RELATION BETWEEN TWO GEOMETRICALLY DEFINED BASES IN REPRESENTATIONS OF $GL_n$

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To the memory of Iosif Donin

1. THE TWO BASES

1.1. Introduction. Following [Gi] and [BG], one can realize the irreducible finite-dimensional representation of $GL_n$, corresponding to a certain Young diagram, in the top cohomology of (the union of) Springer fibers over a nilpotent matrix, whose Jordan decomposition corresponds to this diagram. We will review this construction in Sect. 1.3 below. In particular, the set of irreducible components of these Springer fibers provides a basis for this representation. We call it the Springer basis.

On the other hand, we have the theory of geometric Langlands duality, which realizes the category of finite-dimensional representations of any reductive group $G$ in terms of spherical perverse sheaves on the affine Grassmannian of the Langlands dual group, $Gr_G$. In particular, by taking the top (and, in fact, the only non-zero) cohomology with compact supports of a given irreducible spherical perverse sheaf $IC^\lambda$, corresponding to a dominant coweight $\lambda$, along a semi-infinite orbit $S(\mu) \subset Gr_G$, corresponding to a coweight $\mu$, we obtain a vector space, which is canonically identified with the weight space $V^\lambda(\mu)$, where $V^\lambda$ is the irreducible representation of $G$ with highest weight $\lambda$. Therefore, the set of irreducible components of the intersection of $S(\mu)$ with the support of $IC^\lambda$, provides a basis for $V^\lambda(\mu)$. We call it the Mirković-Vilonen (MV for short) basis.

Therefore, it is natural to ask whether for $G = GL_n$ (in which is case $G$ is also $GL_n$), the two bases for $V^\lambda(\mu)$ coincide. The purpose of this note is to prove this fact.

Let us indicate the strategy of the comparison. First, we interpret each given weight space $V^\lambda(\mu)$ as a multiplicity space of a given finite-dimensional representation of $GL_n$ in the tensor product of several other ones. This set of multiplicities also acquires a basis via the interpretation of tensor product in terms of convolution of spherical perverse sheaves on the affine Grassmannian. One shows (cf. Sect. 1.4) that this basis tautologically coincides with the Springer basis.

Thus, we have to relate the two bases, both of which are defined in terms of the affine Grassmannian. The main idea is to view the two appearances of the affine Grassmannian separately, and in fact to work on the product of two copies of $Gr_{GL_n}$, thought of as $Gr_{GL_n} \times GL_n$.

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1A clarification is in order: in [BG] two such geometric realizations were considered. The first realization is the one considered [Gi], and it has been recently shown to be connected with the one of Nakajima, cf. [Ma, Sa]. The realization considered in the present note is the second one, and it is related to the first one by the Fourier-Deligne transform on the Lie algebra $gl_n$. Therefore, in this paper we do not claim any relation between Nakajima’s realization of finite-dimensional representations and the one of Mirković-Vilonen, which is discussed below.
Finally, we interpret $GL_n \times GL_n$ as a Levi subgroup of $GL_{2n}$, and the comparison of bases becomes the corollary of the computation of the intersection cohomology of the Zastava space, which was carried out in [BFGM].

1.2. Conventions. We will work with varieties over an algebraically closed field $k$ of characteristic 0, but the whole discussion carries over to the characteristic $p$ situation with only minor modifications. The theory of sheaves that we will consider can be either holonomic $D$-modules, or constructible $\ell$-adic sheaves, or, if $k = \mathbb{C}$, constructible sheaves in the analytic topology with characteristic 0 coefficients. In terms of notation, we opt for $\ell$-adic sheaves. We do not know what part of the present discussion carries over to the situation, when coefficients of our sheaves are torsion.

The methods used in this paper rely substantially on the theory of spherical perverse sheaves on the affine Grassmannian. We will not review this theory here, but rather refer the reader to [MV]. In addition, we will assume familiarity with the results of the paper [BFGM].

1.3. The Springer basis. Let $V$ be an $n$-dimensional vector space over $\overline{\mathbb{Q}}_\ell$, and we will consider $GL(V) = GL_n^\infty$ as an algebraic group over $\overline{\mathbb{Q}}_\ell$. We fix a decomposition of $V$ as a direct sum of 1-dimensional subspaces $V = V_1 \oplus \ldots \oplus V_n$. This defines a maximal torus $\mathbb{G}_m^\times n \simeq T^\vee \subset GL(V)$.

For a non-negative integer $d$ we will be interested in representations of $GL(V)$ that appear as direct summands in $V^\otimes d$. We will call such representations and their highest weights $d$-positive. The set of weights $\mu$ of $T^\vee$ that appear in $V^\otimes d$ identifies with the set of $n$-tuples of non-negative integers $\overrightarrow{d} = (d_1, \ldots, d_n)$ such that $\sum d_i = d$; it will be denoted by $\Lambda^d$. The subset of dominant coweights among $\Lambda^d$ will be denoted by $\Lambda^d_{d^+}$; explicitly it consists of $n$-tuples $\overrightarrow{d}$ satisfying $d_1 \geq \ldots \geq d_n$. As is well-known, the set $\Lambda^d_{d^+}$ is in bijection with each of the following sets:

(a) Nilpotent conjugacy classes in the Lie algebra $\mathfrak{gl}_d$ over $k$ with no more than $n$ Jordan blocks.
(b) Representations of the symmetric group $\Sigma_d$ over $\overline{\mathbb{Q}}_\ell$, corresponding to Young diagrams with no more than $n$ rows.

