Defining relations of the tame automorphism group of polynomial algebras in three variables

U. U. Umirbaev  
Eurasian National University,  
Astana, 010008, Kazakhstan  
e-mail: umirbaev@yahoo.com

Abstract

We describe a set of defining relations of the tame automorphism group $TA_3(F)$ of the polynomial algebra $F[x_1, x_2, x_3]$ in variables $x_1, x_2, x_3$ over an arbitrary field $F$ of characteristic 0.

Mathematics Subject Classification (2000): Primary 14R10, 14J50, 14E07; Secondary 14H37, 14R15.

Key words: polynomial algebras, automorphism groups, defining relations.

1 Introduction

Let $A_n = F[x_1, x_2, \ldots, x_n]$ be the polynomial algebra in the variables $x_1, x_2, \ldots, x_n$ over a field $F$, and let $GA_n(F) = Aut A_n$ be the automorphism group of $A_n$. Let $\phi = (f_1, f_2, \ldots, f_n)$ denote an automorphism $\phi$ of $A_n$ such that $\phi(x_i) = f_i$, $1 \leq i \leq n$. An automorphism

$$\sigma(i, \alpha, f) = (x_1, \ldots, x_{i-1}, \alpha x_i + f, x_{i+1}, \ldots, x_n),$$

where $0 \neq \alpha \in F$, $f \in F[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$, is called elementary. The subgroup $TA_n(F)$ of $GA_n(F)$ generated by all elementary automorphisms is called the tame automorphism group, and the elements of this subgroup are called the tame automorphisms of $A_n$. Nontame automorphisms of $A_n$ are called wild.

It is well known [4, 8, 9, 10] that the automorphisms of polynomial algebras and free associative algebras in two variables are tame. It was recently proved in [13, 14, 15, 16] that the well-known Nagata automorphism (see [12])

$$\sigma = (x + (x^2 - yz)z, y + 2(x^2 - yz)x + (x^2 - yz)^2z, z)$$

1
of the polynomial algebra $F[x, y, z]$ over a field $F$ of characteristic 0 is wild.

It is well known (see, for example [2]) that the groups of automorphisms of polynomial algebras and free associative algebras in two variables are isomorphic and have a nice representation as a free product of groups. Similar results are well known for two dimensional Cremona groups [6, 7, 19]. In combinatorial group theory (see, for example [18]) there are several well-known descriptions of the group of automorphisms of free groups by generators and defining relations. Main part of the investigations of the automorphism groups in the case of free linear algebras consist in finding a system of generators. In 1968 P.Cohn [1] proved that the automorphisms of free Lie algebras with a finite set of generators are tame.

In this paper we describe a set of defining relations of the tame automorphism group $TA_3(F)$ of the polynomial algebra $A = F[x_1, x_2, x_3]$ over a field $F$ of characteristic 0. The discovered set of defining relations (18)–(20) is new and very natural for free linear algebras. This opens the way to the representation theory of the automorphism groups of free associative algebras, free Lie algebras and the others. Using the obtained presentation of $TA_3(F)$ we proved (see [17]) that the well-known Anick automorphism of the free associative algebra in three variables is wild.

The paper is organized as follows. In Section 2 we give some preliminaries from [14, 15]. It was proved in [15] that every tame automorphism of the polynomial algebra $A$ of degree greater than 3 admits either an elementary reduction or a reduction of types I–IV. In this paper we use some extension of the definition of automorphisms admitting a reduction of type IV in the sense of [15]. Furthermore, in Section 3 we define essential reductions of the tame automorphisms and study the uniqueness of these reductions. In particular, it is proved that every tame automorphism admits only one type of the essential reductions. A set of defining relations of the tame automorphism group $TA_3(F)$ is given in Section 4.

The results of this paper were first published in [17].

2 Reducions of tame automorphisms

Let $F$ be an arbitrary field of characteristic 0, and let $A = F[x_1, x_2, x_3]$ be the polynomial algebra in the variables $x_1, x_2, x_3$ over $F$. As in [14], we will identify the algebra $A$ with the corresponding subspace of the free Poisson algebra $P = P\langle x_1, x_2, x_3 \rangle$. The highest homogeneous part $\overline{f}$ and the degree $\deg f$ can be defined in an ordinary way. Note that

$$\overline{fg} = \overline{f} \overline{g}, \quad \deg(fg) = \deg f + \deg g, \quad \deg[f, g] \leq \deg f + \deg g.$$

If $f_1, f_2, \ldots, f_k \in A$, then we denote by $\langle f_1, f_2, \ldots, f_k \rangle$ the subalgebra of $A$ generated by these elements.

We will often use the terminology and results of [15]. Therefore, let us make an agreement that Corollary 2A, Lemma 4A, Proposition 5A, and Theorem 7A mean Corollary 2, Lemma 4, Proposition 5, and Theorem 7 of [15], respectively.
Let us remind now the more necessary definitions and statements from [14, 15].

**Lemma 2.1** [14] Let \( f, g, h \in A \). Then the following statements are true:

1. \([f, g] = 0\) if and only if \( f \) and \( g \) are algebraically dependent.
2. Suppose that \( f, g, h \notin F \) and \( m = \deg[f, g] + \deg h, n = \deg[g, h] + \deg f, k = \deg[h, f] + \deg g \). Then \( m \leq \max(n, k) \). If \( n \neq k \), then \( m = \max(n, k) \).

The following statements are well known (see [2]):

(F1) If \( a \) and \( b \) are homogeneous algebraically dependent elements of the algebra \( A \), then there exists an element \( z \in A \) such that \( a = \alpha z^n, b = \beta z^m \) and \( \alpha, \beta \in F \). The subalgebra \( \langle a, b \rangle \) is single generated if and only if \( m|n \) or \( n|m \).

(F2) Let \( f, g \in A \) be such that \( f \) and \( g \) are algebraically independent. If \( h \in \langle f, g \rangle \), then \( h \in \langle f, g \rangle \).

A pair of elements \( f, g \) of the algebra \( A \) is called **reduced** if \( f \notin \langle g \rangle \) and \( g \notin \langle f \rangle \). A **reduced pair of algebraically independent elements** \( f, g \in A \) is called **\(*\)-reduced** if \( f \) and \( g \) are algebraically dependent.

Consider a \(*\)-reduced pair of elements \( f, g \) of the algebra \( A \) and let \( n = \deg f < m = \deg g \). Put \( p = \frac{n}{(n, m)}, s = \frac{m}{(n, m)} \); and

\[
N = N(f, g) = \frac{mn}{(m, n)} - m - n + \deg[f, g] = mp - m - n + \deg[f, g],
\]

where \((n, m)\) is the greatest common divisor of \( n \) and \( m \). Note that \((p, s) = 1\), and since \( f \) and \( g \) are algebraically dependent, there exists an element \( a \in A \) such that \( \overline{f} = \beta a^p \) and \( \overline{g} = \gamma a^s \). Sometimes we will call a \(*\)-reduced pair of elements \( f, g \) also a **\( p \)-reduced** pair. Assume that \( G(x, y) \in F[x, y] \). It was proved in [14] that if \( \deg_y(G(x, y)) = pq + r, 0 \leq r < p \), then

\[
\deg(G(f, g)) \geq qN + mr, \tag{1}
\]

and if \( \deg_x(G(x, y)) = sq_1 + r_1, 0 \leq r_1 < s \), then

\[
\deg(G(f, g)) \geq q_1N + nr_1. \tag{2}
\]

**Corollary 2.1** [15] Assume that \( G(x, y) \in F[x, y] \) and \( h = G(f, g) \). Consider the following conditions:

(i) \( \deg h < N(f, g) \);

(ii) \( \deg_y(G(x, y)) < p \);

(iii) \( h = \sum_{i, j} \alpha_{ij} f^i g^j \), where \( \alpha_{ij} \in F \) and \( in + jm \leq \deg h \) for all \( i, j \);

(iv) \( \overline{h} \in \langle \overline{f}, \overline{g} \rangle \).
Then (i) ⇒ (ii) ⇒ (iii) ⇒ (iv).

Lemma 2.2 [15] There exists a polynomial \( w(x, y) \in F[x, y] \) of the type
\[
w(x, y) = y^p - \alpha x^s - \sum \alpha_{ij} x^i y^j, \quad ni + mj < mp,
\]
which satisfies the following conditions:
1. \( \deg w(f, g) < pm \);
2. \( w(f, g) \notin (f, g) \).

A polynomial \( w(x, y) \) satisfying the conditions of Lemma 2.2 is called a derivative polynomial of the \(*\)-reduced pair \( f, g \).

Corollary 2.2 [15] If \( h \in \langle f, g \rangle \setminus F \) and \( \deg h < n \), then \( h = \lambda w(f, g), \; 0 \neq \lambda \in F \), where \( w(x, y) \) is a derivative polynomial of the pair \( f, g \).

Corollary 2.3 [15] If \( w(x, y) \) is a derivative polynomial of the pair \( f, g \), then
\[
\begin{align*}
\deg \left( \frac{\partial w}{\partial x} (f, g) \right) &= n(s - 1), \\
\deg \left( \frac{\partial w}{\partial y} (f, g) \right) &= m(p - 1).
\end{align*}
\]

Let \( \theta = (f_1, f_2, f_3) \) be an arbitrary automorphism of the algebra \( A \). The number
\[
\deg \theta = \deg f_1 + \deg f_2 + \deg f_3
\]
is called the degree of \( \theta \).

Recall that an elementary transformation of a triple \( (f_1, f_2, f_3) \) is, by definition, a transformation that changes only one element \( f_i \) to an element of the form \( \alpha f_i + g \), where \( 0 \neq \alpha \in F \) and \( g \in \langle \{f_j|j \neq i\} \rangle \). The notation
\[
\theta \rightarrow \tau
\]
means that the triple \( \tau \) is obtained from \( \theta \) by a single elementary transformation. An automorphism \( \theta \) is called elementarily reducible or admits an elementary reduction if there exists \( \tau \in GA_3(F) \) such that \( \theta \rightarrow \tau \) and \( \deg \tau < \deg \theta \). The element \( f_i \) of the automorphism \( \theta \) which was changed in \( \tau \) to an element of less degree is called reducible and we will say also that \( f_i \) is reduced in \( \theta \) by the automorphism \( \tau \).

Definition 2.1 [15] Let \( \theta = (f_1, f_2, f_3) \) be an automorphism of \( A \) such that \( \deg f_1 = 2n \), \( \deg f_2 = ns \), \( s \geq 3 \) is an odd number, \( 2n < \deg f_3 \leq ns \), and \( f_3 \notin (f_1, f_2) \). Suppose that there exists \( 0 \neq \beta \in F \) such that the elements \( g_1 = f_1 \) and \( g_2 = f_2 - \beta f_3 \) satisfy the conditions:
(i) $g_1,g_2$ is a 2-reduced pair and $\deg g_1 = \deg f_1$, $\deg g_2 = \deg f_2$;

(ii) the element $f_3$ of the automorphism $(g_1,g_2,f_3)$ is reduced by an automorphism $(g_1,g_2,g_3)$ with the condition $\deg [g_1,g_3] < \deg g_2 + \deg [g_1,g_2]$.

Then we will say that $\theta$ admits a reduction of type I and the automorphism $(g_1,g_2,g_3)$ will be called a reduction of $\theta$ of type I with an active element $f_3$.

**Definition 2.2** [15] Let $\theta = (f_1,f_2,f_3)$ be an automorphism of $A$ such that $\overline{f}_1$ and $\overline{f}_3$ are linearly independent, $\deg f_1 = 2n$, $\deg f_2 = 3n$, and $\frac{3n}{2} < \deg f_3 \leq 2n$. Suppose that there exist $\alpha, \beta \in F$, where $(\alpha,\beta) \neq 0$, such that the elements $g_1 = f_1 - \alpha f_3$ and $g_2 = f_2 - \beta f_3$ satisfy the conditions (i) and (ii) of Definition 2.1. Then we will say that $\theta$ admits a reduction of type II and the automorphism $(g_1,g_2,g_3)$ will be called a reduction of $\theta$ of type II with an active element $f_3$.

**Definition 2.3** [15] Let $\theta = (f_1,f_2,f_3)$ be an automorphism of $A$ such that $\deg f_1 = 2n$, and either $\deg f_2 = 3n$, $n < \deg f_3 \leq \frac{3n}{2}$, or $\frac{5n}{3} < \deg f_2 \leq 3n$, $\deg f_3 = \frac{3n}{2}$. Suppose that there exist $\alpha, \beta_1, \beta_2 \in F$, where $(\alpha,\beta_1,\beta_2) \neq 0$, such that the elements $g_1 = f_1 - \alpha f_3$ and $g_2 = f_2 - \beta_1 f_3 - \beta_2 f_3^2$ satisfy the conditions:

(i) $g_1,g_2$ is a 2-reduced pair and $\deg g_1 = 2n$, $\deg g_2 = 3n$;

(ii) the element $f_3$ of the automorphism $(g_1,g_2,f_3)$ is reduced by an automorphism $(g_1,g_2,g_3)$ with the condition $\deg g_3 < n + \deg [g_1,g_2]$.

Then we will say that $\theta$ admits a reduction of type III and the automorphism $(g_1,g_2,g_3)$ will be called a reduction of $\theta$ of type III with an active element $f_3$.

The next remark can be extracted from the proofs of Propositions 1A, 2A, and 3A.

**Remark 2.1**

(i) $\deg [g_1,g_2] \leq 2n$ in the conditions of Definition 2.1;

(ii) $\deg [g_1,g_2] \leq n$ in the conditions of Definition 2.2;

(iii) $\deg [g_1,g_2] \leq \frac{n}{2}$ in the conditions of Definition 2.3.

