Monochromatic interfacial wave propagating over one and two bar(s)

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Abstract. Wave propagation is an interesting subject to be modelled and studied in oceanography. Interfacial wave is one of waves that can occur between two fluid-layers of different density. Since the complexity of the problem, simplification is made by considering a specific type of wave, i.e. monochromatic. In propagating, the transmission and reflection of the wave are considered as the effect of disturbing of the fluid depth. It can be made by putting a bar on the bottom of the channel. The dimension of the bar plays an important role in obtaining the minimum amplitude of the transmitted wave. This is then continued to determine the distance between two bars to get the minimum amplitude. The result was presented in the MSA-conference.

1. Introduction
Wave propagation in ocean is an interesting phenomena for entertainment or some attractions, but also carrying energy that can destroy places where the waves are passed by. As references, wave propagation can be read with various mathematical model, such as in Benjamin [1] (KdV equation), Joseph [2] and Kubota, et. al. [3] (Intermediate Long Wave (ILW) equation), Benjamin [4], Davis and Acrivos [5] and Ono [6] (Benjamin-Ono (BO) equation), Matsuno [7], Choi and Camassa [8, 9], and Camassa, et.al. [10] (combination between KdV and ILW equations).

Meanwhile, when the waves approach a beach, they need to be controlled so that they are not dangerous surrounding. Study on wave propagation indicates that energy is proportional to the amplitude squared, and amplitude of the wave can be reduced when the wave passes over a bar, since part of the wave will be reflected and some is transmitted. This can be seen for example in Mei [11]. Therefore, to control waves we can propose by putting some bars along the bottom and we simulate to calculate the percentage of reduction of the amplitude after passing those bars.

In this paper, we consider monochromatic wave, but involving the effect of the fluid above the ocean. The density of air is not neglected, so that we can consider the air is another fluid laying above the ocean, and the wave is then called as interfacial wave. Tuck and Wiryanto [12] modelled the interfacial wave into a single equation called Composite Long Wave equation, similar to BO equation, but involving Hilbert transform. They obtained a periodic interfacial wave as the solution of the model. In propagating, the wave satisfies a relation called dispersion relation, the relation between the frequency and the wavenumber. Wiryanto [13] derived that relation based on potential function of fluid particle moving on flat bottom in one system fluid, but then Wiryanto and Jamhuri [14], also Lusia et. al. [15], extended the derivation of the dispersion relation of interfacial wave in two-fluid system. The dispersion relation, for
one and two-fluid system, is mainly depends on the fluid depth, so that when the wave propagates passing over a bar, it can change wave-number. This is then used to observe the break up of the wave into reflection and transmission.

The dispersion relation in formulating the transmitted wave is presented in this paper. Since the wave propagates over one or two bars, the water depth changes and each area contains two waves in different direction. Therefore, at the changing depth, we use the continuation of surface and flux. When these conditions are applied we can calculate the amplitude of each wave. So that we can determine the reduction of the amplitude after the wave passing over the bars. Moreover, we can also determine the optimal dimension of the bar and the distance between two bars, giving the minimum amplitude of the transmitted wave. This paper summaries [13], [14] and [16].

2. Problem Description

2.1. Dispersion relation

We consider wave propagation between two fluids with different density in a canal. The lower fluid has density $\rho_1$ with depth $h_1$. Another fluid with density $\rho_2$ and having infinite depth lies down above the lower layer. A bar is placed at the bottom of the canal. Therefore, the wave propagating over the bar is partly transmitted and reflected. We illustrate this wave propagation and the medium in Figure 1.

![Figure 1. Sketch of the interfacial wave and coordinates](image)

As the coordinates, we choose the $x$-axis is along the interface with $x = 0$ at the left side of the bar. The vertical $y$-axis is perpendicular to the horizontal one. So, the wave can be denoted by $y = \eta(x, t)$ as the elevation of the still water interface.

In case the bottom is flat and the wave is monochromatic, i.e. $\eta(x, t) = Ae^{-i(kx-\omega t)}$ with constant amplitude $A$, the relation between the wave number $k$ and the frequency $\omega$ is given by

$$
\frac{\omega^2}{k} \left[-\rho_1 \left( e^{-h_1k} \frac{1}{\sinh(h_1k)} + 1 \right) + \rho_2 \left( e^{-h_2k} \frac{1}{\sinh(h_2k)} + 1 \right) \right] = (\rho_2 - \rho_1)g
$$

where $g$ is the acceleration of gravity. The derivation of (1) can be seen in [15]. When the upper fluid is not involved, we can take $\rho_2 \rightarrow 0$, and (1) agrees with the dispersion relation for one layer, as derived by Wiryanto [13]

$$
\frac{\omega^2}{k} = g \tanh h_1 k
$$
Meanwhile, for infinite depth of the upper layer, (1) becomes
\[
\left( \frac{\omega^2}{k} \right) \left[ -\rho_1 \left( e^{-b_1 k} \sinh(h_1 k) + 1 \right) + \rho_2 \right] = (\rho_2 - \rho_1) g
\] (2)

This dispersion relation is then used to calculate the amplitude of the transmitted and reflected waves.

