Categorical Bernstein operators and the Boson-Fermion correspondence

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Abstract
We prove a conjecture of Cautis and Sussan providing a categorification of the Boson-Fermion correspondence as formulated by Frenkel and Kac. We lift the Bernstein operators to infinite chain complexes in Khovanov’s Heisenberg category \( \mathcal{H} \) and from them construct categorical analogues of the Kac-Frenkel fermionic vertex operators. These fermionic functors are then shown to satisfy categorical Clifford algebra relations, solving a conjecture of Cautis and Sussan. We also prove another conjecture of Cautis and Sussan demonstrating that the categorical Fock space representation of \( \mathcal{H} \) is a direct summand of the regular representation by showing that certain infinite chain complexes are categorical Fock space idempotents. In the process, we enhance the graphical calculus of \( \mathcal{H} \) by lifting various Littlewood-Richardson branching isomorphisms to the Karoubian envelope of \( \mathcal{H} \).

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1 Introduction

Building upon his ideas to categorify quantum groups and associated TQFTs, Igor Frenkel conjectured that conformal field theory could be categorified as well. As a starting point, he suggested that the Boson-Fermion correspondence should admit a categorification. One of the first breakthroughs came when Khovanov categorified the simplest Heisenberg algebra (see Sect. 4, [1,25]).\(^1\) Shortly after, Cautis and Licata [2] defined Heisenberg categories associated to every affine Dynkin diagram. Using this framework, they defined categorical vertex operators lifting the homogeneous realization of the basic representation of quantum affine algebras within the Frenkel-

\(^1\) In his original paper, Khovanov only proved injectivity \( h \hookrightarrow K_0(\mathcal{H}) \) and conjectured surjectivity. The full bijection was not proved until much later in a recent paper by Brundan, Savage and Webster [1].

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Kac-Segal construction [11,33]. Specifically, they lifted vertex operators to infinite chain complexes in a Heisenberg category, inducing 2-representations of quantum affine algebras. The Boson-Fermion correspondence as formulated by Frenkel [13] and Kac [23] is analogous to the Frenkel-Kac-Segal construction in that it also defines vertex operators from a Heisenberg algebra. However, in the Boson-Fermion correspondence these vertex operators induce an action of the Clifford algebra instead of the quantum affine algebra. Motivated by this, Cautis and Sussan [6] conjectured a categorical formulation of the Boson-Fermion correspondence within the framework of Khovanov’s Heisenberg category $\mathcal{H}$ [25]. One of our main results is the proof of this conjecture (see Theorem 5.10).

Our proof relies on a family of chain complexes in the homotopy category of $\mathcal{H}$ which lift the Bernstein operators introduced by Zelevinsky [39]. These operators create and annihilate Schur functions when acting on the ring of symmetric functions [27]. We show that these categorical Bernstein operators create and annihilate Specht modules when acting on categorical Fock space.

We also prove a series of Cautis-Sussan conjectures [6] regarding certain unbounded chain complexes analogous to those defined in [5]. In that paper, Cautis, Licata, and Sussan prove that these functors satisfy braid group relations, thus showing that integrable 2-representations of the Heisenberg algebra always induce braid group actions. In this article we prove that the analogous complexes defined by Cautis and Sussan are Fock space idempotents (see Sect. 7). This is in line with the decategorified picture where any $\mathfrak{h}$-module generated by highest weight vectors decomposes into a direct sum of infinitely many copies of the trivial module; these functors are categorical analogues of orthogonal projectors onto a summand of Fock space.

1.1 The Boson-Fermion correspondence and its related algebras

Dirac’s sea of electrons is a theoretical model for describing a system of fermionic quantum particles with infinitely many energy states. In this model energy states are indexed by half integers and are assumed to be entirely occupied below and unoccupied above certain values. By the Pauli exclusion principle no two particles may occupy the same energy state. Mathematically this is captured by the following construction. Let $v_i$ denote a particle in energy level $i$ with $i \in \mathbb{Z} + \frac{1}{2}$ and let $\mathbb{k}$ be a field of characteristic zero. If a vector $v_{i_m} \wedge v_{i_{m-1}} \wedge \ldots$ with $i_m > i_{m-1} > \ldots$ denotes a system where only the energy levels $i_m, i_{m-1}, \ldots$ are filled, then a semi-infinite monomial is a vector of this form satisfying $i_s = i_{s-1} + 1$ for all $s \ll 0$. Fermionic Fock space is the $\mathbb{k}$-linear span of all semi-infinite monomials (it is also known as the semi-infinite wedge $\bigwedge^\infty$)[31,34,36]. This space carries a natural grading by charge known as the principal gradation. On the other hand, Bosonic Fock space $\mathcal{B}$ is defined as a graded sum of infinitely many copies of the ring of symmetric functions $\text{Sym}$. Moreover, because Schur functions are a basis for $\text{Sym}$ and are indexed by partitions, then $\mathcal{B}$ has a natural basis indexed by charged partitions. As is the case for $\mathcal{F}$, the charge in $\mathcal{B}$ keeps track of the grading. Since to each half infinite monomial there is a canonical way of assigning a charged partition, it can be shown that Fermionic and Bosonic Fock space are isomorphic as graded vector spaces. Thus, Bosonic and
Fermionic Fock spaces may be identified [23,31,36]. This isomorphism is the first part of the Boson-Fermion correspondence (see Theorem 2.1).

The second part of the Boson-Fermion correspondence concerns the action of the infinite dimensional Clifford and Heisenberg algebras on Fock space. In particular, Fermionic Fock space carries a natural action the infinite dimensional Clifford algebra $\mathcal{C} \mathcal{L}$ given by a family of creation and annihilation operators $\psi_i$ and $\psi_i^*$ which satisfy the anti-commutation relations

$$
\psi_i \psi_j^* + \psi_j^* \psi_i = \delta_{i,j} \quad \psi_i \psi_j + \psi_j \psi_i = 0 \quad \psi_i^* \psi_j^* + \psi_j^* \psi_i^* = 0.
$$

Likewise, Bosonic Fock space has an action of the infinitely generated Heisenberg algebra $\mathfrak{h}$, whose generators $\alpha_n$ and $\alpha_{-n}$ with $n \in \mathbb{Z}_{\geq 0}$ satisfy the commutation relations

$$
\alpha_n \alpha_m - \alpha_m \alpha_n = n \delta_{n,-m}.
$$

As in [3,25], we will instead work with alternate generators for $\mathfrak{h}$ denoted by $p^\lambda$ and $q^\lambda$ where $\lambda$ ranges over all row and column partitions. Given the isomorphism between Bosonic and Fermionic Fock space, there is an induced action of the Clifford algebra on $\mathcal{B}$. The second part of the Boson-Fermion correspondence states that under this isomorphism the action of the operators $\psi_i$ and $\psi_i^*$ on $\mathcal{B}$ can be expressed in terms of the generators of the Heisenberg algebra [23]. More specifically, identifying $\mathcal{B}$ and $\mathcal{F}$ under this isomorphism and denoting by $\psi_i$ and $\psi_i^*$ the images of the fermionic operators on $\mathcal{B}$, the formal power series $\psi(z) : = \sum_{i \in \mathbb{Z}} \psi_i z^i$ and $\psi_i^*(z) : = \sum_{i \in \mathbb{Z}} \psi_i^* z^{-i}$ satisfy the following equalities [23, Theorem 14.10]:

$$
\psi(z) = z^{\alpha_0} t \exp \left( \sum_{m \geq 1} \frac{z^m}{m} \alpha_{-m} \right) \exp \left( - \sum_{m \geq 1} \frac{z^{-m}}{m} \alpha_m \right) \quad (1.2)
$$

$$
\psi^*(z) = t^{-1} z^{-\alpha_0} \exp \left( - \sum_{m \geq 1} \frac{z^m}{m} \alpha_{-m} \right) \exp \left( \sum_{m \geq 1} \frac{z^{-m}}{m} \alpha_m \right). \quad (1.3)
$$

In particular, the correspondence allows us to express the induced action of $\mathcal{C} \mathcal{L}$ on $\mathcal{B}$ in terms of the action of $\mathfrak{h}$. Operators of the form in (1.2) and (1.3) are known as vertex operators [10,11,13,21,22]. They form an integral part in the study of conformal field theories and many other areas of mathematical physics [15,16,24,28]. The process of constructing the Fermionic fields $\psi(z)$ and $\psi^*(z)$ from the Bosonic generators $\alpha_n, \alpha_{-n}$ is known as fermionization. The reverse process which expresses the action of the Heisenberg algebra on Fermionic Fock space as formal power series in terms of the generators of $\mathcal{C} \mathcal{L}$ is known as bosonization. It entails the other side of the Boson-Fermion correspondence [23].

1.2 The Boson-Fermion correspondence categorified

Geissinger [18] showed that the Hopf algebra of symmetric functions $\text{Sym}$ could be categorified by modules over $\mathbb{k}[S_n]$ for any field $\mathbb{k}$ of characteristic zero. If $\mathcal{K}_0$ denotes
the split Grothendieck ring of the category, then \( \text{Sym} \cong \bigoplus_{n \geq 0} K_0(\mathbb{k}[S_n]-\text{mod}) \) via the Frobenius character map. In this isomorphism, the algebra and coalgebra structure of \( \text{Sym} \) correspond to induction and restriction functors on \( \bigoplus_n \mathbb{k}[S_n]-\text{mod} \).

Building upon these ideas, Khovanov constructed the first categorification of the Heisenberg algebra \( \mathfrak{h} \) [25]. Khovanov’s Heisenberg category \( \mathcal{H} \) is a diagrammatic, additive, monoidal category with generating objects \( P \) and \( Q \) and whose action on \( \bigoplus_n \mathbb{k}[S_n]-\text{mod} \) is given by induction and restriction functors (see Sect. 4). When taking the Grothendieck group, the generating objects of \( \mathcal{H} \) descend to generators of \( \mathfrak{h} \) given by \( p \) and \( q \). Since the Hopf algebra of symmetric functions is categorified by \( \bigoplus_n \mathbb{k}[S_n]-\text{mod} \) and \( B \) consists of infinitely many copies of \( \text{Sym} \), it is natural to define categorical Fock space \( \mathcal{V}_{\text{Fock}} \) as the direct sum of copies of \( \bigoplus_n \mathbb{k}[S_n]-\text{mod} \) indexed by integers. We will encode this in terms of infinite vectors over \( \bigoplus_n \mathbb{k}[S_n]-\text{mod} \) with finitely many nonzero entries.

Categorification by nature produces integral structures when decategorified. If an algebra \( \mathcal{A} \) arises as the Grothendieck ring of some (reasonable) category, it will usually inherit an integral basis given by the isomorphism classes of indecomposable objects in \( \mathcal{C} \). In the Heisenberg algebra, there is an identification between its generators and symmetric functions which precisely exemplifies this fact. When acting on \( \text{Sym} \), the action of \( p^\lambda \) and \( q^\lambda \), when \( \lambda \) is a row or column partition, is given by multiplication by elementary and complete symmetric functions and their adjoints, both of which are generators over \( \mathbb{Z} \) for \( \text{Sym} \). However, the action of the generators \( \alpha_n \) and \( \alpha_{-n} \) is through power sum symmetric functions which generate \( \text{Sym} \) over \( \mathbb{Q} \). This is the fundamental reason behind the use of the \( p^{(n)} \) and \( q^{(n)} \) generators for \( \mathfrak{h} \) in this paper and why, in Khovanov’s formulation, these are the generators that lift to indecomposable objects in \( \mathcal{H} \). Under this identification, the vertex operators \( \psi \) and \( \psi^* \) from Kac’s formulation in equations (1.2) and (1.3) can be rewritten as infinite series in terms of \( p^\lambda \) and \( q^\lambda \). Consequently, the action of the Fermionic fields on \( B \) is given by shifted products of the Bernstein operators \( B_a \) and \( B^*_a \) (see Sect. 2.5, [27,39]).

Cautis and Sussan conjectured a categorical Boson-Fermion correspondence where the generators of the Clifford algebra, when realized as vertex operators, lift to infinite complexes in the homotopy category of Khovanov’s Heisenberg category \( \mathcal{K}(\mathcal{H}) \) [6]. They proposed that these functors satisfy categorical analogues of the Clifford algebra relations (2.6) up to homotopy. In this paper we prove a slightly modified but equivalent formulation of their conjecture.

To this effect, we introduce the categorical Bernstein operators, indexed by \( a \in \mathbb{Z} \), as the following unbounded complexes in \( \mathcal{K}(\mathcal{H}) \)^2:

\[
B_a := \left( \cdots \to p^{(x+a)}Q^{(1^x)} \to p^{(x+a-1)}Q^{(1^{x-1})} \to \cdots \right) \quad \in \mathcal{K}^{-}(\mathcal{H})
\]

\[
B^*_a := \left( \cdots \to p^{(1^y)}Q^{(y+a)} \to p^{(1^{y+1})}Q^{(y+a+1)} \to \cdots \right) \quad \in \mathcal{K}^{+}(\mathcal{H})
\]

whose differentials are given by adjunction maps. When taking the Grothendieck ring these functors recover the original Bernstein operators and their properties. In

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2 The complexes \( B_a \), \( B^*_a \) are equivalent to the complexes \( C_a \), \( C^*_a \) originally defined by Cautis and Sussan via the involution \( a \mapsto -a \) and a homological grading shift of \([a]\).
particular, the categorical Bernstein operators have a natural action on $\bigoplus_n \mathbb{k}[S_n]$-mod induced by the action of $\mathcal{H}$. In Theorems 6.6, 6.8, 6.14, and 6.15 we show these functors satisfy categorical lifts of the commutation relations fulfilled by the Bernstein operators $B_a$ and $B^*_a$ [27, Section 1.4]. In particular, we prove the following theorem (see Theorem 5.5).

**Theorem 1.1** The categorical Bernstein operators are creation and annihilation functors for Specht modules. That is, for any Specht module $S_\lambda$ associated to partition $\lambda = (\lambda_k, \ldots, \lambda_1) \vdash n$ with $\lambda_k \geq \cdots \geq \lambda_1 \geq 0$ and $\mathbb{k}$ the trivial module over $\mathbb{k}[S_n]$-mod, then $B_{\lambda_k}B_{\lambda_{k-1}} \ldots B_{\lambda_1}(\mathbb{k}) \simeq S_\lambda$ and $B^*_{\lambda_k} \cdots B^*_{\lambda_{k-1}}B^*_k(S_\lambda) \simeq \mathbb{k}$.

As evidenced by Eqs. (1.2) and (1.3), the action of the vertex operators on $\mathcal{B}$ does not preserve the charge but instead expresses $\psi_i$ and $\psi^*_i$ in terms of Bernstein operators and a charge function $z^{a_0}$. In their original conjecture, Cautis and Sussan account for the charge operator by incorporating an additional variable $q$ to keep track of the principal gradation. The approach taken here is different. Denote by $\text{Mat}_{\mathbb{Z} \times \mathbb{Z}}(\mathcal{K}(\mathcal{H}))$ the category of unbounded matrices with certain row-finiteness conditions whose entries are chain complexes in $\mathcal{K}(\mathcal{H})$ (see Definition 5.1). Within this framework, we define the charge operators $Q, Q^{-1}$ in $\text{Mat}_{\mathbb{Z} \times \mathbb{Z}}(\mathcal{K}(\mathcal{H}))$ that act on $V_{\text{Fock}}$ by raising and lowering the charge of each entry in a vector (see Sect. 5.2). Employing these charge operators and infinite matrices of categorical Bernstein operators we introduce categorical analogues of the Fermionic vertex operators given in (1.2) and (1.3). The Fermionic functors $\Psi_i$ and $\Psi^*_i$ are defined as unbounded matrices in $\text{Mat}_{\mathbb{Z} \times \mathbb{Z}}(\mathcal{K}(\mathcal{H}))$ whose entries consist of $B_a$ and $B^*_a$ for varying values of $a \in \mathbb{Z}$ and whose action on $V_{\text{Fock}}$ is given by matrix multiplication. This differs from the approach taken by Cautis and Sussan since they defined the action of $\Psi$ and $\Psi^*$ pointwise on each $v \in \bigoplus \mathbb{k}[S_n]$-mod. In Theorem 5.10 we prove these categorical vertex operators $\Psi_i$ and $\Psi^*_i$ satisfy categorical analogues of the Clifford algebra relations (1.1) and thus lift the Kac-Frenkel formulation of the Boson-Fermion correspondence.

**Theorem 1.2** (Categorical Boson-Fermion correspondence) The Fermionic functors $\Psi_i$ and $\Psi^*_i$ satisfy the following relations in $\text{Mat}_{\mathbb{Z} \times \mathbb{Z}}(\mathcal{K}(\mathcal{H}))$:

1. $(\Psi_i)^2 \simeq 0$ and $\Psi_i \Psi_j \simeq \begin{cases} \Psi_j \Psi_i[-1] & \text{if } i > j \\ \Psi_j \Psi_i[1] & \text{if } i < j \end{cases}$
2. $(\Psi_i^*)^2 \simeq 0$ and $\Psi_i^* \Psi_j^* \simeq \begin{cases} \Psi_j^* \Psi_i^*[-1] & \text{if } i > j \\ \Psi_j^* \Psi_i^*[1] & \text{if } i < j \end{cases}$

Moreover, the following relations also hold in $\text{Mat}_{\mathbb{Z} \times \mathbb{Z}}(\mathcal{K}^{\text{ch}}(\mathcal{H}))$:

3. $\Psi_i \Psi_j^* \simeq \begin{cases} \Psi_j^* \Psi_i[1] & \text{if } i > j \\ \Psi_j^* \Psi_i[-1] & \text{if } i < j \end{cases}$
4. there exist distinguished triangles $\Psi_i \Psi_i^* \rightarrow 1 \rightarrow \Psi_i^* \Psi_i$ and $\Psi_i^* \Psi_i \rightarrow 1 \rightarrow \Psi_i \Psi_i^*$.

In the original theorem by Kac, the Boson-Fermion correspondence is proven by letting the vertex operators act on Fock space. Since the action is integrable the infinite
series become finite. In our categorification this additional finiteness constraint is not needed for proving relations (1) and (2). These isomorphisms hold between unbounded complexes in the homotopy category of $\mathcal{H}$ with no finiteness conditions imposed at all. Consequently, our result is stronger in the sense that these relations hold in full generality for the functors in $\text{Mat}_{\mathbb{Z} \times \mathbb{Z}}(\mathcal{K}(\mathcal{H}))$. This is because relation (1) only involves complexes that are bounded below and relation (2) only involves complexes that are bounded above. These boundedness conditions are lacking in relations (3) and (4) and cause very serious convergence issues. Specifically, $B_a^\circ$ is bounded below whereas $B_a^\ast$ is bounded above, so relations (3) and (4) consist of homotopy equivalences between chain complexes which are unbounded in both directions. Such complexes do not generally behave well with the usual methods from homological algebra. Things are further complicated by Khovanov’s Heisenberg category being neither graded nor Krull-Schmidt. To overcome this obstruction, we prove isomorphisms (3) and (4) in the context of an arbitrary finite quotient of $\mathcal{H}$ (see Definition 5.2). This allows us to work over the homotopy category of bounded complexes $K^b(\mathcal{H})$, effectively eliminating all convergence issues. Given that the action of $\Psi_i$ and $\Psi_i^\ast$ on $V_{\text{Fock}}$ is integrable, when the construction is decategorified this additional constraint simply recovers the condition that the correspondence holds when acting on Fock space.

To prove Theorem 1.2 we augment Khovanov’s diagrammatic framework to include the images of Young symmetrizers in the Karoubi envelope and construct explicit diagrammatic isomorphisms realizing certain Littlewood-Richardson decompositions (see Sect. 4.3). We hope that these isomorphisms are the beginning of a thick calculus, or complete diagrammatic description of the Karoubi envelope of Khovanov’s Heisenberg category. This is of independent interest since the objects of this Karoubi envelope are natural constructions on induction and restriction functors between modules of the symmetric group.

We also make extensive use of some homological algebra tools for infinite complexes very recently described by Elias and Hogancamp [9]. As was previously noted, manipulating unbounded complexes introduces many technical complications and subtleties that are nonexistent when the chain complexes are finite. In order to derive the desired homotopy equivalences we work with two different notions of the tensor product of complexes normally absent in usual study of homotopy categories.

With these homological techniques we prove a series of commutation relations for the Bernstein functors $B_a$ and $B_a^\ast$, and from those derive the desired categorical Clifford relations stated in Theorem 1.2. Although we give explicit isomorphisms lifting the Clifford relations, we postpone the study of the Hom spaces between the Clifford generators for future work. It is our hope that this leads to a direct diagrammatic categorification of an infinite dimensional Clifford algebra.

In recent related work Frenkel, Penkov, and Serganova propose a categorification of the Boson-Fermion correspondence via the representation theory of $sl(\infty)$ [12]. In their paper they introduce certain chain complexes over the category of tensor modules and prove that when passing to the Grothendieck ring these complexes satisfy the Clifford algebra anticommutation relations. While we anticipate their complexes and ours will be related, the precise connection is unclear. On the one hand, their result is only at the level of the Grothendieck ring. On the other hand, their category not semisimple
and their complexes are not biadjoint. Thus their functors can only be defined in one
direction.

In another related work [35] Tian explores the correspondence from the reverse
direction, that is of bosonization. His approach uses contact geometry and a modifica-
tion of Khovanov’s Heisenberg category. In this context, he constructs functors in
terms of generators of a certain Clifford category and proves they satisfy the commu-
tation relations of a Heisenberg algebra. Once again, his categorical Fock space is not
semisimple and his construction does not admit biadjointness. Nonetheless, given its
connection to Khovanov’s original work, one can surmise a relation with ours.

1.3 Structure of the paper

In Sect. 2 we explain the decategorified picture and define the Heisenberg and Clif-
ford algebras. We also present the vertex operator formulation of the Boson-Fermion
correspondence and explain its relation to symmetric functions and Bernstein opera-
tors. In Sect. 3 we establish some notation and describe the tools from homological
algebra necessary to manipulate infinite complexes. Section 4 provides a summary of
the diagrammatics of Khovanov’s Heisenberg category and expands the calculus by
recalling explicit expressions of generalized Young idempotents. This mainly consists
of constructing various Littlewood-Richardson branching isomorphisms within the
Karoubian envelope of $\mathcal{H}$. The entire categorified construction of the Boson-Fermion
correspondence is explained in Sect. 5. This includes the categorical Bernstein opera-
tors, their relations, and the fermionic functors $\Psi_i$ and $\Psi_i^*$. Due to their technical
complexity, many of the proofs for the relations of the categorical Bernstein operators
are presented in Sect. 6. Finally, in Sect. 7 we prove that certain unbounded chain
complexes, $\Sigma^\pm$ in $\mathcal{K}(\mathcal{H})$, satisfy particular properties conjectured by Cautis and Sus-
san and are thus categorified Fock space idempotents. At the end there is an appendix
presenting worked out examples of the commutation relations between the categorical
Bernstein operators.

2 Fermionic vertex operators and the Boson-Fermion
correspondence

Henceforth let $k$ be a field of characteristic zero. In this section we loosely follow [23,
Chapter 14] and refer the reader to this source and [36] for a more detailed exposition
of these topics.

2.1 Fock space

Given an infinite sequence of integers $i_n \in \mathbb{Z} + \frac{1}{2}$ such that $i_1 > i_2 > \cdots$ and
$i_n = i_{n-1} - 1$ for $n$ large enough, a vector of the form $v = i_1 \wedge i_2 \wedge i_3 \wedge \cdots$ is
called a semi-infinite monomial. Since wedge products anti-commute and are zero
for repeated entries, any nonzero semi-infinite wedge product is equal to a signed
multiple of a semi-infinite monomial. The $k$-linear span of all such vectors is known
Theorem 2.1

Theorem 2.1 also defines Bosonic Fock space also admits a principal gradation as the ring \( B = \mathbb{K}[p_1, p_2, \ldots; t, t^{-1}] \). This space also admits a principal gradation \( B = \bigoplus_{c \in \mathbb{Z}} B_c \) where each \( B_c \cong \mathbb{K}[p_1, p_2, \ldots]^{t^c} \). By identifying each \( p_n \) with the \( n^{th} \) power sum symmetric function then each \( B_c \) is isomorphic to the ring of symmetric functions.

To any vector \( v \) of charge \( c \) in \( F_c \) we can assign a partition \( \lambda(v) = (\lambda_1, \lambda_2, \ldots, \lambda_n) \) given by \( \lambda_j = i_j - c + j - \frac{1}{2} \). The ring of symmetric functions has a \( \mathbb{Z} \)-basis given by Schur functions. These functions are indexed by partitions which allows us to identify \( B_c \) with \( F_c \) for each \( c \in \mathbb{Z} \). Since this holds for all charges, this bijection yields the first part of the Boson-Fermion correspondence \([23, \text{Section 14}]\).

