A bijective proof of the enumeration of maps in higher genus.

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Abstract

Bender and Canfield proved in 1991 that the generating series of maps in higher genus is a rational function of the generating series of planar maps. In this paper, we give the first bijective proof of this result. Our approach starts with the introduction of a canonical orientation that enables us to construct a bijection between 4-valent bicolourable maps and a family of unicellular blossoming maps.

1 Introduction

A map of genus $g$ is a proper embedding of a graph in $S_g$, the torus with $g$ holes. Planar maps (or maps of genus 0) have been studied extensively since the pioneering work of Tutte in the sixties [23]. In a series of work, Tutte obtained remarkable formulas for many families of maps. His techniques rely on some recurrence relations for maps and some clever manipulations of generating series. They were extended in the late eighties to the case of maps with higher genus by Bender and Canfield, who first obtained the asymptotic number of maps on any orientable surface of genus $g$ [3] and then obtained in [2] the following stronger result:

Theorem 1.1 (Bender and Canfield [2]). For any $g \geq 0$, the generating series $M_g(z)$ of maps of genus $g$ enumerated by edges is a rational function of $z$ and $\sqrt{1 - 12z}$.

Since then, many other approaches have been developed, illustrating deep connections of maps with various fields of algebra and mathematical physics (e.g. [22, 16, 13]).

The main purpose of this paper is to provide the first bijective proof of theorem 1.1 for $g \geq 2$. Our proof starts with the well-known bijection between general maps and so-called 4-valent bicolourable maps. In the planar case, Schaeffer exhibits in [20] a constructive bijection between 4-valent planar maps and some so-called blossoming trees. The blossoming tree associated to a map is one of its spanning trees, decorated by some half-edges, that enables to reconstruct the “missing edges”. In genus $g > 0$, the natural counterpart of trees are unicellular maps (i.e. maps with only one face) and we obtain in this work the following generalization of Schaeffer’s result (the terminology is introduced in section 3.1):

Theorem 1.2. There exists a constructive bijection between rooted maps of genus $g$ with $n$ edges and well-rooted well-labeled well-oriented 4-valent unicellular blossoming maps with $n$ vertices.

The enumeration of maps now boils down to the much easier enumeration of this family of unicellular maps. As a byproduct we obtain a bijective proof of theorem 1.1.

Let us now put our work in context of the existing literature. In the planar case, there are numerous bijections between maps and some families of decorated trees. Two main trends emerge in these bijections. Either the decorated trees are some blossoming trees as already described (e.g. [20, 8, 13]) or the trees are decorated by some integers that capture some metric properties of the maps (e.g. [21, 9]). Bijections of the

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latter type have been successfully extended to higher genus \[11,10,17\]. Unfortunately, with these techniques the generating series of maps can be expressed as a rational function of some auxiliary functions whose degree of algebraicity is higher than the known enumerative results. The situation is much different in the case of bijections with blossoming trees, and apart from the recent work \[12\] which presents a bijection between simple triangulations of genus 1 (with some additional constraints) and a family of blossoming unicellular maps, there was, previously to our work, no other extension of the existing bijections.

Let us end with an important connection to our work. As emphasized by Bernardi \[4\] in the planar case and generalized by Bernardi and Chapuy \[5\], a map endowed with a spanning unicellular embedded graph (whose genus can be smaller than the genus of the initial surface) can also be viewed as a map endowed with an orientation of its edges with specific properties. The general theory of \(\alpha\)-orientations developed by Felsner in the planar case \[14\] has been successfully combined with the result of \[11\] to give general bijective schemes in the planar case \[14\], \[17\], which enables to recover the previously known bijections. It would be highly desirable to obtain systematic bijective schemes in higher genus by combining Bernardi and Chapuy’s result together with the theory of \(c\)-orientations introduced by Propp \[19\] or its extension by Felsner and Knauer \[15\]. The main difficulty to tackle is to characterize the orientations that produce spanning unicellular embedded graph whose genus matches the genus of the original surface. Our work, presented here only in the case of bicolorable 4-valent maps for lack of space, but which can be extended straightforwardly to all bicolorable maps, can also be seen as an important first step in that direction and we hope to be able to extend to other families of maps in some future work.

**Organization of the paper:** In section \[2\] we recall definitions about maps and orientations and state Propp’s theorem adapted to our setting. In section \[3\] we define an explicit and constructive bijection between 4-valent bicolorable maps and a family of unicellular maps. Finally, in section \[4\] we analyse this family, and prove that its generating series admits a rational expression involving the generating series of planar maps.

