We investigate an equivariant generalization of Morse theory for a general class of integrable models. In particular, we derive equivariant versions of the classical Poincaré-Hopf and Gauss-Bonnet-Chern theorems and present the corresponding path integral generalizations. Our approach is based on equivariant cohomology and localization techniques, and is closely related to the formalism developed by Matthai and Quillen in their approach to Gaussian shaped Thom forms.
1. Introduction

Equivariant cohomology is presently attracting much interest both in Physics and Mathematics. This is largely due to its relevance to various localization formulas originally introduced by Duistermaat and Heckman, and its connections to cohomological topological field theories. More recently, equivariant cohomology has also found applications in the investigation of $W$-gravity and the formalism is also relevant in a geometric loop space approach to Poincaré supersymmetric theories.

The original localization formula by Duistermaat and Heckman [1], [2] (for a review, see [3]) concerns exponential integrals over a $2n$ dimensional symplectic manifold $M$ i.e. classical partition functions of the form

$$Z = \frac{1}{(2\pi)^n} (i\phi)^n \int \frac{1}{n!} \omega^n e^{i\phi H} = \frac{1}{(2\pi)^n} \int \exp\{i\phi(H + \omega)\}$$  \hspace{1cm} (1)

where $\omega$ is the symplectic two-form. If the Hamiltonian $H$ determines a global symplectic action of a circle $S^1 \sim U(1)$ - or more generally the global action of a torus - on the manifold with isolated and nondegenerate fixed points $p$, the integral (1) localizes to these points,

$$Z = \left(\frac{2\pi}{i\phi}\right)^n \sum_{dH=0} \exp\left\{\frac{\pi}{4} \eta_H \right\} \frac{\sqrt{\det|\omega_{\mu\nu}|}}{\sqrt{\det|\partial_{\mu\nu}H|}} \exp\{i\phi H\}$$  \hspace{1cm} (2)

Here $\eta_H$ is the $\eta$-invariant of the matrix $\partial_{\mu\nu}H$ when viewed as a linear operator on $TM_p$, i.e. if we denote the dimensions of the eigenspaces of the matrix $\partial_{\mu\nu}H$ at a (non-degenerate) critical point $p$ by $\dim T^+_p$ and $\dim T^-_p$,

$$\eta_H = \dim T^+_p - \dim T^-_p$$

As explained by Berline and Vergne [4], the integration formula (2) can be interpreted in the context of equivariant cohomology. In the case of a circle action, the pertinent equivariant cohomology is described by an equivariant exterior derivative

$$d_H = d + i_H$$
where \( i_H \) is the (nilpotent) contraction operator along the Hamiltonian vector field of \( H \). The operator \( d_H \) squares to the Lie derivative of the circle action,

\[
\mathcal{L}_H = di_H + i_H d
\]

which implies that it is nilpotent on the subcomplex \( \Lambda_H \) of \( U(1) \) invariant exterior forms

\[
\mathcal{L}_H \Lambda_H = 0
\]

Furthermore, since

\[
d_H(H + \omega) = 0
\]

we conclude that \( H + \omega \) can be viewed as an equivariant extension of the symplectic two-form \( \omega \). In particular, the integrand in (1) is equivariantly closed and the integration formula (2) can be seen as a consequence of an equivariant version of Stokes theorem.

An infinite dimensional generalization of the Duistermaat-Heckman formula was presented by Atiyah and Witten \[5\]. They considered loop space equivariant cohomology described by the loop space differential operator

\[
Q \dot{z} = d + i_{\dot{z}}
\]

and they were interested in evaluating a supersymmetric path integral that describes the Atiyah-Singer index of a Dirac operator on a Riemannian manifold. The crucial idea in their work is that the fermionic bilinear in the supersymmetric action can be interpreted as a loop space symplectic two-form, and integration over the fermions yields the loop space Liouville measure. Their approach was generalized by Bismut \[6\] to twisted operators, and to the computation of the Lefschetz number of a Killing vector field acting on the manifold.

In \[7\], \[8\] the action of four dimensional topological Yang-Mills theory was interpreted in terms of equivariant cohomology and Weil algebra. In \[9\] Atiyah and Jeffrey gave an interpretation of its partition function as a regularized Euler class on a vector bundle associated to the bundle \( \mathcal{A} \to \mathcal{A}/\mathcal{G} \) of connections on a principal bundle. This interpretation of cohomological topological field theories is an infinite dimensional generalization \[9\], \[10\] of the formalism developed earlier.
by Matthai and Quillen [11], who explained how representatives of the Thom class
of vector bundles can be constructed using equivariant cohomology. They also
applied this formalism to establish a direct connection between the Poincaré-Hopf
and Gauss-Bonnet-Chern theorems in classical Morse theory using localization
methods.

In a series of papers [12]-[19] the quantum mechanics of circle actions of isome-
tries on symplectic manifolds was considered using a different method of loop space
localization. The derivation of the twisted Atiyah-Singer index theorem and its
generalizations was also considered in this formalism [14], [15] and the ensuing
equivariant cohomology and loop space symplectic geometry was applied to for-
mulate general Poincaré-supersymmetric quantum field theories in a geometrical
framework [14], [16].

In the present paper we shall be interested in applying the equivariant coho-
mology and localization techniques developed in [12]-[19] to investigate certain
geometrical aspects of quantum integrability, for the general class of Hamiltonians
that determine the global action of a circle on the phase space. In particular, we
explain how loop space equivariant cohomology can be used to construct novel
quantum mechanical partition functions that are based on this family of Hamilto-
nians. We show how these partition functions can be evaluated exactly by local-
ization methods. Our final results are integrals of equivariant characteristic classes
over a moduli space which describes the classical configurations of the underlying
dynamical system.

Our main results can be viewed as an equivariant version of classical Morse
theory. In particular, we explain how loop space equivariant generalizations of
the Poincaré-Hopf and Gauss-Bonnet-Chern theorems can be derived, with the
equivariance determined by the Hamiltonian dynamics of $H$. Our formalism is
closely related to the Matthai-Quillen formalism, and in a sense it can be viewed
as an ”equivariantization” of their work.

In order to describe in more detail the results that we shall derive here, we first
recall some aspects of classical Morse theory (from the present point of view see
e.g. [3], [10], [20]). For this we consider critical points $p$ of a smooth function $F$ -
the Morse function - on a compact oriented 2n dimensional manifold $\mathcal{M}$,

$$dF_{|p} = 0$$

If these critical points $p \in \mathcal{M}$ of $F$ are isolated and non-degenerate, the Poincaré-Hopf theorem states that the Euler characteristic $\chi(\mathcal{M})$ of $\mathcal{M}$ is related to these critical points by

$$\chi(\mathcal{M}) = \sum_{i=0}^{2n} (-1)^i \dim H^i(\mathcal{M}; \mathbb{R}) = \sum_{dF = 0} \text{sign}(\det \left| \frac{\partial^2 F(p)}{\partial x^\mu \partial x^\nu} \right|)$$ \hspace{1cm} (3)

with $x^\mu$ local coordinates around $p$. In particular, (3) means that the sum over the critical points of $F$ is a topological invariant of the manifold, independently of the function $F$.

Here we shall be interested in generalizations of (3) to sums that are of the form

$$\sum_{dF = 0} \text{sign}(\det \left| \frac{\partial^2 F(p)}{\partial x^\mu \partial x^\nu} \right|) \cdot \exp\{i\phi F(p)\}$$ \hspace{1cm} (4)

where $\phi$ is a parameter. We shall find that such generalizations of (3) are also related to an invariant of the manifold $\mathcal{M}$ which is an equivariant version of the Euler class. Furthermore, we shall explain how sums like (4) arise in the loop space, and we also present the appropriate degenerate versions.

In section 2. we shall introduce some background material on equivariant cohomology. In section 3. we present a derivation of the Duistermaat-Heckman integration formula (2) and generalize it to the degenerate case. This derivation introduces techniques that we use in the subsequent sections to discuss more general localizations formulas. In section 4. we introduce the supersymmetric complex $S^*\mathcal{M}$ and explain how the Matthai-Quillen formalism follows. We also generalize this formalism by equivariantizing it with respect to a Hamiltonian $H$ that generates the action of a circle on the phase space. In section 5. we use our formalism to derive equivariant generalizations of the Poincaré-Hopf and Gauss-Bonnet-Chern theorems in the finite dimensional context. In section 6. we return to the Duistermaat-Heckman integration formula, by generalizing it to the loop space. The techniques we introduce in this section are then applied in the last two
sections to localize loop space integrals over the supersymmetric complex \( S^*M \). First, in section 7, we derive the Poincaré-Hopf and Gauss-Bonnet-Chern theorems for an arbitrary, non-gradient vector field. In section 8, we then extend these theorems to the equivariant context determined by a nontrivial Hamiltonian dynamics.

2. Equivariant cohomology on a symplectic manifold

In this section we introduce relevant concepts in equivariant cohomology. We shall be interested in a \( 2n \) dimensional compact symplectic manifold \( M \), with local coordinates \( x^\mu \) and Poisson bracket

\[
\{x^\mu, x^\nu\} = \omega^{\mu\nu}(x)
\]

Here \( \omega^{\mu\nu} \) is the inverse matrix to the closed symplectic two-form on \( M \),

\[
\omega = \frac{1}{2} \omega_{\mu\nu} dx^\mu \wedge dx^\nu \\
d\omega = 0
\]

so that locally we can introduce a one-form \( \vartheta \) called the symplectic potential such that

\[
\omega = d\vartheta
\]

We are interested in the equivariant cohomology \( H^*_G(M) \) associated with the symplectic action of a Lie group \( G \) on the manifold \( M \),

\[
G \times M \rightarrow M
\]

If the action of \( G \) is free i.e. the only element of \( G \) which acts trivially is the unit element, the coset \( M/G \) is well defined and the \( G \)-equivariant cohomology of \( M \) coincides with the ordinary cohomology of the coset,

\[
H^*_G(M) = H^*(M/G)
\]

In the case of non-free \( G \)-actions, more elaborated methods are needed to compute the equivariant cohomology. Three different approaches have been introduced to
model $H^*_G(\mathcal{M})$, using differential forms on $\mathcal{M}$ together with polynomial functions and forms on the Lie algebra $\mathfrak{g}$ of $G$. The two classical approaches are the Cartan and Weil models, and they are interpolated by the BRST model as described e.g. in [19], [21]. In the following we shall mainly need the Cartan model.

