Relaxed commutant lifting: an equivalent version
and a new application

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Abstract

This paper presents a few additions to commutant lifting theory. An operator interpolation problem is introduced and shown to be equivalent to the relaxed commutant lifting problem. Using this connection a description of all solutions of the former problem is given. Also a new application, involving bounded operators induced by $H^2$ operator-valued functions, is presented.

0 Introduction

Let $\mathcal{U}$ and $\mathcal{Y}$ be Hilbert spaces, and let $\mathcal{F}$ be a subspace of $\mathcal{U}$. In this paper we consider the following problem. Given a contraction

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} : \mathcal{F} \to \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix},$$

(0.1)

find a (all) contraction(s) $\Gamma$ from $\mathcal{U}$ into $H^2(\mathcal{Y})$ satisfying the equation

$$E_\mathcal{Y} \omega_1 + S_\mathcal{Y} \Gamma \omega_2 = \Gamma|_\mathcal{F}.$$  

(0.2)

Here and in the sequel we use the convention that for any Hilbert space $\mathcal{N}$ the symbol $S_\mathcal{N}$ denotes the forward shift on the Hardy space $H^2(\mathcal{N})$ and $E_\mathcal{N}$ denotes the embedding of $\mathcal{N}$ into $H^2(\mathcal{N})$ defined by $(E_\mathcal{N} n)(\lambda) \equiv n$. Furthermore, $\Gamma|_\mathcal{F}$ stands for the restriction of $\Gamma$ to $\mathcal{F}$ viewed as an operator from $\mathcal{F}$ into $H^2(\mathcal{Y})$.

A contraction $\Gamma$ from $\mathcal{U}$ into $H^2(\mathcal{Y})$ satisfying equation (0.2) will be called a solution to the interpolation problem defined by the contraction $\omega$ in (0.1).

We shall show that the problem stated above can be reformulated as a relaxed commutant lifting problem. On the other hand, as we know from [4], the relaxed commutant lifting problem can be reduced to an interpolation
problem defined by a special contraction \( \omega \) of the form (0.1). Hence it follows that the above problem and the relaxed commutant lifting problem are equivalent in the sense that the one problem can be reduced to the other and conversely.

To state the main results we need some additional notation. Throughout \( \mathcal{L}(\mathcal{U}, \mathcal{Y}) \) stands for the space of all (bounded linear) operators from \( \mathcal{U} \) into \( \mathcal{Y} \). By \( \mathcal{H}^2(\mathcal{U}, \mathcal{Y}) \) we denote the space of all \( \mathcal{L}(\mathcal{U}, \mathcal{Y}) \)-valued functions that are analytic on \( \mathbb{D} \) such that the Taylor coefficients \( H_0, H_1, H_2, \ldots \) of the function \( H \) at zero satisfy the constraint \( \sum_{n=0}^{\infty} \|H_n u\|^2 < \infty \) for each \( u \in \mathcal{U} \). Given such a function \( H \), the formula

\[
(\Gamma u)(\lambda) = H(\lambda)u, \quad \lambda \in \mathbb{D}, \quad u \in \mathcal{U},
\]  

(0.3)
defines an operator \( \Gamma \) from \( \mathcal{U} \) into the Hardy space \( \mathcal{H}^2(\mathcal{Y}) \), which we shall refer to as the \emph{operator defined by} \( H \). Conversely, if \( \Gamma \) is an operator from \( \mathcal{U} \) into \( \mathcal{H}^2(\mathcal{Y}) \), then there is a unique \( H \in \mathcal{H}^2(\mathcal{U}, \mathcal{Y}) \) such that (0.3) holds, and in this case we call \( H \) the \emph{defining function} of \( \Gamma \).

Replacing \( \Gamma \) in (0.2) by its defining function we see that our problem has the following alternative formulation: find all \( H \in \mathcal{H}^2(\mathcal{U}, \mathcal{Y}) \) satisfying

\[
\omega_1 + \lambda H(\lambda)\omega_2 = H(\lambda)|_F, \quad \lambda \in \mathbb{D},
\]  

(0.4)
and such that the operator defined by \( H \) is a contraction. In this case we also say that \( H \) is a \emph{solution to the interpolation problem defined by the contraction} \( \omega \) in (0.1).

The connection with relaxed commutant lifting mentioned above allows us to use Theorem 1.1 in [6] (cf., Theorem 0.1 in [5]) to prove the following theorem.

**Theorem 0.1** Let \( \omega \) be a contraction as in (0.1). Then \( H \in \mathcal{H}^2(\mathcal{U}, \mathcal{Y}) \) is a solution to the interpolation problem defined by the contraction \( \omega \) if and only if \( H \) is given by

\[
H(\lambda) = \Pi_{\mathcal{Y}} Z(\lambda)(I - \lambda \Pi_{\mathcal{U}} Z(\lambda))^{-1}, \quad \lambda \in \mathbb{D},
\]  

(0.5)
where \( Z \) is an arbitrary Schur class function from \( \mathcal{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U}) \) satisfying the constraint \( Z(\lambda)|_F = \omega \) for each \( \lambda \in \mathbb{D} \). Here \( \Pi_{\mathcal{Y}} \) and \( \Pi_{\mathcal{U}} \) are the orthogonal projections from the Hilbert space direct sum \( \mathcal{Y} \oplus \mathcal{U} \) onto \( \mathcal{Y} \) and \( \mathcal{U} \), respectively.

Recall that for Hilbert spaces \( \mathcal{H} \) and \( \mathcal{K} \) the Schur class \( \mathcal{S}(\mathcal{H}, \mathcal{K}) \) consists of all \( \mathcal{L}(\mathcal{H}, \mathcal{K}) \)-valued functions \( F \), analytic on \( \mathbb{D} \), such that \( \sup_{\lambda \in \mathbb{D}} \|F(\lambda)\| \leq 1 \).
In general, the map $Z \mapsto H$ defined by Theorem 0.1 is not one-to-one. In fact, using the connection with relaxed commutant lifting and Theorem 1.2 in [6] we shall derive the following result.

