Non-Euclidean Newtonian cosmology

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Abstract
We formulate and solve the problem of Newtonian cosmology under the assumption that the absolute space of Newton is non-Euclidean. In particular, we focus on the negatively-curved hyperbolic space, $H^3$. We point out the inequivalence between the curvature term that arises in the Friedmann equation in Newtonian cosmology in Euclidean space and the role of curvature in the $H^3$ space. We find the generalisation of the inverse-square law and the solutions of the Newtonian cosmology that follow from it. We find the generalisations of the Euclidean Michell ‘black hole’ in $H^2$ and show that it leads to different maximum force and area results to those we have found in general relativity. We show how to add the counterpart of the cosmological constant to the gravitational potential in $H^3$ and explore the solutions and asymptotes of the cosmological models that result. We also discuss the problems of introducing compact topologies in Newtonian cosmologies with non-negative spatial curvature.

Keywords: cosmology, non-Euclidean geometry, Newtonian gravity

1. Introduction

When Newton formulated his theory of gravity he assumed time to be linear, with a rate of flow that was unchangeable. He assumed space to be the absolute, unchanging, and Euclidean: ‘the divine sensorium’ [1]. All motion was played out on this Euclidean space without influencing the geometry of space or the rate of flow of time. Newton’s assumption of 3d Euclidean geometry, $E^3$, was tied to the belief that had existed for centuries that Euclidean geometry was part of the ultimate truth about the nature of the Universe. It was not just a mathematical model. If a philosopher or theologian was challenged that their quest for ultimate truth was doomed to fail because the human mind was too limited, then they could just point to Euclid’s geometry as an example of our success in finding a part of the ultimate truth [2].

Yet we can see that there was no need to formulate Newton’s theory of gravity on non-Euclidean space. In this paper we are going to look at what happens to cosmology when Newtonian gravity is formulated on Lobachevsky’s space of constant negative curvature—ie hyperbolic 3-space, $H^3$, in modern terminology. The situation in a positively curved space of constant curvature will also be briefly discussed.
The subject of Newtonian cosmology has a long history, beginning with Milne and McCrea’s deduction of the counterpart to the Friedmann universes [8]. The general relationship between Newtonian and relativistic gravity has been partly obscured because most of the studies of the former have been carried out in the context of homogeneous and isotropic universes where the incompleteness of Newtonian gravity is hidden. When anisotropies are present the lack of any propagation equations for the shear tensor components is notable [9]. As a result, the rigorous mathematical studies of the asymptotic behaviour of the unbound Newtonian N-body problem obtain results for the radius of gyration (or moment of inertia) of the system but give no information about the shape anisotropy [10].

One of the features we will highlight is the difference between Newtonian cosmology on an absolute curved space of constant curvature and the Newtonian version of the general relativistic Friedmann universes where the metric curvature constant \( k \) no longer describes any curvature because Newtonian cosmology in that Milne–McCrea context is always Euclidean and the constant labelled by \( k \) arises from the conservation of the total energy of motion. The possibility of non-Euclidean Newtonian cosmologies is ignored in all textbook expositions of general relativistic cosmology that use the Newtonian formulation of cosmology to motivate it.

Newton’s inverse-square law in \( E^3 \) has a form that is inversely proportional to the area of a 3d ball of the required curvature: \( 4\pi r^2 \). So, in \( E^3 \), we have the familiar force law, \( F(r) \), and gravitational potential\(^1\), \( \Phi(r) \)

\[
F \propto \frac{1}{r^2}, \quad \Phi \propto \frac{1}{r}. \tag{1}
\]

Similarly, in \( H^3 \), using the area element, \( 4\pi R^2 \sinh^2(r/R) \), we have

\[
F \propto \frac{1}{R^2 \sinh^2(r/R)}, \quad \Phi \propto \coth(r/R), \tag{2}
\]

