Asymptotic analysis of a multiclass queueing control problem under heavy-traffic with model uncertainty

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Abstract

We study a multiclass M/M/1 queueing control problem with finite buffers under heavy-traffic where the decision maker is uncertain about the rates of arrivals and service of the system and by scheduling and admission/rejection decisions acts to minimize a discounted cost that accounts for the uncertainty. The main result is the asymptotic optimality of a $\mu$-type of policy derived via underlying stochastic differential games studied in [16].

Under this policy, with high probability, rejections are not performed when the workload lies below some cut-off that depends on the ambiguity level. When the workload exceeds this cut-off, rejections are carried out and only from the buffer with the cheapest rejection cost weighted with the mean service rate in some reference model. The allocation part of the policy is the same for all the ambiguity levels. This is the first work to address a heavy-traffic queueing control problem with model uncertainty.

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1 Introduction

We consider a multiclass M/M/1 queueing model under diffusion-scaled heavy-traffic where the decision maker (DM) is uncertain about the parameters and acts to optimize an overall cost that accounts for this uncertainty. The model consists of a server that at any time instant her effort is allocated by the DM to customers from several number of classes. Customers of each class are kept in a finite buffer. Apart from the scheduling control, upon arrival of a customer, the DM has to decide whether to reject it or to assign it to the buffer that corresponds to its class type. The DM has ambiguity about the rates of arrivals and the mean service times. The cost accounts for the ambiguity, the holding of customers in the buffers, and rejections of new arrivals.
The problem without ambiguity, also referred to as the risk-neutral problem, was analyzed by Atar and Shifrin in [6], under the framework of G/G/1. Plambeck et al. studied in [33] a similar non-robust problem with time constraints instead of the finite buffers constraints. In these problems as well as in many other classical models of queueing control problems (QCPs), see e.g., [8, 12, 13] and the references therein, there is a fixed random model; that is, the DM is certain about the evolution of the system, which, moreover does not change in time. Such an assumption is not realistic, and a robust analysis is desirable.

We assume that based on the available data, the DM has a reference model in mind, which, up to some degree, describes the situation she is facing. To model the uncertainty about the reference model, the DM takes into account other models and penalizes them based on their deviation from the reference model. The penalization depends then on how averse the DM is to ambiguity. Such ambiguity models are sometimes referred to as model uncertainty or Knightian uncertainty, see e.g., [30, 21, 20, 7] and in the context of queueing systems see [23, 11, 29]. We allow for different levels of model uncertainty for each of the arrival processes as well as for each of the service processes. More specifically, we consider the following cost function, which the DM aims to minimize.

$$
\sup_Q \left\{ \mathbb{E}^Q \left[ \int_0^\infty e^{-\rho t} \left( \hat{h} \cdot X(t) dt + \hat{r} \cdot dR(t) \right) \right] - \sum_{i=1}^I \frac{1}{\kappa_{1,i}} L^\rho(Q_{1,i} \| P_{1,i}) - \sum_{i=1}^I \frac{1}{\kappa_{2,i}} L^\rho(Q_{2,i} \| P_{2,i}) \right\},
$$

where $I$ is the number of classes. The vectors $\hat{h}$ and $\hat{r}$ stand for the holding and the rejection costs, respectively. The $I$-dimensional processes $X$ and $R$ represent the queue lengths and the rejection processes, respectively. The supremum is taken over the product measures $Q = \prod_{i=1}^I Q_{1,i} \times Q_{2,i}$ (with some additional technical conditions), where the components with the indices 1 (resp., 2) refer to the arrival (resp., service) processes. The function $L^\rho$ is a discounted variant of the Kullback–Liebler divergence and measures how far away $Q_{j,i}$ from the reference measure $P_{j,i}$ is. The parameters $\kappa_{j,i} > 0$ are the ambiguity parameters, which penalize the deviation from the reference measures. Since the QCP is formulated under heavy-traffic, throughout the paper we consider a sequence of queueing systems labeled by the scaling parameter. This parameter is omitted above in order to make the presentation clear and concise.

To tackle QCPs under heavy-traffic one often solves a limiting control problem associated with a Brownian motion, called Brownian control problem (BCP) and uses its solution to construct an asymptotically optimal policy in the QCP. This concept was first introduced by [22]; for further reading on BCPs see e.g., [8, 12, 13] and the references therein. In our case, the cost function given above suggests that the BCP is in fact a stochastic game. The players in this game are the DM and the nature, which according to their goals are referred to as the minimizer and the maximizer, respectively. Interpreting the roles of the processes from the QCP to a multidimensional stochastic differential game (MSDG), the minimizer controls the server’s effort allocation and the admission/rejection, while the maximizer is free to choose a probability measure and is penalized in accordance to the deviation of the chosen measure from the reference measure. This game was analyzed in [16], where it was also shown that a state-space collapse property holds. This is done by considering a reduced stochastic differential
game (RSDG), which emerges from the workload process, and showing that the games share
the same value and that given equilibrium in either one of the games one can construct an
equilibrium in the other game. Further properties of the games, such as dependency on the
ambiguity parameters are also given there.

This paper is devoted to the connection between the QCP and the BCPs (namely, the MSDG
and RSDG); we show that the value function of the QCP is approximately the value function
of the BCPs and that the minimizer’s optimal strategy from the MSDG leads to an asymptotically
stationary optimal policy in the QCP. Roughly speaking, this strategy suggests that the DM
should use a $c\mu$ type of rule and fill in the buffers in accordance to their holding costs without
using rejections, unless a cut-off level of workload that depends on the ambiguity level has
been reached and then to use rejections only from the buffer with the cheapest rejection cost,
weighted with the mean service rate, until the workload level goes below the cut-off. More
specifically, let $\mu_i$ be the mean service rate of class $i$ customers under the reference model
and recall that $h_i$ is the holding cost per customer of class $i$. The DM should prioritized
the classes in the order of $\left\{\frac{h_i\mu_i}{\mu_i}\right\}_{i=1}^n$, where the lowest priority is given to the class with the
lowest $\frac{h_i\mu_i}{\mu_i}$ among the classes for which the buffers are not ‘almost’ full. As for the admission
control part of the policy, whenever the workload level remains below the mentioned cut-off
use rejections only if there is a new arrival to a full buffer; as is shown, the probability of such
an event vanishes with the scaling parameter. If the workload level exceeds the cut-off, the
DM rejects all incoming customers to the class with the lowest $\frac{r_i}{\mu_i}$. This policy (with different
cut-off level) was shown to be asymptotically optimal in the risk-neutral setup and in a QCP
with the same mechanism but with the moderate-deviation heavy-traffic regime (instead of
the diffusion scaling) and a risk-sensitive cost criteria. The latter QCP models a situation of
a ‘very’ risk averse DM. The only difference between the policies in the three models is in the
the position of the cut-off point. Specifically, the allocation policy is the same. This shows us
the usefulness of the allocation policy as it is robust to ambiguity. The optimality of the same
policy (with differences only in the cut-off level) in these three models is not obvious due to
the existence of the maximizer, which leads to a non-stationary problem. For further reading
about the moderate-deviation heavy-traffic regime see [1, 10, 2, 5, 3, 4] and in the context of
our paper also the discussion in [16, p. 3]. The current paper does not aim to establish a
rigorous relationship between the QCP with the model uncertainty described below and the
moderate-deviation heavy-traffic regime QCP

We now make some comments on proof techniques. The proof is divided into two parts:
showing that the value function of the RSDG forms an asymptotic lower bound for the QCP
and that the expected cost associated with the candidate policy is asymptotically bounded
above by the value function of the RSDG. Keep in mind that the RSDG shares the same value
as the MSDG. Also, recall that we consider a sequence of queueing systems. For the lower
bound we assign the maximizer a strategy that is driven from the equilibrium strategy in the
MSDG and using it to show that for any sequence of strategies used by the minimizer, the
expected cost is bounded below by the value of the RSDG. By its structure the maximizer’s
strategy preserves the critical load of the system. The main technical difficulty in this part
is that the sequence of strategies of the minimizer is arbitrary and due to the nature of the
control in the QCP, which is replaced by a singular control in the BCP, compactness arguments

\[ 3 \]
do not apply here. In the risk-neutral case, where there is no maximizing player, Atar and Shifrin [6] managed to bypass this issue by \( C \)-tightness arguments applied to the integrands of the relevant processes and later on taking the derivatives of the implied limiting processes. In our setup, the maximizer’s strategy depends on the scaled queue lengths process and not on its integrand. Therefore, the \( C \)-tightness of the sequence of the scaled queue lengths processes is required. As a result, the integrand-derivative method cannot be applied in our case. We use the \textit{time-stretching} method, which was introduced by Meyer and Zheng in [32] and studied in the same framework by Kurtz in [26]. In the context of stochastic control the method was first used by Kushner and Martins in [31, 27] and was adopted in [12, 14, 15]. We set up a random time transformation for any system such that the controls are Lipschitz continuous with Lipschitz constant 1. Then we can apply \( C \)-tightness arguments for the sequence of the relevant time-stretched processes (including the scaled queue lengths process) and obtain limiting processes. Using an inverse time transformation we go back to the original scale. Finally, we connect between the costs associated with the time rescaled limiting processes and the value of the RSDG.

To show that the expected cost associated with the candidate policy is asymptotically bounded above by the value function of the RSDG we do the following. We consider an arbitrary sequence of strategies for the maximizer. This sequence of probability measures is not forced to satisfy the critical load condition. Therefore, at first step we show that it is ‘too costly’ for the maximizer to have a big deviation from the reference measure and thus restrict the maximizer to strategies that in average do not deviate much from the reference measure (Proposition 4.2). Then, in Proposition 4.3 we adapt a state-space collapse result from the risk-neutral case to ours and show that under the sequence of strategies chosen by the maximizer, the underlying stochastic dynamics of the scaled queue lengths lie close to a specific path, so that the properties of the policy mentioned before the previous paragraph hold. At this point one can use \( C \)-tightness arguments to conclude the convergence of the relevant dynamics including the holding and rejection cost parts. However, in order to estimate the change of measure penalty given by the Kullback–Liebler divergence one shall have another reduction and ‘truncate’ the maximizer’s strategies such that the critically load condition is almost surely preserved. Then we show that the relevant processes and all the cost components associated with the two levels of restrictions of the maximizer’s strategies are close to each other (Proposition 4.4). This approximation relies also on the state-space collapse. Thus, the rest of the analysis is performed in the more convenient case, where the critically load condition holds and the expected cost is shown to be asymptotically bounded from above by the value function of the RSDG. For this, we reduce to one-dimensional dynamics and estimate from below the change of measure penalty and together with the convergence of the holding and rejections cost components, we conclude the upper bound.

The paper is organized as follows. Section 2 presents the model. Section 3 collects a few results from [16] required for the proof. In Section 4 we provide and prove the main result (Theorem 4.1), which states that the QCPs value converges to that of the BCP from [16] and an asymptotically optimal policy is provided. Some auxiliary results appear in the Appendix.
1.1 Notation

We use the following notation. For \(a, b \in \mathbb{R}\), \(a \land b = \min\{a, b\}\) and \(a \lor b = \max\{a, b\}\). For a positive integer \(k\) and \(c, d \in \mathbb{R}^k\), \(c \cdot d\) denotes the usual scalar product and \(\|c\| = (c \cdot c)^{1/2}\). We denote \([0, \infty)\) by \(\mathbb{R}_+\). The infimum of the empty set is taken to be \(-\infty\). For subintervals \(I_1, I_2 \subseteq \mathbb{R}\) and \(m \in \{1, 2\}\) we denote by \(C(I_1, I_2)\), \(C^m(I_1, I_2)\), and \(D(I_1, I_2)\) the space of continuous functions [resp., functions with continuous derivatives of order \(m\), functions that are right-continuous with finite left limits (RCLL)] mapping \(I_1 \rightarrow I_2\). The space \(D(I_1, I_2)\) is endowed with the usual Skorohod topology. For \(T, \delta > 0\) and a function \(f : \mathbb{R}_+ \rightarrow \mathbb{R}^k\), \(\|f\|_T = \sup_{t \in [0, T]} \|f(t)\|\), while osc\(_T\)\((f, \delta) = \sup\{\|f(u) - f(t)\| : 0 \leq u \leq t \leq (u + \delta) \land T\}\). For any RCLL processes \(X, Y\), the quadratic variation of \(X\) is denoted by \([X]\) and the quadratic covariation of \(X\) and \(Y\) is denoted by \([X, Y]\).

2 The queueing model

We consider a QCP with customers of \(I\) different classes that arrive at the system to get serviced by a single server. Upon arrival the customers are queued in \(I\) buffers with finite capacity in accordance to their class. Processor sharing is allowed and the server may serve up to \(I\) customers at a time, where two customers from the same class cannot be served simultaneously. We study the system under heavy-traffic. Hence, we consider a sequence of systems, indexed by the scaling parameter \(n \in \mathbb{N}\).

2.1 The reference model

For every \(i \in \{I\} = \{1, \ldots, I\}\) and \(n \in \mathbb{N}\) we consider the probability spaces \((\Omega^I_n, G^I_n, \mathbb{P}^n)\) and \((\Omega^I_{2,i}, G^I_{2,i}, \mathbb{P}^n_{2,i})\) that respectively support a Poisson process \(A_n^i\) with a given rate \(\lambda_n^i\) and a Poisson process \(S_n^i\) with rate \(\mu_n^i\). The process \(A_n^i\) counts the number of arrivals to the \(i\)-th buffer and \(S_n^i(t)\) stands for the number of service completions of customers of class \(i\) that had service was given to class \(i\) for \(t\) units of time. Denote \(A_n = (A_n^i)_{i=1}^I\) and \(S_n = (S_n^i)_{i=1}^I\).

We consider a different level of uncertainty about each of the \(2I\) components of \((A_n, S_n)\), and thus set the following reference probability space that supports these processes

\[
(\Omega^n, G^n, \mathbb{P}^n) := \left( \prod_{i=1}^I (\Omega^I_{1,i} \times \Omega^n_{2,i}) \right) \otimes \prod_{i=1}^I (G^n_{1,i} \otimes G^n_{2,i}) \otimes \left( \prod_{i=1}^I (\mathbb{P}^n_{1,i} \times \mathbb{P}^n_{2,i}) \right),
\]

where \(\otimes = \bigotimes_{i=1}^I (G^n_{1,i} \otimes G^n_{2,i}) = (G^n_{1,1} \otimes G^n_{2,1}) \otimes \ldots \otimes (G^n_{1,I} \otimes G^n_{2,I})\).

By the structure of the probability space, under the measure \(\mathbb{P}^n\), the \(2I\) processes \(A_n^i, S_n^i, \ldots, A_n^I, S_n^I\) are mutually independent. Moreover, the distribution of \(A_n^i\) (resp., \(S_n^i\)) under \(\mathbb{P}^n\) is identical to its distribution under \(\mathbb{P}^n_{1,i}\) (resp., \(\mathbb{P}^n_{2,i}\)).

Let \(U^n = (U^n_i)_{i=1}^I\) be an RCLL process taking values in \(\mathbb{U} = \{x = (x_1, \ldots, x_I) \in [0, 1]^I : \sum x_i \leq 1\}\), where its \(i\)-th component \(U^n_i(t)\) represents the fraction of effort the server dedicates to class \(i\) (recall that process sharing is allowed). Then

\[
T_n^i(t) := \int_0^t U^n_i(s)ds, \quad t \in \mathbb{R}_+,
\]
gives the time the server dedicates to class $i$ customers that are present in the system at time $t$ and the number of class $i$ job completions by time $t$ is thus $S^n_i(T^n_i(t))$. This is a Cox process with rate $\mu^n_i U^n_i$.

We allow rejections of customers (only) upon arrival and a rejected customer will never return to the system. The number of rejections from class $i$ until time $t$ is denoted by $R^n_i(t)$ and satisfies,

$$R^n_i(t) = \int_0^t z^n_i(s)dA^n_i(s), \quad t \in \mathbb{R}_+,$$

for some process $z^n_i$.

Denote by $X^n_i(t)$ the number of class $i$ customers in the system at time $t$. Then, the balance equation is given by

$$X^n_i(t) = X^n_i(0) + A^n_i(t) - S^n_i(T^n_i(t)) - R^n_i(t), \quad t \in \mathbb{R}_+. \tag{2.2}$$

For simplicity, we assume that $X^n_i(0)$ is deterministic. We use the notation $L^n = (L^n_i)_{i=1}^I$ for $L \in \{X,R,T\}$. Since $U^n$ is an RCLL process, and by construction, so are $A^n$ and $S^n$, we conclude that $X^n$ and $R^n$ are RCLL as well. The capacity of buffer $i$ is given by $\hat{b}_i^n := b_i n^{1/2}$ for some constant $\hat{b}_i \in (0, \infty)$, $i \in [I]$.

We now introduce the diffusion scaling and the heavy-traffic condition. First, we assume that

$$\lambda^n_i := \lambda_i n + \hat{\lambda}_i n^{1/2} + o(n^{1/2}), \quad \mu^n_i := \mu_i n + \hat{\mu}_i n^{1/2} + o(n^{1/2}), \tag{2.3}$$

for some fixed constants $\lambda_i, \mu_i \in (0, \infty)$ and $\hat{\lambda}_i, \hat{\mu}_i \in \mathbb{R}$. Moreover, the system is assumed to be critically loaded, that is, $\sum_{i=1}^I \rho_i = 1$, where $\rho_i := \lambda_i/\mu_i$, $i \in [I]$.

The process $(U^n, R^n)$ is regarded as a control in the $n$-th system and is now given rigorously.

**Definition 2.1 (admissible control for the decision maker, QCP)** An admissible control for the minimizer for any initial state $X^n(0)$ is a process $(U^n, R^n)$ taking values in $\mathbb{U} \times \mathbb{R}_+^I$ that satisfies the following,

(i) $(U^n, R^n)$ is adapted to the filtration $G^n_i = G^n(t) := \sigma\{A^n_i(s), S^n_i(T^n_i(s)), \quad i \in [I], s \leq t\}$ and has RCLL sample paths;

(ii) the processes $\{R^n\}_n$ are nondecreasing;

(iii) for each $i \in [I]$ and $t \in \mathbb{R}_+$,

$$X^n_i(t) = 0 \quad \text{implies} \quad U^n_i(t) = 0;$$

(iv) the buffer constraints $X^n_i(t) \in [0, \hat{b}_i^n], \quad t \in \mathbb{R}_+, \quad i \in [I]$, hold.

The first condition expresses the fact that the DM makes her decision based on past observations. The second condition follows since rejections are accumulated. The third condition asserts that service cannot be given to an empty buffer. We denote the set of admissible controls for the DM in the $n$-th system by $\mathcal{A}^n(X^n(0))$. 

6
2.2 The optimization problem with model uncertainty

Recall that we consider a DM that is uncertain about the underlying reference probability measure \( \mathbb{P}^n \), or in other words, she suspects that the rates/intensities \( \{\lambda_{j,i}^n\}_{i=1}^I \) and \( \{\mu_{j,i}^n\}_{i=1}^I \) may be unspecified or may even change over the time. Therefore, instead of optimizing under the reference measure \( \mathbb{P}^n \), she considers a set of candidate measures (provided in the sequel) and penalizes their deviation from \( \mathbb{P}^n \). The penalization is done by using a discounted variant of the Kullback–Leibler divergence. More explicitly, the QCP is set up as a stochastic game that models a type of worst case scenario. The players are: the DM that chooses a policy that minimizes a cost and a maximizing player also referred to as the nature or maximizer, who has access to the policy chosen by the minimizer and to the history. The nature is penalized for deviating from the reference model.

We are interested in a cost that accounts for the scaled queue lengths and rejections, in addition to the uncertainty about the model. Denote the scaled headcount process and the scaled rejection count by

\[
\hat{X}^n(t) := n^{-1/2}X^n(t) \quad \text{and} \quad \hat{R}^n(t) := n^{-1/2}R^n(t), \quad t \in \mathbb{R}_+.
\]

Fix a discount factor \( \rho > 0 \), vectors of holding and rejection costs \( \hat{h}, \hat{r} \in (0, \infty)^I \), and ambiguity parameters \( \kappa := (\kappa_{1,i}, \kappa_{2,i})_{i=1}^I \in (0, \infty)^{2I} \). The DM is facing the following robust optimization problem:

\[
J^n(X^n(0), U^n, R^n, \hat{Q}^n; \kappa),
\]

where

\[
J^n(X^n(0), U^n, R^n, \hat{Q}^n; \kappa) :=
\]

\[
\mathbb{E}^{\hat{Q}^n} \left[ \int_0^\infty e^{-\rho t} \left( \hat{h} \cdot \hat{X}^n(t) dt + \hat{r} \cdot \hat{R}^n(t) \right) \right] - \sum_{i=1}^I \frac{1}{\kappa_{1,i}} L_1^\rho(\hat{Q}^n_{1,i} \| \mathbb{P}^n_{1,i}) - \sum_{i=1}^I \frac{1}{\kappa_{2,i}} L_2^\rho(\hat{Q}^n_{2,i} \| \mathbb{P}^n_{2,i}),
\]

and

\[
L_1^\rho(\hat{Q}^n_{1,i} \| \mathbb{P}^n_{1,i}) := \mathbb{E}^{\hat{Q}^n_{1,i}} \left[ \int_0^\infty \rho e^{-\rho t} \log \frac{d\hat{Q}^n_{1,i}(t)}{d\mathbb{P}^n_{1,i}(t)} dt \right],
\]

\[
L_2^\rho(\hat{Q}^n_{2,i} \| \mathbb{P}^n_{2,i}) := \mathbb{E}^{\hat{Q}^n_{2,i}} \left[ \int_0^\infty \rho e^{-\rho t} \log \frac{d\hat{Q}^n_{2,i}(t)}{d\mathbb{P}^n_{2,i}(t)} dT^n(t) \right].
\]

The last two sums in (2.4) are referred to as the change of measure penalty. When \( \kappa_{j,i} \) is ‘small’ (resp., ‘big’) we say that there is a weak (resp., strong) ambiguity about the rates of the processes \( A^n_i \) and \( S^n_i(T^n_i) := S^n_i(T^n_i(\cdot)) \). The idea is that for small \( \kappa_{j,i} \)’s there is a big punishment per unit of deviation from the reference measure and therefore, the measures \( \hat{Q}^n_{j,i} \) and \( \mathbb{P}^n_{j,i} \) should be close to each other and as a consequence also the relevant expectations. However, one needs to make sure that the total punishment given by \( \frac{1}{\kappa_{j,i}} L_1^\rho(\hat{Q}^n_{j,i} \| \mathbb{P}^n_{j,i}) \) is also small. In [16] Theorems 5.1 and 5.2 we show that as the ambiguity parameters converge to zero, the stochastic differential games, which are provided in the same paper and are summarized

[16]
below in Section 3 converge to the risk-neutral BCP studied in [6]. Therefore, our problem indeed models ambiguity with respect to (w.r.t.) the risk-neutral model.

The set of candidate measures \( \hat{Q}^n(X^n(0)) \) consists of all the product measures \( \hat{Q}_n = \prod_{i=1}^I (\hat{Q}_{1,i} \times \hat{Q}_{2,i}) \) that for every \( i \in [I] \) and \( t \in \mathbb{R}_+ \) satisfy

\[
\begin{align*}
\frac{d\hat{Q}_1^n(t)}{d\hat{P}_1^n(t)} &= \exp \left\{ \int_0^t \log \left( \frac{\psi_{1,i}^n(s)}{\lambda_{i}^n} \right) dA_{i}^{n}(s) - \int_0^t (\psi_{1,i}^n(s) - \lambda_{i}^n) ds \right\}, \\
\frac{d\hat{Q}_2^n(t)}{d\hat{P}_2^n(t)} &= \exp \left\{ \int_0^t \log \left( \frac{\psi_{2,i}^n(s)}{\mu_{i}^n} \right) dS_{i}^{n}(T_{i}^{n}(s)) - \int_0^t (\psi_{2,i}^n(s) - \mu_{i}^n) dT_{i}^{n}(s) \right\},
\end{align*}
\]

for \( \{\Omega^n_t\} \)-predictable measurable and positive processes \( \psi_{j,i}^n, j \in \{1, 2\} \), satisfying \( \int_0^t \psi_{j,i}^n(s) ds < \infty \) \( \mathbb{P} \)-almost surely (a.s.), for every \( t \in \mathbb{R}_+ \). These conditions assure, first, that the right-hand sides in (2.6) are \( \mathbb{P}_{1,i} \)-martingales, \( j \in \{1, 2\} \) and second, that under the measure \( \hat{Q}_{1,i} \) (resp., \( \hat{Q}_{2,i} \)), \( A_{i}^{n} \) (resp., \( S_{i}^{n}(T_{i}^{n}) \)) is a counting process with intensity \( \psi_{1,i}^n \) (resp., \( \psi_{2,i}^{n} U_{i}^{n} \)). Notice also that under the measures \( \hat{Q}_{j,i} \), \( j \in \{1, 2\} \), the critically load condition might be violated since we do not restrict the intensities \( \{\psi_{j,i}^n\}_{j,i,n} \) in such a way. However, as we argue in (4.12), Proposition 4.2 and Section 4.3.1, such changes of measures are ‘too costly’ and will be avoided by the maximizer so that ‘in average’, \( \psi_{1,i}^n(t) = \lambda_{i}^n + \mathcal{O}(n^{1/2}) \) and \( \psi_{2,i}^n(t) = \mu_{i}^n + \mathcal{O}(n^{1/2}) \).

