Classification trees in a box extent lattice

Laura Veres
Institute of Mathematics, University of Miskolc
3515 Miskolc-Egyetemváros, Hungary
matvlaura@gmail.com

Abstract.
In this paper we show that during an elementary extension of a context each of the classification trees of the newly created box extent lattice can be obtained by the modification of the classification trees of the box extent lattice of the original, smaller context. We construct also an algorithm which, starting from a classification tree of the box extent lattice of the smaller context \((H, M, I \cap H \times M)\), gives a classification tree of the extended context \((G, M, I)\) which contains the new elements inserted. The effectiveness of the method is that it ensures that there is enough to know the original context, the classification tree of the box extent lattice and its box extents, we do not need a new box extension of the extended context mesh elements (except for one, which is the new element box extension).

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1.1 Preliminaries: box lattice, extent lattice

A context (see [2]) is a triple \((G, M, I)\) where \(G\) and \(M\) are sets and \(I \subseteq G \times M\) is a binary relation. The elements of \(G\) and \(M\) are called objects and respectively attributes of the context. The relation \(gIm\) means that the object \(g\) has the attribute \(m\). A small context can be easily represented by a cross table, i.e., by a rectangular table, the rows of which are headed by the object names and the columns are headed by the attribute names. A cross in the intersection of the row \(g\) and the column \(m\), means that the object \(g\) has the attribute \(m\).

For all sets \(A \subseteq G\) and \(B \subseteq M\) we define

\[ A' = \{ m \in M \mid gI m \text{ for all } g \in A \}, \]
\[ B' = \{ g \in G \mid gI m \text{ for all } m \in B \}. \]

A concept of the context \((G, M, I)\) is a pair \((A, B)\), in which \(A' = B\) and \(B' = A\), and \(A \subseteq G, B \subseteq M\). \(L(G, M, I)\) denotes the set of all concepts of the context \((G, M, I)\).

\(L(G, M, I)\) can be endowed with the structure of a complete lattice defining the join and meet of concepts as follows:
\[ \bigwedge_{i \in I} (A_i, B_i) = \left( \bigcup_{i \in I} A_i, \left( \bigcap_{i \in I} B_i \right)^\prime \prime \right) \]
\[ \bigvee_{i \in I} (A_i, B_i) = \left( \left( \bigcap_{i \in I} A_i \right)^\prime \prime, \bigcup_{i \in I} B_i \right) \]

The lattice \((L(G, M, I), \wedge, \vee)\) will be called the concept lattice of the context \((G, M, I)\).

An extent partition of a formal context \((G, M, I)\) is a partition of \(G\), all classes of which are concept extents. Clearly, the trivial partition \(\{G\}\) is an extent partition. Note that, since the intersection of extents always yields an extent, the common refinements of extent partitions are still extent partitions. Therefore, the extent partitions of \((G, M, I)\) form a complete \(\wedge\)-subsemilattice of the partition lattice of \(G\), and thus a complete lattice which will be denoted with \(\text{Ext}(G, M, I)\). In particular, there is always a finest extent partition of the context denoted with \(\pi\).

The zero element of a complete lattice \(L\) and all elements that are contained in some classification system of \(L\) are called the box elements of \(L\). The set of all box elements of \(L\) is denoted by \(\text{Box}(L)\), and \((\text{Box}(L), \leq)\) is a poset obtained by restricting the partial order of \(L\) to the box elements. If every nonzero element of \(L\) is a join of atoms of \(L\), then \(L\) is called an atomistic lattice.

In [10] the following was shown: If \(L\) is a complete lattice in which every element is a join of some completely join-irreducible elements, then \((\text{Box}(L), \leq)\) is a complete atomistic lattice.

In [3] the box extents of a context \((G, M, I)\) were characterized if \(E\) belongs to some extent partition of \((G, M, I)\) or \(E = \varnothing\). The set of all box extents of \(L\) is denoted with \(\text{Box}(L)\).

Let \(L\) be a bounded lattice. A set \(X \subseteq L\) is called CD-independent, if for any \(x, y \in X\) either \(x \leq y\) or \(y \leq x\) or \(x \wedge y = 0\) (any elements of \(X\) are comparable or disjoint). Maximal CD-independent sets (with respect to \(\subseteq\)) are called CD-bases.

