MSW Instantons

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Abstract

We analyze M5-instantons in F-theory, or equivalently D3-instantons with varying axio-dilaton, in the presence of 7-brane gauge groups. The chiral two-form on the M5-brane plays an important role, because it couples the M5 brane to vector multiplets and charged chiral fields. The chiral two-form does not have a semi-classical description. However if the worldvolume of the M5 admits a fibration over a curve with surface fibers, then we can reduce the worldvolume theory to an ‘MSW’ CFT by shrinking the surface. For this class of MSW instantons, we can use heterotic methods to do computations. We explain this in some detail using the physical gauge approach. We further compare M5-instantons with D3-instantons in perturbative type IIb and find some striking differences. In particular, we show that instanton zero modes tend to disappear and constraints from chirality on instanton contributions to the superpotential evaporate for finite string coupling.
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1. The $M5$ instanton in $F$-theory

The superpotential in an $F$-theory compactification is independent of the Kähler moduli, to all orders in large volume perturbation theory. However, dependence on the Kähler moduli may arise non-perturbatively. As the leading corrections to the superpotential, instanton corrections are important for a number of issues in phenomenological models. For instance we may use them for generating scales, like the scale of supersymmetry breaking, or for lifting flat directions. In $F$-theory models, the natural objects generating such corrections are $D3$-instantons, as they remain $D3$-instantons under $SL(2, \mathbb{Z})$ transformations. Other instantons, such as worldsheet/$D1$-instantons of perturbative type IIB, are normally absent in $F$-theory due to $SL(2, \mathbb{Z})$ monodromies.

$F$-theory vacua are essentially supergravity backgrounds. Rules for computing corrections due to Euclidean solitons in supergravity have been proposed by [1, 2, 3]. The basic idea is to compute a correlator of vertex operators in the twisted worldvolume theory of the instanton. Since the worldvolume theory of a $D3$-brane is the maximally supersymmetric Yang-Mills theory, naively this means we should compute correlators in this Yang-Mills theory, or perhaps some twisted version thereof.

However it is not really correct to conclude that $SL(2, \mathbb{Z})$ monodromies do not affect the worldvolume theory. The coupling constant of the Yang-Mills theory is the axio-dilaton, which varies from point to point over the worldvolume and is usually multi-valued, because the discriminant locus typically intersects the worldvolume. This means that generically the worldvolume theory cannot be discussed in terms of a weakly coupled Yang-Mills theory with only electric degrees of freedom, as circling around a branch cut turns electric into magnetic degrees of freedom.

As always in $F$-theory, the way to deal with the branch cuts is to switch to different variables [4]. The clearest way to define an $F$-theory vacuum is by taking a limit of an $M$-theory vacuum on an elliptically fibered Calabi-Yau four-fold with $G_4$-flux. Essentially this entails a change of variable from the axio-dilaton to the coefficients of the Weierstrass equation, which are defined globally without branch cuts. The $D3$ instanton with varying coupling constant in $F$-theory lifts to an $M5$ instanton in $M$-theory, wrapping the elliptic fiber. Therefore a computation on a $D3$-brane with varying axio-dilaton can be reformulated as a computation in the $M5$ worldvolume theory. The $M5$-brane theory describes both electric and magnetic degrees of freedom, and is not weakly coupled.

The $M5$ worldvolume theory is the six-dimensional $(0, 2)$-theory, with five scalars, a self-dual two-form, and sixteen superpartner fermions. Unlike a $D3$-brane in perturbative IIB, it is a chiral theory. It must have somewhat exotic properties, as expected from the above observations, and this complicates the computations. The fermions and scalars are relatively straightforward. The fermion kinetic terms on the $M5$ are schematically of the
form
\[ \int_{M5} d^6 x \theta (\mathcal{D} + G) \theta \]

As we will briefly review later, the analysis of fermion zero modes is straightforward in the absence of $G$-flux, and has been addressed even in backgrounds with $G$-flux. Although global $G$-fluxes are hard to work with, one can easily find examples where they don’t affect the fermionic zero modes. The (twisted) scalars are also manageable.

Unfortunately the chiral two-form is a whole other story. This field acts as the book-keeping device that allows us to keep track of the electric and magnetic gauge fields on the $D3$-brane together with the monodromies, but it does not admit a conventional Lagrangian description. Its partition function can be related to a theta function through holomorphic factorization [5]. However expressing this explicitly in terms of the compactification data is not straightforward, and involves transcendental methods rather than algebraic ones. In general it also appears to suffer from a holomorphic anomaly.

One may consider taking a Sen limit [6] and comparing with perturbative IIb, where issues are understood more clearly. In this limit, the string scale is parametrically small compared to the 10d Planck scale, and one could hope to use the method of Ganor strings [7]. However the limit is not necessarily smooth or may not even exist [8]. Moreover, $F$-theory has fewer symmetries, and so one suspects that many more instantons should contribute in $F$-theory to a given amplitude. But in the IIb limit one frequently seems to find too many zero modes and the amplitude vanishes. This indicates that we perhaps missed some effects by looking at instantons from a $g_s = 0$ perspective. One would prefer to learn to compute directly at finite string coupling, without reference to a IIb limit.

On the other hand, without the chiral two-form we are clearly missing a crucial part of the story. In $F$-theory models with non-trivial gauge groups, charged matter fields are certain simultaneous fluctuations of the three-form field $C_3$ and the metric. The $M5$-instanton couples to these modes through the chiral two-form. In the IIb limit, this yields the 37-strings which couple charged chiral fields to the $D3$-instanton. So for various phenomenological applications, where we are interested in the coupling to charged chiral fields, it is crucial to incorporate the chiral two-form.

In order to make progress, we are going to make a simplifying assumption. A general $M5$-brane in $F$-theory only admits an elliptic fibration. We will consider situations where the worldvolume of the $M5$-brane also admits a surface fibration over a complex curve $Q$:

\[ S \rightarrow M5 \]
\[ \downarrow \]
\[ Q \] (1.2)

An instanton correction relevant for superpotential terms is of the form
\[ \Delta W \sim f(\Phi) e^{-T} \] (1.3)
where $T$ is a Kähler modulus and $\Phi$ denotes complex structure moduli. Because the prefactor $f(\Phi)$ does not depend on the Kähler moduli, we may do the computations in the limit that $S$ shrinks to zero. In this limit, the $M5$ collapses to a string, and the worldvolume theory of the $M5$ reduces to an effective “MSW” CFT [9] on $Q$. The chiral two-form reduces to chiral right- or left-moving scalars, which may be fermionized. The partition function is identified with a section of a determinant line bundle, and its zeroes can be understood as certain Dolbeault cohomology groups. Therefore in this class of examples we can be pretty explicit, while still retaining the essential features of $M5$-instantons, like a varying axio-dilaton with branch cuts on the worldvolume.

This reasoning applies for instance to $F$-theory models with a heterotic dual, where the $M5$ wraps a $K3$-surface and the MSW CFT is simply the worldvolume theory of the heterotic string, and global $G$-fluxes are also fairly well understood. The $D3$-instantons can then be described as worldsheet instantons of the MSW CFT.

Having mapped the problem to a heterotic-like set-up, one could now try to apply heterotic techniques to compute instanton correlators. From the $F$-theory perspective it is natural to try and apply the rules of [1] et al to compute instanton correlators, instead of using the old worldsheet techniques. The MSW string is generally not fundamental and so the old worldsheet methods should not be necessary. In fact such an approach was already proposed by Witten [3] quite a while ago. We will elaborate on this idea and see how it can be used to compute instanton correlators of charged fields.

We also compare the $M5$ results with $D3$-instantons in the IIb limit, in some special cases where the limit exists. We find that the zero mode structure for finite string coupling differs from the IIb weak coupling limit. The main reason this happens is that there are extra $U(1)$ gauge symmetries in type IIb that are broken non-perturbatively. As a result there are fewer selection rules for finite string coupling. For instance we find that an apparent conflict between chiral matter and instanton contributions to the superpotential in type IIb evaporates in $F$-theory. This has interesting consequences for model building [10, 11, 12]. Along the way we include a discussion of reducible brane configurations, which applies more broadly than the context encountered here.

A number of results on $D3$-instantons in type IIb and $T$-dual set-ups have appeared in recent years. See [13] for an extensive review of the literature on $D$-instantons in perturbative type II theories, and [14] for configurations close to those discussed here. Our discussion is meant to address the interaction of $D3$-instantons with mutually non-local 7-branes.

Our ‘heterotic’ approach to $M5$-instantons and a number of our results were announced at the workshop “GUTS and Strings” in Munich [15]. Several related works appeared during the course of this work, see [16, 17] and [18].
2. Instantons in the heterotic string

2.1. Green-Schwarz approach

In the Green-Schwarz formalism, the worldsheets of instanton amplitudes look quite similar to those of $D$-instantons, so one may try to use $D$-instanton inspired techniques to calculate worldsheets of instanton effects. In [3] this is called the physical gauge or Green-Schwarz approach.

Consider a curve $Q$ in the heterotic Calabi-Yau $Z$ with bundle $V$. The physical degrees of freedom living on $Q$ are as follows. The left-movers form an $SO(32)$ or $E_8 \times E_8$ current algebra. We will use the fermionic formulation. Then the left-moving fermions live in

$$\lambda \in \Gamma (Q, \mathcal{V}(-1)|_Q)$$

(2.1)

The structure group of $\mathcal{V}$ is the manifest symmetry group that is visible in the fermionic formulation. For the $SO(32)$ heterotic string, $\mathcal{V}$ is an $SO(32)$ bundle. For the $E_8 \times E_8$ heterotic string, which is the case we will be interested in here, $\mathcal{V}$ denotes an $SO(16) \times SO(16)$ bundle. The fermions satisfy a Majorana condition in Minkowski signature, but we have to complexify them when we work in Euclidean signature.

For the right-movers, we use GS variables $(X, \Theta)$ and restrict to physical gauge, fixing world-sheet reparametrizations by setting the longitudinal coordinates $X^l/ (z) = z$ and using kappa symmetry to set $P_+ \Theta = 0$. Then the remaining right-moving bosons live in

$$X^\perp \in \Gamma (Q, \bar{\mathcal{O}} \oplus \bar{\mathcal{O}} \oplus \bar{\mathcal{N}})$$

(2.2)

and the right-moving fermions, collectively called $\Theta^\perp$, live in

$$\chi^m_{\alpha} \in \Gamma (Q, \bar{\mathcal{N}}|_Q) \otimes S^-$$

$$\theta^\alpha \in \Gamma (Q, \bar{\mathcal{O}}|_Q) \otimes S^+ = \mathbb{C}^2$$

$$\theta^2_{\alpha} \in \Gamma (Q, \bar{\Omega}_2|_Q) \otimes S^+$$

(2.3)

We have adopted the notation of [19], where the bar indicates right-moving fields which we take to be anti-holomorphic. The zero modes are given by the global holomorphic sections for the left-movers, or the global anti-holomorphic sections for the right-movers. At least we have four bosonic zero modes, for the four translation symmetries broken by the instanton, and two right-moving fermionic zero modes.

The fermion bilinears in the 4d supergravity action are given by [20]

$$e^{K/2}[W \psi \gamma \psi + D_i W \chi^i \gamma \psi + D_i D_j W \chi^i \chi^j]$$

(2.4)
The contribution of an instanton wrapped on \( Q \) to the superpotential can be obtained by computing a correction to the \( \langle \chi \chi \rangle \) correlator. The end result is surprisingly simple; the contribution to the superpotential is simply given by the world-volume partition function, obtained by integrating out all the physical degrees of freedom on \( Q \):

\[
Z[A, g, B] = \int dX^{\perp} d\Theta^{\perp} d\lambda e^{-S_Q[A, g, B]} \tag{2.5}
\]

It is convenient to factor out the universal zero modes:

\[
\Delta S_{4d} = \int d^4xd^2\theta \Delta W, \quad \Delta W = \int d\hat{X} d\hat{\Theta} d\lambda e^{-S_Q[A, g, B]} \tag{2.6}
\]

We get a non-zero contribution only when \( Q \) is a rational curve. Higher genus curves would carry additional \( \theta^\alpha \bar{z}^\bar{\alpha} \) zero modes, and integrating over them leads to multi-fermion \( F \)-terms but not superpotential contributions.

Let us further assume that our instanton is isolated, i.e. it has only the universal zero modes and no more. As usual in supersymmetric theories, we only need to evaluate the classical action and the one-loop determinant around the classical solution:

\[
\Delta W = \frac{\text{Pfaff}'(D_F)}{\sqrt{\det(D_B)}} e^{-T_Q} \tag{2.7}
\]

Here we defined the Kähler modulus as

\[
T_Q = \int_Q J - i \int_Q B \tag{2.8}
\]

The Pfaffian in (2.7) arises because we complexified the fermions in Euclidean space, so we must take a square root to factor out the extra contributions. A prime denotes omission of zero modes. Specializing this to our isolated instanton, we get the following contribution to the space-time superpotential [3]:

\[
\Delta W \sim \frac{\text{Pf}(\bar{\partial}_{\mathcal{V}(-1)})}{(\det \bar{\partial}_{\mathcal{O}(-1)})^2 (\det \bar{\partial}_{\mathcal{O}})^2} e^{-T_Q} \tag{2.9}
\]

Here \( \mathcal{V} \) is assumed to be an \( SO(32) \) bundle. For \( E_8 \times E_8 \), the natural guess is to replace the determinant of \( \bar{\partial}_{\mathcal{V}(-1)} \) by the partition function of level one \( E_8 \times E_8 \) Kac-Moody algebra. The partition function depends on a choice of (perturbative) vacuum, i.e. a choice of background metric, \( B \)-field and gauge field.
2.2. Calculating the Pfaffian

The denominator in (2.9) can be computed easily but does not depend on massless chiral fields. Let us therefore focus on the partition function of the left-movers, the Pfaffian, which does depend on them. As a partition function of fermions, the Pfaffian suffers from an anomaly:

\[ Z_\psi[A + d\lambda] = e^{\frac{2\pi}{h} \int_{\Sigma} \lambda F} Z_\psi[A] \]  

(2.10)

where \( n \) is the number of species. Therefore the value of the Pfaffian at a given point on the moduli space does not make sense. The Pfaffian is not a function, but rather a section of a certain line bundle over the moduli space.

Nevertheless, there is invariant information in the Pfaffian. Although the value of a section of a line bundle is not well-defined, the zero locus of this section is well-defined, in fact it determines the section up to rescaling. Therefore one can still compute the moduli dependence of the superpotential up to an overall scalar. This strategy was successfully carried out in several examples by [21] and further in [22].

Let us consider the zero locus of the Pfaffian. It vanishes when the moduli are such that \( \hat{\partial}_{\mathcal{V}(-1)} \) develops zero modes, i.e. when \( H^0(Q, \mathcal{V}(-1)|_Q) \) is non-trivial. Any bundle on \( \mathbb{P}^1 \) decomposes as a sum of line bundles, so we may write

\[ \mathcal{V}|_Q = \bigoplus_i \mathcal{O}(a_i) \oplus \mathcal{O}(-a_i) \]  

(2.11)

(The \( a_i \) come in \( \pm \) pairs since our \( \mathcal{V} \) is assumed to be orthogonal.) Using the well-known formula for the Dolbeault cohomology of line bundles on \( \mathbb{P}^1 \), we find that

\[ H^0(Q, \mathcal{V}(-1)|_Q) = \sum |a_i| \]  

(2.12)

Hence left-moving zero modes are absent if and only if all \( a_i \) vanish, i.e. \( \mathcal{V}|_Q \) is a trivial bundle [23]. This argument also applies to \( E_8 \times E_8 \) bundles, as long as the holonomy is contained in \( SO(16) \times SO(16) \).

We assume that \( Z \) is elliptically fibered with section \( \sigma_{B_2} \), and \( V \) is constructed through a spectral cover \((C, L)\). We consider only rational curves that are contained in the zero section \( \sigma_{B_2} \). For these we have

\[ V|_Q = \pi_{C_*} L|_Q, \quad V(-1)|_Q = \pi_{C_*} L(-F)|_Q \]  

(2.13)

where \( F \) is the class of the elliptic fiber inside the surface \( \pi_Z^{-1}(Q) \). (One complication that arises for more general rational curves \( Q \) that are not necessarily contained in \( \sigma_{B_2} \) is that the direct image above has to be replaced by a Fourier-Mukai transform. A more significant issue is that quite often rigid curves \( Q \) that are contained in \( \sigma_{B_2} \) are the unique
effective representatives, in the threefold $Z$, of their cohomology class, while more general rigid curves $Q$ that are not contained in $\sigma_{B_2}$ often come in large collections that all represent the same cohomology class in $Z$.) In our case, let us define the spectral curve $\Sigma_{37}$ to be given by

$$\Sigma_{37} = \pi_{Z}^{-1}(Q) \cap C$$

(2.14)

We will explain the reason for this notation in section 4. By applying the direct image, we find that

$$H^0(Q, V(-1)|_Q) = H^0(\Sigma_{37}, L(-F)|_{\Sigma_{37}}),$$
$$H^0(Q, V^*(-1)|_Q) = H^1(\Sigma_{37}, L(-F)|_{\Sigma_{37}})^*$$

(2.15)

Note that since $c_1(V) = 0$, from the Riemann-Roch formula on $Q$ it follows that the ranks of these two Dolbeault cohomology groups must be equal, so fermion zero modes always come in pairs. From Riemann-Roch on $\Sigma_{37}$ we see that $L(-F)$ is a line bundle of degree $g - 1$, where $g$ is the genus of $\Sigma_{37}$. The vanishing locus of the Pfaffian corresponds to the locus where the Dolbeault cohomology groups on $\Sigma_{37}$ valued in the line bundle $L(-F)|_{\Sigma_{37}}$ are non-zero.

The fact that $L$ has vanishing holomorphic Euler characteristic (i.e. degree $g - 1$), rather than vanishing degree, is crucial also for the interpretation of the Pfaffian via theta functions. In our set-up varying the bundle $V$ results in variation of the complex structure of $\Sigma_{37}$ as well as the induced line bundle on it. The line bundle $L$ on $C$ typically has no moduli, but by varying the spectral cover, the restriction of $L$ to $\Sigma_{37}$ may vary. Hence we are dealing with a map

$$\mathcal{M}(C, L) \to \mathcal{U},$$

(2.16)

from the space $\mathcal{M}(C, L)$ parametrizing the pairs $C, L$ to a "universal Picard variety" $\mathcal{U}$ which fibers over the moduli space $\mathcal{M}_g$ of genus $g$ Riemann surfaces, the fiber over $C$ being the torus $\text{Pic}^{g-1}(C)$ parametrizing all complex line bundles of degree $g - 1$ on $C$. When we thus allow $\tau$ (or equivalently, the period matrix) to vary, it becomes important that the Pfaffian really computes $\theta/\eta$, with the $1/\eta$ being the contribution of the massive modes. Here $\theta$ becomes a function of $z$ and $\tau$, while $\eta$ is a function of $\tau$ alone. The vanishing locus of the Pfaffian corresponds to the inverse image of the theta divisor, and $\eta$ becomes part of the undetermined scaling.

The algebraic approach is based on determinant line bundles, following Grothendieck-Knudsen-Mumford and Deligne. It is known to be equivalent [25] to the transcendental approach outlined above, but is often much more amenable to explicit calculations. The algebraic description of the Pfaffian also makes it clear that we get a polynomial on $\mathcal{M}(C, L)$.

