\textbf{C}^{1,\alpha} \textit{REGULARITY OF SOLUTIONS TO PARABOLIC MONGE-AMP\'ERE EQUATIONS}

PANAGIOTA DASKALOPOULOS* AND OVIDIU SAVIN**

\textbf{Abstract.} We study interior $C^{1,\alpha}$ regularity of viscosity solutions of the parabollic Monge-Ampère equation

$$u_t = b(x, t) \left( \det D^2 u \right)^p,$$

with exponent $p > 0$ and with coefficients $b$ which are bounded and measurable. We show that when $p$ is less than the critical power $\frac{1}{n-2}$ then solutions become instantly $C^{1,\alpha}$ in the interior. Also, we prove the same result for any power $p > 0$ at those points where either the solution separates from the initial data, or where the initial data is $C^{1,\beta}$.

1. \textbf{Introduction}

In this paper we investigate interior regularity of viscosity solutions of the parabollic Monge-Ampère equation

$$u_t = b(x, t) \left( \det D^2 u \right)^p,$$

with exponent $p > 0$ and with coefficients $b$ which are bounded measurable and satisfy

$$\lambda \leq b(x, t) \leq \Lambda$$

for some fixed constants $\lambda > 0$ and $\Lambda < \infty$. We assume that the function $u$ is convex in $x$ and increasing in $t$.

Equations of the form of (1.1) appear in geometric evolution problems and in particular in the motion of a convex $n$-dimensional hyper-surface $\Sigma^t$ embedded in $\mathbb{R}^{n+1}$ under Gauss curvature flow with exponent $p$, namely the equation

$$\frac{\partial P}{\partial t} = K^p N$$

where each point $P$ moves in the inward direction $N$ to the surface with velocity equal to the $p$-power of its Gaussian curvature $K$. If we express the surface $\Sigma^t(t)$

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locally as a graph \( x_{n+1} = u(x, t) \), with \( x \in \Omega \subset \mathbb{R}^n \), then the function \( u \) satisfies the parabolic Monge-Ampère equation

\[
(1.4) \quad u_t = \frac{(\det D^2 u)^p}{(1 + |\nabla u|^2)^{(n+2)p-1}}.
\]

Since any convex solution satisfies locally the bound \( |\nabla u| \leq C \), equation (1.4) becomes of the form (1.1).

The case \( p = 1 \) corresponds to the well studied Gauss curvature flow which was first introduced by W. Firey in [9] as a model for the wearing process of stones. It follows from the work of Tso [15] that uniformly strictly convex hyper-surfaces will become instantly \( C^\infty \) smooth and they remain smooth up to their vanishing time \( T \). However, convex surfaces which are not necessarily uniformly strictly convex, may not become instantly strictly convex and smooth (c.f. [12], [5]) and their regularity poses an interesting problem that we will investigate in this paper.

Equations of the form (1.3) for different powers of \( p > 0 \) were studied by B. Andrews in [1] (see also in [6]). He showed that when \( p \leq 1/n \) any convex hypersurface will become instantly strictly uniformly convex and smooth.

It can be seen from radially symmetric examples that, when \( p > 1/n \), surfaces evolving by (1.3) or (1.1) may have a flat side that persists for some time before it disappears. These surfaces are of class \( C^{1,\gamma_p} \) with \( \gamma_p := \frac{p}{np-1} \). Since \( \gamma_p < 1 \) if \( p > \frac{1}{n+1} \), solutions which are not strictly convex fail, in general, to be of class \( C^{1,1} \) in this range of exponents. In particular, solutions to the Gauss curvature flow (\( p = 1 \)) with flat sides are no better than \( C^{1,\frac{1}{n+1}} \) while the flat sides persist. The \( C^{1,\alpha} \) regularity of solutions of (1.3) for any \( p > 0 \) will be addressed in this work.

In dimension \( n = 2 \), the regularity for the Gauss curvature flow (\( p = 1 \)) is well understood. It follows from the work of B. Andrews in [2] that, in this case, all surfaces become instantly of class \( C^{1,1} \) and remain so up to a time when they become strictly convex and therefore smooth, before they contract to a point. Also, it follows from the works of the first author with Hamilton [7] and Lee [8] that \( C^{1,1} \) is the optimal regularity here, as can be seen from evolving surfaces \( \Sigma^2_t \) in \( \mathbb{R}^3 \) with flat sides. The optimal regularity of surfaces with flat sides and interfaces was further discussed in [7, 8].

We mention that \( C^{1,\alpha} \) and \( W^{2,p} \) interior estimates were established by Gutiérrez and Huang in [11] for equations similar to (1.1) for \( p = -1 \) and by Huang and Lu for \( p = \frac{1}{n} \). However, their work requires uniform convexity of the initial data and strict monotonicity of the function on the lateral boundary.
If \( w \) is a solution to the Monge-Ampère equation

\[
\det D^2 w = 1, \quad x \in \Omega \subset \mathbb{R}^n,
\]

then \( u(x, t) = w(x) + t \) solves equation (1.1) with \( b \equiv 1 \) for any \( p \). The question of regularity for the Monge-Ampère equation is closely related to the strict convexity of \( w \). Strict convexity does not always hold in the interior as it can be seen from a classical example due to Pogorelov [14]. However, Caffarelli [3] showed that if the convex set \( D \) where \( w \) coincides with a tangent plane contains at least a line segment then all extremal points of \( D \) must lie on \( \partial \Omega \). We prove the parabolic version of this result for solution of (1.1). Our result says that, if at a time \( t \) the convex set \( D \) where \( u \) equals a tangent plane contains at least a line segment then, either the extremal points of \( D \) lie on \( \partial \Omega \) or \( u(\cdot, t) \) coincides with the initial data on \( D \) (see Theorem 5.3). The second behavior occurs for example in those solutions with flat sides. In other words, a line segment in the graph of \( u \) at time \( t \) either originates from the boundary data at time \( t \) or from the initial data.

We prove a similar result for angles instead of line segments, which is crucial for our estimates. We show that if at a time \( t \) the solution \( u \) admits a tangent angle from below then either the set where \( u \) coincides with the edge of the angle has all extremal points on \( \partial \Omega \) or the initial data has the same tangent angle from below (see Theorem 6.1).

The \( C^{1, \alpha} \) regularity is closely related to understanding whether or not solutions separate instantly away from the edges of a tangent angle of the initial data. It turns out that when \( p > \frac{1}{n-2} \) the set where \( u \) coincides with the edge of the angle may persist for some time (see Proposition 4.8), hence \( C^1 \) regularity does not hold in this case without further hypotheses. If \( p < \frac{1}{n-2} \) we prove that, at any time \( t \) after the initial time, solutions are \( C^{1, \alpha} \) in the interior of any section of \( u(\cdot, t) \) which is included in \( \Omega \) (see Theorem 8.1). For the critical exponent \( p = \frac{1}{n-2} \) we show that solutions are \( C^1 \) with a logarithmic modulus of continuity for the gradient (see Theorem 8.2).

In the case of any power \( p > 0 \) we prove \( C^{1, \alpha} \) estimates at all points \((x, t)\) where \( u \) separates from the initial data (see Theorem 8.4). Also, if we assume that the initial data is \( C^{1, \beta} \) in some direction \( e \) then we show that the solution is \( C^{1, \alpha} \) in the same direction \( e \) for all later times (see Theorem 8.3).

In particular, our methods can be applied for solutions with flat sides. If the initial data has a flat side \( D \subset \mathbb{R}^n \), then solutions are \( C^{1, \alpha} \) for all later times in the
interior of $D$. A similar statement holds for solutions that contain edges of tangent angles: they are $C^{1,\alpha}$ along the direction of the edge for all later times. To be more precise we state these results below.

**Theorem 1.1.** Let $u$ be a viscosity solution of \( (1.1) \) in $\Omega \times [0, T]$ with $u(x, 0) \geq 0$ in $\Omega$, $u(x, 0) \geq 1$ on $\partial \Omega$. There exists $\alpha > 0$ depending on $n, \lambda, \Lambda, p$ such that

a) $u(x, t)$ is $C^{1,\alpha}$ in $x$ at all points $(x, t)$ with $x$ an interior point of the set \( \{ u(x, 0) = 0 \} \) and $u(x, t) < 1$.

b) If $u(x, 0) \geq |x_n|$ then $u(x, t)$ is $C^{1,\alpha}$ in the $x'$ variables at all points $((x', 0), t)$ with $x'$ an interior point of the set \( \{ x' : u((x', 0), 0) = 0 \} \) and $u(x, t) < 1$.

We finally remark that the equations \( (1.1) \) for negative and positive powers are in some sense dual to each other. Indeed, if $u$ is a solution of \( (1.1) \) and $u^*(\xi, t)$ is the Legendre transform of $u(\cdot, t)$ then

$$ u^*_t = -\tilde{b}(\xi, t)(\det D^2 u^*)^{-p}, \quad \lambda \leq \tilde{b}(\xi, t) \leq \Lambda. $$

The paper is organized as follows. In section 2 we introduce the notation and some geometric properties of sections of convex functions. In sections 3 we derive estimates for subsolutions and supersolutions. In section 4 we discuss the separation of solutions away from constant solutions such as planes and angles. In sections 5 and 6 we discuss the geometry of lines and angles. In section 7 and 8 we quantify the results of section 6 and prove the main theorems concerning $C^{1,\alpha}$ regularity.

## 2. Preliminaries

We use the standard notation $B_r(x_0) := \{ x \in \mathbb{R}^n : |x - x_0| < r \}$ to denote the open ball of radius $r$ and center $x_0$, and we write shortly $B_r$ for $B_r(0)$. Also, given a point $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, we denote by $x'$ the point $x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}$.

Throughout the paper we refer to positive constants depending on $n, \lambda, \Lambda$ and $p$ as universal constants. We denote them by abuse of notation as $c$ for small constants and $C$ for large constants, although their values change from line to line. If a constant depends on universal constants and other parameters $d, \delta$ etc. then we denote them by $c(d, \delta), C(d, \delta)$.

We use the following definition to say that a function is $C^{1,\alpha}$ in a pointwise sense.

**Definition 2.1.** A function $w$ is $C^{1,\alpha}$ at a point $x_0$ if there exists a linear function $l(x)$ and a constant $C$ such that

$$ |w(x) - l(x)| \leq C|x - x_0|^{1+\alpha}. $$
A function is $C^{1,\alpha}$ in a set $D$ if it is $C^{1,\alpha}$ at each $x \in D$.

A function is $C^{1,\alpha}$ at a point $x_0$ in the direction $e$ if it is $C^{1,\alpha}$ at $x_0$ when restricted to the line $x_0 + se$, $s \in \mathbb{R}$.

Next we introduce the notion of a section. We denote by $S_h(x, t) \subset \mathbb{R}^n$ a section at height $h$ of the function $u$ at the point $(x, t)$ defined by

$$S_h(x, t) := \{ y \in \Omega : u(y, t) \leq u(x, t) + p_h \cdot (y - x) + h \},$$

for some $p_h \in \mathbb{R}^n$. Sometimes, in order to simplify the notation, we denote such sections as $S_h, S_h(t)$ whenever there is no possibility of confusion.

We define the notion of $d$-balanced convex set with respect to a point.

**Definition 2.2** ($d$-balanced convex set). A convex set $S$ with $0 \in S$ is called $d$-balanced with respect to the origin, if there exists a linear transformation $A$ (which maps the origin into the origin) such that

$$B_1 \subset A S \subset B_d.$$  

Clearly, the notion of $d$-balanced set around 0 is invariant under linear transformations. Next we recall

**John’s lemma** Every convex set in $\mathbb{R}^n$ is $C_n$-balanced with respect to its center of mass, with $C_n$ depending only on $n$.

It is often convenient to consider sections at a point $x$ that have $x$ as center of mass. We denote such sections by $T_h(x, t)$ instead of $S_h(x, t)$. The existence of centered sections is due to Caffarelli [4].

**Theorem.** [Centered sections] Let $w$ be a convex function defined on a bounded convex domain $\Omega$. For each $x_0 \in \Omega$, and $h > 0$ there exists a centered $h$-section $T_h(x_0)$ at $x_0$

$$T_h(x_0) := \{ w(x) < w(x_0) + h + p_h \cdot (x - x_0) \}$$

(for some $p_h \in \mathbb{R}^n$) which has $x_0$ as its center of mass.

The following simple observations follow from the definition of $d$-balanced sets and will be used throughout the paper.