**Theorem 2.1 (Boson-Fermion correspondence part 1, [23, Section 14.10])** For any fixed \( c \in \mathbb{Z} \), the map \( \sigma : F_c \to B_c \) given by sending a vector \( v \) of charge \( c \) in \( F_c \) to \( t^c s_{\lambda(v)} \) in \( B_c \) is a grading preserving isomorphism of vector spaces. Thus, \( F \cong B \).

### 2.2 Heisenberg algebra

The infinitely generated Heisenberg algebra is a central player in the representation theory of affine Lie algebras and quantum field theories. To any \( \mathbb{Z} \)-lattice one can associate a Heisenberg algebra; in this article we focus on the case of \( \mathbb{Z} \) with pairing \( (1, 1) = 1 \).

**Definition 2.2** The Heisenberg algebra \( \mathfrak{h} \) is the associative, unital, \( \mathbb{K} \)-algebra with \( \mathbb{Z} \)-basis \( p^{(m)}, q^{(m)} \) for \( m \in \mathbb{N}_+ \) satisfying the commutation relations:

- \( p^{(m)} p^{(n)} = p^{(m)} p^{(n)} \)
- \( q^{(m)} q^{(n)} = q^{(m)} q^{(n)} \)
- \( q^{(m)} p^{(n)} = \sum_{k \geq 0} p^{(m-k)} q^{(n-k)} \).

Alternatively, \( \mathfrak{h} \) could also be presented by the generators \( p^{(1^m)} \) and \( q^{(1^m)} \) for \( m \in \mathbb{N} \), satisfying analogous relations to those above.

**Proposition 2.3** ([25, Sec. 2.1]) The following relations hold in \( \mathfrak{h} \):

\[
q^{(n)} p^{(1^m)} = p^{(1^m)} q^{(n)} + p^{(1^{m-1})} q^{(n-1)} \quad \text{and} \quad q^{(1^n)} p^{(m)} = p^{(m)} q^{(1^n)} + p^{(m-1)} q^{(1^{n-1})}.
\]  

(2.1)

If we denote by \( \mathfrak{h}^+ \) and \( \mathfrak{h}^- \) the unital subalgebras of \( \mathfrak{h} \) generated by the \( q^{(n)} \) and \( p^{(n)} \) for \( n \in \mathbb{N}_+ \), respectively, then there is an isomorphism of vector spaces \( \mathfrak{h} \cong \mathfrak{h}^+ \otimes \mathfrak{h}^- \). Furthermore if \( \mathbb{I} \) denotes the trivial representation of \( \mathfrak{h}^+ \), then the induced
representation $\text{Ind}_{\mathfrak{h}^+_+}^\mathfrak{h}(1)$ is the unique irreducible representation of $\mathfrak{h}$ known as the \textit{Fock space representation}. This representation is faithful and isomorphic as a vector space to the ring of symmetric functions.

\textbf{Remark 2.4} Those familiar with Heisenberg algebras will notice the previous presentations differ from the usual definition. More generally, the Heisenberg algebra shows up as a collection of oscillators in the following way. Define the oscillator algebra as the algebra generated by $\alpha_n, n \in \mathbb{Z}$ and central element $\hbar \in \mathbb{k}$ satisfying the commutation relations:

$$\alpha_n \alpha_m - \alpha_m \alpha_n = n \delta_{n,-m} \hbar \quad \text{and} \quad \hbar \alpha_n - \alpha_n \hbar = 0.$$  \hfill (2.2)

This algebra can be seen as an algebra of differential operators by defining its action on Bosonic Fock space as follows; For any $f = f(p_1, p_2, \ldots) t^c \in B_c$ and $n > 0$ set:

$$\alpha_0 (f) := \frac{\partial f}{\partial t} \quad \alpha_n (f) := n \frac{\partial f}{\partial p_n} \quad \alpha_{-n} (f) := p_n f \quad \hbar (f) := \hbar f.$$ \hfill (2.3)

In particular, if $\hbar \neq 0$ this action yields an irreducible representation of $B$. Moreover if we set $\hbar = 1$ and consider $\alpha_n$ for $n \in \mathbb{Z} \setminus \{0\}$ then we recover the usual formulation for the Heisenberg algebra $\mathfrak{h}$. When seen as an algebra of differential operators, $\mathfrak{h}$ is often referred to as the \textit{Weyl algebra}. Thus, $B$ is an infinite dimensional irreducible representation of $\mathfrak{h}$, equal to infinitely many copies of the irreducible Fock space representation.

\textbf{Proposition 2.5} (c.f. [25]) \textit{The following relations hold inside $\mathfrak{h}$:}

$$\sum_{m \in \mathbb{N}} p^{(m)} z^m = \exp \left( \sum_{n \in \mathbb{N}} \frac{z^n}{n} \alpha_n \right) \quad \text{and}$$

$$\sum_{m \in \mathbb{N}} (-1)^m p^{(m)} z^m = \exp \left( - \sum_{n \in \mathbb{N}} \frac{z^n}{n} \alpha_n \right) \hfill (2.4)$$

$$\sum_{m \in \mathbb{N}} q^{(m)} z^m = \exp \left( \sum_{n \in \mathbb{N}} \frac{z^n}{n} \alpha_{-n} \right) \quad \text{and}$$

$$\sum_{m \in \mathbb{N}} (-1)^m q^{(m)} z^m = \exp \left( - \sum_{n \in \mathbb{N}} \frac{z^n}{n} \alpha_{-n} \right). \hfill (2.5)$$

\textbf{2.3 Clifford algebra}

To any vector space $V$ over a field $\mathbb{k}$ equipped with a quadratic form $Q(v)$ we can associate an infinite dimensional \textit{Clifford algebra} $\text{CL}(V, Q) := T(V) / J$ where $J$ is its 2-sided ideal generated by terms $v \otimes v - Q(v) 1$. Specifically, given operators $\psi_i$
and $\psi^*_i$ for $i \in \mathbb{Z}$, we are interested in the Clifford algebra associated to the vector space $V = \bigoplus_i \mathbb{k}\psi_i + \bigoplus_i \mathbb{k}\psi^*_i$ with $Q(\psi_i + \psi^*_j) = \delta_{i,j}1$ and zero otherwise.\footnote{For completeness, we mention that its unique irreducible module $F_{\Omega}$, i.e., its spin module, is given by the maximal isotropic subspace $U = \bigoplus_{i\leq 0} \mathbb{k}\psi_i + \bigoplus_{i>0} \mathbb{k}\psi^*_i$ with the property that $U(\overline{0}) = 0$. This fact will not be used here.}

**Definition 2.6** The Clifford algebra $\mathbb{CL}$ is the associative, unital, $\mathbb{k}$-algebra with generators $\psi_i$ and $\psi^*_i$ for $i, j \in \mathbb{Z}$ satisfying the anticommutation relations,

\[ \psi_i \psi_j^* + \psi_j^* \psi_i = \delta_{j,i} \quad \psi_i \psi_j + \psi_j \psi_i = 0 \quad \psi_i^* \psi_j^* + \psi_j^* \psi_i^* = 0. \tag{2.6} \]

**Proposition 2.7** Fermionic Fock space has an action of the Clifford algebra $\mathbb{CL}$ given by

\[
\psi_j(i_1 \wedge i_2 \wedge \cdots) := \begin{cases} 
0 & \text{if } j = \frac{1}{2} \text{ for some } s, \\
(-1)^s i_1 \wedge i_2 \wedge \cdots i_s \wedge (j - \frac{1}{2}) \wedge i_{s+1} \wedge \cdots & \text{if } i_s > j - \frac{1}{2} > i_{s+1}.
\end{cases}
\tag{2.7}
\]

\[
\psi^*_j(i_1 \wedge i_2 \wedge \cdots) := \begin{cases} 
0 & \text{if } j \neq \frac{1}{2} \text{ for all } s, \\
(-1)^{s+1} i_1 \wedge i_2 \wedge \cdots i_{s-1} \wedge i_{s+1} \wedge \cdots & \text{if } i_s = j - \frac{1}{2}.
\end{cases}
\tag{2.8}
\]

Hence, when acting on $\mathcal{F}$ these operators, known as free fermions, are adjoint to each other and act like creation and annihilation operators on $\mathcal{F}$.

In essence, $\psi_i$ and $\psi^*_i$ raise and lower the charge of $v$ by creating and eliminating electrons in the $i$th energy state. Moreover, since for any vectors $v, w \in \mathcal{F}$ there is always a finite sequence of $\psi_i$’s and $\psi^*_i$’s whose action on $v$ will yield $w$, the representation of the Clifford algebra induced by this action is also irreducible.

### 2.4 Connections with symmetric functions

The ring of symmetric functions $\text{Sym}$ has various bases indexed by partitions. We will particularly consider Schur functions $s_\lambda$ and the elementary $e_\lambda$, complete $h_\lambda$, and power sum $p_\lambda$ symmetric functions. This ring has a natural inner product with the property that $\langle s_\lambda, s_\mu \rangle = \delta_{\mu, \lambda}$. Since $\text{Sym}$ acts on itself via multiplication, then for each $f \in \text{Sym}$ we can define $f^\perp \in \text{End}(\text{Sym})$ as the adjoint operator on $\text{Sym}$ with respect to this inner product, that is $\langle f^\perp u, v \rangle = \langle u, f v \rangle$ for any $u, v \in \text{Sym}$. In this way, for each integer $n \in \mathbb{N}$ Macdonald considers the operators $e_n^\perp, h_n^\perp, p_n^\perp \in \text{End}(\text{Sym})$ and introduces the following generating functions \cite[Section 1.2]{Macdonald}.

\[
P(z) = \sum_{n \geq 0} p_n z^{n-1} \quad E(z) = \sum_{n \geq 0} e_n z^n \quad H(z) = \sum_{n \geq 0} h_n z^n \tag{2.9}
\]

\[
P^\perp(z) = \sum_{n \geq 0} p_n^\perp z^{n-1} \quad E^\perp(z) = \sum_{n \geq 0} e_n^\perp z^n \quad H^\perp(z) = \sum_{n \geq 0} h_n^\perp z^n \tag{2.10}
\]
These functions satisfy the relations $E(z) = \exp\left(\sum_{n \geq 1} (-1)^{n-1} \frac{pn}{n} z^n\right)$ and $H(z) = \exp\left(\sum_{n \geq 1} \frac{pn}{n} z^n\right)$.

The Bernstein operators $B_a$ and $B^*_a$ are the coefficients of $z^a$ for each $a \in \mathbb{Z}$ in the following formal power series\(^4\) [39]:

$$B(z) = \sum_{a \in \mathbb{Z}} B_a z^a := H(z) E\left(-z^{-1}\right) \quad B^*(z) = \sum_{a \in \mathbb{Z}} B^*_a z^a := E\left(-z^{-1}\right) H^\perp(z)$$

$$= \sum_{a \in \mathbb{Z}} \sum_{n,m \in \mathbb{N}} (-1)^m h_n e_m^\perp \quad = \sum_{a \in \mathbb{Z}} \sum_{n,m \in \mathbb{N}} (-1)^n e_n h_m^\perp$$

The Bernstein operators act like creation and annihilation operators for Schur functions $s_\lambda$ in the sense that for any partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, we have $s_\lambda = B_{\lambda_1} \ldots B_{\lambda_n}(1)$ and $B^*_{\lambda_1} \ldots B^*_{\lambda_n}(s_\lambda) = 1$ [27, Section 1.5]. We prove a categorical analogue of these relations in Theorem 5.5.

**Proposition 2.8** The Heisenberg algebra $\mathfrak{h}$ acts on the ring of symmetric functions.

**Proof** This is immediate from Eq. (2.3). \[\square\]

From the definition of $p_n^\perp$ it can be deduced that $p_n^\perp$ acts on symmetric functions by $\frac{\partial}{\partial p_n}$ (see [27, pg. 76]). Thus, combining the action in Eq. (2.3) and Proposition 2.5 then we can identify the various generators of $\mathfrak{h}$ and $\text{Sym}$ in the following manner:

$$p^{(n)} \rightarrow h_n \quad q^{(n)} \rightarrow h_n^\perp \quad \alpha_{-n} \rightarrow p_n$$

$$p^{(1^r)} \rightarrow e_n \quad q^{(1^r)} \rightarrow e_n^\perp \quad \alpha_n \rightarrow p_n^\perp$$

(2.13)

Moreover, since $\text{Sym}$ can be expressed as $\mathbb{Z}[h_1, h_2, \ldots]$, $\mathbb{Z}[e_1, e_2, \ldots]$, and $\mathbb{Q}[p_1, p_2, \ldots]$ it follows that $p^{(n)}$ and $q^{(n)}$ generate the integral form of the Heisenberg algebra $\mathfrak{h}_\mathbb{Z}$ whereas the $\alpha_n$ are generators for $\mathfrak{h}_\mathbb{Q} = \mathfrak{h}_\mathbb{Z} \otimes \mathbb{Q}$.

### 2.5 The Boson-Fermion correspondence

Given Theorem 2.1 we will henceforth identify Fermionic and Bosonic Fock space and assume all operators are acting on $\mathcal{B}$. This isomorphism allows us to express the action of $\mathcal{CL}$ on $\mathcal{B}$ in such a way that the Clifford algebra action induced from Proposition 2.7 can be entirely recovered from the Heisenberg algebra action in Proposition 2.8. In particular, we will construct fermions $\psi_i$, $\psi^*_i$ in terms of bosons $\alpha_n$, by a process known as fermionization. To this effect, we introduce the operators on $\mathcal{B}$

$$\Gamma_+(z) := \exp\left(\sum_{n \geq 1} \frac{z^{-n}}{n} \alpha_n\right) \quad \text{and} \quad \Gamma_-(z) := \exp\left(\sum_{n \geq 1} \frac{z^n}{n} \alpha_{-n}\right)$$

---

\(^4\) Macdonald uses the notation $B^\perp_a$ instead of $B^*_a$. 
which satisfy the commutation relation $\Gamma_{-}(y)\Gamma_{+}(z) = (1 - z\frac{\partial}{\partial z})\Gamma_{+}(z)\Gamma_{-}(y)$. Operators of this form are known as halves of vertex operators. By Eqs. (2.4) and (2.5) we immediately obtain the following identities:

$$\Gamma_{-}(z) = \sum_{m \in \mathbb{N}} p^{(m)} z^{m} \quad \Gamma_{-}(z)^{-1} = \sum_{m \in \mathbb{N}} (-1)^{m} p^{(m)} z^{m}$$

$$\Gamma_{+}\left(\frac{1}{z}\right) = \sum_{m \in \mathbb{N}} q^{(m)} z^{m} \quad \Gamma_{+}\left(\frac{1}{z}\right)^{-1} = \sum_{m \in \mathbb{N}} (-1)^{m} q^{(m)} z^{m}$$

The operators $\Gamma_{+}(z)$ and $\Gamma_{-}(z)$ are adjoint operators in the sense that if we consider the generating series $\Gamma_{\pm}(z) = \sum_{n \geq 0} \Gamma_{\pm} z^{\pm n}$ then $\Gamma_{n}^+$ and $\Gamma_{n}^-$ are adjoint operators on $\mathcal{B}$ [23, pg 315]. This statement follows immediately from the fact that $p^{(m)}$ and $q^{(m)}$ are adjoint operators in $\mathfrak{h}$.

Thus, if we define the Fermionic fields as $\psi(z) := \sum_{i \in \mathbb{Z}} \psi_{i} z^{i}$ and $\psi^{*}(z) := \sum_{i \in \mathbb{Z}} \psi_{i}^{*} z^{-i}$ and the charge operator on $\mathcal{B}$ by $z^{\pm a_{0}}(t^{c} s_{\lambda}) := z^{\pm c} s_{\lambda}$ then the second part of the Boson-Fermion correspondence can be stated as follows.

**Theorem 2.9** (Boson-Fermion correspondence part 2, [23, 14.10]) The operators $\Gamma_{-}$ and $\Gamma_{+}$ satisfy:

$$\psi(z) = z^{a_{0}} t \Gamma_{-}(z) \Gamma_{+}(z)^{-1} \quad \text{and} \quad \psi^{*}(z) = t^{-1} z^{-a_{0}} \Gamma_{-}(z)^{-1} \Gamma_{+}(z).$$

Under the identification from (2.13), the generating series from (2.9), (2.10), and (2.11) can be seen as operators on Bosonic Fock space $\mathcal{B}$. Hence we may write $\Gamma_{-}(z) = H(z)$ and $\Gamma_{+}(z)^{-1} = E^{-1}(-z)^{-1}$. Consequently, $\Gamma_{-}(z)\Gamma_{+}(z)^{-1} = H(z) E^{-1}(-z)^{-1} = B(z)$ and $\Gamma_{-}(z)^{-1}\Gamma_{+}(z) = E(-z) H^{-1}(z)^{-1} = B^{*}(z)^{-1}$ where for any $a \in \mathbb{Z}$ the Bernstein operators $B_{a}$ and $B_{a}^{*}$ on $\mathcal{B}$ are given by:

$$B_{a} := \sum_{m-n=a} (-1)^{m} p^{(n)} q^{(m)} \quad \text{and} \quad B_{a}^{*} := \sum_{m-n=a} (-1)^{n} q^{(m)} p^{(n)}.$$

Thus, for any $t^{c} s_{\lambda} \in \mathcal{B}_{c}$ with charge $c \in \mathbb{Z}$ the action of $\psi(z)$ and $\psi^{*}(z)$ is given by:

$$\psi(z)(t^{c} s_{\lambda}) = z^{a_{0}} t \sum_{a \in \mathbb{Z}} B_{a}(t^{c} s_{\lambda}) z^{a} = z^{c+1} t^{c+1} \sum_{a \in \mathbb{Z}} B_{a}(s_{\lambda}) z^{a} = \sum_{i \in \mathbb{Z}} t^{c+1} B_{i-c-1}(s_{\lambda}) z^{i}$$

$$\psi^{*}(z)(t^{c} s_{\lambda}) = t^{-1} z^{-a_{0}} \sum_{a \in \mathbb{Z}} B_{a}^{*}(t^{c} s_{\lambda}) z^{-a} = t^{-1} z^{-c} \sum_{a \in \mathbb{Z}} B_{a}^{*}(s_{\lambda}) z^{-a} = \sum_{i \in \mathbb{Z}} t^{-1} B_{i-c}^{*}(s_{\lambda}) z^{-i}.$$
Consequently, the condition that \( \psi_i \) and \( \psi^*_i \) satisfy the Clifford algebra relations from (2.6) is equivalent to the Bernstein operators satisfying certain anticommutation relation. Specifically, through a direct computation we find the equations on the left-hand side below hold if and only if the relations on the right-hand side also hold.

\[
\begin{align*}
\psi_i \psi^*_j + \psi^*_i \psi_j & = \delta_{i,j} & \iff & & B_{i-c} B^*_{j-c} + B^*_{j-c-1} B_{i-c-1} = \delta_{i,j} \\
\psi_i \psi_j + \psi_j \psi_i & = 0 & \iff & & B_{i-c-2} B_{j-c-1} + B_{j-c-2} B_{i-c-1} = 0 \\
\psi^*_i \psi^*_j + \psi^*_j \psi^*_i & = 0 & \iff & & B_{i-c+1} B^*_{j-c} + B^*_{j-c+1} B_{i-c} = 0.
\end{align*}
\] (2.16) (2.17) (2.18)

The relations on the right hand side involving the Bernstein operators are proven by Macdonald in [27, Sec. 1.5] in the context of symmetric functions.

### 3 Homological algebra for infinite complexes

We begin by establishing some conventions used throughout the paper. Suppose \( C \) is an additive, monoidal category and \( A \) and \( B \) are two chain complexes over \( C \). Then:

- The homological degree of each chain group \( A_i \) of \( A \) will always be made explicit. That is, we will think of \( A_i \) as sitting in homological degree zero and \( A_i \) as sitting in homological degree \( i \). In this way, we can compactly present all the chain groups of \( A \) by \( \bigoplus_i A_i = \bigoplus_i A_i \).
- The differentials \( d_A \) decrease homological degree. That is, they are maps \( d_A : A_{i+1}[i+1] \to A_i[i] \).
- Given a chain complex \( \cdots \to X \xrightarrow{d} Y \xrightarrow{d} Z \xrightarrow{d} \cdots \) such that \( Y \) lives in homological degree \( s \), we will always use the notation on the right rather than the left.

\[
\left( \cdots \to X \xrightarrow{d} Y \xrightarrow{d} Z \xrightarrow{d} \cdots \right) = \left( \cdots \to X[s+1] \xrightarrow{d} Y[s] \xrightarrow{d} Z[s-1] \xrightarrow{d} \cdots \right)
\]

- For any \( s \in \mathbb{Z} \) we denote by \( A[s] \) the chain complex \( A \) with a homological degree shift of \( s \). That is, \( A[s] \) is the complex whose \( i \)th chain group \( (A[s])_i \) is equal to \( A_{i-s} \) and whose differentials are the same as in \( A \) (modulo a sign). In particular, \( A[1] \) indicates a homological shift up by 1 (i.e. to the left) whereas \( A[-1] \) is a shift down by one (i.e. to the right). Thus, we denote the chain groups of \( A[s] \) by

\[
\left( \bigoplus_i A_i[i] \right)[s] = \bigoplus_i A_i[i+s]
\]

and write the chain complex \( A[s] = \cdots \xrightarrow{(-1)^r d_A} A_{i+1}[i+1+s] \xrightarrow{(-1)^r d_A} A_i[i+s] \xrightarrow{(-1)^r d_A} \cdots \), for indexing reasons apparent in Sect. 6, compactly in...
either of the following ways,

\[ A[s] = \bigoplus_i A_i[s + i], (-1)^s d_A \right\} = \{A_i[s + i], (-1)^s d_A\}. \]

We now recall some definitions and present the general machinery needed to perform the upcoming computations on bi-infinite chain complexes. We follow Elias-Hogancamp closely and refer the reader to Section 4 of [9] for all the statements below in their full generality.

A chain map \( f : A \to B \) is a collection of maps \( f_i : A_i \to B_i \) such that \( d_B f_i - f_{i-1} d_A = 0 \) for all \( i \). We say two chain maps \( f, g : A \to B \) are homotopic, denoted by \( f \simeq g \) if there exists a homotopy \( H : A \to B[-1] \) with the property that \( H \circ d_A + d_B \circ H = f - g \). If such maps exist, we say that \( A \) and \( B \) are homotopy equivalent. In particular, we say a chain complex \( A \) is nullhomotopic if there exists a homotopy \( H \) such that \( 1_A \simeq 0 \).

The homotopy category of \( C \) is denoted by \( \mathcal{K}(C) \). Its objects are chain complexes over \( C \) and its morphisms are chain maps up to homotopy. Thus two complexes are isomorphic in \( \mathcal{K}(C) \), denoted \( A \simeq B \), if they are homotopy equivalent. Let \( \mathcal{K}^+(C), \mathcal{K}^-(C) \), and \( \mathcal{K}^b(C) \) denote the full subcategories of \( \mathcal{K}(C) \) of chain complexes which are bounded above, bounded below, and bounded above and below respectively.

For any chain map \( f : A \to B \), the mapping cone of \( f \) is the chain complex \( \text{Cone}(f) := A[1] \oplus B \) whose differential is given by \( \begin{bmatrix} -d_A & 0 \\ f & d_B \end{bmatrix} \). In particular, a triangle in \( \mathcal{K}(C) \) is distinguished if and only if it is isomorphic to

\[ A \xrightarrow{f} B \xrightarrow{\iota} \text{Cone}(f) \xrightarrow{\pi} A[1] \]

where both \( \iota \) and \( \pi \) are canonical inclusion and projection maps associated to mapping cones. Thus, we say \( \mathcal{K}(C) \) (and consequently \( \mathcal{K}^+(C), \mathcal{K}^-(C) \), and \( \mathcal{K}^b(H) \)) is a triangulated category.

