A Discontinuous Coarse Spaces (DCS) Algorithm for Cell Centered Finite Volumes based Domain Decomposition Methods: the DCS-RJMin algorithm

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Abstract

In this paper, we introduce a new coarse space algorithm, the “Discontinuous Coarse Space Robin Jump Minimizer” (DCS-RJMin), to be used in conjunction with one-level domain decomposition methods (DDM). This new algorithm makes use of Discontinuous Coarse Spaces (DCS), and is designed for DDM that naturally produce discontinuous iterates such as Optimized Schwarz Methods (OSM). This algorithm is suitable both at the continuous level and for cell-centered finite volume discretizations. At the continuous level, we prove, under some conditions on the parameters of the algorithm, that the difference between two consecutive iterates goes to 0. We also provide numerical results illustrating the convergence behavior of the DCS-RJMin algorithm.

Key words: discontinuous coarse space, optimized Schwarz method.

1 Introduction

Due to the ever increasing parallelism in modern computers, and the ever increasing affordability of massively parallel calculators, it is of utmost importance to develop algorithms that are not only parallel but scalable. In this paper, we are interested in Domain Decomposition Methods (DDM) which are one way to parallelize the numerical resolution of Partial Differential Equations (PDE).

In Domain Decomposition Methods, the whole domain is subdivided in several subdomains and a computation unit is assigned to each subdomain. In this paper, we only consider non-overlapping domain decompositions. The numerical solution is then computed in parallel inside each subdomain with artificial boundary conditions. Then, subdomains exchange information between each other. This process is reapplied until convergence. In practice, such a scheme, called iterative DDM, should be accelerated
using Krylov methods. However, for the purpose of analyzing an algorithm, it can be interesting to work directly with the iterative algorithm itself as Krylov acceleration is so efficient it can hide away small design problems in the algorithm.

In one-level DDM, only neighboring subdomains exchange information. Most classical DDM are one-level. While one-level DDM can be very efficient and converge in a few iterations, they are not scalable: convergence can never occur before information has propagated between the two furthest apart subdomains. I.E., a one level DDM must iterate at least as many times as the diameter of the connectivity graph of the domain decomposition. Typically, if $N$ is the number of subdomains, this means at least $O(N)$ iterations for one-dimensional problems, $O(\sqrt{N})$ for two-dimensional ones and $O(\sqrt[3]{N})$ for three-dimensional ones. For DDM to be scalable, some kind of global information exchange is needed. The traditional approach to achieve such global information exchange is adding a coarse space to a pre-existing one-level DDM.

To the author knowledge, the first use of coarse spaces in Domain Decomposition Methods can be found in [16]. Because coarse spaces enable global information exchange, scalability becomes possible. Well known methods with coarse spaces are the two-level Additive Schwarz method [3], the FETI method [13], and the balancing Neumann-Neumann methods [12] [4] [14]. Coarse spaces are also an active area of research, see [2] [15] for high contrast problems. It is not trivial to add an effective coarse space to one-level DDM that produce discontinuous iterates such as Optimized Schwarz Methods, see [6] [7], and [5] chap.5.

In [9], the authors introduced the idea of using discontinuous coarse spaces. Since many DDM algorithms produce discontinuous iterates, the use of discontinuous coarse corrections is needed to correct the discontinuities between subdomains. In that proceeding, one possible algorithm, the DCS-DMNV (Discontinuous Coarse Space Dirichlet Minimizer Neumann Variational), was described at the continuous level and at the discrete level for Finite Element Methods on a non-overlapping Domain Decomposition. In [17], a similar method, the DCS-DGLC algorithm was proposed. Both the DCS-DMNV and the DCS-DGLC are well suited to finite element discretizations. Also, a similar approach was proposed in [8] for Restricted Additive Schwarz (RAS), an overlapping DDM.