In fact we can use the algebraic description to calculate the moduli dependence of the Pfaffian directly. Suppose we have a family of Riemann surfaces parametrized by $s \in S$. In the algebraic description the determinant line bundle over $S$ is defined as

$$\mathcal{D}et = \det(H^0(\Sigma_s, L_s)) \det(H^1(\Sigma_s, L_s))^{-1}$$

(2.17)
Now suppose that we can establish an exact sequence of the form

\[ 0 \to H^0(\Sigma_s, L_s) \to W_1 \xrightarrow{f} W_2 \to H^1(\Sigma_s, L_s) \to 0 \]  

(2.18)

or some variation thereof. Then by general properties of determinant lines, we get an isomorphism

\[ \mathcal{D}et = \det(W_1)^{-1} \det(W_2) \]  

(2.19)

and a canonical section \( \det f \). Since the algebraic and analytic approaches agree, we may take

\[ Pf = \det \bar{\partial} = \det f \]  

(2.20)

The authors of [21] arrive at the following exact sequence

\[ 0 \to H^0(\Sigma_{37}, L(-F)) \to W_1 \xrightarrow{f} W_2 \to H^1(\Sigma_{37}, L(-F)) \to 0 \]  

(2.21)

with

\[ W_1 = H^1(\pi^{-1}Q, L(-F - \Sigma_{37})), \quad W_2 = H^1(\pi^{-1}Q, L(-F)) \]  

(2.22)

Here \( f \) is multiplication by the section which vanishes at \( \Sigma_{37} \), i.e. the equation of \( \Sigma_{37} \) in \( \pi^{-1}Q \), and the last map is restriction to \( \Sigma_{37} \). Calculating \( f \) by pushing down to \( Q \), one finds explicit formulae for the Pfaffian as a function of the moduli [21].

Let us try to restate this in a more physical and less mathematical manner. We will go through this because we get very similar objects as in the \( M5 \)-brane story, but in a more familiar setting. We have some left-moving fermions on \( Q \), coupled to a gauge bundle, and we want to compute their partition function (or at least the moduli dependence). Since \( V|_{\Sigma_{\mathcal{B}2}} = \pi C_s L \), we can locally think of \( V \) as \( n \) copies of \( L \), where \( n \) is the degree of the spectral cover, but then the \( n \) fermions coupled to each copy will have branch cuts. We can eliminate the branch cuts by thinking of the \( n \) fermions as a single fermion living on a \( n \)-fold cover of \( Q \), which is what we have called \( \Sigma_{37} \), and coupled to \( L \). Therefore we want to compute the partition function of a chiral fermion on \( \Sigma_{37} \) coupled to \( L \):

\[ Z_\psi[A] \propto \int d\psi e^{-\int_{\Sigma_{37}} d^2z \sqrt{g} \bar{\psi} \partial_\psi \phi} \]  

(2.23)

as well as a second fermion on \( \Sigma_{37} \), obtained by lifting the fermion on \( Q \) that is coupled to \( V^* \). The line bundle should be flat in order for the partition function to make sense. In the fermionic description this arises because if the line bundle is not flat, then there are always chiral fermion zero modes which can only be absorbed by extra insertions in the path integral. In the bosonized description, this arises because the interaction term may be rewritten as

\[ \int_{\Sigma} d^2z \partial_\phi A_\phi = -\int_{\Sigma} d^2z \phi F_{\phi z} \]  

(2.24)
which implies that there is a tadpole for the chiral boson. The line bundle $L(-F)$ that we found above actually has degree $g - 1$, but we should morally think of it as $\tilde{L} \otimes K_{\Sigma_{37}}^{1/2}$, where $\tilde{L}$ is flat.

Partition functions of chiral fermions on a Riemann surface are well-known objects, and we could take several points of view. Our interest here is in the fact that up to a non-vanishing factor, they are given by theta-functions with characteristics:

$$Z_\psi \propto \Theta_{[\beta]}^{[\alpha]}(\tau|\nu) \quad (2.25)$$

For instance, on a genus one Riemann surface we have the well-known expressions

$$Z_\psi = q^{-1/24 + \frac{i}{2} \alpha^2} e^{2\pi i \alpha \beta} \prod (1 + e^{2\pi i \beta} q^{m - \frac{1}{2} + \alpha z})(1 + e^{-2\pi i \beta} q^{m - \frac{1}{2} - \alpha z}) = \frac{\Theta_{[\beta]}^{[\alpha]}(\tau|\nu)}{\eta(\tau)} \quad (2.26)$$

with $q = \exp(2\pi i \tau)$ and $z = \exp(2\pi i \nu)$. This generalizes to higher genus Riemann surfaces and also to $M5$-branes. Let us review some aspects of the theory of theta-functions.

We fix a basis $\{A_i, B_j\}$ of one-cycles on $\Sigma_{37}$, with the following intersection properties:

$$A_i \cap A^j = B_i \cap B_j = 0, \quad A_i \cap B_j = \delta_j^i \quad (2.27)$$

Let us also fix a basis $\omega^j$ of holomorphic one-forms with the property

$$\int_{A_i} \omega_j = \delta_j^i \quad (2.28)$$

The period matrix is defined as

$$\tau_{ij} = \int_{B_i} \omega_j \quad (2.29)$$

Now let us fix a base point $p_0$ on $\Sigma_{37}$. Then we get a multi-valued map from $\Sigma$ to $\mathbb{C}^g$ by

$$p \rightarrow \int_{p_0}^p \omega_j \quad (2.30)$$

To get a single valued map on $\Sigma$, we have to make the identifications

$$\bar{x} \sim \bar{x} + (\bar{a} + \bar{b} \cdot \tau), \quad \bar{a}, \bar{b} \in \mathbb{Z}^g \quad (2.31)$$

The periodic identifications by the lattice $\Lambda = \mathbb{Z}^g + \mathbb{Z}^g \tau$ make $\mathbb{C}^g$ into a torus, which is called the Jacobian of $\Sigma$:

$$J(\Sigma) = \mathbb{C}^g / \Lambda \quad (2.32)$$
Said differently, a closed one-form $A$ on $\Sigma$ defines a point on $H^1(\Sigma, \mathbb{R})/H^1(\Sigma, \mathbb{Z}) \sim T^{2g}$. The Hodge $*$-operator satisfies $*^2 = -1$, so it defines a complex structure on $H^1$:

$$*A^{1,0} = +iA^{1,0}, \quad *A^{0,1} = -iA^{0,1} \quad (2.33)$$

The induced a complex structure on $T^{2g}$ gives us $J(\Sigma)$.

The Jacobian naturally comes with several additional structures. Given a metric on $\Sigma$, we get a translationally invariant metric on $J(\Sigma)$ given by

$$g_J(A, A) = \int_\Sigma A \wedge *A \quad (2.34)$$

The associated Kähler form

$$\omega(A, A') = \int_\Sigma A \wedge A' \quad (2.35)$$

defines a ‘principle polarization’ on $J$, i.e. it is a symplectic form in $H^2(J, \mathbb{Z})$ such that $J(\Sigma)$ has volume equal to one.

Any holomorphic line bundle on $\Sigma_{37}$ is determined by the flux (i.e. its first Chern class), and by the Wilson lines, i.e. the periods of a holomorphic connection. Let us assume that the flux vanishes, so that the fermion partition function is defined. Then the connection is a closed form, and the Wilson lines determine a unique point on the Jacobian; the identifications by $\Lambda$ are due to the large gauge transformations. So we may think of the Jacobian as the moduli space of flat connections on $\Sigma_{37}$. Therefore the partition function naturally ‘lives’ on $J(\Sigma)$.

Now recall that the partition function of a chiral fermion on $\Sigma_{37}$ is not a function on the Jacobian but a section of a line bundle $\mathcal{L}$. The curvature of this line bundle is in fact $2\pi \omega$. From the index theorem on $J(\Sigma)$ and positivity of $\omega$, it follows that $\mathcal{L}$ has exactly one section, which we want to identify with $Z_\psi$. But to fix $\mathcal{L}$ we also need to specify the holonomies.

To do this we first consider the moduli space of degree $g - 1$ line bundles. Here we have a canonical section, namely the Riemann theta function. Its zero set, called the theta-divisor, corresponds to degree $g - 1$ line bundles with a section. Its translates by a spin structure yields the theta-functions with characteristics. They have the right vanishing behaviour [24]. Since for each spin structure we had a line bundle with a unique section, the partition function is uniquely determined to be proportional to the associated theta-function with characteristics on the Jacobian.

In our case actually we are not given $\tilde{L}_{flat}$ and $K_{\Sigma}^{1/2}$ separately, but rather the tensor product $\tilde{L}_{flat} \otimes K_{\Sigma}^{1/2} = L(-F)$ which is defined unambiguously. Since $L(-F)$ has degree $g - 1$, as discussed above there is actually a unique choice of theta-function, the Riemann
theta-function, which we interpret as the partition function of our fermions. The theta-function vanishes along a divisor on \( J(\Sigma_{37}) \), the theta-divisor, which corresponds to the locus where the line bundle \( L(-F)|_{\Sigma_{37}} \) has a section, i.e. where we get fermion zero modes.

2.3. Coupling to supergravity

Although the partition function for chiral fermions suffered from a gauge anomaly, the combined partition function, including all the worldsheet fields and using a proper definition of the \( B \)-field, must be anomaly free. So it seems that modulo possible \( R \)-anomalies, the superpotential should be an ordinary function on the moduli space. However when we couple to supergravity, in order for the action to be even classically invariant under Kähler transformations, we also need accompany the Kähler transformations with a certain local \( U(1)_R \) transformation. As a result of this, when \( M_{Pl} \) is finite, the superpotential is still a section of a non-trivial line bundle over the moduli space. Let us briefly sketch some of the structure of \( N = 1 \) supergravity Lagrangians that lead to this conclusion (see eg. [26]).

We can couple the globally supersymmetric Lagrangian for the matter fields to the linearized supergravity multiplet through the Noether currents with coupling \( 1/M_{Pl} \). This action is invariant under local supersymmetry up to terms of order \( 1/M_{Pl} \). We then add further terms to the Lagrangian and supersymmetry variations so that supersymmetry is preserved to this order, and so on. One ends up with a Lagrangian in which the fermion kinetic terms contain the following covariant derivative:

\[
D_i = \nabla_i + \frac{1}{2} q \kappa^2 \partial_i K, \quad \kappa^2 = \frac{1}{M_{Pl}^2}
\]  

(2.36)

Here \( \nabla \) is the ordinary covariant derivative including Kähler Christoffel symbols for the non-trivial metric on the sigma model, and \( q \) is \(-1\) for chiral fermions and \(+1\) for gauginos and the gravitino.

Now under a Kähler transformation, we have

\[
\kappa^2 K(z, \bar{z}) \rightarrow \kappa^2 K(z, \bar{z}) + f(z) + f^*(\bar{z}), \quad q\kappa^2 \partial_i K \rightarrow q\kappa^2 \partial_i K + q \partial_i f
\]  

(2.37)

In other words, although the Lagrangian is invariant under local supersymmetry variation, it is not invariant under local Kähler transformations. So in order to make the Lagrangian well-defined, we have to cancel the Kähler variation of the covariant derivative by accompanying it with a chiral \( U(1)_R \) rotation on the fermions.

As a result, the superpotential, which has \( R \)-charge two under such a rotation, is not a function on the moduli space, but rather a section of a line bundle \( \mathcal{L}_K \) on the moduli space, the line bundle being defined by the transformation properties under Kähler
transformations above. The curvature of this line bundle is given by the Kähler form [27]

\[ c_1(\mathcal{L}_K) = \frac{i}{2\pi} \partial \bar{\partial} K / M_{Pl}^2 \]  

(2.38)

Moreover, this class should be quantized.

### 2.4. Instanton correlators

Suppose now instead we want to compute corrections to specific correlators. To do this, we go back to the partition function (2.5). It depends on a choice of vacuum, which involves a choice of background \( A \) on \( Z \). Let us consider the partition function as a functional of the background field \( A \):

\[ Z[A] = \int dX \perp d\Theta \perp d\lambda e^{-S[A]} \]  

(2.39)

In light-cone gauge, the couplings of the world-sheet fields to the background gauge field and gaugino are given by

\[ I = \int d^2z \, Tr(A \bar{z} J_z) + \frac{1}{4} (\Theta \Gamma \bar{z} \Gamma^m \Theta) \, Tr(F_{m \bar{\alpha}} J_z) + j \bar{z} \bar{\alpha} \, Tr(\xi \bar{\alpha} J_z) \]  

(2.40)

Here \( \bar{\alpha} \) refers to ten-dimensional spinor indices, and \( j \bar{\alpha} \sim \bar{\partial} \bar{z} X_M \Gamma^M \Theta \bar{\alpha} \) is the supersymmetry current in light-cone gauge. We will be interested mainly in isolated rational curves, in which case the four-Fermi term vanishes.

The zero modes \( \delta A \) are tangent vectors to the space of gauge fields modulo gauge transformations at the chosen base point. Since the unbroken supersymmetry generators relate bosonic and fermionic wave-functions, we may collect them in \( 4d \, N = 1 \) supermultiplets:

\[ \delta A \bar{z} = (\phi + \sqrt{2} \theta^a \chi_a) \delta A \bar{z} \]  

(2.41)

Thus the coupling of the worldsheet fields to the zero modes of \( (A, \xi) \) may be written as

\[ I \rightarrow \int d^2z \, Tr(\delta A \bar{z} J_z) \]  

(2.42)

By differentiating, we then find the correlation functions:

\[ D_1 \ldots D_n W = \int d\hat{X} \, d\hat{\Theta} \, d\lambda \, I_1 \ldots I_n \, e^{-S[A]} \]  

(2.43)
By $W$ in this section we really mean $\Delta W$, the contribution to the superpotential due to a worldsheet instanton wrapped on $Q$. The derivatives appearing in the above expression are the covariant derivatives, so unfortunately we cannot directly interpret this amplitude as computing a coupling in the superpotential. In general one expects that if the superpotential coupling is not forbidden due to too many fermion zero modes, it will be non-zero. For instance if there are no left-moving zero modes on the instanton, then the instanton generically contributes to all possible couplings in the superpotential.

But there is also a way to check whether a given term is non-zero and we are not missing any hidden cancellations. Namely we can try to tune the bundle moduli so that we get extra left-moving fermionic zero modes. For instance suppose we can tune the bundle moduli to get two left-moving fermionic zero modes. For those values of the moduli, the instanton then does not generate a contribution to $W$, and

$$D_i W = \partial_i W + \kappa^2 \partial_i K W = \partial_i W$$

so the $n = 1$ case of (2.43) can be interpreted as directly computing a superpotential term. By holomorphy, if this superpotential term is non-vanishing, then it remains non-vanishing after a generic deformation. Similarly if $\langle W \rangle = \langle \partial_k W \rangle = 0$ then

$$\langle D_2 D_1 W \rangle = \langle \partial_2 D_1 W + \kappa^2 \partial_2 K D_1 W + \Gamma_{21}^k D_k W \rangle = \langle \partial_2 \partial_1 W \rangle$$

and the instanton correlator directly computes a term in the superpotential. Again by holomorphy we conclude that if it does not vanish for those special values of the moduli where we get extra instanton zero modes, then it cannot vanish for generic bundle moduli either.

A second situation where we can apply (2.43) is when there are ‘chiral’ fermion zero modes on the instanton. All our fermions were chiral anyways, but if $V$ is an $SU(n)$ bundle then some left-movers are coupled to $V$ and others to $V^*$, so we can ask for the net number of left-moving fermions valued in $V$ minus the number of left-moving fermions valued in $V^*$. Equivalently, we can ask for the net charge violation of the left-moving $U(1)$ symmetry. As we already observed, this is governed by an index and vanishes when $\Sigma_{37}$ is irreducible, so there is no net number of chiral fermions in the above sense. However it may happen that $\Sigma_{37}$ is reducible, with an equal and opposite number of ‘chiral’ zero modes on each irreducible piece. This can happen for example if the vacuum admits an unbroken space-time $U(1)$ symmetry and a gauged shift symmetry. In such a situation, again some of the lower-point correlators are guaranteed to vanish, effectively turning some of the covariant derivatives in (2.43) into ordinary derivatives. We’ll discuss an example of this type in section 2.7.

\footnote{This issue was essentially previously encountered in [28]: because two-point functions were non-vanishing there, one could not get a clean computation of the three-point functions. Here also if one proceeds to calculate higher point functions anyways, one finds the correlator is not well-defined on Dolbeault cohomology classes and contact terms are needed.}
We would like to use (2.43) to check for instanton corrections to specific superpotential couplings. As a first example, let us consider heterotic models with with an SU(3) bundle, which yields $E_6$ GUT models in four dimensions. We decompose the adjoint according to

$$248 = (1, 78) + (3, 27) + (\bar{3}, \bar{27}) + (8, 1)$$

Chiral fields live in $H^1(Z, \text{adj}(E_8))$. Using the above decomposition, one finds that the charged chiral fields are counted by

$$H^1(Z, V) \otimes 27, \quad H^1(Z, \Lambda^2 V) \otimes \bar{27}$$

Let us see what kind of couplings we can compute.

We use the fermionic formulation of the $E_8 \times E_8$ theory. In this formulation, the 32 left-moving fermions transform are split into two sets, each only transforming manifestly under an $SO(16) \subset E_8$. Only bundles with holonomy contained in $SO(16) \times SO(16)$ can be described in these variables. Embedding the $SU(3)$ holonomy group in $SO(16)$, the 16 of $SO(16)$ splits as

$$16 = (3, 1)_{-1} + (\bar{3}, 1)_{+1} + (1, 10)_{0}$$

of $SU(3) \times SO(10) \times U(1)$. We will label the fermion indices as $i, \bar{i}, a$ accordingly. In a (2,2) model, the $U(1)$ may be identified with the left-moving $U(1)_R$ symmetry.

Under $SO(10) \times U(1) \subset E_6$, the 27 decomposes as

$$27 = 10_{-1} + 16_{1/2} + 1_2$$

The vertex operators are of the form

$$w(z)_I \wedge J^I = \delta A_{z,I}(z) d\bar{z} \wedge J^I$$

For the 27 states in the NS sector they are given by

$$V_{10} = w_{z,i}(z) d\bar{z} \lambda^i \lambda^a, \quad V_1 = w_{z,i}(z) d\bar{z} g^{\bar{j}} \epsilon_{\bar{j} \bar{k} \bar{l}} \lambda^\bar{k} \lambda^\bar{l}$$

Let us also write the vertex operators for the bundle moduli, which are of the form

$$V_m = w(z) d\bar{z} T^I_{ij} \lambda^i \lambda^j$$
We first consider generic $SU(3)$ bundles. Generically the restriction of the bundle to $Q$ is balanced, i.e.
\[ V|_Q = O(0) + O(0) + O(0) \]  
(2.53)
Recall that the left-moving instanton zero modes are counted by
\[ H^0(Q, V(-1)|_Q) \]  
(2.54)
where $V$ is the rank 16 vector bundle transforming as (2.48). Therefore in this case there are no left-moving zero modes. As we already saw previously in section 2.2, this implies that the instanton generically contributes to the zero-point function (the partition function). Therefore barring unexpected cancellations, which would be hard to see and non-generic, the instanton will contribute to all the couplings appearing in the superpotential, but it would be hard to compute them directly using (2.43) due to appearance of covariant derivatives rather than ordinary derivatives. Let us now assume that we can tune the moduli to get extra left-moving zero modes.

