MONOMIAL BASES AND BRANCHING RULES

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ABSTRACT. Following a question of Vinberg, a general method to construct monomial bases in finite-dimensional irreducible representations of a reductive Lie algebra \( g \) was developed in a series of papers by Feigin, Fourier, and Littelmann. Relying on this method, we construct monomial bases of multiplicity spaces associated with the restriction of the representation to a reductive subalgebra \( g_0 \subset g \). As an application, we produce new monomial bases for representations of the symplectic Lie algebra associated with a natural chain of subalgebras. One of our bases is related via a triangular transition matrix to a suitably modified version of the basis constructed earlier by the first author. In type \( A \), our approach shows that the Gelfand–Tsetlin basis and the canonical basis of Lusztig have a common PBW-parameterisation. This implies that the transition matrix between them is triangular. We show also that a similar relationship holds for the Gelfand–Tsetlin and the Littelmann bases in type \( A \).

INTRODUCTION

A general method to construct monomial bases in finite-dimensional irreducible representations of a reductive Lie algebra \( g \) has been developed in a series of papers by E. Feigin, G. Fourier, and P. Littelmann \([7, 8, 9]\) following a question and initial examples of E. Vinberg. In accordance with this method, one chooses a triangular decomposition \( g = n^- \oplus h \oplus n^+ \) and a basis \( \{f_1, \ldots, f_N\} \) of the nilpotent Lie algebra \( n^- \) consisting of root vectors. Let \( V(\lambda) \) be a finite-dimensional irreducible \( g \)-module and let \( v_\lambda \in V(\lambda) \) be a highest weight vector. By introducing special orderings on monomials in the basis elements \( f_i \) it is possible to specify conditions on the powers \( \alpha_i \) so that the vectors

\[
f_1^{\alpha_1} \ldots f_N^{\alpha_N} v_\lambda
\]

form a basis of \( V(\lambda) \). Such conditions are given in an explicit form for types \( A \) and \( C \) in \([7]\) and \([8]\), respectively. A unified approach is presented in \([9]\).

One of the features of the initial solutions \([7, 8]\) is that a homogeneous order on the monomials was used, which means that the degrees are compared first. In such a setup, the sequence of factors is not significant. By now there is a tremendous development in the area, with both geometric and combinatorial applications, and numerous variations have

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been studied, see e.g. [4, 6] and references therein. Of particular interest and importance are connections with the Littelmann bases [14] and with the PBW-type versions of the canonical basis of Lusztig [15, 16], see [6, Sect. 11&12].

Our goal in this paper is to adjust the FFLV method to construct bases of the multiplicity spaces associated with the restriction of \( V(\lambda) \) to a reductive subalgebra \( g_0 \). Given a finite-dimensional irreducible \( g_0 \)-module \( V'(\mu) \), the corresponding multiplicity space is defined by

\[
U(\lambda, \mu) = \text{Hom}_{g_0}(V'(\mu), V(\lambda)).
\]

Note that \( U(\lambda, \mu) \) is isomorphic to the subspace \( V(\lambda)_{\mu}^+ \) of \( g_0 \)-highest weight vectors in \( V(\lambda) \) of weight \( \mu \) and we have a vector space decomposition

\[
(0 \cdot 1) \quad V(\lambda) \cong \bigoplus_{\mu} V(\lambda)_{\mu}^+ \otimes V'(\mu).
\]

Hence, if some bases of the spaces \( V(\lambda)_{\mu}^+ \) and \( V'(\mu) \) are produced, then the decomposition (0·1) yields the natural tensor product basis of \( V(\lambda) \).

The celebrated Gelfand–Tsetlin bases [10, 11] for representations of the general linear and orthogonal Lie algebras are obtained by iterating this procedure and applying it to the subalgebras of the chains

\[
\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \cdots \subset \mathfrak{gl}_n \quad \text{and} \quad \mathfrak{o}_2 \subset \mathfrak{o}_3 \subset \cdots \subset \mathfrak{o}_N.
\]

The multiplicity spaces \( V(\lambda)_{\mu}^+ \) corresponding to the pairs of orthogonal and symplectic Lie algebras \( \mathfrak{o}_{N-2} \subset \mathfrak{o}_N \) and \( \mathfrak{sp}_{2n-2} \subset \mathfrak{sp}_{2n} \) turned out to carry representations of certain quantum algebras originally introduced by Olshanski [21] and which are known as twisted Yangians. The Yangian representation theory together with the theory of Mickelsson algebras developed in the work by Zhelobenko [23, 24, 25] have lead to a construction of bases of the Gelfand–Tsetlin type for representations of the orthogonal and symplectic Lie algebras; see review paper [19] and book [20, Ch. 9] for a detailed exposition of these results, as well as a discussion of various approaches to constructions of Gelfand–Tsetlin-type bases in the literature.

The Zhelobenko theory allows one to describe the multiplicity spaces \( V(\lambda)_{\mu}^+ \) corresponding to the pair \( g_0 \subset g \) as linear spans of lowering operators obtained via the action of the extremal projector \( p \) associated with the Lie algebra \( g_0 \). Our main general result provides precise choices of those operators to form a basis of \( V(\lambda)_{\mu}^+ \). These choices are made in the spirit of the FFLV method and rely on some special monomial order. In more detail, we will assume that \( g_0 \subset g \) is a reductive subalgebra normalised by \( \mathfrak{h} \). Then \( g_0 \) inherits the triangular decomposition \( g_0 = n_0^+ \oplus \mathfrak{h}_0 \oplus n_0^- \) with \( n_0^+ = n^\pm \cap g_0 \) and \( \mathfrak{h}_0 = \mathfrak{h} \cap g_0 \). Let \( n^- = n_0^- \oplus \mathfrak{r} \) be an \( \mathfrak{h} \)-stable vector space decomposition. We describe a family of admissible monomials \( m \in \mathfrak{U}(\mathfrak{r}) \) such that the elements \( pmv_\lambda \) form a basis of the multiplicity space \( V(\lambda)_{\mu}^+ \).
In order to obtain a basis for \( V(\lambda) \) inductively, it suffices to produce first a basis for a quotient of \( V(\lambda) \) that is isomorphic to \( U(\lambda, \mu) \) in some natural way. This idea is used in [14] and [12]. In the latter, an answer to Vinberg’s question for the orthogonal Lie algebra is given. We formalise the method that can be called the “FFLV-branching” in Section 1 and as an application produce a new answer to Vinberg’s question in type \( C \) in Section 2.

Recall that finite-dimensional irreducible representations of \( \mathfrak{gl}_n \) are parameterised by their highest weights \( \lambda = (\lambda_1, \ldots, \lambda_n) \) which are \( n \)-tuples of complex numbers satisfying the conditions \( \lambda_i - \lambda_{i+1} \in \mathbb{Z}_+ \) for all \( i = 1, \ldots, n - 1 \). Later on we will need to work simultaneously with irreducible highest weight representations of \( \mathfrak{sp}_{2n} \) and \( \mathfrak{gl}_n \). To avoid a confusion we will denote such representations of \( \mathfrak{gl}_n \) by \( L(\lambda) \) while keeping the notation \( V(\lambda) \) in the context of general complex reductive Lie algebras and in the particular case of the symplectic Lie algebras. Thus \( L(\lambda) \) is generated by a nonzero vector \( v_\lambda \) such that

\[
E_{ij} v_\lambda = 0 \quad \text{for} \quad 1 \leq i < j \leq n, \quad \text{and} \quad E_{ii} v_\lambda = \lambda_i v_\lambda \quad \text{for} \quad 1 \leq i \leq n,
\]

where the \( E_{ij} \) denote the standard basis elements of \( \mathfrak{gl}_n \). A Gelfand–Tsetlin pattern \( \Lambda \) associated with \( \lambda \) is an array of row vectors

\[
\begin{array}{cccc}
\lambda_{n1} & \lambda_{n2} & \cdots & \lambda_{nn} \\
\lambda_{n-11} & \cdots & \lambda_{n-1,n-1} \\
\cdots & \cdots & \cdots \\
\lambda_{21} & \lambda_{22} \\
\lambda_{11}
\end{array}
\]

where the upper row coincides with \( \lambda \) and the following conditions hold

\[
\lambda_{ki} - \lambda_{k-1,i} \in \mathbb{Z}_+, \quad \lambda_{k-1,i} - \lambda_{k,i+1} \in \mathbb{Z}_+, \quad i = 1, \ldots, k - 1
\]

for each \( k = 2, \ldots, n \).

Let \( \{\xi_\Lambda\} \) be the Gelfand–Tsetlin basis for \( L(\lambda) \), see Section 3 for its detailed description.

**Theorem A.** Let \( \{\pi_\Lambda\} \) be the set of vectors

\[
\pi_\Lambda = E_1^{\lambda_{11} - \lambda_{12}} \cdots E_1^{\lambda_{11} - \lambda_{1n}} E_2^{\lambda_{22} - \lambda_{23}} \cdots E_2^{\lambda_{22} - \lambda_{2n}} \cdots E_n^{\lambda_{n1} - \lambda_{n2}} \cdots E_n^{\lambda_{n1} - \lambda_{nn}} v_\lambda,
\]

where \( \Lambda \) runs over all Gelfand–Tsetlin patterns associated with \( \lambda \). Then \( \{\pi_\Lambda\} \) is a PBW-parametrisation of \( \{\xi_\Lambda\} \), i.e., there is an order \( \succ \) on \( \{\Lambda\} \) such that

\[
\xi_\Lambda = \sum_{\Lambda' \succ \Lambda} d_{\Lambda,\Lambda'} \pi_{\Lambda'},
\]

where \( d_{\Lambda,\Lambda'} \) is a nonzero complex number. In particular, \( \{\pi_\Lambda\} \) is a basis for \( L(\lambda) \).
Theorem A is proven in Section 3 by essentially repeating the argument used by Zhelobenko in [23, Theorem 7] and [24, Lemma 2]. We also indicate briefly how it follows from the FFLV-theory.

The basis described in Theorem A is not new. With a slightly different, but combinatorially equivalent, description it appeared in [22] as a PBW-parameterisation of the canonical basis of Lusztig [15, 16]. Therefore the theorem provides a link between the Gelfand–Tsetlin and the canonical bases, see Corollary 3.4. The same basis is described in [17, Theorem 2.6].

We will regard the symplectic Lie algebra \( \mathfrak{sp}_{2n} \) as a subalgebra of \( \mathfrak{gl}_{2n} \) and we will number the rows and columns of \( 2n \times 2n \) matrices with the indices \(-n, \ldots, -1, 1, \ldots, n\). Accordingly, the zero value will be omitted in the summation or product formulas. The Lie algebra \( \mathfrak{sp}_{2n} \) is spanned by the elements \( F_{ij} \) with \(-n \leq i, j \leq n\), defined by

\[
F_{ij} = E_{ij} - \text{sgn } i \text{ sgn } j E_{-j, -i}.
\]

For any \( n \)-tuple of nonpositive integers \( \lambda = (\lambda_1, \ldots, \lambda_n) \) satisfying the conditions

\[
\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n
\]

the finite-dimensional irreducible representation \( V(\lambda) \) of the Lie algebra \( \mathfrak{sp}_{2n} \) with the highest weight \( \lambda \) is generated by a nonzero vector \( v_\lambda \) such that

\[
F_{ij} v_\lambda = 0 \quad \text{for} \quad -n \leq i < j \leq n, \quad \text{and}
\]

\[
F_{ii} v_\lambda = \lambda_i v_\lambda \quad \text{for} \quad 1 \leq i \leq n.
\]

Define a type C pattern \( \Lambda \) associated with \( \lambda \) as an array of the form

\[
\begin{array}{cccccc}
\lambda_{n1} & \lambda_{n2} & \lambda_{n3} & \cdots & \lambda_{nn} \\
\lambda'_{n1} & \lambda'_{n2} & \lambda'_{n3} & \cdots & \lambda'_{nn} \\
\lambda_{n-11} & \lambda_{n-12} & \cdots & \lambda_{n-1,n-1} \\
\lambda'_{n-11} & \lambda'_{n-12} & \cdots & \lambda'_{n-1,n-1} \\
\cdots & \cdots & \cdots \end{array}
\]

\[
\begin{array}{cc}
\lambda_{11} & \\
\lambda'_{11} & \\
\end{array}
\]

such that \( \lambda_{ni} = \lambda_i \) for \( i = 1, \ldots, n \), the remaining entries are all nonpositive integers and the following inequalities hold:

\[
\lambda'_{k1} \geq \lambda_{k1} \geq \lambda'_{k2} \geq \lambda_{k2} \geq \cdots \geq \lambda'_{k,k-1} \geq \lambda_{k,k-1} \geq \lambda'_{kk} \geq \lambda_{kk}
\]

for \( k = 1, \ldots, n \), and

\[
\lambda'_{k1} \geq \lambda_{k-11} \geq \lambda'_{k2} \geq \lambda_{k-12} \geq \cdots \geq \lambda'_{k,k-1} \geq \lambda_{k-1,k-1} \geq \lambda'_{kk}
\]
for \( k = 2, \ldots, n \).

**Theorem B.** The vectors

\[
\theta_\lambda = \prod_{k=1}^{n} \left( F_{k, -k}^{-\lambda_k} \prod_{i=1}^{k-1} F_{k, i-i}^{\lambda_{k-1-i}} F_{i, -i}^{-\lambda_i} \right) v_\lambda
\]

parameterised by all type C patterns \( \Lambda \) associated with \( \lambda \) form a basis of \( V(\lambda) \).

The proof of Theorem B is given in Section 2.1, it is derived from our general results on monomial bases of multiplicity spaces. In Section 2.2, we present another basis of \( U(\lambda, \mu) \) with somewhat more complicated conditions on the exponents of the monomials, which can be extended inductively to a basis of \( V(\lambda) \). Furthermore, in Section 4, we produce a certain modified version \( \zeta_\Lambda \) of the basis of \( V(\lambda) \) constructed in [18] and derive explicit formulas for the action of generators of the Lie algebra \( sp_{2n} \) in this basis. Then we demonstrate in Section 5 that the bases \( \theta_\Lambda \) and \( \zeta_\Lambda \) are related via a triangular transition matrix. This also gives another proof of Theorem B.

