Distinguishability, Ensemble Steering, and the No-Signaling Principle

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Distinguishability in a physical theory characterizes and dictates fundamental limitations of information tasks performed in the theory. Here we formalize the distinguishability in a general framework of non-signaling theories, namely generalized probabilistic theories, in terms of minimum-error state discrimination by using convex optimization techniques. We provide a geometric formulation for finding optimal discrimination, and show that the distinguishability is generally a global property assigned to ensembles of states rather than other details of a state space or pairwise relations. We then show, if ensemble steering is possible within a generalized probabilistic theory, the distinguishability is determined by the no-signaling principle in general. This elucidates a measurement principle in physics, which is the Born rule in the quantum case, as a consequence that ensemble steering on states and the no-signaling principle on probabilities are compatible within a theory.

In the recent years, extensive efforts have been made to derive information principles that characterize multipartite probabilistic correlations which can be produced from quantum systems. Since the no-signaling principle itself is not sufficient for the purpose [1, 2], further principles are sought, in particular from information-theoretic views, so that the information capabilities of quantum theory can be identified among non-signaling theories. While remarkable results, such as the information causality that dictates most of bipartite correlations [3] as well as its extension to multi-partite systems [4], have been obtained, it still remains challenging to have principles that would give the full characterization [5].

Contrast to these cases, for operational tasks on single-party quantum systems, the no-signaling principle has been a powerful tool to guide tightly the capabilities. Here, the no-signaling constraint is imposed on probability distributions of two parties in the following way: two parties, Alice and Bob, sharing entangled states and having no communication between them, Alice’s measurement does not cause instantaneous communication with Bob who may apply some local operations to learn about Alice’s measurement. Then, no matter what operations or measurement Bob may perform, capabilities of the local operations are constrained such that the no-signaling principle is satisfied. In fact, it has been shown that the no-signaling principle dictates tightly the capabilities of quantum cloning [4], quantum dynamics [2] or the distinguishability in terms of minimum-error quantum state discrimination [8]. To be precise, for these operations, if the performance is more efficient than what can be obtained within quantum theory, the distinguishability on quantum states would be so improved that a faster-than-light communication would be established.

In fact, it is worth to emphasize that the distinguishability itself contains and reflects fundamentally non-trivial properties of the underlying quantum state space in general. For instance, it has been shown that the distinguishability is indeed a global property that cannot be reduced to pairwise relations of states [6, 10]. In this way, the result in Ref. [6] implies that the global property over the quantum state space is dictated by the no-signaling principle on measurement outcomes, however, without specifically referring to how the measurement is particularly postulated. Note also that the distinguishability is generally a useful theoretical tool in quantum information theory, in particular, in cryptographic applications.

The present work considers the distinguishability in a most fundamental setting of general non-signaling theories, namely, in the framework of generalized probabilistic theories (GPTs). We aim to characterize the distinguishability in terms of fundamental properties of physical systems, such as particular properties existing in physical sources and information principles for their outcomes. This is generally of fundamental interest but, more importantly, has significance for practical quantum information applications in order that fundamental elements in quantum foundations are directly linked to and then applied to information advantages in applications, for instance, along the line of device-independent quantum information processing.

The main results are summarized as follows. We first formalize the distinguishability in terms of minimum-error state discrimination in GPTs. As a state space of a GPT forms convex, the distinguishability can be naturally introduced within the convex optimization framework. We formalize the distinguishability in the form of the so-called complementarity problem that generalizes the convex optimization, and provide the geometric formulation of minimum-error state discrimination in GPTs. In the geometric formulation, the distinguishability is characterized only by the geometry of given states, without referring to optimal measurement over effects. As an example, we show minimum-error discrimination in
the polygon systems in Refs. [11, 12] and present how the geometric formulation can be applied. With the formulation established, we show that if ensemble steering [13] is allowed in a GPT, the distinguishability is determined by the no-signaling principle in general. This shows that the distinguishability is determined as a consequence that ensemble steering on states and the no-signaling principle on probabilities are compatible within a theory. This also elucidates the measurement postulate of quantum theory as the condition by which ensemble steering on states and non-signaling tasks on measurement outcomes are compatible and coexist peacefully.