**Remark 2.3.** Assume that the $h$-section of $w$, $S_h(x_0) = \{ w(x) < w(x_0) + h + p_h \cdot (x - x_0) \}$, is $d$-balanced around $x_0$. Then,

$$-d h < w(x) - (w(x_0) + h + p_h \cdot (x - x_0)) < 0, \quad \text{in} \ S_h(x_0).$$
Also, if we assume \( w \geq 0 \) and \( w(x_0) = 0 \), then \( w(x) \leq d h \) for all \( x \in S_h(x_0) \).

Next lemma proves the existence of certain balanced sections which are compactly included in the domain of definition. It says that if we have a \( d \)-balanced section \( S_h \) which is compactly included in \( \Omega \), then we can find \( C_n d \)-balanced sections for all smaller heights than \( h \) that are included in \( S_h \).

**Lemma 2.4.** (a) Assume that \( w \) is a convex function defined on a set \( \Omega \subset \mathbb{R}^n \) with \( w(0) = 0 \) such that \( S_1 := \{ x : w(x) < 1 \} \subset \subset \Omega \) is \( d \)-balanced around 0. Then, there exists a constant \( C_n > 0 \) depending only on \( n \), such that for every \( h < 1 \) we can find a section \( S_h \) at height \( h \) with \( S_h \subset S_1 \) and \( S_h \) is \( C_n d \)-balanced around 0.

(b) Let us denote by \( r(x) \) the volume of the maximal ellipsoid centered at \( x \) that is included in \( S_1 \). Then, there exists a number \( C_n > 0 \) such that the section \( S_h \) in part (a) is either \( C_n \)-balanced around 0 or \( r(x^*) \geq 2 r(0) \) where \( x^* \) is the center of mass of \( S_h \).

**Proof.** a) For \( h < 1 \) fixed, consider the section \( S_h \) at height \( h \) that has 0 as its center of mass. If \( S_h \subset S_1 \) we have nothing to prove. Assume not and let’s say \( S_h = \{ w(x) < h + \alpha e_n \cdot x \} \), for some \( \alpha > 0 \).

We decrease the slope \( \alpha \) continuously till we obtain the section \( S_{h,t} := \{ w < h + t e_n \cdot x \} \) for which the set

\[
\{(x,x_{n+1}) : x \in \overline{S}_{h,t}, \ x_{n+1} = h + t e_n \cdot x \}
\]

becomes tangent to the hyper-plane \( x_{n+1} = 1 \) at a point \((x_0,1)\). We will show that \( S_{h,t} \) satisfies (a) and (b).

Clearly \( S_{h,t} \subset S_1 \). At the point \( x_0 \) we have \( x_0 \in \partial S_1 \) and

\[
S_1 \subset \{(x-x_0) \cdot e_n \leq 0 \}.
\]

Since \( S_1 \) is \( d \)-balanced, we may assume that \( B_1 \subset S_1 \subset B_d \) hence \( 1 \leq x_0 \cdot e_n \). Also, \( S_h \cap \{x_n = 0\} = S_{h,t} \cap \{x_n = 0\} \), hence the section \( S_{h,t} \) is already \( C_n \)-balanced in \( x' := (x_1, \cdots, x_{n-1}) \) around 0.

Since \( t \leq \alpha \), the center of mass \( x^* \) of \( S_{h,t} \) satisfies \( x^* \cdot e_n \leq 0 \). This together with \( x_0 \cdot e_n \geq 1 \) and \( x_0 \in \partial S_{h,t} \subset B_d \), implies that \( S_{h,t} \) is \( C_n d \)-balanced around 0 in all the directions.
b) If we assume that
\[ -x^* \cdot e_n \leq C_0(n) x_0 \cdot e_n \]
then we obtain that \( S_{h,t} \) is \( C(n, C_0(n)) \) balanced with respect to 0. Assume now that (2.1) doesn’t hold and denote by \( E \) the maximum volume ellipsoid centered at 0 which is included in \( S_1 \). After an affine transformation we have the following:

\[ E = B_1 \subset S_1, \quad S_{h,t} \subset \{ (x - x_0) \cdot e_n \leq 0 \}, \quad x_0 \in \partial S_{h,t} \]

and
\[ -x^* \cdot e_n > C_0(n) x_0 \cdot e_n \geq C_0(n) \]
which implies that \( |x^*| \geq C_0(n) \). Since \( x^* \) is the center of mass of \( S_{h,t} \) and \( 0 \in S_{h,t} \) we see from John’s lemma that \( (1 + c_n) x^* \in S_{h,t} \subset S_1 \). Hence if \( C_0(n) \) is sufficiently large we can find an ellipsoid of volume 2 centered at \( x^* \) and included in the convex set generated by \( (1 + c_n) x^* \) and \( B_1 \). This convex set is contained in \( S_1 \), and this concludes the proof of part (b). \( \square \)

3. Estimates for subsolutions and supersolutions

In this section we use the scaling of the equation to derive estimates for viscosity subsolutions and supersolutions of

\[ \lambda (\det D^2 u)^p \leq u_t \leq \Lambda (\det D^2 u)^p, \quad x \in \Omega. \]

Throughout the paper we assume that \( u \) is convex in \( x \), increasing in \( t \) and the domain \( \Omega \) is convex and bounded.

Let us now introduce the scaling of equation (3.1). Given an affine transformation \( A := \mathbb{R}^n \rightarrow \mathbb{R}^n \) and \( h > 0, m > 0 \) positive constants, the function

\[ v(x, t) := \frac{1}{h} u(A x, m t) \]

is a solution of equation (3.1) provided

\[ m = \frac{(\det A)^{2p}}{h^{np-1}}. \]

The equation is not affected by adding or subtracting a linear function in \( x \). For this reason we write our comparison results using constant functions instead of linear functions.
Lemma 3.1. Let \( u \) be a viscosity subsolution in \( B_1 \) i.e.

\[
    u_t \leq \Lambda (\det D^2 u)^p
\]

with

\[
    u(0,0) \geq -1, \quad u(x,0) \leq 0 \text{ on } \partial B_1.
\]

Then

\[
    u(0,t) \geq -2 \quad \text{for } t \geq -c,
\]

with \( c > 0 \) universal.

Proof. If \( u(0,-c) \leq -2 \) then, by convexity, \( u \) at time \( -c \) is below the cone generated by \((0,-2)\) and \( \partial B_1 \) i.e

\[
    u(x,-c) \leq -2 + 2|x| \quad \text{in } B_1.
\]

This implies that \( u \leq w \) on the boundary of the parabolic cylinder \( B_1 \times [-c,0] \) for

\[
    w(x,t) := m(t+c) + 2|x|^2 - \frac{3}{2} \quad \text{with } m = \Lambda^4 np.
\]

Since \( w_t = \Lambda (\det D^2 w)^p \) we obtain by the maximum principle that \( u(0,0) \leq w(0,0) \) and we reach a contradiction by choosing \( c = 1/(4m) \).

\( \square \)

Remark 3.2. The conclusion can be replaced by \( u(0,t) \geq -(1 + \delta) \) for \( t \geq -c(\delta) \).

The scaling of the equation and the previous lemma give the following:

Proposition 3.3. Assume that \( u \) is a viscosity subsolution in a convex set \( S \) with center of mass \( 0 \). If

\[
    u(0,0) \geq -h, \quad u(x,0) \leq 0 \text{ on } \partial S,
\]

then

\[
    u(0,t) \geq -2h \quad \text{for } t \geq -c \frac{|S|^{2p}}{h^{np-1}},
\]

with \( c \) universal.

Proof. From John’s lemma there exists a linear transformation \( A \) such that

\[
    B_1 \subset A^{-1}S \quad \text{with} \quad \det A \geq c(n)|S|.
\]

The proposition follows by applying Lemma 3.1 to the rescaled solution

\[
    v(x,t) := \frac{1}{h} u(Ax, mt), \quad m = \frac{(\det A)^{2p}}{h^{np-1}}.
\]

\( \square \)
Remark 3.4. We obtain a slightly different version of Proposition 3.3 by requiring $S$ to be only $d$-balanced around the origin and by replacing the conclusion by $u(0, t) \geq -(1 + \delta)h$. In this case we need to take the constant $c = c(d, \delta)$ depending also on $d$ and $\delta$ as can be seen from the proofs of Lemma 3.1 and Proposition 3.3.

Remark 3.5. In general we apply Proposition 3.3 at a point $(x_0, t_0)$ in an $h$-section $S_h = S_h(x_0, t_0)$ which is $d$-balanced around $x_0$ to conclude that
\[ u(x_0, t) \geq u(x_0, t_0) - h \quad \text{for} \quad t \geq t_0 - c \frac{|S_h|^{2p}}{h^{np-1}}. \]

Remark 3.6. At a given point we can apply the Proposition directly in the sections given by its tangent plane. Indeed, taking $S$ to be the set $S_h = S_h(0, 0) := \{ u < h + P(x) \}$, $P(x) := u(0, 0) + \nabla u(0, 0) \cdot x$ we conclude that $u(x^*, t) \geq P(x^*) - 2h$ with $x^*$ the center of mass of $S_h$. This, by John’s lemma, implies a bound in whole $S_h$
\[ u(x, t) \geq P(x) - C(n) h, \quad \text{for all} \quad x \in S_h, \quad t \geq t_0 - c \frac{|S_h|^{2p}}{h^{np-1}}, \]

with $C(n)$ depending only on the dimension.

Corollary 3.7. Assume that $u$ is a bounded subsolution of equation (3.1) in the cylinder $Q_1 := B_1 \times [-1, 0]$. Then, $u$ is uniformly Hölder continuous in time $t$ on the cylinder $Q_{1/2} := B_{1/2} \times [-1/2, 0]$, namely $u \in C^{1, \beta}(Q_{1/2})$, with $\beta = 1/(np+1)$.

Proof. Since $u$ is bounded on $Q_1$, the convexity of $u(\cdot, t)$ implies that $|\nabla u|$ is bounded by a constant $M$ in $Q_{3/4}$. Then, by Proposition 3.3 applied in $B_h(x)$, with $x \in B_{1/2}$ and $h < 1/4$, we have
\[ (3.2) \quad -2M h \leq u(x, t) - u(x, 0) \leq 0 \quad \text{if} \quad -c \frac{|B_h(x)|^{2p}}{h^{np-1}} \leq t \leq 0. \]

Taking $t = -c_1 h^{np+1}$, we find that for all $t$ small enough
\[ |u(x, t) - u(x, 0)| \leq C(M) t^{1/(np+1)} \]
from which the desired result readily follows. \hfill \Box

As a consequence we obtain compactness of viscosity solutions.

Corollary 3.8. A sequence of bounded solutions of (3.1) in $\Omega \times [-T, 0]$ has a subsequence that converges uniformly on compact sets to a solution of the same equation.
Next we discuss the case of supersolutions.

**Lemma 3.9.** Let $u$ be a viscosity supersolution in $S \subset B_1$ i.e.

$$u_t \geq \lambda (\det D^2 u)^p$$

with

$$u(x, 0) \geq -1 \text{ in } S, \quad u(x, 0) \geq 0 \text{ on } \partial S.$$  

Then

$$u(x, t) \geq -\frac{1}{2} \text{ for } t \geq C,$$

with $C > 0$ universal.

**Proof.** The lemma follows by comparison of our solution $u$ with the function

$$w(x, t) = \frac{1}{2} (|x|^2 - 1) + \lambda (t - C)$$

on the cylinder $S \times [0, C]$. The function $w$ is a solution of the equation $w_t = \lambda (\det D^2 w)^p$ and, since $S \subset B_1$, satisfies $w \leq 0$ on $\partial S(0) \times [0, C]$. In addition, by choosing $C = 1/\lambda$, we have $w(x, 0) \leq -1 \leq u(x, 0)$ for $x \in S$. The comparison principle implies $u(x, C) \geq w(x, C) \geq -1/2$ in $S$.

Remark 3.10. We can replace $-1/2$ by $-\delta$ in the lemma above by taking $C = C(\delta)$ depending also on $\delta$.

**Remark 3.11.** If we assume that $S$ is $d$-balanced around 0 and $u(0, 0) = -1$, $u(x, 0) = 0$ on $\partial S$, then the same conclusion holds by taking $C = C(d)$ depending also on $d$. Indeed, in this case we obtain $u(x, 0) \geq -C(d)$ for all $x \in S$ and the desired conclusion follows as before.

