A general symmetry preserving reduction scheme and normal form for dynamical systems with a compact symmetry group

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Abstract

We present a generalized Lyapunov Schmidt (ls) reduction scheme for diffeomorphisms living on a finite dimensional real vector space $V$ which transform under real one dimensional characters $\chi$ of an arbitrary compact group with linear action on $V$. Moreover we prove a normal form theorem, such that the normal form still has the desirable transformation properties with respect to $\chi$.

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1 Introduction

We develop a framework to study bifurcation of \(q\)-periodic points from a fixed point in families of \(\chi\)-equivariant maps, for a given integer \(q \geq 1\). The main question asks for the solutions of an equation of the form \(\Phi^q_\lambda(x) = x\) near the fixed point, \(x_0\), for \(\lambda\) close to a given critical value \(\lambda_0\). Mainly aiming at generalizing the work of A. Vanderbauwhede \[13, 17\] and M.C. Ciocci \[3\] to a broader class of maps, our approach consists of a combined use of Lyapunov Schmidt (LS) reduction and normal form (NF) reduction. The presented reduction scheme is structure-preserving and solves the conjectures proposed in \[3\]. Note that \[3\] itself can be seen as a special case of the theory presented here. The basic idea of the procedure is as follows. The periodic points we seek are determined by the zeros of a map \(\psi\) on an orbit space \(Y\). By the Implicit Function Theorem, we reduce \(\psi = 0\) to \(\psi|_U = 0\) where \(U\) is a subspace of \(Y\), larger than the nullspace of the linearisation of \(\psi\) at the bifurcation point. When the map is put into normal form (in particular, it commutes with the semisimple part of the linearisation at the bifurcation point) up to order \(k\), the reduced map, and thus the determining equations for the bifurcations, can be expressed explicitly in terms of this normal form without going through the details of the reduction. As we will explain later, the reduced equations retain the full \(\mathbb{Z}_q\)-equivariance associated with the LS reduction and also inherit the ‘structure’ of the original map. In this respect, our method is a powerful tool that guarantees as much symmetry as possible in the reduced equations. One further feature of the method is that it doesn’t require any condition on the linearisation of the map at the equilibrium (except invertibility), and can therefore be applied when this linearisation has purely imaginary eigenvalues of any multiplicity, and satisfying any number of resonance conditions. We emphasize that the symmetry induced by the structure of the original map and the natural \(\mathbb{Z}_q\)-equivariance associated with the LS reduction play an important role in analyzing the reduced equations and may imply symmetry properties of the corresponding solutions. For related works and examples we refer for example to \[8, 10, 5\]. As our framework applies to problems where one need solve an equation of the form \(P^q(x) = x\) for some map \(P\), it turns out to be particularly useful when investigating the existence and bifurcations of subharmonics from a given periodic orbit in a parameterized family of autonomous systems. Also, the LS reduction can be used, in combination with \((\mathbb{Z}_q^*)\) equivariant singularity theory, to study the geometry of resonance tongues obtained by Hopf bifurcation from a fixed point of a map. For such a study in the general case (i.e. no structure) we refer to \[2\].

In the following section we specify our setting.

2 Preliminaries

We will consider two types of compact groups \((\mathcal{G},.)\) with continuous linear action on a finite dimensional real vectorspace \(V\). On the one hand \(\mathcal{G}\) is a compact continuous group with left invariant Haar measure \(\mu\), and on the other hand \(\mathcal{G}\) is a finite discrete group. Let \(\chi : \mathcal{G} \to \mathbb{R}\) be a one dimensional \((\mathbb{Z}_q^*)\) real group character \(^1\), and let \(\psi_\lambda : V \to V\) be a smooth family of local diffeomorphisms defined on a neighborhood of zero. The map \(\psi_\lambda\) is called \(\chi\)-equivariant if

\[
g \circ \psi_\lambda \circ g^{-1} = \psi_\chi(g), \quad \forall g \in \mathcal{G},
\]

with \(\lambda\) belonging to an open neighborhood \(O\) of zero in \(\mathbb{R}^m\). Remark that for any \(\mathcal{G}\) and one-dimensional character \(\chi\), \(|\chi(g)| = 1\) for any \(g \in \mathcal{G}\). We shall call \(g \in \mathcal{G}\) with \(\chi(g) = 1\) a symmetry of \(\psi\), and \(h \in \mathcal{G}\) with \(\chi(h) = -1\) a reversing symmetry. In the sequel we’ll focus on the case \(\mathcal{G}\) discrete for simplicity. But the results admit a straightforward generalization to the continuous case. The only observation here is that one should consider averaging over \(\mathcal{G}\). That is, \(\int\) instead of \(\sum\) (see for example Remark \[4,4\]).

\(^1\chi\) is irreducible if the corresponding representation is. Moreover \(\chi\) satisfies the product law \(\chi(gh) = \chi(g)\chi(h)\) for all \(g, h \in \mathcal{G}\) if the representation is one-dimensional.
Remark 2.1. It is reasonable to think that when a continuous group \( G \) can be written as \( G_0 \rtimes T \), where \( G_0 \) is the unit component of \( G \) and \( T \) is a finite subgroup\(^2\), one could try to gauge out the \( G_0 \) part by choosing appropriate coordinates so that only the discrete part plays a role yielding possible non-trivial real characters. For example, \( O(2) = SO(2) \times \mathbb{Z}_2 \) where \( 0 \in \mathbb{Z}_2 \) corresponds to the matrices of \( \det= 1 \) and \( 1 \) corresponds to the matrices with \( \det= -1 \), the product \( \times \) is the semidirect product. Any real one-dimensional character of \( O(2) \) is the identity on \( SO(2) \) and \( \pm 1 \) on \( SO(2) \times \{1\} \). Therefore, the reduction problem, we want to deal with, might reduce to a discrete problem: i.e., by choosing adapted coordinates one can gauge out the \( SO(2) \) factor and work only with \( \mathbb{Z}_2 \). Notice also that the \( \mathbb{Z}_2 \) factor also restricts the possible complex one-dimensional representations of the \( SO(2) \) factor that one could choose when the considered family of diffeomorphisms is generated by a vector field.

We assume that

\[
\psi_\lambda(0) = 0, \quad \forall \lambda \in \mathcal{O} \quad \text{and} \quad A_0 := D_x\psi_\lambda(0) \text{ is invertible.} \tag{2}
\]

Remark 2.2. It should be noted that we do not require that there exists a reversing symmetry that is an involution.

The goal of this paper is threefold. On the one hand we prove a reduction theorem to study \( g \)-periodic points of \( \psi_\lambda \) in a neighborhood of the fixed point \( 0 \in \mathbb{R}^n \) for \( \lambda \in \mathcal{O} \), (Theorem 3.1). On the other hand, using the terminology introduced in [12], we give an inner product (nilpotent) normal form for the diffeomorphisms \( \hat{\psi}_\lambda \) satisfying the hypotheses [1] - [2], (Theorem 3.1). Finally, when the original map is put in normal form, Proposition 3.1 shows that the reduced map can be approximated without going through the details of the reduction itself.

3 Reduction

Let \( \psi_\lambda : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) satisfy the assumptions given above. Our interest is with the \( g \)-periodic points of \( \psi_\lambda \) in a neighborhood of \( 0 \in \mathbb{R}^n \) for \( \lambda \in \mathcal{O} \). That is, we look for solutions of the equation

\[
x = \psi_\lambda^n(x)
\]

for \((x, \lambda)\) close to \((0, 0) \in V \times \mathbb{R}^m\). Denote by \( S_\lambda^n \) the solution set, then obviously \( \psi_\lambda \) defines a \( \mathbb{Z}_q \)

action on \( S_\lambda^n \). In order to find \( S_\lambda^n \), we lift the equation (3) to the \( nq \)-dimensional vectorspace \( Y_q \) of \( g \)-periodic sequences, where \( n = \dim(V) \). That is:

\[
Y_q = \{ (x_k)_{k \in \mathbb{Z}} \mid x_{k+q} = x_k, \text{ for all } k \in \mathbb{Z} \}.
\]

Let \( \hat{\psi}_\lambda \) denote the lift of \( \psi_\lambda \) to \( Y_q \), that is

\[
\hat{\psi}_\lambda((x_i)_{i \in \mathbb{Z}}) = (\psi_\lambda(x_i))_{i \in \mathbb{Z}}
\]

and similarly, for all \( g \in \mathcal{G} \), let \( \hat{g} \) be given by

\[
\hat{g}((x_i)_{i \in \mathbb{Z}}) = (g(x_{\lambda(g)i}))_{i \in \mathbb{Z}}.
\]

Define also the left shift \( \sigma \) on \( Y_q \) by

\[
\sigma((x_i)_{i \in \mathbb{Z}}) = (x_{i+1})_{i \in \mathbb{Z}}.
\]

Then, equation (3) is equivalent to

\[
\hat{\psi}_\lambda(x) = \sigma(x),
\]

where \( x \equiv (x_i)_{i \in \mathbb{Z}} \in Y_q \).

