Homogenization and Filtration and Seismic Acoustic Problems in Thermo-elastic Porous Media

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Abstract. A linear system of differential equations describing a joint motion of thermoelastic porous body and thermofluid occupying porous space is considered. Although the problem is linear, it is very hard to tackle due to the fact that its main differential equations involve non-smooth oscillatory coefficients, both big and small, under the differentiation operators. The rigorous justification, under various conditions imposed on physical parameters, is fulfilled for homogenization procedures as the dimensionless size of the pores tends to zero, while the porous body is geometrically periodic. As the results, we derive Biot’s like system of equations of thermo-poroelasticity, system of equations of thermo-viscoelasticity, or decoupled system consisting of non-isotropic Lamé’s equations for thermoelastic solid and Darcy’s system of filtration for thermofluid, depending on ratios between physical parameters. The proofs are based on Nguetseng’s two-scale convergence method of homogenization in periodic structures.

Key words: Biot’s equations, Stokes equations, Lamé’s equations, two-scale convergence, homogenization of periodic structures, thermo-poroelasticity, thermo-visco-elasticity.

Introduction

In this article a problem of modelling of small perturbations in thermoelastic deformable solid, perforated by a system of channels (pores) filled with thermofluid, is considered. The solid component of such a medium has a name of skeleton, and the domain, which is filled with a fluid, is named a porous space. The exact mathematical model of such of medium consists of the classical equations of momentum, energy and mass balance, which are stated in Euler variables, of the equations determining stress fields and thermodynamics law in both solid and liquid phases, and of an endowing relation

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determining behavior of the interface between liquid and solid components. The latter relation expresses the fact that the interface is a material surface, which amounts to the condition that it consists of the same material particles all the time. Clearly the above stated original model is a model with an unknown (free) boundary. The more precise formulation of the nonlinear problem is not in focus of our present work. Instead, we aim to study the problem, linearized at the rest state. In continuum mechanics the methods of linearization are developed rather deeply. The so obtained linear model has been studied in [11]. Further we refer to this model as to model (NA). In this model the characteristic function of the porous space $\bar{\chi}$ is a known function for $t > 0$. It is assumed that this function coincides with the characteristic function of the porous space $\bar{\chi}$, given at the initial moment.

In dimensionless variables (without primes)

$$x' = Lx, \quad t' = \tau t, \quad w' = Lw, \quad \theta' = \frac{L}{\tau v_s} \theta$$

the differential equations of model (NA) in the domain $\Omega \in \mathbb{R}^3$ for the dimensionless displacement vector $w$ of the continuum medium and the dimensionless temperature $\theta$, have the form:

$$\alpha_\tau \tilde{\rho} \frac{\partial^2 w}{\partial t^2} = \text{div}_x P + \tilde{\rho} F,$$

$$\alpha_\tau \tilde{c}_p \frac{\partial \theta}{\partial t} = \text{div}_x (\bar{\alpha}_\kappa \nabla_x \theta) - \bar{\alpha}_\theta \frac{\partial}{\partial t} \text{div}_x w + \Psi,$$

$$P = \bar{\chi} \alpha_\mu \mathbb{D} \left( x, \frac{\partial w}{\partial t} \right) + (1 - \bar{\chi}) \alpha_\lambda \mathbb{D}(x, w) - (q + \pi) \mathbb{I},$$

$$q = p + \frac{\alpha_\nu}{\alpha_p} \frac{\partial p}{\partial t} + \bar{\chi} \alpha_{\beta f} \theta,$$

$$p + \bar{\chi} \alpha_p \text{div}_x w = 0,$$

$$\pi + (1 - \bar{\chi}) (\alpha_{\eta} \text{div}_x w - \alpha_{\theta s} \theta) = 0.$$

Here and further we use notations

$$\mathbb{D}(x, u) = (1/2) \left( \nabla_x u + (\nabla_x u)^T \right),$$

$$\bar{\rho} = \bar{\chi} \rho_f + (1 - \bar{\chi}) \rho_s, \quad \bar{c}_p = \bar{\chi} c_{pf} + (1 - \bar{\chi}) c_{ps},$$

$$\alpha_\kappa = \bar{\chi} \alpha_{\kappa f} + (1 - \bar{\chi}) \alpha_{\kappa s}, \quad \bar{\alpha}_\theta = \bar{\chi} \alpha_{\theta f} + (1 - \bar{\chi}) \alpha_{\theta s}.$$
The dimensionless vector of distributed mass forces and dimensionless mean densities of the solid and the fluid at rest (all without primes), respectively, are given by the formulas

\[ F' = gF, \quad \rho_s' = \rho_0 \rho_s, \quad \rho_f' = \rho_0 \rho_f, \]

and the dimensionless ratios by the formulas

\[
\begin{align*}
\alpha_{\tau} &= \frac{L}{g \tau^2}, & \alpha_\mu &= \frac{2\mu}{\tau g \rho_0 L}, \\
\alpha_\nu &= \frac{(\nu - \frac{2}{3} \mu)}{\tau g \rho_0 L}, & \alpha_\eta &= \frac{(\eta - \frac{2}{3} \lambda)}{g \rho_0 L}, \\
\alpha_{\xi s} &= \frac{v_s^2 \tau g \rho_0 L}{\nu_s \tau g \rho_0 L}, & \alpha_{\xi f} &= \frac{\mu \tau g \rho_0 L}{v_s^2 \tau g \rho_0 L}, \\
\alpha_\lambda &= \frac{2\lambda}{g \rho_0 L}, & \alpha_p &= \frac{2 c_{fp} + \rho_f' c_{fp} \rho_f' (\rho_f')^2}{g \rho_0 L}, \\
\alpha_{\theta s} &= \frac{\rho_s \phi \rho_s \phi^2}{\rho_0 v_s^2}, & \alpha_{\theta f} &= \frac{\rho_f \phi \rho_f \phi^2}{v_s \tau g \rho_0}, \\
c_{pf} &= -\frac{c_{sf} \phi \rho_f' \phi^2}{\rho_0 v_s^2}, & c_{ps} &= -\frac{c_{sf} \phi \rho_s' \phi^2}{\rho_0 v_s^2},
\end{align*}
\]

(0.7)

where all dimensionless constants on the left-hand sides of relations are positive and all constants on the right-hand sides of relations are described in [11].

We endow model (NA) with initial data

\[ w|_{t=0} = w_0, \quad \frac{\partial w}{\partial t}|_{t=0} = v_0, \quad \theta|_{t=0} = \theta_0, \quad x \in \Omega \]

(0.8)

and homogeneous boundary conditions

\[ w = 0, \quad \theta = 0, \quad x \in S = \partial \Omega, \quad t \geq 0. \]

(0.9)

From the purely mathematical point of view, the corresponding initial-boundary value problem for model (NA) is well-posed in the sense that it has a unique solution belonging to a suitable functional space on any finite temporal interval (see [11]). However, in view of possible applications, for example, for developing numerical codes, this model is ineffective due to its sophistication even if a modern supercomputer is available. The differential equations of the model involve frequently oscillating non-smooth coefficients,
which have structures of linear combinations of the function $\overline{\chi}$. These coefficients undergo differentiation with respect to $x$ and besides may be very big or very small quantities as compared to the main small parameter $\varepsilon$. In the model under consideration we define the dimensionless size of the pores $\varepsilon$ as the characteristic size of pores $l$ divided by the characteristic size $L$ of the entire porous body:

$$\varepsilon = \frac{l}{L}.$$ 

Therefore a question of finding an effective approximate models is vital. Since the model involves the small parameter $\varepsilon$, the most natural approach to this problem is to derive models that would describe limiting regimes arising as $\varepsilon$ tends to zero. Such an approximation significantly simplifies the original problem and at the same time preserves all of its main features. But even this approach is too hard to work out, and some additional simplifying assumptions are necessary. In terms of geometrical properties of the medium, the most appropriate is to simplify the problem postulating that the porous structure is periodic. Further by model $(NB)^\varepsilon$ we will call model $NA$ supplemented by this periodicity condition. Thus, our main goal now is a derivation of all possible homogenized equations in the model $(NB)^\varepsilon$.

Remark 0.1. The particular case of the model $(NA)$ when the system $(0.1) - (0.6)$ is decoupled ($\alpha_{\theta f} = \alpha_{\theta s} = 0$) we call as model $(A)$. Correspondingly, the same particular case of the model $(NB)^\varepsilon$ we call as model $(B)^\varepsilon$.

The first research with the aim of finding limiting regimes in the case when the skeleton was assumed to be an absolutely rigid isothermal body was carried out by E. Sanchez-Palencia and L. Tartar. E. Sanchez-Palencia [14, Sec. 7.2] formally obtained Darcy’s law of filtration using the method of two-scale asymptotic expansions, and L. Tartar [14, Appendix] mathematically rigorously justified the homogenization procedure. Using the same method of two-scale expansions J. Keller and R. Burridge [3] derived formally the system of Biot’s equations [4] from the model $(B)^\varepsilon$ in the case when the parameter $\alpha_{\mu}$ was of order $\varepsilon^2$, and the rest of the coefficients were fixed independent of $\varepsilon$. It is well-known that the various modifications of Biot’s model are bases of seismic acoustics problems up-to-date. This fact emphasizes importance of comprehensive study of model $(NA)$ and model $(NB)^\varepsilon$ one more time. J. Keller and R. Burridge also considered model $(B)^\varepsilon$ under assumption that all the physical parameters were fixed independent of $\varepsilon$, and formally derived as the result a system of equations of viscoelasticity.
Under the same assumptions as in the article [3], the rigorous justification of Biot’s model was given by G. Nguetseng [13] and later by A. Mikelić, R. P. Gilbert, Th. Clopeaut, and J. L. Ferrin in [5, 6, 7]. Also A. Mikelić et al derived a system of equations of viscoelasticity, when all the physical parameters were fixed independent of $\varepsilon$. The most general case of the model $B^\varepsilon$ has been studied in [10]. In these works, Nguetseng’s two-scale convergence method [9, 12] was the main method of investigation of the model $(B)^\varepsilon$.

In the present work by means of the same method we investigate all possible limiting regimes in the model $(NB)^\varepsilon$. This method in rather simple form discovers the structure of the weak limit of a sequence $\{z^\varepsilon\}$ as $\varepsilon \searrow 0$, where $z^\varepsilon = u^\varepsilon v^\varepsilon$ and sequences $\{u^\varepsilon\}$ and $\{v^\varepsilon\}$ converge as $\varepsilon \searrow 0$ merely weakly, but at the same time function $u^\varepsilon$ has the special structure $u^\varepsilon(x) = u(x/\varepsilon)$ with $u(y)$ being periodic in $y$.

Moreover, this method allows to establish asymptotic expansions of a solution of the model $(NB)^\varepsilon$ in the form

$$w^\varepsilon(x, t) = \varepsilon^{\beta_f} \left( w^f_0(x, t) + \varepsilon w^f_1(x, t, x/\varepsilon) + o(\varepsilon) \right),$$

where $w^f_0(x, t)$ is a solution of the homogenized (limiting) problem, $w^f_1(x, t, y)$ is a solution of some initial-boundary value problem posed on the generic periodic cell of the porous space, and exponent $\beta$ is defined by dimensionless parameters of the model. In some situations expansion (0.10) has more complicated form like

$$w^\varepsilon(x, t) = \varepsilon^{\beta_f} \left( w^f_0'(x, t) + \varepsilon w^f_1(x, t, x/\varepsilon) + o(\varepsilon) \right),$$

in the liquid component and

$$w^\varepsilon(x, t) = \varepsilon^{\beta_s} \left( w^s_0(x, t) + \varepsilon w^s_1(x, t, x/\varepsilon) + o(\varepsilon) \right),$$

in the rigid component. For example, if $\alpha_\mu = \varepsilon^2$ and $\alpha_\lambda = \varepsilon^{-1}$, then $\beta_f = -1$, $w^f_0 = 0$ and $\beta_s = 1$.

Suppose that all dimensionless parameters depend on the small parameter $\varepsilon$ and there exist limits (finite or infinite)

$$\lim_{\varepsilon \searrow 0} \alpha_\mu(\varepsilon) = \mu_0, \quad \lim_{\varepsilon \searrow 0} \alpha_\lambda(\varepsilon) = \lambda_0, \quad \lim_{\varepsilon \searrow 0} \alpha_\tau(\varepsilon) = \tau_0.$$

We restrict our consideration by the cases when when $\tau_0 < \infty$ and either of
the following situations has place:

\[
\begin{align*}
(I) & \quad \mu_0 = 0, \quad 0 < \lambda_0 < \infty; \\
(II) & \quad 0 \leq \mu_0 < \infty, \quad \lambda_0 = \infty; \\
(III) & \quad 0 < \mu_0, \lambda_0 < \infty.
\end{align*}
\]

If \( \tau_0 = \infty \) then, re-normalizing the displacement vector and temperature by setting

\[ w \to \alpha \tau w, \quad \theta \to \alpha \tau \theta, \]

we reduce the problem to one of the cases (I)–(III).

In the present paper we show that for the first case the limiting regime is a two-velocity and two-temperature continuum media, which is described by different types of Biot’s like system of equations of poro-elasticity coupled with corresponding heat equations (equations of thermo-poro-elasticity), or one-velocity and one-temperature continuum media, which is described by non-isotropic Lamé’s system of equations coupled with corresponding heat equation (equations of thermo-elasticity) (theorem 2.2). For the second case the limiting regime is a two-temperature continuum media, which is described as a first approximation by different types of Darcy’s like system of equations of filtration for the velocity of the liquid component, coupled with corresponding heat equations (for this first approximation the solid component behaves yourself as absolutely rigid body) and as a second approximation – non-isotropic Lamé’s system of elasticity for the re-normalized displacements of the solid component, or two-velocity and two-temperature continuum media, which is described by Biot’s like system of equations of thermo-poro-elasticity for the re-normalized displacements (theorem 2.3). Finally, for the last case the limiting regime is a one-velocity and one-temperature continuum media, which is described by system of non-local equations of thermo-visco-elasticity or non-local equations of thermo-elasticity (theorem 2.4).

§1. Model \((NB)^\varepsilon\)

1.1. Geometry of porous space. In the model \((NB)^\varepsilon\) the porous space has geometrically periodic structure. Its formal description is as follows [7, 2].

Firstly a geometric structure inside a pattern unit cell \( Y = (0, 1)^3 \) is defined. Let \( Y_s \) be a ‘solid part’ of the cell \( Y \). The ‘liquid part’ \( Y_f \) is its open
complement. Set $Y^k_s := Y_s + k$, $k \in \mathbb{Z}^3$, the translation of $Y_s$ on an integer-valued vector $k$. Union of such translations along all $k$, $E_s := \bigcup_{k \in \mathbb{Z}^3} Y^k_s$ is the 1-periodic repetition of $Y_s$ all over $\mathbb{R}^3$. Let $E_f$ be the open complement of $E_s$ in $\mathbb{R}^3$. The following assumptions on geometry of $Y_s$ and $E_f$ are accepted.

(i) $Y_s$ is an open connected set of strictly positive measure with a Lipschitz boundary, and $Y_f$ also has strictly positive measure on $Y$. 

(ii) $E_f$ and $E_s$ are open sets with $C^{0, 1}$-smooth boundaries. The set $E_f$ is locally situated on one side of the boundary $\partial E_f$, and the set $E_s$ is locally situated on one side of the boundary $\partial E_s$ and connected.