where the constant \( R \) is the curvature radius. In the Euclidean space limit \( R \to \infty \), the formulae equation (2) reduce as expected to equation (1). This expression for the area element was first given by Lobachevsky in 1835 in the form \( \pi (e^r - e^{-r})^2 \) in reference [3]. The connection with inverse-square laws of physics was also appreciated by Bolyai after 1832 (but surprisingly not by Gauss) and the analytic form for the Newtonian potential in \( H^3 \) was given by Schering in 1870 [6]. One of the motivations for these deviations from the \( E^3 \) inverse-square law of Newton was the desire to avoid divergence of the total mass, gravitational potential, gravitational force and tidal force at infinity. This was possible by changing the inverse-square law of Newton or by assuming space was not \( E^3 \) [4]. The history of these developments and the application of non-Euclidean geometry to Bertrand’s problem for potentials that give circular orbits for the Kepler problem, Newton’s spherical property, and the force laws given above is rather confused with incorrect attributions to first studies; a good overview of the history is given in references [5, 7] and we will not repeat it here. The case of a positively curved space of constant curvature, \( S^3 \), is a simple complex transform that changes the sinh to sin, so yields

\[
F \propto \frac{1}{R^2 \sin^2(r/R)}, \quad \Phi \propto \cot(r/R). \tag{3}
\]

\(^1\)If you want \( \Phi(r) \to 0 \) as \( r \to \infty \) then the constant of integration in the Poisson equation can be used to produce a solution \( \Phi(r) = -\frac{GM}{R} \coth(r/R) + \frac{GM}{r} \) with that behaviour but we shall not use that form.
2. Solutions of the Poisson equation in $H^3$

We formulate Newtonian cosmology on a fixed non-Euclidean space by solving Poisson’s equation

$$\nabla^2 \Phi = 4\pi G \rho, \quad (4)$$

with $\nabla^2$ defined on a negatively curved space in $(r, \theta, \phi)$ coordinates by

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right). \quad (5)$$

Specialising to $\Phi(r)$, the solution to $\nabla^2 \Phi = 0$ or $\nabla^2 \Phi = 4\pi G \delta(x) \delta(y) \delta(z)$ for the gravitational action on a test mass, $M$, where the arguments of the delta functions are given by

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta, \quad (6)$$

is

$$\Phi(r) = -\frac{GM}{R} \coth\left(\frac{r}{R}\right), \quad F(r) = -\frac{\partial \Phi}{\partial r} = -\frac{GM}{R^2 \sinh^2(r/R)} \quad (7)$$

to obtain the generalisation of Newton’s inverse-square law for the acceleration at a distance $r$ from a mass $M^2$. This approaches the Euclidean results $\Phi = -GM/r$ and $F = -GM/r^2$ when $R \gg r$. We can use this equation to obtain the generalised Friedmann equation for a Newtonian Universe of constant negative curvature:

$$\ddot{r} = -\frac{GM}{R^2 \sinh^2(r/R)} \quad (8)$$

and integrate this equation to obtain the generalised Friedmann equation for a Newtonian Universe of constant negative curvature containing matter with zero pressure (the cases of non-zero pressure could be investigated by changing $\rho \rightarrow \rho + 3p$ in Poisson’s equation):

$$\frac{\dot{r}^2}{r^2} = \frac{2GM}{Rr^2} \coth\left(\frac{r}{R}\right) - \frac{K}{r^2}, \quad (9)$$

with $K$ an integration constant. Note that this is not the same as the general-relativistic Friedmann equation for a cosmology with a space of constant negative curvature (although it reduces to it in the Euclidean $r \ll R$ limit).

From equation (9) with $K = 0$ we can find the scale factor of an expanding Universe $r(t)$ from

$$\int \sqrt{\tanh(r/R)} dr = \sqrt{\frac{GM}{R}} (t + t_0), \quad (10)$$

hence

$$\tanh^{-1}\left(\sqrt{\tanh(r/R)}\right) - \tan^{-1}\left(\sqrt{\tanh(r/R)}\right) = \sqrt{\frac{GM}{R}} (t + t_0). \quad (11)$$

To repeat the calculation for spherical space change sinh to sin). In $N$ space dimensions, $H^N$, the required solution of Laplace’s equation is $\Phi(r) \propto \coth^{N-2}(r/R)$ for $N > 2$ and $\ln[\tanh(r/R)]$ when $N = 2$.\footnote{To repeat the calculation for spherical space change sinh to sin). In $N$ space dimensions, $H^N$, the required solution of Laplace’s equation is $\Phi(r) \propto \coth^{N-2}(r/R)$ for $N > 2$ and $\ln[\tanh(r/R)]$ when $N = 2.$}
We have $t_0 = 0$ if we set $r(0) = 0$ and there is an initial singularity in the density $\rho \propto r^{-3} \to \infty$ as $r \to 0$.