2.2. Wave amplitude

We consider an incoming wave of amplitude \( A \). In its propagation, the wave passes a bar of height \( d \) and length \( L \). So, there are three regions:
1. in the left side of the bar \( x < 0 \),
2. above the bar \( 0 < x < L \),
3. in the right side of the bar \( x > L \).

Since the fluid depth changes, the wave number at each region also change satisfying (2) for given \( \omega \). Also in region 1 and 2 there are two waves travel in different direction and different amplitude. Therefore, we can express the interface elevation
\[
\eta(x, t) = \begin{cases} 
Ae^{-i(k_1 x - \omega t)} + A_r e^{-i(k_1 x + \omega t)}, & x < 0 \\
B_t e^{-i(k_2 x - \omega t)} + B_r e^{-i(k_2 x + \omega t)}, & 0 < x < L \\
C_t e^{-i(k_3 x - \omega t)}, & L < x 
\end{cases}
\] (3)

\( A \) is the amplitude of the incoming wave, \( A_r \) is the amplitude of the reflected wave and \( k_1 \) is the wave number corresponding to fluid depth \( h_{11} \) of lower layer in the region 1. Similar to \( B_t, B_r \) and \( k_2 \) represent the amplitude of the transmitted wave, reflected wave and the wave number corresponding to fluid depth \( h_{12} \) of lower layer in the region 2. \( C_t \) is the amplitude of the transmitted wave, and \( k_3 \) is the wave number in region 3, the same fluid depth in region 1.

Our task is to determine \( A_r, B_t, B_r, \) and \( C_t \), that can be calculated from physical situation should be satisfied, i.e. the elevation and the flux must be continue across \( x = 0 \) and \( x = L \). Note that flux is the total horizontal velocity. Wiryanto [13] formulated these into
\[
\begin{align*}
A + A_r &= B_t + B_r \\
B_t e^{-ik_2 L} + B_r e^{ik_2 L} &= C_t e^{-ik_1 L} \\
k_1(A - A_r) &= k_2(B_t - B_r) \\
k_1(B_t e^{-ik_2 L} - B_r e^{ik_2 L}) &= k_2 C_t e^{-ik_1 L}
\end{align*}
\] (4)

When we can solve that problem, we then observe \( C_t \) for various \( L \). Wiryanto [13] showed that there is a relation between both quantities periodically, the same value \( C_t \) for many values \( L \). The shortest \( L \) that gives smallest value of \( C_t \) is then called \( L_{opt} \), denoted by \( L_{opt} \).

Now we extend for two bars on the bottom of the channel, with distance \( S \). So, there are five regions. Similar following the formulation above, each region contains two waves with different direction and different amplitude, except the last region containing only the transmitted wave, namely \( E_t \). The interface elevation is then expressed by extended from (3), i.e.
\[
\eta(x, t) = \begin{cases} 
A e^{-i(k_1 x - \omega t)} + A_r e^{-i(k_1 x + \omega t)}, & x < 0 \\
B_t e^{-i(k_2 x - \omega t)} + B_r e^{-i(k_2 x + \omega t)}, & 0 < x < L \\
C_t e^{-i(k_1 x - \omega t)} + C_r e^{-i(k_1 x + \omega t)}, & L < x < L + S \\
D_t e^{-i(k_2 x - \omega t)} + D_r e^{-i(k_2 x + \omega t)}, & L + S < x < 2L + S \\
E_t e^{-i(k_3 x - \omega t)}, & 2L + S < x
\end{cases}
\] (5)

Here we use the same length (\( L \)) and the same height (\( h_{11} - h_{12} \)) of the bars.

All type of amplitudes is formulated from elevation and flux continuation at each place where the fluid depth changes, We adopt from Wiryanto & Mungkasi [16].
\[ A + A_r = B_t + B_r \]
\[ B_t e^{-ik_2L} + B_r e^{ik_2L} = C_t e^{-ik_1L} + C_r e^{ik_1L} \]
\[ C_t e^{-ik_1(L+S)} + C_r e^{ik_1(L+S)} = D_t e^{-ik_2(L+S)} + D_r e^{ik_2(L+S)} \]
\[ D_t e^{-ik_2(2L+S)} + D_r e^{ik_2(2L+S)} = E_t e^{-ik_1(2L+S)} \]
\[ k_2(A - A_r) = k_1(B_t - B_r) \]
\[ k_2(C_t e^{-ik_1(L+S)} - C_r e^{ik_1(L+S)}) = k_1(D_t e^{-ik_2(L+S)} - D_r e^{ik_2(L+S)}) \]
\[ k_2(D_t e^{-ik_2(2L+S)} - D_r e^{ik_2(2L+S)}) = k_2E_t e^{-ik_1(2L+S)} \]

In this step, we are interested in determining \( S \) optimal, denoted by \( S_{opt} \), i.e. the smallest distance between two bars that gives minimum value of transmitted amplitude \( E_t \), by using \( L_{opt} \). We present the result in the next section.

