Robust Hybrid Zero-Order Optimization Algorithms with Acceleration via Averaging in Continuous Time

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Abstract

We study novel robust zero-order algorithms with acceleration for the solution of real-time optimization problems. In particular, we propose a family of derivative-free dynamics that can be universally modeled as singularly perturbed hybrid dynamical systems with resetting mechanisms. From this family of dynamics, we synthesize four algorithms designed for convex, strongly convex, constrained, and unconstrained zero-order optimization problems. In each case, we establish semi-global practical asymptotic or exponential stability results, and we show how to obtain well-posed discretized algorithms that retain the main properties of the original hybrid dynamics. Given that existing averaging theorems for singularly perturbed hybrid systems are not directly applicable to our setting, we derive a new averaging theorem that relaxes some of the existing assumptions in the literature. This allows us to make a clear link between the KL bounds that characterize the rates of convergence of the hybrid dynamics and their average dynamics. We also show that our results are applicable to non-hybrid zero-order dynamics, thus providing a unifying framework for hybrid and non-hybrid zero-order algorithms based on averaging theory. The superior performance of the hybrid algorithms compared to the traditional schemes that do not incorporate acceleration is illustrated via numerical examples.

1 Introduction

In this paper, we study algorithms for the solution of optimization problems of the form

\[
\min f(z) \quad \text{subject to} \quad z \in \mathcal{F},
\]

where \( f : \mathbb{R}^n \to \mathbb{R} \) is a smooth cost function, and \( \mathcal{F} \subset \mathbb{R}^n \) is a nonempty, closed, and convex set. Unlike standard model-based optimization problems considered in the literature, e.g., [26,5], our key assumption is that the mathematical forms of the cost function \( f(z) \) and its gradient are unknown. Indeed, we treat the cost function as a black-box, where only measurements of \( f(z) \) are available for the algorithms. This setting precludes the implementation of off-the-shelf first and second-order optimization algorithms, and instead calls for \textit{zero-order} methods that are gradient and Hessian-free. Algorithms of this form have seen a renewed interest in the control’s community due to the increasing complexity of the optimization problems that emerge in several feedback architectures [31]. For example, robust model-free optimization methods based on finite difference approximations have been studied in [42] using tools from differential inclusions and Lyapunov stability theory. Periodic and aperiodic sampled-data techniques for steady state model-free optimization have also been studied in [21] and [35]. Extremum seeking (ES) dynamics, a class of zero-order optimization algorithms based on averaging theory, have also been extensively studied in [27,2,3,18,41,19] and [30] for ODEs, and in [34,32] for hybrid inclusions.

One of the main designing goals in feedback-based optimization algorithms is to generate dynamical systems that induce fast rates of convergence for a general class of cost functions. This challenge has motivated in part the development of accelerated optimization algorithms that incorporate momentum in the dynamics, see for instance [26,40,48,12,11,14,46], and references therein. In the context of zero-order methods, discrete-time algorithms with momentum have been developed in [10], [9], and [8]. In the continuous-time domain, ES dynamics with momentum have been studied in [23] and [24]. However, no acceleration properties have been established so far for these type of ES dynamics, and the tradeoffs that may emerge between robustness and acceleration properties, recently studied in [33,25,7] for first order methods, remain mostly unexplored in continuous-time zero-order methods.

Motivated by this background, we present in this paper a novel class of zero-order optimization algorithms with fast rates of convergence based on averaging theory for hybrid dynamical systems (HDS). In particular, the main contributions of this paper are as follows:

(1) We present in Section 3.1 a novel class of \textit{robust} ES al-
accelerated discrete-time optimization algorithms [29], but setting and restarting mechanisms that have been used in the literature of ES for constrained optimization. Moreover, our theoretical results are general enough for the design and analysis of other hybrid ES dynamics that go beyond those considered in this paper.

The rest of this paper is organized as follows: Section 2 presents the preliminaries. Section 3 presents the main optimization algorithms, as well as their stability properties. Section 4 presents some numerical simulations that illustrate our theoretical results. Section 5 presents the averaging results needed for the analysis of the algorithms, Section 6 presents the stability analysis of the hybrid optimization dynamics, and finally Section 7 ends with some conclusions.

2 Preliminaries

The set of (nonnegative) real numbers is denoted by $(\mathbb{R}_{\geq 0})$. We use $\mathbb{B}$ to denote a closed unit ball of appropriate dimension, $\rho \mathbb{B}$ to denote a closed ball of radius $\rho > 0$, and $X+\rho \mathbb{B}$ to denote the union of all sets obtained by taking a closed ball of radius $\rho$ around each point in the set $X$. We use $S^2 \subset \mathbb{R}^2$ to denote the unit circle. For a compact set $A$ we use $|x|_A := \inf_{y \in A} \| y-x \|$. We use $X^n := X \times X \times \ldots \times X$ to denote the $n$-carthesian product of $X$. The $n \times n$ identity matrix is denoted as $I_n$, the vector of $n$-dimension with all entries equal to $c \in \mathbb{R}$ is defined as $c_n$, and $e_i$ is the unitary vector of appropriate dimension with $i^{th}$ entry equal to 1. A continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $K$ if $\alpha$ is zero at zero and strictly increasing. A function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $L$ if it is continuous, non-increasing, and converging to zero as its argument grows unbounded. A function $\beta$ is of class $KL$ if it is of class $K$ in its first argument, and of class $L$ in its second argument. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be radially unbounded if $f(x) \to \infty$ as $|x| \to \infty$.

In this paper, we use the framework of HDS [17] to study optimization algorithms. A HDS is modeled by the equations

\begin{align}
\dot{x} & = F(x), \\
x & \in C,
\end{align}

\begin{align}
\dot{x} & = G(x), \\
x & \in D,
\end{align}

where $x \in \mathbb{R}^n$ is the state, $F : \mathbb{R}^n \to \mathbb{R}^n$ is called the flow map, and $G : \mathbb{R}^n \to \mathbb{R}^n$ is called the jump map. The sets $C$ and $D$, called the flow set and the jump set, respectively, describe the set of points where the system evolves according to (2a), or (2b), respectively. The data of the HDS is defined as $H := \{ C, F, D, G \}$. Systems of the form (2) generalize purely continuous-time systems and purely discrete-time systems. Namely, continuous-time dynamical systems can be seen as a HDS of the form (2) with $D = \emptyset$, while discrete-time dynamical systems correspond to the case when $C = \emptyset$. Solutions of (2) are defined on hybrid time domains. For a precise definition of hybrid time domains and the concept of solutions in HDS we refer the reader to the Appendix C.

Definition 1. A HDS $H := \{ C, F, D, G \}$ is said to satisfy the Basic Conditions if $C$ and $D$ are closed sets, $F$ and $G$ are continuous functions, $C \subset \text{dom}(F)$ and $D \subset \text{dom}(G)$. □
In order to study the stability and convergence properties of our algorithms, we will use the following two notions:

**Definition 2** A compact set $A \subset \mathbb{R}^n$ is said to be uniformly globally asymptotically stable (UGAS) for a HDS $\mathcal{H}$ if there exists a $\beta \in \mathcal{K}L$ such that every solution of $\mathcal{H}$ satisfies

$$|x(t, j)|_A \leq \beta(|x(0, 0)|_A, t + j),$$

(3)

for all $(t, j) \in \text{dom}(x)$. When $\beta(r, s) = c_1 r \exp(-c_2 s)$ for some $c_1, c_2 > 0$, we say that the system is uniformly globally exponentially stable (UGES).

**Definition 3** For a parameterized HDS $\mathcal{H}_{\delta_1, \delta_2}$, a compact set $A \subset \mathbb{R}^n$ is said to be semi-globally practically asymptotically stable (SGPAS) as $(\delta_1, \delta_2) \to 0^+$ with $\beta \in \mathcal{K}L$ if there exists a $\beta \in \mathcal{K}L$ such that for each compact set $K \subset \mathbb{R}^n$ and each $\nu > 0$ there exists a $\delta_1^* > 0$ such that for each $\delta_1 \in (0, \delta_1^*)$ there exists a $\delta_2 > 0$ such that for each $\delta_2 \in (0, \delta_2^*)$ every solution $x_{\delta_1, \delta_2}$ of $\mathcal{H}_{\delta_1, \delta_2}$ with $x_{\delta_1, \delta_2}(0, 0) \in K$ satisfies

$$|x_{\delta_1, \delta_2}(t, j)|_A \leq \beta(|x_{\delta_1, \delta_2}(0, 0)|_A, t + j) + \nu,$$

(4)

for all $(t, j) \in \text{dom}(x_{\delta_1, \delta_2})$. When $\beta(r, s) = c_1 r \exp(-c_2 s)$ for some $c_1, c_2 > 0$, we say that the system is semi-globally practically exponentially stable (SGPES).

Note that when $D = \emptyset$, Definition 2 reduces to the classic UGAS notion for continuous-time systems, e.g., [20, Lemma 4.5]. Also, Definition 3 can be extended to parameterized hybrid systems with any number of parameters, i.e., $\mathcal{H}_{\delta_1, \delta_2, \ldots, \delta_p}$, $p \in \mathbb{Z}_{\geq 1}$. Of particular interest to us are the cases $p \in \{1, 2, 3\}$.

### 3 Robust Algorithms for Model-Free Optimization

We wish to design zero-order optimization algorithms that generate trajectories that converge to a $\nu$-neighborhood of the set of solutions of (1), given by

$$A_f := \{z^* \in F : f(z^*) \leq f(z), \forall z \in F\},$$

(5)

where $\nu > 0$ can be arbitrarily small. We are particularly interested in algorithms with desirable robustness properties with respect to small disturbances of arbitrary frequency acting on the states and dynamics of the system. We call this property structural robustness.

**Definition 4 (Structural Robustness)** Let $\mathcal{H}$ be a hybrid dynamical system of the form (2) rendering UGAS (resp. SGPAS as $\delta \to 0^+$) a compact set $A$ with $\beta \in \mathcal{K}L$. Let $\varepsilon : \text{dom}(e) \to \mathbb{R}^n$ be a measurable function satisfying $\sup_{t \geq 0} |e(t)| \leq \bar{\varepsilon}$. We say that $\mathcal{H}$ is Structurally Robust if the perturbed system

$$\begin{align*}
    x + \varepsilon &\in C, & \dot{x} &= F(x + \varepsilon) + \varepsilon, \\
    x + \varepsilon &\in D, & x^+ &= G(x + \varepsilon) + \varepsilon,
\end{align*}$$

(6a)

renders the set $A$ SGPAS as $\bar{\varepsilon} \to 0^+$ (resp. SGPAS as $(\delta, \bar{\varepsilon}) \to 0^+$) with $\beta \in \mathcal{K}L$. □

As noted in [33, Ex. 1], some accelerated optimization dynamics may not satisfy the robustness notion of Definition 4. Indeed, it can be shown that even for smooth strongly convex cost functions, the time-varying Nesterov’s ODE, given by $\dot{x} + \frac{2\kappa}{\tau^2} \nabla f(x) = 0$, $\tau = 1$, and studied in [40], can be rendered unstable under arbitrarily small disturbances on the gradient. This tension between robustness and acceleration, also discussed recently in [48, 7] and [25], makes the design of structurally robust zero-order accelerated optimization algorithms not trivial. In this paper, we will address this challenge by harnessing the frameworks of averaging theory and HDS.

#### 3.1 Zero-Order Hybrid Optimization Algorithms

Consider a family of algorithms modeled as a HDS with states $(x, \tau, \mu) \in \mathbb{R}^{n+m+1+2n}$, with continuous-time dynamics given by

$$\begin{pmatrix}
    \dot{x} \\
    \dot{\tau} \\
    \dot{\mu}
\end{pmatrix} = F(x, \tau, \mu) := \begin{pmatrix}
    F_x(x, \mu, f(z)) \\
    F_\tau \\
    \frac{1}{\tau} R \mu
\end{pmatrix},$$

(7)

which are allowed to evolve whenever the states satisfy the condition

$$(x, \tau, \mu) \in C := \mathbb{R}^{n+m} \times \mathcal{T}_C \times \mathbb{S}^n.$$  

(8)

The discrete-time dynamics of the HDS are defined as

$$\begin{pmatrix}
    x^+ \\
    \tau^+ \\
    \mu^+
\end{pmatrix} = G(x, \tau, \mu) := \begin{pmatrix}
    G_x(x) \\
    T_{\min} \\
    \mu
\end{pmatrix},$$

(9)

where are allowed to evolve whenever the states satisfy the condition

$$(x, \tau, \mu) \in D := \mathbb{R}^{n+m} \times \mathcal{T}_D \times \mathbb{S}^n.$$  

(10)

In this HDS, the state $x := [x_1^T, x_2^T] \in \mathbb{R}^{n+m}$ has two main components, with $x_1$ acting as the main state, and $x_2$ acting as an auxiliary variable that can be used to incorporate momentum or dual variables. The state $\tau$ models a scalar clock that coordinates the flows and jumps in the system. The state $\mu$ models a vector of periodic exploration signals evolving on the $n$-torus $\mathbb{S}^n$. The constant $\varepsilon > 0$ is a tunable parameter, and the matrix $R \in \mathbb{R}^{2n \times 2n}$ in (7) is block diagonal, with $\delta^{th}$ block given by $R_{\ell} := 2\pi\kappa_\ell \cdot [-\varepsilon_2, \varepsilon_1] \in \mathbb{R}^{2 \times 2}$. The parameters $\kappa_\ell \in \mathbb{R}_{>0}$ are chosen to satisfy the following Assumption.

