Neutrinos in the Simplest Little Higgs Model

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Abstract: The simplest little Higgs model based on a $SU(3)$ global symmetry contains a $SU(3)_{\text{weak}}$ triplet and a singlet per a generation in the lepton sector. A neutral component of the triplet and the singlet turn into a neutral vector-like $SU(2)_L$ singlet after electroweak symmetry breaking while the other neutral component of the triplet is the SM neutrino. At tree level, Yukawa couplings of the lepton sector not only allow the neutral vector-like lepton to couple to the SM neutrino, but also give them a Dirac mass. Majorana mass terms for the SM neutrinos and their partners arise at one loops, leading to neutrino flavor mixing in addition to neutrino-heavy neutral lepton mixing.

Keywords: Neutrino Physics, Beyond Standard Model.
1. Introduction

The little Higgs models, as an alternative solution of the “little hierarchy problem” of the Standard Model (SM), revive the idea of the Higgs doublet being a pseudo Goldstone boson of some global symmetry which is spontaneously broken at a TeV scale. In the little Higgs models the SM Higgs is a part of the scalar multiplet(s) of some global symmetry and the SM Higgs mass parameter depends logarithmically on the UV cutoff scale of the global symmetry. Among the various little Higgs models, there is the simplest little Higgs model (SLHM) \[1, 2\] based on a simple $SU(3)$ global symmetry. The study of precision data in the gauge and quark sector set constraints on the parameters of the simplest little Higgs model \[1, 2\].

The SLHM contains two distinct features compared with the littlest Higgs model (LHM) \[3\] which is most intensively examined among the little Higgs models. The first feature is that the SLHM has no counter partner(s) to the SM Higgs scalar. The existence of the SM Higgs counter partner in the little Higgs models is required to cancel dangerous quadratically divergent contribution to the SM Higgs mass squared parameters coming from the SM Higgs loops. In the LHM, the complex $SU(2)_L$ triplet scalar play a role of the counter partner to the SM Higgs doublet. However, in the SLHM the absence of the quadratically divergent contribution to the Higgs mass parameter from the Higgs loops is excused by the fact that the SM Higgs is contained in a pair of Higgs multiplets rather than in a Higgs multiplet.

The second feature, barely noticed before, is that the SM neutrino in a generation accompanies a neutral vector-like lepton just as the top quark teams up with a heavy vector-like quark. The heavy top partners lead to the large Yukawa couplings for the top quark. Unlike the heavy top quark, the SM left-handed neutrinos do not have right-handed pairs. Furthermore, they have extremely small masses, which has been convinced by neutrino flavor
oscillation experiments as well as cosmological and astrophysical observations. The standard formalism for neutrino oscillations is based on oscillations among the three left-handed neutrinos ($\nu_e, \nu_\mu, \nu_\tau)_L$, which is referred to as first-class oscillations. In the SLHM, the coupling of the SM neutrino to its heavy partner further give rise to transitions between the SM neutrinos and their heavy partners, which is dubbed as second-class oscillations.\[7\]

The object of this letter is to make a detailed analysis of the second-class oscillations as well as the first-class oscillations within the framework of the SLHM. To do so, we first inspect the origin of the SM neutrino masses which in general arise from Dirac or Majorana mass terms in various extensions of the SM. In the SLHM, though right-handed neutral leptons are present in the neutral lepton sector they are not Dirac-paired with the SM (left-handed) neutrinos. Rather they are Dirac-paired with the partners of the SM neutrinos. Therefore it is naively expected that the SM neutrino mass in the SLHM arises only from Majorana mass terms. We show that radiative corrections give rise to the Majorana mass terms, leading to not only the SM neutrino masses but also the SM neutrino mixing angles. We further search for other consequences of the scenario at weak energy scale.

The rest of the letter is organized as follows: In section 2 we briefly review the Higgs sector of the SLHM. In section 3 we discuss the Yukawa couplings of the lepton sector in one generation case. In section 4 we study the mechanism of the SM neutrino masses in one generation case. In section 5 we extend the mechanism of the SM neutrino masses to three generation case and investigate both the first-class and second-class oscillations. In section 6 we give a summary and provide an outlook for the neutrino physics within the SLHM.

2. Higgs sector

The simplest little Higgs model based on the $SU(3)$ simple global symmetry has the initial electroweak gauge structure $SU(3)_{\text{weak}} \times U(1)_X$ which is broken to the SM $SU(2)_L \times U(1)_Y$. The symmetry breaking is triggered by the VEV’s of a pair of triplets $\Phi_1, \Phi_2$, which transform as $(3, -\frac{1}{3})$ under the $SU(3)_{\text{weak}} \times U(1)_X$. They are parameterized non-linearly as follows:

\[
\begin{align*}
\Phi_1 &= \exp \left\{ i \Theta \frac{f_2}{f_1} \right\} \begin{pmatrix} 0 \\ 0 \\ f_1 \end{pmatrix} \equiv \exp \{ i \Theta \cot \beta / f \} \begin{pmatrix} 0 \\ 0 \\ f \sin \beta \end{pmatrix}, \\
\Phi_2 &= \exp \left\{ -i \Theta \frac{f_1}{f_2} \right\} \begin{pmatrix} 0 \\ 0 \\ f_2 \end{pmatrix} \equiv \exp \{ -i \Theta \tan \beta / f \} \begin{pmatrix} 0 \\ 0 \\ f \cos \beta \end{pmatrix}
\end{align*}
\]

where $f_1 \equiv f \sin \beta$, $f_2 \equiv f \cos \beta$, and the $\Theta$ NGB matrix is

\[
\Theta = \frac{\eta}{\sqrt{2}} + \begin{pmatrix} 0 & 0 & h \\ 0 & 0 & h^\dagger \\ h^\dagger & 0 & 0 \end{pmatrix}.
\]

