Extended set of Majorana spinors, a new dispersion relation, and a preferred frame

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Abstract: We aspire to fulfill Majorana’s original goal of bringing full symmetry between the charged and fundamentally neutral particles. We present a description of fundamentally neutral particles without any reliance on Dirac spinors. We show that the extended set Majorana spinors (a) describe a Wigner class of fermions in which the charge conjugation and the parity operators commute, rather than anticommute, and (b) support two type of dispersion relations, the usual $E = \pm \sqrt{p^2 + m^2}$, and a new $E = -2m \pm \sqrt{p^2 + m^2}$. The latter may be preferred over the former on minimization of energy requirement and offers a natural source for observed matter-antimatter asymmetry in the universe. The ensuing physical interpretation requires existence of a preferred frame which may be identified with the cosmic neutrino background.

Keywords: Space-Time Symmetries, Neutrino Physics.

Written to celebrate birthday of Dr. I. S. Ahluwalia-Khalilova

*CONACyT (Mexico) is acknowledged for funding this research through Project 32067-E. IUCAA (India) is acknowledged for its hospitality, and a senior visiting professorship which supported part of this work.
1. Introduction

Early in December 2001, the long sought-after signal from experiments on neutrinoless double beta decay was finally reported by the Heidelberg-Moscow (HM) collaboration [1, 2]. In its most natural explanation the HM events suggest neutrinos to be fundamentally neutral particles in the sense of Majorana [3]. However, fundamental neutrality of particles is not a neutrino specific property. For example, in supersymmetric theories a host of similarly neutral particles are required to exist.

The discovery of fundamentally neutral particle of spin one half should be considered a major step towards a possible discovery of supersymmetry as I noted at Beyond the Desert 2002, see [3, 4]. This assertion, in part, arises from our ability to construct Table 1. In it the diagonal Wigner blocks are the ordinary fermionic-matter and bosonic-gauge fields. The top off-diagonal block is one of the recent key results. Supersymmetric fermionic gauge particles live in that block. The bottom off-diagonal block is populated by bosonic matter fields and awaits experimental confirmation. It was constructed in a 1993 paper

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3Parenthetically, we note that initial concerns of C. E. Aalseth et al. seem to have been fully attended. Of the nine original questions raised by C. E. Aalseth et al. six have been withdrawn by the authors themselves and the remaining have now been attended in detail by H. V. Klapdor-Kleingrothaus et al. in Ref. [4]. The C. E. Aalseth et al. “Comment,” after extensive revisions, is now published, see Ref. [4].
Table 1: The diagonal Wigner blocks are the ordinary fermionic matter and bosonic gauge fields, while the off-diagonal blocks refer to new structure in spacetime.

of mine with Johnson and Goldman cited here as Ref. [7]. The gauge aspect of the top off-diagonal block tentatively refers either to the gauginos, or for neutrinos, should they be confirmed to be Majorana, it should be interpreted as an internal fermionic line as it appears in neutrinoless double beta decay. The $C$ and $P$ operators belong to the indicated representation space for each of the Wigner blocks. Possibility, but without explicit construction (for which we take credit), of such blocks is due to Wigner, and his colleagues [8].

The existing description of Majorana particles has a pivotal reliance on Dirac spinors [3] – a circumstance I, and many other scholars who concern themselves with such questions, consider unsatisfactory. The competing reformulation of McLennan and Case [9, 10], which avoids its dependence on Dirac spinors, is on the other hand incomplete. It only carries two degrees freedom to span a four dimensional $(1/2, 0) \oplus (0, 1/2)$ representation space and cannot address a whole range of relevant questions.

For sometime, therefore, we have been attempting to understand the type of particles Majorana envisaged [11, 12]. This paper is the first exposition in the series which we think places Majorana’s vision fully at par with that of Dirac on charged particles.

2. Review of Dirac Construct

To better appreciate the construct that we present let us first briefly review the structure of Dirac’s construct. This will also help us set up the notation. A Dirac spinor, in Weyl representation, is

$$\psi(p) = \begin{pmatrix} \phi_R(p) \\ \phi_L(p) \end{pmatrix}$$

(2.1)

where the massive Weyl spinors $\phi_R(p)$ transforms as $(1/2, 0)$ representation-space objects, and massive Weyl spinors $\phi_L(p)$ transforms as $(0, 1/2)$ representation-space objects. Our notation is very close to that of Ryder [13]. For the ease of reference we shall mark needed observations with $O_n$, with $n = 1, 2, 3, \ldots$.

$O_1$: The first thing the reader should explicitly note is that the Dirac spinors are constructed by taking both $\phi_R(p)$ and $\phi_L(p)$ to be in same helicity. Contrary is required for Majorana spinors (see below).

Following the usual particle-antiparticle nomenclature, the following holds in addition:
In the rest frame, characterized by $\mathbf{p} = 0$, for particles there is no relative phase between $\phi_R(0)$ and $\phi_L(0)$, i.e., $\phi_R(0) = \phi_L(0)$. Whereas, for antiparticles $\phi_R(0)$ and $\phi_L(0)$ carry opposite phase i.e., $\phi_R(0) = -\phi_L(0)$.

This last property has only been noted explicitly in the last decade. The momentum-space wave equation satisfied by the spinors thus constructed follows uniquely from [13, 16],

$$\phi_R(0) = \pm \phi_L(0),$$

and

$$\phi_R(0) = \kappa^{(\frac{1}{2}, 0)} \phi_R(0), \quad \phi_L(0) = \kappa^{(0, \frac{1}{2})} \phi_L(0).$$

Here,

$$\kappa^{(\frac{1}{2}, 0)} = \exp \left( + \frac{\sigma}{2} \cdot \varphi \right) = \sqrt{\frac{E + m}{2m}} \left( I + \frac{\sigma \cdot \mathbf{p}}{E + m} \right),$$

$$\kappa^{(0, \frac{1}{2})} = \exp \left( - \frac{\sigma}{2} \cdot \varphi \right) = \sqrt{\frac{E + m}{2m}} \left( I - \frac{\sigma \cdot \mathbf{p}}{E + m} \right),$$

with the boost parameter defined as:

$$\cosh(\varphi) = \frac{E}{m}, \quad \sinh(\varphi) = \frac{\lvert \mathbf{p} \rvert}{m}, \quad \hat{\varphi} = \frac{\mathbf{p}}{\lvert \mathbf{p} \rvert}.$$  

These wave equations are,

$$(\gamma^\mu p_\mu \mp m I) \psi(\mathbf{p}) = 0.$$  

Here, $I$ are $n \times n$ identity matrices. Their dimensionality being apparent from the context in which they appear. The $\gamma^\mu$ have their standard Weyl-representation form:

$$\gamma^0 = \begin{pmatrix} O & I \\ I & O \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} O & -\sigma^i \\ \sigma^i & O \end{pmatrix}.$$  

For consistency of notation, $O$ here represents a $n \times n$ null matrix (in the above equation, $n = 2$). Letting, $p_\mu = i \hbar \partial_\mu$, and, $\psi(x) = \exp \left( \mp \frac{i}{\hbar} n_\mu x^\mu \right) \psi(\mathbf{p})$, with upper sign for particles, and lower sign for antiparticles, one obtains the configuration space Dirac equation:

$$(i \gamma^\mu \partial_\mu - m I) \psi(x) = 0.$$  

See, my book review of Ryder’s text on quantum field theory in Ref. [14], and Gaioli and Garcia-Alvarez’s American Journal of Physics disposition in Ref. [15]. The genesis of this observation in fact begins with Ref. [7] formally, and a conversation between myself and Christoph Burgard at Texas A&M University when we both were there as students.

Note [7, 15, 14], the necessity of minus sign in Eq. (2.2). This sign in “group theoretical derivations” of Dirac equation has been consistently missed (see, e.g., [13] p.44 in 1987 edition, and pp. 168-169 of Ref. [17]. The minus is necessary to have antiparticles in the momentum space. The antiparticles, in the spacetime description with minus sign dropped, appear only after “two mistakes which cancel one mistake of omitting the indicated sign.”

