SOME EXPLICIT EXPRESSIONS CONCERNING FORMAL GROUP LAWS

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Abstract. This paper provides some explicit expressions concerning the formal group laws of the following cohomology theories: BP, The Brown-Peterson cohomology, \( G(s) \), the cohomology theory obtained from \( BP \) by putting \( v_i = 0 \) for all \( i \geq 1 \) with \( i \neq s \), the Morava \( K \)-theory and the Abel cohomology. The coefficient ring of \( p \)-typization of the universal Abel formal group law is computed in low dimensions. One consequence of the latter is that even though the Abel formal group laws come from Baas-Sullivan theory, the corresponding \( p \)-typical version does not.

1. Introduction

Let us start with some necessary definitions. The main reference for formal group laws is [4], see also [15].

A formal group law over a commutative ring with unit \( R \) is a power series \( F(x, y) \in R[[x, y]] \) satisfying

(i) \( F(x, 0) = F(0, x) = x \),
(ii) \( F(x, y) = F(y, x) \),
(iii) \( F(x, F(y, z)) = F(F(x, y), z) \).

Let \( F \) and \( G \) be formal group laws. A homomorphism from \( F \) to \( G \) is a power series \( \nu(x) \in R[[x]] \) with constant term 0 such that

\[ \nu(F(x, y)) = G(\nu(x), \nu(y)). \]

It is an isomorphism if \( \nu'(0) \) (the coefficient at \( x \)) is a unit in \( R \), and a strict isomorphism if the coefficient at \( x \) is 1.

If \( F \) is a formal group law over a commutative \( \mathbb{Q} \)-algebra \( R \), then it is strictly isomorphic to the additive formal group law \( x + y \). In other words, there is a strict isomorphism \( l(x) \) from \( F \) to the additive formal group law, called the logarithm of \( F \), so that \( F(x, y) = l^{-1}(l(x) + l(y)) \). The inverse to logarithm is called the exponential of \( F \).

The logarithm \( l(x) \in R \otimes \mathbb{Q}[[x]] \) of a formal group law \( F \) is given by

\[ l(x) = \int_0^x \frac{dt}{\omega(t)}, \quad \omega(x) = \frac{\partial F(x, y)}{\partial y}(x, 0). \]

There is a ring \( L \), called the universal Lazard ring, and a universal formal group law \( F(x, y) = \sum a_{ij} x^i y^j \) defined over \( L \). This means that for any formal group law \( G \) over any commutative ring with unit \( R \) there is a unique ring homomorphism \( r : L \to R \) such that \( G(x, y) = \sum r(a_{ij}) x^i y^j \).

The formal group law of geometric cobordism was introduced in [13]. Following Quillen we will identify it with the universal Lazard formal group law as it is proved in [14] that the coefficient ring of complex cobordism \( MU_* = \mathbb{Z}[x_1, x_2, ...], |x_i| = 2i \) is naturally isomorphic as a graded ring to the universal Lazard ring.

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For a power series of the form $m(x) = x + m_1x^2 + m_2x^3 + \ldots$, its composition inverse $e(x) = x + e_1x^2 + e_2x^3 + \ldots$ is given by

$$
e_n = \sum_{k_1, k_2, \ldots, \geq 0, k_1+2k_2+3k_3+\ldots=n} (-1)^{k_1+k_2+\ldots} \frac{(n+k_1+k_2+\ldots)!}{(n+1)!k_1!k_2!\ldots} m_1^{k_1}m_2^{k_2}\ldots
$$

Coefficients of the corresponding formal group $F(x, y) = e(m(x) + m(y)) = x + y + \alpha_{11}xy + \ldots$ are thus given by

$$
\alpha_{ij} = \sum_{\nu_1+2\nu_2+\ldots=i+j-1} \sum_{i_0+2i_1+3i_2+\ldots=i, j_0+2j_1+3j_2+\ldots=j, i_1+j_1+k_1=\nu_1, \nu_2+j_2+k_2=\nu_2, \ldots} (-1)^{k_1+k_2+\ldots} \frac{(i_0+j_0+k_1+i_1+j_1+k_2+\ldots-1)!}{i_0!j_0!k_1!i_1!j_1!k_2!\ldots} m_1^{\nu_1}m_2^{\nu_2}\ldots
$$

The rest of the paper is organized as follows. In Section 2 we give some explicit expressions concerning universal $p$-typical formal group law $F_{BP}$. In Section 3 we consider the Morava case $F_{G(s)}$. Section 4 is devoted to Abel universal formal group law $F_{Ab}$. Finally in Section 5 we consider $p$-typization of $F_{Ab}$ and compute its coefficient ring in low dimensions. Our computation imply that even though $F_{Ab}$ come from Baas-Sullivan theory, the corresponding $p$-typical version does not. The main reference for the last result is the original article by Baas [1] and for bordism theory the book by Stong [18]. Sullivan defined a similar notion of singular manifolds in his work on the Hauptvermutung [19].