**Assumption 1** For each $\ell \in \{1, 2, \ldots, n\}$ the parameter $\kappa_\ell$ is a positive rational number, and $\kappa_\ell \neq \kappa_j$ for all $j \neq \ell$. □
The dynamics of the HDS (7)-(10) have two main components, which are closely related to the ideas of exploration and exploitation:

**Exploration:** The continuous-time dynamics of \( \mu \), corresponding to \( n \) uncoupled oscillators, each one evolving on \( S^1 \subset \mathbb{R}^2 \), are in charge of providing sufficient exploration to the system. During flows, these dynamics generate a vector of signals \( \mu \) with tunable frequency \( \varepsilon^{-1} \), and odd entries given by

\[
\mu_i(t) = \Psi_i(t)^\top \psi_{0,i} \quad i \in \{1, 3, 5, \ldots, 2n - 1\},
\]

where \( \psi_{0,i} := [\mu_i(0), \mu_{i+1}(0)]^\top \) and

\[
\Psi_i(t) := \left[ \cos\left(\frac{2\pi t}{\varepsilon \kappa_{\psi_i}}\right), \sin\left(\frac{2\pi t}{\varepsilon \kappa_{\psi_i}}\right) \right]^\top.
\]

During the jumps (9), the state \( \mu \) is kept constant. The signals (11) will be added to the argument of \( f(z) \) in order to facilitate an online approximation of \( \nabla f \) via averaging theory. The following Lemma will be fundamental in order to achieve this task. The proof is presented in the Appendix.

**Lemma 1** Suppose that Assumption 1 holds and let \( \hat{\mu} \) be defined as in (14). Then, there exists a \( T > 0 \) such that every solution \( \mu : \mathbb{R}_{\geq 0} \to \mathbb{R}^{2n} \) of \( \dot{\mu} = R\mu \) with \( \mu(0) \in S^n \) satisfies:

\[
\frac{1}{kT} \int_0^{kT} \hat{\mu}(\tau)\hat{\mu}(\tau)^\top d\tau = \frac{1}{2} I_n, \quad \int_0^{kT} \hat{\mu}(\tau)d\tau = 0_n, \tag{13}
\]

for any \( k \in \mathbb{Z}_{>0} \), where \( \hat{\mu} \) is the vector of odd entries of \( \mu \), i.e.,

\[
\hat{\mu} := [\mu_1, \mu_3, \mu_5, \ldots, \mu_{2n-1}]^\top \in \mathbb{R}^n. \tag{14}
\]

**Exploitation:** The hybrid dynamics of the states \((x, \tau)\), which depend on the mappings \((F_x, F_\tau, G_x)\), as well as the sets \((T_D, T_C)\), are in charge of the exploitation in the optimization algorithm, and they will be designed based on the qualitative structure of \( f \) and \( F \) in (1). In this paper, we will focus on four main qualitative cases: I) convex cost functions with no constraints, II) strongly convex functions with no constraints, III) strongly convex functions with equality constraints, and IV) strongly convex functions with inequality constraints. For each case, the hybrid dynamics of the states \((x, \tau)\) will be designed to guarantee convergence of the state component \( x_1 \) to a neighborhood of the optimal set (5).

Based on these ideas, exploration and exploitation is simultaneously achieved by defining the input \( z \) in (7) as

\[
z := x_1 + a\hat{\mu}, \tag{15}
\]

where the amplitude \( a \in \mathbb{R}_{>0} \) is a tunable parameter. Equation (15) implies that the time domain of \( z \) will be the same hybrid time domain of the states \((x, \tau, \mu)\). Since we will design the sets \((T_D, T_C)\) such that \( T_D \subset T_C \), the stability and convergence properties of the HDS (7)-(10) will be studied with respect to a universal compact set of the form

\[
A := A_x \times T_C \times \mathbb{S}^n, \tag{16}
\]

where \( A_x \subset \mathbb{R}^{n+m} \) is a compact set having the property that its projection in \( \mathbb{R}^n \) coincides with the set of solutions of (1), i.e.,

\[
\{x_1 \in \mathbb{R}^n : (x_1, x_2) \in A_x\} = A_f. \tag{17}
\]

We proceed to characterize, for each particular case, the mappings \((F_x, F_\tau, G_x)\) and the sets \((T_C, T_D)\) of the HDS (7)-(10).

### 3.1.1 Case I: Unconstrained Convex Optimization

We first consider the case when \( F := \mathbb{R}^n \), and the cost function \( f \) satisfies the following Assumption:

**Assumption 2** The cost function \( f(z) \) is twice continuously differentiable, convex, radially unbounded, and satisfies at least one of the following conditions: (a) The function \( f(z) \) has a unique minimizer. (b) The gradient \( \nabla f(z) \) is globally Lipschitz.

Based on Assumption 2, we consider the HDS (7)-(10) with \( m = n \), and the mappings

\[
F_x := \begin{pmatrix}
\frac{2}{\tau}(x_2 - x_1) \\
-c\tau f(z)\hat{\mu}
\end{pmatrix}, \quad F_\tau = \frac{1}{2}, \quad G_x := \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}, \tag{18}
\]

where \( c > 0 \) is a tunable gain, \( z \) is given by (15), and

\[
T_C := [T_{\min}, T_{\max}], \quad T_D := [T_{\med}, T_{\max}], \tag{19}
\]

with \( T_{\max} \geq T_{\med} > T_{\min} \). This hybrid system incorporates momentum during the flows via the state \( x_2 \), and it generates multiple solutions, including periodic and aperiodic solutions. In particular, it allows jumps whenever \( \tau \geq T_{\med} \), but no later than when \( \tau = T_{\max} \). Since \( T_{\med} > T_{\min} \), after the first jump this system will necessarily flow for at least \( T_{\max} - T_{\min} \) seconds before jumps are again allowed. Thus, the HDS has no purely discrete or Zeno solutions. For the case when \( T_{\max} = T_{\med} \) the jumps are periodic. In order to get suitable online approximations of \( \nabla f \), the parameter \( a > 0 \) in (18) is selected to be the same of (15).

The following theorem characterizes the convergence, stability and robustness properties of the HESwA (7)-(10) with respect to the compact set (16), with \( A_x \) defined as

\[
A_x := \{x \in \mathbb{R}^{2n} : x_1 = x_2, \ x_1 \in A_f\}. \tag{20}
\]

The proof is presented in Section 6.
**Theorem 1** Consider the set $\mathcal{A}$ in (16) with $A_x$ given by (20). Suppose that Assumptions 1-2 hold, and consider the HESwA (7)-(10) with maps $(F_x, F_\tau, G_x)$ given by (18), and sets $T_C$ and $T_D$ given by (19). Then, the following holds:

(a) The HDS satisfies the Basic Conditions, and all maximal solutions have an unbounded time domain.

(b) There exists a $\beta_1 \in KL$ such that the set $\mathcal{A}$ is SGFAS as $(a, \omega^{-1}) \to 0$ with $\beta_1$.

(c) For each compact set $K_0 \subseteq \mathbb{R}^{2n}$ such that the $\mathcal{A}_x \subseteq K_0$, and for each $\nu > 0$, there exists $\delta_1^\nu > 0$ such that for all $\delta_1 \in (0, \delta_1^\nu)$ there is $\delta_1 \in (0, a^\nu)$ such that for all $\alpha \in (0, a^\nu)$, all solutions with $x(0, 0) \in K_0$ and all $(t, j) \in \text{dom}(x, \tau, \mu)$, the following bound holds:

$$f(z(t, j)) - f(A_f) \leq \frac{\nu_j}{(t - \delta_1^\nu)^2} + \nu, \quad t > \delta_1^\nu,$$

where $\delta_1^\nu = \inf\{t \geq 0 : (t, j) \in \text{dom}(x, \mu, \tau)\} + \delta_1$, and $\nu_j \to 0^+$ is a monotonically decreasing sequence (defined for each solution) satisfying $\nu_0 < \nu^*$. □

In words, items (a) and (b) of Theorem 1 establish that by tuning the parameters $(a, \nu)$ in the HDS (7)-(10), one can guarantee uniform convergence of the state $x_1$, and therefore $z$ via (15), from compact sets of initial conditions to arbitrarily small $\nu$-neighborhoods of $A_f$. On the other hand, item (c) describes a semi-acceleration property: For each $j \in \mathbb{Z}_{\geq 0}$, and all $t > \delta_1^\nu$, the sub-optimality measure $f(z) - f(A_f)$ will decrease at a rate of $O_j(1/ (t - \delta_1^\nu)^2)$ outside a $\nu$-neighborhood of $f^*$. For instance, if one selects $\tau(0, 0) = T_{\text{min}}$, then $j = 0$, $\delta_1^\nu = \delta_1$, and the semi-acceleration bound $O(1/ (t - \delta_1^\nu)^2)$ will hold during the interval of flow $(\delta_1, T_{\text{med}} - T_{\text{min}}]$, which can be made arbitrarily large by the choice of $T_{\text{med}}$, and arrive close to the complete first interval of flow $[0, T_{\text{med}} - T_{\text{min}}]$ by decreasing $\delta_1$.

**Remark 1** (Connections with time-varying Nesterov’s ODE) In order to achieve the semi-acceleration property for non-strongly convex functions, the flow map in (18) of the HESwA is designed to be intrinsically related to the time-varying Nesterov’s ODE [40]. Indeed, when $k_1 = 1$, by considering the change of variables $x = x_1$ and $x_2 = x - 0.5\tau \dot{x}$, the flows of the average system, obtained by averaging (18) along the solutions of $\mu$, are characterized by the dynamics

$$\ddot{x} + \frac{2 + \tau}{\tau} \dot{x} + 4c \nabla f(x) = 0. \quad (22)$$

For the case when $\tau = 1$, $c = 0.25$, $T_{\text{min}} = 0$, $T_{\text{med}} = \infty$, and $\dot{x}(0) = \tau(0) = 0$, equation (22) corresponds to the time-varying ODE studied in [40]. For the case when $c = 1$, $\tau = 0.5$ and $\tau(0) \geq 1$, equation (22) corresponds to the ODE recently studied in [50]. When $\tau = 0$ equation (22) reduces to the Heavy-Ball algorithm [1] studied in the context of ES in [23].

Based on Remark 1, our HESwA can be seen as a zero-order hybrid time-invariant regularization of the Nesterov’s ODE that not only minimizes the sub-optimality measure $f(z) - f(A_f)$ at the rate (21) during each interval of flow, but which also renders SGFAS the set $A_x$ by persistently resetting the clock $\tau$. For this HDS, there are clear tradeoffs between robustness and acceleration: as $T_{\text{med}} \to \infty$ and the intervals of flow are larger, the robustness margins $\tilde{c}$ of the perturbed system (6) shrink to zero, as the HDS starts to approximate the time-varying Nesterov’s ODE with no hybrid regularization [33, Ex. 1]. Thus, the tuning of the parameter $T_{\text{med}}$ is critical in order to obtain a good tradeoff between longer periods of flow with acceleration (21) and larger margins of robustness.

While for convex functions the HESwA (7)-(10) induces a semi-acceleration property during the flows, for cost functions that are additionally strongly convex and have a globally Lipschitz gradient, a simple modification of the jump map of (18) guarantees uniform global exponential convergence.

### 3.1.2 Case II: Unconstrained Strongly Convex Optimization

We now consider the following assumption on $f(z)$:

**Assumption 3** The cost function $f(z)$ is twice continuously differentiable, $\theta$-strongly convex, and has $L$-Lipschitz gradient, i.e., it satisfies:

- $|\nabla f(z_1) - \nabla f(z_2)| \leq L|z_1 - z_2|, \quad \forall z_1, z_2 \in \mathbb{R}^n$, (23a)

- $f(z_1) - f(A_f) \geq \frac{\theta}{2}|z - z^*|^2, \quad \forall z \in \mathbb{R}^n$, (23b)

for some $[\theta, L]^T \in \mathbb{R}^{2n}$. □

For cost functions satisfying Assumption 3, we consider the HESwA (7)-(10) with $m = n$, and the mappings

$$F_x := \begin{pmatrix} \frac{2}{\tau}(x_2 - x_1) \\ -\frac{4c}{\sqrt{\tau}} f(z) \tilde{\mu} \end{pmatrix}, \quad F_\tau = \frac{1}{2}, \quad G_x := \begin{pmatrix} x_1 \\ x_1 \end{pmatrix}, \quad (24)$$

where $c > 0$ is again a tunable gain, $z$ is given by (15), and

$$T_C := \{T_{\text{min}}, T_{\text{max}}\}, \quad T_D := \{T_{\text{max}}\}, \quad (25)$$

with $T_{\text{max}} > T_{\text{min}}$. These sets and the dynamics of the clock $\tau$ induce periodic jumps with period equal to $T_{\text{max}} - T_{\text{min}}$. For this HDS, the solutions are unique, and its stability, convergence, and robustness properties are characterized by the following Theorem. The proof is presented in Section 6.

**Theorem 2** Consider the set $\mathcal{A}$ given by (16) with $A_x$ given by (20). Suppose that Assumptions 1 and 3 hold, and consider the HESwA (7)-(10) with maps $(F_x, F_\tau, G_x)$ given by...
(24), and sets $\mathcal{T}_C$ and $\mathcal{T}_D$ given by (25). If the parameters $(T_{\min}, T_{\max}, c)$ are selected such that

$$T_{\max}^2 - T_{\min}^2 \geq \frac{1}{2ghc},$$

then the following holds:

(a) The HDS satisfies the Basic Conditions, and all maximal solutions have an unbounded time domain.

(b) There exists an exponential $\beta_2 \in KL$ such that the compact set $A$ is SGPES as $(a, \omega^{-1}) \rightarrow 0$ with $\beta_2$.

