Sobolev Inequalities In Manifolds With Asymptotically Nonnegative Curvature

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ABSTRACT. Using the ABP-method as in a recent work by Brendle [10], we establish some sharp Sobolev and isoperimetric inequalities for compact domains and submanifolds in a complete Riemannian manifold with asymptotically non-negative curvature. These inequalities generalize those given by Brendle in the case of complete Riemannian manifolds with nonnegative curvature.

1 Introduction

It is known that Sobolev inequalities, as an important analytic tool in geometric analysis, have close connections with isoperimetric inequalities. The classical isoperimetric inequality for a bounded domain \( D \) in \( \mathbb{R}^n \) says that

\[
n^n |B^n||D|^{n-1} \leq |\partial D|^n
\]

where \( B^n \) denotes the unit ball in \( \mathbb{R}^n \), and the equality holds if and only if \( D \) is a ball. There have been numerous works generalizing this inequality to different settings (cf. [32], [14], [15]).

The isoperimetric inequalities on minimal surfaces or minimal submanifolds have a long history. For example, [13], [14], [22], [34], [28], [35], [36] investigated the isoperimetric inequality on minimal surfaces under various conditions, while the famous Michael-Simon Sobolev inequality for general dimensions ([5], [31]) implies an isoperimetric inequality for minimal submanifolds, but with a non-sharp constant. It is conjectured that any \( n \)-dimensional minimal submanifold \( \Omega \) of \( \mathbb{R}^N \) satisfies the classical isoperimetric inequality: \( n^n |B^n||\Omega|^{n-1} \leq |\partial \Omega|^n \) with equality holds if and only if \( \Omega \) is a ball in an \( n \)-plane of \( \mathbb{R}^N \). Recently, S. Brendle [9], inspired by the ABP method as in [11] and [37], established a Michael-Simon-Sobolev type inequality on submanifolds of arbitrary dimension and codimension, which is sharp if the codimension is at most 2. In particular,
his result implies a sharp isoperimetric inequality for minimal submanifolds in Euclidean space of codimension at most 2. Later, Brendle [10] also generalized his results in [9] to the case that the ambient space is a Riemannian manifold with nonnegative curvature. In [23], F. Johne gave a sharp Sobolev inequality for manifolds with nonnegative Bakry-Émery Ricci curvature, which generalizes Brendle’s results in [10]. In [7], Balogh and Krisály proved a sharp isoperimetric inequality in metric measure spaces satisfying CD(0, N) condition which implies the sharp isoperimetric inequalities in [10] and [23]. Moreover, they also obtained a sharp $L^p$-Sobolev inequality for $p \in (1, n)$ on manifolds with nonnegative Ricci curvature and Euclidean volume growth. In a recent preprint [6], the authors also investigated sharp and rigid isoperimetric comparison theorems in RCD($K, N$) metric measure spaces.

In this paper, we generalize Brendle’s results in [10] to the case that the ambient space has asymptotically nonnegative curvature. The notion of asymptotically nonnegative curvature was first introduced by U. Abresch [1]. Some important geometric, topological and analysis problems have been investigated for this kind of manifolds (cf. [2], [3], [24, 25], [30, 29], [40], [21], [8], [39], etc). Now we recall its definition as follows. Let $\lambda: [0, +\infty) \to [0, +\infty)$ be a nonnegative and nonincreasing continuous function satisfying

\begin{equation}
(1.1) \quad b_0 := \int_0^{+\infty} s\lambda(s)ds < +\infty,
\end{equation}

which implies

\begin{equation}
(1.2) \quad b_1 := \int_0^{+\infty} \lambda(s)ds < +\infty.
\end{equation}

A complete noncompact Riemannian manifold $(M, g)$ of dimension $n$ is said to have asymptotically nonnegative Ricci curvature (resp. sectional curvature) if there is a base point $o \in M$ such that

\begin{equation}
(1.3) \quad \text{Ric}_q(\cdot, \cdot) \geq -(n-1)\lambda(d(o,q))g \quad (\text{resp. } \text{Sec}_q \geq -\lambda(d(o,q))))
\end{equation}

where $d(o, q)$ is the distance function of $M$ relative to $o$. Clearly, this notion includes the manifolds whose Ricci (resp. sectional) curvature is either nonnegative outside a compact set or asymptotically flat at infinity. In particular, if $\lambda \equiv 0$ in (1.3), then this becomes the case treated in [10].

Let $h(t)$ be the unique solution of

\begin{equation}
(1.4) \quad \begin{cases}
    h''(t) = \lambda(t)h(t), \\
    h(0) = 0, h'(0) = 1.
\end{cases}
\end{equation}

By ODE theory, the solution $h(t)$ of (1.4) exists for all $t \in [0, +\infty)$. According to [40] (see also Theorem 2.14 in [33]), the function

\[ \frac{|\{q \in M : d(o,q) < r\}|}{n|B^n| \int_0^r h^{n-1}(t)dt} \]
is a non-increasing function on \([0, +\infty)\) and thus we may introduce the asymptotic volume ratio of \(M\) by

\[
\theta := \lim_{r \to +\infty} \frac{|\{q \in M : d(o,q) < r\}|}{n|B^n| \int_0^r h^{n-1}(t) dt},
\]

with \(\theta \leq 1\). In particular, we have \(|\{q \in M : d(o,q) < r\}| \leq |B^n|e^{(n-1)b_0}r^n\).

First, by combining the method in [10] with some comparison theorems, we establish a Sobolev type inequality for a compact domain in a Riemannian manifold with asymptotically nonnegative Ricci curvature as follows.

**Theorem 1.1.** Let \(M\) be a complete noncompact \(n\)-dimensional manifold of asymptotically nonnegative Ricci curvature with respect to a base point \(o \in M\). Let \(\Omega\) be a compact domain in \(M\) with boundary \(\partial \Omega\), and let \(f\) be a positive smooth function on \(\Omega\). Then

\[
\int_{\partial \Omega} f + \int_{\Omega} |Df| + 2(n-1)b_1 \int_{\Omega} f \geq n|B^n|^\frac{1}{n} \theta^n \left( \frac{1 + b_0}{e^{2r_0b_1} + b_0} \right) \left( \int_{\Omega} f^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}},
\]

where \(r_0 = \max\{d(o,x) : x \in \Omega\}\), \(\theta\) is the asymptotic volume ratio of \(M\) given by (1.5) and \(b_0, b_1\) are defined in (1.1) and (1.2).

The following result characterizes the case of equality in Theorem 1.1:

**Theorem 1.2.** Let \(M\) be a complete noncompact \(n\)-dimensional manifold of asymptotically nonnegative Ricci curvature with respect to a base point \(o \in M\). Let \(\Omega\) be a compact domain in \(M\) with boundary \(\partial \Omega\), and let \(f\) be a positive smooth function on \(\Omega\). If

\[
\int_{\partial \Omega} f + \int_{\Omega} |Df| + 2(n-1)b_1 \int_{\Omega} f = n|B^n|^\frac{1}{n} \theta^n \left( \frac{1 + b_0}{e^{2r_0b_1} + b_0} \right) \left( \int_{\Omega} f^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}},
\]

where \(r_0 = \max\{d(o,x) : x \in \Omega\}\), \(\theta\) is the asymptotic volume ratio of \(M\) given by (1.5) and \(b_0, b_1\) are defined in (1.1) and (1.2). Then \(b_0 = b_1 = 0\), \(M\) is isometric to Euclidean space, \(\Omega\) is a ball, and \(f\) is constant.