Domains $\Omega^\varepsilon_s$ and $\Omega^\varepsilon_f$ are intersections of the domain $\Omega$ with the sets $\mathbb{R}_s^3$ and $\mathbb{R}_f^3$, where the sets $\mathbb{R}_s^3$ and $\mathbb{R}_f^3$ are periodic domains in $\mathbb{R}^3$ with generic cells $\varepsilon Y_s$ and $\varepsilon Y_f$ of the diameter $\varepsilon$, respectively.

Union $\bar{Y}_s \cup \bar{Y}_f$ is the closed cube $\bar{Y} = \{y \in \mathbb{R}^3, 0 \leq y_i \leq 1, i = 1, 2, 3\}$, and the interface $\Gamma^\varepsilon = \partial \Omega^\varepsilon_s \cap \partial \Omega^\varepsilon_f$ is the $\varepsilon$-periodic repetition of the boundary $\varepsilon \gamma = \varepsilon \partial Y_f \cap \partial Y_s$ all over $\omega$.

Further by $\bar{\chi} = \chi^\varepsilon$ we will denote the characteristic function of the porous space.

For simplicity we accept the following constraint on the domain $\Omega$ and the parameter $\varepsilon$.

**Assumption 1.1.** Domain $\Omega$ is cube, $\Omega := (0, 1)^3$, and quantity $1/\varepsilon$ is integer, so that $\Omega$ always contains an integer number of elementary cells $Y^\varepsilon_i$.

Under this assumption, we have

$$\bar{\chi}(x) = \chi^\varepsilon(x) = \chi(x/\varepsilon),$$

$$\bar{c}_p = c^\varepsilon_p(x) = \chi^\varepsilon(x)c_{pf} + (1 - \chi^\varepsilon(x))c_{ps},$$

$$\bar{\rho} = \rho^\varepsilon(x) = \chi^\varepsilon(x)\rho_f + (1 - \chi^\varepsilon(x))\rho_s,$$

$$\bar{\alpha}_\kappa = \alpha^\varepsilon_\kappa(x) = \chi^\varepsilon(x)\alpha_{\kappa f} + (1 - \chi^\varepsilon(x))\alpha_{\kappa s},$$

$$\bar{\alpha}_\theta = \alpha^\varepsilon_\theta(x) = \chi^\varepsilon(x)\alpha_{\theta f} + (1 - \chi^\varepsilon(x))\alpha_{\theta s},$$

where $\chi(y)$ is the characteristic function of $Y_f$ in $Y$.

We say that a porous space is disconnected (isolated pores) if $\gamma \cap \partial Y = \emptyset$.

**1.2. Generalized solutions in the model $(NB)^\varepsilon$.** Equations (0.1)-(0.6) are understood in the sense of distributions theory. They involve the
equations (0.1) – (0.6) in a usual sense in the domains $\Omega_f^\varepsilon$ and $\Omega_s^\varepsilon$ and the boundary conditions

$$
[\vartheta] = 0, \quad [w] = 0, \quad x_0 \in \Gamma^\varepsilon, \ t \geq 0, \quad (1.2)
$$

$$
[P] = 0, \quad [\alpha^\varepsilon_x \nabla_x \vartheta] = 0, \quad x_0 \in \Gamma^\varepsilon, \ t \geq 0 \quad (1.3)
$$
on the interface $\Gamma^\varepsilon$, where

$$
[\varphi](x_0) = \varphi(s)(x_0) - \varphi(f)(x_0),
$$

$$
\varphi(s)(x_0) = \lim_{x \to x_0 \in \Omega^\varepsilon_s} \varphi(x), \quad \varphi(f)(x_0) = \lim_{x \to x_0 \in \Omega^\varepsilon_f} \varphi(x).
$$

There are various equivalent in the sense of distributions forms of representation of equations (0.1) – (0.2) and boundary conditions (1.2) – (1.3). In what follows, it is convenient to write them in the form of the integral equalities.

**Definition 1.1.** Five functions $(w^\varepsilon, \theta^\varepsilon, p^\varepsilon, q^\varepsilon, \pi^\varepsilon)$ are called a generalized solution of **model (NB)$^\varepsilon$** if they satisfy the regularity conditions in the domain $\Omega_T = \Omega \times (0, T)$

$$
w^\varepsilon, \mathbb{D}(x, w^\varepsilon), \text{div}_x w^\varepsilon, q^\varepsilon, p^\varepsilon, \frac{\partial p^\varepsilon}{\partial t}, \pi^\varepsilon, \theta^\varepsilon, \nabla_x \theta^\varepsilon \in L^2(\Omega_T), \quad (1.4)
$$

boundary conditions (0.9) in the trace sense, equations

$$q^\varepsilon = p^\varepsilon + \alpha_p \frac{\partial p^\varepsilon}{\partial t} + \chi^\varepsilon \alpha_{\theta f} \theta^\varepsilon, \quad (1.5)
$$

$$p^\varepsilon + \chi^\varepsilon \alpha_p \text{div}_x w^\varepsilon = 0, \quad (1.6)
$$

$$\pi^\varepsilon + (1 - \chi^\varepsilon)(\alpha_\eta \text{div}_x w^\varepsilon - \alpha_{\theta s} \theta^\varepsilon) = 0, \quad (1.7)
$$
a.e. in $\Omega_T$, and the integral equalities

$$
\int_{\Omega_T} \left( \alpha_x \rho^\varepsilon w^\varepsilon \cdot \frac{\partial^2 \varphi}{\partial t^2} - \chi^\varepsilon \alpha_\mu \mathbb{D}(x, w^\varepsilon) : \mathbb{D}(x, \frac{\partial \varphi}{\partial t}) - \rho^\varepsilon \mathbf{F} \cdot \nabla \varphi + \{(1 - \chi^\varepsilon)\alpha_\lambda \mathbb{D}(x, w^\varepsilon) - (q^\varepsilon + \pi^\varepsilon)\mathbb{I} \} : \mathbb{D}(x, \varphi) \right) dx dt +
$$

$$
\int_{\Omega} \alpha_x \rho^\varepsilon \left( w_0^\varepsilon \cdot \frac{\partial \varphi}{\partial t}|_{t=0} - v_0^\varepsilon \cdot \varphi|_{t=0} \right) dx = 0 \quad (1.8)
$$
for all smooth $\varphi = \varphi(x,t)$ such that $\varphi|_{\partial \Omega} = \varphi|_{t=T} = \partial \varphi / \partial t|_{t=T} = 0$ and
\begin{align*}
\int_{\Omega_T} \left( (\alpha_{C} \varepsilon \theta^c + \alpha_{D} \varepsilon \omega^c) \frac{\partial \xi}{\partial t} - \alpha_{E} \varepsilon \nabla \theta^c \cdot \nabla_x \xi + \Psi \xi \right) d\mathbf{x} dt \\
+ \int_{\Omega} \left( (\alpha_{C} \varepsilon \theta^c + \alpha_{D} \varepsilon \omega^c) \xi|_{t=0} \right) d\mathbf{x} = 0
\end{align*}
(1.9)
for all smooth $\xi = \xi(x,t)$ such that $\xi|_{\partial \Omega} = \xi|_{t=T} = 0$.

In (1.8) by $A : B$ we denote the convolution (or, equivalently, the inner tensor product) of two second-rank tensors along the both indexes, i.e., $A \cdot B = \text{tr} (B^* \circ A) = \sum_{i,j=1}^{3} A_{ij} B_{ij}$.

The first result on existence and uniqueness of solutions to model $(\mathcal{N}B)^\varepsilon$ is the following lemma (1.1):

**Lemma 1.1.** Whenever sequences $\{\sqrt{\alpha} \chi(1 - \chi^\varepsilon) \nabla \mathbf{w}_0^\varepsilon\}$, $\{\sqrt{\alpha} \Psi^0\}$, $\{\sqrt{\alpha} \theta_0^\varepsilon\}$, $\{\sqrt{\alpha} \chi^\varepsilon \nabla \mathbf{w}_0^\varepsilon\}$, $\{\sqrt{\alpha} \varepsilon^\chi \div \mathbf{w}_0^\varepsilon\}$ are uniformly bounded in $L^2(\Omega)$ with respect to $\varepsilon$ and $\mathbf{F}, \Psi, \in L^2(\Omega_T)$, model $(\mathcal{N}B)^\varepsilon$ has a unique generalized solution $(\mathbf{w}^\varepsilon, \theta^c, p^\varepsilon, q^\varepsilon, \nu^\varepsilon)$ in the sense of Definition 1.1 and

\begin{align*}
\max_{0<t<T} \left( \sqrt{\alpha} || \mathbf{w}^\varepsilon(t) ||_{2,\Omega_T} + \sqrt{\alpha} || \nabla \mathbf{w}^\varepsilon(t) ||_{2,\Omega_T} \right) \\
+ \sqrt{\alpha} || \mathbf{w}^\varepsilon(t) ||_{2,\Omega_T} + \sqrt{\alpha} || \nabla \mathbf{w}^\varepsilon(t) ||_{2,\Omega_T} + \sqrt{\alpha} || \theta^c(t) ||_{2,\Omega_T} \\
+ \sqrt{\alpha} || \nabla_x \theta^c ||_{2,\Omega_T} + \sqrt{\alpha} || (1 - \chi^\varepsilon) \nabla \theta^c ||_{2,\Omega_T} \\
+ \sqrt{\alpha} || \chi^\varepsilon \div \mathbf{w}^\varepsilon ||_{2,\Omega_T} + \sqrt{\alpha} || \chi^\varepsilon \nabla \mathbf{w}^\varepsilon ||_{2,\Omega_T} \leq C_0 \sqrt{\alpha}.
\end{align*}
(1.10)

where $C_0$ is a constant independent of $\varepsilon$.

§2. Formulation of the main results

Suppose additionally that there exist limits (finite or infinite)

\begin{align*}
\lim_{\varepsilon \searrow 0} \alpha_{p}(\varepsilon) = p_s, & \quad \lim_{\varepsilon \searrow 0} \alpha_{\nu}(\varepsilon) = \nu_0, \quad \lim_{\varepsilon \searrow 0} \alpha_{\theta f}(\varepsilon) = \theta_{0f}, \quad \lim_{\varepsilon \searrow 0} \alpha_{\theta s}(\varepsilon) = \theta_{0s}, \\
\lim_{\varepsilon \searrow 0} \alpha_{q}(\varepsilon) = q_0, & \quad \lim_{\varepsilon \searrow 0} \alpha_{\theta s}(\varepsilon) = \theta_{0s}, \quad \lim_{\varepsilon \searrow 0} \alpha_{\theta f}(\varepsilon) = \theta_{0f}, \quad \lim_{\varepsilon \searrow 0} \alpha_{\theta f}(\varepsilon) = \theta_{1f}, \\
\lim_{\varepsilon \searrow 0} \alpha_{\mu}(\varepsilon) = \mu_1, & \quad \lim_{\varepsilon \searrow 0} \varepsilon_0^2 \frac{\alpha_{p}}{\alpha_{\mu}} = p_1, \quad \lim_{\varepsilon \searrow 0} \varepsilon_0^2 \frac{\alpha_{\theta s}}{\alpha_{\mu}} = \theta_{1s}, \quad \lim_{\varepsilon \searrow 0} \varepsilon_0^2 \frac{\alpha_{\theta f}}{\alpha_{\mu}} = \theta_{1f},
\end{align*}

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Assumption 2.1. Dimensionless parameters in the model \((NB)\) satisfy restrictions

\[
0 < \tau_0 + \mu_1, \quad \kappa_0, \quad \kappa_1, \quad \lambda_0, \quad p_\times, \quad \eta_0;
\]

\[
\tau_0, \quad \kappa_1, \quad \mu_0, \quad \nu_0 < \infty.
\]

In what follows all parameters may take all permitted values. For example, if \(\tau_0 = 0\) or \(\beta_{0s} = 0\), then all terms in final equations containing these parameters disappear.

The following Theorems 2.1–2.4 are the main results of the paper.

Theorem 2.1. Assume that conditions of lemma \(L\) hold, \(\partial F / \partial t \in L^2(\Omega_T)\), and that \(\{w^\varepsilon, \theta^\varepsilon, q^\varepsilon, p^\varepsilon, \pi^\varepsilon\}\) is a generalized solution in Model \((NB)\).

The following assertions hold true:

(I) If

\[
\max_{0 \leq t \leq T} \|w^\varepsilon(t)\| + \sqrt{\alpha_\mu} \varepsilon \|\nabla w^\varepsilon(t)\| + \|\nabla w^\varepsilon(t)\| \leq C_0,
\]

\[
\|\theta^\varepsilon\|_{2,\Omega_T} + \sqrt{\alpha_{\kappa_1}} \varepsilon \|\nabla \theta^\varepsilon\|_{2,\Omega_T} \leq C_0,
\]

\[
\|\nabla \theta^\varepsilon\|_{2,\Omega_T} + \|\nabla \theta^\varepsilon\|_{2,\Omega_T} \leq C_0.
\]

(II) If

\[
\lambda_0 = \kappa_0 = \mu_1 = \infty, \quad 0 < \lambda_1, \quad p_1, \quad \eta_1, \quad \beta_{1s}, \quad \beta_1, \quad \kappa_{1s} \leq \infty,
\]

and sequences \(\epsilon^{-1} \sqrt{\alpha_\mu} \alpha_\mu (1 - \chi^\varepsilon) \nabla w_0^\varepsilon, \epsilon^{-1} \sqrt{\alpha_\tau} \alpha_\mu \theta_0^\varepsilon, \epsilon^{-1} \sqrt{\alpha_\kappa} \alpha_\mu \alpha_\kappa \chi^\varepsilon \ \div w_0^\varepsilon, \epsilon^{-1} \sqrt{\alpha_\kappa} \alpha_\mu \alpha_\kappa \chi^\varepsilon \ \div w_0^\varepsilon\) are uniformly bounded with respect to \(\varepsilon\) in \(L^2(\Omega)\), then estimates (2.1), (2.2), (2.3) hold for pressures \(q^\varepsilon, p^\varepsilon, \pi^\varepsilon\) and re-normalized displacements and temperature

\[
w^\varepsilon \to \epsilon^{-2} \alpha_\mu w^\varepsilon, \quad \theta^\varepsilon \to \epsilon^{-2} \alpha_\mu \theta^\varepsilon.
\]
with re-normalized parameters

\[
\alpha_\mu \to \varepsilon^2, \quad \alpha_\lambda \to \varepsilon^2 \frac{\alpha_\lambda}{\alpha_\mu}, \quad \alpha_\tau \to \varepsilon^2 \frac{\alpha_\tau}{\alpha_\mu}, \quad \alpha_\nu \to \varepsilon^2 \frac{\alpha_\nu}{\alpha_\mu},
\]

\[
\alpha_{\theta f} \to \varepsilon^2 \frac{\alpha_{\theta f}}{\alpha_\mu}, \quad \alpha_{\theta s} \to \varepsilon^2 \frac{\alpha_{\theta s}}{\alpha_\mu}, \quad \alpha_\eta \to \varepsilon^2 \frac{\alpha_\eta}{\alpha_\mu},
\]

\[
\alpha_{\kappa f} \to \varepsilon^2 \frac{\alpha_{\kappa f}}{\alpha_\mu}, \quad \alpha_{\kappa s} \to \varepsilon^2 \frac{\alpha_{\kappa s}}{\alpha_\mu}, \quad \alpha_p \to \varepsilon^2 \frac{\alpha_p}{\alpha_\mu}.
\]

(III) If \(\lambda_0 = \eta_0 = \infty, \quad \mu_1 < \infty\).