So, e.g., in Eqs. (2.4) and (2.5), the $I$ stand for $2 \times 2$ identity matrices; while in Eq. (2.7) $I$ is a $4 \times 4$ identity matrix.
It should be noted that in Eq. (2.6), $E > 0$. Whereas, $\text{Det} (\gamma^\mu p_\mu \mp m I) = 0$ yields $E = \pm \sqrt{m^2 + p^2}$. At this stage, going back to original paper of Dirac, Dirac did not consider this as an internal inconsistency and did not discard the $E < 0$ through a constraint. He could have done so in a covariant manner and hardly any one would raised an objection. The lesson is simple: one should not impose mathematical constraints to satisfy one’s physical intuition. Had Dirac taken the path of physical intuition, rather than opting for a mathematically imposed inevitability, a local $U(1)$ gauge theory based on such a (covariant) theory would have been mathematically pathological, and physically it would have missed Lamb shift, not to say antiparticles. We have tried to teach the same lesson in a different context in Ref. [18, 16] to the exasperation of many distinguished people. The same lesson shall be seen to be important for Majorana particles now: a four dimensional $(1/2, 0) \oplus (0, 1/2)$ representation space — whether it be Dirac, or Majorana — requires four degrees of freedom (i.e., four independent spinors).

The derivation of Dirac equation as outlined here carries a quantum mechanical aspect in allowing for the fact the the two Weyl spaces may carry a relative phase, in the sense made explicit above; and concurrently a relativistic element via the Lorentz transformation properties of the Weyl spinors. In turn the very existence of the latter depends on existence of left and right spacetime $SU(2)$s:

$$SU(2)_R: \quad A = \frac{1}{2} (J + iK), \quad SU(2)_L: \quad B = \frac{1}{2} (J - iK).$$

(2.9)

The $J$ and $K$ represent the generators of rotations and boosts for the any of the relevant finite dimensional representation space which may be under consideration. From the womb of this structure emerges a new symmetry, i.e., that of charge conjugation. The operator associated with this symmetry shall be now written in a slightly unfamiliar form (so as to fully exploit it for understanding Majorana particles):

$$C = \begin{pmatrix} 0 & i\Theta \\ -i\Theta & 0 \end{pmatrix} K.$$

(2.10)

Here, operator $K$ complex conjugates any Weyl spinor that appears on its right, and $\Theta$ is the Wigner’s spin-1/2 time reversal operator

$$\Theta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (2.11)$$

5The physical intuition may ask for $E > 0$, or a definite spin for particles, etc. Such constraints may have limited validity in a classical framework. But in a quantum framework interactions shall, in general, induce transitions between classically allowed and classically forbidden sectors.

6That these four degrees of freedom may be complex, or real (in a given realization), is a different matter. We shall not contest if one wishes to say that in “Majorana representation (realization)” Dirac spinors has eight real degrees of freedom, and Majorana has four.

7For an arbitrary spin it is defined by the property $\Theta J \Theta^{-1} = -J^\ast$. We refrain from identifying $\Theta$ with “$-i\sigma_2$,” as is done implicitly in all considerations on the subject – see, e.g., Ref. [19] – because such an identification does not exist for higher-spin $(j, 0) \oplus (0, j)$ representation spaces. The existence of Wigner time reversal operator for all $j$, allows, for fermionic $j$’s, the introduction of $(j, 0) \oplus (0, j)$ neutral particle spinors.
It is readily seen that the standard form, \( C = -\gamma^2 K \), is recovered. It is important to note, in the context of the derivation of a wave equation for the extended set of Majorana spinors (presented below), that \( (1/2, 0) \oplus (0, 1/2) \) boost operator, \( \kappa_{(1/2, 0)} \oplus \kappa_{(0, 1/2)} \), and the \( (1/2, 0) \oplus (0, 1/2) \)-space charge conjugation operator, \( C \), commute.

So particles and antiparticles are offsprings of a fine interplay between the quantum realm and the realm of spacetime symmetries. Here, we have made it transparent. The operation of \( C \) takes, up to a spinor-dependent global phase, the particle spinors into antiparticle spinors and vice versa – see, Eq. (51) of Appendix G. Keeping with our pedagogic style, we note: The Dirac spinors are thus not eigenspinors of the Charge conjugation operator. That honor belongs to extended set of Majorana spinors which we introduce below.

Since our task is a logical one, rather than a historical one, we now ask what are the eigenspinors of the \( C \) operator? The answer which we shall obtain is: a set of four spinors; two of which are identical to massive McLennan-Case spinors (and are called Majorana spinors in literature), and other two which are new. The set of four spinors representing fundamentally neutral spinors shall be called extended set of Majorana spinors. Towards the task of obtaining the extended set of Majorana spinors we shall follow a path which sheds maximal light on the various relative phases involved. At the same time our procedure takes due note of the the transformation properties of the right and left transforming components of these spinors and makes them manifest while at the same time emphasizing their helicity orientations. One may look at our defined task as to first build counterpart of Dirac’s \( \psi(p) \) and then to fully outline the properties of this counterpart in a way that makes it stand in its own right.

3. Extended set of Majorana spinor: Eigenspinors of charge conjugation operator

In the spirit just outlined our task begins with the observation that

\[
\left( \kappa_{(0,1/2)} \right)^{-1} = \left( \kappa_{(1/2,0)} \right)^\dagger, \quad \left( \kappa_{(1/2,0)} \right)^{-1} = \left( \kappa_{(0,1/2)} \right)^\dagger. \tag{3.1}
\]

Further, \( \Theta \), the Wigner’s spin-1/2 time reversal operator, has the property

\[
\Theta [\sigma/2] \Theta^{-1} = - [\sigma/2]^*, \tag{3.2}
\]

When combined, these observations imply that \( \Theta \): (a) If \( \phi_L(p) \) transforms as a left handed spinor, then \( (\zeta_\lambda \Theta) \phi_L^*(p) \) transforms as a right handed spinor – where, \( \zeta_\lambda \) is an unspecified phase; (b) If \( \phi_R(p) \) transforms as a right handed spinor, then \( (\zeta_\rho \Theta)^* \phi_R^*(p) \) transforms as a left handed spinor – where, \( \zeta_\rho \) is an unspecified phase. As a consequence, the following spinors belong to the \( (1/2, 0) \oplus (0, 1/2) \) representation space :

\[
\lambda(p) = \begin{pmatrix} (\zeta_\lambda \Theta) \phi_L^*(p) \\ \phi_L(p) \end{pmatrix}, \quad \rho(p) = \begin{pmatrix} \phi_R(p) \\ (\zeta_\rho \Theta)^* \phi_R^*(p) \end{pmatrix}. \tag{3.3}
\]
Demanding $\lambda(p)$ and $\rho(p)$ to be self/anti-self conjugate under $C$,

$$C\lambda(p) = \pm \lambda(p), \quad C\rho(p) = \pm \rho(p), \quad (3.4)$$

restricts the phases, $\zeta_\lambda$ and $\zeta_\rho$, to two values:

$$\zeta_\lambda = \pm i, \quad \zeta_\rho = \pm i. \quad (3.5)$$

The plus sign in the above equation yields self conjugate, $\lambda^S(p)$ and $\rho^S(p)$ spinors; while the minus sign results in the anti-self conjugate spinors, $\lambda^A(p)$ and $\rho^A(p)$.

In the rest frame, it is clear that the phase relationship between the right handed and left handed Weyl components of the fundamentally neutral spinors is profoundly different from their Dirac counterparts. Furthermore, as shall be explicitly shown immediately, the two Weyl components of a fundamentally neutral spinor must carry opposite helicities. That is, Majorana spinors cannot be eigenspinors of the helicity operator. This, we believe, is important for making a physical picture of neutrinoless double beta decay in which a neutrino is emitted as a particle at one vertex and absorbed as antiparticle at another vertex. In other words, the fact that fundamentally neutral spinors are not single helicity objects, but invite an interpretation of dual helicity spinors, makes processes such as neutrinoless double beta decay possible.

