Abstract

We use new over-convergent $p$-adic exponential power series, inspired by work of Pulita [12], to build self-dual normal basis generators for the square root of the inverse different of certain abelian weakly ramified extensions of an unramified extension $K$ of $\mathbb{Q}_p$. These extensions, whose set we denote by $\mathcal{M}$, are the degree $p$ subextensions over $K$ of $M_{p,2}$, the maximal abelian totally, wildly and weakly ramified extension of $K$, whose norm group contains $p$. Our construction follows Pickett’s [10], who dealt with the same set $\mathcal{M}$ of extensions of $K$, but does not depend on the choice of a basis of the residue field $k$ of $K$. Instead it furnishes a one-to-one correspondence, commuting with the action of the Galois group of $K/\mathbb{Q}_p$, from the projective space of $k$ onto $\mathcal{M}$. We describe very precisely the norm group of the extensions in $\mathcal{M}$. When $K \neq \mathbb{Q}_p$, their compositum $M_{p,2}$ yields an interesting example of non abelian weakly ramified extension of $\mathbb{Q}_p$, with Galois group isomorphic to a wreath product. Finally we show that, with a slight modification, our over-convergent exponential power series endow certain differential modules with a Frobenius structure, generalising a result of Pulita. Unfortunately, they then lose the property we need to build self-dual normal basis generators, hence the desirable link between Galois module structure and differential modules is not yet obtained.

Introduction

Exponential power series over $p$-adic rings are very useful objects in a number of different fields. They have recently been used in the description of rank one $p$-adic differential equations over the Robba ring of a $p$-adic field [12] and to obtain results concerning integral Galois module structure in wildly ramified extensions of both local and global fields [10] [11]. In this paper we generalise and modify some previous constructions of exponential power series and explore their applications.

Let $E/F$ be a finite odd degree Galois extension of number fields, with Galois group $G$ and rings of integers $\mathfrak{O}_E$ and $\mathfrak{O}_F$. From Hilbert’s formula for the valuation of the different $\mathfrak{d}_{E/F}$, see [15] IV §2 Prop.4, we know that $\mathfrak{d}_{E/F}$ will have an even valuation at every prime
ideal of \( \mathfrak{O}_E \) and we can thus define the square-root of the inverse different \( \mathcal{A}_{E/F} \) to be the unique fractional \( \mathfrak{O}_E \)-ideal such that
\[
\mathcal{A}_{E/F}^2 = \mathfrak{D}_{E/F}^{-1}.
\]
In [2], Erez proved that \( \mathcal{A}_{E/F} \) is locally free over \( \mathfrak{O}_F[G] \) if and only if \( E/F \) is at most weakly ramified, i.e., when the second ramification groups are trivial at every prime. This gives \( \mathcal{A}_{E/F} \) the uncommon property that it is locally free in certain wildly ramified extensions and as such, the question of describing whether it is free over the group ring \( \mathbb{Z}[G] \), as Taylor famously achieved for the ring of integers of a tamely ramified extension [16], raises new difficulties. Despite partial results, see [2, 11, 17, 18, 19], this question remains open.

We now let \( p \) be a rational prime and let \( \gamma \) be a root of the polynomial \( X^p - 1 + p \) in a fixed algebraic closure \( \bar{\mathbb{Q}}_p \) of the field of \( p \)-adic numbers \( \mathbb{Q}_p \). Dwork’s \( p \)-adic exponential power series is defined as
\[
E_\gamma(X) = \exp(\gamma X - \gamma X^p).
\]
This power series was originally introduced by Dwork in his study of the zeta function of hypersurfaces [1]. The convergence properties of this power series can be used to endow a certain \( p \)-adic differential module with a so-called Frobenius structure, see Section 5, and such modules have become central objects in the subjects of \( p \)-adic differential equations and \( p \)-adic arithmetic geometry.

In [10] Pickett demonstrated how special values of Dwork’s power series can be used to construct arithmetic Galois module generators in certain wildly ramified extensions of local fields contained in Lubin-Tate division fields. In [11] Pickett and Vinatier use these constructions to make further progress with the conjecture that \( \mathcal{A}_{E/F} \) is free over \( \mathbb{Z}[G] \) when \( E/F \) is weakly ramified and of odd degree.

This new approach marks the first progress to this goal for quite some time; we now ask whether further progress can be made using other techniques arising in the study of \( p \)-adic differential equations and whether any insight can be gained in \( p \)-adic analysis with this new perspective. We note that with the use of Fröhlich’s Hom-description of the locally free class group, see [5], much of the work required for results in the global theory can be carried out at a local level. We focus here on the local behaviour with the view that a complete understanding of this will lead to results at a global level.

The main inspiration for this paper has come from recent work of Pulita [12], where he generalises Dwork’s power series to a class of exponentials, each with coefficients in a Lubin-Tate extension of \( \mathbb{Q}_p \): Let \( f(X) \in \mathbb{Z}_p[X] \) be some Lubin-Tate polynomial with respect to the uniformising parameter \( p \), i.e.,
\[
f(X) \equiv X^p \pmod{p\mathbb{Z}_p[X]} \quad \text{and} \quad f(X) \equiv pX \pmod{X^2\mathbb{Z}_p[X]}.
\]
Let \( \{\omega_i\}_{i \geq 0} \) be a coherent set of roots associated to \( f(X) \), namely a sequence of elements of \( \bar{\mathbb{Q}}_p \) such that \( f(\omega_i) = \omega_{i-1} \) and \( \omega_0 = 0 \neq \omega_1 \). Then define the \( n \)th Pulita exponential as
\[
E_n(X) = \exp\left( \sum_{i=0}^{n-1} \frac{\omega_{n-i}(X^{p^i} - X^{p^{i+1}})}{p^i} \right).
\]
These power series endow certain $p$-adic differential modules with Frobenius structures and Pulita uses these modules to categorise all rank one solvable $p$-adic differential equations. As Pulita observes, restricting to the uniformising parameter $p$ means that only Lubin-Tate division fields arising from a formal group isomorphic to the multiplicative formal group are considered, see [14, §3] for an overview of the main theorems in Lubin-Tate theory.

This paper is divided into five sections. In Section 1 we set up the general framework. Assume $p$ is an odd rational prime and $K$ is an unramified extension of $\mathbb{Q}_p$ with residue field $k$ of cardinality $q = p^d$. A finite extension of $K$ is said to be weakly ramified if it is Galois and has a trivial second ramification group (in the lower numbering). We first recall some properties of Galois groups of Lubin-Tate division fields over $K$. Then, given a uniformising parameter $\pi$ of $K$, we identify the maximal abelian totally, wildly and weakly ramified extension of $K$, such that $\pi$ is a norm from this extension, as the largest $p$-extension $M_{\pi,2}$ contained in the second Lubin-Tate division field $K_{\pi,2}$ relative to $\pi$.

In Section 2 we describe how values of modifications of Pulita’s exponentials can be used to construct Galois module generators in the degree $p$ subextensions of $M_{p,2}/K$. Let $\mu_{q-1} = \{ \mu \in \bar{\mathbb{Q}}_p : \mu^{p-1} = 1 \}$, then for each $u \in \mu_{q-1}$, let $f_u(X) = X^p + upX$ and let $\omega_u$ be a root of $f_u(X) - \gamma$ (recall that $\gamma$ is a root of $f_1(X/X)$. We prove the over-convergence (see Notations and Conventions for a definition) of the power series

$$E_{u,2}(X) = \exp \left( \omega_u X - u \omega_u X^p + \frac{\gamma X^p - \gamma X^{p^2}}{p} \right).$$

We then show that if $v \in \mu_{q-1}$ and $u = v^{1-p}$, then $E_{u,2}(v)$ generates a Kummer extension $L_u$ over $K(\gamma) = K(\zeta_p)$ (where $\zeta_p$ is any primitive $p$th root of unity), such that $L_u/K$ is cyclic of degree $p(p-1)$. We show that its only subextension $M_u$ of degree $p$ over $K$ is contained in $M_{p,2}$, hence weakly ramified, and that

$$\alpha_u = \frac{\sum_{s \in S} E_{u,2}(v)^s}{p}$$

is a self-dual integral normal basis generator for $A_{M_u/K}$, the square-root of the inverse different of $M_u/K$, where $S = \mu_{p-1} \cup \{0\}$. These basis generators are more natural than the ones in [10] as they do not rely on a fixed basis of $k$ over $\mathbb{F}_p$.

In Section 3 we explicitly describe the norm group of $M_u/K$, namely

$$N(M_u/K) = \langle p \rangle \times \mu_{q-1} \times \exp(pv^{-p}Z),$$

where $Z = \{ x \in \mathfrak{O}_K : Tr_{K/\mathbb{Q}_p} x \in p\mathbb{Z}_p \}$ ($\mathfrak{O}_K$ is the valuation ring of $K$). It follows that the map $v \mapsto M_u$ (recall that $u = v^{1-p}$) is, after natural identifications, a one-to-one correspondence between the projective space $\mathbb{P}(k)$ of the residue field $k$ and the set $\mathcal{M}$ of subextensions of $M_{p,2}/K$ of degree $p$.

In Section 4 we study $M_{p,2}$ as an absolute extension of $\mathbb{Q}_p$. This extension is easily checked to be Galois; when $[K : \mathbb{Q}_p] > 1$, it yields an interesting example of a non-abelian weakly ramified extension. We give a complete description of its Galois group, showing that it is
isomorphic to the regular wreath product of a cyclic group of \( p \) elements with a cyclic group of \( d \) elements, \( C_p \wr C_d \). We also study the action of \( \text{Gal}(K/\mathbb{Q}_p) \) on some subextensions of \( K_{p,2}/K \) and, as a corollary, we get that the one-to-one correspondence of Section 3 commutes with the action of \( \text{Gal}(K/\mathbb{Q}_p) \).

Our study of exponential power series from the viewpoint of Galois module structure has led to the proof of the over-convergence of power series that generalise the original constructions of Pulita. Namely, in Section 5 we prove the over-convergence of the power series

\[
E_{u,n}(X) = \exp \left( \sum_{i=0}^{n-1} \frac{\omega_{u,n-i}(X^{p^i} - uX^{p^{i+1}})}{p^i} \right)
\]

for all \( u \in \mathbb{Z}_p^\times \) and \( n \in \mathbb{N} \), where \( \{\omega_{u,i}\}_{i \geq 0} \) is a coherent set of roots associated to a Lubin-Tate polynomial with respect to the uniformising parameter \( u_p \). This removes Pulita’s restriction that the Lubin-Tate formal group be isomorphic to the multiplicative formal group. We then show that these exponentials again furnish certain \( p \)-adic differential modules with Frobenius structures.

