CORRECTION TO A THEOREM OF SCHOENBERG

CARL JOHAN RAGNARSSON, WESLEY WAI SUEN, AND DAVID G. WAGNER

Abstract. A well-known theorem of Schoenberg states that if $f(z)$ generates a PF$_r$ sequence then $1/f(-z)$ generates a PF$_r$ sequence. We give two counterexamples which show that this is not true, and give a correct version of the theorem. In the infinite limit the result is sound: if $f(z)$ generates a PF sequence then $1/f(-z)$ generates a PF sequence.

1. The Bad News.

Theorem 1.2 in Chapter 8 of Karlin’s book [2] implies the following:

**Theorem A.** Let $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ be power series with real coefficients such that $g(z) = 1/f(-z)$. For any positive integer $r$, the Toeplitz matrix of $f$ is totally positive up to order $r$ if and only if the Toeplitz matrix of $g$ is totally positive up to order $r$.

Theorem A first appears in work of Schoenberg et al. in the early 1950s [1, 4, 5]. The bad news is that Theorem A is false.

First let’s review the definitions. For a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$, the *Toeplitz matrix* of $f$ is the infinite matrix $T[f]$, indexed by pairs of integers, with entries

$$ T[f]_{ij} := \begin{cases} a_{j-i} & \text{if } j - i \geq 0, \\ 0 & \text{if } j - i < 0. \end{cases} $$

An infinite matrix $M$ is *totally positive up to order* $r$ when every minor of $M$ of order at most $r$ is nonnegative. This condition is abbreviated TP$_r$. If $M$ is TP$_r$ for all $r$ then $M$ is *totally positive*, abbreviated TP.

The matrix $T[f]$ is TP$_1$ if and only if the coefficients of $f(z)$ are nonnegative. If $T[f]$ is TP$_2$ then the sequence of coefficients $a_0, a_1, \ldots$ *has no internal zeros*: if $0 \leq h < i < j$ and $a_h a_j \neq 0$, then $a_i \neq 0$.

1991 Mathematics Subject Classification. 15A48; 15A45, 15A57, 05E05.

Key words and phrases. Total positivity, Pólya frequency sequence, skew Schur function.
0. Also, if $T[f]$ is $TP_2$ then the sequence of coefficients $a_0, a_1, \ldots$ is logarithmically concave: if $j \geq 1$ then $a_j^2 \geq a_{j-1}a_{j+1}$. Nongativity of the remaining 2–by–2 minors of $T[f]$ follow from these two conditions. That is, the Toeplitz matrix $T[f]$ is $TP_2$ if and only if the sequence of coefficients $a_0, a_1, \ldots$ is nonnegative, has no internal zeros, and is logarithmically concave.

Our first counterexample to Theorem A is the polynomial $f(z) = 1 + 4z^2 + 3z^3$. By the preceding paragraph, one sees easily that $T[f]$ is $TP_2$. Elementary calculation with linear recurrence relations yields

$$g(z) = \frac{1}{1 - 4z + 3z^2 - z^3} = 1 + 4z + 13z^2 + 41z^3 + 129z^4 + 406z^5 + \cdots.$$  

Since $129^2 - 41 \cdot 406 = -5 < 0$, the Toeplitz matrix $T[g]$ is evidently not $TP_2$. Theorem A is false.

With hindsight, one notices that the coefficients of $f(z) = 1 + z + 2z^2$ are nonnegative, but that

$$g(z) = \frac{1}{1 - z + 2z^2} = 1 + z - z^2 - 3z^3 - 4z^5 + 5z^5 + \cdots$$

has negative coefficients. Thus, $T[f]$ is $TP_1$ while $T[g]$ is not $TP_1$. This is a rather glaring counterexample to Theorem A.

2. The Good News.

The good news is that Theorem A can be fixed.