For $\lambda \in \Lambda^d_{d^+}$, we denote by $V^\lambda$ the corresponding irreducible representation of $GL(V)$, normalized so that its highest weight line is identified with $\otimes V_i^{d_i}$. In particular, for $\lambda$ of the form $(d, 0, \ldots, 0)$, the corresponding $V^\lambda$ is the symmetric power $\text{Sym}^d(V)$. We have

\begin{equation}
V^\lambda \simeq (\rho^\lambda \otimes V^\otimes d)^{\Sigma_d},
\end{equation}

where $\rho^\lambda$ is the corresponding irreducible representation of $\Sigma_d$.

Let us fix $\lambda \in \Lambda^d_{d^+}$, and $\mu \in \Lambda^d$, with $\mu = \overrightarrow{\lambda} = (d'_1, \ldots, d'_n)$. By $V^\lambda(\mu)$ we will denote the $\mu$-weight space in $V^\lambda$. Let us recall the construction of a basis in $V^\lambda(\mu)$, following [BG].

Let $\text{Fl}(\mu)$ denote the partial flag variety, classifying flags $0 = M_0 \subset M_1 \subset \ldots \subset M_n = k^d$ in the standard $d$-dimensional space $k^d$, such that $\dim(M_i/M_{i-1}) = d'_i$. We will denote by $A^\lambda$ an arbitrary element in the nilpotent conjugacy class in $\mathfrak{gl}_d(k)$ corresponding to $\lambda$. Let $\text{Fl}(\mu)^{A^\lambda}$ be the Springer fiber over $A^\lambda \subset \mathfrak{gl}_d$, i.e., the subscheme of fixed points of $A^\lambda$ acting naturally on $\text{Fl}(\mu)$. As is well-known, the dimension of this scheme is $d^\lambda = \Sigma d_i \cdot i - d$.

According to [BG], $H^{2d^\lambda}(\text{Fl}(\mu)^{A^\lambda}, \overline{\mathbb{Q}}_\ell) \otimes \left( \bigotimes_i V_i^{d'_i} \right)$ identifies naturally with $V^\lambda(\mu)$. Hence, the set $B_{spr}^\lambda(\mu)$ of irreducible components of $\text{Fl}(\mu)^{A^\lambda}$ provides a basis for $V^\lambda(\mu)$, tenesorized by the inverse of the line $\otimes V_i^{d'_i}$. (This basis does not depend on the choice of an element $A^\lambda$ in the corresponding nilpotent orbit, since its centralizer in $GL_d$ is connected.)
1.4. The Springer basis via convolution. Fix another \( n \)-dimensional vector space \( E \) over \( \mathbb{Q}_d \), and consider the corresponding group \( GL(E) \). For each \( \lambda \in \Lambda^{d,+} \), we set \( E^\lambda \) to be the irreducible representation of \( GL(E) \), defined as

\[
\text{Hom}_{\Sigma_d}(\rho^\lambda, E^{\otimes d}),
\]

where \( \rho^\lambda \) is as in \([1] \).

Consider the \( \mathbb{Q}_d \)-vector space \( \text{Sym}^d(V \otimes E) \) as a representation of \( GL(V) \times GL(E) \). We have:

\[
\text{Sym}^d(V \otimes E) = \text{Hom}_{\Sigma_d}(\mathbb{Q}_d, (V \otimes E)^{\otimes d}) \simeq \text{Hom}_{\Sigma_d \times \Sigma_d}(\text{Ind}_{\Sigma_d}^{\Sigma_d}(\mathbb{Q}_d), (V \otimes E)^{\otimes d}) \simeq \bigoplus_{\lambda \in \Lambda^{d,+}} (V^{\otimes d} \otimes \rho^\lambda)^{\Sigma_d} \otimes \text{Hom}_{\Sigma_d}(\rho^\lambda, E^{\otimes d}) \simeq \bigoplus_{\lambda \in \Lambda^{d,+}} V^\lambda \otimes E^\lambda.
\]

In particular, we obtain an isomorphism of \( GL(V) \)-representations:

\[
(2) \quad \text{Hom}_{GL(E)}(E^\lambda, \text{Sym}^d(V \otimes E)) \simeq V^\lambda.
\]

Fix now a weight \( \mu = (d'_1, \ldots, d'_n) \) as above. We have the binomial formula

\[
(3) \quad \text{Hom}_{T^\mu} \left( \bigotimes_i V_i^{d'_i}, \text{Sym}^d(V \otimes E) \right) \simeq \text{Sym}^{d'_1}(E) \otimes \ldots \otimes \text{Sym}^{d'_n}(E).
\]

Hence, we obtain that

\[
(4) \quad V^\lambda(\mu) \simeq \text{Hom}_{GL(E)}(E^\lambda, \text{Sym}^{d'_1}(E) \otimes \ldots \otimes \text{Sym}^{d'_n}(E)) \otimes \left( \bigotimes_i V_i^{d'_i} \right).
\]

Let now \( \text{Gr}_E \) be the affine Grassmannian of the group dual to \( GL(E) \). We will think of it as the ind-scheme, classifying lattices \( M \) in \( k^n((t)) \). We let \( \text{Gr}_E^- \) denote the “negative” part, i.e., the subscheme, consisting of lattices that contain the standard lattice \( M_0 = k^n[[t]] \). For a non-negative integer \( d \), we let \( \text{Gr}_E^{d,-} \) denote the connected component of \( \text{Gr}_E^- \) corresponding to lattices such that \( \dim_k(M/M_0) = d \).