We begin with the framework in GPTs where notions of states, measurement, and their relations are generalized [11, 14, 15]. We here mainly refer to the mathematical framework in Ref. [15]. The set of states, denoted by Ω, consists of all possible states that a system can be prepared in. Any probabilistic mixture of states, i.e., \( p\{w_1, \ldots, w_{N}\} \in \Omega \) for \( w_1, w_2 \in \Omega \) and probability \( p \) is also a state, i.e. Ω is convex. A general mapping from states to probabilities is called effects and described by linear functionals \( \Omega \to [0,1] \). A measurement denoted by \( s \) corresponds to set of effects, \( E^{(s)} = \{ e_x^{(s)} \}_{x=1}^n \), by which the probability of getting outcome \( x \) given a state \( w \) is, \( p(x|s) = e_x^{(s)}[w] \). A unit effect \( u \) means a measurement that occurs, that is, \( u[w] = 1 \) for all \( w \in \Omega \), so that for any measurement \( s \), it holds \( \sum_x e_x^{(s)} = u \). As effects are dual to the state space, they are also convex.

Minimum-error state discrimination [16] can be described as a game of two parties, Alice and Bob. Suppose a set of states agreed by them in advance, and Alice prepares a system in one of \( N \) states with some probability and gives it to Bob. If Bob makes a correct guess, their score is given 1, otherwise 0. Their goal is to maximize the average score over all measurements.

We write the states by \( \{ w_x \}_{x=1}^{N} \) and Alice’s a priori probabilities by \( q_x \), \( x=1, \ldots, N \), and then by \( q_x, w_x \) altogether. Bob has to find optimal measurement to maximize the score. We write by \( p_B|A(x|y) = e_x[w_y] \) the probability that Bob makes a guess \( x \) when state \( w_y \) is given by Alice. Now, the goal is to find the guessing probability in the following,

\[
p_{\text{guess}} := \max_{x=1}^{N} q_x p_B|A(x|x) = \max_{x=1}^{N} q_x e_x[w_x] \tag{1}
\]

where the maximization runs over all effects. Note that GPTs are generally not self-dual, meaning an isomorphism between two spaces does not exist in general [12], and thus state and effect spaces are generally distinct.

The sole fact that state and effect spaces are convex allows us to formalize the distinguishability in the convex optimization framework [17]. For states \( \{ q_x, w_x \}_{x=1}^{N} \), we take the form in Eq. (1) as the primal problem denoted by \( p^* \) and derive its dual \( d^* \), as follows,

\[
p^* = \max \{ \sum_{x=1}^{N} q_x e_x[w_x] \mid e_x \geq 0 \forall x, \sum_{x=1}^{N} e_x = u \} \tag{2}
\]

\[
d^* = \min \{ u[K] \mid K \geq q_x w_x, x = 1, \ldots, N \} \tag{3}
\]

where inequalities mean the order relation in the convex set: by \( e_x \geq 0 \), it is meant that \( e_x[w] \geq 0 \) for all \( w \in \Omega \), and by \( K \geq q_x w_x \), that \( e[K - q_x w_x] \geq 0 \) for all effects \( e \).

The property called the strong duality holds true in the above, meaning that both solutions from primal and dual problems are equal, i.e. \( p^* = d^* \). This follows from the so-called Slater’s constraint quantification in convex optimization. A sufficient condition for the strong duality is the strict feasibility, i.e., the existence of a strictly feasible point of parameters: for instance, primal parameters \( \{ e_x = u/N \}_{x=1}^{N} \) are in the case, since \( e_x[w_y] > 0 \) \( \forall x, y \) and \( \sum_x e_x = 1 \).

In another approach called complementarity problem that generalizes convex optimization, optimality conditions of a given optimization problem are directly analyzed. It deals with both primal and dual parameters in Eqs. (2) and (3), and consequently is not considered more efficient in numerics. The advantage, however, lies at the fact that generic structures existing in the problem are exploited. The optimality conditions can be summarized by the so-called Karush-Kuhn-Tucker (KKT) conditions, which are constraints listed in Eqs. (2) and (3), together with the followings,

\[
\text{(Symmetry parameter) \quad K = q_x w_x + r_x d_x, \ \forall x \tag{4}}
\]

\[
\text{(Orthogonality) \quad e_x[r_x d_x] = 0, \ \forall x, \tag{5}}
\]

where \( r_x \in [0,1] \) for all \( x \), and \( \{ d_x \}_{x=1}^{N} \) which we call complementary states are normalized, i.e. \( u[d_x] = 1 \).