(c) For each compact set $K_0 \subset \mathbb{R}^{2n}$ such that $A_z \subset K_0$, and each $\nu > 0$ there exists $a^* > 0$ such that for all $a \in (0, a^*)$ there exists $\varepsilon > 0$ such that for all $x \in (0, \varepsilon)$ every solution with $x(0, 0) \in K_0$ induces the bound:

$$|z(t, j)|_{A_z} \leq c_2 |x(0, 0)|_{A_x} \exp \left(-c_3 (t + j)\right) + \nu, \quad (27)$$

for all $(t, j) \in \text{dom}(x, \tau, \mu)$, where $c_2, c_3 > 0$. □

The result of Theorem 2 establishes that condition (26) is sufficient to achieve exponential convergence of $z$ to a neighborhood of $A_f$, where the constants $c_2$ and $c_3$ are characterized in Section 6.2.2. When $T_{\min} + T_{\max} > 1$, condition (26) is satisfied by a standard dwell-time condition of the form $T_{\max} - T_{\min} > (2\varepsilon)\tau^{-1}$. Note, however, that since $\tau$ is unknown, this condition cannot be verified a priori. Optimal choices of the period $\Delta T = T_{\max} - T_{\min}$ can also be studied as in [33,29]. However, these choices usually require exact knowledge of the constants of (23), and therefore they are difficult to implement in model-free algorithms such as the HESwA.

Remark 2: Since the flow map $F_z \times F_\tau$ in (24) is the same as in (18), the resulting HESwA can be seen as a hybrid time-invariant regularization that periodically resets the clock $\tau$ and the speed $\dot{x}_1$. Such regularizations are known to improve the transient performance in model-based first-order optimization dynamics with acceleration [29,33]. However, to the knowledge of the authors, stability, robustness, and convergence certificates for model-free zero-order hybrid optimization dynamics with acceleration were absent in the literature.

The setting of zero-order HDS can also be used to study optimization algorithms modeled as ODEs. Next, Cases III and IV leverage this property.

3.1.3 Case III: Strongly Convex Optimization with Linear Equality Constraints

We now consider the case when the feasible set has the form

$$\mathcal{F} = \{z \in \mathbb{R}^n : Az = b\}, \quad b \in \mathbb{R}^m, \quad A \in \mathbb{R}^{m \times n}, \quad (28)$$

and satisfies the following Assumption:

Assumption 4: The matrix $A$ is full row rank, $\kappa_1 I \leq AA^T \leq \kappa_2 I$, for some $\kappa_1, \kappa_2 > 0$.

For problem (1) with feasible sets of the form (28), we consider zero-order dynamics (7)-(10) with mappings

$$F_z := \left(-\frac{2}{\alpha} f(z) \tilde{\mu} - k_1 A^T x_2 \right), \quad F_\tau = 0, \quad G_z := x, \quad (29)$$

where $z$ is again given by (15). Given that these dynamics are independent of $\tau$ we now define the sets $\mathcal{T}_C$ and $\mathcal{T}_D$ as follows:

$$\mathcal{T}_C := \{T_{\min}\}, \quad \mathcal{T}_D := \emptyset, \quad (30)$$

which implies that the jump set of the resulting HDS is empty and therefore the system never jumps (i.e., $\tau = 0$ always holds), which implies that solutions can be completely parameterized by the continuous-time index $t$, i.e., as a regular ODE. The dynamics (29) can be seen as a type of zero-order Primal-Dual dynamics [36,15]. For this system we study the stability properties of the set

$$A_z := \{(z_1^*, z_2^*) \in \mathbb{R}^{n+m} : L(z_1^*, x_2) \leq L(z_1^*, z_2^*) \leq L(x_1, z_2^*) , \quad \forall x_1 \neq z_1^*, x_2 \neq z_2^* \}, \quad (31)$$

which is the set of saddle points of the Lagrangian

$$L(x_1, x_2) = f(x_1) + x_2^T (Ax_1 - b). \quad (32)$$

The following theorem establishes a semi-global practical exponential stability result for the HESwA.

Theorem 3: Consider the set $A$ given by (16) with $A_z$ given by (31) with Lagrangian (32). Suppose that Assumptions 1, 3, and 4 hold, and consider the HESwA (7)-(10) with mappings $(F_z, F_\tau, G_z)$ given by (29), and sets $\mathcal{T}_C$ and $\mathcal{T}_D$ given by (30). Then, the following holds:

(1) The HDS satisfies the Basic Conditions, every maximal solution has an unbounded time domain, and the set (31) is a singleton and satisfies (17).

(2) There exists an exponential $\beta_3 \in KL$ such that set $A$ is SGPES as $(a, \omega^{-1}) \rightarrow 0$ with $\beta_3$.

(3) For each compact set $K_0 \subset \mathbb{R}^{n+m}$ such that $A_z \subset K_0$, and each $\nu > 0$ there exists $a^* > 0$ such that for all $a \in (0, a^*)$ there exists $\varepsilon > 0$ such that for all $x \in (0, \varepsilon)$ every solution with $x(0) \in K_0$ induces the bound:

$$|z(t)|_{A_z} \leq c_4 |x(0)|_{A_x} \exp \left(-c_5 t\right) + \nu, \quad (33)$$

for all $t \geq 0$, where $c_4, c_5 > 0$.

Primal-dual zero-order dynamics that are similar to (29) are also considered in [13,49]. However, for strongly convex functions the results of [13,49] establish only an asymptotic result. Theorem 3 establishes that this result is, indeed, exponential, and, as it will be shown in Section 3.2, it also provides robustness guarantees under small bounded disturbances.
3.1.4 Case IV: Strongly Convex Cost Function with Inequality Constraints

We now consider strongly convex functions with feasible set

\[ F := \{ z \in \mathbb{R}^n : A z \leq b \}, \quad A \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^m. \]  

(34)

In this case, the mappings \( F_x \) and \( G_x \) are now defined as:

\[ F_x := \left( -\frac{1}{2} f'(z) \mu - k_1 \sum_{j=1}^m H_j(x) A_j, \sum_{j=1}^m (H_j(x) - x_{2,j}) e_j \right), \quad F_r = 0, \quad G_x := \{ x \}, \]

(35)

where \( A^T := [A_1, A_2, \ldots, A_m], A_i \in \mathbb{R}^n \) for all \( i, b := [b_1, b_2, \ldots, b_m]^T \), \( H_j(x) = \max (A_j^T x - b_j + x_{2,j}, 0) \), and the sets \( \mathcal{T}_C \) and \( \mathcal{T}_D \) are again given by (30), i.e., the system has no jumps. The flow map in (35) is related to a class of zero-order Augmented Primal-Dual Gradient Dynamics [36] defined with respect to the Augmented Lagrangian

\[ \mathcal{L}(x_1, x_2) := f(x_1) + \sum_{j=1}^m H_j(x). \]  

(36)

The following Theorem also establishes a semi-global practical exponential stability result for Case IV.

**Theorem 4** Consider the set \( \mathcal{A} \) given by (16) generated from the set \( \mathcal{A}_n \) given by (31) with Augmented Lagrangian (36). Suppose that Assumptions 1, 3, and 4 hold and consider the HESwA (7)-(10) with maps \( (F_x, F_r, G_x) \) given by (35), and sets \( \mathcal{T}_C \) and \( \mathcal{T}_D \) defined with respect to the Augmented Lagrangian. Then, the following holds:

1. The HDS satisfies the Basic Conditions, every maximal solution has an unbounded time domain, and the set of saddle points is a singleton and satisfies (17).
2. There exists an exponential \( \beta \in \mathcal{KL} \) such that set \( \mathcal{A} \) is SGPES as \( (a, \omega^{-1}) \to 0 \) with \( \beta \).
3. For each compact set \( K_0 \subset \mathbb{R}^{n+m} \) such that \( \mathcal{A} \subset K_0 \), and each \( \nu > 0 \) there exists sufficiently small \( a^* \) such that for all \( a \in (0, a^*) \) there exists \( \varepsilon^* \) such that for all \( \varepsilon \in (0, \varepsilon^*) \) every solution with \( x(0, 0) \in K_0 \) induces the bound:

\[ |z(t)|_{A_t} \leq c_0 |x(0)|_{A_0} \exp (-c_7 t) + \nu, \]  

(37)

for all \( t \geq 0 \), where \( c_6, c_7 > 0 \).

\[ \square \]

3.2 Structural Robustness

Since all the HESwA designed in the previous subsections satisfy the Basic Conditions, and also render the compact set \( \mathcal{A} \) SGPAS as \( (a, \omega^{-1}) \to 0 \), we can now directly establish structural robustness properties.

**Corollary 5** Under the corresponding Assumptions of Theorems 1, 2, 3, and 4, the family of HDS model by (7)-(10) are structurally robust in the sense of Definition 4.

\[ \square \]

Structural robustness allow us to guarantee that arbitrarily small disturbances \( \varepsilon \) acting on states and dynamics will not dramatically modify the stability properties of all the HESwA. To the knowledge of the authors, this result is novel in the setting of zero-order hybrid optimization algorithms with acceleration. Indeed, Corollary 5 follows directly by the following Proposition, which is a modest extension of [17, Lem. 7.20] for the case when a nominal HDS renders a compact set SGPAS instead of UGAS. The proof is almost identical to the proof of [17, Lem. 7.20] and it is presented in the Appendix for the sake of completeness.

**Proposition 6** Suppose that a \( \delta \)-parameterized hybrid system \( \mathcal{H}_\delta := \{ C_{\delta}, D_{\delta}, F_{\delta}, G_{\delta} \} \) satisfies the Basic Conditions for each \( \delta > 0 \), and that it renders a compact set \( \mathcal{A} \) SGPAS as \( \delta \to 0^+ \) with \( \beta \in \mathcal{KL} \). Then, the \( \rho \)-inflated system \( \mathcal{H}_{\delta, \rho} := \{ C_{\delta, \rho}, D_{\delta, \rho}, F_{\delta, \rho}, G_{\delta, \rho} \} \) with data:

\[ F_{\delta, \rho}(x) := \overline{\cup} F_{\delta}((x + \rho B) \cap C_{\delta}) + \rho B \]  

(38a)

\[ G_{\delta, \rho}(x) := \{ v \in \mathbb{R}^n : v \in g + \rho B, g \in G_{\delta}((x + \rho B) \cap D_{\delta}) \} \]  

(38b)

\[ C_{\delta, \rho} := \{ x \in \mathbb{R}^n : (x + \rho B) \cap C_{\delta} \neq \emptyset \} \]  

(38c)

\[ D_{\delta, \rho} := \{ x \in \mathbb{R}^n : (x + \rho B) \cap D_{\delta} \neq \emptyset \} \]  

(38d)

renders the set \( \mathcal{A} \) SGPAS as \( (\delta, \rho) \to 0 \) with \( \beta \in \mathcal{KL} \).

\[ \square \]

3.3 Stable Discretization via Forward-Euler and Consistent Runge-Kutta

Hybrid systems satisfying the Basic Conditions and having suitable stability properties are “stable” under Euler and Runge-Kutta discretizations [37, Thm. 3.5]. In this section, we leverage this property to construct suitable discretizations for the HESwA (7)-(10) with step size \( h > 0 \), such that the resulting zero-order discretized system with overall state \( \bar{x}_h = (x_h, \tau_h, \mu_h) \), and dynamics

\[ \bar{x}_h \in C_h, \quad \bar{x}_h^+ = F_h(\bar{x}_h), \]  

(39a)

\[ \bar{x}_h \in D_h, \quad \bar{x}_h^+ = G_h(\bar{x}_h), \]  

(39b)

retains the \( \mathcal{KL} \) convergence bounds of the original zero-order hybrid dynamics, up to a time scaling of \( t = \ell h \), where \( \ell \) is the discrete-time index of the discretized flows (39a). In order to do this, we rely on the notion of well-posed hybrid simulators, introduced in [37].

**Definition 5** The discretized HESwA obtained from (7)-(10), and denoted by \( \mathcal{H}_h \), is said to be well-posed if the discretized data \( (F_h, C_h, G_h, D_h) \) satisfies the following conditions:

(a) \( F_h \) is such that, for each compact set \( K \subset \mathbb{R}^{n+m+1+2n} \), there exists a function \( \rho \in \mathcal{KL} \) and \( h^* > 0 \) such that for each \( \bar{x}_h \in C_h \cap K \) and each \( h \in (0, h^*) \)

\[ F_h(\bar{x}_h) \subset \bar{x}_h + h \overline{\cup} F(\bar{x}_h + \rho(h) B) + h \rho(h) B. \]
(b) $G_h$ is such that for any decreasing sequence $h_i \to 0$ we have that $G_0 = G(\bar{x}_h)$, where $G_0$ is the graphical limit [17, Def. 5.18] of $G$. Let $D = \sum_{k=1}^{S} b_k = 1$, and $S = \{1, 2, \ldots, \bar{s}\}$, $\bar{s} \in \mathbb{Z}_{\geq 1}$. Then, by [37, Ex. 4.8 & 4.9] the forward-Euler method $F_h(\bar{x}_h) := \bar{x}_h + hF(\bar{x}_h)$ and the $S$-Order Runge-Kutta (RK) discretization method, defined as

$$F_h(\bar{x}_h) := \bar{x}_h + h \sum_{k=1}^{S} b_k F(g_k), \quad g_k = \bar{x}_h + h \sum_{\ell=1}^{i-1} a_{\ell j} F(g_{\ell}),$$

generate mappings $F_h$ that satisfy condition (a) in Definition 5. Since in the HDS (7)-(10) the clock $\tau$ flows in the interval $[T_{\min}, T_{\max}]$, initial conditions satisfying $\tau(0, 0) < T_{\max}$ and located arbitrarily close to $T_{\max}$ could lead to discretized flows $\tau^h = \tau_h + h F_\tau(\tau_h) > T_{\max}$ that prematurely leave the flow set without hitting the jump set $T_{\max}$. In order to avoid this, we can consider a discretized jump set given by

$$D_h := D \cup \{ \bar{x}_h : y \in C_h, \bar{x}_h = F_h(y) \notin C \},$$

(40)

which inflates the nominal jump set $D = \mathbb{R}^{n+m} \times T_D \times S^n$ in order to include the points that leave the discretized flow set $C_s$ after a discretized flow. This discretized flow set is defined as $C_h := \mathbb{R}^{n+m} \times T_C \times (\mathbb{S}^n + \rho(h))$, where $\rho \in \mathcal{K}_\infty$.

