Fresnel coefficients as hyperbolic rotations

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(Dated: October 28, 2018)

We describe the action of a plane interface between two semi-infinite media in terms of a transfer matrix. We find a remarkably simple factorization of this matrix, which enables us to express the Fresnel coefficients as a hyperbolic rotation.

Keywords: Fresnel formulas, hyperbolic rotations, reflection and transmission of light waves.

I. INTRODUCTION

Reflection and transmission at a discontinuity are perhaps the first wavelike phenomena that one encounters in any undergraduate physics course. The physics underlying this behavior is well understood: mismatched impedances generate the reflected and transmitted waves, while the application of the proper boundary conditions at the discontinuity provide their corresponding amplitude coefficients \[.\] Moreover, this general framework facilitates a unified treatment for all the types of waves appearing in Nature.

For light waves the impedance is proportional to the refractive index. Accordingly, the behavior of light at the plane interface between two semi-infinite media are derived in most optics textbooks \[\text{[2, 3, 4]}\]. The resulting amplitude coefficients are described by the famous Fresnel formulas. It seems almost impossible to say anything new about these Fresnel formulas. However, a quick look at the indexes of this Journal \[\text{[5]}\], among others \[\text{[6, 7]}\], immediately reveals a steady flow of papers devoted to subtle aspects of this problem, which shows that the topic is far richer than one might naively expect.

In this paper we reelaborate once again on this theme. We present the action of any interface in terms of a transfer matrix, and we find a hitherto unsuspectedly simple factorization of this matrix. After renormalizing the field amplitudes, such a factorization leads us to introduce a new parameter in terms of which Fresnel formulas appear as a hyperbolic rotation.

As our teaching experience demonstrates, the students have troubles in memorizing Fresnel formulas due to their fairly complicated appearance. In this respect, there are at least two reasons that, in our opinion, bear out the interest of our contribution: first, Fresnel formulas appear in the new variables as a hyperbolic rotation that introduces remarkable simplicity and symmetry. Second, this formalism is directly linked to other fields of physics, mainly to special relativity, which is more than a curiosity \[\text{[8, 9, 10]}\].

II. THE INTERFACE TRANSFER MATRIX

Let two homogeneous isotropic semi-infinite media, described by complex refractive indices \(N_0\) and \(N_1\), be separated by a plane boundary. The \(Z\) axis is chosen perpendicular to the boundary and directed as in Fig. 1.

We assume an incident monochromatic, linearly polarized plane wave from medium 0, which makes an angle \(\theta_0\) with the \(Z\) axis and has amplitude \(E_0^{(+)}\). The electric field is either in the plane of incidence (denoted by superscript \(\parallel\)) or perpendicular to the plane of incidence (superscript \(\perp\)). This wave splits into a reflected wave \(E_0^{(r)}\) in medium 0, and a transmitted wave \(E_0^{(t)}\) in medium 1 that makes and angle \(\theta_1\) with the \(Z\) axis. The angles of incidence \(\theta_0\) and refraction \(\theta_1\) are related by the Snell’s law

\[
N_0 \sin \theta_0 = N_1 \sin \theta_1.
\]

If media 0 and 1 are transparent (so that \(N_0\) and \(N_1\) are real numbers) and no total reflection occurs, the angles \(\theta_0\) and \(\theta_1\) are also real and the above picture of how a plane wave is reflected and refracted at the interface is simple. However, when either one or both media is absorbing, the angles \(\theta_0\) and \(\theta_1\) become, in general, complex and the discussion continues to hold only formally, but the physical picture of the fields becomes complicated \[\text{[11]}\].

We consider as well another plane wave of the same frequency and polarization, and amplitude \(E_1^{(-)}\), incident from medium 1 at an angle \(\theta_1\), as indicated in Fig. 1. In the same way, we shall denote by \(E_1^{(+)}\) and \(E_1^{(t)}\) the reflected and transmitted amplitudes of the corresponding waves.

The complex amplitudes of the total output fields at opposite points immediately above and below the interface will be called \(E_0^{(t)}\) and \(E_1^{(t)}\), respectively. The wave vectors of all waves lie in the plane of incidence and when the incident fields are \(\parallel\) or \(\perp\) polarized, all plane waves excited by the incident ones have the same polarization.