2. Formal group law in Brown-Peterson cohomology

Denote by $F_{BP}(x, y) = \sum \alpha_{ij}x^iy^j$ the formal group law of Brown-Peterson cohomology $BP$ [8] and let $\log_{BP}(x) = x + l_1x^2 + l_2x^3 + \ldots$ be the logarithm of $F_{BP}$. The first choice of the generators of $BP_* = \mathbb{Z}(p)[v_1, v_2, \ldots]$, $|v_i| = 2(p^n - 1)$ was given by Hazewinkel [10]. The coefficients $l_i \in BP_* \otimes \mathbb{Q}$ are related to the Hazewinkel generators $v_i \in BP_* \leftrightarrow BP_* \otimes \mathbb{Q}$ through the recursive equation [10] [15]

$$pl_n = v_n + v_{n-1}^p l_1 + v_{n-2}^p l_2 + \ldots + v_1^{p^{n-1}} l_{n-1}.
$$

The following is easily checked by explicit computation:

**Proposition 2.1.** An explicit solution of (2.1), i.e. an expression of $l_n$ through the $v_k$ is given by

$$l_n = \sum_{k=1}^{n} \sum_{n_1, \ldots, n_k \geq 0, n_1+\ldots+n_k = n} v_{n_1}v_{n_2}^p v_{n_3}^{p+n_2} \ldots v_{n_k}^{p+n_1+\ldots+n_k - 1} / p^k.
$$

Thus there are on the whole $2^{n-1}$ summands for $l_n$. For example,

$$l_4 = v_4/p + v_1 v_3^p / p^2 + 2 v_2 v_3^p / p^2 + 3 v_1 v_2 v_3^p / p^3 + v_1^2 v_3^p / p^3 + 2 v_1 v_2^p v_3^p / p^3 + v_1 v_2^{p^2} v_3^p / p^3 + v_1 v_2 v_3 v_4^p / p^4 + v_1^2 v_2 v_3 v_4^p / p^4 + v_1^3 v_2 v_3 v_4^p / p^4 + v_1^4 v_2 v_3 v_4^p / p^4.
$$
Proposition 2.2. The coefficient $\alpha_{ij}$ of the formal group law $F_{BP}$ at $x^iy^j$ is given by

$$\alpha_{ij} = \sum_{(p-1)(v_1+p^{-1}v_2+p^{-2}v_3+\ldots)=i+j-1} \sum_{i_0+j_0+i_1+j_1+k_1+i_2+j_2+k_2+\ldots=1} (-1)^{k_1+k_2+\ldots} \frac{(i_0+j_0+i_1+j_1+k_1+i_2+j_2+k_2+\ldots-1)!}{i_0!j_0!i_1!j_1!k_1!i_2!j_2!k_2!\ldots} \nu_1\nu_2\nu_3\ldots;$$

in particular, $\alpha_{ij}$ is nonzero only when $i+j-1$ is a multiple of $p-1$.

Proof. Taking in (1.1) $l(x) = \log_{BP}(x)$ with $m_{p,k-1} = l_k$ and all other $m_i$ equal to zero, we obtain the desired expression. $\square$

It follows that for any $0 < k < p$ the coefficient of $\alpha_{k(p^{-k})p^n}$ at $l_{n+1}$ is equal to $-\left(\frac{p^{n+1}}{k_p^n}\right)$. Moreover we know from (2.1) that $l_{n+1} = \frac{1}{p}v_{n+1}$ + decomposables. Hence we have

$$v_{n+1} = -\frac{p}{(p^{n+1})} \alpha_{k(p^{-k})p^n} + \text{decomposables}.$$

The binomial coefficient $\left(\frac{p^{n+1}}{k_p^n}\right)$ is divisible by $p$ and not divisible by $p^2$: Recall that for any $m$ the $p$-adic valuation of $m$! (i.e. the largest power of $p$ dividing $m!$) is given by

