ON CERTAIN CLASSES OF HARMONIC FUNCTIONS DEFINED BY THE FRACTIONAL DERIVATIVES

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Abstract. In this paper we have introduced two new classes $HM(\beta, \lambda, k, \nu)$ and $\overline{HM}(\beta, \lambda, k, \nu)$ of complex valued harmonic multivalent functions of the form $f = h + \overline{g}$, satisfying the condition

$$\text{Re} \left\{ (1 - \lambda) \frac{\Omega^\nu f}{z} + \lambda(1 - k) \frac{(\Omega^\nu f)'}{z'} + \lambda k \frac{(\Omega^\nu f)''}{z''} \right\} > \beta, \quad (z \in D)$$

where $h$ and $g$ are analytic in the unit disk $D = \{ z : |z| < 1 \}$. A sufficient coefficient condition for this function in the class $HM(\beta, \lambda, k, \nu)$ and a necessary and sufficient coefficient condition for the function $f$ in the class $\overline{HM}(\beta, \lambda, k, \nu)$ are determined. We investigate inclusion relations, distortion theorem, extreme points, convex combination and other interesting properties for these families of harmonic functions.

1. Introduction

Let $u, v$ be real harmonic function in a simply connected domain $\Omega$, then the continuous function $f = u + iv$ defined in $\Omega$ is said to be harmonic in $\Omega$. If $f = u + iv$ is harmonic in $\Omega$ then there exist analytic functions $G, H$ such that $u = \text{Re } G$ and $v = \text{Im } H$, therefor $f = u + iv = h + \overline{g}$ where $h = \frac{G + H}{2}$, $\overline{g} = \frac{G - H}{2}$ and we call $h$ and $g$ analytic part and co-analytic part of $f$ respectively. The jacobian of $f$ is given by $J_f|z| = |h'(z)|^2 - |g'(z)|^2$, also we show by $w(z)$ the dilatation function for $f$ and define $w(z) = \frac{g'(z)}{h'(z)}$. Lewy [6], Clunie and Small [3] have showed that the mapping $z \rightarrow f(z)$ is sense preserving and injective in $\Omega$ if and only if $J_f|z| > 0$ in $\Omega$. The function $f = h + \overline{g}$ is said to be univalent in $\Omega$ if the mapping $z \rightarrow f(z)$ is sense preserving and injective in $\Omega$. Denote by $H$ the class of all harmonic functions $f = h + \overline{g}$ that are univalent and sense preserving in the open unit disk $D$ where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n \quad |b_1| < 1.$$  

With normalization conditions $f(0) = 0$, $f_z(0) = 1$ where $f_z(0)$ denotes the partial derivative of $f(z)$ at $z = 0$. In case $g = 0$ this class reduces to the class of $S$ consisting of all analytic univalent functions.

Definition 1.1. (See [7] and [9]) Let the function $f(z)$ be analytic in a simply-connected region of the $z$-plane containing the origin. The fractional derivative of $f$ of order $\nu$ is defined by

$$D^\nu f(z) = \frac{1}{\Gamma(1 - \nu)} \frac{d}{dz} \int_0^1 \frac{f(\zeta)}{(z - \zeta)^\nu} d\zeta, \quad 0 \leq \nu < 1$$

where the multiplicity of $(z - \zeta)^\nu$ is removed by requiring $\log(z - \zeta)$ to be real when $z - \zeta > 0$.

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Making use of fractional derivative and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [8] introduced the operator \( \Omega^\nu_z : A_0 \longrightarrow A_0 \) defined by

\[
\Omega^\nu_z f(z) := \Gamma(2 - \nu)z^\nu D^\nu f(z) \quad \nu \neq 2, 3, 4, \ldots
\]

where \( A_0 \) denote the class of functions which are analytic in the unit disk \( D \), satisfying normalization conditions \( f(0) = f'(0) - 1 = 0 \).

It is easy to see that

\[
\Omega^\nu_z f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(2 - \nu)\Gamma(n + 1)}{\Gamma(n + 1 - \nu)} a_n z^n. \quad f \in A_0
\]

**Definition 1.2.** Suppose that \( f = h + \overline{g} \) where \( h \) and \( g \) are in (1.1), define \( \Omega^\nu_z f(z) = \Omega^\nu_z h(z) + \Omega^\nu_z g(z) \).

