Modelling Derivatives Pricing Mechanisms with Their Generating Functions

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Abstract. In this paper we study dynamic pricing mechanisms of financial derivatives. A typical model of such pricing mechanism is the so-called $g$–expectation defined by solutions of a backward stochastic differential equation with $g$ as its generating function. Black-Scholes pricing model is a special linear case of this pricing mechanism. We are mainly concerned with two types of pricing mechanisms in an option market: the market pricing mechanism through which the market prices of options are produced, and the ask-bid pricing mechanism operated through the system of market makers. The later one is a typical nonlinear pricing mechanism. Data of prices produced by these two pricing mechanisms are usually quoted in an option market.

We introduce a criteria, i.e., the domination condition (A5) in (2.5) to test if a dynamic pricing mechanism under investigation is a $g$–pricing mechanism. This domination condition was statistically tested using CME data documents. The result of test is significantly positive. We also provide some useful characterizations of a pricing mechanism by its generating function.

Keywords: BSDE, dynamic pricing mechanism, $g$–expectation, $g$-martingale, dynamic risk measures.

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1 Introduction

How to quantitatively describe the pricing mechanism of a market of derivatives is a very interesting problem. A model of dynamic pricing mechanism of derivatives is formulated (see (A1)–(A4) in the next section) to characterize this pricing behavior.

We are mainly concerned with two types of pricing mechanisms in an option market: the market pricing mechanism which outputs the trading prices of options and the bid–ask pricing mechanism operated according the system of market makers. We stress here that, in our point of view, the ask prices and the bid prices quoted in a market are determined by a single pricing mechanism. The difference of a ask price and the corresponding bid price, called bid–ask spread, reflects the nonlinearity of this mechanism. The data of prices of above mentioned two pricing systems is usually systematically quoted in the internet thus the models under our investigation can be statistically tested. We hope that our modelling can also be applied to describe the pricing mechanism of some other financial institutions.

The well-known Black–Scholes formula is a typical model of dynamic pricing mechanism of derivatives. It is a linear pricing mechanism. In fact, the prices produced by this mechanism is solved by a linear Backward Stochastic Differential Equation (BSDE). This means that the corresponding generating function \( g \) of the BSDE is a linear function. Nonlinear pricing model by BSDE was proposed in [24] (cf. [27]). In this paper we show that each well-defined BSDE with a fixed generating function \( g \) forms a dynamic pricing mechanism, called \( g \)-expectation and that the behaviors of this mechanism are perfectly characterized by the behaviors of \( g \). Several conditions of equivalence provided in this paper will be very helpful to characterize and to find the generating function, or in some other circumstances, to regulate or to design a pricing mechanism.

A very interesting problem is how to design a test procedure to verify whether an existing pricing mechanism of derivatives is a \( g \)-expectation. We will present the following result: if a dynamic pricing mechanism is uniformly dominated by a \( g_\mu \)-expectation with a sufficiently large number \( \mu \) for the function \( g_\mu = \mu(|y|+|z|) \), then it is a \( g \)-expectation. This domination inequality (2.5) has been applied as a testing criteria in our data analysis. The results strongly support that both the market pricing mechanism and the bid–ask pricing mechanism under our investigation can be modelled as \( g \)-expectations, and that the bid–ask prices are then produced by this single mechanism.

The paper is organized as follows: in Section 2 we present the notion of \( g \)-expectation and show that, for each well-defined function \( g \) it satisfies the basic conditions (A1)–(A4) of a dynamic pricing mechanism of derivatives. We then show that, a dynamic pricing mechanism dominated by a \( g_\mu \)-expectation, i.e.,
(2.5) is satisfied, is a $g$–expectation. In Section 3, we will present some equivalent conditions to show that the behaviors of a $g$–expectation are perfectly reflected by its generating function $g$. We also provide some examples and explain how to statistically find the function $g$ by testing the input–output data of prices.

In Appendix 4.2 we apply the crucial domination inequality (2.5) to test the market pricing mechanisms and the bid–ask pricing mechanisms of S&P500 index future options and S&P500 index options, using data of parameter files provided by CME and CBOE. The result supports that they are $g$–expectations.

Application of the dynamic expectations and pricing mechanisms is to risk measures. Axiomatic conditions for a (one step) coherent risk measure was introduced by Artzner, Delbaen, Eber and Heath 1999 [2] and, for a convex risk measure, by Föllmer and Schied (2002) [29]. Rosazza Gianin (2003) studied dynamic risk measures using the notion of $g$–expectations in [55] (see also [50], [3], [4]) in which an additional condition of cash translatability is assumed.

2 The pricing mechanisms and $g$–pricing mechanisms by BSDE

Let us consider a market of financial derivatives in which the price $(S_t)_{t \geq 0}$ of the underlying assets is driven by a $d$–dimensional Brownian motion $(B_t)_{t \geq 0}$ in a probability space $(\Omega, \mathcal{F}, P)$. Here $S$ is an $m$–dimensional process, namely the number of the underlying assets is $m$. We assume that the past information $\mathcal{F}_t^S$ of the price $S$ before $t$ coincides with that of the Brownian motion: $\mathcal{F}_t^S = \sigma\{S_s, s \leq t\} = \mathcal{F}_t := \sigma\{B_s, s \leq t\}$.

A derivative $X$ with maturity $T$ is an $\mathcal{F}_T$–measurable and square–integrable random value called maturity value is denoted by $X \in L^2(\mathcal{F}_T)$. The market price $Y_t$ of this derivative at time $t < T$ is assumed to be in $L^2(\mathcal{F}_t)$.

Let us consider a BSDE model of a pricing mechanism of derivatives, where $Y_t$ is the solution of the following BSDE:

$$Y_t = X + \int_t^T g(s,Y_s,Z_s)ds - \int_t^T Z_s dB_s. \quad (2.1)$$

Here $(Y,Z)$ a pair of the adapted processes to be solved, $g$ is a given function $g : (\omega,t,y,z) \in \Omega \times [0, \infty) \times R \times R^d \rightarrow R$.

We call $g$ the generating function of the BSDE. It satisfies the following basic assumptions for each $\forall y, \bar{y} \in R$ and $z, \bar{z} \in R^d$,

$$\begin{cases} g(\cdot, y, z) \in L^2_F(0,T), & \forall T \in (0, \infty), \\ |g(t,y,z) - g(T,\bar{y},\bar{z})| \leq \mu(|y - \bar{y}| + |z - \bar{z}|). \end{cases} \quad (2.2)$$

It is important to consider the following special situation:

$$\begin{cases} (a) & g(\cdot, 0, 0) \equiv 0, \\ (b) & g(\cdot, y, 0) \equiv 0, \forall y \in R. \end{cases} \quad (2.3)$$
Obviously (b) implies (a). This BSDE (2.1) was introduced by Bismut [6], [7] for the case where \( g \) is a linear function of \((y, z)\). [40, Pardoux-Peng, 1990] obtained the following fundamental result: for each \( X \in L^2(\mathcal{F}_T) \), there exists a unique square–integrable adapted solution \((Y, Z)\) of the BSDE (2.1). The following notion of \( g \)-expectations was introduced by [44, Peng 1997a] and [45, Peng 1997].

