Partial classification of the Baumslag-Solitar group von Neumann algebras

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Abstract

We prove that the rational number $|n/m|$ is an invariant of the group von Neumann algebra of the Baumslag-Solitar group $BS(n,m)$. More precisely, if $L(BS(n,m))$ is isomorphic with $L(BS(n',m'))$, then $|n'/m'| = |n/m|^{\pm 1}$. We obtain this result by associating to abelian, but not maximal abelian, subalgebras of a II$_1$ factor, an equivalence relation that can be of type III. In particular, we associate to $L(BS(n,m))$ a canonical equivalence relation of type III$_{|n/m|}$.

1. Introduction and statement of the main result

Some of the deepest open problems in functional analysis center around the classification of group von Neumann algebras $L(G)$ associated with certain natural families of countable groups $G$. In the case of the free groups, this becomes the famous free group factor problem asking whether $L(F_n) \cong L(F_m)$ when $n, m \geq 2$ and $n \neq m$. For property (T) groups with infinite conjugacy classes (icc), this leads to Connes’s rigidity conjecture ([Co75]) asserting that an isomorphism $L(G) \cong L(\Lambda)$ between the property (T) factors entails an isomorphism $G \cong \Lambda$ of the groups.

As a consequence of Connes’s uniqueness theorem of injective II$_1$ factors ([C73]), the group von Neumann algebra $L(G)$ of an amenable icc group $G$ is isomorphic with the unique hyperfinite II$_1$ factor $R$. In the nonamenable case, many nonisomorphic groups $G$ are known to have nonisomorphic group von Neumann algebras $L(G)$. Nevertheless, concerning the classification of group von Neumann algebras of natural families of groups, e.g. lattices in simple Lie groups, little is known. A notable exception however is [CH88] where it is shown that for $n \neq m$, lattices in $Sp(n,1)$, respectively $Sp(m,1)$, have nonisomorphic group von Neumann algebras.

Since 2001, Popa has been developing a new arsenal of techniques to study II$_1$ factors, called deformation/rigidity theory. This theory has provided several classes $G$ of groups such that an isomorphism $L(G) \cong L(\Lambda)$ with both $G, \Lambda \in G$ entails the isomorphism $G \cong \Lambda$. By [Po04], this holds in particular when $G$ is the class of wreath product groups of the form $\mathbb{Z}/2\mathbb{Z} \wr \Gamma$ with $\Gamma$ an icc property (T) group.

In [IPV10], the first $W^*$-superrigidity theorems for group von Neumann algebras were discovered, yielding icc groups $G$ such that an isomorphism $L(G) \cong L(\Lambda)$ with $\Lambda$ an arbitrary countable group, implies that $G \cong \Lambda$. The groups $G$ discovered in [IPV10] are generalized wreath products of a special form. In [BV12], it was then shown that one can actually take $G = (\mathbb{Z}/2\mathbb{Z})^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$ with $\Gamma$ ranging over a large family of nonamenable groups including the free groups $F_n, n \geq 2$.

In this article, we apply Popa’s deformation/rigidity theory to partially classify the group von Neumann algebras of the Baumslag-Solitar groups $BS(n,m)$. Recall that for all $n, m \in \mathbb{Z} - \{0\}$, this group is defined as the group generated by $a$ and $b$ subject to the relation $ba^n b^{-1} = a^m$. So, $BS(n,m) := \langle a,b \mid ba^n b^{-1} = a^m \rangle$.

The Baumslag-Solitar groups were introduced in [BS62] as the first examples of finitely presented non-Hopfian groups. Ever since, they have been used as examples and counterexamples for numerous group theoretic phenomena. Therefore, it is a natural problem to classify the group von Neumann algebras $L(BS(n,m))$.

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Supported by ERC Starting Grant VNALG-200749, Research Programme G.0639.11 of the Research Foundation – Flanders (FWO) and K.U.Leuven BOF research grant OT/08/032.
Whenever \(|n| = 1\) or \(|m| = 1\), the group \(BS(n,m)\) is solvable, hence amenable. So we always assume that \(|n| \geq 2\) and \(|m| \geq 2\). In that case, \(BS(n,m)\) contains a copy of the free group \(F_2\) and hence, is nonamenable. In \(\text{Mo91}\), the Baumslag-Solitar groups were classified up to isomorphism: \(BS(n,m) \cong BS(n',m')\) if and only if \(\{n,m\} = \{\varepsilon n',\varepsilon m'\}\) for some \(\varepsilon \in \{-1,1\}\). So, up to isomorphism, we only consider 2 \(\leq n \leq |m|\). Finally by \(\text{St05}\) Exemple 2.4, the group \(BS(n,m)\) is icc if and only if \(|n| \neq |m|\). Therefore, we always assume that \(2 \leq n < |m|\).

Using Popa’s deformation/rigidity theory and in particular his spectral gap rigidity (\(\text{Po06}\)) and the work on amalgamated free products (\(\text{IPP05}\)), several structural properties of the II\(_1\) factors \(M = L(BS(n,m))\) were proven. In particular, it was shown in \(\text{Fi10}\) that \(M\) is not solid, that \(M\) is prime and that \(M\) has no Cartan subalgebra. More generally, it is proven in \(\text{Fi10}\) that any amenable regular von Neumann subalgebra of \(M\) must have a nonamenable relative commutant.

Our main result is the following partial classification theorem for the Baumslag-Solitar group von Neumann algebras \(L(BS(n,m))\). Whenever \(M\) is a II\(_1\) factor and \(t > 0\), we denote by \(M^t\) the amplification of \(M\). Up to unitary conjugacy, \(M^t\) is defined as \(p[M_n(\mathbb{C}) \otimes M]p\) where \(p\) is a projection satisfying \((\text{Tr} \otimes \tau)(p) = t\). The II\(_1\) factors \(M\) and \(N\) are called stably isomorphic if there exists a \(t > 0\) such that \(M \cong N^t\).

**Theorem A.** Let \(n,m,n',m' \in \mathbb{Z}\) such that \(2 \leq n < |m|\) and \(2 \leq n' < |m'|\). If \(L(BS(n,m))\) is stably isomorphic with \(L(BS(n',m'))\), then \(\frac{n}{|m|} = \frac{n'}{|m'|}\).

Note that Theorem A formally resembles, but is independent of, the results in \(\text{Ki11}\) on orbit equivalence relations of essentially free ergodic probability measure preserving actions of Baumslag-Solitar groups, especially \(\text{Ki11}\) Proposition B.2 and Theorem 1.2]. It would be very interesting to find a framework that unifies both types of results.

We prove our Theorem A by associating a canonical equivalence relation to \(L(BS(n,m))\) and proving that it is of type III\(_{n/|m|}\). More precisely, assume that \((M,\tau)\) is a von Neumann algebra with separable predual, equipped with a faithful normal tracial state. Whenever \(A \subset M\) is an abelian von Neumann subalgebra, the normalizer

\[
N_M(A) := \{u \in U(M) \mid uAu^* = A\}
\]

induces a group of trace preserving automorphisms of \(A\). Writing \(A = L^\infty(X,\mu)\) with \(\mu\) being induced by \(\tau|_A\), the corresponding orbit equivalence relation is a countable probability measure preserving (pmp) equivalence relation on \((X,\mu)\).

