Biseparability of noisy pseudopure, W and GHZ states using conditional quantum relative Tsallis entropy

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Abstract We employ the conditional version of sandwiched Tsallis relative entropy to determine \( 1 : N - 1 \) separability range in the noisy one-parameter families of pseudopure and Werner-like \( N \)-qubit W, GHZ states. The range of the noisy parameter, for which the conditional sandwiched Tsallis relative entropy is positive, reveals perfect agreement with the necessary and sufficient criteria for separability in the \( 1 : N - 1 \) partition of these one parameter noisy states.

Keywords Bipartite separability criterion · Entropic approach for separability · Noisy pseudopure states · Noisy Werner-like W-, GHZ states · Conditional version of quantum relative Tsallis entropy

1 Introduction

Entropic characterization of separability in mixed composite states has attracted significant attention [1–20]. Nielsen and Kempe [21] brought forth the remarkable feature that the subsystems of an entangled state may exhibit more disorder than the whole system—unlike a separable state, which is emphatically more disordered globally than locally. Consequent to this, the von Neumann conditional entropy \( S(B|A) = \)
\[ S(\rho_{AB}) - S(\rho_A) \] of a pure entangled bipartite state is negative. Negative von Neumann conditional entropies would only offer sufficient but not necessary condition for identifying entanglement in mixed states. For instance, the two-qubit Werner state \[ \rho_{AB} = I_4(1-x)/4 + x |\Psi\rangle\langle\Psi|, \quad 0 \leq x \leq 1, \quad |\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \] known to be separable in the range \( 0 \leq x \leq \frac{1}{3} \) and entangled in the range \( 1/3 < x \leq 1 \). But positive von Neumann conditional entropy \( S(B|A) \geq 0 \) results in the separability range \( 0 \leq x \leq 0.747 \) for the two-qubit Werner state. Generalized conditional entropies, such as Rényi and Tsallis entropies, offer more sophisticated tools to detect entanglement in mixed composite systems [1–20]. In fact, the conditional version of the Tsallis entropy

\[ S^T_q(A|B) = \frac{1}{q-1} \left[ 1 - \frac{\text{Tr} \rho_{AB}^q}{\text{Tr} \rho_B^q} \right], \quad (1) \]

was employed by Abe and Rajagopal [10] to obtain the separability range \( 0 \leq x \leq \frac{1}{3} \) for the two-qubit Werner state (identified in the limit of \( q \to \infty \) for which the conditional Tsallis entropy \( S^T_q(A|B) \) is positive). The separability criterion using the Abe–Rajagopal q-conditional entropy, \( (AR\text{-criterion}) \) was found to yield separability ranges matching with the positivity under partial transpose (PPT) criterion [24,25] in some one-parameter families of noisy states [17]

Entropic separability criterion received a further impetus recently with the introduction of the generalized non-commutative conditional sandwiched Tsallis relative entropy (CSTRE), which is shown [19,20] to be superior than the Abe–Rajagopal (AR) version of conditional Tsallis entropy in witnessing entanglement. In fact, the sandwiched (non-commuting) form of the Rényi relative entropy introduced in Refs. [26–28] led to an analogous sandwiched form of the Tsallis relative entropy of a density operator \( \rho \) and a positive operator \( \sigma \), given by [19],

\[ \tilde{D}^T_q(\rho||\sigma) = \frac{\text{Tr} \left\{ \left( \sigma^{1-q/2q} \rho \sigma^{1-q/2q} \right)^q \right\} - 1}{q-1}. \quad (2) \]

The Tsallis relative entropy \( \tilde{D}^T_q(\rho||\sigma) \) is zero if and only if \( \rho = \sigma \).

The new version of the Tsallis relative entropy \( \tilde{D}^T_q(\rho||\sigma) \) reduces to the traditional relative Tsallis entropy \( D^T_q(\rho||\sigma) \)

\[ D^T_q(\rho||\sigma) = \frac{\text{Tr} \left[ \rho^{q} \sigma^{1-q} \right] - 1}{q-1} \quad (3) \]

when \( \rho \) and \( \sigma \) commute with each other.