Taking \(f = 1\) in Theorem 1.1, we obtain a sharp isoperimetric inequality:

**Corollary 1.3.** Let \(M, \Omega, r_0, \theta, b_0, b_1\) be as in Theorem 1.1. Then

\[
|\partial \Omega| \geq \left( n|B^n|^\frac{1}{n} \theta^n \left( \frac{1 + b_0}{e^{2r_0b_1} + b_0} \right) \right)^{\frac{n}{n-1}} - 2(n-1)b_1 |\Omega|^\frac{n}{n-1}. \]

Furthermore, the equality holds if and only if \(M\) is isometric to Euclidean space and \(\Omega\) is a ball.
Remark 1.4. If $M$ has nonnegative Ricci curvature, then $b_0 = b_1 = 0$ and Corollary 1.3 becomes

$$|\partial \Omega| \geq n|B^n|^\frac{1}{n} \theta^\frac{1}{n},$$

which was first given by V. Agostiniani, M. Fogagnolo, and L. Mazziari [4] in dimension 3 and obtained by S. Brendle [10] for any dimension, see also [18] for related results in $3 \leq n \leq 7$.

Similarly, we may establish a Sobolev type inequality for a compact submanifold (possibly with boundary) in a Riemannian manifold with asymptotically nonnegative sectional curvature as follows.

**Theorem 1.5.** Let $M$ be a complete noncompact $(n+p)$-dimensional manifold of asymptotically nonnegative sectional curvature with respect to a base point $o \in M$. Let $\Sigma$ be a compact $n$-dimensional submanifold of $M$ (possibly with boundary $\partial \Sigma$), and let $f$ be a positive smooth function on $\Sigma$. If $p \geq 2$, then

$$\int_{\partial \Sigma} f + \int_{\Sigma} \sqrt{|D^\Sigma f|^2 + f^2|H|^2} + 2nb_1 \int_{\Sigma} f \geq n \left( \frac{(n+p)|B^{n+p}|}{p|B^p|} \right) \frac{1}{n} \theta^\frac{1}{n} \left( \frac{1 + b_0}{e^{2b_1 + b_0}} \right)^{\frac{n+p-1}{n}} \left( \int_{\Sigma} f^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}},$$

where $r_0 = \max\{d(o,x) | x \in \Sigma\}$, $H$ is the mean curvature vector of $\Sigma$, $\theta$ is the asymptotic volume ratio of $M$ given by (1.5) and $b_0, b_1$ are defined in (1.1) and (1.2).

Note that $(n + 2)|B^{n+2}| = 2|B^2||B^n|$. Hence, we obtain the following Sobolev type inequality for codimension 2:

**Corollary 1.6.** Let $M$ be a complete noncompact $(n+2)$-dimensional manifold of asymptotically nonnegative sectional curvature with respect to a base point $o \in M$. Let $\Sigma$ be a compact $n$-dimensional submanifold of $M$ (possibly with boundary $\partial \Sigma$), and let $f$ be a positive smooth function on $\Sigma$. Then

$$\int_{\partial \Sigma} f + \int_{\Sigma} \sqrt{|D^\Sigma f|^2 + f^2|H|^2} + 2nb_1 \int_{\Sigma} f \geq n|B^n|^\frac{1}{n} \theta^\frac{1}{n} \left( \frac{1 + b_0}{e^{2b_1 + b_0}} \right)^{\frac{n+1}{n}} \left( \int_{\Sigma} f^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}},$$

where $r_0 = \max\{d(o,x) | x \in \Sigma\}$, $H$ is the mean curvature vector of $\Sigma$, $\theta$ is the asymptotic volume ratio of $M$ given by (1.5) and $b_0, b_1$ are defined in (1.1) and (1.2).

The following result characterizes the case of equality in Corollary 1.6:
Theorem 1.7. Let $M, \Sigma, f, r_0, H, \theta, b_0, b_1$ as in Corollary 1.6. If
\[
\int_{\partial \Sigma} f + \int_{\Sigma} \sqrt{|Df|^2 + f^2|H|^2} + 2nb_1 \int_{\Sigma} f \\
= n|B^n|^{\frac{1}{n}} \theta^{\frac{1}{n}} \left( \frac{1 + b_0}{c^{2n} + b_0} \right)^{\frac{n+1}{n}} \left( \int_{\Sigma} f^{\frac{n}{n-1}} \right)^{\frac{n}{n-1}}.
\]
Then $b_0 = b_1 = 0$ and $M$ is isometric to Euclidean space, $\Sigma$ is a flat ball, and $f$ is constant.

Letting $f = 1$ in Corollary 1.6, we obtain a sharp isoperimetric inequality for minimal submanifolds of codimension 2 as follows.

Corollary 1.8. Let $M$ be a complete noncompact $(n+2)$-dimensional manifold of asymptotically nonnegative sectional curvature with respect to a base point $o \in M$. Let $\Sigma$ be a compact $n$-dimensional minimal submanifold of $M$ (possibly with boundary $\partial \Sigma$). Then
\[
|\partial \Sigma| \geq n \left( |B^n|^{\frac{1}{n}} \theta^{\frac{1}{n}} \left( \frac{1 + b_0}{c^{2n} + b_0} \right)^{\frac{n+1}{n}} - 2b_1 |\Sigma|^{\frac{2}{n}} \right) |\Sigma|^{\frac{1}{n}},
\]
where $r_0 = \max \{ d(o, x) | x \in \Sigma \}$, $\theta$ is the asymptotic volume ratio of $M$ given by (1.5) and $b_0, b_1$ are defined in (1.1) and (1.2). Furthermore, the equality holds if and only if $M$ is isometric to Euclidean space and $\Sigma$ is a flat ball.

It is obvious that the above inequalities are nontrivial only when $\theta > 0$. We say that a complete Riemannian manifold with asymptotically nonnegative (Ricci) curvature has maximal volume growth if $\theta > 0$. Examples of such manifolds may be found in [1], [19], [12], [26, 27], and the first case of Theorem 1.2 in [38], etc.

## 2 The case of domains

Let $(M, g)$ be a complete noncompact $n$-dimensional Riemannian manifold of asymptotically nonnegative Ricci curvature with respect to a base point $o \in M$. Let $\Omega$ be a compact domain in $M$ with smooth boundary $\partial \Omega$ and $f$ be a smooth positive function on $\Omega$. Without loss of generality, we assume hereafter that $\Omega$ is connected.

By scaling, we may assume that
\[
\int_{\partial \Omega} f + \int_{\Omega} |Df| + \int_{\Omega} 2(n-1)b_1 f = n \int_{\Omega} f^{\frac{n}{n-1}}.
\]
Due to (2.1) and the connectedness of $\Omega$, we can find a solution of the following Neumann boundary problem
\[
\begin{cases}
\text{div}(fDu) = nf^{\frac{n}{n-1}} - 2(n-1)b_1 f - |Df|, & \text{in } \Omega, \\
\langle Du, \nu \rangle = 1, & \text{on } \partial \Omega,
\end{cases}
\]
where $\nu$ is the outward unit normal vector field along $\partial \Omega$. By standard elliptic regularity theory (see Theorem 6.31 in [20]), we know that $u \in C^{2,\gamma}$ for each $0 < \gamma < 1$.