Then for displacements \(w^\varepsilon\) and temperature \(\theta^\varepsilon\) hold true estimates \((2.1), (2.2)\) and if

\[
p_* < \infty, \quad \beta_{0f} < \infty,
\]

then for the pressures \(q^\varepsilon\) and \(p^\varepsilon\) in the liquid component hold true estimates \((2.3)\).

If instead of restriction \((2.4)\) hold true conditions

\[
0 < p_2, \eta_2 < \infty; \quad \Psi = 0; \quad F = \nabla \Phi, \quad \frac{\partial \Phi}{\partial t}, \quad |\frac{\partial F}{\partial t}| \in L^2(\Omega_T),
\]

then are valid estimates

\[
\max_{0 \leq t \leq T} \| (1 - \chi^\varepsilon) \nabla_x (\alpha_\lambda w^\varepsilon) \| + |\chi^\varepsilon \text{div}_x(\alpha_\lambda w^\varepsilon)| + \| (1 - \chi^\varepsilon) \text{div}_x(\alpha_\lambda w^\varepsilon) \| + \sqrt{\alpha_\kappa} \alpha_\lambda \| (1 - \chi^\varepsilon) \nabla_x (\alpha_\lambda \theta^\varepsilon) \|_{2, \Omega_T} \leq C_0,
\]

which imply estimates \((2.3)\) if \(\beta_{0f}, \beta_{0s} < \infty\).

In \((2.1) - (2.6)\) \(C_0\) is a constant independent of the small parameter \(\varepsilon\).

Note, that for the last case \(\{\lambda_0 = \eta_0 = \infty, \mu_1 < \infty\}\) we can get same estimates \((2.1)-(2.3)\) and \((2.6)\), if instead restrictions \((2.5)\) and assumption \(2.2\) we assume restrictions \((2.4)\) together with
Assumption 2.3.

1) \( F = F^\varepsilon(x,t)(1 - \chi^\varepsilon), \quad \Psi = \Psi^\varepsilon(x,t)(1 - \chi^\varepsilon), \quad \eta_2 < \infty; \)

2) sequences \( \{\alpha_\varepsilon(1-\chi^\varepsilon)\nabla w_0^\varepsilon\} \), \( \{\sqrt{\alpha_r}\alpha_\varepsilon v_0^\varepsilon\} \), \( \{\sqrt{\alpha_p}\alpha_\varepsilon(1-\chi^\varepsilon)\text{div}_x w_0^\varepsilon\} \) and \( \{\sqrt{\alpha_p}\alpha_\varepsilon\varepsilon\text{div}_x w_0^\varepsilon\} \) are uniformly bounded with respect to \( \varepsilon \) in \( L^2(\Omega) \) and sequences \( \{F^\varepsilon\}, \{\partial F^\varepsilon/\partial t\}, \{\Psi^\varepsilon\} \) and \( \{\partial \Psi^\varepsilon/\partial t\} \) are uniformly bounded with respect to \( \varepsilon \) in \( L^2(\Omega_T) \).

Theorem 2.2. Assume that the hypotheses in Theorem 2.1 hold, and

\[
\mu_0 = 0; \quad \lambda_0, \quad \rho_0, \quad \beta_0, \quad \beta_0 < \infty. \tag{2.7}
\]

Then functions \( \theta^\varepsilon \) and \( \vartheta^\varepsilon \) admit an extension \( \theta^\varepsilon \) and \( \vartheta^\varepsilon \) respectively from \( \Omega_{s,T}^\varepsilon = \Omega^\varepsilon_s \times (0,T) \) into \( \Omega_T \) such that the sequences \( \{\theta^\varepsilon\} \) and \( \{\vartheta^\varepsilon\} \) converge strongly in \( L^2(\Omega_T) \) and weakly in \( L^2((0,T);W_2^1(\Omega)) \) to the functions \( \theta \) and \( \vartheta \) respectively. At the same time, sequences \( \{\theta^\varepsilon\}, \{\vartheta^\varepsilon\}, \{p^\varepsilon\}, \{q^\varepsilon\} \), and \( \{\pi^\varepsilon\} \) converge weakly in \( L^2(\Omega_T) \) to \( w, \theta, p, q, \) and \( \pi \), respectively.

The following assertions for these limiting functions hold true:

1) If \( \mu_1 = \infty \) then \( w = u, \quad \theta = \vartheta \) and the weak limits \( u, \vartheta, \) and \( p, q, \) and \( \pi \) satisfy in \( \Omega_T \) the initial-boundary value problem

\[
\tau_0 \partial^2 u \partial t^2 + \nabla (q + \pi) - \dot{\rho} F = \div_x \{\lambda_0 A_s^0 : D(x,u) + B_s^0 (\text{div}_x u - \frac{\beta_0}{\eta_0} \vartheta) + B_s^0 q\}, \tag{2.8}
\]

\[
(\tau_0 \dot{c}_p + \frac{\beta_0^2}{\eta_0} (1-m)) \frac{\partial \vartheta}{\partial t} - \frac{\beta_0}{\eta_0} \frac{\partial p}{\partial t} - \frac{\beta_0}{\eta_0} \frac{\partial \pi}{\partial t} = \div_x (B^\theta \cdot \nabla \vartheta) + \Psi, \tag{2.9}
\]

\[
\frac{1}{\eta_0} \pi + C_0^s : D(x,u) + a_0^s (\text{div}_x u - \frac{\beta_0}{\eta_0} \vartheta) + a_0^s q = 0, \tag{2.10}
\]

\[
\frac{1}{\eta_0} p + \frac{1}{\eta_0} \pi + \text{div}_x u - (1-m) \frac{\beta_0}{\eta_0} \vartheta = 0, \tag{2.11}
\]

\[
q = p + \nu_0 p_s \frac{1}{\partial t} + \beta_0 \vartheta, \tag{2.12}
\]

where

\[
\dot{\rho} = m \rho_f + (1-m) \rho_s, \quad \dot{c}_p = m c_p f + (1-m) c_{ps}, \quad m = \int_Y \chi(y) dy.
\]
The symmetric strictly positively defined constant fourth-rank tensor $A_0$, constant matrices $C_0, B_0 B_1$, strictly positively defined constant matrix $B^\vartheta$ and constants $a_0^s, a_1^s$ are defined below by Eqs. (5.35) - (5.37) and (5.40).

Differential equations (2.8)-(2.12) are endowed with initial conditions at $t = 0$ and $x \in \Omega$:

$$(\tau_0 + \beta_{0\vartheta})(\vartheta - \vartheta_0) = 0, \quad \tau_0(u - u_0) = \tau_0(\frac{\partial u}{\partial t} - v_0) = 0; \quad (2.13)$$

and boundary conditions

$$\vartheta(x,t) = 0, \quad u(x,t) = 0, \quad x \in S, \quad t > 0. \quad (2.14)$$

(II) If the porous space is disconnected, then $w = u$ and strong and weak limits $u, \vartheta, p, q, \pi$ together with a weak limit $\theta^f$ of the sequence $\{\chi^\varepsilon \theta^\varepsilon\}$ satisfy in $\Omega_T$ equations (2.8), (2.10)–(2.11), equation

$$q = p + \nu_0 \beta_{1\vartheta} \frac{\partial p}{\partial t} + \beta_0 \theta^f, \quad (2.15)$$

and heat equation

$$\tau_0 c_f \frac{\partial \theta^f}{\partial t} + (\tau_0 c_p + \beta_{0\vartheta}^2 \eta_0)(1 - m) \frac{\partial \vartheta}{\partial t} - \beta_0 \beta_{1\vartheta} \frac{\partial p}{\partial t} - \beta_{0\vartheta} \frac{\partial \pi}{\partial t} =$$

$$\text{div}_x(B^\vartheta \cdot \nabla \vartheta) + \Psi. \quad (2.16)$$

Here $\theta^f$ is defined below by Eqs. (5.42)–(5.48) and $A_0^s, C_0^s B_0^s B_1^s B^\vartheta a_0^s a_1^s$ are the same as in (2.8), continuity equation (2.10), continuity equation 1

$$(p^s + \eta_0 \pi + \text{div}_x w^f - (1 - m) \beta_{0s} \eta_0 \vartheta) = (m - 1) \text{div}_x u, \quad (2.18)$$

The problem is endowed with initial and boundary conditions (2.13)-(2.14).

(III) If $\mu_1 < \infty$ then strong and weak limits $u, \vartheta, w^f, \theta^f, p, q$ and $\pi$ of the sequences $\{u^\varepsilon\}, \{\vartheta^\varepsilon\}, \{\chi^\varepsilon w^\varepsilon\}, \{\chi^\varepsilon \theta^\varepsilon\}, \{p^\varepsilon\}, \{q^\varepsilon\}$ and $\{\pi^\varepsilon\}$ satisfy the initial-boundary value problem in $\Omega_T$, consisting of the balance of momentum equation

$$\tau_0 \left( \rho_f \frac{\partial^2 w^f}{\partial t^2} + \rho_s(1 - m) \frac{\partial^2 u}{\partial t^2} \right) + \nabla(q + \pi) - \hat{\rho}F = \text{div}_x\{\lambda_0 A_0^s \nabla x, u \} + B_0^s(\text{div}_x u - \frac{\beta_{0s}}{\eta_0} \vartheta) + B_1^s q\},$$

where $A_0^s, B_0^s$ and $B_1^s$ are the same as in (2.8), continuity equation (2.10), continuity equation 1.
state equation (2.15), heat equation (2.16) and Darcy’s law in the form
\[
\frac{\partial w^f}{\partial t} = m \frac{\partial u}{\partial t} + \int_0^t B_1(\mu_1, t - \tau) \cdot (\nabla x q + \rho_f F - \tau_0 \rho_f \frac{\partial^2 u}{\partial \tau^2})(x, \tau) d\tau \tag{2.19}
\]
in the case of \(\tau_0 > 0\) and \(\mu_1 > 0\), Darcy’s law in the form
\[
\frac{\partial w^f}{\partial t} = \frac{\partial u}{\partial t} + B_2(\mu_1) \cdot (\nabla x q + \rho_f F) \tag{2.20}
\]
in the case of \(\tau_0 = 0\) and, finally, Darcy’s law in the form
\[
\frac{\partial w^f}{\partial t} = B_3 \cdot \frac{\partial u}{\partial t} + \frac{1}{\tau_0 \rho_f} (mI - B_3) \cdot \int_0^t (\nabla x q + \rho_f F)(x, \tau) d\tau \tag{2.21}
\]
in the case of \(\mu_1 = 0\). The problem is supplemented by boundary and initial conditions (2.13)–(2.14) for the displacement \(u\) and temperature \(\vartheta\) of the rigid component and by the boundary condition
\[
w^f(x, t) \cdot n(x) = 0, \quad (x, t) \in S = \partial \Omega, \quad t > 0, \tag{2.22}
\]
for the displacement \(w^f\) of the liquid component. In Eqs. (2.19)–(2.21) \(n(x)\) is the unit normal vector to \(S\) at a point \(x \in S\), and matrices \(B_1(\mu_1, t)\), \(B_2(\mu_1)\), and \(B_3\) are given in [10].

**Theorem 2.3.** Assume that the hypotheses in Theorem 2.1 hold, and that
\[
\lambda_0 = \infty. \tag{2.23}
\]

**(I)** If \(\gamma_{0s}, \beta_{0f}, \mu_1 < \infty\) and one of conditions (2.4) or (2.5) holds true, then sequences \(\{\chi^{\varepsilon} w^{\varepsilon}\}\), \(\{\chi^{\varepsilon} \theta^{\varepsilon}\}\), \(\{p^{\varepsilon}\}\), and \(\{q^{\varepsilon}\}\) converge weakly in \(L^2(\Omega_T)\) to \(w^f\), \(\theta^f\), \(p\), and \(q\) respectively. The functions \(w^f\) and \(\theta^f\) admit an extension \(u^\varepsilon\) and \(\vartheta^\varepsilon\) respectively from \(\Omega^\varepsilon \times (0, T)\) into \(\Omega_T\) such that the sequences \(\{w^\varepsilon\}\) and \(\{\theta^\varepsilon\}\) converge strongly in \(L^2(\Omega_T)\) and weakly in \(L^2((0, T); W^1_2(\Omega))\) to zero and \(\vartheta\) respectively and

1) if \(\tau_0 > 0\) and \(\mu_1 > 0\), then functions \(w^f\), \(\theta^f\), \(\vartheta\), \(p\) and \(q\) solve the problem \((F_1)\) which consist of the state equation (2.15), continuity equation
\[
\frac{1}{p^*} p + \text{div}_x w^f = 0, \tag{2.24}
\]
heat equation

\[ \tau_0(c_p f \frac{\partial \theta^f}{\partial t} + c_p s (1 - m) \frac{\partial \theta}{\partial t}) - \frac{\beta_0 f \partial p}{p_s \partial t} = \text{div}_x (B^0 \cdot \nabla \vartheta) + \Psi, \quad (2.25) \]

Darcy’s law in the form

\[ \frac{\partial w^f}{\partial t} = m v_0(x) + \int_0^t B_1(\mu_1, t - \tau) \cdot (-\nabla_x q + \rho_f F(x, \tau)) d\tau; \quad (2.26) \]

2) if \( \tau_0 = 0 \) and \( \mu_1 > 0 \), then functions \( w^f, \theta^f, \vartheta, p \) and \( q \) solve the problem \( (F_2) \), which consist of equations \( (2.15), (2.24), (2.25) \) and Darcy’s law in the form

\[ \frac{\partial w^f}{\partial t} = B_2(\mu_1) \cdot (-\nabla_x q + \rho_f F); \quad (2.27) \]

finally,

3) if \( \tau_0 > 0 \) and \( \mu_1 = 0 \), then functions \( w^f, \theta^f, \vartheta, p \) and \( q \) solve the problem \( (F_3) \), which consist of equations \( (2.15), (2.24), (2.25) \) and Darcy’s law in the form

\[ \frac{\partial w^f}{\partial t} = m v_0(x) + \frac{1}{\tau_0 \rho_f} (m I - B_3) \cdot \int_0^t (-\nabla_x q + \rho_f F(x, \tau)) d\tau. \quad (2.28) \]

In Eqs. \( (2.26)-(2.28) \) matrices \( B_1(\mu_1, t), B_2(\mu_1), \) and \( B_3 \) are the same as in Theorem \( 2.2 \) and \( \theta^f \) is defined below by Eqs. \( (5.42)-(5.48) \).

Problems \( F_1-F_3 \) are endowed with boundary condition \( (2.22) \) for the displacement in the liquid component and initial condition \( (2.13) \), where \( \beta_{0s} = 0 \), and boundary condition \( (2.14) \) for the temperature in the solid component.