We will denote by \( \text{Conv}^k(\text{Gr}_E) \) the \( k \)-fold convolution diagram

\[
\xymatrix{ \text{Gr}_E \ast \ldots \ast \text{Gr}_E \ar[r]^{p_k} & \text{Gr}_E; }
\]

which is in fact isomorphic to the \( k \)-th direct power of \( \text{Gr}_E \). For a collection of non-negative integers \( \vec{d} = d'_1, \ldots, d'_k \), we will denote by \( \text{Conv}^\vec{d}^- \text{(Gr}_E) \) the subscheme of \( \text{Conv}^k(\text{Gr}_E) \), which classifies \( k \)-tuples of lattices \( (M_1, \ldots, M_k) \), such that \( M_{i-1} \subset M_i \) and \( \dim_k(M_i/M_{i-1}) = d'_i \).

Let \( \text{Sph}_E \) denote the category of spherical perverse sheaves on \( \text{Gr}_E \). For a dominant weight \( \lambda \) of \( GL(E) \) we will denote by \( \text{Gr}_E^\lambda \) (resp., \( \text{Gr}_E^- \)) the corresponding Schubert variety (resp., its closure), i.e., the \( GL_E \)-orbit of the diagonal matrix with entries \( (t^{-d_1}, \ldots, t^{-d_k}) \). It is easy to see that \( \lambda \) belongs to \( \Lambda^{d,+} \) if and only if \( \text{Gr}_E^\lambda \) is contained in \( \text{Gr}_E^{d,-} \). Let \( \text{IC}_E^\lambda \) denote the corresponding irreducible object of \( \text{Sph}_E \).

For \( \lambda \) of the form \( (d', 0, \ldots, 0) \), we have \( \text{Gr}_E^\lambda = \text{Gr}_E^{d,-} \), and it is rationally smooth, i.e., the IC sheaf on it is the shifted constant sheaf, and we will use for it the short-hand notation \( \text{IC}_E^{d'} \).

Let us recall now the basic fact that the category \( \text{Sph}_E \) is a tensor category under convolution, denoted \( \mathcal{F}_1 \ast \mathcal{F}_2 \rightarrow \mathcal{F}_1 \ast \mathcal{F}_2 \), and as such it is equivalent to the category of finite-dimensional representations of \( GL(E) \), once we identify \( E \) with the standard \( n \)-dimensional subspace. Under this equivalence, the object \( \text{IC}_E^\lambda \) goes over to the irreducible representation with highest weight \( \lambda \), with the trivialized highest weight line. In particular, for \( \lambda \in \Lambda^{d,+} \), the corresponding representation of \( GL(E) \) identifies with \( E^\lambda \).
Hence, in view of (4), for $\lambda \in \Lambda^{d,+}$,

$$V^\lambda(\mu) \simeq \text{Hom}_{Sp(k)}(\text{IC}_{E}^{\lambda}, \text{IC}_{E}^{d_1} \otimes \cdots \otimes \text{IC}_{E}^{d_s}) \otimes \left( \bigotimes_i V_i^{d_i} \right).$$

(5)

The latter Hom space can be interpreted as follows. Choose a point $M \in \text{Gr}_E^{\lambda}$, and consider the scheme

$$p_n^{-1}(M) \cap \text{Conv}^d(\text{Gr}_E) \subset \text{Conv}^d(\text{Gr}_E).$$

This scheme is of dimension $d^\lambda$ and its top ($= 2d^\lambda$-th) cohomology identifies canonically with the above Hom. However, it is easy to see that this scheme identifies also with the Springer fiber $\text{Fl}(\mu)^{\lambda}$. Indeed, let us denote by $\mathcal{N}_n^{d}$ the variety of nilpotent $d \times d$-matrices, which have no more than $n$ Jordan blocks, and let $\mathcal{N}_n^{d} \subset \mathcal{N}_n^{d} \times \text{Fl}(\mu)$ be the sub-scheme consisting of pairs $(A \in \mathcal{N}_n^{d}, (0 = M_0 \subset M_1 \subset \cdots \subset M_n = k^d) \in \text{Fl}(\mu))$, such that $A(M_i) \subset M_i$. Then we have a natural smooth and surjective map of stacks $\text{Gr}_{E}^{d,-} \to \mathcal{N}_n^{d}/\text{GL}_d$, and a Cartesian diagram

$$\text{Conv}^d(\text{Gr}_E) \longrightarrow \mathcal{N}_n^{d}/\text{GL}_d$$

$$\downarrow$$

$$\text{Gr}_{E}^{d,-} \longrightarrow \mathcal{N}_n^{d}/\text{GL}_d.$$
Theorem 1.6. There exists a natural bijection $B^{\lambda \Phi}_{Spr}(\mu) \simeq B^{\lambda \Lambda}_{MV}(\mu)$, such that the corresponding basis vectors in $V^{\lambda}(\mu) \otimes \left( \bigotimes V_{i}^{d_{i}} \right)^{-1}$ coincide.

