Pairwise thermal entanglement in the Ising-XYZ diamond chain structure in an external magnetic field

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Abstract – Quantum entanglement is one of the most fascinating types of correlation that can be shared only among quantum systems. The Heisenberg chain is one of the simplest quantum chains which exhibits a rich entanglement feature, due to the fact that the Heisenberg interaction is quantum coupling in the spin system. The two particles were coupled through XYZ coupling or simply called as two-qubit XYZ spin, which are the responsible for the emergence of thermal entanglement. These two-qubit operators are bonded to two nodal Ising spins, and this process is repeated infinitely resulting in a diamond chain structure. We will discuss the two-qubit thermal entanglement effect on the Ising-XYZ diamond chain structure. The concurrence could be obtained straightforwardly in terms of two-qubit density operator elements; using this result we study the thermal entanglement, as well as the threshold temperature where entangled state vanishes. The present model displays a quite unusual concurrence behavior, e.g., the boundary of two entangled regions becomes a disentangled region, and this is intrinsically related to the XY-anisotropy in the Heisenberg coupling. Although a similar property had been found for only two qubits, here we show it for the case of a diamond chain structure, which reasonably represents real materials.

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Introduction. – In the last decade, many efforts were dedicated to characterizing qualitatively and quantitatively the entanglement properties of condensed-matter systems which are the natural candidate to apply to quantum communication, as well as quantum information. In this sense, it is quite interesting to study the entanglement of solid-state systems such as spin chains [1]. The Heisenberg chain is one of the simplest quantum chains which exhibit a rich entanglement feature, due to the fact that the Heisenberg interaction is a nonlocal correlation between quantum systems [2].

Quantum entanglement, with its applications to quantum phase transitions of strongly correlated spin systems and its experimental implementation in optical lattices, was considered, in particular, for one-dimensional systems. Diverging entanglement length without quantum phase transition was found in a localizable entanglement (LE) for valence bond solids (VBSs), since the correlation length remains finite [3]. This is a rather new and remarkable result regarding the entanglement properties of VBS quantum spin ground states. A theory for localizable entanglement was developed based on matrix product states from the density matrix renormalization group (DMRG) method and applied to VBS states [4]. In ref. [5], an experimental implementation was proposed for VBSs of the spin-1 Heisenberg Hamiltonians and ladders, and a method was proposed directly to measure quantum observables that are not accessible in standard materials in condensed matter.

Motivated by real materials such as Cu$_3$(CO$_3$)$_2$(OH)$_2$, known as azurite, which is an interesting quantum antiferromagnetic model described by the Heisenberg model on a generalized diamond chain. Honecker et al. [6] studied the dynamic and thermodynamic properties for this model. Moreover, the thermodynamics of the Ising-Heisenberg model on a diamond-like chain was also widely discussed in refs. [7–10]. The motivation to search for the Ising-XYZ diamond chain model is based on some recent works, i.e., the experiments of the natural mineral azurite, the theoretical calculations of the Ising-XXZ model, as well as the experimental result of the exchange dimers (interstitial sites) parameter and their descriptions
of the various theoretical models. The 1/3 magnetization plateau, the double peaks both in the magnetic susceptibility and specific heat, was observed in the experimental measurements [11,12]. It should be noted that the dimers (interstitial sites) are exchanged much more strongly than the nodal sites. Since the dimer interaction is much stronger than the rest, it can be represented as an exactly solvable Ising-Heisenberg model. In addition, the experimental data on the magnetization plateau coincides with the Ising-Heisenberg model [7,13,14].

Recently, several investigations have focused on thermal entanglement with Heisenberg coupling qubits as well as they have assumed some finite chain structure. The thermal entanglement of the isotropic Heisenberg spin chain has been studied in the absence [15] and in the presence of entanglement behavior [18], such as more than one threshold temperature. On the other hand, an unusual property of entanglement also was considered in an alternating Ising and Heisenberg spins in a simple one-dimensional chain [19], where although at zero temperature there is no evidence of entanglement, then a small amount of concurrence arises indicating that the system has a thermal entanglement between two Heisenberg spins. Moreover, this result was confirmed by the theoretical model [20] by the Gibbs-Bogoliubov approach (Heisenberg-Ising model) with the experimental results of the natural material azurite [12].

