Conic support measures

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Abstract

The conic support measures localize the conic intrinsic volumes of closed convex cones in the same way as the support measures of convex bodies localize the intrinsic volumes of convex bodies. In this note, we extend the ‘Master Steiner formula’ of McCoy and Tropp, which involves conic intrinsic volumes, to conic support measures. Then we prove Hölder continuity of the conic support measures with respect to the angular Hausdorff metric on convex cones and a metric on conic support measures which metrizes the weak convergence.

Keywords: Conic support measure, conic intrinsic volume, Master Steiner formula, Hölder continuity

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1 Introduction

It is well known that the intrinsic volumes, which play a prominent role in the Brunn–Minkowski theory and in the integral geometry of convex bodies, have local versions in the form of measures. One type of these, the area measures of convex bodies, was introduced independently by Aleksandrov [1] and by Fenchel and Jessen [12]. The other series, the curvature measures, was defined and thoroughly studied by Federer [11] (more generally, for sets of positive reach). A common generalization, the support measures, of which area and curvature measures are marginal measures, was apparently first introduced (though not under this name) and used in [25]. Support measures have their name from the fact that they are defined on Borel sets of support elements; a support element of a convex body $K$ is a pair $(x, u)$, where $x$ is a boundary point of $K$ and $u$ is an outer unit normal vector of $K$ at $x$. Characterization theorems of the three series of measures are proved in [22], [24], [15], with integral-geometric applications in [23, formula (4.3)], [24, formula (7.1)], [15 Theorem 3.1]. Integral-geometric formulas for support measures appear in [27], [28], [15]. We refer also to the surveys [26], [29] on curvature and surface area measures, the surveys [33], [17] on integral geometric formulas, and to Chapter 4 in the book [30].

In spherical integral geometry, the role of the Euclidean intrinsic volumes is taken over by the spherical intrinsic volumes. Their equivalent counterparts for closed convex cones in $\mathbb{R}^d$ are known as conic intrinsic volumes. In spherical space, the local integral geometry of convex bodies, involving spherical analogues of support measures and curvature measures, was developed by Glasauer [13]; a short summary appears in [14]. Parts of this, as far as they are relevant for stochastic geometry, are reproduced in [32, Section 6.5].

The spherical kinematic formula, together with the spherical Gauss–Bonnet theorem, can be used to answer the following question, in terms of a formula involving conic intrinsic volumes. If $C, D$ are closed convex cones in $\mathbb{R}^d$, and if a uniform random rotation $\Theta$ is applied to $D$, what is the probability that $C$ and $\Theta D$ have a non-trivial intersection? This question,
mainly for polyhedral cones, and subsequent studies of conic intrinsic volumes, have recently played a central role in various investigations on conic optimization and signal demixing under certain random models (see, e.g. [4, 5, 7, 16, 19]). This caused renewed interest in spherical integral geometry and was an incentive to reconsider parts of it from the conic viewpoint, to develop new proofs and to extend some results; see [3, 6, 31]. We mention that also a local version of the conic kinematic formula has proved useful for applications (see [5]). It may, therefore, be the right time for a more detailed study of the conic support measures, which are the local versions of the conic intrinsic volumes. In the following, we begin such a study. Our first main aim is to localize the ‘Master Steiner formula’ of McCoy and Tropp [20]; see Theorem 3.4. Then we prove a Hölder continuity property of the conic support measures, with respect to the angular Hausdorff metric on the convex cones and a suitable metric on the conic support measures, which metrizes the weak convergence. The result is Theorem 4.1.

2 Preliminaries on convex cones

We work in the \(d\)-dimensional real vector space \(\mathbb{R}^d (d \geq 2)\), with scalar product \(\langle \cdot, \cdot \rangle\) and induced norm \(\| \cdot \|\). The origin of \(\mathbb{R}^d\) is denoted by \(o\), and the unit sphere by \(S^{d-1}\). The total spherical Lebesgue measure of \(S^{d-1}\) is given by \(\omega_d = 2\pi^{d/2}/\Gamma(d/2)\). By \(C^d\) we denote the set of closed convex cones (with apex \(o\)) in \(\mathbb{R}^d\). Occasionally, we have to exclude the cone \(\{o\}\); therefore, we introduce the notation \(C^*_d\) for the set of cones in \(C^d\) of positive dimension. For \(C \in C^d\), the dual cone is defined by

\[
C^\circ = \{ x \in \mathbb{R}^d : \langle x, y \rangle \leq 0 \text{ for all } y \in C \}.
\]

Writing \(C^{\circ\circ} = (C^\circ)^\circ\), we have \(C^{\circ\circ} = C\). The nearest point map or metric projection \(\Pi_C : \mathbb{R}^d \to C\) associates with \(x \in \mathbb{R}^d\) the unique point \(\Pi_C(x) \in C\) for which

\[
\| x - \Pi_C(x) \| = \min\{|x - y| : y \in C\}.
\]

It satisfies \(\Pi_C(\lambda x) = \lambda \Pi_C(x)\) for \(x \in \mathbb{R}^d\) and \(\lambda \geq 0\).

For two vectors \(x, y \in \mathbb{R}^d\), we denote by \(d_a(x, y)\) the angle between them, thus

\[
d_a(x, y) = \arccos \left( \frac{x}{\|x\|}, \frac{y}{\|y\|} \right) \quad \text{if } x, y \neq o.
\]

Restricted to unit vectors, this yields the usual geodesic metric on \(S^{d-1}\). The definition is supplemented by \(d_a(o, o) = 0\) and \(d_a(x, o) = d_a(o, x) = \pi/2\) for \(x \in \mathbb{R}^d \setminus \{o\}\). Let \(C \in C^*_d\). The angular distance \(d_a(x, C)\) of a point \(x \in \mathbb{R}^d\) from \(C\) is defined by

\[
d_a(x, C) := \arccos \frac{\|\Pi_C(x)\|}{\|x\|} \quad \text{if } x \neq o,
\]

and \(d_a(o, C) = \pi/2\). Thus,

\[
d_a(x, C) = \begin{cases} 
\min\{d_a(x, y) : y \in C\} & \text{if } x \notin C^\circ, \\
\pi/2 & \text{if } x \in C^\circ.
\end{cases}
\]

From the Moreau decomposition (Moreau [21])

\[
\Pi_C(x) + \Pi_{C^\circ}(x) = x, \quad \langle \Pi_C(x), \Pi_{C^\circ}(x) \rangle = 0
\]
for $C \in \mathcal{C}^d$ and $x \in \mathbb{R}^d$ it follows that

\[
d_{a}(x, C) + d_{a}(x, C^o) = \pi/2 \quad \text{for } x \in \mathbb{R}^d \setminus (C \cup C^o).
\] (1)

For $C \in \mathcal{C}_s^d$ and $\lambda \geq 0$, we define the angular parallel set of $C$ at distance $\lambda$ by

\[
C^a_{\lambda}:= \{x \in \mathbb{R}^d : d_{a}(x, C) \leq \lambda\}.
\]

The angular Hausdorff distance of $C, D \in \mathcal{C}_s^d$ is then defined by

\[
\delta_{a}(C, D) := \min \{\lambda \geq 0 : C \subseteq D^a_{\lambda}, \ D \subseteq C^a_{\lambda}\}.
\]

Similarly as for the usual Hausdorff metric, one shows that this defines a metric on $\mathcal{C}_s^d$. With respect to this metric, polarity is a local isometry.