**Theorem 0.2** Let $H \in \mathcal{H}^2(U, \mathcal{Y})$ be a solution to the interpolation problem defined by the contraction $\omega$ in (0.1), and let $\Gamma$ from $U$ into $H^2(\mathcal{Y})$ be the operator defined by $H$. Then the set of all $Z \in \mathcal{S}(U, \mathcal{Y} \oplus U)$ satisfying $Z(\lambda)|_{\mathcal{F}} = \omega$ for each $\lambda \in \mathbb{D}$ and such that (0.5) holds is parameterized by the set

$$\{ C \in \mathcal{S}(\mathcal{D}_\Gamma, \mathcal{D}_\Gamma) \mid C(\lambda)\mathcal{D}_\Gamma|_{\mathcal{F}} = \mathcal{D}_\Gamma \omega_2 \text{ for each } \lambda \in \mathbb{D} \}. \quad (0.6)$$

The parameterization referred to in the preceding theorem can be made more explicit. Indeed, let $H \in \mathcal{H}^2(U, \mathcal{Y})$ be a solution to the interpolation problem defined by the contraction $\omega$ in (0.1), and let the operator $\Gamma$ defined by $H$ be a contraction. Then given $C \in \mathcal{S}(\mathcal{D}_\Gamma, \mathcal{D}_\Gamma)$ we define

$$Z_C(\lambda) = \begin{bmatrix} 2H(\lambda)(W(\lambda) + I)^{-1} \\ \lambda^{-1}(W(\lambda) - I) \end{bmatrix}, \quad \lambda \in \mathbb{D}, \quad (0.7)$$

where

$$W(\lambda) = \Gamma^*(I + \lambda S^*_\mathcal{Y})(I - \lambda S^*_\mathcal{Y})^{-1}\Gamma +$$

$$+ \mathcal{D}_\Gamma(I + \lambda C(\lambda))(I - \lambda C(\lambda))^{-1}\mathcal{D}_\Gamma, \quad \lambda \in \mathbb{D}. \quad (0.8)$$

We shall see that the map $C \mapsto Z_C$ induces a one-to-one map from the set (0.6) onto the set of all $Z \in \mathcal{S}(U, \mathcal{Y} \oplus U)$ satisfying $Z(\lambda)|_{\mathcal{F}} = \omega$ for each $\lambda \in \mathbb{D}$ and such that (0.5) holds.

As a new application we shall use Theorem 0.1 to prove the following result.

**Theorem 0.3** Let $H \in \mathcal{H}^2(U, \mathcal{Y})$, and let $\Theta \in \mathcal{S}(\mathcal{E}, U)$ be inner such that $\Theta(0) = 0$. Put $\mathcal{H} = H^2(\mathcal{U}) \ominus \Theta H^2(\mathcal{E})$. In order that the map $f \mapsto Hf$ defines a contraction from $\mathcal{H}$ into $H^2(\mathcal{Y})$ it is necessary and sufficient that $H$ is given by

$$H(\lambda) = \Pi_{\mathcal{Y}} Z(\lambda)(I - \Theta(\lambda)\Pi_{\mathcal{E}} Z(\lambda))^{-1}, \quad \lambda \in \mathbb{D}, \quad (0.9)$$

where $Z$ is an arbitrary Schur class function from $\mathcal{S}(U, \mathcal{Y} \oplus \mathcal{E})$.

For $\Theta(\lambda) = \lambda^N$ the matrix-valued version of the above theorem can been found in [1, 2] and for operator-valued functions in [7]. For the scalar case, with $\Theta(\lambda) = \lambda$, the result goes back to [9], page 490.
The paper consists of three sections (not counting the present introduction). The first section has a preliminary character. Here we recall how the relaxed commutant lifting problem can be reduced to an interpolation problem of the type defined above. In the second section we prove Theorems 0.1 and 0.2 using relaxed commutant lifting. In the third section Theorem 0.3 is proved.

1 Preliminaries about relaxed commutant lifting

This section has a preliminary character. We recall the relaxed commutant lifting problem and how this problem can be reduced to an interpolation problem defined by a contraction of the form (0.1).

We begin with some terminology. A quintet \( \{A, T', U', R, Q\} \) consisting of five Hilbert space operators is called a data set if the operator \( A \) is a contraction mapping \( \mathcal{H} \) into \( \mathcal{H}' \), the operator \( U' \) on \( \mathcal{K}' \) is a minimal isometric lifting of the contraction \( T' \) on \( \mathcal{H}' \), and \( R \) and \( Q \) are operators from \( \mathcal{H}_0 \) to \( \mathcal{H} \) satisfying the following constraints:

\[
T'AR = AQ \quad \text{and} \quad R^*R \leq Q^*Q. \tag{1.1}
\]

Without loss of generality we can and shall assume that \( U' \) is the Sz.-Nagy-Schäffer (minimal) isometric lifting of \( T' \). The latter means (see [8]) that

\[
U' = \begin{bmatrix}
T' \\
E_{\mathcal{D}_T'}, D_{T'} \\
S_{\mathcal{D}_T'}
\end{bmatrix}
\]

on \( \mathcal{K}' = \begin{bmatrix} \mathcal{H}' \\
H^2(\mathcal{D}_T') \end{bmatrix} \). \hspace{1cm} (1.2)

Given this data set the relaxed commutant lifting problem (RCL problem) is to find all contractions \( B \) from \( \mathcal{H} \) to \( \mathcal{K}' \) such that

\[
\Pi_{\mathcal{K}'} B = A \quad \text{and} \quad U'BR = BQ. \tag{1.3}
\]

Here \( \Pi_{\mathcal{K}'} \) is the orthogonal projection from \( \mathcal{K}' \) onto \( \mathcal{H}' \) viewed as an operator from \( \mathcal{K}' \) into \( \mathcal{H}' \). In this case we refer to \( B \) as a solution to the RCL problem for the data set \( \{A, T', U', R, Q\} \).