\( \Pi \) If

\[ \kappa_{0s}, \quad \beta_{0f}, \quad \beta_{0s}, \quad \mu_1 < \infty \]

and conditions \( (2.5) \) together with assumption \( 2.2 \) hold true, then the sequence \( \{\alpha_\lambda u^\varepsilon\} \) converges strongly in \( L^2(\Omega_T) \) and weakly in \( L^2((0, T); W^1_2(\Omega)) \) to function \( u \), the sequence \( \{\pi^\varepsilon\} \) converges weakly in \( L^2(\Omega_T) \) to the function \( \pi \) and sequence \( \{\vartheta^\varepsilon\} \) converges strongly in \( L^2(\Omega_T) \) and weakly in \( L^2((0, T); W^1_2(\Omega)) \) to zero. The limiting functions \( u \) and \( \pi \) satisfy the boundary value problem in the domain \( \Omega \)

\[ 0 = \text{div}_x \{A_0^s : D(x, u) + B_0^s(\text{div}_x u) + B_1^s q - (q + \pi) \cdot I\} + \rho F, \quad (2.29) \]
\[ \frac{1}{\eta^2} \pi + C_0^s \cdot \mathbb{D}(x, u) + a_0^s(\text{div}_x u) + a_1^s q = 0, \quad (2.30) \]

where the function \( q \) is referred to as given. It is defined from the corresponding of Problems \( F_1 - F_3 \) (the choice of the problem depends on \( \tau_0 \) and \( \mu_1 \)), where \( \vartheta = 0 \). The symmetric strictly positively defined constant fourth-rank tensor \( A_0^s \), matrices \( C_0^s, B_0^s \) and \( B_1^s \) and constants \( a_0^s \) and \( a_1^s \) are defined below by formulas (5.35) - (5.37), in which we have \( \eta_0 = \eta_2 \) and \( \lambda_0 = 1 \).

This problem is endowed with the homogeneous boundary conditions for the displacement \( u \) of the solid component.

(III) If
\[ \beta_{2f}, \beta_{2s}, \kappa_{2s}, \kappa_{2s}^{-1}, \lambda_1, \ p_2, \ \eta_2 < \infty \]
and assumption 2.2 hold true, then after re-normalization
\[ \tilde{w}^\varepsilon = \alpha \lambda w^\varepsilon, \quad \tilde{\theta}^\varepsilon = \alpha \vartheta^\varepsilon \]
we arrive at the problem, considered in Theorem 2.2 where
\[ \tilde{\tau}_0 = 0, \quad \tilde{\mu}_1 = \frac{1}{\lambda_1}, \quad \tilde{\lambda}_0 = 1, \quad \tilde{p}_s = p_2, \quad \tilde{\eta}_0 = \eta_2, \]
\[ \tilde{\beta}_{0s} = \beta_{2s}, \quad \tilde{\beta}_{0f} = \beta_{2f}, \quad \tilde{\kappa}_{0f} = \kappa_{2f}, \quad \tilde{\kappa}_{0s} = \kappa_{2s}. \]

(IV) Let
\[ \mu_1 = \infty, \quad 0 < \lambda_1, \quad p_1, \quad \eta_1, \quad \beta_{1s}, \quad \beta_{1f}, \quad \kappa_{1s} < \infty \]
and conditions (II) of Theorem 2.1 hold. Then re-normalizing by
\[ \tilde{w}^\varepsilon = \alpha \mu \varepsilon^{-2} w^\varepsilon, \quad \tilde{\theta}^\varepsilon = \alpha \varepsilon^{-2} \theta^\varepsilon \]
we arrive at the assumptions of Theorem 2.2 where
\[ \tilde{\tau}_0 = 0, \quad \tilde{\mu}_1 = 1, \quad \tilde{\lambda}_0 = \lambda_1, \quad \tilde{p}_s = p_1, \quad \tilde{\eta}_0 = \eta_1, \]
\[ \tilde{\beta}_{0s} = \beta_{1s}, \quad \tilde{\beta}_{0f} = \beta_{1f}, \quad \tilde{\kappa}_{0f} = \kappa_{1f}, \quad \tilde{\kappa}_{0s} = \kappa_{1s}. \]

(V) If
\[ \mu_1 = \lambda_1 = \infty, \quad 0 < p_1, \quad \eta_1, \quad \beta_{1s}, \quad \beta_{1f}, \quad \kappa_{1s} < \infty, \]
then the corresponding problem for displacements \( \{\alpha \varepsilon^{-2} w^\varepsilon\} \) and temperature \( \{\alpha \varepsilon^{-2} \theta^\varepsilon\} \) has been considered in parts (I)-(II) of the present theorem.
Theorem 2.4. Assume that conditions of lemma [1.1] hold and
\[ \lambda_0, \, \kappa_0, \, p_*, \, \eta_0, \, \beta_0, \, \beta_0 < \infty, \, 0 < \mu_0. \]

Then weak limits \( w, \theta, \, p, \, q \) and \( \pi \) of sequences \( \{w^\varepsilon\}, \{\theta^\varepsilon\}, \{p^\varepsilon\}, \{q^\varepsilon\} \) and \( \{\pi^\varepsilon\} \) satisfy in \( \Omega_T \) the following initial-boundary value problem:

\[ \tau_0 \rho \frac{\partial^2 w}{\partial t^2} + \nabla (q + \pi) = \text{div}_x \left( A_2 : \mathcal{D}(x, \frac{\partial w}{\partial t}) + A_3 : \mathcal{D}(x, w) \right) + (2.31) \]

\[ B_4 \text{div}_x w + B_4^0 \theta + \int_0^t \left( A_4(t - \tau) : \mathcal{D}(x, w(x, \tau)) \right) + \]

\[ B_5(t - \tau) \text{div}_x w(x, \tau) + B_5^0(t - \tau) \theta(x, \tau) \right) d\tau \right) + \hat{\rho} F, \quad (2.32) \]

\[ q = p + \frac{\nu_0}{p_*} \frac{\partial p}{\partial t} + \beta_0 m \theta, \quad (2.33) \]

\[ \frac{1}{p_*} p + m \text{div}_x w = - \int_0^t \left( C_2(t - \tau) : \mathcal{D}(x, w(x, \tau)) \right) + \]

\[ a_2(t - \tau) \text{div}_x w(x, \tau) + a_2^0(t - \tau) \theta(x, \tau) d\tau. \quad (2.34) \]

\[ \frac{1}{\eta_0} \pi + (1 - m)(\text{div}_x w - \frac{\beta_0 s}{\eta_0} \theta) = - \int_0^t \left( a_2^0(t - \tau) \theta(x, \tau) + \right) \]

\[ C_3(t - \tau) : \mathcal{D}(x, w(x, \tau)) + a_3(t - \tau) \text{div}_x w(x, \tau) \right) d\tau, \quad (2.35) \]

The problem is supplemented by corresponding initial and boundary conditions \((2.13) - (2.14)\) for \( \theta \) and \( w \).

In \((2.31) - (2.35)\) \( A_2, A_3 \) and \( A_4 \) - fourth-rank tensors, \( B_4, B_5, B_4^0, B_5^0, B_2^0, B_3^0 \) - matrices and \( a_2, a_3, a_2^0 \) and \( a_2^0 \) - scalars. The exact expressions for these objects are given below by formulas \((7.38) - (7.46)\). The matrix \( B_0^0 \) is strictly positively defined.

If the porous space is connected then \( A_2 \) is strictly positively defined symmetric tensor.

If the porous space is disconnected, which is a case of isolated pores, then \( A_2 = 0 \) and the system \((2.31)\) degenerates into nonlocal non-isotropic Lam‘e’s system with strictly positively defined and symmetric tensor \( A_3 \).
3. Preliminaries

3.1. Two-scale convergence. Justification of Theorems \[2.1\]-\[2.4\] relies on systematic use of the method of two-scale convergence, which had been proposed by G. Nguetseng \[12\] and has been applied recently to a wide range of homogenization problems (see, for example, the survey \[9\]).

Definition 3.1. A sequence \(\{\phi^\varepsilon\} \subset L^2(\Omega_T)\) is said to be two-scale convergent to a limit \(\phi \in L^2(\Omega_T \times Y)\) if and only if for any 1-periodic in \(y\) function \(\sigma = \sigma(x,t,y)\) the limiting relation

\[
\lim_{\varepsilon \to 0} \int_{\Omega_T} \phi^\varepsilon(x,t)\sigma(x,t,x/\varepsilon) \, dx \, dt = \int_{\Omega_T} \int_Y \phi(x,t,y)\sigma(x,t,y) \, dy \, dx \, dt \quad (3.1)
\]

holds.

Existence and main properties of weakly convergent sequences are established by the following fundamental theorem \[12, 9\]:

Theorem 3.1. (Nguetseng’s theorem)

1. Any bounded in \(L^2(Q)\) sequence contains a subsequence, two-scale convergent to some limit \(\phi \in L^2(\Omega_T \times Y)\).

2. Let sequences \(\{\phi^\varepsilon\}\) and \(\{\varepsilon \nabla_x \phi^\varepsilon\}\) be uniformly bounded in \(L^2(\Omega_T)\). Then there exist a 1-periodic in \(y\) function \(\phi = \phi(x,t,y)\) and a subsequence \(\{\phi^\varepsilon\}\) such that \(\phi, \nabla_y \phi \in L^2(\Omega_T \times Y)\), and \(\phi^\varepsilon\) and \(\varepsilon \nabla_x \phi^\varepsilon\) two-scale converge to \(\phi\) and \(\nabla_y \phi\), respectively.

3. Let sequences \(\{\phi^\varepsilon\}\) and \(\{\nabla_x \phi^\varepsilon\}\) be bounded in \(L^2(Q)\). Then there exist functions \(\phi \in L^2(\Omega_T)\) and \(\psi \in L^2(\Omega_T \times Y)\) and a subsequence from \(\{\phi^\varepsilon\}\) such that \(\psi\) is 1-periodic in \(y\), \(\nabla_y \psi \in L^2(\Omega_T \times Y)\), and \(\phi^\varepsilon\) and \(\nabla_x \phi^\varepsilon\) two-scale converge to \(\phi\) and \(\nabla_x \phi^\varepsilon(x,t) + \nabla_y \psi(x,t,y)\), respectively.

Corollary 3.1. Let \(\sigma \in L^2(Y)\) and \(\sigma^\varepsilon(x) := \sigma(x/\varepsilon)\). Assume that a sequence \(\{\phi^\varepsilon\} \subset L^2(\Omega_T)\) two-scale converges to \(\phi \in L^2(\Omega_T \times Y)\). Then the sequence \(\sigma^\varepsilon \phi^\varepsilon\) two-scale converges to \(\sigma \phi\).

3.2. An extension lemma. The typical difficulty in homogenization problems while passing to a limit in Model \((NB)^\varepsilon\) as \(\varepsilon \to 0\) arises because of the fact that the bounds on the gradient of displacement \(\nabla_x w^\varepsilon\) may be distinct in liquid and rigid phases. The classical approach in overcoming this difficulty consists of constructing of extension to the whole \(\Omega\) of the
displacement field defined merely on $\Omega_s$. The following lemma is valid due to the well-known results from \[1, 8\]. We formulate it in appropriate form:

**Lemma 3.1.** Suppose that assumptions of Sec. 1.2 on geometry of periodic structure hold, $\psi^\varepsilon \in W^1_2(\Omega^\varepsilon_s)$ and $\psi^\varepsilon = 0$ on $S^\varepsilon_s = \partial \Omega^\varepsilon_s \cap \partial \Omega$ in the trace sense. Then there exists a function $\sigma^\varepsilon \in W^1_2(\Omega)$ such that its restriction on the sub-domain $\Omega^\varepsilon_s$ coincide with $\psi^\varepsilon$, i.e.,

\[
(1 - \chi^\varepsilon(x))(\sigma^\varepsilon(x) - \psi^\varepsilon(x)) = 0, \quad x \in \Omega,
\]

and, moreover, the estimate

\[
\|\sigma\|_{2,\Omega} \leq C\|\psi\|_{2,\Omega^\varepsilon}, \quad \|\nabla_x \sigma\|_{2,\Omega} \leq C\|\nabla_x \psi\|_{2,\Omega^\varepsilon}
\]

(3.3)

hold true, where the constant $C$ depends only on geometry $Y$ and does not depend on $\varepsilon$.

**3.3. Friedrichs–Poincaré’s inequality in periodic structure.** The following lemma was proved by L. Tartar in \[14\] Appendix. It specifies Friedrichs–Poincaré’s inequality for $\varepsilon$-periodic structure.

**Lemma 3.2.** Suppose that assumptions on the geometry of $\Omega^\varepsilon_f$ hold true. Then for any function $\varphi \in W^1_2(\Omega^\varepsilon_f)$ the inequality

\[
\int_{\Omega^\varepsilon_f} |\varphi|^2\,dx \leq C\varepsilon^2 \int_{\Omega^\varepsilon_f} |\nabla_x \varphi|^2\,dx
\]

(3.4)

holds true with some constant $C$, independent of $\varepsilon$.

**3.4. Some notation.** Further we denote

1) $\langle \Phi \rangle_Y = \int_Y \Phi \,dy$, $\langle \Phi \rangle_{Y_f} = \int_{Y_f} \Phi \,dy$, $\langle \Phi \rangle_{Y_s} = \int_{Y_s} \Phi \,dy$.

2) If $\mathbf{a}$ and $\mathbf{b}$ are two vectors then the matrix $\mathbf{a} \otimes \mathbf{b}$ is defined by the formula

$$\langle \mathbf{a} \otimes \mathbf{b} \rangle \cdot \mathbf{c} = \mathbf{a}(\mathbf{b} \cdot \mathbf{c})$$

for any vector $\mathbf{c}$.

3) If $B$ and $C$ are two matrices, then $B \otimes C$ is a forth-rank tensor such that its convolution with any matrix $A$ is defined by the formula

$$(B \otimes C) : A = B(C : A)$$

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4) By $I^{ij}$ we denote the $3 \times 3$-matrix with just one non-vanishing entry, which is equal to one and stands in the $i$-th row and the $j$-th column.

5) We also introduce

$$J^{ij} = \frac{1}{2}(I^{ij} + I^{ji}) = \frac{1}{2}(e_i \otimes e_j + e_j \otimes e_i),$$

where $(e_1, e_2, e_3)$ are the standard Cartesian basis vectors.

§4. Proof of Theorem 2.1

4.1. Let

$$\lambda_0, \ p_*, \ \eta_0, \ \beta_{0f}, \ \beta_{0s} < \infty.$$

If restriction $\tau_0 > 0$ holds, then (2.1)–(2.3) follow from the lemma 1.1.

