Partition Functions of Supersymmetric Gauge Theories in Noncommutative $\mathbb{R}^{2D}$ and their Unified Perspective

Akifumi Sako†, Toshiya Suzuki∗

† Department of Mathematics, Faculty of Science and Technology, Keio University
3-14-1 Hiyoshi, Kohoku-ku, Yokohama 223-8522, Japan

∗ Department of Physics, Faculty of Science, Ochanomizu University
2-1-1 Otsuka, Bunkyo-ku, Tokyo 112-8610, Japan

E-mail: † sako@math.keio.ac.jp
∗ tsuzuki@phys.ocha.ac.jp

Abstract

We investigate cohomological gauge theories in noncommutative $\mathbb{R}^{2D}$. We show that vacuum expectation values of the theories do not depend on noncommutative parameters, and the large noncommutative parameter limit is equivalent to the dimensional reduction. As a result of these facts, we show that a partition function of a cohomological theory defined in noncommutative $\mathbb{R}^{2D}$ and a partition function of a cohomological field theory in $\mathbb{R}^{2D+2}$ are equivalent if they are connected through dimensional reduction. Therefore, we find several partition functions of supersymmetric gauge theories in various dimensions are equivalent. Using this technique, we determine the partition function of the $\mathcal{N} = 4$ U(1) gauge theory in noncommutative $\mathbb{R}^4$, where its action does not include a topological term. The result is common among (8-dim , $\mathcal{N} = 2$), (6-dim , $\mathcal{N} = 2$), (2-dim , $\mathcal{N} = 8$) and the IKKT matrix model given by their dimensional reduction to 0-dim.
I Introduction

The first break through of the recent calculation technology for $\mathcal{N} = 2$ supersymmetric Yang-Mills theories is brought by Nekrasov [21, 22]. After [21], many kinds of developments appear in $\mathcal{N} \geq 2$ supersymmetric Yang-Mills theories and string theories corresponding to them. From those analysis, it is found that different dimension theories are related each other [4, 17, 18, 34, 36]. There is more example that the different dimensional theories are connected to each other. For example, Dijkgraaf and Vafa show that some correlation functions in matrix theories and $\mathcal{N} = 1$ Yang-Mills theories are equivalent [5]. It goes on and on. These facts imply the existence of some kind of unified perspectives. One of the ideas to explain the unification is the ’tHooft’s large $N$ gauge theory and string correspondence. Until now, many investigations from this point of view are reported. Meanwhile, the large $N$ gauge theories are similar to noncommutative theories in the operator formalism in some infinite dimensional Hilbert space with discrete basis. In this article, we suggest a simple way to understand the reason why partition functions of various dimensional supersymmetric gauge theories are given as same form or have relations with each other. The basic idea of the way is given in [27, 28, 29]. Cohomological gauge theories in Euclidian spaces are invariant under the noncommutative parameter shifting, as we will see it in the next section. When we take the large noncommutative parameter limit, kinetic terms become irrelevant like dimensional reduction, then the partition function is essentially computable by using lower dimensional theories. From this fact, we will explain that partition functions in various dimensions are equivalent.

Here is the organization of this article. In section II, invariance of cohomological field theories in noncommutative $\mathbb{R}^{2D}$ (N.C. $\mathbb{R}^{2D}$ for short) under deformation of noncommutative parameters will be proved formally. This invariance is not usual symmetry, because the action is deformed. Nevertheless, expectation values and partition functions are invariant. Particularly, we will treat the $\mathcal{N} = 2$ and $\mathcal{N} = 4$ Yang-Mills theories in N.C. $\mathbb{R}^4$ as examples. In section III, universality of the partition functions will be investigated. By using the result of section II, we will show that the several partition functions in different dimensions are equivalent. (In appendix B, concrete discussions for some models will be given again.) In section IV, by the technique of section II we will calculate the partition function of the $\mathcal{N} = 4$ U(1) gauge theory in N.C. $\mathbb{R}^4$ without the terms proportional to the instanton number $\int F \wedge F$. This partition function is equal to partition functions of several dimensions. In section V, the moduli space of $\mathcal{N} = 4$ U(1) gauge theory in N.C. $\mathbb{R}^4$ will be discussed. The partition function of $\mathcal{N} = 4$ U(1) theory with $\int F \wedge F$ will be investigated, too. In section VI we will summarize this article.
II  N.C. Cohomological Yang-Mills Theory

In this section, we investigate some important properties of the cohomological Yang-Mills theories in N.C. $\mathbb{R}^{2D}$ whose noncommutativity is defined as

$$[x^\mu, x^\nu] = i\theta^{\mu\nu},$$

where the $\theta^{\mu\nu}$ is an element of an antisymmetric matrix and called noncommutative parameter.

Since action functionals of cohomological field theories are defined by BRS-exact functionals like $\hat{\delta}\Psi[\phi_i]$, where $\hat{\delta}$ is a some BRS operator and $\{\phi_i\}$ represent all considered fields and $\Psi$ is a some fermionic functional, the partition function of the cohomological field theory is invariant under any infinitesimal variation $\delta'$ which commutes (or anti-commutes) with the BRS transformation:

$$\hat{\delta}\delta' = \pm\delta'\hat{\delta},$$

$$\delta' Z_\theta = \int \prod_i D\phi_i \delta' \left( -\int dx^{2D}\hat{\delta}\Psi \right) \exp(-S_\theta) = \pm \int \prod_i D\phi_i \delta \left( -\int dx^{2D}\delta'\Psi \right) \exp(-S_\theta) = 0.$$  \hspace{1cm} (2)

Let $\delta_\theta$ be the infinitesimal deformation operator of the noncommutative parameter $\theta$ which operates as

$$\delta_\theta \theta^{\mu\nu} = \delta\theta^{\mu\nu},$$

where $\delta\theta^{\mu\nu}$ are some infinitesimal anti-symmetric two form elements. If $\delta_\theta$ and BRS operator $\hat{\delta}$ commute each other, then the partition function is invariant. Indeed, there is some examples such that $\hat{\delta}\delta_\theta = \delta_\theta\hat{\delta}$, and partition functions are calculated by using this property \cite{27, 28, 29}.

In this article, cohomological Yang-Mills theories in noncommutative Euclidian spaces are discussed. If there is a gauge symmetry, the BRS-like transformation is slightly different from the one of non-gauge theory. The BRS-like symmetry is not nilpotent but

$$\delta^2 = \delta_{g,\theta},$$

where $\delta_{g,\theta}$ is a gauge transformation operator deformed by some noncommutative deformation method like the star product $\ast_\theta$. As occasion arises, the gauge transformation $\delta_{g,\theta}$ is defined as one including global symmetry transformations. The partition function of the noncommutative cohomological field theory is invariant under changing noncommutative parameters when the BRS transformation does not depend on the noncommutative...
parameters, because the BRS transformation \( \hat{\delta} \) and the \( \theta \) deformation \( \delta_{\theta} \) commute. Conversely, when the definition of the BRS-like operator \( \hat{\mathcal{H}} \) depends on the noncommutative parameter \( \theta \), then \( \hat{\delta} \) and \( \delta_{\theta} \) do not commute:

\[
\delta_{\theta} \hat{\delta} \neq \hat{\delta} \delta_{\theta} \Rightarrow \delta_{\theta} \hat{\delta} = \hat{\delta} \delta_{\theta} \tag{5}
\]

where \( \hat{\delta}' \) is a BRS-like operator that generates the same transformations as the original BRS-like operator \( \hat{\delta} \), except for the square. The square of \( \hat{\delta} \) is defined by

\[
\hat{\delta}'^2 = \delta_{g,\theta} + \delta_{\theta} \tag{6}
\]

Since the gauge symmetry is defined by using noncommutative parameter \( \theta_{\mu \nu} + \delta \theta_{\mu \nu} \) after the \( \delta_{\theta} \) operation, this difference arises. This fact makes a little complex problem to prove the \( \theta \)-shift invariance of noncommutative cohomological Yang-Mills theory in comparison with the case of non-gauge theory.

Note that the essential point of this problem is not nilpotent property changing, but \( \theta \) dependence of the definition of the BRS operator. (In fact, we can construct the BRS operator for the cohomological Yang-Mills theory as a nilpotent operator \( \mathcal{H} \).)

However, we can prove the invariance of the partition function of cohomological Yang-Mills theory in N.C. \( \mathbb{R}^{2D} \) under the noncommutative parameter deformation. For simplicity, we take

\[
(\theta_{\mu \nu}) = \bigoplus_i \epsilon_{2i-1,2i} \theta = \theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]

where \( \epsilon_{2i-1,2i} \) is an antisymmetric tensor such that \( \epsilon_{2i-1,2i} = -\epsilon_{2i,2i-1} = 1 \), and we restrict the \( \theta \) deformation to

\[
\theta \to \theta + \delta \theta.
\]

In the following, we only use operator formalisms to describe the noncommutative field theory, therefore the fields are operators acting on the Hilbert space \( \mathcal{H} \). Then differential operators \( \partial_\mu \) are expressed by using commutation brackets \( -i \theta_{\mu \nu}^{-1} [x^\nu, \cdot] \equiv [\hat{\partial}_\mu, \cdot] \) and \( \int d^{2D} x \) is replaced with \( det(\theta)^{1/2} Tr_{\mathcal{H}} \). Therefore the noncommutative parameter deformation is equivalent with replacing \( -i \theta_{\mu \nu}^{-1} [x^\nu, \cdot] \) and \( det(\theta)^{1/2} Tr_{\mathcal{H}} \) by \( -i (\theta + \delta \theta)^{-1}_{\mu \nu} [x^\nu, \cdot] \) and \( det(\theta + \delta \theta)^{1/2} Tr_{\mathcal{H}} \), respectively.

Let us consider Donaldson-Witten theory (topological twisted \( \mathcal{N} = 2 \) Yang-Mills theory) on N.C. \( \mathbb{R}^4 \). This theory is constructed by bosonic fields \((A_{\mu}, H^+_{\mu \nu}, \bar{\phi}, \phi)\) and fermionic fields \((\psi_{\mu}, \chi^+_{\mu \nu}, \eta)\), where \((A_{\mu}, H^+_{\mu \nu}, \bar{\phi})\) and \((\psi_{\mu}, \chi^+_{\mu \nu}, \eta)\) are supersymmetric (BRS) pairs,

\[
\begin{align*}
\chi^+_{\mu \nu}, H^+_{\mu \nu} & \in \Omega^{2,+}(\mathbb{R}^4, \text{ad} P), \\
\psi_{\mu} & \in \Omega^{1}(\mathbb{R}^4, \text{ad} P), \\
\eta, \bar{\phi}, \phi & \in \Omega^{0}(\mathbb{R}^4, \text{ad} P).
\end{align*}
\tag{7}
\]
Their ghost numbers are assigned as \((A_\mu, \chi^\mu, H^\mu, \psi_\mu, \eta, \phi) = (0, -1, 0, 1, -1, -2, 2)\). The BRS-like operator is defined by

\[
\begin{align*}
\hat{\delta} A_\mu &= \psi_\mu, \\
\hat{\delta} \chi^\mu &= H^\mu, \\
\hat{\delta} \psi_\mu &= D_\mu \phi, \\
\hat{\delta} \phi &= 0,
\end{align*}
\]

where the covariant derivative is defined by

\[
D_\mu := \hat{\partial}_\mu + i A_\mu, \quad \hat{\partial}_\mu := -i \theta^{-1} \partial_\mu.
\]

When we consider only the case of N.C. \(\mathbb{R}^{2D}\), field theories are expressed by the Fock space formalism. (See appendix \([\Delta]\).) In the Fock space representation, fields are expressed as

\[
A_\mu = \sum A_{n_1 n_2}^{m_1 m_2} |m_1 m_2\rangle \langle n_1 n_2|, \quad \psi_\mu = \sum \psi_{n_1 n_2}^{m_1 m_2} |n_1 n_2\rangle \langle m_1 m_2|, \quad \text{etc.}
\]

Therefore, the above BRS transformations are expressed as

\[
\hat{\delta} A_{n_1 n_2}^{m_1 m_2} = \psi_{n_1 n_2}^{m_1 m_2}, \quad \cdots.
\]

Let us define gauge fermions as

\[
\begin{align*}
\Psi &= Tr_{\mathcal{H}} tr \left[ 2 \chi^\mu (-i F^{\mu\nu} + \frac{1}{2} H^{\mu\nu}) \right], \\
\Psi_{\text{proj}} &= -Tr_{\mathcal{H}} tr \left[ \bar{\phi} D_\mu \psi_\mu \right],
\end{align*}
\]

then the action functional is given by

\[
S = Tr_{\mathcal{H}} L(A_\mu, \ldots; \hat{\partial}_{z_1}, \hat{\partial}_{z_2})
\]

\[
= Tr_{\mathcal{H}} tr \hat{\delta}(\Psi + \Psi_{\text{proj}})
\]

\[
= Tr_{\mathcal{H}} tr \left( F^{+2} - 4i \chi^{+\mu\nu} D_\mu \psi_\nu - \eta D_\mu \psi_\mu + i \phi \{ \chi^{+\mu}, \chi^{+\nu} \} \right.
\]

\[
\left. -i \bar{\phi} \{ \psi_\mu, \psi_\mu^\nu \} - \bar{\phi} D_\mu \phi \right).
\]

where \(tr\) is trace for gauge group. In this article, we omit to note \(det(\theta)^{1/2}\) that is an overall factor, for economy of space. Let us change the dynamical variables as

\[
A_\mu \to \frac{1}{\sqrt{\theta}} \tilde{A}_\mu, \quad \psi_\mu \to \frac{1}{\sqrt{\theta}} \tilde{\psi}_\mu, \quad \bar{\phi} \to \frac{1}{\theta} \tilde{\phi}, \quad \eta \to \frac{1}{\theta} \tilde{\eta},
\]

\[
\chi^{+\mu\nu} \to \frac{1}{\theta} \tilde{\chi}^{+\mu\nu}, \quad H^{+\mu\nu} \to \frac{1}{\theta} \tilde{H}^{+\mu\nu}, \quad \phi \to \tilde{\phi}.
\]

Note that this changing does not cause nontrivial Jacobian from the path integral measure because of the BRS symmetry. Then, the action is rewritten as

\[
S \to \frac{1}{\theta^2} \tilde{S}, \quad L(A_\mu, \ldots; \hat{\partial}_{z_1}, \hat{\partial}_{z_2}) \to \frac{1}{\theta^2} L(\tilde{A}_\mu, \ldots; -a^i_1, a_i)
\]

Here the action in the lefthand side depends on \(\theta\) because the derivative is given by \(\hat{\partial}_{z_i} = -\sqrt{\theta^{-1}} [a^i_1, \ ]\) and so on. In contrast, the action \(\tilde{S}\) in the righthand side does not
depend on $\theta$ because all $\theta$ parameters are factorized out. Using the BRS symmetry or the fact of eq. (2), it is proved that the partition function is invariant under the deformation of $\theta$, because $\delta_\theta Z = -2(\delta \theta)\theta^{-3}(\bar{S}) = 0$. $Tr_{\text{H2dim}}(\phi F + \frac{1}{2}\psi \wedge \psi)$ and $tr\phi^3$ are known as observables of Donaldson-Witten theory. They are rewritten as $\frac{1}{\theta}Tr_H tr(\phi \bar{F} + \frac{1}{2}\bar{\psi} \wedge \psi)$ and $tr\phi^3$. We use $O$ to represent such observables, then $\delta_\theta (O) = 0$ are proved in a similar way to the proof of $\delta_\theta Z = 0$. Therefore, invariance of Donaldson-Witten theory under $\theta \to \theta + \delta \theta$ is proved.

We can discuss the topological twisted $\mathcal{N} = 4$ Yang-Mills theory in noncommutative $\mathbb{R}^4$ similarly a) b). There are additional fields $(B^+_{\mu\nu}, c, H_\mu)$ and $(\psi^+_{\mu\nu}, \bar{\eta}, \chi_\mu)$, where $(B^+_{\mu\nu}, c, H_\mu)$ are bosonic fields and $(\psi^+_{\mu\nu}, \bar{\eta}, \chi_\mu)$ are fermionic fields, where $B^+_{\mu\nu}, \psi^+_{\mu\nu} \in \Omega^{2+}(\mathbb{R}^4, \text{ad}P)$. They are supersymmetric partners, and the BRS multiplets are expressed by the following diagram.