\(^2\)This decomposition of \( \mathcal{G} \) as a semidirect product \( \mathcal{G} = G_0 \rtimes T \), with \( G_0 \) the unit component of \( \mathcal{G} \) and \( T \) a finite subgroup, is always possible when \( \mathcal{G}/G_0 \) is abelian. [4].
Remark 3.1. The following properties are easy to verify.

1- the ‘hat’ operator, \( \hat{\cdot} \), is a group representation, i.e., \( \hat{g_1g_2}(y) = \hat{g_1}\hat{g_2}(y) \) for all \( g_1, g_2 \in \mathcal{G} \) and \( y \in Y_q \),

2- \( \hat{\psi}_\lambda \) is \( \sigma \)-equivariant, i.e., \( \hat{\psi}_\lambda(\sigma(y)) = \sigma \circ \hat{\psi}_\lambda(y) \) for all \( y \in Y_q \),

3- \( \hat{g} \circ \hat{\psi}_\lambda \circ \hat{g}^{-1} = \hat{\psi}_{\lambda(g)} \) for all \( g \in \mathcal{G} \),

4- \( \sigma^q = id \) and \( \hat{g} \circ \sigma = \sigma^{\lambda(g)} \circ \hat{g} \).

Let \( \hat{A}_0 = D\hat{\psi}_0(0) \) denote the lift of \( A_0 \) to \( Y_q \), and let \( A_0 = N_0 + S_0 \) be the Jordan-Chevalley decomposition of \( A_0 \) (e.g. [3]). Then,

\[ \hat{A}_0 = \hat{S}_0 + \hat{N}_0 \]

is the Jordan-Chevalley decomposition of \( \hat{A}_0 \) where

- \( \sigma\hat{S}_0 = \hat{S}_0\sigma \) and \( \sigma\hat{N}_0 = \hat{N}_0\sigma \),
- \( \hat{g} \circ \hat{S}_0 = \hat{S}_0^{\lambda(g)} \circ \hat{g} \).

A straightforward application of the (classical) (1s) reduction [14] to equation \( \hat{\Phi}_\lambda(y) = \sigma \cdot y \) [4] would result in a bifurcation equation of the form \( E(v, \lambda) = 0 \), where \( E(\cdot, \lambda) \) is a \( \mathbb{Z}_q \)-equivariant map from \( ker(D\hat{\Phi}_0(0) - \sigma) \) into a complement of \( Im(D\hat{\Phi}_0(0) - \sigma) \) satisfying \( E(0, \lambda) = 0 \) and \( D_{\sigma}E(0, 0) = 0 \). In the case where \( A_0 = D_{\chi}\Phi_0(0) \) and hence also its lift \( \hat{A}_0 = D\hat{\Phi}_0(0) \) are non-semisimple, the details of the reduction strongly depend on the nilpotent part of \( A_0 \). Since we do not want to impose any restriction on \( A_0 \) except that it has to be invertible, we perform a LS reduction with respect to the semisimple part \( \hat{S}_0 \) of \( A_0 \), cf. [3] [13] [4]. Recall that ‘semisimple’ means complex diagonalizable.

Using the decomposition

\[ Y_q = ker\left( \hat{S}_0 - \sigma \right) \oplus Im\left( \hat{S}_0 - \sigma \right), \]

as starting point for the reduction of [14], we prove the following result. Note that this theorem proves the conjecture stated in [5] and generalizes [3] [7]. The result also applies to the reversible-equivariant systems as considered in [11].

Theorem 3.1 (Reduction Result). Let \( \psi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) be a local family of \( \chi \)-equivariant diffeomorphisms, satisfying [14] and [2]. Let \( q \geq 1 \), and let \( A_0 = S_0 + N_0 \) be the Jordan Chevalley decomposition of \( A_0 \). Define the reduced phase space \( \hat{U} \) by

\[ \hat{U} := ker(S^c_0 - I). \]

Then there exist a family of (reduced) diffeomorphisms \( \psi_{r,\lambda} : \hat{U} \to \hat{U} \) and a map \( x^* : \hat{U} \times \mathbb{R}^m \to \mathbb{R}^n \) such that for each sufficiently small \( \lambda \in \mathbb{R}^m \) the following properties hold:

(i) \( \psi_{r,\lambda}(0) = 0 \), \( D_u\psi_{r,\lambda=0} = A_0|_U \), \( x^*(0, 0) = 0 \), and, for all \( \tilde{u} \in \hat{U} \), \( D_u x^*(0, 0) \cdot \tilde{u} = \tilde{u} \); 

(ii) \( \psi_{r,\lambda} \) is \( \mathbb{Z}_q \)-equivariant: \( \psi_{r,\lambda}(S_0u) = S_0\psi_{r,\lambda}(u) \);

(iii) \( \psi_{r,\lambda} \) is \( \chi \)-equivariant: \( g \circ \psi_{r,\lambda} \circ g^{-1} = \psi_{r,\lambda}^{\lambda(g)} \);

(iv) for sufficiently small \( (x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m \) the point \( x \) is \( q \)-periodic under \( \psi_{\lambda} \) if and only if \( x = x^*(u, \lambda) \), where \( u \in \hat{U} \) is \( q \)-periodic under \( \psi_{r,\lambda} \);

(v) for sufficiently small \( (u, \lambda) \in \hat{U} \times \mathbb{R}^m \) the point \( u \) is \( q \)-periodic under \( \psi_{r,\lambda} \) if and only if

\[ \psi_{r,\lambda}(u) = S_0u. \]
Moreover, let \( B : U \times \mathbb{R}^n \to U \) be defined by
\[
B(u, \lambda) := S_0^{-1}\psi_{r, \lambda}(u) - S_0\psi_{r, \lambda}^{-1}(u),
\]
then

(iii) a point \((u, \lambda) \in U \times \mathbb{R}^m\) is a solution of equation (6) if and only if it is a solution of
\[
B(u, \lambda) = 0; \tag{7}
\]

(iv) the map \( B(\cdot, \lambda) \) is such that \( B(S_0u, \lambda) = S_0B(u, \lambda) \) and \( B(gu, \lambda) = \chi(g)B(u, \lambda) \).
\[
(8)
\]

The rest of this section is devoted to the prove of Theorem 3.1

Lemma 3.1. Let the subspaces \( U \subseteq \mathbb{R}^n \) and \( \hat{U} \subseteq \mathcal{Y}_q \) be defined by:
\[
U := \ker\left( S_0^0 - I \right) \subseteq \mathbb{R}^n, \quad \hat{U} := \ker\left( \hat{S}_0 - \sigma \right) \subseteq \mathcal{Y}_q. \tag{9}
\]
Then, the mapping
\[
\xi : U \to \hat{U}, \quad \xi(u) := (S_0^i u)_{i \in \mathbb{Z}} \tag{10}
\]
is an isomorphism which satisfies the following properties:

(i) \( \xi(S_0u) = \hat{S}_0\xi(u) = \sigma \cdot \xi(u), \quad u \in U \);

(ii) \( \xi(A_0u) = \hat{A}_0\xi(u), \quad u \in U \);

(iii) \( \hat{g} \cdot \xi(u) = \xi(gu), \quad u \in U \);

(iv) \( \mathcal{Y}_q = \xi(U) \oplus \text{Im}(\hat{S}_0 - \sigma) \), and this decomposition is invariant under \( \hat{A}_0, \hat{S}_0, \) and \( \sigma \).

Note also that \( U \) is invariant under \( S_0 \) and \( A_0 \) and that \( \left( \hat{A}_0 - \sigma \right) \) is invertible on \( \text{Im}(\hat{S}_0 - \sigma) \).