**Definition 3.1** Let \( T \subset \mathcal{K}(C) \) be a full triangulated subcategory and \( I \subset \mathbb{Z} \) some indexing set. A bi-complex in \( T \) is a collection \( \{A^i, d_i, d^i\}_{i \in I} \) with \( A^i \in T \) such that for each \( i \in I \), \( d_i = d_A^i : A^i_j \to A^i_{j-1} \) and \( d^i : A^i_j \to A^i_{j-1} \) is a morphism of chain complexes satisfying:

\[(d_i)^2 = 0, \quad (d^i)^2 = 0 \quad \text{and} \quad d_id^i + d^id_i = 0.\]

**Remark 3.2** In this work we will only consider \( T = \mathcal{K}(C), \mathcal{K}^-(C), \mathcal{K}^+(C), \) or \( \mathcal{K}^b(C) \). Moreover, the relations in Definition 3.1 indicate equalities on the nose and not equalities up to homotopy equivalence.
Given a bi-complex \( \{ A^i, d_i, d^i \}_{i \in I} \) in \( T \), we define its total complex and its completed total complex as the chain complexes given by:

\[
\text{Tot}^\oplus \{ A^i, d_i, d^i \}_{i \in I} := \left\{ \bigoplus_i A^i, d \right\} \quad \text{with differential} \quad d = \sum d_i + d^i
\]

\( (3.1) \)

\[
\text{Tot}^\prod \{ A^i, d_i, d^i \}_{i \in I} := \left\{ \prod_i A^i, d \right\} \quad \text{with differential} \quad d = \sum d_i + d^i
\]

\( (3.2) \)

We say \( \{ A^i, d_i, d^i \} \) is \( T \)-locally finite if \( \text{Tot}^\oplus \{ A^i, d_i, d^i \} \) and \( \text{Tot}^\prod \{ A^i, d_i, d^i \} \) exist in \( T \) and are isomorphic.

**Definition 3.3** Given any additive, monoidal category \( C \) and \( A, B \in K(C) \) we define the tensor product of \( A \) and \( B \) as the chain complex

\[
A \otimes B := \text{Tot}^\oplus \left\{ A \otimes B^j, d_j, d^j \right\}_{j \in J} \quad \text{with} \quad d_j := d_A \otimes 1_{B^j} \quad \text{and} \quad d^j := 1_A \otimes d_{B^j}
\]

and the completed tensor product as the chain complex

\[
A \widehat{\otimes} B := \text{Tot}^\prod \left\{ A \otimes B^j, d_j, d^j \right\}_{j \in J} \quad \text{with} \quad d_j := d_A \otimes 1_{B^j} \quad \text{and} \quad d^j := 1_A \otimes d_{B^j}.
\]

When \( A \) is a single term complex, i.e. \( A = \left( \cdots 0 \xrightarrow{d} A_i[i] \xrightarrow{d} 0 \cdots \right) \) for some \( i \in \mathbb{Z} \) and \( A_i \in C \), we will write \( A_i B[i] \) to denote \( A \otimes B \) with differential \( 1_A \otimes d_B \) and omit any explicit mention of the type of tensor product since the bi-complex \( \{ A \otimes B^j, d_j, d^j \} \) is trivially \( T \)-locally finite.

In light of this new terminology, we pause to make a few technical but important remarks that will be used in Sect. 6.

**Remark 3.4** Due to the symmetry in Definition 3.3 we can equivalently define the tensor product and the completed tensor product in the following ways:

\[
A \otimes B := \text{Tot}^\oplus \left\{ A^i \otimes B, d_i, d^i \right\}_{i \in I} \quad \text{with} \quad d_i := 1_{A^i} \otimes d_B \quad \text{and} \quad d^i := d_{A^i} \otimes 1_B
\]

\[
A \widehat{\otimes} B := \text{Tot}^\prod \left\{ A^i \otimes B, d_i, d^i \right\}_{i \in I} \quad \text{with} \quad d_i := 1_{A^i} \otimes d_B \quad \text{and} \quad d^i := d_{A^i} \otimes 1_B
\]

When the distinction between these formulations is needed for \( A \otimes B \) (or equivalently for \( A \widehat{\otimes} B \)), we will refer to \( A^i \otimes B \) as the \( i^{th} \) row and \( A \otimes B^j \) as the \( j^{th} \) column of the bi-complex.

**Remark 3.5** While it is clear that for finite complexes these definitions agree, i.e. bounded complexes are \( T \)-locally finite, the distinction between direct sums and direct
products is crucial when "tensoring" infinite complexes. This is because depending on certain boundedness conditions of \(A\) and \(B\) their tensor product and their completed tensor product are often not homotopy equivalent or even well defined. In particular, if \(A \in C^+(C)\) and \(B \in C^-(C)\) then \(\{A \otimes B_j, d_j, d^j\}\) is not necessarily \(K(C)\)-locally finite.

Given a bi-complex \(\{A^i, d_i, d^i\}_{i \in I}\) where each subcomplex \(A^i\) is given by \(\bigoplus_j A^i_j, d_i\}, we say the bi-complex is homologically locally finite if for any \(k \in \mathbb{Z}\) the sum \(\bigoplus_{i+j=k} A^i_j\) is finite and thus isomorphic to \(\prod_{i+j=k} A^i_j\). In this case the following proposition holds.

**Proposition 3.6** ([9, Prop. 4.10]) Let \(\{A^i, d_i, d^i\}\) be a bi-complex such that each subcomplex \(A^i = \bigoplus A^i_j, d_i\} \in \mathcal{T}\) and that for each fixed homological degree \(j\) as objects inside \(C\) we have,

\[
\bigoplus_{i \in \mathbb{Z}} A^i_j \cong \prod_{i \in \mathbb{Z}} A^i_j.
\]

Then \(\bigoplus_i A^i \cong \prod_i A^i\) and thus \(\{A^i, d_i, d^i\}\) is \(\mathcal{T}\)-locally finite.

In particular, if \(A, B \in K^+(C)\) (resp. \(K^-(C)\)) then \(\{A \otimes B_j, d_j, d^j\}\) is homologically locally finite and so, by Proposition 3.6, also \(K^+(C)\)-locally finite (resp. \(K^-(C)\)-locally finite).

Lastly, we present our main tools. We refer the reader to [2, Lemma 6.1] and [9, Prop. 4.20] for their respective proofs.

**Lemma 3.7** (Gaussian elimination) Let \(X, Y, Z, W, U, V\) be six objects in an additive category and consider the complex \([\cdots \to U \xrightarrow{u} X \oplus Y \xrightarrow{f} Z \oplus W \xrightarrow{v} V \to \ldots]\) where \(f = \begin{pmatrix} C & D \\ A & B \end{pmatrix}\) and \(u, v\) are arbitrary morphisms. If \(D : Y \to Z\) is an isomorphism, then the following chain complexes are homotopic.

\[
\begin{align*}
\cdots \to U & \xrightarrow{u} X \oplus Y \xrightarrow{f} Z \oplus W \xrightarrow{v} V \to \ldots \\
\cong \cdots \to U & \xrightarrow{u} X \xrightarrow{A-BD^{-1}C} W \xrightarrow{v|W} V \to \ldots
\end{align*}
\]

**Proposition 3.8** (Simultaneous simplifications) Let \(I \subset \mathbb{Z}\) and suppose \(\{A^i, d_i, d^i\}_{i \in I}\) and \(\{B^i, b_i, b^i\}_{i \in I}\) are bi-complexes in \(\mathcal{T}\) such that for each \(i \in I\), \(A^i\) and \(B^i\) are homotopy equivalent.

i) If \(I\) is bounded below then \(\text{Tot}^\oplus \{A^i, d_i, d^i\}_{i \in I} \cong \text{Tot}^\oplus \{B^i, b_i, b^i\}_{i \in I}\).

ii) If \(I\) is bounded above then \(\text{Tot}^\prod \{A^i, d_i, d^i\}_{i \in I} \cong \text{Tot}^\prod \{B^i, b_i, b^i\}_{i \in I}\).

In particular, if \(A_i\) and \(B_i\) are homotopic by a sequence of Gaussian eliminations and \(\mathcal{T} = K^+(C)\) or \(K^-(C)\) then Proposition 3.8 applies and all Gaussian eliminations can be simultaneously performed.
4 Extended graphical calculus for Khovanov’s Heisenberg category

4.1 Khovanov’s Heisenberg category

For any commutative unital ring $\mathbb{k}$, Khovanov introduced a categorical framework for the Heisenberg algebra $\mathfrak{h}$. This framework consists of a monoidal category $\mathcal{H}$ which is the Karoubi envelope of a category $\mathcal{H}'$ whose definition we now sketch (see [25] for more details).

The additive, $\mathbb{k}$-linear, monoidal category $\mathcal{H}'$ is generated by two objects $Q_+$ and $Q_-$. An object of $\mathcal{H}'$ is a finite direct sum of tensor products $Q_\varepsilon := Q_{\varepsilon_1} \otimes \cdots \otimes Q_{\varepsilon_k}$ where $\varepsilon = (\varepsilon_i)$ is a sequence with $\varepsilon_i \in \{+, -\}$ for each $i$. The empty sequence $\emptyset$ corresponds to the monoidal unit and is denoted $1$. If $\varepsilon_i = +$ or $\varepsilon_i = -$ for all $i$ we denote its corresponding object by $Q_{+k}$ or $Q_{-k}$ respectively. For any sequences $\varepsilon, \varepsilon'$, the space of homomorphisms $\text{Hom}_{\mathcal{H}'}(Q_\varepsilon, Q_{\varepsilon'})$ is the $\mathbb{k}$-module described by certain string diagrams with relations.

Specifically, we denote $Q_+$ as an upward pointing arrow and $Q_-$ as a downward pointing arrow. Monoidal composition of such objects is given by sideways concatenation. By convention, composition of morphisms is done vertically and read bottom to top. When not presented diagrammatically we will use “$\otimes$” and “$\circ$” to denote horizontal and vertical composition, respectively. The morphisms between such objects are generated by the crossings, caps, and cups displayed below:

\[
\begin{align*}
\begin{array}{c}
\xrightarrow{\text{crossing}} \quad \text{or} \quad \xleftarrow{\text{crossing}} \quad \text{or} \quad \xrightarrow{\text{cap}} \quad \text{or} \quad \xleftarrow{\text{cup}}
\end{array}
\end{align*}
\]

(4.1)

The crossing on the left is a morphism $Q_{++} \rightarrow Q_{++}$ whereas the rightmost cap is a morphism $Q_{+-} \rightarrow 1$. These morphisms are isotopy invariant and satisfy the relations below (note that (4.2) and (4.5) also hold for downward pointing strands).

\[
\begin{align*}
\begin{array}{c}
\xrightarrow{\text{crossing}} = \uparrow \quad \text{or} \quad \xleftarrow{\text{crossing}} = \uparrow \quad \text{or} \quad \xrightarrow{\text{cap}} = \uparrow \quad \text{or} \quad \xleftarrow{\text{cup}} = \uparrow
\end{array}
\end{align*}
\]

(4.2)

(4.5)

\[
\begin{align*}
\begin{array}{c}
\xrightarrow{\text{crossing}} = \downarrow \quad \text{or} \quad \xleftarrow{\text{crossing}} = \downarrow \quad \text{or} \quad \xrightarrow{\text{cap}} = \downarrow \quad \text{or} \quad \xleftarrow{\text{cup}} = \downarrow
\end{array}
\end{align*}
\]

(4.3)

(4.6)

\[
\begin{align*}
\begin{array}{c}
\xrightarrow{\text{crossing}} = 1, \quad (4.4) \quad \xleftarrow{\text{crossing}} = 0 \quad (4.7)
\end{array}
\end{align*}
\]

(4.4)

(4.7)

*Khovanov’s Heisenberg category* $\mathcal{H}$ is defined as the Karoubian envelope of $\mathcal{H}'$. Thus, the objects of $\mathcal{H}$ are given by pairs $(Q_\varepsilon, e)$ where $Q_\varepsilon$ is an object in $\mathcal{H}'$ and $e \in \text{End}_{\mathcal{H}'}(Q_\varepsilon)$ an idempotent, i.e. $e^2 = e$. The space $\text{Hom}_{\mathcal{H}}(Q_\varepsilon, e) \cdot (Q_{\varepsilon'}, e')$ consists of morphisms $f \in \text{Hom}_{\mathcal{H}'}(Q_\varepsilon, Q_{\varepsilon'})$ such that $e'f = fe$. A map
\( f \in \text{Hom}_\mathcal{H}((Q_\varepsilon, e), (Q_{\varepsilon'}, e')) \) is an isomorphism in \( \mathcal{H} \) if there exists \( f' \in \text{Hom}_\mathcal{H}((Q_{\varepsilon'}, e'), (Q_\varepsilon, e)) \) such that \( ff' = id_{Q_\varepsilon} \) and \( f'f = id_{Q_{\varepsilon'}} \). This category is clearly \( k \)-linear, additive, and monoidal. Moreover, it categorifies the integral form of the Heisenberg algebra.

**Theorem 4.1** ([25, Thm. 1] [1, Thm. 1.1]) There is a ring isomorphism \( \phi: \mathfrak{h} \rightarrow K_0(\mathcal{H}) \).

Set \( P := (Q_+, id) \) and \( Q := (Q_-, id) \), and denote them diagrammatically by an upward and downward pointing strand respectively. Then by (4.2) and (4.5) there exist canonical homomorphisms:

\[
\mathbb{K}[S_n] \rightarrow \text{End}_{\mathcal{H}'}(Q_+^n) \quad \text{and} \quad \mathbb{K}[S_n] \rightarrow \text{End}_{\mathcal{H}'}(Q_-^n).
\]

In particular, if \( \mathbb{K} \) is a field of characteristic zero then for any partition \( \lambda \vdash n \) we denote the set of standard Young tableau by \( \text{SYT}(\lambda) \), then for \( T \in \text{SYT}(\lambda) \) the image of its Young symmetrizer \( e_T \) is an idempotent in \( \text{End}_{\mathcal{H}'}(Q_+^n) \) [14,17]. Consequently, for each \( T \in \text{SYT}(\lambda) \) we set:

\[
P^T := (Q_+, e_T) \quad \text{and} \quad Q^T := (Q_-, e_T).
\]

If \( T_{row} \) is the standard row ordered filling of \( \lambda \), then for any other \( T \in \text{SYT}(\lambda) \) there exists a permutation \( \sigma \in S_n \) such that \( T_{row} = \sigma T \). That is, any standard Young tableau can be transformed into the row ordered filling by applying some permutation of the entries. However, by the inclusions above we have that \( S_n \subset \text{Aut}(Q_\pm^n) \) and so the map \( \sigma : (Q_+^n, e_T) \rightarrow (Q_+^n, e_{T_{row}}) \) is an isomorphism in \( \text{Kar}(\mathcal{H}') \). Consequently, if \( T, T' \in \text{SYT}(\lambda) \) then \( e^T \) and \( e^{T'} \) are conjugate and so \( P^T \cong P^{T'} \) in \( \mathcal{H} \). Thus, set

\[
P^\lambda := (Q_+, e_{T_{row}}) \quad \text{and} \quad Q^\lambda := (Q_-^n, e_{T_{row}})
\]

as the canonical representatives for the isomorphism classes of \( P^T \) and \( Q^T \) with \( T \in \text{SYT}(\lambda) \).

### 4.2 Diagramatics for young idempotents

For any partition \( \lambda \vdash n \), let \( \lambda^t \) denote the transposed partition whose rows are given by the columns of \( \lambda \). Denote by \( T_{row} \) and \( T_{col} \) the standard row and column ordered fillings of \( \lambda \) respectively.

Since \( P^\lambda \) will always denote the idempotent corresponding to the tableau with the standard row filling, if we label the leftmost strand on the top of the diagram by 1 and the rightmost strand \( n \), then the diagram corresponding to \( \lambda \) will consist of symmetrizing all strands whose labels lie in the same row in \( \lambda \) and antisymmetrizing the strands whose entries lie in the same column of \( \lambda \). The objects corresponding to \( (n) \) and \( (n)^t = (1^n) \) in \( \mathcal{H} \) are the complete symmetrizer and complete antisymmetrizer depicted by white and dark gray boxes. In line with Khovanov and Cvitanović, we
depict them below. The diagrams for $Q^{(n)}$ and $Q^{(n)^t}$ are identical but with downward arrows instead.

\[
P^{(n)} := \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{array}
\quad P^{(n)^t} := \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{array}
\] (4.8)

More generally, for any partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ with transpose $\lambda^t = (\lambda_1^*, \ldots, \lambda_l^*)$ let $\sigma \in S_n$ be a permutation such that $T_{row} = \sigma T_{col}$. Following Cvitanović [7] we depict the Young idempotent $P^\lambda$ in $\mathcal{H}$ in the following manner:

\[
P^\lambda = \alpha_\lambda 
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{array}
\quad = \alpha_\lambda
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{array}
\] (4.9)

Then the coefficient $\alpha_\lambda$ ensures the object is idempotent and is defined by the formula:

\[
\alpha_\lambda := \frac{\prod_{i=1}^k |\lambda_i| \prod_{j=1}^l |\lambda_j^*|}{Y} \quad \text{where} \quad Y = \frac{n!}{\#\text{SYT}(\lambda)}.
\]

Clearly, $\alpha_\lambda$ is the same for all SYT($\lambda$) and satisfies $\alpha_\lambda = \alpha_{\lambda^t}$. Such detail is rarely needed, so henceforth we will instead utilize the following notation to represent (4.9). Note that the coefficient $\alpha_\lambda$ is built into this notation.

\[
P^\lambda = \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{array}
\quad Q^{\lambda^t} = \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{array}
\] (4.10)

Moreover, we will often group strands together into “thick” lines $\uparrow$ to denote more than one strand at a time and reserve “thin” lines $\uparrow$ to indicate strands of thickness exactly equal to 1.
**Example 4.2** Suppose \( \lambda = (a, b) \), so that \( \lambda' = (2^b, 1^{a-b}) \). Then \( p^\lambda \) in (4.10) is depicted by the diagram:

\[
p^{(a,b)} = \frac{(2!)^b(a-b+1)}{a+1}
\]

Symmetrizers and antisymmetrizers are elements of the group algebra of the symmetric group and satisfy the following relations (see [7,17]):

\[
\begin{align*}
\begin{array}{c}
\text{and} \quad \begin{array}{c}
\end{array}
\end{array} = \begin{array}{c}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\text{and} \quad \begin{array}{c}
\end{array}
\end{array} = \begin{array}{c}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\text{and} \quad \begin{array}{c}
\end{array}
\end{array} = 0 = \begin{array}{c}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\left( \begin{array}{c}
\end{array} \right) = \frac{1}{n} \left\{ \begin{array}{c}
\end{array} + n-1 \begin{array}{c}
\end{array} \right\}
\end{align*}
\]

\[
\begin{align*}
\left( \begin{array}{c}
\end{array} \right) = \frac{1}{n} \left\{ \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} + \cdots + \begin{array}{c}
\end{array} \right\}
\end{align*}
\]

\[
\begin{align*}
\left( \begin{array}{c}
\end{array} \right) = \frac{1}{n} \left\{ \begin{array}{c}
\end{array} - n-1 \begin{array}{c}
\end{array} \right\}
\end{align*}
\]

\[
\begin{align*}
\left( \begin{array}{c}
\end{array} \right) = \frac{1}{n} \left\{ \begin{array}{c}
\end{array} - \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} - \cdots + (-1)^{n-1} \begin{array}{c}
\end{array} \right\}
\end{align*}
\]

**4.3 Littlewood-Richardson branching isomorphisms**

In this section we provide categorifications of several branching rules for Schur, elementary, and complete symmetric polynomials and their adjoints by constructing explicit isomorphisms in \( \mathcal{H} \) that realize these decompositions. We begin with a classic result from the representation theory of the symmetric group. We refer the reader to [14] for more details and [7, Section 9.4.2] for a proof of the following theorem.
**Theorem 4.3** Given partitions \( \lambda, \mu \vdash n \), let \( \Delta_\lambda = \# \text{STY}(\lambda) \) be the dimension of the irreducible representation of \( S_n \) corresponding to \( \lambda \). For any \( T \in \text{SYT}(\lambda) \) and \( T' \in \text{SYT}(\mu) \) the following relations holds in \( \mathcal{H} \):

\[
P^{\otimes n} = \bigoplus_{T \in \text{SYT}(\lambda)} p^T \cong \bigoplus_{\lambda \vdash n} (p^\lambda)^{\oplus \Delta_\lambda} \tag{4.19}
\]

\[
1_pT \circ 1_pT' \cong \begin{cases} 1 & \text{if } \lambda = \mu \text{ and } T = T', \\ 0 & \text{else.} \end{cases} \tag{4.20}
\]

Given any two standard fillings \( T, T' \) of \( \lambda, \mu \vdash n \) let \( \delta \) denote the Kronecker delta of the corresponding standard Young tableau. Then equations (4.19) and (4.20) are diagrammatically the following statements.

\[
\uparrow \downarrow n \quad = \quad \bigoplus_{\lambda \vdash n \atop T \in \text{SYT}(\lambda)} \quad \uparrow \quad \quad \quad \uparrow \downarrow T \tag{4.21}
\]

\[
\uparrow \downarrow T \quad = \quad \delta_{\lambda, \mu} \delta_{T, T'} \quad \uparrow \downarrow T' \tag{4.22}
\]

**Proposition 4.4** Given any \( n, m \in \mathbb{N} \), the following relation holds in \( \mathcal{H}' \):

\[
\sum_{0 \leq s \leq \min(n, m) \atop s \text{-matchings}} \sum_{s \text{-matchings}} \quad = \quad \sum_{s \text{-matchings}} \tag{4.23}
\]

where the sum ranges over all possible ways of selecting \( s \) distinct \( Q_- \) and \( s \) distinct \( Q_+ \) and pairing them with a cap and cup for all values \( 0 \leq s \leq \min(n, m) \). On the
left, the map is the identity morphism for $Q_{-n} \otimes Q_{+m}$. On the right, each map is an endomorphism of $Q_{-n} \otimes Q_{+m}$ that factors through $Q_{+m-s} \otimes Q_{-n-s}$ for each value of $s$.

Consequently, the following isomorphism holds in $\mathcal{H}$:

$$Q^{\otimes n} P^{\otimes m} \cong \bigoplus_{s=0}^{\min(n,m)} \binom{n}{s} \binom{m}{s} P^{\otimes (m-s)} Q^{\otimes (n-s)}$$

**Proof** The proof follows from a straightforward inductive argument on $n$ and $m$. The case $n = m = 1$ is relation (4.3). Inducting first on $n$ consists of tensoring $QP$ on the left by $Q$. In this way, we see that the identity morphism of $Q^{\otimes n} P$ equals the sum of $n+1$ maps. The first is the obvious crossing map, which crosses the upward strand across all downward strands to its left and then back. The remaining $n$ maps are given by all $n$ distinct ways of paring $P$ with a $Q$ by a cup on top and a cap on the bottom. The second induction follows identically by tensoring $Q^{\otimes n} P$ with $P$ on the right $m$ times and inducting on $m$.

In $\mathcal{H}$ the top half of the diagrams are morphisms between $P^{\otimes (m-s)} Q^{\otimes (n-s)}$ and $Q^{\otimes n} P^{\otimes m}$ and the bottom half are maps in the reverse direction. Since for any $0 \leq s \leq \min(n, m)$ the number of $s$-matchings between $P$’s and $Q$’s is precisely $(s!) \binom{n}{s} \binom{m}{s}$, the relation in $\mathcal{H}$ follows.

**Notation:** In the remainder of the paper we make extensive use of the following notation:

- Given any object $X \in \mathcal{H}$, denote by $1_X$ the identity morphism on $X$ and depict it by vertical oriented lines with no crossings.
- For any indexing set $I$ and any two indices $s, t \in I$, denote by $\delta_{s,t}$ the Kronecker delta function.

With these isomorphisms in hand, we will now present various relations for the tensor products of $Q^\lambda$ and $P^\mu$ for varying partitions $\lambda$ and $\mu$. We begin by a Proposition due to Khovanov [25, Section 2.2].

**Proposition 4.5** For any integers $n, m \geq 1$, define the morphisms $\rho_1, \rho_2, \iota_1, \iota_2$ in $\mathcal{H}$ as follows:

$$\rho_1 := \begin{array}{c} \hline \hline \hline m & m-1 & m-2 & \cdots & 1 \end{array}, \quad \rho_2 := \begin{array}{c} \hline \hline \hline n & n-1 & n-2 & \cdots & 1 \end{array}, \quad \iota_1 := \begin{array}{c} \hline \hline \hline n & n-1 & n-2 & \cdots & 1 \end{array}, \quad \iota_2 := \begin{array}{c} \hline \hline \hline m & m-1 & m-2 & \cdots & 1 \end{array}$$

These maps satisfy the relations $1_Q^{\otimes n} \rho_1 = \iota_1 \circ \rho_1 + \iota_2 \circ \rho_2$ and $\rho_1 \circ \iota_5 = \delta_{s,t} 1_s$ in $\mathcal{H}$. Consequently, the isomorphism $Q^{\otimes n} P^{\otimes m} \cong \rho(m) Q^{\otimes n} \oplus Q^{\otimes (m-1)}$ holds in $\mathcal{H}$. 

