Partner groups and quantum motion algebras

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Abstract. In this paper an extension of formalism of partner groups is proposed. The partner groups as an algebraic tool for description of laboratory and intrinsic variables are shortly introduced. A natural generalisation of partner groups is recognized as the Quantum Motion Algebra. This algebra is based on a group of motions \( G \) of a given quantum system. An example of the limiting case of of the state space \( L^2(G, d\mu(g)) \) is also considered.

1. Introduction
In classical physics the moments of inertia allow to distinguish among spherical, axial and dihedral symmetry of the Hamiltonian of a rotating body. They give sufficient information for description of rotating body behavior, but they say nearly nothing about shape of this body and, in fact, about intrinsic geometric symmetries of this physical system. In general, there is no rigid body in reality and there is always nonzero (even if extremely small) coupling of rotational motion to other degrees of freedom often dependent on shape of the body. Because of that the shape and other intrinsic degrees of freedom have an influence on symmetry of such system and its dynamics. For example, one tries to consider now theoretically and experimentally the higher geometrical nuclear symmetries than traditional spherical, axial and dihedral \([1]\). This analysis can be done by use of a very popular parametrization in nuclear physics represented by multipole moments of mass distribution, e.g. the moments of inertia correspond to the mass quadrupole moments. As an example, one can analyse a possibility that some nuclei have tetrahedral symmetry. In this case all three moments of inertia are equal and from this point of view such nucleus should
be treated as spherical, what is not true. Such differences are important because experiments in molecular physics show that the shape symmetries of a body have an important influence on energy spectrum and the transition selection rules in dynamics of such objects.

In the following we develop an idea how to apply the concept of partner groups to construct algebraic models describing both: the laboratory and intrinsic properties of a given physical system.

One way of constructing such models is using of the intrinsic frame concept. The idea of intrinsic frame in quantum physics is not a trivial issue. The problem of definition of intrinsic frame in physics was developed, among others, by C. Eckart [2, 3]. In more abstract way by T. Iwai [4] or by G. F. Filippov and others in [5]. To interpret our results we use an intuitive definition of intrinsic frame as a frame which is attached to the body and which moves together with this body. Closer analysis of intrinsic frame properties one can find also in [6, 7, 8].

To introduce an idea of description of a physical system in an intrinsic frame we use, as an illustrative example, a very well known model of a rotating nucleus [9]. In this model the nucleus can be described in the laboratory frame a set of variables, so called laboratory deformation parameters. Usually, the intrinsic frame is taken in such a way that all axes of this frame coincide with the main axes of the ellipsoid of inertia. After transformation to such intrinsic rotating frame, the rotational motion of a nucleus can be described by two kinds of variables: the intrinsic deformation parameters which are scalars in respect to rotations and the Euler angles which describe relative position of the laboratory frame and the intrinsic frame. Higher symmetries, in this kind of models, can be introduced by extension of space of deformation variables. This procedure extends the space of variables and makes analysis much more complicated.

Another way to consider higher symmetries without changing of space of variables, is an extension of the Hamiltonian operator by introducing more complicated terms which simulate other kinds of interactions and symmetries. A good example of such extension, which has a wide application range, is called the generalized rotor [10, 11, 12].

The main idea of the generalized rotor is based on a construction of appropriate irreducible tensors built from the angular momentum operators defined in both: the laboratory $I = \{I_-, I_0, I_+\}$ and the intrinsic $\bar{I} = \{\bar{I}_-, \bar{I}_0, \bar{I}_+\}$ frames. Note that both sets of operators commute:

$$T_{\lambda\mu}(n) = (((I \otimes I_2 \otimes I_3 \otimes \cdots \otimes I)_{n-1} \otimes I)_{\lambda\mu},$$

$$\bar{T}_{\lambda\mu}(n) = (((\bar{I} \otimes \bar{I}_2 \otimes \bar{I}_3 \otimes \cdots \otimes \bar{I})_{n-1} \otimes \bar{I})_{\lambda\mu},$$

