Brane Lorentz Symmetry from Lorentz Breaking in the Bulk

O. Bertolami¹,² and C. Carvalho¹,²
¹ Departamento de Física, Instituto Superior Técnico, Avenida Rovisco Pais 1, 1049-001 Lisboa, Portugal
² Centro de Física dos Plasmas, Instituto Superior Técnico, Avenida Rovisco Pais 1, 1049-001 Lisboa, Portugal
E-mail: orfeu@cosmos.ist.utl.pt, ccarvalho@ist.edu

Abstract. We propose the mechanism of spontaneous symmetry breaking of a bulk vector field as a way to generate the selection of bulk dimensions invisible to the standard model confined to the brane. By assigning a non-vanishing vacuum value to the vector field, a direction is singled out in the bulk vacuum, thus breaking the bulk Lorentz symmetry. We present the condition for induced Lorentz symmetry on the brane, as phenomenologically required.

1. Introduction

The prediction of extra dimensions, which the brane-confined standard model is oblivious to, suggests that early enough in the history of the Universe a dimensional selection would have taken place, implying the breaking of the Lorentz symmetry at the level of the higher dimensional bulk spacetime and the possible generation of a different geometry along the directions orthogonal to the brane.

The breaking of the Lorentz symmetry can be realized by spontaneous breaking of a symmetry in the presence of a vector field $B_\mu$ living in the bulk and subject to a potential $V(B^2)$. The acquisition by the bulk vector field $B_\mu$ of a non-vanishing vacuum value and the consequent assignment to the bulk vacuum of an intrinsic direction determined by $\langle B_\mu \rangle$ induce the breaking of rotation invariance and thus of Lorentz symmetry in the bulk. The brane, regarded as the locus of the observable universe embedded in the higher dimensional bulk spacetime, would be expected to share a subgroup of the symmetries of the bulk preserved in a manner akin to the Goldstone mechanism. The overly tested Lorentz symmetry of the observable universe imposes that the brane must stand as a vacuum solution where the Lorentz invariance would be a residual symmetry of the spontaneously broken symmetry [1].

The possibility of violation of Lorentz invariance has nevertheless been widely discussed in the recent literature (see e.g. [2]). Spontaneous breaking of Lorentz symmetry may arise in the context of string/M-theory due to the existence of non-trivial solutions in string field theory [3], in loop quantum gravity [4] and in noncommutative field theories [5], or via the spacetime variation of fundamental couplings [6]. Lorentz violation modifications of the dispersion relations via five dimensional operators for fermions have also been considered and constrained [7].

¹ Based on a talk presented by O. Bertolami at D.I.C.E. 2006, Piombino, Italy.
Consequently, this putative breaking will have astrophysical [8, 9] implications, and in what concerns ultra-high energy cosmic rays, one can establish that Lorentz symmetry holds up to about $2 \times 10^{-25}$ [9]. The way to relate the breaking of Lorentz symmetry to gravity has been considered in Ref. [10], and solutions as well as implications have been discussed in Ref. [11]. In Ref. [12], a connection between the cosmological constant and the violation of Lorentz invariance has been conjectured. For a general discussion on the relation between spacetime symmetries and higher dimensions, the reader is referred to Ref. [13].

In this contribution we report on a recent study on how spontaneous Lorentz violation in the bulk has repercussions on the brane and how it can be constrained. We consider a bulk vector field coupled non-minimally to the graviton which, upon acquiring a non-vanishing expectation value in the vacuum, introduces spacetime anisotropies in the gravitational field equations through the coupling with the graviton [10] [11]. First we derive the bulk field equations and project them parallel and orthogonal to the brane. We then establish how to derive brane quantities from bulk quantities by adopting Fermi normal coordinates with respect to the directions on the brane and continuing into the bulk using the Gauss normal prescription. Finally, we obtain the conditions for the Lorentz invariance to be a symmetry on the brane.

2. Notation Definition

We begin by parameterizing the world-sheet in terms of coordinates $x^A = (t_b, x_b)$ intrinsic to the brane. Using the chain rule, we may express the brane tangent and normal unit vectors in terms of the bulk basis as follows:

$$\hat{e}_A = \frac{\partial}{\partial x^A} = X^\mu_A \frac{\partial}{\partial x^\mu} = X^\mu_A \hat{e}_\mu,$$
$$\hat{e}_N = \frac{\partial}{\partial n} = N^\mu \frac{\partial}{\partial x^\mu} = N^\mu \hat{e}_\mu,$$  
(1)

with

$$g_{\mu\nu}N^\mu N^\nu = 1, \quad g_{\mu\nu}X^\nu_A = 0,$$  
(2)

where $g$ is the bulk metric

$$g = g_{\mu\nu} \hat{e}_\mu \otimes \hat{e}_\nu = g_{AB} \hat{e}_A \otimes \hat{e}_B + g_{AN} \hat{e}_A \otimes \hat{e}_N + g_{NB} \hat{e}_N \otimes \hat{e}_B + g_{NN} \hat{e}_N \otimes \hat{e}_N.$$  
(3)