The first condition, symmetry parameter, follows from the Lagrangian stability and shows that for any discrimination problem e.g. \( \{ q_x, w_x \}_{x=1}^{N} \), there exists a single parameter \( K \) which is decomposed into \( N \) different ways with given states and complementary states \( \{ r_x, d_x \}_{x=1}^{N} \). Then, the second condition in Eq. (5) from the complementarity slackness characterizes optimal effects by the orthogonality relation between complementary states and optimal effects. These generalize optimality conditions for quantum states in Refs. [18, 19] to all GPTs, see also different forms of optimality conditions [10].

Primal and dual parameters satisfying the KKT conditions are automatically optimal parameters by which solutions are obtained in the primal and the dual problems. Moreover, for the problem here, recall that the strong duality holds i.e. \( p^* = d^* \). Conversely, the fact that the strong duality holds in Eqs. (2) and (3) implies the existence of optimal parameters which satisfy KKT conditions and give the guessing probability in Eq. (1).

Note that we derive all these from the convexity in KKT conditions. We now formalize a geometric method of finding optimal discrimination in GPTs. We first remark that, in
optimality conditions in Eqs. (4) and (5), constraints for states and effects are separated. The symmetry parameter $K$ is characterized on a state space and gives the guessing probability, see Eq. (3), i.e. $p_{\text{guess}} = u[K] = q_x + r_x$. Then, the guessing probability can be found by searching complementary states $\{r_x, d_x\}_{x=1}^N$ fulfilling Eq. (4) on the state space. This can be described in a systematic way, as follows. Let us define a polytope denoted by $\mathcal{P}(\{q_x, w_x\}_{x=1}^N)$ of given states in the state space: each vertex of the polytope corresponds to unnormalized state $q_x w_y$ for $x = 1, \cdots, N$. Then, the polytope of complementary states, $\mathcal{P}(\{r_x, d_x\}_{x=1}^N)$, is in fact immediately congruent to $\mathcal{P}(\{q_x, w_x\}_{x=1}^N)$ in the state space, since the following holds from Eq. (4),

$$q_x w_x - q_y w_y = r_x d_y - r_x d_x,$$

for all $x, y$, (6)

which shows that corresponding lines of two polytopes $\mathcal{P}(\{q_x, w_x\}_{x=1}^N)$ and $\mathcal{P}(\{r_x, d_x\}_{x=1}^N)$ are of equal lengths and anti-parallel. Then, from the underlying geometry of the state space, one can find complementary states by putting two congruent polytopes such that the condition in Eq. (4) holds. Optimal effects can be found from the orthogonal relation in Eq. (5), accordingly.

For a priori probabilities given as $q_x = 1/N$, the guessing probability becomes even simpler. First, it follows $r_x = r_y$ for all $x, y$: this is obtained from the expression $p_{\text{guess}} = q_x + r_x$ for any $x$, see Eqs. (4) and (5). Denoted by $r := r_x$ for all $x$, the guessing probability is now

$$p_{\text{guess}} = \frac{1}{N} + r,$$

where the expression of $r$ follows from the condition in Eq. (5) with a distance measure $\|\cdot\|$ that can be defined in the state space. The parameter $r$ has a meaning as the ratio between two polytopes, $\mathcal{P}(\{1/N, w_x\}_{x=1}^N)$ of given states, and $\mathcal{P}(\{d_x\}_{x=1}^N)$ of only complementary states.