There are two BRS-like operators $\hat{\delta}_+$ and $\hat{\delta}_-$ because of the R-symmetry of the $\mathcal{N} = 4$. The $\hat{\delta}_+$ transformations are given by

$$
\hat{\delta}_+ B^+_{\mu\nu} = \psi^+_{\mu\nu} \ , \ \psi^+_{\mu\nu} = i[B^+_{\mu\nu}, \phi] \tag{14}
$$

$$
\hat{\delta}_+ \chi_\mu = H_\mu \ , \ \hat{\delta}_+ H_\mu = i[\chi_\mu, \phi] \ , \ \hat{\delta}_+ c = \bar{\eta} \ , \ \hat{\delta}_+ \bar{\eta} = i[c, \phi], \tag{15}
$$

and the same transformations as (3) for other fields. The action of the topological twisted $\mathcal{N} = 4$ Yang-Mills theory without the $\tau \int F \wedge F$ is

$$
S = Tr_H tr \hat{\delta}_+ \{ \chi^+_{\mu\nu} (H^{+\mu\nu} - i(F^{+\mu\nu} - i(B^+_{\mu\nu}, B^+_{\nu\sigma}) \delta^{\rho\sigma} - i[B^+_{\mu\nu}, c])) \\
+ Tr_H tr \hat{\delta}_+ \{ \chi^\rho (H_\rho - i(-2D^\mu B^+_{\mu\rho} - D_\rho c)) \} \\
+ Tr_H tr \hat{\delta}_+ \{ i[\phi, \bar{\phi}] \bar{\eta} - i[\bar{\eta}, \bar{\phi}] + i[B^+_{\mu\nu}, \bar{\phi}] \psi^+_{\mu\nu} + (D_\mu \bar{\phi}) \psi_\mu \} \tag{16}
$$

For this action, we change the variables as

$$
B^+_{\mu\nu} \to \frac{1}{\sqrt{\theta}} \tilde{B}^+_{\mu\nu}, \ \psi^+_{\mu\nu} \to \frac{1}{\sqrt{\theta}} \tilde{\psi}^+_{\mu\nu}, \ c \to \frac{1}{\sqrt{\theta}} \tilde{c}, \ \bar{\eta} \to \frac{1}{\sqrt{\theta}} \tilde{\eta}, \ 
$$

$$
\chi_\mu \to \frac{1}{\theta} \tilde{\chi}_\mu, \ H_\mu \to \frac{1}{\theta} \tilde{H}_\mu
$$

with (12), then $S \to \frac{1}{\theta^2} \tilde{S}$, and $\tilde{S}$ does not depend on $\theta$. At last, invariance of the $\mathcal{N} = 4$ topological twisted theory under $\theta \to \theta + \delta \theta$ is proved as same as Donaldson-Witten theory.

---

a) There are many kinds of topological twisted theories of $\mathcal{N} = 4$ Yang-Mills theory. We only consider Vafa-Witten type theory.
It is worth commenting on the topological term $\int F \wedge F$ that exists in usual Vafa-Witten theory but now is removed. This term is not written by a BRS exact term, so we cannot adapt above discussion to the topological term. But, it is natural that we expect that $\int F \wedge F$ is invariant under the $\theta$ shift. Indeed, for instanton solutions constructed from noncommutative deformed ADHM data, we have proof of invariance of instanton number under $\theta$ shift\(^{13, 30}\). This is why, we expect that the partition functions or vacuum expectation values are still invariants even if the action of the cohomological Yang-Mills theories include $\int F \wedge F$. (See also section \(V\))

By applying these facts for several physical models, some interesting information can be found. For example, as we will see soon, we can show that the partition function of the noncommutative cohomological gauge theory and the partition function of the IKKT matrix model have a correspondence. This correspondence is not only for certain classical background theory as we saw in\(^{10}\). The reason is as follows. The IKKT matrix model is constructed as dimensional reduction of the 10 dimensional super $U(N)$ Yang-Mills theory with large $N$ limit\(^{11, 12}\). This dimensional reduction is regarded as the large noncommutative parameter limit ($\theta \rightarrow \infty$ in section \(V\)). Taking the large $N$ limit of the matrix model is similar to considering the Yang-Mills theories on noncommutative Moyal space, i.e. matrices are regarded as linear operators acting on the Hilbert space caused from noncommutativity. By the way, the noncommutative cohomological Yang-Mills model on Moyal space in the large $\theta$ limit is almost the same as the model of Moore, Nekrasov and Shatashvili (MNS)\(^{20}\). MNS show that the partition function is calculated by the cohomological matrix model in\(^{20}\) and related works are seen in\(^{4, 11, 32}\). This cohomological matrix model is almost equivalent to the IKKT matrix model. That is why we can produce similar result by using N.C.cohomological Yang-Mills theories. To show these facts concretely, we will calculate the partition function of $\mathcal{N} = 4$ d=4 $U(1)$ theory on N.C.$\mathbb{R}^4$ in section \(IV\) by using the facts given in this section.

### III Universality of Partition Functions

In this section, we show that the large $\theta$ limit is equivalent to dimensional reduction. From this fact, we find the universal perspective for the partition functions of supersymmetric Yang-Mills theories in N.C.$\mathbb{R}^{2D}$.

In the previous section, we consider the case of $\mathbb{R}^4$. There is two independent noncommutative parameters $\theta^1, \theta^2$ for the N.C.$\mathbb{R}^4$, that is to say, after choosing proper coordinate noncommutative parameters are expressed as

\[
(\theta^{\mu\nu}) = \begin{pmatrix}
0 & \theta^1 & 0 & 0 \\
-\theta^1 & 0 & 0 & 0 \\
0 & 0 & 0 & \theta^2 \\
0 & 0 & -\theta^2 & 0
\end{pmatrix}.
\] (17)
In the discussion of the previous section, we take noncommutative parameter shift co-
incidently, that is \( \theta^1 \equiv \theta^2 \equiv \theta \rightarrow \theta + \delta \theta \). However, we can shift \( \theta^1, \theta^2 \) independently without changing partition functions and vacuum expectations. Further, this discussion is extended to other dimensional theories.

Let us consider more general cases than N.C.\( \mathbb{R}^4 \). Let noncommutative parameter
matrix of N.C.\( \mathbb{R}^{2D} \) be \( (\theta^{\mu
u}) = \sum_{i=1}^{2} \theta^i e^{2i-1, 2i} \). In the large \( \theta \) limit, terms with derivative
operators \( \partial_{x^{2i}} := -i(\theta^{i})^{-1}[x_{2i-1}, \cdot] \) and \( -\partial_{x^{2i-1}} := -i(\theta^{i})^{-1}[x_{2i}, \cdot] \) become irrelevant in
lagrangians. If the partition function and the VEV of arbitrary observables of the coho-
logical field theory are well defined, the terms including \( \partial_{x^{2i}} \) or \( \partial_{x^{2i-1}} \) are possible to
be removed. (In appendix \[\ref{app:2} \] concrete discussions and details are given.) In the complex
coordinate expression, the terms including \( z_i \) and \( \bar{z}_i \) derivatives are omitted. Meanwhile,
an arbitrary operator is expressed as

\[
\hat{O} = \sum_{n_1, m_1} \cdots \sum_{n_D, m_D} \mathcal{O}_{m_1 \cdots m_D}^{n_1 \cdots n_D} \langle n_1, \cdots, n_D \rangle \langle m_1, \cdots, m_D \rangle,
\]

by using fock space basis. (See appendix A.) We consider a quantum theory of infinite
dimensional matrix model, and \( \mathcal{O}_{m_1 \cdots m_D}^{n_1 \cdots n_D} \) is a variable of path integration. Then we cannot
distinguish dynamical variables

\[
\mathcal{O}_{m_1 \cdots m_{i-1} m_{i+1} \cdots m_D}^{n_1 \cdots n_D} \langle n_1, \cdots, n_{i-1}, n_{i+1}, \cdots, n_D \rangle \langle m_1, \cdots, m_{i-1}, m_{i+1}, \cdots, m_D \rangle
\]

from \( \mathcal{O}_{m_1 \cdots m_D}^{n_1 \cdots n_D} \langle n_1, \cdots, n_D \rangle \langle m_1, \cdots, m_D \rangle \) because both of them are infinite dimensional
matrices. From the facts that there is no \( \partial_{z_i} \) or \( \partial_{\bar{z}_i} \) and it is impossible to distinguish
dynamical variables living in \( \mathbb{R}^{2D} \) from variables in \( \mathbb{R}^{2D-2} \), then we conclude that the
large \( \theta \) limit is equivalent to the dimensional reduction corresponding to \( x^{2i-1} \) and \( x^{2i} \)
directions.

We have to note two points, here. The first point is that naive path integrals contain
zero mode integrals. To make story precise, let us define the zero mode here. Let \( \{ \phi_i \} \) be a set of fields and \( S[\phi_i] \) be an action functional of a considered theory. Here, we
define the zero mode \( \phi_i^0 \) by \( S[\phi_i^0] = 0 \). To make the partition functions be well defined,
we manage the zero modes, in general. But it is difficult that the dealing with the zero
modes is discussed in general terms. To avoid this difficulty, the discussion of the zero
mode integrals are taken up in the individual cases. In section \[\ref{sec:5} \] we will closely study
the handling of the zero modes for the case of \( \mathcal{N} = 4 \) U(1) gauge theory in N.C.\( \mathbb{R}^4 \).

The second point is that there might be BPS solutions that become singular at \( \theta^i \rightarrow \infty \)
limit. To the authors’ knowledge, such solutions have never been known until now, but we
can not deny their existence. Since we can not estimate its contribution to the vacuum
expectations when we calculate them at the large \( \theta \) limit, we have to rule out such sin-
gular configurations when we construct the correspondence between finite \( \theta^i \) and infinite
\( \theta^i \).
As a summary of these arguments, the following claim is obtained.

**Claim**

Let $Z_{2D}$ and $\langle O \rangle_{2D}$ be a partition function and vacuum expectation value of $O$ of a cohomological field theory in $N.C.\mathbb{R}^{2D}$ with $D \geq 1$ such that $\delta_\theta Z_{2D} = 0$ and $\delta_\theta \langle O \rangle_{2D} = 0$. Here, zero mode integrals and contributions from BPS solutions that become singular at large noncommutative parameter limit are removed from the path integral of $Z_{2D}$ and $\langle O \rangle_{2D}$. Let $Z_{2D-2}$ and $\langle O \rangle_{2D-2}$ be the partition function and vacuum expectation value of $O$ of a noncommutative cohomological field theory in $N.C.\mathbb{R}^{2D-2}$, where they are given by dimensional reduction of $Z_{2D}$ and $\langle O \rangle_{2D}$. Then,

$$Z_{2D} = Z_{2D-2}, \quad \langle O \rangle_{2D} = \langle O \rangle_{2D-2},$$

i.e. the partition functions of such theories do not change under dimensional reduction from $2D$ to $2D-2$.

From this claim, we find that following partition functions of Super Yang-Mills theories on $N.C.\mathbb{R}^{2D}$ are equivalent:

$$Z_{N=2}^{8 \dim} = Z_{N=2}^{6 \dim} = Z_{N=4}^{4 \dim} = Z_{N=8}^{2 \dim} = Z_{**}^{0 \dim},$$

where $Z_{N=J}^{I \dim}$ is a partition function of the $N = J$ super Yang-Mills theory in noncommutative $\mathbb{R}^{I \dim}$ with arbitrary gauge group. They are obtained by dimensional reduction of the 8 dimensional $N = 2$ super Yang-Mills theory. Note that the topological terms in the actions of above theories should be removed because the topological terms is not universal between the different dimensional theories. The proof of (20) is as follows. In the $\mathbb{R}^{2D}$, a topological twist exists at any time for $N \geq 2$. Using the topological twist, the partition functions are described as the one of cohomological field theories. Therefore, $Z_{N=2}^{8 \dim}$ is invariant under $\theta$-shift and satisfies the condition of the above claim. After all, (20) is obtained. We will calculate the partition functions concretely in the case of $U(1)$ in the next section.

It is worth adding some comments about above models. We consider noncommutative Euclidean spaces. $N = 4$ super Yang-Mills theory in $N.C.\mathbb{R}^4$ is given as follows. At first, we construct the 4-dimensional $N = 4$ super Yang-Mills theory by dimensional reduction of the 10 dimensional $N = 1$ super Yang-Mills defined on Minkowski space with $SO(9,1)$ symmetry. In 4-dim, spinor in Euclidean space is defined as well as the spinor in Minkowski space. Therefore, we can construct the $N = 4$ super Yang-Mills theory in $\mathbb{R}^4$ by formally replacing the metric, gamma matrices and so on. Since the $\theta$-shift invariance of $Z_{N=4}^{4 \dim}$ was shown explicitly in section 11 (see also appendix 13), theories connected to the $N = 4 \text{ d = 4}$ super Yang-Mills theory through the dimensional reduction appear in (20).

This discussion is valid not only for the $N=4$ case. For example, we saw that the $\theta$-shift invariance of $Z_{N=2}^{4 \dim}$ in section 11. Then, the similar relation should exist :

$$Z_{N=2}^{4 \dim} = Z_{N=4}^{2 \dim} = Z_{**}^{0 \dim}.$$
Let us summarize this section. Universality of partition functions and vacuum expectation values of observables of N.C. cohomological field theories are discussed. From the claim, we found that $\mathcal{N} \geq 2$ supersymmetric models or cohomological field theories in N.C. $\mathbb{R}^{2D}$ are invariant under dimensional reduction from $2D$ to $2D - 2$. In the following section, we will apply these facts to concrete calculations.

**IV $\mathcal{N} = 4 \ U(1)$ Gauge Theory in N.C. $\mathbb{R}^4$**

In this section, we calculate the partition function of the topological twisted $\mathcal{N} = 4 \ U(1)$ gauge theory in N.C. $\mathbb{R}^4$, without the topological term $\int F \wedge F$ in its action.

We perform the calculation in the $\theta \to \infty$ limit. The reason why we take this limit is as follows. As explained in section III, the partition function and other correlation functions of cohomological field theories on noncommutative spaces are invariant under the shift transformation of the noncommutative parameter $\theta$. So we obtain the exact result by taking $\theta \to \infty$ limit. Also this limit makes the calculation executable.

In the operator formalism, field theories in N.C. $\mathbb{R}^4$ are expressed as infinite dimensional matrix models whose symmetry is $U(N) \ (N \to \infty)$. The size of matrices appearing in this model is infinite. To perform the calculation, we introduce a cut off for the matrix size. In addition, this matrix model contains trace parts which correspond to zero modes in $\theta \to \infty$. Therefore we must carefully treat the trace parts to make the path integral well-defined.

In subsection IV-i, we give the action of the topological twisted $\mathcal{N} = 4 \ U(1)$ gauge theory in N.C. $\mathbb{R}^4$ in the operator formalism, i.e. in terms of infinite dimensional matrices. In subsection IV-ii, we truncate the size of the matrices into finite size, a finite integer $N$.

In subsection IV-iii, we explain that the truncated $N \times N$ matrix model action obtained in the previous subsection is equivalent to the dimension reduction of the 10 dim. $\mathcal{N} = 1 \ U(N)$ super Yang-Mills action to 0 dim. This $U(N)$ matrix model contains traceless parts and trace parts. In subsection IV-iv, we calculate the partition function of the traceless sector. The traceless sector is a $SU(N)$ matrix model. The partition function of this $SU(N)$ matrix model was obtained by MNS. By modifying their arguments, we evaluate the $N \to \infty$ limit of the partition function of the traceless sector. In subsection IV-v, we introduce extra parts into the matrices to handle the trace parts which are zero modes. The extra parts and trace parts are the next leading terms in the $1/\sqrt{\theta}$ expansion.

In subsection IV-vi, the calculation of the trace sector is performed. Our result is presented at the end of this section.
From (22) or (23), we find the BPS equations. For example, \( U \) vanish in the adjoint representation are described by commutators of matrices and all commutators of the topological twisted gauge theories. This \( U \) bosonic matrices acting on state vectors of the Hilbert space \( H \).

In the Fock space formalism, i.e. in terms of (infinite dimensional) matrices, the action \( S^\text{N=4} \) is expressed as

\[
S^\text{N=4}_{\text{4dim}} = Tr_{\mathcal{H}} \hat{\delta}_+ \left[ +\chi^+_{\mu\nu} \{ H^+_{\mu\nu} - i(F^+_{\mu\nu} - i[ B^+_{\mu\rho}, B^+_{\rho\nu}] \delta^{\rho\sigma} - i[ B^+_{\mu\nu}, c]) \} \\
+ \chi^\mu \{ H_{\mu} - i(-2D^\nu B^+_{\nu\mu} - D_{\mu}c) \} \\
+ i[\phi, \bar{\phi}] \eta - i\bar{\eta}[c, \bar{c}] + i[B^+_{\mu\nu}, \bar{\phi}] \psi^+_{\mu\nu} + (D_{\mu}\bar{\phi}) \psi_{\mu} \right].
\]  

(22)

After acting \( \hat{\delta}_+ \), (22) is rewritten as

\[
S^\text{N=4}_{\text{4dim}} = Tr_{\mathcal{H}} \left[ H^+_{\mu\nu} \{ H^+_{\mu\nu} - i(F^+_{\mu\nu} - i[ B^+_{\mu\rho}, B^+_{\rho\nu}] \delta^{\rho\sigma} - i[ B^+_{\mu\nu}, c]) \} \\
+ \chi^+_{\mu\nu} \{ -i[\chi^+_{\mu\nu}, \phi] + i(2D_{\mu}\psi_{\nu} - 2i[ B^+_{\mu\rho}, \psi^+_{\nu\rho}] \delta^{\rho\sigma} - i[\psi^+_{\mu\nu}, c] - i[B^+_{\mu\nu}, \bar{\eta}] \} \\
+ H^\mu \{ H_{\mu} - i(-2D^\nu B^+_{\nu\mu} - D_{\mu}c) \} \\
+ \chi^\mu \{ -i[\chi^+_{\mu}, \phi] - i(2D^\nu \psi^+_{\nu\mu} + 2i[\psi^+_{\nu\mu}, B^+_{\nu\mu}] - D_{\nu}\bar{\eta} + i[\psi^+_{\nu\mu}, c]) \} \\
+ [\phi, \bar{\phi}]^2 + [c, \bar{\phi}][c, \bar{\phi}] + [B^+_{\mu\nu}, \bar{\phi}][B^+_{\mu\nu}, \phi] + D^\mu \bar{\phi} D_{\mu}\phi \\
+ i[\phi, \eta]\eta + i\bar{\eta}[\bar{\eta}, \bar{\phi}] + i\bar{\eta}[c, \eta] + i[\psi^+_{\mu\nu}, \bar{\phi}] \psi^+_{\mu\nu} + i[B^+_{\mu\nu}, \eta] \psi^+_{\mu\nu} \\
+ D\eta \psi^+_{\mu} + i[\psi^+_{\mu}, \bar{\phi}] \psi_{\mu} \right].
\]  

(23)

From (22) or (23), we find the BPS equations. For example,

\[
F^+_{\mu\nu} - i[ B^+_{\mu\rho}, B^+_{\nu\sigma}] \delta^{\rho\sigma} - i[ B^+_{\mu\nu}, c] = 0, \\
-2D^\nu B^+_{\nu\mu} - D_{\mu}c = 0.
\]  

(24)

In the following, we calculate the partition function \( Z^\text{N=4}_{\text{4dim}} \) formally defined as

\[
Z^\text{N=4}_{\text{4dim}} = \int Df e^{-S^\text{N=4}_{\text{4dim}}[f]},
\]  

(25)

where \( f \) means the all matrices \( A_{\mu}, \psi_{\mu}, \cdots \). Also we use \( f_{\text{boson}} \) and \( f_{\text{fermion}} \) to denotes bosonic matrices \( A_{\mu}, H_{\mu}, \cdots \) and fermionic matrices \( \psi_{\mu}, \chi_{\mu}, \cdots \), respectively.

