Characterization of (semi-)Eberlein compacta using retractional skeletons

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Abstract. We study retractions associated to suitable models in compact spaces admitting a retractional skeleton and find several interesting consequences. Most importantly, we provide a new characterization of Valdivia compacta using the notion of retractional skeletons, which seems to be helpful when characterizing their subclasses. Further, we characterize Eberlein and semi-Eberlein compacta in terms of retractional skeletons and show that our new characterizations give an alternative proof of the fact that a continuous image of an Eberlein compact is Eberlein as well as new stability results for the class of semi-Eberlein compacta, solving in particular an open problem posed by Kubiś and Leiderman.

1. Introduction. The study of the class of compact spaces that admit a retractional skeleton was initiated in [24], where the authors proved that a compact space is Valdivia if and only if it admits a commutative retractional skeleton. Later, in [22] a notion similar to retractional skeletons in the context of Banach spaces was introduced: namely, the notion of projectional skeletons. In some sense, those notions are dual to each other. More precisely, if a compact space $K$ admits a retractional skeleton, then $C(K)$ admits a projectional skeleton, and if a Banach space $X$ admits a projectional skeleton, then $(B_{X^*}, w^*)$ admits a retractional skeleton. The class of Banach (compact) spaces with a projectional (retractional) skeleton was deeply investigated from various perspectives and nowadays we have quite a rich family of natural examples and interesting results related to various fields of mathematics such as topology [26], Banach space theory [15], theory of von Neumann algebras [3] or JBW$^*$-triples [4]. Quite surprisingly, also the

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One of the recent streams in the area is to describe some classes of Banach (compact) spaces using the notion of projectional (retractional) skeletons; see e.g. [24, 9, 15, 22] where the characterizations of Plichko spaces (and Valdivia compacta), WLD spaces (and Corson compacta), Asplund spaces, WLD+Asplund spaces and WCG spaces were given.

The two main results of this paper (Theorems A and B) are characterizations of Eberlein and semi-Eberlein compacta, respectively, using the notion of retractional skeletons. Let us recall that given a set $I$ we define

$$c_0(I) := \{ x \in \mathbb{R}^I : (\forall \varepsilon > 0) |\{ i \in I : |x(i)| > \varepsilon \}| < \omega \} \subset \mathbb{R}^I$$

and that a compact space $K$ is Eberlein if it homeomorphically embeds into $c_0(I)$ for some set $I$. This is a central concept in Banach space theory, as it is known that a compact space is Eberlein if and only if it is homeomorphic to a weakly compact set of a Banach space (see [1] or [14, Corollary 13.19]). For the notion of shrinkingness we refer the reader to Definition 28.

**Theorem A.** Let $K$ be a compact space. Then the following conditions are equivalent:

1. $K$ is Eberlein.
2. There exist a bounded set $A \subset C(K)$ separating the points of $K$ and a retractional skeleton $s = (r_s)_{s \in \Gamma}$ on $K$ such that $s$ is $A$-shrinking.
3. There exist a countable family $\mathcal{A}$ of subsets of $B_{C(K)}$ and a full retractional skeleton $s = (r_s)_{s \in \Gamma}$ on $K$ such that

   a. for every $A \in \mathcal{A}$ there exists $\varepsilon_A > 0$ such that $s$ is $(A, \varepsilon_A)$-shrinking, and
   b. for every $\varepsilon > 0$ we have $B_{C(K)} = \bigcup \{ A \in \mathcal{A} : \varepsilon_A < \varepsilon \}$.

Recall that a compact space $K$ is Eberlein if and only if $C(K)$ is WCG if and only if $C(K)$ is a subspace of a WCG space, thus Theorem A is naturally connected with the characterization of WCG Banach spaces and their subspaces in [15]. Moreover, from Theorem A one may deduce that continuous images of Eberlein compacta are Eberlein (see Remark 43 below). Quite a few steps of our proofs seem to be much more flexible and we believe that those may be used to find characterizations of other natural subclasses of Valdivia compacta (the most important in this respect is probably Theorem D below). This is witnessed by the characterization of semi-Eberlein compacta presented in Theorem B. Recall that, following [23], we say a compact space $K$ is semi-Eberlein if there exists a homeomorphic embedding...
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$h : K \to \mathbb{R}^I$ such that $h^{-1}[c_0(I)]$ is dense in $K$. We denote by $D(s)$ the set induced by a retractional skeleton $s$ (see Definition [1]).

**Theorem B.** Let $K$ be a compact space. Then the following conditions are equivalent:

1. $K$ is semi-Eberlein.
2. There exist a dense subset $D \subset K$, a bounded set $A \subset C(K)$ separating the points of $K$ and a retractional skeleton $s = (r_s)_{s \in \Gamma}$ on $K$ with $D \subset D(s)$ such that
   (a) $s$ is $A$-shrinking with respect to $D$, and
   (b) $\lim_{s \in \Gamma'} r_s(x) \in D$ for all $x \in D$ and every up-directed subset $\Gamma'$ of $\Gamma$.
3. There exist a dense set $D \subset K$, a countable family $A$ of subsets of $B_c(K)$ and a retractional skeleton $s = (r_s)_{s \in \Gamma}$ on $K$ with $D \subset D(s)$ such that
   (a) for every $A \in A$ there exists $\varepsilon_A > 0$ such that $s$ is $(A, \varepsilon_A)$-shrinking with respect to $D$,
   (b) for every $\varepsilon > 0$ we have $B_c(K) = \bigcup \{ A \in A : \varepsilon_A < \varepsilon \}$, and
   (c) $\lim_{s \in \Gamma'} r_s(x) \in D$ for all $x \in D$ and every up-directed subset $\Gamma'$ of $\Gamma$.

Finally, using Theorem B we provide new structural results for the class of semi-Eberlein compacta, answering in particular the second part of [23, Question 6.6] in the affirmative. The most important new stability results are summarized below.

**Theorem C.** Let $K$ be a semi-Eberlein compact space.

- If $L$ is an open continuous image of $K$ and it has densely many $G_\delta$-points, then $L$ is semi-Eberlein.
- If $K$ is moreover Corson and $L$ is a continuous image of $K$, then $L$ is semi-Eberlein.

As mentioned previously, many steps of the proofs of Theorems A and B are of independent interest and we believe they could be used when trying to characterize other subclasses of Valdivia compacta, which opens quite a wide area of potential further research. This is outlined in Section 7.

Let us now briefly describe the content of each section, emphasizing the general steps mentioned above.

Section 2 contains basic notations and some preliminary results.

In Section 3 we consider retractions associated to (not necessarily countable) suitable models. The most important outcome is Theorem 15 where we summarize the properties of canonical retractions associated to suitable models. As an easy consequence, in Proposition 17 we give a very general method of obtaining a continuous chain of retractions on a compact space admitting a retractional skeleton. This part is essentially known, as similar results were obtained e.g. in [5, Lemma 2.5] (using other methods than
suitable models), but our approach is in a certain sense much more flexible (most importantly because it may be combined with other statements involving suitable models) and we actually use this flexibility later. As a corollary we show in Theorem 21 that we may in a certain way combine properties of countably many retractional skeletons.

In Section 4, inspired by the proof of [5, Theorem 2.6], we aim at seeing as concretely as possible the “Valdivia embedding” of compact spaces with a commutative retractional skeleton. As a consequence we obtain the following result which might be thought of as the fourth main result of the whole paper. The most important part which we use later is $\text{(i)} \Rightarrow \text{(iv)}$.

**Theorem D.** Let $K$ be a compact space and $s = (r_s)_{s \in \Gamma}$ be a retractional skeleton on $K$. Then the following conditions are equivalent:

(i) $D(s)$ is induced by a commutative retractional skeleton.
(ii) There exists a subskeleton of $s$ which is commutative.
(iii) There exist a subskeleton $s_2 = (r_s)_{s \in \Gamma''}$ of $s$ and a dense set $D \subset D(s)$ such that for every up-directed set $\Gamma'' \subset \Gamma'$ and every $x \in D$ we have $\lim_{s \in \Gamma''} r_s(x) \in D$.

Moreover, if $\lambda \geq 1$ and $A \subset \lambda B_{C(K)}$ is a closed, symmetric and convex set separating the points of $K$ such that $f \circ r_s \in A$ for all $f \in A$ and $s \in \Gamma$, then those conditions are also equivalent to the following one:

(iv) There exists $\mathcal{H} \subset A$ such that the mapping $\varphi: K \to [-1, 1]^\mathcal{H}$ defined by $\varphi(x)(h) := \frac{h}{\lambda}(x)$ for $h \in \mathcal{H}$ and $x \in K$ is a homeomorphic embedding and $\varphi[D(s)] \subset \Sigma(\mathcal{H})$.

Note that Theorem D provides a characterization of Valdivia compacta, since a compact space is Valdivia if and only if it admits a commutative retractional skeleton.

In Section 5 we prove (slightly more general versions of) Theorems A and B. Section 6 is devoted to applications (in particular to the proof of Theorem C) and Section 7 is devoted to open problems and remarks.

**2. Notation and preliminary results.** We use standard notations from topology and Banach space theory as can be found in [12] and [14].

For a set $I$, we define

$$\Sigma(I) := \{x \in \mathbb{R}^I : |\text{suppt}(x)| \leq \omega\},$$

where $\text{suppt}(x) = \{i \in I : x(i) \neq 0\}$ is the support of $x$. Given a subset $S$ of $I$ we denote the characteristic function of $S$ by $1_S$.

All topological spaces are assumed to be Tikhonov. Let $T$ be a topological space. A subset $S \subset T$ is said to be countably closed if $\overline{C} \subset S$ for every countable subset $C \subset S$. We denote by $w(T)$ the weight of $T$, by $C(T,T)$
the set of continuous functions from $T$ to $T$ and by $\beta T$ the Čech–Stone compactification of $T$. If $T$ is compact, then as usual $\mathcal{C}(T)$ denotes the Banach algebra of real-valued continuous functions defined on $T$, endowed with the supremum norm. Moreover, if $\mathcal{A} \subset \mathcal{C}(T)$, we denote by $\text{alg}(\mathcal{A})$ the algebraic hull of $\mathcal{A}$ in $\mathcal{C}(T)$. Recall that a compact space $T$ is said to be Valdivia if there is a homeomorphic embedding $h : T \to \mathbb{R}^I$ such that $h^{-1}[\Sigma(I)]$ is dense in $T$; we refer to [20] for a survey of this subject.

Let $(\Gamma, \leq)$ be an up-directed partially ordered set. We say that a sequence $(s_n)_{n \in \omega}$ of elements of $\Gamma$ is increasing if $s_n \leq s_{n+1}$ for every $n \in \omega$. We say that $\Gamma$ is $\sigma$-complete if for every increasing sequence $(s_n)_{n \in \omega}$ in $\Gamma$ there exists $\sup_n s_n$ in $\Gamma$. We say that $\Gamma' \subset \Gamma$ is cofinal in $\Gamma$ if for every $s_0 \in \Gamma$ there is $s \in \Gamma'$ with $s \geq s_0$. If $\Gamma$ is $\sigma$-complete and $A \subset \Gamma$, we denote by $A_\sigma$ the smallest $\sigma$-closed subset of $\Gamma$ containing $A$. Notice that, by [21] Proposition 2.3], if $A$ is up-directed, then $A_\sigma$ is up-directed.

**Definition 1.** Following [11], a retractional skeleton in a countably compact space $K$ is a family $\mathfrak{s} = (r_s)_{s \in \Gamma}$ of continuous retractions on $K$ indexed by an up-directed, $\sigma$-complete partially ordered set $\Gamma$, such that

(i) $r_s[K]$ is a metrizable compact space for each $s \in \Gamma$,
(ii) if $s, t \in \Gamma$, $s \leq t$, then $r_s = r_t \circ r_s = r_s \circ r_t$,
(iii) if $s(n) \in \omega$ in $\Gamma$, if $s = \sup_n s_n \in \Gamma$, then $r_s(x) = \lim_{n \to \infty} r_{s_n}(x)$ for every $x \in K$,
(iv) for every $x \in K$, $x = \lim_{s \in \Gamma} r_s(x)$.

We say that the set $\bigcup_{s \in \Gamma} r_s[K]$ is induced by the retractional skeleton $\mathfrak{s}$ and we denote it by $D(\mathfrak{s})$. We say that $\mathfrak{s}$ is commutative if $r_s \circ r_t = r_t \circ r_s$ for all $s, t \in \Gamma$; and $\mathfrak{s}$ is full if $D(\mathfrak{s}) = K$.

The following preliminary result will be used frequently. It seems to be new even though it may be known to experts.

**Lemma 2.** Let $K$ be a compact space. Suppose that $K$ has a retractional skeleton $\mathfrak{s} = (r_s)_{s \in \Gamma}$. Let $\Gamma' \subset \Gamma$ be an up-directed subset. Then the mapping $R_{\Gamma'} : K \to K$ defined by $R_{\Gamma'}(x) = \lim_{s \in \Gamma'} r_s(x)$ is a continuous retraction and $R_{\Gamma'}[K] = \bigcup_{s \in \Gamma'} r_s[K]$. Moreover, the following hold.

(i) If $\Gamma'$ is countable, then $s = \sup \Gamma'$ exists and $R_{\Gamma'} = r_s$.
(ii) If $\mathcal{M}$ is an up-directed subset of $\mathcal{P}(\Gamma)$ such that each $M \in \mathcal{M}$ is up-directed, then $\lim_{M \in \mathcal{M}} R_M(x) = R_{\bigcup \mathcal{M}}(x)$, $x \in K$.
(iii) For all $s \in (\Gamma')_\sigma$ we have $r_s[K] \subset R_{\Gamma'}[K]$ and $r_s \circ R_{\Gamma'} = R_{\Gamma'} \circ r_s$.
(iv) $(r_s|_{R_{\Gamma'}[K]})_{s \in (\Gamma')_\sigma}$ is a retractional skeleton on $R_{\Gamma'}[K]$ with induced set $D(\mathfrak{s}) \cap R_{\Gamma'}[K]$.
(v) If $\mathfrak{s}$ is commutative, then $D(\mathfrak{s}) \cap R_{\Gamma'}[K] = R_{\Gamma'}[D(\mathfrak{s})]$. 
Proof. Let us start by proving that the mapping \( R_{\Gamma'} \) is well-defined. To do so, fix \( x \in K \) and suppose that \( (r_s(x))_{s \in \Gamma'} \) is an infinite set (otherwise the assertion is trivial). Since \( K \) is compact, there exists a cluster point \( x_1 \in K \) for the net \( (r_s(x))_{s \in \Gamma'} \). Let us show that it is unique. Indeed, suppose \( x_1 \neq x_2 \) are two such cluster points. Let \( U_1, U_2 \subset K \) be open subsets such that \( x_1 \in U_1, x_2 \in U_2 \) and \( \overline{U_1} \cap \overline{U_2} = \emptyset \). Let \( (s_n)_{n < \omega}, (t_n)_{n < \omega} \subset \Gamma' \) be increasing sequences such that \( s_n \leq t_n \leq s_{n+1}, r_{s_n}(x) \in U_1, \) and \( r_{t_n}(x) \in U_2, \) for every \( n \in \omega \). Since \( \Gamma \) is \( \sigma \)-complete, we have \( \sup_{n \in \omega} s_n = \sup_{n \in \omega} t_n = s \in \Gamma \). Then \( r_s(x) \in \overline{U_1} \cap \overline{U_2}, \) a contradiction. Therefore \( R_{\Gamma'} \) is well-defined.

The map \( R_{\Gamma'} \) is continuous. Indeed, let \( (x_\lambda)_{\lambda \in \Lambda} \) be a net converging to \( x \in K \). Up to taking a subnet we may assume that \( R_{\Gamma'}(x_\lambda) \) converges to \( y \). Suppose for a contradiction \( y \neq R_{\Gamma'}(x) \). Then there are open subsets \( U, V \subset K \) with \( y \in U \) and \( R_{\Gamma'}(x) \in V \) such that \( \overline{U} \cap \overline{V} = \emptyset \). We find recursively increasing sequences \( (s_n)_{n < \omega} \subset \Gamma' \) and \( (\lambda_n)_{n < \omega} \subset \Lambda \) such that \( r_{s_k}(x_{\lambda_i}) \in V \) if \( i \geq k \) and \( r_{s_k}(x_{\lambda_i}) \in U \) if \( i < k \).

Let us sketch the recursion here. Since \( R_{\Gamma'}(x_\lambda) \to y \), there exists \( \lambda_0 \in \Lambda \) such that \( R_{\Gamma'}(x_\lambda) \in U \) for every \( \lambda \geq \lambda_0 \). Since \( r_s(x) \xrightarrow{\Gamma'} R_{\Gamma'}(x) \), there exists \( s_0 \in \Gamma' \) such that \( r_t(x) \in V \) for every \( t \geq s_0 \). Since \( r_s(x_{\lambda_0}) \xrightarrow{\Gamma'} R_{\Gamma'}(x_{\lambda_0}) \in U \), there exists \( s_1 \geq s_0 \) such that \( r_t(x_{\lambda_0}) \in U \) for every \( t \geq s_1 \). By the continuity of \( r_s \), we have \( r_{s_1}(x_{\lambda_0}) \to r_{s_1}(x) \in V \); hence there exists \( \lambda_1 \geq \lambda_0 \) such that \( r_{s_1}(x\lambda_i) \in V \) for every \( \lambda \geq \lambda_1 \). We proceed recursively in the obvious way.

