Singularity-free Bianchi spaces with nonlinear electrodynamics

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Abstract

In this paper we present an analysis to determine the existence of singularities in spatially homogeneous anisotropic universes filled with nonlinear electromagnetic radiation. These spaces are conformal to Bianchi spaces admitting a three parameter group of motions G\(_3\). For these models we study geodesic completeness. It is shown that with nonlinear electromagnetic field some of the Bianchi spaces are geodesically complete (G\(_3\)VIII). The analysis of the Raychaudhuri equation reveals that the BI field is responsible for the absence of singularity. When certain topology is assumed, Bianchi G\(_3\)IX may present geodesics that are imprisoned. In the linear limit (Maxwell field) the spacetimes are singular and they do not admit a cosmological interpretation in all the range of the time coordinate.

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1. Introduction

General relativity breaks down at certain points called singularities. The singularity theorems, that were shown for the first time 30 years ago by Stephen Hawking, Robert Geroch and Roger Penrose \cite{Hawking1973}, gave the bad news that under certain conditions singularities of the spacetime would appear in the future or in the past. An important subject that is not yet determined is the nature of the singularities of the spacetime in general relativity.

There are several criteria that tell us that a spacetime is singular, such as the divergence of the curvature invariants. However, the implication does not work in the opposite direction: if the curvature invariants do not diverge, we do not know for sure if the spacetime is regular. The common characteristic of singular spacetimes is the existence of incomplete causal geodesics. We shall adopt as a definition of the absence of singularities the completeness of causal geodesics (\(g\)-completeness), i.e. no observer in free fall leaves the spacetime in a finite proper time. We note, however, that not even geodesic completeness guarantees the absence of...
singularities (see a counterexample by Geroch [2] of a geodesically complete spacetime with incomplete non-geodesic curves).

In [3] Mars and Senovilla considered non-diagonal inhomogeneous cosmologies with a perfect fluid, and studied their separability. Among the derived solutions they found a class of singularity-free solutions. One particular case, Bianchi IX, is geodesically complete and satisfies the dominant energy condition. This particular model provides an example of how the presence of matter can help to avoid the initial singularity while keeping physical compelling dominant energy conditions.

Recently, more studies about singularity-free Bianchi type-IX models have arisen; for example, Berger used the method of consistent potentials [4] to explain how a minimally coupled (classical) scalar field can suppress mixmaster oscillations [5] when approaching the singularity of Bianchi IX cosmology. Toporensky and Ustiansky studied the possibility of non-singular transition from contraction to expansion in the dynamics of the Bianchi IX cosmological model with minimally coupled massive real scalar field [6].

Moreover, nonlinear or logarithmic electromagnetic Lagrangians coupled to gravity have been studied in an attempt to remove some of the singularities associated with charged black holes. In fact, there have been found solutions corresponding to regular black holes with nonlinear electrodynamics with Lagrangians that depend only on one of the electromagnetic invariants [7].

In this paper, we study singularities in spatially homogeneous but anisotropic spaces where the energy–momentum tensor corresponds to a (source-free) nonlinear electromagnetic field, in particular, the Born–Infeld field. The energy–momentum tensor satisfies the dominant energy condition and the matter does not diverge; moreover, the Weyl invariants are regular. These conditions point to the absence of singularities provided the spaces pass the test of geodesic completeness. The assumed spacetime includes particular cases of Bianchi spaces. The analysis of the geodesics leads us to conclude that some of the studied Bianchi spaces avoid the singularity, like Bianchi G 3VIII, while Bianchi G3II and Bianchi G3IX present incomplete geodesics; however, some geodesics, characterized by certain momentum in one direction, are complete. Spaces such as Bianchi G3III and G3I as well as a particular case of Kantowski–Sachs present singularity in the curvature, i.e. the invariants diverge after a finite time.

The solutions investigated in this paper have already been discussed. An approximate solution was presented in [8] and the possibility of nonlinear electrodynamics (NLED) as a source of inflation was addressed in [9]. The singularity structure is studied in the present paper.

Before entering into the main subject, we give a summary of NLED and energy conditions in section 2. In section 3 the studied solutions of the coupled Einstein–Born–Infeld (EBI) equations are introduced along with comments on their geometric structure and scalar invariants. In section 4 we analyse the geodesics and the nature of the singularities that appear in the models. In section 5 we address the limit of linear electrodynamics (Einstein–Maxwell); these solutions are singular and the cosmological interpretation does not hold all the time since the metric changes its signature after some time. In section 6 we discuss the Raychaudhuri equation for both cases, BI and the linear limit, and this clarifies that the BI field is responsible for preventing the occurrence of the singularity. In the final section we draw some conclusions.

2. Born–Infeld nonlinear electrodynamics

One of the motivations to explore spacetimes with nonlinear electromagnetic fields is that in early epochs of the universe the magnetic fields exceeded some $10^{15}$ G. At such values
of the electromagnetic field, the interaction of photons with themselves becomes important and classical electrodynamics is not valid anymore [10]. To overcome the use of quantum electrodynamics (QED) we shall employ the Born–Infeld theory.

The nonlinear electrodynamics theory presented by Born and Infeld (1934) possesses a Lagrangian structure that is similar to an effective action in quantum electrodynamics. This fact was noted by Euler and Heisenberg soon after the proposal by Born and Infeld and later on it was remarked by Schwinger. In the papers by Heisenberg and Euler [11] and by Drummond and Hathrell [12], it was demonstrated that the Born–Infeld Lagrangian has the same dependence on the electromagnetic invariants as QED for the one-loop approach. On the other hand, based on very general considerations, Schwinger [13] obtained an expression for the Lagrangian that coincides, within numerical factors of order unity, with the BI Lagrangian in the same approximation. Therefore, EBI theory may give us, at least qualitatively, some features of early universes that could not be derived within classical electrodynamics.

Moreover, lately the BI-type action has arisen in string theory: it happens that open strings with Dirichlet boundary conditions (D-branes) are described by an effective action of Dirac–Born–Infeld that is related to the effective action of BI type in the theory of open strings. The action to low energies corresponding to these objects, called branes, can be written like a term of BI plus a term of the Weiss–Zumino or Chern–Simons type [14]. The BI strings are solitons that represent terminal strings on branes.

On the other hand, in relation to cosmological models with electromagnetic fields, Brill [15] studied Einstein–Maxwell solutions in a homogeneous and nonisotropic universe. His solution is a generalization of Taub’s vacuum universe and represents a closed universe of topology $R \times S^3$ filled with gravitational and electromagnetic radiation. This universe expands in two spatial directions and contracts along the third one; moreover, its invariants are finite. Also Hughston and Jacobs [16] investigated the behaviour of source-free, homogeneous, electromagnetic fields in Bianchi-type cosmologies. They showed that any pure electric (magnetic) field is compatible with types VIII, IX, V, VI ($h \neq 0$) and VI ($h \neq -1$). Types III, VI ($h = -1$, $h = 0$) allow one independent component of the magnetic field. Type II fits with two independent magnetic field components while type I puts no constraints on the magnetic field components. Note that results in [16] were obtained for Maxwell fields in the Bianchi geometry, noncoupled to Einstein equations. In a previous paper, Hacyan [17] investigated the effect of BI electrodynamics in the initial singularity of a Bianchi I space. It turned out in that case that the presence of a BI field smoothed the oscillations of the metric functions when approaching the singularity. In the cases addressed in the present paper, Bianchi types G3I, G3III as well as Kantowski–Sachs admit an electric or magnetic nonlinear field (not both) and G3II, G3VIII and G3IX admit both components; moreover, the presence of the electromagnetic field prevents the occurrence of singularities in some of them.

### 2.1. The Born–Infeld field parametrization

We describe in what follows the main characteristics of the BI formalism used to determine the EBI solutions.