In usual commutative spaces, \( U(1) \) gauge theories are free if all matters belong to the adjoint representation, because the gauge interactions between the fields belonging to the adjoint representation are described by commutators of matrices and all commutators vanish in the \( U(1) \) case. However, in noncommutative spaces, the noncommutativity of the multiplication induces the \( U(1) \) gauge theories to non-Abelian \( U(N) \) \((N \rightarrow \infty)\) like gauge theories. This \( U(N) \) \((N \rightarrow \infty)\) is identified with the unitary transformation group acting on state vectors of the Hilbert space \( \mathcal{H} \). \(^{b)\}

\(^{b)\}It is well known fact that the \( U(\infty) \) is different from \( \lim_{N \rightarrow \infty} U(N) \), in the meaning of the topology. In this article, we perform the all calculation by using \( \lim_{N \rightarrow \infty} U(N) \), and there is denying that some extra collections appear from the difference. However, there is no doubt about validity of calculation of \( U(N) \) \((N \rightarrow \infty)\) as a good approximation even in the case.

10
Let us consider to take the $\theta \to \infty$ limit in the calculation of the partition function $Z_{N=4}^{4\text{dim}}$. We can evaluate the partition function exactly in this limit, as explained in section II. In the $\theta \to \infty$ limit we naively expect that all differential terms in the action vanish and dimensional reduction occur as we saw in section III. Therefore, we can perform the calculation by using a matrix model in 0 dim. space whose symmetry is $U(N)$ ($N \to \infty$). We define the action in 0 dim spacetime as

$$S_{MM}^\infty = S_{N=4}^{4\text{dim}}|_{\theta \to \infty} : U(N) \ (N \to \infty) \text{ matrix model},$$

then, we find $Z_{N=4}^{4\text{dim}}$ is equal to the partition function of the matrix model (26)

$$Z_{N=4}^{4\text{dim}} = \frac{1}{\text{Vol} U(N)(N \to \infty)} \int \mathcal{D} f e^{-S_{MM}^\infty}. \quad (27)$$

To calculate the partition function of this infinite dimensional $U(N) \ (N \to \infty)$ matrix model (26), we need to overcome the following problems.

(i) The size of the matrices is infinite. To perform the calculation, we truncate the size of the matrices into a finite integer $N$.

(ii) The matrices contain trace parts. These trace parts play a role of zero modes. To make the path integral well-defined, we must carefully treat the trace parts.

In the rest of this section, we solve these problems and obtain the partition function (27).

**IV-ii Cut off for matrix size**

In this subsection, we truncate the size of the matrices, to calculate the partition function.

The Hilbert space of the $\mathcal{N} = 4 \ U(1)$ gauge theory on N.C.$\mathbb{R}^4$ is constructed by a Fock space

$$\mathcal{H} = \bigoplus_{n_1=0,n_2=0}^{n_1=\infty,n_2=\infty} \mathbb{C} |n_1,n_2\rangle. \quad (28)$$

We introduce a cut off, a finite integer $N_c$, and truncate the Hilbert space into a finite dimensional subspace $\mathcal{H}_N$ whose dimension is $N$. We can perform such truncation in several ways. For example, $\mathcal{H}_N$ is defined by

$$\mathcal{H}_N = \bigoplus_{n_1=0,n_2=0}^{n_1=N_c,n_2=N_c} \mathbb{C} |n_1,n_2\rangle. \quad (29)$$

For this case

$$\dim \mathcal{H}_N = N = (N_c + 1)^2, \quad (30)$$

and the unit matrix of $\mathcal{H}_N$ is given as

$$1_N = \bigoplus_{n_1=0,n_2=0}^{n_1=N_c,n_2=N_c} |n_1,n_2\rangle \langle n_1,n_2| \quad (31)$$
The results and calculations do not depend on the definition of the cut off in the following discussion. (See appendix A.) Therefore we do not use concrete expression of the example \( \text{(29)} \). By definition,

\[
Tr_{\mathcal{H}} 1_N = \dim \mathcal{H}_N = N. \tag{32}
\]

For later use, we define \( \mathcal{I} \) as

\[
\mathcal{I} = \frac{1}{\sqrt{N}} 1_N, \tag{33}
\]

which satisfies

\[
Tr_{\mathcal{H}} \mathcal{I} \mathcal{I} = 1. \tag{34}
\]

We truncate the infinite dimensional matrices appearing in \( \text{(26)} \) into finite dimensional \( N \times N \) matrices. We use the symbol \( f_N \) to denote the \( N \times N \) truncation of \( f \). For example of \( \text{(29)} \), if

\[
f = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f_{m1m2}^{n1n2} |n_1, n_2\rangle \langle m_1, m_2|,
\]

then

\[
f_N = \sum_{n=0}^{N_c} \sum_{m=0}^{N_c} f_{m1m2}^{n1n2} |n_1, n_2\rangle \langle m_1, m_2|.
\]

Now we consider the finite dimensional \( N \times N \) matrix model \( S^N_{MM} \) which is obtained by the truncation from \( \text{(26)} \)

\[
S^N_{MM} = S^\infty_{MM} |\text{N x N truncation}. \tag{35}
\]

The partition function of the truncated matrix model \( \text{(35)} \) is defined by

\[
Z^{4\dim|N}_N = \frac{1}{Vol.U(N)} \int \mathcal{D}f_N e^{-S^N_{MM}}. \tag{36}
\]

\( N \times N \) matrix \( f_N \) is decomposed into the traceless part and the trace part

\[
f_N = f^{su} + f^{tr}, \tag{37}
\]

where \( f^{su} \) is the traceless part and \( f^{tr} \) is the trace part. The traceless part \( f^{su} \) is expanded by the generators of the Lie algebra \( su(N) \)

\[
f^{su} = \sum_{a=1}^{N^2-1} f^{(a)} \tau^a, \quad \tau^a \in su(N), \tag{38}
\]

and \( f^{tr} \) is proportional to \( \mathcal{I} \)

\[
f^{tr} = f^{(1)} \mathcal{I}. \tag{39}
\]

The basis, \( \tau^a \) and \( \mathcal{I} \), satisfy the following orthonormal conditions

\[
Tr_{\mathcal{H}} \tau^a \tau^b = \delta^{ab} \quad , \quad Tr_{\mathcal{H}} \mathcal{I} \mathcal{I} = 1 \quad , \quad Tr_{\mathcal{H}} \tau^a \mathcal{I} = 0. \tag{40}
\]
In the naive $\theta \to \infty$ limit (i.e. 0 dimension reduction), (35) contains no trace part $f^{tr}$ c)

$$Z^N_{MM} = Z^N_{MM}|_{\text{traceless}} \times \int \mathcal{D}f^{tr},$$

(41)

where $Z^N_{MM}|_{\text{traceless}}$ is defined by

$$Z^N_{MM}|_{\text{traceless}} = \frac{1}{\text{Vol.}SU(N)} \int \mathcal{D}f^{su} e^{-S^N_{MM}[f^{su}]}.$$

(42)

So the trace part $f^{tr}$ plays the role of the zero mode such that $S^N_{MM}[f^{tr}] = 0$. To make the path integral well-defined, we must carefully treat it. For other handling the zero modes, see for example [33]. However we postpone this task for the moment. First, we concentrate on the traceless sector. Before the calculation of the traceless sector, we explain the equivalence between (35) and the action considered in [20] in the next subsection.

IV-iii Relation to the work of MNS and IKKT

To explain that the equivalence between (35) and the action considered in [20], we first recall the fact that the dimensional reduction model from the $D = 10, \mathcal{N} = 1$ super Yang-Mills theory to 0 dimension can be reformulated into a cohomological matrix model [11, 20].

The 0 dimension matrix model given by dimensional reduction from the $D = 10, \mathcal{N} = 1$ super Yang-Mills theory is expressed as

$$S_{N=1}^{10 \to 0 \text{ dim}} = \text{tr} \left( \frac{1}{4} [A_M, A_N]^2 + \frac{i}{2} \bar{\Psi} \Gamma^M [A_M, \Psi] \right),$$

(43)

where $A_M$ is gauge vector fields and $M, N$ takes $1 \cdots 10$ for the 10 dimension Euclid space, or $0, 1 \cdots 9$ for the 10 dimension Minkowski spacetime. $\Psi$ is a Majorana-Weyl spinor of the 10 dimension spacetime. It contains real 16 components d) .

In [11, 20], it is shown that (43) can be reformulated into a cohomological matrix model. The mapping rules between them are as follows [20]. $A_M$ are arranged into complex matrices $\phi$ and $B_i$ ($i = 1, \ldots, 4$),

$$B_i = A_{2i-1} + iA_{2i}, \quad \text{(for } i = 1, 2, 3),$$

$$B_4 = A_9 + iA_8,$$

$$\phi = A_7 + iA_{10},$$

(44)

and $\Psi$ are arranged as

$$\Psi \rightarrow (\psi_i, \psi^\dagger_i) \oplus \bar{\chi} \oplus \eta,$$

(45)

c) Precisely speaking, the trace part of the auxiliary fields appear in [35]. After integrating out the auxiliary fields, no trace part appears in [35].

d) Note that there is no Majorana-Weyl spinor in 10 dim Euclidean space. So, if we consider 10 dim model, we should take Minkowski spacetime. To obtain low dimensional Euclidean model, we first perform dimensional reduction from 10 dim. to lower dim, and then carry out Wick rotation.
where $\vec{\chi}$ belongs to the 7 representation of $Spin(7)$. Introducing the bosonic auxiliary matrices $\vec{H}$, we can rewrite (43) into a cohomological form

$$S_{MNS} = tr \hat{\delta} \left( \frac{1}{16} \eta[\phi, \bar{\phi}] - i\vec{\chi} \cdot \vec{E} + \vec{\chi} \cdot \vec{H} + \frac{1}{4} \sum_{a=1}^{8} \Psi_a [A_a, \bar{\phi}] \right),$$

(46)

where $\vec{E}$ is defined by

$$\vec{E} = \left( [B_i, B_j] + \frac{1}{2} \epsilon_{ijkl} [B_k^1, B_l^1] \right) (i < j), \sum_i [B_i, B_i^1].$$

(47)

The BRS transformation rules are given as

$$\hat{\delta} A_a = \Psi_a, \; \hat{\delta} \Psi_a = [\phi, A_a],$$

$$\hat{\delta} \vec{\chi} = \vec{H}, \; \hat{\delta} \vec{H} = [\phi, \vec{\chi}],$$

$$\hat{\delta} \phi = \eta, \; \hat{\delta} \eta = [\phi, \bar{\phi}],$$

$$\hat{\delta} \bar{\phi} = 0.$$  

(48)

From (46) and (48), the following BPS equations are obtained

$$\vec{E} = 0, \; [\phi, \bar{\phi}] = 0, \; [A_a, \phi] = 0.$$  

(49)

One can show that (46) is equivalent to (26), by using the following correspondence

$$(\phi, \; c, \; \bar{\phi}) \iff \left( \sqrt{2} \varphi_{34}, \; i \frac{1}{\sqrt{2}} (\varphi_{14} - \varphi_{23}), \; \sqrt{2} \varphi_{12} \right)$$

$$(B_{\mu \nu}^{11}, \; B_{\mu \nu}^{12}, \; B_{\mu \nu}^{22}) \iff \left( \sqrt{2} \varphi_{13}, \; -\frac{1}{\sqrt{2}} (\varphi_{14} + \varphi_{23}), \; \varphi_{24} \right),$$

(50)

where $\varphi$ is defined by

$$\varphi_{k4} = -\varphi_{4k} = \frac{1}{\sqrt{2}} (A_{k+4} + i A_{k+7}), \; \varphi_{ij} = (\epsilon^{ijk} \varphi_{k4})^*, \; k = 1, 2, 3.$$  

(51)

Remark that the equivalence among (33), (43) and (46) holds for both $U(N)$ group and $SU(N)$ group.

By choosing gauge group $SU(N)$ and setting $N$ to be a finite integer, we obtain the equivalence between (35) and (46)

$$S_{MNS, traceless}^{N, finite}_{gauge group: SU(N)},$$

(52)

Therefore

$$Z_{MM, traceless}^N = Z_{MNS, gauge group: SU(N)}^{N, finite}.$$  

(53)
where

\[
Z_{MNS|_{\text{gauge group } SU(N)}}^{N: \text{finite}} = \frac{1}{\text{Vol. } SU(N)} \int \mathcal{D} f^{su} \exp \left\{ -S_{MNS}[f^{su}]^{N: \text{finite}}_{\text{gauge group } SU(N)} \right\}.
\] (54)

MNS obtained the partition function \(^{(54)}\) \(^{(20)}\) \(^{(e)}\).

On the other hand, by choosing gauge group \(U(N)\) and taking the \(N \rightarrow \infty\) limit, the action \(^{(43)}\) becomes the IKKT matrix model \(^{(12)}\)

\[
S_{IKKT} = \lim_{N \rightarrow \infty} S^{10 \rightarrow 0}_{\text{dim}}|_{\text{gauge group } U(N)}.
\] (55)

So, we obtain the equivalence between \(^{(26)}\) and \(^{(55)}\) :

\[
S_{MM}^\infty = S_{IKKT}.
\] (56)

**IV-iv Calculation of traceless sector**

As explained in the previous subsection the partition function \(^{(12)}\) is calculated in \(^{(20)}\). Their result tells us that

\[
Z_{MM}^N|_{\text{traceless}} = \sum_{d|N} \frac{1}{d^2},
\] (57)

where the summation is taken over all divisor \(d\) of the finite integer \(N\).

One might expect that to obtain the contribution of the traceless part \(f^{su}\) to \(Z_{MM}^{4 \text{ dim}}\), one take the \(N \rightarrow \infty\) limit,

\[
Z_{MM}^\infty|_{\text{traceless}} = \lim_{N \rightarrow \infty} Z_{MM}^N|_{\text{traceless}}.
\] (58)

However \(N \rightarrow \infty\) limit in the right-hand side of \(^{(57)}\) is not well-defined. The reason is as follows. We see that the right-hand side of \(^{(57)}\) is finite;

\[
\sum_{d|N} \frac{1}{d^2} < \sum_{n=1}^{N} \frac{1}{n^2} < 1 + \int_{1}^{\infty} dx \frac{1}{x^2} = 2.
\] (59)

But \(\sum_{d|N} \frac{1}{d^2}\) is not a monotonically increasing function of \(N\). So it does not converge. For example, if we constrain \(N\) to be prime numbers,

\[
\lim_{N \rightarrow \infty} \sum_{d|N} \frac{1}{d^2} = \lim_{N \rightarrow \infty} (1 + N^{-2}) = 1.
\] (60)

If we constrain \(N = 2^{N'}\),

\[
\lim_{N \rightarrow \infty} \sum_{d|N} \frac{1}{d^2} = \lim_{N \rightarrow \infty} \sum_{n=0}^{N'} 2^{-2n} = \frac{4}{3}.
\] (61)

\(^{(e)}\)See also \(^{(6)}\) where the partition function of the D-instanton model was calculated.
Therefore, we must give the proper definition of $N \to \infty$ limit.

To find a prescription which leads the definite answer of the $N = \infty$ case, let us recall the argument of \[20\] where the result \[57\] is concluded for a finite $N$.

(i) The authors of \[20\] separated the coupling constant $g$ to $g$, $\tilde{g}$ and $\hat{g}$ \footnote{In the righthand side of \[46\], we omitted the coupling constant $g$.},

$$S_{MNS} \to tr \, \delta \left( \frac{1}{16\tilde{g}} \eta[\phi, \bar{\phi}] - i \vec{\chi} \cdot \vec{E} + g\vec{\chi} \cdot \vec{H} + \frac{1}{4\hat{g}} \sum_{a=1}^{8} \Psi_{a}[A_{a}, \bar{\phi}] \right). \quad (62)$$

(ii) They deformed the action by redefining $\mathcal{E}_{ij}$, the $(6 \oplus \bar{6})_{\tau}$ part of $\vec{E}$, as

$$\mathcal{E}_{ij} = \Phi_{ij} - \frac{1}{2} \epsilon_{ijkl} \Phi_{kl}^{\dagger}, \quad \Phi_{ij} = [B_{i}, B_{j}] - m\epsilon_{ijk}B_{k}, \quad (63)$$

where $m$ is a mass parameter. This mass deformation corresponds to the supersymmetry breaking from $\mathcal{N} = 4$ to $\mathcal{N} = 1$ in the picture of 4 dimensional space.