2. The construction

2.1. The proof of Theorem 1.6 will be based on a construction involving the affine Grassmannian of the group dual to $GL(V) \times GL(E)$.

Consider the product $Gr_{V}^{d_{+}} \times Gr_{E}^{d_{-}}$, and consider now the scheme, denoted $P^{d}_{loc}$, that classifies the data of triples $(M', M, \alpha)$, where $(M', M) \in Gr_{V}^{d_{+}} \times Gr_{E}^{d_{-}}$, and $\alpha$ is an isomorphism of $k[[t]]$-modules $M_{0}'/M' \simeq M/M_{0}$. Let $\pi$ denote the natural projection $P^{d}_{loc} \to Gr_{V}^{d_{+}} \times Gr_{E}^{d_{-}}$, and $\pi_{V}$ (resp., $\pi_{E}$) the further projection onto the $Gr_{V}^{d_{+}}$ (resp., $Gr_{E}^{d_{-}}$) factor.

Note that the map $\pi_{V}$ is smooth: indeed, the fiber of $\pi_{V}$ over a given point $M' \in Gr_{V}^{d_{+}}$ is the set of extensions $0 \to M_{0} \to M \to M_{0}'/M' \to 0$, which are torsion-free as $k[[t]]$-modules, i.e., it embeds into $\text{Ext}^{1}(M_{0}'/M', M_{0})$.

For $\lambda \in \Lambda^{d_{+}}$ let $\Pi^{\lambda}_{loc}$ denote the subscheme $\pi_{V}^{-1}(\text{Gr}_{V}^{\lambda}) \subset P^{d}_{loc}$, and for a weight $\mu$, let us denote by $P^{d}_{loc}(\mu)$ the subscheme $\pi_{V}^{-1}(S(\mu))$. Finally, let us denote by $\Pi^{\lambda}_{loc}(\mu)$ the intersection $\Pi^{\lambda}_{loc} \cap P^{d}_{loc}(\mu)$.

Proposition 2.2. The map $\pi_{E}$, restricted to $P^{d}_{loc}(\mu)$, factors naturally through a map $\pi_{E}^{\text{Conv}} : P^{d}_{loc}(\mu) \to \text{Conv}_{-}(\text{Gr}_{E})$, followed by $p_{n}$, where $\mu = \sum (d_{i}, \ldots, d_{n})$.

Proof. For $M' \in Gr_{V}^{d_{+}}$, let $N'$ denote the quotient $M'/M'$, viewed as a $k[[t]]$-module. The flag $0 = M_{0}' \subset M_{1}' \subset \ldots \subset M_{n}' = M'$ induces on $N'$ a flag of $k[[t]]$-modules $0 = N'^{0} \subset N'^{1} \subset \ldots \subset N'^{n} = N'$, such that $\dim_{k}(N'^{i}/N'^{i-1}) = d_{i}'$. Therefore, for any $(M', M, \alpha) \in \Pi^{\lambda}_{loc}(\mu)$, we obtain a sequence of lattices

$$M_{0} \subset M_{1} \subset \ldots \subset M_{n} = M$$

with $M_{i}/M_{i-1} \simeq N'^{i}/N'^{i-1}$, and hence of dimension $d_{i}'$, i.e., we obtain the sought-for point of $\text{Conv}_{-}(\text{Gr}_{E})$.

Lemma 2.3. The map $\pi_{E}^{\text{Conv}} : P^{d}_{loc}(\mu) \to \text{Conv}_{-}(\text{Gr}_{E})$ constructed above is smooth. Every fiber is isomorphic to non-empty open subset of a vector space of dimension $\sum_{i=1, \ldots, n} d_{i}'(n-i+1)$.

Proof. The proof is a corollary of the following observation. Let $N'^{\bullet}$ be a flag of torsion modules over $k[[t]]$, with the $i$-th subquotient of dimension $d_{i}'$. Let $M'^{\bullet}_{0}$ be a flag of locally free $k[[t]]$-modules, with the $i$-th subquotient being locally free of rank 1. (We can consider the above data over an arbitrary scheme of parameters.)

Consider the scheme classifying filtered maps $M'^{\bullet}_{0} \to N'^{\bullet}$. This is a vector scheme of dimension $\sum_{i=1, \ldots, n} d_{i}'(n-i+1)$. The fibers of the map of interest are isomorphic to an open subscheme in the above scheme of filtered maps, corresponding to the condition that the maps $M'^{i}_{0}/M'^{i-1}_{0} \to N'^{i}/N'^{i-1}$ are surjective for all $i = 1, \ldots, n$.

