Heat Kernel Asymptotic Expansion on Unbounded Domains with Polynomially Confining Potentials

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In this paper we analyze the small-$t$ asymptotic expansion of the trace of the heat kernel associated with a Laplace operator endowed with a spherically symmetric polynomially confining potential on the unbounded, $d$-dimensional Euclidean space. To conduct this study, the trace of the heat kernel is expressed in terms of its partially resummed form which is then represented as a Mellin-Barnes integral. A suitable contour deformation then provides, through the use of Cauchy’s residue theorem, closed formulas for the coefficients of the asymptotic expansion. The general expression for the asymptotic expansion, valid for any dimension and any polynomially confining potential, is then specialized to two particular cases: the general quartic and sextic oscillator potentials.

I. INTRODUCTION

The study of the spectrum of elliptic, second-order differential operators acting on suitable functions defined on a Riemannian manifold has attracted much attention throughout the years. The reason for such widespread interest lies in the fact that important information can be extracted from the spectrum of these operators. For instance, the analysis of the spectrum of a Laplace-type operator defined on a manifold provides useful information about its geometry [12, 13, 17]. In addition, the one-loop effective action for quantum fields can be constructed from the spectrum of the operator that describes the dynamics of the field fluctuations [8]. Although in the general case the spectrum of a Laplace-type operator is not explicitly known a detailed study of the entire spectral sequence can be performed by using spectral functions [20]. One of the most widely used spectral functions, which is also the main subject of this work, is the heat kernel.

In order to define the heat kernel, we introduce a smooth Riemannian manifold $\mathcal{M}$, and an Hermitian vector bundle over $\mathcal{M}$ represented by $\mathcal{V}$. We denote by $\mathcal{P}$ a Laplace-type operator acting on the space of smooth sections of $\mathcal{V}$, namely $C^\infty(\mathcal{V})$. For $t > 0$ the one-parameter family of operators

$$U(t) = \exp(-t\mathcal{P})$$

forms a semigroup of bounded operators on $L^2(\mathcal{V})$, the space of square integrable sections of $\mathcal{V}$, called the
heat semigroup. If we denote by $\varphi_n \in L^2(V)$, with $n \in \mathbb{N}^+$, the eigenfunctions of $P$ and by $\lambda_n$ the corresponding eigenvalues, counted with their algebraic multiplicity, we can define the heat kernel associated with the heat semigroup (1.1) as

$$U(t|x, x') = \sum_{n=1}^{\infty} e^{-t\lambda_n} \varphi_n(x) \otimes \varphi_n^*(x'),$$

which satisfies the following parabolic partial differential equation

$$(\partial_t + P) U(t|x, x') = 0,$$

with the initial condition

$$U(0|x, x') = \delta(x, x'),$$

where $\delta(x, x')$ denotes the covariant delta function, and appropriate boundary conditions if $\partial \mathcal{M} \neq \emptyset$. It can be proved [12] that the heat semigroup is of trace-class and, hence, its trace can be written as

$$\text{Tr}_{L^2} e^{-tP} = \int_{\mathcal{M}} \text{Tr}_V U(t|x, x) dvol,$$

where $\text{Tr}_V$ represents the vector bundle trace, and $dvol$ is the volume element of the Riemannian manifold. The integral in (1.5) is then referred to as the trace of the heat kernel.

Unfortunately, it is not possible to obtain an explicit expression for the heat kernel and its trace for a general case. Its exact form can only be found when the spectrum of the operator is explicitly known, or when the background geometry is highly symmetric. It can be proved [13, 29, 30], however, that for Laplace-type operators on smooth compact manifolds, with or without boundary, there exists a small-$t$ expansion of the associated heat kernel. The universal coefficients of this expansion are constructed from geometric invariants of the underlying manifold [12]. The first studies involving the coefficients of the asymptotic expansion were performed for the Laplacian on a smooth compact Riemannian manifold in [21, 22]. Since the appearance of these seminal works, the analysis of the asymptotic expansion of the heat kernel has attracted widespread interest and, as a consequence, its coefficients have been systematically evaluated for several specific cases. Very efficient methods have been developed over the years with the purpose of computing these coefficients; we will not indulge, here, in the exposition of these techniques, however the interested reader can find an exhaustive report in [32] which also contains a thorough list of references.

The vast majority of work on the asymptotic expansion of the heat kernel has been performed for elliptic, differential operators defined on compact domains with or without boundary [13, 20, 32]. Complete results are also available for the coefficients the heat kernel expansion of a Laplace-type operator endowed with an
integrable potential defined on an unbounded domain. In this case the computation is performed by utilizing scattering theory and the appropriate Jost function or, equivalently, the phase shift [20, 23].

On the other hand, the heat kernel expansion on unbounded domains with confining potentials is somewhat more involved to analyze and, therefore, remained a much less investigated case. In fact, when considering non-standard situations, such as unbounded domains with confining potentials, the heat kernel asymptotic expansion does not display the usual structure [7]. In addition, while in the case of bounded domains the coefficients of the expansion of the trace of the heat kernel can be obtained by simply integrating the local coefficients, this operation becomes meaningless in unbounded domains with non-integrable potentials. It is worth mentioning, however, that the case of a one-dimensional Laplace operator endowed with an harmonic oscillator potential is well understood. In this situation the eigenvalues are explicitly known and the trace of the associated heat kernel can be easily computed. The small-\(t\) asymptotic expansion can then be straightforwardly evaluated by performing a Taylor expansion in the neighborhood of \(t=0\) (see e.g. Section IV).

It is clear, from the previous remarks, that standard methods are not suitable for the evaluation of the coefficients of the heat kernel asymptotic expansion for Laplace-type operators on unbounded domains when confining potentials are present. In fact, for the above mentioned class of problems results about the expansion of the trace of the heat kernel are limited to the detailed analysis of its leading term and first few subleading terms [7]. This does not seem to be surprising considering that the difference between the bounded case and the unbounded one with confining potential can already be recognized at the level of the leading term of the asymptotic expansion. Moreover, the first few terms of the asymptotic expansion are shown to be sufficient, for instance, for the study of Bose-Einstein condensation under the influence of very general external conditions [18, 19] and on manifolds with non-trivial topology [10]. Finally, another reason for the availability of only limited results regarding the heat kernel in the unbounded case is represented by the fact that in order to understand the large-\(n\) behavior of the eigenvalues \(\lambda_n\) of a suitable operator it is only necessary to find the leading term of the asymptotic expansion of the trace of its associated heat kernel [24, 33].

Although some useful information can be extracted from just the knowledge of the leading term of the asymptotic expansion of the trace of the heat kernel it would be of particular interest, at least from a mathematical point of view, to have a method which would produce the complete asymptotic expansion. It is, hence, the main purpose of this work to provide a first step towards the development of a systematic theory, which is still lacking [7], for the evaluation of the coefficients of the heat kernel asymptotic expansion for Laplace-type operators on unbounded domains with confining potentials. Here we will be mainly focused on the Laplacian on \(\mathbb{R}^d\) with the presence of a spherically symmetric polynomially confining potential. This
framework only describes the wide class of isotropic problems but the method presented in this work should provide a platform from which additional research on non-homogeneous cases can be developed.

The outline of the paper is as follows. In the next Section we begin our analysis with the resummed form of the asymptotic expansion of the trace of the heat kernel $K(t)$. By utilizing a Mellin-Barnes integral we then find a contour integral representation for $K(t)$. In the subsequent Section a suitable contour deformation is performed in order to find explicit expressions for the coefficients of the asymptotic expansion. The general formulas are then specialized to the cases of the Laplacian in three dimensions with quartic and sextic oscillator potentials. The Conclusions point out the main results and provide a few directions for further research.