The Born–Infeld (BI) nonlinear electromagnetic theory is self-consistent and satisfies all natural requirements. Its Lagrangian depends on the BI constant $b$, the maximum field strength; it also depends in nonlinear form on the electromagnetic invariants, $P$ and $Q$, of the antisymmetric tensor $P_{\mu \nu}$, the generalization of the electromagnetic tensor $F_{\mu \nu}$; the BI Lagrangian is given by

$$L_{\text{BI}} = -\frac{1}{2} P_{\mu \nu} F_{\mu \nu} + \mathcal{H}(P, Q).$$

\(\mathcal{H}(P, Q)\)
where $\mathcal{H}(P, Q)$ is the so-called structural function. The structural function is constrained to satisfy some physical requirements: (i) the correspondence with the linear theory of Maxwell in the limit when $b$ goes to infinity ($\mathcal{H}(P, Q) = P + O(P^2, Q^2)$); (ii) the parity conservation ($\mathcal{H}(P, Q) = \mathcal{H}(P, -Q)$); (iii) the positive definiteness of the energy density ($\mathcal{H}_{\mu \nu} P^\mu p^\nu > 0$) and (iv) the requirement of the timelike nature of the energy flux vector ($P \mathcal{H}_{\mu \nu} P^\mu p^\nu - Q \mathcal{H}_{\mu \nu} q^\mu q^\nu > 0$).

Conditions (iii) and (iv) amount to the fulfilment of the dominant energy condition (DEC) [1]: $T_{\mu \nu} u^\mu u^\nu > 0$, for every time direction $u^\mu$ ($u^\mu u_\mu < 0$) and $T_{\mu \nu} u^\mu$ is a nonspacelike vector.

The energy–momentum tensor in terms of the structural function is given by

$$4\pi T_{\mu \nu} = \mathcal{H}_{\mu \rho} P^{\rho} P_\nu - 8\pi(2P \mathcal{H}_{,\rho} p^\rho + Q \mathcal{H}_{,\rho} q^\rho - \mathcal{H}).$$

(2)

The structural function for the BI field is given by

$$\mathcal{H}(P, Q) = b^2 - \sqrt{b^4 - 2b^2 P + Q^2}.$$  

(3)

In the linear limit that is obtained by taking $b \to \infty$, $\mathcal{H} = P$ and $P_{\mu \nu} = F_{\mu \nu}$. $P_{\mu \nu}$ and $F_{\mu \nu}$ are related through the material or constitutive equations:

$$F_{\mu \nu} = \mathcal{H}_{\mu \rho} P^{\rho} + \mathcal{H}_{\nu \rho} \tilde{P}^{\rho},$$

(4)

where $\tilde{P}^{\mu \nu} = \frac{1}{2} \epsilon^{\mu \nu \alpha \beta} P_{\alpha \beta}$ is the dual of $P_{\mu \nu}$ (defined such that $\tilde{f}_{\mu \nu} = f_{\mu \nu}$). An extensive treatment on nonlinear electrodynamics in curved spacetimes can be consulted in [18].

In the null tetrad formalism [19], and dealing with a Petrov type-D metric, one can always align the directions of the real null vectors $e_3$ and $e_4$ along the double Debeer–Penrose (DP) vectors. We shall assume that the eigenvectors of $F_{ab}$ (and consequently the ones of $P_{ab}$) are aligned in the direction of the DP vectors. Hence the nonvanishing components of $F_{ab}$ are $F_{12}$ and $F_{34}$ ($P_{12}$ and $P_{34}$). We shall adopt the following parametrization,

$$P_{12} = iH, \quad P_{34} = D,$$

(5)

where $D$ is the electric displacement and $H$ is the magnetic field; the corresponding invariants are

$$P = \frac{P^{ab} P_{ab}}{4} = -\frac{D^2 - H^2}{2}, \quad Q = \frac{P^{ab} \tilde{P}_{ab}}{4} = iHD.$$  

(6)

We note that in Plebanski’s formalism [18], the pseudo invariant $Q$ is purely imaginary. To establish the correspondence with Maxwell electrodynamics we must project along a timelike vector $\eta^\beta$: $D^\alpha = \eta_\beta P^{\alpha \beta}, \tilde{H}^\alpha = -i\eta_\beta P^{\alpha \beta}, E^a = \eta_\beta F^{\alpha \beta}, B^a = -i\eta_\beta F^{\alpha \beta}$, to obtain the electric and magnetic intensities and inductions; all these vectors are real (for instance the invariant $D^\alpha H_\alpha = 4HD$).

In parametrization (5) the structural function turns out to be

$$\mathcal{H} = b^2 - \sqrt{b^4 + b^2(D^2 - H^2)} - H^2D^2$$

$$= b^2 - \sqrt{(b^2 + D^2)(b^2 - H^2)}.$$  

(7)

The parametrization (5) has the advantage that there are only two nonvanishing components of the energy–momentum tensor

$$T_{12} = 2b^2 \left[1 - \frac{b^2 + D^2}{\sqrt{(b^2 + D^2)(b^2 - H^2)}}\right] = 2b^2(1 - e^{-v}),$$  

$$T_{34} = 2b^2 \left[1 - \frac{b^2 - H^2}{\sqrt{(b^2 + D^2)(b^2 - H^2)}}\right] = 2b^2(1 - e^v).$$

(8)
where we have defined
\[ e^\nu = \sqrt{\frac{b^2 - H^2}{b^2 + D^2}}. \] (9)

The condition that \( e^\nu \) be real imposes the restriction that \( H < b \), i.e. the fields do not reach the maximum allowed field \( b \).

2.2. Energy conditions associated with Born–Infeld field

In this subsection we show that the strong energy condition may be violated in spaces with nonlinear Born–Infeld electrodynamics as a result of the nonvanishing trace of the Born–Infeld energy–momentum tensor that can be negative.

The strong energy condition is stated as: for any timelike vector \( V^a \), \( V^a V_a = -1 \), \( R_{ab} V^a V^b \geq 0 \), where \( R_{ab} \) is the Ricci tensor.

We consider the relation coming from Einstein equations contracting with a timelike vector \( V^a \):
\[ R_{ab} V^a V^b = 8\pi (T_{ab} V^a V^b + T/2), \] (10)
where \( T \) is the trace of \( T_{ab} \) and \( R = R^a_a = -8\pi T \). In the null tetrad formalism, for metrics of Petrov type D, the components of \( T_{ab} \) can be aligned such that only \( T_{12} \) and \( T_{34} \) are nonvanishing; moreover, substituting in (10) the expression of \( T_{ab} \) in terms of the structural function (2), we have
\[ R_{ab} V^a V^b = 8\pi (2T_{12} V^1 V^2 + 2T_{34} V^3 V^4) + 4\pi T \]
\[ = 2[H, P] (P_{12}^2 - P_{34}^2) - 2(PH, P + QH, Q - H)] V^1 V^2 + 2[H, P] (P_{12}^2 - P_{34}^2) V^1 V^2 + 4\pi T. \] (11)

For the BI energy–momentum tensor the trace is \( 8\pi T = -8(PH, P + QH, Q - H) \). We can choose the timelike vector such that \( V^1 V^2 = V^3 V^4 \) (for instance, \( V^a = [1/2, -1/2, 1/2, -1/2] \)), obtaining
\[ R_{ab} V^a V^b = 2(PH, P + QH, Q - H). \] (12)

On the other hand, the dominant energy condition (DEC) that states ‘for every timelike vector, \( V^a \), \( T_{ab} V^a V^b \geq 0 \) and \( T_{ab} V^a \) is a nonspacelike vector’ [1] is indeed fulfilled by the BI energy–momentum tensor, equations (2)–(3). DEC means that to any observer the local energy density appears nonnegative and the local energy flow vector is nonspacelike. In the BI case DEC is a requirement of a physically reasonable nonlinear structural function \( H \) and its fulfilment is guaranteed by conditions (iii) and (iv) as stated in section 2.1. The dominant energy condition imposed on the Born–Infeld structural function is consistent with \( (PH, P + QH, Q - H) > 0 \). Therefore, in equation (12), \( R_{ab} V^a V^b < 0 \) and SEC is violated as a consequence of the nonvanishing negative trace of \( T_{ab} \). In the Maxwell case, being \( T = 0 \), the weak energy condition \( (T_{ab} V^a V^b > 0) \) leads directly to the fulfillment of SEC, as can be seen from (10) with \( T = 0 \).