Let \(L\) be a lattice with the smallest element 0. A set \(O = \{a_i | i \in I\}, I \neq \varnothing\) of nonzero elements of \(L\) is called a disjoint set or orthogonal system, if \(a_i \wedge a_j = 0, i \neq j\). \(O\) is a maximal orthogonal system, if there is no other orthogonal system \(O'\) of \(L\) containing \(O\) as a proper subset. Notice that if \(S_1 = \{a_i | i \in I\}, I \neq \varnothing\) and \(S_2 = \{b_j | j \in J\}, 2\)

In [4] was introduced the notion of CD-independent sets in an arbitrary poset, we will define it in a lattice as follows:

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$J \neq \emptyset$ are two orthogonal systems, then $S_1 \leq S_2$, if for each $i \in I$ there exists $j(i) \in J$ such that $a_i \leq b_{j(i)}$. We denoted with $\text{Ort}(L)$ the set of all orthogonal systems, $(\text{Ort}(L), \leq)$ is evidently a poset, moreover a lattice.

**Theorem 1.1:** ([4],[12]) Let $L$ be a lattice, and $T$ a CD-base in $L$. 
(i) Then there exists a chain $C = \{S_\lambda \mid \lambda \in \Lambda\}$ in $\text{Ort}(L)$, such that $T = \bigcup_{\lambda \in \Lambda} S_\lambda$. 
(ii) Any CD-base $T$ is the union of disjoint sets (orthogonal systems) belonging to a maximal chain.

2. Classification trees and complete classification trees

The classification trees are used in many fields: lattice theory, data mining, group technology problems. Classification trees are used for clustering the objects by their attributes, and they appear in some clustering problems originated in Group Technology. Several independent definitions are for classification trees in lattice theory; we will define it as a special chain and also we will show some construction theorems for them.

First we will relive the notion of classification trees and the relation between classification trees, disjoint (orthogonal) systems and CD-independent sets [11].

Let $L$ be a lattice with the greatest element $1 \in L$ and $T \subseteq L \setminus \{0\}$, $T \neq \emptyset$ a subset of it. $T$ is a directed tree, if $[t) \cap T$ is a chain for each $t \in T$ and $1 \in T$.

Now strengthening the above condition we obtain the notion of a classification tree:

**Definition 2.1.** $T$ is a classification tree, if $[x) \cap T$ is a chain (nonempty) for each $x \in L \setminus \{0\}$. $H$ is a maximal classification tree, if there is no other classification tree which contains it as a proper subset.

Observe that in the definition of a classification tree the condition $[t) \cap T$ is a chain is replaced with a stronger one: $[x) \cap T$ is a chain (nonempty). As a consequence, we get that any classification tree is also a directed tree but in addition for any $x, y \in X$ either $x \wedge y = 0$ or $x, y$ are comparable. Hence any classification tree is CD-independent, and any maximal classification tree is a CD-base. Moreover in [4,10] are proved the following assertions, which are also true for classification trees and CD-independent sets:

**Proposition 2.2.** Let $L$ be a bounded lattice and $T \subseteq L$ a nonempty subset of it. Then the following are equal:

(i) $T$ is a maximal classification tree in $L$;
(ii) $T \cup \{0\}$ is a CD-base of $L$. 

Remark 2.3: (i) If $L$ is an atomistic lattice, $A(L)$ the set of its all atoms then any CD-base as well as any maximal classification trees of $L$ contains $A(L)$.

(ii) If $T$ is a classification tree and we add atoms to it, it remains a classification tree. (If $S \subseteq A(P)$, then $T \cup S$ is a classification tree.) [11]

The notion of context and classification trees appears in Group Technology problems. This engineering discipline exploits similarities between technological objects and divides them into relatively homogeneous groups, extent partitions (classes) in order to optimize manufacturing processes. The concept of classification tree appear also in the Group Technology. In Group Technology by the term "classification tree" we refer to a tree made of extents belonging to the context. This trees have the property that any maximal antichain selected from the tree is a cover of the set $G$ with extents.

Obvious that basis, the following definition results:

Definition 2.4. A classification tree $T \subseteq \text{Ext}(G, M, I)$ is called a complete classification tree if for any maximal antichains $\{E_i| i \in I\} \subseteq T$ we have $\bigcup_{i \in I} E_i = G$. $T$ is a maximal complete classification tree, if there is no other complete classification tree which contains it as a proper subset.

It's obvious that all elements of such a complete classification tree are box extents, ie $T$ is a classification tree in the lattice of box extents.

Proposition 2.5. Let $(G, M, I)$ be a finite context, $\text{Ext}(G, M, I)$ the extent lattice of the context and $\mathcal{B}(G, M, I)$ the lattice of the box extents of the context. Then the followings are equal:

(i) $T \subseteq \text{Ext}(G, M, I)$ is a complete classification tree;

(ii) $T$ is a classification tree $\mathcal{B}(G, M, I)$ and each maximal antichain of it $\{E_i| i \in I\} \subseteq T$ is a complete orthogonal system in $\mathcal{B}(G, M, I)$.

