Field-dependent BRST-antiBRST Transformations in Yang–Mills and Gribov–Zwanziger Theories

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Abstract

We introduce the notion of finite BRST-antiBRST transformations, both global and field-dependent, with a doublet \( \lambda_a, a = 1, 2 \), of anticommuting Grassmann parameters and find explicit Jacobians corresponding to these changes of variables in Yang–Mills theories. It turns out that the finite transformations are quadratic in their parameters. At the same time, exactly as in the case of finite field-dependent BRST transformations for the Yang–Mills vacuum functional, special field-dependent BRST-antiBRST transformations, with \( s_a \)-potential parameters \( \lambda_a = s_a \Lambda \) induced by a finite even-valued functional \( \Lambda \) and by the anticommuting generators \( s_a \) of BRST-antiBRST transformations, amount to a precise change of the gauge-fixing functional. This proves the independence of the vacuum functional under such BRST-antiBRST transformations. We present the form of transformation parameters that generates a change of the gauge in the path integral and evaluate it explicitly for connecting two arbitrary \( R_\xi \)-like gauges. For arbitrary differentiable gauges, the finite field-dependent BRST-antiBRST transformations are used to generalize the Gribov horizon functional \( h \), given in the Landau gauge, and being an additive extension of the Yang–Mills action by the Gribov horizon functional in the Gribov–Zwanziger model. This generalization is achieved in a manner consistent with the study of gauge independence. We also discuss an extension of finite BRST-antiBRST transformations to the case of general gauge theories and present an ansatz for such transformations.

Keywords: BRST-antiBRST Lagrangian quantization, gauge theories, Yang–Mills theory, Gribov–Zwanziger theory, field-dependent BRST-antiBRST transformations

1 Introduction

Contemporary quantization methods for gauge theories \cite{1, 2, 3, 4} are based primarily on the special supersymmetries known as BRST symmetry \cite{5, 6, 7} and BRST-antiBRST symmetry \cite{8, 9, 10, 11}. They are characterized by the presence of a Grassmann-odd parameter \( \mu \) and two Grassmann-odd parameters \( (\mu, \bar{\mu}) \), respectively. In the framework of the Sp(2)-covariant schemes of generalized Hamiltonian \cite{12, 13} and Lagrangian \cite{15, 16} quantization (see also
the parameters \((\mu, \bar{\mu}) = (\mu_1, \mu_2) = \mu_a\) form an \(Sp(2)\)-doublet. These infinitesimal odd-valued parameters may be regarded as constants and thus used to derive the Ward identities. They may also be chosen as field-dependent functionals and thus used to establish the gauge-independence of the corresponding vacuum functional in the path integral approach.

BRST transformations with a finite field-dependent parameter in Yang–Mills theories, whose quantum action is constructed by the Faddeev–Popov rules \([19]\), were first introduced in \([20]\) by means of a functional equation for the parameter in question, and used to provide the path integral with such a change of variables that would allow one to relate the quantum action in a certain gauge with the one given in a different gauge; see also \([21]\). This equation, as well as a similar equation \([22]\) for the finite parameter of a field-dependent BRST transformation in the generalized Hamiltonian formalism, has not been solved in the general setting of the problem. Namely, the corresponding equation \((4.13)\) in \([20]\), or equation \((3.6)\) in \([22]\), for the Jacobian \(J\) of a change of variables given by infinitesimal field-dependent BRST transformations with an odd-valued functional \(\Theta(\phi(\kappa))\) allows one to express an additional contribution \(S_1\) to the quantum action in terms of \(\Theta(\phi(0))\), but has not been solved neither in the form \(S_1 = S_1(\Theta(\phi(0)))\), for an unknown quantity \(S_1\), nor in the form \(S_1 = S_1(\Theta(\phi(0)))\), for an unknown quantity \(\Theta(\phi(0))\). Instead, a series of particular cases having the form of an ansatz for the functional \(S_1\) have been examined, and a solution of the above-mentioned equation was found without any explicit calculation of the Jacobian for the change of variables induced by finite field-dependent BRST transformations. On the other hand, there emerges the problem of establishing a relation of the Faddeev–Popov action in a certain gauge with the action in a different gauge, by using a change of variables induced by a finite field-dependent BRST transformation. This problem was solved for the first time in the case of linear and quadratic gauges in \([20]\) and for the class of general gauges in \([23]\), thereby providing an exact relation between a finite parameter and a finite variation of the gauge-fixing condition in terms of the gauge Fermion. There it was established that the Jacobian of any finite field-dependent BRST transformation reproduces BRST-exact terms, which can be entirely absorbed into the gauge-fixed part of the BRST-invariant Faddeev–Popov action, corresponding to a certain change of the gauge \(\Delta \psi\), so that the vacuum functional \(Z_{\psi + \Delta \psi}\), resulting from the above change of variables, coincides with the initial vacuum functional \(Z_{\psi}\) and should be regarded as a vacuum functional with the same BRST-exact classical (renormalized) action, having, however, a gauge-fixed (BRST-exact) action given by a different gauge, \(\psi + \Delta \psi\). In particular, this implies the conservation of the number of physical degrees of freedom in a given Yang–Mills theory with respect to finite field-dependent BRST transformations. This means the impossibility of relating the Yang–Mills theory to a theory whose action may contain, in addition to the Faddeev–Popov action, some BRST non-invariant terms (such as the Gribov horizon functional in the Gribov–Zwanziger theory \([31]\), having additional degrees of freedom as compared to the Yang–Mills theory) in the same configuration space.

The solution of a similar problem for arbitrary dynamical systems with first-class constraints in the generalized Hamiltonian formalism \([7, 27, 28]\) has been recently proposed in \([29]\). For general gauge theories, which may possess a reducible gauge symmetry and/or an open gauge algebra, an exact Jacobian corresponding to a change of variables given by field-dependent BRST transformations in the path integral constructed according to the Batalin–Vilkovisky (BV) procedure \([30]\) was obtained in \([31]\) and shown to be identical with the Jacobian of the Yang–Mills theory. The

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1. \(\Theta'(\phi(\kappa))\) depends on a numerical parameter, \(\kappa\), so that the finite field-dependent BRST transformations with the odd-valued functional \(\Theta(\phi(0))\) are obtained from \(\Theta'(\phi(\kappa))\) by \(\Theta(\phi(0)) = \frac{1}{\kappa} \Theta'(\phi(\kappa))dk\).

2. The property of gauge independence for the vacuum functional in the Yang–Mills theory with an action constructed by the Faddeev–Popov recipe \([19]\), or with an action constructed by the Batalin–Vilkovisky (BV) procedure \([30]\), uses an explicit form of the above Jacobian.

3. Instead of a local Gribov–Zwanziger horizon functional \(S_{\gamma}\) in \((3.3)\), there exists a relation \([24]\) by finite field-dependent BRST transformations to a BRST-invariant model with the functional \(\Sigma_{\gamma}\) in \((3.6)\), being a Yang–Mills theory defined in an appropriate configuration space.
study of \[31\] extends the results of \[23\] to first-rank theories with a closed algebra and solves the problem of gauge-independence for gauge theories with the so-called soft breaking of BRST symmetry. This problem was raised in \[32\] to study the problem of Gribov copies \[33\] by using various gauges in the Gribov–Zwanziger approach \[31\]; for recent progress, see \[35, 36, 37, 38, 39\].

On the other hand, there emerges the problem of finding a correspondence of the quantum action in the BRST-antiBRST invariant Lagrangian quantization \[15, 16, 17\], where gauge is introduced by a Bosonic gauge-fixing functional, \(F\), with the quantum action of the same theory in a different gauge, \(F + \Delta F\), for a finite value \(\Delta F\), by using a change of variables in the vacuum functional. This problem has not been solved even in theories of Yang–Mills type. Note that finite field-dependent antiBRST transformations in Yang–Mills theories were considered in \[25\] in the same way as in the case of BRST transformations \[20\], so as to relate the antiBRST invariant quantum action of a Yang–Mills theory in different gauges by using an ansat for a term introduced to the quantum action in order to satisfy an infinitesimal functional equation for the transformation parameter. The study of \[26\] proposed finite two-parametric BRST-antiBRST transformations (“mixed”, by the terminology \[26\]): \(\delta_m \phi = -s_a \Theta_1 + s_a \Theta_2\) in (3.7), including field-dependent ones, which form a Lie superalgebra; however, without any parameters, constant and/or field-dependent, being quadratic in \(\Theta_1, \Theta_2\) (allowing one to consider BRST-antiBRST transformations as group transformations), which prohibits the complete BRST-antiBRST invariance of the quantum action in Yang–Mills theories and similarly in more general gauge theories. Therefore, this leads immediately to the problem of finding a solution for the above functional equation, since the latter does not “feel” the finite polynomial character of the parameters \(\Theta_1, \Theta_2\), and therefore prohibits the gauge independence of the vacuum functional under finite field-dependent BRST-antiBRST transformations even for functionally-dependent parameters (see footnote 6).

A similar problem in the Sp (2)-covariant generalized Hamiltonian formalism \[12, 13\] remains unsolved as well. We expect that the solution of these problems in the Lagrangian and Hamiltonian quantization schemes for gauge theories should be based on the concept of finite BRST-antiBRST transformations with an Sp (2)-doublet of Grassmann-odd parameters \(\mu_a(\phi)\) depending on the field variables. This would allow one to generate the Gribov horizon functional by using different gauges in a way consistent with the gauge-independence of the path integral, based on the Gribov–Zwanziger prescription \[31\] and starting from the BRST-antiBRST invariant Yang–Mills quantum action in the Landau gauge.

Motivated by these reasons, we intend to address the following issues, paying our attention primarily to the Yang–Mills theory in Lagrangian formalism:

1. introduction of finite BRST-antiBRST transformations, being polynomial in powers of a constant Sp (2)-doublet of Grassmann-odd parameters \(\lambda_a\) and leaving the quantum action of the Yang–Mills theory invariant to all orders in \(\lambda_a\);

2. definition of finite field-dependent BRST-antiBRST transformations, being polynomial in powers of an Sp (2)-doublet of Grassmann-odd functionals \(\lambda_a(\phi)\) depending on the classical Yang–Mills fields, the ghost-antighost fields, and the Nakanishi–Lautrup fields; calculation of the Jacobian related to a change of variables by using a special class of such transformations with \(s_a\)-potential parameters \(\lambda_a(\phi) = s_a \lambda(\phi)\) for a Grassmann-even functional \(\Lambda(\phi)\) and Grassmann-odd generators \(s_a\) of BRST-antiBRST transformations;

3. solution of the so-called compensation equation for an unknown functional \(\Lambda\) generating the Sp (2)-doublet \(\lambda_a\) with the purpose of establishing a relation of the Yang–Mills quantum action \(S_F\) in a gauge determined by a gauge Boson \(F\) with the quantum action \(S_F + \Delta F\) in a different gauge \(F + \Delta F\);

\[^{4}\text{For the recent progress achieved in this area since the appearance of the present work in arXiv, see footnote 11 in Discussion.}\]
4. explicit construction of the parameters $\lambda_a$ of finite field-dependent BRST-antiBRST transformations generating a change of the gauge in the path integral within a class of linear $R_\xi$-like gauges realized in terms of Bosonic gauge functionals $F(\xi)$, with $\xi = 0, 1$ corresponding to the Landau and Feynman (covariant) gauges, respectively;

5. construction of the Gribov horizon functional $h_\xi$ in arbitrary $R_\xi$-like gauges by means of finite field-dependent BRST-antiBRST transformations starting from a known BRST-antiBRST non-invariant functional $h$, given in the Landau gauge and realized in terms of the Bosonic functional $F(0)$.

The present work is organized as follows. In Section 2 we remind the general setup of the BRST-antiBRST Lagrangian quantization of general gauge theories and list its basic ingredients. In Section 3 we introduce the notion of finite BRST-antiBRST transformations, both global and local (field-dependent). We find an explicit Jacobian corresponding to this change of variables in theories of Yang–Mills type and show that, exactly as in the case of field-dependent BRST transformations for the Yang–Mills vacuum functional, the field-dependent transformations amount to a precise change of the gauge-fixing functional. In Section 4 we present the form of transformation parameters that generates a change of the gauge and evaluate it for connecting two arbitrary $R_\xi$-like gauges in Yang–Mills theories. In Section 5 the Gribov horizon functional in an arbitrary $R_\xi$-like gauge, and generally in any differentiable gauge, is determined with the help of respective finite field-dependent BRST-antiBRST transformations. In Discussion, we make an overview of our results and outline some open problems. In particular, we discuss an extension of finite BRST-antiBRST transformations to the case of general gauge theories and present an ansatz for such transformations. In Appendix A we study the group properties of finite field-dependent BRST-antiBRST transformations. In Appendix B we present a detailed calculation of the Jacobian corresponding to the finite, both global (Appendix B.1) and field-dependent (Appendix B.2), BRST-antiBRST transformations. Appendix C is devoted to calculations involving the BRST-antiBRST invariant Yang–Mills action in $R_\xi$-gauges.

We use DeWitt’s condensed notations \cite{40}. By default, derivatives with respect to the fields are taken from the right, and those with respect to the corresponding antifields are taken from the left; otherwise, left-hand and right-hand derivatives are labelled by the subscripts “l” and “r”, respectively: $F_A$ stands for the right-hand derivative $\delta F / \delta \phi^A$ of a functional $F = F(\phi)$ with respect to $\phi^A$. The raising and lowering of Sp (2) indices, $s^a = \varepsilon^{ab} s_b$, $s_a = \varepsilon_{ab} s^b$, is carried out with the help of a constant antisymmetric second-rank tensor $\varepsilon^{ab}$, $\varepsilon^{ac} \varepsilon_{cb} = \delta^a_c$, subject to the normalization condition $\varepsilon^{12} = 1$. The Grassmann parity and ghost number of a quantity $A$, assumed to be homogeneous with respect to these characteristics, are denoted by $\varepsilon(A)$, gh($A$), respectively.

## 2 General Setup for BRST-antiBRST Lagrangian Quantization

The BRST-antiBRST Lagrangian quantization of general gauge theories \cite{15,16,17} involves a set of fields $\phi^A$ and a set of corresponding antifields $\phi^*_{Aa}$ ($a = 1, 2$), $\bar{\phi}_A$, where the doublets of antifields $\phi^*_{Aa}$ play the role of sources to the BRST and antiBRST transformations, while the antifields $\bar{\phi}_A$ are the sources to the mixed BRST and antiBRST transformations, with the following distributions of the Grassmann parity and ghost number:

$$
\varepsilon(\phi^A) \equiv \varepsilon_A \; , \; \varepsilon(\phi^*_{Aa}) = \varepsilon_A + 1 \; , \; \varepsilon(\bar{\phi}_A) = \varepsilon_A \; , \; \text{gh}(\phi^*_{Aa}) = (-1)^a - \text{gh}(\phi^A) \; , \; \text{gh}(\bar{\phi}_A) = -\text{gh}(\phi^A) \; .
$$

(2.1)

The configuration space of fields $\phi^A$ is identical with that of the BV formalism \cite{30} of covariant quantization and is determined by the properties of the initial classical theory. Namely, we consider an initial classical theory of fields $A^i$, $\varepsilon(A^i) \equiv \varepsilon_i$, with an action $S_0(A)$ invariant under gauge transformations,

$$
\delta A^i = R^i_{\alpha a}(A) \xi^\alpha \implies S_0, i(A) R^i_{\alpha a}(A) = 0 \; ,
$$

(2.2)
where $R^i_{\alpha_0}(A)$ are generators of the gauge transformations, $\varepsilon(R^i_{\alpha_0}) = \varepsilon_i + \varepsilon_{\alpha_0}$, and $\zeta^{\alpha_0}$ are arbitrary functions of the space-time coordinates, $\varepsilon(\zeta^{\alpha_0}) = \varepsilon_{\alpha_0}$. The generators $R^i_{\alpha_0}(A)$ form a gauge algebra \[30\] with the relations

\[
R^i_{\alpha_0,j}(A) R^j_{\beta_0}(A) = - (1)^{\varepsilon_{\alpha_0} \varepsilon_{\beta_0}} R^i_{\beta_0,j}(A) R^j_{\alpha_0}(A) = - R^i_{\alpha_0,j}(A) M^{\alpha_0}_{\beta_0} (A),
\]

\[
F^\alpha_{\alpha_0} = - (1)^{\varepsilon_{\alpha_0} \varepsilon_{\beta_0}} M^{\alpha_0}_{\beta_0} = - (1)^{\varepsilon_{\alpha_0} \varepsilon_{\beta_0}} M^{\alpha_0}_{\beta_0} = - (1)^{\varepsilon_{\alpha_0} \varepsilon_{\beta_0}} M^{\alpha_0}_{\beta_0}.
\]

In the case the vectors $R^i_{\alpha_0}(A)$, enumerated by the index $\alpha_0$, are linearly independent, the theory is irreducible; otherwise it is reducible. Depending on the (ir)reducibility of the generators of gauge transformations, the specific structure of the configuration space $\phi^A$ is described by the set of fields

\[
\phi^A = (A^i, B^{\alpha_s|a_1...a_s}, C^{\alpha_s|a_0...a_s}) , \quad s = 0, 1, ..., L
\]

where the ghost $C^{\alpha_s|a_0...a_s}$ and auxiliary $B^{\alpha_s|a_1...a_s}$ fields form symmetric Sp (2) tensors, being irreducible representations of the Sp (2) group, with the corresponding distribution \[16\] of the Grassmann parity and ghost number. These fields absorb the pyramids of ghost-antighost and Nakanishi–Lautrup fields of a given (ir)reducible gauge theory, where $L$ in \[24\] is the corresponding stage of reducibility \[30\], and $L = 0$ stands for an irreducible theory.

