Abstract: In a recent paper [16], boundary contributions in BCFW recursion relations have been related to roots of amplitudes. In this paper, we make several analyses regarding to this problem. Firstly, we use different ways to re-derive boundary BCFW recursion relations given in [16]. Secondly, we generalize factorization limits to $z$-dependent ones, where information of roots is more transparent. Then, we demonstrate our analysis with several examples. In general, relations from factorization limits cannot guarantee to find explicit expressions for roots.

Keywords: Root, Factorization.
## Contents

1. Introduction

2. A new derivation of on-shell recursion relations with boundary contributions

3. Strategy to find roots

4. $z$-parameterized factorization limits
   4.1 Conventions and useful results
   4.2 Poles with nontrivial $z$-dependence
      - 4.2.1 The pole $\langle i|k \rangle = 0$ from the cut $P_{iI}$ with $k \in I$
      - 4.2.2 The pole $[j|k] = 0$ from the cut $P_{jJ}$ with $k \in J$
   4.3 Poles without $z$-dependence
      - 4.3.1 The $[i|j] \to 0$ limit
      - 4.3.2 The $\langle i|j \rangle \to 0$ limit

5. Examples
   5.1 Example I—MHV amplitudes
      - 5.1.1 Poles without $z$-dependence
      - 5.1.2 Poles with $z$-dependence
      - 5.1.3 The $P_{12}$ pole
   5.2 Example II—The Einstein-Maxwell Theory
      - 5.2.1 The four-point amplitude $M_4(1_1^-\gamma, 2_1^+, 3_1^{-2}, 4_1^{+2})$
      - 5.2.2 The four-point amplitude $M_4(1_1^-\gamma, 2_2^+, 3_1^-, 4_1^{-2})$
      - 5.2.3 The five-point amplitude $M(1_1^{-1}, 2_2^{+1}, 3_3^{-1}, 4_4^{+1}, 5_5^{-2})$
   5.3 Example III—The six-gluon amplitude $M_6(1_1^-, 2_1^-, 3_1^-, 4_1^+, 5_1^+, 6_1^+)$
      - 5.3.1 Poles under factorization limits without $z$-dependence
      - 5.3.2 Poles under factorization limits with $z$-dependence
      - 5.3.3 True values of roots

6. Conclusion
1. Introduction

The standard method to calculate scattering amplitudes in quantum field theory relies on Feynman diagrams. However, such computations become extremely complex with increasing external particles. Naturally, more efficient methods are desired. Among all developed methods, the on-shell method is particularly promising. Unitarity-cut method \[ \text{[1, 2, 3, 4, 5, 6, 7, 8]} \] is very powerful for the one-loop as well as higher loop calculations. On-shell recursion relations \[ \text{[9, 10]} \] are not only very useful for practical calculations but also helpful to understand many properties of quantum field theories.

In the derivation of on-shell recursion relation, one expresses the amplitude as an analytic function \( M(z) \) of single complex variable \( z \) with momenta deformation \( p_i \rightarrow p_i - zq, p_j \rightarrow p_j + zq \), where \( q^2 = q \cdot p_i = q \cdot p_j = 0 \). The function \( M(z) \) has single poles at finite positions of \( z \) as well as possibly a multiple pole at \( z = \infty \) (i.e., the boundary). The behavior of \( M(z) \) around finite single poles can be analyzed by factorization properties. The behavior of \( M(z) \) around \( z = \infty \) is not well understood. In many examples, with proper choice of deformed pair \( (p_i, p_j) \), \( M(z) \rightarrow 0 \) when \( z \rightarrow \infty \), thus boundary contributions can be avoided (i.e., \( z = \infty \) is not a pole). However, if \( M(z) \rightarrow C_0 + C_1 z + ... C_k z^k \) with \( k \geq 0, z = \infty \) is a pole. To get amplitudes under these circumstances, we have to know the value of \( C_0 \) which yields the boundary contribution.

There are field theories in which nontrivial boundary contributions cannot be avoided, no matter which deformed pair is chosen. Familiar examples are the \( \lambda \phi^4 \) theory and theories with Yukawa couplings. Several proposals have been made to deal with boundary contributions. The first \[ \text{[11, 12]} \] is to add auxiliary fields such that boundary contributions for the enlarged theory are zero. By proper reduction one gets desired amplitudes. The second \[ \text{[13, 14, 15]} \] is to carefully analyze Feynman diagrams and isolate boundary contributions within them. With these information, boundary contributions can be evaluated directly or recursively. The third \[ \text{[16]} \] is to express boundary contributions in terms of roots of amplitudes. Generically, we can write

\[
M(z) = \sum_{\alpha=1}^{N_p} \frac{a_\alpha}{z - z_\alpha} + \sum_{l=0}^{v} C_l z^l \tag{1.1}
\]

where \( a_\alpha \) can be calculated by using factorization properties while the \( C_l \)’s are related to boundary behaviors. \( M(z) \) can be rewritten as \( P_{N_p+v}(z)/\prod_{\alpha}(z - z_\alpha) \). \( P_{N_p+v}(z) \) is a polynomial of \( z \) of degree \( N_p + v \), so there are \( N_p + v \) roots of \( M(z) \). Expressing \( C_l \)’s in terms of \( v + 1 \) roots, one relates boundary contributions to the latter.

This translation to roots is nice. Following upon that, Benincasa and Conde \[ \text{[17]} \] have discussed the extension of the constructible notion initiated in \[ \text{[1]} \]. In this paper, we would like to explore several issues for this proposal. The first issue is how practical this procedure is for real calculations. The second one is as the following. Starting from a physical amplitude \( M(z = 0) \), after the \( z \)-deformation we will arrive at the form \[ \text{[11]} \] with given power \( v \) and unique \( C_l \). However, one can always add an arbitrary polynomial \( zf(z) \) to get a new function \( \tilde{M}(z) = M(z) + zf(z) \). Both \( \tilde{M}(z) \) and \( M(z) \) yield the same amplitude when
$z = 0$, but $\tilde{M}(z)$ may have a different set of roots from $M(z)$. This ambiguity matters especially when we try to construct amplitudes recursively, i.e., starting from lower-point amplitudes to find higher-point amplitudes. Certain principle will be needed to infer roots of $n$-point amplitudes if we know only roots of $m$-point amplitudes with $m < n$.

In section 2, we re-derive the on-shell recursion relation with boundaries in [16] via a new method. The key here is the shuffling of roots. Presented in section 3 are the factorization limits that can be used to deal with roots. Following this brief discussion, we analyze the $z$-parameterized factorization limit carefully in section 4. We will consider poles with and without $z$-dependence, respectively and then use different efficient limits to construct consistent conditions from factorization and boundary BCFW relations. One obtains some information about roots under these limits, but not enough to determine them precisely in general. Our analysis is thus inconclusive. In section 5, several examples are calculated to demonstrate these general discussions. Finally, we conclude in section 6.

2. A new derivation of on-shell recursion relations with boundary contributions

We present in this section a new derivation of on-shell recursion relations with boundary contributions, in contrast with the one given in [16]. The BCFW recursion relation with boundary contributions can be written as

$$M_n(z) = \sum_{k \in P^{I \cup J}} \frac{M_L(z_k)M_R(z_k)}{P^2_k(z)} + C_0 + \sum_{l=1}^v C_l z^l,$$

(2.1)

where we have assumed that $i \in k$ so $P^2_k(z) = (-2P_k \cdot q)(z - z_k)$ with $z_k = P^2_k/2P_k \cdot q$. Pulling all denominators together, one has

$$M_n(z) = \frac{c \prod_{l=1}^{n_I}(z - w_l)^{m_l}}{\prod_{k=1}^{N_p} P^2_k(z)} \prod_{l=1}^{n_J}(z - w^j_l),$$

(2.2)

where $w_l$ are roots of $M_n(z)$.

Unlike results without boundary contributions, (2.1) has single poles at finite locations of $z$ and a pole at $z = \infty$ of degree $v + 1$ as well. To completely determine $M_n(z)$, we need to determine not only residues of single poles at finite locations, but also coefficients related to the pole of degree $v$ at $z = \infty$.

With $N_z \geq N_p$ in (2.2), we can split roots into two groups $I, J$ with number of roots $n_I$ and $n_J$ ($N_z = n_I + n_J$) respectively. If $n_I < N_p$, we can write ($w_l$ is a root of multiplicity $m_l$)

$$M_n(z) = \frac{c \prod_{l=1}^{n_I}(z - w_l)^{m_l}}{\prod_{k=1}^{N_p} P^2_k(z)} \prod_{l=1}^{n_J}(z - w^j_l),$$

(2.3)

where $c_k$’s are unknown $z$-independent coefficients. Plugging (2.3) back to (2.2), we have

$$M_n(z) = \sum_{k=1}^{N_p} \frac{c_k}{P^2_k(z)} \prod_{l=1}^{n_J}(z - w^j_l).$$

(2.4)
Performing a contour integration of (2.3) and (2.4) around a single pole $z_k$, one obtains
\[
\frac{M_L(z_k)M_R(z_k)}{(-2P_k \cdot q)} = \frac{c_k}{(-2P_k \cdot q) \prod_{l=1}^{n}\left(z_k - w_l\right)} \Rightarrow c_k = \frac{M_L(z_k)M_R(z_k)}{\prod_{l=1}^{n}\left(z_k - w_l\right)}. \tag{2.5}
\]
Plugging (2.5) into (2.4), we have
\[
M_n(z) = \sum_{k \in P^{(i,j)}} \frac{M_L(z_k)M_R(z_k)}{P_k^2(z)} \prod_{l=1}^{n}\left(z_k - w_l\right) \tag{2.6}
\]
which is similar to what given in [16], but with new features.

In (2.6), the splitting of roots into two groups is arbitrary as long as $n_\mathcal{I} < N_p$. If $n_\mathcal{I} \leq N_p - 2$, there will be extra consistent relations. Taking $n_\mathcal{I} = N_p - 2$ and expanding (2.6) into the form of (2.1), one has
\[
M_n(z) = \sum_{k \in P^{(i,j)}} \frac{M_L(z_k)M_R(z_k)}{(-2P_k \cdot q)(z - z_k)} \left(\frac{(z - z_k)^{v+2}}{\prod_{l=1}^{n}\left(z_k - w_l\right)} + \frac{(z - z_k)^{v+1} \sum_{l}(z_k - w_l)}{\prod_{l=1}^{n}\left(z_k - w_l\right)} + \ldots + 1\right). \tag{2.7}
\]
The coefficient of the $z^{v+1}$ term is $\sum_{k \in P^{(i,j)}} M_L(z_k)M_R(z_k)/(-2P_k \cdot q)\prod_l(z_k - w_l)$. It should be zero and this results in a consistent condition. To avoid such extra consistent conditions and to deal with only minimum number of roots, we will take $n_\mathcal{I} = N_p - 1$ from now on and obtain
\[
M_n(z) = \sum_{k \in P^{(i,j)}} \frac{M_L(z_k)M_R(z_k)}{P_k^2(z)} \prod_{l=1}^{v+1}\left(z_k - w_l\right), \tag{2.8}
\]
which is the expression presented in [10] and will be the starting point of most our discussion.

Also, the $v+1$ roots $w_l$ in (2.8) can be chosen arbitrarily from the total $N_z$ roots. For practical purposes, $w_l$ should be chosen with certain discretion, instead of being left totally arbitrary. This is related to the issue raised in the introduction, namely, the arbitrariness in defining $M_n(z)$.

Having established (2.8), we can get the boundary BCFW recursion relation by setting $z = 0$
\[
M_n = \sum_{k \in P^{(i,j)}} \frac{M_L(z_k)M_R(z_k)}{P_k^2} \prod_{l=1}^{v+1}\left(w_l\right) = 1 + \left(\frac{z}{z_k - w_l} + 1\right) = 1 + \sum_{s=1}^{v+1}(z - z_k)^s \sum_{\sigma=1}^{v+1} \frac{1}{\prod_{\sigma=1}^{v+1}(z_k - w_{\sigma})}. \tag{2.10}
\]
where the sum $\sum'$ is over all $C_{v+1}^s$ possible selections of $s$ $(z_k - w_{\sigma})$-factors from all $(v+1)$ factors. Thus we have
\[
C_l = \sum_{k \in P^{(i,j)}} \frac{M_L(z_k)M_R(z_k)}{(-2P_k \cdot q)} \sum_{s=1}^{v+1} \frac{d^s(z - z_k)^s - 1}{dz^s} \bigg|_{z=0} \sum_{\sigma=1}^{v+1} \frac{1}{\prod_{\sigma=1}^{v+1}(z_k - w_{\sigma})}, \quad l = 0, 1, \ldots, v \tag{2.11}
\]
We now address two more points before ending this section. Firstly, the divergent degree $v$ is a function of $n$ in general quantum field theory (except gauge theory, gravity theory or other well-defined renormalizable theories). If one adds an interaction vertex with arbitrary number of external fields, the divergent degree $v$ will be modified. Secondly, both poles and roots are important to determine tree level amplitudes. Poles are local property and easier to determine while roots are (quasi)global property and harder to deal with. In general, roots depend on the choice of deformed pair, helicity configuration and other detail information.

### 3. Strategy to find roots

To find the $n$-point amplitude via (2.9), the crucial point is to find its roots. Starting with the known roots of $m$-point amplitudes, we hope to find the roots of $n$-point amplitude ($m < n$), recursively. How to do so?

It was suggested in [16] to consider consistent conditions, obtained from various collinear or multiple particle factorization channels. Here the higher-point amplitude consists of the product of two lower-point amplitudes, from which we may infer information of roots under factorization limits.