**Remark 2.1** (i) Notice that the right-hand side (r.h.s.) of (2.6) are martingales, and therefore, there exist probability measures \( \hat{Q}_{j,i}^n \), \( j \in \{1, 2\} \), such that \( \hat{Q}_{j,i}^n | \Omega^n_t \) satisfies (2.6) for all \( t \in \mathbb{R}_+ \), see [37, Lemma 4.2].

(ii) Notice that the QCP is a stochastic game that models a type of worst case scenario. The minimizer chooses a strategy and the maximizer, which is penalized for deviating from the reference measure, responses to this strategy by choosing a worst case scenario. For further reading about the structure of the information in control problems with model uncertainty, the reader is referred to [37].

## 3 The BCPs

We now present two BCPs that have the form of stochastic differential games that approximate the QCP and that were fully analyzed in [16]. One game is \( I \)-dimensional as the QCP, and the other is one-dimensional with the workload process as its underlying state. We also provide some of the results from [16] that are relevant to the present paper.

### 3.1 The multidimensional stochastic differential game (MSDG)

To derive the game we introduce some scaled processes and constants in addition to \( \hat{X}^n \) and \( \hat{R}^n \) introduced earlier. Set for every \( i \in [I] \) and \( t \in \mathbb{R}_+ \)

\[
\begin{align*}
\hat{A}_{i}^{n}(t) &:= n^{-1/2}(A_{i}^{n}(t) - \lambda_{i}^{n} t), \\
\hat{S}_{i}^{n}(t) &:= n^{-1/2}(S_{i}^{n}(t) - \mu_{i}^{n} t), \\
\hat{Y}_{i}^{n}(t) &:= \mu_{i}^{n} n^{-1/2}(\rho_{i} t - T_{i}^{n}(t)), \\
\hat{m}_{i}^{n} &:= n^{-1/2}(\lambda_{i}^{n} - \rho_{i} \mu_{i}^{n}).
\end{align*}
\]
We use the notation \( \hat{L}^n = (\hat{L}^n_t)_{t=1}^I \) for \( L \in \{X, R, A, S, Y, m\} \). The scaled version of (2.2) is given by,

\[
\hat{X}^n(t) = \hat{X}^n(0) + \hat{m}^n t + \hat{A}^n(t) - \hat{S}^n(t) + \hat{Y}^n(t) - \hat{R}^n(t), \quad t \in \mathbb{R}_+,
\]

An admissible policy satisfies

\[
\hat{X}^n(t) \in \mathcal{X} := \prod_{i=1}^I [0, \hat{b}_i], \quad t \in \mathbb{R}_+, \text{ } \mathbb{P}^n-\text{a.s.}, \quad (3.3)
\]

Under the reference measure \( \mathbb{P}^n \), \( \{(\hat{A}^n, \hat{S}^n)\}_n \) weakly converges a 2I-dimensional \((0, \hat{\sigma})\)-Brownian motion, where \( \hat{\sigma} := \text{Diag} (\lambda_1^{1/2}, \ldots, \lambda_I^{1/2}) \). As we show rigorously in the proof of Lemma 4.4, \( \hat{Y}^n \) is of order one as \( n \to \infty \). Hence its definition implies that \( T^n(t) \to (\rho_1, \ldots, \rho_I)t \), \( t \in \mathbb{R}_+ \), and therefore, under \( \mathbb{P}^n \), \( \{(\hat{A}^n_i - \hat{S}^n_i(t))_{i=1}^I\}_n \) weakly converges to an I-dimensional \((0, \sigma)\)-Brownian motion, where \( \hat{S}^n_i(t) := \hat{S}^n_i(T^n_i(\cdot)), i \in [I] \), and

\[
\hat{\sigma} = (\hat{\sigma}_{ij}) := \text{Diag}((2\lambda_1)^{1/2}, \ldots, (2\lambda_I)^{1/2}).
\]

Recall that in the QCP, an admissible control is of the form \((U^n, R^n)\). Notice that \((\hat{Y}^n(t), \hat{R}^n(t))\) is uniquely determined by \((U^n(s), R^n(s))_{0 \leq s \leq t}\). Hence, we consider an instantaneous control \((\hat{Y}, \hat{R})\).

**Definition 3.1 (admissible controls, MSDG)** An admissible control for the minimizer for any initial state \( \hat{x}_0 \in \mathcal{X} \) is a filtered probability space

\[
(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P}) := \left( \prod_{i=1}^I \Omega^i, \mathcal{F}^1 \otimes \cdots \otimes \mathcal{F}^I, \{\mathcal{F}_t\}, \prod_{i=1}^I \mathbb{P}_i \right),
\]

that supports a process \((\hat{Y}, \hat{R})\) taking values in \((\mathbb{R}_+)^2\) with RCLL sample paths adapted to the filtration \( \{\mathcal{F}_t\} \), where \((\Omega^i, \mathcal{F}^i, \{\mathcal{F}^i_t\}, \mathbb{P}_i) \) supports a one-dimensional standard Brownian motion \( \hat{B}_i \) adapted to the filtration \( \{\mathcal{F}^i_t\} \), \( i \in [I] \). Moreover, assume that the following properties hold:

(i) for every \( i \in [I] \) and \( 0 \leq s < t \), \( \hat{B}_i(t) - \hat{B}_i(s) \) is independent of \( \mathcal{F}^i_s \) under \( \mathbb{P}_i \);

(ii) \( \theta \cdot \hat{Y} \) and \( \hat{R}_i \), \( i \in [I] \) are nonnegative and nondecreasing, where

\[
\theta := (\mu_1^{-1}, \ldots, \mu_I^{-1});
\]

(iii) The controlled process satisfies \( \hat{X}(t) = \hat{x}_0 + \hat{m}t + \hat{\sigma}\hat{B}(t) + \hat{Y}(t) - \hat{R}(t) \) and \( \hat{X}(t) \in \mathcal{X}, t \in \mathbb{R}_+, \mathbb{P}-\text{a.s.} \), where \( \hat{m} := \lim_n \hat{m}^n = \hat{\lambda} - \rho_i \hat{\mu}_i \) and \( \hat{B} = (\hat{B}_i)_{i=1}^I \).

An admissible control for the maximizer is a product measure \( \hat{Q} = \prod_{i=1}^I \hat{Q}_i \), where each \( \hat{Q}_i \) is defined on \((\Omega^i, \mathcal{F}^i, \{\mathcal{F}^i_t\})\), such that

\[
\frac{d\hat{Q}_i(t)}{d\mathbb{P}_i(t)} = \exp \left\{ \int_0^t \hat{\psi}_i(s)d\hat{B}_i(s) - \frac{1}{2} \int_0^t \hat{\psi}_i^2(s)ds \right\}, \quad t \in \mathbb{R}_+,
\]

for an \( \{\mathcal{F}_t\} \)-progressively measurable process \( \hat{\psi} = (\hat{\psi}_1, \ldots, \hat{\psi}_I) \) satisfying

\[
\mathbb{E}^\mathbb{P}\left[ \int_0^\infty e^{-\alpha s} \hat{\psi}_i^2(s)ds \right] < \infty \quad \text{and} \quad \mathbb{E}^\mathbb{P}\left[ e^{\frac{1}{2} \int_0^t \hat{\psi}_i^2(s)ds} \right] < \infty, \quad t \in \mathbb{R}_+, i \in [I].
\]

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The motivation for Condition (ii) is that the rejection process (in the QCP) and also $\theta^n \cdot \hat{Y}$ are nondecreasing, where
\[ \theta^n := (1/\mu^n_1, \ldots, 1/\mu^n_I). \]

Denote by $\hat{A}(\hat{x}_0)$ and (resp., $\hat{Q}(\hat{x}_0)$) the set of all admissible controls for the minimizer (resp., maximizer), given the initial condition $\hat{x}_0$.

In [16, (2.17)] it is argued that the controlled process can alternatively be written as
\[ \hat{X}(t) = \hat{x}_0 + \hat{m}t + \int_0^t \hat{\sigma} \hat{\psi}(s) ds + \hat{\sigma} \hat{B}(t) + \hat{Y}(t) - \hat{R}(t), \quad t \in \mathbb{R}_+, \tag{3.6} \]
where $\hat{B}(t) := \hat{B}(t) - \int_0^t \hat{\psi}(s) ds$, $t \in \mathbb{R}_+$, is an $\{F_t\}$-dimensional standard Brownian motion under $\hat{Q}$.

Pay attention that the difference between the processes $\hat{A}_n$ and $\hat{S}_n(T^n)$ approximately behaves like Brownian motion (without drift under $P_i$ and with drift under $\hat{Q}_i$). Hence, we consider only $I$ changes of measures instead of $2I$. For this, in the cost function we consider new ambiguity parameters,
\[ \epsilon_i := \frac{1}{2} (\kappa_{1,i} + \kappa_{2,i}), \quad i \in [I]. \tag{3.7} \]

The intuition behind this form is given in [16, p. 10] and is rigorously justified in (4.88). Set $\epsilon = (\epsilon_i)_i$. The cost associated with the initial condition $\hat{x}_0$ and the strategies $(\hat{Y}, \hat{R})$ and $\hat{Q}$ is given by
\[ \hat{J}(\hat{x}_0, \hat{Y}, \hat{R}, \hat{Q}; \epsilon) := \mathbb{E}^{\hat{Q}} \left[ \int_0^\infty e^{-\hat{\rho}t} \left( \hat{h} \cdot \hat{X}(t) dt + \hat{r} \cdot d\hat{R}(t) \right) \right] - \sum_{i=1}^I \frac{1}{\epsilon_i} L^\theta(\hat{Q}_i||P_i), \tag{3.8} \]
where
\[ L^\theta(\hat{Q}_i||P_i) := \mathbb{E}^{\hat{Q}_i} \left[ \int_0^\infty \hat{\rho} e^{-\hat{\rho}t} \log \frac{d\hat{Q}_i(t)}{dP_i(t)} dt \right]. \tag{3.9} \]

The DM faces the following robust optimization problem
\[ \hat{V}(\hat{x}_0; \epsilon) = \inf_{(\hat{Y}, \hat{R}) \in \hat{A}(\hat{x}_0)} \sup_{\hat{Q} \in \hat{Q}(\hat{x}_0)} \hat{J}(\hat{x}_0, \hat{Y}, \hat{R}, \hat{Q}; \epsilon) \]

### 3.2 The reduced stochastic differential game (RSDG)

The $I$-dimensional game can be reduced to a game with one-dimensional dynamics by projecting the controlled process onto the workload vector $\theta$. This game is referred to as the reduced stochastic differential game (RSDG).

To introduce the game, we need the following notation,
\[ x_0 := \theta \cdot \hat{x}_0, \quad m := \theta \cdot \hat{m}, \quad \sigma := \|\theta \hat{\sigma}\|, \quad \epsilon := \frac{1}{\sigma^2} \sum_{i=1}^I (\theta \hat{\sigma})^2_i \epsilon_i, \quad b := \theta \cdot \hat{b}, \tag{3.10} \]
where $\hat{b} = (\hat{b}_1, \ldots, \hat{b}_I)$. 

In Section 4.1 asymptotically satisfies this condition. In [16, (2.23)] it is argued that rejections are performed only from this class. The candidate asymptotically nearly optimal service rate. In fact, as was shown in [16, Theorem 4.1], in an equilibrium of the MSDG, the index $i$ below.

This form will turn out to be useful in the approximation procedure, see (4.35) and (4.89) below.

Definition 3.2 (admissible controls, RSDG) An admissible control for the minimizer for any initial state $x_0 \in [0, b]$ is a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ that supports a one-dimensional standard Brownian motion $B$ and a process $(Y, R)$ taking values in $\mathbb{R}^2_+$ with RCLL sample paths, both adapted to the filtration $\{\mathcal{F}_t\}$ and satisfy the following properties:

(i) for every $0 \leq s < t$, $B(t) - B(s)$ is independent of $\mathcal{F}_s$ under $\mathbb{P}$;
(ii) $Y$ and $R$ are nonnegative and nondecreasing;
(iii) The controlled process satisfies $X(t) = x_0 + mt + \sigma B(t) + Y(t) - R(t)$, and $X(t) \in [0, b]$, $t \in \mathbb{R}_+$, $\mathbb{P}$-a.s.

An admissible control for the maximizer is a measure $\mathbb{Q}$ defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$ such that

$$\frac{d\mathbb{Q}(t)}{d\mathbb{P}(t)} = \exp\left\{\int_0^t \psi(s) dB(s) - \frac{1}{2} \int_0^t \psi^2(s) ds\right\}, \quad t \in \mathbb{R}_+,$$

for an $\{\mathcal{F}_t\}$-progressively measurable process $\psi$ satisfying

$$\mathbb{E}^\mathbb{P}\left[\int_0^\infty e^{-\theta s} \psi^2(s) ds\right] < \infty \quad \text{and} \quad \mathbb{E}^\mathbb{P}\left[e^{\frac{1}{2} \int_0^t \psi^2(s) ds}\right] < \infty \quad \text{for every } t \in \mathbb{R}_+. \quad (3.12)$$

Denote by $\mathcal{A}(x_0)$ (resp., $\mathcal{Q}(x_0)$) the set of all admissible controls for the minimizer (resp., maximizer), given the initial condition $x_0$. The cost associated with the initial condition $x_0$ and the controls $(Y, R)$ and $\mathbb{Q}$ is given by

$$J(x_0, Y, R, \mathbb{Q}; \varepsilon) := \mathbb{E}^\mathbb{Q}\left[\int_0^\infty e^{-\theta t} (h(X(t)) dt + rdR(t))\right] - \frac{1}{\varepsilon} L^0(\mathbb{Q}||\mathbb{P}),$$

where

$$h(x) := \min\{\hat{h} \cdot \xi : \xi \in \mathcal{X}, \theta \cdot \xi = x\}, \quad (3.13)$$

$$r := \min\{\hat{r} \cdot q : q \in \mathbb{R}^I_+, \theta \cdot q = 1\}, \quad (3.14)$$

and $L^0(\mathbb{Q}||\mathbb{P})$ is given by (3.9) with $(\mathcal{Q}_i, \mathbb{P}_i)$ replacing $(\mathcal{Q}_i, \mathbb{P}_i)$. By the convexity of $\mathcal{X}$ it follows that $h$ is convex. In fact, $h$ is piecewise linear and Lipschitz continuous. Moreover, $h(x) \geq 0$ for $x \geq 0$ and equality holds if and only if $x = 0$. Therefore, $h$ is strictly increasing. In [6, page 568] it is shown that there is $i^* \in [I]$ such that

$$r = r_{i^*} \mu_{i^*} := \min\{\hat{r}_i \mu_i : i \in [I]\}. \quad (3.15)$$

The index $i^*$ stands for the class with the smallest rejection cost, weighted with the mean service rate. In fact, as was shown in [16, Theorem 4.1], in an equilibrium of the MSDG, rejections are performed only from this class. The candidate asymptotically nearly optimal policy in Section 4.1 asymptotically satisfies this condition. In [16, (2.23)] it is argued that

$$L^0(\mathbb{Q}||\mathbb{P}) = \mathbb{E}^\mathbb{Q}\left[\int_0^\infty e^{-\theta t} \psi^2(t) dt\right]. \quad (3.16)$$

This form will turn out to be useful in the approximation procedure, see (4.35) and (4.89) below.

The value function is given by

$$V(x_0; \varepsilon) = \inf_{(Y, R) \in \mathcal{A}(x_0)} \sup_{\mathbb{Q} \in \mathcal{Q}(x_0)} J(x_0, Y, R, \mathbb{Q}; \varepsilon). \quad (3.17)$$
3.3 Properties of the games

The RSDG admits a simple optimal strategy for the minimizer that enforces the workload to stay in a specific interval of the form $[0, \beta]$ with minimal effort. To rigorously define such a strategy we make use of the Skorokhod map on an interval. Fix $\beta > 0$. For any $\eta \in D(\mathbb{R}_+, \mathbb{R})$ there exists a unique triplet of functions $(\chi, \zeta_1, \zeta_2) \in D(\mathbb{R}_+, \mathbb{R}^3)$ that satisfies the following properties:

(i) for every $t \in \mathbb{R}_+$, $\chi(t) = \eta(t) + \zeta_1(t) - \zeta_2(t)$;
(ii) $\zeta_1$ and $\zeta_2$ are nondecreasing, $\zeta_1(0-) = \zeta_2(0-) = 0$, and

$$\int_0^\infty 1_{[0, \beta]}(\chi(t))d\zeta_1(t) = \int_0^\infty 1_{[0, \beta]}(\chi(t))d\zeta_2(t) = 0,$$

where $1_F(x) = 1$ if $x$ belongs to the set $F$ and 0 otherwise. We denote by $\Gamma_{[0, \beta]}(\eta) = (\Gamma_{[0, \beta]}^1, \Gamma_{[0, \beta]}^2, \Gamma_{[0, \beta]}^3)(\eta) = (\chi, \zeta_1, \zeta_2)$. See [25] for existence and uniqueness of the solution, and continuity and further properties of the map. In particular, we have the following.

**Lemma 3.1** There exists a constant $c_S > 0$ such that for every $t > 0$, $\beta, \tilde{\beta} > 0$ and $\omega, \tilde{\omega} \in D(\mathbb{R}_+, \mathbb{R})$,

$$\|\Gamma_{[0, \beta]}(\omega)(t) - \Gamma_{[0, \beta]}(\tilde{\omega})(t)\|_t \leq c_S(\|\omega - \tilde{\omega}\|_T + |\beta - \tilde{\beta}|).$$

**Definition 3.3** Fix $x_0, \beta \in [0, b]$. The strategy $(Y, R)$ is called a $\beta$-reflecting strategy if for every $\eta \in C(\mathbb{R}_+, \mathbb{R})$ one has $(X, Y, R)(\eta) = \Gamma_{[0, \beta]}(\eta)$, with $X(0) = x_0$.

The next proposition provides a characterization of the value function of the RSDG and an equilibrium.

**Proposition 3.1** (Theorems 3.1 and 4.1 in [16]) Fix $\varepsilon \in (0, \infty)$. The value function $V(\cdot; \varepsilon)$ is the unique $C^2([0, b], \mathbb{R})$ solution of

$$\begin{cases}
[f''(x) + H(x, f(x), f'(x))] \land f'(x) \land [r - f'(x)] = 0, & x \in (0, b), \\
f'(0) = 0, & f'(b) = r,
\end{cases}
$$

(3.18)

where

$$H(x, y, z) := \frac{2}{\sigma^2} \left( mz + \frac{1}{2} \sigma^2 \varepsilon z^2 - gy + h(x) \right).$$

Moreover, set

$$\beta_\varepsilon = \inf \{ x \in (0, b) : V'(x; \varepsilon) = r \},
$$

(3.19)

where $V'(x; \varepsilon)$ is the derivative of $x \mapsto V(x; \varepsilon)$. Then the $\beta_\varepsilon$-reflecting strategy is admissible and optimal for the minimizer and $V(\cdot) = V(\cdot; \varepsilon)$ satisfies,

$$\begin{cases}
V''(x) + H(x, V(x), V'(x)) = 0, & 0 \leq x \leq \beta_\varepsilon, \\
r - V'(x) = 0, & \beta_\varepsilon \leq x \leq b, \\
V'(0) = 0.
\end{cases}
$$

(3.20)

Moreover, let $Q_{V} = Q_{V(\cdot; \varepsilon)}$ be the measure driven by $\psi_V(t) := \varepsilon V'(X(t); \varepsilon)$. Then, the $\beta_\varepsilon$ strategy and the measure $Q_V$ form an equilibrium. That is,

$$V(x_0; \varepsilon) = \sup_{Q \in \mathcal{Q}(x_0)} J(x_0, Y_{\beta_\varepsilon}, R_{\beta_\varepsilon}, Q; \varepsilon) = J(x_0, Y_{\beta_\varepsilon}, R_{\beta_\varepsilon}, Q_{V}; \varepsilon) = \inf_{(Y, R) \in \mathcal{A}(x_0)} J(x_0, Y, R, Q_V; \varepsilon).$$
From the definition of (3.18) and the proposition above we get the following corollary, which is given for reference purposes.

**Corollary 3.1** For any \( \varepsilon \in (0, \infty) \), \( 0 \leq V'(\cdot; \varepsilon) \leq r \).

The strategy of the maximizer in the RSDG under equilibrium plays an important role in the asymptotics given in Section 4.2.

**Equilibrium in the MSDG.** Given an equilibrium in the RSDG, one can construct an equilibrium in the MSDG. This construction is summarized now. Without loss of generality assume that

\[
    h_1 \mu_1 \geq h_2 \mu_2 \geq \cdots \geq h_I \mu_I. \tag{3.21}
\]

Recall that \( b := \theta \cdot (\hat{b}_1, \ldots, \hat{b}_I) \). Given \( x \in [0, b] \), let \((j, v)(x)\) be the unique pair that is determined by

\[
x = \sum_{i=j+1}^{I} \theta_i \hat{b}_i + \theta_j v, \quad j \in [I], \quad v \in [0, \hat{b}_j),
\]

and for \( x = b \), take \((j, v)(b) = (1, \hat{b}_1)\). Let \( \gamma : [0, b] \to \mathcal{X} \) be the function given by

\[
    \gamma(x) = \sum_{i=j+1}^{I} \hat{b}_i e_i + ve_j, \tag{3.22}
\]

where \( \{e_1, \ldots, e_I\} \) is the standard basis of \( \mathbb{R}^I \). The curve \( \gamma(x), x \in [0, b] \) is continuous and located on the edges of \( \mathcal{X} \), see Figure 1. The idea is as follows, recall that the components of \( \check{Y} = (\check{Y}_i)_{i=1}^I \) can be positive or negative, as long as \( \theta \cdot \check{Y} \) is nonnegative and nondecreasing. Now, as the workload changes in the interval \([0, b)\), the minimizer can use only the process \( \check{Y} \), without the need of the process \( \hat{R} \), so that \( \check{X} \) moves along the curve of \( \gamma \). As we argue now, the process \( \check{R} \) should be active only when the workload exceeds the level \( \beta_\varepsilon \) and only through the coordinate with the cheapest rejection cost, weighted with the service rate. More explicitly, let \((Y_{\beta_\varepsilon}, R_{\beta_\varepsilon})\) be an optimal reflecting strategy for the minimizer. That is,

\[
    (X_{\beta_\varepsilon}, Y_{\beta_\varepsilon}, R_{\beta_\varepsilon})(t) = \Gamma_{[0, \beta_\varepsilon]}(x_0 + m \cdot + \sigma B(\cdot))(t) \quad t \in \mathbb{R}_+.
\]

Also, let \( Q_V = Q_V(\cdot; \varepsilon) \) be the measure driven by \( \psi_V = \psi_V(\cdot; \varepsilon) \) (see Proposition 3.1). Set the strategy \((\check{Y}_{\beta_\varepsilon}, \check{R}_{\beta_\varepsilon})\) by

\[
    \check{R}_{\beta_\varepsilon}(t) := R_{\beta_\varepsilon}(t) \mu_{i^*} e_{i^*} \quad \text{and} \quad \check{Y}_{\beta_\varepsilon}(t) := X_{\beta_\varepsilon}(t) - \check{x}_0 - \check{m}t - \check{\sigma} B(t) + \check{R}_{\beta_\varepsilon}(t), \quad t \in \mathbb{R}_+,
\]

where \( i^* \) is given immediately after (3.15) and for any \( t \in \mathbb{R}_+ \),

\[
    \check{X}_{\beta_\varepsilon}(t) := \gamma(X_{\beta_\varepsilon}(t)).
\]

Moreover, let \( \hat{Q}_V \) be the measure associated with \( \hat{\psi}_V(t) = (\hat{\psi}_{V,1}(t), \ldots, \hat{\psi}_{V,I}(t)) \), given by

\[
    \hat{\psi}_{V,i}(t) := \frac{\sigma \psi_V(t)(\theta \hat{\sigma}) e_i}{\sum_{j=1}^{I} (\theta \hat{\sigma})^2 \hat{\psi}_j}, \quad i \in [I], \quad t \in \mathbb{R}_+. \tag{3.23}
\]
Figure 1: The graphs refer to the case $I = 3$, $\hat{h} = (1.5/2, 3/2)$, $\mu = (3, 1, 3/2)$, and $(\hat{b}_1, \hat{b}_2, \hat{b}_3) = (4, 7, 6)$. The graph to the left stands for the workload levels. The curve of the function $\gamma$ is in bold in the graph to the right. The workload levels with the lower case letters are $j = 0$, $k = \hat{b}_3/\mu_3 = 4$, $l = \hat{b}_3/\mu_3 + \hat{b}_2/\mu_2 + 1$, and $n = \hat{b}_3/\mu_3 + \hat{b}_2/\mu_2 + \hat{b}_1/\mu_1 = b = 37/3$. They respectively correspond to the upper case letters: $J = (0, 0, 0)$, $K = (0, 0, \hat{b}_3/\mu_3) = (0, 0, 4)$, $L = (0, \hat{b}_2/\mu_2, \hat{b}_3/\mu_3) = (0, 7, 4)$, $M = (1, \hat{b}_2/\mu_2, \hat{b}_3/\mu_3) = (1, 7, 4)$, and $N = (\hat{b}_1/\mu_1, \hat{b}_2/\mu_2, \hat{b}_3/\mu_3) = (4/3, 7, 4)$. If for example $m = \beta_e$, then the process $\bar{X}_{\beta_e}$ moves continuously along the bold curve between the points $J$ and $M$, where it is reflected at these two points.