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Here the field $h$ is an $SU(2)_L$ doublet which is identified with the SM Higgs doublet, $\eta$ is a real $SU(2)_L$ singlet scalar with no vev. The mass of $\eta$ is assumed to be of the electroweak scale and the complete analysis of the physics associated with $\eta$ is given in Ref. [8].

There are yet seven degrees of freedom in $\Phi_1$ and $\Phi_2$ but we have omitted them in Eq. (2.1) by intention. The reason is as follows: five degrees of freedom are eaten by the gauge fields during the symmetry breaking of $SU(3)_{\text{weak}} \times U(1)_X \rightarrow SU(2)_L \times U(1)_Y$ acting as the longitudinal components of the broken gauge fields and the rest, two degrees of freedom are the massive excitation modes along the two vev’s directions with their masses being of order $f$. Thus we can take into account only five pseudo Goldstone bosons, $h$ and $\eta$, below the scale $f$.

3. One generation case

We now take into account the leptonic sector in the SLHM. For simplicity, we consider only the first generation in this section. The SM first lepton generation is embedded in $SU(3)_{\text{weak}} \times U(1)_X$ representations as follows:

$$\Psi_L = (3, -\frac{1}{3}) \quad e_R = (1, 1) \quad n_R = (1, 0)$$

where the triplet $\Psi_L = (iL, n_L)^T$ embraces the left-handed $SU(2)_L$ doublet, $L = (\nu_{eL}, e_L)^T$ while the right-handed $SU(3)_{\text{weak}}$ singlets $e_R$ and $n_R$ are the Dirac partners of $e_L$ and $n_L$ in the triplet, respectively. Note that the SM neutrino in the triplet has no Dirac partner.

The existence of the two triplet scalars allows us to construct the Yukawa couplings, yielding mass terms for the fermions in the model. In particular, the mass terms for the SM charged lepton $e$ and neutral lepton $n$ arise from the interactions of the form:

$$-\mathcal{L}_{\text{yuk}} = \lambda^e n_R \Phi_1^\dagger \Psi_L + \frac{\lambda^e}{f} e_R \Phi_1^\dagger \Phi_2^j \Psi_L^k \epsilon_{ijk} + \text{h.c.},$$

where $\lambda^e$ and $\lambda^n$ are set to be real, which is allowed by redefining phases of $n_R$ and $e_R$, respectively. As in the quark sector, one may also take into account another term like $n_R \Phi_2^\dagger \Psi_L$ which is obtained by replacing $\Phi_1^j$ by $\Phi_2^j$ in the first term of (3.2). However, we need not include it because it does not change the physics associated with neutrinos which we shall discuss. To make the physics simple, we ignore the interaction from now on.

The second component in the lepton triplet, $e_L$, and the charged singlet, $e_R$, have a Dirac mass after the SM Higgs (the first component of $\Phi_1$) gets a vev, $\langle h \rangle = (\frac{v}{\sqrt{2}}, 0)$:

$$\frac{\lambda^e}{f} e_R \Phi_1^\dagger \Phi_2^j \Psi_L^k \epsilon_{ijk} + \text{h.c.} \rightarrow -\mathcal{L}_{\text{Dirac}} = \lambda^e \frac{v}{\sqrt{2}} e_R e_L + \text{h.c.},$$

---

1. We shall assert that $n_L^c \neq n_R$ after we introduce the SM neutrino mass.

2. There are two different embeddings of the quarks and leptons in Ref. [1]. Though the quark charges under the 3-3-1 gauge group are different between the two embeddings, the lepton charges are the same in both.
yielding the electron mass, \( m_e = \lambda^e v/\sqrt{2} \), as in the SM. On the other hand, a linear combination of \( \nu_{eL} \) and \( n_L \) have Dirac masses via the neutral singlet, \( n_R \), as the first and third components of \( \Phi_1 \) acquire vevs, \( \langle \Phi_1 \rangle = (i f_2/f (h), f_1)^T \), respectively:

\[
\lambda^n n_R \Phi_1^\dagger \Psi_L + \text{h.c.} \rightarrow \lambda^n f_1 n_R n_L + \lambda^n f_2 \frac{v}{\sqrt{2}} n_R \nu_{eL} + \text{h.c.}
\]

\[
= \lambda^n f_1 n_R \left( \sin \beta n_L + \frac{v}{f \sqrt{2}} \cos \beta \nu_{eL} \right) + \text{h.c.}
\]

(3.4)