In the space of fields and antifields $(\phi^A, \phi^*_{\beta_0}, \phi_A)$, one introduces the basic object of the BRST-antBRST Lagrangian scheme, being an even-valued functional $S = S(\phi, \phi^*, \phi_A)$ subject to an Sp (2)-doublet of the generating equations \[15\]

\[
\frac{1}{2} (S, S)^a + V^a S = i\hbar \Delta^a S \iff \tilde{\Delta}^a \exp [(i/\hbar) S] = 0 , \quad \tilde{\Delta}^a = \Delta^a + (i/\hbar) V^a .
\]

Here, $\hbar$ is the Planck constant, whereas the extended antibracket $(\cdot, \cdot)^a$ and the operators $\Delta^a, V^a$ are given by

\[
(\cdot, \cdot)^a = \frac{\delta \cdot}{\delta \phi^A} \frac{\delta \cdot}{\delta \phi^*_{\beta_0}} - \frac{\delta \cdot}{\delta \phi^*_{\beta_0}} \frac{\delta \cdot}{\delta \phi^A} , \quad \Delta^a = (-1)^{\varepsilon_A} \frac{\delta}{\delta \phi^A} \frac{\delta}{\delta \phi^*_{\beta_0}} , \quad V^a = \varepsilon_{ab} \phi^*_{\beta_0} \frac{\delta}{\delta \phi_{\beta_0}} .
\]

The properties of the operators $\Delta^a, V^a, \tilde{\Delta}^a$ and those of the extended antibracket $(\cdot, \cdot)^a$ were investigated in \[15\]. The study of \[17\] proved the existence of solutions to \[24\] with the boundary condition $S|_{\phi^* = \phi_A = 0} = S_0$ in the form of an expansion in powers of $\hbar$ and described the arbitrariness in solutions, which is controlled by a transformation generated by the operators $\tilde{\Delta}^a$, connecting two solutions and describing the gauge-fixing procedure. A solution $S = S(\phi, \phi^*, \phi_A)$ of the generating equations \[24\] allows one to construct an extended (due to the antifields) generating functional of Green’s functions $Z(J, \phi^*, \phi_A)$ for the fields $\phi^A$ of the total configuration space \[15\], namely,

\[
Z(J, \phi^*, \phi_A) = \int d\phi \exp \left\{ \frac{i}{\hbar} \left[ S_{\text{ext}} (\phi, \phi^*, \phi_A) + J A (\phi^A) \right] \right\} .
\]

Hence, the generating functional of Green’s functions $Z(J) = Z(J, \phi^*, \phi_A)|_{\phi^* = \phi_A = 0}$ is given by

\[
Z(J) = \int d\phi \exp \left\{ \frac{i}{\hbar} \left[ S_{\text{eff}} (\phi) + J A (\phi^A) \right] \right\} , \quad S_{\text{eff}} (\phi) = S_{\text{ext}} (\phi, \phi^*, \phi_A) |_{\phi^* = \phi_A = 0} ,
\]

where $J_A, \varepsilon(J_A) = \varepsilon_A$, are external sources to the fields $\phi^A$, and $S_{\text{ext}} = S_{\text{ext}} (\phi, \phi^*, \phi_A)$ is an action constructed with the help of an even-valued gauge-fixing functional $F = F(\phi)$:

\[
\exp [(i/\hbar) S_{\text{ext}}] = \tilde{U} \exp [(i/\hbar) S] , \quad \text{with} \quad \tilde{U} = \exp \left( F_A \frac{\delta}{\delta \phi_A} + \frac{i\hbar}{2} \varepsilon_{ab} \frac{\delta}{\delta \phi^*_{\beta_0}} F_{AB} \frac{\delta}{\delta \phi^*_{\beta_0}} \right).
\]

Due to the commutativity of $\tilde{\Delta}^a$ and $\tilde{U}$, the gauge-fixing procedure retains the form of the generating equations \[25\],

\[
\tilde{\Delta}^a \exp [(i/\hbar) S_{\text{ext}}] = 0 .
\]
A possible choice of the gauge-fixing functional $F(\phi)$ has the form of the most general $\text{Sp}(2)$-scalar being quadratic in the ghost and auxiliary fields $[15]$. Introducing a set of auxiliary fields $\pi^Aa$ and $\lambda^A$,

$$
\varepsilon(\pi^Aa) = \varepsilon_A + 1, \quad \varepsilon(\lambda^A) = \varepsilon_A, \quad \text{gh}(\pi^Aa) = -(\lambda^A)^a + \text{gh}(\phi^A), \quad \text{gh}(\lambda^A) = \text{gh}(\phi^A),
$$

(2.11)

one can represent $Z(J)$ as a functional integral in the extended space of variables $[15]$:

$$
Z(J) = \int d\Gamma \exp\left\{ \frac{i}{\hbar} \left[ S + \phi_{Aa}^* \pi^Aa + (\phi_A - F_A) \lambda^A - (1/2) \varepsilon_{ab} \pi^Aa F_{AB} \pi^Bb + J_A \phi^A \right] \right\},
$$

(2.12)

where $d\Gamma = d\phi \, d\phi^* \, d\bar{\phi} \, d\lambda \, d\pi$ is the integration measure.

An important property of the integrand in (2.12) for $J_A = 0$ is its invariance under the following infinitesimal transformations of global supersymmetry:

$$
\delta \left( \phi^A, \phi_{Aa}^*, \bar{\phi}_A, \pi^Aa, \lambda^A \right) = (\pi^Aa \mu_a, \mu_a S_A, \varepsilon_{ab} \mu_a \phi_{Ab}^*, -\varepsilon_{ab} \lambda^A \mu_b, 0),
$$

(2.13)

where $\mu_a$ is a doublet of constant anticommuting Grassmann parameters, $\mu_a \mu_b + \mu_b \mu_a \equiv 0$. The transformations (2.13) realize the BRST-antiBRST transformations in the extended space $(\phi^A, \phi_{Aa}^*, \bar{\phi}_A, \pi^Aa, \lambda^A)$.

The symmetry of the integrand in (2.12) for $J_A = 0$ under the transformations (2.13) with constant infinitesimal $\mu_a$ allows one to derive the following Ward identities in the extended space:

$$
J_A \langle \pi^Aa \rangle_{F,J} = 0,
$$

(2.14)

for $\langle \mathcal{O} \rangle_{F,J} = Z^{-1}(J) \int d\Gamma \mathcal{O} \exp\left\{ \frac{i}{\hbar} \left[ S + \phi_{Aa}^* \pi^Aa + (\phi_A - F_A) \lambda^A - (1/2) \varepsilon_{ab} \pi^Aa F_{AB} \pi^Bb + J_A \phi^A \right] \right\},
$$

where the expectation value of a functional $\mathcal{O}(\Gamma)$ is given in the extended space parameterized by $\Gamma$ with a gauge $F(\phi)$ in the presence of external sources $J_A$. To obtain (2.14), we subject (2.12) to a change of variables $\Gamma \to \Gamma + \delta \Gamma$ with $\delta \Gamma$ given by (2.13) and use the equations (2.5) for $S$. At the same time, with allowance for the equivalence theorem [11], the transformations (2.13) permit one to establish the independence of the $S$-matrix from the choice of a gauge. Indeed, suppose $Z_F \equiv Z(0)$ and change the gauge, $F \to F + \Delta F$, by an infinitesimal value $\Delta F$. In the functional integral for $Z_{F+\Delta F}$ we now make the change of variables (2.13). Then, choosing the parameters $\mu_a$ as

$$
\mu_a = -\frac{i}{2\hbar} \varepsilon_{ab} (\Delta F)_{,A} \pi^{Ab},
$$

(2.15)

we find that $Z_{F+\Delta F} = Z_F$, and therefore the $S$-matrix is gauge-independent.

For the purpose of a subsequent treatment of Yang–Mills theories, we need the particular case of solutions to the generating equations (2.5) given by a functional $S = S(\phi, \phi^*, \bar{\phi})$ linear in the antifields. Namely, we assume

$$
S = S_0 + \phi_{Aa}^* X^{Aa} + \bar{\phi}_A Y^A,
$$

(2.16)

which implies

$$
S_{0,i} X^{ia} = 0, \quad X_{A,B}^{Aa} X^{Bb} = \varepsilon^{ab} Y^A, \quad Y_{,A}^B X^{Aa} = 0, \quad X_{,A}^{Aa} = 0
$$

(2.17)

and allows one to present $S$ in the form

$$
S = S_0 + \phi_{Aa}^* \left( s^a \phi^A \right) - \frac{1}{2} \bar{\phi}_A \left( s^2 \phi^A \right), \quad s^2 \equiv s_a s^a,
$$

(2.18)

where $s^a$ are generators of BRST-antiBRST transformations,

$$
\delta \phi^A = (s^a \phi^A) \mu_a, \quad s^a \phi^A = X^{Aa},
$$

(2.19)
and $s^2$ are generators of mixed BRST-antiBRST transformations,

$$\delta^2 \phi^A = s^a \left( s^b \phi^A \mu_b \right) \mu_a = -\frac{1}{2} \left( s^2 \phi^A \right) \mu^2, \quad s^2 \phi^A = \varepsilon_{ab} X_{BA} X^{Bb} = -2Y^A. \quad (2.20)$$

The explicit form of $X^{Aa}$ and $Y^A$ for theories of Yang–Mills type was found in [15] and is given in Appendix C.

For a solution of (2.5) linear in the antifields, integration in (2.12) over $\phi^A_{\lambda A}$, $\tilde{\phi} A$, $\pi^A$, $\lambda^A$ is trivial [15]:

$$Z(J) = \int d\phi \exp \left\{ \frac{i}{\hbar} \left[ S_F(\phi) + J_A \phi^A \right] \right\}, \quad (2.21)$$

where

$$S_F(\phi) = S_0(A) + F_A Y^A - (1/2) \varepsilon_{ab} X^{Aa} F_{AB} X^{Bb}, \quad (2.22)$$

which can also be established directly by inserting the solution (2.16) into (2.21).

The quantum action $S_F(\phi)$ can be presented in terms of a mixed BRST-antiBRST variation,

$$S_F(\phi) = S_0(A) - (1/2) s^2 F(\phi), \quad (2.23)$$

where the operators $s^a$, acting on an arbitrary functional $V = V(\phi)$ of any Grassmann parity, define a BRST-antiBRST analogue of the Slavnov variation, $s^a V = V_A \left( s^a \phi^A \right)$. Thus defined operators $s^a$ are anticommuting, $s^a s^b + s^b s^a \equiv 0$, for any $a, b = 1, 2$,

$$s^a s^b V = \varepsilon^{ab} W, \quad W \equiv (1/2) \varepsilon_{ab} V_{BA} X^{Aa} X^{Bb} (-1)^{e_B} V_A Y^A, \quad s^a s^b V = (1/2) \varepsilon^{ab} s^2 V, \quad W = (1/2) s^2 V, \quad (2.24)$$

and therefore nilpotent, $s^a s^b s^c \equiv 0$, which proves the invariance of $S_F$ given by (2.23) under the infinitesimal transformations (2.19),

$$\delta S_F = (S_F)_A \delta \phi^A = (s^a S_F) \mu_a = (s^a S_0) \mu_a - \frac{1}{2} \left( s^a s^2 F \right) \mu_a = 0,$$

by virtue of the condition $s^a S_0 = S_0, X^a_0, Y^A = 0$ from (2.17), being a consequence of the Noether identities (2.22).

In view of the condition $X^{Aa}_{\lambda A} = 0$ from (2.17), the integration measure in (2.21) is also invariant under the transformations (2.19), which ensures the invariance of the integrand in (2.21) for $J_A = 0$ under (2.19). By analogy with the previous consideration, this allows one to establish the Ward identities for $Z(J)$ in (2.21),

$$J_A \langle s^a \phi^A \rangle_{F,J} = J_A \langle X^{Aa}(\phi) \rangle_{F,J} = 0 \quad \text{for} \quad \langle \mathcal{O} \rangle_{F,J} = Z^{-1}(J) \int d\phi \mathcal{O}(\phi) \exp \left\{ \frac{i}{\hbar} \left[ S_F(\phi) + J_A \phi^A \right] \right\}, \quad (2.25)$$

as well as the independence of the $S$-matrix from the choice of a gauge. Indeed, suppose $Z_F \equiv Z(0)$ in (2.22) and change the gauge $F \rightarrow F + \Delta F$ by an infinitesimal value $\Delta F$. Then, making in $Z_{F+\Delta F}$ the change of variables (2.19) with the field-dependent infinitesimal parameters

$$\mu_a = \frac{i}{2\hbar} \varepsilon_{ab} (\Delta F)_A X^{Ab} = \frac{i}{2\hbar} (s^a \Delta F), \quad (2.26)$$

being a particular case of the field-dependent BRST-antiBRST transformations studied in the following section, we find $Z_{F+\Delta F} = Z_F$, which establishes the gauge-independence of the $S$-matrix.

### 3 Finite BRST-antiBRST Transformations and their Jacobians

Let us introduce finite transformations of the fields $\phi^A$ with a doublet $\lambda_a$ of anticommuting Grassmann parameters, $\lambda_a \lambda_b + \lambda_b \lambda_a = 0$,

$$\phi^A \rightarrow \phi'^A = \phi^A + \Delta \phi^A = \phi'^A (\phi | \lambda), \quad \text{so that} \quad \phi'^A (\phi | 0) = \phi^A. \quad (3.1)$$
In the general case, such transformations are quadratic in the parameters, due to \( \lambda_a \lambda_b \lambda_c \equiv 0 \),

\[
\phi'^A (\phi|\lambda) = \phi^A (\phi|0) + \left[ \frac{\partial}{\partial \lambda_a} \phi^A (\phi|\lambda) \right] \lambda_a \left[ \frac{\partial}{\partial \lambda_b} \phi^A (\phi|\lambda) \right] \lambda_b , \tag{3.2}
\]

which implies

\[
\Delta \phi^A = Z^{Aa} \lambda_a + (1/2) Z^A \lambda^2 , \quad \text{where} \quad \lambda^2 \equiv \lambda_a \lambda^a , \tag{3.3}
\]

for certain functions \( Z^{Aa} = Z^{Aa} (\phi) \), \( Z^A = Z^A (\phi) \), corresponding to the first- and second-order derivatives of \( \phi^A (\phi|\lambda) \) with respect to \( \lambda_a \) in \( \lambda_a \).

In view of the obvious property of nilpotency \( \Delta \phi^A_1 \cdots \Delta \phi^A_n \equiv 0 \), \( n \geq 3 \), an arbitrary functional \( F (\phi) \) under the above transformations \( \phi^A \to \phi'^A + \Delta \phi^A \) can be expanded as

\[
F (\phi + \Delta \phi) = F (\phi) + F_A (\phi) \Delta \phi^A + (1/2) F_{AB} (\phi) \Delta \phi^B \Delta \phi^A . \tag{3.4}
\]

Based on (3.1)–(3.4), we now introduce \textit{finite BRST-antiBRST transformations} as invariance transformations of the quantum action \( S_F (\phi) \) given by (2.23) under finite transformations of the fields \( \phi^A \), such that

\[
S_F (\phi + \Delta \phi) = S_F (\phi) , \quad \left[ \frac{\partial}{\partial \lambda_a} \Delta \phi^A \right]_{\lambda=0} = s^a \phi^A \quad \text{and} \quad \left[ \frac{\partial}{\partial \lambda_a} \frac{\partial}{\partial \lambda_b} \Delta \phi^A \right]_{\lambda=0} = \frac{1}{2} \varepsilon^{ab} s^2 \phi^A , \tag{3.5}
\]

which implies \( Z^{Aa} = s^a \phi^A = X^{Aa} \) and \( Z^A = (1/2) s^2 \phi^A = -Y^A \), according to (2.19)–(2.20), (3.3).