We illustrate the method with the polygon systems shown in Refs. 11, 12. We consider the case of four states, which is of particular interest as its bipartite non-signaling extension can show the maximally non-local correlations. Four states $\{w_x\}_{x=1}^4$ and measurement $\{E(x)\}_{x=1}^4$ with $E(x) = \{e_0(x), e_1(x)\}$ are given as

$$w_x = \begin{pmatrix} \cos \frac{2\pi x}{4} \\ \sqrt{\cos \frac{\pi}{4}} \sin \frac{2\pi x}{4} \\ \sqrt{\cos \frac{\pi}{4}} \sin \frac{2\pi x}{4} \\ \sqrt{\cos \frac{\pi}{4}} \end{pmatrix}, \quad e_0(x) = \frac{1}{2} \begin{pmatrix} \cos (2\pi x - 1) / 4 \\ \cos \pi / 4 \\ \sin (2\pi x - 1) / 4 \\ \cos \pi / 4 \end{pmatrix},$$

and $e_1(x) = u - e_0(x)$, where the unit effect $u = (0, 0, 1)^T$, with the Euclidean inner product for $p(a|x) = e_1(x)[u]$.

For four states $\{1/4, w_x\}_{x=1}^4$, we find the guessing probability and optimal measurement as follows. Exploiting the underlying geometry (cf. see Fig 2. in Ref. 12), the polytope $\mathcal{P}(\{1/4, w_x\}_{x=1}^4)$ forms a rectangle and also note $r = 1/4$ from Eq. (7). By using the state space geometry, one can see that

$$K = \frac{1}{4} w_x + \frac{1}{4} w_{x+2}, \text{ for } x = 1, 2, 3, 4, \text{ mod } 4, \quad (9)$$

and thus, $p_{\text{guess}} = 1/2$. From the geometry, complementary states are $\{1/4, d_x = w_{x+2}\}_{x=1}^4$. Note that these four states are analogous to cases in quantum theory: for pairs of orthogonal qubit states, the guessing probability is also given by $1/2$ 10. This shows that the guessing probability is independent to non-locality in non-signaling theories, along a similar conclusion drawn in Ref. 20.

By putting $e_x = e_0^x$ in Eq. (3) for $x = 1, \cdots, 4$, optimal measurements which give the guessing probability are the followings: i) $\{e_x/2\}_{x=1}^4$, ii) $\{e_1, e_3\}$, or iii) $\{e_2, e_4\}$. In the case ii), effect on $e_1$ ($e_3$) means that given state is either $w_1$ or $w_4$ ($w_2$ or $w_3$), since $e_1[w_2] = e_1[w_3] = 0$. Once effect on $e_1$ ($e_3$) is given, one randomly guesses either $w_1$ or $w_4$ ($w_2$ or $w_3$), and the guessing probability is obtained $1/2$. The case iii) works in a similar way. From the example, it is shown that the followings are properties not only in quantum cases but also among GPTs: i) optimal measurement is generally not unique 10, and ii) no-measurement sometimes gives an optimal strategy 10 21.

We now move to a bipartite extension, which is specified in an operational way that ensemble steering is possible within the theory. This means that Alice can steer any decomposition of Bob’s ensemble. In quantum theory, the steering was firstly asserted by Schrödinger 22 and then, with specification to a bipartite Hilbert space, formalized as the so-called Gisin-Hughston-Jozsa-Wooters theorem 23. Note that ensemble steering does not yet single out quantum theory among GPTs 13. We also distinguish the extension from the purification lemma which fully characterizes quantum theory 24.

In what follows, we apply the theoretical tool developed so far, and show that for any GPTs endowed with ensemble steering, the distinguishability is immediately determined by a way of excluding instantaneous communication. We first derive a bound to the optimal distinguishability in a given GPT by the non-signaling condition, and prove that the bound is tight, i.e. it can be achieved within the given GPT. The result is independent to particular properties of a state space.

Let us incorporate state discrimination to the following non-signaling framework. Let $\{q_x, w_x\}_{x=1}^N$ denote the states we are interested in discriminating among. Suppose Alice steers the ensemble of Bob, denoted by $w_x$, in $N$ different decompositions. That is, the ensemble has $N$ different decomposition as $w_B = w_B^{(x)}$ for $x = 1, \cdots, N$ where

$$w_B^{(x)} = p_x w_x + (1 - p_x) c_x, \text{ with } p_x = \frac{p_x}{\sum_{x'=1}^N p_{x'}}$$

with some states $\{c_x\}_{x=1}^N$ and probabilities $\{p_x\}_{x=1}^N$. By
ensemble steering, it is meant that any of the $N$ decompositions of Bob’s ensemble can be prepared by Alice’s steering. Since Bob holds an identical ensemble, his measurement gains no knowledge about which decomposition is given, until Alice announces about her steering. The non-signaling condition is thus naturally imposed.

