MODULI OF ROOTS OF HYPERBOLIC POLYNOMIALS AND DESCARTES’ RULE OF SIGNS

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ABSTRACT. A real univariate polynomial with all roots real is called hyperbolic. By Descartes’ rule of signs for hyperbolic polynomials (HPs) with all coefficients nonvanishing, a HP with \( c \) sign changes and \( p \) sign preservations in the sequence of its coefficients has exactly \( c \) positive and \( p \) negative roots. For \( c = 2 \) and for degree 6 HPs, we discuss the question: When the moduli of the 6 roots of a HP are arranged in the increasing order on the real half-line, at which positions can be the moduli of its two positive roots depending on the positions of the two sign changes in the sequence of coefficients?

Key words: real polynomial in one variable; hyperbolic polynomial; sign pattern; Descartes’ rule of signs

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1. Introduction

The classical Descartes’ rule of signs states that given a real univariate degree \( d \) polynomial \( P \), the number \( \text{pos} \) of its positive roots is not larger than the number \( c \) of the sign changes in the sequence of its coefficients. In the present paper we consider only polynomials with all coefficients nonvanishing and with positive leading coefficients. Thus the number of sign changes for the polynomial \( P(-x) \) is equal to the number \( p \) of sign preservations for the polynomial \( P(x) \) hence one has \( \text{neg} \leq p \), where \( \text{neg} \) stands for the number of negative roots of \( P \). If \( P \) is hyperbolic, i.e. with all roots real, then the conditions

\[
c + p = d = \text{pos} + \text{neg} , \quad \text{pos} \leq c \text{ and } \text{neg} \leq p ,
\]

imply \( \text{pos} = c \) and \( \text{neg} = p \). We consider only the generic case when all moduli of roots are distinct. Suppose that the \( d \) moduli of roots are arranged in the increasing order on the positive half-line. Then one can formulate the following problem:

Problem 1. Knowing the positions of the sign changes and sign preservations in the sequence of coefficients of a given hyperbolic polynomial (HP), what positions can occupy the moduli of its positive roots in this arrangement?

For \( c = 1 \), the exhaustive answer to this problem (for any degree \( d \)) is given in [11]. Also in [11] one can find the answer to Problem 1 for \( d \leq 5 \) and \( c = 2 \). In the present paper we give the answer to it for \( d = 6 \) and \( c = 2 \).

Remark 1. For \( d \leq 5 \), it is sufficient to study Problem 1 for \( c \leq 2 \), because the polynomial \((-1)^dP(-x)\) has \( p \) sign changes and \( c \) sign preservations in the sequence of its coefficients. Thus if \( P(x) \) has more than 2 sign changes, then \((-1)^dP(-x)\) has not more than 2 sign changes and one can consider \((-1)^dP(-x)\) instead of \( P(x) \).
For \( d \geq 6 \) this is not so. In particular, for \( d = 6 \), HPs with \( c = 3 \) remain such when \( P(x) \) is replaced by \((-1)^d P(-x)\).

The paper is structured as follows. In Section 2 we recall some definitions and results of [11] and we mention some problems related to Problem [11]. In Section 3 we first remind which cases have to be considered for \( d = 6 \) according to the signs of the coefficients of the HP, and then we resolve Problem [11] in each of these cases.

2. Definitions and known results

Definition 1. A sign pattern (SP) of length \( d+1 \) is a sequence of \( d+1 \) signs + or −.

We say that the polynomial \( P := x^d + \sum_{j=0}^{d-1} a_j x^j \) defines the SP \((+, \text{ sgn}(a_{d-1}), \ldots, \text{ sgn}(a_0))\). We consider polynomials with positive leading coefficients, so the SPs we deal with begin with +. In the proofs we use also SPs some of whose components are zeros.

Notation 1. We denote the moduli of the two positive roots of a HP (with two sign changes in the sequence of its coefficients) by \( 0 < \beta < \alpha \). By \( 0 < \gamma_1 < \cdots < \gamma_{d-2} \) we denote the moduli of its negative roots. By \((a, b, w)\), \( a, b, w \in \mathbb{N} \cup 0, a+b+w = d-2 \) we denote the case when \( \gamma_a < \beta < \gamma_{a+1} \) and \( \gamma_{a+b} < \alpha < \gamma_{a+b+1} \) setting \( \gamma_0 := 0 \) and \( \gamma_{d-1} := \infty \). By \( \Sigma_{m,n,q} \), for \( c = 2 \) (resp. by \( \Sigma_{m,n} \), for \( c = 1 \)) we denote the SP consisting of \( m \) pluses followed by \( n \) minuses followed by \( q \) pluses, where \( m+n+q = d+1 \) (resp. consisting of \( m \) pluses followed by \( n \) minuses, where \( m+n = d+1 \)). For \( c=1 \), the moduli of the positive and negative roots are denoted by \( a \) and \( \gamma_1 < \cdots < \gamma_{d-1} \); the case \( (a, b) \), \( a+b = d-1 \), means \( \gamma_a < \alpha < \gamma_{a+1} \). The notation \((a, b, w)\) and \((a, b)\) is different from the one used in [11].

Definition 2. (1) Given a SP one defines its corresponding canonical arrangement as follows. The increasing order of moduli of positive and negative roots on the real half-line coincides with the order of sign changes and sign preservations respectively when the SP is read backward. Example: for \( d = 6 \) and \( c = 2 \), the SP \( \Sigma_{2,4,1} = (+,+,−,−,−,+,+) \) when read backward gives the following order of sign changes and sign preservations: \((c,p,p,p,c,p)\). Hence the canonical arrangement defined by this SP is \( \beta < \gamma_1 < \gamma_2 < \gamma_3 < \alpha < \gamma_4 \) which is the case \((0,3,1)\).