In [16], the factorization limit is always taken for physical amplitude $M_n(z = 0)$ given in (2.9). Since the deformation is on-shell, i.e., $M_n(z)$ is an on-shell amplitude for every $z$, the factorization limit can actually be taken for $z$-parameterized amplitude $M_n$ given in (2.8). In other words, we should have the following consistent condition for any $z$

$$
\lim_{P_2^2(z) \to 0} P_\alpha(z)^2 \sum_{k \in P^{(i,j)}} \frac{M_L(z_k)M_R(z_k)}{P_k^2(z)} \prod_{l=1}^{v+1} \frac{z - w_l}{z_k - w_l} = M_L(z)M_R(z). \tag{3.1}
$$

Conditions (3.1) are much stronger, because both sides are functions of $z$, not merely their values at $z = 0$, to be compared. However, condition $P_\alpha(z)^2 = 0$ holds only for a specific value of $z$ in general, as $P_\alpha^2(z) = P_\alpha^2 - 2zP_\alpha \cdot q$ if $p_i$ is to not allow to change with $z$. Thus conditions (3.1) can not be imposed for general channels, except for two particle channels and channels do not contain $p_i, p_j$. These issues are to be investigated carefully in the next section.

The need of careful distinction between two factorization limits in (3.1) relates closely to the second issue raised in the introduction. In many examples, due to constraints from $z$-parameterized factorization limits, the $z$-dependent amplitude has not much freedom to add a polynomial $zf(z)$. At the same time, conditions (3.1) constrain the roots $w_t$ selected in (2.8) and (2.9).

Constraints from factorization limits provide some information of roots, but these constraints are not enough to get complete answers for roots, as to be shown in the example of the six-gluon amplitude. Our analysis is thus inconclusive.

### 4. $z$-parameterized factorization limits

We now discuss $z$-parameterized factorization limits (3.1) carefully, following [18, 16]. We will mainly
concern the collinear limit (i.e., two particle channel), although we know that for some theories, there is no two particle channel (i.e., there is no on-shell three-point amplitude) with BCFW-recursion relation, such as in the $\lambda \phi^4$ theory.

We will start by fixing notations and collecting some useful results for latter discussion, then proceed by the analysis of various channels one by one.

4.1 Conventions and useful results

Choosing the $(i,j)$-pair the deformation is given as

$$
 p_i \rightarrow p_i - z q, \quad p_j \rightarrow p_j + z q, \quad q^2 = q \cdot p_i = q \cdot p_j = 0.
$$

(4.1)

This deformation works for massive or massless theories. Here we consider only massless theories, thus $q$ can be solved directly. We will choose $q = \lambda_i \lambda_j$ by the $[i|j]$-deformation

$$
 \tilde{\lambda}_i \rightarrow \tilde{\lambda}_i - z \tilde{\lambda}_j, \quad \lambda_j \rightarrow \lambda_j + z \lambda_i.
$$

(4.2)

With this convention, if particle $i$ is in the set $\alpha$, $P_{iI}^2(z) = P_{iI}^2(z) = 0$ will result in $z_{iI} = P_{iI}^2 / 2P_i \cdot q = P_{iI}^2 / \langle i|P_{iI}|j \rangle$. Therefore the contribution from this cut to (2.9) is

$$
 T_{iI} = M_L(\tilde{p}_i,-\tilde{p}_{iI}) \frac{1}{P_{iI}^2} M_R(\tilde{p}_{iI},\tilde{p}_j) \prod_{l=1}^{v+1} \frac{w_l}{w_l - z_{iI}},
$$

(4.3)

and the $z$-parameterized amplitude $M_n(z)$ is

$$
 M_n(z) = \sum_I M_L(\tilde{p}_i,-\tilde{p}_{iI}) \frac{1}{P_{iI}^2(z)} M_R(\tilde{p}_{iI},\tilde{p}_j) \prod_{l=1}^{v+1} \frac{w_l - z}{w_l - z_{iI}}.
$$

(4.4)

Since $p_i, P_{iI}$ take the shifted momenta in (4.4), original physical poles inside $M_L, M_R$ (e.g. $P_{iI}^2, \alpha \subset I$), will be modified and become spurious ones

$$
 P_{\alpha I}^2 \rightarrow (P_{\alpha I} - z_{iI} q)^2 = P_{\alpha I}^2 - P_{iI}^2 \frac{2P_{\alpha I} \cdot q}{2P_i \cdot q} = \frac{\langle i|P_{\alpha I}(P_{\alpha I} - P_{iI})P_{iI}|j \rangle}{\langle i|P_{iI}|j \rangle}.
$$

(4.5)

Such spurious pole will show up at two and only two places, corresponding to cuts $z_{iI}$ and $z_{i\alpha}$, and will cancel each other. Generally, spurious poles are not physical. However, some spurious poles can become physical. For example, if the set $\alpha$ contains only one single particle $k$ in (4.3), the spurious pole will factorize as $\langle i|k|(P_{\alpha I} - P_{iI})P_{iI}|j \rangle$, which contains the physical pole $\langle i|k \rangle = 0$.

On-shell three-point amplitude will be important for late discussions when we study the two-particle channel. As shown in (11), massless three-point amplitudes are uniquely determined by spin symmetry and Lorentz symmetry and are given by following forms

$$
 M_3^h(a,b,c) = (a|b)^{h_a-h_b-h_c} (b|c)^{h_b-h_c-h_a} (c|a)^{h_c-h_a-h_b}, \quad \text{if} \quad h_a + h_b + h_c \geq 0,
$$

(4.6)

$$
 M_3^h(a,b,c) = [a|b]^{h_c+h_a+h_b} [b|c]^{h_a+h_b+h_c} [c|a]^{h_b+h_c+h_a}, \quad \text{if} \quad h_a + h_b + h_c \geq 0.
$$

(4.7)
If the total helicity \( h = \sum h_i \) is positive/negative, only anti-holomorphic/holomorphic part is nonzero. If \( h \) is zero, both are allowed. The mass dimension of expressions (4.6) and (4.7) is \( |h| \). To get the overall mass dimension +1 for three-point amplitude, we have to add a coupling constant \( \kappa \) of the dimension \( 1 - |h| \), i.e., \( \text{dim}(\kappa) = 1 - |h| \).

As we mentioned earlier, poles can be divided into two categories: those with nontrivial \( z \)-dependence and those without. We now discuss these two categories one by one. Among poles without nontrivial \( z \)-dependence, \( P_{ij} \) may or may not exist. For example, for color-ordered gluon amplitude, if \( i, j \) are not nearby there is no pole \( P_{ij} \).

### 4.2 Poles with nontrivial \( z \)-dependence

Poles of this category can be denoted as \( P_{iI} \) with \( j \notin I \) (or \( P_{jJ} \) with \( i \notin J \)). From discussions above, we find that if \( I, J \) contain two or more particles, pole \( P_{iI} \) (or \( P_{jJ} \)) shows up only in one term of (4.3) and the \( z \)-independence factorization limit is trivially true as \( z_{iI} \to 0 \). If we do not allow external momenta to change with \( z \), this limit can not be reached for all \( z \).

However, there is an exception for the \( z \)-dependent factorization limit. It is the two particle channel \( P_{ik} \) (or \( P_{jk} \)). The reason is that the collinear limit of massless theory for two particle channel can take either \( \langle i|k \rangle \to 0 \) or \( |i|k \rangle \to 0 \) (but only one choice for massive theory). Following the deformation (4.2),

\[
P_{ik}^2(z) = \langle i|k \rangle (|k|i - z \langle k|j \rangle)
\]

(4.8)

which vanishes for all \( z \) if \( \langle i|k \rangle \to 0 \). Thus, we have a \( z \)-dependent factorization limit \( P_{ik}^2(z) \to 0 \) with the choice \( \langle i|k \rangle \to 0 \) \footnote{The \( z \)-independent factorization limit \( |i|k \rangle \to 0 \) will be satisfied automatically, but there is no \( z \)-dependent factorization limit of \( |i|k \rangle - z \langle j|k \rangle \to 0 \)}. Similarly, one has a \( z \)-dependent factorization limit \( P_{jk}^2(z) \to 0 \) from \( |j|k \rangle \to 0 \).

Now we are going to find out where poles may show up when \( \langle i|k \rangle \to 0 \) or \( |j|k \rangle \to 0 \). The first possible place is in the cut \( s_{ik} \) in (2.3). That is, in the term

\[
M_3^{(h)}(\hat{i}, k, -\hat{P}_{ik}) \frac{1}{s_{ik}} M_{n-1}(... \prod I_l \frac{w_l}{w_l - z_{ik}}
\]

where \( z_{ik} = |k|i / \langle k|j \rangle \). Because \( \lambda_i \sim \lambda_k \sim \lambda_{P_{ik}} \) in the \( M_3^h \) part, there will be a contribution \( \langle i|k \rangle \to (h_i + h_k + h_P) \).

There is one \( \langle i|k \rangle \) from the pole \( s_{ik} \), but it will be cancelled by other \( |i|k \rangle \) factors from \( M_3^{(h)}(\hat{i}, k, -\hat{P}_{ik}) \) if \( h_i + h_k + h_P \leq -1 \). As a result, there may not be a \( \langle i|k \rangle = 0 \) pole. This is true in many theories, such as pure gauge or gravity theory, so it will be assumed from now on. Similar argument shows that the cut \( s_{jk} \) in (2.4) does not give the pole \( |j|k \rangle = 0 \) in general. Having excluded above possibility, we are left with only one choice: spurious poles. Fortunately, as we have mentioned after (4.3), these two singularities do appear as factor in spurious poles \( T_{ij} \) with \( k \in I \) (notice that we do not have the pole \( |i|k \rangle \) shows up in these spurious poles for consistence) or \( T_{jJ} \) with \( k \in J \).
For latter purposes, we now write down two factorization limits from general principles

\[
D_{ik}(z) = \lim_{(i|k) \to 0} P_{ik}^2(z)M_n(z) = M_3^0(i(z), k, -P_{ik}^{-h_{ik}}(z))M_{n-1}(P_{ik}^{h_{ik}}(z), .., j(z), ..),
\]

\[
D_{jk}(z) = \lim_{[j|k] \to 0} P_{jk}^2(z)M_n(z) = M_3^b(j(z), k, -P_{jk}^{-h_{jk}}(z))M_{n-1}(P_{jk}^{h_{jk}}(z), .., i(z), ..),
\]

which will be compared with limits from (2.8). Expressions (4.9) and (4.10) can be further simplified. For example, in (4.9) one can write \(|k| = |i\rangle \langle i| \mu / \langle i| \mu\rangle\) where \(\mu\) is an arbitrary auxiliary spinor. Thus, \(P_{ik}(z) = |i\rangle (|i\rangle - z |j\rangle + |k]\langle k| \mu / \langle i| \mu\rangle)\) and one can get from (4.7)\(^2\)

\[
M_3^0(i(z)^{h_i}, h_{ik}, -P_{ik}^{-h_{ik}}(z)) = |i - z|k\rangle^{h_i + h_k - h_{ik}} (-)^{2h_{ik}} \left( \frac{|k| \mu}{i| \mu} \right)^{-h_k + h_i - h_{ik}}.
\]

Similarly with \(|k| = |j| [j| \mu / |j| \mu|, P_{jk}(z) = (|j\rangle + z |i\rangle + |k]\langle k| \mu / |j| \mu)\rangle |j\rangle\), \(^3\) one can get

\[
M_3^b(j(z)^{h_j}, h_{jk}, -P_{jk}^{-h_{jk}}(z)) = (-)^{2h_{jk}} (j + z |i\rangle)^{-(h_j + h_k - h_{jk})} \left( \frac{|k| \mu}{j| \mu} \right)^{h_k + h_{jk} - h_j}.
\]

4.2.1 The pole \(\langle i|k\rangle = 0\) from the cut \(P_{il}\) with \(k \in I\)
The factorization limit from (4.4) is given by

\[
\lim_{(i|k) \to 0} P_{ik}^2(z)M_n(z) = \sum_{k \in I} \left[ \lim_{(i|k) \to 0} P_{ik}^2(z)M_L(\tilde{P}_i(z_{il}), -\tilde{P}_i(z_{il})) \right] \frac{M_R(\tilde{P}_i(z_{il}), \tilde{P}_j(z_{il}))}{P_{il}^2(z)} \prod_{l=1}^{v+1} \frac{w_l - z}{w_l - z_{il}},
\]

where the sum is over all sets \(I\) containing \(k\) and at least one of other particles. Using the factorization limit of \(M_L\)

\[
\lim_{(i|k) \to 0} P_{ik}^2(z_{il})M_L(\tilde{g}(z_{il}), k, ..., -\tilde{P}_{i\alpha}(z_{il})) = M_3^{(a)}(\tilde{g}(z_{il}), k, ..., -\tilde{P}_{i\alpha}(z_{il}))M(\tilde{P}_{ik}(z_{il}), ..., -\tilde{P}_{i\alpha}(z_{il}))
\]

where \(I = \alpha \bigcup k\) and the notation \(z_{il}\) emphasizes that momenta are taken at the shifted value \(z = z_{il}\). Putting this back to (4.13) we have

\[
R_{ik}(z) = \sum_{I} M_3^{(a)}(\tilde{g}(z_{il}), k, ..., -\tilde{P}_{i\alpha}(z_{il})) \left\{ \frac{|i(z)|k}{\tilde{g}(z_{il})|k} M(\tilde{P}_{ik}, ..., -\tilde{P}_{i\alpha}) \frac{1}{P_{il}^2(z)} M_R(\tilde{P}_i, \tilde{P}_j) \prod_{l=1}^{v+1} \frac{w_l - z}{w_l - z_{il}} \right\},
\]

which should be equal to the \(D_{ik}\) in (4.4). Comparing these two functions of \(z\), we now try to find (1) the number of roots and how it changes under the limit; (2) the values of roots and their behavior under the limit.