Let us denote by $B^{\lambda}_{Univ}(\mu)$ the set of irreducible components of the scheme $\Pi^{\lambda}_{loc}(\mu)$.
Proposition 2.4. We have canonical bijections
\[ B^\lambda_{MV}(\mu) \leftrightarrow B^\lambda_{niv}(\mu) \to B^\lambda_{Spr}(\mu) \]

Proof. The map \( B^\lambda_{niv}(\mu) \to B^\lambda_{Spr}(\mu) \) comes from the map \( \pi^\text{conv}_E \), constructed above. The fact that it defines a bijection on the corresponding sets of irreducible components follows from Lemma 2.3.

Let us now construct the map \( B^\lambda_{niv}(\mu) \to B^\lambda_{MV}(\mu) \). Note that if \((\mathcal{M}', \mathcal{M}, \alpha) \in \mathcal{P}d_{loc}^d\), with \( \mathcal{M}' \in \text{Gr}^\lambda \) and \( \mathcal{M} \in \text{Gr}^\lambda_E \), then \( \lambda = \lambda' \), because a Schubert cell contained in \( \text{Gr}^{d,+} \) is determined by the isomorphism type of the quotient \( k[[t]] \)-module.

Therefore, the morphism \( \pi_V \) restricted to \( \mathcal{P}^\lambda_{loc} \) maps to \( \text{Gr}^\lambda \), and is a smooth map with non-empty fibers. Therefore, this map induces a bijection between the set of irreducible components of \( \mathcal{P}^\lambda_{loc}(\mu) \) and that of \( S(\mu) \cap \text{Gr}^\lambda \).

2.5. The above proposition identifies \( B^\lambda_{MV}(\mu) \) and \( B^\lambda_{Spr}(\mu) \) as sets. The rest of the paper is devoted to the identification of the corresponding basis vectors.

Lemma 2.6. The complex \( \pi(Q_{\ell}) \) is concentrated in the perverse cohomological degrees \( \leq 2 \cdot d \cdot n \).

The lemma follows from the interpretation of the scheme \( \mathcal{P}^d_{loc} \) as the central fiber of the Zastava space, explained in the next section, combined with Proposition 5.7 of [BFGM].

Let \( \text{Sph}_{V \times E} \) denote the category of spherical perverse sheaves on the affine Grassmannian \( \text{Gr}_V \times \text{Gr}_E \). This is a tensor category, equivalent to the category of representations of the group \( GL(V) \times GL(E) \). The top \((=2 \cdot d \cdot n)\) cohomology perverse sheaf of the above lemma is clearly spherical; let us denote it by \( \mathcal{F}^d_{loc} \).

For a weight \( \mu \) let us denote by \( q(\mu) \) the projection \( S(\mu) \times \text{Gr}_E \to \text{Gr}_E \). For \( \mathcal{F} \in \text{Sph}_{V \times E} \) consider its (usual) inverse image to \( S(\mu) \times \text{Gr}_E \), followed by the direct image with compact supports with respect to \( q(\mu) \), cohomologically shifted by \( [d'(\mu)] \). Slightly abusing the notation, we will denote the resulting functor \( \text{Sph}_{V \times E} \to \text{Sph}_E \) by \( \mathcal{F} \mapsto q(\mu)(\mathcal{F})[d'(\mu)] \). (This is indeed a perverse sheaf, i.e., no lower perverse cohomology appears, by [Mav], Theorem 3.5.)

In terms of the equivalence \( \text{Sph}_{V \times E} \simeq \text{Rep}(GL(V) \times GL(E)) \), the above functor corresponds to sending a representation to the weight \( \mu \) component with respect to the maximal torus of \( GL(V) \), i.e., for \( \mu = \overrightarrow{d} = (d'_1, ..., d'_n) \) this functor sends a representation \( W \) of \( GL(V) \times GL(E) \) to the \( GL(E) \)-representation \( \text{Hom}_{\text{Rep}} \left( \bigotimes_i V_i^{d'_i}, W \right) \).

Using Proposition 2.2 we obtain that there exists a canonical isomorphism
\[ q(\mu)(\mathcal{F}^d_{loc})[d'(\mu)] \simeq \text{IC}^{d'_1}_{E} \ast ... \ast \text{IC}^{d'_n}_{E} \tag{6} \]

The following assertion will be proved in the next section:

Theorem 2.7.
(a) Under the equivalence \( \text{Rep}(GL(V) \times GL(E)) \simeq \text{Sph}_{V \times E} \) the object \( \mathcal{F}^d_{loc} \) goes over to \( \text{Sym}^d(V \otimes E) \).
(b) Under this identification, the isomorphism of \( \text{IC} \) coincides with that of \( \mathcal{K} \).

2.8. We will now deduce Theorem 2.7 from Theorem 2.6. By (2) and Theorem 2.6(a),
\[ \mathcal{F}^d_{loc} \simeq \bigoplus_{\lambda \in \Lambda^{d,+}} \text{IC}^\lambda_{V} \boxtimes \text{IC}^\lambda_{E} \tag{7} \]
Consider the vector space
\[ \tilde{V}^\lambda(\mu) := \text{Hom}_{Sph}^\lambda \left( \text{IC}_{E}^\lambda, \Omega(\mu)!((\mathcal{F}_{loc}^d)[2d!(\mu)]) \right). \]

On the one hand, by (7), we have:
\[ \tilde{V}^\lambda(\mu) \cong H_c^d(\mu) (S(\mu), \text{IC}_{V}^\lambda) \cong V^\lambda(\mu) \otimes \left( \bigotimes_i V_i^d \right)^{-1}. \]

On the other hand, by (6) and (4),
\[ V^\lambda(\mu) \cong \text{Hom}_{Sph}^\lambda \left( \text{IC}_{E}^\lambda, \text{IC}_{E}^d \times \ldots \times \text{IC}_{E}^d \right) \cong V^\lambda(\mu) \otimes \left( \bigotimes_i V_i^d \right)^{-1}. \]

However, by Theorem 2.7(b), the resulting two isomorphisms \( \tilde{V}^\lambda(\mu) \cong V^\lambda(\mu) \otimes \left( \bigotimes_i V_i^d \right)^{-1} \) coincide.