More generally, for the SP \( \Sigma_{m,n,q} \), the corresponding canonical arrangement defines the case \((q-1,n-1,m-1)\).

(2) We say that a SP \( \Sigma_{m,n,q} \) (resp. \( \Sigma_{m,n} \)) is realizable in the case \((a, b, w)\) (resp. \((a, b)\)) if there exists a degree \( d \) HP with \( c = 2 \) and \( p = d-2 \) (resp. with \( c = 1 \) and \( p = d-1 \)) defining this SP, with distinct moduli of its roots which define the case \((a, b, w)\) (resp. the case \((a, b)\)). We can also say that the case \((a, b, w)\) is realizable with the SP \( \Sigma_{m,n,q} \).

The following result can be found in [11]:

Theorem 1. (1) Every SP has a canonical realization, i.e. is realizable in the case defined by its canonical arrangement. The SPs \( \Sigma_{1,d}, \Sigma_{d,1}, \Sigma_{1,d-1,1} \) and \( \Sigma_{m,1,d-m} \) are realizable only in this case.

(2) The SPs \( \Sigma_{d-2,2,1} \) and \( \Sigma_{d-3,3,1} \) are realizable in the case \((1,0,d-3)\). They are realizable only in this case and in cases of the kind \((0,b,w), b+w = d-2 \). For \( n \geq 4 \), the SP \( \Sigma_{m,n,1} \) is realizable only in cases of the kind \((0,b,w), b+w = d-2 \).

Remark 2. Given a degree \( d \) polynomial \( P(x) \) we define its reverted polynomial \( P^R(x) \) as \( P^R(x) := x^d P(1/x) \). It is clear that if the SP \( \Sigma_{m,n,q} \) (resp. \( \Sigma_{m,n} \)) is realizable
in the case \((a, b, w)\) (resp. \((a, b)\)) by the HP \(P(x)\), then the SP \(\Sigma_{q,n,m}\) (resp. \(\Sigma_{n,m}\)) is realizable in the case \((w, b, a)\) (resp. \((b, a)\)) by the polynomial \(\text{sgn}(P(0))P_R(x)\).

Another problem about real univariate polynomials (not necessarily hyperbolic) inspired by Descartes’ rule of signs is the following one:

**Problem 2.** Suppose that a SP with \(c\) sign changes and \(p\) sign preservations is given. For which pairs of nonzero integers \((\text{pos}, \text{neg})\) satisfying the conditions \(\text{pos} \leq c, \text{neg} \leq p\) and \(c - \text{pos} \in 2\mathbb{N} \cup \{0\} \ni p - \text{neg}\) do there exist such polynomials defining the given SP and having exactly \(\text{pos}\) positive and \(\text{neg}\) negative roots, all distinct?

The problem seems to have been formulated for the first time in [2]. Its exhaustive answer for \(d \leq 8\) is to be found in [7], [1], [5] and [8] (and this answer is not trivial). An interesting particular case for \(d = 11\) is considered in [9]. In [6] a tropical analog of Descartes’ rule of signs is formulated. Different aspects of the theory of HPs are exposed in [10]. For metric inequalities involving moduli of roots of polynomials see [3] and [4].

3. Resolution of Problem 1 for \(d = 6\)

For \(d = 6\), the SPs with \(c = 2\) which need to be considered are the following ones: \(\Sigma_{1, 5, 1}, \Sigma_{m, 1, q}\) \((m + q = 6)\), \(\Sigma_{k, 6-k, 1}\) \((k = 2, 3\) and 4\)), \(\Sigma_{2, 3, 2}\) and \(\Sigma_{3, 2, 2}\). The remaining SPs are obtained from these ones by reversion, see Remark 2. We consider different SPs in the subsequent subsections. In Subsection 3.7 we summarize the results.

In the proofs we use the following notation:

**Notation 2.** We remind that we consider HPs having four negative and two positive roots. These roots are denoted by

\[
\xi_1 < \xi_2 < \xi_3 < \xi_4 < 0 < \xi_5 < \xi_6 .
\]

For the roots of the derivatives we use the notation

\[
\zeta_1 < \zeta_2 < \zeta_3 < \zeta_4 < \zeta_5 , \ \zeta_j < \xi_j < \xi_{j+1}
\]

(the latter inequalities result from Rolle’s theorem). One has \(\zeta_3 < 0 < \zeta_4\), if the last but first sign of the SP is +, and \(\zeta_4 < 0 < \zeta_5\), if it is −.

**Remark 3.** In the proofs we use one-parameter deformations of given HPs in which the deformation parameter is considered only for these nonnegative values for which the deformed polynomial is hyperbolic.

**Remark 4.** For \(n\) fixed, one could ask the question what the possible values of the quantity \(b\) can be. As we shall see, for \(d = 6\) and \(n = 2\), one has \(b \leq 4\). The following example (with \(d = 7\) and \(n = 2\)) shows that \(b\) can attain the value 5:

\[
D := (x - 0.9)(x + 0.98)(x + 0.99)(x + 1)(x + 1.01)(x + 1.02)(x - 1.1) = x^7 + 3x^6 + 0.9895x^5 - 5.0505x^4 - 5.09899496x^3 + 0.90101496x^2 + 2.94951496x + 0.9895050396 .
\]

The polynomial \(D\) defines the SP \(\Sigma_{3, 2, 3}\) and realizes the case \((0, 5, 0)\). To obtain such examples with arbitrarily large values of \(d \geq 7\) (and with \(n = 2\)) it suffices to
multiply $D$ by polynomials of the form $\prod_{\mu=1}^{\nu^*} (1 + \varepsilon_{\mu} x) \prod_{\nu=\mu+1}^{\mu^*+\nu^*} (x + \varepsilon_{\nu})$, where $\varepsilon_i$ are small positive quantities. Indeed, for $1 \leq \mu \leq \mu^*$, the moduli of the negative roots $-1/\varepsilon_{\mu}$ are large whereas for $\mu^* + 1 \leq \mu \leq \mu^* + \nu^*$, the moduli of the negative roots $-\varepsilon_{\mu}$ are small; the polynomial then defines the SP $\Sigma_{m,\mu^*,\nu^*}$ and realizes the case $(\nu^*,\mu^*)$.

