Bifurcation Analysis in an Interrupted Dynamical System with State Dependent Input*

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This paper addresses the first step to investigate the bifurcation phenomena observed in the current-controlled DC-DC converter with thermoelectric module, and therefore we propose a simple interrupted dynamical system which simulates dynamic behavior of the DC-DC converter with thermoelectric module in nature. The dynamic behavior of the proposed system can be described by one dimensional topology, and therefore, we can define the return map. By using the return map, the rigorous bifurcation analysis can be possible. Because the proposed system simulates dynamic behavior of the converter, the clarified circuit fundamental characteristics are applicable to the converters in nature.

First, we show an interrupted electric circuit with state dependent input that simulates the switching operation of a current-controlled DC-DC converter with a thermoelectric power generation module as a power supply. Next, we define the return map for analyzing bifurcation phenomena observed in the circuit. Finally, the bifurcation phenomena occurring in the circuit are observed using the return map, and the qualitative properties of this circuit are clarified.

1. Introduction

Interrupted dynamical systems have subsystems which are switched from one to other depending on its own state and a periodic interval[1]. The typical example of interrupted dynamical system includes power conversion circuits such as converters or inverters in electrical field, and vibrating systems such as impact dampers in mechanical systems, and so on. The power conversion circuit includes switching devices and diodes, and therefore, circuit equation changes one to another through the switching action. The interrupted characteristic causes nonlinearity, and bifurcation phenomena occur in the circuit depending on the circuit parameter. The bifurcation phenomena observed in power conversion circuits have been studied rigorously since the 1990s[2–4].

Interrupted dynamical system exhibits rich bifurcation phenomena, which are classified as local or global bifurcations. It is important to calculate characteristic multiplier of the Jacobian matrix for understanding the dynamical mechanism of appearance of the local bifurcations such as the period-doubling bifurcation, saddle-node bifurcation, and Neimark-Sacker bifurcation, and so on[5–7]. On the other hand, the global bifurcations include the grazing, sliding, and border-collision bifurcations[8,9]. The border-collision bifurcation occurs when a part of the solution, which is the Poincaré mapping point, collides with a border. Analyzing the bifurcation phenomena is important for understanding fundamental characteristic of the interrupted dynamical system. Therefore, much research has been reported on the qualitative nature of circuits through bifurcation analysis since decades ago.

On the other hand, activities to reduce CO2 emissions have been intensifying in recent years. There are many options to solve this kind of problem, and we focus on thermoelectric power generation systems in this paper. Maybe, thermoelectric generation systems are not the mainstream of distributed power generation, but it is a research theme that is expected to open up fields for industrial applications in the future[10]. The thermoelectric power generation module requires DC-DC converter for power matching with the load as it many of the other power supplies. Moreover, the thermoelectric power generation...
module has low voltage and high current output characteristics which means that current-controlled DC-DC converter is good for extracting a certain amount of current in order to extract the maximum power from the device. However, there are no research for studying dynamic behavior of the current-controlled DC-DC converter with thermo-electric module in detail. Therefore, the motivation of this paper is to investigate the qualitative characteristic of this kind of circuit via bifurcation analysis. We consider that the results of this paper contribute to development of the circuit theory, and it is important from academic point of view. On the other hand, the power conversion efficiency decreases via the bifurcation phenomena for understanding dynamic behavior of the circuit is important from engineering point of view[11,12].

The circuit equation of DC-DC converters is described by two or more dimensional topology, and it often makes analysis difficult. On the other hand, there are some papers which study interrupted electric circuit in which the circuit equations are described in one dimensional topology, it is possible to easily analyze bifurcation phenomena that occur in the circuits using return maps or one-parameter bifurcation diagrams[13–16]. This paper addresses the first step to investigate the bifurcation phenomena observed in the current-controlled DC-DC converter with thermo-electric module, and therefore we propose a simple interrupted dynamical system which simulates dynamic behavior of the DC-DC converter with thermo-electric module in nature. The dynamic behavior of the proposed system can be described by one dimensional topology, and therefore, we can define the return map. By using the return map, the rigorous bifurcation analysis can be possible. Because the proposed system simulates dynamic behavior of the converter, the clarified circuit fundamental characteristics are applicable to the converters in nature.

First, we show an interrupted electric circuit with state dependent input that simulates the switching operation of a current-controlled DC-DC converter with a thermo-electric power generation module as a power supply. Next, we define the return map for analyzing bifurcation phenomena observed in the circuit. Finally, the bifurcation phenomena occurring in the circuit are observed using the return map, and the qualitative properties of this circuit are clarified.

