CORRECT SOLVABILITY OF NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS IN ORLICZ SPACES.
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Abstract. We prove in this article the well posedness of non-linear Ordinary Differential Equations (ODE) of first and second order in Orlicz spaces with unbounded domain of definition.

Key Words. Orlicz spaces, ordinary non-linear differential equations, equivalent norms.

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0. Introduction. Statement of problem.

Let us consider the first order non-linear ODE of a view:

(1) \[ \frac{dy}{dx} - q(y) = g(x), \ x \in R. \]

or (non-linear Sturm-Liouville equation)

(2) \[ \frac{d^2 y}{dx^2} - q(x, y) = g(x), \ x \in R. \]

It is proved in the works ([2]; [3], [4], [5]) that under some conditions (necessary conditions and sufficient conditions) the equations (1) and (2) without boundary conditions are correct solvable in the spaces \( L_p(R) \) and consequently, under some simple additional conditions in the correspondent Sobolev spaces \( W_1^p(R), W_2^p(R) \).

Our goal is some generalizations of those results on the Orlicz spaces \( L(N) \) with \( N - \) Orlicz function \( N = N(u) \) instead classical functions \( |u|^p, \ p = \text{const} \geq 1 \), in particular, on the Orlicz spaces \( L(N) \) without the so-called \( \Delta_2 \) condition, e.g. Exponential Orlicz Spaces (EOS). This allow us to find some new properties of solutions, for instance, to prove the exponential integrability of solutions and its derivatives.

Probably, it is very interest to describe all the Orlicz spaces for which the equations (1), (2) are correct solvable. The statement of this problem belongs to L.Shuster. Now this general problem is open, but we can prove the correct solvability of our equations on two important classes of Orlicz spaces.

Recall here that the equation (non-linear, in general case) \( Ay = g \) is called correct solvable (or, in other hand, well posed) (more exactly, Lipshitz correct solvable) on the Banach space \( B_1 \) into the space \( B_2 \), if for all \( g \in B_1 \) the solution \( y = A^{-1}g \) there exists, is unique, belongs to the space \( B_2 \) and

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\[ \|A^{-1}g\|B_2 \leq C\|g\|B_1; \quad \|A^{-1}g_1 - A^{-1}g_2\|B_2 \leq C\|g_1 - g_2\|B_1, \]

where \( C = \text{const} \) does not depend on \( g, g_1, g_2 \).

1. Description of using Orlicz spaces.

We will consider two kinds of Orlicz spaces on the real line \( R \) with usually (unbounded) Lebesque measure \( \mu \). Recall here that if the function \( N = N(u), u \in R^1 \) is some \( N \) – Orlicz function (even, downwards convex, \( N(u) \geq 0, N(u) = 0 \iff u = 0 \), strong increasing in the self - line \( R^1_+ \), etc.), then the Orlicz norm \( \|f\|L(N) \) of a (measurable) function \( f : R \to R \) relative to the \( N \) – Orlicz function \( N = N(u) \) may be defined by the formula

\[ \|f\|L(N) = \inf\{k, k > 0, I(N(|f|/k)) \leq 1\}. \]

In this paper \( I(f) = \int_R f(x) \, dx \). As a particular case, if \( N(u) = |u|^p, p = \text{const} \geq 1 \) we obtain the classical \( L_p = L_p(R) \) spaces with the norm

\[ |f|_p \overset{\text{def}}{=} I^{1/p}|f|^p. \]

A. We define a class \( \Delta \) as a set of all Orlicz spaces with correspondent \( N \) – function \( N = N(u) \) belonging to all the classes \( \Delta_2(\infty), \nabla_2(\infty) \) in the terminology of the book [1], p. 22 - 24. By definition,

\[ \Delta_2(\infty) = \{N : \exists k < \infty, \forall u \geq 0 \Rightarrow N(2u)/N(u) < k\}; \]

\[ \nabla_2(\infty) = \{N : \exists l > 1, \forall u > 0 \Rightarrow N(2u) \leq N(lu)/(2l)\}. \]

or, briefly, \( N(\cdot) \in \Delta \overset{\text{def}}{=} \Delta_2(\infty) \cap \nabla_2(\infty) \). For example, let \( N(u) = N(m, r; u) \overset{\text{def}}{=} |u|^m (\log^r(\exp(m + |r|) + |u|)), m = \text{const} > 1, r = \text{const} \in R \), then \( N(\cdot) \) is some \( N \) – Orlicz function such that \( N(\cdot) \in \Delta \).