Then we obtain

\[
\Omega^\nu_z f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(2 - \nu)\Gamma(n + 1)}{\Gamma(n + 1 - \nu)} a_n z^n + \sum_{n=1}^{\infty} \frac{\Gamma(2 - \nu)\Gamma(n + 1)}{\Gamma(n + 1 - \nu)} b_n z^n.
\]

By making use of Definition 1.2, we introduce a new class of harmonic univalent functions in the unit disk \( D \) as in definition 1.3.

**Definition 1.3.** Let \( \mathcal{HM}(\beta, \lambda, k, \nu) \quad (0 \leq k \leq 1, \ 0 < \beta \leq 1, \ 0 \leq \lambda, \ 0 \leq \nu < 1) \) be the class of functions \( f \in \mathcal{H} \) satisfying the following inequality:

\[
\text{Re} \left\{ (1 - \lambda) \frac{\Omega^\nu f}{z} + \lambda(1 - k) \frac{\Omega^\nu f'}{z'} + \lambda k \frac{\Omega^\nu (f'')}{z''} \right\} > \beta, \quad (z = re^{i\theta})
\]

where

\[
z' = \frac{\partial}{\partial \theta} (re^{i\theta}), \quad z'' = \frac{\partial}{\partial \theta} (z'),
\]

and

\[
(\Omega^\nu f(z))' = \frac{\partial}{\partial \theta} (\Omega^\nu f(z)) = iz(\Omega^\nu h(z))' - iz^2(\Omega^\nu g(z))',
\]

\[
(\Omega^\nu f(z))'' = \frac{\partial^2}{\partial \theta^2} (\Omega^\nu f(z))' = -z(\Omega^\nu h(z))' - z^2(\Omega^\nu h(z))'' - z^2(\Omega^\nu g(z))'' - z^2(\Omega^\nu g(z))''',
\]

also we denote by \( \text{HM}(\beta, \lambda, k, \nu) \) the subclass of \( \mathcal{HM}(\beta, \lambda, k, \nu) \) consisting of functions \( f = h + \overline{g} \) such that

\[
h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad g(z) = \sum_{n=1}^{\infty} |b_n| z^n, \quad |b_1| < 1.
\]

In [9] H. M. Srivastava and S. Owa investigated this class with \( D^\nu f(z) \) instead of \( \Omega^\nu f(z) \) where \( D^\nu f(z) \) is the Ruscheweyh derivative of \( f \), for \( p \)-valent harmonic functions. This class in special cases involve the works studied by the previous authors such as Bhosuturmath and Swamy [2], Ahuja and Jahangiri [1,5].

In this paper the coefficient inequalities for the classes \( \mathcal{HM}(\beta, \lambda, k, \nu) \) and \( \overline{\mathcal{HM}}(\beta, \lambda, k, \nu) \) are obtained also some other interesting properties of these classes are investigated.

2. **Coefficient Bounds**

In the first theorem we give the sufficient condition for \( f \in \mathcal{H} \) to be in the class \( \mathcal{HM}(\beta, \lambda, k, \nu) \).

**Theorem 2.1.** Let \( f \in \mathcal{H} \), and

\[
\sum_{n=2}^{\infty} \phi(n, k, \lambda, \nu)|a_n| + \sum_{n=1}^{\infty} |\psi(n, k, \lambda, \nu)||b_n| < 1 - \beta,
\]
where

\begin{align}
\phi(n,k,\lambda,\nu) &= \frac{[1 + \lambda(n-1)(1+nk)]\Gamma(n+1)\Gamma(2-\nu)}{\Gamma(n+1-\nu)}, \\
\psi(n,k,\lambda,\nu) &= \frac{[1 - \lambda(n+1)(1-nk)]\Gamma(n+1)\Gamma(2-\nu)}{\Gamma(n+1)}
\end{align}

then \( f \in \mathcal{HM}(\beta,\lambda,k,\nu) \). The result is sharp for the function \( f(z) \) given by

\[
f(z) = z + \sum_{n=2}^{\infty} \frac{\gamma^n \Gamma(n+1-\nu)z^n}{[1 + \lambda(n-1)(1+nk)]\Gamma(n+1)\Gamma(2-\nu)}
+ \sum_{n=1}^{\infty} \frac{\delta^n \Gamma(n+1-\nu)}{[1 - \lambda(n+1)(1-nk)]\Gamma(n+1)\Gamma(n-\nu)} z^n,
\]

where \( \sum_{n=2}^{\infty} |\gamma_n| + \sum_{n=2}^{\infty} |\delta_n| = 1 - \beta \).

