Source Reconstruction as an Inverse Problem

Norman Gray

Department of Physics and Astronomy, University of Glasgow, Glasgow, G12 8QQ, UK

Iain J Coleman

British Antarctic Survey, Madingley Road, Cambridge, CB3 0ET, UK

Abstract. Inverse Problem techniques offer powerful tools which deal naturally with marginal data and asymmetric or strongly smoothing kernels, in cases where parameter-fitting methods may be used only with some caution. Although they are typically subject to some bias, they can invert data without requiring one to assume a particular model for the source. The Backus-Gilbert method in particular concentrates on the tradeoff between resolution and stability, and allows one to select an optimal compromise between them. We use these tools to analyse the problem of reconstructing features of the source star in a microlensing event, show that it should be possible to obtain useful information about the star with reasonably obtainable data, and note that the quality of the reconstruction is more sensitive to the number of data points than to the quality of individual ones.

1. Introduction

Where once all the interest in microlensing was in the details of the lensing population, there is now an increasing interest in lensing events as probes of the stellar sources. From this point of view, once the lens' geometrical details have been worked out, the event can be used as a 'super-telescope', providing otherwise completely unattainable resolution of the surfaces of distant stellar disks.

Initially, analyses assumed that the microlensing source star could be taken to be a point source, and the first discussion of 'finite source effects' was in the context of a problem – Witt & Mao (1994) asked at what point the point-source approximation would break down; Nemiroff & Wickramasinghe (1994) put the question more positively, asking what information could be obtained from the distortions to the light curve which finite-source effects would cause.

It is not merely intensity information which can be obtained from events. Simmons, Newsam, & Willis (1995) discuss the information which can be extracted when polarization measurements are made of microlensing events, and in (Newsam et al. 1998) show how even relatively poor polarization data can substantially improve fits of source parameters.
Although the basic microlensing effect is achromatic, the fact that stars have different limb-darkening profiles in different colours means that a lens differentially amplifying the disk will produce a chromatic effect. Other workers (Valls-Gabaud 1995; Sasselov 1996; Valls-Gabaud 1998) have discussed how one might obtain such chromaticity information. It is even possible to discuss how one might observe the signatures of stellar spots (Heyrovský & Sasselov 1999; Bryce & Hendry 2000).

The usual way in which source structure is detected is by applying a model-fitting (equivalently, parameter-fitting) algorithm to the observed data, to obtain the best-fit parameters of a suitable limb-darkening model; this is the approach used, for example, by the MACHO collaboration (Alcock et al. 1997) and the PLANET collaboration (Albrow et al. 1999) to make the first detections of limb-darkening in microlensing events. It is also the approach which underlies the insightful error analysis by Gaudi & Gould (1999).

A parameter-fitting algorithm essentially consists of a mechanism for systematically moving through parameter space, repeatedly solving the ‘forward problem’ – calculating the data to be predicted from a given limb-darkening profile – until the predicted data is optimally close to the data actually observed. Here we want to suggest that, because of the fact that the underlying source function is convolved through a broad and asymmetric amplification kernel, a model-fitting approach is potentially problematic, and that this recovery problem is more naturally addressed using the well-established technology of inverse problems.

We plan to discuss the merits of inverse problem techniques in general, and the Backus-Gilbert method in particular, and exemplify the possibilities by inverting simulated microlensing data to recover limb-darkening and limb-polarization effects. We will see that we are able to discuss explicitly and robustly the tradeoff between resolution and stability which is implicit in any such inversion, including recoveries obtained by model-fitting.

2. Background

The geometry we consider is as shown in Fig. 1. The amplification function is the familiar one.

$$A(\xi) = \frac{1}{2} \left( \xi + \frac{1}{\xi} \right), \quad \xi = \left( 1 + \frac{4}{\zeta^2} \right)^{1/2}, \quad \zeta^2 = r^2 + s^2 - 2rs \cos(\chi - \phi).$$ (1)

Denote the intensity on the stellar surface $I(r)$, and the Stokes parameter by $Q(r, \chi) = -P(r) \cos 2\chi$, where $P(r)$ is the polarization of the stellar surface and we are assuming that the surface is rotationally symmetric. In the case of a microlensing event, we cannot directly resolve details of the lensed source, and must therefore measure integrals over the source surface. We immediately obtain

$$I(s(t), \phi(t)) = \int_0^\infty I(r) A_I(r; s, \phi) \, dr$$ (2)

$$Q(s(t), \phi(t)) = \int_0^\infty P(r) A_Q(r; s, \phi) \, dr$$ (3)
Source Reconstruction as an Inverse Problem

