Graphs with no $2\delta + 1$ cycle

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Abstract
Dirac proved that any graph with minimum vertex degree $\delta$ contains either a cycle of length at least $2\delta$ or a Hamilton cycle. Motivated by this result, we characterize those graphs having no cycle longer than $2\delta$.

1 Introduction
Dirac [1952] proved that any 2-connected graph with minimum vertex degree $\delta$ contains either a cycle of length at least $2\delta$ or a Hamilton cycle. In this paper, we shall characterize those graphs with minimum vertex degree $\delta$ which have no cycle of length greater than $2\delta$. This characterization was also motivated by Ali and Staton [1996] in which similar results were given for graphs with no path exceeding $2\delta + 1$. In particular, as a case in their work, they prove that if the number of vertices of a non-hamiltonian 2-connected graph is exactly $2\delta + 1$, then the graph must be isomorphic to the join of a graph on $\delta$ vertices and a totally disconnected graph on $\delta + 1$ vertices. Here, we will show that this same type of structure is present when the number of vertices exceeds $2\delta + 1$.

2 Preliminaries

Unless specified otherwise, the terminology used here will follow Bondy and Murty [1976]. In particular, a graph may have loops and parallel edges. If $v$ is a vertex of a graph $G$, then the set of neighbors of $v$ will be denoted $N(v)$ or $N_G(v)$, and the minimum vertex degree of $G$ will be denoted $\delta$ or $\delta(G)$. If $S$ is a subset of the vertex set of $G$, then $G[S]$ denotes the induced subgraph of $G$ on $S$. The totally disconnected graph on $m$ vertices will be denoted $N_m$, and any subgraph of $K_n$ will be denoted $H_n$. The join of two graphs $G$ and $H$ is the graph $G \vee H$ and is obtained by taking disjoint copies of $G$ and $H$ and adding edges joining every vertex of $G$ to every vertex of $H$. In particular, if $|V(G)| = n$
and \(|V(H)| = m\), then the graph \(G \vee H - (E(G) \cup E(H))\) is isomorphic to \(K_{n,m}\). Finally, if \(x_1x_2 \ldots x_n\) is a path \(P\) in a graph \(G\), then \(P[x_i, x_j], P[x, x_j], P(x_i, x_j],\) and \(P(x_i, x_j)\), will denote the subpaths \(x_1x_2 \ldots x_{i-1}x_i, x_ix_{i+1} \ldots x_j, x_{i+1}x_{i+2} \ldots x_j,\) and \(x_{i+1}x_{i+2} \ldots x_j\), respectively.

## 3 The Main Theorem

To establish the main result of this paper, we shall use the following theorem of Dirac \[1952\].

**Theorem 3.0.1** Let \(G\) be a simple 2-connected graph with minimum vertex degree \(\delta\) and suppose \(|V(G)| \geq 3\). Then \(G\) contains either a cycle of length at least \(2\delta\) or a Hamilton cycle.

Motivated by this result, we will now prove the following characterization for the graphs having no cycle longer than \(2\delta\).

**Theorem 3.0.2** Let \(G\) be a simple 2-connected graph with minimum vertex degree \(\delta\). If \(G\) has no cycle of length at least \(2\delta + 1\), then \(G\) is Hamiltonian or \(G \cong H_\delta \vee N_m\) where \(m > \delta\).

**Proof.** Let \(G\) be a graph with no Hamilton cycle and no cycle of length at least \(2\delta + 1\). By Theorem 3.0.1, \(G\) has a cycle \(C\) of length \(2\delta\). Now, since \(G\) is not Hamiltonian, there is a vertex \(z\) in \(V(G) - V(C)\), and by Menger’s Theorem \[1927\], there are two paths \(Q_1\) and \(Q_2\) from \(z\) to \(C\) that have only the vertex \(x\) in common and such that each \(Q_i\) meets \(C\) in exactly one vertex \(z_i\).