We remark that as differential modules with Frobenius structure and modules with integral Galois structure are such important objects in different subject areas, it is very desirable to obtain a direct link between them. Unfortunately, despite using similar power series to study these two structures, we cannot provide a satisfactory link with the results in this paper. We hope that future work will explore this potential connection further.

**Notation and conventions**

We begin by fixing some notation and conventions which we will use throughout this paper.

- The set \( \mathbb{N} \) is equal to the set of positive integers \( \{1, 2, 3, \ldots\} \). For \( n \in \mathbb{N} \) we denote the cyclic group of order \( n \) by \( C_n \).

- Let \( p \) denote an odd prime and let \( \bar{\mathbb{Q}}_p \) be a fixed algebraic closure of \( \mathbb{Q}_p \). For any \( n \in \mathbb{N} \) we denote by \( \mu_n \) the group of \( n \)th roots of unity contained in \( \bar{\mathbb{Q}}_p^\times \).

- We let \( | \cdot |_p \) be the normalised absolute value on \( \bar{\mathbb{Q}}_p \) such that \( |p|_p = p^{-1} \) and denote by \( \mathbb{C}_p \) the completion of \( \bar{\mathbb{Q}}_p \) with respect to \( | \cdot |_p \). We say that a power series with coefficients in some algebraic extension of \( \mathbb{Q}_p \) is over-convergent if it converges with respect to \( | \cdot |_p \) on some disc \( \{x \in \bar{\mathbb{Q}}_p : |x|_p < 1 + \epsilon \} \) for some \( \epsilon > 0 \). We note that an over-convergent function is convergent on the unit disc \( \{x \in \bar{\mathbb{Q}}_p : |x|_p \leq 1 \} \).

- For any extension \( L/\mathbb{Q}_p \) considered we will always assume \( L \) is contained in \( \bar{\mathbb{Q}}_p \) and we will denote by \( \mathfrak{O}_L, \mathfrak{P}_L \) and \( k_L \) its valuation ring, valuation ideal and residue field respectively; we identify \( k_{\bar{\mathbb{Q}}_p} = \mathbb{Z}_p/p\mathbb{Z}_p \) with the field of \( p \) elements, \( \mathbb{F}_p \). If \( \pi \) is a uniformising parameter of \( \mathfrak{O}_L \), we denote by \( L_{\pi,n} \) the \( n \)th Lubin-Tate division field with respect to \( \pi \), see [11, §3.6], and we set \( L_{\pi} = \bigcup_{n \in \mathbb{N}} L_{\pi,n} \).
Throughout we let $K/\mathbb{Q}_p$ be an unramified extension of degree $d \in \mathbb{N}$, we let $k_K = k$ and $q = p^d = |k|$. We know that $k^\times \cong \mu_{q-1} \subset K$ and using the Teichmüller lifting we will sometimes think of $k$ as lying in $\mathcal{O}_K$ by identifying it with $\mu_{q-1} \cup \{0\}$. We let $\Sigma$ be the Galois group of $K/\mathbb{Q}_p$; $\Sigma$ is cyclic and generated by the unique lifting $\sigma$ of the Frobenius automorphism of the residue field extension.

We denote by $K^{ab}$ the maximal abelian extension of $K$ contained in $\bar{\mathbb{Q}}_p$ and let $\theta_K : K^\times \to \text{Gal}(K^{ab}/K)$ be the Artin reciprocity map. For any extension $L/K$ with $L \subseteq K^{ab}$ we let $\theta_{L/K}$ be the composition of the Artin map with restriction of automorphisms to $L$; we denote by $N_{L/K}$ the norm map from $L$ to $K$ and by $N(L/K)$ the norm group $N(L/K) = N_{L/K}(L^\times)$.

1 Galois groups in Lubin-Tate towers

Let $\pi$ denote a uniformising parameter of $K$. Recall that for $n \in \mathbb{N}$, the $n$th Lubin-Tate division field $K_{\pi,n}$ is an abelian totally ramified extension of $K$, of norm group $N(K_{\pi,n}/K) = \langle \pi \rangle \times (1 + \mathfrak{p}_K^n)$ and Galois group $\text{Gal}(K_{\pi,n}/K) \cong K^\times/N(K_{\pi,n}/K) \cong \mu_{q-1} \times (1 + \mathfrak{p}_K)/(1 + \mathfrak{p}_K^n)$. If further $m \leq n$ in $\mathbb{N}$, then $K \subset K_{\pi,m} \subseteq K_{\pi,n}$ and

$$ \text{Gal}(K_{\pi,n}/K_{\pi,m}) \cong 1 + \mathfrak{p}_K^m/1 + \mathfrak{p}_K^n. \quad (1) $$

See [6] for details, in particular Proposition 7.3 and Lemma 7.4 (note that $K_{\pi,n}$ is denoted there by $K_{n-1}$).

**Proposition 1.1** Let $n, m \in \mathbb{N}$ with $m \leq n$, then

$$ \text{Gal}(K_{\pi,n}/K_{\pi,m}) \cong \mathcal{O}_K/p^{n-m}\mathcal{O}_K \cong \bigoplus_{i=1}^d \left( \mathbb{Z}/p^{n-m}\mathbb{Z} \right). $$

Further, any subextension of $K_{\pi}/K_{\pi,1}$ with Galois group of exponent $p^{m-1}$ is contained in $K_{\pi,m}$.

**Proof.** Since $K/\mathbb{Q}_p$ is unramified and $p \geq 3$, we have $\mathfrak{p}_K \subseteq \{ x \in \mathbb{C}_p : |x|_p < p^{-\frac{1}{p-1}} \}$. Using the Proposition in [8, IV.1] and the fact that $K$ is complete, the exponential power series gives a group isomorphism from the additive group $\mathfrak{p}_K$ to the multiplicative group $1 + \mathfrak{p}_K$, with inverse map the logarithmic power series. For any $i \in \mathbb{N}$, this isomorphism yields

$$ \exp(\mathfrak{p}_K^i) = \exp(p^{i-1}\mathfrak{p}_K) = \exp(\mathfrak{p}_K)^{p^{i-1}} \cong (1 + \mathfrak{p}_K)^{p^{i-1}}. $$

By [15, XIV Proposition 9] we know that $(1 + \mathfrak{p}_K)^{p^{i-1}} \cong 1 + \mathfrak{p}_K^i$. Combining these results we have $\mathfrak{p}_K^i \cong 1 + \mathfrak{p}_K^i$ and thus, using (1):

$$ \text{Gal}(K_{\pi,n}/K_{\pi,m}) \cong \mathfrak{p}_K^m/\mathfrak{p}_K^n = p^m\mathcal{O}_K/p^n\mathcal{O}_K \cong \mathcal{O}_K/p^{n-m}\mathcal{O}_K. $$

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The first result now follows using a \( \mathbb{Z}_p \)-basis of \( \mathcal{O}_K \). Note that \( K_{\pi,n}/K_{\pi,1} \) has Galois group of exponent \( p^{n-1} \).

Suppose \( L \) is a subextension of \( K_{\pi}/K_{\pi,1} \) with Galois group of exponent \( p^{n-1} \), then \( L \subseteq K_{\pi,n} \) for some integer \( n \), with \( n \geq m \) by the previous result, namely \( K_{\pi,m} \subseteq K_{\pi,n} \). Any subgroup \( H \) of \( G = \text{Gal}(K_{\pi,n}/K_{\pi,1}) \) such that \( 1 + H \) is of exponent \( p^{m-1} \) contains \( G^{p^{m-1}} = \{ g^{p^{m-1}} : g \in G \} \), which is of index \( p^{(m-1)d} = [K_{\pi,m} : K_{\pi,1}] \) in \( G \), so that:

\[
\text{Gal}(K_{\pi,n}/K_{\pi,1}) \subseteq \text{Gal}(K_{\pi,n}/K_{\pi,1})^{p^{m-1}},
\]

and \( \text{Gal}(K_{\pi,n}/K_{\pi,m}) \subseteq \text{Gal}(K_{\pi,n}/L) \), hence \( L \subseteq K_{\pi,m} \) as required. \( \blacksquare \)

It follows from the proof that \( (1 + \mathfrak{U}_K)/(1 + \mathfrak{U}_K^n) \cong (C_{q^n})^d \) has order \( q^{n-1} \), thus \( \text{Gal}(K_{\pi,n}/K) \) is the direct product of \( \mu_{q-1} \), which is cyclic of order prime to \( p \), by an abelian \( p \)-group.

**Definition 1.2** We define \( M_{\pi,n} \) to be the unique subextension of \( K_{\pi,n}/K \) such that \( [M_{\pi,n} : K] = q^{n-1} \).

Note that \( K_{\pi,n} = M_{\pi,n}K_{\pi,1} \) and \( \text{Gal}(M_{\pi,n}/K) \cong \text{Gal}(K_{\pi,n}/K_{\pi,1}) \) is of exponent \( p^{n-1} \). The subextensions of \( M_{\pi,n}/K \) are characterised as follows.

**Corollary 1.3** Let \( M \) be a finite abelian extension of \( K \) with \( \pi \in N(M/K) \), then \( M \subseteq M_{\pi,n} \) if and only if \( \text{Gal}(M/K) \) is of exponent dividing \( p^{n-1} \).

**Proof.** Note that since \( \pi \in N(M/K) \), \( M \subseteq K_\pi \) by Lubin-Tate theory. If \( M \subseteq M_{\pi,n} \), the result is clear by Galois theory. Suppose that \( \text{Gal}(M/K) \) is of exponent dividing \( p^{n-1} \), so that \( M/K \) is a \( p \)-extension, then \( M \) is linearly disjoint with \( K_{\pi,1} \) and the compositum \( M' = MK_{\pi,1} \) has Galois group over \( K \) isomorphic to \( \text{Gal}(M/K) \), thus \( M \subseteq M' \subseteq K_{\pi,n} \) by Proposition 1.1. The result follows since \( M/K \) is a \( p \)-extension. \( \blacksquare \)

Note also that \( N(M_{\pi,n}/K) \) is a subgroup of \( K^\times \) containing \( N(K_{\pi,n}/K) \) as a subgroup of index \( q-1 \); it follows that

\[
N(M_{\pi,n}/K) = \langle \pi \rangle \times \mu_{q-1} \times (1 + \mathfrak{U}_K^n).
\]

If \( n \geq 2 \), then \( \mathfrak{D}_K^\times \not\subseteq N(M_{\pi,n}/K) \) and \( c(M_{\pi,n}/K) \), defined as the smallest integer \( m \) such that \( 1 + \mathfrak{U}_K^n \subseteq N(M_{\pi,n}/K) \), equals \( n \). In the following, we are mostly interested in the case \( n = 2 \). We characterise \( M_{\pi,2} \) as the maximal wildly (i.e., non tamely) and weakly ramified abelian extension of \( K \) in \( K_\pi \).