To make calculations clear, we will take three steps:

\(^2\)\(h_i + h_k - h_{ik} \geq 0\) in this case.

\(^3\)\(h_j + h_k - h_{jk} \leq 0\) in this case.

\(^4\)Note that while calculating \(M_3^{a/h}\) we have assumed a particular ordering, which may be different from real situation. However, when we compare the direct factorization limit with the one obtained from (4.4), the ordering ambiguity will be canceled at both sides.
• **Step One:** One difference between $R_{ik}$ and $D_{ik}$ is that the $M_3^a$ inside $R_{ik}(z)$ depends on cuts $I$, while the $M_3^a$ inside $D_{ik}(z)$ is universal. Thus using (4.7) we can rewrite

$$M_3^a(i(z_I)|h_i, k^h, -P_{ik}^{-h_i}(z_I)) = \left( \frac{[i(z_I)|k]}{[i(z)|k]} \right)^{h_i + h_k - h_i} M_3^a(i(z)|h_i, k^h, -P_{ik}^{-h_i}(z))$$

and

$$\frac{R_{ik}(z)}{M^a_3(i(z)|h_i, k^h, -P_{ik}^{-h_i}(z))} = \sum_{k \in I} \left\{ \left( \frac{[i(z_I)|k]}{[i(z)|k]} \right)^{h_i + h_k - h_i} \frac{M(\hat{P}_{ik} \ldots - \hat{P}_{iI})M_R(\hat{P}_{iI}, \hat{p}_j)}{P^2_{iI}(z)} \prod_{l=1}^{v+1} \frac{w_l - z}{w_l - z_{iI}} \right\}.$$ 

Identifying $R_{ik}(z)$ with $D_{ik}(z)$ we obtain a consistent condition

$$M_{n-1}(P_{ik}^{h_i}(z), \ldots, j(z), \ldots) = \sum_I \left\{ \left( \frac{[i(z_I)|k]}{[i(z)|k]} \right)^{h_i + h_k - h_i} \frac{M(\hat{P}_{ik} \ldots - \hat{P}_{iI})M_R(\hat{P}_{iI}, \hat{p}_j)}{P^2_{iI}(z)} \prod_{l=1}^{v+1} \frac{w_l - z}{w_l - z_{iI}} \right\} \quad (4.15)$$

where

$$P_{ik}(z) = |i\rangle \left( |i| - z |j\rangle + |k\rangle \frac{\langle k|\mu\rangle}{\langle i|\mu\rangle} \right), \quad p_j(z) = (|j| + z |i\rangle) |j\rangle, \quad p_l(z) = |i\rangle (|i| - z |j\rangle). \quad (4.16)$$

• **Step Two:** With a little calculation, one sees that $M_{n-1}(P_{ik}^{h_i}(z), \ldots, j(z), \ldots)$ in (4.15) is the $z$-dependent amplitude with the BCFW-deformation $|P_{ik}|j\rangle$ (since under the limit $P_{ik}$ is null, its spinor and anti-spinor components are well defined), so it can be expanded

$$M_{n-1}(P_{ik}^{h_i}(z), \ldots, j(z), \ldots) = \sum_I \frac{M(\hat{P}_{ik} \ldots - \hat{P}_{iI})M_R(\hat{P}_{iI}, \hat{p}_j)}{P^2_{iI}(z)} \prod_{l=1}^{\tilde{v}+1} \frac{w_l - z}{w_l - z_{iI}} \quad (4.17)$$

where the number of roots is $\tilde{v} + 1$ now. Comparing (4.17) with (4.15), we observe that\(^5\):

- (a-1) When $h_i + h_k - h_{ik} - 1 = 0$, as in the case of gauge theory, the number of roots of $n$-point amplitude is the same as the number of roots of $n - 1$-point amplitude. In other words, the number of roots is independent of number of particles.

- (a-2) When $h_i + h_k - h_{ik} - 1 = 1$, which includes the case of gravity theory, there are two possibilities. In one case, $n$-point amplitude has one more root than $n - 1$-point amplitude. In the other case, they have the same number of roots, but there is a nontrivial cancelations.

For gravity theory, the second possibility is realized and the nontrivial cancelation has been discussed carefully in [18] (eq.(75)) as the bonus relation [19, 20].

\(^5\)Notice that $h_i + h_k - h_{ik}$ is always a non-negative integer.
• **Step Three:** The consistent condition under the factorization limit $\langle i|k \rangle \to 0$ is summarized as

$$
\sum I \left\{ \left( \frac{|i(z_i)|^2}{|i(z)|^2} \right)^{h_i+h_k-h_{jk}-1} M(\hat{P}_{i1},...,\hat{P}_{iI})M_R(\hat{P}_{j1},\hat{P}_{j}) \prod_{l=1}^{\bar{v}+1} \frac{w_l-z}{w_l-z_{IL}} \right\} 
$$

$$
= \sum I M(\hat{P}_{i1},...,\hat{P}_{iI})M_R(\hat{P}_{j1},\hat{P}_{j}) \prod_{l=1}^{\bar{v}+1} \frac{w_l-z}{w_l-z_{IL}} .
$$

(4.18)

If one has only one term in the sum, which may happen for low point amplitudes, we will arrive at

$$
\left( \frac{|i(z_i)|^2}{|i(z)|^2} \right)^{h_i+h_k-h_{jk}-1} \prod_{l=1}^{\bar{v}+1} \frac{w_l-z}{w_l-z_{IL}} = \prod_{l=1}^{\bar{v}+1} \frac{w_l-z}{w_l-z_{IL}} .
$$

(4.19)

If there are more than one term in the sum, (4.19) could be true for each term but unlikely.

**Potentially extra singularities at $\langle i|k \rangle = 0$**

Above discussions have a small loop hole, i.e., there are some potential contributions we have overlooked. For example, for $n = 4$, there is no term in the summation in (1.13) and there must be some place to provide the needed contribution. The potential contribution comes from following term

$$
T_{jk}(z) = M_{3}^{(a)}(\hat{j},k,-\hat{P}_{jk}^{-h_{jk}}) \frac{1}{(\langle j|k \rangle + z \langle i|k \rangle)\langle k|j \rangle} M_{n-1}(\hat{P}_{jk}^{h_{jk}},...) \prod_{l=1}^{v} \frac{w_l-z}{w_l-z_{jk}}
$$

(4.20)

where

$$
z_{jk} = -\frac{\langle j|k \rangle}{\langle i|k \rangle} , \quad |\bar{i} \rangle = \frac{|P_{ij}|k}{\langle i|k \rangle} , \quad |\bar{j} \rangle = |k \rangle, \quad \langle \bar{i}|j \rangle = \left( \frac{\langle i|j \rangle}{\langle i|k \rangle} \right) \left( \langle j| + \frac{\langle i|k \rangle}{\langle i|j \rangle} \right) .
$$

(4.21)

Notice that under the limit $\langle i|k \rangle \to 0$, $z_{jk}$, $|\bar{j} \rangle$, and $|\bar{i} \rangle$ are all going to infinity. The $M_{3}^{(a)}$ part now becomes

$$
M_{3}^{(a)}(\hat{j},k,-\hat{P}_{jk}^{-h_{jk}}) \sim |j|k|^{h_{k}+h_{j}-h_{jk}} (-)^{-2h_{jk}} \left( \frac{\langle i|k \rangle}{\langle i|j \rangle} \right)^{-h_{k}-h_{jk}+h_{j}} .
$$

(4.22)

The understanding of $M_{n-1}(\hat{P}_{jk}^{h_{jk}},\bar{i},...)$ part can be given as following. Define the “initial momenta”

$$
p_{i}^{\text{init}} = |i \rangle \left( |i \rangle + \frac{\langle j|k \rangle}{\langle i|j \rangle} |k \rangle \right) , \quad p_{j}^{\text{init}} = |j \rangle \left( |j \rangle + \frac{\langle i|k \rangle}{\langle i|j \rangle} |k \rangle \right) , \quad p_{k}^{\text{init}} + p_{i}^{\text{init}} = p_{i} + p_{j} + p_{k}
$$

(4.23)

and use them to do the $|i^{\text{init}}|p_{j}^{\text{init}}\rangle$-deformation

$$
|p_{i}^{\text{init}} \rangle \rightarrow |p_{i}^{\text{init}} \rangle - z |P_{j}^{\text{init}} \rangle , \quad |P_{j}^{\text{init}} \rangle \rightarrow |P_{jk}^{\text{init}} \rangle + z |P_{i}^{\text{init}} \rangle ,
$$

(4.24)

---

6Related singular behavior is that if $P_{ij}^{2} \rightarrow 0$, then $P_{ij}^{2} \rightarrow 0$, thus we will have $|1 \rangle \sim |2 \rangle$ and $|3 \rangle \sim |4 \rangle$ at same time.

7Notice the unusual definition of the spinor and anti-spinor components of $\hat{P}_{jk}$. The reason for this definition will be clear from later discussions.
then it is easy to see that when we set $z = -\frac{i|k\rangle}{(i|k\rangle}$, we produce right spinor variables (4.21).

In (4.20), one finds three contributions for the overall power of factor $\langle i|k\rangle$: (1) $M_3$ gives a power of $-h_k - h_{jk} + h_j$; (2) If $M_{n-1} \sim z^t$ at the infinity under the deformation $\left[ i^{h_i} P^{h_{jk}}_{jk} \right]$, there is factor $\langle i|k\rangle^{-t}$; (3) The $\prod w_l - z/(w_l - z_jk)$ could give another power of $\langle i|k\rangle^\nu$ with $\nu \leq v + 1$. This happens when root $w_l$ is finite under the limit, thus $(w_l - z)/(w_l - z_jk) \rightarrow \langle i|k\rangle (w_l - z)/\langle j|k\rangle$. Collecting all factors together, we have finally

$$\langle i|k\rangle^{n_{ik}} \equiv \langle i|k\rangle^{-h_k - h_{jk} + h_j - t + \nu}$$

(4.25)

If $n_{ik}$ is non-negative, there is no contribution in the limit $\langle i|k\rangle \rightarrow 0$. If $n_{ik}$ is negative, it does give non-zero contribution. To have a finite factorization limit, one needs $n_{ik} = -1$, from which $\nu$ could be obtained. However, without a general expression for $M_{n-1}$, detailed informations can be only inferred in explicit example.

4.2.2 The pole $[j|k] = 0$ from the cut $P_{j,l}$ with $k \in J$

This part parallels to the discussion of $\langle i|k\rangle = 0$. The factorization limit from (2.8) gives

$$\lim_{[j|k] \rightarrow 0} P^2_{j,k}(z) M_{n}(z) = R_{jk}(z)$$

and

$$= \sum_{k \in J} M_3^{(h)}(j(z_j), k, -\tilde{P}_{jk}(z_j, j)) \left\{ \frac{\langle j(z)|k\rangle}{\langle j(z)|j\rangle} \frac{1}{P^2_{j,k}(z)} M(\tilde{P}_{jk}(z_j, j), ..., -\tilde{P}_{j,l}(z_j, j)) \right\}$$

$$M_R(\tilde{P}_{j,l}(z_j, j), \tilde{P}_{j,l}(z_j, j)) \prod_{l=1}^{v+1} \frac{w_l - z}{w_l - z_j}$$

(4.26)

where the sum is over all $J$ containing $k$ and at least another particle. Consistent conditions can again be obtained by comparing $R_{jk}$ with $D_{jk}$ from (4.10).

- **Step One:** Rewrite

$$M_3^{(h)}(j(z_j), k, -P_{jk}^{-h_{jk}}(z_j, j)) = \left( \frac{\langle j(z_j)|k\rangle}{\langle j(z)|k\rangle} \right)^{-h_j + h_k - h_{jk}} M_3^{(h)}(j(z_j), k, -P_{jk}^{-h_{jk}}(z_j))$$

(4.27)

and compare with the $D_{jk}(z)$ in (4.10), one has the following consistent condition

$$M_{n-1}(P_j^{h_{jk}}(z), ..., i(z), ...)$$

$$= \left( \frac{\langle j + z_i|k\rangle}{\langle j + z_i|j\rangle} \right)^{-h_j + h_k - h_{jk}} M(\tilde{P}_{jk}(z_j), ..., -\tilde{P}_{j,l}(z_j, j)) M_R(\tilde{P}_{j,l}(z_j, j), \tilde{P}_{j,l}(z_j, j)) \prod_{l=1}^{v+1} \frac{w_l - z}{w_l - z_j}$$

(4.28)

here

$$P_{jk}(z) = \left( |j\rangle + z |i\rangle + |k\rangle \frac{k|j\rangle}{|j|j\rangle} \right) |j\rangle, \quad p_j(z) = (|j\rangle + z |i\rangle) |j\rangle, \quad p_i(z) = |i\rangle (|i\rangle - z |j\rangle).$$

(4.29)
• **Step Two:** $M_{n-1}(P_{jk}^{hjk}(z),..,i(z),..)$ is obtained by the $[i|P_{jk}]$-deformation and can be expanded as

$$M_{n-1}(P_{jk}^{hjk}(z),..,i(z),..) = \sum_J \frac{M(\hat{P}_{jk}(z_{J}),..., -\hat{P}_{j,J}(z_{J})) M_R(\tilde{p}_{j,J}(z_{J}))}{P_{jk}^{hjk}(z)} \prod_{l=1}^{v+1} \frac{w_l - z}{w_l - z_{j,J}}.$$  

(4.30)

Now the number of zero is $\tilde{v} + 1$. If $h_j + h_k - h_{jk} + 1 = 0$, the number of roots of $n$ point amplitude is the same as that of $n - 1$ point amplitude. If $h_j + h_k - h_{jk} + 1 \leq -1$, we should study carefully about how the match is realized.