As in [10], we set
\[ U := \{ x \in \Omega \setminus \partial \Omega : |Du(x)| < 1 \}. \]

For any $r > 0$, let
\[ A_r = \{ \bar{x} \in U : ru(x) + \frac{1}{2}d(x, \exp(rDu(\bar{x})))^2 \geq ru(\bar{x}) + \frac{1}{2}r^2|Du(\bar{x})|^2, \forall x \in \Omega \}. \]

Define a transport map $\Phi_r : \Omega \to M$ for each $r > 0$ by
\[ \Phi_r(x) = \exp_x(rDu(x)), \forall x \in \Omega. \]

Using the regularity of the solution $u$ of the Neumann problem, we known that the map $\Phi_r$ is of class $C^{1,\gamma}, 0 < \gamma < 1$.

**Lemma 2.1.** Assume that $x \in U$. Then we have
\[ \frac{1}{n} \Delta u \leq f^{\frac{n}{n-1}} - 2\left( \frac{n-1}{n} \right)b_1. \]

**Proof.** Using the Cauchy-Schwarz inequality and the property that $|Du| < 1$ for $x \in U$, we get
\[ -\langle Df, Du \rangle \leq |Df|. \]

In terms of (2.2), we derive that
\[ f \Delta u = nf^{\frac{n}{n-1}} - 2(n-1)b_1f - |Df| - \langle Df, Du \rangle \leq nf^{\frac{n}{n-1}} - 2(n-1)b_1f. \]

This proves the assertion. \(\square\)

The proofs of the following three lemmas are identical to those for Lemmas 2.2-2.4 in [10] without any change for the case of asymptotically nonnegative Ricci curvature. So we omit them here.

**Lemma 2.2.** The set
\[ \{ q \in M : d(x, q) < r, \forall x \in \Omega \} \]

is contained in $\Phi_r(A_r)$.

**Lemma 2.3.** Assume that $\bar{x} \in A_r$, and let $\bar{\gamma}(t) := \exp_x(tDu(\bar{x}))$ for all $t \in [0, r]$. If $Z$ is a smooth vector field along $\bar{\gamma}$ satisfying $Z(r) = 0$, then
\[ (D^2u)(Z(0), Z(0)) + \int_0^r \left( |D_tZ(t)|^2 - R(\bar{\gamma}'(t), Z(t), \bar{\gamma}'(t), Z(t)) \right) dt \geq 0. \]
Lemma 2.4. Assume that $\bar{x} \in A_r$, and let $\bar{\gamma}(t) := \exp_\bar{x}(tDu(\bar{x}))$ for all $t \in [0,r]$. Moreover, let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $T_{\bar{x}}M$. Suppose that $W$ is a Jacobi field along $\bar{\gamma}$ satisfying

$$\langle D_t W(0), e_j \rangle = (D^2 u)(W(0), e_j), \quad 1 \leq j \leq n.$$ 

If $W(\tau) = 0$ for some $\tau \in (0,r)$, then $W$ vanishes identically.

Now, we give two comparison results for later use.

Lemma 2.5. Let $G$ be a continuous function on $[0, +\infty)$ and let $\phi, \psi \in C^2([0, +\infty))$ be solutions of the following problems

$$\begin{cases}
\phi'' \leq G \phi, & t \in (0, +\infty), \\
\phi(0) = 1, \phi'(0) = b,
\end{cases} \quad \begin{cases}
\psi'' \geq G \psi, & t \in (0, +\infty), \\
\psi(0) = 1, \psi'(0) = \tilde{b},
\end{cases}$$

where $b, \tilde{b}$ are constants and $\tilde{b} \geq b$. If $\phi(t) > 0$ for $t \in (0, T)$, then $\psi(t) > 0$ in $(0, T)$ and

$$\frac{\phi'}{\phi} \leq \frac{\psi'}{\psi} \quad \text{and} \quad \psi \geq \phi \quad \text{on } (0, T).$$

Proof. Set $\beta = \sup\{t : \psi(t) > 0 \text{ in } (0,t)\}$ and $\tau = \min\{\beta, T\}$, so that $\phi$ and $\psi$ are both positive in $(0, \tau)$. The function $\psi' \phi - \psi \phi'$ is continuous on $[0, +\infty)$, nonnegative at $t = 0$, and satisfies

$$(\psi' \phi - \psi \phi')' = \psi'' \phi - \psi \phi'' \geq G(t)\psi \phi - G(t)\psi \phi = 0,$$

in $(0, \tau)$. Thus $\psi' \phi - \psi \phi' \geq 0$ on $[0, \tau)$, which implies

$$(2.3) \quad \frac{\psi'}{\psi} \geq \frac{\phi'}{\phi} \quad \text{in } [0, \tau).$$

Integrating (2.3) between 0 and $t$ $(0 < t < \tau)$ yields

$$\phi(t) \leq \psi(t), \quad \text{in } [0, \tau).$$

Since $\phi > 0$ in $[0, \tau)$ by assumption, this forces $\tau = T$. \hfill \Box

Lemma 2.6. Let $G$ be a nonnegative continuous function on $[0, +\infty)$ satisfying $\int_0^{+\infty} G\,dt < +\infty$. Let $h_1, h_2 \in C^2([0, +\infty))$ be solutions of the following problems

$(2.4) \quad \begin{cases}
h_1'' = G h_1, & t \in (0, +\infty), \\
h_1(0) = 0, h_1'(0) = 1,
\end{cases} \quad \begin{cases}
h_2'' = G h_2, & t \in (0, +\infty), \\
h_2(0) = 1, h_2'(0) = 0.
\end{cases}$

Then we have

$$\lim_{{t \to \infty}} \frac{h_2}{h_1} = \lim_{{t \to \infty}} \frac{h_2'}{h_1'} \leq \int_0^{+\infty} G\,dt < \infty.$$
Proof. From (2.4), we derive 
\[ (h_2 h'_1 - h_1 h'_2)'(t) \equiv 0, \]
and thus
\[ (2.5) \quad (h_2 h'_1 - h_1 h'_2)(t) \equiv 1 \]
in view of the initial values for \( h_1 \) and \( h_2 \). By derivation, one can find
\[ \left( \frac{h_2}{h_1} \right)' = \frac{h'_2 h_1 - h'_1 h_2}{h_1^2} = \frac{-1}{h_1^2} < 0, \]
which implies that \( \lim_{t \to +\infty} \frac{h_2(t)}{h_1(t)} \) exists. It is easy to show that
\[ 0 \leq \left( \frac{h'_2}{h'_1} \right)' = \frac{G(h_2 h'_1 - h_1 h'_2)}{(h_1')^2} \leq \frac{G}{(1 + \int_0^t sG(s)ds)^2} \leq G, \]
so we get
\[ \frac{h'_2(t)}{h'_1(t)} \leq \int_0^{+\infty} G \, dt. \]
By Lemma 2.13 in [33], we have \( h_1(t) \geq t \). Consequently, using (2.5) and \( h'_1 = 1 + \int_0^t G h_1 ds \), we obtain
\[ (2.6) \quad \frac{h_2}{h_1} = \frac{h'_2}{h'_1} + \frac{1}{h_1 h'_1} \leq \int_0^{+\infty} G \, dt + \frac{1}{t}, \quad t \in (0, \infty). \]
Letting \( t \to \infty \), we have
\[ \lim_{t \to \infty} \frac{h_2}{h_1} = \lim_{t \to \infty} \frac{h'_2}{h'_1} \leq \int_0^{+\infty} G \, dt. \]

The next result is useful to study the growth of various balls on \( M \) when their radii approach to infinity.