**Lemma 2.1.** If $C, D \in \mathcal{C}_s^d$ are cones $\neq \mathbb{R}^d$ with $\delta_{a}(C, D) < \pi/2$, then

\[
\delta_{a}(C^o, D^o) = \delta_{a}(C, D).
\]

The proof given by Glasauer [13 Hilfssatz 2.2] for the spherical counterpart is easily translated to the conic situation.

The following two lemmas estimate the distance between the projections $\Pi_C(x)$ and $\Pi_D(x)$, either Euclidean or angular, if $C$ and $D$ have small angular Hausdorff distance.

**Lemma 2.2.** Let $C, D \in \mathcal{C}_s^d$ and $x \in \mathbb{R}^d$. Then

\[
\|\Pi_C(x) - \Pi_D(x)\| \leq \|x\|\sqrt{10\delta_{a}(C, D)}.
\]

**Proof.** Let $B_{\|x\|}$ denote the ball with center $o$ and radius $\|x\|$, and let $K := C \cap B_{\|x\|}$ and $L := D \cap B_{\|x\|}$. We have $\Pi_C(x) \in B_{\|x\|}$ and hence $\Pi_C(x) = p(K, x)$, where $p(K, \cdot)$ denotes the nearest-point map of the convex body $K$ (as in [30 Sec. 1.2]); similarly $\Pi_D(x) = p(L, x)$. Elementary geometry together with a rough estimate shows that the Euclidean Hausdorff distance of $K$ and $L$ satisfies $\delta(K, L) \leq \delta_{a}(C, D)\|x\|$. Now [30 Lem. 1.8.11] yields

\[
\|\Pi_C(x) - \Pi_D(x)\| = \|p(K, x) - p(L, x)\| \leq \sqrt{10\|x\|\delta(K, L)} \leq \|x\|\sqrt{10\delta_{a}(C, D)},
\]

as asserted.

**Lemma 2.3.** Let $C, D \in \mathcal{C}_s^d$ and $x \in \mathbb{R}^d$. Suppose there exists $\varepsilon > 0$ with $\delta_{a}(C, D) \leq \pi/2 - \varepsilon$, $d_{a}(x, C) \leq \pi/2 - \varepsilon$, $d_{a}(x, D) \leq \pi/2 - \varepsilon$. Then

\[
\cos d_{a}(\Pi_C(x), \Pi_D(x)) \geq 1 - c_{\varepsilon}\delta_{a}(C, D)
\]

with $c_{\varepsilon} := 2(\pi^{-1} + \tan(\pi/2 - \varepsilon))$.

**Proof.** Put $\delta_{a}(C, D) = \delta_{a}$. If $x \in C \cup D$, say $x \in C$, then $d_{a}(\Pi_C(x), \Pi_D(x)) = d_{a}(x, \Pi_D(x)) \leq \delta_{a}$ and hence

\[
\cos d_{a}(\Pi_C(x), \Pi_D(x)) \geq \cos \delta_{a} \geq 1 - \frac{2}{\pi}\delta_{a} \geq 1 - c_{\varepsilon}\delta_{a},
\]

as asserted. We can, therefore, assume in the following that $x \notin C \cup D$.

We have $\Pi_C(x) \neq o$, since $d_{a}(x, C) < \pi/2$. Set $p := \Pi_C(x)/\|\Pi_C(x)\|$. Similarly, $\Pi_D(x) \neq o$, and we set $q := \Pi_D(x)/\|\Pi_D(x)\|$. Let $e := \Pi_{C^o}(x)/\|\Pi_{C^o}(x)\|$ (we have $\Pi_{C^o}(x) \neq o$, since $x \notin C$).
We can assume that there is a two-dimensional unit sphere $S^2$ containing the three unit vectors $p, q, e$ ($S^2 \subseteq \mathbb{R}^{d-1}$ if $d \geq 3$; and if $d = 2$, we embed $\mathbb{R}^2$ in $\mathbb{R}^3$). We have $x_0 := x/\|x\| \in S^2$, since $x$ is a linear combination of $p$ and $e$. Put $d_a(x, C) = d_a$ (this is also the spherical distance of $x_0$ and $p$). The spherical cap in $S^2$ with center $x_0$ and spherical radius $d_a + \delta_a$ contains $q$ (since $d_a(x, \Pi_C(x)) = d_a$ and there is a point $z \in D$ with $d_a(z, \Pi_C(x)) \leq \delta_a$). On the other hand, $d_a(q, C) \leq \delta_a$ since $q \in D$. By (1), $d_a(q, C) + d_a(q, C^\circ) = \pi/2$. Since $e \in C^\circ$, we get 
\[
d_a(q, e) \geq d_a(q, C^\circ) = \frac{\pi}{2} - d_a(q, C) \geq \frac{\pi}{2} - \delta_a.
\]
Therefore, the largest possible angular distance between $p$ and $q$ is attained if 
\[
q = w \cos \delta_a + e \sin \delta_a \quad \text{with } w \in e^\perp \cap S^2
\]
and 
\[
\langle x_0, q \rangle = \cos(d_a + \delta_a).
\]
If $q$ satisfies these equations, then 
\[
\cos(d_a + \delta_a) = \langle x_0, q \rangle = \langle p \cos d_a + e \sin d_a, w \cos \delta_a + e \sin \delta_a \rangle = \langle p, w \rangle \cos d_a \cos \delta_a + \sin d_a \sin \delta_a.
\]
Consequently, 
\[
\cos d_a(p, q) = \langle p, q \rangle = \langle p, w \rangle \cos \delta_a = \frac{1}{\cos d_a} [\cos(d_a + \delta_a) - \sin d_a \sin \delta_a] = \cos \delta_a - 2(\tan d_a) \sin \delta_a \geq 1 - \frac{2}{\pi} \delta_a - 2(\tan d_a) \delta_a.
\]
From this, the assertion follows.