Since \( \Gamma \) is \( \sigma \)-complete, \( s = \sup_{k \in \omega} s_k \) belongs to \( \Gamma \). Hence \( r_{s_k}(x_{\lambda_i}) \) converges to \( r_s(x_{\lambda_i}) \in \overline{U} \) for every \( i \in \omega \). Moreover, by compactness we have \( \bigcap_{\lambda \in \omega} (x_{\lambda_i})_{i \geq k} \neq \emptyset \), so we may pick \( \tilde{x} \in \bigcap_{\lambda \in \omega} (x_{\lambda_i})_{i \geq k} \). We observe that \( r_{s_k}(\tilde{x}) \in V \) for every \( k \in \omega \), hence \( r_s(\tilde{x}) \in V \). On the other hand, \( r_{s_k}(x_{\lambda_i}) \to r_s(x_{\lambda_i}) = \overline{U} \) for every \( i \in \omega \); therefore \( r_s(\tilde{x}) \in \overline{U} \), a contradiction. Thus, \( R_{\Gamma'} \) is continuous.

Let us check that \( R_{\Gamma'} \) is a retraction. Indeed, pick \( x \in K \). Then
\[
R_{\Gamma'}(R_{\Gamma'}(x)) = \lim_{t \in \Gamma'} \left( \lim_{s \in \Gamma'} r_t \left( \lim_{s \in \Gamma'} r_s(x) \right) \right) = \lim_{t \in \Gamma'} \left( \lim_{s \in \Gamma'} r_t(r_s(x)) \right) = \lim_{t \in \Gamma'} r_t(r_s(x)) = r_t(x) = R_{\Gamma'}(x).
\]

Finally, for all \( s \in \Gamma' \) and \( x \in r_s[K] \) we have \( R_{\Gamma'}(x) = \lim_{t \in \Gamma', t \geq s} r_t(r_s(x)) = x \) so \( \bigcup_{s \in \Gamma'} r_s[K] \subset R_{\Gamma'}[K] \); the other inclusion follows from the definition of \( R_{\Gamma'} \).

It remains to prove the “Moreover” part. We first observe (see [21] proof of Proposition 2.3 for more details) that \( (\Gamma')_\sigma \) is directed, \( \sigma \)-closed and \( (\Gamma')_\sigma = \bigcup_{\alpha < \omega_1} B_\alpha \), where

- \( B_0 = \Gamma' \);
- \( B_{\alpha+1} = B_\alpha \cup \{ \sup t_n : (t_n) \) is an increasing sequence in \( B_\alpha \} \);
- \( B_\lambda = \bigcup_{\alpha < \lambda} B_\alpha \) if \( \lambda < \omega_1 \) is a limit ordinal.
(i) If $I'$ is countable, then we can find an increasing sequence $(s_n)_{n \in \omega}$ from $I$ with $\sup_n s_n = s = \sup I'$. Then, since $\{s_n : n \in \omega\}$ is cofinal in $I'$, we obtain $R_{I'} = R_{\{s_n : n \in \omega\}} = r_s$.

(ii) Suppose that $\mathcal{M} \subset \mathcal{P}(I)$ is up-directed and each $M \in \mathcal{M}$ is up-directed. Put $M_\infty := \bigcup_{M \in \mathcal{M}} M$, fix $x \in K$ and an open set $U$ such that $R_{M_\infty}(x) \in U$. Let $V$ be an open neighborhood of $R_{M_\infty}(x)$ such that $V \subset U$. Then there exists $s_0 \in M_\infty$ such that $r_s(x) \in V$ for every $s \in M_\infty$ with $s \geq s_0$. By the definition of $M_\infty$, there exists $M_0 \in \mathcal{M}$ such that $s_0 \in M_0$. If $M \in \mathcal{M}$ and $M_0 \subset M$, then $s_0 \in M$. This implies that $\{s \in M : s \geq s_0\}$ is cofinal in $M$ and so

$$R_M(x) = \lim_{s \in M} r_s(x) = \lim_{s \in M, s \geq s_0} r_s(x) \in \overline{V} \subset U.$$ 

This shows that $\lim_{M \in \mathcal{M}} R_M(x) = R_{M_\infty}(x)$.

(iii) We prove inductively that $r_s[K] \subset R_{I'}[K]$ and $r_s \circ R_{I'} = R_{I'} \circ r_s$ for all $\alpha < \omega_1$ and $s \in B_\alpha$. Pick $s \in B_\beta$. Then $r_s[K] = R_{I'}[K]$, since $R_{I'}[K] = \bigcup_{s \in I'} r_s[K]$. Moreover, for $x \in K$ we have

$$r_s(R_{I'}(x)) = \lim_{t \in I', t \geq s} r_s(r_t(x)) = \lim_{t \in I', t \geq s} r_t(r_s(x)) = R_{I'}(r_s(x)).$$

Now, fix $\alpha < \omega_1$ and suppose that the result holds for every $\gamma < \alpha$. If $\alpha$ is a limit ordinal, then the induction hypothesis easily implies that the result also holds for $\alpha$. Suppose that $\alpha = \gamma + 1$. Let $s \in B_\alpha$, $x \in r_s[K]$ and $(s_n)_{n \in \omega} \subset B_\gamma$ be such that $\sup_n s_n = s$. By the induction hypothesis, we have $R_{I'}(s_n(x)) = r_s(x)$ for every $n \in \omega$, and therefore

$$R_{I'}(x) = \lim_{n \in \omega} R_{I'}(s_n(x)) = \lim_{n \in \omega} r_s(x) = r_s(x) = x.$$ 

With a similar argument, we also conclude that $r_s \circ R_{I'} = R_{I'} \circ r_s$.

(iv) First, we claim that for all $x \in R_{I'}[K]$ we have $\lim_{s \in (I')_\sigma} r_s(x) = x$. Indeed, since $(I')_\sigma$ is up-directed, we find that $R_{I'}[K] = \bigcup_{s \in (I')_\sigma} r_s[K]$, which implies that $R_{I'}[K] \subset R_{I'}[K]$ and therefore if $x \in R_{I'}[K]$, then $x = R_{I'}[K] = \lim_{s \in (I')_\sigma} r_s(x)$.

By (iii) and the previous claim, it is easy to see that $s' := (r_s|_{R_{I'}[K]}{s \in (I')_\sigma}$ is a retractional skeleton on $R_{I'}[K]$ with $D(s') = \bigcup_{s \in (I')_\sigma} r_s[R_{I'}[K]] \subset D(s) \cap R_{I'}[K]$. On the other hand, since $D(s)$ is Fréchet–Urysohn (see [22, Theorem 32]), for every $x \in D(s) \cap R_{I'}[K]$ there is a sequence $(s_n)_{n \in \omega}$ in $I'$ with $r_{s_n}(x) \to x$ and therefore $x \in D(s')$, because $D(s')$ is a countably closed set. Thus, $D(s') = D(s) \cap R_{I'}[K]$.

(v) If $(r_s)_{s \in I}$ is commutative, then for every $s \in I$ and $x \in K$ we have

$$R_{I'}(r_s(x)) = \lim_{t \in I'} r_t(r_s(x)) = r_s\left(\lim_{t \in I'} r_t(x)\right) \in D(s),$$

which implies $R_{I'}[D(s)] \subset D(s)$ and so $R_{I'}[D(s)] = D(s) \cap R_{I'}[K]$. $\blacksquare$
3. Retractions associated to suitable models. The most important results concerning projectional skeletons were originally proved in [22] using the so-called “method of suitable countable models” which replaces inductive constructions by “suitable countable models”. The presentation of this method was simplified in [6] and later it was also used in the context of spaces admitting retractional skeletons (see e.g. [7] or [11]). Here we further generalize and investigate this method. The main difference of our approach is that we do not restrict consideration to countable models. The main outcome of this section is that for every (not necessarily countable) suitable model we can define a canonical retraction associated to this model. Those canonical retractions will be extensively used in the remainder of the paper.

Properties of retractions associated to suitable models are summarized in Theorem 15 and in Proposition 17 we obtain a continuous chain of retractions associated to suitable models with very pleasant properties. As an example of an application we show in Theorem 21 that we may in a certain way combine properties of countably many retractional skeletons.

3.1. Preliminaries. Here we settle the notation and give some basic observations concerning suitable models. We refer the interested reader to [6] and [11], where more details about this method may be found (warning: in [6, 11] only countable models were considered, while here we consider suitable models which are not necessarily countable).

Any formula in set theory can be written using the symbols $\in$, $=$, $\land$, $\lor$, $\neg$, $\rightarrow$, $\leftrightarrow$, $\exists$, $\forall$, $(,)$, $[,]$ and symbols for variables. On the other hand, it would be very laborious and pointless to use only the basic language of set theory. For example, we often write $x < y$ and we know that in fact this is a shortcut for a formula $\phi(x, y, <)$ with all free variables shown. Thus, in what follows we will use this extended language of set theory as we are accustomed to, having in mind that the formulas we work with are actually sequences of symbols from the list mentioned above.

Let $N$ be a fixed set and $\phi$ be a formula. Then the relativization of $\phi$ to $N$ is the formula $\phi^N$ which is obtained from $\phi$ by replacing each quantifier “$\exists x$” by “$\exists x \in N$” (and if we extend our language of set theory by the symbol “$\forall$” then we also replace each quantifier “$\forall x$” by “$\forall x \in N$”).

If $\phi(x_1, \ldots, x_n)$ is a formula with all free variables shown, then $\phi$ is absolute for $N$ if

$$\forall a_1, \ldots, a_n \in N \quad (\phi^N(a_1, \ldots, a_n) \leftrightarrow \phi(a_1, \ldots, a_n)).$$

**Definition 3.** Let $\Phi$ be a finite list of formulas and $X$ be any set. Let $M \supset X$ be a set such that each $\phi$ from $\Phi$ is absolute for $M$. Then we say that $M$ is a suitable model for $\Phi$ containing $X$, written $M \prec (\Phi; X)$.

Note that suitable models always exist.
Theorem 4 (see [25, Theorem IV.7.8]). Let $\Phi$ be a finite list of formulas and $X$ be any set. Then there exists a set $R$ such that $R \prec (\Phi; X)$ and $|R| \leq \max(\omega, |X|)$, and moreover, for every countable set $Z \subset R$ there exists $M \prec (\Phi; Z)$ and $M$ is countable.

The fact that a certain formula is absolute for $M$ will always be used exclusively in order to satisfy the assumption of the following lemma. Using this lemma we can force the model $M$ to contain all the needed objects created (uniquely) from elements of $M$. We give the well-known proof for the convenience of the reader.

Lemma 5. Let $\phi(y, x_1, \ldots, x_n)$ be a formula with all free variables shown and let $M$ be a set that is absolute for $\phi$ and for $\exists y \phi(y, x_1, \ldots, x_n)$. If $a_1, \ldots, a_n \in M$ are such that there exists a set $u$ satisfying $\phi(u, a_1, \ldots, a_n)$, then there exists a set $v \in M$ satisfying $\phi(v, a_1, \ldots, a_n)$. Moreover, if there exists a unique set $u$ such that $\phi(u, a_1, \ldots, a_n)$ holds, then $u \in M$.

Proof. From the absoluteness of the formula $\exists y \phi(y, x_1, \ldots, x_n)$, follows that there exists $v \in M$ such that $\phi^M(v, a_1, \ldots, a_n)$. Therefore the absoluteness of the formula $\phi(y, x_1, \ldots, x_n)$ implies that $\phi(v, a_1, \ldots, a_n)$ holds. Moreover, if $u$ is the only set such that $\phi(u, a_1, \ldots, a_n)$ holds, then $v = u$ and thus $u \in M$. □

Convention 6. Whenever we say “for any suitable model $M$ (the following holds ...)” we mean that “there exists a finite list of formulas $\Phi$ and a countable set $Y$ such that for every $M \prec (\Phi; Y)$ (the following holds ...)”.

If $M$ is a suitable model and $\langle X, \tau \rangle$ is a topological space (or $\langle X, d \rangle$ is a metric space, or $\langle X, +, \cdot, \| \cdot \| \rangle$ is a normed linear space) then we say that $M$ contains $X$ if $\langle X, \tau \rangle \in M$, or $\langle X, d \rangle \in M$, or $\langle X, +, \cdot, \| \cdot \| \rangle \in M$, respectively.

The following summarizes certain easy observations. For the proofs we refer the reader to [6, Sections 2 and 3], where it is assumed that $M$ is countable but this fact is not used in the proofs.

Lemma 7. For any suitable model $M$ the following hold:

1. $\mathbb{Q}, \omega, \mathbb{R} \in M$ and $M$ contains the usual operations and relations on $\mathbb{R}$.
2. For every function $f \in M$ we have $\text{Dom } f \in M$, $\text{Rng } f \in M$ and $f[M \cap \text{Dom } f] \subset M$.
3. For every finite set $A$ we have $A \in M$ if and only if $A \subset M$.
4. For every countable set $A \in M$ we have $A \subset M$. Moreover, if $\kappa \in M$ is a cardinal and $\kappa \subset M$ then for every $A \in M$ with $|A| \leq \kappa$ we have $A \subset M$.
5. For any natural number $n > 0$ and sets $a_1, \ldots, a_n$ we have $\{a_1, \ldots, a_n\} \subset M$ if and only if $\langle a_1, \ldots, a_n \rangle \in M$.
6. If $A, B \in M$, then $A \cap B \in M$, $B \setminus A \in M$ and $A \cup B \in M$. 
(7) If \( M \) contains a normed linear space \( X \), then \( X \cap M \) is a linear subspace of \( X \).

Some further easy observations are summarized in the following:

**Lemma 8.** For any suitable model \( M \) the following hold:

1. If \((\Gamma, \leq)\) is up-directed and \((\Gamma, \leq) \in M\), then \( \Gamma \cap M \) is up-directed.
2. If \( f, g \in M \) are functions and \( f \circ g \) is well-defined, then \( f \circ g \in M \).
3. If \( f \in M \) is a function which is one-to-one, then \( f^{-1} \in M \).
4. If \( f \in M \) is a function and \( X \in M \) is a subset of \( \text{Dom} f \), then \( f[M \cap X] = M \cap f[X] \).
5. If \( A \) and \( B \) are sets and \( A, B \in M \), then \( B^A \in M \) and \( A \times B \in M \).
6. For every set \( I \in M \) and \( X \subset \mathbb{R}^I \) with \( X \in M \) we have \( \pi \in M \), where \( \pi : I \to \mathbb{R}^X \) is the mapping given for \( i \in I \) and \( x \in X \) by \( \pi(i)(x) := x(i) \).
7. Let \( X \subset \Sigma(I) \) be such that \( I \in M \). Then \( \text{suppt}(x) \subset M \) for every \( x \in X \cap M \).
8. If \((X, \tau)\) is a topological space with \( \{X, \tau\} \subset M \), then \( \{C(X), +, ⋅, \otimes\} \subset M \) (where \( \cdot \) is multiplication by real numbers and \( \otimes \) is pointwise multiplication of functions). Moreover, if \( X \) is a compact space then \( M \) contains the normed linear space \( C(X) \), \( C(X) \cap M \) is a closed subalgebra of \( C(X) \) and \( 1 \in C(X) \cap M \).
9. If \((K, \tau)\) is a compact space, \( A \subset C(K) \) separates the points of \( K \) and \( \{K, \tau, A\} \subset M \), then \( \text{alg}((A \cup \{1\}) \cap M) = \overline{C(K)} \cap M \).
10. If \((K, \tau)\) is a compact space and \( K' \subset K \) is closed and metrizable with \( \{K', \tau, K\} \in M \), then \( C(K) \cap M \) separates the points of \( K' \) and \( K' \subset K \cap M \).
11. If \((K, \tau)\) is a compact space, \( D \subset K \) a dense subset with \( \{K, D, \tau\} \subset M \) and \( f \in C(K) \cap M \), then \( \|f\| = \|f|_{D \cap M} \| \).

**Proof.** Let \( S \) and \( \Phi \) be the countable set and the list of formulas from the statement of Lemma\(^7\), where \( \Phi \) is enriched by the formulas (and their subformulas) marked by \((*)\) in the proof below. Let \( M \prec (\Phi; S) \). Then \( M \) satisfies \((1)\), \((2)\), \((3)\), \((5)\), and \((6)\). Indeed those items follow easily from Lemma\(^5\) and the absoluteness of the following formulas (and their subformulas):

\[
\forall u, v \in \Gamma \ \exists w \in \Gamma \ \ (w \geq u, v), \quad (*)
\]
\[
\exists h \ (h = f \circ g), \quad (*)
\]
\[
\exists h \ (h = f^{-1}), \quad (*)
\]
\[
\exists W \ (W = B^A), \quad (*)
\]
\[
\exists W \ (W = B \times A), \quad (*)
\]
\[
\exists \pi \in (\mathbb{R}^X)^I \ (\forall i \in I \ \forall x \in X : \pi(i)(x) = x(i)). \quad (*)
\]

\( \Box \)
By Lemma 7.2, we have \( f[M \cap \text{Dom} f] \subset M \) so in particular \( f[M \cap X] \subset M \cap f[X] \). For the other inclusion pick \( x \in f[X] \cap M \). Using Lemma 5 and the absoluteness of the formula (and its subformulas)

\[
\exists y \in X \quad (f(y) = x),
\]

we find \( y \in M \cap X \) with \( f(y) = x \) and so \( x \in f[M \cap X] \).

Pick \( x \in X \cap M \). Using Lemma 5 and the absoluteness of the formula (and its subformulas)

\[
\exists D \subset I \quad (i \in D \iff x(i) \neq 0),
\]

we obtain \( \text{suppt}(x) \in M \). Since \( \text{suppt}(x) \) is a countable set, by Lemma 7.4 we deduce that \( \text{suppt}(x) \subset M \).