**Proposition 3.2 (Theorem 4.1 in [16])** The value functions of both games coincide and moreover, the strategies $(\hat{Y}_{\beta_e}, \hat{R}_{\beta_e})$ and $\hat{Q}_V$ form an equilibrium in the MSDG.

The construction of the nearly optimal policy in the QCP relies on the equilibrium strategy of the minimizer in the MSDG we just described.

4 Asymptotics

4.1 Nearly optimal policy

As mentioned in the previous section, the policy relies on the function $\gamma$ that maps workloads to points on a minimizing curve on the set $\mathcal{X}$. The pre-limit process is discrete and therefore, we use a neighborhood of the minimizing curve. We show that with high probability it is possible to ‘almost’ trace the minimizing curve without rejections, unless the workload level $\beta_e$ is reached in which case rejections occur, and only from the buffer with the cheapest rejection cost.

Again, without loss of generality, we assume that the classes are labeled in such a way that (3.21) holds. Recall the definition of the cut-off parameter $\beta_e$. Fix $\delta_0 > 0$ and let $\hat{a} = (\hat{a}_1, \ldots, \hat{a}_I)$ be given by $\hat{a}_i = \hat{b}_i - \delta_0$, $i \in [I]$, and $a = \beta_e \land (\theta \cdot \hat{a}) < b$. Note that if $\delta_0$ is sufficiently small then $a = \beta_e$ (unless $\beta_e = b$). Set the function $\gamma^a : [0, b] \to \mathcal{X}$ as follows. For $x \in [0, \theta \cdot \hat{a})$, the variables $j = j(x)$ and $v = v(x)$ are determined via

$$
  x = \sum_{i=j+1}^I \theta_i \hat{a}_i + \theta_j \xi, \quad j \in [I], \quad v \in [0, \hat{a}_j),
$$

(4.1)
and let
\[ \gamma^a(x) = \sum_{i=j+1}^{I} \hat{a}_i e_i + v e_j. \]

On the interval \([\theta \cdot \hat{a}, b]\) define \(\gamma^a\) as the linear interpolation between the points \((\theta \cdot \hat{a}, \hat{a})\) and \((b, \hat{b})\).

Recall the definition of \(h\) from (3.13) and let
\[ h^a(x) := \min\{\hat{h} \cdot \xi : \xi \in \mathcal{X}, \theta \cdot \xi = x, \xi_i \leq \gamma^a_i(x), i \in [I]\} = \hat{h} \cdot \gamma^a(x), \quad x \in [0, \theta \cdot a]. \]

Also, set
\[ \omega_1(\delta_0) = \sup_{[0, \theta \cdot \hat{a}]} |h^a - h|. \quad (4.2) \]

By the choice of \(a\) it is clear that \(\omega_1(0+) = 0\).

**Rejection policy:** In case that a class-\(i\) arrival occurs at a time \(t\) when \(\hat{X}^n(t-)+n^{-1/2} > \hat{b}_i\), then it is rejected. Such rejections are called *forced rejections*. Whenever \(\theta \cdot X^n \geq a\), all class-\(i^*\) (see the paragraph preceding (3.15)) arrivals are rejected, and these rejections are called *overload rejections*. Apart from that, no rejections occur from any class.

**Service policy:** For each \(\hat{x} = (\hat{x}_1, \ldots, \hat{x}_I) \in \mathcal{X}\) define the class of low priority
\[ L(\hat{x}) = \max\{i : \hat{x}_i < a_i\}, \]
provided \(\hat{x}_i < \hat{a}_i\) for some \(i\), and set \(L(\hat{x}) = I\) otherwise. The complement set is the set of high priority classes:
\[ H(\hat{x}) = [I] \setminus L(\hat{x}). \]

When at least one class among \(H(\hat{x})\) is not empty, the class \(L(\hat{x})\) receives no service, and all classes within \(H(\hat{x})\) that are not empty receive service at a fraction proportional to their traffic intensities. Namely, denote \(H^+(\hat{x}) = \{i \in H(x) : x_i > 0\}\), and define \(\rho'(\hat{x}) \in \mathbb{R}^I\) as
\[ \rho'_i(\hat{x}) = \begin{cases} 0, & \text{if } \hat{x} = 0, \\ \frac{\rho_i^1}{\sum_{k \in H^+(\hat{x})} \rho_k}, & \text{if } H^+(\hat{x}) \neq \emptyset, \\ e_I, & \text{if } \hat{x}_i = 0 \text{ for all } i < I \text{ and } \hat{x}_I > 0, \end{cases} \quad (4.3) \]
where recall that \(e_I = (0, \ldots, 0, 1) \in \mathbb{R}^I\). Note that \(H^+(\hat{x}) = \emptyset\) can only happen if \(\hat{x}_i = 0\) for all \(i < I\), which is covered by the first and last cases in the above display. Then for each \(t \in \mathbb{R}\),
\[ U^n(t) = \rho'(\hat{X}^n(t)). \quad (4.4) \]

Note that when \(H^+(\hat{x}) \neq \emptyset\),
\[ \rho'_i(\hat{x}) > \rho_i \quad \text{for all } i \in H^+(\hat{x}). \quad (4.5) \]

That is, all prioritized classes receive a fraction of effort strictly greater than the respective traffic intensity. Also note that \(\sum_i U^n_i = 1\) whenever \(\hat{X}^n\) is nonzero. This is therefore a work conserving policy.
Theorem 4.1 Assuming that \( x_0 := \lim_{n \to \infty} \theta^n \cdot \hat{X}^n(0) \) exists, then,
\[
\lim_{n \to \infty} V^n(X^n(0); \kappa) = V(x_0; \varepsilon). \tag{4.6}
\]
Moreover, for every \( n \in \mathbb{N} \), denote the policy constructed above by \((U^n(a), R^n(a))\). Then,
\[
\limsup_{n \to \infty} \sup_{\hat{\mathcal{Q}}^n} J^n(X^n(0), U^n(a), R^n(a), \hat{\mathcal{Q}}^n; \kappa) \leq V(x_0; \varepsilon) + \omega_2(\beta, -a), \tag{4.7}
\]
where \( \omega_2(0+) = 0 \).

By a diagonalization argument, one can deduce an asymptotically optimal policy generated from \( U^n(a) \). The proof of the theorem takes place in the next two sections. In Section 4.2 we show that the game’s value function \( V \) bounds from below the liminf of the QCP’s value functions. That is,
\[
\lim_{n \to \infty} V^n(X^n(0); \kappa) \geq V(x_0; \varepsilon). \tag{4.8}
\]
In Section 4.3 we prove (4.7). Together, we obtain (4.6).

4.2 Proof of (4.8)

For every \( t \in \mathbb{R}_+ \) and \( i \in [I] \), set
\[
\hat{\psi}^n(t) := \frac{(\theta \hat{\sigma})_i e_i \sigma^2 \varepsilon}{\sum_{j=1}^I (\theta^n \hat{\sigma})_j^2 e_j} V'(\theta \cdot \hat{X}^n(t -); \varepsilon) \tag{4.9}
\]
\[
\hat{\psi}^n_{1,i}(t) := \frac{\kappa_{1,i} \sqrt{2}}{\kappa_{1,i} + \kappa_{2,i}} \hat{\psi}^n(t), \quad \hat{\psi}^n_{2,i}(t) := -\frac{\kappa_{2,i} \sqrt{2}}{(\kappa_{1,i} + \kappa_{2,i}) \rho_i^{1/2}} \hat{\psi}^n(t), \tag{4.10}
\]
and also
\[
\psi^*_n(t) := \lambda^*_i + \hat{\psi}^n_{1,i}(t)(\lambda_i n)^{1/2} \quad \text{and} \quad \psi^*_n(t) := \mu^*_i + \hat{\psi}^n_{2,i}(t)(\mu_i n)^{1/2}. \tag{4.11}
\]

Also, let \( \{\hat{\mathcal{Q}}^n_{j,i}\}_{j,i,n} \) be the relevant measures defined as in (2.6). Notice that from Corollary 3.1 it follows that all the processes mentioned in (4.9)–(4.10) are uniformly bounded by some constant. Namely, there exists a constant \( C_0 > 0 \) such that for every \( j \in \{1, 2\}, i \in [I], n \in \mathbb{N}, \) and \( t \in \mathbb{R}_+ \),
\[
|\hat{\psi}^n_{j,i}(t)| < C_0. \tag{4.12}
\]

We now simplify the change of measure penalty from (2.5). Since \( A^n_i(\cdot) - \int_0^\cdot \hat{\psi}^n_{i,j}(s) ds \) is a
where under the measure \( \hat{Q}^n_{1,i} \), we get the following sequence of equations:

\[
L_1^\theta(\hat{Q}^n_{1,i} || P^n_{1,i}) \\
= E^{\hat{Q}^n_{1,i}} \left[ \int_0^\infty \rho e^{-\theta t} \left( \int_0^t \log \left( \frac{\psi_{1,i}(s)}{\lambda_i^n} \right) dA_i^n(s) - \int_0^t (\psi_{1,i}(s) - \lambda_i^n) ds \right) dt \right] \\
= E^{\hat{Q}^n_{1,i}} \left[ \int_0^\infty \rho e^{-\theta t} \left( \int_0^t \log \left( \frac{\psi_{1,i}(s)}{\lambda_i^n} \right) (dA_i^n(s) - \psi_{1,i}(s) ds) \right. \right. \\
\left. \left. \quad \quad \quad \quad \quad + \int_0^t \left\{ \psi_{1,i}(s) \log \left( \frac{\psi_{1,i}(s)}{\lambda_i^n} \right) - \psi_{1,i}(s) + \lambda_i^n \right\} ds \right\} dt \right] \\
= E^{\hat{Q}^n_{1,i}} \left[ \int_0^\infty \rho e^{-\theta t} \left\{ \psi_{1,i}(t) \log \left( \frac{\psi_{1,i}(t)}{\lambda_i^n} \right) - \psi_{1,i}(t) + \lambda_i^n \right\} dt \right],
\]

where the last equality follows by changing the order of integration. Similar calculations apply to the change of measure penalty associated with the service time. Set \( y^n = \hat{\psi}_{1,i}(t) (\lambda_i^n)^{1/2} / \lambda_i^n \). Then, \( |y^n| \leq C_1 n^{-1/2} + o(n^{-1/2}) \). Noticing that \( |(1 + y^n) \log(1 + y^n) - y^n| \) is uniformly bounded over \( n \) and recalling (4.11), we get that there exists a constant \( C_2 > 0 \), independent of \( n \) and \( t \), such that,

\[
\sum_{i=1}^I 1 \frac{L_1^\theta(\hat{Q}^n_{1,i} || P^n_{1,i})}{\kappa_{1,i}} + \sum_{i=1}^I 1 \frac{L_2^\theta(\hat{Q}^n_{2,i} || P^n_{2,i})}{\kappa_{2,i}} \leq C_2.
\]

Fix an arbitrary sequence of controls \( \{\hat{Y}^n, \hat{R}^n\} \). Without loss of generality, we may assume that for every \( n \in \mathbb{N} \),

\[
J^n(X^n(0), U^n, \hat{R}^n, \hat{Q}^n; \kappa) < V(x_0; \varepsilon) + 1. \tag{4.14}
\]

Therefore, for every \( t \in \mathbb{R}_+ \),

\[
e^{-\theta t} E^{Q^n} \left[ \hat{r} \cdot \hat{R}^n \right] \leq E \left[ \int_0^t e^{-\theta s} \hat{r} \cdot d\hat{R}^n(s) \right] < V(x_0; \varepsilon) + C_2 + 1. \tag{4.15}
\]

This property serves us in Lemma 4.1 below when we claim tightness of a time-rescaled version of \( \hat{R}^n \).

Our goal here is to obtain the cost associated with the maximizer’s equilibrium measure in the RSDG. Recall that the intensity of \( S^n_i(T^n_i) \) is \( \psi^n_{2,i} U^n_i \) and that \( dT^n_i(s) = U^n_i(s) ds \). Then, under the measure \( \hat{Q}^n \), the dynamics of \( \hat{X}^n \) satisfy

\[
\hat{X}^n_i(t) = \hat{X}^n_i(0) + \hat{m}^n_i t + \hat{A}^n_i(t) \hat{D}^n_i(t) + \hat{Y}^n_i(t) - \hat{R}^n_i(t) \\
+ \lambda_i^{1/2} \int_0^t \hat{\psi}_{1,i}(s) ds - \mu_i^{1/2} \int_0^t \hat{\psi}_{2,i}(s) dT^n_i(s),
\]

where

\[
\hat{A}^n_i(t) := n^{-1/2} \left( A^n_i(t) - \int_0^t \hat{\psi}_{1,i}(s) ds \right) \tag{4.17}
\]
are \( G^n_i \)-martingales under \( \hat{Q}^n \), where recall that \( G^n_i \) is given in Definition 2.1.(i). The latter claim follows by standard martingale techniques, see e.g., the arguments given in the proof of Theorem 3.4 in [28]. Set \( \hat{A}^n = (\hat{A}_1^n, \ldots, \hat{A}_I^n) \) and similarly \( \hat{D}^n = (\hat{D}_1^n, \ldots, \hat{D}_I^n) \). Recalling (3.6) and (4.9)–(4.10), we expect to obtain a limiting process of the form

\[
\text{Theorem 3.4 in [28]. Set } \hat{\sigma} = \hat{A} + \frac{1}{2} \hat{D}
\]

and

\[
\hat{D}_i^n(t) := n^{-1/2} \left( S^n_i(T^n_i(t)) - \int_0^t \psi^n_{2,i}(s)dT^n_i(s) \right) \quad (4.18)
\]

are \( G^n_i \)-martingales under \( \hat{Q}^n \), where recall that \( G^n_i \) is given in Definition 2.1.(i). The latter claim follows by standard martingale techniques, see e.g., the arguments given in the proof of Theorem 3.4 in [28]. Set \( \hat{A}^n = (\hat{A}_1^n, \ldots, \hat{A}_I^n) \) and similarly \( \hat{D}^n = (\hat{D}_1^n, \ldots, \hat{D}_I^n) \). Recalling (3.6) and (4.9)–(4.10), we expect to obtain a limiting process of the form

\[
\hat{X}_i(t) = \hat{X}_i(0) + \hat{m}_i t + \hat{\sigma}_i (\theta \hat{\sigma})_i \epsilon_i \int_0^t V'(\theta \cdot \hat{X}(s); \epsilon) ds + \hat{\sigma}_i \hat{D}^{Q_V}_i(t) + \hat{Y}_i(t) - \hat{R}_i(t), \quad t \in \mathbb{R}_+,
\]

(4.19)

where \( Q_V \) is the measure associated with \( \hat{\psi}_V \) given in (3.23) and \( \hat{D}^{Q_V} = (\hat{D}_1^{Q_V}, \ldots, \hat{D}_I^{Q_V}) \) is an \( I \)-dimensional standard Brownian motion under the measure \( \hat{Q}_V \).

Since \( \{\hat{Y}^n, \hat{R}^n\}_n \) is an arbitrary sequence of singular controls (up to the restriction in (4.14)), one cannot expect this sequence to be tight. Therefore, as mentioned in the introduction, we use time stretching in order to prove the lower bound. At this point, it is worth mentioning that Atar and Shifrin [6] managed to bypass this issue by arguing \( C \)-tightness of the integrands of the relevant processes. Repeating the same arguments, one may show that the integrands of \( \hat{X}_i^n, \hat{Y}_i^n \), and \( \hat{R}_i^n \) are \( C \)-tight. Since \( V' \) is bounded, see Corollary 3.1 we get that the sequence \( \{ \int_0^t (\hat{\psi}_1^n(s) - \hat{\psi}_{2,i}^n(s)) ds \}_n \) is \( C \)-tight. However, since we wish to obtain the specific integral given in (4.19), we still need to argue tightness of \( \hat{X}^n \). Therefore, a time-stretching method is used. The idea is as follows. Take for example the process \( \hat{R}^n \). In Section 4.2.1 we stretch the time (using the same transformation for all the processes) and generate a process \( \tilde{R}^n \) in such a way that \( \{\tilde{R}^n\} \) is Lipschitz-continuous with Lipschitz constant 1 and therefore tight and converges to a process \( \tilde{R} \). Then in Section 4.2.2 we go back to the original scale by an inverse time transformation and get the process \( \tilde{R} \), which is used in Section 4.2.3 to get the value function of the RSDG.

### 4.2.1 Time rescaling

For every \( n \in \mathbb{N} \) define

\[
\tau^n(t) := t + \theta^n \cdot \hat{R}^n(t) + \theta^n \cdot \hat{Y}^n(t), \quad t \in \mathbb{R}_+.
\]

(4.20)

Since \( \hat{R}^n \) and \( \theta^n \cdot \hat{Y}^n \) are nondecreasing and RCLL (see (3.1)) it follows that \( \tau^n \) is strictly increasing and RCLL. Moreover, for every \( 0 \leq s \leq t \), \( \tau^n(t) - \tau^n(s) \geq t - s \). The time rescaled process is given by

\[
\tilde{\tau}^n(t) := \inf\{s \geq 0 : \tau^n(s) > t\}, \quad t \in \mathbb{R}_+.
\]

Notice that \( \tilde{\tau}^n \) is nondecreasing and continuous. Also, \( \tau^n(\tilde{\tau}^n(t)) = t \) and

\[
0 \leq \tilde{\tau}^n(t) < s \quad \text{if and only if} \quad \tau^n(s) > t \geq 0.
\]

(4.21)
Define also the following rescaled processes

\[ \tilde{L}^n(t) := \tilde{L}^n(\tilde{\tau}^n(t)), \quad \tilde{A}^n(t) := \tilde{A}^n(\tilde{\tau}^n(t)), \quad \tilde{D}^n(t) := \tilde{D}^n(\tilde{\tau}^n(t)), \quad \tilde{T}^n(t) := T^n(\tilde{\tau}^n(t)), \]

for \( L = X, Y, \) and \( R. \)

**Lemma 4.1** The sequence of processes

\[ \{(\tilde{A}^n, \tilde{D}^n, \tilde{A}^n, \tilde{D}^n, \tilde{X}^n, \tilde{Y}^n, \tilde{R}^n, \tilde{T}^n)\}_n \]

is \( C \)-tight. Let \((\tilde{A}, \tilde{D}, \tilde{A}, \tilde{D}, \tilde{X}, \tilde{Y}, \tilde{R}, \tilde{T})\) be a limit of a weakly convergent subsequence. Then, for every \( t \in \mathbb{R}_+ \) one has,

\[ \tilde{X}_i(t) = \tilde{X}_i(0) + \tilde{m}_i\tilde{\tau}(t) + \tilde{\sigma}_i(\theta\tilde{\tau}), \varepsilon_i \int_0^t V(\theta \cdot \tilde{X}(s); \varepsilon) d\tilde{\tau}(s) + \tilde{\sigma}_i\tilde{B}_i(t) + \tilde{Y}_i(t) - \tilde{R}_i(t), \] \( (4.22) \)

\[ \tilde{A}(t) = \tilde{A}(\tilde{\tau}(t)), \quad \tilde{D}(t) = \tilde{D}(\tilde{\tau}(t)), \quad \tilde{T}(t) := (\rho_1, \ldots, \rho_I)\tilde{\tau}(t), \] \( (4.23) \)

where \( \tilde{B} = (\tilde{B}_1, \ldots, \tilde{B}_I) = \tilde{\sigma}^{-1}(\tilde{A} - \tilde{D}) \) is a martingale w.r.t. its own filtration and with quadratic variation \( \tilde{\tau}(\cdot) \tilde{I} \), where \( \tilde{I} \) is the identity matrix of order \( I \times I. \)

**Proof.** The \( C \)-tightness of \( \{(\tilde{\tau}^n, \tilde{Y}^n, \tilde{R}^n)\}_n \) follows by the following observation, which relies on the identity \( \tau^n(\tilde{\tau}^n(t)) = t \) and the definition of \( \tau^n. \)

\[ t - s = \tau^n(\tilde{\tau}^n(t)) - \tau^n(\tilde{\tau}^n(s)) \]

\[ = \tilde{\tau}^n(t) - \tilde{\tau}^n(s) + \theta^n \cdot \tilde{R}^n(t) - \theta^n \cdot \tilde{R}^n(s) + \theta^n \cdot \tilde{Y}^n(t) - \theta^n \cdot \tilde{Y}^n(s). \] \( (4.24) \)

Next, since the sequence \( \{(\tilde{\tau}^n, \tilde{Y}^n)\}_n \) is \( C \)-tight, we get by \( (4.1) \) and the limit \( \mu^n; n^{-1/2} \to \infty, \)

that \( \tilde{T}^n \xrightarrow{\mathcal{D},(4.23)} \tilde{T}. \)

From \( (4.16), \) the \( C \)-tightness of \( \{(\tilde{\tau}^n, \tilde{Y}^n, \tilde{R}^n)\}_n, \) and \( (4.12), \) the \( C \)-tightness of \( \{(\tilde{X}^n)\}_n \) follows once we show that \( \{(\tilde{A}^n, \tilde{D}^n)\}_n \) is \( C \)-tight. Recalling that \( \{(\tilde{\tau}^n)\}_n \) is \( C \)-tight, then in order to prove the latter statement, it is sufficient to show the \( C \)-tightness of \( \{(\tilde{A}^n, \tilde{D}^n)\}. \)

For every \( i \in [I] \) and \( n \in \mathbb{N}, \) the processes \( \psi^n_{1,i} \) and \( \psi^n_{2,i}U^n_{i} \) are the intensities of \( A^n_{i} \) and \( D^n_{i} \) := \( S^n_{i}(T^n_{i}) \), respectively. Denote

\[ (W^n_1, \ldots, W^n_{2I}) := (A^n_1, \ldots, A^n_I, D^n_1, \ldots, D^n_I), \]

\[ \tilde{W}^n = (W^n_1, \ldots, W^n_{2I}) := (\tilde{A}^n_1, \ldots, \tilde{A}^n_I, \tilde{D}^n_1, \ldots, \tilde{D}^n_I), \]

\[ \tilde{W}^n = (W^n_1, \ldots, W^n_{2I}) := (\tilde{A}^n_1, \ldots, \tilde{A}^n_I, \tilde{D}^n_1, \ldots, \tilde{D}^n_I). \]

Therefore, for every \( j \in \{1, \ldots, 2I\}, \) \( \tilde{W}^n_{j} \) is a martingale w.r.t. its own filtration, under the measure \( \tilde{Q}^n. \) Notice that the quadratic variation of \( \tilde{W}^n_{j} \) satisfies,

\[ [\tilde{W}^n_j] = \frac{1}{n} W^n_j, \] \( (4.25) \)
which is of order $t$ thanks to (4.11), (4.12), and (2.3). Fix an arbitrary $T > 0$ and a stopping time $\tau$. Then, by Burkholder-Davis-Gundy inequality (see [34, Theorem 48]), and (4.25), we get,
\[
\left( \mathbb{E}^{\tilde{Q}^n}[\tilde{W}_j^n(\pi + \cdot) - \tilde{W}_j^n(\pi)]^2 \right) \leq \mathbb{E}^{\tilde{Q}^n}[\tilde{W}_j^n(\pi + \cdot) - \tilde{W}_j^n(\pi)]^2 \leq C_3 \mathbb{E}^{\tilde{Q}^n}[\tilde{W}_j^n(\pi)]_{\delta} \leq C_4 \delta
\]
for some constants $C_3, C_4 > 0$, independent of $n$, $\delta$, and $\pi$. Therefore, Aldous criterion for tightness holds (see e.g., [9, Theorem 16.10]). Since the jumps of these processes are of order $O(n^{-1/2})$, any limit process has continuous paths with probability 1 and $C$-tightness of $\{\tilde{W}_j^n\}_n$ is proved.