This paper is organized as follow: in the next section we present the Ising-XYZ model on a diamond chain and its phase diagram at zero temperature. Further, in the third section, we present the exact thermodynamic solution of the model. In the fourth section, we have discussed the thermal entanglement of the Heisenberg reduced density operator of the model, such as concurrence and threshold temperature. Finally, the last section contains the concluding remarks.

The Ising-XYZ chain on diamond chain structure. – The thermal entanglement of the Ising-Heisenberg diamond chain was already discussed. Here we extend this model in agreement with the motivation discussed above. Therefore, let us consider an Ising-XYZ diamond chain structure as illustrated schematically in fig. 1. Thus, the Ising-XYZ Hamiltonian becomes

\[ H = -\sum_{i=1}^{N} \left[ J(1+\gamma)\sigma_{ui,i}^x\sigma_{hi,i}^z + J(1-\gamma)\sigma_{ui,i}^y\sigma_{hi,i}^y \
+ J_\gamma\sigma_{ui,i}^z\sigma_{hi,i}^x + J_0(\sigma_{ui,i}^+ + \sigma_{ui,i}^-)(S_i + S_{i+1}) \
+ h(\sigma_{ui,i}^x + \sigma_{hi,i}^x) + \frac{h}{2}(S_i + S_{i+1}) \right], \]

where \( \sigma_{ui,i}^\alpha, 1 \leq \alpha \leq \beta \) are the Pauli matrices with \( \alpha = \{ x, y, z \} \), and \( S \) corresponds to the Ising spins, whereas \( \gamma \) is the XY-anisotropy parameter.

After diagonalizing the XYZ term, we have the following eigenvalues for XYZ dimer in terms of nodal spin chain \( \mu = S_i + S_{i+1} \):

\[ \varepsilon_{1,4} = -h\frac{\mu}{2} - J\frac{z}{4} \pm \Delta(\mu), \]
\[ \varepsilon_{2,3} = -h\frac{\mu}{2} + J\frac{z}{4}, \]

wherein 
\[ \Delta(\mu) = \sqrt{(h + J_0\mu)^2 + \frac{1}{4}J_0^2\gamma^2}, \]
with the corresponding eigenvectors in terms of the standard basis \( \{|\downarrow\rangle, |\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle\} \) given, respectively, by

\[ |\varphi_{1,4}\rangle = N_\pm (\alpha_{\pm}(\uparrow\downarrow) + |\downarrow\rangle), \]
\[ |\varphi_{2,3}\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle \pm |\downarrow\rangle), \]

where \( \alpha_{\pm} = \frac{J_0}{2h+2J_0\mu\pm\Delta(\mu)} \), and \( N_{\pm} = \frac{1}{\sqrt{1+\alpha_{\pm}}} \)

It is important to recall that the pairwise entanglement between the Heisenberg spins at zero temperature is entangled for any Hamiltonian parameters, unless for the limiting case in which it becomes a disentangled region, which will be illustrated in the zero-temperature limit of the phase transition.

This model is somewhat opposite to that proposed in ref. [19], where at zero temperature there was no thermal entanglement.

Phase diagram of Ising-XYZ on diamond chain. Here, we start our discussion regarding the phase diagram at zero temperature. First we assume that the chain is in the absence of magnetic field. Thus, we illustrate the phase diagram in fig. 2(a), as a function of \( J_0/J \) against \( \gamma \) assuming a fixed value \( J_z/J = 0 \) and \( h/J = 0 \). Here we show two phases: one Ising spin ferromagnetic with Heisenberg spin modulated ferromagnetic simply denoted by modulated ferromagnetic (FMF) phase, and the Ising spin frustrated region (IFR).

The explicit representations of FMF states are expressed as

\[ |\text{FMF}_i\rangle = \prod_{i=1}^{N} (|\varphi_{4}\rangle \otimes |+)\rangle. \]
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\[ |\text{FMF}_2\rangle = \prod_{i=1}^{N} |\varphi_2\rangle_i \otimes |\gamma\rangle_i, \]  
(8)
and the ground-state energies are given by

\[ E_{\text{FMF}_1} = -\frac{h}{2} - \sqrt{\left(h + J_0\right)^2 + \frac{1}{4}J^2\gamma^2}, \]  
(9)
\[ E_{\text{FMF}_2} = \frac{h}{2} - \sqrt{\left(h - J_0\right)^2 + \frac{1}{4}J^2\gamma^2}. \]  
(10)

It is worth noting that FMF$_1$ and FMF$_2$ are degenerated at null magnetic field, which we denote simply by FMF region. At first glance the states FMF$_1$ and FMF$_2$ seem equivalent under total spin inversion; however, the state \( |\varphi_2\rangle \) under spin inversion leads to a different state. Therefore, the states FMF$_1$ and FMF$_2$ are no longer equivalent.