More generally, we can consider the set of partial isometries

\[
\{u \in M \mid u^*u \text{ and } uu^* \text{ are projections in } A' \cap M \text{ and } uAu^* = Auu^*\}.
\]

Every such partial isometry induces a partial automorphism of \(A\) and hence a partial automorphism of \((X,\mu)\). We denote by \(R(A \subset M)\) the equivalence relation generated by all these partial automorphisms. When \(A \subset M\) is maximal abelian, i.e. \(A' \cap M = A\), then \(R(A \subset M)\) coincides with the orbit equivalence relation induced by the normalizer \(N_M(A)\). In particular, in that case the equivalence relation \(R(A \subset M)\) preserves the probability measure \(\mu\).

If however \(A \subset M\) is not maximal abelian, the partial automorphisms of \(A\) induced by the partial isometries in the set \(\{1\}\) need not be trace preserving. So in general, \(R(A \subset M)\) can be an equivalence relation of type III.

Our main technical result is Theorem 3.3 below, roughly saying the following. If \(A,B \subset M\) are abelian subalgebras such that \(Z(A' \cap M) = A\) and \(Z(B' \cap M) = B\), and if there exist intertwining bimodules \(A \triangleleft B\) and \(B \triangleleft A\) (in the sense of Popa, see \(\text{Po03}\) and Theorem 2.3 below), then the equivalence relations \(R(A \subset M)\) and \(R(B \subset M)\) must be stably isomorphic. In particular, their types must be the same.
In Section 2.1, we apply this to $M = L(BS(n, m))$ and $A$ equal to the abelian von Neumann subalgebra generated by the unitary $u_A$. We prove that $R(A ⊂ M)$ is the unique hyperfinite ergodic equivalence relation of type III$_n/m$.

The proof of Theorem 1 can then be outlined as follows. First we note that the von Neumann algebra $A' ∩ M$ is nonamenable. Conversely if $Q ⊂ M$ is a nonamenable subalgebra, it was proven in [CH08], using spectral gap rigidity ([P06]) and the structure theory of amalgamated free product factors ([IPP05]), that $Q' ∩ M ∼ A$. So, up to intertwining-by-bimodules, the position of $A$ inside $M$ is “canonical.” Therefore a stable isomorphism between $L(BS(n, m))$ and $L(BS(n', m'))$ will preserve, up to intertwining-by-bimodules, these canonical abelian subalgebras. Hence their associated equivalence relations are stably isomorphic and, in particular, have the same type. This gives us the equality $n/m = n'/m'$.

2. Preliminaries

We denote by $(M, τ)$ a von Neumann algebra equipped with a faithful normal tracial state $τ$. We always assume that $M$ has a separable predual. If $B$ is a von Neumann subalgebra of $(M, τ)$, we denote by $E_B$ the unique trace preserving conditional expectation of $M$ onto $B$.

Whenever $x ∈ M$ is a normal element, we denote by $supp(x)$ its support, i.e. the smallest projection $p ∈ M$ that satisfies $xp = x$ (or equivalently, $px = x$).

Let $R$ be a countable nonsingular (i.e. measure class preserving) equivalence relation on the standard probability space $(X, μ)$. We denote by $[R]$ the full pseudogroup of $R$, i.e. the pseudogroup of all partial nonsingular automorphisms $φ$ of $X$ such that the graph of $φ$ is contained in $R$. We denote the domain of $φ$ by $dom(φ)$ and its range by $ran(φ)$. We denote by $[x]$ the equivalence class of $x ∈ X$.

Assume that also $R'$ is a countable nonsingular equivalence relation on the standard probability space $(X', μ')$. The equivalence relations $R$ and $R'$ are called

- isomorphic, if there exists a nonsingular isomorphism $Δ : X → X'$ such that $Δ([x]) = [Δ(x)]$ for almost every $x ∈ X$ ;
- stably isomorphic, if there exist Borel subsets $Z ⊂ X$ and $Z' ⊂ X'$ that meet almost every orbit and if there exists a nonsingular isomorphism $Δ : Z → Z'$ such that $Δ([x] ∩ Z) = [Δ(x)] ∩ Z'$ for almost every $x ∈ Z$.

2.1. HNN extensions and Baumslag-Solitar groups

Let $G$ be a group, $H < G$ a subgroup and $θ : H → G$ an injective group homomorphism. The HNN extension $HNN(G, H, θ)$ is defined as the group generated by $G$ and an additional element $t$ subject to the relation $θ(h) = tht^{-1}$ for all $h ∈ H$. So,

$$HNN(G, H, θ) = ⟨G, t \mid θ(h) = tht^{-1} \text{ for all } h ∈ H⟩.$$ 

Elements of $HNN(G, H, θ)$ can be canonically written as “reduced words” using as letters the elements of $G$ and the letters $t^±1$. More precisely, we have the following lemma.

Lemma 2.1 (Britton’s lemma, [Br63]). Consider the expression $g = g_0 t_1^{n_1} g_1 t_2^{n_2} \cdots t_k^{n_k} g_k$ with $k ≥ 0$, $g_0, g_k ∈ G$ and $g_1, \ldots, g_{k−1} ∈ G − \{e\}$. We call this expression reduced if the following two conditions hold:

- for every $i ∈ \{1, \ldots, k−1\}$ with $n_i > 0$ and $n_{i+1} < 0$, we have $g_i ∉ H$,
- for every $i ∈ \{1, \ldots, k−1\}$ with $n_i < 0$ and $n_{i+1} > 0$, we have $g_i ∉ θ(H)$. 

3
If the above expression for \( g \) is reduced, then \( g \neq e \) in the group \( \text{HNN}(G,H,\theta) \), unless \( k = 0 \) and \( g_0 = e \). In particular, the natural homomorphism of \( G \) to \( \text{HNN}(G,H,\theta) \) is injective.

Recall from the introduction that the Baumslag-Solitary group \( \text{BS}(n,m) \) is defined for all \( n, m \in \mathbb{Z} - \{0\} \) as

\[
\text{BS}(n,m) := \langle a, b \mid ba^nb^{-1} = a^m \rangle .
\]

It is one of the easiest examples of an HNN extension. We also recall from the introduction that the Baumslag-Solitary group \( \text{BS}(n,m) \) with \( 2 \leq n < |m| \) is a Hilbert space endowed with commuting normal \(*\)-homomorphisms \( \pi : M \to B(\mathcal{H}) \) and \( \varphi : N^{op} \to B(\mathcal{H}) \). For \( x \in M, y \in N \) and \( \xi \in \mathcal{H} \), we write \( x\xi y \) instead of \( \pi(x)\varphi(y^{op})(\xi) \). We denote an \( M\)-\( N \)-bimodule \( \mathcal{H} \) by \( _M\mathcal{H}_N \). We call an \( M\)-\( N \)-bimodule bifinite if it is finitely generated both as a left Hilbert \( M \)-module and a right Hilbert \( N \)-module.