Proof. (i)$\Rightarrow$(ii) First we will prove that $T$ is a classification tree in the box extent lattice $\mathcal{B}(G, M, I)$. Since $\mathcal{B}(G, M, I)$ is a subsemilattice of $\text{Ext}(G, M, I)$ and it contains the smallest element $(\emptyset)$ of $\text{Ext}(G, M, I)$, it is enough to prove that each element $A \in T$ is a box extent. As $T$ is finite, then $A$ must be an element of a maximal antichain $\{E_i| i \in I\} \subseteq T$, ie. $A = E_k$ for some $k \in I$. Since the elements of an antichain are incomparable, for any two elements $E_i, E_j \in T$, $i, j \in I$ we have $E_i \cap E_j = \emptyset$. As $T$ is a complete classification tree in $\text{Ext}(G, M, I)$, we get $\bigcup_{i \in I} E_i = G$. Then $\{E_i| i \in I\}$ is an extent partition of $G$. Hence $A = E_k$ is a box extent. From the above follows that any maximal antichain $E_i$, $i \in I$ is a part of an orthogonal system, so it is an orthogonal system itself. Obviously, this orthogonal system can not be extended with a new element $E_0 \in \mathcal{B}(G, M, I)$, $E_0 \neq \emptyset$. If we
extended it, then $E_i \cap E_0 = \emptyset$ for each $i \in I$, from which follows that $E_0 = G \cap E_0 = \left( \bigcup_{i \in I} E_i \right) \cap E_0 = \emptyset$, contradiction.

(ii)$\Rightarrow$(i) If $T$ is a classification tree in the lattice $B(G, M, I)$, then it is also a classification tree in $\text{Ext}(G, M, I)$. Assume that (ii) is satisfied, then it is enough to prove that for every maximal antichain $\{E_i| i \in I\} \subseteq T$ we have $\bigcup_{i \in I} E_i = G$. By our hypothesis $\{E_i| i \in I\}$ is a complete orthogonal system. Let $g \in G$ and $g$ the box extents of it. As $\{E_i| i \in I\} \cup g$ is not an orthogonal system, then $g \cap E_k \neq \emptyset$ for some $k \in I$. Since $g$ is an atom in the lattice of box extents, implies that $g \subseteq E_k$. Hence $G \subseteq \bigcup_{i \in I} E_i$. As $\bigcup_{i \in I} E_i \subseteq G$, $\bigcup_{i \in I} E_i = G$. □

**Corollary 2.6.** $\mathcal{T} \subseteq B(G, M, I)$ is a maximal classification tree in $B(G, M, I)$ if and only if $\mathcal{T}$ is a maximal complete classification tree is the extent lattice $\text{Ext}(G, M, I)$.

**Proof.** Let $\mathcal{T}$ be a maximal classification tree in $B(G, M, I)$. Then $\mathcal{T}$ contains all atoms of the lattice $B(G, M, I)$. Let $S$ be a maximal antichain in $\mathcal{T}$. As $\mathcal{T}$ is a CD-independent set, then $S$ is an orthogonal system. We have to prove that $S$ is a complete orthogonal system.

We denote with $A(B(G, M, I))$ the set of all atoms of the lattice $B(G, M, I)$. It is easy to see that $A(B(G, M, I))$ is a complete orthogonal system. In [4] was proved that an orthogonal system $O$ is complete if $A(L) \leq O$. (where $A(L)$ is the set of all atoms of the lattice $L$ and $O$ is an orthogonal system). Thus it’s enough to prove that $A(B(G, M, I)) \leq S$. Indeed, if this inequation is not satisfied then there exists an atom $a$ in $B(G, M, I)$ which is not smaller then any element of $S$. If $a \in \mathcal{T}$, then $S \cup \{a\}$ is an antichain in $\mathcal{T}$, which is a contradiction. Thus we have $A(B(G, M, I)) \leq S$ and $S$ is a maximal orthogonal system. By the Proposition 2.5 $\mathcal{T}$ is a complete classification tree in the lattice $\text{Ext}(G, M, I)$. Now, let $\mathcal{F} \subseteq \text{Ext}(G, M, I)$ be a complete classification tree which contains $\mathcal{T}$. Hence, using Proposition 2.5, $\mathcal{F}$ is also a classification tree in the lattice $B(G, M, I)$ and $\mathcal{T}$ by definition is a maximal classification tree in $B(G, M, I)$, so we obtain $\mathcal{F} = \mathcal{T}$. Thus $\mathcal{T}$ is a maximal complete classification tree in $\text{Ext}(G, M, I)$.

Conversely, assume that $\mathcal{T} \subseteq \text{Ext}(G, M, I)$ is a maximal complete classification tree. Hence using Proposition 2.5 $\mathcal{T}$ is also a classification tree in the box extent lattice $B(G, M, I)$ Thus it is a subset of a maximal classification tree $\mathcal{M}$ of $B(G, M, I)$. Therefore, $\mathcal{M}$ is a complete classification tree in the lattice $\text{Ext}(G, M, I)$ and $\mathcal{T} \subseteq \mathcal{M}$ is maximal, so we obtain $\mathcal{T} = \mathcal{M}$.

