Stress intensity factor of circumferential periodic cracks on tube with stiffened plates under tension

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Abstract. Fracture is one of the most common failure of engineering structure and the stress intensity factor is the key parameter used to decide whether the structure will crack or not. This essay is based on the concept of conservation law and the tension theory in mechanics of material. It analyzes the stress intensity factor of circumferential periodic cracks on tube with stiffened plates, it obtains the specific expression of stress intensity factor of circumferential periodic cracks on tube with stiffened plates under tension. Finite element method is used for checking the stress intensity factor of cracks and proving the accuracy of expression’s results. The method which proposed in this essay is easy to calculate and can get the closed solution.

1. Introduction
For the crack problems of infinite two-dimensional and three-dimensional elastic bodies, a closed analytical solution of the stress intensity factor (SIF) can be given relatively conveniently, when considering the object boundary influence on the crack, which is limited in 2D and 3D cracks, such as unilateral crack, tube with stiffened plates, to give the analytical solution which can satisfying the boundary conditions is very difficult, this must depend on numerical methods. In recent years, the \(J_2\)-integral theory was proposed by Xie et al\([1]\), which is derived from the three-dimensional conservation law and the principle of virtual work. Because of the finite and infinite boundary crack problem, the \(J_2\)-integral method greatly facilitates the analysis of stress intensity factors\([2\gu 7]\).

The tube with stiffened plates is the components which commonly used in engineering, while circumferential periodic cracks on tube with stiffened plates and circular cracks on tube are typical belong to complex three-dimensional crack structures. In this paper, A method of solving the SIFs of the circumferential periodic cracks on tube with stiffened plates and circular cracks on tube by the conservation law and the tension theory in material mechanics is established.

2. Configuration of circumferential periodic cracks on tube with stiffened plates
There has \(2m\) stiffened plates which full of the periodic cracks in a circular tube. Figure 1 gives the case of \(m=2\). For the thin-walled tube, it has the characteristics of the three-dimensional shell and the slender beam, and the crack on the stiffened plate has the characteristics of the two-dimensional stress field. So the stress intensity factor can be obtained by the conservation law and the tension theory in the material mechanics.
Figure 1. Periodic cracks in eight fins of fin-shaped shells under tension.

3. $J_2$-Integral and stress intensity factors

Considering a three-dimensional strain field, for which the displacement vector $u_i$ depends on $x_1, x_2, x_3$, the conservation law $J_j$-integral can be defined as [7]:

$$J_j = \int_A \left( w u_j - T_i u_{ij} \right) dA = 0, \quad j = 1, 2, 3. \quad (1)$$

The $A$ in equation (1) is an arbitrary closed surface, in the solids closed by which there are no defects; $w$ is the strain energy density; $T_i$ is the stress vector acting on the outer side of $A$; $n$ is the unit outward normal to $A$. For a two-dimensional deformation field. Figure 2 gives a two-dimensional crack model of unit thickness (the integral path is a curve in the $x_1$-$x_2$ plane). Let $S$ represent to a closed contour $S_{abc}$ within the K-dominant region around upper crack tip. Note that $S_{ab}$ is a straight line and $S_{bc}$ is a quarter of circle. As such, the following results can be obtained by calculating the stress and displacement next to crack tip

$$J_2 = \int_{S_{abc}} \left( w n_2 - T_i u_{i2} \right) ds = \frac{\left( 1 - \mu^2 \right) K_j^2}{2\pi E}. \quad (2)$$

Figure 2. Integration path around crack tip region.

For the closed path $S_{abc} = S_{abc} + S_{ac}$, from the conservation law [6]. It follows that

$$\int_{S_{abc}} \left( w n_2 - T_i u_{i2} \right) ds = \int_{S_{abc}} \left( w n_2 - T_i u_{i2} \right) ds + \int_{S_{ac}} \left( w n_2 - T_i u_{i2} \right) ds = \frac{\left( 1 - \mu^2 \right) K_j^2}{2\pi E} + \int_{S_{ac}} \left( w n_2 - T_i u_{i2} \right) ds = 0, \quad (3)$$

then
where $\mu$ represents the Poisson’s ratio and $E$ the elastic modulus.