$$\left\lfloor \frac{m}{p} \right\rfloor + \left\lfloor \frac{m}{p^2} \right\rfloor + \left\lfloor \frac{m}{p^3} \right\rfloor + \cdots,$$

where $[x]$ denotes the integer part of $x$. Using this respectively for $m = p^{n+1}$, $m = k_p^n$ and $m = (p-k)p^n$ gives the $p$-adic valuation of $\left(\frac{p^{n+1}}{k_p^n}\right) = \frac{p^{n+1}}{(k_p^n)!((p-k)p^n)!}$ as follows

$$\left\lfloor \frac{p^{n+1}}{p} \right\rfloor + \left\lfloor \frac{p^{n+1}}{p^2} \right\rfloor + \left\lfloor \frac{p^{n+1}}{p^3} \right\rfloor + \cdots + \left\lfloor \frac{p^{n+1}}{p^{n+1}} \right\rfloor - \left\lfloor \frac{k_p^n}{p} \right\rfloor - \left\lfloor \frac{k_p^n}{p^2} \right\rfloor - \left\lfloor \frac{k_p^n}{p^3} \right\rfloor - \cdots - \left\lfloor \frac{k_p^n}{p^n} \right\rfloor - \left\lfloor \frac{(p-k)p^n}{p} \right\rfloor - \left\lfloor \frac{(p-k)p^n}{p^2} \right\rfloor - \left\lfloor \frac{(p-k)p^n}{p^3} \right\rfloor - \cdots - \left\lfloor \frac{(p-k)p^n}{p^n} \right\rfloor = 1$$

Therefore the coefficient $\frac{p}{(k_p^n)}$ belongs to $\mathbb{Z}_p$.

As an application one can construct particular polynomial generators for $BP_*$ as follows.

Corollary 2.3. One can construct polynomial generators for $BP_*$ from coefficients $\alpha_{ij}$ of the universal $p$-typical formal group law

$$BP_* \cong \mathbb{Z}_p[\alpha_{k_0p^{-k_0}}, \alpha_{k_1p^{-k_1}}p, \alpha_{k_2p^{-k_2}}(p-k_2)p^2, \cdots]$$

for any $0 < k_n < p$, $n = 0, 1, 2, \cdots$.

Here are some examples of expression of the Hazewinkel generators through the elements $\alpha_{k(p^{-k})p^n}$:

- for $p = 2$,

  $$v_1 = -\alpha_{11},$$
  $$v_2 = -\frac{1}{3}\alpha_{22} + \frac{4}{3}\alpha_{21}^3,$$
  $$v_3 = -\frac{1}{35}\alpha_{44} + \frac{302}{315}\alpha_{41}\alpha_{22} - \frac{170}{63}\alpha_{11}^7,$$
- for $p = 3$,

\[
v_1 = -\alpha_{12},
\]
\[
v_2 = -\frac{1}{28}\alpha_{36} + \frac{27}{28}\alpha_{12},
\]
\[
v_3 = -\frac{1}{1562275}\alpha_{9,18} + \frac{90115407}{17147530400}\alpha_{12\alpha_{36}}^{5} + \frac{2781961973}{17147530400}\alpha_{12\alpha_{36}}^{3}
\]
\[
- \frac{32851611811}{3429506080}\alpha_{12\alpha_{36}}^{9} - \frac{20612623337247}{17147530400}\alpha_{12}^{13},
\]

etc.

3. THE MORAVA CASE

To apply the above formulæ to calculation of the Morava $K$-theories, consider the theories $G(s)$ obtained from $BP$ by putting $v_i = 0$ for all $i \geq 1$ with $i \neq s$. Since for $G(s)$ theory $v_s$ plays the rôle of just a bookkeeping variable, let us drop it, i. e. put $v_s = 1$. Thus the logarithm of $G(s)$ can be just written as

\[
x + \frac{x^{p^r}}{p} + \frac{x^{p^{2s}}}{p^2} + \frac{x^{p^{3s}}}{p^3} + \cdots.
\]

Thus taking in Proposition $2.2$ $l_{ks} = 1/p^k$ and $l_i = 0$ otherwise, we readily obtain a formula for the coefficients.