The amplitudes \(E_0^{(-)}\) and \(E_1^{(+)}\) are then given by

\[
E_0^{(-)} = E_0^{(r)} + E_1^{(t)} = r_{01} E_0^{(+)} + t_{10} E_1^{(-)},
\]

\[
E_1^{(+)} = E_0^{(t)} + E_1^{(r)} = t_{01} E_0^{(+)} + r_{10} E_1^{(-)},
\]

where \(r_{01}\) and \(t_{01}\) are the Fresnel reflection and transmission coefficients for the interface 01, and \(r_{10}\) and \(t_{10}\) refer to the corresponding coefficients for the interface 10. These Fresnel coefficients are determined by demanding that across the boundary the tangential components of \(\mathbf{E}\) and \(\mathbf{H}\) should be continuous \[\text{[3]}\]. For nonmagnetic media
they are given by
\[ r_{01} = \frac{N_1 \cos \theta_0 - N_0 \cos \theta_1}{N_1 \cos \theta_0 + N_0 \cos \theta_1}, \]
\[ t_{01} = \frac{2N_0 \cos \theta_0}{N_1 \cos \theta_0 + N_0 \cos \theta_1}, \]
for both basic polarizations. It is worth noting that, although these equations are written for electromagnetic waves, it is possible to translate all the results for particle-wave scattering, since there is a one-to-one correspondence between the propagation in an interface between two media of electromagnetic waves and of the non-relativistic particle waves satisfying Schrödinger equation [12].

The linearity revealed by Eqs. (3) suggests the use of 2 × 2 matrix methods. However, Eqs. (4) links output to input fields, while the standard way of treating this topic is by relating the field amplitudes at each side of the interface. Such a relation is expressed as [13, 14]
\[
\begin{pmatrix}
E_0^{(+)} \\
E_0^{(-)}
\end{pmatrix} = I_{01} \begin{pmatrix}
E_1^{(+)} \\
E_1^{(-)}
\end{pmatrix}.
\]

The choice of these column vectors is motivated from the optics of layered media, since it is the only way of calculating the field amplitudes at each side of every layer by an ordered product of matrices.

We shall call \( I_{01} \) the interface transfer matrix and, from Eqs. (4), is given by
\[ I_{01} = \frac{1}{t_{01}} \begin{pmatrix}
1 & -r_{10} \\
t_{01}t_{10} - r_{01}r_{10} & 1
\end{pmatrix}. \]

By using a matrix formulation of the boundary conditions [14] one can factorize the transfer matrix \( I_{01} \) in the new and remarkable form [13] (that otherwise one can also check directly using the Fresnel formulas)
\[ I_{01}^{\parallel} = R^{-1}(\pi/4) \begin{pmatrix}
\cos \theta_1 / \cos \theta_0 & 0 \\
0 & N_1 / N_0
\end{pmatrix} R(\pi/4), \]
\[ I_{01}^{\perp} = R^{-1}(\pi/4) \begin{pmatrix}
(N_1 \cos \theta_1) / (N_0 \cos \theta_0) & 0 \\
0 & 1
\end{pmatrix} R(\pi/4), \]
where
\[ R(\pi/4) = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & -1 \\
1 & 1
\end{pmatrix}. \]

This applies to both basic polarizations by the simple attachment of a label to all the coefficients and constitutes an alternative algebraic demonstration of the well-known Stokes relations without resorting to the usual time-reversal argument [12, 14]. Similar results can be also derived in particle scattering from the unitarity requirement on the S matrix. However, note that the equality \( |r_{10}| = |r_{01}| \) implied by Eq. (4) can become counterintuitive when applied to particle reflection, since one might expect stronger reflection for particle waves moving up in a potential gradient than for those going down. In fact, these relations, as emphasized by Lekner [12], ensure that the reflectivity is exactly the same in the two cases, unless there is total internal reflection.

In summary, these Stokes relations allow one to write [13, 14]
\[ I_{01} = \frac{1}{t_{01}} \begin{pmatrix}
1 & r_{01} \\
r_{01} & 1
\end{pmatrix}. \]

It is worth noting that the inverse matrix satisfies \( I_{01}^{\perp} = I_{10} \) and then describes the interface taken in the reverse order. The physical meaning of these matrix manipulations is analyzed in Section III.