Lemma 2.7. Let \( h \) be the solution of (1.4). Then
\[ \lim_{t \to +\infty} \frac{h(t - C)}{h(t)} = 1 \quad \text{and} \quad \lim_{t \to +\infty} \frac{h(tC)}{h(t)} = C, \]
where \( C \) is any positive constant.
Proof. From Lemma 2.13 in [33], we know \( t \leq h(t) \leq e^{b_0} t \), and thus

\[
(2.7) \quad h'(t) = 1 + \int_0^t \lambda h \, dt \leq 1 + b_0 e^{b_0}.
\]

Clearly (2.7) means that \( h' \) is nondecreasing and bounded from above. Consequently we have

\[
\lim_{t \to +\infty} \frac{h(t-C)}{h(t)} = \lim_{t \to +\infty} \frac{h'(t-C)}{h'(t)} = 1
\]

and

\[
\lim_{t \to +\infty} \frac{h(tC)}{h(t)} = \lim_{t \to +\infty} \frac{C h'(tC)}{h'(t)} = C.
\]

We are now turning to the proof of Theorem 1.1.

**Proof of Theorem 1.1.** For any \( r > 0 \) and \( \bar{x} \in A_r \), let \( \{e_1, \ldots, e_n\} \) be an orthonormal basis of the tangent space \( T_{\bar{x}}M \). Choosing the geodesic normal coordinates \( (x^1, \ldots, x^n) \) around \( \bar{x} \), such that \( \frac{\partial}{\partial x^i} = e_i \) at \( \bar{x} \). Let \( \bar{\gamma}(t) := \exp_{\bar{x}}(tDu(\bar{x})) \) for all \( t \in [0, r] \). For \( 1 \leq i \leq n \), let \( E_i(t) \) be the parallel transport of \( e_i \) along \( \bar{\gamma} \). For \( 1 \leq i \leq n \), let \( X_i(t) \) be the Jacobi field along \( \bar{\gamma} \) with the initial conditions of \( X_i(0) = e_i \) and

\[
\langle D_tX_i(0), e_j \rangle = (D^2 u)(e_i, e_j), \quad 1 \leq j \leq n.
\]

Let \( P(t) = (P_{ij}(t)) \) be a matrix defined by

\[
P_{ij}(t) = \langle X_i(t), E_j(t) \rangle, \quad 1 \leq i, j \leq n.
\]

From Lemma 2.4, we known \( \det P(t) > 0, \forall t \in [0, r] \). Obviously, \( |\det D\Phi_t(\bar{x})| = \det P(t) > 0 \) for \( t \in [0, r] \). Let \( S(t) = (S_{ij}(t)) \) be a matrix defined by

\[
S_{ij}(t) = R(\bar{\gamma}'(t), E_i(t), \bar{\gamma}'(t), E_j(t)), \quad 1 \leq i, j \leq n,
\]

where \( R \) denotes the Riemannian curvature tensor of \( M \). By the Jacobi equation, one can obtain

\[
(2.8) \quad \begin{cases} 
P''(t) = -P(t)S(t), & t \in [0, r], \\
P_{ij}(0) = \delta_{ij}, \quad P_{ij}'(0) = (D^2 u)(e_i, e_j).
\end{cases}
\]

Let \( Q(t) = P(t)^{-1}P'(t), t \in (0, r) \). Using (2.8), a simple computation yields

\[
\frac{d}{dt} Q(t) = -S(t) - Q^2(t),
\]
where \( Q(t) \) is symmetric. The assumption of asymptotically nonnegative Ricci curvature gives

\[
\frac{d}{dt}[\text{tr}Q(t)] + \frac{1}{n}[\text{tr}Q(t)]^2 \leq \frac{d}{dt}[\text{tr}Q(t)] + \text{tr}[Q^2(t)] \\
= -\text{tr}S(t) \\
\leq (n - 1)|Du(\bar{x})|^2\lambda(d(o, \bar{\gamma}(t))),
\]

where \( o \) is the base point. Using triangle inequality and the definition of \( A_r \), it is easy to see that

\[
d(o, \bar{\gamma}(t)) \geq |d(o, \bar{x}) - d(\bar{x}, \bar{\gamma}(t))| = |d(o, \bar{x}) - t|Du(\bar{x})|.
\]

Set
\[
g = \frac{1}{n}\text{tr}Q, \\
\Lambda_{\bar{x}}(t) = \frac{(n - 1)}{n}|Du(\bar{x})|^2\lambda(|d(o, \bar{x}) - t|Du(\bar{x})|).
\]

Noting that \( \lambda \) is nonincreasing, it follows from (2.8), (2.9), (2.10) that

\[
\begin{cases}
g'(t) + g(t)^2 \leq \Lambda_{\bar{x}}(t), & t \in (0, r), \\
g(0) = \frac{1}{n}\Delta u(\bar{x}).
\end{cases}
\]

If we take \( \phi = e^{\int_0^t g(s)ds} \), then \( \phi \) satisfies

\[
\begin{cases}
\phi'' \leq \Lambda_{\bar{x}}(t)\phi, & t \in (0, r), \\
\phi(0) = 1, \phi'(0) = \frac{\Delta u(\bar{x})}{n}.
\end{cases}
\]

Next, we denote by \( \psi_1, \psi_2 \) the solutions of the following problems

\[
\begin{cases}
\psi''_1 = \Lambda_{\bar{x}}(t)\psi_1, & t \in (0, r), \\
\psi_1(0) = 0, \psi'_1(0) = 1,
\end{cases} \\
\begin{cases}
\psi''_2 = \Lambda_{\bar{x}}(t)\psi_2, & t \in (0, r), \\
\psi_2(0) = 1, \psi'_2(0) = 0.
\end{cases}
\]

Similar to the proof of (2.6), it is easy to verify that

\[
\frac{\psi_2}{\psi_1}(r) \leq \int_0^\infty \Lambda_{\bar{x}}(t) \, dt + \frac{1}{r} \leq 2\left(1 - \frac{1}{n}\right)b_1|Du(\bar{x})| + \frac{1}{r}.
\]

Since \(|Du(\bar{x})| < 1\), we obtain

\[
\frac{\psi_2}{\psi_1}(r) \leq 2\left(1 - \frac{1}{n}\right)b_1 + \frac{1}{r}.
\]
Using Lemma 2.13 in [33] and (2.12), we deduce that

\[
\psi_1(t) \leq \int_0^t e^{\int_0^s \Lambda_x(\tau) d\tau} \, ds
\]

\[
\leq t e^{\int_0^\infty \Lambda_s(\tau) d\tau}
\]

\[
= t e^{\int_0^{r_0} \int_0^\infty \lambda(|d(o,\bar{x})-w|) dw}
\]

\[
\leq t e^{\frac{n-1}{n}(2r_0b_1+b_0)},
\]

where \(r_0 = \max\{d(o,x) | x \in \Omega\}\).