**Theorem 1.4** Let \( M \) be a finite abelian extension of \( K \) with \( M \neq K \) and \( \pi \in N(M/K) \), then \( M/K \) is wildly and weakly ramified if and only if \( M \subseteq M_{\pi,2} \). In particular, \( M_{\pi,2}/K \) is weakly ramified.

**Proof.** We first show that \( M_{\pi,2}/K \) is weakly ramified. Let \( g_i \) denote the order of the \( i \)th ramification subgroup of \( \text{Gal}(M_{\pi,2}/K) \) (in the lower numbering). One has \( c(M_{\pi,2}/K) = 2 = \frac{q_0 + q_1}{q_0} \), so the first assertion follows from [6, Coro. to Lemma 7.14].
Consider $M$ as in the Proposition, and note that $M/K$ is totally ramified since $\pi \in N(M/K)$. Suppose $M \subseteq M_{\pi,2}$, then $M/K$ is a $p$-extension, hence is wildly ramified; since $M_{\pi,2}/K$ is weakly ramified, it follows from Herbrand’s theorem [15] IV.3 Prop. 14 that the same holds for $M/K$, as in the proof of [17] Prop. 2.2. Assume finally that $M/K$ is wildly and weakly ramified, namely $h_1 \neq h_2 = 1$, where $h_i$ denotes the order of the $i$th ramification subgroup of $\text{Gal}(M/K)$, hence $h_0 = h_1$ by [15] IV.2 Coro. 2 to Prop. 9. It follows that $M/K$ is a $p$-extension and that $c(M/K) = 2$, thus $\langle \pi \rangle \times \mu_{q-1} \times (1 + \mathfrak{P}_K^2) \subseteq N(M/K)$. We get that $K^\times/\langle \pi \rangle \times \mu_{q-1} \times (1 + \mathfrak{P}_K^2) \cong (1 + \mathfrak{P}_K)/(1 + \mathfrak{P}_K^2) \cong (C_p)^d$ surjects onto $K^\times/N(M/K) \cong \text{Gal}(M/K)$, which is thus of exponent $p$, and we conclude using Corollary [13].

2 Galois modules in Lubin-Tate extensions

In [10] Pickett uses $p$th roots of special values of Dwork’s exponential power series to describe specific elements in extensions $M/K$ such that: $M/K$ is weakly ramified, $[M : K] = p$ and $p \in N(M/K)$. An element $\alpha_M \in M$ is constructed such that

$$\text{Tr}_{M/K}(g(\alpha_M), h(\alpha_M)) = \delta_{g,h}$$

for all $g, h \in \text{Gal}(M/K)$, where $\delta$ is the Kronecker delta, and such that $\alpha_M$ generates $\mathcal{A}_{M/K}$, the square-root of the inverse different of $M/K$, as an integral Galois module:

$$\mathcal{A}_{M/K} = \mathfrak{D}_K[\text{Gal}(M/K)]\alpha_M .$$

Namely $\alpha_M$ is a self-dual normal basis generator for $\mathcal{A}_{M/K}$.

In this section we explain how to modify Pulita’s exponentials $E_2(X)$, described in the Introduction, to construct alternative self-dual normal basis generators for $\mathcal{A}_{M/K}$. This construction is more canonical than the former as it does not rely on a specific choice of basis of the residue field extension $k/F_p$.

2.1 A new over-convergent exponential power series

We recall that Dwork’s power series is defined as $E_\gamma(X) = \exp(\gamma X - \gamma X^p)$, where $\gamma$ is a root of the polynomial $X^{p-1} + p$. It is over-convergent and converges to a primitive $p$th root of unity when evaluated at any $z \in \mu_{p-1}$, see [9] Ch. 14 §2-3. We let $\zeta_p = E_\gamma(1)$ and note that $K(\gamma)/K$ is Kummer of degree $p - 1$, totally ramified with uniformising parameter $\gamma$. We also have $K(\gamma) = K(\zeta_p)$ and we will now denote this field by $K'$.

**Definition 2.1** For each $u \in \mu_{q-1}$, we define $f_u(X) \in \mathfrak{D}_K[X]$ as

$$f_u(X) = X^p + upX .$$
and let $\omega_u$ be a root of the Eisenstein polynomial $f_u(X) - \gamma \in O_{K'}[X]$. We then let $L_u = K'(\omega_u)$ and define

$$E_{u,2}(X) = \exp \left( \frac{\omega_u X - u\omega_u X^p + \gamma X^p - \gamma X^{p^2}}{p} \right).$$

Note that $\omega_u$ is also a root of the Eisenstein polynomial $f_u(X)^{p-1} + p$ with coefficients in $O_K$, so $\omega_u$ generates $L_u$ over $K$ as well, namely $L_u = K(\omega_u)$.

In the case $u = 1$, we see that $E_{1,2}(X)$ is equal to Pulita’s exponential $E_2(X)$ as described in the Introduction, with $f(X) = f_1(X)$. Therefore, from [12, Theorem 2.5], we know $E_{1,2}(X)$ is over-convergent. We deduce the following:

**Theorem 2.2** For each $u \in \mu_{q-1}$, the power series $E_{u,2}(X)$ is over-convergent.

**Proof.** For $u \in \mu_{q-1}$, we consider the power series

$$A_u(X) = \exp \left( \frac{f_1^2(\omega_u X)}{p^2} \right)$$

and note that $A_u(X) = A_1(\frac{\omega_u}{\omega_1} X)$ is over-convergent if and only if $A_1(X)$ is over-convergent. We note that $\gamma^{p-1} = -p$ and $\omega_u^p = \gamma - up\omega_u$; we make the following derivation:

$$f_1^2(\omega_u X) = (\omega_u^p X^p + p\omega_u X)^p + p(\omega_u^p X^p + p\omega_u X) = ((\gamma - up\omega_u)X^p + p(\gamma - up\omega_u)X^p + p\omega_u X) = (\gamma X^p + p\omega_u(X - uX^p)) + p(\gamma X^p + p\omega_u(X - uX^p)) = \gamma^p X^{p^2} + p^3 B_u(X) + p^2 \omega_u(X - uX^p)$$

for some $B_u(X) \in O_K[\omega_u][X]$. Therefore,

$$A_u(X) = \exp \left( \omega_u X - \omega_u X^p + \gamma X^p - \gamma X^{p^2} \right) + pB_u(X).$$

The power series $\exp(pB_u(X))$ is over-convergent and therefore $E_{u,2}(X)$ is over-convergent if and only if $A_u(X)$ is over-convergent. From [12, Theorem 2.5] we know that $E_{1,2}(X)$ is over-convergent, which implies $A_1(X)$ and thus $A_u(X)$ are over-convergent. Therefore, $E_{u,2}(X)$ is over-convergent.

\[\blacksquare\]

### 2.2 Kummer and self-dual normal basis generators

**Theorem 2.3** Let $v \in \mu_{q-1}$ and set $u = v^{1-p}$, then $E_{u,2}(v)$ is a Kummer generator of $L_u/K'$; further the extension $L_u/K$ is cyclic of degree $p(p - 1)$. 

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Proof. A straightforward calculation shows that:

\[ \mathcal{E}_{u,2}(X)^p = \exp(p\omega_u X) \exp(p\omega_u u X^p)^{-1} E_\gamma(X^p), \]

where both \( \exp(p\omega_u X) \) and \( \exp(p\omega_u u X^p) \) are over-convergent. Further, \( v = uv^p \), and so \( \mathcal{E}_{u,2}(v)^p = E_\gamma(v^p) \), which clearly belongs to \( K' = K(\gamma) \).

By the properties of Dwork’s power series one has \( E_{\gamma}(v^p) \equiv 1 + v^p\gamma \) mod \( \gamma^2 \mathcal{O}_K \), so \( E_{\gamma}(v^p) \in 1 + \mathfrak{P}_K' \setminus 1 + \mathfrak{P}_K^{2p} \). Note that \( K' = \langle \gamma \rangle \times \mu_{q-1} \times 1 + \mathfrak{P}_K' \) and that \( (1 + \mathfrak{P}_K')^p \subseteq 1 + \mathfrak{P}_K^{2p} \), hence \( E_{\gamma}(v^p) \notin (K'^\times)^p \). It follows that \( K'(\mathcal{E}_{u,2}(v)) / K \) is Kummer of degree \( p \).

Since clearly \( K'(\mathcal{E}_{u,2}(v)) \subseteq K(\omega_u) = L_u \), a comparison of the degrees then gives equality and it follows that \( L_u / K \) is cyclic of degree \( p \).

We now show that \( L_u / K \) is Galois. Let \( \delta \in \text{Gal}(K'/K) \), then \( \delta : \gamma \mapsto z\gamma \) for some \( z \in \mu_{p-1} \). Any \( w \in \mathcal{O}_K \) is fixed by \( \delta \) and contained in the disc of convergence of \( E_{\gamma}(X) = \exp(\gamma X - \gamma X^p) = \sum_{n=0}^{+\infty} e_n X^n \). One has:

\[
\sum_{n=0}^{+\infty} \delta(e_n) X^n = \exp(\delta(\gamma)X - \delta(\gamma)X^p) = \exp(\gamma(zX) - \gamma(zX^p)) = E_{\gamma}(zX),
\]

which implies

\[
\delta(E_\gamma(w)) = \delta \left( \sum_{n=0}^{+\infty} e_n w^n \right) = \sum_{n=0}^{+\infty} \delta(e_n) w^n = E_{\gamma}(zw) = E_{\gamma}(w)^z,
\]

using [11, Lemma 4.2]. Let \( \tilde{L}_u \) be the Galois closure of \( L_u / K \) and let \( \tilde{\delta} \in \text{Gal}(\tilde{L}_u / K) \) be such that \( \tilde{\delta}|_{K'} = \delta \). We observe that

\[
\tilde{\delta}(\mathcal{E}_{u,2}(v))^p = \tilde{\delta}(\mathcal{E}_{u,2}(v)^p) = \delta(E_{\gamma}(v^p)) = E_{\gamma}(v^p)^z = \mathcal{E}_{u,2}(v)^{p^z}.
\]

We must then have, for some \( \zeta \in \mu_p \):

\[
\tilde{\delta}(\mathcal{E}_{u,2}(v)) = \zeta \mathcal{E}_{u,2}(v)^z, \tag{2}
\]

which means that all the Galois conjugates of \( \mathcal{E}_{u,2}(v) \) over \( K \) are contained in \( L_u \), and so \( L_u = K(\mathcal{E}_{u,2}(v)) \) is Galois over \( K \).

