Potential Energy in Quantum Gravity.

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Abstract

We give a general expression for the static potential energy of the gravitational interaction of two massive particles, in terms of an invariant vacuum expectation value of the quantized gravitational field. This formula holds for functional integral formulations of euclidean quantum gravity, regularized to avoid conformal instability. It could be regarded as the analogue of the Wilson loop of gauge theories and allows in principle, through numerical lattice simulations or other approximation techniques, non perturbative evaluations of the potential or of the effective coupling constant.

1 Introduction.

The present paper is concerned with the problem of the energy of the gravitational field. This energy has been under investigation since the birth of General Relativity and some issues, like the determination of the total energy of a field configuration, have been settled in a rigorous way in the ADM formalism [1] or through Noether’s theorem [2]. Other points, however, like the possibility of “localizing” the gravitational energy, are still obscure (see [3] and references therein).

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Here we shall concentrate on a special case, namely that of the static potential energy of the gravitational interaction of two massive bodies. The gauge-invariant description of this interaction in terms of quantum fields is a crucial issue for any physically reasonable theory of quantum gravity.

This potential energy will turn out to be related to the vacuum average of a simple gauge-invariant functional of the field, a kind of correlation between “scalar” Wilson lines. While evaluation of this average for a weak field on a flat background yields the Newton potential in the usual form \[3\], non-perturbative evaluations could show modifications in the effective coupling constant or in the dependence of the potential on the distance between the particles.

Like in the case of the Wilson loop, our formula can be implemented quite naturally on a lattice version of the theory, in order to allow numerical simulations (Hamber and Williams, \[4\]). The first results are very interesting, showing that in the strong-gravity region of the phase space of lattice theory the potential is indeed Yukawa-like, with a mass parameter which decreases towards the critical point (where the average curvature vanishes). The effective value of Newton constant has been estimated too.

This paper extends and refines an earlier work \[5\]. The outline is the following. In Section 2 we recall in brief the general-relativistic view about the gravitational potential energy. In Section 3 we give a formula which, treating two masses as external sources for a quantized gravitational field on a flat background, allows to write the potential energy of their interaction. This is done using a known technique of euclidean quantum field theory \[6\]. In Section 4 we generalize the formula to the case of “strong” gravity, introducing a definition of the source which avoids the use of background metric. Finally, in Section 5 we discuss the compatibility of the background formula with the general one and the possible behaviour of the effective coupling constant.

## 2 Potential Energy Versus ADM Energy.

In this Section we work in (3+1) dimensions and follow the conventions of Weinberg \[7\]. Starting from next Section, the metric will be euclidean. The Einstein action is given by

\[
S_{\text{Einst}} = -\frac{1}{16\pi G} \int d^4x \sqrt{g(x)} R(x)
\]

and the action of a material particle of mass \(m\) is

\[
S_{\text{Mat.}} = -m \int dp \sqrt{-g_{\mu\nu}[x(p)] \dot{x}^\mu(p) \dot{x}^\nu(p)},
\]

where \(x^\mu(p)\) is the trajectory of the particle and \(p\) is any parameter. The dots will always denote differentiation with respect to the parameter. Finally, the Einstein equations have the form

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu}.
\]

Let us now decompose the metric in the traditional way

\[
g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)
\]

and denote the linearized Einstein equations in the harmonic gauge as

\[
K^{\rho\sigma}_{\mu\nu} h_{\rho\sigma} = T_{\mu\nu}.
\]

The inverse of the kinetic operator \(K\) is the well-known Feynman-De Witt propagator

\[
K^{-1}_{\mu\nu\rho\sigma}(x - y) = \frac{2G}{\pi} \frac{\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho} - \eta_{\mu\nu} \eta_{\rho\sigma}}{(x - y)^2 - i\epsilon}.
\]
Let us compute the Newton potential starting from the preceding equations. This is an elementary calculation, but the integral one encounters will be useful in the following. The field produced by a generic four-momentum source $T_{\rho\sigma}$ in the linearized approximation is given by

$$h_{\mu\nu}(x) = \int d^4y [K^{-1}]_{\mu\nu}(x-y) T_{\rho\sigma}(y); \quad (7)$$