Figure 1. Geometry of a lensing event. The projected path of the lens has impact parameter \( l \), and the path is parameterised by polar coordinates \( s(t) \) and \( \phi(t) \), relative to the centre of the source, projected into the lens plane. Any point in that plane can be given in polar coordinates \( r \) and \( \chi \), and this point is a distance \( \zeta \) from the centre of the lens. All dimensions are normalised to the Einstein radius in the source plane. The angles \( \phi \) and \( \chi \) are taken with respect to the line joining the source to the lens’ point of closest approach.

where the amplification kernels are

\[
\tilde{A}_I = r \int_0^{2\pi} A(r, \chi; s, \phi) \, d\chi \quad (4)
\]

\[
\tilde{A}_Q = -r \int_0^{2\pi} \cos 2\chi A(r, \chi; s, \phi) \, d\chi. \quad (5)
\]

Note that the kernel \( \tilde{A}_I \) is a factor \( 2\pi r \) times the angular average of the amplification function, and the functions \( I(s, \phi) \) and \( Q(s, \phi) \) have the dimensions of flux rather than intensity.

Analytic expressions for these angle-averaged amplification functions have been obtained by numerous people. Schneider & Wagoner (1987) and Gaudi & Gould (1999) perform the average for approximate forms of the amplification function, Heyrovský & Loeb (1997) deal with an elliptical source for a slightly restricted class of source functions. Gray (2000) produced integrals for the exact amplification function, for axisymmetric sources, obtaining

\[
\tilde{A}_I(r; s, \phi) = r(2\pi + I_1 - I_2), \quad \tilde{A}_Q(r; s, \phi) = -r \cos 2\phi(Q_1 - Q_2)
\]

where \( I_1, I_2, Q_1 \) and \( Q_2 \) are elliptic integrals whose arguments have an algebraic dependence on \( r \) and \( s \). Although it is helpful, it is not in fact necessary to have analytic forms for the angle-averaged amplification functions, and the following analysis would be just as successful if these could be obtained only numerically.

Equations (2) and (3) are one-dimensional integral equations – the classic form of an inverse problem. In this context, the function \( I(r) \) is termed the
underlying or source function, and the function \( \hat{A}_I(r; s) \) the kernel. The data is the set of values \( I(s(t_i), \phi(t_i)) \) for some set of times \( t_i \).

The use of inverse problem techniques is relatively rare in this branch of astrophysics. Mineshige & Yonehara (1999) used the technique to map the einstein cross accretion disk using a hypothetical caustic crossing, and Wambsganss (2000) points out a number of other uses in the field of cosmological microlensing. Coleman, Gray, & Simmons (1998) use a similar technique with eclipsing binaries. The method described in this paper is discussed at greater length in Gray & Coleman (2000).

3. Inverse problem techniques vs. parameter fitting

Parameter fitting is the most appropriate technique in the case where (a) there is no doubt about the most appropriate model to use, so that the aim is simply to recover model parameters, and (b) when the problem kernel is not ill-conditioned. When the model itself is open to dispute, or the observational situation means that the kernel is ill-conditioned, then any parameter fit must be done extremely cautiously if it is not to be deceptive.

Ill-conditioning can be characterised in several ways. Fundamentally, an ill-conditioned kernel maps a large volume of parameter space to a small volume of data space; a strongly smoothing (nearly flat) kernel would be an example of this. This implies that the inversion is highly unstable, so that a tiny, noise-induced, change in the data (\( I(t) \) in Eqn. (2)) could, after a na"ıve inversion, be taken to indicate a radically different recovered function \( I(r) \).

Inverse problem techniques – also known as 'non-parametric' or 'model-free' techniques – dispense with a parameterised model, and instead approach the problem from the question ‘how much information does this kernel permit to be recovered from this data?’ (see, for example, Craig & Brown (1986) for a general introduction). They are most natural in the case of marginal data, or an asymmetric or broad kernel. These techniques are typically associated with Bayesian approaches to data analysis, and deal with the instability of the inversion by adding prior information, such as the supposition that the underlying function be smooth (in the case of inversion by regularisation) or otherwise featureless (in the case of maximum-entropy inversion). This explicit addition of prior information inevitably makes inverse problem techniques suffer from bias; in a parameter-fitting algorithm, the model acts as implicit prior information, so that route is not free of bias either.