**Proof.** Suppose

\[
E(z) = (1 - \lambda) \frac{\Omega^n f(z)}{z} + \lambda(1 - k) \frac{(\Omega f(z))'}{z'} + \lambda k \frac{(\Omega f(z))''}{z''}.
\]

It suffices to show that \( |1 - \beta + E(z)| \geq |1 + \beta - E(z)| \). A simple calculation by substituting for \( h \) and \( g \) in \( E(z) \) shows

\[
E(z) = 1 + \sum_{n=2}^{\infty} \frac{[1 + \lambda(n-1)(1+nk)]\Gamma(n+1)\Gamma(2-\nu)}{\Gamma(n+1-\nu)} a_n z^{n-1}
+ \sum_{n=1}^{\infty} \frac{[1 - \lambda(n+1)(1-nk)]\Gamma(n+1)\Gamma(2-\nu)}{\Gamma(n+1-\nu)} b_n z^n,
\]

Considering (2.1) and (2.2) we have

\[
\phi(n,k,\lambda,\nu) = n(n-1)[1 + \lambda(n-1)(1+nk)]B(n-1,2-\nu),
\]

and

\[
\psi(n,k,\lambda,\nu) = n(n-1)[1 - \lambda(n+1)(1-nk)]B(n-1,2-\nu),
\]

where \( B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \) is the familiar Beta function. Then we obtain

\[
E(z) = 1 + \sum_{n=2}^{\infty} \phi(n,k,\lambda,\nu) a_n z^{n-1} + \sum_{n=1}^{\infty} \psi(n,k,\lambda,\nu) b_n z^n.
\]
Now we have

\[
|1 - \beta + E(z)| - |1 + \beta - E(z)|
\]

\[
= |2 - \beta + \sum_{n=2}^\infty \phi(n, k, \lambda, \nu)a_n z^{n-1} + \sum_{n=1}^\infty \psi(n, k, \lambda, \nu)b_n \frac{\phi_n}{z}|,
\]

\[
-|\beta - \sum_{n=2}^\infty \phi(n, k, \lambda, \nu)a_n z^{n-1} - \sum_{n=1}^\infty \psi(n, k, \lambda, \nu)b_n \frac{\phi_n}{z}|,
\]

\[
\geq 2 - \beta + \sum_{n=2}^\infty \phi(n, k, \lambda, \nu)|a_n||z|^{n-1} + \sum_{n=1}^\infty |\psi(n, k, \lambda, \nu)||b_n||\frac{\phi_n}{z}|,
\]

\[
-\beta - \sum_{n=2}^\infty \phi(n, k, \lambda, \nu)|a_n||z|^{n-1} - \sum_{n=1}^\infty |\psi(n, k, \lambda, \nu)||b_n||\frac{\phi_n}{z}|,
\]

\[
= 2 - 2\beta - 2\sum_{n=2}^\infty \phi(n, k, \lambda, \nu)|a_n||z|^{n-1} - 2\sum_{n=1}^\infty |\psi(n, k, \lambda, \nu)||b_n||\frac{\phi_n}{z}|,
\]

\[
> 2 - 2\beta - 2\sum_{n=2}^\infty \phi(n, k, \lambda, \nu)|a_n||z|^{n-1} - 2\sum_{n=1}^\infty |\psi(n, k, \lambda, \nu)||b_n||\frac{\phi_n}{z}|
\]

\[
\geq 0,
\]

and the proof is complete. \(\square\)

In our next theorem we obtain the necessary and sufficient coefficients condition for the \(f \in \mathcal{H}\) to be in \(\mathcal{HM}(\beta, \lambda, k, \nu)\).