B. An other very important class of \( N \) – Orlicz functions are so - called \( EOF = \) Exponential Orlicz Functions and correspondent Exponential Orlicz Spaces \( EOS \) will be considered. Let \( \varphi = \varphi(z), z \geq 1 \) be some continuous function such that the function \( h(y) = h_\varphi(y) := \varphi(\exp y), y \geq 0 \) is strong increasing, downward convex and

\[ (3) \quad \sum_{k=3}^{\infty} \exp(h(k) - h(k + 1)) < \infty. \]

The set of all those function we will denote \( \Phi; \Phi = \{\varphi\} \).

For example, put \( \varphi(z) = \varphi_{m,r}(z) = z^m \log^r((\exp(m + |r|) + z]], \) but here \( m = \text{const} \in (0, \infty); \) or \( \varphi(z) = \varphi_\beta(z) = \log^{1+\beta}(2 + z), \beta = \text{const} > 0. \) Then \( \varphi_{m,r}(\cdot) \in \Phi, \varphi_\beta(\cdot) \in \Phi. \)

Let also \( \alpha = \text{const} \geq 1, \varphi \in \Phi. \) We denote by \( \exp_{\alpha} \varphi(z) \) the following continuous function: at \( z \in [0, C_1] \Rightarrow \exp_{\alpha} \varphi(z) = C_2 z^\alpha, \) and at \( z \in (C_1, \infty) \Rightarrow \exp_{\alpha} \varphi(z) = \exp(\varphi(z). \)
Let us prove at first the existence of the constants $C_1 = C_1(\alpha, \phi), C_2 = C_2(\alpha, \phi)$ such that $\exp_\alpha(\cdot) \in EOF$. We will use the so-called Young-Fenchel, or Legendre transform:

$$h^*(w) = \sup_{y \geq C} (yw - h(y)) = y_0w - h(y_0), \ w \geq C_1 = C_1(C),$$

the value $y_0 = y_0(w)$ there exists and is unique for all sufficiently large values $w$.

We must only prove that the constant $C_2$ there exists and is nontrivial:

$$C_2 = \inf_{z \geq C_1} \exp \varphi(z)/z^\alpha = \exp \left(- \sup_{z \geq C_1} (\alpha \log z - \varphi(z)) \right) = \exp \left(- \sup_{y \geq \exp(C_1)} (\alpha y - h(y)) \right) = \exp (-h^*(\alpha));$$

therefore the constant $C_2 = C_2(\alpha, \varphi(\cdot))$ there exists and is nontrivial: $C_2 \in (0, \infty)$.

Further we will choose the constants $C_1, C_2$ only such that the function $\exp_\alpha \varphi(\cdot) \in EOF.$

**Definition.** We define the so-called Exponential Orlicz Space (EOS) $B(\alpha; \varphi)$, where $\alpha = \text{const} \geq 1$, $\varphi(\cdot) \in \Phi$ as the Orlicz space on the set $R^1$ with the correspondent $N-$Orlicz function

$$N(u) = N(\alpha; \varphi, u) = \exp_\alpha \varphi(u)$$

and the correspondent Orlicz norm $\|f\|L(\exp_\alpha(\varphi)) = \|f\|B(\alpha; \varphi)$ and will say this spaces as EOS = Exponential Orlicz Spaces.

In the case $\varphi(z) = \varphi_m(z) = z^m, z, m > 0$ we will write simply $\|f\|B(\alpha, m) = \|f\|B(\alpha; \varphi_m)$, i.e. here

$$N(z) = N(\alpha, m, z) = \exp_\alpha (z^m), \ z \geq 0.$$

Note that at $\alpha = m(l + 1), \ l = 0, 1, 2, \ldots$ the equivalent $N-$function for $N(\alpha, m; u)$ may be constructed by the formula (see [9], [10], p. 13-14)

$$N(\alpha, m; u) \sim \exp (|u|^m) - \sum_{k=0}^l |u|^{mk}/k!.$$  

In the case $m = \infty$ the space $B(\alpha, \infty)$ may be defined as a projective limit of the spaces $B(\alpha, m)$ at $m \to \infty$. But $B(\alpha, \infty)$ is isomorphic to the space of all bounded (mod $\mu$) and integrable with power $\alpha$ functions: $\|f\|B(\alpha, \infty) \leq C_1(\alpha) [\|f\| + \text{vrai\text{max}}_{x \in R} |f(x)|] \leq C_2(\alpha)\|f\|B(\alpha, \infty)$.