Estimation of $w^\varepsilon$ and $\theta^\varepsilon$ in the case $\tau_0 = 0$ is not simple, and we outline it in more detail. As usual, we obtain the basic estimates if we multiply equations for $w^\varepsilon$ by $\partial w^\varepsilon / \partial t$, equation for $\theta^\varepsilon$ by $\theta^\varepsilon$, sum the result and then integrate by parts all obtained terms. The only two terms $F \cdot \partial w^\varepsilon / \partial t$ and $\Psi \cdot \theta^\varepsilon$ heed additional consideration here. First of all, on the strength of Lemma 3.1, we construct an extension $u^\varepsilon$ of the function $w^\varepsilon$ from $\Omega^\varepsilon_s$ into $\Omega^\varepsilon_f$ such that $u^\varepsilon = w^\varepsilon$ in $\Omega^\varepsilon_s$, $u^\varepsilon \in W^1_2(\Omega)$ and

$$\|u^\varepsilon\|_{2,\Omega} \leq C\|\nabla x u^\varepsilon\|_{2,\Omega} \leq \frac{C}{\sqrt{\alpha_\lambda}}\|(1 - \chi^\varepsilon)\sqrt{\alpha_\lambda} \nabla x w^\varepsilon\|_{2,\Omega}.$$

After that we estimate $\|w^\varepsilon\|_{2,\Omega}$ with the help of Friedrichs–Poincaré's inequality in periodic structure (Lemma 3.2) for the difference $(u^\varepsilon - w^\varepsilon)$:

$$\|w^\varepsilon\|_{2,\Omega} \leq \|u^\varepsilon\|_{2,\Omega} + \|u^\varepsilon - w^\varepsilon\|_{2,\Omega} \leq \|u^\varepsilon\|_{2,\Omega} + C\varepsilon\|\chi^\varepsilon \nabla x (u^\varepsilon - w^\varepsilon)\|_{2,\Omega} \leq \|u^\varepsilon\|_{2,\Omega} + C\varepsilon\|\chi^\varepsilon \nabla x w^\varepsilon\|_{2,\Omega} \leq \frac{C}{\sqrt{\alpha_\lambda}}\|(1 - \chi^\varepsilon)\sqrt{\alpha_\lambda} \nabla x w^\varepsilon\|_{2,\Omega} + C(\varepsilon\alpha_\mu^{\frac{1}{2}})\|\chi^\varepsilon \sqrt{\alpha_\mu} \nabla x w^\varepsilon\|_{2,\Omega}.$$

The same method we apply for the temperature $\theta^\varepsilon$: there is an extension $\vartheta^\varepsilon$ of the function $\theta^\varepsilon$ from $\Omega^\varepsilon_s$ into $\Omega^\varepsilon_f$ such that $\vartheta^\varepsilon = \theta^\varepsilon$ in $\Omega^\varepsilon_s$, $\vartheta^\varepsilon \in W^1_2(\Omega)$ and

$$\|\vartheta^\varepsilon\|_{2,\Omega} \leq C\|\nabla x \vartheta^\varepsilon\|_{2,\Omega} \leq C\|\nabla x \vartheta^\varepsilon\|_{2,\Omega} \leq \frac{C}{\sqrt{\alpha_{\varepsilon s}}}\|(1 - \chi^\varepsilon)\sqrt{\alpha_{\varepsilon s}} \nabla x \theta^\varepsilon\|_{2,\Omega}.$$
Next we pass the derivative with respect to time from $\partial w^\varepsilon / \partial t$ to $\rho^\varepsilon F$ and bound all obtained new terms in a usual way with the help of Hölder and Grownwall’s inequalities. Thus we have got estimates like estimates (1.10), but with right-hand side independent of $\tau_0$ (or independent of $\varepsilon$).

4.2. The proof of this part of the theorem is obvious, because the renormalization reduces this case to the case of $\mu_1 = 1$ and $\tau_0 = 0$, which has been already considered above.

4.3. Let $\lambda_0 = \infty$, $\mu_1 < \infty$ and conditions (2.4) hold true. It is obvious that estimates (2.1), (2.2) and estimates (2.3) for the pressures $q^\varepsilon$ and $p^\varepsilon$ in the liquid component are still valid.

If conditions (2.5) together with assumption 2.2 hold true, then the desired estimates (2.6) follow from the basic equations for $\alpha_\lambda w^\varepsilon$ and $\alpha_\lambda \theta^\varepsilon$ in the same way as in the case of estimates (2.1)–(2.3). The main difference here is in the term $\rho^\varepsilon F \cdot \alpha_\lambda \partial w^\varepsilon / \partial t$, which now transforms to

$$\Upsilon \equiv \rho_f F \cdot \alpha_\lambda \frac{\partial w^\varepsilon}{\partial t} + (\rho_f - \rho_f)(1 - \chi^\varepsilon) F \cdot \alpha_\lambda \frac{\partial w^\varepsilon}{\partial t}.$$  

The integral of first term in $\Upsilon$ transforms as

$$\rho_f \int_0^t \int_{\Omega} \nabla \Phi \cdot \alpha_\lambda \frac{\partial w^\varepsilon}{\partial \tau} dxd\tau = -\rho_f \int_0^t \int_{\Omega} \Phi \alpha_\lambda \text{div}_x \frac{\partial w^\varepsilon}{\partial \tau} dxd\tau$$

$$= -\rho_f \int_0^t \int_{\Omega} (\chi^\varepsilon \cdot \Phi \alpha_\lambda \text{div}_x w^\varepsilon + (1 - \chi^\varepsilon) \cdot \Phi \alpha_\lambda \text{div}_x w^\varepsilon) dxd\tau$$

$$= -\rho_f \int_0^t \int_{\Omega} (\chi^\varepsilon \cdot \Phi \alpha_\lambda \text{div}_x w^\varepsilon + (1 - \chi^\varepsilon) \cdot \Phi \alpha_\lambda \text{div}_x u^\varepsilon) dxd\tau +$$

$$\rho_f \int_0^t \int_{\Omega} (\chi^\varepsilon \cdot \Phi \alpha_\lambda \text{div}_x w^\varepsilon + (1 - \chi^\varepsilon) \cdot \Phi \tau \alpha_\lambda \text{div}_x u^\varepsilon) dxd\tau$$

and is bounded with the help of terms

$$\int_{\Omega} (\chi^\varepsilon (\alpha_\mu \alpha_\lambda^{-1}) (\alpha_\lambda \text{div}_x w^\varepsilon)^2 + (1 - \chi^\varepsilon)|\alpha_\lambda \nabla_x u^\varepsilon|^2) d\mathbf{x},$$

which appear in the basic identity after using the continuity equations.

The integral of the second term in $\Upsilon$ is bounded with the help of the term

$$\alpha_\lambda^2 \int_{\Omega} (1 - \chi^\varepsilon)|\nabla_x u^\varepsilon|^2 d\mathbf{x}$$
in the same way as before.
Estimates (2.3) follow now from (2.6) if \( \beta_0 f, \beta_0 s < \infty \). \( \square \)

§5. Proof of Theorem 2.2

5.1. Weak and two-scale limits of sequences of displacement, temperatures and pressures. On the strength of Theorem 2.1 the sequences \( \{\theta^\varepsilon\}, \{p^\varepsilon\}, \{q^\varepsilon\}, \{\pi^\varepsilon\} \) and \( \{w^\varepsilon\} \) are uniformly in \( \varepsilon \) bounded in \( L^2(\Omega_T) \). Hence there exist a subsequence of small parameters \( \{\varepsilon > 0\} \) and functions \( \theta, p, q, \pi \) and \( w \) such that

\[
\theta^\varepsilon \to \theta, \quad p^\varepsilon \to p, \quad q^\varepsilon \to q, \quad \pi^\varepsilon \to \pi, \quad w^\varepsilon \to w
\]

weakly in \( L^2(\Omega_T) \) as \( \varepsilon \searrow 0 \).

Due to Lemma 3.1 there is a function \( u^\varepsilon \in L^\infty((0,T);W_1^2(\Omega)) \) such that

\[ u^\varepsilon = w^\varepsilon \text{ in } \Omega_\ast \times (0,T), \]

and the family \( \{u^\varepsilon\} \) is uniformly in \( \varepsilon \) bounded in \( L^\infty((0,T);W_1^2(\Omega)) \). Therefore it is possible to extract a subsequence of \( \{\varepsilon > 0\} \) such that

\[ u^\varepsilon \to u \text{ weakly in } L^2((0,T);W_2^2(\Omega)) \]

as \( \varepsilon \searrow 0 \).

Applying again the same lemma 3.1 we conclude that there is a function \( \vartheta^\varepsilon \in L^2((0,T);W_2^1(\Omega)) \) such that \( \vartheta^\varepsilon = \theta^\varepsilon \) in \( \Omega_\ast \times (0,T) \), and the family \( \{\vartheta^\varepsilon\} \) is uniformly in \( \varepsilon \) bounded in \( L^2((0,T);W_2^1(\Omega)) \). Therefore it is possible to extract a subsequence of \( \{\varepsilon > 0\} \) such that

\[ \vartheta^\varepsilon \to \vartheta \text{ weakly in } L^2((0,T);W_2^1(\Omega)) \]

as \( \varepsilon \searrow 0 \).

Moreover, 

\[ \chi^\varepsilon \alpha_\mu [x, w^\varepsilon] \to 0, \quad \chi^\varepsilon \alpha_\infty f \nabla \theta^\varepsilon \to 0 \quad (5.1) \]

as \( \varepsilon \searrow 0 \).

Relabelling if necessary, we assume that the sequences converge themselves.

On the strength of Nguetseng’s theorem, there exist 1-periodic in \( y \) functions \( \Theta(x,t,y), P(x,t,y), \Pi(x,t,y), Q(x,t,y), W(x,t,y), \Theta^s(x,t,y) \) and \( U(x,t,y) \) such that the sequences \( \{\theta^\varepsilon\}, \{p^\varepsilon\}, \{\pi^\varepsilon\}, \{q^\varepsilon\}, \{w^\varepsilon\}, \{\nabla x \vartheta^\varepsilon\} \) and \( \{\nabla x u^\varepsilon\} \) two-scale converge to \( \Theta(x,t,y), P(x,t,y), \Pi(x,t,y), Q(x,t,y), W(x,t,y), \nabla_x \vartheta + \nabla_y \Theta^s(x,t,y) \) and \( \nabla_x u + \nabla_y U(x,t,y) \), respectively.
Note that the sequence \( \{ \text{div}_x w^\varepsilon \} \) weakly converges to \( \text{div}_x w \) and \( \vartheta, |u| \in L^2((0, T); W^{1,2}_{0}(\Omega)) \). Last assertion for disconnected porous space follows from inclusion \( \vartheta^\varepsilon, |u^\varepsilon| \in L^2((0, T); W^{1,2}_{0}(\Omega)) \) and for the connected porous space it follows from the Friedrichs–Poincaré’s inequality for \( u^\varepsilon \) and \( \vartheta^\varepsilon \) in the \( \varepsilon \)-layer of the boundary \( S \) and from convergence of sequences \( \{u^\varepsilon\} \) and \( \{\vartheta^\varepsilon\} \) to \( u \) and \( \vartheta \) respectively strongly in \( L^2(\Omega_T) \) and weakly in \( L^2((0, T); W^{1,2}_{0}(\Omega)) \).

5.2. Micro- and macroscopic equations I.

**Lemma 5.1.** For all \( x \in \Omega \) and \( y \in Y \) weak and two-scale limits of the sequences \( \{\theta^\varepsilon\}, \{p^\varepsilon\}, \{\pi^\varepsilon\}, \{q^\varepsilon\}, \{w^\varepsilon\}, \{\nabla_x \vartheta^\varepsilon\} \) and \( \{\nabla_x u^\varepsilon\} \) satisfy the relations

\[
Q = \frac{1}{m} \chi q, \quad P = \frac{m}{p_*} \partial P/\partial t + \beta_0 f \chi \Theta; \tag{5.2}
\]
\[
\frac{1}{\eta_0} \Pi + (1 - \chi)(\text{div}_x u + \text{div}_y U) - \frac{\beta_0}{\eta_0} \partial \vartheta = 0; \tag{5.3}
\]
\[
\text{div}_y W = 0; \tag{5.4}
\]
\[
W = \chi W + (1 - \chi) u; \tag{5.5}
\]
\[
\Theta = \chi \Theta + (1 - \chi) \vartheta; \tag{5.6}
\]
\[
q = p + \nu_0 p_*^{-1} \partial p/\partial t + \beta_0 f \theta f, \quad \theta f = \langle \Theta \rangle_{Y_f}; \tag{5.7}
\]
\[
\frac{1}{\eta_0} \pi + (1 - m) \text{div}_x u + \langle \text{div}_y U \rangle_{Y_s} - (1 - m) \frac{\beta_0}{\eta_0} \vartheta = 0; \tag{5.8}
\]
\[
\frac{1}{\eta_0} \pi + \frac{1}{p_*} p + \text{div}_x w - (1 - m) \frac{\beta_0}{\eta_0} \vartheta = 0. \tag{5.9}
\]

**Proof.** The weak and two-scale limiting passage in Eq. (1.6) yield that Eq. (5.7) and the second equation in (5.2).

In order to prove first equation in (5.2), into Eq. (1.8) insert a test function \( \psi^\varepsilon = \varepsilon \psi(x, t, x/\varepsilon) \), where \( \psi(x, t, y) \) is an arbitrary 1-periodic and finite on \( Y_f \) function in \( y \). Passing to the limit as \( \varepsilon \downarrow 0 \), we get

\[
\nabla_y Q(x, t, y) = 0, \quad y \in Y_f. \tag{5.10}
\]

Next, fulfilling the two-scale limiting passage in the equality

\[
(1 - \chi^\varepsilon) p^\varepsilon = 0
\]

we arrive at

\[
(1 - \chi) P = 0
\]
which along with Eqs. (5.10) and second equation in (5.2) justifies first equation in (5.2).

Eqs. (5.3), (5.4), (5.8), and (5.9) appear as the results of two-scale limiting passages in Eqs. (1.6) and (1.7) with the proper test functions being involved. Thus, for example, Eq. (5.8) is just a subsequence of Eq. (5.3) and Eq. (5.9) is a result of two-scale convergence in the sum of Eq. (1.6) and Eq. (1.7) with the test functions independent of the “fast” variable $x/\varepsilon$. Eq. (5.4) is derived quite similarly if we represent Eq. (1.6) in the form

$$
\frac{1}{\alpha_p} p^\varepsilon + \text{div}_x w^\varepsilon = (1 - \chi^\varepsilon)\text{div}_x u^\varepsilon, \quad (5.11)
$$
multiply by an arbitrary function $\psi^\varepsilon = \varepsilon \psi(x, t, x/\varepsilon)$ and then pass to the limit as $\varepsilon \searrow 0$.

In order to prove Eqs. (5.5)–(5.6) it is sufficient to consider the two-scale limiting relations in

$$(1 - \chi^\varepsilon)(w^\varepsilon - u^\varepsilon) = 0, \quad (1 - \chi^\varepsilon)(\theta^\varepsilon - \vartheta^\varepsilon) = 0.$$
Lemma 5.3. For all \((x,t) \in \Omega_T\) the relations
\[
\begin{align*}
\triangle_y \Theta^s &= 0, \quad y \in Y_s, \\
\frac{\partial \Theta^s}{\partial n} &= -\nabla_x \vartheta \cdot n, \quad y \in \gamma
\end{align*}
\] (5.15) (5.16)
hold true.

Now we pass to the macroscopic equations for the solid displacements.

Lemma 5.4. Let \(\dot{\rho} = m \rho_f + (1 - m) \rho_s, \quad w^f = \langle W \rangle_{Y_f}\). Then functions \(u, w^f, q, \pi, \theta^f, \vartheta\) satisfies in \(\Omega_T\) the system of macroscopic equations
\[
\begin{align*}
\tau_0 \rho_f \frac{\partial^2 w^f}{\partial t^2} + \tau_0 \rho_s (1 - m) \frac{\partial^2 u}{\partial t^2} - \dot{\rho} F &= \mu_1 \Delta_y V - \nabla y R - \nabla_x q, \quad y \in Y_f, \\
\tau_0 c_p f \frac{\partial \theta^f}{\partial t} + (\tau_0 c_p s + \frac{\beta_0^2}{\eta_0})(1 - m) \frac{\partial \vartheta}{\partial t} - \frac{\beta_0 f}{p_*} \frac{\partial p}{\partial t} - \frac{\beta_0 s}{\eta_0} \frac{\partial \pi}{\partial t} &= \kappa_1 \mu_1 \Delta_y \Theta + \beta_0 f \frac{\partial P}{\partial t} + \Psi, \quad y \in Y_f
\end{align*}
\] (5.17) (5.18)

Proof. Eqs. (5.17) and (5.18) arise as the limit of Eqs. (1.8) and (1.9) with test functions being finite in \(\Omega_T\) and independent of \(\varepsilon\). In Eq. (1.9) we have used continuity equations (1.6) and (1.7).