**Proposition 3.1.** The coefficient $\alpha_{ij}$ of the formal group law $F_{G(s)}$ at $x^iy^j$ is given by

\[
\alpha_{ij} = \sum_{(p^r-1)(v_1 + \frac{x^{p^r-1}}{p} + v_2 + \frac{x^{p^r-1}}{p} + v_3 + \cdots) = i + j - 1} \sum_{i_0 + p^{r}i_1 + p^{2s}i_2 + \cdots = i, j_0 + p^{r}j_1 + p^{2s}j_2 + \cdots = j, i_1 + j_1 + k_1 = i, i_2 + j_2 + k_2 = v_2, \cdots} (-1)^{k_1 + k_2 + \cdots} \frac{(i_0 + j_0 + i_1 + j_1 + k_1 + i_2 + j_2 + k_2 + \cdots + 1)!}{i_0! j_0! i_1! j_1! k_1! i_2! j_2! k_2! \cdots p! + 2! \cdots}.
\]

Another convenient form of this formula is

\[
\alpha_{i,k(p^r-1)+1-i} = \sum_{i_0 + p^{r}i_1 + p^{2s}i_2 + \cdots = i, j_0 + p^{r}j_1 + p^{2s}j_2 + \cdots = k(p^r-1)+1-i, k_1 + \frac{x^{p^r-1}}{p}k_2 + \frac{x^{2s}}{p}k_3 + \cdots = k-i_1 - j_1 - \frac{x^{p^r-1}}{p}(i_2 + j_2) - \frac{x^{2s}}{p}(i_3 + j_3) - \cdots} (-1)^{k_1 + k_2 + \cdots} \frac{(i_0 + j_0 + i_1 + j_1 + k_1 + i_2 + j_2 + k_2 + \cdots + 1)!}{i_0! j_0! i_1! j_1! k_1! i_2! j_2! k_2! \cdots p! + 1 + 2 + \cdots}.
\]

All other $\alpha_{ij}$ being zero.

Morava $K$-theory formal group is then reduction of the above modulo $p$.

Another way of computing the latter formal group is via the Ravenel recursive relation [15] involving Witt symmetric functions,

\[
F(x_1, x_2, \cdots) = F(W^{(1)}(x_1), W^{(p)}(x_1)p^{r-1}, W^{(p^2)}(x_1)p^{2(r-1)}, \cdots)
\]

where $W^{(n)}$ are the symmetric functions defined by

\[
\frac{\sum x^n}{n} = \sum_{d | n} W^{(d)}(x_1)^d.
\]
for all \( n \). For example, with two variables one has
\[
\begin{align*}
W^{(1)} &= x + y, \\
W^{(2)} &= -xy, \\
W^{(3)} &= -(x^2y + xy^2), \\
W^{(4)} &= -(x^3y + 2x^2y^2 + xy^3), \\
W^{(5)} &= -(x^4y + 2x^3y^2 + 2x^2y^3 + xy^4), \\
W^{(6)} &= -(x^5y + 3x^4y^2 + 4x^3y^3 + 3x^2y^4 + xy^5), \\
&\cdots
\end{align*}
\]

Some examples of how this recursion actually works: for given \( p \) and \( s \) let us denote \( w_k = W^{(p^k)}(x, y) \); then

For \( p = 2, s = 1 \)
\[
F(x, y) = w_0(= x + y) + w_1(= xy) + w_0w_1(= x^2y + xy^2) + (w_2 + w_0^2w_1)(= 0)
+ (w_0w_2 + w_0^2w_1)(= x^4y + xy^4) + (w_1w_2 + w_0^4w_1)(= x^4y^2 + x^2y^4)
+ w_0w_1w_2(= x^5y^2 + x^4y^3 + x^3y^4 + x^2y^5) + \cdots;
\]

For \( p = 2, s = 2 \)
\[
F(x, y) = w_0(= x + y) + w_1(= x^2y^2) + w_0^2w_1(= x^6y^4 + x^4y^6) + w_2(= x^{12}y^4 + x^4y^{12})
+ w_0^6w_1^4(= x^{14}y^8 + x^{12}y^{10} + x^{10}y^{12} + x^8y^{14}) + (w_4w_1^6 + w_0^{12}w_1^4)(= x^{20}y^8 + x^8y^{20}) + \cdots;
\]

For \( p = 2, s = 3 \)
\[
F(x, y) = w_0(= x + y) + w_1(= x^4y^4) + w_0^4w_1^4(= x^{20}y^{16} + x^{16}y^{20}) + w_2(= x^{48}y^{16} + x^{16}y^{48})
+ w_0^{48}w_1^{16}(= x^{112}y^{64} + x^{96}y^{80} + x^{80}y^{96} + x^{64}y^{112}) + \cdots;
\]