II. TRACE OF THE HEAT KERNEL IN UNBOUNDED DOMAINS

We focus, here, on the analysis of the asymptotic expansion of the trace of the heat kernel $K(t)$ for the operator $L = -\Delta + V(x)$ acting on suitable functions defined on the $d$-dimensional Euclidean space $\mathbb{R}^d$. The operator $\Delta$ denotes the Laplacian, and the function $V(x)$ represents a polynomially confining potential. Moreover, we will assume that $V(x) \in L_\text{loc}^\infty(\mathbb{R}^d)$ is a measurable locally bounded real-valued and positive function in $\mathbb{R}^d$. Under these assumptions it can be proved [5] that the operator $L = -\Delta + V(x)$ on $\mathbb{R}^d$ has a discrete spectrum, bounded from below in which each element has finite multiplicity. By denoting by $\lambda_n$ the eigenvalues and by $\varphi_n(x)$ the corresponding eigenfunctions, the local heat kernel can be written as (1.2).

In this case there exists an asymptotic expansion of the local heat kernel of the form [29, 30]

$$K(t|x, x) \sim \sum_{k=0}^{\infty} t^{k-d/2} a_k(x),$$

where the coefficients $a_k(x)$ are expressed in terms of positive integer powers of the potential and its derivatives. The asymptotic expansion for the trace of the heat kernel is, then, obtained by integrating (2.1) over the entire domain. In the case of unbounded domains and confining potentials this operation cannot be performed since the resulting integrals are clearly divergent. Consequently, instead of the expansion (2.1) we consider its partially resummed form [16, 26]

$$K(t|x, x) = \frac{1}{(4\pi t)^{d/2}} e^{-tV(x)} \sum_{k=0}^{\infty} t^k A_k(x),$$

where now the coefficients $A_k(x)$ do not contain powers of the potential $V(x)$.

The local coefficients in the asymptotic expansion (2.2) can be computed by utilizing a variety of methods. One of the most straightforward ones is based on the covariant Fourier transform [2–4, 25]. This method is quite general and allows for the evaluation of the asymptotic expansion of the local heat kernel.
for the operator $-\Delta + V(x)$ defined on a smooth Riemannian manifold $\mathcal{M}$. In this framework the local heat kernel can be written as

$$K(t|x, x') = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ik(x-x')} \left( \exp \left\{ -i t e^{-ik\cdot x} \left[ -\Delta + V(x) \right] e^{ik\cdot x} \right\} \cdot 1 \right) d^d k . \quad (2.3)$$

The evaluation of the action of the operator $-\Delta + V(x)$ on the exponential leads to the result [3]

$$K(t|x, x') = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ik(x-x')} \left( \exp \left\{ -t \left[ |k|^2 - 2i k \cdot \nabla_j - \Delta + V(x) \right] \right\} \cdot 1 \right) d^d k , \quad (2.4)$$

where we have denoted by $\nabla_k$ the covariant derivative. By changing the integration variable $k \to k/\sqrt{t}$ and by taking the coincidence limit $x \to x'$ one finds

$$K(t|x, x) = \frac{1}{(4\pi t)^d} \int_{\mathbb{R}^d} e^{-|k|^2} \left( \exp \left\{ 2i \sqrt{t} k \cdot \nabla_j + t\Delta - tV(x) \right\} \cdot 1 \right) d^d k . \quad (2.5)$$

The small-$t$ asymptotic expansion (2.2) can, then, be obtained from (2.5) by using the Volterra series [3]

$$\exp \left\{ 2i \sqrt{t} k \cdot \nabla_j + t\Delta - tV(x) \right\} = e^{-tV(x)} \left\{ 1 + \sum_{n=1}^{\infty} \int_0^1 d\tau_n \int_0^{\tau_n} d\tau_{n-1} \cdots \int_0^{\tau_2} d\tau_1 e^{t\nabla_j V(x)} (2i \sqrt{t} k \cdot \nabla_j + t\Delta) e^{-t\tau_1 V(x)} \right\} , \quad (2.6)$$

and the following Gaussian integrals, with $n \in \mathbb{N}$, [3]

$$\frac{1}{\pi^{d/2}} \int_{\mathbb{R}^d} e^{-|k|^2} k_{j_1} \cdots k_{j_{2n}} d^d k = 0 , \quad \frac{1}{\pi^{d/2}} \int_{\mathbb{R}^d} e^{-|k|^2} k_{j_1} \cdots k_{j_{2n}} d^d k = \frac{(2n)!}{2^n n!} g_{j_1 j_2} \cdots g_{j_{2n-1} j_{2n}} , \quad (2.7)$$

where the parentheses () denote complete symmetrization and $g_{ij}$ represents the Riemannian metric on $\mathcal{M}$. The procedure described above gives explicit expressions for the coefficients $A_k(x)$, in more details one obtains, for the first few of them, [7, 26]

$$A_0(x) = 1 , \quad A_1(x) = 0 ,$$

$$A_2(x) = -\frac{1}{6} \Delta V(x) , \quad A_3(x) = -\frac{1}{60} \Delta^2 V(x) + \frac{1}{12} \nabla_j V(x) \nabla^i V(x) ,$$

$$A_4(x) = -\frac{1}{840} \Delta^3 V(x) + \frac{1}{72} (\Delta V(x))^2 + \frac{1}{90} \nabla_i \nabla_j V(x) \nabla^i \nabla^j V(x) + \frac{1}{30} \nabla_i V(x) \nabla^i \Delta V(x) . \quad (2.8)$$

Higher order coefficients can be computed with the help of an algebraic computer program. Clearly, the coefficients found in (2.8), valid for any smooth manifold, are easily specialized to $\mathbb{R}^d$ which is the case we consider here.

By integrating the expression (2.2) over $\mathbb{R}^d$ we obtain the asymptotic expansion of the $L^2$-trace of the heat kernel as follows

$$K(t) = \frac{1}{(4\pi t)^d} \sum_{k=0}^{\infty} \frac{t^k}{k!} \int_{\mathbb{R}^d} A_k(x) e^{-tV(x)} d^d x , \quad (2.9)$$
where, due to the fact that \( V(x) > 0 \) for \( x \in \mathbb{R}^d \), the presence of the negative exponent guarantees the convergence of the integral when \( t > 0 \). To obtain the complete small-\( t \) asymptotic expansion of \( K(t) \) we need to analyze the small-\( t \) asymptotic expansion of the integral appearing in (2.9). In order to simplify this analysis we assume that \( V(x) \) belongs to the class of spherically symmetric potentials. A general polynomially confining potential which is also spherically symmetric can be written, in hyperspherical coordinates \((r, \theta_1, \cdots, \theta_{d-1})\), as

\[
V(r) = \sum_{j=0}^{N} c_j r^{2j}, \tag{2.10}
\]

where \( c_j \in \mathbb{R} \) with the assumption that \( c_0 \geq 0 \) and that \( c_N \neq 0 \). It is worth noting that since the potential depends only on the radial coordinate \( r \), so will the coefficients (2.8). With the last remark in mind the asymptotic expansion (2.9) in hyperspherical coordinates reads

\[
K(t) = \frac{S_d}{(4\pi t)^{\frac{d}{2}}} \sum_{k=0}^{\infty} \int_{0}^{\infty} r^{d-1} A_k(r) e^{-tV(r)} dr, \tag{2.11}
\]

where \( S_d = 2\pi^{d/2}/\Gamma(d/2) \) is the result of the integration over the angular coordinates.

In order to find the small-\( t \) asymptotic expansion of \( K(t) \), we need to explicitly compute the integral in (2.11). This can be accomplished by studying the functional dependence of the coefficients \( A_k \) from the radial coordinate \( r \). The general form of \( A_k(r) \) can be identified by utilizing dimensional arguments [20]. In this framework the coefficients \( A_k(r) \) have dimension \( l^{2k} \) where \( l \) represents a unit of length [12, 13, 20]. By denoting by \( p_V \) the powers of \( V(r) \) and by \( p_\nabla \) the powers of the derivative \( \nabla_k \) we have for each coefficient \( A_k(r), k \geq 0 \), the relation [20]

\[
2p_V + p_\nabla = 2k, \tag{2.12}
\]

which holds true since the potential \( V(r) \) and the derivative \( \nabla_k \) have dimension \( l^2 \) and \( l \), respectively.

