Calculation of next-to-leading QCD corrections to $b \to sg$

Christoph Greub and Patrick Liniger*

*Work partially supported by Schweizerischer Nationalfonds.

Institut für Theoretische Physik, Universität Bern,
CH–3012 Bern, Switzerland

Abstract

In this paper a detailed standard model (SM) calculation of the $O(\alpha_s)$ virtual corrections to the decay width $\Gamma(b \to sg)$ is presented ($g$ denotes a gluon). Also the complete expressions for the corresponding $O(\alpha_s)$ bremsstrahlung corrections to $b \to sg$ are given. The combined result is free of infrared and collinear singularities, in accordance with the KLN theorem. Taking into account the existing next-to-leading logarithmic (NLL) result for the Wilson coefficient $C_{8}^{\text{eff}}$, a complete NLL result for the branching ratio $B_{\text{NLL}}(b \to sg)$ is derived. Numerically, we obtain $B_{\text{NLL}} = (5.0 \pm 1.0) \times 10^{-3}$, which is more than a factor of two larger than the leading logarithmic result $B_{\text{LL}} = (2.2 \pm 0.8) \times 10^{-3}$. The impact of these corrections on the inclusive charmless hadronic branching ratio $\overline{B}_{g}$ of $B$-mesons, which can be used to extract $|V_{ub}/V_{cb}|$ in the context of the SM, is shown to be of similar importance as NLL corrections to $b$-quark decay modes with three quarks in the final state. Finally, the impact of the NLL corrections to $b \to sg$ on $\overline{B}_{g}$ is investigated in scenarios, where the Wilson coefficient $C_{8}$ is enhanced by new physics.
I. INTRODUCTION

The theoretical predictions for inclusive decay rates of $B$-mesons rest on solid grounds due to the fact that these rates can be systematically expanded in powers of $\Lambda_{\text{QCD}}/m_b$ \cite{1,2}, where the leading term corresponds to the decay width of the underlying $b$-quark decay. As the power corrections only start at $O(\Lambda_{\text{QCD}}^2/m_b^2)$, they affect these rates by at most a few percent. Theoretically, spectator effects of order $16\pi^2(\Lambda_{\text{QCD}}/m_b)^3$ \cite{3,4} could be larger \cite{4}, but for the decay rates of $B^{\pm}$ and $B^0$ they are experimentally known to be at the percent level as well \cite{5}. Thus the accuracy of the theoretical predictions is mainly controlled by our knowledge of the perturbative corrections to the free quark decays.

The inclusive charmless hadronic decays, $B \rightarrow X_d$, where $X_d$ denotes any hadronic charmless final state, are an interesting subclass of the decays mentioned above; as pointed out in ref. \cite{6}, a measurement of the corresponding branching ratio would allow the extraction of the presently poorly known ratio $|V_{ub}/V_{cb}|$, where $V_{ub}$ and $V_{cb}$ are elements of the Cabibbo-Kobayashi-Maskawa (CKM) matrix. At the quark level, there are decay modes with three-body final states, viz. $b \rightarrow q'\bar{q}q$, ($q' = u, d, s; q = d, s$) and the modes $b \rightarrow qg$, with two-body final state topology, which contribute to the charmless decay width at leading logarithmic (LL) accuracy. Calculations of next-to-leading logarithmic (NLL) corrections to the three-body decay modes were started already some time ago in ref. \cite{7}, where radiative corrections to the current-current diagrams of the operators $O_1$ and $O_2$ were calculated, together with NLL corrections to the Wilson coefficients. Later, Lenz. et al. included in a first step \cite{8} the contributions of the penguin diagrams associated with the operators $O_1$ and $O_2$, and in a second step \cite{6} the same authors also included one-loop penguin diagrams of the penguin operators $O_3, ..., O_6$; also the effects of the chromomagnetic operator $O_8$ were taken into account to the relevant precision needed for a NLL calculation. Up to contributions from current-current type corrections to the penguin operators, the NLL calculation for the three quark final states is now complete.

In the numerical evaluations of the charmless hadronic branching ratio, the two body decay modes $b \rightarrow qg$ were added in refs. \cite{8,6} at the LL precision, as the full NLL predictions were missing. To fill this gap, we recently wrote a short letter where NLL results for the branching ratio $\mathcal{B}(b \rightarrow sg)$ were presented \cite{9}. In the present work, we describe in detail the non-trivial two-loop NLL calculation, which led to the results in \cite{9}. As the NLL corrections enhance $\mathcal{B}(b \rightarrow sg)$ by more than a factor of 2, we also analyze in the present paper their impact on the charmless hadronic branching ratio.

The decay $b \rightarrow sg$ gained a lot of attention in the last years. For a long time the theoretical predictions for both, the inclusive semileptonic branching ratio $\mathcal{B}_{sl}$ and the charm multiplicity $n_c$ in $B$-meson decays were considerably higher than the experimental values \cite{10}. An attractive hypothesis, which would move the theoretical predictions for both observables into the direction favored by the experiments, assumed the rare decay mode $b \rightarrow sg$ to be enhanced by new physics.

After the inclusion of the complete NLL corrections to the decay modes $b \rightarrow c\bar{q}q$ and $b \rightarrow c\bar{q}q$ ($q = d, s$) \cite{11}, the theoretical prediction for the semileptonic branching ratio and the charm multiplicity \cite{4} are
\[ B^{\text{th}}_{\text{sl}} = (11.7 \pm 1.4 \pm 1.0)\% , \quad n^\text{th}_c = 1.20 \pm 0.06 , \]

where the second error in \( B^{\text{th}}_{\text{sl}} \) takes into account the spectator effects estimated in ref. [1]. The experimental results from measurements at the \( \Upsilon(4S) \) resonance and those from the \( Z^0 \) resonance at LEP and SLD were recently summarized [12] to be

\[ B^\Upsilon(4S) = (10.45 \pm 0.21)\% , \quad n^\Upsilon(4S) = 1.14 \pm 0.06 , \]
\[ B^{Z^0} = (10.79 \pm 0.25)\% , \quad n^{Z^0} = 1.17 \pm 0.04 . \]

We would like to stress that in the theoretical results the renormalization scale was taken in the interval \([m_b/4, 2m_b]\). If one only considers \( \mu \in [m_b/2, 2m_b] \), the theoretical predictions would only have marginal overlap with experimental data. This implies that there is still room for enhanced \( b \to sg \). We therefore also illustrate in this paper the influence of the NLL corrections to \( b \to sg \) on the charmless hadronic branching ratio in scenarios where the Wilson coefficient \( C_8 \) is enhanced by new physics.

We also would like to mention that the component \( b \to sg \) of the charmless hadronic decays is expected to manifest itself in kaons with high momenta (of order \( m_b/2 \)), due to its two body nature [13]. Some indications for enhanced \( b \to sg \) in this context were reported by the SLD collaboration [14]. For a review of other hints for enhanced \( b \to sg \), the reader is referred to [15].

Within the SM, the LL prediction for the branching for \( b \to sg \) is known to be \( \mathcal{B}(b \to sg) \approx 0.2\% \) [16]. The process \( b \to sgg \), which gives a NLL contribution to the inclusive charmless decay width has already been studied in the literature [14][18]. In [18] a complete calculation was performed in regions of the phase space which are free of collinear infrared singularities. Putting suitable cuts, the branching ratio for \( b \to sgg \) was found to be of the order \( 10^{-3} \) in these phase space regions. A complete calculation requires the calculation of a regularized version for the decay width \( \Gamma(b \to sgg) \) in which infrared and collinear singularities become manifest. Only after adding the virtually corrected decay width \( \Gamma(b \to sg) \) a meaningful physical result is obtained. In addition, as we will see later, also the tree level contribution of the operator \( O_8 \) to the decays \( b \to sf \bar{f} \), with \( f = u, d, s \), has to be included.

The remainder of this paper is organized as follows: In section [II], we review the theoretical framework and discuss the steps needed for a NLL calculation for \( \mathcal{B}(b \to sg) \). Section [II] is devoted to the calculation of the virtual corrections to the matrix elements \( \langle sg|O_{1,2}|b \rangle \), including renormalization, while section [IV] deals with virtual corrections to \( \langle sg|O_8|b \rangle \). In section [V] the virtual corrections to the decay width \( \Gamma(b \to sg) \) are calculated. In section [VI] the decay width \( \Gamma(b \to sf \bar{f}) \) is given. Sections [VII] and [VIII] deal with the gluon bremsstrahlung matrix elements \( \langle sgg|O_{1,2,8}|b \rangle \) and the corresponding decay width, respectively. The analytic results for the NLL branching ratio \( \mathcal{B}(b \to sg) \) can be found in section [IX] while the numerical evaluations are presented in section [X]. Section [XI] deals with the impact of the NLL corrections to \( \mathcal{B}(b \to sg) \) on the charmless hadronic branching ratio in the standard model, while in section [XII] similar questions are addressed in scenarios where the Wilson coefficient \( C_8 \) is enhanced by new physics. We conclude with a short summary in section [XIII]. An explicit parametrization of the NLL Wilson coefficient \( C_8^{\text{eff}}(m_b) \) is given in appendix [A].
II. THE EFFECTIVE HAMILTONIAN

We use the framework of an effective low–energy theory with five quarks, obtained by integrating out the heavy degrees of freedom, which in the SM are the $t$–quark and the $W$–boson. We take into account operators up to dimension six and we put $m_s = 0$. In this approximation the effective Hamiltonian relevant for radiative decays and $b \to sg(g)$ reads

$$\mathcal{H}_{\text{eff}} = -\frac{4G_F}{\sqrt{2}} V_{ts}^* V_{tb} \sum_{i=1}^{8} C_i(\mu) O_i(\mu),$$  \hspace{1cm} (3)$$

where $G_F$ is the Fermi coupling constant and $C_i(\mu)$ are the Wilson coefficients evaluated at the scale $\mu$; $V_{ts}$ and $V_{tb}$ are matrix elements of the Cabibbo-Kobayashi-Maskawa (CKM) matrix. The operators $O_i$ read \[^{19}\]

$$O_1 = (\bar{s}_L \gamma^\mu T^A c_L)(\bar{c}_L \gamma^\mu T^A b_L), \quad O_2 = (\bar{s}_L \gamma^\mu c_L)(\bar{c}_L \gamma^\mu b_L),$$

$$O_3 = (\bar{s}_L \gamma^\mu b_L) \sum_q (\bar{q} \gamma^\mu q), \quad O_4 = (\bar{s}_L \gamma^\mu T^A b_L) \sum_q (\bar{q} \gamma^\mu T^A q),$$

$$O_5 = (\bar{s}_L \gamma^\mu \gamma^\nu \gamma^\rho b_L) \sum_q (\bar{q} \gamma^\mu \gamma^\nu \gamma^\rho q), \quad O_6 = (\bar{s}_L \gamma^\mu \gamma^\nu \gamma^\rho T^A b_L) \sum_q (\bar{q} \gamma^\mu \gamma^\nu \gamma^\rho T^A q),$$

$$O_7 = \frac{e}{16\pi^2} \overline{m}_b(\mu) (\bar{s}_L \sigma^{\mu\nu} b_R) F_{\mu\nu}, \quad O_8 = \frac{g_s}{16\pi^2} \overline{m}_b(\mu) (\bar{s}_L \sigma^{\mu\nu} T^A b_R) G^A_{\mu\nu}.$$  \hspace{1cm} (4)

In the dipole operators $O_7$ ($O_8$), $e$ and $F_{\mu\nu}$ ($g_s$ and $G^A_{\mu\nu}$) denote the electromagnetic (strong) coupling constant and field strength tensor, respectively. $T^A$ $(A = 1, \ldots, 8)$ are $SU(3)$ color generators; $L = (1 - \gamma_5)/2$ and $R = (1 + \gamma_5)/2$ stand for left- and right-handed projectors. In eq. (4), $\overline{m}_b(\mu)$ is the running $b$–quark mass in the $\overline{\text{MS}}$ scheme at the renormalization scale $\mu$. Henceforth, $\overline{m}_q(\mu)$ and $m_q$ denote $\overline{\text{MS}}$ running and pole masses, respectively. To first order in $\alpha_s$, these masses are related through:

$$\overline{m}_q(\mu) = m_q \left(1 + \frac{\alpha_s(\mu)}{\pi} \ln \frac{m^2_q}{\mu^2} - \frac{4\alpha_s(\mu)}{3\pi} \right).$$  \hspace{1cm} (5)$$

It is well-known that QCD corrections to the decay rate for $b \to s\gamma$ bring in logarithms of the mass ratios $m_t/m_W$ and $m_b/m_t$. The same is true for the process $b \to sg$: QCD corrections to this process induce terms of the form $\alpha_s \alpha^n_s \ln^m (m_b/M)$, where $M = m_t$ or $m_W$ and $m \leq n$ (with $m, n = 0, 1, 2, \ldots$).

One can systematically resum these large terms by renormalization group techniques. Usually, one matches the full standard model theory with the effective theory at a scale of order $m_W$. At this scale, the large logarithms generated by matrix elements in the effective theory are the same ones as in the full theory. Consequently, the Wilson coefficients only pick up formally small QCD corrections. Using the renormalization group equation, the Wilson coefficients are then calculated at the scale $\mu = \mu_b \approx m_b$, at which the large logarithms are contained in the Wilson coefficients, while the matrix elements of the operators are free of them.

So far the decay rate for $b \to sg$ has been systematically calculated only to leading logarithmic (LL) accuracy, i.e., for $m = n$.

A consistent calculation for $b \to sg$ at LL precision requires the following steps:
1) the extraction of the Wilson coefficients from a matching calculation of the full standard model theory with the effective theory at the scale $\mu = \mu_W$ to order $\alpha_s^0$; $\mu_W$ denotes a scale of order $m_W$ or $m_t$;

2) a renormalization group treatment of the Wilson coefficients, using the anomalous-dimension matrix to order $\alpha_s^1$;

3) the calculation of the decay matrix elements $\langle sg|C_i O_i|b\rangle$ at the scale $\mu = \mu_b$ to order $g_s$; $\mu_b$ denotes a scale of order $m_b$. We note that the matrix elements associated with the four Fermi operators ($i = 1 - 6$) can be absorbed into the effective Wilson coefficient $C_{8}^{\text{eff}}$, when working at LL precision. In the naive dimensional regularization scheme (NDR), which we use in this paper, one obtains

$$C_{8}^{\text{eff}} = C_8 + C_3 - \frac{1}{6} C_4 + 20 C_5 - \frac{10}{3} C_6 .$$

From the analogous decay $b \to s \gamma$ it is well-known that next-to-leading logarithmic (NLL) corrections drastically reduce the large renormalization scale dependence of the LL branching ratio. This implies, in particular, that the NLL corrections are relatively large, at least for certain scales (within the usually considered range $m_b/2 \leq \mu_b \leq 2m_b$). Motivated by the situation in this analogous process, we present in this paper a systematic calculation of the NLL corrections to $b \to sg$.

The principal organization of such a calculation is straightforward: Each of the three steps listed above has to be improved by going to the next order in $\alpha_s$: 1) The matching has to be calculated including $\alpha_s$ corrections; 2) the renormalization group treatment of the Wilson coefficients has to be performed using the anomalous dimension matrix to order $\alpha_s^2$; 3) finally, the order $\alpha_s$ corrections to the decay matrix elements have to be worked out. We note that this step involves both, the calculation of virtual- and bremsstrahlung corrections to $b \to sg$.