**Definition 2.1** We denote by \( \mathbb{E}^g_{t,T}[\cdot] := Y_t^t \):

\[
\mathbb{E}^g_{t,T}[\cdot] : L^2(\mathcal{F}_T) \to L^2(\mathcal{F}_t), \quad 0 \leq t \leq T < \infty. \tag{2.4}
\]

\((\mathbb{E}^g_{t,T}[\cdot])_{0 \leq t \leq T < \infty}\) is called \( g \)-expectation, or \( g \)-pricing mechanism.

As an example, we consider the following Black–Scholes pricing mechanism:

**Example 2.2 (Black–Scholes Pricing Mechanism)** Consider a financial market consisting of 2 underlying assets: one bond and one stock. We denote by \( S_0(t) \) the price of the bond and by \( S(t) \) the price of the stock at time \( t \). We assume that \( S_0(t) \) satisfies an ordinary differential equation: \( dS_0(t) = r_t S_0(t) dt \), and \( S(t) \) is the solution of the following stochastic differential equation (SDE) with 1–dimensional Brownian motion \( B \) (i.e., \( d = 1 \)) as driven noise:

\[
dS(t) = S(t)(b_t dt + \sigma_t dB_t), \quad S(0) = p.
\]

Here \( r_t \) is the interest rate, \( b_t \) the rate of the expected return and \( \sigma_t \) the volatility of the stock at the time \( t \). \( r_t, b_t, \sigma_t \) and \( \sigma^{-1} \) are assumed to be \( \mathcal{F}_t \)-measurable and uniformly bounded. Black and Scholes have solved the problem of the market pricing mechanism of an European call option \( X = (S_T - k)^+ \) and put option \( X = (k - S_T)^+ \), where \( k \) is the strike price, under the assumption that \( r, b \) and \( \sigma \) are constant. Their main idea can be easily adapted to our slightly more general situation for a derivative \( X \in L^2(\mathcal{F}_T) \) with maturity \( T \). Consider an investor with the following investment portfolio at a time \( t \leq T \): he has \( n_0(t) \) bonds and \( n(t) \) stock, i.e., he invests \( n_0(t)S_0(t) \) in bond and \( n(t)S(t) \) in the stock. We define by \( Y_t \) the investor’s wealth invested in the market at time \( t \):

\[
Y_t = n_0(t)P_0(t) + n(t)P(t).
\]

We make the so called “self-financing assumption”:

\[
dY_t = n_0(t)dS_0(t) + n(t)dS(t)
\]
or

\[
dY_t = [r_t Y_t + (b_t - r_t)\pi(t)] dt + \sigma_t \pi_t dB_t.
\]

We denote \( g(t, y, z) := -r_t y - (b_t - r_t)\sigma_t^{-1} z \). Then, by denoting \( Z_t = \sigma_t \pi_t(t) \), the above equation is

\[
-dY_t = g(t, Y_t, Z_t) dt - Z_t dB_t.
\]
We observe that the above function $g$ satisfies (2.2). It follows from the existence and uniqueness theorem of BSDE that for each derivative $X \in L^2(F_T)$, there exists a unique adapted solution $(Y, Z)$ with the terminal condition $Y_T = X$. This result of existence and uniqueness is economically meaningful: in order to replicate the derivative $X$ at the maturity $T$, the investor needs and only needs to invest the $Y_t$ at the present time $t$ and then, during the time interval $s \in [t, T]$, to perform the portfolio strategy $\pi(s) = \sigma_s^{-1}Z_s$. Furthermore, by Comparison Theorem of BSDE, if he wants to replicate a derivative $\bar{X}$ with the same maturity $T$ which is bigger than $X$ (i.e., $\bar{X} \geq X$ and $P(\bar{X} \geq X) > 0$) then he must invest more than $Y_t$ at the time $t$. This means there is no arbitrage opportunity. In this situation $Y_t = \mathbb{E}^q_{t,T}[X]$ is called the Black–Scholes price, and $(\mathbb{E}^q_{t,T}[])_{0 \leq t \leq T < \infty}$ is called the corresponding Black–Scholes pricing mechanism. We observe that the generating function $g$ satisfies (a) of condition (2.3).

**Example 2.3** The following problem was considered in [5], [14] and [24]: the investor is allowed to borrow money at time $t$ at an interest rate $R_t > r_t$. The amount borrowed at time $t$ is equal to $(Y_t - \pi(t))^-$. In this case the wealth process $Y$ still satisfies BSDE:

$$-dY_t = g(t, Y_t, Z_t)dt - Z tdW_t,$$

with $g(t, y, z) := -r_t y - (b_t - r_t)\sigma_t^{-1}z + (R_t - r_t)(y - \sigma_t^{-1}z)^-$. This derives a $g$–pricing mechanism with a sub–additive generating function $g$.

Similar equations appear in continuous trading with short sales constraints with different risk premium for long and short positions (cf. [37], [32] and [24]). In this case $g(t, y, z) := -r_t y - (b_t - r_t)\sigma_t^{-1}z + k_t z^-$. We observe that in each of the above three examples, $g$ is sub-additive in $(y, z)$.

The following result, obtained in [50]-Theorem 3.4, explains why this $g$–expectation is a good candidate to model a dynamic pricing mechanism of derivatives:

**Proposition 2.4** Let the generating function $g$ satisfies (2.2) and (2.3)–(a). Then the above defined $g$–expectation $\mathbb{E}_t^g[\cdot]$ is a dynamic pricing mechanism of derivatives, i.e., it satisfies, for each $t \leq T < \infty$, $X, \bar{X} \in L^2(F_T)$,

(A1) $\mathbb{E}_t^g[X] \geq \mathbb{E}_{t,T}^g[X]$, a.s., if $X \geq \bar{X};$

(A2) $\mathbb{E}_{t,T}^g[X] = X;$

(A3) $\mathbb{E}_{s,T}^g[\mathbb{E}_{t,T}^g[X]] = \mathbb{E}_{s,T}^g[X]$; for $s \leq t;$

(A4) $\mathbb{I}_A \mathbb{E}_{s,T}^g[X] = \mathbb{E}_{s,T}^g[\mathbb{I}_A X]$, $\forall A \in F_t,$

where $\mathbb{I}_A$ is the indicator function of $A$, i.e., $\mathbb{I}_A(\omega)$ equals to 1, when $\omega \in A$ and 0 otherwise.

**Remark 2.5** (A1) and (A2) are economically obvious conditions for a pricing mechanism. Condition (A3) means that, at the time $s$, the random value $\mathbb{E}_{s,T}^g[X]$ can be regarded as a maturity value with maturity $t$. The price of this derivative at $s$ is $\mathbb{E}_{s,t}^g[\mathbb{E}_{t,T}^g[X]]$. It must be the same as the price $\mathbb{E}_{s,T}^g[X]$ of $X$ at $s$. 

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Remark 2.6 The meaning of condition (A4) is that, since at time \( t \), the agent knows the value of whether \( I_A \) is 1 or 0. When \( I_A = 1 \), then the price \( \mathbb{E}_{t,T}^{g_t}[1_A X] \) of \( 1_A X \) must be the same as \( \mathbb{E}_{t,T}^{g_t}[X] \), otherwise \( 1_A X = 0 \), so it worthies 0.