3.1. The SPs $\Sigma_{1,5,1}$ and $\Sigma_{m,1,q}$, $m + q = 6$. These two cases have only canonical realizations, see Theorem [1]. This means that for the HPs realizing the SP $\Sigma_{1,5,1}$ one has $\beta < \gamma_1 < \gamma_2 < \gamma_3 < \gamma_4 < \alpha$ (*). This is the case $(0,4,0)$.

**Example 1.** Consider the polynomial $P^* := (x - 0.01)(x + 0.25)^4(x - 1)$, i.e.

$$P^* = x^6 - 0.01x^5 - 0.625x^4 - 0.30625x^3 - 0.05546875x^2 - 0.0033203125x + 0.0000390625 .$$

This polynomial defines the SP $\Sigma_{1,5,1}$. Hence for $\varepsilon > 0$ small enough, the polynomial

$$(x - 0.01)(x + 0.25 - 2\varepsilon)(x + 0.25 - \varepsilon)(x + 0.25 + \varepsilon)(x + 0.25 + 2\varepsilon)(x - 1)$$

also defines the SP $\Sigma_{1,5,1}$ and has six distinct real roots whose moduli satisfy conditions (*).

**Example 2.** For $m = 1, 2$ and $3$, the following HPs define the corresponding SPs $\Sigma_{m,1,q}$:

$$(x + 0.01)^4(x - 1)^2 = x^6 - 1.96x^5 + 0.9206x^4 + 0.038804x^3 + 0.00059201x^2 + 0.00000398x + 10^{-8} ,$$

$$(x + 0.01)^3(x - 1)^2(x + 4) = x^6 + 2.03x^5 - 6.9397x^4 + 3.790601x^3 + 0.117902x^2 + 0.001193x + 0.0000039804$$

and

$$(x + 0.01)^2(x - 1)^2(x + 4)^2 = x^6 + 6.02x^5 + 1.1201x^4 - 23.9794x^3 + 15.5201x^2 + 0.3176x + 0.0016 .$$

The reverted of these HPs define these SPs with $m = 5$, $4$ and $3$ respectively. One can define one-parameter deformations of these HPs in which the multiple roots split into simple real ones while preserving the signs of the roots and of the coefficients of the corresponding polynomial (as this is done in Example [1]). For small nonzero values of the deformation parameter, the deformations are canonical realizations of the corresponding SPs.

3.2. The SP $\Sigma_{2,4,1}$. For any HP realizing this SP, one has $\beta < \gamma_1$, see part (2) of Theorem [1]. Hence a priori the realizable cases are of the form $(0,b,4-b)$, $0 \leq b \leq 4$. The ones with $b = 2, 3$ and $4$ are realizable:
Notation 2. Consider for a HP defining the SP $\Sigma$ two things happens:

Suppose that the HP $P$ realizes the SAP $\Sigma_{2,4,1}$ in the case $(0, 0, 4)$, see Notation 2. This means that

$$
(0, 2, 2) : (x - 0.001)(x + 0.3)(x + 0.4)(x - 1)(x + 1.01)(x + 1.02) = \\
x^6 + 1.729x^5 - 0.16053x^4 - 1.6063012x^3 - 0.83950954x^2 - 0.122782884x + 0.000123624,
$$

$$
(0, 3, 1) : (x - 0.001)(x + 0.3)(x + 0.4)(x + 1)(x - 1.01)(x + 1.02) = \\
x^6 + 1.709x^5 - 0.19491x^4 - 1.6229468x^3 - 0.84194086x^2 - 0.122780436x + 0.000123624,
$$

$$
(0, 4, 0) : (x - 0.001)(x + 0.3)(x + 0.4)(x + 1)(x + 1.01)(x - 1.02) = \\
x^6 + 1.689x^5 - 0.22889x^4 - 1.6393128x^3 - 0.84432446x^2 - 0.122778036x + 0.000123624.
$$

Proposition 1. The case $(0, 0, 4)$ is not realizable with the SAP $\Sigma_{2,4,1}$.

Proof. Indeed, suppose that the polynomial $P$ realizes the SAP $\Sigma_{2,4,1}$ in the case $(0, 0, 4)$. Hence $\xi_4 < 0 < \xi_5$ and $-\xi_4 = \gamma_1 > \alpha = \xi_6$, see Notation 2. This means that

$$
\xi_1 > -\xi_2 > -\xi_3 > -\xi_4 > \xi_6 > \xi_5.
$$

The HP $P'$ defines the SP $\Sigma_{2,4}$. It follows from Corollary 1 that for $d = 5$, the cases (3, 1) and (2, 2) are realizable with the SAP $\Sigma_{2,4}$, but the cases (1, 3) and (0, 4) are not. This is a contradiction with (3.1).

Proposition 2. The case $(0, 1, 3)$ is not realizable with the SP $\Sigma_{2,4,1}$.

Proof. Suppose that the HP $P$ realizes the SP $\Sigma_{2,4,1}$ in the case $(0, 1, 3)$. We use Proposition 2. Consider for $t \geq 0$ the one-parameter deformation $P_t := P + tx^4(x - \xi_6)$. As $t$ increases, the root $\xi_6$ does not change, $\xi_1$, $\xi_3$ and $\xi_5$ decrease while $\xi_2$ and $\xi_4$ increase; the SP does not change. For some value $t_0 > 0$ of $t$, at least one of the two things happens:

A) the roots $\xi_2$ and $\xi_3$ coalesce;

B) one has $|\xi_4| = |\xi_5|$, i.e. $-\xi_4 = \xi_5$.