(iii) They again separated the coupling constants $g$ and $\hat{g}$ as

$$g\vec{\chi} \cdot \vec{H} \to g' \sum_{i<j} \chi_{ij} H_{ij} + g'' \chi_{7} H_{7},$$

$$\frac{1}{4\hat{g}} \sum_{a=1}^{8} \Psi_{a}[A_{a}, \bar{\phi}] \to \frac{1}{4\hat{g}'} \sum_{a=1}^{6} \Psi_{a}[A_{a}, \bar{\phi}] + \frac{1}{4\hat{g}''} \sum_{a=7,8} \Psi_{a}[A_{a}, \bar{\phi}]. \quad (64)$$

(iv) Then, they took the following limit,

$$g' \to 0 \quad \text{and} \quad \hat{g}' \to 0. \quad (65)$$

Notice that each term in the action is BRS exact. So the partition function is independent of separated coupling constants $g'$, $g''$, $\cdots$. By taking \[65\], the partition function is dominated by configurations around solutions of the following fixed point equations

$$[B_{i}, B_{j}] = m\epsilon_{ijk}B_{k}, \quad [B_{4}, B_{i}] = 0, \quad [B_{4}, \phi] = 0, \quad i = 1, 2, 3. \quad (66)$$

where $B_{i}$, $B_{4}$ and $\phi$ are all $N \times N$ matrices.

(v) The solution of \[66\] is given by

$$(B_{i})_{N \times N} = (L_{i})_{a \times a} \otimes 1_{d \times d},$$

$$(B_{4})_{N \times N} = 1_{a \times a} \otimes (B_{4})_{d \times d}, \quad (\phi)_{N \times N} = 1_{a \times a} \otimes \phi_{d \times d}. \quad (67)$$

where $a$ is a divisor of $N$ and $d$ is the quotient of $N$ by $a$, and $(L_{i})_{a \times a}$ denotes the generator of the $SU(2)$ group in the $a \times a$ representation. Of course, there are other solutions of
where $N = \sum_{l=1}^{k} N_l$, $N_l = a_l \times d_l$. However as mentioned in [20], these solutions do not contribute to the partition function. The solutions (68) contain bosonic zero modes, corresponding to extra $U(1)$ parts $tr_{N_l} \phi, \cdots$, and they are accompanied by their fermionic partners. The fermionic partners play a role of fermionic zero modes, and they vanish the path integral. So the solutions (68) do not contribute to the partition function.

(vi) In the above coupling limit (65) the authors integrated out $B_i$ and corresponding fermionic partners around the solutions (67) by the Gaussian integral. The Gaussian integrals from bosons and the one from fermions cancel each other, so they produce no non-trivial factor. The resulting effective action is a matrix model of $d \times d$ matrices, $B_4$, its fermionic partner and $\phi$.

(vii) The partition function of this $d \times d$ matrix model, we call it $Z_d$, is given by

$$Z_d = \frac{1}{d^2},$$

which is another result obtained in the same paper [20]. The partition function $Z_{MM|\text{traceless}}^N$ is given by the sum of the contributions from the solutions, $Z_d = \frac{1}{d^2}$, so they concluded (57).

Now let us turn to the $N \to \infty$ case. Our basic strategy is that taking large $N$ limit is done after calculations with finite $N$. However, the result depends on the definition of the large $N$ limit as mentioned above. To find the proper definition of the large $N$ limit, we consider a naive $N = \infty$ case. That is, we do not take the $N \to \infty$ limit after obtaining the result of the finite $N$ case, but we take the matrices as $\infty \times \infty$ from the starting point for a moment. For the case of $\infty \times \infty$ matrix, the steps (i)-(iv) need no change, but the step (v) should be reconsidered. In $\infty \times \infty$ matrix, we can embed a solution which has a direct product of an arbitrary finite dimensional matrix and an infinite dimensional matrix. Therefore we obtain solutions,

$$\begin{align*}
(B_i)_{\infty \times \infty} &= (L_i)_{\infty \times \infty} \otimes 1_{d \times d}, \\
(B_4)_{\infty \times \infty} &= 1_{\infty \times \infty} \otimes (B_4)_{d \times d}, \\
(\phi)_{\infty \times \infty} &= 1_{\infty \times \infty} \otimes \phi_{d \times d}.
\end{align*}$$

(70)
Now \( d \) takes all natural numbers, and \( (L_i)_{\infty \times \infty} \) are the generator of the \( SU(2) \) group in the \( \infty \times \infty \) representations. Solutions, like (68), again do not contribute to the partition function. Moreover, one can construct other types of solutions,

\[
(B_i)_{\infty \times \infty} = (L_i)_{a \times a} \otimes 1_{\infty \times \infty}, \quad (B_4)_{\infty \times \infty} = 1_{a \times a} \otimes (B_4)_{\infty \times \infty}, \quad (\phi)_{\infty \times \infty} = 1_{a \times a} \otimes \phi_{\infty \times \infty},
\]

and

\[
(B_i)_{\infty \times \infty} = (L_i)_{\infty \times \infty} \otimes 1_{\infty \times \infty}, \quad (B_4)_{\infty \times \infty} = 1_{\infty \times \infty} \otimes (B_4)_{\infty \times \infty}, \quad (\phi)_{\infty \times \infty} = 1_{\infty \times \infty} \otimes \phi_{\infty \times \infty}.
\]

The step (vi), integrating out of \( B_i \) and their fermionic partners, again produces no non-trivial factor, because the cancellation of the Gaussian integrals between bosons and fermions holds for the case of infinite dimensional integral. Therefore the partition function \( Z_{MM}^\infty \) is given by the sum of contributions from the solutions (70, 71, 72),

\[
Z_{MM}^\infty = Z_{MM}^{(\infty \times d)} + Z_{MM}^{(a \times \infty)} + Z_{MM}^{(\infty \times \infty)},
\]

where the first term in the righthand side comes from (70), the second from (71) and the third from (72). \( Z_{MM}^{(\infty \times d)} \) is still given by the sum of \( Z_d = \frac{1}{d^2} \), as the step (vii), but in this \( N = \infty \) case \( d \) runs all natural numbers \( \mathbb{N} \). On the other hand, it is natural to expect that \( Z_{MM}^{(a \times \infty)} \) and \( Z_{MM}^{(\infty \times \infty)} \) vanish, because

\[
Z_{MM}^{(a \times \infty)} \sim \sum_{d \to \infty} \lim_{d \to \infty} \frac{1}{d^2} = 0, \quad Z_{MM}^{(\infty \times \infty)} \sim \sum_{d \to \infty} \lim_{d \to \infty} \frac{1}{d^2} = 0,
\]

if (69) is valid for \( d = \infty \). So we conclude

\[
Z_{MM}^\infty |_{\text{traceless}} = \sum_{d \in \mathbb{N}} \frac{1}{d^2} = \zeta(2) = \frac{\pi^2}{6}.
\]

From these considerations, we propose the following definition of the large \( N \) limit. Let \( N(n_i, k) \) be

\[
N(n_i, k) \equiv \prod_{i=1}^{k} P_i^{n_i},
\]

where \( P_i \) are ordered prime numbers, i.e. \( P_1 = 2 < P_2 = 3 < \cdots < P_k \), and \( k \) and \( n_i \) are positive integers. We define the large \( N \) limit by

\[
\lim_{N \to \infty} \equiv \lim_{k \to \infty} \lim_{n_i \to \infty}.
\]
Using this definition, we reproduce the same result as (75),

\[ Z_{MM}^{\infty}|\text{traceless} = \lim_{k \to \infty} \lim_{n_i \to \infty} \sum_{l_i=0}^{n_i} \frac{1}{(\prod_{i=1}^k P_i^{l_i})^2} = \prod_{i=1}^\infty \frac{1}{1 - P_i^{-2}} = \zeta(2) = \frac{\pi^2}{6}. \tag{78} \]

**IV-v Introduction of extra terms**

In this section, we deal with the zero mode problem. The origin of this problem is the fact that no trace part appears in (35). The reason why all trace parts vanish in (35) is that we drop all differential terms in the $\theta \to \infty$ limit. To solve the zero mode problem, we keep the next leading terms including the trace parts in the $1/\sqrt{\theta}$ expansion.

Let us explain the outline of our calculation. To keep the next leading term, we bring back some extra part $f^{ex}$ living in the outside of $H_N$. The definition of $f^{ex}$ is given later in this subsection. Roughly speaking, $f^{ex}$ are matrices appearing in kinetic terms $\frac{1}{\theta} f^{ex} \Box f^{tr}$ in (23). By keeping $f^{ex}$, the part of (22) or (23) which includes the trace part $f^{tr}$ does not vanish:

\[ S_{tr\oplus ex}[f^{tr}, f^{ex}] = S_{N=4}^{\dim}\text{trace part} - O(1/\theta^{1+\epsilon}) \neq 0, \tag{79} \]

where $\epsilon$ is an arbitrary positive real number. Then the partition function of (79) is well-defined

\[ Z_{tr\oplus ex} = \int Df^{tr} Df^{ex} e^{-S_{tr\oplus ex}[f^{tr}, f^{ex}]} : \text{well-defined}. \tag{80} \]

We suppose $f^{ex}$ has the following expansion form

\[ f^{ex} = \sum_{\mu=1}^4 f(\mu) T_\mu. \tag{81} \]

$T_\mu$ is essentially defined by the commutator of $\hat{\partial}_\mu$ and $1_N$. The precise definition of $T_\mu$ is as follows. First of all, we define $T_\mu$ as the commutator of $\hat{\partial}_\mu$ and $1_N$ i.e.

\[ T_\mu = [\hat{\partial}_\mu, 1_N]. \tag{82} \]

In the Fock space formalism, $\hat{\partial}_\mu$ is given as

\[ \hat{\partial}_1 = \frac{1}{\sqrt{2\theta^1}} (a_1 - a_1^\dagger) , \quad \hat{\partial}_2 = \frac{-i}{\sqrt{2\theta^1}} (a_1 + a_1^\dagger), \]

\[ \hat{\partial}_3 = \frac{1}{\sqrt{2\theta^2}} (a_2 - a_2^\dagger) , \quad \hat{\partial}_4 = \frac{-i}{\sqrt{2\theta^2}} (a_2 + a_2^\dagger), \tag{83} \]

\[ \text{It is well known and will be seen in section V that the partition functions of this case is the sum of the Euler number of the moduli space, } \chi(M_k) \text{ which takes a rational number in general. So one may expect that } Z_{MM}^{\infty}|\text{traceless} \text{ is given by a rational number. However the summation is an infinite one, then it could take an irrational number, } \frac{\pi^2}{6}. \]
where \(a_i\) is the annihilation operator and \(a_i^\dagger\) is the creation operator. Given the definition of \(1_N\), for example (31), we obtain

\[
T_1 = \frac{\sqrt{N+1}}{\sqrt{2\theta^i}} \left( - \sum_{n_2=0}^N |N, n_2\rangle \langle N+1, n_2| - \sum_{n_2=0}^N |N+1, n_2\rangle \langle N, n_2| \right),
\]

\[
T_2 = -i \frac{\sqrt{N+1}}{\sqrt{2\theta^i}} \left( - \sum_{n_2=0}^N |N, n_2\rangle \langle N+1, n_2| + \sum_{n_2=0}^N |N+1, n_2\rangle \langle N, n_2| \right),
\]

\[
T_3 = \frac{\sqrt{N+1}}{\sqrt{2\theta^2}} \left( - \sum_{n_1=0}^N |n_1, N\rangle \langle n_1, N+1| - \sum_{n_1=0}^N |n_1, N+1\rangle \langle n_1, N| \right),
\]

\[
T_4 = -i \frac{\sqrt{N+1}}{\sqrt{2\theta^2}} \left( - \sum_{n_1=0}^N |n_1, N\rangle \langle n_1, N+1| + \sum_{n_1=0}^N |n_1, N+1\rangle \langle n_1, N| \right).
\]

(84)

Using (84), we can show

\[
Tr_H T_\mu T_\nu = \frac{N}{\theta^{i(\mu)}} \delta_{\mu\nu},
\]

(85)

where \(i(\mu) = [(\mu + 1)/2]\) with the symbol \([\ ]\) indicating a Gaussian symbol. \(\mathcal{T}_\mu\) is defined by

\[
\mathcal{T}_\mu = \frac{\sqrt{\theta^{i(\mu)}}}{\sqrt{N}} T_\mu,
\]

(86)

to satisfy

\[
Tr_H \mathcal{T}_\mu \mathcal{T}_\nu = \delta_{\mu\nu}.
\]

(87)

Here we list some formulas about \(\mathcal{I}\) and \(\mathcal{T}_\mu\), which will be used in the calculation of the partition function. They are

\[
Tr_H \mathcal{I} \mathcal{I} = 1 , \quad Tr_H \mathcal{T}_\mu \mathcal{T}_\nu = \delta_{\mu\nu} , \quad Tr_H \mathcal{I} \mathcal{T}_\mu = 0 ,
\]

(88)

\[
Tr_H \mathcal{I} [\hat{\partial}_\mu, \mathcal{I}] = 0 , \quad Tr_H \mathcal{I} [\hat{\partial}_\mu, \mathcal{T}_\nu] = -\frac{1}{\sqrt{\theta^{i(\mu)}}} \delta_{\mu\nu},
\]

\[
Tr_H \mathcal{T}_\mu [\hat{\partial}_\nu, \mathcal{I}] = +\frac{1}{\sqrt{\theta^{i(\mu)}}} \delta_{\mu\nu} , \quad Tr_H \mathcal{T}_\mu [\hat{\partial}_\nu, \mathcal{T}_\rho] = 0 ,
\]

(89)

and

\[
Tr_H \mathcal{I} [\mathcal{I}, \mathcal{I}] = 0 , \quad Tr_H \mathcal{I} [\mathcal{I}, \mathcal{T}_\mu] = 0 ,
\]

\[
Tr_H \mathcal{I} [\mathcal{T}_\mu, \mathcal{T}_\nu] = +\frac{i \theta^{i(\mu)}}{\sqrt{N}} \theta^{-1}_{\mu\nu} , \quad Tr_H \mathcal{T}_\mu [\mathcal{T}_\nu, \mathcal{T}_\rho] = 0.
\]

(90)

For the proof of (88), (89) and (90), see the appendix A. Note that these formulas do not depend on the detail of the definition of the cut off or (31).
Remark that, in the $N \to \infty$ limit, $Tr_{\mathcal{H}}[\mathcal{T}_\mu, \mathcal{T}_\nu]$ vanishes,

$$\lim_{N \to \infty} Tr_{\mathcal{H}}[\mathcal{T}_\mu, \mathcal{T}_\nu] = 0.$$ (91)

We will use this $N \to \infty$ behavior to reduce the calculation of the partition function to the Gaussian integral.

IV-vi Calculation of trace and extra sector

Now, let us calculate the partition function (80). First of all, we list the quantities appearing in the calculation. Because the model is constructed as a balanced topological field theory, it is natural to classify them into the BRS multiplets. For \{\(A_\mu, H_\mu, \psi_\mu, \chi_\mu, H_\mu\),

\[
\begin{array}{ccccccc}
\delta_+ & \psi_{\mu(1)} & , & \phi_\mu(\alpha) & H_\mu(1) & , & H_\mu(\alpha) \\
\delta_- & A_\mu(1) & , & A_\mu(\alpha) & \chi_\mu(1) & , & \chi_\mu(\alpha)
\end{array}
\]

(92)

and for \{\(B_{\mu\nu}^+, \psi_{\mu\nu}^+, \chi_{\mu\nu}^+, H_{\mu\nu}^+\),

\[
\begin{array}{ccccccc}
\delta_+ & \psi_\mu^{\mu(1)} & , & \psi_\mu^{\mu(\alpha)} & H_{\mu\nu}^+(1) & , & H_{\mu\nu}^+(\mu) \\
\delta_- & B_{\mu\nu(1)}^+ & , & B_{\mu\nu(\mu)}^+ & \chi_{\mu\nu(1)}^+ & , & \chi_{\mu\nu(\mu)}^+
\end{array}
\]

(93)

Note that $A_{\mu(1)}$ and $A_{\mu(\alpha)}$ are coefficients of $\mathcal{I}$ and $\mathcal{T}_\alpha$ i.e. $A_\mu = A_{\mu(1)} \mathcal{I} + \sum_{su(N)} A_{\mu(\alpha)} \mathcal{T}_\alpha$, and other fields are noted by similar manner.

It is necessary to comment on the net components of \{\(A_{\mu(\alpha)}, \psi_{\mu(\alpha)}, \chi_{\mu(\alpha)}, H_{\mu(\alpha)}\)\} in (92) and \{\(B_{\mu\nu(\alpha)}^+, \psi_{\mu\nu(\alpha)}^+, \chi_{\mu\nu(\alpha)}^+, H_{\mu\nu(\alpha)}^+\)\} in (93). In the following, we use the term $(\mu, \nu)$ selfdual which means that $A_{\mu(\nu)}$ satisfies $A_{\mu(\nu)} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} A_{\rho(\sigma)}$.