4. Circumferential periodic cracks on tube with inner stiffened plates under tension

4.1. A tube full of periodic cracks on the stiffened plates

The Figure 1 show a tube full of periodic cracks on the stiffened plates, $b$ is the height of the stiffened plate, $t$ is the thickness of the stiffened, $a$ is the length of the periodic crack, $2R$ is the intermediate diameter of tube, $t_0$ is the thickness of tube. $N$ is the axial force. The Figure 1 also show a closed surface $A_{closed}$,

$$A_{closed} = A^+ + A^- + A_{in} + A_{out} + A_c$$

$A^+$ is the cracked cross-section; $A^-$ is the remote uncracked cross-section; $A_{in}$ is the inner surface of shell; $A_{out}$ is the outer surface of shell; $A_c$ is the sum of the eight crack surfaces. On the free surface $A_{in}$ and $A_{out}$, $T_1 = 0$ and $n_2 = 0$. Then the following results can be found

$$\iint_{A_{in}} (w n_z - T u_{1z}) dA = 0, \quad (6)$$

$$\iint_{A_{out}} (w n_z - T u_{1z}) dA = 0. \quad (7)$$

As the fin-shaped shells possess beam characteristics, stresses and displacements in the $J_2$-integral can be given by elementary mechanics. Then for the surfaces $A^+$ and $A^-$, it is not difficult to get:

$$\iint_{A^+} (w n_z - T u_{1z}) dA = -\frac{N}{2} \bar{\mu}^+_{22}, \quad (8)$$

$$\iint_{A^-} (w n_z - T u_{1z}) dA = \frac{N}{2} \bar{\mu}^-_{22}, \quad (9)$$

$\bar{\mu}^+_{22}$ is the axial strain of fin-shaped shells, $\bar{\mu}^-$ is the strain energy density per unit length. All the quantities in equations (8) and (9) can be calculated by the beam theory. The axial strain of remote uncracked cross-section is

$$\bar{\mu}^-_{22} = \frac{N}{EA}, \quad (10)$$
where the moment of inertia $A = 2\pi R t_0 + 2mbt$. If the crack is formed from an ellipse with $d \to 0$ as shown in Figure 3, the fin-shaped shell can be regarded as a variable cross-section beam, and the average axial strain at the cross section becomes

$$\bar{\mu}_{22} = \lim_{d \to 0} \frac{1}{d} \int_{d}^{b} N \frac{dx_2}{EA} \frac{1}{1 - \gamma_1 \left( \frac{a}{b} \right) \left( 1 - \left( \frac{x_2}{d} \right)^2 \right)^{\gamma_2}} \left( \frac{1 + \gamma_1 \left( \frac{a}{b} \right)}{1 - \gamma_1 \left( \frac{a}{b} \right)} \right)^{2 \gamma}$$

where

$$\gamma_1 \left( \frac{a}{b} \right) = \frac{2mta}{2mRt_0 + 2mbt} = \frac{\left( \frac{a}{b} \right)}{1 + \left( \frac{\pi}{m} \right) \left( \frac{R}{b} \right) \left( \frac{t_0}{t} \right)}.$$ (12)

Then, from equations (3)-(12), the $J_2$-integral over the closed surface can be expressed as

$$\frac{m}{\pi E} \left( 1 - \mu^2 \right) K_i^2 + \oint_{\lambda_s} w dA = \frac{N}{2} \left( \bar{\mu}_{22} - \bar{\mu}_{22} \right).$$ (13)

### 4.2. Stress intensity factor

Substituting the equation (10) and equation (11) into equation (13), the $J_2$-integral over the closed surface can be expressed as

$$\frac{m}{\pi E} \left( 1 - \mu^2 \right) K_i^2 + 2m \oint_{\lambda_s} w dA = \frac{N}{2EA} \left[ \frac{2}{\gamma_1 \left( \frac{a}{b} \right) \left( 1 - \gamma_1 \left( \frac{a}{b} \right) \right)^{\gamma_2}} \arctan \left( \frac{1 + \gamma_1 \left( \frac{a}{b} \right)}{1 - \gamma_1 \left( \frac{a}{b} \right)} \right)^{\gamma_2} - \frac{\pi}{2\gamma_1 \left( \frac{a}{b} \right) - 1} \right].$$ (14)