At this point a few remarks are in order. The coefficients \( A_k(r) \) in (2.12) are polynomials in the derivatives of \( V(r) \). This implies, given the form of \( V(r) \) in (2.10), that \( A_k(r) \) are, in turn, polynomials in the radial coordinate \( r \). Since, as already mentioned earlier, \( A_k(r) \) contains no powers of \( V(r) \) we have to require that \( p_\nabla > 0 \). In addition, since derivatives of higher powers of the potential, namely \( V^n(r) \), can be written as a linear combination of derivatives of \( V(r) \), the independent invariants in \( A_k(r) \) that can be constructed with the potential \( V(r) \) have to satisfy the constraint \( p_V \leq p_\nabla \). From the relation (2.12) it is not difficult to see that the coefficients \( A_k(r) \) contain an even number of derivatives and, therefore, have the general form

\[
A_k(r) = \sum_{l=0}^{\gamma_k} \Omega_k^l r^{2l}, \tag{2.13}
\]
where \( \Omega^k \) are real coefficients computable from (2.8) and \( \gamma_k \) is an integer which can be found by using (2.12). The coefficient \( \gamma_k \) provides the highest power of \( r \) appearing in \( A_k(r) \) which, for a polynomial potential, is obtained by maximizing \( p_V \in \mathbb{N}^+ \) within the constraints \( p_V \leq p_N \) and (2.12). For \( k = 0 \) and \( k = 1 \) we use the expression for \( A_0 \) and \( A_1 \) in (2.8) to conclude that \( \gamma_0 = \gamma_1 = 0 \). For \( k \geq 2 \), the maximum value of \( p_V \) is attained at \( \bar{p}_V = [2k/3] \) and the corresponding value of \( p_N \in \mathbb{N}^+ \), satisfying the above mentioned constraints, is \( \bar{p}_N = 2(k - [2k/3]) \), where, here and in the rest of this work, \([x]\) denotes the integer value of \( x \). From the above remarks we can conclude that for a polynomial potential of the form (2.10) the highest power of \( r \) entering \( A_k(r) \) is \( 2N\bar{p}_V - \bar{p}_N \) which implies that

\[
\gamma_k = \left\lfloor \frac{2k}{3} \right\rfloor (N + 1) - k , \tag{2.14}
\]

for \( k \geq 2 \).

At this point we substitute the general form of \( A_k(r) \), found in (2.13), in the expression for the asymptotic expansion (2.11) to obtain

\[
K(t) = \frac{S_d}{(4\pi t)^{d/2}} \sum_{k=0}^{\infty} t^k \sum_{l=0}^{\gamma_k} \Omega^k_l \int_0^\infty r^{d+2l-1} e^{-tV(r)} dr . \tag{2.15}
\]

Since \( t > 0 \) and \( V(r) > 0 \) we use the Mellin-Barnes representation of the exponential function

\[
e^{-tV(r)} = \frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} \Gamma(s) r^{-s} (V(r))^{-s} ds , \tag{2.16}
\]

with \( \delta \in \mathbb{R}^+ \), to obtain

\[
K(t) = \frac{S_d}{(4\pi t)^{d/2}} \sum_{k=0}^{\infty} t^k \sum_{l=0}^{\gamma_k} \Omega^k_l \int_0^\infty r^{d+2l-1} \left( \frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} \Gamma(s) r^{-s} (V(r))^{-s} ds \right) dr . \tag{2.17}
\]

Now, for \( \Re(s) > s_0 \) with \( s_0 \in \mathbb{R}^+ \) large enough we can change the order of integration in (2.17) to get

\[
K(t) = \frac{S_d}{(4\pi t)^{d/2}} \sum_{k=0}^{\infty} t^k \sum_{l=0}^{\gamma_k} \frac{\Omega^k_l}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} \Gamma(s) r^{-s} \left( \int_0^\infty r^{d+2l-1} (V(r))^{-s} dr \right) ds , \tag{2.18}
\]

where \( \delta > s_0 \) and is chosen so that the integration contour lies to the right of all the poles of the integrand.

The expression (2.18) represents a very suitable starting point for the evaluation of the full asymptotic expansion of the trace of the heat kernel. In fact, once the integral over the radial coordinate is computed, the small-\( t \) expansion of \( K(t) \) in (2.18) is obtained by closing the integration contour to the left and by utilizing Cauchy’s residue theorem.

**III. ASYMPTOTIC EXPANSION OF THE TRACE OF THE HEAT KERNEL**

The integral over the radial coordinate \( r \) in (2.18) can be treated by rewriting it as a sum of two terms

\[
\int_0^\infty r^{d+2l-1} (V(r))^{-s} dr = \int_0^1 r^{d+2l-1} (V(r))^{-s} dr + \int_1^\infty r^{d+2l-1} (V(r))^{-s} dr , \tag{3.1}
\]
where the two integrals can be easily computed by utilizing the small-$r$ expansion of $(V(r))^{-s}$ in the first and the large-$r$ expansion of $(V(r))^{-s}$ in the second. We will focus, now, on the first integral for which the small-$r$ expansion of $(V(r))^{-s}$ is needed. By using the binomial series \[14\]

$$\left(a + b\right)^{-s} = \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \left(-1\right)^n \frac{\Gamma(s + n)}{n!} a^{-s-n} b^n,$$  \hspace{1cm} (3.2)

with $b < a$, and the expression \[2.10\] for $V(r)$ we obtain the following expansion valid for $r < 1$

$$\left(V(r)\right)^{-s} = \frac{c_0^{-s}}{\Gamma(s)} \sum_{n=0}^{\infty} \left(-1\right)^n \frac{\Gamma(s + n)}{n!} \left(\sum_{j=1}^{N} \frac{c_j}{c_0} r^2j\right)^n,$$  \hspace{1cm} (3.3)

where we have assumed that $c_0 \neq 0$. In order to further expand the $n$-th power of the polynomial that appears in (3.3) in terms of powers of $r$ we rewrite it in the form

$$\left(\sum_{j=1}^{N} \frac{c_j}{c_0} r^2j\right)^n = r^{2n} \left(\sum_{j=0}^{N-1} \frac{c_{j+1}}{c_0} r^2j\right)^n,$$  \hspace{1cm} (3.4)

At this point we use the following relation

$$\left(\sum_{j=0}^{M} c_j r^2j\right)^n = \sum_{l=0}^{M} D^n_l r^{2l},$$  \hspace{1cm} (3.5)

obtained from the multinomial expansion \[1.27\], with the coefficients $D^n_l$ given by

$$D^n_l = \sum_{n_0=l/M}^{l} \sum_{l-n_0, \cdots, l-n_M=0}^{n_0, \cdots, n_M} \frac{\left(n_0\right)\cdots\left(l-n_0-\cdots-n_M\right)\Gamma\left(l-n_0-\cdots-n_M\right)}{\Gamma\left(n_0\right)\cdots\Gamma\left(l-n_0-\cdots-n_M\right)} c_0^{-n_0} c_1^{n_1} \cdots c_M^{n_M},$$  \hspace{1cm} (3.6)

to get the expression

$$\left(\sum_{j=1}^{N} \frac{c_j}{c_0} r^2j\right)^n = r^{2n} \sum_{j=0}^{(N-1)n} \omega_j^n r^{2j},$$  \hspace{1cm} (3.7)

where

$$\omega_j^n = c_0^{-n} \sum_{n_0=\left\lceil j/(N-1)\right\rceil}^{j} \sum_{n_{N-3}=0}^{n_1} \left(n_0\right)\cdots\left(j-n_0-\cdots-n_{N-3}\right)\Gamma\left(j-n_0-\cdots-n_{N-3}\right) c_0^{-n_0} c_1^{-n_1} \cdots c_{N-1}^{-n_{N-1}} c_N^{-n_N}.$$  \hspace{1cm} (3.8)

Note that according to (3.8), when $N = 1$, the only well defined term is $\omega_0^n = c_0^{-n} c_1^n$. This what we expect from (3.7) in the trivial case $N = 1$.