One can easily verify the consistency of definition (3.5) by considering the equation, implied by \( \Delta S_F = 0 \),

\[
(S_F)_A X^{Aa} \lambda_a - \frac{1}{2} Y^A \lambda^2 + \frac{1}{2} (S_F)_{AB} X^{Bb} \lambda_b - \frac{1}{2} Y^B \lambda^2 \left( X^{Aa} \lambda_a - \frac{1}{2} Y^A \lambda^2 \right) = 0 . \tag{3.6}
\]

Taking into account the fact \( \lambda_a \lambda^2 = \lambda^4 \equiv 0 \), the invariance relations \((S_F)_A X^{Aa} = 0\), and their differential consequences \((S_F)_{AB} X^{Bb} \lambda_a X^{Aa} \lambda_a = (S_F)_{A} Y^A \lambda^2\), implied by the relations \( Y^A = (1/2) X^{Aa} X^{Bb} \varepsilon_{ba} \) from (2.20), we find that the above equation is satisfied identically:

\[
(S_F)_A X^{Aa} \lambda_a - \frac{1}{2} (S_F)_A Y^A \lambda^2 + \frac{1}{2} (S_F)_{AB} X^{Bb} \lambda_b X^{Aa} \lambda_a \equiv 0 . \quad \text{Explicity, the finite BRST-antiBRST transformations can be presented as} \ \ [3.7]
\]

\[
\Delta \phi^A = X^{Aa} \lambda_a - \frac{1}{2} Y^A \lambda^2 = (s^a \phi^A) \lambda_a + \frac{1}{4} (s^2 \phi^A) \lambda^2 , \tag{3.7}
\]

which implies that the finite variation \( \Delta \phi^A \) includes the generators of BRST-antiBRST transformations \( (s^1, s^2) \), as well as their commutator \( s^2 = \varepsilon_{ab} s^a s^b = s^1 s^2 - s^2 s^1 \).

According to (2.21), (3.4), (3.7) and \( \lambda_a \lambda^2 = \lambda^4 \equiv 0 \), the variation \( \Delta F (\phi) \) of an arbitrary functional \( F (\phi) \) under the finite BRST-antiBRST transformations is given by

\[
\Delta F = F_A X^{Aa} \lambda_a - \frac{1}{2} F_A Y^A \lambda^2 + \frac{1}{2} F_{AB} X^{Bb} \lambda_b X^{Aa} \lambda_a
\]

\[
= \left( F_A X^{Aa} \right) \lambda_a + \frac{1}{2} \left[ \frac{1}{2} \varepsilon_{ab} F_{BA} X^{Aa} X^{Bb} (-1) \varepsilon_{a} - F_A Y^A \right] \lambda^2 = (s^a F) \lambda_a + \frac{1}{4} (s^2 F) \lambda^2 . \tag{3.8}
\]

This relation allows one to study the group properties of finite BRST-antiBRST transformations \( \delta_1, \delta_2 \), with account taken for the fact that these transformations do not form a Lie superalgebra, nor a vector superspace structure, due to the presence of the term which is quadratic in \( \lambda_a \). Namely, we have (for details, see Appendix A)

\[
\Delta (\delta_1, \delta_2) F = (s^a \delta_2 (F) \lambda_{a(1)} + \frac{1}{4} (s^2 \delta_2 (F) \lambda_{a(2)} = (s^a F) \theta_{(1,1)} + \frac{1}{4} (s^2 F) (\theta_{(1,2)} , (3.9)
\]
for certain functionals $\delta \varphi (1, 2) = \varphi_{(1, 2)} (\phi)$ and $\theta (1, 2) = \theta_{(1, 2)} (\phi)$, constructed explicitly in (A.7), (A.8) from the parameters of finite transformations, which are generally field-dependent, $\lambda_{(j)} = \lambda_{(j)} (\phi)$, for $j = 1, 2$. Therefore, the commutator of finite variations has the form

$$\left[ \Delta_{(1)}, \Delta_{(2)} \right] F = (s^a F) \varphi_{(1, 2)}[\alpha] + \frac{1}{4} (s^2 F) \theta_{(1, 2)}, \quad \delta \varphi_{(1, 2)} \equiv \varphi_{(1, 2)} - \varphi_{(2, 1)}, \quad \theta_{(1, 2)} \equiv \theta_{(1, 2)} - \theta_{(2, 1)}, \quad (3.10)$$

where $\varphi_{(1, 2)}$, $\theta_{(1, 2)}$ are given explicitly by (A.11), (A.12) and possess the symmetry properties $\varphi_{(1, 2)} = - \varphi_{(2, 1)}$, $\theta_{(1, 2)} = - \theta_{(2, 1)}$. In particular, assuming $F (\phi) = \phi^A$ in (3.10), we have

$$\left[ \Delta_{(1)}, \Delta_{(2)} \right] \phi^A = (s^a \phi^A) \varphi_{(1, 2)}[\alpha] + \frac{1}{4} (s^2 \phi^A) \theta_{(1, 2)} . \quad (3.11)$$

In general, the commutator (3.11) of finite non-linear transformations (3.7) does not belong to the class of these transformations, due to the opposite symmetry properties of $\varphi_{(1, 2)}$ and $\theta_{(1, 2)}$,

$$\varphi_{(1, 2)} \equiv \varphi_{(2, 1)}, \quad \theta_{(1, 2)} \equiv - \theta_{(2, 1)}, \quad (3.12)$$

which reflects the fact that a finite BRST-antiBRST transformation looks as a group element, i.e., not as an element of a Lie superalgebra; however, the linear approximation $\Delta^\text{lin} \phi^A = (s^a \phi^A) \lambda_a$ to a finite transformation $\Delta \phi^A = \Delta^\text{lin} \phi^A + O (\lambda^2)$ does form an algebra. Indeed, due to (A.9), (A.11), (A.12), we have

$$\left[ \Delta^\text{lin}_{(1)}, \Delta^\text{lin}_{(2)} \right] F = \Delta^\text{lin}_{(1, 2)} F = (s^a F) \lambda_{(1, 2)}[\alpha], \quad \lambda_{(1, 2)} = (s_b \lambda_{(1)}^b) \lambda_{(2)}^b - (s_b \lambda_{(2)}^b) \lambda_{(1)}^b . \quad (3.13)$$

Thus, the construction of finite BRST-antiBRST transformations (3.7) reduces to the usual BRST-antiBRST transformations (2.19), $\delta \phi^A = \Delta^\text{lin} \phi^A$, linear in the infinitesimal parameter $\mu_a = \lambda_a$, as one selects in (3.7) the approximation that forms an algebra with respect to the commutator.

Let us now consider the modification of the integration measure $d \phi \rightarrow d \phi'$ in (2.21) under the finite transformations $\phi^A \rightarrow \phi'^A = \phi^A + \Delta \phi^A$, with $\Delta \phi^A$ given by (3.7),

$$d \phi' = d \phi \left( \text{Sdet} \left( \frac{\delta \phi'}{\delta \phi} \right) \right), \quad \text{with Sdet} \left( \frac{\delta \phi'}{\delta \phi} \right) = \text{Sdet} (\mathbb{I} + M) = \exp \left[ \text{Str ln} (\mathbb{I} + M) \right] \equiv \exp (\mathbb{3}) , \quad (3.14)$$

where the Jacobian $\exp (\mathbb{3})$ has the form

$$\mathbb{3} = \text{Str ln} (\mathbb{I} + M) = - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{Str} (M^n), \quad \text{for Str} (M^n) = (M^n)^A A (-1)^A \quad \text{and} \quad M^n_A = \frac{\delta (\Delta \phi^A)}{\delta \phi^B} . \quad (3.15)$$

In the case of global finite transformations, corresponding to $\lambda_a = \text{const}$, the integration measure remains invariant (for details, see Appendix B.3.1)

$$\mathbb{3} (\phi) = 0 \quad \Rightarrow \quad \left( \text{Sdet} \left( \frac{\delta \phi'}{\delta \phi} \right) = 1 \quad \text{and} \quad d \phi' = d \phi \right) . \quad (3.16)$$

Due to the invariance of the quantum action $S_F = S_0 + (1/2) s^a s_a F$ under $\phi^A \rightarrow \phi'^A$ the above implies that the integrand with the vanishing sources $I_{\phi} \equiv d \phi \exp [(i/\hbar) S_F]$ in (2.21) is also invariant, $I_{\phi'} = I_{\phi}$, under the transformations (3.7), which justifies their interpretation as finite BRST-antiBRST transformations.

As we turn to finite field-dependent transformations, let us examine the particular case $\lambda_a (\phi) = s_a \Lambda (\phi)$ with a certain even-valued potential, $\Lambda = \Lambda (\phi)$, which is inspired by infinitesimal field-dependent BRST-antiBRST transformations with the parameters (2.20). In this case, the integration measure takes the form (relation (3.18) is deduced

---

Notice that the parameters $\lambda_a$, $a = 1, 2$, in the case $\lambda_a = s_a \Lambda$ are not functionally independent: $s^1 \lambda_1 + s^2 \lambda_2 = - s^2 \Lambda$.  


in Appendix \[B.2\]

\[
\Im (\phi) = -2\ln [1 + f (\phi)], \quad \text{with} \quad f (\phi) = -\frac{1}{2} s^2 \Lambda (\phi), \quad \text{for} \quad s^a s_a = -s^2, \quad (3.17)
\]

\[
d\phi' = d\phi \exp \left[ \frac{i}{\hbar} (-i\hbar \Im) \right] = d\phi \exp \left\{ \frac{i}{\hbar} \left[ i\hbar \ln \left( 1 + \frac{1}{2} s^a s_a \Lambda \right) \right] ^2 \right\}. \quad (3.18)
\]

In view of the invariance of the quantum action \( S_F (\phi) \) under \( (3.7) \), the change \( \phi^A \to \phi'^A = \phi^A + \Delta \phi^A \) induces in \( (2.21) \) the following transformation of the integrand with the vanishing sources, \( \mathcal{I}_\phi \equiv d\phi \exp [(i/\hbar) S_F (\phi)]:\)

\[
\mathcal{I}_{\phi + \Delta \phi} = d\phi \exp [\Im (\phi)] \exp [(i/\hbar) S_F (\phi + \Delta \phi)] = d\phi \exp \{(i/\hbar) [S_F (\phi) - i\hbar \Im (\phi)]\}, \quad (3.19)
\]

whence

\[
\mathcal{I}_{\phi + \Delta \phi} = d\phi \exp \left\{ (i/\hbar) \left[ S_F (\phi) + i\hbar \ln \left( 1 + s^a s_a \Lambda (\phi) \right) /2 \right] ^2 \right\}. \quad (3.20)
\]

Due to the explicit form of the initial quantum action \( S_F = S_0 + (1/2) s^a s_a F \), the BRST-antiBRST-exact contribution \( i\hbar \ln \left( 1 + s^a s_a \Lambda /2 \right) ^2 \) to the action \( S_F \), resulting from the transformation of the integration measure, can be interpreted as a change of the gauge-fixing functional made in the original integrand \( \mathcal{I}_\phi \),

\[
i\hbar \ln \left( 1 + s^a s_a \Lambda /2 \right) ^2 = s^a s_a (\Delta F/2) \quad (3.21)
\]

\[
\implies \mathcal{I}_{\phi + \Delta \phi} = d\phi \exp \{(i/\hbar) [S_F + (1/2) s^a s_a (F + \Delta F)]\} = \mathcal{I}_\phi |_{F \to F + \Delta F}, \quad (3.22)
\]

for a certain \( \Delta F (\phi) \), whose relation to \( \Lambda (\phi) \) is discussed below. In other words, the field-dependent transformations with the parameters \( \lambda_a = s_a \Lambda \) amount to a **precise change of the gauge-fixing functional**. As a consequence, the integrand in \( (2.21) \) for \( J_A = 0 \), corresponding to the quantum action \( S_{F + \Delta F} = S_0 + (1/2) s^a s_a (F + \Delta F) \) with a modified gauge-fixing functional, is invariant under both the infinitesimal, \( \delta \phi^A \), and finite, \( \Delta \phi^A \), BRST-antiBRST transformations, with constant parameters \( \mu_a \) and \( \lambda_a \) in \( (2.19) \) and \( (3.7) \), respectively.

Let us denote by \( T^{(\Delta F)} \) the operation that transforms an integrand \( \mathcal{I}_\phi^{(F)} \) into \( \mathcal{I}_\phi^{(F + \Delta F)} \), corresponding to the respective gauge-fixing functionals \( F \) and \( F + \Delta F \),

\[
T^{(\Delta F)} : \mathcal{I}_\phi^{(F)} \to \mathcal{I}_\phi^{(F + \Delta F)}, \quad (3.23)
\]

which implies an additive composition law:

\[
T^{(\Delta F_1)} \circ T^{(\Delta F_2)} = T^{(\Delta F_2)} \circ T^{(\Delta F_1)} = T^{(\Delta F_1 + \Delta F_2)} . \quad (3.24)
\]

As we denote by \( \Lambda^{(\Delta F)} \) the gauge-fixing functional corresponding to \( \Delta F \), there follow the properties

\[
\ln \left( 1 + s^a s_a \Lambda^{(\Delta F_1 + \Delta F_2)} /2 \right) ^2 = \ln \left( 1 + s^a s_a \Lambda^{(\Delta F_1)} /2 \right) ^2 + \ln \left( 1 + s^a s_a \Lambda^{(\Delta F_2)} /2 \right) ^2 , \quad \Lambda^{(0)} = 0 , \quad (3.25)
\]

implying relations between \( s^2 \Lambda^{(\Delta F_1 + \Delta F_2)} \) and \( s^2 \Lambda^{(\Delta F_j)} \) for \( j = 1, 2 \), as well as between \( s^2 \Lambda^{(-\Delta F)} \) and \( s^2 \Lambda^{(\Delta F)} \):

\[
s^2 \Lambda^{(\Delta F_1 + \Delta F_2)} = s^2 \left( \Lambda^{(\Delta F_1)} + \Lambda^{(\Delta F_2)} \right) - s^2 \Lambda^{(\Delta F_1)} \left( s^2 \Lambda^{(\Delta F_2)} /2 \right) , \quad (3.26)
\]

\[
s^2 \Lambda^{(-\Delta F)} = - \left( s^2 \Lambda^{(\Delta F)} \right) \left[ 1 - \left( s^2 \Lambda^{(\Delta F)} /2 \right) ^{-1} \right] . \quad (3.27)
\]

The relation \( (3.21) \) between the potential \( \Lambda (\phi) \) and the variation \( \Delta F (\phi) \) of the gauge-fixing functional can be considered as a compensation equation (for the unknown functional \( \Delta F (\phi) \), with a given \( \Lambda (\phi) \), and vice versa),

\[
i\hbar \ln \left( 1 + s^a s_a \Lambda (\phi) /2 \right) ^2 = s^a s_a \Delta F (\phi) /2 , \quad (3.28)
\]
whose solution, up to BRST-antiBRST-exact terms, has the form

$$\Delta F (\phi) = 2i\hbar \Lambda (\phi) \left(s^a s_a \Lambda (\phi)\right)^{-1} \ln \left(1 + s^a s_a \Lambda (\phi)/2\right)^2 .$$  \hspace{1cm} (3.29)$$

The relation (3.28) can be inverted as an equation for $\Lambda (\phi)$, namely,

$$s^a s_a \Lambda = 2 \left[\exp \left(\frac{1}{4i\hbar} s^a s_a \Delta F\right) - 1\right] .$$  \hspace{1cm} (3.30)$$

Up to BRST-antiBRST-exact terms, its solution reads

$$\Lambda = 2\Delta F (s^a s_a \Delta F)^{-1} \left[\exp \left(\frac{1}{4i\hbar} s^a s_a \Delta F\right) - 1\right] = \frac{1}{2i\hbar} \Delta F \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left(\frac{1}{4i\hbar} s^a s_a \Delta F\right)^n ,$$  \hspace{1cm} (3.31)$$

whence

$$\lambda_a = s_a \Lambda = \frac{1}{2i\hbar} (s_a \Delta F) \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left(\frac{1}{4i\hbar} s^b s_b \Delta F\right)^n$$

$$= \frac{1}{2i\hbar} (s_a \Delta F) \left[1 + \frac{1}{2!} \left(\frac{1}{4i\hbar} s^b s_b \Delta F\right) + \frac{1}{3!} \left(\frac{1}{4i\hbar} s^b s_b \Delta F\right)^2 + \frac{1}{4!} \left(\frac{1}{4i\hbar} s^b s_b \Delta F\right)^3 + \ldots \right].$$  \hspace{1cm} (3.32)$$

In particular, the first order of $\lambda_a = \mu_a$ in powers of $\Delta F$ has the form

$$\mu_a = -\frac{i}{2\hbar} (s_a \Delta F) .$$  \hspace{1cm} (3.33)$$

Using (3.32), one can construct a finite BRST-antiBRST transformation that connects two quantum theories of Yang–Mills type corresponding to some gauge-fixing functionals $F$ and $F + \Delta F$ for a given finite variation $\Delta F$. The symmetry of the integrand in (2.21) for $J_A = 0$ under the transformations (3.7) allows one to establish the independence of the $S$-matrix from the choice of a gauge. Indeed, suppose $Z_F \equiv Z(0)$ and change the gauge $F \to F + \Delta F$ by a finite value $\Delta F$. In the functional integral for $Z_{F+\Delta F}$ we now make the change of variables (3.37). Then, selecting the parameters $\lambda_a = \mu_a$ to meet the condition

$$i\hbar \ln \left(1 + s^a s_a \Lambda/2\right)^2 = -(1/2) s^a s_a \Delta F ,$$  \hspace{1cm} (3.34)$$

cf. (3.28), we find that $Z_{F+\Delta F} = Z_F$, whence, due to the equivalence theorem [41], the $S$-matrix is gauge-independent.