To obtain the metric induced on the brane we expand the bulk basis vectors in terms of the coordinates intrinsic to the brane and keep the doubly brane tangent components only. It follows that

$$g_{AB} = X^\mu_A X^\nu_B g_{\mu\nu}$$  
(4)

is the $(3 + 1)$-dimensional metric induced on the brane by the $(4 + 1)$-dimensional bulk metric $g_{\mu\nu}$. The induced metric with upper indices is defined by the relation

$$g_{AB} g^{BC} = \delta^C_A.$$  
(5)

It follows that we can write any bulk tensor field as a linear combination of mutually orthogonal vectors on the brane, $\hat{e}_A$, and a vector normal to the brane, $\hat{e}_N$. We illustrate the example of a vector $B_\mu$ and a tensor $T_{\mu\nu}$ bulk fields as follows

$$B = B_A \hat{e}_A + B_N \hat{e}_N,$$  
(6)
$$T = T_{AB} \hat{e}_A \otimes \hat{e}_B + T_{AN} \hat{e}_A \otimes \hat{e}_N + T_{NB} \hat{e}_N \otimes \hat{e}_B + T_{NN} \hat{e}_N \otimes \hat{e}_N.$$  
(7)
Derivative operators decompose similarly. We write the derivative operator $\nabla$ as

$$\nabla = (X^\mu_A + N^\mu) \nabla_\mu = \nabla_A + \nabla_N. \quad (8)$$

Fixing a point on the boundary, we introduce coordinates for the neighbourhood choosing them to be Fermi normal. All the Christoffel symbols of the metric on the boundary are thus set to zero, although the partial derivatives do not in general vanish. The non-vanishing connection coefficients are

$$\begin{align*}
\nabla_A \hat{e}_B &= -K_{AB} \hat{e}_N, \\
\nabla_A \hat{e}_N &= +K_{AB} \hat{e}_B, \\
\nabla_N \hat{e}_A &= +K_{AB} \hat{e}_B, \\
\nabla_N \hat{e}_N &= 0,
\end{align*} \quad (9)$$

as determined by the Gaussian normal prescription for the continuation of the coordinates off the boundary.

For the derivative operator $\nabla \nabla$ we find that

$$\begin{align*}
\nabla \nabla &= g^{\mu\nu} \nabla_\mu \nabla_\nu \\
&= g^{AB} [(X_A^\mu \nabla_\mu)(X_B^\nu \nabla_\nu) - X_A^\mu \nabla_\mu X_B^\nu \nabla_\nu] + g^{NN} [(N^\mu \nabla_\mu)(N^\nu \nabla_\nu) - N^\mu \nabla_\mu N^\nu \nabla_\nu] \\
&= g^{AB} [\nabla_A \nabla_B + K_{AB} \nabla_N] + \nabla_N \nabla_N. \quad (10)
\end{align*}$$

We can now decompose the Riemann tensor, $R_{\mu\nu\rho\sigma}$, along the tangent and the normal directions to the surface of the brane as follows

$$\begin{align*}
R_{ABCD} &= R^{(\text{ind})}_{ABCD} + K_{AC}K_{BD} - K_{AC}K_{BD}, \\
R_{NBCD} &= K_{BC:D} - K_{BD:C}, \\
R_{NBND} &= K_{BC}K_{DC} - K_{BC:N}, \quad (11)\end{align*}$$

from which we find the decomposition of the Einstein tensor, $G_{\mu\nu}$, obtaining the Gauss-Codacci relations

$$\begin{align*}
G_{AB} &= G^{(\text{ind})}_{AB} + 2K_{AC}K_{CB} - K_{AB}K - K_{AB,N} - \frac{1}{2}g_{AB} \left(3K_{CD}K_{DC} - K^2 - 2K_N\right), \quad (14) \\
G_{AN} &= K_{AB:B} - K_{;A}, \quad (15) \\
G_{NN} &= \frac{1}{2} \left(-R^{(\text{ind})} - K_{CD}K_{DC} + K^2\right). \quad (16)
\end{align*}$$

3. Bulk Vector Field Coupled to Gravity

In order to study the gravitational effects of the breaking of Lorentz symmetry in a braneworld scenario, we consider a bulk vector field $B$ with a non-minimal coupling to the graviton in a five-dimensional anti-de Sitter space. The Lagrangian density consists of the Hilbert term, the cosmological constant term, the kinetic and potential terms for $B$ and the $B$–graviton interaction term, as follows

$$\mathcal{L} = \frac{1}{\kappa_{(5)}^2} R - 2\Lambda + \xi B^\mu B^\nu R_{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} - V(B^\mu B_\mu \pm b^2), \quad (17)$$

where $B_{\mu\nu} = \nabla_\mu B_\nu - \nabla_\nu B_\mu$ is the tensor field associated with $B_\mu$ and $V$ is the potential which induces the spontaneous global symmetry breaking when the $B$ field is driven to the minimum at $B^\mu B_\mu \pm b^2 = 0$, $b^2$ being a real positive constant. Here, $\kappa_{(5)}^2 = 8\pi G_N = M_{Pl}^3$, $M_{Pl}$ is the five-dimensional Planck mass and $\xi$ is a dimensionless coupling constant that we have inserted.
to track the effect of the interaction\(^2\). In the cosmological constant term \(\Lambda = \Lambda_\text{B} + \Lambda_\text{f}\) we have included both the bulk vacuum value \(\Lambda_\text{B}\) and that of the brane \(\Lambda_\text{f}\), described by a brane tension \(\sigma\) localized on the locus of the brane, \(\Lambda_\text{f} = \sigma \delta(N)\).