Let \(\psi(t) = \psi_2(t) + \frac{1}{n} \Delta u(\bar{x}) \psi_1(t)\). Using Lemma 2.5, one can get

\[
\frac{1}{n} \text{tr} Q(t) = \frac{\phi'}{\phi} \leq \frac{\psi'}{\psi}, \quad \forall t \in (0, r).
\]

Thus,

\[
\frac{d}{dt} \log \det P(t) = \text{tr} Q(t) \leq \frac{n \psi'}{\psi}.
\]

Consequently, (2.15) implies

\[
| \det D\Phi_t(\bar{x}) | = \det P(t) \leq \psi^n(t) = (\psi_2(t) + \frac{1}{n} \Delta u(\bar{x}) \psi_1(t))^n
\]

for all \(t \in [0, r]\). This gives

\[
| \det D\Phi_t(\bar{x}) | \leq \left( \frac{\psi_2(r)}{\psi_1(r)} + \frac{1}{n} \Delta u(\bar{x}) \right)^n \psi_1^n(r)
\]

for any \(\bar{x} \in A_r\). Note that \(0 \leq \phi \leq \psi\). Using (2.13), (2.14) and Lemma 2.1, we derive that

\[
| \det D\Phi_t(\bar{x}) | \leq e^{(n-1)(2r_0b_1+b_0)} \left( \frac{n}{n-1} \right)^{b_1} + \frac{1}{r} + \frac{1}{n} \Delta u(\bar{x}) \right)^n \psi_1^n(r)
\]

(2.16)

\[
\leq e^{(n-1)(2r_0b_1+b_0)} \left( \frac{1}{r} + \int \frac{1}{n-1}(\bar{x}) \right)^n r^n
\]

for any \(\bar{x} \in A_r\). Moreover, by (1.4), we obtain \(h(t) \geq t\) and

\[
\lim_{t \to \infty} h'(t) = 1 + \int_0^\infty h(s) \lambda(s) \, ds \geq 1 + \int_0^\infty s \lambda(s) \, ds = 1 + b_0.
\]

Combining Lemma 2.2, (2.16) with the formula for change of variables in multiple integrals, we find that

\[
| \{ q \in M : d(x, q) < r \text{ for all } x \in \Omega \} |
\]

\[
\leq \int_{A_r} | \det D\Phi_r |
\]

\[
\leq \int e^{(n-1)(2r_0b_1+b_0)} \left( \frac{1}{r} + \int \frac{1}{n-1}(\bar{x}) \right)^n r^n.
\]
For $r > r_0$, the triangle inequality implies that

$$B_{r-r_0}(o) \subset \{q \in M : d(x,q) < r \text{ for all } x \in \Omega \} \subset B_{r+r_0}(o).$$

From (1.5), (2.19) and Lemma 2.7, it is easy to show that

$$|B^n|_\theta = \lim_{r \to +\infty} \frac{B_{r-r_0}(o)}{n \int_0^{r-r_0} h(t)^{n-1} dt} \int_0^{r-r_0} h(t)^{n-1} dt$$

$$\leq \lim_{r \to +\infty} \frac{|\{q \in M : d(x,q) < r \text{ for all } x \in \Omega \}|}{n \int_0^{r} h(t)^{n-1} dt} \int_0^{r} h(t)^{n-1} dt$$

$$\leq \lim_{r \to +\infty} \frac{B_{r+r_0}(o)}{n \int_0^{r+r_0} h(t)^{n-1} dt} \int_0^{r+r_0} h(t)^{n-1} dt$$

$$= |B^n|_\theta.$$

Dividing (2.18) by $n \int_0^{r} h(t)^{n-1} dt$ and sending $r \to \infty$, it follows from (2.17) and (2.20) that

$$|B^n|_\theta \leq e^{(n-1)(2r_0b_1+b_0)} \int_\Omega f \frac{n}{n-1} \lim_{r \to \infty} \frac{r^n}{n \int_0^{r} h(t)^{n-1} dt}$$

$$= e^{(n-1)(2r_0b_1+b_0)} \int_\Omega f \frac{n}{n-1} \lim_{r \to \infty} \frac{1}{h'(t)^{n-1}}$$

$$\leq \left(\frac{e^{2r_0b_1+b_0}}{1 + b_0}\right)^{n-1} \int_\Omega f \frac{n}{n-1}.$$

Hence we obtain

$$\int_{\partial \Omega} f + \int_\Omega |Df| + 2(n-1)b_1 \int_\Omega f \geq n|B^n|_\theta \left(\frac{1 + b_0}{e^{2r_0b_1+b_0}}\right)^{\frac{n}{n-1}} \left(\int_\Omega f \frac{n}{n-1}\right)^{\frac{n-1}{n}}.$$

\[\square\]

**Proof of Theorem 1.2.** Suppose the equality of Theorem 1.1 holds. Then we have equalities in (2.13) and (2.17) which force $\lambda \equiv 0$. Thus $M$ has nonnegative Ricci curvature. The assertion follows immediately from Theorem 1.2 in [10]. \[\square\]

### 3 The case of submanifolds

In this section, we assume that the ambient space $M$ is a complete noncompact $(n+p)$-dimensional Riemannian manifold of asymptotically nonnegative sectional curvature with respect to a base point $o \in M$. Let $\Sigma \subset M$ be a compact submanifold of dimension $n$ with or without boundary, and $f$ be a positive smooth function on $\Sigma$. Let $D$ denote
the Levi-Civita connection of $M$ and let $D^\Sigma$ denote the induced connection on $\Sigma$. The second fundamental form $B$ of $\Sigma$ is given by
\[
\langle B(X, Y), V \rangle = \langle \tilde{D}_XY, V \rangle,
\]
where $X, Y$ are the tangent vector fields on $\Sigma$, $V$ is a normal vector field along $\Sigma$. The mean curvature vector of $\Sigma$ is defined by $H = \text{tr}B$.

We only need to treat the case that $\Sigma$ is connected. By scaling, we can assume that
\[
\int_{\partial \Sigma} f + \int_{\Sigma} \sqrt{|D^\Sigma f|^2 + f^2|H|^2} + 2nb_1 \int_{\Sigma} f = n \int_{\Sigma} f^{\frac{n}{n-1}}.
\]

By the connectedness of $\Sigma$ and (3.1), there exists a solution of the following Neumann boundary problem
\[
(3.2) \begin{cases}
\text{div}_\Sigma (f D^\Sigma u) = n f^{\frac{n}{n-1}} - 2nb_1 f - \sqrt{|D^\Sigma f|^2 + f^2|H|^2}, & \text{in } \Sigma, \\
(D^\Sigma u, \nu) = 1, & \text{on } \partial \Sigma,
\end{cases}
\]
where $\nu$ is the outward unit normal vector field of $\partial \Sigma$ with respect to $\Sigma$. Note that if $\partial \Sigma = \emptyset$, then the boundary condition in (3.2) is void. By standard elliptic regularity theory (see Theorem 6.31 in [20]), we know that $u \in C^{2,\gamma}$ for each $0 < \gamma < 1$.