The case with $K \neq 0$ does not allow equation (9) to be integrated exactly. In the $r \to \infty$ limit we have with $K = -1$:

$$\dot{r}^2 \to \frac{2GM}{R} \{1 + 2 \exp[-2r/R]\} - K,$$

and to leading order we have

$$\frac{1}{2} \dot{r}^2 \to \frac{GM}{R} - K,$$

so for $K = -1$

$$r \propto t,$$

and the solution approaches the Milne Universe, as in general relativity and Euclidean Newtonian gravity.

### 3. Non-Euclidean black holes

It is possible to follow the Newtonian logic of Michell [11] and Laplace [12] (see reference [13] for a detailed history) in the case of Newtonian gravity on $H^3$. When $K = 0$ we can create a Newtonian ‘black hole’ by seeking the dimensions of a spherical gravitating object with escape velocity equal to that of light: $\dot{r}^2 = c^2 = 2GM/cR \coth(r/R)$. Its radius is

$$r_{bh} = R \tanh^{-1} \left( \frac{2GM}{Rc^2} \right) = R \ln \left( \frac{1 + \frac{2GM}{Rc^2}}{1 - \frac{2GM}{Rc^2}} \right),$$

for the principal branch $-1 < \frac{2GM}{Rc^2} < 1$ and $r_{bh} \to 2GM/c^2$ in the Euclidean limit $R \to \infty$. As in Michell’s Newtonian black hole, the radius $r_{bh}$ is not an event horizon of no return as in general relativity. It is a surface that requires an escape velocity of $c$ in order to escape all the way to infinity. Any finite distance from the black hole can be reached with a suitable launch speed.

If the surface area of the ‘black hole’ is $4\pi R^2 \sinh^2(r_{bh}/R)$ then this might still be a measure of entropy, as in general relativity:

$$A = 4\pi R^2 \sinh^2(r_{bh}/R) = \frac{4\pi R^2}{\coth^2(r_{bh}/R) - 1} = \frac{4\pi R^2}{\frac{R^2}{c^4} - 1} = \ldots$$

\(16\pi G^2 M^2 R^2 = \frac{4\pi R_S^2}{1 - R_S^2/R^2} \approx 4\pi R_S^2 (1 + R_S^2/R^2 + \ldots) > A_s,

where $R_s = 2GM/c^2$ is the usual Schwarzschild radius and $A_s = 4\pi R_s^2$.

To next order $\tanh^{-1} x = x + x^3/3 + \ldots$

$$r = R_S \left[ 1 + \frac{R_S^2}{3R_s^2} + \ldots \right],$$

where $R_S < R$ and

$$R_S = \frac{2GM}{c^2}.$$
4. Maximum force

Barrow and Gibbons [15] have shown that in general relativity there is strong evidence that there exists a maximum force, equal to $c^4/4G$ in three space dimensions (but not in higher dimensions). This is the Planck unit of force but Planck’s constant does not appear and so it has fundamental classical significance, like the Planck unit of velocity, $c$, and the magnetic moment to angular momentum ratio [16]. It is interesting to ask what becomes of this bound in $H^3$. A simple guide is to use the fact that when two Schwarzschild black holes touch horizons, the force between their centres gives the maximum force$^3$. Using two $H^3$ Newtonian black holes of mass $M$ described by equation (15), the maximum force between them at separation $r_{bh}$ is

$$F = \frac{GM^2}{R^2 \sinh^2 \left[ \tanh^{-1} \frac{2GM}{Rc^2} \right]}.$$  \hfill (19)

As $R \to \infty$, we recover the general relativity limit [15], as $F \to \frac{c^4}{4G}$. Keeping higher-order corrections as $R \to \infty$, we have

$$F \to \frac{GM^2}{R^2 \sinh^2 \left[ \frac{2GM}{Rc^2} \left( 1 + \frac{4G^2M^2}{3R^2c^4} \right) \right]},$$  \hfill (20)

and so, to second order,

$$F \to \frac{c^4}{4G} \left[ 1 - \frac{8G^2M^2}{3R^2c^4} \right] < \frac{c^4}{4G},$$  \hfill (21)

and the maximum force goes to zero when $R = \sqrt{\frac{2}{3}}R_s$.

From equation (19), we see this reduction in the maximum force holds in general because $\tanh^{-1}(x) > x$ on $0 < x < 1$. Hence, we see that the candidate maximum force is reduced from its value in the $E^3$ Newtonian and general relativistic cases examined in reference [15].