when the source is a particle of mass $m$ at rest in the origin, the only non vanishing component of $T_{\rho\sigma}$ is

$$T_{00}(y) = m \delta^3(y), \quad (8)$$

so we have

$$h_{00}(x^0, x) = \int d^4y [K^{-1}]_{00}(x-y) m \delta^3(y)$$

$$= m \int_{-\infty}^{+\infty} dy^0 \int d^3y \frac{\frac{-2G}{|x-y|^2} \delta^3(y)}{(x^0 - y^0)^2 - i\epsilon}$$

$$= m \int_{-\infty}^{+\infty} dy^0 \frac{2G}{x^2 - (x^0 - y^0)^2 - i\epsilon} = \frac{2mG}{|x|}. \quad (9)$$

This is the correct result, since in the newtonian approximation we have

$$g_{00} = -1 - 2V, \quad (10)$$

where $V$ is the Newton potential.

More generally, one can use the ADM formula. In classical General Relativity the total energy (mechanical + gravitational) of a physical system is given by the ADM mass formula

$$E = -\frac{1}{16\pi G} \int \left( \frac{\partial h_{ij}(x)}{\partial x^i} - \frac{\partial h_{ij}(x)}{\partial x^j} \right) n^i r^2 d\Omega, \quad (11)$$

where the integral is computed on a surface at spacelike infinity. We shall briefly review here the connection between this energy and the static gravitational potential.

A generic static metric $g_{\mu\nu}$ can be written at spatial infinity in the form

$$g_{00} \simeq -1 + \frac{2M_1 G}{|x|} + O \left( \frac{1}{|x|^2} \right); \quad (12)$$

$$g_{0i} \simeq O \left( \frac{1}{|x|^2} \right); \quad (13)$$

$$g_{ij} \simeq \delta_{ij} + \frac{2M_2 G}{|x|} x_i x_j + O \left( \frac{1}{|x|^2} \right). \quad (14)$$

Performing the integral (11) with $h_{ij}$ given by (14) one sees that $M_2$ is the ADM energy (“total mass”). On the other hand, $M_1$ is the mass observed by measuring the newtonian force at infinity. Substituting (12) - (14) into Einstein’s equations $R_{\mu\nu} = 0$, it is easy to see that $M_1 = M_2$. This is a quite natural result [1]. In other words it means that, according to special relativity conceptions, the source of the newtonian field is not only the mass, but also the energy density. For instance, in the gravitational collapse of a star a part of the gravitational energy is converted into kinetic energy and eventually this energy is employed to produce heavier elements from hydrogen or helium. If we disregard the radiation emitted into space, the newtonian field far away from the star remains unchanged during the whole process and the same holds for the ADM mass, which is a conserved quantity.
The gravitational potential energy can be found, by definition, assuming a static distribution of matter and computing the metric it generates at infinity. This has been done by Murchada and York for a spherical matter distribution of uniform density, using conformal transformations and a special formulation of the initial-value equations of General Relativity. For a sphere of (small) density $\rho$ and unit radius they found the right newtonian gravitational binding energy, namely the ADM mass is given in this case by (reintroducing the radius $R$ and the velocity of light $c$)

$$M_{\text{TOT}} = \frac{4}{3} \pi R^3 \rho - \frac{1}{c^2} \frac{16}{15} \pi^2 \rho^2 G R^5 + O(\rho^3).$$

(15)

Remembering that $M_{\text{TOT}}$ also represents the effective source of the newtonian field, we see that the second term in (15) gives rise to a deviation from the famous law which states the independence of the potential on the radius of the source. Nevertheless, this effect is usually unobservable, due to the very small factor $c^{-2}$.