Set $Q := P_{t_0}$. If A) takes place, then we consider the one-parameter deformation $Q_s := Q - s(x - \xi_5)^2x^2$, $s \geq 0$. As $s$ increases, the double root $\xi_2 = \xi_3$ does not change, $\xi_1$ and $\xi_5$ decrease while $\xi_4$ and $\xi_6$ increase; the SP does not change. Then for some $s = s_0 > 0$, either B) or C) takes place, with

C) one has $|\xi_6| = |\xi_2|$, i.e. $-\xi_2 = \xi_6$.

We denote by AB) “A) followed by B)”. Thus if A) takes place, then there exists a HP defining the SP $\Sigma_{2,4,1}$ for which either AB) or AC) takes place.

Suppose that B) takes place. Consider the one-parameter deformation $Q_u := Q + uQ^*, Q^* := (x^2 - \xi_5^2)(x - \xi_6)$, $u \geq 0$. The roots $\xi_4$, $\xi_5$ and $\xi_6$ do not change, $\xi_1$ and $\xi_3$ decrease while $\xi_2$ increases. One has
\[ Q^* = x^3 - \xi_6 x^2 - (\xi_4)^2 x + \xi_4^3 \xi_6 = x^3 - U_2 x^2 - U_1 x + U_0, \quad U_j > 0. \]

Hence as \( u \) increases, the signs of the last three coefficients of \( Q_u \) do not change and the first three coefficients do not change at all. The coefficient of \( x^3 \) cannot become negative. Indeed, \( Q_u \) is hyperbolic and four sign changes in the sequence of its coefficients means four positive roots which is not the case. Thus \( u \) can be increased only until for some value \( u_0 \), \( A \) takes place, so one can assume that either \( A \) (or \( B \)) which is the same) or \( AC \) takes place.

Set \( R := Q_{u_0} \). Suppose that \( AB \) takes place. Consider the one-parameter deformation \( R_v := R - v R^*, \ R^* := (x - \xi_2)^2 (x^2 - \xi_4^2), \ v \geq 0 \). One has

\[ R^* = x^4 - 2 \xi_2 x^3 + (\xi_2^2 - \xi_4^2) x^2 + 2 \xi_2 \xi_4 x - \xi_2^2 \xi_4^2 = x^4 + R_3 x^3 + R_2 x^2 - R_1 x - R_0, \quad R_j > 0. \]

Thus in the deformation \( R_v \) only the sign of the linear term can change. The roots \( \xi_i, \ 2 \leq i \leq 5 \), do not change, \( \xi_1 \) decreases and \( \xi_6 \) increases, so for some value \( \nu_0 > 0 \) either \( C \) or \( D \) takes place, with

D) the coefficient of \( x \) equals 0.

Hence if \( AB \) takes place, then it suffices to consider the possibility \( ABC \) or \( ABD \) to take place.

Suppose that \( AC \) takes place. Consider the deformation \( R_r := R + r R^\dagger, \ r \geq 0, \)

\[ R^\dagger := (x + \xi_6)^2 (x - \xi_4) = x^3 + \xi_6 x^2 - \xi_4 x - \xi_6^2. \]

The roots \( \xi_2 = \xi_3 \) and \( \xi_6 \) do not change, \( \xi_1 \) and \( \xi_5 \) decrease while \( \xi_4 \) increases. In the deformation \( R_r \) the coefficient of \( x^3 \) or the one of \( x^2 \) or both of them cannot become positive, because this would mean at least three sign changes, i.e. at least three positive roots which is impossible. The constant term of \( R_r \) cannot vanish. Indeed, in this case after rescaling one can set \( \xi_6 = 1 \) hence \( R_r = x T, \) where

\[ T = (x + 1)^2 (x - 1)(x + g)(x + h), \quad g = -\xi_1 > 1, \quad h = -\xi_4 \in (0, 1). \]

Observe that the SP \( \Sigma_{2,4,1} \) implies that \( R_r(0) < 0 \) hence it is \( \xi_5 \) and not \( \xi_4 \) that vanishes. The coefficient of \( x^3 \) in \( T \) equals \( -1 + g + h + gh > 0 \) whereas it must be negative. However, as \( R^\dagger(0) < 0 \), for \( r = -R(0)/R^\dagger(0) > 0, \) one has \( R_r(0) = 0, \) i.e. for \( r = r_0, \) the constant term does vanish. This contradiction shows that one cannot have \( AC \). So one cannot have \( ABC \) either and only the possibility \( ABD \) remains.

We rescale the variable \( x \) so that \( \xi_4 = -1 = -\xi_5. \) We set \( \xi_2 = \xi_3 = -g, \ g > 1, \) \( \xi_6 = A > 1, \) \( \xi_1 = -B, \) \( B > 1. \) So we consider the polynomial

\[ F := (x + g)^2 (x^2 - 1)(x + B)(x - A) = x^6 + F_5 x^5 + \cdots + F_0. \]

As \( F_1 = g(-gB + gA + 2AB), \) the condition \( F_1 = 0 \) yields \( g = g_0 := 2AB/(B - A). \) Hence \( B > A, \) because \( g > 0. \) One gets

\[ F_4 |_{g=g_0} = (-2A^2B^2 - B^2 + 2AB - A^2 + 3AB^3 + 3A^3B)/(-B + A)^2. \]

