A Fast Algorithm for Partial Fraction Decompositions

Guoce Xin

Department of Mathematics
Brandeis University
Waltham MA 02454-9110

Abstract

We obtain two new algorithms for partial fraction decompositions; the first is over algebraically closed fields, and the second is over general fields. These algorithms take \( O(M^2) \) time, where \( M \) is the degree of the denominator of the rational function. The new algorithms use less storage space, and are suitable for parallel programming. We also discuss full partial fraction decompositions.

Key words: partial fraction, quotient ring

1 Introduction

The partial fraction decomposition of a one-variable rational function is very useful in mathematics. For example, it is crucial to obtain the partial fraction decomposition of a rational function in order to integrate it. Kovacic’s algorithm \text{Kovacic (1986)} for solving the differential equation \( y''(x) + r(x)y(x) = 0 \), where \( r(x) \) is rational, requires the full partial fraction expansion of \( r(x) \) over the complex numbers.

The classical algorithm for partial fraction expansion relies on the following theorem. To make it simple, we consider rational functions in \( \mathbb{C}(t) \).

\textbf{Theorem 1.1} \( \text{If } a_1, \ldots, a_n \text{ are } n \text{ distinct complex numbers, } m_1, \ldots, m_n \text{ are positive integers, and the degree of } p(t) \text{ is less than } m_1 + \cdots + m_n, \text{ then there} \)

\text{Email address: maxima@brandeis.edu} (Guoce Xin).
are unique complex numbers $A_{i,j}$, where $1 \leq i \leq n$ and $1 \leq j \leq m_i$, such that

$$
\frac{p(t)}{(t-a_1)^{m_1} \cdots (t-a_n)^{m_n}} = \sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{A_{i,j}}{(t-a_i)^j}.
$$

(1.1)

The classical algorithm multiplies both sides by the denominator, and then equates coefficients to solve a large system of linear equations for the $A_{i,j}$'s.

The key observation for our first algorithm is that linear transformations will preserve the structure of a partial fraction expansion. This reduces the problem of finding all $A_{i,j}$ to finding only $A_{1,j}$ and assuming that $a_1 = 0$. For finding $A_{1,j}$, we use the unique Laurent series expansion at $t = 0$.

Denote by $F(t)$ the left side of equation (1.1). Let $M$ be the degree of the denominator of $F(t)$, which is $m_1 + m_2 + \cdots + m_k$. Compared with the classical algorithm for obtaining the partial fraction decomposition of $F(t)$, our new algorithm has three advantages. This comparison is under the assumption of fast multiplication of (usually rational) numbers. In the following, when we say that an algorithm takes $O(M)$ time, we mean that the algorithm will do $O(M)$ multiplications.

(1) The new algorithm is fast. The classical algorithm needs to solve $M$ linear equations in $M$ unknowns, which takes $\Omega(M^3)$ time using the Gaussian elimination algorithm. See (Sedgewick, 1988, p. 540, Property 37.1). But our algorithm only takes $O(M^2)$ time.

(2) The new algorithm requires little storage space. The classical algorithm needs to record all of the $M^2$ coefficients in these $M$ linear equations. So the storage space is about $\Omega(M^2)$. But our new algorithm needs only to record two polynomials of degree $m$, where $m$ is the maximum of the $m_i$'s. So the storage space is at most $O(M)$.

(3) The new algorithm computes the partial fraction expansion at different $a_i$'s separately, so it is more suitable for parallel programming.

For partial fraction decompositions in a general field $K$, we also have a fast algorithm by working in some quotient rings. The new algorithm applies to finding the full partial fraction decompositions and to evaluating generalized Dedekind sums. The theory gives an efficient algorithm for MacMahon’s partition analysis (Xin, 2004, Ch. 2.5).

2 Partial Fraction Decompositions Over a General Field

Let $K$ be any field, and let $t$ be a variable. It is well-known that the ring of polynomials $K[t]$ has many nice properties. Here we use the fact that $K[t]$ is
a unique factorization domain. We will use a quotient ring to derive a formula for partial fraction decomposition, which is the basis of our new algorithms.

In what follows, the degree of an element \( r \in K[t] \), denoted by \( \deg(r) \), is the degree of \( r \) as a polynomial in \( t \). The degree of the 0 polynomial is treated as \(-\infty\). We start with the division theorem in \( K[t] \).

**Theorem 2.2 (Division Theorem)** Let \( D, N \in K[t] \) and suppose \( D \neq 0 \). There is a unique pair \((p, r)\) such that \( p, r \in K[t], N = Dp + r, \) and \( \deg r < \deg D \).

The \( r \) above is called the remainder of \( N \) when divided by \( D \).

A rational function \( N/D \) with \( N, D \in K[t] \) is said to be proper if \( \deg N < \deg D \). A proper rational function is simply called a proper fraction. The unit 1 is not proper, but 0 is considered to be proper. It is clear that the sum of proper fractions is a proper fraction, and the product of proper fractions is a proper fraction. But the set of all proper fractions does not form a ring, for 1 does not belong to it.

By the division theorem, any rational function \( N/D \) can be uniquely written as the sum of a polynomial and a proper fraction. Such a decomposition is called a ppfracion (short for polynomial and proper fraction) of \( N/D \). If \( N = Dp + r \) with \( \deg(r) < \deg(D) \), then \( N/D = p + r/D \) is a ppfraction. We denote by \( \text{Poly}(N/D) \) the polynomial part of \( N/D \), and by \( \text{Frac}(N/D) \) the fractional part of \( N/D \).