Let \( A \) and \( B \) be abelian von Neumann algebras. We denote by \( \text{Pliso}(A,B) \) the set of all partial isomorphisms from \( A \) to \( B \), i.e. isomorphisms \( \alpha : Aq \to Bp \), where \( q \in A \) and \( p \in B \) are projections.

The following is a well known result.

**Lemma 2.2.** Let \( A \) and \( B \) be abelian von Neumann algebras. Then every bifinite \( A\)-\( B \)-bimodule \( _A\mathcal{H}_B \) is isomorphic to a direct sum of bimodules of the form \( _A\mathcal{H}(\alpha)_B \) with \( \alpha \in \text{Pliso}(A,B) \).

We finally recall Popa’s intertwining-by-bimodules theorem.

**Theorem 2.3** ([Po03, Theorem 2.1 and Corollary 2.3]). Let \( (M, \tau) \) be a tracial von Neumann algebra and let \( A, B \subset M \) be possibly nonunital von Neumann subalgebras. Denote their respective units by \( 1_A \) and \( 1_B \). The following three conditions are equivalent:

1. \( 1_A L^2(M) 1_B \) admits a nonzero \( A\)-\( B \)-subbimodule that is finitely generated as a right \( B \)-module.
2. There exist nonzero projections \( p \in A, q \in B \), a normal unital \(*\)-homomorphism \( \psi : pAp \to qBq \) and a nonzero partial isometry \( v \in pMq \) such that \( av = v\psi(a) \) for all \( a \in pAp \).
3. There is no sequence of unitaries \( u_n \in \mathcal{U}(A) \) satisfying \( ||E_B(xu_n y^*)||_2 \to 0 \) for all \( x, y \in 1_B M 1_A \).

If one of these equivalent conditions holds, we write \( A \prec_M B \).

**2.3. Quasi-regularity**

Let \( (M, \tau) \) be a tracial von Neumann algebra and \( N \subset M \) a von Neumann subalgebra. We denote by \( \text{QN}_M(N) \) the quasi-normalizer of \( N \) inside \( M \), i.e. the unital \(*\)-algebra defined by

\[
\left\{ a \in M \mid \exists b_1, \ldots, b_k \in M, \exists d_1, \ldots, d_r \in M \text{ such that } Na \subset \sum_{i=1}^{k} b_i N \text{ and } aN \subset \sum_{j=1}^{r} N d_j \right\} .
\]

We call \( N \subset M \) quasi-regular if \( \text{QN}_M(N)^\prime \prime = M \).
If $A, B \subset M$ are abelian von Neumann subalgebras, we define $Q_M(A, B)$ as
\[ Q_M(A, B) := \{ v \in M \mid vv^* \in A' \cap M, v^*v \in B' \cap M \text{ and } Av = vB \} . \]
Whenever $v \in Q_M(A, B)$, we define $q_v = \text{supp}(E_A(vv^*))$ and $p_v = \text{supp}(E_B(v^*v))$, and we denote by $\alpha_v : Aq_v \to Bp_v$ the unique $*$-isomorphism satisfying $av = v\alpha_v(a)$ for all $a \in Aq_v$.

Note that the set $Q_M(A, B)$ can be $\{0\}$. In Lemma 2.4, we will see that $Q_M(A, B) \neq \{0\}$ if and only if there exists a bifinite $A$-$B$-submodule $\mathcal{A} \mathcal{H}_B$ of $AL^2(M)_B$.

We denote $Q_M(A, A)$ by $Q_M(A)$.

**Lemma 2.4.** Let $(M, \tau)$ be a tracial von Neumann algebra and $A, B \subset M$ abelian von Neumann subalgebras. Then the following statements hold.

1. If $\alpha \in \text{Pliso}(A, B)$ and if $\theta : \mathcal{A} \mathcal{H}(\alpha)_B \to AL^2(M)_B$ is an $A$-$B$-bimodular isometry, then there exists a partial isometry $v' \in Q_M(A, B)$ such that $\alpha = \alpha_{v'}$ and such that $\theta(\mathcal{H}(\alpha)) \subset \overline{v'(B' \cap M)} |v' |^2 \subset \overline{\text{span}} |v' |^2 Q_M(A, B)$.

2. Every bifinite $A$-$B$-submodule $\mathcal{A} \mathcal{H}_B$ of $AL^2(M)_B$ is contained in $\overline{\text{span}} |v' |^2 Q_M(A, B)$.

3. $Q_M(A)^\prime prime = QN_M(A)^\prime prime$.

4. We have $Q_M(A, B) \neq \{0\}$ if and only if $AL^2(M)_B$ admits a nonzero bifinite $A$-$B$-submodule.

**Proof.** 1. Let $\alpha : Aq \to Bp$ be an element of $\text{Pliso}(A, B)$. Define $\xi := \theta(p) \in L^2(M)$ and let $\xi = v' |\xi |$ be its polar decomposition. For all $a \in A$, we have $a\xi = \xi\alpha(a)$ and hence, $av = v\alpha_v(a)$. Furthermore $p = \text{supp}(E_B(v^*v))$ and $q = \alpha^{-1}(p) = \text{supp}(E_A(vv^*))$. So we find that $v \in Q_M(A, B)$ and $\alpha = \alpha_{v'}$.

Because $|\xi | \in L^2(B' \cap M)$, we have that $\xi = v' |\xi |$ is an element of $\overline{v'(B' \cap M)} |v' |^2$. And since $p$ generates $\mathcal{A} \mathcal{H}(\alpha)_B$ as a right Hilbert $B$-module, we have proven statement 1.

2. Let $\mathcal{A} \mathcal{H}_B$ be a bifinite $A$-$B$-submodule of $AL^2(M)_B$.

By Lemma 2.2, $\mathcal{A} \mathcal{H}_B$ is isomorphic to a direct sum of bimodules of the form $\mathcal{A} \mathcal{H}(\alpha_i)_B$ with $\alpha_i \in \text{Pliso}(A, B)$. Using statement 1 of the lemma, we find that $\mathcal{H}$ is generated by subspaces of $\overline{\text{span}} |v' |^2 Q_M(A, B)$. This proves statement 2.

3. By definition, we have $Q_M(A)^\prime prime \subset QN_M(A)^\prime prime$. On the other hand, $AL^2(QN_M(A)^\prime prime)_A$ is a direct sum of bifinite $A$-$A$-submodules of $AL^2(M)_A$. So by statement 2, we have that $L^2(QN_M(A)^\prime prime) \subset \overline{\text{span}} |v' |^2 (Q_M(A))$. Therefore we conclude that $QN_M(A)^\prime prime = Q_M(A)^\prime prime$.