Using Lemma 5 and the absoluteness of the following formulas (and its subformulas):

\[
\exists C(X) \in \mathbb{R}^X \quad (\forall f \in \mathbb{R}^X : f \in C(X) \iff f \text{ is continuous}),
\]

\[
\exists + \in C(X)^{C(X) \times C(X)} \quad (\forall f, g \in C(X) \forall x \in X : +(f, g)(x) = f(x) + g(x)),
\]

\[
\exists \cdot \in C(X)^{\mathbb{R} \times C(X)} \quad (\forall \alpha \in \mathbb{R} \forall f \in C(X) \forall x \in X : \cdot(\alpha, f)(x) = \alpha f(x)),
\]

\[
\exists \otimes \in C(X)^{C(X) \times C(X)} \quad (\forall f, g \in C(X) \forall x \in X : \otimes(f, g)(x) = f(x)g(x)),
\]

we obtain \( C(X) \in M \) and \( \{+ , \cdot , \otimes \} \subset M \). Moreover, if \( X \) is a compact space, then using Lemma 5 and the absoluteness of the formula (and its subformulas)

\[
\exists \| \cdot \|_\infty \in \mathbb{R}^{C(X)} \quad (\forall f \in C(X) : \| \cdot \|_\infty(f) = \sup_{x \in X} |f(x)|),
\]

we see that \( M \) contains the normed linear space \( C(X) \). Thus, by Lemma 7.7, \( C(X) \cap M \) is a closed subspace of \( C(X) \) and, since \( \otimes \subset M \), \( C(X) \cap M \) is closed under multiplication and therefore \( C(X) \cap M \) is closed under multiplication as well. Finally, using Lemma 5 and the absoluteness of the formula (and its subformulas)

\[
\exists f \in C(X) \quad (\forall x \in X : f(x) = 1),
\]

we deduce that \( 1 \in C(X) \cap M \).

By \( \exists $, \( C(K) \cap M \) is a closed subalgebra of \( C(K) \) that contains \( (A \cup \{1\}) \cap M \), so we have \( \text{alg}((A \cup \{1\}) \cap M) \subset C(K) \cap M \). For the other inclusion, pick \( f \in C(K) \cap M \). By Lemma 5 and the absoluteness of the formula (and its subformulas)

\[
\exists A \subset A \quad (A \text{ is countable and } f \in \text{alg}(A \cup \{1\})),
\]

there is a countable set \( A \subset A \) with \( A \in M \) and \( f \in \text{alg}(A \cup \{1\}) \). By Lemma 7.4, we have \( A \subset A \cap M \). Therefore, since \( 1 \in M \), we obtain \( \text{alg}((A \cup \{1\}) \cap M) = \text{alg}((A \cap M) \cup \{1\}) \supset C(K) \cap M \).
By (8), Lemma 5 and the absoluteness of the formula (and its subformulas)

\[ \exists A \subset \mathcal{C}(K) \quad (A \text{ is countable and separates the points of } K'), \quad (*) \]

there is a countable set \( A \subset \mathcal{C}(K) \) with \( A \in M \) which separates the points of \( K' \). By Lemma 7, we have \( A \subset \mathcal{C}(K) \cap M \) so \( \mathcal{C}(K) \cap M \) separates the points of \( K' \). Therefore, since by (8) the set \( \mathcal{C}(K) \cap M \) is a closed algebra containing constant functions, the Stone–Weierstrass theorem ensures that the set \( \{ f|_{K'} : f \in \mathcal{C}(K) \cap M \} \) is dense in \( \mathcal{C}(K') \), which implies that \( \{ f|_{K'} : f \in \mathcal{C}(K) \cap M \} \) is dense in \( \mathcal{C}(K') \) and therefore \( \{ f^{-1}(-1/2, 1/2)\cap K' : f \in \mathcal{C}(K) \cap M \} \) is an open basis of \( K' \). Moreover, for every \( f \in \mathcal{C}(K) \cap M \), using Lemma 5 and the absoluteness of the formula (and its subformulas)

\[ \exists x \in f^{-1}(-1/2, 1/2) \cap K', \quad (*) \]

we have \( f^{-1}(-1/2, 1/2) \cap (K' \cap M) \neq \emptyset \) for every \( f \in \mathcal{C}(K) \cap M \) and therefore the set \( K' \cap M \) is dense in \( K' \).

Since \( D \subset K \) is a dense set, we see that \( ||f|| = ||f|_D|| \). It follows from (8) that \( ||\cdot|| \in M \) and so \( ||f|| \in M \). Therefore, using Lemma 5 and the absoluteness of the formula (and its subformulas)

\[ \forall n \in \omega \exists x \in D \quad (||f|| - 1/n < |f(x)| < ||f|| + 1/n), \quad (*) \]

we deduce that for every \( n \in \omega \), there exists \( x_n \in D \cap M \) with \( |f(x_n)| \to ||f|| \).

### 3.2. Retractions associated to suitable models

Here we show that in a compact space with a retractional skeleton, for every suitable model there is an associated canonical retraction (see Definition 12). The main outcome of this subsection is Theorem 15, where the properties of the canonical retraction are summarized.

Lemmas 9 and 10 are inspired by Lemma 4.7, where something similar was proved for suitable models which are countable.

**Lemma 9.** For every suitable model \( M \) the following holds: Let \( X \) be a set and let \( A \subset \mathbb{R}^X \) satisfy \( \{X\} \cup A \subset M \). Consider the mapping \( q_M : X \to \mathbb{R}^A \) defined for \( x \in X \) by \( q_M(x)(f) := f(x) \) for \( f \in A \). Then for every \( B \subset X \) with \( B \in M \) we have \( q_M[B] \subset q_M[B \cap M] \).

**Proof.** In this proof we will use the identification of any \( n \in \omega \) with the set \( \{0, \ldots, n-1\} \). Further, denote by \( B \) the set of all open intervals with rational endpoints and by \( B^{<\omega} \) the set of all functions whose domain is some \( n \in \omega \) and whose values are in \( B \).

Let \( S \) be the countable set from the statement of Lemma 7 enriched by \( \{B, B^{<\omega}\} \) and let \( \Phi \) be the list of formulas from the statement of Lemma 7 enriched by the formulas (and their subformulas) marked by \((*)\) in the proof below. Let \( M \prec (\Phi; S \cup \{X\} \cup A) \).
Characterization of Eberlein compacta

Fix $B \subset X$ with $B \in M$, a point $x \in B$ and a basic neighborhood of a point $q_M(x)$; that is, let us pick finitely many functions $F \subset A$ and a sequence of rational intervals such that $f(x) \in I_f, f \in F$, and consider the neighborhood

$$N := \{ y \in \mathbb{R}^A : y(f) \in I_f \text{ for every } f \in F \}.$$ 

By Lemma 7(3), we have $F \in M$, and by the absoluteness of the formula

$$\exists n \in \omega \exists \eta \ (\eta \text{ is a bijection between } n \text{ and } F) \quad (*)$$

and its subformulas, there is $n \in \omega$ and a bijection $\eta \in M$ between $n$ and $F$.

Let us further define the mapping $\xi : n \rightarrow \mathcal{B}$ by $\xi(i) = I_{\eta(i)}$. Since $\xi \in \mathcal{B}^{<\omega} \in M$, it follows from Lemma 7(4) that $\xi \in M$. By Lemma 5 and the absoluteness of the formula (and its subformulas)

$$\exists x \in B \ (\forall i < n : \eta(i)(x) \in \xi(i)),$$

there is a point $x_0 \in B \cap M$ such that $q_M(x_0) \in N$; hence, $q_M[B \cap M]$ is dense in $q_M[B]$.

**Lemma 10.** For every suitable model $M$ the following holds: Let $(K, \tau)$ be a compact space and $D \subset K$ be a dense subset with $\{K, D, \tau\} \subset M$. If $\mathcal{C}(K) \cap M$ separates the points of $D \cap \overline{M}$, then there exists a unique retraction $r_M : K \rightarrow D \cap \overline{M}$ such that $f = f \circ r_M$, for every $f \in \mathcal{C}(K) \cap M$. Moreover, in this case

1. For any $x \in K$ and $A \subset \mathcal{C}(K)$ separating the points of $K$ with $A \in M$, $r_M(x)$ is the unique point from $D \cap \overline{M}$ satisfying $f(r_M(x)) = f(x)$ for every $f \in A \cap M$.
2. If $B \subset D$ and $B \in M$, then $r_M[B] = B \cap \overline{M}$.

**Proof.** Let $S$ and $\Phi$ be the unions of the sets and the lists of formulas from the statements of Lemmas 7–9. Let $M \prec (\Phi; S \cup \{K, D, \tau\})$ be such that $\mathcal{C}(K) \cap M$ separates the points of $D \cap \overline{M}$. By Lemma 8[8], we have $\mathcal{C}(K) \in M$.

Let us consider the mapping $q_M : K \rightarrow \mathbb{R}^{\mathcal{C}(K) \cap M}$ given by $q_M(x) = (f(x))_{f \in \mathcal{C}(K) \cap M}, x \in K$. Then $q_M$ is continuous and, by the assumption, $q_M|_{D \cap \overline{M}}$ is one-to-one; hence, $q_M|_{D \cap \overline{M}}$ is a homeomorphic embedding. Moreover, whenever $B \subset D$ is such that $B \in M$, then by Lemma 9 we have $q_M[B \cap \overline{M}] \supset q_M[B]$, which implies $q_M[B] = q_M[B \cap \overline{M}]$.

Now, put $r_M := (q_M|_{D \cap \overline{M}})^{-1} \circ q_M$. Then it is a continuous retraction with $r_M[K] = D \cap \overline{M}$. Moreover, for every $x \in K$,

$$r_M(x) = (q_M|_{D \cap \overline{M}})^{-1} \circ q_M(x) = y,$$

where $y \in K$ is the unique point such that $y \in D \cap \overline{M}$ and $g(y) = g(x)$ for every $g \in \mathcal{C}(K) \cap M$. Hence, for $f \in \mathcal{C}(K) \cap M$ we have

$$f(r_M(x)) = f(y) = f(x).$$
In order to see that $r_M$ is unique, let us consider another retraction $r': K \to D \cap M$ satisfying $f = f \circ r'$ for every $f \in \mathcal{C}(K) \cap M$. Then, for every $x \in K$, and every $f \in \mathcal{C}(K) \cap M$, we have $f(r_M(x)) = f(x) = f(r'(x))$; hence, since $\mathcal{C}(K) \cap M$ separates the points of $r_M[K]$, we see that $r_M(x) = r'(x)$. Since $x \in K$ was arbitrary, we have $r_M = r'$. Moreover, given $y \in D \cap M$ such that $f(y) = f(x)$ for every $f \in \mathcal{A} \cap M$, where $\mathcal{A} \subset \mathcal{C}(K)$ is a set separating the points of $K$ with $\mathcal{A} \in M$, we deduce that $f(y) = f(x)$ for $f \in \text{alg}((\mathcal{A} \cup \{1\}) \cap M) = \mathcal{C}(K) \cap M$ (the last equality follows from Lemma 8(9)) and so $y = r_M(x)$.

Finally, if $B \subset D$, such that $B \in M$, then by the above we have $q_M[B] = q_M[B \cap M]$ and so $r_M[B] = r_M[B \cap M] = B \cap M$.  

Let us note that a compact space $K$ admits a retractional skeleton if and only if there exists a dense set $D \subset K$ such that for every suitable model $M$ which is moreover countable, the set $\mathcal{C}(K) \cap M$ separates the points of $D \cap M$ (see e.g. [7, Theorem 4.9] or [19, Theorem 19.16]; that is, the assumption of Lemma 10 is satisfied for suitable models which are countable). The following shows that we do not need to assume countability of the model.

**Proposition 11.** For every suitable model $M$ the following holds: If $(K, \tau)$ is a compact space and $D \subset K$ is a subset of a set induced by a retractional skeleton with $\{D, K, \tau\} \subset M$, then $\mathcal{C}(K) \cap M$ separates the points of $D \cap M$.

**Proof.** Let $S$ be the union of the sets from the statements of Lemmas 7 and 8 and let $\Phi$ be the union of the lists of formulas from those statements enriched by the formulas (and their subformulas) marked by $(\ast)$ in the proof below. Let $M \prec (\Phi; S \cup \{K, D, \tau\})$.

By Lemma 5 and the absoluteness of the formula (and its subformulas) $\exists \Gamma \exists \leq \exists r \quad (D \text{ is a subset of the set induced by})$

the retractional skeleton $\{r(s): s \in \Gamma\}$,

there exist $\Gamma, \leq, r \in M$ such that $\{r(s): s \in \Gamma\}$ is a retractional skeleton on $K$ inducing a set containing $D$. For $s \in \Gamma$ we will write $r_s$ instead of $r(s)$. By Lemma 8(1), the set $\Gamma \cap M$ is up-directed. Hence by Lemma 2 there exists a continuous retraction $R_M: K \to K$ defined by $R_M(x) := \lim_{s \in \Gamma \cap M} r_s(x)$ for every $x \in K$. Using the absoluteness of the formula (and its subformulas)

$\forall u \in D \exists s \in \Gamma \quad (u \in r_s[K])$,

we deduce that $D \cap M \subset R_M[K]$ and therefore $D \cap M \subset R_M[K]$. Now fix $x, y \in D \cap M$ with $x \neq y$. Since $x = \lim_{s \in \Gamma \cap M} r_s(x)$ and $y = \lim_{s \in \Gamma \cap M} r_s(y)$, there exists $s \in \Gamma \cap M$ such that $r_s(x) \neq r_s(y)$. By Lemma 7(2), we have $r(s) = r_s \in M$ and $r_s[K] \in M$. Thus, by Lemma 8(10), there exists $f \in \mathcal{C}(K) \cap M$ such that $f(r_s(x)) \neq f(r_s(y))$. Now using Lemma 8(12), we obtain
$g = f \circ r_s \in \mathcal{C}(K) \cap M$ and $g(x) \neq g(y)$. Thus, $\mathcal{C}(K) \cap M$ separates the points of $D \cap M$. ■

The retraction constructed in Lemma 10 (whose assumption is satisfied by Proposition 11 in compact spaces admitting a retractional skeleton) will be the key to our considerations. Let us give it a name.

**Definition 12.** Let $K$ be a compact space and let $D \subset K$ be a dense subset that is contained in the set induced by a retractional skeleton. Given a set $M$, we say that $r_M$ is the canonical retraction associated to $M$, $K$ and $D$ if it is the unique retraction on $K$ satisfying $r_M[K] = D \cap M$ and $f = f \circ r_M$, for every $f \in \mathcal{C}(K) \cap M$. We say that a set $M$ admits canonical retraction if the canonical retraction associated to $M$, $K$ and $D$ exists.

When $D = K$ we say that $M$ admits the canonical retraction $r_M$ associated to $M$ and $K$.

The properties of canonical retractions associated to suitable models are summarized in Theorem 15. We need two lemmas first.

**Lemma 13.** For every suitable model $M$ the following holds: Let $(K, \tau)$ be a compact space and $A \subset \mathcal{C}(K)$ be a set separating the points of $K$ with $\{A, K, \tau\} \subset M$. Then for every compact set $K' \subset K$, $A \cap M$ separates the points of $K'$ if and only if $\mathcal{C}(K) \cap M$ separates the points of $K'$.

**Proof.** Let $S$ and $\Phi$ be the set and the list of formulas from the statements of Lemmas 7 and 8. Let $M \not\prec (\Phi; S \cup \{K, \tau, A\})$. In order to get a contradiction, let us assume that $\mathcal{C}(K) \cap M$ separates the points of $K'$ but $A \cap M$ does not separate the points of $K'$. Then alg$(\mathcal{A} \cap M \cup \{1\})$ does not separate the points of $K'$ either (because if there are $x \neq y$ with $f(x) = f(y)$ for every $f \in \mathcal{A} \cap M$, then also $g(x) = g(y)$ for every $g$ of the form $g = a_0 + \sum_{i=1}^n a_i \prod_{j=1}^m f_{ij}$). But this is indeed a contradiction, because using Lemma 8, we conclude that $\text{alg}((\mathcal{A} \cap M \cup \{1\}) = \overline{\mathcal{C}(K) \cap M}$. ■

**Lemma 14.** For every suitable model $M$ the following holds: Let $(K, \tau)$ be a compact space and let $D \subset K$ be a dense subset that is contained in the set induced by a retractional skeleton such that $\{K, D, \tau\} \subset M$. Then the mapping $\Phi : \overline{\mathcal{C}(K) \cap M} \to \mathcal{C}(D \cap M)$ defined by $\Phi(f) := f|_{D \cap M}$ for every $f \in \overline{\mathcal{C}(K) \cap M}$ is a surjective isometry.

**Proof.** Let $S$ and $\Phi$ be the unions of the countable sets and the finite lists of formulas from the statements of Lemma 12 and Proposition 11. Let $M \not\prec (\Phi; S \cup \{K, D, \tau\})$. By Lemma 12, we have $\|f\| = \|f|_{D \cap M}\|$ for every $f \in \mathcal{C}(K) \cap M$, so the mapping $\Phi|_{\mathcal{C}(K) \cap M}$ is an isometry, which implies that $\Phi$ is also an isometry. It remains to show that it is surjective. By Lemma 12, $\mathcal{C}(K) \cap M$ is a closed subalgebra of $\mathcal{C}(K)$ and so the image of $\Phi$ is a closed subalgebra of $\mathcal{C}(D \cap M)$ which, by Proposition 11, separates the
Whenever \(\varphi\) and Proposition 11 and Proposition 11 and Lemma 10 that statements enriched by the formulas (and their subformulas) marked by \(r\) in the proof below. Let \(M\) be a compact space and let \(K \subset D\) be a dense subset that is contained in the set induced by a retractional skeleton with \(\{K, D, \tau\} \subset M\). Then there exists a unique retraction \(r_M : K \to D \cap M\) with \(r_M[K] = D \cap M\) and \(f = f \circ r_M\) for every \(f \in C(K) \cap M\). Moreover, for this retraction \(r_M\) the following hold:

(i) Whenever \(A \subset C(K)\) separates the points of \(K\) and \(A \subset M\), then for every \(x \in K\), \(r_M(x)\) is the unique point from \(D \cap M\) satisfying \(f(r_M(x)) = f(x)\) for every \(f \in A \cap M\).