It was proven recently that the proof of convergence for Schwarz found in [11] [1] can be extended to the Discrete Optimized Schwarz algorithm with cell centered finite volume methods, see [10]. It would be interesting to have a discontinuous coarse space algorithm that is suited to cell centered finite volumes. Unfortunately, neither the DCS-DMNV algorithm nor the DCS-DGLC algorithm are practical for cell centered finite volume methods: the stiffness matrix necessary to compute the coarse correction isn’t as sparse as one would intuitively believe. In this paper, our main goal is to describe one family of algorithms making use of discontinuous coarse spaces but suitable for cell centered finite volumes discretizations.

In [2] we briefly recall the motivations behind the use of discontinuous coarse space. In [4] we present the DCS-RJMin algorithm. In [4] we prove that under some conditions on the algorithm parameter, the $L^2$-norm of the difference between two consecutive iterates goes to zero. Finally, we present numerical results in [5].
2 Optimized Schwarz and Discontinuous Coarse Spaces

Let’s consider a polygonal domain $\Omega$ in $\mathbb{R}^2$. As a simple test case, we wish to solve
\[
\eta u - \Delta u = f \text{ in } \Omega,
\]
\[
u = 0 \text{ on } \partial \Omega.
\]

Without a coarse space, the Optimized Schwarz Method is defined as

**Algorithm 2.1** (Coarseless OSM).

1. Set $u^0_i$ to either the null function or to the coarse solution.
2. Until convergence
   (a) Set $u^{n+1}_i$ as the unique solution to
   \[
   \eta u^{n+1}_i - \Delta u^{n+1}_i = f \text{ in } \Omega_i,
   \]
   \[
   \frac{\partial u^{n+1}_i}{\partial n_i} + p u^{n+1}_i = \frac{\partial u^n_j}{\partial n_i} + p u^n_j \text{ on } \partial \Omega_i \cap \partial \Omega_j,
   \]
   \[
u^{n+1}_i = 0 \text{ on } \partial \Omega_i \cap \partial \Omega.
   \]

In practical applications, such an algorithm should be accelerated using Krylov methods. However, studying the iterative (Richardson) version can give mathematical insight on the convergence speed of the Krylov accelerated algorithm.

The main shortcoming of the coarseless Optimized Schwarz methods is the absence of direct communication between distant subdomains. To get a scalable algorithm, one can use a coarse space. A general version of a coarse space method for the OSM is

**Algorithm 2.2** (Generic OSM with coarse space).

1. Set $u^0_i$ to either the null function or to the coarse solution.
2. Until convergence
   (a) Set $u^{n+1}_i$ as the unique solution to
   \[
   \eta u^{n+1/2}_i - \Delta u^{n+1}_i = f \text{ in } \Omega_i,
   \]
   \[
   \frac{\partial u^{n+1/2}_i}{\partial n_i} + p u^{n+1/2}_i = \frac{\partial u^n_j}{\partial n_i} + p u^n_j \text{ on } \partial \Omega_i \cap \partial \Omega_j,
   \]
   \[
u^{n+1/2}_i = 0 \text{ on } \partial \Omega_i \cap \partial \Omega.
   \]
   (b) Compute in some way a coarse corrector $U^{n+1}$ belonging to the coarse space $X$, then set
   \[
u^{n+1} = U^{n+1} + U^{n+1}.
   \]

More important than the algorithm used to compute the coarse correction $U^{n+1}$ is the choice of an adequate coarse space itself. The ideas presented in [9] still apply. In particular, the coarse space should contain discontinuous functions and the discontinuities of the coarse corrector should be located at the interfaces between subdomains.
For these reasons, we suppose the whole domain $\Omega$ is meshed by either a coarse triangular mesh or a cartesian mesh $\mathcal{T}_H$ and we use each coarse cell of $\mathcal{T}_H$ as a subdomain $\Omega_i$ of $\Omega$. The optimal theoretical coarse space $\mathscr{A}$ is the set of all functions that are solutions to the homogenous equation inside each subdomain: for linear problems, the errors made by any iterate are guaranteed to belong to that space. With an adequate algorithm to compute $U^{n+1}$, the coarse space $\mathscr{A}$ gives a convergence in a single coarse iteration. Unfortunately this complete coarse space is only practical for one dimensional problems as it is of infinite dimension in higher dimensions. One should therefore choose a finite dimensional subset $X_d$ of $\mathscr{A}$.