By using (3.3) and (3.7) in the integral over $[0, 1]$ in (3.1) we obtain

$$\int_0^1 r^{d+2l-1} (V(r))^{-s} \, dr = \frac{c_0^{-s}}{\Gamma(s)} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(s + n)}{n!} \sum_{j=0}^{(N-1)n} \frac{\omega_j^n}{d + 2l + 2j + 2n},$$  \hspace{1cm} (3.9)
which represents an analytic function for $s \in \mathbb{C}$.

We consider, now, the large-$r$ expansion of the function $(V(r))^{-s}$. By using the expression (2.10) and by factoring the highest power of $r$ we obtain

$$
\left( \sum_{j=0}^{N} c_j r^{2j} \right)^{-s} = c_N^{-s} r^{-2N} \left( 1 + \sum_{j=0}^{N-1} c_j r^{2j-2N} \right)^{-s} .
$$

(3.10)

Since we are considering the case $r > 1$, for $j = 0, \ldots, N - 1$ the quantity $r^{2j-2N} < 1$ and, hence, we use once again the binomial expansion (3.2) to get

$$
\left( \sum_{j=0}^{N} c_j r^{2j} \right)^{-s} = c_N^{-s} \frac{\Gamma(s)}{(n)} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(n+s)}{n!} r^{-2Ns} \left( \sum_{j=0}^{N-1} \frac{c_j r^{2j-2N}}{c_N} \right)^{-n} - 1 .
$$

(3.11)

By using the relation (3.5) we find the expansion

$$
\left( \sum_{j=0}^{N-1} \frac{c_j r^{2j-2N}}{c_N} \right)^{-n} = r^{-2Nn} \sum_{j=0}^{(N-1)n} \tilde{\omega}_j^n r^j ,
$$

(3.12)

with

$$
\tilde{\omega}_j^n = c_N^{-n} \sum_{n_0=1/(j(N-1))}^{j-n_0} \cdots \sum_{n_{N-3}=1}^{j-n_0-\ldots-n_{N-4}} \frac{(n_0) n_1 \cdots (n_{N-3})}{(j-n_0-\ldots-n_{N-3})} c_0^{-n-n_0} c_1^{-n_0-n_1} \cdots c_{N-2}^{-2Nn_3+n_0+\ldots+n_{N-4}-j-n_0-\ldots-n_{N-3}} c_N^{-N-1} .
$$

(3.13)

By employing the results (3.11) and (3.12) in the integral over $[1, \infty)$ in (3.1) we obtain

$$
\int_1^{\infty} r^{d+2l-1} (V(r))^{-s} dr = \frac{c_N^{-s} \Gamma(s)}{(n)} \sum_{n=0}^{\infty} \frac{(-1)^n+1 \Gamma(n+s)}{n!} \sum_{j=0}^{(N-1)n} \frac{\tilde{\omega}_j^n}{d + 2l + 2j - 2Ns - 2Nn} ,
$$

(3.14)

which represents a meromorphic function of $s$ possessing only isolated simple poles at $s = -n + (2l + 2j + d)/2N$.

By choosing $\tilde{\delta}$ in (2.18) to be larger than the rightmost pole of (3.14), namely $\tilde{\delta} > (d + 2l)/2N$, we can write the contour integral in (2.18) as a sum of two terms

$$
I(t, l) = I_1(t, l) + I_2(t, l) ,
$$

(3.15)

where

$$
I_1(t, l) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{(-1)^n (N-1)n}{n!} \sum_{j=0}^{(N-1)n} \frac{\omega_j^n}{d + 2l + 2j + 2N} \int_{\tilde{\omega}_j^{\delta+i\infty}}^{\tilde{\omega}_j^{-i\infty}} \Gamma(s+n) r^{-s} c_0^{-s} ds ,
$$

(3.16)

represents the contribution coming from (3.9), and

$$
I_2(t, l) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{(-1)^n+1 (N-1)n}{n!} \sum_{j=0}^{(N-1)n} \tilde{\omega}_j^n \int_{\tilde{\omega}_j^{-i\infty}}^{\tilde{\omega}_j^{\delta+i\infty}} \Gamma(s+n) \frac{r^{-s} c_0^{-s}}{d + 2l + 2j - 2Ns - 2Nn} ds ,
$$

(3.17)
represents, instead, the contribution arising from (3.14).

In order to obtain the small-$t$ asymptotic expansion of the integral in (3.16) we close the integration contour to the left. Such contour encloses the simple poles of the integrand located at the points $s = -n - m$ with $m \in \mathbb{N}$. Hence, by using Cauchy’s residue theorem we get

$$\frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \Gamma(s+n)t^{-s} c_0^{-s} \, ds = c_0^m n! \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} c_0^m t^m .$$

(3.18)

To compute the integral in (3.17) we proceed in the same way by closing the integration contour to the left. In this case the enclosed simple poles of the integrand are positioned at $s = -n + (d + 1/2 + j)/2N$ and at $s = -n - m$ with $m \in \mathbb{N}$. Cauchy’s residue theorem then gives

$$\frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \Gamma(s+n) \frac{t^{-s} c_N^{-s}}{d + 2l + 2j - 2Ns - 2Nn} \, ds = -\frac{1}{2N} n^m \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} c_N^m t^m .$$

(3.19)

The substitution of the small-$t$ expansions (3.18) and (3.19) in the expressions (3.16) and (3.17), respectively, leads to the results

$$I_1(t, l) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left( \sum_{j=0}^{N-1} \frac{\omega_j^n}{d + 2l + 2j + 2n} \right) c_0^m n! \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} c_0^m t^m .$$

(3.20)

and

$$I_2(t, l) = \left( c_N^m \right)^{d+2j} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} c_N^m \sum_{j=0}^{N-1} \frac{\omega_j^n}{d + 2l + 2j} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} c_N^m \sum_{j=0}^{N-1} \frac{\omega_j^n}{d + 2l + 2Nn + 2j} .$$

(3.21)

It is important to mention, at this point, that the assumption $c_0 \neq 0$, made to obtain (3.3), was necessary in order to develop a well defined procedure for the evaluation of $I_1(t, l)$. However, the explicit expression found for $I_1(t, l)$ in (3.20) is valid for $c_0 = 0$ and, therefore, the previous assumption of non-vanishing $c_0$ is henceforth no longer needed.