The following Proposition shows that the convergence properties of the set $A$, given by (16), are maintained for the discretized HESwA $H_h := \{C_h, F_h, D_h, G\}$, provided the step size is sufficiently small. The proof is a straightforward combination of [37, Lemma 5.1] with Proposition 6 and the stability results of Theorems 1-4, and therefore it is omitted for the sake of space.

**Proposition 7** Consider the discretized system $H := \{C_h, F_h, D_h, G\}$ with state $\bar{x}_h = (x_h, \tau_h, \mu_h)$, where $F_h$ is given either by the Forward Euler or $S$-Order RK discretization, and $D_h$ is given by (40). Suppose that Assumption 1 holds, as well as the Assumptions of Sections 3.1.1-3.1.4 for their respective HESwA (7)-(10). Then, the set $A$ is SGPAS as $(a, \omega^{-1}, h) \to 0^+$ with KL bound

$$[\bar{x}_h(\ell, j)]_A \leq \beta_i([\bar{x}_h(0, 0)]_A, \ell h + j) + \nu,$$

(41)

for all $(\ell, j) \in \text{dom}(\bar{x}_h)$, where $\ell \in \mathbb{Z}_{\geq 0}$ is the index of the discretized flows, and $\beta_i$ is the KL function generated by item (b) in Theorem i, with $i \in \{1, 2, 3, 4\}$. □

Since in Cases II, III, and IV, the KL bound $\beta_i$ is exponential, Proposition 7 guarantees a geometric rate of convergence for the discretized HESwA. To guarantee completeness of solutions, the jump set can be slightly modified as

$$G_h = (G^T_x, T_{\min}, \text{proj}_{S^n}(\mu_h))^T,$$

which sends $\mu_h$ back to $S^n$ whenever there is a jump. For $h$ sufficiently small and $j > 1$, jumps will always be triggered by the clock $\tau_h$.

**4 Numerical Examples**

**4.1 Non Strongly Convex Functions**

Consider the function $f(z) = 0.25(z - 1)^4$, which is smooth, radially unbounded, convex, but not strongly convex. For this function we apply the HESwA with mappings and sets (18) and (19), respectively. We simulate the hybrid system by using a discretized HDS (39) with discretized flow map obtained via 4-order Runge-Kutta method, discretized jump map (40), and discretization step size $h = 1 \times 10^{-4}$. For all the simulations we considered the initial conditions $x_1(0,0) = 2, x_2(0,0) = 2, \tau(0,0) = 0.1 = T_{\min}$, and the parameters $k_1 = 1, a = 0.01$ and $\varepsilon = 0.02$. Figure 1 shows the evolution in time of three different solutions with parameters: (a) $T_{\text{med}} = 0.5, T_{\text{max}} = 0.5$; (b) $T_{\text{med}} = 15, T_{\text{max}} = 20$; (c) $T_{\text{med}} = 55, T_{\text{max}} = 60$. We also plot a solution of the standard gradient descent-based ES dynamics [27,3] using the same parameters $(a, \varepsilon)$. As shown in the Figure, all solutions converge to a neighborhood of the optimal point $z^*$. However, as shown by the inset, the rate of convergence is dramatically different for each solution. In particular, while
solutions (b) and (c) converge to a small neighborhood of \( z^* \) in approximately 20 seconds, the solution of the classic ES algorithm requires almost 2000 seconds to reach the same small neighborhood. This is consistent with the fact that solutions (b) and (c) exploit the acceleration property (21) before \( \tau \) is reset at 15 and 55 seconds, respectively. We also plotted solution (a), which resets the clock after only 0.5 seconds of flow. This resetting rule essentially approximates an algorithm with a constant \( \tau \approx T_{\text{min}} \), i.e., the time-invariant Heavy Ball ODE [1]. In this case we obtain the slow rate of convergence shown with green line in Figure 1. This suggests that better transient performance is achieved by selecting large values of \( T_{\text{med}} \). However, as discussed in Section 3.1.1, in the limiting case when \( T_{\text{med}} \to \infty \) the HESwA behaves as the time-varying Nesterov’s ODE with no resetting, which can be rendered unstable under small disturbances [33, Ex. 1]. The top plots of Figure 2 illustrate this instability, which emerges when \( T_{\text{med}} > 5 \times 10^4 \) and after adding a small disturbance \( e(t) \) to the term \( f(z)\dot{\mu} \) in the flows (18). The disturbance is a small squared periodic signal with frequency of \( 1 \times 10^{-4} \) Hz and amplitude of \( 1 \times 10^{-2} \). On the other hand, the stable behavior shown in the bottom plots correspond to the case \( T_{\text{med}} = T_{\text{max}} = 25 \). These simulations illustrate the importance of the resetting mechanism in ES algorithms with acceleration.

4.2 Strongly Convex Functions

We now consider the function \( f(z) = \frac{1}{2} z^T A z \), \( A = [0.5, -1; 1, 0.5] \), whose symmetric form has Hessian matrix given by \( \nabla^2 f(z) = 0.5I_2 \). We compare the trajectories \( x_1 \) generated by the HESwA (7)-(10) with mappings (24) and sets (25) versus the trajectory generated by the classic gradient descent-based ES dynamics [2]. Figure 3 shows both trajectories evolving over the level sets of the cost function. It can be observed that the HESwA exhibits significant less oscillations compared to the classic ES algorithm. The evolution in time of \( x_1 \) is shown in Figure 4 for different values of \( T_{\text{max}} \). As it can be observed, under appropriate tuning of \( T_{\text{max}} \), the hybrid dynamics significantly outperform the classic gradient descent-based algorithm. Figure 4 also shows the impact of the initial conditions on the transient performance of the system. In particular, the initial conditions satisfying \( x_1(0) = x_2(0) \) generate the smallest overshoot in the system, which is consistent with the initialization used in the existing model-based optimization dynamics with acceleration used in the machine learning literature, e.g. [40] and [48].

5 Analysis: Part 1 - Averaging Theory

In order to prove the main results of this paper, we first develop some auxiliary stability results for a class of singularly perturbed HDS [47,45] that fits the structure of our algorithms. In particular, we consider hybrid systems with states \( (\phi, \chi) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \) and continuous-time dynamics parameterized by two different constants \( \varepsilon > 0 \) and \( \delta > 0 \). The parameter \( \varepsilon > 0 \) induces a multi-time scale behavior by singularly perturbing the flow map. The parameter \( \delta \) parametrizes the stability properties of the slow dynamics. The singularly perturbed HDS is modeled as:

\[
\dot{\phi} = f^\varepsilon_\delta(\phi, \chi), \quad \dot{\chi} = \frac{1}{\varepsilon} f_\chi(\phi, \chi), \quad (\phi, \chi) \in C \times \Psi \quad (42a)
\]

\[
\phi^+ \in G_\phi(\phi, \chi), \quad \chi^+ = \chi, \quad (\phi, \chi) \in D \times \Psi, \quad (42b)
\]

where \( \phi \in \mathbb{R}^{n_1}, \chi \in \mathbb{R}^{n_2}, C, D \subset \mathbb{R}^n, \Psi \subset \mathbb{R}^{n_2}, f_\phi : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^n, f_\chi : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}, G_\phi : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1} \) is a set-valued mapping, and \( [\delta, \varepsilon] \in \mathbb{R}^2_+ \). For the sake of generality we will allow set-valued jump maps, as well as stability results that are local with respect to some basin of attraction. We make the following regularity assumption on the data of system (42).

**Assumption 5** For each \( \delta > 0 \) the following holds: The sets \( C \) and \( D \) are closed, the functions \( f^\varepsilon_\delta \) and \( f_\chi \) are continuous and defined on \( (\phi, \chi) \in C \times \Psi \), the set-valued mapping \( G_\phi \) is outer-semicontinuous [17, Def. 5.9] and locally bounded [17, Def. 5.14], \( \Psi \) is compact, and for each \( (\phi, \chi) \in D \times \Psi \) the set \( G_\phi(\phi, \chi) \) is not empty.

To analyze the HDS (42) we consider the flows expressed in the time variables \( (\tau, j) \), where \( \tau = t/\varepsilon \), i.e.,

\[
\frac{d\phi}{d\tau} = \varepsilon f^\varepsilon_\delta(\phi, \chi), \quad \frac{d\chi}{d\tau} = f_\chi(\phi, \chi), \quad (\phi, \chi) \in C \times \Psi \quad (43a)
\]

\[
\phi^+ \in G_\phi(\phi, \chi), \quad \chi^+ = \chi, \quad (\phi, \chi) \in D \times \Psi. \quad (43b)
\]

For this system we define the **boundary layer dynamics**.

---

Fig. 3. Evolution in \( \mathbb{R}^2 \) of the solutions of the derivative-free hybrid dynamics, and the solutions of the classic gradient descent-based extremum seeking algorithm.

Fig. 4. Evolution in time of the component \( x_1 \) of the solutions of the derivative-free hybrid dynamics, and the solutions of the classic gradient descent-based algorithm.
Definition 6 The boundary layer dynamics of the hybrid system (43) are given by

\[
\frac{d\phi_{bl}}{dt} = 0, \quad \frac{d\chi_{bl}}{dt} = f_{\chi}(\phi_{bl}, \chi_{bl}), \quad (\phi_{bl}, \chi_{bl}) \in C \times \Psi, \quad (44)
\]

which ignores the jumps, and “freezes” \(\phi\) by setting \(\varepsilon = 0\). \(\square\)

Similar to existing results in singular perturbation and averaging theory, e.g., [20, Ch. 10 & 11], [44], [47], our goal is to establish stability properties for the singularly perturbed hybrid system (42) based on a simplified average system obtained by averaging the dynamics of \(\phi\) along the solutions of \(\chi\). This idea is captured by the following assumption.

Assumption 6 For each \(\delta > 0\) there exists a continuous function \(f_A^\delta : \mathbb{R}^n \to \mathbb{R}\) such that for each compact set \(K \subset C \times \Psi\) there exists a class-L function \(\sigma_{K, \delta}\) such that, for each \(L > 0\), \(\phi \in C \cap K\), each function \(\chi_{\delta} : [0, L] \to \Psi\) satisfying \(\chi_{\delta} = f_A^\delta(\phi, \chi_{\delta})\), the following holds:

\[
\frac{1}{L} \int_0^L \left( f_A^\delta(\phi, \chi_{\delta}(s)) - f_A^\delta(\phi) \right) ds \leq \sigma_{K, \delta}(L). \quad (45)
\]

Using the mapping \(f_A^\delta\) defined in (45) we now define the average hybrid system of (42):

Definition 7 The average hybrid system of the HDS (42) with boundary layer dynamics (44) has state \(y \in \mathbb{R}^n\), and is given by

\[
y = f_A^\delta(y), \quad y \in C, \quad y^+ \in G_A(y), \quad y \in D, \quad (46)
\]

where \(G_a := \{ v_1 \in \mathbb{R}^n : (v_1, v_2) \in G_{\phi}(\phi, \chi), (z, v_2) \in \Psi \times \mathbb{R}^n \}\).

Finally, we assume that the average system (46) satisfies the following semi-global practical stability property with respect to a compact set \(A_{\phi}\).

Assumption 7 There exists a nonempty compact set \(A_{\phi} \subset \mathbb{R}^n\), an open set \(B_{A_{\phi}} \supset A_{\phi}\), and a class \(\mathcal{K}\mathcal{L}\) function \(\beta\) such that for each proper indicator \(\omega(\cdot)\) for \(A_{\phi}\) on \(B_{A_{\phi}}\), each compact set \(K_0 \subset B_{A_{\phi}}\), and each \(\nu > 0\), there exists a \(\delta > 0\) such that for all \(\delta \in (0, \delta^*)\), all solutions of (46) with \(y(0, 0) \in K_0\) satisfy the bound:

\[
\omega(y(t, j)) \leq \beta(\omega(0(0)), t + j) + \nu. \quad (47)
\]

for all \((t, j) \in \text{dom}(y)\). \(\square\)

Using Assumptions 5, 6, and 7 the following two results are obtained. Proposition 8 is a straightforward extension of [47, Thm. 1] that concerns closeness of the \(\phi\)-component of the solutions of (42) to the solutions of (46) on compact time domains. Theorem 9 links the stability properties of system (42) to the stability properties of the average system (46) by establishing SGPS as \((\delta, \varepsilon) \to 0^+\) of the set \(A_{\phi} \times \Psi\).

Proposition 8 Suppose that the HDS (42) satisfies Assumptions 5, 6, and 7. Let \(K_0 \subset B_{A_{\phi}}\) and let \(\delta^* > 0\) be such that for each \(\delta \in (0, \delta^*)\) all solutions of (46) with \(y(0, 0) \in K_0\) do not have finite escape times. Then, for each \(\delta \in (0, \delta^*), each \rho > 0, and any strictly positive real numbers \(T, J\) there exists \(\varepsilon^* > 0\) such that for each \(\varepsilon \in (0, \varepsilon^*)\) and each solution \(\phi\) to system (42) with \(\phi(0, 0) \in K_0\), there exists some solution \(y\) to the average system (46) with \(y(0, 0) \in K_0\) such that \(\phi\) and \(y\) are \((T, J, \rho)\)-close in the sense of [17, Def. 5.23].