The first relation is obtained via a diagrammatic computation and the use of (4.3). The second follows from using (4.15) and (4.17), and then realizing that by (4.7) the diagrams are nonzero if and only if \( s = t \).

The analogous statement \( Q^{(n)} p^{(m)} \cong p^{(m)} Q^{(n)} \oplus p^{(m-1)} Q^{(n-1)} \) also holds in \( \mathcal{H} \).

The remainder of the section is devoted to enhancing the calculus for Khovanov’s Heisenberg category by constructing explicit isomorphisms for various Littlewood-Richardson branching rules.

**Proposition 4.6** For any integers \( n, m \geq 1 \) and \( 0 \leq s \leq \min(m, n) \), define the morphisms \( \rho_s \) and \( \iota_s \) in \( \mathcal{H} \) as follows:

\[
\rho_s := \binom{m-s}{n-s} \binom{n}{m-s} s! \quad \iota_s := \binom{n}{s} \binom{m}{s} \quad (4.24)
\]

These morphisms satisfy \( \rho_s \circ \iota_t = \delta_{s,t} \cdot 1_{p^{(n-t)} q^{(m-t)}} \) and \( 1_{q^{(m)} p^{(n)}} = \sum_s \iota_s \circ \rho_s \).

Consequently, the following isomorphisms hold in \( \mathcal{H} \):

\[
Q^{(m)} p^{(n)} \cong \bigoplus_{s=0}^{\min(m,n)} p^{(n-s)} Q^{(m-s)} \quad \text{and} \quad Q^{(m)} p^{(n)} \cong p^{(n)} Q^{(m)} \oplus Q^{(m-1)} p^{(n-1)}.
\]

**Proof** The relation \( \rho_s \circ \iota_t = \delta_{s,t} \cdot 1_{p^{(n-t)} q^{(m-t)}} \) follows from exploding the symmetrizers \( n \) and \( m \) in (4.15) and noting that either all terms (when \( s \neq t \)) or all but one term (when \( s = t \)) will contain a right twist curl. Thus, by (4.6) and (4.7) the result will be either zero or the identity. Now, if we compose the morphisms in Eq. (4.23) with the symmetrizers for \( n \) and \( m \) on top and for \( m-s \) and \( n-s \) on the bottom we obtain \( \rho_s \). The reverse composition yields \( \iota_s \). Hence, the second relation \( 1_{q^{(m)} p^{(n)}} = \sum_s \iota_s \circ \rho_s \) follows directly from Proposition 4.4.

Proposition 4.6 holds in far more generality. We refer the reader to Theorem 9.2 in [30] for details.

**Proposition 4.7** For any integers \( n \geq 1 \) and \( m > 1 \) define the following morphisms in \( \mathcal{H} \):

\[
\iota_{n,m} := \binom{n}{(m, 1^n)} = \frac{(n)(m+1)}{n+m} \quad (4.25)
\]
These morphisms satisfy the relations

\[ (i) \quad \rho_{n,m}^n \circ t_{m-1}^{n,m} = 1_{P(n,1^m)} \quad \text{and} \quad \rho_{n,m}^{m-1} \circ t_{m-1}^{n,m} = 1_{P(n+1,1^{m-1})} \]

\[ (ii) \quad \rho_{n,m}^n \circ t_{m-1}^{n,m} = 0 \quad \text{and} \quad \rho_{n,m}^{m-1} \circ t_{m-1}^{n,m} = 0 \]

\[ (iii) \quad 1_{P(n)} P^{(m)} t = \rho_{n,m}^n \circ t_{m-1}^{n,m} + t_{m-1}^{n,m} \circ \rho_{n,m}^{m-1} \]

Consequently, the isomorphism \( P^{(n)} P^{(m)^1} \cong P^{(n,1^m)} \oplus P^{(n+1,1^{m-1})} \) holds in \( \mathcal{H} \).

**Proof** Relation (i) follows directly from composing the diagrams and Theorem 4.3. Relation (ii) follows from Eq. (4.14), since composing the respective morphisms will connect a symmetrizer and an antisymmetrizer by more than one strand. Relation (iii) is a consequence of pre and post composing the diagrams in Proposition 4.3 by symmetrizers for \( n \) and antisymmetrizers for \( m \). In particular, given any \( \lambda = (\lambda_1, \ldots, \lambda_k) \vdash (n + m) \) suppose we post-compose the diagram of \( P^\lambda \) with the symmetrizer for \( n \) on the rightmost \( n \) strands and the antisymmetrizer for \( m \) on the leftmost \( m \) strands. If \( \lambda_1 < n \) then all the strands of \( \lambda_1 \) and at least one strand of \( \lambda_2 \) will be connected to the symmetrizer for \( n \) above them. However, since every strand in \( \lambda_2 \) is antisymmetrized with one strand from \( \lambda_1 \), then there will exist an antisymmetrizer than is connected to the symmetrizer for \( n \) by more than one strand. By relation (4.14) the diagram would then be zero. Likewise, if \( \lambda_1 > n + 1 \) or \( \lambda_i > 1 \) for any \( i > 1 \) then the antisymmetrizer for \( m \) would be connected to a symmetrizer for some \( \lambda_i \) by more than one strand. Thus, \( \lambda_1 = n \) or \( n + 1 \) and \( \lambda_i = 1 \) for all \( i > 1 \). Consequently, the only partitions that are not zero are \( \lambda = (n, 1^m) \) and \( (n + 1, 1^{m-1}) \).
Proposition 4.8 For any \(0 \leq s \leq \min(n,m)\) define the morphisms \(\iota^m_s: p^{(m+n-s,s)} \to p^{(m)}p^{(n)}\) and \(\rho^s_m: p^{(m)}p^{(n)} \to p^{(m+n-s,s)}\) in \(\mathcal{H}\) as follows:

\[
\iota^m_s := \begin{cases} 
\begin{array}{c}
\begin{array}{c}
\text{(m)}
\end{array}
\begin{array}{c}
\text{(n)}
\end{array}
\end{array} & m < n \\
\begin{array}{c}
\begin{array}{c}
\text{(n)}
\end{array}
\begin{array}{c}
\text{(m+n-s,s)}
\end{array}
\end{array} & m \geq n
\end{cases}
\]

\(\rho^s_m := \begin{cases} 
\begin{array}{c}
\begin{array}{c}
\text{(m+n-s,s)}
\end{array}
\begin{array}{c}
\text{n}
\end{array}
\end{array} & m < n \\
\begin{array}{c}
\begin{array}{c}
\text{(m+n-s,s)}
\end{array}
\begin{array}{c}
\text{m}
\end{array}
\end{array} & m \geq n
\end{cases}
\]

These maps satisfy the relations \(\rho^s_m \circ \iota^m_s = \delta_{s,t} \cdot 1_{p^{(m+n-s,s)}}\) and \(1_{p^{(m)}p^{(n)}} = \sum_{s=0}^{\min(n,m)} \iota^m_s \circ \rho^s_m\). Hence, the following isomorphism holds in \(\mathcal{H}\):

\[p^{(m)}p^{(n)} \cong \bigoplus_{s=0}^{\min(m,n)} p^{(m+n-s,s)}\]

**Proof** The relation \(\rho^s_m \circ \iota^m_s = \delta_{s,t} \cdot 1_{p^{(m+n-s,s)}}\) follows from (4.22). Likewise, \(1_{p^{(m)}p^{(n)}} = \sum_{s=0}^{\min(n,m)} \iota^m_s \circ \rho^s_m\) is a consequence of pre and post composing the diagrams from Eq. (4.21) with symmetrizers for \(m\) and \(n\) and then noting that the only possible partitions that are nonzero have shape \(\lambda = (n + m - s, s)\) for \(0 \leq s \leq \min(n,m)\). Hence, by Theorem 4.3 the result follows. The final statement, \(p^{(m)}p^{(n)} \cong \bigoplus_{s=0}^{\min(m,n)} p^{(m+n-s,s)}\) is immediate from Theorem 4.3. \(\square\)

**Notation:** Given partitions \(\mu \vdash n\) and \(\lambda \vdash n - 1\), we write \(\lambda = \mu - \square\) if \(\lambda\) can be obtained from \(\mu\) by removing a box (equivalently, \(\mu = \lambda + \square\) if \(\mu\) can be obtained from \(\lambda\) by adding a box). Hence, if \(\mu = (\mu_1, \ldots, \mu_m)\) and \(\lambda = (\lambda_1, \ldots, \lambda_k)\) then \(m = k\) or \(k + 1\) and there will exist an integer \(1 \leq s \leq \min(m,k)\) such that \(\mu_i = \lambda_s + 1\) and \(\mu_i = \lambda_i\) for all \(i \neq s\) (if \(m = k + 1\) set \(\lambda_{k+1} = 0\)). In this case, we write \(\lambda = \mu(s^-)\) or equivalently \(\mu = \lambda(s^+)\). More generally, when \(v\) is a row or column partition of size \(k\) we will write \(\lambda \setminus \mu = v\) whenever \(\mu\) can be obtained from \(\lambda\) by removing \(k\) boxes in such a way that none of the boxes lie in the same column or row, respectively.

Proposition 4.9 For any \(\mu = (\mu_1, \ldots, \mu_m) \vdash n\) and for each \(\mu(s^-) = \mu - \square\), define the following morphisms in \(\mathcal{H}\):

\[
\rho_0 := \begin{array}{c}
\begin{array}{c}
\text{\mu}
\end{array}
\end{array} \quad \rho_s := \begin{array}{c}
\begin{array}{c}
\text{\mu(s^-)}
\end{array}
\end{array} \quad u_0 := \begin{array}{c}
\begin{array}{c}
\text{\mu}
\end{array}
\end{array} \quad \iota_s := \begin{array}{c}
\begin{array}{c}
\text{\mu(s^-)}
\end{array}
\end{array}
\]

(4.30)
These morphisms satisfy the relations:

\[ \rho_s \circ t_r = \delta_{s,r} 1_{\mu(s^-)} \quad \text{and} \quad 1_{Q^\mu P} = \sum_{\mu - \Box} t_s \circ \rho_s. \]

Consequently, the following isomorphism holds in \( \mathcal{H} \):

\[ Q^\mu P \cong P Q^\mu \bigoplus_{\lambda = \mu - \Box} Q^\lambda. \]

**Proof** The result follows from an explicit computation of the composition of the diagrams. For \( s = 0 \), the identity \( \rho_s \circ t_r = \delta_{s,r} 1_{\mu(s^-)} \) is immediate from (4.6). For any other values of \( s, t \) we expand the idempotent corresponding to \( \mu \) and then use (4.7) to observe that the only nonzero diagram occurs precisely when \( s = t \), in which case by (4.22) from Theorem 4.3 the result follows. The identity \( 1_{Q^\mu P} = \sum_{s=0}^m t_s \circ \rho_s \) is a consequence of once again composing the diagrams, expanding the idempotent for \( \mu(s+) \) and then using Proposition 4.4 to deduce the desired equality in \( \mathcal{H}' \) and the resulting isomorphism in \( \mathcal{H} \). \( \square \)

**Proposition 4.10** For any \( \lambda = (\lambda_1, \ldots, \lambda_k) \vdash n - 1 \) and each \( \lambda(s^+) = \lambda + \Box \), define the morphisms \( t_s : P^\lambda(s^+) \to P^\lambda P \) and \( \rho_s : P^\lambda P \to P^\lambda(s^+) \) in \( \mathcal{H} \) as follows:

\[
\begin{align*}
\iota_s &:= \lambda \downarrow \lambda(s^+) \\
\rho_s &:= \lambda(s^+) \downarrow \lambda
\end{align*}
\]

These morphisms satisfy the relations: \( \rho_s \circ t_r = \delta_{s,r} 1_{\lambda(s^+)} \) and \( 1_{P^\lambda P} = \sum_{\lambda + \Box} t_s \circ \rho_s. \) Thus, the following isomorphism holds in \( \mathcal{H} \):

\[ P^\lambda P \cong \bigoplus_{\mu = \lambda + \Box} P^\mu. \]

**Proof** By composing the diagrams and expanding the idempotent for \( \lambda \) by relation (4.22) in Theorem 4.3 we find that \( \rho_s \circ t_r \) is either zero for \( s \neq t \) or the identity for \( s = t \). When composing in the other direction, we instead expand the idempotent corresponding to \( \lambda(s^+) \) with (4.15) and (4.17). We then repeatedly apply (4.14) and (4.12) to simplify and cancel the summands. Finally, we use Theorem 4.3 to ascertain the relation \( 1_{P^\lambda P} = \sum_{\lambda + \Box} t_s \circ \rho_s \) holds in \( \mathcal{H}' \). The final isomorphism in \( \mathcal{H} \) follows immediate from these results by Theorem 4.3. \( \square \)
5 A categorical Boson-Fermion correspondence

Definition 5.1 For any category $\mathcal{C}$ denote by $\text{Mat}_{Z \times Z}(\mathcal{C})$ the category with objects given by formal $Z \times Z$ matrices $X = (X_{i,j})_{i,j \in Z}$ with $X_{i,j} \in \mathcal{C}$ satisfying the condition that $X_{i,j} = 0$ for $|i - j| \gg 0$. Its morphisms also consist of infinite matrices and are given by $(f_{i,j})_{i,j \in Z} \in \text{Hom}_{\text{Mat}(\mathcal{C})}(X, Y)$ with entries $f_{i,j} \in \text{Hom}_\mathcal{C}(X_{i,j}, Y_{i,j})$. Composition of morphisms is given by entry-wise composition of the matrices, so that $(f \circ g)_{i,j} := f_{i,j} \circ g_{i,j}$.

Likewise, define $\text{Mat}_{Z \times 1}(\mathcal{C})$ by restricting the conditions in the definition above to column vectors. Thus, $\text{Mat}_{Z \times 1}(\mathcal{C})$ consists of infinite dimensional vectors on $\mathcal{C}$ with finitely many nonzero entries.

If $\mathcal{C}$ is additive and monoidal then $\text{Mat}_{Z \times Z}(\mathcal{C})$ inherits the additive and monoidal structure by taking entry-wise direct sums and the usual multiplication of matrices. Moreover, if $\mathcal{C}$ acts on an additive category $\mathcal{A}$, there is an induced action of $\text{Mat}_{Z \times Z}(\mathcal{C})$ on $\text{Mat}_{Z \times 1}(\mathcal{A})$ given by multiplying the matrix with the column vector in the usual way and then taking the direct sums of the resulting actions inside $\mathcal{A}$. That is, for any row $[\ldots X_{i,j_1} \ldots X_{i,j_k} \ldots]$ of $X \in \text{Mat}_{Z \times Z}(\mathcal{C})$ and $[\ldots v_{j_1} \ldots v_{j_k} \ldots]^T \in \text{Mat}_{Z \times 1}(\mathcal{A})$ then $[\ldots X_{i,j_1} \ldots X_{i,j_k} \ldots] \cdot [\ldots v_{j_1} \ldots v_{j_k} \ldots]^T = \bigoplus_{j \in Z} X_{i,j}(v_j)$ where $X_{i,j}(v_j)$ is given by the action of $C$ on $\mathcal{A}$. Since only finitely many $v_j$ are non-zero the sum is finite and thus this action is well defined. Thus, for an additive category $\mathcal{C}$, the categories $\text{Mat}_{Z \times 1}(\mathcal{C})$ and $\text{Mat}_{Z \times Z}(\mathcal{C})$ are isomorphic (equivalent) to a direct sum of $\mathbb{N}$ copies of $\mathcal{C}$.

In particular, if $\mathcal{C} = \mathcal{K}(\mathcal{H})$ then $X, Y \in \text{Mat}_{Z \times Z}(\mathcal{K}(\mathcal{H}))$ are infinite matrices whose entries $X_{i,j}, Y_{i,j}$ are infinite chain complexes in $\mathcal{K}(\mathcal{H})$. Consequently their product, denoted by $XY$, is given by multiplying the matrices in the usual way and has entries of the form $\bigoplus_j X_{i,j} \otimes Y_{j,k}$. The term $X_{i,j} \otimes Y_{j,k}$ is the usual tensor product of the chain complexes $X_{i,j}, Y_{j,k}$ up to homotopy. As above, the condition $X_{i,j} = 0$ for $|i - j| \gg 0$ ensures that the direct sum of complexes $\bigoplus_j X_{i,j} Y_{j,k}$ is always finite. Thus, the entries of $XY$ consist of finite direct sums of infinite complexes up to homotopy.

In fact, many of the properties of $\mathcal{K}(\mathcal{H})$ can be extended to $\text{Mat}_{Z \times Z}(\mathcal{K}(\mathcal{H}))$. For instance, given $X, Y, Z \in \text{Mat}_{Z \times Z}(\mathcal{K}(\mathcal{H}))$ and a family of morphisms

$$d_{i,j} : X_{i,j} \rightarrow Y_{i,j} \quad \text{and} \quad d'_{i,j} : Y_{i,j} \rightarrow Z_{i,j}$$

with the property that $d'_{i,j}d_{i,j} = 0$ for all $i, j \in \mathbb{Z}$, then the morphisms $d := (d_{i,j}) \in \text{Hom}_{\text{Mat}(\mathcal{K}(\mathcal{H}))}(X, Y)$ and $d' := (d'_{i,j}) \in \text{Hom}_{\text{Mat}(\mathcal{K}(\mathcal{H}))}(Y, Z)$ are well defined and satisfy $d' \circ d = 0$. Hence, it makes sense to consider the following sequence in $\text{Mat}_{Z \times Z}(\mathcal{K}(\mathcal{H}))$:

$$X \xrightarrow{d} Y \xrightarrow{d'} Z$$

Thus, given appropriate maps $d_{i,j}$ we can consider chain complexes on $\text{Mat}_{Z \times Z}(\mathcal{K}(\mathcal{H}))$. Moreover, if $X_{i,j} \simeq Y_{i,j}$ in $\mathcal{K}(\mathcal{H})$ for all $i, j \in \mathbb{Z}$, then we say $X$ is homotopy equivalent to $Y$ in $\text{Mat}_{Z \times Z}(\mathcal{K}(\mathcal{H}))$ and write $X \simeq Y$. Likewise, for any $s \in \mathbb{Z}$ we write $X[s]$ to denote the matrix with entries $X_{i,j}[s]$ for each $i, j \in \mathbb{Z}$. 


At the decategorified level, recall that by Proposition 2.8 the Fock space representation of $\mathfrak{g}$ is isomorphic to $\text{Sym}$ and thus Bosonic Fock space as a module over $\mathfrak{g}$ is isomorphic to the direct sum of infinitely many copies of the Fock space representation. Motivated by this define categorical Fock space as the additive category

$$V_{Fock} := \text{Mat}_{\mathbb{Z} \times 1} \left( \bigoplus_n \mathbb{k}[S_n]-\text{mod} \right),$$

where any $v = (v_c)_{c \in \mathbb{Z}} \in \text{Ob}(V_{Fock})$ has only finitely many nonzero entries, and whose morphisms consist of vectors with entries given by $\bigoplus_n \mathbb{k}[S_n]-\text{module maps}$. Moreover, we make the following generalization.

**Definition 5.2** Given any $n \in \mathbb{N}$, define the category

$$\mathcal{H}_n := \bigoplus_{m \in \mathbb{Z}} (\mathbb{k}[S_m], \mathbb{k}[S_n]) \text{-bimod.} \quad (5.1)$$

So that, $\mathcal{H}_0 \cong \bigoplus_n \mathbb{k}[S_n]-\text{mod}$ and $V_{Fock} \cong \text{Mat}_{\mathbb{Z} \times 1}(\mathcal{H}_0)$.

**Remark 5.3** Since $P$ and $Q$ are induction and restriction functors on $\bigoplus_n \mathbb{k}[S_n]-\text{mod}$, by considering the left action of $P$ and the right action of $Q$ on the trivial module $\mathbb{k}$ we see that $\mathcal{H}_n$ can equivalently be defined as the category whose objects and morphisms are the same as in $V$, but subject to the condition that any object of the form $M \otimes Q^n$ for any $M \in \text{Obj}(V)$ is zero. This category is the universal categorical Fock space since it maps to any other categorification of Fock space. For more details see [6, Section 7].

### 5.1 Categorical Bernstein operators

Motivated by the Bernstein operators defined in (2.14), we make the following definition.

**Definition 5.4** The categorical Bernstein operators are infinite chain complexes in $K(V)$ given by:

$$B_a := \left\{ \bigoplus_{x-y=a} P^{(x)}Q^{(y)}[y], d_a \right\} \quad \text{with} \quad d_a := \begin{array}{c}\uparrow \downarrow x \downarrow \uparrow y \end{array} \quad \text{and} \quad a \in \mathbb{Z},$$

so that $B_a = \left( \cdots \to P^{(x)}Q^{(y)}[y] \to P^{(x-1)}Q^{(y-1)}[y-1] \to \cdots \right)$, and

$$B_a^* := \left\{ \bigoplus_{y-x=a} P^{(x)}Q^{(y)}[-x], d_a^* \right\} \quad \text{with} \quad d_a^* := \begin{array}{c}\uparrow \downarrow x \downarrow \uparrow y \end{array} \quad \text{and} \quad a \in \mathbb{Z},$$

so that $B_a^* = \left( \cdots \to P^{(x)}Q^{(y)}[-x] \to P^{(x+1)}Q^{(y+1)}[-x-1] \to \cdots \right)$. 

Since $x, y \geq 0$, the complex $B_a \in \mathcal{K}^-(\mathcal{H})$ is unbounded above and supported in non-negative homological degrees whereas $B_a^\ast \in \mathcal{K}^+(\mathcal{H})$ is unbounded below and supported in non-positive homological degrees. These functors are biadjoint in $\mathcal{K}(\mathcal{H})$ since $P$ and $Q$ are biadjoint in $\mathcal{H}$.

As in the decategorified picture, their action on $V_{Fock}$ is integrable. That is, for any $v \in \bigoplus \mathbb{C}[S_n]$-mod, since $Q$ is the restriction functor there is an $N \in \mathbb{N}$ such that $Q^{\otimes n}(\mathbb{k}) = 0$ for all $n > N$. Hence for any $a \in \mathbb{Z}$, the complexes $B_a(v)$ and $B_a^\ast(v)$ are finite and belong to $\text{Mat}_{Z\otimes 1}(\mathcal{K}^b(\mathcal{H}))$.

In [27, pg. 96] Macdonald shows the Bernstein operators act on $\text{Sym}$ like annihilation and creation operators for Schur functions. We lift that behavior and prove the categorical Bernstein operators satisfy the corresponding property for Specht modules.

**Theorem 5.5** The categorical Bernstein operators are creation and annihilation functors for Specht modules. That is, for any Specht module $S_\lambda$ associated to partition $\lambda = (\lambda_k, \ldots, \lambda_1) \vdash n$ with $\lambda_k \geq \cdots \geq \lambda_1 \geq 0$ and $\mathbb{k}$ the trivial module over $\mathbb{k}[S_n]$-mod, the categorical Bernstein operators satisfy $B_{\lambda_k}B_{\lambda_{k-1}} \cdots B_{\lambda_1}(\mathbb{k}) \simeq S_\lambda$ and $B_{\lambda_1}^\ast \cdots B_{\lambda_{k-1}}^\ast B_{\lambda_k}^\ast (S_\lambda) \simeq \mathbb{k}$.