(ii) Whenever \((\Gamma, \leq)\) is up-directed and \(\sigma\)-complete and \(r : \Gamma \to C(K, K)\) is a mapping such that \(s = \{r(s) : s \in \Gamma\}\) is a retractional skeleton on \(K\) with \(D \subset D(s)\) and \(\{r, \Gamma, \leq\} \subset M\), then:

(a) for every \(s \in \Gamma \cap M\), \(r(s)[K] \subset D \cap M\);
(b) we have

\[
r_M(x) = \lim_{s \in \Gamma \cap M} r(s)(x), \quad x \in K;
\]

c) if \(M\) is countable, then \(r_M = r(s)\) for \(s = \sup \Gamma \cap M\);

d) \(\{r(s)[r_M[K]] : s \in (\Gamma \cap M)_\sigma\}\) is a retractional skeleton on \(r_M[K]\) with induced set \(D(s) \cap r_M[K]\);

e) if \(s\) is commutative, then \(r_M[D(s)] = D(s) \cap r_M[K]\).

(iii) Whenever \(h : K \to L\) is a surjective homeomorphism with \(h \in M\), then \(r := h \circ r_M \circ h^{-1}\) is the unique retraction on \(L\) such that \(r[L] = h[D] \cap M\) and \(f \circ r = f\) for every \(f \in C(L) \cap M\).

(iv) \(w(r_M[K]) \leq |M|\).

**Proof.** Denote by \(B\) the set of all open intervals with rational endpoints. Let \(S\) be the union of the sets from the statements of Lemmas 7, 8, 10, 14 and Proposition 11 and let \(\Phi\) be the union of the lists of formulas from those statements enriched by the formulas (and their subformulas) marked by \((\ast)\) in the proof below. Let \(M < (\Phi; S \cup \{K, D, \tau\})\). It follows directly from Proposition 11 and Lemma 10 that \(r_M\) exists and satisfies (i).

Let \(\{r, \Gamma, \leq\} \subset M\) be as in (ii). For \(s \in \Gamma \cap M\), by Lemma 7(2) we know that \(r(s)[K] \subset M\) and thus Lemma 8(10) ensures that \(r(s)[K] \subset r(s)[K] \cap M\). Moreover, for every \(x \in r(s)[K] \cap M\), by the countable tightness of \(D(s)\) the following formula holds:

\[
\exists C \subset D \quad (C \text{ is countable and } x \in C).
\]

Thus, by Lemma 5 there exists a countable set \(C \subset D\) with \(C \subset M\) (which implies \(C \subset M\)) such that \(x \in C\), which implies that \(x \in D \cap M\), so \(r(s)[K] \subset r(s)[K] \cap M \subset D \cap M\) and (a) holds. By Lemma 8(1), \(\Gamma \cap M\) is
up-directed and so, by Lemma \[2\] the limit \( \lim_{s \in \Gamma \cap M} r(s)(x) =: R_M(x) \) exists for every \( x \in K \).

We claim that \( R_M = r_M \). Note that due to the uniqueness of \( r_M \), it is enough to show that \( R_M[K] \subseteq \overline{D \cap M} \) and that \( f \circ R_M = f \) for every \( f \in \mathcal{C}(K) \cap M \). By the definition of \( R_M \), using \([1]\) we observe that \( R_M[K] \subseteq \overline{D \cap M} \). Moreover, for every \( x \in D \cap M \), by Lemma \[3\] and the absoluteness of the formula (and its subformulas)

\[
\exists s \in \Gamma \quad (r_s(x) = x),
\]

there is \( s \in \Gamma \cap M \) with \( r_s(x) = x \) and so \( x \in R_M[K] \). Thus, we find that \( R_M[K] = \overline{D \cap M} \). Pick \( f \in \mathcal{C}(K) \cap M \). Since \( (r_s)_{s \in \Gamma \cap M} \) converges pointwise to \( R_M \), using \([21]\) Lemma \[5.2\] we conclude that \( (f \circ r_s)_{s \in \Gamma \cap M} \) converges in norm to \( f \circ R_M \). Therefore \( f \circ R_M \in \overline{C(K) \cap M} \), since it follows from Lemmas \[7\] and \[8\] that \( f \circ r_s \in \mathcal{C}(K) \cap M \) for every \( s \in \Gamma \cap M \). Thus \( f \) and \( f \circ R_M \) are two functions from \( \overline{C(K) \cap M} \) which have the same values on \( \overline{D \cap M} \) and so, by Lemma \[14\], they are equal. This proves the claim and establishes (b), which, by Lemma \[2\], implies (c)–(e).

The proof of (iii) is easy once we realize that by Lemma \[8\] we have \( h[D \cap M] = h[D] \cap M \) and \( f \circ h \in \mathcal{C}(K) \cap M \) for every \( f \in \mathcal{C}(L) \cap M \). We omit the straightforward details.

Finally, (iv) follows from Proposition \[11\] since \( R_M[K] = \overline{D \cap M} \). \( \blacksquare \)

**3.3. Families of canonical retractions.** Here we study families of canonical retractions associated to suitable models. Those are more or less straightforward consequences of Theorem \[15\]. The most important for what follows is Proposition \[17\] which will be repeatedly used later.

**Lemma 16.** There is a countable set \( S \) and a finite list of formulas \( \Phi \) such that the following holds: Let \( (K, \tau) \) be a compact space, \( (\Gamma, \leq) \) be an up-directed set, \( r : \Gamma \to \mathcal{C}(K, K) \) be a mapping such that \( s := \{r(s) : s \in \Gamma\} \) is a retractional skeleton on \( K \) and let \( D \subset D(s) \) be dense in \( K \). Put \( S' = \bigcup \{K, \tau, D, \Gamma, \leq, r\} \). Then every \( M \prec (\Phi; S') \) admits the canonical retraction \( r_M \) associated to \( M, K \) and \( D \). Moreover, we have the following.

1. If \( M, N \prec (\Phi; S') \) and \( M \subset N \), then \( r_M \circ r_N = r_N \circ r_M = r_M \).
2. Let \( \mathcal{M} \) be an up-directed set with \( M \prec (\Phi; S') \) for every \( M \in \mathcal{M} \), and let \( M_\infty := \bigcup_{M \in \mathcal{M}} M \). Then \( M_\infty \prec (\Phi; S') \) and \( \lim_{M \in \mathcal{M}} r_M(x) = r_{M_\infty}(x) \), for every \( x \in K \).
3. If \( \mathcal{U} \) is a basis of \( \tau \), \( \mathcal{M} \) is an up-directed set with \( M \prec (\Phi; S') \) for every \( M \in \mathcal{M} \), and \( \mathcal{U} \subset \bigcup_{M \in \mathcal{M}} M \), then \( \lim_{M \in \mathcal{M}} r_M(x) = x \) for every \( x \in K \).

**Proof.** The existence of \( S \) and \( \Phi \) follows from Theorem \[15\]. Let us prove the “Moreover” part using the additional properties of canonical retractions established in Theorem \[15\].
Since $r_M[K] = D \cap M \subset D \cap N = r_N[K]$, we have $r_M = r_N \circ r_M$. Furthermore, for every $f \in C(K) \cap M \subset C(K) \cap N$ we have

$$f(r_M(x)) = f(x) = f(r_N(x)) = f(r_M(r_N(x))), \quad x \in K,$$

and so $r_M(x) = r_M(r_N(x))$ for every $x \in K$.

2. Since $M$ is up-directed, it follows from Lemma 2.1 and Lemma 5 that $M_\infty \prec (S'; \Phi)$. Now, combining Theorem \ref{thm:main} with Lemmas \ref{lem:main} and \ref{lem:main}, we obtain $\lim_{M \in M} r_M(x) = r_{M_\infty}(x)$, $x \in K$.

3. Pick $x \in K$, $U \in U$ such that $x \in U$ and find $V \in U$ with $x \in V \subset \overline{V} \subset U$. Since $V \in U$ and $U \cup \bigcup_{M \in M} M$, there exists $M_0 \in M$ such that $V \in M_0$. Now fix $M \in M$ with $M_0 \subset M$. It follows from Lemma \ref{lem:main} that $V \cap D \in M$. Therefore Lemma \ref{lem:main} ensures that $r_M(x) \in r_M[\overline{V \cap D}] \subset V \cap D \subset U$, since $x \in \overline{V \cap D}$.

Proposition 17. There exist a countable set $S$ and a finite list of formulas $\Phi$ such that the following holds: Let $(K, \tau)$ be a compact space, $(\Gamma, \leq)$ be an up-directed set and $\tau : \Gamma \to C(K, K)$ be a mapping such that $S = \{r(s) : s \in \Gamma\}$ is a retractional skeleton on $K$. Let $\kappa := w(K)$ and $U : \kappa \to \tau$ be such that $\{U(i) : i < \kappa\}$ is an open basis of $\tau$. Put $S' := S \cup \{K, D(S), \Gamma, \leq, r, \tau, U\}$. Let $(M_\alpha)_{\alpha \leq \kappa}$ be a sequence of sets satisfying

1. $M_\alpha \prec (\Phi; S')$ for every $\alpha \in [0, \kappa]$,
2. $|M_\alpha| \leq \max(\omega, |\alpha|)$ for every $\alpha \in [0, \kappa]$,
3. $M_{\alpha+1} \supset M_\alpha \cup \{\alpha\}$ for every $\alpha \in [0, \kappa]$,
4. $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$ if $\alpha \in (0, \kappa]$ is a limit ordinal.

Then for every $\alpha \in [0, \kappa]$ there exists the canonical retraction $r_\alpha$ associated to $M_\alpha$, $K$ and $D(S)$ and the following hold:

1. For every $\alpha < \beta$, we have $r_\alpha \circ r_\beta = r_\beta \circ r_\alpha = r_\alpha$.
2. $r_\alpha(x) \to x$ for every $x \in K$.
3. Let $\alpha \leq \kappa$, let $\eta : [0, \alpha) \to \kappa$ be an increasing function and let $\xi \leq \kappa$ be a limit ordinal with $\sup_{\beta < \alpha} \eta(\beta) = \xi$. Then $\lim_{\beta < \alpha} r_\eta(\beta)(x) = r_\xi(x)$ for every $x \in K$.
4. $r_\kappa = \text{id}$.
5. For $\alpha \in [0, \kappa]$, we have $w(r_\alpha[K]) \leq \max(\omega, |\alpha|)$ and $(r_s|_{r_\alpha[K]})_{s \in (\Gamma \cap M_\alpha)_{\sigma}}$ is a retractional skeleton on $r_\alpha[K]$ with induced set $D(S) \cap r_\alpha[K]$.
6. If $A$ is a closed subset of $C(K)$ and $f \circ r_s \in A$ for every $f \in A$ and $s \in \Gamma$, then $\{f \circ r_\alpha : f \in A, \alpha \leq \kappa\} \subset A$.
7. If $x, y$ are distinct points in $K$ then $\beta := \min \{\alpha < \kappa : r_\alpha(x) \neq r_\alpha(y)\}$ exists and it is a successor ordinal or $\beta = 0$.
8. For every $\alpha \leq \kappa$, the set $\Gamma \cap M_\alpha$ is up-directed and

$$r_\alpha(x) = \lim_{s \in (M_\alpha \cap \Gamma)} r_s(x)$$

for every $x \in K$.  


Moreover, if \( s \) is full or commutative, then \( r_\alpha[D(s)] \subset D(s) \) for every \( \alpha \in [0, \kappa] \), and if \( D \subset D(s) \) is such that \( r_\alpha[D] \subset D \) for every \( \alpha \in [0, \kappa] \), then we also have the following:

(R11) The sets \( \{ r_\alpha(x) : \alpha < \kappa \} \) and \( \{ \alpha < \kappa : r_\alpha(x) \neq r_{\alpha+1}(x) \} \) are countable for every \( x \in D \).

\[ \text{Proof.} \] Let \( S \) and \( \Phi \) be the unions of the sets and the lists of formulas from the statements of Theorem 15 and Lemma 16. Then the existence of \( r_\alpha \), \( \alpha \in [0, \kappa] \), follows from Theorem 15. Now, (R1) follows immediately from Lemma 16. (R2) and (R3) follow from Lemma 16 as well (using for (R2) the fact that \( \{ U(i) : i < \kappa \} \subset \bigcup \alpha < \kappa M_\alpha \) and for (R3) the fact that \( \bigcup \beta \leq \alpha \bigcup M_{\eta(\beta)} = M_\xi \)). (R4) follows from (R2) and (R3) applied to \( \eta(i) = i \), \( i < \kappa \), and \( \xi = \kappa \). (R5) and (R10) follow from Theorem 15. For (R6) we observe that by Theorem 11(ii) the net of continuous retractions \( (s_r)_{s \in \Gamma \cap M_\alpha} \) converges pointwise to the continuous retraction \( r_\alpha \) and so [21, Lemma 5.2] ensures that the net \( (f \circ r_s)_{s \in \Gamma \cap M_\alpha} \) converges in norm to \( f \circ r_\alpha \) for every \( f \in C(K) \), which implies (R6). For (R7) we observe that by (R2) there is \( i < \kappa \) such that \( r_i(x) \neq r_i(y) \) so \( \beta \) is well-defined, and if \( \beta \neq 0 \) then it is a successor ordinal by (R3). For (R8) it suffices to apply Lemma 3[1] and Theorem 15(ii) (R9) follows from Lemma 2.[iii]

Moreover, if \( s \) is full then obviously \( r_\alpha[D(s)] = r_\alpha[K] \subset K = D(s) \), and if it is commutative then Theorem 11[iii] ensures that \( r_\alpha[D(s)] \subset D(s) \).

(R11) Pick \( x \in D \) and note that to prove that \( \{ r_\alpha(x) : \alpha < \kappa \} \) is countable, it suffices to show that for every strictly increasing function \( \eta : [0, \omega_1] \rightarrow \kappa \), there is \( \zeta < \omega_1 \) with \( r_{\eta(\zeta)}(x) = r_{\eta(\beta)}(x) \) for every \( \zeta < \beta < \omega_1 \). Let \( \eta : [0, \omega_1] \rightarrow \kappa \) be a strictly increasing function and set \( \xi := \sup_{\beta < \omega_1} \eta(\beta) \).

By (R3) we have \( r_\xi(x) = \lim_{\beta < \omega_1} r_{\eta(\beta)}(x) \). Hence, since \( r_\xi[D] \subset D \) and \( D \) has countable tightness (see [22, Theorem 32]), there is a \( \zeta < \omega_1 \) with \( r_\xi(x) \in r_{\eta(\zeta)}[K] \); so, for \( \zeta \leq \beta < \omega_1 \), using (R1) we obtain \( r_{\eta(\beta)}(x) = r_{\eta(\beta)}(r_\xi(x)) = r_\xi(x) \). Finally, to conclude that the set \( \{ \alpha < \kappa : r_\alpha(x) \neq r_{\alpha+1}(x) \} \) is countable, note that the mapping \( \varphi(\alpha) = r_\alpha(x) \) is an injection from this set into \( \{ r_\alpha(x) : \alpha < \kappa \} \). Indeed, suppose on the contrary that there exist \( \alpha, \beta < \kappa \) with \( \alpha \neq \beta \), \( r_\alpha(x) \neq r_{\alpha+1}(x) \) and \( r_\beta(x) \neq r_{\beta+1}(x) \) such that \( r_\alpha(x) = r_\beta(x) \).

Without loss of generality, we may assume that \( \alpha < \beta \). Then applying the map \( r_{\alpha+1} \), by (R1) we obtain \( r_\alpha(x) = r_{\alpha+1}(r_\alpha(x)) = r_{\alpha+1}(r_\beta(x)) = r_{\alpha+1}(x) \), which is a contradiction. \( \blacksquare \)
3.4. Application: passing to a subskeleton. Here we introduce the notion of a (weak) subskeleton and show that for a countable family of retractional skeletons inducing the same set there is a common weak subskeleton (see Theorem 21).

**Definition 18.** Let $K$ be a compact space and let $s = (r_s)_{s \in \Gamma}$ be a retractional skeleton on $K$. We say that $(r_s)_{s \in \Gamma'}$ is a subskeleton of $s$ if $\Gamma' \subset \Gamma$ is a $\sigma$-closed and cofinal subset.

It is easy to see that every subskeleton is a retractional skeleton.

**Definition 19.** Let $(r_s)_{s \in \Gamma}$ be a retractional skeleton on a compact space $K$. We say that $(R_i)_{i \in \Lambda}$ is a weak subskeleton of $(r_s)_{s \in \Gamma}$ if $(R_i)_{i \in \Lambda}$ is a retractional skeleton on $K$ and there exists a mapping $\varphi : \Lambda \to \Gamma$ such that

- $R_i = r_{\varphi(i)}$ for every $i \in \Lambda$;
- $\varphi$ is $\omega$-monotone, that is, if $i, j \in \Lambda$ with $i \leq j$, then $\varphi(i) \leq \varphi(j)$, and if $(i_n)_{n \in \omega}$ is an increasing sequence from $\Lambda$, then $\sup_n \varphi(i_n) = \varphi(\sup_n i_n)$;
- $\{\varphi(i) : i \in \Lambda\}$ is cofinal in $\Gamma$.

Clearly, every subskeleton of a retractional skeleton is also a weak subskeleton. Some basic properties of weak subskeletons are summarized below.

**Fact 20.** Let $(r_s)_{s \in \Gamma}$ be a retractional skeleton on a compact space $K$, and $(R_i)_{i \in \Lambda}$ be a weak subskeleton of $(r_s)_{s \in \Gamma}$. Then

- $(R_i)_{i \in \Lambda}$ induces the same subset as $(r_s)_{s \in \Gamma}$;
- if $(S_j)_{j \in \Delta}$ is a weak subskeleton of $(R_i)_{i \in \Lambda}$, then it is a weak subskeleton of $(r_s)_{s \in \Gamma}$.

The following result shows that we may concentrate properties of countably many retractional skeletons into one skeleton, which is moreover generated by suitable models.