The first two identities in (4.23) follow by the convergence $(\tilde{W}_j^n, \tilde{W}_j^n, \tilde{r}_j^n) \Rightarrow (\tilde{W}, \tilde{r})$, which in turn follows by the tightness of $\{(\tilde{W}_j^n, \tilde{W}_j^n, \tilde{r}_j^n)\}_n$. Finally, the quadratic variation of $\tilde{B}$ follows by (4.11), (4.23), and the martingale central limit theorem, see [19, Theorem 7.1.4].

By reducing to a subsequence and by Skorokhod’s representation theorem (see [9, Theorem 6.7]), we may assume without loss of generality that there is a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}^*, \tilde{Q}^*)$ that supports the sequence of processes $\{(\tilde{A}_j^n, \tilde{D}_j^n, \tilde{\tilde{A}}_j^n, \tilde{\tilde{D}}_j^n, \tilde{X}_j^n, \tilde{r}_j^n, \tilde{Y}_j^n, \tilde{R}_j^n, \tilde{\tilde{R}}_j^n)\}_n$, which in turn follows by (4.26), and the martingale central limit theorem, see [19, Theorem 7.1.4].

By reducing to a subsequence and by Skorokhod’s representation theorem (see [9, Theorem 6.7]), we may assume without loss of generality that there is a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}^*, \tilde{Q}^*)$ that supports the sequence of processes $\{(\tilde{A}_j^n, \tilde{D}_j^n, \tilde{\tilde{A}}_j^n, \tilde{\tilde{D}}_j^n, \tilde{X}_j^n, \tilde{r}_j^n, \tilde{Y}_j^n, \tilde{R}_j^n, \tilde{\tilde{R}}_j^n)\}_n$, and (4.26) holds.

Throughout the rest of Section 4.2 we consider the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}^*, \tilde{Q}^*)$ and w.l.o.g. occasionally assume that (4.26) is at force.

The following lemma states that the time rescaled process $\tilde{\tau}(t)$ grows to infinity together with $t$. The Lemma plays an important role in the proof of Proposition 4.1 below.

**Lemma 4.2**

\[
\lim_{t \to \infty} \tilde{\tau}(t) = \infty, \quad \tilde{Q}^*-a.s.
\]

**Proof.** Fix $0 < s < t$. From (4.21), it follows that
\[
\tilde{Q}^*(\tilde{\tau}^n(t) < s) = \tilde{Q}^*(\tau^n(s) > t) = \tilde{Q}^*(s + \theta^n \cdot \hat{R}_n(s) + \theta^n \cdot \hat{Y}_n(s) > t)
\]
\[
\leq \frac{1}{t - s} \mathbb{E}^{\tilde{Q}^*}[\theta^n \cdot \hat{R}_n(s) + \theta^n \cdot \hat{Y}_n(s)].
\]

From (4.12), (4.16), and the inequality $T^n(u) \leq u$, $u \in \mathbb{R}_+$ we get that there is a constant $C_5 > 0$ independent of $n$ and $t$, and such that
\[
\hat{Y}_i^n(s) \leq C_5(1 + \hat{R}_i^n(s) + \tilde{A}_i^n(s) - \tilde{D}_i^n(s)).
\]

Recalling that $\tilde{A}_i^n - \tilde{D}_i^n$ is a martingale, and (4.15), we get that the last term in (4.27) is bounded above by $\frac{C_5}{\varepsilon}$ for some $C_6 > 0$, independent of $n$ and $t$. Now, since the events $\{\tilde{\tau}(t) > s\}_t$ are decreasing with $t$, we get that
\[
\tilde{Q}^* \left( \lim_{t \to \infty} \tilde{\tau}(t) < s \right) = \lim_{t \to \infty} \tilde{Q}^*(\tilde{\tau}(t) < s).
\]
Using the convergence in law $\tilde{\tau}^n \overset{d}{\Rightarrow} \tilde{\tau}$, and since $Q^*(\tilde{\tau}^n(t) < s) \leq \frac{C_n}{t^8}$, we conclude that $$Q^*(\lim_{t \to \infty} \tilde{\tau}(t) < s) \leq \lim_{t \to \infty} \limsup_{n \to \infty} Q^*(\tilde{\tau}^n(t) < s) = 0.$$ 

\[ \square \]

4.2.2 Back to the original scale

We now define the inverse of $\tilde{\tau}$, which brings the limit processes back to the original scale. Set

$$\tau(t) := \inf\{s \geq 0 : \tilde{\tau}(s) > t\}, \quad t \in \mathbb{R}_+.$$ 

One can verify that $\tau$ is right-continuous and strictly increasing. Moreover, $\lim_{t \to \infty} \tau(t) = \infty$ $Q^*$-a.s. and from Lemma 4.2 for every $t \in \mathbb{R}_+$, $\tau(t) < \infty$ a.s. Finally, for every $t \in \mathbb{R}_+$, $\tilde{\tau}(\tau(t)) = t$, $\tau(\tilde{\tau}(t)) \geq t$, and

$$0 \leq \tilde{\tau}(s) \leq t \quad \text{if and only if} \quad \tau(t) \geq s \geq 0.$$

The time-transposed processes are defined as follows:

$$L^*(\cdot) := \tilde{L}(\tau(\cdot)), \quad L \in \{X, A, D, Y, R, T\}.$$ 

From (4.22), the equality $\tilde{\tau}(\tau(t)) = t$, and Lemma A.1 we have for every $t \in \mathbb{R}_+$,

$$X^*_t(t) = X^*_t(0) + m_t t + \hat{\sigma}_i(t) \hat{\varepsilon}_i \int_0^t V^\prime(\theta \cdot X^*(s); \hat{\varepsilon}) ds + \hat{\sigma}_i B^*_t(t) + Y^*_t(t) - R^*_t(t), \quad (4.28)$$

where

$$B^* = (B^*_1, \ldots, B^*_d) := \hat{\sigma}^{-1}(A^* - D^*).$$

We now turn to showing that the processes within (4.28) satisfy the properties of Definition 3.1 for an appropriate filtration, where the hats are replaced by the superscript *. Property (ii) in Definition 3.1 follows since the processes $\tilde{Y}^n$ and $\tilde{R}^n$ satisfy an equivalent condition, see the paragraph that comes after the definition. The rest of the properties will follow once we show that $B^*$ is a standard Brownian motion w.r.t. the chosen filtration, see (3.6). Thus, we now turn to define the relevant filtration. Set $\tilde{G}^*_t = \tilde{G}^*(t) := \sigma((\tilde{X}(s), \tilde{A}(s), \tilde{D}(s), \tilde{Y}(s), \tilde{R}(s)), 0 \leq s \leq t)$. Notice that for every $0 \leq s < t < \infty$, $\{\tau(s) < t\} = \{\tilde{\tau}(t) > s\} \in \tilde{G}^*(t)$. Therefore, $\tau(s)$ is an optional time for $\tilde{G}^*(t)$. From [24, Corollary 2.4], $\tau(s)$ is a stopping time for the complete right-continuous filtration $\tilde{G}_t = \tilde{G}(t) := \tilde{G}^*(t+) \vee \mathcal{N}$, where $\mathcal{N}$ is the collection of $Q^*$-null sets. Using the monotonicity of $t \mapsto \tau(t)$, we get that $\tilde{G}^*_t := \tilde{G}(\tau(t))$ is a filtration.

Proposition 4.1 The process $B^*$ is an $I$-dimensional standard Brownian motion under the filtration $\tilde{G}^*_t$.

Proof. We use the Lévy characterization for Brownian motions. Namely, we show that $B^*(t)$ and $B^*(t)(B^*(t))^\top - It$ are continuous local martingales w.r.t. $\tilde{G}^*$, where recall that $I$ is the identity matrix of order $I \times I$. 

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To simplify the presentation, set

\[ \tilde{B}^n = (\tilde{B}_1^n, \ldots, \tilde{B}_n^n) = \hat{\sigma}^{-1}(\tilde{A}^n - \tilde{D}^n), \quad \hat{B} = (\hat{B}_1, \ldots, \hat{B}_t) = \hat{\sigma}^{-1}(\tilde{A} - \tilde{D}). \]

We start by arguing the continuity. From \[ (4.26), \] \((\tilde{B}^n, \tilde{\tau}^n, \tilde{B}^n) \to (\hat{B}, \tilde{\tau}, \hat{B}) \) \(\mathbb{Q}^\ast\)-a.s., u.o.c. and therefore, \( \hat{B}(\cdot) = \hat{B}(\tilde{\tau}(\cdot)) \), \(\mathbb{Q}^\ast\)-a.s., u.o.c. Thus, \( B^\ast(\cdot) = \hat{B}(\tilde{\tau}(\cdot)) = \hat{B}(\tilde{\tau}(\tilde{\tau}(\cdot))) = \hat{B}(\cdot) \), which is continuous \(\mathbb{Q}^\ast\)-a.s., see Lemma \[ 4.1. \] The proof that \( B^\ast(t)(B^\ast(t))^\top - It \) is a local martingale follows by the same lines of the proof that \( B^\ast(t) \) is a local martingale. We start with the proof of the latter, and then add the missing details for the former. Recall the definition of \( \mathcal{G}^n \) from Definition \[ 2.1 (i). \]

Fix \( t, s \in \mathbb{R}_+ \) and \( n \in \mathbb{N} \). Notice that \( \{ \tilde{\tau}^n(s) \leq t \} = \{ \tau^n(t) \geq s \} = \{ t + \theta \cdot \tilde{R}^n(t) + \theta \cdot \tilde{Y}^n(t) \geq \tau^n(s) \} \). Thus, \( \tilde{\tau}^n(s) \) is a \( \mathcal{G}^n \)-stopping time. Recall that \( \tilde{A}^n \) and \( \tilde{D}^n \) are \( \mathcal{G}^n \)-martingales, then the optional sampling theorem (see Problem 1.3.24 in [20]) yields that \( \tilde{B}^n(t + \cdot) = \tilde{B}^n(\tilde{\tau}(t + \cdot)) \) is a \( \mathcal{G}^n(\tilde{\tau}^n(t)) \)-martingale. As a consequence, for every \( i \in [I] \) and every \( \mathcal{G}^n(\tilde{\tau}^n(t)) \)-measurable random variable \( \zeta^n \),

\[ \mathbb{E}^{\mathbb{Q}^n}[\tilde{B}^n(t + s) - \tilde{B}^n(t) \zeta^n] = 0. \]

Recall that \( t \mapsto \tilde{\tau}^n(t) \) is nondecreasing and therefore \( \mathcal{G}^n(\tilde{\tau}^n(t)) \) is a filtration. Now, for every bounded continuous function \( g \),

\[ g(\tilde{A}^n(s_m), \tilde{D}^n(s_m), \tilde{Y}^n(s_m), \tilde{R}^n(s_m), \tilde{\tau}^n(s_m); 0 \leq s_m \leq t, m = 1, \ldots, k) = g(\tilde{A}^n(\tilde{\tau}^n(s_m)), \tilde{D}^n(\tilde{\tau}^n(s_m)), \tilde{Y}^n(\tilde{\tau}^n(s_m)), \tilde{R}^n(\tilde{\tau}^n(s_m)), \tilde{\tau}^n(s_m); 0 \leq s_m \leq t, m = 1, \ldots, k) \]

is \( \mathcal{G}^n(\tilde{\tau}^n(t)) \)-measurable, where \( k \in \mathbb{N} \), and \( \{s_m\} \) is an arbitrary sequence. Combining the last two displays one gets,

\[ \mathbb{E}^{\mathbb{Q}^n}[g(\tilde{A}^n(s_m), \tilde{D}^n(s_m), \tilde{Y}^n(s_m), \tilde{R}^n(s_m), \tilde{\tau}^n(s_m); 0 \leq s_m \leq t, m = 1, \ldots, k) \times (\tilde{B}^n_i(t + s) - \tilde{B}^n_i(t))] = 0. \quad (4.29) \]

From \[ (4.16), \]

\[ \tilde{X}^n_i(t) = \tilde{X}^n_i(0) + \tilde{m}^n_i \tilde{\tau}^n(t) + \tilde{A}^n_i(t) - \tilde{D}^n_i(t) + \tilde{Y}^n_i(t) - \tilde{R}^n_i(t) \]

\[ + \lambda_i^{1/2} \int_0^{\tilde{\tau}^n(t)} \hat{\psi}^n_i(s)ds - \mu_i^{1/2} \int_0^{\tilde{\tau}^n(t)} \hat{\psi}^n_i(s)d\tilde{I}^n_i(s). \quad (4.30) \]

From \[ (4.12), (4.30), \] and the bound \( T^n(u) \leq u, u \in \mathbb{R}_+ \), we get that there are constants \( C_7, C_8 > 0 \), independent of \( t \), such that for sufficiently large \( n \) one has

\[ |\tilde{B}^n_i(t)| \leq C_7(\theta^n \cdot \tilde{R}^n_i(t) + \theta^n \cdot \tilde{Y}^n_i(t) + \tilde{\tau}^n_i(t) + 1) \leq C_8(t + 1), \quad t \in \mathbb{R}_+. \]

where the last inequality follows by the same arguments leading to \[ (4.24). \] From \[ (4.26), (4.29), \] and the bounded convergence theorem, we get that,

\[ \mathbb{E}^{\mathbb{Q}^n}[g(\tilde{A}(s_m), \tilde{D}(s_m), \tilde{Y}(s_m), \tilde{R}(s_m), \tilde{\tau}(s_m); 0 \leq s_m \leq t, m = 1, \ldots, k) \times (\tilde{B}_i(t + s) - \tilde{B}_i(t)))] = 0. \quad (4.31) \]
which implies that $\tilde{B}$ is a $\tilde{G}_t$-martingale.

Next, for every $i \in [I]$, we show that the composition $B^*_i(t) = \tilde{B}_i(\tau(t))$ is a $G^*_i$-local martingale. For this, we need to define the following stopping times. Fix $M > 0$ and set,

$$\tilde{\pi}_{i,M} := \inf\{t \geq 0 : \tilde{B}_i(t) > M\},$$

$$\pi_{i,M} := \inf\{t \geq 0 : B_i(t) > M\}.$$

One can verify that $\tau(\pi_{i,M}) = \tilde{\pi}_{i,M}$.

We now show that $B^*_i(t \wedge \pi_{i,M})$, which by definition equals $\tilde{B}_i(\tau(t \wedge \pi_{i,M}))$, is a $G^*_i$-martingale. Since $\lim_{M \to \infty} \pi_{i,M} = \infty$, $Q^*$-a.s., we conclude that $B^*$ is a $G^*_t$-local martingale. For this, set $\tilde{B}_{i,M} := \tilde{B}_i(\cdot \wedge \tilde{\pi}_{i,M})$. We use the optional sampling theorem given in [19, Theorem 2.2.13], which (in adaptation to our notation) states that if for every $t \in \mathbb{R}_+$,

$$\mathbb{E}^{Q^*} |\tilde{B}_{i,M}(\tau(t \wedge \pi_{i,M}))| < \infty \quad (4.32)$$

and

$$\lim_{T \to \infty} \mathbb{E}^{Q^*} |\tilde{B}_{i,M}(T)|1_{\{\tau(t) > T\}} = 0, \quad (4.33)$$

then for every $0 \leq s \leq t$,

$$\mathbb{E}^{Q^*} \left[ \tilde{B}_{i,M}(\tau(t \wedge \pi_{i,M})) | G^*(s \wedge \pi_{i,M}) \right] = \mathbb{E}^{Q^*} \left[ \tilde{B}_{i,M}(\tau(t \wedge \pi_{i,M})) | \tilde{G}(\tau(s \wedge \pi_{i,M})) \right]$$

$$= \tilde{B}_{i,M}(\tau(s \wedge \pi_{i,M})),$$

where we used the identity $G^*(t) = \tilde{G}(\tau(t))$. Therefore, $\tilde{B}_{i,M}(\tau(t \wedge \pi_{i,M}))$ is a $G^*_i$-martingale. Notice that

$$B^*_i(t \wedge \pi_{i,M}) = \tilde{B}_i(\tau(t \wedge \pi_{i,M})) = \tilde{B}_i(\tau(t) \wedge \tau(\pi_{i,M})) = \tilde{B}_i(\tau(t) \wedge \tau(\pi_{i,M}) \wedge \pi_{i,M})$$

$$= \tilde{B}_i(\tau(t \wedge \pi_{i,M}) \wedge \pi_{i,M}) = \tilde{B}_{i,M}(\tau(t \wedge \pi_{i,M}))$$

and therefore, $B^*_i(t \wedge \pi_{i,M})$ is a $G^*_i$-martingale. Indeed, the first and last equalities follow by the definitions of $B^*$ and $\tilde{B}_{i,M}$, respectively. The second and the forth equalities follow by the monotonicity of the function $t \mapsto \tau(t)$. Finally, the third equality follows since $\tau(\pi_{i,M}) = \tilde{\pi}_{i,M}$.

We now prove that Properties (4.32) and (4.33) hold. Property (4.32) follows by the definition of $\tilde{B}_{i,M}$. To prove Property (4.33) notice that

$$\mathbb{E}^{Q^*} \left[ |\tilde{B}_{i,M}(T)|1_{\{\tau(T) > T\}} \right] \leq M \mathbb{E}^{Q^*} \left[ |\tilde{B}_{i,M}(T)|1_{\{\tau(T) \leq T\}} \right] \leq M Q^*(\tau(T) \leq T).$$

Now, from Lemma 4.2 the l.h.s. of the above approaches 0 as $T \to \infty$ and (4.33) is proven.

We end the proof by providing the missing arguments for the proof that $B^*(t)(B^*(t))^\top - It$ is a martingale. One may go over the proof and replace the $\tilde{B}_i(t)$’s, $i \in [I]$ with $\tilde{N}_{ij}(t) = \tilde{B}_i(t)\tilde{B}_j(t) - \delta_{ij}t$, $i, j \in [I]$. The only difference between the proofs lies in proving that $\tilde{N}(t) = \tilde{B}(t)(B(t))^\top - It$ is a $G_t$-martingale. Or equivalently, in showing that (4.31) holds with $\tilde{N}_{ij}$, replacing $\tilde{B}_i$. To this end, recall that $\tilde{Q}^n = \prod_{i=1}^I (\tilde{Q}_{1,i}^n \times \tilde{Q}_{2,i}^n)$. Thus, $\{A^n_1, \ldots, A^n_1, D^n_1, \ldots, D^n_1\}$
are mutually independent under $\hat{Q}^n$ and therefore also under $Q^*$. Moreover, using the continuity of $\tilde{\tau}^n$ and the notation $\Delta L(t) = L(t) - L(t-)$, we get that for any $i, j \in [I]$, $i \neq j$, the following equalities hold $Q^*$-a.s.,

$$[\tilde{B}_i^n, \tilde{B}_j^n](t) = \sum_{0 \leq s \leq t} \Delta \tilde{B}_i^n(s) \Delta \tilde{B}_j^n(s) = \frac{1}{n} \sum_{0 \leq s \leq t} \Delta B_i^n(\tilde{\tau}^n(s)) \Delta B_j^n(\tilde{\tau}^n(s)) = 0$$

and

$$[\tilde{B}_i^n, B_i^n](t) = [\hat{B}_i^n](t) = \sum_{0 \leq s \leq t} (\Delta \tilde{B}_i^n(s))^2 = \frac{1}{n} \sum_{0 \leq s \leq t} (\Delta A_i^n(\tilde{\tau}^n(s)))^2 + (\Delta D_i^n(\tilde{\tau}^n(s)))^2$$

$$= \frac{1}{2n\lambda_i} (A_i^n(\hat{\tau}(t)) + D_i^n(\hat{\tau}(t))) \xrightarrow{n \to \infty} \hat{\tau}(t),$$

where the limit holds by the strong law of large numbers. Since $\tilde{B}_i^n(t)\tilde{B}_j^n(t) - [\tilde{B}_i^n, \tilde{B}_j^n](t)$ is a $\mathcal{G}^n(\hat{\tau}(t))$-martingale, by taking the limit $n \to \infty$, one deduces that \([4.31]\) holds with $\tilde{\lambda}_{ij}$ replacing $B_i$.