**2. Main results.** We consider at first the differential equation (1), or, by notation, $y = Q[g]$, if obviously the solution $y$ there exists and is unique. We suppose $q(0) = 0$ and

$$0 < m = m(q) \overset{\text{def}}{=} \inf_{x \neq z} |q(x) - q(z)|/|x - z| \leq$$
sup |q(x) - q(z)|/|x - z| \equiv M(q) = M < \infty.

Recall also here the definition of Orlicz - Sobolev norms and correspondent Orlicz - Sobolev spaces: $W_k(L(N))$, $k = 1, 2, \ldots$ consists on all the measurable functions $y = y(x)$ with finite norm

\[ ||y||_{W_k(L(N))} = \sum_{l=1}^{\infty} ||d^l y/dx^l||_{L(N)} + ||y||_{L(N)}. \]

**Theorem 1.** Suppose that $N(\cdot) \in \Delta$ or $N(\cdot) = B(\alpha; \varphi)$ for some $\alpha \geq 1$, $\varphi \in \Phi$. Then the problem (1) is Lipschitz correct solvable on the space $L(N)$ into the space $W_1(L(N))$:

\[ ||Q[g]||_{W_1(L[N])} \leq C ||g||_{L[N]}, \quad C = C(m, M, N(\cdot)) \in (0, \infty); \]

\[ ||Q[g_1] - Q[g_2]||_{W_1(L[N])} \leq C(m, M, N(\cdot)) ||g_1 - g_2||_{L[N]}. \]

Note that the assertion (5) is some generalization of main result of paper [2]. It is proved at the same place that the conditions (4) are necessary and sufficient for (5) even for the spaces $L_p$.

We consider now the problem 2. Denote the solution of equation (2) by $y = S[g]$ again in the Orlicz space $L[N](R) = \{g\}$. (We will prove further the existence and uniqueness of $S[g]$.)

Let us introduce for the finite measurable function $v = v(x, y)$, $x, y \in R$ of two variables with condition $v(x, 0) = 0$ the following norm:

\[ |||v||| = \sup_x \sup_{y \neq z} |v(x, y) - v(x, z)|/|y - z|. \]

We denote by $V$ the Banach space of all the (measurable) functions $v$ such that $v(x, 0) = 0$ with finite norm $|||v|||$ : $V = \{v : |||v||| < \infty\}$. We suppose that the measurable function $q(x, y)$ has a properties: 1) $q(x, y) \geq 0$; 2) $\exists v \in V, \exists q_0(\cdot) \in L^1$, $q_0 \geq 0$, $v(x, 0) = 0$,

\[ q(x, y) = q_0(x)y + v(x, y). \]

3) We define also $d = d(x)$ as a (unique) non-negative solution of equation

\[ \int_0^{\sqrt{2}} dt \int_{x-t}^{x+t} q_0(\xi)d\xi = 2, \]

and denote

\[ A = \inf_{x \in R} d(x), \quad B = \sup_{x \in R} d(x), \]

\[ \nu(B) = 8 \exp(-1/e)B \max(B, 1). \]
Theorem 2. Suppose $B < \infty$ and

\[ |||v||| < 1/\nu(B). \]

Assume again that $N(\cdot) \in \Delta$ or $N(\cdot) = B(\alpha; \varphi)$ for some $\alpha \geq 1$, $\varphi \in \Phi$. We assert that for all functions $N(\cdot)$ and potentials $q(x, y)$ which satisfies our conditions the problem (2) is well-posed on the space $L_\beta = L_\beta(R)$, where $\beta \geq \alpha$, into the space $L[N]$:

\[ ||S[g]||L[N] \leq C_4(q(\cdot)) ||g||L_\beta, \]