Because of the free action of the crack surface with $\lambda_{ch}$ just out of the $K$-dominant region, the integral in the left hand side of equation (14) is a small quantity and can be neglected. Hence, the normalized stress intensity factors can be found as

$$f_i \left( \frac{a}{b} \right) = \frac{K_i}{\sigma_0 \sqrt{\pi b}} = \left[ \frac{\left( \frac{a}{b} \right)}{\gamma_1 \left( \frac{a}{b} \right) \left( 1 - \mu^2 \right)} \left( \frac{2}{\gamma_1 \left( \frac{a}{b} \right) \left( 1 - \gamma_1 \left( \frac{a}{b} \right) \right)^{\gamma_2}} \arctan \left( \frac{1 + \gamma_1 \left( \frac{a}{b} \right)}{1 - \gamma_1 \left( \frac{a}{b} \right)} \right)^{\gamma_2} - \frac{\pi}{2\gamma_1 \left( \frac{a}{b} \right) - 1} \right]^{1/2}.$$ (15)

where $\sigma_0 = N/A$.

### 4.3. Comparison between numerical examples and results

This section will give a discussion on the comparison between the normalized stress intensity factor expressed by equation (15) and the results from the finite element method (EFM). Computations were used XFEM in the commercial software Abaqus [8-9], in which the C3D20 type element was employed. The elastic modulus is $E=200GPa$ and Poisson’s ratio is $\mu=0.3$. The length of tube with stiffened plates is $500mm$, $m=2$, $2R=200mm$, $t=10mm$, $t_0=3mm$, $b=60mm$, Figure 5 is show the finite element model and meshes and Figure 6 is used to compare the results of equation (15) with the finite element analysis.
Figure 4. The finite element model and meshes.

Figure 5. The comparisons between the present results and the finite element results

\( R/b=1.67, \pi/m=1.57, t_0/t=0.3 \)

5. Circular crack on inner surface of tube under tension

5.1. Circular crack on inner surface of tube

The \( i \)-th crack surface \( A_{ci} \)

When the number of stiffened plates in the tube tends to be infinite \((m \rightarrow \infty)\), the stiffened plates cover the inner surface of the tube, and the cracks on each stiffened plate will become circular cracks on inner surface of tube. As shown in Figure 6, \( 2R_0 \) is the intermediate diameter of tube, \( t_2 \) is the thickness of tube. \( 2\pi R_0 \alpha \) is the area of the cracks on the tube and \( 2\pi R t_2 \) is the area of remote uncracked cross-section. Hence,
\[ \gamma_1(a/b) = \left( \frac{R_0}{R} \right) \frac{a}{t_2}. \]  

(16)

The normalized stress intensity factor is the same as equation (15).

5.2. Compare results

Tada’s\[8\] solution to the stress intensity factor of the circular crack on inner surface of tube under tension is:

\[ K_I = \sigma_0 \sqrt{\pi a} f_1 \left( \frac{r_1}{r_0}, \frac{a}{t_2} \right) \]  

(17)

To facilitate the comparison between the solution of this paper and the Tada’s, equation (17) can changed to

\[ K_I = \sigma_0 \sqrt{\pi a} f_2 \left( \frac{r_1}{r_0}, \frac{a}{t_2} \right) \]  

(18)

Where

\[ f_2 \left( \frac{r_1}{r_0}, \frac{a}{t_2} \right) = \sqrt{\frac{a}{t_2}} f_1 \left( \frac{r_1}{r_0}, \frac{a}{b} \right) \]  

(19)

Figure 7. The Comparisons between the present results and Tada’s.

When the number of stiffened plates in the tube tends to be infinite (m→∞), the comparisons between the present results and Tada’s are shown in Figure 7, which shows that the stress intensity factor solution obtained in this paper agrees well with the existing results.

6. Conclusions

Circumferential periodic cracks on tube with stiffened plates and circular cracks on surface of tube is a typical three-dimensional problem, in this paper, the conservation law and tension theory of material mechanics are used to study the SIFs of Circumferential periodic cracks on tube with stiffened plates, then, used XFEM in Abaqus checking the expression’s results. Based on the expression of the stress intensity factor of Circumferential periodic cracks on tube with stiffened plates, it’s easy to deriving the expression of stress intensity factor of circular cracks on surface of tube and the results agree well with the existing results.
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