Finally we have that $\mathcal{T}$ is a maximal classification tree in the lattice $B(G, M, I)$. □
3. Classification trees in the lattice of box extents

In what follows we will consider a formal context $K = (G, M, I)$ with a concrete and fixed technical meaning, where $G$ and $M$ are finite and nonempty sets, $G$ denotes a fixed set of technical objects, $M$ denotes a fixed set of some possible, technically relevant properties, and for any $g \in G$ and any $m \in M$, $gI m$ means that the object (part) $g$ has the property $m$. Additionally we suppose that the context does not contain rows or columns filled with only zeros. A full zero column means that the corresponding property not held by any of the parts, so it is irrelevant; a full zero row corresponds to a part $g$ possessing none of the properties from our list.

Let $K = (G, M, I)$ be a context and $K_H = (H, M, I \cap H \times M)$ a subcontext of it, where $H \subseteq G$ and there exists a $z \in G$ such that $H = G \setminus \{z\}$. In this section we use the results from the article [3], in which the authors studied what how will change the box extents after a one-object extension of the context. In [3] was shown that the intersection of box extents is also a box extent. In [3] was also proved that any extent partition of the subcontext is also an extent partition of $K$ and the box extents of the subcontext are also box extents of $K$ and the following propositions:

**Proposition 3.1.** If $\pi = \{A_k \mid k \in K\}$ is an extent partition of $K$ then

$$\pi_H = \{A_k \cap H \mid k \in K\} \setminus \{\emptyset\}$$

is an extent partition of the subcontext $(H, M, I \cap H \times M)$. (If $\pi$ is the finest extent partition of $K$ then $\pi_H$ is called the restriction of the extent partition $\pi$ and conversely, $\pi_H$ is not necessarily the finest extent partition of $H$.)

**Corollary 3.2.** [3] If $E$ is a box extent of $(G, M, I)$ then $E \cap H$ is a box extent of $(H, M, I \cap H \times M)$.

**Corollary 3.3.** [10] $\mathcal{B}(G, M, I)$ is a complete atomistic lattice. The atomic box extents are the classes of the finest extent partition $\pi_\square$.

**Proposition 3.4.** [3] If $E$ is a box extent of $(G, M, I)$ and $H = G \setminus \{z\}$ for some $z \in G$ then

1. $E$ is a box extent of $(H, M, I \cap H \times M)$ with $E \cap z^{\square\square} = \emptyset$ or
2. $E \setminus \{z\}$ is a box extent of $(H, M, I \cap H \times M)$.

**Proposition 3.5.** [3] If $H = G \setminus \{z\}$ then $A$ is a class of finest extent partition of $\pi_\square$ of $(G, M, I)$ if and only if

1. either $A = z^{\square\square}$, or
2. $A$ is a class, disjoint from $z^{\square\square}$, of the finest extent partition of the subcontext $(H, M, I \cap H \times M)$. 6
Theorem 3.6. [3] Let \((G, M, I)\) be a context, \(E\) a box extent of the subcontext \((H, M, I \cap H \times M)\) with \(H = G \setminus \{z\}\). Then

(i) \(E\) is a box extent of \((G, M, I)\) if and only if \(z^\square \cap E'' = \emptyset\);
(ii) \(E'^* = E \cup \{z\}\) is a box extent of \((G, M, I)\) if and only if \(z^\square \setminus \{z\} \subseteq E\) and \((E \cup \{z\})'' = E \cup \{z\}\).

Remark: In other words, in view of the theorem below, we have two possibilities:

1) \(E\) is also box extent in the new context iff \(z^\square \cap E'' = \emptyset\);
2) or \(E \cup \{z\}\) is a box extent in the new context iff \(z^\square \setminus \{z\} \subseteq E\) and \((E \cup \{z\})'' = E \cup \{z\}\).

\[E'^* = \begin{cases} E & \text{if } z^\square \cap E'' = \emptyset \\ E \cup \{z\} & \text{else} \end{cases}\]

In the following we show that during an elementary extension of the context each of the classification trees of the newly created box extent lattice can be obtained by the modification of the classification trees of the box extent lattice of the original, smaller context.

Proposition 3.7. Let \(K_H = (H, M, I \cap H \times M)\) be a subcontext of \(K = (G, M, I)\) and \(T\) a classification tree in the box extent lattice \(\mathcal{B}(K)\). Then the set

\[T_H = \{E \cap H \mid E \in T\}\]

is a classification tree in the box lattice \(\mathcal{B}(K_H)\).