The distinguishability on $\{q_x, w_x\}_{x=1}^N$ is then constrained by the non-signaling condition as follows. Assume that Bob optimizes measurement to guess which state among $\{w_x\}_{x=1}^N$ exists in his ensemble. The strategy is, once state $w_x$ is found, he concludes his ensemble is in the decomposition $w_B^{(x)}$, see Eq. (11), by which he also guesses Alice’s steering. Then, by the no-signaling condition, discrimination among states $\{w_x\}_{x=1}^N$ must be constrained so that Bob would not learn Alice’s steering better than the random guess.

We now derive an upper bound to the guessing probability by the no-signaling condition. Let $P_{B|x} (x|y)$ denote the probability that, while Alice has actually steered ensemble $w_B^{(x)}$, Bob concludes his ensemble in $w_B^{(y)}$ by discriminating among $\{q_x, w_x\}_{x=1}^N$. The no-signaling condition (11) implies the following constraint

$$\sum_{x=1}^N P_{B|x} (x|x) \leq 1. \quad (11)$$

The derivation is shown in Supplementary Material. Or, if the condition is not fulfilled, one can explicitly construct a superluminal communication protocol [8]. Then, recall Bob’s strategy of guessing Alice’s steering: to guess about Alice’s steering, he attempts to distinguish ensemble decompositions $\{w_B^{(x)}\}_{x=1}^N$ by exploiting optimal discrimination of states $\{w_x\}_{x=1}^N$ existing in the ensemble.

If Alice has steered Bob’s ensemble $w_B^{(x)}$, Bob’s correct conclusion happens when i) $w_x$ is given, which appears with probability $p_x$, and ii) measurement gives a correct answer, that is, with probability $p_x P_{B|x} (x|x)$. In the strategy, there can be contribution in measurement from the other state $c_x$ in the ensemble with probability $1 - p_x$. Thus, it holds, $p_x P_{B|x} (x|x) \leq P_{B|x} (x|x)$. In addition, recall that measurement is optimized for discrimination among $\{q_x, w_x\}_{x=1}^N$, since the a priori probability for state $w_x$ among $\{w_x\}_{x=1}^N$ is given by $q_x$, see Eq. (10). From the no-signaling condition in Eq. (11), we have

$$\sum_{x=1}^N p_x P_{B|x} (x|x) \leq 1,$$

from which we have

$$p_{\text{guess}} = \max_x \sum_{x=1}^N p_x P_{B|x} (x|x) \leq \frac{1}{p_1 + \cdots + p_N}. \quad (12)$$

Thus, a upper bound to the distinguishability is obtained from the no-signaling condition, and expressed in terms of parameters $\{p_x\}_{x=1}^N$ of steering each state in $\{w_x\}_{x=1}^N$.

We then show that the bound is indeed tight, i.e. it can be achieved within a given GPT. We show the tightness by proving that, for any set $\{q_x, w_x\}_{x=1}^N$, the optimal discrimination characterized by the KKT conditions implies the existence of both an identical ensemble in Eq. (10) and effects achieving the bound in Eq. (12).

Recall the general method of optimal discrimination, the existence of a symmetry parameter $K$ that completely characterizes the optimal distinguishability, see Eq. (1). The parameter has $N$ decompositions with complementary states $\{r_x, d_x\}_{x=1}^N$. Its normalization $\tilde{K} = K/u[K]$ shows, for each $x$,

$$\tilde{K} = p_x w_x + (1 - p_x) d_x,$$

with $p_x = q_x/u[K]$. This corresponds to ensemble steering in Eq. (10). Recall the dual problem in Eq. (9) which gives the guessing probability in GPTs, as $p_{\text{guess}} = u[K]$. Using the identity $\sum_{x=1}^N q_x = 1$ and the relation $p_x = q_x/u[K]$, the solution in the dual problem can be computed as, $u(K) = (\sum_{x=1}^N p_x)^{-1}$. This shows that the bound in Eq. (12) is already achieved within a given GPT, and hence the tightness is shown. In addition, optimal effects also exist with complementary states, see Eq. (5).