The violation of SEC is not exceptional; we recall that minimally coupled scalar field violates SEC and indeed curvature-coupled scalar field theory also violates SEC. This point has been discussed recently in [20] and [21]. The fulfilling of SEC is related to the convergence of neighbouring geodesics, so the aspect of interest for us is that the violation of this condition could imply the absence of singularities. In section 5 the Raychaudhuri equation is analysed in this respect.
3. Spatially homogeneous spacetimes with nonlinear electromagnetic field

Since the energy–momentum tensor has two nonvanishing eigenvalues, equation (8), it may be coupled to spacetimes with two preferred directions, i.e. spacetimes of type D in Petrov classification. In [8] solutions were found to the coupled Einstein–Born–Infeld equations for spatially homogeneous and anisotropic spaces; among them there are several cases of Bianchi metrics.

The studied line element is of the form

\[ \dot{\varphi}(t)^2 ds^2 = -\frac{dr^2}{s(t)} + s(t)(dx + 2l z dy)^2 + \frac{dz^2}{h(z)} + h(z) dy^2, \]  

(13)

where \( x \) and \( y \) are ignorable coordinates, \( h = 1 - \epsilon z^2 \) with \( \epsilon \) and \( l \) being constants; \( s = s(t) \) and \( \varphi = \varphi(t) \) are the metric functions that are determined from the Einstein–Born–Infeld coupled equations. Choosing the null tetrad in the form:

\[ e^1 = \sqrt{2\dot{\varphi}} \left[ \frac{dz}{\sqrt{h}} + i\sqrt{h} dy \right], \]
\[ e^2 = \sqrt{2\dot{\varphi}} \left[ \frac{dr}{\sqrt{s}} + \sqrt{s}(dx + 2l z dy) \right], \]
\[ e^3 = \sqrt{2\dot{\varphi}} \left[ -\frac{dr}{\sqrt{s}} + \sqrt{s}(dx + 2l z dy) \right], \]
\[ e^4 = \sqrt{2\dot{\varphi}} \left[ -\frac{dt}{\sqrt{s}} + \sqrt{s}(dx + 2l z dy) \right], \]

(14)

the metric function \( s(t) \) turns out to be

\[ s(t) = \varphi^3 \dot{\varphi} \left[ C_1 + 2b^2 \int \left( \frac{1 - e^\nu}{\varphi^2 \phi^4} \right) dt + \epsilon \int \frac{dt}{\varphi^2 \phi^2} \right], \]

(15)

where a dot symbolizes a derivative with respect to time \( t \). \( C_1 \) is an integration constant related to the vacuum case, \( b \) is the BI parameter and \( \epsilon \) is the constant curvature parameter. Moreover, \( \phi(t) \) must satisfy the differential equation \( \ddot{\varphi} + l^2 \varphi = 0 \).

The system of EBI equations admits two possible solutions for the electromagnetic function \( \nu(t) \), equation (9),

\[ e^{\nu(t)} = \sqrt{1 \pm \phi^4(t)}. \]

(16)

Both solutions present regions in which \( s(t) > 0 \) passing then to the opposite sign, \( s(t) < 0 \). The regions with \( s(t) > 0 \) admit a cosmological interpretation as spatially homogeneous spaces. For \( s(t) < 0 \) the interpretation of (13) corresponds to stationary spaces (the \( t \) direction changes from timelike to spacelike). We shall study the case \( e^\nu = \sqrt{1 - \phi^4} \). The condition \( e^\nu \) is real imposes the restriction that \( \phi^4 < 1 \) and it determines the range of the coordinate \( t \). The electromagnetic fields are homogeneous and constant on spacelike hypersurfaces \( \{ t = \text{const} \} \).

\[ D(t) = \phi^2(t) \cos \left[ 2l \int \frac{dt}{\sqrt{1 - \phi^4}} \right], \quad B(t) = -\phi^2(t) \sin \left[ 2l \int \frac{dt}{\sqrt{1 - \phi^4}} \right], \]

(17)

or in terms of the electromagnetic tensor,

\[ \phi^2(t) F_{z y} = B(t) = e^\nu H(t), \quad \phi^2(t) F_{t z} = E(t) = e^{-\nu} D(t). \]

(18)

The line element (13) can be written in terms of a synchronous time coordinate if we transform \( \frac{dt}{\varphi(t)} = dT^2 \) and with \( \sigma^1 = dx + 2l z dy \):

\[ \phi^2 ds^2 = dS^2 = -dT^2 + s(\sigma^1)^2 + \frac{dz^2}{h} + h dy^2. \]

(19)
Written in this form, Bianchi types can be identified according to Table 1 (cf equations (13.1) and (13.2) in [22]). In the case Bianchi G3IX the metric (19) is a particular case of Taub’s universe (cf equations (6.29) or (8.5) in [23]).

### 3.1. Bianchi cases included

The spacetime (13) possesses four Killing vectors. The four generator group \( G_4 \) has invariant subgroups \( G_3 \) that can be spatial rotations or translations (\( S_3 \) or \( T_3 \)). In particular, we shall pay attention to the hypersurface–homogeneous spacetimes that can be identified as Bianchi spaces I, II, III, VIII and IX and one of them as Kantowski–Sachs.

The Killing vectors \( X_a \) are classified according to the values of the curvature parameter \( \epsilon \) that can be 1, 0, \(-1\):

(i) \( \epsilon = 0 \), \( h = 1 \),

\[
X_1 = \partial_y, \quad X_2 = \partial_x, \\
X_3 = -\partial_x / 2l + y \partial_z, \\
X_4 = z \partial_y - y \partial_z + l(y^2 - z^2) \partial_x.
\]  

(ii) \( \epsilon = -1 \), \( h = 1 + z^2 \),

\[
X_1 = \partial_y, \quad X_2 = \partial_x, \\
X_3 = \sqrt{h} \cos y \partial_z - \frac{\sinh y}{\sqrt{h}}(2ly \partial_x + z \partial_y), \\
X_4 = \sqrt{h} \sinh y \partial_z - \frac{\cosh y}{\sqrt{h}}(2ly \partial_x + z \partial_y).
\]  

(iii) \( \epsilon = 1 \), \( h = 1 - z^2 \),

\[
X_1 = \partial_y, \quad X_2 = \partial_x, \\
X_3 = \sqrt{h} \cos y \partial_z - \frac{\sin y}{\sqrt{h}}(2ly \partial_x - z \partial_y), \\
X_4 = \sqrt{h} \sin y \partial_z + \frac{\cos y}{\sqrt{h}}(2ly \partial_x - z \partial_y).
\]  

Special cases of Bianchi metrics with a particular topology can be obtained as follows:

(i) \( \epsilon = 0 \). In the metric (13) with \( \epsilon = 0 \) we obtain \( h(z) = 1 \) and the line element is

\[
\phi(t)^2 ds^2 = -\frac{dt^2}{s(t)} + s(t)(dx + 2l z dy)^2 + dz^2 + dy^2.
\]
where the ranges of all the coordinates are from $-\infty$ to $\infty$, which is a non-compact spacetime with topology $\mathbb{R}^4$ that corresponds to Bianchi II. If $l = 0$, the metric becomes diagonal and can be identified as a Bianchi I,

$$\phi(t)^2 ds^2 = -\frac{dr^2}{s(t)} + s(t)dx^2 + dz^2 + dy^2.$$  \hspace{1cm} (24)

(ii) $\epsilon = -1$. In the metric (13) we transform $z \mapsto \sinh \theta$ obtaining $h(\theta) = \cosh^2 \theta$ and the line element becomes

$$\phi(t)^2 ds^2 = -\frac{dr^2}{s(t)} + s(t)(dx + 2l \sinh \theta \, dy)^2 + d\theta^2 + \cosh^2 \theta \, dy^2,$$  \hspace{1cm} (25)

where the ranges of all the coordinates are from $-\infty$ to $\infty$; this is a non-compact spacetime with topology $\mathbb{R}^4$ that corresponds to Bianchi VIII. If $l = 0$ the metric corresponds to a Bianchi III space.