From the above results we see that \( \mathbb{E}^g \) is a good candidate to be a dynamic pricing mechanism. The following result provides a criteria to test if a dynamic pricing mechanism is a \( g \)-expectation. The proof can be found in [52].

Definition 2.7 A system of mappings \( (\mathbb{E}_{t,T}[\cdot])_{0 \leq t \leq T < \infty} \)

\[
\mathbb{E}_{t,T}[X] : X \in L^2(\mathcal{F}_T) \mapsto L^2(\mathcal{F}_t)
\]

is called a dynamic pricing mechanism of derivatives if it satisfies (A1)–(A4) (with \( \mathbb{E}[\cdot] \) in the place of \( \mathbb{E}^g[\cdot] \)).

Theorem 2.8 Let \( \mathbb{E}_{t,T}[\cdot]_{0 \leq t \leq T < \infty} \) be an dynamic pricing mechanism. If there exists a sufficiently large constant \( \mu > 0 \), such that the following domination criteria is satisfied

\[
(A5): \quad \mathbb{E}_{t,T}[X] - \mathbb{E}_{t,T}[^X] \leq \mathbb{E}_{t,T}^{g_{\mu}}[X - X].
\]

(2.5)

\( \mathbb{E}^{g_{\mu}} \) is a \( g \)-expectation with the generating function \( g_{\mu} \) defined by

\[
g_{\mu}(y, z) := \mu |y| + \mu |z|, \quad (y, z) \in \mathbb{R} \times \mathbb{R}^d. \tag{2.6}
\]

Then there exists a unique generating function \( g(\omega, t, y, z) \) satisfying (2.2) and (2.3)-(a) such that, for each \( t \leq T \) and for each derivative \( X \in L^2(\mathcal{F}_T) \), we have

\[
\mathbb{E}_{t,T}[X] = \mathbb{E}_{t,T}^{g_t}[X], \tag{2.7}
\]

namely \( \mathbb{E} \) is a \( g \)-expectation.

Remark 2.9 This theorem also implies that, for a generating function \( g \) satisfying (2.2) and (2.3)-(a), the corresponding \( g \)-expectation \( \mathbb{E}^g \) is also dominated by \( \mathbb{E}^{g_{\mu}} \), i.e., (A5) is satisfied. This can be also directly proved by using the comparison theorem of BSDE.

Remark 2.10 It turns out that the domination condition (2.5) becomes a crucial criteria to test whether a dynamic pricing mechanism of derivatives is a \( g \)-expectation. We provide a test in Appendix 4.2 to use market data to check the inequality (2.5).

Remark 2.11 This deep result has non-trivially generalized the main result of [13], theoretically and practically, where a special case \( g = g(t, z) \) with \( g(s, 0) \equiv 0 \) is considered. The \( g \)-expectation originally introduced in [45] corresponds such situation of “zero interest rate”. (cf. Proposition 3.8, or [50]).
Markovian pricing mechanisms

We limit ourselves to consider, for each fixed maturity $T$, the derivatives $X$ depending only on the price $S_T$, i.e., $X$ is a path independent derivative. $X$ is then in the class of $X = \Phi(S_T)$ with $\Phi \in L^2(S_T)$

where $L^2(S_T)$ denotes the collection of all real functions $\Phi$ defined on $\mathbb{R}^n$ such that $\Phi(S_T) \in L^2(F_T)$. A dynamic pricing mechanism $E$ is called a Markovian pricing mechanism if for each $0 \leq t \leq T < \infty$ and $\Phi \in L^2(S_T)$ there exists $\phi \in L^2(S_t)$ such that $E^t_T[\Phi(S_T)] = \phi(S_t)$. In other words, the price of a path-independent option by a Markovian pricing mechanism is still path-independent.

Example 2.12 We consider a situation where the underlying price $S$ is a diffusion process:

$$dS_t = b(S_t)dt + \Lambda(S_t)dB_t, \quad S_0 = s_0 \in \mathbb{R}^n.$$ 

where $b$ and $\Lambda$ are given Lipschitz functions of $\mathbb{R}^n$ valued on $\mathbb{R}^n$ and $\mathbb{R}^{n \times d}$ respectively. If a generating function $g$ has the following form:

$$g(t,y,z) = f(S_t,y,z),$$

where $f$ is a Lipschitz function of $(s,y,z) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d$. By the nonlinear Feynman–Kac formula introduced in [42, Peng 1991], [43, Peng 1992] and developed in [41, Pardoux-Peng 1992], for each option $X = \Phi(S_T)$ with smooth function $\Phi$ the price of the related $g$-expectation is

$$E^g_{t,T}[\Phi(S_T)] = u(t,S_t)$$

where $u : \mathbb{R}_+ \times \mathbb{R}^n \mapsto \mathbb{R}$ is the (viscosity) solution of the following PDE defined on $(t,s) \in [0,T] \times \mathbb{R}^n$:

$$\frac{\partial u}{\partial t} + \frac{1}{2} \sum_{i,j=1}^n (\Lambda(s)\Lambda^T(s))_{ij} \frac{\partial^2 u}{\partial s_i \partial s_j} + \sum_{i=1}^n b_i(s) \frac{\partial u}{\partial s_i} + f(s,u,\Lambda^T(s)\nabla u) = 0$$

with terminal condition $u(T,s) = \Phi(s)$. If $S_t$ is a 1-dimensional geometric Brownian motion, i.e., $\Lambda(s) = \sigma s$ and $b(s) = \mu s$, then the above PDE becomes

$$\frac{\partial u}{\partial t} + \frac{1}{2} \sigma^2 s^2 \frac{\partial^2 u}{\partial s^2} + \mu s \frac{\partial u}{\partial s} + f(s,u,\sigma^2 s \nabla u) = 0.$$ 

The Black–Scholes formula corresponds to the case $f = -ry - (\mu - r)s^{-1}z$. We then have

$$\frac{\partial u}{\partial t} + \frac{1}{2} \sigma^2 s^2 \frac{\partial^2 u}{\partial s^2} - ru + rz \frac{\partial u}{\partial s} = 0, \quad u(T,s) = \Phi(s).$$
3 Characterization of $g$-pricing mechanism by its generating function $g$

For a pricing mechanism, it is important to distinguish the selling price and buying price of a same pricing mechanism, corresponding to the ask price and the bid price if the mechanism under investigation is generated through the system of market makers of an option market (cf. [33] Sec. 6.5 and [39]). If $E_{t,T}[X]$ is the ask price at the time $t$ of a derivative $X$ with maturity $T$, then the bid price must be $-E_{t,T}[-X]$ and we have, in general,

$$E_{t,T}[X] > -E_{t,T}[-X].$$

Here we stress our point of view that, in fact, the ask price and bid price are produced by a single mechanism, called bid–ask pricing mechanism of market makers. Our result of data analysis to test the criteria (A5) of the domination condition (2.5) strongly supports this point of view. Moreover, this analysis also supports our point of view that, for a well–developed market, there exist a function $g$ satisfying Lipschitz condition (2.2) such that the corresponding ask-bid pricing mechanism is modeled by the $g$–expectation $E^g[\cdot]$.