The $G$-action on $\mathcal{M}$ is generated by vector fields $\mathcal{X}_\alpha$, $\alpha = 1, ..., m$ that realize the commutation relations of the Lie-algebra $\mathfrak{g}$,

$$[\mathcal{X}_\alpha, \mathcal{X}_\beta] = f^{\alpha\beta\gamma} \mathcal{X}_\gamma$$

with $f^{\alpha\beta\gamma}$ the structure constants of $\mathfrak{g}$. With $\mathcal{X}$ a generic vector field on $\mathcal{M}$, we denote contraction along $\mathcal{X}$ by $i_\mathcal{X}$. In particular, the basis of contractions corresponding to the Lie algebra generators $\{\mathcal{X}_\alpha\}$ is denoted by $i_{\mathcal{X}_\alpha} \equiv i_\alpha$. The pertinent Lie-derivatives are

$$\mathcal{L}_\alpha = di_\alpha + i_\alpha d$$

where $d$ is the exterior derivative on the exterior algebra $\Lambda(\mathcal{M})$ of the manifold $\mathcal{M}$. They generate the $G$-action on $\Lambda(\mathcal{M})$,

$$[\mathcal{L}_\alpha, \mathcal{L}_\beta] = f^{\alpha\beta\gamma} \mathcal{L}_\gamma$$

We shall assume that the action of $G$ is symplectic so that

$$\mathcal{L}_\alpha \omega \equiv di_\alpha \omega = 0 \quad \text{for all } \alpha$$

Provided the one-forms $i_\alpha \omega$ are exact (for this the triviality of $H^1(\mathcal{M}, \mathbb{R})$ is sufficient), we can then introduce the corresponding momentum map

$$H_G : \mathcal{M} \to \mathfrak{g}^*$$

where $\mathfrak{g}^*$ is the dual Lie algebra. This yields a one-to-one correspondence between the vector fields $\mathcal{X}_\alpha$ (and corresponding Lie-derivatives $\mathcal{L}_\alpha$) and certain functions $H_\alpha$ on $\mathcal{M}$, the components of the momentum map

$$H_G \equiv \phi^\alpha H_\alpha$$

where $\{\phi^\alpha\}$ is a (symmetric) basis of the dual Lie algebra $\mathfrak{g}^*$, and

$$i_\alpha \omega = -dH_\alpha$$
or in local coordinates
\[ \mathcal{X}_\alpha = \omega^{\mu\nu} \partial_\mu H_\alpha \partial_\nu \]

In the sequel we shall only consider the simplest example of a group action on
the symplectic manifold \( \mathcal{M} \), the action of a circle \( \mathcal{G} = U(1) = S^1 \). It is determined
by a momentum map \( H \) corresponding to a Hamiltonian vector field \( \mathcal{X}_H \) as the
only generator of the Lie algebra \( u(1) \) of \( U(1) \),
\[ \mathcal{X}_H = \omega^{\mu\nu} \partial_\mu H \partial_\nu \]

We introduce the following equivariant exterior derivative operator on \( \mathcal{M} \),
\[ d_H = d + \phi i_H \quad (5) \]

Here the factor \( \phi \) is a real parameter that we interpret as a generator of the algebra
of polynomials on \( u(1) \) i.e. as a generator of the symmetric algebra \( S(u(1)^*) \) over
the dual of the Lie algebra of \( U(1) \). Thus the operator (5) acts on the complex
\( S(u(1)^*) \otimes \Lambda(\mathcal{M}) \). Since
\[ d_H^2 = \phi (d i_H + i_H d) = \phi \mathcal{L}_H \]

we conclude that on the \( U(1) \) invariant subcomplex \( (S(u(1)^*) \otimes \Lambda(\mathcal{M}))^{U(1)} \) the
action of \( d_H \) is nilpotent and defines the \( U(1) \)-equivariant cohomology of \( \mathcal{M} \) as the
\( d_H \)-cohomology of \( (S(u(1)^*) \otimes \Lambda(\mathcal{M}))^{U(1)} \). Since the operations of evaluating \( \phi \) and
formation of cohomology commute for abelian group actions the results coincide
independently of the interpretation of \( \phi \), and for notational simplicity we shall in
the following usually set \( \phi = 1 \). This model for the \( U(1) \)-equivariant cohomology
of \( \mathcal{M} \) is the abelian Cartan model.

In the following we shall need a canonical realization of the various operations
on the algebra \( S(u(1)^*) \otimes \Lambda(\mathcal{M}) \). For this we introduce canonical momentum
variables \( p_\mu \) which are conjugate to the coordinates \( x^\mu \) of the original symplectic
manifold, identify the basis of one-forms \( dx^\mu \) with anticommuting \( \eta^\mu \) and realize
the contraction operator acting on \( \eta^\mu \) canonically by \( \bar{\eta}_\mu \), using Poisson brackets
\[ \{ p_\mu, x^\nu \} = \{ \bar{\eta}_\mu, \eta^\nu \} = \delta^\nu_\mu \quad (6) \]
In terms of these variables the exterior derivative, contraction and Lie derivative can be realized by the Poisson bracket actions of

\[
\begin{align*}
\mathbf{d} & = \mathbf{p}_\mu \eta^\mu \\
\mathcal{L}_\mathbf{H} & = \mathbf{X}_\mathbf{H}^\mu \eta^\mu + \eta^\mu \partial_\mu \mathbf{X}_\mathbf{H}^\nu \eta^\nu
\end{align*}
\]

Finally, since \((\phi = 1)\)

\[
d_H(H + \omega) = dH + i_H \omega = 0
\]

we conclude that \(H + \omega\) is an element of \(H^*_U(1)(\mathcal{M})\), that is it determines an equivariant cohomology class. This is an equivalence class consisting of elements in \(\Lambda(\mathcal{M})\) which are linear combinations of zero- and two-forms and can be represented as

\[
H + \omega + d_H \psi
\]

where \(\psi\) is in \(\Lambda_H(\mathcal{M})\) i.e. it satisfies

\[
\mathcal{L}_H \psi = 0
\]

3. Duistermaat-Heckman Integration Formula

The integration formula by Duistermaat and Heckman concerns the exact evaluation of the classical partition function

\[
Z = \int \omega^n e^{i\phi H}
\]

where \(H\) is a hamiltonian function that determines the global symplectic action of \(S^1 \sim U(1)\) on the phase space \(\mathcal{M}\), and in physical applications \(\phi\) is identified as the inverse temperature. The integration measure is the phase space Liouville measure which is a canonically invariant measure on \(\mathcal{M}\).

If the critical points of \(H\) are isolated and nondegenerate, the integration formula by Duistermaat and Heckman states that \(\int \omega^n e^{i\phi H}\) localizes to the critical points.
of \( H \),

\[
Z = \sum_{dH=0} \frac{\sqrt{\det|\omega_{\mu\nu}|}}{\sqrt{\det|\partial_{\mu}H|}} \exp\{i\phi H\}
\]  \( (10) \)

where for simplicity we have included the phase factor that appears in \( (3) \) to the definition of the determinant of \( H \), and we have also absorbed the additional factors that appear in \( (2) \) to the normalization of \( Z \).

In this section we shall present a derivation of the integration formula \( (10) \) and explain how it generalizes to the case where the critical points of \( H \) are degenerate. In the subsequent sections we then use the techniques that we introduce here to derive new integration formulas for integrals that are more general than \( (9) \).

Using the fact that integration picks up the \( 2n \)-form, modulo an overall normalization we can write \( (9) \) as

\[
Z = \int \exp\{i\phi(H + \omega)\}
\]  \( (11) \)

or in local coordinates

\[
Z = \int \exp\{i\phi(H + \omega) + \frac{1}{2} \eta^\mu \omega_{\mu\nu} \eta^\nu\}
\]

In order to prove the integration formula \( (10) \), we introduce the following generalization of \( (11) \)

\[
Z_\lambda = \int \exp\{i\phi(H + \omega) + \lambda d_H \psi\}
\]  \( (12) \)

Here \( \psi \) is a one-form and \( \lambda \) is a parameter. We shall first argue that if \( \psi \) is in the \( H \)-invariant subspace,

\[
\mathcal{L}_H \psi = 0
\]  \( (13) \)

the integral \( (12) \) does not depend on \( \lambda \). Since the integrals \( (11) \) and \( (12) \) coincide for \( \lambda \rightarrow 0 \), this implies that these integrals coincide for all values of \( \lambda \). By evaluating \( (12) \) in the \( \lambda \rightarrow \infty \) limit we then obtain the integration formula \( (10) \).

Notice in particular, that the \( \lambda \)-independence of \( (12) \) means that \( (11) \) only depends on the equivalence class that \( H + \omega \) determines in the equivariant cohomology \( H^*_{U(1)}(\mathcal{M}) \).

In order to establish the \( \lambda \)-independence of \( (12) \), we consider an infinitesimal variation \( \lambda \rightarrow \lambda + \delta\lambda \) and show that

\[
Z_\lambda = Z_{\lambda + \delta\lambda}
\]
For this we introduce the following infinitesimal change of variables in (12):

\[ x^\mu \rightarrow x^\mu + \delta x^\mu = x^\mu + \delta \psi \cdot d_H x^\mu \]
\[ \eta^\mu \rightarrow \eta^\mu + \delta \eta^\mu = \eta^\mu + \delta \psi \cdot d_H \eta^\mu = \eta^\mu - \delta \psi \chi_H^\mu \]

(14)

where

\[ \delta \psi = \delta \lambda \cdot \psi \]

As a consequence of (8) and (13) the exponential in (12) is invariant under the change of variables (14). However, the Jacobian is nontrivial:

\[ dx d\eta \rightarrow (1 + d_H \delta \psi) dx d\eta \sim \exp(d_H(\delta \psi))dx d\eta = \exp(\delta \lambda d_H \psi)dx d\eta \]

Hence

\[ Z_\lambda = \int dx d\eta \exp\{i \phi (H + \omega) + \lambda d_H \psi + \delta \lambda d_H \psi\} = Z_{\lambda + \delta \lambda} \]

and the classical partition function (11) depends only on the equivalence class that \( H + \omega \) determines in the equivariant cohomology \( H^*_{U(1)}(\mathcal{M}) \).

The \( \lambda \)-independence of (12) implies (10): For this we first observe that since the group \( U(1) \) is compact we may construct a metric tensor \( g_{\mu\nu} \) on \( \mathcal{M} \) for which the canonical flow of \( H \) is an isometry,

\[ \mathcal{L}_H g = 0 \]

(15)

or in components,

\[ \chi_H^\mu \partial_\mu g_{\rho\nu} + g_{\mu\rho} \partial_\nu \chi_H^\mu + g_{\nu\rho} \partial_\mu \chi_H^\mu = 0 \]

Such a metric is obtained by selecting an arbitrary Riemannian metric \( g_{\mu\nu} \) on \( \mathcal{M} \), and averaging it over the group \( U(1) \). A converse is also true: Since \( \mathcal{M} \) is compact the isometry group of \( g_{\mu\nu} \) must be compact.