To obtain explicit expressions for $\lambda(p)$, we first write down the rest spinors. These are:

$$\lambda^S(0) = \begin{pmatrix} +i \Theta \phi^*_L(0) \\ \phi_L(0) \end{pmatrix}, \quad \lambda^A(0) = \begin{pmatrix} -i \Theta \phi^*_R(0) \\ \phi_L(0) \end{pmatrix}. \quad (3.7)$$

Next, we choose the $\phi_L(0)$ to be helicity eigenstates,

$$\sigma \cdot \hat{p} \phi^\pm_L(0) = \pm \phi^\pm_L(0), \quad (3.8)$$

and concurrently note that$^9$

$$\sigma \cdot \hat{p} \Theta \left[ \phi^\pm_L(0) \right]^* = \mp \Theta \left[ \phi^\pm_L(0) \right]^*. \quad (3.9)$$

That is, $\Theta \left[ \phi^\pm_L(0) \right]^*$ has opposite helicity of $\phi^\pm_L(0)$. Since $\sigma \cdot \hat{p}$ commutes with the boost operator $\kappa(1/2, 0)$ the above result applies for all momenta. In conjunction with the definition of the neutral spinors we are thus lead to the result that neutral spinors are not single

$^8$Since $i\Theta = \sigma_2$, we may write:

$$\lambda(p) = \begin{pmatrix} \pm \sigma_2 \phi^*_L(p) \\ \phi_L(p) \end{pmatrix}, \quad \rho(p) = \begin{pmatrix} \phi_R(p) \\ \mp \sigma_2 \phi^*_R(p) \end{pmatrix}. \quad (3.6)$$

where the upper sign is for self conjugate spinors, and the lower sign yields the antself conjugate spinors. The $\lambda^S(p)$ thus turn out to be identical to the standard textbook, or McLennan-Case, Majorana spinors. $\lambda^A(p)$, are new and mathematically orthogonal to $\lambda^S(p)$. The use, and linear dependence of the $\rho(p)$ on $\lambda(p)$, shall be discussed as we proceed further.

$^9$See Appendix A for derivation of Eq. (3.9). The explicit forms of $\phi^\pm_L(0)$ are given in Appendix B.
helicity objects. Instead, they invite an interpretation of dual helicity spinors. In the process we are led to four rest spinors. Two of which are self-conjugate,

\[ \lambda^S_{\{+,\}}(0) = \left( +i \Theta \left[ \phi^+_L(0) \right]^* \right), \quad \lambda^S_{\{+,-\}}(0) = \left( +i \Theta \left[ \phi^-_L(0) \right]^* \right), \]

and the other two, which are anti-self conjugate,

\[ \lambda^A_{\{-,\}}(0) = \left( -i \Theta \left[ \phi^+_L(0) \right]^* \right), \quad \lambda^A_{\{-,+\}}(0) = \left( -i \Theta \left[ \phi^-_L(0) \right]^* \right). \]

The first helicity entry refers to the \((1/2, 0)\) transforming component of the \(\lambda(p)\), while the second entry encodes the helicity of the \((0, 1/2)\) component. The boosted spinors are now obtained via the operation:

\[ \lambda_{\{h,-h\}}(p) = \left( \kappa(\frac{1}{2}, 0) \bigoplus \kappa(0, \frac{1}{2}) \right) \lambda_{\{h,-h\}}(0). \]

In the boosts, we replace \(\sigma \cdot p\) by \(\sigma \cdot \hat{p} |p|\), and then exploit Eq. (3.9). After simplification, Eq. (3.12) yields:

\[ \lambda^S_{\{-,\}}(p) = \sqrt{E + m^2} \left( 1 - \frac{|p|}{E + m} \right) \lambda^S_{\{+,\}}(0), \]

which, in the massless limit, identically vanishes; while

\[ \lambda^S_{\{+,\}}(p) = \sqrt{E + m^2} \left( 1 + \frac{|p|}{E + m} \right) \lambda^S_{\{-,\}}(0), \]

does not. We hasten to warn the reader that one should not be tempted to read the two different prefactors to \(\lambda^S(0)\) in the above expressions as the boost operator that appears in Eq. (3.12). For one thing, there is only one (not two) boost operator(s) in the \((1/2, 0) \oplus (0, 1/2)\) representation space. The simplification that appears here is due to a fine interplay between Eq. (3.9), the boost operator, and the structure of the \(\lambda^S(0)\). Similarly, the anti-self conjugate set of the boosted spinors reads:

\[ \lambda^A_{\{-,\}}(p) = \sqrt{E + m^2} \left( 1 - \frac{|p|}{E + m} \right) \lambda^A_{\{+,\}}(0), \]

\[ \lambda^A_{\{+,\}}(p) = \sqrt{E + m^2} \left( 1 + \frac{|p|}{E + m} \right) \lambda^A_{\{-,\}}(0). \]

In the massless limit, the first of these spinors identically vanishes, while the second does not.

4. Majorana Dual

For any \((1/2, 0) \oplus (0, 1/2)\) spinor \(\xi(p)\), whether it be Majorana or Dirac, the Dirac dual spinor \(\bar{\xi}(p)\) is defined as:

\[ \bar{\xi}(p) = \xi^\dagger(p) \gamma^0 \]
With the Dirac dual, the Majorana spinors have an imaginary definite bi-orthogonal norm (see Appendix C). Recalling the implicitly-contained lessons in quantization of the Dirac field we wish to introduce a dual which is appropriate for the extended set of Majorana spinors. The new dual must have the property that it yields an invariant real definite norm. In addition, the new dual must secure a positive definite norm for two of the four spinors contained in the extended set of Majorana spinors, and negative definite norm for the remaining two. A unique, up to a relative sign, definition of such a dual, which we call *Majorana dual*, for each of the spinors is:

\[ \lambda^S(p) : \bar{\lambda}^S(p) = + [\rho^A(p)]^\dagger \gamma^0 \]  
(4.2)

\[ \lambda^A(p) : \bar{\lambda}^A(p) = - [\rho^S(p)]^\dagger \gamma^0, \]  
(4.3)

where the \( \rho(p) \) are given in Appendix D. With Majorana dual so defined, we then have,

\[ \bar{\lambda}_\alpha^S(p) \lambda^S_{\alpha'}(p) = + 2m \delta_{\alpha \alpha'}, \]  
(4.4)

\[ \bar{\lambda}_\alpha^A(p) \lambda^A_{\alpha'}(p) = - 2m \delta_{\alpha \alpha'}. \]  
(4.5)

The completeness relation,

\[ \frac{1}{2m} \sum_\alpha \left[ \lambda^S_{\alpha}(p) \bar{\lambda}^S_{\alpha}(p) - \lambda^A_{\alpha}(p) \bar{\lambda}^A_{\alpha}(p) \right] = \mathbb{I}, \]  
(4.6)

clearly shows the non-trivial mathematical necessity of the anti-self conjugate spinors (cf. observation \( O_3 \)). Equations (4.4), (4.5), and (4.6) have their direct counterpart in Dirac’s construct – see, Eqs. (24), (25), and (26) in Appendix F.

5. A Master wave equation for spinors

Appendix F shows that extended set of Majorana spinors do not satisfy Dirac equation. Therefore, to study time evolution of neutral particle spinors we need appropriate wave equation. This we do in the following manner. First, we obtain the momentum-space wave equation satisfied by the \( \lambda(p) \) spinors. The time evolution then follows by careful, but simple, implementation of the “\( p_\mu \to i\partial_\mu \)” prescription.

We seek a momentum-space wave equation for a general \((1/2, 0) \oplus (0, 1/2)\) spinor\(^{10}\)

\[ \xi(p) = \begin{pmatrix} \chi^{(\frac{1}{2}, 0)}(p) \\ \chi^{(0, \frac{1}{2})}(p) \end{pmatrix}. \]  
(5.1)

In particle’s rest frame, where, \( p = 0 \), by definition,

\[ \chi^{(\frac{1}{2}, 0)}(0) = A \chi^{(0, \frac{1}{2})}(0). \]  
(5.2)

\(^{10}\)Since the method we use, and the results we obtain, appear somewhat unusual we exercise extra care in presenting our derivation. We, therefore, present a unified method which applies not only to the extended set of Majorana spinors but it applies equally well to other cases (such as the Dirac formalism). The method is a generalization of the textbook procedure \[5\] with corrections noted in Refs. \[6, 13, 14, 15\].