For \( p = 3, s = 1 \)
\[
F(x, y) = w_0(= x + y) + w_1(= -x^2y - xy^2) + (-w_0^2w_1)(= -x^4y - xy^4)
+ (-w_0w_2 + w_0^4w_1)(= -x^6y - x^4y^3 - x^3y^4 - xy^6) + \cdots;
\]

For \( p = 3, s = 2 \)
\[
F(x, y) = w_0(= x + y) + w_1(= -x^6y^3 - x^3y^6)
+ (-w_0^6w_1^3)(= x^{24}y^9 - x^{21}y^{12} + x^{18}y^{15} + x^{15}y^{18} - x^{12}y^{21} + x^9y^{24}) + \cdots
\]

Let us give the following two approximations to the formal group law of Morava \( K \)-theory.

**Proposition 3.2.** (see [2]) For the formal group law in mod \( p \) Morava \( K \)-theory \( K^*(s) \) at prime \( p \) and \( s > 1 \) we have
\[
F(x, y) \equiv x + y - v_s \sum_{0 < j < p} p^{-1}(\binom{p}{j})(x^{p^{s-j}})^j (y^{p^{s-1}})^{p-j}
\]
modulo \( x^{p^{2(s-1)}} \) (or modulo \( y^{p^{2(s-1)}} \)).

**Proof.** As above it is convenient to put \( v_s = 1 \) in the formal group law
\[
F(x, y) = F(x + y, v_sW^{(p)}(x, y)^{p^{s-1}}, v_s^2W^{(p^2)}(x, y)^{p^{2(s-1)}}, \ldots),
\]
where \( W(p) \) is the homogeneous polynomial of degree \( p^i \) defined above and \( e_i = (p^i - 1)/(p^i - 1) \). In particular \( W^{(1)} = x + y \),
\[
W(p) = - \sum_{0 < j < p} p^{-1} \binom{p}{j} x^j y^{p-j},
\]
and \( W(p) \notin (x^p, y^p) \).

Then for \( s > 1 \) we can reduce modulo the ideal \( (x^{2^{(s-1)}}, y^{2^{(s-1)}}) \) and get
\[
F(x, y) = F(x + y, W(p)(x, y)^{p^{s-1}}) = F(x + y + W(p)(x, y)^{p^{s-1}}, W(p)(x + y, W(p)(x, y)^{p^{s-1}}, \ldots)
\]
and modulo \( (x^{2^{(s-1)}}, y^{2^{(s-1)}}) \) we have
\[
F(x, y) \equiv x + y + W(p)(x, y)^{p^{s-1}}.
\]

One has also the following

**Proposition 3.3.** *(see [3])* Let \( p = 2 \). Then
\[
F(x, y) = x + y + (xy + (x + y)(xy)^{2^{(s-1)}})^{2^{(s-1)}} \text{ modulo } ((x + y)xy)^{2^{(s-2)}}.
\]

4. **The Abel formal group law and its p-typization.**

The Abel formal group law \( F_{Ab} \) is defined as the universal formal group that can be written in the form

\[
F_{Ab} = xR(y) + yR(x), \text{ where } R(x) = 1 + \frac{a_1}{2}x + a_2x^3 + a_3x^3 + \cdots
\]

Note that associativity requirement on \( F_{Ab} \) imposes the equation
\[
x(R(yR(z) + R(y)z) - R(y)R(z)) = (R(xR(y) + R(x)y) - R(x)R(y))z.
\]

To satisfy this equation one must uniquely determine all the \( a_i \) through \( a_1 \) and \( a_2 \), which might be arbitrary. For example, one has

\[
\begin{align*}
    a_3 &= \frac{2}{3}a_1 a_2, \\
a_4 &= \frac{1}{2}a_1^2 a_2 - \frac{1}{2}a_2^2, \\
a_5 &= -\frac{2}{5}a_3 a_2 + 16a_1 a_2^2, \\
a_6 &= \frac{1}{3}a_1^4 a_2 - \frac{29}{18}a_1^2 a_2^2 + \frac{1}{2}a_2^3, \\
a_7 &= -\frac{2}{7}a_5 a_2 + \frac{74}{35}a_3 a_2^3 - \frac{64}{35}a_1 a_3^3, \\
a_8 &= \frac{1}{4}a_1^6 a_2 - \frac{103}{40}a_1^4 a_2^2 + \frac{751}{180}a_1^2 a_2^3 - \frac{5}{8}a_2^4, \\
a_9 &= \frac{2}{9}a_1^7 a_2 + \frac{944}{315}a_1^5 a_2^2 - \frac{21632}{2835}a_1^3 a_2^3 + \frac{1024}{315}a_1 a_2^4,
\end{align*}
\]

etc.

