On the Bergmann Energy-Momentum Complex of a Charged Regular Black Hole

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Abstract

We use the Bergmann energy-momentum complex to calculate the energy of a charged regular black hole. The energy distribution is the same as we obtained in the Einstein prescription. Also, we get the expression of the energy in the Bergmann prescription for a general spherically symmetric space-time of the Kerr-Schild class.

Keywords: energy momentum-complex, charged regular black hole
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1 Introduction

The subject of energy-momentum localization in general relativity continues to be an open one because there is no given yet a generally accepted expression for the energy-momentum density. Even they are coordinate dependent various energy-momentum complexes give the same energy distribution for a given space-time. Aguirregabiria, Chamorro and Virbhadra [1] obtained that the energy-momentum complexes of Einstein [2], Landau and Lifshitz [3], Papapetrou [4] and Weinberg [5] give the same result for the energy distribution for any Kerr-Schild metric. Also, recently Virbhadra [6] investigated if these definitions lead to the same result for the most general non-static

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spherically symmetric metric and found they don’t give the same expression for the energy distribution. He concluded that only the energy-momentum complex of Einstein still gives the same energy distribution when the calculations are performed in the Kerr-Schild Cartesian and Schwarzschild Cartesian coordinates.

Chang, Nester and Chen [7] showed that the energy-momentum complexes are actually quasilocal and legitimate expression for the energy-momentum. Very important is the Cooperstock [8] hypothesis which states that the energy and momentum are confined to the regions of non-vanishing energy-momentum tensor of the matter and all non-gravitational fields.

In this paper we calculate the energy distribution of a charged regular black hole using the Bergmann energy-momentum complex [9]. Also, we obtain the expression of the energy in the Bergmann prescription for a general static spherically symmetric space-time of the Kerr-Schild class. We show that the Bergmann energy-momentum complex is a good tool for evaluating the energy distribution. We use geometrized units \((G = 1, c = 1)\) and follow the convention that Latin indices run from 0 to 3.

## 2 Energy in the Bergmann prescription

E. Ayón-Beato and A. García (ABG) [10] gave recently a solution to the coupled system of the Einstein field and equations of the nonlinear electrodynamics. This is a singularity free black hole solution with mass \(M\) and electric charge \(q\). Also, the metric at large distance behaves as the Reissner-Nordström (RN) solution. The usual singularity of the RN solution, at \(r = 0\), has been smoothed out and now it simply corresponds to the origin of the spherical coordinates. The line element is given by

\[
ds^2 = A(r) \, dt^2 - B(r) \, dr^2 - r^2 \left( d\theta^2 + \sin^2 \theta \, d\varphi^2 \right),
\]

where

\[
A(r) = B^{-1}(r) = 1 - \frac{2M}{r} \left( 1 - \tanh \left( \frac{q^2}{2M} \right) \right). \tag{2}
\]

If the electric charge vanishes we reach the Schwarzschild solution. At large distances (2) resembles to the Reissner-Nordström solution and can be written

\[
A(r) = B^{-1}(r) = 1 - \frac{2M}{r} + \frac{q^2}{r^2} - \frac{q^6}{12M^2r^4} + O \left( \frac{1}{r^6} \right). \tag{3}
\]
We show that the solution given by (1) can be transformed to another form of a general spherically symmetric space-time of the Kerr-Schild class. For the Kerr-Schild class space-times the metrics $g_{ik}$ have the form

$$g_{ik} = \eta_{ik} - H l_i l_k,$$

(4)

where $\eta_{ik} = \text{diag}(1, -1, -1, -1)$ is the Minkowski metric. $H$ represents the scalar field and $l_i$ is a null, geodesic and shear free vector field in the Minkowski space-time. We also have

$$\eta^{ab} l_a l_b = 0,$$

$$\eta^{ab} l_{i,a} l_b = 0,$$

$$(l_{a,b} + l_{b,a}) l^a c \eta_{bc} - \left( l^a a \right)^2 = 0.$$  

(5)

For the metric given by (1) we make the transformation

$$u = t + \int A^{-1} (r) dr$$

(6)

and we have

$$dt = du - A^{-1} (r) dr.$$  

(7)

We obtain

$$dt^2 = du^2 + A^{-2} (r) dr^2 - 2 A^{-1} (r) dr du.$$  

(8)

With (8) the metric given by (1) becomes

$$ds^2 = A (r) du^2 - 2 du dr - r^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right).$$ 

(9)

This metric corresponds to the particular static case of the general non-static spherically symmetric space-time of the Kerr-Schild class considered by Virbhadra [6] (see Eq. (28) in his paper), for the evaluation of the energy in the Einstein, Landau and Lifshitz, Papapetrou and Weinberg prescriptions.