3 The ‘Master Steiner formula’, localized

First we consider polyhedral cones. Let $\mathcal{P}C^d \subset C^d$ denote the subset of polyhedral cones, and $\mathcal{P}C^d_\ast = \mathcal{P}C^d \cap C^d_\ast$. For $P \in \mathcal{P}C^d$ and $k \in \{0, \ldots, d\}$, we denote by $\mathcal{F}_k(C)$ the set of $k$-dimensional faces of $C$ (which are again polyhedral cones). The $k$-skeleton of $C$ is defined by 
\[
\text{skel}_k(C) = \bigcup_{F \in \mathcal{F}(C)} \text{relint } F,
\]
where relint denotes the relative interior.

We denote by $\mathcal{B}(X)$ the $\sigma$-algebra of Borel sets in a topological space $X$ and define the ‘conic $\sigma$-algebra’ 
\[
\hat{\mathcal{B}}(\mathbb{R}^d) = \{A \in \mathcal{B}(\mathbb{R}^d) : \lambda a \in A \text{ for } a \in A \text{ and } \lambda > 0\}
\]
and the ‘biconic $\sigma$-algebra’ (terminology from [4], see also [3]) 
\[
\hat{\mathcal{B}}(\mathbb{R}^d \times \mathbb{R}^d) = \{\eta \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d) : (\lambda x, \mu y) \in \eta \text{ for } (x, y) \in \eta \text{ and } \lambda, \mu > 0\},
\]
where $\mathbb{R}^d \times \mathbb{R}^d$ is endowed with the product topology. Clearly, these are $\sigma$-algebras. We denote by $\mathbf{g}$ a standard Gaussian random vector in $\mathbb{R}^d$, and $\mathbb{P}$ denotes probability. In the following form, the definition of conic support measures was suggested in [3] (1.25)].
Definition 3.1. The conic support measures \( \Omega_0(C, \cdot), \ldots, \Omega_d(C, \cdot) \) of the polyhedral cone \( C \) are defined by

\[
\Omega_k(C, \eta) = \mathbb{P}\{ \Pi_C(g) \in \text{skel}_k C, (\Pi_C(g), \Pi_C^\infty(g)) \in \eta \}, \quad \eta \in \hat{B}(\mathbb{R}^d \times \mathbb{R}^d).
\]

It is clear that \( \Omega_k(C, \cdot) \) is a measure. Its total measure is the \( k \)th conic intrinsic volume, denoted by \( \nu_k(C) = \Omega_k(C, \mathbb{R}^d \times \mathbb{R}^d) \).

First we deal with polyhedral cones, and for these we extend the ‘Master Steiner formula’ of McCoy and Tropp [20] from conic intrinsic volumes to conic support measures. Let \( f : \mathbb{R}^d_+ \rightarrow \mathbb{R}_+ \) be a measurable function. Let \( \mathbb{E} \) denote expectation and \( \mathbf{1}_\eta \) the indicator function of \( \eta \). For a cone \( C \in \mathcal{P}C^d \) we define a measure \( \varphi_f(C, \cdot) \) by

\[
\varphi_f(C, \eta) := \mathbb{E} \left[ f(\|\Pi_C(g)\|^2, \|\Pi_C^\infty(g)\|^2) \cdot \mathbf{1}_\eta(\Pi_C(g), \Pi_C^\infty(g)) \right]
\]

for \( \eta \in \hat{B}(\mathbb{R}^d \times \mathbb{R}^d) \). The following theorem expresses this measure as a linear combination of conic support measures, with coefficients depending on the function \( f \).

Theorem 3.1. Let \( C \in \mathcal{P}C^d \) be a polyhedral cone. Let \( f : \mathbb{R}^d_+ \rightarrow \mathbb{R}_+ \) be a measurable function such that \( \varphi_f(C, \eta) \) is finite for all \( \eta \in \hat{B}(\mathbb{R}^d \times \mathbb{R}^d) \). Then

\[
\varphi_f(C, \eta) = \sum_{k=0}^d I_k(f) \cdot \Omega_k(C, \eta)
\]

(2)

for \( \eta \in \hat{B}(\mathbb{R}^d \times \mathbb{R}^d) \), where the coefficients are given by

\[
I_k(f) = \varphi_f(L_k, \mathbb{R}^d \times \mathbb{R}^d)
\]

(3)

with an (arbitrary) \( k \)-dimensional subspace \( L_k \) of \( \mathbb{R}^d \). Explicitly,

\[
I_k(f) = \frac{\omega_k \omega_{d-k}}{\sqrt{2\pi}} \int_0^\infty \int_0^\infty f(r^2, s^2) e^{-\frac{1}{2}(r^2+s^2)} r^{k-1} s^{d-k-1} ds \, dr
\]

for \( k = 1, \ldots, d-1 \) and

\[
I_0(f) = \frac{\omega_d}{\sqrt{2\pi}} \int_0^\infty f(0, s^2) e^{-\frac{s^2}{2}} s^{d-1} ds,
\]

\[
I_d(f) = \frac{\omega_d}{\sqrt{2\pi}} \int_0^\infty f(r^2, 0) e^{-\frac{r^2}{2}} r^{d-1} dr.
\]

The proof need not be carried out here, since it proceeds by an obvious modification of the proof given by McCoy and Tropp [20] for the global case (that is, for \( \eta = \mathbb{R}^d \times \mathbb{R}^d \)). Using the notation of [20], the modification consists in replacing, in the proof of [20] Lemma 8.1], the function \( 1_{\text{relint}(F)}(u) \) by \( 1_{\text{relint}(F)}(u) \mathbf{1}_\eta(u, w) \). The continuity of the function \( f \) that is assumed in [20] Lemma 8.1] is not needed at this stage.