To do this we need a few facts about symmetric functions – see Macdonald [3] for details. Let $e_1, e_2, \ldots$ and $h_1, h_2, \ldots$ be indeterminates which are algebraically independent over the field $\mathbb{Q}$ of rational numbers, and form the generic power series $E(t) := 1 + \sum_{n=1}^{\infty} e_n t^n$ and $H(t) := 1 + \sum_{n=1}^{\infty} h_n t^n$. By imposing the single relation $E(t) = H(-t)^{-1}$ one can determine each $e_n$ as a polynomial in the $h_n$-s, and conversely. The indeterminates $\{h_n\}$ remain algebraically independent over $\mathbb{Q}$, as do the indeterminates $\{e_n\}$. The ring $\Lambda$ of polynomials with integer coefficients in these indeterminates is the ring of symmetric functions.

Since the indeterminates $\{h_n\}$ are algebraically independent and generate $\Lambda$, a homomorphism $\varphi : \Lambda \to R$ from $\Lambda$ to another ring $R$ is determined by its values $\{\varphi(h_n)\}$. A real power series $f(z) = 1 + \sum_{n=0}^{\infty} a_n z^n$ determines such a homomorphism $\varphi_f : \Lambda \to \mathbb{R}$ by $\varphi_f(h_n) := a_n$. Notice that if $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ is such that $g(z) = 1/f(-z)$ then $\varphi_f(e_n) = b_n$ and $\varphi_g(e_n) = a_n$.

The set of all integer partitions, partially ordered by inclusion of Ferrers diagrams, is called Young’s lattice and denoted by $\mathcal{Y}$. For $\mu \leq \lambda$
in $\mathcal{Y}$ there is a symmetric function $s_{\lambda/\mu}$ called a \textit{skew Schur function}. Without loss of generality we may assume not only that $\mu \leq \lambda$ in $\mathcal{Y}$, but also that $\mu$ has strictly fewer parts than $\lambda$ and that the largest part of $\mu$ is strictly smaller than the largest part of $\lambda$. We will denote this relation by $\mu < \lambda$ in $\mathcal{Y}$. The formulae we need are the Jacobi–Trudy formula and its dual form:

$$s_{\lambda'/\mu'} = \det(e_{\lambda_i-i+j-\mu_j}) \quad \text{and} \quad s_{\lambda/\mu} = \det(h_{\lambda_i-i+j-\mu_j}).$$

The order of these determinants is the number of parts of $\lambda$. The notation $\lambda'$ denotes the partition conjugate to $\lambda$. If $f(z)$ and $g(z)$ are real power series such that $g(z) = 1/f(-z)$ then

$$\varphi_f(s_{\lambda/\mu}) = \varphi_g(s_{\lambda'/\mu'}) \quad \text{and} \quad \varphi_g(s_{\lambda/\mu}) = \varphi_f(s_{\lambda'/\mu'}).$$

Consider the submatrix $M$ of $T[f]$ supported on rows $\{i_1 < i_2 < \cdots < i_r\}$ and columns $\{j_1 < j_2 < \cdots < j_r\}$. If $j_k < i_k$ for any $1 \leq k \leq r$ then $\det(M) = 0$, so we may assume that $j_k \geq i_k$ for all $1 \leq k \leq r$. If $j_1 = i_1$ or $j_r = i_r$ then $\det(M)$ reduces by Laplace expansion to a smaller minor of $T[f]$. Thus we may assume as well that $j_1 > i_1$ and $j_r > i_r$. A minor satisfying all these conditions is called an \textit{essential minor} of $T[f]$. It is clear that $T[f]$ is TP$_r$ if and only if every essential minor of $T[f]$ of order at most $r$ is nonnegative.

Every essential minor of $T[f]$ has the form $\varphi_f(s_{\lambda/\mu}) = \det(a_{\lambda_i-i+j-\mu_j})$ for some $\mu < \lambda$ in $\mathcal{Y}$. To see this, let $\det(M)$ be an essential minor of $T[f]$ supported on rows $\{i_1 < i_2 < \cdots < i_r\}$ and columns $\{j_1 < j_2 < \cdots < j_r\}$. For each $1 \leq k \leq r$ let $\lambda_k := j_r - i_k + k - r$. The inequalities $\lambda_1 \geq \cdots \lambda_r > 0$ are easily seen, so that $\lambda$ is an integer partition with $r$ parts. For each $1 \leq k \leq r$ let $\mu_k := j_r - j_k + k - r$. One can check that $\mu$ is an integer partition with $m = r - 1$ parts, that $\mu < \lambda$ in $\mathcal{Y}$, and that $\det(M) = \det(a_{\lambda_i-i+j-\mu_j})$. This construction can be reversed, so that every $\varphi_f(s_{\lambda/\mu})$ is an essential minor of $T[f]$. In this way the skew Schur functions can be regarded as “generic essential Toeplitz minors”.

