Identification of material stiffness and damping in vibrating plates using full-field measurements

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Abstract. This study concerns the identification of complex stiffness components (ie, stiffness and damping) on isotropic vibrating plates. One of the main difficulties in damping measurements comes from the connection of the specimen to its environment (joints, clamps etc…) causing extra energy dissipation. The present procedure based on full field slope measurements from a deflectometry set up together with an inverse identification technique called the Virtual Fields Method is capable of identifying the material damping regardless of the dissipation coming from the specimen boundary conditions. This paper will present some experimental results confirming this statement.

1. Introduction
The measurement of material elastic stiffness and damping parameters, essential for the prediction of the vibrating or vibro-acoustic behaviour of a large range of structures, is common in material testing laboratories. The identification of the stiffness parameters is usually performed using tension, bending or torsion tests on rectangular coupons leading to simple stress states that can be expressed as functions of the specimen geometry and the applied load through a closed-form solution of the mechanical problem. Nevertheless, these procedures exhibit certain drawbacks. First, experimental boundary conditions must comply with that of the mechanical model, which is not always easy to achieve. Then, only a small number of parameters can be retrieved from a specific test because of the very simple stress state. As a result, several tests usually have to be performed to identify the full set of material parameters, increasing the cost of the procedure. As an alternative, several authors have tried to use the resonance frequencies of bending plates to identify the full stiffness tensor. More recently, the above approach was refined by using not only modal frequencies but also mode shapes.

An alternative to these methods was suggested by Grédiac et al. [1] making use of the measurement of slope fields at the surface of bent plates and performing the identification through a particular application of the principle of virtual work, the so-called Virtual Fields Method (VFM). This procedure uses the global equilibrium of the observed part of the coupon. One advantage of this technique is that stiffnesses are obtained directly (no iterations, no optimization scheme) and that restrictions on specimen geometry and boundary conditions are less critical than with other methods.

The measurement of damping parameters is a more complex problem than stiffness because of all the parasitic dissipation that is usually added in a classical mechanical test. One another main advantage of the present extension of this method is to greatly minimize the parasitic effects of the boundary conditions.
2. Theory – Virtual Fields Method (VFM.)

2.1. Considered case

The considered coupon of the isotropic test material is a rectangular thin plate (thickness $h$), free on its external boundary and clamped approximately in its centre to an excitation device which imposes a sinusoidal driving movement in the out-of-plane direction (inertial excitation). The external boundary of the plate is free. Assuming linear viscoelastic behaviour of the material and thin plate theory, the vibrating response of the plate is pure harmonic bending at the same frequency [2]. This response can be described by the actual deflection field $w(x,y,t)$ such that:

$$w(x,y,t) = \Re[(w_0 + j\omega t) \cdot \exp(j\omega t)]$$

2.2. Global equilibrium

The plate can be virtually divided in two parts: a small zone called $\Pi$ surrounding the clamping area and the second part $\Omega$ which is free on its external boundary. The virtual border between these two parts $\Pi$ and $\Omega$ shall be called $\Gamma$. In the case of small perturbations, the local equation of equilibrium can be written at any point $M$ of $\Omega$ (using the convention for summation on repeated indices):

$$\sigma_{ij} + f_i = \rho \cdot \gamma_i$$  \hspace{1cm} (1)

where $\sigma$ is the stress tensor, $f$ the vector of volume forces, $\rho$ the density and $\gamma$ the acceleration vector (with $i$ and $j$ belong to $\{1,2,3\}$).

Each term can be multiplied by a $u^*$ function which is user selected and must be continuous over $\Omega$ and differentiable. This function can be seen as a displacement field which is virtually imposed all over $\Omega$:

$$\sigma_{ij} u_i^* + f_i u_i^* = \rho \gamma_i u_i^*$$  \hspace{1cm} (2)

Equation (2) can then be integrated over $\Omega$ and leads to:

$$\int_{\Omega} \sigma_{ij} u_i^* dV + \int_{\Omega} f_i u_i^* dV = \int_{\Omega} \rho \gamma_i u_i^* dV$$  \hspace{1cm} (3)

Using the divergence theorem for the calculation of the first term and assuming $\varepsilon^*$ is the virtual strain field related to the virtual displacement field $u^*$, Eq. 3 can be rearranged in a more useful expression:

$$-\int_{\Omega} \sigma_{ij} \varepsilon_j^* dV + \int_{\partial\Omega} T_i u_i^* dS + \int_{\Omega} f_i u_i^* dV = \int_{\Omega} \rho \gamma_i u_i^* dV$$  \hspace{1cm} (4)

where $T$ is the vector of boundary tractions over $\partial\Omega$. It can be seen each term of Eq. 4 could be viewed as a virtual work under the action of the virtual displacement $u^*$ and the related $\varepsilon^*$ virtual strain. Respectively from the left to the right one can find, the virtual work of the internal forces, of the junction forces along the $\Gamma$ border, of the volume forces and finally of the acceleration forces. Due to the assumption of the linear behaviour of the plate material, the acceleration of the harmonic response field of the plate can be simply expressed as $\gamma_i = \omega^2 u_i$. Finally assuming that the volume forces can be neglected, the global equilibrium of the part $\Omega$, free along its external boundary, can be written as:

$$-\int_{\Omega} \sigma_{ij} \varepsilon_j^* dV + \int_{\Gamma} T_i u_i^* dS = -\rho \omega^3 \int_{\Omega} w^* w^* dV$$  \hspace{1cm} (5)

where $w$ is the out-of-plane deflection (it is assumed that the inertial forces generated by the in-plane deformations can be neglected since they are at least one order of magnitude lower than $w$) and $w^*$ the virtual out-of-plane deflection.
2.3. Virtual Fields Method (V.F.M.)