**Proof** Suppose $n \in \mathbb{N}$ and $\lambda = (\lambda_k, \ldots, \lambda_1) \vdash n$. Since $Q$ acts by zero on $\mathbb{k}$, by Proposition 4.4 for any $y \geq 0$

$$Q^{\otimes y}p^{\otimes n}(\mathbb{k}) \cong \bigoplus_{s=0}^{\min(y,n)} p^{\otimes (n-s)}Q^{\otimes (y-s)}(\mathbb{k}) \cong \begin{cases} p^{\otimes (n-y)}(\mathbb{k}) & 0 \leq y \leq n \\ 0 & n < y \end{cases}$$

Composing with the canonical projection and inclusion maps for any $y \leq n$

$$Q^{(y)}p^\lambda \hookrightarrow Q^{\otimes y}p^n \twoheadrightarrow p^{\otimes (n-y)} \rightarrow p^\mu$$

then, when acting on $\mathbb{k}$, there is an isomorphism $Q^{(y)}p^\lambda(\mathbb{k}) \cong \bigoplus_{\lambda \vdash (1^y)} p^{\otimes (n-a)}p^\mu(\mathbb{k})$ where the sum is taken over all ways of removing $y$ boxes in distinct rows from $\lambda$. Moreover, we also know that $p^{(y+a)}p^\mu = \bigoplus_{\gamma \vdash (y+a)} p^\gamma$ where the sum ranges over all ways of adding $y + a$ boxes in distinct columns to $\mu$. Hence, for any $a \geq \lambda_n$ the action of $B_a$ on $p^\lambda(\mathbb{k})$ reduces to following bounded complex

$$B_a p^\lambda(\mathbb{k}) \simeq \bigoplus_{\lambda \vdash (1^n)} p^{(n+a)}p^\mu(\mathbb{k}) \rightarrow \cdots \rightarrow \bigoplus_{\lambda \vdash (1^y)} p^{(y+a)}p^\mu(\mathbb{k}) \rightarrow \bigoplus_{\lambda \vdash (1^{y-1})} p^{(y+a-1)}p^\mu(\mathbb{k}) \rightarrow \cdots \rightarrow p(\mathbb{k})p^\mu(\mathbb{k})$$

$$\simeq \bigoplus_{\lambda \vdash (1^n)} p^\gamma(\mathbb{k}) \rightarrow \cdots \rightarrow \bigoplus_{\gamma \vdash (1^y)} p^\gamma(\mathbb{k}) \rightarrow \cdots \rightarrow \bigoplus_{\gamma \vdash (1^{y-1})} p^\gamma(\mathbb{k}).$$

Composing the differential of $B_a$ with the projection and injection maps above we find that the differential from $\bigoplus_{\lambda \vdash (1^n)} p^{(y+a)}p^\mu(\mathbb{k}) \rightarrow \bigoplus_{\lambda \vdash (1^{y-1})} p^{(y+a-1)}p^\mu(\mathbb{k})$ for each $\gamma$ with $\gamma \vdash (y + a)$ and $\mathbb{k}$ with $\lambda \vdash (1^y)$ is
given by the leftmost diagram below. The subsequent equivalences are obtained by the following steps.

1. Expanding the idempotent for \( \lambda \) the map is nonzero if and only if \( \mu \backslash \nu = 1 \). That is, \( \mu = \nu + \Box \) so that \( v_i = \mu_i - 1 \) for some index \( i \) and \( v_j = \mu_j \) for all other \( j \neq i \). After some computations the differential reduces to a nonzero multiple of the second diagram.

2. By fully expanding the symmetrizers and anti-symmetrizers for \( y + a - 1, \mu, \nu \) and \( y + a \) the map is equivalent to a sum of multiples \( c_\sigma \) of \( 1_{P^\beta} \circ \sigma \circ 1_{P^\gamma} \) over some subset \( J \in S_{|\gamma|} \).

3. Since \( 1_{P^\beta} \circ \sigma \circ 1_{P^\gamma} = 1_{P^{T_{row}}} \circ 1_{P^{T'}} \) with \( T_{row} \in \text{SYT}(\beta) \) and \( T' = \sigma T_{row} \in \text{SYT}(\gamma) \) then by Eq. (4.22) from Theorem 4.3 the sum is nonzero only when \( \beta = \gamma \) and \( \sigma = id \). Thus, the third diagram is equivalent to a scalar multiple of the last diagram below.

Thus, these maps are isomorphisms whenever \( \beta = \gamma \) and zero otherwise. By Lemma 3.7 there is a homotopy equivalence

\[
B_a P^\lambda (k) \simeq 0 \rightarrow \cdots \rightarrow 0 \rightarrow 0 \cdots \rightarrow P^{(a, \lambda)} (k) \simeq P^{(a, \lambda_n, \ldots, \lambda_1)} (k).
\]

Inductively, we obtain \( B_{\lambda_n} B_{\lambda_{n-1}} \cdots B_{\lambda_1} (k) \simeq B_{\lambda_n} P^{(\lambda_{n-1}, \ldots, \lambda_1)} (k) \simeq P^\lambda (k) \simeq S_\lambda \) for any partition \( \lambda \). The dual statement \( B^*_\lambda \cdots B^*_1 B^*_1 S_\lambda \simeq k \) follows from the biadjointness of \( B_a \) and \( B^*_a \).

Recall that by (2.16), (2.17), and (2.18) the condition that the fermionic vertex operators satisfy the Clifford algebra relations in the Boson-Fermion correspondence is equivalent to the Bernstein operators satisfying certain anticommutation relations. Thus, in order to categorify this construction we prove analogous relations for the categorical Bernstein operators. We now state these relations but the proofs, due to their technical complexity, are deferred until Sect. 6.

**Theorem 5.6** Given any \( a, b \in \mathbb{Z} \), the categorical Bernstein operators satisfy the following chain homotopy relations in \( \mathcal{K}(\mathcal{H}) \):

\[
\cdots \rightarrow 0 \rightarrow 0 \cdots \rightarrow P^{(a, \lambda)} (k) \simeq P^{(a, \lambda_n, \ldots, \lambda_1)} (k) \simeq P^\lambda (k) \simeq S_\lambda \]

Recall that by (2.16), (2.17), and (2.18) the condition that the fermionic vertex operators satisfy the Clifford algebra relations in the Boson-Fermion correspondence is equivalent to the Bernstein operators satisfying certain anticommutation relations. Thus, in order to categorify this construction we prove analogous relations for the categorical Bernstein operators. We now state these relations but the proofs, due to their technical complexity, are deferred until Sect. 6.

**Theorem 5.6** Given any \( a, b \in \mathbb{Z} \), the categorical Bernstein operators satisfy the following chain homotopy relations in \( \mathcal{K}(\mathcal{H}) \):
Theorem 5.7 Given any \( a, b \in \mathbb{Z} \) and \( n \in \mathbb{N} \), the categorical Bernstein operators satisfy the following chain homotopy relations in \( \mathcal{K}(\mathcal{H}_n) \):

\[
B_{a+1} \otimes B^{*}_{b+1} \cong \begin{cases} 
B^*_b \otimes B_a[-1] & a < b \\
B^*_b \otimes B_a[1] & a > b 
\end{cases}
\]

Moreover if \( a \geq 0 \) then \( B^*_a \otimes B_a \cong \text{Cone} \left( B_{a+1} \otimes B^{*}_{a+1} \to 1 \right) \) and if \( a < 0 \) then \( B_{a+1} \otimes B^*_{a+1}[1] \cong \text{Cone} \left( 1 \to B^*_a \otimes B_a \right) \). Hence, there are distinguished triangles

\[
B_{a+1} \otimes B^*_{a+1} \to 1 \to B^*_a \otimes B_a \quad \text{and} \quad B^*_a \otimes B_a \to 1 \to B_{a+1} \otimes B^*_{a+1}.
\]

5.2 A categorical Boson-Fermion correspondence

We begin by defining certain categorical operators on \( \text{Mat}_{\mathbb{Z} \times \mathbb{Z}}(\mathcal{K}(\mathcal{H})) \).

Corollary 5.8 For integers \( i \in \mathbb{Z} \) define the operators \( B_i \) and \( B^*_i \) in \( \text{Mat}_{\mathbb{Z} \times \mathbb{Z}}(\mathcal{K}(\mathcal{H})) \) as infinite diagonal matrices with entries given by:

\[
(B_i)_{n,m} := \begin{cases} 
B_i \quad m = n \\
0 & \text{else} 
\end{cases} \quad (B^*_i)_{n,m} := \begin{cases} 
B^*_i \quad m = n \\
0 & \text{else}.
\end{cases}
\]

These functors satisfy the following relations in \( \text{Mat}_{\mathbb{Z} \times \mathbb{Z}}(\mathcal{K}(\mathcal{H})) \):

\[
B_{i-1} B_i := \begin{cases} 
B_{i-1} B_i[-1] & i > j \\
B_{i-1} B_i[1] & i < j \\
0 & i = j
\end{cases} \quad \text{and} \quad B^*_i B^*_j := \begin{cases} 
B^*_i B^*_j[-1] & i > j \\
B^*_i B^*_j[1] & i < j \\
0 & i = j.
\end{cases}
\]

Additionally, in \( \text{Mat}_{\mathbb{Z} \times \mathbb{Z}}(\mathcal{K}(\mathcal{H}_n)) \) they also satisfy the relations:

\[
B_{i+1} B^*_j + 1 \cong \begin{cases} 
B^*_j B_i[-1] & i < j \\
B^*_j B_i[1] & i > j
\end{cases}
\]

\[
B_{i+1} B^*_i \to \text{Id} \to B^*_i B_i \quad \text{and} \quad B^*_i B^*_i \to \text{Id} \to B_i B^*_i \quad \text{are distinguished triangles.}
\]

Proof Since by Definition 5.1 morphisms in \( \text{Mat}_{\mathbb{Z} \times \mathbb{Z}}(\mathcal{K}(\mathcal{H})) \) are given by matrices whose entries are chain homomorphisms and whose composition is given by composing the morphisms in \( \mathcal{K}(\mathcal{H}) \) entry-wise, then the relations are an immediate consequence of Theorems 5.6 and 5.7. Moreover, since any \( v = (v_c)_{c \in \mathbb{Z}} \) only finitely
many nonzero entries, the action of \( B_i \) and \( B_i^\ast \) on \( V_{Foc} \) is integrable and thus well defined.

We define the charge functor \( Q \in \text{Mat}_{\mathbb{Z} \times \mathbb{Z}}(\mathcal{K}(\mathcal{H})) \) as the infinite dimensional matrix with 1’s on the lower off-diagonal and zeroes elsewhere. Likewise, we define \( Q^{-1} \in \text{Mat}_{\mathbb{Z} \times \mathbb{Z}}(\mathcal{K}(\mathcal{H})) \) as the infinite dimensional matrix with 1’s on the upper off-diagonal and zeroes everywhere else. Functor \( Q \) raises the charge by shifting the indices of \( v = (\ldots, v_{c-1}, v_c, v_{c+1}, \ldots)^T \in V_{Foc} \) down by one. Analogously, \( Q^{-1} \) does the opposite. These functors are mutual inverses since \( QQ^{-1} = Q^{-1}Q = \text{Id} = Q^{-1}Q \).

A straightforward computation shows that for any integer \( i \in \mathbb{Z} \), the functors \( B_i \), \( B_i^\ast \), and \( Q \) satisfy the following commutation relations in \( \text{Mat}_{\mathbb{Z} \times \mathbb{Z}}(\mathcal{K}(\mathcal{H})) \):

\[
B_i = QB_i^{-1}Q^{-1} \quad B_i^\ast = QB_i^\ast Q^{-1}
\]  

(5.8)

We can now introduce the categorical Fermionic creation and annihilation operators as functors in \( \text{Mat}_{\mathbb{Z} \times \mathbb{Z}}(\mathcal{K}(\mathcal{H})) \) and present the main result, a categorification of Theorem 2.9.

Definition 5.9 Given any \( i \in \mathbb{Z} \), the Fermionic functors are the operators in \( \text{Mat}_{\mathbb{Z} \times \mathbb{Z}}(\mathcal{K}(\mathcal{H})) \) given by

\[
\Psi_i := B_iQ \quad \text{and} \quad \Psi_i^\ast := Q^{-1}B_i^\ast.
\]  

(5.9)

Theorem 5.10 (Categorical Boson-Fermion correspondence) For any \( i, j \in \mathbb{Z} \) the Fermionic functors satisfy the following relations in \( \text{Mat}_{\mathbb{Z} \times \mathbb{Z}}(\mathcal{K}(\mathcal{H})) \):

\begin{enumerate}
\item \( (\Psi_i)^2 \simeq 0 \) and \( \Psi_i\Psi_j \simeq \begin{cases} \Psi_j\Psi_i[-1] & \text{if } i > j \\ \Psi_j\Psi_i[1] & \text{if } i < j \end{cases} \)
\item \( (\Psi_i^\ast)^2 \simeq 0 \) and \( \Psi_i^\ast\Psi_j^\ast \simeq \begin{cases} \Psi_j^\ast\Psi_i^\ast[-1] & \text{if } i > j \\ \Psi_j^\ast\Psi_i^\ast[1] & \text{if } i < j \end{cases} \)
\end{enumerate}

Moreover, for any \( n \in \mathbb{N} \) the following relations also hold in \( \text{Mat}_{\mathbb{Z} \times \mathbb{Z}}(\mathcal{K}(\mathcal{H}_n)) \):

\begin{enumerate}
\item \( \Psi_i\Psi_j \simeq \begin{cases} \Psi_j\Psi_i[-1] & \text{if } i < j \\ \Psi_j\Psi_i[1] & \text{if } i > j \end{cases} \)
\item \( \Psi_i\Psi_j^\ast \rightarrow \text{Id} \rightarrow \Psi_i^\ast\Psi_i \) and \( \Psi_i^\ast\Psi_i \rightarrow \text{Id} \rightarrow \Psi_i\Psi_i^\ast \) are distinguished triangles.
\end{enumerate}

Proof The result follows easily from Corollary 5.8. Given any \( i, j \in \mathbb{Z} \) we use the commutation relations from (5.8) and compute the required compositions.

\[
\Psi_i\Psi_j = B_iQB_jQ = QB_i^{-1}B_jQ \quad \Psi_i^\ast\Psi_j^\ast = Q^{-1}B_i^\ast Q^{-1}B_j^\ast = Q^{-1}Q^{-1}B_{i+1}^\ast B_j^\ast
\]

\[
\Psi_j\Psi_i = B_jQB_iQ = QB_{j-1}B_iQ \quad \Psi_j^\ast\Psi_i^\ast = Q^{-1}B_j^\ast Q^{-1}B_i^\ast = Q^{-1}Q^{-1}B_{j+1}^\ast B_i^\ast
\]

Since the charge operators \( Q \) and \( Q^{-1} \) are invertible, then relations (1) and (2) of the theorem follow directly from (5.5) in Corollary 5.8. Similarly we have,

\[
\Psi_i\Psi_j^\ast = B_iQQ^{-1}B_j^\ast = B_iB_j^\ast
\]
\[ \Psi_j^* \Psi_i = Q^{-1} B_j^* B_i Q = Q^{-1} B_i^* B_{i-1} = Q^{-1} Q B_{j-1}^* B_{i-1} = B_{j-1}^* B_{i-1}. \]

Once again, relations (3) and (4) follow directly from (5.6) and (5.7) in Corollary 5.8. □

6 Properties of categorical Bernstein operators

**Notation:** We will often sum over indices \( t \geq \max(a_1, \ldots, a_n) \) or \( \min(a_1, \ldots, a_n) \geq t \). For notational simplicity, we will simply denote this by \( t \geq (a_1, \ldots, a_n) \) or \((a_1, \ldots, a_n) \geq t\).

### 6.1 Proof of Theorem 5.6

In this section we will compute the tensor products \( B_{b-1} \otimes B_a \) and \( B_{a-1} \otimes B_b \) for integers \( a, b \). In order to do this we appeal to Definition 3.3 and compute the total complex of certain bi-complexes. Since \( B_{b-1} \otimes B_a = \text{Tot}^\oplus \{ P(x) Q(y)^t B_a[y], d_{b-1} \otimes 1_{B_b}, (-1)^x 1_{B_{b-1}} \otimes d_a \}_{x-y=a} \), we first prove certain homotopy equivalences for the subcomplexes \( P(x) Q(y)^t B_a \) for fixed integers \( x, y \). With these equivalences, Lemma 3.7, and Proposition 3.8 we derive the desired categorical commutation relations for \( B_{b-1} \otimes B_a \) and \( B_{a-1} \otimes B_b \) stated in Theorem 5.6. Thus, we begin with a sequence of technical lemmas.

**Lemma 6.1** Given any fixed \( a \in \mathbb{Z} \) and \( n \in \mathbb{N} \), the chain complex \( Q^{(n)^t} B_a \) with differential \( 1_n \otimes d_a \) is homotopy equivalent to \( \text{Cone}(D) \) where \( D \) is the chain map:

\[
\begin{cases}
\bigoplus_{x \geq (n, -a, 0)} p(x+a) Q(x, n)^t [x-1], d_x = \begin{cases}
0, & \text{for } x = n \\
\cdots & \text{for } x \neq n \text{ and zero otherwise.}
\end{cases} \\
\end{cases}
\]

\[ D = \begin{cases}
\bigoplus_{x' \geq (0, -a+1)} q(x'+a-1) Q(x, -a)^t [x'], d_{x'} = \begin{cases}
0, & \text{for } x' = n \\
\cdots & \text{for } x' \neq n \text{ and zero otherwise.}
\end{cases} \\
\end{cases} \]

**Proof** We begin by applying the isomorphism from Propositions (4.5) and (4.8) to each chain group of \( Q^{(n)^t} B_a = \{ Q^{(n)^t} P(x+a) Q(x)^t [x], 1_n \otimes d_a \}_{x \geq (0, -a)} \).
\[ Q^{(n)f} \otimes B_a \simeq (\ldots \to P^{(x+a)} Q^{(n)f} Q^{(x)f} [x] \oplus p^{(s+a-1)} Q^{(n-1)f} Q^{(s)f} [x] \to \ldots) \]
\[ \simeq (\ldots \to \bigoplus_{s=0}^{(n,x)} p^{(x+a)} Q^{(n+x-s,s)} [x] \oplus \bigoplus_{s=0}^{(n-1,x)} p^{(x+a-1)} Q^{(n-1+x-s,s')} [x] \to \ldots) \]

Pre and post-composing with these isomorphisms, for each fixed index \( s \) in homological degree \( x \), the differential on each summand of the resulting chain complex is a morphism

\[ p^{(x+a)} Q^{(n+x-s,s)} [x] \to (\delta_{s',s} + \delta_{s',s-1}) p^{(x+a-1)} Q^{(n+x-1-s',s')} [x - 1] \]
\[ d : \bigoplus \to \bigoplus \]
\[ p^{(x+a-1)} Q^{(n+x-1-s,s')} [x] \to (\delta_{s',s} + \delta_{s',s-1}) p^{(x+a-2)} Q^{(n+x-2-s',s')} [x - 1] \]

given explicitly by the matrix

\[
d = \begin{pmatrix}
\delta_{s',s} + \delta_{s',s-1} & \delta_{s,s'} \\
0 & \delta_{s',s} + \delta_{s',s-1}
\end{pmatrix}_{s' \leq x}
\]

The differential from \( p^{(x+a-1)} Q^{(n-1+x-s,s)} [x] \to p^{(x+a-1)} Q^{(n+x-1-s',s')} [x - 1] \) on the top right corner of (6.1) equals \( \rho^{s'}_{n-2} \circ p^{a-2}_n \). By Proposition 4.8 it is an isomorphism if and only if \( s = s' \) and is zero otherwise. Thus, the map between homological degrees \( x \) and \( x - 1 \) given below is zero whenever \( s \neq s' \).

\[
d : \bigoplus_{s=0}^{(n-1,x)} p^{(x+a-1)} Q^{(n-1+x-s,s)} [x] \to \bigoplus_{s'=0}^{(n,x-1)} p^{(x+a-1)} Q^{(n+x-1-s',s')} [x - 1].
\]

Specifically, for \( n > x \) this map becomes

\[
d : \bigoplus_{s=0}^{x} p^{(x+a-1)} Q^{(n-1+x-s,s)} [x] \to \bigoplus_{s'=0}^{x-1} p^{(x+a-1)} Q^{(n+x-1-s',s')} [x - 1].
\]

Since the terms with \( 0 \leq s \leq x - 1 \) in homological degree \( x \) are bijectively mapped onto the terms with \( 0 \leq s' \leq x - 1 \) in homological degree \( x - 1 \), then for each
homological degree $x < n$ the only terms that are not canceled by Lemma 3.7 are $p(x+a-1)Q^{(n-1,x)'}[x]$. Moreover, since the maps

$$p(x+a-1)Q^{(n-1,x)'}[x] \to \bigoplus_{s' = 0}^{x-1} p(x+a-1)Q^{(n+x-1-s',s')'}[x - 1]$$

are zero, then the Gaussian elimination will not alter the existing arrow from $p(x+a-1)Q^{(n-1,x)'}[x] \to p(x+a-2)Q^{(n-1,x-1)'}[x - 1]$. As a result, there is a chain homotopy equivalence

$$\left\{ Q^{(n)'} p(x+a) Q^{(x)'} [x], d_x \right\}_{n > x \geq (0,-a)} \simeq \left\{ p(x+a-1)Q^{(n-1,x)'}[x], d_x \right\}_{n > x \geq (0,-a+1)}.$$

Likewise, if $x \geq n$ then

$$d : \bigoplus_{s=0}^{n-1} p(x+a)Q^{(n+x-s,s)'}[x + 1] \to \bigoplus_{s'=0}^{n-1} p(x+a)Q^{(n+x-s',s')'}[x].$$

By the same argument as before we obtain that $\{ Q^{(n)'} p(x+a) Q^{(x)'} [x], d_x \}_{x \geq n} \simeq \{ p(x+a)Q^{(x,n)'} [x], d_x \}_{x \geq n}$.

When considering $d : (Q^{(n)'} B_n)_{n} \to (Q^{(n)'} B_n)_{n-1}$, however, the situation is different since we have an isomorphism between all terms in the source and target as follows

$$d : \bigoplus_{s=0}^{n-1} p(n+a-1)Q^{(2n-1-s,s)'}[n] \to \bigoplus_{s'=0}^{n-1} p(n+a-1)Q^{(2n-1-s',s')'}[n - 1].$$

Therefore, the Gaussian elimination in this homological degree alters the differential from $p(n+a)Q^{(n,n)'}[n] \to p(n+a-1)Q^{(n-1,n)'}[n - 1]$, which by Lemma 3.7 is given by the diagram for $D$ at $x = n$ in the statement of the Lemma 6.1.

Putting all this together we see that between homological degrees $n$ and $n - 1$ we have,

$$Q^{(n)'} B_n \simeq \ldots \xrightarrow{d_x} p(n+a)Q^{(n,n)'}[n] \xrightarrow{D} p(n+a-2)Q^{(n-1,n-1)'}[n - 1] \xrightarrow{d_x} \ldots.$$

Define a new chain map $D : \{ p(x+a)Q^{(x,n)'}[x-1], d_x \}_{x \geq n} \to \{ p(x+a-1)Q^{(n-1,x)'}[x], d_x \}_{n > x \geq (0,-a+1)}$ by setting $D_x = 0$ for all $x \neq n$ and $D_n$ equal to the previous diagram from Lemma 6.1. Thus, we obtain a homotopy equivalence between $Q^{(n)'} B_n$ and Cone($D$).
Lemma 6.2  Given any $b \in \mathbb{Z}$ and $n \in \mathbb{N}$, the chain complex $B_{b-1}P^{(n)}$ with differential $d_{b-1} \otimes 1_n$ is homotopy equivalent to Cone $(D)$, where $D$ is the chain map:

$$
\begin{align*}
\bigoplus_{y \geq (n,0,b-1)} p^{(y,n)}Q^{(y-b+1)'}[y-b], d_y \bigg\} \xrightarrow{D} \bigg\{ \bigoplus_{y' \geq (0,b)} p^{(n-1,y')}Q^{(y'-b)'}[y'-b+1], d'_y \bigg\}
\end{align*}
$$

with $d_y = \begin{cases} 1_n & y = n \\ 0 & \text{otherwise} \end{cases}$, $d'_y = \begin{cases} 1_n & y = n \\ 0 & \text{otherwise} \end{cases}$, and $D = \begin{cases} 0 & \text{for } y = n \\ 1_n & \text{otherwise} \end{cases}$.

Proof The proof is identical to Lemma 6.1. \qed

Since $B_{b-1} \otimes B_a$ is the total complex of the bi-complex with columns indexed by the homological degrees of $B_a$ and whose rows are indexed by the homological degrees of $B_{b-1}$, then combining (5.2) and Definition 3.3 we have:

$$
B_{b-1} \otimes B_a = \text{Tot}^\oplus \left\{ B_{b-1}P^{(x)}Q^{(x-a)'}[x-a], d_{b-1} \otimes 1_x, (-1)^x 1_{B_{b-1}} \otimes d_a \right\}_{x \geq (a,0)}.
$$

Since the indexing set of the bi-complex is $I = \{ x \geq \max(0, a) \}$ which is bounded below, by Proposition 3.8 we can apply Lemma 6.2 to each column in $B_{b-1} \otimes B_a$ and simultaneously reduce all the columns of the bi-complex. The simultaneous chain homotopies, however, alter the differentials along the rows in a nontrivial manner which Lemma 6.2 does not address. In the following lemma we investigate the bi-complex further.