Recall the following well-known result in algebra.

**Lemma 2.3** Let \( N, D \in K[t] \) with \( D \neq 0 \). If \( D = D_1 \cdots D_k \) is a factorization of \( D \) in \( K[t] \), and all the \( D_i \) are pairwise relatively prime, then \( N/D \) can be uniquely written as

\[
\frac{N}{D} = p + \frac{r_1}{D_1} + \cdots + \frac{r_k}{D_k},
\]

where \( r_i \) is a polynomial of degree smaller than \( \deg(D_i) \) for all \( i \), and \( p \) equals the polynomial part of \( N/D \).

We call the above decomposition the ppfraction expansion of \( N/D \) with respect to \((D_1, \ldots, D_k)\). To find such decomposition, it suffices to find \( p \) and \( r_1, \ldots, r_k \).

It is easy to find \( p \). In finding \( r_i \), it is convenient to use the concept of quotient ring \( K[t]/\langle D_i \rangle \), where \( \langle D_i \rangle \) is the ideal generated by \( D_i \).

Recall that \( D' + \langle D \rangle \) has a multiplicative inverse in \( K[t]/\langle D \rangle \) if and only if \( D' \) is relatively prime to \( D \). Moreover, if \( \frac{1}{D'} = \frac{r}{D} + \frac{r'}{D'} \), which is not necessarily a ppfraction expansion, then \( 1/D' + \langle D \rangle = r + \langle D \rangle \). The last fact we will use is that for any polynomials \( N_1 \) and \( N_2, N_1 + \langle D \rangle = N_2 + \langle D \rangle \) if and only if
\[ \text{Frac}(N_1/D) = \text{Frac}(N_2/D). \]

Suppose that \( D = D_1D' \) and that \( D_1 \) and \( D' \) are relatively prime. Then we have a ppfraction of \( N/D \) with respect to \((D_1, D')\):

\[ N/D = \text{Poly}(N/D) + r_1/D_1 + r'/D'. \]

In such a decomposition, we call \( r_1/D_1 \) the \textit{fractional part of} \( N/D \) \textit{with respect to} \( D_1 \), and denote it by \( \text{Frac}(N/D, D_1) \). If \( D_1 = (t - a)^m \) for some \( a \in K \), then we simply denote it by \( \text{Frac}(N/D, t = a) \). Clearly \( \text{Frac}(N/D, 1) \) is always 0, and \( \text{Frac}(N/D, D_1) \) is always a proper fraction with denominator \( D_1 \). Also we have the following relation:

\[ r_1 + \langle D_1 \rangle = N/D' + \langle D_1 \rangle. \]

Thus to find \( r_1 \), we pick a representative of \( N/D' + \langle D_1 \rangle \), and then find its remainder when divided by \( D_1 \).

\textbf{Theorem 2.4} For any \( N, D \in K[t] \) with \( D \neq 0 \), if \( D_1, \ldots, D_k \in K[t] \) are pairwise relatively prime, and \( D = D_1 \cdots D_k \), then

\[ \frac{N}{D} = \text{Poly}\left(\frac{N}{D}\right) + \text{Frac}\left(\frac{N}{D}, D_1\right) + \cdots + \text{Frac}\left(\frac{N}{D}, D_k\right) \]

is the ppfraction expansion of \( N/D \) with respect to \((D_1, \ldots, D_k)\). Moreover, if \( 1/(D_1D_i) = s_i/D_1 + p_i/D_i \), then

\[ \text{Frac}(N/D, D_1) = \text{Frac}(Ns_2s_3\cdots s_k/D_1). \]

\textbf{PROOF.} For the first part, suppose that

\[ \frac{N}{D} = p + \frac{r_1}{D_1} + \cdots + \frac{r_k}{D_k} \tag{2.2} \]

is the ppfraction expansion of \( N/D \) with respect to \((D_1, \ldots, D_k)\). Let \( D' = D_2 \cdots D_k \). Then \( D_1 \) and \( D' \) are relatively prime and \( r_2/D_2 + \cdots + r_k/D_k = r'/D' \) is a proper fraction with denominator \( D' \). By the uniqueness of ppfraction of \( N/D \) with respect to \((D_1, D')\), we have \( r_1/D_1 = \text{Frac}(N/D, D_1) \). Similarly \( r_i/D_i = \text{Frac}(N/D, D_i) \) for all \( i \).

For the second part, multiplying both sides of equation (2.2) by \( D \), and thinking of this as an identity in the quotient ring \( K[t]/\langle D_1 \rangle \), we get

\[ N + \langle D_1 \rangle = pD_1D' + r_1D_2 \cdots D_k + \cdots + r_kD_1 \cdots D_{k-1} + \langle D_1 \rangle = r_1D' + \langle D_1 \rangle. \]
Now multiply both sides of the above equation by \(1/D' + \langle D_1 \rangle\), we get

\[
\frac{r_1}{D_1} + \langle D_1 \rangle = \frac{N}{D_2 \cdots D_k} + \langle D_1 \rangle = Ns_2 s_3 \cdots s_k + \langle D_1 \rangle.
\]

Therefore

\[
\frac{r_1}{D_1} = \frac{r_1}{D_1} = \frac{Ns_2 s_3 \cdots s_k}{D_1}.
\]

Theorem 2.4 is the basis of our new algorithms. Let \(M\) be the degree of the denominator of a rational function. We will give an \(O(M^2)\) algorithm for finding the partial fraction decomposition based on the above theorem.