3. Results

The reflected and transmitted interfacial-waves are observed from an incoming wave of amplitude \( A \) in this section. We first discuss the result for one bar submerged beneath the water. In our calculation, we use the acceleration of gravity \( g = 10 \), wave frequency \( \omega = 2 \), the lower fluid depth \( h_{11} = 3.5 \), and the bar height \( d = 1.5 \) \((h_{12} = 2)\). The lower fluid density is \( \rho_1 = 1 \).

From those quantities, (2) with \( \rho_2 = 0.1 \) gives the wave number \( k_1 = 0.5132 \) corresponding to \( h_{11} \) and \( k_2 = 0.6809 \) corresponding to \( h_{12} \) the region above the bar. We then calculate the amplitudes: \( A_r \), \( B_t \), \( B_r \), and \( C_t \) satisfying (4), for giving the length \( L \) of the bar. But basically we are interested in determining \( C_t \), proportional to the amplitude of the incoming wave \( A \). In other word we choose \( A = 1 \).

For \( L = 2.3 \), we obtain \( A_r = 0.2754 \) and \( C_t = 0.9881 \). For other \( L \), we can repeat the calculation. Plot of \( C_t \) versus \( L \), shown in Figure 2, indicates that there is minimum value of \( C_t \). This value can be obtained for some \( L \), but for the smallest \( L \) we then call as the \( L \) optimal, denoted by \( L_{opt} \). In this case, we obtain \( L_{opt} = 3.0 \) giving \( C_t = 0.9856 \). Similar curve in Fig.2 can be obtained for different density of the upper fluid, for example when we use \( \rho_2 = 0.49 \), the optimal length of the bar is \( L_{opt} = 2.73 \) giving \( C_t = 0.9769 \). We can see the effect of the upper fluid density to the reducing of the transmitted amplitude.

![Figure 2. Plot of \( C_t \) (vertical) versus \( L \) (horizontal)](image-url)
Now, we continue to observe the transmitted wave after passing two bars. From our previous calculation, for $\rho_2 = 0.1$, we put two bars of optimal length $(L_{opt} = 3.0)$ side by side and with distance $S$ between them. We then calculate the transmitted amplitude $E_t$ after passing those two bars by solving (6). When the distance $S$ is 1.5, (6) gives $E_t = 0.9777$ and the reflected amplitude is $A_r = 0.21000$. Meanwhile, for $S = 3.6$, we obtain the transmitted amplitude $E_t = 0.9445$. Compare to other value of $S$, for example $S = 4$, we have $E_t = 0.9465$. What is the optimal of $S$ giving minimum $E_t$? To answer this question, we can calculate for various $S$ and collects the data for $E_t$. The plot of $E_t$ versus $S$ is given in Figure 3. The optimal $S$ is at 3.55 giving $E_t = 0.9445$.

We can compare our model of two bars by considering the bar separately. When the incoming wave passes the first bar, it reduces the amplitude becoming 0.9858. This is then used as a new incoming wave that passes to the second bar. The amplitude re-reduces again becoming quadrat of that number, i.e. 0.9718. Calculation of separate bar reduces smaller amplitude than using the optimal distance $S_{opt}$.

To see the effect of the upper-layer density, the above procedure can be repeated for various $\rho_2$. We present the result in following table

**Table 1.** The optimal length of bars and the optimal distance between two bars for various upper density

| $\rho_2$ | $L_{opt}$ | $S_{opt}$ |
|----------|----------|----------|
| 0.05     | 3.02     | 3.57     |
| 0.1      | 3.00     | 3.55     |
| 0.2      | 2.94     | 3.52     |
| 0.3      | 2.88     | 3.48     |
| 0.4      | 2.81     | 3.44     |
| 0.5      | 2.72     | 3.39     |

Meanwhile, by increasing $\rho_2$ the transmitted wave has smaller amplitude.

**4. Conclusion**

We have observed an interfacial wave of type monochromatic passing over one and two bars. The amplitude reduction is observed and the dimension of the bar plays an important role in obtaining the maximum reduction. The distance between two bars is also parameter that can be optimized. But all of those is also effected to the ratio of the fluid-layer density.
Acknowledgement
The research of the problem was supported by Research Gant from Indonesian Government Bandung, year 2017 and 2018. Therefore, the author acknowledged.

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