We select

\[ \psi = i_H g = g_{\mu\nu} \chi_H^{\mu} \eta^{\nu} \]

(16)

As a consequence of (13),

\[ \mathcal{L}_H \psi = 0 \]

and

\[ d_H \psi = K + \Omega \]
and the integral
\[ Z = \int \exp\{i\phi(H + \omega) - \frac{\lambda}{2}(K + \Omega)\} \] (17)
is independent of $\lambda$. Here we have defined
\[ K = g_{\mu\nu}\chi_\mu^\mu\chi_\nu^\nu \]
and
\[ \Omega_{\mu\nu} = -\Omega_{\nu\mu} = \frac{1}{2}[\partial_\mu(g_{\nu\rho}\chi_\rho^\rho) - \partial_\nu(g_{\mu\rho}\chi_\rho^\rho)] = \frac{1}{2}[\nabla_\mu(g_{\nu\rho}\chi_\rho^\rho) - \nabla_\nu(g_{\mu\rho}\chi_\rho^\rho)] \] (18)
which is called the Riemannian momentum map [3]. Here $\nabla_\mu$ is the covariant derivative
\[ \nabla_\mu(g_{\nu\rho}\chi_\rho^\rho) = \partial_\mu(g_{\nu\rho}\chi_\rho^\rho) - \Gamma_{\mu\rho}^\sigma(g_{\sigma\rho}\chi_\rho^\rho) \]
and $\Gamma_{\mu\rho}^\sigma$ is the Levi-Civita connection
\[ \Gamma_{\mu\rho}^\sigma = \frac{1}{2}g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \]
and the last relation in (18) follows from antisymmetry.

We note that since $\Omega$ determines a closed two-form on $\mathcal{M}$ it can be viewed as a (degenerate) symplectic two-form. Furthermore, we find that $(H, \omega)$ and $(K, \Omega)$ defines a bi-hamiltonian pair in the sense that their classical trajectories coincide,
\[ \Omega_{\mu\nu}\dot{x}^\nu = \partial_\mu K = \Omega_{\mu\nu}\omega^{\nu\rho}\partial_\mu H \]
which is consistent with the classical integrability of $H$.

Explicitly, (17) is
\[ Z = \int dx d\eta \exp\{i\phi(H + \omega) - \frac{\lambda}{2}g_{\mu\nu}\chi_\mu^\mu\chi_\nu^\nu - \frac{\lambda}{2}\Omega_{\mu\nu}\eta^{\nu}\eta^\nu\} \] (19)
and the integration formula (10) follows immediately when we recall that
\[ \delta(\alpha x) = \frac{1}{|\alpha|}\delta(x) = \lim_{\lambda \to \infty} \sqrt{\frac{\lambda}{2\pi}} e^{-\frac{\lambda}{2}(\alpha x)^2} \]
which localizes (19) onto (10),
\[ Z = \int dx d\eta \frac{\text{Pf}(\Omega_{\mu\nu})}{\sqrt{\det \|g_{\mu\nu}\|}} \delta(\chi_\mu^\mu) e^{i\phi(H + \omega)} \]
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Here we have used
\[ \partial_\mu \chi_\nu^\mu = \omega^\nu_\sigma \partial_\mu \partial_\sigma H \]
on the critical points \( dH = 0 \) and included the phase factor that appears in (2) to the definition of the determinant of \( \partial_\mu \nu \).

We now generalize (10) for Hamiltonians \( H \) with degenerate critical points. We denote the critical submanifold of \( H \) in \( \mathcal{M} \) by \( \mathcal{M}_0 \) and by \( \mathcal{N}_\perp \) its normal bundle in \( \mathcal{M} \). In a neighborhood near \( \mathcal{M}_0 \) we write the local coordinates \( x^\mu \) as
\[ x^\mu = \hat{x}^\mu + \delta x^\mu \]
where \( \hat{x}^\mu \) are local coordinates in \( \mathcal{M}_0 \),
\[ \chi_\nu^\mu(\hat{x}) = 0 \]
and \( \delta x^\mu \) are local coordinates in \( \mathcal{N}_\perp \). Similarly, we introduce
\[ \eta^\mu = \hat{\eta}^\mu + \delta \eta^\mu \]
where \( \hat{\eta}^\mu \) are one-forms in \( T^* \mathcal{M}_0 \) and \( \delta \eta^\mu \) are one-forms in \( T^* \mathcal{N}_\perp \). In particular, the \( \hat{\eta}^\mu \) satisfy
\[ \Omega_{\mu\nu}(\hat{x})\hat{\eta}^\nu = 0 \]
for all \( \mu \).

For large \( \lambda \) the integral (19) localizes exponentially to the vicinity of \( \mathcal{M}_0 \), and consequently we can extend integration over all values of \( \delta x^\mu \). We introduce the following change of variables,
\[ x^\mu = \hat{x}^\mu + \delta x^\mu \rightarrow \hat{x}^\mu + \frac{1}{\sqrt{\lambda}} \delta x^\mu \]
\[ \eta^\mu = \hat{\eta}^\mu + \delta \eta^\mu \rightarrow \hat{\eta}^\mu + \frac{1}{\sqrt{\lambda}} \delta \eta^\mu \]
(22)
Since the Jacobians for the \( \delta x^\mu \) and for the \( \delta \eta^\mu \) cancel, the corresponding Jacobian is trivial.
Consider the last term in (19),
\[ \frac{\lambda}{2} \eta^\mu \Omega_{\mu \nu} \eta^\nu \] (23)
If we perform the change of variables (22) and expand (23) in powers of \( \sqrt{\lambda} \) we find that the term which is proportional to \( \lambda \)
\[ \hat{\eta}^\mu \Omega_{\mu \nu} \hat{\eta}^\nu \]
vanes by (21). Similarly, the terms which are proportional to \( \sqrt{\lambda} \)
\[ \hat{\eta}^\mu x^\rho \partial_\rho \Omega_{\mu \nu} \eta^\nu + \delta \eta^\mu \Omega_{\mu \nu} \delta \eta^\nu \]
vanish as a consequence of (21), (21) and the Lie derivative condition (15) for the metric tensor.

We shall now proceed to investigate the \( O(1) \) contribution in this expansion,
\[ \frac{1}{2} \hat{\eta}^\mu \hat{\eta}^\nu \delta x^\rho \partial_\rho \Omega_{\mu \nu} \eta^\nu + \delta \eta^\mu \Omega_{\mu \nu} \delta \eta^\nu + \delta \eta^\mu \Omega_{\mu \nu} \delta \eta^\nu \]
A lengthy but straightforward calculation reveals, that these terms can be combined into
\[ \frac{1}{2} \delta \eta^\mu \Omega_{\mu \nu} \delta \eta^\nu + \frac{1}{2} \delta x^\sigma \Omega^\nu_{\rho \sigma} R_{\rho \sigma \mu \nu} \hat{\eta}^\rho \hat{\eta}^\sigma \delta x^\nu \] (24)
where
\[ R^\rho_{\sigma \mu \nu} = \partial_\mu \Gamma^\rho_{\nu \sigma} - \partial_\nu \Gamma^\rho_{\mu \sigma} + \Gamma^\rho_{\mu \kappa} \Gamma^\kappa_{\nu \sigma} - \Gamma^\rho_{\nu \kappa} \Gamma^\kappa_{\mu \sigma} \]
is the Riemann tensor. Hence in the \( \lambda \to \infty \) limit (23) reduces to (24).

Similarly, we expand the second term in (19), i.e.
\[ \frac{\lambda}{2} g_{\mu \nu} \chi_\mu^\nu \chi_\nu \]
in powers of \( \sqrt{\lambda} \), after we first perform the change of variables (22). We find, that in the \( \lambda \to \infty \) limit the only term that survives is
\[ \frac{\lambda}{2} g_{\mu \nu} \chi_\mu^\nu \chi_\nu \to \frac{1}{2} \delta x^\mu \Omega^\mu_{\rho \sigma} \delta x^\nu \] (25)
From (23) and (24) we conclude that in the \( \lambda \to \infty \) limit the integrals over \( \delta x^\mu \) and \( \delta \eta^\mu \) in (14) become Gaussian, and evaluating these integrals we get
\[ Z = \int_{\mathcal{M}_0} d\delta x d\delta \eta \exp \{ i\phi(H + \omega) \} \cdot \frac{\text{Pf}(\Omega_{\mu \nu})}{\sqrt{\det [\Omega_{\mu \nu} + R^\sigma_{\nu \rho \kappa} \eta^\rho \eta^\kappa]}} \]
where the det and Pf are evaluated over the normal bundle \( \mathcal{N}_\perp \). This result generalizes (10) to the degenerate case. Indeed, if we take the limit where \( \mathcal{M}_0 \) becomes a set of isolated and nondegenerate critical points and carefully account for the sign of the Pfaffian in (26), we find that in this limit (26) reproduces (10).

If we set \( H = 0 \) in (26) we recognize in the numerator the Chern class of the symplectic two-form \( \omega \) and in the denominator the Euler class of the curvature two-form \( R_{\mu\nu} \). With nonzero \( H \), we can then identify \[ \text{Ch}(H + \omega) = \exp\{i\phi(H + \omega)\} \]

as the equivariant Chern class of \( \mathcal{M}_0 \) and

\[ \text{E}(\Omega_{\mu\nu} + R_{\mu\nu\rho\sigma}\eta^\rho\eta^\sigma) = \text{Pf}(\Omega_{\mu\nu} + R_{\mu\nu\rho\sigma}\eta^\rho\eta^\sigma) \]

as the equivariant Euler class of \( \mathcal{M}_0 \), evaluated over the normal bundle \( \mathcal{N}_\perp \). Equivariant characteristic classes \([3]\) are generalizations of ordinary characteristic classes to the equivariant context, and provide representatives of equivariant cohomology. In particular, we note that equivariant characteristic classes are independent of the connection which implies that (26) is independent of the metric tensor consistently with our general arguments.

4. \( S^*M \) and generalization of the Matthey-Quillen formalism

We shall now proceed to explain how the previous construction can be extended to derive more general integration formulas. For this, instead of integrals over the cotangent bundle \( \mathcal{M} \otimes T^*\mathcal{M} \) with local coordinates \( x^\mu \) and \( \eta^\mu \) we shall in the following consider integrals that are defined over all four variables \( x^\mu, \eta^\mu, p_\mu, \bar{\eta}_\mu \) that we have introduced in (16).