Finally, 4 is an immediate consequence of 2. \hfill \Box

We end this subsection with the following lemma, clarifying why later, we will consider abelian subalgebras $A \subset M$ satisfying $Z(A' \cap M) = A$. Note that since $A$ is abelian, the condition $Z(A' \cap M) = A$ is equivalent with the “bicommutant” property $(A' \cap M)' \cap M = A$.

**Lemma 2.5.** Let $(M, \tau)$ be a tracial von Neumann algebra and $A, B, C \subset M$ abelian von Neumann subalgebras. If $v \in Q_M(A, B)$, $w \in Q_M(B, C)$ and if $Z(B' \cap M) = B$, then there exists an element $u \in Q_M(A, C)$ such that $\alpha_u \circ \alpha_v = \alpha_w$.

**Proof.** Choose $v \in Q_M(A, B)$ and $w \in Q_M(B, C)$. Note that $vbw \in Q_M(A, C)$ for every $b \in B' \cap M$ and $\alpha_{vbw} = \alpha_u \circ \alpha_v |Aq_{wbw}$. We claim that $\bigvee_{b \in B' \cap M} q_{bw} = \alpha_v^{-1}(q_w p_v)$.

Since $(B' \cap M)' \cap M = B$, we have that for every $x \in M$, the projection $\text{supp}(E_B(xv^*))$ equals the projection of $L^2(M)$ onto the closed linear span of $(B' \cap M)xM \subset L^2(M)$. Since $vq_w = \alpha_v^{-1}(q_w p_v)v$,
it follows that
\[ \text{span}^{||\cdot||_2}((A' \cap M)\nu(B' \cap M)wM) = \text{span}^{||\cdot||_2}((A' \cap M)\nu q_wM) \]
\[ = \text{span}^{||\cdot||_2}(\alpha_v^{-1}(q_wp_v)(A' \cap M)\nu M) \]
\[ = \alpha_v^{-1}(q_wp_v)q_vL^2(M) = \alpha_v^{-1}(q_wp_v)L^2(M) \]

Since for every \( b \in B' \cap M \), we have that \( q_{vbw} \) equals the projection onto \( \text{span}^{||\cdot||_2}(A' \cap M)vbwM \), we have proven our claim that \( \alpha_v^{-1}(q_wp_v) \) equals the projection onto \( \text{span}^{||\cdot||_2}(A' \cap M)vbwM \).

By cutting down with appropriate projections, we find \( b_n \in B' \cap M \) such that the projections \( q_{vbw} \) are orthogonal and sum up to \( \alpha_v^{-1}(q_wp_v) \). In particular, the left supports, resp. right supports, of the elements \( vbww \) are orthogonal. So we can define \( u = \sum_n vb_nw \). It follows that \( u \in Q_M(A, C) \) and \( \alpha_w \circ \alpha_v = \alpha_u \).

2.4. The type of an ergodic nonsingular countable equivalence relation

Let \( \mathcal{R} \) be a nonsingular ergodic countable Borel equivalence relation on a standard probability space \((X, \mu)\). The Radon-Nikodym 1-cocycle of \( \mathcal{R} \) is the almost everywhere uniquely defined Borel map \( \omega : \mathcal{R} \to \mathbb{R} \) such that
\[ \omega(\varphi(x), x) = \log\left(\frac{d\mu \circ \varphi(x)}{d\mu}(x)\right) \quad \text{for all } \varphi \in [[\mathcal{R}]] \text{ and almost every } x \in \text{dom } \varphi. \]

Note that \( \omega \) satisfies the 1-cocycle relation \( \omega(x, z) = \omega(x, y) + \omega(y, z) \) for almost every \((x, y, z) \in \mathcal{R}^2 := \{(x, y, z) \in X^3 \mid (x, y), (y, z) \in \mathcal{R}\} \). One then defines the Maharam extension \( \mathcal{R} \) of \( \mathcal{R} \) as the equivalence relation on \((X \times \mathbb{R}, \mu \times \exp(-t)dt)\) defined by
\[ (x, t) \sim (y, s) \quad \text{if and only if } (x, y) \in \mathcal{R} \text{ and } t - s = \omega(x, y). \]

Note that \( \mu \times \exp(-t)dt \) is an infinite invariant measure for \( \mathcal{R} \). Denote the von Neumann algebra of all \( \mathcal{R} \)-invariant functions in \( L^\infty(X \times \mathbb{R}) \) by \( L^\infty(X \times \mathbb{R})^\mathcal{R} \). Since \( \mathcal{R} \) was assumed to be ergodic, one can easily check that the action of \( \mathcal{R} \) on \( L^\infty(X \times \mathbb{R})^\mathcal{R} \) given by translation of the second variable, is also ergodic. Depending on how this action of \( \mathcal{R} \) looks like, we define as follows the type of \( \mathcal{R} \).

- I or II, if the action is conjugate with \( \mathbb{R} \wr \mathbb{R} \);
- III_\lambda (0 < \lambda < 1), if the action is conjugate with \( \mathbb{R} \wr \mathbb{R}/\mathbb{Z} \log(\lambda) \);
- III_1, if the action is on one point;
- III_0, if the action is properly ergodic.

Remark 2.6. Denote by \( L(\mathcal{R}) \) the von Neumann algebra associated with \( \mathcal{R} \). Denote by \( \varphi \) the normal semifinite faithful state on \( L(\mathcal{R}) \) that is induced by \( \mu \). Finally denote by \( (\sigma_t^\varphi)_{t \in \mathbb{R}} \) its modular automorphism group. There is a canonical identification \( L(\mathcal{R}) \cong L(\mathcal{R}) \rtimes_{\sigma^\varphi} \mathbb{R} \). Under this identification, the dual action of \( \mathbb{R} \) on \( L(\mathcal{R}) \rtimes_{\sigma^\varphi} \mathbb{R} \) corresponds to the action of \( \mathbb{R} \) on \( L(\mathcal{R}) \) that we defined above. Also, the center of \( L(\mathcal{R}) \rtimes_{\sigma^\varphi} \mathbb{R} \) corresponds to \( L^\infty(X \times \mathbb{R})^\mathcal{R} \). Altogether it follows that the type of the equivalence relation \( \mathcal{R} \) coincides with the type of the factor \( L(\mathcal{R}) \).