In the particular case of an infinitesimal variation $\Delta F$, condition (3.34) produces, in virtue of (3.33), precisely the form (2.20) of field-dependent parameters $\lambda_a = \mu_a$ in the framework of infinitesimal BRST-antiBRST transformations.

As we identify $\lambda_a = s_a \Lambda$ with a solution of (3.28), $\Lambda^{(\Delta F)} \equiv \Lambda (\Delta F)$, the representation (2.21) describes the dependence of the functional $Z_F (J)$ on a finite variation of the gauge:

$$\Delta Z_F (J) = \frac{i}{\hbar} Z_F (J) \left(J_A \left[(s^a \phi^A) s_a \Lambda (-\Delta F) + \frac{1}{4} (s^2 \phi^A) \left[s \Lambda (-\Delta F)\right]^2 + \frac{i}{4\hbar} \varepsilon_{ab} (s^a \phi^A) J_B (s^b \phi^B) \left[s \Lambda (-\Delta F)\right]^2\right]\right)_{F,J} ,$$  \hspace{1cm} (3.35)$$

where $\Delta Z_F (J) \equiv Z_{F+\Delta F} (J) - Z_F (J)$. The above relation (3.35) generalizes the gauge-dependence of $Z(J)$ in Yang–Mills type theories to the case of finite variations of the gauge.

### 4 Correspondence between Gauges in Yang–Mills Theories

In this section, we consider the Yang–Mills theory, given by the action

$$S_0 (A) = -\frac{1}{4} \int d^D x \, F_{\mu \nu}^m F^{m \mu \nu} , \quad \text{for} \quad F_{\mu \nu}^m = \partial_\mu A_\nu^m - \partial_\nu A_\mu^m + f^{mnl} A_\mu^n A_\nu^l ,$$  \hspace{1cm} (4.1)$$

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with the Lorentz indices \( \mu, \nu = 0, 1, \ldots, D-1 \), the metric tensor \( \eta_{\mu \nu} = \text{diag}(-, +, \ldots, +) \), and the totally antisymmetric \( su(N) \) structure constants \( f^{lmn} \) for \( l, m, n = 1, \ldots, N^2 - 1 \).

The action (4.1) is invariant under the gauge transformations

\[
\delta A^m_\mu (x) = D^m_\mu (x) \zeta^m (x) = \int d^D y \, R^m_\mu (x; y) \zeta^m (y) , \quad D^m_\mu = \delta^m_n \partial_\mu + f^{mln} A^l_\mu ,
\]

with arbitrary Bosonic functions \( \zeta^m (y) \) in \( \mathbb{R}^{1,D-1} \), the covariant derivative \( D^m_\mu \), and the generators \( R^m_\mu (x; y) = R^m_\alpha \) of the gauge transformations, the condensed indices being \( i = (\mu, m, x), \alpha = (n, y) \). The generators \( R^i_\alpha \) in (4.2) form a closed gauge algebra with \( M^i_{\alpha \beta} = 0 \) in (2.3), whereas the structure coefficients \( F^i_{\alpha \beta} \) arising in (2.3) are given by

\[
F^i_{\alpha \beta} = f^{lmn} \delta(x-z) \delta(y-z) , \quad \text{for} \quad \alpha = (m, x) , \beta = (n, y) , \gamma = (l, z) .
\]

The total configuration space of fields \( \phi^A \) and the corresponding antifields \( \phi^*_A, \bar{\phi}_A \) of the theory are given by

\[
\phi^A = (A^{\mu m}, B^m, C^{ma}) , \quad \phi^*_A = (A^a_{\mu m}, B^a_{m}, C^{ab}) , \quad \bar{\phi}_A = (\bar{A}^m_{\mu}, \bar{B}^m, \bar{C}^m_{A}) .
\]

With allowance made for (2.1), the Grassmann parity and ghost number assume the values

\[
\varepsilon (\phi^A) \equiv (0, 0, 1) , \quad \text{gh} (\phi^A) = \left( 0, 0, (-1)^{a+1} \right) .
\]

The generating equations (2.3) with the boundary condition \( S|_{\phi^* = \bar{\phi} = 0} = S_0 \) are solved by a functional linear in the antifields (for details, see (C.23), (C.4) in Appendix C)

\[
S = S_0 + \int d^D x \left( A^{\mu m}_{1} X^1_{\mu m} + B^m_{2} X^2_{\mu m} + C^{ab} X^3_{\mu m} + \bar{A}^m_{\mu} Y^1_{\mu m} + \bar{C}^m_{a} Y^2_{\mu m} \right) ,
\]

where the functionals \( X^{Aa} = \delta S/\delta \phi^*_A = (X^1_{\mu m}, X^2_{\mu m}, X^3_{\mu m}) \) and \( Y^A = \delta S/\delta \bar{\phi} = (Y^1_{\mu m}, Y^2_{\mu m}, Y^3_{\mu m}) \) are given by

\[
X^1_{\mu m} = D^{\mu m} C^m , \quad Y^1_{\mu m} = D^{\mu m} B^m + \frac{1}{2} f^{mnl} C^{la} D^{mbk} \varepsilon_{ba} ,
\]

\[
X^2_{\mu m} = -\frac{1}{2} f^{mnl} B^l C^m - \frac{1}{12} f^{mnl} f^{irs} C^{sh} C^{ra} C^{nc} \varepsilon_{cb} , \quad Y^2_{\mu m} = 0 ,
\]

\[
X^3_{\mu m} = -\varepsilon_{ab} B^m - \frac{1}{2} f^{mnl} C^{da} C^m , \quad Y^3_{\mu m} = f^{mnl} B^l C^m + \frac{1}{6} f^{mnl} f^{irs} C^{sh} C^{ra} C^{nc} \varepsilon_{cb} .
\]

Hence, the finite BRST-antiBRST transformations \( \Delta \phi^A = X^{Aa} \lambda_a - (1/2) Y^A \lambda^2 \chi^2 \) read as follows:

\[
\Delta A^m_\mu = D^m_\mu C^m \lambda_a - \frac{1}{2} \left( D^m_\mu B^m + \frac{1}{2} f^{mnl} C^{da} D^{mbk} \varepsilon_{ba} \right) \lambda^2 ,
\]

\[
\Delta B^m = -\frac{1}{2} \left( f^{mnl} B^l C^m + \frac{1}{6} f^{mnl} f^{irs} C^{sh} C^{ra} C^{nc} \varepsilon_{cb} \right) \lambda_a ,
\]

\[
\Delta C^{ma} = \left( \varepsilon^{ab} B^m - \frac{1}{2} f^{mnl} C^{da} C^m \right) \lambda_b - \frac{1}{2} \left( f^{mnl} B^l C^m + \frac{1}{6} f^{mnl} f^{irs} C^{sh} C^{ra} C^{nc} \varepsilon_{cb} \right) \lambda^2 ,
\]

where the approximation linear in \( \lambda_a = \mu_a \) produces the infinitesimal BRST-antiBRST transformations \( \delta \phi^A = X^{Aa} \mu_a = (s^a \phi^A) \mu_a \).

To construct the generating functional of Green’s functions \( Z(J) \) in (2.21), we choose the gauge functional \( F = F (\phi) \) to be diagonal in \( A^{\mu m}, C^{ma} \), namely,

\[
F(A, C) = - \frac{1}{2} \int d^D x \, (\alpha A^m_\mu A^{mn}_\mu + \beta \varepsilon_{ab} C^m_\mu C^{mb}_\mu) .
\]

The quantum action \( S_F (\phi) \) corresponding to this gauge-fixing functional reads (see (C.5), (C.22) in Appendix C)

\[
S_F (A, B, C) = S_0 (A) + (1/2) s^a s_a F (A, C) = S_0 (A) + S_{gf} (A, B) + S_{gh} (A, C) + S_{add} (C) ,
\]
where the gauge-fixing term $S_{gf}$, the ghost term $S_{gh}$, and the interaction term $S_{add}$, quartic in $C^{ma}$, are given by
\begin{equation}
S_{gf} = \int d^Dx \left[ \alpha (\partial^\mu A^m_\mu) - \beta B^m \right] B^m, \quad S_{gh} = \frac{\alpha}{2} \int d^Dx \left( \partial^\mu C^{ma} \right) D^m_{\mu} C^{nb} \varepsilon_{ab}, \quad S_{add} = \frac{\beta}{24} \int d^Dx \, f^{mnl} f^{trs} C^{sa} C^{rc} C^{mb} C^{md} \varepsilon_{ab} \varepsilon_{cd}. \tag{4.13}
\end{equation}

Let us examine the choice of the coefficients $\alpha$, $\beta$ leading to $R_\xi$-like gauges. Namely, in view of the contribution $S_{gf}$ to the quantum action $S_F$,
\begin{equation}
S_{gf} = \int d^Dx \left[ \alpha (\partial^\mu A^m_\mu) - \beta B^m \right] B^m, \tag{4.15}
\end{equation}
we impose the conditions
\begin{equation}
\alpha = 1, \quad \beta = -\frac{\xi}{2}. \tag{4.16}
\end{equation}
Thus, the gauge-fixing functional $F(\xi) = F(\xi) (A, C)$ corresponding to an $R_\xi$-like gauge can be chosen as
\begin{equation}
F(\xi) = \frac{1}{2} \int d^Dx \left( -A^m_\mu A^{\mu m} + \frac{\xi}{2} \varepsilon_{ab} C^{ma} C^{mb} \right), \quad \text{so that} \quad F(0) = -\frac{1}{2} \int d^Dx \, A^m_\mu A^{\mu m} \quad \text{and} \quad F(1) = \frac{1}{2} \int d^Dx \left( -A^m_\mu A^{\mu m} + \frac{\xi}{2} \varepsilon_{ab} C^{ma} C^{mb} \right), \tag{4.17}
\end{equation}
where the gauge-fixing functional $F(0) (A)$ induces the contribution $S_{gf} (A, B)$ to the quantum action that arises in the case of the Landau gauge $\chi(A) = \partial^\mu A^m_\mu = 0$ for $(\alpha, \beta) = (1, 0)$ in $\text{[4.15]}$, whereas the functional $F(1) (A, C)$ corresponds to the Feynman (covariant) gauge $\chi(A, B) = \partial^\mu A^m_\mu + (1/2) B^m = 0$ for $(\alpha, \beta) = (1, -1/2)$ in $\text{[4.15]}$.

Let us find the parameters $\lambda_a = s_a \Lambda$ of a finite field-dependent BRST-antiBRST transformation that connects an $R_\xi$ gauge with an $R_{\xi + \Delta \xi}$ gauge, according to (3.32), where
\begin{equation}
\Delta F(\xi) = F(\xi + \Delta \xi) - F(\xi) = \frac{\Delta \xi}{4} \varepsilon_{ab} \int d^Dx \, C^{ma} C^{mb}. \tag{4.19}
\end{equation}
Explicitly,
\begin{equation}
\delta \left( \Delta F(\xi) \right) = s^a \left( \Delta F(\xi) \right) \mu_a = \frac{\Delta \xi}{2} \varepsilon_{ba} \int d^Dx \, C^{mb} \delta C^{ma}, \tag{4.20}
\end{equation}
where $\delta C^{ma} = (\varepsilon^{ab} B^m - (1/2) f^{mnl} C^{la} C^{nb}) \mu_b$ is the linear part of the finite BRST-antiBRST transformation (4.11), which implies
\begin{equation}
s^a \left( \Delta F(\xi) \right) = \frac{\Delta \xi}{2} \varepsilon_{bc} \int d^Dx \, C^{mb} \left( \varepsilon^{ca} B^m - \frac{1}{4} f^{mnl} C^{lc} C^{ma} \right). \tag{4.21}
\end{equation}
In order to calculate $s^a s_a \left( \Delta F(\xi) \right)$, we remind that
\begin{equation}
\frac{1}{2} s^a s_a F(\xi) = S_{gf} + S_{gh} + S_{add} |_{\alpha = 1, \beta = -\xi/2} = \int d^Dx \left\{ \left( \partial^\mu A^m_\mu \right) + \frac{\xi}{2} B^m \right\} B^m + \frac{1}{2} \left( \partial^\mu C^{ma} \right) D^m_{\mu} C^{nb} \varepsilon_{ab} - \frac{\xi}{48} f^{mnl} f^{trs} C^{sa} C^{rc} C^{mb} C^{md} \varepsilon_{ab} \varepsilon_{cd}, \tag{4.22}
\end{equation}
whence
\begin{equation}
s^a s_a \left( \Delta F(\xi) \right) = \frac{\Delta \xi}{2} \int d^Dx \left( B^m B^m - \frac{1}{24} f^{mnl} f^{trs} C^{sa} C^{rc} C^{mb} C^{md} \varepsilon_{ab} \varepsilon_{cd} \right). \tag{4.23}
\end{equation}
Finally, the functionals $\lambda_a (\phi)$ that connect an $R_\xi$-like gauge to an $R_{\xi + \Delta \xi}$-like gauge are given by (3.32)
\begin{equation}
\lambda_a = \frac{\Delta \xi}{4i \hbar} \varepsilon_{ab} \int d^Dx \, B^n C^{nb} \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left[ \frac{1}{4i \hbar} \Delta \xi \int d^Dy \left( B^n B^u - \frac{1}{24} f^{uwt} f^{irss} C^{se} C^{rp} C^{mq} C^{nd} \varepsilon_{ed} \varepsilon_{pq} \right) \right]^n. \tag{4.24}
\end{equation}
In particular, the first order of $\lambda_a = \mu_a$ in powers of $\Delta F(\xi)$ has the form

$$\mu_a = -\frac{i}{2\hbar} s_a \Delta F(\xi) = -\frac{i\Delta \xi}{4\hbar} \varepsilon_{ab} \int d^Dx \, B^m C^{mb} \, .$$

(4.25)

We have thus solved the problem of reaching any gauge in the family of $R_\xi$-like gauges, starting from a certain gauge encoded in the path integral by a functional $F(\xi)$, within the framework of BRST-antiBRST quantization for Yang–Mills theories by means of finite BRST-antiBRST transformations with field-dependent parameters $\lambda_a$ in (4.24).

Generally, if the BRST-antiBRST invariant quantum action $S_F$ of a Yang–Mills theory is given in terms of a gauge induced by a gauge-fixing functional $F_0$, then, in order to reach the quantum action $S_F$ in terms of another gauge induced by a gauge-fixing functional $F$, it is sufficient to make a change of variables in the path integral (2.21) with $S_F$, given by a finite field-dependent BRST-antiBRST transformation with an Sp(2)-doublet of the odd-valued functionals

$$\lambda_a(F - F_0) = \frac{1}{2\hbar} [s_a(F - F_0)] \sum_{n=0}^{\infty} \frac{1}{(n + 1)!} \left( \frac{1}{4\hbar} s_b(F - F_0) \right)^n \, .$$

(4.26)

In particular, if we choose $F_0 = F(\xi)$, with $F(\xi)$ given by (4.17), then the above relation (4.26) describes the transition from an $R_\xi$-like gauge to a gauge parameterized by an arbitrary gauge-fixing functional $F = F(A,B,C)$.