By varying the action with respect to the metric, we obtain the Einstein equation

\[
\frac{1}{\kappa(5)} G_{\mu \nu} + \Lambda g_{\mu \nu} - \xi L_{\mu \nu} - \xi \Sigma_{\mu \nu} = \frac{1}{2} T_{\mu \nu},
\]

(18)

where

\[
L_{\mu \nu} = \frac{1}{2} g_{\mu \nu} B^\rho B^\sigma R_{\rho \sigma} - (B_\mu B^\rho R_{\rho \mu} + R_{\mu \rho} B^\rho B_\nu),
\]

(19)

\[
\Sigma_{\mu \nu} = \frac{1}{2} [\nabla_\mu \nabla_\rho(B_\nu B^\rho) + \nabla_\nu \nabla_\rho(B_\mu B^\rho) - \nabla^2(B_\mu B_\nu) - g_{\mu \nu} \nabla_\rho \nabla_\sigma(B^\rho B^\sigma)]
\]

(20)

are the contributions from the interaction term and

\[
T_{\mu \nu} = B_\mu B_\nu + 4V' B_\mu B_\nu + g_{\mu \nu} \left[\frac{1}{4} B_{\rho \sigma} B^{\rho \sigma} - V\right]
\]

(21)

is the contribution from the vector field for the stress-energy tensor. For the equation of motion for the vector field \(B\), obtained by varying the action with respect to \(B_\mu\), we find that

\[
\nabla^\nu (\nabla_\nu B_\mu - \nabla_\mu B_\nu) - 2V' B_\mu + 2\xi B^{\nu} R_{\mu \nu} = 0,
\]

(22)

where \(V' = dV/dB^2\).

We now proceed to project the equations parallel and orthogonal to the surface of the brane. Following the prescription used in the derivation of the Gauss-Codacci relations, we derive the components of the stress-energy tensor and of the interaction terms. For the stress-energy tensor we find that

\[
T_{AB} = B_{AC} B_{B}^C + B_{AN} B_{B}^N + 4V' B_A B_B + g_{AB} \left[-\frac{1}{4} (B_{CD} B^{CD} + 2B_{CN} B^{CN}) - V\right],
\]

\[
T_{AN} = B_{AC} B_{B}^C + 4V' B_A B_N,
\]

\[
T_{NN} = B_{NC} B_{B}^C + 4V' B_N B_B + g_{NN} \left[-\frac{1}{4} (B_{CD} B^{CD} + 2B_{CN} B^{CN}) - V\right],
\]

(23)

and for the interaction source terms that

\[
L_{AB} = \frac{1}{2} g_{AB} \left[B^C B^D \left( R^{(\text{ind})}_{CD} + 2K_{CE} K_{ED} - K_{CD} K - K_{DN}\right) 
+ 2B^C B^N (K_{EC;E} - K_{C}) + B^N B^N (K_{CD} K_{DC} - K_{N})\right] 
- B_A \left[ B^C \left( R^{(\text{ind})}_{CB} + 2K_{CE} K_{EB} - K_{CB} K - K_{CB;N}\right) + B^N (K_{EB;E} - K_B)\right] 
- \left[ \left( R^{(\text{ind})}_{AC} + 2K_{AE} K_{EC} - K_{AC} K - K_{AC;N}\right) B^C + (K_{EA;E} - K_A) B^N\right] B_B,
\]

\[
L_{AN} = -B_A \left[ B^C \left( K_{EC;E} - K_{C}\right) + B^N (K_{EH} K_{EH} - K_{N})\right] 
- \left[ \left( R^{(\text{ind})}_{AC} + 2K_{AE} K_{EC} - K_{AC} K - K_{AC;N}\right) B^C + (K_{EA;E} - K_A) B^N\right] B_N,
\]

\[
L_{NN} = -2B_N \left[ B^C \left( K_{EC;E} - K_{C}\right) + B^N (K_{EH} K_{EH} - K_{N})\right],
\]

(24)

\(^2\) In Ref. [\textsuperscript{1}] we used \(\xi = 1\), which enabled a simplification of the results without loss in generality of the purpose of the paper. Here, however, we shall keep \(\xi\) free.
The equation of motion for the vector field and parallel out by the substitution of the matching conditions. Considering the brane as a vector field in terms of quantities measured on the brane. The induced equations on the brane δshell of thickness 2K relates the induced geometry with the localization of the induced stress-energy in generate singular distributions on the brane. Integration of these terms in the coordinate normal (Σ
abla
abla) projected components after the singular terms across the brane are subtracted
\[ (\nabla A)K_{E,D} - K_{K,C}B_{C}B_{D} - (2K_{CD}K_{DC} - K^{2})B_{N}B_{N} \].