Lemma 2.7. Let \( \mathcal{R} \) be a nonsingular ergodic countable Borel equivalence relation on the standard probability space \((X, \mu)\). Denote by \( \omega \) its Radon-Nikodym 1-cocycle. If the essential image \( \text{Im}(\omega) \) of \( \omega \) equals \( \log(\lambda)\mathbb{Z} \) for some \( 0 < \lambda < 1 \) and if the kernel \( \text{Ker}(\omega) \) of \( \omega \) is an ergodic equivalence relation, then \( \mathcal{R} \) is of type III_\lambda.
Proof. Since Ker(ω) is an ergodic equivalence relation on \((X, \mu)\), we have
\[
L^\infty(X \times \mathbb{R})^\mathbb{R} \subset L^\infty(X \times \mathbb{R})^{\text{Ker}(\omega)} = 1 \otimes L^\infty(\mathbb{R})
\]
For a given \(F \in L^\infty(\mathbb{R})\), we have that \(1 \otimes F\) is \(\mathbb{R}\)-invariant if and only if \(F\) is invariant under translation by the essential image of \(\omega\). So,
\[
L^\infty(X \times \mathbb{R})^\mathbb{R} = 1 \otimes L^\infty(\mathbb{R}/\log(\lambda)\mathbb{Z})
\]
\(\square\)

3. Equivalence relations associated to subalgebras that are abelian, but not maximal abelian

Throughout this section, we fix a tracial von Neumann algebra \((M, \tau)\) with separable predual. We also fix an abelian von Neumann subalgebra \(A \subset M\) satisfying \(\mathcal{Z}(A' \cap M) = A\). Choose a standard probability space \((X, \mu)\) such that \(A = L^\infty(X, \mu)\). For every nonsingular partial automorphism \(\varphi\) of \((X, \mu)\), we denote by \(\alpha_\varphi\) the corresponding partial automorphism of \(A\).

We first prove that \(Q_M(A)\) induces a nonsingular countable Borel equivalence relation \(\mathcal{R}(A \subset M)\) on \((X, \mu)\). For this, we introduce the notation
\[
\mathcal{G}(A \subset M) := \{\alpha_v \mid v \in Q_M(A)\}
\]

Proposition 3.1. There exists a nonsingular countable Borel equivalence relation \(\mathcal{R}\) on \((X, \mu)\) with the following property: a nonsingular partial automorphism \(\varphi\) of \(X\) satisfies \(\alpha_\varphi \in \mathcal{G}(A \subset M)\) if and only if \((x, \varphi(x)) \in \mathcal{R}\) for a.e. \(x \in \text{dom}(\varphi)\).

Moreover, \(\mathcal{R}\) is essentially unique: if a nonsingular countable Borel equivalence relation \(\mathcal{R}'\) on \((X, \mu)\) satisfies the same property, then there exists a Borel subset \(X_0 \subset X\) with \(\mu(X - X_0) = 0\) and \(\mathcal{R}|_{X_0} = \mathcal{R}'|_{X_0}\).

We denote \(\mathcal{R}(A \subset M) := \mathcal{R}\). The equivalence relation \(\mathcal{R}(A \subset M)\) is ergodic if and only if \(Q_M(A)''\) is a factor.

Before proving Proposition 3.1, we introduce some terminology and a lemma. To every \(\alpha \in \text{PAut}(A)\) are associated the support projections \(q_\alpha, p_\alpha \in A\) such that \(\alpha: Aq_\alpha \rightarrow Ap_\alpha\) is a \(*\)-isomorphism. Assume that \(\alpha \in \text{PAut}(A)\) and \(F \subset \text{PAut}(A)\). We say that \(\alpha\) is a gluing of elements in \(F\), if there exists a sequence of elements \(\alpha_n \in F\) and projections \(q_n \in A\) such that \(q_n = \sum_n q_n\) and such that \(q_n \leq q_\alpha\) and \(\alpha|_{Aq_n} = \alpha_n|_{Aq_n}\) for all \(n\).

Lemma 3.2. Let \(J \subset Q_M(A)\) and \(v \in Q_M(A)\) such that \(v \in \text{span}^l_{\|\cdot\|_2} J\). Then \(\alpha_v\) is a gluing of elements in \(\{\alpha_w \mid w \in J\}\).

Proof. By a standard maximality argument, it suffices to prove that for every nonzero projection \(q \in Aq_v\), there exists a nonzero subprojection \(q_0 \in Aq\) and \(w \in J\) such that \(q_0 \leq q_w\) and \(\alpha_v|_{Aq_0} = \alpha_w|_{Aq_0}\).

So fix a nonzero projection \(q \in Aq_v\). It follows that \(qE_A(v^*v) \neq 0\). Since \(v \in \text{span}^l_{\|\cdot\|_2} J\), we can pick a \(w \in J\) such that \(qE_A(v^*v) \neq 0\). Define \(q_1 := \text{supp}(E_A(v^*v))\) and note that \(q_1 \subseteq q_v\) and \(q_w = q_v\). Also note that \(q_0 = 0\). For all \(a \in A\), we have
\[
\alpha_v^{-1}(ap_v)v^*v = va^*v = vv^*a^{-1}(ap_w)
\]
Applying the conditional expectation onto \(A\) and using that \(A\) is abelian, we find that
\[
\alpha_v^{-1}(ap_v)q_1 = \alpha_w^{-1}(ap_w)q_1 \quad \text{for all} \quad a \in A
\]
This means that \(\alpha_v|_{Aq_1} = \alpha_w|_{Aq_1}\). We put \(q_0 := qq_1\). We already showed that \(q_0 \neq 0\). Since \(q_0 \leq q_1\), we have that \(\alpha_v|_{Aq_0} = \alpha_w|_{Aq_0}\). \(\square\)
Proof of Proposition 3.1. We say that a subpseudogroup \( \mathcal{G} \subset \text{PAut}(A) \) is of countable type if there exists a countable subset \( \mathcal{J} \subset \mathcal{G} \) such that every \( \alpha \in \mathcal{G} \) is a gluing of elements in \( \mathcal{J} \). To prove the first part of the proposition, we must show that \( \mathcal{G}(A \subset M) \) is a subpseudogroup of countable type of \( \text{PAut}(A) \). From Lemma 2.23, it follows that \( \mathcal{G}(A \subset M) \) is indeed a subpseudogroup. Since \( M \) has a separable predual, we can choose a countable \( \alpha \)-basis of \( \mathcal{G}(A \subset M) \). By Lemma 3.24 every \( \alpha \in \mathcal{G}(A \subset M) \) is a gluing of elements in \( \{ \alpha_w \mid w \in \mathcal{J} \} \). Hence \( \mathcal{G}(A \subset M) \) is of countable type. So the first part of the proposition is proven and we can essentially uniquely define the nonsingular countable Borel equivalence relation \( \mathcal{R} \) on \( (X, \mu) \).

Since \( A' \cap M \subset \text{QN}_M(A)'' \) and since we assumed that \( (A' \cap M)' \cap M = A \), the center of \( \text{QN}_M(A)'' \) is a subalgebra of \( A \). By Lemma 2.43, we have \( \text{QN}_M(A)'' \cong \text{QM}(A)'' \). Therefore,

\[
\mathcal{Z}(\text{QN}_M(A)''') = \{ a \in A \mid av = va \text{ for all } v \in \text{QM}(A) \}.
\]

The right hand side equals \( A^R \), the subalgebra of \( \mathcal{R} \)-invariant functions in \( A \). So \( \mathcal{R} \) is ergodic if and only if \( \text{QN}_M(A)'' \) is a factor.

