Abstract. This is the last of a trilogy of papers on triangle centers. A fairly obscure “conformal center of gravity” is computed for the class of all isosceles triangles. This calculation appears to be new. A byproduct is the logarithmic capacity or transfinite diameter of such, yielding results consistent with Haegi (1951).

Before discussing triangles, let us give both a review of [1] and a preview involving a simpler region in the plane.

Let $\Omega = \{ x + iy \in \mathbb{C} : y > 0, x^2 + y^2 < 1 \}$, the interior of the upper half-disk of unit radius. Let $\Delta$ denote the (full) disk of unit radius and $\Sigma$ denote the infinite horizontal strip of width $\pi$. Define a function $\ell : \Omega \to \Sigma$ by

$$\ell(z) = \ln \left( \frac{(1 + z)^2}{(1 - z)^2} \right).$$

The conformal map $f_w : \Omega \to \Delta$ given by

$$f_w(z) = \frac{\exp(\ell(z)) - \exp(\ell(w))}{\exp(\ell(z)) - \exp(\ell(w))}$$

satisfies $f_w(w) = 0$; it is well-known that $\ln |f_w(z)|$ is Green’s function for $\Omega$. We deduce that

$$h(w) = \lim_{z \to w} \left| \frac{f_w(z)}{z - w} \right| = \left| \frac{\exp(\ell(w))\ell'(w)}{\exp(\ell(w)) - \exp(\ell(w))} \right|$$

where $\ell'$ denotes the derivative of $\ell$. Restricting attention to the $y$-axis only, we have

$$h(iy) = \frac{\frac{2(1+iy)}{(1-iy)^2} + \frac{2(1+iy)^2}{(1-iy)^4}}{(1+iy)^2 - (i+y)^2} = \frac{1 + iy^2}{2y (1 - y^2)}$$

for $0 < y < 1$. Minimizing this expression, it follows that

$$iy_0 = i\sqrt{-2 + \sqrt{5}} = (0.4858682717566456781828638...)i$$

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is the inner conformal center (what was called the least capacity point in [1]) for
Ω. Also,
\[
\frac{1}{h(iy_0)} = \sqrt{-22 + 10\sqrt{5}} = 0.6005662120015552157733894...
\]
is the maximum inner radius of Ω [3, 4]. This concludes our review.

Let Ωc denote the complement of the closure of Ω. Inverting a function [5]
\[
g(z) = \frac{1}{z} = \frac{\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right) (1 + \frac{1}{w})^{2/3} + \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) (1 - \frac{1}{w})^{2/3}}{(1 + \frac{1}{w})^{2/3} - (1 - \frac{1}{w})^{2/3}}
\]
in terms of w yields
\[
\frac{(1 + \frac{1}{w})^{2/3}}{(1 - \frac{1}{w})^{2/3}} = \frac{\frac{\sqrt{3}}{2} + \frac{i}{2} + \frac{1}{z}}{\frac{\sqrt{3}}{2} + \frac{i}{2} - \frac{1}{z}} = m(z).
\]
The conformal map \( g : \Delta \to \Omega^c \)
\[
g(z) = \frac{1 + 2m(z)^{3/2} + m(z)^3}{-1 + m(z)^3} = \frac{4}{3\sqrt{3}z} + \frac{2i}{3\sqrt{3}} + O(z)
\]
satisfies \( g(0) = \infty \), has positive leading Laurent coefficient, and is unique in this
regard. The constant term of the series expansion
\[
\frac{2i}{3\sqrt{3}} = (0.3849001794597505096727658...)i
\]
is the outer conformal center, which clearly lies in Ω but is not the same as the
inner conformal center. A motivating feature is
\[
\frac{1}{2\pi} \int_0^{2\pi} g((1 - \varepsilon)e^{it}) dt = \frac{2i}{3\sqrt{3}}
\]
for \( \varepsilon > 0 \), but the literature is small [6, 7, 8, 9, 10]. The leading coefficient (of 1/z)
is the outer radius of Ω [3, 11]:
\[
\frac{4}{3\sqrt{3}} = 0.769803589195010193455317...
\]
which is also known as the logarithmic capacity or transfinite diameter of \( \Omega \). An alternative definition is \(^{12}\)

\[
\lim_{n \to \infty} \max_{z_1, z_2, \ldots, z_n \in \Omega} \left( \prod_{j<k} |z_j - z_k| \right)^{\frac{2}{n(n-1)}},
\]

that is, the maximal geometric mean of pairwise distances for \( n \) points in \( \Omega \), in the limit as \( n \to \infty \). This constitutes a fascinating collision of ideas from potential theory; replacing a geometric mean by an arithmetic mean seems to be an open computational issue.