5. Adding a “cosmological constant”

The addition of the cosmological constant in the Euclidean Newtonian problem amounts to the determination of the two forms of the gravitational potential $\Phi(r)$ that obey the Newton spherical property and make the external gravity field of a sphere equal to that of a point mass at its centre of equal mass. There are two potentials of this sort

$$\Phi \propto \frac{1}{r} \quad \text{and} \quad \Phi \propto \Lambda r^2,$$  \hfill (22)

Both of these solutions were known to Newton and appear in the Principia, but it was Laplace who showed that the general solution for the spherical property for a potential in $E^3$ is the linear combination of these two potentials, yielding [14],

$$\Phi = -\frac{GM}{r} + \frac{\Lambda r^2}{6}.$$  \hfill (23)

$^3$It is also the maximum cosmic string tension that arises when the deficit angle caused by the string equals $2\pi$. 


In order to find the second potential in $H^3$ that is the counterpart of the $\Phi \propto \Lambda r^2$ in $E^3$, we can use the elegant analysis of the general Bertrand problem for closed circular orbits in static spherically symmetric made by Perlick [17] and elaborated by Ballesteros et al [18]. Writing the spatial metric for a general static spherically symmetric metric as
\[
dl^2 = h(r)^2 dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\]
the radial Laplacian operator for the radial potential is then
\[
\nabla^2 \Phi(r) = \frac{1}{r^2 h(r)} \frac{d}{dr} \left( \frac{r^2}{h(r)} \frac{d\Phi}{dr} \right).
\]
Hence, the solution of $\nabla^2 \Phi(r) = 0$ is
\[
\Phi(r) = \int \frac{h(\bar{r})}{\bar{r}^2} d\bar{r}.
\]
The counterparts of equation (22) in general are again the inverse squares of each other, as in $E^3$ [18]:
\[
\Phi_1(r) = A_1 \left( \int r^{-2} h(r) dr + B_1 \right),
\]
\[
\Phi_2(r) = A_2 \left( \int r^{-2} h(r) dr + B_2 \right)^{-2},
\]
where $A_1, A_2, B_1$ and $B_2$ are constants. For the space of constant curvature, $K$, familiar from general relativistic cosmology, we have
\[
\Phi_1(r) = \sqrt{r^{-2} - K}, \quad \Phi_2(r) = (r^{-2} - K)^{-1}.
\]
If we introduce the distance function to $r = 0$ on the curved space, $\rho_K = r = R \tanh(r/R)$, then we have for $K = -1$, noting that
\[
K = -\frac{1}{R^2},
\]
we obtain the inverse-square pair [17–19].
\[
\Phi_1 = -\frac{GM}{R} \coth \left( \frac{\rho_K}{R} \right), \quad \Phi_2 = \frac{\Lambda R^2}{6} \tanh^2 \left( \frac{\rho_K}{R} \right),
\]
where the constant of proportionality has been labelled $\Lambda/6$ so as to recover the $E^3$ proportionality constant in equation (23) in the Euclidean limit $R \to \infty$, where $\Phi_1 \propto -\frac{1}{\rho_K}$ and $\Phi_2 \propto \rho_K^2$. Thus, $\Phi_2$ gives us the $H^3$ equivalent of the $\Lambda$-term potential giving the Newtonian

For the positively curved space ($K = +1$) the same results hold with the transforms $\coth \to \cot$ and $\tanh \to \tan$. 
spherical property and Bertrand’s circular orbits like Newton’s potentials in equation (22) in $E^3$.

What happens to the cosmological equations now? The force law (using the $r$ coordinate now) is

$$F = -\nabla \Phi = \frac{-GM}{R^2 \sinh^2(r/R)} + \frac{\Lambda R}{3 \cosh^2(r/R)} \tanh(r/R). \tag{32}$$

At small $r$, we recover the $E^3$ Newtonian result for the force on a test particle $m$

$$F \to -\frac{GMm}{r^2} + \frac{m \Lambda r}{3}. \tag{33}$$

There is a counterpart to the Newtonian de Sitter solution if we keep only the second potential ($\Phi_2$) on the right-hand side of equation (32)

$$\ddot{r} = \frac{\Lambda R}{3 \cosh^2(r/R)} \tanh(r/R). \tag{34}$$

So at large $r \gg R$,

$$\ddot{r} \to \frac{\Lambda r}{3}, \tag{35}$$

and we recover the de Sitter asymptote, $r(t) = \exp[t \sqrt{\Lambda/3}]$.