**Lemma 3.0.3** \(Q_1\) and \(Q_2\) must not meet consecutive vertices on \(C\).

**Proof.** Suppose \(Q_1\) and \(Q_2\) meet \(C\) at \(z_1\) and \(z_2\) where \(z_1z_2\) is an edge of \(C\). Then \(G\) has a cycle \(C'\) formed by the subpath \(C - \{z_1z_2\}\) together with the path \(z_1z_2\). Therefore, \(C'\) contains \(2\delta + 1\) vertices of \(C\), namely, all the vertices in \(V(C) \cup \{z\}\); a contradiction. \(\Box\)

Since there are paths from every vertex of \(V(G) - V(C)\) to \(C\), choose a longest path \(P\) subject to the condition that one of the endvertices of \(P\) lies on \(C\) and no other vertex of \(P\) lies on \(C\). We label the endvertex of \(P\) that is not in \(V(C)\) as \(x\) and the vertex in \(V(P) \cap V(C)\) is labelled \(y\). Now, since \(P\) is a longest path of this type, it is clear that every neighbor of \(x\) lies in \(V(P) \cup V(C)\). Moreover, we have the following lemma.

**Lemma 3.0.4** \(x\) has a neighbor in \(V(C) - y\).

**Proof.** Suppose not. Then every neighbor of \(x\) lies on \(P\). Since \(G\) is 2-connected, it is clear that there is a vertex of \(P\) adjacent to some vertex on \(C\). Viewing \(P\) as a directed path from \(x\) to \(y\), let \(z\) be the first vertex on \(P\) that is adjacent to a vertex, \(z'\), of \(C\), and
let $u_x$ be the first neighbor of $x$ after $z$ on $P$. Now, $C$ is partitioned into two paths from $z'$ to $y$, and we arbitrarily label them $C_1$ and $C_2$. Since $C$ has size exactly $2\delta$, one of $C_1$ and $C_2$ has at least $\delta$ edges. Without loss, assume that $C_1$ has at least $\delta$ edges.

Now, let $P'$ be the path containing $z'z$, $P[z,x]$, $xu_x$, and $P[u_x,y]$. Notice that $P'$ contains all the vertices of $P$ except those on $P(z,u_x)$. Since there are no neighbors of $x$ on $P(z,u_x)$, it is clear that in addition to $x$ itself, $P'$ contains all the neighbors of $x$; thus, $P'$ contains at least $\delta+1$ vertices from $P$ as well as the vertex $z'$ on $C$. Since $C_1$ contains $\delta+1$ vertices, we can combine $C_1$ and $P'$ to obtain a cycle $C'$ of size $|V(C_1)| + |V(P')| - 2 \geq (\delta+1) + (\delta+2) - 2 = 2\delta+1$; a contradiction. □

Having proved Lemma 3.0.4, let the vertices of $C - y$ that are adjacent to $x$ be labelled $x_1, x_2, \ldots, x_k$ in cyclic order on $C$, where $y$ lies between $x_k$ and $x_1$ on $C$, and relabel $y$ as $x_{k+1}$. It is clear that the cycle $C$ is partitioned into $k+1$ internally disjoint paths $C(x_1), C(x_2), \ldots, C(x_{k+1})$ where $C(x_i)$ is the subpath of $C$ from $x_i$ to $x_{i+1}$ that contains no other vertex in $\{x_1, x_2, \ldots, x_{k+1}\}$; here $C(x_{k+1})$ is the subpath of $C$ from $x_{k+1}$ to $x_1$. If $k = 1$, then $C$ can be partitioned into two paths from $x_1$ to $x_2$. We arbitrarily label one of these $C(x_1)$ and the other $C(x_2)$. Since $G$ has no cycle containing $2\delta+1$ vertices, it is clear by Lemma 3.0.3 that each $C(x_i)$ contains at least one vertex in its interior, and we let the vertex of $C(x_i)$ that is adjacent to $x_i$ be labelled $a_i$.

