nabla^o(f + g)(x) \), as needed. By definition
\[ f(x)f^o(y_1) = \langle x, y_1 \rangle - 1, \quad g(x)g^o(y_2) = \langle x, y_2 \rangle - 1. \]
Since \( x \in \text{int}(\text{dom}(f) \cap \text{dom}(g)) = \mathbb{R}^n \) we have that \( f^o(y_1)g^o(y_2) \neq 0 \). Thus
\[ f(x) + g(x) = \langle x, \frac{y_1}{f^o(y_1)} + \frac{y_2}{g^o(y_2)} \rangle - \left( \frac{1}{f^o(y_1)} + \frac{1}{g^o(y_2)} \right). \]
Rearrange to get
\[ (f(x) + g(x)) \left( f^o(y_1)^{-1} + g^o(y_2)^{-1} \right)^{-1} = \langle x, z \rangle - 1. \]
Using Lemma 8.1 we have that
\[ (f + g)^o(z) \leq (f^o(y_1)^{-1} + g^o(y_2)^{-1})^{-1} \]
so that
\[ (f + g)(x)(f + g)^o(z) \leq \langle x, z \rangle - 1. \]
The opposite inequality holds by the definition of \( (fg)^o \), so we get equality, which means \( z \in \partial^o(f + g)(x) \), as claimed. \( \Box \)

Note that in the proof we obtained the formula
\[ (f + g)^o((1 - \lambda)\nabla^o f(x) + \lambda \nabla^o g(x)) = (f^o(\nabla^o f(x))^{-1} + g^o(\nabla^o g(x))^{-1})^{-1} \]
for \( \lambda = \frac{f^o(\nabla^o f(x))}{f^o(\nabla^o f(x)) + g^o(\nabla^o g(x))} \), which is similar to a corresponding formula for inf-convolution.

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