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1. **The FFLV approach to the branching problem**

Let \( g \) be a complex reductive Lie algebra and \( V(\lambda) \) be an irreducible finite-dimensional \( g \)-module. Fix a triangular decomposition \( g = n^- \oplus h \oplus n^+ \). The Lie algebra \( n^- \) has a standard basis consisting of root vectors \( \{f_1, \ldots, f_N\} \). We choose a total *monomial order* on the monomials \( m \in S(n^-) \) in this basis. Recall that a monomial order is a *total* order satisfying the following two conditions:

- \( 1 \leq m \) for each monomial \( m \),
- if \( m_1 \leq m_2 \) and \( m_3 \leq m_4 \), then \( m_1m_3 \leq m_2m_4 \);

i.e., a monomial order is compatible with multiplication. The order leads to a filtration on \( V(\lambda) \) as follows. Let \( v_\lambda \in V(\lambda) \) denote a highest weight vector. The enumeration of root vectors defines a sequence \( f_1, f_2, \ldots, f_N \). Making use of this enumeration, or, equivalently, of this sequence, we identify \( m = \prod_{i=1}^{N} f_i^{a_i} \in S(n^-) \) with the product \( f_1^{a_1} \cdots f_N^{a_N} \in U(n^-) \), which we denote by the same letter \( m \). In the expression \( mv_\lambda \), the symbol \( m \) stands for an element of \( U(n^-) \). A monomial \( \tilde{m} \in S(g) \) is *essential* if \( \tilde{m}v_\lambda \) does not lie in the linear span of \( \{mv_\lambda\} \) with \( m < \tilde{m} \). Let \( E_s(V(\lambda)) = E_s(\lambda) \) denote the set of essential monomials related to \( V(\lambda) \). As was observed in [9], \( \{mv_\lambda \mid m \in E_s(\lambda)\} \) is a basis of \( V(\lambda) \) by the very construction.
For any two finite-dimensional irreducible $g$-modules $V(\lambda)$ and $V(\lambda')$, one has the inclusion

\begin{equation}
\mathcal{E}s(\lambda)\mathcal{E}s(\lambda') \subset \mathcal{E}s(\lambda + \lambda'),
\end{equation}

see [9, Prop. 2.11]. The proof of that proposition works for any, not necessarily homogeneous, monomial order. However, the authors remark in the proof that they are using a homogeneous order and therefore can assume that the root vectors commute. For completeness, we briefly outline the argument.

Suppose that $m = \prod_{i=1}^{N} f_{i}^{a_{i}}$ is essential for $\lambda$ and $m' = \prod_{i=1}^{N} f_{i}^{a'_{i}}$ is essential for $\lambda'$. Set $\tilde{m} = mm'$ in $S(n^{-})$. As an element of $\mathcal{U}(n^{-})$, the monomial $\tilde{m}$ is equal to the product $f_{N}^{a_{N}+a'_{N}} \cdots f_{1}^{a_{1}+a'_{1}}$. Let $v = v_{\lambda} \otimes v_{\lambda'}$ be a highest weight vector of $V_{\lambda+\lambda'} \subset V_{\lambda} \otimes V_{\lambda'}$. Then we have

\[ \tilde{m}v \in \prod_{i=1}^{N} \left( \frac{a_{i} + a'_{i}}{a_{i}} \right) mv_{\lambda} \otimes m'v_{\lambda'} + (V_{\lambda} \otimes \langle \hat{m}v_{\lambda'} \mid \hat{m} < m' \rangle_{C} + \langle \hat{m}v_{\lambda} \mid \hat{m} < m \rangle_{C} \otimes V_{\lambda'}). \]

From this one can conclude that $\tilde{m} \in \mathcal{E}s(\lambda + \lambda')$.

The main novelty of our approach to the branching problem is that we combine the FFLV method with the more classical theory of Zhelobenko. In particular, the extremal projector will be playing a major role.

1.1. The extremal projector. Let $\Delta^{+}$ be the set of positive roots of $g$ which is determined by the triangular decomposition so that $n^{+}$ (resp., $n^{-}$) is spanned by the root vectors $e_{\alpha}$ (resp., $f_{\alpha}$) with $\alpha \in \Delta^{+}$. Consider the $sl_{2}$-triples $\{f_{\alpha}, h_{\alpha}, e_{\alpha}\} \subset g$ and assume that the roots are normalised to satisfy the condition $\alpha(h_{\alpha}) = 2$. Set

\[ p_{\alpha} = 1 + \sum_{k=1}^{\infty} f_{\alpha}^{k} e_{\alpha}^{k} \frac{(-1)^{k}}{k!(\rho(h_{\alpha}) + 1) \cdots (\rho(h_{\alpha}) + k)}, \]

$\rho$ is the half sum of the positive roots. This expression is regarded as an element of the algebra of formal series of monomials

\[ f_{\alpha_{1}}^{r_{1}} \cdots f_{\alpha_{N}}^{r_{N}} e_{\alpha_{1}}^{k_{1}} \cdots e_{\alpha_{N}}^{k_{N}} \quad \text{with} \quad (k_{1} - r_{1})\alpha_{1} + \cdots + (k_{N} - r_{N})\alpha_{N} = 0 \]

with coefficients in the field of fractions of the commutative algebra $\mathcal{U}(\mathfrak{h})$. Choose a numbering of positive roots, $\alpha_{1}, \ldots, \alpha_{N}$. A total order on $\Delta^{+}$ is said to be normal if either $\alpha < \alpha + \beta < \beta$ or $\beta < \alpha + \beta < \alpha$ for each pair of positive roots $\alpha, \beta$ such that $\alpha + \beta \in \Delta$. Choose a normal order $\alpha_{1} < \cdots < \alpha_{N}$ and set

\[ p = p_{\alpha_{1}} \cdots p_{\alpha_{N}}. \]

The element $p$ is independent of the choice of a normal order and is known as the extremal projector; see Asherova, Smirnov, and Tolstoy [1], [2]. A more detailed description of its
properties can be found in the work by Zhelobenko [24, 25]. In particular, \( p \) is characterised by the properties \( p^2 = p \) and
\[
(1.2) \quad e_\alpha p = pf_\alpha = 0 \quad \text{for all} \quad \alpha \in \Delta^+.
\]

1.2. The specifics of branching. A subalgebra \( q \subset g \) is a reductive subalgebra if \( q \) is reductive and the centre of \( q \) consists of \( \text{ad}_g \)-semisimple elements.

Let \( g_0 \subset g \) be a reductive subalgebra normalised by \( h \). Then \( g_0 \) inherits the triangular decomposition, \( g_0 = n_0^+ \oplus h_0 \oplus n_0^- \), where \( n_0^\pm = n^\pm \cap g_0 \). In order to see the branching rules \( g \downarrow g_0 \), we need a certain special monomial order. Let \( n^- = n_0^- \oplus r \) be the \( h \)-stable decomposition. Write \( m = m_0 m_1 \), where \( m_0 \in S(n_0^-) \) and \( m_1 \in S(r) \). Having two monomials \( m = m_0 m_1 \) and \( m' = m'_0 m'_1 \), we first compare \( m_1 \) with \( m'_1 \) and if \( m_1 < m'_1 \), then \( m < m' \). If \( m_1 = m'_1 \), then we compare \( m_0 \) with \( m'_0 \). The order on the \( S(n_0^-) \)-factors is of no particular importance. When identifying \( m_0 m_1 \in S(n^-) \) with an element of \( u(n^-) \), we take a monomial from \( u(n_0^-) u(r) \).

Set \( u_+(r) := r u(r) \) and let \( m_1 \in u_+(r) \) be a monomial having our chosen sequence of factors. The most crucial restriction on the monomial order is that
\[
(1.3) \quad x m_1 v_\lambda = [x, m_1] v_\lambda \in \langle \tilde{m} v_\lambda \mid \tilde{m} \in S(n^-), \tilde{m} < m_1 \rangle_{\mathbb{C}}
\]
for each dominant weight \( \lambda \) and each \( x \in n_0^+ \). We will assume that it is satisfied. If \( \tilde{m} < m_1 \) and \( m_1 \in S(r) \), then \( \tilde{m} = \tilde{m}_0 \tilde{m}_1 \), where \( \tilde{m}_1 < m_1 \). Therefore (1.3) implies that
\[
(1.4) \quad X m_1 v_\lambda \in \langle \tilde{m} v_\lambda \mid \tilde{m} \in S(n^-), \tilde{m} < m_1 \rangle_{\mathbb{C}}
\]
for each dominant weight \( \lambda \) and each \( x \in u(g_0) n_0^+ \).

**Lemma 1.1.** Suppose that \( [n_0^+ \oplus r, r] \subset r \). Then there is a natural way to guarantee that (1.3) is satisfied. Namely, one has to compare the \( h \)-weights \( \nu, \nu' \) of \( m_1, m'_1 \in S(r) \) first and say that if \( \nu - \nu' \in \Delta^+ \), then \( m_1 < m'_1 \).

**Proof.** We may assume that \( x \in n_0^+ \) is a root vector corresponding to a positive root \( \beta \) of \( g \). In this case the weight of \( x m_1 \) is \( \nu + \beta \). By the assumptions on \( r \), \( x m_1 v_\lambda \in \langle \tilde{m} v_\lambda \mid \tilde{m} \in S(r) \rangle_{\mathbb{C}} \), where the weight of each \( \tilde{m} \) equals \( \tilde{\nu} = \nu + \beta \). Hence here \( \tilde{m} < m_1 \) as required. \( \square \)

Let \( p \) be the extremal projector associated with \( g_0 \). Set \( N' = \dim n_0^- \). Suppose that \( w \in V(\lambda) \) is a weight vector such that \( pw \) is well-defined. Then \( pw \) is equal to \( w \) plus a finite linear combination of expressions
\[
f_{\alpha_1}^{r_1} \cdots f_{\alpha_N'}^{r_N'} e_{\alpha_{N'}}^{k_{N'}} \cdots e_{\alpha_1}^{k_1} w,
\]
where \( k_1 + \ldots + k_{N'} > 0 \). By (1.4), we have
\[
(1.5) \quad p m_1 v_\lambda \in m_1 v_\lambda + \langle \tilde{m} v_\lambda \mid \tilde{m} < m_1 \rangle_{\mathbb{C}}
\]
whenever $pm_1 v_\lambda$ is well-defined (that is, the values of the denominators occurring in $pm_1 v_\lambda$ are not zero).

Recall that $V(\lambda)^+_\mu = (V(\lambda)^{n_0^+})_\mu$ stands for the subspace of $g_0$-highest weight vectors in $V(\lambda)$ of $h_0$-weight $\mu$.

**Proposition 1.2.** Keep the above notation and the assumptions on the monomial order. Then $pm_1 v_\lambda$ is well-defined for each $m_1 \in E s(\lambda) \cap S(\frak{t})$ and the set of vectors

$$\{pm_1 v_\lambda \mid m_1 \in E s(\lambda) \cap S(\frak{t})\}$$

is a basis of the subspace $V(\lambda)^+ = V(\lambda)^{n_0^+} = \bigoplus_\mu V(\lambda)^{s_\mu}$.

**Proof.** One observes easily that $V(\lambda)^+$ is spanned by $pm v_\lambda$, where $m \in E s(\lambda)$ and the $h_0$-weight of $mv_\lambda$ is dominant for $g_0$. If $m \not\in S(\frak{t})$, then $pm = 0$ by (1.2) and our assumption on the sequence of factors in $\frak{u}(\frak{n}^-)$. It remains to prove that the vectors in question are well-defined and linearly independent.

Assume that $pm_1 v_\lambda$ is not well-defined for some $m_1 \in E s(\lambda) \cap S(\frak{t})$. Then the weight of $u = m_1 v_\lambda$ is not dominant for $g_0$. Let $\{e, h, f\} \subset g_0$ be an $sl_2$-triple such that $e \in n_0^+$ is a simple root vector and $hu = -du$ for some $d > 0$. By the standard $sl_2$-theory, which includes classification of the finite-dimensional $sl_2$-modules, there is $k = d + 2k'$ such that $u$ lies in $\bigoplus_{t=d}^k S^t \mathbb{C}^2$ up to an isomorphism. Therefore one can find elements $a(t) \in \mathbb{C}$ such that

$$u = \sum_{t=d}^{d+k'} a(t) f^t e^t u.$$ 

Here each $e^t m_1 v_\lambda$, and hence also each $f^t e^t m_1 v_\lambda$, lies in $\langle \tilde{m} v_\lambda \mid \tilde{m} < m_1 \rangle_{\mathbb{C}}$, see (1.4). Therefore $m_1$ is not essential for $\lambda$.

Assume finally that a non-trivial linear combination of $pm_1 v_\lambda$ with $m_1 \in E s(\lambda) \cap S(\frak{t})$ is equal to zero. Then by (1.5), the largest monomial appearing in it with a non-zero coefficient is not essential for $\lambda$.

The inclusion (1.1) justifies the following definition.

**Definition 1.3.** The subset

$$\Gamma = \Gamma_{\frak{g} \downarrow \frak{g}_0} := \{(\lambda, m_1) \mid m_1 \in E s(\lambda) \cap S(\frak{t})\} \subset h^* \times S(\frak{t}),$$

where $\lambda$ is dominant, is called the branching semigroup of $\frak{g} \downarrow \frak{g}_0$. Set also $\Gamma(\lambda) = \{m_1 \mid (\lambda, m_1) \in \Gamma\}$.