It is also possible to find the following corrections to (15), proportional to $\rho^3$, $\rho^4$, ... They denote the existence of general-relativistic corrections to the potential energy $m_1 m_2 G/r$. For instance, the term proportional to $\rho^3$ would contribute to $M_{\text{TOT}}$ a term of the form

$$\Delta M \propto \frac{1}{c^4} \rho^3 G^2 R^7 + O(\rho^4).$$

(16)

In the case of a source constituted by two pointlike bodies of masses $m_1$ and $m_2$, kept at rest at a fixed distance $r$, the method of solution mentioned above is not applicable. From eq. (15) we may infer that the ADM mass is given in this case by

$$M_{\text{TOT}} = m_1 + m_2 - \frac{1}{c^2} \frac{G m_1 m_2}{r} + o \left( \frac{1}{c^2} \right).$$

(17)

There are, however, no relativistic corrections to the two-body potential coming from (16), because the corresponding potential does not admit a continuum limit. For instance, if we try to integrate a potential of the form $G^2 m_1^{3/2} m_2^{3/2} / r^2$ to obtain the term proportional to $\rho^3$, we find that the binding energy of the sphere depends on the way it has actually been put together. This means that the correction proportional to $\rho^3$ comes from a three particles potential, and so on for $\rho^4$ etc.

3 Quantum formula for the potential energy on a flat background.

The same result we found in the preceding Section using the classical equations of motion can be obtained in a completely different way. It is known that the ground state energy of a system described by an action $S_0[\phi] = \int d^4x L(\phi(x))$ in the presence of external sources $J(x)$ can in euclidean quantum field theory be expressed as

$$E = \lim_{T \to \infty} -\frac{\hbar}{T} \log \frac{\int d[\phi] \exp \left\{ -\hbar^{-1} \left[ \int d^4x L(\phi(x)) + \int d^4x \phi(x) J(x) \right] \right\}}{\int d[\phi] \exp \left\{ -\hbar^{-1} \int d^4x L(\phi(x)) \right\}}$$

$$= \lim_{T \to \infty} -\frac{\hbar}{T} \log \left\langle \exp \left\{ -\hbar^{-1} \int d^4x \phi(x) J(x) \right\} \right\rangle,$$

(18)

where, outside the interval $(-\frac{1}{2}T, \frac{1}{2}T)$, the source has been switched off.
This equation has been proved exactly in perturbation theory for the case of a linear local coupling between the field and the external source, but it can be generalized assuming that in any case the vacuum-to-vacuum transition amplitude is given by

$$
\langle 0^+|0^- \rangle_J = \frac{\int d[\phi] \exp \left\{ -\hbar^{-1}(S_0[\phi] + S_{\text{Inter.}}[\phi, J]) \right\}}{\int d[\phi] \exp \left\{ -\hbar^{-1} S_0[\phi] \right\}}
= \langle \exp \{ \hbar^{-1} S_{\text{Inter.}}[\phi, J] \} \rangle.
$$

(19)

In fact, inserting a complete set of energy eigenstates we can write

$$
\langle 0^+|0^- \rangle_J = \langle 0|e^{-HT/\hbar}|0 \rangle = \sum_n \langle 0|e^{-HT/\hbar}|n \rangle \langle n|0 \rangle = \sum_n \langle 0|n \rangle^2 e^{-E_n T/\hbar}.
$$

(20)

The smallest energy eigenvalue $E_n$ corresponds to the ground state, and in the limit $T \to \infty$ it dominates the sum. So taking the logarithm and multiplying by $(-\hbar/T)$ we obtain that energy. This is a well-known technique in QCD (see for instance [9]).

In the case of a weak gravitational field quantized on a flat background, we may consider the source constituted by two masses $m_1$, $m_2$, placed at rest near the origin at a distance $L$ each from the other (see eq. (23)). The action of this system is

$$
S = -\frac{1}{16\pi G} \int d^4x \sqrt{g(x)} R(x) - m_1 \int_{-\frac{T}{2}}^{\frac{T}{2}} dt_1 \sqrt{g_{\mu\nu}[x(t_1)]} \dot{x}^\mu(t_1) \dot{x}^\nu(t_1) - m_2 \int_{-\frac{T}{2}}^{\frac{T}{2}} dt_2 \sqrt{g_{\mu\nu}[y(t_2)]} \dot{y}^\mu(t_2) \dot{y}^\nu(t_2),
$$

(21)

where the trajectories $x^\mu(t_1)$ and $y^\mu(t_2)$ of the particles with respect to the background are simply given by

$$
x^\mu(t_1) = \left( t_1, \frac{L}{2}, 0, 0 \right); \quad y^\mu(t_2) = \left( t_2, \frac{L}{2}, 0, 0 \right).
$$

(22)