The general formula for \( a_n \), i.e., the addition law for \( F_{Ab}(x, y) \), has been obtained by V. M. Buchsteber in [3].
Theorem 4.1. (see [5], Theorem 35.) Let \( a = a_1, \ b = -2a_2 \). Over the ring \( \mathbb{Q}[a, b] \), the formal group law \( F_{ab}(x, y) \) is expressed as

\[
F_{ab}(x, y) = x + y + b\left( \sum_{n=2}^{\infty} A_n(x^ny + xy^n) \right),
\]

where \( A_2 = -\frac{1}{g}, \ A_3 = \frac{1}{3}a \),

\[
A_n = \frac{\delta_n}{n!} \prod_{j=2}^{[n/2]} [j(n-j)a^2 + (n - 2j + 1)^2b], \quad n \geq 4,
\]

and

\[
\delta_n = \begin{cases} 
-(2s - 1) & \text{if } n = 2s, \\
2(s + 1)sa & \text{if } n = 2s + 1.
\end{cases}
\]

See also [5] for earlier description of the addition law for \( F_{ab}(x, y) \).

The logarithm of this formal group law is given by

\[
\log_{ab}(x) = x + \sum_{i \geq 1} m_i x^{i+1} = \int_0^x \frac{dt}{1 + a_1t + a_2t^2 + a_3t^3 + \cdots}.
\]

Thus for example

\[
\begin{align*}
m_1 &= -\frac{1}{2}a_1, \\
m_2 &= \frac{1}{3}(a_1^2 - a_2), \\
m_3 &= -\frac{1}{4}a_1(a_1^2 - \frac{8}{3}a_2), \\
m_4 &= \frac{1}{5}(a_1^2 - \frac{1}{3}a_2)(a_1^2 - \frac{9}{2}a_2), \\
m_5 &= -\frac{1}{6}a_1(a_1^2 - a_2)(a_1^2 - \frac{32}{5}a_2), \\
m_6 &= \frac{1}{7}(a_1^2 - \frac{1}{6}a_2)(a_1^2 - \frac{9}{2}a_2)(a_1^2 - \frac{25}{3}a_2), \\
m_7 &= -\frac{1}{8}a_1(a_1^2 - \frac{8}{15}a_2)(a_1^2 - \frac{3}{2}a_2)(a_1^2 - \frac{72}{7}a_2), \\
m_8 &= \frac{1}{9}(a_1^2 - \frac{1}{10}a_2)(a_1^2 - a_2)(a_1^2 - \frac{25}{7}a_2)(a_1^2 - \frac{49}{4}a_2), \\
m_9 &= -\frac{1}{10}a_1(a_1^2 - \frac{1}{3}a_2)(a_1^2 - \frac{32}{21}a_2)(a_1^2 - \frac{9}{2}a_2)(a_1^2 - \frac{128}{9}a_2), \\
\end{align*}
\]

etc.

An evidence for the following general formula is that all the coefficients occurring at \( a_2 \) are of the form \( \frac{2(1+k)^2}{g^4} \).

Proposition 4.2. Let \( \log_{ab}(x) = x + \sum_{i \geq 1} m_i x^{i+1}, \ m_i \in \mathbb{Q}[a_1, a_2] \) be the logarithmic series of the universal Abel formal group law. Then one has

\[
m_{n-1} = \frac{1}{n} \prod_{j=1}^{n-1} \left( \frac{n - 2i}{\sqrt{j(n-j)}} \sqrt{2a_2 - a_1} \right);
\]

An equivalent formula is

\[
m_n = \frac{1}{n+1} \prod_{r=\frac{2}{g^{2r}}} \left( \frac{2r}{\sqrt{1 - r^2}} \sqrt{2a_2 - a_1} \right).
\]
The corresponding two-parameter genus $MU_* \to \mathbb{Q}[a,b]$ generalizes classical Todd genus of the multiplicative formal group law $F_m(x,y) = x + y + txy$ over coefficient ring $\mathbb{Z}[t,t^{-1}]$. The exponential of the Abel formal group law is

$$\exp_{AB}(t) = \frac{e^{\alpha t}(e^{\beta t} - 1)}{\sqrt{\beta}} = \frac{e^{at} - e^{bt}}{a-b},$$

where $\alpha = a_1/2, \beta = 2a_2 + 1/4a_1^2, a = \alpha + \sqrt{\beta}, b = \alpha - \sqrt{\beta}$.