Proof: Since for each \(\delta > 0\) the singularly perturbed HDS (42) satisfies all the assumptions needed to apply [47, Thm. 1], it only needs to be shown that there exists a \(\delta^* > 0\) such that the system has no finite escape times from \(K_0\). Indeed, by Assumption 7 for each compact set of initial conditions \(K_0\) there exists \(\delta^* > 0\) such that for all \(\delta \in (0, \delta^*)\) all the solutions satisfy (47), which precludes finite escape times from \(K_0\). \(\blacksquare\)

Theorem 9 Suppose that the HDS (42) satisfies Assumptions 5, 6, and 7. Then, for each proper indicator \(\omega\) for \(A_{\phi}\) on \(B_{A_{\phi}}\), each compact set \(K_0 \subset B_{A_{\phi}}\), and each \(\nu > 0\) there exists \(\delta^* > 0\) such that for each \(\delta \in (0, \delta^*)\) there exists \(\varepsilon^* > 0\) such that for all \(\varepsilon \in (0, \varepsilon^*)\) all solutions of (46) with \(\phi(0, 0) \in K_0\) satisfy:

\[
\omega(\phi(t, j)) \leq \beta(\omega(0(0)), t + j) + \nu. \quad (48)
\]

for all \((t, j) \in \text{dom}(\phi)\). \(\square\)

Proof: The proof is similar to the proofs of [47, Thm. 2] and [45, Thm. 2]. Let \(K_0 \subset B_{A_{\phi}}\) and \(\nu > 0\) be given. Let \(\omega : B_{A_{\phi}} \to \mathbb{R}_{\geq 0}\) be a proper indicator for \(A_{\phi}\) with respect to \(B_{A_{\phi}}\). Define the set \(K_1 := \{ \phi \in B_{A_{\phi}} : \omega(\phi) \leq \beta(\max_{y \in K}\omega(y), 0) + 1 \}\), and

\[
K := K_1 \cup G_A(K_1 \cap D). \quad (49)
\]

Since \(K_1\) is compact, and \(G_a\) is outer semicontinuous and locally bounded, the set \(K\) is compact. Moreover \(K \subset B_{A_{\phi}}\) since \(\omega\) is a proper indicator and \(G_a\) is an OSC mapping that maps \(B_{A_{\phi}} \cap D \to B_{A_{\phi}}\). Let \(\varepsilon > 0\) be such that, for all \(\phi \in K, all y \in K + \varepsilon B \) with \(|\phi - y| \leq \varepsilon\), and all \(s \geq 0\), the following holds:

\[
\omega(\phi) \leq \omega(y) + \frac{\nu}{3}, \quad \beta(\omega(y), s) \leq \beta(\omega(\phi), s) + \frac{\nu}{3} \quad (50)
\]

Such \(\varepsilon > 0\) always exists given that \(\beta \in \mathcal{K}\mathcal{L}\) and that \(\beta, \omega\) are continuous functions. Using Proposition 6, there exists a \(\delta^* > 0\) such that for all \(\delta \in (0, \delta^*), there exists a \(p^* \in (0, \varepsilon_1)\) such that for all \(\rho \in (0, p^*)\) all solutions \(y(\cdot, 0)\) of the \(p\)-inflation of system (46) with \(y(0, 0) \in K\) satisfy for all \((t, j) \in \text{dom}(y)\) the following bound:

\[
\omega(y(t, j)) \leq \beta(\omega(0(0)), t + j) + \frac{\nu}{3}. \quad (51)
\]
Let $\mu \geq 0$ and consider the extended hybrid dynamical system, constructed from (42), with auxiliary state $\eta \in \mathbb{R}^{n_1}$ and $K$-restricted flow and jump set, given by:

\[
\begin{align*}
\dot{\phi} &= f^\delta_\phi(\phi, \chi), \quad \dot{\chi} = \frac{1}{\varepsilon} f_x(\phi, \chi), \\
\dot{\eta} &= \frac{1}{\varepsilon} \left[ f^\delta_\phi(\phi, \chi) - f^\varepsilon_\mu(\phi) - \mu \eta \right], \quad \in (C \cap K) \times \Psi \times \mathbb{R}^n.
\end{align*}
\]  
\[
(52a)
\]

\[
\phi^+ \in G_\phi(\phi, \chi), \quad \eta^+ = \phi^+ + \eta^+ \in (D \cap K) \times \Psi \times \mathbb{R}^n.
\]

\[
(52b)
\]

Since for each $\delta > 0$ all the assumptions needed to apply [47, Lemma 4] are satisfied, the next Lemma follows directly by [47, Lemma 4].

**Lemma 10** Suppose that the HDS (42) satisfies Assumptions 5 and 6. Then, for each $\delta \in (0, \delta^*)$ and each $\rho > 0$ there exists $[\varepsilon^*, \lambda^*] \subset \mathbb{R}_+$ such that, for all $\varepsilon \in (0, \varepsilon^*)$, each solution $(\phi, \chi, \eta)$ of system (52) with $\eta(0,0) = 0$ satisfies $\lambda \|\eta(t,j)\| \leq \rho$, for all $(t,j) \in \text{dom}(\phi, \chi)$.

Let $K$, $\delta$, and $\rho$ generate $(\varepsilon, \lambda)$ via Lemma 10. Let $\varepsilon^* := \min\{\rho, \varepsilon, \lambda\}$ and let $\varepsilon \in (0, \varepsilon^*)$. For each solution $(\phi, \chi, \eta)$ of (52) let us define $y := \phi - \varepsilon \eta$. Since $\eta^+ = 0$, we obtain $\dot{y} = \phi - \varepsilon \eta = f^\delta_\mu(\phi) + \lambda \eta$, and $y^+ = \phi^+ + \eta^+ \in G_\phi(\phi)$ with

\[
G_\phi(\phi) := \{ v_1 \in \mathbb{R}^{n_1} : (v_1, v_2) \in G_\phi(\phi, \chi), (\chi, v_2) \in \Psi \times \mathbb{R}^{n_2} \}.
\]

Since $\phi = y + \varepsilon \eta$, we have that

\[
\dot{y} = f^\delta_\mu(y + \varepsilon \eta) + \lambda \eta, \quad y + \varepsilon \eta \in C, \quad y^+ \in G_\mu(y + \varepsilon \eta), \quad y + \varepsilon \eta \in D,
\]

and by the choice of $\varepsilon^*$ above we have that $\dot{y} \in f^\delta_\mu(y + \rho \mathbb{R}) + \rho \mathbb{R}$, when $y \in C_\mu$, and $y^+ \in G_\mu(y + \rho \mathbb{R})$, when $y \in D_\mu$, where the sets $C_\mu$ and $D_\mu$ correspond to the $\rho$-inclusions constructed as in (38c) and (38d). Therefore, for each solution $(\phi, \chi, \eta)$ of system (52) with $(\phi(0,0), \chi(0,0)) \in K_0 \times \Psi$ and $\eta(0,0) = 0$, the trajectory $y := \phi - \varepsilon \eta$ is a solution to the $\rho$-inclusion (38) of the average system (46), and since $\delta \in (0, \delta^*)$ the trajectory $y$ also satisfies the bound (51). Since $\rho \leq \varepsilon^*$, the inequalities (50) hold, and all solutions $(\phi, \chi)$ to the system (52) with $(\phi(0,0), \chi(0,0)) \in K_0 \times \Psi$ satisfy for all $(t,j) \in \text{dom}(\phi, \chi)$ the following bounds:

\[
\omega(\phi(t,j)) \leq \omega(y(t,j)) + \frac{\mu}{3} \leq \beta(\omega(y(0,0)), t + j) + \frac{2\mu}{3} \leq \beta(\omega(0,0), t + j) + \nu.
\]

\[
(53)
\]

Since $\mu \in (0,1)$, by the inequality (53), each solution of (52) with $(\phi(0,0)) \in K_0$ remains in the compact set $K_{\varepsilon} := \{ \phi \in \mathbb{R}^{n_1} : \omega(\phi) \leq \beta(\max_{\varepsilon \in K} \omega(\phi), 0) + \nu \}$, which is contained in the interior of the set $K$. We now use the properties of the solutions of the K-restricted system (52) to derive conclusions about the solutions of the original HDS (42). Indeed, since $K_0 \subset K$ a solution of (42) with $(\phi(0,0), \chi(0,0)) \in K_0 \times \Psi$ must agree with a solution of (52) for all $(t,j) \in \text{dom}(\phi, \chi)$ such that $\phi(t,j) \in K$. However, using the definition of $K$ in (49) and the $KL$ bound (53) we have that all solutions of (42) with $(\phi(0,0), \chi(0,0)) \in K_0 \times \Psi$ remain in the set $(K_0 \cup G(\phi \subset K \times \Psi) \subset K \times \Psi$. This implies that inequality (53) holds for all $(t,j) \in \text{dom}(\phi, \chi)$, which establishes the result.

6 **Analysis: Part 2 - Algorithmic Stability**

In this section we use Theorem 9 to prove Theorems 1, 2, 3, and 4. In particular, we show that the zero-order hybrid optimization dynamics can be written as a singularly perturbed system of the form (42) with $\delta = a$, and that all the assumptions required to apply Theorem 9 hold. Indeed, by construction, it can be seen that for small values of $\varepsilon > 0$ the HDS (7)-(10) is a singularly-perturbed hybrid dynamical system of the form (42) with $\phi = (x, \tau)$, $\mu = C = \mathbb{R}^{n+m} \times \mathcal{T}_C$, $D = \mathbb{R}^{n+m} \times \mathcal{T}_D$, and $\Psi = \mathbb{S}^n$. By construction of the dynamics, Assumption 5 is satisfied since for each $a > 0$ all the mappings $F_x$ and $G_x$ are continuous and defined in $C \times \Psi$ and $D \times \Psi$, respectively, and the sets $\mathcal{T}_C$ and $\mathcal{T}_D$ are closed.

6.1 **Average Hybrid Systems**

We now show that the zero-order optimization dynamics satisfy Assumption 6. Indeed, since the cost function $f$ is at least twice continuously differentiable, the Taylor expansion of $f(x_1 + a\bar{\mu})$ around $x_1$ is well-defined and given by $f(x_1 + a\bar{\mu}) = f(x_1) + a\bar{\mu}^T \nabla f(x_1) + O(a^2)$. Substituting in (18), (24), (29), and (35), and using the fact that $|\bar{\mu}| \leq 1$, we obtain mappings $F_x := (F_x^T, F_x^T) \in (F_x^T, F_x^T) \in (F_x^T, F_x^T)$ with components:

(a) For Cases I and II:

\[
F_{x_1} = -4\tau \left( \frac{f(x_1)}{\mu} + \bar{\mu}^T \nabla f(x_1) \right) + O(a),
\]

(b) Case III:

\[
F_{x_1} = -2 \left( \frac{f(x_1)}{\mu} + \bar{\mu}^T \nabla f(x_1) \right) - A^T x_2 + O(a),
\]

(c) For Case IV:

\[
F_{x_1} = -2 \left( \frac{f(x_1)}{\mu} + \bar{\mu}^T \nabla f(x_1) \right) - \sum_{j=1}^m H_j(x) A_j + O(a),
\]

For each case, we define the following average mappings $f^a_\mu$:

(a) For Cases I and II:

\[
f^a_\mu(x, \tau) := \begin{cases} 
\frac{2}{\tau} (x_2 - x_1) \\
-2\tau \nabla f(x_1) + O(a)
\end{cases}, \quad \frac{1}{2}.
\]

(b) For Case III:

\[
f^a_\mu(x, \tau) := \begin{cases} 
-\nabla f(x_1) - A^T x_2 + O(a) \\
A x_1 - b
\end{cases}.
\]

\[
(55)
\]
The average system has state $\bar{x} = \frac{1}{\rho} \sum_{i=1}^{\rho} x_i$.

6.2 Stability Analysis of Average Systems

For each particular Case, we show that this system satisfies Assumption 1.

6.2.1 Case I:

For each particular Case, we show that this system satisfies Assumption 1.

Lemma 2 Under Assumption 1, the mappings (54)-(56) satisfy Assumption 6 with $\phi := [x^T, \tau]^T$, $\chi_{bl} := \mu$, $\delta := a$, $f_\phi^0 := F_x \times F_\tau$, and $f_x = R_\mu$.

Proof: Using the definitions of the Lemma, the result of Lemma 1, for each $a > 0$ there exists a $\hat{T} > 0$ such that for each $(x, \tau) \in (\mathbb{R}^n \times \mathcal{T}_C) \cap K$ with $K \subset \mathbb{R}^{n+1}$ compact, the following holds for each case:

$$
\frac{1}{kT} \int_0^{kT} \left[ \begin{array}{c} F_x(x, \tau, \mu(s)) \\ F_\tau \\
\end{array} \right] - f_\phi^0(x, \tau) \right] ds = 0.
$$

for all $k \in \mathbb{Z}_{\geq 0}$. Since any $L \in \mathbb{R}_{>0}$ can be written as $L = k\hat{T} + \bar{L}$ where $|\bar{L}| \leq \hat{T}$, it suffices to consider the integral

$$
\int_0^L \left[ \begin{array}{c} F_x(x, \tau, \mu(s)) \\ F_\tau \\
\end{array} \right] - f_\phi^0(x, \tau) \right] ds.
$$

By the proof of Lemma 1 in the Appendix, the integrals of $\mu(s)$ and $\mu(s)\mu(s)^T$ are bounded on any finite time. Thus, by the construction of $F_x$ and the fact that $(x, \tau) \in K$ with $K$ compact, and the continuity of $f_\phi^0(x, \tau)$ and $H_j$, as well as the smoothness of $f$, there exists $\tilde{M}_{k,a} > 0$ such that

$$
\left| \int_0^L \left[ \begin{array}{c} F_x(x, \tau, \mu(s)) \\ F_\tau \\
\end{array} \right] - f_\phi^0(x, \tau) \right| \leq \tilde{M}_{k,a}.
$$

Thus, the bound (45) holds with $L = k\hat{T} + \bar{L}$ and $\sigma_{K,H}(L) = M_{k,a}/L$.

6.2.2 Case II:

The average system has state $y \in \mathbb{R}^{2n+1}$ and dynamics (61), where $f_A^y$ is defined as in (54), $G_A$ is defined as in (60), and $\mathcal{T}_C, \mathcal{T}_D$ are defined as in (19). For this HDS we have the following Lemma.

