On triangulating $k$-outerplanar graphs

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Abstract

A $k$-outerplanar graph is a graph that can be drawn in the plane without crossing such that after $k$-fold removal of the vertices on the outer-face there are no vertices left. In this paper, we study how to triangulate a $k$-outerplanar graph while keeping its outerplanarity small. Specifically, we show that not all $k$-outerplanar graphs can be triangulated so that the result is $k$-outerplanar, but they can be triangulated so that the result is $(k + 1)$-outerplanar.

1 Introduction

A planar graph is a graph $G = (V, E)$ that can be drawn in the plane without crossing. Given such a drawing $\Gamma$, the faces are the connected pieces of $\mathbb{R}^2 - \Gamma$; the unbounded piece is called the outer-face. A planar drawing can be described by giving for each vertex the clockwise order of edges at it, and by saying which edges are incident to the outer-face; we call this a combinatorial embedding.

Assume that a planar drawing $\Gamma$ has been fixed. Define $L_1$ to be the vertices incident to the outer-face, and define $L_i$ for $i > 1$ recursively to be the vertices on the outer-face of the planar drawing obtained when removing the vertices in $L_1, \ldots, L_{i-1}$. We call $L_i$ (for $i \geq 1$) the $i$th onion peel of drawing $\Gamma$. A graph is called $k$-outerplanar if it has a planar drawing that has most $k$ onion peels. The outer-planarity of a planar graph $G$ is the smallest $k$ such that $G$ is $k$-outerplanar.

A triangulated graph is a planar graph for which all faces (including the outer-face) are triangles. A triangulated disk is a planar graph for which the outer-face is a simple cycle and all inner faces (i.e., faces that are not the outer-face) are triangles. It is well-known that any planar graph can be triangulated, i.e., we can add edges to it without destroying planarity so that it becomes triangulated.

Sometimes it is of interest to triangulate a planar graph while maintaining other properties. For example, any planar graph without separating triangles can be triangulated without creating separating triangles [2], with the exception of graphs with a universal vertex. Any

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planar graph can be triangulated so that the maximum degree increases by at most a constant \[10\]. Any planar graph \(G\) can be triangulated such that the result has treewidth at most \(\max\{3, \text{tw}(G)\}\) \[3\]. Also, following the proof of Heawood’s 3-color theorem \[8\], one can easily show that any 3-colorable planar graph can be made triangulated by adding edges and vertices such that the result is 3-colorable.

In this paper, we investigate whether a planar graph can be triangulated without changing its outer-planarity. We show first that this is not true. For example, a 4-cycle has outer-planarity 1, but the only way to triangulate it is to create \(K_4\), which has outer-planarity 2. (We give more complicated examples for higher outer-planarity in Section 2.) However, if we are content with “only” converting the graph to a triangulated disk, then it is always possible to do so without increasing the outer-planarity (see Section 3). In consequence, any \(k\)-outerplanar graph can be triangulated so that its outer-planarity is at most \(k + 1\). In Section 4, we use our triangulations to give a different proof of the well-known result \[6\] that \(k\)-outerplanar graphs have treewidth at most \(3k - 1\).

2 Triangulating \(k\)-outerplanar graphs

In this section, we show that not all planar graphs can be triangulated while maintaining the outer-planarity.

**Theorem 1.** For any \(k \geq 1\), there exists a triangulated disk \(G\) with \(O(k)\) vertices that is \(k\)-outerplanar, but any triangulation of \(G\) has outer-planarity at least \(k + 1\).

**Proof.** For \(k = 1\), the graph \(K_4\) with one edge deleted is a suitable example. For \(k > 1\), we first define an auxiliary graph \(T_i\) as follows. \(T_1\) consists of a single triangle \(t_1\). \(T_i\), for \(i > 1\), is obtained by taking a triangle \(t_i\) and inserting a copy of \(T_{i-1}\) inside it; then add a 6-cycle between triangles \(t_i\) and \(t_{i-1}\). In other words, \(T_i\) consists of \(i\) nested triangles. Clearly graph \(T_i\) is 3-connected and has \(i\) onion peels if \(t_i\) is the outer-face. See Figure 1 (left).

We now define graph \(G\) to consist of four copies of \(T_k\), in the embedding with \(t_k\) on the outer-face, and connect them so that the outer-face contains two vertices of each copy of \(t_k\). The inner faces “between” the four copies of \(T_k\) are triangulated arbitrarily. See Figure 1 (right). Notice that the first and second onion peel will contain (in each copy of \(T_k\)) all vertices of \(t_k\) and \(t_{k-1}\). Therefore the \(i\)th onion peel (for \(2 \leq i \leq k\)) contains \(t_{k-i}\) and hence \(G\) is \(k\)-outerplanar. It is also a triangulated disk and has \(12k\) vertices.