The principle of the Virtual Fields Method is to replace the stress components in the above equation by the actual elastic strains through the constitutive equations which parameters are to be identified. Then, by selecting appropriate virtual fields, $u^*(x,y)$, it is possible to derive equations relating the materials constitutive parameters to integral functions of the actual strains.

In the present case, the only external forces acting on $\Omega$ are the connection forces on the $\Gamma$ border. If a virtual displacement field $u^*$ is selected such that it cancels out the virtual work of these connection forces then the second term of Eq. 5 is null and only the first and the last integrals of the latter remain. It can be shown that the first term can be expressed according to the Love-Kirchhoff theory, and Eq. 5 can then be rewritten as:

$$-\oint (D_j + j\omega B_j) k^*_i k^*_i dS = -\rho h \alpha^2 \oint w^* w^* dS$$

where $w$ and $k$ are the actual deflection and curvatures fields to be measured on the surface $S$ of $\Omega$ whereas $w^*$ and the corresponding $k^*$ are the virtual deflection and curvature fields to be selected for the identification (here, $i$ and $j$ belong to {1,2,6} with the usual rule of contracted indices). $[D]$ and $[B]$ are respectively the elastic bending stiffness and viscous damping matrices, whose $D_{11}$, $D_{12}$, $B_{11}$ and $B_{12}$ components are the unknown material parameters. It must be pointed out that this relationship is valid whatever the excitation frequency, at resonance or out of resonance.

By separating the real and imaginary parts of Eq. 6 it can be shown that the latter can be split into two independent equations, presented in Eq. 7, where $G_p$ and $H_p$ denote combinations of products between virtual and actual curvatures. These quantities are computed using the real or imaginary parts of the actual curvature fields according to whether the $p$ index is $re$ or $im$.

$$D_{11} G^{re} + D_{12} H^{re} - \omega B_{11} G^{im} - \omega B_{12} H^{im} = \rho h \alpha^2 \oint w^{re} w^* dS$$
$$D_{11} G^{im} + D_{12} H^{im} + \omega B_{11} G^{re} + \omega B_{12} H^{re} = \rho h \alpha^2 \oint w^{im} w^* dS$$

with

$$G^p = \oint (k_1^p k_1^* + k_2^p k_2^* + \frac{1}{2} k_6^p k_6^*)dS; \quad H^p = \oint (k_2^p k_1^* + k_1^p k_2^* - \frac{1}{2} k_6^p k_6^*)dS$$

The introduction of two independent deflection virtual fields $w_1^*$, $w_2^*$ and related virtual curvatures fields $k_1^*$, $k_2^*$ and $k_6^*$ leads to a linear system of four equations whose unknowns are the unknown parameters: $D_{11}$, $D_{12}$, $B_{11}$ and $B_{12}$.

2.4. Virtual fields selection

The selection of the virtual fields is a major issue for the identification method. It is carried out using three successive criteria.

- First, using piecewise virtual fields [3], virtual deflection and curvature fields are generated such that they are null along the $\Gamma$ border leading to the cancellation of the virtual work of the connection forces.
- Then, to ensure independence between the four equations issued from Eq. 3, a particular application of a work by Grédiac et al [4] is used to select the ‘special’ virtual deflection fields $w_{1(2)}^*$, $w_{2(3)}^*$ which verify particular values of $G_p$ and $H_p$ and lead to very simple calculations of the parameters.
- The final selection is achieved using an adaptation of the work by Avril et al [5] providing ‘optimized’ virtual fields which minimize the effects of the measurement noise on the identification results.
3. Experimental results

3.1. Set-up

As presented in the previous section, the application of the Virtual Fields Method to the present case requires an excitation arrangement providing an inertial out-of-plane excitation of the plate which is free at its boundaries and a measurement set-up based on an optical method providing the curvature fields and the out-of-plane displacement fields on the whole surface of the plate. These measurements must be taken at two particular times: in-phase and at $\pi/2$ lag with the driving movement of the plate. These two particular positions correspond respectively to the real and imaginary parts of the displacement field of the coupon.

Tests have been carried out using an acrylic 200 x 160 x 3 mm$^3$ specimen. The sinusoidal out of plane inertial excitation is provided by a dedicated device where the coupon is clamped in its centre between the end of the driving rod of the device and a rigid steel washer of the same diameter, tightened by an axial screw. An accelerometer mounted on the rod is used to control the imposed driving movement (see Fig. 1).