In conclusion, we have developed and established a general method of distinguishing states in GPTs. This generalizes i) the geometric formulation [10] and ii) optimality conditions [18] [19] in the quantum case to GPTs. The formulation is also illustrated with an example, the four-state polygon system, a particularly interesting case where the bipartite extension shows the maximally non-local correlations [11] [12]. It is also shown that the distinguishability and the non-locality are independent resources, along a similar conclusion in the quantum case [20]. We also remark that in GPTs i) measurement for the optimal discrimination is generally not unique and ii) no-measurement sometimes give an optimal strategy, along the results in quantum cases in Refs. [10] [16] [21].

Then, with the general formalism developed, we have shown that for GPTs where ensemble steering is possible, the distinguishability can be tightly determined by no-signaling condition. We remark that, although the distinguishability itself explains a fundamental limitation in measurement on states via a measurement principle postulated in given GPTs, it is shown that the distinguishability is dictated by the relation between fundamental principles, ensemble steering and the no-signaling condition. This also finds a physical meaning of Gleason’s theorem [20] in quantum theory as the expression such that, in Hilbert space, both ensemble steering on states and the no-signaling principle on measurement outcomes are compatible and peacefully co-exist.

Finally, we note that the tight relation among the distinguishability, ensemble steering, and the no-signaling principle does not imply quantum theory uniquely yet. The reason may be even deeper, however, it is also worth to mention that the distinguishing task applies a single setting of measurement, differently from the non-locality test [1]. If quantum theory can be uniquely found among
non-signaling theories by replacing the distinguishability with some other, the task would apply more measurement settings. We leave it open to find some operational tasks tightly related to fundamental principles such that quantum theory is uniquely characterized. In this way, we also envisage information-theoretic characterizations of quantum theory.

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Appendix I. Convex optimization framework

We show the derivation of primal and dual problems for optimal state discrimination in GPTs. Recall that both state and effect spaces are convex. For convenience, we follow the formalism and notations in Ref. [17]. The problem to maximize the success probability can be written as the primal problem as,

\[
\begin{align*}
\min & \quad f(\{e_x\}_{x=1}^N) = - \sum_{i=1}^N q_x e_x [w_x] \\
\text{subject to} & \quad e_x \geq 0 \quad \forall x, \quad \sum_x e_x = u,
\end{align*}
\]

where by \(e_i \geq 0\) it is meant that \(e_x[w] \geq 0\) for all \(w \in \Omega\). Note that the above problem is feasible as the set of parameters satisfying constraints is not empty. It is also strictly feasible with parameters with \(e_x = u/N\) for all \(x\).

To derive the dual problem, the Lagrangian can be constructed as,

\[
\mathcal{L}(\{e_x\}_{x=1}^N, \{d_x\}_{x=1}^N, K) = f(\{e_x\}_{x=1}^N) - \sum_x e_x [d_x] + (\sum_x e_x - u)[K],
\]

with \(\{d_x \geq 0\}_{x=1}^N\) and \(K\) are dual parameters. The dual problem is then obtained by solving the following,

\[
g(\{d_x\}_{x=1}^N, K) = \min_{e_x} \mathcal{L}(\{e_x\}_{x=1}^N, \{d_x\}_{x=1}^N, K), \quad (14)
\]

for which the Lagrangian can be further evaluated as,

\[
\mathcal{L}(\{e_x\}_{x=1}^N, \{d_x\}_{x=1}^N, K) = \sum_x e_x[K - q_x w_x - d_x] - u[K].
\]

Then, the minimization in Eq. (14) is to be, \(-u[K]\) if \(K = q_x w_x + d_x\) for all \(x\), otherwise \(-\infty\). Thus, we have \(d_x = K - q_x w_x\) for each \(x\). Since \(d_x \geq 0\) i.e. \(e[d_x] \geq 0\) for all effects \(e\), we write this by, \(K \geq q_x w_x\) for each \(x\). The dual problem is therefore as follows.

\[
\begin{align*}
\max & \quad -u(K) \quad \text{(or, min } u[K]) \\
\text{subject to} & \quad K \geq q_x w_x \quad \forall x.
\end{align*}
\]

The inequality means an order relation in a convex cone, which is determined by effects, i.e. \(e[K - q_x w_x] \geq 0\) for all effects \(e\). Note also that the dual problem is also strictly feasible, with \(K = \sum_x q_x w_x\).