where the symbol $(\hat{Q}_{\lambda_1} \otimes \hat{Q}_{\lambda_2})_\lambda$ denotes set of components of irreducible spherical tensor of the rank $\lambda$, which is obtained from the following combination of tensors and the Clebsch-Gordan coefficient of the group $SO(3)$

$$(\hat{Q}_{\lambda_1} \otimes \hat{Q}_{\lambda_2})_{\lambda\mu} = \sum_{\mu_1\mu_2} (\lambda_1\mu_1 \lambda_2\mu_2|\lambda\mu)\hat{Q}_{\lambda_1\mu_1}\hat{Q}_{\lambda_2\mu_2}. \tag{2}$$

The appropriate linear combinations of the tensors (1) are used to build operators invariant in respect to required symmetries. In both cases, the laboratory and the intrinsic operators act in the same space of quantum states, i.e. they work in the space of functions defined on the domain of Euler angles (or other parametrization of rotational group). Moreover, both sets of operators commute $[T_{\lambda\mu}, \bar{T}_{\lambda'\mu'}] = 0$.

Summarizing, there are two interesting approaches: (i) traditional formalism, where the algebraic model is constructed in the laboratory frame, with the laboratory symmetries applied, and later on transformed to the intrinsic frame and (ii) the formalism in which the partner groups are used to construct the model for required symmetry. The second approach is more
intuitive/physical because it clearly distinguishes between laboratory and intrinsic part of the physical system Hamiltonian.

There are possible two structures which allow to construct algebraic models based on the partner groups $G \times \overline{G}$.

First one represents a direct construction in the quantum state space $L^2(G, d\mu(\theta))$, where $d\mu(\theta)$ is left (right) invariant measure on the group $G$.

The second idea is based on the so called Gelfand–Naimark–Segal (GNS) construction which allows to create the quantum space of an algebraic model from more sophisticated theories, e.g., many-body approach. In this case, the main problem is a construction of Quantum Motion Algebra (QMA(G)) and the quantum state space generated by the appropriate physical functional. From mathematical point of view this is a construction of a Hilbert space structure on an algebra with involution by applying the GNS theorem. Note that, the first method is a special limiting case of GNS construction and we consider only this more general case.

Next section contains definitions and the most important mathematical tools which are needed for reading of this article. The third section describes the Quantum Motion Algebras and the corresponding GNS constructions. These results can be also interpreted as generalization of the popular Generator Coordinate Method [13, 14, 15]. Note that the Generator Coordinate Method generates the space of states starting from a family of pure states and the Quantum Motion Algebra approach allows to use the density matrix as a generating state.

2. Partner groups

The concept of partner groups is based on use of the laboratory group $G$ and the intrinsic group $\overline{G}$ as the direct product $G \times \overline{G}$ acting in the common state space.

The idea of intrinsic group is introduced in [16]. From mathematical point of view it is not a new concept because the laboratory and intrinsic groups can be considered as two realizations of the same group structure $G$ by the left and right shift actions. The formal definitions of laboratory, intrinsic and partner groups are as follows:

For each element $g$ of group $G$, one can define the corresponding operator $\overline{g}$ in the group linear space $L_G$ as

$$\overline{g}S = Sg \quad \text{for all } S \in L_G.$$  \hspace{1cm} (3)

The groups formed by the collection of the operators $\overline{g}$ is called the intrinsic group $\overline{G}$ of the group $G$.

In this paper the group $G$ will be referred to as the laboratory group. The groups $G$ and $\overline{G}$ we call the partners groups.

The most important property of partner groups is the following commutation relation

$$[G, \overline{G}] = 0.$$ \hspace{1cm} (4)

This relation implies that intrinsic properties defined by intrinsic group are independent of the properties of the laboratory group. The second important fact about these groups is the anti-isomorphism between these two structures.

$$\phi_G : \overline{G} \rightarrow G, \quad \phi_G(\overline{g}) = g \text{ and } \phi_G(\overline{g_1 \overline{g_2}}) = g_2g_1.$$ \hspace{1cm} (5)

This means that both structure are very similar and most properties of the laboratory group $G$ and the intrinsic group $\overline{G}$ coincide, e.g., representations of both groups are related, they have the same Clebsch–Gordan coefficient etc.