As a conclusion to our preview, the Appell \( F_1 \) function can be written as a definite integral \(^{13, 14}\)

\[
F_1(a, b, b', c; x, y) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 s^{a-1}(1-s)^{c-a-1}(1-sx)^{-b}(1-sy)^{-b'} \, ds,
\]

\(|x| < 1, \quad |y| < 1, \quad \text{Re}(c) > \text{Re}(a) > 0\)

as well as a double hypergeometric series

\[
F_1(a, b, b', c; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m!n!} \frac{\Gamma(a+m+n)}{\Gamma(a)} \frac{\Gamma(b+m)}{\Gamma(b)} \frac{\Gamma(b'+n)}{\Gamma(b')} \frac{\Gamma(c)}{\Gamma(c+m+n)} x^m y^n.
\]

Its numerical implementation in Mathematica is crucial to everything that follows.

1. **ISOSCELES TRIANGLES**

Let

\[
T_\theta = \{ x + iy \in \mathbb{C} : 0 < x < \cos(\theta/2), \quad y < \tan(\theta/2)x, \quad y > -\tan(\theta/2)x \},
\]

the interior of an isosceles triangle with apex angle \( 0 < \theta < \pi \) located at the origin. Two sides of unit length meet there; the third (vertical) side has \( x \)-intercept \( \cos(\theta/2) \) and length \( 2 \sin(\theta/2) \).

To construct a conformal map \( g : \Delta \to T_\theta^c \) requires two steps. First, define

\[
f(z) = \int_{z_0}^{z} \frac{(\zeta - a_1)^{\mu_1}(\zeta - a_2)^{\mu_2}(\zeta - a_3)^{\mu_3}}{\zeta^2} \, d\zeta
\]

on \( \Delta \), where \( a_1, a_2, a_3 \) are prevertices of the unit circle mapping onto vertices of \( T_\theta \), and \( \pi(1 + \mu_1) \), \( \pi(1 + \mu_2) \), \( \pi(1 + \mu_3) \) are exterior angles at the corresponding vertices.
Also, $z_0$ is some point of $\Delta$ other than 0, and the integral is taken along any curve in $\Delta$ joining $z_0$ to $z$ not passing through 0 (it does not matter which). Clearly

$$\mu_1 = \frac{\pi - \theta}{\pi}, \quad \mu_2 = \mu_3 = \frac{\pi + \theta}{2\pi}.\$$

The choice of point $a_1$ is arbitrary; here let $a_1 = -1$. The remaining two points $a_2, a_3$ must satisfy the constraint \[15, 16\]

$$\mu_1/a_1 + \mu_2/a_2 + \mu_3/a_3 = 0$$
in order that $f(0) = \infty$. Thus

$$a_2 = \frac{\pi - \theta + 2i\sqrt{\pi\theta}}{\pi + \theta}, \quad a_3 = \frac{\pi - \theta - 2i\sqrt{\pi\theta}}{\pi + \theta}$$
work for our purposes. We further choose $z_0 = -1$, so that

$$f(z) = \int_{-1}^{z} \frac{(\zeta + 1)^{1 - \theta/\pi} (\zeta^2 - 2 \frac{\pi - \theta}{\pi + \theta} \zeta + 1)^{(\pi + \theta)/(2\pi)}}{\zeta^2} d\zeta.$$

The image of $\{a_1, a_2, a_3\}$ under $f$ evidently lies in the left half plane – needing rotation by $\pi$ – plus rescaling so that the vertical triangle side has the proper length. This second step is achieved by defining

$$g(z) = -\frac{2\sin(\theta/2)}{\text{Im} f(a_2) - \text{Im} f(a_3)} f(z).$$

For the scenario $\theta = \pi/2$, it is true that the coefficient

$$-\frac{\sqrt{2}}{\text{Im} f(a_2) - \text{Im} f(a_3)} = 0.4756344438799819320567570... ≈ \frac{3^{3/4}}{2^{7/2} \pi^{3/2}} \Gamma\left(\frac{1}{4}\right)^2 = \kappa$$
to high numerical precision. This expression (the outer radius of an isosceles right triangle) is well-known and is a special case of a more general formula due to Haegi \[3, 17\]. More on this will be given soon.