The proof of Lemma 3.1 is grosso modo analogous to the proof of Lemma A.2 in \[3\]. We should only verify (iii):
\[
\hat{g}\xi(u) = \hat{g}\left( S_0^i u \right)_{i \in \mathbb{Z}} = \left( g S_0^i \chi(g) u \right)_{i \in \mathbb{Z}} = \left( S_0^i g u \right)_{i \in \mathbb{Z}} = \xi(gu).
\]
Obviously, according to item (iv) in the lemma above, \( \sigma \cdot y = \xi(S_0u) + \sigma \cdot v \), where \( v \in \text{Im}(\hat{S}_0 - \sigma) \) and \( u \in U \). It follows that equation (11) splits into a system of two equations
\[
\begin{align*}
S_0u &= \Psi\lambda(u, v) \quad \text{(a)} \\
\sigma \cdot v &= \Sigma\lambda(u, v) \quad \text{(b)}
\end{align*}
\]
(11)
where the maps
\[
\Psi\lambda : U \times \text{Im}(\hat{S}_0 - \sigma) \to U \quad \text{and} \quad \Sigma\lambda : U \times \text{Im}(\hat{S}_0 - \sigma) \to \text{Im}(\hat{S}_0 - \sigma)
\]
are uniquely determined by the relation
\[
\hat{\psi}_\lambda(\xi(u) + v) = \xi(\Psi\lambda(u, v)) + \Sigma\lambda(u, v). \tag{12}
\]
One calculates that,

\[
\begin{align*}
\Psi_{\lambda}(0, 0) &= 0, \\
\Sigma_{\lambda}(0, 0) &= 0, \\
D_u\Psi_{0}(0, 0) &= A_{0}|_{U}, \\
D_v\Psi_{0}(0, 0) &= 0, \\
D_u\Sigma_{0}(0, 0) &= 0, \\
D_v\Sigma_{0}(0, 0) &= \hat{A}_{0}\big|_{\text{Im}(\mathcal{S}_{0} - \sigma)}.
\end{align*}
\]

So, the Implicit Function Theorem applies, hence there exists a unique mapping \( v^*: U \times \mathbb{R}^m \to \text{Im}(\mathcal{S}_{0} - \sigma) \), smooth near the origin, with \( v^*(0, 0) = 0 \) and equation (11)(b) holds for all \((u, v, \lambda) \in U \times \text{Im}(\mathcal{S}_{0} - \sigma) \times \mathbb{R}^m\) if and only if \( v = v^*(u, \lambda) \). Observe that the \( \mathbb{Z}_q \)-equivariance of \( \hat{\psi}_{\lambda} \) decomposes as

\[
\Psi_{\lambda}(S_{0}u, \sigma \cdot v) = S_{0}\Psi_{\lambda}(u, v),
\]
and

\[
\Sigma_{\lambda}(S_{0}u, \sigma \cdot v) = \sigma \cdot \Sigma_{\lambda}(u, v).
\]

Moreover, one has that \( v^*(0, \lambda) = 0 \), for \( \lambda \in \mathbb{R}^m \) near 0, and \( D_u v^*(0, 0) = 0 \). Then, uniqueness of the solution and (14) imply that \( v^*(S_{0}u, \lambda) = \sigma \cdot v^*(u, \lambda) \).

Substituting the solution \( v_{\lambda}^*(u) \) into the equation (11)(a) gives the determining equation

\[
S_{0}u = \psi_{r, \lambda}(u)
\]

where the reduced map \( \psi_{r}: U \times \mathbb{R}^m \to U \) is defined by

\[
\psi_{r, \lambda}(u) := \Psi_{\lambda}(u, v_{\lambda}^*(u)).
\]

The following lemma summarizes some basic properties of the reduced map.

**Lemma 3.2.** The map \( \psi_{r, \lambda} \) defined by (16), is such that

(i) \( \psi_{r, 0}(0) = 0 \), \( \lambda \in \mathbb{R}^m \) and \( D_u \psi_{r, \lambda}(0) = A_{0}|_{U} \);

(ii) \( \psi_{r, \lambda} \) is \( \mathbb{Z}_q \)-equivariant: \( \psi_{r, \lambda}(S_{0}u) = S_{0}\psi_{r, \lambda}(u) \), for all \((u, \lambda) \in U \times \mathbb{R}^m\).

(iii) \( \psi_{r, \lambda} \) is \( \chi \)-equivariant: \( g \circ \psi_{r, \lambda} \circ g^{-1} = \psi_{r, \lambda}^{\chi(g)} \).

The proof of Lemma 3.2 follows from the definitions and the remark that the equation

\[
\hat{\psi}_{\lambda}(\xi(u_1) + v) = \xi(u_2) + \sigma v
\]

holds for all \( u_1, u_2 \in U \), \( v \in \text{Im}(\hat{S}_{0} - \sigma) \) if and only if \( v = v_{\lambda}^*(u_1) \) and \( u_2 = \psi_{r, \lambda}(u_1) \).

From here on, the proof of Theorem 3.1 uses the same arguments as in Theorem 1 of [3] and it is therefore omitted.

Note that:

\[
\hat{g}v_{\lambda}^*(u) = \sigma \frac{1 - \chi(u_1)}{\chi(u_2)} v_{\lambda}\left(g\psi_{r, \lambda}^{\frac{1 - \chi(u_1)}{\chi(u_2)}}(u)\right).
\]
3.1 Reduced map via Normal Form

As we will prove later in Section 5, we may assume that, up to a near-identity transformation, the map \( \psi_\lambda \) has the form

\[
\psi_\lambda(x) = \psi_{\text{NF}}^\lambda(x) + R_{k+1}(x, \lambda),
\]

with

\[
\psi_{\text{NF}}^\lambda(S_0x) = S_0\psi_{\text{NF}}^\lambda(x),
\]

and

\[
R_{k+1}(x, \lambda) = O(|x|^{k+1}),
\]

as \( x \to 0 \) uniformly in \( \lambda \). Now, assuming that \( \psi_\lambda \) has been put in the form (17), application of the \( \mathbf{ls} \) reduction scheme as developed above yields that \( \psi_{r,\lambda} \) can be approximated as follows.

**Proposition 3.1.** Assume (2) and (17) then the maps \( x^*: U \times \mathbb{R}^m \to \mathbb{R}^n \) and \( \psi_{r,\lambda}: U \times \mathbb{R}^m \to U \) given by Theorem 3.1 are such that

\[
x^*(u, \lambda) = u + O(|u|^{k+1}) \quad \text{and} \quad \psi_{r,\lambda}(u) = \psi_{\text{NF}}^\lambda(u) + O(|u|^{k+1})
\]

as \( u \to 0 \), uniformly for \( \lambda \) in some neighborhood of the origin of \( \mathbb{R}^m \). Moreover, \( D_u\psi_{r,\lambda}(0, \lambda) = D_x\psi_\lambda^\text{NF}(0)|_U = A_\lambda|_U \); so the eigenvalues of \( D_u\psi_{r,\lambda}(0) \) coincide with the eigenvalues of \( A_\lambda \) which are close to \( q \text{'th roots of unity.}

**Remark 3.2.** In Sec. 5 we show that things can be arranged so that \( \psi_{\text{NF}} \) satisfies some extra constraints involving the nilpotent part of \( D\psi_0(0) \). But, this is of no particular use in determining the reduced diffeomorphism \( \psi_{r,\lambda} \). However, it might be useful when dealing with bifurcation problems at multiple resonances, see [4] for an example.

4 Linear Normal Form

Recall from linear algebra (cf. e.g. [3]) that an operator \( A \in GL(n,\mathbb{R}) \) admits a unique semisimple-unipotent (su) decomposition \( A = S \exp(N) \), where \( S \) is semisimple, \( N \) is nilpotent and \( SN = NS \). Goal of this section is to prove that any \( A \in GL_G^\chi(n,\mathbb{R}) \) close to a given \( A_0 \in GL_G^\chi(n,\mathbb{R}) \) can be normalised to the form \( A = S_0e^{N_0} + B \), where \( A_0 = S_0e^{N_0} \) is the SU-decomposition of \( A_0 \) and \( S_0B = BS_0, BN_0^T = N_0^TB \). Such normal form depends on the choice of a suitable inner product for \( gl(\mathbb{R}, n) \), whose existence is proved first (see Lemma 4.1 and Lemma 4.2 below).

We start with some technical details.