Originally, we introduced \( p_\mu \) as a canonical realization of the local basis for the tangent bundle \( T\mathcal{M} \) and \( \bar{\eta}_\mu \) as a canonical realization of the local basis of the
contraction dual bundle of \( T^*M \). In the following we shall instead interpret these variables as follows: We view \( x^\mu \) and \( \bar{\eta}_\mu \) as local coordinates on a supermanifold that we denote by \( S^*M \). In analogy with (7), we interpret \( \eta^\mu \) as (part of) a local basis for the cotangent bundle of \( S^*M \) with the identification \( \eta^\mu \sim dx^\mu \). However, instead of viewing \( p_\mu \) as a local basis for the tangent bundle \( TS^*M \) we shall in the following interpret \( p_\mu \) as (part of) a local basis for the cotangent bundle of \( S^*M \) with the identification \( p_\mu \sim d\bar{\eta}_\mu \). The pertinent exterior derivative is

\[
d = \eta^\mu \frac{\partial}{\partial x^\mu} + p_\mu \frac{\partial}{\partial \bar{\eta}_\mu}
\]

and it is a nilpotent operator on the exterior algebra \( \Lambda(S^*M) \).

We also introduce a basis \( i_\mu, \pi^\mu \) of contractions on the exterior algebra \( \Lambda(S^*M) \), dual to the basis \( \eta^\mu, p_\mu \) of one-forms,

\[
i_\mu \eta^\nu = \pi^\nu p_\mu = \delta^\nu_\mu
\]

In order to construct the Matthai-Quillen formalism we consider a conjugation of \( d \) by a functional \( \Phi \),

\[
d \Phi \rightarrow e^{-\Phi} d e^\Phi = d + [d, \Phi] + \frac{1}{2}[[d, \Phi], \Phi] + ...
\]

Since this conjugation is an invertible transformation, the cohomologies of (27) and (28) coincide. Using the Levi-Civita connection \( \Gamma^\rho_{\mu\nu} \) we define

\[
\Phi = -\Gamma^\rho_{\mu\nu} \pi^\nu \eta^\rho \bar{\eta}_\rho
\]

which yields for the conjugated exterior derivative

\[
d = \eta^\mu \frac{\partial}{\partial x^\mu} + (p_\mu + \Gamma^\rho_{\mu\nu} \eta^\nu \bar{\eta}_\rho) \frac{\partial}{\partial \bar{\eta}_\mu} + (\Gamma^\rho_{\mu\nu} p_\rho \pi^\nu - \frac{1}{2} R^\rho_{\mu\sigma\nu} \eta^\nu \eta^\sigma \bar{\eta}_\rho) \pi^\mu
\]

In particular, we have

\[
\begin{align*}
\text{d}x^\mu &= \eta^\mu \\
\text{d}\eta^\mu &= 0 \\
\text{d}p_\mu &= \Gamma^\rho_{\mu\nu} p_\rho \pi^\nu - \frac{1}{2} R^\rho_{\mu\sigma\nu} \eta^\nu \eta^\sigma \bar{\eta}_\rho \\
\text{d}\bar{\eta}_\mu &= p_\mu + \Gamma^\rho_{\mu\nu} \eta^\nu \bar{\eta}_\rho
\end{align*}
\]
where we identify the transformation laws of the standard \((N = 1)\) deRham sup-

ersymmetric quantum mechanics, with \(p\mu\) the auxiliary field (see e.g. \([20]\)). As
explained in \([10]\), this means that \((30)\) can be related to the Matthai-Quillen formal-

ism \([11]\). In particular, the corresponding quantum mechanical path integral
determines an infinite dimensional version of the Matthai-Quillen formalism \([10]\)
and reduces to the original, finite dimensional Matthai-Quillen formalism by a reg-

ularization procedure. As explained in \([10]\) we can use such a path integral and
localization methods to derive the Gauss-Bonnet-Chern and Poincaré-Hopf the-

orems of classical Morse theory. We shall not reproduce these derivations here, but
refer to \([10]\) for details. Instead we shall now proceed to generalize the previous
construction of the Matthai-Quillen formalism to include the action of a nontrivial
vector field on \(S^*M\), which corresponds to the global circle action on the original
symplectic manifold \(M\) generated by our Hamiltonian \(H\).

We first recall the canonical realization \([7]\) of the Lie derivative along the
Hamiltonian vector field \(\mathcal{X}_H\) on the exterior algebra \(\Lambda(M)\),

\[
\mathcal{L}_H = \mathcal{X}_H^\mu p_\mu + \eta^\nu \partial_\nu \bar{\mathcal{X}}_H^\mu \bar{\eta}_\nu
\]

Notice that since it acts by the Poisson brackets \([8]\), it is in fact defined on the
exterior algebra \(\Lambda(S^*M)\). In particular, on our canonical variables the Poisson
bracket action of \(\mathcal{L}_H\) is

\[
\begin{align*}
\mathcal{L}_H x^\mu &= \mathcal{X}_H^\mu \\
\mathcal{L}_H \eta^\mu &= \eta^\nu \partial_\nu \mathcal{X}_H^\mu \\
\mathcal{L}_H \bar{\eta}_\mu &= -\partial_\mu \mathcal{X}_H^\nu \bar{\eta}_\nu \\
\mathcal{L}_H p_\mu &= -\partial_\mu \mathcal{X}_H^\nu p_\nu - \eta^\nu \partial_\mu \bar{\mathcal{X}}_H^\mu \bar{\eta}_\nu
\end{align*}
\]

In order to generalize the Matthai-Quillen formalism, as a first step we realize this
action by a Lie derivative that acts as a differential operator on the exterior algebra
\(\Lambda(S^*M)\). For this we define the following vector field on \(TS^*M\)

\[
\mathcal{X} = \mathcal{X}_H^\nu \frac{\partial}{\partial x^\nu} - \bar{\eta}_\mu \partial_\mu \mathcal{X}_H^\mu \frac{\partial}{\partial \bar{\eta}_\nu}
\]

and introduce the corresponding nilpotent contraction operator on \(\Lambda(S^*M)\),

\[
i_{\mathcal{X}} = \mathcal{X}_H^\mu i_\mu - \bar{\eta}_\mu \partial_\mu \mathcal{X}_H^\mu \pi^\nu
\]
and the corresponding equivariant exterior derivative

\[ Q_X = d + i_X = \eta^\nu \frac{\partial}{\partial x^\nu} + p_\mu \frac{\partial}{\partial \bar{\eta}_\mu} + X_H^\mu i_\mu - \bar{\eta}_\nu \partial_\nu \lambda^\mu = \eta^\nu \partial_\nu \lambda^\mu - \bar{\eta}_\nu \partial_\nu \lambda^\mu \]  

(32)

We find that the ensuing Lie-derivative

\[ L_X = di_X + i_X d \]

\[ = X_H^\mu \frac{\partial}{\partial x^\mu} + \eta^\nu \partial_\nu \lambda_H^\mu i_\mu - \partial_\mu \lambda_H^\nu \bar{\eta}_\nu \frac{\partial}{\partial \bar{\eta}_\mu} - (\partial_\mu \lambda_H^\nu p_\nu + \eta^\nu \partial_\nu \lambda_H^\mu \bar{\eta}_\mu) \pi^\mu \]  

(33)

then reproduces the transformation laws (31) on the variables (\(x^\mu, \bar{\eta}_\mu, \eta^\mu, p_\mu\)).

Next, we introduce the conjugation (28) for \(Q_X\) using the functional \(\Phi\) defined in (29). This yields for the equivariant exterior derivative (32)

\[ Q_X \rightarrow e^\Phi Q_X e^\Phi = \eta^\mu \frac{\partial}{\partial x^\mu} + (p_\mu + \Gamma^\sigma_\mu_\nu \eta^\nu \bar{\eta}_\sigma) \frac{\partial}{\partial \bar{\eta}_\mu} + X_H^\mu i_\mu + \]

\[ + (\Gamma^\rho_\mu_\nu \eta^\nu - \frac{1}{2} R^\rho_\mu_\nu_\sigma \eta^\sigma \bar{\eta}_\rho - X_H^\rho \Gamma^\rho_\mu_\nu \bar{\eta}_\nu + \partial_\nu \lambda^\rho \pi^\mu) \]

and for the Lie-derivative (33) we find

\[ L_X = X_H^\mu \frac{\partial}{\partial x^\mu} + \eta^\nu \partial_\nu \lambda_H^\mu i_\mu - \partial_\mu \lambda_H^\nu \bar{\eta}_\nu \frac{\partial}{\partial \bar{\eta}_\mu} - p_\nu \partial_\mu \lambda^\nu \pi^\mu \]

\[ - \eta^\nu \bar{\eta}_\rho (X_H^\sigma \partial_\sigma \Gamma^\rho_\mu_\nu + \partial_\nu \lambda_H^\rho \Gamma^\rho_\mu_\sigma + \partial_\mu \lambda_H^\nu \Gamma^\rho_\sigma_\nu - \Gamma^\sigma_\mu_\nu \partial_\sigma \lambda_H^\rho + \partial_\sigma \lambda_H^\rho) \pi^\mu \]  

(34)

Here we recognize in the last term

\[ X_H^\sigma \partial_\sigma \Gamma^\rho_\mu_\nu + \partial_\nu \lambda_H^\rho \Gamma^\rho_\mu_\sigma + \partial_\mu \lambda_H^\nu \Gamma^\rho_\sigma_\nu - \Gamma^\sigma_\mu_\nu \partial_\sigma \lambda_H^\rho + \partial_\sigma \lambda_H^\rho \]

the Lie-derivative of the connection one-form \(\Gamma_\mu\) in the original manifold \(\mathcal{M}\) along the Hamiltonian vector field \(X_H\). Since \(X_H\) generates the global action of a circle on \(\mathcal{M}\) which leaves the metric tensor \(g_{\mu \nu}\) invariant, we conclude that on \(\mathcal{M}\) we can set

\[ \mathcal{L}_H \Gamma_\mu = 0 \]  

(35)

Consequently we find that (34) simplifies into

\[ L_X = X_H^\mu \frac{\partial}{\partial x^\mu} + \eta^\nu \partial_\nu \lambda_H^\mu i_\mu - \partial_\mu \lambda_H^\nu \bar{\eta}_\nu \frac{\partial}{\partial \bar{\eta}_\mu} - p_\nu \partial_\mu \lambda_H^\nu \pi^\mu \]  

(36)
and instead of (31) we have the following transformation laws for the local variables on $S^*\mathcal{M}$,

\[
\begin{align*}
L_X x^\mu &= \mathcal{X}_H^\mu \\
L_X \bar{\eta}_\mu &= -\partial_\mu \mathcal{X}_H^\nu \bar{\eta}_\nu \\
L_X \eta^\mu &= \eta^\nu \partial_\nu \mathcal{X}_H^\mu \\
L_X p_\mu &= -p_\nu \partial_\mu \mathcal{X}_H^\nu
\end{align*}
\] (37)

In particular, comparing (31) and (37) we observe that in (31) the first three have the appropriate covariant forms for the transformation of a coordinate ($x^\mu$), one-form ($\eta^\mu$) and its dual ($\bar{\eta}_\mu$) on $\mathcal{M}$ respectively under a coordinate transformation generated by the vector field $\mathcal{X}_H$, and the inhomogeneity in the last term reflects the fact that under a general coordinate transformation a derivative transforms inhomogeneously. However, in (37) all four transformations are generally covariant. In particular, the last one has the appropriate homogeneous form of the generally covariant transformation law for a covariant derivative on $\mathcal{M}$.