### III. Renormalization of Field Amplitudes

From Eqs. (5) one directly obtain that, for both basic polarizations, we have
\[ \det I_{01}^{\parallel} = \det I_{01}^{\perp} = \frac{N_1 \cos \theta_1}{N_0 \cos \theta_0} \neq 1. \]

For the reasons that will become clear in Section IV, it is adequate to renormalize the field amplitudes to ensure that the transfer matrix has always unit determinant. To this end, let us define
\[ e_0^{(\pm)} = \sqrt{N_0 \cos \theta_0} E_0^{(\pm)}, \]
\[ e_1^{(\pm)} = \sqrt{N_1 \cos \theta_1} E_1^{(\pm)}. \]
Accordingly, the action of the interface is described now by
\[
\left( \begin{array}{c} e_{0}^{(+)} \\ e_{0}^{(-)} \end{array} \right) = i_{01} \left( \begin{array}{c} e_{1}^{(+)} \\ e_{1}^{(-)} \end{array} \right),
\]
where the renormalized interface matrix is
\[
i_{01} = R^{-1}(\pi/4) \left( \begin{array}{cc} 1/\xi_{01} & 0 \\ 0 & \xi_{01} \end{array} \right) R(\pi/4)
\]
\[
= \frac{1}{2} \left( \begin{array}{cc} \xi_{01} + 1/\xi_{01} & \xi_{01} - 1/\xi_{01} \\ \xi_{01} - 1/\xi_{01} & \xi_{01} + 1/\xi_{01} \end{array} \right),
\]
and the factor \( \xi_{01} \) has the values
\[
\xi_{01}^\parallel = \sqrt{N_1 \cos \theta_0 \cos \theta_1} = \frac{\sin(2\theta_0)}{\sin(2\theta_1)},
\]
\[
\xi_{01}^\perp = \sqrt{N_0 \cos \theta_0 \cos \theta_1} = \frac{\tan \theta_1}{\tan \theta_0}.
\]

Other way of expressing these relations is
\[
\frac{\xi_{01}^\parallel}{\xi_{01}^\perp} = \frac{\cos \theta_0}{\cos \theta_1},
\]
\[
\frac{\xi_{01}^\parallel}{\xi_{01}^\perp} = \frac{N_1}{N_0}.
\]

It is now evident from Eq. (15) that the renormalized interface matrix satisfies \( \det i_{01} = +1 \), as desired. Moreover, by taking into account the general form given in Eq. (11), we can reinterpret \( i_{01} \) in terms of renormalized Fresnel coefficients as
\[
i_{01} = \frac{1}{t_{01}} \left( \begin{array}{cc} 1 & \hat{r}_{01} \\ \hat{r}_{01} & 1 \end{array} \right),
\]
where
\[
\hat{r}_{01} = \frac{\xi_{01} - 1/\xi_{01}}{\xi_{01} + 1/\xi_{01}},
\]
\[
\hat{t}_{01} = \frac{2}{\xi_{01} + 1/\xi_{01}},
\]
which satisfy
\[
\hat{r}_{01}^2 + \hat{t}_{01}^2 = 1.
\]

This relation does not trivially reduce to the conservation of the energy flux on the interface, because the complex reflection and transmission coefficients appear in the form \( \hat{r}_{01}^2 \) and \( \hat{t}_{01}^2 \) instead of \( |r_{01}|^2 \) and \( |t_{01}|^2 \). In fact, it can be seen as a consequence of the renormalization factors appearing in the definition (13) that project the direction of the corresponding wave vector onto the normal to the boundary.

The Fresnel coefficients can be obtained from the renormalized ones as
\[
r_{01} = \hat{r}_{01},
\]
\[
t_{01} = \frac{\sqrt{N_1 \cos \theta_1}}{N_0 \cos \theta_0} \hat{t}_{01}.
\]

It is clear from Eqs. (19) that the single parameter \( \xi_{01} \) gives all the information about the interface, even for absorbing media or when total reflection occurs. We have \( i_{01} = i_{10} \); that is, the inverse also describes the interface taken in the reverse order. Thus, \( \xi_{10} = 1/\xi_{01} \) and it follows that
\[
\hat{r}_{01} = -\hat{r}_{10},
\]
\[
\hat{t}_{01} = \hat{t}_{10}.
\]

In Fig. 2 we have plotted the behavior of \( \xi_{01}^\parallel \) and \( \xi_{01}^\perp \) as a function of the angle of incidence \( \theta_0 \), for an interface air-glass \( (N_0/N_1 = 2/3) \) and, for the purpose of comparison, the corresponding values of \( r_{01}^\parallel \) and \( r_{01}^\perp \). The discussion about these amplitude coefficients and the corresponding phase shifts can be developed much in the same way as it is done in most of the undergraduate optics textbooks.