Each \( \delta \in \text{Gal}(K'/K) \) has \( p \) liftings in \( \text{Gal}(L_u / K) \), given by the various choices of the \( p \)-th root of unity \( \zeta \) in \( \langle 2 \rangle \). Suppose \( \delta \) generates \( \text{Gal}(K'/K) \) and let \( \tilde{\delta} \) now denote the lifting of \( \delta \) corresponding to \( \zeta = 1 \), then \( \tilde{\delta} \) generates a subgroup of \( \text{Gal}(L_u / K) \) of order \( p - 1 \). Recall that \( \zeta_p \) is a primitive \( p \)-th root of unity, and note that \( \tilde{\delta}(\zeta_p^z) = \zeta_p^z \) from [10, Lemma 10]. Let \( \phi \) denote the generator of \( \text{Gal}(L_u / K') \) defined by \( \phi(\mathcal{E}) = \zeta_p \mathcal{E} \), where we let \( \mathcal{E} = \mathcal{E}_{u,2}(v) \) for ease of notation. Then \( \phi \circ \tilde{\delta}(\mathcal{E}) = \zeta_p^z \mathcal{E}^z = \tilde{\delta} \circ \phi(\mathcal{E}) \), namely \( \text{Gal}(L_u / K) \) is abelian, hence the result.

The description of the extensions \( L_u / K \) given in Theorem 2.3 is only available when \( u \in \mu_{q-1} \) is a \((p-1)\)-th power, namely when \( u \) belongs to the subgroup \( (\mu_{q-1})^{p-1} = \mu_{q-1}^{\frac{1}{p-1}} \).
of \(\mu_{q-1}\). If this is not the case, we are not able to prove that \(L_u/K’\) is Galois. However, we shall see below that the extensions described in the theorem are the only ones of interest to us, see Remark \[3.2\].

From now on, we thus restrict ourselves to the units \(u \in (\mu_{q-1})^{p-1}\). It amounts to the same as restricting to the units \(v \in \mu_{q-1}/\mu_{p-1}\), since \(v \mapsto v^{1-p}\) yields an isomorphism between \(\mu_{q-1}/\mu_{p-1}\) and \((\mu_{q-1})^{p-1}\). We remark that \(\sigma\) acts on both these groups and commutes with this isomorphism. Let \(\mathbb{P}(k) = k^x/\mathbb{F}_p^x\) denote the projective space of \(k\) considered as an \(\mathbb{F}_p\)-vector space of dimension \(d\); in the following we identify \(\mathbb{P}(k)\) with \(\mu_{q-1}/\mu_{p-1}\) through the Teichmüller lifting.

**Definition 2.4** For any \(u \in (\mu_{q-1})^{p-1}\), we let \(M_u\) be the unique subextension of \(L_u/K\) of degree \(p\).

**Proposition 2.5** Let \(u \in (\mu_{q-1})^{p-1}\), then \(M_u\) is the splitting field of \(\psi_u(X) = X(X + up)^{p-1} + p\) and
\[
M_u = K(\omega_u^{p-1}) \subseteq M_{p,2}.
\]

**Proof.** By Theorem \[2.3\] \(L_u = K(\omega_u)\) is the splitting field over \(K\) of the polynomial
\[
f_1(f_u(X)) = (X^p + upX)^p + p(X^p + upX).
\]
Therefore \(L_u\) also splits the Eisenstein polynomial \(X^{p-1}(X^{p-1} + up)^{p-1} + p\) which is the minimum polynomial of \(\omega_u\) over \(K\). This implies that the minimum polynomial of \(\omega_u^{p-1}\) over \(K\) is \(\psi_u(X) = X(X + up)^{p-1} + p\). The equality \(M_u = K(\omega_u^{p-1})\) now follows as this polynomial is of degree \(p\) and \(L_u/K\) has only one subextension of degree \(p\). Further, \(N_{M_u/K}(\omega_u^{p-1}) = p\) and \(\text{Gal}(M_u/K)\) has exponent \(p\), therefore \(M_u \subseteq M_{p,2}\) from Corollary \[1.3\].

It follows that \(M_u/K\) is weakly ramified by Proposition \[1.4\] hence its square-root of the inverse different admits a self-dual normal basis generator, by \[2. Theorem 1\] and \[3. Corollary 4.7\].

**Theorem 2.6** Let \(v \in \mathbb{P}(k)\), set \(u = v^{1-p} \in (\mu_{q-1})^{p-1}\), and let \(\mathcal{A}_{M_u/K}\) denote the unique fractional ideal of \(\mathcal{O}_{M_u}\) whose square is equal to the inverse different of \(M_u/K\). Set \(S = \mu_{p-1} \cup \{0\}\), then:
\[
\alpha_u = \frac{1 + Tr_{L_u/M_u}(\mathcal{E}_{u,2}(v))}{p} = \sum_{s \in S} \frac{\mathcal{E}_{u,2}(v)^s}{p}
\]
is a self-dual integral normal basis generator for \(\mathcal{A}_{M_u/K}\).

**Proof.** As in the proof of the previous theorem we know that \(\text{Gal}(L_u/M_u) = \langle \tilde{\delta} \rangle\), where \(\tilde{\delta}(\mathcal{E}_{u,2}(v)) = \mathcal{E}_{u,2}(v)^2\) for some primitive \((p-1)\)th root of unity \(z\). This proves the equivalence of the presentations of \(\alpha_u\).

In order to use the methods in \[10\] we highlight the following three properties of \(\mathcal{E}_{u,2}(v)\):

1. \(\mathcal{E}_{u,2}(v)\) is a Kummer generator of \(L_u/K’\).
2. From [9, Ch. 14 §2] we know that $E_\gamma(v^p) = 1 + v^p \gamma \mod \gamma^2$. Therefore, $E_\gamma(v^p) - 1 = \mathcal{E}_{u,2}(v)^p - 1$ is a uniformising parameter of $K'$.

3. The involution element of $\text{Gal}(L_u/M_u)$ maps $\mathcal{E}_{u,2}(v)$ to $\mathcal{E}_{u,2}(v)^{-1}$.

The proof that $\alpha_u$ generates a self-dual normal basis for $\mathcal{A}_{M_u/K}$ now follows the exact method of proof of [10 Theorem 12].

3 The degree $p$ subextensions of $M_{p,2}/K$

Recall that $M_{p,2}$ is the unique subfield of $K_{p,2}$ such that $[M_{p,2} : K] = q$, with norm group:

$$N(M_{p,2}/K) = \langle \pi \rangle \times \mu_{q-1} \times 1 + \mathfrak{P}_K^2.$$ 

Specifically $M_{p,2} = M_{w_{p,2}}$ for all $w \in \mu_{q-1}$. Since $\text{Gal}(M_{p,2}/K) \cong (C_p)^d$, we know that $M_{p,2}/K$ has $\frac{q-1}{p-1} = 1 + p + \cdots + p^{d-1}$ subextensions of degree $p$. Recall that for each $u \in (\mu_{q-1})^{p-1}$ we have defined an extension $M_u/K$ of degree $p$, such that $M_u \subseteq M_{p,2}$, see Proposition 2.5.

We now fix $u \in (\mu_{q-1})^{p-1}$ and calculate the norm group $N(M_u/K)$ of $M_u/K$. As $M_u \subseteq M_{p,2}$, from standard local class field theory we know that

$$N(M_{p,2}/K) = \langle p \rangle \times \mu_{q-1} \times 1 + \mathfrak{P}_K^2 \subseteq N(M_u/K). \quad (3)$$

Furthermore, the Artin map $\theta$ of $K$ yields the isomorphism:

$$K^\times /N(M_u/K) \cong \text{Gal}(M_u/K) \cong C_p. \quad (4)$$

We introduce the $\mathbb{Z}_p$-module $Z$ defined by:

$$Z = \{ x \in \mathcal{O}_K : Tr_{K/Q_p}(x) \in p\mathbb{Z}_p \}.$$ 

Note that $Z \subseteq \mathcal{O}_K$ since $K/Q_p$ is unramified. Further $\mathfrak{P}_K \subseteq Z$, but $Z$ needs not be an ideal of $\mathcal{O}_K$. In fact the next result implies that $[\mathcal{O}_K : Z] = p$, whereas $[\mathcal{O}_K : \mathfrak{P}_K] = p^d$, hence $\mathfrak{P}_K \not\subseteq Z$.

We recall from above that as $p \geq 3$ and $K/Q_p$ is unramified, for any $n \in \mathbb{N}$ the power series $\exp$ and $\log$ give us group isomorphisms between $1 + \mathfrak{P}_K^n$ with the multiplicative structure and $\mathfrak{P}_K^n$ with the additive structure.

**Theorem 3.1** Let $v \in \mathbb{P}(k)$ then

$$N(M_{v_{1-p}}/K) = \langle p \rangle \times \mu_{q-1} \times \exp(pv^{-p}Z).$$

As a consequence, the set $\{ M_{v_{1-p}} : v \in \mathbb{P}(k) \}$ gives a complete set of intermediate fields of $M_{p,2}/K$ of degree $p$ over $K$. Further let $v, v_1 \neq v_2 \in \mathbb{P}(k)$, then $M_{v_{1-p}} \subseteq M_{v_1^{-p}M_{v_2^{-p}}}$ if and only if $v$ belongs to the line of $\mathbb{P}(k)$ through $v_1$ and $v_2$.  

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Remark 3.2 Before proving this theorem we address a question posed in [44, Remark 4.4]. Namely, it was stated that it would be desirable to describe a generating set for the degree \( p \) extensions of \( K' \) contained in \( K_{p,2} \) that does not depend on the choice of a basis of \( k \) over \( \mathbb{Z}_p / p\mathbb{Z}_p \). This theorem, together with Theorem 2.3, shows that \( \{ E_{v_1-p,2}(v) : v \in \mathbb{P}(k) \} \) is such a set.

Proof. Set \( u = v^{1-p} \), so \( u \) is well defined in \((\mu_{q-1})^{p-1}\). We already know by equality (3) that \( \langle p \rangle \times \mu_{q-1} \subseteq N(M_u/K) \). We first show that \( \exp(pv^{-p}Z) \subseteq N(M_u/K) \), then that \( \langle p \rangle \times \mu_{q-1} \times \exp(pv^{-p}Z) \) is of index \( p \) in \( K^\times \). This is enough to obtain equality using (4).

We let \( \eta \in \mu_{q-1} \) be such that \( \eta \mod p \) generates a normal basis of \( k/\mathbb{F}_p \). It yields the following decompositions of \( \mathcal{O}_K \) and \( Z \).

Lemma 3.3 One has the decompositions: \( \mathcal{O}_K = \eta\mathbb{Z}_p \oplus \bigoplus_{i=1}^{d-1} (\eta^{p^i} - \eta^{p^{i+1}})\mathbb{Z}_p \) and \( Z = p\eta\mathbb{Z}_p \oplus \bigoplus_{i=1}^{d-1} (\eta^{p^i} - \eta^{p^{i+1}})\mathbb{Z}_p \).