The relation of mass eigenstate \( (\hat{\nu}_{eL}, \hat{n}_L) \) to weak eigenstate \( (\nu_{eL}, n_L) \) is given by

\[
\hat{n}_L = \cos \theta n_L + \sin \theta \nu_{eL}, \quad \hat{\nu}_{eL} = -\sin \theta n_L + \cos \theta \nu_{eL}
\]

(3.5)

with the mixing angle

\[
\sin \theta \equiv \frac{\sqrt{2} f \cos \beta}{\sqrt{\sin^2 \beta + \frac{v^2}{2 f^2} \cos^2 \beta}}, \quad \cos \theta \equiv \frac{\sin \beta}{\sqrt{\sin^2 \beta + \frac{v^2}{2 f^2} \cos^2 \beta}}.
\]

(3.6)

Note that the mixing angle is independent of the Yukawa coupling \( \lambda^n \), and vanishes in the limits of either \( v/f \rightarrow 0 \) or \( \beta \rightarrow \pi/2 \).

The neutral fields \( \hat{n}_L \) and \( n_R \) consist of the neutral heavy lepton:

\[
\mathcal{L}_{\text{Dirac}} = -m_n n_R \hat{n}_L + \text{h.c.}
\]

(3.7)

with a Dirac mass

\[
m_n = \lambda^n f \sqrt{\sin^2 \beta + \frac{v^2}{2 f^2} \cos^2 \beta},
\]

(3.8)

which is determined almost by the coupling \( \lambda^n \) because the global symmetry breaking parameter \( f \) is assumed to be a TeV range in little Higgs models. Experimentally, the neutral heavy lepton have a larger mass than the weak gauge bosons, satisfying the neutral heavy lepton mass limits \( 9 \). Assuming that the vev parameter is \( f \lesssim 10 \text{ TeV} \) we can set a lower bound on the coupling, \( \lambda^n \gtrsim 10^{-2} \).

From Eq. (3.5) we see that second-class oscillations between \( \nu_{eL} \) and \( n_L \) can be present so that the chance of finding the signature of an neutral heavy lepton is proportional to the mixing angle, \( \sin \theta \). But due to a large mass of the neutral heavy lepton second-class oscillations are not present in neutrino experiments. Instead, a way to probe \( \sin \theta \), is to measure decays of the neutral heavy lepton via charged currents like \( n \rightarrow W \ell_L \) or neutral currents like \( n \rightarrow Z \nu_L \).

As for the SM neutrino the Yukawa interactions in Eq. (3.2) leaves the SM neutrino to be massless. To explain nonzero mass for the SM neutrino favored in the neutrino oscillation experiments, we explore not only the origin of the neutrino mass but also a UV completion of

\[\text{In case that the Yukawa interaction for } n \text{ is given by } \lambda^n n^c \Phi_1^\dagger \Psi_L \text{ instead of } \lambda^n n^c \Phi_1^\dagger \Psi_L, \text{ the mass } m_n \text{ is obtained by exchanging } \sin \beta \leftrightarrow \cos \beta \text{ in Eq. (3.8).}\]
the simplest little Higgs model. But one should keep in mind that the UV cutoff of little Higgs models is about $10 \sim 100$ TeV, which implies that the symmetry breaking scale associated with the SM neutrino masses is so low that the seesaw mechanism is of no use. Therefore, we reckon that the SM neutrino masses will stem from another mechanism which yields a sufficiently small mass for the neutrino.

4. Mechanism of Majorana masses

We now turn to our attention to the origin of the SM neutrino masses in the context of the simplest little Higgs model. One may think of a Dirac mass term as a source of the SM neutrino mass due to the presence of the right-handed neutral singlet field, $n_R$. But we have already shown that the right-handed singlet does not contribute to a Dirac mass for the SM neutrino. Rather, it becomes the Dirac pair with the heavy neutral partner of the SM neutrino. Furthermore, the SM neutrinos in other little Higgs models seems to acquire a Majorana mass rather than a Dirac mass [10, 11, 12]. Accordingly tiny SM neutrino masses in the simplest little Higgs model are expected to be Majorana, which was first considered in Ref. [13]. In what follows we shall further scrutinize the mechanism of the SM neutrino mass in Ref. [13]. But for simplicity we consider one generation case in this section and shall extend to three generation case in Section 5.

First of all, we add a Majorana mass term for $SU(3)_{\text{weak}}$ singlet, $n_R$, to the Lagrangian:

$$\mathcal{L}_{\text{Majorana}} = m_{\phi} n_R n^c_R + h.c., \quad (4.1)$$

whose origin is due to a singlet Higgs or a bare mass term but is unknown in the context of the simplest little Higgs model. A UV completion of the model should give an explanation for its presence. We assert a mass hierarchy, $m_{\phi} \ll m_n$, to maintain the vector-like property of heavy neutral leptons in the simplest little Higgs model. Thus the smallness of $m_{\phi}$ is expected to be achieved by quantum effects in a UV completion of the model.

Next, we introduce Majorana mass term to $SU(2)_L$ singlet $n_L$:

$$\mathcal{L}_{\text{Majorana}} = m_{\nu} \nu^c_L \nu_L + h.c. \quad (4.2)$$

Since both $\nu^c_L$ and $n_L$ belongs to a $SU(3)_{\text{weak}}$ triplet, the presence of a Majorana mass term for $n_L$ is expected to be accompanied with a Majorana mass term for $\nu^c_L$:

$$\mathcal{L}_{\text{Majorana}} = m_{\nu} \nu^c_L \nu^c_{\nu} L + h.c. \quad (4.3)$$

---

Figure 1: Diagram for Majorana mass matrix of $\Psi_L$. 