A rational dynamic pricing mechanism also possesses some other important properties, such as convexity, sub-additivity. See [2], [3], [4], [11], [26], [27], [24], [29], [31], [46], [55], [35], [36] among many others. We will see that the generating function $g$ perfectly reflects the behavior of $E^g$. This may play an important role to statistically find $g$ by using the corresponding data of prices. In the following we provide several theoretical results with proofs given in Appendix. This problem was treated also by [55], [35] and [36].

**Proposition 3.1** Let $g$, $\bar{g} : (\omega, t, y, z) \in \Omega \times [0, \infty) \times R \times R^d \to R$ be two generating functions satisfying (2.2). Then the following two conditions are equivalent:

(i) $g(\omega, t, y, z) \geq \bar{g}(\omega, t, y, z), \forall (y, z) \in R \times R^d, dP \times dt$ a.s.

(ii) The corresponding $g$–pricing mechanisms $E^g[\cdot]$ and $E^{\bar{g}}[\cdot]$ satisfy

$$E^g_{t,T}[X] \geq E^{\bar{g}}_{t,T}[X], \forall 0 \leq t \leq T < \infty, \forall X \in L^2(F_T).$$

In particular, $E^g[X] \equiv E^{\bar{g}}[X]$ if and only if $g \equiv \bar{g}$.

**Corollary 3.2** The following two conditions are equivalent:

(i) The generating function $g$ satisfies, for each $(y, z) \in R \times R^d$,

$$g(t, y, z) \geq -g(t, -y, -z), \text{ a.e., a.s.},$$

(ii) $E^g_{t,T}[\cdot] : L^2(F_T) \hookrightarrow L^2(F_t)$ satisfies, for each $0 \leq t \leq T$,

$$E^g_{t,T}[X] \geq -E^g_{t,T}[-X], \quad X \in L^2(F_T).$$
Proof. We denote \( \bar{g}(t,y,z) := -g(t,-y,-z) \) and compare the following two BSDEs:

\[
Y_t = X + \int_t^T g(s,Y_s,Z_s)ds - \int_t^T Z_sdB_s, \ t \in [0,T],
\]

and

\[
\bar{Y}_t = -X + \int_t^T g(s,\bar{Y}_s,\bar{Z}_s)ds - \int_t^T \bar{Z}_sdB_s, \ t \in [0,T].
\]

or, with \( \bar{Y} = -\bar{Y}, \bar{Z} = -\bar{Z} \)

\[
\bar{Y}_t = X + \int_t^T g(s,\bar{Y}_s,\bar{Z}_s)ds - \int_t^T \bar{Z}_sdB_s, \ t \in [0,T].
\]

From the above Proposition it follows that \( \mathbb{E}_{t,T}^g[\cdot] \geq \mathbb{E}_{t,T}^{\bar{g}}[\cdot] \) iff \( g \geq \bar{g} \). This with \( \mathbb{E}_{t,T}^g[X] = -\mathbb{E}_{t,T}^{\bar{g}}[-X] \) yields (i) \( \Leftrightarrow \) (ii). \( \blacksquare \)

**Proposition 3.3** The following two conditions are equivalent:

(i) The generating function \( g = g(t,y,z) \) is convex (resp. concave) in \((y,z)\), i.e., for each \((y,z)\) and \((\bar{y},\bar{z})\) in \(R \times R^d\) and for a.e. \( t \in [0,T] \),

\[
g(t,\alpha y + (1-\alpha)\bar{y},\alpha z + (1-\alpha)\bar{z}) \leq \alpha g(t,y,z) + (1-\alpha)g(t,\bar{y},\bar{z}), \text{ a.s.}
\]

(resp. \( g(t,\alpha y + (1-\alpha)\bar{y},\alpha z + (1-\alpha)\bar{z}) \geq \alpha g(t,y,z) + (1-\alpha)g(t,\bar{y},\bar{z}), \text{ a.s.} \)).

(ii) The corresponding pricing mechanism \( (\mathbb{E}_{t,T}^g[\cdot])_{0 \leq t \leq T < \infty} \) is convex (resp. concave), i.e., for each fixed \( \alpha \in [0,1] \), we have

\[
\mathbb{E}_{t,T}^g[\alpha X + (1-\alpha)\bar{X}] \leq \alpha \mathbb{E}_{t,T}^g[X] + (1-\alpha)\mathbb{E}_{t,T}^g[\bar{X}], \text{ a.s.} \quad (3.1)
\]

(resp. \( \mathbb{E}_{t,T}^g[\alpha X + (1-\alpha)\bar{X}] \geq \alpha \mathbb{E}_{t,T}^g[X] + (1-\alpha)\mathbb{E}_{t,T}^g[\bar{X}], \text{ a.s.} \))

for each \( t \leq T \), and \( X, \bar{X} \in L^2(\mathcal{F}_T) \).

**Proposition 3.4** The following two conditions are equivalent:

(i) The generating function \( g \) is positively homogenous in \((y,z)\) in \(R \times R^d\), i.e.,

\[
g(t,\lambda y,\lambda z) = \lambda g(t,y,z), \text{ a.e., a.s.}
\]

(ii) The corresponding pricing mechanism \( \mathbb{E}_{t,T}^g[\cdot] : L^2(\mathcal{F}_T) \longrightarrow L^2(\mathcal{F}_t) \) is positively homogenous: for each \( 0 \leq t \leq T \), i.e., \( \mathbb{E}_{t,T}^g[\lambda X] = \lambda \mathbb{E}_{t,T}^g[X] \), for each \( \lambda \geq 0 \) and \( X \in L^2(\mathcal{F}_T) \).

From the above two propositions we immediately have

**Corollary 3.5** The following two conditions are equivalent:

(i) The generating function \( g \) is sub-additive: for each \((y,z)\), \((\bar{y},\bar{z})\) in \(R \times R^d\),

\[
g(\omega,t,y + \bar{y},z + \bar{z}) \leq g(\omega,t,y,z) + g(\omega,t,\bar{y},\bar{z}), \text{ dt } \times dP, \text{ a.s.}
\]

(ii) The corresponding pricing mechanism \( \mathbb{E}_{t,T}^g[\cdot] : L^2(\mathcal{F}_T) \longrightarrow L^2(\mathcal{F}_t) \) is sub-additive: for each \( 0 \leq t \leq T \) and \( X, \bar{X} \in L^2(\mathcal{F}_T) \)

\[
\mathbb{E}_{t,T}^g[X + \bar{X}] \leq \mathbb{E}_{t,T}^g[X] + \mathbb{E}_{t,T}^g[\bar{X}].
\]
Proposition 3.6 The generating function \( g \) is independent of \( y \) if and only if the corresponding \( g \)-expectation satisfies the following “cash translatability” property: for each \( t \leq T \),
\[
\mathbb{E}^g_{t,T}[X + \eta] = \mathbb{E}^g_{t,T}[X] + \eta, \text{ a.s., for each } X \in L^2(F_T), \eta \in L^2(F_t).
\]

We consider the following self-financing condition:
\[
\mathbb{E}^g_{t,T}[0] = 0, \forall 0 \leq t \leq T.
\]

Proposition 3.7 \( \mathbb{E}^g[\cdot] \) satisfies the self-financing condition if and only if its generating function \( g \) satisfies (2.3)-(a).