The notion of partner groups is related to the idea of intrinsic frame. This property was exploited in nuclear collective models [17, 18].

The laboratory group $G$ is usually treated as a group of motions (symmetries are a subgroup of this group of motions), characteristic for the physical model under consideration. It usually acts in a configuration space of the physical system.
3. Quantum motion algebras

Because the partner groups are defined by its action on the group algebra it is convenient to use this property directly. In this case a natural method which allows to construct the appropriate quantum state space (Hilbert space) is the method implied by Gelfand, Naimark and Segal theorem, (GNS) [19].

For this purpose we have to construct a group algebra with involution which we call the Quantum Motion Algebra generated by the group of motions $G$ and we denote it by QMA($G$). After determining of a linear functional on this algebra we can use the GNS method to construct the appropriate quantum state space. This functional, appropriate for physical applications, is explicitly defined below.

Let $G$ be a locally compact topological group and $d\mu(g)$ be a left invariant Haar measure on this group (an analogous construction can be done for the right Haar measure). Below, the group $G$ is called the generating group. In this construction the generating group and the corresponding intrinsic group should be considered as groups generating characteristic motions of analysed quantum system. The elements of required algebra with involution $\mathcal{A} = \text{QMA}(G)$ are defined as follows:

$$S = \int_G d\mu(g)F(g)g + \sum_G \tau(g)g, \quad F \in L^1(G,d\mu(g)), \quad \tau \in l^1(G),$$  

where $L^1(G,d\mu(g))$ represents the space of absolutely integrable functions and $l^1(G)$ is the space of absolutely convergent sequences. In most applications to facilitate calculations the functions $u(g)$ and $\tau(g)$ are chosen to belong to smaller spaces $F \in L^1(G,d\mu(g)) \cap L^2(G,d\mu(g))$ and $\tau \in l^1(G) \cap l^2(G)$.

The involution is linear operation defined as

$$S^* = \int_G d\mu(g)F(g)^\ast g^{-1} + \sum_G \tau(g)^\ast g^{-1} = \int_G d\mu(g)\Delta_G(g^{-1})F(g^{-1})^\ast g + \sum_G \tau(g^{-1})^\ast g,$$

where $\Delta_G(g)$ is a modulus function of the group $G$ fixed by the equation

$$\int_G d\mu(g)f(gh^{-1}) = \Delta_G(h)\int_G d\mu(g)f(g).$$

In Eq. (7) we have used the useful inversion formula

$$\int_G d\mu(g)\Delta_G(g^{-1})f(g) = \int_G d\mu(g)f(g^{-1}).$$

The discrete sum over the group ensures, it is important in physical applications, that the state from which the quantum state space is generated also belongs to this state space, though it is not always needed.

In the following we consider only the subalgebra of the full QMA obtained by neglecting all discrete elements, i.e. all discrete functions in (6) are equal to zero, $\tau(g) \equiv 0$. More information about QMA with full form of element $S$ one can find in [20].

The multiplication and addition operations of elements of this algebra are defined in a natural way as multiplication and addition of linear operators, for example

$$S_1 \circ S_2 = \left( \int_G d\mu(g)F_1(g)g \right) \circ \left( \int_G d\mu(g')F_2(g')g' \right) = \int_G d\mu(g)d\mu(g')F_1(g)F_2(g^{-1}g'),$$

$$S_1 + S_2 = \int_G d\mu(g)(F_1(g) + F_2(g))g.$$
where invariance of the measure $\mu(A)$ was used.