**Lemma 2.3** Let $\theta = (f_1,f_2,f_3)$ satisfy the conditions of Definition 2.3. Then the following statements are true:

(i) if $a \in < f_2,f_3 >$ and $\deg a \leq \frac{5n}{2}$, then $a \in < f_3 >$;

(ii) if $a \in < f_1,f_3 >$ and $\deg a \leq \frac{7n}{2}$, then $\pi \in < \overline{f}_1,\overline{f}_3 >$;

(iii) if $a \in < f_1,f_2 >$ and $\deg a < 2n$, then $a = \delta \in F$. 

5
We omit the proof of this lemma, since later in Lemma 2.4 we consider analogous statements for a more complicated case.

The next definition is some extension of the definition of automorphisms admitting a reduction of type IV in the sense of [15].

**Definition 2.4** Let \( \theta = (f_1, f_2, f_3) \) be an automorphism of \( A \) such that \( \deg f_3 \leq \frac{3n}{2} \) and \( \deg \theta \leq \frac{13n}{2} \). Suppose that there exist \( \alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \beta_4 \in F \) such that the elements \( g_1 = f_1 - \alpha_1 f_3 - \alpha_2 f_3^2 \) and \( g_2 = f_2 - \beta_1 f_3 - \beta_2 f_3^2 - \beta_3 f_1 f_3 - \beta_4 f_3^3 \) satisfy the conditions:

(i) \( g_1, g_2 \) is a 2-reduced pair and \( \deg g_1 = 2n, \deg g_2 = 3n; \)

(ii) there exists an element \( g_3 \) of the form

\[
g_3 = f_3 - \gamma w(g_1, g_2), \quad 0 \neq \gamma \in F,
\]

where \( w(x, y) \) is a derivative polynomial of the 2-reduced pair \( g_1, g_2 \), such that

(a) \( \deg g_3 = \frac{3n}{2} \) and \( \deg [g_1, g_3] < 3n + \deg [g_1, g_2]; \)

(b) there exists \( 0 \neq \mu \in F \) such that \( \deg (g_2 - \mu g_3^2) \leq 2n. \)

Then we will say that \( \theta \) admits a reduction of type IV and the automorphism \( (g_1, g_2 - \mu g_3^2, g_3) \) will be called a reduction of \( \theta \) of type IV with an active element \( f_3. \)

We will also use Definitions 2.1–2.4, admitting a permutation of the components of \( (f_1, f_2, f_3). \)

It is difficult to find examples of automorphisms illustrating the definitions 2.1–2.4. An example of an automorphism admits a reduction of type I was constructed in [15, p. 204, Example 1]. At the moment we have no example of an automorphism admits a reduction of types II–IV and the corresponding question was also formulated in [15, p. 225, Problem 1].

**Proposition 2.1** Let \( \theta = (f_1, f_2, f_3) \) be an automorphism of \( A \) satisfying the conditions of Definition 2.4. Then the following statements are true:

(1) \( \deg [g_1, g_2] \leq \frac{n}{2}, \deg f_3 \geq n + \deg [g_1, g_2]; \)

(2) \( 2n \leq \deg f_1 < \frac{5n}{2}, \frac{5n}{2} + \deg [g_1, g_2] \leq \deg f_2 < \frac{7n}{2}; \)

(3) \( \deg [f_1, f_3] = \deg [g_1, f_3] > 3n, \deg [f_2, f_3] > 3n; \)

(4) \( \deg f_2 + \deg f_3 > 4n; \)

(5) if \( (\beta_3, \beta_4) = 0 \), then either \( \deg f_2 = 3n \) or \( \deg f_3 = \frac{3n}{2}; \)

(6) the coefficients \( \alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \beta_4 \) are uniquely defined;

(7) if \( (\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \beta_4) \neq 0 \), then \( \deg [f_1, f_2] > 3n. \)
Proof. Since \( \deg g_3, \deg f_3 \leq \frac{3n}{2} \), and \( \gamma \neq 0 \), Definition 2.4 gives \( \deg w(g_1, g_2) \leq \frac{3n}{2} \). By the definition of a derivative polynomial, applying Corollary 2.1 we get
\[
\frac{3n}{2} \geq \deg w(g_1, g_2) \geq N(g_1, g_2) = n + \deg [g_1, g_2].
\]

Consequently, \( \deg [g_1, g_2] \leq \frac{n}{2} \). According to Corollary 2.3 we have \( \deg \frac{\partial w}{\partial y}(g_1, g_2) = 3n \). By Definition 2.4, we have also
\[
\deg [g_1, g_3] < 3n + \deg [g_1, g_2] = \deg [g_1, g_2] \frac{\partial w}{\partial y}(g_1, g_2).
\]

Since
\[
[f_1, f_3] = [g_1, f_3] = [g_1, g_3] + [g_1, g_2] \frac{\partial w}{\partial y}(g_1, g_2),
\]
comparing the degrees of elements here we find that
\[
\deg [f_1, f_3] = \deg [g_1, f_3] = 3n + \deg [g_1, g_2].
\]
Since \( \deg g_1 = 2n \), it follows also that \( \deg g_3 \geq n + \deg [g_1, g_2] \). Consequently, \( \deg f_3 < \deg g_1 = 2n < \deg f_3^2 \), which gives \( \deg f_1 \geq 2n \).

Obviously,
\[
\deg [g_1, g_2] + \deg f_3 < \deg [g_1, f_3] + \deg g_2.
\]

Then by Lemma 2.1(2) we obtain
\[
\deg [g_2, f_3] + \deg g_1 = \deg [g_1, f_3] + \deg g_2,
\]
i.e.
\[
\deg [g_2, f_3] = 4n + \deg [g_1, g_2].
\]

Next we have
\[
\deg [g_2, f_3] < 4n + 2 \deg [g_1, g_2] \leq \deg [f_1, f_3] f_3.
\]

Since
\[
[f_2, f_3] = [g_2, f_3] + \beta_3 [f_1, f_3] f_3,
\]
comparing the degrees of elements we get
\[
\deg [f_2, f_3] \geq 4n + \deg [g_1, g_2].
\]

Then
\[
\deg f_2 \geq 4n + \deg [g_1, g_2] - \deg f_3 \geq \frac{5n}{2} + \deg [g_1, g_2].
\]

7
Assume that \((\beta_3, \beta_4) \neq 0\). Then it is easy to check that \(\deg (\beta_3 f_1 + \beta_4 f_2^2) f_3 > 3n\). Since \(\deg f_3 < \deg f_2^2 \leq \deg g_2 = 3n\), we have \(\deg f_2 > 3n\). Now if \((\beta_3, \beta_4) = 0\), then we have \(\deg f_2 \leq 3n\). And the inequality \(\deg f_2 < 3n\) is possible only if \(\deg f_3 = \frac{3n}{2}\). It remains to note that the inequality \(\deg f_2 + \deg f_3 > 4n\) is fulfilled in both cases. Since \(\deg \theta \leq \frac{13n}{2}\), this gives also \(\deg f_1 < \frac{5n}{2}\). Note that the inequality \(\deg f_1 + \deg f_3 > 3n\) implies that \(\deg f_2 < \frac{7n}{2}\).

We have

\[
\begin{align*}
&f_1 = g_1 + \alpha_1 f_3 + \alpha_2 f_3^2, \\
&f_2 = g_2 + \beta_1 f_3 + \beta_2 f_3^2 + \beta_3 f_1 f_3 + \beta_4 f_3^3 \\
&= g_2 + \beta_1 f_3 + (\beta_2 + \alpha_1 \beta_3) f_3^2 + \beta_3 g_1 f_3 + (\beta_4 + \alpha_2 \beta_3) f_3^3.
\end{align*}
\]

Consequently,

\[
[f_1, f_2] = [g_1, g_2] + \beta_1 [g_1, f_3] + \alpha_1 [f_3, g_2] + (2\beta_2 + \alpha_1 \beta_3) [g_1, f_3] f_3 \\
+ \beta_3 g_1 [g_1, f_3] + 2\alpha_2 f_3 [f_3, g_2] + (3\beta_4 + \alpha_2 \beta_3) [g_1, f_3] f_3^2.
\]

Note that

\[
\begin{align*}
\deg [g_1, g_2] < 3n &< \deg [g_1, f_3] < \deg [f_3, g_2] < \deg [g_1, f_3] f_3 \\
&< \deg g_1 [g_1, f_3] < \deg [f_3, g_2] f_3 < \deg [g_1, f_3] f_3^2.
\end{align*}
\]

Consequently, \(3\beta_4 + \alpha_2 \beta_3 = 0\) or

\[
[f_1, f_2] = (3\beta_4 + \alpha_2 \beta_3) [g_1, f_3] f_3^2 = (3\beta_4 + \alpha_2 \beta_3) [f_1, f_3] f_3^2,
\]

i.e. \(3\beta_4 + \alpha_2 \beta_3\) is uniquely defined. Now as in the proofs of Propositions 1A, 2A, and 3A we can easily deduce the statement (6) of Proposition 2.1. If \(\deg [f_1, f_2] \leq 3n\), then obviously

\[
\beta_1 = \alpha_1 = 2\beta_2 + \alpha_1 \beta_3 = \beta_3 = 2\alpha_2 = 3\beta_4 + \alpha_2 \beta_3 = 0,
\]

i.e. \((\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \beta_4) = 0\). □

**Lemma 2.4** Let \(\theta = (f_1, f_2, f_3)\) satisfy the conditions of Definition 2.4. Then the following statements are true:

(i) if \(a \not< f_2, f_3\) and \(\deg a \leq \frac{5n}{2}\), then \(a \not< f_3\);

(ii) if \(a \not< f_1, f_3\) and \(\deg a \leq \frac{7n}{2}\), then \(a \not< f_1, f_3\);

(iii) if \(a \not< f_1, f_2\) and \(\deg a < 2n\), then we have \(a = \delta \in F\) if \((\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \beta_4) \neq 0\), while \(a = \delta w(g_1, g_2) + \lambda\) otherwise, where \(w(x, y)\) is a derivative polynomial of the 2-reduced pair \(g_1, g_2\).
Proposition 2.1 we have \( \deg f \geq\), the subalgebra \(< f_2, f_3 > \) does not change if we replace \( f_2 \) by \( g_2 + \beta_3 g_1 f_3 \). So, we can assume that \( f_2 = g_2 + \beta_3 g_1 f_3 \). If \( \mathcal{T}_2 \) and \( \mathcal{T}_3 \) are algebraically independent, then by (F2) we have \( \mu \in < \mathcal{T}_2, \mathcal{T}_3 > \). By Proposition 2.1, we have \( \deg f_2 > \frac{5n}{2} \). Consequently, \( a \in < f_3 > \). Suppose that \( \mathcal{T}_2 \) and \( \mathcal{T}_3 \) are algebraically dependent. If \( \beta_3 \neq 0 \), then \( f_2, f_3 \) is a \(*\)-reduced pair. If \( \beta_3 = 0 \), then either \( f_2, f_3 \) is a \(*\)-reduced pair, or there exists \( \mu \in F \) such that \( \mathcal{T}_2 = \mu \mathcal{T}_3^2 \). In the last case we can assume that \( f_2 = g_2 - \mu f_3^2 \). Then again \( f_2, f_3 \) is a \(*\)-reduced pair. Since

\[
N(f_2, f_3) > \deg [f_2, f_3] > 4n > \frac{5n}{2} \geq \deg a,
\]

the inequality \( \deg a \geq N(f_2, f_3) \) leads to a contradiction. Consequently, \( \deg a < N(f_2, f_3) \). Since \( \deg f_2 > \frac{5n}{2} \geq \deg a \), by Corollary 2.1 we get again \( a \in < f_3 > \).

Suppose that \( a \in < f_1, f_3 > \). As above we can assume that \( f_1 = g_1 \). If \( \mathcal{T}_1 \) and \( \mathcal{T}_3 \) are algebraically dependent, then \( g_1, f_3 \) is a \( p \)-reduced pair. Consider the case when

\[
\frac{7n}{2} \geq \deg a \geq N(g_1, f_3) = (p - 1) \deg g_1 - \deg f_3 + \deg [g_1, f_3].
\]

If \( p \geq 3 \), then with regard to Proposition 2.1 we have \( N(g_1, f_3) > 5n \). If \( p = 2 \), then this is possible only if

\[
\deg g_1 = 2n = 3k, \quad \deg f_3 = 2k \leq \frac{3n}{2}.
\]

Consequently, repeatedly applying Proposition 2.1 gives

\[
N(g_1, f_3) = k + \deg [g_1, f_3] > \frac{2n}{3} + 3n = \frac{11n}{3} > \frac{7n}{2}.
\]

Thus \( \deg a < N(g_1, f_3) \). Then Corollary 2.1 gives \( \overline{a} \in < \overline{g_1}, \overline{f_3} > \).

Now suppose that \( a \in < f_1, f_2 > \) and \( \deg a < 2n \). If \( (\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \beta_4) \neq 0 \), then by Proposition 2.1 we have \( \deg [f_1, f_2] > 3n \). By Corollary 2.1, from here we deduce that \( \mathcal{T}_1 \) and \( \mathcal{T}_2 \) are algebraically independent, then \( a \in < \mathcal{T}_1, \mathcal{T}_2 > \). Since \( \deg f_1, \deg f_2 \geq 2n \), it follows that \( a = \delta \in F \). If \( (\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \beta_4) = 0 \), then Corollary 2.2 gives the statement (iii) of the lemma. \( \square \)

It is easy to deduce from Proposition 2.1 that if \( (\alpha_2, \beta_3, \beta_4) = 0 \), then Definition 2.4 gives exactly the automorphisms admitting a reduction of type IV in the sense of [15].

**Theorem 2.1** [15] Let \( \theta = (f_1, f_2, f_3) \) be a tame automorphism of the polynomial algebra \( A = F[x_1, x_2, x_3] \) over a field \( F \) of characteristic 0. If \( \deg \theta > 3 \), then \( \theta \) admits either an elementary reduction or a reduction of one of the types I–IV.