The first two steps are already available in the literature. The order $\alpha_s$ matching of the dipole operators $O_7$ and $O_8$ was calculated in refs. [20], while the matching conditions and the anomalous dimension matrix for the four Fermi operators have been calculated by several groups [21]. These calculations were done in the “old operator basis”, introduced by Grinstein et al. [22]. The most difficult part, the order $\alpha_s^2$ mixing of the four-Fermi operators into the dipole operators requires the calculation of three loop diagrams [19]. In order to perform a consistent NDR calculation (i.e. with anticommuting $\gamma_5$), the old operator basis was replaced by the new one displayed in eq. (4). The full $8 \times 8$ anomalous dimension matrix, the corresponding matching conditions and the definition of the evanescent operators is given in ref. [19] and is repeated in appendix A of the present paper.

Step 3), the calculation of the virtual $O(\alpha_s)$ corrections to the matrix elements $M_i = \langle sg|O_i|b\rangle$, as well as the evaluation of the gluon bremsstrahlung process $b \to sgg$, is performed the first time in the present paper. As illustrated in table 4, the LL Wilson coefficients $C_{3}^{0}(\mu_b), \ldots, C_{6}^{0}(\mu_b)$ are much smaller than $C_{1}^{0}(\mu_b)$ and $C_{2}^{0}(\mu_b)$. We therefore only calculate $M_1$, $M_2$ and $M_8$ together with the corresponding bremsstrahlung corrections. As $M_1$ and $M_2$ vanish at one-loop (i.e. without QCD corrections), only the leading order pieces, $C_{1}^{0}(\mu_b)$ and $C_{2}^{0}(\mu_b)$, appearing in the decomposition
\[ C_i(\mu_b) = C_0^i(\mu_b) + \frac{\alpha_s(\mu_b)}{4\pi} C_1^i(\mu_b) \]  \tag{7}

of the NLL Wilson coefficients \( C_1(\mu_b) \) and \( C_2(\mu_b) \) are needed. On the other hand, the operator \( O_8 \) contributes to \( M_8 \) already at tree-level. Consequently the full NLL Wilson coefficient \( C_{\text{eff}}^8(\mu_b) \) is needed. The numerical value of the NLL piece \( C_{\text{eff}}^8 \) (defined as in eq. (6) and (7)) is also given in table I, while the analytic form is relegated to appendix A.

| \( \alpha_s \) | \( \mu = m_W \) | \( \mu = 9.6 \text{ GeV} \) | \( \mu = 4.8 \text{ GeV} \) | \( \mu = 2.4 \text{ GeV} \) |
|---|---|---|---|---|
| 0.121 | 1.82 | 0.218 | 0.271 |
| \( C^0_1 \) | 0.0 | -0.335 | -0.497 | -0.711 |
| \( C^0_2 \) | 1.0 | 1.012 | 1.025 | 1.048 |
| \( C^0_3 \) | 0.0 | -0.002 | -0.005 | -0.010 |
| \( C^0_4 \) | 0.0 | -0.042 | -0.067 | -0.103 |
| \( C^0_5 \) | 0.0 | 0.0002 | 0.0005 | 0.001 |
| \( C^0_6 \) | 0.0 | 0.0005 | 0.001 | 0.002 |
| \( C^0_7 \) | -0.192 | -0.285 | -0.324 | -0.371 |
| \( C^0_8 \) | -0.096 | -0.136 | -0.150 | -0.166 |
| \( C^0_{\text{eff}} \) | -0.196 | -0.280 | -0.314 | -0.356 |
| \( C^0_{\text{eff}} \) | -0.097 | -0.135 | -0.149 | -0.165 |
| \( C^1_{\text{eff}} \) | -2.166 | -1.318 | -1.098 | -0.950 |
| \( C^2_{\text{eff}} \) | -0.118 | -0.154 | -0.168 | -0.186 |

TABLE I. Wilson coefficients \( C_i^0(\mu) \) \( (i = 1, \ldots, 8) \), \( C_{\text{eff}}^1 \) and \( C_{\text{eff}}^2 \) (see eq. (6) in the text) at the matching scale \( \mu = m_W = 80.33 \) GeV and at three other scales, \( \mu = 9.6 \) GeV, \( \mu = 4.8 \) GeV and \( \mu = 2.4 \) GeV. For \( \alpha_s(\mu) \) (in the \( \overline{\text{MS}} \) scheme) we used the two-loop expression with 5 flavors and \( \alpha_s(m_Z) = 0.119 \). The entries correspond to the pole top quark mass \( m_t = 175 \) GeV.

### III. VIRTUAL CORRECTIONS TO \( O_1 \) AND \( O_2 \)

In this section we present the calculation of the matrix elements of the operator \( O_1 \) and \( O_2 \) for \( b \to sg \) up to order \( \alpha_s \) in the NDR scheme. The one-loop \( (\alpha_s^0) \) matrix elements vanish and we must consider several two-loop contributions. Since they involve ultraviolet singularities also counterterm contributions are needed. These are easy to obtain, because the operator renormalization constants \( Z_{ij} \) are known with enough accuracy from the order \( \alpha_s \) anomalous dimension matrix \[19\].

#### A. Regularized two-loop matrix elements of \( O_1 \) and \( O_2 \)

For the following discussion it is useful to define the operators \( \hat{O}_1 \) and \( \hat{O}_2 \):
FIG. 1. Graphs associated with the operators $\hat{O}_1$ and $\hat{O}_2$. The wavy lines represent gluons; the real gluons are understood to be attached to the circle-crosses.

FIG. 2. Building block $I_\beta$ (with an off-shell gluon) for the diagrams a), b), e) and f) in fig. 1 and building block $J_{\alpha\beta}$ for the diagrams c) and d) in fig. 1. $g^*$ and $g$ denote an off-shell and an on-shell gluon, respectively.

\[
\hat{O}_1 = 2O_1 + \frac{1}{3}O_2 \quad ; \quad \hat{O}_2 = O_2 .
\]  
(8)

$\hat{O}_1$ and $\hat{O}_2$ are nothing but the current-current operators in the old basis \[22\]:

\[
\hat{O}_1 = (\bar{s}_L\gamma_\mu c_{L\beta}) (\bar{c}_{L\beta}\gamma^\mu b_{L\alpha}) , \quad \hat{O}_2 = (\bar{s}_{L\beta}\gamma_\mu c_{L\beta}) (\bar{c}_{L\alpha}\gamma^\mu b_{L\alpha}) .
\]  
(9)

We now present the calculation of the matrix elements $\hat{M}_i = \langle s g | \hat{O}_i | b \rangle$: The dimensionally regularized matrix element $\hat{M}_2$ is obtained by calculating the two-loop diagrams a) – h) shown in fig. 1.

We start with the calculation of the diagrams a) – f) in fig. 1 in which the virtual gluon connects the charm quark in the loop with an external fermion leg. The main steps of the calculation are the following: We first calculate the Fermion loops in the individual diagrams, i.e., the 'building blocks' $I_\beta$ and $J_{\alpha\beta}$ shown in fig. 2; $J_{\alpha\beta}$ denotes the sum of the two diagrams on the right.

We work in $d = 4 - 2\epsilon$ dimensions; the results of the building blocks are presented as integrals over Feynman parameters after integrating over the (shifted) loop-momentum.

\[\text{The diagrams g) and h) are much easier to calculate than those in a) – f), because } m_c \text{ is the only scale in the corresponding integrals.}\]
Then we insert these building blocks into the full two-loop diagrams. Using one more Feynman parametrization, we calculate the integral over the second loop-momentum. As the remaining Feynman parameter integrals contain rather complicated denominators, we do not evaluate them directly. At this level we also do not expand in the regulator $\epsilon$. The heart of our procedure which will be explained more explicitly below, is to represent these denominators as complex Mellin-Barnes integrals \[23\]. After inserting this representation and interchanging the order of integration, the Feynman parameter integrals are reduced to well-known Euler Beta-functions. Finally, the residue theorem allows to write the result of the remaining complex integral as the sum over the residues taken at the pole positions of Beta- and Gamma-functions; this naturally leads to an expansion in the ratio $z = (m_c/m_b)^2$, which numerically is about $z \approx 0.1$.

We express the diagram on the left in fig. 2 (denoted by $I_{\beta}$) in a way convenient for inserting into the two-loop diagrams. As we will use $\overline{\text{MS}}$ subtraction later on, we introduce the renormalization scale in the form $\mu^2 \exp(\gamma_E)/(4\pi)$, where $\gamma_E \approx 0.577$ is the Euler constant. Then, $\overline{\text{MS}}$ corresponds to subtracting the poles in $\epsilon$. In the NDR scheme, $I_{\beta}$ is given by\(^2\)

\[
I_{\beta}^A = -\frac{g_s^2}{4\pi^2} \Gamma(\epsilon) \mu^{2\epsilon} \exp(\gamma_E \epsilon) (1 - \epsilon) \exp(i\pi\epsilon) \left[ r_{\beta\beta}^2 - r^2 \gamma_{\beta} \right] L \times \int_0^1 \left[ x(1-x) \right]^{1-\epsilon} \left[ r^2 - \frac{m_c^2}{x(1-x)} + i\delta \right]^{-\epsilon},
\]

where $r$ is the four-momentum of the (off-shell) gluon, $m_c$ is the mass of the charm quark propagating in the loop and the term $i\delta$ is the ”$\epsilon$-prescription”. The free index $\beta$ will be contracted with the gluon propagator when inserting the building block into the two-loop diagrams a), b), e) and f) in fig. 1. Note that $I_{\beta}$ is gauge invariant in the sense that $r_{\beta}^2 I_{\beta} = 0$.

Next we give the sum of the two diagrams on the right in fig. 2, using the decomposition in \[13\]. The on-shell gluon has momentum $q$, color $A$ and polarization $\alpha$ (therefore we drop the terms $q^2$ and $q_{\alpha}$), while the off-shell gluon has momentum $r$, color $B$ and polarization $\beta$. This building block, denoted by $J_{\alpha\beta}^{AB}$, can be decomposed with respect to the color structure as

\[
J_{\alpha\beta}^{AB} = T_{\alpha\beta}^+(q,r) \left[ T^A, T^B \right] + T_{\alpha\beta}^-(q,r) \left[ T^A, T^B \right].
\]

The quantities $T_{\alpha\beta}^+(q,r)$ and $T_{\alpha\beta}^-(q,r)$ read

\[
T_{\alpha\beta}^+(q,r) = \frac{g_s^2}{32\pi^2} \left[ E(\alpha, \beta, r) \Delta i_5 + E(\alpha, \beta, q) \Delta i_6 - E(\beta, r, q) \frac{r_{\alpha}}{(qr)} \Delta i_{23}
\]

\[
- E(\alpha, r, q) \frac{r_{\beta}}{(qr)} \Delta i_{25} - E(\alpha, r, q) \frac{q_{\beta}}{(qr)} \Delta i_{26} \right] L,
\]

\[
T_{\alpha\beta}^-(q,r) = \frac{g_s^2}{32\pi^2} \left[ E(\alpha, \beta, r) \Delta i_5 + E(\alpha, \beta, q) \Delta i_6 - E(\beta, r, q) \frac{r_{\alpha}}{(qr)} \Delta i_{23}
\]

\[
- E(\alpha, r, q) \frac{r_{\beta}}{(qr)} \Delta i_{25} - E(\alpha, r, q) \frac{q_{\beta}}{(qr)} \Delta i_{26} \right] L.
\]

\(^2\)The fermion/gluon and the fermion/photon couplings are defined according to the covariant derivative $D = \partial + ig_s T^B A^B + ieQA$ where $T^B = \lambda^B/2$ are the SU(3) generators.
\[
T_{\alpha\beta}(q, r) = \frac{g_s^2}{32\pi^2} \left[ \gamma g_{\alpha\beta} \Delta i_2 + \gamma g_{\alpha\beta} \Delta i_3 + \gamma r_\alpha \Delta i_8 + \gamma r_\beta \Delta i_{11} + \gamma q_{\beta} \Delta i_{12} + \gamma r_\alpha r_\beta \Delta i_{15} + \gamma r_\alpha q_\beta \Delta i_{17} + \gamma r_\alpha r_\beta \Delta i_{19} + \gamma r_\alpha q_\beta \Delta i_{21} \right] L. \tag{13}
\]

The matrix \( E \) in eq. (12) is defined as
\[
E(\alpha, \beta, r) = \gamma_\alpha \gamma_\beta \gamma - \gamma_\alpha r_\beta + \gamma_\beta (r_\alpha) - \gamma g_{\alpha\beta}. \tag{14}
\]

In a four-dimensional context these \( E \) quantities can be reduced to expressions involving the Levi-Civit\`a tensor, i.e., \( E(\alpha, \beta, \gamma) = -i \varepsilon_{\alpha\beta\gamma\mu} \gamma^\mu \gamma_5 \) (in the Bjorken-Drell convention). The dimensionally regularized expressions for the \( \Delta i \) functions read

\[
\Delta i_5 = -4 B^+ \int_S dx \, dy \, C^{-1-\epsilon}[4(qr)x^2 y\epsilon - 4(qr)x y\epsilon - 2r^2 x^3 \epsilon + 3r^2 x^2 \epsilon - r^2 x \epsilon + 3x^2 C - C] \tag{15}
\]

\[
\Delta i_6 = 4 B^+ \int_S dx \, dy \, C^{-1-\epsilon}[4(qr)x^2 y\epsilon - 4(qr)x y\epsilon - 2r^2 x^2 y\epsilon + 2r^2 x^2 \epsilon + 2r^2 x y\epsilon - 2r^2 x \epsilon + 3y^2 C - C] \tag{16}
\]

\[
\Delta i_{23} = -\Delta i_{26} = 8 B^+(qr) \epsilon \int_S dx \, dy \, C^{-1-\epsilon} xy \tag{17}
\]

\[
\Delta i_{25} = -8 B^+(qr) \epsilon \int_S dx \, dy \, C^{-1-\epsilon} x(1 - x) \tag{18}
\]

\[
\Delta i_2 = 4 B^- \int_S dx \, dy \, C^{-1-\epsilon}(1 - x) \left[ 4(qr)x y\epsilon - 2r^2 x^2 \epsilon + r^2 x \epsilon + C \right] \tag{19}
\]

\[
\Delta i_3 = 4 B^- \int_S dx \, dy \, C^{-1-\epsilon}[4(qr)x^2 y\epsilon - 4(qr)x y\epsilon - 2r^2 x^2 y\epsilon + 2r^2 x^2 \epsilon + 2r^2 x y\epsilon - 2r^2 x \epsilon + y^2 C - C]\tag{20}
\]

\[
\Delta i_8 = -4 B^- \int_S dx \, dy \, C^{-1-\epsilon}[4(qr)x^2 y\epsilon + 2(qr)x y\epsilon - 2r^2 x^3 \epsilon + 2r^2 x^2 \epsilon + 2r^2 x \epsilon + x C + C] \tag{21}
\]

\[
\Delta i_{11} = 4 B^- \int_S dx \, dy \, C^{-1-\epsilon}(1 - x) \left[ 4(qr)x y\epsilon - 2(qr)x \epsilon - 2r^2 x^2 \epsilon + r^2 x \epsilon + C \right] \tag{22}
\]

\[
\Delta i_{12} = 4 B^- \int_S dx \, dy \, C^{-1-\epsilon}[4(qr)x^2 y\epsilon + 2(qr)x y\epsilon - 2r^2 x^2 y\epsilon - 2r^2 x^2 \epsilon + 2r^2 x \epsilon + y C + C] \tag{23}
\]

\[
\Delta i_{15} = 16 B^-(qr) \epsilon \int_S dx \, dy \, C^{-1-\epsilon} x^2(1 - x) \tag{24}
\]

\[
\Delta i_{17} = -8 B^-(qr) \epsilon \int_S dx \, dy \, C^{-1-\epsilon} xy(1 - 2x) \tag{25}
\]

\[
\Delta i_{19} = 8 B^-(qr) \epsilon \int_S dx \, dy \, C^{-1-\epsilon} x(1 - x - y + 2yx) \tag{26}
\]

\[
\Delta i_{21} = 8 B^-(qr) \epsilon \int_S dx \, dy \, C^{-1-\epsilon} xy(1 - 2y) \tag{27}
\]

where \( C, C^{-1-\epsilon} \) and \( B^\pm \) are given by
\[ C = m_c^2 - 2xy(qr) - x(1 - x)r^2 - i\delta \]
\[ C^{-1-\epsilon} = -\exp(\frac{i\pi\epsilon}{x(1-x)})^{-1-\epsilon} \left[ r^2 + \frac{2y(qr)}{1-x} - \frac{m_c^2}{x(1-x)} + i\delta \right]^{-1-\epsilon} \] (28)

\[ B^+ = (1 + \epsilon)\Gamma(\epsilon) \exp(\gamma_E\epsilon) \mu^{2\epsilon}, \quad B^- = (\epsilon - 1)i\Gamma(\epsilon) \exp(\gamma_E\epsilon) \mu^{2\epsilon}. \] (29)

The range of integration in \((x, y)\) is restricted to the simplex \(S\), i.e., \(0 \leq y \leq (1 - x)\) and \(0 \leq x \leq 1\).