As in [10], we define
\[
U : = \{ x \in \Sigma \setminus \partial \Sigma : |D^\Sigma u(x)| < 1 \}, \\
E : = \{ (x, y) : x \in U, y \in T_x\Sigma, |D^\Sigma u(x)|^2 + |y|^2 < 1 \}.
\]

For each $r > 0$, we denote by $A_r$ the set of all points $(\bar{x}, \bar{y}) \in E$ satisfying
\[
ru(x) + \frac{1}{\bar{r}}d(x, \exp_{\bar{x}}(rD^\Sigma u(\bar{x})) + \bar{y})^2 \geq ru(x) + \frac{1}{\bar{r}}d^2(|D^\Sigma u(\bar{x})|^2 + |\bar{y}|^2)
\]
for all $x \in \Sigma$. Define the transport map $\Phi_r : T^\perp \Sigma \to M$ for each $r > 0$ by
\[
\Phi_r(x, y) = \exp_x(rD^\Sigma u(x) + ry)
\]
for all $x \in \Sigma$ and $y \in T^\perp_x \Sigma$. The regularity of $u$ implies that $\Phi_r$ is of class $C^{1,\gamma}$, $0 < \gamma < 1$.

**Lemma 3.1.** Assume that $(x, y) \in E$. Then we have
\[
\frac{1}{n}(|\Delta_\Sigma u(x) - \langle H(x), y \rangle | \leq f^{\frac{n}{n-1}}(x) - 2b_1.
\]

**Proof.** Combining $|D^\Sigma u(x)|^2 + |y|^2 < 1$ with Cauchy-Schwarz inequality, we obtain
\[
- \langle D^\Sigma f(x), D^\Sigma u(x) \rangle - f(x) \langle H(x), y \rangle
\leq \sqrt{|D^\Sigma f(x)|^2 + f(x)^2|H(x)|^2} \sqrt{|D^\Sigma u(x)|^2 + |y|^2}
\leq \sqrt{|D^\Sigma f(x)|^2 + f(x)^2|H(x)|^2}.
\]
In terms of (3.2) and (3.3), one derives that
\[ f(x)\Delta_\Sigma u(x) - f(x)\langle H(x), y \rangle \]
\[ = nf(x)\pi^n - 2nb_1f - \sqrt{|D^\Sigma f(x)|^2 + f(x)^2|H(x)|^2} \]
\[ - \langle D^\Sigma f(x), D^\Sigma u(x) \rangle - f(x)\langle H(x), y \rangle \]
\[ \leq nf(x)\pi^n - 2nb_1f. \]

The proof is completed. \( \Box \)

The following three lemmas are due to Brendle (Lemmas 4.2, 4.3, 4.5 in [10]). Their proofs are independent of the curvature condition of ambient space too.

**Lemma 3.2.** For each \( 0 \leq \sigma < 1 \), the set
\[ \{ q \in M : \sigma r < d(x, q) < r, \forall x \in \Sigma \} \]
is contained in the set
\[ \Phi_r(\{(x, y) \in A_r : |D^\Sigma u(x)|^2 + |y|^2 > \sigma^2\}). \]

**Lemma 3.3.** Assume that \((\bar{x}, \bar{y}) \in A_r\), and let \( \bar{\gamma}(t) := \exp_2(tD^\Sigma u(\bar{x}) + t\bar{y}) \) for all \( t \in [0, r] \). If \( Z \) is a smooth vector field along \( \bar{\gamma} \) satisfying \( Z(0) \in T_{\bar{x}}\Sigma \) and \( Z(r) = 0 \), then
\[ ((D^\Sigma)^2 u)(Z(0), Z(0)) - \langle B(Z(0), Z(0)), \bar{y} \rangle \]
\[ + \int_0^r (|\tilde{D}_t Z(t)|^2 - \tilde{R}(\gamma'(t), Z(t), \gamma'(t), Z(t)))dt \geq 0. \]

**Lemma 3.4.** Assume that \((\bar{x}, \bar{y}) \in A_r\), and let \( \bar{\gamma}(t) := \exp_2(tD^\Sigma u(\bar{x}) + t\bar{y}) \) for all \( t \in [0, r] \). Let \( \{e_1, \ldots , e_n\} \) be an orthonormal basis of \( T_{\bar{x}}\Sigma \). Suppose that \( W \) is a Jacobi field along \( \bar{\gamma} \) satisfying \( W(0) \in T_{\bar{x}}\Sigma \) and \( \langle \tilde{D}_t W(0), e_j \rangle = ((D^\Sigma)^2 u)(W(0), e_j) - \langle B(W(0), e_j), \bar{y} \rangle \) for each \( 1 \leq j \leq n \). If \( W(\tau) = 0 \) for some \( \tau \in (0, r) \), then \( W \) vanishes identically.

Now we begin the proof of Theorem 1.4.

**Proof of Theorem 1.4.** For any \( r > 0 \) and \((\bar{x}, \bar{y}) \in A_r\), let \( \{e_i\}_{1 \leq i \leq n} \) be any given orthonormal basis in \( T_{\bar{x}}\Sigma \). Choose a normal coordinate system \( (x^1, \ldots , x^n) \) on \( \Sigma \) around \( \bar{x} \) such that \( \frac{\partial}{\partial x^i} = e_i \) at \( \bar{x} \) (\( 1 \leq i \leq n \)). Let \( \{e_\alpha\}_{n+1 \leq \alpha \leq n+p} \) be an orthonormal frame field of \( T^\perp \Sigma \) around \( \bar{x} \) such that \( ((D^\Sigma)^\perp e_\alpha)_x = 0 \) for \( n+1 \leq \alpha \leq n+p \), where \( (D^\Sigma)^\perp \) denotes the normal connection in the normal bundle \( T^\perp \Sigma \) of \( \Sigma \). Any normal vector \( y \) around \( \bar{x} \) can be written as \( y = \sum_{\alpha=n+1}^{n+p} y^\alpha e_\alpha \), and thus \( (x^1, \ldots , x^n, y^{n+1}, \ldots , y^{n+p}) \) becomes a local coordinate system on the total space of the normal bundle \( T^\perp \Sigma \).
Let $\bar{\gamma}(t) := \exp_{\bar{x}}(tD^2u(\bar{x}) + t\bar{y})$ for all $t \in [0, r]$. For each $1 \leq A \leq n + p$, we denote by $E_A(t)$ the parallel transport of $e_A(\bar{x})$ along $\bar{\gamma}$. For each $1 \leq i \leq n$, let $X_i$ be the Jacobi field along $\bar{\gamma}$ with the following initial conditions

$$X_i(0) = e_i,$$

(3.4) \quad $\langle \bar{D}_tX_i(0), e_j \rangle = ((D^2)^2u)(e_i, e_j) - \langle B(e_i, e_j), \bar{y} \rangle$, \quad $1 \leq j \leq n$,

$$\langle \bar{D}_tX_i(0), e_\beta \rangle = \langle B(e_i, D^2u(\bar{x})): e_\beta \rangle$, \quad $n + 1 \leq \beta \leq n + p$.

For each $n + 1 \leq \alpha \leq n + p$, let $X_\alpha$ be the Jacobi field along $\bar{\gamma}$ satisfying

(3.5) \quad $X_\alpha(0) = 0, \quad \bar{D}_tX_\alpha(0) = e_\alpha$.