Lemma 11 Under Assumption 2, the HDS (61) renders the set $A_\phi := A_x \times [T_{\min}, T_{\max}]$ SGPAS as $a \to 0^+$.

Proof: In the first step of the proof we neglect the $O(a)$ perturbation term in the mapping (54), and we establish UGAS of the set $A_x \times [T_{\min}, T_{\max}]$. In the second step, we use the robustness properties of well-posed hybrid systems (e.g., [17, Thm. 7.21]) to establish SGPAS as $a \to 0^+$ for the original $O(a)$-perturbed system.

Step 1: Let $f^* = \min_{x \in \mathbb{R}^n} f(x_1)$ and consider the Lyapunov function

$$
V(y) = \frac{1}{4}|y_2 - y_1|^2 + \frac{1}{4}|y_2|^2 + c_0(y_1 - f^*),
$$

which is radially unbounded and positive definite with respect to $A$. Using the definition of $\bar{y}$ and denoting $z^* := \bar{F}_{A_{j}^{-1}}(x_2)$ the projection of $x_2$ on $A_{j}$ [15, pp. 19], we obtain that $V(y) = \nabla V^T \bar{y}$ is given by

$$
\bar{V} \leq -\frac{1}{2}|y_2 - y_1|^2 - \epsilon \left( \nabla f(y_1)^T (y_1 - z^*) - f(y_1) + f^* \right),
$$

which by convexity implies that $\bar{V} \leq 0$ for all $y \in C$. Moreover, when $A_{j}$ is a singleton Lemma 3 in the Appendix implies that $\bar{V} < 0$ for all $y \in C \setminus A_{j}$. On the other hand, when $A_{j}$ is not a singleton, but $\nabla f(z)$ is globally Lipschitz, the right hand side of (63) can be further upper bounded as $\bar{V} \leq -\frac{1}{2}|y_2 - y_1|^2 - \frac{\epsilon}{2m|\nabla f(y_1)|^2}$, and by the definition of $A$ and convexity, this implies that $\bar{V} < 0$ for all $y \in C \setminus A_{j}$. In addition, during jumps the change $\Delta V(y) := V(y^+ - V(y)$ in the Lyapunov function satisfies

$$
\Delta V(y) = -c(f(y_1) - f(z^*))|y_2|^2 - T_{\min}^2 < 0,
$$

for all $y \in D$. Inequalities (63) and (64) imply that $A$ is stable [16, Thm. 23]. Since the Lyapunov function does not increase during jumps, and it is strictly decreasing during flows, it follows that there is no complete solution $y$ such that $V(y(t, j)) = k$ for all $(t, j) \in \text{dom}(\phi)$ and for any $k > 0$. Therefore, by [16, Thm. 23] there exists a $\beta_1 \in \mathcal{K}$ such that the set $A$ is UGAS.

Step 2: Since the HDS (61) with $O(a) := 0$ satisfies the Basic Conditions, and $A$ is UGAS, by [17, Thm. 7.21] the original $O(a)$-perturbed system (61) renders the set $A$ SGP-AS as $a \to 0^+$ with $\beta_1 \in \mathcal{K}$.

6.2.2 Case II:

The average system has state $y \in \mathbb{R}^{2n+1}$ and dynamics (61), where $f_A^y$ is defined as in (54), $G_A$ is defined as in (60), and $\mathcal{T}_C, \mathcal{T}_D$ are defined as in (25). For this HDS we have the following Lemma.

Lemma 12 Under Assumption 3, the HDS (61) renders the set $A_\phi := A_x \times [T_{\min}, T_{\max}]$ SGPES as $a \to 0^+$.
**Proof:** We follow the same two steps as in the proof of Lemma 11, and we use the fact that \( A_f = \{ z^* \} \) is a singleton, and that \( |y|^2_{A_e} = |y_1 - z^*|^2 + |y_2 - z^*|^2 \) for all \( y \in C \cup D \).

**Step 1:** Neglecting the \( O(a) \) perturbation, using the Lyapunov function (62), and using strong convexity and the global Lipschitz property of \( \nabla f \), we obtain the lower bound
\[
\frac{1}{4} |y_2 - z^*|^2 + cT^2_{\text{min}} \frac{\theta}{2} |y_1 - z^*|^2 \leq V(z) \tag{65}
\]
and the upper bound
\[
V(z) \leq \frac{1}{4} |y_2 - y_1|^2 + \frac{1}{4} |y_2 - z^*|^2 + cT^2_{\text{max}} \frac{L}{2} |y_1 - z^*|^2, \tag{66}
\]
for all \( y \in C \cup D \). Using the triangle inequality and the fact that \( |z| \geq 0 \implies 2ab \leq a^2 + b^2 \) for any reals \( a, b \), we obtain that \( |y_2 - y_1|^2 \leq 2 |y_1|^2_{A_e} \). Defining
\[
\epsilon := 0.25 \min \{ 1, 2cT^2_{\text{min}} \theta \}, \ c := 0.25 \max \{ 3, 6cT^2_{\text{max}} L \},
\]
and using (65) and (66) we obtain
\[
\epsilon |y|^2_{A_e} \leq V(y) \leq \epsilon |y|^2_{A_e}. \tag{67}
\]

Since the flow map is the same as in Case I, the derivative \( \dot{V} \) is still given by (63). Thus, using strong convexity and the definition of \( T_C \) we can further bound \( V \) as follows:
\[
\dot{V}(z) \leq -\frac{1}{T_{\text{max}}} |y_2 - y_1|^2 - cT^2_{\text{min}} \frac{\theta}{2} |y_1 - z^*|^2
\]
\[
\leq -\frac{1}{T_{\text{max}}} |y_2 - y_1|^2 - cT^2_{\text{min}} \frac{\theta}{2} |y_1 - z^*|^2
\]
\[
\leq -\rho \left( |y_2 - y_1|^2 + |y_1 - z^*|^2 \right) - cT^2_{\text{min}} \frac{\theta}{4} |y_1 - z^*|^2,
\]
for all \( y \in C \), where \( \rho := \min \{ \frac{1}{T_{\text{max}}}, 0.25cT^2_{\text{min}} \} \). We also have that \( |y_2 - z^*|^2 \leq 2(|y_2 - y_1|^2 + |y_1 - z^*|^2) \), which implies that \( \dot{V}(y) \) can be upper bounded as
\[
\dot{V}(z) \leq -0.5\tilde{\rho}|x_2 - z^*|^2 - cT^2_{\text{min}} \frac{\mu}{4} |x_1 - z^*|^2, \ \forall y \in C.
\]

This inequality implies that \( \dot{V}(y) \leq -\rho |y|^2_{A_e} \), where \( \rho \) is defined as \( \rho := \min \{ 0.5\tilde{\rho}, 0.25cT^2_{\text{min}} \theta \} \), and using the upper bound of (68) we obtain \( \dot{V}(y) \leq -\epsilon V(y) \) for all \( y \in C \).

Define \( \gamma \defeq 1 - \frac{T^2_{\text{min}}}{T^2_{\text{max}}} - \frac{cT^2_{\text{min}}}{2c\theta T^2_{\text{max}}} \). Due to condition (26) we have that \( \gamma \in (0, 1) \). In fact, note that
\[
T^2_{\text{max}} - T^2_{\text{min}} > \frac{1}{2c\theta} \implies 1 - \frac{T^2_{\text{min}}}{T^2_{\text{max}}} - \frac{1}{2c\theta T^2_{\text{max}}} > 0,
\]
and \( \gamma < 1 \) since \( T_{\text{max}} > T_{\text{min}} > 0 \) and \( \theta, c > 0 \). Let
\[
\lambda_2 := \min \left\{ \frac{\rho}{c}, -\log(1 - \gamma) \right\},
\]
and note that since \( \gamma \in (0, 1) \) we have that \( \log(1 - \gamma) < 0 \), which implies that \( \lambda_2 > 0 \). Since \( \lambda_2 \leq \frac{2}{\tilde{\rho}} \), we finally obtain:
\[
\dot{V}(y) \leq -\lambda_2 V(y), \ \forall y \in C. \tag{69}
\]
During the jumps, the Lyapunov function satisfies
\[
\Delta V(y) = \frac{1}{4} |y_1 - z^*|^2 + cT^2_{\text{min}} (f(y_1) - f^*) - \frac{1}{4} |y_2 - y_1|^2
\]
\[
- \frac{1}{4} |y_2 - z^*|^2 - cr^2(f(y_1) - f^*).
\]
Using strong convexity we can bound \( \Delta V \) as follows
\[
\Delta V(y) \leq \frac{1}{2\mu} (f(y_1) - f^*) + cT^2_{\text{min}} (f(y_1) - f^*)
\]
\[
- \frac{1}{4} |y_2 - y_1|^2 - \frac{1}{4} |y_2 - z^*|^2 - cr^2(f(y_1) - f^*).
\]
Factorizing \( cr^2(f(y_1) - f^*) \) we obtain
\[
\Delta V(z^+) \leq \frac{1}{4} |y_2 - y_1|^2 = \frac{1}{4} |y_2 - z^*|^2
\]
\[
- cr^2(f(y_1) - f^*) \left( 1 - T^2_{\text{min}} \frac{\mu}{T^2_{\text{max}}} - \frac{1}{2c\theta T^2_{\text{max}}} \right),
\]
and using the definition of \( \gamma \), the fact that \( 0 < \gamma < 1 \), and that \( \tau = T_{\text{max}} \) in the jump set, we obtain
\[
\Delta V(z) \leq -\gamma \left( |y_1 - y_1|^2 + \frac{1}{4} |y_2 - z^*|^2 + cr^2(f(y_1) - f^*) \right),
\]
\[
\leq -\gamma V(y), \ \forall y \in D.
\]
Thus, during jumps the Lyapunov function satisfies
\[
V(y^+) \leq (1 - \gamma) V(y) \leq \exp(-\lambda_2) V(y), \tag{70}
\]
for all \( y \in D \), where the last inequality follows by the fact that \( \lambda_2 \leq -\log(1 - \gamma) \) which implies that \( -\lambda_2 \geq \log(1 - \gamma) \). Since \( \exp(\cdot) \) is an increasing function, this implies that \( \exp(-\lambda_2) \geq (1 - \gamma) \). By [43, Thm. 1], inequalities (68), (69), and (70), imply that the HDS (61) with \( O(a) = 0 \) renders the set \( \mathcal{A}_e \) GUES. Using again [17, Thm. 7.21] we obtain SGPES as \( a \to 0^+ \) for the \( O(a) \)-perturbed system (61). \( \blacksquare \)

### 6.2.3 Case III:

The average system has a state \( y \in \mathbb{R}^{n+m+1} \), and the hybrid dynamics (61), where \( f^a_t \) is defined as in (55), \( G_A \) is defined as in (60), and \( T_C, T_D \) are defined as in (25). For this HDS we have the following Lemma.

**Lemma 13** Under Assumption 4, the HDS system (61) renders the set \( \mathcal{A}_\phi \defeq \mathcal{A}_e \times \{ T_{\text{min}} \} \) SGPES as \( a \to 0^+ \). \( \square \)

**Proof:** Since \( y_3(t) = T_{\text{min}} \) for all \( t \geq 0 \), and the dynamics of \( (y_1, y_2) \) and \( y_3 \) are uncoupled, it suffices to study the properties of \( y_1, y_2 \) with respect to \( \mathcal{A}_e \). Neglecting
the $O(a)$-perturbation, and ignoring the jumps, by [36, Lemma 1] there exists a quadratic Lyapunov function $V(\tilde{y}) = \tilde{y}^T P_2 \tilde{y}$, $P_2 > 0$, $\tilde{y} := [(y_1 - z^*)^T, (y_2 - \lambda^*)^T]^T$. By [36, Lemma 2] there exists $\lambda_3 \in \mathbb{R}_{>0}$ such that $V(\tilde{y}) \leq -\lambda_3 V(\tilde{y})$. Therefore, the point $y = z^*$ is UGES. Since the average HDS with $O(a) = 0$ is well-posed, [17, Thm. 7.21] establishes SGPES as $a \to 0^+$ for the original $O(a)$-perturbed average system. ■

6.2.4 Case IV:

The average system has state $y \in \mathbb{R}^{n+m+1}$, and hybrid dynamics (61), where $f_A^A$ is defined as in (56), $G_A$ is defined as in (60), and $T_C, T_D$ are defined as in (25). For this HDS we have the following Lemma.

**Lemma 14** Under Assumption 4, the HDS system (61) renders the set $\mathcal{A}_0 := \mathcal{A}_x \times \{T_{min}\}$ SGPES as $a \to 0^+$.

**Proof:** Since $y_A(t) = T_{min}$ for all $t \geq 0$ and the dynamics of $(y_1,y_2)$ and $y_A$ are uncoupled, it suffices to study the properties of $y_1,y_2$ with respect to $\mathcal{A}_x$. Neglecting the $O(a)$-perturbation, by [36, Lemmas 3 & 4], there exists a quadratic Lyapunov function $V = \tilde{y}^T P_2 \tilde{y}$, $P_2 > 0$, $\tilde{y} := [(y_1 - z^*)^T, (y_2 - \lambda^*)^T]^T$ and some $\lambda_4 \in \mathbb{R}_{>0}$ such that $V(\tilde{y}) \leq -\lambda_4 V(\tilde{y})$, during flows of the system. Therefore, the point $y = z^*$ is UGES. Since the average HDS with $O(a) = 0$ is well-posed, [17, Thm. 7.21] establishes SGPES as $a \to 0^+$ for the original $O(a)$-perturbed average system. ■

Lemmas 11, 12, 13, and 14 imply that the average hybrid system (61) satisfies Assumption 7 for all cases. Thus, by Theorem 9, the original hybrid dynamics (7)-(10) render the set $\mathcal{A} := \mathcal{A}_x \times T_C \times S^n$ SGPAS as $(a,\varepsilon) \to 0^+$ with the same $\mathcal{KL}$ function as the average system. Completeness of solutions follows by the absence of finite escape times, the fact that $\tau^* \in T_C$, and that every solution can flow for at most $T_{max} - T_{min}$ before entering the jump set. These facts establish items (a) and (b) of Theorems 1, 2, 3, and 4.