We are now ready to evaluate the two-loop diagrams. Due to the absence of extra singularities in the limit of vanishing strange quark mass, we set \(m_s = 0\) from the very beginning.

In ref. [24] the detailed calculation of one of the diagrams in fig. [1a] was presented for \(b \to s\gamma\). As all the other diagrams, which involve the building block \(I_{\beta}\), i.e., a), b), e) and f) in fig. [1], can be computed in a very similar way, we prefer to concentrate on the diagrams involving the building block \(J_{\alpha\beta}\). As an example in this class, we concentrate on the diagrams d) in fig. [1], which we redisplay in fig. [3] in order to set up the notation for the momenta.

![Feynman diagram for the Mellin-Barnes example](image)

FIG. 3. Feynman diagram for the Mellin-Barnes example. The momentum and the polarization vector of the emitted gluon are denoted by \(q\) and \(\varepsilon\), respectively.

The sum \(\hat{M}_2(d)\) of the two diagrams can be decomposed into a color symmetric part \(\hat{M}_2^+(d)\) and a color antisymmetric part \(\hat{M}_2^-(d)\) according to

\[ \hat{M}_2(d) = \hat{M}_2^+(d) + \hat{M}_2^-(d) \] (30)

with

\[ \hat{M}_2^-(d) = g_s (-i) f^{ABC} T^B T^C \mu^{2\epsilon} \frac{e^{\gamma_E}}{(4\pi)^\epsilon} \frac{1}{i} \int \frac{d^d r}{(2\pi)^d} \bar{u}(p') \left( T_{\alpha\beta} \varepsilon^a \right) \frac{\not{q} + \not{p} + m_b}{r^2 + 2(pr)} \gamma^\beta u(p) \frac{1}{r^2} \] (31)

where \(T_{\alpha\beta}^+\) and \(T_{\alpha\beta}^-\) are given in eqs. (12) and (13), respectively. As the calculation of \(\hat{M}_2^+(d)\) is nothing but a repetition of the \(b \to s\gamma\) case, we concentrate on \(\hat{M}_2^-(d)\) in the following. All the \(\Delta i\) quantities in \(T_{\alpha\beta}^-\) contain the factor \(C^{-1-\epsilon}\), whose explicit form is given in eq. (28). \(\hat{M}_2^-(d)\) can be written in the form
\[ \hat{M}_2^-(d) = \frac{g_s^3}{32\pi^2} (-i) f^{A B C} T^B T^C \]

\[
\mu^{2\epsilon} e^{\epsilon\gamma} \frac{1}{(4\pi)^\epsilon} \int \frac{d^dT}{(2\pi)^d} \bar{u}(p') P(r) u(p) \left[ -\exp(\epsilon)\frac{[x(1-x)]^{-1+\epsilon}}{D_1 D_2 D_3^{1+\epsilon}} \right], \tag{32}
\]

with \( D_1 = (r^2 + 2(pr)), \ D_2 = r^2, \ D_3 = r^2 + 2(qr)y/(1-x) - m_c^2/(x(1-x)) \). The symbol \( P(r) \) is a matrix in Dirac space, which depends in a polynomial way on the integration variable \( r \). In the next step, the three propagators \( D_1, D_2 \) and \( D_3 \) in the denominator are Feynman parametrized as

\[
\frac{1}{D_1 D_2 D_3^{1+\epsilon}} = \frac{\Gamma(3+\epsilon)}{\Gamma(1+\epsilon)} \int_s \frac{dudw u'}{[s + 2(pr)u + 2(qr)y w/(1-x) - m_c^2 w/(x(1-x))] + i\epsilon}^{1+\epsilon} \tag{33}
\]

with \( 0 \leq w \leq 1 - u \) and \( 0 \leq u \leq 1 \). Then the integral over the loop momentum \( r \) is performed. At this level, a four dimensional integral over the Feynman parameters \( (x, y; u, w) \) remains. It is useful for the following to perform the substitutions

\[
x \to x'; \quad y \to -\frac{(1-x')(1-w'-y')}{w'}; \quad u \to (1-w')u'; \quad w \to u'w'. \tag{34}
\]

The new variables then run in the intervals

\[
x', u', w' \in [0,1]; \quad y \in [1-w',1]. \tag{35}
\]

Taking into account the corresponding Jacobian and omitting the primes (' of the integration variables, \( \hat{M}_2^-(d) \) can be cast into the form

\[
\hat{M}_2^- (d) = \frac{g_s^3}{32\pi^2} (-i) f^{A B C} T^B T^C \int dx dy dw d\bar{u}(p') \left[ F_1 \frac{\bar{C}}{C^{2\epsilon}} + F_2 \frac{1}{C^{2\epsilon}} + F_3 \frac{1}{C^{1+2\epsilon}} \right] u(p) \tag{36}
\]

where \( F_1, F_2 \) and \( F_3 \) are matrices in Dirac space depending on the Feynman parameters \( x, y, u, w \). Note that this expression is understood to be written in such a way that \( F_1, F_2 \) and \( F_3 \) are independent of \( m_c \). The charm quark mass then only enters through \( \bar{C} \), which reads

\[
\bar{C} = m_c^2 uy(1-w) + m_c^2 \frac{w}{x(1-x)} \tag{37}
\]

In what follows, the ultraviolet \( \epsilon \) regulator remains a fixed, small positive number.

The central point of our procedure is to use now the Mellin-Barnes representation of the denominators that look like propagators \( (1/(k^2 - M^2)^\lambda) \) \(^{[25]}\), which is given by \( (\lambda > 0) \)

\[
\frac{1}{(k^2 - M^2)^\lambda} = \frac{1}{(k^2)^\lambda} \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int ds (-M^2/k^2)^s \Gamma(-s) \Gamma(\lambda + s). \tag{38}
\]
The symbol $\gamma$ denotes the integration path which is parallel to the imaginary axis (in the complex $s$-plane) hitting the real axis somewhere between $-\lambda$ and 0. In this formula, the "momentum squared" $k^2$ is understood to have a small positive imaginary part.

In our approach, we use formula (38) in order to simplify the remaining Feynman parameter integrals in eq. (36) where we represent the factors $1/\bar{C}^2\varepsilon$ and $1/\bar{C}^{1+2\varepsilon}$ as Mellin-Barnes integrals using the identifications

$$k^2 \leftrightarrow m_b^2 w y (1-w) \ ; \ M^2 \leftrightarrow \frac{-m_c^2 w}{x(1-x)} . \ (39)$$

By interchanging the order of integration, we first carry out the integrals over the Feynman parameters for any given fixed value of $s$ on the integration path $\gamma$. These integrals are basically the same as for the massless case $m_c = 0$ (in eqs. (36) and (37)) up to the factor

$$\left[ \frac{w}{u y (1-w) x(1-x)} \right]^s \left( \frac{m_c^2}{m_b^2} \right)^s . \ (40)$$

Note that the functions $F_1$, $F_2$ and $F_3$ are such that the Feynman parameter integrals exist if the integration path $\gamma$ is properly chosen. In the terms involving $F_2$ and $F_3$ in eq. (39), the path must be chosen such that $-\varepsilon < \text{Re}(s) < 0$; in the terms involving $F_1$ the situation is slightly more complicated: $\bar{C}$ in the numerator should be replaced by the r.h.s of eq. (37). For the terms proportional to $m_b^2$ the path has to be chosen as for the $F_2$ and $F_3$ contributions. The terms proportional to $m_c^2$, however, lead to Feynman parameter integrals which do not converge for values of $s$ on this path. It turns out that the path has to be chosen such that $-2\varepsilon < \text{Re}(s) < -\varepsilon$ in order to have convergent integrals for these terms.

We would like to mention that the variable substitutions in eq. (34) were constructed in such a way that all the Feynman parameter integrals are either elementary or of the form

$$\int_0^1 dx \, x^p (1-x)^q = \beta(p+1, q+1).$$

For the $s$ integration we use the residue theorem after closing the integration path in the right $s$-halfplane. According to the above discussion, the residue at $s = -\varepsilon$ has to be taken into account in the terms proportional to $m_b^2$. In the other terms, however, the residue at $s = -\varepsilon$ must not be taken into account. The other poles inside the integration contour are located at

$$s = 0, 1, 2, 3, \ldots, \quad s = 1 - \varepsilon, 2 - \varepsilon, 3 - \varepsilon, \ldots, \quad s = 1 - 2\varepsilon, 2 - 2\varepsilon, 3 - 2\varepsilon, \ldots, \quad s = 1/2 - 2\varepsilon, 3/2 - 2\varepsilon, 5/2 - 2\varepsilon, \ldots, \quad s = 1 - 3\varepsilon, 2 - 3\varepsilon, 3 - 3\varepsilon, \ldots . \ (41)$$

The other two-loop diagrams are evaluated similarly. The non-trivial Feynman integrals can always be reduced to $\beta$-functions after suitable substitutions.

The sum over the residues naturally leads to an expansion in $z = (m_c^2/m_b^2)^s$ in eq. (40). This expansion, however, is not a Taylor series; it also involves
logarithms of $z$, which are generated by the expansion in $\epsilon$. A generic diagram which we denote by $G$ has then the form

$$G = c_0 + \sum_{n,m} c_{nm} z^n \ln^m z, \quad z = \frac{m_c^2}{m_b^2}. \quad (42)$$

The power $n$ in eq. (42) is in general a natural multiple of $1/2$ and $m$ is a natural number including 0. In the explicit calculation, the lowest $n$ turns out to be $n = 1$. This implies the important fact that the limit $m_c \to 0$ exists.

From the structure of the poles one can see that the power $m$ of the logarithm is bounded by 4, independent of the value of $n$. For a detailed explanation, we refer to [24]. As in this reference, we retain all terms up to $n = 3$ in our results.

Unlike in $b \to s\gamma$, the diagrams in the individual figures are not gauge invariant. This statement holds even for the sum of all the diagrams in a) – f) in fig. [1]. A gauge invariant result is only obtained after including the diagrams in g) and h)[1]. We would like to mention that the diagrams analogous to g) also exist for $b \to s\gamma$. Their sum, however, vanishes in this case. As there are no gauge invariant subsets, we only present the result which is obtained by summing all diagrams a) – h) in fig. [1]. The result for $\hat{M}_2 = \langle sg|\hat{O}_2|b\rangle$ reads (using $z = (m_c/m_b)^2$ and $L = \ln z$):

$$\hat{M}_2 = \frac{1}{2592} \frac{\alpha_s}{\pi} \langle sg|O_8|b\rangle_{\text{tree}} \left( \frac{m_b}{\mu} \right)^{-4\epsilon} \left\{ - \frac{384}{\epsilon} - 2170 - 54\pi^2 + z[48816 - 252\pi^2 + (22680 - 1620\pi^2)L] 
+ 2808L^2 + 612L^3 - 6480\zeta(3) \right.
- 12672\pi^2 z^{3/2} + z^2[66339 + 1872\pi^2 + (-40446 + 1512\pi^2)L] 
+ 6642L^2 - 1008L^3 + 7776\zeta(3)]
+ z^3[-3420 - 60\pi^2 - 6456L + 7884L^2] 
+ 24\pi i[ - 28 + z(549 - 24\pi^2 + 153L + 72L^2) 
+ z^2(-432 + 30\pi^2 + 54L - 90L^2) + z^3(-259 + 192L)] \right\}. \quad (43)$$

In this expression, the symbol $\zeta$ denotes the Riemann Zeta function, with $\zeta(3) \approx 1.2021$; The symbol $\langle sg|O_8|b\rangle_{\text{tree}}$ denotes the tree level matrix element of the operator $O_8$. As such, it contains the running $b$-quark mass and the running strong coupling constant, both evaluated at the scale $\mu$ (see eq. (4)). However, as the corrections to $O_2$ are explicitly proportional to $\alpha_s$, we are allowed (modulo higher order terms) to identify the running $b$-quark mass with the pole mass $m_b$; in the same spirit we can identify the strong coupling constant with $g_s(m_b)$. With this interpretation, which we will use in the following, $\langle sg|O_8|b\rangle_{\text{tree}}$ is a scale independent quantity, reading

\footnote{We thank M. Neubert for making us aware of these diagrams.}
\[ \langle sg|O_8|b\rangle_{\text{tree}} = m_b \frac{g_s(m_b)}{8\pi^2} \tilde{u}(p') \not\! qRT^A u(p) \quad . \] (44)

We now turn to the matrix elements of the operator \( \hat{O}_1 \). Due to the specific color structure it is straightforward to see that only the diagrams e) and f) in fig. 1 yield a non-vanishing contribution, which is generated by the color symmetric part of the building block \( J_{\alpha\beta} \) in eq. (11). The complete regularized result for \( \hat{M}_1 = \langle sg|\hat{O}_1|b\rangle \) reads

\[
\hat{M}_1 = \frac{1}{96} \alpha_s \langle sg|O_8|b\rangle_{\text{tree}} \left( \frac{m_b}{\mu} \right)^{-4\epsilon} \{ -\frac{18}{\epsilon} - 87 + z[120 - 16\pi^2 + (120 - 36\pi^2)L] +12L^2 + 4L^3 - 144\zeta(3) \\
+ z^2[84 + 32\pi^2 - 24\pi^2 L] -12L^2 + 4L^3 + z^3[-56 - 12\pi^2 + 96L - 36L^2] -4\pi i[3 + z(-24 + 2\pi^2 - 6L - 6L^2) +z^2(-6 + 2\pi^2 + 12L - 6L^2) - 12z^3] \} .
\] (45)

The regularized matrix elements \( M_1 \) and \( M_2 \) of \( O_1 \) and \( O_2 \) in the operator basis (4) are related to \( \hat{M}_1 \) in eq. (45) and \( \hat{M}_2 \) in eq. (43) as follows:

\[
M_1 = \frac{1}{2} \hat{M}_1 - \frac{1}{6} \hat{M}_2 \quad ; \quad M_2 = \hat{M}_2 .
\] (46)

The operators mix under renormalization and thus the counterterm contributions must be taken into account. As we are interested in this section in contributions to \( b \to sg \) which are proportional to \( C_1 \) and \( C_2 \), we have to include, in addition to the two-loop matrix elements of \( C_1O_1 \) and \( C_2O_2 \), also the one-loop matrix elements of the four Fermi operators \( C_i\delta Z_{ij}O_j \) (\( i = 1, 2; j = 1, 2, \ldots, 6 \)) and the tree level contribution of the magnetic operator \( C_i\delta Z_{18}O_8 \) (\( i = 1, 2 \)). In the NDR scheme the only non-vanishing contributions to \( b \to sg \) come from \( j = 4, 8 \) only. The operator renormalization constants \( Z_{ij} \) are obtained from the leading order anomalous dimension matrix in the literature [19]. The entries needed in our calculation are

\[
\delta Z_{14} = -\frac{\alpha_s}{36\pi\epsilon} , \quad \delta Z_{18} = \frac{167\alpha_s}{2592\pi\epsilon} .
\] (47)

\(^4\)Note that the effective anomalous dimension matrix \( \gamma^{0,\text{eff}} \) given in [19] has to be converted into \( \gamma^0 \), before the relevant \( \delta Z \)-factors can be read off.
\[ \delta Z_{24} = \frac{\alpha_s}{6\pi\epsilon}, \quad \delta Z_{28} = \frac{19\alpha_s}{108\pi\epsilon}. \]  

(48)

The counterterm contributions \( M_{1ct} \) and \( M_{2ct} \) proportional to \( C_1 \) and \( C_2 \) are then given by

\[ M_{1ct} = \langle sg|\delta Z_{14} O_4 + \delta Z_{18} O_8 |b\rangle = \left( \frac{\alpha_s}{216\pi} \frac{1}{\epsilon} \left( \frac{m_b}{\mu} \right)^{-2\epsilon} + \frac{\alpha_s}{\pi} \frac{167}{2592} \frac{1}{\epsilon} \right) \langle sg|O_8 |b\rangle_{\text{tree}}. \]  

(49)

\[ M_{2ct} = \langle sg|\delta Z_{24} O_4 + \delta Z_{28} O_8 |b\rangle = \left( -\frac{\alpha_s}{36\pi} \frac{1}{\epsilon} \left( \frac{m_b}{\mu} \right)^{-2\epsilon} + \frac{\alpha_s}{\pi} \frac{19}{108} \frac{1}{\epsilon} \right) \langle sg|O_8 |b\rangle_{\text{tree}}. \]  

(50)

We note that there are no one-loop contributions to the matrix element for \( b \to sg \) from the counterterms proportional to the evanescent operators \( P_{11} \) and \( P_{12} \) given in appendix A of ref. [19].