(9)

\[ ||S[g_1] - S[g_2]||L[N] \leq C_4 ||g_1 - g_2||L_\beta. \]

Suppose in addition that

\[ \sup_x q_0(x) = C_5 < \infty. \]

Then the problem 2 is correct solvable in the space $L_\beta$ into the space $W_2(L(N))$:

\[ ||S[g]||W_2(L(N)) \leq C_4(q(\cdot)) ||g||L_\beta, \]

\[ ||S[g_1] - S[g_2]||W_2(L(N)) \leq C_4 ||g_1 - g_2||L_\beta. \]

Theorem 3. Suppose that in the problem (2) $v(x, y) = 0$ (linear equation) and

\[ A \overset{\text{def}}{=} \inf_x d(x) > 0. \]

Then for all $\alpha > 1$, $\beta > \alpha$, $\delta \in (0, \beta - \alpha)$ the problem (2) is ill-posed in the space $L_{\beta - \delta}$ into the space $L_\beta$. Namely, $\forall \beta > \alpha$, $\delta \in (0, \beta - \alpha)$ $\exists g(\cdot) \in L_{\beta - \delta}$ $\Rightarrow S[g] \notin L_\beta$.

Theorem 2 is some generalization of main result of paper [3]. It is obtained in [4] in linear case $v = 0$ the criterion of correct solvability (2) in the spaces $L_p \rightarrow L_p$, $p \geq 1$.

Remark 1. We can notice the difference between equations of first order ODE (1) and second order ODE (2). In first case the right - side of equation must belong to the Orlicz space $L(N)$, in the second case $g(\cdot)$ must belong only to the $L_\beta(R)$ space.

3. Auxiliary result. Denote $\psi(p) = \psi(p; \varphi) = \exp (h^*(p)/p)$. Let us introduce a new Banach space $G(\alpha; \varphi)$, $\alpha \geq 1$, $\varphi \in \Phi$, as a set of all measurable functions $f : R \rightarrow R$ with finite norm

\[ ||f||G(\alpha, \varphi) \overset{\text{def}}{=} \sup_{p \geq \alpha} |f|_p / \psi(p) < \infty. \]
Theorem 4. We propose that the norms \( \| \cdot \|_B(\alpha; \varphi) \) and \( \| \cdot \|_G(\alpha; \varphi) \) are equivalent: \( \exists C_3, C_4 = C_3, C_4(\alpha, \varphi) \in (0, \infty) \Rightarrow \)

\[
C_3 \| f \|_G(\alpha; \varphi) \leq \| f \|_B(\alpha; \varphi) \leq C_4 \| f \|_G(\alpha; \varphi).
\]

Proof of theorem 4. Assume at first that \( \| f \|_B(\alpha; \varphi) < \infty \). Without loss of generality we can suppose

\[
I(\exp_\alpha |f|) = 1.
\]

Let us introduce the function

\[
\gamma(p) = \gamma_\alpha(p) = \sup_{z>0} z^p / \exp_\alpha \varphi(z).
\]

We have for the values \( p \geq \alpha \) and some \( C_1 = C_1(\alpha, \varphi(\cdot)) \in (0, \infty) \):

\[
\gamma(p) \leq \max \left[ \max_{z \in [0, C_1]} C^p z^{p-\alpha}, \sup_{z \geq C_1} z^p \exp(-\varphi(z)) \right] \leq \max \left[ C_2^{p-\alpha}, \exp(\sup_{z \geq C_1} (p \log z - \varphi(z))) \right] = \max \left[ C_3^{p-\alpha}, \exp(\sup_{v \geq C_3} (pv - h(y))) \right] = \max \left[ C_4^{p-\alpha}, \exp h^*(p) \right] \leq C_4^p(\alpha) \exp h^*(p).
\]

Following, for the values \( p \geq \alpha \) we have: \( z \geq 0 \Rightarrow \)

\[
z^p \leq \gamma(p) \exp_\alpha \varphi(z) \leq C_4^p(\alpha) \psi^p(p) \exp_\alpha \varphi(z).
\]