Based on the generalized non-commutative version \( \tilde{D}^T_q(\rho||\sigma) \) of the Tsallis relative entropy, one obtains the corresponding CSTRE to be of the form, \( \tilde{D}^T_q(\rho_{AB}||I_A \otimes \rho_B) \) of a composite bipartite state \( \rho_{AB} \) and the positive operator \( I_A \otimes \rho_B \) (where \( \rho_B \) is the subsystem state \( \rho_B = \text{Tr}_A[\rho_{AB}] \), and \( I_A \) denotes the identity matrix in the Hilbert space of the subsystem A) as [19],

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\[
\widetilde{D}_q^T (\rho_{AB} \| I_A \otimes \rho_B) = \frac{\widetilde{Q}_q (\rho_{AB} \| I_A \otimes \rho_B) - 1}{1 - q}.
\]

(4)

Here, we have denoted,

\[
\widetilde{Q}_q (\rho_{AB} \| I_A \otimes \rho_B) = \text{Tr} \left\{ \left[ (I_A \otimes \rho_B)^{\frac{1-q}{2q}} \rho_{AB} (I_A \otimes \rho_B)^{\frac{1-q}{2q}} \right]^q \right\} = \sum \lambda_i^q,
\]

where \(\lambda_i\) are the eigenvalues of the sandwiched matrix \((I_A \otimes \rho_B)^{\frac{1-q}{2q}} \rho_{AB} (I_A \otimes \rho_B)^{\frac{1-q}{2q}}\). Non-positive values of the CSTRE \(\widetilde{D}_q^T (\rho_{AB} \| I_A \otimes \rho_B)\) with \(q > 1\), i.e.,

\[
\widetilde{D}_q^T (\rho_{AB} \| I_A \otimes \rho_B) = \frac{(\sum \lambda_i^q) - 1}{1 - q} < 0
\]

(5)

imply entanglement (see Refs. [19,20]).

When the subsystem \(\rho_B\) is maximally mixed, the CSTRE \(\widetilde{D}_q^T (\rho_{AB} \| I_A \otimes \rho_B)\) reduces to the commutative version viz., the AR q-conditional Tsallis entropy \(S_q^T (A | B)\) of (1). In our earlier papers [19,20] we had investigated bipartite separability of one parameter noisy symmetric multiqubit systems based on the non-positivity of both AR conditional entropy and the corresponding CSTRE. We had shown [19,20] that whenever the subsystem is not maximally mixed, the CSTRE criterion yields stricter separability range than that obtained through the commutative AR version. In this article, we extend the CSTRE criterion to witness entanglement in noisy one parameter families of the \(N\)-qubit pseudopure states [29] and the \(N\)-qubit generalizations of Werner-like one parameter states [22] involving W, GHZ states. We show that the non-commutative CSTRE criterion is both necessary and sufficient to detect entanglement in the \((1 : N - 1)\) partitions of the one parameter noisy multiqubit states explored here.

2 Pseudopure \(N\)-qubit W and GHZ states

The pseudopure (PP) families of states are formed by mixing any pure state with white noise [29]. They have played a crucial role in demonstrating quantum information processing possibilities in liquid state NMR spectroscopy [30,31]. In Ref. [29], different measures of quantum correlations of bipartite \(d \times d\) PP states of the form

\[
\rho_{\phi}^{pp} (x) = \frac{1 - x}{d^2 - 1} [(I_d \otimes I_d) - |\phi\rangle\langle \phi|] + x|\phi\rangle\langle \phi|
\]

(6)

(where \(|\phi\rangle\) is any arbitrary \(d \times d\) pure entangled state and \(0 \leq x \leq 1\) denotes the noisy parameter) are examined. Here we investigate entanglement in the \((1 : N - 1)\) bipartition of the \(N\)-qubit PP states, constructed using W and GHZ states, based on the CSTRE approach.
The one parameter family of $N$-qubit pseudopure states

$$\rho_{W_N}^{PP}(x) = \frac{1-x}{2^N-1} \left( I_2^{\otimes N} - |W_N\rangle\langle W_N| \right) + x |W_N\rangle\langle W_N|$$

obtained by considering the pure state $|\phi\rangle$ in (6) to be the $N$-qubit W state:

$$|W_N\rangle = \frac{1}{\sqrt{N}} \left[ |1102\cdots0_N\rangle + |0112\cdots0_N\rangle + \cdots + |010203\cdots1_N\rangle \right]$$

and the $d \times d$ matrix $I_d \otimes I_d$ replaced by its multiqubit counterpart $I_2^{\otimes N}$.