C. Renormalized matrix elements of \( O_1 \) and \( O_2 \)

Adding the regularized two-loop result in eq. (43) and the counterterm in eq. (50), we find the renormalized result for \( M_2 \) in the NDR scheme:

\[ M_2^{\text{ren}} = \langle sg|O_8 |b\rangle_{\text{tree}} \frac{\alpha_s}{4\pi} \left( \ell_2 \ln \frac{m_b}{\mu} + r_2 \right), \]  

(51)

with

\[ \ell_2 = \frac{70}{27}. \]  

(52)

\[ \text{Re}(r_2) = \frac{1}{648} \{ -2170 - 54\pi^2 + z[48816 - 252\pi^2 + (22680 - 1620\pi^2)L \\
+ 2808L^2 + 612L^3 - 6480\zeta(3)] \\
- 12672\pi^2z^{3/2} + z^2[66339 + 1872\pi^2 + (-40446 + 1512\pi^2)L \\
+ 6642L^2 - 1008L^3 + 7776\zeta(3)] \\
+ z^3[-3420 - 60\pi^2 - 6456L + 7884L^2] \} \]

\[ \text{Im}(r_2) = \frac{\pi}{27} \{ -28 + z[549 - 24\pi^2 + 153L + 72L^2] \\
+ z^2[-432 + 30\pi^2 + 54L - 90L^2] + z^3[-259 + 192L] \} \]  

(53)

Here, \( \text{Re}(r_2) \) and \( \text{Im}(r_2) \) denote the real and the imaginary part of \( r_2 \), respectively. The quantity \( z \) is defined as \( z = (m_c^2/m_b^2) \) and \( L = \ln(z) \).
Similarly, we obtain the renormalized version of $M_1$ by adding the regularized two-loop result in eq. (46) and the counterterm in eq. (49); we find

$$M_1^{\text{ren}} = \langle s g |O_8| b \rangle_{\text{tree}} \frac{\alpha_s}{4 \pi} \left( \ell_1 \ln \frac{m_b}{\mu} + r_1 \right), \quad (54)$$

with

$$\ell_1 = \frac{173}{162} \quad (55)$$

$$\text{Re}(r_1) = -\frac{1}{3888} \{ 4877 - 54\pi^2 + 36z[1086 + 29\pi^2 + (360 + 36\pi^2)L \\
+ 51L^2 + 8L^3 + 144\zeta(3)] \\
- 12672\pi^2 z^{3/2} + 9z^2[6615 - 80\pi^2 + (-4494 + 384\pi^2)L \\
+ 864L^2 - 148L^3 + 864\zeta(3)] \\
+ 12z^3[93 + 76\pi^2 - 1186L + 900L^2] \}$$

$$\text{Im}(r_1) = \frac{\pi}{324} \{ 25 + 6z[75 + \pi^2 + 24L - 3L^2] \\
+ 6z^2[-171 + 19\pi^2 + 72L - 57L^2] + 2z^3[-421 + 192L] \} \quad (56)$$

In figs. we show the real and the imaginary parts of $r_2$ and $r_1$, respectively. For $z \geq 1/4$ the imaginary parts must vanish exactly; indeed we see from these plots that the imaginary parts based on the expansion retaining terms up to $z^3$ indeed vanish at $z = 1/4$ to high accuracy.
FIG. 5. Real and imaginary part of $r_1$ in the NDR scheme (from eq. (56)).

IV. VIRTUAL CORRECTIONS TO $O_8$

In this section we calculate the order $\alpha_s$ virtual corrections to the matrix element

$$M_8 = \langle sg|O_8|b\rangle.$$  (57)

As the contributing Feynman graphs in fig. 6 are one loop diagrams, the computation of $M_8$ is straightforward. We use dimensional regularization for both, the ultraviolet and the infrared singularities. Singularities which appear in the situation where the virtual gluon becomes almost real and collinear with the emitted gluon are also regulated dimensionally; on the other hand, those singularities where the almost real internal gluon is collinear with the $s$-quark, are regulated with a small strange quark mass $m_s$; the latter manifest themselves in logarithmic terms of the form $\ln(\rho)$, where $\rho = (m_s/m_b)^2$.

We were able to separate the ultraviolet $1/\epsilon$ poles from those which are of infrared (and/or collinear) origin. For ultraviolet poles we use the symbol $1/\epsilon$ in the following, while collinear and infrared poles are denoted by $1/\epsilon_{IR}$.

When working in Feynman gauge for the gluon propagator, the individual diagrams contributing to $M_8$ have the following infrared/collinear properties (the letters refer to the diagrams in fig. 6): a) and b) are free of infrared and collinear singularities; c) has combined infrared/collinear singularities of the form $1/\epsilon^2_{IR}$ or $\ln(\rho)/\epsilon_{IR}$ as well as $1/\epsilon_{IR}$ poles; d) has combined infrared/collinear singularities of the form $1/\epsilon^2_{IR}$ as well as $1/\epsilon_{IR}$ poles; e) has a collinear singularity of the form $\ln(\rho)$; f) is free of infrared and collinear singularities; g) has a combined collinear and infrared singularity of the form $\ln(\rho)/\epsilon_{IR}$ as well as collinear singularities of the form $\ln^2(\rho)$ and $\ln(\rho)$; h) has an infrared singularity of the form $1/\epsilon_{IR}$; more precisely, this diagram is proportional to the combination $(1/\epsilon - 1/\epsilon_{IR})$. 

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FIG. 6. Diagrams associated with the operator \( O_8 \). The real gluon can be attached to any of the circle-crosses on the fermion lines.

As the results of the individual diagrams are not very instructive, we only give their sum:

\[
M_8 = \frac{\alpha_s}{4\pi} f_s \langle sg|O_8|b\rangle_{\text{tree}},
\]

with

\[
f_s = \left[ -\frac{3}{\epsilon_{\text{IR}}^2} - \frac{(4\ln(\rho) + 9 + 9i\pi)}{3\epsilon_{\text{IR}}} + \frac{11}{3\epsilon} \right] \left( \frac{m_b}{\mu} \right)^{-2\epsilon} + \frac{1}{3} \left[ \frac{59\pi^2}{12} + 1 - 8\ln(\rho) + 2\ln^2(\rho) - 8i\pi \right].
\]

We would like to mention that we did not include diagrams with self energy insertions in the external legs. As we work in an on-shell renormalization scheme with respect to quark and gluon fields, such diagrams are cancelled against counterterm contributions.

A. Counterterms to the \( O_8 \) contribution

The counterterm is generated by expressing the bare quantities in the tree-level matrix element of \( O_8 \) by their renormalized counterparts. It has the structure

\[
M_8^{\text{ct}} = \delta R \langle sg|O_8|b\rangle_{\text{tree}},
\]

where the factor \( \delta R \) is given by

\[
\delta R = \sqrt{Z_2(m_b)} \sqrt{Z_2(m_s)} \sqrt{Z_3} Z_{g_s} Z_{m_b} Z_{88} - 1.
\]

\( Z_2(m_b), Z_2(m_s) \) and \( Z_3 \) denote the on-shell wave function renormalization factors of the \( b \)-quark, the \( s \)-quark and the gluon, respectively. \( Z_{g_s} \) and \( Z_{m_b} \) denote the \( \overline{\text{MS}} \) renormalization
constants for the strong coupling constant $g_s$ and the $b$-quark mass factor, which appear explicitly in the definition of the operators (see eq. (61)). Finally, $Z_{88}$ is the renormalization factor of the operator $O_8$.

The explicit form of $Z_2(m)$ reads

$$Z_2(m) = 1 - \frac{\alpha_s}{3\pi} \left( \frac{m}{\mu} \right)^{-2\epsilon} \left[ \frac{1}{\epsilon} + \frac{2}{\epsilon_{\text{IR}}} + 4 \right]$$

(62)

where we again separated infrared and ultraviolet poles. For $Z_3$ we get in the on-shell scheme:

$$Z_3 = 1 + \frac{\alpha_s}{2\pi} \frac{5}{2} \left( \frac{1}{\epsilon} - \frac{1}{\epsilon_{\text{IR}}} \right) - \frac{\alpha_s}{2\pi} \frac{1}{3} \sum_f \left[ \frac{1}{\epsilon} - 2 \ln \frac{m_f}{\mu} \right].$$

(63)

The sum in this formula run over the five flavors $f = u, d, c, s, b$. For $Z_{mb}$ and $Z_{88}$ (see ref. [19]) one obtains

$$Z_{mb} = 1 - \frac{\alpha_s}{4\pi} \frac{4}{\epsilon} ; \quad Z_{88} = 1 + \frac{\alpha_s}{4\pi} \frac{14}{3\epsilon} .$$

(64)

Finally, the renormalization constant for the strong coupling constant reads

$$Z_{g_s} = 1 - \frac{\alpha_s}{4\pi} \left[ \frac{11}{3\epsilon} - \frac{N_f}{3} \right] \frac{1}{\epsilon} ; \quad N_f = 5 .$$

(65)

Inserting the various $Z$ factors in eq. (61), one obtains

$$\delta R = -\frac{\alpha_s}{4\pi} \left[ \frac{11}{3\epsilon} + \frac{31}{6\epsilon_{\text{IR}}} - 8 \ln \frac{m_b}{\mu} - \frac{2}{3} \sum_f \ln \frac{m_f}{\mu} + \frac{16}{3} - 2 \ln \rho \right].$$

(66)

**B. Renormalized matrix element of $O_8$**

Adding the regularized matrix element of $O_8$ in eq. (58) and the counterterm contribution $M_8^{\text{ct}}$ in eq. (60), one obtains the renormalized result

$$M_8^{\text{ren}} = \frac{\alpha_s}{4\pi} f_8^{\text{ren}} \langle s g | O_8 | b \rangle_{\text{tree}} ,$$

(67)

with

$$f_8^{\text{ren}} = \left[ \frac{3}{\epsilon_{\text{IR}}} - \left( \frac{8 \ln(\rho) + 49 + 18i\pi}{6\epsilon_{\text{IR}}} \right) \right] \left( \frac{m_b}{\mu} \right)^{-2\epsilon} \frac{29}{3} \ln \frac{m_b}{\mu} + \frac{2}{3} \sum_f \ln \frac{m_f}{\mu} + 5 \frac{59\pi^2}{36} - \frac{2}{3} \ln \rho + \frac{2}{3} \ln^2 \rho - \frac{8}{3} i\pi .$$

(68)
In eq. (68) the sum runs over the five flavors $f = u, d, c, s, b$, and $\rho = (m_s/m_b)^2$. We anticipate that the singular terms of the form $1/\epsilon_\text{IR}^2$, $1/\epsilon_\text{IR}$ and $\ln \rho$ in eq. (68) will cancel (at the level of the decay width) against the corresponding singularities present in the gluon bremsstrahlung corrections to $b \to sg$. On the other hand, the logarithmic terms $\ln (m_f/\mu)$, which also represent some kind of singularities for the light flavor $f = u, d, s$ are not cancelled by the gluon bremsstrahlung process. Keeping in mind that these terms originate from the renormalization factor $Z_3$ of the gluon field, i.e., from gluon self energy diagrams in which these flavors propagate, it is expected that these logarithms will cancel against the logarithms present in the decay rate $\Gamma (b \to sf\bar{f})$ with $f = u, d, s$. To cancel these unphysical terms, we will include the $O_8$ contribution to this process in section $V$.