Now let \(G'\) be any triangulation of \(G\). Since there are three vertices on the outer-face of \(G'\), there exists one copy \(C\) of \(T_k\) that does not have any vertex on the outer-face. In consequence (since \(T_k\) is 3-connected), the embedding of \(C\) induced by \(G'\) must have \(t_k\) as its outer-face. The first onion peel of \(G'\) contains no vertex of \(C\). In consequence, at least \(k + 1\) onion peels are required before all vertices of \(C\) are removed, and the outer-planarity of \(G'\) is at least \(k + 1\). \(\Box\)
Figure 1: (Left) Graph $T_3$. (Right) A 3-outer planar graph which cannot be triangulated and stay 3-outerplanar. Thick edges indicate an outer-face-rooted spanning forest of height 2 (defined formally in Section 3).

### 3 Converting to triangulated disks

In this section, we aim to show that we can triangulate inner faces without increasing the outer-planarity. To our knowledge, this result was not formally described in the literature before (though Lemma 3.11.1 in [4] has many of the crucial steps for it.) From now on, let $G$ be a $k$-outerplanar graph with the planar embedding and outer-face fixed such that it has onion peels $L_1, L_2, \ldots, L_k$. We first compute a special spanning forest of $G$ (after adding some edges). We need some preliminary results

**Observation 1.** If $v \in L_i$ (for some $i > 1$), then some incident face of $v$ contains vertices in $L_{i-1}$.

**Proof.** Since $v$ is in $L_i$ and not in $L_{i-1}$, it is not on the outer-face of the graph $H_{i-1}$ induced by $L_{i-1} \cup L_i \cup L_{i+1} \cup \ldots$. Therefore all incident faces of $v$ (in $H_{i-1}$) are inner faces. But since $v$ is on the outer-face after deleting $L_{i-1}$, at least one of its incident faces merges with the outer-face when removing $L_{i-1}$. Therefore at least one incident face of $v$ contains a vertex from $L_{i-1}$.

**Observation 2.** We can add edges (while maintaining planarity) such that every vertex in $L_i$, $i > 1$ has a neighbor in $L_{i-1}$.

**Proof.** Add edges in any inner face $f$ as follows: Let $w$ be the vertex of $f$ contained in the onion peel with smallest index among all vertices of $f$ (breaking ties arbitrarily.) For any
vertex \( v \neq w \) of \( f \), add an edge \( (v, w) \) if it did not exist already. Clearly this maintains planarity since all new edges can be drawn inside face \( f \).

By Observation 1, every vertex \( v \in L_i \) (for \( i > 1 \)) had an incident face \( f_v \) that contained a vertex in \( L_{i-1} \). When applying the above procedure to face \( f_v \), some vertex \( w \) in \( L_{i-1} \) is made adjacent to \( v \), unless \( (w, v) \) already was an edge. Either way, afterwards \( v \) has the neighbor \( w \in L_{i-1} \).

A spanning forest of \( G \) is a subgraph that contains all vertices of \( G \) and has no cycles. We say that a spanning forest is outer-face-rooted if every connected component of it contains exactly one vertex on the outer-face. We say that an outer-face-rooted spanning forest \( F \) has height \( h \) if every vertex \( v \) has distance (in \( F \)) at most \( h \) to an outer-face vertex. See also Figure 1.

**Lemma 1.** Let \( G \) be a \( k \)-outerplanar graph. The we can add edges to \( G \) (while maintaining planarity) such that \( G \) has an outer-face-rooted spanning forest of height at most \( k - 1 \).

**Proof.** First add edges as in Observation 2. Now any vertex \( v \) in \( L_i \), \( i \geq 1 \) has distance at most \( i - 1 \) from some vertex in \( L_1 \): This holds by definition for \( i = 1 \), and holds by induction for \( i > 1 \), since vertex \( v \) has a neighbor \( w \) in \( L_{i-1} \) and \( w \) has distance at most \( i - 2 \) to some vertex in \( L_1 \).

Now perform a breadth-first search, starting at all the vertices on the outer-face \( L_1 \). The resulting breadth-first search tree \( F \) (which is a forest, since we start with multiple vertices) has one component for each outer-face vertex. Since breadth-first search computes distances from its start-vertices, each vertex has distance at most \( k - 1 \) from a root of \( F \) and so \( F \) has height at most \( k - 1 \). 

**Lemma 2.** Let \( G \) be a planar graph that (for some fixed planar embedding and outer-face) has an outer-face-rooted spanning forest \( F \) of height \( k - 1 \). Then \( G \) is \( k \)-outerplanar.