**Lemma 3.0.5** There is no path from $a_i$ to $a_j$ avoiding $x \cup C$.

**Proof.** Suppose $Q$ is a path joining $a_i$ to $a_j$ and avoiding $x \cup C$. It is easy to see that $G$ has a cycle $C'$ formed by the edges of $C - \{a_ia_j, x_{a_i}\}$ together with the path $x_{x_{a_i}}$ and the edges of $Q$. Thus, $C'$ contains the vertices of $C$ and $x$; a contradiction. □

Now, consider $C(x_k)$ and $C(x_{k+1})$. These paths have the vertex $x_{k+1}$ as an endvertex. If one of these paths has less than $|P|$ internal vertices, then we can delete the path from $C$ and replace it with the path $P$ and the edge $xx_k$ or $xx_1$ creating a longer cycle. Thus, each of $C(x_k)$ and $C(x_{k+1})$ has length at least $|P| + 1$. So, the length of $C(x_j)$ is greater than or equal to $|P| + 1$ for $j \in \{k, k+1\}$, and by Lemma 3.0.3 the length of $C(x_i)$ is at least 2 for $i \in \{1, \ldots, k-1\}$. This implies that $|C| \geq 2(|P| + 1) + (k-1)2$ which means that $2\delta \geq 2|P| + 2k$. Therefore, $\delta \geq |P| + k$.

Now, since $d_G(x) \geq |P| + k$ and $x$ is adjacent to exactly $k$ vertices in $V(C) - x_{k+1}$, the vertex $x$ must be adjacent to every vertex on $P - x$. This implies that $d_G(x) = |P| + k$ which in turn means that $\delta = |P| + k$. Furthermore, this restriction shows that each $C(x_i)$ has length 2 when $i \in \{1, \ldots, k-1\}$ and that both $C(x_k)$ and $C(x_{k+1})$ have length $|P| + 1$.

**Lemma 3.0.6** $|P| = 1$. Moreover, $\delta = k + 1$ and $N_G(x) = \{x_1, \ldots, x_{k+1}\}$.

**Proof.** Suppose $|P| > 1$ and let the vertices of $P$ from $x$ to $x_{k+1}$ be labelled $u_1, u_2, \ldots, u_{|P|}$ in this order where $u_{|P|} = x_{k+1}$. Since $x_{x_{k+1}}$ is an edge, we have a path $P_i = P - \{u_i, u_{i+1}\} \cup xu_{i+1}$ for each $u_i$, and this path is a longest path having one endvertex on $C$ and the other end being $u_i$. Since $d_G(u_i) \geq \delta$, the vertex $u_i$ must be adjacent to at least $k$ vertices on
$C - x_{k+1}$, and using Lemmas 3.0.3 and 3.0.5, it is easy to see that the neighbors of $u_i$ on $C - x_{k+1}$ must be exactly the vertices in $\{x_1, x_2, \ldots, x_k\}$. So, $u_i$ must be adjacent to every member of $P_i$. But, $(C - \{a_1\}) \cup \{x_2xu_1x_1\}$ is a $2\delta + 1$ cycle, since $u_1 \neq x_{k+1}$. \end{proof}

Now, by Lemma 3.0.6, every path from a vertex in $V(G) - V(C)$ to $V(C)$ must have length 1; so, since $G$ is connected, every vertex in $V(G) - V(C)$ has a neighbor in $V(C)$. If $x'$ and $x''$ are distinct members of $V(G) - V(C)$, then $x'x''$ is not an edge of $G$; otherwise if $y$ is a neighbor of $x'$ on $C$ then $yx'x''$ is a path of length 2 having only one common vertex with $C$. We conclude that $G[V(G) - V(C)]$ is totally disconnected.

Upon combining Lemmas 3.0.3, 3.0.5, and 3.0.6 we see that $N(x) = \{x_1, \ldots x_{k+1}\}$ for every $x$ in $V(G) - V(C)$. Moreover, the set of neighbors for each $a_i$ is $\{x_1, \ldots x_{k+1}\}$ since $a_i,a_j$ is not an edge of $G$ and $\delta = k + 1$. Therefore, $G[\{a_1, \ldots , a_{k+1}\} \cup (V(G) - V(C))]$ is totally disconnected with $k + 1 + |V(G) - V(C)| = m$ vertices. Since $V(G) - V(C)$ has at least one vertex, $m$ exceeds $\delta$, and the theorem is established. \end{proof}

**Corollary 3.0.7** Let $G$ be a 2-connected graph with at least $2\delta + 1$ vertices. Then, $G$ has no cycle of length at least $2\delta + 1$ if and only if $G \cong H_\delta \lor N_m$ where $m > \delta$.

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