Further we need the following proposition.

**Proposition 2.2** Let \( (f_1, f_2, f_3) \) be a tame automorphism of the algebra \( A \) satisfying the following conditions:
(i) $f_1, f_2$ is a 2-reduced pair and $\deg f_1 = 2n$, $\deg f_2 = sn$, where $s \geq 3$ is an odd number;

(ii) $\deg f_3 = m < sn$ and $f_3$ is an irreducible element of the automorphism $(f_1, f_2, f_3)$.

Then one of the following statements is true:

(1) $m < n(s - 2) + \deg[f_1, f_2]$;

(2) $(f_1, f_2, f_3)$ admits a reduction of type IV with the active element $f_3$, and the coefficients $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \beta_4$ of Definition 2.4 are equal to 0;

(3) $\deg[f_1, f_3] < 3n + \deg[f_1, f_2]$ and there exists $\mu \in F$ such that $\deg(f_2 - \mu f_3^2) \leq 2n$.

Proof. The conditions of this proposition coincide with the conditions of Propositions 4A and 5A if we take into account the fact that the tame automorphisms of the algebra $A$ are already simple and every elementarily reducible element is simple reducible in the sense of [15]. It is easy to check that the proofs of Propositions 4A and 5A give also the proof of Proposition 2.2. $\square$

3 On the uniqueness of reductions

Every tame automorphism $\theta$ has a sequence of elementary transformations

$$(x_1, x_2, x_3) = \theta_0 \to \theta_1 \to \ldots \to \theta_{k-1} \to \theta_k = \theta. \quad (3)$$

Put $d = \max\{\deg \theta_i \mid 0 \leq i \leq k\}$. The number $d$ will be called the width of the sequence (3). The minimal width of all sequences of the type (3) for $\theta$ will be called the width of the automorphism $\theta$.

Let $\theta \in TA_3(F), \deg \theta > 3$. An automorphism $\phi$ will be called an essential reduction of $\theta$ if there exists a sequence of elementary transformations

$$\phi \to \theta_0 \to \theta_1 \to \ldots \to \theta_{t-1} \to \theta_t = \theta \quad (4)$$

such that $d(\phi) < d(\theta)$ and $\deg(\theta_i) \leq d(\theta)$, where $0 \leq i \leq t$.

The minimal number $t$ for which there exists an essential reduction $\phi$ of $\theta$ of the type (4) will be called the height of $\theta$, and the corresponding sequence (4) will be called a minimal essential reduction of $\theta$. If $t = 0$, then we will say that $\theta$ admits an essential elementary reduction.

Later we will see that the elementary reductions of an automorphism admitting a reduction of type III or IV are not essential.

If $\deg \theta = 3$, then we put $t = \infty$. Introduce a lexicographic order on the set of all pairs $(d, t)$, where $d$ is the width and $t$ is the height of some tame automorphism, by putting $(d_1, t_1) < (d_2, t_2)$ if either $d_1 < d_2$ or $d_1 = d_2, t_1 < t_2$. 
Lemma 3.1 If (4) is a minimal essential reduction of \( \theta \), then \( \deg \theta_0 = d(\theta) \) and

\[(d(\phi), t(\phi)) < (d(\theta_0), t(\theta_0)) < (d(\theta_1), t(\theta_1)) < \ldots < (d(\theta_i), t(\theta_i)) = (d(\theta), t(\theta)).\]

Proof. If \( \deg \theta_0 < d(\theta) \), then (4) gives \( d(\theta_0) < d(\theta) \). In this case instead of \( \phi \) we can take \( \theta_0 \), which contradicts the minimality of (4). In addition, by the minimality of (4), \( \phi \) is also a minimal essential reduction of each \( \theta_i \), \( 0 \leq i \leq k \). Consequently,

\[\deg \theta_0 = d(\theta_i), \ 0 \leq i \leq k; \ t(\theta_{i+1}) = t(\theta_i) + 1, \ 0 \leq i \leq t-1,\]

which gives the statement of the lemma. \( \square \)

Let \( d \) be the width and \( t \) be the height of \( \theta \). The sequence (3) will be called a minimal representation of the automorphism \( \theta \) if

\[\theta_{k-t-1} \to \theta_{k-t} \to \theta_{k-t+1} \to \ldots \to \theta_k = \theta\]

is a minimal essential reduction of \( \theta \) and \( \deg \theta_i < d(\theta), \ 0 \leq i \leq k-t-2 \).

So, every minimal essential reduction of a tame automorphism can be continued to a minimal representation.

**Proposition 3.1** Let \( \theta \in TA_3(F) \), \( \deg \theta > 3 \), let \( d \) be the width and \( t \) be the height of \( \theta \), and let (4) be a minimal essential reduction of \( \theta \). Then the following statements hold:

(a) If \( \theta \) admits a reduction of type I, then \( d = \deg \theta, \ t = 1 \), and in the conditions of Definition 2.1 we have

\[\theta_{t-1} = (f_1, \eta g_2 + \kappa g_1 + \nu, f_3), \ \eta \neq 0.\] (5)

(b) If \( \theta \) admits a reduction of type II, then \( d = \deg \theta \) and in the conditions of Definition 2.2 the automorphism \( \theta_{t-1} \) and the height \( t \) will be calculated in the following way:

1. if \( \alpha \beta \neq 0 \), then \( t = 2 \) and \( \theta_{t-1} \) can be written down simultaneously in the form (5) and in the form

\[\theta_{t-1} = (\rho g_1 + \sigma, f_2, f_3), \ \rho \neq 0;\] (6)

2. if \( \alpha = 0 \), then \( t = 1 \) and \( \theta_{t-1} \) has the form (5);

3. if \( \beta = 0 \), then \( t = 1 \) and \( \theta_{t-1} \) has the form (6).

(c) If \( \theta \) admits a reduction of type III, then in the conditions of Definition 2.3 we have \( d = 5n + \deg f_3 \) and the automorphism \( \theta_{t-1} \) and the height \( t \) will be calculated in the following way:

1. if \( \alpha \neq 0 \) and \( (\beta_1, \beta_2) \neq 0 \), then \( t = 2 \) and \( \theta_{t-1} \) can be written down simultaneously in the form (5) and (6);

2. if \( \alpha = 0 \), then \( t = 1 \) and \( \theta_{t-1} \) has the form (5);
(3) if \((\beta_1, \beta_2) = 0\), then \(t = 1\) and \(\theta_{t-1}\) has the form (6).

(d) If \(\theta\) admits a reduction of type IV, then in the conditions of Definition 2.4 we have \(d = \frac{13n}{2}\) and the automorphism \(\theta_{t-1}\) and the height \(t\) will be calculated in the following way:

1. if \((\alpha_1, \alpha_2) \neq 0\) and \((\beta_1, \beta_2, \beta_3, \beta_4) \neq 0\), then \(t = 3\) and \(\theta_{t-1}\) can be written down simultaneously in the form (5) and (6);
2. if \((\alpha_1, \alpha_2) = 0\) and \((\beta_1, \beta_2, \beta_3, \beta_4) \neq 0\), then \(t = 2\) and \(\theta_{t-1}\) has the form (5);
3. if \((\alpha_1, \alpha_2) \neq 0\) and \((\beta_1, \beta_2, \beta_3, \beta_4) = 0\), then \(t = 2\) and \(\theta_{t-1}\) has the form (6);
4. if \((\alpha_1, \alpha_2) = 0\) and \((\beta_1, \beta_2, \beta_3, \beta_4) = 0\), then \(t = 1\) and \(\theta_{t-1} = (g_1, g_2, \rho g_3 + \sigma)\).

(e) In the other cases \(\theta\) admits an essential elementary reduction, i.e. \(d = \deg \theta\) and \(t = 0\).

Proof. Let \(\theta \in TA_3(F)\) and let \(d\) be the width and \(t\) be the height of \(\theta\). Denote by \((d_1, t_1)\) the value of \((d, t)\) indicated in Proposition 3.1. We first show that

\[(d, t) \leq (d_1, t_1)\]  \hspace{1cm} (7)

by induction on \(\deg \theta\). So, we can assume that the inequality (7) is true for automorphisms of less degree.

We next proceed with the case study. The cases when \(\theta\) satisfies one of the conditions (a), (b), (c), and (d) are similar. Therefore, we give a proof in the case (d). In this case \(\theta\) admits a reduction of type IV, and we adopt all the conditions and notation of Definition 2.4. Consider the sequence of elementary transformations

\[
\phi = (g_1, g_2 - \mu g_3, g_3) \rightarrow (g_1, g_2, g_3) \rightarrow (g_1, g_2, f_3) \rightarrow (f_1, g_2, f_3) \rightarrow (f_1, f_2, f_3). \]  \hspace{1cm} (8)

Since \(\deg \phi < \deg \theta\), it follows that the inequality (7) is valid for \(\phi\).

We will show that

\[d(\phi) < \deg \theta.\]  \hspace{1cm} (9)

First of all we will give a standard argument that deduces (7) from (9). In the sequence (8) the automorphism \((g_1, g_2, g_3)\) has the maximal degree and \(\deg (g_1, g_2, g_3) = \frac{13n}{2} = d_1\). If (9) is fulfilled, then we get \(d \leq d_1\). If \(d < d_1\), then (7) is fulfilled. Suppose that \(d = d_1\). Then the sequence (8) gives that \(\phi\) is an essential reduction of \(\theta\). Consequently, \(t \leq 3\). We have \(f_1 = g_1\) if \((\alpha_1, \alpha_2) = 0\), and \(f_2 = g_2\) if \((\beta_1, \beta_2, \beta_3, \beta_4) = 0\). In these cases, excluding \((f_1, g_2, f_3)\) from (8), we get \(t \leq 2\). If \((\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \beta_4) = 0\), then excluding also \((g_1, g_2, f_3)\) from (8), we get \(t \leq 1\). Thus, in all the cases \(t \leq t_1\), and (7) is true.

If \(\phi\) satisfies one of the conditions (a), (b), and (e) of the proposition, then by (7) we have \(d(\phi) \leq \deg (\phi)\), which gives (9).

We have

\[
\deg g_1 = 2n, \quad \deg (g_2 - \mu g_3^2) \leq 2n, \quad \deg g_3 = \frac{3n}{2}. \]  \hspace{1cm} (10)

12
Suppose that \( \phi \) admits a reduction of type III or IV. According to (10), the element \( g_1 \) has the highest degree among the components of \( \phi \). Therefore the active element of the reduction is \( g_2 - \mu g_3^2 \) or \( g_3 \). If \( g_3 \) is the active element of the reduction, then there exists \( m \) such that

\[
m < \deg g_3 \leq \frac{3m}{2}, \quad 2m \leq \deg (g_2 - \mu g_3^2) < \frac{5m}{2}, \quad \frac{5m}{2} < \deg g_1 < \frac{7m}{2}.
\]

Taking account of (10), from here we deduce the inequalities

\[
2m \leq 2n, \quad \frac{3n}{2} \leq \frac{3m}{2}, \quad \frac{5m}{2} < \deg g_1 = 2m,
\]

which are in a contradiction.

If \( g_2 - \mu g_3^2 \) is the active element of the reduction of \( \phi \), then there exists \( m \) such that

\[
m < \deg (g_2 - \mu g_3^2) \leq \frac{3m}{2}, \quad 2m \leq \deg g_3 < \frac{5m}{2}, \quad \frac{5m}{2} < \deg g_1 < \frac{7m}{2}.
\]

Consequently, \( 2m \leq \frac{3n}{4} \), i.e. \( m \leq \frac{3n}{8} \). By (7), we have \( d(\phi) \leq \frac{13m}{2} \leq \frac{39n}{8} < 5n \). From Proposition 2.1 we have \( \deg \theta > 6n \), i.e. the inequality (9) is fulfilled.

Now, consider the case (e). If \( \theta \) does not admit reductions of types I–IV, then according to Theorem 2.1 \( \theta \) admits an elementary reduction. Let \( \phi \) be an elementary reduction of \( \theta \). If \( \phi \) satisfies one of the conditions (a), (b), and (e), then by (7) we have \( d(\phi) \leq \deg \phi < \deg \theta \), i.e. the inequality (9) is fulfilled.

Assume that \( \phi \) admits a reduction of type IV. Temporarily we assume that \( \phi = (f_1, f_2, f_3) \) and that \( (f_1, f_2, f_3) \) satisfies the conditions of Definition 2.4. According to (7), we have \( d(\phi) \leq \frac{13m}{2} \). Put also \( \theta = (q_1, q_2, q_3) \). If \( \deg \theta > \frac{13n}{2} \), then (9) is fulfilled. Therefore we may assume that

\[
\deg \theta \leq \frac{13n}{2}.
\]

We first consider the case in which

\[
q_1 = \rho f_1 + a, \quad q_2 = f_2, \quad q_3 = f_3, \quad \rho \neq 0, \quad a \in <f_2, f_3>.
\]

Changing \( \rho g_1 \) to \( g_1 \), we may assume that \( \rho = 1 \). Proposition 2.1 gives \( \deg f_2 + \deg f_3 > 4n \) and \( \deg f_1 < \frac{5n}{2} \). Then from (11) we obtain \( \deg q_1 < \frac{5n}{2} \). Consequently, \( \deg a < \frac{5n}{2} \). By Lemma 2.4 we have \( a \in <f_3> \) and

\[
a = \alpha_0 + \alpha_1 f_3 + \alpha_2 f_3^2.
\]

Consequently,

\[
q_1 = g_1 + \alpha_0 + (\alpha_1 + \alpha_1') f_3 + (\alpha_2 + \alpha_2') f_3^2.
\]

Changing \( g_1 + \alpha_0' \) to \( g_1 \), we may assume that \( \alpha_0' = 0 \). Then \( \theta \) admits a reduction of type IV, which contradicts the condition (e).
Let
\[ q_1 = f_1, \quad q_2 = \eta f_2 + a, \quad q_3 = f_3, \quad \eta \neq 0, \quad a \in \langle f_1, f_3 \rangle. \] (13)

As above we can take \( \eta = 1 \). By Proposition 2.1 we have \( \deg f_1 + \deg f_3 > 3n \) and \( \frac{5m}{2} < \deg f_2 < \frac{7n}{2} \). Applying (11) we find that \( \deg q_2 < \frac{7n}{2} \), i.e. \( \deg a < \frac{7n}{2} \). By Lemma 2.4 we obtain \( a \in \langle f_1, f_3 \rangle \). Now it is easy to deduce that
\[ a = \gamma_0 + \gamma_1 g_1 + \beta_1' f_3 + \beta_2' f_3^2 + \beta_3' f_1 f_3 + \beta_4' f_3^3. \]

Thus,
\[ q_2 = g_2 + \gamma_0 + \gamma_1 g_1 + (\beta_1 + \beta_1') g_1 (\beta_2 + \beta_2') f_3^2 + (\beta_3 + \beta_3') f_1 f_3 + (\beta_4 + \beta_4') f_3^3. \]

Changing \( g_2 + \gamma_0 + \gamma_1 g_1 \) to \( g_2 \), we may assume that \( \gamma_0 = \gamma_1 = 0 \). Then \( \theta \) admits a reduction of type IV.