3.1. Backus-Gilbert

The particular technique we have used is the Backus-Gilbert method. This works by allowing us to explicitly trade off recovery resolution against stability. We very briefly outline the method here; there is a fuller account in, for example, Parker (1977), and an astrophysical example in Loredo & Epstein (1989).

Given a kernel \( K(r; s_i) \), underlying function \( u(r) \), data \( F(s_i) \), and noise \( n_i \), the general 1-d inverse problem can be written as

\[
F(s_i) = \int u(r)K(r, s_i)dr + n_i.
\]
We suppose that we can find ‘response kernels’ $q_i(r)$ which permit us to form an estimate $\hat{u}$ of the underlying function as a weighted average of the data:

$$\hat{u}(r) = \sum_i q_i(r) F(s_i).$$  \hfill (6)

This is a random variable, but we can relate its mean to the underlying function through an ‘averaging kernel’ $\Delta$:

$$\langle \hat{u}(r) \rangle = \int \Delta(r, r') u(r') \, dr'.$$  \hfill (7)

Ideally, this kernel would be the Dirac delta function, and the estimator $\hat{u}$ would perfectly track the underlying function. Since the underlying function is (of course) unknown, we cannot use Eqn. (7) directly, but this definition of $\Delta(r, r')$ allows us to define $\text{Width}[\Delta]$ and $\text{Var}[\hat{u}(r)]$, which depend only on the kernel $K(r; s_i)$ and the noise $n_i$. This means that we can explicitly trade off improved recovery resolution (narrower $\text{Width}[\Delta]$) against improved stability (smaller $\text{Var}[\hat{u}(r)]$), with the relative weighting parametrised by smoothing parameter $\lambda$. For each value of $\lambda$ we can analytically obtain a set of coefficients $q_i(r)$ for Eqn. (6).

Note that this technique is an analysis of the kernel, rather than a particular data set, which means (a) the understanding we gain of how much information is available from a data set is portable both to other inverse problem techniques, and also to parameter-fitting approaches, and (b) the analysis can be done, and an optimal $\lambda$ selected, prior to any data being collected, given a set of demands on the required resolution and stability. A feature of the Backus-Gilbert method is that the response kernels $q_i(r)$ has a dependence on the radial parameter $r$, so that the analysis needs to be redone for each $\hat{u}(r)$ we wish to recover (at a cost of potentially large matrix inversions each time). This means that the Backus-Gilbert method (at least in its simplest form) is not an obvious choice for a data reduction pipeline, but it has the compensation that the optimal tradeoff can be chosen differently for different values of $r$.

4. Recovery of limb-darkening and limb-polarization profiles

In Figure 3, we show the tradeoff curves for the polarization kernel, Eqn. (3), for a variety of numbers of data points, and choices of smoothing parameter $\lambda$ (there is a similar curve for limb-darkening, Eqn. (2)). Recovery quality can be improved either by increasing the number of data points, or by adjusting the times at which data is taken. In the figure, the line marked ‘$n = 25$ (uneven)’ shows the tradeoff curve for data taken unevenly, with observations clustered at points the analysis shows to be particularly sensitive; it is clear that the quality of the recovery is sensitive to both the number of data points and the manner in which they are obtained.

The bottom left of the plot corresponds to high resolution and high stability, the top left to recoveries with high resolution and low stability (in the limit, picking a single data point and scaling it), and the bottom right to low resolution and high stability (in the limit, simply averaging all the data).
Figure 2. Resolution versus stability for various numbers of data points, and choices of smoothing parameter $\lambda$. The box in the lower left hand corner shows the region of ‘acceptable’ recoveries.

The question of what counts as ‘adequate’ resolution and stability can only be decided in the context of a particular stellar model (though this does not make the Backus-Gilbert analysis, or the trade-off diagram in Figure 2, model-dependent). For a pure electron atmosphere, Chandrasekhar calculated a limb-polarization of 11%, suggesting that the variance of the limb-polarization recovery may be of order 10% at most. Estimating the width of a typical limb-polarization profile suggests a similar upper bound for the resolution in Figure 2. The figure therefore indicates what value of the tradeoff parameter $\lambda$ we should pick, and therefore what is the best resolution and variance we can expect to achieve with a given number of data points.