Since \( U' \) is given by (1.2), we have \( \mathcal{K}' = \mathcal{H}' \oplus H^2(\mathcal{D}_T') \), and an operator \( B \) from \( \mathcal{H} \) into \( \mathcal{H}' \oplus H^2(\mathcal{D}_T') \) is a contraction satisfying the first identity in (1.3) if and only if \( B \) can be represented in the form

\[
B = \begin{bmatrix}
A \\
\Gamma D_A
\end{bmatrix} : \mathcal{H} \to \begin{bmatrix} \mathcal{H}' \\
H^2(\mathcal{D}_T') \end{bmatrix}, \tag{1.4}
\]
where Γ is a contraction from \( D_A \) into \( H^2(D_{T'}) \). Note that Γ and B in (1.4) define each other uniquely. Moreover, given (1.4) the second identity in (1.3) holds if and only if Γ satisfies the equation

\[ E_{D_{T'}, A} AR + S_{D_{T'}, A} \Gamma DR = \Gamma DAQ. \]  

(1.5)

Therefore, with \( U' \) as in (1.2), the RCL problem for \( \{ A, T', U', R, Q \} \) is equivalent to the problem of finding all contractions Γ from \( D_A \) into \( H^2(D_{T'}) \) such that (1.5) holds.

Equation (1.5) can be rewritten as an equation of the form (0.2). To see this one first observes that, because of (1.1), for each \( h \in H_0 \) we have

\[
\| D_A Qh \|_2 = \| Qh \|_2^2 - \| AQh \|_2^2 \geq \| Rh \|_2^2 - \| T'ARh \|_2^2 \\
= \| ARh \|_2^2 - \| T'ARh \|_2^2 + \| Rh \|_2^2 - \| ARh \|_2^2 \\
= \| D_{T'} ARh \|_2^2 + \| D_A Rh \|_2^2.
\]

Hence the identity

\[
\omega D_A Qh = \begin{bmatrix} D_{T'} ARh \\ D_A Rh \end{bmatrix}, \quad h \in H_0,
\]

uniquely defines a contraction \( \omega \) from \( \mathcal{F} = D_A QH \) into \( D_{T'} \oplus D_A \). We refer to this contraction as the contraction underlying the data set \( \{ A, T', U', R, Q \} \).

Using this contraction equation (1.5) can equivalently be represented as

\[ E_{D_{T'}, A} \omega_1 + S_{D_{T'}, A} \Gamma \omega_2 = \Gamma|_{\mathcal{F}}, \]  

(1.6)

where \( \omega_1 \) is the contraction mapping \( \mathcal{F} \) into \( D_{T'} \) determined by the first component of \( \omega \) and \( \omega_2 \) is the contraction mapping \( \mathcal{F} \) into \( D_A \) determined by the second component of \( \omega \). Summarizing the above discussion we arrive at the following conclusion.

With \( U' \) equal to the Sz.-Nagy-Schäffer isometric lifting of \( T' \), the RCL problem for \( \{ A, T', U', R, Q \} \) is equivalent to the problem of finding all contractions Γ from \( D_A \) into \( H^2(D_{T'}) \) satisfying (1.6), where

\[
\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}: \mathcal{F} \rightarrow \begin{bmatrix} D_{T'} \\ D_A \end{bmatrix},
\]

is the contraction underlying the given data set. Moreover, the map \( \Gamma \mapsto B \) determined by (1.4) provides a one-to-one correspondence between the solutions of the interpolation problem defined by \( \omega \) and the solutions of the RCL problem for \( \{ A, T', U', R, Q \} \). In particular, any RCL problem reduces to a problem of the type considered in the introduction.
2 Proofs of Theorems 0.1 and 0.2

Throughout this section \( \mathcal{U} \) and \( \mathcal{Y} \) are Hilbert spaces, \( \mathcal{F} \) is a subspace of \( \mathcal{U} \), and the operator \( \omega \) in (0.1) is a contraction. We associate with \( \omega \) a lifting data set.

**Proposition 2.1** Let \( \omega \) be a contraction as in (0.1). Put

\[
\tilde{A} = \begin{bmatrix} I_{\mathcal{Y}} & 0 \\ 0 & 0 \end{bmatrix} \text{ on } \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix}, \quad \tilde{T}' = \begin{bmatrix} 0 & 0 \\ 0 & I_{\mathcal{U}} \end{bmatrix} \text{ on } \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix}
\]

\[
\tilde{R} = \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} : \mathcal{F} \to \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix}, \quad \tilde{Q} = \begin{bmatrix} 0 \\ \Pi_{\mathcal{F}}^* \end{bmatrix} : \mathcal{F} \to \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix},
\]

\[
\tilde{U}' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & I_{\mathcal{U}} & 0 & 0 \\ E_{\mathcal{Y}} & 0 & S_{\mathcal{Y}} & 0 \\ H^2(\mathcal{Y}) \end{bmatrix} \text{ on } \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix}.
\]

Then \( \{\tilde{A}, \tilde{T}', \tilde{U}', \tilde{R}, \tilde{Q}\} \) is a data set, and the underlying contraction is precisely the given contraction \( \omega \). Furthermore, \( \tilde{U}' \) is the Sz-Nagy-Schäffer isometric lifting of \( \tilde{T}' \).

Here \( \Pi_{\mathcal{F}} \) stands for the orthogonal projection of \( \mathcal{U} \) onto \( \mathcal{F} \) viewed as a map from \( \mathcal{U} \) into \( \mathcal{F} \), and hence \( \Pi_{\mathcal{F}}^* \) is the canonical embedding of \( \mathcal{F} \) into \( \mathcal{U} \).