4.1 Technicalities

Let \( \chi_1, \chi_2 \) be one-dimensional real characters of a compact finite discrete group \( G \). Define the associated projection operators \( P_G^{\chi_1}, P_G^{\chi_2}: gl(n,\mathbb{R}) \to gl(n,\mathbb{R}) \) by

\[
P_G^{\chi_1}(A) := \frac{1}{|G|} \sum_{g \in G} \chi_1(g)gAg^{-1}.
\]

**Remark 4.1.** In the continuous case one should consider

\[
P_G^{\chi_1}(A) := \int_G \chi_1(g)gAg^{-1}.
\]

If \( \chi_1 \) and \( \chi_2 \) are orthogonal characters, i.e., \( \sum_{g \in G} \chi_1(g^{-1})\chi_2(g) = 0 \) then

\[
P_G^{\chi_1} P_G^{\chi_2} = 0.
\]
Define the subset $\text{gl}^\lambda_G(n, \mathbb{R}) \subset \text{gl}(n, \mathbb{R})$ by
\[
\text{gl}^\lambda_G(n, \mathbb{R}) := P_\lambda^G(\text{gl}(n, \mathbb{R})) = \{ A \in \text{gl}(n, \mathbb{R}) | g A g^{-1} = \chi(g) A \}. 
\]
and the set $\text{GL}^\lambda_G(n, \mathbb{R})$ by
\[
\text{GL}^\lambda_G(n, \mathbb{R}) := \{ A \in \text{GL}(n, \mathbb{R}) | g A g^{-1} = A^{\chi(g)} \}. 
\]

**Remark 4.2.** Note that $\text{GL}^\lambda_G(n, \mathbb{R})$ is in general only an algebraic variety and not a manifold. However, this does not occur in the case of reversible systems as considered in \([\text{R}]\). For example, $\text{GL}^\lambda_G(n, \mathbb{R})$ is, in the simple case $n = 2$, $\mathcal{G} = \{ I, R \}$ with $R \in \text{GL}(2, \mathbb{R})$ such that $R^2 = I$, and $\chi(R) = -1$, diffeomorphism equivalent to a hyperboloyde.

For $A \in \text{GL}(n, \mathbb{R})$, define the adjoint action $Ad(A)$ of $A$ on $\text{gl}(n, \mathbb{R})$ by
\[
Ad(A)B = ABA^{-1}. 
\]
We need to study the action of $Ad(A_0^{-1}) - I$ on the space $\text{gl}^\lambda_G(n, \mathbb{R})$ of all $\chi$-equivariant operators, where $A_0 \in \text{GL}^\lambda_G(n, \mathbb{R})$. Therefore, we introduce the group $\mathcal{G}^\chi(A_0)$,
\[
\mathcal{G}^\chi(A_0) = \{ x | x \in \mathcal{G} \text{ and } \chi(x) = 1, \text{ or } x = g A_0 \text{ where } g \in \mathcal{G}, \chi(g) = -1 \}. 
\]
A straightforward calculation shows that
\[
Ad(A_0^{-1}) - I : \text{gl}^\lambda_G(n, \mathbb{R}) \rightarrow \text{gl}^{1\chi}_{\mathcal{G}^\chi(A_0)}(n, \mathbb{R}). 
\]
Given a character $\alpha$ of the group $\mathcal{G}$, we can define an associated character $\tilde{\alpha}$ of the group $\mathcal{G}^\chi(A_0)$ as follows:
\begin{itemize}
  \item $\tilde{\alpha}(h) = \alpha(h)$, if $\chi(h) = 1$ and $h \in \mathcal{G}$;
  \item $\tilde{\alpha}(gA_0) = \alpha(g)$, if $\chi(g) = -1$ and $g \in \mathcal{G}$.
\end{itemize}
With this notation, one shows that the following generalization of (22) holds:
\[
Ad(A_0^{-1}) - I : \text{gl}^\lambda_G(n, \mathbb{R}) \rightarrow \text{gl}^{1\chi}_{\mathcal{G}^\chi(A_0)}(n, \mathbb{R}). 
\]

### 4.2 Inner Product

**Lemma 4.1.** Let $S_0 \in \text{GL}^\chi_G(n, \mathbb{R})$ be semisimple and $\chi$ be as before, then there exists a scalar product on $\mathbb{R}^n$ (with corresponding involution $*$) such that $S_0$ is normal, i.e., $S_0 S_0^* = S_0^* S_0$, and $g S_0 = (S_0^*)^{\chi(g)} g$, for all $g \in \mathcal{G}$.

**Proof.** Denote by $\sigma(S_0)$ the spectrum of $S_0$ and by $(\cdot, \cdot)$ the usual scalar product on $\mathbb{R}^n$. Note that $S_0 \in \text{GL}^\chi_G$ implies that the spectrum of $S_0$ consists of $\pm 1$’s, or pairs $\{ \lambda, \overline{\lambda} \} \in \mathbb{C}$ with $|\lambda| = 1$, or quadruples $\{ \lambda, \lambda^{-1}, \overline{\lambda}, \overline{\lambda}^{-1} \}$. Since $S_0$ is semisimple, the spectral decomposition holds, i.e.,
\[
\mathbb{C}^n = \bigoplus_{\lambda \in \sigma(S_0)} V_\lambda
\]
where $V_\lambda$ denotes the eigenspace corresponding to the eigenvalue $\lambda$. Define a scalar product $(\cdot, \cdot)$ on $\mathbb{R}^n$ as follows:
\[
(x, x') := \sum_{\lambda \in \sigma(S_0) \cap \mathbb{R}} (v, v'_\lambda)_{\lambda} + \sum_{\lambda \in \sigma(S_0), \text{Im}(\lambda) > 0} (w, w'_\lambda)_{\lambda},
\]
where $v, v'_\lambda \in V_\lambda \cap \mathbb{R}^n$, $w, w'_\lambda \in W_\lambda := (V_\lambda \oplus V_{\overline{\lambda}}) \cap \mathbb{R}^n$, and moreover $x$ decomposes uniquely as $x = \sum_{\lambda \in \sigma(S_0) \cap \mathbb{R}} v_\lambda + \sum_{\lambda \in \sigma(S_0), \text{Im}(\lambda) > 0} w_\lambda$ and similarly $x'$ does. Our aim is to determine $(\cdot, \cdot)_{\lambda}$ such that $S_0$ is normal with respect to this scalar product.
When $\lambda \in \mathbb{R}$, any choice of scalar product is good since $S_0|_{V_\lambda} = \lambda V_\lambda$. So, the first part of the proof is completed when our assertion is shown for $S_0|_{W_\lambda}$, $\Im \lambda > 0$. Let $\lambda = \alpha + i\beta$ with $\beta > 0$, then $V_\lambda \oplus V_\lambda^*$ is the kernel of the operator $(S_0 - \alpha I)^2 + \beta^2 I$. Define

$$J_\lambda = \frac{1}{\beta}(S_0 - \alpha I)|_{W_\lambda},$$

and let $(\cdot, \cdot)$ be some scalar product on $W_\lambda$. Note that $J_\lambda^2 = -I_{W_\lambda}$ and define $(\cdot, \cdot)_\lambda$ by

$$(w, w')_\lambda := \frac{1}{2}[(w, w') + (J_\lambda w, J_\lambda w')], \quad w, w' \in W_\lambda.$$  

If $\ast$ denotes the involution defined by this scalar product, then $J_\lambda^* J_\lambda = I_{W_\lambda}$, i.e., $J_\lambda$ is orthogonal which implies that $J_\lambda = -J_\lambda^*$. Also, $S_0|_{W_\lambda}$ is normal since $S_0|_{W_\lambda} = \alpha I_{W_\lambda} + \beta J_\lambda$. Moreover, $S_0^*|_{W_\lambda}$ maps $W_\lambda$ into itself.

Suppose $\lambda$ is a real eigenvalue of $S_0$, then either $gV_\lambda = V_\lambda$ or $gV_\lambda = V_{\lambda^{-1}}$ depending on whether $\chi(g) = 1$ or $\chi(g) = -1$ respectively. In both cases, it is obvious that $gS_0^*|_{V_\lambda} = gS_0|_{V_\lambda} = S_0^{\chi(g)}|_{V_{\lambda^{-1}}}$, $g|_{V_\lambda} = (S_0^*|_{V_{\lambda^{-1}}})^{\chi(g)}|_{V_\lambda}$ and we are only left to prove this equality for eigenvalues with a strictly positive imaginary part. If $\chi(g) = 1$, then $gW_\lambda = W_\lambda$ and therefore $gS_0|_{W_\lambda} = S_0|_{W_\lambda}$. But then $g|_{W_\lambda}$ commutes with $S_0|_{W_\lambda}$ since $S_0|_{W_\lambda}$ is normal. In case $\chi(g) = -1$, $gW_\lambda = W_{\lambda^{-1}}$, $J_{\lambda^{-1}} = \frac{1}{\beta}(S_0^{-1} - \alpha I)|_{W_{\lambda^{-1}}}$ and

$$S_0|_{W_{\lambda^{-1}}} = \frac{\alpha}{\alpha^2 + \beta^2} I|_{W_{\lambda^{-1}}} - \frac{\beta}{\alpha^2 + \beta^2} J_{\lambda^{-1}}.$$  

Straightforward calculation reveals that $gJ_\lambda g^{-1}|_{W_{\lambda^{-1}}} = J_{\lambda^{-1}}$ and therefore $gS_0|_{W_{\lambda^{-1}}} = (\alpha I|_{W_{\lambda^{-1}}} - \beta J_{\lambda^{-1}}) = (S_0^*|_{W_{\lambda^{-1}}})^{-1}$.