Although the expansions (3.20) and (3.21) contain only powers of the small parameter $t$, they do not yet represent proper asymptotic expansions. To solve this problem it is necessary to rearrange (3.20) and (3.21) in terms of increasing powers of $t$. In the expression (3.20) we multiply the two series and gather the terms with the same power of $t$ to obtain

$$I_1(t, l) = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} U_p l^p c_0^p t^p .$$

(3.22)
where \( U^l_p \), found by using the Cauchy product formula \([31]\), has the form

\[
U^l_p = \sum_{j=0}^{p} \binom{p}{j} \left( \sum_{q=0}^{(N-1)j} \omega^q \right) d + 2l + 2q + 2j .
\]

A simple application of the Cauchy product formula to the sums that appear in \([3.21]\) will provide an expression for \( I_2(t,l) \) in terms of increasing powers of \( t \). For the first term in \([3.21]\) we have

\[
\frac{(c_N t)^{-dN/2}}{2N} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} c_N^p \sum_{j=0}^{(N-1)n} \tilde{\omega}^j \Gamma \left( \frac{d + 2l + 2j}{2N} \right) c_N^p t^{-\frac{j}{N}}
\]

\[
= \frac{(c_N t)^{-dN/2}}{2N} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} c_N^p \left( \sum_{j=0}^{(N-1)n} \tilde{\omega}^j \Gamma \left( \frac{d + 2l + 2j}{2N} \right) c_N^p t^{-\frac{j}{N}} \right) = \frac{E^l_p}{2N} \sum_{p=0}^{\infty} E^l_p c_N^p t^{-\frac{p}{N}} , \quad (3.24)
\]

where \( E^l_p \) is obtained by collecting the coefficients corresponding to the power \( t^{p/N} \). By setting \( p = Nn - j \) and by noting that the index \( j = \{0, 1, \ldots, (N - 1)n\} \), we can conclude that the coefficient of \( t^{p/N} \) is the sum of terms for which the index \( n \) takes values in the interval \([p/N], \ldots, p\) and the index \( j \) satisfies the constraint \( j = Nn - p \). In more details we have

\[
E^l_p = \frac{1}{2N} \sum_{n=0}^{p} \frac{(-1)^n}{n!} \tilde{\omega}^n_{Nn-p} \Gamma \left( \frac{d + 2l + 2Nn - 2p}{2N} \right) .
\]

In the second term of \([3.21]\) we multiply the two series to obtain

\[
\sum_{n=0}^{\infty} \frac{(-1)^n N}{n!} c_N^p \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{j=0}^{(N-1)m} \tilde{\omega}^m_j \Gamma \left( \frac{d + 2l + 2Nm + 2j}{2N} \right) c_N^p t^{m} = \sum_{p=0}^{\infty} \frac{(-1)^{p+1}}{p!} B^l_p c_N^p t^p , \quad (3.26)
\]

where the coefficients \( B^l_p \) are given by the expression

\[
B^l_p = \sum_{j=0}^{p} \binom{p}{j} \sum_{q=0}^{(N-1)j} \tilde{\omega}^q .
\]

By utilizing the results obtained in \([3.24]\) and \([3.26]\) we can finally write the small-\( t \) expansion of the integral \( I_2(t,l) \) as follows

\[
I_2(t,l) = (c_N t)^{-dN/2} \sum_{p=0}^{\infty} E^l_p c_N^p t^{-\frac{p}{N}} - \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} B^l_p c_N^p t^p .
\]

The expansions \([3.22]\) and \([3.28]\) together with the relation \([3.15]\) allow us to write the trace of the heat kernel in \([2.18]\) as

\[
K(t) = K_1(t) + K_2(t) + K_3(t) ,
\]

where

\[
K_1(t) = \frac{S_d}{(4\pi t)^{\frac{d}{2}}} \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \left( \sum_{l=0}^{\gamma} \Omega^l_{kl} U^l_p \right) c_N^p t^p ,
\]

\[
K_2(t) = \frac{S_d}{(4\pi t)^{\frac{d}{2}}} \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \left( \sum_{l=0}^{\gamma} \Omega^l_{kl} U^l_p \right) c_N^p t^p .
\]

\[
K_3(t) = \frac{S_d}{(4\pi t)^{\frac{d}{2}}} \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \left( \sum_{l=0}^{\gamma} \Omega^l_{kl} U^l_p \right) c_N^p t^p .
\]
\[ K_2(t) = \frac{S_d \gamma_N^d}{(4\pi)^2 t^{d+\gamma_N}} \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{l=0}^{\gamma_N} \Omega_l^k \gamma_N^{d-l} \sum_{p=0}^{\infty} E_p^k \gamma_N^{d-p}, \]  
\[ (3.31) \]

and

\[ K_3(t) = -\frac{S_d \gamma_N^d}{(4\pi)^2 t^{d+\gamma_N}} \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \left( \sum_{l=0}^{\gamma_N} \Omega_l^k \gamma_N^{d-l} \right) c_p^k \gamma_N^{d-p}. \]
\[ (3.32) \]

It is, then, clear from the definition (3.29) that in order to obtain the small-\( t \) asymptotic expansion of \( K(t) \) it is sufficient to find the small-\( t \) expansion of each of the terms \( K_1(t), K_2(t), \) and \( K_3(t) \).

First, we analyze the terms \( K_1(t) \) and \( K_3(t) \). Since they have the same functional form we consider their sum

\[ K_1(t) + K_3(t) = \frac{S_d \gamma_N^d}{(4\pi)^2 t^{d+\gamma_N}} \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \left( \sum_{l=0}^{\gamma_N} \Omega_l^k \gamma_N^{d-l} \right) c_p^k \gamma_N^{d-p} \cdot \]  
\[ (3.33) \]

Before reorganizing the two series in terms of one in ascending powers of the parameter \( t \), it is first convenient to explicitly evaluate the coefficient in curly bracket.

From (3.23) and the explicit expression (3.8) for \( \omega_j^p \) we find

\[ c_p^0 U_p^l = \sum_{j=0}^{p} \sum_{q=0}^{(N-1)n} \sum_{n_0=q(N-1)}^{q} \sum_{n_1=0}^{q-n_0} \cdots \sum_{n_{N-3}=0}^{q-n_0-\ldots-n_{N-4}} \left( \begin{array}{c} p \\ j \end{array} \right) \left( \begin{array}{c} j \end{array} \right) \cdots \left( \begin{array}{c} n_{N-3} \\ n_0 \\ n_1 \end{array} \right) \left( q-n_0-\ldots-n_{N-3} \right) \]  
\[ \frac{e_0^p \gamma_N^d}{\gamma_N^d} c_1^0 \cdots c_{N-1} \gamma_N^{d-0} \cdots c_{N-1} \gamma_N^{d-N-2} \cdot \]  
\[ (3.34) \]

By setting \( q - n_0 - n_1 - \ldots - n_{N-3} = n_{N-2} \) we can rewrite the sum in (3.34) as follows

\[ c_p^0 U_p^l = \sum_{j=0}^{p} \sum_{n_0=0}^{N} \sum_{n_1=0}^{n_0} \cdots \sum_{n_{N-3}=0}^{n_{N-2}} \left( \begin{array}{c} p \\ j \end{array} \right) \left( \begin{array}{c} j \end{array} \right) \cdots \left( \begin{array}{c} n_{N-3} \\ n_0 \\ n_1 \end{array} \right) \left( q-n_0-\ldots-n_{N-3} \right) \]  
\[ \frac{e_0^p \gamma_N^d}{\gamma_N^d} c_1^0 \cdots c_{N-1} \gamma_N^{d-0} \cdots c_{N-1} \gamma_N^{d-N-2} \cdot \]  
\[ (3.35) \]

Now, we define \( p - j = y_0, j - n_0 = y_1, n_{k-1} - n_k = y_{k+1} \) with \( 1 \leq k \leq N - 2 \), and \( n_{N-2} = y_N \) so that \( y_0 + \cdots + y_N = p \). In terms of the newly introduced indexes the sum in (3.35) reads

\[ c_p^0 U_p^l = \sum_{y_0=0}^{y_N} \sum_{y_1=0}^{y_N} \cdots \sum_{y_N=0}^{y_N} \frac{p!}{y_0!y_1!\cdots y_N!} \frac{e_0^y \gamma_N^d}{\gamma_N^d} c_1^y \cdots c_{N-1}^y \gamma_N^{d-0} \cdots c_{N-1}^y \gamma_N^{d-N-2} \cdot \]  
\[ (3.36) \]