**Proof.** The operators \( \tilde{A} \) and \( \tilde{T}' \) are orthogonal projections and hence contractions. Observe that \( \tilde{T}' \tilde{A} \) and \( \tilde{A} \tilde{Q} \) are both zero operators. Furthermore, note that \( R = \omega \) is a contraction defined on \( \mathcal{F} \) and \( \tilde{Q}^* \tilde{Q} \) is the identity operator on \( \mathcal{F} \). From these remarks we see that

\[
\tilde{T}' \tilde{A} \tilde{R} = \tilde{A} \tilde{Q} \quad \text{and} \quad \tilde{R}^* \tilde{R} \leq \tilde{Q}^* \tilde{Q}.
\] (2.1)

Next, observe that

\[
D_{\tilde{T}'} = \begin{bmatrix} I_{\mathcal{Y}} & 0 \\ 0 & 0 \end{bmatrix} \text{ on } \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix}.
\]

Thus we can identify \( D_{\tilde{T}'} \), with the space \( \mathcal{Y} \). With this identification in mind it is straightforward to check that \( \tilde{U}' \) is the Sz-Nagy-Schäffer isometric lifting of \( \tilde{T}' \). It follows that \( \{\tilde{A}, \tilde{T}', \tilde{U}', \tilde{R}, \tilde{Q}\} \) is a data set. Notice that in this case the space \( \mathcal{H}_0 \) appearing in the definition of a data set is equal to the space \( \mathcal{F} \). Using

\[
D_{\tilde{A}} = \begin{bmatrix} 0 & 0 \\ 0 & I_{\mathcal{U}} \end{bmatrix} \text{ on } \begin{bmatrix} \mathcal{Y} \\ \mathcal{U} \end{bmatrix},
\]
we see that
\[ D_A \hat{Q} = \hat{Q}, \quad D_T \hat{A} \hat{R} = \omega_1, \quad D_A \hat{R} = \omega_2. \]

It is then easy to show that the contraction \( \omega \) in (0.1) is precisely the contraction underlying the data set \( \{ \hat{A}, \hat{T}', \hat{U}', \hat{R}, \hat{Q} \} \).

**Proof of Theorem 0.1.** Let \( \omega \) be a contraction as in (0.1), and let \( \{ \hat{A}, \hat{T}', \hat{U}', \hat{R}, \hat{Q} \} \) be the data set constructed in Proposition 2.1. Since \( \omega \) is the contraction underlying this data set and \( \hat{U}' \) is Sz-Nagy-Schäffer isometric lifting of \( \hat{T}' \), we know (see the conclusion at the end of the previous section) that an operator \( \Gamma : \mathcal{U} \to H^2(\mathcal{Y}) \) is a solution of interpolation problem defined by the contraction \( \omega \) if and only if the operator \( B = \begin{bmatrix} \hat{A} \\ \Gamma D_A \end{bmatrix} : \mathcal{Y} \oplus \mathcal{U} \to \begin{bmatrix} \mathcal{Y} \oplus \mathcal{U} \\ H^2(\mathcal{Y}) \end{bmatrix} \) (2.2)

is a solution to the RCL problem for the data set \( \{ \hat{A}, \hat{T}', \hat{U}', \hat{R}, \hat{Q} \} \). Recall (using canonical identifications) that \( D \hat{T}' \) and \( D \hat{A} \) are equal to \( \mathcal{Y} \) and \( \mathcal{U} \), respectively. But then Theorem 1.1 in [6] tells use that \( B \) in (2.2) is a solution to the RCL problem for the data set \( \{ \hat{A}, \hat{T}', \hat{U}', \hat{R}, \hat{Q} \} \) if and only if the defining function \( H \) of \( \Gamma \) is given by

\[ H(\lambda) = \Pi_Y Z(\lambda)(I - \lambda \Pi_U Z(\lambda))^{-1}, \quad \lambda \in \mathbb{D}, \]

where \( Z \) is an arbitrary Schur class function from \( \mathcal{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{U}) \) satisfying the constraint \( Z(\lambda)|_F = \omega \) for each \( \lambda \in \mathbb{D} \). From these two “if and only if” statements Theorem 0.1 follows.

**Proof of Theorem 0.2.** Let \( H \in H^2(\mathcal{U}, \mathcal{Y}) \) be a solution to the interpolation problem defined by the contraction \( \omega \) in (0.1), and let \( \Gamma \) from \( \mathcal{U} \) into \( H^2(\mathcal{Y}) \) be the operator defined by \( H \). Then \( \Gamma \) is a contraction satisfying (0.2). Hence for each \( f \in \mathcal{F} \) we have

\[
\| D_T f \|^2 = \| f \|^2 - \| \Gamma f \|^2 = \| f \|^2 - \| E_{\mathcal{Y}} \omega_1 f \|^2 - \| S_{\mathcal{Y}} \Gamma \omega_2 f \|^2
\]

\[
= \| f \|^2 - \| \omega_1 f \|^2 - \| \omega_2 f \|^2 + \| \omega_2 f \|^2 - \| \Gamma \omega_2 f \|^2
\]

\[
\geq \| D_T \omega_2 f \|^2.
\] (2.3)

The above calculation shows that there exists a (unique) contraction \( \Omega \) mapping \( \mathcal{F} = \overline{D_T \mathcal{F}} \) into \( D_T \) such that

\[
\Omega D_T|_F = D_T \omega_2.
\]

(2.4)

Now let \( \mathcal{S}'(D_T, D_T) \) be the set defined by (0.6). Let \( C \) be a function in \( \mathcal{S}(D_T, D_T) \). Using the identity (2.4) we see that the function \( C \) belongs to
$S'(D_\Gamma, D_\Gamma)$ if and only if $C(\lambda)D_\Gamma|_F = \Omega D_\Gamma|_F$. The latter identity can be rewritten as $C(\lambda)|_{D_\Gamma} = \Omega$. We conclude that

$$S'(D_\Gamma, D_\Gamma) = \{ C \in S(D_\Gamma, D_\Gamma) \mid C(\lambda)|_{D_\Gamma} = \Omega \text{ for each } \lambda \in \mathbb{D} \}. \quad (2.5)$$

In other words, for the data set considered here the set (0.6) is precisely the set $S_\Omega(D_\Gamma, D_\Gamma)$ appearing in Theorem 1.2 of [6]. Next, recall that

$$B = \begin{bmatrix} \tilde{A} \\ \Gamma D_\tilde{A} \end{bmatrix} : Y \oplus U \to \begin{bmatrix} Y \oplus U \\ H^2(Y) \end{bmatrix}$$

is a solution to the RCL problem for the data set $\{\tilde{A}, \tilde{T}', \tilde{U}', \tilde{R}, \tilde{Q}\}$. But then, using that the set (0.6) is equal to the right hand side of (2.5), we can apply Theorem 1.2 in [6] to complete the proof. \qed

**Corollary 2.2** Assume that $\omega$ in (0.1) is an isometry such that $\omega_2 F$ is dense in $U$. Then the map $Z \mapsto H$ defined by Theorem 0.1 is one-to-one, and (0.5) provides a proper parameterization of all solutions to the interpolation problem defined by $\omega$.