Nevertheless, with no magnetic field, the model illustrates a frustrated region, which is represented as

\[ |\text{IFR}\rangle = \prod_{i=1}^{N} |\varphi_2\rangle_i \otimes |S\rangle_i, \]  
(11)

with ground-state energy given by

\[ E_{\text{IFR}} = -\frac{|J|}{2}. \]

Let us observe that the IFR phase has a residual entropy \( S_0 = \kappa_B \ln(2) \), the term 2 comes from two possible orientations of \( S \) in eq. (11). The green curve contouring the IFR region given by \( \gamma = \sqrt{1 - 4J_0^2} \) has a residual entropy \( S_0 = \kappa_B \ln(3) \), while the blue point corresponds to a residual entropy \( S_0 = \kappa_B \ln(4) \).

A similar phase diagram at zero temperature is displayed in fig. 2(b) as a function of \( J_0/J \) and \( h/J \) for a fixed value \( \gamma = 0.95 \) and \( J_z/J = 0 \), the thick red line has a residual entropy \( S_0 = \kappa_B \ln(2) \), and the green (gray) circle corresponds to a frustrated point with residual entropy given by \( S_0 = \kappa_B \ln(3) \), while the dashed curve given by \( h/J = \pm \frac{\sqrt{10}}{2} \) (top/bottom, respectively) has a residual entropy \( S_0 = \kappa_B \ln(2) \).

In fig. 3(a) there is displayed the phase diagram at zero temperature \( \gamma \) against \( h/J \) for a fixed value \( J_0/J = -0.3 \) and \( J_z/J = 0 \), the thick red line corresponds to the Ising frustrated (IFR) region in a zero magnetic field, while the dashed line corresponds to a non-zero Heisenberg frustrated (HFR) phase. Furthermore, there are displayed three phases: one FAF phase, one FMF$_1$ phase, and another FMF$_2$ phase, given by states (7) and (8). Similarly, the phase diagram at zero temperature is displayed in fig. 3(b) as a function of \( \gamma \) and \( h/J \) for a fixed value \( J_0/J = -0.3 \) and \( J_z/J = 0.3 \); the thick red line, the dashed line and the green (gray) circle have the same meaning as for the case (a).

An explicit representation of the FAF state is expressed as

\[ |\text{FAF}\rangle = \prod_{i=1}^{N} |\varphi_2\rangle_i \otimes |\gamma\rangle_i, \]  
(12)
and its ground-state energy is given by \( E_{\text{FAF}} = -\frac{J + h}{2} \).

Particularly, when \( \gamma = \sqrt{1 - 4(J_0 + h)^2} \), we obtain the frustration state with magnetic field, given by the state

\[ |\text{HFR}\rangle = \prod_{i=1}^{N} |\varphi_2(4)\rangle_i \otimes |\gamma\rangle_i, \]  
(13)
with its corresponding ground-state energy given by

\[ E_{\text{HFR}} = -\frac{|J| + h}{2}. \]

It is worth mentioning that the HFR phase (dashed line in figs. 2 and 3) has also a residual entropy \( S_0 = \kappa_B \ln(2) \). At low temperatures residual entropies have common features with the thermal entanglement. Comparing fig. 3(b), and fig. 4(a) in the next section one can observe that fact.

One can also easily verify that the \( ab \)-dimers are entangled in all regions illustrated in the phase diagram, only the limiting cases could lead to disentangled regions such as those discussed for the Ising-XXZ chain [21].