\[ \Box \]

**Lemma 4.2.** Let $S_0$, $\mathcal{G}$ and $\chi$ be as before, then there exists a scalar product on $\mathbb{R}^n$ (with corresponding involution $\ast$) such that $S_0S_0^* = S_0^*S_0$ and $g^* g = I$ for all $g \in \mathcal{G}$

**Proof.** Denote by $(\cdot, \cdot)$ and $T$ the scalar product and associated involution given by Lemma 4.1. The remainder of the proof then consists in showing that the scalar product $(\cdot, \cdot)$ defined by

$$\langle x, y \rangle = \sum_{g \in \mathcal{G}} (gx, gy)$$

has all the desired properties. Clearly, $g^* g = I$ for all $g \in \mathcal{G}$, since the right translation on $\mathcal{G}$ defined by $g$ is a bijection. Therefore, we should only prove that $S_0^* S_0 = S_0 S_0^*$, which is a direct consequence of the following calculation:

$$\langle S_0 x, S_0 y \rangle = \sum_{g \in \mathcal{G}} (gS_0 x, gS_0 y)$$

$$= \sum_{g \in \mathcal{G}} (gx, (S_0^{\chi(g)})^T S_0^{\chi(g)} gy)$$

$$= \sum_{g \in \mathcal{G}} (gx, S_0^{\chi(g)} (S_0^{\chi(g)})^T gy)$$

$$= \sum_{g \in \mathcal{G}} (gS_0^T x, gS_0^T y)$$

$$= \langle S_0^T x, S_0^T y \rangle$$

and the direct observation that $S_0^T = S_0^*$. \[ \Box \]
Remark 4.3. If $G$ is a compact continuous group with left invariant Haar measure $dg$, one may define a $G$-invariant inner product on $\mathbb{R}^n$ by averaging an inner product for $\mathbb{R}^n$ over $G$:

$$\langle x, y \rangle = \int_G (hx, hy) dg, \quad x, y \in \mathbb{R}^n.$$ 

It can then be shown that every real representation is isomorphic to an orthogonal representation. So, one obtains the counterparts of Lemma’s 4.1 and 4.2 for the continuous case.

4.3 Linear Nilpotent Normal Form

Lemma 4.3. Let $A_0 = S_0 e^{N_0}$ be the SU-decomposition of $A_0 \in GL^\chi_G(n, \mathbb{R})$. Then, $\text{Ad}(A_0^{-1}) - I : U_1 \to V_1$ is an isomorphism as well as $\text{Ad}(A_0^{-1}) - I : U_2 \to V_2$, where

$$U_1 = \text{Im}(\text{Ad}(S_0^{-1}) - I) \cap \text{gl}^1_{G^\chi}(n, \mathbb{R}),$$
$$U_2 = \text{Im}(\text{Ad}(S_0^{-1}) - I) \cap \text{gl}^1_{G^\chi}(n, \mathbb{R}),$$
$$V_1 = \text{Im}(\text{Ad}(S_0^{-1}) - I) \cap \text{gl}^1_{G^\chi(A_0)}(n, \mathbb{R}),$$
$$V_2 = \text{Im}(\text{Ad}(S_0^{-1}) - I) \cap \text{gl}^1_{G^\chi(A_0)}(n, \mathbb{R}).$$

Proof. We know that $(\text{Ad}(A_0^{-1}) - I)$ is invertible on $\text{Im}(\text{Ad}(S_0^{-1}) - I)$ and maps $U_1$ in $V_1$ and $U_2$ into $V_2$ injectively. We show that $\text{Ad}(A_0^{-1} - I)$ is an isomorphism from $U_2$ to $V_2$ by proving surjectivity.

Let $\psi \in \text{gl}^1_{G^\chi(A_0)}(n, \mathbb{R}) \cap \text{Im}(\text{Ad}(S_0^{-1}) - I)$ and let $\tilde{\psi} \in \text{Im}(\text{Ad}(S_0^{-1}) - I)$ be such that $\psi = (\text{Ad}(A_0^{-1}) - I)\tilde{\psi}$ (as $\psi$ exists and is unique). Then, one calculates that

$$\psi = \text{P}^1_{G^\chi(A_0)}(\psi) = \text{P}^1_{G^\chi(A_0)}(\text{Ad}(A_0^{-1}) - I)\tilde{\psi} = ((\text{Ad}(A_0^{-1}) - I))\text{P}^1_{G^\chi} \tilde{\psi}.$$

So, if we can show that $\text{P}^1_{G^\chi} \tilde{\psi} \in \text{Im}(\text{Ad}(S_0^{-1}) - I)$ then $\text{P}^1_{G^\chi} \tilde{\psi} = \tilde{\psi}$ by uniqueness and the surjectivity is proved. Now, we know that there exists some $\tilde{\psi} \in \text{gl}(n, \mathbb{R})$ such that $\tilde{\psi} = (\text{Ad}(S_0^{-1}) - I)\psi$. Using the fact that $S_0^{-1} \in GL^\chi_G(n, \mathbb{R})$ one then obtains that

$$\text{P}^1_{G^\chi} \tilde{\psi} = \text{P}^1_{G^\chi} ((\text{Ad}(S_0^{-1}) - I)\tilde{\psi}) = (\text{Ad}(S_0^{-1}) - I)\text{P}^1_{G^\chi(S_0^{-1})} \tilde{\psi},$$

and the result follows. \qed

Lemma 4.4. Given $A_0 \in GL^\chi_G(n, \mathbb{R})$, let $A_0 = S_0 e^{N_0}$ be its SU-decomposition. Then there exist a neighborhood $\Omega$ of $A_0$ in $GL^\chi_G(n, \mathbb{R})$ and a map $\phi : \Omega \to \text{gl}^1_{G^\chi}(n, \mathbb{R}) \cap \text{Im}(\text{Ad}(S_0^{-1}) - I)$ such that $\phi(A_0) = 0$ and

$$\text{Ad}(e^{\phi(A)}) A = A_0 e^{B(A)},$$

where $B(A) \in \ker(\text{Ad}(S_0) - I) \cap \text{gl}^\chi_{G^\chi(A_0)}$ and $B(A_0) = 0$.

Proof. Define $f : \text{gl}^1_{G^\chi}(n, \mathbb{R}) \times GL^\chi_G(n, \mathbb{R}) \to \text{gl}^\chi_{G^\chi(A_0)}(n, \mathbb{R})$ by the relation

$$\text{Ad}(e^{\phi}) A = A_0 e^{f(\phi, A)}.$$ (24)

That is, $f(\phi, A) := \log(g(\phi, A))$ with $g(\phi, A) := A_0^{-1} e^{\phi} A e^{-\phi}$. The map $f$ is well-defined and smooth for $(\phi, A)$ near $(0, A_0)$ since $g(0, A_0) = I$. Also, $\text{Ad}(e^{\phi}) A \in GL^\chi_G(n, \mathbb{R})$ and

$$f(0, A_0) = 0$$
$$D_{\phi} f(0, A_0) = (\text{Ad}(A_0^{-1}) - I)|_{\text{gl}^1_{G^\chi}(n, \mathbb{R})} \in C(\text{gl}^1_{G^\chi}(n, \mathbb{R}), \text{gl}^\chi_{G^\chi(A_0)}(n, \mathbb{R})).$$
Consider the splitting
\[
gl_{\tilde{\mathfrak{g}}^\chi(A_0)}(n, \mathbb{R}) = \left( gl_{\tilde{\mathfrak{g}}^\chi(A_0)}(n, \mathbb{R}) \cap \mathrm{Im}(\mathrm{Ad}(S_0^{-1}) - I) \right)
+ \left( gl_{\tilde{\mathfrak{g}}^\chi(A_0)}(n, \mathbb{R}) \cap \ker(\mathrm{Ad}(S_0^{-1}) - I) \right)
\]
(25)
and let \( \pi : gl_{\tilde{\mathfrak{g}}^\chi(A_0)}(n, \mathbb{R}) \to \left( gl_{\tilde{\mathfrak{g}}^\chi(A_0)}(n, \mathbb{R}) \cap \mathrm{Im}(\mathrm{Ad}(S_0^{-1}) - I) \right) \) the corresponding projection on the first factor. Define
\[
h := \pi \cdot f|_{gl_{\tilde{\mathfrak{g}}^\chi(A_0)}(n, \mathbb{R}) \times GL^0_{\mathfrak{g}}(n, \mathbb{R})}.
\]
Then, \( h(0, A_0) = 0 \) and the operator \( D_0 h(0, A_0) \) is an isomorphism between \( gl_{\tilde{\mathfrak{g}}^\chi(A_0)}(n, \mathbb{R}) \cap \mathrm{Im}(\mathrm{Ad}(S_0^{-1}) - I) \) and \( gl_{\tilde{\mathfrak{g}}^\chi(A_0)}(n, \mathbb{R}) \cap \ker(\mathrm{Ad}(S_0^{-1}) - I) \). The Implicit Function Theorem then implies that there exists \( \tilde{\phi} : GL^0_{\mathfrak{g}}(n, \mathbb{R}) \to gl_{\tilde{\mathfrak{g}}^\chi(A_0)}(n, \mathbb{R}) \) with \( \tilde{\phi}(A_0) = 0 \) such that \( h(\tilde{\phi}(A), A) = 0 \). Hence, setting \( B(A) := f(\tilde{\phi}(A), A) \in \ker(\mathrm{Ad}(S_0) - I) \cap gl_{\tilde{\mathfrak{g}}^\chi(A_0)}(n, \mathbb{R}) \) proves the lemma.