So we have

$$
E_{\text{B-G}} = \lim_{T \to \infty} -\frac{\hbar}{T} \log \frac{\int d[h] \exp \left\{ -\hbar^{-1} \left[ S_{\text{Einst.}} + m_1 \int_{-\frac{T}{2}}^{\frac{T}{2}} dt_1 \sqrt{1 + h_{00}[x(t_1)]} + m_2 \int_{-\frac{T}{2}}^{\frac{T}{2}} dt_2 \sqrt{1 + h_{00}[y(t_2)]} \right] \right\}}{\int d[h] \exp \left\{ -\hbar^{-1} S_{\text{Einst.}} \right\}}
= \lim_{T \to \infty} -\frac{\hbar}{T} \log \left\{ \exp \left\{ -\hbar^{-1} \left[ m_1 \int_{-\frac{T}{2}}^{\frac{T}{2}} dt_1 \sqrt{1 + h_{00}[x(t_1)]} + m_2 \int_{-\frac{T}{2}}^{\frac{T}{2}} dt_2 \sqrt{1 + h_{00}[y(t_2)]} \right] \right\} \right\}.
$$

(23)

We call this expression “B-G” (background-geometry) formula. It has the property that the time $T$ is referred to the background geometry. It follows in particular that the average is sensitive also to fields $h$ which are pure gauge modes and carry no curvature. For further details on this point compare next Section and the discussion in Section 5.

By standard perturbation techniques it is straightforward to see that for weak fields and to lowest order in $G$ eq. (23) reduces to

$$
E_{\text{B-G}} = m_1 + m_2 + \lim_{T \to \infty} \frac{\hbar}{T} \log \left\{ 1 + \frac{m_1 m_2}{4\hbar^2} \int_{-\frac{T}{2}}^{\frac{T}{2}} dt_1 \int_{-\frac{T}{2}}^{\frac{T}{2}} dt_2 \langle h_{00}[x(t_1)] h_{00}[y(t_2)] \rangle \right\}
$$

5
\[ m_1 + m_2 + \lim_{T \to \infty} \frac{1}{4} \int_{-T}^{T} dt_1 \int_{-T}^{T} dt_2 K_{0000}^{-1}(\tau_1 - \tau_2, L, 0, 0) \]

\[ = m_1 + m_2 - \frac{m_1 m_2 G}{L}. \]  

(24)

In the last step we have used the classical equation (9). The next term in the perturbative series would be the first quantum correction, proportional to \( \hbar G^2 \).

Eq. (23), like the corresponding ones in QED or QCD, has the physically appealing feature of showing how the force between the sources ultimately arises from the exchange of massless gravitons. However, let us make a closer comparison with electrodynamics. In that case the analogue of the functional integral which appears in the logarithm of (23) has the form \[ \langle \exp \{ g \int_{-T}^{T} dt A_0(x(t)) - g \int_{-T}^{T} dt A_0(y(t)) \} \rangle. \]  

(25)

(The two charges have been chosen to be opposite: \( q_1 = g, q_2 = -g \).) Reversing the direction of integration in the second integral and closing the contour at infinity, one is able to show that the quantity (25) coincides with the Wilson loop of a single charge \( g \), thus giving a gauge invariant expression for the potential energy.

In gravity this is not possible: we may imagine that an expression like (25) could be obtained in the first-order formalism (with \( A_0 \) replaced by the tetrad \( e_0^a \)), but the masses necessarily have the same sign, so the loop cannot be closed.

4 General case.

We aim now at giving a quantum formula for the potential which does not rely on a fixed background. We assume that a functional integral for euclidean gravity exists, denoted by

\[ Z = \int d[g] \exp \{-\hbar^{-1} S_{\text{Einst}}[g]\} \]  

(26)

and require that all the field configurations in this functional integral are asymptotically flat.

In practice it is known that the euclidean Einstein action is not bounded from below, so one has to be careful in treating it. The action needs to be regularized in some way; in the actual lattice simulations, this happens thank to the \( R^2 \) term and to the measure (see [4] and references).

Since we have no background, the quantities \( T \) and \( L \) which enter in the definition of the source must now refer to the dynamical metric \( g \), configuration by configuration, in the functional integral (26).