Obviously, the formalism that we have developed here can be viewed as a generalization - or equivariantization - of the Matthai-Quillen formalism [11], [10], and we now proceed to apply it to derive new integration formulas.

5. Equivariant Morse theory

We shall first apply our generalization of the Matthai-Quillen formalism to derive a new finite dimensional integration formula for integrals on $\Lambda(S^*\mathcal{M})$ that are of the form

\[
Z = \int dx dp d\eta d\bar{\eta} \exp\{i\phi(H + \omega) + Q_X \psi\}
\] (38)

Here the integration measure is the Liouville measure in the extended phase space ($x^\mu, p_\mu, \eta^\mu, \bar{\eta}_\mu$) with Poisson brackets (3). This is an invariant integration measure on $\Lambda(S^*\mathcal{M})$. The Hamiltonian $H$ and the symplectic two-form $\omega$ are as before, i.e. defined in the original phase space $\mathcal{M}$ so that $H$ depends only on $x^\mu$ while $\omega$ is a function of $x^\mu$ and a bilinear in $\eta^\mu$. The equivariant exterior derivative $Q_X$ is
defined in (32) and \( \psi \) is an element in the subspace of \( \Lambda(S^{*}\mathcal{M}) \) that satisfies
\[
L_{X}\psi = 0
\] (39)
where \( L_{X} \) is the Lie derivative defined in (36). In particular, since (36) assumes (35) i.e. that the Lie derivative of the Levi-Civita connection one-form \( \Gamma_{\mu} \) on \( \mathcal{M} \) vanishes, we again take the Hamiltonian \( H \) to be a canonical generator of a global circle action on the original phase space \( \mathcal{M} \).

Since
\[
Q_{X}(H + \omega) = 0
\]
we conclude using our earlier arguments that if \( \psi \) is in the subspace (40) the integral (38) is invariant under such local variations of \( \psi \) that are in this subspace. Hence the integral (38) only depends on the equivariant cohomology classes of \( Q_{X} \). However, since \( H \) and \( \omega \) do not depend on the variables \( p_{\mu} \) and \( \bar{\eta}_{\mu} \) we can not naively set \( \psi \to 0 \) since in this limit the integral (38) is not properly defined. This integral is properly defined only if the \( Q_{X}\psi \)-term depends appropriately also on the variables \( p_{\mu} \) and \( \bar{\eta}_{\mu} \). Thus we expect that (38) does not coincide with the Duistermaat-Heckman integral (10); the equivariant cohomology on \( \Lambda(\mathcal{M}) \) does not describe the equivariant cohomology on \( \Lambda(S^{*}\mathcal{M}) \).

In order to evaluate (38), we need to construct an appropriate functional \( \psi \). For this, we observe that since the basic variables transform in the homogeneous and generally covariant manner (37) under the Lie derivative, any generally covariant quantity which is built from \( p_{\mu}, \eta^{\mu}, \bar{\eta}_{\mu} \) and invariant tensors on \( \mathcal{M} \) such as the metric tensor \( g_{\mu\nu} \), the Hamiltonian vector field \( X_{H}^{\mu} \), the symplectic two-form \( \omega_{\mu\nu} \), the covariant derivative \( \nabla_{\mu} \) etc. automatically satisfies the Lie-derivative condition (40) on \( S^{*}\mathcal{M} \). Notice that this is in a marked contrast with the Duistermaat-Heckman case, where we have no general rule for the construction of functionals \( \psi \) that satisfy the condition (13) on \( \mathcal{M} \).

We shall first assume that the critical points of \( H \) are isolated and nondegenerate. We select
\[
\psi = \frac{1}{2} X_{H}^{\mu} \bar{\eta}_{\mu}
\] (40)
As a generally covariant quantity, this automatically satisfies the condition (10).
Explicitly, 

$$Q_X\psi = p_\mu X^\mu_H + \eta^\mu \nabla_\mu X_H^{\nu} \bar{\eta}_\nu$$

where $\nabla_\mu$ denotes the covariant derivative with respect to the $H$-invariant Levi-Civita connection $\Gamma_\mu$ on $\mathcal{M}$. Substituting in the integral (38) we get

$$Z = \int dx dp d\eta d\bar{\eta} \exp\{i\phi(H + \omega) + p_\mu X^\mu_H + \eta^\mu \nabla_\mu X_H^{\nu} \bar{\eta}_\nu\}$$

We evaluate the $p_\mu$ integrals and the integrals over the anticommuting variables $\eta^\mu$ and $\bar{\eta}_\mu$. The result is

$$Z = \int dx e^{i\phi H} \delta(X^\mu_H) \det|\partial_\mu X_H^{\nu}| = \sum_{dH=0} e^{-i\phi H} \text{sign}(\det|\partial_\mu H|)$$

where on the r.h.s. we recognize an equivariant version of the quantity that appears in the Poincaré-Hopf theorem.

Next we consider

$$\psi = g^{\mu\nu} p_\mu \bar{\eta}_\nu$$

As a generally covariant quantity, this again satisfies the condition (40) and we have

$$Q_X\psi = g^{\mu\nu} p_\mu p_\nu - \frac{1}{2} R^\mu_{\nu\rho\sigma} \eta^\nu \eta^\rho \bar{\eta}_\sigma - \nabla_\nu X_H^{\mu} g^{\nu\rho} \bar{\eta}_\mu \bar{\eta}_\rho$$

When we substitute this in (38), evaluate the Gaussian integral over $p_\mu$ and the integral over the anticommuting $\bar{\eta}_\mu$ we get

$$Z = \int dx d\eta \exp\{i\phi (H + \omega)\} \cdot \text{Pf}[\nabla_\nu X_H^{\mu} + \frac{1}{2} R^\mu_{\nu\rho\sigma} \eta^\rho \eta^\sigma]$$

We identify this as an equivariant version of the quantity that appears in the Gauss-Bonnet-Chern theorem.

Combining (41) and (42) we obtain the following equivariant generalization of the familiar relation between the Poincaré-Hopf and Gauss-Bonnet-Chern theorems,

$$\sum_{dH=0} e^{-i\phi H} \text{sign}(\det|\partial_\mu H|) = \int dx d\eta e^{i\phi (H + \omega)} \text{Pf}[\nabla_\nu X_H^{\mu} + \frac{1}{2} R^\mu_{\nu\rho\sigma} \eta^\rho \eta^\sigma]$$

In particular, in the $\phi \to 0$ limit (43) reduces to the standard relation

$$\sum_{dH=0} \text{sign}(\det|\partial_\mu H|) = \int dx d\eta \text{Pf}[\frac{1}{2} R^\mu_{\nu\rho\sigma} \eta^\rho \eta^\sigma]$$
between the Poincaré-Hopf and Gauss-Bonnet-Chern theorems. As we have explained in the previous section, in this limit we also reproduce the finite dimensional Matthai-Quillen formalism. (Notice that in taking the $\phi \to 0$ limit in the r.h.s. of (43) we have used the fact, that the integral picks up the top-form of the Pfaffian.)

In order to generalize to the case where the critical point set $\mathcal{M}_0$ of the Hamiltonian $H$ is degenerate we first observe that both quantities in the r.h.s. of (43) are equivariantly closed on $\Lambda(\mathcal{M})$, 

$$d_H \exp\{i\phi(H + \omega)\} = d_H \text{Pf}[\nabla_\nu \mathcal{X}^\mu_H + \frac{1}{2} R^\mu_{\nu\rho\sigma} \eta^\rho \eta^\sigma] = 0$$

Indeed, as we have explained in section 3. these quantities are the equivariant generalizations of the Chern class and Euler class on $\mathcal{M}$, respectively. Using our standard arguments we then conclude that the following generalization of the integral in (43) over $\mathcal{M}$,

$$Z = \int dxd\eta \exp\{i\phi(H + \omega)\} \cdot \text{Pf}[\nabla_\nu \mathcal{X}^\mu_H + \frac{1}{2} R^\mu_{\nu\rho\sigma} \eta^\rho \eta^\sigma]$$ (44)

is independent of the functional $\psi$ and coincides with (43), provided $\psi$ satisfies the Lie-derivative condition

$$\mathcal{L}_H \psi = 0$$

on $\Lambda(\mathcal{M})$. If we select (44) and repeat the computation that yielded the degenerate version (26) of the Duistermaat-Heckman integration formula, we find that the contribution from the $d_H \psi$-term in (44) cancels the Pfaffian in (44) except for the contribution that comes from the evaluation of the determinant over the normal bundle $\mathcal{N}_0$ of the critical submanifold $\mathcal{M}_0$ of $H$. Consequently (44) reduces to the following integral over the critical submanifold $\mathcal{M}_0$ of the Hamiltonian $H$,

$$Z = \int_{\mathcal{M}_0} dxd\eta \exp\{i\phi(H + \omega)\} \cdot \text{Pf}[\nabla_\nu \mathcal{X}^\mu_H + \frac{1}{2} R^\mu_{\nu\rho\sigma} \eta^\rho \eta^\sigma]$$

which can be viewed as an equivariant version of the Gauss-Bonnet-Chern formula in the degenerate case.

6. Duistermaat-Heckman formula in the loop space
We shall now proceed to generalize the previous results to loop space i.e. path integrals. We again start by first considering the loop space generalization of the Duistermaat-Heckman integration formula, and in the subsequent sections we continue to more general loop space integration formulas.