IV. THE INTERFACE AS A HYPERBOLIC ROTATION

The definition of the renormalized transfer matrix for an interface in Eq. (15) may appear, at first sight, rather artificial. In this Section we shall interpret its meaning by recasting it in an appropriate form that will reveal the origin of the rotation matrices \( R(\pi/4) \).

To simplify as much as possible the discussion, let us assume that we are dealing with an interface between two transparent media when no total reflection occurs. In this relevant case, the Fresnel reflection and transmission coefficients, and therefore \( \xi_{01} \), are real numbers. Let us introduce a new parameter \( \zeta_{01} \) by
\[
\xi_{01} = \exp(\zeta_{01}/2).
\]

Then, the action of the interface can be expressed as
\[
\left( \begin{array}{c} e_{0}^{(+)} \\ e_{0}^{(-)} \end{array} \right) = \left( \begin{array}{cc} \cosh(\zeta_{01}/2) & \sinh(\zeta_{01}/2) \\ \sinh(\zeta_{01}/2) & \cosh(\zeta_{01}/2) \end{array} \right) \left( \begin{array}{c} e_{1}^{(+)} \\ e_{1}^{(-)} \end{array} \right),
\]
where the renormalized Fresnel coefficients can be written now as
\[
\hat{r}_{01} = \tanh(\zeta_{01}/2),
\]
\[
\hat{t}_{01} = \frac{1}{\cosh(\zeta_{01}/2)}.
\]
Given the importance of this new reformulation of the action of an interface, some comments seem pertinent: it is clear that the reflection coefficient can be always expressed as a hyperbolic tangent, whose addition law is simple. In fact, such an important result was first derived by Khashan [16] and is the origin of several approaches for treating the reflection coefficient of layered structures [17], including bilinear or quotient functions [18], that are just of the form [19]. However, the transmission coefficient for these structures seems to be (almost) safely ignored in the literature, because it behaves as a hyperbolic secant, whose addition law is more involved.

Now the meaning of the rotation $R(\pi/4)$ can be put forward in a clear way. To this end, note that the transformation (24) is formally a hyperbolic rotation of angle $\xi_{01}/2$ acting on the complex field variables $[e^{(+)}_0,e^{(-)}_0]$. As it is usual in hyperbolic geometry [19], it is convenient to study this transformation in a coordinate frame whose new axes are the bisecting lines of the original one. In other words, in this frame whose axes are rotated $\pi/4$ respect to the original one, the new coordinates are

$$
\begin{pmatrix}
\tilde{e}^{(+)}_0 \\
\tilde{e}^{(-)}_0
\end{pmatrix} = R(\pi/4) \begin{pmatrix}
e^{(+)}_0 \\
e^{(-)}_0
\end{pmatrix}
$$

(26)

for both 0 and 1 media, and the action of the interface is represented by the matrix

$$
\begin{pmatrix}
e^{(+)}_0 \\
e^{(-)}_0
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & \xi_{01}
\end{pmatrix} \begin{pmatrix}
\tilde{e}^{(+)}_0 \\
\tilde{e}^{(-)}_0
\end{pmatrix},
$$

(27)

which is a squeezing matrix that scales $\tilde{e}^{(+)}_1$ down to the factor $\xi_{01}$ and $\tilde{e}^{(-)}_1$ up by the same factor.

Furthermore, the product of these complex coordinates remains constant

$$
|e^{(+)}_0|e^{(-)}_0 = |e^{(+)}_0|e^{(-)}_0
$$

(28)

or

$$
|e^{(+)}_0|^2 - |e^{(-)}_0|^2 = |e^{(+)}_0|^2 - |e^{(-)}_0|^2,
$$

(29)

which appears as a fundamental invariant of any interface. In these renormalized field variables it is nothing but the hyperbolic invariant of the transformation. When viewed in the original field amplitudes it reads as

$$
N_0 \cos \theta_0 \{|E^{(+)}_0|^2 - |E^{(-)}_0|^2\} = N_1 \cos \theta_1 \{|E^{(+)}_1|^2 - |E^{(-)}_1|^2\}
$$

(30)

which was assumed as a basic axiom by Vigoureux and Grossel [3].