6.3 Convergence Bounds

6.3.1 Case I: Semi-Acceleration

Let $K_0$ and $\nu$ be given. Consider the average nominal hybrid system $H^A$ with state $y$, the average $O(a)$-perturbed hybrid system $H^A_0$ with state $y^A$, and the original hybrid system $H$ with state $(x,\mu,\tau)$. Define the set $K_0 = K_0 + \mathbb{B}$ and the quantity $m := \max_{x_0 \in K_0} [x_0|_{\mathcal{A}_x}]$. Let $\beta^A_1$ be the $\mathcal{KL}$-bound that characterizes the UGAS property of the average nominal system $H^A$, which was established in Section 6.2.1. Define the set:

$$K_1 = \{x \in \mathbb{R}^{2n} : |x|_{\mathcal{A}_x} \leq \beta^A_1 (m,0) + 1\}.$$  (71)

Since $K_1$ is compact, there exists $M > 0$ such that $K_1 \subseteq MB$. By uniform continuity of $f(\cdot)$ on compact sets, there exists $\delta^*_1 \in (0,\min\{1,T_{med} - T_{min}\})$ such that $|r_1 - r_2| \leq \frac{\delta^*_1}{4} \Rightarrow (f(r_1) - f(r_2))^2 \leq \frac{\delta^*_1}{4}$, for all $r_1, r_2 \in MB$ and $r_1 \in (0,\delta^*_1)$ and let $T^* > 1$ be such that for all $w \geq T^*$ we have that $\beta(m,w) < \frac{\delta^*_1}{4}$. Such $T^*$ always exists because $\beta(m,\cdot)$ is a class-$\mathcal{L}$ function. Let $T := T^* + 1$. Then, by their respective properties of UGAS, SGPAS as $a \to 0^+$, and SGPAS as $(a,\varepsilon) \to 0^+$, and by Proposition (8), there exists $a^* \in (0,\frac{\delta_1^*}{4})$ such that for all $a \in (0,a^*)$ there exists $\varepsilon^* \in (0,1)$ such that for all $\varepsilon \in (0,\varepsilon^*)$ the following properties hold: (a) All solutions $y$ of $H^A$ with $y_0(0,0), y_2(0,0)) \in K_0$ satisfy $y(t) \in MB \times TC$ for all $(t,j) \in dom(y)$. (b) All solutions $y_a$ of $H^A_a$ with $(y_a(0,0), y_2(0,0)) \in K_0$ satisfy $y_a(t,j) \in MB \times TC$ for all $(t,j) \in dom(y_a)$. (c) All solutions $(x,\mu,\tau)$ of $H$ with $(x(0,0), \mu(0,0)) \in K_0 + \mathbb{B}$ exists a solution $y \in H^A$ with $y_0(0,0), y_2(0,0)) \in K_0$ that is $(T,\frac{\delta^*_1}{4})$-close. (d) For each solution $y$ of $H^A_a$ with $y_a(0,0) \in K_0 + \mathbb{B}$ there exists a solution $y \in H^A$ with $y_0(0,0), y_2(0,0)) \in K_0$ that is $(T,\frac{\delta^*_1}{4})$-close to $x$.

By the second part of Property (c), the uniform continuity of $f$ on compact sets, and the choice of $\delta_1$, we obtain

$$f(x_1(t,j)) - f^* \leq \frac{\nu}{T},$$  (72)

for all $(t,j) \in dom(x,\mu,\tau)$ such that $t+j \geq T^*$. Now, to establish the transient bound for the sub-optimality measure, we use the Lyapunov analysis of system $H^A$ and properties (d) and (e).

**Step 1:** By the stability analysis of Section 6.2.1, we have that $V \leq 0$ during flows of system $H^A$, which implies that $V(y(t,j))$ does not increase during flows, i.e., $V(y(t,j)) \leq V(y(t',j))$ for all $t \geq t'$ such that $(t,j),(t',j) \in dom(y)$. Let $s_j = \inf\{t \geq 0 : (t,j) \in dom(y)\}$. Using the structure of the Lyapunov function (62), for each $j$ such that $(t,j) \in dom(y)$ the following bound holds:

$$f(y_1(t,j)) - f^* \leq \frac{V(y(s_j,j),j)}{y_2^2(t,j)}, \quad \forall t > s_j.$$  (73)

Since $y_A = 0.5$, integrating in the interval $[s_j,\tilde{t}]$ and substituting in (73), gives

$$f(y_1(t,j)) - f^* \leq \frac{V(y(s_j,j),j)}{y_2^2(t,j)} \leq \frac{c_j}{(t-s_j)^2},$$  (74)

for all $t > s_j$, where $c_j := 4V(y(s_j,j),j)$ and where the last inequality follows by the fact that $y_2(s_j,j) \geq T_{min} > 0$. By the analysis of Section 6.2.1, we know that for each solution $y$ the sequence $\{c_j\}_{j=1}^\infty$ is monotonically decreasing to $0^+$.

**Step 2:** By Property (d), for each solution of $H^A_a$ with $y_a(0,0) \in K_0 + \mathbb{B}$ there exists a solution $y \in H^A$ with $y_0(0,0) \in K_0$ that satisfies the following: For all $(t,j) \in
dom(ya) with \( \hat{t} + j < \hat{T} \), there exists \( \hat{t}' \) such that \((\hat{t}', j) \in \text{dom}(y)\), \(|\hat{t} - \hat{t}'| \leq \frac{\tilde{\alpha}}{4}\) and \([ya(\hat{t}, j) - y(\hat{t}', j)] \leq \frac{\tilde{\beta}}{4}\).

Using again the uniform boundedness of \( y \) and \( ya \), the uniform continuity of \( f \) on \( MB \), and the choice of \( \delta_1 \), we have that \([ya(\hat{t}, j) - y(\hat{t}', j)] \leq \frac{\tilde{\beta}}{4} \implies f(ya(\hat{t}, j)) \leq f(y(\hat{t}', j)) + \frac{\tilde{\beta}}{4}\). Since every solution \( y \) of \( H \) satisfies the bound (74), and since \(|\hat{t} - \hat{t}'| \leq \frac{\tilde{\alpha}}{4}\), for every solution \( ya \) there exists \( \{c_j\} \) such that for each \( j \) such that \((\hat{t}, j) \in \text{dom}(ya)\) and \( \hat{t} + j \leq \hat{T} \) the following bound holds:

\[
f(ya(\hat{t}, j)) - f^* \leq \frac{4c_j}{(\tilde{\alpha} - \delta_1 - s_j)^2} + \frac{\nu}{3}, \quad \forall \hat{t} > \frac{\delta_1}{4} + \tilde{s}_j,
\]

(75)

where \( \tilde{s}_j = \inf\{\hat{t} \geq 0 : (\hat{t}, j) \in \text{dom}(ya)\} \). Defining \( s_j^0 = \inf\{t \geq 0 : (t, j) \in \text{dom}(ya)\} \) and using again \((\hat{t}, \frac{\tilde{\alpha}}{4})\)-closeness of solutions between \( y \) and \( ya \), we have that \( |s_j - s_j^0| \leq \frac{\tilde{\beta}}{4}, \) which implies that

\[
f(ya(\hat{t}, j)) - f^* \leq \frac{4c_j}{(\tilde{\alpha} - \delta_1 - s_j^0)^2} + \frac{\nu}{4}, \quad \forall \hat{t} > \frac{\delta_1}{2} + s_j^0.
\]

(76)

**Step 3:** By Property (e), for each solution \( \check{x} \) of \( H \) with \( x(0, 0) \in K_0 \) there exists a solution \( ya \) of \( H_a \) with \((ya(0, 0), ya(0, 0)) \in K_0 \) that satisfies the following: For all \((t, j) \in \text{dom}(\check{x})\) with \( t < \hat{T} \), there exists \( \hat{t}' \) such that \((\hat{t}', j) \in \text{dom}(ya)\), \( |t - \hat{t}'| \leq \frac{\tilde{\alpha}}{4}\), and \( |\check{x}(t, j) - ya(\hat{t}', j)| \leq \frac{\tilde{\beta}}{4}. \) Using this property, defining \( s_j^0 = \inf\{t \geq 0 : (t, j) \in \text{dom}(\check{x})\} \) and following the exact same procedure as in the previous step, we obtain the bound

\[
f(x(t, j)) - f^* \leq \frac{4c_j}{(t - t_0)^2} + \frac{2\nu}{4}, \quad \forall t > s_j^0 + \tilde{\delta}_1,
\]

(77)

which holds for all \((t, j) \in \text{dom}(x, \mu, \tau)\) such that \( t + j \leq \hat{T} \).

Defining \( \tilde{\delta}_j = \inf\{t \geq 0 : (t, j) \in \text{dom}(x, \mu, \tau)\} + \tilde{\delta}_1, \) and combining the bounds (77) and (72), we obtain

\[
f(x(t, j)) - f^* \leq \frac{4c_j}{(t - t_0)^2} + \frac{3\nu}{4}, \quad t > t_0 + \tilde{\delta}_j,
\]

(78)

which holds for all \((t, j) \in \text{dom}(x, \mu, \tau)\). Finally, since \( a < \frac{\tilde{\alpha}}{4}, |\mu(t, j)| \leq 1 \) for all \((t, j) \in \text{dom}(x, \mu, \tau)\), and using again the uniform continuity of \( f \) on \( MB \), we obtain \( f(x(t) + a\mu) \leq f(x(t)) + \frac{\nu}{4}. \) Combining this inequality with (78), the final bound (21) follows by using the definition of \( z \) in (15).

### 6.4 Cases II, III, and IV: Exponential Decrease

The SGPES results of Lemmas 12, 13 and 14, imply that in each Case i, every solution of the average HDS satisfies the bound (47) with \( \omega(y) = |ya|_{\mathcal{A}_s}, \) and \( \beta_i \in \mathcal{K}\mathcal{L} \) given by \( \beta_i(r, s) = \alpha_i, \exp(-\alpha_i, s) r, \) where \( \alpha_{1, i}, \alpha_{2, i} > 0, i \in \{2, 3, 4\}. \) The bounds (27), (33) and (37) follow now directly by Theorem 9, the definition of \( z \) in (15), and the triangle inequality.

### 7 CONCLUSIONS AND OUTLOOK

In this paper, we studied a class of novel zero-order dynamics for real-time optimization in problems where the cost function is only accessible by measurements. We presented four algorithms which can all be modeled as singularly perturbed hybrid dynamical systems. We established stability and convergence results for convex, strongly convex, constrained, and unconstrained optimization problems. We extended these results to discretized hybrid systems obtained via forward Euler or Runge Kutta methods with small step sizes. For all our algorithms, we established structural robustness properties with respect to arbitrarily small bounded disturbances, which could be time-varying and of adversarial nature. In order to obtain our main results, we developed a novel averaging theorem for hybrid dynamical systems that generate an average system that renders a compact set SGPAS instead of UGAS. This theorem is instrumental for the analysis of fast hybrid extremum seeking controllers that go beyond those studied in this paper, including algorithms that incorporate Hessian damping without any explicit estimation of the Hessian [38,4].