The equation of motion for the vector field B decomposes similarly as follows, respectively parallel
\[ \nabla C (\nabla C B_{A} - \nabla A B_{C}) + \nabla N (\nabla N B_{A} - \nabla A B_{N}) + 2K_{AC} (\nabla C B_{N} - \nabla N B_{C}) + K (\nabla N B_{A} - \nabla A B_{N}) - 2V' B_{A} + 2\xi [B_{C} (R_{AC}^{(nd)} + 2K_{AD}K_{DC} - K_{AC}K - K_{AC,N}) + B_{N} (K_{AC,C} - K_{A})] = 0, \]

and orthogonal to the brane
\[ \nabla C (\nabla C B_{N} - \nabla N B_{C}) - 2V' B_{N} + 2\xi [B_{C} (K_{CD,D} - K_{C}) + B_{N} (K_{CD}K_{CD} - K_{C,N})] = 0. \]
equations of motion for $\mathbf{B}$ which, due to the coupling of $\mathbf{B}$ to gravity, will also be used as complementary conditions for the dynamics of the metric on the brane.

Combining the Gauss-Codacci relations with the projections of the stress-energy tensor and the interaction source terms, we integrate the $(AB)$ component of the Einstein equation in the coordinate normal to the brane to obtain the matching conditions for the extrinsic curvature across the brane, i.e. the Israel matching conditions. We find that

\[
\frac{1}{\kappa^{(5)}} [-K_{AB} + g_{AB}\kappa] = -g_{AB}\sigma
\]

\[
+ \frac{\xi}{2} \left[ \nabla_A(B_BN) + \nabla_B(B_AB_N) - \nabla_N(B_AB_B) \\
+ 4(B_A B_C K_{CB} + K_{AC} B_C B_B) - 2K_{AB} B_N B_N \\
+ g_{AB} (-2\nabla_C(B_C B_N) - \nabla_N(B_N B_N) + K_{CD} B_C B_D - K_B N B_N) \right].
\]

These provide boundary conditions for ten of the fifteen degrees of freedom. Five additional boundary conditions are provided by the junction conditions for the $(AN)$ and $(NN)$ components of the projection of the Einstein equations. From inspection of the $(AN)$ component, we note that

\[
G_{AN} = K_{AB} B_N - K_{iA} = -\nabla_B \left( \int_{-\delta}^{+\delta} dN G_{AB} \right) = -\kappa^{(5)}_N \nabla_B T_{AB} = 0
\]

which vanishes due to conservation of the induced stress-energy tensor $T_{AB}$ on the brane. From integration of the $(NN)$ component in the normal direction to the brane, we find the following junction condition

\[
\nabla_C (B_C B_N) + 3K_B N_B - K_{CD} B_C B_D = 0,
\]

which we substitute back in, obtaining

\[
\frac{1}{\kappa^{(5)}} \left( \frac{1}{2} (\Delta^{(i)N} - K_{CD} K_{CD} + \kappa^2) + \Lambda^{(5)} \right)
= \frac{1}{2} \left[ -\frac{1}{4} B_{CD} B_{CD} + 4V' B_N B_N - V \right]
+ \frac{\xi}{2} \left[ -\nabla_C \nabla_D (B_C B_D) - \nabla_C \nabla_A (B_N B_N) + \frac{12}{2\xi - 1} B_N (\nabla_C \nabla_C B_N - 2V' B_N) \\
+ 2(K_{CD} - g_{CD} K) \nabla_C (B_D B_N) + 2K_{CD} B_D (\nabla_C B_N) + K_{CD} C_D B_D B_N \\
+ \left( 7K_{CD} K_{CD} - \kappa^2 \right) B_N B_N + ((6\xi + 1)K_{CE} K_{ED} + K_{CD}) B_C B_D \right].
\]

However, the Israel matching conditions also contain terms which depend on the prescription for the continuation of $\mathbf{B}$ out of the brane and into the bulk, namely $\nabla_N B_A$ and $\nabla_N B_N$. The five additional boundary conditions required are those for the vector field $\mathbf{B}$. In Ref. [14] the boundary conditions for bulk fields were derived subject to the condition that modes emitted by the brane into the bulk do not violate the gauge defined in the bulk. Here, however, we integrate the $(A)$ and $(N)$ components of the equation of motion for $\mathbf{B}$, Eq. (26) and Eq. (27) respectively, to find the corresponding junction condition for $B_A$ and for $B_N$ across the brane. From Eq. (26) we have that

\[
\int_{-\delta}^{+\delta} dN [\nabla_N (\nabla_N B_A - \nabla_A B_N) - 2\xi K_{AC} (\nabla_N B_C) - 2\xi B_C K_{AC,N}] = 0.
\]
If $\delta$ is sufficiently small, the difference between $K_{AB,N}$ and $K_{AB,B}$ is negligibly small. It follows that, in the limit where $\delta \to 0$, we can assume that $\nabla_N \approx \partial_N$. It then follows that

$$\nabla_N B_A - \nabla_A B_N - 2\xi K_{AC} B_C = 0.$$  \hfill (33)