Therefore

\[
|f|^p \leq C^p(\alpha) \psi^p(p) \exp_\alpha (h^*(|f|)), \quad |f|_p \leq C(\alpha; \varphi) \psi(p),
\]

\[
\| f \|_G(\alpha, \varphi) \leq C(\alpha; \varphi(\cdot)) < \infty.
\]

Inverse, assume that

\[
|f|_p^p \leq \exp (h^*(p)), \quad p \geq \alpha.
\]

We have by virtue of Chebyshev inequality for all the values \( w \geq C_5 \):

\[
T(|f|, w) \overset{\text{def}}{=} \mu \{ x : |f(x)| > w \} \leq \exp (h^*(p) - p \log w),
\]

After the minimization of the right - side over \( p \), \( p \geq C \), we receive for \( w \geq C_2 \):

\[
T(|f|, w) \leq \exp (-h^{**}(\log w)) = \exp (-h(\log w))
\]
on the basis of theorem of Fenchel - Moraux. We conclude for the value of
\( \varepsilon = \exp(-2) \), choosing \( W(k) = \exp(k) \) and denoting
\[
U(k) = U(|f|, k) = \{ x : W(k) \leq |f(x)| < W(k + 1) \} :
\]
\[
I(\exp(\varphi_a(\varepsilon|f|))) \leq C + \sum_{k=3}^\infty \int_{U(k)} \exp(\varphi(\varepsilon|f|)) \, dx \leq
\]
\[
C + \sum_{k=3}^\infty \exp(\varepsilon(k) - h(k + 1)) < \infty
\]
by virtue of condition 3. This completes the proof of theorem 4.

For example, let \( N(u) = N_{\alpha,m}(u) = \exp_a(\varphi_m(u)) = \exp_a|u|^m \). It follows from
theorem 4 that
\[
||f||_{L(N_{\alpha,m})} < \infty \iff \sup_{p \geq \alpha} |f|_p \, p^{-1/m} < \infty,
\]
or equally
\[
\exists \varepsilon > 0, \ I(\exp_a(\varepsilon|f|)) < \infty \iff \sup_{p \geq \alpha} |f|_p \, p^{-1/m} < \infty.
\]

**Notice.** Let us introduce the weight Lorentz norm:
\[
||f||_{\mathbf{B}}(\alpha; \varphi) = \sup_{p \geq \alpha} ||f||_{p,b} / \psi(p),
\]
where \( ||f||_{p,b} \) is the Lorentz norm (more exactly, seminorm):
\[
||f||_{p,b} = \left[ \int_0^\infty T_{p/b}(|f|, x) \, dx^b \right]^{1/b},
\]
\( p \in [1, \infty), \ b \in [1, \infty] \), where if \( b = \infty \) then
\[
||f||_{p,\infty} = \sup_{x \geq 0} \left( x \, T_{1/p}(|f|, x) \right).
\]

It is easy to prove using the embedding theorem for the Lorentz spaces as well as by proving of theorem 4 that all the norms
\[
|| \cdot ||_{B(\alpha; \varphi)}, \ || \cdot ||_{G(\alpha; \varphi)}, \ || \cdot ||_{\mathbf{B}}(\alpha; \varphi)
\]
are equivalent with constants does not depending on \( b \).

Note than if we consider the Orlicz space \( L(N) \), \( N \in EOF \) on the arbitrary measurable space \( (\Omega, F, \mu) \) with finite measure \( \mu \), the result of theorem 4 is known (see [12], p.341).
Proof of theorem 1. Let us consider at first the case \( N \in \Delta \). Let \( g \in L(N), \ N \in \Delta \). We will use the main result of paper [2]:

$$\exists C(m, M) \in (0, \infty), \ \forall p \geq 1 \ |Q[g]|_p \leq C(m, M) \ |g|_p,$$

and

$$|Q[g_1] - Q[g_2]|_p \leq C(m, M) \ |g_1 - g_2|_p,$$

where we denote for this problem \( y = Q[g] \). It follows from (12) that the operator \( Q \) is correctly defined and bounded as operator \( L_p \rightarrow L_p, \ p \geq 1 \). The first proposition of theorem 1 follows from Ryan’s theorem ([1], p. 193).