A first, simple recipe for generalizing the B-G equation (23) along this lines is the following. Let us consider two massive bodies (with \( m_1 = m_2 \), for simplicity) which are kept at a fixed invariant distance \( L \) from each other by some device and “fall” through a given field configuration \( g \). Suppose that the motion is started and stopped in the asymptotically flat region. If \( T \) is the total proper time measured along the trajectory of the first mass (with \( T \gg L \)), the proper time measured along the trajectory of the other mass will be, say, \( T + \alpha[g] \).

Then the action of this system leads, after integration over all field configurations, to the energy

\[ E_{T-G} = \lim_{T \to \infty} -\frac{\hbar}{T} \log \langle \exp \{-\hbar^{-1} m(2T + \alpha[g])\} \rangle \]

\[ = 2m + \lim_{T \to \infty} -\frac{\hbar}{T} \log \langle \exp \{-\hbar^{-1} m\alpha[g]\} \rangle. \]  

(27)

We call this expression “T-G” (total geometry) formula. Unlike the B-G formula (eq. 23), it refers only to geometrical, invariant quantities.
If we suppose that the field configurations contributing to the average are not too singular, then $\alpha$ is small
and we may expand the exponential, finding (note that $\langle \alpha[g] \rangle$ obviously vanishes by symmetry)

$$E_{T-G} = 2m + \lim_{T \to \infty} - \frac{m^2}{2\hbar T} \langle \alpha^2[g] \rangle + ...$$  \hspace{1cm} (28)$$
Thus the interaction energy is always negative in this case. Also, eq. (28) gives an intuitive picture of the
gravitational attraction: it arises because keeping two masses at some fixed distance increases their action, due
to the vacuum fluctuations of the gravitational field.

Now, we turn to a more careful description of the whole procedure above, and eliminate the (apparent! –
thank to the average) asymmetry between the two bodies, assigning the total proper time $T$ to the geodesic of
their center of mass.

Let us suppose that a field configuration is given. We consider a geodesic line of length $T$
which starts at
an arbitrary point in the “past” asymptotically flat region with unit timelike velocity.

To fix the ideas, this curve could be written in its first part as

$$\xi^\mu(\tau) = \left( -\frac{1}{2} T + \tau, 0, 0, 0 \right); \hspace{1cm} 0 \leq \tau \leq \bar{\tau},$$  \hspace{1cm} (29)$$
where $\tau$ is the proper time measured along the curve and we have chosen the remaining coordinates of the
starting point to be equal to $(0, 0, 0)$ (this is an irrelevant arbitrariness, since at the end we shall integrate over
all the configurations of the field). As usual, $T$ denotes a very long time interval. After a time $\approx \bar{\tau}$ the curve
enters the region of spacetime where the gravitational field is non vanishing. It continues as a geodesic, which
means that $\xi^\mu(\tau)$ satisfies the equation

$$\Gamma^\rho_{\mu\nu}[\xi(\tau)] \frac{d}{d\tau} \xi^\mu(\tau) \frac{d}{d\tau} \xi^\nu(\tau) + \ddot{\xi}^\rho(\tau) = 0,$$  \hspace{1cm} (30)$$
where $\Gamma^\rho_{\mu\nu}$ is the Christoffel symbol of the metric. The curve terminates at $\tau = \frac{1}{2} T$, again in the flat region.

Let us then take in the initial point $\xi^\mu(0)$ a unit vector $q^\mu(0)$, orthogonal to $\dot{\xi}^\mu(0)$ (for instance, in our
example, $q^\mu(0) = (0, 1, 0, 0)$), and define a vector $q^\rho(\tau)$ along the curve $\xi^\mu(\tau)$ by parallel transport of $q^\mu(0)$. We remind that $\dot{\xi}^\mu(\tau)$, being the tangent vector of a geodesic, is parallel transported along the geodesic itself,
and that the parallel transport preserves the norms and the scalar products. Then the following relations hold
along the curve

$$\dot{\xi}^\mu(\tau) \frac{d}{d\tau} \xi^\nu(\tau) g_{\mu\nu}[\xi(\tau)] = -1;$$  \hspace{1cm} (31)$$
$$q^\mu(\tau) \frac{d}{d\tau} q^\nu(\tau) g_{\mu\nu}[\xi(\tau)] = 1;$$  \hspace{1cm} (32)$$
$$\dot{\xi}^\mu(\tau) q^\nu(\tau) g_{\mu\nu}[\xi(\tau)] = 0.$$  \hspace{1cm} (33)$$