• **Step Three:** If the sum over $J$ has only one term, we will have

$$\left(\frac{(j + z_{j,J}i|k)}{(j + z_i|k)}\right)^{-(h_j + h_k - h_{jk})-1} \prod_{l=1}^{v+1} \frac{w_l - z}{w_l - z_{j,J}} = \prod_{l=1}^{\tilde{v}+1} \frac{w_l - z}{w_l - z_{ik}}.$$  

(4.31)

Again we cannot be certain whether (4.31) is true for every possible cut.

**Potentially extra singularities at pole $[j|k] = 0$**

A possible contribution comes from the following term\(^8\)

$$T_{ik}(z) = M_3^{(h)}(i, k, -\hat{P}_{ik}^{-h_{ik}}) \frac{1}{s_{ik}(z)} M_{n-1}(\hat{P}_{ik}^{h_{ik}}, \tilde{p}_{ik},..) \prod_{l=1}^{v+1} \frac{w_l - z}{w_l - z_{ik}}.$$  

(4.32)

with definitions of various quantities as

$$z_{ik} = \frac{|i|}{|j|}, \quad \hat{z} = |k|, \quad \hat{j} = |j| + \frac{|i|}{|j|}, \quad \hat{P}_{ik} = \left(|k| \frac{|j|}{|j|} + |i| \right) \left(|k| \frac{|j|}{|j|} \right).$$  

(4.33)

Under the limit $[j|k] \to 0$, $z_{ik}, \hat{j}$ and $\hat{i}$ are all going to infinity. As in the case of $\langle i|k \rangle$, we now count the singularity power. The first factor comes from

$$M_3^{(h)}(i^{h_i}(z), k^{h_k}, -\hat{P}_{ik}^{-h_{ik}}(z)) = (-)^{2h_{jk}} \langle i|k \rangle^{-(h_i + h_k - h_{ik})} \left(\frac{|j|}{|j|} \right)^{h_k + h_{ik} - h_i}.$$  

(4.34)

The second one comes from the infinity behavior $z^t$ of $M_{n-1}(\hat{P}_{ik}^{h_{ik}}, \tilde{p}_{ik},..)$ with “initial momenta”

$$p_j^{\text{init}} = |j| - \frac{|i|}{|j|} |k|, \quad \hat{p}_{ik}^{\text{init}} = \left(|k| \frac{|j|}{|j|} + |i| \right) |j|, \quad P_{ik}^{\text{init}} + p_j^{\text{init}} = p_i + p_j + p_k$$  

(4.35)

and the $[P_{ik}^{\text{init}}|j^{\text{init}}]$ deformation\(^9\)

$$\hat{p}_{j^{\text{init}}}^{\text{init}} \to |p_{j^{\text{init}}}^{\text{init}}| + z|P_{j^{\text{init}}}^{\text{init}}|, \quad \hat{p}_{j^{\text{init}}}^{\text{init}} \to |P_{j^{\text{init}}}^{\text{init}}| - z|P_{j^{\text{init}}}^{\text{init}}|.$$  

(4.36)

The third one comes from $\prod_l (w_l - z)/(w_l - z_{jk})$ of power $\nu \leq v + 1$. Collecting these together, the final power is

$$[j|k]^{n_{jk}} \equiv [j|k]^{h_k + h_{ik} - h_i - t + \nu}$$  

(4.37)

If $n_{jk}$ is non-negative, there is no contribution in the limit of $[j|k] \to 0$. If it is negative, it does give non-zero contribution and $\nu$ can be determined by requiring $h_k + h_{ik} - h_i - t + \nu = -1$.

\(^8\)In fact, if the summation over $J$ is empty, this term must contribute to get consistent result.

\(^9\)Setting $z = [i|k]/[j|k]$, we reproduce right spinor variables in (4.33).
4.3 Poles without \(z\)-dependence

Among poles without \(z\)-dependence, two-particle pole \(P_{ij} = 0\) will be particularly important. The general pole \(P_\alpha\) here appears in cuts \(T_{ij}\) (4.3) with \(\alpha \subset I\) or \(\alpha \subset i,j, I\). Under the limit \(P_\alpha^2 \to 0\), we have

\[
\lim_{P_\alpha^2 \to 0} P_\alpha^2 M_n(z) = \lim_{P_\alpha^2 \to 0} -2 \left\{ \sum_{\alpha \in I} M_L(\hat{p}_i, -\hat{p}_j) \frac{1}{P_{ij}^2(z)} M_R(\hat{P}_{ij}, \hat{p}_j) \prod_{l=1}^{v+1} \frac{w_l - z}{w_l - z_{il}} \right\}
\]

\[
= \sum_{\alpha \in I} M(\{\alpha\}, -P_\alpha) \tilde{M}_L(\hat{p}_i, -\hat{P}_{ij}, P_\alpha) \frac{1}{P_{ij}^2(z)} M_R(\hat{P}_{ij}, \hat{p}_j) \prod_{l=1}^{v+1} \frac{w_l - z}{w_l - z_{il}}
\]

\[
+ \sum_{\alpha \in i,j} M_L(\hat{p}_i, -\hat{P}_{ij}) \frac{1}{P_{ij}^2(z)} M_R(\hat{P}_{ij}, \hat{p}_j, P_\alpha) M(\{\alpha\}, -P_\alpha) \prod_{l=1}^{v+1} \frac{w_l - z}{w_l - z_{il}}
\]

\[
\equiv M(\{\alpha\}, -P_\alpha) M_{n-\{\alpha\}}(z)
\] (4.38)

The number of roots of \(M_{n-\{\alpha\}}\) should be the same or less than that in \(M_n\). They are the same only if no \(w_l \to \infty\) when \(P_\alpha^2 \to 0\). Therefore, values of these non-divergent roots \(w_l\) under the limit \(P_\alpha^2 \to 0\) could be read out from lower point amplitudes \(M_{n-\{\alpha\}}(z)\) with the same \(z\)-deformation.

Now some remarks on (4.38) about the two particle channel \(P_{k_1 k_2}\). The term \(M_L(\hat{i}, k_1, k_2, -\hat{P}_{ik_1 k_2})\) as given in (4.4) will give zero contribution under \([k_1 | k_2] \to 0\). Since \(z = |k_1 | i ] / [k_1 | j\) under the limit, so

\[
\tilde{\lambda}_i = \lambda_i - z \tilde{\lambda}_j = \lambda_k \frac{|i | j ]}{[k_1 | j}\]

and

\[
\tilde{\lambda}_k \sim \tilde{\lambda}_k \sim \tilde{\lambda}_k \sim \lambda_k \sim \lambda_{\hat{P}_{ik_1 k_2}}.
\]

Thus

\[
\lim_{[k_1 | k_2] \to 0} P_{k_1 k_2}^2 M(\hat{i}, k_1, k_2, -\hat{P}_{ik_1 k_2}) \to M_3(k_1, k_2, -P_{ik_1 k_2}) M_3(P_{ik_1 k_2}, -\hat{P}_{ik_1 k_2}, \hat{i}) \to 0
\] (4.39)

because the product in (4.38) must be \(M_3^2 \times M_3^h\) and \(M_3^a = 0\). This null contribution was explained in [18], while \(M_3(\hat{i}, k_1, -\hat{P}_{ik_1}) M_{n-1}(-\hat{P}_{ik_1}, k_2, j, \ldots)\) (plus the term with \(k_1 \leftrightarrow k_2\)) provides an extra contribution. Similar subtleties arise when \((k_1 | k_2) \to 0\) and more can be found in [18].

Now we turn to the pole at \(P_{ij} = 0\). It does not appear explicitly in the recursion relation (4.4). Since \(P_{ij}(z) = P_{ij}\) for all \(z\), factorization limits exists for all values of \(z\):

\[
\lim_{[i | j] \to 0} P_{ij}^2 M_n \to M_3^h(i(z), j(z), -P_{ij}^{-h_{ij}}) M_{n-1}(P_{ij}^{-h_{ij}}, \ldots)
\] (4.40)
and
\[
\lim_{(i|j) \to 0} P_{ij}^2 M_n \to M_3^g (i(z), j(z), -P_{ij}^{-h_{ij}}) M_{n-1} (P_{ij}^{+h_{ij}}, ...). \tag{4.41}
\]
Depending on the helicity configuration \( h_i, h_j \), both limits can be nontrivial or one of them be trivial, for example, if \( h_i, h_j \) make \( A^h(i, j, -P_{ij}) = 0 \) no matter what the helicity of \( P_{ij} \) is. The two singularities \( \langle i|j \rangle \to 0 \) and \( [i|j] \to 0 \) do not come from spurious poles \( \langle i|P_{\alpha \alpha} (P_{\alpha} - P_{\alpha}) P_{\alpha l}|j \rangle \) or \( \langle i|P_{l l} (P_{l l} - P_{\alpha}) P_{\alpha l}|j \rangle \), but the soft limit of \( \hat{p}_i \) or \( \hat{p}_j \) in following two types of cut contributions (other terms do not give soft limit and wanted singularities)

\[
T_{ik}(z) = M_3^h (\hat{g}^{h_i}, k^{h_k}, -P_{ik}^{-h_{ik}}) \frac{1}{s_{ik}(z)} M_{n-1} (\hat{P}_{ik}^{h_{ik}}, \hat{g}^{h_j}, ..., \hat{s}_{ik}) \prod_l \frac{w_l - z}{w_l - z_{ik}}
\]
\[
T_{jk}(z) = M_3^g (\hat{g}^{h_j}, k^{h_k}, -P_{jk}^{-h_{jk}}) \frac{1}{s_{jk}(z)} M_{n-1} (\hat{P}_{jk}^{h_{jk}}, \hat{g}^{h_i}, ..., \hat{s}_{jk}) \prod_l \frac{w_l - z}{w_l - z_{jk}}. \tag{4.42}
\]
In \( T_{ik}(z) \)

\[
z_{ik} = \frac{[k|i]}{[k|j]}, \quad |\hat{i}\rangle = \lambda_k \frac{[i|j]}{[k|j]}, \quad \hat{P}_{ik} = \lambda_k \left( \lambda_i [i|j] + \lambda_k \right), \quad \hat{p}_j = |j\rangle \left( |j\rangle + \frac{[k|i]}{[k|j]} |i\rangle \right) \tag{4.43}
\]
which gives \( |\hat{i}\rangle \to 0 \) under the limit \( [i|j] \to 0 \). In \( T_{jk}(z) \)

\[
z_{jk} = -\langle j|k \rangle, \quad |\hat{j}\rangle = |i\rangle + \langle j|k \rangle |j\rangle, \quad |\hat{j}\rangle = |k\rangle \left( |j\rangle \frac{[i|j]}{[i|k]} + |j\rangle \right) \tag{4.44}
\]
thus \( |\hat{j}\rangle \to 0 \) under the limit \( \langle i|j \rangle \to 0 \).

### 4.3.1 The \([i|j] \to 0\) limit

We now compare the factorization limit of \( T_{ik}(z) \) with \(1(40)\). The \( z \)-independent part of \( M_3 \) in \( T_{ik}(z) \) is

\[
M_3^h (\hat{g}, k, -P_{ik}^{-h_{ik}}) \sim (-)^{2h_{ik}} \langle i|k \rangle^{-h_i + h_k - h_{ik}} \left( \frac{[i|j]}{[k|j]} \right)^{h_i + h_k - h_{ik}} \sim (-)^{2h_{ik}} \langle i|k \rangle \delta_{ik}^h \left( \frac{[i|j]}{[k|j]} \right)^{2h_i + \delta_{ik}^h}. \tag{4.45}
\]
where we have used \(4.43\) and defined

\[
\delta_{ik}^h = -(h_i + h_k - h_{ik}) \geq 0. \tag{4.46}
\]

The \( z \)-dependent part of \( M_3 \) in \(1(40)\) is

\[
M_3^g (i(z), j(z), -P_{ij}^{-h_{ij}}) \sim (-)^{2h_{ij}} \langle i|j \rangle^{-h_i + h_j - h_{ij}} \left( \frac{[\mu|i]}{[\mu|j]} - z \right)^{h_i + h_j - h_{ij}} \sim (-)^{2h_{ij}} \langle i|j \rangle \delta_{ij}^h \left( \frac{[\mu|i]}{[\mu|j]} - z \right)^{2h_i + \delta_{ij}^h}. \tag{4.47}
\]
where we have used

\[
P_{ij} = \left( |j\rangle + |i\rangle \frac{[\mu|i]}{[\mu|j]} \right) |j\rangle \tag{4.48}
\]
and defined

\[ \delta_{ij}^h = -(h_i + h_j - h_{ij}) \geq 0. \]  
(4.49)

Now to the \(M_{n-1}\) part of \(T_{ik}\). Under the limit \([i|j] \to 0\), \(\hat{p}_j \to p_i + p_j = P_{ij}\) and \(\hat{P}_{ik} \to p_k\), the \(M_{n-1}\) in \(T_{ik}\) becomes \(M_{n-1}(p_{ik}^{h_{ik}}, p_{ij}^{h_{ij}}, \ldots)\). To link it with \(M_{n-1}(P_{ij}^{h_{ij}}, p_k^{h_k}, \ldots)\) in (4.40), we define

\[ M_{n-1}(p_{ik}^{h_{ik}}, p_{ij}^{h_{ij}}, \ldots) = \mathcal{H}_{n-1}^{(ijk)} M_{n-1}(P_{ij}^{h_{ij}}, p_k^{h_k}, \ldots) \]  
(4.50)

where the function \(\mathcal{H}_{n-1}^{(ijk)}\) is \(z\)-indepedent.