If \(D = \prod_{i=1}^{m} p_i \cdots p_k \cdot p_k^{m_k}\), where \(a \in K\), is a factorization of \(D\) into monic primes in \(K[t]\), then \(p_1^{m_1}, \ldots, p_k^{m_k}\) are pairwise relatively prime. Let \(D_i = p_i^{m_i}\), and let \(r_i\) be a polynomial with \(\deg(r_i) < \deg(D_i)\). Then every \(r_i/D_i\) can be uniquely written in the form \(\sum_{j=1}^{m_j} A_j/p_i^j\) with \(\deg(A_j) < \deg(p_i)\) for all \(j\). The partial fraction expansion of \(N/D\) is the result of applying the above decomposition to the partial fraction of \(N/D\) with respect to \((D_1, \ldots, D_k)\). In this case, we can use the following lemma to reduce the problem to computing only the partial fraction expansion of \(1/(p_ip_j)\) for all \(i \neq j\).

**Lemma 2.5** Let \(p, q \in K[t]\) be relatively prime polynomials. If \(r\) and \(s\) are two polynomials such that \(1/(pq) = r/p + s/q\), then for any positive integers \(m, n\),

\[
\frac{1}{p^m q^n} = \frac{1}{p^m} \sum_{i=0}^{m-1} \binom{m+i}{i} r^m s^i p^i + \frac{1}{q^n} \sum_{j=0}^{n-1} \binom{n+j}{j} r^j s^m q^j.
\]

**(2.3)**

**PROOF.** Using the formula \(1/(pq) = r/p + s/q\), we have

\[
\frac{1}{p^m q^n} = \frac{1}{pq} \cdot \frac{1}{p^{m-1} q^{n-1}} = \frac{r}{p^m q^{n-1}} + \frac{s}{p^{m-1} q^n}.
\]

If we let \(A(m, n) = 1/(p^m q^n)\), then the above equation is equivalent to

\[
A(m, n) = rA(m, n-1) + sA(m-1, n).
\]

Using this recursive relation, we can express \(A(m, n)\) in terms of \(A(0, j)\) and \(A(i, 0)\), where \(1 \leq j \leq n\) and \(1 \leq i \leq m\).

Either using induction or a combinatorial argument, we can easily get

\[
A(m, n) = \sum_{i=0}^{m-1} \binom{m+i}{i} r^m s^i A(m-i, 0) + \sum_{j=0}^{n-1} \binom{n+j}{j} r^j s^m A(0, n-j).
\]

Equation (2.3) is just a restatement of the above equation.
3 Partial Fraction Decompositions in $\mathbb{C}(t)$

In this section, $K$ is an algebraically closed field (e.g., the field of complex numbers $\mathbb{C}$). Partial fraction decomposition in this situation is simple, since every polynomial in $K[t]$ can be written as a product of linear factors $t - a$ for $a \in K$.

The key idea to our new algorithm is that linear transformation will not change the structure of a partial fraction decomposition. This can be illustrated by the following example.

The partial fraction expansion of $f(t)$ is $A/(t - a) + B/(t - b)$ if and only if the partial fraction expansion of $f(t + c)$ is $A/(t + c - a) + B/(t + c - b)$. So we can compute the partial fraction expansion of $f(t + a)$, and after that, replace $t$ with $t - a$.

Let $b \in K$ and let $\tau_b$ be the transformation defined by $\tau_b f(t) = f(t + b)$ for any $f(t) \in K[t]$ or $f(t) \in K(t)$. Then $\tau_b$ is clearly an automorphism of $K[t]$ and of $K(t)$, and its inverse is $\tau_{-b}$. The following properties can be easily checked for any $p, q \in K[t]$ and $b \in K$.

1. $p$ is prime in $K[t]$ if and only if $\tau_b p$ is.
2. $\tau_b \gcd(p, q) = \gcd(\tau_b p, \tau_b q)$.
3. $\deg(\tau_b p) = \deg(p)$.
4. $p/q$ is a proper fraction if and only if $\tau_b p/q$ is.

Thus for any $N, D \in K[t]$ with $D \neq 0$, $N/D = p + r_1/D_1 + \cdots + r_k/D_k$ is the partial fraction expansion of $N/D$ if and only if $\tau_b N/D = (\tau_b p) + (\tau_b r_1/D_1) + \cdots + (\tau_b r_k/D_k)$ is a partial fraction expansion of $\tau_b N/D$. The partial fraction expansion can be obtained by first computing the partial fraction expansion of $\tau_b N/D$, then applying $\tau_{-b}$ to the result. Choosing $b$ appropriately can simplify the computation. The above argument gives us the following lemma.

**Lemma 3.6** For any $N, D, D_1 \in K[t]$ with $D \neq 0$, $D/D_1 \in K[t]$, and $\gcd(D_1, D/D_1) = 1$, we have

$$\Frac(N/D, D_1) = \tau_{-b} \Frac(\tau_b N/D, \tau_b D_1).$$

Let $[t^m]$ be the map from $K[[t]]$ to $K[t]$ given by replacing $t^n$ with 0 for all $n \geq m$. More precisely,

$$[t^m] \sum_{n \geq 0} a_n t^n = \sum_{n=0}^{m-1} a_n t^n.$$
where $a_i \in K$ for all $i$. The following properties can be easily checked for all $f, g \in K[[t]]$.