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Here, the $2 \times 2$ matrix $A$ encodes $C$, $P$, and $T$ properties of the spinor and is left unspecified at the moment except that we require it to be invertible. We envisage the most general form of $A$ to be a unitary matrix with determinant $\pm 1$:

$$A_{\pm} = \begin{pmatrix} a e^{i\phi_a} & \sqrt{\pm 1 - a^2} e^{i\phi_b} \\ -\sqrt{\pm 1 - a^2} e^{-i\phi_b} & a e^{-i\phi_a} \end{pmatrix},$$  \tag{5.3}$$

with $a$, $\phi_a$, and $\phi_b$ real. The plus sign yields Determinant of $A$ to be $+1$, while the minus sign yields it to be $-1$. Once $\chi^{(\frac{1}{2},0)}(0)$ and $\chi^{(0,\frac{1}{2})}(0)$ are specified the $\chi^{(\frac{1}{2},0)}(p)$ and $\chi^{(0,\frac{1}{2})}(p)$ follow from,

$$\chi^{(\frac{1}{2},0)}(p) = \kappa^{(\frac{1}{2},0)} \chi^{(\frac{1}{2},0)}(0), \tag{5.4}$$

$$\chi^{(0,\frac{1}{2})}(p) = \kappa^{(0,\frac{1}{2})} \chi^{(0,\frac{1}{2})}(0). \tag{5.5}$$

Below, we shall need their inverted forms also. These we write as follows:

$$\chi^{(\frac{1}{2},0)}(0) = \left(\kappa^{(\frac{1}{2},0)}\right)^{-1} \chi^{(\frac{1}{2},0)}(p), \tag{5.6}$$

$$\chi^{(0,\frac{1}{2})}(0) = \left(\kappa^{(0,\frac{1}{2})}\right)^{-1} \chi^{(0,\frac{1}{2})}(p). \tag{5.7}$$

Equation (5.2) implies,

$$\chi^{(0,\frac{1}{2})}(0) = A^{-1} \chi^{(\frac{1}{2},0)}(0)$$  \tag{5.8}$$

which on immediate use of (5.6) yields,

$$\chi^{(0,\frac{1}{2})}(0) = A^{-1} \left(\kappa^{(\frac{1}{2},0)}\right)^{-1} \chi^{(\frac{1}{2},0)}(p).$$  \tag{5.9}$$

However, since

$$\left(\kappa^{(\frac{1}{2},0)}\right)^{-1} = \kappa^{(0,\frac{1}{2})} \tag{5.10}$$

we have:

$$\chi^{(0,\frac{1}{2})}(0) = A^{-1} \kappa^{(0,\frac{1}{2})} \chi^{(\frac{1}{2},0)}(p). \tag{5.11}$$

Similarly,

$$\chi^{(\frac{1}{2},0)}(0) = A \kappa^{(\frac{1}{2},0)} \chi^{(0,\frac{1}{2})}(p). \tag{5.12}$$

Substituting for $\chi^{(\frac{1}{2},0)}(0)$ from Eq. (5.12) in Eq. (5.4) and re-arranging gives:

$$- \chi^{(\frac{1}{2},0)}(p) + \kappa^{(\frac{1}{2},0)} A \kappa^{(\frac{1}{2},0)} \chi^{(0,\frac{1}{2})}(p) = 0; \tag{5.13}$$

while similar use of Eq. (5.11) in Eq. (5.5) results in:

$$\kappa^{(0,\frac{1}{2})} A^{-1} \kappa^{(0,\frac{1}{2})} \chi^{(\frac{1}{2},0)}(p) - \chi^{(0,\frac{1}{2})}(p) = 0. \tag{5.14}$$
The last two equations when combined into a matrix form result in the momentum-space master equation for \( \xi(p) \),

\[
\begin{pmatrix}
-\mathbb{I} & \kappa(\frac{1}{2},0)A\kappa(\frac{1}{2},0) \\
\kappa(0,\frac{1}{2})A^{-1}\kappa(0,\frac{1}{2}) & -\mathbb{I}
\end{pmatrix} \xi(p) = 0.
\]

(5.15)

Thus, the momentum-space equation for \( \xi(p) \) is entirely determined by the boosts \( \kappa(\frac{1}{2},0) \) and \( \kappa(0,\frac{1}{2}) \) and the CPT-property encoding matrix \( A \). Inserting \( A \) from Eq. (5.3) into (5.15), we evaluate the determinant of the operator

\[
\mathcal{O} = \begin{pmatrix}
-\mathbb{I} & \kappa(\frac{1}{2},0)A\kappa(\frac{1}{2},0) \\
\kappa(0,\frac{1}{2})A^{-1}\kappa(0,\frac{1}{2}) & -\mathbb{I}
\end{pmatrix},
\]

(5.16)

and find it to be:

\[
\text{Det}[\mathcal{O}] = \frac{(m^2 + p^2 - (2m + E)^2)^2 (m^2 + p^2 - E^2)^2}{(2m(E + m))^4},
\]

(5.17)

where \( p = |p| \). The wave operator, \( \mathcal{O} \), supports two type of spinors. Those associated with the usual dispersion relation,

\[
E^2 = m^2 + p^2, \quad \text{multiplicity} = 4
\]

(5.18)

and those associated with:

\[
E = \begin{cases}
-2m - \sqrt{m^2 + p^2}, & \text{multiplicity} = 2 \\
-2m + \sqrt{m^2 + p^2}, & \text{multiplicity} = 2
\end{cases}
\]

(5.19)

The origin of the new dispersion relation must certainly lie, or at least we suspect it to be so, in the new \( U(2) \) phases matrix. When \( A_\pm \) equals \( \pm \mathbb{I} \), i.e. when we confine to spin-1/2 charged particles, only the usual dispersion relation gets invoked. For the extended set of Majorana spinors the situation is more subtle as we shall soon discuss.

There are also data-dictated reasons which suggest a possible violation of Lorentz symmetry, and appearance of new dispersion relations. See, e.g., the work of Mavromatos and Lehnert [20, 21]. A recent review on theories with varying speed of light is by Magueijo [22]. Though some interpretational question still remain to be resolved [24, 25, 26], the work of Amelino-Camelia on the subject [23] when extended to Majorana particles shall also carry a preferred frame and result in a similar deformation as noted above. We also draw our reader’s attention to systematic work of Mattingly, Jacobson, and Liberati on violation of Lorentz symmetry and dispersion relations [27]. The basic thread running through all these works is that in one way or to another theory and observations suggest a modification of \( E = \pm \sqrt{m^2 + p^2} \). In our work such a modification appears without invoking nonlinear realizations of the spacetime symmetries, and without introducing any additional assumptions.
5.1 Dirac equation

To give confidence to our reader in the physical content of the Master equation we now apply it to the charged particle spinors of Dirac formalism. Once we do that we shall return to the task of constructing momentum-space wave equation for the $\lambda(p)$.

The $A$ can be read off from the Dirac rest spinors. However, we remind the reader, that the writing down of the Dirac rest spinors, as shown by Weinberg and also by our independent studies, follows from the following two requirements: (a) The conservation of parity \cite{28, 14, 16}; and that (b) in a quantum field theoretic framework, the Dirac field describe fermions \cite{28}. These physical requirements determine $A$ to be:

$$A = \begin{cases} + \mathbb{I}, & \text{for } u(p) \text{ spinors} \\ - \mathbb{I}, & \text{for } v(p) \text{ spinors} \end{cases}, \quad (5.20)$$

and correspond to $A_+$ with $a = 1$, $\phi_a = 0$, and $a = 1$, $\phi_a = \pi$, respectively, with $\phi_b$ remaining arbitrary. The subscript on $A$ simply represents that its determinant is plus unity.