Proof. In [3] was proved that if \(E\) is a box extent of the context \((G, M, I)\) then \(E \cap H\) is the box extent of the subcontext \((H, M, I \cap H \times M)\). Thus for any \(E \in T\), \(E \cap H\) is a box extent of the context \(K_H\), so \(T_H \subseteq \mathcal{B}(K_H)\). Since \(G \in T\), then \(H = G \cap H \in T_H\) and also \(H\) is the largest element of the box extent lattice \(\mathcal{B}(K_H)\). We have to prove that \(T_H\) is a classification tree. It is enough to prove that \(T_H\) is a CD-independent set of the lattice \(\mathcal{B}(K_H)\). Let \(E_1 \cap H\) and \(E_2 \cap H\) be two incomparable elements of \(T_H\). Then \(E_1 \cap H \neq \emptyset\) and \(E_2 \cap H \neq \emptyset\) and \(E_1, E_2 \in T\) are also incomparable. Since \(T\) is classification tree and also a CD-independent set, we have \(E_1 \cap E_2 = \emptyset\). Then \((E_1 \cap H) \cap (E_2 \cap H) = E_1 \cap E_2 \cap H = \emptyset\). Thus we proved that \(T_H\) is also a CD-independent set in the lattice \(\mathcal{B}(K_H)\). Therefore \(T_H\) is a classification tree in \(\mathcal{B}(K_H)\). \(\square\)

Further we show how to construct a classification tree after a one-object extension of a context:

Theorem 3.8. Let \(K_H = (H, M, I \cap H \times M)\) be subcontext of a finite context \((G, M, I)\) such that \(H = G \setminus \{z\}\), \(z^\square \neq \{z\}\). Let \(T\) be a classification tree in the box extent lattice \(\mathcal{B}(K_H)\). Then:

(i) \(T^{(1)} = \{E \in T \mid E \in \mathcal{B}(G, M, I)\}\) is an order ideal in \(T\) and
(ii) \(T^{(2)} = \{E \in T \mid E \cup \{z\} \in \mathcal{B}(G, M, I)\}\) a finite chain in \(T\) and \(T^{(1)} \cap T^{(2)} = \emptyset\);
(ii) \( \mathcal{T}^* = \mathcal{T}^{(1)} \cup \{ E \cup \{ z \} \mid E \in \mathcal{T}^{(2)} \} \) is a classification tree in the lattice \( \mathcal{B}(G, M, I) \).

(iii) If \( \mathcal{T} \) contains all the atoms of the box extent lattice \( \mathcal{B}(K_H) \), then \( \mathcal{T}^* \cup \{ z \} \) is a classification tree in \( \mathcal{B}(G, M, I) \), which contains all the atoms of it.

Proof. (i) Let \( E \in \mathcal{T}^{(1)} \) and \( F \subseteq E \). Since \( E \) is a box extent of \( (G, M, I) \) we get \( z \supseteq E \cap E' = \emptyset \).

Since \( F' \subseteq E' \) we have \( z \cup E' = \emptyset \), which in view of Theorem 3.6 means that \( F \subseteq \mathcal{T}^{(1)} \). Thus \( \mathcal{T}^{(1)} \) is an order ideal in \( \mathcal{T} \).

We have to prove now that \( \mathcal{T}^{(2)} \) is a finite chain in \( \mathcal{T} \). We take the set \( C = \{ E \in \mathcal{T} \mid z \supseteq \{ z \} \subseteq E \} \). Obviously, \( H \in C \), then \( C \neq \emptyset \). As \( z \supseteq \{ z \} \neq \emptyset \) we get \( E \supseteq z \). Thus \( \mathcal{T}^{(1)} \) and \( C \) have no common elements. Take \( E \in C \) and \( F \in \mathcal{T} \). Since \( E \subseteq F \), we have \( z \supseteq \{ z \} \supseteq F \), so \( F \in C \). Therefore \( C \) is an order filter of \( \mathcal{H} \) and since \( H \) is finite and \( C \subseteq \mathcal{H} \) is lower bounded by its minimal elements. We show that \( C \) has only one minimal element \( E_1 \in C \). Assume that \( E_2 \in C \) is a minimal element in \( C \) and \( E_1 \neq E_2 \). Since \( E_1, E_2 \in \mathcal{T} \) and \( E_1, E_2 \) are incomparable, we have \( E_1 \cap E_2 = \emptyset \), which is a contradiction because \( z \supseteq \{ z \} \subseteq E_1 \cap E_2 \) and \( z \supseteq \{ z \} \neq \emptyset \) by hypothesis. Thus \( E_1 \) is the smallest element of \( C \) and \( C \) is equal to \( \{ E_1 \} \cap \mathcal{T} \). Since \( \mathcal{T} \) is a classification tree \( C \) must be a chain. Consider now the set \( \mathcal{T}^{(2)} \).