(iii) $\epsilon = 1$. In the metric (13) we transform $z \mapsto \sin \theta$ and we obtain $h(\theta) = \cos^2 \theta$ and the line element as

$$\phi(t)^2 ds^2 = -\frac{dr^2}{s(t)} + s(t)(dx + 2l \sin \theta \, dy)^2 + d\theta^2 + \cos^2 \theta \, dy^2,$$  \hspace{1cm} (26)

where the ranges of the coordinates are $-\infty < t < \infty$, $0 \leq \theta \leq \pi$; if $0 \leq y \leq 2\pi$, $0 \leq x \leq 4\pi$ it is a compact spacetime with the topology $\mathbb{R} \times SO(3)$ that corresponds to Bianchi IX. For $l = 0$ in metric (13) and changing coordinates to $z \mapsto \cos \theta$ and with $\epsilon = 1$ we obtain $h(\theta) = \sin^2 \theta$ and the line element transforms into

$$\phi(t)^2 ds^2 = -\frac{dr^2}{s(t)} + s(t)dx^2 + d\theta^2 + \sin^2 \theta \, dy^2;$$  \hspace{1cm} (27)

it is a spatially homogeneous solution which admits no simply transitive $G_3$ that can be identified as a Kantowski–Sachs space.

The following results in subsections 3.2 and 3.3 were derived in [8] but are included here for completeness.

### 3.2. Scalar polynomial invariants

For these solutions the only nonvanishing Weyl scalar is given by

$$\Psi_2 = \frac{\phi^2}{12}(\ddot{s} - 8sl\dot{s}^2 + 2\epsilon + 6l\dot{s}).$$  \hspace{1cm} (28)

The scalar polynomial invariants constructed from $\Psi_2$ are given by $I = 3\Psi_2^2$ and $J = -\Psi_2^3$, in such a manner that if $\Psi_2$ is unbounded then both $I$ and $J$ must be unbounded and a curvature singularity occurs. The regularity and boundedness of $\Psi_2$ depends on the proper for $\ddot{s}(t), \dot{s}(t)$ and $s(t)$. In case of a suspected singular point, we must check if $\Psi_2$ is regular when evaluated there. If $\Psi_2$ remains bounded then we proceed to investigate if geodesics are complete.

One could think that regularity of the curvature invariants might be helpful to avoid singularities but there are known examples of spacetimes with regular invariants, such as Taub–NUT, that enclose geodesics that are not complete in their affine parametrization due to the phenomenon called imprisoned incompleteness [1], in which a nonspacelike curve as it follows to its future enters and remains within a compact set. In spite Hawking himself asserting that ‘the kind of behaviour which occurs in Taub–NUT space cannot happen if there is some matter present’, when studying a spacetime with regular invariants we must show that in fact geodesics are complete to be able to assert that the space is singularity-free.
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1. The metric functions $s(t)$ when the BI field is present (continuous curve) and in the linear limit (Maxwell field) (dashed curve) are shown. In general, the presence of the BI field smooths oscillations of the metric function. The plot corresponds to Bianchi G3IX, the values of the constants are $l = 1, A = 0.8, C_1 = 0.5, b = 0.85, \epsilon = 1$ and the value of the constant linear field is $C = 0.85$.

3.3. Linear limit

The linear (Maxwell) limit can be obtained from the solution equation (15) by taking the limit $b \to \infty$. In this limit the metric function $s(t)$ is

$$s(t) = \phi^3 \phi\left[C_1 + C^2 \int \frac{dt}{\phi^2} + \epsilon \int \frac{dt}{\phi^2}\right],$$

(29)

where $C^2 = H^2 + D^2 = B^2 + E^2$ is related to the energy density of the field; in this case the electromagnetic field is given by

$$B = C \sin(2\alpha t), \quad E = C \cos(2\alpha t),$$

(30)

where $\alpha$ is a constant. In figure 1 are displayed both metric functions, the linear (Maxwell) and Born–Infeld, the values of the constants correspond to a case of Bianchi G3IX. In general, the BI field smooths oscillations of the metric function $s(t)$. We shall address geodesic completeness for this case in section 5.

4. Analysis of singularity structure

Hawking and Ellis [1] discussed Bianchi I spaces and gave a theorem asserting that singularities will occur in all non-empty spatially homogeneous models in which the timelike convergence condition $(R_{\mu\nu} u^\mu u^\nu \geq 0)$ is satisfied. However, as we showed in section 2.2, in the studied case the timelike convergence condition (or SEC) is not fulfilled. Therefore, the possibility of the absence of singularity persists.

For our Bianchi spaces with nonlinear electrodynamics there are two branches characterized by the value of the constant $l$ that appears in the non-diagonal term of the metric, $(dx + 2l z dy)$. In the case $l = 0$ the conformal function $\phi(t)$ fulfils the equation $\dot{\phi} = 0$, then $\phi = At + B$, where $A$ and $B$ are constants. In the case $l \neq 0$ the equation for $\phi(t)$ is $\dot{\phi} + l^2 \phi = 0$, with solutions $\phi = A \cos(\alpha t + B)$, $A$ and $B$ being constants. The analytical expression for the metric function $s(t)$ depends on the conformal function $\phi(t)$ and its behaviour is different in each case.

We will consider a spacetime as singularity-free when it is geodesically causal complete (g-completeness) that means intuitively that an observer in free fall does not leave the spacetime in a finite proper time; equivalently, that every geodesic can be extended to arbitrary values of its affine parameter. The demonstration of geodesic completeness is not simple in most
cases [20]. Nevertheless, the analysis of the geodesic extension can be simplified when there exist constants of motion associated with Killing vectors.

The method that we use to analyse the completeness is to obtain the first order equations for the derivatives of the coordinates with respect to an affine parameter \( \tau, x, t, y, z, \) using the constants of motion. Then showing that the first derivatives are bounded one concludes that the corresponding geodesic curves \( x(\tau), t(\tau), y(\tau), z(\tau) \), are complete [24].

We shall analyse geodesic completeness of the spacetimes in the following subsections, first for the class of non-diagonal metrics \( l \neq 0 \) that includes Bianchi G3II, G3VIII and G3IX and in the last subsection for the diagonal metrics Kantowski–Sachs and Bianchi spaces G3I and G3III.

4.1. Bianchi G3II, G3VIII and G3IX

In the case that \( l \neq 0 \) the equation for \( \phi \) is \( \ddot{\phi} + \frac{l^2}{A^2} \phi = 0 \), with solution \( \phi = A \cos(\epsilon t + B) \), \( A, B \) being constants; the sector \((x, y)\) of the metric is non-diagonal and the cases included are Bianchi G3VIII and G3IX for \( \epsilon = -1, 1 \), respectively and G3II when \( \epsilon = 0 \).

The metric function \( s(t) \), equation (15), can be integrated in exact form; with \( \phi = A \cos(t) \) (taking \( l = 1 \) and \( B = 0 \)) we obtain a cumbersome expression in terms of \( \sin t, \cos t \) and elliptic functions,

\[
s(t) = -A^4 C_1 \cos^3 t \sin t + \epsilon (\cos^4 t - \cos^2 t \sin^2 t) + \frac{2b^2}{A^2} \left\{ \cos^4 t - \frac{5}{3} \cos^2 t \sin^2 t - \frac{1}{3} \sin^2 t \right\}
- \frac{1}{6} (2 \cos 2t + \cos 4t) \sqrt{(2 - A^2 - A^2 \cos^2 t)(2 + A^2 + A^2 \cos^2 t)}
+ \frac{8}{3} A^2 \cos^4 t \sin^2 t \sqrt{(2 + A^2 + A^2 \cos^3 t) \over (2 - A^2 - A^2 \cos^2 t)}
- \frac{2}{3} \sin t \cos^3 t \left[ \sqrt{1 + A^2 E[\alpha, \beta]} - \left( \frac{4 - A^4}{4 - A^4} \right) F[y, \beta] \right],
\]

\[
\alpha = \arcsin \left( \frac{\sqrt{2} \sin t}{\sqrt{2 - A^2 - A^2 \cos 2t}} \right), \quad \beta = \frac{2A^2}{1 + A^2},
\gamma = \arcsinh \left( \frac{(1 - A^2)\sqrt{2 + A^2 + A^2 \cos 2t}}{2 \sin t} \right),
\]

where \( E \) and \( F \) are the elliptic integrals of second and first kind, respectively. From the expression (31), it is clear that \( s(t) \) is a periodic and bounded function. There are values of \( t \) for which the metric function \( s(t) \) is null and where the metric (13) is singular. For \( \phi(t) = A \cos(t) \), these values are \( t = (2n + 1)\pi/2, n = 0, 1, 2, \ldots \); at those points there might be a singularity. There are additional zeros when negative terms cancel the positive ones; however, the position of these zeros depends on the balance between the distinct parameters, \( C_1, b^2/A^2, A^2 \) and \( \epsilon \).