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These Majorana mass terms are achieved at loop level by diagrams as shown in Fig. 1 where the vertex-cross at bottom stands for Majorana mass for $n_R$ while the two vertex-crosses at top represent the vev of $\Phi_2$. The black square in Fig. 1 represents a coupling involved with a dimension four operator, $|\Phi_1^\dagger \Phi_2|^2$ which gives rise to the logarithmic divergent contributions to the Higgs mass at one-loop [14]. Diagrams in Fig. 2 and Fig. 3 show how the operator $|\Phi_1^\dagger \Phi_2|^2$ is generated at one-loop. The diagram in Fig. 2 generates a operator of the form:

$$\frac{g_4^4}{16\pi^2}|\Phi_1^\dagger \Phi_2|^2 \log(\Lambda^2/f^2),$$

(4.4)

where $\Lambda \sim 4\pi f$ is a UV cutoff. This diagram contributes to the SM Higgs mass term, $-f^2/16\pi^2 h^\dagger h$, leading to a lower value of the SM Higgs mass. Therefore, "$\mu$"-term is included to enhance the SM Higgs mass parameter:

$$\mathcal{L} = \mu^2 |\Phi_1^\dagger \Phi_2| + h.c.$$  \hspace{1cm} (4.5)

The diagram in Fig. 3 also contributes to the SM Higgs mass, giving rise to a operator of the form:

$$\lambda_1^2 \lambda_2^2/16\pi^2 |\Phi_1^\dagger \Phi_2|^2 \log(\Lambda^2/f^2).$$

(4.6)

Note that fermion loop contributions are proportional to the two Yukawa couplings squared and the top quark sector has the largest Yukawa couplings. Thus the leading fermion loop contribution comes from the top sector and all the other quark contributions are safely negligible compared with the top contribution 4. In what follows, we sum up all the four $\Phi$'s interactions and rewrite it as the effective operator:

$$\mathcal{L} = \kappa |\Phi_1^\dagger \Phi_2|^2,$$

(4.7)

where $\kappa$ is a dimensionless coupling whose value is of order $10^{-2}$.

In summary the Majorana masses for the SM neutrino and its heavy partner are generated at two-loop level and the corresponding low-energy effective dimension five operator is given by

$$\mathcal{L}_5 = \frac{(\lambda_n^2)}{\Lambda_\nu} (\Phi_2^\dagger \Psi_L) (\Phi_2^{\dagger} \Psi_L) + h.c.,$$

(4.8)

with

$$\frac{1}{\Lambda_\nu} \approx \frac{\kappa}{16\pi^2 f^2 m_\nu}.$$  \hspace{1cm} (4.9)

After substituting the vev of $\Phi_2$ in Eq. (4.8), $\mathcal{L}_5$ becomes nothing but the Majorana mass matrix of the SM neutrino and its heavy partner:

$$-\mathcal{L}_{Majorana} \left( \begin{array}{c} \nu_{eL} \\ n_{L} \end{array} \right) \left( \begin{array}{cc} (\lambda_n^2/2 \cos^2 \beta) & (\lambda_n^2/2 \sin 2\beta) \\ (\lambda_n^2/2 \sin 2\beta) & (\lambda_n^2/\cos^2 \beta) \end{array} \right) \left( \begin{array}{c} \nu_{eL} \\ n_{L} \end{array} \right) + h.c.$$  \hspace{1cm} (4.10)

4For the first two generations, $\lambda_1 \ll \lambda_2 \sim 1$ in the first model [1].
From Eqs. (4.2) and (4.3) we read that
\[ m_{\Box} \approx \left( \frac{\lambda n}{\Lambda_{\nu}} \right)^2 f^2 \sin^2 \beta, \quad m_{\nu} \approx \left( \frac{\lambda n}{\Lambda_{\nu}} \right)^2 \frac{v^2}{2} \cos^2 \beta. \]  

(4.11)

Furthermore, there is a mixing term between \( \nu_{eL} \) and \( n_{L} \), whose dimensionful coupling is
\[ \hat{m} \approx -\left( \frac{\lambda n}{\Lambda_{\nu}} \right)^2 \frac{v f}{\sqrt{2}} \sin 2\beta. \]  

(4.12)

Note that the relative minus sign in Eq. (4.12) compared with Eq. (4.11) originates from the relative phase difference between \( \Phi_1 \) and \( \Phi_2 \), which is manifest in Eq. (2.1)\(^5\).