**Proof.** Using the fact that $\omega$ is an isometry, we see from (2.3) that the operator $\Omega$ in (2.5) is also an isometry. In particular, the space $\Omega F_\Gamma$ is a closed subspace of $D_\Gamma$. Since $\omega_2 F$ is dense in $U$, the space $D_\Gamma \omega_2 F$ is dense in $D_\Gamma$. By (2.4) the space $D_\Gamma \omega_2 F$ is contained in $\Omega F_\Gamma$. Thus $\Omega F_\Gamma$ is also dense in $D_\Gamma$. But $\Omega F_\Gamma$ is closed in $D_\Gamma$, and therefore $\Omega F_\Gamma \subseteq D_\Gamma$. Hence $\Omega$ is a unitary operator from $F_\Gamma$ onto $D_\Gamma$. This implies that the set defined by the right hand side of (2.5), or equivalently the set (0.6), consists of one element only. Thus if $\omega$ is an isometry and $\omega_2 F$ is dense in $U$, then the map $Z \mapsto H$ defined by Theorem 0.1 is one-to-one. \qed

From Theorem 1.2 in [6] it follows that the map $C \mapsto Z_C$ given by (0.7) and (0.8) is well defined and induces a one-to-one map from the set (0.6) onto the set of all $Z \in S(U, Y \oplus U)$ satisfying $Z(\lambda)|_F = \omega$ for each $\lambda \in \mathbb{D}$ and such that (0.5) holds.

### 3 Proof of Theorem 0.3

Throughout this section $H \in H^2(U, Y)$, and $\Theta \in S(E, U)$ is inner such that $\Theta(0) = 0$. Furthermore, $\mathcal{H} = H^2(U) \ominus \Theta H^2(E)$. Our aim is to prove Theorem 0.3. We begin with some auxiliary results.
Lemma 3.1  Let $\Phi = \lambda^{-1} \Theta$, and put $H_0 = H^2(U) \ominus \Phi H^2(E)$. Then

$$H = E_\mathcal{U} U \oplus \lambda H_0, \quad H = H_0 \oplus \Phi E_\mathcal{E} E. \quad (3.1)$$

Proof. As usual, given any inner function $\alpha \in S(\mathcal{X}, \mathcal{Y})$, we shall denote the space $H^2(\mathcal{Y}) \ominus \alpha H^2(\mathcal{X})$ by $H(\alpha)$. The two identities in (3.1), then follow from the rule (see, e.g., Theorem X.1.9 in [3]) that for two inner functions $\alpha \in S(\mathcal{X}, \mathcal{Y})$ and $\beta \in S(\mathcal{Y}, \mathcal{X})$ we have

$$H(\beta \alpha) = H(\beta) \oplus \beta H(\alpha).$$

Indeed, we apply this rule twice. First with $\alpha(\lambda) = \Psi(\lambda)$ and $\beta(\lambda) = \lambda I_U$, and next with $\alpha(\lambda) = \lambda I_\mathcal{E}$ and $\beta(\lambda) = \Psi(\lambda)$. Note that in both cases $\beta \alpha = \Theta$. With the first choice of $\alpha$ and $\beta$ we get the first identity in (3.1), and the second choice yields the second identity in (3.1). □

The above lemma allows us to define the following auxiliary operators:

$$R : H_0 \to H, \quad Rh_0 = h_0; \quad Q : H_0 \to H, \quad Qh_0 = \lambda h_0 \quad (h_0 \in H_0). \quad (3.2)$$

Note that the operators $R$ and $Q$ are isometries.

Lemma 3.2  Let $\Gamma : H \to H^2(\mathcal{Y})$ be a (bounded linear) operator, and put

$$K(\lambda) = E_\mathcal{Y}^* (I - \lambda S_\mathcal{Y}^*)^{-1} \Gamma E_\mathcal{U}, \quad \lambda \in \mathbb{D}. \quad (3.3)$$

Then $K \in \mathcal{H}^2(\mathcal{U}, \mathcal{Y})$, and the operator $\Gamma$ satisfies $S_\mathcal{Y} \Gamma R = \Gamma Q$ if and only if for each $f \in H$ we have

$$(\Gamma f)(\lambda) = K(\lambda) f(\lambda), \quad \lambda \in \mathbb{D}. \quad (3.4)$$

Proof. According to the first identity in (3.1) the operator $E_\mathcal{U}$ maps $\mathcal{U}$ into $H$. Thus $\Gamma E_\mathcal{U}$ is well-defined, and hence the same holds true for $K$. Obviously, $K$ is an $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued function which is analytic on $\mathbb{D}$. Let $K_n$ be the $n$-th coefficient of the Taylor expansion of $K$ at zero. Take $u \in \mathcal{U}$. Then $K_n u = E_\mathcal{Y}^* (S_\mathcal{Y}^*)^n \Gamma E_\mathcal{U} u = (\Gamma E_\mathcal{U} u)_n$, where $(\Gamma E_\mathcal{U} u)_n$ is the $n$-th coefficient of the Taylor expansion of the $\mathcal{Y}$-valued function $\Gamma E_\mathcal{U} u$ at zero. Therefore

$$(\Gamma E_\mathcal{U} u)(\lambda) = K(\lambda) u, \quad u \in \mathcal{U}, \quad \lambda \in \mathbb{D}. \quad (3.5)$$

Thus $K \in \mathcal{H}^2(\mathcal{U}, \mathcal{Y})$, and $K$ is the defining function for the operator $\Gamma E_\mathcal{U}$. 