5.3. Micro- and macroscopic equations II.

Lemma 5.5. If \(\mu_1 = \infty\) then \(u = w\) and \(\theta = \vartheta\).

Proof. In order to verify, it is sufficient to consider the differences \((u^\varepsilon - w^\varepsilon)\) and \((\theta^\varepsilon - \vartheta^\varepsilon)\) and apply Friedrichs–Poincar’e’s inequality, just like in the proof of Theorem 2.1.

Lemma 5.6. Let \(\mu_1 < \infty\) and \(V = \chi \partial W / \partial t\). Then
\[
\begin{align*}
\tau_0 \rho_f \frac{\partial V}{\partial t} - \rho_f F &= \mu_1 \Delta_y V - \nabla y R - \nabla_x q, \quad y \in Y_f, \\
\tau_0 c_p f \frac{\partial \Theta}{\partial t} &= \kappa_1 \mu_1 \Delta_y \Theta + \frac{\beta_0 f}{p_*} \frac{\partial P}{\partial t} + \Psi, \quad y \in Y_f, \\
V &= \frac{\partial u}{\partial t}, \quad \Theta = \vartheta, \quad y \in \gamma
\end{align*}
\] (5.19) (5.20) (5.21)
in the case $\mu_1 > 0$, and

\[
\tau_0 \rho_f \frac{\partial V}{\partial t} = -\nabla_y R - \nabla_x q + \rho_f F, \quad y \in Y_f, \tag{5.22}
\]

\[
\tau_0 \rho_c \frac{\partial \Theta}{\partial t} = \beta_0 \frac{\partial P}{\partial t} + \Psi, \quad y \in Y_f, \tag{5.23}
\]

\[
(\chi W - u) \cdot n = 0, \quad y \in \gamma. \tag{5.24}
\]

in the case $\mu_1 = 0$.

In Eq. (5.24) $n$ is the unit normal to $\gamma$.

**Proof.** Differential equations (5.19) and (5.22) follow as $\varepsilon \searrow 0$ from integral equality (1.8) with the test function $\psi = \varphi(x\varepsilon^{-1}) \cdot h(x, t)$, where $\varphi$ is solenoidal and finite in $Y_f$.

The same arguments apply for the Eq. (5.20) and Eq. (5.23). The only difference here is that we use the continuity equation (1.6) to exclude the term $\chi^\varepsilon \text{div}_x (\partial w^\varepsilon / \partial t)$.

First boundary condition in (5.21) is the consequence of the two-scale convergence of $\{\alpha_1^{\varepsilon} \nabla_x w^\varepsilon\}$ to the function $\mu_1 \nabla_y W(x, t, y)$. On the strength of this convergence, the function $\nabla_y W(x, t, y)$ is $L^2$-integrable in $Y$. As above we apply the same argument to the second boundary condition in (5.21). The boundary conditions (5.24) follow from Eq. (5.4). \hfill \Box

**Lemma 5.7.** If the porous space is disconnected, which is the case of isolated pores, then $u = w$.

**Proof.** Indeed, in the case $0 \leq \mu_1 < \infty$ the systems of equations (5.4), (5.19), and (5.21), or (5.4), (5.22), and (5.24) have the unique solution $V = \partial u / \partial t$. \hfill \Box

### 5.4. Homogenized equations I.

**Lemma 5.8.** If $\mu_1 = \infty$ then $w = u$, $\theta = \vartheta$ and the weak limits $u$, $\vartheta$, $p$, $q$, and $\pi$ satisfy in $\Omega_T$ the initial-boundary value problem

\[
\tau_0 \rho \frac{\partial^2 u}{\partial t^2} + \nabla (q + \pi) - \rho F = \tag{5.25}
\]

\[
div_x \{\lambda_0 A_0^s \cdot D(x, u) + B_0^s (\div u - \eta_0 \vartheta) + B_1^s q\},
\]

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\[(\tau_0 \dot{c}_p + \frac{\beta_{0s}^2}{\eta_0}(1-m)) \frac{\partial \vartheta}{\partial t} \frac{\partial \vartheta}{\partial t} - \frac{\beta_{0f}}{p_*} \frac{\partial p}{\partial t} - \frac{\beta_{0s}}{\eta_0} \frac{\partial \pi}{\partial t} = \text{div}_x (B^\theta \cdot \nabla \vartheta) + \Psi, \quad (5.26)\]

\[
\frac{1}{\eta_0} \pi + C_0^* : \mathbb{D}(x, u) + a_0^s (\text{div}_x u - \frac{\beta_{0s}}{\eta_0} \vartheta) + a_1^s q = 0, \quad (5.27)
\]

\[
\frac{1}{p_*} p + \frac{1}{\eta_0} \pi + \text{div}_x u - (1-m) \frac{\beta_{0s}}{\eta_0} \vartheta = 0, \quad (5.28)
\]

\[
q = p + \nu_0 p_*^{-1} \frac{\partial p}{\partial t} + \beta_{0f} \nu_0 \vartheta, \quad (5.29)
\]

where the symmetric strictly positively defined constant fourth-rank tensor \(A_0^s\), constant matrices \(C_0^*, B_0^*, B_1^s\), strictly positively defined constant matrix \(B^\theta\) and constants \(a_0^s, a_1^s\) are defined below by formulas \((5.35) - (5.37), (5.40)\).

Differential equations \((5.25) - (5.26)\) are endowed with initial conditions at \(t = 0\) and \(x \in \Omega\)

\[
(\tau_0 + \beta_{0s})(\vartheta - \vartheta_0) = 0, \quad \tau_0 (u - u_0) = \tau_0 \frac{\partial u}{\partial t} - v_0 = 0; \quad (5.30)
\]

and boundary conditions

\[
\vartheta(x, t) = 0, \quad u(x, t) = 0, \quad x \in S, \quad t > 0. \quad (5.31)
\]

**Proof.** In the first place let us notice that \(u = w\) and \(\theta = \vartheta\) due to Lemma \(5.5\).

The homogenized equations \((5.25)\) follow from the macroscopic equations \((5.17)\), after we insert in them the expression

\[
\langle \mathbb{D}(y, U) \rangle_{Y_s} = A_1^s : \mathbb{D}(x, u) + B_0^s (\text{div}_x u - \frac{\beta_{0s}}{\eta_0} \vartheta) + B_1^s q.
\]

In turn, this expression follows by virtue of solutions of Eqs.\((5.3)\) and \((5.14)\) on the pattern cell \(Y_s\). Indeed, setting

\[
U = \sum_{i,j=1}^3 U^{ij}(y)D_{ij} + U_0(y) (\text{div}_x u - \frac{\beta_{0s}}{\eta_0} \vartheta) + \frac{1}{m} U_1(y) q
\]

and

\[
\Pi = \lambda_0 \sum_{i,j=1}^3 \Pi^{ij}(y)D_{ij} + \Pi_0(y) (\text{div}_x u - \frac{\beta_{0s}}{\eta_0} \vartheta) + \frac{1}{m} \Pi_1(y) q,
\]

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where
\[ D_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \]
we arrive at the following periodic-boundary value problems in \( Y \):
\[
\begin{align*}
\text{div}_y \{(1 - \chi)(\nabla(y, U^{ij}) + J^{ij}) - \Pi^{ij} \cdot \mathbb{I}\} &= 0, \\
\frac{1}{\eta_0} \Pi^{ij} + (1 - \chi)\text{div}_y U^{ij} &= 0; \quad \text{(5.32)} \\
\text{div}_y \{\lambda_0(1 - \chi)\nabla(y, U_0) - \Pi_0 \cdot \mathbb{I}\} &= 0, \\
\frac{1}{\eta_0} \Pi_0 + (1 - \chi)(\text{div}_y U_0 + 1) &= 0; \quad \text{(5.33)} \\
\text{div}_y \{\lambda_0(1 - \chi)\nabla(y, U_1) - (\Pi_1 + \chi) \cdot \mathbb{I}\} &= 0, \\
\frac{1}{\eta_0} \Pi_1 + (1 - \chi)\text{div}_y U_1) &= 0. \quad \text{(5.34)}
\end{align*}
\]

On the strength of the assumptions on the geometry of the pattern “liquid” cell \( Y_s \), problems (5.32)–(5.34) have unique solution, up to an arbitrary constant vector. In order to discard the arbitrary constant vectors we demand
\[
\langle U^{ij} \rangle_{Y_s} = \langle U_0 \rangle_{Y_s} = \langle U_1 \rangle_{Y_s} = 0.
\]
Thus
\[
A_0^s = \sum_{i,j=1}^{3} J^{ij} \otimes J^{ij} + A_1^s, \quad A_1^s = \sum_{i,j=1}^{3} \langle (1 - \chi)D(y, U^{ij}) \rangle_{Y} \otimes J^{ij}. \quad \text{(5.35)}
\]
Symmetry and strict positiveness of the tensor \( A_0^s \) have been proved in [10].

Finally, Eqs. (5.27)–(5.29) for the pressures follow from Eqs. (5.7)–(5.9) and
\[
B_0^s = \langle \nabla(y, U_0) \rangle_{Y_s}, \quad B_1^s = \frac{1}{m} \langle \nabla(y, U_1) \rangle_{Y_s}, \quad a_1^s = \frac{1}{m} \langle \text{div}_y U_1 \rangle_{Y_s}, \quad \text{(5.36)}
\]
and
\[
a_0^s = 1 - m + \langle \text{div}_y U_0 \rangle_{Y_s}, \quad C_0^s = \sum_{i,j=1}^{3} \langle \text{div}_y U^{ij} \rangle_{Y_s} J^{ij}. \quad \text{(5.37)}
\]
Now for \( i = 1, 2, 3 \) we consider the model problems
\[
\begin{align*}
\Delta_y \Theta_i^s &= 0, \quad \mathbf{y} \in Y_s, \\
\frac{\partial \Theta_i^s}{\partial n} &= -e_i \cdot \mathbf{n}, \quad \mathbf{y} \in \gamma
\end{align*}
\]
(5.38)}
and put

$$\Theta^s = \sum_{i=1}^{3} (\Theta_i^s \otimes e_i) \cdot \nabla_x \vartheta.$$  \hfill (5.39)

Then $\Theta^s$ solves the problem (5.15)–(5.16) and

$$B^\theta = \kappa_0 (1 - m) I + \sum_{i=1}^{3} (\nabla_y \Theta_i^s) y_i \otimes e_i.$$  \hfill (5.40)

All properties of the matrix $B^\theta$ are well known (see [14], [8]).

**Lemma 5.9.** If the porous space is disconnected, then $w = u$ and the weak limits $\theta^f$, $\vartheta$, $p$, $q$, and $\pi$ satisfy in $\Omega_T$ equations (5.25), (5.27)–(5.28), (5.7), where $A_0^s$, $C_0^s$, $B_0^s$, $B_0^\theta$, $a_0^s$ and $a_1^s$ are the same as in Lemma 5.8, and heat equation

$$\tau_0 c_{pf} \frac{\partial \theta^f}{\partial t} + (\tau_0 c_{ps} + \beta_0^2 p_s \eta_0)(1 - m) \frac{\partial \vartheta}{\partial t} - \frac{\beta_0}{p_s} \frac{\partial p}{\partial t} - \frac{\beta_0}{\eta_0} \frac{\partial \pi}{\partial t} = \text{div}_x (B^\theta \cdot \nabla \vartheta) + \Psi.$$  \hfill (5.41)

If $\beta_0 = 0$, then for the case $\mu_1 > 0$, $\tau > 0$

$$\theta^f(x, t) = m \vartheta(x, t) + \int_{0}^{t} b_f^\theta(t - \tau) h(x, \tau) d\tau,$$  \hfill (5.42)

where

$$h(x, t) = \Psi(x, t) - \tau_0 c_{pf} \frac{\partial \vartheta}{\partial t}(x, t);$$

for the case $\tau = 0$

$$\theta^f(x, t) = m \vartheta(x, t) - c_f^\theta \Psi(x, t);$$  \hfill (5.43)

and, finally, for the case $\mu_1 = 0$

$$\theta^f(x, t) = m \vartheta_0(x) + \frac{m}{\tau_0 c_{pf}} \int_{0}^{t} \Psi(x, \tau) d\tau.$$  \hfill (5.44)

Here $b_f^\theta(t)$ and $c_f^\theta$ are defined below by formulas (5.45), (5.46).

If $\nu_0 = 0$, then $\theta^f$ is defined by the same formulas (5.42)–(5.46) where $\tau_0 c_{pf}$ is replaced by $(\tau_0 c_{pf} + \beta_0^2 p_s \eta_0)$ and $\Psi$ is replaced by $(\Psi + \frac{1}{m} q)$.

If $\nu_0 > 0$ and $\beta_0 > 0$, then $\theta^f$ is defined by (5.42) where $h(x, t)$ and $b_f^\theta(t)$ are defined below by formulas (5.47) and (5.48).

The problem is endowed with initial and boundary conditions (5.30)–(5.31).
Proof. The only one difference here with the previous lemma is the heat equation for \( \vartheta \), because \( \theta \neq \vartheta \). The function \( \theta^f \) now is defined from microscopic equations (5.2), (5.20) and (5.21).

If \( \beta_0 = 0 \), then the system (5.2), (5.20) and (5.21) is decoupled and \( \theta^f \) is represented by (5.42)–(5.44) where

\[
\begin{align*}
\bar{b}^\theta_{1,f}(t) &= \langle \Theta^f_{1} \rangle_{Y_f}, & c^\theta_{1,f} &= \langle \Theta^f_{0} \rangle_{Y_f}, \\
\end{align*}
\]

and \( \Theta^f_{1}, \Theta^f_{0} \) are solutions of periodic in \( y \) problems

\[
\begin{align*}
\tau_0 c^f p \frac{\partial \Theta^f_{1}}{\partial t} &= \kappa_1 \mu_1 \Delta_y \Theta^f_{1}, & y \in Y_f, \\
\Theta^f_{1}(y,0) &= 1, & y \in Y_f; & \Theta^f_{1} = 0, & y \in \gamma,
\end{align*}
\]

and

\[
\begin{align*}
\kappa_1 \mu_1 \Delta_y \Theta^f_{0} &= 1, & y \in Y_f; & \Theta^f_{0} = 0, & y \in \gamma.
\end{align*}
\]

The pressure \( p \) is defined from (5.7).