Using this information in the Master equation (5.15), along with the explicit expressions for $\kappa(\frac{1}{2},0)$ and $\kappa(0,\frac{1}{2})$, yields:

$$\left( \begin{array}{cc} -\mathbb{I} & \exp(\sigma \cdot \varphi) \\ \exp(-\sigma \cdot \varphi) & -\mathbb{I} \end{array} \right) u(p) = 0, \quad (5.21)$$

$$\left( \begin{array}{cc} \mathbb{I} & \exp(\sigma \cdot \varphi) \\ \exp(-\sigma \cdot \varphi) & \mathbb{I} \end{array} \right) v(p) = 0. \quad (5.22)$$

Exploiting the fact that $\sigma^2 = \mathbb{I}$, and using the definition of the boost parameter $\varphi$ given in Eqs. (2.6), the exponentials that appear in the above equation take the form,

$$\exp(\pm \sigma \cdot \varphi) = \frac{(E \mathbb{I} \pm \sigma \cdot p)}{m}. \quad (5.23)$$

Using these expansions in Eqs. (5.21) and (5.22), multiplying both sides of the resulting equations by $m$, using $p_\mu = (E, -p)$, and introducing $\gamma^\mu$ as in Eqs. (2.8), gives Eqs. (5.21) and (5.22) the form

$$(p_\mu \gamma^\mu - m \mathbb{I}) u(p) = 0, \quad (5.24)$$

$$(p_\mu \gamma^\mu + m \mathbb{I}) v(p) = 0. \quad (5.25)$$

These are the well-known momentum space wave equations for the charged particle spinors (i.e. the Dirac equations). The linearity of these equations in $p_\mu$ is due to form of $A$, and the property of Pauli matrices, $\sigma^2 = \mathbb{I}$ – see, Eq. (5.23). The Det $[p_\mu \gamma^\mu - m \mathbb{I}] = 0$ yields dispersion relation (5.18) only.

5.2 Wave equation for the extended set of Majorana spinors

The requirement that the $\lambda(p)$ be eigenstates of the charge conjugation operator completely determines $A$ for the neutral particle spinors to be:\footnote{This is slightly non trivial but can be extracted from explicit forms of $\lambda(0)$ given in Eqs. (3.10) and (3.11) and by making use of the information given in Appendix B.}

$$A = \zeta_\lambda \Theta \beta, \quad (5.26)$$
where

$$\beta = \begin{pmatrix} \exp(i\phi) & 0 \\ 0 & \exp(-i\phi) \end{pmatrix}. \tag{5.27}$$

Explicitly,

$$A_S^* = \begin{pmatrix} 0 & -ie^{-i\phi} \\ ie^{i\phi} & 0 \end{pmatrix}, \quad A_A^* = \begin{pmatrix} 0 & ie^{-i\phi} \\ -ie^{i\phi} & 0 \end{pmatrix}. \tag{5.28}$$

The noted $A$’s correspond to the following choice of the parameters $\{a, \phi_a, \phi_b\}$: $a = 0$, $\phi_b = -\phi + \pi$ and $a = 0$, $\phi_b = -\phi$, respectively, with $\phi_a$ remaining arbitrary. The subscript on $A$ is to remind that its determinant is minus unity. This difference – summarized in Table 2 – in $A$, for Dirac and Majorana spinors, does not allow the $\lambda(p)$ to satisfy the Dirac equation. Following the same procedure as in Sec. 5.1, and using

$$\exp\left( \pm \frac{\sigma \cdot \varphi}{2} \right) = \frac{(E + m) \mathbb{I} \pm \sigma \cdot p}{\sqrt{2m(E + m)}}, \tag{5.29}$$

we obtain, instead:

$$\left[ (p_\mu \gamma^\mu + m\gamma^0) \tilde{A} (p_\mu \gamma^\mu + m\gamma^0) - 2m(E + m) \mathbb{I} \right] \lambda(p) = 0; \tag{5.30}$$

where

$$\tilde{A} = \begin{pmatrix} \mathbb{I} & A \\ A^{-1} & \mathbb{I} \end{pmatrix}. \tag{5.31}$$

As a check we verify that $\text{Det} \left[ (p_\mu \gamma^\mu + m\gamma^0) \tilde{A} (p_\mu \gamma^\mu + m\gamma^0) - 2m(E + m) \mathbb{I} \right] = 0$ yields dispersion relation (5.18) and also (5.19).

| SPINOR TYPE | Det[$A$] | $a$ | $\phi_a$ | $\phi_b$ |
|------------|----------|----|----------|----------|
| Dirac $u(p)$ | $+1$ | $1$ | $0$ | arbitrary |
| Dirac $v(p)$ | $+1$ | $1$ | $\pi$ | arbitrary |
| Majorana $\lambda^S(p)$ | $-1$ | $0$ | arbitrary | $-\phi + \pi$ |
| Majorana $\lambda^A(p)$ | $-1$ | $0$ | arbitrary | $-\phi$ |

Table 2: The parameters $\{a, \phi_a, \phi_b\}$. See text.

6. Physical Interpretation

The existence of (5.19), and lack of manifest covariance of Eq. (5.30), seem to be deeply connected. Following the stated spirit of this paper outlined in observation marked $\mathcal{O_3}$ of Sec. 2, we refrain from simply overlooking the situation.\textsuperscript{12} Here we offer what appears to

\textsuperscript{12}I thank Abhay Ashtekar and Naresh Dadhich for discussions at IUCAA. In part, the content of this sections reflects that discussion.
be the most natural physical interpretation. To put forward our interpretation, we define two wave operators,

\[ O_S = \left[ (p_\mu \gamma^\mu + m \gamma^0) \tilde{A} (p_\mu \gamma^\mu + m \gamma^0) - 2m(E + m) \right]_{A=A^S}, \]  
\[ O_A = \left[ (p_\mu \gamma^\mu + m \gamma^0) \tilde{A} (p_\mu \gamma^\mu + m \gamma^0) - 2m(E + m) \right]_{A=A^A}. \]  

By construction,

\[ O_S \lambda^S(p) = 0, \quad O_A \lambda^A(p) = 0. \]  

for \( E = \pm \sqrt{p^2 + m^2} \), and assuming \( E = \pm \sqrt{p^2 + m^2} \) we verify that,

\[ O_S \lambda^A(p) \neq 0, \quad O_A \lambda^A(p) \neq 0. \]  

However, if we assert \( E = -2m \pm \sqrt{p^2 + m^2} \), a brute force calculation shows the result:

\[ O_S \lambda^A(p) = 0, \quad O_A \lambda^A(p) = 0. \]  

This role reversal of self and anti-self conjugate sectors runs miraculously through the whole structure. For instance, imposing \( E = -2m \pm \sqrt{p^2 + m^2} \) results in (cf., results of Sec. 4, paying careful attention to the sub- and super-scripts S and A):

\[ \bar{\lambda}^A_\alpha(p) \lambda^A_{\alpha'}(p) = + 2m \delta_{\alpha \alpha'}, \]  
\[ \bar{\lambda}^S_\alpha(p) \lambda^S_{\alpha'}(p) = - 2m \delta_{\alpha \alpha'}. \]  

The completeness relation also formally “interchanges” self and anti-self conjugate sectors,

\[ \frac{1}{2m} \sum_\alpha \left[ \lambda^A_\alpha(p) \bar{\lambda}^A_{\alpha}(p) - \lambda^S_\alpha(p) \bar{\lambda}^S_{\alpha}(p) \right] = \mathbb{I}, \]  

Our interpretation, then, is the following conjecture: Majorana particles, in order to minimize energy of the system, physically realize the new dispersion relation. The massless limit of the \( \lambda^S(p) \) and \( \lambda^S(p) \), when coupled with results of Appendix G, suggest that we identify self conjugate sector with Majorana particles, while the anti-self conjugate sector describes Majorana antiparticles. The observed cosmological matter-antimatter asymmetry then may be a result of \( S-A \) asymmetry contained in the new dispersion relation. The lack of manifest covariance of the wave equation for the extended set of Majorana spinors reflects, and defines, a preferred frame which we identify with that of cosmic neutrino background. It shall then be an experimental challenge to decipher if the frame of cosmic microwave background coincides with that of cosmic neutrino background.

The fact that the new dispersion relation may have escaped experimental detection may simply be related to the fact direct experiments with neutrinos are notoriously difficult and that neutrino masses \[1, 2\] are orders of magnitude smaller than their charged counterparts.
7. Additional properties of the extended set of Majorana spinors

In Majorana realization ("representation"), Appendix E shows that the self conjugate \( \lambda^S(p) \) are real, while antiself conjugate \( \lambda^A(p) \) are pure imaginary. The commutativity of C and P for the extended set of Majorana spinors is established in Appendix G, and agrees with the old results of Foldy and Nigam [29]. These results allow to construct top right block of Table 1. The parity properties of the extended set of Majorana spinors is given in Appendix H. We have followed this format so as not to impede the general flow of arguments and placing some of the results in the Appendices is by no means intended to diminish their relative significance.