Let now \( N(\cdot) = B(\alpha; \varphi) \) for some \( \alpha \geq 1, \ \varphi \in \Phi \) and let \( g(\cdot) \in B(\alpha; \varphi) \). By virtue of theorem 4

$$|Q[g]|_p = |y|_p \leq C_1 \ C(m, M) \ ||g||B(\alpha; \varphi) \ \psi(p).$$

Again from theorem 4 follows

$$||Q[g]|B(\alpha, \varphi) = ||y||B(\alpha; \varphi) \leq C_1 \ C_2 \ C(m, M) \ ||g||B(\alpha; \varphi),$$

and analogously

$$||Q[g_1] - Q[g_2]|B(\alpha; \varphi) \leq C_3(\alpha, \varphi, m, M) \ ||g_1 - g_2||B(\alpha; \varphi).$$

This completes the proof of theorem 1.

Proof of theorem 2. Part 1. We consider here the linear case, i.e. \( v(x, y) = 0 \). We can denote by \( y_0 = S[g] \) the solution of linear equation

$$d^2y_0(x)/dx^2 - q_0(x) \ y_0(x) = g(x), \ \lim_{x \rightarrow \infty} y_0(x) = 0,$$

as long as \( y_0 \) there exists and is unique ([4], [5]). The first part of this theorem is proved analogously to the proof of theorem 1, since (see [4])

$$|S[g]|_p \leq C \ |g|_p, \ |S[g_1] - S[g_2]|_p \leq C \ |g_1 - g_2|_p.$$

Part 2. Further, we will denote by \( \Gamma(t, x) \) the Green’s function for the linear equation (2):

$$y_0(x) = \int_R \Gamma(x, t) \ g(t) \ dt.$$

We will use the fine result of paper [5]: the function \( \Gamma(\cdot, \cdot) \) there exists, is unique, and

$$\Gamma(x, t) = \sqrt{\rho(x) \rho(t)} \ exp \left( -0.5 \left| \int_x^t d\xi/\rho(\xi) \right| \right),$$

where

$$2^{-3/2}d(x) \leq \rho(x) \leq 2^{-1/2}d(x).$$
It follows from (13) and condition (8)

\[ \Gamma(x, t) \leq B \sqrt{2} \exp \left(-2^{-3/2}|t - x|/B\right). \]

Therefore

\[ |y_0(x)| \leq B \sqrt{2} \int \exp \left(-2^{-3/2}|t - x|/B\right) g(t) \, dt = s \ast g(x), \quad s(x) = B \sqrt{2} \exp \left(-2^{-3/2}|x|/B\right), \]

and the symbol \( f \ast g \) denotes the usually convolution for the function defined on \( R \). Using the Young inequality for convolution we obtain for all the values \( r \geq \beta \):

\[ |S[g]|_r \leq |g|_\beta \cdot \sup_{p \in [1, \beta/(\beta - 1))] |s|_p, \]

where at \( \beta = 1 \Rightarrow \beta/(\beta - 1) = +\infty \).

It is easy to calculate that

\[ \sup_{p \geq 1} |s|_p \leq 8 \exp(-1/e)B \max(B, 1) = \nu(B) < \infty, \]

following,

\[ \sup_{p \geq \alpha} |S[g]|_p \leq \nu(B) |g|_\beta, \]

and \( \forall \varphi \in \Phi, \alpha \geq 1, \beta \geq \alpha \)

\[ (15) \quad ||S[g]||G(\alpha; \varphi) \leq C(\alpha; \varphi) \nu(B) |g|_\beta. \]

**Part 3.** Let us consider in this section the non-linear case \( v \neq 0 \). We can rewrite the equation (2) on the form

\[ y = W[y], \quad W[y](x) = S[g](x) + \int_R \Gamma(x, z) v(z, y(z)) \, dz. \]

Since \( |v(x, y)| \leq C|y| \), it is evident that the (non-linear) operator \( W[\cdot] \) has the property: \( W : L_\beta \to L_\beta \), i.e.

\[ ||W[\cdot]||(L_\beta \to L_\beta) < \infty \]

and we have by virtue of inequality (14):

\[ W[y_1](x) - W[y_2](x) = \int_R \Gamma(x, z) [v(z, y_1(z)) - v(z, y_2(z))] \, dz; \]

\[ |W[y_1](x) - W[y_2](x)| \leq ||v|| \int_R s(x - z) |y_1(z) - y_2(z)| \, dz; \]

\[ |W[y_1] - W[y_2]|_\beta \leq ||v|| \nu(B) |y_1 - y_2|_\beta. \]
Therefore, the operator $W[\cdot]$ satisfies the contraction property in the space $L_\beta$. Following, there exists the fixed point of $W[\cdot]$ in the space $L_\beta$.