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Next assume that $\Gamma$ satisfies the intertwining relation $S_2\Gamma R = \Gamma Q$. From (3.3) we see that (3.4) holds for $f = E_\mathcal{U} u$ with $u \in \mathcal{U}$ arbitrary. Indeed, for $f = E_\mathcal{U} u$ we have $f(\lambda) \equiv u$, and hence
\[
K(\lambda)f(\lambda) = K(\lambda)u = E_\mathcal{U}^\ast(I - \lambda S_2^\ast)^{-1}(\Gamma E_\mathcal{U} u)(\lambda) = (\Gamma f)(\lambda).
\]
Using the first equality in (3.1), we see that it suffices to prove (3.4) for $f = \lambda h_0$ with $h_0 \in \mathcal{H}_0$. However for such a function $f$ we have
\[
\Gamma f = \Gamma Q h_0 = S_2\Gamma R h_0 = S_2 \Gamma h_0 = \lambda \Gamma h_0, \quad K(\lambda)f(\lambda) = \lambda K(\lambda)h_0(\lambda).
\]
Therefore, it suffices to prove (3.4) for $h_0 \in \mathcal{H}_0$.

Take $h_0 \in \mathcal{H}_0$. Note that the identity operator on $H^2(\mathcal{U})$ is equal to $S_\mathcal{U} S_\mathcal{U}^* + P_\mathcal{U}$, where $P_\mathcal{U}$ is the orthogonal projection of $H^2(\mathcal{U})$ onto $E_\mathcal{U} \mathcal{U}$. The first identity in (3.1) shows that $E_\mathcal{U} \mathcal{U} \subset \mathcal{H}$. Thus $\Gamma P_\mathcal{U}$ is well defined. Since $\mathcal{H}_0$ is invariant under $S_\mathcal{U}^*$, we see that $S_\mathcal{U} S_\mathcal{U}^* h_0$ belongs to $\mathcal{H}$, and hence
\[
\Gamma h_0 = \Gamma S_\mathcal{U} S_\mathcal{U}^* h_0 + \Gamma P_\mathcal{U} h_0 = \Gamma Q S_\mathcal{U}^* h_0 + \Gamma P_\mathcal{U} h_0
\]
\[
= S_2 \Gamma R S_\mathcal{U}^* h_0 + \Gamma P_\mathcal{U} h_0 = S_2 \Gamma S_\mathcal{U}^* h_0 + \Gamma P_\mathcal{U} h_0.
\]
Since $h_0$ is an arbitrary element of $\mathcal{H}_0$, we get
\[
\Gamma h_0 - S_2 \Gamma S_\mathcal{U}^* h_0 = \Gamma P_\mathcal{U} h_0, \quad h_0 \in \mathcal{H}_0.
\]
By induction, using $\mathcal{H}_0$ is invariant under $S_\mathcal{U}^*$, the preceding identity yields
\[
\Gamma h_0 = \sum_{\nu=0}^n S_\mathcal{U}^\ast(\Gamma P_\mathcal{U})(S_\mathcal{U}^\ast)^\nu h_0 + S_\mathcal{U}^\ast(\Gamma(S_\mathcal{U}^\ast)^{(n+1)} h_0, \quad h_0 \in \mathcal{H}_0,
\]
for $n = 0, 1, 2, \ldots$ Now fix $\lambda \in \mathbb{D}$. From (3.3) we know that
\[
(\Gamma P_\mathcal{U} f)(\lambda) = (\Gamma E_\mathcal{U} E_\mathcal{U}^\ast f)(\lambda) = K(\lambda)E_\mathcal{U}^\ast f, \quad f \in H^2(\mathcal{U}).
\]
Hence for $h_0 \in \mathcal{H}_0$ we have
\[
(\Gamma h_0)(\lambda) = \sum_{\nu=0}^n \lambda^\nu K(\lambda)E_\mathcal{U}^\ast (S_\mathcal{U}^\ast)^\nu h_0 + \lambda^{n+1} (\Gamma(S_\mathcal{U}^\ast)^{(n+1)} h_0)(\lambda)
\]
\[
= K(\lambda)\sum_{\nu=0}^n \lambda^\nu E_\mathcal{U}^\ast (S_\mathcal{U}^\ast)^\nu h_0 + \lambda^{n+1} (\Gamma(S_\mathcal{U}^\ast)^{(n+1)} h_0)(\lambda). \quad (3.5)
\]
Note that for $n \to \infty$ the function $\Gamma(S_\mathcal{U}^\ast)^{(n+1)} h_0$ converges to zero in the norm of $H^2(\mathcal{V})$, and hence the same holds true for $S_\mathcal{U}^\ast(\Gamma(S_\mathcal{U}^\ast)^{(n+1)} h_0)$. It
follows that the second term in the right side of (3.5) converges to zero when \( n \to \infty \). Furthermore, for \( n \to \infty \) the vector \( \sum_{\nu=0}^{n} \lambda^{\nu} \mathcal{E}_{\nu}^{*} (S_{\nu}^{*})^{\nu} h_{0} \) converges to \( h_{0}(\lambda) \) in the norm of \( \mathcal{U} \). Hence the first term in the right hand side of (3.5) converges to \( K(\lambda)h_{0}(\lambda) \) when \( n \to \infty \). Thus we have proved that (3.4) holds.

To prove the converse implication. Assume that (3.4) holds. Let \( h_{0} \in \mathcal{H}_{0} \). Then for each \( \lambda \in \mathbb{D} \) we have

\[
(\Gamma Q h_{0})(\lambda) = K(\lambda)\lambda h_{0}(\lambda) = \lambda K(\lambda)h_{0}(\lambda) = (S_{\lambda} \Gamma h_{0})(\lambda) = (S_{\lambda} \Gamma R h_{0})(\lambda).
\]

Since \( h_{0} \) is an arbitrary element in \( \mathcal{H}_{0} \), we see that \( S_{\lambda} \Gamma R = \Gamma Q \). □

Next we put the problem into the setting of our alternative version of the relaxed commutant lifting problem. Put \( \mathcal{F} = \lambda \mathcal{H}_{0} \), and define

\[
\omega = \begin{bmatrix} \omega_{1} \\ \omega_{2} \end{bmatrix} : \mathcal{F} \to \begin{bmatrix} \mathcal{Y} \\ \mathcal{H} \end{bmatrix}, \quad \omega_{1} = 0, \quad \omega_{2}(\lambda h_{0}) = h_{0} (h_{0} \in \mathcal{H}_{0}).
\]

(3.6)

Note that \( \omega_{2} Q = R \), and hence a contraction \( \Gamma \) from \( \mathcal{H} \) into \( H^{2}(\mathcal{Y}) \) satisfies the intertwining relation \( S_{\lambda} \Gamma R = \Gamma Q \) if and only if \( S_{\lambda} \Gamma \omega_{2} = \Gamma |_{\mathcal{F}} \). Since \( \omega_{1} = 0 \), we are now ready to prove the main result.