By specialization, we obtain the local spherical Steiner formula, in a conic version. For this, let \( C \in \mathcal{P}C^d_+ \) and \( \eta \in \hat{B}(\mathbb{R}^d \times \mathbb{R}^d) \). For \( 0 \leq \lambda < \pi/2 \), we define the angular local parallel set

\[
M^\lambda(C, \eta) := \{ x \in \mathbb{R}^d : d_\eta(x, C) \leq \lambda, (\Pi_C(x), \Pi_C^\infty(x)) \in \eta \}.
\]

(4)

This is a Borel set. By \( \gamma_d \) we denote the standard Gaussian measure on \( \mathbb{R}^d \).
Theorem 3.2. Let \( C \in \mathcal{P}C^d \). The Gaussian measure of the angular local parallel set \( M^a_\lambda(C, \eta) \), for \( 0 \leq \lambda < \pi/2 \), is given by

\[
\gamma_d(M^a_\lambda(C, \eta)) = \sum_{k=1}^{d} g_k(\lambda) \cdot \Omega_k(C, \eta),
\]

where

\[
g_k(\lambda) = \frac{\omega_k \omega_{d-k}}{\omega_d} \int_{0}^{\lambda} \cos^{k-1} \varphi \sin^{d-k-1} \varphi \, d\varphi
\]

for \( k = 1, \ldots, d-1 \) and \( g_d \equiv 1 \).

Theorem 3.2 follows from Theorem 3.1 by choosing the function \( f \) defined by

\[
f(a, b) := \begin{cases} 
1 & \text{if } a \leq b \tan^2 \lambda, \\
0 & \text{otherwise,}
\end{cases}
\]

for which

\[
\varphi_f(C, \eta) = \mathbb{P}\{d_a(g, C) \leq \lambda, (\Pi_C(x), \Pi_C^{-1}(x)) \in \eta\} = \gamma_d(M^a_\lambda(C, \eta)).
\]

Theorem 3.2 is equivalent to the local spherical Steiner formula, proved by Glasauer [13, Satz 3.1.1].

We need weak convergence to extend the conic support measures to general convex cones. For a function \( f : S^{d-1} \times S^{d-1} \to \mathbb{R} \), we define its homogeneous extension by

\[
f_h(x, y) := f \left( \frac{x}{\|x\|}, \frac{y}{\|y\|} \right) \quad \text{for } x, y \in \mathbb{R}^d \setminus \{0\}.
\]

For finite measures \( \mu, \mu_i \) on \( \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d) \), the weak convergence \( \mu_i \overset{w}{\to} \mu \) is defined by each of the following equivalent conditions (for the equivalence see, e.g., [8, p. 176]):

(a) For all continuous functions \( f : S^{d-1} \times S^{d-1} \to \mathbb{R} \),

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} f_h(x, y) \, \mu_i(d(x, y)) \to \int_{\mathbb{R}^d \times \mathbb{R}^d} f_h(x, y) \, \mu(d(x, y))
\]
as \( i \to \infty \);

(b) For every open set \( \eta \subset \mathbb{R}^d \times \mathbb{R}^d \),

\[
\mu(\eta) \leq \liminf_{i \to \infty} \mu_i(\eta),
\]

and

\[
\mu(\mathbb{R}^d \times \mathbb{R}^d) = \lim_{i \to \infty} \mu_i(\mathbb{R}^d \times \mathbb{R}^d).
\]

If we now define

\[
\mu_\lambda(C, \eta) = \gamma_d(M^a_\lambda(C, \eta)), \quad \eta \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d)
\]

for given \( C \in \mathcal{C}^d \) and \( \lambda \geq 0 \), then \( \mu_\lambda(C, \cdot) \) is a finite measure on \( \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d) \), and \( \mu_\lambda \) depends weakly continuously on \( C \). The counterpart to this fact for Euclidean convex bodies is [30, Thm. 4.1.1], and Glasauer [13, Hilfssatz 3.1.3] has carried this over to spherical space. His argument can easily be adapted to the conic setting. Convergence of cones refers to the angular Hausdorff metric.
Lemma 3.1. Let $0 \leq \lambda < \pi/2$. Let $C, C_i \in \mathcal{C}_*^d$ ($i \in \mathbb{N}$) be cones with $C_i \to C$ as $i \to \infty$. Then $\mu_\lambda(C_i, \cdot) \xrightarrow{w} \mu_\lambda(C, \cdot)$.

With the aid of this lemma, the following theorem can be proved. It is deliberately formulated close to [30, Thm. 4.2.1], to show the analogy.

Theorem 3.3. To every cone $C \in \mathcal{C}_*^d$ there exist finite positive measures $\Omega_0(C, \cdot), \ldots, \Omega_d(C, \cdot)$ on the $\sigma$-algebra $\mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d)$ such that, for every $\eta \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d)$ and every $\lambda$ with $0 \leq \lambda < \pi/2$, the Gaussian measure of the angular local parallel set $M_\lambda^*(C, \eta)$ is given by

$$\mu_\lambda(C, \eta) = \sum_{k=1}^d g_k(\lambda) \cdot \Omega_k(C, \eta)$$

with

$$g_k(\lambda) = \frac{\omega_k}{\omega_d} \int_0^\lambda \cos^{k-1} \sin^{d-k-1} \varphi \, d\varphi$$

for $k = 1, \ldots, d-1$ and $g_d \equiv 1$.

The mapping $C \mapsto \Omega_k(C, \cdot)$ (from $\mathcal{C}_*^d$ into the space of finite measures on $\mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d)$) is a weakly continuous valuation.

For each $\eta \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d)$, the function $\Omega_k(\cdot, \eta)$ (from $\mathcal{C}_*^d$ to $\mathbb{R}$) is measurable.

The proof just requires to reformulate Glasauer’s [13] arguments in the conical setting.

Our aim is to extend Theorem 3.1 to general convex cones. For this, we need the following lemma.

Lemma 3.2. Let $f : \mathbb{R}_+^d \to \mathbb{R}_+$ be continuous and bounded. Let $C, C_i \in \mathcal{C}_*^d$ ($i \in \mathbb{N}$) be cones with $C_i \to C$ as $i \to \infty$. Then $\varphi_f(C_i, \cdot) \xrightarrow{w} \varphi_f(C, \cdot)$.

