A NEW PROOF OF ROTH'S THEOREM ON ARITHMETIC PROGRESSIONS

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Abstract. We present a proof of Roth’s theorem that follows a slightly different structure to the usual proofs, in that there is not much iteration. Although our proof works using a type of density increment argument (which is typical of most proofs of Roth’s theorem), we do not pass to a progression related to the large Fourier coefficients of our set (as most other proofs of Roth do). Furthermore, in our proof, the density increment is achieved through an application of a quantitative version of Varnavides’s theorem, which is perhaps unexpected.

1. Introduction

Given an integer \( N \geq 1 \), let \( r_3(N) \) denote the size of any largest subset \( S \) of \( [N] := \{1, ..., N\} \) for which there are no solutions to

\[
x + y = 2z, \ x, y, z \in S, \ x \neq y;
\]

in other words, \( S \) has no non-trivial three term arithmetic progressions.

In the present paper we give a proof of Roth’s theorem \([4]\) that, although iterative, uses a more benign type of iteration than most proofs.

Theorem 1.1. We have that \( r_3(N) = o(N) \).

Roughly, we achieve this by showing that \( r_3(N)/N \) is asymptotically decreasing. We will do this by starting with a set \( S \subseteq [N], |S| = r_3(N) \), such that \( S \) has no three term progressions, and then convolving it with a measure on a carefully chosen three term arithmetic progression \( \{0, x, 2x\} \). The set \( T \) where this convolution is positive will be significantly larger than \( S \), yet will have very few three term arithmetic progressions. We will thus be able to deduce, using a quantitative version of a theorem of Varnavides \([8]\), that \( r_3(N)/N \) is much smaller than \( r_3(M)/M \) for some \( M = (\log N)^{1/16-o(1)} \). It is easy to see that this implies that \( r_3(N) = o(N) \). Alas, the upper bound that our method will produce for \( r_3(N) \) is quite poor, and is of the quality \( r_3(N) \ll N/\log^*(N) \), which nonetheless is the sort of bounds produced by the “triangle-deletion” proof of Roth’s theorem \([5]\).

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Many of the other proofs of Roth’s theorem, in particular [1], [3], [6], and [7], make use of similar convolution ideas\footnote{In the case of Szemerédi’s argument [7], the convolution is disguised, but after the dust has settled, one will see that he convolves with a measure on a very long arithmetic progression. In the case of [3] and [6], the arguments can be directly expressed in terms of convolution with a measure supported on a long arithmetic progression.}; however, none of these methods convolve with such a short progression as ours (three terms only), and none use the result of Varnavides to achieve a density increment. Furthermore, it seems that our method can be generalized to any context where: (1) the number of three term progressions in a set depends only on a small number of Fourier coefficients; and, (2) one has a quantitative version of Varnavides’s theorem. This might prove especially useful in certain contexts, because the particular sets on which our method achieves a density increment (via Varnavides) are unrelated to the particular additive characters where the Fourier transform of $S$ is “large”\footnote{That is, the progression to which we pass with each iteration is unrelated to the additive characters where $\hat{1}_S$ is “large”}.

## 2. Notation

We shall require a modicum of notation: given a function $f : \mathbb{F}_p \to [0, 1]$, we write

$$\Lambda(f) := \mathbb{E}_{x,d \in \mathbb{F}_p} f(x)f(x + d)f(x + 2d)$$

(where $\mathbb{E}$ represents an averaged sum; thus the $\mathbb{E}$ above represents $p^{-2} \sum$). Thus $\Lambda$ gives an average of $f$ over three term arithmetic progressions; when $f$ is the indicator function of a set $A$, this is just the number of progressions in $A$ divided by $p^2$. We shall make use of the Fourier transform $\hat{f} : \mathbb{F}_p \to \mathbb{C}$ of a function $f$, given by

$$\hat{f}(r) := \mathbb{E}_{x \in \mathbb{F}_p} f(x)e^{2\pi irx/p},$$

as well as the easily-verified Parseval’s identity

$$\sum_{r \in \mathbb{F}_p} |\hat{f}(r)|^2 = \mathbb{E}_x |f(x)|^2.$$

It is also easy to check that

$$\Lambda(f) = \sum_{r \in \mathbb{F}_p} \hat{f}(r)^2 \hat{f}(-2r). \quad (2.1)$$

Given a set $T \subseteq \mathbb{F}_p$, we shall furthermore use the notation

$$\Lambda(T) := \Lambda(1_T).$$

Finally, the notation $\|t\|_T$ will be used to denote the distance from $t$ to the nearest integer.
3. Proof of Theorem 1.1

Let

$$\kappa := \limsup_{N \to \infty} \frac{r_3(N)}{N}.$$  

We shall show that \( \kappa = 0 \), which will prove the theorem.