### 4.2.3 Asymptotic lower bound

We are now ready to analyze the cost function. We start with a lower bound for the limit of the Kulback-Leibler divergences from (2.5). Recall (4.13) and consider $y^n = \hat{\psi}^n_{1,i}(t)(\lambda_i n)^{1/2}/\lambda_i^n$. By (4.9)–(4.10) and Corollary 3.1 $y^n \geq 0$. Notice that for every $y \geq 0$,

$$(1 + y) \log(1 + y) - y \leq \frac{1}{2} y^2. \quad (4.34)$$

Since $\lambda_i n/\lambda_i^n \to 1$ as $n \to \infty$, we get from (4.13) that

$$L_1^\theta(\hat{Q}^n_{1,i}||\mathbb{P}^n_{1,i}) \leq \mathbb{E}^Q\left[\frac{1}{2} \int_0^{\infty} e^{-\theta t}(\hat{\psi}^n_{1,i}(t))^2 dt\right] + o(1).$$

The same calculation yields,

$$L_2^\theta(\hat{Q}^n_{2,i}||\mathbb{P}^n_{2,i}) \leq \mathbb{E}^Q\left[\frac{1}{2} \int_0^{\infty} e^{-\theta t}(\hat{\psi}^n_{2,i}(t))^2 dt\rho_i(t)\right] + \mathbb{E}^Q\left[\int_0^{\infty} e^{-\theta t}(\hat{\psi}^n_{2,i}(t))^2 d(T^n_i(t) - \rho_i t)\right] + o(1).$$

The last expectation above is of order $o(1)$. Indeed, by (4.9)–(4.10), $(\hat{\psi}^n_{2,i}(t))^2 = c_i(V'(\theta \cdot \tilde{X}^n(t-));\varepsilon))^2$ for some $c_i > 0$. From Lemma A.1

$$\int_0^{\infty} e^{-\theta t}(\hat{\psi}^n_{2,i}(t))^2 d(T^n_i(t) - \rho_i t) = \int_0^{\infty} e^{-\theta t}(\hat{\psi}^n_{2,i}(\tilde{\tau}^n(t)))^2 d(T^n_i(\tilde{\tau}^n(t)) - \rho_i \tilde{\tau}^n(t))$$

$$= c_i \int_0^{\infty} e^{-\theta t}(V'(\theta \cdot \tilde{X}^n(t-)))^2 d(T^n_i(\tilde{\tau}^n(t)) - \rho_i \tilde{\tau}^n(t)).$$

From Lemmas [A.1] and [A.2] the last integral converges to 0, $\hat{Q}^n$-a.s. Although the mentioned lemma is stated for finite time interval, the discounted cost, the boundedness of $V'$ and the
bound $T^n(t) \leq t$, allow us to take the integral’s upper limit to be infinity. Now the uniform boundedness of the integral implies that the expectation of the last integral above converges to 0 as well. Using the representations above together with (3.7), (3.10), and (4.9)–(4.10), it follows that

$$
\sum_{i=1}^{I} \frac{1}{K_1,i} L^E_1(\hat{Q}^n_{1,i}||P^n_{1,i}) + \sum_{i=1}^{I} \frac{1}{K_2,i} L^E_2(\hat{Q}^n_{2,i}||P^n_{2,i}) \leq \frac{1}{2\varepsilon} \mathbb{E}^{Q^*} \left[ \int_{0}^{\infty} e^{2\sigma^2(V'(\theta^n \cdot \hat{X}^n(t); \varepsilon))^2} dt \right] + o(1).
$$

(4.35)

Notice that $\hat{X}^n$ has only countable number of jumps during the time interval $[0, \infty)$ and therefore we could replace $\hat{X}^n(t^-)$ by $\hat{X}^n(t)$ without affecting the integral. From (2.4), the above, and Lemma A.1, one has

$$
J^n(X^n(0), U^n, R^n, \hat{Q}^n; \kappa) \geq \mathbb{E}^{Q^*} \left[ \int_{0}^{\infty} e^{2\sigma^2(V'(\theta^n \cdot \hat{X}^n(t); \varepsilon))^2} dt + \hat{r} \cdot d\hat{R}^n(t) \right] + o(1)
$$

(4.36)

Recall that $\hat{X}^n$ and $V'$ are uniformly bounded, and also that $\hat{R}^n$ is nondecreasing, continuous, and bounded on any compact time interval. Then from (4.26), Lemma A.2, and the bounded convergence theorem, we get that for every $s \in \mathbb{R}_+$, one has

$$
\lim_{n \to \infty} \mathbb{E}^{Q^*} \left[ \int_{0}^{s} e^{-\rho^2(t)} \left\{ [\hat{h} \cdot \hat{X}^n(t) - (\varepsilon\sigma V'(\theta^n \cdot \hat{X}^n(t); \varepsilon))^2/(2\varepsilon)] d\hat{\tau}^n(t) + \hat{r} \cdot d\hat{R}^n(t) \right\} \right] = \mathbb{E}^{Q^*} \left[ \int_{0}^{s} e^{-\rho^2(t)} \left\{ [\hat{h} \cdot \hat{X}(t) - (\varepsilon\sigma V'(\theta^n \cdot \hat{X}(t); \varepsilon))^2/(2\varepsilon)] d\hat{\tau}(t) + \hat{r} \cdot d\hat{R}(t) \right\} \right].
$$

Since $\hat{h}$ and $\hat{r}$ have positive entries and $0 \leq V' \leq r$,

$$
\liminf_{n \to \infty} \mathbb{E}^{Q^*} \left[ \int_{0}^{s} e^{-\rho^2(t)} \left\{ [\hat{h} \cdot \hat{X}^n(t) - (\varepsilon\sigma V'(\theta^n \cdot \hat{X}^n(t); \varepsilon))^2/(2\varepsilon)] d\hat{\tau}^n(t) + \hat{r} \cdot d\hat{R}^n(t) \right\} \right] 
\geq \mathbb{E}^{Q^*} \left[ \int_{0}^{s} e^{-\rho^2(t)} \left\{ [\hat{h} \cdot \hat{X}(t) - (\varepsilon\sigma V'(\theta^n \cdot \hat{X}(t); \varepsilon))^2/(2\varepsilon)] d\hat{\tau}(t) + \hat{r} \cdot d\hat{R}(t) \right\} 
- (2\varepsilon)^{-1}(\varepsilon\sigma r)^2 \int_{s}^{\infty} e^{-\rho^2(t)} d\hat{\tau}(t) \right].
$$

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Using again the uniform bound of $\tilde{X}^n$ and $V'$, that $\tilde{R}^n$ is nondecreasing, and Lemma 4.2 taking $s \to \infty$ in the above, we get

$$
\liminf_{n \to \infty} \mathbb{E}^Q \left[ \int_0^\infty e^{-\varrho t^n(t)} \left\{ [\hat{h} \cdot \tilde{X}^n(t) - (\varepsilon \sigma V'(\theta^n \cdot \tilde{X}^n(t); \varepsilon))^2/(2\varepsilon)]d\tilde{r}^n(t) + \hat{r} \cdot d\tilde{R}^n(t) \right\} \right] 
$$

(4.37)

$$
\geq \mathbb{E}^Q \left[ \int_0^\infty e^{-\varrho t^n(t)} \left\{ [\hat{h} \cdot \tilde{X}(t) - (\varepsilon \sigma V'(\theta \cdot \tilde{X}(t); \varepsilon))^2/(2\varepsilon)]d\tilde{r}(t) + \hat{r} \cdot d\tilde{R}(t) \right\} \right] 
$$

$$
= \mathbb{E}^Q \left[ \int_0^\infty e^{-\varrho t} \left\{ [\hat{h} \cdot X^*(t) - (\varepsilon \sigma V'(\theta \cdot X^*(t); \varepsilon))^2/(2\varepsilon)]dt + \hat{r} \cdot dR^*(t) \right\} \right] 
$$

$$
\geq \mathbb{E}^Q \left[ \int_0^\infty e^{-\varrho t} \left\{ [\hat{h}(\theta \cdot X^*(t)) - (\varepsilon \sigma V'(\theta \cdot X^*(t); \varepsilon))^2/(2\varepsilon)]dt + \hat{r} \cdot dR^*(t) \right\} \right],
$$

where we used Lemma A.1 to get the equality. Indeed, recall that $\hat{r}(\tau(t)) = t$, $X^*(t) = \tilde{X}(\tau(t))$, $R^*(t) = \tilde{R}(\tau(t))$, and Lemma 4.2. The last inequality follows since the definitions of $h$ and $r$ (see (3.13)–(3.14) and $[16$, (2.45)–(2.46)) $h(\theta \cdot X^*(t)) \leq \hat{h} \cdot X^*(t)$ and $\int_0^\infty e^{-\varrho t}r(\theta \cdot R^*(t)) \leq \int_0^\infty e^{-\varrho t}\hat{r} \cdot dR^*(t)$.

Denote $B := \left( \sum_{i=1}^I (\bar{\theta} \hat{\sigma}; B^n) / \sigma \right)$ and $L := \theta \cdot L^*$, for $L \in \{X, Y, R\}$. Then, from (4.22), together with (3.7), (3.10), and the limit $x_0 = \lim_{n \to \infty} \theta^n \cdot \hat{X}^n(0)$, we have that

$$X(t) = x_0 + mt + \sigma \int_0^t \varepsilon \sigma V'(X(s); \varepsilon)ds + \sigma B(t) + Y(t) - R(t), \quad t \in \mathbb{R}_+.
$$

From Proposition 4.1, we get that $B$ is a standard one-dimensional Brownian motion w.r.t. $\mathcal{G}^*_t$. Hence, (3.16) and the definition of $Q_V$ in Proposition 3.1 imply that the last expectation in the sequence of relations in (4.37) equals $J(x_0, Y, R, Q_V; \varepsilon)$. Together with (4.36) we have,

$$
\liminf_{n \to \infty} J^n(X^n(0), U^n, R^n, \hat{Q}^n; \kappa) \geq J(x_0, Y, R, Q_V; \varepsilon) \geq \bar{V}(x_0; \varepsilon).
$$

Since the sequence of policies $\{(U^n, R^n)\}_n$ is arbitrary, it follows that

$$
\liminf_{n \to \infty} V^n(X^n(0); \kappa) \geq \bar{V}(x_0; \varepsilon).
$$

\qed

4.3 Proof of (4.7)

In this section we consider a sequence of arbitrary strategies for the maximizer and show that in the limit, the candidate policy for the DM is bounded above by the value function of the RSDG. We start with considering an arbitrary sequence of strategies for the maximizer that is not too costly. Then in Section 4.3.2 we show that by using the candidate measure, the dynamics of the buffers’ sizes stay close to $\gamma$ from (3.22). In Section 4.3.3 we asymptotically bound the expected cost; in order to estimate the change of measure penalty, we truncate the processes $\{\psi_{j,i}^n\}_{n,j,i}$ and show that by doing this the penalty does not change much.
4.3.1 The maximizer’s perspective

Consider an arbitrary sequence of measures chosen by the maximizer in the QCP, \( \{\hat{Q}^n\}_{n \in \mathbb{N}} \), where each \( \hat{Q}^n \in \hat{Q}^n(X^n(0)) \). Recall that every measure \( \check{Q}^n \in \check{Q}^n(X^n(0)) \) is associated with the processes \( \{\psi^n_{j,i}\}_{j,i} \), see (2.6). These processes stand for the ‘new’ intensity of the processes \( \{A^n_i\}_{i,n} \) and \( \{S^n_i(T^n_i)\}_{i,n} \). To simplify the notation and some of the arguments, we consider one probability space \( (\Omega, \mathcal{G}, \hat{Q}) \) that supports the processes \( \{(A^n_i, S^n_i(T^n_i))\}_{i,n} \) and underwhich, for every \( n \in \mathbb{N} \), the relevant intensities are \( \{\psi^n_{j,i}\}_{j,i,n} \). However, occasionally, when we want to emphasize the relevant measure, we use the measures \( \{\hat{Q}^n_{i,j}\}_{n,j,i} \) and \( \{\hat{Q}^n\}_n \).

Notice that the same calculation given in (4.3) is valid here as well, so

\[
L_1^\hat{Q}(\hat{Q}^n_{1,i}||\hat{P}^n_{1,i}) = \mathbb{E}^{\hat{Q}_{1,i}} \left[ \int_0^\infty e^{-\theta t} \left( \psi^n_{1,i}(t) \log \left( \frac{\psi^n_{1,i}(t)}{\lambda^n_i} \right) - \psi^n_{1,i}(t) + \lambda^n_i \right) dt \right],
\]

\[
L_2^\hat{Q}(\hat{Q}^n_{2,i}||\hat{P}^n_{2,i}) = \mathbb{E}^{\hat{Q}_{2,i}} \left[ \int_0^\infty e^{-\theta t} \left( \psi^n_{2,i}(t) \log \left( \frac{\psi^n_{2,i}(t)}{\mu^n_i} \right) - \psi^n_{2,i}(t) + \mu^n_i \right) dT^n_i(t) \right].
\]

We now show that without any loss, the maximizer can be restricted to measures \( \hat{Q}^n \in \hat{Q}^n(X^n(0)) \), which are ‘not too far away’ from the reference measure \( \hat{P}^n \). The idea behind it, as will be provided rigorously in the proof below, is that by changing the rate, the rejection cost will contribute at most a linear cost, while the penalty for the change of measure is superlinear. Without loss of generality, we may and will assume that

\[
J^n(X^n(0), U^n(a), R^n(a), \hat{Q}^n; \kappa) \geq \hat{V}(x_0; \varepsilon) - 1.
\]

Define

\[
\hat{\psi}^n_{1,i}(t) := (\lambda^n_i)^{-1/2} (\psi^n_{1,i}(t) - \lambda^n_i), \quad \text{and} \quad \hat{\psi}^n_{2,i}(t) := (\mu^n_i)^{-1/2} (\psi^n_{2,i}(t) - \mu^n_i).
\]

Recall that in the previous subsection the maximizer was given a specific strategy for which \( \sup_{t,j,i} |\hat{\psi}^n_{j,i}(t)| \) was bounded. Since we consider now a sequence of arbitrary strategies for the maximizer, it does not hold in this case. However, as we show in Proposition 4.2 and in Section 4.3.3, this is approximately the case.

**Proposition 4.2** There exists \( M > 0 \) such that for every \( i \in [I] \),

\[
\mathbb{E}^{\hat{Q}} \left[ \int_0^\infty e^{-\theta t} \left( |\hat{\psi}^n_{1,i}(t)| dt + |\hat{\psi}^n_{2,i}(t)| dT^n_i(t) \right) \right] + \int_0^\infty e^{-\theta t} \left( (\hat{\psi}^n_{1,i}(t))^2 dt + (\hat{\psi}^n_{2,i}(t))^2 dT^n_i(t) \right) \leq M
\]

and

\[
\mathbb{E}^{\hat{Q}} \left[ \int_0^\infty e^{-\theta t} \left( \psi^n_{1,i}(t) \log \left( \frac{\psi^n_{1,i}(t)}{\lambda^n_i} \right) - \psi^n_{1,i}(t) + \lambda^n_i \right) dt \right] + \int_0^\infty e^{-\theta t} \left( \psi^n_{2,i}(t) \log \left( \frac{\psi^n_{2,i}(t)}{\mu^n_i} \right) - \psi^n_{2,i}(t) + \mu^n_i \right) dT^n_i(t) \leq M.
\]
Proof. The second bound follows from the first one together with inequality (4.34). Thus, we only prove the first one. As argued in [6], at the bottom of page 595, for every $t \in \mathbb{R}_+$,
\[
\|\hat{R}_n(t)\| \leq C(1 + t + \|\hat{A}_n\|_t + \|\hat{D}_n\|_t),
\]
where in the above expression, and in the rest of the proof, $C$ refers to a finite positive constant that is independent of $n$ and $t$ and which can change from one line to the next. Denote
\[
\varphi^n(t) := \sum_{i=1}^{\ell} \left( |\hat{\psi}_{1,i}^n(t)| + |\hat{\psi}_{2,i}^n(t)|U_i^n(t) \right), \quad t \in \mathbb{R}_+.
\]
Recall the definitions of $\hat{A}_n$, $\hat{D}_n$ given in (4.17) and (4.18), then
\[
\|\hat{R}_n(t)\| \leq C\left( 1 + t + \int_0^t \varphi^n(s) ds + \|\hat{A}_n\|_t + \|\hat{D}_n\|_t \right). \tag{4.41}
\]
Applying Burkholder–Davis–Gundy inequality to $\hat{A}_n$ and $\hat{D}_n$, and noticing that $n^{-1}|\hat{\psi}_{j,i}^n| \leq C(1 + n^{-1/2}|\hat{\psi}_{j,i}^n|)$, we have
\[
\mathbb{E}_{\hat{Q}}[\|\hat{A}_n\|^2] \leq Cn^{-1} \mathbb{E}_{\hat{Q}}\left[ \int_0^t \hat{\psi}_{1,i}^n(s) ds \right] \leq C\left( t + n^{-1/2} \mathbb{E}_{\hat{Q}}\left[ \int_0^t \hat{\psi}_{1,i}^n(s) ds \right] \right), \tag{4.42}
\]
\[
\mathbb{E}_{\hat{Q}}[\|\hat{D}_n\|^2] \leq Cn^{-1} \mathbb{E}_{\hat{Q}}\left[ \int_0^t \hat{\psi}_{2,i}^n(s) dT_i^n(s) \right] \leq C\left( t + n^{-1/2} \mathbb{E}_{\hat{Q}}\left[ \int_0^t \hat{\psi}_{2,i}^n(s) dT_i^n(s) \right] \right).
\]
Hence,
\[
\mathbb{E}_{\hat{Q}}[\|\hat{R}_n(t)\|] \leq C\left( 1 + t + \mathbb{E}_{\hat{Q}}\left[ \int_0^t \varphi^n(s) ds \right] \right). \tag{4.43}
\]
An application of integration by parts yields
\[
\int_0^\infty e^{-\varrho t} \cdot d\hat{R}_n(t) = \left[ -\rho^{-1}e^{-\varrho t} \cdot \hat{R}_n(t) \right]_{t=0}^{t=\infty} + \int_0^\infty \rho e^{-\varrho t} \cdot \hat{R}_n(t) dt, \tag{4.44}
\]
and changing the order of integration implies
\[
\int_0^\infty e^{-\varrho t} \int_0^t \varphi^n(s) ds = \int_0^\infty \rho e^{-\varrho t} \varphi^n(t) dt.
\]
Notice that the first term in (4.44) is non-positive. Taking expectation on both sides of it and using the bound (4.43), we get that
\[
\mathbb{E}_{\hat{Q}}\left[ \int_0^\infty e^{-\varrho t} \cdot d\hat{R}_n(t) \right] \leq C\left( 1 + \mathbb{E}_{\hat{Q}}\left[ \int_0^\infty \rho e^{-\varrho t} \left\{ \int_0^t \varphi^n(s) ds \right\} dt \right] \right)
= C\left( 1 + \mathbb{E}_{\hat{Q}}\left[ \int_0^\infty \rho e^{-\varrho t} \varphi^n(t) dt \right] \right).
\]

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Clearly, \( \hat{h} \cdot \hat{X}^n \) is bounded. Then (4.39) and the last bound yield that
\[
E\hat{Q}\left[ \int_0^\infty e^{-\rho t} \left( (\hat{\psi}_{1,i}^n(t))^2 dt + (\hat{\psi}_{2,i}^n(t))^2 dT^n_i(t) \right) \right] \\
\leq C_0 \left( 1 + E\hat{Q}\left[ \int_0^\infty e^{-\rho t} \left( |\hat{\psi}_{1,i}^n(t)| dt + |\hat{\psi}_{2,i}^n(t)| dT^n_i(t) \right) \right] \right),
\]
(4.45)
for some \( C_0 > 0 \), independent of \( n \). Since the function \( z \mapsto z^2 \) is superlinear, there is a constant \( C_1 \in \mathbb{R} \) such that for any \( z \in \mathbb{R}, z^2 \geq C_1 + 2C_0z \), and thus
\[
E\hat{Q}\left[ \int_0^\infty e^{-\rho t} \left( (\hat{\psi}_{1,i}^n(t))^2 dt + (\hat{\psi}_{2,i}^n(t))^2 dT^n_i(t) \right) \right] \\
\geq C_1 + 2C_0 \left( E\hat{Q}\left[ \int_0^\infty e^{-\rho t} \left( |\hat{\psi}_{1,i}^n(t)| dt + |\hat{\psi}_{2,i}^n(t)| dT^n_i(t) \right) \right] \right).
\]
Together with (4.45), we obtain that,
\[
E\hat{Q}\left[ \int_0^\infty e^{-\rho t} \left( |\hat{\psi}_{1,i}^n(t)| dt + |\hat{\psi}_{2,i}^n(t)| dT^n_i(t) \right) \right] \leq 1 - C_1/C_0,
\]
and the result holds by another application of (4.45) and the bound above.

Define
\[
\hat{\Psi}_{1,i}^n(\cdot) := \int_0^{\cdot} \hat{\psi}_{1,i}^n(t) dt, \quad \text{and} \quad \hat{\Psi}_{2,i}^n(\cdot) := \int_0^{\cdot} \hat{\psi}_{2,i}^n(t) dT^n_i(t),
\]
and set \( \hat{\Psi}_i^n = (\hat{\Psi}_{1,i}^n; i \in [I]). \)

**Lemma 4.3** The sequence \( \{(\hat{\Psi}_1^n, \hat{\Psi}_2^n)\}_n \) is \( C \)-tight.

**Proof.** From Proposition 4.2 it follows immediately that for every \( T > 0 \),
\[
\lim_{K \to \infty} \lim_{n \to \infty} \hat{Q}\left( \| (\hat{\Psi}_1^n, \hat{\Psi}_2^n) \|_T \geq K \right) = 0.
\]
Fix \( \delta, \eta, K > 0 \) and for every \( n \in \mathbb{N} \) set
\[
P^n := \int_0^t e^{-\rho s} \left( (\hat{\psi}_{1,i}^n(s))^2 ds + (\hat{\psi}_{2,i}^n(s))^2 dT^n_i(s) \right). \tag{4.46}
\]
Clearly, for any \( K > 0 \),
\[
\hat{Q}\left( \text{osc}_T((\hat{\Psi}_1^n, \hat{\Psi}_2^n), \delta) \geq \eta \right) = \hat{Q}\left( \text{osc}_T((\hat{\Psi}_1^n, \hat{\Psi}_2^n), \delta) \geq \eta, P^n > K \right) + \hat{Q}\left( \text{osc}_T((\hat{\Psi}_1^n, \hat{\Psi}_2^n), \delta) \geq \eta, P^n \leq K \right) \tag{4.47}
\]
Proposition 4.2 implies that
\[
\lim_{K \to \infty} \limsup_{\delta \to 0^+} \limsup_{n \to \infty} \hat{Q}\left( \text{osc}_T((\hat{\Psi}_1^n, \hat{\Psi}_2^n), \delta) \geq \eta, P^n > K \right) \leq \lim_{K \to \infty} \limsup_{n \to \infty} \hat{Q}\left( P^n > K \right) = 0. \tag{4.48}
\]
We now examine the second term on the r.h.s. of (4.47). On the event \( \{ P_n \leq K \} \), Jensen’s inequality implies that there exists a constant \( C_T > 0 \), independent of \( n \), such that for every \( 0 \leq s < t \leq T \),
\[
\| (\hat{\Psi}_1^n, \hat{\Psi}_2^n)(t) - (\hat{\Psi}_1^n, \hat{\Psi}_2^n)(s) \|_2^2 \leq C_T(t - s)P_n \leq C_TK(t - s).
\]
As a result,
\[
\lim_{\delta \to 0^+} \lim_{n \to \infty} \hat{Q} \left( \text{osc}_T((\hat{\Psi}_1^n, \hat{\Psi}_2^n), \delta) \geq \eta, P_n \leq K \right) = 0.
\]
Combining it with (4.48), we get
\[
\lim_{\delta \to 0^+} \lim_{n \to \infty} \hat{Q} \left( \text{osc}_T((\hat{\Psi}_1^n, \hat{\Psi}_2^n), \delta) \geq \eta \right) = 0,
\]
and the \( C \)-tightness is established.

### 4.3.2 Staying close to the minimizing curve

We consider a set of one-dimensional processes. It is obtained by multiplying the scaled processes by \( \theta^n \). For its definition denote
\[
\hat{W}^n := \hat{A}^n - \hat{D}^n + \hat{m}^n, \quad (4.49)
\]
and
\[
W^{x,n} := \theta^n \cdot (\hat{W}^n), \quad X^{x,n} := \theta^n \cdot \hat{X}^n, \quad Y^{x,n} := \theta^n \cdot \hat{Y}^n, \quad R^{x,n} := \theta^n \cdot \hat{R}^n. \quad (4.50)
\]
Moreover, set \( \psi_1^{x,n} := \sigma^{-1} \sum_{i=1}^I \theta^n \lambda_i^{1/2} \psi_1^{n,i}, \quad \psi_2^{x,n} := \sigma^{-1} \sum_{i=1}^I \theta^n \mu_i^{1/2} \psi_2^{n,i}, \) and
\[
\Psi_1^{x,n}(\cdot) := \int_0^\cdot \psi_1^{x,n}(s)ds, \quad \Psi_2^{x,n}(\cdot) := \int_0^\cdot \psi_2^{x,n}(s)dT_i^n(s). \quad (4.51)
\]
The identity from (4.16) is valid here as well and can be expresses as
\[
\hat{X}_i^n(t) = \hat{X}_i^n(0) + \hat{m}_it + \hat{A}_i^n(t) - \hat{D}_i^n(t) + \hat{Y}_i^n(t) - \hat{R}_i^n(t) + \lambda_i^{1/2} \hat{\Psi}_1^{x,i,n}(t) - \mu_i^{1/2} \hat{\Psi}_2^{x,i,n}(t). \quad (4.52)
\]
As a result,
\[
X^{x,n}(t) = X^{x,n}(0) + W^{x,n}(t) + Y^{x,n}(t) - R^{x,n}(t) + \sigma(\Psi_1^{x,n}(t) - \Psi_2^{x,n}(t)), \quad t \in \mathbb{R}_+. \quad (4.53)
\]
The next lemma states that under the candidate policy given in Section 4.1 the service time of every buffer converges to its traffic intensity.