The characteristic behaviour of the functions \( s(t), \dot{s}(t), \ddot{s}(t) \) is illustrated in figure 2. The three functions \( s(t), \dot{s}(t), \ddot{s}(t) \) are periodic and bounded. As a consequence, the Weyl scalar \( \Psi_2 \), equation (28) and the derived invariants are also finite over all the range \(-\infty < t < \infty \). We shall investigate the completeness of geodesics to elucidate the behaviour of the spacetime at the zeros of \( s(t) \).

For each Killing vector \( X_i \) there exists a conserved quantity \( x_i^\mu X_\mu = x_i^\mu g_{\mu\nu}X^\nu = \text{const} \), where \( x^\mu(\tau) \) is a geodesic parametrized by \( \tau \). Since \( \partial_\tau \) and \( \partial_\nu \) are Killing vectors, it implies
that there are constants $K_1$, $K_2$ given by

$$K_1 = g_{xx}x,\tau + g_{xy}y,\tau, \quad K_2 = g_{yy}y,\tau + g_{yx}x,\tau. \quad (32)$$

From equation (32) we determine $x,\tau$ and $y,\tau$:

$$y,\tau = \frac{\phi^2}{h}(K_2 - MK_1), \quad x,\tau = \frac{\phi^2K_1}{s} - My,\tau, \quad (33)$$

where $M = 2lz$. From equation (33) we can analyse completeness for $x(\tau)$ and $y(\tau)$. All the coefficients in equation (33) are finite except for the first term in $x,\tau$, since there will be divergences if $s/\phi^2$ is null. We shall analyse this term in detail. Also note that $h(\pm 1) = 1 - \epsilon z^2 = 0$ in the case $\epsilon = 1$; however in this case we transform $z \mapsto \cos \theta$ and the divergence is associated with the spherical coordinates; we analyse it in detail in section 4.2.

Taking the limit $t \to \pi/2$,

$$\lim_{t \to \pi/2} \left(\frac{s}{\phi^2}\right) = -\epsilon,$$

therefore, only in the case that $\epsilon = 0$, the geodesics $x(\tau)$ may be incomplete. This case corresponds to spaces Bianchi II. Even in this case complete geodesics may exist when the constant $K_1 = 0$ that means,

$$K_1 = \frac{s}{\phi^2}(x,\tau + My,\tau) = 0, \quad (35)$$

in this case $x,\tau = -MY,\tau$ and the geodesics $x(\tau)$ are complete, since the equation for $y,\tau$ is given in terms of bounded coefficients and $M = 2lz$ is bounded (for bounded $z$) as well. However, the generic behaviour when $\epsilon = 0$ is geodesically uncomplete.

Moreover, note from (34) that in the case $\epsilon = 1$, the limit is negative, meaning that $s(t)$ undergoes a change in sign and that $s(t) = 0$ for some $t < \pi/2$. At that point the geodesics $x(\tau)$ may be incomplete for $\epsilon = 1$, unless $K_1 = 0$. In this case (of complete geodesics) if one considers the topology $R \times S^3$ for Bianchi G1IX then $0 < x < 4\pi$ and $x$ reaches a finite value as $\tau$ increases. This geodesic ‘wraps’ around the $x$-direction an infinite number of times. It corresponds to the ‘phenomena’ known as imprisoned incompleteness, described by Hawking (see paragraph 8.5 in [1]) for Taub–NUT spaces. This pathological behaviour arises essentially because $x$ is identified at 0 and $4\pi$, a consequence if Bianchi-type IX is assumed to be compact.
To obtain the equations for \( t, \tau \) and \( z, \tau \) we shall use the rest of the Killing vectors, quoted in section 3.1. These Killing vectors are different depending on the values of the curvature parameter \( \epsilon \). We separate them into three cases: \( \epsilon = 1, -1, 0 \).

For \( \epsilon = 1 \) the Killing vectors equation (22) imply the existence of two conserved quantities \( K_3 \) and \( K_4 \) that allow us to determine \( z, \tau \) as

\[
z_{\tau}^2 + \phi^4 (zK_2 - 2lK_1)^2 - \phi^4 h(K_3^2 + K_4^2) = 0.
\]

(36)

On the other hand, we can also use the line element to obtain \( t, \tau \),

\[
ds^2 = \delta = \frac{z_{\tau}^2}{\phi^4 h} + \frac{h}{\phi^4 s^2} - \frac{t_{\tau}^2}{\phi^4 s} + \frac{s}{\phi^2} (x_{\tau} + My_{\tau})^2,
\]

(37)

where \( \delta = 1 \) for spacelike geodesics, \( 0 \) for null and \(-1 \) if the geodesics are timelike.

Substituting the constants of motion we have

\[
t_{\tau}^2 = s\phi^4 [(K_3^2 + K_4^2) - (zK_2 - 2lK_1)^2/h + (K_2 - K_1M)^2/h] + \phi^4 K_3^2 - \phi^2 s\delta.
\]

(38)

In the equations for \( t, \tau \) and \( z, \tau \) all the coefficients are bounded, then the geodesic curves \( t(\tau) \) and \( z(\tau) \) are complete.

In an analogous way \( t, \tau \) and \( z, \tau \) can be determined for the case \( \epsilon = -1 \). Using the Killing vectors \( X_3 \) and \( X_4 \), equation (21), we obtain

\[
z_{\tau}^2 = \phi^4 (zK_2 + 2lK_1)^2 + \phi^4 h(K_3^2 - K_4^2),
\]

\[
t_{\tau}^2 = s\phi^4 [(K_3^2 - K_4^2) + (zK_2 + 2lK_1)^2/h + (K_2 - K_1M)^2/h] + \phi^4 K_3^2 - \phi^2 s\delta.
\]

(39)

For \( \epsilon = 0 \) we have from the motion constants that imply the Killing vectors in equation (20),

\[
z_{\tau}^2 = 4l\phi^4 h^2 [lK_3^2 - K_4K_1 - K_1(lK_1z^2 - K_2z)],
\]

\[
t_{\tau}^2 = s\phi^4 [4l^2 h(K_3 - K_1y)^2 + (K_2 - K_1M)^2/h] + \phi^4 K_3^2 - \phi^2 s\delta.
\]

(40)

In the three cases, \( \epsilon = 1, -1, 0 \), the geodesics \( t(\tau) \) and \( z(\tau) \) are complete.

4.2. Extended manifold

If a value of \( t \) is found where \( s(t) = 0 \), another coordinate system has to be used, i.e. the metric has to be extended and geodesics must be analysed in the extended metric. Although the metric is singular at \( t = \pi/2 \), in \((t, x, y, z)\) coordinates, no scalar polynomials of the curvature tensor diverge as \( t \to \pi/2 \); this suggests that there is not a real singularity at \( t = \pi/2 \), but rather that it is a result of a bad choice of coordinates. First rescaling the \( x \)-coordinate as \( x \mapsto 2lx \) and dropping primes the metric (13) can be written as

\[
\phi^2 ds^2 = -dt^2/s + 4l^2 s(dx + zdz)^2 + dz^2/h + hdy^2.
\]

(41)

Then we define an extended line element by means of

\[
t \mapsto t', \quad x \mapsto x' = \frac{1}{2l} \int_0^t \frac{dt}{s(t)}.
\]

(42)

The integral in the transformation is a Riemannian integral, since the number of zeros of \( s(t) \) is numerable and the transformation \((x, t) \mapsto (x', t')\) is well defined (the Jacobian is 1). Dropping primes, the extended metric can be written as

\[
\phi^2 ds^2 = 4l^2 s(dx + zdz)^2 - 4l \, dt(dx + zdz) + \frac{dz^2}{h} + hdy^2.
\]