3.2. Slope measurements by deflectometry

To avoid noise problems on curvatures arising from a double spatial differentiation of measured deflection fields, measurement of the slope field is preferred through deflectometry [6,7]. Images are frozen using a flash triggered by the driving movement with a 0 or $\pi/2$ lag to get the images respectively related to the real or imaginary parts of the response.

Deflection and curvature fields are respectively obtained from spatial numerical integration and differentiation of polynomial fits to the slope fields (see Fig. 2).
3.3. Identification results
Identification has been carried out using Eq. 7 with a set of two displacement virtual fields selected such that they are null along the rectangular border \( \Gamma \) of the \( \Pi \) zone surrounding the clamping area (central area in white on Fig. 2). The two stiffness parameters \( D_{11}, D_{12} \) and the two viscous damping parameters \( B_{11}, B_{12} \) of the tested material are first extracted. According to \[8\], material elastic properties, Young modulus \( E \) and Poisson ratio \( \nu \) and their related loss factors \( \eta = \tan(\delta) \) and \( \eta_{\nu} \) are presented in Table 1 below for five excitation frequencies. It must be noticed the 100 Hz excitation is close to a resonance of the tested plate and the four other frequencies are out of resonance. The identified values are compared with the Young modulus and its loss factor obtained from, respectively, vibrating tests performed by the authors in the same frequency range on a beam and classical DMTA tests. Coefficients of variation resulting from eight measurements at each tested frequency are noted in italic blue.

| Table 1 – First identification results |
|--------------------------------------|
| \( E \) (GPa) | 70 Hz | 80 Hz | 90 Hz | 100 Hz | 110 Hz | Beam/DMTA |
| --- | --- | --- | --- | --- | --- | --- |
| \( 4.62 \) | 1.8% | 4.63 | 1.3% | 4.71 | 1.2% | 4.63 | 2.4% | 4.69 | 0.6% | 4.9 | 2.4% |
| \( \nu \) | 0.33 | 1.3% | 0.33 | 0.5% | 0.33 | 0.8% | 0.34 | 1.7% | 0.34 | 0.6% |
| \( \eta = \tan(\delta) \) | 0.065 | 2.9% | 0.052 | 1.8% | 0.050 | 1.8% | 0.064 | 7.1% | 0.068 | 0.3% | 0.054 | 3.5% |
| \( \eta_{\nu} \) | -0.024 | 10% | 0.019 | 4.7% | -0.014 | 6% | -0.018 | 9.1% | 0.021 | 0.8% |

The results for the elastic constants seem to be consistent and no significant differences are noticeable between the results issued from measurements at resonance or out of resonance. Some improvements are needed to increase the accuracy of image freezing in the time domain. The present trigger process is probably responsible for the higher scatter in the two loss factors compared to the elastic constants.

4. Insensitivity to boundary conditions
As mentioned in section 2, the selection of appropriate virtual fields leads to the cancellation of the virtual works of the junction forces along the \( \Gamma \) border. Therefore if dissipation phenomena occur in the clamping area, this does not affect the global equilibrium of the \( \Omega \) observed part of the coupon as mentioned in Eq. 6. As a consequence, the identification process should not be disturbed.

To prove this very important feature of the method, a second experimentation was carried out where the same coupon was alternatively mounted at the end of the driving rod, first rigidly clamped as previously described in section 3. Secondly the same fixture is used with two rubber washers (1.5 mm thick) inserted between each face of the plate and the rigid parts of the driving device. In the first case, due to the rigid clamping, the displacement of the rod and the driving movement of the plate are the same. In the second case, a miniature accelerometer was mounted directly on the rear face of the plate, as closely as possible to the clamping area (see Fig. 3). This transducer provides a good evaluation of the out of plane solid movement of the plate. The comparison between the two acceleration signals provided by this accelerometer and the one mounted on the driving rod shows the amplitude of the solid motion of the plate is 14% lower than the amplitude of the movement of the rod and the phase lag is 34°. These results are obviously due to the dissipative behaviour of the rubber washers.
Fig. 3 – Details of second tested attachment of the plate

Identifications were carried out using images taken at 100 Hz with the two configurations. The results are presented in Table 2 in the same manner as above.

Table 2 – Identification results at 100 Hz with the two different attachments

|                  | 100 Hz | Initial clamping | Rubber washers |
|------------------|--------|------------------|----------------|
| E (GPa)          | 4.63   | 4.73             |
| υ                | 0.34   | 0.33             |
| η=E=tan(δ)       | 0.064  | 0.055            |
| ηυ               | -0.017 | -0.018           |

These results confirm that only the material properties of the Ω area of interest are identified without effects of the external conditions, which is one of the strengths of the method.

5. Conclusion
Previous studies have shown the principle of the Virtual Fields Method applied to vibrating plates and demonstrated the experimental feasibility. The last tests have proved that the method greatly minimizes the parasitic effects due to experimental conditions. This property obviously deserves major attention since it is of great interest for the identification of material damping. Very important potential applications of the method concern the identification of local material damping in a structure (plate or shell with stiffeners, for instance) and the possibility of obtaining local damping values at different locations of a plate or structure to be used as damage indicators. This is presently underway on composite plates.

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