In Figure 3, we show two recoveries of a limb-darkening profile from synthetic data with different noise standard-deviation. Note (a) that the optimal recovered variance varies across the disk; (b) that the recovery is biased – towards a flat profile in this case – with the bias being greatest at the centre of the disk and at the limb; (c) that both the bias and the variance are least around 70% of the projected radius, which is particularly fortunate since it is in this region that limb-darkening profiles tend to be most sensitive to parameters such as $g$, and where the profiles of giants differ most from those of normal stars (Hauschildt 2000); and (d) that the quality of the recovery is surprisingly insensitive to the quality of the data.

5. Discussion

One of the aims of this paper is to emphasise the seriousness of the ill-conditioning of the source reconstruction problem, and hence the desirability of using an an-
Figure 3. Recoveries of a limb-darkening profile (solid line) with 100 data points, and impact parameter, einstein radius and stellar radius all equal. The points plotted with boxes and solid error bars are recovered from simulated data with $\sigma = 0.002$, and the points with triangles and dotted error bars from data with $\sigma = 0.1$. After Coleman (1998).

analytical technique which starts by examining that ill-conditioning, goes on to discuss what information is nonetheless recoverable, and only then produces the numerical information which is the point of the exercise. Of course, the same questions can be asked using a parameter-fit approach, but less naturally, since such approaches assume, in a sense, that the information is recoverable, with the result that problems can only be uncovered post hoc, by an intelligent examination of goodness-of-fit measures, or by tracing the propagation of errors through a calculation.

The second aim is to use this inverse problem approach to analyse the kernel which turns the underlying limb-darkening and limb-polarization functions into microlensing intensity and polarization data. It turns out that the information is indeed recoverable with adequate uncertainties but, depending on the quality of the data available, one many have to make significant compromises over the resolution one is prepared to accept. It is also possible to use this approach to analyse the effect of modifications in the way the data is collected, and discover that this can indeed significantly improve the data’s effective quality.

References

Albrow, M. D., Beaulieu, J.-P., Caldwell, J. A. R., Dominik, M., Greenhill, J., Hill, K., Kane, S., Martin, R., et al., 1999, ApJ, 522, 1011
Alcock, C., Allsman, R. A., Alves, D., Axelrod, T. S., Becker, A., C., Bennett, D. P., Cook, K. H., Freeman, et al., 1997, ApJ, 491, L11
Gray and Coleman

Bryce, H. & Hendry, M. A. 2000, in this proceedings
Coleman, I. J. 1998, PhD thesis, University of Glasgow
Coleman, I. J., Gray, N., & Simmons, J. F. L. 1998, A&AS, 131, 187
Craig, I. D. & Brown, J. C. 1986, Inverse Problems in Astronomy (Adam Hilger)
Gaudi, B. S. & Gould, A. 1999, ApJ, 513, 619
Gray, N. 2000, Gravitational Microlensing Source Limb-Darkening and Limb-Polarization, I: Angle-Averaged Amplification Functions [arXiv:astro-ph/0001359]
Gray, N. & Coleman, I. J. 2000, Gravitational Microlensing Source Limb-Darkening and Limb-Polarization, II: The Inverse Problem (in preparation)
Hauschildt, P. H. 2000, in this proceedings
Heyrovský, D. & Loeb, A. 1997, ApJ, 490, 38
Heyrovský, D. & Sasselov, D. 1999, Detecting Stellar Spots by Gravitational Microlensing, Tech. rep., Harvard University
Loredo, T. J. & Epstein, R. I. 1989, ApJ, 336, 896
Mineshige, S. & Yonehara, A. 1999, PASJ, 51, 497
Nemiroff, R. J. & Wickramasinghe, W. A. D. T. 1994, ApJ, 424, L21
Newsam, A. M., Simmons, J. F. L., Hendry, M. A., & Coleman, I. J. 1998, New Astron. Reviews, 42, 121
Parker, R. L. 1977, Ann. Rev. Earth Planet. Sci., 5, 35
Sasselov, D. D. 1996, in 12th IAP Astrophysics Colloquium, July 1996, Paris, ed. R. Ferlet
Schneider, P. & Wagoner, R. V. 1987, ApJ, 314, 154
Simmons, J. F. L., Newsam, A. M., & Willis, J. P. 1995, MNRAS, 276, 182
Valls-Gabaud, D. 1995, in Large scale structure in the universe, Proceedings of an international workshop, Potsdam, Germany, edited by J.P. Muecket, S. Gottloeber and V. Mueller 1995 (World Scientific Co.), 326
Valls-Gabaud, D. 1998, MNRAS, 294, 747
Wambsganss, J. 2000, in this proceedings
Witt, H. J. & Mao, S. 1994, ApJ, 430, 505