(i) \{\(A_{\mu(\alpha)}, \cdots\)\} have not sixteen but four components. Three of them satisfy the selfdual relation and the rest one is $A_\mu(\mu)$:

\[
\{A_{\mu(\nu)} \mid A_{\mu(\nu)} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} A_{\rho(\sigma)} \ (\mu, \nu) \text{ selfdual} \} \text{ and } \{\sum_{\mu=1}^{4} A_{\mu(\mu)}\}.
\]

(94)

(ii) \{\(B_{\mu\nu(\mu)}^+, \cdots\)\} have four components corresponding to $B_{\mu\nu(\mu)}^+$

\[
B_{\mu\nu(\mu)}^+ = \sum_{\mu=1}^{4} B_{\mu\nu(\mu)}^+.
\]

(95)
On the other hand, \( \{ \phi, c, \tilde{\phi}, \tilde{\eta}, \eta \} \) contain only trace parts

\[
\begin{align*}
\phi(1) & \quad \hat{\delta}_- \downarrow \quad \tilde{\eta}(1) \\
\tilde{\delta}_+ & \quad \hat{\delta}_- \downarrow \quad \hat{\eta}(1) \\
\tilde{c}(1) & \quad \hat{\delta}_- \downarrow \quad \eta(1) \\
\tilde{\phi}(1) & \quad \hat{\delta}_+ \\
\end{align*}
\]

Later we obtain the Gaussian action \((106-110, 111-115, 119, 120)\). For example in \((106)\) we find a term proportional to

\[
\chi^+_{\mu\nu}(1)(A_{\nu(\mu)} - A_{\mu(\nu)}).
\]

From this and other terms in \((106-110, 111-115, 119, 120)\), we see that the net components \((94, 95, 96)\) should be taken to remove the zero modes.

Taking the net components \((94, 95, 96)\) and using \((98, 99)\), we obtain

\[
S_{tr\oplus ex} = Tr_H \hat{\delta}_+ \quad \left[ +\chi^+_{\rho(\mu)} T^\rho \{H^+_{\mu(\sigma)} T^\sigma - ([\hat{\delta}_-, A_{\nu(1)} I] - [\hat{\delta}_+, A_{\mu(1)} I])] \right.
\]

\[
+\chi^+_{\mu(1)} I \{H^+_{\mu(1)} I - ([\hat{\delta}_-, A_{\nu(\rho)} T^\rho] - [\hat{\delta}_+, A_{\mu(\rho)} T^\rho]\}
\]

\[
+\chi^+_{\rho(\mu)} T^\rho \{H^+_{\mu(\sigma)} T^\sigma - (-2[\hat{\delta}_-, B^+_{\nu(\rho)} T^\rho] - [\hat{\delta}_+, c(1) I])] \}
\]

\[
+\chi^+_{\rho(1)} I \{H^+_{\mu(1)} I - (-2[\hat{\delta}_-, B^+_{\nu(\rho)} T^\rho] - [\hat{\delta}_+, c(1) T^\nu]\}
\]

\[
- [\hat{\delta}_-, A_{\mu(1)} I] \psi_{\mu(\nu)} T^\nu \quad ]
\]

\[
+ O(N^{-\frac{1}{2}}).
\]

Note that \( B^+_{\nu(\mu)} \) looks 12 components but only \( B^+_{\nu(\mu)} \) proportional terms survive in \( Tr_H \chi^+_{\mu}(1) I \hat{\delta}^\nu, B^+_{\nu(\mu)} T^\rho \). In the \( N \to \infty \) limit, only quadratic terms survive \(^h\)

\[
S_{tr\oplus ex}^\infty = \lim_{N \to \infty} S_{tr\oplus ex} : \text{quadratic action.}
\]

The action \((98)\) has the following gauge symmetry,

\[
\delta_{\text{gauge}} A_{\mu(\nu)} = \frac{1}{\sqrt{\theta(\mu)}} \delta_{\mu\nu} \varphi(1).
\]

Note that the gauge parameter \( \varphi \) contains only one component \( \varphi(1) \)

\[
\varphi = \varphi(1) I.
\]

\(^h\) Alternatively, we can take the weak coupling limit in the calculation. In general, partition functions of cohomological field theories are independent of coupling constants. So they can be evaluated exactly in the weak coupling limit.
Now we give the BRS transformation rules for $f(1)$ and $f(\mu)$. Except for $A_{\mu(\nu)}, \psi_{\mu(\nu)}$ and $A_{\mu(1)}, \psi_{\mu(1)}$,

$$
\hat{\delta}_+ B(\nu) = F(\nu), \quad \hat{\delta}_+ F(\nu) = 0,
$$
$$
\hat{\delta}_+^2 B(\nu) = 0, \quad \hat{\delta}_+^2 F(\nu) = 0,
$$

(101)

and

$$
\hat{\delta}_+ B(1) = F(1), \quad \hat{\delta}_+ F(1) = 0,
$$
$$
\hat{\delta}_+^2 B(1) = 0, \quad \hat{\delta}_+^2 F(1) = 0
$$

(102)

where $B$ denotes the bosonic matrix and $F$ denotes the fermionic one.

For $A_{\mu(\nu)}, \psi_{\mu(\nu)}$ and $A_{\mu(1)}, \psi_{\mu(1)}$,

$$
\hat{\delta}_+ A_{\mu(\nu)} = \psi_{\mu(\nu)}, \quad \hat{\delta}_+ \psi_{\mu(\nu)} = \frac{1}{\sqrt{\theta(\mu)}} \delta_{\mu\nu} \phi_1,
$$
$$
\hat{\delta}_+^2 A_{\mu(\nu)} = \frac{1}{\sqrt{\theta(\mu)}} \delta_{\mu\nu} \phi_1, \quad \hat{\delta}_+^2 \psi_{\mu(\nu)} = 0,
$$

(103)

and

$$
\hat{\delta}_+ A_{\mu(1)} = \psi_{\mu(1)}, \quad \hat{\delta}_+ \psi_{\mu(1)} = 0,
$$
$$
\hat{\delta}_+^2 A_{\mu(1)} = 0, \quad \hat{\delta}_+^2 \psi_{\mu(1)} = 0.
$$

(104)

For simplicity, in this section we set $\theta^1 = \theta^2 = \theta$ in the following. Using (88,89) and (103-104), (98) is shown to be

$$
S_{\infty}^\infty = S_{\infty}^\infty \text{boson} \oplus S_{\infty}^\infty \text{fermion},
$$

(105)

where

$$
S_{\infty}^\infty \text{boson} = + H^{+ \mu\nu}_1 \{ H^{+ \mu\nu}_1 + \frac{i}{\sqrt{\theta}} (A_{\mu(\nu)} - A_{\mu(\nu)}) \}
$$
$$
+ H^{+ \mu\nu}_1 \{ H^{+ \mu\nu}_1 - \frac{i}{\sqrt{\theta}} (\delta^{\alpha}_\mu A_{\nu(\alpha)} - \delta^{\alpha}_\nu A_{\mu(\alpha)}) \}
$$
$$
+ H^{+ \mu}_1 \{ H^{+ \mu}_1 + \frac{i}{\sqrt{\theta}} (-2B^{+}_{\alpha\mu(\alpha)}) \}
$$
$$
+ H^{+ \mu}_1 \{ H^{+ \mu}_1 - \frac{i}{\sqrt{\theta}} (-2B^{+}_{\alpha\mu(1)} + \delta_{\mu\alpha} c(1)) \}
$$
$$
+ \frac{4}{\sqrt{\theta}} \bar{\phi}(1) \phi(1),
$$

(106) (107) (108) (109) (110)
\[ S_{\text{tr} \oplus \text{ex}}^{\infty \ \text{fermion}} = -\frac{i}{\sqrt{\theta}} \chi^+_{(1) \mu} (\psi_{\nu(\mu)} - \psi_{\mu(\nu)}) \]  
\[ -\frac{2i}{\sqrt{\theta}} \chi^+_{(\alpha) \mu} \psi_{\mu(1)} \]  
\[ + \frac{i}{\sqrt{\theta}} \chi^\mu_{(1)} (2\psi^+_{\alpha(\mu)}) \]  
\[ - \frac{i}{\sqrt{\theta}} \chi^\mu_{(\alpha)} (2\psi^+_{\alpha(\mu)} + \delta_{\mu\alpha}\eta(1)) \]  
\[ + \frac{i}{\sqrt{\theta}} \eta(1) \psi^\mu_{(\mu)}. \]  

Now we fix the gauge symmetry \[ (99) \]. We introduce the ghost \( \rho \), the anti-ghost \( \bar{\rho} \) and the Nakanishi-Lautrup field \( b \). Their ghost number are assigned as \( (+1, -1, 0) \) for \( (\rho, \bar{\rho}, b) \), respectively. BRS transformations for \( \{ \bar{\rho}, b, \rho \} \) are defined as
\[ \hat{\delta}_+ b = \rho, \hat{\delta}_+ \rho = 0, \hat{\delta}_+ \bar{\rho} = 0. \]  

Because the gauge symmetry is given by \[ (99) \], \( \{ \bar{\rho}, b, \rho \} \) contain only the trace parts.

Let us introduce a gauge fixing action by
\[ S_{\text{g.f.}} = Tr \mathcal{H} \hat{\delta}_+ \left[ \bar{\rho}(1) \mathcal{I}(b(1) \mathcal{I} + [\hat{\partial}^\mu, A^\mu_{(\nu)} \mathcal{T}^\nu]) \right]. \]  

To get the BRS exact action including the gauge fixing action, let us deform the BRS transformation rules for \( A_{\mu(\nu)}, \psi_{\mu(\nu)} \) \[ (113) \] as
\[ \hat{\delta}_+ A_{\mu(\nu)} = \psi_{\mu(\nu)} + \frac{1}{\sqrt{\theta}} \delta_{\mu\nu}\rho(1) \]  
\[ \hat{\delta}_+ \psi_{\mu(\nu)} = + \frac{1}{\sqrt{\theta}} \delta_{\mu\nu}\phi(1). \]  

\[ (117) \] is rewritten into
\[ S_{\text{g.f.}} = +b(1)(b(1) - \frac{1}{\sqrt{\theta}} A_{\mu(\mu)}) \]  
\[ + \frac{4}{\theta} \bar{\rho}(1)\rho(1) + \frac{1}{\sqrt{\theta}} \bar{\rho}(1) \psi_{\mu(\mu)}. \]

We list degrees of the Gaussian integral in \[ (106)-(10), (111)-(115) \] and \[ (119)-(20) \].

| Degree | \( H^+_{\mu(\nu)} \) | \( A_{\mu(\nu)} \) | \( \mu, \nu \) selfdual |
|--------|------------------|----------------|------------------|
| 3 + 3  | \[ (106) \]      |                |                  |
| 4 + 4  | \[ (117) \]      | \( A_{\mu(1)} \) |                  |
| 4 + 4  | \[ (108) \]      | \( B^\alpha_{\alpha(\mu)} \) |                  |
| 3 + 1 + 3 + 1 | \( H_{\mu(\alpha)} \) | \( H_{\mu(\mu)} \) | \( B^+_{\alpha(1)} \) | \( c(1) \) |
| 1 + 1  | \( \phi(1) \) | \( \phi(1) \) |                  |
| 1 + 1  | \( b(1) \) | \( A_{\mu(\mu)} \) |                  |

24
From fermions

\begin{align*}
3 + 3 & \quad \chi^+_{\mu\nu(1)} , \; \psi_{\nu(\mu)} (\mu, \nu) \text{ selfdual} \quad \text{in } (111) \\
4 + 4 & \quad \chi^+_{(\alpha)\mu} , \; \psi_{\mu(1)} \quad \text{in } (112) \\
4 + 4 & \quad \chi^+_{(1)\mu} , \; \psi^+_{(a)\mu} \quad \text{in } (113) \\
3 + 1 + 3 + 1 & \quad \chi_{(\nu)\mu} (\mu, \nu) \text{ selfdual} , \; \chi^+_{(\mu)(1)} , \; \psi^+_{\alpha\mu(1)} , \; \bar{\eta}(1) \quad \text{in } (114) \\
1 + 1 & \quad \eta(1) , \; \psi^+_{(\mu)} \quad \text{in } (115) \\
1 + 1 & \quad \rho(1) , \; \bar{\rho}(1) \quad \text{in } (120)
\end{align*}

From (121) and (122), we see that the path integral contains no zero mode, so we obtain a definite partition function. We adopt a standard path integral measure, which is largely expressed by

\[ Df = \prod df_{\text{boson}} \sqrt{2\pi} \prod df_{\text{fermion}}, \quad (123) \]

where \( f_{\text{boson}} \) denotes a bosonic field and \( f_{\text{fermion}} \) denotes a fermionic field. For the precise definition of \( Df \) and the validity of this choice, see the next subsection and appendix C.

Then \( Z_{\text{tr@ex}}^\infty \) is calculated as

\[ Z_{\text{tr@ex}}^\infty = \int Df e^{-(S_{\text{tr@ex}}^\infty + S_{g.f.})} = 1. \quad (124) \]

\section*{IV-vii Results and remarks of this section}

From (78) and (124), we conclude that the partition function of the \( N = 4 \) \( U(1) \) gauge theory on N.C. \( \mathbb{R}^4 \) is given by

\[ Z_{\text{dim}}^{\text{4dim}} = Z_{\text{MM}}^\infty = Z_{\text{MM}}^\infty |_{\text{traceless}} \times Z_{\text{tr@ex}}^\infty = \frac{\pi^2}{6}. \quad (125) \]

We comment on the universality of partition function, (20). Our calculation consists of largely two steps. In the first step the traceless part is treated, then in the second step the trace and extra parts are managed. The first step is manifestly dimensionally independent, because after the dimensional reduction to 0 dim all actions of (8-dim, \( \mathcal{N} = 2 \)), (6-dim, \( \mathcal{N} = 2 \)), (4-dim, \( \mathcal{N} = 4 \)), and (2-dim, \( \mathcal{N} = 8 \)) are the same as the IKKT matrix model action. On the other hand, the calculations in the second step may seem to depend on the dimension of the model, since we keep the derivatives, \( \partial_\mu \). However, the same result \( Z_{\text{tr@ex}}^\infty = 1 \) is expected to be universal. The reason is as follows. The second step, introducing the extra part and fixing the gauge symmetry (99), is a kind of regularization of the zero mode integral. As expected from other regularization method, for example, naively dropping the trace part, equivalent to dividing the path integral measure by the \( U(1) \) gauge volume, the regularization should produce a trivial factor 1. In our regularization method, this is implemented by the supersymmetry. Also, as
explained in appendix C, our regularization is valid for all of (8-dim, \( N = 2 \)), (6-dim, \( N = 2 \)), (4-dim, \( N = 4 \)) and (2-dim, \( N = 8 \)). Then we conclude

\[
Z_{N=2}^{8 \text{dim}} = Z_{N=2}^{6 \text{dim}} = Z_{N=4}^{4 \text{dim}} = Z_{N=8}^{2 \text{dim}} = \frac{\pi^2}{6}.
\] (126)

Finally, we make a remark relating the mathematical significance of (126). In topological field theories, the path integral can be decomposed into finite dimensional integrals of the moduli space defined by the BPS equations and infinite dimensional integrals of fluctuations around each vacuum. The absolute value of the infinite dimensional integrals of the fluctuations should be normalized to 1 to make the partition functions well-defined, then only the integrals of the moduli space remain. (See also appendix C.) If the moduli space is compact, the remained moduli integrals produce a definite number, which is the Euler number of some vector bundle over the moduli space. In the case of this section, each \( 1/d^2 \) in (123) corresponds to the Euler number. In this light, our prescription above, adopting the measure (123) to obtain (124), is an almost unique choice, though it may seem to be chosen by hand. Also, for the traceless part, the similar prescription is performed in 20). To conclude, the result \( \pi^2/6 \) is decided without ambiguity and has an absolute meaning as a topological invariant.

V Moduli Space and Instanton Number

In this section, we concentrate on the relation between the moduli space of the Monads and the partition function of the \( \mathcal{N} = 4 \) supersymmetric Yang-Mills theory. The partition function of Vafa-Witten theory is given by the generating function of the Euler number of the some vector bundle over the moduli space with sign \( \pm 1 \):

\[
Z = \sum_{k=1}^{\infty} \epsilon_k \chi(\mathcal{M}_k)q^k
\] (127)

\[
q^k = e^{2\pi i k \tau}, \quad \epsilon_k = \pm 1.
\] (128)

Here \( \tau \) is the complex coupling constant and \( \mathcal{M}_k \) is the moduli space defined by

\[
\{ A, B, c| F^{+\mu\nu} - i[B^+_{\mu\rho} B^+_{\nu\rho}] - i[B^+_{\mu\nu}, c] = 0, 2D^\mu B^+_{+\mu\rho} + D_\rho c = 0 \}/\mathcal{G},
\] (129)

where \( \mathcal{G} \) is the gauge transformation group. In addition, if \( \chi^{\mu\nu}, \chi^\mu \) zero-modes are sections of the cotangent bundle of \( \mathcal{M}_k \), then \( \chi(\mathcal{M}_k) \) is the Euler number of \( \mathcal{M}_k \). Particularly, the base 4-fold satisfies the vanishing theorem in 35), then the moduli space is identified with the instanton moduli space with its instanton number \( k \). Therefore, it is important to investigate the \( \mathcal{M}_k \).

It is natural to assume that the topology of the moduli space does not change under the \( \theta \)-shift. After dimensional reduction (large \( \theta \) limit), let us replace variables as [44],
and (51). As operators, fields are infinite dimensional matrices. If matrix size of these \(B_i\) is cut off at \(N\), BPS eqs. (24) are replaced by hyperkahler momentum maps

\[
\mu_C := [B_i, B_j] + \frac{1}{2} \epsilon_{ijkl} [B_k^l, B_l^j] = 0,
\]

\[
\mu_R := 4 \sum_{i=1}^{N} [B_i, B_i^\dagger] = 0,
\]

then the moduli space is determined by

\[
\mathcal{M}_N = (\mu_C^{-1}(0) \cap \mu_R^{-1}(0))/U(N).
\]

It is known that the solutions of eqs. (130) include the solutions of simultaneous ADHM eqs. (26). \(\theta\) deformation realizes the continuous connection between (129) and (131). This is a direct correspondence between BPS equations of noncommutative field theory and Monads by means of changing the noncommutative parameter.