We focus on finding the $1 : N - 1$ separability range of the W family of PP states $\rho_{W_N}^{PP}(x)$ using CSTRE criterion. For this purpose, an evaluation of the eigenvalues $\lambda_i(x)$ of the sandwiched matrix

$$\left( I_2 \otimes \sigma_{W_{N-1}}^{PP}(x) \right)^{\frac{1-q}{2^q}} \rho_{W_N}^{PP}(x) \left( I_2 \otimes \sigma_{W_{N-1}}^{PP}(x) \right)^{\frac{1-q}{2^q}},$$

where $\sigma_{W_{N-1}}^{PP}(x) = \text{Tr}_1[\rho_{W_N}^{PP}(x)]$ denotes the $N - 1$ qubit subsystem of $\rho_{W_N}^{PP}(x)$, needs to be carried out. We obtain the following explicit structure of the eigenvalues $\lambda_i$ (for $N \geq 3$):

$$\lambda_1 = (2)^{\frac{1-q}{q}} \left( \frac{1-x}{2^N-1} \right)^{\frac{1}{q}} \left( 2^N - 4 \right) \text{-fold degenerate};$$

$$\lambda_2 = \left( \frac{1-x}{2^N-1} \right) \left[ (2N-1) + \left( \sum_{j=3}^{N} 2^{j-1} - 2(N-2) \right) \frac{x}{N(2^N-1)} \right]^{\frac{1-q}{q}},$$

$$\lambda_3 = \left( \frac{1-x}{2^N-1} \right) \left[ (N+1) + \left( \sum_{j=3}^{N} 2^{j-1} + (N-2)(2^N-2) \right) \frac{x}{N(2^N-1)} \right]^{\frac{1-q}{q}},$$

$$\lambda_{4/5} = \left[ N \left( 2^N - 1 \right) \right]^{\frac{1-q}{q}} \left( \frac{1}{2} \right) \times \left[ \alpha a + \beta b \pm \sqrt{(\alpha a + \beta b)^2 + 8N^2(2^N-1)x(x-1)\alpha \beta} \right].$$

where

$$\alpha = \left[ 2N-1 + \left( \sum_{j=3}^{N} 2^{j-1} - 2(N-2) \right) x \right]^{\frac{1-q}{q}},$$

$$\beta = \left[ N+1 + \left( \sum_{j=3}^{N} 2^{j-1} + (N-2)(2^N-2) \right) x \right]^{\frac{1-q}{q}}.$$
Table 1  Comparison of the $1 : N - 1$ separability range of the state $\rho_{W_N}^{PP}(x)$, for $N = 3, 4, 5, 6$ obtained through different separability criteria

| Number of qubits ($N$) | von Neumann conditional entropy | AR q-conditional entropy | CSTRE | PPT |
|------------------------|---------------------------------|--------------------------|-------|-----|
| 3                      | 0.7390                          | 0.3636                   | 0.3083| 0.3083|
| 4                      | 0.6963                          | 0.25                     | 0.1807| 0.1807|
| 5                      | 0.6723                          | 0.1621                   | 0.1014| 0.1014|
| 6                      | 0.6621                          | 0.1                      | 0.0552| 0.0552|

Substituting these eigenvalues $\lambda_i$ in (5), a numerical estimation of the $1 : N - 1$ CSTRE separability range for $N = 3, 4, 5, 6$ has been carried out. This results in the separability range for the noisy parameter $x$ to be $(0, 0.3083), (0, 0.1807), (0, 0.1014), (0, 0.0552)$ in the $1 : 2, 1 : 3, 1 : 4, 1 : 5$ partitions of the noisy state $\rho_{W_N}^{PP}(x)$ with $N = 3, N = 4, N = 5, N = 6$, respectively. The results obtained based on the CSTRE along with the corresponding cutoff value of the parameter $x$ obtained using the AR- and the PPT criteria are listed in Table 1. This offers a direct comparison of different approaches, each leading to the threshold values of the parameter $x$ (beyond which the noisy state is found to be entangled). From Table 1 it is clearly seen that, for the noisy state $\rho_{W_N}^{PP}(x)$, CSTRE provides better separability range than the AR-criterion. Moreover, the CSTRE separability range matches identically with the PPT separability range.