**Theorem 21.** Let $K$ be a compact space and let $(r^n_s)_{s \in \Gamma_n}$, $n \in \omega$, be a sequence of retractional skeletons on $K$ inducing the same set $D$. Then there exists a retractional skeleton which is a weak subskeleton of $(r^n_s)_{s \in \Gamma_n}$ for every $n \in \omega$. Moreover, for every countable set $S$ and every finite list of formulas $\Phi$, there exists a family $\mathcal{M}$ consisting of countable suitable models for $\Phi$ containing $S$ such that every $M \in \mathcal{M}$ admits the canonical retraction $r_M$ associated to $M$, $K$ and $D$, and $(r_M)_{M \in \mathcal{M}}$ is a weak subskeleton of $(r^n_s)_{s \in \Gamma_n}$ for every $n \in \omega$, where the ordering on $\mathcal{M}$ is given by inclusion.

**Proof.** Let $\Gamma_n = (\Gamma_n, \leq_n)$ and $r_n : \Gamma_n \to C(K, K)$ be such that $r_n(s) := r^n_s$ for all $n \in \omega$ and $s \in \Gamma_n$. Let $\tau$ be the topology on $K$. Let $S'$ be the union of $S$ and the countable set from the statement of Theorem 15 enriched by $\{K, D, \tau, r_n, \Gamma_n, \leq_n : n \in \omega\}$ and let $\Phi'$ be the union of $\Phi$ and the list of formulas from the statement of Theorem 15. By Theorem 4, there is a set...
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Let \( R \supset S' \cup \tau \cup \bigcup_{n \in \omega} \Gamma_n \) such that \( R \prec (\Phi'; S') \) and for every countable set \( Z \subset R \) there is a countable set \( M(Z) \subset R \) satisfying \( M(Z) \prec (\Phi'; Z) \). Set

\[
\mathcal{M} = \{ M \in [R]^\omega : M \prec (\Phi'; S') \},
\]

ordered by inclusion. The \( \sigma \)-completeness of \( \mathcal{M} \) follows from [6, Lemma 2.4]. To see that \( \mathcal{M} \) is up-directed, let \( N_1, N_2 \in \mathcal{M} \). Then \( M(N_1 \cup N_2) \in \mathcal{M} \) and \( N_1 \cup N_2 \subset M(N_1 \cup N_2) \). By Theorem 15 every \( M \in \mathcal{M} \) admits the canonical retraction \( r_M \) associated to \( M, K \) and \( D \). Note that for every \( U \in \tau \) there is \( M \in \mathcal{M} \) with \( U \subseteq M \) (it suffices to put \( M = M(\{U \} \cup S') \) and so \( \tau \cap \bigcup \mathcal{M} = \tau \) is indeed a basis of the topology \( \tau \). Therefore it follows from Theorem 14(iv) and Lemma 16 that \( (\tau_M)_{M \in \mathcal{M}} \) is a retractional skeleton on \( K \). Now for every \( n \in \omega \), let \( \varphi_n : \mathcal{M} \to \Gamma_n \) be the mapping defined by \( \varphi_n(M) := \sup(\Gamma_n \cap M) \). By Theorem 13(ii) we have \( r_M = r_n \varphi_n(M) \) for every \( n \in \omega \). Now, let \( (M_k)_{k \in \omega} \subset \mathcal{M} \) be an increasing sequence. Then it is easy to see that \( \sup_k \varphi_n(M_k) = \varphi_n(M_\omega) \), where \( M_\omega = \bigcup_{k \in \omega} M_k \). It remains to prove that the set \( \{ \varphi_n(M) : M \in \mathcal{M} \} \) is cofinal in \( \Gamma_n \) for each \( n \in \omega \). Let \( n \in \omega \) and \( s \in \Gamma_n \). Then there exists \( M \in \mathcal{M} \) such that \( s \in M \) (it suffices to put \( M = M(\{s \} \cup S') \)). Therefore \( \varphi_n(M) = \sup(\Gamma_n \cap M) \geq s \).

4. Valdivia embedding of compact spaces admitting a commutative skeleton. Giving a compact space \( K \) that admits a commutative retractional skeleton \( \mathfrak{s} \), it is known that there exists a homeomorphic embedding \( h : K \to [-1, 1]^I \) such that \( h[D(\mathfrak{s})] \subset \Sigma(I) \) (that is, \( K \) is Valdivia and \( D(\mathfrak{s}) \) is a \( \Sigma \)-subset of \( K \) ). The main aim of this section is to have a very concrete and very flexible way of understanding the mapping \( h \), which is the topic handled in Subsection 4.2, where the proof of Theorem \( \mathbb{D} \) is given. Apart from Theorem \( \mathbb{D} \) we would like to highlight Theorem 27 which gives a new characterization of Valdivia compacta using suitable models.

4.1. Canonical retractions associated to suitable models in Valdivia compact spaces. The goal here is to obtain the following concrete description of the canonical retractions associated to suitable models in Valdivia compact spaces.

**Lemma 22.** For every suitable model \( M \) the following holds: Let \( D \subset \Sigma(I) \) be such that \( K = \overline{D} \) is compact and \( \{K, D, I, \tau\} \subset M \) (where \( \tau \) is the topology on \( K \) ). Then the mapping \( r : K \to K \) defined by \( r(x) = x|_{I \cap M} \) for \( x \in K \) is the canonical retraction associated to \( M, K \) and \( D \).

This was in a certain sense most probably well-known for countable models (see e.g. [24, Lemma 2.4]); here we show that the situation is the same for uncountable models as well. The remainder of this subsection is more-or-less devoted to the proof of Lemma 22. We start with two preliminary results.
Lemma 23. For every suitable model $M$ the following holds: Let $X \subset \Sigma(I)$ be such that $\{X, I\} \subset M$. Then

$$X \cap M = \{x|_{I \cap M} : x \in X\}.$$  

Proof. Let $S$ and $\Phi$ be the unions of the sets and the lists of formulas from the statements of Lemmas 7 and 8, and let $M \prec (\Phi; S \cup \{X, I\})$. Let $\pi : I \to \mathbb{R}^X$ be the mapping given by $\pi_i(x) = x(i)$ for all $x \in X$ and $i \in I$. By Lemmas 7 and 8, we have $\pi \in M$ and $A := \pi[I \cap M] \subset M$. Let $q_M : X \to \mathbb{R}^A$ be the mapping from Lemma 9 that is, for $x \in X$ we have $q_M(x)(\pi_i) = x(i)$. Consider the mapping $\varphi : \mathbb{R}^A \to \mathbb{R}^I$ given for $x \in \mathbb{R}^A$ by $\varphi(x)(i) := x(\pi_i)$ if $i \in I \cap M$, and $\varphi(x)(i) := 0$ if $i \in I \setminus M$. It is easy to see that $\varphi$ is continuous. By Lemma 9, we have $q_M[X] \subset q_M[X \cap M]$, which implies that

$$\{x|_{I \cap M} : x \in X\} = \varphi(q_M[X]) \subset \varphi(q_M[X \cap M]) = \{x|_{I \cap M} : x \in X \cap M\}.$$  

Thus, it suffices to note that for every $x \in X \cap M$ we have $x|_{I \cap M} = x$, which follows from the fact that the support of every $x \in X \cap M$ is contained in $M$ (see Lemma 8). \hfill \blacksquare

The following is well-known. We did not find an exact reference, but proof of Theorem 6.1 can be hinted on (for the key step see also Lemma 1.2]). For the convenience of the reader we present a short argument based on our previous considerations.

Lemma 24. Let $K \subset \mathbb{R}^I$ be a compact space and let $D := \Sigma(I) \cap K$ be dense in $K$. Put

$$\Gamma := \{A \in [I]^{\leq \omega} : x|_A \in K \text{ for every } x \in K\}$$

and for every $A \in \Gamma$ define $r_A : K \to K$ by $r_A(x) := x|_A$ for $x \in K$. Then $(r_A)_{A \in \Gamma}$ is a commutative retractional skeleton on $K$ inducing the set $D$.

Proof. It is obvious that each $r_A$ is a continuous retraction with $r_A[K]$ metrizable. For any $A, B \in \Gamma$, we have $A \cap B \in \Gamma$ and $r_A \circ r_B = r_{A \cap B}$, which implies that $r_A \circ r_B = r_B \circ r_A$. Given an increasing sequence $(A_n)_{n \in \omega}$ from $\Gamma$ and $x \in K$, we have $r_{A_n}x \to x|_{\bigcup_{n \in \omega} A_n}$ and so $A_\infty := \bigcup_{n \in \omega} A_n \in \Gamma$ and $r_{A_\infty}x \to r_{A_\infty}x$. Let us now observe that for every $x \in D$ there is $A \in \Gamma$ with $r_Ax = x$. Indeed, any $x \in D$ has a countable support, so it suffices to see that for every countable $E \subset I$ there is $A \in \Gamma$ with $E \subset A$. Indeed, by Theorem 4 and Lemma 23 (applied to $X = D$), there exists a countable set $M$ such that $E \subset M \cap I$ and $\{x|_{I \cap M} : x \in D\} = D \cap M$, which implies that $M \cap I \in \Gamma$. Finally, the cofinality of $\Gamma$ in $[I]^{\leq \omega}$ implies that the net $(r_A)_{A \in \Gamma}$ converges pointwise to the identity in $K$. Thus, $\Gamma$ is cofinal and $\sigma$-closed in $[I]^{\leq \omega}$ (in particular $\Gamma$ is a $\sigma$-complete up-directed set) and $(r_A)_{A \in \Gamma}$ is a commutative retractional skeleton on $K$ inducing the set $D$. \hfill \blacksquare
Proof of Lemma 22. Let \( S \) and \( \Phi \) be the unions of the countable sets and the finite lists of formulas from the statements of Lemmas \( \mathbb{7}, \mathbb{8} \) and Lemma \( \mathbb{23} \). Pick \( M < (\Phi; S \cup \{ K, D, I, \tau \}) \). Since \( D \) is dense in \( K \) and contained in the set induced by a retractional skeleton (see e.g. Lemma \( \mathbb{24} \)), by Theorem \( \mathbb{15} \) \( M \) admits the canonical retraction \( r_M \) associated to \( M, K \) and \( D \).

By Lemma \( \mathbb{23} \), using the continuity of the mapping \( K \ni x \mapsto x|_{I \cap M} \) and compactness of \( K \), we have \( \overline{D \cap M} = \{ x|_{I \cap M} : x \in K \} \) and so the retraction \( r \) is well-defined, continuous and \( r[K] = \overline{D \cap M} \). Let \( \pi : I \to \mathbb{R}^K \) be the mapping given for \( i \in I \) and \( x \in K \) as \( \pi(i)(x) := x(i) \). By Lemma \( \mathbb{3}(4,6) \), we have \( \pi \in M \) and \( \pi[I] \cap M = \pi[I \cap M] \). Since we obviously have \( f \circ r = f \) for every \( f \in \pi[I \cap M] = \pi[I] \cap M \) with \( \pi[I] \in M \) separating the points of \( K \), using Theorem \( \mathbb{15} \) we obtain \( r = r_M \).

4.2. Valdivia embedding. This subsection is devoted to the proof of Theorem \( \mathbb{D} \); the reasoning is based on \( \mathbb{5} \) proof of Theorem 2.6]. Quite surprisingly, the inductive argument not only gives us the “Valdivia embedding” (that is, Theorem \( \mathbb{D}(i) \Rightarrow (iv) \)), but also provides us with a new characterization of Valdivia compacta (that is, Theorem \( \mathbb{D}(i) \Rightarrow (iii) \)) which we use later. Let us also highlight that there is an analogue of this new characterization in the language of suitable models: see Theorem \( \mathbb{27} \). We start with a lemma.

Lemma 25. Let \( K \) be a compact space and let \( \mathcal{A} \subset C(K) \) be a set separating the points of \( K \). Then there exists \( \mathcal{A}' \subset \mathcal{A} \) with \( |\mathcal{A}'| = w(K) \) which separates the points of \( K \).

Proof. First, since by the Stone–Weierstrass theorem \( \text{alg}(\mathcal{A} \cup \{1\}) \) is dense in \( C(K) \), we easily observe that \( \{ f^{-1}(-1/2, 1/2) : f \in \text{alg}(\mathcal{A} \cup \{1\}) \} \) is a basis for the topology of \( K \). Thus, by \( \mathbb{12} \) Theorem 1.1.15, there is \( \mathcal{F} \subset \text{alg}(\mathcal{A} \cup \{1\}) \) with \( |\mathcal{F}| = w(K) \) such that \( \{ f^{-1}(-1/2, 1/2) : f \in \mathcal{F} \} \) is a basis for the topology of \( K \). Pick \( \mathcal{A}' \subset \mathcal{A} \) such that \( |\mathcal{A}'| = w(K) \) and \( \mathcal{F} \subset \text{alg}(\mathcal{A}' \cup \{1\}) \). Then \( \mathcal{A}' \) separates the points of \( K \), because otherwise \( \text{alg}(\mathcal{A}' \cup \{1\}) \) would not separate the points of \( K \), contradicting the fact that \( \{ f^{-1}(-1/2, 1/2) : f \in \mathcal{F} \} \) is a basis of the topology.

Proof of Theorem \( \mathbb{D} \). \( (i) \Rightarrow (ii) \) Let \( \mathfrak{s}_2 \) be the commutative retraction skeleton on \( K \) inducing \( D(\mathfrak{s}) \). Then by Theorem \( \mathbb{21} \) there is a weak sub-skeleton of both \( \mathfrak{s} \) and \( \mathfrak{s}_2 \), which easily implies that there is a cofinal subset \( \Gamma'' \subset \Gamma \) such that \( r_s \circ r_t = r_t \circ r_s \) for all \( s, t \in \Gamma'' \). Thus, it suffices to let \( \Gamma' = (\Gamma'')_\sigma \).

\( (ii) \Rightarrow (iii) \) Let \( \mathfrak{s}_2 = (r_s)_{s \in \Gamma'} \) be a commutative subskeleton of \( \mathfrak{s} \). Pick an up-directed set \( \Gamma'' \subset \Gamma' \) and \( x \in D(\mathfrak{s}) = D(\mathfrak{s}_2) \). By Lemma \( \mathbb{2} \) the limit \( \lim_{s \in \Gamma''} r_s(x) \) exists. Since there exists \( s_0 \in \Gamma' \) such that \( x = r_{s_0}x \), using the
commutativity we obtain
\[
\lim_{s \in I''} r_s(x) = \lim_{s \in I''} r_s(r_{s_0} x) = r_{s_0} \left( \lim_{s \in I''} r_s(x) \right) \in D(s).
\]
Thus, \( \mathfrak{s}_2 \) satisfies \( \text{(iii)} \) with \( D = D(\mathfrak{s}) \).

\( \text{(iii)} \Rightarrow \text{(iv)} \) First, we may without loss of generality assume that \( \mathfrak{s}_2 = \mathfrak{s} \).
We will prove the result by induction on \( \kappa := w(K) \). We may without loss of generality assume that \( \lambda = 1 \). If \( \kappa = \omega \), then by Lemma 25, there exists a countable set \( H \subset A \) which separates the points of \( K \) and this set does the job. So let us assume that the result holds for every compact space of weight strictly smaller than \( \kappa \). Proposition 17 together with Theorem 4 implies the existence of sets \( (M_\alpha)_{\alpha \leq \kappa} \) satisfying \( \text{(Ra)} \)–\( \text{(Rd)} \) and retractions \( (r_\alpha)_{\alpha \leq \kappa} \) satisfying \( \text{(R1)} \)–\( \text{(R11)} \). Note that using \( \text{(R8)} \) we obtain \( r_\alpha[D] \subset D \).

For every \( \alpha < \kappa \), define
\[
\mathcal{A}_\alpha := \{ f \in C(\sigma_r[K]) : f \circ r_\alpha \in \mathcal{A} \}.
\]
It is easy to see that, for every \( \alpha < \kappa \), the set \( \mathcal{A}_\alpha \) is symmetric, closed, convex and bounded. The fact that \( \mathcal{A}_\alpha \) separates the points of \( r_\alpha[K] \) follows from \( \text{(R6)} \) and \( \text{(R9)} \) implies that \( f \circ r_s \in \mathcal{A}_\alpha \), for every \( f \in \mathcal{A}_\alpha \) and every \( s \in (\Gamma \cap M_\alpha)_{\sigma} \). For every \( \alpha < \kappa \), define \( D_\alpha := D \cap r_\alpha[K] \subset D(s) \cap r_\alpha[K] \).

Since \( r_\alpha[D] \) is dense in \( r_\alpha[K] \) and \( r_\alpha[D] \subset D_\alpha \), we see that \( D_\alpha \) is dense in \( r_\alpha[K] \). Therefore the induction hypothesis and \( \text{(R5)} \) imply that there are sets \( T_\alpha \subset \mathcal{A}_\alpha \) such that the mapping \( \varphi_\alpha : r_\alpha[K] \to [-1,1]^{T_\alpha} \) given by \( \varphi_\alpha(x)(t) := t(x) \), \( t \in T_\alpha \) and \( x \in r_\alpha[K] \), is a homeomorphic embedding and \( \varphi_\alpha[D_\alpha] \subset \Sigma(T_\alpha) \), for every \( \alpha < \kappa \). We may assume that \( T_\alpha \cap T_\beta = \emptyset \) for \( \alpha \neq \beta \).