Turning now to next issue, let us study the partition function whose action functional includes the topological term. In section IV, we perform the calculation with the action functional which does not include the term of \(\tau \int F \wedge F\) (or \(\tau \text{Tr}_H F \wedge F\)). In the MNS calculation, they use the mass deformation to decompose the theory to more simple ones whose partition function is given by \(1/d^2\) in (57). (See section 7 in [20], and section IV-iv in this article.) This mass deformation causes supersymmetry breaking from \(\mathcal{N} = 4\) to \(\mathcal{N} = 1\). \(B_1, B_2, B_3\) become massive, and \(B_4, B_4^\dagger\) and \(\phi, \bar{\phi}\) are left for massless fields. If we consider this mass deformation in the finite \(\theta\) theory, we find that gauge fields are given from \(B_4, B_4^\dagger\) and \(\phi, \bar{\phi}\) as 4-dim theory, because the massless fields correspond to the unbroken gauge fields. In the reduced theory after integrating out \(B_1, B_2\) and \(B_3\), fixed point loci are defined by

\[
[B_4, B_4^\dagger] = 0, \quad [\phi, \bar{\phi}] = 0, \quad [B_4, \phi] = 0,
\]

where \(B_4, B_4^\dagger\) and \(\phi, \bar{\phi}\) are \(d \times d\) matrices where \(d\) is a divisor of \(N\) and is appearing in the argument of (57). Furthermore, contributions for the partition function are given by isolated fixed points, as MNS mentioned in the end of section 5 in [20]. At least one of \(B_4\) and \(\phi\) is the rank \(d\) when \(B_4\) and \(\phi\) are solutions of the fixed point equations and the fixed points contribute to the path integral. Because if rank \(< d\) then there are zero modes of the equations

\[
[\delta B_4, B_4^\dagger] + [B_4, \delta B_4^\dagger] = 0, \quad [\delta \phi, \bar{\phi}] + [\phi, \delta \bar{\phi}] = 0, \quad [\delta B_4, \phi] + [B_4, \delta \phi] = 0,
\]

where these equations are given by variation of (132). These zero modes mean that the fixed point loci are non-zero dimension and path integrals vanish by the fermionic zero modes. With attention to these points, if we specify the instanton numbers corresponding to solutions of (132) labeled by \(d\), then we determine the partition function whose action
A hint to speculate the instanton number is ADHM correspondence. The solutions of (132) is included in the set of solutions of noncommutative deformed ADHM eqs. corresponding to d instanton, i.e.

\[
[B_4, B_4^\dagger] + [\phi, \bar{\phi}] + II^\dagger - J^\dagger J = 0, \quad [B_4, \phi] + IJ = 0,
\]

where \(I\) and \(J^\dagger\) are \(d\)-dim. vectors. This is ADHM equations of noncommutative U(1) theories under the condition of noncommutativity \(\theta^1 = -\theta^2\)\(^2\). Here we have to fix \(I\) and \(J^\dagger\) to compare (134) with (132) as

\[
I = 0_d, \quad J^\dagger = 0_d,
\]

where \(0_d\) is 0 vector of \(d\)-dim. Then, the solutions of (134) are given by the solutions of (132). From this observation, we find that the moduli space of \(B_4, B_4^\dagger\) and \(\phi, \bar{\phi}\), which are gauge fields in this case, is the submanifold in instanton moduli space of instanton number \(d\).

Therefore, someone might think it is not so strange to expect that the instanton number is given as \(-\frac{\det(\theta)}{16\pi^2} \text{Tr}_R F \wedge F = d\), where the gauge fields correspond to \(B_4, B_4^\dagger\) and \(\phi, \bar{\phi}\), and we conjecture that the partition function of the \(\mathcal{N} = 4\) U(1) gauge theory in noncommutative \(\mathbb{R}^4\) with the topological term \(\tau \int F \wedge F\) is given by \(\tilde{Z}_{\mathcal{N}=4, \tau}^{4\text{dim}} = \sum_{d=0}^{\infty} \frac{1}{d^2} e^{2\pi i \tau d}\).

However, It would still be unwise to conclude \(\tilde{Z}_{\mathcal{N}=4, \tau}^{4\text{dim}} = \sum_{d=0}^{\infty} \frac{1}{d^2} e^{2\pi i \tau d}\), because the direct corresponding with the instanton number and \(B_4, B_4^\dagger\) and \(\phi, \bar{\phi}\) fixed point locus labeled by \(d\) is unknown. Meanwhile, it might be possible to investigate this conjecture from Montonen and Olive duality\(^{19,35}\) if such a duality of noncommutative version exists. (See also\(^8\).) For example, if we assume that the partition function takes the form as

\[
\tilde{Z}_{\mathcal{N}=4, \tau}^{4\text{dim}} = \sum_{d=1}^{\infty} \frac{1}{d^2} e^{2\pi i \tau (d)} ,
\]

where \(k(d)\) is an instanton number depending on \(d\), restriction to the modular like form

\[
\tilde{Z}_{\mathcal{N}=4, 1/\tau}^{4\text{dim}} = \pm \left(\frac{\tau}{i}\right)^n \tilde{Z}_{\mathcal{N}=4, \tau}^{4\text{dim}}
\]

might determine \(k(d)\), where \(n\) is a some number. Unfortunately, we do not know how to chose a suitable modular like form , and above naive conjecture \(\tilde{Z}_{\mathcal{N}=4, \tau}^{4\text{dim}} = \sum_{d=0}^{\infty} \frac{1}{d^2} e^{2\pi i \tau d}\) does not satisfy this condition. Anyway, further investigations are necessary to determine the contribution of the topological term.

At the end of this section, we consider the dimensional reduction of the theory with topological terms. In the discussions in section\(^{\text{III}}\) we use cohomological field theory without topological terms like \(\int F \wedge F\), and some of them are not supersymmetric gauge theories in the meaning of the usual supersymmetry. Now, let us consider the case including topological terms. As an example, let us consider the 4 dimensional case whose action...
is given by cohomological terms and instanton number:

\[ S = \int \mathrm{tr} \, \hat{\delta} \Psi + (i \vartheta / 8 \pi^2) \int \mathrm{tr} \, F \wedge F. \]

Let us consider perturbation around classical background fixed by instanton number, i.e.,

\[ A_\mu = A_\mu^{(k)} + \delta A_\mu^{(k)}, \quad \int \mathrm{tr} \, F(A^{(k)}) \wedge F(A^{(k)}) = 8\pi^2 k. \]

The partition function is given by

\[ Z^{4\text{dim}} = \sum_k e^{2\pi i rk} \int \mathcal{D}\delta A^{(k)} \cdots e^{-\int \mathrm{tr} \, \hat{\delta} \Psi^{(k)}}, \]

where \( \mathcal{D}\delta A^{(k)} \cdots \) is the path integral measure of the all fields and the functional \( \Psi^{(k)} \) depends on both \( A^{(k)} \) and \( \delta A^{(k)} \). The BRS transformations are induced from (8) and so on as \( \hat{\delta}(\delta A^{(k)}) = \psi^{(k)} \) etc. So we have

\[ Z^{4\text{dim}} = \sum_k e^{2\pi i rk} Z^{4\text{dim}}_k, \]

where \( Z^{4\text{dim}}_k \) is the perturbative partition function of the 4 dimension theory without the topological term. The action \( \hat{\delta} \Psi^{(k)} \) is still given by a BRS exact term. The arguments for the \( \theta \)-shift invariance of the path integral are valid for \( \int \mathcal{D}\delta A^{(k)} \cdots e^{-\int \mathrm{tr} \, \hat{\delta} \Psi^{(k)}}. \) Then the dimensional reduction of the perturbative partition functions arises at the large \( \theta \) limit;

\[ Z^{4\text{dim}} = Z^{2\text{dim}} = Z^{0\text{dim}}, \tag{138} \]

where \( Z^{4\text{dim}}_k \) and \( Z^{2\text{dim}}_k \) are possible to be described by partition functions of 2 and 0 dimension field theories, respectively. Therefore, we find that the universality of the perturbative partition functions \( Z^{4\text{dim}}_k, Z^{2\text{dim}}_k \) and \( Z^{0\text{dim}}_k \), similar to the claim in section III.

Now let us discuss the possibility that (138) means a universality of the partition functions of usual supersymmetric theories in various dimensions. Consider the weighted sum of \( Z^{4\text{dim}}_k, Z^{2\text{dim}}_k \) and \( Z^{0\text{dim}}_k \) with weight \( e^{2\pi i rk} \),

\[ \sum_k e^{2\pi i rk} Z^{4\text{dim}}_k = \sum_k e^{2\pi i rk} Z^{2\text{dim}}_k = \sum_k e^{2\pi i rk} Z^{0\text{dim}}_k. \tag{139} \]

\( \sum_k e^{2\pi i rk} Z^{4\text{dim}}_k \) is equal to \( Z^{4\text{dim}} \), the partition of the 4 dimension supersymmetric theory including the topological term. On the other hand, the meanings of the weighted sums \( \sum_k e^{2\pi i rk} Z^{2\text{dim}}_k \) and \( \sum_k e^{2\pi i rk} Z^{0\text{dim}}_k \) are obscure. It is unclear that they have the meaning of the partition functions of some lower dimension theories. If they can be interpreted as the partition functions of some supersymmetric theories in lower dimensions, (139) means a universality of the partition functions of supersymmetric theories in various dimensions.

To answer whether this statement is true or not, we need to clarify the following questions.

(i) Is the number \( k \) expressed in terms of lower dimension theories?
(ii) Is the number \( k \) interpreted as a topological invariant? And does it characterize classical solutions of the lower dimension theories?
(iii) Is the total action, the sum of the action defining \( Z^{2\text{dim}}_k \) or \( Z^{0\text{dim}}_k \) and the action giving the number \( k \), equivalent to a supersymmetric action in lower dimension?

At this time, we can only make a few comments on question (i). We calculated the large
\( \theta \) limit of the elongated N.C. \( U(1) \) \( k \)-instanton, that is the reduction from 4 dimension to 2 dimension of the solution. (For construction of the elongated N.C. \( U(1) \) \( k \)-instanton, see [14].) For this case, we can show that \( k \) is expressed in terms of 2 dimension theories, 

\[
\theta_1 Tr F_{z_1, \bar{z}_1} = -k .
\]  

(140)

It may be that this fact implies that the number \( k \) is expressed in terms of lower dimension theories. However, we have no concrete answer to question (ii) and question (iii) at this time.

VI Conclusions and Discussions

We investigated cohomological gauge theories in N.C. \( \mathbb{R}^{2D} \). We saw that vacuum expectation values of the theories do not depend on noncommutative parameters, and the large noncommutative parameter limit is equivalent to the dimensional reduction. As a result of these facts, we showed that two types of cohomological theories defined in N.C. \( \mathbb{R}^{2D} \) and N.C. \( \mathbb{R}^{D+2} \) are equivalent, if they are connected through dimensional reduction. Therefore, we found several partition functions of noncommutative supersymmetric Yang-Mills theories in various dimensions are equivalent, when they are connected by dimensional reduction from \( 2+2D \) to \( 2D \). Using this technique and requiring some natural assumptions, we determine the partition function of the \( \mathcal{N} = 4 \) U(1) gauge theory in N.C. \( \mathbb{R}^{4} \), where the action does not include the topological term \( \tau \int F \wedge F \), and the result is equivalent to the partition function of \( (8\text{dim}, \mathcal{N} = 2) \), \( (6\text{dim}, \mathcal{N} = 2) \), \( (2\text{dim}, \mathcal{N} = 8) \) and the IKKT matrix model given by their dimensional reduction to 0 dim. The case including the topological term was discussed, too.

Let us list some left problems below. In this article, concrete partition functions are given for the \( \mathcal{N} = 4 \) U(1) gauge theory in N.C. \( \mathbb{R}^{4} \) and the series connecting to it by dimensional reduction. So, we are interested in N.C.non-abelian cases. To calculate them, we have to find some new formulation like MNS, because we know the partition function concerning \( su(N) \) but we need it for \( su(N) \times su(M) \) for \( U(M) \) theory.

Next, we had qualitative observation of \( \mathcal{N} = 2 \) 4-dim case but we do not do quantitative approach. So, we have to do the more detail analysis for the \( \mathcal{N} = 2 \) super Yang-Mills cases. We saw in section [V] after taking large \( \theta \) limit, moduli space is described by Monads in \( \mathcal{N} = 4 \) 4-dim case. From the analogy with \( \mathcal{N} = 4 \) 4-dim case, direct and smooth connections between noncommutative instanton moduli spaces and ADHM spaces might be given in \( \mathcal{N} = 2 \) 4-dim case.

Other important problems are applications to the various fuzzy spaces, \( T^{d}_{\theta} \), \( \mathbb{C}P^{d}_{N} \), and so on. Since these noncommutative spaces are expressed by finite dimensional Hilbert spaces, the dimensional reduction will not occur at the large \( \theta \) limit despite omitting kinetic terms.

Wide spread applications of the technology of this article are going to happen in many cases other than above subjects. All of them are left for future works.
Acknowledgements
We would like to thank Takuya Miyazaki for useful remarks and discussion. Discussions during the workshop “An International Meeting Noncommutative Geometry, K-theory and Physics 2005” were useful to complete this work.

A Fock Space

Let us consider N.C. $\mathbb{R}^{2D}$. First of all, we introduce following operators,

$$
\begin{align*}
  a_i &\equiv \frac{z_i}{\sqrt{\theta^{2i-1,2i}}} , & z_i &\equiv \frac{1}{\sqrt{2}}(x^{2i-1} + i x^{2i}), \\
  a_i^\dagger &\equiv \frac{\bar{z}_i}{\sqrt{\theta^{2i-1,2i}}} , & \bar{z}_i &\equiv \frac{1}{\sqrt{2}}(x^{2i-1} - i x^{2i}),
\end{align*}
$$

(141)

where $i$ runs from 1 to $D$, and $a_i$ and $a_i^\dagger$ satisfy

$$
[a_i, a_j^\dagger] = \delta_{ij}.
$$

(142)

We often use the symbol $\theta^i$ defined as

$$
\theta^i = +\theta^{2i-1,2i} = -\theta^{2i,2i-1}.
$$

(143)

The Hilbert space is constructed as the Fock space,

$$
\mathcal{H} = \bigoplus \mathbb{C} |n_1, \ldots, n_D\rangle ,
$$

(144)

$$
|n_1, \ldots, n_D\rangle \equiv \frac{(a_1^\dagger)^{n_1} \cdots (a_D^\dagger)^{n_D}}{\sqrt{n_1! \cdots n_D!}} |0, \ldots, 0\rangle .
$$

(145)

$a_i$ and $a_i^\dagger$ operate on $|n_1, \ldots, n_D\rangle$ as follows

$$
\begin{align*}
  a_i |n_1, \ldots, n_D\rangle &= \sqrt{n_i} |n_1, \ldots, n_i - 1, \ldots, n_D\rangle , \\
  a_i^\dagger |n_1, \ldots, n_D\rangle &= \sqrt{n_i + 1} |n_1, \ldots, n_i + 1, \ldots, n_D\rangle .
\end{align*}
$$

(146)

$|n_1, \ldots, n_D\rangle$ are the eigenstates of the number operator $\hat{n}_i \equiv a_i^\dagger a_i$,

$$
\hat{n}_i |n_1, \ldots, n_D\rangle = n_i |n_1, \ldots, n_D\rangle .
$$

(147)

Arbitrary operator has following expression;

$$
\hat{O} = \sum_{n_1, m_1} \cdots \sum_{n_D, m_D} O_{n_1 \cdots n_D}^{m_1 \cdots m_D} |n_1, \ldots, n_D\rangle \langle m_1, \ldots, m_D|.
$$

Let us consider $2D = 4$ case. The Hilbert space $\mathcal{H}$ is expanded by the Fock basis $|n_1, n_2\rangle$,

$$
\mathcal{H} = \bigoplus \mathbb{C} |n_1, n_2\rangle ,
$$

(148)

$$
|n_1, n_2\rangle \equiv \frac{(a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2}}{\sqrt{n_1! n_2!}} |0, 0\rangle .
$$

(149)
\(a_1^\dagger\) and \(a_i\) are expressed as

\[
a_1^\dagger = \sum_{n_1=0}^{\infty} \sqrt{n_1+1}|n_1+1,n_2\rangle \langle n_1,n_2|, \quad a_2^\dagger = \sum_{n_2=0}^{\infty} \sqrt{n_2+1}|n_1,n_2+1\rangle \langle n_1,n_2|,
\]

\[
a_1 = \sum_{n_1=0}^{\infty} \sqrt{n_1+1}|n_1,n_2\rangle \langle n_1+1,n_2|, \quad a_2 = \sum_{n_2=0}^{\infty} \sqrt{n_2+1}|n_1,n_2\rangle \langle n_1,n_2+1|.
\]

(148)

The finite dimensional truncation \(\mathcal{H}_N\) can be defined by several ways. One definition of \(\mathcal{H}_N\) is given by

\[
\mathcal{H}_N = \bigoplus_{n_1=0,n_2=0}^{N_c} \mathbb{C}|n_1,n_2\rangle,
\]