V. VIRTUAL CORRECTIONS TO THE DECAY WIDTH FOR $b \to sg$

We are now ready to write down the renormalized version of the matrix $M^\text{ren}(b \to sg)$ element for $b \to sg$, where the virtual order $\alpha_s$ corrections are included. We obtain:

$$M^\text{ren}(b \to sg) = \frac{4G_F i}{\sqrt{2}} V_{ts}^* V_{tb} \left\{ C^\text{eff}_8 + \frac{\alpha_s}{4\pi} \left[ C^\text{1}\text{eff}_8 (\ell_1 \ln \frac{m_b}{\mu} + r_1) + C^0\text{eff}_8 (\ell_2 \ln \frac{m_b}{\mu} + r_2) + C^\text{0,eff}_8 f^\text{ren}_8 \right] \right\} \langle sg|O_8(\mu)|b\rangle_{\text{tree}}. $$

(69)

The quantities $\ell_1, r_1, \ell_2, r_2,$ and $f^\text{ren}_8$ are given in eqs. (65), (54), (52), (53) and (68), respectively. As eq. (69) shows, $C^\text{eff}_8$ is the only Wilson coefficient needed to NLL precision. For the following, it is useful to decompose it as

$$C^\text{eff}_8 = C^\text{0,eff}_8 + \frac{\alpha_s}{4\pi} C^\text{1,eff}_8. $$

(70)

The symbol $\langle sg|O_8(\mu)|b\rangle_{\text{tree}}$ in eq. (69) denotes the tree level matrix element of $O_8(\mu)$, which contains the running $b$-quarks mass and the strong running coupling constant at the scale $\mu$. In order to get expressions where the $b$-quark mass enters as the pole mass, and the strong coupling constant enters as $g_s(m_b)$, we rewrite $\langle sg|O_8(\mu)|b\rangle_{\text{tree}}$ as

$$\langle sg|O_8(\mu)|b\rangle_{\text{tree}} = \langle sg|O_8|b\rangle_{\text{tree}} \left[ 1 + \frac{2\alpha_s}{\pi} \ln \frac{m_b}{\mu} - \frac{4\alpha_s}{3\pi} + \frac{\alpha_s}{4\pi} \beta_0 \ln \frac{m_b}{\mu} \right] ; \quad \beta_0 = \frac{23}{3}, $$

(71)

where we made use of eqs. (4) and (A18). The symbol $\langle sg|O_8|b\rangle_{\text{tree}}$ then stands for the tree level matrix element of $O_8$ in which $m_b(\mu)$ and $g_s$ have to be identified with the pole mass $m_b$ and $g_s(m_b)$, respectively. (See also the discussion after eq. (13) and eq. (14)). Inserting eqs. (70) and (71) into eq. (69) we obtain:

$$M^\text{ren}(b \to sg) = \frac{4G_F i}{\sqrt{2}} V_{ts}^* V_{tb} \left\{ C^\text{0,eff}_8 + \frac{\alpha_s}{4\pi} \left[ C^\text{1,eff}_8 + (8 + \beta_0) \ln \frac{m_b}{\mu} C^\text{0,eff}_8 - \frac{16}{3} C^\text{0,eff}_8 + C^\text{0,eff}_1 (\ell_1 \ln \frac{m_b}{\mu} + r_1) + C^\text{0,eff}_2 (\ell_2 \ln \frac{m_b}{\mu} + r_2) + C^\text{0,eff}_8 f^\text{ren}_8 \right] \right\} \langle sg|O_8|b\rangle_{\text{tree}}. $$

(72)
To obtain the decay width $\Gamma^{\text{virt}}$ from $M^{\text{ren}}(b \to sg)$ is straightforward. We get:

$$\Gamma^{\text{virt}} = \frac{\alpha_s(m_b) m_b^5}{24\pi^4} |G_F V_{ts}V_{tb}|^2 \left\{ \left( \frac{C_8^{0,\text{eff}}}{m_b} \right)^2 + \frac{\alpha_s}{4\pi} C_8^{0,\text{eff}} \left[ 2 C_8^{1,\text{eff}} + 2(8 + \beta_0) \ln \frac{m_b}{\mu} C_8^{0,\text{eff}} - \frac{32}{3} C_8^{0,\text{eff}} + 2 C_1^{0,\text{eff}} \ln \frac{m_b}{\mu} + \text{Re}(r_1) \right] + 2 C_2^{0,\text{eff}} \ln \frac{m_b}{\mu} + \text{Re}(r_2) \right\} \left( 1 - \frac{1}{4}(\pi^2 - 16) + 2 \right) \left( m_b^2 - m_f^2 - 2 \right) \right\} . \quad (73)$$

We note that due to the infrared poles present in $f_8^{\text{ren}}$ the phase space integrations have been done consistently in $d = 4 - 2\epsilon$ dimensions, which leads to the last two extra factors in the last term in eq. (73). The other factor, $(1 - \epsilon)$, in the last term in eq. (73), is due to the fact that all the $(d - 2)$ possible transverse polarizations of the emitted gluon were taken into account.

VI. $O_8$ CONTRIBUTION TO THE DECAY WIDTH $\Gamma(b \to sff)$

As discussed at the end of section IV, we should take into account the contribution of the operator $O_8$ to the process $b \to sff$ $(f = u, d, s)$, in order to cancel the unphysical logarithms of the form $\ln(m_f/\mu)$ in the virtual corrections to $b \to sg$. The $O_8$ contribution to the decay width $\Gamma_8(b \to sff)$ yields

$$\Gamma_8(b \to sff) = \frac{m_b^5}{72\pi^5} |G_F V_{ts}V_{tb} C_8^{0,\text{eff}}|^2 \alpha_s^2 \left[ \ln \frac{m_b}{2m_f} - \frac{2}{3} \right]. \quad (74)$$

Comparing this result with $\Gamma^{\text{virt}}$ in eq. (73), we see explicitly, that the mentioned logarithms indeed cancel.

VII. MATRIX ELEMENTS FOR GLUON BREMSSTRAHLUNG

In this section we discuss the gluon bremsstrahlung corrections to $b \to sg$, i.e. the matrix element for the process $b \to sgg$, associated with the operators $\hat{O}_1, \hat{O}_2$ and $\hat{O}_8$. For literature on the analogous corrections to $b \to s\gamma$, we refer to [26].

A. Bremsstrahlung associated with $\hat{O}_1$ and $\hat{O}_2$

We first discuss the matrix element of $\hat{O}_2$. There are two diagrams contributing; they are displayed in d) and e) of fig. [4]. The sum of diagram d) and the one with the two gluons interchanged is denoted by $\tilde{J}_{\alpha\beta}$. Its analytic form is obtained by putting $r^2 = 0$ and $r_\beta = 0$ in the expression for $J_{\alpha\beta}$ in eq. (13):
\[ J_{\alpha\beta}^{AB} = T_{\alpha\beta}^+(q,r) \left\{ T^A, T^B \right\} + T_{\alpha\beta}^-(q,r) \left[ T^A, T^B \right] . \] (75)

This expression is understood to be contracted with the polarization vectors \( \varepsilon^\alpha(q) \) and \( \varepsilon^\beta(r) \) of the gluons. The diagram e) in fig. [4], denoted by \( S_{\alpha\beta}^{AB} \), is color antisymmetric and can be written as

\[ S_{\alpha\beta}^{AB} = S_{\alpha\beta}^- \left[ T^A, T^B \right] , \] (76)

where \( S_{\alpha\beta}^- \) reads \((t = (2qr)/m_c^2)\)

\[ S_{\alpha\beta}^- = \frac{g_2^2}{32\pi^2} \left\{ \frac{4}{3} \left( \frac{m_c}{\mu} \right)^2 - \frac{4}{3} - 8G_1(t) + 8G_2(t) \right\} \left[ \gamma g_{\alpha\beta} - \frac{\alpha q}{(qr)} + \frac{\mu}{(qr)} \right] . \] (77)

The functions \( G_i(t) \) \((i = -1, 0, 1, ...)\) are defined as

\[ G_i(t) = \int_0^1 dx \ x^i \ln[1 - tx(1 - x) - i\delta] . \] (78)

The Ward identities \( r^\alpha T_{\alpha\beta}^+ = q^\alpha T_{\alpha\beta}^+ = 0 \), stated in [24], imply that

\[ T_{\alpha\beta}^+ = \frac{g_2^2}{32\pi^2} \left\{ E(\alpha, \beta, r) - E(\alpha, \beta, q) - E(\beta, r, q) + E(\alpha, r, q) \right\} L \Delta_{i23} . \] (79)

General considerations (or a straightforward calculation which makes use of the explicit expressions for the functions \( G_i \) and \( \Delta_{i\alpha} \)) imply the Ward identities

\[ r^\alpha (T_{\alpha\beta}^- + S_{\alpha\beta}^-) = 0 ; \quad q^\alpha (T_{\alpha\beta}^- + S_{\alpha\beta}^-) = 0 , \] (80)

which can be used to cast \((T_{\alpha\beta}^- + S_{\alpha\beta}^-)\) into the simple form

\[ T_{\alpha\beta}^- + S_{\alpha\beta}^- = \frac{g_2^2}{32\pi^2} \left( \gamma - \frac{\alpha q}{(qr)} - g_{\alpha\beta} \right) L \Delta_{i17} . \] (81)

To summarize, the matrix element \( \hat{M}_{2}^{\text{brems}} = \langle sgg | \hat{O}_2 | b \rangle \) can be written as

\[ \hat{M}_{2}^{\text{brems}} = T_{\alpha\beta}^+ \left\{ T^A, T^B \right\} + \left( T_{\alpha\beta}^- + S_{\alpha\beta}^- \right) \left[ T^A, T^B \right] , \] (82)

where \( T_{\alpha\beta}^+ \) and \( (T_{\alpha\beta}^- + S_{\alpha\beta}^-) \) are given in eqs. (79) and (81), respectively. The functions \( \Delta_{i23} \) and \( \Delta_{i17} \) occurring in these expressions, can be written in terms of \( G_0(t) \) and \( G_{-1}(t) \) \((t = (2qr)/m_c^2)\) defined in eq. (78):

\[ \Delta_{i23} = -2 \frac{t + 6G_{-1}(t) - 12G_0(t)}{t} ; \quad \Delta_{i17} = -3 \frac{t + 6G_{-1}(t) - 12G_0(t)}{t} . \] (83)
FIG. 7. Bremsstrahlung diagrams associated to $O_8$ and $\hat{O}_2$. Circle-crosses denote possible gluon emissions. Note that picture a) actually represents four Feynman diagrams (obtained by interchanging the gluons) and the one in d) represents two diagrams (again: including the interchange of the gluons).

The explicit form of $G_{-1}(t)$ and $G_0(t)$ is given in appendix \[3\]. Note that these results are ultraviolet finite. As the subsequent phase space integrals do not generate infrared singularities, it is consistent to retain terms up to order $\epsilon^0$ only in eq. (82).

Due to the specific color structure of the operator $\hat{O}_1$, the diagram e) in fig. 7 does not contribute and the color antisymmetric part encoded in $\bar{T}_{-\alpha\beta}$ is also absent. The matrix element $\hat{M}_1^{\text{brems}} = \langle sgg | \hat{O}_1 | b \rangle$ is therefore proportional to $\bar{T}_{+\alpha\beta}$, reading

$$\hat{M}_1^{\text{brems}} = \bar{T}_{+\alpha\beta} \delta^{AB} \delta^{ab};$$

(84)

$A, B$ and $a, b$ are the color indices of the gluons and the quarks, respectively.

B. Bremsstrahlung associated with $O_8$

The Feynman diagrams contributing to the matrix element $M_8^{\text{brems}} = \langle sgg | O_8 | b \rangle$ are shown in a), b) and c) in fig. 7. Similar to $\hat{M}_2^{\text{brems}}$ in eq. (82), one can decompose $M_8^{\text{brems}}$ into a color symmetric- and a color antisymmetric part:

$$M_8^{\text{brems}} = R^+_{\alpha\beta} \{ T^A, T^B \} + R^-_{\alpha\beta} \left[ T^A, T^B \right].$$

(85)

The diagrams shown in b) and c) only contribute to $R^-_{\alpha\beta}$, while the diagrams in a) contribute to both $R^+_{\alpha\beta}$ and $R^+_{\alpha\beta}$. As the calculation of these tree level diagrams is straightforward, we do not give the explicit expressions for $R^+_{\alpha\beta}$ and $R^-_{\alpha\beta}$.

VIII. DECAY WIDTH FOR $b \rightarrow sgg$

The total matrix element $M^{\text{brems}}(b \rightarrow sgg)$ can be written as

$$M^{\text{brems}} = \frac{4G_F i}{\sqrt{2}} V_{ts} V_{tb} \left[ \hat{C}_1 \hat{M}_1^{\text{brems}} + \hat{C}_2 \hat{M}_2^{\text{brems}} + C_8^{\text{eff}} \hat{M}_8^{\text{brems}} \right],$$

(86)
where the three terms on the r.h.s., given in eqs. (84), (82) and (85), correspond to
the contributions of the operators $\hat{O}_1$, $\hat{O}_2$ and $O_8$, respectively. The coefficients $C_1$ and $C_2$ are
understood to be the following linear combinations of the Wilson coefficients $C_1$ and $C_2$
appearing in the effective Hamiltonian [3]:

$$\hat{C}_1 = \frac{1}{2} C_1 \; ; \; \hat{C}_2 = C_2 - \frac{1}{6} C_1 ;$$

(87)

We note that in eq. (86) only the leading order pieces of the Wilson coefficients are needed.

The expression for the decay width reads in d dimensions:

$$d\Gamma^{\text{brems}}(b \to sgg) = \frac{1}{2m_b} \int (2\pi)^d \delta^d(p-p'-q-r) |M^{\text{brems}}|^2 d\mu(p')d\mu(q)d\mu(r) ,$$

(88)

where $p$, $p'$, $q$, $r$ are the four-momenta of the $b$-quark, $s$-quark, and the gluons. $|M^{\text{brems}}|^2$ is
obtained by squaring the matrix element $M^{\text{brems}}$, followed by summing/averaging over spins
and color of the final/initial state particles. The factor $(1/2)$ due to the two gluons in the
final state is also absorbed there.

The phase space integrals are plagued with infrared and collinear singularities. Configurations
with one gluon flying collinear to the $s$-quark are regulated by a small strange quark
mass $m_s$, while configurations with two collinear gluons, or one soft gluon are dimensionally
regularized. As in the calculations of the virtual corrections, we write the dimension as
d = 4 - 2\epsilon. (Note that \epsilon has to be negative in order to regulate the phase space integrals).

When squaring $M^{\text{brems}}$ in eq. (88), nine terms are generated, which we denote for obvious
reasons by $(\hat{O}_1, \hat{O}_1^*)$, $(\hat{O}_1, \hat{O}_2^*)$, $(\hat{O}_1, \hat{O}_8^*)$, $(\hat{O}_2, \hat{O}_1^*)$, $(\hat{O}_2, \hat{O}_2^*)$, $(\hat{O}_2, \hat{O}_8^*)$, $(O_8, \hat{O}_1)$, $(O_8, \hat{O}_2)$, and
$(O_8, O_8^*)$. It turns out that all terms except $(O_8, O_8^*)$ are free of infrared and collinear
singularities. We therefore can put $m_s = 0$ in these terms and evaluate the phase space integrals in d = 4 dimensions. Denoting this finite contribution to the decay width by $\Gamma^{\text{brems}}_{\text{fin}}$, we get:

$$\Gamma^{\text{brems}}_{\text{fin}} = \frac{8 |G_F V_{ts}^* V_{tb}|^2}{64\pi^3 m_b} \cdot \frac{1}{12} \frac{\alpha_s^2}{64\pi^2} \int \! dE_q dE_r (\tau_{11}^+ + \tau_{22}^+ + \tau_{22}^- + \tau_{12}^+ + \tau_{18}^+ + \tau_{28}^+ + \tau_{28}^-).$$

(89)

The superscripts (+) and (−) on the various $\tau$-quantities refer to color even and color odd
contributions, respectively. The result is represented as a two dimensional integral over the
energies $E_q$ and $E_r$ of the gluons in the rest frame of the $b$-quark. $E_q$ and $E_r$ vary in the
range

$$E_q \in \left[0, \frac{m_b}{2}\right] ; \quad E_r \in \left[\frac{m_b}{2} - E_q, \frac{m_b}{2}\right].$$

(90)

The various $\tau$-quantities, in which all the scalar products are understood to be expressed in
terms of $E_q$ and $E_r$, read:

$$\tau_{11}^+ = \hat{C}_1^2 \cdot 24 |\Delta i_{23}|^2 2m_b^2 [m_b^2 - 2(qr)]$$

$$\tau_{22}^+ = \hat{C}_2^2 \cdot \frac{28}{3} |\Delta i_{23}|^2 2m_b^2 [m_b^2 - 2(qr)]$$
\[\tau_{22}^{-} = \hat{C}_2^2 12 |\Delta i_{17}|^2 2 [16 (pq)^2 - 16 (pq) (qr) - 8 m_b^2 (pq) + 6 m_b^2 (qr) + m_b^4] \]
\[\tau_{12}^{+} = 2 \hat{C}_1 \hat{C}_2 8 |\Delta i_{23}|^2 2 m_b^2 [m_b^2 - 2 (qr)] \]
\[\tau_{18}^{+} = 2 \hat{C}_1 C_{8, eff} 8 \text{Re}(\Delta i_{23}) 16 m_b^2 (qr) \]
\[\tau_{28}^{+} = 2 \hat{C}_2 C_{8, eff} 28 \frac{3}{5} \text{Re}(\Delta i_{23}) 16 m_b^2 (qr) \]
\[\tau_{28}^{-} = 2 \hat{C}_2 C_{8, eff} 12 \text{Re}(\Delta i_{17}) (-4 m_b^2) \left[ m_b^4 (pq) + m_b^4 (pr) - 2 m_b^2 (pq)^2 - 2 m_b^2 (pr)^2 \right. \]
\[-2 m_b^2 (pq) (pr) + 4 (pq)^2 (pr) + 4 (pr)^2 (pq)] / [(pq) (pr)] \]  

\[(91)\]

were the functions \(\Delta i_{17}\) and \(\Delta i_{23}\) are given in eq. (83). As these functions are rather complicated, the integrals over \(E_q\) and \(E_r\) are done numerically.