In view of these neutral lepton mass terms, the usual manner of proceeding here is to arrange mass matrix for the neutral leptons in the following 3 \( \times \) 3 matrix,
\[ \Phi^\dagger \Phi \quad \text{potential arises from fermion loop contributions which give log}- \] divergent contribution to the Higgs mass.

\[ -L_{\text{DM}} = (\nu^c_L, \, n^c_L, \, n_R) M_{\text{DM}} \begin{pmatrix} \nu_L \\ n_L \\ n_R \end{pmatrix} + h.c. \]  

(4.13)

with
\[ M_{\text{DM}} \equiv \begin{pmatrix} \frac{(\lambda n)^2 v^2}{2} \cos^2 \beta & -\frac{(\lambda n)^2 v f}{2\sqrt{2}} \sin 2\beta \frac{\lambda n}{\sqrt{2}} \cos \beta \\ -\frac{(\lambda n)^2 v f}{2\sqrt{2}} \sin 2\beta & \frac{(\lambda n)^2}{\sqrt{2}} \cos \beta & \frac{\lambda n}{\sqrt{2}} f \sin \beta \\ \frac{\lambda n}{\sqrt{2}} f \sin \beta & \lambda n f \sin \beta & m_{\Box} \end{pmatrix}. \]  

(4.14)

Since there is a mass hierarchy among the elements of \( \mathcal{M} \), \( \lambda n f > \lambda n v \gg m_{\Box} \gg \{ m_{\nu}, \hat{m}, m_{\Box} \} \) we can evaluate physical neutrino mass by computing the product of the three non-zero eigenvalues of \( M_{\text{DM}} \)
\[ \det M_{\text{DM}} = -\frac{(\lambda n)^4 f^2 v^2 \sin^2 2\beta}{2\Lambda_{\nu}}. \]  

(4.15)

We compare this result with the product of the two heavy lepton masses in Eq. (3.8)
\[ \det' M_{\text{Dirac}} = -(\lambda n)^2 \left( \frac{v^2}{2} \cos^2 \beta + f^2 \sin^2 \beta \right). \]  

(4.16)

To first order in the neutrino mass, the neutral heavy lepton masses are unchanged, and we end up with
\[ m_{\nu} = \frac{\det M_{\text{DM}}}{\det' M_{\text{Dirac}}} = \frac{(\lambda n)^2}{2\Lambda_{\nu}} \frac{v^2 \sin^2 2\beta}{2f^2 \cos^2 \beta + \sin^2 \beta}. \]  

(4.17)

From the fact that the heaviest SM neutrino mass is expected to be of order 0.1 eV, we can set a rough upper limit on the Majorana mass of \( n_R \):
\[ m_{\Box} \lesssim 1 \times \left[ \frac{10^{-2}}{\lambda n} \right]^2 \left[ \frac{10^{-2}}{\kappa} \right] \left[ \frac{f}{1 \text{TeV}} \right]^2 \text{GeV}, \]  

(4.18)

which guarantees the previous assumption, \( m_{\Box} \ll m_n \). It also implies that the heavier the mass \( m_n \) is the smaller the mass \( m_{\Box} \) is.

\(^5\)It will turn on the first-class oscillations in the framework of the three lepton generation.
5. Three generation case

Enlarging one generation to three we construct the lepton Yukawa couplings in three generations. The general Yukawa couplings for the three generations are given by extending the coupling numbers in Eq. (3.2) to the $3 \times 3$ coupling matrices:

$$-\mathcal{L}_{\text{yuk}} = [\lambda^n]_{ab} n_{aR} \Phi_1^+ \Psi_b + \frac{[\lambda^n]_{ab}}{f} \ell_{aR} \Phi_1^j \Phi_2^k \Psi_b^L \epsilon_{ijk} + h.c. \quad (5.1)$$

where $\Psi_a^L = (i\nu_a, i\ell_a, n_a)^T_L$, $n_aR$ and $\ell_aR$ are a left-handed lepton triplet, a right-handed neutral singlet and a right-handed charged singlet under SU(3)$_{\text{weak}}$ in the $a$-th generation, respectively. In what follows square bracket represents a $3 \times 3$ matrix.

We first consider mass eigenstates of the charged leptons by diagonalizing the $3 \times 3$ Yukawa coupling matrix, $[\lambda^n]$, which is transformed by two unitary matrices $U_R$ and $U_L$:

$$U_R^T [\lambda^n] U_L = \text{diag}[\lambda^e, \lambda^\mu, \lambda^\tau]. \quad (5.2)$$

The charged lepton fields in weak eigenstate, $\ell_L \equiv (\ell_1, \ell_2, \ell_3)^T_L$ and $\ell_R \equiv (\ell_1, \ell_2, \ell_3)^R$, are related to the fields in mass eigenstates, $\hat{\ell}_L$ and $\hat{\ell}_R$ as follows:

$$\ell_R = U_R \hat{\ell}_R, \quad \ell_L = U_L \hat{\ell}_L. \quad (5.3)$$

Thus we find that the three charged leptons have the Dirac masses as in the SM,

$$(m_e, m_\mu, m_\tau) = \frac{v}{\sqrt{2}} (\lambda^e, \lambda^\mu, \lambda^\tau). \quad (5.4)$$

As for the neutral leptons we consider the first term of the Lagrangian in Eq. (5.1). We expand it in powers of $1/f$,

$$-\mathcal{L} = [\lambda^n]_{ab} n_{aR} \Phi_1^+ \Psi_b + h.c. \rightarrow [\lambda^n]_{ab} \frac{n_{aR}}{f} \left( \sin \beta n_{bL} + \frac{\nu}{\sqrt{2} f} \cos \beta \nu_{bL} \right) + h.c. \quad (5.5)$$