5 Gribov–Zwanziger Action in $R_\xi$-like Gauges

Let us extend the construction of the Gribov horizon [33] to the case of a BRST-antiBRST invariant Yang–Mills theory in a way consistent with the gauge-independence of the $S$-matrix. To this end, we examine the sum of the Yang–Mills quantum action (4.12) in the Landau gauge $\partial^\mu A^m_\mu = 0$ (with the gauge-fixing functional $F(0)$ in (4.18) corresponding to the case $\alpha = 1$, $\beta = 0$) and the non-local horizon functional [33]

$$h(A) = \gamma^2 \int d^Dx \left( \int d^Dy \, f^{mrl}_\mu(x) \left( K^{-1} \right)^{mn}(x;y) f^{nsl}_\mu(y) + D (N^2 - 1) \right) \, .$$

(5.1)

where $K^{-1}$ is the inverse,

$$\int d^Dz \, (K^{-1})^{ml}(x;z)(K)^{ln}(z;y) = \int d^Dz \, (K^{-1})^{nl}(x;z)(K)^{lm}(z;y) = \delta^{mn}\delta(x-y) \, ,$$

(5.2)

of the Faddeev–Popov operator $K$ induced by the gauge-fixing functional $F(\xi \to 0)$ corresponding to the Landau gauge $\partial^\mu A^m_\mu = 0$ in the BRST approach,

$$K^{mn}(x;y) = (\delta^{mn}\partial^2 + f^{mnl}_\mu \partial_\mu) \delta(x-y) \, , \quad K^{mn}(x;y) = K^{nm}(y;x) \, ,$$

(5.3)

whereas $\gamma \in \mathbb{R}$ is the so-called thermodynamic, or Gribov, parameter [34], introduced in a self-consistent way by the gap equation for an analogue $S_h$ of the Gribov–Zwanziger action in the BRST-antiBRST approach:

$$\frac{\partial}{\partial \gamma} \left\{ \frac{\hbar}{i} \ln \left[ \int D\phi \, \exp \left( \frac{i}{\hbar} S_h \right) \right] \right\} = \frac{\partial \mathcal{E}_{\text{vac}}}{\partial \gamma} = 0 \, .$$

(5.4)

In (5.4), we have used the definition of the vacuum energy $\mathcal{E}_{\text{vac}}$ and introduced a modified quantum action for the Gribov–Zwanziger model as an additive extension of the Yang–Mills quantum action $S_{F_0}$ (4.12) in the Landau gauge:

$$S_h(\phi) = S_{F_0}(\phi) + h(\phi) \, , \quad F_0 = F(0) \, ,$$

(5.5)

The action $S_h(\phi)$ is not invariant under the finite BRST-antiBRST transformations:

$$\Delta S_h = \Delta h = (s^2 h) \lambda_a + \frac{1}{4} (s^2 h) \lambda^2 \neq 0 \, ,$$

(5.6)
Indeed, according to $\Delta \phi^A = (s^a \phi^A) \lambda_n + (1/4) (s^2 \phi^A) \lambda^2$, with allowance for (4.8)–(4.10), (4.2), we have

$$s^a h = \gamma^2 f^{mnk} f^{kn} \int d^D x \ d^D y \left[ 2 D_{\mu}^k C^l a (x) (K^{-1})^{mn} (x; y) - f_{\mu\nu} \int d^D x' \ d^D y' \ A_{\mu}^a (x) (K^{-1})^{mn} (x; y') K^{tl} (x'; y') C^l a (y') (K^{-1})^{vn} (y'; y) \right] A^a (y) \tag{5.7}$$

and

$$s^2 h = \gamma^2 f^{mnk} f^{kn} \int d^D x \ d^D y \left\{ 4 \left( -D_{\mu}^t B^t + \frac{1}{2} f^{tlt} C^a b D_{\mu}^t C^{a b} \right) (x) (K^{-1})^{mn} (x; y) A^{a} (y) + 2 \varepsilon_{ab} D_{\mu}^t C^a (x) (K^{-1})^{mn} (x; y) D^{t b} C^b (y) - 4 \varepsilon_{ab} f_{\mu\nu} \int d^D x' \ d^D y' \ D_{\mu}^t C^a (x) (K^{-1})^{mn} (x; y') K^{tw} (x'; y') C^{t a} (y') (K^{-1})^{vn} (y'; y) + f_{\mu\nu} \int d^D x' \ d^D y' \ A_{\mu}^a (x) \left[ -\varepsilon_{ab} f^{u'v' t} \int d^D x'' \ d^D y'' \ (K^{-1})^{mu'} (x; x'') K^{tv'} (x''; y'') C^{t a} (y'') \times (K^{-1})^{v' n} (y''; y) - \varepsilon_{ab} f^{u'v' t} \int d^D x'' \ d^D y'' \ (K^{-1})^{mu'} (x; x'') K^{tv'} (x''; y'') \times C^{t a} (y'') \ e^{ab} K^{m n} (y''; y) + 2 \varepsilon_{ab} f^{u'v' t} \int d^D x'' \ d^D y'' \ (K^{-1})^{mu'} (x; x'') K^{tv'} (x''; y'') \times C^{t a} (y'') \ e^{ab} K^{m n} (y''; y) \right] A^a (y) \right\} \tag{5.8}$$

where we have used the identity

$$s^a K^{mn} (x; y) = f^{mnk} K^{k a} (x; y) C^a (y) \tag{5.9}$$

To determine the horizon functional for a general $R_{\xi}$-like gauge in the BRST-antiBRST description, we propose

$$h_\xi = h + \frac{1}{2 i \hbar} (s^a h) \left( s a \Delta F_\xi \right) \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left( \frac{1}{4 i \hbar} s^b s b \Delta F_\xi \right)^n$$

$$- \frac{1}{16 \hbar^2} \left( s^a h \right) (s a \Delta F_\xi)^2 \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left( \frac{1}{4 i \hbar} s^b s b \Delta F_\xi \right)^n \right]^2 \tag{5.10}$$

Here, $s^a h$ and $s^2 h$ are given by (5.7), (4.8), while $s_a \Delta F_\xi$ and $s^a s_a \Delta F_\xi$ are given by (4.21), (4.22) for $\Delta \xi = \xi$, whereas the Sp(2)-doublet $\lambda_\xi^a (\phi)$ of field-dependent anticommuting parameters in (4.24) relates the Landau gauge to an arbitrary $R_{\xi}$-like gauge:

$$\Delta F_\xi = F_\xi - F_0 = \frac{\xi}{4} \varepsilon_{ab} \int d^D x \ C^{ma} C^{mb} \tag{5.11}$$

$$s_a \Delta F_\xi = \frac{\xi}{2} \varepsilon_{ab} \int d^D x \ B^{ma} C^{mb} \tag{5.12}$$

$$s^a s_a \Delta F_\xi = \xi \int d^D x \left( B^{ma} B^{mb} - \frac{1}{24} f^{m n l} f^{l s} C^{m a} C^{n c} C^{m b} C^{m d} \varepsilon_{ab} \varepsilon_{cd} \right) \tag{5.13}$$

In particular, the approximation linear in $\xi$ implies $\lambda_\xi^a (\phi) = s^a \Lambda_\xi (\phi)$ for $\Lambda_\xi (\phi) = \frac{\xi}{4 i \hbar} \varepsilon_{ab} \int d^D x \ C^{ma} C^{mb}$,

$$h_\xi (\phi) = h \left( A \right) + \frac{\xi}{4 i \hbar} \varepsilon_{ab} \gamma^2 f^{m n l} f^{l s} \int d^D x \ d^D y \left[ 2 D_{\mu}^k C^{la} (x) (K^{-1})^{mn} (x; y) - f^{m n l} n' \int d^D x' \ d^D y' \ A_{\mu}^s (x) \times (K^{-1})^{m n'} (x; x') K^{s l t'} (x'; y') C^{t a} (y') (K^{-1})^{n' n} (y'; y) \right] A^{a} (y) \int d^D z \left( B^w C^{w b} \right) \tag{5.14}$$
Notice that even the approximation to \( h_\xi (\phi) \) being linear in powers of \( \xi \) is different from the proposal \([37]\) for the horizon functional given by \( R_\xi \)-gauges in terms of field-dependent BRST transformations, which reflects the Sp(2)-symmetric character of the dependence of \( h_\xi (\phi) \) on the ghost and antighost fields \( C^{ma} \).

The proposal \([5, 10]\) for the Gribov horizon functional in a general \( R_\xi \)-gauge is consistent with the study of gauge-independence for the generating functional of Green’s functions, determined for a BRST-antiBRST extension of the Gribov–Zwanziger model as follows:

\[
Z_{GZ, F_0}(J) = \int d\phi \exp \left\{ \frac{i}{\hbar} \left[ S_h(\phi) + J_A\phi^A \right] \right\} .
\]  

(5.15)

Indeed, making in the path integral for \( Z_{GZ, F_0}(J) \) a change of variables being a finite field-dependent BRST-antiBRST transformation with the parameters \( \lambda^a_\xi (\phi) \) given by \([4.26]\), where \( \Delta \xi = \xi \), we find, due to the fact that the Yang–Mills quantum action \( S_{F_0}(\phi) \) transforms to \( S_{F_\xi}(\phi) \), with \( F_\xi = F_\xi(\xi) \),

\[
Z_{GZ, F_\xi}(J) = \int d\phi \exp \left\{ \frac{i}{\hbar} \left[ S_{F_\xi}(\phi) + h_\xi (\phi) + J_A\phi^A + J_A\Delta \phi^A \right] \right\} ,
\]

(5.16)

where \( h_\xi (\phi) \) in \([5.10]\) corresponds to an \( R_\xi \)-gauge. As a result, we have

\[
Z_{GZ, F_\xi}(J) = Z_{GZ, F_\xi}(J) \left[ 1 + \frac{i}{\hbar} J_A \left( \langle (s^a_\xi \phi^A)s_a\Lambda(\Delta F(\xi)) \rangle \right)_{F_0,J} + \frac{i}{4\hbar} J_A \left[ \langle (s^2\phi^A)^2 \rangle + \frac{i}{\hbar} \varepsilon^{ab}(s^a \phi^A)J_B(s^b \phi^B) \langle \Lambda(\Delta F(\xi)) \rangle^2 \right]_{F_0,J} \right] ,
\]

(5.17)

where the vacuum expectation value is computed with respect to \( Z_{GZ, F}(J) \). The relation \([5.17]\) implies that neither the functional \( Z_{GZ, F_\xi}(J) \) nor the \( S \)-matrix depends on the gauge (parameter \( \xi \)) at the extremals given by \( J_A = 0 \). This justifies our proposal for the horizon functional in the form \([5.10]\). At the same time, we note that the Gribov–Zwanziger model in BRST-antiBRST quantization encounters the problem of unitarity, since the gauge degrees of freedom, being non-dynamical in the Yang–Mills theory, should now be regarded as dynamical ones, due to the explicit form of the horizon functional \( h_\xi (\phi) \).

Finally, it is possible to construct a Gribov horizon functional \( h_F(\phi) \) in any differential gauge, induced by a gauge-fixing functional \( F(\phi) \), starting from the horizon functional \( h(A) \) in the Landau gauge, corresponding to the gauge-fixing functional \( F_0(A) \). To this end, it is sufficient to make a change of variables in the path integral \([5.15]\), given by a finite field-dependent BRST-antiBRST transformation with the Sp(2)-doublet \( \lambda_a(F - F_0) \) of odd-valued functionals given by \([4.26]\). Thus, the functional \( h_F(\phi) \) reads as follows:

\[
h_F = h + \frac{1}{2i\hbar} \langle s^a_{\hbar}(F - F_0) \rangle \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left( \frac{1}{4i\hbar} s^a{s_b}(F - F_0) \right)^n
\]

\[-\frac{1}{16\hbar^2} \langle s^2_{\hbar}(F - F_0) \rangle^2 \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left( \frac{1}{4i\hbar} s^a{s_b}(F - F_0) \right)^n \right] .
\]

(5.18)

Generally, a finite change \( F \to F + \Delta F \) of the gauge condition induces a finite change of any functional \( G_F(\phi) \), so that in the reference frame corresponding to the gauge \( F + \Delta F \) it can be represented according to \([3.8], [1.20]\),

\[
G_{F + \Delta F} = G_F + \langle s^a G_F \rangle \lambda_a(\Delta F) + \frac{1}{4} \langle s^2 G_F \rangle \lambda_a(\Delta F) \lambda_a(\Delta F) ,
\]

(5.19)
which is an extension of the infinitesimal change \( G_F \to G_F + \delta G_F \) induced by a variation of the gauge, \( F \to F + \delta F \),

\[
G_{F+\delta F} = G_F - \frac{i}{2\hbar} (s^a G_F) (s_a \delta F),
\]

(5.20)
corresponding, in the case \( G_F(A) \), to the gauge transformations, with the functions \( \zeta^m(x) \) given below

\[
\delta G_F = G_{F+\delta F} - G_F = \int d^Dx \frac{\delta G_F}{\delta A_{m\mu}(x)} D^{mn\mu} \zeta^n(x), \quad \text{where} \quad \zeta^m(x) = -\frac{i}{2\hbar} C^{ma}(x)(s_a \delta F).
\]

(5.21)
Due to the presence of the term with \( s^2 G_F \) in a finite gauge variation of a functional \( G_F(A) \) depending only on the classical fields \( A^{m\mu} \), the representation (5.19) is more general than the one that would correspond to the usual Lagrangian BRST approach (see relation (17) in [39]), having the form similar to (5.21), and thus also to (5.20).

We emphasize that the suggested method of using the finite field-dependent BRST-antiBRST transformations with the purpose of finding the Gribov–Zwanziger horizon functional in any differential gauge, starting from the Gribov–Zwanziger theory in the Landau gauge, is valid in perturbation theory and preserves the number of physical degrees of freedom, without entering into contradiction with the result of [24] in the BRST setting of the problem. However, it is impossible to solve this problem (in particular, in the Yang–Mills theory) in terms of finite field-dependent BRST-antiBRST transformations [26], in view of the absence of a term being quadratic in powers of the odd-valued parameters, since the corresponding Yang–Mills quantum action fails to be BRST-antiBRST invariant, and the Jacobian of the corresponding change of variables with odd-valued functionally-dependent parameters does not generate terms which are entirely BRST-antiBRST-exact. These terms change the BRST-antiBRST-exact part of the action, as well as the extremals; however, they do not affect the number of physical degrees of freedom.

6 Discussion

In the present work, we have proposed the concept of finite BRST-antiBRST transformations for Yang–Mills theories in the Sp(2)-covariant Lagrangian quantization [13] [16], realized in the form [35] [37], being polynomial in powers of a constant Sp(2)-doublet of anticommuting Grassmann parameters \( \lambda_a \) and leaving the quantum action of the Yang–Mills theory invariant to all orders in \( \lambda_a \). In the case of constant \( \lambda_a \), the set of finite BRST-antiBRST transformations forms an Abelian two-parametric Lie supergroup with the elements \( g(\lambda) = \exp (\mathbb{S}^a \lambda_a) = (1 + \mathbb{S}^a \lambda_a + \frac{1}{4} \mathbb{S}^a \mathbb{S}^b \lambda_b^2) \), so that \( \Delta \phi^A = \phi^A [\exp (\mathbb{S}^a \lambda_a) - 1] \), where \( G^a \mathbb{S}^a \equiv s^a G \), for any \( G = G(\phi) \). Secondly, this ensures exact invariance of the integrand in the generating functional of Green’s functions \( Z_F(J) \) with vanishing external sources \( J_A \) and also allows one to obtain the Ward identities.

We have determined the finite field-dependent BRST-antiBRST transformations as polynomials in the Sp(2)-doublet of Grassmann-odd functionals \( \lambda_a(\phi) \), depending on the whole set of fields that compose the configuration space of Yang–Mills theories, and have also calculated the Jacobian [31-32] corresponding to this change of variables by using a special class of transformations with \( s_a \)-potential parameters \( \lambda_a(\phi) = s_a \Lambda(\phi) \) for a Grassmann-even functional \( \Lambda(\phi) \) and Grassmann-odd generators \( s_a \) of BRST-antiBRST transformations.

In comparison with finite field-dependent BRST transformations in Yang–Mills theories [23], in which a change of the gauge corresponds to a unique field-dependent parameter (up to BRST-exact terms), it is only functionally-dependent finite field-dependent BRST-antiBRST transformations with \( \lambda_a = s_a \Lambda(\Delta F) \) that are in one-to-one correspondence with \( \Delta F \). We have found a solution \( \Lambda(\Delta F) \) to the so-called compensation equation (3.28) for an unknown functional \( \Lambda(\Delta F) \) generating an Sp(2)-doublet \( \lambda_a \), in order to establish a relation of the Yang–Mills quantum action \( S_F \) in a certain gauge determined by a gauge Boson \( F \) with the action \( S_F + \Delta F \) induced by a different gauge \( F + \Delta F \). This makes it possible to investigate the problem of gauge-dependence for the generating functional \( Z_F(J) \) under a finite change of the gauge in the form [33-35], leading to the gauge-independence of the physical \( S \)-matrix.
In terms of the potential Λ inducing the finite field-dependent BRST-antiBRST transformations, we have explicitly constructed \[ \delta S = \delta \phi^A \mu_a + \beta \mu_a , \] for finite BRST transformations in section 3, in fact, applies to any infinitesimal symmetry transformations \( \delta \phi = X^A \mu_a = (s^A \phi^A) \mu_a \), with anticommuting parameters \( \mu_a, a = 1, 2 \), for a certain functional \( S_F (\phi) \), such that \( \delta S_F (\phi) = 0 \), and does not involve any subsidiary conditions on \( X^A \) and the corresponding \( s^a \), since the construction is achieved only by using \( Y^A = (1/2) X^A_B X^{Bb} \epsilon^e_{ba} \) in (3.7), according to (2.20). Let us apply this to the vacuum functional \( Z(0) \) of a general gauge theory, given by the path integral \[ \Gamma^F = (\phi^A, \phi^*_A, \bar{\phi}_A, \pi^A, \lambda^A) \],

\[
Z(0) = \int d\Gamma \exp \left[ \frac{i}{\hbar} S_F (\Gamma) \right] , \quad S_F = S + \phi^*_A \pi^A + (\bar{\phi}_A - F_A \lambda^A - (1/2) \epsilon^{ab} \pi^A F_{AB} \pi^B , \quad (6.1)
\]

where the integrand \( \mathcal{I}_F = d\Gamma \exp \left[ \frac{i}{\hbar} S_F (\Gamma) \right] \) is invariant, \( \delta \mathcal{I}_F = 0 \), under the global infinitesimal BRST-antiBRST transformations (2.13), \( \delta \Gamma^F = (s^A \Gamma^F) \mu_a \), with the corresponding generators \( s^a \),

\[
\delta \Gamma^F = (s^A \Gamma^F) \mu_a = \delta \left( \phi^A, \phi^*_A, \bar{\phi}_A, \pi^A, \lambda^A \right) = \left( \pi^A, \delta_{\pi^A} S_A (-1)^{\epsilon^A}, \epsilon^{ab} \phi^*_A (1) \epsilon^{A+1}, \epsilon^{ab} \lambda^A, 0 \right) \mu_a . \quad (6.2)
\]

We have solved this problem in our recent works [13, 14].