Comparing (4.40) with \(\sum_k P_{ij}^2 T_{ik}(z)\), one has

\[
\sum_k (-)^{2h_{ik}} \langle i|k \rangle \delta_{ik}^h \left( \frac{[i|j]}{[k|j]} \right)^{2h_i + \delta_{ij}^h} \frac{s_{ij}}{s_k(z)} M_{n-1}(p_{ik}^{h_{ik}}, P_{ij}^{h_{ij}}, \ldots) \prod_l \frac{w_l - z}{w_l - z_{ik}} \\
= (-)^{2h_{ij}} \langle i|j \rangle \delta_{ij}^h \left( \frac{[\mu|i]}{[\mu|j]} - z \right)^{2h_i + \delta_{ij}^h} M_{n-1}(P_{ij}^{h_{ij}}, p_k^{h_k}, \ldots)
\]

which can be simplified to (the sign comes from possible different color ordering)

\[
\pm 1 = \sum_k (-)^{2h_{ik} - 2h_{ij}} \langle i|k \rangle \delta_{ik}^h \langle i|j \rangle \delta_{ij}^h \left( \frac{[i|j]}{[k|j]} \right)^{2h_i + \delta_{ij}^h} \left( \frac{[\mu|i]}{[\mu|j]} - z \right)^{-2h_i + \delta_{ij}^h} \frac{s_{ij}}{s_k(z)} \mathcal{H}_{n-1}^{(ijk)} \prod_l \frac{w_l - z}{w_l - z_{ik}}. 
\]  
(4.51)

Here are some points on (4.51):

- **\(z\)-dependence**: Note \(s_{ik}(z) = \langle i|k \rangle ([i|i] / [k|j] - z)\) and \([k|i] / [k|j] = [\mu|i] / [\mu|j]\) for any \(\mu\) under the limit. To make the right-handed side of (4.51) \(z\)-independent, the number of finite \(w_l\) must be

\[ N_{\text{zero}}^h = 1 + 2h_i + \delta_{ij}^h. \]  
(4.52)

For each \(k\), the \(z\)-dependence of the factors \(([k|i] / [k|j] - z)^{-1} ([\mu|i] / [\mu|j] - z)^{-2h_i + \delta_{ij}^h} \prod_l (w_l - z)\) are the same, so they can be pulled out uniformly through the summation.

- **Universal behavior**: A consequence of the above \(z\)-dependence is that values of roots will be

\[ w_l \to \frac{[t_l|i]}{[t_l|j]} \]  
(4.53)

under the limit \([i|j] \to 0\).

- **Further simplification**: Using (4.53) we have \(w_l - z_{ik} = [t_l|k] [i|j] / [t_l|j] [k|j]\), thus (4.51) can be further simplified to

\[
\pm 1 = \sum_k (-)^{2h_{ik} - 2h_{ij}} \langle i|k \rangle \delta_{ik}^h \langle i|j \rangle \delta_{ij}^h \left( \frac{[i|j]}{[k|j]} \right)^{2h_i + \delta_{ij}^h} \frac{s_{ij}}{s_k(z)} \mathcal{H}_{n-1}^{(ijk)} N_{\text{zero}}^h \prod_{l=1}^{N_{\text{zero}}^h} \frac{[t_l|j]}{[t_l|k]} \frac{[i|j]}{[i|j]}. 
\]  
(4.54)
• A special case: In cases such as pure gauge or gravity theory, \( \delta_{ik}^h = \delta_{ij}^h \) (though we cannot assume so in general). Under these circumstances, we can simplify further

\[
\pm 1 = \sum_k \frac{\langle i|k \rangle \delta_{ik}^h}{\langle i|j \rangle \delta_{ij}^h} \sum_{l=1}^{N_{z\text{zero}}} \frac{[t_l|j]}{[t_l|k]} H_{n-1} \prod_{l} [t_l|j] \prod_{l} [t_l|k].
\]

\[ (4.55) \]

4.3.2 The \( \langle i|j \rangle \to 0 \) limit

Now compare the factorization limit of \( T_{jk}(z) \) with \( (4.41) \). The discussion will be brief due to its similarity to the previous one. The \( z \)-independent part of \( M_3 \) in \( T_{jk}(z) \) is

\[
M_3^a(\hat{g}_z, k, -P_{jk}^{-h_{jk}}) \sim (-)^{-2h_{jk}} [j|k]^{(h_j + h_k - h_{jk})} \left( \frac{\langle i|j \rangle}{\langle i|k \rangle} \right)^{-h_j - h_k + h_{jk}} \sim (-)^{-2h_{jk}} [j|k]^{\delta_{ij}^h} \left( \frac{\langle i|j \rangle}{\langle i|k \rangle} \right)^{-2h_j + \delta_{ij}^h}.
\]

\[ (4.56) \]

where \( \delta_{ij}^a = (h_j + h_k - h_{jk}) \geq 0 \). The \( z \)-independent part of \( M_3 \) in \( (4.40) \) is

\[
M_3^a(\hat{g}_z, j(z), -P_{ij}^{-h_{ij}}) \sim (-)^{-2h_{ij}} [i|j]^{(h_i + h_j - h_{ij})} \left( \frac{\langle \mu|j \rangle}{\langle \mu|i \rangle} + z \right)^{h_i - h_{ij} - h_j} \sim (-)^{-2h_{ij}} [i|j]^{\delta_{ij}^h} \left( \frac{\langle \mu|j \rangle}{\langle \mu|i \rangle} + z \right)^{\delta_{ij}^a - 2h_j}.
\]

\[ (4.57) \]

where \( \delta_{ij}^a = (h_i + h_j - h_{ij}) \geq 0 \). To link \( M_{n-1}(P_{k}^{h_{jk}}, P_{ij}^{h_{ij}}, \ldots) \) in \( T_{jk} \) with \( M_{n-1}(P_{i}^{h_{ij}}, P_{k}^{h_{jk}}, \ldots) \) in \( (4.40) \), we define

\[
M_{n-1}(P_{k}^{h_{jk}}, P_{ij}^{h_{ij}}, \ldots) = \frac{H_{n-1}^{(ijk)}}{H_{n-1}^{(ijk)}} M_{n-1}(P_{i}^{h_{ij}}, P_{k}^{h_{jk}}, \ldots).
\]

\[ (4.58) \]

The comparison of \( (4.40) \) with \( \sum_k P_{ij}^2 T_{ik}(z) \) leads to following equation

\[
\pm 1 = \sum_k (-)^{-2h_{jk} + 2h_{ij}} [j|k]^{\delta_{ik}^h} \left( \frac{\langle i|j \rangle}{\langle i|k \rangle} \right)^{-2h_j + \delta_{ij}^h} \left( \frac{\langle \mu|j \rangle}{\langle \mu|i \rangle} + z \right)^{-(\delta_{ij}^a - 2h_j)} \frac{s_{ij}}{s_{jk}(z)} H_{n-1}^{(ijk)} \prod_{l} \frac{w_l - z}{w_l - z_{jk}}.
\]

\[ (4.59) \]

from which one observes:

- \textbf{z-dependence}: To cancel the \( z \)-dependence on the right-handed side, the number of finite \( w_l \) is

\[
N_{z\text{zero}}^a = 1 + \delta_{ij}^a - 2h_j.
\]

\[ (4.60) \]

- \textbf{Universal behavior}: Values of root will be

\[
w_l \to -\frac{\langle t_l|j \rangle}{\langle t_l|i \rangle}.
\]

\[ (4.61) \]

under the limit \( \langle i|j \rangle \to 0 \).
• **Further simplification:** Using \( w_l - z_{jk} = \langle j | i \rangle \langle t_l | k \rangle / \langle i | k \rangle \langle t_l | i \rangle \), (4.59) can be simplified further to

\[
\pm 1 = \sum_k \left( \frac{[j | k]}{[i | j]} \right)^{\delta_{jk}} \frac{\langle i | j \rangle}{\langle i | k \rangle} s_{ij} H_{n-1}^{ijk} \prod_{l=1}^{N_{zero}} \langle i | k \rangle \langle t_l | i \rangle \langle t_l | k \rangle . \tag{4.62}
\]

• **A special case:** Assuming \( \delta_{jk} = \delta_{ij} \), we have

\[
\pm 1 = \sum_k \left( \frac{[j | k]}{[i | j]} \right)^{-\delta_{j}} \frac{\langle i | j \rangle}{\langle i | k \rangle} s_{ij} H_{n}^{ijk} \prod_{l=1}^{N_{zero}} \langle i | k \rangle \langle t_l | i \rangle . \tag{4.63}
\]

### 5. Examples

Listed in this section are examples [16] to demonstrate previous general discussions about roots. We will show (1) how the number of roots behaves under various \( z \)-dependent factorization limits; (2) how to infer values of roots from rational function of \( z \) under \( z \)-dependent factorization limits, when possible; (3) finally, to show the limitation of our approach in the case of six-gluon amplitudes.

#### 5.1 Example I– MHV amplitudes

We start with the simplest case, the MHV amplitudes \( M_n(-,+,−,+,..+,+) \) with deformation \( \lambda_1 \rightarrow \lambda_1 + z\lambda_2, \tilde{\lambda}_2 \rightarrow \tilde{\lambda}_2 - z\tilde{\lambda}_1 \), or the \([2|1]\)-deformation. There is only one pole and one gets from (2.8)

\[
M_n(z) = M^0_n(n^+,\tilde{\lambda}^−,−P^+) \frac{1}{s_{n1}(z)} M(P^−,\tilde{\lambda}^+3^−,..,(n−1)^+) \prod_{l=1}^{v+1} \frac{w_l - z}{w_l - z_α}
\]

\[
= \frac{-1}{\langle 1|2 \rangle \langle 2|3 \rangle ... \langle n|1 \rangle} \left( \frac{\langle n|3 \rangle \langle 1|2 \rangle}{\langle n|2 \rangle} \right)^4 \frac{\langle 1|n \rangle}{\langle 1|n \rangle + z \langle 2|n \rangle} \prod_{l=1}^{v+1} \frac{w_l - z}{w_l - z_α} \tag{5.1}
\]

with \( z_α = -\langle 1|n \rangle / \langle 2|n \rangle \).

#### 5.1.1 Poles without \( z \)-dependence

The \( M_{n-1}(z) \) part in (4.38) is \( \langle \hat{1}|3 \rangle \langle 1|2 \rangle ... \langle a - 1|P_{a,a+1} \rangle \langle P_{a,a+1}|a + 2 \rangle ... \langle n|\hat{1} \rangle \). The collinear limits are\(^{10} \langle a^+(a + 1)^+ \rangle \rightarrow 0 \) with \( 4 \leq a \leq n - 1 \). Under this limit and with the same deformation \([2|1]\), we find a root \( w_l = -\langle 1|3 \rangle / \langle 2|3 \rangle \) of multiplicity 4. In the original amplitude without taking the limit, we should have

\[
w_l = -\frac{\langle 1|3 \rangle}{\langle 2|3 \rangle} (1 + f_l) \tag{5.2}
\]

where \( f_l \) should be constrained by several physical requirements: (1) \( f_l \) should have a factor \( \langle a|a+1 \rangle \) to give the root under the limit; (2) \( f_l \) should be helicity neutral for all particles, of either the form \( s_{a,a+1} \)

\(^{10}\)There is no multiple-particle channel and only one nontrivial choice in the limit \( \langle a|a + 1 \rangle \rightarrow 0 \).
or the combination \( \langle a|a+1 \rangle \langle t|s \rangle / \langle a|s \rangle \langle a+1|t \rangle \) of spinors \( \lambda_t, \lambda_s \); (3) \( f_l \) should be dimensionless; (4) \( f_l \) should be consistent with all different choices \( \langle a|a+1 \rangle \rightarrow 0 \); (5) there must be no un-physical pole from \( \prod_{l=1}^{v+1} w_l/(w_l - z_\alpha) \) when \( f_l \) is included. Consistent with these requirements, there is simple solutions \( f_l = 0 \), for \( l = 1, 2, 3, 4 \).

### 5.1.2 Poles with \( z \)-dependence

Here \( s_{n1} = 0 \) and \( s_{23} = 0 \) are pole of \( z \)-dependence. For \( s_{n1} = 0 \), the limit \( \langle n|1 \rangle \rightarrow 0 \) is automatically satisfied by (2.5), while \( M_{n-1}(P_{n1}^+, 2^+, 3^-, \ldots) = 0 \) under the limit \( \langle n|1 \rangle \rightarrow 0 \). These are trivial. For \( s_{23} \), the \( [2|3] \rightarrow 0 \) limit is trivial and we will focus on the \( (2|3) \rightarrow 0 \) limit. A new feature arises under this limit. The true root \( w_l = -\langle 1|3 \rangle / \langle 2|3 \rangle \rightarrow \infty \) and \( (w_l - z)/(w_l - z_{n1}) \rightarrow 1 \), so the degree of \( z \) is reduced in the combination. Let’s see how this happen.