2. Interrupted Dynamical System

Figure 1 is a circuit that simulates the switching operation of a current-controlled DC-DC converter that includes a thermo-electric power generation module. The main circuit contains resistors $R$, capacitor $C$, power supply $E$, and converter $E_{SDI}$. Also, this circuit is a simple circuit in which the switch is switched by the output of the flip-flop.

Figure 2 shows an example of the waveform behavior. In the figure, $E_{SDI}$ denotes output characteristic of the state-dependent input. It is clear that $E_{SDI}$ decreases during the capacitor voltage $v(t)$ increases. This output characteristic, which we call state-dependent input, is the same with thermoelectric module as reported in Ref.[17].

The circuit equation when the switches are connected to position A and B in Fig. 1 is as follows.

$$\frac{dE}{dt} = \begin{cases} E - v & \text{SW is A} \\ E_{SDI} - v & \text{SW is B} \end{cases} \quad (1)$$

Here, it is assumed that the clock pulse is input with a period $T$. The switches are controlled by comparators, periodic pulses from a clock, and flip-flops. Also, $E_{SDI}$ simulates the output characteristics of the thermo-electric power generation module which is expressed by

$$E_{SDI}(v) = -av + b, \quad (2)$$

where $a$ and $b$ are arbitrary values. The converter converts the voltage received by the capacitor into $E_{SDI}$ and outputs it. Assume the dimensionless variables as follows.

$$v = Ex, \ t = RCt, \ \frac{b}{E} = \gamma, \ \sigma = a + 1, \ T' = RC$$

Therefore, (1) is rewritten by

$$\frac{dx}{d\tau} = \begin{cases} 1 - x & \text{SW is A} \\ -\sigma x + \gamma & \text{SW is B} \end{cases} \quad (4)$$

When the switches are A and B, the solution of the circuit equation are rewritten by

![Fig. 1 Circuit model](image1)

![Fig. 2 Example of waveform behavior](image2)
In the following, note that $T'$ is rewritten as $T$ in the following analysis. The waveform $x(\tau)$ reaches the threshold, the switch switches from position A to B. If the clock is applied when the switch is at position A, switching operation is not performed.

In order to analyze bifurcations phenomena in this circuit, we define return maps. The waveform during the duration of clock interval can be classified by using the border $D$.

$$D = (x_t - 1)e^T + 1$$

Using the return maps and the border $D$, the behavior of the circuit given in Fig. 1 can be completely described. Hereinafter, the time when the clock is applied is $\tau = kT$, let $x_k$ be an initial value at this time, and define a return map.

If $x_k \leq D$ is satisfied, the switch maintains the state of A until the next clock is applied. Therefore, the return map is defined as follows.

$$x_{k+1} = (x_k - 1)e^{-T} + 1$$

On the other hand, when $x_k > D$ is satisfied, the switch changes from position A to B at $\tau = kT + \tau_A$, where

$$\tau_A = \frac{x_k - 1}{x_t - 1}.$$}

Therefore, the return map can be defined by the following equation.

$$x_{k+1} = \frac{1}{\sigma}\left\{\left(\sigma x_k - \gamma\right)\frac{x_k - 1}{x_t - 1}e^{-T} + \gamma\right\}$$

Consequently, the return map is completely defined as follows:

$$x_{k+1} = F(x_k)$$

$$= \begin{cases} f_\alpha(x_k) = (x_k - 1)e^{-T} + 1 & x_k \leq D \\ f_\beta(x_k) = \frac{1}{\sigma}\left\{\left(\sigma x_k - \gamma\right)\frac{x_k - 1}{x_t - 1}e^{-T} + \gamma\right\} & x_k > D \end{cases}$$

Assume that the periodic solution satisfies the following conditions:

$$F^m(x_k) - x_k = 0,$$ (11)

and

$$F^{l}(x_k) - x_k \neq 0,$$ (12)

where $l < m$ and $l > 1$. We call period-$m$ solution if these equations are satisfied.

Moreover, stability of the period-$m$ solution can be calculated based on the slope of the return map because this is one dimensional interrupted dynamical system. The slope of the return map can be defined as follows.

$$\frac{dF}{dx_k}(x_k) = \begin{cases} \alpha = e^{-T} & x_k \leq D \\ \beta = \frac{(\sigma x_k - \gamma)e^{-T}}{\sigma(x_t - 1)} & x_k > D \end{cases}$$ (13)

Figure 3 shows an example of the return map. The return map is a kind of tent map that changes piecewise linearly. The border $D$ in (6) corresponds to the corner of the return map and becomes an indifferrentiable point. This border causes border-collision bifurcation into the system behavior.