Proof. For $x \in \mathbb{R}^d$ we have

$$\Pi_{C_i}(x) \to \Pi_C(x), \quad \Pi_{C_i}^\circ(x) \to \Pi_C^\circ(x) \quad (i \to \infty),$$

which follows from Lemma 2.2. This implies

$$\lim_{i \to \infty} f(\|\Pi_{C_i}(x)\|^2, \|\Pi_{C_i}^\circ(x)\|^2) = f(\|\Pi_C(x)\|^2, \|\Pi_C^\circ(x)\|^2).$$

(9)

Let $\eta \subset \mathbb{R}^d \times \mathbb{R}^d$ be open. If $(\Pi_C(x), \Pi_C^\circ(x)) \in \eta$, we have $(\Pi_{C_i}(x), \Pi_{C_i}^\circ(x)) \in \eta$ for almost all $i$. Thus,

$$1_\eta(\Pi_C(x), \Pi_C^\circ(x)) \leq 1_\eta(\Pi_{C_i}(x), \Pi_{C_i}^\circ(x))$$

for almost all $i$. We deduce that

$$f(\|\Pi_C(x)\|^2, \|\Pi_C^\circ(x)\|^2) 1_\eta(\Pi_C(x), \Pi_C^\circ(x))$$

$$\leq \liminf_{i \to \infty} f(\|\Pi_{C_i}(x)\|^2, \|\Pi_{C_i}^\circ(x)\|^2) 1_\eta(\Pi_{C_i}(x), \Pi_{C_i}^\circ(x)).$$

Fatou’s lemma shows that

$$\varphi_f(C, \eta) \leq \liminf_{i \to \infty} \varphi_f(C_i, \eta).$$

Further, (9) together with the dominated convergence theorem gives

$$\lim_{i \to \infty} \varphi_f(C_i, \mathbb{R}^d \times \mathbb{R}^d) = \varphi_f(C, \mathbb{R}^d \times \mathbb{R}^d).$$

Both relations together yield the assertion. \qed
We can now extend Theorem [3.1] to general convex cones.

**Theorem 3.4.** Let \( C \in \mathcal{C}^d \) be a convex cone. Let \( f : \mathbb{R}^2_+ \to \mathbb{R}_+ \) be a measurable function such that

\[
\varphi_f(C, \eta) := \mathbb{E} \left[ f(\|\Pi_C(g)\|^2, \|\Pi_{C^\circ}(g)\|^2) \cdot 1_\eta(\Pi_C(g), \Pi_{C^\circ}(g)) \right]
\]

is finite for all \( \eta \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d) \). Then

\[
\varphi_f(C, \eta) = \sum_{k=0}^{d} I_k(f) \cdot \Omega_k(C, \eta)
\]

(10)

for \( \eta \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d) \), where the coefficients are as in Theorem [3.1].

**Proof.** For given \( C \in \mathcal{C}^d \), there is a sequence of polyhedral cones \( C_i \in \mathcal{P}C^d \) converging to \( C \), and by Theorem [3.1] we have

\[
\varphi_f(C_i, \cdot) = \sum_{k=0}^{d} I_k(f) \cdot \Omega_k(C_i, \cdot)
\]

for \( i \in \mathbb{N} \), where \( I_k(f) = \varphi_f(L_k, \mathbb{R}^d \times \mathbb{R}^d) \). We assume first that \( f \) is continuous. Then \( \varphi_f(C_i, \cdot) \xrightarrow{w} \varphi_f(C, \cdot) \) by Lemma [3.2] and \( \Omega_k(C_i, \cdot) \xrightarrow{w} \Omega_k(C, \cdot) \) by Theorem [3.3]. We conclude that

\[
\varphi_f(C, \cdot) = \sum_{k=0}^{d} I_k(f) \cdot \Omega_k(C, \cdot).
\]

(11)

We modify the argumentation of McCoy and Tropp [20]. We fix a Borel set \( \eta \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d) \). Let \( h : \mathbb{R}^2_+ \to \mathbb{R} \) be bounded and continuous. We decompose \( h = h^+ - h^- \) with bounded continuous functions \( h^+, h^- : \mathbb{R}^2_+ \to \mathbb{R}_+ \) and define \( \varphi_h(C, \eta) = \varphi_{h^+}(C, \eta) - \varphi_{h^-}(C, \eta) \). Then we can write

\[
\varphi_h(C, \eta) = \mathbb{E} \left[ h(\|\Pi_C(g)\|^2, \|\Pi_{C^\circ}(g)\|^2) \cdot 1_\eta(\Pi_C(g), \Pi_{C^\circ}(g)) \right]
\]

\[
= \int_{B_\eta} h(\|\Pi_C(x)\|^2, \|\Pi_{C^\circ}(x)\|^2) \gamma_d(dx)
\]

\[
= \int_{\mathbb{R}^d_+} h(s, t) \mu_\eta(d(s, t)),
\]

where

\[
B_\eta := \{ x \in \mathbb{R}^d : (\Pi_C(x), \Pi_{C^\circ}(x)) \in \eta \}
\]

and where \( \mu_\eta \) is the pushforward of the restriction \( \gamma_d|_{\mathbb{R}^d_+} \) under the mapping \( x \mapsto (\|\Pi_C(x)\|^2, \|\Pi_{C^\circ}(x)\|^2) \). Denoting by \( \mu_k \) the pushforward of \( \gamma_d \) under the mapping \( x \mapsto (\|\Pi_{L_k}(x)\|^2, \|\Pi_{L_k^\circ}(x)\|^2) \) (where \( L_k \) is the subspace used in (3)), we have

\[
\sum_{k=0}^{d} \varphi_h(L_k, \mathbb{R}^d \times \mathbb{R}^d) \cdot \Omega_k(C, \eta) = \sum_{k=0}^{d} \left( \int_{\mathbb{R}^d_+} h(s, t) \mu_k(d(s, t)) \right) \Omega_k(C, \eta).
\]

Therefore, (11) gives

\[
\int_{\mathbb{R}^d_+} h \, d\mu_\eta = \int_{\mathbb{R}^2_+} h \, \left( \sum_{k=0}^{d} \Omega_k(C, \eta) \mu_k \right).
\]

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Since this holds for all bounded, continuous real functions \( h \) on \( \mathbb{R}^2_+ \), it follows (e.g., from [9, Lemma 30.14]) that

\[
\mu_\eta = \sum_{k=0}^{d} \Omega_k(C, \eta) \mu_k.
\]

Integrating a nonnegative, measurable function \( f \) on \( \mathbb{R}^2_+ \) with respect to these measures gives the assertion. \( \square \)

4 Hölder continuity of the conic support measures

The fact that the conic support measures are weakly continuous will now be improved, by establishing Hölder continuity with respect to a metric which metrizes the weak convergence. This is in analogy to the case of support measures of convex bodies, which was treated in [18].