**Theorem 2.2.** Let \(f \in \mathcal{H}\) then \(f \in \mathcal{HM}(\beta, \lambda, k, \nu)\) if and only if

\[
\sum_{n=2}^\infty \phi(n, k, \lambda, \nu)|a_n| + \sum_{n=1}^\infty |\psi(n, k, \lambda, \nu)||b_n| < 1 - \beta.
\]

**Proof.** Since \(\mathcal{HM}(\beta, \lambda, k, \nu) \subset \mathcal{HM}((\lambda, \beta, \nu)\) then the "if" part of theorem follows from Theorem 2.1, for "only if" part we show that if the condition (2.3) dose not hold then \(f \notin \mathcal{HM}(\beta, \lambda, k, \nu)\). Let \(f \in \mathcal{HM}(\beta, \lambda, k, \nu)\) then we have

\[
0 \leq \Re \left\{ (1 - \lambda) \frac{\Omega f(z)}{\nu} + \lambda(1 - k) \frac{(\Omega f(z))^\prime}{z^\prime} + \lambda k \frac{(\Omega f(z))''}{z''} - \beta \right\}
\]

\[
= \Re \left\{ 1 - \beta - \sum_{n=2}^\infty \phi(n, k, \lambda, \nu)a_n Z^{n-1} - \sum_{n=1}^\infty \psi(n, k, \lambda, \nu)b_n \frac{\phi_n}{z} \right\}.
\]

This inequality holds for all values of \(z\) for which \(|z| = r < 1\) so we can choose the values of \(z\) on positive real axis such that \(0 \leq z = r < 1\) therefore we get the followin inequality

\[
0 \leq 1 - \beta - \sum_{n=2}^\infty \phi(n, k, \lambda, \nu)|a_n|r^{n-1} - \sum_{n=1}^\infty |\psi(n, k, \lambda, \nu)||b_n||r^n - 1|.
\]

Now by letting \(r \to 1^+\) we have

\[
0 \leq 1 - \beta - \sum_{n=2}^\infty \phi(n, k, \lambda, \nu)|a_n| - \sum_{n=1}^\infty |\psi(n, k, \lambda, \nu)||b_n|.
\]

If the condition (2.3) dose not hold then the right hand of (2.4) is negative for \(r\) sufficiently close to 1. Thus there exists a \(z_0 = r_0 \in (0, 1)\) for which the right hand of (2.4) is negative. This contradicts the required condition for \(f \in \mathcal{HM}(\beta, \lambda, k, \nu)\) and so the proof is complete. \(\square\)
Putting $\lambda = 0$ in Theorem 2.2 we get:

**Corollary 2.1.** $f \in \overline{HM}(\beta, 0, k, \nu) = \left\{ f : \ Re \left( \frac{\Omega f(z)}{z} \right) > \beta \right\}$ if and only if

$$
\sum_{n=1}^{\infty} n(n-1)B(n-1,2-\nu)|a_n| + \sum_{n=1}^{\infty} n(n-1)B(n-1,2-\nu)|b_n| < 1 - \beta.
$$

Putting $\lambda = 1$ in Theorem 2.2 we have:

**Corollary 2.2.** $f \in \overline{HM}(\beta, 1, k, \nu) = \left\{ f : \ Re \left( (1-k)\frac{\Omega f(z)'}{z'} + k\frac{\Omega f(z)'}{z''} \right) > \beta \right\}$ if and only if

$$
\sum_{n=1}^{\infty} n^2(n-1)(1-k+nk)B(n-1,\nu)|a_n| + \sum_{n=1}^{\infty} n^2(n-1)|nk+k-1|B(n-1,2-\nu)|b_n| < 1 - \beta.
$$

Putting $k = 1$ in Theorem 2.2 we have:

**Corollary 2.3.** $f \in \overline{HM}(\beta, \lambda, 1, \nu) = \left\{ f : \ Re \left( (1-\lambda)\frac{\Omega f(z)'}{z'} + \lambda\frac{\Omega f(z)'}{z''} \right) > \beta \right\}$ if and only if

$$
\sum_{n=1}^{\infty} n(n-1)[1+\lambda(n^2-1)]B(n-1,2-\nu)(|a_n| + |b_n|) < 1 - \beta.
$$

Finally putting $k = 0$ in Theorem 2.2 we obtain:

**Corollary 2.4.** $f \in \overline{HM}(\beta, \lambda, 0, \nu) = \left\{ f : \ Re \left( (1-\lambda)\frac{\Omega f(z)'}{z'} + \lambda\frac{\Omega f(z)'}{z''} \right) > \beta \right\}$ if and only if

$$
\sum_{n=1}^{\infty} n(n-1)[1\lambda(n-1)]B(n-1,2-\nu)|a_n| + \sum_{n=1}^{\infty} n(n-1)[1\lambda(n+1)]B(n-1,2-\nu)|b_n| < 1 - \beta.
$$

**Theorem 2.3.** $f \in \overline{HM}(\beta, \lambda, k, \nu)$ if and only if

$$
(2.5) \quad f(z) = t_1z + \sum_{n=2}^{\infty} t_nf_n(z) + \sum_{n=1}^{\infty} s_ng_n(z) \ (z \in D),
$$
where $t_i \geq 0$, $s_i \geq 0$, $t_1 + \sum_{n=2}^{\infty} t_n + \sum_{n=1}^{\infty} s_n = 1$ and

$$
\begin{align*}
    f_n(z) &= z - \frac{1-\beta}{\phi(n,k,\lambda,\nu)}z^n, \\
    g_n(z) &= z + \frac{1-\beta}{|\psi(n,k,\lambda,\nu)|}z^n.
\end{align*}
$$

**Proof.** Let $f$ be of the form (2.5) then we have

$$
\begin{align*}
f(z) &= t_1z + \sum_{n=2}^{\infty} t_n \left( z - \frac{1-\beta}{\phi(n,k,\lambda,\nu)}z^n \right) + \sum_{n=1}^{\infty} s_n \left( z + \frac{1-\beta}{|\psi(n,k,\lambda,\nu)|}z^n \right) \\
&= z - \sum_{n=2}^{\infty} \frac{1-\beta}{\phi(n,k,\lambda,\nu)}t_nz^n + \sum_{n=1}^{\infty} \frac{1-\beta}{|\psi(n,k,\lambda,\nu)|}s_nz^n.
\end{align*}
$$

Therefore we have

$$
\begin{align*}
\sum_{n=2}^{\infty} \phi(n,k,\lambda,\nu) \frac{1-\beta}{\phi(n,k,\lambda,\nu)}t_n + \sum_{n=1}^{\infty} |\psi(n,k,\lambda,\nu)| \frac{1-\beta}{|\psi(n,k,\lambda,\nu)|}s_n \\
&= (1-\beta) \left[ \sum_{n=2}^{\infty} t_n + \sum_{n=1}^{\infty} s_n \right] = (1-\beta)(1-t_1) < 1 - \beta.
\end{align*}
$$
This shows that \( f \in \overline{\mathcal{HM}}(\beta, \lambda, k, \nu) \). Conversely suppose that \( f \in \overline{\mathcal{HM}}(\beta, \lambda, k, \nu) \) letting
\[
t_1 = 1 - \sum_{n=2}^{\infty} t_n - \sum_{n=1}^{\infty} s_n,
\]
where
\[
t_n = \frac{\phi(n, k, \lambda, \nu)}{1 - \beta} |a_n|, \quad s_n = \frac{\psi(n, k, \lambda, \nu)}{1 - \beta} |b_n|.
\]
We obtain
\[
f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + \sum_{n=1}^{\infty} |b_n| \overline{z}^n
\]
\[
= z - \sum_{n=2}^{\infty} \frac{1 - \beta}{\phi(n, k, \lambda, \nu)} t_n z^n + \sum_{n=1}^{\infty} \frac{1 - \beta}{\psi(n, k, \lambda, \nu)} s_n \overline{z}^n.
\]
\[
= z - \sum_{n=2}^{\infty} (z - f_n(z)) t_n + \sum_{n=1}^{\infty} (g_n(z) - z) s_n
\]
\[
= \left(1 - \sum_{n=2}^{\infty} t_n - \sum_{n=1}^{\infty} s_n\right) z + \sum_{n=2}^{\infty} t_n f_n(z) + \sum_{n=1}^{\infty} s_n g_n(z)
\]
\[
= t_1 z + \sum_{n=2}^{\infty} t_n f_n(z) + \sum_{n=1}^{\infty} s_n g_n(z).
\]
This completes the proof. \( \square \)