Shortly after the publication of the present work, we have become aware of the more recent study [15] of finite BRST transformations in the BV formalism.

We have solved this problem in detail [16], including the case of Yang–Mills theories.
In this connection, let us determine finite BRST-antiBRST transformations, $\Gamma^p \to \Gamma^p + \Delta \Gamma^p$, parameterized by anticommuting parameters $\lambda_a$, $a = 1, 2$, as follows:

$$I^{(F)}_{\Gamma + \Delta \Gamma} = I^{(F)}_{\Gamma}, \quad \left[ \frac{\delta}{\delta \lambda_0} \Delta \Gamma^p \right]_{\lambda = 0} = \sigma^a \Gamma^p \quad \text{and} \quad \left[ \frac{\delta}{\delta \lambda_0} \frac{\delta}{\delta \lambda_b} \Delta \Gamma^p \right] = \frac{1}{2} \varepsilon^{ab} \sigma^2 \Gamma^p, \quad \text{where} \quad \sigma^2 = \sigma_a \sigma^a. \quad (6.3)$$

Thus determined finite BRST-antiBRST symmetry transformations for the integrand $I^{(F)}_\Gamma$ in a general gauge theory have the form $(\lambda^{pa} = \sigma^a \Gamma^p$ and $Y^p = (1/2) \lambda^{pa} \lambda^{qb} \varepsilon_{ba} = - (1/2) \sigma^2 \Gamma^p)$

$$\Delta \Gamma^p = \lambda^{pa} \lambda_a - \frac{1}{2} \varepsilon^{ab} \delta \phi^A = (\sigma^a \Gamma^p) \lambda_a + \frac{1}{4} \left( \frac{\delta^2 S}{\delta \phi^A_\xi \delta \phi^B_\xi} - \frac{\delta^2 S}{\delta \phi^b_\xi \delta \phi^a_\xi} \right) (1 - \varepsilon^a) \lambda_a, \quad I^{(F)}_{\Gamma + \Delta \Gamma} = I^{(F)}_\Gamma, \quad (6.4)$$

or, in terms of the components,

$$\Delta \phi^A = \pi^{Aa} \lambda_a + \frac{1}{2} \lambda^A \lambda^2, \quad \Delta \phi^A = \varepsilon^{ab} \lambda_a \phi^*_A + \frac{1}{2} S_A \lambda^2, \quad \Delta \lambda^A = 0, \quad \Delta \phi^*_A = \lambda_a S_A + \frac{1}{4} (-1)^{\varepsilon^A} \left[ \varepsilon_{ab} \frac{\delta^2 S}{\delta \phi^A_\xi \delta \phi^b_\xi} \frac{\delta \phi^*_A}{\delta \phi^b_\xi} + \varepsilon_{ab} \frac{\delta^2 S}{\delta \phi^a_\xi \delta \phi^b_\xi} \frac{\delta \phi^*_A}{\delta \phi^a_\xi} \right] (1 - \varepsilon^B) \lambda^2. \quad (6.5)$$

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**Appendix**

### A Group Properties of Finite BRST-antiBRST Transformations

In this Appendix, in order to clarify the relations (3.9)–(3.13) of Section 3, we examine the composition of finite variations $\Delta(1)\Delta(2)$ acting on an arbitrary functional $F = F(\phi)$, with the variation $\Delta F$ given by (3.8),

$$\Delta F = (s^a F) \lambda_a + \frac{1}{4} (s^2 F) \lambda^2. \quad (A.1)$$

Using the readily established Leibnitz-like properties of the generators of BRST-antiBRST transformations, $s^a$ and $s^2$, acting on the product of any functionals $A, B$ with definite Grassmann parities,

$$s^a (AB) = (s^a A) B (-1)^{\varepsilon_B} + A (s^a B) \quad \text{and} \quad s_a (AB) = (s_a A) B (-1)^{\varepsilon B} + A (s_a B), \quad s^2 (AB) = (s^2 A) B - 2 (s_a A) (s^a B) (-1)^{\varepsilon_B} + A (s^2 B), \quad \text{for} \quad s^2 = s_a s^a, \quad (A.2)$$

and the identities

$$s^a s^b = (1/2) \varepsilon^{ab} s^2 \quad \text{and} \quad s_a s_b = - s_b s_a = (1/2) \delta^{ab} s^2 \quad \text{and} \quad s^a s^b s^c = 0, \quad (A.3)$$
with the notation $UV \equiv U_{a}V^{a} = -U^{a}V_{a}$ for pairing up any $\text{Sp}(2)$-vectors $U^{a}$, $V^{a}$, we obtain

$$s^{a} (\Delta F) = s^{a} \left[ (s^{b} F) \lambda_{b} + \frac{1}{4} (s^{2} F) \lambda^{2} \right] = s^{a} \left[ (s^{b} F) \lambda_{b} + (1/4) s^{a} [(s^{2} F) \lambda^{2}] \right]$$

$$= - (s^{a} s^{b} F) \lambda_{b} + (s^{b} F) (s^{a} \lambda_{b}) + (1/4) (s^{2} F) (s^{a} \lambda^{2})$$

$$= - (1/2) (s^{2} F) \lambda^{a} - (s F) (s^{a} \lambda) + (1/4) (s^{2} F) (s^{a} \lambda^{2})$$

(A.4)

and

$$s^{2} (\Delta F) = s^{2} \left[ (s^{b} F) \lambda_{b} + \frac{1}{4} (s^{2} F) \lambda^{2} \right] = s^{2} \left[ (s^{b} F) \lambda_{b} + \frac{1}{4} s^{2} [(s^{2} F) \lambda^{2}] \right]$$

$$= 2 (s_{a} s^{b} F) (s^{a} \lambda_{b}) + (s^{b} F) (s^{2} \lambda_{b}) + \frac{1}{4} (s^{2} F) (s^{2} \lambda^{2})$$

$$= - (s^{2} F) (s \lambda) - (s F) (s^{2} \lambda) + \frac{1}{4} (s^{2} F) (s^{2} \lambda^{2}) .$$

(A.5)

Therefore, $\Delta_{(1)} \Delta_{(2)} F$ is given by

$$\Delta_{(1)} \Delta_{(2)} F = (s^{a} \Delta_{(2)} F) \lambda_{(1)a} + \frac{1}{4} (s^{2} \Delta_{(2)} F) \lambda^{2}_{(1)}$$

$$= \left[ - (1/2) (s^{2} F) \lambda^{2}_{(2)} - (s F) (s^{a} \lambda_{(2)}) + (1/4) (s^{2} F) (s^{a} \lambda^{2}_{(2)}) \right] \lambda_{(1)a}$$

$$+ \frac{1}{4} \left[ (s^{2} F) (s \lambda_{(2)}) - (s F) (s^{2} \lambda_{(2)}) + \frac{1}{4} (s^{2} F) (s^{2} \lambda^{2}_{(2)}) \right] \lambda^{2}_{(1)}$$

$$\equiv (s^{a} F) \vartheta_{(1),2} + \frac{1}{4} (s^{2} F) \theta_{(1),2} ,$$

(A.6)

whence

$$\vartheta_{(1),2} = - (s \lambda^{2}_{(2)}) \lambda_{(1)} + \frac{1}{4} (s^{2} \lambda^{2}_{(2)}) \lambda^{2}_{(1)} ,$$

(A.7)

$$\theta_{(1),2} = \left[ 2 \lambda_{(2)} - (s \lambda^{2}_{(2)}) \right] \lambda_{(1)} - \left[ (s \lambda_{(2)}) - \frac{1}{4} (s^{2} \lambda^{2}_{(2)}) \right] \lambda^{2}_{(1)} .$$

(A.8)

Hence, the commutator of finite variations reads

$$[\Delta_{(1)}, \Delta_{(2)}] F = (s^{a} F) \vartheta_{[1],[2]} + \frac{1}{4} (s^{2} F) \theta_{[1],[2]} .$$

(A.9)

Finally, using the identity

$$\lambda_{(2)} \lambda_{(1)} - \lambda_{(1)} \lambda_{(2)} = \lambda_{(2)a} \lambda^{a}_{(1)} - \lambda_{(1)a} \lambda^{a}_{(2)} = \lambda_{(2)a} \lambda^{a}_{(1)} - \lambda_{(2)a} \lambda^{a}_{(1)} = 0 ,$$

(A.10)

we obtain

$$\vartheta^{a}_{(1),2} = \vartheta^{a}_{(1),2} - \vartheta^{a}_{(2),1} = \left( s \lambda^{a}_{(1)} \right) \lambda_{(2)} - \left( s \lambda^{a}_{(2)} \right) \lambda_{(1)} - \frac{1}{4} \left[ (s^{2} \lambda^{a}_{(1)}) \lambda^{2}_{(2)} - (s^{2} \lambda^{a}_{(2)}) \lambda^{2}_{(1)} \right] ,$$

(A.11)

$$\theta^{a}_{(1),2} = \theta^{a}_{(1),2} - \theta^{a}_{(2),1} = \left[ (s \lambda^{2}_{(1)}) \lambda_{(2)} - (s \lambda^{2}_{(2)}) \lambda_{(1)} \right] + \left[ (s \lambda_{(1)}) \lambda^{2}_{(2)} - (s \lambda_{(2)}) \lambda^{2}_{(1)} \right]$$

$$+ \frac{1}{4} \left[ (s^{2} \lambda^{2}_{(2)}) \lambda^{2}_{(1)} - (s^{2} \lambda^{2}_{(1)}) \lambda^{2}_{(2)} \right] .$$

(A.12)

In particular, the linear approximation $\Delta^{\text{lin}} F = (s^{a} F) \lambda_{a}$, $\Delta F = \Delta^{\text{lin}} F + O (\lambda^{2})$, implies [3.13].

**B Calculation of Jacobians**

In this Appendix, we present the calculation of the Jacobian [3.14], [3.15], induced in the functional integral [2.21] by the finite BRST-antiBRST transformations [3.7] with an $\text{Sp}(2)$-doublet of anticommuting parameters $\lambda_{a}$, considering the global case, $\lambda_{a} = \text{const}$, and the case of field-dependent functionals $\lambda_{a} (\phi)$ of a special form, $\lambda_{a} (\phi) = s_{a} \Lambda (\phi)$. 

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B.1 Constant Parameters

Let us assume $\lambda_a$ to be constant parameters in (3.7) and consider an even matrix $M$ in (3.15) with the elements $M^A_B$, 
$\varepsilon (M^A_B) = \varepsilon_A + \varepsilon_B$.

$$M^A_B = \frac{\delta (\Delta \phi^A)}{\delta \phi_B} = (Q_1)_B^A + R^A_B,$$

with

$$(Q_1)_B^A = \frac{\delta X^{Aa}}{\delta \phi_B} \lambda_a (-1)^{\varepsilon_B} \quad \text{and} \quad R^A_B = -\frac{1}{2} \frac{\delta Y^A}{\delta \phi_B} \lambda^2.$$

(B.1)

Notice the fact that $Q_1 \sim \lambda_a$, $R \sim \lambda^2$, which, in view of the nilpotency properties $\lambda_a \lambda^2 = \lambda^3 \equiv 0$, implies

$$\text{Str} (M^n) = \text{Str} (Q_1 + R)^n = \begin{cases} \text{Str} (Q_1 + R) = \text{Str} (R), & n = 1, \\ \text{Str} (Q_1^2) = 2 \text{Str} (R), & n = 2, \\ 0, & n > 2. \end{cases}$$

(B.2)

Indeed, due to the relations $X^{Aa}_A = 0$ in (2.17), we have

$$\text{Str} (Q_1) = (Q_1)_A^A (-1)^{\varepsilon_A} = \frac{\delta X^{Aa}}{\delta \phi_A} \lambda_a = 0. \quad (B.3)$$

Next, let us examine $\text{Str} (Q_1^2)$:

$$\text{Str} (Q_1^2) = (Q_1)_A^A (-1)^{\varepsilon_A} = \frac{\delta X^{Aa}}{\delta \phi_B} \lambda_a \frac{\delta X^{Bb}}{\delta \phi_A} \lambda_b (-1)^{\varepsilon_B} = \frac{\delta X^{Aa}}{\delta \phi_B} \frac{\delta X^{Bb}}{\delta \phi_A} \lambda_b \lambda_a (-1)^{\varepsilon_A}. \quad (B.4)$$

Differentiating the relation $X^{Aa}_B X^{Bb} = \varepsilon^{ab} Y^A$ in (2.17) with respect to $\phi^A$, we find

$$\frac{\delta}{\delta \phi_B} \left( \frac{\delta X^{Aa}}{\delta \phi_A} \right) X^{Bb} (-1)^{\varepsilon_B} + \frac{\delta X^{Aa}}{\delta \phi_B} \frac{\delta X^{Bb}}{\delta \phi_A} + \varepsilon^{ab} \frac{\delta Y^A}{\delta \phi_A} = 0.$$ 

Then, due to the relation $X^{Aa}_A = 0$ in (2.17), we have

$$\frac{\delta X^{Aa}}{\delta \phi_B} \frac{\delta X^{Bb}}{\delta \phi_A} = \varepsilon^{ab} \frac{\delta Y^A}{\delta \phi_A}, \quad (B.5)$$

and therefore

$$\text{Str} (Q_1^2) = \varepsilon^{ab} \frac{\delta Y^A}{\delta \phi_A} \lambda_b \lambda_a (-1)^{\varepsilon_A} = -\frac{\delta Y^A}{\delta \phi_A} \lambda^2 (-1)^{\varepsilon_A} = 2 \text{Str} (R). \quad (B.6)$$

Thus, the Jacobian $\exp (3)$ in (3.15) is given by

$$\exp (3) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{Str} (M^n) = \text{Str} (M) - \frac{1}{2} \text{Str} (M^2) = \text{Str} (R) - \frac{1}{2} \text{Str} (Q_1^2) \equiv 0,$$

(B.7)

which proves (3.16).

B.2 Field-dependent Parameters

In the case of field-dependent parameters $\lambda_a (\phi) = s_a \Lambda (\phi)$ from (3.7), given by an even-valued potential $\Lambda (\phi)$, let us consider an even matrix $M$ in (3.15) with the elements $M^A_B$,

$$M^A_B = \frac{\delta (\Delta \phi^A)}{\delta \phi_B} = P^A_B + Q^A_B + R^A_B,$$

with

$$Q^A_B = (Q_1)_B^A + (Q_2)_B^A,$$

(B.8)

for

$$P^A_B = X^{Aa} \frac{\delta \lambda_a}{\delta \phi_B}, \quad (Q_1)_B^A = \lambda_a \frac{\delta X^{Aa}}{\delta \phi_B} (-1)^{\varepsilon_B} + 1, \quad (Q_2)_B^A = \lambda_a Y^A \frac{\delta \lambda_a}{\delta \phi_B} (-1)^{\varepsilon_A} + 1, \quad R^A_B = -\frac{1}{2} \lambda^2 \frac{\delta Y^A}{\delta \phi_B}. \quad (B.9)$$

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Using the property

\[ \text{Str} (AB) = \text{Str} (BA), \]  

(B.10)

which takes place for any even matrices \( A, B \), and the fact that the occurrence of \( R \sim \lambda^2 \) in \( \text{Str} (M^n) \) more than once yields zero, \( \lambda^4 \equiv 0 \), we have

\[ \text{Str} (M^n) = \text{Str} (P + Q + R)^n = \sum_{k=0}^{1} C^n_k \text{Str} \left[ (P + Q)^{n-k} R^k \right], \quad C^n_k = \frac{n!}{k!(n-k)!}. \]  

(B.11)

Furthermore,

\[ \text{Str} (P + Q + R)^n = \text{Str} (P + Q)^n + n \text{Str} \left[ (P + Q)^{n-1} R \right] = \text{Str} (P + Q)^n + n \text{Str} (P^{n-1} R), \]  

(B.12)

since any occurrence of \( R \sim \lambda^2 \) and \( Q \sim \lambda \) simultaneously entering \( \text{Str} (M^n) \) yields zero, owing to \( \lambda_n \lambda^2 = 0 \), as a consequence of which \( R \) can only be coupled with \( P^{n-1} \).