Now we consider the case in which
\[ q_1 = f_1, \quad q_2 = f_2, \quad q_3 = \rho f_3 + a, \quad \rho \neq 0, \quad a \in \langle f_1, f_2 \rangle. \] (14)

Changing \( \rho g_3 \) to \( g_3 \), we may assume that \( \rho = 1 \). Proposition 2.1 gives \( \deg f_1 + \deg f_2 > \frac{9m}{2} \). By (11), from here we get \( \deg q_3 < 2n \), i.e. \( \deg a < 2n \). If \( (\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \beta_4) \neq 0 \), then according to Lemma 2.4 we obtain \( a = \delta \in F \), i.e. \( \theta \) admits a reduction of type IV. If \( (\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \beta_4) = 0 \), then \( f_1 = g_1, \quad f_2 = g_2 \), and by Lemma 2.4 we have
\[ a = \delta w(g_1, g_2) + \sigma. \]

Then
\[ q_3 = g_3 + (\gamma + \delta) w(g_1, g_2) + \sigma. \]

Changing \( g_3 + \sigma \) to \( g_3 \), we can take \( \sigma = 0 \). If \( \gamma + \delta \neq 0 \), then \( \theta \) admits a reduction of type IV. If \( \gamma + \delta = 0 \), then \( \theta = (g_1, g_2, g_3) \) and instead of \( \phi \) we can take
\[ \phi' = (g_1, g_2 - \mu q_3, q_3). \]

Recall that, considering the case when \( \theta \) admits a reduction of type IV, we have simultaneously proved that
\[ d(\phi') < \frac{13n}{2}. \]

Since \( \deg(\theta) = \frac{13m}{2} \), the inequality (9) is also fulfilled.

If \( \phi \) admits a reduction of type III, then as above we can assume that \( \phi = (f_1, f_2, f_3) \) and that \( (f_1, f_2, f_3) \) satisfies the conditions of Definition 2.3. This case can be settled by analogy to the case when \( \phi \) admits a reduction of type IV. We only note that instead of (11) in this case we have a stronger inequality
\[ \deg \theta \leq 5n + \deg f_3. \]
The inequality (7) is thus proved if \( \theta \) satisfies the conditions (e). Therefore we assume that the inequality (7) is proved.

Now we begin a full proof of Proposition 3.1. Assume that the statement of the proposition is not true. Let \( \theta \) be an automorphism with minimal \((d, t)\) which does not satisfy the statement of the proposition. Note that if \( \theta \) satisfies the condition (e), then we have nothing to prove.

Restrict ourselves to the case when \( \theta \) admits a reduction of type IV. Let \( \theta \) satisfy the conditions of Definition 2.4. Put \( \theta_{t-1} = (q_1, q_2, q_3) \). Since
\[
(d(\theta_{t-1}), t(\theta_{t-1})) < (d, t)
\]
by Lemma 3.1, it follows that Proposition 3.1 is valid for \( \theta_{t-1} \). By (7) we have \( d = d(\theta) \leq \frac{13n}{2} \). Then
\[
d(\theta_{t-1}) \leq \frac{13n}{2}.
\]
Suppose that \((q_1, q_2, q_3)\) has the form (12). Repeating the same arguments which were given after (12), we can assume that
\[
q_1 = g_1 + (\alpha_1 + \alpha'_1)f_3 + (\alpha_2 + \alpha'_2)f^2_3,
\]
i.e. \( \theta_{t-1} \) admits a reduction of type IV.

Let \((\beta_1, \beta_2, \beta_3, \beta_4) \neq 0\). If \((\alpha_1 + \alpha'_1, \alpha_2 + \alpha'_2) \neq 0\), then according to the statement (d) of Proposition 3.1 we have \( d(\theta_{t-1}) = \frac{13n}{2} \) and \( t(\theta_{t-1}) = 3 \). Since \( \deg \theta \leq \frac{13n}{2} \) from here we get \( d = \frac{13n}{2} \) and \( t = 4 \), which contradicts (7). Consequently, \((\alpha_1 + \alpha'_1, \alpha_2 + \alpha'_2) = 0\).

In this case Proposition 3.1 gives \( d(\theta_{t-1}) = \frac{13n}{2} \) and \( t(\theta_{t-1}) = 2 \). Hence, \( d = \frac{13n}{2} \) and \( t = 3 \). If \((\alpha_1, \alpha_2) \neq 0\), then Proposition 3.1 is valid. If \((\alpha_1, \alpha_2) = 0\), then the inequality (7) is not fulfilled. It means that \( \theta_{t-1} \) is obtained from \( \theta \) by another elementary transformation!

Now let \((\beta_1, \beta_2, \beta_3, \beta_4) = 0\). As above we have \((\alpha_1 + \alpha'_1, \alpha_2 + \alpha'_2) = 0\). Then \( d(\theta_{t-1}) = \frac{13n}{2} \) and \( t(\theta_{t-1}) = 1 \). Consequently, \( d = \frac{13n}{2} \) and \( t = 2 \). If \((\alpha_1, \alpha_2) \neq 0\), then Proposition 3.1 is valid, and if \((\alpha_1, \alpha_2) = 0\), then the inequality (7) is not fulfilled.

The above discussion is standard and the other cases can be examined similarly. \( \square \)

**Corollary 3.1** Reductions of types I–IV are essential minimal reductions.

**Corollary 3.2** If \( \theta \in TA_3(F) \) admits a reduction of types I–IV, then the type of this reduction is uniquely defined.

**Proof.** Let \( \theta = (f_1, f_2, f_3) \) admit a reduction of types I–IV. By Proposition 3.1, the active element \( f_3 \) of this reduction is uniquely characterized as a component of \( \theta \) which does not change at a minimal reduction before appearing in an automorphism \((g_1, g_2, f_3)\) with the property \( \deg [g_1, g_2] \leq \deg f_3 \). The last inequality is a generalization of the estimates of the degree of \([g_1, g_2]\) in Remark 2.1 and in Proposition 2.1. In addition, if \( \deg f_1 \leq \deg f_2 \), then the roles of elements in Definitions 2.1–2.4 are uniquely defined.

15
Now the reductions of types I, II, and III can be easily distinguished among themselves by the degree of \( \deg f_3 \).

Assume that \( \theta \) admits a reduction of type IV and satisfies the conditions of Definition 2.4. If \( \alpha_2 \neq 0 \), then \( \overline{f_1} = \alpha_2 \overline{f_3}^2 \). Consequently, \( f_1 \) is an elementarily reducible element of \( \theta \). Note that if \( \theta \) admits a reduction of types I–III, then \( \theta \) does not admit such an elementary reduction. If \( \alpha_2 = 0 \), then \( \deg f_1 = 2n \) and \( \deg f_3 \leq \frac{3n}{2} \). Consequently, \( \theta \) does not admit a reduction of type I or II. If \( \theta \) admits a reduction of type III, then the elements \( g_1 \) and \( g_2 \) are uniquely defined for both reductions. We have \( g_3 = f_3 - \gamma w(g_1, g_2) \). Since \( \deg [g_1, g_3] < 3n + \deg [g_1, g_2] \) and \( \deg [g_1, w(g_1, g_2)] = 3n + \deg [g_1, g_2] \), it follows that \( \gamma \) is also uniquely defined from the equality \( [g_1, \overline{f_3}] = \gamma [g_1, w(g_1, g_2)] \). Now if \( \deg g_3 < \frac{3n}{2} \), then \( \theta \) admits a reduction of type III, and if \( \deg g_3 = \frac{3n}{2} \), then \( \theta \) admits a reduction of type IV. So, \( \theta \) does not admit reductions of types III and IV simultaneously. \( \blacksquare \)

As we can see from the proof of Corollary 3.2, not only the type of a reduction of \( \theta \) but also the elements \( g_1 \) and \( g_2 \) are uniquely defined. The element \( g_3 \) in Definitions 2.3 and 2.4 is uniquely defined up to a summand from the field \( F \), since it is with this exactness that the element \( w(g_1, g_2) \) is defined. In Definition 2.1 instead of \( g_3 \) we can take any element of the form \( \rho g_3 + H(g_1) \), where \( 0 \neq \rho \in F \) and \( \deg H(g_1) \leq \deg g_3 \). Furthermore, in Definition 2.2 instead of \( g_3 \) we can take any element of the form \( \rho g_3 + \gamma \), where \( 0 \neq \rho \in F \).

Proposition 3.1 and Corollary 3.2 immediately give the next stronger form of Theorem 2.1.

**Theorem 3.1** Let \( \theta \in TA_3(F) \), \( \deg \theta > 3 \). Then \( \theta \) admits only one type of the following reductions: an essential elementary reduction or a reduction of types I–IV.

Besides, from the proof of Proposition 3.1 we can extract the next corollary.

**Corollary 3.3** Let \( \theta \in TA_3(F) \), \( \deg \theta > 3 \). If \( \theta \) admits a reduction of types I–IV, then every elementary reduction of \( \theta \) again admits a reduction of the same type.

In fact, the elementary reductions considered in this corollary are not essential.

Note that if \( \theta \) admits a reduction of type I or II, then \( \theta \) does not admit any elementary reduction. If \( \theta \) admits a reduction of type III, then in the conditions of Definition 2.3 \( \theta \) admits an elementary reduction if and only if \( \overline{f_2} = \varsigma \overline{f_3}^2 \), \( \deg f_3 = \frac{3n}{2} \), and \( \deg f_2 = 3n \). Assume that \( \theta \) admits a reduction of type IV. If \( (\alpha_2, \beta_3, \beta_4) = 0 \) in the conditions of Definition 2.4, then \( \theta \) also admits an elementary reduction if and only if \( \overline{f_2} = \varsigma \overline{f_3}^2 \), \( \deg f_3 = \frac{3n}{2} \), and \( \deg f_2 = 3n \). If \( (\alpha_2, \beta_3, \beta_4) \neq 0 \), then \( \theta \) admits obvious elementary reductions.

**4 Defining relations of the group of tame automorphisms**
Let $A_n = F[x_1, x_2, \ldots, x_n]$ be the polynomial algebra over a field $F$ with the set of variables $X = \{x_1, x_2, \ldots, x_n\}$. Then the group $TA_n(F)$ is generated by all elementary automorphisms

$$
\sigma(i, \alpha, f) = (x_1, \ldots, x_{i-1}, \alpha x_i + f, x_{i+1}, \ldots, x_n),
$$

where $0 \neq \alpha \in F$, $f \in F[X \setminus \{x_i\}]$.

Our aim in this section is to describe defining relations of the group $TA_n(F)$ with respect to the generators (17). It is easy to check that

$$
\sigma(i, \alpha, f)\sigma(i, \beta, g) = \sigma(i, \alpha\beta, \beta f + g).
$$

(18)

Note that from here we obtain trivial relations

$$
\sigma(i, 1, 0) = id, \ 1 \leq i \leq n.
$$

If $i \neq j$ and $f \in F[X \setminus \{x_i, x_j\}]$, then we have also

$$
\sigma(i, \alpha, f)^{-1}\sigma(j, \beta, g)\sigma(i, \alpha, f) = \sigma(j, \beta, \sigma(i, \alpha, f)^{-1}(g)).
$$

(19)

Consequently, if $i \neq j$ and $f, g \in F[X \setminus \{x_i, x_j\}]$, then the automorphisms $\sigma(i, \alpha, f)$, $\sigma(j, \beta, g)$ commute.

For every pair $k, s$, where $1 \leq k \neq s \leq n$, we define a tame automorphism $(ks)$ by putting

$$(ks) = \sigma(s, -1, x_k)\sigma(k, 1, -x_s)\sigma(s, 1, x_k).$$

Note that the automorphism $(ks)$ of the algebra $A_n$ changes only the positions of the variables $x_k$ and $x_s$. Now it is easy to see that

$$
\sigma(i, \alpha, f)^{(ks)} = \sigma(j, \alpha, (ks)(f)),
$$

(20)

where $x_j = (ks)(x_i)$.

Let $G(A_n)$ be the abstract group with generators (17) and defining relations (18)–(20).

**Lemma 4.1** The subgroup of $G(A_n)$ generated by all elements $(ks)$, where $1 \leq k \neq s \leq n$, is isomorphic to the symmetric group $S_n$.