By construction, the set \( B^\lambda_{Univ}(\mu) \) of irreducible components of \( \mathcal{F}_{loc}^\lambda(\mu) \) defines a basis in \( \tilde{V}^\lambda(\mu) \). We claim that under the identification of (8), this basis goes over to the basis given by \( B^\lambda_{Sph}(\mu) \), where the bijections on the level of underlying sets are given by Proposition 2.4. Clearly, this would imply the assertion of the Theorem 1.6.

The first assertion follows readily from the fact that that the map
\[ \pi_V : \mathcal{F}_{loc}^\lambda(\mu) \to \left( S(\mu) \cap \text{Gr}_{V}^\lambda \right), \]
considered in the proof of Proposition 2.4, is a smooth fibration. The second assertion follows from the corresponding property of the morphism
\[ \pi^{\text{Conv}}_E : \mathcal{F}_{loc}^\lambda(\mu) \to \left( \text{Conv}_{E}(\text{Gr}_{E}) \cap p_{k}^{-1}(\text{Gr}_{E}) \right), \]
given by Lemma 2.23.

3. Proof of Theorem 2.7 and Zastava spaces

3.1. Let \( G \) be the group \( GL_{2n} \), thought of as the dual group of \( GL(V \oplus E) \), let \( P \) be the maximal parabolic, whose Levi quotient \( M \) is \( GL_n \times GL_n \), thought of as the group dual to \( GL(V) \times GL(E) \), and the unipotent radical \( N \cong \text{Mat}_{n,n} \). Following [BFGM], Sect. 2.2, we will consider the enhanced Zastava space, corresponding to the pair \((G,P)\). It depends on a parameter \( d \in \mathbb{Z}^+ \), and we will denote it here by \( \mathcal{F}^d_{\text{glob}} \). Let us remind the definition in the form adapted to the present situation:

Let \( X \) be a smooth algebraic curve (not necessarily complete). Let \( M_0 \) and \( M_0' \) be the trivial rank \( n \)-vector bundles on \( X \). The scheme \( \mathcal{F}^d_{\text{glob}} \) classifies the data of
\[ (0 \to M_0 \to A \to M_0' \to 0, M, M', i, j), \]
where \( 0 \to M_0 \to A \to M_0' \to 0 \) is a short exact sequence of vector bundles; \( M, M' \) are rank \( n \) vector bundles; \( i \) and \( j \) are maps of coherent sheaves
\[ i : M' \to A \text{ and } j : A \to M, \]
such that \( j \circ i = 0 \), and such that the compositions
\[ \beta := j \circ i_0 : M_0 \to M \text{ and } \beta' := j_0 \circ i : M' \to M_0' \]
are both injective. Finally, we require that resulting meromorphic isomorphism \( \det(A) \simeq \det(M) \otimes \det(M') \) be globally regular.

In other words, a point of \( \mathbb{P}^d_{\text{glob}} \) can be thought of as a diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & M_0 & \rightarrow & A & \rightarrow & M'_0 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \downarrow & \\
 & & & & M. & & & & \\
\end{array}
\]

In particular, we have an isomorphism \( A/(M_0 \oplus M') \simeq M'_0/M' \) and a map \( A/(M_0 \oplus M') \rightarrow M/M_0 \). We will denote by \( \alpha \) the resulting map \( M'_0/M' \rightarrow M/M_0 \).

Let Mod\(_d^G\) denote the scheme that classifies the data of pairs \((M, \beta)\), where \( M \) is a rank \( n \) vector bundle, and \( \beta \) is an injective map of coherent sheaves \( M_0 \hookrightarrow M \), such that the quotient torsion sheaf has length \( d \). We have a natural map \( s_E : \text{Mod}^d_E \rightarrow X(d) \). Similarly, we introduce the scheme Mod\(_d^V\), which classifies lower modifications of \( M'_0 \) of length \( d \). We obtain a natural map \( \pi : \mathbb{P}^d_{\text{glob}} \rightarrow \text{Mod}^d_V \times X(d) \text{Mod}^d_E \).

Note that the data of \(((M, \beta), (M', \beta'), \alpha)\), such that \( s_E(M, \beta) = s_V(M', \beta') \in X(d) \), is equivalent to a point of \( \mathbb{P}^d_{\text{glob}} \). Indeed, one reconstructs \( A \) as the preimage under the map \( M'_0 \oplus M \rightarrow M'_0/M' \oplus M/M_0 \) of the graph of \( M'_0/M' \), embedded by means of \( \text{id} \oplus \alpha \).