Let \( N \geq 2 \) be an integer, and then let \( p \) be a prime number satisfying

$$2N < p < 4N.$$ 

The fact that such a \( p \) exists is of course the content of Bertrand’s postulate.

Let \( S \subset [N] \) be a set free of three term progressions with \( |S| = r_3(N) \). Thinking of \( S \) as a subset of \( \mathbb{F}_p \) in the obvious way, we shall write \( f = 1_S : \mathbb{F}_p \to \{0, 1\} \) for the indicator function of \( S \). Let

$$R := \{ r \in \mathbb{F}_p : |\hat{f}(r)| \geq (2 \log \log p / \log p)^{1/2} \}.$$ 

By Parseval’s identity, this set of large Fourier coefficients cannot be too big; certainly,

$$|R| \leq \log p / 2 \log \log p.$$ 

We may therefore dilate these points of \( R \) to be contained in a short part of \( \mathbb{F}_p \). Indeed, by Dirichlet’s box principle there is an integer dilate \( x \) satisfying

$$0 < x < p^{1-(1/(|R|+1))} \leq p / \log p,$$

such that for all \( r \in R \) we have

$$\|xr/p\|_\tau \leq p^{-1/(|R|+1)} \leq 1 / \log p.$$ 

(3.1)

Taking such an \( x \), define

$$B := \{0, x, 2x\},$$ 

and define \( h \) to be the normalised indicator function for \( B \), given by

$$h(n) := p1_B(n)/3.$$ 

Then convolve \( f \) with \( h \) to produce the new function

$$g(n) := (f * h)(n) = (f(n) + f(n - x) + f(n - 2x))/3.$$ 

Since

$$\hat{f}(r) - \hat{g}(r) = \hat{f}(r)(1 - \hat{h}(r)),$$

it is easy to check using (3.1) that for all \( r \in \mathbb{F}_p \)

$$|\hat{f}(r) - \hat{g}(r)| \ll (\log \log p / \log p)^{1/2}.$$ 

From this, along with the Cauchy-Schwarz inequality, Parseval’s identity, and equation (2.1), one can quickly deduce that

$$|\Lambda(f) - \Lambda(g)| \ll (\log \log p / \log p)^{1/2},$$ 

and therefore since \( \Lambda(f) \ll 1/p \) (because \( S \) is free of three term arithmetic progressions), we deduce

$$\Lambda(g) \ll (\log \log p / \log p)^{1/2}.$$ 

(3.2)
Define 
\[ T := \{ n \in \mathbb{F}_p : g(n) > 0 \}, \]
and note that from (3.2), along with the obvious fact that \( \Lambda(T) \ll \Lambda(g) \), we have
\[ \Lambda(T) \ll (\log \log p / \log p)^{1/2}. \] (3.3)

Furthermore, since \( S \) is free of three term progressions even in \( \mathbb{F}_p \), we must have that \( g(n) \leq 2/3 \) for all \( n \in \mathbb{F}_p \). Thus \( 1_T(n) \geq 3g(n)/2 \) for all \( n \), immediately implying that \( |T| \geq 3|S|/2 \). The set \( T \) would thus serve our purposes if it was not for the fact that it is not necessarily contained in \( [N] \). However, since \( x \leq p/\log p \), we certainly have the inclusion \( T \subset [N + 2p/\log p] \). So, if we let \( T' \) be those elements of \( T \) lying in \( [N] \), then
\[ |T'| = |T| - O(N/\log N) \]
and \( \Lambda(T') \leq \Lambda(T) \).

Hence, for \( N \) large enough,
\[ |T'| \geq 4|S|/3 \]
(unless of course \( r_3(N) = O(N/\log N) \), but then we would be happy anyway).