The scaling of the equation and the previous lemma give the following:

**Proposition 3.12.** Let $u$ be a supersolution in $\Omega$ and assume

$$u(x, 0) \geq 0, \quad \text{and } S_h := \{u(x, 0) < h\} \subset \subset \Omega.$$  

Then

$$u(\cdot, t) \geq \frac{h}{2}, \quad \text{for } t \geq C \frac{|S_h|^{2p}}{h^{np-1}},$$

with $C$ universal.
Proof. Let $A$ be a linear transformation such that $A^{-1}S_h \subset B_1$ so that $\det A \leq C(n)|S_h|$. We then apply the previous lemma to the re-scaled solution

$$v_h = \frac{1}{h} u(Ax, mt) - 1, \quad m = \frac{(\det A)^{2p}}{h^{np-1}}.$$

\[\square\]

Remark 3.13. In view of Remark 3.11 we obtain a version of Proposition 3.12 for sections $S_h = S_h(x_0, t_0)$ which are $d$-balanced around $x_0$ and are compactly included in $\Omega$, and conclude that

$$u(x_0, t) \geq u(x_0, t_0) + (1 - \delta)h \quad \text{for} \quad t \geq t_0 + C(\delta, d)\frac{|S_h|^{2p}}{h^{np-1}}.$$

4. SEPARATION FROM CONSTANT SOLUTIONS

In this section we consider the case when the solution $u$ at the initial time $t = 0$ is above a given function $w$ depending only on $n - 1$ variables, $u$ and $w$ coincide at the origin, and $u > w$ on $\partial \Omega$. We investigate whether $u$ separates from $w$ instantaneously for positive times, i.e $u(0, t) > w(0)$ for all $t > 0$. Of particular interest is the case of angles given by $w = |x_n|$.

Throughout this section we assume that $u(x, 0) \geq 0$. For $h > 0$ we will consider the sub-level set $S_h(t)$ of our solution $u(\cdot, t)$ in $\Omega$ which is defined as

$$S_h(t) := \{x \in \Omega : u(x, t) < h\}.$$

We will also consider balls $B'_\rho \subset \mathbb{R}^{n-1}$, namely

$$B'_\rho := \{x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1} : |x'| < \rho\}.$$

Proposition 4.1. Let $u$ be a supersolution in $\Omega \times [0, T]$ with $u \geq 0$ at $t = 0$. Assume that $S_h(0) \cap \{x_n < 2 \beta\}$ is compactly included in $\Omega$ and is included as well in the cylinder $\{0 < x_n < 2\beta\} \times S'$ for a bounded domain $S' \subset \mathbb{R}^{n-1}$ and two positive constants $h > 0, \beta > 0$. Then,

$$S_{h/4}(t_0) \subset \{x_n > \beta\}, \quad \text{for} \quad t_0 = C \frac{(\beta |S'|)^{2p}}{h^{np-1}}.$$

with $C$ universal.

Proof. We apply Proposition 3.3 for

$$\tilde{u} = u + \frac{h}{2\beta} x_n$$

and see that $\tilde{u} \geq u \geq 0$. Also $\{\tilde{u}(x, 0) < h\}$ is compactly included in $\Omega$ and is included in $\{0 < x_n < 2\beta\} \times S'$. We conclude that $\tilde{u}(x, t_0) \geq \frac{3}{4}h$ with $t_0$
given above. This implies that if $x_n \leq \beta$ then $u(x,t_0) \geq \tilde{u}(x,t_0) - \frac{h}{2} \geq \frac{h}{4}$ hence $S_{h/4}(t_0) \subset \{x_n > \beta\}$. □

From Proposition 4.1 we obtain the following corollary.

**Corollary 4.2.** Let $u$ be a supersolution in $\Omega \times [0,T]$ and assume that

$$u(x,0) \geq w(x') \geq 0, \quad u(0,0) = w(0) = 0, \quad u(x,0) > 0 \quad \text{on } \partial\Omega,$$

for a function $w$ defined on $\mathbb{R}^{n-1}$. Suppose that $w$ satisfies

(4.1) $$\frac{a_{h_j}^{2p}}{h_j^{np-1}} \rightarrow 0, \quad \text{for a sequence } h_j \rightarrow 0.$$

with

$$a_h := |\{w(x') < h\} \cap \pi_n(\Omega)|, \quad \text{where } \pi_n(x) := x'.$$

Then,

$$u(0,t) > 0 \quad \text{for any } t > 0.$$

**Proof.** Let $h > 0$ be small such that the sub-level sets $S_h(0)$ of $u$ is compactly supported in $\Omega$. Since $u \geq w$ we obtain that

$$S_h(0) \subset \{(w(x') < h) \cap \pi_n(\Omega)) \times [b,\infty)\},$$

for some $b < 0$ (since $0 \in S_h(0)$). We apply Proposition 4.1 for $h_j \leq h$ (hence $S_{h_j}(0) \subset S_h(0)$), with $\beta = -b$. We conclude that

$$S_{h_j/4}(t_j) \subset \{x_n > 0\}, \quad t_j = C \beta^{2p} \frac{a_{h_j}^{2p}}{h_j^{np-1}},$$

and obtain $u(0,t_j) \geq h_j/4 > 0$ for a sequence $t_j \rightarrow 0$. □

**Remark 4.3.** If $p > 1/2$ and the sequence above is bounded, then the conclusion of Corollary 4.2 still holds true.

Next we investigate the case when $w$ is identically 0.

**Proposition 4.4.** Let $u$ be a supersolution in $\Omega \times [0,T]$ with $p \leq 1/n$. Assume that $u \geq 0$ at $t = 0$ and $u(x,0) > 0$ on $\partial\Omega$. Then, $u > 0$ in $\Omega$ for any $t > 0$.

**Proof.** For $p < 1/n$ the proposition follows from Corollary 4.2.

Let $p = 1/n$. Assume that for $h > 0$ small we have $S_h(0) \subset B_\rho$ for some $\rho$ in $0 < \rho \leq 1$, and $S_h(0)$ is compactly supported in $\Omega$. We first show that for $\beta \in (0,\rho]$ small, we have

(4.2) $$S_{h/4}(t_0) \subset B_{\rho-\beta}(0), \quad \text{for } t_0 = C \beta^{1+\frac{1}{n}}.$$
To this end, we will apply Proposition 4.1 for each \( x_0 \in \partial B_{\rho} \) in the direction \((-x_0)\).

Let us assume for simplicity that \( x_0 = (0, \cdots, 0, -\rho) \). Then, since \( S_h(0) \subset B_{\rho} \), we have

\[
S_h(0) \cap \{-\rho < x_n < -\rho + 2\beta\} \subset B'_{2\sqrt{\beta}} \times (-\rho, -\rho + 2\beta).
\]

Applying Proposition 4.1, we obtain that

\[
S_{h/4}(t_0) \subset \{x_n > -\rho + \beta\}, \quad \text{for } t_0 = C (\beta^{\frac{n-1}{2}})^{2/n}.
\]

and (4.2) readily follows.

We will now use (4.2) to show that \( u > 0 \) for \( t > 0 \). Let \( t > 0 \) and fixed. Choose \( \beta := 1/k > 0 \) with \( k \) the smallest integer so that \( C \beta^{\frac{2}{n-2}} \leq t \), with \( C \) the constant from (4.2). Using this \( \beta \) we repeat the argument above \( k \) times, starting at \( \rho = 1 \), to conclude that

\[
S_{h/4^k}(t_0) \subset B_{1-k\beta}, \quad \text{for } t_0 = C k \beta^{\frac{n+1}{2}}.
\]

This shows that \( S_{h/4^k}(t_0) = \emptyset \), for \( t_0 = C \beta^{\frac{n+1}{2}} \leq t \) hence \( u(\cdot, t) \geq h/4^k > 0 \). \( \square \)

**Remark 4.5.** For \( p > 1/n \) there exist radial solutions with a flat side that persists for some time.

**Remark 4.6.** In the proof we showed in fact that if \( u \geq 0 \), \( u(x, 0) \geq h \) on \( \partial \Omega \), then \( u(\cdot, t) \geq h e^{-Ct^{-n}} \)

for some \( C \) universal.

In the next results we investigate the case of angles i.e when \( w(x) = |x_n| \). First proposition shows that \( u \) separates instantly from the edge of the angle when the exponent \( p \leq \frac{1}{n-2} \). The second proposition shows that this is not the case when \( p > \frac{1}{n-2} \).

**Proposition 4.7.** Assume \( u \) is a supersolution, and \( p \leq \frac{1}{n-2} \). If \( u(x, 0) \geq |x_n| \) and \( u(x, 0) > 0 \) on \( \partial \Omega \), then \( u > 0 \) for any \( t > 0 \).

**Proof.** If \( p > \frac{1}{n-2} \) then the proposition follows from Corollary 4.2 since \( a_h \leq Ch \).

Let \( p = \frac{1}{n-2} \). Since \( u \geq |x_n| \) we may assume without loss of generality that \( S_h(0) \subset B'_1 \times [-h, h] \). For each \( x'_0 \in B'_1 \) we apply Proposition 4.1 in the direction \((-x'_0)\), in a manner similar to that used in Proposition 4.4 to show that

\[
S_{h/4}(t_0) \subset B'_{1-\beta} \times [-h/4, h/4], \quad \text{for } t_0 = C \frac{(\beta |S'|)^{2p}}{h^{p-1}}.
\]
Notice that this time $|S'| = h |B''_r|$, where $B''_r$ is an $n - 2$ dimensional ball, hence (since $p = \frac{1}{n - 2}$)
\[ t_0 \geq C \frac{(h \beta)^{2p}}{h^{np-1}} = C\beta^{\frac{n-2}{2}}. \]
Now the proof continues as in the proof of Proposition 4.4 and we obtain
\[ u(\cdot, t) \geq he^{-Ct^{\frac{n-2}{2}}} . \]
\[ \square \]

**Proposition 4.8.** There exists a non-trivial solution $u$ of equation
\[ u_t = (\det D^2 u)^p, \quad \text{on } \mathbb{R}^n \times [0, \infty) \]
for which $u(x, 0) \geq |x_n|$ and $u(0, t) = 0$, for all $t \in [0, \delta]$, for some $\delta > 0$.

**Proof.** We will seek for a solution $u$ of the form
\[ (4.4) \quad u(x, t) = f(t) v \left( \frac{x}{g(t)} \right) \]
for some functions $f = f(t)$ and $v = v(y)$. The function $u$ satisfies (4.3) if and only if
\[ (-f') \left( \frac{x}{t} \right) \nabla v \left( \frac{x}{t} \right) - w = f^{-n}p (\det D^2 v)^p. \]
We pick a function $f$ which satisfies
\[ (4.5) \quad -f' = f^{-np}. \]
Solving (4.5) gives us
\[ (4.6) \quad f(t) = [(1 + np)(T - t)]^{\frac{1}{1-np}} \]
for any constant $T > 0$. We will next show that there exists a function $v = v(y)$ such that
\[ (4.7) \quad y \cdot \nabla v - v = (\det D^2 v)^p, \quad v(y) \geq |y_n|, \quad v(0) = 0. \]
The existence of such a function $v$ implies the claim of our proposition. To this end, we seek for $v$ of the form
\[ (4.8) \quad v(y', y_n) = \tilde{v}(|y'|, y_n) = \varphi(|y'|) g \left( \frac{y_n}{\varphi(|y'|)} \right), \]
with $g(s) \geq |s|$. A direct computation shows that,
\[ \tilde{v}_1 = \varphi' g - \varphi' \frac{y_n}{\varphi} g' = \varphi' (g - s g'), \quad \tilde{v}_2 = g' \left( \frac{y_n}{\varphi} \right) = g'(s) \]
with \( s = y_n/\phi \). Also,

\[
\begin{align*}
\tilde{v}_{11} &= \phi'' (g - s g') + \phi' \frac{y_n}{\phi^2} \phi', \\
\tilde{v}_{12} &= -\frac{\phi'}{\phi} s g'', \\
\tilde{v}_{22} &= \frac{1}{\phi} g''.
\end{align*}
\]

Using that \( y_n/\phi = s \), we get

\[
y \cdot \nabla v - v = |y'| \phi' (g - s g') + y_n g' - \phi g = (|y'| \phi' - \phi) (g - s g'),
\]

and also,

\[
\det D^2 v = \frac{\phi''}{\phi} g'' (g - s g')^{n-1} \left( \frac{\phi'}{|y'|} \right)^{n-2}.
\]

Separating the functions \( g \) and \( \phi \), we conclude that \( v \) satisfies (4.5), if

\[
g'' (g - s g')^{n-1 - \frac{1}{p}} = 1 \quad \text{and} \quad \phi'' \left( \frac{\phi'}{|y'|} \right)^{n-2} = (|y'| \phi' - \phi)^{\frac{2}{p}} \phi.
\]

For the second equation we seek for a solution in the form \( \phi(r) = C_{n,p} r^\beta \) with \( \beta > 1 \). We find that \( \phi \) satisfies the above equation if

\[
(\beta - 2) (n - 1) = \frac{\beta}{p} + \beta
\]

which after we solve for \( \beta \) yields to

\[
\beta = \frac{2 (n - 1)}{(n - 2 - 1/p)}.
\]

Since we need \( \beta > 1 \), we have to restrict ourselves to the exponents \( p > \frac{1}{n-2} \).