However the inequalities \( B > A > 1 \) imply \( 2AB^3 > 2A^2B^2, \) \( AB^3 > B^2 \) and \( AB > A^2, \) i.e. \( F_4 |_{g=g_0} > 0 \) which is in contradiction with the SP \( \Sigma_{2,4,1}. \) This contradiction proves the proposition.
3.3. The SP $\Sigma_{3,3,1}$. According to part (2) of Theorem 1 the HPs realizing this SP either satisfy the inequalities $\gamma_1 < \beta < \alpha < \gamma_2 < \gamma_3 < \gamma_4$, which is the case $(1, 0, 3)$ realizable by the HP

$$(x + 0.98)(x - 0.99)(x - 1)(x + 2.05)(x + 2.1)(x + 40) =$$

$$x^6 + 43.14x^5 + 124.7533x^4 - 41.23068x^3$$

$$-294.614531x^2 - 0.116529x + 167.06844,$$

or they realize one of the cases $(0, b, 4-b)$, $0 \leq b \leq 4$. These cases are also realizable:

$$(0, 0, 4) : (x - 0.1)(x - 9)(x + 9.6)(x + 9.7)(x + 9.8)(x + 9.9) =$$

$$x^6 + 29.9x^5 + 216.35x^4 - 1448.135x^3$$

$$-24185.4276x^2 - 78877.71684x + 8131.05216,$$

$$(0, 1, 3) : (x - 0.1)(x + 0.99)(x - 1)(x + 1.01)(x + 1.02)(x + 40) =$$

$$x^6 + 41.92x^5 + 76.6179x^4 - 9.305492x^3 - 81.6975778x^2 - 32.613922x + 4.079592,$$

$$(0, 2, 2) : (x - 0.1)(x + 0.99)(x + 0.995)(x - 1)(x + 1.02)(x + 40) =$$

$$x^6 + 41.905x^5 + 76.00425x^4 - 9.835474x^3$$

$$-81.0232111x^2 - 32.0695689x + 4.019004,$$

$$(0, 3, 1) : (x - 0.1)(x + 0.99)(x + 0.995)(x + 0.999)(x - 1)(x + 40) =$$

$$x^6 + 41.884x^5 + 75.145665x^4 - 10.55580655x^3$$

$$-80.08192694x^2 - 31.3281913x + 3.9362598,$$

$$(0, 4, 0) : (x - 0.1)(x + 9.6)(x + 9.7)(x + 9.8)(x + 9.9)(x - 10) =$$

$$x^6 + 28.9x^5 + 177.45x^4 - 2014.585x^3$$

$$-27835.3426x^2 - 87541.52424x + 9034.5024.$$

3.4. The SP $\Sigma_{4,2,1}$. For the HPs realizing this SP there exist two possibilities, see Theorem 1. The first of them is to satisfy the inequalities $\gamma_1 < \beta < \alpha < \gamma_2 < \gamma_3 < \gamma_4$, and this is the case $(1, 0, 3)$ realizable by the polynomial

$$(x + 1)(x - 1.5)(x - 1.6)(x + 10)(x + 11)(x + 12) =$$

$$x^6 + 30.9x^5 + 292x^4 + 539.1x^3 - 2946.2x^2 - 55.2x + 3168.$$
The second is one of the cases \((0, b, 4 - b), 0 \leq b \leq 4\), to be realizable. For \(b = 0, 1\) and \(2\) we provide examples of HPs realizing the corresponding cases:

\[
(0, 0, 4) : (x - 1)(x - 4)(x + 5)(x + 6)(x + 100)(x + 101) = \\
x^6 + 207x^5 + 11285x^4 + 56273x^3 - 233286x^2 - 1046480x + 1212000 ,
\]

\[
(0, 1, 3) : (x - 1)(x + 2)(x - 4)(x + 5)(x + 100)(x + 101) = \\
x^6 + 203x^5 + 10481x^4 + 15957x^3 - 216482x^2 - 214160x + 404000 ,
\]

\[
(0, 2, 2) : (x - 1)(x + 2.1)(x + 3)(x - 4)(x + 100)(x + 101) = \\
x^6 + 2001.1x^5 + 1.0011849 \times 10^6x^4 + 69673.7x^3 - 1.52373859 \times 10^7x^2 \\
-1.10606748 \times 10^7x + 2.52252 \times 10^7 .
\]

**Proposition 3.** The case \((0, 4, 0)\) is not realizable with the SP \(\Sigma_{4,2,1}\).

**Proof.** Suppose that the SP \(\Sigma_{4,2,1}\) is realizable in the case \((0, 4, 0)\) by the HP \(P\). Then the SP \(\Sigma_{4,2,4}\) is realizable in the case \((0, 4, 0)\) by the HP \(Q := P^R\). For the roots of \(Q\) and \(Q^R\) we use Notation [2] The HP \(Q^R\) defines the SP \(\Sigma_{4,2,3}\) and as \(|\xi| \in (\xi_5, \xi_6), 1 \leq \nu \leq 4, \) one has \(|\xi| > \xi_4, k = 1, 2\) and \(3\). One has \(\xi_5 > \xi_4\); the position of \(\xi_5\) w.r.t. \(|\xi|, 1 \leq k \leq 3\), cannot be specified.