Thermodynamics. – The Ising-XYZ diamond chain can be solved exactly using the decoration transformation [22–25] together with the usual transfer matrix technique [26]. So, let us start considering the partition function as follows:

\[ Z = \text{tr} \left( e^{-\beta H} \right), \]  
(14)
where \( \beta = 1/\kappa_B T \), with \( \kappa_B \) being the Boltzmann constant and \( T \) is the absolute temperature, and the Hamiltonian
\( H \) is given by eq. (1). The transfer matrix of the model can be expressed by
\[
T = \begin{bmatrix}
w(2) & w(0) \\
w(0) & w(-2)
\end{bmatrix},
\]
where the Boltzmann factors is expressed by
\[
w(\mu) = 2e^{\frac{\beta\mu}{\lambda}} \left[ e^{\frac{\beta\Delta}{\lambda}} \cosh \left( \frac{\beta h}{\lambda} \right) + e^{\frac{-\beta\Delta}{\lambda}} \cosh \left( \beta \Delta(\mu) \right) \right],
\]
and the transfer matrix eigenvalues, become
\[
\lambda_{\pm} = w(2) + w(-2) \pm \sqrt{(w(2) - w(-2))^2 + 4w(0)^2}.
\]

To study the thermodynamic properties we will use the exact free energy per unit cell in the thermodynamic limit
\[
f = \frac{1}{\beta} \lim_{N \to \infty} \ln \frac{Z}{N} = -\frac{1}{\beta} \ln \lambda_+.
\]

Next, let us calculate the thermal entanglement behaviour between the ab-dimers of our model.

**Pairwise thermal entanglement.** – Now let us start our discussion regarding the quantum entanglement of the Ising-XYZ diamond chain, recalling that a two-qubits-with-XYZ coupling was discussed in ref. [18]. As a measure of entanglement for two arbitrary mixed states of dimers, we use the quantity called concurrence [27], which is defined in terms of reduced density matrix \( \rho \) of two mixed states
\[
C(\rho) = \max\{0, 2\lambda_{\text{max}} - \text{tr}\sqrt{\rho}\},
\]
assuming \( R = \rho \sigma^y \otimes \sigma^y \rho^* \sigma^y \otimes \sigma^y \), where \( \lambda_{\text{max}} \) is the largest eigenvalue of the matrix \( \sqrt{\rho} \) and \( \rho^* \) represents the complex conjugate of matrix \( \rho \), with \( \sigma^y \) being the Pauli matrix.

Consequently the concurrence between ab-dimers becomes
\[
C = \max\{0, |\rho_{14}| - |\rho_{22}\rho_{33}| - |\rho_{23}| - \sqrt{\rho_{11}\rho_{44}}\},
\]
where the \( \rho_{ij} \) are the elements of the density matrix. For the case of an infinite chain, the reduced density operator elements [28] could be expressed in terms of the correlation function between two entangled particles [29],
\[
\rho_{11} = \frac{1}{4} + \langle \sigma^z_1 \sigma^z_1 \rangle + \langle \sigma^z_1 \rangle,
\]
\[
\rho_{22} = \rho_{33} = \frac{1}{4} - \langle \sigma^z_1 \sigma^z_1 \rangle,
\]
\[
\rho_{44} = \frac{1}{4} + \langle \sigma^z_2 \sigma^z_2 \rangle - \langle \sigma^z_1 \rangle,
\]
\[
\rho_{14} = \langle \sigma^z_1 \sigma^z_2 \rangle - \langle \sigma^z_1 \rangle \langle \sigma^z_2 \rangle,
\]
\[
\rho_{23} = \langle \sigma^z_2 \sigma^z_1 \rangle + \langle \sigma^z_2 \rangle \langle \sigma^z_1 \rangle.
\]

It is worth mentioning that the zero-temperature entanglement for ab-dimers is maximally entangled in the FAF region \( C = 1 \), while in the FMF region the concurrence depends on the Hamiltonian parameters, which is given by
\[
C = \frac{|\rho_{14}|}{\sqrt{\rho_{11}\rho_{44}} + \rho_{22}\rho_{33}}.
\]
However, at finite temperature, each expected value becomes a temperature-dependent quantity. These quantities are expressed as
\[
\langle \sigma^z_1 \sigma^z_2 \rangle = e^{\beta \Delta(1)} \frac{\Delta(1)}{2} e^{-\beta \Delta(1)} \sinh(\beta/2) \left( \frac{\tan(\beta/2)}{\Delta(1)} \right),
\]
\[
\langle \sigma^y_1 \sigma^y_2 \rangle = e^{\beta \Delta(1)} \frac{\Delta(1)}{2} e^{-\beta \Delta(1)} \sinh(\beta/2) \left( \frac{\tan(\beta/2)}{\Delta(1)} \right),
\]
\[
\langle \sigma^z_1 \rangle = e^{\beta \Delta(1)} \frac{\Delta(1)}{2} e^{-\beta \Delta(1)} \sinh(\beta/2) \left( \frac{\tan(\beta/2)}{\Delta(1)} \right),
\]
\[
\langle \sigma^z_2 \rangle = e^{\beta \Delta(1)} \frac{\Delta(1)}{2} e^{-\beta \Delta(1)} \sinh(\beta/2) \left( \frac{\tan(\beta/2)}{\Delta(1)} \right).
\]

Alternatively, we can obtain also an equivalent result using the approach described in ref. [21].