Note that the above objects depend on the basis of $\frak{n}^-$, on the monomial order, and on the sequence of factors in $\frak{u}(\frak{n}^-)$. A standard procedure for calculating $\Gamma$ is to consider first small values of $\lambda$, like the fundamental weights $\varpi_i$, obtain enough elements in $\Gamma(\lambda)$, and then compare the cardinality with the dimension of $V(\lambda)^+$. However, this approach can produce a description of $\Gamma$ only if the semigroup is finitely generated. It is conjectured
in [6] that \( \Gamma \) is always finitely generated in our context as well as in a less restrictive one considered there. Partial positive results in this direction are obtained in [6, Sect. 12].

**Example 1.4.** As we will see below, the semigroup \( \Gamma = \Gamma_{s_k, s_{\lambda_0}} \) is generated by the pairs \( (\varpi_i, m_1) \) with \( m_1 \in \Gamma(\varpi_i) \) and \( 1 \leq i < n \).

### 1.3. Inductive bases for \( V(\lambda) \)

Next we show how branching rules lead to constructions of FFLV-type bases.

**Proposition 1.5.** We have \( m_0 m_1 \in \mathcal{E}_s(\lambda) \) if and only if \( m_1 \in \Gamma_{\varphi_{\mu_0}}(\lambda) \) and \( m_0 \in \mathcal{E}_s(\mu) \), where \( \mu = \mu(m_1 v_\lambda) \) is the weight of \( m_1 v_\lambda \) w.r.t. \( \mathfrak{h}_0 \).

**Proof.** Suppose first that \( m_0 m_1 \in \mathcal{E}_s(\lambda) \). If \( m_1 \) is not essential for \( \lambda \), then

\[
m_1 v_\lambda = \sum_k A(k) a_0(k) a_1(k) v_\lambda
\]

for some \( A(k) \in \mathbb{C} \), some monomials \( a_0(k) \in \mathcal{U}(\mathfrak{n}_0^-) \) and \( a_1(k) \in \mathcal{U}(\mathfrak{r}) \), and \( a_1(k) < m_1 \) for all \( k \). In this case \( m_0 a_0(k) a_1(k) < m_0 m_1 \) for each \( k \) and hence \( m_0 m_1 \) is not essential, a contradiction.

If \( m_0 \notin \mathcal{E}_s(\mu) \), then

\[
m_0 p m_1 v_\lambda = \sum_k B(k) b_0(k) p m_1 v_\lambda
\]

for some \( B(k) \in \mathbb{C} \), some monomials \( b_0(k) \in \mathcal{U}(\mathfrak{n}_0^-) \), and we have \( b_0(k) < m_0 \) for each \( k \). Since \( m_1 v_\lambda \) is the leading term of \( p m_1 v_\lambda \) by (1.5), we conclude that \( m_0 m_1 \) is not essential, a contradiction.

Now we know that

\[
|\mathcal{E}_s(\lambda)| \leq \sum_{m_1 \in \Gamma(\lambda)} |\mathcal{E}_s(\mu(m_1 v_\lambda))| = \dim V(\lambda).
\]

Since also \( |\mathcal{E}_s(\lambda)| = \dim V(\lambda) \), we can conclude that each product \( m_0 m_1 \), where \( m_1 \) and \( m_0 \) are essential for \( \lambda \) and \( \mu \), respectively, is essential for \( \lambda \). This completes the proof. \( \square \)

**Remark 1.6.** One can also give a direct proof for the inclusion \( \mathcal{E}_s(\mu) \Gamma_{\varphi_{\mu_0}}(\lambda) \subset \mathcal{E}_s(\lambda) \) avoiding dimension reasons.

### 1.4. The Gelfand–Tsetlin order in type A

Here we show how effortlessly the FFLV method leads to a construction of the basis described in Theorem A.

Take \( \mathfrak{g} = \mathfrak{gl}_n \) and \( \mathfrak{g}_0 = \mathfrak{gl}_{n-1} \) that is the span of \( E_{ij} \) with \( 1 \leq i, j < n \). Then \( \mathfrak{r} \) is the linear span of \( E_{n,k} \) with \( 1 \leq k < n \). Note that \( [\mathfrak{r}, \mathfrak{r}] = 0 \). Hence the sequence of factors in \( m_1 \in \mathcal{U}(\mathfrak{n}^-) \) is of no significance. The \( \mathfrak{h}_0 \)-weights of \( E_{n,k} \) with \( 1 \leq k < n \) are linearly independent. If \( m_1 \neq \bar{m}_1 \) and \( p m_1 v_\lambda \neq 0 \), then \( p m_1 v_\lambda \neq p \bar{m}_1 v_\lambda \). The branching \( \mathfrak{gl}_n \rightarrow \mathfrak{gl}_{n-1} \) is multiplicity free, which is the key point of [10]. Given a highest weight \( \mu \) such that \( U(\lambda, \mu) = \text{Hom}_{\mathfrak{g}_0}(V'(\mu), V(\lambda)) \neq 0 \), there is a unique way to write the corresponding
\( m_1 \in \mathcal{E}s(\lambda) \), which exists by Proposition 1.2. Since the branching rules are well-known, the description of \( \Gamma(\lambda) \) results from Proposition 1.2 immediately. Write \( \lambda = (\lambda_1, \ldots, \lambda_n) \) with \( \lambda_k - \lambda_{k+1} \in \mathbb{Z}_+ \) for \( k = 1, \ldots, n-1 \).

**Corollary 1.7.** For each monomial order satisfying the assumptions of Section 1.2,

\[
\Gamma_{\mathfrak{gl}_n, \mathfrak{gl}_n-1}(\lambda) = \{ E_{\alpha_1}^{\alpha_1} \cdots E_{n,n-1}^{\alpha_{n-1}} \mid \alpha_k \leq \lambda_k - \lambda_{k+1} \}.
\]

Hence, the semigroup \( \Gamma_{\mathfrak{gl}_n, \mathfrak{gl}_n-1} \) is generated by the sets \( \{(\varpi_k, 1), (\varpi_k, E_{n,k})\} \) with \( 1 \leq k < n \) together with the 1-dimensional representations of \( \mathfrak{gl}_n \).

The central elements of \( \mathfrak{gl}_n \) act on \( L(\lambda) \) as scalars and any 1-dimensional representation of \( \mathfrak{sl}_n \) is trivial. Thereby the statement of Example 1.4 follows from Corollary 1.7. For the sake of brevity, one says also that \( \Gamma_{\mathfrak{sl}_n, \mathfrak{gl}_n-1} \) is generated by the fundamental weights or by \( \Gamma_{\mathfrak{sl}_n, \mathfrak{gl}_n-1}(\varpi_i) \).

An example of a suitable, i.e., satisfying (1:3), monomial order on \( \mathfrak{sl}(\tau) \) is the lexicographical order on \( E_{\alpha_1}^{\alpha_1} \cdots E_{n,n-1}^{\alpha_{n-1}} \), which is also the right lexicographical order on the tuples \( (\alpha_{n-1}, \ldots, \alpha_1) \).

The elements of \( \Gamma_{\mathfrak{gl}_n, \mathfrak{gl}_n-1}(\lambda) \) can be parameterised by the Gelfand–Tsetlin patterns \( \Lambda \), as defined in (0:3). Each such \( \Lambda \) corresponds to the monomial

\[
m_1(\Lambda) = E_{n,n-1}^{\lambda_n-1} \cdots E_{n,1}^{\lambda_1-1}.
\]

Arguing inductively with the use of Proposition 1.5, we restrict \( L(\lambda) \) further to \( \mathfrak{gl}_{n-2}, \mathfrak{gl}_{n-3} \), and so on. Taking the sequence of factors

\[
E_{2,1}^{\alpha_{3,1}} E_{3,1}^{\alpha_{3,2}} \cdots E_{n,1}^{\alpha_{n,1}} E_{n,n-1}^{\alpha_{n,n-1}}
\]

in \( \mathfrak{u}(\mathfrak{g}) \) and the lexicographical order at each step we obtain the basis of Theorem A. An alternative way to express this basis is to write

\[
\mathcal{E}s(\lambda) = \left\{ \prod_{i,j} E_{i,j}^{\alpha_{i,j}} \mid \alpha_{i,j} = \lambda_j - \lambda_{j+1} + \sum_{k=i+1}^n (\alpha_{k,j+1} - \alpha_{k,j}) \right\}.
\]

This is the set of inequalities given in [22, Introduction]. The same inequalities are used in [3, Sect. 6] for a description of a different, but related, basis.

The inductive argument shows also that the semigroup \( \Gamma = \Gamma_{\mathfrak{sl}_n, \{0\}} \) is generated by \( \Gamma(\varpi_k) \) with \( 1 \leq k < n \).

The next example is crucial for the symplectic case.

**Example 1.8.** Consider \( \mathfrak{gl}_{n-1} \subset \mathfrak{gl}_{n+1} \) embedded as the middle \( (n-1) \times (n-1) \)-square. For elements of \( \mathfrak{u}(\mathfrak{r}) \), we are using the following sequence of root vectors:

\[
\prod_{k=2}^n E_{n+1,k}^{\alpha_{n+1,k}} \prod_{k=2}^{n+1} E_{k,1}^{\alpha_{k,1}}.
\]
The monomial order is given by the right lexicographical order on the tuples
\[(\alpha_{n+1,n}, \ldots, \alpha_{n+1,2}, \alpha_{2,1}, \ldots, \alpha_{n+1,1}).\]

Here \(\Gamma_{gl_{n+1}\oplus gl_{n-1}}(\lambda) = \Gamma_{sl_{n+1}\oplus gl_{n-1}}(\lambda)\) is equal to
\[
\left\{ \prod_{k=2}^{n} E_{n+1,k}^{\alpha_{k+1,k}} \prod_{k=2}^{n+1} E_{k,k}^{\alpha_{k,k}} \mid \alpha_{k+1,1} \leq \lambda_k - \lambda_{k+1} \quad \text{and} \quad \alpha_{n+1,k} \leq \lambda_k - \lambda_{k+1} + \alpha_{k,1} - \alpha_{k+1,1} \right\}.
\]

The branching semigroup \(\Gamma_{gl_{n+1}\oplus gl_{n-1}}\) is generated by 1-dimensional representations of \(gl_{n+1}\) and by the essential monomials of the fundamental weights. Record that
\[
\begin{align*}
\Gamma_{gl_{n+1}\oplus gl_{n-1}}(\varpi_1) &= \{1, E_{2,1}, E_{n+1,2} E_{2,1}\}; \\
\Gamma_{gl_{n+1}\oplus gl_{n-1}}(\varpi_k) &= \{1, E_{k+1,1}, E_{n+1,k}, E_{n+1,k+1} E_{k+1,1}\} \quad \text{if} \ 2 \leq k < n; \\
\Gamma_{gl_{n+1}\oplus gl_{n-1}}(\varpi_n) &= \{1, E_{n+1,1}, E_{n+1,n}\}.
\end{align*}
\]

If we replace \(gl_{n+1}\) with \(sl_{n+1}\), then the 1-dimensional representations disappear from the generating set.

2. **Symplectic branching rules**

In this section we take \(g = sp_{2n}\) and use the presentation of the symplectic Lie algebra defined in the Introduction. The subalgebra \(g_0 = sp_{2n-2}\) is spanned by the elements \(F_{ij}\) with \(-n+1 \leq i, j \leq n-1\). Let \(\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+\) be the triangular decomposition, where \(\mathfrak{h}\) is the Cartan subalgebra of \(\mathfrak{g}\) with the basis \(\{F_{11}, \ldots, F_{nn}\}\), while the subalgebra \(\mathfrak{n}^+\) (resp., \(\mathfrak{n}^-\)) is spanned by the elements \(F_{ij}\) with \(i < j\) (resp., \(i > j\)). We have a vector space decomposition \(\mathfrak{n}^- = \mathfrak{n}_0^- \oplus \mathfrak{r}\), where \(\mathfrak{r} = \langle F_{n,i} \mid i < n \rangle_\mathbb{C}\) is a Heisenberg Lie algebra and \([\mathfrak{r}, \mathfrak{r}]\) is spanned by \(F_{n,-n}\). The elements from different pairs \((F_{n,i}, F_{i,-n}), (F_{n,j}, F_{j,-n})\) commute with each other and \([F_{n,i}, F_{i,-n}] = F_{n,-n}\), where \(F_{n,-n}\) is a central element of \(\mathfrak{r}\). Note that \([\mathfrak{g}_0, \mathfrak{r}] \subset \mathfrak{r}\).

2.1. **The Gelfand–Tsetlin-type order in the symplectic case.** We will describe a rather elaborate monomial order on \(S(\mathfrak{r})\) suggested by the structure of the branching semigroup of Example 1.8.

**Definition 2.1.** Define a monomial order on \(S(\mathfrak{r})\) by the following rule. The monomial
\[
F_{n,-n}^{\alpha_1} F_{n,-1}^{\alpha_2} F_{n,-2}^{\alpha_3} \cdots F_{n,-n+1}^{\alpha_n} F_{n,1}^{\alpha_{n+1}} \cdots F_{n,n-1}^{\alpha_{2n-1}}
\]
given by \(\bar{\alpha} = (\alpha_1, \ldots, \alpha_{2n-1})\) is smaller than the monomial given by \(\bar{\alpha}' = (\alpha'_1, \ldots, \alpha'_{2n-1})\) if and only if either \(\nu - \nu' \in \Delta^+\) for the \(\mathfrak{h}\)-weights \(\nu, \nu'\) of these monomials or \(\nu - \nu' \not\in \Delta\) and \(\bar{\alpha} < \bar{\alpha}'\) in the lexicographical order.

**Lemma 2.2.** Choose the sequence of factors in \(U(\mathfrak{r})\) as in (2.1). Then the monomial order of Definition 2.1 satisfies (1.3).