Using Lemma 3.4, we known that $\{X_A(t)\}_{1 \leq A \leq n + p}$ are linearly independent for each $t \in (0, r)$.

Let $P(t) = (P_{AB}(t))$ and $S(t) = (S_{AB}(t))$ be the matrices given by

$$P_{AB}(t) = \langle X_A(t), E_B(t) \rangle,$$

$$S_{AB}(t) = \bar{R}(\bar{\gamma}(t), E_A(t), \bar{\gamma}'(t), E_B(t))$$

for $1 \leq A, B \leq n + p$ and $t \in [0, r]$, where $\bar{R}$ denotes the Riemannian curvature tensor of $M$. Using the Jacobi equation and the initial conditions (3.4), (3.5), we have

(3.6) \quad $P''(t) = -P(t)S(t)$,

$$P_{AB}(0) = \begin{bmatrix} \delta_{ij} & 0 \\ 0 & 0 \end{bmatrix},$$

$$P'_{AB}(0) = \begin{bmatrix} ((D^2)^2u)(e_i, e_j) - \langle B(e_i, e_j), \bar{y} \rangle & \langle B(e_i, D^2u(\bar{x})): e_\beta \rangle \\ 0 & \delta_{\alpha\beta} \end{bmatrix}.$$

Set $Q(t) = P(t)^{-1}P'(t)$, $t \in (0, r)$. By (3.6), a simple computation yields

(3.7) \quad $\frac{d}{dt}Q(t) = -S(t) - Q^2(t)$,

where $Q(t)$ is symmetric. For the matrices $P(t), Q(t)$, it is easy to derive their following asymptotic expansions (cf. [10])

(3.8) \quad $P(t) = \begin{bmatrix} \delta_{ij} + O(t) & O(t) \\ O(t) & t\delta_{\alpha\beta} + O(t^2) \end{bmatrix}$,

$$Q(t) = \begin{bmatrix} ((D^2)^2u)(e_i, e_j) - \langle B(e_i, e_j), \bar{y} \rangle + O(t) & O(1) \\ O(1) & \frac{1}{t}\delta_{\alpha\beta} + O(1) \end{bmatrix}.$$
as \( t \to 0^+ \). In terms of (3.7) and the curvature assumption for \( M \), we deduce

\[
\frac{d}{dt} Q_{AA}(t) + Q_{AA}(t)^2 \leq \frac{d}{dt} Q_{AA}(t) + \sum_{B=1}^{n+p} Q_{AB}Q_{BA}(t)
\]

(3.9)

\[
= -S_{AA}(t) \\
\leq (|D^2 u(\bar{x})|^2 + |\bar{y}|^2 - \langle D^2 u(\bar{x}) + \bar{y}, e_A \rangle^2)\lambda(d(o, \bar{\gamma}(t)))
\]

\[
\leq (|D^2 u(\bar{x})|^2 + |\bar{y}|^2 - \langle D^2 u(\bar{x}) + \bar{y}, e_A \rangle^2)\lambda(|d(o, \bar{x}) - t|D^\Sigma u(\bar{x}) + \bar{y}|)
\]

for \( 1 \leq A \leq n+p \), where the last inequality follows from the following triangle inequality

\[
d(o, \bar{\gamma}(t)) \geq |d(o, \bar{x}) - d(\bar{x}, \bar{\gamma}(t))| = |d(o, \bar{x}) - t|D^\Sigma u(\bar{x}) + \bar{y}|.
\]

For \( 1 \leq A \leq n + p \), we set

\[
\Lambda_{x,A}(t) = (|D^2 u(\bar{x})|^2 + |\bar{y}|^2 - \langle D^2 u(\bar{x}) + \bar{y}, e_A \rangle^2)\lambda(|d(o, \bar{x}) - t|D^\Sigma u(\bar{x}) + \bar{y}|).
\]

Then we have

\[
\left\{ \begin{array}{l} Q''_u(t) + Q_u(t)^2 \leq \Lambda_{x,i}(t), \quad t \in (0, r), \\ \lim_{t \to 0^+} Q_u(t) = \lambda_i, \end{array} \right.
\]

where \( \lambda_i = P''_{ii}(0) \). Let \( \phi_i \) be defined by

\[
\phi_i(t) = e^{\int_0^t Q_u(r)dr}.
\]

Then \( \phi_i \) satisfies

(3.10)

\[
\left\{ \begin{array}{l} \phi''_i \leq \Lambda_{x,i}\phi_i, \quad t \in (0, r), \\ \phi_i(0) = 1, \phi'_i(0) = \lambda_i. \end{array} \right.
\]

Next, we denote by \( \psi_{1,i}, \psi_{2,i} \) solutions to the following problems

(3.11)

\[
\left\{ \begin{array}{l} \psi''_{1,i} = \Lambda_{x,i}\psi_{1,i}, \quad t \in (0, r), \\ \psi_{1,i}(0) = 0, \psi'_{1,i}(0) = 1, \end{array} \right. \quad \left\{ \begin{array}{l} \psi''_{2,i} = \Lambda_{x,i}\psi_{2,i}, \quad t \in (0, r), \\ \psi_{2,i}(0) = 1, \psi'_{2,i}(0) = 0. \end{array} \right.
\]

Similar to the proof of (2.6), (2.13) and (2.14), we obtain

(3.12)

\[
\frac{\psi_{2,i}}{\psi_{1,i}}(r) \leq \int_0^{+\infty} \Lambda_{x,i}(t) \, dt + \frac{1}{r}
\]

\[
\leq 2b_1 \frac{|D^\Sigma u(\bar{x})|^2 + |\bar{y}|^2 - \langle D^\Sigma u(\bar{x}) + \bar{y}, e_i \rangle^2}{\sqrt{|D^\Sigma u(\bar{x})|^2 + \bar{y}^2}} + \frac{1}{r}
\]

\[
\leq 2b_1 \sqrt{|D^\Sigma u(\bar{x})|^2 + \bar{y}^2} + \frac{1}{r}
\]
\[ (3.13) \quad \psi_{1,i}(t) \leq te^{|D^{2\psi}(\bar{x}_i)|^2+|y|^2-(D^{2\psi}(\bar{x}_i)+\bar{y},e_i)|^2/(2r_{ob1}+b_0)}, \quad t \in (0, r), \]

where \( r_0 = \max\{d(o,x)|x \in \Sigma\} \). Using Lemma 2.5, one can find from (3.10) and (3.11) that

\[ (3.14) \quad Q_{ii}(t) = \frac{\phi'_i(t)}{\phi_i(t)} \leq \psi_{2,i}^i(t) + \lambda_i \psi_{1,i}(t). \]

Similarly we obtain from (3.8) and (3.9) that

\[
\begin{cases} 
Q'_{\alpha\alpha}(t) + Q_{\alpha\alpha}(t)^2 \leq \Lambda_{\bar{x},\alpha}(t), & t \in (0, r), \\
Q_{\alpha\alpha}(t) = \frac{1}{t} + O(1), & \text{as } t \to 0^+ 
\end{cases}
\]

for \( n + 1 \leq \alpha \leq n + p \). Set \( \phi_{\alpha}(t) = te^{\int_0^t (Q_{\alpha\alpha}(\tau) - \frac{1}{\tau})d\tau} \). Then \( \phi_{\alpha} \) satisfies

\[
\begin{cases} 
\phi''_{\alpha} \leq \Lambda_{\bar{x},\alpha}\phi_{\alpha}, & t \in (0, r), \\
\phi_{\alpha}(0) = 0, \phi'_{\alpha}(0) = 1. 
\end{cases}
\]