Next we find an even function \( g \), convex of class \( C^{1,\alpha} \), that solves the ODE for \( g \) in the viscosity sense and for which \( g(s) = |s| \) for large values of \( s \). Rewriting the ODE and the conditions above in terms of the Legendre transform \( g^* \) of \( g \) we find

\[
(g^*)'' = |g^*|^{n-1-1/p} \quad \text{in} \quad [-1,1], \quad g^*(1) = g^*(-1) = 0.
\]

The existence of \( g^* \) follows by scaling the negative part of any even solution \( \bar{g} \) to the ODE above, i.e \( g^*(t) = a \bar{g}(t/b) \) for appropriate constants \( a \) and \( b \). We obtain the function \( g \) by taking the Legendre transform of \( g^* \).

\[ \square \]

Remark 4.9. Proposition 4.8 shows that in the Gauss curvature flow (1.3) with exponent \( p \), if the initial data is a cube, then the edges \((n-1\text{-dimensional})\) move instantaneously if and only if \( p \leq \frac{1}{n-2} \). In the particular case of the classical Gauss curvature flow with \( p = 1 \), the edges of the cube move instantaneously if and only \( n \leq 3 \).
5. The geometry of lines

Our goal in this section is to prove Theorem 5.3, which constitutes the parabolic version of the result of Caffarelli for Monge-Ampere equation. Theorem 5.3 deals with extremal points of the set \( \{ u = 0 \} \) for a nonnegative solution \( u \) of (3.1). We begin by giving the definition of an extremal point of a convex set (cf. in [10], Chapter 5).

**Definition 5.1.** Let \( D \) be a convex subset of \( \mathbb{R}^n \). The point \( x_0 \in \partial D \) is an extremal point of \( D \) if \( x_0 \) is not a convex combination of other points in \( D \).

We now give the main results of this section. The first Theorem states that a constant segment in time can be extended backward all the way to the initial data.

**Theorem 5.2.** Let \( u \) be a solution of equation (3.1) on \( \Omega \times [-T, 0] \). Assume \( u(0,t) = 0 \) for \( t \in [-\delta, 0] \) and there exists a section \( S_{h_0}(0) := \{ u(x,0) < h_0 + p_{h_0} \cdot x \} \) at \( (0,0) \) that is compactly supported in \( \Omega \). Then \( u(0,t) = 0 \) for all \( t \in [-T, 0] \).

The second Theorem states that if the graph of \( u \) at a given time coincides with a tangent plane in a set \( D \) that has an extremal point in \( \Omega \), and \( D \) contains at least a line segment, then \( u \) agrees with the initial data on \( D \).

In other words, a line segment at a given time either originates from the boundary data at that particular time or from the data at the initial time.

**Theorem 5.3.** Let \( u \) be a solution of equation (3.1) on \( \Omega \times [-T, 0] \), for some convex domain \( \Omega \subset \mathbb{R}^n \). Suppose that at time \( t = 0 \) we have \( u \geq 0 \), and the set

\[
D := \{ u(x,0) = 0 \}
\]

contains a line segment and \( D \) has an extremal point in \( \Omega \). Then,

\[
 u(x,-T) = 0, \quad \text{for all } x \in D.
\]

As a consequence of the theorems above we obtain the following:

**Corollary 5.4.** Assume \( u \) is defined in \( \Omega \times [-T, 0] \) and \( u(x,-T) \geq 0 \) on \( \partial \Omega \). Then \( u \) is strictly convex in \( x \) and strictly increasing in \( t \) at all points \( (x,t) \) that satisfy \( u(x,-T) < u(x,t) < 0 \).

We first prove Theorem 5.2.
Proof of Theorem 5.2. By continuity of $u$ the section

$$S_{h_{0}}(-\sigma) := \{u(x, -\sigma) < h_{0} + p_{h_{0}} \cdot x\}$$

at $(0, -\sigma)$ is also compactly included in $\Omega$ for a small $\sigma \in [0, \delta]$. Let $d$ be sufficiently large so that $S_{h_{0}}(-\sigma)$ is $d$-balanced around 0. By Lemma 2.4 for each $h \leq h_{0}$ we can find a section $S_{h}(-\sigma)$ which is $C_{n}d$-balanced around 0. We apply Proposition 3.12 (see Remark 3.13) and use that $u(0, 0) - u(-\sigma, 0) = 0 < h/2$ to conclude

$$\sigma \leq C(d) \frac{|S_{h}(-\sigma)|^{2p}}{h^{np-1}}.$$

Assume next that $u(0, -t_{0}) = 0$, for some $t_{0} > \sigma$. We apply Proposition 3.3 (see Remark 3.4) at $(0, -t_{0})$ in the set $S := S_{h}(-\sigma)$ and conclude

$$u(0, t) \geq -h, \quad \text{for} \quad t \geq -t_{0} - c(d) \sigma.$$

Using the bound on $\sigma$ we find that $u(0, t) = 0$ for $t \geq -t_{0} - c(d) \sigma$ and the conclusion follows.

Next lemma is the key step in the proof of Theorem 5.3.

Lemma 5.5. Assume $u(e_{n}, 0) = 0$ for $s \in [0, 2]$, and for some $t_{0} > 0$

$$u(e_{n}, -t_{0}) \geq -h, \quad T_{0_{h}}(0, -t_{0}) \subset B_{\delta} \subset \subset \Omega,$$

where $T_{0_{h}}(0, -t_{0})$ is the centered section at $0$ at time $-t_{0}$. Then

$$u(e_{n}, -Mt_{0}) \geq -2h, \quad \text{with} \quad M = 1 + c\delta^{-2p}, \quad (c \text{ universal}).$$

Proof. Since $u(2e_{n}, -t_{0}) \leq u(2e_{n}, 0) = 0$, the convexity of $u(\cdot, -t_{0})$ implies that $u(0, -t_{0}) \geq -2h$. We apply Proposition 3.12 (see Remark 3.13) in the section

$$T_{0_{h}} := T_{0_{h}}(0, -t_{0}) = \{u(x, -t_{0}) < u(0, -t_{0}) + 6h + p_{0_{h}} \cdot x\}$$

and conclude that

$$t_{0} \leq C \frac{|T_{0_{h}}|^{2p}}{h^{np-1}}.$$

Indeed, otherwise we obtain $u(0, 0) \geq h$ which contradicts the hypothesis. Since $T_{0_{h}} \subset B_{\delta}$ and has 0 as center of mass, we find

$$|T_{0_{h}}| \leq C\delta |T'_{0_{h}}|, \quad \text{where} \quad T'_{0_{h}} := \{x' \in \mathbb{R}^{n-1} \mid (x', 0) \in T_{0_{h}}\},$$

for some $C$ depending only on $n$. Using the inequality for $t_{0}$ we conclude

$$\frac{|T'_{0_{h}}|^{2p}}{h^{np-1}} \geq c\delta^{-2p} t_{0}.$$
Now we apply Proposition 3.3 (see Remark 3.4) for the function
\[  \tilde{u} = u - p'_{6h} \cdot x' - 6h \]
in the convex set \( S \) which is the convex hull generated by the \( n-1 \) dimensional set \( T_{6h} \times \{0\} \) and the segment \( [0,2e_n] \). Notice that \( \tilde{u} \) is negative at time \(-t_0\) in \( S \) and \( \tilde{u}(e_n,-t_0) \geq -7h \). Since \( S \) is \( d \)-balanced with respect to \( e_n \) with \( d \) depending only on \( n \) we conclude that

\[ \tilde{u}(e_n,t) \geq -8h \quad \text{for} \quad t \geq -t_0 - c(2|T_{6h}|)^{2p}\frac{1}{h^{np-1}}, \]
with \( c \) universal. Using (5.1) we find \( u(e_n,t) \geq -2h \) if \( t \geq -t_0(1+c\delta^{-2p}) \).

\[ \square \]

Proof of Theorem 5.3. Assume for simplicity that \( 0 \in \Omega \) is an extremal point for \( D \) and \( 2e_n \in D \). We want to prove that \( u(2e_n,-T) = 0 \).

Fix \( \delta > 0 \) small, smaller than a universal constant to be specified later. There exists \( \sigma > 0 \) depending on \( u \) and \( \delta \) such that

\[ T_{6h}(0,-t) \subset B_\delta \subset \Omega \quad \text{for all} \quad h, t \in [0,\sigma]. \]

Indeed, otherwise we can find a sequence of \( h_n, t_n \) tending to 0 for which the inclusion above fails. In the limit we obtain that 0 can be written as a linear combination of two other points in \( D \) (one of them outside \( B_\delta \)) and contradict that 0 is an extremal point.

First we show that \( u(x,-\sigma) = 0 \) on the line segment \([0,2e_n]\). Using the Holder continuity of \( u \) in \( t \) at the point \((e_n,0)\) we find that for small \( t_0 > 0 \),

\[ u(e_n,-t_0) \geq -h := -C(u)t_0^{\frac{1}{np+1}}. \]

We can apply Lemma 5.5 inductively and conclude that as long as \( M^{k-1}t_0 \leq \sigma \), \( 2^{k-1}h \leq \sigma \) then

\[ u(e_n,-M^{k-1}t_0) \geq -2^kh. \]

We choose \( \delta \) small enough so that \( M = 1 + c\delta^{-2p} > 4np+1 \). Then

\[ 2^kh \leq C(u)2^{-k}(M^{k-1}t_0)^{\frac{1}{np+1}} \leq C(u)2^{-k}(M\sigma)^{\frac{1}{np+1}}. \]

This shows that if we start with \( t_0 \) small enough then \( M^{k-1}t_0 \leq \sigma \) implies \( 2^{k-1}h \leq \sigma \) and moreover, as \( t_0 \to 0 \) then \( 2^kh \to 0 \) as well. We conclude that \( u(e_n,-\sigma) = 0 \) hence \( u(x,-\sigma) = 0 \) on the line segment \([0,2e_n]\).

Now we can use Theorem 5.2 for the points \((se_n,0)\) for small \( s \geq 0 \) which are included in a compact section at the origin at time \( t = 0 \). Since \( u(se_n,t) = 0 \) for
t ∈ [−σ, 0], we conclude that \( u(se_n, -T) = 0 \) for small \( s \). Then convexity in \( x \) and monotonicity in \( t \) imply \( u(x, -T) = 0 \) on the segment \([0, 2e_n]\).

\[ \square \]

6. The geometry of angles

Our goal in this section is to prove the analogue of Theorem 5.3 for angles. That is, if \( u : \Omega \times [-T, 0] \to \mathbb{R} \) is a solution to (3.1) for which the graph of \( u \) at time \( t = 0 \) has a tangent angle from below, then this angle originates either from the initial data \( u(\cdot, -T) \) or from the boundary data on \( \partial \Omega \) at time \( t = 0 \).

Throughout this section we will denote by \( x' \) points \( x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1} \) and by \( x = (x', x_n) \) points in \( \mathbb{R}^n \). Our result states as follows.

**Theorem 6.1.** Let \( u : \Omega \times [-T, 0] \to \mathbb{R} \) be a solution of equation (3.1) with \( \Omega \subset \mathbb{R}^n \). Assume that at time \( t = 0 \), we have \( u(0, 0) = 0 \), \( u(x, 0) \geq |x_n| \) and \( 0 \) is an extremal point for the set \( D := \{ x \in \Omega : u(x, 0) = 0 \} \). Then, \( u(x, -T) \geq |x_n| \).

The proof of Theorem 6.1 is more involved than that of Theorem 5.3. We introduce the following convenient notation.