Proof. The “if” part is obvious. The “only if part”:
\[
Y_t := \mathbb{E}^g_{t,T}[0] \equiv 0, \text{ implies } Y_t \equiv 0 \equiv y + \int_t^T g(s, 0, Z_s)ds - \int_t^T Z_s dB_s, \ t \in [0, T].
\]
Thus \( Z_t \equiv 0 \) and then \( g(t, 0, Z_t) = g(t, 0, 0) \equiv 0. \)

“Zero-interest rate” condition:
\[
\mathbb{E}^g_{t,T}[\eta] = \eta, \forall 0 \leq t \leq T < \infty, \eta \in L^2(F_t).
\]

Proposition 3.8 \( \mathbb{E}^g[\cdot] \) satisfies the zero-interest rate condition if and only if its generating function \( g \) satisfies (2.3)-(b).

Proof. For a fixed \( y \in \mathbb{R} \), we consider \( Y_t := \mathbb{E}^g_{t,T}[y] \equiv y \). Let \( Z_t \) be the corresponding Itô’s integrand in \( Y \):
\[
Y_t = y + \int_t^T g(s, Y_s, Z_s) - \int_t^T Z_s dB_s = y.
\]
But this is equivalent to
\[
Y_t \equiv y, \ Z_s \equiv 0, \ g(s, y, 0) \equiv 0.
\]

For each \( \bar{z}^{i_0} \in L^2_T(0, T) \)
\[
\mathbb{E}^g_{t,T}[X] + \int_0^t \bar{z}^{i_0}_s dB^{i_0}_s = \mathbb{E}^g_{t,T}[X + \int_0^T \bar{z}^{i_0}_s dB^{i_0}_s] \tag{3.2}
\]

Proposition 3.9 Condition (3.2) holds if and only if \( g(t, y, z) \) does not depends on the \( i_0 \)-th component \( z^{i_0} \) of \( z \in \mathbb{R}^d \).

Proposition 3.10 The following condition are equivalent:

(i) For each \( 0 \leq t \leq T \) and \( X \in L^2(F_T) \), the \( g \)-pricing mechanism \( \mathbb{E}^g_{t,T}[X] \) is a deterministic number;

(ii) The corresponding pricing generating function \( g \) is a deterministic function of \( (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \).
The proof is similar as the others. We omit it.

**Example 3.11** An interesting problem is: if we know that a pricing mechanism under our investigation is a $g$–expectation $\mathbb{E}^g$, how to find the generating function $g$? If we limited ourselves to only take data of prices quoted by markets, this is still an open problem. We now consider a case of “toy model” where $g$ depends only on $z$, i.e., $g = g(z): \mathbb{R}^d \to \mathbb{R}$. We will find such $g$ by the following testing method. Let $\tilde{z} \in \mathbb{R}^d$ be given. We denote $Y_s := \mathbb{E}_{s,T}^g[\tilde{z} \cdot (B_T - B_t)]$, $s \in [t,T]$, where $t$ is the present time. It is the solution of the following BSDE

$$Y_s = \tilde{z} \cdot (B_T - B_t) + \int_s^T g(Z_u)du - \int_s^T Z_u dB_u, \quad s \in [t,T].$$

It is seen that the solution is $Y_s = \tilde{z} \cdot (B_s - B_t) + \int_s^T g(\tilde{z})ds$, $Z_s \equiv \tilde{z}$. Thus

$$\mathbb{E}_{t,T}^g[\tilde{z} \cdot (B_T - B_t)] = Y_t = g(\tilde{z})(T-t),$$

or

$$g(\tilde{z}) = (T-t)^{-1}\mathbb{E}_{t,T}^g[\tilde{z} \cdot (B_T - B_t)], \quad (3.3)$$

Thus the function $g$ can be tested as follows: at the present time $t$: if the valuation $\mathbb{E}_{t,T}^g[\tilde{z} \cdot (B_T - B_t)]$ of (a toy model of) derivative $\tilde{z} \cdot (B_T - B_t)$ is obtained, then $g(\tilde{z})$ is explicitly given by (3.3). We observe that, in the case where $S$ is a geometric Brownian motion, $B_T - B_t$ can be expressed as a function of $S_T/S_t$. But this cannot be applied to a general situation.

**Remark 3.12** The above test is also applied for the case $g: [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$, or for a more general situation $g = \gamma y + g_0(t, z)$.

An interesting problem is, in general, how to find the generating function $g$ by a testing of the input–output behavior of $\mathbb{E}^g[\cdot]$? Let $b: \mathbb{R}^n \to \mathbb{R}^n$, $\sigma: \mathbb{R}^n \to \mathbb{R}^{n \times d}$ be two Lipschitz functions. For each $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$, we consider the SDE of the form

$$X^{t,x}_s = x + \int_t^s b(X^{t,x}_u)du + \int_t^s \sigma(X^{t,x}_u)dB_u, \quad s \geq t.$$  

This SDE is regarded as the equation of the price of the underlying stock. The following result was obtained in Proposition 2.3 of [8].

**Proposition 3.13** We assume that the generating function $g$ satisfies (2.2). We also assume that, for each fixed $(y,z)$, $g(\cdot, y, z) \in D^2_{F_t}(0, T)$ (the space of all $\mathcal{F}_t$–adapted processes with RCLL paths). Then for each $(t,x,p,y) \in [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, we have

$$L^2 - \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathbb{E}_{t,t+\varepsilon}^g[y + p \cdot (X^{t,x}_{t+\varepsilon} - x)] - y = g(t, y, \sigma^T(x)p) + p \cdot b(x).$$
4 Appendix

4.1 Proofs of Propositions 3.1–3.10

We begin with introducing some technique lemmas. The first one is called decomposition theorem of \(\mathbb{E}^\mu\)-supermartingale. The proof can be find in [47] and [50].

**Proposition 4.1** We assume (2.2). Let \(Y \in D_T^2(0, T)\) be an \(\mathbb{E}^\mu\)-supermartingale, namely, for each \(0 \leq s \leq t \leq T\),

\[
\mathbb{E}^\mu_{s,t}[Y_t] \leq Y_s.
\]

Then there exists a unique \(\mathcal{F}_t\)-adapted increasing and RCLL process \(A \in D_T^2(0, T)\) (thus predictable) with \(A_0 = 0\), such that, \(Y\) is the solution of the following BSDE:

\[
Y_t = Y_T + (A_T - A_t) + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \quad t \in [0, T].
\]

Let a function \(f : (\omega, t, y, z) \in \Omega \times [0, T] \times R \times R^d \rightarrow R\) satisfy the same Lipschitz condition (2.2) as for \(g\). For each fixed \((t, y, z) \in [0, T] \times R \times R^d\), we consider the following SDE of Itô’s type defined on \([t, T]\):

\[
Y^{t, y, z}_s = y - \int_t^s f(r, Y^{t, y, z}_r, z)dr + z \cdot (B_s - B_t) \quad (4.1)
\]

We have the following classical result of Itô’s SDE.