For the coefficient \( c_p^N B_p^l \) we use the definition (3.27) and the expression (3.13) to write

\[ c_p^N B_p^l = \sum_{j=0}^{p} \sum_{y_0=0}^{y_N} \sum_{y_1=0}^{y_N} \cdots \sum_{y_N=0}^{y_N} \left( \begin{array}{c} p \\ j \end{array} \right) \left( \begin{array}{c} j \end{array} \right) \cdots \left( \begin{array}{c} n_{N-3} \\ n_0 \\ n_1 \end{array} \right) \left( q-n_0-\ldots-n_{N-3} \right) \]  
\[ \frac{e_0^p \gamma_N^d}{\gamma_N^d} c_1^{N-1} \cdots c_{N-2}^{N-1} \gamma_N^{d-0} \cdots c_{N-2}^{N-1} \gamma_N^{d-N-2} \cdot \]  
\[ (3.37) \]
We set, once again, \( q - n_0 - n_1 - \ldots - n_{N-2} = n_{N-3} \) and rewrite the sum in (3.37) as

\[
c_N^p B_p^l = \sum_{n_0=0}^{p} \sum_{n_1=0}^{n_0} \cdots \sum_{n_{N-2}=0}^{n_{N-3}} \left( p \right)_{n_0} \left( j \right)_{n_1} \cdots \left( n_{N-3} \right)_{n_{N-2}} c_0^{n_0} c_1^{n_1} \cdots c_{N-2}^{n_{N-2}} c_{N-1}^{n_{N-2}} c_N^{p-j} . \tag{3.38}
\]

By defining, then, \( j - n_0 = y_0, n_{k-1} - n_k = y_k \) with \( 1 \leq k \leq N - 2, n_{N-2} = y_N, \) and \( p - j = y_N \) the expression in (3.38) can be written as

\[
c_N^p B_p^l = \sum_{y_0, y_1, \ldots, y_N} \left( p! \right)_{y_0!y_1! \cdots y_N!} c_0^{y_0} c_1^{y_1} \cdots c_N^{y_N} , \tag{3.39}
\]

Since the expressions in (3.36) and (3.39) are equal we can conclude that \( c_N^p U_p^l = c_N^p B_p^l \) and, therefore,

\[
K_1(t) + K_3(t) = 0 , \tag{4.0}
\]

which indicates that the trace of the heat kernel \( K(t) \) reduces to just the term \( K_2(t) \).

A proper small-\( t \) asymptotic expansion for \( K_2(t) \) is obtained by gathering all the terms with the same power of \( t \) and then organizing them in a single asymptotic series in increasing powers of \( t \). To this end, we multiply the two innermost sums in the result (3.31) to obtain

\[
\sum_{l=0}^{y_k} \Omega^k c_N^{y_k} t^{y_k} \sum_{p=0}^{\infty} E_p^l c_N^{y_k} t^{y_k} = t^{y_k} \sum_{l=0}^{y_k} \Omega^k c_N^{y_k} \left( t^{y_k} \right)^{y_k-l} \sum_{p=0}^{\infty} E_p^l c_N^{y_k} \left( t^{y_k} \right)^p = \sum_{q=0}^{\infty} M_q^k \left( t^{y_k} \right)^q , \tag{3.41}
\]

where we have set \( q = -l + y_k + p \). Since in the previous expression \( l = \{0, \ldots, y_k\} \), the coefficient \( M_q^k \) of \( t^{q+y_k} \) is obtained as the sum of terms for which the index \( p = \{\max(0, q - y_k), \ldots, q\} \) and the index \( l = p - q + y_k \), namely

\[
M_q^k = c_N^{y_k} \sum_{p=\max(0, q-y_k)}^{q} \Omega^k c_N^{y_k} E_p^{p-q+y_k} . \tag{3.42}
\]

The formulas obtained in (3.41) and (3.42) allow us to represent \( K_2(t) \) in (3.31) as follows

\[
K_2(t) = \frac{S d \alpha^{y_k}}{(4\pi)^{1/2} t^{y_k + 1/2}} \sum_{k=0}^{\infty} \sum_{q=0}^{\infty} M_q^k \left( t^{y_k} \right)^q . \tag{3.43}
\]

The above expression, however, requires further manipulation since terms with the same power of \( t \) still need to be collected. To this end, we rewrite the two series in (3.43) as

\[
\sum_{k=0}^{\infty} \sum_{q=0}^{\infty} M_q^k \left( t^{y_k} \right)^q = \sum_{k=0}^{\infty} \left( t^{y_k} \right)^{y_k} \sum_{q=0}^{\infty} M_q^k \left( t^{y_k} \right)^q . \tag{3.44}
\]

Since \( y_0 = y_1 = 0 \) and, according to (2.13), \( y_k = \left[ 2k/3 \right] (N + 1) - k \) for \( k \geq 2 \), we have that

\[
\sum_{k=0}^{\infty} \left( t^{y_k} \right)^{N-k-y_k} \sum_{q=0}^{\infty} M_q^k \left( t^{y_k} \right)^q = \sum_{q=0}^{\infty} M_q^0 \left( t^{y_k} \right)^q + \sum_{q=N}^{\infty} \left( M_q^0 + M_{q-N} \right) \left( t^{y_k} \right)^q + \sum_{k=2}^{\infty} \left( t^{y_k} \right)^{k(N+1)} \sum_{q=0}^{\infty} M_q^k \left( t^{y_k} \right)^q , \tag{3.45}
\]
where \( t_k = k - [2k/3] \), for \( k \geq 2 \), is defined by the equation \( Nk - \gamma_k = t_k(N + 1) \). At this point we notice that \( t_k \) with \( k \geq 2 \) represents an increasing sequence of positive integers for which \( t_2 = t_3 = 1 \) and \( t_{3n} = t_{3n-1} = t_{3n-2} = n \) for \( n \geq 2 \). This last remark applied to (3.45) gives

\[
\sum_{k=0}^{\infty} \left( \frac{t}{t + N} \right)^{Nk-\gamma_k} \sum_{q=0}^{\infty} M_q^k \left( \frac{t}{t + N} \right)^q = \sum_{k=0}^{\infty} \left( \frac{t}{t + N} \right)^{k(N+1)} \sum_{q=0}^{\infty} \tilde{M}_q^k \left( \frac{t}{t + N} \right)^q ,
\]

where

\[
\tilde{M}_q^0 = M_q^0 , \quad 0 \leq q \leq N - 1 , \quad \tilde{M}_q^0 = M_q^0 + M_{q-N}^1 , \quad q \geq N ,
\]

\[
\tilde{M}_q^1 = M_q^2 + M_q^3 , \quad q \geq 0 ,
\]

\[
\tilde{M}_q^k = M_q^{3k} + M_q^{3k-1} + M_q^{3k-2} , \quad k \geq 2 , \quad q \geq 0 .
\]

By employing the Cauchy product formula in (3.46) we then obtain

\[
\sum_{k=0}^{\infty} \left( \frac{t}{t + N} \right)^{k(N+1)} \sum_{q=0}^{\infty} \tilde{M}_q^k \left( \frac{t}{t + N} \right)^q = \sum_{j=0}^{\infty} C_j t^j ,
\]

where the coefficients of the expansion have the form

\[
C_j = \sum_{p=0}^{\infty} \tilde{M}_p^j \left( t - p(N+1) \right) .
\]

Since, according to (3.29) and (3.40), \( K_2(t) = K(t) \) we utilize the equations (3.44) and (3.46) together with the result obtained in (3.50) to finally provide the small-\( t \) asymptotic expansion of the trace of the heat kernel \( K(t) \) when a spherically symmetric polynomially confining potential is present, namely

\[
K(t) = \frac{S_d e^{-\frac{d}{2}}}{(4\pi)^{\frac{d}{2}} t^{\frac{d}{2}} + \frac{\pi}{2}} \sum_{j=0}^{\infty} C_j t^j ,
\]

with the coefficients of the expansion given by (3.51).