(149)

where \(N_c\) is a finite integer number. By the definition, we obtain

\[
\text{dim. of } \mathcal{H}_N = (N_c + 1)^2 = N,
\]

(150)

and

\[
1_N = \sum_{n_1=0,n_2=0}^{N_c} |n_1,n_2\rangle \langle n_1,n_2|.
\]

(151)

Another definition of \(\mathcal{H}_N\) is given by

\[
\mathcal{H}_N = \bigoplus_{n_1=0,n_2=0}^{N_c} \mathbb{C}|n_1,n_2\rangle.
\]

(152)

In this case,

\[
\text{dim. of } \mathcal{H}_N = \frac{(N_c + 1)(N_c + 2)}{2} = N,
\]

(153)

and

\[
1_N = \sum_{n_1=0,n_2=0}^{N_c} |n_1,n_2\rangle \langle n_1,n_2|.
\]

(154)

By using the definition of \(1_N\), (151) or (154), and the following expressions of the differential operators \(\hat{\partial}_\mu\) in terms of \(a_1^\dagger\) and \(a_i\),

\[
\hat{\partial}_1 = \frac{1}{\sqrt{2\theta^1}}(a_1 - a_1^\dagger), \quad \hat{\partial}_2 = \frac{-i}{\sqrt{2\theta^1}}(a_1 + a_1^\dagger),
\]

\[
\hat{\partial}_3 = \frac{1}{\sqrt{2\theta^2}}(a_2 - a_2^\dagger), \quad \hat{\partial}_4 = \frac{-i}{\sqrt{2\theta^2}}(a_2 + a_2^\dagger),
\]

(155)
Given the definition of $\mathcal{H}_N$, for example by (31), we obtain

\begin{align*}
[a_1, 1_N] &= -\sqrt{N + 1} \sum_{n_2=0}^{N} |N, n_2\rangle\langle N + 1, n_2|, \\
[a_1^*, 1_N] &= +\sqrt{N + 1} \sum_{n_2=0}^{N} |N + 1, n_2\rangle\langle N, n_2|, \\
[a_2, 1_N] &= -\sqrt{N + 1} \sum_{n_1=0}^{N} |n_1, N\rangle\langle n_1, N + 1|, \\
[a_2^*, 1_N] &= +\sqrt{N + 1} \sum_{n_1=0}^{N} |n_1 + 1, n_1\rangle\langle n_1, N|.
\end{align*}

(156)

From (156) and (155), we obtain

\begin{align*}
T_1 &= \frac{1}{\sqrt{2} \theta_1} \left( -\sqrt{N + 1} \sum_{n_2=0}^{N} |N, n_2\rangle\langle N + 1, n_2| \\
&\quad -\sqrt{N + 1} \sum_{n_2=0}^{N} |N + 1, n_2\rangle\langle N, n_2| \right), \\
T_2 &= \frac{-i}{\sqrt{2} \theta_1} \left( -\sqrt{N + 1} \sum_{n_2=0}^{N} |N, n_2\rangle\langle N + 1, n_2| \\
&\quad +\sqrt{N + 1} \sum_{n_2=0}^{N} |N + 1, n_2\rangle\langle N, n_2| \right), \\
T_3 &= \frac{1}{\sqrt{2} \theta_2} \left( -\sqrt{N + 1} \sum_{n_1=0}^{N} |n_1, N\rangle\langle n_1, N + 1| \\
&\quad -\sqrt{N + 1} \sum_{n_1=0}^{N} |n_1, N + 1\rangle\langle n_1, N| \right), \\
T_4 &= \frac{-i}{\sqrt{2} \theta_2} \left( -\sqrt{N + 1} \sum_{n_1=0}^{N} |n_1, N\rangle\langle n_1, N + 1| \\
&\quad +\sqrt{N + 1} \sum_{n_1=0}^{N} |n_1, N + 1\rangle\langle n_1, N| \right).
\end{align*}

(157)

Using (151) and (157), we can show

\begin{align*}
Tr_{\mathcal{H}} 1_N 1_N &= N, \\
Tr_{\mathcal{H}} T_\mu T_\nu &= +\frac{1}{\theta_1^2} N \delta_{\mu\nu}, \\
Tr_{\mathcal{H}} 1_N T_\mu &= 0.
\end{align*}

(158)
Also, we can obtain

\[ \text{Tr}_{\mathcal{H}} \mathbf{1}_N [\hat{\partial}_\mu, \mathbf{1}_N] = 0 \ , \ \text{Tr}_{\mathcal{H}} \mathbf{1}_N [\hat{\partial}_\mu, T_\nu] = -\frac{N}{\theta^i} \delta_{\mu\nu}, \]

\[ \text{Tr}_{\mathcal{H}} T_\mu [\hat{\partial}_\nu, \mathbf{1}_N] = +\frac{N}{\theta^i} \delta_{\mu\nu} \ , \ \text{Tr}_{\mathcal{H}} T_\mu [\hat{\partial}_\nu, T_\mu] = 0 \ , \] (159)

and

\[ \text{Tr}_{\mathcal{H}} \mathbf{1}_N [\mathbf{1}_N, \mathbf{1}_N] = 0 \ , \ \text{Tr}_{\mathcal{H}} \mathbf{1}_N [\mathbf{1}_N, T_\mu] = 0, \]

\[ \text{Tr}_{\mathcal{H}} \mathbf{1}_N [T_\mu, T_\nu] = +iN\theta^{-1}_{\mu\nu} \ , \ \text{Tr}_{\mathcal{H}} T_\mu [T_\nu, T_\rho] = 0. \] (160)

Let us define \( \mathcal{I} \) and \( \mathcal{T}_\mu \) as,

\[ \mathcal{I} = \frac{1}{\sqrt{N}} \mathbf{1}_N, \] (161)

and

\[ \mathcal{T}_\mu = \sqrt{\theta^i} \hat{T}_\mu. \] (162)

By definition,

\[ \mathcal{T}_\mu = \sqrt{\theta^i} [\hat{\partial}_\mu, \mathcal{I}]. \] (163)

Using \( \mathcal{I} \) and \( \mathcal{T}_\mu \), (158), (159) and (160) are rewritten into

\[ \text{Tr}_{\mathcal{H}} \mathcal{I} \mathcal{I} = 1 \ , \ \text{Tr}_{\mathcal{H}} \mathcal{T}_\mu \mathcal{T}_\nu = \delta_{\mu\nu} \ , \ \text{Tr}_{\mathcal{H}} \mathcal{I} \mathcal{T}_\mu = 0, \] (164)

\[ \text{Tr}_{\mathcal{H}} \mathcal{I} [\hat{\partial}_\mu, \mathcal{I}] = 0 \ , \ \text{Tr}_{\mathcal{H}} \mathcal{I} [\hat{\partial}_\mu, \mathcal{T}_\nu] = -\frac{1}{\sqrt{\theta^i}} \delta_{\mu\nu}, \]

\[ \text{Tr}_{\mathcal{H}} \mathcal{T}_\mu [\hat{\partial}_\nu, \mathcal{I}] = +\frac{1}{\sqrt{\theta^i}} \delta_{\mu\nu} \ , \ \text{Tr}_{\mathcal{H}} \mathcal{T}_\mu [\hat{\partial}_\nu, \mathcal{T}_\rho] = 0 \ , \] (165)

and

\[ \text{Tr}_{\mathcal{H}} \mathcal{I} \mathcal{I} \mathcal{I} = 0 \ , \ \text{Tr}_{\mathcal{H}} \mathcal{I} \mathcal{I} \mathcal{T}_\mu = 0, \]

\[ \text{Tr}_{\mathcal{H}} \mathcal{I} \mathcal{T}_\mu \mathcal{T}_\nu = +\frac{i\theta^i}{\sqrt{N}} \theta^{-1}_{\mu\nu} \ , \ \text{Tr}_{\mathcal{H}} \mathcal{T}_\mu \mathcal{T}_\nu \mathcal{T}_\rho = 0. \] (166)

The same formulae as (161), (165) and (166) hold for the case of (152). The difference between the definitions of \( \mathcal{H}_N \)'s, (149) and (152), are absorbed in dim. of \( \mathcal{H}_N \).

It is worthwhile to notice that the independence of the precise definitions of \( \mathcal{H}_N \) holds generally. The proof is done by using the discrete version of Stokes’s theorem for the boundary of the finite truncated Fock space (13, 30).
B Large \( \theta \) limit

In this article, we removed the terms including \( \partial_\mu = -i\theta^{-1}_{\mu\nu}[x^\nu, *] \) in the lagrangian when we calculated the partition function without zero mode integrals in the large \( \theta \) limit. If we consider some specific fixed function \( f(x) \), then expression of \( \partial_\mu f(x) = -i\theta^{-1}_{\mu\nu}[x^\nu, f(x)] \) is not changed by taking large \( \theta \) limit because \([x^\nu, f(x)]\) becomes large with \( \theta \). Therefore, someone might think that the process of removing terms including \( \partial_\mu \) is not correct. However, we have to recall that our lagrangian is changed by \( \theta \) variation and then the equations of motion and BPS equations are changed. Then the solutions of the equations, which make much contribution to the partition functions and vacuum expectation values, are changed by \( \theta \) changing. It follows that the terms including derivatives become irrelevant. In this section, we show concretely the validity of taking the terms including \( \partial_\mu = -i\theta^{-1}_{\mu\nu}[x^\nu, *] \) away from lagrangians at the large \( \theta \) limit.

The BPS eqs. in this paper are given by differential equations of first order ;

\[
\begin{equation}
\sum_{i,I} c_{iI,k} \partial_z f_I + V_k(f_J) = 0 ,
\end{equation}
\]

where \( f_I \) are fields, \( V_k(f_I) \) are some quadratic polynomial in \( f_I \) and \( c_{iI,k} \) are some constants. \( k = 1, \ldots, n \), where \( n \) is the number of elements of \( f_I \) minus degree of gauge freedom. For example, BPS eqs. of \( \mathcal{N} = 4 \) 4-dim. gauge theory are

\[
\begin{equation}
F^{+\mu\nu} - i[B^+_{\mu\rho}, B^+_{\nu\rho}] - i[B^+_{\mu\nu}, c] = 0 ,
\end{equation}
\]

Let us consider (167) by using the Fock basis ;

\[
B_k(\hat{f}_I, \theta) \equiv c_{iI,k}^+ \frac{1}{\sqrt{\theta}} [a_i, \hat{f}_I] + c_{iI,k}^- \frac{1}{\sqrt{\theta}} [\hat{a}_i^*, \hat{f}_I] + V_k(\hat{f}_J) = d_{i,k} \frac{1}{\theta} .
\]

Here \( d_{i,k} \frac{1}{\theta} \) are constants derived from \([\partial_z, \partial_{\bar{z}}]\). For example, eqs. of \( \mathcal{N} = 4 \) 4-dim. cases are given by

\[
\begin{equation}
P^+_{\mu\nu\rho\tau}[\hat{D}^\rho, \hat{D}^\tau] + [B^+_{\mu\rho}, B^+_{\nu\rho}] + [B^+_{\mu\nu}, c] = i \left( P^+_{\mu\nu\rho\tau}(\theta^{-1})^{\rho\tau} \right) ,
\end{equation}
\]

\[
2[\hat{D}^\mu, B^+_{\mu\rho}] + [\hat{D}_\rho, c] = 0 ,
\]

where \( P^+_{\mu\nu\rho\tau} \) is self-dual projection operator and \( \hat{D}_\mu = \hat{\partial}_\mu + iA_\mu \). When we take \( \theta^{\mu\nu} \) as (17), the righthand side of (170) is rewritten as

\[
P^+_{\mu\nu\rho\tau}(\theta^{-1})^{\rho\tau} = -\frac{\varepsilon_{\mu\nu}}{2} \left( \frac{1}{\theta^2} + \frac{1}{\theta^2} \right) ,
\]

\[
\begin{pmatrix}
0 & 1 \\
-1 & 0 \\
0 & 1 \\
-1 & 0
\end{pmatrix}
\]
\( \hat{f}_I \) is a operator representation of \( f_I \), i.e. 
\[
\hat{f}_I = \sum (f_I)_{n_1,...,n_D}^{m_1,...,m_D} |n_1,...,n_D\rangle \langle m_1,...,m_D|
\]
In this representation, the BPS eqs. are just simultaneous quadratic equations, and the noncommutative parameters \( \theta^i \) appear in only first 2 terms and right hand side in (169). Note that solutions of (169) depend on \( \theta^i \) but variables \( (f_I)_{n_1,...,n_D}^{m_1,...,m_D} \) themselves do not depend on \( \theta \). For this reason, BPS equations are truncated to
\[
B_k(\hat{f}_I, \infty) \equiv V_k(\hat{f}_I) = 0 , \quad (172)
\]
at the \( \theta \to \infty \) limit. Such truncations have been discussed in many works, see for example (9, 10, 15). Thus, it becomes clear that terms including \( \partial_\mu = -i\theta^{-1}_\mu [x^\nu , *] \) in the lagrangian become irrelevant at the large \( \theta \) limit.

However, the above discussion is insufficient for the proof which justifies removing terms including \( \partial_\mu \). Because we assume the convergency of path integral which has not been confirmed when we formally prove that partition functions and vacuum expectation values of observables do not depend on \( \theta \). Therefore, we have to check our models satisfying the convergency conditions. To understand this statement, let us consider the following example.

Let \( f_i \) be dynamical variables and assume that action functional take the following form :
\[
S_\epsilon[f] = S_0 + \epsilon S_1 , \quad (173)
\]
where \( \epsilon \) is a some constant, \( S_0 \) and \( S_\epsilon \) are BRS exact actions, and they do not depend on \( \epsilon \). Let us expand the partition function as
\[
Z_\epsilon = \int \mathcal{D} f e^{-S_\epsilon} \quad (174)
\]
\[
= \int \mathcal{D} f e^{-S_0} (1 - \epsilon S_1 + \frac{1}{2} \epsilon^2 S_1^2 - \cdots) \quad (175)
\]
and introduce
\[
Z_0 = \int \mathcal{D} f e^{-S_0} . \quad (176)
\]
If \( e^{-S_0} \) damp the integrand, the integral
\[
\int \mathcal{D} f e^{-S_0} \epsilon^n S_1^n , \quad \text{for } n \geq 1 \quad (177)
\]
is well defined. Then, \( Z_\epsilon \) does not depend on \( \epsilon \), i.e.
\[
Z_\epsilon = Z_0 ,
\]
because \( S_1^n \) is a BRS exact term and
\[
\int \mathcal{D} f e^{-S_0} \epsilon^n S_1^n = 0 , \quad \text{for } n \geq 1 .
\]
Therefore, we found that we have to verify that (176) is well defined and $e^{-S_0}$ damp integrands for proof of $\epsilon$ independence.

To get a feeling for how all of this should work out, consider simple toy models. At first, let us consider the toy model given by Vafa and Witten in section 2 of 35). Let $x, y, H_1$ and $H_2$ be real bosonic variables, and $\psi_x, \psi_y, \chi_1$ and $\chi_2$ be fermionic variables. We define BRS transformations by

$$
\hat{\delta}x = \psi_x, \quad \hat{\delta}y = \psi_y, \quad \hat{\delta}\chi_1 = H_1, \quad \hat{\delta}\chi_2 = H_2.
$$

Consider following action

$$
S^{toy1}_\epsilon = \hat{\delta}\{\chi_1(H_1 + 2i(x^2 - \epsilon - y^2)) + \chi_2(H_2 + 2i(2xy))\},
$$

$$
= S^{toy1}_0 + \epsilon S^{toy1}_1,
$$

where

$$
S^{toy1}_0 = \hat{\delta}\{\chi_1(H_1 + 2i(x^2 - y^2)) + \chi_2(H_2 + 2i(2xy))\},
$$

$$
S^{toy1}_1 = \hat{\delta}\chi_1.
$$

$e^{-S^{toy1}_0}$ makes the integral

$$
\int Df e^{-S^{toy1}_0} \epsilon^n (S^{toy1}_1)^n, \text{ for } n \geq 1
$$

be well defined, and

$$
Z^{toy1}_\epsilon = Z^{toy1}_0 \equiv \int Df e^{-S^{toy1}_0}.
$$

Indeed, we can easily perform the direct calculations of the partition functions $Z^{toy1}_\epsilon$ and $Z^{toy1}_0$, respectively, and their results reproduce (181). Note that degeneracy of the solutions does not affect the independence of $\epsilon$. In this case, when $\epsilon \neq 0$ equations are given by $x^2 - \epsilon - y^2 = 0$ and $2xy = 0$, then the solutions are given as $(x, y) = (\pm \sqrt{\epsilon}, 0)$. These two sets of solutions become degenerate in $\epsilon \to 0$. Despite such singularities, path integrals moderate them, and the partition function is smooth at $\epsilon = 0$.