**Definition 6.2.** For negative times \( t \leq 0 \) we say that

\((h, \alpha) \in A_t(u) \subset \mathbb{R}^2_+\)

if there exist vectors \( q_1, q_2 \in \mathbb{R}^n \) such that

\[ u(x, t) \geq u(0, 0) - h + \max \{ q_1 \cdot x, q_2 \cdot x \} \]

in \( \Omega \) and \((q_1 - q_2) \cdot e_n \geq \alpha\). Whenever there is no possibility of confusion we write \( A_t \) instead of \( A_t(u) \).

**Remark 6.3.** The statement \((h, \alpha) \in A_t \) is in fact a one-dimensional condition on \( u(x, t) \). It says that, when restricted to the line \( se_n \), we can find a certain angle below the graph of \( u(\cdot, t) \). The vertex of the angle is at distance \( h \) below \( u(0, 0) \) at the origin and the difference in the slopes of the lines that form the angle is \( \alpha \).

Clearly, if \((h, \alpha) \in A_{t_1} \) then \((h, \alpha) \in A_t \) for all \( t \geq t_1 \). The statement \((h, \alpha) \in A_t \) remains true if we add to \( u \) a linear function in \( x \) or if we perform an affine transformation in the \( x \) variable that leaves \( e_n \) invariant.

Next proposition is the key step in proving Theorem 6.1 and later for obtaining interior \( C^{1,\alpha} \) estimates.
Proposition 6.4. Let $u$ be a solution of equation (3.1) with $u(0,0) = 0$. Assume that at time $-t_0$, $(t_0 > 0)$ the solution $u$ satisfies for a fixed constant $C_0$ and a parameter $\delta \leq 1$:

i. $(h, \alpha) \in A_{-t_0}$ and $(C_0 h, (1 + \delta) \alpha) \notin A_{-t_0}$, and

ii. there exists a section (at distance $h$ from the origin)

$$S_h := \{u(x, -t_0) < h + q \cdot x\}$$

of $u(\cdot, -t_0)$ which is $d$-balanced with respect to the origin and is compactly supported in $\Omega$.

Then,

$$\left(C_0^2 h, \frac{\alpha}{1 + \frac{\delta}{2}}\right) \in A_{-t}, \quad \text{for} \quad t_0 \leq t \leq t_0 + c(d) \delta^{-2p} t_0$$

for some $c(d) > 0$.

Remark 6.5. From the proof we will see that we can take the constant $C_0 = 100$.

Proof. Since $(h, \alpha) \in A_{-t_0}$, we have $u(x, t) \geq -h + \max\{q_1 \cdot x, q_2 \cdot x\}$, for some vectors $q_1, q_2$. Without loss of generality, we may assume that $q_1, q_2$ have only components in the $e_n$ direction. This reduction is possible by first subtracting the linear map $\frac{q_1 + q_2}{2} \cdot x$ and then performing a linear transformation that leaves $e_n$ invariant. Thus, assume that

$$u(x, -t_0) \geq -h + \frac{\alpha}{2} |x_n|.$$ 

Since $S_h$ is $d$-balanced, the inequality above and Remark 2.3 imply that

$$S_h \subset \{|x_n| < 4d \frac{h}{\alpha}\}.$$

Thus, if $S_h' := S_h \cap \{x_n = 0\}$, we have

$$|S_h| \leq Cd \frac{h}{\alpha} |S_h'|.$$

Since $u(0,0) = 0$ and $u(0, -t_0) \geq -h$, Proposition 3.12 implies that

$$t_0 \leq C(d) \frac{|S_h|^{2p}}{h^{\alpha p - 1}},$$

and from the previous estimate we have

$$t_0 \leq C(d) \frac{|S_h'|^{2p}}{h^{\alpha p - 1}}.$$
On the other hand, since \((C_0 h, (1 + \delta) \alpha) \notin A_{-t_0}\) there exists \(s_1 e_n \in \Omega\) with \(s_1 > 0\), such that
\[
u(s_1 e_n, -t_0) < -C_0 h + \frac{\alpha}{2} (1 + 2 \delta) s_1.
\]
Otherwise the angle with vertex at \(-C_0\) and lines of slopes \(-\alpha/2, \alpha/2 + \delta \alpha\) would be below the graph of \(\nu(x, -t_0)\) on the line \(x = s e_n\) and we reach a contradiction.

Since \(\nu(s_1 e_n, -t_0) \geq -h + \frac{\alpha}{2} s_1\), the above yields the bound
\[
s_1 \geq \frac{(C_0 - 1) h}{\alpha \delta} := s_0.
\]
Moreover, since \(\nu(0, -t_0) \leq \nu(0, 0) \leq 0\) and
\[
u(s_1 e_n, -t_0) < -C_0 h + \frac{\alpha}{2} (1 + 2 \delta) s_1 < \frac{\alpha}{2} (1 + 2 \delta) s_1
\]
the convexity of \(\nu(\cdot, -t_0)\) implies that
\[
u(s e_n, -t_0) < \frac{\alpha}{2} (1 + 2 \delta) s, \quad \forall s \in [0, s_0] \subset [0, s_1].
\]
Hence, if \(s \in [0, s_0]\), then
\[
u(s e_n, -t_0) < \frac{\alpha}{2} s + \alpha \delta s_0 = (C_0 - 1) h + \frac{\alpha}{2} s.
\]
Recalling that \(S_h := \{\nu(x, -t_0) < h + q' \cdot x' + q_n x_n\}\), it follows from the above discussion that the set
\[
\{\nu(x, -t_0) < (C_0 - 1) h + q' \cdot x' + \frac{\alpha}{2} x_n\}
\]
contains the convex set \(\tilde{S}\) which is generated by \(S'_h := S_h \cap \{x_n = 0\}\) and the segment \([0, s_0 e_n]\). It follows from the convexity of \(\tilde{S}\) that
\[
(6.2) \quad |\tilde{S}| \geq c_n |S'_h| s_0 = c_n \left|S'_h\right| \frac{(C_0 - 1) h}{\alpha \delta}
\]
for some universal \(c_n > 0\).

We apply Proposition 3.3 (see Remark 3.4) for \(\tilde{S}\) which is \(Cd\)-balanced around \(s_0 e_n/2\) and with \(\tilde{h} = C_0 h, \tilde{\delta} = 1/30\), and find that (since \(C_0 \geq 100\))
\[
u(s_0 e_n/2, -t) \geq -h + \frac{\alpha}{2} \frac{s_0}{2} - \frac{C_0 h}{30} \geq \frac{\alpha}{2} (1 - \frac{\delta}{5}) \frac{s_0}{2}
\]
for
\[
-t_0 - c(d) \frac{|\tilde{S}|}{h^{n+1}} \leq -t \leq -t_0.
\]
Observing that a similar consideration holds for negative \(x_n\) and using (6.2) we conclude
\[
\nu(\pm s_0 e_n/2, -t) \geq \frac{\alpha}{2} (1 - \frac{\delta}{5}) \frac{s_0}{2}
\]
for
\[-t_0 - c(d) \frac{\left( |S'_h| \left( \frac{(C_0 - 1)h}{\alpha \delta} \right)^{2p} \right)}{h^{np-1}} \leq -t \leq -t_0,\]

or, from (6.1), for
\[-t_0 - c(d)\delta^{-2p}t_0 \leq -t \leq -t_0.\]

It follows that for such \( t \) we have (since \( u(0, -t) \leq 0) \nabla u(\pm \frac{s_0 e_n}{2}, -t) \cdot (\pm e_n) \geq \frac{\alpha}{2} \left( 1 - \frac{\delta}{2} \right). \)

Setting \( \tilde{q}_1 = \nabla u(\frac{s_0 e_n}{2}, -t) \) and \( \tilde{q}_2 = \nabla u(-\frac{s_0 e_n}{2}, -t) \),
we obtain
\[(\tilde{q}_1 - \tilde{q}_2) \cdot e_n \geq \alpha \left( 1 - \frac{\delta}{2} \right) \geq \frac{\alpha}{1 + \frac{s}{2}} \]
since \( \delta \leq 1 \). From the convexity of \( u(\cdot, -t) \) and the inequalities
\[u(s_0 e_n, -t) \leq u(s_0 e_n, -t_0) \leq \frac{\alpha}{2} s_0 + (C_0 - 1) h \]
\[u(s_0 e_n, -t) \geq \frac{\alpha}{2} s_0 - \frac{C_0 - 1}{20} h \]
we conclude that the tangent planes at \( \pm \frac{s_0 e_n}{2} \) for \( u(\cdot, -t) \) are above \(-2C_0h\) (and therefore \(-C_0^2h\)) at the origin. This implies that
\[
\left( C_0^2 h, \frac{\alpha}{1 + \frac{s}{2}} \right) \in A_{-t}, \quad \text{if} \quad t_0 \leq t \leq t_0 + c(d)\delta^{-2p}t_0
\]
which finishes the proof of the proposition. \qed

Remark 6.6. If hypothesis ii) is satisfied only for a time \(-\tilde{t}\) with \( \tilde{t} \leq t_0 \) i.e
\[S_h := \{ u(x, -\tilde{t}) \leq h + q \cdot x \} \subset \subset \Omega \quad \text{and} \quad S_h \text{ is } d\text{-balanced around } 0,\]
then the same conclusion holds in the smaller time interval
\[
\left( h, \frac{\alpha}{1 + \frac{s}{2}} \right) \in A_{-t}, \quad \text{for} \quad t_0 \leq t \leq t_0 + c(d)\delta^{-2p} \tilde{t}.
\]
Indeed, the only difference appears when estimating \( |S'_h| \) from below: in (6.1) we have to replace the left hand side \( t_0 \) by \( \tilde{t} \).
Remark 6.7. If \( x^* \) denotes the center of mass of the \( d \)-balanced section \( S_h \) at time \(-t_0\), then it follows from the proof of Proposition 6.4 and Remark 2.3 that
\[
u(x,-t) \geq \nu(x^*,-t_0) - C(d) h + \max_{i=1,2} \{ \tilde{q}_i \cdot (x-x^*) \}
\]
for \( t_0 \leq t \leq t_0 + c(d) \delta^{-2p} t_0 \), with
\[
(\tilde{q}_1 - \tilde{q}_2) \cdot e_n \geq \frac{\alpha}{1 + \delta^2}, \quad \tilde{q}_i = \nabla \left( \frac{S\ell e_n}{2}, -t \right).
\]
In other words, if \( \tilde{u} \) is the translation of \( u \) defined by
\[
\tilde{u}(x, t) = \nu(x + x^*, t - t_0) - \nu(x^*, -t_0)
\]
then
\[
\left( C(d) h, \frac{\alpha}{1 + \delta^2} \right) \in A_{-t}(\tilde{u}), \quad \text{for } 0 \leq t \leq c(d) \delta^{-2p} t_0.
\]

We will now proceed to the proof of Theorem 6.1.

Proof of Theorem 6.1. We will denote throughout the proof by \( u_0 := \nu(\cdot, 0) \). Since \( u_0 \geq 0 \), and \( 0 \) is an extremal point for the set \( D = \{ u_0 = 0 \} \) we can find (as in the proof of Theorem 5.3) \( \sigma_0 := \sigma_0(u) > 0 \) small, depending on \( u \), such that if \( 0 \leq h, t \leq \sigma_0 \) then the section
\[
T_{h,-t} := \{ \nu(x, -t) \leq h + q \cdot x \}
\]
of \( \nu(\cdot, -t) \) that has \( x = 0 \) as center of mass is compactly supported in \( \Omega \). Thus, by John’s lemma \( T_{h,-t} \) is \( C_n \)-balanced with respect to the origin.

Let \( 0 < \delta < \delta_0 \) with \( \delta_0 \) small universal constant to be made precise later. Without loss of generality we may assume that \( u_0 \) is tangent to \( |x_n| \) on the line \( x' = 0 \) at the origin, i.e. we have
\[
\lim_{x_n \to 0^+} \frac{u_0(0, x_n)}{x_n} = 1 \quad \text{and} \quad \lim_{x_n \to 0^-} \frac{u_0(0, x_n)}{x_n} = -1.
\]
Hence, by taking \( \sigma_1 = \sigma_1(\delta, u) \) smaller than \( \sigma_0 \), depending also on \( \delta \), we can assume that
\[
\left( \frac{h}{2} (1 + \frac{\delta}{2}) \right) \notin A_0, \quad \text{for } \frac{h}{2} \leq \sigma_1.
\]
Choose \( h << \sigma_1 \). Since \( u_0 \) is Lipschitz in say \( B_a \subset \Omega \) with \( |\nabla u_0| < 1/a \), for some small \( a \) we find (using Proposition 3.3) that at time \(-t_0\), given by
\[
t_0 := c(a) \frac{h^{2p}}{h^{np-1}} = c(a) h^{np+1}
\]
the we have \( u(x, -t_0) \geq u_0(x) - h \) for \( x \in B_a \). This easily implies

\[
(h, \alpha) \in A_{-t_0}, \quad \alpha := 2 \left(1 - \frac{1}{a} h\right).
\]

Also notice that

\[
\left(\tilde{h}, \alpha (1 + \delta)\right) \notin A_0, \quad \text{if } h, \tilde{h} \leq \sigma_2 = \sigma_2(a, \sigma_1).
\]

We choose \( \delta_0 \) such that

\[
M^2 := c(C_n) \delta^{-2p} \geq c(C_n) \delta^{-2p} := C_0^{10(np+1)}
\]

where \( c(d) \) is the constant that appears in Proposition 6.4.