3. Convolution and Convex combinations

In the present section we investigate the convolution properties of the class \( \overline{\mathcal{HM}}(\beta, \lambda, k, \nu) \). The convolution of two harmonic function \( f_1 \) and \( f_2 \) given by
\[
(3.1) \quad f_1(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + \sum_{n=1}^{\infty} |b_n| \overline{z}^n, \quad f_2(z) = z - \sum_{n=2}^{\infty} |c_n| z^n + \sum_{n=1}^{\infty} |d_n| \overline{z}^n,
\]
is defined by
\[
(3.2) \quad (f_1 \ast f_2)(z) = z - \sum_{n=2}^{\infty} |a_n c_n| z^n + \sum_{n=1}^{\infty} |b_n d_n| \overline{z}^n.
\]

**Theorem 3.1.** For \( 0 \leq \beta < \alpha < 1 \) let \( f_1, f_2 \) be of the form (3.1) such that for every \( n, \ |c_n| < 1, \ |d_n| < 1. \) If \( f_1, f_2 \in \overline{\mathcal{HM}}(\alpha, \lambda, k, \nu) \) then
\( f_1 \ast f_2 \in \overline{\mathcal{HM}}(\alpha, \lambda, k, \nu) \subset \mathcal{HM}(\beta, \lambda, k, \nu) \).

**Proof.** Considering (3.2) we have
\[
\sum_{n=2}^{\infty} \phi(n, k, \lambda, \nu) |a_n c_n| + \sum_{n=1}^{\infty} |\psi(n, k, \lambda, \nu)| |b_n d_n|
\]
\[
< \sum_{n=2}^{\infty} \phi(n, k, \lambda, \nu) |a_n| + \sum_{n=1}^{\infty} |\psi(n, k, \lambda, \nu)| |b_n|
\]
\[
< 1 - \alpha,
\]
and the proof is complete. \( \square \)

In the last theorem we examine the convex combination properties of the elements of \( \overline{\mathcal{HM}}(\beta, \lambda, k, \nu) \).

**Theorem 3.2.** The class \( \overline{\mathcal{HM}}(\beta, \lambda, k, \nu) \) is closed under convex combination.
Proof. Suppose that
\[ f_i(z) = z - \sum_{n=2}^{\infty} |a_{n,i}|z^n + \sum_{n=1}^{\infty} |b_{n,i}|z^n, \quad i = 1, 2, \ldots \]
then the convex combinations of \( f_i \) may be written as
\[ \sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{n=2}^{\infty} \left( \sum_{i=1}^{\infty} t_i|a_{n,i}| \right) z^n + \sum_{n=1}^{\infty} \left( \sum_{i=1}^{\infty} t_i|b_{n,i}| \right) z^n, \]
where \( \sum_{i=1}^{\infty} t_i = 1, \quad 0 \leq t_i \leq 1 \).
Since
\[ \sum_{n=2}^{\infty} \phi(n,k,\lambda,\nu)|a_{n,i}| + \sum_{n=1}^{\infty} \psi(n,k,\lambda,\nu)||b_{n,i}| < 1 - \beta, \]
so we have
\[ \sum_{n=2}^{\infty} \phi(n,k,\lambda,\nu) \left( \sum_{i=1}^{\infty} t_i|a_{n,i}| \right) + \sum_{n=1}^{\infty} \psi(n,k,\lambda,\nu) \left( \sum_{i=1}^{\infty} t_i|b_{n,i}| \right) \]
\[ = \sum_{i=1}^{\infty} t_i \left( \sum_{n=2}^{\infty} \phi(n,k,\lambda,\nu)|a_{n,i}| + \sum_{n=1}^{\infty} \psi(n,k,\lambda,\nu)||b_{n,i}| \right) \]
\[ < (1 - \beta) \sum_{i=1}^{\infty} t_i = 1 - \beta. \]
This shows that \( \sum_{i=1}^{\infty} t_i f_i(z) \in \mathcal{H}(\beta, \lambda, k, \nu) \) and the proof is complete. \( \square \)

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