Proposition 6.3  Given any $a, b \in \mathbb{Z}$, the chain complex $B_{b-1} \otimes B_a \in \mathcal{K}(\mathcal{H})$ is homotopy equivalent to $\text{Tot}^\oplus \left\{ \bigoplus_{y \geq (b-1)} D_{x,y} \otimes D_{x,y}, d_{x,y}, d^x \right\}_{x \geq (a,0)}$ defined by:

$$
D_{x,y} := \bigoplus_{s=0}^{(y,x)} \bigoplus_{r=0}^{(y,b,x-a)} p^{(x+y,s,s)}Q^{(x+y+1-a-b-r)'}[x+y-a-b+1]
$$

and

$$
D_{x,y} := \bigoplus_{s=0}^{(y,x-1)} \bigoplus_{r=0}^{(y-b,x-a)} p^{(x+y-1,s,s)}Q^{(x+y-a-b-r)'}[x+y-a-b+1]
$$
where \( d_y : \mathcal{F}^\Pi_{x,y} \oplus \mathcal{F}^\Pi_{x,y} \rightarrow \mathcal{F}^\Pi_{x,y-1} \oplus \mathcal{F}^\Pi_{x,y-1} \) and \( d^x : \mathcal{F}^\Pi_{x,y} \oplus \mathcal{F}^\Pi_{x,y} \rightarrow \mathcal{F}^\Pi_{x-1,y} \oplus \mathcal{F}^\Pi_{x-1,y} \)
onumber on each summand \( s, r \mapsto s', r' \) is given by:

\[
\begin{align*}
\left(\begin{array}{c}
d_y|_{s,r} \\
0 \\
\end{array}\right) & = \\
\left(\begin{array}{c}
\delta_{x-s}b_{y,r}r' \\
0 \\
\end{array}\right), \\
\left(\begin{array}{c}
d^x|_{s,r} \\
0 \\
\end{array}\right) & = \\
\left(\begin{array}{c}
\delta_{x-s}b_{y,r}r' \\
0 \\
\end{array}\right).
\end{align*}
\]

In particular, the restriction \( d^x : \mathcal{F}^\Pi_{x,y} \rightarrow \mathcal{F}^\Pi_{x,y-1} \) is injective for \( y - b + 1 < x - a \), surjective for \( y - b + 1 > x - a \), and bijective for \( y - b + 1 = x - a \). Likewise, the restriction \( d_y : \mathcal{F}^\Pi_{x,y} \rightarrow \mathcal{F}^\Pi_{x,y-1} \) is surjective for \( x > y \), injective for \( x < y \), and bijective for \( x = y \).

**Proof** Applying the chain homotopy given by the isomorphism \( Q^{(y-b+1)'}p(x) \cong p(x)Q^{(y-b+1)'} \oplus p(x-1)Q^{(y-b)'} \) in Proposition 4.6 to \( B_{b-1} \otimes B_a \),

\[
B_{b-1} \otimes B_a = \text{Tor}^B \left\{ \bigoplus_{y \geq (0, b-1)} p(y)Q^{(y-b+1)'}p(x)Q^{(x-a)'}[x + y - a - b + 1], d_{b-1} \otimes 1_x, (1-y)1_{b-1} \otimes d_a \right\}_{x \geq (a, 0)}
\]

\[
\cong \text{Tor}^B \left\{ \bigoplus_{y \geq (0, b-1)} \mathcal{F}^\Pi_{x,y} \oplus \mathcal{F}^\Pi_{x,y}, d_y', (-1)^yd^x \right\}_{x \geq (a, 0)}
\]

with chain groups and differentials \( d_y' : \tilde{\mathcal{F}}^\Pi_{x,y} \oplus \mathcal{F}^\Pi_{x,y} \rightarrow \tilde{\mathcal{F}}^\Pi_{x,y-1} \oplus \mathcal{F}^\Pi_{x,y-1} \) and \( d^x' : \mathcal{F}^\Pi_{x,y} \oplus \tilde{\mathcal{F}}^\Pi_{x,y} \rightarrow \mathcal{F}^\Pi_{x-1,y} \oplus \tilde{\mathcal{F}}^\Pi_{x-1,y} \) given by,

\[
\tilde{\mathcal{F}}^\Pi_{x,y} := p(x)p(x)Q^{(y-b+1)'}Q^{(x-a)'}[x + y - a - b + 1] \\
\mathcal{F}^\Pi_{x,y} := p(x)p(x)Q^{(y-b)'}Q^{(x-a)'}[x + y - a - b + 1]
\]
Theorem 6.4 Let $a, b \in \mathbb{Z}$ with $a < b$ and for each integer $k \geq 2 - b - a$ define $\mathcal{G}_k$ as the following object in $\mathcal{H}$:

$$
\mathcal{G}_k := \bigoplus_{x + y = k + a + b, x > y \geq (b,0)} p(x - 1, y) \left( \bigoplus_{r = y - b + 1} Q(x + y - a - b - r, r)\right) \bigoplus_{x + y = k + a + b, x \geq b, y \geq (a,0)} p(x - 1, y) Q(x - b, y) \bigoplus_{y \geq (a,0)} Q(y - a)\right.
$$

(6.2)

and let $d_k$ be given by:

$$
d_k := \sum_{x + y = k + a + b} \sum_{r = y - b + 1} \sum_{x > y \geq (b,0)} (x - b, y - a) (x - b, y - r)\left. + (-1)^y (x - 1, y - b) y - a (x - 1, y - b) y - a \right) \left( - \right) (6.3)
$$
Then, the following homotopy equivalence holds in $\mathcal{K}(\mathcal{H})$:

$$B_{b-1} \otimes B_a \simeq \ldots \xrightarrow{d_{k+2}} G_{k+1}[k+1] \xrightarrow{d_{k+1}} G_k[k] \xrightarrow{d_{k-1}} \ldots$$ (6.5)

**Proof** Let $x > y$ and $\mathcal{F}^\Pi_{x,y}$, $\mathcal{F}^\Pi_{x,y}$, $d_y : \mathcal{F}^\Pi_{x,y} \to \mathcal{F}^\Pi_{x-1,y}$, and $d^x : \mathcal{F}^\Pi_{x,y} \to \mathcal{F}^\Pi_{x-1,y}$ be defined as in Proposition 6.3. Then, for any integer $1 \leq j < x - y - a + b$ consider the following restrictions of the differentials

$$d_{y+j} : \mathcal{F}^\Pi_{x-j,y+j} \to \mathcal{F}^\Pi_{x-j,y+1+j}, \quad d^{x-j} : \mathcal{F}^\Pi_{x-j,y+j} \to \mathcal{F}^\Pi_{x-1-j,y-j}.$$

By Proposition 6.3, $d_{y+j}$ is surjective for $1 \leq j \leq \frac{x-y}{2}$ and injective for $\frac{x-y}{2} < j < x - y$. Consequently, if we restrict $d_{y+j}$ to the summands corresponding to $0 \leq s \leq y$ for each $j$, then $d_{y+j} : \mathcal{F}^\Pi_{x-j,y+j} \to \mathcal{F}^\Pi_{x-j,y+j-1}$ is an isomorphism for all $1 \leq j < x - y$. Likewise, $d^{x-j}$ is injective for $0 \leq j \leq \frac{x-y-a+b}{2}$ and surjective for $\frac{x-y-a+b}{2} < j < x - y - a + b$. Thus, if we restrict $d^{x-j}$ to the summands with $0 \leq r \leq y - b$ for each $j$, then the map $d^{x-j} : \mathcal{F}^\Pi_{x-j,y+j} \to \mathcal{F}^\Pi_{x-1-j,y+j}$ is an isomorphism for all $0 \leq j < x - y - a + b$.

Now for each $1 \leq j < x - y$ and $0 \leq s \leq y$, the map $d_{y+j} : \mathcal{F}^\Pi_{x-j,y+j} \to \mathcal{F}^\Pi_{x-j,y+j-1}$ is invertible. Moreover since $x - y < x - y - a + b$, then $d^{x-j} : \mathcal{F}^\Pi_{x-j,y+j} \to \mathcal{F}^\Pi_{x-1-j,y+j}$ is a bijection for all $0 \leq r \leq y - b$ and $1 \leq j < x - y$. Thus, the composition:

$$d_j := d^{x-j} \circ (d_{y+j-1})^{-1} \circ d^{x-j-1} \circ \ldots \circ (d_{y+1})^{-1} \circ d^x$$ (6.6)

is well defined and exists for all $1 \leq j < x - y$ and $0 \leq s \leq y$. Graphically, these maps correspond to following the darker blue and green lines in an upward zigzag in Fig. 2.

This map can also be envisioned as the following sequence of compositions:

$$\mathcal{F}^\Pi_{x,y} \to \mathcal{F}^\Pi_{x-1,y} \to \mathcal{F}^\Pi_{x-1,y+1} \to \ldots \to \mathcal{F}^\Pi_{x-j,y+1+j} \to \mathcal{F}^\Pi_{x-j,y+j} \to \mathcal{F}^\Pi_{x=(j+1),y+j}$$

By Proposition 6.3 we know that $B_{b-1} \otimes B_a \simeq \text{Tor} \left\{ \bigoplus_{y \geq (0,b-1)} \mathcal{F}^\Pi_{x,y}, \mathcal{F}^\Pi_{x,y} \right\}_{x \geq (a,0)}$. Moreover, by Lemma 6.2 each subcomplex $\left\{ \bigoplus_{y \geq (0,b-1)} \mathcal{F}^\Pi_{x,y}, \mathcal{F}^\Pi_{x,y} \right\}_{x \geq (a,0)}$. Moreover, by Lemma 6.2 each subcomplex $\left\{ \bigoplus_{y \geq (0,b-1)} \mathcal{F}^\Pi_{x,y}, \mathcal{F}^\Pi_{x,y} \right\}_{x \geq (a,0)}$.
Fig. 2  The bi-complex for $B_{b-1} \otimes B_a$ with $d_y$ denoted in green and $d^x$ denoted in blue. Terms in the same homological degree lie in the same column.

$d_y \{ \}$ is homotopy equivalent to Cone$(D \otimes 1)$ where $D \otimes 1 : F^I_x \to F^{II}_x$ is the chain map between the chain complexes below with $D$ defined as in Lemma 6.2.

$F^I_x := \begin{cases} \bigoplus_{y \geq (0, b-1, x)} P(y, x)Q^{y+1-b^I}Q^{x-a^I} [x + y - a - b], \\ \end{cases}$

4.8 \begin{align*} & \bigoplus_{y \geq (0, b-1, x)} F^I_{x, y}|_{s=x} [-1], d_y \bigg) \\
\end{align*}

$F^{II}_x := \begin{cases} \bigoplus_{y \geq (0, b)} P(x-1, y)Q^{y-b}Q^{x-a} [x + y - a - b + 1], \\ \end{cases}$

4.8 \begin{align*} & \bigoplus_{y \geq (0, b)} F^{II}_{x, y}|_{s=y}, d_y \bigg) \\
\end{align*}

These homotopies affect the existing differentials $d^x$ along the rows of the bi-complex in the following way:

- If $y \geq x$ then $d^x : F^I_x \to F^I_{x-1}$ is given by $d^I : F^I_{x, y} \to F^{II}_{x-1, y}$ defined in Proposition 6.3. Since $F^{II}_{x, y}$ does not exist for $y \geq x$ then all the previous arrows $d^x : F^I_{x, y} \to F^I_{x-1, y} \oplus F^{II}_{x-1, y}$ disappear.
Fig. 3  The bi-complex $B_{b-1} \otimes B_a$ for $a < b \leq 0$. The differentials $d_{x-y}$ for $y = 0$ (left) and $y = 1$ (right) are presented in upward red arrows. The differential $d_x$ are the diagonal blue arrows, $d_y$ the thin horizontal green arrows, and $D$ the thick dark green arrows. Terms in the same column have equal homological degree, which is denoted by the number above shifted by $a + b - 1$.

- If $x > y$ then the differentials $d^x : F^I_{x} \to F^I_{x-1}$ are given by $d^x : F^I_{x,y} \to F^I_{x-1,y}$ from Proposition 6.3. If also $y \neq x - 1$ then the terms contained in the image of $d^x : F^I_{x,y} \mid_{s=y} \to F^I_{x-1,y}$ are all canceled under Lemma 6.2. Hence, the image of $F^I_{x} \mid_{s=y}$ in $F^I_{x-1}$ under $d^x$ is zero except for when $y = x - 1$, in which case by Proposition 6.3 $d^x : p(x-1,x-1)Q(x-a)Q(x-1-b) \to p(x-1,x-1)Q(x-1-a)Q(x-b)$ is an isomorphism for all but one summand in the source.

When $x > y$ the process also creates certain new arrows which we now describe (see Fig. 2). Since the only value of $j$ for which $F^I_{x-j,y-j} \mid_{s=x-j-1}$ is not eliminated by the homotopies in Lemma 6.2 is $j = x - y - 1$, then the simultaneous reductions generate the maps $d_{x-y-1} : F^I_{x,y} \mid_{s=y} \to F^I_{x-1,y-1} \mid_{s=y}$ given by the iterated compositions of $d^{x-j}$ and $(d_{y+j})^{-1}$ in (6.6) (see Fig. 3). In particular, each $d_{x-y-1}$ is the composition of maps which are either isomorphisms or zero. Hence, for all $x \geq \max(1,b+1)$ and $x \geq y \geq \max(0,a)$ there is an isomorphism between $F^I_{x,y} \mid_{s=y}$ and $F^I_{x',y'} \mid_{s=x'}$ by sending $x' \mapsto y$ and $y' \mapsto x - 1$.

Taking the total complex of the bi-complex we see that, for each integer $k \geq 2 - a - b$, the complex $B_{b-1} \otimes B_a$ is homotopy equivalent to a complex whose chain group in homological degree $k$ is given by

$$(B_{b-1} \otimes B_a)_k \simeq \bigoplus_{x'+y'=k+a+b-1} F^I_{x',y'} \mid_{s=x'} \bigoplus_{x+y=k+a+b-1} F^I_{x,y} \mid_{s=y}$$

$$\simeq \bigoplus_{x+y=k+a+b} F^I_{x,y-1} \mid_{s=y} \bigoplus_{x+y=k+a+b} F^I_{x,y} \mid_{s=y}.$$
which are either multiples of the identity or zero. Since $b > a$, these maps are injective. Performing Gaussian elimination on these isomorphisms $F_{x}^{II}$ is eliminated for all $x$. Moreover, since the source of these morphisms consists only of summands of $F_{x}^{II}$ and the target only of summands of $F_{y}^{I}$, then these reductions can be applied iteratively along increasing homological degrees. Therefore, the only terms that are left are certain summands of $F_{y}^{I}$ for $x > y$ described below. By comparing them with the source and target of (6.7) it is straightforward to see that these homotopies do not alter the differentials $d_{x-1} \mapsto d_{y}$ and $d_{y} \mapsto d_{x-1}$ on $F_{y,x-1}$ for $x > y$. Hence, in homological degree $k$ each chain group

\[
\bigoplus_{x+y=k+a+b} F_{y,x-1} | s=y \oplus \bigoplus_{x+y=k+a+b-1} F_{x,y} | s=y
\]

can be reduced to the following object in $\mathcal{H}$

\[
\mathcal{G}_k := \bigoplus_{x+y=k+a+b \atop x > y \geq (b,0)} p(x-1,y) \left( \bigoplus_{r=y+1} Q(x+y-a-b-r,r) \right) + \bigoplus_{x+y=k+a+b \atop x > y \geq (a,0)} p(x-1,y) Q(x-b) Q(y-a)
\]

with differential $d_k := \sum_{x+y=k+a+b} d_{x-1} + (-1)^y d^y$ given by

\[
d_{x-1} = \sum_{r=y-b+1}^{x-1} (x-b,y-a) (\cdots, r)
\]

and

\[
d^y = \sum_{r=y-b+1}^{x-1} (x-b,y-a) (\cdots, r)
\]

for $x > y \geq (b,0)$ (6.8)
Then the following homotopy equivalence holds in $K$:

$$
G_{b-1} \otimes B_a \simeq \ldots \xrightarrow{d_{k+2}} G_{k+1}[k+1] \xrightarrow{d_{k+1}} G_{k}[k] \xrightarrow{d_{k-1}} \ldots
$$

(6.10)

Theorem 6.5 Let $a, b \in \mathbb{Z}$ with $a > b$ and for each integer $k \geq 1 - a - b$ define $G'_k \in \mathcal{H}$,

$$
G'_k := \bigoplus_{x+y=k+a+b \atop y \geq (a,0)} p^{(y-1,x)} \left( \bigoplus_{r=x-a+1} \sum_{x+y=k+a+b} (-1)^x (x-b,y-a) \right) \bigoplus_{y \geq a} Q^{(y-a)^j} Q^{(x-y-a-b-r,r,y)}
$$

(6.11)

and let $d'_k$ be given by:

$$
d'_k := \sum_{x+y=k+a+b \atop y \geq (a,0)} (-1)^x \left. \sum_{r=x-a+1} \right. (x-b,y-a) \bigoplus_{y \geq a} Q^{(y-a)^j} Q^{(x-y-a-b-r,r,y)}
$$

(6.12)

Then the following homotopy equivalence holds in $K(\mathcal{H})$: 

$$
B_{b-1} \otimes B_a \simeq \ldots \xrightarrow{d'_{k+1}} G'_k[k+1] \xrightarrow{d'_k} G'_{k-1}[k] \xrightarrow{d'_{k-1}} \ldots
$$

(6.14)

Proof This proof is analogous to Theorem 6.4 with some minor differences which we now sketch. Let $x < y$ and for integers $1 < j \leq y-x-b+a$ and $\mathcal{F}^x_{x,y}, \mathcal{F}^y_{x,y}, d_y, \text{ and } d^x$ defined as in Proposition 6.3 consider the restrictions of the maps

$$
d_{y-j} : \mathcal{F}^y_{x+j,y-j} \rightarrow \mathcal{F}^x_{x+j,y-j-1} \quad \text{and} \quad d^x_{y-j} : \mathcal{F}^y_{x+j,y-j} \rightarrow \mathcal{F}^x_{x+j+1,y-j-1}.
$$

As before, restricting the summands to $0 \leq s \leq x$ and $0 \leq r < y-x+b-a$, we obtain that $d_y$ and $d^x$ are bijections for $1 < j \leq y-x$. Since $b < a$ then $y-x < y-x-b+a$. 

For $d'_k$ and $d'_{k-1}$, the proof follows a similar construction.
Thus, the composition
\[ d_j := d^{x-j} \circ (d_{y+j-1})^{-1} \circ d^{x-j-1} \cdots (d_{y+1})^{-1} \circ d^x \]  
(6.15)
is well defined and exists for all exists for all \( 1 < j \leq y - x \) and \( 0 \leq s \leq x \). These maps correspond to following the darker blue and green lines in an downward zigzag in Fig. 2. Equivalently, we can envision them as the following sequence of compositions:

\[ F_{x+j,y-j} \rightarrow F_{x+j-1,y-j} \rightarrow F_{x+j-1,y-j+1} \rightarrow \cdots \rightarrow F_{x+1,y-2} \rightarrow F_{x+1,y-1} \rightarrow F_{x,y-1} \]

Once again by Proposition 6.3 \( B_{b-1} \otimes B_a \simeq \text{Tor}^{\mathbb{Z}}_{1} \left( \bigoplus_{y \geq 0, b-1} F_{x,y} \otimes F_{x,y}', d_y, d^x \right) \) with each subcomplex \( \bigoplus_{y \geq 0, b-1} F_{x,y} \otimes F_{x,y}', d_y \) is homotopy equivalent to the mapping cone of the chain map \( D \otimes 1 : F^x \rightarrow F^x_x \) where, as before:

\[ F^x_x \simeq \bigoplus_{y \geq 0, b-1, x} F_{x,y} \]  
and

\[ F^x_x \simeq \bigoplus_{y \geq 0, b-1} F_{x,y} \]

Since \( a > b \), this time if we apply Lemma 6.2 the new arrows arise for \( j = y - x \). Hence, for all \( x \geq \max(1,b+1) \) and \( x \geq y \geq \max(0,a) \) there is an isomorphism between \( F^x_{x', y'} |_{s = y'} \) and \( F^x_{x,y-1} |_{s = x} \) by sending \( x' \mapsto y \) and \( y' \mapsto x \).

Thus, if we take the total complex of the reduced bi-complex then for each integer \( k \geq 2 - a - b \), the complex \( B_{b-1} \otimes B_a \) is homotopy equivalent to a complex whose chain group in homological degree \( k \) is

\[
\begin{align*}
(B_{b-1} \otimes B_a)_k & \simeq \bigoplus_{x+y=k+a+b-1} F^x_{x,y} |_{s = x} \bigoplus_{x+y'=k+a+b-1} F^x_{x', y'} |_{s = y'} \\
& \simeq \bigoplus_{x+y=k+a+b-1} F^x_{x,y} |_{s = x} \bigoplus_{y+x=k+a+b-1} F^x_{y,x} |_{s = x}.
\end{align*}
\]

In particular, for any fixed \( k \) such that \( x + y = k + a + b - 1 \) and \( y > x \) the new arrows from (6.15) analogous to the morphism from (6.7) are given by:

\[
\bigoplus_{y \geq (1,a)} F^x_{y,x} |_{s = x} [d_{y-1}] \rightarrow \bigoplus_{y \geq (1,a+1)} F^x_{x,y-1} |_{s = x}
\]  
(6.16)
Since \( a > b \) these maps are surjective. Performing Gaussian elimination across all homological degrees \( F^\Pi_x,y \) is eliminated for all \( y \geq (0, b - 1) \) with \( y > x \geq (0, a) \). Hence, in homological degree \( k \) each chain group

\[
\bigoplus_{x + y = k + a + b - 1 \atop x \geq (0, a), \ y \geq (0, b - 1, x)} F^\Pi_{x,y} \bigg|_{s=x} \oplus \bigoplus_{y + x = k + a + b - 1 \atop y \geq (1, b + 1), \ y' \geq (0, b)} F^\Pi_{y,x} \bigg|_{s=x}
\]

can be reduced to the following

\[
G'_{k-1} := \bigoplus_{x + y = k - 1 + a + b \atop y > x \geq (a, 0)} p(y-1,x) \left( \bigoplus_{r = x-a+1} Q(x+y-a-b-r, r)' \right) \bigoplus_{x + y = k - 1 + a + b \atop y > x \geq (b, 0)} p(y-1,x) Q(x-b)' Q(y-a)'.
\]

Thus, we have the following homotopy equivalence

\[
B_{b-1} \otimes B_a \simeq \cdots \xrightarrow{d_{k+1}'} G'_k[k+1] \xrightarrow{d_k'} G'_{k-1}[k] \xrightarrow{d_{k-1}'} \cdots \tag{6.17}
\]

with \( d_k' = \sum_{x + y = k + a + b} (-1)^x d^x + d_y \) defined analogously as for Theorem 6.5. \( \square \)

Combining the last two theorems we prove the first of the categorical Bernstein operator relations.

**Theorem 6.6** For any integers \( a, b \in \mathbb{Z} \), the categorical Bernstein operators satisfy the following homotopy relations in \( \mathcal{K}(\mathcal{H}) \):

\[
B_{a-1} \otimes B_a \simeq 0 \quad \text{and} \quad B_{a-1} \otimes B_b \simeq \begin{cases} B_{b-1} \otimes B_a[+1] & a < b \\ B_{b-1} \otimes B_a[-1] & a > b. \end{cases}
\]

**Proof** The first homotopy, \( B_{a-1} \otimes B_a \simeq 0 \) follows from setting \( a = b \) in Eq. (6.7) within the proof of Theorem 6.5. That is, the isomorphisms between homological degrees \( k \) and \( k - 1 \) are given by

\[
\bigoplus_{x \geq (1, a+1) \atop x > y \geq (0, a)} F^\Pi_{x,y} \bigg|_{s=y} \xrightarrow{[d_{e-1}]} \bigoplus_{x \geq (1, a+1) \atop x > y \geq (0, a)} F^\Pi_{y,x} \bigg|_{s=y}. \tag{6.18}
\]

Since \( Q^\nu \otimes Q^\lambda \simeq Q^\lambda \otimes Q^\nu \) for any partitions \( \nu, \lambda \), performing Gaussian eliminations across all homological degrees, all the chain groups cancel one another. Thus, \( B_{a-1} \otimes B_a \) is contractible.
Now, suppose $a < b$. Then by Theorem 6.5, $B_{a-1} \otimes B_b \simeq \{ \bigoplus_{k \in \mathbb{Z}} G'_k[k+1], d'_k \}$. Relabeling, so that $x \mapsto y$ and $y \mapsto x$ in (6.11) and comparing with 6.2 we immediately see $G'_k \mapsto G_k$ and

$$d'_k = \sum_{x+y=k+a+b} (-1)^x d^x + d_{y-1} \mapsto \sum_{x+y=k+a+b} (-1)^y d^y + d_{x-1} = d_k.$$  

Since by Theorem 6.5 we know $B_{b-1} \otimes B_a \simeq \{ \bigoplus_{k \in \mathbb{Z}} G_k[k], d_k \}$, it immediately follows that $B_{b-1} \otimes B_a[a+1] \simeq B_{a-1} \otimes B_b$.