**Lemma 6.8.** As long as \( M^k t_0 \leq \sigma_0 \) and \( C_0^{3k+1} h \leq \sigma_2 \), there exists \( 0 \leq m \leq k \) such that

\[
(6.5) \quad \left(C_0^{3k-m} h, \alpha \frac{1}{1 + \frac{\delta}{2}} \cdots \frac{1}{1 + \frac{\delta}{2m}}\right) \in A_{-M^k t_0}.
\]

**Proof.** We will use induction in \( k \). When \( k = 0 \) we take \( m = 0 \) and we use (6.4).

Assume now that the statement holds for \( k \) and let \( m \) be the smallest so that (6.5) holds. If \( m > 0 \), then

\[
\left(C_0^{3k-(m-1)} h, \alpha \frac{1}{1 + \frac{\delta}{2}} \cdots \frac{1}{1 + \frac{\delta}{2(m-1)}}\right) \notin A_{-M^k t_0}.
\]

Combining this with (6.5), and applying Proposition 6.4 we find that

\[
\left(C_0^{3k-m+2} h, \alpha \frac{1}{1 + \frac{\delta}{2}} \cdots \frac{1}{1 + \frac{\delta}{2m+1}}\right) \in A_{-t}, \quad \text{if } t \leq M^{k+2} t_0
\]

which proves (6.5) for the pair \((k+1, m+1)\).

If \( m = 0 \), then \((C_0^{3k} h, \alpha) \in A_{-M^k t_0}\). On the other hand, since \( C_0^{3k+1} h \leq \sigma_2 \) we have \((C_0^{3k+1} h, \alpha (1 + \delta)) \notin A_0\), thus

\[
(C_0^{3k+1} h, \alpha (1 + \delta)) \notin A_{-M^k t_0}.
\]

Hence, by Proposition 6.4

\[
(C_0^{3k+2} h, \alpha \frac{1}{1 + \frac{\delta}{2}}) \in A_{-t}
\]

for \( t \leq M^{k+2} t_0 \) which again proves (6.5) for the pair \((k+1, 1)\). This concludes the proof of the lemma. \(\square\)
We will now finish the proof of the theorem. Since $M \geq C_0 \alpha^{(np+1)} t_0 = c h^{np+1}$ we see that for the last $k$ for which $M^k t_0 \leq \sigma_0$ we satisfy

$$C_0^{3k+1} h \leq C_0 M^{k+2} h \leq C(\sigma_0) h^{2} < \sigma_2$$

if $h \ll \sigma_2$ is sufficiently small. Also, if $\delta$ is chosen small, depending on $\sigma_0$ and $T$, for the last $k$ we also have $M^{k+2} t_0 \geq T$. We conclude from the lemma above that

$$(\sigma_0) h^{2}, \alpha e^{-\delta}) \in A_{-T}$$

and by letting $h \to 0$ we obtain

$$(0, 2 e^{-\delta}) \in A_{-T}.$$ 

Finally, letting $\delta \to 0$ we conclude that $(0, 2) \in A_{-T}$ which proves the theorem. \(\square\)

7. \(C^{1,\alpha}\) REGULARITY - I

In the next two sections we establish \(C^{1,\alpha}\) interior regularity of solutions to (3.1). They are based on quantifying the result of Theorem 6.1. In the elliptic case \(C^{1,\alpha}\) regularity is obtained by a compactness argument. However, in our setting compactness methods would only give \(C^1\) continuity for exponents $p \leq \frac{1}{n-2}$. The reason for this is that in the parabolic setting it is more delicate to normalize a solution in space and time.

The main result of this section is the following Theorem (see Definition 2.1).

**Theorem 7.1.** Let $u$ be a solution to (3.1) in $\Omega \times [-T, 0]$ and assume there exists a section of $u(x, 0)$ which is $d$-balanced around $0$ and is compactly supported in $\Omega$.

a) If the initial data $u(x, -T)$ is $C^{1,\beta}$ at $0$ in the $e$ direction then $u(x, 0)$ is $C^{1,\alpha}$ at the origin in the $e$ direction with $\alpha = \alpha(\beta, d)$ depending on $\beta$, $d$ and the universal constants.

b) If $u(0, 0) > u(0, -T)$ then $u(x, 0)$ is $C^{1,\alpha}$ at the origin with $\alpha = \alpha(d)$ depending on $d$ and the universal constants.

Part b) will be improved in Theorem 8.4 in which we show that $\alpha$ can be taken to be a universal constant. As a consequence we obtain Theorem 1.1.

**Proof of Theorem 7.1.** In view of Remark 2.3 at a point $(x, t)$ for which $u(x, t) \leq c_n$, with $c_n$ small depending only on $n$ the centered section $T_h(x, t)$ at $x$, for small $h$, is compactly supported in $\Omega$. Clearly $u(x, 0)$ is $C^{1,1}$ at an interior point of the set $\{u(x, 0) = 0\}$. Thus we can apply Theorem 7.1 with $d$ depending only on $n$ and
\( \beta = 1 \) and obtain the desired result. If \( c_n < u(x, t) < 1 \) then we can apply directly Theorem 8.4 and obtain the same conclusion. The second part of the theorem follows similarly.

\[ \square \]

The following simple lemma gives the relation between the sets \( A_t \) defined in Definition 6.2 and \( C^{1,\alpha} \) regularity. Its proof is straightforward and is left to the reader.

**Lemma 7.2.** Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a convex function with \( f(0) = 0 \) and let \( q \) be a sub-gradient of \( f \) at \( x = 0 \). If, for some \( x \), we have \( f(x) - q \cdot x \geq a \cdot |x|^{1+\alpha} \), then

\[ (h, a \cdot |x|^{1+\alpha}) \in A(f) \]

with \( h = a \cdot |x|^{1+\alpha} \). Conversely, if for some number \( h \), (7.1) holds, then

\[ f(x) - q \cdot x \geq \frac{a}{4^{\alpha+1}} |x|^{1+\alpha} \]

for some \( x \) with \( |x| = 4 \cdot (\frac{h}{a})^{\frac{1}{1+\alpha}} \).

As a consequence we obtain the following useful corollary.

**Corollary 7.3.** The function \( u(x, 0) \) is \( C^{1,\alpha} \) at \( 0 \) in the \( e_n \) direction if and only if

\[ (h, Ch^{\frac{1}{\alpha+1}}) \notin A_0 \]

for some large \( C \) and for all small \( h \).

Theorem 7.1 will follow from the following lemma.

**Lemma 7.4.** Assume that \( u : \Omega \times [-T, 0] \rightarrow \mathbb{R} \) is a solution of (3.1) such that \( u(0, 0) = 0, u(x, -T) > 1 \) on \( \partial \Omega \), and

\[ B_{\frac{1}{2}} \subset \{ u(x, 0) < 1 \} \subset \{ u(x, -T) < 1 \} \subset B_1. \]

Choose \( \delta_0(d) \) sufficiently small, so that

\[ c(C_n d) \delta_0^{-2p} = C_0^{12(np+1)} := M, \]

where \( c(C_n d) \) and \( C_0 \) are the constants from Proposition 6.4 and \( C_n \) the constant from Lemma 2.4. Assume also that \( (C_0^{-k}, (1 + \delta_0)^{-l}) \in A_{-t_0} \), for some \( k \geq 0 \) and some \( 0 < t_0 \leq T \).

There exists a constant \( C_1(d) > 0 \) such that if \( m_0 \) is an integer satisfying

\[ 3m_0 \leq k - l - C_1(d) \quad \text{and} \quad M^{m_0} t_0 \geq T, \]
then
\[
(C_0^{-k}, (1 + \delta_0)^{-l-C_1(d)}) \in A_{-T}.
\]

**Proof.** Define \(\eta : \mathbb{N} \to \mathbb{Z}\) as

\[
(C_0^{-k}, (1 + \delta_0)^{\eta(k) - 1}) \in A_{-t_0} \quad \text{but} \quad (C_0^{-k}, (1 + \delta_0)^{-\eta(k)}) \notin A_{-t_0}.
\]

Clearly,

i) \(\eta\) is nondecreasing i.e \(\eta(k+1) \geq \eta(k)\),

ii) \(\eta(0) \geq -C_1(d)\), and

iii) \(\eta(k) < l\) (by assumption).

For each integer \(m\) with \(0 \leq m \leq k - l - C_1(d)\), we define \(s_m\) as the largest \(s\), \(0 \leq s \leq k\) that satisfies

\[
\eta(k-s) \leq l + 3m - s.
\]

Notice that we satisfy the inequality above when \(s = 0\) and the opposite inequality when \(s = k\). We obtain:

\[
(7.4) \quad \eta(k-s_m) = l + 3m - s_m, \quad \text{thus} \quad s_m - 3m \leq l + C_1(d).
\]

Also, from the definition of \(s_m\) we find that \(s_{m+1} \geq s_m + 3\).

**Claim:** There exists \((r_1, r_2, r_3) \in \mathbb{Z}^3, r_i \geq 0, such that**

\[
(7.5) \quad \left(C_0^{-r_1-k}, (1 + \delta_0)^{r_2-l} \frac{1}{(1 + \delta_0)^{r_3}} \right) \in A_{-t_m}, \quad t_m = M^m t_0
\]

with

\[
(7.6) \quad r_1 - r_2 + r_3 = 3m, \quad r_3 \leq m, \quad r_1 + r_3 \leq s_m ( \Leftrightarrow 0 \leq r_2 \leq s_m - 3m).
\]

**Proof of Claim:** In order to simplify the notation, instead of (7.5) we write

\[
(r_1, r_2, r_3) \in A_{-t_m}
\]

We will use induction on \(m\). For \(m = 0\) the claim holds from our assumption \((C_0^{-k}, (1 + \delta_0)^{-l}) \in A_{-t_0}\), if \((r_1, r_2, r_3) = (0, 0, 0)\).

Assume now that the claim holds for \(m\). Consider the pairs

\[
(r_1 + s, r_2, r_3 - s), \quad \text{if} \quad 0 \leq s \leq r_3
\]

\[
(r_1 + s, r_2 + s - r_3, 0), \quad \text{if} \quad s \geq r_3
\]
where \((r_1, r_2, r_3)\) comes from the induction step \(m\). When \(s = 0\) the first pair belongs to \(A_{-t_m}\), by the induction hypothesis, and when \(s = s_m - r_1\) the second pair doesn’t belong to \(A_{-t_m}\), since for that choice of \(s\) the second pair is
\[
(\begin{array}{c} C_0^{s_m-k} (1+\delta_0)^{-(l+3m-s_m)} \end{array}, (1+\delta_0)^{-\eta(k-s_m)}) \notin A_{-t_0}
\]
from the definition of the function \(\eta\) given above. Note that for \(s = r_3\) the two pairs are the same.

It follows that either there exists an \(s < r_3\) such that
\[(r_1 + s, r_2, r_3 - s) \in A_{-t_m} \quad \text{and} \quad (r_1 + s + 1, r_2, r_3 - s - 1) \notin A_{-t_m}\]
or, there exists an \(r_3 \leq s < s_m - r_1\) such that
\[(r_1 + s, r_2, r_3 - s, 0) \in A_{-t_m} \quad \text{and} \quad (r_1 + s + 1, r_2 + s + 1 - r_3, 0) \notin A_{-t_m}.\]

In either case we can apply Proposition 6.4. Indeed, the hypothesis (7.2) and Lemma 2.4 imply the existence of a section \(S_h\) of \(u(\cdot, t)\) that satisfies ii) in Proposition 6.4 for any \(h \leq 1\) and any \(t \in [-T, 0]\). More precisely, \(S_h\) is \(C_n\)-balanced section around 0 and it is compactly supported in \(\Omega\). We conclude that either \((r_1 + s + 2, r_2, r_3 - s + 1)\) for some \(0 \leq s < r_3\) or \((r_1 + s + 2, r_2 + s - r_3, 1)\) for some \(s \geq r_3\) belongs to \(A_{-Mt_m}\). Notice that in both cases the sum of the first and third component is less than \(s_m + 3 \leq s_{m+1}\). This concludes the proof of the claim.