We now turn to the \((O_8, O_8^*)\) contribution, denoted by \(\Gamma_{88}^{\text{brems}}\). Without going too much into the details, we would like to mention that some care has to be taken when summing over the \((d - 2)\) transverse polarizations of the gluons. These sums are of the form

\[\sum_{r=1}^{d-2} \varepsilon_\mu (k) \varepsilon_\nu^* (k) = -g_\mu\nu + k_\mu f_\nu + k_\nu f_\mu, \]  

where the vector \(f\) satisfies the condition \((f k) = 1\), with \(k\) being the four-momentum of the gluon. It turns out that both terms involving \(f\) on the r.h.s in eq. (92) contribute to the color antisymmetric part of \(\Gamma_{88}^{\text{brems}}\). After a lengthy, but straightforward calculation, we obtain (with \(\rho = (m_s / m_b)^2\))

\[\Gamma_{88}^{\text{brems}, +} = \frac{7 \alpha_s (C_{8, eff})^2}{96 \pi} V \left( \frac{m_b}{\mu} \right)^{4 \epsilon} \left[ 8 + 4 \ln \rho \epsilon_{\text{IR}} - 2 \ln^2 \rho + 6 \ln \rho + 18 - \frac{4 \pi^2}{3} \right] \]  

\[(93)\]

for the color symmetric part, and

\[\Gamma_{88}^{\text{brems}, -} = \frac{\alpha_s (C_{8, eff})^2}{16 \pi} V \left( \frac{m_b}{\mu} \right)^{4 \epsilon} \left[ 24 \epsilon_{\text{IR}}^2 + \frac{80 + 6 \ln \rho}{\epsilon_{\text{IR}}} - 3 \ln^2 \rho + 9 \ln \rho + 299 - 26 \pi^2 \right] \]  

\[(94)\]

for the color antisymmetric part. \(V\) is defined as

\[V = \frac{\alpha_s m_b^5}{24 \pi^2} |G_F V_{ts} V_{tb}|^2. \]  

\[(95)\]

The total decay with for \(b \to sgg\) is then given by

\[\Gamma^{\text{brems}} (b \to sgg) = \Gamma_{\text{fin}}^{\text{brems}} + \Gamma_{88}^{\text{brems}, +} + \Gamma_{88}^{\text{brems}, -}, \]  

\[(96)\]

where the three terms on the r.h.s. are given in eqs. (89), (93) and (94).
In this section we combine the decay widths for the virtually corrected process \(b \rightarrow sg\) and the bremsstrahlung process \(b \rightarrow sgg\) to the decay width, which we call \(\Gamma_{\text{NLL}}(b \rightarrow sg)\). We also absorb in this quantity the \(O_8\) induced contribution to the process \(b \rightarrow sf\), where \(f = u, d, s\), as discussed at the end of section [IV] and in section [VI]. The expression for \(\Gamma\), which contains the lowest order contribution to the decay width for \(b \rightarrow sg\), together with its virtual corrections, may be found in eq. (73). The result for the bremsstrahlung process, \(\Gamma^{\text{brems}}\) is given in eq. (74). Putting together the individual pieces, we obtain

\[
\Gamma_{\text{NLL}}(b \rightarrow sg) = \frac{\alpha_s(m_b) m_b^5}{24\pi^4} |G_F V_{ts} V_{tb}|^2 \left\{ \left( C_{8}^{0,\text{eff}} \right)^2 + \frac{\alpha_s}{4\pi} C_{8}^{0,\text{eff}} \left[ 2 C_{8}^{1,\text{eff}} - \frac{32}{3} C_{8}^{0,\text{eff}} \right] \
+ 2 C_{1}^{0} [\ell_1 \ln \frac{m_b}{\mu} + \text{Re}(r_1)] + 2 C_{2}^{0} [\ell_2 \ln \frac{m_b}{\mu} + \text{Re}(r_2)] \
+ 2 C_{8}^{0,\text{eff}} [(\ell_8 + 8 + \beta_0) \ln \frac{m_b}{\mu} + r_8] \right\} + \Gamma_{\text{fin}}^{\text{brems}},
\]

where \(\Gamma_{\text{fin}}^{\text{brems}}\), given in eq. (89), contains all the bremsstrahlung corrections except those originating from the \((O_8, O_8^*)\) interference. The quantities \(\ell_1, r_1, \ell_2\) and \(r_2\) stem from the virtual corrections; they are given in eqs. (92), (93), (52) and (73), respectively. On the other hand, \(\ell_8\) and \(r_8\) contain information from the real part of the virtual corrections, encoded in \(\text{Re}(f_{8}^{\text{rem}})\); the contributions from the \((O_8, O_8^*)\) interference of the gluon bremsstrahlung process; and the \(O_8\) contribution to the process \(b \rightarrow sf\): The explicit expressions for \(\ell_8\) and \(r_8\) (which is real by definition) read

\[
\ell_8 = -\frac{19}{3}; \quad r_8 = \frac{1}{18} \left[ 351 - 19\pi^2 - 36 \ln 2 + 6 \ln \frac{m_c^2}{m_b^2} \right].
\]

We would like to stress that all scale dependent quantities in eq. (97) are understood to be evaluated at the scale \(\mu\), unless indicated explicitly in the notation.

To prepare the discussion on the numerical size of the NLL QCD corrections, it is useful to cast the final result (97) into another form:

\[
\Gamma_{\text{NLL}}(b \rightarrow sg) = \frac{\alpha_s(m_b) m_b^5}{24\pi^4} |G_F V_{ts} V_{tb}|^2 |\bar{D}|^2 + \Gamma_{\text{fin}}^{\text{brems}},
\]

with

\[
\bar{D} = C_{8}^{0,\text{eff}} + \frac{\alpha_s}{4\pi} \left[ C_{8}^{1,\text{eff}} - \frac{16}{3} C_{8}^{0,\text{eff}} + C_{1}^{0} [\ell_1 \ln \frac{m_b}{\mu} + r_1] \
+ C_{2}^{0} [\ell_2 \ln \frac{m_b}{\mu} + r_2] + C_{8}^{0,\text{eff}} [(\ell_8 + 8 + \beta_0) \ln \frac{m_b}{\mu} + r_8] \right].
\]
The modulus square of $D$ is understood to be taken in the same way as in the virtual
contributions, i.e., by systematically discarding the $O(\alpha_s^2)$ term. In this sense, the quantity
$D$ can be viewed as an effective matrix element.

We would like to mention that $\ell_1, \ell_2$ and $(\ell_8 + 8 + \beta_0)$ are identical to the anomalous
dimension matrix elements $\gamma_{18}^{0,\text{eff}}, \gamma_{28}^{0,\text{eff}}$, and $\gamma_{88}^{0,\text{eff}}$, respectively. This is of course what has
to happen: Only in this case the leading scale dependence of $C_8^{0,\text{eff}}(\mu)$ gets compensated by
the second term in eq. (100).

The NNL branching ratio $B_N^{\text{NNL}}(b \to sg)$ is then obtained as

$$B_N^{\text{NNL}}(b \to sg) = \frac{\Gamma_N^{\text{NNL}}(b \to sg)}{\Gamma_{\text{sl}}} B_{\text{sl}}^{\text{exp}},$$

(101)

where $B_{\text{sl}}^{\text{exp}}$ denotes the experimental semileptonic branching ratio of the $B$-meson. $\Gamma_{\text{sl}}$ stands
for the theoretical expression of the semileptonic decay width of the $B$-meson. Neglecting
non-perturbative corrections of the order $(\Lambda_{\text{QCD}}/m_b)^2$, $\Gamma_{\text{sl}}$ reads (with $x_c = (m_c/m_b)$)

$$\Gamma_{\text{sl}} \approx \Gamma(b \to c\bar{e}\nu_e) = \frac{G_F^2 m_b^5}{192\pi^3} |V_{cb}|^2 g(x_c) \left[ 1 + \frac{\alpha_s(\mu_b)}{2\pi} h_{\text{sl}}(x_c) + O(\alpha_s^2) \right],$$

(102)

where the phase space function $g(x_c)$ reads

$$g(x_c) = 1 - 8 x_c^2 - 24 x_c^4 \ln x_c + 8 x_c^6 - x_c^8.$$

(103)

The analytic expression for $h_{\text{sl}}(x_c)$ can be found in ref. [27]. The approximation

$$h_{\text{sl}}(x_c) = -3.341 + 4.05 (x_c - 0.3) - 4.3 (x_c - 0.3)^2$$

(104)

holds to an accuracy of 1 permille in the relevant range $0.2 \leq x_c \leq 0.4$.

We note that in the numerical analysis of $B_N^{\text{NNL}}(b \to sg)$ we systematically expand the
expression for the branching ratio (101) in $\alpha_s$, dropping terms of $O(\alpha_s^2)$.

A short remark concerning the LL branching ratio is in order: For the decay width
$\Gamma^{\text{LL}}(b \to sg)$, we use the expression

$$\Gamma^{\text{LL}}(b \to sg) = \frac{\alpha_s(\mu) m_b^5}{24\pi^3} |G_F V_{ts}^* V_{tb}|^2 \left( C_8^{\text{LL,eff}}(\mu) \right)^2.$$

(105)

The LL branching ratio for $b \to sg$ is then obtained as in eq. (101), but by discarding the
radiative corrections in $\Gamma_{\text{sl}}$.

**X. NUMERICAL RESULTS FOR THE COMBINED BRANCHING RATIO**

Before we present the numerical result for the branching ratio $B_N^{\text{NNL}}(b \to sg)$, we discuss
the sizes of the various NLL corrections at the level of the function $\bar{D}$, defined in eq. (100)
FIG. 8. Scale ($\mu$) dependence of the function $\bar{D}$ (see eq. (100)) in various approximations: The long-dashed line shows $C_8^{0,\text{eff}}$; the short-dashed line corresponds to putting $r_1 = r_2 = r_8 = 0$; the dotted line is obtained by only putting $r_2 = 0$; the solid line shows the full function $\bar{D}$. See text. (anticipating that the finite bremsstrahlung corrections in eq. (99) are relatively small).

We already mentioned that the terms containing the explicit logarithms of the ratio $(m_b/\mu)$ get compensated by the scale dependence of the first term on the r.h.s. of eq. (100). This feature can be observed in fig. 8, when comparing the two dashed lines. The long-dashed line represents only the first term $C_8^0$ of the function $\bar{D}$, while the short-dashed line shows $\bar{D}$, in which $r_1$, $r_2$ and $r_8$ are put to zero. As expected, the short-dashed line has a milder $\mu$-dependence. When switching on also $r_1$ and $r_8$ (but keeping $r_2 = 0$), the resulting curve, shown by the dotted line, stays close to the short-dashed curve and the scale dependence remains mild. However, when switching on also $r_2$, the situation changes drastically. The resulting solid line, which represents the full NLL $\bar{D}$ function, implies that the term containing the two-loop quantity $r_2$, induces a large NLL correction. As this large correction term contains a factor $\alpha_s(\mu) C_2(\mu)$, it is of no surprise, that the NLL prediction for the function $\bar{D}$ suffers from a relatively large scale dependence, as illustrated by the solid line.

The NLL branching ratio $B_{\text{NLL}}(b \to s g)$ is then obtained as described in section 11. The result is shown by the solid line in fig. 4. For the input values, we take: $m_b = (4.8 \pm 0.2)$ GeV, $(m_c/m_b) = 0.29 \pm 0.02$, $\alpha_s(m_Z) = 0.119 \pm 0.003$, $|V_{ts}^*V_{ub}/V_{cb}|^2 = 0.95 \pm 0.03$, $B_{\text{exp}}^3 = (10.49 \pm 0.46)$%, and $m_t^{\text{pole}} = (175 \pm 5)$ GeV. As the scale dependence is rather large, we did not take into account the error due to the uncertainties in the input parameters. Based on
FIG. 9. Branching ratio $B(b \to sg)$ as a function of the scale $\mu$ in various approximations: The dashed and the solid lines show the LL and the NLL predictions, respectively; the dotted line is obtained by putting $r_1 = r_2 = r_8 = \Gamma_{\text{brems}}^{\text{fin}} = 0$ in the NLL expression for $\Gamma_{\text{NLL}}^{\text{NLL}}(b \to sg)$ in eq. (99). See text.

In fig. 9, we obtain the NLL branching ratio

$$B_{\text{NLL}}(b \to sg) = (5.0 \pm 1.0) \times 10^{-3}.$$  \hfill (106)

We would like to stress that the NLL corrections drastically enhance the LL value (see dashed line in fig. 9), for which one obtains

$$B_{\text{LL}}(b \to sg) = (2.2 \pm 0.8) \times 10^{-3}.$$  \hfill (107)

As already mentioned in the discussion of the function $\bar{D}$, the main enhancement is due to the virtual- and bremsstrahlung corrections to $b \to sg$, calculated in this paper. At the level of the branching ratio, this fact is illustrated by the dotted line in fig. 9, which is obtained by discarding $\Gamma_{\text{brems}}^{\text{fin}}$ and by switching off $r_1$, $r_2$ and $r_8$ in the expression for $\Gamma_{\text{NLL}}^{\text{NLL}}(b \to sg)$ (see eq. (99)).

The largest uncertainty due to the physical input parameters on $B_{\text{NLL}}(b \to sg)$ results from the charm quark mass. The dependence of $B_{\text{NLL}}(b \to sg)$ on $m_c$ is illustrated in fig. 10, where $x_c = m_c / m_b$ is varied between 0.27 and 0.31. Choosing $\mu = m_b$, the resulting uncertainty amounts to $\sim \pm 6\%$.