The Dirac mass terms for the neutral leptons are present so that Eq. (5.5) may be rewritten in more suggestive form:

$$-\mathcal{L}_{\text{Dirac}} = \left( \begin{array}{c} \nu_L^c \\bar{n}_L \\bar{n}_R \end{array} \right) \bar{M}_D \left( \begin{array}{c} \nu_L \\ n_L \\ n_R^c \end{array} \right) + h.c., \quad (5.6)$$

where the three neutral leptons are

$$\nu_L = \begin{pmatrix} \nu_{1L} \\ \nu_{2L} \\ \nu_{3L} \end{pmatrix}, \quad n_L = \begin{pmatrix} n_{1L} \\ n_{2L} \\ n_{3L} \end{pmatrix}, \quad n_R = \begin{pmatrix} n_{1R}^c \\ n_{2R}^c \\ n_{3R}^c \end{pmatrix}, \quad (5.7)$$
and the $9 \times 9$ Dirac mass matrix is

$$
M_D \equiv \begin{pmatrix}
[0] & [0] & [\lambda_n^\dagger \frac{v}{\sqrt{2}} \cos \beta] \\
[0] & [0] & [\lambda_n^\dagger f \sin \beta] \\
[\lambda_n^\dagger \frac{v}{\sqrt{2}} \cos \beta] & [\lambda_n^\dagger f \sin \beta] & [0]
\end{pmatrix}.
$$

We diagonalize the mass matrix $M_D$ by a unitary transformation:

$$
W_0^\dagger M_D W_0 = M_{\text{diag}},
$$

where a transformation matrix $W_0$ is given by

$$
W_0 = \begin{pmatrix}
\cos \theta[1] & \sin \theta[1] & 0 \\
-\sin \theta[1] & \cos \theta[1] & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
[1] & [0] & [0] \\
[0] & V_L & [0] \\
[0] & V_R & [0]
\end{pmatrix} \begin{pmatrix}
[1] & [0] & [0] \\
[0] & -\frac{1}{\sqrt{2}} [1] & \frac{1}{\sqrt{2}} [1] \\
\frac{1}{\sqrt{2}} [1] & \frac{1}{\sqrt{2}} [1] & [0]
\end{pmatrix},
$$

with $V_R$ and $V_L$ being $3 \times 3$ unitary transformation matrices which diagonalize $[\lambda_n^\dagger]$:

$$
V_R^\dagger [\lambda_n^\dagger] V_L = \text{diag} [\lambda_1^n, \lambda_2^n, \lambda_3^n].
$$

The diagonal matrix $M_{\text{diag}}$ is given by

$$
M_{\text{diag}} = \text{diag} [0, 0, 0, -\lambda_1^n, -\lambda_2^n, -\lambda_3^n, \lambda_1^n, \lambda_2^n, \lambda_3^n].
$$

Here we, for simplicity, assume that $\lambda_1^n, \lambda_2^n$ and $\lambda_3^n$ are all real and further are all different, $\lambda_1^n \neq \lambda_2^n \neq \lambda_3^n$. Note that the first three zeros in $M_{\text{diag}}$ imply massless SM neutrinos while the remaining three pairs of mass eigenvalues give rise to Dirac masses for the three heavy neutral leptons. Using Eq. (5.10) we can write the mass eigenstates $(\tilde{\nu}_L, \hat{n}_L, \hat{n}_R)$ in the linear combinations of the weak eigenstates ($\nu_L, n_L, n_R^c$):

$$
\begin{pmatrix}
\tilde{\nu}_L \\
\hat{n}_L \\
\hat{n}_R^n
\end{pmatrix} = \begin{pmatrix}
\cos \theta[1] & -\sin \theta[1] & 0 \\
-\frac{1}{\sqrt{2}} V_L \sin \theta[1] & -\frac{1}{\sqrt{2}} V_L \cos \theta[1] & \frac{1}{\sqrt{2}} V_R[1] \\
\frac{1}{\sqrt{2}} V_L \sin \theta[1] & \frac{1}{\sqrt{2}} V_L \cos \theta[1] & -\frac{1}{\sqrt{2}} V_R[1]
\end{pmatrix} \begin{pmatrix}
\nu_L \\
n_L \\
n_R^c
\end{pmatrix}.
$$

In order to make the SM neutrino acquire masses we include the Majorana mass terms for the neutral leptons to the Lagrangian in Eq. (5.1), as in Section 4. As we enlarge one generation to three the Majorana masses in Eqs. (4.1), (4.11) and (4.12) are replaced with $3 \times 3$ Majorana mass matrices, respectively:

$$
-\mathcal{L}_{\text{Majorana}} = \left( \begin{pmatrix}
\nu_L \\
n_L \\
n_R^c
\end{pmatrix} \begin{pmatrix}
\nu_L \\
n_L \\
n_R^c
\end{pmatrix} \right) M_M \begin{pmatrix}
\nu_L \\
n_L \\
n_R^c
\end{pmatrix} + h.c.,
$$