Let \( \mathcal{P}^d_{\text{glob}} \) denote the open subscheme of \( \mathbb{P}^d_{\text{glob}} \) corresponding to the condition that the map \( \alpha \) is an isomorphism. Note that this is equivalent to each of the following two conditions: that \( M' \rightarrow A \) is a bundle map, or that \( A \rightarrow M \) is surjective. Therefore, the scheme \( \mathcal{P}^d_{\text{glob}} \) classifies the data of a \( G \)-bundle on \( X \), together with a reduction to \( P \) and a reduction to \( N \), which are transversal at the generic point of the curve. We will denote by \( \pi \) the restriction of the morphism \( \mathcal{P} \) to this open subscheme.

Let now \( x \in X \) be some chosen point, and \( t \) a local coordinate. Note that the preimage of \( d \cdot x \in X(d) \) under the map \( \text{Mod}^d_V \times \text{Mod}^d_E \rightarrow X(d) \) identifies with \( \text{Gr}^{d,+}_V \times \text{Gr}^{d,-}_E \); and the preimage of the latter subscheme under \( \mathcal{P}^d_{\text{glob}} \rightarrow \text{Mod}^d_V \times \text{Mod}^d_E \) identifies naturally with the scheme \( \mathcal{P}^d_{\text{loc}} \), introduced in the previous section.

3.2. Let \( \text{IC}^d_{\mathcal{P}^d_{\text{glob}}} \) denote the intersection cohomology sheaf on \( \mathcal{P}^d_{\text{glob}} \). According to [BFGM], Cor. 3.8, the open subset \( \mathcal{P}^d_{\text{glob}} \) is smooth, so \( \text{IC}^d_{\mathcal{P}^d_{\text{glob}}} \simeq \mathbb{Q}[2 \cdot d \cdot n] \). Consider the direct image

\[
\mathcal{P}^d_{\text{glob}} := \pi_!(\text{IC}^d_{\mathcal{P}^d_{\text{glob}}}).
\]

The following is the result of application of the machinery of [BFGM], Theorem 4.5, Proposition 5.2 and Theorem 5.9 of loc.cit. to our pair \((G, P)\):
Theorem 3.3.
(a) $\mathcal{F}_\text{glob}^d$ is isomorphic to the intersection cohomology sheaf of $\text{Mod}_V^d \times \text{Mod}_E^d$.
(b) The natural map $\pi_1(\mathbb{F}[2 \cdot d \cdot n])|_{d \cdot x} \to \mathcal{F}_\text{glob}^d|_{d \cdot x}$ induces an isomorphism
$$\mathcal{F}_\text{loc}^d \simeq \mathcal{F}_\text{glob}^d|_{d \cdot x}[-d].$$

Theorem 3.3 readily implies point (a) of Theorem 2.7. Indeed, let $\mathcal{O}^{(d)}$ denote the open subset of $X^{(d)}$, corresponding to multiplicity-free divisors, and let $\mathcal{O}^d$ be its preimage in the $d$-th Cartesian power of $X$. We have:

$$\text{(Mod}_V^d \times \text{Mod}_E^d)_X^{(d)} \times \mathcal{O}^d \simeq (\text{Mod}_V^d \times \text{Mod}_E^d)_X^{(d)} \times \mathcal{O}^d.$$

For each point $x \in X$, the object of $\text{Rep}(GL(V) \times GL(E))$, corresponding to $\text{IC}_{\text{Mod}_V^d \times \text{Mod}_E^d}_X[-1]|_x \simeq \mathbb{F}[\text{Gr}_V^d \times \text{Gr}_E^d][2(n-1)] \in \text{Sph}_V \times E$ is $V \otimes E$. Now the assertion of Theorem 2.7(a) follows from Lemma 4.3 of [BFGM].

3.4. It remains to prove point (b) of Theorem 2.7. We fix a flag $0 = M_{0}^0 \subset M_{1}^0 \subset \ldots \subset M_{n}^0 = M_{0}^0$ of sub-bundles in $M_{0}^0$ with $M_{0}^0$ being of rank $i$. For a weight $\mu = (d_1', \ldots, d_n')$ let $S_{\text{glob}}(\mu)$ be a locally closed subscheme of $\text{Mod}_V^d$ consisting of pairs $(M', \beta : M' \hookrightarrow M_{0}^0)$, such that the induced filtration $M'$ on $M'$ is such that each subquotient $(M_{0}^0/M_{i-1}^0)(M_{i}/M_{i-1}^0)$, which is a torsion sheaf on $X$, is of length $d_i'$. We have a natural map $\mathcal{F}(\mu) : S_{\text{glob}}(\mu) \rightarrow X^\mu := X^{(d_1')} \times \ldots \times X^{(d_n')}$. 

By analogy with the local situation, we will denote by $\mathcal{F} \mapsto q(\mu)_!(\mathcal{F})|_{d'(\mu)}$ the functor from the derived category on $\text{Mod}_V^d \times \text{Mod}_E^d$ to that on $X^\mu \times \text{Mod}_E^d$, given by restriction to $S_{\text{glob}}(\mu) \times \text{Mod}_E^d$, followed by the $[d'(\mu)]$-shifted direct image onto $X^\mu \times \text{Mod}_E^d$.