Our key result is that the function $f(z)$ possesses an exact representation. Let

$$\xi(z) = \frac{(\sqrt{\pi} - i\sqrt{\theta}) (z + 1)}{2\sqrt{\pi}}, \quad \eta(z) = \frac{(\sqrt{\pi} + i\sqrt{\theta}) (z + 1)}{2\sqrt{\pi}},$$

$$\varphi(z) = \left[\frac{\pi (z - 1)^2 + \theta (z + 1)^2}{\pi + \theta}\right]^{(\pi + \theta)/(2\pi)},$$
\[
\psi(z) = \left[ \frac{(\pi - 1)^2 + \theta(z + 1)^2}{\pi + \theta} \right]^{(\pi - \theta)/(2\pi)},
\]
\[
\delta(z) = \left[ -\sqrt{\pi}(z - 1) - i\sqrt{\theta}(z + 1) \right]^{(\pi - \theta)/(2\pi)} \left[ -\sqrt{\pi}(z - 1) + i\sqrt{\theta}(z + 1) \right]^{(\pi - \theta)/(2\pi)}.
\]
Then we have
\[
f(z) = (z + 1)^{1 - \theta/\pi} \left\{ -\frac{\varphi(z)}{z} + \frac{2^{\theta/\pi}(\pi + \theta)/(2\pi)}{2\pi^2 + \pi\theta - \theta^2} \frac{\delta(z)}{\psi(z)} \right. \]
\[
\left. -2(2\pi - \theta)F_1 \left( \frac{1 - \theta}{\pi}, \frac{\pi - \theta}{2\pi}, 2 - \frac{\theta}{\pi}; \xi(z), \eta(z) \right) \right. + \]
\[
(\pi + \theta)(z + 1)F_1 \left( 2 - \frac{\theta}{\pi}, \frac{\pi - \theta}{2\pi}, 3 - \frac{\theta}{\pi}; \eta(z), \xi(z) \right) \right\}
\]
as can be easily proved after-the-fact by differentiation. (Our before-the-fact technique consisted of examining rational multiples of \( \theta \) in Mathematica, seeking recognizable patterns.) This integral evaluation appears to be new.

Returning to the \( \theta = \pi/2 \) scenario,
\[
g(z) = \frac{\kappa}{z} + \lambda + O(z)
\]
as \( z \to 0 \), where
\[
\lambda = \kappa \frac{5^{5/4}}{3^{3/4}} \left[ 2F_1 \left( \frac{1}{2}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{2 - i\sqrt{2}}{4}, \frac{2 + i\sqrt{2}}{4} \right) - F_1 \left( \frac{3}{2}, \frac{1}{4}, \frac{1}{2}, \frac{5}{4}, \frac{2 + i\sqrt{2}}{4}, \frac{2 - i\sqrt{2}}{4} \right) \right]
\]
\[
= 0.5045039334500261012764068...
\]
is the outer conformal center of \( T_{\pi/2} \). We wonder whether this expression for \( \lambda \) can be simplified, for example, as a ratio of gamma or Gauss hypergeometric function values.

As a corollary, let \( \tilde{T} = \{x + iy \in \mathbb{C} : x > 0, \ y > 0, \ x + y < 1\} \), the initial triangle examined in [1, 18]. The outer conformal center of \( \tilde{T} \) is simply
\[
\left( \frac{1 + i}{\sqrt{2}} \right) \lambda = (0.3567381524778001406751307\ldots)(1 + i)
\]
which is not the same as the inner conformal center \((0.301\ldots)(1 + i)\).

We mention finally that the outer conformal center of \( T_{\pi/3} \) (an equilateral triangle) is \( 1/\sqrt{3} \), that is, it coincides with the centroid of \( T_{\pi/3} \). No other scenarios with such recognizable \( \lambda \) have been found.
2. Haegi’s Formula