For our application, the following theorem is crucial. It says that \( \mathcal{R}(A \subset M) \) remains the same, up to stable isomorphism, if we replace \( A \) by an abelian subalgebra \( B \) that has a mutual intertwining bimodule into \( A \).

**Theorem 3.3.** Let \( M \) be a II\(_1\) factor with separable predual. Let \( A, B \subset M \) be abelian, quasi-regular von Neumann subalgebras satisfying \( \mathcal{Z}(A' \cap M) = A \) and \( \mathcal{Z}(B' \cap M) = B \). If \( A \vartriangleleft_M B \) and \( B \vartriangleleft_M A \), then the equivalence relations \( \mathcal{R}(A \subset M) \) and \( \mathcal{R}(B \subset M) \) are stably isomorphic.

**Proof.** Since \( A, B \) are quasi-regular and since \( A \vartriangleleft_M B \) as well as \( B \vartriangleleft_M A \), there exists a nonzero bifinite \( A\cdot B \)-bimodule of \( L^2(M) \). So by Lemma 2.44, there exists a nonzero element \( v \in \text{QM}(A, B) \) with corresponding \( \alpha_v \in \text{PAut}(A, B) \). Using the notation in (2) and using Lemma 2.5, we find that

\[
\alpha_v \circ \beta \circ \alpha_v^{-1} \in \mathcal{G}(B \subset M) \quad \text{for all } \beta \in \mathcal{G}(A \subset M) \quad \text{and}
\]

\[
\alpha_v^{-1} \circ \gamma \circ \alpha_v \in \mathcal{G}(A \subset M) \quad \text{for all } \gamma \in \mathcal{G}(B \subset M).
\]

So \( \alpha_v \) implements a stable isomorphism between \( \mathcal{R}(A \subset M) \) and \( \mathcal{R}(B \subset M) \).

The following lemma will allow us to easily compute \( \mathcal{R}(A \subset M) \) in concrete examples.

**Lemma 3.4.** Let \( (M, \tau) \) be a tracial von Neumann algebra and \( A \subset M \) an abelian von Neumann subalgebra satisfying \( \mathcal{Z}(A' \cap M) = A \). Let \( \mathcal{F} \subset M \) be a subset such that

- \( M = (\mathcal{F} \cup \mathcal{F}^* \cup (A' \cap M))'' \),
- as an \( A\cdot A \)-bimodule, \( \text{span} \{ ||\cdot||^2 A\mathcal{F}A \} \) is isomorphic to a direct sum of bimodules of the form \( \mathcal{A}\mathcal{H}(\alpha_n)_A \) with \( \alpha_n \in \text{PAut}(A) \).

Choose nonsingular partial automorphisms \( \varphi_n \) of \( (X, \mu) \) such that \( \alpha_n = \alpha_{\varphi_n} \) for all \( n \). Up to measure zero, \( \mathcal{R}(A \subset M) \) is generated by the graphs of the partial automorphisms \( \varphi_n \).

**Proof.** We again use the notation (2). By Lemma 2.41, we find \( v_n \in \text{QM}(A) \) such that \( \alpha_n = \alpha_{v_n} \) and

\[
\text{span} \{ ||\cdot||^2 A\mathcal{F}A \} \subset \text{span} \{ v_n(A' \cap M) \mid n \in \mathbb{N} \}.
\]  
(3)

In particular, we have \( \alpha_n \in \mathcal{G}(A \subset M) \). Choose nonsingular partial automorphisms \( \varphi_n \) of \( (X, \mu) \) such that \( \alpha_n = \alpha_{\varphi_n} \) for all \( n \).
Denote by $\mathcal{R}$ the smallest (up to measure zero) equivalence relation on $(X, \mu)$ that contains the graphs of all the partial automorphisms $\phi_n$. By the previous paragraph, we know that $\mathcal{R}$ is a subequivalence relation of $\mathcal{R}(A \subset M)$. Denote by $\mathcal{J}$ the set of all products of elements in

$$\{v_n \mid n \in \mathbb{N}\} \cup \{v_n^* \mid n \in \mathbb{N}\} \cup (A' \cap M).$$

By construction, the graph of every $\alpha_w$, $w \in \mathcal{J}$, belongs to $\mathcal{R}$. Combining our assumption that $M = (\mathcal{F} \cup \mathcal{F}^* \cup (A' \cap M))''$ with $[3]$, it follows that span $\mathcal{J}$ is $\| \cdot \|_2$-dense in $L^2(M)$. By Lemma 3.2, every $\alpha \in \mathcal{G}(A \subset M)$ is a gluing of elements in $\{\alpha_w \mid w \in \mathcal{J}\}$. So the graph of every $\alpha \in \mathcal{G}(A \subset M)$ belongs to $\mathcal{R}$ a.e. Hence $\mathcal{R}$ equals $\mathcal{R}(A \subset M)$ almost everywhere.

We finally note in the following proposition that every nonsingular countable Borel equivalence relation $\mathcal{R}$ arises as $\mathcal{R}(A \subset M)$.

**Proposition 3.5.** Let $\mathcal{R}$ be a nonsingular countable Borel equivalence relation. Then there exists a quasi-regular inclusion of an abelian von Neumann algebra $A$ in a tracial von Neumann algebra $(M, \tau)$ satisfying $\mathcal{Z}(A' \cap M) = A$ and such that $\mathcal{R} \cong \mathcal{R}(A \subset M)$.

**Proof.** Let $\mathcal{R}$ be a nonsingular countable Borel equivalence relation on the standard probability space $(X, \mu)$. Denote by $(P, \text{Tr})$ the unique hyperfinite $\Pi_\omega$ factor and choose a trace-scaling action $(\alpha_t)_{t \in \mathbb{R}}$ of $\mathbb{R}$ on $P$. This means that $\text{Tr} \circ \alpha_t = e^{-t} \text{Tr}$. The corresponding action of $\mathcal{R}$ on $L^2(P)$ will also be denoted by $(\alpha_t)$. We denote by $\omega : \mathcal{R} \to \mathbb{R}$ the Radon-Nikodym 1-cocycle of $\mathcal{R}$ (see Section 2.4).

In the same way as with the Maharam extension of a nonsingular group action, the equivalence relation $\mathcal{R}$ admits a natural trace preserving action on $L^\infty(X) \otimes P$. We denote by $(\mathcal{M}, \text{Tr})$ the crossed product. For completeness, we recall the construction of $(\mathcal{M}, \text{Tr})$. To every $\varphi \in \mathbb{R}$, we associate the operator $W_\varphi$ on $L^2(\mathcal{R}, L^2(P))$ given by

$$(W_\varphi \xi)(x, y) = \begin{cases} \alpha_{\omega(x, \varphi^{-1}(x))}(\xi(\varphi^{-1}(x), y)) & \text{if } x \in \text{dom}(\varphi^{-1}) \, , \\ 0 & \text{otherwise,} \end{cases}$$

for every $\xi \in L^2(\mathcal{R}, L^2(P))$. One checks that $W_\varphi W_\psi = W_{\varphi \circ \psi}$ and $W_\varphi^* = W_{\varphi^{-1}}$.