where the $9 \times 9$ Majorana mass matrix is

$$
M_M = \begin{pmatrix}
[m_\nu] & [\tilde{m}]^\dagger & [0] \\
[\tilde{m}] & [m_\tau] & [0] \\
[0] & [0] & [m_\mu]
\end{pmatrix}.
$$
Here we make a simplifying assumption that \( m_\diamond = m_\diamond \text{diag}[1, 1, 1] \). This allows us to present our results in more suggestive form. But it does not affect in any essential way the physics which we shall draw. With the assumption we can write the 3 \( \times \) 3 Majorana mass matrices in Eq. (5.15) as follows:

\[
\begin{align*}
[m_\nu] &\approx \frac{[\lambda^n]^\dagger[\lambda^n]}{\Lambda_\nu} \frac{v^2}{2} \cos^2 \beta, \\
[m_\bar{\sigma}] &\approx \frac{[\lambda^n]^\dagger[\lambda^n]}{\Lambda_\nu} f^2 \sin^2 \beta, \\
[\hat{m}] = [\hat{m}]^\dagger &\approx -\frac{[\lambda^n]^\dagger[\lambda^n]}{\Lambda_\nu} \frac{v f}{2\sqrt{2}} \sin 2\beta.
\end{align*}
\]

(5.16)

(5.17)

(5.18)

Note that all the three matrices are proportional to \([\lambda^n]^\dagger[\lambda^n]\) but differ only in vev’s. Now the full 9 \( \times \) 9 mass matrix is given by \( M_D + M_M \). Comparing with \( M_D \), we observe that the elements in \( M_M \) are much smaller than those in \( M_D \), so that the perturbation theory can be applied to approximately compute the nonzero SM neutrino masses. Accordingly \( M_D \) and \( M_M \) act as the unperturbed and perturbed mass matrix in the perturbation theory, respectively. However, since there still remain threefold degenerate zero eigenvalues, we in particular use the perturbation method for the degenerate case.

Let us carry out the perturbation for the degenerate case. First of all, we need to find the projection operator \( P_0 \) onto the three SM neutrino states among the nine neutral lepton states. From Eq. (5.10), \( P_0 \) is given by

\[
P_0 = \begin{pmatrix}
\cos^2 \theta[1] & -\cos \theta \sin \theta[1] & 0 \\
-\cos \theta \sin \theta[1] & \sin^2 \theta[1] & [0] \\
[0] & [0] & [0]
\end{pmatrix}.
\]

(5.19)

As a result, the mass eigenvalues of the SM neutrinos to the first order are just the nonzero roots of the characteristic equation in variable \( \Delta \),

\[
\text{det}[P_0 M_M P_0 - \Delta I] = 0,
\]

(5.20)

where \( I \) is the identity matrix. From Eqs. (5.16), (5.17) and (5.18), the nonzero roots of the characteristic equation are

\[
\Delta_i = m_i' = \frac{2 \Lambda_\nu}{v^2} \frac{v^2 \sin^2 2\beta}{\sin^2 2\beta + \frac{v^2 f}{2\sqrt{2}} \cos^2 \beta} \quad \text{with} \quad i = 1, 2, 3,
\]

(5.21)

which are the SM neutrino masses to the first order. Remind that the neutrino are massless to the zeroth order in the perturbation theory. In addition, the characteristic equation gives the mixing angles for the SM neutrinos to the zeroth order

\[
\hat{\nu}_L = V_L (\cos \theta \nu_L - \sin \theta \bar{n}_L).
\]

(5.22)
From Eq. (5.21), it is straightforward to see that the SM neutrino masses are proportional to $(\lambda_n)^2$, yielding a mass relation between the SM neutrinos and heavy neutral leptons as follows:

$$m_1^\nu : m_2^\nu : m_3^\nu \approx m_{n_1}^2 : m_{n_2}^2 : m_{n_3}^2.$$  \hspace{1cm} (5.23)

This implies that the neutrino mass hierarchy is revealed by the heavy neutral lepton mass hierarchy. Therefore, measuring of the heavy neutral lepton masses at the International Linear Collider (ILC) can tell the neutrino mass hierarchy that is to be disclosed at the neutrino experiments in the near future.

Using Eq. (5.13) and (5.22), we can construct a full $9 \times 9$ unitary leptonic mixing matrix to the zeroth order

$$W_0 = \begin{pmatrix}
V_L^\dagger \cos \theta [1] & -\frac{1}{\sqrt{2}} V_L^\dagger \sin \theta [1] & \frac{1}{\sqrt{2}} V_L^\dagger \sin \theta [1] \\
-\frac{1}{\sqrt{2}} V_L^\dagger \sin \theta [1] & \frac{1}{\sqrt{2}} V_L^\dagger \cos \theta [1] & -\frac{1}{\sqrt{2}} V_L^\dagger \cos \theta [1] \\
0 & 1 & 0 \\
\end{pmatrix}. \hspace{1cm} (5.24)$$

Thus the weak eigenstates of the neutral leptons are given by the linear combinations of the mass eigenstates

$$\begin{pmatrix}
\nu_L \\
n_L \\
n_R \\
\end{pmatrix} = W_0 \begin{pmatrix}
\hat{\nu}_L \\
\hat{n}_L \\
\hat{n}_R \\
\end{pmatrix}. \hspace{1cm} (5.25)$$

From Eq. (5.24), the MNS mixing matrix is given as

$$U_{MNS} = \cos \theta V_L^\dagger, \hspace{1cm} (5.26)$$

which is obviously not unitary due to $\cos \theta \neq 1$. Furthermore, neutrino-heavy neutral lepton mixing angles are $\sin \theta V_L^\dagger$ which is obviously not small. Does it conflict with the experimental data on the neutrino oscillations? It is not. In fact, we can not extract the value of $\cos \theta$ from the neutrino oscillation experiments. Due to the tremendous mass difference between the SM neutrinos and the heavy neutral leptons, such a mixing yields heavy neutral lepton decays rather than neutrino-heavy neutral lepton oscillations.