For the line element (9) we can use the transformations

$$u = T + r,$$

$$x = r \sin \theta \cos \varphi,$$

$$y = r \sin \theta \sin \varphi,$$

$$z = r \cos \theta$$  

(10)
and we obtain
\[ ds^2 = dT^2 - dx^2 - dy^2 - dz^2 - (1 - A) \times \left[ dT + \frac{xdx + ydy + zdz}{r} \right]^2. \]  
(11)

This is a Kerr-Schild class metric with \( H = 1 - A \) and \( l_i = \left( 1, \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right) \).

The Bergmann energy-momentum complex \[9\] is given by
\[ B^{ik} = \frac{1}{16\pi} \frac{\partial (g^{ik}H^{km})}{\partial x^m} \]  
(12)
where
\[ H^{km} = \frac{g_{ln}}{\sqrt{-g}} \left[ -g (g^{kn}g^{mp} - g^{mn}g^{kp}) \right]_{,p} \]  
(13)
and with \( H^{ikm} = g^{il}H^{km} \). The energy and momentum are given by
\[ P^i = \iiint B^{i0} dx^1 dx^2 dx^3. \]  
(14)

Using the Gauss theorem we have
\[ P^i = \frac{1}{16\pi} \iiint H^{i0\alpha} n_\alpha dS, \]  
(15)
where \( n_\alpha = \left( \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right) \) are the components of a normal vector over an infinitesimal surface element \( dS = r^2 \sin \theta d\theta d\varphi \).

For the metric given by (11) we obtain the nonvanishing \( H^{i0\alpha} \) components
\[ H^{001} = \frac{-2x (-1 + A (r))}{r^2}, \]  
\[ H^{002} = \frac{-2y (-1 + A (r))}{r^2}, \]  
\[ H^{003} = \frac{-2z (-1 + A (r))}{r^2}. \]  
(16)

Now, using (15) and (16) we get the energy distribution for the space-time described by (9)
\[ E (r) = \frac{r}{2} (1 - A (r)). \]  
(17)
For the (ABG) black hole, with (2) and (17) we obtain

\[ E(r) = M \left( 1 - \tanh \left( \frac{q^2}{2Mr} \right) \right) \]  

(18)

and

\[ E(r) = M - \frac{q^2}{2r} + \frac{q^6}{24 r^3 M^2} - \frac{q^{10}}{240 M^4 r^5} + O \left( \frac{1}{r^6} \right). \]  

(19)

Also, (19) can be written

\[ E(r) = E_{RN}(r) + \frac{q^6}{24 r^3 M^2} - \frac{q^{10}}{240 M^4 r^5} + O \left( \frac{1}{r^6} \right). \]  

(20)

The term \( E_{RN}(r) \) represents the energy of the Reissner-Nordström solution that corresponds to the Penrose [25] quasi-local mass definition.

With the notations \( E' = \frac{E(r)}{M}, \ Q = \frac{q}{M} \) and \( R = \frac{r}{M} \) we have \( E' = 1 - \tanh(\frac{Q^2}{2R}) \). We plot the expression of \( E' \) in FIGURE 1 (\( E' \) on Z-axis is plotted against \( R \) on X-axis and \( Q \) on Y-axis).

### 3 Discussion

Bondi [24] gave his opinion that a nonlocalizable form of energy is not admissible in relativity.

Many results recently obtained [11]-[23] sustain that the energy-momentum complexes can give reasonable results.

Chang, Nester and Chen [7] argued that every energy-momentum complex is associated with a legitimate Hamiltonian boundary term, and, because of this the energy-momentum complexes are quasilocal and acceptable. Each is the energy-momentum density for some physical situation. This Hamiltonian approach to quasilocal energy-momentum rehabilitates the energy-momentum complexes.

Using an adequate coordinate transformation we get for the line element (1) the form given by (9). For this form of metric we calculate the energy distribution with the energy-momentum complex of Bergmann. We use Kerr-Schild Cartesian coordinates. The expression of the energy is the particular static case of the result obtained by Virbhadra [3] (see Eq. (31) therein), for a general non-static spherically symmetric space-time of the Kerr-Schild class, using the Einstein, Landau and Lifshitz, Papapetrou and Weinberg
prescriptions. Also, in his earlier paper Virbhadra [1] showed that several energy-momentum complexes give the same result for a Kerr-Schild class metric. Tod obtained the same expression for the energy as obtained by Virbhadra using the Penrose [23] quasi-local mass definition (see in [1]).

For the (ABG) charged regular black hole the energy distribution calculated in the Bergmann prescription depends on the mass $M$ and electric charge $q$. The first two terms in the expression of the energy represent the energy distribution of the Reissner-Nordström solution. The other terms are due to the nonlinearity effect. The result is the same as we get in the Einstein prescription [24] for the metric given by (1) and (2).

We made the calculations using Kerr-Schild Cartesian coordinates and the Bergmann energy-momentum complex provides for the metric given by (9) the same expression for the energy distribution as the Einstein, Landau and Lifshitz, Papapetrou and Weinberg energy-momentum complexes. Our result sustain the importance of the energy-momentum complexes in the evaluation of the energy distribution of a given space-time.

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