Now, we put \( T = T_0 \cup \bigcup_{\alpha < \kappa} T_{\alpha+1} \) and define \( \varphi : K \to [-1,1]^{T} \) by
\[
\varphi(x)(t) := \begin{cases}
\frac{1}{2} (\varphi_{\alpha+1}(r_{\alpha+1}(x))(t) - \varphi_{\alpha+1}(r_\alpha(x))(t)), & t \in T_{\alpha+1}, \\
\varphi_0(r_0(x))(t), & t \in T_0.
\end{cases}
\]
Then \( \varphi \) is of course continuous. Let us verify that it is one-to-one. Indeed, if \( x, y \) are distinct points from \( K \) then by \( \text{(R7)} \) there is a minimal ordinal \( \alpha_0 < \kappa \) for which \( r_{\alpha_0}(x) \neq r_{\alpha_0}(y) \) and either \( \alpha_0 = 0 \) or \( \alpha_0 \) is a successor ordinal. If \( \alpha_0 = 0 \), then there exists \( t \in T_0 \) such that \( \varphi_0(r_0(x))(t) \neq \varphi_0(r_0(y))(t) \) and so we have \( \varphi(x)(t) \neq \varphi(y)(t) \). Otherwise, \( \alpha_0 = \beta_0 + 1 \) for some \( \beta_0 < \kappa \) and there is \( t \in T_{\alpha_0} \) such that \( \varphi_{\alpha_0}(r_{\alpha_0}(x))(t) \neq \varphi_{\alpha_0}(r_{\alpha_0}(y))(t) \). Moreover, since \( \alpha_0 \) is minimal, we have \( r_{\beta_0}(x) = r_{\beta_0}(y) \), hence \( \varphi(x)(t) \neq \varphi(y)(t) \) and so \( \varphi(x) \neq \varphi(y) \). Thus, \( \varphi \) is a homeomorphic embedding.

Let us show that \( \varphi[D(s)] \subset \Sigma(T) \). Indeed, by \( \text{(R11)} \) for every \( x \in D \) the set \( \{ \alpha < \kappa : r_{\alpha+1}(x) \neq r_\alpha(x) \} \) is countable. Moreover, since \( r_\alpha[D] \subset D_\alpha \), the induction hypothesis ensures that the supports of \( \varphi_0(r_0(x)), \varphi_{\alpha+1}(r_{\alpha+1}(x)) \) and \( \varphi_{\alpha+1}(r_\alpha(x)) \) are countable. Therefore the support of \( \varphi(x) \) is countable and \( \varphi[D] \subset \Sigma(T) \). Moreover, since \( D \) is dense in \( K \), by Lemma 24.
there is a commutative retractional skeleton $s_2$ on $\varphi[K]$ such that $D(s_2) = \varphi[K] \cap \Sigma(T)$. Since $D(s_2) \supseteq \varphi[D]$, by [7, Lemma 3.2] we find that $\varphi[D(s)] = D(s_2) \subset \Sigma(T)$.

Now, let us show that $\pi_t \circ \varphi \in \mathcal{A}$ for every $t \in T$. Firstly, we note that $\pi_t \circ \varphi_\alpha \in \mathcal{A}_\alpha$, for every $t \in T_\alpha$ and every $\alpha < \kappa$. If $t \in T_0$, then $\pi_t \circ \varphi = \pi_t \circ \varphi_0 \circ r_0 \in \mathcal{A}$. Pick $\alpha < \kappa$ and $t \in T_{\alpha+1}$. Then, in the same way as above, $\pi_t \circ \varphi_{\alpha+1} \circ r_{\alpha+1} \in \mathcal{A}$ and therefore using (R6) we obtain

$$\pi_t \circ \varphi = \frac{1}{2}(\pi_t \circ \varphi_{\alpha+1} \circ r_{\alpha+1} - \pi_t \circ \varphi_{\alpha+1} \circ r_{\alpha+1} \circ r_\alpha) \in \mathcal{A}.$$ 

Omitting some indices, we may assume that the mapping $T \ni t \mapsto f_t := \pi_t \circ \varphi \in \mathcal{A}$ is one-to-one and so $\mathcal{H} := \{f_t : t \in T\}$ does the job.

(iv) $\Rightarrow$ (i) By Lemma 24 there is a commutative retractional skeleton $s_2$ on $\varphi[K]$ such that $D(s_2) = \varphi[K] \cap \Sigma(I)$. Since $\varphi[D(s)] \subset D(s_2)$, by [7, Lemma 3.2] we have $\varphi[D(s)] = D(s_2)$ and so the set $D(s)$ is induced by a commutative retractional skeleton.

(iii) $\Rightarrow$ (i) This follows from (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) applied to the set $A := B_{\mathcal{C}(K)}$.

The following corollary might be well-known, but let us mention it for future reference.

**Corollary 26.** Let $K$ be a compact space and let $s$ be a full retractional skeleton on $K$. Then there exists a commutative subskeleton of $s$.

**Proof.** Apply Theorem 15(iii) $\Rightarrow$ (ii) to $D := K$. ■

The proof of Theorem 15 also gives us the following.

**Theorem 27.** Let $K$ be a compact space and let $D$ be a set induced by a retractional skeleton on $K$. Then the following are equivalent:

(a) $D$ is induced by a commutative retractional skeleton.
(b) For every suitable model $M$,

$$\forall x \in D \; \exists y \in D \cap D \cap M \; \forall f \in \mathcal{C}(K) \cap M : \; f(x) = f(y).$$

**Proof.** (a) $\Rightarrow$ (b). By Theorem 15 there is a finite list $\Phi$ of formulas and a countable set $S$ (depending on the compact space $K$ and the set $D$) such that for any $M \prec \langle \Phi; S \rangle$, $M$ admits the canonical retraction $r_M$ and we have $r_M[D] \subset D$. Then (b) follows from Theorem 15(i) applied to $A := \mathcal{C}(K)$.

(b) $\Rightarrow$ (a) follows from the fact that in the proof of Theorem 15(iii) $\Rightarrow$ (iv) we used condition (iii) only to ensure that for a suitable model $M_\alpha$ we have $r_\alpha[D] \subset D$, which by Theorem 15(i) follows from condition (b) above. ■

5. **Characterization of (semi-)Eberlein compacta.** Here, we apply the results of the preceding sections and characterize (semi-)Eberlein compacta using the notion of an $\mathcal{A}$-shrinking retractional skeleton.
**Definition 28.** Let $K$ be a countably compact space. Let $\emptyset \neq A \subset \mathcal{C}(K)$ be a bounded set. The pseudometric $\rho_A$ on $K$ is given as

$$\rho_A(k, l) := \sup_{f \in A} |f(k) - f(l)|, \quad k, l \in K.$$ 

If $(r_s)_{s \in \Gamma}$ is a retractional skeleton on $K$ and $D \subset K$, we say that $(r_s)_{s \in \Gamma}$ is $A$-shrinking with respect to $D$ if for every $x \in D$ and every increasing sequence $(s_n)_{n \in \omega}$ in $\Gamma$ with $s := \sup_{n \in \omega} s_n$, we have $\lim_{n \in \omega} \rho_A(r_s(x), r_s(x)) = 0$. If $(r_s)_{s \in \Gamma}$ is $A$-shrinking with respect to $K$, then we just say that $(r_s)_{s \in \Gamma}$ is $A$-shrinking.

Finally, given $\varepsilon > 0$ we say that $(r_s)_{s \in \Gamma}$ is $(A, \varepsilon)$-shrinking with respect to $D$ if for every $x \in D$ and every increasing sequence $(s_n)_{n \in \omega}$ in $\Gamma$ with $s := \sup_{n \in \omega} s_n$, we have $\limsup_{n \in \omega} \rho_A(r_s(x), r_s(x)) \leq \varepsilon$.

Note that if the nonempty bounded set $A$ separates the points of $K$, then $\rho_A$ is a metric on $K$.

The aim of this section is to prove the following result, from which Theorems A and B easily follow.

**Theorem 29.** Let $K$ be a compact space and let $D \subset K$ be a dense set. Consider the following conditions:

(i) There exists a homeomorphic embedding $h : K \to [-1, 1]^I$ such that $h[D] = c_0(I) \cap h[K]$.

(ii) There exist a bounded set $A \subset \mathcal{C}(K)$ separating the points of $K$ and a retractional skeleton $s = (r_s)_{s \in \Gamma}$ on $K$ with $D \subset D(s)$ such that

(a) $s$ is $A$-shrinking with respect to $D$, and
(b) $\lim_{s \in \Gamma'} r_s(x) \in D$ for every $x \in D$ and every up-directed subset $\Gamma'$ of $\Gamma$.

(iii) There exist a countable family $\mathcal{A}$ of subsets of $\mathcal{B}_C(K)$ and a retractional skeleton $s = (r_s)_{s \in \Gamma}$ on $K$ with $D \subset D(s)$ such that

(a) for every $A \in \mathcal{A}$ there exists $\varepsilon_A > 0$ such that $s$ is $(A, \varepsilon_A)$-shrinking with respect to $D$,
(b) for every $\varepsilon > 0$ we have $B_{\mathcal{C}(K)} = \bigcup \{ A \in \mathcal{A} : \varepsilon_A < \varepsilon \}$, and
(c) $\lim_{s \in \Gamma'} r_s(x) \in D$, for every $x \in D$ and every up-directed subset $\Gamma'$ of $\Gamma$.

(iv) There exists a homeomorphic embedding $h : K \to [-1, 1]^J$ such that $h[D] \subset c_0(J)$.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).

Let us first give some comments.
Remark 30. The notion of a shrinking retractional skeleton is inspired by [15], where the definition of a shrinking projectional skeleton was given and WCG Banach spaces were characterized using this notion.

Given a retractional skeleton \((r_s)_{s \in \Gamma}\) on a compact space \(K\), it is well-known that \((P_s)_{s \in \Gamma}\) defined by \(P_s(f) := f \circ r_s, \ s \in \Gamma\), is a projectional skeleton on \(C(K)\) (see e.g. [21] Proposition 5.3). Moreover, if \(\emptyset \neq \mathcal{A} \subset C(K)\) is a bounded set and \((P_s)_{s \in \Gamma}\) is \(\mathcal{A}\)-shrinking in the sense of [15] Definition 16], it is not very difficult to observe that \((r_s)_{s \in \Gamma}\) is \(\mathcal{A}\)-shrinking in the sense of Definition 28. It is not clear whether the converse holds as it is (at least formally) a stronger condition. Thus, Theorem A allows us in a certain sense to strengthen the implication (ii)⇒(i) from [15] Theorem 21. Since the other implication is easier, Theorem A may be thought of as a topological counterpart and in a sense also a strengthening of [15] Theorem 21 in the context of \(C(K)\) spaces.

Notice that the shrinkingness of a retractional skeleton is not a specific property of a particular skeleton. First, observe that any \(\mathcal{A}\)-shrinking retractional skeleton is also full whenever \(\mathcal{A} \subset C(K)\) separates the points of \(K\). This is generalized in the following.

Lemma 31. Let \(K\) be a compact space, \(\mathcal{A} \subset C(K)\) be a bounded set separating the points of \(K\) and let \(s = (r_s)_{s \in \Gamma}\) be a retractional skeleton on \(K\) which is \(\mathcal{A}\)-shrinking with respect to a set \(D\) with \(D \supset D(s)\). Then \(D = D(s)\) and \(\lim_{s \in \Gamma'} r_s(x) \in D\) for every \(x \in D\) and every up-directed subset \(\Gamma'\) of \(\Gamma\).

Proof. Fix \(x \in D\) and an up-directed set \(\Gamma' \subset \Gamma\). Since \(s\) is \(\mathcal{A}\)-shrinking with respect to \(D\), it is not difficult to observe (see e.g. [15] Proposition 20) that there exists an increasing sequence \((s_n)_{n \in \omega}\) in \(\Gamma'\) with \(s = \sup_n s_n \in \Gamma\) such that \(\rho_\mathcal{A}\lim_{t \in \Gamma'} r_t(x) = r_s(x)\). Therefore, since \(\lim_{t \in \Gamma'} r_t(x)\) exists, we obtain \(f(\lim_{t \in \Gamma'} r_t(x)) = f(r_s(x))\) for every \(f \in \mathcal{A}\). Since \(\mathcal{A}\) separates the points of \(K\), we deduce that \(\lim_{t \in \Gamma'} r_t(x) = r_s(x) \in D(s) \subset D\). Finally, for \(\Gamma' = \Gamma\) we obtain \(x = \lim_{s \in \Gamma'} r_s(x) \in D(s)\) and so \(D \subset D(s)\).

Lemma 32. Let \(K\) be a compact space and \(\mathcal{A} \subset C(K)\) be a bounded set separating the points of \(K\). If there exists an \(\mathcal{A}\)-shrinking retractional skeleton on \(K\), then every full retractional skeleton on \(K\) admits a weak subskeleton which is \(\mathcal{A}\)-shrinking and commutative.

Proof. By Lemma 31 there exists an \(\mathcal{A}\)-shrinking and full retractional skeleton \((\tilde{r}_i)_{i \in I}\) on \(K\). Moreover, by Corollary 26 we may without loss of generality assume that \((\tilde{r}_i)_{i \in I}\) is commutative. Now, let \((r_s)_{s \in \Gamma}\) be a full retractional skeleton on \(K\). By Theorem 21 there exists a retractional skeleton \((R_i)_{i \in \Lambda}\) which is a weak subskeleton of both \((r_s)_{s \in \Gamma}\) and \((\tilde{r}_i)_{i \in I}\). It is easy to see that \((R_i)_{i \in \Lambda}\) is commutative and \(\mathcal{A}\)-shrinking. ■
In the remainder of this section we provide the proof of Theorem 29. Let us start with the implication (i)⇒(ii).

**Lemma 33.** Let $K$ be a compact space and $D \subseteq K$ be a dense subset. If there exists a homeomorphic embedding $h : K \to [-1,1]^I$ such that $h[D] = h[K] \cap c_0(I)$, then there are $A \subseteq B_{C(K)}$, $s = (r_s)_{s \in I}$ such that (ii) in Theorem 29 holds with $A$, $s = (r_s)_{s \in I}$ and moreover $f \circ r_s \in A$ for all $f \in A$ and $s \in I$.

**Proof.** We may assume that $K \subseteq [-1,1]^I$, $D = K \cap c_0(I)$. Pick the commutative retractional skeleton $(r_A)_{A \in I}$ from Lemma 24 and put $S := \{ \pi_i|_K : i \in I \} \cup \{ \emptyset \} \subset C(K)$. Clearly $S$ is bounded and separating the points of $K$. Moreover $D$ is obviously contained in $K \cap \Sigma(I)$ (which is the set induced by $(r_A)_{A \in I}$) and it is easy to observe that if $A \in \Gamma$ and $f \in S$, then $f \circ r_A \in S$.

Now, we show that $(r_A)_{A \in I}$ is $S$-shrinking with respect to $D$. Pick $x \in D$ and an increasing sequence $(A_n)_{n \in \omega}$ in $\Gamma$ and put $A = \sup_n A_n = \bigcup_{n \in \omega} A_n$. Fix $\epsilon > 0$ and let $n_0 \in \omega$ be such that $\{ i : x(i) > \epsilon \} \subseteq A_{n_0}$. Then for every $n \ge n_0$ we obtain $r_{A_n} x(i) - r_A x(i) = 1_{A \setminus A_n} \cdot x(i)$, therefore for every $i \in I$ we have $|r_{A_n} x(i) - r_A x(i)| < \epsilon$; hence $\sup_{f \in S} |f(r_{A_n} x) - f(r_A x)| < \epsilon$.

Finally, we note that whenever $\Gamma' \subset \Gamma$ is up-directed and $x \in D$, then $y := \lim_{A \in \Gamma'} r_A x$ exists by Lemma 2 and moreover if $i \in \text{suppt}(y)$, then $y(i) = x(i)$. Therefore, $y \in K \cap c_0(I) = D$. ■

The most demanding in the proof of Theorem 29 is the implication (iii)⇒(iv). We start with an easy observation.

**Lemma 34.** Let $K$ be a compact space, $A \subset C(K)$ be a bounded set, $\epsilon > 0$ and $D$ be a subset of $K$. Suppose that $(r_s)_{s \in I'}$ is a retractional skeleton on $K$ that is $(A,\epsilon)$-shrinking with respect to $D$. For an up-directed set $\Gamma' \subset \Gamma$ let $R_{\Gamma'}$ be as in Lemma 2. Then for every $x \in D$ we have the following.

**(Sa)** If $\Gamma' \subset \Gamma$ is up-directed, then there exists $s_0 \in I'$ such that

$$\rho_A(R_{\Gamma'}(x), r_s(x)) \le 7\epsilon, \quad s \ge s_0, s \in I'.$$

**(Sb)** If $M \subset \mathcal{P}(\Gamma')$ is up-directed and each $M \in M$ is up-directed, then there exists $M_0 \in M$ such that for every $M \in M$ with $M \supseteq M_0$,

$$\rho_A(R_M(x), R_M(x)) \le 14\epsilon.$$

**Proof.** Pick $x \in D$.

**(Sa)** First, let us observe that there exists $s_0 \in I'$ such that

$$\rho_A(r_s(x), r_{s_0}(x)) < 3\epsilon, \quad s \ge s_0, s \in I'.$$

Indeed, if this is not the case we inductively construct an increasing sequence $(s_n)$ in $I'$ with $\rho_A(r_{s_n}(x), r_{s_{n+1}}(x)) \ge 3\epsilon$ for $n \in \mathbb{N}$, which contradicts $(A,\epsilon)$-shrinkingness.
Pick \( f \in \mathcal{A} \) and \( s_1 \geq s_0 \), \( s_1 \in \Gamma' \). Since \( \lim_{s \in \Gamma'} r_s(x) = R_{\Gamma'}(x) \), there exists \( s_2 \geq s_1 \), \( s_2 \in \Gamma' \) with \( |f(r_{s_2}(x)) - f(R_{\Gamma'}(x))| < \epsilon \). Therefore, by \([1]\),
\[
|f(r_{s_1}(x)) - f(R_{\Gamma'}(x))| \leq \rho_A(r_{s_1}(x), r_{s_0}(x)) + \rho_A(r_{s_0}(x), r_{s_2}(x)) + \epsilon < 7\epsilon.
\]
Since \( f \in \mathcal{A} \) was arbitrary, this proves \([Sa]\).