**Proof of Theorem 0.3.** Let \( H \in H^{2}(\mathcal{U}, \mathcal{Y}) \), and let us assume that the map \( f \mapsto Hf \) defines a contraction from \( \mathcal{H} \) into \( H^{2}(\mathcal{Y}) \). Denote this contraction by \( \Gamma \). Then (3.3) holds with \( H \) in place of \( K \), and Lemma 3.2 shows that the contraction \( \Gamma \) satisfies the intertwining relation \( S_{\lambda} \Gamma R = \Gamma Q \), and thus \( S_{\lambda} \Gamma \omega_{2} = \Gamma |_{\mathcal{F}} \). Since \( \omega_{1} = 0 \), we know from Theorem 0.1 that \( H \) is given by

\[
H(\lambda) = F(\lambda)(I_{\mathcal{H}} - \lambda G(\lambda))^{-1} \mathcal{E}_{\mathcal{U}}, \quad \lambda \in \mathbb{D},
\]

(3.7)

where

\[
W = \begin{bmatrix} F \\ G \end{bmatrix} \in \mathcal{S}(\mathcal{H}, \begin{bmatrix} \mathcal{Y} \\ \mathcal{H} \end{bmatrix}), \quad W(\lambda)|_{\mathcal{F}} = \omega \ (\lambda \in \mathbb{D}).
\]

(3.8)

Conversely, let \( H \in H^{2}(\mathcal{U}, \mathcal{Y}) \) be given by (3.7), where \( F \) and \( G \) are such that (3.8) holds. Then, again using Theorem 0.1, we know that there exists a contraction \( \Gamma \) from \( \mathcal{H} \) into \( H^{2}(\mathcal{Y}) \) such that

\[
(\Gamma h)(\lambda) = F(\lambda)(I_{\mathcal{H}} - \lambda G(\lambda))^{-1} h, \quad h \in \mathcal{H}, \quad \lambda \in \mathbb{D}.
\]

Moreover, \( S_{\lambda} \Gamma \omega_{2} = \Gamma |_{\mathcal{F}} \), and hence \( \Gamma \) satisfies the intertwining relation \( S_{\lambda} \Gamma R = \Gamma Q \). It follows that \( H(\lambda) = (\Gamma \mathcal{E}_{\mathcal{U}})(\lambda) \) for each \( \lambda \in \mathbb{D} \). The fact
that $\Gamma$ satisfies the intertwining relation $S_Y \Gamma R = \Gamma Q$ allows us to again apply Lemma 3.2. We conclude that $(\Gamma h)(\lambda) = H(\lambda)h(\lambda)$ for each $h \in \mathcal{H}$ and each $\lambda \in \mathbb{D}$. Thus the map $f \mapsto Hf$ induces a contraction from $\mathcal{H}$ into $H^2(\mathcal{Y})$ as desired.

From the previous results we see that it remains to show that the representations given by (3.7), (3.8) and by (0.9), with $Z \in \mathcal{S}(\mathcal{U}, \mathcal{Y} \oplus \mathcal{E})$, are equivalent. Consider the spaces

$$
\mathcal{F} = \lambda \mathcal{H}_0, \hspace{1em} \mathcal{G} = \mathcal{H} \otimes \mathcal{F} = \mathcal{E}_\mathcal{U} \mathcal{U}, \hspace{1em} \mathcal{F}' = \mathcal{H}_0, \hspace{1em} \mathcal{G}' = \mathcal{H} \otimes \mathcal{F}' = \Phi \mathcal{E}_\mathcal{E}. \hspace{1em} (3.9)
$$

Let $F$ and $G$ be as in (3.8). Fix $z \in \mathbb{D}$. Since $G(z)|_F = \omega_2$ and $\omega_2$ is an isometry which maps $\mathcal{F}$ onto $\mathcal{F}'$, we know that $G(z)\mathcal{G} \subset \mathcal{G}' = \Phi \mathcal{E}_\mathcal{E}$. Thus given $u \in \mathcal{U}$, we have $G(z)E_{\mathcal{U}}u = \Phi \mathcal{E}_\mathcal{E}e(z)$ for some $e(z) \in \mathcal{E}$. Let $M_\Phi$ be the operator of multiplication by $\Phi$ acting from $\mathcal{H}^2(\mathcal{E})$ onto $H^2(\mathcal{U})$. The fact that $\Phi$ is inner is equivalent to the statement that $M_\Phi$ in an isometry. Put $C(z) = E_{\mathcal{E}'}M_\Phi^*G(z)E_{\mathcal{U}}$. Then

$$
C(z)u = E_{\mathcal{E}'}M_\Phi^*G(z)E_{\mathcal{U}}u = E_{\mathcal{E}'}M_\Phi^*\Phi \mathcal{E}_\mathcal{E}e(z) = E_{\mathcal{E}'}\mathcal{E}_\mathcal{E}e(z) = e(z).
$$

We conclude that $G(z)E_{\mathcal{U}} = \Phi \mathcal{E}_\mathcal{E}C(z)$. From the definition of $C(z)$ it is clear that $C(z)$ is a bounded linear operator from $\mathcal{U}$ into $\mathcal{E}$. Moreover, $C(z)$ depends analytically on $z \in \mathbb{D}$.