**Proof.** By (18) and (19), we have

$$
(ks)^2 = \sigma(s, -1, x_k)\sigma(k, 1, -x_s)\sigma(s, 1, x_k)\sigma(s, -1, x_k)\sigma(k, 1, -x_s)\sigma(s, 1, x_k)
$$

$$
= \sigma(s, -1, x_k)\sigma(k, 1, -x_s)\sigma(s, -1, 0)\sigma(k, 1, -x_s)\sigma(s, 1, x_k)
$$

$$
= \sigma(s, -1, x_k)\sigma(s, -1, 0)\sigma(k, 1, -x_s)\sigma(s, 1, x_k)
$$

$$
= \sigma(s, -1, x_k)\sigma(k, 1, -x_s)\sigma(s, 1, x_k) = \sigma(s, 1, -x_k)\sigma(s, 1, x_k) = id.
$$
Then (20) gives

\[(ks)^{(sk)} = \sigma(s, -1, x_k)^{(sk)} \sigma(k, 1, -x_s)^{(sk)} \sigma(s, 1, x_k)^{(sk)} = \sigma(k, 1, x_s) \sigma(s, 1, -x_k) = (sk),\]

i.e. \((ks) = (sk)\). Now it is not difficult to deduce from (18)–(20) that

\[[ij], (ks)] = id, \quad (ik)^{(is)} = (ks),\]

where \(i, j, k, s\) are all distinct. It is immediate that the given relations imply the defining relations of the group \(S_n\) with respect to the system of generators \((i i + 1)\), where \(1 \leq i \leq n - 1\), which are indicated in [3]. □

By Lemma 4.1, the elements of the symmetric group \(S_n\) can be identified with elements of \(G(A_n)\). Note that (20) can be rewritten as

\[\sigma(i, \alpha, f)^{\pi} = \sigma(\pi^{-1}(i), \alpha, \pi^{-1}(f)),\]

where \(\pi \in S_n\).

It is well known that the group of affine automorphisms \(Af_n(F)\) of the algebra \(A_n\) is generated by all affine elementary automorphisms.

**Lemma 4.2** The relations (18)–(20) for elementary affine automorphisms are defining relations of the group \(Af_n(F)\).

**Proof.** Let \(\varphi\) be a product of elementary affine automorphisms. Suppose that \(\varphi = id\). By (18) and (19), we can represent \(\varphi\) in the form

\[\varphi = \sigma(1, 1, \alpha_1)\sigma(2, 1, \alpha_2) \ldots \sigma(n, 1, \alpha_n)\varphi',\]

where \(\varphi'\) is a product of elementary linear automorphisms. Obviously, \(\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0\). Therefore we can assume that \(\varphi\) is a product of elementary linear automorphisms. By (18) and (19), we can easily represent \(\varphi\) in the form

\[\varphi = \sigma(1, \alpha_1, 0)\sigma(2, \alpha_2, 0) \ldots \sigma(n, \alpha_n, 0)\varphi',\]

where \(\varphi'\) is a product of elementary automorphisms of the type \(\sigma(i, 1, f)\). By (18)–(20), we have

\[
\begin{align*}
\sigma(k, \alpha, 0) &= \sigma(s, \alpha, 0)^{(ks)} \\
&= \sigma(s, -1, x_k)\sigma(k, 1, -x_s)^{(sk)} \sigma(s, 1, x_k)^{(sk)} \sigma(s, 1, x_k)^{(sk)} \\
&= \sigma(s, -1, x_k)\sigma(k, 1, -x_s)\sigma(s, 1, x_k) \\
&= \sigma(s, -1, x_k)\sigma(s, -\alpha, 0)\sigma(s, 1, (1 - \alpha)x_k)^{(sk)} \\
&= \sigma(s, -1, x_k)\sigma(s, -\alpha, 0)\sigma(s, 1, (1 - \alpha)x_k)^{(sk)} \\
&= \sigma(s, -1, x_k)\sigma(s, -\alpha, 0)\sigma(s, 1, (1 - \alpha)x_k)^{(sk)} \\
&= \sigma(s, -1, x_k)\sigma(s, -\alpha, 0)\sigma(s, 1, (1 - \alpha)x_k)^{(sk)}.
\end{align*}
\]

By this relation, we can represent \(\varphi\) in the form

\[\varphi = \sigma(n, \beta_n, 0)\varphi',\]
where $\varphi'$ is a product of elementary linear automorphisms of the form $\sigma(i, 1, f)$. Hence $\beta_n = 1$. Note that $\sigma(i, 1, f)$ can be represented as a product of automorphisms

$$X_{ij}(\lambda) = \sigma(j, 1, \lambda x_i), \quad \lambda \in F, \quad i \neq j.$$  

(21)

Thus, we can assume that $\varphi$ is a product of automorphisms of the form (21).

Let $G$ be the subgroup of $TA_n(F)$ generated by all automorphisms of the form (21).

We define a map

$$J : G \to SL_n(F),$$

where $J(\psi)$ is the Jacobian matrix of $\psi \in G$. It is easy to check that

$$J(X_{ij}(\lambda)) = E_{ij}(\lambda),$$

and that $J$ is an isomorphism of the groups.

Now it is sufficient to prove that every relation of the group $SL_n(F)$ is a corollary of (18)–(20). Obviously, (18)–(19) cover the Steinberg relations (see, for example [11]).

Besides, according to [11], we need to check the relation

$$\{u, v\} = id, \quad 0 \neq u, v \in F,$$

where

$$\{u, v\} = h_{ij}(uv)h_{ij}(u)^{-1}h_{ij}(v)^{-1},$$

$$h_{ij}(u) = w_{ij}(u)w_{ij}(-1),$$

$$w_{ij}(u) = X_{ij}(u)X_{ji}(-u^{-1})X_{ij}(u).$$

Applying (18)–(20) we have

$$w_{ij}(u) = \sigma(j, 1, ux_i)\sigma(i, 1, -u^{-1}x_j)\sigma(j, 1, ux_i)$$

$$= \sigma(j, 1, ux_i)\sigma(i, u, 0)\sigma(i, 1, -x_j)\sigma(i, u^{-1}, 0)\sigma(j, 1, ux_i)$$

$$= \sigma(i, u, 0)\sigma(j, 1, x_i)\sigma(i, 1, -x_j)\sigma(j, 1, x_i)\sigma(i, u^{-1}, 0)$$

$$= \sigma(i, u, 0)\sigma(j, -1, 0)\sigma(i, 1, -x_j)\sigma(j, 1, x_i)\sigma(i, u^{-1}, 0)$$

$$= \sigma(i, u, 0)\sigma(j, -1, 0)(ij)\sigma(i, u^{-1}, 0) = (ij)\sigma(i, u, 0)(ij)\sigma(j, -1, 0)(ij)\sigma(i, u^{-1}, 0)$$

$$= (ij)\sigma(j, u, 0)(ij)\sigma(i, u^{-1}, 0)$$

Consequently,

$$h_{ij}(u) = w_{ij}(u)w_{ij}(-1) = (ij)\sigma(j, u, 0)(ij)\sigma(j, -1, 0)\sigma(i, 1, 0)$$

$$= \sigma(j, u, 0)(ij)\sigma(i, -u^{-1}, 0)(ij)\sigma(j, -1, 0)$$

$$= \sigma(i, u, 0)\sigma(j, -u^{-1}, 0)\sigma(j, -1, 0) = \sigma(i, u, 0)\sigma(j, u^{-1}, 0).$$

19
Hence

\[ \{u, v\} = h_{ij}(uv)h_{ij}(u)^{-1}h_{ij}(v)^{-1} = \sigma(i, uv, 0)\sigma(j, (uv)^{-1}, 0)\sigma(i, u, 0)\sigma(j, u^{-1}, 0)\sigma(i, v, 0)\sigma(j, v^{-1}, 0) = id. \]

Thus we can say that every relation of the group \( SL_n(F) \) follows from (18)–(20). \( \square \)

Assume that

\[ \theta = \phi_1\phi_2 \ldots \phi_r \in TA_n(F), \quad (22) \]

where \( \phi_i, 1 \leq i \leq r \), are elementary automorphisms. Put

\[ \psi_i = \phi_1\phi_2 \ldots \phi_i, \quad 0 \leq i \leq r. \]

In particular, we have

\[ \psi_r = \theta, \quad \psi_0 = id. \]

To (22) corresponds the sequence of elementary transformations

\[ id = \psi_0 \xrightarrow{\varphi_1} \psi_1 \xrightarrow{\varphi_2} \psi_2 \xrightarrow{\varphi_3} \ldots \xrightarrow{\varphi_r} \psi_r = \theta. \quad (23) \]

Note that the representations (22) and (23) of the automorphism \( \theta \) are equivalent. If (23) is a minimal representation of \( \theta \), then the representation (22) will be also called a minimal representation of \( \theta \).

**Theorem 4.1** Let \( F \) be a field of characteristic 0. Then the relations (18)–(20) are defining relations of the group \( TA_3(F) \) with respect to the generators (17).

**Plan of the proof.** Assume that

\[ \varphi_1\varphi_2 \ldots \varphi_k = id = (x_1, x_2, x_3), \quad (24) \]

where \( \varphi_i, 1 \leq i \leq k \), are elementary automorphisms. Put

\[ \theta_i = \varphi_1\varphi_2 \ldots \varphi_i, \quad 0 \leq i \leq k. \]

In particular, we have \( \theta_0 = \theta_k = (x_1, x_2, x_3) \). To (24) corresponds the sequence of elementary transformations

\[ id = \theta_0 \xrightarrow{\varphi_1} \theta_1 \xrightarrow{\varphi_2} \ldots \xrightarrow{\varphi_k} \theta_k = id. \quad (25) \]

Denote by \( d = \max\{\deg \theta_i | 0 \leq i \leq k\} \) the width of the sequence (25). Let \( i_1 \) be the minimal number and \( i_2 \) be the maximal number which satisfy the equations \( \deg \theta_{i_1} = d \) and \( \deg \theta_{i_2} = d \). Put \( q = i_2 - i_1 \). The pair \((d, q)\) will be called the exponent of the relation (24).
To prove the theorem, we show that (24) follows from (18)–(20). Assume that our theorem is not true. Call a relation of the form (24) trivial if it follows from (18)–(20). We choose a nontrivial relation (24) with the minimal exponent \((d, q)\) with respect to the lexicographic order. To arrive at a contradiction, we show that (24) is also trivial.

If \(d = 3\), then Lemma 4.2 gives the triviality of the relation (24). Therefore we can assume that \(d > 3\).

Our plan is to change the product (24) by using (18)–(20) and to obtain a new sequence (25) whose exponent is strictly less than \((d, q)\). The proof of the theorem will be completed by Lemmas 4.3–4.10.

Denote by \(t = \left\lfloor \frac{q}{2} \right\rfloor\) the integral part of \(\frac{q}{2}\). Put also
\[
\phi = \theta_{i_1+t-1}, \quad \theta = \theta_{i_1+t}, \quad \tau = \theta_{i_1+t+1}.
\]

Then we have
\[
\phi \phi_{i_1+t} \theta \phi_{i_1+t+1} \tau.
\]

Lemma 4.3 The following statements are true:

1. \(d\) is the width of \(\theta\), \(t\) is the height of \(\theta\), and

\[
\theta = \phi_1\phi_2 \ldots \phi_{i_1+t}
\]

is a minimal representation of \(\theta\).

2. If \(q\) is an even number, then

\[
\theta = \phi_{k-1}^{-1} \phi_{k-1}^{-1} \cdots \phi_{i_1+t+1}^{-1}
\]

is also a minimal representation of \(\theta\).

3. If \(q\) is an odd number, then \((d(\tau), t(\tau)) = (d, t)\) and

\[
\tau = \phi_{k-1}^{-1} \phi_{k-1}^{-1} \cdots \phi_{i_1+t+2}^{-1}
\]

is a minimal representation of \(\tau\). Moreover, in (24) the product (27) can be replaced by an arbitrary minimal representation of \(\theta\).

Proof. Assume that \((d(\theta), t(\theta)) < (d, t)\) and let (22) be a minimal representation of \(\theta\). Then (24) is a consequence of the equalities

\[
\phi_1\phi_2 \ldots \phi_{i_1+t}\phi_{r\phantom{1}}^{-1} \phi_2^{-1} \phi_1^{-1} = id,
\]

\[
\phi_1\phi_2 \ldots \phi_r\phi_{i_1+t+1} \phi_{k-1}^{-1} \phi_k = id.
\]
To (28) corresponds the sequence of elementary transformations

\[(x_1, x_2, x_3) \rightarrow \theta_1 \rightarrow \ldots \rightarrow \theta_{i+t} = \theta = \psi_r \rightarrow \psi_{r-1} \ldots \rightarrow \psi_1 \rightarrow (x_1, x_2, x_3),\]

and to (29) corresponds

\[(x_1, x_2, x_3) \rightarrow \psi_1 \rightarrow \ldots \rightarrow \psi_t = \theta_{i+t} \rightarrow \theta_{i+t+1} \rightarrow \ldots \rightarrow \theta_{k-1} \rightarrow (x_1, x_2, x_3).\]

Since \((d(\theta), t(\theta)) < (d, t)\), it follows that (28) and (29) have exponents strictly less than \((d, q)\). This gives the first statement of the lemma.

It is obvious that

\[\varphi_k^{-1} \varphi_{k-1} \ldots \varphi_1^{-1} = id\]

has the same exponent \((d, q)\). Applying the first statement of the lemma to this relation, we get the second statement of the lemma, as well as the minimality of the representation of \(\tau\) if \(q\) is an odd number. If \(q\) is an odd number, then (28) has exponent strictly less than \((d, q)\), and (29) has the exponent \((d, q)\). Consequently, (24) and (29) are equivalent modulo (18)–(20). Thus \(\theta\) can be changed by an arbitrary minimal representation in (29).

\[\square\]

**Lemma 4.4** If \(t \geq 1\), then the relation (24) is trivial.

**Proof.** Our aim is to change the product \(\varphi_{i+t}\varphi_{i+t+1}\) by (18)–(20) and to get a new sequence (25) whose exponent is strictly less than \((d, q)\), i.e. to show that (24) is trivial.