8. Conclusion

The standard Dirac field is written as,

\[
\psi^{\text{charged}}(x) = \int \frac{d^3p}{2\pi^3} \sum_{\sigma=+,-} \left[ a_{\sigma}(p) u_{\sigma}(p) e^{-p_\mu x^\mu} + b_{\sigma}^\dagger(p) v_{\sigma}(p) e^{+p_\mu x^\mu} \right].
\] (8.1)

By identifying \( b_{\sigma}^\dagger(p) \) with \( a_{\sigma}^\dagger(p) \) one obtains the Majorana field

\[
\psi^{\text{neutral}}(x) = \int \frac{d^3p}{2\pi^3} \sum_{\sigma=+,-} \left[ a_{\sigma}(p) u_{\sigma}(p) e^{-p_\mu x^\mu} + a_{\sigma}^\dagger(p) v_{\sigma}(p) e^{+p_\mu x^\mu} \right].
\] (8.2)

Such a description makes Majorana particles subordinate to Dirac’s representation space in which the particle and antiparticle spinors are the basis spinors and endow the space with very specific C, P, and T properties. Inspired by recent Heidelberg-Moscow results [1, 2], we have aspired to fulfill Majorana’s original goal by bringing full symmetry between the charged and fundamentally neutral particles. We constructed a complete set of Majorana spinors and unearthed their properties. This allows for introduction of,

\[
\nu^{\text{neutral}}(x) = \int \frac{d^3p}{2\pi^3} \sum_{\alpha=\{+,+\}, \{+-\}, \{-+,+\}} \left[ c_{\alpha}(p) \lambda^S_{\alpha}(p) e^{-p_\mu x^\mu} + c_{\alpha}^\dagger(p) \lambda^A_{\alpha}(p) e^{+p_\mu x^\mu} \right],
\] (8.3)

and

\[
\nu^{\text{charged}}(x) = \int \frac{d^3p}{2\pi^3} \sum_{\alpha=\{+,+\}, \{+-\}, \{-+,+\}} \left[ d_{\alpha}(p) \lambda^S_{\alpha}(p) e^{-p_\mu x^\mu} + d_{\alpha}^\dagger(p) \lambda^A_{\alpha}(p) e^{+p_\mu x^\mu} \right].
\] (8.4)

Describing charged particles by \( \nu^{\text{charged}}(x) \) may appear absurd. But it is this “absurdity” of \( \psi^{\text{neutral}}(x) \) that led us to this paper. In fact neither of the four fields defined above are absurd at any level. Deciphering their physical content and realization is a matter of further theoretical work, and prediction of phenomenologically distinct phenomena may open up a new experimental arena. It is already clear that such a venture may be full of unexpected surprises. We have discovered some surprises already and have summarized the same in the Abstract and argued in the paper.
Acknowledgments

Parts of this paper are based on Concluding Remarks and an Invited Talk presented at “Physics beyond the Standard Model: Beyond the Desert 2002” and archived as Ref. [6].

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Appendix A: Derivation of Eq. (3.9)

Complex conjugating Eq. (3.8) gives,

$$\sigma^* \cdot \hat{p} \left[ \phi^\pm_L(0) \right]^* = \pm \left[ \phi^\pm_L(0) \right]^*.$$  \hspace{1cm} (5)

Substituting for $\sigma^*$ from Eq. (3.2) then results in,

$$\Theta \sigma \Theta^{-1} \cdot \hat{p} \left[ \phi^\pm_L(0) \right]^* = \mp \left[ \phi^\pm_L(0) \right]^*.$$  \hspace{1cm} (6)

But $\Theta^{-1} = -\Theta$. So,

$$-\Theta \sigma \Theta \cdot \hat{p} \left[ \phi^\pm_L(0) \right]^* = \mp \left[ \phi^\pm_L(0) \right]^*.$$  \hspace{1cm} (7)
Or, equivalently,

\[ \Theta^{-1} \sigma \Theta \cdot \hat{p} \left[ \phi^+_L(0) \right]^* = \mp \left[ \phi^+_L(0) \right]^*. \tag{8} \]

Finally, left multiplying both sides of the preceding equation by \( \Theta \), and moving \( \Theta \) through \( \hat{p} \), yields Eq. (3.9).

**Appendix B: The \( \phi^+_L(0) \)**

Representing the unit vector along \( p \), as,

\[ \hat{p} = \left( \sin(\theta) \cos(\phi), \sin(\theta) \sin(\phi), \cos(\theta) \right), \tag{9} \]

the \( \phi^+_L(0) \) take the explicit form:

\[ \phi^+_L(0) = \sqrt{m} e^{i \vartheta_1} \left( \cos(\theta/2) e^{-i \phi/2} \begin{pmatrix} \sin(\theta/2) \sin(\phi) \\ - \cos(\theta/2) e^{i \phi/2} \end{pmatrix} \right), \tag{10} \]

\[ \phi^-_L(0) = \sqrt{m} e^{i \vartheta_2} \left( \sin(\theta/2) e^{-i \phi/2} \begin{pmatrix} \sin(\theta/2) \cos(\phi) \\ - \cos(\theta/2) e^{i \phi/2} \end{pmatrix} \right). \tag{11} \]

In this paper we take \( \vartheta_1 \) and \( \vartheta_2 \) to be zero.

**Appendix C: Bi-orthonormality relations for \( \lambda(p) \) spinors**

On setting \( \vartheta_1 \) and \( \vartheta_2 \) to be zero — a fact that we explicitly note [11, 12] — we find the following *bi-orthonormality* relations for the self-conjugate spinors,

\[ \overline{\lambda}_S^{(-,+)}(p) \lambda_S^{(-,+)}(p) = 0, \quad \overline{\lambda}_S^{(+,-)}(p) \lambda_S^{(+,-)}(p) = +2im, \tag{12} \]

\[ \overline{\lambda}_S^{(-,+)}(p) \lambda_S^{(+,-)}(p) = -2im, \quad \overline{\lambda}_S^{(+,-)}(p) \lambda_S^{(-,+)}(p) = 0. \tag{13} \]

Their counterpart for antiself-conjugate spinors reads,

\[ \overline{\lambda}_A^{(-,+)}(p) \lambda_A^{(-,+)}(p) = 0, \quad \overline{\lambda}_A^{(+,-)}(p) \lambda_A^{(+,-)}(p) = -2im, \tag{14} \]

\[ \overline{\lambda}_A^{(-,+)}(p) \lambda_A^{(+,-)}(p) = +2im, \quad \overline{\lambda}_A^{(+,-)}(p) \lambda_A^{(-,+)}(p) = 0, \tag{15} \]

while all combinations of the type \( \overline{\lambda}_A^{(A)}(p) \lambda_S^{(A)}(p) \) and \( \overline{\lambda}_S^{(A)}(p) \lambda_A^{(A)}(p) \) identically vanish. We take note that the bi-orthogonal norms of the Majorana spinors are intrinsically *imaginary*. The associated completeness relation is:

\[ -\frac{1}{2im} \left( \left[ \lambda_S^{(-,+)}(p) \overline{\lambda}_S^{(-,+)}(p) - \lambda_S^{(+,-)}(p) \overline{\lambda}_S^{(+,-)}(p) \right] \\
- \left[ \lambda_A^{(-,+)}(p) \overline{\lambda}_A^{(-,+)}(p) - \lambda_A^{(+,-)}(p) \overline{\lambda}_A^{(+,-)}(p) \right] \right) = \mathbb{I}. \tag{16} \]

**Appendix D: The \( \rho(p) \) spinors**
Now, \((1/2, 0) \oplus (0, 1/2)\) is a four dimensional representation space. Therefore, there cannot be more than four independent spinors. Consistent with this observation, we find that the \(\rho(p)\) spinors are related to the \(\lambda(p)\) spinors through the following identities:

\[
\begin{align*}
\rho^S_{\{-,\,+\}}(p) &= -i\lambda^A_{\{+,\,-\}}(p), \\
\rho^S_{\{+,\,-\}}(p) &= +i\lambda^A_{\{-,\,+\}}(p), \\
\rho^A_{\{-,\,+\}}(p) &= +i\lambda^S_{\{+,\,-\}}(p), \\
\rho^A_{\{+,\,-\}}(p) &= -i\lambda^S_{\{-,\,+\}}(p).
\end{align*}
\] (17)