If \( \nu_0 > 0 \) and \( \beta_0 > 0 \), then we look for the solution of (5.2), (5.20) and (5.21) in the form

\[
\begin{align*}
\Theta(x,t,y) &= \vartheta(x,t) + \int_0^t \Theta^f(y,t-\tau) h(x,\tau) d\tau, \\
P(x,t,y) &= h_0(x,t) + \int_0^t P^f(y,t-\tau) h(x,\tau) d\tau.
\end{align*}
\]

Functions \( h(x,t) \) and \( h_0(x,t) \) solve a system

\[
\begin{align*}
\beta_0 \left( \frac{\partial h_0}{\partial t} + h \right) - \tau_0 c^f p h &= \tau_0 c^f p \frac{\partial \vartheta}{\partial t} - \Psi, \\
\nu_0 \left( \frac{\partial h_0}{\partial t} + h \right) + h_0 &= \frac{1}{m} q - \beta_0 \vartheta,
\end{align*}
\]

and 1-periodic in \( y \) functions \( \Theta^f(y,t) \) and \( P^f(y,t) \) solve a problem

\[
\begin{align*}
\tau_0 c^f p \frac{\partial \Theta^f}{\partial t} &= \kappa_1 \mu_1 \Delta_y \Theta^f + \frac{\beta_0}{p_*} \frac{\partial P^f}{\partial t}, & y \in Y_f, \\
P^f + \nu_0 \frac{\partial P^f}{\partial t} + \beta_0 \Theta^f &= 0, & y \in Y_f;
\end{align*}
\]
\[ \tau_0(\Theta^f - 1)|_{t=0} = (P - 1)|_{t=0} = 0, \quad y \in Y_f; \quad \mu_1 \Theta^f = 0, \quad y \in \gamma. \]

This problem has a unique solution due to estimates

\[
\int_0^t \int_{Y_f} \{x \mu_1 |\nabla \Theta^f(y, \tau)|^2 + \frac{\nu_0}{p_*^2} |\frac{\partial P^f(y, \tau)}{\partial t}|^2 \}dy \, d\tau + \\
\int_{Y_f} \{ \frac{1}{2} \tau_0 c_p |\Theta^f(y, t)|^2 + \frac{1}{2p_*} |P^f(y, t)|^2 \} dy = \frac{m}{2} (\tau_0 c_p + \frac{1}{p_*}).
\]

Thus, \( \theta^f \) is defined by (5.42) where \( h(x, t) \) is defined by (5.47) and

\[
b^f(t) = (\Theta^f)|_{Y_f}.
\] (5.48)

5.5. Homogenized equations II.

Let \( \mu_1 < \infty \). In the same manner as above, we verify that the weak limit \( u \) of the sequence \( \{u^\varepsilon\} \) satisfies some initial-boundary value problem like problem (5.25)–(5.31) because, in general, the weak limit \( w \) of the sequence \( \{w^\varepsilon\} \) differs from \( u \). More precisely, the following statement is true.

**Lemma 5.10.** If \( \mu_1 < \infty \) then the weak limits \( u, w^f, \theta^f, \vartheta, p, q, \) and \( \pi \) of the sequences \( \{u^\varepsilon\}, \{x^\varepsilon w^\varepsilon\}, \{x^\varepsilon \theta^\varepsilon\}, \{\vartheta^\varepsilon\}, \{p^\varepsilon\}, \{q^\varepsilon\}, \) and \( \{\pi^\varepsilon\} \) satisfy the initial-boundary value problem in \( \Omega_T \), consisting of the balance of momentum equation

\[
\tau_0 (\rho_f \frac{\partial^2 w^f}{\partial t^2} + \rho_s (1 - m) \frac{\partial^2 u}{\partial t^2}) + \nabla(q + \pi) - \hat{\rho} F = \text{div}_x \{\lambda_0 A^s_0 : D(x, u) + B^s_0 \text{div}_x u + B^s_1 q\},
\]

where \( A^s_0, B_0^s \) and \( B_1^s \) are the same as in (5.25), the continuity equations (5.27), equation

\[
\frac{1}{p_*} \rho + \frac{1}{\eta_0} \pi + \text{div}_x w^f + \frac{(1 - m) \beta_0}{\eta_0} \vartheta = (m - 1) \text{div}_x u,
\] (5.50)
equation (5.7), heat equation (5.41) and Darcy’s law in the form

\[
\frac{\partial w^f}{\partial t} = \frac{\partial u}{\partial t} + \int_0^t B_1 (\mu_1 t - \tau) \cdot (-\nabla q + \rho_f F - \tau_0 \rho_f \frac{\partial^2 u}{\partial t^2})(x, \tau) d\tau
\] (5.51)
in the case of $\tau_0 > 0$ and $\mu_1 > 0$, Darcy’s law in the form
\[ \frac{\partial w^f}{\partial t} = \frac{\partial u}{\partial t} + B_2(\mu_1) \cdot (-\nabla x q + \rho_f F) \] (5.52)
in the case of $\tau_0 = 0$ and, finally, Darcy’s law in the form
\[ \frac{\partial w^f}{\partial t} = B_3 \cdot \frac{\partial u}{\partial t} + \frac{1}{\tau_0 \rho_f} (mI - B_3) \cdot \int_0^t (-\nabla x q + \rho_f F)(x, \tau) d\tau \] (5.53)
in the case of $\mu_1 = 0$. The problem is supplemented by boundary and initial conditions (5.30)–(5.31) for the displacement $u$ and temperature $\vartheta$ of the rigid component and by the boundary condition
\[ w^f(x, t) \cdot n(x) = 0, \quad (x, t) \in S = \partial \Omega, \quad t > 0, \] (5.54)
for the displacement $w^f$ of the liquid component. In Eqs. (5.51)–(5.54) $n(x)$ is the unit normal vector to $S$ at a point $x \in S$, and matrices $B_1(\mu_1, t)$, $B_2(\mu_1)$, and $B_3$ are given in [10].

The proof of these statements repeats the proof of Lemma 5.8 in [10].

§6. Proof of Theorem 2.3

6.1. Weak and two-scale limits of sequences of displacements, temperatures and pressures.

I. Let
\[ \chi_{0s}, \quad \beta_{0f}, \quad \beta_{0s}, \quad \mu_1 < \infty \]
and one of the conditions (2.4) or (2.5) holds true. Then on the strength of Theorems 2.1 and 3.1 we conclude that sequences $\{\chi^\varepsilon w^\varepsilon\}$, $\{\chi^\varepsilon \vartheta^\varepsilon\}$, $\{p^\varepsilon\}$ and $\{q^\varepsilon\}$ two-scale converge to $\chi(y) W(x, t, y)$, $\chi(y) \Theta(x, t, y)$, $P(x, t, y)$ and $Q(x, t, y)$ and weakly converge in $L^2(\Omega_T)$ to $w^f$, $\vartheta^f$, $p$ and $q$ respectively. A sequence $\{u^\varepsilon\}$, where $u^\varepsilon(x, t)$ is an extension of $w^\varepsilon(x, t)$ from the domain $\Omega^\varepsilon_s$ into domain $\Omega$, strongly converges in $L^2(\Omega_T)$ and weakly in $L^2((0, T); W^1_2(\Omega))$ to zero and a sequence $\{\vartheta^\varepsilon\}$, where $\vartheta^\varepsilon(x, t)$ is an extension of $\vartheta^\varepsilon(x, t)$ from the domain $\Omega^\varepsilon_s$ into domain $\Omega$, strongly converges in $L^2(\Omega_T)$ and weakly in $L^2((0, T); W^1_2(\Omega))$ to $\vartheta$. Moreover a sequence $\{\nabla_x \vartheta^\varepsilon(x, t)\}$ two-scale converge to $\nabla_x \vartheta^\varepsilon(x, t) + \nabla_y \Theta^\varepsilon(x, t, y)$.

II. If
\[ \chi_{0s}, \quad \beta_{0f}, \quad \beta_{0s}, \quad \mu_1 < \infty \]
and conditions (2.3) together with assumption 2.2 hold true, then due to estimates (2.3) and (2.6) the sequences \( \{\alpha \lambda u^\varepsilon\} \) and \( \{\vartheta^\varepsilon\} \) converge strongly in \( L^2(\Omega_T) \) and weakly in \( L^2((0,T);W^1_2(\Omega)) \) to a function \( u \) and zero respectively, and the sequence \( \{\pi^\varepsilon\} \) converges weakly in \( L^2(\Omega_T) \) to a function \( \pi \).

III. If

\[
\gamma_2s, \quad \gamma_2s^{-1}, \quad \beta_{2f}, \quad \beta_{2s}, \quad \lambda_1, \quad p_2, \quad \eta_2 < \infty
\]

and assumption 2.2 holds true, then due to estimates (2.1)–(2.3) for the renormalized displacements and temperature sequences \( \{\chi^\varepsilon \alpha \lambda w^\varepsilon\}, \{\chi^\varepsilon \alpha \lambda \theta^\varepsilon\}, \{p^\varepsilon\}, \{q^\varepsilon\} \) and \( \{\pi^\varepsilon\} \) weakly converge in \( L^2(\Omega_T) \) to \( \tilde{w}^f, \tilde{\theta}^f, \tilde{p}, \tilde{q} \) respectively and sequences \( \{\alpha \lambda u^\varepsilon\} \) and \( \{\alpha \lambda \vartheta^\varepsilon\} \) converge strongly in \( L^2(\Omega_T) \) and weakly in \( L^2((0,T);W^1_2(\Omega)) \) to functions \( \tilde{u} \) and \( \tilde{\vartheta} \) respectively.

IV. If \( \mu_1 = \infty \) and

\[
0 < \lambda_1, \quad p_1, \quad \eta_1, \quad \beta_{1s}, \quad \beta_{1f}, \quad \gamma_{1s} < \infty,
\]

then on the strength of part 2) of Theorem 2.1 the sequences \( \{\alpha_{\mu \varepsilon}^{-2} \chi^\varepsilon w^\varepsilon\} \), \( \{\alpha_{\mu \varepsilon}^{-2} \chi^\varepsilon \theta^\varepsilon\}, \{p^\varepsilon\}, \{q^\varepsilon\} \) and \( \{\pi^\varepsilon\} \) weakly converge in \( L^2(\Omega_T) \) to \( \tilde{w}^f, \tilde{\theta}^f, \tilde{p}, \tilde{q} \) and \( \tilde{\pi} \) respectively, and the sequences \( \{\alpha_{\mu \varepsilon}^{-2} u^\varepsilon\} \) and \( \{\alpha_{\mu \varepsilon}^{-2} \vartheta^\varepsilon\} \) converge strongly in \( L^2(\Omega_T) \) and weakly in \( L^2((0,T);W^1_2(\Omega)) \) to functions \( \tilde{u} \) and \( \tilde{\vartheta} \) respectively.

As before in previous section §5, we conclude that

\[
\vartheta, |u|, \tilde{\vartheta}, |\tilde{u}| \in L^2((0,T);W^1_2(\Omega)).
\]

6.2. Homogenized equations.

I. We construct a closed system of equations for the velocity \( \partial w^f/\partial t \) in the liquid component pressures \( p \) and \( q \) as before (proof of Theorem 2.2), consisting of state equation (2.15), continuity equation (2.24), heat equation (2.25), modifications of Darcy’s law (5.51)–(5.53) (in which we have \( u(x,t) = 0 \)), boundary condition (5.54) for \( \partial w^f/\partial t \) and boundary and initial conditions (5.30), (5.31) for the temperature of the solid component. To derive continuity equation (2.24) we use equation (5.11) and to derive a homogenized heat equation (2.25) we rewrite the heat equation (1.9) using continuity equation (1.6). Note that in the previous section to get the same homogenized equation for the temperature of the solid component we have used both continuity equations (1.6) and (1.7). Here we do not express the term \( (1 - \chi^\varepsilon) \nabla (\partial u^\varepsilon/\partial t) \), because it converges to zero.

We entitle the above described systems as Problem \( F_1, F_2, \) or \( F_3 \) depending on the relations between parameters \( \tau_0 \) and \( \mu_1 \).
1) $\tau_0 > 0, \mu_1 > 0$ for the problem $F_1$;
2) $\tau_0 = 0$ for the problem $F_2$;
3) $\mu_1 = 0$ for the problem $F_3$.

II. In this situation we observe that the limiting displacements in the rigid skeleton are equal to zero. In order to find a more accurate asymptotic of the solution of the original model, we use again the re-normalization. Namely, let

$$w^\varepsilon \to \alpha_\lambda w^\varepsilon.$$

Then new displacements satisfy the same problem as displacements before re-normalization, but with new parameters

$$\alpha_\lambda \to \alpha_\lambda^{-1}, \quad \alpha_\tau \to \alpha_\tau^{-1}.$$

Thus we arrive at the assumptions of Theorem 2.2: the limiting functions $u(x, t), \pi(x, t), \Pi(x, t, y)$, and $U(x, t, y)$ satisfy the system of micro- and macroscopic equations (5.3), (5.14), (5.8) and (5.17), in which the pressure $q$ is given by virtue of one of Problems $F_1$–$F_3$, and $\beta_0 = 0, \tau_0 = 0, \eta_0 = \eta_2, \lambda_0 = 1$.

Hence for $u(x, t)$ and $\pi(x, t)$ there hold true the homogenized momentum equation (2.29), homogenized continuity equation (2.30) and a homogeneous boundary condition for $u$. 

§7. Proof of Theorem 2.4

7.1. Weak and two-scale limits of sequences of displacement, temperature and pressures.

On the strength of Theorem 2.1 the sequences $\{w^\varepsilon\}, \{\theta^\varepsilon\}, \{p^\varepsilon\}$ and $\{\pi^\varepsilon\}$ are uniformly in $\varepsilon$ bounded in $L^2(\Omega_T)$. Then there exist a subsequence from $\{\varepsilon > 0\}$ and functions $p, \pi, q, w$ such that as $\varepsilon \to 0$

$$w^\varepsilon \to w, \quad \theta^\varepsilon \to \theta, \quad p^\varepsilon \to p, \quad q^\varepsilon \to q, \quad \pi^\varepsilon \to \pi \quad L^2(\Omega_T). \quad (7.1)$$

Moreover, since $\lambda_0, \mu_0 > 0$ then the bounds (2.1)–(2.3) imply

$$\nabla_x w^\varepsilon \to \nabla_x w, \quad \nabla_x \theta^\varepsilon \to \nabla_x \theta \quad \text{weakly in } L^2(\Omega_T). \quad (7.2)$$

Due to limiting relations (7.1), (7.2) and Ngutseng’s theorem, there exist one more subsequence from $\{\varepsilon > 0\}$ and 1-periodic in $y$ functions $P(x, t, y)$, $\Pi(x, t, y), Q(x, t, y), \Theta(x, t, y)$ and $U(x, t, y)$ such that the sequences $\{p^\varepsilon\}$,
\{\pi^\varepsilon\}, \{q^\varepsilon\}, \{\nabla \theta^\varepsilon\} and \{\nabla w^\varepsilon\} two-scale converge as \varepsilon \searrow 0 respectively to \(P, \Pi, Q, \nabla_x \theta + \nabla_y \Theta\), and \(\nabla_x w + \nabla_y U\).