If instead $a > b$, then in an identical manner by exchanging $x, y$ in Eq. (6.2) of Theorem 6.5 we see that $B_{a-1} \otimes B_b \simeq \{ \bigoplus_{k \in \mathbb{Z}} G'_k[k], d_k \}$. Since by Theorem 6.5, we have $B_{b-1} \otimes B_a \simeq \{ \bigoplus_{k \in \mathbb{Z}} G'_k[k+1], d'_k \}$ then again we obtain the homotopy equivalence $B_{b-1} \otimes B_a[a-1] \simeq B_{a-1} \otimes B_b$. $\blacksquare$

**Remark 6.7** Since $\mathcal{H}$ is not Krull-Schmidt none of the presentations above are necessarily minimal. In particular, via an analogous argument using Lemma 6.1 instead of Lemma 6.2, one can reduce the bi-complex and reprove Theorem 6.6 using completely different but still homotopy equivalent presentations of the chain complex $B_{b-1} \otimes B_a$.

In an identical manner, we can show the analogous statements for all the Lemmas, Propositions and Theorems for the adjoint categorical Bernstein operators. In particular, we can derive the following theorem.

**Theorem 6.8** For any integers $a, b \in \mathbb{Z}$, the following categorical Bernstein relations hold in $\mathcal{K}(\mathcal{H})$.

$$B^*_{a+1} \otimes B^*_a \simeq 0 \quad \text{and} \quad B^*_{a+1} \otimes B^*_b \simeq \begin{cases} B^*_{b+1} \otimes B^*_a[a+1] & a < b \\ B^*_{b+1} \otimes B^*_a[a-1] & a > b. \end{cases}$$

**Proof** The result follows from an identical proof to that of Theorem 6.6 by exchanging symmetrizers and antisymmetrizers and applying the dual statements of the isomorphisms. $\blacksquare$

### 6.2 Proof of Theorem 5.7

In this section we present the proofs for the remaining two relations for the categorical Bernstein operators. We consider the bi-complexes for the various tensor products of $B_a$ and $B^*_b$. This case differs vastly from Theorem 6.6 in that $B_a \in \mathcal{K}^- (\mathcal{H})$ whereas $B^*_b \in \mathcal{K}^+ (\mathcal{H})$. Thus, when considering the corresponding bi-complexes we will see that they are not $\mathcal{K}(\mathcal{H})$-locally finite. In particular, we will show that $B_{a+1} \otimes B^*_{b+1}$ and $B_b \otimes B^*_a$ arise as the total complex and the completed total complex of the same bi-complex. Unfortunately, the bi-complex is not homologically locally-finite and thus Proposition 3.6 does not hold. By setting certain additional finiteness conditions, however, the tensor product and the completed tensor product do agree and the desired result is obtained.

We begin by considering the two complexes $Q^{(m)} B^*_b$ and $Q^{(n)} B_a$ for integers $m, n \in \mathbb{N}$. 
**Proposition 6.9** Given any \( k \in \mathbb{N} \), the chain complex

\[
\bigoplus_{x=A}^{B} Q^{(k-x)}Q^{(x)}[-x], \ d = \begin{cases} 
\cdots & \cdots \\
\cdots & \cdots \\
\cdots & \cdots \\
\cdots & \cdots \\
\cdots & \cdots \\
\end{cases}
\]

is homotopy equivalent to one of the following (single term) complexes with zero differential,

\[
\bigoplus_{x=A}^{B} Q^{(k-x)}Q^{(x)}[-x], \ d = \begin{cases} 
\cdots & \cdots \\
\cdots & \cdots \\
\cdots & \cdots \\
\cdots & \cdots \\
\cdots & \cdots \\
\end{cases}
\]

\[
\xrightarrow{\rho_{x-1} \circ d \circ t_{x-1}^{Q^{(x)}}, \ rho_{x-1}^{Q^{(x)}} \circ d \circ t_{x-1}^{Q^{(x)}} \circ d \circ t_{x-1}^{Q^{(x)}} \circ d \circ t_{x-1}^{Q^{(x)}}} \]

**Proof** Applying Proposition 4.7 to each chain group will force the differentials of the resulting chain complex to be either isomorphisms or zero. In particular, the composition of the map \( d : Q^{(x)}Q^{(y)} \rightarrow Q^{(y)}Q^{(x)} \) defined as above with the projection and injection morphisms from Proposition 4.7 satisfy

\[
\rho_{y,x-1}^{Q^{(x)}} \circ d \circ t_{x-1}^{Q^{(x)}} = 1_{Q^{(y)}Q^{(x-1)}}, \rho_{x-1}^{Q^{(x)}} \circ d \circ t_{x-1}^{Q^{(x)}} = 0, \rho_{x-1}^{Q^{(x)}} \circ d \circ t_{x-1}^{Q^{(x)}} = 0,
\]

and \( \rho_{x-1}^{Q^{(x)}} \circ d \circ t_{x-1}^{Q^{(x)}} = 0. \)

Since the complex is finite we can use Lemma 3.7 across each homological degree so that only terms on the endpoints of the bi-complex are left. When \( A = 0 \) or \( B = k \) the terms at \( x = A \) and \( x = B \), respectively, do not decompose further. Hence, these terms cancel completely when applying Gaussian elimination. Consequently, the complex is contractible unless \( A > 0 \) or \( B < k \) in which case only the point with \( x = A \) or \( x = B \) remain.

**Notation:** For any set \( J \subset \mathbb{Z} \) let \( 1_{J}^{+} := \begin{cases} 
1; & z \in J \\
0; & \text{else.} \end{cases} \) denote its characteristic function and \([0, N] \) denote the set of all integers \( 0 \leq z \leq N \).

**Lemma 6.10** For any \( n, m \in \mathbb{N} \) let \( D_{m} := \begin{cases} 
\cdots & \cdots \\
\cdots & \cdots \\
\cdots & \cdots \\
\cdots & \cdots \\
\cdots & \cdots \\
\cdots & \cdots \\
\end{cases} \) and \( D_{n} := \begin{cases} 
\cdots & \cdots \\
\cdots & \cdots \\
\cdots & \cdots \\
\cdots & \cdots \\
\cdots & \cdots \\
\cdots & \cdots \\
\end{cases} \). Then, the following chain homotopies hold in \( K(\mathcal{H}) \):
is the total complex of bi-complex:

\[ Q^{(m)^T} B^*_b \cong \text{Cone} \left( \bigoplus_{R \geq (a+1,0)} \rho(R) Q^{(n+1,1R-a-1)}[R-a-1], d = a \right) \]

\[ Q^n B_a \cong \text{Cone} \left( \bigoplus_{R \geq (a+1,0)} \rho(R) Q^{(n+1,1R-a-1)}[R-a-1], d = a \right) \]

**Proof** Since both claims are proved similarly we provide details only for the first homotopy equivalence. Given \( m \in \mathbb{N} \) we apply Proposition (4.7) to \( Q^{(m)^T} B_b \). At the level of chain groups we have the isomorphisms

\[
Q^{(m)^T} B^*_b \cong \left\{ (m, w) \mid \bigoplus_{s=0}^w P^{(w-s)^T} Q^{(m-s)^T} Q^{(w)}[-w], 1 \otimes d^*_b \right\}_{v-w=b}.
\]

Letting \( R := w - s \) we have

\[
Q^{(m)^T} B^*_b \cong \left\{ w \mid \bigoplus_{R \geq (0,w-m)} P^{(R)^T} Q^{(m+R-w)^T} Q^{(b+w)}[-w], 1 \otimes d^*_b \right\}_{w \geq (0,-b)}.
\]

In particular, the differentials are now given by:

\[
d_w : P^{(R)^T} Q^{(m+R-w)^T} Q^{(b+w)}[-w] \rightarrow P^{(R)^T} Q^{(m+R-w-1)^T} Q^{(b+w+1)}[-w-1]
\]

\[
d^R : P^{(R)^T} Q^{(m+R-w)^T} Q^{(b+w)}[-w] \rightarrow P^{(R+1)^T} Q^{(m+R-w)^T} Q^{(b+w+1)}[-w-1]
\]

\[
d_w = - \quad d^R = \begin{array}{c}
\begin{array}{c}
\text{R}
\end{array}
\end{array}
\]

(6.19)

Thus for each fixed \( R \geq (-m-b, 0) \) we may identify subcomplexes \( \{ A_R(m), d_w \} \in K^b(\mathcal{H}) \) defined by:

\[
A_R(m) := \bigoplus_{w \geq (-b,R)} P^{(R)^T} Q^{(m+R-w)^T} Q^{(b+w)}[-w].
\]

Since \( d^R : A_R(m)[1] \rightarrow A_{R+1}(m) \) is a chain map such that \((d^R)^2 = 0\), then \( Q^{(m)} B^*_b \) is the total complex of bi-complex:
\[
Q^{(m)} R_b^* \simeq \text{Tot}^{\oplus} \{ A_R(m), d_R, d^R \}_{R \geq (0, -m-b)} \tag{6.20}
\]

Since each \( A_R(m) \) is finite we can apply Proposition 6.9 to each row of the bi-complex. Specifically, \( Q^{(m')} R_b^* \simeq \text{Tot}^{\oplus} \{ A_R(m), d_R, d^R \}_{R \geq (0, -m-b)} \), then if we take \( A = \max(-b, R) \) and \( B = k = R + m \) in Proposition 6.9, each \( A_R(m) \) will be contractible whenever \( -b \geq R > -b - m \) and isomorphic to a single term with \( w = R \) whenever \( R > -b \) and \( w = -b \) when \( R = -b - m \). Because \( R = w - s \) these terms correspond to \( s = 0 \) for \( R > -b \) and \( s = m \) when \( R = -b - m \). Consequently, each subcomplex \( A_R(m) \) is homotopy equivalent to one of the following:

\[
A_R(m) \simeq \begin{cases} 
p^{(R)} Q^{(R+b, 1m)} [-R], & R > -b \ (w = R) \\
p^{(-b-m)} [b], & R = -b - m \ (w = -b) \tag{6.21} \\
0, & -b \geq R > -b - m. 
\end{cases}
\]

Since \( A_R(m) \) is bounded for all \( R, m \) then the bi-complex is homologically locally finite and by Proposition 3.6:

\[
Q^{(m')} R_b^* \simeq \text{Tot}^{\oplus} \{ A_R(m), d_R, d^R \}_{R \geq (0, -m-b)} \simeq \text{Tot}^{\Pi} \{ A_R(m), d_R, d^R \}_{R \geq (0, -m-b)}. 
\]

Since \( R \geq (0, -m - b) \), then the indexing set of the homological degrees is bounded above, so by Proposition 3.8 we can simultaneously simplify each row in \( \text{Tot}^{\Pi} \{ A_R(m), d_R, d^R \}_{R \geq (0, -m-b)} \).

The simultaneous reductions will affect the arrows in the following manner. If \( R = -b - m \geq 0 \) then \( A_{-b-m}(m) \) will consist of a single term \( p^{(-b-m)} [b] \), denoted by the top most node in Fig. 4, whose new differential is obtained by following the zigzag down \( d^R \) (vertical blue arrows) and back along \( d_R \) (horizontal red arrows) until reaching the circled node in \( A_{R+1}(m) \). Following the diagram in Fig. 4, we can see that since any new morphism must point downward and to the left, this zig-zag will eventually terminate and thus no other arrows are created.

Consequently, the differential \( d^R \) is not modified but simply restricted to the terms that do not cancel under the Gaussian elimination. When \( 0 \leq R < -b \) we also have a new map from \( A_{-b-m}(m) \rightarrow A_{R+1}(m) \) corresponding to \( D : p^{(-b-m)} [b] \rightarrow p^{(-b+1)} Q^{(m+1)} [b - 1] \) and given by \( D = \)

Therefore, we have the equivalences

\[
Q^{(m')} R_b^* = \text{Tot} \{ A_R(m), d_R, d^R \} \\
\simeq \text{Cone} \left( \bigoplus_{0 \leq R \leq m} p^{(-b-m)} [b - 1] \right) \\
D \left( \bigoplus_{R \geq (-b+1, 0)} p^{(R)} Q^{(R+b, 1m)} [-R], \right).
\]
Fig. 4 The bicomplex from Lemma 6.10 with rows $\mathcal{A}_R(m)$ for various values of $R$

\[ A_{-b-m}(m) \]

\[ \cdots \]

\[ A_{R-1}(m) \]

\[ A_R(m) \]

\[ A_{R+1}(m) \]

\[ \cdots \]

\[ 0 \leq R < -b \]

\[ R = \max(-b, 0) \]

\[ R > -b > 0 \]

\[ \dfrac{D_y}{R \geq -b, 0} \]

\[ \sum \rho(y+a+1) \rho(-b-y-1) [b + y + 1] \]

\[ d_y = \]

\[ d^y = (-1)^R \]

\[ \text{with} \]

\[ D_y = \]

\[ d_y = \]

\[ d^y = \]

\[ \text{and} \]

\[ \text{Theorem 6.11} \text{ For any } a, b \in \mathbb{Z} \text{ the homotopy equivalence } B_{a+1} \otimes B_{b+1}^* \simeq \text{Tot}^\oplus \{ \mathcal{A}^y, d_y, d^y \}_{y \geq (0, -a-1)} \text{ holds in } K(H) \text{ where the chain complex } \{ \mathcal{A}^y, d_y \} \simeq \text{Cone}(D_y) \text{ with} \]

\[ \sum \rho(y+a+1) \rho(-b-y-1) [b + y + 1] \]

\[ \dfrac{D_y}{R \geq -b, 0} \]

\[ \sum \rho(y+a+1) \rho(R)^y Q^{(R+b, 1)} [y - R], d_y \]

\[ d_y = \]

\[ d^y = (-1)^R \]

\[ \text{and} \]
**Proof** Once again, we consider \( B_{a+1} \otimes B^*_b \) as a bi-complex so that,

\[
B_{a+1} \otimes B^*_b = \operatorname{Tot}^\oplus \left\{ p^{(y+a+1)} Q^{(y)} B^{*}_{b+1}, 1_y \otimes d^{*}_{b+1}, (-1)^{R} d_{a+1} \otimes 1_{b+1} \right\}_{y \geq \max(0, -a-1)}
\]

By tensoring \( Q^{(y)} B^{*}_{b+1} \) with \( p^{(y+a+1)} \) on the right we can apply Lemma 6.10 to each row of the bi-complex and obtain: \( p^{(y+a+1)} Q^{(y)} B^{*}_{b+1} \simeq \operatorname{Cone}(D_y) \), where \( D_y \) is the chain map:

\[
\begin{align*}
&1_{y}^{(0,-b-1)} p^{(y+a+1)} p^{(-b-y-1)'} [b + y] \\
&\quad \mapsto \\
&\quad \bigoplus_{R \geq (-b,0)} p^{(y+a+1)} p^{(R)'} Q^{(R+b+1,1')} [y - R], \quad d_y = 1_{p^{(y+a+1)}} \otimes d
\end{align*}
\]

where \( d \) is defined as in Lemma 6.10.

As always, we must address the effect of the simultaneous homotopies on \( d^{y} \). We claim that \( d^{y} \) is given by \( (-1)^{R} d_{a+1} \otimes 1_{b+1} \) restricted to the terms in the reduced complexes. This is equivalent to saying that the reductions do not interfere with the existing differentials. Indeed, if such a map were generated it would necessarily originate in a term that is not canceled under Lemma 6.10. Thus, it must have the form \( p^{(y+a+1)} p^{(R)'} Q^{(R+b+1,1')} \) for \( R > \max(0, -b - 1) \). We will show that the set of all its images under \( d_{a+1} \otimes 1_{b+1} \) have no preimages under the isomorphisms in \( 1_y \otimes d_{b+1} \). Since any new morphism would necessarily have to trace back through these isomorphisms, the previous statement would prove that zigzags originating in \( p^{(y+a+1)} p^{(R)'} Q^{(R+b+1,1')} \) cannot exist.

Since \( p^{(y+a+1)} p^{(R)'} Q^{(R+b+1,1')} \leftrightarrow p^{(y+a+1)} p^{(R)'} Q^{(y)'} Q^{(R+b+1)} \), consider the compositions:

\[
\begin{align*}
p^{(y+a+1)} p^{(R)'} Q^{(y)'} Q^{(R+b+1)} &\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \}
Given diagrammatically by,

\[
\begin{array}{c}
\text{Diagram 1}
\end{array}
\]

\[
\begin{array}{c}
\text{Diagram 2}
\end{array}
\]

\[
\delta_{s,0}
\]

Consequently, the map \( P^{(y+a+1)} p(R)^{y} Q^{(R+b+1,1)^{y}} \rightarrow P^{(y+a)} p(R-s)^{y} Q^{(y-1-s)^{y}} Q^{(R+b+1)} \) is nonzero if and only if \( s = 0 \). From Proposition 6.9 we know that the composition below is an isomorphism if and only if \( s' = s - 1 \), but since \( s' \geq 0 \) then when \( s = 0 \) no such \( s' \) exists.

\[
p(y+a) p(R-1-s')^{y} Q^{(y-1-s')^{y}} Q^{(R+b-1)} \quad p(y+a) p(R-s)^{y} Q^{(y-1-s)^{y}} Q^{(R+b+1)}
\]

\[
\downarrow
\]

\[
p(y+a) Q^{(y-1)^{y}} p(R-1)^{y} Q^{(R+b)} \xrightarrow{1_{y} \otimes d^{y}_{b+1}} p(y+a) Q^{(y-1)^{y}} p(R)^{y} Q^{(R+b+1)}
\]

This is precisely saying that image of \( P^{(y+a+1)} p(R)^{y} Q^{(R+b+1,1)^{y}} \) in \( P^{(y+a)} Q^{(y-1)^{y}} p(R)^{y} Q^{(R+b+1)} \) under \( d_{a+1} \otimes 1_{b} \) cannot be pulled back isomorphically onto any summand of \( P^{(y+a)} Q^{(y-1)^{y}} p(R)^{y} Q^{(R+b)} \) under \( 1_{y} \otimes d^{y}_{b+1} \), i.e. no new arrows are created under the simultaneous simplification of Lemma 6.10.

Consequently, the differential \( d^{y} \) is \( d_{a+1} \otimes 1_{b+1} \) composed and precomposed with the isomorphisms that exchange \( P^{x} \) and \( Q^{y} \) (Proposition 4.6) and merge the \( Q \)'s (Proposition 4.7), then restricted to the remaining terms under the reductions from Lemma 6.10. After some diagrammatic computations this differential simplified to the desired diagram \( d^{y} \) given in the theorem.

**Theorem 6.12** For any \( a, b \in \mathbb{Z} \) the homotopy equivalence \( B_{b}^{a} \simeq B_{a} \simeq \text{Tot}^{\pi} \{ A^{v}, d_{v}, d^{v} \}_{v \geq (a,b)} \) holds in \( K(H) \) where the chain complex \( \{ A^{v}, d_{v} \} \simeq \text{Cone}(D_{v}) \) with \( D_{v} \) the chain map:

\[
\left\{ \bigoplus_{R \geq (a+1)} p^{(v-b)^{y}} p^{(R)^{y} Q^{(R+b+1,1)^{y}}} [R - a + b - v - 1], d_{v} \right\} \xrightarrow{D_{v}} 1^{v}_{[0,a]} p^{(v-b)^{y}} p^{(a-v)} [b - v]
\]
with

\[ D_v = \begin{pmatrix} v & v+1 & v \end{pmatrix}, \quad d_v = (-1)^b - v \]

\[ d^v = \begin{pmatrix} v+2 & R \end{pmatrix}, \quad B_{a+1} \otimes B^*_b \]

and

\[ P(v-b)^{v} Q(v) B_a, (-1)^b - v 1 \otimes d_a, \]

\[ d^b \otimes 1 \}_{v \geq 0}^{(0,b)}. \]

The result follows from an identical argument to Theorem 6.11 with the following minor variations. Since \( P(v-b)^{v} Q(v) \) sits in homological degree \( b - v \) an increase in \( v \) corresponds to a decrease in the homological degree of the term. Consequently, \( v \) is bounded below by max\((0, b)\) and hence the bi-complex is bounded above. Since we are dealing with the completed tensor product, by Proposition 3.8 we can simultaneously apply Lemma 6.10 to each subcomplex \( P(v-b)^{v} Q(v) B_a \) and arrive at the desired homotopy equivalence.

\[ \square \]

The previous result would not hold if we swapped the completed tensor product with the usual tensor product. This is because the usual total complex of this bi-complex would not be well-defined after performing the simultaneous reductions on each subcomplex.

**Remark 6.13** Via identical arguments, the dual statements for Lemma 6.10 and Theorems 6.12 and 6.11 also hold.

Now, as in Remark 3.5, when considering tensor products of complexes in \( K^-(\mathcal{H}) \) and \( K^+(\mathcal{H}) \), the bi-complexes that arise are generally not \( K(\mathcal{H}) \)-locally finite. To remedy this we shift our paradigm from \( K(\mathcal{H}) \) to \( K(\mathcal{H}_n) \) for \( n \in \mathbb{N} \) arbitrary. Recall that \( \mathcal{H}_n \) is equivalent to \( \mathcal{H} \) quotiented by the maximal ideal generated by \( RQ^{\otimes n} \) for \( R \in \mathcal{H} \). This additional condition has two immediate effects:

1. The chain complexes \( B_a \) and \( B^*_b \) become finite over \( K(\mathcal{H}_n) \) and are consequently contained in \( K^b(\mathcal{H}) \).
2. The bi-complexes are bounded and thus homologically locally finite so Proposition 3.6 applies and the usual tensor product and completed tensor product are the same.

Thus, we have the following theorem.

**Theorem 6.14** For any \( n \in \mathbb{N} \), the following homotopy equivalences hold in \( K(\mathcal{H}_n) \).

\[
B_{a+1} \otimes B^*_b \simeq \begin{cases} B^*_b \otimes B_a[-1] & a < b \\ B^*_b \otimes B_a[+1] & a > b. \end{cases}
\]
Proof Suppose \( a < b \). Then by Theorem 6.11 and 6.12:

\[
\mathcal{A}^y = 1^y_{[0,-b-1]} \mathcal{P}^{(y+a+1)} \mathcal{P}^{(-b-y-1)} [b + y + 1] \\
\rightarrow \bigoplus_{R \geq (-b,0)} \mathcal{P}^{(y+a+1)} \mathcal{P}^{(R)^+} \mathcal{Q}^{(R+b+1,1^y)} [y - R], d_y \}
\]

\[
\mathcal{A}^v = \bigoplus_{R \geq (0,a+1)} \mathcal{P}^{(v-b)^+} \mathcal{P}^{(R)^+} \mathcal{Q}^{(v+1,1^{R-a-1})} [R - a + b - v], d_v \}
\]

\[
\rightarrow 1_v^{[0,a]} \mathcal{P}^{(v-b)^+} \mathcal{P}^{(a-v)} [b - v]
\]

However, \( \mathcal{P}^{(y+a+1)} \mathcal{P}^{(-b-y-1)} [b + y + 1] \) and \( \mathcal{P}^{(v-b)^+} \mathcal{P}^{(a-v)} [b - v] \) are nonzero if and only if \(-b - 1 \geq y \geq -a - 1\) and \( a \geq v \geq b\), which implies \( a \geq b\), but \( b > a\) so neither of these terms can exist. Thus,

\[
\mathcal{B}_{a+1} \otimes \mathcal{B}_{b+1}^* \simeq \text{Tot}^\oplus \bigoplus_{R \geq (-b,0)} \mathcal{P}^{(y+a+1)} \mathcal{P}^{(R)^+} \mathcal{Q}^{(R+b+1,1^y)} [y - R], d_y, d^y \}
\]

\[
\mathcal{B}_{b}^* \otimes \mathcal{B}_{a} \simeq \text{Tot}^\prod \bigoplus_{R' \geq (0,a+1)} \mathcal{P}^{(v-b)^+} \mathcal{P}^{(R)^+} \mathcal{Q}^{(v+1,1^{R-a-1})} [R' - a + b - v], d_v, d^v \}
\]

where the differentials are given by

\[
d_y = (-1)^y \begin{bmatrix}
R+1 \\
R+2 \\
(R+b+1,1^y)
\end{bmatrix}, \quad d^y = (-1)^R \begin{bmatrix}
R+y+1 \\
R+1 \\
(R+b+1,1^y)
\end{bmatrix}
\]

\[
d_v = (-1)^{b-v} \begin{bmatrix}
v-b \\
0 \\
(v+1,1^{R-a-1})
\end{bmatrix}, \quad d^v = \begin{bmatrix}
0 \\
0 \\
(v+1,1^{R-a-1})
\end{bmatrix}
\]

By Remark 3.4 we can rewrite \( \mathcal{B}_{b}^* \otimes \mathcal{B}_{a} \) in term of its rows instead of its columns, so that:

\[
\mathcal{B}_{b}^* \otimes \mathcal{B}_{a} \simeq \text{Tot}^\prod \bigoplus_{R' \geq (0,a+1)} \mathcal{P}^{(v-b)^+} \mathcal{P}^{(R)^+} \mathcal{Q}^{(v+1,1^{R'-a-1})} [R' - a + b - v], d_{R'}, d^{R'} \}
\]
Where \( d_{R'} = d^u \) and \( d^{R'} = d^v \). So then, if we send \( R' \rightarrow y + a + 1 \) and \( v - b \rightarrow R \) in \( B_b^* \otimes B_a \) and recall that \( P^jP \cong PP^j \), we have:

\[
B_b^* \otimes B_a \cong \text{Tot}^\Pi \left\{ \bigoplus_{R \geq (0,-b)} p^{(y+a+1)} p^{(R')} Q^{(R'+b+1,1)} [y + 1 - R], d_y, d^v \right\}
\]

where the differentials are exactly those obtained for \( B_{a+1} \otimes B_{b+1}^* \).