This formal group law is named by V. M. Buchstaber because of Abel’s study of a functional equation that this exponential satisfies. The coefficient ring $\Lambda_{AB}$ of $F_{AB}$ and its localizations at primes have been computed in [6].

The following ”numerical” characterization of $\Lambda_{AB}$ is given in [9].

**Proposition 4.3.** $\Lambda_{AB}$ consists of those symmetric polynomials in $\mathbb{Q}[a,b]$ such that $f(kt,lt) \in \mathbb{Z}[t, (k-l)^{-1}]$ for any integers $k, l$ such that $k \neq l$.

It is pointed out in [11], [12] that the Sheffer sequence associated with $\exp_{AB}(t)$ gives the Gould polynomials $G_k(x,u,v)$ for $u = \alpha, v = \sqrt{\beta}$.

**Proposition 4.4.** The logarithm $\log_{AB}(t)$ of the Abel universal formal group law, i.e. the inverse to the exponent $\exp_{AB}(t) = \frac{e^{at}(e^{bt} - 1)}{v}$, is the generating function for $G_k(x,u,v)$

$$\log_{AB}(t) = \sum_{k \geq 1} \frac{\partial G_k(x,u,v)}{\partial x}(0,u,v)v^k t^k$$

and can be given by applying $\frac{\partial e^{x\log_{AB}(t)}}{\partial x}$ at $x = 0$ to the left and right sides of the equation

$$\sum_{k \geq 0} G_k(x,u,v) v^k k! t^k = e^{x\log_{AB}(t)}.$$

Now one can easily deduce from [17]

**Proposition 4.5.** The logarithmic series $\log_{AB}(t)$ of the universal Abel formal group law is given by

$$\log_{AB}(t) = \sum_{k \geq 1} \frac{1}{v} \left(\frac{(v + uk)/(v)}{k - 1}\right) v^k k! t^k,$$

where $u = \alpha$, and $v = \sqrt{\beta}$.

Some few terms of $\log_{AB}(t)$ are

$$\log_{AB}(t) = t - \frac{2u + v^2}{2} + \frac{(3u + v)(3u + 2v)}{3!} t^3 - \frac{(4u + v)(4u + 2v)(4u + 3v)}{4!} t^4 + \cdots.$$
So it is straightforward to calculate its coefficients and images of the Hazewinkel generators $v_n$ under the homomorphism from $BP_* = \mathbb{Z}(p)[v_1, v_2 \cdots]$, $|v_n| = 2(p^n - 1)$ to $\mathbb{Z}_p[a_1, a_2]$ which classifies this $p$-typical formal group law. For example, one has for $p = 2$:

$v_1 \mapsto -a_1,$
$v_2 \mapsto \frac{4}{3} a_1 a_2,$
$v_3 \mapsto \frac{284}{105} a_1^3 a_2 - \frac{808}{105} a_1^3 a_2^2 + \frac{128}{35} a_1 a_2^3,$
$v_4 \mapsto \frac{184108}{45045} a_1^3 a_2 - \frac{1108792}{15015} a_1 a_2^2 + \frac{4521344416}{10135125} a_1^3 a_2^3 - \frac{1265861152}{1126125} a_1^2 a_2^4 + \frac{107918689792}{91216125} a_1^5 a_2^5,$

etc.

Using these expressions, we can then compute generators of the ideal in $BP_*$ which is the kernel of the classifying map, thus presenting the coefficient ring of our two-parameter formal group by generators and relations.

The explicit computation of these relations for $p = 2$ in low degrees motivates the following

**Conjecture 4.6.** The coefficient ring of the 2-typization of the universal Abel formal group law is isomorphic to the quotient

$$\Lambda = \mathbb{Z}(2)[v_1, v_2 \cdots]/R,$$

with $|v_i| = 2(2^n - 1)$ and all generating relations are given by $v_1 v_i^2 v_j^2 \sim P$, $1 \leq i < j$, where $P$ consists of monomials not divisible by any $v_1 v_i^2 v_j^2$, with $1 \leq i' < j'$.