A few remarks are in order at this point. It is well known that the leading small-\( t \) behavior of the trace of the heat kernel \( \mathcal{K}(t) \) for a Laplace-type operator on a \( d \)-dimensional compact manifold \( \mathcal{M} \) with or without boundary is [12, 20]

\[
\mathcal{K}(t) \sim \frac{1}{(4\pi)^{\frac{d}{2}}} \text{Vol} \mathcal{M} ,
\]

namely the leading term is \( O(t^{-d/2}) \). In the unbounded case with a polynomially confining potential studied here the form of the leading term differs from the one described above. In fact, from the explicit expansion (3.52) the leading behavior of \( K(t) \) is provided by the term \( C_0 \). By using the relations (3.51), (3.47), (3.42), and (3.25) together with the explicit expression of \( S_d \) given below (2.11) one obtains for the leading term

\[
K(t) \sim \frac{2^{1-d} \Gamma \left( \frac{d}{N} \right)}{2 N c_N \Gamma \left( \frac{d}{2} \right) t^{\frac{d}{2}}} ,
\]
which is the same as the one obtained in [7] once we set $c_N = 1$. In this case the leading term is of order $O(t^{-d/2-d/2N})$ and depends not only on the dimension of the underlying manifold but also on the degree $2N$ of the polynomial modeling the potential.

We would also like to point out that since the coefficient $c_0$ in the potential function (2.10) plays the role of a mass term or spectral parameter, the coefficients of the expansion of the trace of the heat kernel for the case of a pure potential, with no mass term, can be obtained from the ones in this Section by simply performing the limit $c_0 \to 0$.

IV. SPECIFIC CONFINING POTENTIALS

In this Section we use the general results obtained earlier to find the coefficients of the asymptotic expansion of the trace of the heat kernel for specific spherically symmetric polynomial potentials defined on the Euclidean space $\mathbb{R}^d$. According to the expression (3.52) the coefficients of the asymptotic expansion (3.51) are written in terms of $M_p^0$ which have been introduced in the process of organizing the various small-$t$ expansions in increasing powers of $t$. Once the dimension of the underlying Euclidean space and the polynomial potential have been specified, the coefficients $M_p^0$ can be found with a simple computer program.

The simplest and most studied example is represented by a $d$-dimensional spherically symmetric harmonic oscillator potential $V(r) = cr^2$, with $c > 0$. In this case the eigenvalues of the associated operator are known explicitly and they are $\lambda_{n_1,\ldots,n_d} = \sqrt{c(n_1 + \cdots + n_d + d/2)}$ with $n_1,\ldots,n_d \geq 0$. The trace of the heat kernel can be computed in closed form and coincides with the partition function of the $d$-dimensional harmonic oscillator, namely

$$K(t) = \frac{1}{\left[2 \sinh \left(\frac{\sqrt{c}}{t}\right)\right]^d}. \quad (4.1)$$

Although the small-$t$ asymptotic expansion for $K(t)$ can be found by using the method developed in this work, it can be more easily obtained from the Taylor series of the function in the denominator. For this reason the details of the expansion will not be shown here and, instead, we will consider the next non-trivial example, namely the spherically symmetric quartic oscillator potential. In this case the potential has the general form

$$V(r) = c_0 + c_1 r^2 + c_2 r^4, \quad (4.2)$$

where we assume that $V(r)$ is defined on $\mathbb{R}^3$. By using the explicit form (4.2) of the potential $V(r)$ in the
expression (2.8) it is not difficult to obtain from (2.13) the following coefficients \( \Omega_i^j \)

\[
\Omega_0^0 = 1 , \quad \Omega_0^1 = 0 ,
\]

\[
\Omega_0^2 = -c_1 , \quad \Omega_1^2 = -\frac{1}{3} c_2 ,
\]

\[
\Omega_0^3 = -2c_2 , \quad \Omega_1^3 = \frac{1}{3} c_1^2 , \quad \Omega_2^3 = \frac{4}{3} c_1 c_2 , \quad \Omega_3^3 = \frac{4}{3} .
\]

In addition, the expression (3.13) with \( N = 2 \) gives

\[
\tilde{\omega}_n^l = (n_l)^c_n - l_0 c_l - n_2 .
\]

By setting \( d = 3 \) and \( N = 2 \) in (3.25) and by using the above formula for \( \tilde{\omega}_n^l \) we can write the coefficients \( E_l^p \) as

\[
E_l^p = \frac{1}{4} \sum_{n=[\frac{d}{2}]}^{p} \frac{(-1)^n}{n!} \binom{n}{2n-p} \Gamma \left( \frac{2l+4n-2p+3}{4} \right) c_0^{p-n} c_1^{2n-2p} c_2^{-n} .
\]

By recalling the general result (3.52) and by noticing that \( S_3 = 4\pi^2 \), the asymptotic expansion of the trace of the heat kernel for the quartic oscillator potential reads

\[
K(t) = \frac{\sqrt{\pi}}{2c_2^{3/4} t^{9/4}} \sum_{j=0}^{\infty} C_j t^j ,
\]

where the coefficients \( C_j \) of the expansion are obtained from (3.51) by using (4.7) and (3.42). The explicit expression for the first few of them is

\[
C_0 = \frac{1}{4} \Gamma \left( \frac{3}{4} \right) , \quad C_1 = -\frac{c_1}{16 \sqrt{\pi}} \Gamma \left( \frac{1}{4} \right) ,
\]

\[
C_2 = \frac{1}{c_2} \left( \frac{3}{32} c_1^2 - \frac{1}{4} c_0 c_2 \right) \Gamma \left( \frac{3}{4} \right) ,
\]

\[
C_3 = -\frac{1}{c_2^{3/2}} \left( \frac{5}{384} c_1^3 - \frac{1}{16} c_0 c_1 c_2 + \frac{5}{48} c_2^2 \right) \Gamma \left( \frac{1}{4} \right) ,
\]

\[
C_4 = \frac{1}{c_2} \left( \frac{7}{512} c_1^4 - \frac{3}{32} c_0 c_1^2 c_2 + \frac{3}{16} c_1 c_2^2 + \frac{1}{8} c_2^2 \right) \Gamma \left( \frac{3}{4} \right) ,
\]

and

\[
C_5 = -\frac{1}{c_2^{5/2}} \left( \frac{3}{2048} c_1^5 - \frac{5}{384} c_0 c_1^3 c_2 + \frac{13}{384} c_1^2 c_2^2 + \frac{1}{32} c_0^2 c_1 c_2^2 - \frac{5}{48} c_0 c_2^3 \right) \Gamma \left( \frac{1}{4} \right) .
\]
As a further example we consider the spherically symmetric sextic oscillator potential

\[ V(r) = c_0 + c_1 r^2 + c_2 r^4 + c_3 r^6 , \]  

(4.14)
defined on \( \mathbb{R}^3 \). The coefficients \( \Omega_j^i \) associated with the above potential are

\[ \Omega_0^0 = 1 , \quad \Omega_0^1 = 0 , \]  

(4.15)

\[ \Omega_0^2 = -c_1 , \quad \Omega_1^2 = - \frac{10}{3} c_2 , \quad \Omega_2^2 = -7c_3 , \]  

(4.16)

\[ \Omega_0^3 = -2c_2 , \quad \Omega_1^3 = -14c_3 + \frac{1}{3} c_1^2 , \quad \Omega_2^3 = \frac{4}{3} c_1 c_2 , \quad \Omega_3^3 = \frac{4}{3} c_2^3 + 2c_1 c_3 \]  

(4.17)

\[ \Omega_4^3 = 4c_2 c_3 , \quad \Omega_2^3 = 3c_3^2 , \]  