An arbitrary triangle with sides $a$, $b$, $c$ and opposite angles

$$
\alpha = \arccos \left( \frac{b^2 + c^2 - a^2}{2bc} \right),
$$

$$
\beta = \arccos \left( \frac{a^2 + c^2 - b^2}{2ac} \right),
$$

$$
\gamma = \arccos \left( \frac{a^2 + b^2 - c^2}{2ab} \right)
$$

has area, circumradius and logarithmic capacity given by

$$
A = \sqrt{\frac{a + b + c - a + b + c a - b + c a + b - c}{2}},
$$

$$
R = \frac{abc}{\sqrt{(a + b + c)(b + c - a)(c + a - b)(a + b - c)}},
$$

$$
\kappa = \frac{A}{4\pi^2 q(\alpha/\pi)q(\beta/\pi)q(\gamma/\pi)R}
$$

where

$$
q(x) = \frac{1}{\Gamma(x)} \sqrt{\frac{x^x}{(1-x)^{1-x}}},
$$

Under the special circumstances that $a = b = 1$ and $c = 2 \sin(\theta/2)$, we have

$$
\kappa(\theta) = \frac{\sqrt{\pi + \theta}}{8\pi^{3/2}} \left( \frac{\pi + \theta}{4\theta} \right)^{\theta/(2\pi)} \sin(\theta/2)^2 \Gamma \left( \frac{\theta}{\pi} \right) \Gamma \left( \frac{\pi - \theta}{2\pi} \right).
$$

for the isosceles triangles $T_\theta$. Over such triangles, the one with maximal $\kappa$ has $\theta = 2.5360873621...$, which seems not to have been noticed before. Over the family of all triangles with fixed $A$, the one with minimal $\kappa$ is equilateral, as proved by Pólya & Szegő [3, 19, 20]. If we fix perimeter rather than area, then (to the contrary) the equilateral triangle provides maximal $\kappa$.

3. Addendum: 30°-60°-90° Triangle

Define $T = \{x + iy \in \mathbb{C} : x > 0, y > 0, \sqrt{3}x + y < \sqrt{3}\}$. Proceeding as before, we obtain

$$
\mu_1 = \frac{1}{2}, \quad \mu_2 = \frac{2}{3}, \quad \mu_3 = \frac{5}{6}.
$$
The choice of point \(a_1\) is arbitrary; here let \(a_1 = 1\). From \(\mu_1/a_1 + \mu_2/a_2 + \mu_3/a_3 = 0\), we deduce that
\[
a_2 = i, \quad a_3 = \frac{3}{5} - \frac{4}{5}i,
\]
work for our purposes. Choosing \(z_0 = 1\), it follows that
\[
f(z) = \int_1^z \frac{(\zeta - 1)^{1/2}(\zeta - i)^{2/3}(\zeta + (\frac{3}{5} + \frac{4}{5}i))^{5/6}}{\zeta^2} d\zeta.
\]
An exact representation for \(f(z)\) in terms of the Appell \(F_1\) function is possible. From this, we can verify the outer radius expression \[3, 17\]
\[
\kappa = \frac{5^{5/12}}{2^{10/3}\pi^2} \Gamma \left(\frac{1}{3}\right)^3 = 0.3779137429709558321024882...\]
to high numerical precision, but have not yet determined the outer conformal center of \(T\).

4. Addendum: 6-9-13 Triangle

The Schwarz-Christoffel toolbox for Matlab \[21, 22\] makes numerical computations of a conformal map feasible. For the triangle with vertices
\[
0, \quad 6, \quad -\frac{13}{3} + \frac{4\sqrt{35}}{3}i
\]
the following code:

\begin{verbatim}
    p = polygon([0 6 -13/3+(4*sqrt(35)/3)*i])
    f = extermap(p,scmapopt('Tolerance',1e-18))
    p = parameters(f)
    format long
    p.prevertex
\end{verbatim}
gives
\[
\mu_1 = 0.659, \quad \mu_2 = 0.207, \quad \mu_3 = 0.132,
\]
\[
a_1 = 1, \quad a_2 = 0.0163 - 0.9998i, \quad a_3 = -0.4069 + 0.9134i.
\]
Closed-form expressions for these exponents and prevertices are possible yet cumbersome. The same is true for the outer radius \(\kappa = 3.805336\). Determining the outer conformal center (even approximately) remains open. Figures 1, 2, 3 provide conformal map images of ten evenly-spaced concentric circles in the disk; orthogonal trajectories are also indicated. We leave the task of exploring whether outer conformal centers belong in Kimberling’s database \[23\] to someone else.
Figure 1: Images of ten concentric circles, center at 0.356 + (0.356)i.
Figure 2: Images of ten concentric circles, center unknown.
Figure 3: Images of ten concentric circles, center unknown.
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