7.2. Micro- and macroscopic equations.

Lemma 7.1. Two-scale limits of the sequences \(\{p^\varepsilon\}, \{\pi^\varepsilon\}, \{q^\varepsilon\}, \{\nabla \theta^\varepsilon\}\) and \(\{\nabla w^\varepsilon\}\) satisfy in \(Y_T = Y \times (0, T)\) the following relations

\[
\frac{1}{\eta_0} \Pi + (1 - \chi)(\text{div}_x w + \text{div}_y W) - \frac{\beta_{0s}}{\eta_0} \theta = 0; \quad (7.3)
\]

\[
\frac{1}{p_*} P + \chi (\text{div}_x w + \text{div}_y W) = 0; \quad (7.4)
\]

\[
Q = P + \frac{\nu_0}{p_*} \frac{\partial P}{\partial t} + \beta_{0f} \chi \theta; \quad (7.5)
\]

\[
\text{div}_y (\chi \mu_0(\mathbb{D}(x, \frac{\partial w}{\partial t}) + \mathbb{D}(y, \frac{\partial W}{\partial t})) + (1 - \chi) \lambda_0(\mathbb{D}(x, w) + \mathbb{D}(y, W))) \tag{7.6}
\]

\[
- \nabla_y (Q + \Pi) = 0;
\]

\[
\text{div}_y (\chi s_0 f (\nabla_x \theta + \nabla_y \Theta) + (1 - \chi) s_{0s} (\nabla_x \theta + \nabla_y \Theta)) = 0. \tag{7.7}
\]

Lemma 7.2. The weak limits \(p, \pi, q, w\) satisfy in \(\Omega_T\) the following system of macroscopic equations:

\[
\frac{1}{\eta_0} \pi + (1 - m) \text{div}_x w + \langle \text{div}_y W \rangle_{Y_s} - (1 - m) \frac{\beta_{0s}}{\eta_0} \theta = 0; \quad (7.8)
\]

\[
\frac{1}{p_*} p + m \text{div}_x w + \langle \text{div}_y W \rangle_{Y_f} = 0; \quad (7.9)
\]

\[
q = p + \frac{\nu_0}{p_*} \frac{\partial p}{\partial t} + \beta_{0f} m \theta; \quad (7.10)
\]

\[
\tau_0 \dot{\rho} \frac{\partial^2 w}{\partial t^2} + \nabla (q + \pi) - \rho F = \text{div}_x (\mu_0(m \mathbb{D}(x, \frac{\partial w}{\partial t}) + \langle \mathbb{D}(y, \frac{\partial W}{\partial t}) \rangle_{Y_f}) + \lambda_0((1 - m)\mathbb{D}(x, w) + \langle \mathbb{D}(y, W) \rangle_{Y_f})); \tag{7.11}
\]

\[
(\tau_0 \dot{c}_p + \frac{\beta_{0s}^2}{\eta_0}(1 - m)) \frac{\partial \theta}{\partial t} - \frac{\beta_{0f} \rho}{p_*} \frac{\partial p}{\partial t} - \frac{\beta_{0s} \theta}{\eta_0} \frac{\partial \Pi}{\partial t} - \Psi = \tag{7.12}
\]

\[
\text{div}_x (s_0 f (m \nabla_x \theta + \langle \nabla_y \Theta \rangle_{Y_f}) + s_{0s}((1 - m) \nabla_x \theta + \langle \nabla_y \Theta \rangle_{Y_s}));
\]

\[
\text{div}_x (s_0 f (m \nabla_x \theta + \langle \nabla_y \Theta \rangle_{Y_f}) + s_{0s}((1 - m) \nabla_x \theta + \langle \nabla_y \Theta \rangle_{Y_s})).
\]
Proofs of these statements are the same as in lemmas 5.1 – 5.4.

7.3. Homogenized equations.

**Lemma 7.3.** Weak limits $p$, $\pi$, $q$, $\theta$ and $w$ satisfy in $\Omega_T$ the following system of homogenized equations:

\[
\begin{align*}
\tau_0 \rho \frac{\partial^2 w}{\partial t^2} + \nabla (q + \pi) &= \text{div}_x (A_2 : \mathbb{D}(x, \frac{\partial w}{\partial t}) + A_3 : \mathbb{D}(x, w)) + (7.13) \\
B_4 \text{div}_x w + B_1^0 \theta + \int_0^t (A_4(t - \tau) : \mathbb{D}(x, w(x, \tau))) + \\
B_5(t - \tau) \text{div}_x w(x, \tau) + B_2^0(t - \tau)\theta(x, \tau) \, d\tau) + \hat{\rho} F, \\
q &= p + \nu_0 \frac{\partial p}{p_* \partial t} + \beta_0 m \theta, \\
\frac{1}{p_*} p + m \text{div}_x w &= - \int_0^t (C_2(t - \tau) : \mathbb{D}(x, w(x, \tau))) + \\
a_2(t - \tau) \text{div}_x w(x, \tau) + a_1^0(t - \tau)\theta(x, \tau) \, d\tau.
\end{align*}
\]

\[
\begin{align*}
\frac{1}{\eta_0} \pi + (1 - m)(\text{div}_x w - \frac{\beta_0 w}{\eta_0} \theta) &= - \int_0^t (a_2^0(t - \tau)\theta(x, \tau) + \\
C_3(t - \tau) : \mathbb{D}(x, w(x, \tau))) + a_3(t - \tau) \text{div}_x w(x, \tau) \, d\tau, \\
(\tau_0 \rho \pi p + \frac{\beta_0^2}{\eta_0} (1 - m) \frac{\partial \theta}{\partial t}) - \frac{\beta_0 f}{p_*} \frac{\partial \rho}{\partial t} - \frac{\beta_0 \rho}{\eta_0} \frac{\partial \pi}{\partial t} &= \text{div}_x (B_0^0 \cdot \nabla_x \theta) + \Psi. 
\end{align*}
\]

Here $A_2$, $A_3$, $A_4$ – fourth-rank tensors, $B_4$, $B_5$, $B^0_0$, $B^0_1$, $B^0_2$, $C_2$, $C_3$ - matrices and $a_2$, $a_3$, $a_1^0$, $a_2^0$ - scalars. The exact expressions for these objects are given below by formulas (7.38)–(7.46).

**Proof.** Let

\[
Z(x, t) = \mu_0 \mathbb{D}(x, \frac{\partial w}{\partial t}) - \lambda_0 \mathbb{D}(x, w), \quad Z_{ij} = e_i \cdot (Z \cdot e_j),
\]

\[
z(x, t) = \sum_{i=1}^3 z_i(x, t)e_i = (\kappa_0 f - \kappa_0 f)\nabla_x \theta, \quad z_0(x, t) = \text{div}_x w.
\]

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As usual we look for the solution of the system of microscopic equations (7.3)–(7.7) in the form

\[ W = \int_0^t \left[ W^0(y, t - \tau)z_0(x, \tau) + W^\theta(y, t - \tau)\theta(x, \tau) \right] d\tau \]

\[ + \sum_{i,j=1}^3 W^{ij}(y, t - \tau)Z_{ij}(x, \tau) \] (7.18)

\[ P = \chi \int_0^t \left[ P^0(y, t - \tau)z_0(x, \tau) + P^\theta(y, t - \tau)\theta(x, \tau) \right] d\tau \]

\[ + \sum_{i,j=1}^3 P^{ij}(y, t - \tau)Z_{ij}(x, \tau) \] (7.19)

\[ Q = \chi (Q_0(y) \cdot z_0(x, t) + \sum_{i,j=1}^3 Q_0^{ij}(y)Z_{ij}(x, t)) \]

\[ + Q^0_0(y)\theta(x, t) + \int_0^t \left[ Q^0(y, t - \tau)z_0(x, \tau) + Q^\theta(y, t - \tau)\theta(x, \tau) \right] d\tau \]

\[ + \sum_{i,j=1}^3 Q^{ij}(y, t - \tau)Z_{ij}(x, \tau) \] (7.20)

\[ \Pi = (1 - \chi) \int_0^t \left[ \sum_{i,j=1}^3 \Pi^{ij}(y, t - \tau)Z_{ij}(x, \tau) + \Pi^0(y, t - \tau)\theta(x, \tau) \right] d\tau \]

\[ + Q^0_0(y)\theta(x, t) + \Pi^\theta(y, t - \tau)\theta(x, \tau) \] (7.21)

\[ \Theta = \sum_{i=1}^3 \Theta^i(y)z_i(x, t) \] (7.22)

where 1-periodic in \( y \) functions \( W^0, W^\theta, W^{ij}, P^0, P^\theta, P^{ij}, Q^0, Q^\theta, Q_0, Q_0^{ij}, Q^0, Q^\theta, Q^{ij}, \Pi^0, \Pi^\theta, \Pi^{ij}, \Theta^i \) satisfy the following periodic initial-boundary value problems in the elementary cell \( Y \):
Problem (I)

\[ \text{div}_y \left( \chi(\mu_0 \mathbb{D}(y, \frac{\partial W^{ij}}{\partial t}) + (1 - \chi)(\lambda_0 \mathbb{D}(y, W^{ij}) - ((1 - \chi)\Pi^{ij} + \chi Q^{ij}) \mathbb{I})) \right) = 0; \]  
\[ \frac{1}{p_*} P^{ij} + \chi \text{div}_y W^{ij} = 0, \quad Q^{ij} = P^{ij} + \frac{\nu_0}{p_*} \frac{\partial P^{ij}}{\partial t}, \]  
\[ \frac{1}{\eta_0} \Pi^{ij} + (1 - \chi)\text{div}_y W^{ij} = 0, \quad W^{ij}(y, 0) = W^{ij}_0(y); \]  
\[ \text{div}_y \left( \chi(\mu_0 \mathbb{D}(y, W^{0ij}) + J^{ij} - Q^{ij}_0 \mathbb{I}) \right) = 0, \]  
\[ \chi(Q^{ij}_0 + \nu_0 \text{div}_y W^{ij}_0) = 0. \]

Problem (II)

\[ \text{div}_y \left( \chi(\mu_0 \mathbb{D}(y, \frac{\partial W^0}{\partial t}) + (1 - \chi)(\lambda_0 \mathbb{D}(y, W^0) - ((1 - \chi)\Pi^0 + \chi Q^0) \mathbb{I})) \right) = 0; \]  
\[ \chi \left( \frac{1}{p_*} P^0 + \text{div}_y W^0 + 1 \right) = 0, \quad Q^0 = P^0 + \frac{\nu_0}{p_*} \frac{\partial P^0}{\partial t}; \]  
\[ (1 - \chi) \left( \frac{1}{\eta_0} \Pi^0 + \text{div}_y W^0 + 1 \right) = 0; \]  
\[ W^0(y, 0) = W^0_0(y), \quad \text{div}_y \left( \chi(\mu_0 \mathbb{D}(y, W^0_0) - Q_0 \mathbb{I}) \right) = 0, \]  
\[ \chi(Q_0 + \nu_0 (\text{div}_y W^0_0 + 1)) = 0. \]

Problem (III)

\[ \text{div}_y \left( \chi(\mu_0 \mathbb{D}(y, \frac{\partial W^\theta}{\partial t}) + (1 - \chi)(\lambda_0 \mathbb{D}(y, W^\theta) - ((1 - \chi)\Pi^\theta + \chi Q^\theta) \mathbb{I})) \right) = 0; \]  
\[ \chi \left( \frac{1}{p_*} P^\theta + \text{div}_y W^\theta \right) = 0, \quad Q^\theta = P^\theta + \frac{\nu_0}{p_*} \frac{\partial P^\theta}{\partial t} + \beta_0 f \chi; \]  
\[ (1 - \chi) \left( \frac{1}{\eta_0} \Pi^\theta + \text{div}_y W^\theta - \frac{\beta_0 s}{\eta_0} \right) = 0; \]  
\[ W^\theta(y, 0) = W^\theta_0(y), \quad \text{div}_y \left( \chi(\mu_0 \mathbb{D}(y, W^\theta_0) - Q^\theta_0 \mathbb{I}) \right) = 0, \]  
\[ \chi(Q^\theta_0 + \nu_0 (\text{div}_y W^\theta_0 - \beta_0 f)) = 0. \]

Problem (IV)

\[ \text{div}_y \left( \chi \varphi_0 f + (1 - \chi) \varphi_0 s \right) \nabla_y \Theta^i + \chi e_i = 0. \]
Then
\[ A_2 = \mu_0 m \sum_{i,j=1}^{3} J^{ij} \otimes J^{ij} + \mu_0 A_0^f, \]
\[ A_0^f = \mu_0 \sum_{i,j=1}^{3} \langle \mathbb{D}(y, W_0^{ij}(y,t)) \rangle Y_f \otimes J^{ij}; \]  
\[ (7.38) \]
\[ A_3 = \lambda_0(1 - m) \sum_{i,j=1}^{3} J^{ij} \otimes J^{ij} - \lambda_0 A_0^f + \mu_0 A_1^f(0), \]
\[ A_4(t) = \mu_0 \frac{d}{dt} A_1^f(t) - \lambda_0 A_1^f(t); \]
\[ (7.39) \]
\[ A_1^f(t) = \sum_{i,j=1}^{3} \{ \mu_0 \langle \mathbb{D}(y, \frac{\partial W^{ij}}{\partial t}(y,t)) \rangle Y_f + \lambda_0 \langle \mathbb{D}(y, W^{ij}(y,t)) \rangle Y_s \} \otimes J^{ij}; \]
\[ (7.40) \]
\[ B_5(t) = \mu_0 \langle \mathbb{D}(y, \frac{\partial W^0}{\partial t}(y,t)) \rangle Y_f + \lambda_0 \langle \mathbb{D}(y, W^0(y,t)) \rangle Y_s; \]
\[ (7.41) \]
\[ C_2(t) = -C_3(t) = \sum_{i,j=1}^{3} \langle \text{div}_y W^{ij}(y,t) \rangle Y_f J^{ij}; \]
\[ (7.42) \]
\[ a_2(t) = -a_3(t) = \langle \text{div}_y W^0(y,t) \rangle Y_f, \quad B_4 = \mu_0 \langle \mathbb{D}(y, W_0^0(y)) \rangle Y_f; \]
\[ a_1^\theta(t) = -a_2^\theta(t) = \langle \text{div}_y W^\theta(y,t) \rangle Y_f, \quad B_1^\theta = \mu_0 \langle \mathbb{D}(y, W_0^\theta(y)) \rangle Y_f; \]
\[ (7.43) \]
\[ B_2^\theta(t) = \mu_0 \langle \mathbb{D}(y, \frac{\partial W^\theta}{\partial t}(y,t)) \rangle Y_f + \lambda_0 \langle \mathbb{D}(y, W^\theta(y,t)) \rangle Y_s, \]
\[ (7.44) \]
\[ B_0^\theta = \hat{\kappa}_0 \mathbb{I} + \sum_{i=1}^{3} \{ \kappa_{0f} \langle \nabla \Theta^i \rangle Y_f + \kappa_{0s} \langle \nabla \Theta^i \rangle Y_s \} \otimes e_i, \]
\[ (7.45) \]
where \( \hat{\kappa}_0 = m \kappa_{0f} + (1 - m) \kappa_{0s}. \)  
\[ \square \]

**Lemma 7.4.** Tensors \( A_2, A_3 \) and \( A_4, \) matrices \( B_4, B_5, B_0^\theta, B_1^\theta, B_2^\theta, C_2 \) and \( C_3 \) and scalars \( a_2, a_3, a_1^\theta \) and \( a_2^\theta \) are well-defined and infinitely smooth in time. The matrix \( B_0^\theta \) is strictly positively defined.

If a porous space is connected, then the symmetric tensor \( A_2 \) is strictly positively defined. For the case of disconnected porous space (isolated pores) \( A_2 = 0 \) and the tensor \( A_2 \) becomes strictly positively defined.

The proof of this lemma, except statement about a matrix \( B_0^\theta, \) one can find in [10]. All properties of the matrix \( B_0^\theta \) are well-known (see [14], [8]).
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