The order of the minor $\varphi_f(s_{\lambda/\mu})$ of $T[f]$ is the number of parts of $\lambda$. This implies the following:

\begin{itemize}
  \item[(a)] The Toeplitz matrix $T[f]$ is TP$_r$ if and only if $\varphi_f(s_{\lambda/\mu}) \geq 0$ for all $\mu < \lambda$ in $\mathcal{Y}$ for which $\lambda$ has at most $r$ parts.
  
  Similarly,
  \item[(b)] The Toeplitz matrix $T[g]$ is TP$_r$ if and only if $\varphi_g(s_{\lambda/\mu}) \geq 0$ for all $\mu < \lambda$ in $\mathcal{Y}$ for which $\lambda$ has at most $r$ parts.
  
  If $g(z) = 1/f(-z)$ then, since $\varphi_g(s_{\lambda/\mu}) = \varphi_f(s_{\lambda'/\mu'})$, condition (b) is equivalent to
  \item[(c)] The Toeplitz matrix $T[g]$ is TP$_r$ if and only if $\varphi_f(s_{\lambda'/\mu'}) \geq 0$ for all $\mu < \lambda$ in $\mathcal{Y}$ for which $\lambda$ has at most $r$ parts.
\end{itemize}
Or, in other words,
\[(d) \] The Toeplitz matrix $T[g]$ is TP$_r$ if and only if $\varphi_f(s_{\lambda/\mu}) \geq 0$ for all $\mu \prec \lambda$ in $Y$ for which $\lambda$ has largest part at most $r$.
Comparing (a) and (d) we see that the two conditions in Theorem A are closely related, but not equivalent.

Interpreting $\varphi_f(s_{\lambda/\mu})$ as a minor of $T[f]$, bounding the number of parts of $\lambda$ corresponds to bounding the order of the minor. What corresponds to bounding the largest part of $\lambda$? For the submatrix $M$ of $T[f]$ supported on rows $\{i_1 < i_2 < \cdots < i_r\}$ and columns $\{j_1 < j_2 < \cdots < j_r\}$, define the level of $M$ to be $\ell := j_r - i_1 + 1 - r$.
The level of a minor of $T[f]$ is the level of the submatrix of which it is the determinant. The Toeplitz matrix $T[f]$ is totally positive up to level $\ell$ when every minor of $T[f]$ of level at most $\ell$ is nonnegative. This condition is abbreviated TP$'_\ell$. If $T[f]$ is TP$'_\ell$ for all $\ell$ then $T[f]$ is totally positive, TP.

**Theorem B.** Let $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$ be power series with real coefficients such that $g(z) = 1/f(-z)$. For any positive integer $r$, $T[f]$ is totally positive up to level $r$ if and only if $T[g]$ is totally positive up to order $r$.

Notice that in the limit as $r \to \infty$ we get the equivalence: $T[f]$ is TP if and only if $T[g]$ is TP. This is the most important consequence of Theorem A in the literature, and it is a huge relief that it survives.

**References**

[1] M. Aissen, I.J. Schoenberg, and A. Whitney, *On the generating functions of totally positive sequences*, Proc. Nat. Acad. Sci. U.S.A. 37 (1952), 303–307.
[2] S. Karlin, “Total Positivity, vol. I,” Stanford U.P., Stanford CA, 1968.
[3] I.G. Macdonald, “Symmetric Functions and Hall Polynomials” (2nd ed.), Oxford U.P., Oxford, 1995.
[4] I.J. Schoenberg, *On smoothing operations and their generating functions*, Bull. Amer. Math. Soc. 59 (1953), 199–230.
[5] I.J. Schoenberg, *On the zeros of the generating functions of multiply positive sequences and functions*, Ann. Math. 62 (1955), 447–471.

**Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1**

*E-mail address:* wwsuen@math.uwaterloo.ca

*E-mail address:* dgwagner@math.uwaterloo.ca