In fig. 4(a) there is illustrated a density-plot concurrence in the low-temperature limit \( T/J = 0.01 \) with fixed value \( J_0/J = -0.3 \) and \( J_z/J = 0.3 \), as a function of \( \gamma \) and \( h/J \). The darkest region corresponds to a higher concurrence \( C = 1 \), while the white region represents the disentangled region \( C = 0 \). This plot follows the pattern of the phase diagram displayed in fig. 3(a). Whereas in fig. 4(b) there is illustrated a density-plot concurrence in the low-temperature limit \( T/J = 0.01 \) with fixed value \( J_z/J = 0 \) and \( \gamma = 0.95 \), as a function of \( J_0/J \) and \( h/J \). This plot also follows the pattern of the phase diagram illustrated in fig. 2(b). It is worth mentioning that the disentangled region has similar behaviour to those found in ref. [18] for null magnetic field and with external magnetic field as in ref. [30], both for the case of two qubits with XYZ coupling.

For the purpose of giving a more detailed behavior of the concurrence as a function of \( \gamma \) and \( h/J \), in fig. 5 there is illustrated the concurrence as a sequence of temperatures higher than those in fig. 4(a), assuming the concurrence for same fixed parameters \( J_0/J = -0.3 \) and \( J_z/J = 0.3 \).
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In fig. 5(a) the thermal entanglement is illustrated at \( T/J = 0.02 \), basically the entangled region still remains (fig. 4(a)), displaying the change of the disentangled region. In fig. 5(b) this is illustrated for \( T/J = 0.04 \), showing how the entangled region “deteriorates” and how the disentangled region is modified. Similarly in fig. 5(c) the concurrence is illustrated for \( T/J = 0.08 \), where the entangled region increasingly deteriorates. Lastly in fig. 5(d) the concurrence is illustrated for \( T/J = 0.16 \), at this temperature the entangled region was highly modified compared to that in fig. 4(a), showing only a nearly straight valley. In summary, we illustrate how the thermal entanglement “deteriorates” (vanishes) as far as the temperature increases.

**Threshold temperature.** We follow a similar definition of the critical temperature for the threshold temperature \( T_{th} \), i.e., disorder-order-disorder (or inverse) sequences of transition (reentrant phenomenon) have been observed in many systems. The reentrant phase transition has been observed for the first time in polymer gels \([31]\), frustrated spin-gas models \([32]\), Rochelle salts \([33]\), and in a predator-prey model \([34]\). Quantum reentrant phase transitions (disentangled (D)-entangled (E)-disentangled (D)-entangled (E)-disentangled (D) regions) of the concurrence vs. magnetic field have been noticed in the Lipkin-Meshkov-Glick model \([35]\).

In fig. 6(a) there is illustrated the concurrence as a function of \( T/J \) and \( J_0/J \) assuming a fixed value \( J_z/J = 0 \), \( \gamma = 0.95 \) and \( h/J = 0.27 \). Here we can observe how the concurrence vanishes and for a higher temperature a thermal entanglement arises. Thereafter, the entanglement for even a higher temperature disappears. In fig. 6(b) there is depicted the concurrence as a function of \( T/J \) and \( J_0/J \) assuming a fixed value \( J_0/J = 0.3 \) and \( J_z/J = 0.3 \). (a) \( T/J = 0.02 \). (b) \( T/J = 0.04 \). (c) \( T/J = 0.08 \). (d) \( T/J = 0.16 \).

**Fig. 5:** (Color online) Thermal entanglement deterioration as a function of \( \gamma \) and \( h/J \) when temperature increases and with fixed \( J_0/J = -0.3 \) and \( J_z/J = 0.3 \). (a) \( T/J = 0.02 \). (b) \( T/J = 0.04 \). (c) \( T/J = 0.08 \). (d) \( T/J = 0.16 \).

**Fig. 6:** (Color online) Thermal entanglement deterioration when temperature increases. (a) For \( J_z/J = 0 \), \( \gamma = 0.95 \) and \( h/J = 0.27 \). (b) For \( J_z/J = 0.3 \), \( \gamma = 0.6 \) and \( h/J = 0.35 \).