Using these identities, one may immediately obtain the bi-orthonormality and completeness relations for the \(\rho(p)\) spinors. In the massless limit, \(\rho^S_{\{+,\,-\}}(p)\) and \(\rho^A_{\{-,\,+\}}(p)\) identically vanish. A particularly simple orthonormality, as opposed to bi-orthonormality, relation exists between the \(\lambda(p)\) and \(\rho(p)\) spinors:

\[
\begin{align*}
\bar{\lambda}^S_{\{-,\,+\}}(p)\rho^A_{\{-,\,+\}}(p) &= -2m = \bar{\lambda}^A_{\{-,\,+\}}(p)\rho^S_{\{-,\,+\}}(p) \\
\bar{\lambda}^S_{\{+,\,-\}}(p)\rho^A_{\{+,\,-\}}(p) &= -2m = \bar{\lambda}^A_{\{+,\,-\}}(p)\rho^S_{\{+,\,-\}}(p).
\end{align*}
\] (19)

An associated completeness relation also exists, and it reads:

\[
-\frac{1}{2m} \left( \left[ \lambda^S_{\{-,\,+\}}(p)\overline{\rho}^A_{\{-,\,+\}}(p) + \lambda^S_{\{+,\,-\}}(p)\overline{\rho}^A_{\{+,\,-\}}(p) \right] \\
+ \left[ \lambda^A_{\{-,\,+\}}(p)\overline{\rho}^S_{\{-,\,+\}}(p) + \lambda^A_{\{+,\,-\}}(p)\overline{\rho}^S_{\{+,\,-\}}(p) \right] \right) = \mathbb{I}.
\] (21)

The results of this section are in spirit of Refs. [30, 31, 32].

The completeness relation (16) confirms that a physically complete theory of fundamentally neutral particle spinors must incorporate the self as well as antself conjugate spinors. However, one has a choice. One may either work with the set \(\{\lambda^S(p), \lambda^A(p)\}\), or with the physically and mathematically equivalent set, \(\{\rho^S(p), \rho^A(p)\}\). One is also free to choose some appropriate combinations of neutral particle spinors from these two sets.

Appendix E: Extended set of Majorana spinors in Majorana realization

The \(\lambda^{S,A}(p)\) obtained above are in Weyl realization (subscripted by, \(W\). In Majorana realization (subscripted by, \(M\)) these spinors are given by:

\[
\lambda^S_M(p) = \mathcal{S} \lambda^S_W(p),
\] (22)

where

\[
\mathcal{S} = \frac{1}{2} \begin{pmatrix}
\mathbb{I} + i\Theta & \mathbb{I} - i\Theta \\
-(\mathbb{I} - i\Theta) & \mathbb{I} + i\Theta
\end{pmatrix}.
\] (23)

Calculations show \(\lambda^S_M(p)\) are real, while \(\lambda^A_M(p)\) are pure imaginary.

Appendix F: The \(\lambda(p)\) do not satisfy Dirac equation \(^{13}\)

\(^{13}\)The main result of this Appendix is a re-rendering of a proof given my M. Kirchbach[30]. Any mistake, if any, that the reader may notice is entirely due to our failure.
The bi-orthonormality relations (12–15) and the completeness relation (16) are counterpart of the following relations for the charged, i.e. Dirac, particle spinors:

\[ \bar{u}_h(p) u_{h'}(p) = +2m \delta_{hh'} , \tag{24} \]
\[ \bar{v}_h(p) v_{h'}(p) = -2m \delta_{hh'} , \tag{25} \]
\[ \frac{1}{2m} \left[ \sum_{h=\pm1/2} u_h(p) \bar{u}_h(p) - \sum_{h=\pm1/2} v_h(p) \bar{v}_h(p) \right] = \mathbb{I} . \tag{26} \]

Furthermore, if one wishes (with certain element of hazard to become apparent below), one can write the the momentum-space extended set of Majorana spinors \( \{ \lambda^S(p), \lambda^A(p) \} \), in terms of Dirac spinors in momentum-space, \( \{ u(p), v(p) \} \). This task is best accomplished by introducing the following notation:

\[ d_1 \equiv u_+(p), \ d_2 \equiv u_-(p), \ d_3 \equiv v_+(p), \ d_4 \equiv v_-(p) , \tag{27} \]
\[ m_1 \equiv \lambda^S_{(-,+)}(p), \ m_2 \equiv \lambda^S_{(+,-)}(p), \ m_3 \equiv \lambda^A_{(-,+)}(p), \ m_4 \equiv \lambda^A_{(+,-)}(p) . \tag{28} \]

Then, the extended set of Majorana spinors can be written as,

\[ m_i = \sum_{j=1}^4 \Omega_{ij} d_j , \quad i = 1, 2, 3, 4 \tag{29} \]

where

\[ \Omega_{ij} = \begin{cases} + (1/2m) d_j m_i, & \text{for } j = 1, 2 \\ - (1/2m) d_j m_i, & \text{for } j = 3, 4 \end{cases} \tag{30} \]

In matrix form, the \( \Omega \) reads:

\[ \Omega = \frac{1}{2} \begin{pmatrix} \mathbb{I} & -i \mathbb{I} & -i \mathbb{I} & -i \mathbb{I} \\ i \mathbb{I} & \mathbb{I} & i \mathbb{I} & -\mathbb{I} \\ i \mathbb{I} & i \mathbb{I} & \mathbb{I} & i \mathbb{I} \\ -i \mathbb{I} & i \mathbb{I} & -i \mathbb{I} & -\mathbb{I} \end{pmatrix} . \tag{31} \]

Equations (29) and (31) immediately tell us that a spinor in the extended set of Majorana spinors is a linear combination of the Dirac particle and antiparticle spinors. In momentum space, the Dirac spinors are annihilated by \( (\gamma^\mu p_\mu \pm m) \),

\[ \begin{cases} \text{For particles:} & (\gamma^\mu p_\mu - m) u(p) = 0 , \\ \text{For antiparticles:} & (\gamma^\mu p_\mu + m) v(p) = 0 . \end{cases} \tag{32} \]

Since the mass terms carry opposite signs, hence are different for the particle and antiparticle, the spinors in the extended set of Majorana spinors cannot be annihilated by \( (\gamma^\mu p_\mu - m) \), or, by \( (\gamma^\mu p_\mu + m) \). Moreover, in the configuration space, since the time evolution of the of \( u(p) \) occurs via \( \exp(-ip_\mu x^\mu) \) while that for \( v(p) \) spinors occurs via \( \exp(+ip_\mu x^\mu) \) one cannot naively go from momentum-space expression (29) to its configuration space counterpart. In fact several conceptual and technically subtle hazards are
confronted if one begins to mix the two set of spinors. One ought to develop the theory of fundamentally neutral particle spinors entirely in its own right. We thus end this digression by making part of the above argument more explicit. For that purpose we introduce:

\[
M \equiv \begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix}, \quad D \equiv \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}, \quad \Lambda \equiv \begin{pmatrix} \gamma_{\mu} p^{\mu} & 0 & 0 & 0 \\ 0 & \gamma_{\mu} p^{\mu} & 0 & 0 \\ 0 & 0 & \gamma_{\mu} p^{\mu} & 0 \\ 0 & 0 & 0 & \gamma_{\mu} p^{\mu} \end{pmatrix}. \tag{33}
\]

In this language, equation (29) becomes

\[
M = \Omega D. \tag{34}
\]

Now, applying from left the operator \(\Lambda\) and using, \(\left[\Lambda, \Omega\right] = 0\), we get

\[
\Lambda M = \Omega \Lambda D. \tag{35}
\]

But, Eqs. (32) imply

\[
\Lambda D = \begin{pmatrix} m \mathbb{1} & 0 & 0 & 0 \\ 0 & m \mathbb{1} & 0 & 0 \\ 0 & 0 & -m \mathbb{1} & 0 \\ 0 & 0 & 0 & -m \mathbb{1} \end{pmatrix} D. \tag{36}
\]

Therefore, on using \(D = \Omega^{-1} M\) we obtain,

\[
\Lambda M = \Omega (r.h.s. \ of \ Eq. \ 36) \Omega^{-1} M. \tag{37}
\]