In the Duistermaat-Heckman case we are interested in evaluating the standard path integral

$$Z = \int [dx] \sqrt{\det |\omega_{\mu\nu}|} \exp\{i \int_0^T \vartheta_{\mu} \dot{x}^\mu - H\}$$

In the following all time integrals will be implicit, and e.g. instead of (46) we write simply

$$d = \eta^\mu \partial_\mu$$

Similarly various other quantities on $M$ can be lifted to $LM$. For example, by defining the loop space symplectic two-form

$$\Omega_{\mu\nu}(t, t') = \omega_{\mu\nu} \delta(t - t')$$

we have a symplectic structure on $LM$.

We interpret the bosonic part of the action in (45) as a Hamiltonian functional on the loop space. The corresponding loop space hamiltonian vector field has
components
\[ X^\mu_S = \dot{x}^\mu - \omega^{\mu\nu} \partial_\nu H = \dot{x}^\mu - X^\mu_H \]  \hspace{1cm} (47)

In particular, its critical points coincide with the classical trajectories.

We introduce a basis \( i_\mu \) of loop space contractions dual to the basis of one-forms \( \eta^\mu(t) \),
\[ i_\mu(t) \eta^\nu(t') = \delta^\nu_\mu(t - t') \]
and define the loop space equivariant exterior derivative along the loop space vector field \((47)\),
\[ d_S = d + i_S = \eta^\mu \partial_\mu + X^\mu_S i_\mu \]  \hspace{1cm} (48)

The corresponding Lie-derivative is
\[ L_S = di_S + i_S d = X^\mu_S \partial_\mu + \eta^\mu \partial_\mu X^\nu_S i_\nu \]  \hspace{1cm} (49)

We introduce the invariant subcomplex of loop space exterior algebra where the Lie derivative \((49)\) vanishes. In this subspace \((48)\) is then nilpotent and determines loop space equivariant cohomology. In particular, the action in \((45)\) is equivariantly closed,
\[ d_S(\theta_\mu \dot{x}^\mu - H + \frac{1}{2} \eta^\mu \omega_{\mu\nu} \eta^\nu) = 0 \]
and determines an element of the corresponding equivariant cohomology class.

Again, we conclude \([12]\) that if we add to the exponential in \((45)\) a \( d_S \) exact term,
\[ S \to \int \theta_\mu \dot{x}^\mu - H + \frac{1}{2} \eta^\mu \omega_{\mu\nu} \eta^\nu + d_S \psi \]
where \( \psi \) is in the nilpotent subspace
\[ L_S \psi = 0 \]  \hspace{1cm} (50)

the corresponding path integral
\[ Z_\psi = \int [dx][d\eta] \exp\{i \int \theta_\mu \dot{x}^\mu - H + \frac{1}{2} \eta^\mu \omega_{\mu\nu} \eta^\nu + d_S \psi\} \]  \hspace{1cm} (51)

is invariant under local variations of \( \psi \) and coincides with \((45)\). By selecting \( \psi \) properly the path integral can then be evaluated by the localization method.
In order to enumerate the different possibilities, we consider the following example,

\[ Z = \int [d\cos(\theta)] [d\phi] \exp\left\{ i \int_0^T j \cos(\theta) \dot{\phi} - j \cos(\theta) \right\} \]  

(52)

This path integral is defined on the sphere \( S^2 \) and yields the character of \( SU(2) \) in the spin-\( j + \frac{1}{2} \) representation \( [22] \).

The loop space Hamiltonian vector field (47) of the action in (52) has two components,

\[ \mathcal{X}_\theta = \dot{\theta} \]
\[ \mathcal{X}_\phi = \dot{\phi} - 1 \]  

(53)

For \( T \neq 2\pi n \) the only \( T \)-periodic critical trajectories of (53) coincide with the critical points of the Hamiltonian \( H = j \cos(\theta) \),

\[ \theta = 0, \pi \]

Consequently for \( T \neq 2\pi n \) the critical point set of the action in (52) is isolated and nondegenerate. On the other hand, for \( T = 2\pi n \) we have \( T \)-periodic classical solutions for any initial value of \( \theta \) and \( \phi \) and the critical point set of the classical action coincides with the classical phase space \( S^2 \).

From this example we can abstract the following generic properties: The critical points of the vector field (17) i.e. classical solutions

\[ \mathcal{X}^\mu = \dot{x}^\mu - \omega^{\mu\nu} \partial_\nu H = 0 \]  

(54)

with the boundary conditions \( x^\mu(T) = x^\mu(0) = x_0^\mu \) form a submanifold \( \mathcal{M}_0 \) of \( \mathcal{M} \), the moduli space of classical solutions. For the class of Hamiltonians we consider i.e. Hamiltonians that determine the action of a circle on the phase space this moduli space can be characterized as follows:

- There are in general discrete values of \( T \) for which the classical solutions (54) admit nontrivial periodic solutions \( x^\mu(0) = x^\mu(T) \) for any initial condition \( x^\mu(0) = x_0^\mu \) in a compact submanifold \( \mathcal{M}_0 \) of \( \mathcal{M} \). In the example above with \( T = 2\pi n \), this submanifold coincides with the original phase space \( \mathcal{M} \).
- For *generic* values of $T$ the periodic solutions with $x^\mu(0) = x^\mu(T)$ can only exist if $x^\mu(0) = x^\mu_0$ is a point on the critical submanifold of $H$. In particular, the classical equations of motion reduce to

$$\dot{x}^\mu = \omega^{\mu\nu} \partial_\nu H = 0$$

and consequently the moduli space $\mathcal{M}_0$ of $T$-periodic solutions in this case coincides with the critical point set of $H$, which is a submanifold of $\mathcal{M}$.

In order to localize (51), we need to construct an appropriate $\psi$. For this we lift the $H$-invariant metric $g_{\mu\nu}$ on $\mathcal{M}$ to the loop space and define

$$\psi = \frac{\lambda}{2} g_{\mu\nu} \chi^\mu_S \eta^\nu$$

As a consequence of (13) the Lie-derivative condition (50) is satisfied. The evaluation of the $\lambda \to \infty$ limit follows closely (that in section 3.: We write the loop space variable $x^\mu(t)$ as

$$x^\mu(t) = \hat{x}^\mu(t) + \delta x^\mu(t)$$

where $\hat{x}^\mu(t)$ solves the classical equations of motion

$$\chi^\mu_S(\hat{x}^\mu) = \partial_t \hat{x}^\mu - \omega^{\mu\nu}(\hat{x}) \partial_\nu H(\hat{x}) = 0$$

and $\delta x^\mu(t)$ denotes the fluctuations. Similarly, we define

$$\eta^\mu(t) = \hat{\eta}^\mu(t) + \delta \eta^\mu(t)$$

where the $\hat{\eta}^\mu(t)$ are zeroes of the loop space Riemannian momentum map

$$\Omega_{\mu\nu} = \frac{1}{2} \left\{ \frac{\delta}{\delta x^\mu} (g_{\nu\rho} \chi^\rho_S) - \frac{\delta}{\delta x^\nu} (g_{\mu\rho} \chi^\rho_S) \right\}$$

$$\Omega_{\mu\nu}(\hat{z}) \hat{\eta}^\nu = 0$$

In particular this implies that the $\hat{\eta}^\mu$ are *Jacobi fields*, i.e. satisfy the fluctuation equation

$$[\delta^\mu_\nu \partial_t - \partial_\nu \chi^\mu_S(\hat{x})] \hat{\eta}^\nu = 0$$

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We define the loop space measure in (51) by

\[ [dx][d\eta] = d\hat{x}^\mu(t)d\hat{\eta}^\mu(t) \prod d\delta x^\mu(t)d\delta \eta^\mu(t) \]

and introduce the path integral change of variables

\[ x^\mu(t) \rightarrow \dot{x}^\mu(t) + \frac{1}{\sqrt{\lambda}} \delta x^\mu(t) \]

\[ \eta^\mu(t) \rightarrow \hat{\eta}^\mu(t) + \frac{1}{\sqrt{\lambda}} \delta \eta^\mu(t) \]

The corresponding Jacobian in the path integral measure is one. We are interested in the \( \lambda \rightarrow \infty \) limit. By generalizing the computation in section 3. to the loop space, we find that in this limit the path integral (51) reduces to an integral over the moduli space \( M_0 \) of classical solutions,

\[ Z = \int d\hat{x}(t)d\hat{\eta}(t) \exp\left\{ i \int_0^T \partial_\mu \dot{x}^\mu - H(\dot{x}) + \frac{1}{2} \hat{\eta}^\mu \omega_{\mu\nu} \hat{\eta}^\nu \right\} \text{Pf} ||| \delta_{\mu\nu} \omega||| \]

(55)

Here the Pfaffian is evaluated over the fluctuation modes \( \delta x^\mu \), and \( R^\mu_{\nu} \) is the Riemannian curvature of the metric \( g_{\mu\nu} \) evaluated on the classical solution \( \hat{x}^\mu(t) \).

Notice in particular, that the measure in ( ) is an invariant measure over the moduli space of classical solutions which is itself a symplectic manifold.

In the limit, where we assume that the solutions to (54) are isolated and nondegenerate (for example if the critical points of \( H \) itself are isolated and nondegenerate and if the period \( T \) is such that the boundary condition \( x^\mu(0) = x^\mu(T) \) only admits constant loops as solutions to the classical equations of motion) we can further reduce (55) to

\[ Z = \sum_{dS=0} \frac{\exp\{iS(\dot{x})\}}{\sqrt{\det ||\delta_{\mu\nu} S||}} \]

(56)

In conclusion, we have here reduced the path integral to an integral over the moduli space of classical solutions. In general this moduli space has a complicated, \( T \) dependent structure. We shall now proceed to derive an alternative integration formula for (43) which is applicable independently of the structure of the moduli space of classical solutions. It also has the advantage, that it can be directly
generalized to loop space equivariant Morse theory, as we shall see in the following sections.

We select $$\psi = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \eta^\nu$$

As a consequence of (15) the condition (50) is satisfied and the corresponding action (51) is

$$S = \int \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + (\psi - \frac{1}{2} g_{\mu\nu} \chi_H^\nu) \dot{x}^\mu - H - \frac{1}{2} \eta^\mu (g_{\mu\nu} \partial_\nu + g_{\nu\rho} \dot{x}^\rho \Gamma^\sigma_{\mu\rho}) \eta^\sigma + \frac{1}{2} \eta^\mu \omega_{\mu\nu} \eta^\nu$$

where $$\Gamma^\sigma_{\mu\rho}$$ is again the (metric) Levi-Civita connection. By the $$\psi$$ independence, the path integral (51) remains invariant under the scaling $$g_{\mu\nu} \rightarrow \lambda g_{\mu\nu}$$ of the metric.