Proof. Since \( K/\mathbb{Q}_p \) is unramified, the set of conjugates of \( \eta \) under \( \Sigma \) forms a basis \((\eta^{p^i})_{0 \leq i \leq d-1}\) for \( \mathcal{O}_K \) over \( \mathbb{Z}_p \). A determinant computation shows that \( (\eta, \eta^p - \eta^{p^2}, \eta^{p^2} - \eta^p, \ldots, \eta^{p^{d-1}} - \eta) \) is also a basis of \( \mathcal{O}_K \) over \( \mathbb{Z}_p \), hence the decomposition of \( \mathcal{O}_K \). Any element of \( \bigoplus_{i=1}^{d-1} (\eta^{p^i} - \eta^{p^{i+1}})\mathbb{Z}_p \) has zero trace (this is in fact the kernel of the trace restricted to \( \mathcal{O}_K \)), so \( Z \supseteq p\eta\mathbb{Z}_p \oplus \bigoplus_{i=1}^{d-1} (\eta^{p^i} - \eta^{p^{i+1}})\mathbb{Z}_p \). Suppose \( x = \sum_{i=0}^{d-1} x_i \eta^{p^i} \in Z \), where each \( x_i \in \mathbb{Z}_p \), then \( Tr_{K/\mathbb{Q}_p}(x) = (\sum_{j=0}^{d-1} x_j) Tr_{K/\mathbb{Q}_p}(\eta) \in p\mathbb{Z}_p \).

Let \( \overline{y} \) denote the reduction modulo \( p\mathcal{O}_K \) of \( y \in \mathcal{O}_K \), then \( \overline{Tr_{K/\mathbb{Q}_p}(\eta)} = Tr_{k/\mathbb{F}_p}(\overline{\eta}) \neq 0 \), since \( \overline{\eta} \) generates a normal basis, hence \( p \) divides \( \sum_{j=0}^{d-1} x_j \). Therefore,

\[
x = \left( \sum_{j=0}^{d-1} x_j \right) \eta + \sum_{i=1}^{d-1} \left( \sum_{j=0}^{d-1} x_j \right) (\eta^{p^i} - \eta^{p^{i+1}}) \in p\eta\mathbb{Z}_p \oplus \bigoplus_{i=1}^{d-1} (\eta^{p^i} - \eta^{p^{i+1}})\mathbb{Z}_p
\]

as required. ■

We get \( \exp(pv^{-p}Z) = \exp(p^2v^{-p}\eta\mathbb{Z}_p) \times \prod_{i=1}^{d-1} \exp(pv^{-p}(\eta^{p^i} - \eta^{p^{i+1}})\mathbb{Z}_p) \). Further \( p^2v^{-p}\eta\mathbb{Z}_p \subseteq \mathfrak{P}_K^2 \) hence \( \exp(p^2v^{-p}\eta\mathbb{Z}_p) \subseteq N(M_u/K) \) thanks to (4). The inclusion \( \exp(pv^{-p}Z) \subseteq N(M_u/K) \) will thus be a consequence of the next result.

Lemma 3.4 Let \( w \in \mathbb{K}^\times \), then the multiplicative subgroup of \( 1 + \mathfrak{P}_K \) given by \( \exp(pv^{-p}(w - w^p)\mathbb{Z}_p) \) is a subgroup of \( N(M_u/K) \).

Proof. Let \( x \in \mathbb{Z}_p \), we then have

\[
\exp(pv^{-p}(w - w^p)x) \equiv 1 + pv^{-p}(w - w^p)x \mod \mathfrak{P}_K^2
\]

and as subgroups of \( 1 + \mathfrak{P}_K \),

\[
\exp(pv^{-p}(w - w^p)\mathbb{Z}_p)(1 + \mathfrak{P}_K^2) = \langle 1 + pv^{-p}(w - w^p) \rangle (1 + \mathfrak{P}_K^2).
\]
We know that $1 + \mathfrak{P}_K^2$ is a subgroup of $N(M_u/K)$, therefore it is now sufficient to show that some element congruent to $1 + pv^p(w - w^p) \mod \mathfrak{P}_K^2$ belongs to $N(M_u/K)$. We see that

$$\psi_v\left(\frac{-u}{w}\right) = \frac{-u}{w} + up\right)^{p-1} + p$$

and therefore the norm of $1 + \omega_u^{-1}wv^{-1}$ will equal

$$\frac{-u}{w} + up\right)^{p-1} + p \equiv -(\frac{-u}{w})^p + p(1 - \frac{1}{w^{p-1}}) \mod \mathfrak{P}_K^2.$$  

We then see that

$$N_{M_u/K}\left(-\frac{w}{v}(1 + \omega_u^{-1}wv^{-1})\right) = -(\frac{w}{v})^p(N_{M_u/K}(1 + \omega_u^{-1}wv^{-1})$$

$$\equiv 1 + pv^p(w - w^p) \mod \mathfrak{P}_K^2,$$

which proves the result. ■

We now compute the index $I$ of $\langle p \rangle \times \mu_{q-1} \times \exp(pv^{-p}Z)$ in $K^\times = \langle p \rangle \times \mu_{q-1} \times 1 + \mathfrak{P}_K$. One has:

$$I = [\exp(\mathfrak{P}_K) : \exp(pv^{-p}Z)] = [\mathfrak{P}_K : pv^{-p}Z] = [\mathcal{O}_K : Z] = p,$$

using the decompositions in Lemma 3.3. This completes the norm group computation.

The extension $M_{v^1}$ of $K$ only depends on the class of $v \in \mu_{q-1} \bmod \mu_{p-1}$, so the set $\{M_{v^1} : v \in \mu_{q-1}\}$ contains at most $\frac{q-1}{p-1}$ extensions of $K$. This is the number of subextensions of $M_{p,2}/K$ of degree $p$, so we are left with showing that $M_{v^1} \neq M_{w^1}$ whenever $v \neq w$ in $\mathbb{P}(k)$. This is equivalent to showing that the norm groups of these extensions are different, namely that $v^{-p}Z \neq w^{-p}Z$ whenever $v \neq w$ in $\mathbb{P}(k)$.

For $x \in \mathcal{O}_K$, we denote by $\overline{x}$ its reduction modulo $p\mathcal{O}_K$.

**Lemma 3.5** Let $v \in \mu_{q-1}$, then

$$v^{-p}Z = \{x \in \mathcal{O}_K : Tr_{k/\mathbb{F}_p}(v^p\overline{x}) = 0\},$$

so the reduction of $v^{-p}Z$ modulo $p$ is the kernel of the linear form $y \mapsto Tr_{k/\mathbb{F}_p}(v^p y)$ on the $\mathbb{F}_p$-vector space $k$.

**Proof.** Let $x \in \mathcal{O}_K$, then $x \in v^{-p}Z$ if and only if $Tr_{K/\mathbb{Q}_p}(v^p x) = Tr_{k/\mathbb{F}_p}(v^p \overline{x}) = 0$, which yields the result. ■

The trace form $T(v, v') = Tr_{k/\mathbb{F}_p}(vv')$ is a non degenerate bilinear pairing on $k \times k$, so it induces an isomorphism $v \mapsto T(v, \cdot)$ between $k$ and its dual. It follows that, for $v, v' \in k^\times$, $T(v, \cdot)$ and $T(v', \cdot)$ have the same kernel if and only if $v/v' \in \mathbb{F}_p^\times$. Suppose $v \neq w$ in $\mathbb{P}(k)$ then the same holds for $v^p$ and $w^p$, so $v^{-p}Z$ and $w^{-p}Z$ are different modulo $p$, hence are different. It follows that $\{M_{v^1} : v \in \mathbb{P}(k)\}$ is the complete set of subextensions of degree $p$ of $M_{p,2}/K$.

By local class field theory one has $M_{v^1} \subset M_{v^1}M_{v^1}M_{v^1}$ if and only if $N(M_{v^1}/K) \supset N(M_{v^1}/K) \cap N(M_{v^1}/K)$, namely if and only if $v^{-p}Z \supset v^{-p}Z \cap v^{-p}Z$, where with a slight
abuse of notation we now see \( v, v_1, v_2 \) as elements of \( \mu_{q-1} \) (since any of their liftings can equally be chosen). By Lemma 3.5, this is also equivalent to

\[
\forall w \in k, \quad Tr_{k/F}(v^p w) = 0 = Tr_{k/F}(v_1^p w) \Rightarrow Tr_{k/F}(v^p w) = 0.
\]

This implication is clearly true if there exist \( a, b \in F_p \) such that \( v = av_1 + bv_2 \), namely if \( v \) (mod \( \mu_{p-1} \)) belongs to the line of \( P(k) \) through \( v_1 \) (mod \( \mu_{p-1} \)) and \( v_2 \) (mod \( \mu_{p-1} \)). Since there are exactly \( p + 1 \) points on this line, which all correspond to a different subextension of \( M_{v_1^{-p}}M_{v_2^{-p}} \) of degree \( p \), and that there are exactly \( p + 1 \) such subextensions, we see that \( M_{v_1^{-p}} \subset M_{v_1^{-p}}M_{v_2^{-p}} \) implies that \( v \) belongs to the line of \( P(k) \) through \( v_1 \) and \( v_2 \). We have therefore proved the declared equivalence.

This ends the proof of Theorem 3.1.

**Corollary 3.6** Let \( v \in P(k) \) and set \( u = v^{1-p} \), then

\[
\text{Gal}(M_u/K) = \langle \theta_{M_u/K}(\exp(\eta v^{-p}p)) \rangle = \langle \theta_{M_u/K}(1 + \eta v^{-p}p) \rangle.
\]

**Proof.** Restricting the elements in \( \text{Gal}(K^{ab}/K) \) to their action on \( M_u \) gives the short exact sequence:

\[
1 \to N(M_u/K) \to K^\times \xrightarrow{\theta_{M_u/K}} \text{Gal}(M_u/K) \to 1.
\]

Using the decompositions in Lemma 3.3 we see that

\[
K^\times /N(M_u/K) \cong \exp(\eta v^{-p}p\mathbb{Z}_p) / \exp(\eta v^{-p}p^2\mathbb{Z}_p)
\]

which proves the first equality. The second equality comes from the fact that \( \langle \exp(\eta v^{-p}p) \rangle(1 + \mathfrak{P}_K) = \langle 1 + \eta v^{-p}p \rangle(1 + \mathfrak{P}_K^2) \) and \( (1 + \mathfrak{P}_K^2) \subseteq N(M_u/K) \).

4 The absolute extension \( M_{p,2}/\mathbb{Q}_p \)

**Proposition 4.1** Suppose \([K : \mathbb{Q}_p] > 1\), then \( M_{p,2}/\mathbb{Q}_p \) is a non-abelian weakly ramified extension of local fields.