XI. NUMERICAL EVALUATION OF THE CHARMLESS DECAY RATE

In this section we investigate the impact of the NLL QCD corrections to $b \to sg$ on the inclusive hadronic charmless decay rate of the $\bar{B}$ meson. At the quark level, we take into account the hadronic processes
FIG. 10. NLL Branching ratio $B_{\text{NLL}}(b \to sg)$ as a function of the scale $\mu$ for the three value of the ratio $x_c = m_c/m_b$. See text.

\[ b \to q'q'q ; \quad b \to sg , \]  

(108)

where $q = d, s$ and $q' = u, d, s$. As we do not distinguish between $\Delta S = 0$ and $\Delta S = 1$ contributions, we can safely neglect the CKM suppressed decay mode $b \to dg$. More precisely, we calculate the CP-averaged branching ratio

\[ \overline{B}_d = \frac{\Gamma(b \to X_d) + \Gamma(b \to X_{d'})}{2 \Gamma_{\text{sl}}} \overline{B}_{\text{sl}}^\text{exp} , \]  

(109)

where $X_d$ stands for the final states listed in eq. (108). In the numerical results for $\overline{B}_d$ we will insert $\Gamma_{\text{sl}}$ as given in eq. (102), i.e., we do not make an $\alpha_s$ expansion of $1/\Gamma_{\text{sl}}$ in eq. (109). The charmless hadronic decay rate $\overline{B}_d$ then reads

\[ \overline{B}_d = \overline{B}_{sg} + \sum_{q = d, s} \sum_{q' = u, d, s} \overline{B}_{q'q'q} . \]  

(110)

While the $O(\alpha_s)$ corrections to semileptonic processes have been known for a long time (see e.g. ref. [27]), the NLL corrections to the hadronic processes in eq. (108) with 3 quarks in the final state had a long history and were completed to a large extent only recently by Lenz et al. [8,6]; however, current-current type corrections to the penguin operators are still missing. To briefly summarize the history, it is useful to decompose the NLL expressions for the decay widths of these processes into various pieces. Taking as an example the process $b \to u\bar{u}d$, we write as in ref. [8]:

\[ \Gamma(b \to u\bar{u}d) = \Gamma(0) + \frac{\alpha_s}{4\pi} \left[ \Delta \Gamma_{cc} + \Delta \Gamma_{\text{peng}} + \Delta \Gamma_8 + \Delta \Gamma_W \right] . \]  

(111)
The first two terms in the square bracket in eq. (111) describe the effect of current-current and penguin diagrams involving the operators \( O_1 \) and \( O_2 \). \( \Delta \Gamma \) likewise contains the matrix element of the operator \( O_8 \). The remaining part \( \Delta \Gamma_W \) of the NLL contribution is made of the corrections to the Wilson coefficients multiplying the tree-level amplitudes in \( \Gamma^{(0)} \). In this approximation, the matrix elements of the penguin operators \( O_3, \ldots, O_6 \) only enter at tree level. As the expressions for the r.h.s. of eq. (111) are explicitly given in ref. [8], we do not give them here. For later reference, we denote this approximation (in lack of a better word) by “approx1”.

Later, in ref. [6], the same authors added the contributions of the penguin diagrams associated with the penguin operators to the decay matrix elements and took into account the interference with the tree level matrix element of the operator \( O_2 \) in the decay width. In addition, they took into account the square of the matrix element of the penguin diagram associated with \( O_2 \). Although being of next-to-NLL, this term is numerically relatively large, as it is multiplied with \( C_2^2 \). These new contributions can be absorbed into the quantity \( \Delta \Gamma_{\text{new}} \), which is understood to be added to the terms in the bracket in eq. (111). As the extraction of \( \Delta \Gamma_{\text{new}} \) from ref. [6] is straightforward, we do not give the explicit expression. This approximation, which contains – up to the current-current type corrections to the penguin operators – the full NLL contribution to the hadronic three body decays, is called “approx2”.

We note that the approximation where only the current-current type corrections \( \Delta \Gamma_{cc} \) were considered together with the shifts \( \Delta \Gamma_W \) in the Wilson coefficients has existed for a long time [7]. We denote this approximation by “approx0” in the numerical discussion.

In table II we present numerical results for the charmless hadronic branching ratio \( B_c \) in the various approximations mentioned above. The process \( b \to sg \), encoded in \( B_{sg} \) in eq. (110) is taken into account in the columns “approx0”, “approx1” and “approx2” at LL precision. The last column includes in addition the NLL corrections to \( b \to sg \) which were calculated in this paper. Table II was produced with the following input parameters:

\[
\begin{align*}
m_b &= (4.8 \pm 0.2) \text{ GeV}, \quad \mu = m_b, \quad (m_c/m_b) = 0.29 \pm 0.04, \\
\alpha_s(m_Z) &= 0.119 \pm 0.003, \quad m_t^{\text{pole}} = (175 \pm 5) \text{ GeV}, \quad B_{sl}^{\exp} = (10.49 \pm 0.46)\% \\
|V_{us}| &= 0.22, \quad |V_{cb}| = 0.038, \quad |V_{ub}/V_{cb}| = 0.095 \pm 0.035, \quad \delta = 60^\circ \pm 30^\circ.
\end{align*}
\]

The central value for \( |V_{ub}/V_{cb}| \) corresponds to the (improved) Wolfenstein parameters \( \rho = 0.20 \) and \( \eta = 0.37 \) [28]. The remaining entries of the CKM matrix are then obtained as described in detail in [29]. We note that the averaged charmless hadronic branching ratio is practically independent of \( \delta \), as already observed in ref. [6].

The numbers in column “approx2” are very similar to those in table 1 of ref. [6]. The small discrepancy is due to the omission of the \( 1/m_b^2 \) power corrections in our work.

Staring from the numbers in column “approx0”, table II illustrates, that the various improvements shown in the other columns are relatively large, tending to increase \( B_c \).

Note, that the authors of refs. [6,7] use the old operator basis [22].
TABLE II. Table for the charmless hadronic branching ratio $B_c$ (in %) in the various approximations discussed in the text. Unless specified explicitly in the first column, the input parameters correspond to the central values in eq. (112).

| input | approx0 | approx1 | approx2 | with NLL $b \rightarrow s g$ |
|-------|---------|---------|---------|---------------------------|
| as in [112] | 1.32 | 1.50 | 1.62 | 1.88 |
| $\mu = m_b/4$ | 3.86 | 3.21 | 3.34 | 3.62 |
| $\mu = m_b/2$ | 2.06 | 2.09 | 2.18 | 2.43 |
| $\mu = 2m_b$ | 0.96 | 1.14 | 1.28 | 1.55 |
| $|V_{ub}/V_{cb}| = 0.06$ | 0.94 | 1.13 | 1.24 | 1.50 |
| $|V_{ub}/V_{cb}| = 0.07$ | 1.03 | 1.22 | 1.33 | 1.59 |
| $|V_{ub}/V_{cb}| = 0.08$ | 1.14 | 1.32 | 1.44 | 1.69 |
| $|V_{ub}/V_{cb}| = 0.09$ | 1.26 | 1.44 | 1.55 | 1.81 |
| $|V_{ub}/V_{cb}| = 0.10$ | 1.39 | 1.57 | 1.69 | 1.94 |
| $|V_{ub}/V_{cb}| = 0.11$ | 1.54 | 1.72 | 1.83 | 2.09 |
| $|V_{ub}/V_{cb}| = 0.12$ | 1.70 | 1.87 | 1.99 | 2.25 |
| $|V_{ub}/V_{cb}| = 0.13$ | 1.87 | 2.05 | 2.16 | 2.42 |
| $x_c = 0.25$ | 1.14 | 1.32 | 1.45 | 1.69 |
| $x_c = 0.27$ | 1.22 | 1.41 | 1.53 | 1.78 |
| $x_c = 0.29$ | 1.32 | 1.50 | 1.62 | 1.88 |
| $x_c = 0.31$ | 1.44 | 1.61 | 1.72 | 1.99 |
| $x_c = 0.33$ | 1.57 | 1.74 | 1.84 | 2.12 |
particular, the NLL corrections to $b \to s g$ are of similar importance as the corrections calculated in \[8,6\].

For $|V_{ub}/V_{cb}| = 0.095$ we obtain the charmless hadronic branching ratio

$$\mathcal{B}_q = \left(1.88_{-0.38}^{+0.60}\right)\%,$$

(113)

where the error corresponds to a variation of $x_c = (m_c/m_b)$ and of the renormalization scale $\mu$ in the ranges $0.25 \leq x_c \leq 0.33$ and $0.5 \leq \mu/m_b \leq 2.0$. The corresponding errors are added in quadrature. The experimental uncertainty in $\alpha_s(m_Z)$ has a smaller impact and the errors due to the remaining input parameters in eq. (112) are negligible. The large renormalization scale dependence of this result is expected to be weakened once the current-current type corrections to the penguin operators are included.

So far, we have considered the **charmless hadronic** branching ratio $\mathcal{B}_q$. To obtain the **total charmless** branching ratio $\mathcal{B}(B \to \text{no charm})$, one has to add twice the charmless semileptonic branching ratio $\mathcal{B}(B \to X_u(\ell \nu_\ell))$, for $\ell = e$ and $\ell = \mu$ [27] (the contribution for $\ell = \tau$, as well as radiative decay modes can be safely neglected):

$$\mathcal{B}(B \to X_u(\ell \nu_\ell)) = (0.17 \pm 0.03)\% \times \left(\frac{|V_{ub}/V_{cb}|}{0.095}\right)^2.$$

(114)

For $|V_{ub}/V_{cb}| = 0.095$, we find

$$\mathcal{B}(B \to \text{no charm}) = \left(2.22_{-0.38}^{+0.60}\right)\%.$$

(115)

The experimental result for the total charmless branching ratio reads

$$\mathcal{B}^{\text{exp}}(B \to \text{no charm}) = (0.2 \pm 4.1)\%,$$

(116)

obtained in ref. [30] from CLEO data [31].

**XII. NUMERICAL PREDICTIONS IN THE PRESENCE OF ENHANCED $C_8^{\text{eff}}$**

As discussed in the introduction, the theoretical prediction of the semileptonic branching ratio and the charm multiplicity are compatible with the experimental findings if the renormalization scale is allowed to be as low as $m_b/4$. Both predictions are, however, at the lower side and therefore an enhancement of the charmless hadronic branching ratio $\mathcal{B}_q$ by new physics would lead to a better agreement. It is therefore still conceivable that $\mathcal{B}_q$ is considerably larger than in the standard model (SM).

In the SM the initial conditions for $C_{3-6}$ and $C_8$ are generated at a scale $\mu = O(m_W)$ by the one-loop $bsg$ vertex function. Due to the fact that the $W$-boson only couples to left-handed quarks, only chromomagnetic operators proportional to $m_b$ (and $m_s$) are generated. In extensions of the SM, however, also chromomagnetic operators where $m_b$ (or $m_s$) is
FIG. 11. Branching ratio $B(b \to sg)$ as a function of $f = C^\text{eff}_8(m_W)/C^\text{eff,SM}_8(m_W)$. For the exact definition of $f$, see eq. (117). The dotted (solid) curve shows the LL (NLL) approximation. The dashed curve is obtained by switching off the matrix elements of the operators $O_1$ and $O_2$.

replaced by the mass of a heavy particle propagating in the loop, can be generated \cite{32}. Such operators potentially lead to large contributions to $b \to sg$. In the following we will perform a model independent analysis of the impact of enhanced $C_8$ on $\overline{B}_d$, emphasizing the role of the NLL corrections to $b \to sg$. We assume that only chromomagnetic operators with the same helicity structure as $O_8$ in the SM are generated which can then be described as a shift in $C_8$. For simplicity, we further assume that the CKM structure of the new contribution is the same as in the SM, hence neglecting the possibility of new CP-violating phases, by assuming the shift in $C_8$ to be real.

In fig. 11 we investigate the impact of enhanced $C^\text{eff}_8(m_W) = C^0\text{eff}_8(m_W) + \alpha_s/(4\pi) C^1\text{eff}_8(m_W)$ on the branching ratio for $b \to sg$. In the NLL approximation for this branching ratio, both, $C^0\text{eff}_8(m_W)$ and $C^1\text{eff}_8(m_W)$ enter. In general, it is expected that the two pieces get different new physics shifts. For purpose of illustration, we assume however that both pieces are the same multiple $f$ of the SM counterparts, i.e., we assume that

$$C^0\text{eff}_8(m_W) = f C^{0,\text{eff,SM}}_8(m_W) ; \quad C^1\text{eff}_8(m_W) = f C^{1,\text{eff,SM}}_8(m_W) .$$

(117)

The dotted curve shows the LL prediction of $B(b \to sg)$ as a function of $f$, while the solid curve shows the NLL prediction. It is expected that for large enhancement factors, the matrix elements of the operators $O_1$ and $O_2$ become unimportant; this feature is illustrated by the dashed line, which is obtained by switching off these matrix elements. The NLL corrections (for large enhancement factors) amount to almost 50% of the LL prediction.

In fig. 12, the impact of enhanced $C_8$ on the charmless hadronic branching ratio $\overline{B}_d$ is illustrated. The dotted curve includes the NLL corrections to the decay modes with three quark in the final state and the LL result for $B(b \to sg)$ (see “approx2” in section XI), while
FIG. 12. Charmless hadronic branching ratio $\mathcal{B}_f$ as a function of $f = C_8^{\text{eff}}(m_W)/C_8^{\text{eff,SM}}(m_W)$. For the exact definition of $f$, see eq. (117). The dotted (solid) curve includes the LL (NLL) approximation for $\mathcal{B}(b \rightarrow g)$. The NLL corrections to the decay modes with three quark in the final state (see “approx2” in section [XI]) are included in both cases.

the solid curve also includes the NLL corrections to $\mathcal{B}(b \rightarrow sg)$. For a given value of $\mathcal{B}_f$ (from an ideal measurement), $C_8(m_W)$ can be measured in principle. To illustrate this, we take the hypothetical value $\mathcal{B}_f = 5\%$. The two solutions for the enhancement factor $f$ are $f = 7$ and $f = -9$ when using the dotted curve; including NLL corrections to $b \rightarrow sg$ (solid curve), enhancement factors with smaller absolute values do the job, viz. $f = 5$ and $f = -8$.

XIII. SUMMARY

In this paper we presented a detailed calculation of the $O(\alpha_s)$ virtual corrections to the decay width $\Gamma(b \rightarrow sg)$. The most difficult part, the two-loop diagrams associated with the operators $O_1$ and $O_2$ which from the numerical point of view play a crucial role, was obtained by using Mellin-Barnes techniques. Also complete expressions for the corresponding $O(\alpha_s)$ bremsstrahlung corrections to $b \rightarrow sg$ were given. The combined result is free of infrared and collinear singularities, in accordance with the KLN theorem.

The renormalized virtually corrected matrix element $\langle sg | O_8 | b \rangle$ contains logarithms of the form $\ln(m_f/\mu) \,(f = u, d, s, c, b)$, which for the light flavors $(u, d, s)$ represent a special kind of singularity. Keeping in mind that these terms originate from the renormalization factor $Z_3$ of the gluon field, i.e., from gluon self energy diagrams in which these flavors propagate, we argued that these singularities cancel against the logarithms present in the decay rate $\Gamma(b \rightarrow s f \bar{f})$ with $f = u, d, s$. We therefore included the $O_8$ contribution to $\Gamma(b \rightarrow s f \bar{f})$ for $f = u, d, s$.  

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Taking into account the existing next-to-leading logarithmic (NLL) result for the Wilson coefficient $C_{7}^{\text{eff}}$, a complete NLL result for the branching ratio $\mathcal{B}^{\text{NLL}}(b \to s g)$ was obtained. Numerically, we found $\mathcal{B}^{\text{NLL}} = (5.0 \pm 1.0) \times 10^{-3}$, which is more than a factor of two larger than the leading logarithmic result $\mathcal{B}^{\text{LL}} = (2.2 \pm 0.8) \times 10^{-3}$.

We then investigated the impact of these corrections on the inclusive charmless hadronic branching ratio $\mathcal{B}_c$ of $B$-mesons. We found that the NLL corrections calculated in this paper are of similar importance as NLL corrections to $b$-quark decay modes with three quarks in the final state, which were presented by Lenz et al. [3,6].

Finally, the impact of the NLL corrections to $b \to s g$ on $\mathcal{B}_d$ was studied in scenarios, where the Wilson coefficient $C_8$ is enhanced by new physics. For a given value of $\mathcal{B}_d$ (from an ideal measurement), $C_8(m_W)$ can be measured in principle. To illustrate this, we took the hypothetical value $\mathcal{B}_d = 5\%$. The two solutions for the enhancement factor $f$ are $f = 7$ and $f = -9$, using the LL approximation for $\mathcal{B}(b \to s g)$; including NLL corrections to $b \to s g$, somewhat smaller enhancement factors ($f = 5$ and $f = -8$) are needed to obtain the hypothetical value $\mathcal{B}_d = 5\%$.