In general, the CSTRE criterion (the inequality (5) in the limit $q \to \infty$) leads to,

$$0 \leq x \leq \frac{N + \sqrt{N - 1}}{N + 2^N \sqrt{N - 1}}$$

for the separability range in the $(1 : N - 1)$ partition of the noisy $N$-qubit PP state $\rho_{W_N}^{PP}(x)$ for $N \geq 3$. Alternately, in the parameter region

$$\frac{N + \sqrt{N - 1}}{N + 2^N \sqrt{N - 1}} < x \leq 1,$$

the CSTRE method witnesses entanglement in the $(1 : N - 1)$ bipartition of the noisy state.

The PP family of states (see (6)) with the pure entangled state $|\phi\rangle$ expressed in terms of the Schmidt coefficients, i.e., $|\phi\rangle = \sum_{i=1}^{d} u_i |i_Ai_B\rangle$, with $u_1 \geq u_2 \geq \cdots \geq u_d \geq 0$ are shown to be separable iff $[29,32]$
\[ 0 \leq x \leq \frac{1 + u_1 u_2}{1 + d^2 (u_1 u_2)} \]  

(11)

For the PP state \( \rho_{WP}^N(x) \) of (7) with \((1 : N - 1)\) bipartition under investigation, the Schmidt coefficients (positive square roots of the eigenvalues of the reduced single qubit subsystem density matrix) of the \(N\)-qubit \(W\) state are given by,

\[ u_1 = \sqrt{\frac{N-1}{N}}, \quad u_2 = \frac{1}{\sqrt{N}}. \]  

(12)

Substituting (12) and replacing \( d^2 \) by \( 2^N \) in (11), we recover the result (10) for the separability range. This establishes that the CSTRE approach serves as both necessary and sufficient to detect entanglement in the \((1 : N - 1)\) partition of the PP state \( \rho_{WP}^N(x) \).

We now proceed to investigate the noisy one parameter family of \(N\)-qubit PP states \( \rho_{PP}^{GHZ,N}(x) \) given by,

\[
\rho_{PP}^{GHZ,N}(x) = \frac{1-x}{2^N-1} \left( I_2^N - |GHZ_N\rangle\langle GHZ_N| \right) + x|GHZ_N\rangle\langle GHZ_N|.
\]

where,

\[ |GHZ_N \rangle = \frac{1}{\sqrt{2}} (|0102\cdots 0 N \rangle + |1112\cdots 1 N \rangle) \]  

(13)

To find the \(1 : N - 1\) separability range of \( \rho_{PP}^{GHZ,N}(x) \) using CSTRE approach, one needs to evaluate the eigenvalues \( \lambda_i \) of the sandwiched matrix \( \left(I_2 \otimes \sigma_{PP}^{GHZ,N-1}(x)\right)^{1-q} \)

\[
\rho_{PP}^{GHZ,N}(x) \left(I_2 \otimes \sigma_{PP}^{GHZ,N-1}(x)\right)^{1-q}, \quad \text{where} \quad \sigma_{PP}^{GHZ,N-1}(x) = \text{Tr}_1[\rho_{PP}^{GHZ,N}(x)] \quad \text{corresponds to the } N - 1 \text{ qubit subsystem of } \rho_{PP}^{GHZ,N}(x). \]

The nonzero eigenvalues \( \lambda_i \) are given below (for \(N \geq 3\)) in (14):

\[
\begin{align*}
\lambda_1 &= \frac{1-x}{2^N-1} \left[ 2 \frac{(1-x)}{2^N-1} \right]^{\frac{1-q}{q}}; \quad (2^N-4)-\text{fold degenerate} \\
\lambda_2 &= \frac{1-x}{2^N-1} \left[ \frac{3 + \left( \sum_{j=3}^{N} \frac{2^{j-1}}{2^j} \right) x}{\sum_{j=1}^{N} 2^j} \right]^{\frac{1-q}{q}}; \quad 3\text{-fold degenerate} \\
\lambda_3 &= x \left[ \frac{3 + \left( \sum_{j=3}^{N} \frac{2^{j-1}}{2^j} \right) x}{\sum_{j=1}^{N} 2^j} \right]^{\frac{1-q}{q}}.
\end{align*}
\]  

(14)