One can compute the generating function

$$\sum_{n=0}^{\infty} \text{rank}_{\mathbb{Z}(2)}(\Lambda_n) t^n.$$

Indeed using the above relations we can eliminate any monomials divisible by some $v_1 v_i^2 v_j^2$. Remaining monomials will then form a basis of $\Lambda$ as a $\mathbb{Z}(2)$ module. These monomials can be subdivided into three disjoint sets as follows:

(a) any monomials not divisible by $v_1$;
(b) monomials of the form $v_1 v_{i_1} v_{i_2} \cdots v_{i_k} v_j^n$ and $v_1^2 v_{i_1} v_{i_2} \cdots v_{i_k} v_j^n$, where $k \geq 0$, $n > 1$, $v_{i_1} v_{i_2} \cdots v_{i_k}$ are pairwise distinct and different from 1 and from $j$, and
(c) monomials $v_1^n v_{i_1} v_{i_2} \cdots v_{i_k}$, where $k \geq 0$, $n > 0$, $v_{i_1} v_{i_2} \cdots v_{i_k}$ are pairwise distinct and not equal to 1.

According to this, the generating function will consist of three summands, namely:
\[ \frac{1}{1-t^3} \frac{1}{1-t^7} \frac{1}{1-t^{15}} \cdots \frac{1}{1-t^{2^{n-1}}} \cdots \]

\[(t + t^2)( \frac{t^6}{1-t^6} (1+t^7)(1+t^{15})(1+t^{31}) \cdots + (1 + t^3) \frac{t^{14}}{1-t^{14}} (1+t^{15})(1+t^{31}) \cdots \]
\[+ (1 + t^3)(1 + t^7) \frac{t^{30}}{1-t^{15}} (1+t^{31}) \cdots \]
\[+ (1 + t^3)(1 + t^7)(1 + t^{31}) \frac{t^{62}}{1-t^{31}} \cdots \]
\[\cdots \]
\[+ (1 + t^3)(1 + t^7) \cdots (1 + t^{2n-1-1}) \frac{t^{2(2^n-1)}}{1-t^{2^n-1}} (1 + t^{2n+1-1}) \cdots \]
\[+ \cdots ), \]

and

\[ \frac{t}{1-t} (1+t^3)(1 + t^7)(1 + t^{15}) \cdots (1 + t^{2^n-1}) \cdots . \]

If we replace here all \((1 - t^{2^n-1})^{-1}\) by \(\frac{1+t^{2^n-1}}{1-t^{2^n-1}}\), in the sum of these three functions one will have a common multiple \((1 + t^3)(1 + t^7)(1 + t^{15}) \cdots (1 + t^{2^n-1}) \cdots\), so that the generating function will be

\[ (1 + t^3)(1 + t^7)(1 + t^{15}) \cdots (1 + t^{2^n-1}) \cdots \left( \frac{1}{1-t^6} \frac{1}{1-t^{14}} \frac{1}{1-t^{30}} \cdots \frac{1}{1-t^{2(2^n-1)}} \cdots \right. \]
\[+ (t + t^2) \left( \frac{1}{1-t^2} + \frac{t^6}{1-t^6} + \frac{t^{14}}{1-t^{14}} \cdots + \frac{t^{2(2^n-1)}}{1-t^{2(2^n-1)}} + \cdots \right) \right) . \]

**Proposition 4.7.** The coefficient ring of the 2-typization of the universal Abel formal group law cannot be realized as the coefficient ring of a cohomology theory as a Brown-Peterson cohomology with singularities.

**Proof.** Reducing modulo 2 the ring \(\Lambda = \mathbb{Z}(2)[v_1, v_2, \cdots] / R\) we have

\[ \Lambda / 2\Lambda \cong F_2[v_1, v_2, v_3, \cdots] / (v_1^3v_2^2, v_1^4v_3^2 + v_1^2v_2^3 + v_1^2, v_1v_3^2v_3 + v_2, v_3^2 + v_2^2v_3^2, \cdots ) . \]

We see that \(v_2^7 \equiv 0 \) modulo \(v_1\), i.e., the last sequence of generating relations is not regular in \(BP_*\).

□

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