The choice of the coarse space $X_d$ is primordial. It should have a dimension that is a small multiple of the number of subdomains. To choose $X_d$, one only need to choose boundary conditions on every subdomain, then fill the interior of each subdomain by solving the homogenous equation in each subdomain. In this paper, we have not tried to optimize $X_d$ and for the sake of simplicity have chosen $X_d$ as the set of all functions in $\mathscr{A}$ with linear Dirichlet boundary conditions on each interface between any two adjacent subdomains.

### 3 The DCS-RJMin Algorithm

We now describe the DCS-Robin Jump Minimizer algorithm:

**Algorithm 3.1 (DCS-RJMin).**

Set $p > 0$ and $q > 0$ and $X_d$ a finite dimensional subspace of $\mathscr{A}$.

Set $u^0$ to either 0 or to the coarse space solution.

Until Convergence

1. Set $u^{n+1/2}$ as the unique solution to

   \[
   \eta u^{n+1/2} - \Delta u^{n+1/2} = f \text{ in } \Omega_i, \\
   \frac{\partial u^{n+1/2}}{\partial \nu_{ij}} + pu^{n+1/2}_i = \frac{\partial u^n}{\partial \nu_{ij}} + pu^n_j \text{ on } \partial \Omega_i \cap \partial \Omega_j, \\
   u_i = 0 \text{ on } \partial \Omega_i \cap \partial \Omega_j. 
   \]

2. Set $U^{n+1}$ in $X_d$ as the unique coarse function that minimizes

   \[
   \sum_{i=1}^N \sum_{j \in \mathcal{N}(i)} \left\| \frac{\partial (u^{n+1/2}_i + U^{n+1}_i)}{\partial \nu_i} + q(u^{n+1/2}_i + U^{n+1}_i) \\
   - \frac{\partial (u^{n+1/2}_j + U^{n+1}_j)}{\partial \nu_j} - q(u^{n+1/2}_j + U^{n+1}_j) \right\|_{L^2(\partial \Omega_i \cap \partial \Omega_j)}^2, 
   \]

   where $\nu_i$ is the outward normal to subdomain $\Omega_i$ and $\mathcal{N}(i)$ the set of all $j$ such that $\Omega_j$ and $\Omega_i$ are adjacent.

3. Set $u^{n+1} := u^{n+1/2} + U^{n+1}$. 

4
4 Partial “Convergence” results for DCS-RJMin

We don’t have a complete convergence theorem for the DCS-RJMin algorithm. However, we can prove the iterates of the DCS-RJMin algorithm are close to converging when \( p = q \):

**Proposition 4.1.** If \( q = p \). Then, the iterates produced by the DCS-RJMin algorithm satisfy \( \lim_{n \to +\infty} \| u_i^{n+1/2} - u_i^n \|_{L^2} = 0 \).

**Proof.** Let \( u \) be the mono-domain solution, set \( e_i^n = u_i^n - u_i \), then, following Lions energy estimates [11],

\[
\eta \int_{\Omega_i} |e_i^{n+1/2} - e_i^n|^2 \text{d}x + \int_{\Omega_i} |\nabla (e_i^{n+1/2} - e_i^n)|^2 \text{d}x
\]

\[
\eta \sum_{\Omega_i} \int_{\Omega_i} |e_i^{n+1/2} - e_i^n|^2 \text{d}x + \int_{\Omega_i} |\nabla (e_i^{n+1/2} - e_i^n)|^2 \text{d}x =
\]