We represent $L^\infty(X) \otimes P = L^\infty(X, P)$ on $L^2(\mathcal{R}, L^2(P))$ by

$$(F \xi)(x, y) = F(x) \xi(x, y) \text{ for all } \xi \in L^2(\mathcal{R}, L^2(P)) \text{ and } F \in L^\infty(X, P).$$

Note that the partial isometries $W_\varphi$, $\varphi \in \mathbb{R}$, normalize $L^\infty(X, P)$ and that

$$(W_\varphi F W_\varphi)(x) = \begin{cases} \alpha_{\omega(x, \varphi(x))}(F(\varphi(x))) & \text{if } x \in \text{dom } \varphi \\ 0 & \text{otherwise.} \end{cases}$$

Define $\mathcal{M}$ as the von Neumann algebra generated by $L^\infty(X, P)$ and the partial isometries $W_\varphi$, $\varphi \in \mathbb{R}$. Denoting by $\Delta \subset \mathcal{R}$ the diagonal subset, the orthogonal projection onto $L^2(\Delta, L^2(P))$ implements a normal faithful conditional expectation $E : \mathcal{M} \to L^\infty(X) \otimes P$ satisfying

$$E(W_\varphi) = \chi_{\{x \mid \varphi(x) = x\}} \otimes 1 \text{ for all } \varphi \in \mathbb{R}.$$ 

The formula $\text{Tr} := (\mu \otimes \text{Tr}) \circ E$ defines a normal semifinite faithful trace on $\mathcal{M}$.

Fix a nonzero projection $q \in P$ with $\text{Tr}(q) = 1$. Define the projection $p \in L^\infty(X) \otimes P$ given by $p = 1 \otimes q$. Write $A := L^\infty(X)p$ and $M := p\mathcal{M}p$. Then $A$ is a quasi-regular abelian von Neumann subalgebra of $M$ and the restriction of $\text{Tr}$ to $M$ gives a normal faithful tracial state $\tau$ on
M. The relative commutant $L^\infty(X)' \cap M$ equals $L^\infty(X) \overline{\otimes} P$. Since $P$ is a factor, it follows that $\mathcal{Z}(A' \cap M) = A$.

We finally prove that $\mathcal{R} \cong \mathcal{R}(A \subset M)$. Write $\mathcal{R} = \bigcup_k \text{graph}(\varphi_k)$, with $\varphi_k \in [\mathcal{R}]$. Then $\varphi_k$ induces an automorphism of $L^\infty(X)$ and hence of $A = L^\infty(X)p$ that we denote by $\beta_k \in \text{Aut}(A)$. Since $P$ is a $\Pi_\infty$ factor and $q \in P$ is a finite projection, we can choose partial isometries $w_n \in P$ such that $\sum_n w_n^*w_n = 1$ and $w_n w_n^* = q$ for all $n$. Define the elements

$$v_{n,k} := (1 \otimes w_n)W_{\varphi_k}p.$$  

All $v_{n,k}$ belong to $Q_M(A)$ and $\alpha_{v_{n,k}}$ equals the restriction of $\beta_k$ to $Ap_{n,k}$ for projections $p_{n,k} \in A$.

Since the sum of all $w_n^*w_n$ equals $1$, we also have that $\bigvee_p p_{n,k} = p$. Therefore the graphs of the partial automorphisms $\alpha_{v_{n,k}}$ generate an equivalence relation that is isomorphic with $\mathcal{R}$.

To conclude the proof, we put $\mathcal{F} := \{v_{n,k} \mid n, k \in \mathbb{N}\}$ and observe that $M = (\mathcal{F} \cup \mathcal{F}^* \cup (A' \cap M))''$. By Lemma 3.1 the equivalence relation $\mathcal{R}(A \subset M)$ is generated by the graphs of the partial automorphisms $\alpha_{v_{n,k}}$. \hfill \Box

4. Proof of Theorem A

Throughout this section, we assume that $n$ and $m$ are integers satisfying $2 \leq n < |m|$. As explained in the introduction, the corresponding groups $BS(n,m)$ form a complete list of the nonamenable icc Baumslag-Solitar groups up to isomorphism.

Throughout this section, we write $M = L(BS(n,m))$ and $A = \{u_a, u_a^*\}'''$. We start with the following observation.

Proposition 4.1. We have that $A \subset M$ is a quasi-regular abelian von Neumann subalgebra satisfying $\mathcal{Z}(A' \cap M) = A$. Moreover, $A' \cap M$ has no amenable direct summand.

Proof. It is clear that $A \subset M$ is a quasi-regular abelian von Neumann subalgebra, because the element $a \in BS(n,m)$ generates an almost normal abelian subgroup of $BS(n,m)$.

To prove that $\mathcal{Z}(A' \cap M) = A$, we define the finite index subalgebra $A_0 := \{u_a^n, u_a^{-n}\}''$ of $A$. We will first prove that $\mathcal{Z}(A_0' \cap M) = A_0$. Afterwards we will show that this implies that $\mathcal{Z}(A' \cap M) = A$.

Define $G := (a^Z, b^{-1}a^Zb) \subset BS(n,m)$. Then $L(G)$ is a subalgebra of $A_0' \cap M$. So $\mathcal{Z}(A_0' \cap M) \subset L(G)' \cap M$. Using Lemma 2.1 one can easily see that $\{g^\gamma g^{-1} \mid g \in G\}$ is an infinite set for every $\gamma \in BS(n,m) - a^{-nZ}$. Therefore $L(G)' \cap M \subset A_0$. This shows that $\mathcal{Z}(A_0' \cap M) \subset A_0$. Since the converse inclusion is obvious, we find that $A_0 = \mathcal{Z}(A_0' \cap M)$.

Since $A_0 \subset A$ has finite index, there exist orthogonal projections $p_j \in A$ such that $Ap_j = A_0p_j$ and $\sum_j p_j = 1$. But then

$$\mathcal{Z}(A' \cap M)p_j = \mathcal{Z}((A' \cap M)p_j) = \mathcal{Z}((Ap_j)' \cap p_jMp_j) = \mathcal{Z}(Ap_j)' \cap p_jMp_j = \mathcal{Z}(p_j(A_0' \cap M)p_j) = \mathcal{Z}(A_0' \cap M)p_j = A_0p_j \subset A$$

Therefore $\mathcal{Z}(A' \cap M) \subset A$. The converse inclusion being obvious, we have proven that $A = \mathcal{Z}(A' \cap M)$.

Using Lemma 2.1 it follows that $G$ is an amalgamated free product of two copies of $Z$ over a copy of $Z$ embedded as $nZ$ and $mZ$ respectively. In particular, $G$ is nonamenable and $L(G)$ has no amenable direct summand. Since $L(G) \subset A_0' \cap M$, it follows that $A_0' \cap M$ has no amenable direct summand either. As above, we have that

$$(A' \cap M)p_j = p_j(A_0' \cap M)p_j$$

for all $j$. Hence $A' \cap M$ has no amenable direct summand. \hfill \Box
We now identify the associated countable equivalence relation $\mathcal{R}(A \subset M)$.