Since \(t \geq 1\), according to Proposition 3.1 \(\theta\) admits a reduction of types I–IV. We restrict ourselves only to the case when \(\theta\) admits a reduction of type IV. The other cases can be considered similarly.

Let \(\theta = (f_1, f_2, f_3)\) satisfy the conditions of Definition 2.4. According to Lemma 4.3, the representation (27) of \(\theta\) is minimal. Then \(\phi\) can be calculated by using Proposition 3.1.

**Case I:** \(q\) is even, \(q = 2t\). According to Lemma 4.3 \(\tau\) can also be calculated by using Proposition 3.1.

Assume that \(t = 2\). By Proposition 3.1 the automorphisms \(\phi\) and \(\tau\) have the same form, i.e. either

\[\phi = (f_1, \eta_1 g_2 + \kappa_1 g_1 + \nu_1, f_3), \quad \tau = (f_1, \eta_2 g_2 + \kappa_2 g_1 + \nu_2, f_3),\]

or

\[\phi = (\rho_1 g_1 + \sigma_1, f_2, f_3), \quad \tau = (\rho_2 g_1 + \sigma_2, f_2, f_3).\]

By (18), in both cases \(\varphi_{i+t}\varphi_{i+t+1}\) can be replaced by an elementary automorphism. Obviously, (24) will then be replaced by a relation whose exponent is strictly less than \((d, q)\).
Assume that \( t = 3 \). According to Proposition 3.1, the automorphisms \( \phi \) and \( \tau \) have the forms (5) and (6). If \( \phi \) and \( \tau \) have the same form (5) (or (6)), then as above, by (18), we obtain the triviality of (24). Suppose that \( \phi \) has the form (5), and \( \tau \) has the form (6). It is immediate that

\[
\varphi_{i_1+t} = \sigma(2, \eta^{-1}, F(x_1, x_3)), \quad \varphi_{i_1+t+1} = \sigma(1, \rho, G(x_3)).
\]

By (19) we get

\[
\varphi_{i_1+t} \varphi_{i_1+t+1} = \sigma(1, \rho, G(x_3)) \sigma(2, \eta^{-1}, F_1(x_1, x_3)).
\]

We replace \( \varphi_{i_1+t} \varphi_{i_1+t+1} \) in (24) according to this equality. Then \( \theta \) in (25) can be changed to

\[
\theta' = (\rho g_1 + \alpha, \eta g_2 + \kappa g_1 + \nu, f_3).
\]

Note that \( \deg \theta' = \frac{13n}{2} \) and after such replacement the exponent \( (d, q) \) of the sequence (25) does not change. As before \( \theta' \) admits a reduction of type IV. But according to Proposition 3.1, in this case we have \( t(\theta') = 1 \). This contradicts Lemma 4.3, since \( t(\theta') < t \).

If \( t = 1 \), then according to Proposition 3.1 we have

\[
\phi = (g_1, g_2, \rho g_3 + \alpha_1), \quad \tau = (g_1, g_2, \rho g_3 + \alpha_2).
\]

By (18), we obtain the triviality of (24) as above.

**Case II: \( q \) is odd, \( q = 2t + 1 \).** If \( (\alpha_1, \alpha_2) = 0 \) and \( (\beta_1, \beta_2, \beta_3, \beta_4) = 0 \) in the conditions of Definition 2.4, then \( \theta = (g_1, g_2, f_3) \). Moreover, according to Proposition 3.1, we have \( t = 1 \) and

\[
\phi = (g_1, g_2, \rho g_3 + \alpha).
\]

Assume that

\[
\tau = (\lambda f_1 + a, f_2, f_3), \quad a \in < f_2, f_3 >.
\]

By Lemma 4.3 we have \( (d(\tau), t(\tau)) = (d, t) \). Since \( d = d(\theta) = \frac{13n}{2} \), from here we get \( \deg \tau \leq \frac{13n}{2} \). With regard to the inequalities of Proposition 2.1, we have \( \deg (\lambda f_1 + a) < \frac{5n}{2} \) and \( \deg a < \frac{5n}{2} \). Applying Lemma 2.4 gives also

\[
a = \alpha_0' + \alpha_1' f_3 + \alpha_2' f_3^2.
\]

Consequently,

\[
\lambda f_1 + a = \lambda g_1 + \alpha_0' + \alpha_1' f_3 + \alpha_2' f_3^2.
\]

If \( (\alpha_1', \alpha_2') \neq 0 \), then \( \tau \) admits a reduction of type IV. Moreover, \( t(\tau) > 1 \), which contradicts the equality \( (d(\tau), t(\tau)) = (d, t) \). Consequently,

\[
\tau = (\lambda g_1 + \alpha_0', g_2, f_3).
\]
We have
\[ \varphi_{i+t} = \sigma(3, \rho^{-1}, F(x_1, x_2)), \quad \varphi_{i+t+1} = \sigma(1, \lambda, \alpha'_0). \]

According to (19), we obtain
\[ \varphi_{i+t}\varphi_{i+t+1} = \sigma(1, \lambda, \alpha'_0)\sigma(3, \rho^{-1}, F_1(x_1, x_2)). \]

After the corresponding replacement, instead of \( \theta \) we have
\[ \theta' = (\lambda g_1 + \alpha'_0, g_2, \rho g_3 + \alpha). \]

Note that \( \theta' \) admits an essential elementary reduction, i.e. \( t(\theta') = 0 \).

Now assume that
\[ \tau = (f_1, \eta f_2 + a, f_3), \quad a \in < f_1, f_3 >. \]  \hspace{1cm} (31)

Using the same arguments as above we get
\[ \tau = (g_1, \eta g_2 + \kappa g_1 + \nu, f_3). \]

We have
\[ \varphi_{i+t} = \sigma(3, \rho^{-1}, F(x_1, x_2)), \quad \varphi_{i+t+1} = \sigma(2, \eta, \kappa x_1 + \nu). \]

The relation (19) gives
\[ \varphi_{i+t}\varphi_{i+t+1} = \sigma(2, \eta, \kappa x_1 + \nu)\sigma(3, \rho^{-1}, F_1(x_1, x_2)). \]

After such replacement, instead of \( \theta \) we have
\[ \theta' = (g_1, \eta g_2 + \kappa g_1 + \nu, \rho g_3 + \alpha), \]

and this automorphism also admits an essential elementary reduction.

If the elementary reduction \( \theta \rightarrow \tau \) replaces the element \( f_3 \) of the automorphism \( \theta \), then applying (18) also gives the triviality of (24).

We now consider the case when \( (\alpha_1, \alpha_2) = 0 \) and \( (\beta_1, \beta_2, \beta_3, \beta_4) \neq 0 \). Then according to Proposition 3.1 we have \( t = 2 \) and \( \phi \) has the form (5), i.e.
\[ \phi = (g_1, \eta g_2 + \kappa g_1 + \nu, f_3). \]

If \( \tau \) has the form (31), then (18) gives the triviality of (24). Assume that \( \tau \) has the form (30). Then by the same discussion as above we get
\[ \tau = (\rho g_1 + \alpha, f_2, f_3). \]

We have
\[ \varphi_{i+t} = \sigma(2, \eta^{-1}, F(x_1, x_2)), \quad \varphi_{i+t+1} = \sigma(1, \rho, \alpha). \]
By (19), we get
\[ \varphi_{i_1+t}\varphi_{i_1+t+1} = \sigma(1, \rho, \alpha)\sigma(3, \eta^{-1}, F_1(x_1, x_2)). \]

After the corresponding replacement, instead of \( \theta \) we obtain
\[ \theta' = (\rho g_1 + \alpha, \eta g_2 + \kappa g_1 + \nu, f_3). \]

Proposition 3.1 gives \( t(\theta') = 1 < t \); a contradiction.

Assume that
\[ \tau = (f_1, f_2, \rho f_3 + a), \quad a \in < f_1, f_2 >. \] (32)

By Proposition 2.1 we have \( \deg f_1 + \deg f_2 > \frac{9n}{2} \). Since \( d(\theta) = \frac{13n}{2} \geq \deg \tau \), we have \( \deg (\rho f_3 + a) < 2n \) and \( \deg a < 2n \). Lemma 2.4 gives \( a = \alpha \in F \). Then \( \varphi_{i_1+t+1} = \sigma(3, \rho, \alpha) \). Hence
\[ \varphi_{i_1+t}\varphi_{i_1+t+1} = \sigma(3, \rho, \alpha)\sigma(2, \eta^{-1}, F_1(x_1, x_2)). \]

After this replacement, \( \theta \) is changed to
\[ \theta' = (g_1, \eta g_2 + \kappa g_1 + \nu, \rho f_3 + a). \]

Proposition 3.1 gives \( t(\theta') = 1 \), which also leads to a contradiction.

The case when \( (\alpha_1, \alpha_2) \neq 0 \) and \( (\beta_1, \beta_2, \beta_3, \beta_4) = 0 \) can be considered analogously.

Assume that \( (\alpha_1, \alpha_2) \neq 0 \) and \( (\beta_1, \beta_2, \beta_3, \beta_4) \neq 0 \). According to Proposition 3.1, we have \( t = 3 \). Now we use the statement (3) of Lemma 4.3 about the arbitrariness of the minimal representation of \( \theta \) in (24). If \( \tau \) has the form (30), then we can assume that \( \phi \) has the form (6), and if \( \tau \) has the form (31), then we can assume that \( \phi \) has the form (5).

By (18), in both cases we can decrease the exponent of (24). If \( \tau \) has the form (32), then using the same arguments we get
\[ \tau = (f_1, f_2, \rho f_3 + \alpha). \]

Assume that \( \phi = (\lambda g_1 + \beta, f_2, f_3) \). Applying (19) to \( \varphi_{i_1+t}\varphi_{i_1+t+1} \) changes \( \theta \) to
\[ \theta' = (g_1, g_2, \rho f_3 + a). \]

Then we have \( t(\theta') = 2 \), which also leads to a contradiction. \( \square \)

Now we begin to consider the most complicated case when \( t = 0 \), i.e. \( q = 0, 1 \). Put \( \theta = (f_1, f_2, f_3) \). According to Lemma 4.3, \( t = 0 \) is the height of \( \theta \), i.e. \( \theta \) admits an essential elementary reduction. Moreover, \( \phi \) is an essential elementary reduction of \( \theta \). If \( q = 0 \), then \( \tau \) is also an essential elementary reduction of \( \theta \). If \( q = 1 \), then we have \( (d(\tau), t(\tau)) = (d, t) \). Consequently, \( \deg \tau \leq \deg \theta = d \). Thus we can assume that
\[ \tau = (f_1, f_2, f), \quad f = \alpha f_3 + a, \quad a = a(f_1, f_2) \in < f_1, f_2 >, \quad \deg a \leq \deg f_3. \] (33)

25
Lemma 4.5 If \( \phi \) reduces the element \( f_3 \) of \( \theta \), then the relation (24) is trivial.

Proof. Applying (18) we can replace \( \varphi_{i+t} \varphi_{i+t+1} \) by an elementary automorphism. Obviously, this replacement also decreases the exponent of (25). \( \square \)

By Lemma 4.5, we can assume that \( \phi \) reduces one of the elements \( f_1 \) and \( f_2 \) of \( \theta \). Without loss of generality, later on we consider that \( \phi \) reduces the element \( f_2 \) of \( \theta \), i.e. \( \phi = (f_1, g_2, f_3) \) and \( \deg g_2 < \deg f_2 \).

Lemma 4.6 If \( \phi' \) reduces the element \( f_2 \) of \( \theta \), then in (26) the automorphism \( \phi \) can be replaced by \( \phi' \).

Proof. According to (18), in this case the elementary transformation \( \phi \to \theta \) can be changed to \( \phi' \to \theta \). Since \( \deg \phi' < \deg \theta = d \), the exponent \( (d, q) \) of the sequence (25) does not change after this replacement. But in the new sequence (25) we have \( \phi' \) instead of \( \phi \). \( \square \)

Taking this lemma into account, we can assume that

\[
\phi = (f_1, g_2, f_3), \quad g_2 = f_2 - b, \quad b \in \langle f_1, f_3 \rangle, \quad \deg g_2 < \deg f_2.
\]  

(34)

Lemma 4.7 If one of the following conditions is fulfilled, then (24) is trivial:

1. \( f_2 \in \langle f_1 \rangle \);
2. \( f_3 \in \langle f_1 \rangle \);
3. \( a \) does not depend on \( f_2 \);
4. \( f_2 = \beta f_3 + \gamma f_1^l \);
5. \( f_1 \) and \( f_3 \) are algebraically independent.

Proof. Assume that \( \overline{f_2} \in \langle \overline{f_1} \rangle \) and \( \overline{f_2} = \beta \overline{f_1}^l \). According to Lemma 4.6, we can suppose that \( g_2 = f_2 - \beta f_1^l \). Then

\[
\varphi_{i+t} = \sigma(2, 1, \beta x_1^l), \quad \varphi_{i+t+1} = \sigma(3, \alpha, a(x_1, x_2)).
\]

By (19), we have

\[
\varphi_{i+t} \varphi_{i+t+1} = \sigma(3, \alpha, a_1(x_1, x_2)) \sigma(2, 1, \beta x_1^l).
\]

After the corresponding replacement in (24), \( \theta \) is replaced by \( \theta' = (f_1, g_2, f) \) in (25). Since \( \deg f_1 + \deg g_2 + \deg f < d \), the exponent of (24) is decreased.