(43)

To proceed with the analysis of geodesics we shall address each case separately depending on the value of \( \epsilon \).
Singularity-free Bianchi spaces with nonlinear electrodynamics

In the case \( \epsilon = -1, h = 1 + z^2 \geqslant 1 \) and the Killing vectors for (43) are

\[
X^a = [X^1, X^4, X^3, X^2],
\]

\[
X_1^a = [0, 1, 0, 0],
\]

\[
X_2^a = [0, 0, 1, 0],
\]

\[
X_3^a = \cosh(y)/\sqrt{h}[0, 1, z, -h \tanh(y)],
\]

\[
X_4^a = \sinh(y)/\sqrt{h}[0, 1, z, -h \coth(y)].
\]

The constants of motion associated with the above Killing vectors are

\[
\phi^2 K_1 = 4l^2 s(x, \tau + zy, \tau) - 2lt, \tau,
\]

\[
\phi^2 K_2 = hy, \tau + zK_1\phi^2,
\]

\[
\phi^2 h(K_3^2 + K_4^2) = -z^2 \phi^2 + h^2(\phi^2 K_1 + zy, \tau)^2.
\]

Besides, from the line element we have

\[
\phi^2 \delta = 4l^2 s(x, \tau + zy, \tau)^2 - 4lt, \tau(x, \tau + zy, \tau) + \frac{z^2 h}{l} + hy, \tau^2,
\]

where \( \delta = 1, 0, -1 \) for spacelike, null and timelike geodesics, respectively. From the above relations we obtain equations for the first derivatives of the coordinates with respect to an affine parameter \( \tau \):

\[
y, \tau = (K_2 - zK_1)\phi^2/h,
\]

\[
z, \tau = \phi^2 \sqrt{(K_1 + zK_2)^2 - h(K_3^2 + K_4^2)},
\]

\[
t, \tau = (\phi^2/2l)\sqrt{K_1^2 + 4l^2 (s/\phi^2)(K\phi^2 - \delta)},
\]

\[
x, \tau = \frac{\phi^2}{4l^2 s}
\]

\[
K_1 \left[ 1 + \sqrt{K_1^2 + 4l^2 s/\phi^2(K\phi^2 - \delta)} \right] - zy, \tau,
\]

where \( K = K_1^2 + K_3^2 - K_2^2 - K_4^2 \). We use again the criteria of boundedness of the first derivatives \( x^a_\tau \) to deduce the completeness of the corresponding geodesic, \( x^a(\tau) \). Divergence could arise in the fourth equation, for \( x, \tau \) if \( s/\phi^2 \) is zero; however, as we have stated in the previous section, the term \( s/\phi^2 \) is zero at \( t = \pi/2 \) only in the case \( \epsilon = 0 \). Therefore, equations (47) have bounded coefficients and the corresponding geodesics are complete. So, in principle, this spacetime is singularity-free.

(ii) For the case \( \epsilon = 0, h = 1 \) and the Killing vectors are

\[
X^a = [X^1, X^4, X^3, X^2], X_1^a = [0, 1, 0, 0], X_2^a = [0, 0, 1, 0], X_3^a = [0, y, 0, -1], X_4^a = [0, y^2 - z^2, 2z, -2y].
\]

Using \( K_i, i = 1, 2, 3, 4 \), the constants associated with the Killing vectors \( X^1 \), we obtain the first derivatives \( x^a_\tau \) as

\[
y, \tau = \phi^2(K_1 - zK_2),
\]

\[
z, \tau = \phi^2(yK_2 - K_1),
\]

\[
t, \tau = (\phi^2/2l)\sqrt{K’^2 - \delta\phi^2},
\]

\[
x, \tau = (4l^2 s/\phi^2)^{-1}[K_2 + \sqrt{K’^2 - \delta\phi^2}] - zy, \tau,
\]

where \( K’ = K_1^2 + K_2^2 - K_3^2 - K_4^2 \). The first three equations (49) have bounded coefficients; however, in the fourth equation, for \( x, \tau \), the term \( s/\phi^2 \) arises that is zero at \( t = \pi/2 \) if \( \epsilon = 0 \). Therefore, at that point \( x, \tau \) diverges, indicating that the corresponding geodesic \( x(\tau) \) may be incomplete at \( t = \pi/2 \). In this case the analysis is not conclusive since the spacetime could
admit a further extension. Even in this case certain null geodesics may be complete, those with \( K_2 = 0 = K' \), since if this is the case, \( x, \tau = -zy, \tau \).

(iii) In the case \( \epsilon = 1 \) it may happen that \( h = 1 - z^2 = 0 \) when \( z = \pm 1 \) and at those points the geodesics may diverge. To clarify the situation we transform \( z \mapsto \cos \theta, x \mapsto \psi, \ y \mapsto \varphi \) and the metric becomes

\[
\phi^2 \, ds^2 = 4I^2\delta(\psi + \cos \theta \, d\varphi)^2 - 4I \, dt(\psi + \cos \theta \, d\varphi) + d\theta^2 + \sin \theta^2 \, dy^2,
\]

where \( (\psi, \varphi, \theta) \) are the Euler angle coordinates with ranges \( 0 \leq \psi < 4\pi, 0 \leq \theta < \pi \) and \( 0 \leq \varphi < 2\pi \). The Killing vectors for (50) are

\[
X^a = \{ X', X^\psi, X^\varphi, X^\theta \},
\]

\[
X^a_1 = [0, \cos \varphi \csc \theta, -\cos \varphi \cot \theta, -\sin \varphi],
\]

\[
X^a_2 = [0, \sin \varphi \csc \theta, -\sin \varphi \cot \theta, \cos \varphi],
\]

\[
X^a_3 = [0, 1, 0, 0],
\]

\[
X^a_4 = [0, 0, 1, 0].
\]

The corresponding constants of motion are

\[
\phi^2 K_1 = \sin \theta \cos \varphi(K_3 \phi^2 - \cos \theta \varphi, \tau) - \sin \varphi \theta, \tau,
\]

\[
\phi^2 K_2 = \sin \theta \sin \varphi(K_3 \phi^2 - \cos \theta \varphi, \tau) + \cos \varphi \theta, \tau,
\]

\[
\phi^2 K_3 = 4I^2\delta(\psi, \varphi, \cos \theta \varphi, \tau) - 2I \zeta, \tau,
\]

\[
\phi^2 K_4 = K_3 \cos \theta \phi^2 + \sin \theta^2 \varphi, \tau.
\]

In their action on \( \theta \) and \( \varphi \) the Killing vectors are the standard generators of infinitesimal rotations of the sphere \( S^3 \). We can therefore use the invariance of the manifold under these transformations to assume that the poles of the spherical coordinates \( \theta, \varphi \) are located at our convenience. If we describe a transformation of \( \theta, \varphi \) coordinates by \( n_a \mapsto A_{ab}nb_b \), where \( A_{ab} \) is a constant \( 3 \times 3 \) orthogonal matrix and \( n_a \) are homogeneous coordinates defined by \( (n_x, n_y, n_z) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) \), then equation (51) will continue to hold if we replace the \( X^a \) by an equivalent set of Killing fields according to \( X_a \mapsto A_{ab}X_b \). This changes the constants to the same rotation, \( K_a \mapsto A_{ab}K_b \). We will therefore choose \( A_{ab} \), in studying any one given geodesic, in such a way that \( K_1 = 0 = K_2 \) (see [25]). So we obtain from the first two equations (52) that

\[
\theta, \csc \theta = 0, \quad \varphi, \varphi = K_3 \phi^2 / \cos \theta,
\]

the first equation implies that \( \theta \) is constant along the geodesic and the second one that \( \varphi, \varphi \) is bounded.