Substituting these eigenvalues \( \lambda_i \) in (5), we numerically evaluate the \(1 : N - 1\) separability range (beyond which the CSTRE is negative and hence imply entanglement) for specific cases \(N = 3, 4, 5, 6\). We obtain the result...
Fig. 1 (Color Online) Implicit plots of $\tilde{D}_q(\rho_{PP|GHZ}^N || I_2 \otimes \sigma_{GHZ,N-1}^{PP}) = 0$ (dashed line) and the Abe–Rajagopal $q$-conditional entropy $S_q^I(A|B) = 0$ (solid line) as a function of $q$ in the 1 : 5 partition of the state $\rho_{GHZ}^{PP}(x)$. This demonstrates the relatively slower convergence of the noisy parameter $x$ to the cutoff value 0.04545 in the case of the CSTRE approach, when compared with that of the AR method. (The quantities plotted are dimensionless)

$[0, 0.3]$, $[0, 0.1666]$, $[0, 0.0882]$, $[0, 0.0454]$ as the separability ranges for the noisy state $\rho_{GHZ}^{PP}(x)$ in its 1 : 2, 1 : 3, 1 : 4, 1 : 5 partitions with $N = 3, 4, 5, 6$, respectively. We verify that these results agree with the ones obtained based on both AR and PPT criteria. It may however be identified that though the CSTRE and AR criteria result in the same separability threshold for the noisy parameter $x$, they approach the cutoff value with different convergence rates, which is depicted in Fig. 1, for the specific case of $N = 6$.

In general for any $N \geq 3$, we obtain the following bound

$$0 \leq x \leq \frac{3}{2^N + 2}$$

in the limit $q \to \infty$, within which the PP state $\rho_{GHZ}^{PP}(x)$ is separable.

This result matches identically with the necessary and sufficient condition (11) for separability (obtained by substituting the Schmidt coefficients associated with the $(1 : N - 1)$ partition of the GHZ state, i.e., $u_1 = u_2 = 1/\sqrt{2}$). Thus, the CSTRE method is found to serve as a necessary and sufficient condition to detect entanglement in the 1 : $N - 1$ partition of the $N$-qubit PP state $\rho_{GHZ}^{PP}(x)$.

3 Werner-like one parameter noisy families of $N$-qubit W and GHZ states

We consider the $N$-qubit generalizations of Werner-like one parameter noisy family of states
\[ \rho_{\Phi_N}(x) = (1 - x) I_{2^N}^{\otimes N} + x |\Phi\rangle \langle \Phi|, \quad 0 \leq x \leq 1 \]  

and investigate the separability in the \((1 : N - 1)\) partition based on CSTRE approach.  

When the pure entangled state \(|\Phi\rangle\) corresponds to the \(N\)-qubit W state (See (7)), we get the noisy state

\[ \rho_{W_N}(x) = (1 - x) I_{2^N}^{\otimes N} + x |W_N\rangle \langle W_N|. \]  

In order to carry on the task of identifying the \(1 : N - 1\) separability range of the state \(\rho_{W_N}(x)\) via the CSTRE method, we evaluate the \(2^N\) eigenvalues \(\lambda_i\) of the ‘sandwiched’ matrix \((I_2 \otimes \sigma_{W_{N-1}})^{1-q} \rho_{W_N}(x) (I_2 \otimes \sigma_{W_{N-1}})^{1-q}\) with \(\sigma_{W_{N-1}}(x) = \text{Tr}[\rho_{W_N}(x)]\) and they are given by

\[ \lambda_1 = \left(1 - x \right) \left[ \frac{1 - x}{2^{N-1}} \right]^{1-q}; \quad (2^N - 4)\text{-fold degenerate} \]

\[ \lambda_2 = \left(1 - x \right) \left[ \frac{N + \left( \sum_{j=3}^{N} 2^{j-2} - (N - 2) \right) x}{N 2^{N-1}} \right]^{1-q}; \]

\[ \lambda_3 = \left(1 - x \right) \left[ \frac{N + \left( \sum_{j=3}^{N} 2^{j-2} + (N - 2) (2^{N-1} - 1) \right) x}{N 2^{N-1}} \right]^{1-q}; \]

\[ \lambda_{4/5} = \frac{1}{4} \left( 2^{N-1} N \right)^{\frac{1}{q}} \left[ \alpha a + \beta b \pm \sqrt{(\alpha a - \beta b)^2 + 2^{2N+2}(N - 1)x^2 \alpha \beta} \right] \]