**Proof.** We first check that \( M_{p,2} \) is Galois over \( \mathbb{Q}_p \). The polynomial \( h(X) = X^q + pX \) is a Lubin-Tate polynomial over \( \mathcal{O}_K \) for uniformising parameter \( p \). The extension \( K_{p,2}/K \) is abelian, of degree \( q(q-1) \) and contains all the roots of the polynomial \( h(h(X)) \). The polynomial

\[
H(X) = h(h(X))/h(X) = (X^q + pX)^{q-1} + p
\]

is Eisenstein, and so irreducible, of degree \( q(q-1) \) and therefore \( K_{p,2}/K \) is the splitting field of \( H(X) \). With a straightforward rearrangement we see that

\[
H(X) = X^{q-1}(X^{q-1} + p) + p = \tilde{H}(X^{q-1})
\]
where $\tilde{H}(X) = X(X + p)^{q-1} + p$ is also Eisenstein, hence irreducible, and of degree $q$. As $M_{p,2}$ is the unique subfield of $K_{p,2}$ of degree $q$ over $K$ it must be the splitting field of $\tilde{H}(X)$.

Both $H$ and $\tilde{H}$ have coefficients invariant under the action of $\text{Gal}(K/\mathbb{Q}_p)$, therefore the extensions $K_{p,2}/\mathbb{Q}_p$ and $M_{p,2}/\mathbb{Q}_p$ must be Galois: let $\omega$ denote a root of $H$, so that $K_{p,2} = K(\omega)$, and let $\tilde{\sigma}$ be a lifting of $\sigma$ in the Galois group of the Galois closure of $K_{p,2}$ over $\mathbb{Q}_p$, then $H(\tilde{\sigma}(\omega)) = \tilde{\sigma}(H(\omega)) = 0$. But $K_{p,2}$ splits $H$, so $\tilde{\sigma}(\omega) \in K_{p,2}$, which therefore is normal over $\mathbb{Q}_p$. The same argument applies to $M_{p,2}$.

Since $K/\mathbb{Q}_p$ is unramified, it now follows from Proposition 1.4 that $M_{p,2}/\mathbb{Q}_p$ is weakly ramified. If $d = [K : \mathbb{Q}_p] > 1$, $M_{p,2}/\mathbb{Q}_p$ can not be abelian since otherwise its ramification index would equal $p$ by [17 Théorème 1.1], and $M_{p,2}/K$ is totally ramified of degree $p^d$. ■

The fact that $M_{p,2}/\mathbb{Q}_p$ is Galois can also be shown by considering that $N(M_{p,2}/K)$ is invariant under $\text{Gal}(K/\mathbb{Q}_p)$.

Let $\eta \in \mu_{q-1}$ be such that $\eta \mod p$ is a normal basis generator of $k/F_p$. Since $K/\mathbb{Q}_p$ is unramified, the set of conjugates of $\eta$ under $\Sigma$ forms a basis $(\eta^{p^i})_{0 \leq i \leq d-1}$ for $\mathfrak{O}_K$ over $\mathbb{Z}_p$.

This means that:

$$\mathfrak{P}_K = p\mathfrak{O}_K = p \sum_{i=0}^{d-1} \eta^{p^i} \mathbb{Z}_p = \sum_{i=0}^{d-1} \eta^{p^i} p \mathbb{Z}_p .$$

**Lemma 4.2** One has: $(1 + \mathfrak{P}_K)/(1 + \mathfrak{P}^2_K) = \left( \prod_{i=0}^{d-1} (1 + \eta^{p^i} p) \right) / (1 + \mathfrak{P}^2_K) .$

**Proof.** The inclusion “$\supset$” is clear, so let us prove “$\subset$”. In view of the above decomposition of $\mathfrak{P}_K$, any $y \in 1 + \mathfrak{P}_K$ can be written as $y = 1 + p \sum_{i=0}^{d-1} y_i \eta^{p^i}$, with $y_i \in \mathbb{Z}_p$, so $y \equiv \prod_{i=0}^{d-1} (1 + py_i \eta^{p^i}) \equiv \prod_{i=0}^{d-1} (1 + py^{p^i} y_i) \mod p^2$. Thus there exists $\alpha \in \mathfrak{O}_K$ such that

$$\prod_{i=0}^{d-1} (1 + py^{p^i} y_i) = y + p^2 \alpha = y(1 + p^2 \alpha y^{-1}) ,$$

and the result follows. ■

Note that $\eta^{p^i}$ only depends on $i$ modulo $d$, so we let $g_i = \theta_{M_{p,2}/K}(1 + \eta^{p^i} p)$ for $i \in \mathbb{Z}/d\mathbb{Z}$. As the Galois group of $M_{p,2}/K$ is isomorphic, via the Artin map, to $(1 + \mathfrak{P}_K)/(1 + \mathfrak{P}^2_K)$, we get:

$$\text{Gal}(M_{p,2}/K) = \langle g_i : i \in \mathbb{Z}/d\mathbb{Z} \rangle = \prod_{i \in \mathbb{Z}/d\mathbb{Z}} \langle g_i \rangle . \quad (5)$$

This gives an explicit description of the isomorphism $\text{Gal}(M_{p,2}/K) \cong (C_p)^d$.

We now describe explicitly the Galois group of $M_{p,2}/\mathbb{Q}_p$. We let $\tilde{\sigma} \in \text{Gal}(K_{p,2}/\mathbb{Q}_p)$ be some lifting of $\sigma$, i.e., $\tilde{\sigma}|_K = \sigma$. Let again $\omega$ be a root of $H(X)$, we then have $K_{p,2} = K(\omega)$ and $M_{p,2} = K(\omega^{q-1})$, so $\tilde{\sigma}$ is determined by its value on $\omega$. We have seen that $\tilde{\sigma}(\omega)$ is one of the roots of $H$. It follows that taking the value at $\omega$ yields a one-to-one correspondence between the liftings of $\sigma$ and the roots of $H$. Consequently, we can pick $\tilde{\sigma}$ such that $\tilde{\sigma}(\omega) = \omega$.  

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Theorem 4.3 Let $\tilde{\sigma} \in \text{Gal}(K_{p,2}/\mathbb{Q}_p)$ be such that $\tilde{\sigma}(\omega) = \omega$ and $\tilde{\sigma}|_K = \sigma$, and consider its restriction, denoted the same, to $M_{p,2}$. With the definitions above,

$$\text{Gal}(M_{p,2}/\mathbb{Q}_p) = \langle \tilde{\sigma}, g_i : i \in \mathbb{Z}/d\mathbb{Z} \rangle.$$ 

Further, $\tilde{\sigma} \circ g_i = g_{i+1} \circ \tilde{\sigma}$ and $\text{Gal}(M_{p,2}/\mathbb{Q}_p)$ is isomorphic to the regular wreath product $C_p \wr C_d$.

For a full definition of wreath products see, for example, [13, Ch.7].

Proof. We first let $\tilde{g}_i = \theta_{K_{p,2}/K}(1 + \eta^p i)$ and note that $g_i = \tilde{g}_i|_{M_{p,2}}$. We know that $1 + \mathfrak{P}_K^2 \subseteq N(K_{p,2}/K)$, and so $\theta_{K_{p,2}/K}(1 + \eta^p i) = \theta_{K_{p,2}/K}((1 + \eta^p i)a)$ for any $a \in 1 + \mathfrak{P}_K$. Clearly $(1 + \eta^p i)^{-1} \equiv (1 - \eta^p i) \mod (1 + \mathfrak{P}_K^2)$, so from standard local class field theory we have

$$\tilde{g}_i(\omega) = \theta_{K_{p,2}/K}(1 + \eta^p i)(\omega) = [1 - \eta^p i](\omega),$$

where $[1 - \eta^p i](X)$ is the Lubin-Tate power series associated to the polynomial $h(X)$ and the unit $(1 - \eta^p i) \in O_K^\times$ (see [14, §3.3 Prop.2] and [14, §3.4 Theorem 3(c)]). Considering the power series expansion of $[1 - \eta^p i](X)$ we see that

$$\tilde{\sigma} \circ \tilde{g}_i(\omega) = \tilde{\sigma}([1 - \eta^p i](\omega)) = [1 - \sigma(\eta^p i)][\tilde{\sigma}(\omega)] = [1 - \eta^{p+1} i](\omega) = \tilde{g}_{i+1}(\omega) = \tilde{g}_{i+1} \circ \tilde{\sigma}(\omega).$$

We must therefore also have $\tilde{\sigma} \circ g_i(\omega^{q^{-1}}) = g_{i+1} \circ \tilde{\sigma}(\omega^{q^{-1}})$.

As $g_i$ acts trivially on $\eta$ for all $i$, we clearly have $g_i \circ \tilde{\sigma}(\eta) = \tilde{\sigma} \circ g_{i+1}(\eta)$. It now follows that $g_i \circ \tilde{\sigma} = \tilde{\sigma} \circ g_{i+1}$, as $M_{p,2} = \mathbb{Q}_p(\omega^{q^{-1}}, \eta)$.

Finally we obtain the isomorphism $\text{Gal}(M_{p,2}/\mathbb{Q}_p) \cong C_p \wr C_d$ by comparing cardinalities.

We finish our study of $K_{p,2}/\mathbb{Q}_p$ by describing the action of the element $\tilde{\sigma} \in \text{Gal}(K_{p,2}/\mathbb{Q}_p)$ on the subfields $L_u$ and $M_u$.

Proposition 4.4 With $v \in \mathbb{P}(k)$ and $u = v^{1-p}$, we have

$$\tilde{\sigma}(L_u) = L_{\sigma(u)} \quad \text{and} \quad \tilde{\sigma}(M_u) = M_{\sigma(u)}.$$ 

Moreover, $M_u/\mathbb{Q}_p$ is Galois if and only if $\sigma(u) = u^p = u$.

Proof. We recall that $L_u = K(\omega_u)$ is the splitting field of the minimum polynomial of $\omega_u$ over $K$, $(X^p + u^p X)^{p-1} + p$. We have

$$(\omega_u^p + up\omega_u)^{p-1} + p = 0,$$
and so
\[(\sigma(\omega_u))^p + \sigma(u)p\sigma(\omega_u))^{p-1} + p = 0 .\]

This means that \(\sigma(\omega_u)\) is a root of \((X^p + \sigma(u)pX)^{p-1} + p\) and as \(L_{\sigma(u)}\) is the splitting field of this polynomial, we must have \(L_{\sigma(u)} = K(\sigma(\omega_u)) = \sigma(L_u)\). The fact that \(\sigma(M_u) = M_{\sigma(u)}\) follows from the same reasoning using \(\omega_u^{p-1}\) and its minimum polynomial in the place of \(\omega_u\).