The lemma follows now from the claim above. Since \(M^{m_0} t_0 \geq T\) and
\[r_1 \leq s_{m_0} \leq l + 3m_0 + C_1(d), \quad r_2 \geq 0,
\]
we conclude that
\[
(\begin{array}{c} C_0^{C_1(d)+l+3m_0-k} (1+\delta_0)^{-l e^{-\delta_0}} \end{array}, (1+\delta_0)^{-\eta(k-s_m)}) \in A_{-T}.
\]

Remark 7.5. If we assume that hypothesis (7.2) holds only on a smaller interval \(t \in [-T_1, 0]\) instead of the full interval \([-T, 0]\) then the same conclusion holds by replacing \(C_1(d)\) with a constant \(C_1(d, T/T_1)\).

The only difference occurs in the inductive step that shows \((r_1, r_2, r_3) \in A_{-t_m}\), and we have to distinguish whether \(t_m \leq T_1\) or \(t_m > T_1\). The case when \(t_m \leq T_1\) is the same and we obtain \(t_{m+1} = Mt_m\) as before. In the case when \(t_m > T_1\) we apply Remark 6.6 of Proposition 6.4 and obtain \(t_{m+1} = t_m + MT_1\). This second
case occurs at most $T/(MT_1) = C(d, T/T_1)$ times and therefore we need to replace $m_0$ by $m_0 + C(d, T/T_1)$.

Remark 7.6. If in the assumption (7.2) we have a constant $a$ instead of 1 i.e

$$B_\frac{1}{2} \subset \{ x : u(x, 0) < a \} \subset \{ x : u(x, -T) < a \} \subset B_1$$

then the conclusion is the same, except that $k \geq 0$ is replaced by $k \geq C(a)$ and $C_1(d)$ is replaced by $C_1(d, a)$.

Indeed, $\tilde{u}(x, t) := \frac{1}{a} u(x, a^{1-np} t)$ satisfies the assumptions of the lemma with $\tilde{t}_0 = a^{np-1} t_0$ and $\tilde{T} = a^{np-1} T$ and $(C_0^{-k+C(a)}, (1 + \delta_0)^{-l-C(a)}) \in A_{-\tilde{t}_0}(\tilde{u})$, hence the conclusion of the lemma follows.

Next we prove Theorem 7.1.

Proof of Theorem 7.1. From the continuity of $u$ we can assume that, after a linear transformation, we have the following situation: $u(0, 0) = 0, u(x, -T_1) > 1$ on $\partial\Omega$ and

$$B_{\frac{1}{2}} \subset \{ u(x, 0) < 1 \} \subset \{ u(x, -T_1) < 1 \} \subset B_1$$

for some small $T_1 \in (0, T]$.

Let $k \geq 0, l$ be integers such that

$$(C_0^{-k}, (1 + \delta_0)^{-l}) \in A_0.$$  

In view of Corollary 7.3 it suffices to show that there exists $\epsilon := \epsilon(d, \beta)$ small (or $\epsilon = \epsilon(d)$ for the second part) such that $l \geq \epsilon k$ for all large $k$. Assume by contradiction that

$$l < \epsilon k$$  

for a sequence of $k \to \infty$.

Then, from the Lipschitz continuity of $u(x, 0)$ in $B_{1/4d}$ and Proposition 3.3 we find (as in the proof of Theorem 6.1) that $(2C_0^{-k}, (1 + \delta_0)^{-l}) \in A_{-t_0}$ or, for $k$ large enough

$$(7.7) \quad (C_0^{-k}, (1 + \delta_0)^{-l-1}) \in A_{-t_0} \quad \text{with} \quad t_0 := c(d)C_0^{-k(np+1)}.$$  

Now we can apply Remark 7.5 and conclude that if

$$3m_0 \leq k - l - C_1$$  

and $M^{m_0} t_0 \geq T$

then

$$(C_0^{C_1+3m_0+l-k}, (1 + \delta_0)^{-l-C_1}) \in A_{-T} \quad \text{with} \quad C_1 = C_1(d, T/T_1).$$
We choose $m_0 = \left\lceil \frac{k}{6} \right\rceil$ to be the smallest integer greater than $k/6$. Clearly both inequalities for $m_0$ are satisfied for $k$ large (we assume $\varepsilon \leq 1/6$) since $M = C_0^{12(np+1)}$ and

$$M^{m_0} t_0 \geq C_0^{2k(np+1)} t_0 \to \infty \quad \text{as} \quad k \to \infty.$$ 

Thus

$$(C^{-k/6}_0, (1 + \delta_0)^{-2k}) \in A_{-T} \quad \text{for a sequence of} \quad k \to \infty.$$ 

We reached a contradiction if $u(0, -T) < 0$ (we choose $\varepsilon = 1/6$).

If we assume that $u(0, -T) = 0$ and $u(x, -T)$ is $C^{1,\beta}$ at 0 in the $e_n$ direction then it follows from Corollary 7.3,

$$\frac{\log C_0}{6} \beta \leq 2\varepsilon \log(1 + \delta_0)$$

and we reach a contradiction again by choosing $\varepsilon(d, \beta)$ small. $\square$

8. $C^{1,\alpha}$ Regularity - II

In this section we prove the main estimates. Let $u$ be a solution defined in $\Omega \times [-T, 0]$ and assume that $u > l(x)$ on $\partial \Omega \times [-T, 0]$ for some linear function $l(x)$. We are interested in obtaining $C^{1,\alpha}$ estimates in $x$ at time $t = 0$ in any compact set $K$ included in the section $\{u(x, 0) < l(x)\}$. Theorem 7.1 gives such estimates but with the exponent $\alpha$ depending also on the distance from $K$ to $\partial \{u(x, 0) < l(x)\}$ which is not desirable.

We can assume that after rescaling we are in the following situation:

(8.1) \(\lambda (\det D^2 u)^p \leq u_t \leq \Lambda (\det D^2 u)^p, \quad \text{in} \quad \Omega \times [-T, 0],\)

(8.2) \(u > 1 \text{ on } \partial \Omega \times [-T, 0], \quad \Omega \subset B_1(y) \text{ for some } y \in \mathbb{R}^n,\)

(8.3) \(u_0(x) := u(x, 0) \text{ satisfies } u_0(0) = 0.\)

First two theorems deal with the case $p < \frac{1}{n-2}$ and $p = \frac{1}{n-2}$. In view of the results of Section 3, $C^{1,\alpha}$ (or $C^1$) continuity is expected for these exponents regardless of the behavior of the initial data at time $-T$.

**Theorem 8.1.** Let $u$ be a solution of (8.1)-(8.3) with $0 < p < \frac{1}{n-2}$ and $T \leq 1$. Then,

$$\|u_0\|_{C^{1,\alpha}(K)} \leq C(K) T^{-\gamma} \quad \text{for any set} \quad K \subset \subset \{u_0(x) < 1\}.\)

The constants $\alpha, \gamma > 0$ are universal (depend only on $n, p, \lambda$ and $\Lambda$), and $C(K)$ depends on the universal constants and the distance between $K$ and $\partial \{u_0(x) < 1\}$.\)
The example in Proposition 4.8 shows that the Theorem 8.1 fails when $p > \frac{1}{n-2}$. For the critical exponent $p = \frac{1}{n-2}$ we obtain a logarithmic modulus of continuity of the gradient.

**Theorem 8.2.** Under the same assumptions and notation as in Theorem 8.1, if $p = \frac{1}{n-2}$, then

$$|\nabla u_0(x) - \nabla u_0(y)| \leq C(K) \|\log|x-y\|^{-\alpha}T^{-\gamma}, \quad \forall x, y \in K.$$  

Next two theorems deal with the case of general exponents $p > 0$. First theorem states that if the initial data $u(x, -T)$ is $C^{1,\beta}$ in the $e$ direction then $u(x, 0)$ is $C^{1,\alpha}$ in the $e$ direction with $\alpha = \alpha(\beta)$.

**Theorem 8.3.** Let $u$ be a solution of (8.1)-(8.3) with $p > 0$. If

$$\partial_e u(\cdot, -T) \in C^\beta(\bar{S}), \quad S := \{u(x, -T) < 1\},$$

for some $\beta > 0$ small, then for any set $K \subseteq \{u_0(x) < 1\}$

$$\|\partial_e u_0\|_{C^\alpha(K)} \leq C(K)\|\partial_e u(\cdot, -T)\|_{C^\beta(\bar{S})}.$$  

The constant $\alpha = \alpha(\beta) > 0$ depends on $\beta$ and the universal constants.

The second Theorem is a pointwise $C^{1,\alpha}$ estimate at points that separated from the initial data at time $-T$.

**Theorem 8.4.** Let $u$ be a solution of (8.1)-(8.3) with $p > 0$. If

$$u(0, 0) - u(0, -T) := a > 0$$

then, there exists $q \in \mathbb{R}^n$ for which

$$|u_0(x) - q \cdot x| \leq C(a) |x|^{1+\alpha}$$

with $\alpha$ universal and $C(a)$ depends on $a$, the distance from 0 to $\partial \{u_0 < 1\}$ and the universal constants.

The theorems above will follow from a refinement of Lemma 7.4. We show that we may choose $\delta_0$ universal in Lemma 7.4 and satisfy the conclusion at a point $\hat{x}$ possibly different from the origin. The key step is to use the part b) of Lemma 2.4.

**Lemma 8.5.** Let $u : \Omega \times [-T, 0] \to \mathbb{R}$ be a solution of (5.1) such that $u > 1$ on $\partial \Omega \times [-T, 0]$ and $u(0, 0) = 0$. Let $E$ be an ellipsoid centered at the origin such that $|E| \geq 2^{-j}|B_1|$ and

$$E \subset \{u(x, 0) < 1\} \subset \{u(x, -T) < 1\} \subset B_1(y).$$
Let \( \delta_0, M \) be universal as they appear in Lemma 7.4 for \( d = C_n \) the constant from Lemma 2.4. Then, there exists a constant \( C(j) \) (depending on universal constants and \( j \)) such that if \( k \geq 0, l \) are integers and

\[
(C_0^{-k}, (1 + \delta_0)^{-l}) \in A_{-t_0} \quad \text{for some } t_0 \in (0, T]
\]

and \( m_0 \) is an integer satisfying

\[
3m_0 \leq k - l - C(j), \quad M^{m_0} t_0 \geq C(j) T,
\]

then we can find \( \tilde{x} \in \{ u(x, -T) < 1 \} \) such that

\[
u(x, -T) \geq \nu(\tilde{x}, -\tilde{t}) - C_0^{C(j) + l + 3m_0 - k} + \max_{i=1,2} \{ q_i \cdot (x - \tilde{x}) \},
\]

with

\[
\tilde{t} = T - \frac{T}{C(j)} (q_2 - q_1) \cdot e_n \geq (1 + \delta_0)^{-l - C(j)}.
\]

Remark 8.6. Another way of stating the conclusion of the lemma is that the translation

\[
\tilde{u}(x, t) := u(x + \tilde{x}, t - \tilde{t}) - u(\tilde{x}, -\tilde{t}), \quad \tilde{t} = T - \frac{T}{C(j)}
\]

satisfies

\[
(C_0^{C(j) + l + 3m_0 - k}, (1 + \delta_0)^{-l - C(j)}) \in A_{-\tilde{t}_0/C(j)}(\tilde{u}).
\]

Proof. The proof is by induction in \( j \).

The case \( j = 1 \) is proved in Lemma 7.4. Indeed, since \( B_{1/2} \subset E \subset B_{1}(y) \subset B_{d/2} \) we see that the hypothesis (7.2) is satisfied and the conclusion holds for \( \tilde{x} = 0 \).

For a general \( j \) we start the proof as before. The only difference here is that we cannot guarantee in the induction step \( m \Rightarrow m + 1 \) that there exists a section at time \( -t_m = -M^{m_0} t_0 \) which is \( C_n d = C_n^2 \) balanced around the origin.