The HP \((Q^R)^R\) defines the SP \(\Sigma_{3,2,1}\) and has roots \(\eta_j = 1/\xi_j\) for which one has \(|\eta| < \eta_4\) and \(\eta_k < 0, k = 1, 2\) and \(3\). Notice that \(\eta_4\) is the largest root \(\eta_j\). Indeed, there are two positive roots \(\eta_4\) and \(\eta_5\), and as \(\xi_5 > \xi_4\), this implies \(\eta_5 < \eta_4\). From part \((2)\) of Theorem [1] and from [11] Proposition 1 (which claims that for \(d = 5\), the case \((0, 3, 0)\) is not realizable with the SP \(\Sigma_{3,2,1}\)) one deduces that this is impossible. \(\square\)

**Proposition 4.** The case \((0, 3, 1)\) is not realizable with the SP \(\Sigma_{4,2,1}\).

**Proof.** Suppose that the HP \(P\) realizes the SP \(\Sigma_{4,2,1}\) in the case \((0, 3, 1)\). We use Notation [2] Consider the polynomial \(V := x^3(x - \xi_2)(x - \xi_3)(x - \xi_4)\). It defines the SP \((+, +, +, +, 0, 0, 0)\). Hence throughout the deformation \(P_t := P + tV, t \geq 0\), the SP does not change, the roots \(\xi_2, \xi_3\) and \(\xi_4\) remain the same, \(\xi_1\) and \(\xi_5\) increase while \(\xi_6\) decreases. Hence for some \(t = t_0 > 0\), at least one of the following things happens:

\begin{itemize}
  \item[A] one has \(|\xi| = |\xi_5|, \) i.e. \(-\xi_4 = \xi_5;\\
  \item[B] one has \(|\xi| = |\xi_6|, \) i.e. \(-\xi_1 = \xi_6;\\
  \item[C] one has \(|\xi| = |\xi_6|, \) i.e. \(-\xi_2 = \xi_6.\\
\end{itemize}

Suppose that A) takes place. Set \(Q := P_{t_0}\) and consider the deformation \(Q_s := Q + sQ^\Delta, Q^\Delta := (x^2 - \xi_4^2)(x^2 - \xi_5^2), s \geq 0.\) The SP defined by the polynomial \(Q^\Delta\) is \((0, 0, +, 0, -, 0, +),\) so the SP does not change throughout the deformation \(Q_s.\) The roots \(\xi_4, \xi_5\) and \(\xi_6\) do not change; \(\xi_1\) increases without reaching \(-\xi_6; \xi_2\) increases and \(\xi_3\) decreases. Hence for some \(s = s_0 > 0,\) one has

\begin{itemize}
  \item[D] the roots \(\xi_2\) and \(\xi_3\) coalesce.
Proposition 5. All cases $a, b, w$, $a + b + w = 4$, $0 \leq a, b, w \leq 4$, are realizable with the SP $\Sigma_{2,3,2}$.

Proof. The SP $\Sigma_{2,3,2}$ is center-symmetric, this is why if this SP is realizable in the case $(a, b, w)$ by a HP $P$, then it is realizable in the case $(w, b, a)$ by the HP $P^R$. We prove the proposition by exhibiting examples of HPs which realize $\Sigma_{2,3,2}$ in the indicated cases. The above observations allow to skip some of the cases.
(1, 0, 3) : \((x + 0.01)(x - 0.1)(x - 1)(x + 1.01)(x + 1.02)(x + 1.03) = x^6 + 1.97x^5 - 0.1253x^4 - 2.067553x^3 - 0.87576764x^2 + 0.097559534x + 0.001061106\),

\((1, 1, 2) : (x + 0.01)(x - 0.1)(x + 0.99)(x - 1)(x + 1.02)(x + 1.03) = x^6 + 1.95x^5 - 0.1445x^4 - 2.045655x^3 - 0.85653356x^2 + 0.095648466x + 0.001040094\),

\((1, 2, 1) : (x + 0.01)(x - 0.1)(x + 0.98)(x + 0.99)(x - 1)(x + 1.03) = x^6 + 1.91x^5 - 0.1817x^4 - 2.001931x^3 - 0.81930584x^2 + 0.091937534x + 0.000099306\),

\((1, 3, 0) : (x + 0.01)(x - 0.1)(x + 0.97)(x + 0.98)(x + 0.99)(x - 1) = x^6 + 1.85x^5 - 0.2345x^4 - 1.936645x^3 - 0.76643456x^2 + 0.086638466x + 0.000941094\),

\((0, 2, 2) : (x - 1)(x + 1.1)(x + 2)(x - 2.1)(x + 2.2)(x + 2.3) = x^6 + 4.5x^5 - 0.25x^4 - 24.205x^3 - 23.6436x^2 + 19.2214x + 23.3772\),

\((0, 1, 3) : (x - 1)(x + 1.1)(x - 2)(x + 2.05)(x + 2.1)(x + 2.15) = x^6 + 4.4x^5 - 0.0425x^4 - 21.8665x^3 - 20.921675x^2 + 17.068025x + 20.36265\),

\((0, 4, 0) : (x - 1)(x + 2.9)(x + 3)(x + 3.1)(x + 3.2)(x - 8) = x^6 + 3.2x^5 - 46.01x^4 - 291.172x^3 - 487.418x^2 + 129.968x + 690.432\),

\((2, 0, 2) : (x + 0.8)(x + 0.9)(x - 1)(x - 5)(x + 5.1)(x + 5.2) = x^6 + 6x^5 - 22.25x^4 - 156x^3 - 72.1556x^2 + 147.9336x + 95.472\),

\((0, 0, 4) : (x - 1)(x - 1.001)(x + 1.002)(x + 1.01)(x + 1.02)(x + 1.1) = x^6 + 2.131x^5 - 0.867672x^4 - 4.26624106x^3 - 1.268949846x^2 + 2.135240980x + 1.136621926\).
Proposition 6. The cases \((a, b, w)\) with \(w \geq 1\) are realizable with the \(SP\ \Sigma_{3,2,2}\).