The statement of part 3 follows from part 1 and (16), as long as all the terms of the right side (16) belong to the space $B(\alpha; \varphi)$; $\alpha \geq 1, \varphi \in \Phi$.

Part 4. Suppose in addition $\sup_{x \in R} q_0(x) = C_6 < \infty$, $v(x, y) = 0$. We can rewrite in this case our equation (2) on the form

$$\frac{d^2 y}{dx^2} = g(x) + q_0(x)y(x),$$

in the spaces $L_\beta \rightarrow L_\beta$, i.e. we assume that $g \in L_\beta$, and following, $y \in L_\beta$. Therefore $d^2 y/dx^2 \in L_\beta$. The last assertion of theorem 2 follows from the classical Kolmogorov’s inequality

$$||dy/dx||_p^2 \leq 16||y||_p ||d^2 y/dx^2||_p;$$

see, for example, [11], p. 49. This completes the proof of theorem 3.

4. Proof of theorem 3. Let in the equation (2) $\alpha > 1, v(x, y) = 0$. It follows from (13) that

(16) \hspace{1cm} \Gamma(x, t) \geq 2^{-3/2}A \exp \left( -2^{-5/2}|t - x|/A \right).

Let $g(x) = g_\beta(x) = (x \log^2 x)^{1/\beta}$ if $x \geq 2$ and $g(x) = 0$, $x < 2$. Then $g \in L_\beta$ and $g \notin L_{\beta - \delta}$ $\forall \delta \in (0, \beta - \alpha)$. We have for the values $x \geq 3$:

$$W[g] \geq C_7 \int_2^\infty \exp(-|x - t|) (t \log^2 t)^{1/\beta} \, dt =$$

$$= C_7 x^{1-1/\beta} \int_{2/x}^\infty \exp(-x|z - 1|) (z \log^2(xz))^{-1/\beta} \, dz.$$

The exact asymptotic of the last integral at $x \to \infty$ may be calculated by means of the Laplace’s method; the critical point $z = 1$. We obtain:

$$W[g](x) \sim C_8 (x \log^2 x)^{-1/\beta}, \quad C_8 = C_8(\beta) \in (0, \infty).$$

Therefore, $W[g] \in L_\beta$ but $\forall \delta \in (0, \beta - \alpha) \Rightarrow W[g] \notin L_{\beta - \delta}$.

Finally, let us consider in addition to the problem (1) and (2) the Pseudodifferential linear operator $P$ in the space $R^n$ with the symbol $P(\xi)$. Assume that $\forall k = 1, 2, \ldots, [n/2] + 1$, ($[x]$ denotes here the integer part of $x$)

$$\sup_{z > 0} \max_{|\xi| \leq k} \int_{|\xi| < |\xi| \leq 2z} \frac{|P(\xi)(\xi)|^2}{|\xi|^{n+k}} \, d\xi < \infty;$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$, $|\xi| = \sum_i |\xi_i|$, $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n)$, $\zeta_j = 0, 1, \ldots$;

$$P^{(\zeta)}(\xi) = \frac{\partial^{(\zeta)} P(\xi)}{\prod_{i=1}^n \partial^{\zeta_i} \xi_i}.$$ 

It is known (see, for example, [13], p. 262 - 270)) that

(17) \hspace{1cm} |Pf|_{L_p(R^n)} \leq C(n, P) \, p \, |f|_{L_p(R^n)}, \quad p > 1.
Suppose $f \in B(\alpha, m)$ for some $\alpha > 1$, $m > 0$. From theorem 4 follows that $Pf \in B(\alpha, m/(m + 1))$, hence $P : B(\alpha, m) \to B(\alpha, m/(m + 1))$ and

$$||P||(B(\alpha, m) \to B(\alpha, m/(m + 1))) < \infty.$$ 

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Concluding remark. Our results (without proof) was announced in [14].

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