Similarly, from Eq. (27) we find that

$$\int_{-\delta}^{+\delta} dN \left[ -\nabla_N \nabla_C B_C - (2\xi - 1)\nabla_N (KB_N) \right] = 0,$$  \hfill (34)

which becomes

$$\nabla_C B_C + (2\xi - 1)KB_N = 0.$$  \hfill (35)

The junction conditions Eq. (33) and Eq. (35) offer the required $(4 + 1)$ boundary conditions respectively for $B_A$ and $B_N$ on the brane. Substituting the junction condition for $B_A$ back in Eq. (26) and using the result from $G_{AN} = 0$, we find for the induced equation of motion for $B_A$ on the brane that

$$\nabla_C (\nabla_C B_A - \nabla_A B_C) + 2\xi K_{AC} (\nabla_C B_N) - 2V' B_A + 2\xi B_C \left( R_{AC}^{(ind)} + 2\xi K_{AD} K_{CD} \right) = 0.$$  \hfill (36)

Similarly, substituting the junction condition for $B_N$ back in Eq. (27) we obtain

$$\nabla_C \nabla_C B_N - 2V' B_N + (2\xi - 1)K (\nabla_N B_N) + B_N K_{CD} K_{CD} = 0.$$  \hfill (37)

Thus, Eq. (33) provides the value at the boundary for $\nabla_N B_A$ whereas Eq. (37) provides that for $\nabla_N B_N$. Using the results derived above in the Israel matching conditions we find that

$$\frac{1}{\kappa^2(5)} \left[ -K_{AB} + g_{AB} K \right] = -g_{AB} \sigma$$

$$+ \frac{\xi}{2} \left[ (\nabla_A B_B)B_N + (\nabla_B B_A)B_N \right]$$

$$+ \frac{\xi B_A B_C K_{CD}}{2} + \frac{\xi K_{AC} BC B_B - K_{AB} B_B B_N}{2}$$

$$+ \xi g_{AB} \left\{ -\nabla_C (B_C B_N) + \frac{1}{2} K_{CD} B_C B_D - \frac{1}{2} K B_N B_N$$

$$+ \frac{1}{2\xi - 1} K \left( \nabla_C \nabla_C B_N - 2V' B_N + (2\xi - 1)K_{CD} K_{CD} B_N \right) \right\}.$$  \hfill (38)

which provide a second order equation for the trace of the extrinsic curvature, $K$. Finally, using Eq. (31) in the $(AB)$ Einstein equation, we find for the Einstein equation induced on the brane that

$$\frac{1}{\kappa^2(5)} \left[ G^{(ind)}_{AB} + 2K_{AC} K_{BC} - K_{AB} K + \frac{1}{2} g_{AB} \left( -R^{(ind)} + 4K_{CD} K_{CD} + 2K^2 \right) \right] + 2g_{AB} \Lambda_{(5)}$$

$$+ \frac{1}{2} \left[ -B_{AC} B_{BC} - 4V' B_A B_B + g_{AB} \left( \frac{1}{4} B_{CD} B_{CD} + 4V' B_N B_N + 2V \right) \right]$$

$$= \frac{\xi}{2} \left\{ \nabla_A \nabla_C (B_B B_C) + \nabla_B \nabla_C (B_A B_C) - \nabla_C \nabla_C (B_A B_B) \right.$$  

$$- 2K_{AC} \nabla_C (B_B B_N) - 2K_{AC} (B_B \nabla_C B_N + B_C \nabla_B B_N)$$

$$- 2K_{BC} \nabla_C (B_A B_N) - 2K_{BC} (B_A \nabla_C B_N + B_C \nabla_B B_N)$$

$$- \frac{8}{2\xi - 1} K \left[ K_{AB} (\nabla_C \nabla_C B_N - 2V' B_N + (2\xi - 1)K_{CD} K_{CD} B_N) + K B_N (\nabla_A B_B + \nabla_B B_A) \right]$$

$$- \nabla_A \nabla_B (B_B B_A) - \nabla_B \nabla_A (B_B B_A).$$
The induced equations of motion become

\[ -2B_AB_C \left( R_{CB}^{(ind)} + 2K_{CD}K_{BD} + (\xi - 1)K_{CB}K \right) \\
-2B_BB_C \left( R_{CA}^{(ind)} + 2K_{AD}K_{CD} + (\xi - 1)K_{AC}K \right) \\
+ (K_{AC;B} + K_{BC;A} - 2K_{AB;C})B_NB_N - 4K_{AB}K_NB_N \\
+ (K_{AC}B_B + K_{BC}B_A)(-5\xi K_{DC}B_D + KB_C) - (4\xi + 2)K_{AC}K_{BD}B_CB_D \]

\[ + \frac{\xi}{2}g_{AB} \left[ -2\nabla_C\nabla_D(B_CB_D) - \nabla_C\nabla(B_NB_N) + \frac{12}{2\xi - 1}B_N(\nabla_C\nabla_CB_N - 2V'B_N) \\
+ 4(K_{CD} - g_{CD}K)\nabla_D(B_CB_N) + 4K_{CD}B_D(\nabla_CB_N) \\
+ B_CB_DB_C^{(ind)} + (9K_{CD}K_{CD} - 2K^2)B_NB_N + (14\xi K_{CE}K_{DE} + KK_{CD})B_CB_D \right]. \quad (39) \]

The results obtained above show both the coupling of the bulk to the brane and the coupling of the vector field \( B \) to the geometry of the spacetime. The first is manifested in the dependence on normal components in the induced equations; the latter is manifested in the presence of terms of the form \( (R_{AB}B_CB_D) \). Terms of the form \( (K_{AB}B_N) \) illustrate both couplings, where \( B_N \) relates with \( K \) and \( B_A \) by Eq. (35). The directional dependence on the \( N \) direction is encapsulated in the extrinsic curvature. In the penultimate line we can substitute the Israel matching condition found above. However, the derivatives of the extrinsic curvature along directions parallel to the brane which appear in the ninth line are not reducible to quantities intrinsic to the brane.