**Proof of the splitting** (25). We prove that the splitting (25) holds. The semisimplicity of \( S_0 \in GL^0_{\mathfrak{g}}(n, \mathbb{R}) \) implies that \( gl(n, \mathbb{R}) = \mathrm{Im}(\mathrm{Ad}(S_0^{-1}) - I) \oplus \ker(\mathrm{Ad}(S_0^{-1}) - I) \). Therefore, each \( \Psi \in gl(n, \mathbb{R}) \) has unique decomposition
\[
\Psi = \Psi_{im} + \Psi_{ker}
\]
(26)
with \( \Psi_{im} \in \mathrm{Im}(\mathrm{Ad}(S_0^{-1}) - I) \) and \( \Psi_{ker} \in \ker(\mathrm{Ad}(S_0^{-1}) - I) \). Recall that \( \ker(\mathrm{Ad}(S_0^{-1}) - I) = \ker(\mathrm{Ad}(S_0) - I) \). For \( \Psi \in gl_{\tilde{\mathfrak{g}}^\chi(A_0)}(n, \mathbb{R}) \) this allows us to write:
\[
\Psi = P_{\tilde{\mathfrak{g}}^\chi(A_0)}^\chi A = P_{\tilde{\mathfrak{g}}^\chi(A_0)}^\chi \Psi_{im} + P_{\tilde{\mathfrak{g}}^\chi(A_0)}^\chi \Psi_{ker}.
\]
Using the fact that \( S_0^{-1} \in GL^0_{\mathfrak{g}}(n, \mathbb{R}) \) and the definitions, one checks that
\[
(\mathrm{Ad}(S_0) - I) P_{\tilde{\mathfrak{g}}^\chi(A_0)}^\chi (\mathrm{Ad}(S_0) - I) = P_{\tilde{\mathfrak{g}}^\chi(A_0)}^\chi (\mathrm{Ad}(S_0) - I) P_{\tilde{\mathfrak{g}}^\chi(A_0)}^\chi.
\]
Therefore, \( P_{\tilde{\mathfrak{g}}^\chi(A_0)}^\chi \Psi_{im} \in \mathrm{Im}(\mathrm{Ad}(S_0^{-1}) - I) \) and \( P_{\tilde{\mathfrak{g}}^\chi(A_0)}^\chi \Psi_{ker} \in \ker(\mathrm{Ad}(S_0) - I) \). The uniqueness of the splitting (25) then implies that \( \Psi_{im} = P_{\tilde{\mathfrak{g}}^\chi(A_0)}^\chi \Psi_{im} \) and \( \Psi_{ker} = P_{\tilde{\mathfrak{g}}^\chi(A_0)}^\chi \Psi_{ker} \). Hence, \( \Psi_{im} \in \mathrm{Im}(\mathrm{Ad}(S_0^{-1}) - I) \cap gl_{\tilde{\mathfrak{g}}^\chi(A_0)}(n, \mathbb{R}) \) and \( \Psi_{ker} \in \ker(\mathrm{Ad}(S_0^{-1}) - I) \cap gl_{\tilde{\mathfrak{g}}^\chi(A_0)}(n, \mathbb{R}) \), which proves the result.

**Proposition 4.1 (Linear Nilpotent Normal Form).** Let \( A_0 \in GL^0_{\mathfrak{g}}(n, \mathbb{R}) \) have SU-decomposition \( A_0 = S_0 e^{N_0} \) and let \( \langle \cdot, \cdot \rangle \) be a scalar product as in Lemma 4. Then, there exist a neighborhood \( \Omega \) of \( A_0 \) in \( GL^0_{\mathfrak{g}}(n, \mathbb{R}) \) and a map \( \phi : \Omega \to gl_{\tilde{\mathfrak{g}}^\chi}(n, \mathbb{R}) \) such that \( \phi(A_0) = 0 \) and
\[
\mathrm{Ad} \left( e^{\phi(A)} \right) A = S_0 e^{N_0 + C(A)},
\]
where \( C(A) \in \ker(\mathrm{Ad}(S_0) - I) \cap \ker(\mathrm{ad}(N_0^\mathfrak{g}^\chi)) \cap gl_{\tilde{\mathfrak{g}}^\chi}(n, \mathbb{R}) \) and \( C(A_0) = 0 \).

**Proof.** From Lemma 4.4, we can assume that there exists a neighborhood \( \Omega^* \subset GL^0_{\mathfrak{g}}(n, \mathbb{R}) \) of \( A_0 \) and a map \( \phi : \Omega^* \to gl_{\tilde{\mathfrak{g}}^\chi}(n, \mathbb{R}) \) such that
\[
\mathrm{Ad}(e^{\tilde{\phi}(A)}) A = S_0 e^{N_0 + B(A)},
\]
where \( B(A) \in \ker(\mathrm{Ad}(S_0) - I) \cap gl_{\tilde{\mathfrak{g}}^\chi}(n, \mathbb{R}) \) and \( B(A_0) = 0 \). Let \( \phi \in \ker(\mathrm{Ad}(S_0) - I) \cap gl_{\tilde{\mathfrak{g}}^\chi}(n, \mathbb{R}) \), then,
\[
\mathrm{Ad}(e^{\phi}) \mathrm{Ad}(e^{\tilde{\phi}(A)}) A = S_0 e^{N_0 + f(\phi, A)},
\]
where \( f : (\ker(\Ad(S_0) - I) \cap g_l^y(n, \mathbb{R})) \times \GL^x_y(n, \mathbb{R}) \to g_l^y(n, \mathbb{R}) \) is given by

\[
f(\phi, A) := \Ad(e^\phi)(N_0 + B(A)) - N_0.
\]

We have that \( f(0, A_0) = 0 \) and that \( D_\phi f(0, A_0) = -\ad(N_0)|_{\ker(\Ad(S_0) - I) \cap g_l^y(n, \mathbb{R})} \) belongs to \( \mathcal{L}(\ker(\Ad(S_0) - I) \cap g_l^y(n, \mathbb{R}), \ker(\Ad(S_0) - I) \cap g_l^y(n, \mathbb{R})) \). Now, by Lemma 4, the operators \( \ad(N_0) \) and \( \ad(N_0^T) \) leave the complementary subspaces \( \ker(\Ad(S_0) - I) \) and \( \text{Im}(\Ad(S_0) - I) \) of \( g_l(n, \mathbb{R}) \) invariant and

\[
(ad(N_0)|_{\ker(\Ad(S_0) - I)})^T = ad(N_0^T)|_{\ker(\Ad(S_0) - I)}.
\]

Also, \( ad(N_0) \) maps \( g_l^y(n, \mathbb{R}) \) on \( g_l^y(n, \mathbb{R}) \) and vice versa. Hence,

\[
\ker(\Ad(S_0) - I) \cap g_l^y(n, \mathbb{R}) = \ker(\Ad(S_0) - I) \cap \text{Im}(\Ad(N_0)) \cap g_l^y(n, \mathbb{R})
\]

\[
\oplus \ker(\Ad(S_0) - I) \cap \ker(\ad(N_0^T)) \cap g_l^y(n, \mathbb{R})
\]

and

\[
\ker(\Ad(S_0) - I) \cap g_l^y(n, \mathbb{R}) = \ker(\Ad(S_0) - I) \cap \ker(\ad(N_0)) \cap g_l^y(n, \mathbb{R})
\]

\[
\oplus \ker(\Ad(S_0) - I) \cap \text{Im}(\ad(N_0^T)) \cap g_l^y(n, \mathbb{R})
\] (27)