**Proof.** Since \([\mathfrak{g}_0 \oplus \mathfrak{r}, \mathfrak{r}] \subset \mathfrak{r}\), the statement follows from Lemma 1.1. \qed
Let \( \tilde{\Gamma} \) be the branching semigroup of \( \mathfrak{g} \downarrow \mathfrak{g}_0 \) defined by the sequence of root vectors as in (2.1) and the monomial order of Definition 2.2.

**Theorem 2.3.** The semigroup \( \tilde{\Gamma} \) is generated by the pairs \((\varpi_i, m_1)\), where \( \varpi_i \) is a fundamental weight and \( m_1 \in \tilde{\Gamma}(\varpi_i) \). Under a suitable identification, \( \tilde{\Gamma} \) is defined by the same inequalities as the semigroup \( \Gamma_{\mathfrak{sl}_{n+1} \downarrow \mathfrak{gl}_{n-1}} \) described in Example 1.8.

**Proof.** We use the bijection between the sets

\[
\{ F_{n,k} \mid -n \leq k < n, \ k \neq 0 \} \quad \text{and} \quad \{ E_{n+1,k}, E_{t,1} \mid 1 \leq k \leq n, \ 2 \leq t \leq n \}
\]

which takes \( F_{n,-n} \) to \( E_{n+1,1} \), the vector \( F_{n,-k} \) with \( 1 \leq k < n \) to \( E_{n+1,n-k+1} \), and \( F_{n,k} \) to \( E_{n+1-k,1} \). Using the same letters, \( \varpi_i \) for the fundamental weights of both \( \mathfrak{sp}_{2n} \) and \( \mathfrak{sl}_{n+1} \), we identify also the highest weights \( \lambda = \sum c_i \varpi_i \) of \( \mathfrak{sp}_{2n} \) and \( \mathfrak{sl}_{n+1} \). Then the standard branching theory assures that \( |\tilde{\Gamma}(\lambda)| = |\Gamma_{\mathfrak{sl}_{n+1} \downarrow \mathfrak{gl}_{n-1}}(\lambda)| \), see e.g. [19] and patterns in the Introduction. Since we have the property \( \tilde{\Gamma}(\lambda) \tilde{\Gamma}(\lambda') \subset \tilde{\Gamma}(\lambda + \lambda') \), see (1.1), it remains to show that the image of each \( \tilde{\Gamma}(\varpi_k) \) is exactly \( \Gamma_{\mathfrak{sl}_{n+1} \downarrow \mathfrak{gl}_{n-1}}(\varpi_k) \). The latter is presented in (1.6). Let \( v_1 \in V(\varpi_1) \) be a highest weight vector.

Take \( \varpi_1 \). Here \( |\tilde{\Gamma}(\varpi_1)| = 3 \). Notice that \( F_{n,n-1} v_1 \neq 0 \) is a highest weight vector of \( \mathfrak{g}_0 \) and that \( F_{n,n-1} \) is the smallest root vector in the monomial order. Therefore \( F_{n,n-1} \in E(\varpi_1) \).

This root vector is mapped to \( E_{2,1} \in \Gamma_{\mathfrak{sl}_{n+1} \downarrow \mathfrak{gl}_{n-1}}(\varpi_1) \). It remains to take care of the second copy of the trivial representation, which one obtains by applying either \( F_{n,-n} \) or \( F_{k,-n} F_{n,k} \) with \( 1 \leq k < n \) to \( v_1 \). The smallest monomial here is \( F_{n-1,-n} F_{n,n-1} \). Since \( F_{n-1,-n} \) is mapped to \( E_{n+1,2} \), we see that the image of \( \tilde{\Gamma}(\varpi_1) \) is exactly \( \Gamma_{\mathfrak{sl}_{n+1} \downarrow \mathfrak{gl}_{n-1}}(\varpi_1) \).

Take next \( \varpi_k \) with \( 2 \leq k < n \). Here \( |\tilde{\Gamma}(\varpi_k)| = 4 \). The root vectors \( F_{k,-n} \) and \( F_{k,n-1} \) are essential for \( \varpi_k \). The root vector \( F_{n,-n} \) is not essential, because it can be replaced by \( F_{n,k-n} F_{k,-n} \), which is smaller. We have also \( F_{n,i} v_k = 0 \) if \( n-k < i \leq n-1 \). Therefore, it remains to choose the smallest monomial among \( F_{n,t} F_{i,-n} \) with \( k-n \leq t \leq -1 \). This is exactly \( F_{k,n-k} F_{k,-n} \). Thus the image of \( \tilde{\Gamma}(\varpi_k) \) is \( \Gamma_{\mathfrak{sl}_{n+1} \downarrow \mathfrak{gl}_{n-1}}(\varpi_k) \).

Finally take \( \varpi_n \), where we have \( \tilde{\Gamma}(\varpi_n) = \{ 1, F_{n,-n}, F_{1,-n} \} \). Note that \( F_{n,-n} \) is mapped to \( E_{n+1,1} \) and \( F_{1,-n} \) to \( E_{n+1,n} \). This finishes the proof. \( \square \)

If a dominant weight \( \lambda = \sum_{k=1}^{n} c_k \varpi_k \) of \( \mathfrak{sp}_{2n} \) is presented by a tuple \((\lambda_1, \lambda_2, \ldots, \lambda_n)\) with \( 0 \geq \lambda_1 \geq \ldots \geq \lambda_n \) as in the Introduction, cf. (0.5), then \( c_1 = \lambda_{n-1} - \lambda_n \), for \( 2 \leq k < n \), we have \( c_k = \lambda_{n-k} - \lambda_{n-k-1} \), and \( c_n = -\lambda_1 \). Consistently, we write \( \mu = (\mu_1, \ldots, \mu_{n-1}) \) with \( 0 \geq \mu_1 \geq \ldots \geq \mu_{n-1} \). Taking this into account and using bijections between the branching semigroups and the corresponding patterns (Gelfand-Tsetlin patterns and type C patterns), we obtain the following statement.

**Corollary 2.4.** The vector space \( V(\lambda)_\mu^+ \) has a basis

\[
\left\{ p_{n,-n} F_{n,-n}^{\lambda_{n-1} - \nu_{n}} F_{n,n+1}^{\lambda_{n-1} - \nu_{n}} \ldots p_{n,k} F_{n,k}^{\lambda_{k-1} - \nu_{k}} F_{n,n+1}^{\lambda_{k} - \nu_{k+1}} \ldots F_{n,-1}^{\lambda_{1} - \nu_{n}} F_{n,1}^{\lambda_{1} - \nu_{2}} v_\lambda \right\}
\]
parameterised by the $n$-tuples $\nu = (\nu_1, \ldots, \nu_n)$ satisfying the betweenness conditions

\begin{align*}
0 \geq \nu_1 \geq \lambda_1 \geq \nu_2 \geq \lambda_2 \geq \cdots \geq \nu_{n-1} \geq \lambda_{n-1} \geq \nu_n \geq \lambda_n, \\
0 \geq \nu_1 \geq \mu_1 \geq \nu_2 \geq \mu_2 \geq \cdots \geq \nu_{n-1} \geq \mu_{n-1} \geq \nu_n.
\end{align*}

(2.2)

Going inductively through the chain of subalgebras

\begin{equation}
\mathfrak{sp}_2 \subset \ldots \subset \mathfrak{sp}_{2n-2} \subset \mathfrak{sp}_{2n}
\end{equation}

and using Proposition 1.5 at each step, we obtain the basis of Theorem B. The chain defines also the branching semigroup $\tilde{\Gamma}_{\mathfrak{sp}_{2n}, \nu(0)}$, where the order of Definition 2.1 and the sequence of factors (2.1) are used at each step.

Remark 2.5. Arguing inductively, one can show that $\tilde{\Gamma}_{\mathfrak{sp}_{2n}, \nu(0)}$ is generated by $\tilde{\Gamma}_{\mathfrak{sp}_{2n}, \nu(0)}(\varpi_k)$ with $1 \leq k \leq n$. This implies that $\tilde{\Gamma}_{\mathfrak{sp}_{2n}, \nu(0)}$ is saturated, i.e., $\tilde{\Gamma}_{\mathfrak{sp}_{2n}, \nu(0)}(N\lambda) = (\tilde{\Gamma}_{\mathfrak{sp}_{2n}, \nu(0)}(\lambda))^N$ for any $N \in \mathbb{N}$ and any dominant weight $\lambda$. In this situation, there is a nice toric degeneration of the complete flag variety in the spirit of [6, Sect. 15] and [9, Sect. 10].

2.2. A different, more natural, order. In this section, it is more convenient to use different indices for the matrix realisation of $\mathfrak{g} = \mathfrak{sp}_{2n}$. Now $\mathfrak{g}$ is the linear span of $F_{ij}$ with $i, j \in \{1, \ldots, 2n\}$, where

\begin{equation}
F_{ij} = E_{ij} - \varepsilon_i \varepsilon_j E_{i', j'}, \quad i' = 2n - i + 1,
\end{equation}

$\varepsilon_i = 1$ for $i \leq n$, and $\varepsilon_i = -1$ for $i > n$. The subalgebra $\mathfrak{g}_0 = \mathfrak{sp}_{2n-2}$ is spanned by the elements $F_{ij}$ with $i, j \in \{2, \ldots, 2n-1\}$. We have $r = \langle F_{2nk} \mid 1 \leq k < 2n \rangle$.

This alternative presentation of $\mathfrak{g}$ requires a change in the convention for tuples $\lambda = (\lambda_1, \ldots, \lambda_n)$. Now $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ unlike the Introduction. Fix highest weights $\lambda = (\lambda_1, \ldots, \lambda_n)$ for $\mathfrak{g}$ and $\mu = (\mu_2, \ldots, \mu_n)$ for $\mathfrak{g}_0$, where we suppose also that $\mu_2 \geq \mu_3 \geq \cdots \geq \mu_n \geq 0$. Assume that the multiplicity space $U(\lambda, \mu)$ is nonzero.

Set $a_i = |\lambda_i - \mu_i|$ for $i \geq 2$ and define the monomial $Y(\mu) = y_1^{a_1} \cdots y_2^{a_2}$ by the rule:

\begin{equation}
y_i = F_{i1} \text{ if } \lambda_i \leq \mu_i \quad \text{and} \quad y_i = F_{2ni} \text{ if } \lambda_i > \mu_i.
\end{equation}

Now use a non-zero vector $\xi_\mu \in V(\lambda)^+_\mu$ defined in formula [20, (9.69)], cf. (4.10). We need the existence of this vector and the computation of its weight w.r.t. to $F_{11}$. In the notation of this section, formula [20, (9.69)] leads to the following

\begin{equation}
F_{11} \xi_\mu = \left( \lambda_1 - \sum_{i=2}^{n} \{2 \max(\lambda_i, \mu_i) - \lambda_i - \mu_i\} \right) \xi_\mu = \left( \lambda_1 - \sum_{i=2}^{n} a_i \right) \xi_\mu.
\end{equation}

Hence, the $\mathfrak{h}$-weight of $\xi_\mu$ coincides with that of the vector $Y(\mu) v_\lambda$.

Remark 2.6. If $m_1 \in S(\mathfrak{r})$ is an eigenvector of $\mathfrak{h}$ of the same weight as $Y(\mu)$, then $m_1$ lies in $\mathbb{C}Y(\mu)$. Thus $\xi_\mu = p Y(\mu) v_\lambda$ and also $\xi_\mu = \prod_{i=2}^{n} (py_i)^{a_i} v_\lambda$, up to non-zero scalar factors.
We would like to find inequalities for \( b, b_n, \ldots, b_2 \) such that the corresponding vectors
\[
pF_{2n}^b (F_{2n} F_{n})^{a_n} \cdots (F_{2n} F_{21})^{a_2} Y(\mu) v_\lambda
\]
form a basis of \( V(\lambda)_\mu^+ \). For this purpose, the most natural monomial order on \( \mathfrak{S}(\tau) \) is suitable.

For a vector \( \bar{\alpha} = (\alpha_2, \ldots, \alpha_{2n-1}) \), set \( |\bar{\alpha}| = \sum_{k=2}^{2n-1} \alpha_k \).

**Definition 2.7.** We say that \( F_{2n}^{\alpha_2} F_{2n}^{\alpha_3} \cdots F_{2n}^{\alpha_{2n-1}} < F_{2n}^{\beta_2} F_{2n}^{\beta_3} \cdots F_{2n}^{\beta_{2n-1}} \) if and only if either \( |\bar{\alpha}| < |\bar{\beta}| \) or \( |\bar{\alpha}| = |\bar{\beta}| \) and there is \( k \) such that \( 2 \leq k \leq 2n \) and
\[
\alpha_k < \beta_k, \ \alpha_i = \beta_i \text{ for all } i < k.
\]

A few remarks on the definition are due.

(1) Since we are comparing the degrees first, the sequence of factors of \( m_1 \in \mathcal{U}(\tau) \) is not significant for being essential.

(2) Independently of the sequence of factors in \( \mathcal{U}(\tau) \), the chosen order satisfies (1-3). Therefore, by Proposition 1.2, the subspace \( V(\lambda)_\mu^+ \) has a basis \( \{pm_1 v_\lambda | m_1 \in \mathcal{E}(\lambda) \cap \mathcal{S}(\tau)\} \).

**Lemma 2.8.** We have
\[
\Gamma(\varpi_1) = \{1, F_{2n1}, F_{21}\},
\]
\[
\Gamma(\varpi_k) = \{1, F_{2n1}, F_{2nk}, F_{k+11}\} \text{ if } 2 \leq k < n,
\]
\[
\Gamma(\varpi_n) = \{1, F_{2n1}, F_{2nn}\}.
\]

**Proof.** The statements can be obtained by direct calculations. \( \square \)

The dimension of \( U(\lambda, \mu) \) is the product of \( n \) positive integers \( (d_1 + 1) \cdots (d_n + 1) \), where
\[
d_i = \min(\lambda_i, \mu_i) - \max(\lambda_{i+1}, \mu_{i+1}),
\]
assuming that \( \min(\lambda_1, \mu_1) = \lambda_1 \) and \( \lambda_{n+1} = \mu_{n+1} = 0 \); see e.g. [19].