Having established (B.12), let us examine \( \text{Str} (P^{n-1} R) \), namely,

\[ \text{Str} (P^{n-1} R) = \begin{cases} \text{Str} (R), & n = 1, \\ 0, & n > 1. \end{cases} \]  

(B.13)

Indeed, due to the contraction property \( P^2 = f \cdot P \implies P^l = f^{l-1} \cdot P \), where \( f \) is an even-valued parameter (for details, see (B.34) below), we have

\[ \text{Str} (P^{n-1} R) = f^{n-2} \text{Str} (PR), \quad n > 1, \]  

(B.14)

\[ \text{Str} (PR) = \text{Str} (RP) = (RP)^A A^{(1)} \varepsilon A = P_B^A P_A^{B(-1)} \varepsilon A = \frac{1}{2} \lambda^2 \left( \frac{\delta Y^A}{\delta \phi^B} X^{Bb} \right) \delta_{b0} (1 - 1) \varepsilon A = 0, \]  

(B.15)

since \( Y^A_B X^{Bb} = 0 \) in (2.17), which implies

\[ \text{Str} (M^n) = \text{Str} (P + Q)^n + n \text{Str} (P^{n-1} R) = \begin{cases} \text{Str} (P + Q) + \text{Str} (R), & n = 1, \\ \text{Str} (P + Q)^n, & n > 1. \end{cases} \]  

(B.16)

so that \( R \) drops out of \( \text{Str} (M^n) \), \( n > 1 \), and enters the Jacobian only as \( \text{Str} (R) \).

Considering the contribution \( \text{Str} (P + Q)^n \) in (B.16), we notice that an occurrence of \( Q \sim \lambda_n \) more then twice yields zero, \( \lambda_n \lambda^2 \lambda^2 = 0 \). A direct calculation for \( n = 2, 3 \) leads to

\[ \text{Str} (P + Q)^n = \sum_{k=0}^{n} C^n_k \text{Str} (P^{n-k} Q^k) = \text{Str} (P^n + nP^{n-1} Q + C^2_n P^{n-2} Q^2) \].  

(B.17)

Next, starting from the case \( n = 4 \), \( \text{Str} (M^4) = \text{Str} (P^4 + 4P^3Q + 4P^2Q^2 + 2PQPQ) \), one can prove that for any \( n \geq 4 \) we have

\[ \text{Str} (P + Q)^n = \text{Str} (P^n + n P^{n-1} Q + n P^{n-2} Q^2 + K_n P^{n-3} Q P Q) \],  

(B.18)

where the coefficients\(^{12} \) \( K_n \) are given by (in particular, \( n = 4, C_4^2 = 6, K_4 = C_4^2 - 4 = 2 \))

\[ K_n = C_n^2 - n, \quad C_n^2 = n(n-1)/2 \implies K_n = n(n-3)/2, \]  

(B.19)

which implies

\[ \frac{C_n^2}{n} - \frac{K_n}{n} = 1, \quad \frac{C_n^2}{n} - \frac{K_{n+1}}{n+1} = \frac{1}{2}, \]  

(B.20)

\(^{12}\)The coefficient \( K_n \) turns out to be the number of monomials in \( (P + Q)^n \) for \( n \geq 4 \) that contain two matrices \( Q \) and cannot be transformed by cyclic permutations under the symbol \( \text{Str} \) of supertrace to the form \( \text{Str}(P^{n-2} Q^2) \).
The proof of (B.18) goes by induction. To this end, suppose that (as in the case \( n = 4 \))

\[
(P + Q)^n = P^n + A_n^{(1)}(P, Q) + B_n^{(2)}(P, Q) + C_n^{(2)}(P, Q),
\]

where

\[
A_n^{(1)} = a_{kl}P^kQP^l, \quad a_n \equiv a_{00} = 1, \quad B_n^{(2)} = b_{kl}P^kQP^l, \quad C_n^{(2)} = c_{kml}P^kQP^mQP^l, \quad m \geq 1,
\]

and \( \text{Str}(A_n^{(1)}) = n \text{Str}(P^{n-1}Q), \text{Str}(B_n^{(2)}) = n \text{Str}(P^{n-2}Q^2), \text{Str}(C_n^{(2)}) = K_n \text{Str}(P^{n-3}QPQ) \).

Then, due to the vanishing of the terms containing \( Q \) more than twice, we have

\[
(P + Q)^{n+1} = P^{n+1} + A_{n+1}^{(1)} + B_{n+1}^{(2)} + C_{n+1}^{(2)},
\]

for

\[
A_{n+1}^{(1)} = P^nQ + A_n^{(1)}P, \quad B_{n+1}^{(2)} = B_n^{(2)} + C_{n+1}^{(2)} = A_n^{(1)}Q + B_n^{(2)}P + C_n^{(2)}P,
\]

where

\[
A_{n+1}^{(1)} = P^nQ + a_{kl}P^kQP^lP \implies a_{n+1} = 1, \quad B_{n+1}^{(2)} = a_{kl}P^kQP^l + B_n^{(2)}P, \quad C_{n+1}^{(2)} = a_{kl}P^kQP^lQ + C_n^{(2)}P, \quad l \geq 1.
\]

Due to the contraction property \( P^2 = f \cdot P \implies P^l = f^{l-1} \cdot P \) in (B.31), the above implies

\[
\text{Str}(A_{n+1}^{(1)}) = (n+1) \text{Str}(P^nQ), \quad \text{Str}(B_{n+1}^{(2)}) = (n+1) \text{Str}(P^nQ^2), \quad \text{Str}(C_{n+1}^{(2)}) = (n-1) \text{Str}(P^{n-2}QPQ) + K_n \text{Str}(P^{n-3}QPQ).
\]

Notice that

\[
K_{n} + n - 1 = \frac{n(n - 3)}{2} + \frac{2n - 2}{2} = \frac{(n+1)(n-2)}{2} = K_{n+1},
\]

which proves the induction.

Recall that the Jacobian \( \exp(\mathcal{L}) \) in (3.15) is given by

\[
\mathcal{J} = \text{Str} \ln(I + M) = - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{Str}(M^n),
\]

where, according to the previous considerations,

\[
\text{Str}(M^n) = \sum_{k=0}^{1} C_n^k \text{Str}(P^{n-k}Q^k) + D_n, \quad n \geq 1,
\]

for

\[
D_n = \begin{cases} 
\text{Str}(R), & n = 1, \\
C_n^2 \text{Str}(P^{n-2}Q^2), & n = 2, 3, \\
(C_n^2 - K_n) \text{Str}(P^{n-2}Q^2) + K_n \text{Str}(P^{n-3}QPQ), & n > 3,
\end{cases}
\]

or, in detail,

\[
\text{Str}(M^n) = \begin{cases} 
\text{Str}(P) + \text{Str}(Q) + \text{Str}(R), & n = 1, \\
\text{Str}(P^n) + C_n^1 \text{Str}(P^{n-1}Q) + C_n^2 \text{Str}(P^{n-2}Q^2), & n = 2, 3, \\
\text{Str}(P^n) + C_n^1 \text{Str}(P^{n-1}Q) + (C_n^2 - K_n) \text{Str}(P^{n-2}Q^2) + K_n \text{Str}(P^{n-3}QPQ), & n > 3.
\end{cases}
\]

First of all, the calculation of the Jacobian is based on the previously established properties (B.6) and (B.3), namely,

\[
\text{Str}(Q_1) = 0, \quad \text{Str}(Q_1^2) = 2 \text{Str}(R).
\]
It has also been established (Appendix B.1) that the quantity $\text{Str}(R)$ in (B.16) cancels the contribution $\text{Str}(Q_1^2)$ to the Jacobian, where these contributions enter in the first and second orders, $\text{Str}(M^1)$ and $\text{Str}(M^2)$, respectively, thus summarily producing an identical zero:

$$\text{Str}(R) - (1/2) \text{Str}(Q_1^2) = 0 .$$  \hfill (B.33)

Therefore, we can exclude $\text{Str}(R)$ and $\text{Str}(Q_1^2)$ from further consideration.

Recalling that $\lambda_a = s_a A$, we can deduce the additional properties

$$P^2 = f \cdot P , \quad QP = (1 + f) \cdot Q_2 , \quad f = -\frac{1}{2} \text{Str}(P) ,$$  \hfill (B.34)

where the quantity $f$ is given by

$$\delta \lambda b \over \delta \phi A X^{Aa} = s^a \lambda_b = \delta_a^b f \implies f = \frac{1}{2} s^a \lambda_a = -\frac{1}{2} s^2 A .$$  \hfill (B.35)

Indeed,

$$(P^2)^A_B = (P)^A_D (P)_B^D = X^{Aa} \left( \frac{\delta \lambda a}{\delta \phi B} X^{Db} \right) \frac{\delta \lambda b}{\delta \phi B} = f \cdot \delta^a_a X^{Aa} \frac{\delta \lambda b}{\delta \phi B} = f \cdot (P)_B^A ,$$

$$\frac{\delta \lambda a}{\delta \phi B} X^{Bb} = s^b \lambda_a = s^b s_a A = \delta_a^b f , \quad f = \lambda_A Y^A - (1/2) \varepsilon_{a b} X^{Aa} \lambda_{A B} X^{Bb} ,$$

$$f = \frac{1}{2} \left( \frac{\delta \lambda a}{\delta \phi A} X^{Aa} \right) = -\frac{1}{2} (P)_A^A (1 + f)^a = -\frac{1}{2} \text{Str}(P) .$$  \hfill (B.36)

As a consequence, we have $QP = (1 + f) \cdot Q_2$, namely, in view of $X^{Aa} X^{Bb} = \varepsilon_{a b} Y^A$ from (2.17),

$$(QP)^A_B = Q^A_D P^D_B = \varepsilon_{a b} Y^A \lambda_a \left( \frac{\delta X^{Aa}}{\delta \phi D} + Y^A \frac{\delta \lambda a}{\delta \phi D} \right) X^{Bd} \frac{\delta \lambda b}{\delta \phi B}$$

$$= \varepsilon_{a b} Y^A \lambda_a \left[ \delta X^{Aa} \left( \frac{\delta \lambda a}{\delta \phi B} \right) + \varepsilon_{a b} Y^A \delta \lambda a \right] \left( \frac{\delta \lambda b}{\delta \phi B} \right) = \varepsilon_{a b} Y^A \lambda_a \left( \frac{\delta \lambda a}{\delta \phi B} \right) X^{Bb}$$

Finally,

$$\text{Str}(P^n) = f^{n-1} \text{Str}(P) = -2 f^n ,$$

$$\text{Str}(P^n-Q) = \begin{cases} \text{Str}(Q) = \text{Str}(Q_2) , & n \geq 1 , \\ f^n-2 \text{Str}(PQ) = f^{n-2} (1 + f) \text{Str}(Q_2) , & n = 1 , \\ f^n-3 \text{Str}(PQ^2) = f^{n-3} \text{Str}(Q(PQ)) = f^{n-3} (1 + f) \text{Str}(Q_1 + Q_2) Q_2 , & n = 2 , \\ f^n-4 \text{Str}(PQQ) = f^{n-4} \text{Str}(Q(PQPQ)) = f^{n-4} (1 + f)^2 \text{Str}(Q_2^2) , & n = 3 , \\ n \geq 4 \end{cases} .$$  \hfill (B.38)

where the term $\text{Str}(Q_2^2)$ has been omitted according to the previous considerations related to (B.33).

We further notice that $\text{Str}(Q_1 Q_2) \neq 0$. Indeed, due to $X^{Bb} X^{Bb} = \varepsilon_{a b} Y^A$ and $Y^A_B X^{Bb} = 0$ in (2.17), we have

$$(Q_1 Q_2)_A^A (1 + f)^a = \lambda_a \left[ \frac{\delta X^{Aa}}{\delta \phi B} Y^B \frac{\delta \lambda^2}{\delta \phi A} \right] = \frac{1}{2} \left( \frac{\delta X^{Aa}}{\delta \phi B} X^{Bb} \right) X^{Bd} \frac{\delta \lambda^2}{\delta \phi A}$$

$$= \frac{1}{2} \lambda_a \delta X^{Aa} \delta X^{Bb} \frac{\delta \lambda^2}{\delta \phi A} = \frac{1}{2} \lambda_a \left( \frac{\delta X^{Aa}}{\delta \phi B} \right) X^{Bd} \frac{\delta \lambda^2}{\delta \phi A}$$

$$= \frac{1}{2} \lambda_a \left[ \delta X^{Aa} \left( \frac{\delta \lambda^2}{\delta \phi B} \right) X^{Bb} \right] \left( \frac{\delta X^{Aa}}{\delta \phi B} \right) X^{Bd} \frac{\delta \lambda^2}{\delta \phi A}$$

$$= \frac{1}{2} \lambda_a \left[ \delta X^{Aa} \left( \frac{\delta \lambda^2}{\delta \phi B} \right) X^{Bb} \right] \delta X^{Aa} \delta \lambda^2 \frac{\delta \lambda^2}{\delta \phi A}.$$  \hfill (B.39)
Besides,
\[ \text{Str} (Q_2^2) = \text{Str}^2 (Q_2) \neq 0 . \]  \hspace{1cm} (B.40)

Indeed,
\[
(Q_2)_{A}^A (-1)^F = \lambda_a Y^A \frac{\delta \lambda^a}{\delta \phi^A} ,
\]  \hspace{1cm} (B.41)
\[
(Q_2)_{B}^A (Q_2)_{A}^B (-1)^F = \left( \lambda_a Y^B \frac{\delta \lambda^a}{\delta \phi^A} \right) \left( \lambda_b Y^A \frac{\delta \lambda^b}{\delta \phi^B} \right) .
\]  \hspace{1cm} (B.42)

Therefore, \( \exists \) in the expression (B.28) for the Jacobian exp (\( \exists \)) has the general structure
\[ \exists = A (f) + B (f | Q_2) + C (f | Q_1 Q_2) , \]  \hspace{1cm} (B.43)
for \( B (f | Q_2) = b_1 (f) \text{Str} (Q_2) + b_2 (f) \text{Str} (Q_2^2) = [b_1 (f) + b_2 (f) \text{Str} (Q_2)] \text{Str} (Q_2) , \) and \( C (f | Q_1 Q_2) = c (f) \text{Str} (Q_1 Q_2) . \)

Let us examine \( A (f) \), namely,
\[ A (f) = - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{Str} (P^n) = 2 \sum_{n=2}^{\infty} \frac{(-1)^n}{n} f^n = -2 \ln (1 + f) . \]  \hspace{1cm} (B.44)

Let us examine the explicit structure of the series related to \( b_1 (f) \): the quantity \( \text{Str} (Q_2) \) derives from \( \text{Str} (P^{n-1} Q) \) for \( n \geq 1 \) in (B.38), and is coupled with the combinatorial coefficient \( C_1^n \). The part of \( \exists \) containing \( \text{Str} (Q_2) \) is given by
\[ b_1 (f) \text{Str} (Q_2) = C_1^n \text{Str} (Q_2) - \sum_{n=2}^{\infty} \frac{(-1)^n}{n} C_1^n f^{n-2} (1 + f) \text{Str} (Q_2) , \]  \hspace{1cm} (B.45)
whence
\[ b_1 (f) = 1 - (1 + f) \sum_{m=0}^{\infty} (-1)^m f^m = 1 - (1 + f) (1 + f)^{-1} \equiv 0 . \]  \hspace{1cm} (B.46)

Let us examine the explicit structure of the series related to \( b_2 (f) \): the quantity \( \text{Str}^2 (Q_2) \) derives from \( \text{Str} (P^{n-2} Q^2) \) for \( n \geq 2 \) in (B.38), coupled with the combinatorial coefficients \( C_2^n \) for \( n = 2, 3 \) and \( (C_2^n - K_n) \) for \( n > 3 \), and also derives from \( \text{Str} (P^{n-3} Q PQ) \) for \( n > 3 \) in (B.38), coupled with the combinatorial coefficients \( K_n \). The part of \( \exists \) containing \( \text{Str}^2 (Q_2) \) reads
\[ b_2 (f) \text{Str}^2 (Q_2) = \frac{(-1)^2}{2} C_2^2 \text{Str}^2 (Q_2) - \frac{(-1)^3}{3} C_3^2 (1 + f) \text{Str}^2 (Q_2) - \sum_{n=4}^{\infty} \frac{(-1)^n}{n} (C_2^n - K_n) f^{n-3} (1 + f) \text{Str}^2 (Q_2) = \sum_{n=4}^{\infty} \frac{(-1)^n}{n} K_n f^{n-4} (1 + f)^2 \text{Str}^2 (Q_2) , \]  \hspace{1cm} (B.47)
whence
\[ b_2 (f) = \frac{1}{2} + (1 + f) - \sum_{n=4}^{\infty} \frac{(-1)^n}{n} \left[ (C_2^n - K_n) f^{n-3} (1 + f) + K_n f^{n-4} (1 + f)^2 \right] \]
\[ = \frac{1}{2} + f - (1 + f) \sum_{n=4}^{\infty} \frac{(-1)^n}{n} (C_2^n f^{n-3} + K_n f^{n-4}) \]
\[ = \frac{1}{2} + f - (1 + f) \left[ \frac{1}{2} - \sum_{m=1}^{\infty} (-1)^m \left( \frac{C_{m+3}}{m + 3} - \frac{K_{m+4}}{m + 4} \right) f^m \right] . \]  \hspace{1cm} (B.48)
By virtue of (B.20), this implies the vanishing of $b_2(f)$, namely,

$$b_2(f) = \frac{1}{2} f + (1 + f) \sum_{m=1}^{\infty} (-1)^m \left( \frac{1}{2} \right) f^m = \frac{1}{2} f + \frac{1}{2} (1 + f) \sum_{m=1}^{\infty} (-1)^m f^m$$

$$= \frac{1}{2} f + \frac{1}{2} (1 + f) \left[ (1 + f)^{-1} - 1 \right] \equiv 0 .$$