The next interesting case is
\[ V|_Q = O(1) + O(-1) + O(0) \]  
(2.55)
In this case we can reliably compute the one-point function. There are two left-moving fermion zero modes, one for $\lambda^1$ and one for $\lambda^2$. The wave functions for both of these zero modes are constant on $Q$. Vertex operators for vector bundle moduli $X$ can absorb the zero modes if they are of the following form when restricted to $Q$:
\[ V_X = w_X d\bar{z} \lambda^1 \lambda^2 \]  
(2.56)
This signifies the presence of a tadpole non-perturbatively:
\[ \partial_X W = \int d\hat{X} d\hat{\Theta} d\lambda V_X e^{-s} \sim \left( \int_Q w_X \right) e^{-T_X} \]  
(2.57)
On the right we left out the one-loop determinants, which are non-zero generically. If these tadpoles are non-vanishing, then computations of non-perturbative corrections to higher order terms in the superpotential can not be done cleanly in this perturbative vacuum.

By a further tuning, we can get a splitting type of the form
\[ V|_Q = O(2) + O(-1) + O(-1) \]  
(2.58)
This happens at complex codimension four in moduli space. In this case there are four left-moving zero modes. An example of this is the standard embedding, for which we
have $V = TZ$. But in the standard embedding we get the same splitting type above on any generic isolated curve. For a more generic $SU(3)$ bundle satisfying (2.58), we would typically get this splitting type only on $Q$.

Zero and one-point functions are vanishing, and we can unambiguously compute a two-point function with two vector bundle moduli (in other words, non-perturbative mass terms for vector bundle moduli). But a small surprise happens: the $27^3$ Yukawa couplings are unpolluted by these non-vanishing two-point functions, so we can also compute them. The reason for this is as follows. Let us denote fields in the $27$ by $\Phi$ and neutral fields by $X$. We look at the correlator

$$ D^3_\Phi W = \partial^3_\Phi W + \kappa^2 \partial^2_\Phi K \partial_\Phi W + \ldots + \Gamma^X_{\Phi \Phi} \partial_\Phi \partial_X W $$

(2.59)

The expansion in Christoffel symbols is quite messy, but the point is that all these terms vanish if $\langle W \rangle = \langle \partial X W \rangle = 0$. Terms like $\langle \partial_\Phi W \rangle$ vanish automatically. As a result,

$$ \langle D^3_\Phi W \rangle = \langle \partial^3_\Phi W \rangle $$

(2.60)

and so the $27^3$ Yukawa couplings can be computed even with only four left-moving zero modes. We will do this computation next. Note that this simplification does not hold for all Yukawa couplings however. For the $27 \cdot 27 \cdot 1$ Yukawa couplings we do have to take account of the non-vanishing two-point functions, as was previously found in [28].

To make things slightly more transparent, let us use homogeneous coordinates $\sigma^\zeta$, $\zeta = 1, 2$ on $Q$. We find the following left-moving instanton zero modes,

$$ \lambda^1 = \sigma^\zeta \alpha^\zeta \\
\lambda^2 = \beta \\
\lambda^3 = \gamma $$

(2.61)

Recall that under $SO(10) \times U(1) \subset E_6$, the $27$ decomposes as

$$ 27 = 10_{-1} + 16_{1/2} + 1_2 $$

(2.62)

so to get the $27^3$ we can try to compute the $16 \cdot 16 \cdot 10$ or the $10 \cdot 10 \cdot 1$ coupling. Both must give the same answer due to the underlying $E_6$ symmetry. Let us first consider the $10 \cdot 10 \cdot 1$ coupling.

There are no bosonic zero modes or right-moving fermionic zero modes other than the universal ones, so the zero mode measure is simply given by

$$ dM = d^2\alpha d\beta d\gamma $$

(2.63)
Actually the true fermion measure also depends on the Pfaffian of the non-zero modes, but again for our purpose this is not too important. We have

\[ d\sigma_i = \epsilon_{\alpha\beta} \sigma_i^\alpha d\sigma_i^\beta = \langle \sigma_i, d\sigma_i \rangle, \quad d^2 \sigma = d\sigma \land d\bar{\sigma} \quad (2.64) \]

Using the vertex operators given above, we get

\[
\partial_1 \partial_2 \partial_3 W \propto \int dM \prod_{i=1}^3 d^2 \sigma_i \left\langle V^{(1)}_{10}(\sigma_1)V^{(2)}_{10}(\sigma_2)V^{(3)}_{1}(\sigma_3) \right\rangle \\
\propto \int dM \prod_{i=1}^3 w(i)(\sigma_i)d^2 \sigma_i (\alpha \cdot \sigma_1)(\alpha \cdot \sigma_2) \frac{1}{\langle \sigma_1, \sigma_2 \rangle} \beta\gamma \\
= \int \prod_{i=1}^3 w(\sigma_i)d^2 \sigma_i \frac{\langle \sigma_1, \sigma_2 \rangle}{\langle \sigma_1, \sigma_2 \rangle} \\
= \left( \int_\mathbb{Q} w(1) \right) \left( \int_\mathbb{Q} w(2) \right) \left( \int_\mathbb{Q} w(3) \right) 
\]

(2.65)

Here the propagator came from the non-zero modes \( \langle \lambda^a(\sigma_1)\lambda^b(\sigma_2) \rangle = \text{Tr}(T^a T^b)/\langle \sigma_1, \sigma_2 \rangle \), and the fermionic integral yielded \( \int d^2 \alpha (\alpha \cdot \sigma_1)(\alpha \cdot \sigma_2) = \langle \sigma_1, \sigma_2 \rangle \).

Similarly we may try to compute the 16 \cdot 16 \cdot 10 coupling. The vertex operators for the 16 create a branch cut for the \( \lambda \)'s. We can eliminate the branch cut by computing on the cylinder; equivalently we can work in a twisted version of the theory. Let us first discuss some aspects of the R sector states appearing in the 248 of \( E_8 \); then by breaking this down to various subgroups, we can get all others as special cases.

Under \( SO(16) \), the 248 of \( E_8 \) splits as

\[ 248 = 120 + 128 \]

(2.66)

where 128 is the positive chirality spinor of \( SO(16) \). To get the 248 \( ^3 \), we compute the 128 \cdot 128 \cdot 120 coupling, which has two R sector states and one NS sector state. Let us assume that the \( SO(16) \) bundle \( \mathcal{V} \) has a Hermitian structure and can be split as \( \mathcal{V} = W + W^* \). We consider the following left-moving current:

\[ J^\text{tot}_L = \sum_{i=1}^8 \lambda^i \lambda^i \]

(2.67)

This corresponds to the ‘diagonal’ \( U(1) \subset SO(16) \). Now the left-moving Ramond ground states form a Clifford algebra. The ‘empty’ state corresponds to the following vertex operator:

\[ \exp \left( \frac{1}{2} \int J^\text{tot}_L \right) \quad (2.68) \]
and the remaining states are found by acting on it with $\lambda^i$, subject to a GSO projection. Hence we will twist by $\frac{1}{2} J_L^{tot}$. This maps the three-point function to a computation on the sphere with only NS sector states. The new NS vertex operators replacing the R sector vertex operators correspond to $(0,1)$ forms valued in $\Lambda^{even} W$, and they naturally sit in the $128$.

After twisting, the left-moving fermions are sections of

$$
\Gamma(Q, W|_Q), \quad \Gamma(Q, W^* \otimes K_Q|_Q)
$$

and the zero modes correspond to global holomorphic sections. From Riemann-Roch, we expect at least eight left-moving zero modes, so the calculation is going to be a bit different from the $10 \cdot 10 \cdot 1$ coupling where we only had four zero modes.

In order to apply this to the case at hand, we further split

$$
W = V + U
$$

where $V$ is our non-trivial $SU(3)$ bundle, and $U$ is the trivial rank five bundle. The fermions $\lambda^i, i = 1,2,3$ couple to $V$, and the fermions $\lambda^b, b = 1,\ldots,5$ couple to $U$. The vertex operators for the $10_{-1} \subset 27$ where listed in (2.50). They correspond to $(0,1)$ forms valued in $V \otimes (U + U^*)$. After twisting, the vertex operators for the $16_{1/2} \subset 27$ are given by

$$
16_{1/2} : 
\begin{align*}
    w(z), & \lambda^i \lambda^b \\
    w(z), & \lambda^i \lambda^{[b_1 b_2 b_3]} \\
    w(z), & \lambda^i \lambda^{[b_1 b_2 b_3 b_4 b_5]}
\end{align*}
$$

(2.71)

Here we used the shorthand notation $\lambda^{[b_1 \ldots b_k]} = \lambda^{b_1} \ldots \lambda^{b_k}$. Also, after spectral flow the $U(1)$ charge shifts from $16_{1/2}$ to $16_{-1}$. These vertex operators are made from $(0,1)$ forms valued in

$$
V \otimes (U + \Lambda^3 U + \Lambda^5 U)
$$

(2.72)

Now when the bundle $V$ has splitting type $\{2, -1, -1\}$ on $Q$, from (2.69) we find three zero modes for $\lambda^1$, and one zero mode for each $\lambda^b_j, j = 1,\ldots,5$. Hence the correlator is simply

$$
\partial_1 \partial_2 \partial_3 W \propto \int dM \prod_{i=1}^3 d^2 \sigma_i V_{16}(\sigma_1)V_{10}(\sigma_2)V_{16}(\sigma_3)
$$

$$
= \int d^3 \lambda^1 \prod_{j=1}^5 d\lambda^b_j \prod_{i=1}^3 d^2 \sigma_i w^{(1)}(\sigma_1)\lambda^i \lambda^{b_1} w^{(2)}(\sigma_2)\lambda^j \lambda^{b_2} w^{(3)}(\sigma_3)\lambda^k \lambda^{[b_3 b_4 b_5]} + \ldots
$$

$$
\propto \left( \int_Q w^{(1)} \right) \left( \int_Q w^{(2)} \right) \left( \int_Q w^{(3)} \right)
$$

(2.73)
The dots in the second line denote the other terms one gets by writing out all the pieces of the vertex operators in the $16$ in (2.71) and in the $10$. To go from the second line to the third line, first we absorbed one $\lambda^1$ zero mode with each of the three vertex operators, so that the $w^{(i)}$ become $(1,1)$ forms on $Q$. For the $\lambda^b$ correlators, the only terms that can be non-zero are the terms which have exactly one $\lambda^b$ for every $j$. This follows from the unbroken gauge symmetry, specifically the selection rules for the $U(1)^5 \subset SO(10) \subset E_6$. Normally the correlator should be neutral under each unbroken $U(1)$, but due to the twisting each $U(1)$ has a background charge and so there is a shift by one. Finally then absorbing the five $\lambda^b$ zero modes picks the singlet in the tensor product $16 \otimes 10 \otimes 16$. (In terms of the decomposition above, this reduces to the familiar GUT group algebra which yields the singlets $10 \cdot 10 \cdot \tilde{5}_h$ and $10 \cdot \tilde{5}_m \cdot \tilde{5}_h$). So the correlator is ultimately completely determined by the zero modes, and we get the same answer as for the $10 \cdot 10 \cdot 1$ coupling, as it had to be by the underlying $E_6$ symmetry.

We may also ask if we get a contribution to the $\overline{27}$. The vertex operators are similar to those in (2.51) but the fermions are conjugated:

$$V_{10} \sim w_{\bar{z},\bar{\lambda}}(z) d\bar{z} \bar{\lambda}^i \lambda^a, \quad V_1 \sim w_{\bar{z},\bar{\lambda}}(z) d\bar{z} g^{ij} \epsilon_{jkl} \lambda^k \lambda^l \quad (2.74)$$

The fermion zero modes are still given by (2.61). We see that it is impossible to absorb the two $\lambda^1$ zero modes. So the instanton contribution to the $\overline{27}$ vanishes. Non-vanishing instanton contributions can only come from instantons with at most two left-moving fermion zero modes.

The calculations we have done here for the Yukawa couplings are slightly simplified versions of calculations originally done in $(2,2)$ models in the world-sheet approach [29]. The coefficient of the Yukawa coupling counts the number of holomorphic maps from $\mathbb{P}^1$ to $Q$ where three marked points are mapped to $D_i \cap Q$, with $D_i$ the Poincaré dual of $w^{(i)}$. As we saw above, this gives the same number as the GS computation. The main differences in the computation are as follows: we do not integrate over the space of maps $\mathbb{P}^1 \to Z$, only over the space of (super)embeddings; and since we fixed the gauge redundancy from the start, there are no (super)ghost correlators. As a result, the GS computation is a bit shorter.

2.6. Couplings in $SU(5)$ models

By further breaking down $E_6$, we can generalize this to smaller gauge groups. We will briefly spell this out for $SU(5)_{\text{GUT}}$. In the fermionic description, the manifest $SO(16)$ is broken to $SU(4) \times U(1)$ by the $SU(5)$ holonomy, and the $16$ of $SO(16)$ is broken as

$$16 = (5, 1)_{-1} + (\bar{5}, 1)_{-1} + (1, 6)_0 \quad (2.75)$$

under $SU(5) \times SU(4) \times U(1)$. Accordingly we split up the 16 indices as $i = 1, \ldots, 5$, $\bar{i} = 1, \ldots, 5$, $a = 1, \ldots, 6$. Thus in this description, only a subgroup $SU(4) \times U(1) \subset$
$SU(5)_{GUT}$ of the GUT group is manifest. We use the following decompositions of the matter representations under $SU(4) \times U(1)$:

$$5 = 4_{1/2} + 1_{-2}, \quad 10 = 4_{-3/2} + 6_1$$ (2.76)

Now let us think about the instanton zero modes. As usual they are counted by

$$H^0(Q, \mathcal{V}(-1)|_Q)$$ (2.77)

where in the present case

$$\mathcal{V} = V \oplus V^* \oplus \bigoplus_{a=1}^6 \mathcal{O}$$ (2.78)

The fermions $\lambda^a$ are sections of $\mathcal{O}(-1)_Q$ and can never have any zero modes. Let us denote by $V$ the $SU(5)$ bundle which breaks $E_8$ to $SU(5)_{GUT}$. Then we can decompose

$$V|_Q \sim \bigoplus_{i=1}^5 \mathcal{O}(a_i), \quad \sum_i a_i = 0$$ (2.79)

Clearly we get essentially the same calculations as for the $E_6$ models, as the zero mode structure is completely determined by the splitting type of the bundle on $Q$. For instance let us try to understand corrections to Yukawa couplings. To get a clean computation, we need to arrange for four left-moving zero modes, i.e.

$$a_i = \{2, -1, -1, 0, 0\}$$ (2.80)

Now we look at corrections to the $10 \cdot 10 \cdot 5$ coupling, which can be computed in the NS sector from the $6_1 \cdot 6_1 \cdot 1_{-2}$. To find the vertex operators, we recall the usual decomposition of the adjoint representation of $E_8$ under $SU(5)_H \times SU(5)_{GUT}$:

$$248 = (24, 1) + (1, 24) + (5, 10) + (\bar{5}, \bar{10}) + (10, \bar{5}) + (\bar{10}, 5)$$ (2.81)

Therefore a state of the $5$ matter representation comes from a generator of the Dolbeault cohomology group $H^1(Z, \Lambda^2 V^*)$. Similarly a state in the $10$ comes from a generator of $H^1(Z, V)$. Hence the vertex operators are of the following form:

$$V_1 = w(z) \bar{z} \partial \bar{z} \lambda^i \lambda^j, \quad V_6 = w(z) \bar{z} \partial \bar{z} \lambda^i \lambda^a$$ (2.82)

Essentially the same calculation as for the $27^3$ coupling above shows that we get a non-zero contribution of the form $(\int_Q w)^3$ of the instanton to the Yukawa coupling, since we have just broken it into smaller pieces. We will not repeat it here.
Similarly we can try to compute the $10 \cdot 5 \cdot 5$ coupling for splitting type $\{2, -1, -1, 0, 0\}$, for instance by evaluating the $6_1 \cdot 4_{-1/2} \cdot 4_{-1/2}$. Again due to the branch cuts it is convenient to compute this in the twisted theory. This is similar to the $16 \cdot 16 \cdot 10$ couplings in $E_6$ models. We again split $V$ as $V = W + W^*$, and $W = V + U$ where $V$ is our non-trivial $SU(5)$ bundle, and $U$ is a trivial rank three bundle. After twisting, vertex operators in the Ramond sector become $(0, 1)$ forms valued in $\Lambda^{{\text{even}}} W = \bigoplus_{p+q={\text{even}}} \Lambda^p V \otimes \Lambda^q U$ (2.83)

I.e. we have vertex operators of the form

\[ 4_{-3/2} : w(z)_{\bar{\xi}, i} d\bar{z} \lambda^{[ib_1]} \quad 4_{-1/2} : w(z)_{\bar{\xi}, ij} d\bar{z} \lambda^{[ij]} \]

and their conjugates. Here we have slightly abused notation, indicating the $U(1)$ charge before spectral flow. After spectral flow we have $4_{-3/2} \rightarrow 4_1$, $4_{-1/2} \rightarrow 4_2$. The calculation now proceeds as before and is completely determined by the zero modes structure:

\[
\partial^3 W \propto \int dM \prod_{i=1}^3 d^2 \sigma_i V_4(\sigma_1)V_6(\sigma_2)V_4(\sigma_3) \\
= \int \prod_{i=1}^3 d^2 \sigma_i w_{\bar{\xi}, z4}(\sigma_1)w_{\bar{\xi}, z5}(\sigma_2)w_{\bar{\xi}, z5}(\sigma_3) \\
= \left( \int_Q w_4 \right) \left( \int_Q w \right) \left( \int_Q w_5 \right) \quad (2.85)
\]

2.7. Unbroken $U(1)$s

The examples we have considered are somewhat uninteresting, in the sense that the instanton generically has no left-moving zero modes, and thus it should contributes to all the gauge invariant couplings in the superpotential. The only really nice quantity to compute in this case is the Pfaffian. An exception to this rule appears in vacua where extra left-moving zero modes are forced on us, such as vacua with an unbroken $U(1)$ symmetry and a gauged shift symmetry. The $U(1)$ symmetry does not necessarily need to be anomalous. We will review some of the story here and state the $F$-theory analogues in section 3.

Let us assume that we have an unbroken $U(1)_X$ symmetry, and a chiral field $X$ charged under it. If the symmetry is anomalous, then the Kähler moduli will shift under a $U(1)_X$ gauge transformation. The Kähler moduli and dilaton fields are defined as

\[
S = e^{-2\phi}V + ia, \quad T_Q = \int_Q J - i \int_Q B_2 \quad (2.86)
\]
Here \( V = \text{vol}(Z) \), \( a = \int_Z B_6 \), \( \phi \) is the 10d dilaton, and volumes are measured in string units. Under a \( U(1)_X \) gauge transformation we have

\[
\delta S \sim i \frac{\text{Tr}(Q^X)}{96\pi^2} \lambda_X
\]  

(2.87)

where the trace runs over all the massless charged fields and \( Q_X \) denotes the charge operator for \( U(1)_X \), and

\[
\delta T_Q \sim i \lambda_X \int_Q F^X = 2\pi i q^X \lambda_X
\]  

(2.88)

where we defined \( F^X = \text{Tr}(T^X F_{E_8}) \). We will mostly assume that \( \text{Tr}(Q^X) = 0 \), although it is not hard to make adjustments. If \( \delta T_Q \neq 0 \), the \( U(1)_X \) will pick up a mass through the St"uckelberg mechanism:

\[
\mathcal{L} \supset (q^X A_{\mu} - \partial_{\mu} \text{Im}(T_Q))^2
\]  

(2.89)

This can be avoided by mixing with another \( U(1) \) coming from the second \( E_8 \). At any rate, here we are mostly interested in implications for instanton contributions to the superpotential.