Consider the \( sl_2 \)-triple \( \{\frac{1}{2} F_{2n1}, \frac{1}{2} F_{11}, \frac{3}{2} F_{12n}\} \). The subalgebra of \( \mathfrak{g} \) spanned by this triple acts on \( U(\lambda, \mu) \) as on \( S^{d_1} \mathbb{C}^2 \otimes \cdots \otimes S^{d_n} \mathbb{C}^2 \). Moreover,
\[
\xi \in V(\lambda)_\mu^+ \cong U(\lambda, \mu)
\]
is a highest weight vector of this representation and its \( F_{11} \)-weight is equal to \( d_1 + \cdots + d_n \).

For a vector \( Y = y_{n}^{a_n} \cdots y_{2}^{a_2} \), where each \( y_i \) is either \( F_{i1} \) or \( F_{2ni} \) and \( a_i \in \mathbb{Z}_{\geq 0} \) are arbitrary, set
\[
\iota_i = \begin{cases} 
0 & \text{if } y_i = F_{i1}, \\
1 & \text{if } y_i = F_{2ni}.
\end{cases}
\]
This defines a vector \( \iota = (\iota_2, \ldots, \iota_n) \), which depends on \( Y \). Set \( \iota_1 = 0 \) and \( a_{n+1} = 0 \).

We have
\[
\lambda = (\lambda_1 - \lambda_2) \varpi_1 + \cdots + (\lambda_{n-1} - \lambda_n) \varpi_{n-1} + \lambda_n \varpi_n.
\]
Set $c_n = \lambda_n$ and $c_k = (\lambda_k - \lambda_{k+1})$ for $k < n$. Suppose that $\xi_\mu = pYv_\lambda \neq 0$ for some $Y$ as above. It is not difficult to see then that $Y = Y(\mu)$ and

$$d_k = c_k - t_ka_k - (1 - t_{k+1})a_{k+1}$$

for each $k \geq 2$. Informally speaking, each $y_i$ in $Y$ decreases $c_k$ by 1 if $y_i \in \Gamma(\varpi_k)$. More formally, if $y_i \in \Gamma(\varpi_k)$, then $a_i \leq c_k$ and therefore $y_i^a_i \in \Gamma(\varpi_k)^c \subset c_k\varpi_k$. Thus, $Y \in \Gamma(\lambda)$. Note that Equation (2.5) defines the numbers $d_k = d_k(Y)$ for each vector $Y$ as above.

The next step is to consider $\Gamma(\varpi_k + \varpi_j)$ with $k \neq j$.

**Lemma 2.9.** Suppose that $j > k$ and $\lambda = \varpi_k + \varpi_j$. Then

$$\mathcal{E}_s(\lambda) \cap S(\tau)^{h_0} = \{1, F_{2n_1}, F_{2n_1}, F_{2n_j}F_{j1}\}.$$

**Proof.** Set $\mu = \lambda|_{h_0}$. Then $\dim U(\lambda, \mu) = 4$. As a representation of $\mathfrak{sl}_2 = (F_{2n_1}, F_{11}, F_{12n})_C$, it decomposes as $\mathbb{C}^3 \oplus \mathbb{C}$. Since $F_{2n_1} \in \Gamma(\varpi_i)$ for each $i$, we have $F_{2n_1}, F_{2n_1}^2 \in \mathcal{E}_s(\lambda)$. It remains to show that $F_{2n_j}F_{j1}$ is essential. In the case $k = j - 1$, this follows from the inclusion (1.1) and Lemma 2.8. Therefore suppose that $k < j - 1$. Then $\dim V(\lambda) = 1$ as one of the numbers 9, 12, and 16, depending on $k$ and $j$. In any case, $\Gamma(\lambda)$ is the disjoint union of three subsets, $X = \{1, F_{2n_1}, F_{2n_k}, F_{2n_j}, F_{k+1}, F_{j+1}\}$, the product $F_{2n_1}(X \setminus \{1\})$, and the subset

$$\{F_{2n_k}F_{2n_j}, F_{2n_k}F_{j+1}, F_{k+1}F_{2n_j}, F_{k+1}F_{j+1}, x\},$$

where $pxv_\lambda \in V(\lambda)_0$ and the $F_{11}$-weight of $x$ is $-2$. Since $F_{2n_1} \in X$, these two conditions on $x$ imply that $x = F_{2n_1}F_{11}$ for some $t \leq n$.

First we show that $t \leq j$. If $j < n$, take $s > j$. Let us regard $V(\varpi_r)$ as a subspace of $\wedge^r \mathbb{C}^{2n}$, where $\mathbb{C}^{2n} = V(\varpi_1)$ is the underlying vector space of the defining representation of $g$. Let $\{v_1, \ldots, v_{2n}\}$ be the standard basis of $\mathbb{C}^{2n}$. Then $v_r = v_1 \wedge \ldots \wedge v_r$ is a highest weight vector of $V(\varpi_r)$. Set $u = F_{2n_k}s_1(v_k \otimes v_j)$. Then $u = \frac{1}{2}F_{2n_1}(v_k \otimes v_j) + u'$, where

$$u' = (v_s \wedge v_2 \wedge \ldots \wedge v_k) \otimes (v_s' \wedge v_2 \wedge \ldots \wedge v_j) + (v_s' \wedge v_2 \wedge \ldots \wedge v_k) \otimes (v_s \wedge v_2 \wedge \ldots \wedge v_j).$$

Here $s' > n \geq s$ and $u' = \frac{1}{2}F_{s'}s\tilde{u}$ for

$$\tilde{u} = (v_s \wedge v_2 \wedge \ldots \wedge v_k) \otimes (v_s \wedge v_2 \wedge \ldots \wedge v_j).$$

Thereby $pu' = 0$ by (1.2), hence $pu = \frac{1}{2}F_{2n_1}(v_k \otimes v_j)$ and $F_{s_1}F_{2n_1}$ is not essential for $\varpi_k + \varpi_j$. We have shown that $x \geq F_{j1}F_{2n_j}$.

Assume that $F_{j1}F_{2n_j}F_{j+1}$ is not essential. Then $w = F_{j1}F_{2n_j}(v_k \otimes v_j)$ lies in the linear span of smaller than $F_{j1}F_{2n_j}$ essential monomials. Each such monomial is of the form $m_0m_{11}$, where $m_1$ has weight $-2$ w.r.t. $F_{11}$ and $m_1 < F_{j1}F_{2n_j}$. This is possible only for $F_{2n_1}, F_{2n_1}F_{k+1},$ and $F_{j+1}F_{k+1}$. 
The decomposition $V(\varphi_1) = \mathbb{C}v_1 \oplus V'(\varphi_1) \oplus \mathbb{C}v_{2n}$ leads to a $\mathfrak{g}_0$-invariant tri-grading on each $V(\varphi_\tau)$. In the tensor product $V(\varphi_k) \otimes V(\varphi_j)$, the vector $F_{2n_j} F_{j+1} (v_k \otimes v_j)$ has non-zero summands of degrees

$$(0, k - 1, 1; 1, j - 1, 0), \quad (0, k, 0; 1, j - 2, 1), \quad (0, k, 0; 0, j, 0).$$

The monomials $F_{2n_j} F_{k+1}$ and $F_{j+1} F_{k+1}$ produce vectors of degrees

$$(0, k, 0; 1, j - 2, 1), \quad (0, k, 0; 0, j, 0), \quad \text{and} \quad (0, k, 0; 0, j, 0).$$

This implies that the summand of degree $(0, k - 1, 1; 1, j - 1, 0)$, which is equal to

$$w = (v_{2n} \land v_2 \land \ldots \land v_k) \otimes (v_1 \land \ldots \land v_j),$$

is written as $a m_0 F_{2n_1} (v_k \otimes v_j)$ for some $a \in \mathbb{C}$ and $m_0 \in \mathcal{U}(\mathfrak{n}_0^-)$. However, $F_{2n_1} (v_k \otimes v_j) = 2(w + \bar{w})$, where $\bar{w} \neq 0$ is of degree $(1, k - 1, 0; 0, j - 1, 1)$. This contradiction finishes the proof.

\(\square\)

**Proposition 2.10.** (i) The defining inequalities for $\Gamma(\lambda)$ in terms of

$$F_{2n_1}^b (F_{2n_2} F_{n_1})^{b_n} \ldots (F_{2n_2} F_{21})^{b_2} y_n^{a_n} \ldots y_2^{a_2}$$

are:

\begin{align*}
(2.6) & \quad 0 \leq d_k, \quad \text{where the numbers } d_k \text{ are given by } (2.5), \\
(2.7) & \quad b_k \leq d_k, \\
(2.8) & \quad b_k \leq d_1 + \sum_{i=2}^{k-1} (d_i - 2b_i) \text{ for each } k \text{ such that } 2 \leq k \leq n; \\
(2.9) & \quad b + 2 \sum_{k=2}^{n} b_k \leq \sum_{i=1}^{n} d_i.
\end{align*}

(ii) The semigroup $\Gamma$ is generated by $\Gamma(\varphi_t)$ and $\Gamma(\varphi_k + \varphi_j)$ with $1 \leq t, k, j \leq n$ and $j > k + 1$.

**Proof.** (i) The inequalities (2.6) are equivalent to $U(\lambda, \mu) \neq 0$, where $\mu$ is the $\mathfrak{n}_0$-weight of $y_n^{a_n} \ldots y_2^{a_2} v_\lambda$. Each weight $\mu$ such that $U(\lambda, \mu) \neq 0$ defines the tuple $\bar{a} = (a_2, \ldots, a_n)$ uniquely. Let $\bar{a}$ be fixed.

Next we show that the number of tuples $(b, b_n, \ldots, b_2) \in \mathbb{Z}_+^n$ satisfying the inequalities (2.7)–(2.9) is equal to $\prod_{i=1}^{n} (d_i + 1) = \dim U(\lambda, \mu)$. We argue by induction on $n$. If $n = 1$, then there is just one inequality $b \leq d_1$. There are $d_1 + 1$ possibilities for $b$.

Suppose that $n = 2$. Then $b_2 \leq d_1, d_2$. Each admissible $b_2$ corresponds to the irreducible $\mathfrak{sl}_2$-submodule of $S^{d_1} \mathbb{C}^2 \otimes S^{d_2} \mathbb{C}^2$ of dimension $d_1 + d_2 + 1 - 2b_2$. If $b_2$ is fixed, then there are exactly $d_1 + d_2 - 2b_2 + 1$ possibilities for $b$. For $n = 2$, the number of tuples $(b, b_2)$ is correct.
Suppose now that \( n > 2 \) and that for \( n - 1 \) there is a bijection between the tuples \( \vec{b} = (b_2, \ldots, b_{n-1}) \) satisfying the inequalities and the irreducible \( sl_2 \)-submodules of 

\[
S^{d_1}C^2 \otimes \ldots \otimes S^{d_{n-1}}C^2
\]

such that the module \( V(\vec{b}) \) corresponding to \( \vec{b} \) is of dimension

\[
\sum_{i=1}^{n-1} d_i + 1 - 2 \sum_{i=2}^{n-1} b_i.
\]

The irreducible submodules of \( V(\vec{b}) \otimes S^{d_n}C^2 \) can be enumerated by integers \( b_n \) such that

\[
0 \leq b_n \leq \min(d_n, \dim V(\vec{b}) - 1).
\]

We can arrange the submodules in such a way that the dimension decreases when \( b_n \) increases. Then \( b_n \), or rather \( (b_2, \ldots, b_{n-1}, b_n) \), corresponds to the summand of dimension

\[
\sum_{i=1}^{n} d_i + 1 - 2 \sum_{i=2}^{n} b_i.
\]

This completes the inductive argument.

In the proof of part (ii) below, we show that each admissible tuple

\[
(a_2, \ldots, a_n, b, b_2, \ldots, b_n)
\]

defines a monomial of \( \Gamma(\lambda) \). Hence by the dimension reasons, (i) holds.

(ii) For convenience, we will identify the monomials \( m_1 \in \Gamma(\lambda) \) with the tuples of their exponents and use additive notation for \( \Gamma(\lambda) \), so that \( \Gamma(\lambda) + \Gamma(\lambda') = \Gamma(\lambda + \lambda') \); see (1.1).

Let \( (\vec{b}, \vec{a}) \) with \( \vec{b} = (b, b_2, \ldots, b_n), \vec{a} = (a_2, \ldots, a_n) \) be an admissible tuple. Recall that each \( y_i \) belongs to a unique \( \Gamma(\varpi_s) \) with \( s = s(i) \). Set \( \tilde{\lambda} = \lambda - \sum_{i=2}^{n} a_i \varpi_{s(i)} \). In view of (2.5), we have \( \tilde{\lambda} = \sum_{i=1}^{n-1} d_i \varpi_i \). The inequalities (2.6) guarantee that \( \tilde{\lambda} \) is a dominant weight of \( \mathfrak{g} \). If \( \vec{b} \), identified with \( (\vec{b}, \vec{0}) \), lies in \( \Gamma(\tilde{\lambda}) \), then

\[
(\vec{b}, \vec{a}) \in \Gamma(\tilde{\lambda}) + \sum_{i=2}^{n} a_i \Gamma(\varpi_{s(i)}) \subset \Gamma(\lambda).
\]

Next we express \( \vec{b} \) as a sum of tuples belonging to sets \( \Gamma(\varpi_i) \) and \( \Gamma(\varpi_k + \varpi_j) \) and show that indeed \( \vec{b} \in \Gamma(\tilde{\lambda}) \).

If all \( d_k \) are zero, then \( \vec{b} = 0 \) and there is nothing to prove. Suppose next that \( d_k \neq 0 \) only for \( k = i \). Then \( b_j = 0 \) for all \( j \geq 2 \) and only \( F_{2n} b \) with \( b \leq d_i \) is left. Here \( b \in d_i \Gamma(\varpi_i) \).