In the following, we fix each circuit parameter as follows.

$$R = 33[k\Omega], \quad C = 47[\mu H], \quad E = 5[V], \quad f = 1[Hz]$$

Note that reference value $x_t$ and switching frequency $f$ are the bifurcation parameter. We calculated the time constant under this circuit parameter. By using this circuit parameter, behavior of the waveform is match well with the converter.

\[\text{Fig. 3 Return map example}\]

### 3. Bifurcation Analysis

Figure 4 shows one-parameter bifurcation diagrams, which is calculated based on the return map. Here, for comparison, we show the one-parameter bifurcation diagram in the circuit without state dependent input, which is calculated as $\sigma = 1.0, \gamma = 0.0$, in panel (a). On the other hand, panels (b) and (c) are the one-parameter bifurcation diagrams in the circuit with state dependent input, where panel (c) shows a partially enlarged view.

Figure 5 shows typical examples of the waveforms and the return maps. Note that Fig. 5 shows periodic solutions and chaotic attractors observed in Fig. 4 (b) and (c). For example, the period-1 solution is observed at $x_t = 0.719$ (see Fig. 5 (a)). The period-1 solution bifurcates to period-2 solution as shown in Fig. 5 (b). Moreover, the non-periodic solution, which may be chaotic attractor, is observed at
When not receiving state dependent input
\( (\sigma = 1.0, \gamma = 0.0) \)

When receiving state dependent input
\( (\sigma = 2.0, \gamma = 0.4) \)

Partial enlargement of (b) \( (\sigma = 2.0, \gamma = 0.4) \)

\[ x_r = 0.876 \] (see Fig. 5 (c)). The non-periodic solution bifurcates to period-3 solution as shown in Fig. 5 (d).

It is clear from Figs. 4 and 5 that the system exhibits rich nonlinear phenomena by changing the bifurcation parameter. The bifurcation phenomena observed in the system are classified into two types, which are period doubling bifurcation and border-collision bifurcation. In the following, we consider the bifurcation mechanism based on the standard techniques of the bifurcation analysis \([2-4]\).

First, we focus on the period doubling bifurcation. This bifurcation occurs when a characteristic multiplier equals to -1. Because our system dynamics is expressed by one dimensional topology, the period doubling bifurcation occurs when the composition of the slope of the return map equals to -1. Therefore, the bifurcation condition can be defined by

\[ \frac{dF^m(x_k)}{dx_k} = -1. \]  \hspace{1cm} (15)

At the bifurcation point, the period-\( m \) solution \( (m \geq 1) \) has \( m - 1 \) mapping points in slope \( \alpha \) and 1 mapping point in slope \( \beta \) in the return map. Therefore, (15) is rewritten by

\[ g(x_r, T) = (e^{-mT} + 1)x_r - \frac{\gamma}{\sigma}e^{-mT} - 1. \]  \hspace{1cm} (16)

The derivation of the bifurcation condition with bifurcation parameter \( x_r \) can be expressed by

\[ \frac{\partial g}{\partial x_r}(x_r, T) = e^{-mT} + 1. \]  \hspace{1cm} (17)

On the other hand, the border-collision bifurcation satisfies the following condition.

\[ F^m(D) = D \]  \hspace{1cm} (18)
At the bifurcation point, the period-$m$ solution ($m \geq 3$) has $m-1$ mapping points in slope $\alpha$ and 1 mapping point in slope $\beta$ in the return map. It is important to define the mapping pattern in the analysis of the border-collision bifurcation. The mapping pattern of the period-$m$ solution at the bifurcation point is expressed as

$$f_{\alpha}^{m-2} \circ f_{\beta} \circ f_{\alpha}(D) = D. \quad (19)$$

Therefore, the bifurcation condition is given by

$$h(x_r,T) = (e^{-(m-1)T} - e^T)x_r - \frac{\gamma}{\sigma}e^{-(m-1)T}x_r - \left(\frac{\gamma}{\sigma} - 1\right)e^{-(m-2)T}x_r = 0. \quad (20)$$

The derivation of the bifurcation condition with bifurcation parameter $x_r$ can be expressed by

$$\frac{\partial h}{\partial x_r}(x_r,T) = e^{-(m-1)T} - e^T. \quad (21)$$

Based on (16) and (17), the bifurcation curve of the period doubling bifurcation can be calculated based on Newton method. Likewise, the bifurcation curve for border-collision bifurcation can be calculated based on (20) and (21).