If we evaluate it in the $$\lambda \rightarrow \infty$$ limit we find [17] that the result is an ordinary integral over the classical phase space $$\mathcal{M}$$,

$$Z = \int dx d\eta \exp\{-iT(H + \frac{1}{2} \eta^\mu \omega_{\mu\nu} \eta^\nu)\} \frac{1}{\sqrt{\det \left| \frac{T}{2}(\Omega^\mu_{\nu} + R^\mu_{\nu\rho\sigma} \eta^\rho \eta^\sigma) \right|}}$$

We evaluate the determinant using e.g. $$\zeta$$-function regularization. The result is

$$Z = \int dx d\eta \exp\{-iT(H + \frac{1}{2} \eta^\mu \omega_{\mu\nu} \eta^\nu)\} \cdot \hat{A} \left[ \frac{T}{2}(\Omega^\mu_{\nu} + R^\mu_{\nu\rho\sigma} \eta^\rho \eta^\sigma) \right] \tag{57}$$

where

$$\hat{A} \left[ \frac{T}{2}(\Omega^\mu_{\nu} + R^\mu_{\nu\rho\sigma} \eta^\rho \eta^\sigma) \right] = \sqrt{\det \left[ \frac{T}{2}(\Omega^\mu_{\nu} + R^\mu_{\nu\rho\sigma} \eta^\rho \eta^\sigma) \right]}$$

is the equivariant $$\hat{A}$$-genus. This is our final integration formula for the path integral (15), in terms of equivariant characteristic classes on the classical phase space $$\mathcal{M}$$. Notice in particular, that here we integrate over the entire phase space $$\mathcal{M}$$ and not only over the moduli space $$\mathcal{M}_0$$ of classical solutions as in (55), which is in general a $$T$$-dependent submanifold of $$\mathcal{M}$$.

Since the equivariant $$\hat{A}$$-genus and the equivariant Chern class are both equivariantly closed with respect to $$d_H$$, we can further reduce (57) to the critical point set $$\mathcal{M}_0$$ of $$H$$ by repeating the steps that led to (29). For this, instead of (57) we consider the more general integral

$$Z_\psi = \int dx d\eta \hat{A} \left[ \frac{T}{2}(\Omega^\mu_{\nu} + R^\mu_{\nu\rho\sigma} \eta^\rho \eta^\sigma) \right] \cdot \exp\{-iT(H + \frac{1}{2} \eta^\mu \omega_{\mu\nu} \eta^\nu) + d_H \psi\} \tag{58}$$
If \( \psi \) is in the invariant subspace of \( L_H \), according to our standard arguments \((57)\) and \((58)\) coincide. If we select \((16)\) and repeat the steps in section 3. we find that \((57)\) localizes to the following integral over the critical point set \( M_0 \)

\[
Z = \int_{M_0} dxd\eta \hat{A} \left[ \frac{T}{2} (\Omega_{\mu\nu} + R_{\mu\nu\rho\sigma} \eta^\rho \eta^\sigma) \right] \cdot \exp \left\{ -iT \left( H + \frac{1}{2} \eta^\mu \omega_{\mu\nu} \eta^\nu \right) \right\} \frac{\text{Pf}(\Omega_{\mu\nu} + R_{\mu\nu\rho\sigma} \eta^\rho \eta^\sigma)}{	ext{Pf}(\Omega_{\mu\nu})}
\]

of the classical Hamiltonian \( H \), which generalizes \((21)\) to path integrals. In particular, if the critical point set \( M_0 \) of \( H \) is isolated and nondegenerate this reduces further to the following generalization of \((11)\),

\[
Z = \sum_{dH=0} \hat{A} \left[ \frac{T}{2} \Omega_{\mu\nu} \right] \cdot \frac{\exp \left\{ -iTH \right\}}{\text{Pf}(\Omega_{\mu\nu})}
\]

7. Classical Morse theory and path integrals

We shall now proceed to generalize the loop space localization methods of the previous section to path integrals that are defined over the supermanifold \( S^*M \). In the present section we shall consider the infinite dimensional Mathai-Quillen formalism described in \([10]\) to derive some standard results in nondegenerate Morse theory. In the following section we then generalize these results to the equivariant and degenerate cases.

The various quantities constructed in section 4. can be directly lifted to the loop space \( L(S^*M) \) over \( S^*M \). In particular, we define the loop space equivariant exterior derivative (in the following time integrals are implicit)

\[
Q_t = \eta^\mu \frac{\partial}{\partial x^\mu} + p_\mu \frac{\partial}{\partial \eta^\mu} + \dot{x}^\mu i_\mu + \dot{\eta}_\mu \pi^\mu
\]

which is equivariant with respect to the natural action of the circle \( S^1 \): \( x^\mu(t) \to x^\mu(t + \tau) \) etc. on \( L(S^*M) \). The corresponding Lie derivative which acts on the exterior algebra over \( L(S^*M) \) is

\[
L_t = Q_t^2 = \dot{x}^\mu \frac{\partial}{\partial x^\mu} + \dot{\eta}_\mu \frac{\partial}{\partial \eta^\mu} + \dot{\eta}^\mu i_\mu + \dot{\eta}_\mu \pi^\mu \equiv \frac{\partial}{\partial t}
\]
We again introduce the conjugation (28), (29) which yields for the equivariant exterior derivative (59)

\[ Q_t^{-} e^{-\Phi} = \gamma^\mu \frac{\partial}{\partial x^\mu} + (p_\mu + \Gamma^\rho_{\mu\nu} \bar{\eta}_\rho) \frac{\partial}{\partial \bar{\eta}_\mu} \]

\[ \dot{x}^\mu \bar{\eta}_\mu + (\bar{\eta}_\mu - \dot{x}^\nu \Gamma^\rho_{\mu\nu} \bar{\eta}_\rho + \Gamma^\rho_{\mu\nu} p_\rho \bar{\eta}^\nu - \frac{1}{2} R^\rho_{\mu\nu\sigma} \eta^\sigma \bar{\eta}^\nu \bar{\eta}_\rho) \pi^\mu \]

while the Lie derivative (60) remains intact.

We are interested in deriving localization formulas for the following path integral over \( L(S^* \mathcal{M}) \)

\[ Z = \int [dx][d\eta][dp][d\bar{\eta}] \exp \{ i \int Q_t \psi \} \] (61)

which is of the standard form of (cohomological) topological path integral [20]. According to our general arguments, (61) is invariant under local variations of \( \psi \) provided these variations are in the subspace

\[ L_t \psi = 0 \] (62)

But as a consequence of (60) any functional \( \psi \) which is single-valued over the loops satisfies (62) since

\[ L_t \psi = \int_0^T dt \partial_t \psi = \psi(T) - \psi(0) \]

We select

\[ \psi = \frac{\lambda}{2} g_{\mu\nu} \dot{x}^\mu \eta^\nu + \frac{\lambda}{2} p_\mu \bar{\eta}_\nu \]

where \( g_{\mu\nu} \) is a metric tensor on the original phase space \( \mathcal{M} \). This gives for the action in (61)

\[ S = \int Q_t \psi = \int \frac{\lambda}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \frac{\lambda}{2} \eta^\mu (g_{\mu\nu} \partial_t + g_{\nu\sigma} \dot{x}^\rho \Gamma^\sigma_{\rho\mu}) \eta^\nu + \frac{1}{2} g^{\mu\nu} p_\mu p_\nu - \frac{1}{2} R^\rho_{\mu\nu\sigma} \eta^\nu \eta^\sigma \bar{\eta}_\rho \bar{\eta}_\mu + \frac{1}{2} (\bar{\eta}_\rho g^{\kappa\mu})(g_{\mu\nu} \partial_t - g_{\nu\sigma} \dot{x}^\rho \Gamma^\sigma_{\rho\mu})(g^{\nu\xi} \bar{\eta}_\kappa) \]

According to our standard arguments the corresponding path integral is independent of \( \lambda \), and we evaluate it in the \( \lambda \to \infty \) limit. For this we introduce the
following local coordinates on the loop space,

\[ x^\mu(t) = x_0^\mu + x_t^\mu \]
\[ \eta^\mu(t) = \eta_0^\mu + \eta_t^\mu \]
\[ p_\mu(t) = p_{\mu 0} + p_{\mu t} \]
\[ \bar{\eta}_\mu(t) = \bar{\eta}_{\mu 0} + \bar{\eta}_{\mu t} \]

(63)

with \( x_0^\mu \), \( \eta_0^\mu \) the constant modes of \( x^\mu(t) \), \( \eta^\mu(t) \) and \( x_t^\mu \), \( \eta_t^\mu \) the \( t \)-dependent fluctuation modes. We define the path integral measure by

\[
[dx][d\eta][dp][d\bar{\eta}] = dx_0^\mu d\eta_0^\mu dp_{\mu 0} d\bar{\eta}_{\mu 0} \prod_t dx_t^\mu d\eta_t^\mu dp_{\mu t} d\bar{\eta}_{\mu t}
\]

and introduce the change of variables

\[
x_t^\mu \to \frac{1}{\sqrt{\lambda}} x_t^\mu \\
\eta_t^\mu \to \frac{1}{\sqrt{\lambda}} \eta_t^\mu
\]

(64)

At least formally, the corresponding Jacobian is trivial. In the \( \lambda \to \infty \) limit we can evaluate the path integrals over all fluctuation modes and the ordinary integrals over the constant modes \( p_{\mu 0} \) and \( \bar{\eta}_{\mu 0} \). In this way we find that (61) reduces to the integral of the Pfaffian of the curvature two-form over the constant modes \( x_0^\mu \) and \( \eta_0^\mu \),

\[
Z = \int dx d\eta \ Pf(R_{\mu \nu \rho \sigma} \eta^\rho \eta^\sigma) = \chi(M)
\]

(65)

i.e. the path integral (61) yields the Euler number of the phase space \( \mathcal{M} \).