**Fig. 7:** Threshold temperature limiting entangled (E) region and disentangled (D) region as a function of the magnetic field, assuming fixed parameters values \( J_0/J = -0.3 \) and \( J_z/J = 0.3 \), for several values of the XY-anisotropy \( \gamma \). (a) \( \gamma = 0 \), (b) \( \gamma = 0.1 \), (c) \( \gamma = 0.4 \), (d) \( \gamma = 0.5 \), (e) \( \gamma = 0.53 \), (f) \( \gamma = 0.535 \), (g) \( \gamma = 0.56 \), (h) \( \gamma = 0.585 \), (i) \( \gamma = 0.6 \).

Assuming a fixed value \( J_z/J = 0.3 \), \( \gamma = 0.6 \) \( h/J = 0.35 \); we can also observe how the concurrence vanishes for a higher temperature and for a bit higher temperature a small thermal entanglement emerges, hereafter the entanglement for a higher temperature disappears definitively.

In fig. 7 there is illustrated the threshold temperature \( T_{th} \) as a function of the magnetic field for several values of the XY-anisotropy \( \gamma \). The curves indicate the threshold temperature, the limiting entangled (E) region and disentangled (D) region as a function of the magnetic field, assuming the fixed parameters values \( J_0/J = -0.3 \) and \( J_z/J = 0.3 \), for several values of the XY-anisotropy \( \gamma \). In fig. 7(a), for \( \gamma = 0 \), it is observed that the threshold temperature upper region corresponds to the entangled (E) region and the bottom region corresponds to the disentangled (D) region. In fig. 7(b), for \( \gamma = 0.1 \), we observe a rise of the threshold temperature \( T_{th} \) for the left side curve. From fig. 7(a), (b), we observe that those threshold temperature curves are highly sensitive for \( \gamma \neq 0 \), because the entanglements are extremely small close to the threshold temperature. In fig. 7(c), for \( \gamma = 0.4 \), the reentrance temperature decreases and also a small disentangled region arises below the entangled region for
\( h/J \lesssim 0.6 \). In fig. 7(d), for \( \gamma = 0.5 \), the disentangled region increases and the reentrance temperature \( T_{\text{th}} \) for the right side curves still remains. In fig. 7(e), for \( \gamma = 0.53 \), the entangled region shrinks even more and we can observe two quite interesting reentrance temperatures. In fig. 7(f), increasing a little bit the XY-anisotropy \( \gamma = 0.535 \), we observe how the reentrance temperature changes. In fig. 7(g), despite being similar to the previous plot for \( \gamma = 0.56 \), there is illustrated how the reentrance temperature vanishes. In fig. 7(h), for \( \gamma = 0.585 \), the threshold temperature behaves similarly to the previous plot (g). In fig. 7(i), for \( \gamma = 0.6 \), the typical reentrance temperature vanishes, although there is a small isolated disentangled region.

**Conclusions.** In summary, we have presented a detailed study of the spin-(1/2) Ising-XYZ chain on a diamond structure, which is an exactly solvable model through the decoration transformation and transfer matrix approach. The phase diagram of the ground-state energy was discussed, displaying a frustrated region and the pairwise thermal entanglement between two Heisenberg spins. It is noteworthy that at zero temperature, there is non-zero concurrence in a vast region for the Hamiltonian parameters, only in the limiting case the entanglement vanishes for some regions, as in the case of the Ising-XXZ diamond chain [21]. Due to the XY-anisotropy Hamiltonian, we found a quite interesting behavior for the thermal entanglement, e.g., the presence of more than one threshold temperature; this phenomenon is influenced by the XY-anisotropy (\( \gamma \)), which means that the entanglement vanishes at threshold temperature and for a higher temperature the thermal entanglement arises again, and finally, for an even higher temperature, the entanglement vanishes definitively. It is worth emphasizing that a similar result was discussed in refs. [18,30], but for only two qubits, and here we propose it for an infinite diamond chain with alternating Ising-Heisenberg spins.

**References**

[1] **O’Connor K. M. and Wootters W. K., Phys. Rev. A, 63** (2001) 052302; Sun Y., Chen Y. and Chen H., Phys. Rev. A, 68 (2003) 044301.

[2] **Kamta G. L. and Starace A. F., Phys. Rev. Lett., 88** (2002) 107901.