By Proposition 3.5 \( \mathcal{T}^{(2)} \subseteq C \), then \( \mathcal{T}^{(2)} \) is a finite chain and \( \mathcal{T}^{(1)} \cap \mathcal{T}^{(2)} = \emptyset \).

(ii) As \( \mathcal{T}^{(1)} \subseteq \mathcal{T} \) we have that \( \mathcal{T}^{(1)} \) is a CD-independent set. Observe that \( \{ E \cup \{ z \} \mid E \in \mathcal{T}^{(2)} \} \) is a chain in \( \mathcal{B}(G, M, I) \). Indeed, let \( E_1 \cup \{ z \} \) and \( E_2 \cup \{ z \} \) be two elements of this set. Since \( \mathcal{T}^{(2)} \) is a chain, then we have \( E_1 \subseteq E_2 \) or \( E_2 \subseteq E_1 \) and \( E_1 \cup \{ z \} \subseteq E_2 \cup \{ z \} \), or conversely \( E_2 \cup \{ z \} \subseteq E_1 \cup \{ z \} \). We show that the set \( \mathcal{T}^{(1)} \cup \{ E \cup \{ z \} \mid E \in \mathcal{T}^{(2)} \} \) is also CD-independent.

Take \( E_1 \in \mathcal{T}^{(1)} \) and \( E_2 \in \mathcal{T}^{(2)} \). As \( E_1, E_2 \in \mathcal{T} \) \( \mathcal{T} \) is a classification tree, we have the following three cases: \( E_1 \subseteq E_2 \) or \( E_2 \subseteq E_1 \) or \( E_1 \cap E_2 = \emptyset \). We have to show that also the sets \( E_1 \) and \( E_2 \cup \{ z \} \) are either comparable or disjoint.

Clearly, in the first case \( E_1 \subseteq E_2 \cup \{ z \} \).

In the second case \( E_2 \subseteq E_1 \) implies \( E_2 \in \mathcal{T}^{(1)} \) and \( \mathcal{T}^{(1)} \) is an order ideal in \( \mathcal{T} \). However, this is impossible, because \( E_2 \in \mathcal{T}^{(2)} \) and \( \mathcal{T}^{(1)} \cap \mathcal{T}^{(2)} = \emptyset \).

Let us consider now the case \( E_1 \cap E_2 = \emptyset \). Since \( E_1 \in \mathcal{T}^{(1)} \) we have \( E_1' \cap z = \emptyset \) and this results also \( E_1 \cap \{ z \} = \emptyset \). Therefore, we obtain \( E_1 \cap (E_2 \cup \{ z \}) = \emptyset \).

Thus we have proved that \( \mathcal{T}^* = \mathcal{T}^{(1)} \cup \{ E \cup \{ z \} \mid E \in \mathcal{T}^{(2)} \} \) is CD-independent.

Since \( G = H \cup \{ z \} \) and \( H \in \mathcal{T} \) (and \( G \in \mathcal{B}(G, M, I) \)), we obtain \( G \in \mathcal{T}^* \). As \( G \) is the greatest element of the lattice \( \mathcal{B}(G, M, I) \), in view of Remark 2.3 \( \mathcal{T}^* \) is a classification tree in \( \mathcal{B}(G, M, I) \).
(iii) Observe that because $z^{□□}$ is an atom in the lattice $\mathcal{B}(G, M, I)$, if we add it to the classification tree $T^*$, then in view of Proposition 2.2 $T^* \cup \{z^{□□}\}$ remains a classification tree in $\mathcal{B}(G, M, I)$.

Finally assume that $T$ contains all the atoms of the lattice $\mathcal{B}(K_H)$. We have to show that the classification tree $T^* \cup \{z^{□□}\}$ contains all the atoms of the lattice $\mathcal{B}(G, M, I)$. Evidently, it contains the atom $z^{□□}$ also. On the other hand the atoms of $\mathcal{B}(G, M, I)$ are blocks of the finest extent partition $\pi_\infty$ of $(G, M, I)$. Then in view of Proposition 3.5 all the other atoms of $\mathcal{B}(G, M, I)$ which are different from $z^{□□}$ are also atoms of the lattice $\mathcal{B}(K_H)$, and belongs to $T$. Therefore, this atoms belongs to $T^{(1)}$ by the construction of $T^{(1)}$. Since $T^{(1)} \subseteq T^* \cup \{z^{□□}\}$, then $T^* \cup \{z^{□□}\}$ contains all the atoms of $\mathcal{B}(G, M, I)$.

**Theorem 3.9.** Let $K_H = (H, M, I \cap H \times M)$ be a subcontext of the finite context $K = (G, M, I)$ such that $H = G \setminus \{z\}$, $z^{□□} \neq \{z\}$. Then:

(i) For each classification tree $T_G \subseteq \mathcal{B}(G, M, I)$ there exists a classification tree $T_H \subseteq \mathcal{B}(K_H)$ such that the equality $T_G = T_H^*$ is satisfied.