Figure 6 shows an example of the two-parameter bifurcation diagram. The bifurcation parameters are $x_r$ and $f$, respectively. These bifurcation parameters are included in the current-controlled DC-DC converter with thermoelectric module. Because the analyzed circuit simulates the switching action of the converter, from the following analytical results, we can understand the bifurcation mechanism observed in the converter. The symbols PD and BC denote the bifurcation curves calculated by the bifurcation condition of the period-doubling bifurcation and the border-collision bifurcation, where subscript means period of the solution. Note that (a) show the two-parameter bifurcation diagram in the system without state dependent input, which is calculated as $\sigma = 1.0, \gamma = 0.0$.

![Two-parameter bifurcation diagrams](image)

Fig. 6 Two-parameter bifurcation diagrams

| $\sigma$, $\gamma$ | $I_1$ | $I_2$ | $I_3$ |
|-------------------|-------|-------|-------|
| $\sigma = 1.0$, $\gamma = 0.0$ | 0.731059 | 0.149797 | 0.042605 |
| $\sigma = 2.0$, $\gamma = 0.4$ | 0.784847 | 0.120638 | 0.034084 |
| $\sigma = 1.5$, $\gamma = 0.4$ | 0.802776 | 0.110585 | 0.031243 |

Table 1 shows the intervals in which periodic solutions exist, where $m \leq 3$ and $f$ is fixed as $f = 1.0$. Let $|l_1|$ be the interval of period-$m$ solution. Thus, the interval of period-1 solution is defined by following equation.

$$I_1 = \left[0, \frac{\gamma e^{-T} + 1}{e^{-T} + 1}\right]. \quad (22)$$

Likewise, the interval of period-2 solution is given by

$$I_2 = \left[\frac{\gamma e^{-T} + 1}{e^{-T} + 1}, \frac{\gamma e^{-2T} + 1}{e^{-2T} + 1}\right]. \quad (23)$$

Moreover, the interval of period-$m$ solution, where $m \geq 3$ is expressed as follows:

$$I_m = \left[\frac{(\frac{\gamma}{\sigma} + \frac{1}{\sigma})e^{-(m-1)T} - 1}{e^{-mT}}, \frac{\gamma e^{-mT} + 1}{e^{-mT} + 1}\right]. \quad (24)$$

Table 1 shows the intervals in which period-$m$ solution exists.
exists, where \( m \leq 3 \) as an example. The table says that the state dependent input extends the interval in which period-1 solution exists, whereas it may contract the interval in which period-\( m \) solution exists.

Moreover, it is clear that the bifurcation point of the period-doubling bifurcation was stabilized in the sense that the value of the characteristic multiplier of the periodic points becomes small by the state-dependent input. This is why the slope of the return map \( \beta \) becomes gentle, and it makes an initial value \( x_k \) large. Because the initial value \( x_k \) becomes large, a part of the initial value tends to collide with the border \( D \) and the border-collision bifurcation occurs. Our circuit simulates the switching action of the DC-DC converter with thermoelectric module. Therefore, we conclude that the thermoelectric module may have the stabilizing effect for the period-doubling bifurcation point, whereas it does not stabilize the border-collision bifurcation point. It can be predicted that existence region of the period-1 solution is extended as affected by the output characteristic of the thermoelectric module in the DC-DC converter, which means that the circuit dynamics tend to keep stable period-one oscillation as reference current increases.

4. Conclusion

In this paper, we proposed a simple interrupted dynamical system which simulates dynamic behavior of the DC-DC converter with thermoelectric module in nature. First, we showed the circuit model. Next, we defined the return map for analyzing bifurcation phenomena observed in the circuit. Finally, the bifurcation phenomena occurring in the circuit were analyzed using the return map, and the qualitative properties of this circuit were clarified. The proposed circuit simulates the switching rule of current-controlled DC-DC converter with thermoelectric module. Therefore, the fundamental characteristic of the proposed circuit is the same with the DC-DC converter in nature. The laboratory experiment is the future work. Moreover, series or parallel connected thermoelectric modules are used in the environment with thermal difference. In this case, we consider that the complicated output characteristic of the thermoelectric module affects the bifurcation structure. In future, we modify the proposed circuit with above output characteristics, and investigate the bifurcation phenomena.

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