Proof. For \(d = 5\) and for the \(SP\ \Sigma_{2,2,2}\), all cases \((a, b, w)\) with \(a + b + w = 3\) are realizable, see [11]. Hence for \(d = 6\), all corresponding cases \((a, b, w + 1)\) are also realizable. Indeed, if the degree 5 \(HP\ \(P(x)\) defines the \(SP\ \Sigma_{2,2,2}\) and realizes the case \((a, b, w)\), then for \(\varepsilon > 0\) small enough, the \(HP\ \((1 + \varepsilon x)P(x)\) defines the \(SP\ \Sigma_{3,2,2}\) and realizes the case \((a, b, w + 1)\). Indeed, the root \(-1/\varepsilon\) has the largest of the moduli of its roots. 

Thus it remains to consider the cases \((a, b, 0)\) with \(a + b = 4\). The case \((0, 4, 0)\) is realizable by the \(HP\)

\[
(x - 1)(x + 1.9)(x + 1.91)(x + 1.92)(x + 2)(x - 2.1) = \\
x^6 + 4.63x^5 + 0.5412x^4 - 24.36394x^3 - 28.469668x^2 + 17.398152x + 29.264256 .
\]

Proposition 7. The case \((4, 0, 0)\) is not realizable with the \(SP\ \Sigma_{3,2,2}\).

Proof. Suppose that the \(HP\ \(P\) realizes the case \((4, 0, 0)\) with the \(SP\ \Sigma_{3,2,2}\). For the roots of \(P\) and \(P'\) we use Notation 2, in particular, \(\xi_4 < 0 < \xi_5\) and \(\xi_3 < 0 < \xi_4\). The polynomial \(P'\) defines the \(SP\ \Sigma_{3,2,1}\) and for its roots one has \(\xi_5 > |\xi_k|, k = 1, 2\) and 3. It follows from part (2) of Theorem 3 and Proposition 1] by which for \(d = 5\), the case \((0, 3, 0)\) is not realizable with the \(SP\ \Sigma_{3,2,1}\) that this is impossible.

Proposition 8. The cases \((1, 3, 0), (2, 2, 0)\) and \((3, 1, 0)\) are not realizable with the \(SP\ \Sigma_{3,2,2}\).

Proof. Suppose that the \(HP\ \(P\) realizes the case \((1, 3, 0)\) or \((2, 2, 0)\) or \((3, 1, 0)\) with the \(SP\ \Sigma_{3,2,2}\). We use Notation 2 Consider the one-parameter deformation

\[
P_t := P + t(x - \xi_1)(x - \xi_6)x^2 , \ t \geq 0 .
\]

The polynomial \((x - \xi_1)(x - \xi_6)x^2\) defines the \(SP\ \((0, 0, +, -,-,0,0)\)\) because \(|\xi_1| < \xi_6\); therefore the \(HP\) does not change throughout the deformation \(P_t\). The roots \(\xi_1\) and \(\xi_6\) do not change, \(\xi_2\) and \(\xi_4\) increase while \(\xi_3\) and \(\xi_5\) decrease. Hence there exists \(t_0 > 0\) such that for \(P_{t_0}\) at least one of the two things holds true:

A) in case \((1, 3, 0)\), one has \(|\xi_4| = |\xi_5|\), i.e. \(-\xi_4 = \xi_5\);

in case \((2, 2, 0)\), one has \(|\xi_5| = |\xi_6|\), i.e. \(-\xi_5 = \xi_6\);

the possibility \(|\xi_2| = |\xi_5|\) also exists, but we consider it only in the case \((3, 1, 0)\), because in both cases \((2, 2, 0)\) and \((3,1,0)\) it gives \(\xi_2 = -\xi_5, -\xi_1 < \xi_6\);

in case \((3, 1, 0)\), one has \(|\xi_2| = |\xi_5|\), i.e. \(-\xi_2 = \xi_5\);

B) the roots \(\xi_2\) and \(\xi_3\) coalesce.

Consider the case \((1, 3, 0)\). Suppose that A) takes place. Set \(Q := P_{t_0}\). Then throughout the deformation

\[
Q_s := Q + s(x^2 - \xi^2_1)(x^2 - \xi^2_5) , \ s \geq 0 ,
\]

the \(SP\) does not change. Indeed, the polynomial \((x^2 - \xi^2_1)(x^2 - \xi^2_5)\) defines the \(SP\ \((0,0,+,0,-,0,+))\). The roots \(\xi_1, \xi_4\) and \(\xi_5\) do not change while \(\xi_6\) decreases (but never becomes equal to \(-\xi_1\)). The root \(\xi_2\) increases while \(\xi_3\) decreases.
Therefore there exists $s_0 > 0$ for which both A) and B) take place. In this case after rescaling of $x$ one can have

$$
\xi_2 = \xi_3 = -1, \ -\xi_4 = \xi_5 = g \in (0,1), \ -\xi_1 = B > 1 \text{ and } \xi_6 = A > B .
$$

The corresponding polynomial equals $(x + B)(x + 1)^2(x^2 - g^2)(x - A)$ and its coefficient of $x^4$ is $2(B - A) + (1 - AB) - g^2 < 0$ which contradicts the SP $\Sigma_{3,2,2}$. Hence A) does not take place.

Suppose that B) takes place. We use again the rescaling of $x$ leading to $\xi_2 = \xi_3 = -1$. Consider the polynomial $S := (x + 1)^2(x - 1/2)$. It defines the SP $(0,0,0,+,+,0,-)$. Then throughout the deformation

$$(3.3) \quad Q_v := Q - vS , \ v \geq 0 ,$$

the SP does not change. The roots $\xi_2$ and $\xi_3$ do not change while $\xi_1$ and $\xi_6$ increase; $\xi_6$ and does not coalesce with $\xi_2$. The root $\xi_5$ increases, if $\xi_5 < 1/2$, decreases, if $\xi_5 > 1/2$, and remains fixed, if $\xi_5 = 1/2$. As A) does not take place, for some $v_0 > 0$,

C) the root $\xi_1$ coalesces with $\xi_2 = \xi_3$.