1. $\left[ t^m \right] (f + g) = \left[ t^m \right] f + \left[ t^m \right] g$.
2. $\left[ t^m \right] (fg) = \left[ t^m \right] (\left[ t^m \right] f \left[ t^m \right] g)$.
3. If $0 < k < m$ then $\left[ t^m \right] t^k f = t^k \left[ t^m - k \right] f$.
4. If $g(0) \neq 0$, then $\left[ t^m \right] f/g = \left[ t^m \right] (\left[ t^m \right] f / \left[ t^m \right] g)$.

The main formula for our algorithm is the following, which is a consequence of Theorem 2.4. But we would like to prove this result by using Laurent series expansion.

**Theorem 3.7** Let $N, D \in K[t]$ and let $D = t^m E$ with $E \in K[t]$ and $E(0) \neq 0$. Then

$$t^m \text{Frac}(N/D, t^m) = \left[ t^m \right] \frac{N(t)}{E(t)}.$$

**PROOF.** Since $E(0) \neq 0$, $t^m$ and $E$ are relatively prime. Let

$$\frac{N(t)}{D(t)} = p(t) + \frac{r(t)}{t^m} + \frac{s(t)}{E(t)} \quad (3.1)$$

be the ppfracion of $N/D$ with respect to $(t^m, E)$. Thus $\text{deg}(r(t)) < m$, and $r(t) = t^m \text{Frac}(N/D, t^m)$.

Because $K(t)$ can be embedded into the field of Laurent series $K((t))$, equation (3.1) is also true as an identity in $K((t))$. On the right-hand side of equation (3.1), when expanded as Laurent series in $K((t))$, the the second term contains only negative powers in $t$, and the other terms contain only nonnegative powers in $t$. Therefore, $r(t)/t^m$ equals the negative part of $N/D$ when expanded as a Laurent series. More precisely, for $i = 1, \ldots, m$, we have

$$\left[ t^{-i} \right] \frac{N(t)}{D(t)} = \left[ t^{-i} \right] \frac{r(t)}{t^m}.$$

This is equivalent to $\left[ t^{m-i} \right] N(t)/E(t) = \left[ t^{m-i} \right] r(t)$ for $i = 1, \ldots, m$. Now $r(t)$ is a polynomial of degree at most $m - 1$, and $N(t)/E(t) \in K[[t]]$, so

$$r(t) = \left[ t^m \right] \frac{N(t)}{E(t)}.$$

**Remark 3.8** The idea of using Laurent series expansion to obtain part of the partial fraction expansion appeared in the proof of [Gessel, 1997, Theorem 4.4].

7
Gessel observed that this same idea can also be used to compute the polynomial part of a rational function, and that it is fast when the polynomial part has small degree.

**Proposition 3.9** If $R(t)$ is a rational function in $K(t)$, then the polynomial part $P(t)$ can be computed by the following equation.

$$t^{-1} P(t^{-1}) = \text{Frac}(t^{-1} R(t^{-1}), t = 0).$$

**PROOF.** Let $R(t) = P(t) + N(t)/D(t)$ be the ppfraction of $R(t)$, and let $p = \text{deg}(P)$, $d = \text{deg}(D)$, and $n = \text{deg}(N)$. Then $n < d$. Now we have

$$t^{-1} R(t^{-1}) = t^{-1} P(t^{-1}) + t^{-1} N(t^{-1})/D(t^{-1}) = t^{-1} P(t^{-1}) + t^{d-n-1} N(t)/(\tilde{D}(t)),$$

where $\tilde{D}(t) = t^d D(t^{-1})$, and similarly for $\tilde{N}(t)$.

Apply ppfraction expansion to the second term. Since $\tilde{D}(t)$ has nonzero constant term, it is relatively prime to $t^{p+1}$. Now it is clear that $t^{-1} P(t^{-1})$ is the fractional part of $t^{-1} R(t^{-1})$ with respect to $t^{p+1}$.

**Example 3.10** It is easy to check that

$$R(t) = \frac{t^3 + 2t^2 - 3t + 4}{t^2 - 4t + 2} = t + 6 + \frac{-8 + 19t}{t^2 - 4t + 2}.$$

Now we compute the polynomial part of $R(t)$ by Proposition 3.9.

$$t^{-1} R(t^{-1}) = \frac{1 + 2t - 3t^2 + 4t^3}{t^2 (1 - 4t + 2t^2)}$$

$$t^2 \text{Frac}(t^{-1} R(t^{-1}), t^2) = [t^2] \frac{1 + 2t - 3t^2 + 4t^3}{(1 - 4t + 2t^2)}$$

$$= [t^2] \frac{1 + 2t}{1 - 4t} = 1 + 6t.$$

So we obtain that the polynomial part of $R(t)$ is $t + 6$.

Note that when expanded as Laurent series in $t$, we have

$$[t^{m_0}] \frac{1}{(t - a_i)^{m_i}} = \sum_{j=0}^{m_0-1} (-1)^{m_i} \binom{m_i - 1 + j}{j} \frac{t^j}{a_i^{m_i+j}}.$$

Hence by Theorem 3.7, we get
Corollary 3.11 Let \( N \in K[t] \) and \( D = t^{m_0}(t - a_1)^{m_1} \cdots (t - a_k)^{m_k} \) with all the \( a_i \)'s distinct and not equal to 0. Then

\[
t^{m_0} \text{Frac}\left( \frac{N}{D}, t^{m_0} \right) = \lceil t^{m_0} \rceil N s_1 \cdots s_k,
\]

where

\[
s_i = \sum_{j=0}^{m_0-1} (-1)^{m_i} \binom{m_i - 1 + j}{j} \frac{t^j}{a_i^{m_i+j}}.
\]

Therefore, combining Theorem 2.4, Lemma 3.6 and Corollary 3.11, we obtain an algorithm for computing the partial fraction decomposition of a proper rational function of the general form

\[
F(t) = \frac{N(t)}{(t - a_1)^{m_1} \cdots (t - a_k)^{m_k}}.
\]

(1) Let \( S := 0 \)
(2) For \( i \) from 1 to \( k \) do \( G(t) := F(t + a_i) \), \( S := S + \tau_{-a_i} \text{Frac}(G(t), t^{m_i}) \) next \( i \).
(3) Return \( S \).