Appendix II. No-signaling condition

The no-signaling condition we have applied is the following

\[
\sum_{x=1}^N P_{B|A}(x|x) \leq 1, \quad (15)
\]

which is not explicitly from other known formulas of the no-signaling constraint e.g. Refs. [11] [25]. Recall that \(P_{B|A}(x|y)\) is the probability that Bob making guess \(x\) once Alice applies steering \(y\).

Let us first recall the non-signaling scenario that we apply to constrain the guessing probability. It is supposed that Alice can steer Bob’s ensemble in \(N\) decompositions, labeled by \(x = 1, \ldots, N\). Note that the ensemble average of Bob remains the same, and by the ensemble steering of Alice, the decomposition of Bob’s ensemble is prepared in one of \(N\) decompositions. This is precisely the way that the no-signaling condition is fulfilled in the communication. Bob therefore has no way to gain advantage to learn about the decomposition, unless Alice announces her choice about the ensemble steering. In what follows, we explain the no-signaling condition in Eq. (15) in detail.

Let \(P_{AB}(a, b|A, B)\) denote the joint probability distribution of Alice and Bob. In the communication scenario, Alice’s choice in the steering is denoted by \(A\), which prepares the decomposition \(w^{(A)}_x\),

\[
u^{(A=x)}_B = p_x w_x + (1 - p_x)c_x,\]

with

\[
\sum_{x'=1}^N p_{x'} = 1
\]

on Bob’s side, see also Eq. (10) in the manuscript. Then, Bob performs measurement according to her measurement setting \(B\) and makes a guess about the steered decomposition from outcome \(b\). The no-signaling condition on the joint probability \(P_{AB}(a, b|A, B)\) means that Bob’s measurement outcomes are independent to Alice’s measurement setting \(A\), that is, for all \(x \neq x'\)

\[
\sum_a P_{AB}(a, b|A = x, B) = \sum_a P_{AB}(a, b|A = x', B). \quad (16)
\]

It is clear that Alice’s choice of steering \(A\) does not affect to the statistics of Bob’s measurement outcomes. In the communication, the no-signaling condition in Eq. (16) is clearly satisfied since Alice’s ensemble steering does not change Bob’s ensemble average itself but its decomposition.
We now derive the condition in Eq. (15) by contradiction. Once Alice applies steering $A$, Bob attempts to guess about Alice’s steering from his measurement outcome $b$, which corresponds to the probability, $P_{B|A}(b|A, B) = \sum_a P_{AB}(a, b|A, B)$. The no-signaling condition implies that Bob’s measurement outcome $b$ for any $b$ must be independent to Alice’s different choice on $A$. From the condition in Eq. (16), this means that it is fulfilled, $P_{B|A}(b|A, B) = P_{B|A}(b|A', B)$ for all $A, A'$. Suppose that for some measurement $B$ of Bob, it is possible that

$$\sum_{x=1}^{N} P_{B|A}(b = x|A = x, B) > 1. \quad (17)$$

That is, given that Alice applies steering $x$ with probability $1/N$, the average probability that Bob makes correct guesses about Alice’s steering from his measurement outcome $b$ is larger than 1.

Here, for the generality, we assume that Bob’s measurement may not generally be complete. This means that

$$\sum_{x=1}^{N} P_{B|A}(b = x|A = x', B) \leq 1, \text{ for all } x' \text{ and } B. \quad (18)$$

From Eqs. (17) and (18), we have the following relation,

$$\sum_{x=1}^{N} (P_{B|A}(x|x, B) - P_{B|A}(x|x', B)) > 0.$$

This means that there exist $x$ and $x'$ such that

$$P_{B|A}(x|x, B) > P_{B|A}(x|x', B). \quad (19)$$

This contradicts to the no-signaling condition in Eq. (10). Or, with probabilities in Eq. (11), one can also construct a faster-than-light communication protocol. Thus, to fulfill the no-signaling condition, it is not possible that Alice and Bob are in the case in Eq. (17). We have shown the no-signaling condition in Eq. (15).