Assume that \( \overline{f_3} \in \langle \overline{f_1} \rangle \) and \( \overline{f_3} = T(\overline{f_1}) \). Put \( g_3 = f_3 - T(f_1) \). According to (19), we have

\[
\varphi_{i+t} = \sigma(2, 1, b(x_1, x_3)) = \sigma(3, 1, -T(x_1)) \sigma(2, 1, b_1(x_1, x_3)) \sigma(3, 1, T(x_1)).
\]

After the corresponding replacement in (24), the elementary transformation \( \phi \to \theta \) is replaced by the sequence of elementary transformations

\[
\phi = (f_1, g_2, f_3) \to (f_1, g_2, g_3) \to (f_1, f_2, g_3) \to (f_1, f_2, f_3) = \theta.
\]
Since $d(\phi), \deg(f_1, g_2, g_3), \deg(f_1, f_2, g_3) < d = \deg \theta$, the new sequence (25) has the same exponent $(d, q)$. However, instead of $\phi$ we have $(f_1, f_2, g_3)$. By Lemma 4.5, we obtain the triviality of (24).

Assume that $a$ does not depend on $f_2$. By (19) we have

$$\varphi_{i_1+t}\varphi_{i_1+t+1} = \sigma(2, 1, b(x_1, x_3))\sigma(3, \alpha, a(x_1)) = \sigma(3, \alpha, a(x_1))\sigma(2, 1, b_1(x_1, x_3)).$$

After the corresponding replacement in (24), instead of $\theta$ we obtain $\theta' = (f_1, g_2, f)$. Since $\deg(f_1, g_2, f) < d$, this replacement also decreases the exponent of (24).

Consider the case when $f_2 = \beta f_3 + \gamma f_1^k$. By Lemma 4.7(1) we can assume that $\beta \neq 0$. By Lemma 4.6 we can also assume that $b = \beta f_3 + \gamma f_1^k$. Consequently,

$$f_2 = g_2 - \beta f_3 - \gamma f_1^k,$$

$$f_3 = g_2 + \beta f_3 + \gamma f_1^k,$$

$$f_3 = -\frac{1}{\beta} g_2 + \frac{1}{\beta} f_2 - \frac{\gamma}{\beta} f_1^k.$$

These equalities justify the sequence of elementary transformations

$$(f_1, f_3, g_2) \rightarrow (f_1, f_2, g_2) \rightarrow (f_1, f_2, f_3) = \theta.$$

We have

$$\varphi_{i_1+t} = \sigma(2, 1, \beta x_3 + \gamma x_1^k)).$$

Applying (18) and (19) we get

$$\varphi_{i_1+t} = \sigma(2, 1, \beta x_3)\sigma(2, 1, \gamma x_1^k)$$

$$= \sigma(2, 1, \beta x_3)\sigma(3, -\beta, x_2)\sigma(3, -\frac{1}{\beta}, \frac{1}{\beta} x_2)\sigma(2, 1, \gamma x_1^k)$$

$$= \sigma(2, 1, \beta x_3)\sigma(3, -\beta, x_2)\sigma(2, 1, \gamma x_1^k)\sigma(3, -\frac{1}{\beta}, \frac{1}{\beta} x_2 - \gamma x_1^k))$$

$$= \sigma(2, 1, \beta x_3)\sigma(3, -\beta, x_2)\sigma(2, 1, \gamma x_1^k)\sigma(3, -\frac{1}{\beta}, \frac{1}{\beta} x_2 - \gamma x_1^k))$$

$$= \sigma(2, 1, \beta x_3)\sigma(3, -\beta, x_2)\sigma(2, 1, \gamma x_1^k)\sigma(3, -\frac{1}{\beta}, \frac{1}{\beta} x_2 - \gamma x_1^k)).$$

Since the transposition $(23) \in S_3$ has a linear representation

$$(23) = \sigma(2, 1, \beta x_3)\sigma(3, -\beta, x_2)\sigma(2, 1, \frac{1}{\beta}, -\frac{1}{\beta} x_3),$$

we obtain

$$\varphi_{i_1+t} = (23)\sigma(2, 1, \beta x_3 + \gamma x_1^k))\sigma(3, -\frac{1}{\beta}, \frac{1}{\beta} x_2 - \gamma x_1^k)).$$
Then

$$\theta = (23)\varphi_1^{(23)}\varphi_2^{(23)} \cdots \varphi_{i+t-1}^{(23)}(2, \beta, x_3 + \gamma x_1^k))\sigma(3, -\frac{1}{\beta}, \frac{1}{\beta}(x_2 - \gamma x_1^k)), \quad (35)$$

where \(\varphi_i^{(23)}\) are elementary automorphisms, according to (20). To (35) corresponds the sequence of elementary transformations

$$(x_1, x_2, x_3) \mapsto (x_1, x_3, x_2) \mapsto \theta_1' \mapsto \theta_2' \mapsto \ldots \mapsto \theta_{i+t-1}''$$

where \(\theta_i'\) is obtained from \(\theta_i\) only by the permutation of the second and the third components, and the automorphism \((x_1, x_2, x_3) \mapsto (x_1, x_3, x_2)\) is a composition of three elementary linear transformations.

If in (24) we replace \(\theta\) by (35), then the exponent of (25) remains the same. But instead of \(\phi\) we have \((f_1, f_2, g_2)\), and Lemma 4.5 gives the triviality of (24).

We now consider the case when \(\overline{f_1}\) and \(\overline{f_3}\) are algebraically independent. Then \(\overline{f_2} = b \in \langle \overline{f_1}, \overline{f_3} \rangle\). By Lemma 4.7(1) we can assume that \(\overline{f_2} \notin \langle \overline{f_1} \rangle\), i.e. \(\overline{f_2}\) depends on \(\overline{f_3}\). Consequently, \(\deg f_3 \leq \deg f_2\). If \(\overline{f_1}\) and \(\overline{f_2}\) are algebraically dependent, then it follows that \(\overline{f_1}\) and \(\overline{f_3}\) are algebraically dependent. Consequently, \(\overline{f_1}\) and \(\overline{f_2}\) are algebraically independent. Then \(\sigma \in \langle \overline{f_1}, \overline{f_2} \rangle\). By Lemma 4.7(3) we can assume that \(a\) necessarily contains \(f_2\). Then \(\deg f_2 \leq \deg a \leq \deg f_3\), i.e. \(\deg f_2 = \deg f_3\). Hence

$$b = \overline{f_2} = \beta \overline{f_3} + \gamma \overline{f_1}.$$  

From the statement (4) of the lemma we obtain that (24) is trivial. \(\square\)

So, by Lemma 4.7, we can assume that \(\overline{f_1}\) and \(\overline{f_3}\) are algebraically dependent and that \(\overline{f_3} \notin \langle \overline{f_1} \rangle\). It remains to consider the following three cases separately:

1. \(f_1, f_3\) is a \(*\)-reduced pair and \(\deg f_1 < \deg f_3\);
2. \(f_1, f_3\) is a \(*\)-reduced pair and \(\deg f_3 < \deg f_1\);
3. \(\overline{f_1} \in \langle \overline{f_3} \rangle\) and \(\deg f_1 > \deg f_3\).

**Lemma 4.8** If \(f_1, f_3\) is a \(*\)-reduced pair and \(\deg f_1 < \deg f_3\), then (24) is trivial.

**Proof.** We first consider the case when \(\deg f_2 > \deg f_3\). If \(\overline{f_1}\) and \(\overline{f_2}\) are algebraically independent, then \(\sigma \in \langle \overline{f_1}, \overline{f_2} \rangle\). Since \(\deg a \leq \deg f_3 < \deg f_2\), we have \(a \in \langle f_1 \rangle\). Lemma 4.7 gives the triviality of (24).

Suppose that \(\overline{f_1}\) and \(\overline{f_2}\) are algebraically dependent. By Lemma 4.7, we can assume that \(\overline{f_2} \notin \langle \overline{f_1} \rangle\). Since \(\deg f_1 < \deg f_3 < \deg f_2\), therefore \(f_1, f_2\) is a \(*\)-reduced pair.

Assume that \(\deg a < N = N(f_1, f_2)\). Since \(\deg a \leq \deg f_3 < \deg f_2\), Corollary 2.1 gives \(a \in \langle f_1 \rangle\). By Lemma 4.7, the relation (24) is trivial.

Therefore we can assume that \(\deg a \geq N\). Then \(\deg f_2 > N = N(f_1, f_2)\). By the definition of \(N = N(f_1, f_2)\), this is possible only if \(f_1, f_2\) is a 2-reduced pair. Consequently, \(\deg f_1 = 2n\), \(\deg f_2 = sn\), where \(s \geq 3\) is an odd number, and moreover

$$N = n(s - 2) + \deg [f_1, f_2] \leq \deg a \leq \deg f_3.$$  

28
Let \((f_1, f_2, g_3)\) be an elementary reduction of \(\theta\) such that \(g_3\) is an irreducible element of \((f_1, f_2, g_3)\). If \(f_3\) is an irreducible element of \(\theta\), then we put \(g_3 = f_3\). Consequently, \((f_1, f_2, g_3)\) satisfies the conditions of Proposition 2.2. Assume that Proposition 2.2(1) is valid for \((f_1, f_2, g_3)\), i.e.

\[
\deg g_3 < n(s - 2) + \deg [f_1, f_2] = N.
\]

Since \(\deg f_3 \geq N\), it means that

\[
f_3 = g_3 + c(f_1, f_2), \quad c(f_1, f_2) = \overline{f_3}.
\]

By Lemma 4.7 we can assume that \(c(f_1, f_2) = \overline{f_3} \notin \langle f_1 \rangle\). Since \(\deg c(f_1, f_2) = \deg f_3 < \deg f_2\), we have \(c(f_1, f_2) \notin \langle f_1, f_2 \rangle\). It is easy to deduce from (1) that \(\deg_x c(x, y)\) equals 2 or 4. Then \(\deg_y \frac{\partial c}{\partial y}(x, y)\) equals 1 or 3, and (1) gives \(\frac{\partial c}{\partial y}(f_1, f_2) \geq \deg f_2\). We have

\[
\deg [f_1, g_3] \leq \deg f_1 + \deg g_3 < ns + \deg [f_1, f_2] = \deg f_2 + \deg [f_1, f_2].
\]

Then

\[
[f_1, f_3] = [f_1, g_3] + [f_1, f_2] \frac{\partial c}{\partial y}(f_1, f_2)
\]

gives that

\[
\deg [f_1, f_3] = \deg [f_1, f_2] \frac{\partial c}{\partial y}(f_1, f_2) \geq \deg [f_1, f_2] + \deg f_2.
\]

Consequently, \(N(f_1, f_3) > \deg f_2\). By (34) we have \(\deg b = \deg f_2 < N(f_1, f_3)\). Corollary 2.1 gives \(\overline{f_2} \in \langle \overline{f_1}, \overline{f_3} \rangle\). Since \(\deg f_1 f_3, \deg f_3^2 > \deg f_2\), we obtain \(\overline{f_2} \in \langle \overline{f_1} \rangle\); a contradiction.

We now assume that one of the statements (2) and (3) of Proposition 2.2 is valid for \((f_1, f_2, g_3)\). Combining both the cases we can say that

\[
f_3 = h_3 + r(f_1, f_2), \quad \deg f_2 = 3n, \quad \deg (f_2 - \beta h_3^2) \leq 2n, \quad \deg [f_1, h_3] < n + \deg [f_1, f_2].
\]

Since \(\deg f_3 < 2n\) and \(\deg h_3 = \frac{3n}{2}\), it follows that \(\deg r(f_1, f_2) < 2n\). By Corollary 2.2, we have \(r(f_1, f_2) = \lambda w(f_1, f_2) + \mu\), where \(w(x, y)\) is a derivative polynomial of the pair \(f_1, f_2\). If \(\lambda \neq 0\), then an immediate calculation gives \(\deg [f_1, f_3] > \deg f_2\). This leads to a contradiction, as above. Then \(\lambda = 0\), and we can assume that \(f_3 = h_3\). Combining (33) and Corollary 2.2 we also conclude that

\[
a = \lambda w(f_1, f_2) + \mu, \quad f = \alpha f_3 + \lambda w(f_1, f_2) + \mu.
\]

If \(\lambda \neq 0\), then \(\tau\) admits a reduction of type IV. Otherwise \(a = \mu \in F\) and we can apply Lemma 4.7.

Now, suppose that \(\deg f_2 \leq \deg f_3\). If \(\overline{f_2} \in \langle \overline{f_1}, \overline{f_3} \rangle\), then \(\overline{f_2} = \beta \overline{f_3} + \gamma \overline{f_1}\). Hence Lemma 4.7 gives the triviality of (24).
Let $\overline{b} = \overline{f_2} \notin \langle \overline{f_1}, \overline{f_3} \rangle$. Since $\deg b = \deg f_2 \leq \deg f_3$, applying Corollary 2.1 gives $\deg f_3 \geq N(f_1, f_3)$. By the definition of $N(f_1, f_3)$, we see that $f_1, f_3$ is a 2-reduced pair, i.e., $\deg f_1 = 2n$ and $\deg f_3 = ns$, where $s \geq 3$ is an odd number. According to Lemma 4.6, we can assume that $g_2$ is an irreducible element of $\phi = (f_1, g_2, f_3)$. Then $\phi$ satisfies the conditions of Proposition 2.2.