In general for this case, we have

$$\dot{r}^2 = \frac{2\Lambda R}{3} \int \frac{\tanh(r/R)}{\cosh^2(r/R)} dr = \frac{2\Lambda R^2}{3} \tanh^2(r/R) + C, \tag{36}$$

where $C$ is an integration constant. The solution for $C = 0$ is

$$\sinh(r/R) = \exp\left[\frac{2\Lambda}{3}(t + t_0)\right] \tag{37}$$

where $t_0$ is an inessential constant. Thus, as $t \to \infty$, we have the Milne asymptote

$$r \propto t \tag{38}$$

This is understandable because the force goes to zero exponentially as $r \to \infty$: the Universe looks like a vacuum solution.

If we keep both potential terms in equation (32), then

$$\frac{\dot{r}^2}{r^2} = \frac{2GM}{Rr^2} \coth(r/R) + \frac{2\Lambda R^2}{3r^2} \tanh^2(r/R) + \frac{C}{r^2}, \tag{39}$$

and as $r \to \infty$, the first (‘matter’) term on the right-hand side (from $\Phi_1$) is dominated by the curvature-like term $C/r^2$ and the $\Lambda/r^2$ terms. We will not unpick the various cases determined by the size and sign of the ratio $\frac{\Lambda R^2}{3r^2}$ but it is a straightforward exercise.
6. Compact topologies

In general relativistic cosmologies we are familiar with the possibility that universes with zero or negative spatial curvature might have finite volume due to a compact topology [20–22]. The simplest example would be a 3-torus topology for the zero-curvature Friedmann Universe. When universes become anisotropic the problem of acceptable topologies becomes much more complicated and there are anisotropic Bianchi type universes where compactification of the topology forces the dynamics to be isotropic [23, 25]. However, unlike in general relativity, the possibility of compactification of a negatively curved or flat space in Newtonian cosmology seems to be prohibited. If we integrate Poisson’s equation over the three-sphere

\[ \frac{1}{4\pi G} \int_V \rho dV = \int_V \nabla^2 \Phi dV = \int_V \text{div} \Phi dV = \int_S \text{grad} \Phi dS = 0 \]  

(40)

The last two steps use the divergence theorem and the fact that the last integral is over the boundary S of the three-sphere, and so is integral is zero when the cosmological constant vanishes since the three-sphere has no boundary. However, this implies that \( \rho = 0 \) everywhere since \( \rho \geq 0 \). If we had included pressure then we would have arrived at the conclusion that \( \rho + 3p = 0 \), [14, 24, 26]. This argument can be applied to our non-Euclidean cosmological model derived from equation (4) to exclude compact spatial topologies in the absence of a cosmological constant.

7. Conclusions

We have formulated the theory of Newtonian gravity on a non-Euclidean absolute space of constant curvature and investigated the case of negatively-curved, 3d hyperbolic space, \( H^3 \), in detail. We have also indicated how the case of positively curved space can be obtained from it. We derived the generalisation of Newton’s inverse-square law in \( H^3 \) and used it to develop a non-Euclidean Newtonian cosmology of the sort first created by Milne and McCrea [8] for Euclidean space and found solutions of the generalised Friedmann equation. We showed how the curvature term of general relativity that shows up in the Newtonian counterpart is different to the terms governing the curvature of space in the \( H^3 \) Newtonian cosmology. We found the structure of Newtonian ‘black holes’ in \( H^3 \) by generalising the arguments first employed by Michell. We also found the modifications to the maximum force condition in general relativistic black holes using the \( H^3 \) ‘black hole’, showing that the maximum force is reduced below its general relativistic value argued for earlier by Barrow and Gibbons [15], while the area of the ‘black hole’ is increased over its general relativistic value. Using the analysis of Perlick [17], we then showed how to add the equivalent of a cosmological constant to non-Euclidean gravity theory by using the Bertrand criteria for closed circular orbits and Newton’s spherical property to generalise the sum of \( r^{-1} \) and \( r^2 \) potentials first found by Laplace to the inverse-square pair \( \coth r \) and \( \tanh^2 r \). We studied the cosmological solutions and asymptotic forms that they display at small and large times when this generalised cosmological constant term is present. Finally, we make some remarks about the impossibility of compact topologies in Newtonian gravity when space is flat or negatively curved.

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