(ii) If $T_G \subseteq \mathcal{B}(G, M, I)$ is a maximal classification tree, then in $\mathcal{B}(K_H)$ exists also a maximal classification tree $M$ such that $T_G = M^*$.

**Proof.** (i) In view of Proposition 3.7 $T_H = \{E \cap H \mid E \in T_G\}$ is a classification tree in $\mathcal{B}(K_H)$. We show that the classification tree $T_H^* = T_H^{(1)} \cup \{E \cup \{z\} \mid E \in T_H^{(2)}\} = \{E \in T_H \mid E \in \mathcal{B}(G, M, I)\} \cup \{E \cup \{z\} \mid E \in T_H, E \cup \{z\} \in \mathcal{B}(G, M, I)\}$

assigned to $T_H$ by Theorem 3.8 is equal to $T_G$. ($T_H^{(1)}, T_H^{(2)}$ was defined in Theorem 3.8).

Let $E \in T_G$ be arbitrary. Then $F = E \cap H \in T_H$ by the definition of $T_H$. We have only two cases:

1. if $E \subseteq H$, then $E = F$;
2. if $E \not\subseteq H$, then $E = F \cup \{z\}$.

In the first case $F \in T_H^{(1)}$ so we have $E = F \in T_H^{(1)} \subseteq T_H^*$.

In the second case as $F \cup \{z\} \in \mathcal{B}(G, M, I)$ we have $E = F \cup \{z\} \in T_H^*$.

In both cases $T_G \subseteq T_H^*$ because $E \in T_H^*$.

Conversely, let $E \in T_H^*$ be arbitrary. By the construction of $T_H^*$ we have two cases or $E \in T_H^{(1)} \subseteq T_H$, or $E = F \cup \{z\}, F \in T_H^{(2)} \subseteq T_H$.

Then by the definition of $T_H^*$ exists an $A \in T_G$ such that in the first case we have $A \cap H = E$, and $A \cap H = F$ in the second case.

First we show that $z \in A$ is not possible.

Indeed, in view of Theorem 3.6, as $E \in \mathcal{B}(K_H)$, $E \cap z^{□□} = E'' \cap z^{□□} = \emptyset$ . On the other hand from $z \in A$ we would get $z^{□□} \subseteq A$ so $z^{□□} \setminus \{z\} \subseteq A \cap H = E$. Combining with the first result this would imply $z^{□□} \setminus \{z\} = \emptyset$, i.e. $z^{□□} = \{z\}$, contradiction.

Thus in the case of $A \cap H = E$ we have $z \notin A$. Therefore, $A \subseteq H$ and $E = A \cap H = A \in T_G$. 

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In the case two \( A \cap H = F \). Observe that \( A \subseteq H \) would imply \( F = A \in \mathcal{B}(G, M, I) \) which means that \( F \in \mathcal{T}_H^{(1)} \). However this is not possible because by the definition \( F \in \mathcal{T}_H^{(2)} \) and \( \mathcal{T}_H^{(2)} \cap \mathcal{T}_H^{(1)} = \emptyset \). Therefore \( A \not\subseteq H \), which means \( z \in A \). Then \( A = F \cup \{z\} = E \). Thus we get \( E = A \in \mathcal{T}_G \).

Since in both cases we proved \( E \in \mathcal{T}_G \), it follows \( \mathcal{T}_H^* = \mathcal{T}_G \).

(ii) Assume that \( \mathcal{T}_G \) is a maximal classification tree in the lattice \( \mathcal{B}(K_H, K_H) \). Since \( \mathcal{T}_G = \mathcal{T}_H^* \), if \( \mathcal{T}_H \) is a maximal classification tree in \( \mathcal{B}(K_H, K_H) \), we done. If \( \mathcal{T}_H \) is not maximal, then in \( \mathcal{B}(K_H) \) exists a maximal classification tree \( \mathcal{M} \) in \( \mathcal{B}(K_H) \), such that \( \mathcal{T}_H \subseteq \mathcal{M} \). Then we have \( \mathcal{T}_G = \mathcal{T}_H^* \subseteq \mathcal{M}^* \). In view of Theorem 3.8 \( \mathcal{M}^* \) is also a classification tree in \( \mathcal{B}(G, M, I) \). As \( \mathcal{T}_G \) is a maximal classification tree by hypothesis, we get \( \mathcal{T}_G = \mathcal{M}^* = \mathcal{T}_H^* \). \( \square \)

4. An algorithm for tree-construction in the box extent lattice

In chapter 3 of the article, we examined that the box extents of a context are not changed with a one-object extension or reduction. We add an object to the context which has attributes from the existing \( M \) attribute set of the context. We showed that during a one-object extension of the context each of the classification trees of the newly created box extent lattice can be obtained by the modification of the classification trees of the box extent lattice of the original, smaller context.