On \( \mathbb{R}^d \times \mathbb{R}^d \), we use the standard Euclidean norm \( | \cdot | \), which is thus also defined on \( S^{d-1} \times S^{d-1} \). For a bounded real function \( f \) on \( S^{d-1} \times S^{d-1} \), we define

\[
|f|_L := \sup_{a \neq b} \frac{|f(a) - f(b)|}{|a - b|}, \quad |f|_\infty := \sup_a |f(a)|.
\]

Let \( \mathcal{F}_{bL} \) be the set of all functions \( f : S^{d-1} \times S^{d-1} \to \mathbb{R} \) with \( |f|_L \leq 1 \) and \( |f|_\infty \leq 1 \). The functions in \( \mathcal{F}_{bL} \) are continuous and hence integrable with respect to every finite Borel measure on \( S^{d-1} \times S^{d-1} \). We define the bounded Lipschitz distance of finite Borel measures \( \mu, \nu \) on \( S^{d-1} \times S^{d-1} \) by

\[
d_{bL}(\mu, \nu) := \sup \left\{ \left| \int_{S^{d-1} \times S^{d-1}} f \, d\mu - \int_{S^{d-1} \times S^{d-1}} f \, d\nu \right| : f \in \mathcal{F}_{bL} \right\}.
\]

This defines a metric \( d_{bL} \). Convergence of a sequence of finite Borel measures on \( S^{d-1} \times S^{d-1} \) with respect to this metric is equivalent to weak convergence of the sequence (see, e.g., [18, Sect. 11.3]). For finite measures \( \mu, \nu \) on the biconic \( \sigma \)-algebra \( \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d) \), we then define

\[
d_{bL}(\mu, \nu) := \sup \left\{ \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} f_h \, d\mu - \int_{\mathbb{R}^d \times \mathbb{R}^d} f_h \, d\nu \right| : f \in \mathcal{F}_{bL} \right\}.
\]

Clearly, this defines a metric \( d_{bL} \) which metrizes the weak convergence of finite measures on the biconic \( \sigma \)-algebra.

In the following theorem, the assumption \( \delta_a(C, D) \leq 1 \) is convenient, but could be relaxed to \( \delta_a(C, D) \leq \pi/2 - \varepsilon \), with some \( \varepsilon > 0 \). The constant \( c \) would then also depend on \( \varepsilon \).

**Theorem 4.1.** Let \( C, D \in \mathcal{C}_d \) be convex cones, and suppose that \( \delta_a(C, D) \leq 1 \). Then

\[
d_{bL}(\Omega_k(C, \cdot), \Omega_k(D, \cdot)) \leq c \delta_a(C, D)^{1/2}
\]

for \( k \in \{1, \ldots, d-1\} \), where \( c \) is a constant depending only on \( d \).

**Proof.** Let \( C \in \mathcal{C}_d \), \( \eta \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d) \), and \( 0 \leq \lambda < \pi/2 \). In contrast to the earlier definitions of \( M_{\lambda}^a(C, \eta) \) and \( \mathcal{C}_d^a \), we now need

\[
M_{0,\lambda}^a(C, \eta) := \{ x \in \mathbb{R}^d : 0 < d_a(x, C) \leq \lambda, (\Pi_C(x), \Pi_C(x)) \in \eta \}\]
and

\[ C^a_{0,\lambda} := C^a_{\lambda} \setminus C = \{x \in \mathbb{R}^d : 0 < d_a(x, C) \leq \lambda\}, \]

further

\[ \nu_{\lambda}(C, \eta) := \gamma_d(M^a_{0,\lambda}(C, \eta)) . \]

Let \( F_\lambda : C^a_{0,\lambda} \to \mathbb{R}^d \times \mathbb{R}^d \) be defined by \( F_\lambda(x) := (\Pi_C(x), \Pi_{C^o}(x)) \) for \( x \in C^a_{0,\lambda} \). Then \( F_\lambda \) is continuous, and for \( \eta \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R}^d) \) we have \( \gamma_d(F^{-1}_\lambda(\eta)) = \nu_{\lambda}(C, \eta) \). Therefore, for any \( \nu_{\lambda}(C, \cdot) \)-integrable, homogeneous function \( f_h \) on \( \mathbb{R}^d \times \mathbb{R}^d \),

\[ \int_{\mathbb{R}^d \times \mathbb{R}^d} f_h \, d\nu_{\lambda}(C, \cdot) = \int_{C^a_{0,\lambda}} f_h \circ (\Pi_C, \Pi_{C^o}) \, d\gamma_d . \]

Now we assume that also \( D \in \mathcal{C}_d^e \) and that \( \delta_u(C, D) \leq 1 \). Let \( f : \mathbb{S}^{d-1} \times \mathbb{S}^{d-1} \to \mathbb{R} \) be a function with \( \|f\|_L \leq 1 \) and \( \|f\|_\infty \leq 1 \). Then

\[
\left| \int_{\mathbb{R}^d \times \mathbb{R}^d} f_h \, d\nu_{\lambda}(C, \cdot) - \int_{\mathbb{R}^d \times \mathbb{R}^d} f_h \, d\nu_{\lambda}(D, \cdot) \right|
\]

\[
= \left| \int_{C^a_{0,\lambda}} f_h \circ (\Pi_C, \Pi_{C^o}) \, d\gamma_d - \int_{D^a_{0,\lambda}} f_h \circ (\Pi_D, \Pi_{D^o}) \, d\gamma_d \right|
\]

\[
\leq \int_{C^a_{0,\lambda} \cap D^a_{0,\lambda}} |f_h \circ (\Pi_C, \Pi_{C^o}) - f_h \circ (\Pi_D, \Pi_{D^o})| \, d\gamma_d
\]

\[ + \int_{C^a_{0,\lambda} \setminus D^a_{0,\lambda}} |f_h \circ (\Pi_C, \Pi_{C^o})| \, d\gamma_d + \int_{D^a_{0,\lambda} \setminus C^a_{0,\lambda}} |f_h \circ (\Pi_D, \Pi_{D^o})| \, d\gamma_d
\]

\[
\leq \int_{C^a_{0,\lambda} \cap D^a_{0,\lambda}} |f_h \circ (\Pi_C, \Pi_{C^o}) - f_h \circ (\Pi_D, \Pi_{D^o})| \, d\gamma_d + \gamma_d(C^a_{0,\lambda} \setminus D^a_{0,\lambda}) + \gamma_d(D^a_{0,\lambda} \setminus C^a_{0,\lambda}).
\]

Let \( x \in C^a_{0,\lambda} \cup D^a_{0,\lambda}, x \neq o \), and write

\[
\frac{\Pi_C(x)}{\|\Pi_C(x)\|} = u, \quad \frac{\Pi_{C^o}(x)}{\|\Pi_{C^o}(x)\|} = u^o, \quad \frac{\Pi_D(x)}{\|\Pi_D(x)\|} = v, \quad \frac{\Pi_{D^o}(x)}{\|\Pi_{D^o}(x)\|} = v^o.
\]