[3] **Verstraete F., Martin-Delgado M. A. and Cirac J. I., Phys. Rev. Lett., 92** (2004) 087201.

[4] **Poppl M., Verstraete F., Martin-Delgado M. A. and Cirac J. I., Phys. Rev. A, 71** (2005) 042306.

[5] **Garcia-Ripoll J. I., Martin-Delgado M. A. and Cirac J. I., Phys. Rev. Lett., 93** (2004) 250405.

[6] **Honecker A., H. S., Peters R. and Ritcher J., J. Phys.: Condens. Matter, 23** (2011) 164211.

[7] **Canova L., Strecka J. and Jascur M., J. Phys.: Condens. Matter, 18** (2006) 4967.

[8] **Lisni B. M., Ukr. J. Phys., 56** (2011) 1237.

[9] **Rojas O., de Souza S. M., Ohanyan V. and Khursudiy M., Phys. Rev. B, 83** (2011) 094430.

[10] **Valverde J. S., Rojas O. and de Souza S. M., J. Phys.: Condens. Matter, 20** (2008) 345208; Rojas O. and de Souza S. M., Phys. Lett. A, 375 (2011) 1295.

[11] **Rule K. C. et al., Phys. Rev. Lett., 100** (2008) 117202.

[12] **Kikuchi H. et al., Phys. Rev. Lett., 94** (2005) 227201; Kikuchi H. et al., Prog. Theor. Phys. Suppl., 159 (2005) 1.

[13] **Ananikian N. S., Ananikyan L. N., Chakhmakhchyan L. A. and Rojas O., J. Phys.: Condens. Matter, 24** (2012) 256001.

[14] **Chakhmakhchyan L., Ananikian N., Ananikyan L. and Burdik C., J. Phys.: Conf. Series, 343** (2012) 012022.

[15] **Wang X., Phys. Rev. A, 66** (2002) 044305.

[16] **Arnesen M. C., Bose S. and Vedral V., Phys. Rev. Lett., 87** (2001) 017901.

[17] **Xiaoguang Wang, Phys. Rev. A, 66** (2002) 034302.

[18] **Rigolin G., Int. J. Quant. Inf., 2** (2004) 393.

[19] **Rojas M., de Souza S. M. and Rojas O., Phys. Rev. A, 89** (2014) 032336.

[20] **Ananikian N., Lazaryan H. and Nalbandyan M., Eur. Phys. J. B, 85** (2012) 223.

[21] **Rojas O., Rojas M., Ananikian N. and de Souza S. M., Phys. Rev. A, 86** (2012) 042330.

[22] **Szyozi L., Prog. Theor. Phys., 6** (1951) 341.

[23] **Fisher M., Phys. Rev., 113** (1959) 969.

[24] **Rojas O., Valverde J. S. and de Souza S. M., Physica A, 388** (2009) 1419; Rojas O. and de Souza S. M., J. Phys. A: Math. Theor., 44 (2011) 245001.

[25] **Strečka J., Phys. Lett. A, 374** (2010) 3718.

[26] **Baxter R. J., Exactly Solved Models in Statistical Mechanics (Academic Press, New York) 1982.

[27] **Wootters W. K., Phys. Rev. Lett., 80** (1998) 2245.

[28] **Bukman D. J., An G. and Van Leeuwen J. M. J., Phys. Rev. B, 43** (1991) 13532.

[29] **Amico L., Osterloh A., Plastina F., Fazio R. and Palma G. M., Phys. Rev. A, 69** (2004) 022304.

[30] **Zhou L., Song H. S., Guo Y. Q. and Li C., Phys. Rev. A, 68** (2003) 024301.

[31] **Katayama S., Hirokawa Y. and Tanaka T., Macro molecules, 17** (1984) 2641; Hirotsu S., Hirokawa Y. and Tanaka T., J. Chem. Phys., 87 (1987) 1392.

[32] **Berker A. N. and Walker J. S., Phys. Rev. Lett., 47** (1981) 1409.

[33] **Levitski R. R., Zachek I. R., Moina A. P. and Andrusyk A. Ya., Condens. Matter Phys., 7** (2004) 111.

[34] **Han S. G., Park S.-C. and Kim B. J., Phys. Rev. E, 79** (2009) 066114.

[35] **Morrison S. and Parkins A. S., Phys. Rev. Lett., 100** (2008) 044043.