where

\[ \alpha = \left[ N + \left( \sum_{j=3}^{N} 2^{j-2} - (N - 2) \right) x \right]^{1-q}; \]

\[ \beta = \left[ N + \left( \sum_{j=3}^{N} 2^{j-2} + (N - 2) (2^{N-1} - 1) \right) x \right]^{1-q}. \]

\[ ^1 \text{Note that both PP and Werner-like noisy one parameter family of states considered here are not permutation symmetric states, as they do not get restricted only to the } N + 1 \text{ dimensional symmetric subspace of the } 2^N \text{ Hilbert space of } N\text{-qubits. This is because the completely random state } I_{2^N}^{\otimes N}/2^N \text{ (which is mixed with the pure symmetric } N\text{-qubit states in } \rho_{PP_{W_N}}^{\text{GHZ}_N}(x), \rho_{PP}_{\text{GHZ}_N}(x), \rho_{W_N}(x), \rho_{\text{GHZ}_N}(x)) \text{ is not confined to the } N + 1 \text{ dimensional subspace of permutation symmetric } N\text{-qubit states. In Refs. [19,20] noisy one parameter families of symmetric } N\text{-qubit states were investigated using AR- and CSTRE criteria.} \]
Table 2  1 : \( N - 1 \) separability threshold value of the noisy parameter \( x \) in the states \( \rho_{W_N}(x) \) for \( N = 3, 4, 5, 6 \), obtained via the positivity of the CSTRE, the von Neumann and the AR conditional entropies, along with the one obtained from the PPT criteria

| Number of qubits | von Neumann conditional entropy | AR q-conditional entropy | CSTRE | PPT |
|------------------|---------------------------------|--------------------------|-------|-----|
| 3                | 0.7018                          | 0.2727                   | 0.2095| 0.2095|
| 4                | 0.6760                          | 0.2                      | 0.1261| 0.1261|
| 5                | 0.6618                          | 0.1351                   | 0.0724| 0.0724|
| 6                | 0.6567                          | 0.0857                   | 0.0402| 0.0402|

\[
a = N + \left( \sum_{j=3}^{N} 2^{j-2} - (N - 2) + 2^{N-1} \right) x,
\]

\[
b = N + \left( \sum_{j=3}^{N} 2^{j-2} + 2^{N-1}(2N - 3) - (N - 2) \right) x.
\]

Substituting these eigenvalues in (5), we numerically estimate the separability ranges in the 1 : 2, 1 : 3, 1 : 4, 1 : 5 bipartitions of the noisy states \( \rho_{W_3}(x), \rho_{W_4}(x), \rho_{W_5}(x), \rho_{W_6}(x) \), respectively. We have tabulated (see Table 2) the separability threshold value of the parameter \( x \) obtained using CSTRE approach, along with the corresponding results from PPT criteria and also those inferred via the positivity of the corresponding von Neumann and the AR conditional entropies. It is readily seen that the result based on the positivity of the CSTRE is stronger than the one obtained from the positivity of the von Neumann, AR conditional entropies. Further, it is observed that the CSTRE result agrees with that identified from the PPT criterion.

In general, the CSTRE approach is found to lead to the separability range

\[
0 \leq x \leq \frac{N}{N + 2^N \sqrt{N - 1}}
\]

for the 1 : \( N - 1 \) partitions of the state \( \rho_{W_N}(x) \) for \( N \geq 3 \). We recall that the noisy \( N \)-qubit state \( \rho_{\Phi_N}(x) \) of (16) is known to be separable iff [32]

\[
0 \leq x \leq \frac{1}{2^N u_1 u_2 + 1}
\]

where \( u_1 \) and \( u_2 \) are the two largest Schmidt coefficients of the pure entangled state \( |\Phi_N\rangle \) under bipartition. In the specific case of (1 : \( N - 1 \)) partition of the state \( \rho_{W_N}(x) \), on substituting the corresponding Schmidt coefficients (see (12)) \( u_1 = \sqrt{\frac{N-1}{N}}, \ u_2 = \frac{1}{\sqrt{N}} \) in (21), one can recognize that the separability range reveals a clear agreement with (20) obtained via the CSTRE approach. This establishes that the CSTRE method serves as necessary and sufficient for inferring separability in this example too.
We continue to investigate the separability in the \((1 : N - 1)\) partition of the noisy Werner-like \(N\)-qubit GHZ state