By the homogeneity of \( f_h \) and the Lipschitz property of \( f \), we have

\[
|f_h \circ (\Pi_C, \Pi_{C^o}) - f_h \circ (\Pi_D, \Pi_{D^o})|(x)
\]

\[
= |f_h(\Pi_C(x), \Pi_{C^o}(x)) - f_h(\Pi_D(x), \Pi_{D^o}(x))|
\]

\[
= |f(u, u^o) - f(v, v^o)|
\]

\[
\leq \|(u, u^o) - (v, v^o)\| \leq \|u - v\| + \|u^o - v^o\|
\]

\[
= 2 \sin \frac{1}{2}d_a(u, v) + 2 \sin \frac{1}{2}d_a(u^o, v^o)
\]

\[
= 2 \sin \frac{1}{2}d_a(\Pi_C(x), \Pi_{D^o}(x)) + 2 \sin \frac{1}{2}d_a(\Pi_{C^o}(x), \Pi_{D^o}(x)),
\]

where \( \|u - v\| = 2 \sin \frac{1}{2}d_a(u, v) \) for unit vectors \( u, v \) was used. By Lemma 2.8,

\[
2 \sin^2 \frac{1}{2}d_a(\Pi_C(x), \Pi_{D^o}(x)) = 1 - \cos d_a(\Pi_C(x), \Pi_{D^o}(x)) \leq c^2 \delta_a(C, D),
\]
with \( \varepsilon = (\pi/2) - 1 \). By Lemma \([2, 1]\), we have \( \delta_a(C^0, D^0) = \delta_a(C, D) \), hence also
\[
2 \sin^2 \frac{1}{2} d_a(\Pi C, \Pi D^0) \leq c \varepsilon \delta_a(C, D).
\]
This yields
\[
\int_{C^0_{\alpha \lambda} \cap D^0_{\alpha \lambda}} |f_h \circ (\Pi C, \Pi C^0) - f_h \circ (\Pi D, \Pi D^0)| \, d\gamma_d \leq b \delta_a(C, D)^{1/2}
\]
with \( b = 2 \sqrt{2C} \).

Write \( \delta_a(C, D) =: \delta \). To estimate \( \gamma_d(C^0_{\alpha \lambda} \setminus D^0_{\alpha \lambda}) \), let \( x \in C^0_{\alpha \lambda} \setminus D^0_{\alpha \lambda} \). Then \( x \in C^\alpha_{\alpha \lambda} \setminus C \) and \( x \notin D^\alpha_{\lambda} \setminus D \). If \( x \in D \), then \( d_a(x, C) \leq \delta \), hence \( x \in C^\alpha_{\lambda} \setminus C \). If \( x \notin D \), but \( x \in C^\alpha_{\lambda} \subseteq (D^\alpha_{\lambda})^a \subseteq D^a_{\lambda + \delta} \), thus \( x \in D^a_{\lambda + \delta} \setminus D^\alpha_{\lambda} \). It follows that
\[
C^0_{\alpha \lambda} \setminus D^0_{\alpha \lambda} \subseteq (C^\alpha_{\lambda} \setminus C) \cup (D^a_{\lambda + \delta} \setminus D^\alpha_{\lambda}).
\]
Therefore,
\[
\gamma_d(C^0_{\alpha \lambda} \setminus D^0_{\alpha \lambda}) \leq \gamma_d(C^\alpha_{\lambda} \setminus C) = \gamma_d(D^a_{\lambda + \delta}) \leq C(d) \delta.
\]
By \([3]\) (with \( \eta = \mathbb{R}^{d} \times \mathbb{R}^{d} \)),
\[
\gamma_d(C^\alpha_{\lambda}) - \gamma_d(C) = \sum_{k=1}^{d-1} g_k(\delta) v_k(C) \leq \sum_{k=1}^{d-1} \frac{\omega_k \omega_d - k}{\omega_d} \cdot \delta =: C(d) \delta,
\]
where \([3]\) and \( v_k \leq 1 \) were observed. Similarly, \( \gamma_d(D^a_{\lambda + \delta}) - \gamma_d(D^\alpha_{\lambda}) \leq C(d) \delta \). Here \( C \) and \( D \) can be interchanged, and we obtain
\[
\gamma_d(C^0_{\alpha \lambda} \setminus D^0_{\alpha \lambda}) + \gamma_d(D^a_{\alpha \lambda} \setminus C^0_{\alpha \lambda}) \leq 4C(d) \delta.
\]
Altogether, we have obtained
\[
\left| \int_{\mathbb{R}^d \times \mathbb{R}^d} f_h \, d\nu(C, \cdot) - \int_{\mathbb{R}^d \times \mathbb{R}^d} f_h \, d\nu(D, \cdot) \right| \leq b \delta_a(C, D)^{1/2} + 4C(d) \delta_a(C, D).
\]
Since this holds for all functions \( f : \mathcal{S}^{d-1} \times \mathcal{S}^{d-1} \to \mathbb{R} \) with \( \|f\|_L \leq 1 \) and \( \|f\|_{\infty} \leq 1 \), and since \( \delta_a(C, D) \leq 1 \), we conclude that
\[
d_b \delta_a(C, D), \nu(C, \cdot), \nu(D, \cdot) \leq b_1 \delta_a(C, D)^{1/2},
\]
with a constant \( b_1 \) depending only on \( d \).