As \(M_{p,2}/\mathbb{Q}_p\) is Galois and \(M_u \subseteq M_{p,2}\), all embeddings of \(M_u\) into \(\bar{\mathbb{Q}}_p\) will be obtained by restricting the action of elements of \(\text{Gal}(M_{p,2}/\mathbb{Q}_p)\) to \(M_u\). We have seen that \(\text{Gal}(M_{p,2}/\mathbb{Q}_p) = \langle \sigma \rangle \) for all \(i \in \mathbb{Z}/d\mathbb{Z}\).

For all \(i\), \(g_i \in \text{Gal}(M_{p,2}/K)\), and so \(g_i\mid_{M_u}\) is an element of \(\text{Gal}(M_u/K) \subseteq \text{Gal}(M_u/\mathbb{Q}_p)\), by Galois theory. Therefore, \(M_u/\mathbb{Q}_p\) is Galois if and only if \(M_u = \sigma(M_u)\). We have just shown that \(\sigma(M_u) = M_{\sigma(u)}\) and from Theorem 3.1 we know that \(M_u = M_{\sigma(u)}\) if and only if \(u = \sigma(u)\) which completes the proof.

Remark 4.5 We know by Theorem 3.1 that the map \(v \mapsto v^{1-p}\) is a one-to-one correspondence from \(\mathbb{P}(k)\) onto the set \(\mathcal{M}\) of degree \(p\) extensions of \(K\) contained in \(K_{p,2}\). We have noticed earlier that \(\sigma\) acts on \(\mathbb{P}(k)\); let it act on \(\mathcal{M}\) through \(\sigma\), then it follows from Proposition 4.4 that the above bijection commutes with the action of \(\sigma\).

It seems natural to ask whether \(\sigma(\alpha_u) = \alpha_{\sigma(u)}\), with suitable choices of \(\omega_u\) and possibly a different choice of \(\sigma\). However, we are not able to answer this question at present.

5 Exponential power series and differential Frobenius structures

We now discuss how Pulita’s method in [12] can be modified to give further results concerning differential modules with Frobenius structure. For each Lubin-Tate formal group over \(\mathbb{Q}_p\), Pulita constructs a set of exponential power series and proves that all exponentials in this set are over-convergent if and only if the formal group is isomorphic to the multiplicative formal group (see [12, Theorem 2.1]), i.e., the formal group is associated to the uniformising parameter \(p\).

In this section we describe Pulita’s main results used in the construction of his exponential power series, omitting some technical details which are described fully in [12]. We describe how a different choice of Witt vector in this construction allows us to relax the condition that the Lubin-Tate formal group must be isomorphic to the multiplicative formal group in order for each of the associated exponentials to be over-convergent. We then describe how these exponentials can be used to describe differential modules with a Frobenius structure.

These results were found during our investigation of exponential power series from the point of view of Galois module structure, and we hoped that they would describe some kind of link between these two subject areas. However, we have not achieved this elusive goal. Indeed, the only power series from this section that can be used directly in the constructions in Section 2 is \(E_{1,2}(X)\) — see Theorem 5.7. It has the very special property that \(E_{1,2}(u)\)
is a primitive \( p^2 \)th root of unity for \( u \in \mu_{p^2} \), and so \( E_{1,2}(X) \) only gives us information about cyclotomic extensions. In order to obtain Galois module generators in extensions not obtained from cyclotomy, we had to modify the power series \( E_{1,2}(X) \) as in Section 2. These modified power series then lose the properties that we use to endow a differential module with a Frobenius structure in this section. Using the methods here, apart from in the cyclotomic case, the power series considered can not give information about both differential Frobenius structure and integral Galois module structure.

5.1 Pulita exponentials arising from different Witt vectors

Definition 5.1 The \( n \)th Witt polynomial is defined as

\[
W_n(X_0, \ldots, X_n) = X_0^n + pX_1^{p^n-1} + \ldots + p^nX_n.
\]

For any ring \( R \), the ring of Witt vectors over \( R \), denoted \( W(R) \), is equal as a set to \( R^{N_\mathbb{Z}} \cup \{0\} \) and endowed with the ring structure such that the map

\[
W : W(R) \to R^{N_\mathbb{Z}} \cup \{0\}
\]

is a ring homomorphism. For \( r = (r_0, r_1, \ldots) \in W(R) \), we define

\[
r^{(n)} = W_{(n)}(r_0, r_1, \ldots, r_n)
\]

and let \( W(r) = \langle r^{(0)}, r^{(1)}, \ldots \rangle \) to be the \( n \)th ghost component and the ghost vector of \( r \) respectively.

See, for example, [4] for full details.

For \( u \in \mathbb{Z}_p^* \) we let \( f_u(X) \in \mathbb{Z}_p[X] \) be such that

\[
f_u(X) \equiv X^p \mod p\mathbb{Z}_p[X] \quad \text{and} \quad f_u(X) \equiv upX \mod X^2\mathbb{Z}_p[X]
\]

and let \( \{\omega_{u,n}\}_{n \geq 0} \) be a coherent set of roots associated to \( f_u(X) \). Note that \( \mathbb{Q}_p(\omega_{u,n}) = (\mathbb{Q}_p)_{up,n} \) is the \( n \)th Lubin-Tate extension of \( \mathbb{Q}_p \) with respect to the uniformising parameter \( up \).

Definition 5.2 We define

\[
[\phantom{X}]: \mathbb{Z}_p[[X]] \to W(\mathbb{Z}_p[[X]])
\]

to be the unique ring homomorphism such that for \( h(X) \in \mathbb{Z}_p[[X]] \), \( [h(X)] \) is the unique Witt vector over \( \mathbb{Z}_p[[X]] \) whose ghost vector is equal to

\[
\langle h(X), h(f_u(X)), h(f_u(f_u(X))), \ldots \rangle.
\]

See [12, Lemma 2.1] for a proof of existence and uniqueness.

Following Pulita, for any finite extension \( L/\mathbb{Q}_p \) and any \( x \in \mathfrak{P}_L \), we then denote \( [h(x)] = [h(X)]|_{X=x} \in W(\mathfrak{O}_L) \).
Remark 5.3 If we take \( h(X) = X \) and \( x = \omega_{u,n} \) in the above definition, then \([\omega_{u,n}] \in W(\mathcal{O}_K(\omega_{u,n}))\) is the unique Witt vector with ghost vector

\[
< \omega_{u,n}, \omega_{u,n-1}, \ldots, \omega_{u,1}, 0, 0, \ldots > .
\]

We denote by \( E(X) = \exp \left( X + \frac{X^p}{p} + \frac{X^{p^2}}{p^2} + \cdots \right) \) the Artin-Hasse exponential and recall that \( E(X) \in 1 + X\mathbb{Z}_p[[X]] \) (see, for example, [4, I §9]).

Definition 5.4 With \( L \) a finite extension of \( \mathbb{Q}_p \), let \( \lambda = (\lambda_0, \lambda_1, \ldots) \in W(\mathcal{O}_L) \). The Artin-Hasse exponential relative to \( \lambda \) is defined as

\[
E(\lambda, X) := \prod_{i \geq 0} E(\lambda_i X^{p^i}) = \exp \left( \lambda^{(0)} X + \frac{\lambda^{(1)} X^p}{p} + \frac{\lambda^{(2)} X^{p^2}}{p^2} + \cdots \right).
\]

We note that the integrality of the Artin-Hasse exponential and the fact that \( \lambda \in W(\mathcal{O}_L) \) imply that \( E(\lambda, X) \in 1 + X\mathcal{O}_L[[X]] \). Further, we observe that for \( u \in \mathbb{Z}_p^\times \)

\[
E([\omega_{u,n}]\lambda, X) = \exp \left( \omega_{u,n}\lambda^{(0)} X + \omega_{u,n-1}\lambda^{(1)} X^p \frac{p^i}{p} + \cdots + \omega_{u,1}\lambda^{(n-1)} X^{p^{n-1}} \frac{p^{n-1}}{p^{n-1}} \right).
\]

We now state two lemmas from [12] that we need to prove our result.

Lemma 5.5 Let \( L \) be a finite extension of \( \mathbb{Q}_p \). Let \( h(X) = \sum_{i \geq 0} a_i X^i \in \mathbb{Z}_p[[X]] \) and let \( \lambda(h(x)) = (\lambda_0, \lambda_1, \ldots) \in W(\mathcal{O}_L) \) with \( x \in \mathfrak{p}_L \) (i.e., \( |x|_p < 1 \)). Then,

\[
|a_0|_p = |p|_p^{r} \quad \text{if and only if} \quad |\lambda_0|_p, \ldots, |\lambda_{r-1}|_p < 1 \quad \text{and} \quad |\lambda_r|_p = 1 .
\]

Moreover, if \( a_0 = 0 \), then \( |\lambda_r|_p < 1 \) for all \( r \).

Proof. [12] Lemma 2.2. □

Lemma 5.6 Let \( \lambda = (\lambda_0, \lambda_1, \ldots) \in W(\mathcal{O}_L) \), then for \( u \in \mathbb{Z}_p^\times \), \( E([\omega_{u,n}]\lambda, X) \) is over convergent if and only if \( |\lambda_0|_p, \ldots, |\lambda_{n-1}|_p < 1 \).

Proof. This is the equivalence of 2 and 3 in [12] Theorem 2.2. □

We now state our over-convergence result.

Theorem 5.7 The power series

\[
E_{u,n}(X) = \exp \left( \sum_{i=0}^{n-1} \frac{\omega_{u,n-i}(X^{p^i} - uX^{p^{i+1}})}{p^i} \right)
\]

is over-convergent for all \( u \in \mathbb{Z}_p^\times \) and \( n \in \mathbb{N} \).
This is a generalisation of the equivalence of 3 and 4 in [12, Theorem 2.5] and is essentially proved in the same way. Letting $u = 1$ reduces to Pulita’s setup (with a shift in the numbering).

**Proof.** Let $u \in \mathbb{Z}_p^\times$ and $n \in \mathbb{N}$, then

$$E_{u,n}(X) = \exp\left(\sum_{i=0}^{n-1} \frac{\omega_{u,n-i}X^{p^i}}{p^i}\right) - \exp\left(\sum_{i=0}^{n-1} \frac{u\omega_{u,n-i}X^{p^i+1}}{p^i}\right)$$

$$= \exp(up\omega_{u,n+1}X)\exp\left(\sum_{i=0}^{n} \omega_{u,n-i+1} \left(\frac{\omega_{u,n-i}}{\omega_{u,n-i+1}} - up\right) \frac{X^{p^i}}{p^i}\right).$$

Letting $h(X) = f_u(X)/X - up$ we then have

$$E_{u,n}(X) = \exp(up\omega_{u,n+1}X) E([\omega_{u,n+1}][h(\omega_{n+1})], X).$$

We know that $\exp(up\omega_{u,n+1}X)$ is over-convergent, and so we need only to check the over-convergence of $E([\omega_{u,n+1}][h(\omega_{n+1})], X)$.