The factorization limit from (5.1) is

\[
\lim_{\langle 2|3 \rangle \rightarrow 0} \left( \frac{[3|2] - z [3|1]}{[2|3]} \right) \frac{\langle n|3 \rangle \langle 1|2 \rangle}{\langle n|2 \rangle} = \frac{\langle n|1 \rangle}{\langle n|2 \rangle} \prod_{l=1}^{v+1} \frac{w_l - z}{w_l - z_\alpha}
\]

while the direct factorization limit is

\[
[2 - z|3] \left( \frac{\langle 3|\mu \rangle}{\langle 2|\mu \rangle} \right)^3 \frac{\langle 1|2 \rangle^3}{\langle 2|4 \rangle \langle 4|5 \rangle \ldots \langle n - 1|n \rangle} \frac{\langle n|1 \rangle}{\langle n|2 \rangle} \frac{1}{\langle n|2 \rangle} = \left( \frac{\langle 3|\mu \rangle}{\langle 2|\mu \rangle} \right)^3 \frac{\langle 1|2 \rangle^3}{\langle 2|4 \rangle}
\]

where \( P_{23}(z) = [2|2 - z|1 + [3|3|\mu \rangle / \langle 2|\mu \rangle] \) has been used. Comparing both we arrive

\[
\frac{1}{\langle 2|3 \rangle \langle 3|4 \rangle} \left( \frac{\langle n|3 \rangle}{\langle n|2 \rangle} \right)^4 \prod_{l=1}^{v+1} \frac{w_l - z}{w_l - z_\alpha} = \left( \frac{\langle 3|\mu \rangle}{\langle 2|\mu \rangle} \right)^3 \frac{\langle 1|2 \rangle^3}{\langle 2|4 \rangle}
\]

from which one has

\[
\prod_{l=1}^{v+1} \frac{w_l - z}{w_l - z_\alpha} = 1
\]

(5.3)

here we have used \( \langle n|3 \rangle / \langle n|2 \rangle = \langle 3|\mu \rangle / \langle 2|\mu \rangle = \langle 3|4 \rangle / \langle 2|4 \rangle \). (5.3) holds when and only when \( w_l \rightarrow \infty \) under the limit. It is tempting to conjecture \( w_l \sim \langle 3|1 \rangle / \langle 3|2 \rangle \). But we shall see, roots need not be a rational function and a general prediction cannot be made.

These conclusions can be reached by general analysis as well. \( M_{n-1}(1^-, P_{23}, 4, \ldots, n) \) in (4.17) has no boundary contributions (with deformation \( [P_{23}|1] \) and \( P_{23}(z) = [2|2 - z|1 + [3|3|\mu \rangle / \langle 2|\mu \rangle] \) and \( p_1(z) = ([1] + [2|2])|1]) \), thus we have \( \bar{v} + 1 = 0 \). We reach (5.3) immediately from (4.19) as it has only one-cut and \( h_2 + h_3 - h_{23} - 1 = 0 \).

### 5.1.3 The \( P_{12} \) pole

The limit \( [2|1] \rightarrow 0 \) is trivial and we consider only \( \langle 2|1 \rangle \rightarrow 0 \). Using \( |P_{12} \rangle = [2|2 + [1|\langle \mu|1 \rangle / \langle \mu|2 \rangle] \) and comparing limits from (2.5) and direct factorization we arrive

\[
\frac{1}{\langle 2|3 \rangle \langle 3|4 \rangle} \left( \frac{\langle n|3 \rangle}{\langle n|2 \rangle} \right)^4 \frac{1}{\langle n|1 \rangle + z \langle n|2 \rangle} \prod_{l=1}^{v+1} \frac{w_l - z}{w_l - z_\alpha} = \left( \frac{\langle \mu|1 \rangle}{\langle \mu|2 \rangle} + z \right)^3 \frac{\langle 2|3 \rangle^3}{\langle n|2 \rangle}
\]

(5.4)
thus \( w_i = -\langle t_i | 1 \rangle / \langle t_i | 2 \rangle \), \( w_i - z_m = \langle 1 | 2 \rangle \langle t_i | n \rangle / \langle 2 | n \rangle \langle t_i | 2 \rangle \). Putting it back to (5.4) we find

\[
\left( \frac{\langle n | 3 \rangle \langle 1 | 2 \rangle}{\langle n | 2 \rangle} \right)^4 \prod_{i=1}^{4} \frac{\langle 2 | n \rangle \langle t_i | 2 \rangle}{\langle 1 | 2 \rangle \langle t_i | n \rangle} = \langle 2 | 3 \rangle^4, \rightarrow \prod_{i=1}^{4} \frac{\langle t_i | 2 \rangle}{\langle t_i | n \rangle} = \langle 3 | 2 \rangle^4 \langle 3 | n \rangle^4
\]

(5.5)

The solution is \( t_i = p_3 \). As expected, it is the right answer.

5.2 Example II—The Einstein-Maxwell Theory

The second example is a theory of photons coupled with gravitons. In addition to three-point graviton amplitudes, there are two extra three-point amplitudes

\[
M_\alpha(1^-_1, 2^+_2, 3^{-2}_3) = \kappa \frac{\langle 3 | 1 \rangle^4}{\langle 1 | 2 \rangle^2}, \quad M_\alpha(1^-_1, 2^+_2, 3^{+2}_g) = \kappa \frac{\langle 3 | 2 \rangle^4}{\langle 1 | 2 \rangle^2}.
\]

(5.6)

We will take the \( [1 | 2] \)-deformation

\[
\tilde{\lambda}_1 \to \tilde{\lambda}_1 - z \tilde{\lambda}_2, \quad \lambda_2 \to \lambda_2 + z \lambda_1.
\]

(5.7)

5.2.1 The four-point amplitude \( M_4(1^-_1, 2^+_2, 3^{-2}_3, 4^{+2}_g) \)

There are two poles \( s_{13} = 0 \) and \( s_{14} = 0 \) in the recursion relation with boundaries, but the pole at \( s_{14} = 0 \) gives no contribution under the deformation in (5.7). Thus we have only one term

\[
M_4(1^-_1, 2^+_2, 3^{-2}_3, 4^{+2}_g)(z) = \frac{[2 | 4]^4}{s_{13}s_{23}^2} \frac{\langle 1 | 3 \rangle^2 \langle 2 | 3 \rangle^2}{\langle 3 | 1 \rangle - z \langle 3 | 2 \rangle} \prod_{l=1}^{v+1} \frac{w_l - z}{w_l - z_\alpha}
\]

(5.8)

where \( z_\alpha = [3 | 1] / [3 | 2] \), \( \overline{1} = [3] [1 | 2] / [3 | 2] \).

Poles with \( z \)-dependence:

Two poles \( P^{14}_{13}(z) = 0 \) and \( P^{14}_{24}(z) = 0 \) exist here. For the pole \( P^{14}_{13}(z) = 0 \), \( [1(z) | 3] \) cannot vanish for arbitrary \( z \), but \( [1 | 3] \) can. In the \( 1^{-1}, 3^{-2} \) helicity configuration, the factorization limit is trivial and this pole gives no information on roots at all. For the pole \( P^{14}_{24}(z) = 0 \), the \( [1 | 4] \to 0 \) limit is not trivial and will be discussed carefully.

There are several nontrivial facts for the limit \( [1 | 4] \to 0 \). Due to momentum conservation, \( z_\alpha = [3 | 1] / [3 | 2] = -\langle 2 | 4 \rangle / \langle 1 | 4 \rangle \to \infty \) under the limit. As the pole \( \langle i | k \rangle = 0 \) discussed in the previous section, there are potentially extra contributions in \( T_{jk}(z) \). Here, the whole contribution comes totally from this extra possibility. Poles \( s_{14} \to 0 \) and \( s_{23} \to 0 \) occurs simultaneously, as they go to zero at the same time. This nontrivial kinematics happens only in four-point amplitudes. Due to this special kinematics, there is in fact no free parameter \( z \), as to be shown shortly.

The factorization limit from general consideration is

\[
I_{direct} = \lim_{\langle 1 | 4 \rangle \to 0} P_4(z)^2 M_4(z) = M^a_3(1^- \langle z \rangle, 4^{+2}, -P^{+1}_{14}(z)) M^h_3(P^{-1}_{14}(z), 2^+, 3^{-2})
\]

\[
= ([4 | 1] - z [4 | 2])^2 \left( \frac{\mu | 4 \rangle}{\langle \mu | 1 \rangle} \right)^{-2} \langle 3 | 2 \rangle + z (3 | 1 \rangle)^2 \left( \frac{\bar{\mu} | 3 \rangle}{\langle \bar{\mu} | 2 \rangle} \right)^{-2}
\]

(5.9)
where we have used
\[ P_{14}(z) = [1] \left( [1] - z [2] + [4] \frac{\langle \mu | 4 \rangle}{\langle \mu | 1 \rangle} \right), \quad P_{23}(z) = \left( [2] + z [1] + [3] \frac{[\bar{\mu} | 3]}{[\bar{\mu} | 2]} \right) [2] \] (5.10)

Since \( [1|4] \to 0 \) implies \( P_{23}(z)^2 \to 0 \), one must then have either \( [2(z)] \sim [3] \) or \( [2] \sim [3] \) for all \( z \) and the sensible choice is \( [2] \sim [3] \).

To get the other factorization limit, we rewrite (5.8) as
\[ M_4(1^-, 2^+, 3^-, 4^+)(z) = \frac{[2|4]^2 (1|3)^4}{s_{13} (1|4)^2} \frac{[3|1]}{[3|2]} \prod_{l=1}^{w+1} \frac{w_l - z}{w_l - z_\alpha} \] (5.11)

As \( z_\alpha \to \infty \), \( w_l - z_\alpha \to -z_\alpha \). To get a finite factorization limit, we need one and only one root. Putting all together we obtain
\[ I_{BCFW} = \lim_{[1|4] \to 0} P_{14}(z)^2 M_4(z) = \frac{[2|4]^2 (1|3)^4}{s_{13} (1|4)^2} \frac{[3|1]}{[3|2]} \prod_{l=1}^{w+1} \frac{w_l - z}{w_l - z_\alpha} \] (5.12)

where \( [2|3] \to 0 \) is used.

Superficially, one fails to see that \( I_{direct} = I_{BCFW} \) since one is a polynomial of \( z \) of degree 4 while the other of degree 2. They are indeed the same, due to the special kinematics of four particles, as we see presently. From \( P_{14}(z) = -P_{23}(z) \), one has
\[ [1] = \alpha \left( [2] + z [1] + [3] \frac{[\bar{\mu} | 3]}{[\bar{\mu} | 2]} \right), \quad [2] = -\alpha^{-1} \left( [1] - z [2] + [4] \frac{\langle \mu | 4 \rangle}{\langle \mu | 1 \rangle} \right) \] (5.13)

which is true if and only if
\[ \frac{[\bar{\mu} | 3]}{[\bar{\mu} | 2]} = -\frac{[1|2]}{[1|3]}, \quad \frac{\langle \mu | 4 \rangle}{\langle \mu | 1 \rangle} = -\frac{[2|1]}{[2|4]} \] (5.14)

and
\[ \alpha^{-1} = z + \frac{[3|2]}{[3|1]} = \left( z - \frac{[4|1]}{[4|2]} \right)^{-1} \] (5.15)

The condition (5.14) is in fact very tricky. \( [3|2]/[3|1] = -[4|1]/[4|2] \) can be obtained from momentum conservation, thus (5.14) gives a relation between \( z \) and external momenta. In other words, due to momentum conservation, \( z \) is not a variable under the factorization limit. (5.14) can be solved by
\[ z + \frac{[3|2]}{[3|1]} = \kappa = z - \frac{[4|1]}{[4|2]}, \quad \kappa = \pm 1. \] (5.16)

Using it we can simplify
\[ I_{BCFW} = -\frac{(2|4)^4 (1|3)^4}{s_{13}^2} \kappa (w - z), \quad I_{direct} = \frac{(2|4)^4 (1|3)^4}{s_{12}^2} \] (5.17)

Finally because \( s_{12} = -s_{13} \), we have
\[ w - z = -\kappa, \quad w = \frac{[4|1]}{[4|2]}, \quad \frac{w}{w - z_{13}} = -\frac{s_{23}}{s_{12}} \] (5.18)

Putting it back to (5.8) we will get the right amplitude.
Poles without $z$-dependence

Now there is only one pole $s_{12} = 0$. From the direct factorization limit, one has

$$\lim_{s_{12} \to 0} s_{12} M_4(z)$$

$$= M_2^a(1^-, 2^+, -P_{12}^{+2})M_3^b(P_{12}^{-2}, 3^-, 4^{+2}) + M_3^h(1^-, 2^+, -P_{12}^{-2})M_3^p(P_{12}^{+2}, 3^-, 4^{+2})$$

$$= \frac{[1|2]^2 (1|3)^6}{(3|4)^2 (4|1)^2} + \frac{[2|4]^6 (1|2)^2}{[3|4]^2 [3|2]^2}$$

(5.18)

where the first term is from the limit $\langle 1|2 \rangle \to 0$ while the second from the limit $[1|2] \to 0$. (5.18) does not depend on $z$ at all.

Now identify (5.18) with (5.8) after multiplying the latter by $s_{12}$. The $z$-independence of (5.18) means that the factor $[3|1]/[3|2] - z$ in denominator of (5.8) should be canceled by one factor $w - z$. That is, there is one root. We may work with two different limits, namely $[1] \sim [2], [3] \sim [4]$ or $[3] \sim [4], [1] \sim [2]$. The natural choice is $w = [4|1]/[4|2]$, the same result as given in (5.17). To work out the matching factors, careful kinematic analysis should be carried out as for the $\langle 1|4 \rangle \to 0$ limit.