**Lemma 4.4** For every \( i \in [I] \), \( \{ T^n_i \} \) converges u.o.c. to \( \bar{T}_i \), where \( \bar{T}_i(t) := \rho_i t, \ t \in \mathbb{R}_+ \). Moreover, \( \{(\hat{A}^n, \hat{D}^n, W^n)\} \) is \( C \)-tight.
Proof. By \((4.52)\) we have,
\[
||\hat{Y}^n||_T \leq ||\hat{X}^n||_T + ||\hat{A}^n||_T + ||\hat{D}^n||_T + ||\hat{m}^n||_T + C_2||\hat{\Psi}^n_1, \hat{\Psi}^n_2||_T + ||\hat{R}^n||_T,
\]
where \(C_2\) depends solely on \(\{(\lambda_i, \mu_i)\}_i\). From \((4.41)\) it follows that there exists a constant \(C_3 > 0\), independent of \(n\) and \(T\), such that for every \(T \in \mathbb{R}_+\),
\[
||\hat{R}^n||_T \leq C_3 \left(1 + T + ||\hat{A}^n||_T + ||\hat{D}^n||_T + ||\hat{\Psi}^n_1, \hat{\Psi}^n_2||_T \right).
\]
Recall that \(||\hat{X}^n||_T\) is bounded and from Lemma 4.3, \(||(\hat{\Psi}^n_1, \hat{\Psi}^n_2)||_T\) is tight. Then, once we show that \(\{(\hat{A}^n, \hat{D}^n)\}_n\) is \(\mathcal{C}\)-tight, from \((4.54)\) and \((4.55)\), we get that \(\{||\hat{Y}^n||_T\}_n\) is tight. Now, by the definitions of \(\hat{Y}^n\) and \(\mu^n\), see \((3.1)\) and \((2.3)\), we get that \(\{T^n_i\}_n\) converges u.o.c. to \(\hat{T}_i\), where \(\hat{T}_i(t) = \rho_it\).

We now argue the \(\mathcal{C}\)-tightness of \(\{(\hat{A}^n, \hat{D}^n)\}_n\). From \((4.42)\) and Proposition 4.2, it follows that for every \(T > 0\),
\[
\lim_{K \to \infty} \lim_{n \to \infty} \hat{Q} \left(\{||\hat{A}^n, \hat{D}^n||_T \geq K\} \right) = 0.
\]
As argued in the proof of Lemma 4.3, it is sufficient to prove that for every \(K > 0\),
\[
\lim_{\delta \to 0} \lim_{n \to \infty} \hat{Q} \left(\{\text{osc}_T((\hat{A}^n, \hat{D}^n)), \delta \geq \eta, P^n \leq K\} \right) = 0,
\]
where \(P^n\) is given in \((4.46)\). This follows since under the event \(\{P^n < K\}\), the jumps of \((\hat{A}^n, \hat{D}^n)\) are of order \(n^{-1/2}\) and therefore, the limit process is continuous.

Finally, the \(\mathcal{C}\)-tightness of \(\{\hat{W}^n\}_n\) follows now by it definition.

We now define another set of processes generated from the last set. Let \(\tau^n\) be the first time a forced rejection occurred in the \(n\)-th system and set
\[
L^{s,n}_\tau := L^{s,n}_\tau (\cdot \wedge \tau^n), \quad L \in \{W, X, Y, R, \Psi_1, \Psi_2\}.
\]
We continue the proof of \((4.7)\) with the case that the initial state lies close to the minimizing curve. That is,
\[
\lim_{n \to \infty} \hat{X}^n(0) - \gamma^a(x_0) = 0 \quad \text{and} \quad \theta^n \cdot \hat{X}^n(0) \in [0, a], \quad n \in \mathbb{N},
\]
where recall that \(x_0 := \theta^n \cdot \hat{X}^n(0)\). As argued in [6] Step 5, p. 596, this condition can be relaxed by considering a stopping time that indicates when the state is ‘sufficiently close’ to the minimizing path. This stopping time converges to zero. Then we may continue from that stopping time, in the same way. The proof follows by the same lines as in case that the initial state lies close to the minimizing curve but with heavier notation and therefore omitted.

We now state a couple of results from [6] that are needed here. The proofs in our case are almost identical, yet require some technical modifications. For completeness of presentation we provide their proofs. A minor modification that can be observed immediately is that we defined \(\hat{W}^n\) and \(W^{s,n}_\tau\) using \(\hat{A}^n\) and \(\hat{D}^n\) instead of \(A^n\) and \(D^n\). We start with the arguments that appear in [6] Step 1, pp. 586–588].

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Lemma 4.5 For every \( n \in \mathbb{N} \),

\[
(a \land X^{o,n}, Y^{o,n}, R^{o,n}) = \Gamma_{[0,a]}(X^{\sharp,n}(0) + W^{o,n} + \sigma(\Psi^{o,n}_1 - \Psi^{o,n}_2) + E^n),
\]

where

\[
E^n := (a \land X^{o,n}) - X^{o,n} \Rightarrow 0.
\]

Moreover, the sequence \( \{(W^{o,n}, X^{o,n}, Y^{o,n}, R^{o,n}, \Psi^{o,n}_1, \Psi^{o,n}_2)\}_n \) is \( C \)-tight.

**Proof.** By the definition of the candidate policy, rejections do not occur when \( X^{o,n} < a \) and the policy is work conserving, namely \( \sum_{i=1}^I U^n_i(t) = 1 \), whenever \( X^n(t) > 0 \). Therefore,

\[
\int_0^\tau\mathbbm{1}\{X^{o,n}(t) < a\}dR^{o,n}(t) = \int_0^\tau\mathbbm{1}\{X^{o,n}(t) > 0\}dY^{o,n}(t) = 0,
\]

and (4.58) holds.

Recall that \( \{(\Psi^{\sharp,n}_1, \Psi^{\sharp,n}_2, \widehat{W}^n)\}_n \) is \( C \)-tight and therefore also \( \{(\Psi^{o,n}_1, \Psi^{o,n}_2, W^{o,n})\}_n \). Hence, in order to show the \( C \)-tightness of all the processes as mentioned in the lemma, it is sufficient to prove that \( E^n \Rightarrow 0 \). Fix \( T > 0 \). It is sufficient to show that as \( n \to \infty \),

\[
\left( \sup_{t \in [0,T]} X^{o,n}(t) - a \right)^+ \Rightarrow 0.
\]

For \( \nu > 0 \) consider the event \( \Omega_T^n := \{\sup_{t \in [0,T]} X^{o,n}(t) > a + \nu\} \). On this event there exist random times \( 0 \leq \tau^n_1 < \tau^n_2 \leq \tau^n_\nu \) such that \( X^{\sharp,n}(\tau^n_1) \leq a + \nu/2 \), \( X^{\sharp,n}(\tau^n_2) \geq a + \nu \) and \( X^{\sharp,n}(t) > a \) for every \( t \in [\tau^n_1, \tau^n_2] \). Using the notation \( \psi^n_{s,t} = L(t) - L(s) \) for every process \( L \) and \( 0 \leq s \leq t \), by (4.53) and the fact that \( Y^{\sharp,n} \) does not increase on an interval for which the system is not empty, we have

\[
(a + \nu) - (a + \nu/2) \leq X^{\sharp,n}|_{[\tau^n_1, \tau^n_2]}
\]

\[
= (W^{\sharp,n} + \sigma(\Psi^{\sharp,n}_1 - \Psi^{\sharp,n}_2))|_{[\tau^n_1, \tau^n_2]} - R^{\sharp,n}|_{[\tau^n_1, \tau^n_2]}
\]

\[
= (W^{\sharp,n} + \sigma(\Psi^{\sharp,n}_1 - \Psi^{\sharp,n}_2))|_{[\tau^n_1, \tau^n_2]} - n^{-1/2}\bar{A}^n_{\nu}([\tau^n_1, \tau^n_2]),
\]

where we used the fact that the policy rejects all class-\( i^* \) arrivals when \( X^{\sharp,n} > a^* \). Fix a sequence \( r^n > 0 \), \( r^n \to 0 \), such that \( n^{1/2}r^n \to \infty \). In case that \( \tau^n_2 - \tau^n_1 < r^n \), one has

\[
\nu/2 \leq (W^{\sharp,n} + \sigma(\Psi^{\sharp,n}_1 - \Psi^{\sharp,n}_2))|_{[\tau^n_1, \tau^n_2]} \leq \text{osc}_T(W^{\sharp,n}, r^n) + \text{osc}_T(\sigma(\Psi^{\sharp,n}_1 - \Psi^{\sharp,n}_2), r^n)
\]

and in case \( \tau^n_2 - \tau^n_1 \geq r^n \), one has

\[
2 \left( \|\sigma(\Psi^{\sharp,n}_1 - \Psi^{\sharp,n}_2)\|_T + \|W^{\sharp,n}\|_T \right) \geq \bar{A}^n_{\nu}|_{[\tau^n_1, \tau^n_2]}n^{-1/2} = \bar{A}^n_{\nu}|_{[\tau^n_1, \tau^n_2]} + n^{-1/2}\int^{\tau^n_2}_{\tau^n_1} \psi_{\nu}(s,t)\,dt
\]

\[
\geq -2\|\bar{A}^n_{\nu}\|_T + C_T r^n n^{1/2},
\]

for some \( C_T > 0 \), independent of \( n \). The last inequality follows since the last interval must be greater than \( C_T r^n n^{1/2} \) for some \( C_T > 0 \). Otherwise, there is a sequence of intervals
\{I^n\}_n with lengths \( r^n \), such that \( \lim_{n \to \infty} (r^n)^{-1} \int_{I^n} \psi_{1,i}^n(t) dt = 0 \) (recall that \( \psi_{1,i}^n > 0 \)). This implies that \( \limsup_{n} (r^n)^{-1} \int_{I^n} (\lambda_i n)^{1/2} \psi_{1,i}^n(t) dt < 0 \), and therefore, for sufficiently large \( n \), \( \int_{I^n} \psi_{1,i}^n(t) dt \leq -C_4 r^n n^{1/2} \), for some \( C_4 > 0 \) independent of \( n \). This implies that \( \int_{I^n} |\psi_{1,i}^n(t)| dt \geq C_4 r^n n^{1/2} \to \infty \) as \( n \to \infty \), in contradiction to Proposition 4.2. Therefore, the last inequality in the above holds.

The tightness of \( \{\hat{A}^n\}_n \) (see Lemma 4.3) and the C-tightness of \( \{(\hat{W}^{\sharp,n}, \Psi_{1}^{\sharp,n}, \Psi_{2}^{\sharp,n})\}_n \) (Lemma 4.4) imply that

\[
\lim_{n \to \infty} \left[ \hat{Q}\left( \text{osc}_T(\hat{W}^{\sharp,n}, r^n) \geq \nu/2 \right) + \hat{Q}\left( \text{osc}_T(\sigma(\Psi_{1}^{\sharp,n} - \Psi_{2}^{\sharp,n}), r^n) \geq \nu/2 \right) \\
+ \hat{Q}\left( \{\|\Psi_{1}^{\sharp,n}, \Psi_{2}^{\sharp,n}\|_T + \|W^{\sharp,n}\|_T + \|A_n\|_T \} \geq \tilde{C} T r^n n^{1/2} \} \right) = 0,
\]

and thus \( \lim_{n \to \infty} \hat{Q}(\Omega^n) = 0 \). Since \( \nu > 0 \) was arbitrary, we obtain (4.59).

\[\square\]

Next we provide a state space collapse result and claim that the multidimensional process \( \hat{X}^n \) lies close to the minimizing curve. Recall the dependency of the candidate policy on the parameter \( \delta_0 \).

**Proposition 4.3** For sufficiently small \( \delta_0 \), the following limit holds u.o.c.

\[
\hat{X}^n - \gamma^a(\hat{X}^{\sharp,n}) \Rightarrow 0,
\]

as \( n \to \infty \) and moreover for every \( T > 0 \), \( \lim_{n \to \infty} \hat{Q}(\tau^n < T) = 0 \).

This is the equivalent of Equation (67) and the conclusion in Step 3 at the bottom of page 593 in [6]. The proof there in given in Step 2, and spans over pages 588–593. We now provide an adaptation of [6, Step 2] to our case. For simplicity we share most of the notation and provide the claims in the same order.

**Proof.** Denote by \( \mathcal{G} = \{\hat{x} \in \mathcal{X} : \theta \cdot \hat{x} \leq a, \hat{x} = \gamma^a(\theta \cdot \hat{x})\} \) the set of points lying on the minimizing curve, and \( \partial^+ \mathcal{X} = \{\hat{x} \in \mathcal{X} : \hat{x}_i = \hat{b}_i \text{ for some } i\} \). These two sets are compact and disjoint. Hence, there exists \( \nu_0 > 0 \) such that for any \( 0 < \nu' < \nu_0 \), \( \mathcal{G}^{\nu'} \cap (\partial^+ \mathcal{X})^{\nu'} = \emptyset \), where for a set \( F \in \mathbb{R}^d \) we denote

\[
F^{\nu'} = \{\hat{x} : \text{dist}(\hat{x}, F) \leq \nu'\}.
\]

In the rest of the proof we consider only \( \nu' \) strictly smaller than \( \nu_0 \). For sufficiently large \( n \), forced rejections occur only when \( \hat{X}^n \) lies in \( (\partial^+ \mathcal{X})^{\nu'} \). As a result, as long as the process \( \hat{X}^n \) lies in \( \mathcal{G}^{\nu'} \), no forced rejections occur. Therefore, \( \sigma^n \leq \tau^n \), where

\[
\hat{\sigma}^n = \hat{\zeta}^n \wedge \zeta^n,
\]

\[
\hat{\zeta}^n = \inf\{t : X^{\hat{\sigma}^n} \geq a + \nu'\}, \quad \zeta^n = \inf\{t : \max_{i \in [I]} |\hat{X}_i^n(t) - \gamma^a(X^{\hat{\sigma}^n}(t))| \geq \nu'\}.
\]

As a result, in order to prove the first limit in the lemma, it is sufficient to show that \( \hat{Q}(\sigma^n < T) \to 0 \), for any small \( \nu' > 0 \) and any \( T \). Fix \( \nu' \) and \( T \). Since \( \sigma^n \leq \tau^n \),

\[
\hat{Q}(\sigma^n < T) = \hat{Q}(\sigma^n < T, \sigma^n \leq \tau^n) \leq \hat{Q}(\hat{\sigma}^n \wedge \zeta^n \leq T \wedge \tau^n) \leq \hat{Q}(\hat{\zeta}^n \leq T \wedge \tau^n) + \hat{Q}(\zeta^n \leq T \wedge \tau^n).
\]

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From (4.60) it follows that \( \hat{Q}(\zeta^n \leq T \wedge \tau^n) \to 0 \) as \( n \to \infty \). It therefore suffices to prove that

\[
\lim_{n \to \infty} \hat{Q}(\zeta^n \leq T \wedge \tau^n) = 0. \tag{4.61}
\]

On \( \zeta^n \leq T \wedge \tau^n \) let \( x^n := X^\sharp,n(\zeta^n) = X^{\circ,n}(\zeta^n) \) and let \( j = j^n \) and \( \xi^n \) be the corresponding components from the representation \((j, \xi)\) of \( x^n \) given in (4.1).

Fix a positive integer \( K = K(\nu') \). Consider the covering of \([0, b]\) by the \( K - 1 \) intervals \( \Xi_k = B(k\nu_1, \nu_1) \), \( k = 1, 2, \ldots, K-1 \), where \( B(x, a) \) denotes \( [x-a, x+a] \) and \( \nu_1 = b/K \). Let also \( \tilde{\Xi}_k = B(k\nu_1, 2\nu_1) \).

Recall that \( X^{\circ,n} \) are \( C \)-tight. Hence, given \( \delta > 0 \) there exists \( \delta' = \delta'(\delta, T, \nu_1) > 0 \) such that for all sufficiently large \( n \),

\[
|X^{\circ,n}(s) - X^{\circ,n}(t)| \leq \nu_1 \text{ for all } s, t \in [0, T], |s-t| \leq \delta', \text{ with probability at least } 1 - \delta. \tag{4.62}
\]

Fix such \( \delta \) and \( \delta' \). Denote by \( T^n := [(\zeta^n - \delta' \wedge 0), \zeta^n] \), and set

\[
\Omega^{n,k} := \{ \zeta^n \leq T \wedge \tau^n, \; x^n \in \Xi_k, \; X^\sharp,n(t) \in \tilde{\Xi}_k \text{ for all } t \in T^n \}.
\]

Then for all large \( n \),

\[
\hat{Q}(\zeta^n \leq T \wedge \tau^n) \leq \delta + \sum_{k=1}^{K-1} \hat{Q}(\Omega^{n,k}), \tag{4.63}
\]

where we used the identity \( X^\sharp,n = X^{\circ,n} \) on \([0, \tau^n] \). We fix \( k \in \{1, \ldots, K-1\} \) and use a similar (but more advanced) argument to the one given in the proof of Lemma 4.5 to analyze \( \Omega^{n,k} \).

The value assigned by the policy to \( U^n \) (see (4.3)) remains fixed as \( X^n \) varies within any of the intervals \((\tilde{\alpha}_j, \tilde{\alpha}_{j+1})\), where \( \tilde{\alpha}_i := \sum_{k=i+1}^{I} \theta_k \tilde{\alpha}_k \), \( i \in [I] \). We now provide four separate cases, that under each one of them, for each \( k \), \( \lim_{n \to \infty} \Omega^{n,k} = 0 \):

(I) \( \tilde{\Xi}_k \subset (0, a) \) and for all \( j \), \( \tilde{\alpha}_j \notin \tilde{\Xi}_k \). Then we consider the cases

(II) \( \tilde{\Xi}_k \subset (0, a) \) but \( \tilde{\alpha}_j \in \tilde{\Xi}_k \) for some \( j \in \{1, 2, \ldots, I-1\} \).

(III) \( 0 \in \tilde{\Xi}_k \).

(IV) \( a \notin \tilde{\Xi}_k \).

There may be additional intervals \( \tilde{\Xi}_k \), but they are all subsets of \((a, \infty)\) and therefore not important for our purpose.

We analyze only case (I) (and afterwards comment on the other ones). This means that in the representation of \( \gamma^n \), the \( j \)-th component is the same for all the points \( x \in \tilde{\Xi}_k \). Note that \( j := j(k) \) depends on \( k \) only, and in particular does not vary with \( n \).

Fix \( i \in \{j+1, \ldots, I\} \) (unless \( i = I \)). We estimate the probability that, on \( \Omega^{n,k} \), \( \zeta^n \leq T \wedge \tau^n \) occurs by having \( \hat{X}^\sharp_i(\zeta^n) - \gamma^a_i(X^\sharp,n(\zeta^n)) \geq \nu' \). More precisely, Since \( i > j \), \( \gamma^a_i(x^n) = a_i \). Then we will show that

\[
\text{for every } \nu'' \in (0, \nu'), \quad \hat{Q}(\Omega^{n,k} \cap \{ \hat{X}^\sharp_i(\zeta^n) > a_i + \nu'' \}) \to 0 \quad \text{as } n \to \infty. \tag{4.64}
\]

Recall the convergence of the initial condition given in (4.34) and that \( \gamma^a \) is continuous. Now, the jumps of \( \hat{X}^\sharp \) are of size \( n^{-1/2} \), and therefore on the event indicated in (4.64) there must exist \( \eta^n \in [0, \zeta^n] \) with the properties that

\[
\hat{X}^\sharp_i(\eta^n) < a_i + \nu''/2, \quad \hat{X}^\sharp_i(t) > a_i \text{ for all } t \in [\eta^n, \zeta^n]. \tag{4.65}
\]
On this event, during the time interval \([\eta^n, \zeta^n]\), \(i\) is always a member of \(\mathcal{H}(\hat{X}^n)\), and therefore by \((4.4) - (4.5)\), \(U^n(t) = \rho(t)(\hat{X}^n(t)) > \rho_0 + c\), for some constant \(c > 0\), independent of \(n\). By the definition of \(\hat{Y}^n\) from \((3.1)\), \(\frac{d}{dt} \hat{Y}^n \leq -\frac{\mu^n}{\sqrt{n}} c\). Set \(\hat{\eta}^n = \eta^n \lor (\zeta^n - \delta')\). Then for every \(t \in [\hat{\eta}^n, \hat{\zeta}^n]\) one has \(\hat{X}^n(t) \in \mathcal{E}_k \subset (0, a)\) and therefore no rejections occur. Combining these facts with \((4.32)\) and the definitions of \(\hat{A}^n, \hat{D}^n\), and \(\hat{W}^n\), given in \((4.17), (4.18),\) and \((4.49)\), we have

\[
\hat{X}_i^n[\hat{\eta}^n, \hat{\zeta}^n] = \left(\hat{W}^n_i + \lambda_i^{1/2} \hat{\Psi}^{n,i}_1 - \mu_i^{1/2} \hat{\Psi}^{n,i}_2\right) [\hat{\eta}^n, \hat{\zeta}^n] - \frac{\mu_i^n}{\sqrt{n}} (\zeta^n - \hat{\eta}^n),
\]

where we used the notation \(L[t, s] = L(t) - L(s)\) for any process \(L\), and \(0 \leq s \leq t\). As in the proof of Lemma \((4.5)\) fix a sequence \(r^n > 0\) with \(r^n \to 0\) and \(r^n \sqrt{n} \to \infty\). If \(\zeta^n - \eta^n < r^n\) and \(n\) is sufficiently large then \(\hat{\eta}^n = \eta^n\), thus by \((4.64)\) and the definition of \(\eta^n, \hat{X}^n[\hat{\eta}^n, \zeta^n] \geq \nu^n/2\). As a result,

\[
onc_T \left(\lambda_i^{1/2} \hat{\Psi}^{n,i}_1 - \mu_i^{1/2} \hat{\Psi}^{n,i}_2, r^n\right) + \osc_T(\hat{W}^n_i, r^n) \geq \left(\hat{W}^n_i + \lambda_i^{1/2} \hat{\Psi}^{n,i}_1 - \mu_i^{1/2} \hat{\Psi}^{n,i}_2\right) [\hat{\eta}^n, \zeta^n] \geq \nu^n/2
\]

must hold. If on the other hand, \(\zeta^n - \eta^n \geq r^n\) then by \((4.66)\),

\[
2 \left(\|\sigma(\hat{\Psi}^{n,i}_1 - \hat{\Psi}^{n,i}_2)\|_T + \|\hat{W}^n_i\|_T\right) \geq \left(\hat{W}^n_i + \lambda_i^{1/2} \hat{\Psi}^{n,i}_1 - \mu_i^{1/2} \hat{\Psi}^{n,i}_2\right) [\hat{\eta}^n, \zeta^n] \geq \frac{c\mu^n}{\sqrt{n}} r^n \geq cr_n \sqrt{n},
\]

for some constant \(c > 0\). Hence the probability in \((4.64)\) is bounded above by

\[
\hat{Q} \left(\osc_T \left(\lambda_i^{1/2} \hat{\Psi}^{n,i}_1 - \mu_i^{1/2} \hat{\Psi}^{n,i}_2, r^n\right) \geq \nu^n/4 \right) + \hat{Q} \left(\osc_T(\hat{W}^n_i, r^n) \geq \nu^n/4\right)
\]

\[
+ \hat{Q} \left(2 \left\{\|\lambda_i^{1/2} \hat{\Psi}^{n,i}_1 - \mu_i^{1/2} \hat{\Psi}^{n,i}_2\|_T + \|\hat{W}^n_i\|_T\right\} \geq cr_n \sqrt{n}\right),
\]

which converges to zero as \(n \to \infty\), by \(C\)-tightness of \(\{\hat{W}^n\}_n\) and \(\left\{\lambda_i^{1/2} \hat{\Psi}^{n,i}_1 - \mu_i^{1/2} \hat{\Psi}^{n,i}_2\right\}_n\). The rest of the proof follows by the same lines as in \((6)\), where again \(\hat{W}^n\) is replaced by \(\hat{W}^n_i + \lambda_i^{1/2} \hat{\Psi}^{n,i}_1 - \mu_i^{1/2} \hat{\Psi}^{n,i}_2\). The properties needed are the \(C\)-tightness of \(\{X^{\omega,n}\}_n\) and \(\{\hat{W}^n\}_n\), the uniform continuity of \(\gamma^n\), the convergence \(\theta^n \to \theta\) and of the initial condition given in \((4.57)\), the boundedness of \(X\), and that the jumps of \(X^n\) are of size \(n^{-1/2}\).