**Notation:** In this article, combinatorial families are named with calligraphic letters, their generating series is the corresponding capital letter, and an object of the family, is usually denoted by the corresponding lower case letter. The size being denoted by \(|\cdot|\), we therefore have for a combinatorial family \(S\): 
\[
S(z) = \sum_{s \in S} z^{|s|}.
\]

## 2 Orientations in higher genus

### 2.1 General

We begin with some definitions about maps. An **embedded graph** is an embedding of a connected graph into a given compact surface, taken up to orientation-preserving homeomorphism of the surface. An embedded graph is **cellularly embedded** if all its faces (connected component of the complement) are homeomorphic to discs. A **map** is a cellularly embedded graph. The set of maps, counted by number of edges, is denoted \(\mathcal{M}\). In this paper we only consider maps embedded on orientable surfaces. **General maps** have no other restriction, and in particular, can have loops or multiple edges. The **genus** of a map is the genus of its underlying surface. All families of maps can be refined by their genus; we denote this refinement by an index indicating the genus, so that for instance \(\mathcal{M}_0\) is the set of planar maps. See figure \[1a\] for an example of map of genus 1.

An adjacency between a face and a vertex is called a **corner**. Note that a single pair vertex-face can give rise to several distinct corners. The **degree** of a face (resp. vertex) is the number of adjacent corners. To get rid of automorphisms, maps are **rooted** at a distinguished **root corner** (whose vertex and face are called **root vertex** and **root face**).

The set of vertices (resp. edges, faces) of \(m\) is denoted \(V_m\) (resp. \(E_m, F_m\)). The number of vertices (resp. edges, faces) of \(m\) is denoted \(v_m\) (resp. \(e_m, f_m\)). The genus of \(m\) is denoted \(g_m\). We recall Euler’s formula: 
\[
v_m - e_m + f_m = 2 - 2g_m.
\]

Since an edge connects two vertices and separates two faces, we can define the **dual map** \(m^\dagger\) of \(m\) by exchanging the role of vertices and faces, and swapping the connection and separation induced by each edge (see figure \[1b\]). The root corner remains the same (but its vertex and its face are exchanged). Note that duality is involutive: \((m^\dagger)^\dagger = m\).
A map is **unicellular** if it has only one face. A **tree** is a map whose underlying graph has no cycle. A map is **4-valent** if all its vertices are of degree 4. Dually, a map is a quadrangulation if all its faces are of degree 4. A map is **bipartite** if its underlying graph is bipartite, which means that its vertices can be properly colored black and white. In particular, a bipartite map has no loop. Dually, a map is **bicolorable** if its faces can be properly colored black and white. The set of bipartite quadrangulations (resp. bicolorable 4-valent maps), counted by number of faces (resp. number of vertices), is denoted \( Q \) (resp. \( V \)).

A map is **Eulerian** if all its vertices have even degree. In the sphere, Eulerian is equivalent to bicolorable. It is not the case anymore in higher genus, where bicolorability is more relevant. Indeed, in addition to having a nicer dual property, it also appears in the following well-known bijection, illustrated in figure 2:

**Proposition 2.1.** General maps of genus \( g \) with \( n \) edges are in bijection with 4-valent bicolorable maps of genus \( g \) with \( n \) vertices, or dually, with bipartite quadrangulations of genus \( g \) with \( n \) faces. The 4-valent map is called the radial map, and the bipartite quadrangulation is the quadrangulated map. Therefore, \( M(z) = V(z) = Q(z) \).

### 2.2 Structure of orientations of a graph

An **orientation** of a map is an orientation of each of its edges. The **dual orientation** of an orientation \( r \) of a map \( m \) is the orientation of \( m^\dagger \) where all dual edges are oriented from the face to the right of the primal.
edge toward the face to its left (see figure 1c). Note that applying duality twice reverses the orientation.

Orientations provides additional structural properties to maps, useful for algorithmic purposes. However, since our final purpose is to study maps without an orientation, it is convenient to assign a generic orientation to maps. Such a generic orientation is obtained as the minimum of a lattice of orientations, as described below.

The geodesic orientation of a bipartite rooted map is the orientation whose edges are all oriented toward the root in terms of graph distance. Along any cycle, forward (resp. backward) edges in this orientation correspond to a distance to the root increasing (resp. decreasing) by 1. The geodesic orientation thus belongs to the set of bipartite orientations, in which there are as many forward edges as backward edges along any cycle. This set is endowed with the vertex-push operation, that changes a sink distinct from the root into a source, by reversing all adjacent edges. Dually, we call bicolorable orientation the dual of a bipartite orientation, and face-flip the dual of a vertex-push. The next result follows from [10, Theorem 1].