In order to relate (61) to the non-degenerate version of the Poincaré-Hopf theorem, we introduce an arbitrary smooth vector field on \( \mathcal{M} \) with components \( V^\mu \) such that its zeroes are isolated and nondegenerate. (Notice that e.g. in (10) only gradient vector fields were considered.) We then select

\[
\psi = \frac{\lambda}{2} g_{\mu \nu} \dot{x}^\mu \eta^\nu + (\dot{x}^\mu + V^\mu) \bar{\eta}_\mu
\]

This yields for the action in (61)

\[
S = \int Q_t \psi
\]

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\[
\lambda^{-2} g_{\mu \nu}(g_{\mu \rho} \partial_t + g_{\nu \sigma} \dot{x}^\rho \Gamma_{\rho \mu}^\sigma) \eta^\nu + p_\mu (\dot{x}^\mu + V^\mu) + \bar{\eta}_\mu (\delta^\nu_\mu \nabla_t + \nabla_x V^\mu) \eta^\nu
\]

We again introduce the change of variables (64) for the fluctuation modes. In the \( \lambda \to \infty \) limit the integral over \( p_\mu(t) \) yields a \( \delta \)-function that localizes the path integral over \( x^\mu(t) \) to the zeroes of \( V^\mu \). The remaining path integrals are Gaussians, and evaluating these we obtain

\[
Z = \sum_{dV = 0} \text{sign}(\det ||\nabla_\mu V^\nu||) \tag{66}
\]

Combining (65) and (66) we then have the familiar Morse theory relation between the Poincaré-Hopf and Gauss-Bonnet-Chern theorems,

\[
\sum_{dV = 0} \text{sign}(\det ||\nabla_\mu V^\nu||) = \int d\tau d\eta \text{Pf}(R^\rho_{\mu \rho \sigma} \eta^\rho \eta^\sigma) \tag{67}
\]

We note that a generalization of (67) to the degenerate case can be derived by directly generalizing the computations in the previous sections.

8. Equivariant Morse theory in loop space

In order to derive path integral versions of the equivariant Poincaré-Hopf and Gauss-Bonnet-Chern theorems we need a loop space version of the equivariant exterior derivative \( Q_X \) in (32). For this, we combine (32) and (59) to the following equivariant exterior derivative on the loop space \( L(S^* M) \),

\[
Q_S = d + i_S = \eta^\mu \frac{\partial}{\partial x^\mu} + p_\mu \frac{\partial}{\partial \bar{\eta}_\mu} + (\dot{x}^\mu - X^\mu_H)i_\mu + (\dot{i}_\mu - \partial_\mu X^\nu_H \bar{\eta}_\nu) \pi^\mu \tag{68}
\]

Notice that the zeroes of the \( i_\mu \)-components of the loop space vector field in (68) yield the equations of motion

\[
\dot{x}^\mu - X^\mu_H = 0
\]

for the classical action

\[
S = \int \partial_\mu \dot{x}^\mu - H
\]
while the zeroes of the $\pi^\mu$-components of the vector field in (68) determines the
Jacobi equation,
\[(\delta^\mu_\nu \partial_t - \partial^\nu \mathcal{X}^\mu_H)\bar{\eta}_\mu = 0\]

We again assume that
\[\mathcal{L}_H \Gamma_\mu = 0\]
and introduce the conjugation (28) which yields for (68)
\[Q_\mathcal{S} e^{-\Phi} = \eta^\mu \frac{\partial}{\partial x^\mu} + (p_\mu + \Gamma^\rho_\mu \eta^\nu \bar{\eta}_\rho) \frac{\partial}{\partial \bar{\eta}_\mu} + (\dot{x}^\mu - \mathcal{X}^\mu_H) i_\mu\]
\[+ \{\Gamma^\rho_\mu p_\rho \eta^\nu - \frac{1}{2} R^\rho_{\mu \sigma \nu} \eta^\sigma \bar{\eta}_\rho - (\dot{x}^\nu - \mathcal{X}^\nu_H) \Gamma^\rho_\nu \bar{\eta}_\rho + (\delta^\rho_\mu \partial_t - \partial^\rho \mathcal{X}^\rho_H) \bar{\eta}_\rho\} \pi^\mu\]
The pertinent conjugated Lie derivative is a linear combination of (33) and (60),
\[L_\mathcal{S} = \partial_t + L_X = \partial_t + \mathcal{X}^\mu_H \frac{\partial}{\partial x^\mu} + \eta^\mu \frac{\partial^\nu. \mathcal{X}^\nu_H i_\nu - \delta^\mu_\nu \bar{\eta}_\nu \frac{\partial}{\partial \bar{\eta}_\mu} - p_\nu \partial^\mu \mathcal{X}^\nu_H \pi^\mu\] (69)
and it determines the action of the vector field in (68) on the exterior algebra over
the loop space $L(S^*\mathcal{M})$.

We are interested in deriving localization formulas for the path integral
\[Z = \int [dx][dp][d\eta][d\bar{\eta}] \exp \{i \int \dot{\eta}^\mu \dot{x}^\mu - H + \frac{1}{2} \eta^\mu \omega^\mu_\nu \eta^\nu + Q_\mathcal{S} \psi\}\] (70)
Since (69) is a linear combination of time translation and (36), we conclude that any
generally covariant functional $\psi$ which is single valued in the loop space satisfies
the $\psi$-independence condition
\[L_\mathcal{S} \psi = 0\] (71)

We shall first derive an interpretation of (70) corresponding to the Poincaré-
Hopf theorem in the case where the critical points of the action $S$ are isolated and
nondegenerate. As we have explained in section 6. this can be the case for example
if the period $T$ is properly selected and the critical point set of the Hamiltonian
$H$ is isolated and nondegenerate.

We first introduce the functional
\[\psi_1 = g^\mu_\nu p_\mu \bar{\eta}_\nu\] (72)
where \( g_{\mu\nu} \) is again a metric tensor on the original phase space \( \mathcal{M} \) which is Lie-derived by the Hamiltonian vector field of \( H \),

\[
\mathcal{L}_H g = 0
\]

As a local and generally covariant quantity \((72)\) then satisfies the Lie-derivative condition \((71)\). A direct computation yields for the last term in the action in \((70)\)

\[
Q_S \psi_1 = g^{\mu\nu} p_\mu p_\nu - \frac{1}{2} R^\rho_{\mu\sigma\nu} \eta^\rho \eta^\sigma \eta^\mu \eta^\nu - \tilde{\eta}_\mu (g^{\mu\nu} \partial_t + \partial_\rho \mathcal{X}^\mu_H g^\rho\nu) \tilde{\eta}_\nu + \tilde{\eta}_\mu g^{\mu\sigma} (\dot{x}^\mu - \mathcal{X}^\mu_H \Gamma^\nu_{\rho\sigma}) \Gamma^\nu_{\rho\sigma} \tilde{\eta}_\nu
\]

Next, we introduce

\[
\psi_2 = \frac{\lambda}{2} g_{\mu\nu} (\dot{x}^\mu - \mathcal{X}^\mu_H) \eta^\nu
\]

where \( \lambda \) is a parameter. This functional is also generally covariant and single valued on the loop space, and consequently it satisfies the condition \((71)\). Explicitly, we find for the last term in \((70)\)

\[
Q_S \psi_2 = \frac{\lambda}{2} g_{\mu\nu} (\dot{x}^\mu - \mathcal{X}^\mu_H) (\dot{x}^\nu - \mathcal{X}^\nu_H) + \frac{\lambda}{2} \eta^\mu \partial_\mu (g_{\nu\rho} \dot{x}^\nu - g_{\nu\rho} \mathcal{X}^\nu_H) \eta^\rho
\]

We substitute both \((72)\) and \((73)\) in \((70)\) and take the \( \lambda \to \infty \) limit. By assuming that the solutions to \((54)\) are isolated and nondegenerate and repeating the localization procedure that led to \((56)\), we find that in this limit \((70)\) reduces to

\[
Z = \sum_{\delta S = 0} \text{sign}(\det |\delta_{\mu\nu} S||) \exp\{i S\}
\]

where the sum is over all classical solutions \((56)\) \( i.e. \) critical points of the classical action \( S \). This result shows, that the path integral \((70)\) indeed can be related to an equivariant version of the Poincaré-Hopf theorem in the loop space.

We shall now consider the following functional

\[
\psi_3 = \frac{\lambda}{2} g_{\mu\nu} \dot{x}^\mu \eta^\nu
\]

As a generally covariant and single valued loop space functional, this satisfies the Lie-derivative condition \((71)\). Explicitly, we get for the last term in \((70)\)

\[
Q_S \psi_3 = \frac{\lambda}{2} (\dot{x}^\mu - \mathcal{X}^\mu_H) g_{\mu\nu} \dot{x}^\nu + \frac{\lambda}{2} \eta^\mu (g_{\nu\rho} \partial_\nu + \dot{x}^\rho g_{\rho\sigma} \Gamma^\sigma_{\nu\mu}) \eta^\nu
\]
We then consider (70) with
\[ \psi = \psi_1 + \psi_3 \]
We introduce the change of variables (64) and repeat the steps that led to (65). In this way we find that as \( \lambda \to \infty \) (70) localizes to the following integral over the original symplectic manifold \( \mathcal{M} \),
\[ Z = \int_{\mathcal{M}} dxd\eta \exp\{-iT(H + \frac{1}{2} \eta^\mu \omega_{\mu\nu} \eta^\nu)\} \text{Pf}\left[\frac{1}{2}(\Omega^\mu_\nu + R^\mu_{\nu\rho\sigma} \eta^\rho \eta^\sigma)\right] \quad (74) \]
where \( \Omega^\mu_\nu \) is again the Riemannian momentum map (18) and \( R^\mu_{\nu\rho\sigma} \) is the curvature two-form on \( \mathcal{M} \). In particular, we find that the path integral (70) coincides with the finite dimensional integral (42).

Since the equivariant Chern class and the equivariant Pfaffian are both equivariantly closed with respect to \( d + iH \) on the manifold \( \mathcal{M} \), we can apply further localization to the integral (74). If \( \mathcal{M}_0 \) again denotes the critical submanifold of \( H \) in \( \mathcal{M} \), following section 3, we then find that (74) reduces further to
\[ Z = \int_{\mathcal{M}_0} dxd\eta \exp\{-iT(H + \omega)\} \text{Pf}\left[\frac{1}{2}(\Omega^\mu_\nu + R^\mu_{\nu\rho\sigma} \eta^\rho \eta^\sigma)\right] \]
which can be further reduced to a sum over the critical points of \( H \)
\[ Z = \sum_{dH = 0} e^{-iTH} \text{sign}(\det ||\partial_{\mu\nu}H||) \]
provided these critical points are isolated and nondegenerate.

9. Conclusions

In conclusion, we have shown how the Matthai-Quillen formalism can be generalized by "equivariantizing" it with respect to a vector field on a supersymmetric complex \( S^*\mathcal{M} \). We have applied this generalization to construct novel (path)integrals that yield equivariant versions of the Poincaré-Hopf and Gauss-Bonnet-Chern theorems in classical Morse theory. These (path)integrals are naturally associated with integrable dynamical systems, suggesting that for a large class of integrable models the quantum theory could be formulated geometrically.
in terms of equivariant cohomology on the classical moduli space of the theory. In particular, our work indicates that there should be an intimate relationship between cohomological topological field theories and quantum integrable models.

Acknowledgements: We thank A. Alekseev, V. Fock and A. Rosly for discussions.
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