\[
\rho_{\text{GHZ}_N}(x) = (1 - x) \frac{I_2 \otimes \sigma_{\text{GHZ}_{N-1}}}{2^N} + x \left| \text{GHZ}_N \right\rangle \left\langle \text{GHZ}_N \right| \tag{22}
\]

using CSTRE criteria. Here, the eigenvalues of the sandwiched matrix \((I_2 \otimes \sigma_{\text{GHZ}_{N-1}})^{1-q} \rho_{\text{GHZ}_N}(x) (I_2 \otimes \sigma_{\text{GHZ}_{N-1}})^{1-q} \), with \(\sigma_{\text{GHZ}_{N-1}}(x) = \text{Tr}_1[\rho_{\text{GHZ}_N}(x)]\), for any \(N \geq 3\) are found to be

\[
\lambda_1 = \left[ \frac{1 - x}{2^N} \right] \left[ \frac{1 - x}{2^{N-1}} \right]^{\frac{1-q}{q}}; \ (2^N - 4)\text{-fold degenerate;}
\]

\[
\lambda_2 = \left[ \frac{1 - x}{2^N} \right] \left[ \frac{1 + (2^N - 2 - 1)x}{2^{N-1}} \right]^{\frac{1-q}{q}}; \ 3\text{-fold degenerate}
\]

\[
\lambda_3 = \left[ \frac{1 + (2^N - 1)x}{2^N} \right] \left[ \frac{1 + (2^N - 2 - 1)x}{2^{N-1}} \right]^{\frac{1-q}{q}}. \tag{23}
\]

Substituting (23) in (5) we find that positivity of CSTRE as \(q \to \infty\) requires the following bounds

\[
0 \leq x \leq \frac{1}{2^{N-1} + 1}. \tag{24}
\]
on the noisy parameter \( x \). This result agrees with the \( 1 : N - 1 \) separability range obtained based on the commutative AR method too in the case of \( \rho_{\text{GHZ}_N}(x) \). However, the convergence toward the threshold value of the parameter \( x \to \frac{1}{2^{N-1}+1} \) in the limit \( q \to \infty \) based on the CSTRE method is slower compared to that of the AR approach. This is illustrated in Fig. 2 in the specific case of \( N = 6 \). Moreover, substituting the Schmidt coefficients \( u_1 = u_2 = \frac{1}{\sqrt{2}} \) associated with the \( (1 : N - 1) \) partition of the GHZ state in (21) reveals that the range given in Eq. (24) for the parameter \( x \) obtained from CSTRE approach is both necessary and sufficient for the separability in the \( (1 : N - 1) \) bipartition of the state \( \rho_{\text{GHZ}_N}(x) \).

4 Conclusion

We have evaluated the \( 1 : N - 1 \) separability range in the noisy \( N \)-qubit states of the PP, W and GHZ family using the CSTRE approach. Our results show that the positivity of the CSTRE in the limit \( q \to \infty \) is both necessary and sufficient criterion for the separability of the \( (1 : N - 1) \) partition of the one parameter family of noisy PP, W and GHZ states.