4. Bulk Vector Field with a Non-vanishing Vacuum Expectation Value

In this section we particularize the formalism developed above for the case when the bulk vector field \( B \) acquires a non-vanishing vacuum expectation value by spontaneous symmetry breaking. The vacuum value generates the breaking of the Lorentz symmetry by selecting the direction orthogonal to the plane of the brane. This selection can be achieved by assigning to the parallel or the orthogonal components of the bulk vector field a non-vanishing vacuum expectation value. The minimum of the potential occupied by the vacuum value is assumed to be also a zero of the potential.

4.1. \( \langle B_A \rangle \neq 0 \) and \( \langle B_N \rangle = 0 \)

Here we consider the case when the parallel component of the vector field with respect to the brane acquires a non-vanishing expectation value, \( \langle B_A \rangle \neq 0 \), whereas the expectation value of the normal component is chosen to vanish on the brane, \( \langle B_N \rangle = 0 \). The junction conditions from the equations for \( B_A, B_N, G_{NN} \) and \( G_{AB} \) reduce respectively to

\[ \nabla_N \langle B_A \rangle - 2\xi K_{AC} \langle B_C \rangle = 0, \quad (40) \]

\[ \nabla_C \langle B_C \rangle = 0, \quad (41) \]

\[ K_{CD} \langle B_C \rangle \langle B_D \rangle = 0, \quad (42) \]

\[ \frac{1}{\kappa^{(5)}_5} [-K_{AB} + g_{AB}K] = -g_{AB} \sigma + \xi \langle B_A \rangle \langle B_C \rangle K_{CB} + \xi \langle B_B \rangle \langle B_C \rangle K_{AC}, \quad (43) \]

and the induced equations of motion become

\[ \nabla_C (\nabla_C \langle B_A \rangle - \nabla_A \langle B_C \rangle) + 2\xi \langle B_C \rangle \left( R_{AC}^{(ind)} + 2\xi K_{AD}K_{DC} \right) = 0 \quad (44) \]

for \( B_A \),

\[ \frac{1}{\kappa^{(5)}_5} \frac{1}{2} \left( -R^{(ind)} - K_{CD}K_{CD} + K^2 \right) + \Lambda_5 \]
for $G_{NN}$ and finally

\[
\frac{1}{\kappa^2(5)} \left[ G_{AB}^{(\text{ind})} + 2K_{AC}K_{BC} - \left( \frac{1}{2} + \xi - 1 \right) K_{AB}K \right] + \frac{g_{AB} \left( R^{(\text{ind})} - 2K_{CD}K_{CD} - (1 - 2(\xi - 1)) K^2 \right)}{2} + g_{AB} \Lambda_{(5)} - \frac{1}{2} \langle B_A \rangle \langle B_{BC} \rangle
\]

\[
= \frac{1}{2} \left[ \left( \frac{1}{2} + \xi - 1 \right) \right] \left[ (B_A \nabla_C \nabla_B \langle B_C \rangle - \nabla_B \langle B_C \rangle) + (B_B \nabla_C \nabla_A \langle B_C \rangle - \nabla_A \langle B_C \rangle) \right]
\]

\[
+ \frac{\xi}{2} \left[ \left( \frac{5}{2} - 2 + \frac{2}{\xi} \right) \left( \langle B_A \rangle \langle B_C \rangle R_{CB}^{(\text{ind})} + \langle B_B \rangle \langle B_C \rangle R_{AC}^{(\text{ind})} \right) - (4\xi + 2)K_{AC}K_{BD} \langle B_C \rangle \langle B_D \rangle \right]
\]

\[
+ \frac{1}{2} \left( 1 - 2(\xi - 1) \right) g_{AB} K\sigma + \frac{\xi}{2} g_{AB} \left[ \langle B_C \rangle \langle B_D \rangle R_{CD}^{(\text{ind})} + 2(\xi - 1)K_{CE}K_{ED} \langle B_C \rangle \langle B_D \rangle \right]
\]

from $G_{AB}$, where we used also the previous results, namely the $G_{NN}$ equation, the Israel matching condition and the $B_A$ equation.