5.2.2 The four-point amplitude $M_4(1^-, 2^+, 3^-, 4^+)$

There is only one pole $s_{14}$ in the recursion relation with boundaries (cut $s_{13}$ yields a null contribution here). With the deformation in (5.7), the boundary amplitude can be written as

$$M_4(1^-, 2^+, 3^-, 4^+)(z) = \frac{[2|4]^2 (1|3)^2}{\langle 1|4 \rangle (4|1) - z [4|2]} \prod_{l=1}^{v+1} \frac{w_l - z}{w_l - z_\alpha}$$

(5.19)

where $z_\alpha = [4|1]/[4|2]$, $\hat{1} = [4|1]/[4|2]$.

Poles with $z$-dependence:

Here one has two poles at $P_{13}^2(z) = 0$ and $P_{14}^2(z) = 0$. For the pole $P_{13}^2(z) = 0$, $[1(z)|3] \to 0$ can not be true for arbitrary $z$, but $\langle 1|3 \rangle \to 0$ can. In the $1^-, 3^-$-helicity configuration, the factorization limit is trivial and this pole gives no information on roots at all. For the pole $P_{14}^2(z)$, the $\langle 1|4 \rangle \to 0$ limit is not trivial. Similar to discussions in the previous subsection, due to momentum conservation and the special kinematics of four particles, $z$ cannot vary, as to be discussed presently.

The factorization limit from (5.13) is

$$I_{BCFW} = \lim_{(1|4) \to 0} P_{14}(z)^2 M_4(z) = [2|4]^2 (1|3)^2 \prod_{l=1}^{v+1} \frac{w_l - z}{w_l - z_\alpha}$$

(5.20)

and the factorization limit from general consideration is

$$I_{direct} = \lim_{(1|4) \to 0} P_{14}(z)^2 M_4(z)$$

$$= M_2^a(1^-(z), 4^+, -P_{14}^{+2}(z))M_3^b(P_{14}^{-2}(z), 2^+, 3^-) + M_3^h(1^-(z), 4^+, -P_{14}^{-2}(z))M_3^p(P_{14}^{+2}(z), 2^+, 3^-)$$

$$= ([4|1] - z [4|2])^2 ([3|2] + z [3|1])^2$$

(5.21)
where \( P_{14}(z) \) and \( P_{23}(z) \) are listed in (5.13). Following reasonings after (5.13), we have to choose \([2] \sim [3]\).

Setting \( I_{\text{direct}} = I_{\text{BCFW}} \) and \( w \to z_\alpha \), the dominator in \( I_{\text{BCFW}} \) must go to zero if the \( z \)-dependent part is consistent. The subtlety again resides in the special kinematics. \( I_{\text{direct}} \) can actually be written as

\[
I_{\text{direct}} = ([4|1] - z [4|2])^2((3|2) + z (3|1))^2 = [4|2]^2 (1|3)^2
\]

(5.22)

Compared with the \( I_{\text{BCFW}} \), one sees that \( w \to \infty \) under this limit.

**Poles without \( z \)-dependent**

There is only pole \( s_{12} = 0 \) in this case. The factorization limit from general theory is

\[
\lim_{s_{12} \to 0} s_{12} M_4(z) = M_3^2(1^-, 2^+, -P_{12}^{-2}) M_3^3(P_{12}^{-2}, 3^-, 4^+) + M_3^3(1^-, 2^+, -P_{12}^{-2}) M_3^3(P_{12}^{-2}, 3^-, 4^+) = [1|2]^2 (3|4)^2 + (1|2)^2 [3|4]^2
\]

(5.23)

where the first term is from the limit \( 1|2 \to 0 \) while the second from the limit \( [1|2] \to 0 \). (5.23) *does not depend on \( z \) at all.*

Now identify (5.23) with (5.19) after multiplying the latter by \( s_{12} \). The \( z \)-independence of (5.23) means that the factor \([4|1]/[4|2] - z\) in denominator of (5.13) should be canceled by one factor \( w - z \) under this limit. That is, there is only one root.

There are two different limits, \([1] \sim [2], [3] \sim [4] \) and \([3] \sim [4], [1] \sim [2] \). The natural choice is \( w = [3|1] / [3|2] = -[4|2] / [4|1] \). This gives naturally \( w \to \infty \) as \( [1|4] \to 0 \).

**5.2.3 The five-point amplitude** \( M(1_\gamma^{-1}, 2_\gamma^{+1}, 3_\gamma^{-1}, 4_\gamma^{+1}, 5_\gamma^{-2}) \)

Written as a BCFW expansion, the five-point amplitude \( M(1_\gamma^{-1}, 2_\gamma^{+1}, 3_\gamma^{-1}, 4_\gamma^{+1}, 5_\gamma^{-2}) \) can be deformed as

\[
M_5(z) = M_3(\hat{1}^{-1}, 5^{-2}, \hat{P}_{15}^{1+}) \frac{1}{s_{15}(z)} M_4(\hat{P}_{15}^{-1}, \hat{2}^{3+}, 3^{-4}, 4^{+}) \prod_{l=1}^{v+1} \frac{w_l - z}{w_l - z_{15}}
\]

\[
+ M_3(\hat{1}^{-1}, 4^{+}, P_{14}^{-2}) \frac{1}{s_{14}(z)} M_4(\hat{P}_{14}^{3+}, \hat{2}^{3+}, 3^{-5}, 5^{-2}) \prod_{l=1}^{v+1} \frac{w_l - z}{w_l - z_{14}}
\]

(5.24)

In the first term \( z_{15} = [5|1] / [5|2], \hat{1} = [5] [1|2] / [5|2] \) and in the second term \( z_{14} = [4|1] / [4|2], \hat{1} = [4] [1|2] / [4|2] \).
Poles with $z$-dependence:

There are four poles resulting from the vanishing of $P_{15}(z)$, $P_{14}(z)$, $P_{25}(z)$ and $P_{23}(z)$, respectively. Other possible poles give trivial factorization limits. For poles $P_{15}(z) = 0$ and $P_{14}(z) = 0$, both $(1|5) \to 0$ and $(1|4) \to 0$ make the factorization limit trivial. They yield nothing at all. $[2|5] \to 0$ and $[2|3] \to 0$ are nontrivial limits, but $(2|z|5) \to 0$ and $(2|z|3) \to 0$ cannot be true for arbitrary $z$.

Consider $[2|5] \to 0$ first. The factorization limit from general theory is

$$I_{\text{direct}} = \lim_{[2|5] \to 0} P_{25}(z)^2 M_5(z)$$

$$= M_4(1^- (z), P_{25}^+(z), 3^-, 4^+) M_3(P_{25}^-(z), 2^-(z), 5^-)$$

$$= \left[ \frac{\mu[2]}{\mu[5]} \right]^{2} \frac{(1|3)[4][2](z)}{(1|4)[2][5][5] - z[2][5]} \prod_{l=1}^{z_{15}+1} \frac{w_l - z}{w_l - z_{15}}$$

where we have used

$$P_{25}(z) = [2] \left( [2] + z [1] + [5] \left[ \frac{\mu[5]}{\mu[1]} \right] \right)$$

The factorization limit from (5.24) is

$$I_{\text{BCFW}} = \lim_{[2|5] \to 0} P_{25}(z)^2 M_4(z)$$

$$= \left\{ \frac{(1|5)[3][5][2][4](z)}{[4][3][2][5][5] - [2][5]} \prod_{l=1}^{z_{15}+1} \frac{w_l - z}{w_l - z_{15}} \right\} \left[ \frac{2|4}{[2][3][4][5]} \right]$$

Identifying $I_{\text{BCFW}}$ with $I_{\text{direct}}$ and noticing that $z_{15} \to \infty$, one gets one root in this limit: $w = -[3|1]/[3|2]$.

The analysis of $[2|3] \to 0$ is analogous that of $[2|5] \to 0$. To make the degrees of $z$ identical in these two factorization limits, the root $w$ here has to become infinity under $[2|3] \to 0$. This implies that there may be an factor $[2|3]^{-1}$ in the denominator.

Poles without $z$-dependent

Now the $z$-independent limits. Poles without $z$-dependence include two categories: $P_{\alpha}^2 \to 0 (\alpha \subset I$ or $\alpha \subset I^c$ ) and $P_{12}^2 \to 0$. And there are three kinds of $P_{\alpha}^2 \to 0$ limits: (1) from the collinear limit $s_{34}$ (where $\lambda_3 \sim \lambda_4$) one obtains a single root $w = -[2|5]/[1|5] = [1|P_{34}]/[2|P_{34}]$ with $M_4(1^-, 2^+, P_{34}^{12}, 5^-)$ in direct factorization limit; (2) from the collinear limit $s_{35}$ (where $\lambda_3 \sim \lambda_5$) one gets a single root $w = -[2|4]/[1|4] = [1|P_{35}]/[2|P_{35}]$ with $M_4(1^-, 2^+, P_{34}^{12}, 4^+)$ in direct limit part; (3) from the collinear limit $s_{45}$ (where $\lambda_4 \sim \lambda_5$) one finds a single root $w = [1|3]/[2|3]$ with $M_4(1^-, 2^+, 3^-, 4^+)$ in direct limit part. From these, we deduce the root’s expression as

$$w = \left[ \frac{1|3}{2|3} \right] (1 + f_1 [3|4][3|5][4|5])$$

(5.28)
with following requirements: (1) \( f_1 [3|4] [3|5] [4|5] \) should be dimensionless; (2) \( f_1 [3|4] [3|5] [4|5] \) should be helicity neutral for all external particles; (3) the \( w/(w - z_\alpha) \) should not produce un-physical pole when we collect all results. One is then led to the natural choice \( w = [1|3] / [2|3] \).

Now the roots under the limit \( P_{12}^2 \to 0 \). \( 1|2 \to 0 \) will lead to a trivial result, while \([1|2] \to 0 \) could result in a solution. Moreover, \( z_{15} \sim z_{14} \to 1 \) under this limit. Comparing two different factorization limit under the limit \([1|2] \to 0 \), one finds that \( w = 1 \).

Together with above discussions, the root in \( M(z) \) is \( w = [3|1] / [3|2] \). Plugging it back to \( M(z = 0) \), one obtains the same amplitude as in [16].

5.3 Example III— The six-gluon amplitude \( M_6(1^−, 2^−, 3^−, 4^+, 5^+, 6^+) \)

In previous examples we can solve roots with the help of their factorization limits. One naturally asks whether this is possible in general. In this subsection, we will use the example of six-gluon amplitude \( M_6(1^−, 2^−, 3^−, 4^+, 5^+, 6^+) \) to show that, generally, knowing values of roots under all factorization limits is not enough to solve them. In fact, values of roots are not even simple rational functions of spinor contraction \([| |], [ | |] \).

The six-gluon amplitude \( M_6(1^−, 2^−, 3^−, 4^+, 5^+, 6^+) \) is given by

\[
M_6(1^−, 2^−, 3^−, 4^+, 5^+, 6^+) = \frac{1}{(5|3 + 4|2)} \left( \frac{(1|2 + 3|4)^3}{[23][34] \langle 5|6 \rangle \langle 6|1 \rangle (p_2 + p_3 + p_4)^2} \right.
\]

\[
+ \left. \frac{(3|4 + 5|6)^3}{[61][12] \langle 3|4 \rangle \langle 4|5 \rangle (p_3 + p_4 + p_5)^2} \right),
\]

For our purpose we will use the deformation-[5|3]\(^{11}\).

\[
|3 \to |3 + z|5\rangle, \quad |5 \to |5 - z|3\rangle.
\]

Under this deformation, the boundary BCFW recursion relation gives following \( z \)-dependent amplitudes

\[
M_6(1^−, 2^−, 3(z^−), 4^+, 5(z^+), 6^+) = \frac{[6|5 + 3|4]^3 (3|5)^3}{[2|3 + 4|5 \langle 4|5 \rangle^4 [6|1] [1|2] P_{345}^2 ((3|4) + z (5|4))} \prod_l \frac{w_l - z}{w_l - z_{34}}
\]

\[
+ \frac{[4|2 + 3|5]^3 (3|5 + 6|1)^3}{[2|3 + 4|5 \langle 3|2 + 4|5 \rangle^3 [2|3 \langle 4|3 \rangle \langle 5|6 \rangle \langle 6|1 \rangle (P_{234}^2 + z [3|2 + 4|5])} \prod_l \frac{w_l - z}{w_l - z_{234}}.
\]

where \( z_{34} = -(4|3) / (4|5) \) and \( z_{234} = -P_{234}^2 / [3|2 + 4|5] \).

The pole structure of six-gluon amplitude is the following. There are three three-particles poles, \( s_{123} = s_{456}, s_{234} = s_{561}, s_{345} = s_{612} \). Among them, the split helicity configuration, \( s_{123} = s_{456} \) is trivial.

\(^{11}\)For deformation [4|3], there is no pole and the recursion relation should be modified accordingly.
For two particle poles we need to consider the holomorphic and anti-holomorphic part. After splitting helicity configurations, nontrivial channels are

\[ [1|2], [2|3], (3|4), [3|4], (4|5), (5|6), (6|1), [6|1] \]  

(5.32)

### 5.3.1 Poles under factorization limits without z-dependence

Here one has one three-particle channel \( P_{612} \) and three two-particle channels \([1|2], \langle 6|1 \rangle, [6|1]\). Since 3, 5 are not nearby, we do not have the pole \( P_{35} \), as mentioned before.

**Pole \( P_{612} \):** When \( P_{216}^2 \rightarrow 0 \), factorization limit leads to

\[ M_3(1^-, 2^-, -P_{612}^+) M_5(P_{12}^-, 3^-, 4^+, 5^+, 6^+) = \frac{\langle 1|2 \rangle^3 [4|5 - z3]^3}{[3|4] \langle 6|1 \rangle (2|P_{612}|3) [5 - z3]P_{612}|6)} \]

which leads to triple roots \( w_i^{(3)} = [4|5] / [4|3] \). This shows that we could find roots without working out detailed comparison, evidencing certain power of \( z \)-dependent factorization limits.