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APPENDIX A: NEXT-TO-LEADING ORDER WILSON COEFFICIENTS

In this appendix we present the explicit formulas which allow to calculate the Wilson coefficients needed in this paper.

In section A1, we give the results for the Wilson coefficients at the matching scale $\mu_W$, which is usually taken to be of order $m_W$. Section A2 is devoted to the Wilson coefficients at the scale $\mu_b$, where $\mu_b$ is of order $m_b$. We give an explicit expression for $C_{8\text{eff}}(\mu_b)$ at NLL, which is new. To make this appendix self-contained, we also repeat the results for the Wilson coefficients $C_{1\text{eff}}(\mu_b)$ and $C_{2\text{eff}}(\mu_b)$ which are needed only to LL precision in our application.

1. NLL Wilson coefficients at the matching scale $\mu_W$

To give the results for the effective Wilson coefficients $C_{i\text{eff}}$ at the matching scale $\mu_W$ in a compact form, we write

$$C_{i\text{eff}}(\mu_W) = C_{i0\text{eff}}(\mu_W) + \frac{\alpha_s(\mu_W)}{4\pi} C_{i1\text{eff}}(\mu_W).$$

(A1)

The LL Wilson coefficients at this scale are well known [33,34].

\begin{align*}
C_{20\text{eff}}(\mu_W) &= 1 \\
C_{i0\text{eff}}(\mu_W) &= 0 \quad (i = 1, 3, 4, 5, 6) \\
C_{70\text{eff}}(\mu_W) &= \frac{x}{24} \left[ -8x^3 + 3x^2 + 12x - 7 + (18x^2 - 12x) \ln x \right] \\
C_{80\text{eff}}(\mu_W) &= \frac{x}{8} \left[ -x^3 + 6x^2 - 3x - 2 - 6x \ln x \right] (x - 1)^4. \\
\end{align*}

(A2)

The coefficients $C_{70\text{eff}}(\mu_W)$ and $C_{80\text{eff}}(\mu_W)$ are functions of $x = m_t^2/m_W^2$. Note that there is no explicit dependence of the matching scale $\mu_W$ in these functions. Whether there is an implicit $\mu_W$–dependence via the $t$–quark mass depends on the precise definition of this mass which has to be specified when going beyond leading logarithms. If one chooses to work with $\bar{m}_t(\mu_W)$, then there is such an implicit $\mu_W$–dependence of the lowest order Wilson coefficient; in contrast, when working with the pole mass $m_t$ there is no such dependence. We choose to express our NLL results in terms of the pole mass $m_t$.

The NLL pieces $C_{i1\text{eff}}(\mu_W)$ of the Wilson coefficients have an explicit dependence on the matching scale $\mu_W$ and for $i = 7, 8$ they also explicitly depend on the actual definition of the $t$–quark mass. Initially, when the heavy particles are integrated out, it is convenient to work out the matching conditions $C_{i1\text{eff}}(\mu_W)$ for $i = 7, 8$ in terms of $\bar{m}_t(\mu_W)$. Using eq. (5),

$^6$Note that $C_i^{\text{eff}}(\mu) = C_i(\mu)$ by definition for $i = 1, ..., 6$. 

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it is then straightforward to get the corresponding result expressed in terms of the pole mass $m_t$. One obtains for $i = 1, ..., 6$:

\[ C_{1,\text{eff}}^i(\mu_W) = 15 + 6 \ln \frac{\mu_W^2}{m_W^2}, \]

\[ C_{4,\text{eff}}^i(\mu_W) = E_0 + \frac{2}{3} \ln \frac{\mu_W^2}{m_W^2}, \]

\[ C_{i,\text{eff}}^i(\mu_W) = 0, \quad (i = 2, 3, 5, 6) \quad (A3) \]

with

\[ E_0 = \frac{x(x^2 + 11x - 18)}{12(x - 1)^3} + \frac{x^2(4x^2 - 16x + 15)}{6(x - 1)^4} \ln x - \frac{2}{3} \ln x - \frac{2}{3}. \quad (A4) \]

For $i = 7, 8$, we split $C_{i,\text{eff}}^i(\mu_W)$ into three terms:

\[ C_{i,\text{eff}}^i(\mu_W) = W_i + M_i \ln \frac{\mu_W^2}{m_W^2} + T_i \left( \ln \left( \frac{m_i^2}{\mu_W^2} \right) - \frac{4}{3} \right). \quad (A5) \]

The first two terms $W_i$ and $M_i$ would be the full result when working in terms of $m_i(\mu_W)$. $T_i$ results when expressing $m_i(\mu_W)$ in terms of the pole mass $m_t$ in the corresponding lowest order coefficients. Thus, for $i = 7, 8$, the term $T_i$ is obtained as

\[ T_i = 8x \frac{\partial C_{i,\text{eff}}^{0, i}(\mu_W)}{\partial x}. \quad (A6) \]

The explicit form of the functions $W_i$, $M_i$ and $T_i$ reads

\[ W_7 = \frac{-16x^4 - 122x^3 + 80x^2 - 8x}{9(x - 1)^4} \ln^2 x + \frac{6x^4 + 46x^3 - 28x^2}{3(x - 1)^5} \ln x \]

\[ + \frac{1646x^4 + 12205x^3 - 10740x^2 + 2509x - 436}{486(x - 1)^4} \ln x \]

\[ W_8 = \frac{-4x^4 + 40x^3 + 41x^2 + x}{6(x - 1)^4} \ln^2 x + \frac{-17x^3 - 31x^2}{2(x - 1)^5} \ln x \]

\[ + \frac{-210x^5 + 1086x^4 + 4893x^3 + 2857x^2 - 1994x + 280}{216(x - 1)^5} \ln x \]

\[ + \frac{737x^4 - 14102x^3 - 28209x^2 + 610x - 508}{1296(x - 1)^4} \ln x \]

\[ M_7 = \frac{82x^5 + 301x^4 + 703x^3 - 2197x^2 + 1319x - 208 - (162x^4 + 1242x^3 - 756x^2)}{81(x - 1)^5} \ln x \]
$$M_8 = \frac{77x^5 - 475x^4 - 1111x^3 + 607x^2 + 1042x - 140 + (918x^3 + 1674x^2) \ln x}{108(x-1)^5}$$

$$T_7 = \frac{x}{3} \left[ \frac{47x^3 - 63x^2 + 9x + 7 - (18x^3 + 30x^2 - 24x) \ln x}{(x-1)^5} \right]$$

$$T_8 = 2x \left[ \frac{-x^3 - 9x^2 + 9x + 1 + (6x^2 + 6x) \ln x}{(x-1)^5} \right].$$

(A7)

The dilogarithm $\text{Li}_2(x)$ is defined by

$$\text{Li}_2(x) = -\int_0^x \frac{dt}{t} \ln(1 - t).$$

(A8)

2. NLL Wilson coefficients at the low scale $\mu_b$

The evolution from the matching scale $\mu_W$ down to the low–energy scale $\mu_b$ is described by the renormalization group equation

$$\mu \frac{d}{d\mu} C_{ji}^{\text{eff}}(\mu) = C_{ji}^{\text{eff}}(\mu) \gamma_{ji}^{\text{eff}}(\mu).$$

(A9)

The initial conditions $C_{ji}^{\text{eff}}(\mu_W)$ for this equation are given in section A1, while the anomalous dimension matrix $\gamma_{ji}^{\text{eff}}$ up to order $\alpha_s^2$ can be found in ref. [19]. For completeness we display the result here. The anomalous dimension matrix can be expanded perturbatively as

$$\gamma_{ji}^{\text{eff}}(\mu) = \frac{\alpha_s(\mu)}{4\pi} \gamma_{ji}^{0,\text{eff}} + \frac{\alpha_s^2(\mu)}{(4\pi)^2} \gamma_{ji}^{1,\text{eff}} + \mathcal{O}(\alpha_s^3)$$

(A10)

where matrix $\gamma_{ji}^{0,\text{eff}}$ is given by

$$\left\{ \gamma_{ji}^{0,\text{eff}} \right\} = 
\begin{pmatrix}
-4 & \frac{8}{3} & 0 & -\frac{2}{9} & 0 & 0 & -\frac{208}{213} & 173 \\
12 & 0 & 0 & \frac{4}{3} & 0 & 0 & \frac{416}{81} & 70 \\
0 & 0 & -52 & 0 & 2 & -176 & \frac{81}{14} & 27 \\
0 & 0 & -\frac{49}{9} & -\frac{100}{9} & \frac{4}{9} & \frac{5}{6} & \frac{243}{152} & -587 \\
0 & 0 & -256 & \frac{3}{9} & 0 & 20 & -6272 & \frac{6596}{27} \\
0 & 0 & -\frac{256}{9} & \frac{56}{9} & \frac{40}{9} & -\frac{2}{3} & 4624 & \frac{4772}{81} \\
0 & 0 & 0 & \frac{32}{3} & 0 & 0 & \frac{32}{3} & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{32}{9} & \frac{28}{3} \\
\end{pmatrix},$$

(A11)

and in the $\overline{\text{MS}}$ scheme with fully anticommuting $\gamma_5$, $\gamma_{ji}^{1,\text{eff}}$ is
\[
\left\{ \gamma^{1, \text{eff}} \right\} = \begin{pmatrix}
-355 & 502 & -1412 & -1369 & 134 & -35 & 818 & 2779 \\
-35 & -28 & -416 & 1280 & 56 & 35 & 508 & 1841 \\
0 & 0 & -4468 & 31469 & 400 & 3373 & 22348 & 10178 \\
0 & 0 & -8158 & 59399 & 269 & 12899 & 17584 & -172471 \\
0 & 0 & -251680 & 128648 & 23836 & 6106 & 1183696 & 2901296 \\
0 & 0 & 58640 & 26348 & -14324 & 2551 & 2480344 & 3296257 \\
0 & 0 & 0 & 0 & 0 & 0 & 4688 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -2192 & 4063
\end{pmatrix} \cdot \quad (A12)
\]

The solution of eq. (A9), obtained through the procedure described in [29], yields for the coefficient \( C^\text{eff}_8(\mu_b) \), which we decompose as

\[
C^\text{eff}_8(\mu_b) = C^0_8(\mu_b) + \frac{\alpha_s(\mu_b)}{4\pi} C^1_8(\mu_b) \ , \quad (A13)
\]

the LL term

\[
C^0_8(\mu_b) = \eta \eta'^2 C^0_8(\mu_W) + \sum_{i=1}^5 h'_i \eta \eta'^i C^0_2(\mu_W) , \quad (A14)
\]

and the NLL contribution

\[
C^1_8(\mu_b) = \eta \eta'^2 C^1_8(\mu_W) + 6.7441 \left( \eta \eta'^2 - \eta \eta'^2 \right) C^0_8(\mu_W) \\
+ \sum_{i=1}^8 \left( e_i' \eta C^1_4(\mu_W) + (f_i' + k_i' \eta) C^0_2(\mu_W) + l_i' \eta C^1_1(\mu_W) \right) \eta a_i . \quad (A15)
\]

The symbol \( \eta \) is defined as \( \eta = \alpha_s(\mu_W)/\alpha_s(\mu_b) \); the vectors \( a_i, a'_i, h'_i, e'_i, f'_i, k'_i \) and \( l'_i \) read

\[
\{ a_i \} = \left\{ \frac{14}{23}, \frac{16}{23}, \frac{6}{23}, \frac{-12}{23} , 0.4086, -0.4230, -0.8994, 0.1456 \right\} \\
\{ a'_i \} = \left\{ \frac{14}{23} , 0.4086, -0.4230, -0.8994, 0.1456 \right\} \\
\{ h'_i \} = \left\{ \frac{313063}{363036} , -0.9135, 0.0873, -0.0571, 0.0209 \right\} \\
\{ e'_i \} = \left\{ 2.1399, 0, 0, 0, -2.6788, 0.2318, 0.3741, -0.0670 \right\} \\
\{ f'_i \} = \left\{ -5.8157, 0, 1.4062, -3.9895, 3.2850, 3.6851, -0.1424, 0.6492 \right\} \\
\{ k'_i \} = \left\{ 3.7264, 0, 0, 0, -3.2247, 0.3359, 0.3812, -0.2968 \right\} \\
\{ l'_i \} = \left\{ 0.2169, 0, 0, 0, -0.1793, -0.0730, 0.0240, 0.0113 \right\} . \quad (A16)
\]

As already mentioned earlier, we neglect the contributions of the operators \( O_3, \ldots, O_6 \) in our analysis for \( b \to s g \), as their Wilson coefficients are rather small. We therefore only list the results for the coefficients \( C^\text{eff}_1(\mu_b) \) and \( C^\text{eff}_2(\mu_b) \), which are needed to LL precision only.
\[ C^{0,\,\text{eff}}_{1}(\mu_b) = \left( \eta_{\frac{5}{3}}^6 - \eta_{-\frac{12}{23}}^6 \right) C^{0,\,\text{eff}}_{2}(\mu_W) \]
\[ C^{0,\,\text{eff}}_{2}(\mu_b) = \left( \frac{2}{3} \eta_{\frac{5}{3}}^6 + \frac{1}{3} \eta_{-\frac{12}{23}}^6 \right) C^{0,\,\text{eff}}_{2}(\mu_W). \]  

(A17)

When calculating NLL results in the numerical analysis, we use the NLL expression for the strong coupling constant:

\[
\alpha_s(\mu) = \frac{\alpha_s(m_Z)}{v(\mu)} \left[ 1 - \frac{\beta_1}{\beta_0} \frac{\alpha_s(m_Z)}{4\pi} \ln \left( \frac{m_Z}{\mu} \right) \right],
\]

(A18)

with

\[
v(\mu) = 1 - \beta_0 \frac{\alpha_s(m_Z)}{2\pi} \ln \left( \frac{m_Z}{\mu} \right), \]

(A19)

where \( \beta_0 = \frac{23}{3} \) and \( \beta_1 = \frac{116}{3} \) (for 5 flavors). However, for LL results we always use the LL expression for \( \alpha_s(\mu) \), i.e., \( \beta_1 \) is put to zero in eq. (A18).

**APPENDIX B: ONE-LOOP FUNCTIONS \( G_{-1}(t) \) AND \( G_0(t) \)**

In this appendix we give the explicit results for the functions \( G_{-1}(t) \) and \( G_0(t) \) needed in eq. (83). Evaluating the integral in eq. (78) for \( i = -1, 0 \), one obtains:

\[
G_{-1}(t) = \begin{cases} 
-\frac{\pi^2}{2} + 2 \ln^2 \left( \frac{\sqrt{t}+\sqrt{t-1}}{2} \right) - 2i\pi \ln \left( \frac{\sqrt{t}+\sqrt{t-1}}{2} \right); & t \geq 4 \\
-\frac{\pi^2}{2} - 2 \arctan^2 \left( \sqrt{\frac{1-t}{t}} \right) + 2\pi \arctan \left( \sqrt{\frac{1-t}{t}} \right); & 0 \leq t \leq 4
\end{cases}
\]

(B1)

\[
G_0(t) = \begin{cases} 
-2 + 2 \sqrt{\frac{t-1}{t}} \ln \left( \frac{\sqrt{t}+\sqrt{t-1}}{2} \right) - i\pi \sqrt{\frac{t-1}{t}}; & t \geq 4 \\
-2 - 2 \sqrt{\frac{1-t}{t}} \arctan \left( \sqrt{\frac{1-t}{t}} \right) + \pi \sqrt{\frac{1-t}{t}}; & 0 \leq t \leq 4
\end{cases}
\]

(B2)
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