Let's assume this fails for a first integer \( m \). By Lemma 2.4 we can find a \( C_1(j) \)-balanced section (with \( C_1(j) > C_n^2 \)) at the time \( -t_m \). The idea is to apply Proposition 6.4 as in the induction step and then to “replace” the origin with the center of mass \( x^* \) of this section. To be more precise, by Remark 6.7, the translation

\[
\tilde{u}(x, t) = u(x^* + x, t - t_m) - u(x^*, -t_m)
\]

satisfies

\[
(C_2(j) C_0^{r_1 - k}, (1 + \delta_0)^{r_2 - l} e^{-\delta_0}) \in A_{-\tilde{t}_0}(\tilde{u})
\]

with

\[
\tilde{t}_0 := c_1(j) t_m = c_1(j) M^{m} t_0
\]
and from (7.4)-(7.6)
\[ r_1 \leq 3m + r_2, \quad 0 \leq r_2 \leq l + C_3(j). \]

Here we assumed that \( T > t_m + \tilde{t}_0 \), otherwise the proof is the same as before by taking \( \tilde{x} = 0 \), and there is no need to change the origin. Notice that \( m_0 > m \) if \( C(j) > 1/c_1(j) \).

The above imply
\[ (C^{-\tilde{k}}, (1 + \delta_0)^{-\tilde{l}}) \in A_{-t_0}(\tilde{u}) \]
with
\[ \tilde{l} := l - r_2 + C_1 \quad \text{and} \quad \tilde{k} = k - (3m + r_2) - C_4(j). \]

Now we apply the induction \((j - 1)\)-step for \( \tilde{u} \). First we set
\[ \tilde{m}_0 := m_0 - m \quad \text{and} \quad \tilde{T} := T - t_m, \]
and we have \( \tilde{T} \geq \tilde{t}_0 \geq c_1(j)t_m \geq c_2(j)T \).

By Lemma 2.4 the maximal ellipsoid centered at the origin and included in the set \( \{ \tilde{u}(x, 0) < \tilde{a} \} \) has volume greater than \( 2^{j-1}|B_1| \). The constant \( \tilde{a} = 1 - u(x^*, -t_m) \) and by Remark 2.3 \( c_3(j) \leq \tilde{a} \leq 1/c_3(j) \). Thus in order to apply the rescaled induction step for \( \tilde{u} \) we need to check that (see Remark 7.6)
\[ \tilde{k} \geq C'(j), \quad 3\tilde{m}_0 \leq \tilde{l} - \tilde{k} - C'(j), \quad M^{\tilde{m}_0} \tilde{t}_0 \geq \tilde{T}C'(j) \]
for some large constant \( C'(j) \).

If \( C(j) \) is sufficiently large then
\[ M^{\tilde{m}_0} \tilde{t}_0 = M^{m_0 - m} c_1(j) M^m t_0 \geq C(j)c_1(j)T \geq C'(j)\tilde{T}, \]
and
\[ \tilde{k} - (\tilde{l} + 3\tilde{m}_0) = k - l - 3(m + m_0) - C_4(j) - C_1 \]
\[ = (k - l - 3m_0) - C_5(j) \]
\[ \geq C(j) - C_5(j) \geq C'(j). \]
\[ (8.5) \]

and also,
\[ \tilde{k} \geq k - (3m + l) - C_5(j) - C_4(j) \]
\[ \geq k - (3m_0 + l) - C_6(j) \geq C(j) - C_6(j) \geq C'(j). \]

From the equality in (8.5), \( \tilde{T} \geq c_2(j)T \) and \( \tilde{l} \leq l + C_1 \) we clearly obtain the desired result when we apply the induction step by choosing \( C(j) \) sufficiently large. \( \square \)
Remark 8.7. If in addition to the hypothesis of the lemma we have
\[ u(0,0) - u(0,-T) \geq a, \]
then
\[ u(\tilde{x},-\tilde{t}) - u(\tilde{x},-T) \geq \frac{a}{C(j)} - C_0^{C(j)+l+3m_0-k}, \quad \text{for } \tilde{t} = T - \frac{T}{C(j)}. \]
This and the conclusion of the lemma imply
\[ a \leq C_0^{C(j)+l+3m_0-k}, \]
with \( C(j) \) a constant larger than the previous ones.

Proof of Remark 8.7. From the proof of Lemma 8.5 we see that when for a certain \( m \) we replace 0 with the center of mass \( x^* \) of the section \( S_h := \{ u(x,-t_m) \leq l(x) \} \) (for \( l \) linear) with \( h = C_0^{r_1-k} \), then
\[ u(x^*,-t_m) - l(x^*) \geq -C(j)h. \]
On the other hand, we have
\[ u(0,-T) - l(0) \leq u(0,-T) - u(0,0) \leq -a, \]
and since \( u(x,-T) - l(x) \) is negative in \( S_h \), at \( x^* \) we have
\[ u(x^*,-T) - l(x) \leq -\frac{a}{C(j)}. \]
In conclusion
\[ u(x^*,-t_m) \geq u(x^*,-T) + \tilde{a}, \quad \tilde{a} := \frac{a}{C(j)} - C(j)h. \]
Since we perform this change of origin at most \( j \) times we obtain the desired result.

Proof of Theorem 8.1. Let
\[ (C_0^{-k},(1+\delta_0)^{-l}) \in A_0, \quad \text{for some } k \geq 0 \]
where \( C_0 \) and \( \delta_0 \) are the constants taken from Lemma 8.3. Let \( E \) be an ellipsoid of volume \( 2^{-j} \) around the origin included in the set \( \{ x : u_0(x) < 1 \} \) where \( j \) depends on \( \text{dist}(k,\partial \{ u_0(x) < 1 \}) \). In view of Lemma 7.2 it suffices to prove the existence of constants \( \epsilon_0 \) and \( C_1 \) universal and \( \tilde{C}(j) \) such that
\[ l \geq \epsilon_0 (k + C_1 \log T - \tilde{C}(j)). \]
Since our assumption \((C_0^{-k}, (1 + \delta_0)^{-l}) \in A_0\) implies that \(l \geq -C_0(j)\) if \(k \geq 0\), it follows that (8.6) is satisfied, for some \(\tilde{C}(j)\), if

\[ k \leq -C_1 \log T + C_1(j), \]

where \(C_1(j)\) will be specified later. Assume, by contradiction that (8.6) does not hold. Thus, since \(T \leq 1\),

(8.7) \[ \epsilon_0 k > l, \quad \text{for some} \quad k > -C_1 \log T + C_1(j) \geq C_1(j). \]

Using the Lipschitz continuity of \(u_0\) we obtain, as in (7.7), that

\[ (C_0^{-k}, (1 + \delta_0)^{-l}) \in A_0 \Rightarrow (C_0^{-k+1}, (1 + \delta_0)^{-l} - C(j) C_0^{-k}) \in A_{-\tilde{t}_0} \]

with \(\tilde{t}_0 = c(j) C_1^{-k(np+1)}\) which implies that

\[ (C_0^{-k+1}, (1 + \delta_0)^{-l-1}) \in A_{-t_0}. \]

We now apply Lemma 8.5 with \(m_0 = \lceil \frac{k}{3} \rceil\) and check that he hypotheses are satisfied. Recall that \(M = C_0^{12(np+1)}\) hence

\[ M^{m_0} t_0 \geq c(j) C_0^{12m_0-k(np+1)} \geq C(j) \geq C(j) T. \]

Also, \(l < \epsilon_0 k\) implies that

(8.8) \[ k \geq \frac{2k}{3} \geq 3m_0 + l + C(j) \]

by choosing \(C_1(j)\) sufficiently large.

Thus, Lemma 8.5 holds. Now we apply the estimate (6.1) for the translation function \(\tilde{u}\) of (8.4) that appears in the conclusion of Lemma 8.5. In our case

\[ \tilde{h} = C_0^{C(j)+3m_0+l-k}, \quad \widetilde{\alpha} = (1 + \delta_0)^{-l-C(j)}, \quad \tilde{t}_0 = \frac{T}{C(j)}. \]

Since \(S_{\tilde{h}} \subset B_1(y)\) we have \(|S_{\tilde{h}}'| \leq C\), hence

\[ \tilde{h}^1-(n-2)p \widetilde{\alpha}^{-2p} \geq \frac{T}{C_2(j)}. \]

Using (8.8) we have

\[ (1 - (n-2)p)(-\frac{k}{3}) \log C_0 + 2pl \log(1 + \delta_0) \geq \log T - C_3(j) \]

or

\[ l - 2\epsilon_0 k \geq C \log T - C_4(j), \quad \epsilon_0 := \frac{(1 - (n-2)p) \log C_0}{12p \log(1 + \delta_0)} \]

and \(C\) universal. We obtain the inequality

\[ \epsilon_0 k \leq C |\log T| + C_4(j) \]
which contradicts our assumption (8.7) if we choose the constants $C_1$ and $C_1(j)$ appropriately. This concludes the proof of the theorem.

We will next sketch the proof of Theorem 8.2 for the case $p = \frac{1}{n-2}$.

Proof of Theorem 8.2. The proof is the same as above with the difference that we need to replace $k$ by $\log k$ in (8.6), i.e. we need to show that there exists $\epsilon_0$ and $C_1$ universal such that

\[
(8.9) \quad l \geq \epsilon_0 (\log k + C_1 \log T) - \tilde{C}(j).
\]

After we apply Lemma 8.5 we know that the translation $\tilde{u}$ is above an angle of opening $\tilde{\alpha}$ at time $-\tilde{t}_0$ and it separates away from it at most a distance $\tilde{h}$ at time 0. Now we use the stronger estimate (rescaled) obtained in Proposition 4.7 instead of (6.1). We find

\[
\tilde{h} \geq c(j) e^{-C\tilde{\alpha}^{-1} \tilde{t}_0 - \frac{2}{2}}.
\]

hence

\[
C_0^{C(j)+3m_0+1-k} \geq e^{-\frac{C(j)(1+\delta_0)t}{T}}.
\]

We obtain

\[
k \leq \frac{C(j) (1 + \delta_0)t}{T C},
\]

or

\[
l \geq 2\epsilon_0 \log k + C \log T - C(j),
\]

and we finish the proof as before. □

We will now proceed with the proof of Theorem 8.3

Proof of Theorem 8.3. We begin by observing that since $u_0(0) = 0$, then

\[
T \leq C(\|u(\cdot, -T)\|_{L^\infty(\bar{S})}).
\]

We want to prove that if $(C_0^{-k}(1 + \delta_0)^{-l}) \in A_0$, for some $k \geq 0$, then

\[
l \geq \epsilon_0 k + C(j, a) \quad \text{with} \quad a := \|\partial_n u(\cdot, -T)\|_{C^{\beta}(\bar{S})}
\]

for some $\epsilon_0$ depending on $\beta$ and universal constants. To show (8.10) we argue similarly as before. If (8.10) doesn’t hold, then

\[
\epsilon_0 k > l, \quad \text{for some} \quad k > C_1(j, a).
\]
We set \( m_0 = \lfloor \frac{k}{6} \rfloor \) and that the hypotheses of Lemma 8.5 are clearly satisfied. We find that
\[
\left( C_0^{C(j)+3m_0+l-k}, (1+\delta_0)^{-l-C(j)} \right) \in A_{-\tilde{T}}(\tilde{u})
\]
from which we conclude that
\[
\left( C_0^{-\frac{k}{C(j)}}, (1+\delta_0)^{-l-C(j)} \right) \in A_{-\tilde{T}}(\tilde{u}).
\]
Using that \( \partial_{e_n} u(\cdot, -T) \in C^\beta \) at \( \hat{x} \) we obtain
\[
\frac{\log(1+\delta_0)}{\log C_0} (l + C(j)) \geq \frac{\beta}{\beta + 1} \frac{k}{3} - C(j, a)
\]
from which we derive a contradiction if \( \epsilon_0(\beta) \) is chosen sufficiently small and \( C_1(j, a) \) is chosen large. This concludes the proof of our theorem. \( \Box \)

We finish with the proof of Theorem 8.4.

Proof of Theorem 8.4. We use the previous notation. It suffices to show that for some \( \epsilon_0 \) universal
\[
l \geq \epsilon_0 k - C(j, a).
\]
From Proposition 8.12 we obtain the bound \( T \leq C(j, a) \). Now the proof is the same as before. In view of the Remark 8.7 our hypothesis implies that
\[
C_0^{C(j)+3m_0+l-k} \geq a,
\]
and the conclusion clearly follows. \( \Box \)
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Department of Mathematics, Columbia University, New York, USA
E-mail address: pdaskalo@math.columbia.edu

Department of Mathematics, Columbia University, New York, USA
E-mail address: savin@math.columbia.edu