It was stated in (Xin, 2004, Ch. 2.4) that the computation of \( \text{Frac}(G(t), t^{m_i}) \) for all \( i \) will take time \( O(k(m_1^{1.58} + m_2^{1.58} + \cdots + m_k^{1.58})) \). However, this estimate does not show \( O(M^2) \) time. For example, \( M = 2k \), and \( m_1 = m_2 = \cdots = m_{k-1} \), and \( m_k = M - k + 1 \).

Theorem 3.12 Let \( M \) be the degree of the denominator of a rational function. The above algorithm for partial fraction decomposition can be executed in \( O(M^2) \) time.

The proof of this theorem, which will be given later, uses the fact that manipulations in \( K[t]/\langle t^m \rangle \) are fast. Now let us estimate the computational time of manipulations in the quotient ring \( K[t]/\langle t^m \rangle \).

The following is a well-known result by the method of divide and conquer. See, e.g., (Sedgewick, 1988, Property 36.1).

Proposition 3.13 Let \( R(m) \) be the time for computing the product of two polynomials of degree less than \( m \). Then \( R(m) = O(m^{1.59}) \).

Remark 3.14 In the proof of Theorem 3.12, we only need the obvious upper bound \( R(m) = O(m^2) \). The above proposition shows that our algorithm can be accelerated.

Since most of our estimations use the method of divide and conquer, it is better to introduce it here. In what follows, we shall always assume that \( m \) is
a power of 2 for simplicity. The estimation of \( R(m) \) follows from the following observation. Bisect \( P(t) \) as \( P(t) = P_1(t) + t^{m/2}P_2(t) \), and bisect \( Q(t) \) as \( Q(t) = Q_1(t) + t^{m/2}Q_2(t) \). Then

\[
PQ = P_1Q_1 + t^{m/2}((P_1 + Q_1)(P_2 + Q_2) - P_1Q_1 - P_2Q_2) + t^mP_2Q_2. \quad (3.2)
\]

which shows that we need only three polynomial multiplications. This gives the recurrence \( R(m) = 3R(m/2) \) and that \( R(m) = O(m^{\log_2 3}) = O(m^{1.59}) \).

**Lemma 3.15** Let \( P(t) \) and \( Q(t) \) be two polynomials of degree \( m - 1 \). Then \( \lceil t^m \rceil P(t)Q(t) \) can be computed in no more than \( R(m) \) time.

The proof of this lemma is trivial.

**Lemma 3.16** The computation of \( \lceil t^m \rceil P/Q \), where \( Q(0) \neq 0 \), takes no more than \( 2R(m) \) time.

**PROOF.** We use the method of divide and conquer. Let \( T(m) \) be the time for the computation in question.

Suppose that \( \lceil t^m \rceil P/Q = Z \). Bisect \( P, Q, Z \) as \( P = P_1 + t^{m/2}P_2 \), \( Q = Q_1 + t^{m/2}Q_2 \), and \( Z = Z_1 + t^{m/2}Z_2 \). Then

\[
\lceil t^m \rceil \frac{P_1 + t^{m/2}P_2}{Q_1 + t^{m/2}Q_2} = Z_1 + t^{m/2}Z_2.
\]

Obviously \( \lceil t^{m/2} \rceil P_1/Q_1 = Z_1 \). So it will take \( T(m/2) \) time to find \( Z_1 \). To find \( Z_2 \), we use the formula (from direct algebraic computation).

\[
Z_2 = \lceil t^{m/2} \rceil \frac{(P_1 - Q_1Z_1)/t^{m/2} + P_2 - Q_2Z_1}{Q_1}.
\]

Therefore, we get the recurrence \( T(m) = T(m/2) + 2R(m/2) + T(m/2) \), where the first summand is for \( Z_1 \), and the rest is for \( Z_2 \). Using this recurrence, it is easy to see that \( T(m) \) is no more than \( 2R(m) \).

To apply the above lemma, we need the expanded representation of \( P(t) \) and \( Q(t) \).

**Lemma 3.17** Suppose that \( Q(t) \) is the product of \( m \) linear factors. Then it takes no more than \( R(m) \) time to expand \( Q(t) \).

**PROOF.** Let \( U(m) \) denote the time for expanding the products of \( m \) linear factors. Factor \( Q(t) \) as \( Q(t) = Q_1(t)Q_2(t) \), where \( Q_1(t) \) consists of the first
m/2 factors. Then it will take $U(m/2)$ time to expand $Q_1(t)$, and $U(m/2)$ time to expand $Q_2(t)$, and then $R(m/2)$ time to get the final expansion. Thus $U(m) = 2U(m/2) + R(m/2)$. This implies that $U(m)$ is approximately equal to $R(m)$.

We might be able to speed up the expansion in the above lemma by the following lemma.

**Lemma 3.18** The expansion of $(t - a)^m$ takes $O(m)$ time.

**PROOF.** This lemma follows from the binomial theorem

$$(t - a)^m = \sum_{i=0}^{m} t^i (-a)^{m-i} \binom{m}{i},$$

and the fact that the ratios of consecutive summands are simple.