Next we consider two masses $m_1$, $m_2$, and a length $L$ which we may regard as infinitesimal, compared to
the scale $T$. We assume that the two masses follow the trajectories $x^\mu(\tau)$ and $y^\mu(\tau)$, respectively, given by

$$x^\mu(\tau) = \xi^\mu(\tau) - L_1 q^\mu(\tau);$$  \hspace{1cm} (34)$$
$$y^\mu(\tau) = \xi^\mu(\tau) + L_2 q^\mu(\tau);$$  \hspace{1cm} (35)$$
where $L_1$ and $L_2$ are two positive lengths such that

$$L_1 + L_2 = L \hspace{1cm} \text{and} \hspace{1cm} -m_1 L_1 + m_2 L_2 = 0.$$  \hspace{1cm} (36)$$
The physical meaning of the preceding geometrical construction is apparent: it represents an observer which falls freely in the center of mass of the system composed by \( m_1 \) and \( m_2 \), while holding the two masses at rest at a distance \( L \) each from the other. This is the generalization of the source introduced in eq. \((22)\) that is naturally dictated by the equivalence principle.

We notice that if the two masses were allowed to fall freely in the field, they would not keep at a constant distance from each other. In fact, as it is well known from the so-called geodesic deviation equation, the distance between two neighboring geodesics varies according to the sign of the curvature in the region they are traversing.

We can reparameterize the two curves \( x^\mu(\tau) \) and \( y^\mu(\tau) \) introducing their proper times \( \tau_1 \) and \( \tau_2 \), respectively. The ratio between the proper time \( \tau_1 \) and the proper time \( \tau \) is given by the equation

\[
d\tau_1 = \sqrt{-g_{\mu \nu}[x(\tau_1)]\dot{x}^\mu(\tau_1)\dot{x}^\nu(\tau_1)}d\tau.
\]

An analogous relation holds for \( \tau_2 \). We agree to adjust the function \( \tau_1(\tau) \) in such a way that \( \tau_1(0) = 0 \). Then we shall denote \( \tau_1(-\frac{1}{2}T) = -\frac{1}{2}T_1' \), \( \tau_1(\frac{1}{2}T) = \frac{1}{2}T_1'' \) and \( T_1 = \frac{1}{2}(T_1' + T_1'') \). For flat geometries we have \( T_1 = T'' = T = T \). Analogous relations hold for \( \tau_2 \).

The energy can then be written as

\[
E_{T-G} = \lim_{T \to \infty} -\frac{\hbar}{T} \log \left\{ \exp \left\{ -\hbar^{-1} [m_1T_1 + m_2T_2] \right\} \right\}. \tag{38}
\]

In order to make contact with eq. \((27)\), suppose now to take \( m_1 = m_2 \) and to “attach” smoothly the ends of the curves \( x(\tau) \) and \( y(\tau) \) to the geodesic \( \xi(\tau) \). The corresponding change of length will have no effect on eq. \((38)\), because it is in no way a function of \( T \). Since a geodesic has minimal length with respect to neighboring curves, we have, for the total times \( T_1 \) and \( T_2 \) as functions of \( L \)

\[
\frac{T_1}{T} = 1 + \frac{1}{2}a[g]L^2 + \frac{1}{6}b[g]L^3 + ... \tag{39}
\]

\[
\frac{T_2}{T} = 1 + \frac{1}{2}a[g]L^2 - \frac{1}{6}b[g]L^3 + ... \tag{40}
\]

where \( a[g] > 0 \), while \( b \) has no definite sign. From this we can find the difference between \( T_1 \) and \( T_2 \), which is the analogue of the quantity \( \alpha[g] \) of eq. \((27)\):

\[
\frac{T_1}{T_2} = 1 + \frac{1}{3}b[g]L^3 + ... \quad \text{that is} \quad \frac{\alpha[g]}{T} \sim b[g]L^3. \tag{41}
\]

5 Discussion.

The B-G formula \((23)\) and the more general T-G formula \((27)\) (which is the one actually implemented in the lattice simulations) must be consistent in the limit of small \( G \). In this Section we are going to clarify this point and work out some consequences.