\[
\sum_{(i,j)} \int_{\Gamma_{ij}} |\nabla (e_i^{n+1/2} - e_i^n)|^2 \text{d}x + \int_{\Gamma_{ij}} |\nabla (e_i^{n+1/2} - e_i^n)|^2 \text{d}x,
\]

where \( \| \cdot \| \) represents a jump across the interface. Since the coarse step of the DCS-RJMin algorithm minimizes the Robin Jumps, we have

\[
\sum_{(i,j)} \int_{\Gamma_{ij}} |\nabla (e_i^{n+1/2} - e_i^n)|^2 \text{d}x + \int_{\Gamma_{ij}} |\nabla (e_i^{n+1/2} - e_i^n)|^2 \text{d}x \leq
\]

\[
\sum_{(i,j)} \frac{1}{4p} \left( \int_{\Gamma_{ij}} \left| \frac{\partial e_i^n}{\partial \nu_i} + pe_i^n \right|^2 \text{d}x - \int_{\Gamma_{ij}} \left| \frac{\partial e_i^{n+1/2}}{\partial \nu_i} + pe_i^{n+1/2} \right|^2 \text{d}x \right),
\]

Summing over \( n \geq 0 \) yields the stated result. \( \square \)
5 Numerical Results

![Convergence plots](image)

Figure 1: Convergence for OSM and DCS-RJmin with $\Omega = [0, 4]^2$, $f(x, y) = 0$ and random initial boundary conditions. Plotting $\log(\|e_{50}\|_\infty / \|e_0\|_\infty)$.

We have implemented the DCS-RJMin algorithm in C++ for cell-centered finite volumes on a cartesian grid. We chose $\Omega = [0, 4] \times [0, 4]$, $\eta = 0$ and iterated directly on the errors by choosing $f = 0$. We initialized the Robin boundary conditions at the interfaces between subdomains at random and performed multiple runs of the DCS-RJMin algorithm for various values of $p$, $q$ and of the number of subdomains. We had $p$ vary from 1.0 to 20.0 with 0.5 increments and $q$ takes the following values $p_m \times 10^{p_e}$ with $p_m$ in $\{1.0, 2.0, 4.0, 8.0\}$ and $p_e$ in $\{0, 1\}$. We consider $2 \times 2$, $4 \times 4$, $6 \times 6$ and $8 \times 8$ subdomains. There are always $20 \times 20$ cells per subdomains. In Figure 1 we plot $\log(\|e_{50}\|_\infty / \|e_0\|_\infty)$ as a function of $p$ for various values of $q$. First, we notice that for each value of $q$, the convergence deteriorates above a certain $p_q$. In fact, for low values of $q$ and high values of $p$, the iterates diverge. For two different values of $q$, the curves are very close when $p$ is smaller than both $p_q$. We also notice than even though we could only prove Proposition 4.1 for the case $p = q$, we observe numerical convergence even when $p \neq q$. In fact $p = q$ is not the numerical optimum. This is to be expected at the intuitive level: for a theoretical proof of convergence, we want the algorithm to keep lowering some functional. The existence of such a functional is likely only if all the substeps of the algorithm are optimized for the same kind of errors. If $p = q$, both the coarse step or the local step will either remove low frequency errors (small $p$ and $q$) or high frequency ones (high $p$ and $q$). An efficient numerical algorithm should have substeps optimized for completely different kind of errors. This is why efficient numerical algorithms are usually the ones for which the convergence proofs are the more difficult.
6 Conclusion

In this paper, we have introduced a new discontinuous coarse space algorithm, the DCS-RJMin, that is suitable for cell-centered finite volume discretizations. The coarse space greatly improve numerical convergence. It would be of great interest to study which is the optimal low-dimensional subspace of all piecewise discontinuous piecewise harmonic functions. Future work also includes the development of a possible alternative to coarse space in order to get scalability: “Piecewise Krylov Methods” where the same minimization problem than the one used in DCS-RJMin is used but where the coarse space are made of piecewise, per subdomain, differences between consecutive one-level iterates.