Using the binomial theorem, it is easy to see the following.

**Lemma 3.19** Suppose the degree of $N(t)$ is less than $M$. Then the expansion of $\lceil t^m \rceil N(t+a)$ takes no more than $O(m \cdot \text{the number of nonzero terms in } N(t))$, which is no more than $O(Mm)$ time.

Now we estimate the computational time for $\text{Frac}(N/D, x = a_i)$ in Theorem 3.7.

**Proposition 3.20** Let $M = m_0 + m_1 + \cdots + m_k$. Then it takes $O(Mm_0^{0.59})$ time to compute $\lceil t^{m_0} \rceil \frac{N(t)}{(t-a_1)^{m_1} \cdots (t-a_k)^{m_k}}$.

**PROOF.** We first expand the denominator by grouping every $(m_0-1)$ factors together. So we have about $(M-m_0)/(m_0-1)$ groups. It takes $R(m_0)$ time for expanding the products for each group, and then about $(M-m_0)/(m_0-1)$ multiplications when taking $\lceil t^{m_0} \rceil$. Thus the total time for this expansion is about $2M/m_0 R(m_0)$.

Denote by $E(t)$ the resulting expansion. Now it will take about $2R(m_0)$ time to compute $\lceil t^{m_0} \rceil \frac{N(t)}{E(t)}$. Therefore the total time for the final answer is $O(M/m_0)R(m_0) = O(Mm_0^{0.58})$.

**Proof of Theorem 3.12.** It will take $O(Mm_i)$ time for finding the expansion of $N(t+a_i)$, and will take $O(Mm_i^{0.58})$ time for finding $\lceil t^{m_i} \rceil t^{m_i} N(t+a_i)/D(t+a_i)$. Therefore, the total time for finding $\text{Frac}(N/D, x = a_i)$ takes $O(Mm_i)$.
time. Summing on all $i$, we see that it takes $O(M^2)$ time to find the partial
fraction decomposition of $N/D$.

This new algorithm also enables us to work with some difficult rational func-
tions by hand.

**Example 3.21** Compute the partial fraction expansion of $f(t)$, where

$$f(t) = \frac{t}{(t + 1)^2(t - 1)^3(t - 2)^5}.$$  

**Solution.** Clearly, the polynomial part of $f(t)$ is 0. Although applying Corol-
larly 3.11 is faster, we compute the fractional part of $f(t)$ at $t = -1$ and
$t = 1$ differently. For the fractional part of $f(t)$ at $t = -1$, we apply $\tau_{-1}$, and
compute $\text{Frac}(f(t - 1), t^2)$ by Theorem 3.7. We have

$$t^2 \text{Frac}(f(t - 1), t^2) = [t^2] \frac{t - 1}{(t - 2)^5(t - 3)} = \frac{t - 1}{(-8 + 12t)(-3^5 + 3^4 \cdot 5t)} = \frac{t - 1}{8 \cdot 3^5(1 - 19/6t)} = \frac{t - 1}{8 \cdot 3^5(1 + 13t/6)}.$$

Thus

$$\text{Frac}(f(t), (t + 1)^2) = -\frac{1}{2^3 \cdot 3^5(t + 1)^2} - \frac{13}{2^4 \cdot 3^6(t + 1)}.$$  

Similarly, we can compute the fractional part of $f(t)$ at $t = 1$. We have

$$t^3 \text{Frac}(f(t + 1), t^3) = [t^3] \frac{t + 1}{(t + 2)^2(t - 1)^5} = \frac{t + 1}{(t^2 + t + 4)(-10t^2 + 5t - 1)} = \frac{t + 1}{-4 + 16t - 21t^2} = -\frac{1}{4} \frac{[t^3]}{(t - 1)^2}.$$  

Thus

$$\text{Frac}(f(t), (t - 1)^3) = -\frac{1}{4(t - 1)^3} - \frac{1}{4(t - 1)^2} - \frac{5}{16(t - 1)}.$$  

12
The fractional part of $f(t)$ at $t = 2$ can be obtained similarly, but it is better to use Corollary 3.11. In fact, this computation becomes quite complicated. Although it is still possible to work by hand, we did use Maple.

\[
\begin{align*}
t^5 \text{Frac}(f(t + 2), t^5) \\
= [t^5](t + 2) \left( \frac{1}{9} - \frac{2t}{27} + \frac{t^2}{27} - \frac{4t^3}{243} + \frac{5t^4}{729} \right) \left( 1 - 3t + 6t^2 - 10t^3 + 15t^4 \right)
= \frac{29}{9} - \frac{19}{27} t + \frac{13}{9} t^2 - \frac{593}{243} t^3 + \frac{2689}{729} t^4.
\end{align*}
\]

Applying Theorem 2.4, we get the partial fraction expansion of $f(t)$, which is too lengthy to be worth giving here.

4 Algorithm for a General Field and Full Partial Fraction Decompositions

When $K$ is a general field, e.g., the field of rational numbers $\mathbb{Q}$, linear transformations will not help. Manipulations in $K[t]/\langle D_1(t) \rangle$ are not as good as the case of $D_1(t) = x^m$. But we still have an $O(M^2)$ algorithm.

**Proposition 4.22** Suppose that $D_1, \ldots, D_k \in K[t]$ are pairwise relatively prime, and $D = D_1 \cdots D_k$. If $\deg(N) < \deg(D)$, then the partial fraction decomposition of $N/D$ with respect to $D_1, \ldots, D_k$ can be computed in $O(\deg(D)^2)$ time.