From the limit \(\lim_n \hat{Q}(\tau^n < T) \to 0\), we conclude that \(\{(W^{z,n}, X^{z,n}, Y^{z,n}, R^{z,n}, \hat{\Psi}^{z,n}_1, \hat{\Psi}^{z,n}_2)\}_n\) is \(C\)-tight.

### 4.3.3 Asymptotic bound for the costs

In this subsection we take advantage of the convergence of the dynamics to the minimizing curve we just argued. We start by showing that the expected holding and rejection components of the cost can be approximated by equivalent components associated with one dimensional dynamics. Then we approximate the one-dimensional dynamics with simpler dynamics for which the intensities used by the maximizer are truncated in some sense. The difference between the expected holding and rejection cost components of these dynamics are shown to be small. The

\[\text{This is where the proof given in [6] requires modification: } \hat{W}^n \text{ is replaced by } \hat{W}^n_i + \lambda_i^{1/2} \hat{\Psi}^{n,i}_1 - \mu_i^{1/2} \hat{\Psi}^{n,i}_2.\]
reason for this reduction is as follows. Although we know that the \( \hat{Q} \)-a.s. absolutely continuous valued processes \( (\hat{\Psi}_1^n, \hat{\Psi}_2^n) \) are \( C \)-tight and therefore have a converging subsequence whose limit is denoted by \( (\hat{\Psi}_1, \hat{\Psi}_2) \), it does not imply that \( \hat{Q} \)-a.s. the paths of \( (\hat{\Psi}_1, \hat{\Psi}_2) \) are absolutely continuous. Hence, we cannot argue for example that \( \hat{\Psi} \) is of the form \( \hat{\Psi}_{1,i} = \int_0^t \hat{\psi}_{1,i}(t) dt \) for some \( \hat{\psi}_{1,i} \). As a consequence, we cannot express the limiting measure for the maximizer, nor the change of measure penalty using \( \hat{\psi} \) as can done for example in (4.38) by substituting \( \psi_{1,i} = \lambda_i^n + \hat{\psi}_{1,i}(\lambda_i n)^{1/2} \). After this reduction we bound the change of measure penalty. Finally, the expected cost associated with the uniformly bounded rates dynamics is approximated by the value function of the RSGD.

**One dimensional reduction.** We start with showing the following uniform bound

\[
\limsup_{n \to \infty} \mathbb{E}^{\hat{Q}} \int_0^\infty e^{-\varepsilon t} (\hat{R}^n(t))^2 dt < \infty. \tag{4.67}
\]

To establish this, recall the bounds in (4.41)–(4.42). As a result,

\[
\mathbb{E}^{\hat{Q}} \left[ \|\hat{R}^n(t)\|^2 \right] \leq C \left( 1 + t^2 + \mathbb{E}^{\hat{Q}} \left[ \|\hat{\Psi}_1^n(t)\|^2 + \|\hat{\Psi}_2^n(t)\|^2 \right] \right),
\]

for some constant \( C > 0 \) independent of \( n \) and \( t \). Together with Proposition 4.2, (4.67) follows.

Now turn to the holding and rejection costs. By the definitions of the rejection mechanism and of \( R^{\varepsilon,n} \) in Section 4.3.2, on the event \( \{\tau^n > T\} \), one has

\[ \hat{R}^n(T) = \hat{R}_{i^*}^n(T) e_{i^*} = \mu_{i^*} R^{\varepsilon,n}(T)e_{i^*}. \]

Relations (4.43) and (4.44) imply that

\[
\mathbb{E}^{\hat{Q}} \left[ \int_0^\infty e^{-\varepsilon t} \cdot d\hat{R}^n(t) \right] = \mathbb{E}^{\hat{Q}} \left[ \int_0^\infty e^{-\varepsilon t} \cdot \hat{R}^n(t) dt \right]. \tag{4.68}
\]

Recall that \( h^a(w) = \hat{h} \cdot \gamma^a(w) \). Then, Proposition 4.3, the boundedness of \( \hat{X}^n \) and \( h^a(X^{\varepsilon,n}) \), the bound \( R^{\varepsilon,n} \leq \|\theta^n\| \|\hat{R}^n\| \), and (4.67) imply that

\[
\lim_{n \to \infty} \mathbb{E}^{\hat{Q}} \left[ \int_0^\infty e^{-\varepsilon t} \{\hat{h} \cdot \hat{X}^n(t) + \varepsilon \hat{\nu} \cdot \hat{R}^n(t)\} dt - \int_0^\infty e^{-\varepsilon t} \{h^a(X^{\varepsilon,n}(t)) + \varepsilon \hat{R}^{\varepsilon,n}(t)\} dt \right] = 0,
\]

where we used the identity \( r = r_{i^*} \mu_{i^*} \), see (3.15). Using the limit \( \lim_{n \to \infty} \hat{Q}(\tau^n < T) \to 0 \) and (4.68), we deduce,

\[
\lim_{n \to \infty} \mathbb{E}^{\hat{Q}} \left[ \int_0^\infty e^{-\varepsilon t} \{\hat{h} \cdot \hat{X}^n(t) + \hat{\nu} \cdot \hat{R}^n(t)\} dt \right] = 0. \tag{4.69}
\]

**Truncated intensities.** We now show that the maximizer can use probability measures for which \( \{\hat{\psi}_{j,i,n}^j\} \) are uniformly bounded from above by a sufficiently large constant \( k \) without too much lost.
Recall that under \( \hat{Q} \), \( A^n_i(\cdot) - \int_0^\cdot \psi^n_{1,i}(t) dt \) and \( S^n_i(T^n_i(\cdot)) - \int_0^\cdot \psi^n_{2,i}(t) dT^n_i(t) \) are martingales. We now truncate the intensities and consider the \( \hat{Q} \) martingales \( A^n_{i,k}(\cdot) - \int_0^\cdot \psi^n_{1,i,k}(t) dt \) and \( S^n_{i,k}(T^n_{i,k}(\cdot)) - \int_0^\cdot \psi^n_{2,i,k}(t) dT^n_{i,k}(t) \), where

\[
\begin{align*}
\psi^n_{1,i,k}(t) &:= \psi^n_{1,i}(t) - (\lambda_i n)^{1/2} \hat{\psi}^n_{1,i}(t) 1_{\{|\hat{\psi}^n_{1,i}(t)| > k\}}, \\
\psi^n_{2,i,k}(t) &:= \psi^n_{2,i}(t) - (\mu_i n)^{1/2} \hat{\psi}^n_{2,i}(t) 1_{\{|\hat{\psi}^n_{2,i}(t)| > k\}},
\end{align*}
\]

and \( T^n_{i,k} = (T^n_{i,k} : i \in [I]) \) is the DM’s control associated with the intensities \( \{\psi^n_{j,i}\}_{j,i} \). Denote \( \hat{\psi}^n_{j,i}(\cdot) := \hat{\psi}^n_{j,i}(\cdot) 1_{\{|\hat{\psi}^n_{j,i}(\cdot)| \leq k\}}, j \in \{1, 2\} \). Clearly, \( |\hat{\psi}^n_{j,i}| \leq k \) and

\[
\begin{align*}
\psi^n_{1,i,k}(t) &= \lambda^n_i + \hat{\psi}^n_{1,i}(t) (\lambda_i n)^{1/2} + o(n^{1/2}), \\
\psi^n_{2,i,k}(t) &= \mu^n_i + \hat{\psi}^n_{2,i}(t) (\mu_i n)^{1/2} + o(n^{1/2}),
\end{align*}
\]

The processes \( (A^n_{i,k}, D^n_{i,k}) \) and \( (A^n, D^n) \) are coupled such that for every \( i \in [I] \),

\[
(A^n_i - A^n_{i,k})(\cdot) - (\lambda_i n)^{1/2} \int_0^\cdot \hat{\psi}^n_{1,i}(t) 1_{\{|\hat{\psi}^n_{1,i}(t)| > k\}} dt,
\]

\[
(D^n_i - D^n_{i,k})(\cdot) - (\mu_i n)^{1/2} \int_0^\cdot \hat{\psi}^n_{2,i}(t) 1_{\{|\hat{\psi}^n_{2,i}(t)| > k\}} dT^n_{i,k}(t)
\]

are martingales.

Define \( \hat{A}^{n,k} := (\hat{A}^{n,k}_i : i \in [I]) \), with \( \hat{A}^{n,k}_i := n^{-1/2}(A^{n,k}_i(\cdot) - \int_0^\cdot \psi^n_{1,i,k}(t) dt) \) and similarly define \( \hat{D}^{n,k} \). For every \( L \in \{W, X, Y, R, \Psi_1, \Psi_2\} \) let \( \hat{L}^{n,k} \) and \( L^{n,k} \) be defined as \( \hat{L}^{n} \) and \( L^{n,k} \), where the intensities \( (\psi^n_{1,i}, \psi^n_{2,i}) \) are replacing \( (\psi^n_{1,i}, \psi^n_{2,i}) \), \( i \in [I] \) and let \( T^n_{i,k} \) be the equivalence of \( T^n_i \) in this setup. Also, let \( \tau^{n,k} \) be the first time a forced rejection occurs in this setup and set \( \hat{L}^{n,k} := L^{n,k} \wedge \tau^{n,k} \). Clearly, Lemmas 4.4, 4.5 and Proposition 4.3 hold in this case as well, where the superscript \( n \) is replaced by \( n, k \). For the sake of exposition, we state all the necessary results here.

As a private case of Lemma 4.4 we get that \( \{T^{n,k}\}_n \) converges u.o.c. to \( \hat{T} = (\hat{T}_1, \ldots, \hat{T}_I) \), where recall that \( \hat{T}_i(t) = \rho_i t, t \in \mathbb{R}_+ \). Proposition 4.3 implies that for sufficiently small \( \delta_0 \), the following limits hold

\[
\lim_{n \to \infty} \hat{X}^{n,k} - \gamma^{\ast}(X^{\hat{z},n,k}) = \hat{Q} \Rightarrow 0,
\]

and for every \( k, T > 0 \),

\[
\hat{Q}(\tau^{n,k} < T) = 0. \quad (4.71)
\]

**Lemma 4.6** For every \( n \in \mathbb{N} \),

\[
(a \wedge X^{0,n,k}, Y^{0,n,k}, R^{0,n,k}) = \Gamma_{[0,a]}(X^{\hat{z},n}(0) + W^{0,n,k} + \sigma(\Psi_1^{0,n,k} - \Psi_2^{0,n,k}) + E^{n,k}), \quad (4.72)
\]

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where
\[ E_{n,k} := (a \wedge X_{n,k}^o) - X_{n,k}^o \Rightarrow 0. \] (4.73)

Moreover, the sequence \( \{(W_{n,k}^o, X_{n,k}^o, Y_{n,k}^o, R_{n,k}^o, \Psi_{1,n,k}^o, \Psi_{2,n,k}^o, \tilde{\Psi}_{1,n,k}^o, \tilde{\Psi}_{2,n,k}^o)\}_n \) is C-tight and any subsequential limit of it \( (\bar{W}_{n,k}^o, \bar{X}_{n,k}^o, \bar{Y}_{n,k}^o, \bar{R}_{n,k}^o, \bar{\Psi}_{1,n,k}^o, \bar{\Psi}_{2,n,k}^o, \bar{\tilde{\Psi}}_{1,n,k}^o, \bar{\tilde{\Psi}}_{2,n,k}^o) \) satisfies \( \bar{Q} \)-a.s.
\[
(X_{n,k}^o, \bar{Y}_{n,k}^o, \bar{R}_{n,k}^o) = \Gamma_{[0,\alpha]}(X_{n,k}^o(0) + \bar{W}_{n,k}^o + \sigma(\bar{\Psi}_{1,n,k}^o - \bar{\tilde{\Psi}}_{2,n,k}^o)), \]
(4.74)
where \( \bar{X}_{n,k}^o(0) = X_{n,k}^o(0) = x_0, \bar{W}_{n,k}^o \) is an \((m, \sigma)\)-Brownian motion w.r.t. the filtration \( \mathcal{F}_t := \sigma\{W_{n,k}^o(s), X_{n,k}^o(s), Y_{n,k}^o(s), R_{n,k}^o(s), \bar{\Psi}_{1,n,k}^o(s), \bar{\Psi}_{2,n,k}^o(s), \bar{\tilde{\Psi}}_{1,n,k}^o(s), \bar{\tilde{\Psi}}_{2,n,k}^o(s) : 0 \leq s \leq t\} \), and
\[
\tilde{\Psi}_{1,k}^o = \sigma^{-1} \sum_{i=1}^I \theta_i \lambda_i^{1/2} \tilde{\Phi}_{1,i}^k, \quad \tilde{\Psi}_{2,k}^o = \sigma^{-1} \sum_{i=1}^I \theta_i \mu_i^{1/2} \tilde{\Phi}_{2,i}^k. \] (4.75)

Moreover, there are processes \( \{\tilde{\Phi}_{j,i}^k\}_{j,i} \), progressively measurable w.r.t. \( \mathcal{G}_t^n \), such that
\[
\tilde{\Phi}_{j,i}^k(\cdot) = \int_0^\cdot \tilde{\phi}_{j,i}^k(t)dt, \quad \tilde{\Phi}_{2,i}^k(\cdot) = \int_0^\cdot \tilde{\phi}_{2,i}^k(t)dt, \] (4.76)
which satisfy \( \max_{j,i} \|\tilde{\phi}_{j,i}^k\|_\infty \leq k \). Finally, \( \bar{Q} \)-a.s.,
\[
\lim_{n \to \infty} \int_0^\infty e^{-\varepsilon t} (\tilde{\psi}_{1,i}^{n,k}(t))^2 dt \geq \int_0^\infty e^{-\varepsilon t} (\tilde{\phi}_{1,i}^k(t))^2 dt, \]
\[
\lim_{n \to \infty} \int_0^\infty e^{-\varepsilon t} (\tilde{\psi}_{2,i}^{n,k}(t))^2 dt T_i^n(t) \geq \int_0^\infty e^{-\varepsilon t} (\tilde{\phi}_{2,i}^k(t))^2 dt. \] (4.77)

Proof. The Skorokhod mapping formulation of the pre-limit processes, the limit in (4.73), and the C-tightness are private cases of Lemma 4.5. This results imply the Skorokhod formulation of the limiting process in (4.74).

The expressions in (4.75) follow since \( \theta^n \to \theta \) and from (4.51),
\[
\psi_{1,n,k}^o(\cdot) := \sigma^{-1} \sum_{i=1}^I \theta_i \lambda_i^{1/2} \psi_{1,i}^{n,k} (\tau_{n,k} \wedge \cdot), \quad \psi_{2,n,k}^o(\cdot) := \sigma^{-1} \sum_{i=1}^I \theta_i \mu_i^{1/2} \psi_{2,i}^{n,k} (\tau_{n,k} \wedge \cdot). \] (4.78)

By definition, for every \( i \in [I] \) and \( j \in \{1, 2\} \), \( |\psi_{j,i}^{n,k}| \leq k \) and therefore, \( \bar{Q} \)-a.s., \( \{\tilde{\psi}_{j,i}^{n,k}\}_{j,i} \) are all Lipschitz-continuous with the same Lipschitz constant \( k \). Since Lipschitz-continuity implies absolutely continuity, from Section IV.17 of [13], we get the existence of progressively measurable processes \( \{\tilde{\phi}_{j,i}^k\}_{j,i} \) such that (4.76) holds.

The bounds in (4.77) follow immediately from Lemma A.3 noticing that by Skorokhod representation theorem the convergence \( (T_{i,n,k}^k, \tilde{\Psi}_{2,i}^k) \Rightarrow (T_i, \Phi_{2,i}^k) \) can be replaced by u.o.c. \( \bar{Q} \)-a.s. convergence.

Finally, we prove that \( \bar{W}_{n,k}^o \) is an \((m, \sigma)\)-Brownian motion. The martingale central limit theorem ([19] Theorem 7.1.4) implies that \( \{(A_{n,k}^o, S_{n,k}^o)\}_n \) converges to a 2I-dimensional \((0, \sigma)\)-Brownian motion, where \( S^n = (S_i^n : i \in [I]) \) and
\[
S_i^n(t) := n^{-1/2} \left( S_i^n(t) - \int_0^t \psi_{2,i}^n(s)ds \right), \quad \tilde{\sigma} = \text{Diag}(\lambda_i^{1/2}, \ldots, \lambda_i^{1/2}, \mu_1^{1/2}, \ldots, \mu_I^{1/2}).
\]
Using a lemma regarding random change of time from [9, p.151] with $T^n_i \to \bar{T}_i$, gives that 
\{(\hat{A}^{n,k}, \hat{D}^{n,k})\} converges to a $(0, \bar{\sigma})$-Brownian motion, where 
\[ \bar{\sigma} := (\lambda_1^{1/2}, \ldots, \lambda_i^{1/2}, \lambda_1^{1/2}, \ldots) \].

As a result from the definitions of $\hat{W}^n, \hat{W}^{s,n,k}$, and $\hat{W}^{c,o,n,k}$ (see their equivalences in (4.49), (4.50), and (4.56)), we get that $\hat{W}^{c,o,k}$ is an $(m, \bar{\sigma})$-Brownian motion w.r.t. its own filtration. The proof that its filtration can be replaced by $\mathcal{F}_t$ follows by the same lines of Proposition 4.1 and therefore omitted.

The next proposition tells us that by truncating the intensities, the expected cost does not change much.

**Proposition 4.4** The following limit holds
\begin{align}
\lim_{k \to \infty} \lim_{n \to \infty} \left[ \mathbb{E}^\mathbb{Q}\left[ \int_0^\infty e^{-\xi t} \{ h^a(X^{o,n}(t)) + \varrho R^{c,n}(t) \} dt \right] - \int_0^\infty e^{-\xi t} \{ h^a(X^{o,n,k}(t)) + \varrho R^{c,n,k}(t) \} dt \right] = 0
\end{align}
and for every $i \in [I]$,
\begin{align}
\lim_{k \to \infty} \lim_{n \to \infty} \left[ \mathbb{E}^\mathbb{Q}\left[ \int_0^\infty e^{-\xi t} \| \hat{W}^{c,o,n,k}(t) \|_t dt \right] - \mathbb{E}^\mathbb{Q}\left[ \int_0^\infty e^{-\xi t} \| \hat{W}^{c,o,n}(t) \|_t dt \right] \right] = 0,
\end{align}
where $\hat{Q}_{1,i}$ and $\hat{Q}_{2,i}$ are the measures associated with $\psi_{1,i}(t) = \lambda_1^0 t + (\lambda_i n)^{1/2} \hat{\psi}_{1,i}(t)$ and $\psi_{2,i}(t) = \mu_1^0 t + (\mu_i n)^{1/2} \hat{\psi}_{2,i}(t)$.

**Proof.** From the representations in (4.58) and (4.72), the limits (4.59) and (4.73), the definitions (4.51), (4.56), and (4.78), and Lemma 3.1 it follows that in order to obtain (4.79), it is sufficient to show that
\begin{align}
\lim_{k \to \infty} \lim_{n \to \infty} \mathbb{E}^\mathbb{Q}\left[ \int_0^\infty e^{-\xi t} \| W^{c,o,n,k} - W^{c,o,n} \|_t dt \right] = 0,
\end{align}
\begin{align}
\lim_{k \to \infty} \lim_{n \to \infty} \mathbb{E}^\mathbb{Q}\left[ \int_0^\infty e^{-\xi t} \| \hat{\psi}_{1,i}^n(t) - \hat{\psi}_{1,i}^n(t) \|_t dt \right] = 0,
\end{align}
\begin{align}
\lim_{k \to \infty} \lim_{n \to \infty} \mathbb{E}^\mathbb{Q}\left[ \int_0^\infty e^{-\xi t} \| \hat{\psi}_{2,i}^n(t) - \hat{\psi}_{2,i}^n(t) \|_t dT^n_i(t) \right] = 0,
\end{align}
Set the functions $f_{1,i}^n : (-\lambda_1 n, (\lambda_i n)^{-1/2}, \infty) \to \mathbb{R}$ and $f_{2,i}^n : (-\mu_i n, (\mu_i n)^{-1/2}, \infty) \to \mathbb{R}$ given by
\begin{align*}
f_{1,i}^n(x) := \left( \lambda_1^0 + (\lambda_i n)^{1/2} x \right) \log \left( 1 + (\lambda_i n)^{1/2} x / \lambda_1^n \right) - (\lambda_i n)^{1/2} x,
\end{align*}
\begin{align*}
f_{2,i}^n(x) := \left( \mu_1^0 + (\mu_i n)^{1/2} x \right) \log \left( 1 + (\mu_i n)^{1/2} x / \mu_1^n \right) - (\mu_i n)^{1/2} x.
\end{align*}
To obtain (4.80), recall (4.38). Then it is sufficient to show that
\begin{align}
\lim_{k \to \infty} \lim_{n \to \infty} \mathbb{E}^\mathbb{Q}\left[ \int_0^\infty e^{-\xi t} \| f_{1,i}^n(\hat{\psi}_{1,i}^n(t)) - f_{1,i}^n(\hat{\psi}_{1,i}^n(t)) \| dt \right] = 0,
\end{align}
\begin{align}
\lim_{k \to \infty} \lim_{n \to \infty} \mathbb{E}^\mathbb{Q}\left[ \int_0^\infty e^{-\xi t} \| f_{2,i}^n(\hat{\psi}_{2,i}^n(t)) - f_{2,i}^n(\hat{\psi}_{2,i}^n(t)) \| dT^n_i(t) \right] = 0.
\end{align}
We start with the limit in (4.81). Simple algebraic manipulation gives the bound
\[ \| W^{*,n,k} - W^{0,n} \|_t \leq C(\| \dot{A}^{n,k} - \dot{A}^n \|_t + \| \ddot{D}^{n,k} - \ddot{D}^n \|_t), \]
where \( C \) refers to a finite positive constant that is independent of \( n \) and \( t \) and which can change from one line to the next. From (4.70) and Burkholder–Davis–Gundy inequality we get that for every \( i \in \{I\} \),
\[
\mathbb{E}^\hat{Q}[\| A_{i}^{n,k} - A_{i}^n \|^2_t] \leq Cn^{-1/2} \mathbb{E}^\hat{Q}\left[ \int_0^t |\hat{\psi}_{1,i}^n(s)| \mathbbm{1}_{\{\hat{\psi}_{1,i}^n(s) > k\}} ds \right].
\]
Now, changing the order of integration gives
\[
\lim_{k \to \infty} \limsup_{n \to \infty} \mathbb{E}^\hat{Q}\left[ \int_0^\infty e^{-\rho t} \| A_{i}^{n,k} - A_{i}^n \|^2 dt \right] \leq \lim_{k \to \infty} \limsup_{n \to \infty} Cn^{-1/2} \mathbb{E}^\hat{Q}\left[ \int_0^\infty e^{-\rho t} \left\{ \int_0^t |\hat{\psi}_{1,i}^n(s)| \mathbbm{1}_{\{\hat{\psi}_{1,i}^n(s) > k\}} ds \right\} dt \right] \leq \lim_{k \to \infty} \limsup_{n \to \infty} Cn^{-1/2} \mathbb{E}^\hat{Q}\left[ \int_0^\infty g e^{-\rho t} |\hat{\psi}_{1,i}^n(t)| \mathbbm{1}_{\{\hat{\psi}_{1,i}^n(t) > k\}} dt \right] = 0.
\]
Pay attention that by Proposition 4.2, the last limit holds even without the \( n^{-1/2} \) term.

The rest of the sufficient limits, namely (4.82), (4.83), (4.84), and (4.85), are treated similarly: notice that
\[
\| \hat{\psi}_{2,i}^{n,k} - \hat{\psi}_{2,i}^n \|_t \leq C \sum_{i=1}^I \int_0^t |\hat{\psi}_{1,i}^n(s)| \mathbbm{1}_{\{\hat{\psi}_{1,i}^n(s) > k\}} ds,
\]
\[
|f_{1,i}^n(\hat{\psi}_{1,i}^{n,k}(t)) - f_{1,i}^n(\hat{\psi}_{1,i}^n(t))| \leq C \sum_{i=1}^I f_{1,i}^n(\hat{\psi}_{1,i}^n(t)) \mathbbm{1}_{\{f_{1,i}^n(\hat{\psi}_{1,i}^n(t)) > k\}},
\]
and similar bounds apply for \( j = 2 \) as well. We may continue in the same way as we did in (4.86), where now the \( n^{1/2} \) term is absent, using the two bounds in Proposition 4.2.