**Lemma 4.2** We assume that \(f\) satisfies the same Lipschitz condition (2.2) as for \(g\). Then there exists a constant \(C\), depending only on \(\mu, T \) and \(E \int_0^T |f(\cdot, 0, 0)|^2 ds\), such that, for each \((t, y, z) \in [0, T] \times R \times R^d\), we have

\[
E[|Y^{t, y, z}_s - y|^2] \leq C(|y|^2 + |z|^2 + 1)(s - t), \quad \forall s \in [t, T]. \quad (4.2)
\]

**Proof.** It is classic that \(E \int_0^T |f(r, Y^{t, y, z}_r, z)|^2 dr \leq C_0(|y|^2 + |z|^2 + 1)\), where \(C_0\) depends only on \(\mu, T\) and \(E \int_0^T |f(\cdot, 0, 0)|^2 ds\). We then have

\[
E[|Y^{t, y, z}_s - y|^2] \leq 2E[\int_t^s |f(r, Y^{t, y, z}_r, z)|^2 dr] + 2|z|^2(s - t)
\]

\[
\leq 2E[\int_t^s |f(r, Y^{t, y, z}_r, z)|^2 dr]^{1/2}(t - s) + 2|z|^2(s - t)
\]

\[
\leq C(|y|^2 + |z|^2 + 1)(s - t).
\]

For each \(n = 1, 2, 3, \cdots\), we set

\[
f^n(s, y, z) := \sum_{i=0}^{2^n-1} f(s, Y_s^{t_i^n,y, z})1_{[t_i^n, t_{i+1}^n]}(s), \quad s \in [0, T] \quad (4.3)
\]

\[
t_i^n = i2^{-n}T, \quad i = 0, 1, 2, \cdots, 2^n. \quad (4.4)
\]

It is clear that \(f^n\) is an \(\mathcal{F}_t\)-adapted process.
Lemma 4.3 For each fixed \((y, z) \in R \times R^d\), \(\{f^n(\cdot, y, z)\}_{n=1}^{\infty}\) converges to \(f(\cdot, y, z)\) in \(L_2^2(0, T)\), i.e.,

\[
\lim_{n \to \infty} E \int_0^T |f^n(s, y, z) - f(s, y, z)|^2 ds = 0. \tag{4.5}
\]

Proof. For each \(s \in [0, T]\), there are some integers \(i \leq 2^n - 1\) such that \(s \in [t^n_i, t^n_{i+1})\). We have, by (4.2)

\[
E[|f^n(s, y, z) - f(s, y, z)|^2] = E[|f(s, Y^{t^n_i, y, z}_s, z) - f(s, y, z)|^2]
\leq \mu^2 E[|Y^{t^n_i, y, z}_s - y|^2]
\leq \mu^2 C(|y|^2 + |z|^2 + 1)2^{-n}T.
\]

Thus \(\{f^n(\cdot, y, z)\}_{n=1}^{\infty}\) converges to \(f(\cdot, y, z)\) in \(L_2^2(0, T)\). ■

Lemma 4.4 If for each \((t, y, z) \in [0, T] \times R \times R^d\), we have

\[
f(\omega, r, Y^{t, y, z}_r, z) \geq 0 \quad \text{(resp. } = 0\text{), } (\omega, r) \in [t, T] \times \Omega, \cdot \ dP \text{ a.s.}
\]

Then, for each \((y, z) \in R \times R^d\),

\[
f(\omega, t, y, z) \geq 0, \quad \text{(resp. } = 0\text{), } (\omega, t) \in [0, T] \times \Omega, \cdot \ dt \text{ a.s.}. \tag{4.6}
\]

Proof. Let us fix \(y\) and \(z\). We define \(f^n(s, y, z)\) as in (4.3). It is clear that,

\[
f^n(r, y, z) \geq 0, \quad \text{(resp. } = 0\text{), } (\omega, r) \in [0, T] \times \Omega, \cdot \ dr \text{ a.s.}
\]

But from Lemma 4.3 we have \(f^n(\cdot, y, z) \to f(\cdot, y, z)\), in \(L_2^2(0, T)\) as \(n \to \infty\). We thus have 4.6. ■

We now can give the proofs of several propositions given in the previous section. The method is very different from [55], [35] and [36] where Proposition 3.13 plays a central role.

Proof of Proposition 3.1 (i)\(\Rightarrow\)(ii) is the well–known comparison theorem of BSDE (cf. [43] and [24]).

(ii)\(\Rightarrow\)(i): For fixed \(t \geq 0\) and \((y, z)\) in \(R \times R^d\), let \(Y^{t, y, z}_r\) be the solution of SDE (4.1) with \(f = g\). From (ii) we have

\[
E^\xi_{r,s}[Y^{t, y, z}_s] \leq E^\xi_{r,s}[Y^{t, y, z}_r] = Y^{t, y, z}_s, \quad t \leq r \leq s.
\]

Thus \((Y^{t, y, z}_s)_{s \geq t}\) is an \(E^\xi\)–supermartingale. From the decomposition theorem, i.e., Proposition 4.1, it follows that there exists an increasing process \(\bar{A}_s\) such

\[
Y^{t, y, z}_s = y - \int_t^s \bar{g}(r, Y^{t, y, z}_r, \bar{Z}_r) dr - \bar{A}_s + \int_t^s \bar{Z}_r dB_r, \quad s \geq t.
\]

This with \(Y^{t, y, z}_r = y - \int_t^s \bar{g}(r, Y^{t, y, z}_r, z) dr + \int_t^s z dB_r\) yields \(\bar{Z}_s \equiv z\) and

\[
g(r, Y^{t, y, z}_r, z) \geq \bar{g}(r, Y^{t, y, z}_r, z), \quad r \geq t.
\]
We then can apply the above Lemma 4.4 to prove that $g \geq \bar{g}$. ■

**Proof of Proposition 3.3** We only prove the convex case.

(i)$\Rightarrow$(ii): For a given $t > 0$, we set $Y_s^X := \mathbb{E}_s^g[X]$, $Y_s^\bar{X} := \mathbb{E}_s^g[\bar{X}]$, $s \in [0, t]$. These two pricing processes solve respectively the following two BSDEs on $[0, t]$:

$$Y_s^X = X + \int_s^t g(r, Y_r^X, Z_r^X)dr - \int_s^t Z_r^X dB_r,$$

$$Y_s^{\bar{X}} = \bar{X} + \int_s^t g(r, Y_r^{\bar{X}}, Z_r^{\bar{X}})dr - \int_s^t Z_r^{\bar{X}} dB_r.$$

Their convex combination: $(Y_s, Z_s) := (\alpha Y_s^X + (1 - \alpha)Y_s^{\bar{X}}, \alpha Z_s^X + (1 - \alpha)Z_s^{\bar{X}})$, satisfies

$$Y_s = \alpha X + (1 - \alpha)\bar{X} + \int_s^t [g(r, Y_r, Z_r) + \psi_r]dr - \int_s^t Z_r dB_r,$$

where we set $\psi_r = \alpha g(r, Y_r^X, Z_r^X) + (1 - \alpha)g(r, Y_r^{\bar{X}}, Z_r^{\bar{X}}) - g(r, Y_r, Z_r)$.

But since the price generating function $g$ is convex in $(y, z)$, we have $\psi \geq 0$. It then follows from the comparison theorem that $Y_s \geq \mathbb{E}_s^g[\alpha X + (1 - \alpha)\bar{X}]$. We thus have (ii).