The proof of this proposition will be given later. Now assume that $D_i = p_i^{a_i}$, and $\deg(D_i) = m_i$. It is easy to show that the partial fraction decomposition of $r_i/D_i$, where $\deg(r_i) < m_i$, can be computed in $O(m_i^2) = O(Mm_i)$ time. Thus the above argument and Proposition 4.22 will give us the following.

**Theorem 4.23** Suppose that $\deg(N) < \deg(D)$, and we are given a factorization $D = p_1^{a_1} \cdots p_k^{a_k}$ of $D$ into primes in $K[t]$. Then the partial fraction decomposition of $N/D$ takes $O(\deg(D)^2)$ time.

In order to prove Proposition 4.22, we need to estimate manipulations in $K[t]/\langle D \rangle$ for a given polynomial $D$. In most situations, we need the unique representative of $N + \langle D \rangle$ that has degree less than $\deg D$. We denote by $\lceil D \rceil N$ this representative, which is also known as the remainder of $N$ when divided by $D$.

The following estimations are obvious. The computational time refers to the number of multiplications of two elements in $K$. Time spent on additions is omitted.

13
Then the computation of Lemma 4.24 exists, where 

Lemma 4.25 Suppose that 
P and Q are two polynomials of degree less than deg(D).
Then the computation of \([D]\) PQ takes no more than 2 deg(D)^2 time.

**Lemma 4.24** Suppose P is relatively prime to D and deg(P) < deg(D). Then the computation of \([D]\) 1/P takes \(O(\text{deg}(D)^2)\) time.

This estimation is obtained by the extended Euclidean algorithm for polynomials. See, e.g., Moenck (1973), which says that an \(O(\text{deg}(D) \log^r(D))\) algorithm exists, where \(r\) is a fixed number.

**Lemma 4.25** Suppose that deg(D_1) = m and deg(D_2 \cdots D_k) = M. Then the computation of \([D_1] D_2 \cdots D_k\) takes no more than \(4(M + m)m\) time.

**PROOF.** Denote by \(V(M)\) the computational time described in the lemma. We shall prove that \(V(M) \leq \max\{4Mm - 2m^2, Mm + m^2\}\), which implies the lemma. The proof is in two parts. The first part deals with the case when \(M \leq 2m\), and the second part deals with the case when \(M \geq m\). Note that there is an overlap.

We first show that the expansion of \(D_2 \cdots D_k\) takes no more than \(M^2/2\) time by induction on \(M\). This claim is clearly true for small \(M\), e.g., \(M = 1, 2\). Now suppose the claim is true for all \(l \leq M\). Then the expansion of \(D_2 \cdots D_k\) can be obtained by first expanding \(D_2 \cdots D_{k-1}\) (of degree \(M_1\)), then multiplying it by \(D_k\) (of degree \(M_2\)). The computational time is (by induction) no more than \(M_1^2/2 + M_1M_2 \leq (M_1 + M_2)^2/2 = M^2/2\). Thus for \(M \leq 2m\), we get \(V(M) \leq M^2/2 + m^2 \leq Mm + m^2\) by expanding \(D_2 \cdots D_k\), and then taking \([D_1]\).

We claim that for all \(M \geq m\), \(V(M) \leq 4Mm - 2m^2\), and prove the claim by induction on \(M\). The claim follows from the inequality \(M^2/2 + m^2 < 4Mm - 2m^2\) when \(M \leq 7m\) by the first part. For \(M \geq 7m\), we can separate \(D_2 \cdots D_k\) into two products of degree \(M_1\) and \(M_2\) respectively. We can assume that \(M_1 \geq m\) and \(M_2 \geq m\), for otherwise, the degree of one of \(D_i\) is larger than \(5m\), in which case the claim is easily seen to be true. Now we compute the remainder of each product, and then compute the resulting product and compute the remainder. This process takes time

\[
V(M_1) + V(M_2) + 2m^2 \leq 4M_1m - 2m^2 + 4M_2m - 2m^2 + 2m^2 = 4Mm - 2m^2.
\]

This completes the proof.
Proposition 4.26 Suppose that \( \deg(D) = M, \deg(N) \leq M, \deg(D_1) = m, \) and that \( D = D_1 \cdots D_k \) is a factorization of \( D \) into relatively prime factors. Then the computation of \( [D_1] \frac{N}{(D_2 \cdots D_k)} \) takes \( O(Mm) \) time.

**Proof.** We first compute \( [D_1] \frac{D_2 \cdots D_k}{N} \), and denote the result by \( D' \). This step takes no more than \( 4Mm \) time. Then we compute \( [D_1] N \), and denote the result by \( N' \). This step takes no more than \( Mm \) time. Finally we compute \( [D_1] \frac{N'}{D'} \). This step takes \( O(m^2) = O(Mm) \) time by Lemma 4.24. So the total time is \( 4Mm + Mm + O(Mm) = O(Mm) \).

**Proof of Proposition 4.22.** By Theorem 2.4, the numerator of \( \text{Frac}(\frac{N}{D}, D_i) \) is given by
\[
 r_i = [D_i] \frac{ND_i}{(D_1D_2 \cdots D_k)}.
\]
The computation of \( r_i \) takes \( O(M \deg(D_i)) \) time by Proposition 4.26. Summing on all \( i \) we get the total computational time for the ppfraction of \( N/D \) with respect to \( D_1, \ldots, D_k \), which is \( \sum_{i=1}^{k} O(Mm_i) = O(M^2) \).