From the result of the previous paragraph we know that

$$
G(z)(E_{\mathcal{U}}u \oplus \lambda h_0) = h_0 \oplus \Phi \mathcal{E}_\mathcal{E}C(z) = E_{\mathcal{U}}v \oplus \lambda k_0,
$$

where

$$
v = E_{\mathcal{U}}^*\Phi \mathcal{E}_\mathcal{E}C(z)u + E_{\mathcal{U}}^*h_0, \hspace{1em} k_0 = S_{\mathcal{U}}^*\Phi \mathcal{E}_\mathcal{E}C(z)u + S_{\mathcal{U}}^*h_0.
$$

Recall that $\mathcal{H} = E_{\mathcal{U}} \mathcal{U} \oplus \lambda \mathcal{H}_0$. Let $J$ be the operator from $E_{\mathcal{U}} \mathcal{U} \oplus \lambda \mathcal{H}_0$ to $\mathcal{U} \oplus \mathcal{H}_0$ defined by $J(E_{\mathcal{U}}u \oplus \lambda h_0) = u \oplus h_0$. Obviously, $J$ is unitary and its inverse is given by $J^{-1}(u \oplus h_0) = E_{\mathcal{U}}u \oplus \lambda h_0$.

It follows that relative to the direct sum decomposition $\mathcal{U} \oplus \mathcal{H}_0$ the operator $JG(z)J^{-1}$ is given by the following $2 \times 2$ operator matrix:

$$
JG(z)J^{-1} = \begin{bmatrix}
E_{\mathcal{U}}^*\Phi \mathcal{E}_\mathcal{E}C(z) & E_{\mathcal{U}}^* \\
S_{\mathcal{U}}^*\Phi \mathcal{E}_\mathcal{E}C(z) & S_{\mathcal{U}}^*
\end{bmatrix}. \hspace{1em} (3.10)
$$

But then

$$
J(I - zG(z))J^{-1} = \begin{bmatrix}
I - zE_{\mathcal{U}}^*\Phi \mathcal{E}_\mathcal{E}C(z) & -zE_{\mathcal{U}}^* \\
-zS_{\mathcal{U}}^*\Phi \mathcal{E}_\mathcal{E}C(z) & I - zS_{\mathcal{U}}^*
\end{bmatrix},
$$

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and hence, using a Schur complement argument, we have

\[ J(I - zG(z))^{-1}J^{-1} = \begin{bmatrix} \Delta(z)^{-1} & * \\ * & * \end{bmatrix}, \]

where

\[ \Delta(z) = I - zE_t^\ast \Phi E \xi C(z) - (zE_t^\ast)(I - zS_t^\ast)^{-1}(-zS_t^\ast \Phi E \xi C(z)) \]
\[ = I - zE_t^\ast \Phi E \xi C(z) + zE_t^\ast(I - zS_t^\ast)^{-1}(I - zS_t^\ast - I) \Phi E \xi C(z) \]
\[ = I - zE_t^\ast(I - zS_t^\ast)^{-1} \Phi E \xi C(z) \]
\[ = I - z\Phi(z)C(z) = I - \Theta(z)C(z). \]

We also know that \( F(z)|_F = 0 \). Thus \( F(z)J^{-1} \) admits the following representation

\[ F(z)J^{-1} = \begin{bmatrix} F_1(z) & 0 \end{bmatrix} : \begin{bmatrix} U \\ H_0 \end{bmatrix} \rightarrow \begin{bmatrix} Y \end{bmatrix}. \]  

(3.11)

But then we have

\[ H(z) = F(z)(I - zG(z))^{-1}E_{td} \]
\[ = F(z)J^{-1}J(I - zG(z))^{-1}J^{-1}E_{td} \]
\[ = \begin{bmatrix} F_1(z) & 0 \end{bmatrix} \begin{bmatrix} \Delta(z)^{-1} & * \\ * & * \end{bmatrix} \begin{bmatrix} I_{td} \\ 0 \end{bmatrix} \]
\[ = F_1(z)\Delta(z)^{-1} = F_1(z)(I - \Theta(z)C(z))^{-1}. \]

Let \( \tau \) be the canonical embedding of \( H_0 \) into \( H \), that is, \( \tau \) is defined by \( \tau h_0 = h_0 \). From (3.10) and (3.11) it follows that

\[ \begin{bmatrix} F(z) \\ G(z) \end{bmatrix} J^{-1} = \begin{bmatrix} F_1(z) \\ \Phi E \xi C(z) \end{bmatrix} \begin{bmatrix} 0 & \tau_{H_0} \\ \tau_{H_0} & \tau_{H_0} \end{bmatrix} : \begin{bmatrix} U \\ H_0 \end{bmatrix} \rightarrow \begin{bmatrix} Y \\ H \end{bmatrix}. \]  

(3.12)

Since \( \text{Im} \tau \) is perpendicular to \( \Phi E \xi E \) we see that for \( h = E_{td}u \oplus \lambda h_0 \) we have

\[ \|F(z)h\|^2 + \|G(z)h\|^2 = \|F_1(z)u\|^2 + \|\Phi E \xi C(z)u\|^2 + \|h_0\|^2 \]

But \( \|h\|^2 = \|u\|^2 + \|h_0\|^2 \), and hence

\[ \|h\|^2 - (\|F(z)h\|^2 + \|G(z)h\|^2) = \|u\|^2 - (\|F_1(z)u\|^2 + \|\Phi E \xi C(z)u\|^2). \]
Since multiplication by $\Phi$ and the map $E_\mathcal{E}$ are isometries, we conclude that
\[
W = \begin{bmatrix} F \\ G \end{bmatrix} \in S(\mathcal{U}, \begin{bmatrix} \mathcal{Y} \\ \mathcal{H} \end{bmatrix}) \iff Z = \begin{bmatrix} F_1 \\ C \end{bmatrix} \in S(\mathcal{U}, \begin{bmatrix} \mathcal{Y} \\ \mathcal{E} \end{bmatrix}).
\]
We have now shown that the representations given by (3.7) and (3.8) imply those given by (0.9), with $Z \in S(\mathcal{U}, \mathcal{Y} \oplus \mathcal{E})$. The reverse implication is obtained by reversing the arguments. □

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