Imposing the condition for the covariant conservation of the vacuum expectation value of the field $B$, $\nabla_A \langle B_C \rangle = 0$ \([\text{10}][\text{11}],\) it follows that $\langle B_{AC} \rangle = \nabla_A \langle B_C \rangle - \nabla_C \langle B_A \rangle = 0$, which enables us to further simplify Eq. (46):

\[
\frac{1}{\kappa^2(5)} \left[ G_{AB}^{(\text{ind})} + 2K_{AC}K_{BC} - \left( \frac{1}{2} + \xi - 1 \right) K_{AB}K \right] + \frac{g_{AB} \left( R^{(\text{ind})} - 2K_{CD}K_{CD} - (1 - 2(\xi - 1)) K^2 \right)}{2} + g_{AB} \Lambda_{(5)}
\]

\[
= \frac{\xi}{2} \left[ \left( \frac{5}{2} - 2 + \frac{2}{\xi} \right) \left( \langle B_A \rangle \langle B_C \rangle R_{CB}^{(\text{ind})} + \langle B_B \rangle \langle B_C \rangle R_{AC}^{(\text{ind})} \right) - (4\xi + 2)K_{AC}K_{BD} \langle B_C \rangle \langle B_D \rangle \right]
\]

\[
+ \frac{1}{2} \left( 1 - 2(\xi - 1) \right) g_{AB} K\sigma + \frac{\xi}{2} g_{AB} \left[ \langle B_C \rangle \langle B_D \rangle R_{CD}^{(\text{ind})} + 2(\xi - 1)K_{CE}K_{ED} \langle B_C \rangle \langle B_D \rangle \right].
\]

Hence, in order to obtain a vanishing cosmological constant and ensure that Lorentz invariance holds on the brane, we must impose respectively that

\[
\Lambda_{(5)} = \frac{1}{2} \left( 1 - 2(\xi - 1) \right) K\sigma
\]

and that

\[
\frac{1}{\kappa^2(5)} \left[ 2K_{AC}K_{BC} - \left( \frac{1}{2} + \xi - 1 \right) K_{AB}K \right]
\]

\[
+ \frac{g_{AB} \left( R^{(\text{ind})} - 2K_{CD}K_{CD} - (1 - 2(\xi - 1)) K^2 \right)}{2}
\]

\[
= \frac{\xi}{2} \left[ \left( \frac{5}{2} - 2 + \frac{2}{\xi} \right) \left( \langle B_A \rangle \langle B_C \rangle R_{CB}^{(\text{ind})} + \langle B_B \rangle \langle B_C \rangle R_{AC}^{(\text{ind})} \right) - (4\xi + 2)K_{AC}K_{BD} \langle B_C \rangle \langle B_D \rangle \right]
\]

\[
+ \frac{\xi}{2} g_{AB} \left[ \langle B_C \rangle \langle B_D \rangle R_{CD}^{(\text{ind})} + 2(\xi - 1)K_{CE}K_{ED} \langle B_C \rangle \langle B_D \rangle \right],
\]

which for $\xi = 1$ reduce to the results presented in Ref. \([\text{11}].\) We observe that there is close relation between the vanishing of the cosmological constant and the maintenance of the Lorentz invariance on the brane. These conditions are enforced so that the higher dimensional signatures
encapsulated in the induced geometry of the brane cancel the Lorentz symmetry breaking inevitably induced on the brane, thus reproducing the observed geometry. The first condition, Eq. (48), can be modified to account for any non-vanishing value for the cosmological constant, as appears to be suggested by the recent Wilkinson Microwave Anisotropy Probe data, by defining the observed cosmological constant $\Lambda$ such that $\Lambda^{(5)} = \Lambda + \tilde{\Lambda}^{(5)}$. A much more elaborate fine-tuning, however, is required for the Lorentz symmetry to be observed on the brane, as described by the condition in Eq. (49). To our knowledge this is a new feature in braneworld models, as in most models Lorentz invariance is a symmetry shared by both the bulk and the brane. Notice that a connection between the cosmological constant and Lorentz symmetry has been conjectured, on different grounds, long ago [12].

4.2. $\langle B_A \rangle = 0$ and $\langle B_N \rangle \neq 0$

Choosing instead $\langle B_A \rangle = 0$ and $\langle B_N \rangle \neq 0$ we would also expect to violate Lorentz symmetry on the brane. However, for $K \neq 0$ the boundary conditions imply that we must have $\langle B_N \rangle = 0$ and thus rendering the vacuum Lorentz symmetric. If, on the other hand, we allow $K = 0$, then the Israel matching conditions yield that $\sigma = 0$, rendering the brane inexistent, with $\langle B_N \rangle$ being but the five dimensional gravitational constant $\langle B_N \rangle = \pm 1/\kappa^{(5)}$.

4.3. $\langle B_A \rangle \neq 0$ and $\langle B_N \rangle \neq 0$

If we consider the general case, with both $B_A$ and $B_N$ acquiring different, constant non-vanishing vacuum expectation values along the directions parallel to the brane, i.e. $\langle B_A \rangle \neq 0$ and $\langle B_N \rangle \neq 0$ and such that $\nabla_B \langle B_A \rangle = \nabla_B \langle B_N \rangle = 0$, we find that for $K \neq 0$ we must have that $\langle B_N \rangle = 0$, thus obtaining the same boundary conditions as those found in Subsection 4.1. Should we allow $K = 0$, then we find, as in Subsection 4.2, that $\sigma = 0$ and that $\kappa^{(5)}$ is defined in terms of both $\langle B_A \rangle$ and $\langle B_N \rangle$ according to $1/\kappa^2^{(5)} = -\lambda [\langle B_A \rangle \langle B_C \rangle K_{CB} + \langle B_B \rangle \langle B_C \rangle K_{CA}] / K_{AB} + \langle B_N \rangle \langle B_N \rangle$.