In particular, \( B_{a+1} \otimes B_{b+1}^* \) and \( B_b^* \otimes B_a \) correspond to the total complex and the completed total complex of the same bi-complex over \( \mathcal{K}(\mathcal{H}_n) \). Thus Proposition 3.6 applies and the result follows.

If \( a > b \) then through an entirely dual construction we derive the following result.

\[
B_b^* \otimes B_a \cong \text{Tot}^\oplus \left\{ \bigoplus_{R' \geq (0,b+1)} p^{(x+1,1)} Q^{(R')} Q^{(x-a)} [b - R' + x - a], d_x, d^x \right\}
\]

\[
B_{a+1} \otimes B_{b+1}^* \cong \text{Tot}^\Pi \left\{ \bigoplus_{R \geq (0,-a,0)} p^{(R+a+1,w)} Q^{(R')} Q^{(w+b+1)} [R - w], d_w, d^w \right\}
\]

Once again, these bi-complexes correspond to the usual and the completed total complex of the same bi-complex. Since we are working over \( \mathcal{K}^b(\mathcal{H}) \) then \( B_{a+1} \otimes B_{b+1}^* \cong B_{a+1} \otimes B_{b+1}^* \). Thus, we obtain \( B_{a+1} \otimes B_{b+1}^* \cong B_b^* \otimes B_a \).

**Theorem 6.15** For any \( n \in \mathbb{N} \), then over \( \mathcal{K}(\mathcal{H}_n) \) we have:

\[
B_a^* \otimes B_a \cong \text{Cone}(B_{a+1} \otimes B_{a+1}^* \rightarrow 1) \quad \text{and} \quad B_{a+1} \otimes B_{a+1}[1] \cong \text{Cone}(1 \rightarrow B_a^* \otimes B_a) \quad (6.22)
\]

\[
B_{a+1} \otimes B_{a+1}^* \cong \text{Cone}(B_a^* \otimes B_a \rightarrow 1) \quad \text{and} \quad B_a^* \otimes B_a[1] \cong \text{Cone}(1 \rightarrow B_{a+1} \otimes B_{a+1}^*) \quad (6.23)
\]

Thus, \( B_{a+1} \otimes B_{a+1}^* \rightarrow 1 \rightarrow B_a^* \otimes B_a \) and \( B_{a+1} \otimes B_{a+1}^* \rightarrow 1 \rightarrow B_{a+1} \otimes B_{a+1}^* \) are distinguished triangles in \( \mathcal{K}(\mathcal{H}_n) \).

**Proof** Since we are working over \( \mathcal{K}(\mathcal{H}_n) \) it follows \( B_a^* \) and \( B_b^* \) are finite and thus all tensor products and completed tensor products agree.

Suppose \( a \geq 0 \). Then by Theorems 6.11 and 6.12 we have

\[
B_{a+1} \otimes B_{a+1}^* \cong \text{Tot}^\oplus \left\{ \text{Cone}(D_y), \begin{pmatrix} d_y & 0 \\ D_y & d^v \end{pmatrix}, d^v \right\}_{y \geq 0}
\]

and \( B_a^* \otimes B_a \cong \text{Tot}^\oplus \left\{ \text{Cone}(D_v), \begin{pmatrix} d_v & 0 \\ D_v & d^v \end{pmatrix}, d^v \right\}_{v \geq a} \). In particular,

\[
\text{Cone}(D_y) = \bigoplus_{R \geq 0} p^{(y+a+1)} p^{(R')} Q^{(R'+a+1,1)} [y - R'], d_y \right\}
\]
\[ \text{Cone}(D_v) = \begin{cases} \bigoplus_{R \geq a+1} P^{(v-a)^t} P^{(R)} Q^{(v+1,1,R-a-1)} [R - v], d_v \end{cases} \overset{D_v}{\rightarrow} \delta_{v,a} 1[0] \]

Moreover, \( D_v \) is zero unless \( v = a \) and \( R = a + 1 \). Thus the bi-complex for \( B_a^* \otimes B_a \) has only one arrow going into \( 1[0] \) and no arrows coming out of it. Consequently, we can write:

\[ B_a^* \otimes B_a \simeq \text{Tot} \begin{cases} \bigoplus_{R \geq a+1} P^{(v-a)^t} P^{(R)} Q^{(v+1,1,R-a-1)} [R - v], d_v, d_v \end{cases} \overset{D_v}{\rightarrow} \delta_{v,a} 1[0] \]

But then, re-indexing so that \( R' \mapsto v - a \) and \( y + a + 1 \mapsto R \) and comparing the differentials we immediately obtain that:

\[ B_a^* \otimes B_a \simeq B_{a+1} \otimes B_{a+1}^*[1] \overset{D_v}{\rightarrow} \delta_{v,a} 1[0] \]

Likewise, if \( a < 0 \) we find that \( \text{Cone}(D_v) \) has no identity terms whereas \( \text{Cone}(D_y) \) has a term equal to \( 1 \) that is the source of one arrow and the target of none. By an identical argument we find that \( B_{a+1}^* \otimes B_{a+1}^* \simeq \text{Cone}(D_y) \). Thus, there exists a distinguished triangle

\[ \cdots \rightarrow B_{a+1} \otimes B_{a+1}^* \rightarrow 1 \rightarrow B_a^* \otimes B_a \rightarrow B_{a+1} \otimes B_{a+1}^*[1] \rightarrow \cdots \]

All the remaining statements follow by dual arguments.

\[ \square \]

### 7 Fock space idempotents

Cautis, Licata, and Sussan studied certain complexes \( \Sigma_i \) in the Heisenberg category from [3] and showed they induced categorical braid group actions [5]. Motivated by this, Cautis and Sussan introduced analogous chain complexes \( \Sigma^- \), \( \Sigma^+ \in K(H) \) and made several conjectures about their properties [6]. In this section we prove these conjectures.

Define the following biadjoint chain complexes in \( K(H) \):

\[ \Sigma^- := \begin{cases} \bigoplus_{\lambda \vdash n} P^{\lambda^t} Q^{\lambda} [n], d^- \end{cases} \in K^-(H) \text{ with} \]

\[ d^- : P^{\mu(s^+)} Q^{\mu^t(s^+)} \overset{t \otimes t}{\rightarrow} P^{\mu} P Q Q^{\mu^t} \overset{1 \otimes \text{adj} \otimes 1}{\rightarrow} P^{\mu} Q^{\mu^t} \]

\[ \Sigma^+ := \begin{cases} \bigoplus_{\lambda \vdash n} P^{\lambda^t} Q^{\lambda} [-n], d^+ \end{cases} \in K^+(H) \text{ with} \]

\[ d^+ : P^{\mu} Q^{\mu^t} \overset{1 \otimes \text{adj} \otimes 1}{\rightarrow} P^{\mu} P Q Q^{\mu^t} \overset{\rho \otimes \rho}{\rightarrow} P^{\mu(s^+)} Q^{\mu^t(s^+)} \]
where $\mu$ is any partition of $n - 1$, the maps $\iota_s$ and $\rho_s$ defined as in (4.31) and $\text{adj}$ equal to the adjunction maps given by the cap and cup morphisms. In particular, we have that

\[ d_n^- = \sum_{\mu} \mu(n-1) \sum_{\mu + \Box} 1_{\mu(s+) \otimes 1_{\mu}} \text{adj}^{(\mu + \Box) \otimes 1}, \]

where $\sum_{\mu + \Box}$ is summing all possible $\mu(s+)$ and $1_{\mu}$ is given by the morphisms in Proposition 4.9.

**Lemma 7.1** The chain complexes $\Sigma^P$ and $Q\Sigma^-$ are contractible.

**Proof** Tensoring $\Sigma^-$ with $P$, by Propositions 4.9 and 4.10 we have the chain group isomorphisms

\[ \bigoplus_{\lambda \vdash n} P^\lambda Q^{\lambda'} P \cong \bigoplus_{\lambda \vdash n} P^\lambda P Q^{\lambda'} \oplus \bigoplus_{\lambda \vdash n} \lambda = \mu + \Box \bigoplus_{\mu \vdash n-1} \mu(s+) P^{\mu(s+)} Q^{\mu'} \].

Let $D_n := \bigoplus_{\lambda \vdash n} P^\lambda P Q^{\lambda'} \cong \bigoplus_{\lambda \vdash n} \bigoplus_{\mu(s+)} P^{\lambda(s+)} Q^{\lambda'}$. Then, $\Sigma^- \cong \{ \bigoplus_n D_n \oplus D_{n-1}, D_n \}$ for some differential $D_n$. In particular, we will show that the differential restricted as follows $D_n : D_{n-1} \to D_{n-1}$ is an isomorphism for all $n > 0$.

First since the differential $d^- \otimes 1 : P^\lambda Q^{\lambda'} P \to P^\mu Q^{\mu'} P$ is zero whenever $\lambda \neq \mu + \Box$, then for any $\mu \vdash n - 1$, $D_n : P^{\mu(s+)} Q^{\mu'} \to P^\mu P Q^{\mu'}$ is given by a nonzero multiple of the map below:

This is because if $s \neq r$, the diagram on the left has a right twist curl which by (4.7) is zero. Consequently, if we denote $\iota_{s,r}$ by $\iota_{s,r} \otimes 1_{\mu'}$ then the restriction $D_n : D_{n-1} \to D_{n-1}$ is given by $D_n = \sum_{\mu \vdash n - 1} (\mu \otimes 1_{\mu}) \otimes 1_{\mu}$. Now consider the map

\[ \widetilde{D}_n := \sum_{\mu \vdash n - 1} \sum_{\mu + \Box} \rho^{\mu + \Box} \otimes 1_{\mu'} : \bigoplus_{\mu \vdash n - 1} P^{\mu} P Q^{\mu'} \to \bigoplus_{\mu + \Box} P^{\mu + \Box} Q^{\mu'}. \]
Computing its composition with $D_n$ we find that

$$D_n \tilde{D}_n = \left( \sum_{\mu' + \square} \left( \sum_{\mu'' + \square} \mu'' \otimes \rho_{\mu'' \otimes 1} \right) \right) \left( \sum_{\mu'' + \square} \mu'' \otimes \rho_{\mu'' \otimes 1} \right)$$

$$= \sum_{\mu' + \square} \left( \sum_{\mu'' + \square} \mu'' \otimes \rho_{\mu'' \otimes 1} \right) \otimes \left( \delta_{\mu', \mu''} \right)$$

$$= \sum_{\mu' + \square} 1_{\mu''} \otimes 1_{\mu'}$$

$$= 1 \oplus_{\mu' + \square} \mu''.$$

Similarly, we can compute that $\tilde{D}_n D_n = 1 \oplus_{\mu' + \square} \mu''$. Thus, $\tilde{D}_n$ and $D_n$ are mutual inverses. Defining the nullhomotopy to be $H_n = \begin{pmatrix} 0 & \tilde{D}_n \\ \tilde{D}_n & 0 \end{pmatrix}$ yields the desired result.

The proof for $Q \Sigma^{-} \cong 0$ follows identically by interchanging the terms in the tensor product and noting that $(\mu + \square)' = \mu' + \square$. \hfill \square

**Lemma 7.2** $\Sigma^{+}P$ and $Q \Sigma^{+}$ are contractible.

**Proof** The result follows from an identical argument to Lemma 7.1. \hfill \square

**Theorem 7.3** (Conjecture 4.1 in [6]) Given any nontrivial partition $\lambda$, the nullhomotopies $\Sigma^{-}P^\lambda \cong 0 \cong \Sigma^{+}P^\lambda$ and $Q^{\lambda} \Sigma^{-} \cong 0 \cong Q^{\lambda} \Sigma^{+}$ hold in $K(\mathcal{H})$. Thus, $\Sigma^{-}$ and $\Sigma^{+}$ are categorical projectors for $V_{Fock}$.

**Proof** Since for any pair of nontrivial partitions $\mu$ and $\lambda = \mu + \square$ there are unique maps $P^\lambda \hookrightarrow PP^\mu$ and $Q^{\lambda} \hookrightarrow Q^\mu Q$ given by $\rho$ in Proposition 4.10. Thus by Lemma 7.1, it follows that $\Sigma^{-}P^\lambda \hookrightarrow \Sigma^{-}PP^\mu \cong 0$. The remaining homotopies follow identically from Lemma 7.2. \hfill \square

**Corollary 7.4** Given complexes:

$$C^{-} \cong \cdots \to C_{k} \to C_{k-1} \to \cdots \to C_{1} \to 1 \in K^{-}(\mathcal{H})$$

$$C^{+} \cong 1 \to C_{-1} \to \cdots \to C_{k-1} \to C_{k} \to \cdots \in K^{+}(\mathcal{H})$$
Such that for all $k > 0$ we have $C_r = \bigoplus_{i \in I} P^{\lambda_i} X Q^{\mu_i}$ where $I \subset \mathbb{N}$ is some finite indexing set, $\lambda_i$ and $\mu_i$ are nontrivial partitions, and $X \in \mathcal{H}$. Then the following homotopy equivalences hold in $K(\mathcal{H})$:

- $\Sigma^- \otimes C^- \simeq \Sigma^- \otimes C^- \otimes \Sigma^-$
- $\Sigma^- \otimes C^+ \simeq \Sigma^- \otimes C^+ \otimes \Sigma^-$
- $\Sigma^+ \otimes C^- \simeq \Sigma^+ \otimes C^- \otimes \Sigma^+$
- $\Sigma^+ \otimes C^+ \simeq \Sigma^+ \otimes C^+ \otimes \Sigma^+$.

In particular, $\Sigma^- \otimes \Sigma^- \simeq \Sigma^-$ and $\Sigma^+ \otimes \Sigma^+ \simeq \Sigma^+$, and thus $\Sigma^\pm$ are idempotents.

**Proof** The homotopies follow directly from Theorem 7.3 and Proposition 3.8. $\square$

**Theorem 7.5** (Conjecture 4.2 in [6]) Given any $n \in \mathbb{N}$, the chain complexes $\Sigma^+$ and $\Sigma^-$ are homotopy equivalent in $K(\mathcal{H}_n)$.

**Proof** Since $\Sigma^\pm$ become finite on $K(\mathcal{H}_n)$ they are homologically locally finite and thus by Proposition 3.6 and Corollary 7.4 it immediately follows that $\Sigma^+ \simeq \Sigma^- \otimes \Sigma^+ \simeq \Sigma^- \otimes \Sigma^+ \simeq \Sigma^-$. $\square$

Since the action of $\Sigma^+$ and $\Sigma^-$ is integrable on categorical Fock space, then Theorems 7.3 and 7.5 imply that both $\Sigma^+$ and $\Sigma^-$ can be used to project onto $\mathcal{H}_0 = \bigoplus \mathbb{k}[S_n]-\text{mod}$. This is in line with the decategorified picture where any $\mathfrak{h}$-module generated by highest weight vectors decomposes into a direct sum of copies of the Fock module.

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**Appendix A: Examples**

**A.1.**

Let $a = 2$ and $b = 3$. By Theorem 6.6 we know that $B_1 \otimes B_3 \simeq B_2 \otimes B_2[1]$. We show explicitly how such an equivalence could be true. In particular, $B_2 \otimes B_2$ is given by total complex of the bi-complex,

\[
\cdots \xrightarrow{d \otimes 1} P^{(4)} Q^{(2)^r} P^{(2)}[2] \xrightarrow{d \otimes 1} P^{(3)} Q P^{(2)}[1] \xrightarrow{d \otimes 1} P^{(2)} P^{(2)}[0] \xrightarrow{1 \otimes d} P^{(3)} Q P^{(3)}[2] \xrightarrow{d \otimes 1} P^{(2)} P^{(3)} Q[1] \xrightarrow{1 \otimes d} P^{(2)} P^{(4)} Q^{(2)^r}[2]
\]
Applying Proposition 4.6 followed by Proposition 4.8 we obtain an isomorphism with the bi-complex,

\[
\begin{array}{c}
\begin{array}{c}
p^{(4)}p^{(2)}Q^{(2)}[2] \\
p^{(4)}PQ[2]
\end{array} \\
\downarrow{\text{d}} & \downarrow{\text{d}}
\end{array}
\begin{array}{c}
\begin{array}{c}
p^{(4)}PQ[1] \\
p^{(3)}P[1]
\end{array} \\
\downarrow{\text{d}} & \downarrow{\text{d}}
\end{array}
\begin{array}{c}
\begin{array}{c}
p^{(3)}P[0] \\
p^{(2,2)}[0]
\end{array} \\
\downarrow{\text{d}} & \downarrow{\text{d}}
\end{array}
\begin{array}{c}
\begin{array}{c}
p^{(3)}p^{(1)}QQ[2] \\
p^{(2,2)}p^{(3)}Q[2]
\end{array} \\
\downarrow{\text{d}} & \downarrow{\text{d}}
\end{array}
\begin{array}{c}
\begin{array}{c}
p^{(2)}P^{(3)}Q[1] \\
p^{(2)}p^{(4)}Q^{(2)}[2]
\end{array} \\
\downarrow{\text{d}} & \downarrow{\text{d}}
\end{array}
\end{array}
\]

Applying Lemma 3.7 along the isomorphisms denoted in red, we see that

\[
B_2 \otimes B_2 \cong \cdots \to P^{(3,2)}Q[1] \to P^{(2,2)}[0].
\]

Likewise, \(B_1 \otimes B_3\) is given by the total complex of the bi-complex,

\[
\begin{array}{c}
\begin{array}{c}
p^{(4)}Q^{(3)}[3] \\
p^{(4)}Q^{(3)}[3]
\end{array} \\
\downarrow{\text{d}} & \downarrow{\text{d}}
\end{array}
\begin{array}{c}
\begin{array}{c}
p^{(3)}p^{(2)}Q^{(2)}[2] \\
p^{(3,3)}Q^{(2)}[2]
\end{array} \\
\downarrow{\text{d}} & \downarrow{\text{d}}
\end{array}
\begin{array}{c}
\begin{array}{c}
p^{(2)}p^{(3)}Q^{(1)}[1] \\
p^{(2,2)}[1]
\end{array} \\
\downarrow{\text{d}} & \downarrow{\text{d}}
\end{array}
\begin{array}{c}
\begin{array}{c}
p^{(4)}Q^{(2)}[3] \\
p^{(3,3)}Q^{(2)}[3]
\end{array} \\
\downarrow{\text{d}} & \downarrow{\text{d}}
\end{array}
\begin{array}{c}
\begin{array}{c}
p^{(5)}Q^{(2)}[3] \\
p^{(4,2)}[3]
\end{array} \\
\downarrow{\text{d}} & \downarrow{\text{d}}
\end{array}
\begin{array}{c}
\begin{array}{c}
p^{(6)}Q^{(3)}[3] \\
p^{(4,2)}[3]
\end{array} \\
\downarrow{\text{d}} & \downarrow{\text{d}}
\end{array}
\end{array}
\]

which after applications of Propositions 4.6 and 4.8 becomes,

\[
\begin{array}{c}
\begin{array}{c}
p^{(4)}p^{(3)}Q^{(3)}[3] \\
p^{(4)}p^{(2)}Q^{(2)}[3]
\end{array} \\
\downarrow{\text{d}} & \downarrow{\text{d}}
\end{array}
\begin{array}{c}
\begin{array}{c}
p^{(4)}p^{(3)}Q^{(2)}[2] \\
p^{(3,3)}Q^{(2)}[2]
\end{array} \\
\downarrow{\text{d}} & \downarrow{\text{d}}
\end{array}
\begin{array}{c}
\begin{array}{c}
p^{(3)}p^{(2)}Q^{(2)}[1] \\
p^{(2,2)}[1]
\end{array} \\
\downarrow{\text{d}} & \downarrow{\text{d}}
\end{array}
\begin{array}{c}
\begin{array}{c}
p^{(3)}p^{(1)}QQ[3] \\
p^{(2,2)}p^{(3)}Q[3]
\end{array} \\
\downarrow{\text{d}} & \downarrow{\text{d}}
\end{array}
\begin{array}{c}
\begin{array}{c}
p^{(2)}p^{(5)}Q^{(2)}[3] \\
p^{(2)}p^{(5)}Q^{(2)}[3]
\end{array} \\
\downarrow{\text{d}} & \downarrow{\text{d}}
\end{array}
\begin{array}{c}
\begin{array}{c}
p^{(5)}Q^{(2)}[3] \\
p^{(4,2)}[3]
\end{array} \\
\downarrow{\text{d}} & \downarrow{\text{d}}
\end{array}
\begin{array}{c}
\begin{array}{c}
p^{(6)}Q^{(3)}[3] \\
p^{(4,2)}[3]
\end{array} \\
\downarrow{\text{d}} & \downarrow{\text{d}}
\end{array}
\end{array}
\]

...
Once again, cancelling summands along the isomorphisms in red and blue, by Lemma 3.7 we have

\[ B_1 \otimes B_3 \simeq \ldots \rightarrow P^{(3,2)} Q[2] \rightarrow P^{(2,2)}[1]. \]

Continuing in this manner along all homological degrees it becomes apparent how one obtains the homotopy equivalence \( B_1 \otimes B_3 \simeq B_2 \otimes B_2[1] \).

**A.2**

Now suppose \( a = b = 2 \) and consider \( B_1 \otimes B_2 \). By Theorem 6.6 this product should be nullhomotopic. To see why, consider the following bi-complex,

\[
\begin{array}{c}
P^{(3)} Q^{(2')} \rightarrow P^{(2)} Q[2] \rightarrow P^{(2)'[0]} \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
P^{(2)} Q^{(3)} \rightarrow P^{(3)} [2] \rightarrow P^{(3)[1]} \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
P^{(4)} Q^{(2')} \rightarrow [2]
\end{array}
\]

Once again, by Propositions 4.6 and 4.8 there is an isomorphism with the bi-complex below, with isomorphisms along the red and blue arrows.

\[
\begin{array}{c}
P^{(3)} Q^{(2')} \rightarrow P^{(2)} Q[2] \rightarrow P^{(2)'[0]} \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
P^{(3)} Q^{(3)} \rightarrow P^{(3)} [2] \rightarrow P^{(3)[1]} \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
P^{(4)} Q^{(2')} \rightarrow [2]
\end{array}
\]

After applying Lemma 3.7 along the red and blue isomorphisms, all chain groups in homological degrees zero and one can be canceled. Continuing in this manner one can eliminate the summands in all degrees, thus obtaining the desired nullhomotopy.

**A.3**

Lastly, let \( a = 1 \) and \( b = 2 \) and suppose \( Q^{(4)} = 0 \). By Theorem 6.14 we know that \( B_1 \otimes B_2^* \simeq B_2^* \otimes B_0 \). In particular, we have that \( B_1 \otimes B_2^* \) is given by the total complex of the bi-complex below,
Thus, after applying Lemma 3.7 along the isomorphism in red, we see that

\[ B_1 \otimes B_2^* \simeq P^{(2)} Q^{(2,1)}[1] \rightarrow P^{(2)}[0] \rightarrow P^{(3)}[-1]. \]

Now, if we consider \( B_1^* \otimes B_0 \) we see it is the total complex of the bi-complex

\[ \bigoplus_{i=1}^{3} d_i \bigoplus \begin{array}{c}
\rightarrow P^{(2)} Q^{(2,1)}[1] \\
\rightarrow P^{(2)}[0] \\
\rightarrow P^{(3)}[-1]
\end{array} \]

which after applying Propositions 4.6 and 4.7 is homotopic to,

\[ B_1^* \otimes B_0 \simeq P^{(2)} Q^{(2,1)}[2] \rightarrow P^{(2)}[1] \rightarrow P^{(3)}[0] \simeq B_1 \otimes B_2^*[1]. \]

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