If \( q^X \neq 0 \) then a worldsheet instanton wrapped on \( Q \) cannot contribute to the superpotential, because the classical exponential factor is not gauge invariant:

\[
e^{-T_Q} \rightarrow e^{-T_Q - 2\pi i q^X \lambda_X}
\]  

(2.90)

Instead we will get a prefactor that is also not invariant and cancels the gauge variation of the exponential. In this way, one may generate superpotential terms which are forbidden at tree level due to gauge invariance. But as long as the charged fields have vanishing expectation value, this does not generate a potential for \( T_Q \).

On the other hand, if \( T_Q \) shifts under a gauge transformation, then we already get a potential perturbatively through the \( D \)-terms. The K"ahler potential is given by

\[
\mathcal{K} = -M_{Pl}^2 \log(S + S^* - 2q^X_S V_X) - 3 M_{Pl}^2 \log \frac{1}{6} \hat{T}_i \hat{T}_j \hat{T}_k d^{ijk}
\]  

(2.91)

with

\[
\hat{T}_i = \text{Re} T_i - q^X_i V_X
\]  

(2.92)

Assuming \( q^X_S \sim \text{Tr}(Q^X) = 0 \), from this K"ahler potential we get

\[
\xi^X = \frac{\partial \mathcal{K}}{\partial V_X} \bigg|_{V_X = 0} \sim \frac{M_{Pl}^2}{\int J \wedge J \wedge J} \int c_1(L^X) \wedge J \wedge J
\]  

(2.93)
The $D$-term potential is given by

$$V_D = \frac{1}{2} Re(S)^{-1} \left( \xi^X - \sum \xi^X |\phi|^2 \right)^2 \quad (2.94)$$

Without further information about the superpotential, we do not know if the $\phi$ can get a VEV. At zero VEVs the modulus $q_i^X K^i j \Re(T_j)$ picks up a mass through the $D$-term potential. This is expected by supersymmetry, because we already saw that the imaginary part picked up a mass through the St"uckelberg mechanism. In this case the mass to be of order $g_s M_s$. If $\phi$ gets a large VEV, then the $U(1)_X$ gauge boson will eat the $\phi$ field instead of a K"ahler modulus, and the K"ahler modulus may become massless again. The massive $U(1)_X$ will have a mass of order the KK scale, and we have to re-expand around the correct vacuum.

Although an instanton wrapped on $Q$ does not contribute to the superpotential, it could generate charged couplings non-perturbatively. Let $X$ denote a chiral superfield with charge $q_X$. One of the simplest couplings that could be generated this way is the Polonyi superpotential

$$W = \mu^2 X e^{-T_Q} \quad (2.95)$$

Actually this is not an honest Polonyi superpotential but rather a mixing between $X$ and $T_Q$. At any rate, let us see how such a term could get generated.

We first review how to construct an extra $U(1)_X$ symmetry. We assume for simplicity that the holonomy is contained in $SU(n)$, and take the heterotic bundle $V$ to be decomposable:

$$V = V' + V'' \quad (2.96)$$

Since $c_1(V) = 0$, we have

$$\det(V') = \det(V'')^{-1} \quad (2.97)$$

The extra $U(1)_X$ symmetry is the subgroup of $SU(n) \subset E_8$ that commutes with the holonomy of $V$. The internal part of the gauge field of this $U(1)_X$ is a connection on a non-trivial line bundle on $Z$, given by $L_X = \det(V')$.

Charged chiral fields correspond to generators of $H^1(Z, \text{ad}(E_8))$. It is not hard to see that the only fields $X$ that are neutral under the GUT group but charged under $U(1)_X$ correspond to a generators of $\text{Ext}^1(V', V'')$ or $\text{Ext}^1(V'', V')$. For definiteness, let us say that $w^X \in \text{Ext}^1(V'', V')$. The vertex operator is of the form

$$w^X = w^X(z) \Gamma_i d\bar{z} T_{ij}^I \lambda_i^I \lambda_j^I \quad (2.98)$$

where $i$ is an index valued in $V'$ and $j$ is an index valued in $V''$.

We assume that $\delta T_Q \neq 0$, so that an instanton wrapped on $Q$ does not contribute to the superpotential. Now we want to check for a coupling of the form $\Delta W = X e^{-T_Q}$. In
this case, we are supposed to compute the correlator

\[ \partial_X W \propto \int dM \int_Q i^* w_1^X \wedge \langle J^I \rangle \] (2.99)

where \( J^I \) is the current \( \lambda^I \bar{\lambda}^I \), which carries a non-zero \( U(1)_X \) charge. Clearly this vanishes unless we have precisely two fermionic left-moving zero modes to kill the current, and absorb the \( U(1)_X \) charge: one zero mode for \( \lambda^I \) for some \( i \), and one for \( \bar{\lambda}^J \) for some \( j \).

As usual, the bundle \( V \) splits when restricted to \( Q \). Since \( \int_Q F^X \neq 0 \), the restriction \( V|_Q \) cannot be balanced. We decompose

\[ V'|_Q = \bigoplus \mathcal{O}(a_i), \quad V''|_Q = \bigoplus \mathcal{O}(b_j), \quad \sum a_i + \sum b_j = 0 \] (2.100)

In the minimal case, we have

\[ \int_Q c_1(V') = \int_Q c_1(L^X) = 1, \quad \int_Q c_1(V'') = -\int_Q c_1(L^X) = -1 \] (2.101)

Generic bundles \( V' \) and \( V'' \) with this property have splitting types

\[ a_i = \{1, 0, 0, \ldots\}, \quad b_j = \{-1, 0, 0, \ldots\} \] (2.102)

for our rational curve \( Q \). This yields precisely the right number of left-moving zero modes on \( Q \). We have essentially met such a splitting type before in section 2.5, however here the presence of zero modes is guaranteed by \( \int_Q F^X \neq 0 \) and the splitting type is preserved under a generic deformation. If \( \int_Q c_1(L^X) > 1 \), there would be additional zero modes and we would not be able to generate a term of the form \( X e^{-T_Q} \), but we might be able to generate \( X^m e^{-T_Q} \) with \( m > 1 \), a Yukawa coupling, or a higher dimension operator like the Weinberg operator.

Suppose that both \( V' \) and \( V'' \) are constructed from a spectral cover. From the point of view of the spectral curve, \( \Sigma_{37} \) has split into two pieces:

\[ \Sigma_{37} = \Sigma'_{37} \cup \Sigma''_{37}, \] (2.103)

where

\[ \Sigma'_{37} = \pi^{-1}_Z(Q) \cap C', \quad \Sigma''_{37} = \pi^{-1}_Z(Q) \cap C'' \] (2.104)

In the minimal case we get one zero mode on \( \Sigma'_{37} \) with charge +1, and one zero mode on \( \Sigma''_{37} \) with charge -1. A somewhat similar splitting phenomenon seems to occur when we take the IIb limit in \( F \)-theory, as we will discuss later.
3. \textit{M5 instantons}

As in the heterotic string or type IIb, the superpotential in $F$-theory satisfies a simple non-renormalization theorem. Let us briefly recall the argument. Supersymmetry implies that the Kähler moduli space is a complex manifold. Given a basis of four-cycles $S_i$ in $B_3$, holomorphic coordinates on the moduli space are given by

$$T_i = \text{vol}(S_i) + i \int_{S_i} C_4$$

with volumes measured in ten-dimensional Planck units. Note that the $T_i$ shift under gauge transformations of the RR field $C_4$, hence such shifts are isometries of the Kähler moduli space. $F$-theory is defined as a large volume expansion, so the small parameter in $F$-theory is a Kähler modulus. For instance, the gauge coupling constant $\alpha_{\text{GUT}}^{-1}$ is identified with the real part of some $T_i$.

Now the superpotential must depend holomorphically on the $T_i$. Then because of the shift symmetry, the superpotential can depend on the $T_i$ only as

$$\exp(-T_i)$$

In other words, the superpotential calculated by algebraic geometry at tree level is actually exact to all orders in the expansion parameter. It can receives corrections only non-perturbatively, for example from $D3$-instantons. This is why $D3$-instantons are important – although small, they provide some of the leading non-vanishing corrections to the superpotential.

We consider a $D3$ instanton wrapped on a four-cycle in $B_3$, the internal manifold of our compactification. A precise way to define an $F$-theory vacuum is to take an $M$-theory vacuum on a Calabi-Yau four-fold $Y_4$ that is elliptically fibered over $B_3$, and then take the limit in which the elliptic fiber shrinks to zero. In this description, the $D3$ instanton in $B_3$ descends from an $M5$ instanton in $Y_4$ wrapping the $T^2$-fiber, and 7-branes are encoded in the geometry of the elliptic fibration.

In this section we discuss how the $M5$-brane sees the background geometry and fluxes. We then discuss some aspects of $M5$ instanton corrections in an $F$-theory compactification, and make contact with section 2.

3.1. Scalars and fermions

The $M5$ worldvolume has $(0, 4)$ supersymmetry and contains a single superconformal tensor multiplet. Under the $SO(4)$ little group and $USp(4) = SO(5) R$-symmetry group,
the supercharges transform as $(1,2; 4)$. The tensor multiplet consists of a self-dual two-form $B^+$ transforming as $(1,3; 1)$, and five scalars $(1,1; 5)$. The fermions live in $(1,2; 4)$ and satisfy a symplectic Majorana condition (in Lorentzian signature). Reduction on $T^2$ yields the $N = 4$ Yang-Mills multiplet.

The fermionic part of the $M5$ action is of the schematic form

$$L^M_{f} = \frac{1}{2} \theta \left[ \nabla + \mathcal{G} \right] \theta$$

In order to understand the fermionic zero modes, we may first consider setting $G$ to zero. The $G$-flux appears as a kind of mass term, so the effect of turning $G$ back on will be to lift some of the would-be fermionic zero modes.

Our $M5$ is wrapped on a divisor $D$ of $Y_4$. Accordingly, we split the normal bundle as $\mathbb{R}^3 \oplus N$ where $N$ is a complex line bundle. Since $N$ is the normal bundle to the divisor $D$ in the Calabi-Yau four-fold, it follows from adjunction that $N$ is the same as the canonical line bundle $K_D$ on $D$. Spinors on the $M5$ with chirality $(-1)^i$ are sections of

$$S^+_D = \bigoplus_{i \text{ even}} \Omega^{(0,i)}(D, \mathbb{R}) \otimes K_D^{1/2}$$

Since the tensor multiplet spinors have positive chirality on $D$, we only take $i$ even. However as mentioned above, the tensor multiplet spinors also transform as a $4$ under the $USp(4)_R$ symmetry group, i.e. the spinor of $SO(5)_R$. When the $M5$ wraps a divisor $D$ in a Calabi-Yau four-fold, this is broken to $SO(3)_R \times U(1)_R$ where the second factor is identified with the structure group of the canonical bundle $K_D$ on $D$. The $4$ of $SO(5)_R$ splits as

$$4 = 2_1^+ \oplus 2_{-1}^-$$

of $SO(3)_R \times U(1)_R$. Therefore, the tensor multiplet spinors actually transform as

$$S^+_D \otimes (K_D^{1/2} \oplus K_D^{-1/2}) = \Omega^{0,0}_{-\frac{1}{2}} \oplus \Omega^{0,2}_{-\frac{1}{2}} \oplus \Omega^{3,0}_{\frac{3}{2}} \oplus \Omega^{3,2}_{\frac{3}{2}}$$

under the $U(3)$ structure group of the metric restricted to the divisor $D$, as well as the $2$ of $SO(3)_R$. The $\pm \frac{1}{2}$ is to remind us of the $U(1)_R$ charges. The zero modes in the absence of flux correspond to global holomorphic sections. Therefore we get two universal fermionic zero modes from $h^{0,0}(M5)$, as well as two non-universal zero modes for each generator of $H^{0,i}(M5)$, $i = 1, 2, 3$.

When we turn the $G$-flux back on, zero modes of Hodge type $(0, 2)$ may get lifted \cite{30, 31}. The flux induces a map

$$\mathcal{G} : \Omega^{0,2}(D) \to \Omega^{2,0}(D)$$

When we turn the $G$-flux back on, zero modes of Hodge type $(0, 2)$ may get lifted \cite{30, 31}. The flux induces a map
Specifically
\[(\mathcal{G}\theta)_{ab} = \Omega_{abc\bar{e}}\theta_{\bar{e}\bar{f}}\] (3.8)
where \(\Omega\) denotes the holomorphic \((4,0)\) form, restricted to \(D\). These equations can get further modified in the presence of flux for the chiral two-form. The surviving fermionic zero modes are in the kernel of this map. However we will mostly be interested in situations where \(H^0,1(M5) = H^0,2(M5) = 0\). In this case there are no fermionic zero modes that could be lifted, and the presence of \(G\)-flux is irrelevant in this regard.

Besides the fermionic zero modes, we also have bosonic zero modes. The five scalars yield sections of \(\Gamma(M5,N) \oplus \mathbb{R}^3\). From this we recover three of the four universal bosonic collective coordinates of the instanton, as well as non-universal bosonic zero modes counted by \(h^0(M5,N) = h^0,3(M5)\). The expectation value of the chiral two-form \(B^+\) through the \(T^2\) decompactifies and gives the remaining universal Euclidean \(\mathbb{R}\) in the \(F\)-theory limit. This zero mode is never lifted by the background \(C_3\)-field as a three-form with zero or two indices in the \(T^2\) does not exist in \(F\)-theory. The remaining contributions of the chiral two-form do not lead to vanishing of the partition function for generic \(C_3\), but they are much more complicated to understand. They will be discussed separately in section 3.3.

3.2. Fluxless backgrounds

Since fluxes are small perturbations in the large volume limit we may first find instantons and their zero modes for zero flux, and then consider turning on the flux as a small perturbation. \(D3\) instantons in the absence of flux were studied by Witten [32]. Let us review some of the results.

The \(D3\) lifts to an \(M5\) on a divisor \(D\) in \(Y_4\). The axionic part of the Kähler modulus \(T_D\) is not invariant under rotations of the normal bundle. Indeed from the 11\(d\) supergravity action of \(C_3\)
\[
\frac{1}{2(2\pi)^2} \int dC_3 \wedge *dC_3 + \frac{1}{2\pi} \int C_3 \wedge I_8 + \frac{1}{6(2\pi)^3} \int C_3 \wedge G \wedge G
\] (3.9)
we find that
\[
\delta C_6 = I_6^{(1)}(\Theta, R) + \frac{1}{4\pi} G \wedge \Lambda
\] (3.10)
so \(T_D\) may shift under gauge and Lorentz transformations. Here \(\Lambda = 2\pi \lambda_X \omega^X\) where \(\omega^X\) is a generator of the coroot lattice, i.e. a generator in \(H^2(Y_4)\) orthogonal to classes in \(\sigma_{B_3}H^4(B_3)\) and \(\pi_Y^*H^6(B_3)\), and \(\lambda_X\) is an infinitesimal gauge transformation for the corresponding \(U(1)_X\).

In [32] it is conjectured that
\[
\delta \Theta_X \int_D C_6 = -\Theta_N \chi(D, \mathcal{O}_D)
\] (3.11)
29
where $\Theta_N$ is an infinitesimal $U(1)$ rotation on $N$, the normal bundle to $D$ in $Y_4$. Let us try to get this directly from (3.10). The anomaly polynomial is given by

$$I_8 = -\frac{1}{48} \left[ \frac{p_1(TD)^2 + p_1(ND)^2 - 2p_1(TD)p_1(ND)}{4} - p_2(TD) - p_2(ND) \right]$$  \hspace{1cm} (3.12)$$

In our situation $ND = N \oplus \mathbb{R}^3$. Since we are interested in a rotation of the normal bundle, we only need to look at the pieces that depend on $N$. (The normal bundle will eventually be identified with the anti-canonical bundle $K_D$, but we do not want to do a simultaneous rotation on $K_D$, as that would change the $R$-charges of the fermions (3.6) and the chiral two-form). Furthermore, $p_2(N)$ vanishes and $p_1(N) = c_1(N)^2$. Then we apply the descent procedure

$$I_8 = dI_7^{(0)}, \quad \delta_{\Theta_N} I_7^{(0)} = dI_6$$  \hspace{1cm} (3.13)$$

The first step is ambiguous, because there is a one parameter family of Chern-Simons forms with exterior derivative given by $p_1(TD)p_1(N)$. We pick one on the criterion that $c_1(N)^3$ should ultimately cancel, as it tends to give fractions with large denominators. This leads to

$$I_6^{(1)} \supset -\Theta_N \frac{1}{4 \cdot 48} \left[ c_1(N)^3 - \frac{1}{2} p_1(TD)c_1(N) \right] = +\Theta_N \frac{1}{48} c_2(TD)c_1(TD)$$  \hspace{1cm} (3.14)$$

Modulo a factor of $-\frac{1}{2}$, this recovers $-\Theta_N$ times the index density for $\chi(D, \mathcal{O}_D)$.

The resulting anomaly of the exponential factor $\exp(-TD)$ is cancelled by the one-loop determinants of the worldvolume fields, since the anomalies of the worldvolume fields cancel with the anomaly inflow. Since the chiral two-form does not transform under normal bundle rotations, the anomaly will come from the fermions. From (3.6) we see that the net anomaly due to fermion zero modes is given by $\chi(D, \mathcal{O}_D)$, which cancels the anomaly from the exponential factor precisely when (3.11) holds. (By contrast, possible gauge anomalies of $\exp(-T_D)$ due to (3.10) are cancelled by the partition function for the chiral two-form on the $M5$-brane, as we will see later).

The four-dimensional supercharges carry charge $\pm \frac{1}{2}$ under such normal bundle rotations, so we identify this $U(1)_R$ with the $4d$ $U(1)_R$ symmetry. In order to contribute to the superpotential, we get the two universal fermionic zero modes which each carry charge one-half, and generically no other fermionic zero modes. Since the anomaly due to the universal zero modes $d^2\theta$ is equal to one, it follows that an $M5$-brane wrapped on $D$ can only contribute to the superpotential if the remaining pre-factor has charge zero, and therefore the exponential factor has the opposite variation, i.e.

$$\chi(D, \mathcal{O}_D) = 1$$  \hspace{1cm} (3.15)$$

30
This analysis applies to smooth divisors. In $F$-theory we are typically interested in cases where $D$ is not smooth. However when we go to $M$-theory we expect that singularities of $Y_4$ can be removed by a simultaneous resolution. Further we would not expect to get extra zero modes in the $F$-theory limit, because it corresponds to varying a Kähler modulus and the one-loop determinants we are interested in don’t have any dependence on the Kähler moduli. Therefore let us perform such a resolution of $Y_4$ and analyse the condition there.