We first recall that expanding the euclidean functional integral \((26)\) from flat space in perturbation theory, one finds that to order \( \hbar G \) all field configurations are pure gauge modes carrying no curvature. \(^1\) In fact, the Wilson loops of the connection vanish to this order, and this cannot be an average effect, as they have the form \( W = (\theta_1^2 + \theta_2^2) \), where \( \theta_1 \) and \( \theta_2 \) are the two angles which describe the \( O(4) \) rotation of vectors by parallel transport around the loops (see \((3)\) \((1)\)).

\(^1\)This also suggests that gravitons are, physically, quite fictitious objects.
As we pointed out in Section 3, the B-G formula is sensitive to such pure gauge modes, and gives in fact to perturbative order $\hbar G$ the right result for the potential. On the other hand, the T-G formula would clearly give no potential energy in flat space: if there is no curvature, the difference between the lengths of the trajectories of the two masses vanishes. So the two equations “work” in a very different way.

In lattice theory, a version of the T-G formula is employed. In their non-perturbative simulations, Hamber and Williams [4] trace two parallel lines with reference to a (dynamical) Regge lattice and then compare the lengths $\sum_i l_i$ of the two lines. As we mentioned in the Introduction, the first results are positive.

In general, the lattice formulation seems to be much more appropriate for handling quantum gravity than the perturbative expansion. It also has the property of being automatically coordinate invariant, and of generating dynamically the flat space limit. The perturbation theory on flat background makes sense instead as an effective theory for small $G$, where $G$ is the effective value of the gravitational constant as measured from the macroscopic Newton force.

In the phase space of lattice theory, specified by the parameters $a$ (the coupling to the $R^2$-term) and $G_{\text{Bare}}$, there is a continuous line which separates a physical “smooth” phase and a collapsed “rough” phase. The true continuum theory is the limit of lattice theory on a point of this line, chosen in such a way that the effective long-distance Newton constant, computed through the lattice version of T-G formula, equals the measured value.

Then we may argue that a “correspondence principle” between the B-G and T-G formulas requires that $G_{\text{Bare}}$ is not as small as the effective value. Namely, for the region of very small $G_{\text{Bare}}$ first order perturbative computations are reliable, so we know there is no curvature in that regime, and consequently the lattice evaluation of T-G formula will give $G = 0$ exactly (or better, apart from terms of order $G_{\text{Bare}}^2$). On the contrary, if $G_{\text{Bare}}$ is not so small, its effective value can emerge from lattice theory with no contradiction with the background computations. Whether this is really what happens, it should become clear as the simulations proceed.

It is a pleasure for me to thank D. Maison for the warm hospitality in Munich, and the A. Von Humboldt Foundation for financial support. Also part of this work was made at M.I.T., Center for Theoretical Physics, and the help of R. Jackiw is gratefully acknowledged.

References

[1] R. Arnowitt, S. Deser and C. Misner, Phys. Rev. 116 (1959) 1322. B. De Witt, Phys. Rev. 160 (1967) 1113.

[2] S. Bak, D. Cangemi and R. Jackiw, Phys. Rev. D 49 (1994) 5173.

[3] G. Modanese, Phys. Lett. B 325 (1994) 354.

[4] H.W. Hamber, Nucl. Phys. B 400 (1993) 347, and references therein; H.W. Hamber and R.M. Williams, report CERN-TH.7314/94, DAMTP-94-49, June 94.

[5] G. Modanese, report CTP 2217, June 93.

[6] K. Symanzik, Comm. Math. Phys. 16 (1970) 48.
[7] S. Weinberg, *Gravitation and cosmology: principles and applications of the general theory of relativity*, J. Wiley, New York, 1972.

[8] N.O. Murchada and J.W. York, *Phys. Rev. D* **10** (1974) 2345;

[9] M. Bander, *Phys. Rep.* **75 C** (1981) 205.

[10] W. Fischler, *Nucl. Phys. B* **129** (1977) 157.

[11] G. Modanese, *Phys. Rev. D* **49** (1994) 6534.