One can further take into account leptonic weak charged currents so as to extract the value of $\cos \theta$. But the mixing angles in the quark sector exhibit the same patterns because the Yukawa couplings in the quark sector are of the same form as those in the lepton sector. Thus we can not see the presence of $\cos \theta$ in leptonic weak decays associated with the SM neutrinos by comparing leptonic charged weak currents with quark charged weak currents, i.e. in the weak coupling ratio, $(ud \rightarrow e\nu_e) / (\mu \nu_\mu \rightarrow e\nu_e)$, $\cos \theta$ cancels out.

Finally, we can extract the angle $\cos \theta$ (or $\sin \theta$) from study of the heavy neutral lepton decays. Classifying various decay modes of the heavy neutral leptons we also find $V_L^\dagger$. With these results, one can confirm the MNS mixing matrix that is obtained at the neutrino oscillations.
6. Summary and outlook

We have analyzed the lepton sector in the simplest little Higgs model which contains vector-like neutral leptons as heavy partners to the SM neutrinos. We emphasize that a naive construction of the neutral lepton sector at tree level yield a large Dirac mass to the heavy neutral leptons but no masses to the SM neutrinos. We have shown that one loop diagrams involved with two Higgs triplets yield small Majorana masses for the neutral leptons including the SM neutrinos, leading to not only neutrino flavor mixing but also neutrino-heavy neutral lepton mixing.

We have evaluated a $9 \times 9$ mixing matrix of the neutral leptons and the SM neutrino masses by the standards of perturbation theory. The ratio of the Majorana masses associated with the neutral leptons to the Dirac masses is so small that the higher order terms of the perturbation theory can be negligible. We therefore feel that our computations, though a rough approximation, does not change the outcomes that we have drawn. We can acknowledge that the heavy neutral leptons are directly associated with the SM neutrinos in the simplest little Higgs model and thus the study of the heavy neutral leptons catches a glimpse of the SM neutrinos and vice versa. For example, the mass hierarchy of the heavy neutral leptons, once revealed at the ILC, will back up that of SM neutrinos which will be found at the neutrino oscillation experiments in the near future. Inversely, using the currently known data on the neutrinos one can predict or restrict the properties of heavy neutral leptons.

There are many paths for future research on the neutrinos in the simplest little Higgs model. One can further investigate phenomena associated with the neutral lepton mixing, such as the various decay modes of the heavy neutral leptons. One can boldly explore the origin of Majorana mass for the $SU(3)_{weak}$ singlet in a UV completion of the simplest little Higgs model. Finally, with the help of our scenario one can investigate the neutrinos in other little Higgs models which embrace heavy neutral leptons, such as different simple little Higgs model based on a larger simple group.

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References

[1] M. Schmaltz, JHEP 0408 (2004) 056 [arXiv:hep-ph/0407143].
[2] M. Schmaltz and D. Tucker-Smith, arXiv:hep-ph/0502182.
[3] M. Gronau, C. N. Leung and J. L. Rosner, Phys. Rev. D 29 (1984) 2539.
[4] G. Marandella, C. Schappacher and A. Strumia, Phys. Rev. D 72 (2005) 035014 [arXiv:hep-ph/0502096].
[5] J. A. Casas, J. R. Espinosa and I. Hidalgo, JHEP 0503, 038 (2005) [arXiv:hep-ph/0502066].

[6] N. Arkani-Hamed, A. G. Cohen, E. Katz and A. E. Nelson, JHEP 0207, 034 (2002) [arXiv:hep-ph/0206021].

[7] S. M. Bilenky and B. Pontecorvo, Lett. Nuovo Cim. 17 (1976) 569.

[8] W. Kilian, D. Rainwater and J. Reuter, Phys. Rev. D 71 (2005) 015008 [arXiv:hep-ph/0411213].

[9] S. Eidelman et al. [Particle Data Group], Phys. Lett. B 592 (2004) 1.

[10] W. Kilian and J. Reuter, Phys. Rev. D 70 (2004) 015004 [arXiv:hep-ph/0311095].

[11] J. Y. Lee, JHEP 0506 (2005) 060 [arXiv:hep-ph/0501118].

[12] T. Han, H. E. Logan, B. Mukhopadhyaya and R. Srikanth, arXiv:hep-ph/0505260.

[13] F. del Aguila, M. Masip and J. L. Padilla, Phys. Lett. B 627 (2005) 131 [arXiv:hep-ph/0506063].

[14] D. E. Kaplan and M. Schmaltz, JHEP 0310 (2003) 039 [arXiv:hep-ph/0302049].