Let us also consider the scheme $\text{Conv}(\text{Mod}_E)$, which classifies the data of sequences $M_0 \subset M_1 \subset \ldots \subset M_n = M$ of rank $n$ vector bundles, embedded into one-another as coherent sheaves, such that each quotient $M_i/M_{i-1}$ is of length $d_i'$. We will denote by $p_\mu$ the forgetful map $\text{Conv}(\text{Mod}_E) \rightarrow \text{Mod}_E^d$. By remembering the supports of the subquotients, we obtain a map $\text{Conv}(\text{Mod}_E) \rightarrow X^\mu$.

Let $\hat{X}^\mu$ denote the preimage of this open subset under $X^\mu \rightarrow X^{(d)}$. Note that we have natural isomorphisms

$$(10) \quad \hat{X}^\mu \times \text{Mod}_E^d \simeq \prod_{i=1, \ldots, n} \hat{X}^{(d_i')} \times \text{Mod}_E^d \simeq \hat{X}^\mu \times \text{Conv}(\text{Mod}_E).$$

Using [Mum], Theorem 3.6, and Theorem 3.3(a) we obtain that $q(\mu)_!(\mathcal{F}_\text{glob}^d)|_{d'(\mu)}$ is a perverse sheaf on $X^\mu \times \text{Mod}_E^d$, and it equals the Goresky-MacPherson extension of its restriction to the open subscheme $\hat{X}^\mu \times \text{Mod}_E^d$.

The latter restriction identifies, in terms of the isomorphism of (10), with the constant perverse sheaf on $\hat{X}^\mu \times \text{Conv}(\text{Mod}_E)$. By the smallness of the map $p_\mu : \text{Conv}(\text{Mod}_E) \rightarrow \text{Mod}_E^d$. 


Mod\(^d_E\), we obtain an isomorphism
\[
q(\mu)_!(\mathcal{F}_\text{glob}^d[\mu]) \simeq (p_n)_!(\mathbb{Q}_\ell[d \cdot n]).
\]
By restricting this isomorphism to the preimage of \(d \cdot x \in X^{(d)}\), we obtain an identification of \(q(\mu)_!(\mathcal{F}_\text{loc}^d[\mu])\) with \(\mathcal{IC}_E^{d' \ast} \ast \mathcal{IC}_E^{d''}\), which coincides with that coming from \([3]\), by the construction of the commutativity constraint on \(Sph_E\) via fusion.

Let us denote by \(P^d_\text{glob}(\mu)\) the preimage of \(S^d_\text{glob}(\mu)\) under the natural projection \(P^d_\text{glob} \to \text{Mod}^d_E\). As in Proposition \([2.2]\) and Lemma \([2.3]\) the map
\[
\pi_E : P^d_\text{glob}(\mu) \to S^d_\text{glob}(\mu) \times \text{Mod}^d_E \to \text{Mod}^d_E
\]
factors naturally through a map
\[
\pi^\text{Conv}_E : P^d_\text{glob}(\mu) \to S^d_\text{glob}(\mu) \times \text{Conv}^{\pi}_E(\text{Mod}_E) \to \text{Mod}^d_E,
\]
followed by \(p_n\). The top perverse cohomology of \((\pi^\text{Conv}_E)_!(\mathbb{Q}_\ell)\) is the constant perverse sheaf on \(\text{Conv}^{\pi}_E(\text{Mod}_E)\). This induces another isomorphism
\[
q(\mu)_!(\mathcal{F}_\text{glob}^d[\mu]) \simeq (p_n)_!(\mathbb{Q}_\ell[d \cdot n]).
\]
The restriction of this isomorphism on the preimage of \(d \cdot x \in X^{(d)}\) gives rise to the isomorphism of \([8]\).

However, the above two isomorphisms between \(q(\mu)_!(\mathcal{F}_\text{glob}^d[\mu])\) and \((p_n)_!(\mathbb{Q}_\ell[d \cdot n])\) coincide, because this is evidently true over the open subscheme \(\mathcal{O}^n \times \text{Mod}^d_E\). Therefore, their restrictions to the fibers over \(d \cdot x \in X^{(d)}\) coincide too, which is what we had to show.

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References

[BFGM] A. Braverman, M. Finkelberg, D. Gaitsgory and I. Mirković, Intersection cohomology of Drinfeld’s compactifications, Selecta Mathematica N.S. 8 (2002), 381–418.
[BG] A. Braverman and D. Gaitsgory, On Ginzburg’s Lagrangian construction of representations of GL\(_n\), Math. Res. Lett. 6 (1999), 195–201.
[Gi] Y. Ginzburg, The Lagrangian construction of the enveloping algebra \(U(sl_n)\), C.R. Acad. Sci Paris, Sér I Math 312 (1991), 907–912.
[Ma] A. Maffei, Quiver varieties of type A, Comment. Math. Helv. 80 (2005), 1–27.
[MiVi] I. Mirković and K. Vilonen, Geometric Langlands duality and representations of algebraic groups over commutative rings, math.RT/0401222
[Sa] A. Savage, On two geometric constructions of \(U(sl_n)\) and its representations, math.RT/0411105

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