(4.18)

which have been computed by using (2.8) and the formula (2.13). In addition to the coefficients \( \Omega_j^i \) we will also need \( \tilde{\omega}_j^i \) which are obtained from (3.13) by setting \( N = 3 \). In more details we have

\[ \tilde{\omega}_j^n = \sum_{n_0=\left\lfloor \frac{n}{2} \right\rfloor}^{\frac{n}{2}} \left( \frac{n}{n_0} \right) \binom{n_0}{l-n_0} c_{n-n_0}^0 c_{n-1} c_{n-2} c_{n-3} \]  

(4.19)

By using (4.19) in (3.25) and then by setting \( d = N = 3 \) one obtains the following expression for the coefficients \( E_p^l \)

\[ E_p^l = \frac{1}{6} \sum_{n=\left\lfloor \frac{3n-p}{2} \right\rfloor}^{\frac{3n-p}{2}} \frac{(-1)^n \left( \begin{array}{c} n \\ n_0 \end{array} \right) \left( \begin{array}{c} n_0 \\ l-n_0 \end{array} \right) \Gamma \left( \frac{2l + 6n - 2p + 3}{6} \right) \binom{2l + 6n - 2p + 3}{6} c_{n-n_0}^0 c_{n_0-n_0} c_{n_0-n_0} c_{n_0-n_0} }{n!} . \]  

(4.20)

Since \( S_3 = 4\pi^2 \), the general result (3.52) gives, when \( d = N = 3 \), the asymptotic expansion of the trace of the heat kernel for the sextic oscillator potential

\[ K(t) = \frac{\sqrt{\pi}}{2 \sqrt{c_3^3} t^2} \sum_{j=0}^{\infty} C_j t^j , \]  

(4.21)

where the coefficients \( C_j \) are computed from (3.51) by using the expression (4.20) and (3.42). The first few coefficients for the sextic oscillator potential are

\[ C_0 = \frac{\sqrt{\pi}}{6} , \quad C_1 = -\frac{c_2}{36 c_3^3} \Gamma \left( \frac{1}{6} \right) , \]  

(4.22)

\[ C_2 = \frac{1}{c_3^{1/3}} \left( \frac{5}{72} c_2^3 - \frac{1}{6} c_1 c_3 \right) \Gamma \left( \frac{5}{6} \right) , \]  

(4.23)

\[ C_3 = -\frac{\sqrt{\pi}}{c_3^2} \left( \frac{1}{48} c_2^3 - \frac{1}{12} c_1 c_2 c_3 + \frac{1}{6} c_0 c_3^3 \right) . \]  

(4.24)
\[ C_4 = \frac{1}{c_3^{8/3}} \left( \frac{91}{31104} c_2^4 - \frac{7}{432} c_1^2 c_3 + \frac{1}{36} c_0 c_2 c_3^2 + \frac{1}{72} c_1^2 c_3^2 - \frac{7}{72} c_3^3 \right) \Gamma \left( \frac{1}{6} \right), \]  
(4.25) 

and

\[ C_5 = -\frac{1}{c_3^{10/3}} \left( \frac{187}{31104} c_2^5 - \frac{55}{1296} c_1 c_2^3 c_3 + \frac{5}{72} c_2^2 c_3^2 + \frac{5}{72} c_0 c_2^2 c_3^2 - \frac{1}{6} c_0 c_1 c_3^3 - \frac{5}{24} c_2^3 c_3^3 \right) \Gamma \left( \frac{5}{6} \right). \]  
(4.26) 

We would like to point out that although only the first five coefficients of the asymptotic expansion of the trace of the heat kernel for the quartic and sextic oscillator potentials have been explicitly provided here higher order coefficients can be easily obtained from the general formulas given in (3.42) and (3.51). In addition, the computation of the coefficients \( C_j \) of higher order requires, in turn, the knowledge of higher order coefficients \( A_k \) in (2.2) which can be obtained in a quite algorithmic way by utilizing the method based on the Volterra series described in Section II. 

V. CONCLUSIONS

In this work we have developed a technique for the explicit computation of the complete small-\( t \) asymptotic expansion of the trace of the heat kernel associated with a Laplace operator defined on \( \mathbb{R}^d \) endowed with a spherically symmetric polynomially confining potential. The results presented in the previous Sections are very general since they hold for any spherically symmetric polynomially confining potential in arbitrary dimension \( d \). The method described in this paper for the computation of the asymptotic expansion of the trace of the heat kernel \( K(t) \) is based on two steps. The first consists in resumming all the powers of the potential that appear in the local asymptotic expansion of the trace of the heat kernel in order to obtain an overall exponential factor depending on the potential. As a second step, the resulting resummed asymptotic expansion of \( K(t) \) is represented in terms of a Mellin-Barnes integral. This complex integral representation proves to be very useful since the small-\( t \) asymptotic expansion of \( K(t) \) is obtained by just closing the integration contour to the left and by using Cauchy residue theorem. The resulting coefficients of the asymptotic expansion of the trace of the heat kernel are given by formulas that are very suitable for implementation in an algebraic computer program.

The trace of the heat kernel of a Laplace operator and the associated spectral zeta function, which is obtained from the Riemann zeta function by replacing the sequence of positive integer with the spectral sequence of the operator, are intimately connected through the Mellin transform \([20]\). This implies that the knowledge of one will provide useful information about the other. In particular, the small-\( t \) asymptotic expansion found in this work can be used to obtain the complete meromorphic structure of the associated spectral zeta function. In this framework, it can be proved that the position of the poles are given by the
powers of the small parameter $t$ appearing in the asymptotic expansion of $K(t)$ [15, 20]. For the case considered in this work, the poles of the associated spectral zeta function, which can be shown to be all simple, are real and positioned at the points $d/2 + d/2N - j/N$, with $j \in \mathbb{N}_0$, which strictly depend on the degree of the polynomial potential (see also [7]). We would like to point out that a more complete analysis, although mainly focused in one dimension, of the spectral zeta function for a Laplace operator endowed with a confining potential can be found in [34–36].

The results presented earlier regarding the expansion of the trace of the heat kernel have been obtained under the assumption that the polynomially confining potential has spherical symmetry. This assumption has been made in order to have an explicitly computable integral in (2.9). If the assumption of spherical symmetry were to be abandoned, then, after representing $e^{-tV(x)}$ in (2.9) in terms of a Mellin-Barnes integral, one would need to evaluate explicitly the resulting integral over $x$ containing, as integrand, the product of the coefficients $A_k(x)$ and $V^{-s}(x)$. If a class of polynomially confining potentials $V(x)$ is found for which the integral mentioned above can be explicitly evaluated as a function of $s$, then the technique developed here can be extended to include anisotropic polynomial potentials.

It would be of particular interest to generalize the results for the asymptotic expansion of $K(t)$ presented in this work to the case of non-polynomial but spherically symmetric confining potentials. Let us recall that the first step of our analysis requires the integral, over the radial variable $r$, containing $V^{-s}(r)$ in (2.18) to be explicitly evaluated. Interestingly, to complete this task we do not need the specific expression of $V(r)$, but, according to (3.1), we only need the complete small-$r$ and large-$r$ expansions of $V(r)$. This last remark implies that the integral on the left hand side of (3.14) could be computed for arbitrary spherically symmetric confining potentials for which the small-$r$ and large-$r$ behaviors are known. It is important to point out that such behaviors have to be of a functional form suitable for the evaluation of the radial integral in closed form. In this situation the procedure described in Section III can be used and the small-$t$ asymptotic expansion of $K(t)$ can be obtained.

The next step in the analysis of the asymptotic expansion of the trace of the heat kernel for confining potentials consists in considering spherically symmetric exponentially confining potentials. It has been shown in [7] that the small-$t$ expansion of $K(t)$ in the presence of exponentially confining potentials is non-standard and contains logarithmic terms similar to the case of non-smooth manifolds (see e.g. [6, 9, 11]). Due to this non-standard behavior of the asymptotic expansion we expect that the technique developed in this work will need to be somewhat modified in order to treat the case of exponentially confining potentials.
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