An explicit evaluation of, \(\mu \equiv \Omega (r.h.s. \ of \ Eq. \ 36) \Omega^{-1}\), reveals it to be,

\[
\mu = \begin{pmatrix} 0 & -im \mathbb{1} & 0 & 0 \\ im \mathbb{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & im \mathbb{1} \\ 0 & 0 & -im \mathbb{1} & 0 \end{pmatrix}. \tag{38}
\]

Thus, finally giving us the result,

\[
\begin{pmatrix} \gamma_{\mu} p^{\mu} & 0 & 0 & 0 \\ 0 & \gamma_{\mu} p^{\mu} & 0 & 0 \\ 0 & 0 & \gamma_{\mu} p^{\mu} & 0 \\ 0 & 0 & 0 & \gamma_{\mu} p^{\mu} \end{pmatrix} \begin{pmatrix} \lambda_{(-,+)}^{S}(p) \\ \lambda_{(+,-)}^{S}(p) \\ \lambda_{(-,+)}^{A}(p) \\ \lambda_{(+,-)}^{A}(p) \end{pmatrix} - im \begin{pmatrix} -\lambda_{(-,+)}^{S}(p) \\ \lambda_{(+,-)}^{S}(p) \\ -\lambda_{(-,+)}^{A}(p) \\ \lambda_{(+,-)}^{A}(p) \end{pmatrix} = 0, \tag{39}
\]

which explicitly establishes the result that \((\gamma_{\mu} p_{\mu} \pm m \mathbb{1})\) do not annihilate the neutral particle spinors.\(^{14}\) The text-book assertions that Majorana mass term is ‘off-diagonal” is a rough translation of this equation.

\(^{14}\)The result contained in the above equation confirms earlier result of Ref. \[31\].
Appendix G: Commutativity of $C$ and $P$ for the extended set of Majorana spinors

The parity operation is slightly subtle for neutral particle spinors. In the $(1/2, 0) \oplus (0, 1/2)$ representation space it reads,

$$P = e^{i\phi_p} \gamma^0 \mathcal{R}. \quad (40)$$

The $\mathcal{R}$ is defined as,

$$\mathcal{R} \equiv \{ \theta \to \pi - \theta, \phi \to \phi + \pi, p \to p \}. \quad (41)$$

This has the consequence that eigenvalues, $h$, of the helicity operator

$$h = \frac{\sigma}{2} \cdot \hat{p} \quad (42)$$

change sign under the operation of $\mathcal{R}$,

$$\mathcal{R} : h \to h' = -h. \quad (43)$$

Furthermore,

$$P u_h(p) = e^{i\phi_p} \gamma^0 \mathcal{R} u_h(p) = e^{i\phi_p} \gamma^0 u_{-h}(-p) = -ie^{i\phi_p} u_h(p). \quad (44)$$

Similarly,

$$P v_h(p) = ie^{i\phi_p} v_h(p). \quad (45)$$

We now require the eigenvalues of the $P$ to be real. This fixes the phase factor,

$$e^{i\phi_p} = \pm i. \quad (46)$$

The remaining ambiguity, as contained in the sign, still remains. It is fixed by recourse to text-book convention by taking the sign on the right-hand side of Eq. (46) of to be positive. This very last choice shall not affect our conclusions (as it should not). The parity operator is thus fixed to be,

$$P = i \gamma^0 \mathcal{R}. \quad (47)$$

Thus,

$$P u_h(p) = + u_h(p), \quad (48)$$

$$P v_h(p) = - v_h(p). \quad (49)$$

The consistency of Eqs. (48) and (49) requires,

CHARGED PARTICLE SPINORS : $P^2 = I$, [cf. Eq.(57)]. \quad (50)

To calculate the anticommutator, $\{C, P\}$, when acting on the $u_h(p)$ and $v_h(p)$ we now need, in addition, the action of $C$ on these spinors. This action can be summarized as follows:

$$C : \begin{cases} u_{+1/2}(p) \to -v_{-1/2}(p), u_{-1/2}(p) \to v_{+1/2}(p), \\ v_{+1/2}(p) \to u_{-1/2}(p), v_{-1/2}(p) \to -u_{+1/2}(p). \end{cases} \quad (51)$$
Using Eqs. (48), (49), and (51) one can readily obtain the action of anticommutator, \{C, P\}, on the four \(u(p)\) and \(v(p)\) spinors. For each case it is found to vanish: \{C, P\} = 0.

The \(P\) acting on the neutral particle spinors yields the result,

\[
P\lambda^S_{\{+,\}}(p) = +i \lambda^A_{\{-,+\}}(p), \quad P\lambda^S_{\{-,+\}}(p) = -i \lambda^A_{\{-,+\}}(p),
\]

(52)

\[
P\lambda^A_{\{+,\}}(p) = -i \lambda^S_{\{-,+\}}(p), \quad P\lambda^A_{\{-,+\}}(p) = +i \lambda^S_{\{-,+\}}(p).
\]

(53)

Following the same procedure as before, we now use (52), (53), and (51) to evaluate the action of the commutator \([C, P]\) on each of the four neutral particle spinors. We find it vanishes for each of them: \([C, P]\) = 0. It confirms the claim we made in Table 1.

The commutativity and anticommutativity of the \(C\) and \(P\) operators is a deeply profound result and it establishes that the theory of neutral and charged particles must be developed in their own rights. This is the task we have undertaken and are developing here in this Paper.

Appendix H: Parity asymmetry for the extended set of Majorana spinors

Unlike the charged particle spinors, Eqs. (52) and (53) reveal that neutral particle spinors are not eigenstates of \(P\). Furthermore, a rather apparently paradoxical asymmetry is contained in these equations. For instance, the second equation in (52) reads:

\[
P\lambda^S_{\{+,\}}(p) = -i \lambda^A_{\{-,+\}}(p).
\]

(54)

Now, in a normalization-independent manner

\[
\lambda^S_{\{+,\}}(p) \propto \left(1 + \frac{|p|}{E + m}\right) \lambda^S_{\{+,\}}(0),
\]

(55)

while

\[
\lambda^A_{\{-,+\}}(p) \propto \left(1 - \frac{|p|}{E + m}\right) \lambda^A_{\{-,+\}}(0).
\]

(56)

Consequently, in the massless/high-energy limit the \(P\)-reflection of \(\lambda^S_{\{+,\}}(p)\) identically vanishes. The same happens to the \(\lambda^A_{\{+,\}}(p)\) spinors under \(P\)-reflection. This situation is in sharp contrast to the charged particle spinors. The consistency of Eqs. (52) and (53) requires \(P^2 = -\mathbb{I}\) and in the process shows that the remaining two, i.e. first and third equation in that set, do not contain additional physical content:

\[
\text{Neutral particle spinors : } \quad P^2 = -\mathbb{I}. \quad [\text{cf. Eq. (54)}].
\]

(57)

That is, for neutral particle spinors:

\[
\text{Neutral particle spinors : } \quad P^4 = \mathbb{I},
\]

(58)

The origin of the asymmetry under \(P\)-reflection resides in the fact that the \((1/2,0) \oplus (1/2,0)\) neutral particles spinors, in being dual helicity objects, combine Weyl spinors of
opposite helicities. However, in the massless limit, the structures of \( \kappa^{(\frac{1}{2},0)} \) and \( \kappa^{(0,\frac{1}{2})} \) force only positive helicity \((1/2, 0)\)-Weyl and negative helicity \((0, 1/2)\)-Weyl spinors to be non-vanishing. For this reason, in the massless limit the neutral particle spinors, \( \lambda_{S}^{\{−, +\}}(p) \) and \( \lambda_{A}^{\{−, +\}}(p) \), carrying negative helicity \((1/2, 0)\)-Weyl and positive helicity \((0, 1/2)\)-Weyl spinors identically vanish.

So we have the following situation: The \((1/2, 0) \oplus (0, 1/2)\) is a \( P \) covariant representation space. Yet, in the neutral particle formalism, it carries \( P \)-reflection asymmetry. This circumstance has a precedence in the Velo-Zwanziger observation, who noted \([32]\), “the main lesson to be drawn from our analysis is that special relativity is not automatically satisfied by writing equations which transform covariantly.” We conjecture that this asymmetry may underlie the phenomenologically known parity violation. Even though the latter is incorporated, by hand, in the standard model of the electroweak interactions its true physical origin has remained unknown.