The proof continues by induction on \( |\vec{b}| = b + b_2 + \ldots + b_n \).
Let \(k < r\) be the smallest integers such that \(d_r, d_k \neq 0\). Note that \(b_2 = \ldots = b_{r-1} = 0\). If all \(b_i\) with \(i \geq 2\) are equal to zero, then \(\tilde{b} = (b, 0, \ldots, 0)\), where \(b \leq \sum_{i=1}^n d_i\). Again, such \(\tilde{b}\) belongs to \(\sum_{i=1}^n d_i \Gamma(\varpi_i) \subset \Gamma(\tilde{\lambda})\). Therefore assume that \(\tilde{b} \neq (b, 0, \ldots, 0)\).

Let \(j \geq r\) be the smallest integer such that \(b_r \neq 0\). We divide our monomial by \(F_{2n}F_{11}\), which is an element of \(\Gamma(\varpi_k + \varpi_j)\) by Lemma 2.9. Note that in case \(j = k + 1\), we have \(F_{2n}F_{11} \in \Gamma(\varpi_j)\). The division corresponds to replacing \(\tilde{b}\) with \(\tilde{b}' = (b, b_2', \ldots, b_n')\), where \(b'_i = b_i\) for \(i \neq j\) and \(b'_j = b_j - 1\). Accordingly, set \(\lambda' = \tilde{\lambda} - (\varpi_k + \varpi_j)\). We have \(\lambda' = \sum_{i=1}^n d_i' \varpi_i\), where \(d'_i = d_i\) for \(i \neq k, j\) and \(d'_i = d_i - 1\) for \(i \in \{k, j\}\). The next task is to see that the inequalities (2.7)–(2.9) hold for \(\tilde{b}'\) and \(\lambda'\).

Consider (2.7). For \(i \neq k, j\), we have \(b'_i = b_i \leq d_i = d'_i\). If \(k = 1\), then there is no \(b_k\). If \(k \geq 2\), then \(b'_k = b_k = 0\) and \(b_k \leq d'_k\). Finally, \(b'_j = b_j - 1 \leq d_j - 1 = d'_j\). These inequalities hold.

Consider (2.8). For \(s < j\), we have \(b_s = 0\). Clearly, the inequalities hold for all such \(s\). For the index \(j\), we have

\[
b'_j \leq \left(\sum_{t=1}^{j-1} d_t\right) - 1 = \sum_{t=1}^{j-1} d'_t = d'_1 + \sum_{t=2}^{j-1} (d'_t - 2b'_t).
\]

For \(s > j\), the new right hand side \(d'_1 + \sum_{t=2}^{s-1} (d'_t - 2b'_t)\) is equal to the old one. Since \(b'_s = b_s\) here, all the inequalities hold.

Finally, consider (2.9). We have

\[
\sum_{i=1}^n d'_i - 2 \sum_{t=2}^n b'_t = \sum_{i=1}^n d_i - 2 \sum_{t=2}^n b_t.
\]

Hence the inequality for \(\tilde{b}\) holds.

Summing up, \(\tilde{b}'\) belongs to \(\Gamma(\lambda')\), because \(|\tilde{b}'| < |\tilde{b}|\), and hence \(\tilde{b}\) belongs to \(\Gamma(\tilde{\lambda})\).

The perspective on \(U(\lambda, \mu)\) taken in this section differs from the usual one. In order to obtain a basis, we have regarded \(U(\lambda, \nu)\) as a direct sum of \(\mathfrak{sl}_2\)-modules instead of a tensor product. On the one side, this leads to a more complicated set of inequalities, on the other, we are getting one more basis.

Set \(\tilde{p} = p_{\mathfrak{sl}_2}p\), where \(p_{\mathfrak{sl}_2}\) is the extremal projector associated with \(\mathfrak{sl}_2 = \langle F_{2n+1}, F_{11}, F_{12n}\rangle_{\mathbb{C}}\) and \(p\) is the projector of \(\mathfrak{g}_0\) as before. Let us restrict \(V(\lambda)\) to \(\mathfrak{g}_0 \oplus \mathfrak{sl}_2\).

**Corollary 2.11.** The subspace \(V(\lambda)^+ \cap V(\lambda)^{F_{2n}}\) has a basis

\[
\{\tilde{p} m_1 v_\lambda \mid m_1 \in \Gamma(\lambda) \text{ is given by exponents } (0, b_2, \ldots, b_n, a_2, \ldots, a_n)\},
\]

i.e., we are taking the subset, where \(b = 0\).
The chain of subalgebras (2.3) can be used in order to extend the basis of Proposition 2.10 to a basis for \( V(\lambda) \).

3. Relations to the Gelfand–Tsetlin basis

We start by recalling a construction of the celebrated basis of Gelfand and Tsetlin [10] for each finite-dimensional irreducible representation \( L(\lambda) \) of \( \mathfrak{gl}_n \) as defined in the Introduction. We refer the reader to the review paper \([19]\) where several such constructions are discussed. To be consistent with the notation of that paper we will now let \( \xi \) denote the highest weight vector of \( L(\lambda) \) (along with \( v_\lambda \)).

Consider the extremal projector \( p \) associated with the Lie algebra \( \mathfrak{gl}_{n-1} \). Recall that the Mickelsson–Zhelobenko algebra \( Z(\mathfrak{gl}_n, \mathfrak{gl}_{n-1}) \) is generated by the elements \( E_{n,n}, pE_{i,n} \) and \( pE_{n,i} \) with \( i = 1, \ldots, n-1 \); see \([19, \text{Sect. 2.3}]\) for the definitions. The lowering operators are elements of the universal enveloping algebra \( U(\mathfrak{gl}_n) \) which can be defined by the formulas

\[
z_{n,k} = pE_{n,k} (h_k - h_{k+1}) \cdots (h_k - h_{n-1}),
\]

where \( h_k = E_{kk} - k + 1 \). By the branching rule, the restriction of \( L(\lambda) \) to the subalgebra \( \mathfrak{gl}_{n-1} \) is isomorphic to the direct sum of irreducible \( \mathfrak{gl}_{n-1} \)-modules \( L'(\mu) \),

\[
L(\lambda)|_{\mathfrak{gl}_{n-1}} \simeq \bigoplus_{\mu} L'(\mu),
\]

summed over the highest weights \( \mu = (\mu_1, \ldots, \mu_{n-1}) \) satisfying the betweenness conditions

\[
\lambda_i - \mu_i \in \mathbb{Z}_+ \quad \text{and} \quad \mu_i - \lambda_{i+1} \in \mathbb{Z}_+ \quad \text{for} \quad i = 1, \ldots, n-1.
\]

The \( \mathfrak{gl}_{n-1} \)-submodule in \( L(\lambda) \) isomorphic to \( L'(\mu) \) is generated by the vector

\[
\xi_\mu = z_{\mu_1}^{\lambda_1-\mu_1} \cdots z_{\mu_{n-1}}^{\lambda_{n-1}-\mu_{n-1}} \xi.
\]

In the next lemma we suppose that each of the highest weights \( \mu \) and \( \mu' \) satisfies conditions (3.2) and we use the lexicographical ordering \( \succ \) on such weights, where for complex numbers \( a \) and \( b \) we assume that \( a \succ b \) if and only if \( a - b \in \mathbb{Z}_+ \).

**Lemma 3.1.** For any given \( \mu \), in the module \( L(\lambda) \) we have

\[
E_{\lambda_1-\mu_1} \cdots E_{\lambda_{n-1}-\mu_{n-1}} \xi = c \xi_\mu + \sum_{\mu' \succ \mu} u(\mu') \xi_{\mu'}
\]

for a nonzero constant \( c \) and some elements \( u(\mu') \in U(\mathfrak{n}_0^-) \), where the sum is taken over the highest weights \( \mu' \) satisfying conditions (3.2).

**Proof.** Starting from the rightmost generator which occurs in the product on the left hand side and proceeding to the left, we use the inversion formula

\[
E_{n,k} = pE_{n,k} + \sum_{k < k_1 < \cdots < k_s < n} E_{k_1,k} E_{k_2,k_1} \cdots E_{k_s,k_{s-1}} pE_{n,k_s} \frac{1}{(h_{k_s} - h_k)(h_{k_s} - h_{k_1}) \cdots (h_{k_s} - h_{k_{s-1}})},
\]
summed over \( s = 1, 2, \ldots \). Arguing by induction, observe that each generator \( E_{nl} \) with \( l \leq k \) commutes with all factors \( E_{k1k}, E_{k2k_1}, \ldots, E_{k_kk_{k-1}} \) so that the proof is completed by using (3.1) and taking into account the fact that the lowering operators \( z_{nk} \) pairwise commute.

The vectors \( \xi_\Lambda \) of the Gelfand–Tsetlin basis \( \{\xi_\Lambda\} \) of \( L(\lambda) \) are parameterised by the patterns \( \Lambda \) defined in the Introduction. They are found by the formula

\[
(3.3) \quad \xi_\Lambda = \prod_{k=2,\ldots,n} \left( z_{k1}^{\lambda_{k-1} - \lambda_{k-11}} \cdots z_{k-k-1}^{\lambda_{k-1k-1} - \lambda_{k-1k-1}} \right) \xi.
\]

Represent each pattern \( \Lambda \) associated with \( \lambda \) as the sequence of its rows:

\[
\Lambda = (\lambda_{n-1}, \ldots, \lambda_1), \quad \bar{\lambda}_k = (\lambda_{k1}, \ldots, \lambda_{kk}),
\]

and consider the lexicographical ordering \( \succ \) on the sequences by using the ordering on the highest weights introduced above. Recall the vectors \( \pi_\Lambda \) defined in Theorem A. We now obtain a proof of this theorem.

**Proposition 3.2.** For each pattern \( \Lambda \) associated with \( \lambda \), in the module \( L(\lambda) \) we have

\[
\pi_\Lambda = \sum_{\Lambda' \succ \Lambda} c_{\Lambda,\Lambda'} \xi_{\Lambda'} \quad \text{and hence} \quad \xi_\Lambda = \sum_{\Lambda' \succ \Lambda} d_{\Lambda,\Lambda'} \pi_{\Lambda'},
\]

for some constants \( c_{\Lambda,\Lambda'} \) and \( d_{\Lambda,\Lambda'} \), whereby \( c_{\Lambda,\Lambda} = d_{\Lambda,\Lambda}^{-1} \neq 0 \).

**Proof.** Due to the inductive structure of the vectors (3.3), the proposition follows by a repeated application of Lemma 3.1. \( \square \)

### 3.1. The PBW-parameterisation of the canonical basis.

The canonical basis for \( V(\lambda) \) constructed by Lusztig [15, 16] has a PBW-parameterisation, which fits into the FFLV-framework.

Let \( \omega_0 = s_{i_1} \cdots s_{i_N} \) be a reduced decomposition of the longest element \( \omega_0 \in W(\mathfrak{g}, \mathfrak{h}) \) of the Weyl group. Define the sequence of positive roots \( \beta_1, \ldots, \beta_N \) by \( \beta_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \), where \( \alpha_r \) is the \( r \)-th simple root. Then \( \beta_t \neq \beta_k \) for \( k \neq t \), see e.g. [6, Sect. 12]. Let \( f_k \) be the negative root vector corresponding to \( \beta_k \). Make use of the right opposite lexicographical order on the monomials \( f_1^{a_1} \cdots f_N^{a_N} \), which means that \( f_1^{a_1} \cdots f_N^{a_N} < f_1^{a'_1} \cdots f_N^{a'_N} \) if and only if there is \( k \) such that \( 1 \leq k \leq N \) and

\[
a_k > a'_k, \quad a_r = a'_r \quad \text{for} \quad r > k.
\]

Use the same sequence of vectors for the elements of \( \mathfrak{u}(\mathfrak{n}^-) \). Then the elements of the canonical basis for \( V(\lambda) \) are in bijection with \( \mathfrak{E}s(\lambda) \). Moreover, if the element \( B(m)v_\lambda \) of the canonical basis corresponds to \( m \in \mathfrak{E}s(\lambda) \), then

\[
(3.4) \quad B(m)v_\lambda \in mv_\lambda + \langle \tilde{m}v_\lambda | \tilde{m} < m \rangle_\mathbb{C},
\]
see e.g. [6, Sect. 12]. Note that we have omitted the “height weighted function $\Psi$” of
[6] on the monomials, because it becomes redundant once one fixes a finite-dimensional
module $V(\lambda)$.

Let $B_\lambda$ be the canonical basis of $V(\lambda)$. Then the dual basis $B_\lambda^* \subset V(\lambda^*)$ is good in the
terminology of [3] by [16, Theorem 4.4].

Example 3.3. Let $g$ be of type $A_{n-1}$. Choose the decomposition $\omega_0 = s_1s_2s_1 \ldots s_{n-1} \ldots s_2s_1$.
Then
$$f_1 \ldots f_N = E_{21}E_{31}E_{32} \ldots E_{n1}E_{n2} \ldots E_{nn-1}.$$ 

The right opposite lexicographical order satisfies the assumptions of Section 1.2 at each
step of the reductions along the Gelfand–Tsetlin chain of subalgebras. Therefore, we get
the basis $\{\pi_\Lambda\}$ described in Theorem A. Note that this basis was obtained in [22] for the
same $\omega_0$ as above.

Keep the assumption $g = gl_n$. By the weight considerations, we have

$$\xi_\mu = d_\mu pE_{n1}^{\lambda_1-\mu_1} \ldots E_{n-n-1}^{\lambda_{n-1}-\mu_{n-1}}v_\lambda$$

for some $d_\mu \in \mathbb{C}^\times$.

Corollary 3.4. For each dominant $\lambda$, the transition matrix between the canonical and the Gelfand–
Tsetlin bases of $L(\lambda)$ is triangular.