In this case $Q_{v_0} = (x + 1)^3(x + g)(x - h)(x - A)$, where $-\xi_4 = g < h = \xi_5 < 1$ and $A = \xi_6 > 1$. The coefficient of $x^4$ in $Q_{v_0}$ equals

$$3 + 3g - 3h - gh - (3 + g - h)A \leq 3 + 3g - 3h - gh - 3 - g + h = 2(g - h) - gh < 0$$

which contradicts the SP $\Sigma_{3,2,2}$. Thus the case $(1,3,0)$ is not realizable with the SP $\Sigma_{3,2,2}$.

Consider the case $(2,2,0)$. Suppose that A) takes place. Throughout the deformation $Q_s$ (see (3.2)) the roots $\xi_1$, $\xi_3$ and $\xi_5$ do not change; $\xi_6$ decreases without becoming equal to $-\xi_1$; $\xi_4$ decreases and $\xi_2$ increases. Hence, for some $s = s_0 > 0$, there are two possibilities. The first is to have A) and B), i.e. one can set

$$\xi_2 = \xi_3 = -1 = -\xi_5 , \ -\xi_4 = g \in (0,1) , \ -\xi_1 = B > 1 \text{ and } \xi_6 = A > B .$$

The corresponding polynomial is $W := (x + 1)^2(x - 1)(x + g)(x + B)(x - A)$ whose coefficient of $x^4$ equals

$$W_4 := -1 + g + B + gB - A - Ag - AB = (B - A)(g + 1) + (g - AB) - 1 < 0$$

which contradicts the SP $\Sigma_{3,2,2}$. The second is to have A) and

D) the root $\xi_4$ coalesces with $\xi_3$.

In this case one sets

$$\xi_3 = \xi_4 = -1 = -\xi_5 , \ -\xi_2 = g > 1 , \ -\xi_1 = B > g \text{ and } \xi_6 = A > B .$$

The corresponding polynomial equals $W$ and its coefficient of $x^4$ is $W_4 < 0$. So suppose that B) takes place. Then A) takes place as well, and this possibility was already rejected.

Consider the case $(3,1,0)$. Suppose that A) takes place. Throughout the deformation $Q_s$ (see (3.2)) the roots $\xi_1$, $\xi_2$ and $\xi_5$ do not change, $\xi_6$ decreases, but
remains larger than $|\xi_1|, \xi_3$ increases and $\xi_4$ decreases. Hence for some $s = s_0 > 0$, one has D). Consider the polynomial $(x^2 - 1)(x + g)^2(x + B)(x - A)$, where

$$-\xi_2 = \xi_5 = 1, \ -\xi_3 = -\xi_4 = g \in (0, 1), \ -\xi_1 = B > 1 \text{ and } \xi_6 = A > B.$$  

The coefficient of $x^4$ equals

$$-1 + g^2 + 2gB - 2Ag - BA = -(1 - g^2) - 2g(A - B) - BA < 0$$

which contradicts the SP $\Sigma_{3,2,2}$.

If B) (but not A)) takes place, then in the deformation $Q_v$ (see 3.3) the roots $\xi_2 = \xi_3$ do not change, $\xi_1$ and $\xi_6$ increase while $\xi_4$ decreases. We admit that $\xi_5$ might increase or decrease or remain fixed. Hence for some $v = v^*$, either D) takes place or

E) one has $|\xi_5| = |\xi_1|$, i.e. $\xi_5 = -\xi_1$.

If B) and D) take place, then one considers the polynomial $(x + 1)^3(x - g)(x + B)(x - A)$, where

$$1 = -\xi_2 = -\xi_3 = -\xi_4, \ 1 < g = \xi_5 < B = -\xi_1 < A = \xi_6.$$  

The coefficient of $x^4$ equals

$$3 - 3g + 3B - gB - 3A + Ag - AB = -3(g - 1) - 3(A - B) - A(B - g) - gB < 0$$

which contradicts the SP $\Sigma_{3,2,2}$.

If B) and E) take place, then one considers the polynomial $(x + 1)^2(x^2 - B^2)(x + h)(x - A)$ with

$$-\xi_4 = h < 1 = -\xi_2 = -\xi_3 < B = \xi_5 = -\xi_1 < A = \xi_6.$$  

The coefficient of $x^4$ equals

$$1 - B^2 + 2h - 2A - Ah = -(B^2 - 1) - 2(A - h) - Ah < 0$$

which is a contradiction with the SP $\Sigma_{3,2,2}$.

The proposition is proved.

\[\square\]

### 3.7. Summarization of the results

In this subsection we summarize the results of the section. For each SP of the left column we give in the column $Y$ the cases $(a, b, w), a + b + w = 4$, which are realizable, and in the column $N$ the ones that are not realizable. We list only cases which are allowed by Theorem 1. The results concerning SPs which are not on the list are obtained by reversion, see Remark 2.
\[
\begin{array}{c|c|c|c}
\text{SP} & \text{Y} & \text{N} \\
(1,5,1) & (0,4,0) & \\
(m,1,q), \ m + q = 6 & (q - 1,0,m - 1) & \\
(2,4,1) & (0,2,2), (0,3,1), (0,4,0) & (0,0,4), (0,1,3) \\
(3,3,1) & (1,0,3), (0,0,4), (0,1,3), (0,2,2), (0,3,1), (0,4,0) & \\
(4,2,1) & (1,0,3), (0,0,4), (0,1,3), (0,3,1), (0,4,0), (0,2,2) & \\
(2,3,2) & \text{all possible cases} & \\
(3,2,2) & \text{all other cases} & (4,0,0), (3,1,0), (2,2,0), (1,3,0) \\
\end{array}
\]

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