**Theorem 2.2.** The vertex-push graph on bipartite orientations of a fixed map is the Hasse diagram of a distributive lattice whose minimum is the geodesic orientation.

Dually, the face-flip graph on bicolorable orientations of a fixed map is the Hasse diagram of a distributive lattice whose minimum is the dual of the geodesic orientation.

**Corollary 2.3.** The dual of the geodesic orientation (called dual-geodesic orientation for short) is the unique bicolorable orientation with no clockwise face (see figure 3).

### 3 Closing and opening maps

#### 3.1 The closure of a blossoming map

A blossoming oriented map $b$ is an oriented map with additional half-edges attached to its corners. These half-edges are oriented and hence can be of two types; an outgoing half-edge is called a bud and an ingoing half-edge is called a leaf. A blossoming map must have as many buds as leaves. The size of a blossoming map is the number of edges plus the number of buds. Blossoming maps are rooted on a bud.

In a blossoming oriented map, the interior degree (resp. blossoming degree, resp. degree) of a vertex is the number of edges adjacent to the vertex (resp. the number of half-edges attached to its corner, resp. the sum of the interior and blossoming degrees). These can be refined into ingoing and outgoing degrees. The interior map $b^*$ is the map $b$ without its buds and leaves.

A blossoming oriented map is said to be well-labeled if its corners are labeled so that:

- the labels of two corners adjacent around a vertex differ by 1, in which case the higher label is to the right of the separating edge,
- the labels of two corners adjacent along an edge coincide, and
- the root bud has labels 0 and 1.

In particular, a well-labeled map has a Eulerian orientation.

Let $b$ be a unicellular blossoming map. The contour word of $b$ is defined as follows: when doing a clockwise tour of the unique face (which means that the face is to the right), starting from the root bud, write $U$ (for up-step) for each bud and $D$ (for down-step) for each leaf. We say that $b$ is well-rooted if its contour word is a Dyck path.

The map $b$ is well-oriented if in a tour of the face starting from the root, each edge is first followed backward and then forward. Note that this does not depend on the direction of the tour. In the case of a tree, this means that it is oriented toward the root.

Unicellular blossoming maps are interesting because they are easier to analyse than maps, but still can encode a map, thanks to the closing algorithm, defined hereafter. Let $b$ be a well-rooted unicellular blossoming map. We write the contour word of $b$ and match its steps by pair upstep/downstep; each up-step $U$ going from height $i$ to $i + 1$ is matched to the first down-step $D$ after $U$ going from height $i + 1$ to $i$. The half-edges corresponding to matched steps are then merged into a single oriented edge. An example of the closing algorithm is given in figure 3.
Lemma 3.1. The closure of a well-rooted well-labeled well-oriented 4-valent unicellular blossoming map of genus g is a rooted 4-valent bicolorable map of genus g with dual-geodesic orientation.

To prove that the resulting orientation is indeed the dual-geodesic orientation, we prove that it is bicolorable and has no clockwise face (see corollary 2.3). The set of well-rooted well-labeled well-oriented 4-valent unicellular blossoming maps is denoted \( \mathcal{O} \).

3.2 The opening of a map

In this section, we prove that the closing operation can actually be turned into a bijection and describe its inverse: the opening algorithm.

Given a rooted oriented map, we define the opening algorithm as follows. We explore the map starting from the root. When we meet an unexplored edge, if it is ingoing, we follow it, if it is outgoing, we cut it and replace it by a bud. When we meet an already explored edge, it was either followed, in which case we follow it back, or cut, in which case we just add a leaf. We stop when we get back to the root. An example of an execution of the closing algorithm is given in figure 3.

An equivalent definition in terms of corners highlights the symmetry between the roles of faces and vertices, which we express in lemma 3.2. To a map \( m \) we associate a reflected map \( \tilde{m} \) which is the same as \( m \) except that we switch the orientation of the underlying surface, which amounts to exchange clockwise and counterclockwise. To a subgraph \( s \) of a graph \( g \) we associate the complement subgraph \( s^\complement \) defined with the same set of vertices along with all edges in \( g \) but not in \( s \). This definition is naturally extended to the complement of a map by preserving the embedding.

Lemma 3.2. Up to a change of orientation of the surface, a map and its dual yield complement submaps by the opening algorithm:

\[
\text{Open}(m)^{\complement} = \left(\text{Open}(\tilde{m})^{\complement}\right)^{\complement}.
\]

Theorem 3.3. The opening algorithm is a bijection from \( \mathcal{V}_g \) to \( \mathcal{O}_g \). Its inverse is the closing algorithm. Therefore, \( \mathcal{V}(z) = \mathcal{O}(z) \).
Proof (sketch). Applying the opening algorithm on the geodesic orientation of a bipartite quadrangulation yields the rightmost breadth-first-search exploration tree, along with its buds and leaves. By lemma 3.2 the result of the opening algorithm applied to a 4-valent bicolorable map is the dual of a leftmost breadth-first-search exploration tree. This proves in particular that the algorithm preserves the genus. The rest follows easily.