Proof. Let $\Lambda$ be a Gelfand–Tsetlin pattern associated with $\lambda$. Consider $\pi_\Lambda = mv_\lambda$, where
$m = m(\Lambda)$ is the same as in Theorem A.

If we use the right opposite lexicographical order as above, then (1.5) holds for all re-
duction steps along the Gelfand–Tsetlin chain of subalgebras. For each step, the analogue
of (3.5) holds as well. Therefore $\pi_\Lambda$ is the leading term of $c_{\Lambda,\lambda}v_\lambda$. In view of (3.4), $\pi_\Lambda$ is also
the leading term of $B(m)\xi$.

Remark 3.5. In the case $n = 3$ the canonical basis is monomial [15, Example 3.4]. Thereby
this particular case of Corollary 3.4 follows by a simple calculation with the use of the
Gelfand–Tsetlin formulas.

Proposition 3.6. There is an enumeration of the elements $\xi_\lambda$ such that the transition matrix
between $B_\lambda^* \subset L(\lambda)$ and $\{\xi_\lambda\}$ is triangular.

Proof. The dual basis $\{\xi_\lambda^*\} \subset L(\lambda^*)$ is the Gelfand–Tsetlin basis of $L(\lambda^*)$ up to a permuta-
tion of its elements and multiplications by non-zero scalars. By Corollary 3.4, the transi-
tion matrix between $B_\lambda^*$ and $\{\xi_\lambda^*\}$ is triangular. Therefore $B_\lambda^*$, and $\{\xi_\lambda\}$ are related by a
triangular matrix as well.

Outside type A, these PBW-type bases become less transparent, see e.g. [22].
Example 3.7. Let \( \mathfrak{g} \) be of type \( C_n \). Choose the decomposition

\[
\omega_0 = s_n s_{n-1} s_n \cdots s_1 s_2 \cdots s_{n-1} s_n s_{n-1} \cdots s_2 s_1.
\]

Then \( f_k \in \mathfrak{sp}_{2n-2} \) for \( k \leq N - 2n + 1 \) and

\[
f_{N-2n+2} \cdots f_N = F_{2n-2} \cdots F_{2n} F_{2n+1} \cdots F_{2n-1}.
\]

It is not difficult to see that such a choice produces a branching semigroup related to \( \mathfrak{sp}_n \downarrow \mathfrak{sp}_{2n-2} \) and that this semigroup is the same as in Section 2.2.

3.2. Monomials in simple root vectors. The bases of Littelmann \([14]\) arise as different parameterisations of the canonical basis. His construction involves branching and produces a basis of \( V(\lambda) \) by applying iterated negative simple root vectors to \( v_\lambda \), see \([14]\) and also \([6, \text{Sect. 11}]\) for a connection with the FFLV-method. In type A, the construction is most transparent \([13], [14, \text{Sect. 5&10}]\).

Set \( f_k = E_{k+1} \). The subspace \( L(\lambda)^+ \) is the linear span of vectors \( pf_{n-1}^{a_{n-1}} \cdots f_1^{a_1} v_\lambda \), where \( a_{n-1} > a_{n-2} \geq \cdots \geq a_1 \). Set \( a_0 = 0 \). By the weight considerations,

\[
\xi_\mu = pf_{n-1}^{a_{n-1}} \cdots f_1^{a_1} v_\lambda \text{ with } a_k - a_{k-1} = \lambda_k - \mu_k,
\]

up to a non-zero scalar. In view of the equality

\[
[f_{n-1}, f_{n-2}, \ldots, f_{k+1}, f_k] = E_{nk},
\]

we can conclude directly, without weight arguments, that

\[
pf_{n-1}^{a_{n-1}} \cdots f_1^{a_1} v_\lambda = p E_{n-1}^{a_{n-1}} \cdots E_1^{a_1} v_\lambda.
\]

A basis of \( L(\lambda) \) is obtained inductively, omitting extremal projectors, so that the basis vectors have the form

\[
f(\Lambda) = f_{1}^{a_{1}^{-1}} f_{2}^{a_{2}^{-2}} f_{1}^{a_{1}^{-1}} \cdots f_{n-1}^{a_{n-1}^{-1}} f_{1}^{a_{1}^{-1}} v_\lambda
\]

and are naturally parameterised by the Gelfand–Tsetlin patterns, see \([14, \text{Corollary 5}]\). In the notation of (0·3),

\[
a_{k,j} = \sum_{i=1}^{j} (\lambda_{n-k+1} i - \lambda_{n-k} i).
\]

Let \( m(\Lambda) \) be the leading term of \( f(\Lambda) \) in the monomial order used in Section 1.4. Combining (1·5) with (3·6), we see that \( m(\Lambda) \xi = \pi_\lambda \) and that again \( \pi_\lambda \) is the leading term of \( c_{\Lambda,\Lambda} \xi_\lambda \). Summing up,

\[
f(\Lambda) \in \pi_\lambda + \langle m \xi \mid m < m(\Lambda) \rangle_C = c_{\Lambda,\Lambda} \xi_\lambda + \langle m \xi \mid m < m(\Lambda) \rangle_C
\]

with a non-zero \( c_{\Lambda,\Lambda} \in \mathbb{C} \). Therefore, the transition matrices between all three bases are triangular.
Remark 3.8. Relations between different monomial bases parameterising the canonical basis are studied in [5]. The fact that the bases \( \{ f(\Lambda) \} \) and \( \{ \pi_\Lambda \} \) are related by a unitary matrix can be deduced from the results of that paper.

Remark 3.9. Let \( \succ \) be the lexicographical order on \( \mathbb{Z}^N \). Choose the enumeration of the basis vectors \( f(\Lambda) \) is such a way that the corresponding sequences

\[ \bar{a} = \bar{a}(\Lambda) = (a_{n-1,1}, a_{n-2,2}, a_{n-2,1}, \ldots, a_{1,1}), \]

see (3.7), are decreasing w.r.t. the order \( \succ \). Then the transition matrix between \( \{ f(\Lambda) \} \) and the canonical basis of \( L(\lambda) \) is upper triangular and unipotent by [14, Prop. 10.3]. Refining the above considerations, one can show that

\[ c_{\Lambda,\Lambda} \xi_\Lambda \in f(\Lambda) + \langle f(\Lambda') \mid \bar{a}(\Lambda') \succ \bar{a} \rangle _c \]

and thus produce a different proof of Corollary 3.4.

4. A Gelfand–Tsetlin-type basis for representations of \( \mathfrak{sp}_{2n} \)

We now aim to prove an analogue of Proposition 3.2 for the symplectic Lie algebra \( \mathfrak{sp}_{2n} \). The vectors \( \theta_\Lambda \) defined in Theorem B turn out to be related to a certain modification of the basis of [18]. In this section we will rely on the exposition in [20, Ch. 9] to produce this modification.

Given a type C pattern \( \Lambda \) associated with \( \lambda \), as defined in the Introduction, set

\[ l_{ki} = \lambda_{ki} - i - \frac{1}{2}, \quad l'_{ki} = \lambda'_{ki} - i + \frac{1}{2}. \]

**Theorem 4.1.** The \( \mathfrak{sp}_{2n} \)-module \( V(\lambda) \) admits a basis \( \zeta_\Lambda \) parameterised by the type C patterns \( \Lambda \) associated with \( \lambda \) such that the action of generators of \( \mathfrak{sp}_{2n} \) in the basis is given by the formulas

\[
F_{kk} \zeta_\Lambda = \left( \sum_{i=1}^{k} \lambda_{ki} + \frac{1}{2} \sum_{i=1}^{k-1} \lambda_{k-1,i} - 2 \sum_{i=1}^{k} \lambda'_{ki} \right) \zeta_\Lambda, \\
F_{k,-k} \zeta_\Lambda = \sum_{i=1}^{k} A_{ki} \zeta_{\Lambda - \delta_{ki}}, \\
F_{-k,k} \zeta_\Lambda = \sum_{i=1}^{k} B_{ki} \zeta_{\Lambda + \delta'_{ki}}, \\
F_{k-1,-k} \zeta_\Lambda = -\sum_{i=1}^{k-1} C_{ki} \zeta_{\Lambda + \delta_{k-1,i}} - \sum_{i=1}^{k} \sum_{j,m=1}^{k-1} D_{kijm} \zeta_{\Lambda - \delta'_{k-1,i} - \delta_{k-1,j} - \delta_{k-1,m}},
\]
where
\[
A_{ki} = \prod_{a=1, a\neq i}^{k} \frac{1}{l_{ka} - l'_{ka}},
\]
\[
B_{ki} = 2 A_{ki} \left( 2 l'_{ki} + 1 \right) \prod_{a=1}^{k} \left( l_{ka} - l'_{ka} \right) \prod_{a=1}^{k-1} \left( l_{k-1a} - l'_{k-1a} \right),
\]
\[
C_{ki} = \frac{1}{2 l_{k-1i} + 1} \prod_{a=1, a\neq i}^{k-1} \left( l_{k-1i} - l_{k-1a} \right) \prod_{a=1, a\neq i}^{k} \left( l_{k-1i} + l_{k-1a} + 1 \right).
\]
and
\[
D_{kijm} = A_{ki} A_{k-1m} C_{kj} \prod_{a=1, a\neq i}^{k} \left( l_{k-1j}^{2} - l_{ka}^{2} \right) \prod_{a=1, a\neq m}^{k-1} \left( l_{k-1j}^{2} - l_{k-1a}^{2} \right).
\]
The arrays \( \Lambda \pm \delta_k \) and \( \Lambda \pm \delta'_k \) are obtained from \( \Lambda \) by replacing \( \lambda_{ki} \) and \( \lambda'_{ki} \) by \( \lambda_{ki} \pm 1 \) and \( \lambda'_{ki} \pm 1 \) respectively. The vector \( \zeta_\Lambda \) is considered to be zero if the array \( \Lambda \) is not a pattern.

Proof. The proof is not essentially different from that of [20, Theorem 9.6.2], so we only point out some key steps and alternative choices made in the arguments.

Suppose that \( \mu = (\mu_1, \ldots, \mu_{n-1}) \) is an \( \mathfrak{sp}_{2n-2} \)-highest weight. The multiplicity space \( V(\lambda)^+_\mu \) is nonzero if and only if the components of \( \lambda \) and \( \mu \) satisfy the inequalities
\[
\lambda_i \geq \mu_{i+1}, \quad i = 1, \ldots, n-2 \quad \text{and} \quad \mu_i \geq \lambda_{i+1}, \quad i = 1, \ldots, n-1.
\]
When it is nonzero, the vector space \( V(\lambda)^+_\mu \) carries an irreducible representation of the twisted Yangian \( Y(\mathfrak{sp}_2) \). By [20, Theorem 9.4.11], this representation is isomorphic to the tensor product,
\[
V(\lambda)^+_\mu \cong L(\alpha_1, \beta_1) \otimes \ldots \otimes L(\alpha_n, \beta_n),
\]
where
\[
\alpha_i = \min\{\lambda_{i-1}, \mu_{i-1}\} - i + \frac{1}{2}, \quad \beta_i = \max\{\lambda_{i}, \mu_{i}\} - i + \frac{1}{2},
\]
assuming that \( \lambda_0 = \mu_0 = 0 \) and \( \max\{\lambda_{n}, \mu_{n}\} \) is understood as being equal to \( \lambda_{n} \). Each factor \( L(\alpha_i, \beta_i) \) is the highest weight \( \mathfrak{gl}_2 \)-module which is extended to the evaluation module over the Yangian \( Y(\mathfrak{gl}_2) \). The coproduct on the Yangian allows one to equip the tensor product in (4.3) with a \( Y(\mathfrak{gl}_2) \)-module structure. This module is then restricted to the subalgebra \( Y(\mathfrak{sp}_2) \subset Y(\mathfrak{gl}_2) \).

The required modification of the construction relies on [20, Corollary 4.3.5] which implies an alternative isomorphism
\[
V(\lambda)^+_\mu \cong L(-\beta_1, -\alpha_1) \otimes \ldots \otimes L(-\beta_n, -\alpha_n).
\]
Although the tensor products in (4.3) and (4.4) are isomorphic as \( Y(\mathfrak{sp}_2) \)-modules, they differ as \( Y(\mathfrak{gl}_2) \)-modules. As we shall see below, the use of the alternative isomorphism leads to a different basis of the multiplicity space \( V(\lambda)^+_\mu \).
The basis vectors of \( V(\lambda)_{\mu}^+ \) will be constructed with the use of the Mickelsson–Zhelobenko algebra \( Z(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-2}) \). The lowering operators are elements of \( Z(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-2}) \) defined by
\[
z_{i,-n} = p F_{i,-n} (f_i - f_{i-1}) \cdots (f_i - f_{-n+1}), \quad i = -n + 1, \ldots, n - 1,
\]
where \( p \) is the extremal projector for \( \mathfrak{sp}_{2n-2} \), and we set
\[
f_i = F_{ii} - i, \quad f_{-i} = -F_{ii} + i,
\]
for all \( i = 1, \ldots, n \). One more lowering operator \( z_{i,-n} \in Z(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-2}) \) is defined by
\[
z_{i,-n} = \sum_{n > i_1 > \cdots > i_s > -n} F_{n i_1} F_{i_1 i_2} \cdots F_{i_s, -n} (f_n - f_{i_1}) \cdots (f_n - f_{i_s}),
\]
where \( s = 0, 1, \ldots \) and \( \{j_1, \ldots, j_k\} \) is the complement to the subset \( \{i_1, \ldots, i_s\} \) in the set \( \{-n + 1, \ldots, n - 1\} \). We will also need an interpolation polynomial \( Z_{n,-n}(u) \) with coefficients in the Mickelsson–Zhelobenko algebra given by
\[
Z_{n,-n}(u) = F_{n,-n} \prod_{i=-n+1}^{n-1} (u + g_i) - \sum_{i=-n+1}^{n-1} z_{n i} z_{i,-n} \prod_{j=-n+1, j \neq i}^{n-1} \frac{u + g_j}{g_i - g_j},
\]
where \( g_i = f_i + 1/2 \) for all \( i \) and we set \( z_{n i} = (-1)^{n-i} z_{i,-n} \). This polynomial is even in \( u \) and has the properties
\[
Z_{n,-n}(-g_i) = z_{n i} z_{i,-n}, \quad i = -n + 1, \ldots, n - 1,
\]
and
\[
Z_{n,-n}(-g_n) = z_{n,-n}.
\]