Note that (by the choice of the scalar product) the operator \( \ad(N_0) \) is an isomorphism from \( \ker(\Ad(S_0) - I) \cap \text{Im}(\ad(N_0^T)) \cap g_l^y(n, \mathbb{R}) \) onto \( \ker(\Ad(S_0) - I) \cap \text{Im}(\ad(N_0)) \cap g_l^y(n, \mathbb{R}) \). Indeed, \( \text{ad}(N_0) P^1 g = P^1 \text{ad}(N_0) \). Let

\[
\pi : \ker(\Ad(S_0) - I) \cap g_l^y(n, \mathbb{R}) \to \ker(\Ad(S_0) - I) \cap \text{Im}(\ad(N_0)) \cap g_l^y(n, \mathbb{R})
\]

be the projection on the first factor associated to the splitting (27), and define the map

\[
g : (\ker(\Ad(S_0) - I) \cap \text{Im}(\ad(N_0^T)) \cap g_l^y(n, \mathbb{R})) \times \GL^x_y(n, \mathbb{R})
\]

\[
\to \ker(\Ad(S_0) - I) \cap \text{Im}(\ad(N_0)) \cap g_l^y(n, \mathbb{R}),
\]

by

\[
g := \pi \cdot f|_{(\ker(\Ad(S_0) - I) \cap \text{Im}(\ad(N_0^T)) \cap g_l^y(n, \mathbb{R})) \times \GL^x_y(n, \mathbb{R})}.
\]

Then, \( g(0, A_0) = 0 \) and \( D_\phi g(0, A_0) \) is an isomorphism. The Implicit Function Theorem then implies the existence of a map \( \phi^* : \Omega \subset \Omega^* \to \ker(\Ad(S_0) - I) \cap \text{Im}(\ad(N_0^T)) \cap g_l^y(n, \mathbb{R}) \) with \( \phi^*(A_0) = 0 \) such that

\[
g(\phi^*(A), A) = 0.
\]

Then, \( f(\phi^*(A), A) \in \ker(\Ad(S_0) - I) \cap \ker(\ad(N_0^T)) \cap g_l^y(n, \mathbb{R}) \). Hence, the result follows.

\section{Normal Form}

Goal of this section is to prove by induction that a map \( \psi_\lambda \) as considered in this paper admits a \( \chi \)-equivariant normal form with constraints involving the nilpotent \( N_0 \). More in details, we first prove that a normalisation as in (19) is possible. We fix then a scalar product in \( \mathbb{R}^n \) as given in Lemma (12) and show that \( \psi_\lambda^{NF} \) can be brought into the form \( \psi_\lambda^{NF} = S_0 \exp(N_0 + X_\lambda) \) with \( N_0 + X_\lambda \in \ker(\Ad(S_0) - I) \cap \Im(\Ad(N_0)) \) and \( DXX_0(0)N_{0}^{T}(x) = N_{0}^{T}X_{0}(x) \).

Let \( \mathcal{H}_k = \mathcal{H}_k(\mathbb{R}^n) \) be the space of polynomial maps homogeneous of degree \( k \), note that \( \mathcal{H}_1 = gl(n, \mathbb{R}) \). Then the Taylor series of \( \Phi \in \text{Diff}_0(\mathbb{R}^n) \) is an element of the space of formal power series \( \prod_{k=1}^{\infty} \mathcal{H}_k \). Also, considering \( A_0 = D\Phi_0(0) \in \mathcal{L}(\mathbb{R}^n) \), it directly follows from the definitions that \( \Ad(A_0) \) induces a linear map \( \mathcal{H}_k \to \mathcal{H}_k \) to be denoted by \( \Ad_k A_0 \). Let \( X_0 \) be the space of all
smooth vectorfields on \( \mathbb{R}^n \) with a fixed point at 0, and let \( X_j^k := \{ X \in X_0 \mid D_x^j X(0) = 0, 1 \leq j \leq k \} \).

The following notation is self explanatory:

\[
X = X_1 + \cdots + X_k \mod X_0^k, \quad \text{with } X_j \in H_j.
\]

Define the operator \( C_k : H_1 \to \mathcal{L}(H_k) \), for all \( X_1 \in H_1, X_k \in H_k \), by

\[
C_k(X_1)X_k := \int_0^1 e^{-sX_1}X_k e^{sX_1} ds.
\]

Observe also that if \( X_1 = 0 \), then
\[
C_k(0) = I_{H_k}.
\]

Let \( P_k \) be the space of all polynomial maps \( p : \mathbb{R}^n \to \mathbb{R}^n, p(0) = 0 \), of degree less or equal to \( k \).

According to our previous notation, \( P_k = X_0 / X_0^k \), \( k \geq 1 \). Set

\[
X[k] := X_1 + \cdots + X_k \in P_k, \quad \text{with each } X_j \in H_j.
\]

We need the following generalisation of the Campbell-Hausdorff formula.

**Lemma 5.1 (\( k \)).** Given \( X \in X_0 \), let \( X_1 := X \mod X_0^1 \) such that \( C_k(X_1) \) is invertible. Then, for any \( Y_k \in H_k \),

(i) \( e^X e^Y = e^{X + C_k(X_1)^{-1}Y} \mod X_0^k \),

(ii) \( e^Y e^X = e^{X + C_k(-X_1)^{-1}Y} \mod X_0^k \).

**Proposition 5.1.** Let \( G \) be a finite compact group with one-dimensional character \( \chi : G \to \mathbb{C} \). Let \( \psi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) be a smooth family of local diffeomorphisms satisfying (8) and (4). Let \( A_0 = S_0 e^{X_0} \in GL_0^1(n, \mathbb{R}) \) be the SU-decomposition of \( A_0 = D_x \psi_0(0) \). Then, for each \( k \geq 1 \) there exists a neighborhood \( \omega_k \) of the origin in \( \mathbb{R}^m \) and a parameter dependent near-identity transformation \( \Phi_{k,\lambda} : \mathbb{R}^n \to \mathbb{R}^n \) with \( \Phi_{k,\lambda}(0) = 0, D_x \Phi_{0}(0) = I \) and \( g \Phi_{k,\lambda}^{-1} = \Phi_{k,\lambda} \) such that the following holds:

\[
\text{Ad}(\Phi_{k,\lambda}) \psi_\lambda = A_0 e^{X[k]_\lambda} \mod X_0^k, \quad \forall \lambda \in \omega_k, \quad (30)
\]

with \( X[k]_\lambda \in \ker(\text{Ad}(S_0) - I) \cap \text{Im}(P_{G^*(A_0)}^X) \subset P_k, \quad \text{and } X_0[k] = 0 \mod X_0^k \).

**Proof.** The proof is by induction on \( k \). For \( k = 1 \) the result follows from Lemma 4.4. The induction step is proved as follows. Assume the result true for \( (k - 1) (k > 1) \). Denoting \( \text{Ad}(\Psi_{k-1,\lambda})(\psi_\lambda) \) again by \( \psi_\lambda \), this means that

\[
\psi_\lambda = A_0 e^{X[k-1]_\lambda} \mod X_0^{k-1},
\]

with \( X[k-1]_\lambda \in \ker(\text{Ad}(S_0) - I) \cap \text{Im}(P_{G^*(A_0)}^X) \subset P_{k-1}, \text{and } X_0[k-1] = 0 \mod X_0^{k-1} \). Then,

\[
\psi_\lambda = A_0 e^{X[k-1]_\lambda + Z_{k,\lambda}} \mod X_0^k,
\]

for some \( Z_{k,\lambda} \in H_k \cap \text{Im}(P_{G^*(A_0)}^X) \). The goal is to show that we can transform \( \psi_\lambda \) further so that \( Z_{k,\lambda} \in \ker(\text{Ad}(S_0) - I) \cap \text{Im}(P_{G^*(A_0)}^X) \). To do so, let \( \Phi_k \) be of the form \( \Phi_k = e^{\phi_k} \) with \( \phi_k \in H_k \cap \text{Im}(P_1^X) \). Then, \( \text{Ad}(e^{\phi_k}) \psi_\lambda \in \text{Im}(P_1^X) \) and

\[
\text{Ad}(e^{\phi_k}) \psi_\lambda = A_0 e^{X[k-1]_\lambda + f(\phi_k, \lambda)} \mod X_0^k,
\]

where \( f_k : H_k \cap \text{Im}(P_1^X) \to H_k \cap \text{Im}(P_{G^*(A_0)}^X) \) is given by

\[
f_k(\phi_k, \lambda) = Z_{k,\lambda} + C_k(-X_{1,\lambda}) \text{Ad}(A_0)^{-1} \phi_k = C_k(X_{1,\lambda})^{-1} \phi_k.
\]
Note that $f_k(\phi_k, \lambda)$ is smooth and well defined near $(0,0)$ and $f_k(0, \lambda) = Z_k, \lambda$. Using the fact that $\text{Ad}_k(A_0^{-1}) - 1$ is an isomorphism between $\mathcal{H}_k \cap \text{Im}(P^X) \cap \text{Im}(\text{Ad}_k(S_0^{-1}) - 1$ and $\mathcal{H}_k \cap \text{Im}(P^X(\mathcal{A}_0)) \cap \text{Im}(\text{Ad}_k(S_0^{-1}) - 1)$ and that
\[
\mathcal{H}_k \cap \text{Im}(P^X(\mathcal{A}_0)) = \mathcal{H}_k \cap \text{Im}(P^X(\mathcal{A}_0)) \cap \text{Im}(\text{Ad}_k(S_0^{-1}) - 1) 
\]
the result follows by the Implicit Function Theorem as in Lemma 4.2.