(ii)$\Rightarrow$(i): Let $Y^{t,y,z}$ be the solution of SDE (4.1) with $f = g$. For fixed $t \in [0, T]$ and $(y, z), (\bar{y}, \bar{z})$ in $R \times R^d$, we have

$$Y_s^{t,y,z} = \mathbb{E}_{r,s}^g[Y_{r,s}^{t,y,z}], \quad Y_s^{t,\bar{y},\bar{z}} = \mathbb{E}_{r,s}^g[Y_{r,s}^{t,\bar{y},\bar{z}}], \quad t \leq r \leq s.$$

We set $Y_s := \alpha Y_s^{t,y,z} + (1 - \alpha)Y_s^{t,\bar{y},\bar{z}}$, $s \in [t, T]$. By (3.1),

$$\mathbb{E}_{r,s}[Y_s] \leq \alpha \mathbb{E}_{r,s}[Y_s^{t,y,z}] + (1 - \alpha)\mathbb{E}_{r,s}[Y_s^{t,\bar{y},\bar{z}}]$$

$$= \alpha Y_r^{t,y,z} + (1 - \alpha)Y_r^{t,\bar{y},\bar{z}} = Y_r.$$

Thus the process $Y$ is a $\mathbb{E}^g$-supermartingale defined on $[t, T]$. It follows from the decomposition theorem, i.e., Proposition 4.1, that, there exists an increasing process $A$ such that

$$Y_s = Y_t - \int_t^s g(r, Y_r, Z_r)dr - A_s + \int_t^s Z_r dB_r.$$

We compare this with

$$Y_s = \alpha Y_s^{t,y,z} + (1 - \alpha)Y_s^{t,\bar{y},\bar{z}}$$

$$= \alpha y + (1 - \alpha)\bar{y} - \int_t^s [\alpha g(r, Y_r^{t,y,z}, z) + (1 - \alpha)g(r, Y_r^{t,\bar{y},\bar{z}}, \bar{z})]dr$$

$$+ (\alpha z + (1 - \alpha)\bar{z}) \cdot (B_s - B_t),$$

It follows that

$$Y_t = \alpha y + (1 - \alpha)\bar{y}, \quad Z_t \equiv \alpha z + (1 - \alpha)\bar{z},$$

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Thus we have
\[ g(s, \alpha Y_s^{t,y,z} + (1-\alpha)Y_s^{t,y,z}, \alpha z + (1-\alpha)z) \leq \alpha g(s, Y_s^{t,y,z}, z) + (1-\alpha)g(s, Y_s^{t,y,z}, z). \]

We then can apply Lemma 4.4 to obtain (i).

**Proof of Proposition 3.4.** (i)⇒(ii) is easy.

(ii)⇒(i): Let \( Y^{t,y,z} \) be the solution of SDE (4.1) with \( f = g \). For fixed \( t \in [0, T) \) and \( (y, z) \) in \( R \times R^d \), we have \( \lambda Y^{t,y,z}_s = E_s^g[\lambda Y^{t,y,z}_T], \) \( s \in [t, T] \). This implies that, there exists a process \( Z^{t,y,z,:} \) such that
\[
\lambda Y^{t,y,z}_s = \lambda y - \int_t^s g(r, \lambda Y^{t,y,z}_r, Z^{t,y,z,:}_r)dr + \int_t^s Z^{t,y,z,:}_r dB_r, \quad s \in [t, T].
\]

Compare this with \( \lambda Y^{t,y,z}_s = \lambda y - \int_t^s \lambda g(r, Y^{t,y,z}_r, z)dr + \int_t^s \lambda zdB_r \), it follows that \( Z^{t,y,z,:} \equiv \lambda z \) and \( \lambda g(r, Y^{t,y,z}_r, z) \equiv g(r, \lambda Y^{t,y,z}_r, z), \) \( r \in [t, T] \). We then can apply Lemma 4.4 to obtain (i).

**Proof of Proposition 3.6** We first prove the “If” part. For each \( (y, z) \) ∈ \( R \times R^d \), we have \( Y^{t,y,z}_s \equiv E_s^g[Y^{t,y,z}_T] = y + E_s^g[Y^{t,y,z}_T - y] \). Let \( \tilde{Y}_s = E_s^g[Y^{t,y,z}_T - y], \) \( s \in [0, T] \) and \( \tilde{Z} \) be the corresponding part of Itô’s integrand. By \( \tilde{Y}_r \equiv y + Y^{t,y,z}_T \) it follows that
\[
y + Y_s = y + Y^{t,y,z}_T + \int_s^T g(r, Y^{t,y,z}_r, z) - \int_s^T zdB_r
\]
\[
= (y + Y^{t,y,z}_T) + \int_s^T g(r, \tilde{Y}_r, \tilde{Z}_r) - \int_s^T \tilde{Z}_rdB_r.
\]

Thus \( \tilde{Z}_r \equiv z \) and
\[
g(r, Y^{t,y,z}_r, z) \equiv g(r, Y^{t,y,z}_r - y, \tilde{Z}_r) \equiv g(r, Y^{t,y,z}_r - y, z).
\]

We then can apply Lemma 4.4 to obtain that, for each \( (y, z) \) ∈ \( R \times R^d \),
\[
g(r, y, z) \equiv g(r, y - y, z) \equiv g(r, 0, z).
\]

Namely, \( g \) is independent of \( y \).

“Only if part”: For each for each \( s \leq t \) and \( X \in L^2(\mathcal{F}_t), \eta \in L^2(\mathcal{F}_s) \), we have
\[
Y_r := E_s^g[X + \eta] = X + \eta + \int_r^t g(u, Z_u)du - \int_s^t Z_u dB_u, \quad r \in [s, t].
\]

Thus \( \tilde{Y}_r := Y_r - \eta \) is a \( g \)-solution on \([s, t]\) with terminal condition \( \tilde{Y}_t = X + \eta \).

This implies
\[
E_s^g[X] + \eta = \tilde{Y}_s = E_s^g[X + \eta].
\]

The proof is complete.
Proof of Proposition 3.9 The “if” part: Since process \( Y_t := E_{g}^T[X] \) solves the following BSDE

\[
Y_t = X + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s,
\]

we have

\[
Y_t + \int_0^t \bar{z}_i^0 dB_i^0 = X + \int_0^t \bar{z}_i^0 dB_i^0 + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T \bar{Z}_s d\bar{B}_s,
\]

where

\[
\bar{Z}_s = \left( Z_s^1, \ldots, Z_s^{i_0-1}, Z_s^{i_0} + \bar{z}_i^0, Z_s^{i_0+1}, \ldots, Z_s^d \right).
\]

But since \( g(s, y, z) \) does not depend the \( i_0 \)-th component of \( z \in \mathbb{R}^d \), we thus have \( g(s, Y_s, Z_s) \equiv g(s, Y_s, \bar{Z}_s) \). Thus

\[
Y_t + \int_0^t \bar{z}_i^0 dB_i^0 = X + \int_0^t \bar{z}_i^0 dB_i^0 + \int_t^T g(s, Y_s, \bar{Z}_s)ds - \int_t^T \bar{Z}_s d\bar{B}_s.
\]

This means that (3.2) holds.