The rest of the equations for the first derivatives of the coordinates with respect to the affine parameter are

\[
t, \tau = (\phi^2 / 2I) \sqrt{K_3 + (4I^2\delta / \sin \theta^2)(K_4 - K_3 \cos \theta)^2 - 4I^2\delta(s/\phi^2)},
\]

\[
\psi, \tau = (4I^2\delta / \phi^2)^{-1}\left[ K_3 + \sqrt{K_3 + (4I^2\delta / \sin \theta^2)(K_4 - K_3 \cos \theta)^2 - 4I^2\delta(s/\phi^2)} \right] - K_3 \phi^2.
\]

We use again the criteria of boundedness of the first derivatives \( x^a_\tau \) to deduce the completeness of the corresponding geodesic, \( x^a(\tau) \). Divergence could arise in the equation for \( \psi, \tau \) if \( s/\phi^2 \) is zero. We have stated in the previous section that the term \( s/\phi^2 \) is zero at \( t = \pi/2 \) only in the case \( \epsilon = 0 \), so in this case there is no divergence of the coefficients at \( t = \pi/2 \). However, according to equation (34), we note that at \( t = \pi/2 \) a change in sign has occurred to \( s(t) \), meaning that \( s(t) = 0 \) at some \( t < \pi/2 \) and there the geodesic may be incomplete. In this case the analysis is not conclusive.
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Figure 3. The plot for the metric function $s(t)$ and its derivatives $\dot{s}(t), \ddot{s}(t)$ is shown for the case $l = 0$. The three functions are bounded at $t = 0$, but $\ddot{s}(t)$ diverges as $\dot{t} \rightarrow 1$. In this graphic the values of the constants are $A = 0.8, C_1 = 0.5, b = 0.8$ and $\epsilon = 1$. For $\epsilon = 0, -1$ the behaviour changes quantitatively.

4.3. Kantowski–Sachs and Bianchi spaces $G_3I$ and $G_3III$

Let us address the case $l = 0$. The included spaces are Kantowski–Sachs and Bianchi spaces $G_3I$ and $G_3III$ for $\epsilon = 1, 0, -1$, respectively.

In this case $\phi = At$ is only a scaling of time coordinate; $s(t)$, equation (15), can be integrated in exact form as

$$s(t) = \frac{2\sqrt{2}}{3} Ab^2 t^3 \left\{ K \left[ \frac{1}{2} \right] - F \left[ \frac{\pi}{2} - \arcsin At, \frac{1}{2} \right] \right\} + \frac{2b^2}{3A^2} \left( \sqrt{1 - A^4 t^2} - 1 \right) + C_1 A^4 t^3 - \epsilon t^2,$$

(55)

where $F$ is the elliptic integral of the first kind and $K$ is the complete elliptic integral of the first kind. We observe from equation (55) that $s(t)$ takes real values in the interval $0 \leq (At)^3 < 1$ and that $s(t = 0) = 0$.

The metric function $s(t)$, and its derivatives $\dot{s}(t)$ and $\ddot{s}(t)$ are finite at $t = 0$, then the Weyl scalar $\Psi_2$, equation (28), and the invariants derived from it are regular at $t = 0$. This is an indication that the singularity at $t = 0$ is not a physical one. However, $\ddot{s}(t)$ diverges as $At \rightarrow 1$ and it implies that the polynomial invariants constructed from $\Psi_2$ diverge as well at $At = 1$, having there a singularity. We shall analyse the geodesics to figure out the singularity structure if it exists. Figure 3 displays the metric function $s(t)$, and its derivatives, $\dot{s}(t)$ and $\ddot{s}(t)$.

The equations for the first derivatives of the coordinates with respect to the affine parameter $\tau$ are equally valid in this case by putting $l = 0, M = 0$ and the metric function $s(t)$ from equation (55). If the first derivatives of $x^\alpha(\tau)$ are bounded it implies that geodesics are complete and the field is nonsingular [24]. The first equations are

$$y,\tau = K_2\phi^2, \quad x,\tau = K_1\phi^2, \quad \phi = \frac{s}{\phi^2}$$

(56)

The term $\frac{\phi^2}{\tau}$ is bounded in all the range of $t$ and diverges at $t = 0$ only when $\epsilon = 0$, because

$$\lim_{t \rightarrow 0} \left( \frac{s}{\phi^2} \right) = -\frac{\epsilon}{A^2},$$

(57)

in such a manner that geodesics $x(\tau)$ are unbounded for Bianchi $G_3I$ ($\epsilon = 0$), since $x,\tau$ diverges for a finite value of the affine parameter. Even for the case $\epsilon = 0$ if the constant $K_1 = 0$ then the geodesics $x(\tau)$ are complete since in this case $x(\tau) = At + B, A, B$ constants, and it is a complete curve.
The coefficients in equations for \( z, \tau \) and \( t, \tau \) are bounded as can be seen from expressions (36)–(40) putting \( l = 0, M = 0 \). On the other hand, in spite that
\[
\lim_{At \to 1} \phi(t) = C_1 A^2 t^2 - C_2 < 0
\]
(58)
all the coefficients in the equations for \( x_\alpha, \tau \), \( x_\alpha = (x, y, t, z) \) are bounded, in particular, the term \( \frac{\phi^2}{\tau} \) is finite there,
\[
\lim_{At \to 1} \left( \frac{s}{\phi^2} \right) = C_1 A + \frac{2\sqrt{2}b^2}{3A^2} \left[ \frac{1}{2} - \frac{2b^2}{3A^2} - \frac{\epsilon}{A^2} \right] \quad \text{(59)}
\]
Then at \( (At \to 1) \) the geodesics affect completeness, however, the range of variation of the coordinate \( t \) and therefore also the range of variation of the affine parameter \( \tau \) is finite, which makes the geodesics uncomplete; moreover, the divergence of the invariants indicates the presence of a singularity as \( At \to 1 \).

5. Solutions with linear electromagnetic field

The conserved quantities derived from the existence of the four Killing vectors are independent of the linear or nonlinear electrodynamics. Then the equations for the derivatives of the coordinates with respect to the affine parameter can be analysed to test completeness if instead of having the BI field, we take the linear limit.

Expressions for the derivatives of the coordinates with respect to the affine parameter, equations (33)–(40), are valid in the linear electromagnetic case with \( s(t) \) given by equation (29). The possibility of divergence appears again in the term \( s/\phi^2 \) in the equation for \( x, \tau \), so we analyse the behaviour of the term \( s/\phi^2 \).

In the case \( l \neq 0 \), integrating the expression for \( s(t) \) equation (29) with \( \phi = A \cos t \) we obtain
\[
s(t) = C_1 \phi^3 + \frac{C^2}{A^2} \left[ \phi^2 - \frac{5}{3} \phi^4 \phi^2 - \frac{A^2}{3} \phi^2 \right] + \frac{\epsilon}{A^2} [\phi^4 - \phi^2 \phi^2].
\]
(60)
where \( C^2 = H^2 + D^2 \) is proportional to the energy density of the electromagnetic field and the limits at \( t \to 0 \) and \( t \to \pi/2 \) are
\[
\lim_{t \to 0} s(t) = C^2 / A^2 + \epsilon, \quad \lim_{t \to \pi/2} s(t) = -C^2 / 3A^2.
\]
(61)

From the previous expressions we see that \( s(t) \) is not zero, neither at \( t = 0 \) nor at \( t = \pi/2 \), however, note that \( s(t = \pi/2) < 0 \). It indicates that \( s(t) \) is not zero at some point of the interval \( (t = 0, t = \pi/2) \), independently of the magnitude of the electromagnetic field. At that point the geodesics may become incomplete. Moreover, when \( s(t) < 0 \) the metric no longer admits a cosmological interpretation.

In the case \( l = 0 \) the term \( s/\phi^2 \) is not zero, neither at \( t = 0 \) nor at \( At = 1 \); in this case \( \phi = At \) and
\[
s(t) = C_1 A^4 t^3 - C^2 - \epsilon t^2,
\]
(62)
and the limits as \( t \to 0, At \to 1 \) are
\[
\lim_{t \to 0} s(t) = -C^2, \quad \lim_{At \to 1} s(t) = C_1 A - C^2 - \epsilon / A^2.
\]
(63)
Note that as \( t \to 0 \), \( s(t) \) is negative; since \( s(t) \approx C_1 A^4 t^3 \), eventually \( s(t) \) will cross the axis to become positive, and geodesics may terminate there, when \( s(t) = 0 \). In the case that \( C_1 < 0 \) and \( \epsilon = 0, 1 \) then \( s(t) \) will remain negative all the time, a cosmological interpretation of the solution then being inappropriate.