By Theorem \([3, 3]\) (and observing that \( M^a_{\alpha \lambda}(C, \eta) \cap C = \emptyset \)), we have
\[
\nu(C, \eta) = \sum_{k=1}^{d-1} g_k(\lambda) \cdot \Omega_k(C, \eta).
\]
Since the coefficient functions \( g_1, \ldots, g_{d-1} \) are linearly independent on \([0, 1]\), we can choose numbers \( 0 < \lambda_1 < \cdots < \lambda_{d-1} < 1 \) and numbers \( a_{ij} \), depending only on \( i \) and \( j \), such that
\[
\Omega_i(C, \cdot) = \sum_{j=1}^{d-1} a_{ij} \nu_{\lambda_j}(C, \cdot) \quad \text{for } i = 1, \ldots, d-1
\]
(details are in Glasauer’s \([13]\) proof of the spherical version of Theorem \([3, 3]\)). Using the definition of the bounded Lipschitz metric, we now obtain
\[
d_b \delta_a(\Omega_i(C, \cdot), \Omega_i(D, \cdot)) \leq \sum_{i=1}^{d-1} |a_{ij}| d_b \delta_a(\nu_{\lambda_j}(C, \cdot), \nu_{\lambda_j}(D, \cdot)) \leq c \delta_a(C, D)^{1/2}
\]
with a suitable constant \( c \). This completes the proof. \(\square\)
References

[1] Aleksandrov, A.D., Zur Theorie der gemischten Volumina von konvexen Körpern, I. Verallgemeinerung einiger Begriffe der Theorie der konvexen Körper (in Russian). *Mat. Sbornik* N. S. 2 (1937), 947–972. (English translation in [2])

[2] Aleksandrov, A.D. [Alexandrov, A.D.], *Selected works. Part I: Selected scientific papers*. (Yu.G. Reshetnyak, S.S. Kutateladze, eds.), transl. from the Russian by P.S. Naidu. Classics of Soviet Mathematics, 4. Gordon and Breach, Amsterdam, 1996.

[3] Amelunxen, D., Measures on polyhedral cones: characterizations and kinematic formulas. (Preprint) [arXiv:1412.1569v2] (2015).

[4] Amelunxen, D., Bürgisser, P., Intrinsic volumes of symmetric cones. (Extended version of [3]) [arXiv:1205.1863] (2012).

[5] Amelunxen, D., Bürgisser, P., Intrinsic volumes of symmetric cones and applications in convex programming. *Math. Program., Ser. A* 149, 105–130 (2015).

[6] Amelunxen, D., Lotz, M., Intrinsic volumes of polyhedral cones: a combinatorial perspective. *Discrete Comput. Geom.* 58 (2017), 371–409.

[7] Amelunxen, D., Lotz, M., McCoy, M.B., Tropp, J.A., Living on the edge: phase transitions in convex programs with random data. *Inf. Inference* 3 (2014), 224–294.

[8] Ash, R.B., *Measure, Integration, and Functional Analysis*. Academic Press, New York, 1972.

[9] Bauer, H., *Maß- und Integrationstheorie*. Walter de Gruyter, Berlin, 1990.

[10] Dudley, R.M., *Real Analysis and Probability*. Cambridge University Press, New York, 2002.

[11] Federer, H., Curvature measures. *Trans. Amer. Math. Soc.* 93 (1959), 418–491.

[12] Fenchel, W., Jessen, B., Mengenfunktionen und konvexe Körper. *Danske Vid. Selskab. Mat.-fys. Medd.* 16, 3 (1938), 31 pp.

[13] Glasauer, S., Integralgeometrie konvexer Körper im sphärischen Raum. Doctoral Thesis, Albert-Ludwigs-Universität, Freiburg i. Br. (1995). Available from: [http://www.hs-augsburg.de/~glasauer/publ/diss.pdf](http://www.hs-augsburg.de/~glasauer/publ/diss.pdf)

[14] Glasauer, S., Integral geometry of spherically convex bodies. *Diss. Summ. Math.* 1 (1996), 219–226.

[15] Glasauer, S., A generalization of intersection formulae of integral geometry. *Geom. Dedicate* 68 (1997), 101–121.

[16] Goldstein, L., Nourdin, I., Peccati, G., Gaussian phase transition and conic intrinsic volumes: Steinig the Steiner formula. *Ann. Appl. Prob.* 27 (2017), 1–47.

[17] Hug, D., Schneider, R., Kinematic and Crofton formulae of integral geometry: recent variants and extensions. In *Homenatge al professor Lluís Santaló i Sors* (C. Barceló i Vidal, ed.), pp. 51 – 80, Universitat de Girona, 2002.
[18] Hug, D., Schneider, R., Hölder continuity for support measures of convex bodies. Arch. Math. 104 (2015), 83–92.

[19] McCoy, M.B., Tropp, J.A., Sharp recovery bounds for convex demixing, with applications. Found. Comput. Math. 14 (2014), 503–567.

[20] McCoy, M.B., Tropp, J.A., From Steiner formulas for cones to concentration of intrinsic volumes. Discrete Comput. Geom. 51 (2014), 926–963.

[21] Moreau, J.J., Décomposition orthogonal d’un espace hilbertien selon deux cones mutuellement polaires. C. R. Acad. Sci. 255 (1962), 238–240.

[22] Schneider, R., Kinematische Berührmaße für konvexe Körper. Abh. Math. Sem. Univ. Hamburg 44 (1975), 12–23.

[23] Schneider, R., Kinematische Berührmaße für konvexe Körper und Integralrelationen für Oberflächenmaße. Math. Ann. 218 (1975), 253–267.

[24] Schneider, R., Curvature measures of convex bodies. Ann. Mat. Pura Appl. 116 (1978), 101–134.

[25] Schneider, R., Bestimmung konvexer Körper durch Krümmungsmaße. Comment. Math. Helvet. 54 (1979), 42–60.

[26] Schneider, R., Boundary structure and curvature of convex bodies. In Contributions to Geometry, Proc. Geometry Symp., Siegen 1978 (J. Tölke, J.M. Wills, eds.), pp. 13–59, Birkhäuser, Basel, 1979.

[27] Schneider, R., Curvature measures and integral geometry of convex bodies, II. Rend. Sem. Mat. Univ. Politecn. Torino 44 (1986), 263–275.

[28] Schneider, R., Curvature measures and integral geometry of convex bodies, III. Rend. Sem. Mat. Univ. Politecn. Torino 46 (1988), 111–123.

[29] Schneider, R., Convex surfaces, curvature and surface area measures. In: Handbook of Convex Geometry, vol. A (P.M. Gruber, J.M. Wills, eds.), North-Holland, Amsterdam 1993, pp. 273–299.

[30] Schneider, R., Convex Bodies: The Brunn–Minkowski Theory. 2nd edn., Encyclopedia of Mathematics and Its Applications, vol. 151, Cambridge University Press, Cambridge, 2014.

[31] Schneider, R., Intersection probabilities and kinematic formulas for polyhedral cones. Acta Math. Hungar. 155 (2018), 3–24.

[32] Schneider, R., Weil, W., Stochastic and Integral Geometry. Springer, Berlin, 2008.

[33] Schneider, R., Wieacker, J.A., Integral geometry. In: Handbook of Convex Geometry, vol. B (P.M. Gruber, J.M. Wills, eds.), North-Holland, Amsterdam 1993, pp. 1349–1390.

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