We would like to distinguish two cases:

(i) The $D_3$ worldvolume is not contained in $\Delta$;

(ii) The $D_3$ worldvolume is contained in $\Delta$.

Let us first consider type $(i)$ under heterotic/$F$-theory duality. Such instantons get mapped to worldsheet instantons. We can relate the Betti numbers (or even the Hodge numbers) through the Leray sequence. Let us discuss this for the case that $S = K3$ or $S = dP_9$, which arises in the stable degeneration limit. The $E_2$ term in the Leray sequence is given by $H^p,Q,R^q,\pi_*Q)$, where $R^q,\pi_*Q$ is a sheaf on $Q$ whose fiber at a point $p$ on $Q$ can roughly be identified with the rational cohomology of $\pi^{-1}(p)$. We work over the rational numbers so that we can neglect torsion effects. Now if the fibration is reasonably well-behaved, then all the fibers are are $dP_3$ surfaces, and hence $R^1$ and $R^3$ vanish identically. In our well-behaved case, we also have that $R^0$ and $R^4$ are globally constant, and so $H^i(Q,R^0) = H^i(Q,R^4) = H^i(Q,Q)$. In particular, $h^0(Q) = h^2(Q) = 1$ and $h^1(Q) = 2g_Q$. Therefore we get following table for $H^pR^q$, starting with degree $(0,0)$ in the bottom left corner end ending with degree $(2,4)$ on the top right:

\[
\begin{array}{ccc}
1 & 0 & H^2(Q,R^2) & 0 & 1 \\
2g_Q & 0 & H^1(Q,R^2) & 0 & 2g_Q \\
1 & 0 & H^0(Q,R^2) & 0 & 1 \\
\end{array}
\] (3.16)

Next we need to look at the $d_2$ differential. By definition, a generator of $H^p(Q,R^q,\pi_*)$ corresponds to a generator $\alpha^{p,q} \in \Omega^p(Q,\Omega^q(S))$ such that

\[d_S\alpha^{p,q} = 0, \quad d_Q\alpha^{p,q} = -d_S\alpha^{p+1,q-1}\] (3.17)

for some $\alpha^{p+1,q-1}$. Since we are really interested in the cohomology of the total differential $d = d_S + d_Q$, we simply check how far we succeeded:

\[d(\alpha^{p,q} + \alpha^{p+1,q-1}) = d_Q\alpha^{p+1,q-1}\] (3.18)

So our candidate $\alpha^{p,q} + \alpha^{p+1,q-1}$ may lift to $d$-cohomology if $d_2(\alpha^{p,q}) \equiv d_Q\alpha^{p+1,q-1}$ vanishes. In our case we see from the table that this differential already vanishes, so the non-zero cohomology groups in the table all correspond to generators of $H^{p+q}(M5,Q)$.
We are mostly interested in instantons with only the two universal fermionic zero modes, i.e. we want \( h^{0,i}(M5) = 0 \) for \( i > 0 \). To get \( h^{0,1} = 0 \) we clearly need \( g_Q = 0 \). To get \( h^{0,3} = 0 \) is more convenient to use the isomorphism \( h^{0,3} = h^0(N) \) and check if the normal bundle has sections, i.e. if the instanton is isolated. The \( h^{0,2} \) must come from \( H^0 R^2 \). If \( S = dP_9 \) we wouldn’t get any. If \( S = K3 \) the monodromy would typically kill it. Thus the arithmetic genus criterion for the M5-brane agrees well with the heterotic picture. On the heterotic side, the instantons that contribute are wrapped on a genus zero curve and isolated. Higher genus curves would contribute to certain multi-fermion terms. Non-isolated curves might possibly still contribute to the superpotential (although probably they do not), but one has to understand how to integrate over the family, just as for the M5-brane.

The meaning of some of the remaining cohomology groups might seem somewhat obscure. For instance \( H^1 R^2 \) could be quite large but hasn’t yet played a role for us. We can get much more insight in this case by using the spectral cover construction. Recall that the data of a \( dP_9 \) fibration over \( Q \) is equivalent to a spectral cover in an elliptic fibration over \( Q \). There is a direct mapping between the cohomology of the spectral curve (previously called \( \Sigma_{37} \) in section 2) and the ‘primitive’ cohomology of the \( dP_9 \) fibration, called the cylinder map [33]:

\[
H^{p,q}(\Sigma_{37}) \to H^{p+1,q+1}_A(M5)
\]  

(3.19)

In particular this maps \( H^{0,1}(\Sigma_{37}) \to H^{1,2}(M5) \) and allows us to directly relate the Jacobian of \( \Sigma_{37} \) to the intermediate Jacobian of the M5. We saw that the Jacobian actually played an important role in the heterotic string, through the partition function of the left-movers. Similarly, a non-zero flux through \( Q \) gets lifted to a non-zero flux though \( \Sigma_{37} \), which gets mapped to a \( G_4 \)-flux through a four-cycle of the M5-brane using the cylinder map. On the heterotic side we saw this corresponds to gauging shift symmetries of the Kähler moduli and modifying the zero mode structure of the worldsheet instanton. We will see the analogous role of the intermediate Jacobian and the flux in the M5-brane story in the next section. Thus eventually all the Hodge numbers of the M5 play some kind of role.

More generally, suppose that we consider instantons of type (i) but the M5 does not admit a surface fibration. Then we can only make use of the Leray for the elliptic fibration:

\[
\begin{align*}
H^0(D_3, R^2) & \quad H^1(D_3, R^2) & \quad H^2(D_3, R^2) & \quad H^3(D_3, R^2) & \quad H^4(D_3, R^2) \\
H^0(D_3, R^1) & \quad H^1(D_3, R^1) & \quad H^2(D_3, R^1) & \quad H^3(D_3, R^1) & \quad H^4(D_3, R^1) \\
H^0(D_3, R^0) & \quad H^1(D_3, R^0) & \quad H^2(D_3, R^0) & \quad H^3(D_3, R^0) & \quad H^4(D_3, R^0)
\end{align*}
\]  

(3.20)

where we used the shorthand \( R^i = R^i \pi_* Q \). We will assume that \( D_3 \) is a rational surface (i.e. del Pezzo, Hirzebruch, or a blow-up thereof), because isolated divisors are often of this type.
For generic models (but still with resolved ALE singularities allowed), $R^0 = Q$; $R^1$ is a varying rank two sheaf generated by the two one-cycles of the $T^2$; and $R^2 = Q + \tilde{R}^2$ where $\tilde{R}^2$ is a torsion sheaf, corresponding to the number of exceptional cycles that we get on the locus with enhanced gauge symmetry. We can use this to simplify the table. We have $H^i(D3, R^2) = 0$ for $i > 2$, $H^i(D3, R^0) = H^i(D3, Q)$, and if $D3$ is a rational surface then $H^1(D3) = H^3(D3) = 0$. Generically we also have $H^0(D3, R^1) = 0$, and by duality $H^4(D3, R^1) = 0$. Under these genericity assumptions, we get

$$
\begin{array}{cccc}
1 + H^0(D3, \tilde{R}^2) & H^1(D3, \tilde{R}^2) & H^2(D3) + H^2(D3, \tilde{R}^2) & 0 \\
0 & H^1(D3, R^1) & H^2(D3, R^1) & H^3(D3, R^1) \\
1 & 0 & H^2(D3) & 0 \\
\end{array}
$$

The remaining entries are expected to be non-vanishing generically. The Betti numbers will depend on the differential $d_2 : H^p R^q \to H^{p+2} R^{q-1}$, which may lift some further generators. Again we have $h^{0,1} = 0$, and $h^{0,3} = 0$ is the instanton is isolated, but it is much harder to make a clear statement about $h^{0,2}$ and it could well be non-zero. On the other hand, $h^{0,2}$ might still get lifted by $G$-fluxes, so we might expect fairly generic isolated $D3$ instantons wrapping a rational surface to contribute to the superpotential. In anticipation of the next section, we note again that even simple $M5$ branes intersecting the discriminant locus typically have a large number of three-cycles.

Next let us consider case (ii). If the fiber type is of type $I_1$, then the elliptic fiber has degenerated to a rational curve. However the curve has a double point and the arithmetic genus of this curve is the same as for an elliptic curve, i.e. one. Therefore $\chi(M5) \sim \chi(D3) \chi(T^2) = 0$. It follows that such an instanton should not contribute to the superpotential in $F$-theory.

This might seem somewhat counterintuitive from the IIb perspective, as a $D3$-instanton on top of a 7-brane in IIb is generally thought to contribute. The extra two fermionic zero modes implied by $\chi = 0$ are said to impose fermionic ADHM constraints, rather than lead to vanishing of the superpotential contribution. It is possible that we should be more careful because the $M5$ is singular. However if the arithmetic genus criterion should be modified, one should argue this directly in the $M$-theory language rather than by comparing with type IIb. The properties of an instanton can easily change in the Sen limit, and one should be careful not to extrapolate beyond the regime of validity. We will see some examples of this in section 4. Moreover if we further degenerate the fiber (which we discuss next), we seem to get results consistent with gauge theory.

Moving on, now let us assume that the fiber type is worse then $I_1$, i.e. the $D3$-instanton wraps the same cycle as a 7-brane with non-abelian gauge group $G$. In this case the $M5$-brane should behave just like a gauge theory instanton. This can be verified from the $M5$-picture, to some extent. The elliptic fiber over the $D3$ splits up into a chain of $P^1$'s, one for each node of the affine Dynkin diagram associated with the Kodaira fiber type, satisfying

$$
\sum d_i [P_i] = [T^2]
$$

(3.22)
where $d_i$ are the Dynkin indices. If the singular locus is of split type, then for each $\mathbb{P}^1$ we get a divisor $D_i$, consisting of a $\mathbb{P}^1$-fibration over the $D3$ with fiber $P_i$. For non-split type we get fewer such divisors, as some of the $P_i$ are related globally by monodromy. The $M5$-brane can wrap each of the $D_i$. In the context of $M$-theory compactified on such a Calabi-Yau four-fold, these $M5$-branes may be identified with the monopoles/fractional instantons of the non-abelian gauge theory.

In three dimensions the $N = 2$ vector multiplet has an adjoint scalar $\Phi$. Vacuum configurations satisfy $[\Phi, \Phi] = 0$, so we may diagonalize $\Phi$. Let us introduce $\text{rank}(G) + 1$ real scalars denoted by $\phi_i$. Geometrically the $\phi_i$ specify the sizes of $P_i$, which can be finite in the $M$-theory context, and they are defined only up to Weyl transformations, which means we can restrict them to take values in a fundamental domain. From (3.22) we have

$$\sum d_i \phi_i = \text{vol}(T^2) \equiv 1/R \quad (3.23)$$

The $\text{rank}(G)$ linear combinations of $\phi_i$ orthogonal to this correspond to the eigenvalues of $\Phi$. Now let us assume that the base is a del Pezzo surface, so that $h^{0,1}(S) = h^{2,0}(S) = 0$ and we get a pure $N = 2$ gauge theory in three dimensions. The superpotential of the three-dimensional gauge theory is given by the partition function. We can again apply the Leray sequence, yielding $h^{0,i}(D_j) = 0$ for $i \neq 0$, so each $D_j$ contributes to superpotential. Further if there are no monodromies among the $P_i$, then $h^3(D_i) = 0$ and the partition function for the chiral two-form is trivial. Hence the partition function is given by [34]

$$W = \sum \exp(-d_i \phi_i/g_2^3) \quad (3.24)$$

Since wrapped $M5$ branes correspond to monopoles, this is reminiscent of the well-known results of Polyakov [35]. Solving for the $F$-terms yields $h^\vee$ vacua, where $h^\vee$ is the dual Coxeter number of the non-abelian gauge group $G$.

The relation of these vacua with the 4d gauge theory vacua arising for $R \to \infty$ is somewhat subtle, but can formally be obtained as follows. We introduce a Lagrange multiplier field $S$ to impose the constraint (3.23). The superpotential becomes

$$W = S(\tau - \sum d_i \phi_i/g_2^3) + \sum \exp(-d_i \phi_i/g_3^2) \quad (3.25)$$

where we used the relation $g_4^2 \sim Rg_3^2$. Upon integrating out the $\phi_i$ and using $\sum d_i = h^\vee - 1$ one obtains

$$W = \tau S + h^\vee S(\log(S/\Lambda^3) - 1) \quad (3.26)$$

which is the Veneziano-Yankielowicz superpotential for the gaugino bilinear $S \sim \text{Tr}(\lambda \lambda)$. It is expressed purely in terms of 4d quantities, so we can take the $3d \to 4d$ limit. We can also integrate out $S$ to get

$$W = -h^\vee \Lambda^3 e^{-\tau/h^\vee} e^{2\pi ik/h^\vee} \quad (3.27)$$
in the $k$th vacuum. Although it looks constant, in field theory the meaning of this superpotential is that $\Delta W$ calculates the tension of domain walls. In a gravity theory, $\Lambda \sim M_{Pl} \exp(-1/b_0 g_4^2(M_{Pl}))$ depends on the moduli and is not a constant.

In the $F$-theory limit, we cannot wrap an $M5$-brane on each $D_i$ separately, but only on the sum $\sum d_i D_i$. Such an $M5$-brane is singular, but the holomorphic Euler character of such a divisor vanishes, so we expect the $M5$-brane does not contribute to the superpotential. Indeed in the four dimensional gauge theory this superpotential is not generated by instantons but by strong dynamics, consistent with the arithmetic genus criterion [32]. If the $M5$ contributed, it would indicate a stringy correction to the gauge theory result, which we believe to be absent (as it would modify the $E_8$ gaugino condensation story for example).

Actually getting such pure gauge groups is somewhat rare. In $M$-theory compactifications to three dimensions, we can certainly construct ALE fibrations to get any desired gauge group. However in order for such a compactification to lift to $F$-theory, the compactification must admit an elliptic fibration, and this imposes some important constraints. More typically, embedding in an elliptic fibration will force the presence of matter curves, where the elliptic fiber further degenerates. Such matter curves are closely related to anomaly cancellation conditions in six dimensions. If the matter curve has genus one or larger, and there is no $G$-flux, then there are massive quarks which can becomes massless at special loci on the moduli space.

Some special cases were studied in [36]. Since the $M5$-branes correspond to monopoles in three dimensions, one can also understand their contributions more directly from the gauge theory perspective [37]. Taking the $R \to \infty$ limit should then yield the 4d superpotential.

3.3. The chiral two-form and holomorphic factorization

The most interesting field propagating on the $M5$-brane is the chiral two-form. Let us first discuss its partition function from a down-to-earth point of view, to see how theta-functions arise. Then we move to a more abstract point of view. Most of the statements and manipulations regarding theta-functions and holomorphic factorization have well-known analogues on higher genus Riemann surfaces. Mostly following [38, 39, 5].

The basic problem with the theory of a chiral two-form is that it can not have a conventional Lagrangian description. Indeed, suppose we try to write one. By self-duality we have

$$\int_{M5} H \wedge *H \propto \int_{M5} H \wedge H = 0$$

We can however write the Lagrangian for the theory of a non-chiral two-form, which contains both a chiral and an anti-chiral two-form, in such a way that the anti-chiral part
decouples from the chiral part. Such an action is given by

$$S = \int_{M5} \frac{1}{2} |H - i^* C_3|^2 - iH \wedge i^* C_3$$

(3.29)

where $H$ is not required to be self-dual. The art is then to compute observables using the non-chiral two-form, and to extract the results for the chiral two-form by holomorphic factorization. Let us discuss this procedure for the partition function.

In the partition function, we have to sum over the fluxes of $B$. Let us choose a dual basis of 3-cycles $\{A_i, B^j\}$ in $H^3(M5, \mathbb{Z})$:

$$A^i \cap B_j = \delta^i_j$$

(3.30)

In a suitable basis, the $*$-operator has the following eigenvalues:

$$*\omega^3,0 = -i \omega^3,0, \quad *\omega^2,1 = +i \omega^2,1, \quad *\omega^1,2 = -i \omega^1,2, \quad *\omega^0,3 = +i \omega^0,3.$$

(3.31)

We take a basis $\omega_{j}$ of $H^2(M5) + H^0,3(M5)$, such that $*\omega_{j} = i \omega_{j}$. Then up to an $SO(n)$ rotation we have

$$\int_{A^i} \bar{\omega}_j = \delta^i_j, \quad \int_{B_j} \bar{\omega}_j = \tau_{ij}$$

(3.32)

where $\tau_{ij} = \tau_{ji}$ is the period matrix.

Now we decompose $H = dB$ into a harmonic piece (the flux), which we can think of as a classical field configuration, and an orthogonal piece which contains the quantum fluctuations. We may expand the fluxes as

$$2\pi n^i = \int_{A^i} H, \quad 2\pi m^j = \int_{B_j} H$$

(3.33)

We further write

$$C_3 = 2\pi z^i \omega_i + c.c$$

(3.34)

To obtain the classical partition function, we must evaluate the path integral on all the classical field configurations:

$$Z_0(\tau, z) = \sum_{n^i,m^j} e^{-S_{cl}[\tau, z, m, n]}$$

(3.35)

In the present context this sum was evaluated carried out in [39]. After Poisson resummation, and removing an anomalous factor, it can be written as a sum of squares of theta functions:

$$Z_0(\tau | z) \sim \sum_{\alpha, \beta} \left| \Theta \left[ \alpha \beta \right](\tau | z) \right|^2$$

(3.36)
where
\[
\Theta_{[\theta]}(\tau|z) = \sum_{Z+\theta} \exp\left(\frac{1}{2} n^i n^j 2\pi i \tau_{ij} + 2\pi i n^i (z_i + \phi_i)\right) \tag{3.37}
\]

Since the action we wrote really describes both the chiral and anti-chiral two-form, we now take a holomorphic square root to get a partition function for the chiral two-form only. However there is no unique choice for the classical partition function. Rather, there is a unique partition function for every choice of spin structure.

In addition, we have to evaluate the path integral over the quantum fluctuations and factorize the answer, formally described in [39]. This does not depend on the choice of spin structure. We get that the partition function for spin structure \(\alpha, \beta\) is of the form
\[
Z^{+[\alpha]}(\tau|z) = \frac{\Theta_{[\alpha]}(\tau|z)}{\Delta^+} \tag{3.38}
\]
where \(1/\Delta^+\) arises from the sum over quantum fluctuations. Since it is nowhere vanishing and independent of the spin structure, much of the interesting information is contained in the classical theta function.

We can also consider the more abstract point of view. The partition function is a section of a line bundle on the intermediate Jacobian, which we need to specify. We can do this the same way as for Riemann surfaces. The Hodge \(\ast\)-operator induces a complex structure on the Jacobian. This is also known as the Weil complex structure. The intersection pairing induces a principal polarization \(\omega\). The line bundle is defined by specifying the phases \(\phi = \pm 1\), subject to
\[
\phi(a + b) = (-1)^{\omega(a, b)} \phi(a) \phi(b), \quad \phi(1) = (-1)^{2\alpha}, \quad \phi(\tau) = (-1)^{2\beta} \tag{3.39}
\]
which give the monodromies, and Chern class \(\omega\), which is positive definite. By the index theorem, for each spin structure we get a unique theta function, corresponding to the unique holomorphic section of the line bundle, which we identify with the partition function.