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A Proof of Lemma 1

Let \( \Psi(t) := \left[ \Psi_1(t)^\top, \Psi_2(t)^\top, \Psi_3(t)^\top, \ldots, \Psi_{2n-1}(t)^\top \right]^\top \in \mathbb{R}^{2n} \) and \( \mu_0 := \left[ \mu_{0,1}^\top, \mu_{0,3}^\top, \mu_{0,5}^\top, \ldots, \mu_{0,2n-1}^\top \right]^\top \in \mathbb{R}^{2n} \), where
the vectors \( \Psi_1(t) \) and \( \mu_{0,i}(t) \) are defined as in (12) with \( k_1/\varepsilon = 1 \). With these definitions in hand, we can write 
\[ \tilde{\mu}(t) = \text{diag}(\mu_0)^\top \Psi(t) \] 
which implies [6, Exercise 260] that 
\[ \int_0^{k_T} \tilde{\mu}(t) \Psi(t) \, dt = \text{diag}(\mu_0)^\top \left( \int_0^{k_T} \Psi(t) \Psi(t) \, dt \right) \text{diag}(\mu_0), \]
where \( \text{diag}(\mu_0) \) is a \((2n \times n)\) block diagonal matrix with diagonal blocks given by \( \mu_{0,i}, i \in \{1,3,5,\ldots, n-1\} \). Thus, it suffices to show the existence of a \( \tilde{T} > 0 \) such that 
\[ \frac{1}{k_T} \int_0^{k_T} \Psi(t) \Psi(t) \, dt = \frac{1}{2} I_{2n}, \quad \int_0^{k_T} \Psi(t) \, dt = 0_{2n}, \quad (A.1) \]
since this would imply that \( \int_0^{k_T} \tilde{\mu}(t) \Psi(t) \, dt = 0.5\text{diag}(\mu_0)^\top \text{diag}(\mu_0) = 0.5 I_{2n} \), where the last equality follows by the fact that \( \mu_{0,i} \) is 1 for all \( i \in \{1,3,\ldots, n\} \) since \( \mu(0) \in \mathbb{S}^n \). To show (A.1) we show the existence of a \( \tilde{T} > 0 \) such that 
\[ \frac{1}{k_T} \int_0^{k_T} \Psi_i(t) \Psi_j(t) \, dt = c_{ij} I_2, \quad \int_0^{k_T} \Psi_i(t) \, dt = 0_2, \quad (A.2) \]
for all \( i,j \in \{1,3,\ldots, n\} \) and all \( k \in \mathbb{Z}_{>0} \), where \( c_{ij} = 0.5 \) for all \( i = j \), and \( c_{ij} = 0 \) for all \( i \neq j \). Indeed, by Assumption 1, the parameters \( \kappa_T \) can be written as \( \kappa_T = \kappa_T^p / \kappa_T^s \), for all \( T \in \{1,2,3,\ldots, n\} \), where \( \kappa_T^p \) and \( \kappa_T^s \) are positive integers. Let \( T_{\ell} := 1/\kappa_T \) and \( T := \Pi_{T_{\ell}} \in \mathbb{Z}_{>0} \). Define \( \tilde{T} := \Pi T_{\ell} \kappa_T \) and let \( \tilde{T} = \text{LCM}(\tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_n) \), where LCM stands for least common multiplier. Such \( \tilde{T} \) is a well defined positive integer and it is unique. Then, by definition of the LCM, for each \( T \) there exists an \( n_T \in \mathbb{Z}_{>0} \) such that 
\[ \tilde{T} = n_T T_\ell, \quad n_\ell := n_T \kappa_T \in \mathbb{Z}_{>0}. \quad (A.3) \]
Using \( \ell = (i + 1)/2 \) and the definition of \( \Psi_i(t) \), we obtain 
\[ \int_0^{k_T} \Psi_i(t) \, dt = \left[ \int_0^{k_T} \cos \left( \frac{2\pi t}{\tilde{T}_\ell} \right) \, dt \right] \left[ \int_0^{k_T} \sin \left( \frac{2\pi t}{\tilde{T}_\ell} \right) \, dt \right] = \left[ \sin \left( \frac{2\pi t}{\tilde{T}_\ell} \right) \right]^{k_{\tilde{\kappa}_T T_\ell}}_{0}, \]
which is equal to \( 0_2 \) for all \( i \in \{1,3,\ldots, 2n-1\}, k \in \mathbb{Z}_{>0} \), as in (A.2). Also, \( \int_0^{k_T} \Psi_i(t) \Psi_j(t) \, dt \) is 
\[ \left[ \int_0^{k_T} \cos \left( \frac{2\pi t}{\tilde{T}_\ell} \right) \cos \left( \frac{2\pi t}{\tilde{T}_\ell} \right) \, dt \right] \left[ \int_0^{k_T} \cos \left( \frac{2\pi t}{\tilde{T}_\ell} \right) \sin \left( \frac{2\pi t}{\tilde{T}_\ell} \right) \, dt \right], \quad (A.4) \]
where \( s = (j + 1)/2 \). When \( i = j \) we have that \( \ell = s \) and the diagonal terms satisfy 
\[ \int_0^{k_T} \cos \left( \frac{2\pi t}{\tilde{T}_\ell} \right) \, dt = \frac{1}{2} \left( t + \frac{\sin(4\pi t/T_\ell)}{4\pi} \right)^{k_{\tilde{\kappa}_T T_\ell}}_0 = \frac{k_{\tilde{\kappa}_T T_\ell}}{2}, \]
while the off-diagonal terms are given by 
\[ \int_0^{k_T} \sin \left( \frac{2\pi t}{\tilde{T}_\ell} \right) \, dt = \frac{1}{2} \left( t - \frac{\sin(4\pi t/T_\ell)}{4\pi} \right)^{k_{\tilde{\kappa}_T T_\ell}}_0 = \frac{k_{\tilde{\kappa}_T T_\ell}}{2}. \]
Thus, when \( i = j \) we have that \( c_{ij} = 0.5 \) in (A.2). On the other hand, when \( i \neq j \), we have that \( \ell \neq s \) and the diagonal terms of (A.4) become 
\[ \int_0^{k_T} \cos \left( \frac{2\pi t}{T_\ell} \right) \cos \left( \frac{2\pi t}{T_\ell} \right) \, dt = \frac{\sin(2\pi T_{\ell}^s)}{4\pi(T_{\ell}^s)} + \frac{\sin(2\pi T_{\ell}^p)}{4\pi(T_{\ell}^p)} \int_0^{k_T} \sin \left( \frac{2\pi t}{T_\ell} \right) \sin \left( \frac{2\pi t}{T_\ell} \right) \, dt = \frac{\sin(2\pi T_{\ell}^s)}{4\pi(T_{\ell}^s)} - \frac{\sin(2\pi T_{\ell}^p)}{4\pi(T_{\ell}^p)} \int_0^{k_T} \cos \left( \frac{2\pi t}{T_\ell} \right) \cos \left( \frac{2\pi t}{T_\ell} \right) \, dt = \frac{\cos(2\pi T_{\ell}^s)}{4\pi(T_{\ell}^s)} - \frac{\cos(2\pi T_{\ell}^p)}{4\pi(T_{\ell}^p)} \int_0^{k_T} \cos \left( \frac{2\pi t}{T_\ell} \right) \sin \left( \frac{2\pi t}{T_\ell} \right) \, dt = \frac{\cos(2\pi T_{\ell}^s)}{4\pi(T_{\ell}^s)} + \frac{\cos(2\pi T_{\ell}^p)}{4\pi(T_{\ell}^p)}, \]
which are also zero by the definition of \( T_{\ell,s}^+, T_{\ell,s}^- \) and (A.3). This establishes that \( c_{ij} = 0 \) in (A.2) whenever \( i \neq j \). \( \square \)

**B Proof of Proposition 6**

Let \( K \in \mathcal{B}_A \) and \( \varepsilon > 0 \) be given. Since \( \omega \) is continuous and grows unbounded as \( x \to \text{bd}(\mathcal{B}_A) \) there exists an \( r_2 > \varepsilon \) such that \( K \subset \{ x \in \mathcal{B}_A : \omega(x) \leq r_2 \} \). Choose \( \nu = \varepsilon/4 \). Then, since the system \( \mathcal{H}_\rho \) is SGP-AS (w.r.t. \( \mathcal{B}_A \)) as \( \delta \to 0^+ \) there exists \( \delta^* > 0 \) such that for all \( \delta \in (0,\delta^*) \) all solutions \( x_\delta \) of \( \mathcal{H}_\delta \) with \( \omega(x_\delta(0,0)) \leq r_2 \) and all \( (t,j) \in \text{dom}(x_\delta) \) the following holds:
\[ \omega(x_\delta(t,j)) \leq \beta(w(x_\delta(0,0)), t+j) + \varepsilon/4. \quad (B.1) \]
Let \( T > 0 \) be large enough such that \( \beta(r_2, t+j) \leq \varepsilon/2 \), for all \( t+j \geq T \).

Claim: There exists a \( \rho^* > 0 \) such that for all \( \rho \in (0,\rho^*], \) all solutions \( x_{\delta,\rho} \) to \( \mathcal{H}_{\delta,\rho} \) with \( \omega(x_{\delta,\rho}(0,0)) \leq r_2 \) and all \( (t,j) \in \text{dom}(x_{\delta,\rho}) \) the following holds:
\[ \omega(x_{\delta,\rho}(t,j)) \leq \beta(w(x_{\delta,\rho}(0,0)), t+j) + \varepsilon/2. \quad (B.2) \]
for all \( t+j \leq 2T \). \( \square \)

By the selection of \( T \) above, the claim implies that \( \omega(x_{\delta,\rho}(t,j)) \leq \varepsilon, \) for all \( 2T > t+j \geq T \), such that
(t, j) ∈ dom(\(x_{\delta, \rho}\)). We can recursively apply this argument restarting the solution and using \(\varepsilon < r_2\) to get \(\omega(x_{\delta, \rho}(t, j)) \leq \varepsilon\) for all \((t, j) \in \text{dom}(x_{\delta, \rho})\) such that \(t + j \geq T\).

To prove the claim, suppose by contradiction that there exists a sequence \(p_i \downarrow 0\) and a sequence of solutions \(x_{\delta, \rho,i}\) to \(H_{\delta, \rho,i}\) with \(\omega(x_{\delta, \rho,i}(0, 0)) \leq m\) and points \((t_i, j_i) \in \text{dom}(x_i)\) with \(t_i + j_i \leq 2T\) such that \((B.2)\) does not hold:

\[
\omega(x_{\delta, \rho,i}(t_i, j_i)) > \beta(\omega(x_{\delta, \rho,i}(0, 0)), t_i + j_i) + \varepsilon/2,
\]

\[
(B.3)
\]

Since \(\omega(x_{\delta, \rho,i}(0, 0)) \leq r_2\), implies that the sequence \(x_{\delta, \rho,i}(0, 0)\) lies in a compact subset of \(E_A\), one can assume that it converges to some point in \(E_A \cap (C \cup D)\). At this point, because of \((B.1)\), the HDS \(H_\delta\) is pre-forward complete. Since this implies that the sequence \(x_{\delta, \rho,i}\) is locally eventually bounded [17, Def. 5.24], and since for each \(\delta > 0\) the system \(H_\delta\) satisfies the Basic Conditions, the graphical limit of the sequence \(x_{\delta, \rho,i}\), denoted by \(x_\delta\), will be a solution to \(H_\delta\). Without loss of generality we can assume that the sequence \((t_i, j_i)\) also converges to some \((t, j) \in \text{dom}(x_\delta)\). Using continuity of \(\omega\) and \(\beta\) and taking the limit as \(i \to \infty\) at both sides of \((B.3)\) we obtain \(\omega(x_\delta(t, j)) > \beta(\omega(x_\delta(0, 0)), t + j) + \frac{\varepsilon}{2}\), which violates \((B.1)\) at the time \((t, j) \in \text{dom}(x_\delta)\). This is a contradiction. 

**Lemma 3** Let \(f : \mathbb{R}^n \to \mathbb{R}\) satisfy Assumption 2 and \(A_f = \{z^*\}\). Consider the set

\[
\mathcal{O} := \{x_1 \in \mathbb{R}^n : (x_1 - z^*)^\top \nabla f(x_1) - (f(x_1) - f^*) = 0\}.
\]

Then, we have that \(\mathcal{O} = A_f\).

**Proof:** Let \(x_1\) be such that \((x_1 - z^*)^\top \nabla f(x_1) - (f(x_1) - f^*) = 0\). Suppose that \(x_1 \neq z^*\). Let \(\alpha_1 := f(x_1)\) and define the set \(\Omega_{\alpha_1} := \{x \in \mathbb{R}^n : f(x) \leq \alpha_1\}\). Since \(z^*\) is optimal, we have that \(f(z^*) = f^* \leq f(x_1)\) and therefore \(A_f \subset \Omega_{\alpha_1}\). Since \(f \in C^2\) we have that \(\nabla f\) is locally Lipschitz, and since \(\Omega_{\alpha_1}\) is compact there exists \(L_{\alpha_1} > 0\) such that

\[
|\nabla f(x_1') - \nabla f(x_1'')| \leq L_{\alpha_1}|x_1' - x_1''| \quad \text{for all } (x_1', x_1'') \in \Omega_{\alpha_1}.
\]

By the convexity and Lipschitz properties in \(\Omega_{\alpha_1}\),

\[
f(z^*) - f(x_1) - \nabla f(x_1)^\top (z^* - x_1) \geq \frac{1}{2L_{\alpha_1}}|\nabla f(x_1)|^2,
\]

but since by assumption the left hand side of the inequality is zero, we must have that \(|\nabla f(x_1)| = 0\), which is a contradiction given that \(x_1 \neq z^*\).

**C Solutions of Hybrid Dynamical Systems**

A general HDS can be written as the hybrid inclusion

\[
\dot{x} \in F(x), \quad x \in C, \quad (C.1a)
\]

\[
x^+ \in G(x), \quad x \in D, \quad (C.1b)
\]

where \(F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n\) and \(G : \mathbb{R}^n \rightrightarrows \) are set-valued mappings (this setting covers the single-valued case of (2), see [17, Ch. 2]). Solutions of \((C.1)\) are defined on hybrid time domains. Under mild assumptions on \((F, C, G, D)\), this allows the use of graphical convergence notions to establish sequential compactness results for the solutions of \((C.1)\) (e.g., the limit of a sequence of solutions is also a solution). A set \(E \subset \mathbb{R} \times \mathbb{Z} \) is called a compact hybrid time domain if \(E = \bigcup_{j=0}^{\infty} ([t_j, t_{j+1}], j)\) for some finite sequence of times \(0 = t_0 \leq t_1, \ldots, t_j\). The set \(E\) is a hybrid time domain if for all \((T, J) \in E\), \(E \cap ([0, T] \times \{0, \ldots, J\})\) is a compact hybrid time domain.

**Definition 8** A function \(x : \text{dom}(x) \to \mathbb{R}^n\) is a hybrid arc if \(dom(x)\) is a hybrid time domain and \(x(t, j)\) is locally absolutely continuous for each \(j\) such that the interval \(I_j := \{t : (t, j) \in \text{dom}(x)\}\) has nonempty interior. A hybrid arc \(x\) is a solution to \((C.1)\) if \((x(0, 0), 0) \in \overline{C} \cup D\), and the following two conditions hold:

1. For each \(j \in \mathbb{Z} \) such that \(I_j\) has nonempty interior: \(x(t, j) \in C\) for all \(t \in \text{int}(I_j)\), and \(\dot{x}(t, j) \in F(x(t, j))\) for almost all \(t \in I_j\).
2. For each \((t, j) \in \text{dom}(x)\) such that \((t, j + 1) \in \text{dom}(x)\): \(x(t, j) \in D\), and \(x(t, j + 1) \in G(x(t, j))\).

**Definition 9** A hybrid solution \(x\) is said to be forward pre-complete if its domain is compact or unbounded, i.e., if the flows do not generate finite escape times. A hybrid solution is said to be forward complete if its domain is unbounded. A hybrid solution is maximal if there does not exist another solution \(\psi \in \mathcal{H}\) such that \(dom(\psi)\) is a proper subset of \(dom(\psi)\), and \(x(t, j) = \psi(t, j)\) for all \((t, j) \in \text{dom}(x)\).