To summarize this discussion at a glance, in Fig. 3 we have plotted the unit hyperbola $|e^{(+)}|^2 - |e^{(-)}|^2 = 1$, assuming real values for all the variables. The interface action transforms then the point 1 into the point 0. The same hyperbola, when referred to its proper axes, appears as $\tilde{e}^{(+)}\tilde{e}^{(-)} = 1/2$.

V. THE PHYSICAL MEANING OF INTERFACE COMPOSITION

To conclude, it seems adequate to provide a physical picture of the matrix manipulations we have performed in this paper. First, the inverse of an interface matrix, as pointed out before, describes the interface taken in the reverse order.

Concerning the product of interface matrices, this operation has physical meaning only when the second medium of the first interface is identical to the first medium of the second one. In this case, let us consider the interfaces 01 and 12. A direct calculation from Eqs. (7) shows that

$$
l_{01} l_{12} = l_{02},
$$

(31)

for both basic polarizations, which is equivalent to the constraints

$$
r_{02} = \frac{r_{01} + r_{12}}{1 + r_{01} r_{12}},
$$

(32)

$$
t_{02} = \frac{t_{01} t_{12}}{1 + r_{01} r_{12}}.
$$

Note that the reflected-amplitude composition behaves as a tanh addition law, just as in the famous Einstein addition law for collinear velocities: no matter what values the reflection amplitudes $r_{01}$ and $r_{12}$ (subject only to $|r_{01}| \leq 1$ and $|r_{12}| \leq 1$) have, the modulus of the composite amplitude $|r_{02}|$ cannot exceed the unity. Alternatively, we have

$$
\tilde{r}_{02} = r_{02} = \tanh(\xi_{02}/2) = \tanh(\xi_{01}/2 + \xi_{12}/2)
$$

(33)

which leads directly to the first one of Eqs. (24).

On the contrary, the transmitted amplitudes composes as a sech, whose addition law is more involved and is of little interest for our purposes here.

Obviously, for this interface composition to be realistic one cannot neglect the wave propagation between interfaces. However, this is not an insuperable drawback. Indeed, let us consider a single layer of a transparent material of refractive index $N_1$ and thickness $d_1$ sandwiched between two semi-infinite media 0 and 2. Let

$$
\beta_1 = \frac{2 \pi}{\lambda} N_1 d_1 \cos \theta_1
$$

(34)

denote the phase shift due to the propagation in the layer, $\lambda$ being the wavelength in vacuum. A standard calculation gives for the reflected and transmitted amplitudes by this layer the Airy-like functions [3]

$$
R_{012} = \frac{r_{01} + r_{12} \exp(-i2\beta_1)}{1 + r_{01} r_{12} \exp(-i2\beta_1)},
$$

(35)

$$
T_{012} = \frac{t_{01} t_{12} \exp(-i\beta_1)}{1 + r_{01} r_{12} \exp(-i2\beta_1)}.
$$


The essential point is that in the limit $\beta_1 = 2n\pi$ $(n = 0, 1, \ldots)$, which can be reached either when $d_1 \to 0$ or when the plate is under resonance conditions, then $R_{012} \to r_{02}$ and $T_{012} \to t_{02}$, and we recover Eqs. (32). This gives perfect sense to the matrix operations in this work.

VI. CONCLUSIONS

We have discussed in this paper a simple transformation that introduces remarkable simplicity and symmetry in the physics of a plane interface. In these new suitable variables the action of any interface appears in a natural way as a hyperbolic rotation, which is the natural arena of special relativity.

This formalism does not add any new physical ingredient to the problem at hand, but allows one to obtain previous results (like Fresnel formulas or Stokes relations) in a particularly simple and elegant way that appears closely related to other fields of physics.

FIG. 1: Wave vectors of the incident, reflected, and transmitted fields at the interface 01.

FIG. 2: Plot of the factor $\xi_{01}$ and $r_{01}$ as functions of the angle of incidence $\theta_0$ (in degrees) for both basic polarizations for an interface air-glass ($N_0 = 1, N_1 = 1.5$). The marked points correspond to the Brewster angle.

FIG. 3: Schematic plot of the hyperbolic rotation performed by the interface 01 that transforms on the unit hyperbola the point 1 into the point 0.

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