Next, we denote by \( \psi_{1,\alpha} \) the unique solution to the following problem

\[ (3.15) \quad \begin{cases} 
\psi''_{1,\alpha} = \Lambda_{\bar{x},\alpha}\psi_{1,\alpha}, & t \in (0, r), \\
\psi_{1,\alpha}(0) = 0, \psi'_{1,\alpha}(0) = 1. 
\end{cases} \]

Similar to (2.14), we derive that

\[ (3.16) \quad \psi_{1,\alpha}(t) \leq e^{e^{\int_0^t (Q_{\alpha\alpha}(\tau) - \frac{1}{\tau})d\tau}}. \]

for \( t \in (0, r) \). By Lemma 2.1 in [33] we have

\[ (3.17) \quad Q_{\alpha\alpha}(t) = \frac{\phi'_{\alpha}(t)}{\phi_{\alpha}(t)} \leq \frac{\psi'_{1,\alpha}(t)}{\psi_{1,\alpha}(t)}. \]

From (3.14) and (3.17), it follows that

\[ (3.18) \quad \frac{d}{dt} \log \det P(t) = \text{tr}(Q(t)) \leq \sum_i \frac{\psi'_{2,i} + \lambda_i \psi_{1,i}(t)}{\psi_{2,i} + \lambda_i \psi_{1,i}(t)} + \sum_{\alpha} \frac{\psi'_{1,\alpha}}{\psi_{1,\alpha}}. \]

Combining (3.11), (3.15) with the asymptotic properties in (3.8), we conclude that

\[ (3.19) \quad \lim_{t \to 0^+} \prod_i (\psi_{2,i}(t) + \lambda_i \psi_{1,i}(t)) \prod_{\alpha} \psi_{1,\alpha}(t) = 1. \]
Integrating (3.18) over \([\varepsilon, t]\) for \(0 < \varepsilon < t\) and using (3.19) by letting \(\varepsilon \to 0^+\), it is easy to show that

\[
|\det D\Phi_t(\bar{x}, \bar{y})| = \det P(t) \leq \prod_i (\psi_{2,i}(t) + \lambda_i \psi_{1,i}(t)) \prod_{\alpha} \psi_{1,\alpha}(t).
\]

Note that \(0 \leq \phi_i \leq (\psi_{2,i} + \lambda_i \psi_{1,i})\) and \(\psi_{1,i} \geq 0\) (\(1 \leq i \leq n\)). Combining (3.13), (3.16) with arithmetic-geometric mean inequality, we obtain

\[
|\det D\Phi_t(\bar{x}, \bar{y})| \leq \left(\frac{1}{n} \sum_i \frac{\psi_{2,i}(t)}{\psi_{1,i}(t)} + \frac{1}{n} (\Delta \Sigma u(\bar{x}) - \langle H(\bar{x}), \bar{y} \rangle)\right)^n \prod_A \psi_{1,A}(t)
\]

which yields by (3.12) that

\[
|\det D\Phi_r(\bar{x}, \bar{y})| \leq (2b_1 \sqrt{|Du(\bar{x})|^2 + \bar{y}^2} + \frac{1}{r})^n \prod_{\alpha} \psi_{1,\alpha}(t)
\]

for all \((\bar{x}, \bar{y}) \in A_r\). Noting that \(\sqrt{|Du(\bar{x})|^2 + \bar{y}^2} < 1\), we derive by Lemma 3.1 and (3.20) that

\[
|\det D\Phi_r(\bar{x}, \bar{y})| \leq \left(\frac{1}{r} + f^{\frac{1}{p-1}}(\bar{x})\right)^{n+p} e^{(n+p-1)(2r_b + b_0)}
\]

for all \((\bar{x}, \bar{y}) \in A_r\). Using Lemma 3.2 and (3.21), one may find in a similar way as the proof of Theorem 1.4 in [10] that

\[
|\{p \in M : \sigma r < d(x, p) < r, \forall x \in \Sigma\}| \leq \frac{p}{2} B^p (1 - \sigma^2) e^{(n+p-1)(2r_b + b_0)} \int_{\Sigma} \left(\frac{1}{r} + f^{\frac{1}{p-1}}(\bar{x})\right)^{n+p} dt
\]

for all \(r > 0\) and all \(0 \leq \sigma < 1\). Similar to the proof of (2.20), one can obtain by using Lemma 2.7 that

\[
\lim_{r \to +\infty} \frac{|\{p \in M : \sigma r < d(x, p) < r, \forall x \in \Sigma\}|}{(n + p) \int_0^r h^{n+p-1} dt} = |B^{n+p}| \theta \lim_{r \to +\infty} \frac{1 - \sigma}{h^{n+p-1}(r)} = |B^{n+p}| (1 - \sigma^{n+p}) \theta.
\]

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Dividing \((3.22)\) by \((n + p) \int_0^r h(t)^{n+p-1} \, dt\) and sending \(r \to +\infty\), we deduce by using (2.17) and (3.23) that

\[
\int_0^r h(t)^{n+p-1} \, dt \quad \text{and sending} \quad r \to +\infty
\]

we deduce by using (2.17) and (3.23) that

\[
(3.24) \quad (n + p)|B^{n+p}|\theta \leq p|B^p| \left( \frac{e^{2r_0 b_1 + b_0}}{1 + b_0} \right)^{n+p-1} \int_{\Sigma} f \frac{n}{n-1}.
\]

for all \(0 \leq \sigma < 1\). Now, if we divide (3.24) by \(1 - \sigma\) and let \(\sigma \to 1\), we have

\[
(3.25) \quad (n + p)|B^{n+p}|\theta \leq p|B^p| \left( \frac{e^{2r_0 b_1 + b_0}}{1 + b_0} \right)^{n+p-1} \int_{\Sigma} f \frac{n}{n-1}.
\]

Hence (3.1) and (3.25) imply that

\[
\int_{\partial \Sigma} f + \int_{\Sigma} \sqrt{|D^2 f|^2 + f^2 |H|^2 + 2nb_1} \int_{\Sigma} f \geq n \left( \frac{(n + p)|B^{n+p}|}{p|B^p|} \right)^{1/2} \theta^{1/2} \left( \frac{1 + b_0}{e^{2r_0 b_1 + b_0}} \right)^{n+p-1} \left( \int_{\Sigma} f \frac{n}{n-1} \right)^{n-1}.
\]

**Proof of Theorem 1.6.** Suppose the equality of Theorem 1.5 holds. Then we have equality in both (2.17) and (3.12) and either one forces \(\lambda \equiv 0\). Thus \(M\) has nonnegative sectional curvature. The assertion follows immediately from Theorem 1.6 in [10].

Finally we would like to mention that we have established a Sobolev type inequality for manifolds with density and asymptotically nonnegative Bakery-Émery Ricci curvature in [16] and a logarithmic Sobolev type inequality for closed submanifolds in manifolds with asymptotically nonnegative sectional curvature in [17].

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