\(\text{(Sb)}\) By \([Sa]\) there exists \( s_0 \in \bigcup \mathcal{M} \) such that \( \rho_A(r_s(x), R_{\bigcup \mathcal{M}}(x)) \leq 7\epsilon \), for every \( s \geq s_0 \), \( s \in \bigcup \mathcal{M} \). Let \( M_0 \in \mathcal{M} \) be such that \( s_0 \in M_0 \). Then, for every \( M \in \mathcal{M} \) with \( M \supset M_0 \), by \([Sa]\) there exists \( s_M \geq s_0 \), \( s_M \in M \) with \( \rho_A(r_{s_M}(x), R_M(x)) \leq 7\epsilon \), which implies that
\[
\rho_A(R_M(x), R_{\bigcup \mathcal{M}}(x)) \leq 14\epsilon.
\]

The following proposition together with Theorem \([D]\) is the core of our argument. The idea to use such a result is related to a characterization of Eberlein compacta by Farmaki \([16\, \text{Theorem 2.9}]\) (see also \([13\, \text{Theorem 10}]\)). However, our methods yield a self-contained proof.

**Proposition 35.** Let \( K \subset [-1,1]^I \) be a compact space and for \( I' \subset I \) define
\[
S_ {K,I'} = \{ \pi_{i|K}: i \in I' \}.
\]
Suppose \( K \) admits a retractional skeleton \( s = (r_s)_{s \in \Gamma} \) such that \( D(s) \subset \Sigma(I) \) and let \( D \subset D(s) \). Assume that there is a countable family \( \mathcal{A} \) of subsets of \( I \) such that
(1) for every \( A \in \mathcal{A} \), there exists \( \epsilon_A > 0 \) such that \( (r_s)_{s \in \Gamma} \) is \((S_{K,A}, \epsilon_A)\)-shrinking with respect to \( D \);
(2) for every \( \epsilon > 0 \), we have \( I = \bigcup \{ A \in \mathcal{A}: \epsilon_A < \epsilon \} \);
(3) \( \lim_{s \in \Gamma'} r_s(x) \in D \) for every \( x \in D \) and every up-directed subset \( I' \) of \( \Gamma \).
Then for every \( \epsilon > 0 \) there is a decomposition \( I = \bigcup_{n=0}^{\infty} I_n^\epsilon \) such that
\[
\forall n \, \forall x \in D : \, |\{ i \in I_n^\epsilon : |x(i)| > \epsilon \}| < \omega.
\]

**Proof.** By \([19\, \text{Proposition 19.5}]\), we may pick a set \( J \subset I \) such that \( |J| = w(K) \) and \( \text{suppt}(x) \subset J \) for every \( x \in K \). By \([7\, \text{Lemma 3.2}]\), we see that \( D(s) = \Sigma(I) \cap K \) and hence Lemma \([24]\) ensures that \( D(s) \) is induced by a commutative retractional skeleton. Therefore it follows from Theorem \([D]\) that we may assume the retractional skeleton \( s \) is commutative. Now, let us prove the result by induction on the weight of \( K \). If \( K \) has countable weight, then the set \( J \) is countable and we may enumerate it as \( J := \{ j_n \}_{n \geq 1} \). For each \( \epsilon > 0 \), let \( I_0^\epsilon = I \setminus J \), and \( I_n^\epsilon = \{ j_n \} \) for every \( n \geq 1 \). Then \( I = \bigcup_{n=0}^{\infty} I_n^\epsilon \) and
\[
\forall n \, \forall x \in K : \, |\{ i \in I_n^\epsilon : |x(i)| > \epsilon \}| = 1.
\]

Now suppose that \( w(K) = \kappa \geq \omega_1 \) and that the result holds for compact spaces of weight less than \( \kappa \). Proposition \([17]\) together with Theorem \([4]\) imply the existence of sets \( (M_\alpha)_{\alpha \leq \kappa} \) satisfying \([Ra]\)–\([Rd]\) and retractions \( (r_\alpha)_{\alpha \leq \kappa} \)
satisfying \([\text{R1}],[\text{R11}]\). We can assume that \(J \subset \bigcup_{\alpha \prec \kappa} M_\alpha\), by replacing \([\text{Rc}]\) by the following (stronger) condition:

\[
M_{\alpha + 1} \prec (\Phi; \{j_\alpha, \alpha \} \cup M_\alpha), \forall \alpha \prec \kappa,
\]

where \(J = \{j_\alpha : \alpha \prec \kappa\}\). By Lemma 22, we may assume that \(r_\alpha(x) = x|_{I \cap M_\alpha}\) for all \(x \in K\) and \(\alpha \prec \kappa\). For each \(\alpha \prec \kappa\), it is easy to see that for every \(A \in \mathcal{A}\) the retractional skeleton \((r_s|_{r_\alpha[K]}).\) \(s \in (I \cap M_\alpha)\) given by \([\text{R5}]\) is \((S_{r_\alpha[K]}, A, \varepsilon_A)-\text{shrinking with respect to the set} D \cap r_\alpha[K] \subset D(s) \cap r_\alpha[K] \subset \Sigma(I) \cap [-1, 1]^I\). Moreover, if \(I' \subset (I \cap M_\alpha)\) is up-directed and \(x \in D \cap r_\alpha[K]\), then using \([\text{R9}]\) we conclude that \(\lim_{s \in I'} r_s(x) \in D \cap r_\alpha[K]\).

Now fix \(\varepsilon > 0\) and let \(I = \bigcup_{n \geq 1} I^\varepsilon_{n,0}\) be the decomposition given by induction hypothesis applied to \(r_0[K]\) (by \([\text{R5}]\) we may indeed apply the inductive hypothesis to \(r_0[K]\)), that is, for every \(y \in D \cap r_0[K]\) and \(n \geq 1\) the set

\[\{i \in I^\varepsilon_{n,0} : |y(i)| > \varepsilon\}\]

is finite. Fix \(\alpha \prec \kappa\), and similarly let \(I = \bigcup_{n \geq 1} I^\varepsilon_{n,\alpha + 1}\) be the decomposition given by the induction hypothesis applied to \(r_{\alpha + 1}[K]\), that is, for every \(y \in D \cap r_{\alpha + 1}[K]\) and \(n \geq 1\) the set

\[\{i \in I^\varepsilon_{n,\alpha + 1} : |y(i)| > \varepsilon\}\]

is finite. For \(A \in \mathcal{A}\) with \(\varepsilon_A < \varepsilon / 14\) define \(I^\varepsilon_{(0,A)} = I \setminus J\) and for every \(n \geq 1\) put

\[I^\varepsilon_{(n,A)} = (A \cap J \cap I^\varepsilon_{n,0} \cap M_0) \cup \bigcup_{\alpha \prec \kappa} (A \cap J \cap I^\varepsilon_{n,\alpha + 1} \cap (M_{\alpha + 1} \setminus M_\alpha)).\]

Note that \(I = \bigcup \{I^\varepsilon_{(n,A)} : n \geq 0, A \in \mathcal{A}, 14 \varepsilon_A < \varepsilon\}\), since \(J \subset \bigcup_{\alpha \prec \kappa} M_\alpha\) and \(I = \bigcup \{A \in \mathcal{A} : 14 \varepsilon_A < \varepsilon\}\).

Given \(x \in D\), \(A \in \mathcal{A}\) with \(\varepsilon_A < \varepsilon / 14\) and \(n \geq 0\), let us show that the set

\[S_{(n,A)} = \{i \in I^\varepsilon_{(n,A)} : |x(i)| > \varepsilon\}\]

is finite. Since \(\text{suppt}(x) \subset J\), we see that \(S_{(0,A)}\) is empty. For \(n \geq 1\), in order to conclude that \(S_{(n,A)}\) is finite it suffices to prove that the set

\[A_{(n,A)} = \{\alpha \prec \kappa : |x(i)| > \varepsilon \text{ for some } i \in A \cap J \cap I^\varepsilon_{n,\alpha + 1} \cap (M_{\alpha + 1} \setminus M_\alpha)\}\]

is finite. Indeed, using \(r_0(x) = x|_{I \cap M_0} = x|_{J \cap M_0}\) we obtain

\[S_{(n,A)} \cap (A \cap J \cap I^\varepsilon_{n,0} \cap M_0) \subset \{i \in I^\varepsilon_{n,0} : |r_0(x)(i)| > \varepsilon\}\]

and therefore, since \(r_0(x) = \lim_{s \in (I \cap M_0)} r_s(x) \in D \cap r_0[K]\), we conclude that \(S_{(n,A)} \cap (A \cap J \cap I^\varepsilon_{n,0} \cap M_0)\) is finite. Similarly, for \(\alpha \prec \kappa\) we have

\[S_{(n,A)} \cap (A \cap J \cap I^\varepsilon_{n,\alpha + 1} \cap M_{\alpha + 1} \setminus M_\alpha) \subset \{i \in I^\varepsilon_{n,\alpha + 1} : |r_{\alpha + 1}(x)|_{M_\alpha} > \varepsilon\} \subset \{i \in I^\varepsilon_{n,\alpha + 1} : |r_{\alpha + 1}(x)(i)| > \varepsilon\}\]

and therefore \(S_{(n,A)} \cap (A \cap J \cap I^\varepsilon_{n,\alpha + 1} \cap M_{\alpha + 1} \setminus M_\alpha)\) is finite.
It remains to prove that $\Lambda_{(n,A)}$ is finite. Suppose it is not, so there is a strictly increasing sequence $(\alpha_k)_{k \geq 1}$ of elements of $\kappa$ and a sequence $(i_k)_{k \geq 1}$ such that $i_k \in A \cap \Gamma \cap I^\varepsilon_{n,\alpha_k+1} \cap (M_{\alpha_k+1} \setminus M_{\alpha_k})$ and $|x(i_k)| > \varepsilon$ for every $k \geq 1$. Put $\alpha = \sup_k \alpha_k$. Then (because $i_k \in M_{\alpha_k+1} \setminus M_{\alpha_k} \subset M_{\alpha} \setminus M_{\alpha_k}$) we have

$$\varepsilon < |x(i_k)| = |r_\alpha(x)(i_k) - r_{\alpha_k}(x)(i_k)| \leq \rho_{S_{(K,A)}}(r_\alpha(x), r_{\alpha_k}(x))$$

for every $k \geq 1$. This is a contradiction, because using (R8) and Lemma 34(Sb) applied to $\mathcal{M} = \{M_{\alpha_k} \cap \Gamma : k \geq 1\}$, we conclude that

$$\limsup_{k \to \infty} \rho_{S_{(K,A)}}(r_\alpha(x), r_{\alpha_k}(x)) \leq 14\varepsilon_A < \varepsilon,$$

since $s$ is $(S_{K,A}, \varepsilon_A)$-shrinking with respect to $D$. ■

The following is based on [13, Theorem 10].

**Proposition 36.** Let $K \subset [-1, 1]^I$ be a compact space and $D$ be a subset of $K$. If for every $\varepsilon > 0$, there exists a decomposition $I = \bigcup_{n \in \omega} I^\varepsilon_n$ such that for every $x \in D$ and every $n \in \omega$ the set

$$\{i \in I^\varepsilon_n : |x(i)| > \varepsilon\}$$

is finite, then there is a homeomorphic embedding $\Phi : K \to [-1, 1]^{I \times \omega}$ such that $\Phi[D] \subset c_0(I \times \omega)$.

**Proof.** Let $k \in \omega$ and define a function $\tau_k : \mathbb{R} \to \mathbb{R}$ by

$$\tau_k(t) = \begin{cases} 
  t + 1/k & \text{if } t \leq -1/k, \\
  0 & \text{if } -1/k \leq t \leq 1/k, \\
  t - 1/k & \text{if } t \geq 1/k.
\end{cases}$$

Then define $\Phi : K \to [-1, 1]^{I \times \omega}$ by

$$\Phi(x)(i, k) = \frac{1}{nk} \tau_k(x(i))$$

if $i \in I^1_n$, $n \in \omega$ and $k \in \omega$. Since the map $\pi(i,k) \circ \Phi : K \to \mathbb{R}$ is continuous for every $(i,k) \in I \times \omega$, the map $\Phi$ is continuous as well. The map is also one-to-one. Indeed, for distinct $x_1, x_2 \in K$ there exists an $i \in I$ with $x_1(i) \neq x_2(i)$. Let $k \in \omega$ be such that $1/k < \max(|x_1(i)|, |x_2(i)|)$ and pick $n \in \omega$ with $i \in I^1_n$. Then $\tau_k(x_1(i)) \neq \tau_k(x_2(i))$, therefore $\Phi(x_1)(i, k) \neq \Phi(x_2)(i, k)$.

It remains to prove that $\Phi[D]$ is contained in $c_0(I \times \omega)$. To do so, let $x \in D$ and fix $\varepsilon > 0$. If $n, k \in \omega$, and $n > 1/\varepsilon$ or $k > 1/\varepsilon$, then $|\Phi(x)(i, k)| < \varepsilon$, for any choice of $i \in I^1_n$. Let $n, k < 1/\varepsilon$ (observe that there are only finitely
many $n$ and $k$ such that this inequality holds). Then

$$\{i \in I_n^{1/k} : |\Phi(x)(i, k)| > \varepsilon \} \subseteq \{i \in I_n^{1/k} : \tau_k(x(i)) \neq 0 \} = \{i \in I_n^{1/k} : x(i) > 1/k \} \cup \{i \in I_n^{1/k} : x(i) < -1/k \} = \{i \in I_n^{1/k} : |x(i)| > 1/k \}.$$ 

Thus, $\{i \in I_n^{1/k} : |\Phi(x)(i, k)| > \varepsilon \}$ is finite and so $\Phi(x) \in c_0(I \times \omega)$. □

**Proof of Theorem 29.** Lemma 3 ensures that (i)$\Rightarrow$(ii).

To prove (ii)$\Rightarrow$(iii) let $A$ and $s = (r_s)_{s \in \Gamma}$ be as in the assumption and let $\lambda \geq 1$ be such that $A \subset \lambda B_{C(K)}$. We may assume that the constant function 1 is a member of $A$. For every $n \in \mathbb{N}$ put

$$A_n := \left\{ \sum_{i=1}^{k} a_i \prod_{j=1}^{n} f_{i,j} : f_{i,j} \in A, \ k \in \mathbb{N}, \ \sum_{i=1}^{k} |a_i| \leq n \right\}$$

and for $m \in \mathbb{N}$ further put

$$A_{n,m} := \left( A_n + \frac{1}{2m} B_{C(K)} \right) \cap B_{C(K)}.$$

Now, we claim that the family $\tilde{A} := \{A_{n,m} : n, m \in \mathbb{N} \}$ and the retractional skeleton $s$ satisfy the condition from (iii). Pick $n, m \in \mathbb{N}$. Then $s$ is $(A_{n,m}, 1/m)$-shrinking with respect to $D$. Indeed, given $x \in D$ and an increasing sequence $(s_k)$ in $\Gamma$ with $s = \sup s_k$, we have

$$\rho_{A_{n,m}}(r_{s_k}(x), r_s(x)) \leq \rho_{A_n}(r_{s_k}(x), r_s(x)) + \frac{1}{m} \leq n^2 \lambda^{n+1} \rho_A(r_{s_k}(x), r_s(x)) + \frac{1}{m},$$

so using the fact that $s$ is $A$-shrinking with respect to $D$, we obtain

$$\limsup_k \rho_{A_{n,m}}(r_{s_k}(x), r_s(x)) \leq \frac{1}{m}.$$ 

Finally, since $\bigcup_{n \in \mathbb{N}} A_n = \text{alg}(A)$ is norm-dense in $C(K)$, we easily observe that $B_{C(K)} = \bigcup_{n \in \mathbb{N}} A_{n,m}$ for every $m \in \mathbb{N}$, from which (b) follows.

To prove (iii)$\Rightarrow$(iv) let $A$ and $s = (r_s)_{s \in \Gamma}$ be as in the assumption. By Theorem D there exists $\mathcal{H} \subset B_{C(K)}$ such that the mapping $\varphi : K \to [-1, 1]^\mathcal{H}$ given by $\varphi(x)(h) := h(x)$ for $h \in \mathcal{H}$ and $x \in K$ is a homeomorphic embedding and $\varphi[D(s)] \subset \Sigma(\mathcal{H})$. For every $s \in \Gamma$, define $q_s = \varphi \circ r_s \circ \varphi^{-1} : \varphi[K] \to \varphi[K]$ and note that the retractional skeleton $(q_s)_{s \in \Gamma}$ is $(S_{\varphi[K], \mathcal{H} \cap A}, \varepsilon_A)$-shrinking with respect to the set $\varphi[D] \subset \varphi[D(s)]$ for every $A \in \mathcal{A}$. Indeed, fix $x \in D$ and an increasing sequence $(s_n)_{n \geq 1}$ of elements of $\Gamma$ with $s = \sup s_n$. Then

$$\rho_{S_{\varphi[K], \mathcal{H} \cap A}}(q_{s_n}(\varphi(x)), q_s(\varphi(x))) = \sup_{h \in \mathcal{H} \cap A} |\varphi(r_{s_n}(x))(h) - \varphi(r_s(x))(h)| = \sup_{h \in \mathcal{H} \cap A} |h(r_{s_n}(x)) - h(r_s(x))| \leq \rho_A(r_{s_n}(x), r_s(x)),$$

where $\rho_A$ denotes the distance function in $A$. □
so since $s$ is $(A, \varepsilon_A)$-shrinking with respect to $D$, $(q_s)$ is $(S_{[\varphi[D], \mathcal{H}(\cap A)], \varepsilon_A}$-shrinking with respect to $\varphi[D]$. Obviously, $\mathcal{H} = \bigcup\{A \cap \mathcal{H} : \varepsilon_A < \varepsilon\}$ for every $\varepsilon > 0$. Finally, for every up-directed set $\Gamma' \subset \Gamma$ and every $x \in D$ we have $\lim_{s \in \Gamma'} q_s(\varphi(x)) = \varphi(\lim_{s \in \Gamma'} r_s(x)) \in \varphi[D]$. Therefore, the result follows from Propositions $35$ and $36$.