4 Enumeration and rationality

4.1 Reducing a map to a scheme

We saw that general maps are in bijection with well-rooted well-labeled well-oriented 4-regular Eulerian unicellular map. However the analysis of such objects is made difficult by the non-locality of a condition such as well-rootedness. The following lemma enables to ignore that condition in the rest of the analysis. The set of rooted well-labeled well-oriented 4-regular unicellular maps, counted by leaves, is denoted \( \mathcal{U} \).

Theorem 4.1. There is a \((n+1)\)-to-2 map from rooted well-labeled well-oriented 4-regular unicellular map with inner map \( m \), to well-rooted well-labeled well-oriented 4-regular unicellular map with interior map \( m \).

Proof (sketch). A method similar to the cyclic lemma implies that there are exactly 2 cyclic permutations of the contour word among \( n+1 \) that yield a Dyck word. To prove theorem 4.1 we therefore need to reroot a non-well-rooted map on one of the 2 special half-edges corresponding to these permutations. The main difficulty is to prove that there exists a unique directed path from this special half-edge to the root, and that reversing it yields a well-labeled map.

Let \( u \) be a rooted well-labeled well-oriented 4-regular unicellular map. Define the extended scheme as the unicellular map of genus \( g \) obtained by iteratively removing all vertices of \( u \) with interior degree 1 along with all half-edges.

The map \( u \) is composed of an extended scheme upon which are attached some half-edges and treelike parts. These treelike parts, with their leaves, are binary trees, oriented towards the root. Furthermore, on each interior vertex of these trees is attached a bud. The set of such trees, counted by leaves, is denoted \( \mathcal{T} \). Its generating series satisfies the recurrence relation \( T(z) = z + 3T(z)^2 \). The generating series of such trees with a marked leaf (or equivalently doubly rooted) is \( z \cdot \frac{\partial T}{\partial z}(z) \).

The pruning procedure is defined as follows: each treelike part is replaced by a half-edge: a root bud if it contains the root, a leaf otherwise (see figure 4 left and middle). The image of \( \mathcal{U} \) by the pruning procedure, counted by leaves, is denoted \( \mathcal{P} \).

Lemma 4.2. The pruning algorithm yields: \( U(z) = z \cdot \frac{\partial T}{\partial z}(z) \cdot P(T(z)) \).

All vertices of the pruned map are of interior degree 2, 3 or 4. We call \( v_2 \), \( v_3 \), and \( v_4 \) the number of such vertices. A quick calculation based on Euler formula gives: \( v_3 + 2v_4 = 4g - 2 \). There are thus a finite number of vertices of degree 3 or 4, the other ones being of degree 2. Vertices of interior degree at least 3 are called scheme vertices, and a half-edge (resp. bud, leaf) attached on a scheme vertex (of interior degree 3) is called a scheme half-edge (resp. bud, leaf). Another computation gives:

Lemma 4.3. If the map is rooted on a scheme bud, there are \( 2g - v_4 - 1 \) scheme leaves and \( 2g - v_4 - 1 \) scheme buds. Otherwise, there are \( 2g - v_3 \) scheme leaves and \( 2g - v_4 - 2 \) scheme buds. In particular \( v_3 > 0 \).

We now proceed to reroot the pruned map on a scheme half-edge. We therefore choose one scheme leaf and mark it. In case the original root is a scheme bud, it is also eligible to be marked. This leaves \( 2g - v_4 \) possible choices. The rerooting is defined similarly to the proof of theorem 4.1 (see figure 4 middle and right). The subset of \( \mathcal{P} \) composed of scheme-rooted maps is denoted \( \mathcal{R} \).

Lemma 4.4. The rerooting algorithm yields: \( P(z) = z \cdot \frac{\partial R}{\partial z}(z) \).
Figure 4: An example of the pruning of an opened map (whose treelike parts are encompassed) with scheme vertices $A$ and $B$; and rerooting on the scheme of the opened map. Again the opposite sides are identified, so that the map is of genus 1.

Figure 5: Reducing a map of $\mathcal{R}$ to a labeled scheme, represented with bigger red labels on vertices and smaller black offset labels on corners, along with the corresponding set of weighted Motzkin paths. The offset graph is represented in purple.