5. Discussion and Conclusions

In this contribution we analysed the spontaneous symmetry breaking of Lorentz invariance in the bulk and its effect on the brane. For this purpose, we considered a bulk vector field subject to a potential which endows the field with a non-vanishing vacuum expectation value, thus allowing for the spontaneous breaking of the Lorentz symmetry in the bulk. This bulk vector field is directly coupled to the Ricci tensor so that, after the breaking of Lorentz invariance, the loss of this symmetry is transmitted to the gravitational sector. We assign a non-vanishing vacuum expectation value first separately to the parallel and orthogonal components of the vector field, finding then that the case where both components attained non-vanishing vacuum expectation values reduced to the previous two cases. The complex interplay between matching conditions and the Lorentz symmetry breaking terms was examined. We found that Lorentz invariance on the brane can be made exact via the dynamics of the graviton, vector field and the extrinsic curvature of the surface of the brane. As a consequence of the exact reproduction of Lorentz symmetry on the brane, we found a condition for the matching of the observed cosmological constant in four dimensions. This tuning does not follow from a dynamical mechanism but is instead imposed by phenomenological reasons only. From this point of view, both the value of the cosmological constant and the induced brane Lorentz symmetry seem to be a consequence of a complex fine tuning. We shall examine further implications of this mechanism in a forthcoming publication where we will also discuss the inclusion of a bulk scalar field [15].
Acknowledgments

C. C. thanks the Portuguese Agency, Fundação para a Ciência e a Tecnologia (FCT), for financial support under the fellowship /BPD/18236/2004. C. C. thanks Martin Bucher, Georgios Kofinas and Rodrigo Olea for useful discussions. The work of O.B. is partially supported by the FCT project POCI/FIS/56093/2004.

References

[1] O. Bertolami and C. Carvalho, Phys. Rev. D74 (2006) 084020.
[2] CPT and Lorentz Symmetry III, Alan Kostelecký, ed. (World Scientific, Singapore, 2005); O. Bertolami, Gen. Rel. Gravitation 34 (2002) 707; O. Bertolami, Lect. Notes Phys. 633 (2003) 96, [hep-ph/0301191]
D. Mattingly, Liv. Rev. Rel. 8 (2005) 5, [gr-qc/0502097] T. Jacobson, S. Liberati and D. Mattingly, Ann. Phys. 321 (2006) 150, [astro-ph/0505267] R. Lehnert, “CPT- and Lorentz-symmetry breaking: a review”, [hep-ph/0611177]
[3] V.A. Kostelecký and S. Samuel, Phys. Rev. D39 (1989) 683; Phys. Rev. Lett. 63 (1989) 224; V.A. Kostelecký and R. Potting, Phys. Rev. D51 (1995) 3923; Phys. Lett. B381 (1996) 89
[4] R. Gambini and J. Pullin, Phys. Rev. D59 (1999) 124021; J. Alfaro, H.A. Morales-Tecotl and L.F. Urrutia, Phys. Rev. Lett. 84 (2000) 2318.
[5] S.M. Carroll, J.A. Harvey, V.A. Kostelecký, C.D. Lane and T. Okamoto, Phys. Rev. Lett. 87 (2001) 141601; O. Bertolami and L. Guisado, Phys. Rev. D67 (2003) 025001; JHEP 0312 (2003) 013; O. Bertolami, Mod. Phys. Lett. A20 (2005) 1359.
[6] V.A. Kostelecký, R. Lehnert and M.J. Perry, Phys. Rev. D68 (2003) 123511; O. Bertolami, R. Lehnert, R. Potting and A. Ribeiro, Phys. Rev. D69 (2004) 083513.
[7] O. Bertolami and J.G. Rosa, Phys. Rev. D71 (2005) 097901.
[8] H. Sato, T. Tati, Prog. Theor. Phys. 47 (1972) 1788; S. Coleman and S.L. Glashow, Phys. Lett. B405 (1997) 249; Phys. Rev. D59 (1999) 116008; L. Gonzales-Mestres, [hep-ph/9905430]
[9] O. Bertolami and C. Carvalho, Phys. Rev. D61 (2000) 103002.
[10] V.A. Kostelecký, Phys. Rev. D69 (2004) 105009; R. Bluhm and V.A. Kostelecký, Phys. Rev. D71 (2005) 065008.
[11] O. Bertolami and J. Páramos, Phys. Rev. D72 (2005) 044001.
[12] O. Bertolami, Class. Quantum Gravity 14 (1997) 2783.
[13] O. Bertolami, “The Adventures of Spacetime”, [gr-qc/0607006]
[14] M. Bucher and C. Carvalho, Phys. Rev. D71 (2005) 083511.
[15] O. Bertolami and C. Carvalho, in preparation.