The “only if” part: For each fixed \((t, y, z)\), let \((Y_{s}^{t,y,z})_{s\geq t}\) be the solution of (4.1) with \( f = g \). We have,

\[
E_{s,T}[Y_{T}^{t,y,z}] - z_i^0 B_{s}^{i_0} = E_{s,T}[Y_{T}^{t,y,z} - z_i^0 B_{T}^{i_0}], \ s \in [t, T].
\]

Since the process \( Y_r := E_{s,T}[Y_{T}^{t,y,z} - z_i^0 B_{T}^{i_0}], r \in [t, s] \), solves the BSDE

\[
Y_{s}^{t,y,z} - z_i^0 B_{s}^{i_0} = Y_s = Y_{T}^{t,y,z} + z_i^0 B_{T}^{i_0} + \int_s^T g(r, Y_r, Z_r)ds - \int_s^T Z_r dB_r.
\]

From which we deduce \( Z_s = \bar{z} := (z_1, \ldots, z_i^{i_0-1}, 0, z_i^{i_0+1}, \ldots, z^d) \) is a solution of (3.2) and thus

\[
g(r, Y_r, Z_r) = g(r, Y_r^{t,y,z}, \bar{z}) = g(r, Y_r^{t,y,z}, z), \ 0 \leq t \leq r \leq T.
\]

It then follows from Lemma 4.4 that

\[
g(t, y, \bar{z}) = g(t, y, z), \ t \geq 0, \ a.e., \ a.s.,
\]

i.e., \( g \) does not depend the \( i_0 \)-th component of \( z \in \mathbb{R}^d \). ■

4.2 Testing the criteria (A5) by market data

With Chen L. and Sun P. of our research group, we proceed a data test for the criteria (A5), i.e., the domination inequality (2.5), to check if a specific pricing mechanism is a \( g \)-expectation, or \( g \)-pricing mechanism \( E_{g}^T \).

We have firstly tested the CME (Chicago Mercantile Exchange)'s market pricing mechanism of derivatives by taking the daily closing prices of options with S&P500 index futures as the underlying asset. The data is obtained from
Among those 12 cases of violations, 5 are singular situations since the total 6,200,828 inequalities, only 17 are against the criteria (4.7). We denote by \( X^i_T = (S_T - k_i)^+ \) (resp. \( Y^i_T = (S_T - k_i)^- \)), the market maturity value of the call (resp. put) option with maturity \( T \) and strike price \( k_i \). The corresponding values of the short positions are \(-X^i_T\) and \(-Y^i_T\). We denote the market price of the corresponding prices of options at time \( t < T \) by \( E^m_{t,T}[X^i_T], E^m_{t,T}[Y^i_T] \) and \( E^m_{t,T}[-X^i_T] \) and \( E^m_{t,T}[-Y^i_T] \) respectively. The inequalities we need to put to the test are, according to (2.5), in the following different combinations, with different \((t,T)\) and different strike prices

\[
\begin{align*}
\text{Call–Call:} & \quad E^m_{t,T}[X^i_T] - E^m_{t,T}[X^j_T] \leq E^{g_u}_{t,T}[X^i_T - X^j_T] \\
\text{Put–Put:} & \quad E^m_{t,T}[Y^i_T] - E^m_{t,T}[Y^j_T] \leq E^{g_u}_{t,T}[Y^i_T - Y^j_T] \\
\text{Call–Put:} & \quad E^m_{t,T}[X^i_T] - E^m_{t,T}[Y^j_T] \leq E^{g_u}_{t,T}[X^i_T - Y^j_T] \\
\text{Put–Call:} & \quad E^m_{t,T}[Y^i_T] - E^m_{t,T}[X^j_T] \leq E^{g_u}_{t,T}[Y^i_T - X^j_T]
\end{align*}
\tag{4.7}
\]

and

\[
\begin{align*}
\text{Call–ShortCall:} & \quad E^m_{t,T}[X^i_T] - E^m_{t,T}[-X^j_T] \leq E^{g_u}_{t,T}[X^i_T + X^j_T] \\
\text{Put–ShortPut:} & \quad E^m_{t,T}[Y^i_T] - E^m_{t,T}[-Y^j_T] \leq E^{g_u}_{t,T}[Y^i_T + Y^j_T] \\
\text{Call–ShortPut:} & \quad E^m_{t,T}[X^i_T] - E^m_{t,T}[-Y^j_T] \leq E^{g_u}_{t,T}[X^i_T + Y^j_T] \\
\text{Put–ShortCall:} & \quad E^m_{t,T}[Y^i_T] - E^m_{t,T}[-X^j_T] \leq E^{g_u}_{t,T}[Y^i_T + X^j_T]
\end{align*}
\tag{4.8}
\]

In the above inequalities the data of the left hand sides is the market prices of options taken from CME parameter files. In our testing the transaction cost is neglected, i.e., we assume that \( E^m_{t,T}[-X] = -E^m_{t,T}[X] \). The right hand sides is the corresponding values of \( g_u \)-expectations. We fix \( \mu = 25 \) uniformly for all tested inequalities. We have calculated all these values on the right hand side by using standard binomial tree algorithm of BSDE. Here an improved version of the algorithms of BSDE proposed Peng and Xu [2005] has been applied to solve the following 1-dimensional BSDE:

\[
y_t = X + \int_t^T \mu(|y_s| + |z_s|)ds - \int_t^T z_s dB_s \tag{4.9}
\]

with different terminal values \( y_T = X^i_T - X^j_T, Y^i_T - Y^j_T, X^i_T - Y^j_T, Y^i_T - X^j_T, X^i_T + X^j_T, Y^i_T + Y^j_T, X^i_T + Y^j_T \), respectively. The closing prices of S&P500 futures options of 69 trading days from year 2000 to 2003 have been put in the test. With the above mentioned combinations, we have tested a total number of 6,200,828 inequalities of (4.7) and (4.8). This means that our BSDE (4.9) have been calculated 6,200,828 times. A very positive result was obtained: among the totally 6,200,828 tested inequalities, only 17 are against the criteria (4.7). Among those 12 cases of violations, 5 are singular situation since they themselves
all violate Axiomatic monotonicity condition (A1). 5 cases are all from the same file cme0701s.par, 2003, Put–Put. They are all the following singular cases:

\[ E_{t,T}^m[(S_T - k_i)^-] > E_{t,T}^m[(S_T - k_j)^-], \quad \text{for } k_i > k_j. \]

The other 12 violations are the cases where the time \( T - t \) is too short (less than 2 days).

Since we have not found available data of bid-ask prices of the above options from CME, we then have tested the bid-ask pricing mechanism of S&P500 index options operated by the system of market makers of CBOE The data source is from Yahoo’s finance quotes of the option prices from 07 December to 08 May 2006. We have collected the prices of 5,000 time points, i.e., 5,000 different \( t \) among 100 trading days. We denote this pricing mechanism by \( E_{t,T}^{mm}[X] \) for the ask price of an option \( X \). According to our point of view the bid price of the same \( X \) is \(-E_{t,T}^{mm}[-X]\) and thus the bid–ask spread is \( E_{t,T}^{mm}[X] + E_{t,T}^{mm}[-X] \). We have tested a total number of 589,360 inequalities of (4.7) and (4.8), with \( E^{mm} \) in the place of \( E^m \). Only 1 case of violation appears.

We will report these test results in details in our forthcoming paper. [9].

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