Now that the map is rooted on a scheme bud, the sequence of edges encountered between two scheme vertices in a tour of the face all have the same orientation. Such a sequence is called a branch. We call merging the procedure that replaces each branch by a single edge with the same orientation (see figure 5). The map we obtain is called the labeled scheme. It is not well-labeled because corners adjacent along an edge do not necessarily have the same label anymore, but the rule around a vertex is still respected. The set of labeled schemes is denoted $L$.

4.2 Analyzing a scheme

For $l \in L$, we now want to determine which maps have $l$ as labeled scheme. Each edge of $l$ should be replaced by a valid branch. However we need to be sure that after replacement, the map is well-labeled, and agrees with the labeling of the scheme.

There are 6 types of vertices of interior degree 2, displayed in figure 6 left. If the bud and leaf are on opposite sides, the label of the corners either increases on both side or decreases on both sides. In the 4 other cases, the half-edges are on the same side, and the label remains the same before and after the vertex. Therefore each type of vertex of interior degree 2 can be represented by a step, depending on the variation of the labels around it: an up-step if the label increases, a down-step if it decreases, and 4 types of stay-steps if it stays the same, represented with a green dot placed accordingly to the position of the bud (see figure 6).
right). These steps are called weighted Motzkin steps, and together they form a weighted Motzkin path. An edge of the labeled scheme going from label $i$ to label $j$ can therefore be replaced by a weighted Motzkin path going from height $i$ to height $j$.

We denote by $D$ the set of weighted Motzkin paths going from 0 to $-1$, that remain non-negative before the last step, counted by length. It satisfies the decomposition equation: $D = z(1 + 4D + D^2)$. We denote by $B$ the set of weighted Motzkin paths going from 0 to 0, counted by length. It satisfies the decomposition equation: $B = 1 + 4B + 2zDB$. After combination with the previous equation, this equation is rewritten as a function of $D$ only: $B = \frac{1+4D+D^2}{1+4D}$.

The generating series of paths going from height $i$ to $j$ is: $B \cdot D^{j-i}$.

### 4.3 Rationality

We conclude by the bijective proof of theorem 1.1 announced in the introduction. In fact, we prove a refinement by unlabeled scheme. An unlabeled scheme is a scheme where we forgot all labels. We denote by $S$ the set of unlabeled schemes. We specialize our classes of maps depending on their scheme, by writing $M^s$ for example.

**Theorem 4.5.** For any $s$ in $S$, the generating series $M^s(t)$ is a rational function of $T(t)$.

Since $S_g$ is finite for any fixed $g$, it implies that $M^g(t) = \sum_{s \in S_g} M^s(t)$ is rational in $T(t)$.

**Proof (sketch).** We derive from the 2 previous sections that

$$M^s(t) = t^{2g-1} \cdot \frac{2}{t} \int_0^t z \cdot \frac{\partial T}{\partial z}(z) \cdot \frac{\partial R^s}{\partial z}(T(z))dz.$$ 

In order to prove that $M^s$ is rational in $T$ (and $t$), it suffices to prove that $R^s(z)$ is rational in $z$. Since $z = \frac{1}{1+4+D}$, any series which is rational in $z$ is rational and symmetric in $D$ (a function $\Psi$ is symmetric in $D$ if $\Psi(D) = \Psi(D^{-1})$). Reciprocally, since the polynomials $(D^{-1} + 4 + D)^k$, for $k \in \mathbb{N}$, generate all symmetric polynomials in $D$, any symmetric and rational function in $D$ is rational in $z$. Hence it is enough to prove that $R^s$ is rational and symmetric in $D$.

In a labeled scheme, the offset label of a corner is defined as the difference between its label and the minimal label of the corners incident to the same vertex. Note that offset labels belong to $\{0, 1, 2\}$. In fact, these labels only depend on the unlabeled scheme. Hence, an unlabeled scheme has 2 different types of edges: if the offset labels are the same (01 or 12) on both sides, the edge is called level. If the labels are 01 on one side and 12 on the other, the edge is called offset toward the second one. We define the offset graph as the oriented sub-graph of the scheme where only the offset edges are kept, along with their orientation. An important step of the proof is to establish that the offset graph of a labeled scheme is acyclic.

To conclude, we observe that $R^s$ can be expressed as a sum over ordered partitions of $\mathcal{E}_s$ (edges of $s$). If $s$ has no offset edge, the symmetry can then be seen as a consequence of Euler-Poincaré’s formula applied to the permutahedron (whose faces are ordered partitions). The rationality for arbitrary unlabeled schemes requires a more detailed analysis of ordered partitions and relies crucially on the acyclicity of the offset graph.
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