The constant term in $h(X)$ is $up - up = 0$ and $|\omega_{u,n+1}|_p < 1$. Therefore, if we let $[h(\omega_{n+1})] = (\lambda_0, \lambda_1, \ldots)$, then from Lemma 5.5 we see that $|\lambda_r|_p < 1$ for all $r$. Over-convergence now follows from Lemma 5.6. ■

### 5.2 Differential modules with Frobenius structure

We now explain briefly how these exponentials give a special structure to certain $p$-adic differential modules. For a detailed account of differential modules with Frobenius structure, see [7, Ch. 5 and Ch. 17]. Recall, specifically, that a *differential ring* is a commutative ring $R$ equipped with a derivation $d : R \to R$, namely an additive map satisfying $d(ab) = ad(b) + bd(a)$ for $a, b \in R$.

**Definition 5.8** A differential module over a differential ring $(R, d)$ is an $R$-module $M$ equipped with an additive map $D : M \to M$ satisfying, for any $a \in R$ and $m \in M$,

$$D(am) = aD(m) + d(a)m.$$

Such a $D$ is called a differential operator on $M$ relative to $d$.

Note that morphisms of $(R, d)$-differential modules are morphisms of $R$-modules that commute with the differential operators.

Recall that $L$ is some finite extension of $\mathbb{Q}_p$.

**Definition 5.9** We define $\mathcal{R}_L$, the Robba ring with coefficients in $L$, to be the ring of bi-directional power series over $L$ which converge on some annulus of outer radius 1, namely

$$\mathcal{R}_L = \left\{ \sum_{n=-\infty}^{\infty} a_nX^n \middle| a_n \in L \text{ for every } n \in \mathbb{Z}, \right.$$

$$\lim_{\infty} |a_n|_p a^n = 0 \text{ for some } 0 < \alpha < 1, \right.$$  

$$\lim_{-\infty} |a_n|_p \rho^n = 0 \text{ for all } 0 < \rho < 1. \right\}.$$
We equip $\mathcal{R}_L$ with the derivation $\partial_X = X \frac{d}{dX}$.

Let $M$ be a free rank one $\mathcal{R}_L$-module. In view of Definition 5.8 any differential operator $D$ on $M$ is defined by giving the image $D(e)$ of a basis $e$ of $M$, namely by specifying $g \in \mathcal{R}_L$ such that $D(e) = ge$.

**Definition 5.10** Let $h \in \mathcal{R}_L$. A free rank one $\mathcal{R}_L$-differential module $(N, \Delta)$ is said to be defined by $\partial_X - h$ if there exists a basis $\varepsilon$ of $N$ such that $\Delta(\varepsilon) = h\varepsilon$.

**Remark 5.11** Suppose that $(M, D)$ is defined by $\partial_X - g$, with basis $e$. If $f \in \mathcal{R}_L$, $e' = fe$ is also a basis of $M$, and

$$D(e') = fD(e) + \partial_X(f)e = \left(g + \frac{\partial_X f}{f}\right)e' ,$$

hence $(M, D)$ is also defined by $\partial_X - (g + \frac{\partial_X f}{f})$, with basis $e'$.

On the other hand, one easily checks that if $(M, D)$ is also defined by $\partial_X - h$ for some $h \in \mathcal{R}_L$, then there exists $f \in \mathcal{R}_L^*$ such that $h = g + \frac{\partial_X f}{f}$.

**Proposition 5.12** Let $(M, D)$ be defined by $\partial_X - g$ then, for any free rank one $\mathcal{R}_L$-differential module $(N, \Delta)$, $(N, \Delta)$ is isomorphic to $(M, D)$ if and only if $(N, \Delta)$ is defined by $\partial_X - g$.

**Proof.** Suppose $(N, \Delta)$ is defined by $\partial_X - g$, and let $\varepsilon$ (resp. $e$) be a basis of $N$ (resp. $M$) such that $\Delta(\varepsilon) = g\varepsilon$ (resp. $D(e) = ge$). Let $\varphi$ be the $\mathcal{R}_L$-morphism $M \rightarrow N$ defined by $\varphi(e) = \varepsilon$, then $\varphi$ is an isomorphism of $\mathcal{R}_L$-modules and, for all $a \in \mathcal{R}_L$,

$$\varphi(D(\varepsilon)) = a\varepsilon + \partial_X(a)\varepsilon = a\Delta(\varepsilon) + \partial_X(a)\varepsilon = D'(\varphi(\varepsilon)),$$

namely $\varphi$ is an isomorphism of $\mathcal{R}_L$-differential modules.

Suppose now that an isomorphism of $\mathcal{R}_L$-differential modules $\varphi : (M, D) \rightarrow (N, \Delta)$ is given. Let again $e$ be a basis of $M$ such that $D(e) = ge$ and set $\varepsilon = \varphi(e)$, then $\varepsilon$ is a basis of $N$ and

$$\Delta(\varepsilon) = \Delta(\varphi(e)) = \varphi(D(e)) = g\varepsilon,$$

as required. \[\square\]

We will refer to any $\mathcal{R}_L$-differential module defined by $\partial_X - g$ as the $\mathcal{R}_L$-differential module defined by $\partial_X - g$. This is a slight abuse of terminology, but is consistent with the literature.

We now recall some facts described in [12, §1.2.3-4]. Note that Pulita defines $\mathcal{R}_L$-differential modules in a different (but equivalent) way. Namely, he defines them as modules over the non-commutative ring $\mathcal{R}_L[\partial_X]$ of differential polynomials with coefficients in $\mathcal{R}_L$ (and the rule $\partial_X a = a\partial_X + \partial_X(a)$).

Let $\varphi : L \rightarrow L$ be a continuous $\mathbb{Q}_p$-automorphism lifting the Frobenius automorphism of $k_L/\mathbb{F}_p$. This extends to a continuous ring endomorphism

$$\phi : \mathcal{R}_L \rightarrow \mathcal{R}_L, \quad \sum a_iX^i \mapsto \sum \varphi(a_i)X^{pi}$$

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known as an absolute Frobenius on $\mathcal{R}_L$. There exists a functor $\phi^*$ from the isomorphism classes of rank one $\mathcal{R}_L$-differential modules to themselves, which sends the $\mathcal{R}_L$-differential module defined by $\partial_X - g(X)$ to the $\mathcal{R}_L$-differential module defined by

$$\partial_X - \frac{X}{\phi(X)} d\phi(g(X)) = \partial_X - p\phi(g(X)).$$

**Definition 5.13** A rank one $\mathcal{R}_L$-differential module $M$ has Frobenius structure (of order 1) if $M \cong \phi^*(M)$.

**Theorem 5.14** Let $u \in \mu_{p-1} \subset \mathbb{Z}_p^\times$, $f_u(X) = X^p + upX$, and let $\{\omega_{u,n}\}_{n \geq 0}$ be a coherent set of roots associated to $f_u(X)$. Then, for all $n \in \mathbb{N}$, the $\mathcal{R}_{\mathbb{Q}_p(\omega_{u,n})}$-differential module defined by

$$\partial_X - \sum_{i=0}^{n-1} \omega_{u,n-i} X^{p^i}$$

has Frobenius structure.

**Proof.** For $k \in \mathbb{N}$, we denote by $f_u^k(X)$ the composition of $k$ copies of $f_u(X)$. From standard Lubin-Tate theory we know that $\mathbb{Q}_p(\omega_{u,n})/\mathbb{Q}_p$ is Galois and of degree $p^n - 1$ and that $\omega_{u,n}$ has minimum polynomial $f_u^n/f_u^{-1}$ (this polynomial has the same degree as the extension and is Eisenstein over $\mathbb{Q}_p$).

For our choices of $u$ and $f_u(X)$, we see that $f_u(uX) = uf_u(X)$, and so $f_u^n/f_u^{-1}(uX) = f_u^n/f_u^{-1}(X)$. This means that $u\omega_{u,n}$ is also a root of $f_u^n/f_u^{-1}$, and $\varphi : \omega_{u,n} \mapsto u\omega_{u,n}$ defines an element of $\text{Gal}(\mathbb{Q}_p(\omega_{u,n})/\mathbb{Q}_p)$, which is therefore a $\mathbb{Q}_p$-linear continuous automorphism of $\mathbb{Q}_p(\omega_{u,n})$. From these definitions, we also have $\varphi(\omega_{u,m}) = u\omega_{u,m}$ for all $m \leq n$. As $\mathbb{Q}_p(\omega_{u,n})/\mathbb{Q}_p$ is totally ramified, the residue field extension is trivial and any $\mathbb{Q}_p$-automorphism of $\mathbb{Q}_p(\omega_{u,n})$ lifts the Frobenius automorphism of $k_{\mathbb{Q}_p(\omega_{u,n})}/\mathbb{F}_p$.

Let $\phi$ be the absolute Frobenius on $\mathcal{R}_{\mathbb{Q}_p(\omega_{u,n})}$ coming from $\varphi$ as described above. Let

$$g(X) = \sum_{i=0}^{n-1} \omega_{u,n-i} X^{p^i}$$

and note that we have $\phi(g(X)) = ug(X^p)$.

We note that $E_{u,n}(X) = E_{u,n}(-X)^{-1}$, and so the over-convergence of $E_{u,n}(X)$, proved in Theorem 5.7 implies that $E_{u,n}(X)$ and $E_{u,n}(-X)$ are contained in $\mathcal{R}_{\mathbb{Q}_p(\omega_{u,n})}$; in particular,

$$E_{u,n}(-X) = \exp \left( \sum_{i=0}^{n-1} \omega_{u,n-i} \left( \frac{uX^{p^i+1}}{p^i} - X^{p^i} \right) \right) \in \mathcal{R}_{\mathbb{Q}_p(\omega_{u,n})}^\times.$$

Let $M$ be the $\mathcal{R}_{\mathbb{Q}_p(\omega_{u,n})}$-differential module defined by $\partial_X - g(X)$ with basis vector $e$. From Remark 5.11 we see that $M$ is also defined by $\partial_X - upg(X^p)$ with basis vector $E_{u,n}(-X)e$. Finally, we observe that as $\phi(g(X)) = ug(X^p)$, the differential module $\phi^*(M)$ is defined by $\partial_X - upg(X^p)$, and so $M \cong \phi^*(M)$. ■
Remark 5.15 A differential module with Frobenius structure is necessarily solvable in the sense of [12, Definition 1.7] (see [12, Remark 1.5]); in [12, Theorem 3.1] Pulita completely describes the isomorphism classes of rank one solvable $R_{Q_p(\omega_u,n)}$-differential modules. Therefore, despite that fact that the differential modules above are different to those explicitly described by Pulita, they must each be contained in one of Pulita’s isomorphism classes.

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