(B.49)

Let us examine the explicit structure of the series related to $c(f)$: the quantity $\text{Str} (Q_1 Q_2)$ derives from $\text{Str} (P^{n-2} Q^2)$ for $n \geq 2$ in (B.38), and is coupled with the combinatorial coefficients $C_n^2$, for $n = 2, 3$, and $C_n^2 - K_n$, for $n > 3$. The part of $\mathcal{S}$ containing $\text{Str} (Q_1 Q_2)$ is given by

$$c(f) \text{Str} (Q_1 Q_2) = - \frac{(1)^2}{2} C_n^2 \text{Str} (2Q_1 Q_2) - \frac{(1)^3}{3} C_n^2 (1 + f) \text{Str} (Q_1 Q_2)$$

$$- \sum_{n=1}^{\infty} \frac{(1)^n}{n} \left( C_n^2 - K_n \right) f^{n-3} (1 + f) \text{Str} (Q_1 Q_2) ,$$

(B.50)

whence

$$c(f) = -1 + (1 + f) - \sum_{n=1}^{\infty} \frac{(1)^n}{n} \left( C_n^2 - K_n \right) f^{n-3} (1 + f) = f - (1 + f) \sum_{n=1}^{\infty} (-1)^n \left( \frac{C_n^2}{n} - \frac{K_n}{n} \right) f^{n-3} .$$

(B.51)

By virtue of (B.20), this implies the vanishing of $c(f)$, namely,

$$c(f) = f - (1 + f) \sum_{n=1}^{\infty} (-1)^n f^{n-3} = f + (1 + f) \sum_{m=1}^{\infty} (-1)^m f^m = f + (1 + f) \left[ (1 + f)^{-1} - 1 \right] \equiv 0 .$$

(B.52)

From the vanishing of all the coefficients $b_1(f), b_2(f), c(f)$, due to (B.36), (B.40), (B.49), we conclude that

$$B(f|Q_2) = b_1(f) \text{Str} (Q_2) + b_2(f) \text{Str} (Q_2^2) \equiv 0 \quad \text{and} \quad C(f|Q_1 Q_2) = c(f) \text{Str} (Q_1 Q_2) \equiv 0 ,$$

(B.53)

and therefore the Jacobian $\exp (\mathcal{S})$ is finally given by

$$\mathcal{S} = A(f) + B(f|Q_2) + C(f|Q_1 Q_2) = A(f) = - 2 \ln (1 + f) \quad \text{for} \quad f = - (1/2) s^2 \Lambda ,$$

(B.54)

which is identical with (B.17).

### C BRST-antiBRST Invariant Yang–Mills Action in $R_\xi$-like Gauges

In this Appendix, we present the details of calculations used in Section 4 to establish a correspondence between the gauge-fixing procedures in the Yang–Mills theory described by a gauge-fixing function $\chi(\phi) = 0$ from the class of $R_\xi$-gauges in the BV formalism [39] and by a gauge-fixing functional $F$ in the BRST-antiBRST quantization [15] [16].

The Yang–Mills theories belong to the class of irreducible gauge theories of rank 1 with a closed algebra, which implies that $M^{ij}_{a\beta} = 0$ in (2.20), and that any solution of the equation $R^i_a X^a = 0$ has the form $X^a = 0$. The corresponding space of fields and antifields $(\phi^A, \phi^*_A, \bar{\phi})$ is given by

$$\phi^A = (A^i, B^a, C^{\alpha \alpha}) , \quad \phi^*_{Aa} = (A^*_{a\alpha}, B^*_a, C^{\alpha \alpha}_{a\beta}) , \quad \bar{\phi} = (\bar{A}_i, \bar{B}_a, \bar{C}_{aa}) ,$$

(C.1)

as we take into account (2.1) and the following distribution of the Grassmann parity and ghost number:

$$\varepsilon(\phi^A) \equiv (\varepsilon_i, \varepsilon_\alpha, \varepsilon_\alpha + 1) , \quad \text{gh}(\phi^A) = \left(0, 0, (-1)^{\varepsilon_\alpha + 1}\right) ,$$

(C.2)
whereas a solution to the generating equations (2.10) with a vanishing right-hand side can be found in the linear form (2.10), \( S = S_0 + \phi A_a X^{Aa} + \tilde{\phi}_A Y^A \), obviously satisfying the boundary condition \( S|_{\phi = \tilde{\phi} = 0} = S_0 \). Here, the functionals \( X^{Aa} \) and \( Y^A \) can be chosen as (15)

\[
X^{Aa} = (X_1^{ia}, X_2^{a}, X_3^{ab}) \ , \quad Y^A = (Y_1^i, Y_2^a, Y_3^{ab}) \ ,
\]

where

\[
X_1^{ia} = R^i_{a} C^{ia} \ , \quad X_2^{a} = - \frac{1}{2} F^{a}_{\beta} B^{\beta} C^{a} - \frac{1}{12} (-1)^{i} \sum (2 F^{a}_{\gamma j} R^i_{\gamma} + F^{a}_{\rho \beta} F^{\rho \beta}) C^{b \alpha} C^{c \epsilon} \varepsilon_{cb} \ , \quad X_3^{ab} = - \varepsilon^{ab} B^a - \frac{1}{2} (-1)^{\rho} F^{a}_{\beta} C^{b \epsilon} C^{\beta a} \ , \quad Y_1^{i} = R^i_{a} B^a + \frac{1}{2} (-1)^{\rho} R^i_{\alpha, \beta} R^i_{\gamma} C^{b \epsilon} C^{a} \varepsilon_{ab} \ , \quad Y_2^{a} = 0 \ , \quad Y_3^{ab} = - 2 X_3^{ab} .
\]

By construction, the functionals \( X^{Aa} = \delta S / \delta \phi_a A_a \) and \( Y^A = \delta S / \delta \tilde{\phi}_A \) obey the properties \( S_0, X^{ia} = 0 \), \( X^{Aa} X^{Bb} = \varepsilon^{ab} Y^A \), \( Y^A X^{Aa} = 0 \). Besides, in Yang–Mills theories the explicit form (1.2), (4.3) of the gauge generators \( R^i_{a} \) and structure coefficients \( F^{a}_{\alpha \beta} = \text{const} \) is such that \( X^{Aa} = (X_1^{ia}, X_2^{a}, X_3^{ab}) \) in (C.3) possess the properties \( X^{Aa}_1 = 0 \), so that the entire set of relations (2.11) is fulfilled, and the solution given by (C.4) satisfies the generating equations (2.5) identically.

As we keep the following consideration restricted to the case of constant structure coefficients, \( F^{a}_{\beta \gamma, j} = 0 \), let us choose the gauge-fixing functional \( F(\phi) \) in the form

\[
F = F(A, C) \ , \quad \frac{\delta^2 F}{\delta A_i \delta A_j} \neq 0 \ , \quad \frac{\delta^2 F}{\delta C^{a \alpha} \delta C^{a \alpha}} \neq 0 .
\]

By virtue of (C.4), the quantum action \( S_F(\phi) \) in (2.22) reads as follows:

\[
S_F = S_0 + \frac{\delta F}{\delta A^i} \left( R^i_{a} B^a + \frac{1}{2} (-1)^{\gamma} R^i_{\alpha, \beta} R^i_{\gamma} C^{b \epsilon} C^{a \alpha} \varepsilon_{ab} \right) - \frac{1}{2} \varepsilon_{ab} \left( R^i_{a} C^{ia} \right) \frac{\delta^2 F}{\delta A^i \delta A^j} \left( R^j_{a} C^{ib} \right) + \frac{\delta F}{\delta C^{a \alpha}} \left( F^{a}_{\gamma \beta} B^{\beta} C^{a} + \frac{1}{6} (-1)^{\rho} F^{a}_{\gamma \sigma} F^{\rho \beta} C^{b \epsilon} C^{\beta a} C^{\gamma c} \varepsilon_{cb} \right) - \frac{1}{2} \varepsilon_{ab} \left( F^{a}_{\gamma \beta} C^{b \epsilon} C^{\gamma a} \right) \frac{\delta F}{\delta C^{a \alpha}} \left( \varepsilon^{bd} B^{\beta} + \frac{1}{2} (-1)^{\rho} F^{b \epsilon} C^{\rho d} C^{\epsilon} \right) .
\]

Using the identity

\[
\frac{\delta F}{\delta A^i} \left( R^i_{a} B^a + \frac{1}{2} (-1)^{\gamma} \varepsilon_{ab} \frac{\delta F}{\delta A^i} \left( R^i_{a} C^{ia} \right) \right) = \varepsilon_{ab} \left( \chi_{a} B^{a} + \frac{1}{2} (-1)^{\gamma} \varepsilon_{ab} \left( \chi_{a} R^i_{a} \right) C^{ib} C^{a} \right) \varepsilon_{ab} \quad \text{for} \quad \varepsilon \left( A^i \right) = \varepsilon \left( A^{ia} \right) + 1 = \varepsilon_i , \quad \varepsilon \left( C^{a b} \right) = \varepsilon \left( C^{a \alpha} \right) + 1 = \varepsilon_{a} .
\]

we obtain

\[
S_F = S_0 + \frac{\delta F}{\delta A^i} \left( A^i - \frac{1}{2} \varepsilon_{ab} \left[ \frac{\delta}{\delta A^i} \left( \frac{\delta F}{\delta A^i} \right) \right] A^{i b} + \frac{\delta F}{\delta C^{a \alpha}} C^{a \alpha} - \frac{1}{2} \varepsilon_{ab} C^{a \alpha c} \left( \frac{\delta}{\delta C^{b d}} \frac{\delta F}{\delta C^{c \alpha}} \right) C^{b d} \right) ,
\]

where

\[
A^i \equiv R^i_{a} B^a , \quad A^{ia} \equiv R^i_{a} C^{ia} , \quad C^{a \alpha} \equiv F^{a}_{\gamma \beta} B^{\beta} C^{a} + \frac{1}{6} (-1)^{\rho} F^{a}_{\gamma \sigma} F^{\rho \beta} C^{b \epsilon} C^{\beta a} C^{\gamma c} \varepsilon_{cb} , \quad \varepsilon \left( A^i \right) = \varepsilon \left( A^{ia} \right) + 1 = \varepsilon_i , \quad \varepsilon \left( C^{a b} \right) = \varepsilon \left( C^{a \alpha} \right) + 1 = \varepsilon_{a} .
\]

For Yang–Mills theories, with the classical action \( S_0 \), gauge generators \( R^i_{a} \) and structure coefficients \( F^{a}_{\alpha \beta} \) given by (1.2), (4.3), (4.3), and with the set of fields \( \phi^A \) given by (4.3), (4.5), the relations (C.8), (C.9) take the form

\[
S_F = S_0 + \int d^3 x \left\{ \frac{\delta F}{\delta A^{m a}} A^{m u} - \frac{1}{2} \varepsilon_{ab} \left[ \frac{\delta}{\delta A^{m u}} \left( \frac{\delta F}{\delta A^{m a}} \right) A^{n b} \right] \right\} + \int d^3 x \left[ \frac{\delta F}{\delta C^{m a}} C^{m a} - \frac{1}{2} \varepsilon_{ab} C^{m a c} \left( \frac{\delta}{\delta C^{n b}} \frac{\delta F}{\delta C^{c m}} \right) C^{n d} \right) ,
\]

(C.10)
Choosing the gauge-fixing functional $F(A, C)$ in the quadratic form (4.11) and using the identities (for arbitrary $su(N)$-vectors $F^m$ and $C^m$)

$$D^m_{\mu} A^{\mu} = \frac{\partial}{\partial x^\mu} (D^m_{\mu} F^m) = - \int d^D x \, F^m D^m_{\mu} G^m,$$  

we have

$$\delta_A F = - \alpha \int d^D x \, A^m_{\mu} \delta A^{\mu},$$

$$\frac{\delta F}{\delta A^m_{\mu}} A^{\mu} = - \alpha \int d^D x \, A^m_{\mu} D^m_{\mu} B^n = \alpha \int d^D x \, (D^m_{\mu} A^{\mu}) B^n = \alpha \int d^D x \, (\partial_{\mu} A^{\mu}) B^n,$$

$$\frac{\delta F}{\delta A^m_{\mu}} A^{\mu a} = - \alpha \int d^D x \, A^m_{\mu} D^m_{\mu} C^m = \alpha \int d^D x \, (\partial_{\mu} A^{\mu}) C^m,$$

whence

$$\delta_A \left( \frac{\delta F}{\delta A_{\mu a} A^{\mu a}} \right) = \alpha \int d^D x \, (\partial_{\mu} A^{\mu}) C^m a = - \alpha \int d^D x \, (\partial_{\mu} C^m a) \delta A^{\mu},$$

$$\int d^D x \left[ \frac{\delta}{\delta A_{\mu a} A^{\mu a}} \right] A_{\mu a} = - \alpha \int d^D x \, (\partial_{\mu} C^m a) D^m_{\mu} C^m a.$$

Next,

$$\delta_C F = - \beta \varepsilon_{ba} \int d^D x \, C^{mb} \delta C^m a \iff \frac{\delta F}{\delta C^m a} = \beta \varepsilon_{ab} C^{mb},$$

$$\int d^D x \frac{\delta F}{\delta C^m a} C^m a = \beta \varepsilon_{ab} \int d^D x \, C^{mb} C^m a = \beta \varepsilon_{ba} \int d^D x \, C^m a \left( f^{mn l} B^l C^m b + \frac{1}{6} f^{mn l} f^{rs C^d C^e C^m e} \right).$$

At the same time,

$$\delta_C \left( \frac{\delta F}{\delta C^{mc} (x)} \right) = \beta \varepsilon_{cd} \delta C^{md} (x) = \beta \varepsilon_{cd} \int d^D y \, \delta^{mn} \delta (y - x) \delta C^{nd} (y),$$

$$\delta \left( \frac{\delta F}{\delta C^{mc} (y)} \right) = \beta \varepsilon_{cd} \delta^{mn} \delta (y - x),$$

whence

$$- \frac{1}{2} \varepsilon_{ab} \int d^D x \, d^D y \, C^{mac} (x) \frac{\delta}{\delta C^{nd} (y)} \left( \frac{\delta F}{\delta C^{mc} (x)} \right) C^{nbd} (y)$$

$$= \frac{1}{2} \varepsilon_{ab} \int d^D x \, d^D y \, C^{mac} (x) \delta \varepsilon_{cd} \delta^{mn} \delta (y - x) \delta C^{nbd} (y)$$

$$= \frac{1}{2} \varepsilon_{ab} \varepsilon_{cd} \int d^D x \left( \varepsilon^{ac} B^m + \frac{1}{2} f^{mn l} C^{dc C^m a} \right) \left( \varepsilon^{bd} B^m + \frac{1}{2} f^{mrs C^d C^e C^m e} \right).$$

Therefore,

$$\int d^D x \left[ \frac{\delta F}{\delta C^m a} - \frac{1}{2} \varepsilon_{ab} C^{mac} \frac{\delta}{\delta C^{nd} (y)} \left( \frac{\delta F}{\delta C^{mc} (x)} \right) C^{nbd} \right]$$

$$= - \beta \varepsilon_{ab} \int d^D x \, C^m a \left( f^{mn l} B^l C^m b + \frac{1}{6} f^{mn l} f^{rs C^d C^e C^m e} \right)$$

$$- \frac{1}{2} \varepsilon_{ab} \varepsilon_{cd} \int d^D x \left( \varepsilon^{ac} B^m + \frac{1}{2} f^{mn l} C^{dc C^m a} \right) \left( \varepsilon^{bd} B^m + \frac{1}{2} f^{mrs C^d C^e C^m e} \right).$$
Finally,

\[ S_F(A, B, C) = S_0(A) + S_1(A, B) + S_2(A, C) + S_3(A, B, C), \]  
\[ \text{(C.22)} \]

where

\[ S_1 = \alpha \int d^D x \ (\partial^\mu A^m_\mu) B^m, \quad S_2 = \frac{\alpha}{2} \varpi_{ab} \int d^D x \ (\partial^\mu C^{ma}) D^{mn}_{\mu} C^{nb}, \]

\[ S_3 = -\beta \varpi_{ab} \int d^D x \ C^{ma} \left( f^{mnl} B^l C^{mb} + \frac{1}{2} f^{mnl} f^{rs} C^{sd} C^{rb} C^{nc} \varpi_{cd} \right) \]

\[ - \frac{\beta}{2} \varpi_{ab} \varpi_{cd} \int d^D x \left( \varepsilon^{ac} B^m + \frac{1}{2} f^{mnl} C^{la} C^{ma} \right) \left( \varepsilon^{bd} B^m + \frac{1}{2} f^{mrs} C^{sd} C^{rb} \right). \]  
\[ \text{(C.23)} \]

By virtue of the identity \( f^{lmn} C^{nb} C^{ma} \varpi_{ab} \equiv 0 \), the quantum action \( \text{(C.22)} \) equals to \( \text{(4.12)} \).

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