There is another well-known version of the Jacobian with a different complex structure, known as the Griffiths Jacobian. It corresponds to an involution with the same eigenvalue for \(H^{3,0}(M5)\) and \(H^{2,1}(M5)\). The Griffiths Jacobian is known to vary nicely in holomorphic families. For Calabi-Yau three-folds for instance this can be seen in the special geometry relations:
\[
\int_{A^i} \Omega = z^i, \quad \int_{B_j} \Omega = \partial F/\partial z^j \tag{3.40}
\]
The differentials \(\partial_i \Omega\) span \(H^{3,0} + H^{2,1}\), and the period matrix \(\tau_{ij} = \partial_i \partial_j F\) depends holomorphically on the \(z^i\). (In special geometry the overall rescaling of the \(z^i\) is considered a
gauge symmetry, but in our situation it is a physical modulus and corresponds to moving the $M5$ in the normal direction). However if $h^{3,0}$ and $h^{2,1}$ are both non-zero, then $\omega$ is not positive definite. As a result, instead of having a section, $L$ has higher cohomology and the theta function does not exist. If we would try to construct it as a series as we did earlier but instead with the indefinite norm, we would find that the series diverges.

The theta functions on the Weil Jacobian do not suffer from this, as we might expect physically. However there is a price to pay; unlike the Griffiths Jacobian, the Weil Jacobian generally does not vary holomorphically in families. As we can see from the Calabi-Yau example above, in the Weil complex structure the period matrix $\tau$ is not a holomorphic function of the moduli. In other words, in general the partition function of a chiral two-form suffers from a holomorphic anomaly. The remaining contributions to the partition function from the scalars and fermions appear to vary holomorphically.

Fortunately for isolated instantons, we have $h^{3,0}(M5) = 0$ and the Weil and Griffiths Jacobians coincide. So we do not get an immediate contradiction with the holomorphy of the superpotential. Non-isolated instantons might still contribute to the superpotential after integration over the family. Perhaps we should take this as evidence that non-isolated instantons in fact do not contribute to the superpotential.

Finally, we should take an appropriate linear combination of these partition functions together with those of the bosons and fermions, so that we end up with a theory that is free from global anomalies as well. The general prescription is not clear to us, but at least in the cases we consider in detail there is a close relation with the heterotic string and so there is a natural expression.

As an aside, the vanishing of the partition function of the chiral two-form cannot be ascribed to its zero modes. The chiral two-form is a bosonic field and has periodic identifications due to gauge invariance, so it behaves as a compact scalar. Furthermore the action does not depend on the VEV through a two-cycle. The integral over these zero modes thus only gives a finite overall factor and cannot cause vanishing of the partition function, so these zero modes can be safely ignored. The one exception to this is the VEV through the elliptic fiber, which becomes non-compact and is identified with the emerging Euclidean $\mathbb{R}$ in the $F$-theory limit.

All our above discussion assumed that $G|_{M5} = 0$ in $H^4(M5)$. Is this necessarily the case? Let us first recall that any $G$-flux in an $F$-theory compactification must be orthogonal to four-cycles of the following two types:

(i) $\sigma_{B_3} \cdot H^2(B_3) \subset H^4(Y_4)$
(ii) $\pi_5^* H^4(B_3) \subset H^4(Y_4)$

Now the class of the cycle wrapped by the $M5$ is itself in $\pi_5^* H^2(B_3)$. Let us assume for now that $i_*i^* G = \delta^2(M5) \wedge G$ is a non-trivial class in $H^6(Y_4)$. We will return to the case that this fails later. Poincaré duality implies that the intersection pairing $H^6(Y_4) \cap H^2(Y_4)$
is non-degenerate. Then modulo torsion,

\[ G|_{M5} \neq 0 \quad \Leftrightarrow \quad \delta^2(M5) \wedge G \wedge \omega \neq 0 \quad (3.41) \]

for some \( \omega \in H^2(Y_4) \). Now \( \omega \) cannot be in \( \pi^*H^2_Y(B_3) \) or \( \sigma_{B_3}H^0(B_3) \), because then \( \delta^2(M5) \wedge \omega \) is a class of type \((i)\) or \((ii)\), and the \( G \)-flux would be automatically orthogonal to it. Any other \( \omega \in H^2(Y_4) \) is in the coroot lattice of the 4d gauge group, i.e. \( \omega = \omega^X \) for some \( U(1)_X \) gauge symmetry. Therefore

\[ G|_{M5} \neq 0 \quad \Leftrightarrow \quad \int_{M5} G \wedge \omega^X \neq 0 \quad (3.42) \]

for some \( \omega^X \) in the coroot lattice.

This leads us to the \( F \)-theory analogue of the gauged shift symmetries that we encountered for the heterotic string in section 2.7 (also discussed in v2 of [40]). Recall from (3.10) that a Kähler modulus \( T_D \) shifts under a \( U(1)_X \) gauge transformation when

\[ \delta T_D = \frac{i}{4\pi} \int_D G \wedge \Lambda = 2\pi i \lambda_X \frac{1}{4\pi} \int_D G \wedge \omega^X = 2\pi i q^X_d \lambda_X \neq 0 \quad (3.43) \]

The low energy Kähler potential must then be of the form

\[ K_T \sim -M_{Pl}^2 \log \frac{1}{6} \cdot \frac{1}{8} (T + T^* - q^X V_X)^3 \quad (3.44) \]

The \( U(1)_X \) in this case picks up a mass through GS couplings to the axion \( \text{Im}(T_D) = \int_D C_6 \) and one finds an Fayet-Iliopoulos term of the form

\[ \xi^X \sim M_{Pl}^2 \text{Vol}(B_3)^{-1} \int_D G \wedge \omega^X \wedge J \quad (3.45) \]

with volumes measured in 10d Planck units. Furthermore, a term in the superpotential of the form

\[ \int d^2\theta e^{-T_D} \quad (3.46) \]

is forbidden by \( U(1)_X \) gauge invariance. So an \( M5 \)-brane wrapped on \( D \) can not contribute to the superpotential, but it can generate couplings that are forbidden in perturbation theory due to the \( U(1)_X \) symmetry.

So let us consider an \( M5 \)-brane with \( G|_{M5} \neq 0 \) in \( H^4(M5) \). In this case there is a tadpole for the chiral two-form, because:

\[ \int_{M5} H \wedge C_3 \sim \int_{M5} B \wedge G_4 \quad (3.47) \]
and hence the partition function vanishes. We already saw the meaning of this in the heterotic setting in section 2.7; in this case there are chiral fermion zero modes, and to get a non-vanishing answer we need to insert some fermionic operators in the partition function to absorb these zero modes.

Here too there are some natural operators we have to insert to get a non-vanishing answer. These are the Wilson surface observables discussed in [39]:

\[ W(Q) = e^{i \int B \wedge Q} \]  

(3.48)

where \( Q \) is a four-form in \( M5 \). Dually we may think of \( Q \) as a two-cycle. Inserting such operators in the path integral:

\[ \langle W(Q_1) \ldots W(Q_n) \rangle \sim \int d[B] e^{i \int B \wedge Q_1} \ldots e^{i \int B \wedge Q_n} e^{-S} \]  

(3.49)

such that

\[ \left[ \frac{G}{2\pi} \right] = [Q_1] + \ldots + [Q_n] \]  

(3.50)

in \( H^4(M5) \), then there is no tadpole and we get a non-vanishing answer. Therefore these Wilson surface observables play the role of fermionic zero modes for the \( M5 \)-brane. Again such correlators can be computed (in principle) in the \( M5 \)-brane theory using holomorphic factorization.

In order to compute correlation functions on the \( M5 \), analogous to section 2, we need the vertex operators on the \( M5 \) brane for deformations of the background corresponding to chiral fields. Chiral fields correspond to some particular simultaneous deformations of \( C_3 \) and the complex structure of the Calabi-Yau four-fold. The Wilson surface observables above should appear as a factor in these \( M5 \) vertex operators. It is not clear that such calculations will be very illuminating so we will not proceed along these lines. Instead we focus on a class of \( M5 \) instantons where the calculations can be be mapped to a more standard problem.

Finally we return to the possibility that \( \delta^2(M5) \wedge G = 0 \) even though \( i^*G \) is a non-trivial class in \( H^4(M5) \). In terms of Poincaré duals, this means that \( [G]_{M5} \in H_2(M5) \) is the boundary of a three-chain \( \Gamma \) in \( Y_4 \). By wrapping an \( M2 \) brane on \( \Gamma \) with the opposite orientation, we can cancel the tadpole for the chiral two-form. However even if this configuration would be supersymmetric, it looks like a subleading effect, so at order \( \exp(-T) \) we do not expect it to contribute to the superpotential.
3.4. M5 wrapped on a four-cycle: the MSW CFT

Suppose the M5 worldvolume is fibered over a Riemann surface Q:

\[ S \rightarrow M5 \hspace{1cm} \downarrow \hspace{1cm} Q \]

(3.51)

The partition function depends only classically on the Kähler moduli, through the exponential factor. As a result, we may change the metric on the M5-brane without affecting the one-loop determinants, as long as we keep the complex structure moduli fixed. Thus we can scale down the fiber so that the M5-brane collapses to a string, and consider the effective theory on this string. This reduces the problem of computing zero modes and the partition function to a problem on Q, where we can address them explicitly.

Let us do the reduction. The scalars simply reduce to non-chiral bosons on Q. As for the fermions, locally on Q we have \( K_{M5} \sim K_S \) and

\[ \Omega^{0.2}(M5) \sim \Omega^{0.1}(Q, \Omega^{0.1}(S)) \oplus \Omega^{0.0}(Q, \Omega^{0.2}(S)) \]

(3.52)

Positive chirality spinors on the M5 are given by even \((0, i)\) forms. Upon reduction and a judicious use of the \( \ast \)-operator on S, we get the following complex fermions

\[
S_Q^+ \otimes \Omega^{0.0}(Q, (\Omega^{0.0}(S) + \Omega^{0.2}(S)) \otimes 2_{\pm \frac{i}{2}}
\]

\[
S_Q^- \otimes \Omega^{0.0}(Q, \Omega^{0.1}(S)) \otimes 2_{\pm \frac{i}{2}}
\]

(3.53)

When scaling down S, we should keep only the ground states on S, i.e. the global holomorphic sections. Thus even degree Dolbeault cohomology on S yield right-moving complex fermions on Q, and odd degree Dolbeault cohomology yields left-moving fermions on Q. The symplectic Majorana condition reduces to an ordinary Majorana condition.

Similarly we can understand the reduction of the chiral two-form. In Euclidean space, the chiral two-form is imaginary self-dual. Expanding in self-dual and anti-self-dual two-forms on S, and taking ground states, we get \( b^+_2(S) \) right-moving chiral bosons, and \( b^-_2(S) \) left-moving chiral bosons. Reduction with one or two indices along Q yields only massive fields.

Finally we should also reduce the coupling to the F-theory three-form \( C_3 \). Locally on Q this yields \( b_2(S) \) gauge fields. The coupling to the chiral two-form reduces as

\[
\int_{M5} H^+ \wedge C_3 \rightarrow \int_Q d^2z \partial_z \phi^+ A_{\bar{z}}
\]

(3.54)

The chiral bosons may be fermionized, and we end up with a collection of chiral fermions coupled to gauge fields. This puts us exactly in the type of situation studied in section 2, where we know how to calculate an instanton correction.
In particular, let us assume that $S$ is a $K3$ surface. In this case we have $b_1(S) = 0$, $h^{0,2} = 1$ and $(b^+_2, b^-_2) = (19, 3)$. This is precisely the Narain data for the heterotic string in 7 dimensions, and together with the five non-chiral scalars and the eight real fermions, we recover all the physical degrees of freedom of the worldsheet theory of the 7d heterotic string.

In the context of $F$-theory, the $K3$ surface has some further special properties. The elliptic fibration on the $K3$ surface allows us to single out a $(1, 1)$ sublattice, generated by the elliptic fiber and a section:

$$v_1 = [T^2], \quad v_2 = [T^2] + [\mathbb{P}^1]$$

Of interest are the masses of membrane BPS states, wrapped on these cycles. On the $M$-theory side, these masses are proportional to the volume of the wrapped cycle in Planck units. On the heterotic side, the $(1, 1)$ sublattice corresponds to the momentum and winding charges $(n_8, m_8)$ of a distinguished $S^1$ on the internal $T^3$. Their masses are given by

$$m(v_1) = 2\pi n_8/R_8, \quad m(v_2) = m_8 R_8/2\pi$$

The BPS states with these charges are the Dabholkar-Harvey states. We can follow the masses of these BPS states on both sides. In the $F$-theory limit of small elliptic fiber with area $A \sim 1/R_8$, the distinguished heterotic $S^1$ decompactifies, and the Narain data reduces to that of the 8d heterotic string, i.e. a lattice of signature $(18, 2)$. Thus we may think of our MSW string as a heterotic string living in the base of a $T^2$-fibration.

If $S = dP_9$ we still get a chiral string, but with a Narain lattice of signature $(9, 1)$. In the $F$-theory context, we further eliminate the $(1, 1)$ sublattice corresponding to the elliptic fiber and a section of the $dP_9$. The resulting lattice of signature $(8, 0)$ is none other than the $E_8$ lattice. This can be thought of as half of an eight-dimensional heterotic string, in the limit that we decouple the winding modes on the internal $T^2$. As we already mentioned, the cylinder map in this case gives an explicit map relating the Jacobians on both sides, as well as the theta functions. We could also consider an $M5$-brane wrapping $S = T^4$. In this case, we get a left-right symmetric string.

\footnote{In order to compare properly, we really have to express the masses in 7d Planck units.}
4. D3-instantons in the IIB weak coupling limit

Let us recap what we have seen so far. D3-instantons in F-theory are strongly coupled objects where both electric and magnetic degrees of freedom are important, so we need some exact techniques in order to deal with them. The situation becomes more transparent when we interpret the D3 as a wrapped M5-instanton.

The worldvolume of the M5 contains a chiral two-form, which is inherently quantum mechanical and does not admit a classical description. Nevertheless there are some calculational techniques, primarily the method of holomorphic factorization. We identified a subclass of M5 instantons where we can further reduce calculations to two dimensions, where we get a heterotic-like CFT description. This connected our discussion on M5 instantons with our earlier discussion on instantons in the heterotic string.

Most of the paper has been fairly conventional and unsurprising, at least from the heterotic point of view. In this section we would like to turn to the problem of understanding instanton corrections from a more IIB like perspective, by taking a Sen limit. This involves changing complex structure moduli, rather than Kähler moduli, and so we might expect some qualitative differences to arise.

In fact there appears to be some tension between the behaviour of F-theory instantons that we saw in this paper and D3 instantons in perturbative IIB, especially in the 37 sector. This can be traced to the fact that the theory develops additional U(1) symmetries as we dial the string coupling to zero. The extra U(1)s impose strong selection rules in the IIB limit, but they do not generalize to F-theory. This is already familiar for other couplings, such as the classical $10 \cdot 10 \cdot 5$ Yukawa coupling in $SU(5)_{GUT}$ models, and we will see it holds for instanton contributions as well.

Our results are therefore completely consistent with effective field theory considerations: everything that is not protected by a symmetry could in principle be generated. To illustrate this, we show that some IIB selection rules on instantons in the presence of chiral matter disappear in F-theory. This means that perturbative type II is not a good guide to these instantons, and further insights on summing the contributions and multi-covers will likely have to come from heterotic instantons.

4.1. General picture

Let us briefly recall some aspects of open string quantization and orientifolds in the ‘upstairs’ picture, following the notation of sections 13.4 and 14.3 of [41]. There is already a discussion of instantons in IIB orientifolds available in the literature, see [13] in particular, but we will give a slightly extended discussion in this subsection and the next. We will use $D3$ to denote instantons in the upstairs picture and $D3$ to denote instantons in the downstairs picture.
The \( \mathcal{D}3 \) branes live in a Calabi-Yau three-fold \( X_3 \), with a holomorphic involution \( \sigma \) mapping \( \Omega^{3,0} \rightarrow -\Omega^{3,0} \). The massless modes are of the form

\[
\psi_{-1/2}^M \ |0\rangle_{NS}, \quad |s_0 s_1 s_2 s_3 s_4\rangle, \quad \prod 2s_i = -1 \quad (4.1)
\]

In a curved background this leads to the following 33 zero modes. First we have the modes whose internal wave functions are given by

\[
\text{Ext}^0(i_sL, i_sL) = H^0(\mathcal{D}3, \mathcal{O}) \quad (4.2)
\]

Here we apply a 4d raising operator \( \psi^\mu_{-1/2} \) to the ground state, giving us the four real scalars \( x^\mu \) describing the position of the instanton. By supersymmetry, the same \( \text{Ext}^0 \) generator defines a state \( |(-\frac{1}{2})^5\rangle \) in the Ramond sector. Together with \( |(+\frac{1}{2})^2(-\frac{1}{2})^3\rangle \) this gives a fermionic mode we call \( \psi^\alpha_33 \). From the complex conjugate of this internal wavefunction (or using the dual \( \text{Ext}^3 \) generator) we also get fermionic zero modes \( |(-\frac{1}{2})(+\frac{1}{2})^4\rangle \) and \( |(+\frac{1}{2})(-\frac{1}{2})(+\frac{1}{2})^3\rangle \) which we call \( \psi^\dot{\alpha}_33 \). Here \( \alpha, \dot{\alpha} \) denote \( SO(4) = SU(2) \times SU(2) \) spinor indices associated to the uncompactified \( R^4 \) directions.

By applying an internal oscillator \( \psi^\dot{\mu}_{-1/2} |0\rangle_{NS} \), we further get 33 zero modes whose internal wave functions are given by

\[
\text{Ext}^1(i_sL, i_sL) = H^0(\mathcal{D}3, K) \oplus H^1(\mathcal{D}3, \mathcal{O}) \quad (4.3)
\]

These zero modes split up into modes that are even or odd under \( \sigma^* \).

The action of parity on the modes is given by

\[
P : \alpha_m \rightarrow \pm e^{i\pi m} \alpha_m \quad (4.4)
\]

using + for NN and – for DD boundary conditions, and

\[
P : \psi_r \rightarrow \pm e^{i\pi r} \psi_r \quad (4.5)
\]

where the ± agrees with the action on \( \alpha_m \) in the Ramond sector, and is opposite in the NS sector. Thus the orientifold action on the zero modes is almost the same as for the 77 zero modes of a 7-brane wrapping the same four-cycle. The only essential difference is that due to changing the boundary conditions from \( DD \) to \( NN \) along \( R^4 \), from (4.5) we get an extra minus sign for some of the Ramond sector states. The action of parity on the universal modes from \( H^0(\mathcal{D}3) \) is then given by

\[
x^i \rightarrow \gamma^{-1} x^i T \gamma, \quad \psi_{\alpha} \rightarrow \gamma^{-1} \psi_{\alpha}^T \gamma, \quad \psi_{\dot{\alpha}} \rightarrow -\gamma^{-1} \psi_{\dot{\alpha}}^T \gamma \quad (4.6)
\]
Hence for the $O(1)$ projection $\gamma = (1)$, we keep $(x^i, \psi_{33}^\alpha)$ but project out $\psi_{33}^\dot{\alpha}$. Similarly, for generators of (4.3) we have

$$A \rightarrow -\gamma^{-1} A^T \gamma, \quad \chi_{\alpha} \rightarrow \gamma^{-1} \chi_{\alpha}^T \gamma, \quad \chi_{\dot{\alpha}} \rightarrow -\gamma^{-1} \chi_{\dot{\alpha}}^T \gamma$$

(4.7)

For instance, to go from $|(-\frac{1}{2})^5\rangle$ to $|(-\frac{1}{2})(+\frac{1}{2})(-\frac{1}{2})^2\rangle$ we apply two raising operators, one with $NN$ and one with $DD$ boundary conditions. This relates the parity action on $\psi_{\alpha}$ and $\chi_{\dot{\alpha}}$, with the extra minus sign from (4.5). One subtlety is that the action on $\psi_{\dot{\alpha}}^2$ is opposite for indices tangent to or normal to the brane. However in (4.3) we have mapped normal bundle valued forms to canonical bundle valued forms by contracting with $\Omega^{3,0}$. Since $\sigma^*\Omega^{3,0} = -\Omega^{3,0}$, the combined action of parity and $\sigma^*$ puts all the generators of (4.3) on the same footing, resulting in the orientifold actions listed above. Thus for even generators of (4.3) we get fermionic zero modes $\chi_{\dot{\alpha}}$, and for odd generators we get bosonic and fermionic zero modes $(A, \chi_{\dot{\alpha}})$, assuming the $O(1)$ projection. Although listed for completeness, we are actually interested in isolated instantons, so we assume there are no such zero modes.