If $\deg_y b(x, y) = k$, then (1) gives $k = 2$ if $s > 3$, and $k = 2, 4$ if $s = 3$. Consequently, $\deg_y (\frac{\partial b}{\partial y}(x, y)) = k - 1$ which equals 1 or 3, and (1) gives also $\deg (\frac{\partial b}{\partial y}(f_1, f_3)) \geq ns$. We have

$$[f_1, f_2] = [f_1, g_2] + [f_1, f_3] \frac{\partial b}{\partial y}(f_1, f_3). \quad (36)$$

Assume that Proposition 2.2(1) is valid for $\phi$, i.e., $\deg g_2 < n(s - 2) + \deg [f_1, f_3]$. Comparing the degrees of elements in (36) we obtain

$$\deg [f_1, f_2] \geq \deg [f_1, f_3] + ns. \quad (37)$$

Consequently, $\deg f_2 \geq n(s - 2) + \deg [f_1, f_3] > n$ and $\overline{f_1} \notin \langle \overline{f_2}, \overline{f_3} \rangle$. Then either $\overline{f_1}$ and $\overline{f_2}$ are algebraically independent or $f_1, f_2$ is a $\ast$-reduced pair. Note that $N(f_1, f_2) > \deg [f_1, f_2] > ns \geq \deg a$. By Corollary 2.1, in both cases we have $a \in \langle \overline{f_1}, \overline{f_2} \rangle$. Since $\deg f_2^3 > \deg f_3 \geq \deg a$ and $\deg f_1 f_2 > \deg f_3$, it follows that

$$a = \beta f_2^2 + \gamma f_2 + G(f_1), \quad \deg(G(f_1)) \leq \deg f_3;$$

moreover, we can have $\beta \neq 0$ only if $\deg f_3 \geq 2 \deg f_2$. Note that the equality $\deg(G(f_1)) = \deg f_3$ is impossible. Consequently, $\deg(G(f_1)) < \deg f_3$. Since $f_1, f_3$ is a 2-reduced pair, it follows that $f_1, \alpha f_3 + G(f_1)$ is also a 2-reduced pair. We have

$$\tau = (f_1, f_2, \alpha f_3 + \beta f_2^2 + \gamma f_2 + G(f_1)).$$

If $(\beta, \gamma) \neq (0, 0)$, then it is easy to check that $(f_1, g_2, \alpha f_3 + G(f_1))$ is a reduction of types I–III of the automorphism $\tau$ with the active element $f_2$. Proposition 3.1 gives $d(\tau) = n(s + 2) + \deg f_2 = d = \deg \theta$ and $t(\tau) \geq 1$. This contradicts the inequality $(d(\tau), t(\tau)) \leq (d, t)$. If $\beta = \gamma = 0$, then we apply Lemma 4.7.

We now consider the case when Proposition 2.2(3) is valid for $\phi$. Then (36) again gives (37). Besides, in this case $s = 3$ and $\deg f_2 > \deg g_2 = \frac{3n}{2}$. Consequently, the same argument as above gives $a = \beta f_2^2 + \gamma f_1 + \delta$. By Lemma 4.7, we can assume that $\beta \neq 0$. Then $(f_1, g_2, \alpha f_3 + \gamma f_1 + \delta)$ is a reduction of type I or II of $\tau$ with the active element $f_2$. The value of $(d(\tau), t(\tau))$, which can be calculated by Proposition 3.1, again contradicts the inequality $(d(\tau), t(\tau)) \leq (d, t)$.

At last we consider the case when Proposition 3.1(2) is valid for $\phi$. Let $(f_1, h_2, f_3 - \delta h_2^2)$ be a reduction of type IV of $\phi$, where

$$g_2 = h_2 + g, \quad g \in \langle f_1, f_3 \rangle \setminus F, \quad \deg(f_3 - \delta h_2^2) \leq 2n.$$

Since $\deg g_2 \leq \deg h_2 = \frac{3n}{2}$, we have $\deg g \leq \frac{3n}{2}$. By Definition 2.4, Proposition 2.2(3) is valid for $(f_1, h_2, f_3)$.
Lemma 4.10

We first consider the case when $\deg f > \frac{3n}{2}$, then by Lemma 4.6 we can assume that $\phi = (f_1, h_2, f_3)$ and this case reduces to the preceding one.

Assume that $\deg f \leq \frac{3n}{2}$. By Proposition 3.1, we have $d(\phi) = \frac{13n}{2}$, $t(\phi) = 1$. Since $d = \deg \theta \leq \frac{13n}{2}$, this contradicts the inequality $(d(\phi), t(\phi)) < (d, t)$.

Lemma 4.9

If $f_1, f_3$ is a \textit{\$}\$-reduced pair and $\deg f_3 < \deg f_1$, then (24) is trivial.

Proof. We first consider the case when $\deg f_1 < \deg f_2$. Since $\deg a \leq \deg f_3 < \deg f_1 < \deg f_2$, applying Lemma 4.7 we conclude that $\pi \notin \langle f_1, f_2 \rangle$. Consequently, $f_1$ and $f_2$ are algebraically dependent. By Lemma 4.7, we assume that $f_2 \notin \langle f_1 \rangle$. Then $f_1, f_2$ is a \textit{\$}\$-reduced pair. Corollary 2.1 gives also $\deg a \geq N(f_1, f_2)$. Combining these inequalities and the definition of $N(f_1, f_2)$ we conclude that $\deg f_1 = 2n$ and $\deg f_2 = 3n$. Consider an elementary reduction $(f_1, f_2, g_3)$ of $\theta$ with an irreducible $g_3$ (assume that $g_3 = f_3$ if $f_3$ is an irreducible element of $\theta$). Then $(f_1, f_2, g_3)$ satisfies the conditions of Proposition 2.2. By the same discussions related to the automorphism $(f_1, f_2, g_3)$ as in the proof of Lemma 4.8, we obtain the triviality of (24).

If $\deg f_1 = \deg f_2$, then by Lemma 4.7 we may assume that $f_1$ and $f_2$ are linearly independent. Then $f_1$ and $f_2$ are algebraically independent. Consequently, $a \in F$.

It remains to consider the case when $\deg f_2 < \deg f_1$. Assume that $f_2 \in \langle f_3 \rangle$. By Lemma 4.7 we can assume that $\langle f_2 \rangle$ and $\langle f_3 \rangle$ are linearly independent. Then $\deg f_2 = t \cdot \deg f_3$, $t \geq 2$. In this case $f_1 \notin \langle f_2 \rangle$, since $f_1, f_3$ is a \textit{\$}\$-reduced pair. Consequently, $f_1, f_3$ is also a \textit{\$}\$-reduced pair. Since $\deg a \leq \deg f_3 < \min \{ \deg f_1, \deg f_2 \}$, applying Corollary 2.1 and Lemma 4.7 we obtain $\deg f_2 = 2n$, $\deg f_1 = 3n$, and

$$\deg f_3 \geq \deg a \geq N(f_1, f_2) = n + \deg [f_1, f_2] > \frac{2n}{t} = \deg f_3,$$

which gives a contradiction.

Therefore we can assume that $\langle f_2 \rangle \notin \langle f_3 \rangle$, i.e. $b = f_2 \notin \langle f_1, f_3 \rangle$. Then Corollary 2.1 gives $N(f_1, f_3) \leq \deg f_2 < \deg f_1$. By the definition of $N(f_1, f_3)$ this is possible only if $f_1, f_3$ is a 2-reduced pair, and $\deg f_3 = 2n$, $\deg f_1 = ns$, where $s \geq 3$ is an odd number. By Lemma 4.6 we can assume that $g_2$ is an irreducible element of $\phi = (f_1, g_2, f_3)$. Then $\phi$ satisfies the conditions of Proposition 2.2. Applying the same part of discussions related to the automorphism $\phi$ as in the proof of Lemma 4.8, we obtain the triviality of (24).

Lemma 4.10

If $f_1 \in \langle f_3 \rangle$ and $\deg f_1 \geq \deg f_3$, then the relation (24) is trivial.

Proof. Assume that $n = \deg f_3$ and $f_1 = \beta f_3^k$, $k \geq 2$. Then $\deg f_1 = nk$. By (19) we have

$$\varphi_{i+1} = \sigma(2, 1, b(x_1, x_2)) = \sigma(1, 1, -\beta x_3^k) \sigma(2, 1, b(x_1, x_2)) \sigma(1, 1, \beta x_3^k).$$

After such replacement, instead of $\phi \to \theta$ we obtain

$$\phi \to (f_1 - \beta f_3^k, g_2, f_3) \to (f_1 - \beta f_3^k, f_2, f_3) \to \theta.$$
Since \( \text{deg} (f_1 - \beta f_3^k, g_2, f_3), \text{deg} (f_1 - \beta f_3^k, f_2, f_3) < d = \text{deg} \theta \), the new sequence (25) has the same exponent. Then instead of \( \phi \) we can take \((f_1 - \beta f_3^k, f_2, f_3)\), i.e. we can assume that \( \phi \) reduces the element \( f_1 \) of \( \theta \). Interchanging the elements \( f_1 \) and \( f_2 \) and applying Lemmas 4.7, 4.8, and 4.9 we can restrict ourselves to the case when \( \overline{f}_2 \in (\overline{f}_3) \), \( \text{deg} f_2 > \text{deg} f_3 \). Then \( \text{deg} f_2 = nr, r \geq 2 \). By Lemma 4.7 we may assume that \( f_1, f_2 \) is a *-reduced pair, i.e. \( \text{deg} f_1 = mp, \text{deg} f_2 = ms, (p, s) = 1, m \geq n \). Hence \( N(f_1, f_2) > m(ps - p - s) > n \geq \text{deg} a \). Corollary 2.1 gives \( a \in (\overline{f}_1, \overline{f}_2) \), i.e. \( a \in F \). \( \square \)

This finishes the proof of Theorem 4.1.

Now we formulate some problems related to the relations (18)–(20) and discuss why it is important to study them.

Let us denote by \( G(A_n) \) the group defined by the system of generators (17) and the system of relations (18)–(20).

**Problem 4.1** Find a canonical form of elements of the group \( G(A_n) \). In particular, is the word problem decidable in this group?

Note that the group \( G(A_2) \cong GA_2(F) \) has a nice representation (see [2]). By Theorem 4.1, if \( F \) is a field of characteristic 0, then \( G(A_3) \cong TA_3(F) \). Note that the group \( GA_n(F) = Aut A_n \) has the decidable word problem. In fact, if \( \phi = (f_1, f_2, \ldots, f_n) \in GA_n(F) \), then \( \phi = id \) iff \( f_i = x_i, 1 \leq i \leq n \). Consequently, the groups \( G(A_2) \) and \( G(A_3) \) have the decidable word problem. But we have no canonical form of elements of \( G(A_3) \).

Now, consider the homomorphism

\[
s_n : G(A_n) \longrightarrow TA_n(F)
\]

which sends the generators (17) to the corresponding automorphisms of \( A_n \).

**Problem 4.2** Find the kernel of \( s_n \).

We know that the kernel of \( s_2 \) is trivial. By Theorem 4.1 the kernel of \( s_3 \) is also trivial if \( char(F) = 0 \). A solution of Problem 4.2 gives a description of the group \( TA_n(F) \) by generators and relations.

**Acknowledgments**

I am grateful to I. Shestakov for thoroughly going over the details of the proofs. I am also grateful to Max-Planck Institute für Mathematik for hospitality and excellent working conditions. I also thank J. Alev, P. Cohn, N. Dairbekov, M. Jibladze, L. Makar-Limanov, D. Wright, and M. Zaidenberg for numerous helpful comments and discussions.
References

[1] P. M. Cohn, Subalgebras of free associative algebras, Proc. London Math. Soc., 56(1964), 618–632.

[2] P. M. Cohn, Free rings and their relations, 2nd Ed., Academic Press, London, 1985.

[3] H. S. M. Coxeter, W. O. J. Mozer, Generators and relations for discrete groups, Springer-Verlag, Berlin, Heidelberg, New York, 1980.

[4] A. G. Czerniakiewicz, Automorphisms of a free associative algebra of rank 2, I, II, Trans. Amer. Math. Soc., 160(1971), 393–401; 171(1972), 309–315.

[5] A. van den Essen, Polynomial automorphisms and the Jacobian conjecture, Progress in Mathematics, 190, Birkhauser verlag, Basel, 2000.

[6] M. Kh. Gizatullin, Defining relations for the Cremona group of the plane, Izv. Akad. Nauk SSSR, Ser. Mat., 46 (1982), no. 5, 909–970; English translation: in Math. USSR Izvestiya, 21 (1983), 211–268.

[7] V. A. Iskovskikh, Proof of a theorem on relations in two dimensional Cremona group, Uspekhi Mat. Nauk, 40 (1985), no. 5 (245), 255–256.

[8] H. W. E. Jung, Uber ganze birationale Transformationen der Ebene, J. reine angew. Math. 184(1942), 161–174.

[9] W. van der Kulk, On polynomial rings in two variables, Nieuw Archief voor Wiskunde, (3)1(1953), 33–41.

[10] L. Makar-Limanov, The automorphisms of the free algebra of two generators, Funkcional. Anal. i Prilozhen., 4(1970), no.3, 107-108; English translation: in Functional Anal. Appl. 4 (1970), 262–263.

[11] J. Milnor, Introduction to algebraic K-theory, Princeton University Press, Princeton, N.J., and Univ. of Tokyo Press, Tokyo, 1971.

[12] M. Nagata, On the automorphism group of $k[x, y]$, Lect. in Math., Kyoto Univ., Kinokuniya, Tokio, 1972.

[13] I. P. Shestakov, U. U. Umirbaev, The Nagata automorphism is wild, Proc. Nat. Acad. Sci. USA, 100 (2003), No. 22, 12561–12563.

[14] I. P. Shestakov and U. U. Umirbaev, Poisson brackets and two generated subalgebras of rings of polynomials, Journal of the American Mathematical Society, 17 (2004), 181–196.
[15] I. P. Shestakov and U. U. Umirbaev, Tame and wild automorphisms of rings of polynomials in three variables, Journal of the American Mathematical Society, 17 (2004), 197–227.

[16] U. U. Umirbaev, I. P. Shestakov, Subalgebras and automorphisms of polynomial rings, Dokl. Akad. Nauk, 386 (2002), no. 6, 745-748.

[17] U. U. Umirbaev, Tame and wild automorphisms of polynomial algebras and free associative algebras, Preprint of Max-Planck Institute für Mathematik MPIM2004-108.

[18] K. Vogtmann, Automorphisms of free groups and outer spaces, Geometriae Dedicata, 94 (2002), 1–31.

[19] D. Wright, Two-dimensional Cremona groups acting on simplicial complexes, Trans. Amer. Math. Soc., 331 (1992), no. 1, 281–300.