6. Applications to the structure of (semi-)Eberlein compacta.
We collect our applications to the structure of (semi-)Eberlein compacta. Most importantly, we prove Theorem C.

6.1. Eberlein compacta. As mentioned above, using Theorem A it is not difficult to show that any continuous image of an Eberlein compact is Eberlein. The reason is that for continuous images of Eberlein compacta it is quite standard to verify condition ([iii]) from Theorem 29. We will not provide here the full argument as it is possible to further generalize this observation (see Remark 43 below). The remainder of this subsection is devoted to the proof of Theorem 39 which is a generalization of Theorem A where instead of compactness we assume countable compactness. First we need a lemma. Recall that every real-valued continuous function defined on a countably compact space $D$ is bounded so we may consider the supremum norm on $C(D)$.

**Lemma 37.** Let $D$ be a countably compact space. Suppose that there exist a bounded set $A \subset C(D)$ separating the points of $D$ and a full retractional skeleton $s = (r_s)_{s \in \Gamma}$ on $D$ such that $f \circ r_s \in A$ for all $f \in A$ and $s \in \Gamma$. Then $A' = \{\beta f : f \in A\}$ separates the points of $\beta D$.

**Proof.** By [11] Proposition 4.5, there exists a retractional skeleton $G = (R_s)_{s \in \Gamma}$ on $\beta D$ such that $D(G) = D$ and $R_s|D = r_s$, for every $s \in \Gamma$. Let $x, y \in \beta D$ be distinct. Since $\lim_{s \in \Gamma} R_s(x) = x$ and $\lim_{s \in \Gamma} R_s(y) = y$, there exists $s \in \Gamma$ such that $R_s(x) \neq R_s(y)$. Since $R_s(x)$, $R_s(y) \in D$, there exists $f \in A$ such that $f(R_s(x)) \neq f(R_s(y))$. Therefore $\beta f(R_s(x)) \neq \beta f(R_s(y))$. It is easy to see that $(\beta f \circ R_s)|D = f \circ r_s$, which implies that $\beta f \circ R_s = \beta(f \circ r_s) \in A'$, since $f \circ r_s \in A$.

**Remark 38.** Note that the assumption “$f \circ r_s \in A$ for all $f \in A$ and $s \in \Gamma$” in Lemma 37 is essential. Indeed, consider $D = [0, \omega_1)$ and $A = \{1_{[0]} \cup \{\alpha + 1, \omega_1\} : \alpha < \omega_1\}$. Then it is easy to see that $A$ separates the points of $D$ and that $D$ admits the full retractional skeleton $(r_\alpha)_{\alpha < \omega_1}$ given by the formula

$$r_\alpha(\beta) = \begin{cases} \beta, & \beta \leq \alpha, \\ \alpha + 1, & \alpha < \beta < \omega_1, \end{cases}$$
for every $\alpha < \omega_1$. However, we have $\beta D = [0, \omega_1]$ and the set
\[
A' = \{\beta 1_{\{0\} \cup [\alpha + 1, \omega_1]} : \alpha < \omega_1\} = \{1_{\{0\} \cup [\alpha + 1, \omega_1]} : \alpha < \omega_1\}
\]
does not separate 0 from $\omega_1$.

**Theorem 39.** Let $D$ be a countably compact space. Then the following conditions are equivalent:

(i) There exists a set $I$ such that $D$ embeds homeomorphically into $(c_0(I), w)$.
(ii) $D$ is an Eberlein compact space.
(iii) There exist a bounded set $A \subset C(D)$ separating the points of $D$ and a full retractional skeleton $s = (r_s)_{s \in \Gamma}$ on $D$ such that

(a) $s$ is $A$-shrinking, and
(b) $f \circ r_s \in A$ for all $f \in A$ and $s \in \Gamma$.

**Proof.** (i)$\Rightarrow$(ii) follows from the classical Eberlein–Šmulian theorem and (ii)$\Rightarrow$(iii) follows from Theorem 29 and Lemma 31. If (iii) holds, pick the corresponding set $A$ and the full retractional skeleton $(r_s)_{s \in \Gamma}$ on $D$. By [11, Proposition 4.5], there exists a retractional skeleton $G = (R_s)_{s \in \Gamma}$ on $\beta D$ such that $D(G) = D$ and $R_s|D = r_s$ for every $s \in \Gamma$. Consider now the set $A' := \{\beta f : f \in A\} \subset C(\beta D)$ and the retractional skeleton $G$. By Lemma 37 $A'$ separates the points of $\beta D$. Obviously, $G$ is $A'$-shrinking with respect to $D$ and it is easy to see that for all $f \in A$ and $s \in \Gamma$ we have $\beta f \circ R_s = \beta (f \circ r_s) \in A'$. By Lemma 31 we see that $\lim_{s \in \Gamma'} R_s(x) \in D$ for every $x \in D$ and every up-directed subset $\Gamma'$ of $\Gamma$. Therefore, Theorem 29 ensures that (i) holds. $\blacksquare$

6.2. Semi-Eberlein compacta. In this subsection we provide new stability results for the class of semi-Eberlein compacta. The most important in this respect is probably Corollary 44 which implies Theorem C.

**Lemma 40.** For every suitable model $M$ the following holds: Let $(K, \tau)$ and $(L, \tau')$ be compact spaces, $D \subset K$ a dense subset that is contained in the set induced by a retractional skeleton, and $\varphi : K \rightarrow L$ a continuous map such that $\varphi[D] \subset L$ is a dense subset that is contained in the set induced by a retractional skeleton. If $\{K, L, \tau, \tau', D, \varphi\} \subset M$, then there are the canonical retractional skeletons $r_M$ and $R_M$ associated to $M$, $K$ and $D$ and to $M$, $L$ and $\varphi[D]$, respectively, and we have $R_M \circ \varphi = \varphi \circ r_M$.

**Proof.** Let $S$ and $\Phi$ be the unions of the countable sets and the finite lists of formulas from the statements of Lemma 8 and Theorem 15. Let $M \prec (\Phi; S \cup \{K, L, \tau, \tau', D, \varphi\})$.

The existence of $r_M$ and $R_M$ follows from Theorem 15. Moreover, given $x \in K$ and $f \in C(L) \cap M$, by Lemma 8 we have $f \circ \varphi \in C(K) \cap M$ and hence by the definition of $r_M$, $f \circ \varphi \circ r_M = f \circ \varphi$, which implies that the point $y = \varphi(r_M(x)) \in \varphi[D \cap M] \subset \varphi[D] \cap M$ satisfies $f(y) = f(\varphi(x))$ for
every \( f \in C(L) \cap M \) and so by the uniqueness property of \( R_M(\varphi(x)) \) (see Theorem 13[1]) we obtain \( R_M(\varphi(x)) = \varphi(r_M(x)) \).

**Theorem 41.** Let \( K \) be a compact space and \( D \subset K \) be a dense subset such that there exists a homeomorphic embedding \( h : K \to [-1,1]^J \) with \( h[D] = c_0(J) \cap h[K] \). Suppose that \( \varphi : K \to L \) is a continuous surjection and \( \varphi[D] \) is a subset of the set induced by a retractional skeleton on \( L \).

Then there is a homeomorphic embedding \( H : L \to [-1,1]^J \) with \( H[\varphi[D]] \subset c_0(I) \). In particular, \( L \) is semi-Eberlein.

**Proof.** By Lemma 33 there exists a set \( A \subset B_{C(K)} \) separating the points of \( K \) and a retractional skeleton \( s = (r_s)_{s \in I} \) on \( K \) with \( D \subset D(s) \) such that \( s \) is \( A \)-shrinking with respect to \( D \) and \( \lim_{s \in I'} r_s(x) \in D \) for every \( x \in D \) and every up-directed subset \( I' \) of \( I \). Using Lemma 40 and an argument similar to the one presented in the proof of Theorem 21, we conclude that there are a countable set \( S \), a finite list of formulas \( \Phi \) and a set \( R \) such that the set

\[
\mathcal{M} = \{ M \in [R]^\omega : M \prec (\Phi,S) \}
\]

ordered by inclusion is up-directed and \( \sigma \)-complete, and moreover

- every \( M \in \mathcal{M} \) admits the canonical retractions \( r_M \) and \( R_M \) associated to \( M \), \( K \) and \( D \) and to \( M \), \( L \) and \( \varphi[D] \), respectively;
- \( s_K := (r_M)_{M \in \mathcal{M}} \) is a weak subskeleton of \( s \) and \( s_L := (R_M)_{M \in \mathcal{M}} \) is a retractional skeleton on \( L \);
- for every \( M \in \mathcal{M} \) we have \( R_M \circ \varphi = \varphi \circ r_M \).

In particular, we see that \( D(s) = D(s_K) \), \( s_K \) is \( A \)-shrinking with respect to \( D \), and \( \lim_{M \in \mathcal{M}'} r_M(x) \in D \) for all \( x \in D \) and every up-directed subset \( \mathcal{M}' \) of \( \mathcal{M} \). We obviously have \( \varphi[D] \subset D(s_L) \).

Consider now the isometric embedding \( \varphi^* : C(L) \to C(K) \) given by \( \varphi^*(f) := f \circ \varphi \) for \( f \in C(L) \). Further, similarly to the proof of Theorem 2[1(ii)]\(\rightarrow\)(iii) for every \( n \in \mathbb{N} \) put

\[
A_n := \left\{ \sum_{i=1}^k a_i \prod_{j=1}^n f_{i,j} : f_{i,j} \in A, k \in \mathbb{N}, \sum_{i=1}^k |a_i| \leq n \right\},
\]

for \( m \in \mathbb{N} \) put

\[
A_{n,m} := \left( A_n + \frac{1}{2^m} B_{C(K)} \right) \cap B_{\varphi^*C(L)}, \quad B_{n,m} := (\varphi^*)^{-1}(A_{n,m}),
\]

and observe that \( B_{C(L)} = \bigcup_{m \in \mathbb{N}} B_{n,m} \) for every \( m \in \mathbb{N} \). Moreover, for all \( n,m \in \mathbb{N} \) the retractional skeleton \( s_L \) is \( (B_{n,m},1/m) \)-shrinking with respect to \( \varphi[D] \). Indeed, given \( x \in D \) and an increasing sequence \( (M_k)_{k \in \mathbb{N}} \) in \( \mathcal{M} \).
with \( M = \sup_{k \in \mathbb{N}} M_k \), we have
\[
\rho_{B_{n,m}}(R_{M_k}(\varphi(x)), R_M(\varphi(x))) = \sup_{f \in \mathcal{C}(L), f \circ \varphi \in \mathcal{A}_{n,m}} |f(R_{M_k}(\varphi(x))) - f(R_M(\varphi(x)))|
\]
\[
\leq \rho_{A_{n,m}}(r_{M_k}(x), r_M(x)) \leq \rho_A(r_{M_k}(x), r_M(x)) + 1/m.
\]

so using the fact that \( s \) is \( \mathcal{A} \)-shrinking with respect to \( D \), we obtain
\[
\limsup_k \rho_{B_{n,m}}(R_{M_k}(x), R_M(x)) \leq 1/m.
\]

Finally, for all \( x \in D \) and every up-directed subset \( M' \) of \( M \) we have
\[
\lim_{M \in M'} R_M(\varphi(x)) = \varphi\left( \lim_{M \in M'} r_M(x) \right) \in \varphi[D].
\]

Hence, application of Theorem 29(iii) \( \Rightarrow (iv) \) finishes the proof. \( \square \)

**Corollary 42.** Let \( K \) be a semi-Eberlein compact space, \( \varphi : K \to L \) be a continuous surjection and \( D \subset K \) be a dense subset such that there exists a homeomorphic embedding \( h : K \to [-1,1]^J \) with \( h[D] = c_0(J) \cap h[K] \). Then \( S := h^{-1}[\Sigma(J)] \) is the unique set induced by a retractional skeleton in \( K \) with \( D \subset S \). Assume that one of the following conditions holds:

1. \( \varphi^* \mathcal{C}(L) = \{ f \circ \varphi : f \in \mathcal{C}(L) \} \) is \( \tau_p(S) \)-closed in \( \mathcal{C}(K) \);
2. the set \( \{(x,y) \in S \times S : \varphi(x) = \varphi(y)\} \) is dense in \( \{(x,y) \in K \times K : \varphi(x) = \varphi(y)\} \).

Then there is a homeomorphic embedding \( H : L \to [-1,1]^I \) with \( H[\varphi[D]] \subset c_0(I) \). In particular, \( L \) is semi-Eberlein.

**Proof.** By Lemma 24 \( \Sigma(J) \cap h[K] \) is induced by a retractional skeleton and so its preimage \( S \) is induced by a retractional skeleton as well. The uniqueness of \( S \) follows from [7, Lemma 3.2]. If (1) holds, then by [8, Theorem 4.5] the set \( \varphi[S] \) is induced by a retractional skeleton and so we may apply Theorem 41. Finally, by [20, Lemma 2.8] condition (2) implies (1). \( \square \)

**Remark 43.** Note that a very particular case is that a continuous image of an Eberlein compact space is Eberlein, since in this case we have \( D = K \) and thus condition (2) in Corollary 42 is trivially satisfied.

The following answers the second part of [23, Question 6.6].

**Corollary 44.** Let \( K \) be a semi-Eberlein compact space, \( \varphi : K \to L \) be a continuous surjection. Assume that one of the following conditions holds:

1. \( K \) is Corson;
2. \( \varphi \) is open and \( K \) or \( L \) has a dense set of \( G_\delta \) points.

Then \( L \) is semi-Eberlein.
Proof. Let $D \subset K$ be a dense subset such that there exists a homeomorphic embedding $h : K \to [-1,1]^J$ with $h[D] = c_0(J) \cap h[K]$ and put $S := h^{-1}[\Sigma(J)]$. Note that $S$ is a dense $\Sigma$-subset of $K$ and, by Corollary 42, it is the unique set induced by a retractional skeleton with $D \subset S$.

If $K$ is Corson, then it admits a full retractional skeleton, so by the uniqueness of $S$ we have $S = K$ and thus condition (2) from Corollary 42 is obviously satisfied.

If $\varphi$ is open and $L$ has dense set of $G_\delta$ points, then by [8, Lemma 6.1] the set $\varphi[S]$ is induced by a retractional skeleton and thus we may apply directly Theorem 41 (note that inspecting the proof of [8, Lemma 6.1] one can observe that even the condition (2) from Corollary 42 is satisfied).

Finally, if $\varphi$ is open and $K$ has dense set of $G_\delta$ points, then it is easy to see that $L$ has dense set of $G_\delta$ points and we may apply the above.

Let us note that if $D$ is a dense $\Sigma$-subset of $K$ and $\varphi : K \to L$ is a continuous retraction, it may happen that $\varphi[D]$ is not a $\Sigma$-subset of $K$; see [20, Remark 3.25]. Thus, it is not possible to directly apply Theorem 41 for the case when $\varphi$ is a retraction and this is basically the reason why we do not know how to answer the first part of [23, Question 6.6] using our methods.

7. Open questions and remarks. In Section 3 we showed that for a countable family of retractional skeletons inducing the same set there is a common weak subskeleton (see Theorem 21). It would be interesting to know whether we can find even a subskeleton (not only a weak one).

**Question 45.** In Theorem 21 is it possible to obtain a subskeleton instead of a weak subskeleton?

When working with retractional skeletons, their index sets are quite mysterious. For Banach spaces with a projectional skeleton, the index set may be chosen to consist of the ranges of the projections involved (ordered by inclusion; see [10, Theorem 4.1]). We wonder whether something similar holds for spaces with a retractional skeleton.

**Question 46.** Let $K$ be a compact space and let $D \subset K$ be induced by a retractional skeleton on $K$. Does there exist a family of retractions $\{r_F : F \in \mathcal{F}\}$ indexed by a family of compact spaces $\mathcal{F}$ ordered by inclusion satisfying the following conditions?

(i) Whenever $(F_n)_{n \in \omega}$ is an increasing sequence from $\mathcal{F}$, then $\sup_n F_n = \bigcup_n F_n \in \mathcal{F}$.

(ii) For every $F \in \mathcal{F}$ we have $r_F[K] = F$.

(iii) $(r_F)_{F \in \mathcal{F}}$ is a retractional skeleton on $K$ inducing the set $D$.

Note that in Proposition 35 we proved a result in a sense very similar to the characterization of Eberlein compacta from [16, Theorem 2.9] (see
We wonder whether an analogue of [13, Theorem 10] holds also in the context of semi-Eberlein compacta. Note that one implication follows from Proposition 36, so a positive answer to the following question would give a characterization of semi-Eberlein compact subspaces of $[-1, 1]^I$.

**Question 47.** Let $K \subset [-1, 1]^I$ be a compact space such that $\Sigma(I) \cap K$ is dense in $K$. Let $K$ be semi-Eberlein. Does there exist $D \subset \Sigma(I) \cap K$ which is dense in $K$ such that for every $\varepsilon > 0$ there exists a decomposition $I = \bigcup_{n=0}^{\infty} I_n^\varepsilon$ satisfying

$$\forall n \in \omega \ \forall x \in D : \ |\{i \in I_n^\varepsilon \mid |x(i)| > \varepsilon\}| < \omega?$$

Finally, we believe that Theorem D is flexible enough to allow one to consider other subclasses of Valdivia compact spaces and characterize them using the notion of retractional skeletons. The reason is that by Theorem D we may consider any set induced by a retractional skeleton to be a subset of $\Sigma(I)$ (where $\mathcal{A} \subset \mathcal{C}(K)$ plays the role of the set $I$); moreover, for subsets of $\Sigma(I)$ several classes of compact spaces were characterized using their evaluations on $I$ (see [13]). Thus, there is enough room for further possible research by considering those classes of compacta and trying to develop the right notion which would give a characterization using retractional skeletons.

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