**Proposition 4.2.** Define the countable Borel equivalence relation $\mathcal{R}_{n,m}$ on the circle $T$ given by

$$\mathcal{R}_{n,m} := \{(y, z) \in T \times T \mid \exists a, b \in \mathbb{N} \text{ such that } y^{(a_n b_n)} = z^{(a_n b_n)}\}.$$

Equip $T$ with its Lebesgue measure $\lambda$ and note that $\mathcal{R}_{n,m}$ is a nonsingular countable Borel equivalence relation on $(T, \lambda)$. We have

1. $\mathcal{R}(A \subset M) \cong \mathcal{R}_{n,m}$ up to measure zero.
2. $\mathcal{R}_{n,m}$ is the unique hyperfinite ergodic countable equivalence relation of type $\text{III}_{n|m}$.

**Proof.** 1. Define $\mathcal{R}_0 := \{(y, z) \in T \times T \mid y^m = z^n\}$. Note that $\mathcal{R}_0 \subset \mathcal{R}_{n,m}$ and that $\mathcal{R}_{n,m}$ is the smallest equivalence relation containing $\mathcal{R}_0$. Define $\pi : \mathcal{R}_0 \to T : \pi(y, z) = y^m$. Note that $\pi$ is $n|m|$-to-$1$. Define the probability measure $\mu$ on $\mathcal{R}_0$ given by

$$\mu(U) = \frac{1}{n|m} \int_T \#(U \cap \pi^{-1}([x])) \ d\lambda(x).$$

For all $k, l \in \mathbb{Z}$, we define the function $P_{k,l} : \mathcal{R}_0 \to T : P_{k,l}(y, z) = y^k z^l$. A direct computation yields a unique unitary

$$T : L^2(\mathcal{R}_0, \mu) \to \text{span} \{u^k u^l \mid a \in A\} : P_{k,l} \mapsto u^k u^l.$$

We turn $L^2(\mathcal{R}_0, \mu)$ into an $L^\infty(T)-L^\infty(T)$-bimodule by the formula

$$(F \cdot \xi \cdot F^t)(y, z) = F(y) \xi(y, z) F^t(z).$$

Under the natural identification of $L^\infty(T)$ and $A$, the unitary $T$ is $A$-$A$-bimodular.

By construction, $A L^2(\mathcal{R}_0, \mu) A$ is isomorphic with a direct sum of bimodules of the form $A \mathcal{H}(\alpha_j) A$ where the union of the graphs of the partial automorphisms $\alpha_j$ equals $\mathcal{R}_0$ and hence generates the equivalence relation $\mathcal{R}_{n,m}$. Applying Lemma 3.3 to $\mathcal{F} = \{u_0\}$, we conclude that $\mathcal{R}(A \subset M) \cong \mathcal{R}_{n,m}$ up to measure zero.

2. As in Section 2.4, denote by $\omega : \mathcal{R}_{n,m} \to \mathbb{R}$ the Radon-Nikodym 1-cocycle. Denote by $\Lambda \subset T$ the subgroup given by

$$\Lambda := \left\{ \exp \left( \frac{2\pi is}{(nm)k} \right) \mid s, k \in \mathbb{N} \right\}.$$

For every $z \in \Lambda$, we denote by $\alpha_z : T \to T$ the rotation $\alpha_z(y) = z y$. We have graph $\alpha_z \subset \mathcal{R}_{n,m}$ for all $z \in \Lambda$. Since all $\alpha_z$ are measure preserving, we actually have graph $\alpha_z \subset \text{Ker}(\omega)$. Since $\Lambda \subset T$ is a dense subgroup, it follows that $\text{Ker}(\omega)$ is an ergodic equivalence relation. In particular, $\mathcal{R}_{n,m}$ is ergodic. A direct computation shows that $\omega(y, z) = \log(n/m)$ for all $(y, z) \in \mathcal{R}_0$. Since $\mathcal{R}_0$ generates the equivalence relation $\mathcal{R}_{n,m}$, it follows that the essential image of $\omega$ equals $\log(n/m) \mathbb{Z}$. Using Lemma 2.7, we conclude that $\mathcal{R}_{n,m}$ is of type $\text{III}_{n|m}$. By construction, $\mathcal{R}_{n,m}$ is hyperfinite.

We are now ready to prove our main theorem.

**Proof of Theorem** Fix for $i = 1, 2$, integers $n_i, m_i \in \mathbb{Z}$ with $2 \leq n_i < |m_i|$. Put $M_i = L(\text{BS}(n_i, m_i))$ and denote by $A_i \subset M_i$ the abelian von Neumann subalgebra generated by $u_a$, where $a \in \text{BS}(n_i, m_i)$ is the first canonical generator. Assume that $M_1$ and $M_2$ are stably isomorphic. We must prove that

$$\frac{n_1}{|m_1|} = \frac{n_2}{|m_2|}.$$  (4)
Interchanging if necessary the roles of $M_1$ and $M_2$, we can take a nonzero projection $p_1 \in A_1$ and a $*$-isomorphism $\alpha : p_1 M_1 p_1 \to M_2$.

We claim that inside $M_2$, we have $\alpha(A_1 p_1) \prec A_2$. From Proposition [L3], we know that

$$P := \alpha(A_1 p_1)' \cap M_2 = \alpha((A_1' \cap M_1)p_1)$$

has no amenable direct summand. By Proposition 3.1 in [Ue07], the HNN extension $M_2$ can be viewed as the corner of an amalgamated free product of tracial von Neumann algebras. Since $P$ has no amenable direct summand, it then follows from [CH08, Theorem 4.2] that $P' \cap M_2 \prec A_2$.

So our claim that $\alpha(A_1 p_1) \prec A_2$ follows.

By symmetry, we also have the intertwining $\alpha^{-1}(A_2) \prec A_1 p_1$ inside $p_1 M_1 p_1$. Applying $\alpha$, we find that $A_2 \prec \alpha(A_1 p_1)$ inside $M_2$.

Having proven that inside $M_2$ we have the intertwining relations $\alpha(A_1 p_1) \prec A_2$ and $A_2 \prec \alpha(A_1 p_1)$, it follows from Theorem [8.3] that the equivalence relations $R(A_1 p_1 \subset p_1 M_1 p_1)$ and $R(A_2 \subset M_2)$ are stably isomorphic. By construction, $R(A_1 p_1 \subset p_1 M_1 p_1)$ is the restriction of $R(A_1 \subset M_1)$ to the support of $p_1$. So we conclude that the equivalence relations $R(A_1 \subset M_1)$ and $R(A_2 \subset M_2)$ are stably isomorphic. In particular, these ergodic nonsingular equivalence relations must have the same type. Using Proposition [1.2] we find [4].

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