Finally we need to quantize the 37 strings. We will assume here that the $D3$ and $D7$ are intersecting, rather than coincident. In the upstairs picture, there are zero modes with internal part given by

$$\text{Ext}^1(i_*L_3, j_*L_7) \sim H^0(\Sigma_{37}, L_3^\vee \otimes L_7 \otimes K_{D3}|\Sigma_{37})$$

(4.8)

and similarly for the 73 strings. This is similar to a $D1 - D9$ system; in the $NS$ sector, the zero point energy is positive, so we do not get any bosonic zero modes. In the Ramond sector the ground state energy vanishes, but the raising operators carry positive energy, so we cannot apply them to the ground state. Hence for each $\text{Ext}^1$ generator we only get a 2d chiral fermion $\lambda$ on the intersection. The orientifold action relates the 37 and 73 zero modes.

We have to be a little careful about the bundle we put on the $D3$-instanton. We assume vanishing $B$-field in our discussion. Recall that due to the Freed-Witten anomaly, the gauge field on a 3-brane does not take values in an ordinary line bundle but in a ‘fake’ line bundle

$$\tilde{L} = L \otimes K^{-1/2}$$

(4.9)

which is not necessarily integer quantized. Now world-sheet parity relates a gauge field to its dual, and hence

$$P : \tilde{L} \rightarrow \tilde{L}^\vee = (L^\vee \otimes K) \otimes K^{-1/2}$$

(4.10)

We can also understand the extra factor of $K$ from the point of view of $D$-brane charges. The coupling to RR fields is given by

$$\int_{X_3} \text{ch}(i_*L) A(X_3)^{1/2} \wedge C$$

(4.11)

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where \( \mathbf{C} \) is the formal sum of RR potentials. The extra factor of \( K \) yields the expected action on the Chern character, viz.

\[
\text{ch}_j(i_*(L^\vee \otimes K)) = (-1)^{j+1} \text{ch}_j(i_*L) \tag{4.12}
\]

This states, amongst other things, that \( D3 \)-brane charges \((j = 3)\) are preserved. If we simply had replaced \( L \) by \( L^\vee \), this would not have been the case.

Now assume we have an irreducible \( O(1) \) instanton. The Euclidean \( D3 \)-brane gets mapped to itself under the orientifold action, so this implies that

\[
\sigma^*(L^\vee \otimes K) = L \Rightarrow L \otimes \sigma^*L = K \tag{4.13}
\]

Thus we can not take the trivial line bundle \( O_{D3} \) on the \( D3 \), but from (4.9) we see that under favourable circumstances it is compatible with ‘setting the gauge field to zero.’ Given a solution, we can obtain further solutions for any generator of \( h^{1,1}(D3)_- \).

Now let us start comparing this with an \( M5 \) instanton in the Sen weak coupling limit. The coefficients of the Weierstrass equation

\[
y^2 = x^3 + fx + g \tag{4.14}
\]

are written as

\[
f = -\frac{1}{48}(b_2^2 - 24\epsilon b_4) \\
g = -\frac{1}{864}(-b_2^2 + 36\epsilon b_2 b_4 - 216\epsilon^2 b_6) \tag{4.15}
\]

with \( b_i \) a section of \( K_{B_3}^{-i} \). Then we take a limit \( \epsilon \to 0 \), so that the generic fiber becomes of type \( I_1 \). In this limit, the elliptic fibration is given by

\[
y^2 = x^3 - \frac{1}{48}b_2^2 - \frac{1}{864}b_2^3 = -\frac{1}{864}(b_2 - 6x)(b_2 + 12x)^2 \tag{4.16}
\]

Introducing a new coordinate \( \tilde{y} = y/(b_2 + 12x) \), we can write this as

\[
\tilde{y}^2 = -\frac{1}{864}(b_2 - 6x) \tag{4.17}
\]

This is the equation of a rational curve. The map \((x, \tilde{y}) \to (x, y)\) identifies the two points

\[
(x, \tilde{y}) = (-b_2/12, \pm \sqrt{-b_2/576}) \tag{4.18}
\]
on each fiber. Hence the generic elliptic fiber has degenerated to a nodal curve, a $\mathbb{P}^1$ with two points identified. The two points that get identified define a double cover $X_3$ over $B_3$, branched over $b_2 = 0$, which is a Calabi-Yau three-fold because $b_2$ is a section of $K_{B_3}^{-2}$. In other words, we get a Calabi-Yau three-fold $X_3$ defined by an equation

$$\xi^2 = b_2$$

(4.19)

and sitting naturally inside the Sen limit of the Calabi-Yau four-fold. We identify $X_3$ with the IIb Calabi-Yau.

The intersection of our $M_5$ with $X_3$ yields a divisor, which we identify with the IIb $D_3$ instanton. This $D_3$-instanton is similarly a branched double cover over a divisor $D_3$ in $B_3$. It is invariant under the orientifold action $\xi \to -\xi$, and generically irreducible. Assuming it intersects the orientifold locus, it should then correspond to an $O(1)$ instanton.

The relation between the $M_5$-brane and the $D_3$ it is fibered over was largely discussed in section 3, and we merely need to take the limit. For the scalars, the normal bundle of the $M_5$ is the pull-back of the normal bundle of the $D_3$, and they have the same sections. There is one extra Euclidean normal direction for the $D_3$, which is identified with $\int_{T^2} B^+$ in the $M_5$-description. In particular the $D_3$ inherits the four universal bosonic zero modes from the $M_5$. If the $D_3$ is not tangential to the orientifold locus, then the normal bundle to $D_3$ is the pull-back of the normal bundle to $D_3$. (Since the normal and canonical bundle are the same upstairs, this also means that the difference between the normal bundle and the canonical bundle to the $D_3$ is given by the intersection with the branch locus). This pull-back may have additional sections, but the even sections (with respect to the orientifold action) are the pull-back of the sections downstairs, and recalling that the orientifold action on $H^0(D_3, N)$ and $H^0(D_3, K)$ differ by a minus sign, from our earlier discussion we see that these are precisely the ones that survive the orientifold projection. The odd sections give rise to fermionic but not bosonic zero modes after orientifold projection.

Next let us consider the fermions. We have essentially already discussed this in section 3, but let us also look at this from the point of view of the Leray sequence for Dolbeault cohomology

$$H^i(M_5, O_{M_5}) \sim H^i(D_3, O_{D_3}) + H^{i-1}(D_3, R^1\pi_* O_{M_5})$$

(4.20)

Here we simplified $R^0\pi_* O_{M_5} \sim H^{0,0}(T^2) \otimes O_{D_3} = O_{D_3}$. Close to the Sen limit, $R^1\pi_* O \sim H^{0,1}(T^2) \otimes O_{D_3}$ is constant almost everywhere on the base. Let us denote the generators of $H^1(T^2, \mathbb{Z})$ by $\alpha, \beta$, subject to the relations $\alpha^2 = \beta^2 = 0$, $\alpha \beta = -\beta \alpha = 1$. Then the $(0,1)$ form on $T^2$ is proportional to

$$\alpha_+ = (\text{Im}\tau)^{-1}(\beta - \bar{\tau} \alpha)$$

(4.21)

The monodromy around an orientifold plane maps $\alpha_+ \to -\alpha_+$, and the monodromy around a $D7$-brane leaves $\alpha_+$ invariant. We expect to see something similar in the Sen
limit, but the elliptic fiber has degenerated to a rational curve and there are no ordinary one-forms left. Instead we have to deal with meromorphic forms. Locally around the pinching $S^1$ we can write the equation of the elliptic curve as

$$xy = \epsilon$$

(4.22)

and the $(1,0)$ form as $dx/x$. In the limit $\epsilon \to 0$, $dx/x$ becomes a meromorphic one-form. Resolving the double point, we get a meromorphic one-form with opposite residue at the two poles. Hence such meromorphic one-forms are related to functions on $X_3$ odd under the involution.

Let us denote by $M5_\epsilon$ the $M5$ at finite $\epsilon$, $M5_0$ its Sen limit, and $\nu : \tilde{M5}_0 \to M5_0$ the normalization which separates the double point. Let us further define $\mathcal{O}_{D3-}$ by

$$p_{D3*}\mathcal{O}_{D3} = \mathcal{O}_{D3} \oplus \mathcal{O}_{D3-}$$

(4.23)

In other words, $\mathcal{O}_{D3-}$ corresponds to functions on $D3$ that are odd under the orientifold involution exchanging the two sheets. Then we have a short exact sequence:

$$0 \to \mathcal{O}_{M5_0} \to \nu_*\mathcal{O}_{\tilde{M5}} \to \mathcal{O}_{D3-} \to 0$$

(4.24)

Applying the $R^i\pi_*$, i.e. taking cohomology along the fiber, we get a long exact sequence which implies the isomorphism

$$R^1\pi_*\mathcal{O}_{M5_0} \cong \mathcal{O}_{D3-}$$

(4.25)

Therefore

$$H^{0,i}(M5) \sim H^i(D3,\mathcal{O}_{D3}) + H^{i-1}(D3,\mathcal{O}_{D3-}) \sim H^{0,i}(\mathcal{D}3)_+ + H^{0,i-1}(\mathcal{D}3)_-$$

(4.26)

in the Sen limit. In section 3 we saw that a generator of $H^{0,i}(M5)$ gives rise to a fermionic zero mode $\psi_\alpha$ for $i$ even and $\psi_\dot{\alpha}$ for $i$ odd. This agrees precisely with the orientifold discussion. In particular, for an isolated and rational $D3$ intersecting an orientifold plane, the only expected non-universal fermionic zero modes come from $h^{0,2}(M5)$.

Finally consider reduction of the chiral two-form. We label the two one-cycles of the $T^2$ by $a$ and $b$. Reducing the chiral two-form yields

$$A^{elec} = \int_a B^+, \quad A^{mag} = \int_b B^+$$

(4.27)

In the limit that the $a$-cycle pinches globally over the worldvolume, we may think of the electric gauge field as elementary and eliminate the magnetic gauge field. However in
general neither can be considered fundamental. The $\text{Sl}(2, \mathbb{Z})$ monodromies generally kill the one-forms of the $T^2$ globally, so a single $D3$-brane in $F$-theory usually does not carry any massless gauge field. Equivalently in terms of the Leray sequence, $H^0 R^1$ is generically zero. This agrees with the IIB description, where a $D3$ intersecting the orientifold locus yields an $O(1)$-instanton. On the other hand if the $D3$ does not intersect the discriminant locus, $H^0 R^1$ is non-zero and we get a $U(1)$ brane, as in type IIB.

Assuming an isolated instanton $h^{0,3}(M5) = 0$, the fluxes all live in $h^{2,1}(M5) + h^{1,2}(M5)$. To understand the fluxes we need to know more about $h^{1,2}(M5)$. Let us look at another Leray sequence:

$$H^{1,i}(M5) = H^i(M5, \Omega^1) \sim H^i(D3, R^0 \pi_* \Omega^1) + H^{i+1}(D3, R^1 \pi_* \Omega^1)$$  \hspace{1cm} (4.28)

Here $R^0 \sim H^{1,0}(T^2) \otimes \mathcal{O}_{D3} + H^{0,0}(T^2) \otimes \Omega^1_{D3}$. Taking the Sen limit, the $(1, 0)$ forms on $T^2$ become sections of the dualizing sheaf $\omega_P \equiv \omega_{\mathcal{M}5_0/D3}$, i.e. meromorphic one-forms that may have poles along the double point. Here we use $P$ to denote the fiber of $\mathcal{M}5_0$ over $D3$ which is a rational curve with double point, and $\tilde{P}$ its normalization. We have a short exact sequence

$$0 \rightarrow \nu_* \omega_{\tilde{P}} \rightarrow \omega_P \rightarrow \mathcal{O}_{D3-} \rightarrow 0$$  \hspace{1cm} (4.29)

The map from $\omega_P$ to $\mathcal{O}_{D3-}$ is the residue map. Taking cohomology along the fiber, we get a long exact sequence that implies

$$R^0 \pi_* \omega_P \approx \mathcal{O}_{D3-}$$  \hspace{1cm} (4.30)

So we can write

$$H^i(D3, R^0 \pi_* \Omega^1) \sim H^{0,i}(\mathcal{D}3)_- + H^{1,i}(\mathcal{D}3)_+$$  \hspace{1cm} (4.31)

For $R^1 \pi_* \omega_P$ we will be very sketchy. The rough intuition is that this is generally one-dimensional, but can jump up along the locus where the $D7$-branes are located, because the elliptic fiber should further degenerate there. The jump in dimension should correspond to the number of coinciding $D7$-branes. Hence we would get

$$H^{i-1}(D3, R^1 \pi_* \Omega^1) \sim H^{1,i-1}(\mathcal{D}3)_- + H^{0,i-1}(\mathcal{D}3)_+ + H^{0,i-1}(\mathcal{D}3 \cap D7) \otimes n_{D7}$$  \hspace{1cm} (4.32)

We want to apply this for $i = 2$. So what do all these pieces correspond to? $h^{0,2}(\mathcal{D}3)_-$ corresponds to a $U(1)$ flux on $\mathcal{D}3$ odd under the involution, but for supersymmetric configurations this must vanish. $h^{1,1}(\mathcal{D}3)_-$ corresponds to an odd $U(1)$ flux $F$ of type $(1, 1)$, which is in principle allowed. Fluxes proportional to $H^{1,2}_+$ and $H^{0,1}_+$ do not exist in $F$-theory. Finally, a flux proportional to $H^{0,i-1}(\mathcal{D}3 \cap D7)$ corresponds to a current $J_{\Sigma} = \partial x \delta^2(\Sigma)$, where $\Sigma$ is the intersection between the $\mathcal{D}3$ and $D7$ branes.

In the IIB weak coupling limit, $C_3$ reduces to a 7-brane gauge field $A_{D7}$ or the IIB two-forms $B_{RR}$ and $B_{NS}$. The $\int C_3 \wedge dB$ coupling on the $M5$-brane reduces to couplings
Thus the chiral two-form has the right couplings to describe the 37 strings in the IIb limit. By quantizing 37 strings using RNS we get \( J_\Sigma \) as a fermion bilinear, as we saw earlier.

Recall that the intermediate Jacobian was defined as \( H^3(M_5, \mathbb{R})/H^3(M_5, \mathbb{Z}) \), and the partition function of the chiral two-form was a theta-function on this Jacobian. Since the flux lattice splits up into \( H^2(D_3, \mathbb{Z}) \) and \( H^1(\Sigma, \mathbb{Z}) \), this means that the partition function should factorize. One piece corresponds to a sum over fluxes in \( h_{1,1}(D_3) \) on the \( D_3 \) worldvolume. The remainder is the partition function for the chiral fermions on the 37 intersections, which is again a theta-function (equivalently Ray-Singer holomorphic torsion). Thus the vanishing behaviour of the chiral two-form partition function is usually controlled by the fermionic 37 zero modes in the IIb limit. If \( h_{1,1} \neq 0 \) then it could also depend on the expectation value of the \( B_{RR} \) and \( B_{NS} \).

### 4.2. Reducible instantons

We would like to make some remarks about reducible \( D_3 \)-branes. The situation here is very similar to degenerate 7-branes or spectral covers. By Fourier-Mukai transform, it is also dual to bundles constructed by extension.

In the previous section we saw that for an irreducible instanton, the internal part of the zero modes is given by \( \text{Ext}^p(i_*L,i_*L) \) in the upstairs picture. Now suppose that we degenerate the \( D_3 \) to a reducible instanton, with the two reducible pieces intersecting over the orientifold locus, either by varying instanton moduli or by varying the ambient Calabi-Yau three-fold:

\[
D_3 = D_3' \cup D_3''
\]  

(4.34)

The sheaf on \( D_3 \) is represented by a pair of line bundles

\[
L_3 = (L_{3'},L_{3''})_{\text{f\-glue}}
\]  

(4.35)

supported on each component. Now generically, even though the support is reducible, these two pieces are not independent, but are glued by an isomorphism along the intersection \( \Sigma_{3'3''} = D_3' \cap D_3'' \):

\[
f_{\text{f\-glue}} \in \text{Hom}(L_{3'}|_\Sigma,L_{3''}|_\Sigma) \quad \text{or} \quad \text{Hom}(L_{3''}|_\Sigma,L_{3'}|_\Sigma)
\]  

(4.36)

This gluing isomorphism may be interpreted as the expectation value of a field localized on the intersection. The orientifold action relates the two line bundles:

\[
\sigma^*(L_{3'} \otimes K_{3'}) = L_{3''}
\]  

(4.37)
Since the orientifold action fixes $\Sigma_{3'3''}$, this means that the gluing map is a section of

$$L_{3'}^\vee \otimes \sigma^* L_{3''} \otimes \sigma^* K_{3'}|_{\Sigma_{3'3''}}$$

(4.38)
or its inverse. We also map $f_{\text{glue}}$ to its dual, so it must actually equal its dual and nowhere vanishing. Then we have $L_{3'}^2|_{\Sigma_{3'3''}} = K_{3'}|_{\Sigma_{3'3''}}$. Furthermore due to the orientifold symmetry we have $K_{3'}|_{\Sigma_{3'3''}} = K_{3''}|_{\Sigma_{3'3''}}$. From the Calabi-Yau condition it then follows that $K_{3'}|_{\Sigma_{3'3''}} = K_{3''}|_{\Sigma_{3'3''}}$ determines a spin structure on $\Sigma_{3'3''}$.

In this situation the $\text{Ext}^0$, which counts the universal instanton zero modes, is unchanged in the degeneration limit. The reason for this is that although the support of the $D3$ becomes reducible, in the limiting configuration the naive zero modes which we find on each irreducible piece separately must be glued along the intersection $D3' \cap D3''$. As a result there is only one $\psi_\alpha$ in the upstairs picture, which is then projected out by the orientifold action. Such a reducible instanton has the same number of universal zero modes as an irreducible instanton (i.e. four bosonic and two fermionic modes) and can contribute equally well to the superpotential. We will call this a reducible $O(1)$ instanton.