**Example 4.27** Compute the fractional part of \( f(t) \) with respect to \( t^2 - t + 2 \), where
\[
f(t) = \frac{t^2}{(t^2 - 2t - 1)^2(t^2 - t + 2)}.
\]

**Solution.** Let \( p(t) = t^2 - t + 2 \). Then we need to compute \( [p(t)] \frac{t^2}{(t^2 - 1)^2(t^2 - t + 2)} \).
In the following computation, we shall always replace \( t^2 \) with \( t - 2 \).
\[
[p(t)] \frac{t^2}{(t^2 - 2t - 1)^2} = [p(t)] \frac{t - 2}{(-t - 3)^2} = [p(t)] \frac{t - 2}{t(t + 1)}
= [p(t)] \frac{-3t + 2}{-28} = -\frac{3t - 2}{-28},
\]
where we used the fact that \( (t + 1)(t - 2) = t^2 - t - 2 = p(t) - 4 \). Therefore
\[
\text{Frac}(f(t), p(t)) = \frac{3t - 2}{-28(t^2 - t + 2)}.
\]

In Maple, the full partial fraction expansion of a rational function will involve a form like
\[
\sum_{\alpha = \text{root of } p(t)} \sum_{j=1}^{m} \frac{h_j(\alpha)}{(t - \alpha)^j},
\]
where \( p(t) \) is a prime polynomial, and \( h_j(t) \) will be a polynomial of degree less than \( \deg(p(t)) \). This expansion is useful in some situations. We can get this kind of expansion by applying Theorem 3.7. This is best illustrated by an example.
Example 4.28 \textit{Compute the full partial fraction expansion of } \( f(t) \), \textit{where}

\[ f(t) = \frac{t}{(t^2 - t - 1)^2(t^2 - t + 2)}. \]

\textbf{Solution.} Suppose that \( \alpha \) is a root of the prime polynomial \( p(t) := t^2 - t - 1 \). Since \( K(\alpha) \) is a field, and \( \alpha^2 = \alpha + 1 \), we can use this relation to get rid of all terms containing \( \alpha^n \) for \( n \geq 2 \). Because \( p(t) \) is a prime polynomial, \( \alpha \) can only be a simple root of \( p(t) \). Then \( t \) divides \( p(t + \alpha) \) and \( p(t + \alpha)/t \) has nonzero constant term. In the present example,

\[ p(t + \alpha) = (t + \alpha)^2 - (t + \alpha) - 1 = t(t + 2\alpha - 1). \]

Note that the constant term of \( p(t + \alpha) \) is always 0.

Clearly, \( \tau_\alpha(t^2 - t + 2) \) has nonzero constant term, for otherwise \( t^2 - t + 2 \) will not be relatively prime to \( p(t) \). In the present situation,

\[ (t + \alpha)^2 - (t + \alpha) + 2 = t^2 + (2\alpha - 1)t + 3. \]

By Lemma 3.6 and Theorem 3.7, we can work in \( K(\alpha)[[t]] \).

\[ [t^2] \ t^2 f(t + \alpha) = [t^2] \frac{t + \alpha}{(t + 2\alpha - 1)^2(t^2 + (2\alpha - 1)t + 3)} \]
\[ = \frac{1}{15} [t^2] \frac{t + \alpha}{1 + (2\alpha - 1)11t/15} \]
\[ = \frac{1}{15} [t^2] (t + \alpha)(1 - 11(2\alpha - 1)t/15) \]
\[ = \frac{1}{15} \alpha + \frac{(-11\alpha + 7)}{15^2} t. \]

Thus the fractional part of \( f(t) \) at \( \alpha \) that satisfies \( p(\alpha) = 0 \) can be written as

\[ \frac{\alpha}{15(t - \alpha)^2} + \frac{(7 - 11\alpha)}{225(t - \alpha)}. \]

Similarly, the fractional part of \( f(t) \) at \( \beta \) that satisfies \( \beta^2 - \beta + 2 = 0 \) can be written as

\[ \left( \frac{4}{63} - \frac{1}{63} \beta \right) (t - \beta)^{-1}. \]

Together with the fact that the polynomial part of \( f(t) \) is clearly 0, the full partial fraction expansion of \( f(t) \) is hence

\[ f(t) = \sum_{\alpha^2 - \alpha - 1 = 0} \left[ \frac{\alpha}{15(t - \alpha)^2} + \frac{(7 - 11\alpha)}{225(t - \alpha)} \right] + \sum_{\beta^2 - \beta + 2 = 0} \frac{4 - \beta}{63(t - \beta)}. \]
Of course we can first compute the partial fraction decomposition of $f(t)$ and then compute its full partial fraction decomposition.

**Acknowledgment.** The author is very grateful to his advisor Ira Gessel.

**References**

Gessel, I. M., 1997. Generating functions and generalized Dedekind sums. Elec. J. Comb. 4 (2), Wilf Festschrift, R11.

Kovacic, J. J., 1986. An algorithm for solving second order linear homogeneous differential equations. J. of Symbolic Computation 13, 3–43.

Moenck, R. T., 1973. Fast computation of gcds. In: Proceedings of the fifth annual ACM symposium on Theory of computing. Austin, Texas, United States, pp. 142–151.

Sedgewick, R., 1988. Algorithms, 2nd Edition. Addison-Wesley, New York.

Xin, G., 2004. The Ring of Malcev-Neumann Series and The Residue Theorem. Ph.D. thesis, Brandeis University.