**Theorem 5.1 (Nilpotent Normal Form).** Let $\mathcal{G}$ be a finite compact group with one-dimensional character $\chi : \mathcal{G} \to \mathbb{C}$. Let $\psi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be a smooth family of local diffeomorphisms satisfying (7) and (6). Let $A_0 = S_0 e^{X_0} \in GL_\mathbb{C}(n, \mathbb{R})$ be the SU-decomposition of $A_0$, and let $(\cdot, \cdot)$ be a scalar product as in Lemma 4.1. Then, for each $k \geq 1$ there exists a neighborhood $\omega_k$ of the origin in $\mathbb{R}^n$ and a parameter dependent near-identity transformation $\Phi_{k, \lambda} : \mathbb{R}^n \to \mathbb{R}^n$ with $\Phi_{k, \lambda}(0) = 0$, $D_x \Phi_0(0) = 1$ and $g \Phi_{k, \lambda} g^{-1} = \Phi_{k, \lambda}$ such that the following holds:
\[
\text{Ad}(\Phi_{k, \lambda}) \psi_\lambda = S_0 e^{X_0 + X_\lambda} \mod \mathcal{A}_0^k, \quad \forall \lambda \in \omega_k. \tag{31}
\]
with
\[
X_\lambda(0) = 0, \quad DX_\lambda(0) = 0, \\
S_0 \cdot X_\lambda = X_\lambda \cdot S_0, \\
DX_\lambda(x) X^T_0(x) = X^T_0 X_\lambda(x), \\
(N_0 + X_\lambda) \cdot g = \chi(g) g \cdot (N_0 + X_\lambda). \tag{32}
\]

**Proof.** By Proposition 5.1 we can assume that after transformation $\psi_\lambda$ is of the form
\[
\psi_\lambda = S_0 e^{X_0 + X_\lambda^{[k]}} \mod \mathcal{A}_0^k,
\]
where $X_\lambda^{[k]} \in \ker(\text{Ad}(S_0) - 1) \cap \text{Im}(P^X)$ and $X_\lambda^{[k]}(0) = 0, DX_\lambda^{[k]}(0) = 0$. The proof proceeds then by induction on $k$. For $k = 1$, the result follows from Proposition 4.1. For $k > 1$, the induction argument is similar to that of Proposition 5.1 with the following change:
\[
\psi_\lambda = S_0 e^{X_0 + X_\lambda^{[k-1]}} \mod \mathcal{A}_0^{k-1},
\]
with $X_\lambda^{[k-1]} \in \ker(\text{Ad}(S_0) - 1) \cap \ker(\text{ad}(N^T)) \cap \text{Im}(P^X) \subset P_{k-1}$ and $X_\lambda^{[k-1]} = 0 \mod \mathcal{A}_0^1$. Then,
\[
\psi_\lambda = A_0 e^{X_0 + X_\lambda^{[k-1]} + Z_{k, \lambda}} \mod \mathcal{A}_0^k,
\]
for some $Z_{k, \lambda} \in \ker(\text{Ad}_k(S_0) - 1) \cap \text{Im}(P^X)$. The goal is to bring the term $Z_{k, \lambda}$ in $\ker(\text{Ad}_k(S_0) - 1) \cap \ker(\text{ad}(N^T)) \cap \text{Im}(P^X)$ which is achieved by a similar splitting argument as in Proposition 4.1.

6 Example

Consider the involution $R := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathcal{L}(\mathbb{R}^2)$ and the finite compact group $\mathcal{G} = \{I, R\}$ with character $\chi : \mathcal{G} \to \mathbb{R}$ given by $\chi(I) = 1$ and $\chi(R) = -1$. Let $\Phi_1 : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ be the map given by
\[
(x, y) \mapsto \left(\frac{x^3}{y^2}, \frac{x^2}{y}\right),
\]
here $\Omega := \{(x, y) \in \mathbb{R}^2 | x > 0, y > 0\}$ is the open first quadrant. Now, $\Phi_1$ is a local $R$-reversible diffeomorphism, that is, $R \cdot \Phi_1(x, y) = \Phi_1^{-1}(R(x, y))$. Note that the map $\Phi_1$ has a line of fixed
points, \( y = x \), that is invariant under \( R \). Indeed, \( R(x,x) = (x,x) \). We consider the map \( \Phi_1 \) around the fixed point \( P := (1,1) \) and calculate its normal form (up to the third order) using the theory developed in this paper. The linearisation of \( \Phi_1 \) at the fixed point \( P \) is given by

\[
A_0 := \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix} = S_0 + N_0,
\]

where \( S_0 = I \) and \( N_0 := \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} \). Theorem 5.1 tells us that there exists an \( R \)-equivariant (and \( S_0 \)-equivariant) transformation \( g \in \mathcal{H}_2 \) such that

\[
\Ad(e^g) \Phi = S_0 e^{N_0 + X_2} \mod \lambda^3_0,
\]

where \( \Phi(x,y) = \Phi_1(1 + x, 1 + y) - (1,1) \) and \( X_2 \in \mathcal{P}_2 \) satisfies the properties 58. Recall that \( N_0 = \log(I + S_0^{-1} N_0) \). Direct calculations show that admissible \( X_2 \)'s must have the form

\[
X_2(x,y) = (d(x + y)^2, -d(x + y)^2)
\]

for some \( d \in \mathbb{R} \), while \( g(x,y) \in \mathcal{H}_2 \) must have the form

\[
g(x,y) = (ax^2 + bxy + cy^2, cx^2 + bxy + ay^2),
\]

for some \( a, b, c \in \mathbb{R} \). So, the relation 58 implies that \( b = -2c, \ a = -1/2 + c, \ d = 0, \ c = c \). That is, there exists a one-parameter family of maps \( g(x,y) = ((c-1/2)x^2 - 2cxy + cy^2, cx^2 - 2cxy + (c-1/2)y^2) \) that brings \( \Phi \) into the (third order) normal form with \( X_2 = 0 \). Choosing, for example, \( c = 0 \) one has that

\[
\Ad\left( \exp\left( -\frac{1}{2} x^2, \frac{1}{2} y^2 \right) \right) \Phi(x,y) = A_0 \begin{pmatrix} x \\ y \end{pmatrix} \mod \lambda^3_0.
\]

Remark 6.1. Note that the map \( \Phi_1(x,y) = \left( \frac{x^3}{y^2}, \frac{x}{y} \right) \) can be written as the composition \( \Phi_1(x,y) = R \cdot \Psi(x,y) \) where \( \Psi : \Omega \to \mathbb{R}^2 \) is given by \( \Psi(x,y) = (\Psi_1(x,y), \Psi_2(x,y)) = \left( \frac{x^2}{y}, \frac{x}{y^3} \right) \). We show that the map \( \Psi \) is not the time-one map of vector field. Observe that this doesn’t say anything about the map \( \Phi_1 \). The map \( \Psi \) has the same line of fixed points as \( \Phi_1 \); i.e., \( \Psi(x,x) = (x,x) \) and observe that \( \Psi_2(x,y)/\Psi_1(x,y) = x/y \). In polar coordinates we have that

\[
\Psi(r, \theta) = (r \cot^2(\theta), \pi/2 - \theta),
\]

which shows that points in the first quadrant under the first bisectrice \( y = x \) are mapped into points above this line as shown in Fig. 1.

Now, suppose that there exists a vector field \( X \) which is defined in a neighborhood of \((1,1)\) and is such that \( \Psi(x,y) = e^{X}(x,y) \). Then, one sees that an integral curve of \( X \) cannot cross the fixed line \( y = x \) since otherwise one can find points which remain in the same half plane defined by the first bisectrice. Hence, the only remaining possibility is \( X(x,x) = 0 \) which is also impossible by the same argument as above.

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Figure 1: The map $\Psi$ maps points under the bisectrice $y = x$ to points above it.

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