The number of $\text{Ext}^1$ zero modes could jump, in conjunction with $\text{Ext}^2$ (because the Euler character does not jump). We get modes from the $3'3''$ and $3''3''$ sectors, subject to the gluing condition along the intersection. We also get modes in the ‘off-diagonal’ sectors. Since $\sigma^*$ fixes the intersection $\Sigma_{3'3''}$, from our earlier discussion it follows that the spinors on the $3'3''$ intersection live in

$$\Omega^p(\Sigma_{3'3''}, L_{3'}^2 \otimes K_{3'}^{-1/2} \otimes K_{3''}^{-1/2} \otimes K_{\Sigma}^{1/2}) \sim \Omega^p(\Sigma_{3'3''}, K_{3'})$$

(4.39)

where we used the Calabi-Yau condition to get the second expression. Therefore the internal wave functions of $3'3''$ instanton zero modes (before projection) are counted by the global sections of these sheaves. More generally we can use the Ext-groups

$$\text{Ext}^1(\sigma^* P(i_* L_{3'}), i_* L_{3''}) \sim H^0(\Sigma_{3'3''}, K_{3'}|_{\Sigma_{3'3''}})$$

$$\text{Ext}^1(i_* L_{3'}, \sigma^* P(i_* L_{3'})) \sim H^1(\Sigma_{3'3''}, K_{3'}|_{\Sigma_{3'3''}})^*$$

(4.40)

These cohomology groups must have the same rank, as the Euler character does not jump. Each generator gives rise to a certain number of bosonic and fermionic zero modes. We get essentially the field content of a hypermultiplet in a $D1 - D5$ system. We have

$$H^0(\Sigma_{3'3''}, K_{3'}|_{\Sigma_{3'3''}}) \rightarrow X_{3'3''}, \xi_{3'3''}^\alpha, \xi_{3'3''}^\dot{\alpha}$$

$$H^1(\Sigma_{3'3''}, K_{3'}|_{\Sigma_{3'3''}}) \rightarrow Y_{3'3''}, \psi_{3'3''}^\alpha, \psi_{3'3''}^\dot{\alpha}$$

(4.41)
The orientifold action exchanges the $H^0$ and $H^1$ cohomology groups, and thus kills precisely half the zero modes.

Let us consider again a reducible brane configuration but without imposing an orientifold symmetry. It may be that the reducible configuration has additional moduli for
modifying the sheaf along the intersection. In particular, let us consider the case where
the gluing morphism gets turned off. In this case, we end up with a pair of line bundles
supported on the two components of $D3$, either

\[ L_3 = (L_3', L_3'' \otimes \mathcal{O}_{D3}(-\Sigma_{3,3''})) \quad \text{or} \quad L_3 = (L_3 \otimes \mathcal{O}_{D3}(-\Sigma_{3,3''}), L_3'') \]  

(4.42)

depending on the direction of the gluing morphism. The sheaf $L_3$ is now rank two along
the intersection. Notice that with $\tilde{L}_3'' = L_3'' \otimes \mathcal{O}_{D3}(-\Sigma_{3,3''})$ we have

\[ \text{Ext}^1(i_{3}, \tilde{L}_3', i_{3''}, \tilde{L}_3'') = H^0(\Sigma_{3,3''}, L_3' \otimes L_3''|_{\Sigma_{3,3''}}) \]  

(4.43)

so comparing with (4.36) we see that the gluing isomorphism can be interpreted as the
VEV of a field on the intersection, as we said earlier. Conversely it shows that the fields
localized on the intersection do not correspond to deforming the support of the branes.

In the limit of zero gluing map, one would get extra $\text{Ext}^0$ and $\text{Ext}^1$ zero modes. Again
this is familiar from reducible 7-branes or degenerate spectral covers, where such a limit
leads to enhanced gauge symmetry and an extra chiral field. Let us denote the gauge
generators associated to each reducible branch by $(\Lambda_{3'}, \Lambda_{3''})$ and the chiral field on the
intersection corresponding to the zero mode $\delta f_{\text{glue}}$ by $X$. The $\text{Ext}^0$'s generate a relation
among the $\text{Ext}^1$'s:

\[ \delta X = \Lambda_{3'} X - X \Lambda_{3''} \]  

(4.44)

When $X$ has a VEV, it gets eaten by the vector superfield corresponding to $\Lambda_{3'} - \Lambda_{3''}$. 
In the limit of zero VEV we get massless chiral and vector superfields obtained from
unhiggsing this massive vector superfield.

In the present context with $D3$ instantons, we get exactly the same mathematical
structure, but we interpret it as a (reducible) $U(1)$ instanton. However turning the Higgs
VEV on or off looks like an asymmetric operation which is not compatible with orientifolding.

When taking a Sen limit, the $M5$ typically limits to an irreducible $D3$-instanton, but
in special cases one may end up with a $D3$ with reducible support. In particular it may
happen that $b_2|_{D3} = a^2$ so that the $M5$ will factor into two pieces $\xi = \pm a$ in the Sen
limit. It is not easy to follow the gauge fields in the Sen limit, and it seems likely that
generically we get a reducible $O(1)$ instanton in the limit, rather than a $U(1)$ instanton.
If one does get a $U(1)$ instanton, this suggests that 33 zero modes on a reducible $U(1)$
instanton get lifted for finite coupling and we end up with an $O(1)$ instanton in $F$-theory.

4.3. Comparison with heterotic

Let us specifically consider the case of $SU(5)_{\text{GUT}}$ models. In the neighbourhood of the
GUT brane $S_{\text{GUT}}$, defined by the equation $z = 0$, the equations for the 7-branes become
Here the $b_i$ are certain polynomials on $S_{\text{GUT}}$ which specify the local model, and $R$ and $P$ in turn are certain explicitly known polynomials in the $b_i$. The IIB Calabi-Yau three-fold defined by $\xi^2 = b_2$ has non-perturbative conifold singularities at $z = b_5 = b_4 = 0$. A generic $D3$ instanton intersecting the GUT brane will miss these singularities. We will assume this is the case.

In the upstairs picture (before orientifolding), the locus $z = 0$ consists of two pieces, $\xi = \pm b_2^5$, each wrapped by five $D7$-branes. The intersection of the $D3$ with the GUT brane yields a curve $Q$ in the downstairs picture, and lifts to two copies in the upstairs picture, which we denote by $Q'$ and $Q''$. In heterotic/F-theory duals, the curve $Q$ is the same curve on which the heterotic worldsheet instanton is wrapped. The curve $Q$ typically intersects the orientifold locus, i.e. it intersects the matter curve $\Sigma_{10}$ on $S_{\text{GUT}}$.

In perturbative IIB, the GUT group is $U(5)$, and there are charged 37 modes transforming in the fundamental of $U(5)$ located on $Q'$. From the general discussion, the number of these zero modes is computed by the Dolbeault cohomology group

$$H^p(Q', L_7^{-1} \otimes L_7 \otimes K_{D3}|Q')$$

where $p = 1, 2$, and $L_7$ is the $U(1) \subset U(5)$ line bundle on the 7-brane containing $Q'$. The orientifold action relates this to similar cohomology groups on $Q''$. We can further simplify this as follows: if $Q$ is an isolated rational curve, then its normal bundle in $S_{\text{GUT}}$ is typically $O_Q(-1)$. As long as $Q$ is not tangential to the branch locus, the normal bundle of $D3$ is simply the pull-back of the normal bundle to $Q$ in $S_{\text{GUT}}$, so we have $K_{D3}|Q' = N_{D3}|Q' = \pi_{D3/Q}^* O_Q(-1)|Q' = O_{Q'}(-1)$. We will argue in the next section that there exist models with $c_1(L_7)$ very ample. Thus there are examples where 37 zero modes on $Q$ are guaranteed to exist and are even completely chiral, because for sufficiently ample $L_7$ the cohomology groups in (4.46) are non-zero for $p = 0$ and vanish for $p = 1$.

Similarly the intersection of the $D3$-instanton with the flavour $D7$-brane defined by $R + Pz = 0$ yields another curve that we will call $R$ in the downstairs picture. Its double cover is generally irreducible, and we get chiral fermions living on $R$ with charge one under the $U(1)$ gauge group of the flavour 7-brane.

Now let us take $g_s$ finite. In this case we cannot use the Ganor string approach, and the simplest description of the $D3$ instanton is given by the $M5$-brane. We immediately see a number of differences. First of all there is no analogous formula for the 37 zero modes for finite coupling, and in fact as we discussed any vanishing of the partition function is not due to zero modes of the chiral two-form anyways. We should probably not have expected such an analogous formula, because there is no known higher dimensional analogue of the
Bose/Fermi correspondence in two dimensions. The closest relatives to (4.46) appeared in our discussion on $dP_9$ fibered $M5$-branes, where we could reduce the problem to two dimensions. There we encountered Dolbeault cohomology groups on $Q$ and $\Sigma_{37}$, and they were manifestly different from the Dolbeault cohomology groups we encountered in the IIb limit.

Secondly, we found that these heterotic cohomology groups were generically vanishing, even when the IIb cohomology groups can not be. On $Q$ we had $H^0(Q, U(-1)|_Q)$ where $U = \bigoplus_{i=1}^6 \mathcal{O}(0)$, and does not yield any zero modes; on $\Sigma_{37}$ we found no zero modes generically. These facts implied that the chiral two-form partition function is generically non-vanishing, but in the IIb limit it will vanish due to the 37 zero modes discussed above. This shows one has to be careful to stay within the regime of validity of the approximations.

As the notation suggests, the curve $\Sigma_{37}$ (or rather its image under the cylinder map) can be thought of to some extent as the curve where ground states of 37 strings are localized for finite coupling. In the spirit of [3], suppose we intuitively think of the worldsheet instanton as a $D1$-instanton in type I, and the 10d gauge fields as living on an $SO(16)$ 9-brane. Now we $T$-dualize along the $T^2$-fibers of the heterotic Calabi-Yau three-fold $Z$. Then the 9-branes are mapped to 7-branes wrapping the spectral cover, and the $D1$-instanton is mapped to a $D3$-instanton wrapping $\pi_Z^{-1}(Q)$.

The spectral cover for the $16$ of $SO(16)$ has fewer pieces than the spectral cover for $E_8$. For instance if we turn on $SU(5)_H$ holonomy, we have the decomposition

$$16 = (5, 1)_{+1} + (\bar{5}, 1)_{-1} + (1, 6)_0$$

under $SU(5)_H \times SU(4) \times U(1)$. The intersection of the $D3$ instanton with the 7-branes is given by the union of $Q$ with multiplicity six, and two copies of $\Sigma_{37}$ (with different spectral line bundles). We have already seen that integrating out the physical modes on the instanton leads to the theta function of $\Sigma_{37}$, as we would expect when integrating out the fermionic 37 strings.

4.4. Zero modes in the IIb limit and $U(1)$ symmetries

Consider again the IIB limit of one of our vertical $D3$-instantons, which we assume to be irreducible in our discussion, in the case of $SU(5)$ GUT models. In the last subsection we saw that its contribution to the superpotential can vanish in the presence of chiral matter, even it contributes for finite string coupling. Here we would like to explore this behaviour a little more, and expose the role of $U(1)$ symmetries. But first we would like to complete the argument that for some suitable choice of flux in $F$-theory, there is necessarily a mismatch between zero and finite coupling.

To do this we need to have some idea how a flux in $F$-theory turns into a flux in IIb. It is hard to follow the flux in the IIb limit, but we can make a plausible guess. In
heterotic/F-theory duals there is always a canonical flux, that exists for any values of the complex structure moduli. Such a flux is constructed using only knowledge of $c_1(S)$ and $c_1(N_S)$, in fact this is the only discrete data on which the $dP_9$ fibration depends. To simplify life we may take $c_1(N_S)$ proportional to $c_1(S)$, or even $c_1(N_S) = 0$. If we take such a flux in $F$-theory, then the $U(1)$ flux on $S$ we recover in the limit ought to be some multiple of $c_1(S)$, because no other discrete choices were made. We can even fix the precise multiple by comparing the number of chiral generations. Now $-K_S$ is ample, so taking $L_7$ to be a suitable multiple of this (up to a half-integer shift to satisfy quantization constraints) will do the trick. (This also leads to a large number of generations, but that is besides the point here).

Since $c_1(S)$ is ample, it then follows that $\int_Q F \neq 0$ and there is a net number of chiral fermions on the 37 intersection which cannot be paired up. As a result, the instanton cannot contribute to the superpotential in the IIb limit, and naively we would not expect chiral zero modes to disappear under a continuous deformation. But from our earlier results we know that for generic values of the moduli, the instanton does contribute to the superpotential in $F$-theory or the heterotic string. How could this happen?

A possible resolution to this apparent paradox can already be seen in our discussion in section 2.7. Anomaly cancellation guarantees that there are additional 7-branes intersecting the GUT brane. In particular our GUT brane is intersected by a flavour D7-brane, and the intersection of the $D3$-instanton with this flavour brane yielded a second curve $R$ with chiral 37 zero modes localized on it.

By construction, the net $G$-flux through our $M5$-brane is zero. In fact as we discussed in section 3.3, modulo a small caveat the net $G$-flux through an $M5$-brane is generally non-zero only if there are gauged shift symmetries of the Kähler moduli. Therefore the net flux through the sum of these two curves appearing in the $g_s \to 0$ limit must vanish also. It follows that there must also be 37 zero modes of opposite chirality on $R$ that cannot be paired up.

Now consider the effective action for these modes on the instanton background. Even though the chiral fermions are localized on different submanifolds, we should generically expect that they are lifted non-perturbatively. This is familiar for instance from lifting of chiral zero modes in IIa/M-theory models [42], which are localized in different places on the internal manifold. In the present context, these are non-perturbative effects in the effective action on the instanton background, i.e. composite instantons. Indeed, the extra $U(1)$ symmetry on the 7-branes which protects the zero modes at $g_s = 0$ is broken by $\exp(-1/g_s)$ effects.

As we lift to $F$-theory, we cannot really follow the fermion zero modes to finite coupling, as there is no higher dimensional analogue of fermionization. We could however follow the bosonized fermions to finite coupling, so let us discuss this qualitatively. The analogue of fermionic zero modes is a background charge for the chiral two-form.

The two curves $Q$ and $R$ with the “$U(1)$-charged” vanishing cycles on top of them
yield two distinct four-cycles in the M5 worldvolume for $g_s = 0$. Non-vanishing flux through $Q$ and $R$ means there is non-vanishing $G$-flux through these two four-cycles. As we extrapolate to finite coupling, vanishing cycles which are charged under the extra $U(1)$ symmetry of type IIb get related by monodromies, thereby Higgsing the $U(1)$ and giving it a mass of order the KK scale. (See section 2.2 of [40] for a discussion of this). Thus there no longer is any $U(1)$ symmetry to protect the zero modes. The two four-cycles constructed from the two curves $Q$ and $R$ with the “$U(1)$-charged” vanishing cycles on top of them join into a single four-cycle due to the monodromy. The total $G$-flux through any four-cycle in the M5 worldvolume vanishes for finite $g_s$ as observed above. Thus there is no net background charge and hence no obstruction to pairing up all the fermionic 37 zero modes seen in the IIb limit. This strongly suggests that chiral 37 zero modes will generally disappear for finite string coupling.

Let us ponder the implications of this observation for KKLT like moduli-stabilization scenarios.\(^3\) It has been argued in [43] that there is an inherent conflict between the presence of chiral matter and instanton contributions to the superpotential. The argument can be phrased as follows: gauge groups in type IIb are either $U(n)$, $O(n)$ or $USp(n)$. Chiral matter is associated with $U(n)$ groups. In order to get chiral matter, a flux has to be turned on for $U(1) \subset U(n)$. Let us consider the effect of a gauge transformation for this $U(1)$. The RR four-form transforms as

$$\delta_\lambda C_4 \sim \text{Tr}(\lambda F) \delta^2(S)$$  \hspace{1cm} (4.48)

where $S$ is the four-cycle on which the $U(1)$ is localized. By Poincaré duality there is some class $D$ in $H^2(X_3)$ such that\(^4\)

$$\int_{S\cap D} \text{Tr}(F) \neq 0$$  \hspace{1cm} (4.49)

Under a $U(1)$ gauge variation the associated Kähler modulus transforms as

$$\delta_\lambda \text{Im}(T_D) = \delta_\lambda \int_D C_4 = \lambda \int_{S\cap D} \text{Tr}(F) \neq 0$$  \hspace{1cm} (4.50)

Therefore a contribution of the form

$$\int d^2 \theta e^{-T_D}$$  \hspace{1cm} (4.51)

\(^3\)There are various conceptual issues with these scenarios, most notably the use perturbation theory is inconsistent if the number of vacua is indeed finite non-perturbatively. We ignore these issues here and only concentrate on perturbative vacua, where we can take the volume modulus arbitrarily large.

\(^4\)Tr$(F)/2\pi$ should be in the image $H^2(X_3) \rightarrow H^2(S)$, because we want non-zero intersection with the matter curves.
is forbidden by gauge invariance. This is the behaviour we saw in the IIb limit for $SU(5)$ GUTs. It may not be a problem, since in this case the Kähler modulus is in fact eaten by the $U(1)$ vector multiplet, but it certainly affects the potential for the Kähler moduli.

For finite $g_s$ this extra $U(1)$ disappears into the KK tower and the 37 zero modes can pair up, so there is no $U(1)$ selection rule that could forbid such contributions to the superpotential. The longitudinal mode of the massive $U(1)$ is a complex structure modulus, rather than the Kähler modulus $T_D$. But one may wonder if we missed some effect and there is still some a priori conflict with chirality.

Since there is no such conflict in the heterotic string, heterotic/$F$-theory duality predicts that there can be no such issue in $F$-theory either. Let us examine this more closely. The Kähler moduli $T_Q \sim \text{vol}(Q \times P^1)$ on the $F$-theory side can all appear in the superpotential. The last modulus $T_0 \sim \text{vol}(\sigma_{B2})$ looks more problematic; wrapping a $D3$ on the GUT cycle yields a field theoretic instanton and field theory index results seem to indicate zero modes if there is chiral matter. However as we discussed earlier in section 3.3, unless there are gauged shift symmetries, $G|_{M5} = 0$ even in models with chiral matter, so even $T_0$ should appear in the superpotential.

On the heterotic side, a $D3$ instanton wrapped on $\sigma_{B2}$ corresponds to a space-time instanton, and the Kähler modulus turns into the $8d$ string coupling. So it looks like we would have trouble generating an $\exp(-S)$ contribution to the superpotential, where $S$ denotes the heterotic dilaton. However we could also consider instantons in the second $E_8$, where gaugino condensation leads to an $\exp(-S)$ term in the superpotential. (Strictly speaking this contribution is not generated by instantons, but that is not essential here). So there cannot be any a priori conflict with chirality.

We can check this also directly on the $F$-theory side. The $F$-theory analogue of this is as follows: we may also wrap a $D3$ at the infinity section of $B_3$, instead of the zero section. The Kähler modulus for this cycle is $T_\infty \sim T_0 + n_i T_Q^i$. The $G$-flux through this cycle at infinity is zero. There is no need for chiral matter at this location (as we also know from the heterotic string, because it corresponds to the second $E_8$), and so from this $D3$-instanton we can still get a term $\exp(-T_0)$ in the superpotential.

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