References

[1] Bruno Després. Domain decomposition method and the helmholtz problem. In Gary C. Cohen, Laurence Halpern, and Patrick Joly, editors, Mathematical and numerical aspects of wave propagation phenomena, volume 50 of Proceedings in Applied Mathematics Series, pages 44–52. Society for Industrial and Applied Mathematics, 1991.

[2] Victoria Dolean, Frédéric Nataf, Robert Scheichl, and Nicole Spillane. Analysis of a two-level schwarz method with coarse spaces based on local dirichlet to neumann maps. Computational Methods in Applied Mathematics, 12(4):391–414, 2012.

[3] Maksymilian Dryja and Olof B. Widlund. An additive variant of the Schwarz alternating method for the case of many subregions. Technical Report 339, also Ultracomputer Note 131, Department of Computer Science, Courant Institute, 1987.

[4] Maksymilian Dryja and Olof B. Widlund. Schwarz methods of Neumann-Neumann type for three-dimensional elliptic finite element problems. Comm. Pure Appl. Math., 48(2):121–155, February 1995.

[5] Olivier Dubois. Optimized Schwarz Methods for the Advection-Diffusion Equation and for Problems with Discontinuous Coefficients. PhD thesis, McGill University, 2007.

[6] Olivier Dubois and Martin J. Gander. Convergence behavior of a two-level optimized Schwarz preconditioner. In Domain Decomposition Methods in Science and Engineering XXI. Springer LNCSE, 2009.

[7] Olivier Dubois, Martin J. Gander, Sebastien Loisel, Amik St-Cyr, and Daniel Szyld. The optimized Schwarz method with a coarse grid correction. SIAM J. on Sci. Comp., 34(1):A421–A458, 2012.
[8] Martin J. Gander, Laurence Halpern, and Kévin Santugini. A new coarse grid correction for RAS. In Domain Decomposition Methods in Science and Engineering XXI. Springer LNCSE, 2013.

[9] Martin J. Gander, Laurence Halpern, and Kévin Santugini-Repiquet. Discontinuous coarse spaces for dd-methods with discontinuous iterates. In Domain Decomposition Methods in Science and Engineering XXI. Springer LNCSE, 2013.

[10] Martin J. Gander, Felix Kwok, and Kévin Santugini. Optimized Schwarz at cross points: Finite volume case. In preparation, 2013.

[11] Pierre-Louis Lions. On the Schwarz alternating method. III: a variant for nonoverlapping subdomains. In Tony F. Chan, Roland Glowinski, Jacques Périaux, and Olof Widlund, editors, Third International Symposium on Domain Decomposition Methods for Partial Differential Equations, held in Houston, Texas, March 20-22, 1989, pages 202–223, Philadelphia, PA, 1990. SIAM.

[12] Jan Mandel. Balancing domain decomposition. Communications in Numerical Methods in Engineering, 9(3):233–241, mar 1993.

[13] Jan Mandel and Marian Brezina. Balancing domain decomposition for problems with large jumps in coefficients. Math. Comp., 65:1387–1401, 1996.

[14] Jan Mandel and Radek Tezaur. Convergence of a Substructuring Method with Lagrange Multipliers. Numer. Math., 73:473–487, 1996.

[15] Frédéric Nataf, Hua Xiang, Victorita Dolean, and Nicole Spillane. A coarse sparse construction based on local Dirichlet-to-Neumann maps. SIAM J. Sci. Comput., 33(4):1623–1642, 2011.

[16] Roy A. Nicolaides. Deflation conjugate gradients with application to boundary value problems. SIAM J. Num. An., 24(2):355–365, 1987.

[17] Kévin Santugini. A discontinuous galerkin like coarse space correction for domain decomposition methods with continuous local spaces: the dcs-dglc algorithm. 2014. In revision.