**Pole \([1|2]\):** The factorization limit is

\[ M_3(1^-, 2^-, -P_{12}^+) M_5(P_{12}^-, 3^-, 4^+, 5^+, 6^+) = \frac{[1|2]}{[\mu|2]} \frac{(-)(|\mu|1 + 2|3) + z |\mu|1 + 2|5)}{3 + z5|4} \frac{\langle 4|5 \rangle \langle 5|6 \rangle [\mu|1 + 2|6]}{[\mu|1]^2} \]

which leads to triple roots \( w_i^{(3)} = -|\mu|1 + 2|3| / |\mu|1 + 2|5| \).

**Pole \( \langle 6|1 \rangle \):** The factorization limit gives

\[ M_3(6^+, 1^-, -P_{16}^+) M_5(P_{16}^-, 2^-, 3^-, 4^+, 5^+) = \frac{[1|6]}{[\mu|6]} \frac{\langle 1|6 \rangle [4|5 - z3]^3}{[5 - z3]1 + 6|\mu} \frac{\langle 2|1 + 6|\mu \rangle \langle 2|3 \rangle [3|4]}{[\mu|1]^3} \]

which leads to triple roots \( w_i^{(3)} = [4|5] / [4|3] \).

**Pole \([6|1]\):** The factorization limit gives

\[ M_3(6^+, 1^-, -P_{16}^+) M_5(P_{16}^-, 2^-, 3^-, 4^+, 5^+) = \frac{[1|6]}{[\mu|6]} \frac{\langle 1|6 \rangle [2|3 + z5]^3}{[5|1 + 6|\mu} \frac{\langle 2|1 + 6|\mu \rangle \langle 3 + z5|4]}{[\mu|1]^3} \]

which leads to triple roots \( w_i^{(3)} = -(2|3) / (2|5) \).

### 5.3.2 Poles under factorization limits with z-dependence

For this type, pole \( P_{234} \) does not have \( z \)-dependent factorization limit and need not to be discussed. We are left with five two particle poles \([2|3], (3|4), [3|4], (4|5), \) and \((5|6)\). Among them, \([3|4]\) is automatically satisfied by recursion relation.

**Pole \([2|3]\):** The direct factorization gives

\[ M_3(2^-, 3^-, -\hat{P}_{23}^+) M_5(\hat{P}_{23}^-, 4^+, 5^+, 6^+, 1^-) = \frac{\langle 2|3 \rangle + z \langle 2|5 \rangle [\mu|2 + 3|1] + z \langle 1|5 \rangle [3|\mu]}{[2|\mu]} \frac{\langle 4|5 \rangle \langle 5|6 \rangle \langle 6|1 \rangle \langle 5|4 \rangle \langle 5|3]}{[\mu|3]^3} \]

Compared with contribution from the second term of (5.31), we find triple roots \( w_i^{(3)} = -[\mu|2 + 3|1] / [\mu|3] \) \langle 5|1 \rangle \).
Pole \([3|4]\): The direct factorization gives

\[
M_3(\hat{3}^-, 4^+, -\hat{P}_{34}^-)M_5(\hat{P}_{34}^+, \hat{5}^+, 6^+, 1^-, 2^-) = \frac{[\mu|4]^3 \langle 1|2 \rangle^3 (\langle 3|4 \rangle + z \langle 5|4 \rangle)}{[\mu|3] [\mu|3 + 4|5] \langle 5|6 \rangle \langle 6|1 \rangle (-[\mu|4 + 3|2] + z (2|5) [\mu|3])},
\]

which does not have nontrivial \(z\)-dependence (factor \((\langle 3|4 \rangle + z \langle 5|4 \rangle)\) comes from \(s_{34}\)). From our previous discussions, it can happen when and only when \(w \rightarrow \infty\) under the limit.

Pole \([4|5]\): The factorization limit gives

\[
M_5(6^+, 1^-, 2^-, \hat{3}^-, \hat{P}_{45}^+)M_3(-\hat{P}_{45}^-, 4^+, \hat{5}^+) = \frac{(\langle 6|4 + 5|\mu \rangle - z [3|6] \langle \mu|5 \rangle)^2 ([5|4] - z [3|4])}{\langle \mu|4 \rangle \langle \mu|5 \rangle [\langle 6|1 \rangle [1|2] [2|3] [3|4 + 5|\mu])},
\]

which leads to triple roots \(w_i^{(3)} = -[6|4 + 5|\mu) / [6|3 \langle \mu|5\rangle\).

Pole \([5|6]\): The factorization limit gives

\[
M_5(1^-, 2^-, \hat{3}^-, 4^+, \hat{P}_{56}^+)M_3(-\hat{P}_{56}^-, 5^+, 6^+) = \frac{([4|5 + 6|\mu] + z [4|3] \langle \mu|5 \rangle)^3 ([6|5] - z [3|5])}{\langle \mu|6 \rangle \langle \mu|5 \rangle [1|2] [2|3] [3|4] ([1|5 + 6|\mu] - z [3|1] \langle \mu|5\rangle)},
\]

which leads to triple roots \(w_i^{(3)} = -[4|5 + 6|\mu) / [4|3 \langle \mu|5\rangle\).

### 5.3.3 True values of roots

So far, roots have been found under various factorization limits. We wish to find roots without taking the limits, to reproduce known results in \((5.29)\). However, without using the known result \((5.29)\), we are not able to do so. To show why it is so difficult to solve roots with the help of factorization limits, we now discuss roots directly from \((5.29)\).

The numerator from expression \((5.29)\) is given by

\[
N = T_1 + T_2
\]

\[
T_1 = -\langle 4|5 \rangle [2|1] [6|1] s_{345} \langle 4|5 \rangle \langle 1|5 \rangle^3 [4|3]^3 \left(z + \frac{\langle 3|4 \rangle}{\langle 5|4 \rangle} \right) \left(-\frac{[4|P_{23}|1]}{\langle 1|5 \rangle [4|3]} + z \right)^3
\]

\[
T_2 = -\langle 1|6 \rangle \langle 5|6 \rangle [3|2] [4|3] \langle 5|P_{234}|3 \rangle \langle 5|P_{345}|6 \rangle^3 \left(\frac{s_{234}}{[5|P_{234}|3]} + z \right) \left(\frac{\langle 3|P_{345}|6 \rangle}{[5|P_{345}|6]} + z \right)^3
\]

(5.33)
From (5.33) we can read out values of roots under various factorization limits

\[ [1|2] \to 0, \quad w_i^{(3)} = -\frac{[\mu|1 + 2|3]}{[\mu|1 + 2|5]} = -\frac{6|4 + 5|3}{6|3 + 4|5}, \]

\[ \langle 1|6 \rangle \to 0, \quad w_i^{(3)} = \frac{[4|5]}{[4|3]} = \frac{4|2 + 3|1}{[4|3] \langle 1|5 \rangle}, \]

\[ [1|6] \to 0, \quad w_i^{(3)} = -\frac{\langle 2|3 \rangle}{\langle 2|5 \rangle} = -\frac{6|4 + 5|3}{6|3 + 4|5}, \]

\[ P_{216}^2 \to 0, \quad w_i^{(3)} = \frac{[4|5]}{[4|3]} = -\frac{6|4 + 5|3}{6|3 + 4|5}, \]

\[ [2|3] \to 0, \quad w_i^{(3)} = -\frac{[\mu|2 + 3|1]}{[\mu|3] \langle 5|1 \rangle} = \frac{4|2 + 3|1}{[4|3] \langle 1|5 \rangle}, \]

\[ [3|4] \to 0, \quad w_i^{(3)} \to \infty, \]

\[ \langle 5|6 \rangle \to 0, \quad w_i^{(3)} = -\frac{[4|5 + 6|\mu]}{[4|3] \langle \mu|5 \rangle} = \frac{4|2 + 3|1}{[4|3] \langle 1|5 \rangle}, \]

\[ \langle 5|4 \rangle \to 0, \quad w_i^{(3)} = \frac{[6|4 + 5|\mu]}{[6|3] \langle \mu|5 \rangle} = -\frac{6|4 + 5|3}{6|3 + 4|5}. \] (5.34)

The reason why we obtained simple rational expressions for roots is that one of \( T_1, T_2 \) will be zero under these factorization limits. However, for general momentum configurations, \( T_1 \) and \( T_2 \) are not zero, thus we have to solve roots of degree four polynomial. The analytic expression for roots is very complicated and it is not rational function of spinor.\(^{12}\) Because the irrationality, even with information given in (5.34), it is very hard to find explicit expressions.

6. Conclusion

Understanding nontrivial boundary contributions is important in the application of BCFW recursion relations. In [16], they were translated to discussion of roots of amplitudes. In this paper, we have investigated some aspects of roots.

First we re-derived BCFW recursion relations with boundary contributions from a different perspective. Then we generalized the factorization limits to \( z \)-dependent ones, where the behavior of roots under the limit can be seen more clearly. The merits or the demerits of these analyses was illustrated by examples. One sees that information extracted from roots under various factorization limits is valuable, but not powerful enough to guarantee explicit expressions of roots. Our analysis has not been conclusive. We have the feeling that it may not be practical to find the boundary contributions through roots, though it does help to clarify certain theoretical issues, as shown in this paper and in [17].

Roots of amplitudes have not been discussed extensively in quantum field theories. Their roles are still obscure. It may help to understand quantum field theories if they can get more thorough scrutinization. And we believe that they deserve the attention.

\(^{12}\)We have checked this using numerical method by setting all spinor components to be integer number.
Acknowledgements

We are supported by fund from Qiu-Shi, the Fundamental Research Funds for the Central Universities with contract number 2010QNA3015, National Basic Research Program of China (2010CB833000), as well as Chinese NSF funding under contract Nos.10875104, 11031005, 10875103, 11135006, 11125523.

References

[1] L. D. Landau, Nucl. Phys. 13, 181 (1959);
   S. Mandelstam, Phys. Rev. 112, 1344 (1958);
   S. Mandelstam, Phys. Rev. 115, 1741 (1959);
   R. E. Cutkosky, J. Math. Phys. 1, 429 (1960).

[2] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B 435, 59 (1995) [arXiv:hep-ph/9409265].

[3] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B 425, 217 (1994) [arXiv:hep-ph/9403226].

[4] R. Britto, F. Cachazo and B. Feng, Nucl. Phys. B 725, 275 (2005) [arXiv:hep-th/0412103].

[5] R. Britto, E. Buchbinder, F. Cachazo and B. Feng, Phys. Rev. D 72, 065012 (2005) [arXiv:hep-ph/0503132].

[6] C. Anastasiou, R. Britto, B. Feng, Z. Kunszt and P. Mastrolia, Phys. Lett. B 645, 213 (2007) [arXiv:hep-ph/0609191].

[7] D. Forde, Phys. Rev. D 75, 125019 (2007) [arXiv:0704.1835 [hep-ph]].

[8] Z. Bern, J. J. M. Carrasco, L. J. Dixon, H. Johansson and R. Roiban, Phys. Rev. D 82, 125040 (2010) [arXiv:1008.3327 [hep-th]].

[9] R. Britto, F. Cachazo and B. Feng, “New Recursion Relations for Tree Amplitudes of Gluons,” Nucl. Phys. B 715, 499 (2005) [arXiv:hep-th/0412308].

[10] R. Britto, F. Cachazo, B. Feng and E. Witten, “Direct Proof Of Tree-Level Recursion Relation In Yang-Mills Theory,” Phys. Rev. Lett. 94, 181602 (2005) [arXiv:hep-th/0501052].

[11] P. Benincasa, F. Cachazo, “Consistancy Conditions On The S-Matix Of Massless Particles” hep-th/07054305.

[12] R. H. Boels, “No triangles on the moduli space of maximally supersymmetric gauge theory,” JHEP 1005, 046 (2010) [arXiv:1003.2989 [hep-th]].

[13] B. Feng, J. Wang, Y. Wang and Z. Zhang, “BCFW Recursion Relation with Nonzero Boundary Contribution,” JHEP 1001, 019 (2010) [arXiv:0911.0301 [hep-th]].

[14] B. Feng and C. Y. Liu, “A Note on the boundary contribution with bad deformation in gauge theory,” JHEP 1007, 093 (2010) [arXiv:1004.1282 [hep-th]].

[15] B. Feng, Z. Zhang, “Boundary Contributions Using Fermion Pair Deformation,” [arXiv:1109.1887 [hep-th]].

[16] P. Benincasa and E. Conde, “On the Tree-Level Structure of Scattering Amplitudes of Massless Particles,” arXiv:1106.0166 [hep-th].
[17] P. Benincasa and E. Conde, “Exploring the S-Matrix of Massless Particles,” arXiv:1108.3078 [hep-th].

[18] P. C. Schuster and N. Toro, “Constructing the Tree-Level Yang-Mills S-Matrix Using Complex Factorization,” JHEP 0906, 079 (2009) [arXiv:0811.3207 [hep-th]].

[19] P. Benincasa, C. Boucher-Veronneau and F. Cachazo, “Taming Tree Amplitudes In General Relativity,” JHEP 0711, 057 (2007) [arXiv:hep-th/0702032].

[20] N. Arkani-Hamed and J. Kaplan, “On Tree Amplitudes in Gauge Theory and Gravity,” JHEP 0804, 076 (2008) [arXiv:0801.2385 [hep-th]].