Recall that the dimension of the multiplicity space \( V(\lambda)_{\mu}^+ \) equals the number of \( n \)-tuples of integers \( \nu = (\nu_1, \ldots, \nu_n) \) satisfying the betweeness conditions (2.2). Let us set
\[
\gamma_i = \nu_i - i + \frac{1}{2}, \quad i = 1, \ldots, n.
\]
The highest vector of the \( Y(\mathfrak{sp}_2) \)-module \( V(\lambda)_{\mu}^+ \) is given by the formula (it coincides with the vector in [20, (9.69)]) up to a sign:
\[
\xi_{\mu} = \prod_{i=1}^{n-1} \left( z_{n,-n}^{\max(\lambda_i, \mu_i) - \lambda_i} z_{i,-n}^{\max(\lambda_i, \mu_i) - \mu_i} \right) \xi,
\]
so that following the proof of [20, Theorem 9.5.1] and using the isomorphism (4.4) instead of (4.3), we find that the vectors
\[
\prod_{i=1}^{n} Z_{n,-n}(\gamma_i + 1) \cdots Z_{n,-n}(\alpha_i - 1) Z_{n,-n}(\alpha_i) \xi_{\mu}
\]
with \(\nu\) satisfying the betweenness conditions form a basis of \(V(\lambda)^+_\mu\). By repeating the argument of that proof, we can conclude that the vectors

\[
\xi_\nu = \prod_{i=1}^{n-1} z_{n,-i}^{\mu_i-\nu_i+1} z_{-i,-n}^{\lambda_i-\nu_i+1} \cdot Z_{n,-n}(\gamma_1 + 1) \ldots Z_{n,-n}(\alpha_1) \xi
\]

parameterised by the \(n\)-tuples \(\nu\) satisfying the betweenness conditions form a basis of the multiplicity space \(V(\lambda)^+_\mu\).

Taking into account the decomposition (0.1) and applying the same argument to the subalgebras of the chain (2.3), we obtain that the vectors

\[
\xi_\Lambda = \prod_{k=1}^{n} \prod_{i=1}^{k-1} z_{k,-i}^{\lambda_{k,i}-\lambda'_{k,i}+1} z_{-i,-k}^{\lambda'_{k,i}-\lambda_{k,i}+1} \cdot Z_{k,-k}(\lambda_{k,1} + 1/2) \ldots Z_{k,-k}(-1/2) \xi
\]

parameterised by all patterns \(\Lambda\) associated with \(\lambda\) form a basis of the representation \(V(\lambda)\) of \(\mathfrak{sp}_{2n}\). The same calculations as in the proof of [20, Theorem 9.6.2] allow one to get the formulas for the action of the generators of the Lie algebra \(\mathfrak{sp}_{2n}\) in the basis \(\xi_\Lambda\) and then write them in terms of the normalised basis vectors

\[
\zeta_\Lambda = \prod_{k=2}^{n} \prod_{1 \leq i < j \leq k} \frac{1}{(-l'_{k,i} - l'_{k,j} - 1)!} \xi_\Lambda
\]

thus completing the proof. \(\square\)

Remark 4.2. When written for the basis vectors \(\xi_\Lambda\), the matrix elements of the generators of \(\mathfrak{sp}_{2n}\) provided by Theorem 4.1 and those of [20, Theorem 9.6.2] exhibit the following symmetry: the formal replacements \(\Lambda \mapsto -\Lambda\) together with \(l'_{k,i} \mapsto -l'_{k,i}\) and \(l_{k,i} \mapsto -l_{k,i}\) transform the matrix elements from one case to the other.

5. CONNECTION BETWEEN THE MONOMIAL AND GELFAND–TSETLIN-TYPE BASES

We will demonstrate that the transition matrix between the basis \(\theta_\Lambda\) of the \(\mathfrak{sp}_{2n}\)-module \(V(\lambda)\) provided by Theorem B and the basis \(\zeta_\Lambda\) of Theorem 4.1 is triangular.

Using the notation from the previous section, for each \(n\)-tuple \(\nu\) satisfying the betweenness conditions, introduce the vector \(\eta_\nu \in V(\lambda)^+_\mu\) by

\[
\eta_\nu = \prod_{i=1}^{n-1} z_{n,-i}^{\mu_i-\nu_i+1} z_{-i,-n}^{\lambda_i-\nu_i+1} F_{n,-n}^{\nu_i} \xi,
\]

where we let \(\xi\) denote the highest weight vector of \(V(\lambda)\). By a result of Zhelobenko [23, Theorem 6.1], the vectors \(\eta_\nu\) form a basis of \(V(\lambda)^+_\mu\). This fact will also follow from a relationship between the vectors \(\xi_\nu\) and \(\eta_\nu\) as described in the next lemma. We will consider the lexicographical orderings \(>\) on the set of \(n\)-tuples \(\nu\) and on the set of \((n-1)\)-tuples \(\mu\).
Lemma 5.1. For any \( \nu \) we have the relation
\[
\eta_\nu = \sum_{\nu' \succ \nu} c_{\nu, \nu'} \xi_{\nu'}
\]
for some constants \( c_{\nu, \nu'} \), and \( c_{\nu, \nu} \neq 0 \). In particular, the vectors \( \eta_\nu \) form a basis of \( V(\lambda)_\mu^+ \).

Proof. Since \( F_{n,-n} \) commutes with the lowering operators \( z_{n,j} \), the vector (5.1) can be written as \( \eta_\nu = F_{n,-n}^{-\nu_1} \xi_{\nu,0} \), where \( \nu^0 \) is the \( n \)-tuple obtained from \( \nu \) by replacing \( \nu_1 \) with 0. On the other hand, by the formulas of Theorem 4.1 for any \( \nu \) we have
\[
F_{n,-n} \xi_\nu = \sum_{i=1}^{n} \prod_{a=1, a \neq i}^{n} \frac{1}{\gamma^2_{\nu_1} - \gamma^2_{\nu_i}} \xi_{\nu - \delta_i}.
\]
A repeated application of this formula allows us to write \( F_{n,-n}^{-\nu_1} \xi_{\nu,0} \) as a linear combination of the basis vectors \( \xi_{\nu'} \) which clearly has the required form. \( \square \)

Lemma 5.1 implies that the vectors
\[
(5.2) \quad \eta_\Lambda = \prod_{k=1,n} \left( F_{n,-n}^{-\nu_1} \prod_{i=1}^{k-1} z_{k-i}^{-\nu_i} \right) \xi,
\]
parameterised by all type \( C \) patterns \( \Lambda \) associated with \( \lambda \) form a basis of the representation \( V(\lambda) \).

Since the weight \( \mu \) will now be varied, we will denote the vector (5.1) by \( \eta_{\nu, \mu} \). The following lemma is essentially a particular case of [23, Theorem 7] or [24, Lemma 2].

Lemma 5.2. For any given pair \( (\nu, \mu) \) satisfying the betweenness conditions, in the module \( V(\lambda) \) we have
\[
(5.3) \quad F_{n,-n}^{-\nu_1} \prod_{i=1}^{n} F_{n,-n}^{\mu_i - \nu_1} F_{-1,-n}^{-\nu_1} \xi = c \eta_{\nu, \mu} + \sum_{\nu', \mu'} u(\nu', \mu') \eta_{\nu', \mu'}
\]
for a nonzero constant \( c \) and some elements \( u(\nu', \mu') \in U(n_0) \), where the sum is taken over the pairs \( (\nu', \mu') \) satisfying the betweenness conditions, and \( u(\nu', \mu') = 0 \) unless \( \mu' \succ \mu \), or \( \mu' = \mu \) and \( \nu' \succ \nu \).

Proof. Write the product on the left hand side in the order
\[
F_{n,-n+1}^{\mu_n - \nu_n} \cdots F_{n,-1}^{\mu_1 - \nu_1} F_{-1,-n}^{-\nu_1} \xi.
\]
Taking into account that \( F_{n,-k} = F_{k,-n} \) for positive values of \( k \), start from the rightmost generator and proceed to the left by using the inversion formula [20, Lemma 9.2.2] to replace \( F_{i,-n} \) with \( i = -n + 1, \ldots, n - 1 \) by the expression:
\[
F_{i,-n} = pF_{i,-n} + \sum_{i > i_1 > \ldots > i_s > -n} \frac{1}{(f_{i_1} - f_i)(f_{i_2} - f_{i_1}) \cdots (f_{i_s} - f_{i_{s-1}})}.
\]
summed over \( s = 1, 2, \ldots \). Apply relation (4.5) to write the right hand side of the inversion formula in terms of the lowering operators \( z_{k, -n} \). We will use the following property of these operators: \( z_{i, -n} \) and \( z_{j, -n} \) commute for \( i + j \neq 0 \); see [20, Proposition 9.2.5]. Let \( \tilde{n}_0^- \) denote the subalgebra of \( n_0^- \) spanned by the elements \( F_{j, i} \) with \( 1 \leq i < j \leq n - 1 \). The same argument as in the proof of Lemma 3.1 shows that

\[
F_{-\nu}^{\lambda_1, \dots, \lambda_{n-1}, -\nu} \xi = d \eta_{\nu} + \sum_{\sigma \succ \nu} u(\sigma) \eta_{\sigma}
\]

for a nonzero constant \( d \) and some elements \( u(\sigma) \in \mathcal{U}(\tilde{n}_0^-) \), where \( \tilde{\nu} = (\nu_2, \ldots, \nu_n) \). Now we will be applying the inversion formula for positive values of \( i \) and note that each term with \( i_s < 0 \) in the sum on the right hand side contains a generator \( F_{i_k, i_{k+1}} \) with \( i_k > 0 > i_{k+1} \). However, such a generator commutes with all elements \( F_{i, -n} \) for \( i > 0 \). Therefore, all these terms with \( i_s < 0 \) will only contribute to the sum on the right hand side of the expansion (5.3) within the summands of the form \( u(\nu', \mu') \eta_{\nu' \mu'} \) with \( \mu' > \mu \).

On the other hand, for any element \( u \in \mathcal{U}(\tilde{n}_0^-) \) we have the relation

\[
F_{i, -n} u = u F_{i, -n} + \sum_{j=i+1}^{n-1} F_{j, -n} u_j
\]

for certain elements \( u_j \in \mathcal{U}(\tilde{n}_0^-) \). Hence, considering the terms in the inversion formula with the property \( i_s > 0 \), we may conclude that nonzero summands on the right hand side of (5.3) of the form \( u(\nu', \mu) \eta_{\nu' \mu} \) must have the property \( \nu' \succ \nu \) and \( u(\nu, \mu) \) is a nonzero constant.

Consider the vectors \( \xi_\Lambda \in V(\lambda) \) introduced in Section 4. They are parameterised by the type C patterns \( \Lambda \) defined in the Introduction. Represent each pattern \( \Lambda \) associated with \( \lambda \) as the sequence of the rows:

\[
\Lambda = (\tilde{\lambda}_{n-1}, \tilde{\lambda}_{n-2}, \ldots, \tilde{\lambda}_1),
\]

where we set

\[
\tilde{\lambda}_k = (\lambda_{k1}, \ldots, \lambda_{kk}) \quad \text{and} \quad \tilde{\lambda}'_k = (\lambda'_{k1}, \ldots, \lambda'_{kk}).
\]

Introduce the lexicographical ordering \( \succ \) on the sequences \( \Lambda \) by using the lexicographical orderings on the vectors \( \tilde{\lambda}_k \) and \( \tilde{\lambda}'_k \). Recall the vectors \( \theta_\Lambda \) defined in Theorem B. We can now obtain another proof of the theorem.

**Proposition 5.3.** For each type C pattern \( \Lambda \) associated with \( \lambda \), in the module \( V(\lambda) \) we have

\[
\theta_\Lambda = \sum_{\Lambda' \succ \Lambda} c_{\Lambda, \Lambda'} \xi_{\Lambda'}
\]

for some constants \( c_{\Lambda, \Lambda'} \), and \( c_{\Lambda, \Lambda} \neq 0 \). In particular, \( \theta_\Lambda \) is a basis of \( V(\lambda) \).
Proof. We will use an induction on \(n\). Consider the part of the product defining the vector \(\theta \Lambda\) which corresponds to the value \(k = n\). By applying Lemma 5.2 and using the induction hypothesis, we can write \(\theta \Lambda\) as a linear combination of the basis vectors \(\eta_M\) defined in (5.2) so that it contains the vector \(\eta_\Lambda\) with a nonzero coefficient, while the remaining vectors occurring in the linear combination have the property \(M \succ \Lambda\). It remains to expand the vectors \(\eta_M\) as linear combinations of basis vectors \(\xi_{\Lambda'}\) by using Lemma 5.1 which yields the expansion of \(\theta \Lambda\) with the required properties. \(\square\)

Remark 5.4. The inversion formula can be used also for rewriting the basis of Proposition 2.10 in terms of the lowering operators. Therefore the subspace \(V(\lambda)^+\) has a basis

\[
\left\{ F_{2n1}^{b_1 \ldots b_n + \iota_n a_n} z_{n1}^{b_1 + (1-\iota_n)a_n} \ldots z_{2n2}^{b_2 + (1-\iota_2)a_2} v_\lambda \mid a_2, \ldots, a_n, b, b_2, \ldots, b_n \text{ satisfy } (2.6) - (2.9) \right\},
\]

where \(\iota_k \in \{0, 1\}\).

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