We now show that the change of measure penalty in the truncated case can be approximated by a quadratic form. From (4.38), the bound \( \sup_{t \in \mathbb{R}^+} |\psi_{1,i}^{n,k}(t)| \leq k \), and Taylor’s expansion of \( \log(1 + x) \), one has
\[
L_1^\rho(\hat{Q}_{1,i}^{n,k} || \hat{P}_{1,i}^n) = \mathbb{E}^\hat{Q}\left[ \frac{1}{2} \int_0^\infty e^{-\rho t}(\hat{\psi}_{1,i}^{n,k}(t))^2 dt \right] + o(1).
\]
Similarly,
\[
L_2^\rho(\hat{Q}_{2,i}^{n,k} || \hat{P}_{2,i}^n) = \mathbb{E}^\hat{Q}\left[ \frac{1}{2} \int_0^\infty e^{-\rho t}(\hat{\psi}_{2,i}^{n,k}(t))^2 dt \right] + o(1).
\]
From (4.77) it follows that,
\[
\liminf_{n \to \infty} \left[ \sum_{i=1}^I \frac{1}{K_{1,i}} L_1^\rho(\hat{Q}_{1,i}^{n,k} || \hat{P}_{1,i}^n) + \sum_{i=1}^I \frac{1}{K_{2,i}} L_2^\rho(\hat{Q}_{2,i}^{n,k} || \hat{P}_{2,i}^n) \right] \geq \sum_{i=1}^I \frac{1}{K_{1,i}} \mathbb{E}^\hat{Q}\left[ \frac{1}{2} \int_0^\infty e^{-\rho t}(\hat{\phi}_{1,i}^k(t))^2 dt \right] + \sum_{i=1}^I \frac{1}{K_{2,i}} \mathbb{E}^\hat{Q}\left[ \frac{1}{2} \int_0^\infty e^{-\rho t}(\hat{\phi}_{2,i}^k(t))^2 dt \right] + o(1).
\]
The identity
\[
\frac{1}{k_1} \alpha_1^2 + \frac{1}{k_2} \alpha_2^2 \geq \frac{1}{k_1 + k_2} (\alpha_1 - \alpha_2)^2, \quad k_1, k_2 > 0, \quad \alpha_1, \alpha_2 \in \mathbb{R},
\]
implies that
\[
\frac{1}{2 \kappa_{1,i}} (\hat{\phi}_{1,i}^k(t))^2 + \frac{1}{2 \kappa_{2,i}} \rho_i (\hat{\phi}_{2,i}^k(t))^2 \geq \frac{1}{4 \epsilon_i} \left( \hat{\phi}_{1,i}^k(t) - \rho_i^{1/2} \hat{\phi}_{2,i}^k(t) \right)^2,
\]
(4.88)
where recall that \( \epsilon_i = \frac{1}{2} (\kappa_{1,i} + \kappa_{2,i}) \). Set
\[
\phi^{\delta,k} := \sigma^{-1} \sum_{i=1}^I \theta_i \left( \lambda_i^{1/2} \hat{\phi}_{1,i}^k - \mu_i^{1/2} \hat{\phi}_{2,i}^k \rho_i \right).
\]
By Cauchy–Schwarz inequality,
\[
\left[ \sigma^{-2} \sum_{i=1}^I (\theta \hat{\sigma})^2 \epsilon_i \right] \times \sum_{i=1}^I \frac{1}{2 \epsilon_i} \left( \hat{\phi}_{1,i}^k(t) - \rho_i^{1/2} \hat{\phi}_{2,i}^k(t) \right)^2 \geq \frac{1}{2 \epsilon} (\phi^{\delta,k}(t))^2.
\]
Then,
\[
\sum_{i=1}^I \frac{1}{4 \epsilon_i} \left( \hat{\phi}_{1,i}^k(t) - \rho_i^{1/2} \hat{\phi}_{2,i}^k(t) \right)^2 \geq \frac{1}{2 \epsilon} (\phi^{\delta,k}(t))^2,
\]
where recall that \( \epsilon = \sigma^{-2} \sum_{i=1}^I (\theta \hat{\sigma})^2 \epsilon_i \).

From (4.87), (4.88), and the last bound, we obtain that
\[
\liminf_{n \to \infty} \left[ \sum_{i=1}^I \frac{1}{\kappa_{1,i}} L_i^\epsilon (\hat{Q}_{1,i}^{n,k} || \hat{P}_{1,i}^{n}) + \sum_{i=1}^I \frac{1}{\kappa_{2,i}} L_i^\epsilon (\hat{Q}_{2,i}^{n,k} || \hat{P}_{2,i}^{n}) \right] \geq \frac{1}{2 \epsilon} \mathbb{E} \hat{Q} \left[ \int_0^\infty e^{-\nu t} (\phi^{\delta,k}(t))^2 dt \right].
\]
(4.89)

Fix \( \delta_1 > 0 \). Combining the last limit with the ones from (4.69) and (4.79) and recalling (4.67) and the bound \( \| R^{\infty,n} \| \leq \| \theta^n || \hat{R}^n \| \), we get that there is \( k_{\delta_1} > 0 \) such that for every \( k \geq k_{\delta_1} \),
\[
\limsup_{n \to \infty} J^n(X^n \circ 0, U^n \circ a, R^n \circ a, \hat{Q}^{n}; \kappa) \leq \mathbb{E} \hat{Q} \left[ \int_0^\infty e^{-\nu t} \{ h^n(X^{\circ,k}(t)) + \nu R^{\circ,k}(t) - \frac{1}{2 \epsilon} (\phi^{\delta,k}(s))^2 \} dt \right] + \delta_1/2,
\]
(4.90)
where notice that (4.74)–(4.76), and the definition of \( \phi^{\delta,k} \) gives
\[
(X^{\circ,k}, Y^{\circ,k}, R^{\circ,k}) = \Gamma_{[0,a]} \left[ x_0 + W^{\circ,k} + \sigma \int_0^t \phi^{\delta,k}(s) ds \right].
\]
Fix such $k$. Since $a \rightarrow \beta_\varepsilon$ as $\delta_0 \rightarrow 0$ (see the paragraph before (4.1)), Lemma 3.1 and (4.2) imply that
\[
\int_0^\infty e^{-gt} \{ h^a(\bar{X}^{o,k}(t)) + g\bar{R}^{o,k}(t) - \frac{1}{2\varepsilon}(\phi^{\sharp,k}(s))^2 \} dt \\
\rightarrow \tilde{Q} \int_0^\infty e^{-gt} \{ h(\bar{X}(t)) + g\bar{R}(t) - \frac{1}{2\varepsilon}(\phi^{\sharp,k}(s))^2 \} dt,
\]
as $\delta_0 \rightarrow 0$, where
\[
(\bar{X}, \bar{Y}, \bar{R}) = \Gamma_{[0,\beta_\varepsilon]}(x_0 + \bar{W}^{o,k} + \sigma \int_0^\infty \phi^{\sharp,k}(s) ds).
\]

**Obtaining the upper bound.** As argued earlier, to conclude the convergence in expectation of the above it is sufficient to show that
\[
\limsup_{a \rightarrow \beta_\varepsilon} \mathbb{E} \left[ \int_0^\infty e^{-gt} \|\bar{R}^{o,k}(t)\|^2 dt \right] < \infty.
\]
(4.91)
The proof is borrowed from [6] (85) and adapted to our case with the additional stochastic drift $\sigma \int_0^t \phi^{\sharp,k}(s) ds$. Apply Itô’s formula to $(\bar{X}^{o,k})^2$, use the reflection conditions $\int_0^t \bar{X}^{o,k}(s) d\bar{Y}^{o,k}(s) = 0$ and $\int_0^t \bar{X}^{o,k} d\bar{R}^{o,k}(s) = a\bar{R}^{o,k}(t)$, to get
\[
\bar{R}^{o,k}(t) = \frac{1}{2a} \left\{ (\bar{X}^{o,k}(0))^2 - (\bar{X}^{o,k}(t))^2 + 2 \int_0^t \bar{X}^{o,k}(s) [d\bar{W}^{o,k}(s) + \sigma \phi^{\sharp,k}(s) ds] + \sigma^2 t \right\}.
\]
Since $\bar{X}^{o,k}$ and $\sigma \phi^{\sharp,k}(s)$ are bounded, (4.91) follows easily.

Recall the definition of $J$ and $V$ from Section 3.2 and that $\bar{X}^{o,k}(0) = x_0$, then
\[
\lim_{\delta_0 \rightarrow 0} \mathbb{E}^\mathcal{Q} \left[ \int_0^\infty e^{-gt} \{ h^a(\bar{X}^{o,k}(t)) + g\bar{R}^{o,k}(t) - \frac{1}{2\varepsilon}(\phi^{\sharp,k}(t))^2 \} dt \right] \\
= \mathbb{E}^\mathcal{Q} \left[ \int_0^\infty e^{-gt} \{ h(\bar{X}(t)) + g\bar{R}(t) - \frac{1}{2\varepsilon}(\phi^{\sharp,k}(t))^2 \} dt \right] \\
= J(\bar{X}^{o,k}(0), \bar{Y}, \bar{R}, \mathcal{Q}; \varepsilon) \\
\leq V(x_0; \varepsilon),
\]
where $\mathcal{Q}$ is the measure associated with the rate $\phi^{\sharp,k}$. The last inequality follows by the optimality of the minimizer’s reflected strategy, see Proposition 3.2.

Together with (4.90), we obtain that for sufficiently small $\delta_0 > 0$ and large $n$,
\[
J^n(X^n(0), U^n(a), R^n(a); \mathbb{Q}^n; \kappa) \leq V(x_0; \varepsilon) + \delta_1.
\]
Since the constant $\delta_1(> 0)$ and the measures $\{\mathbb{Q}^n\}_n$ were chosen arbitrary, we get (4.7).

**Appendix**

**Lemma A.1 (Theorem IV.4.5 in [34])** Let $u : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a right-continuous function such that $u(0) = 0$. Let $v(t) := \inf \{ s \geq 0 : u(s) > t \}$, $t \in \mathbb{R}_+$. Assume that $v(t) < \infty$ for all
t ∈ R_+. Let f be a nonnegative Borel-measurable function on R_+ and let F : R_+ → R_+ be a right-continuous, nondecreasing function. Then
\[ \int_0^\infty f(s)dF(u(s)) = \int_0^\infty f(v(s-))dF(s), \]
where we use the convention that the contribution to the integrals above at 0 is f(0)F(0).

**Lemma A.2 (Lemma 2.4 in [17]):** Let \( \{\xi_n\}_{n \in \mathbb{N}} \) and \( \xi \) be \( \mathcal{D}(\mathbb{R}, \mathbb{R}_+) \) functions and \( \{\zeta^n\}_{n \in \mathbb{N}} \) and \( \zeta \) be nondecreasing functions in \( \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+) \). Assume that \( \xi^n \to \xi \) u.o.c. in the Skorokhod \( J_1 \) topology and \( \zeta^n \to \zeta \) u.o.c. in the uniform norm topology. Also, for every bounded and continuous \( f : \mathbb{R}_- \to \mathbb{R} \), one has
\[ \int_0^s f(\xi^n(t))d\zeta^n(t) \to \int_0^s f(\xi(t))d\zeta(t), \]
u.o.c. in the uniform norm topology.

**Lemma A.3** Let \( \{f_n\}_{n \in \mathbb{N}} \) and f be bounded integrable functions mapping \( \mathbb{R}_+ \) to \( \mathbb{R} \). Also, let \( \{\zeta^n\}_{n \in \mathbb{N}} \) and \( \zeta \) be nondecreasing and continuous functions mapping \( \mathbb{R}_+ \) to itself such that \( \zeta^n(t) - \zeta^n(s) \leq t - s \) for every \( 0 \leq s \leq t \). Assume that \( \zeta^n \to \zeta \) and \( \int_0^s f^n(t)dt \to \int_0^s f(t)dt \) in the u.o.c. uniform norm topology. Then, for every \( \rho > 0 \),
\[ \liminf_{n \to \infty} \int_0^\infty e^{-\rho t}(f^n(t))^2 d\zeta^n(t) \geq \int_0^\infty e^{-\rho t}(f(t))^2 d\zeta(t). \]

**Proof.** The functions \( \{f^n\}_n \) and f are bounded and \( \zeta^n(t) - \zeta^n(s) \leq t - s \) for every \( 0 \leq s \leq t \). Therefore,
\[ \lim_{T \to \infty} \left( \int_T^\infty e^{-\rho t}(f^n(t))^2 d\zeta^n(t) + \int_T^\infty e^{-\rho t}(f(t))^2 \rho dt \right) = 0. \]

Thus, it is sufficient to prove the required bound with \( T \) replacing \( \infty \) in the upper limit of the integrals.

Fix \( \nu > 0 \). From the assumptions in the lemma, there exists \( n_0 \in \mathbb{N} \), such that for every \( n \geq n_0 \)
\[ \left\| \int_0^T f^n(t)dt - \int_0^T f(t)dt \right\| \leq \nu. \quad (A.1) \]

Every measurable function is the (a.s. w.r.t. Lebesgue measure) pointwise limit of step functions (see [36] Theorem 4.3]). Denote them by \( \{g^m\}_{m \in \mathbb{N}} \). Since f is bounded, we may and will take the step functions to be uniformly bounded. From Egorov’s theorem it follows that there is a set \( B_\nu \) with Lebesgue measure smaller than \( \nu \) such that \( \lim_{m \to \infty} \sup_{s \in [0,T] \setminus B_\nu} |g^m(s) - f(s)| = 0. \)
Fix \( m_0 \) such that
\[ \sup_{s \in [0,T] \setminus B_\nu} |g^{m_0}(s) - f(s)| \leq \nu. \quad (A.2) \]
By the definition of Riemann-Stieltjes integral, for every partition $0 = s_0 < s_1 < \ldots < s_L = T$ with sufficiently small mesh, one has

$$\left| \sum_{l=1}^{L} e^{-\varrho s_{l+1}} (g^{m_0}(s_l))^2 \Delta^n_l - \int_{0}^{T} e^{-\varrho t} (g^{m_0}(t))^2 d\zeta^n(t) \right| \leq \nu. \quad (A.3)$$

W.l.o.g. we may and will take a partition that refines the one that is generated by the steps of the function $g^{m_0}$. Let $\omega_3 : \mathbb{R}_+ \to \mathbb{R}$ stand for a continuous function that satisfies $\omega_3(0+) = 0$, and which can change from one line to the next. Denote $\Delta^n_l := \zeta^n(s_{l+1}) - \zeta^n(s_l)$. From the monotonicity of $t \mapsto e^{-\varrho t}$, Cauchy–Schwartz inequality, and (A.1),

$$\int_{0}^{T} e^{-\varrho t} (f^n(t))^2 d\zeta^n(t) \geq \sum_{l=0}^{L-1} e^{-\varrho s_{l+1}} \int_{s_l}^{s_{l+1}} (f^n(t))^2 d\zeta^n(t) \geq \sum_{l=0}^{L-1} e^{-\varrho s_{l+1}} \left( \frac{1}{\Delta^n_l} \int_{s_l}^{s_{l+1}} f^n(t) d\zeta^n(t) \right)^2 \Delta^n_l \geq \sum_{l=0}^{L-1} e^{-\varrho s_{l+1}} \left( \frac{1}{\Delta^n_l} \int_{s_l}^{s_{l+1}} f(t) d\zeta^n(t) \right)^2 \Delta^n_l + \omega_3(\nu).$$

Define the following set of indexes $E_\nu := \{0 \leq l \leq L - 1 : [s_l, s_{l+1}] \cap B_\nu = \emptyset\}$. Notice that by refining the partition and recalling that the functions $f$ and $g^{m_0}$ are bounded and (A.2), we get

$$= \sum_{l \in E_\nu} e^{-\varrho s_{l+1}} \left( \frac{1}{\Delta^n_l} \int_{s_l}^{s_{l+1}} f(t) d\zeta^n(t) \right)^2 \Delta^n_l + \omega_3(\nu)$$

$$= \sum_{l \in E_\nu} e^{-\varrho s_{l+1}} \left( \frac{1}{\Delta^n_l} \int_{s_l}^{s_{l+1}} g^{m_0}(t) d\zeta^n(t) \right)^2 \Delta^n_l + \omega_3(\nu)$$

$$= \sum_{l=0}^{L-1} e^{-\varrho s_{l+1}} \left( \frac{1}{\Delta^n_l} \int_{s_l}^{s_{l+1}} g^{m_0}(t) d\zeta^n(t) \right)^2 \Delta^n_l + \omega_3(\nu).$$

Recall that the partition refines the one that is generated by the steps of $g^{m_0}$. Then (A.2), and (A.3) give

$$= \sum_{l=0}^{L-1} e^{-\varrho s_{l+1}} (g^{m_0}(s_l))^2 \Delta^n_l + \omega_3(\nu) = \int_{0}^{T} e^{-\varrho t} (g^{m_0}(t))^2 d\zeta^n(t) + \omega_3(\nu)$$

$$= \int_{0}^{T} e^{-\varrho t} (f(t))^2 d\zeta^n(t) + \omega_3(\nu).$$

Since $\nu > 0$ can be arbitrary small, we get

$$\liminf_{n \to \infty} \left[ \int_{0}^{T} e^{-\varrho t} (f^n(t))^2 d\zeta^n(t) - \int_{0}^{T} e^{-\varrho t} (f(t))^2 d\zeta^n(t) \right] \geq 0.$$
From Lemma A.2, we get that
\[
\lim_{n \to \infty} \int_0^T e^{-\varrho t}(f(t))^2 d\zeta_n(t) = \int_0^T e^{-\varrho t}(f(t))^2 d\zeta(t),
\]
and together with the last inequality, the result holds.

\[\square\]

References

[1] R. Atar and A. Biswas. Control of the multiclass $G/G/1$ queue in the moderate deviation regime. *Ann. Appl. Probab.*, 24(5):2033–2069, 2014.

[2] R. Atar and A. Cohen. A differential game for a multiclass queueing model in the moderate-deviation heavy-traffic regime. *Math. Oper. Res.*, 41(4):1354–1380, 2016.

[3] R. Atar and A. Cohen. Asymptotically optimal control for a multiclass queueing model in the moderate deviation heavy traffic regime. *Ann. Appl. Probab.*, to appear, 2017.

[4] R. Atar and G. Mendelson. On the non-Markovian multiclass queue under risk-sensitive cost. *Queueing Systems Theory Appl.*, to appear, 2016.

[5] R. Atar and S. Saha. Optimality of the generalized $c\mu$ rule in the moderate deviation regime. *Queueing Systems*, 87(1):113–130, Oct 2017.

[6] R. Atar, M. Shifrin, et al. An asymptotic optimality result for the multiclass queue with finite buffers in heavy traffic. *Stochastic Systems*, 87(1):113–130, Oct 2017.

[7] E. Bayraktar and Y. Zhang. Minimizing the probability of lifetime ruin under ambiguity aversion. *SIAM J. Control Optim.*, 53(1):58–90, 2015.

[8] S. L. Bell and R. J. Williams. Dynamic scheduling of a system with two parallel servers in heavy traffic with resource pooling: asymptotic optimality of a threshold policy. *Ann. Appl. Probab.*, 11(3):608–649, 2001.

[9] P. Billingsley. *Convergence of Probability Measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., New York, second edition, 1999. A Wiley-Interscience Publication.

[10] A. Biswas. Risk-sensitive control for the multiclass many-server queues in the moderate deviation regime. *Math. Oper. Res.*, 39(3):908–929, 2014.

[11] J. Blanchet, C. Dolan, and H. Lam. Robust rare-event performance analysis with natural non-convex constraints. In *Proceedings of the 2014 Winter Simulation Conference*, pages 595–603. IEEE Press, 2014.

[12] A. Budhiraja and A. P. Ghosh. Diffusion approximations for controlled stochastic networks: an asymptotic bound for the value function. *Ann. Appl. Probab.*, 16(4):1962–2006, 2006.
[13] A. Budhiraja and A. P. Ghosh. Controlled stochastic networks in heavy traffic: convergence of value functions. *Ann. Appl. Probab.*, 22(2):734–791, 2012.

[14] A. Budhiraja and K. Ross. Existence of optimal controls for singular control problems with state constraints. *Ann. Appl. Probab.*, 16(4):2235–2255, 2006.

[15] A. Budhiraja and K. Ross. Convergent numerical scheme for singular stochastic control with state constraints in a portfolio selection problem. *SIAM J. Control Optim.*, 45(6):2169–2206, 2007.

[16] A. Cohen. Stochastic differential games for a multiclass M/M/1 queueing problem with model uncertainty. *Arxiv preprint*, 2017, https://arxiv.org/abs/1702.06479.

[17] J. G. Dai and R. J. Williams. Existence and uniqueness of semimartingale reflecting brownian motions in convex polyhedrons. *Theory of Probability & Its Applications*, 40(1):1–40, 1996.

[18] C. Dellacherie and P.-A. Meyer. *Probabilities and potential*, volume 29 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam-New York; North-Holland Publishing Co., Amsterdam-New York, 1978.

[19] S. N. Ethier and T. G. Kurtz. *Markov processes*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York, 1986. Characterization and convergence.

[20] L. P. Hansen and T. J. Sargent. *Robustness*. Princeton University Press, Princeton, NJ, 2008.

[21] L. P. Hansen, T. J. Sargent, G. Turmuhambetova, and N. Williams. Robust control and model misspecification. *J. Econom. Theory*, 128(1):45–90, 2006.

[22] J. M. Harrison. Brownian models of queueing networks with heterogeneous customer populations. In *Stochastic differential systems, stochastic control theory and applications (Minneapolis, Minn., 1986)*, volume 10 of *IMA Vol. Math. Appl.*, pages 147–186. Springer, New York, 1988.

[23] A. Jain, A. E. B. Lim, and J. G. Shanthikumar. On the optimality of threshold control in queues with model uncertainty. *Queueing Syst.*, 65(2):157–174, 2010.

[24] I. Karatzas and S. E. Shreve. *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.

[25] L. Kruk, J. Lehoczky, K. Ramanan, and S. Shreve. An explicit formula for the Skorokhod map on [0, a]. *Ann. Probab.*, 35(5):1740–1768, 2007.

[26] T. G. Kurtz. Random time changes and convergence in distribution under the Meyer-Zheng conditions. *Ann. Probab.*, 19(3):1010–1034, 1991.

[27] H. J. Kushner and L. F. Martins. Numerical methods for stochastic singular control problems. *SIAM J. Control Optim.*, 29(6):1443–1475, 1991.
[28] H. J. Kushner and L. F. Martins. Limit theorems for pathwise average cost per unit time problems for controlled queues in heavy traffic. *Stochastics Stochastics Rep.*, 42(1):25–51, 1993.

[29] H. Lam. Robust sensitivity analysis for stochastic systems. *Math. Oper. Res.*, 41(4):1248–1275, 2016.

[30] P. J. Maenhout. Robust portfolio rules and asset pricing. *Review of financial studies*, 17(4):951–983, 2004.

[31] L. F. Martins and H. J. Kushner. Routing and singular control for queueing networks in heavy traffic. *SIAM J. Control Optim.*, 28(5):1209–1233, 1990.

[32] P.-A. Meyer and W. A. Zheng. Tightness criteria for laws of semimartingales. *Ann. Inst. H. Poincaré Probab. Statist.*, 20(4):353–372, 1984.

[33] E. Plambeck, S. Kumar, and J. M. Harrison. A multiclass queue in heavy traffic with throughput time constraints: asymptotically optimal dynamic controls. *Queueing Syst.*, 39(1):23–54, 2001.

[34] P. E. Protter. *Stochastic integration and differential equations*, volume 21 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, second edition, 2004. Stochastic Modelling and Applied Probability.

[35] M. Sîrbu. A note on the strong formulation of stochastic control problems with model uncertainty. *Electron. Commun. Probab.*, 19:no. 81, 10, 2014.

[36] E. M. Stein and R. Shakarchi. *Real analysis*, volume 3 of *Princeton Lectures in Analysis*. Princeton University Press, Princeton, NJ, 2005. Measure theory, integration, and Hilbert spaces.

[37] D. W. Stroock. *Lectures on stochastic analysis: diffusion theory*, volume 6 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1987.