Based on Theorem 3.8 we construct an algorithm which, starting from a classification tree of the box extent lattice of the smaller context \( (H, M, I \cap H \times M) \), gives a classification tree of the extended context \( (G, M, I) \) which contains the new elements inserted. The effectiveness of this method is that it ensures that there is enough to know the original context, the classification tree of the box extent lattice and its box extents, we do not need a new box extent of the extended context mesh elements (except for one, which is the box extent of the new element).

For the construction of the classification trees of the box extent lattice we will use the recursive algorithm ORTOFA presented in [12]. For the algorithm ORTOFA we need the original context \( K \) and the box extents which are contained in the matrix \( DS \). The next step is to determine \( z^\diamond \). For finding the extent partition \( E = z^\diamond \), containing the new inserted element we will use Algorithm 2 presented by Köreï A. in [3].

The function \( \text{Lista}_{\text{Fa}} \) first searches for the elements of the order ideal part of our classification tree, i.e. we check if \( z^\diamond \) is smaller than any
box extent stored in the matrix $L$. If this condition is satisfied we put the element in the matrix $S_1$. After that using the KOBJ(KTUL) we verify that the elements of $S_1$ are in fact box extents, and order them with the function. As a last step we find the chain part of our classification tree.

In the algorithm we used two functions:

BENNE (A, B) - verify that the matrix A of the box extents contains or not the elements of B;

BERAK (A, V, m) - a procedure that inserts in A the vector V as a new $m + 1$-th row.

In the function KOBJ(KTUL) the function KTUL finds the common properties of the object set, and the function KOBJ finds the common objects of the attribute set and used this two functions one after the other we got the set $A''$ of the object set $A$.

We use the following matrices:

DS(mxn) contains the box extents,

L(mxn) contains the copy of the box extents,

$S_1$ contains the elements above to $z\Box\Box$,

$S$ contains the chain above to $z\Box\Box$ and $F$ stores the elements below.

```
1. LISTA_FA(DS)
2. L ← DS
3. S1 ← ∅
4. S ← ∅
5. F ← ∅
6. k ← 0
7. for i ← 1 to m
8. do benne ← true
9. for j ← 1 to n
10. do if $z\Box\Box[j] > L[i][j]$
11. then benne ← false
12. if benne
13. then k + +
14. BERAK(S1, L[i], k)
15. for i ← 1 to k
16. do S1[i][n + 1] ← 1
17. A ← S1$^T$
18. A ← KOBJ(KTUL(A, K), K) */here K means the extended context
19. A ← A$^T$
20. l ← 0
21. for i ← 1 to k
22. do dext ← true
23. for j ← 1 to (n + 1)
```
24. \( \text{do if } A[i][j] \neq S1[i][j] \)
25. \( \quad \text{then dext } \leftarrow \text{false} \)
26. \( \quad \text{if dext} \)
27. \( \quad \quad \text{then } l++ \)
28. \( \quad \quad \text{BERAK}(S, S1[i], l) \)
29. \( h \leftarrow 0 \)
30. \( \text{for } i \leftarrow 1 \text{ to } m \)
31. \( \quad \text{do if } \text{BENNE}(L[i], z) \)
32. \( \quad \quad \text{then } h++ \)
33. \( \quad \quad \text{BERAK}(F, L[i], h) \)
34. \( \text{return } S, F \)

**BERAK** \((A, V, m)\)
1. \(m++\)
2. \(\text{for } t \leftarrow 1 \text{ to } n \)
3. \(\text{do } A[m][t] \leftarrow V[t] \)
4. \(\text{return } A, m \)

**BENNE** \((A, B)\)
1. \(\text{bent } \leftarrow \text{true} \)
2. \(\text{for } i \leftarrow 1 \text{ to } n \)
3. \(\text{do if } A[i] > B[i] \)
4. \(\quad \text{then bent } \leftarrow \text{false} \)
5. \(\text{return bent} \)

The effectiveness of the method is that it ensures that there is enough to know the original context, the classification tree of the box extent lattice and its box extents, we do not need a new box extents of the extended context mesh elements (except for one, which is the new element box extent).

The main parts of the process are the dual matrix operations operating cycles. The run-time of this algorithm is \(O(n^2)\) polynomial time, this means the worst run time for a sufficiently large \(n = \max\{n, m, k\}\) value. However, the method contains many one-line instruction (decisions, variable value increase with 1) if the value of \(n\) is not too big, then the running time is linear, \(O(n)\) because the number of steps, the executable instructions are \(n\). In summary, the worst running time is second-order polynomial time.

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