**Proof.** Root each connected component \( T \) of \( F \) at the vertex on the outer-face. Removing the outer-face \( L_1 \) then removes the root of each tree \( T \). After the roots have been removed, all their children appear on the outer-face of what remains. So all children of the roots are in \( L_2 \). (There may be other vertices in \( L_2 \) as well.) Continuing the argument shows that the vertices at distance \( i \) from the roots are in onion peel \( L_{i+1} \) or in one of earlier onion peels \( L_1, \ldots, L_i \). Therefore \( G \) has at most \( k \) non-empty onion peels and it is \( k \)-outerplanar.

**Theorem 2.** Any \( k \)-outerplanar graph \( G \) can be converted into a \( k \)-outerplanar triangulated disk by adding edges.

**Proof.** Add edges to \( G \) (while maintaining planarity) until it has an outer-face-rooted spanning forest \( F \) of height \( k - 1 \) (Lemma 1). While the outer-face is disconnected, add an edge between two vertices on the outer-face of different connected components. While the outer-face has a vertex \( v \) that appears on it multiple times, add an edge between two neighbors of \( v \) on the outer-face. Finally, add more edges to \( G \) (with the standard techniques for triangulating) until all interior faces are triangles. Note that none of these edges additions removes any vertex from the outer-face. So we end with a triangulated disk \( D \) whose
outer-face vertices are the same as the ones on \( G \). In particular, \( F \) is an outer-face-rooted spanning forest of \( D \) as well, and it still has height \( k - 1 \). By Lemma 2 \( D \) is \( k \)-outerplanar as desired.

\[ \square \]

**Corollary 1.** Any \( k \)-outerplanar graph \( G \) can be triangulated such that the result has outer-planarity at most \( k + 1 \).

**Proof.** First convert \( G \) into a triangulated disk \( D \) that is \( k \)-outerplanar. Now pick one vertex \( r \) on the outer-face of \( D \) that has only two neighbors on the outer-face on \( r \). This exists because the outer-face induces a 2-connected outer-planar graph; such graphs have a degree-2 vertex. Make \( r \) adjacent to all other vertices on the outer-face. Clearly the result \( G' \) is a triangulated graph. Also, if \( L'_0, L'_1, \ldots \) are the onion peels of \( G' \), then \( r \in L'_0 \), any neighbors of \( r \) (and in particular therefore all of \( L_1 \)) is in \( L'_0 \cup L'_1 \), and by induction any vertex in \( L_i \) is in \( L'_0 \cup \cdots \cup L'_i \). Therefore \( G' \) has at most \( k + 1 \) onion peels as desired. \[ \square \]

4 **Treewidth of \( k \)-outerplanar graphs**

It is well-known that any \( k \)-outerplanar graph has treewidth at most \( 3k - 1 \) \([5, 6]\) and this bound is tight \([9]\). (We will not review the definition of treewidth here, since we will only use the closely related concept of branchwidth.) This has important algorithmic consequences: many (normally NP-hard) problems can be solved in polynomial time on \( k \)-outerplanar graphs, which allows for a PTAS for many problems in planar graphs (see Baker \([1]\)), or for solving graph isomorphism and related problems efficiently in planar graphs (see Eppstein \([7]\)).

The proof in \([9]\) is non-trivial and in particular requires first converting the \( k \)-outerplanar graph \( G \) into a \( k \)-outerplanar graph \( H \) with maximum degree 3 such that \( G \) is a minor of \( H \). A detailed discussion (and analysis of the linear-time complexity to find the tree decomposition) is given in \([11]\). A second, different, proof can be derived from Tamaki’s theorem \([13]\) that shows that the branchwidth of a graph is bounded by the radius of the face-vertex-incidence graph. But this proof is not straightforward either, as it requires detours into the medial graph and the carving width.

Our result on triangulating \( k \)-outerplanar graphs, in conjunction with some results of Eppstein concerning tree decompositions of graphs with small diameter \([7]\), allows for a different (and in our opinion simpler) proof that every \( k \)-outerplanar graph has treewidth at most \( 3k - 1 \). We explain this in the following.

We first need to define a closely related concept, the *branchwidth*.

**Definition 1.** A branch decomposition of a graph \( G \) is a tree \( T \) that has maximum degree 3, together with an injective assignment of the edges of \( G \) to the leaves of \( T \). In such a branch decomposition, a vertex \( v \) of \( G \) is said to cross an arc \( a \) of \( T \) if two incident edges of \( v \) are assigned to leaves in two different components of \( T - a \). The branch decomposition is said to have width \( w \) if any arc \( a \) of \( T \) is crossed by at most \( w \) vertices. The branchwidth of a graph \( G \) is the minimum width of a branch decomposition of \( G \).
The following lemma relates the branchwidth of a planar graph $G$ to the height of an outer-planar-rooted spanning forest $F$ of $G$. It is strongly inspired by Lemma 4 of [7] (which in turn was inspired by [1]):

**Lemma 3.** Let $G$ be a triangulated disk with an outer-face-rooted spanning forest $F$ of height $h - 1$. Then $G$ has branchwidth at most $2h$.