The nearly general form of functionals required for constructing a quantum state space can be written as

$$\langle \rho; S \rangle := \text{Tr}(S\rho), \quad \text{where } S \in \text{QMA}(G)$$

and $\rho$ is a quantum density operator from which the group of motion generates the whole state space. In fact, to determine the functional (11) it is sufficient to define the function $\langle \rho; g \rangle := \text{Tr}(g\rho)$, where $g \in G$, and extend it on the whole algebra by linearity.

The inner product in the Hilbert space obtained by the GNS construction is induced by the functional (11):

$$\langle S_1 | S_2 \rangle = \langle \rho; S_1^\# \circ S_2 \rangle = \text{Tr}[\rho S_1^\# \circ S_2].$$

The GNS construction of required Hilbert space is a two steps process. Firstly, one have to collect all vectors from the algebra $A$ which have the norm equal to zero.

$$I_\rho = \{ S \in A : \text{Tr}\left[\rho S^\# \circ S\right] = 0 \}.$$ 

The set of such vectors form a left ideal in this algebraic structure. It allows to construct the quotient space $A/I_\rho$ which is a vector space consisted of vectors $|S\rangle \equiv |S + I_\rho\rangle$ (classes of equivalent elements).

The second step is the completion procedure of the vector space $A/I_\rho$ in respect to the inner product $\langle S_1 | S_2 \rangle = \langle \rho; S_1^\# \circ S_2 \rangle$. The Hilbert space $K$ which comes from this construction is the required state space. This space is obtained from a mixed state, represented by the quantum density operator $\rho$, with the action of all elements of the Quantum Motion Algebra. The QMA(G) algebra determines available excitations in the physical system describe by such algebraic model.

The most practical in use are observables constructed directly from elements of the QMA(G) algebra or generators of partner groups. All other operators require direct action on the density operator $\rho$ what, in many cases is much more complicated.

The eigenproblems of observables can be considered as similar to that in the Generator Coordinate Method (GCM) [13]. As in GCM for determination of eigenfunction one can use the variational principle:

$$\delta \left(\langle \rho S^\# HS \rangle - E\langle \rho S^\# S \rangle \right) = 0.$$ 

Variation in respect to the function $F(g)$ gives the integral equation:

$$\int dg' \text{Tr}[\rho g^{-1}Hg'] F(g') = E \int dg' \text{Tr}[\rho g^{-1}g'] F(g'),$$

$$\int dg' \mathcal{H}(g,g') F(g') = E \int dg' \mathcal{N}(g,g') F(g').$$

This equation is known as Hill–Wheeler equation [13].

Another method is a construction of so called natural states (in case of QMA(G) any basis in $K$) and project the Hamiltonian and other observables onto subspace of states spanned by this basis [13].

The most valuable advantage of the QMA space construction is, as it was in case of generalized rotor, the possibility of description of a laboratory and intrinsic group actions (laboratory and intrinsic variable) simultaneously on this space.

It is important to notice, to have consistent formalism one needs to ensure that the structure of the state space $K$ is conserved by the action of partner groups. The ideal $I_\rho$ is invariant in
respect to the left group action by construction. In the following we consider, by assumption, the simplest case in which the ideal \( I_\rho \) is also right invariant, i.e., \( gI_\rho = I_\rho g = I_\rho \). In this case, the classes/vectors \( |S + I_\rho\rangle \) are compatible with both actions. The more general case will be considered in next paper.

The next problem, which is worth to notice, is an action of generators of partner groups in the space \( K \). For this purpose it is sufficient to consider action of generators on the vectors \( |S\rangle \).