In the following section the Raychaudhuri equation will be analysed and it will become clear that timelike congruences will focus to some point the sooner the more intense the electromagnetic field is.

6. Raychaudhuri equation and singularity-free spaces

According to Raychaudhuri, the equation that governs the rate of change of expansion of timelike congruences, \( \theta \), is

\[
\frac{d \theta}{d \lambda} = -R_{ab} V^a V^b + 2 \omega^2 - 2 \sigma^2 - \frac{\theta^2}{3} + V^a \omega_a,
\]

where \( V^a \) is a tangent vector to the geodesics, \( R_{ab} \) is the Ricci tensor, \( \omega \) is the vorticity, \( \sigma \) is the shear and \( V^a \) is the acceleration of the congruence (that is zero for geodesics).

From equation (64) it can be seen that the expansion \( \theta \) of a timelike geodesic congruence with zero vorticity will monotonically decrease along a geodesic if, for any timelike vector \( V^a, R_{ab} V^a V^b \geq 0 \), i.e. if SEC holds. The term involving the Ricci tensor, \( R_{ab} \), in equation (64) induces contraction of the geodesic lines, indicating that the focusing of neighbouring geodesics is unavoidable if SEC is fulfilled since the other terms on the right-hand side of (64) are also negative.

We have shown in section 2.2 that the Born–Infeld field may violate the strong energy condition and this fact could explain why there exist singularity-free solutions when Born–Infeld matter is included in a universe. In what follows, we analyse two congruences and their corresponding Raychaudhuri equation. We show that in the BI case the focusing may not occur due to the nonvanishing rotation.

First we consider the null congruence with tangent vector given by \( t^a = [t^1, t^2, t^3, t^4] = (\varphi/2)/\sqrt{s}, 1/\sqrt{s}, 0, 0 \). This is a null geodesic congruence characterized by a nonnull expansion and rotation given by \( \theta = \varphi \sqrt{s}/2 \) and \( \omega = \varphi \sqrt{s}/2 \), respectively. The shear is zero and \( R_{ab} t^a t^b = 0 \). The Raychaudhuri equation for the null congruence is

\[
\frac{d \theta}{d \lambda} = -R_{ab} t^a t^b + \omega^2 - \frac{\theta^2}{2} - 2 \sigma^2,
\]

\[
= \frac{1}{2} (\varphi^2 s \theta - \dot{s} \varphi).
\]

The right-hand side of the previous equation, \( \frac{\varphi}{2} (\varphi^2 s \theta - \dot{s} \varphi) \), oscillates and does not become infinite. Therefore, a null congruence will not necessarily focus on a point due to the nonvanishing rotation.

We also analyse the timelike geodesic congruence given by the tangent vector field \( V^a = \varphi \sqrt{s} \partial_t \), or in the null tetrad formalism, \( V^3 = V_4 = \frac{1}{\sqrt{s}}, V^1 = V^2 = 0 \). Using this congruence we can check that SEC is fulfilled in the linear (Maxwell) case, while for Born–Infeld case the same condition is violated.

This congruence is geodesic, vorticity-free with expansion and shear given, respectively, by \( \theta = \varphi \dot{s}/(2 \sqrt{s}) - 3 \sqrt{s} \varphi, \sigma^2 = \varphi^2 \dot{s}^2/(6s) \); substituting in equation (64) gives

\[
\frac{d \theta}{d \lambda} = -R_{ab} V^a V^b - 3 s \varphi^2 - \frac{5}{12} \dot{s}^2 (s/\varphi^2)^{-1} + \dot{s} \varphi \varphi.
\]
For the EBI solutions, the two nonvanishing components of the traceless Ricci tensor, in terms of the matter content, are

\[ R_{12} = 2b^2(e^\nu - 1), \quad R_{34} = 2b^2(e^{-\nu} - 1). \]  (67)

In the considered congruence the SEC amounts to

\[ R_{ab}V^aV^b = -R_{34} = 2b^2(1-e^{-\nu}) < 0. \]

Substituting into (66),

\[ \frac{d\theta}{d\lambda} = 2b^2((1 - \phi^4)^{-1/2} - 1) - 3s\dot{\phi} - \frac{5}{12}s^2(s/\phi^2)^{-1} + \dot{s}\phi. \]  (68)

There appears the term \( s/\phi^2 \), as we have shown this term is zero at \( t = \pi/2 \) only for \( \epsilon = 0 \). Therefore, in the case \( \epsilon = 0 \), \( d\theta/d\lambda \rightarrow -\infty \) and this means that the congruences focus at \( t = \pi/2 \), having there a singularity.

For the cases \( \epsilon \neq 0 \), \( s/\phi^2 \neq 0 \), the BI constant \( b \) can be adjusted to beat the negative terms in the Raychaudhuri equation, avoiding then the convergence of the geodesics.

The same analysis for the Maxwell case points to the existence of a singularity. The Ricci tensor projected in the timelike congruence is

\[ R_{ab}V^aV^b = C^2 = B^2 + E^2 \]

and substituting into the Raychaudhuri equation gives

\[ \frac{d\theta}{d\lambda} = -C^2 - 3s\dot{\phi} - \frac{5}{12}s^2(s/\phi^2)^{-1} + \dot{s}\phi. \]  (69)

There is a positive term coming from \( \theta^2 \); however \( \theta^2 > 0 \), in other words, the right-hand side of equation (69) is negative. This means that sooner or later the congruence will focus at some point, and the effect of the electromagnetic field is to accelerate this fate, since the contribution is also negative: \( -C^2 = -(E^2 + B^2) \).

7. Conclusions

We investigated spacetimes with a four-dimensional group of isometries with a three-dimensional subgroup acting transitively on spacelike hypersurfaces, i.e. spatially homogeneous spacetimes anisotropic in one spatial direction. We address the study of geodesic completeness for solutions to nonlinear electromagnetic field coupled with Einstein equations for metrics conformal to Bianchi G3I, G3II, G3III, G3VIII, G3IX and Kantowski–Sachs. With regard to null geodesics it is well known that they are conformally invariant; i.e. the null geodesics associated with \( g_{ab} \) coincide with those in the conformal space \( \tilde{g}_{ab} = \Omega^2g_{ab} \) with the affine parameter \( \tilde{\lambda} \) for \( \tilde{g}_{ab} \)-geodesics related to the affine parameter \( \lambda \) for \( g_{ab} \)-geodesics by \( d\tilde{\lambda}/d\lambda = c\Omega^2 \), \( c = \) constant. Therefore, the character of completeness or incompleteness of null geodesics applies (whenever the ranges of \( \tilde{\lambda} \) and \( \lambda \) remain the same) to the Bianchi spaces that are conformal to the studied metric (equations (13), (19)). The boundedness of the invariants, or its divergence when \( l = 0 \), applies as well to the Bianchi spaces since the Weyl tensor is unchanged by a conformal transformation.

Two families are distinguished: Bianchi G3I, G3III and Kantowski–Sachs on one side and Bianchi G3II, G3VIII and G3IX on the other.

Bianchi G3I, G3II and Kantowski–Sachs only admit one component of the electromagnetic field, electric or magnetic (not both, interchanging roles by a duality rotation). These spaces show divergence in their invariants at a finite time, moreover, the range of the affine parameter being finite, geodesics are uncomplete. At \( t = 0 \) it is shown that geodesic uncompleteness appears for the case \( \epsilon = 0 \).

Type VIII is geodesically complete and then singularity-free. The space VIII resembles the ones studied by Siklos [26] as extensions of Taub–NUT metrics. Type IX has regular
invariants but geodesics may be incomplete. The same applies to Bianchi G_{3II}: not all geodesics are complete.

From the analysis of the Raychaudhuri equation that governs the convergence of neighbouring geodesics, it is clear that the focusing can be avoided due to the BI field that contributes with a positive term. Our results are not in contradiction with the assertion by Hawking about the necessity of singularity in non-empty spatially homogeneous models, since the timelike convergence condition fails to hold in nonlinear electrodynamics.

In the linear limit (Einstein–Maxwell) the solutions are singular for all cases included; this conclusion is also clear from the Raychaudhuri equation that shows that the congruences converge at some time.

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