**Proof.** Let $G^*$ be the dual graph of $G$. Let $T^*$ be a subgraph of $G^*$ defined as follows: $T^*$ contains all vertices of $G^*$ (= faces of $G$), except for the outer-face of $G$. It also contains the duals of all edges of $E$ that are not in $F$ and not on the outer-face of $G$. See also Figure 2 left).

We claim that $T^*$ is a tree. This can be seen as follows. Define $F^+$ to be the subgraph of $G$ formed by the edges of $F$, as well as all but one edge on the outer-face. Since $F$ is an outerface-rooted forest, $F^+$ is a spanning tree of $G$. By the well-known tree-co-tree result ([14], p.289) therefore the duals of the edges not in $F^+$ form a spanning tree $T^+$ of the dual graph. The outer-face-vertex is a leaf in $T^+$ by definition of $F^+$. Deleting this leaf from $T^+$ yields exactly $T^*$, which therefore is a tree.

We will use $T^*$ (with some additions) as the tree for the branch decomposition. See also Figure 2 A node of $T^*$ will be called face-node and denoted $n(f)$ if it corresponds to the inner face $f$ of $G$. Let $T_1$ be the tree obtained from $T^*$ by subdividing each arc $a$ of $T^*$ with an arc-node $n(a)$. Let $T_2$ be the tree obtained from $T_1$ by adding an edge-node $n(e)$ for every edge $e$ of $G$. If the dual edge $e^*$ of $e$ is an arc of $T^*$, then make $n(e)$ adjacent to the arc-node $n(e^*)$; note that $n(e^*)$ had degree 2 before and is used for exactly one $n(e)$, so it has degree 3 now. If the dual edge of $e$ is not in $T^*$, then either $e$ is on the outer-face or $e$ belongs to $F$. In both cases, pick an inner face $f$ incident to $e$ and make $n(e)$ adjacent to $n(f)$. Notice that in $T_2$ node $n(f)$ has at most one incident arc for each edge of $f$, therefore $n(f)$ has degree at most 3.

We use tree $T_2$ for the branch decomposition and assign edge $e$ of $G$ to node $n(e)$. We have already argued that $T_2$ has maximum degree 3, so it is a branch decomposition, and it only remains to analyze its width. Let $a$ be an arc of $T_2$. If $a$ is incident to a node $n(e)$ of $T_2$, then only the vertices of $e$ can cross $a$, so at most $2 \leq 2h$ vertices cross $a$. If $a$ is not incident to a node $n(e)$, then it has the form $(n(f), n(e^*))$ for some inner face $f$ of $G$ and some edge $e = (v_1, v_2)$ that is incident to $f$ and does not belong to $F$.

If $v_1$ and $v_2$ are in different connected components of $F$, then for $j = 1, 2$, let $P_j$ be the path from $v_j$ to the outer-face vertex $r_j$ in $v_j$’s component of $F$. Observe that $P_1$ and $P_2$ are disjoint, and therefore $P_1 \cup \{e\} \cup P_2$ is a path from outer-face to outer-face that splits the inner faces of $G$ into two parts, namely, the two parts corresponding to the two connected components of $T_2 - a$. Any vertex that has incident edges in both those connected components hence must be on $P_1 \cup \{e\} \cup P_2$. But $P_1$ and $P_2$ contain at most $h - 1$ edges each, so there are at most $2h$ vertices that cross $a$. Similarly, if $v_1$ and $v_2$ are in the same connected component of $F$, then let $P$ the path from $v_1$ to $v_2$ in $F$, and observe that $P \cup \{e\}$ forms a cycle that separates the two components of $T_2 - a$. Since $P$ contains at most $2h - 2$ edges, in this case at most $2h - 1$ vertices cross $a$.

So this branch decomposition has width at most $2h$ as desired. 

\[\square\]
Since $tw(G) \leq \max\{1, \lfloor \frac{3}{2}bw(G) \rfloor \} - 1$ for the treewidth $tw(G)$ and branchwidth $bw(G)$ of a graph [12], we therefore have:

**Corollary 2.** Let $G$ be a triangulated disk with a outer-face-rooted spanning forest $F$ of height $h - 1$. Then $G$ has treewidth at most $3h - 1$.

Since adding edges does not decrease the treewidth, therefore by Lemma [1] we have:

**Corollary 3.** Any $k$-outerplanar graph has treewidth at most $3k - 1$.

Following the steps of our proof, it is easy to see that the branch decomposition of width $2k$ can be found in linear time, and from it, a tree decomposition of width $3k - 1$ is easily obtained by following the proof in [12].

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