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References

1. Horodecki, R., Horodecki, P.: Quantum redundancies and local realism. Phys. Lett. A 194, 147 (1994)
2. Cerf, N.J., Adami, C.: Negative entropy and information in quantum mechanics. Phys. Rev. Lett. 79, 5194 (1997)
3. Abe, S., Rajagopal, A.K.: Quantum entanglement inferred by the principle of maximum nonadditive entropy. Phys. Rev. A 60, 3461 (1999)
4. Giovannetti, V.: Separability conditions from entropic uncertainty relations. Phys. Rev. A 70, 012102 (2004)
5. Guhne, O., Lewenstein, M.: Entropic uncertainty relations and entanglement. Phys. Rev. A 70, 022316 (2004)
6. Horodecki, R., Horodecki, P., Horodecki, M.: Quantum \( \alpha \)-entropy inequalities: Independent condition for local realism? Phys. Lett. A 210, 377 (1996)
7. Horodecki, R., Horodecki, M.: Information-theoretic aspects of inseparability of mixed states. Phys. Rev. A 54, 1838 (1996)
8. Tsallis, C.: Possible generalization of Boltzmann–Gibbs statistics. J. Stat. Phys. 52, 479 (1988)
9. Tsallis, C., Mendes, R.S., Plastino, A.R.: The role of constraints within generalized nonextensive statistics. Physica A 261, 534 (1998)
10. Abe, S., Rajagopal, A.K.: Nonadditive conditional entropy and its significance for local realism. Physica A 289, 157 (2001)
11. Tsallis, C., Lloyd, S., Baranger, M.: Peres criterion for separability through nonextensive entropy. Phys. Rev. A 63, 042104 (2001)
12. Abe, S.: Nonadditive information measure and quantum entanglement in a class of mixed states of an \( N^n \) system. Phys. Rev. A 65, 052323 (2002)
13. Rossignoli, R., Canosa, N.: Generalized entropic criterion for separability. Phys. Rev. A 66, 042306 (2002)
14. Rossignoli, R., Canosa, N.: Violation of majorization relations in entangled states and its detection by means of generalized entropic forms. Phys. Rev. A 67, 042302 (2003)
15. Batle, J., Casas, M., Plastino, A.R., Plastino, A.: Conditional q-entropies and quantum separability: a numerical exploration. J. Phys. A 35, 10311 (2002)
16. Batle, J., Plastino, A.R., Casas, M., Plastino, A.: Some features of the conditional q-entropies of composite quantum systems. Eur. Phys. J. B 35, 391 (2003)
17. Prabhu, R., Usha Devi, A.R., Padmanabha, G.: Separability of a family of one-parameter W and Greenberger-Horne-Zeilinger multiqubit states using the Abe-Rajagopal q-conditional-entropy approach. Phys. Rev. A 76, 042337 (2007)
18. Sudha, Usha Devi, A.R., Rajagopal, A.K.: Entropic characterization of separability in Gaussian states. Phys. Rev. A 81, 024303 (2010)
19. Rajagopal, A.K., Sudha, Nayak, A.S., Usha Devi, A.R.: From the quantum relative Tsallis entropy to its conditional form: separability criterion beyond local and global spectra. Phys. Rev. A 89, 012331 (2014)
20. Nayak, A.S., Sudha, Rajagopal, A.K., Usha Devi, A.R.: Bipartite separability of symmetric N-qubit noisy states using conditional quantum relative Tsallis entropy. Physica A 443, 286–295 (2016)
21. Nielsen, M.A., Kempe, J.: Separable states are more disordered globally than locally. Phys. Rev. Lett. 86, 5184 (2001)
22. Werner, R.F.: Quantum states with Einstein–Podolsky–Rosen correlations admitting a hidden-variable model. Phys. Rev. A 40, 4277 (1989)
23. Popescu, S.: Bell’s inequalities versus teleportation: What is nonlocality? Phys. Rev. Lett. 72, 797 (1994)
24. Peres, A.: Separability criterion for density matrices. Phys. Rev. Lett. 77, 1413 (1996)
25. Horodecki, M., Horodecki, P., Horodecki, R.: Separability of mixed states: necessary and sufficient conditions. Phys. Lett. A 223, 1 (1996)
26. Wilde, M.M., Winter, A., Yang, D.: Strong converse for the classical capacity of entanglement-breaking and Hadamard channels via a sandwiched Rényi relative entropy. Commun. Math. Phys. 331, 593 (2014)
27. Müller-Lennert, M., Dupuis, F., Szehr, O., Tomamichel, M.: On quantum Rényi entropies: a new generalization and some properties. J. Math. Phys. 54, 122203 (2013)
28. Tomamichel, M., Berta, M., Hayashi, M.: Relating different quantum generalizations of the conditional Rényi entropy. J. Math. Phys. 55, 082206 (2014)
29. Chitamber, E.: Quantum correlations in high-dimensional states of high symmetry. Phys. Rev. A 86, 032110 (2012)
30. Cory, D., Fahmy, A., Havel, T.: Ensemble quantum computing by NMR spectroscopy. Proc. Natl. Acad. Sci. USA 94, 1634 (1997)
31. Gershenfeld, N.A., Chuang, I.L.: Bulk spin-resonance quantum computation. Science 275, 350 (1997)
32. Vidal, G., Tarrach, R.: Robustness of entanglement. Phys. Rev. A 59, 141 (1999)