We have now created a set \( T' \), significantly larger than \( S \), but with only a few more three term progressions. The following lemma, a quantitative version of Varnavides’s theorem, will help us make use of this information. The notation \( T_3(X) \) denotes the number of three term progressions \( a, a + d, a + 2d \) with \( d \geq 1 \) in a set \( X \) of integers.

**Lemma 3.1.** For any \( 1 \leq M \leq N \), and for any set \( A \subseteq [N] \), we have
\[ T_3(A) \geq \left( \frac{|A|/N - (r_3(M) + 1)/M}{M^4} \right) N^2. \]

Before we prove this, let us see how we can use it to finish the proof of our main theorem. Set \( M := \lfloor (\log p / \log \log p)^{1/16} \rfloor \) and apply the lemma to our set \( T' \) to obtain the estimate
\[ \Lambda(T') \gg \frac{4|S|/3N - (r_3(M) + 1)/M}{M^4}. \]
Comparing this to (3.3) (recalling that \( \Lambda(T') \leq \Lambda(T) \)), we conclude that
\[ r_3(N)/N = |S|/N \leq 3r_3(M)/4M + O((\log \log N / \log N)^{1/4}). \]
Thus \( r_3(N)/N \) is asymptotically decreasing to 0, whence \( \kappa = 0 \).

**Proof of Lemma 3.1.** The result will follow from an averaging procedure essentially contained in [2]. We include the proof here since our formulation is slightly different: we are working over \( [N] \) rather than \( \mathbb{F}_p \), and so we have to take into account the inhomogeneity of \( [N] \).

Let \( k \) be a positive integer. Let \( \mathcal{B} \) denote the collection of length \( M \) arithmetic progressions contained in \( [N] \) with common difference at most \( k \), and let \( \mathcal{B}_d \) denote the subcollection consisting of such arithmetic progressions with common difference \( d \). Throughout this proof we restrict ourselves to progressions with positive common difference.

We first claim that any 3AP (three term progression) in \( [N] \) can occur in at most \( M^2/4 \) progressions in \( \mathcal{B} \). To see this, note that if a 3AP has common difference \( d \), then it can occur in at most \( M - 2 \) progressions of length \( M \) with common difference \( d \). Similarly,
the 3AP can occur in at most $M - 2d/n$ $M$-APs with difference $n$ provided $n$ divides $d$ and $n \geq 2d/(M-1)$, and in no other $M$-APs. Thus the 3AP can occur in no more than
\[
\sum_{1 \leq m \leq (M-1)/2} (M - 2m) \leq M^2/4
\]
members of $B$, as claimed. It follows immediately that
\[
T_3(A) \geq \frac{4}{M^2} \sum_{B \in B} T_3(A \cap B). \tag{3.4}
\]
Now if $B$ is an arithmetic progression of length $M$ and $|A \cap B| > r_3(M)$, then by definition we have $T_3(A \cap B) > 1$. In view of (3.4) our aim shall therefore be to estimate the number of such sets $B$; we shall do this by looking at progressions of fixed common differences. Indeed, for a fixed common difference $d$, every element in the interval $I_d := [(M-1)d + 1, N - (M-1)d]$ is contained in precisely $M$ progressions in $B_d$, and so
\[
\sum_{B \in B} |A \cap B| = \sum_{d \leq k} \sum_{a \in A} \sum_{B \in B_d} 1_B(a) \geq M \sum_{d \leq k} |A \cap I_d|.
\]
Since $|A \cap I_d| \geq |A| - 2(M-1)d$, this quantity is at least $Mk(|A| - 2Mk)$. Now let $C \subset B$ be the set of progressions $B$ for which $|A \cap B| > r_3(M)$. We then have
\[
\sum_{B \in B} |A \cap B| \leq M|C| + r_3(M)|B \setminus C|,
\]
from which it follows that
\[
|C| \geq k(|A| - 2Mk) - |B|r_3(M)/M.
\]
Since $|B_d| = N - (M-1)d$ for each $d$, the total number of progressions $|B|$ is at most $Nk$. Choosing $k = \lceil N/2M^2 \rceil$ we conclude that there must be at least
\[
|C| \geq \left( \frac{|A|/N - r_3(M)/M - 1/M}{4M^2} \right) N^2
\]
sets $B$ for which $|A \cap B| > r_3(M)$. The result thus follows from (3.4). \qed

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