Let us denote by \( h(\theta_n) \) and \( h(\theta_n) \) the one parameter subgroups of the groups \( G \) and \( \hat{G} \), respectively. The corresponding generators we denote by \( \hat{X}_n \) and \( \hat{X}_n \). The required matrix elements read:

\[
\langle S'|\hat{X}_n|S\rangle = Tr[\rho S^n \hat{X}_n S] = i \frac{\partial}{\partial \theta_n} Tr[\rho S^n h(\theta_n) S],
\]

\[
\langle S'|\hat{X}_n|S\rangle = Tr[\rho S^n \hat{X}_n S] = i \frac{\partial}{\partial \theta_n} Tr[\rho S^n S h(\theta_n)].
\]

These matrix elements allow for calculation of matrix representations of an operator which can be expressed in terms of generators \( \hat{X}_n \) and \( \hat{X}_n \). The model Hamiltonian \( H = H(\hat{X}, \hat{X}) \) is a special case of an operator which can be calculated in this way.

### 4. Example, limiting case of the QMA(G) algebra

We mentioned in the introduction that the case of the state space \( \mathcal{K} = L^2(G, d\mu(g)) \) is the limiting case of GNS construction.

To illustrate this limiting procedure we consider the axially symmetric generalized rotor in an axially symmetric external field, e.g., a magnetic field. For this purpose, let us assume the rotation group \( G = SO(3) \) as the group of motion. The partner groups are \( SO(3) \) and \( SO(3) \). The rotor has only rotational degrees of freedom and its Hamiltonian can be written as the operator dependent only on the generators \( \hat{L}_\mu \) and \( \hat{L}_\mu \), \( \mu = 0, \pm 1 \), of the partner groups \( SO(3) \times SO(3) \)

\[
H = H(\hat{L}_z, \hat{L}_z).
\]

The scalar product in the state space generated by the GNS procedure is usually non-local. In this schematic case, where the basic functional \( \langle \rho; g \rangle \) is not obtained from more microscopic theory, we can use a limiting procedure in which we introduce a family of functionals \( \langle \rho; g \rangle \):

\[
\lim_{\epsilon \to 0^+} \langle \rho; g \rangle = \delta_G(g),
\]

where \( \delta_G(g) \) is the delta like distribution on the group \( G \). This functional defines the local scalar product

\[
\langle u|v \rangle = \lim_{\epsilon \to 0^+} \int_{SO(3)} d\mu(g') \int_{SO(3)} d\mu(g) u(g')^* \langle \rho; g^{-1} g \rangle v(g) = \int_{SO(3)} d\mu(g) u(g)^* v(g).
\]

The state space \( \mathcal{K} \) is then identical with the space \( L^2(SO(3), d\mu(g)) \), i.e., the space of square integrable functions on the rotational group. The orthonormal basis \( r^j_{MK}(g) \) in this space can be identified with the Wigner functions \( D^j_{MK}(g) \) on this group

\[
r^j_{MK}(g) = \sqrt{2J + 1} D^j_{MK}(g).
\]

The vectors (22) are eigenvectors of the Hamiltonian (19):

\[
H(\hat{L}_z, \hat{L}_z, \hat{L}_z)r^j_{MK}(g) = H(J^2, M, K)r^j_{MK}(g).
\]
This model can be straightforwardly extended to any axially symmetric rotor in respect to the laboratory frame (external field brakes the rotational symmetry) but with arbitrary intrinsic symmetry:

\[ H = H(\hat{L}_x, \hat{L}_y, \hat{L}_z) \, . \]  

This Hamiltonian has to be diagonalise by means of symbolic computer algebra or numerical methods.

5. Summary
The full Quantum Motion Algebra which elements are defined as in (6) is, in fact, extension of an idea of partner groups acting on a given configuration space of a physical system. This is because it contains both kinds of elements: \( gS \) and \( \bar{g}S = Sg \) which define the left and right shift actions of the group G. The generating functional, determined by more microscopic physical models, on QMA(G) leads to a model state space which is a projection of the full physical state space (for example, projection from the many-body space of a physical problem under consideration). It is very similar to construction of the state space in the Generator Coordinate Method. QMA(G) formalism can be considered as a generalization of this popular method on mixed quantum states. The formalism allows for using of operators which do not belong to this algebra and even cannot be derived from the QMA(G) algebra. However, it seems, this make the formalism more difficult in practice. This problem requires further investigations.

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6. References
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