As the second example, consider the following action

$$
S^{toy2}_\epsilon = \hat{\delta}\{\chi_1(H_1 + 2i(x^2 - \epsilon)) + \chi_2(H_2 + 2i(2xy))\},
$$

$$
= S^{toy2}_0 + \epsilon S^{toy2}_1,
$$

where

$$
S^{toy2}_0 = \hat{\delta}\{\chi_1(H_1 + 2i(x^2)) + \chi_2(H_2 + 2i(2xy))\},
$$

$$
S^{toy2}_1 = \hat{\delta}\chi_1.
$$

37
At first glance, the partition function $Z_{\epsilon}^{\text{toy}} \propto 2 \epsilon \exp(-S_{\text{toy}}^0)$ looks independent of $\epsilon$ from the formal discussion. But $e^{-S_{\text{toy}}^0}$ does not damp the integrals in this case, then $Z_{\epsilon}^{\text{toy}} \propto 2 \epsilon$ depends on $\epsilon$. Indeed,

$$Z_{\epsilon}^{\text{toy}} = \int \frac{dx}{\sqrt{2\pi}} \frac{dy}{\sqrt{2\pi}} \frac{dH_1}{\sqrt{2\pi}} \frac{dH_2}{\sqrt{2\pi}} d\psi x d\psi y d\chi_1 d\chi_2 \exp(-S_{\epsilon}^{\text{toy}}) = 1 + \frac{\epsilon}{2\pi}^{-1/2} + O(\epsilon^2).$$

(183)

These observations show that we have to check the convergency of $e^{-S_0}$ where the action $S_0$ is $\theta$ independent part of the total action, before removing terms including $\theta^{-1}$ from action.

Let us now attempt to investigate the specific case of $\mathcal{N} = 4$ 4-dim. First we consider the case of $\theta^1 = -\theta^2$. This is very special case and we can understand the validity of removing the terms including $\partial_\mu$ not from above discussions but from the following discussions. Using $\theta^1 = -\theta^2$, the BPS eqs (170) and (171) are replaced by

$$P_{\mu\nu\rho\tau}^{+}[\hat{D}_\rho, \hat{D}_\tau] + [B_{\mu\rho}, B_{\nu\sigma}] + [B_{\mu\nu}, c] = 0,$$

(184)

$$2[\hat{D}_\mu, B_{\mu\nu}] + [\hat{D}_\rho, c] = 0.$$

(185)

On the contrary, the BPS eqs. of the large $\theta$ limit are given by

$$-P_{\mu\nu\rho\tau}^{+}[A^{\rho}, A^{\tau}] + [B_{\mu\rho}, B_{\nu\sigma}] \delta^{\rho\sigma} + [B_{\mu\nu}, c] = 0,$$

(186)

$$2[A_\mu, B_{\mu\rho}] + [A_\rho, c] = 0.$$

(187)

(186, 187) are equivalent to (184, 185) with $1/\theta = 0$. The correspondence of these and more general cases are already known in [2] and [31], that is, we can identify (184, 185) and (186, 187) by redifining

$$iA_\mu = \hat{D}_\mu.$$

(188)

This is a trivial one to one correspondence between the large $\theta$ limit and finite $\theta^1 = -\theta^2$ case. Under change of variables (188), the path integral measure does not cause nontrivial Jacobian, then theories characterized by (184, 185) and (186, 187) are equivalent quantum theories. From this correspondence, it is clear that we can remove the terms including $\partial_\mu$ from its action without changing.

Before investigating $\theta^1 \neq -\theta^2$ case, let us consider

$$S_\epsilon = S_0 + \epsilon S_1,$$

$$S_0 = \text{Tr}_{\mathcal{H}} \text{Tr} \hat{\delta}_+ \left\{ \chi_{\mu\nu}^+ \left( H^{+\mu\nu} - (P_{\mu\nu\rho\tau}^{+}[\hat{D}_\rho, \hat{D}_\tau] + [B_{\mu\rho}, B_{\nu\sigma}] \delta^{\rho\sigma} + [B_{\mu\nu}, c]) \right) \right\} + \text{Tr}_{\mathcal{H}} \text{Tr} \hat{\delta}_+ \chi^\rho \left( H_\rho = i(-2[\hat{D}_\mu, B_{\mu\rho}] - [\hat{D}_\rho, c]) \right) + \text{Tr}_{\mathcal{H}} \text{Tr} \hat{\delta}_+ \{ i[\phi, \bar{\phi}] \eta + i\bar{\eta}[c, \bar{\phi}] + i[B^{+\mu\nu}, \bar{\phi}] \psi_\mu^{+\nu} + (\hat{D}_\mu, \bar{\phi}) \psi_\mu \}$$

(189)

$$S_1 = i\chi_{\mu\nu}^+ \varepsilon^{\mu\nu},$$

38
and their partition functions:

\[ Z_{N=4,\epsilon} = \int Df e^{-S_\epsilon}, \quad Z_{N=4,0} = \int Df e^{-S_0}. \]  

(190)

Note that \( S_0 \) is equivalent to the action of the Yang-Mills theory of \( \theta^1 = -\theta^2 \) and IKKT matrix model when its gauge group is U(1). Therefore, it is natural to assume that \( \exp(-S_0) \) damp the path integral of an arbitrary observable. Indeed, this assumption is required in MNS too [20]. From above discussion and this assumption, we can conclude that

\[ Z_{N=4,\epsilon} = Z_{N=4,0}. \]  

(191)

Next step, we consider \( \theta^1 \neq -\theta^2 \) case. Its action is equivalent to (189) if

\[ \epsilon = -\frac{1}{2} \left( \frac{1}{\theta^1} + \frac{1}{\theta^2} \right). \]

Under the above assumption that \( \exp(-S_0) \) damp integrands of path integrals, as we saw in (191), \( Z_{N=4,\epsilon} \) does not depend on \( \epsilon \). Therefore, the partition function of \( \theta^1 \neq -\theta^2 \) case is equal to the partition function of \( \theta^1 = -\theta^2 \) whose BPS eqs. are given by (184, 185), furthermore the partition function is equal to the partition function whose action functional is given by removing derivative terms and its BPS eqs. are given by (186, 187).

In the above discussion, we have closely studied the case of dimensional reduction from \( \mathcal{N} = 4 \) 4-dim. to 0-dim. But it is clear that we can apply the above general discussion to other dimensional cases or the cases of the \( \mathcal{N} = 2 \) 4-dim. model and the series given by its dimensional reduction. All these things make it clear that it is proper procedure to remove the terms including \( \partial_\mu = -i\theta^{-1}_{\mu\nu}[x^\nu, \ast] \) from lagrangians at the large \( \theta \) limit, in the calculations of this article.

C Normalization of the Partition Function

In this appendix, we give the precise definition of the path integral measure to decide the partition function without ambiguity.

As mentioned in section IV-vii, the absolute value of the infinite dimensional integrals of fluctuations around each vacuum should be normalized to be 1. This is implemented by virtue of the supersymmetry.

When we normalize fields appropriately the action of topological field theory has the following form,

\[ S_{TFT} = \int \delta_+ [\chi_i (H_i - iM_{ij}A_i)], \]  

(192)

here we have omitted terms including fields like \( \phi, \bar{\phi}, \eta \), often called “Higgs sector”, for simplicity. The normalization of the Higgs sector is possible to be managed similarly to
other fields when usual gauge fixing is done by using Nakanishi-Lautrup field, ghost and anti-ghost fields. We can see this fact in the latter half of this section devoted to trace and extra parts. Also we have kept only quadratic terms of fluctuations, because the path integral of topological field theories is estimated exactly in the weak coupling limit. \( M_{ij} \) in (192) depends on backgrounds and parameters in general, but as seen below, the \( M_{ij} \)-dependence does not appear in the result up to sign. The BRS transformation rules are given by

\[
\begin{align*}
\hat{\delta}_+ A_i &= \psi_i, \quad \hat{\delta}_+ \psi_i = 0, \\
\hat{\delta}_+ \chi_i &= H_i, \quad \hat{\delta}_+ H_i = 0.
\end{align*}
\] (193)

For \( A_i, \ldots \), we adopt the following path integral measure,

\[
\prod_i \frac{dH_i}{\sqrt{2\pi}} \frac{dA_i}{\sqrt{2\pi}} d\chi_i d\psi_i,
\] (194)

then we obtain

\[
\left| \int \prod_i \frac{dH_i}{\sqrt{2\pi}} \frac{dA_i}{\sqrt{2\pi}} d\chi_i d\psi_i e^{-S_{TF}} \right| = 1.
\] (195)

The \( M_{ij} \)-dependence does not appear due to the supersymmetry.

Now we give a detailed argument for calculations about the trace and extra parts of our model as an example. The action including the trace and extra parts \( S_{tr, ex}^\infty + S_{g.f.} \) is decomposed into two parts, \( S_1 \) and \( S_2 \). \( S_1 \) consists of (106)-(109), (111)-(114), and also \( S_2 \) consists of (110), (115), (119), (120). \( S_2 \) involves the Higgs sector and also includes the gauge fixing terms. \( S_1 \) involves all the rest.

We start with the \( S_1 \) part. \( S_1 \) is represented in the same form as (192), therefore we obtain

\[
\int \prod_i \frac{dH_i}{\sqrt{2\pi}} \frac{dA_i}{\sqrt{2\pi}} d\chi_i d\psi_i e^{-S_1} = 1.
\] (196)

As mentioned above, the \( \theta \)-dependence does not appear.

Let us turn to the \( S_2 \) part. The action is given as

\[
S_2 = \frac{4}{\theta} \phi_1 \phi_1 (197) \\
+ \frac{i}{\sqrt{\theta}} \eta_1 \psi_{(\mu)} (198) \\
+ b_1 \left( b_1 - \frac{1}{\sqrt{\theta}} A_{\mu(\mu)} \right) (199) \\
+ \frac{4}{\theta} \rho_1 \rho_1 + \frac{1}{\sqrt{\theta}} \rho_1 \psi_{(\mu)} (200).
\]

We adopt the following measure,

\[
\frac{d\bar{\phi}(1)}{\sqrt{2\pi}} \frac{d\phi(1)}{\sqrt{2\pi}} d\bar{\rho}(1) d\rho(1) \frac{db(1)}{\sqrt{2\pi}} \frac{dA_{\mu(\mu)}}{\sqrt{2\pi}} d\eta(1) d\psi_{(\mu)} (201)
\]
then we obtain
\[
\int \frac{d\tilde{\phi}_1}{\sqrt{2\pi}} d\phi_1 \frac{d\tilde{\rho}_1}{\sqrt{2\pi}} d\rho_1 \frac{dB_1}{\sqrt{2\pi}} dA_{\mu(\mu)} \frac{d\eta_1}{\sqrt{2\pi}} d\psi_1 e^{\frac{-S_2}{2}} = 1. \tag{202}
\]

Notice that the result (202) is again a consequence of the supersymmetry.

As a result of these normalizations, partition functions of the cohomological field theories are defined as well-defined functions or finite values without ambiguity from infinite dimensional integral.

At the end of this appendix, we should notice a fact relating the dimension-independence of partition function, (20). The gauge symmetry (99) and the gauge fixing term (117) are expected to have the same form for all cases of (8-dim , \( \mathcal{N} = 2 \)), (6-dim , \( \mathcal{N} = 2 \)) , (4-dim , \( \mathcal{N} = 4 \)) and (2-dim , \( \mathcal{N} = 8 \)). So we expect that the trace and extra sector produce a trivial factor 1 for all of those cases.

References

1) H. Aoki, S. Iso, H. Kawai, Y. Kitazawa, A. Tsuchiya , T. Tada, IIB Matrix Model, Prog.Theor.Phys.Suppl.134(1999)47-83, hep-th/9908038.

2) H. Aoki, N. Ishibashi, S. Iso, H. Kawai, Y. Kitazawa, T. Tada, Noncommutative Yang-Mills in IIB Matrix Model, Nucl.Phys. B565 (2000) 176-192, hep-th/9908141.

3) D.Birmingham, N.Blau, M.Rakowski, G.Thomson, Topological Field Theory Phys.Rep.209(1991)129.

4) U. Bruzzo, F. Fucito, J. F. Morales, A. Tanzini, Multi-Instanton Calculus and Equivalent Cohomology, JHEP 0305(2003)054, hep-th/0211108.

5) R. Dijkgraaf, C. Vafa, Matrix Models, Topological Strings, and Supersymmetric Gauge Theories , Nucl.Phys. B644 (2002) 3-20, hep-th/0206255 R. Dijkgraaf, C. Vafa, On Geometry and Matrix Models , Nucl.Phys. B644 (2002) 21-39, hep-th/0207106 R. Dijkgraaf, C. Vafa, A Perturbative Window into Non-Perturbative Physics, hep-th/0208048.

6) N. Dorey, T. J. Hollowood, V. V. Khoze, The D-Instanton Partition Function, JHEP 0103 (2001) 040, hep-th/0011247.

7) T. Eguchi, H. Kanno, Topological Strings and Nekrasov’s formulas, JHEP 0312 (2003) 006, hep-th/0310235.

8) F. Fucito, J. F. Morales, R. Poghossian, Multi instanton calculus on ALE spaces, Nucl.Phys. B703 (2004) 518-536, hep-th/0406243.

9) R. Gopakumar, S. Minwalla and A. Strominger, Noncommutative Solitons, JHEP 05(2000)020, hep-th/0003160.

10) R. Gopakumar, M. Headrick and M. Spradlin, On Noncommutative Multi-solitons, Commun.Math.Phys.233(2003)355-381, hep-th/0103256.
11) S. Hirano, M. Kato, *Topological Matrix Model*, Prog.Theor.Phys.98(1997)1371-1384, hep-th/9708039.

12) N. Ishibashi, H. Kawai, Y. Kitazawa, A. Tsuchiya, *A Large N Reduction Model as Superstring*, Nucl.Phys.B498(1997)467-491, hep-th/9708039.

13) T. Ishikawa, S.-I. Kuroki, A. Sako, *Instanton number calculus on Noncommutative $R^4$*, JHEP 08(2002)028, hep-th/0201196.

14) T. Ishikawa, S.-I. Kuroki and A. Sako, *Elongated U(1) Instantons on Noncommutative $R^4$*, JHEP 0111 (2001) 068, hep-th/0109111.

15) D. P. Jatkar, G. Mandal, S. R. Wadia, *Nielsen-Olesen Vortices in Noncommutative Abelian Higgs Model*, JHEP 0009 (2000) 018, hep-th/0007078.

16) J.M.F. Labastida and C. Lozano, *Mathai-Quillen Formulation of Twisted $N=4$ Supersymmetric Gauge Theories in Four Dimensions* Nucl.Phys. B502 (1997) 741-790, hep-th/9702106.

17) T. Maeda, T. Nakatsu, K. Takasaki, T. Tamakoshi, *Five-Dimensional Supersymmetric Yang-Mills Theories and Random Plane Partitions*, hep-th/0412327.

18) T. Matsuo, S. Matsuura, K. Ohta, *Large N limit of 2D Yang-Mills Theory and Instanton Counting*, hep-th/0406191.

19) C. Montonen and D. Olive, *Magnetic Monopoles as Gauge Particles?*, Phys.Lett.B72(1977)117; P. Goddard, J. Nuyts and D. Olive, *Gauge Theories and Magnetic Charge*, Nucl.Phys.B125(1977)1.

20) G. W. Moore, N. Nekrasov, S. Shatashvili, *D-particle bound states and generalized instantons*, Commun.Math.Phys.209(2000)77-95, hep-th/9803265.

21) N. A. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, Adv.Theor.Math.Phys. 7 (2004) 831-864, hep-th/0206161.

22) N. Nekrasov and A. Okounkov, *Seiberg-Witten theory and random partitions*, hep-th/0306238.

23) N. Nekrasov and A. Schwarz, *Instantons on noncommutative $R^4$ and $(2,0)$ superconformal six dimensional theory*, Commun.Math. Phys.198(1998)689, hep-th/9802068.

24) N.A. Nekrasov, *Trieste lectures on solitons in noncommutative gauge theories*, hep-th/0011095.

25) N. Nekrasov and A. Schwarz, *Instantons on noncommutative $R^4$ and $(2,0)$ superconformal six dimensional theory*, Commun.Math.Phys. 198 (1998) 689-703, hep-th/9802068.

26) J-S. Park, *Monads and D-instantons*, Nucl.Phys. B493 (1997) 198-230, hep-th/9612096.

27) A. Sako, S-I. Kuroki and T. Ishikawa, *Noncommutative Cohomological Field Theory and GMS soliton*, J.Math.Phys.43(2002)872-896, hep-th/0107033.
28) A. Sako, S-I. Kuroki and T. Ishikawa, *Noncommutative-shift invariant field theory*, proceeding of 10th Tohwa International Symposium on String Theory, (AIP conference proceedings 607, 340).

29) A. Sako, *Noncommutative Cohomological Field Theories and Topological Aspects of Matrix models*, Noncommutative Geometry and Physics, p321-355, World Scientific, hep-th/0312120.

30) A. Sako *Instanton number of Noncommutative U(n) gauge theory*, JHEP 04(2003)023, hep-th/0209139.

31) C. Sochichiu, *On the Equivalence of Noncommutative Models in Various Dimensions and Brane Condensation*, JHEP 0008 (2000) 048, hep-th/0007127.

32) F. Sugino, *Cohomological Field Theory Approach To Matrix Strings*, Int.J.Mod.Phys.A14(1999)3979-4002, hep-th/9904122.

33) T. Suyama, A. Tsuchiya, *Exact Result in $N_C = 2$ IIB Matrix Model*, Prog.Theor.Phys.99(1998)321-325, hep-th/9711073.

34) Y. Tachikawa, *Five-dimensional Chern-Simons terms and Nekrasov’s instanton counting*, JHEP 0402 (2004) 050, hep-th/0401184.

35) C. Vafa and E. Witten, *A Strong coupling test of S-duality*, Nucl.Phys.B431(1994)3-77, hep-th/9408074.

36) M. Wijnholt, *Five-Dimensional Gauge Theories and Unitary Matrix Models*, hep-th/0401025.

37) E. Witten, *Topological quantum field theory*, Commun.Math.Phys.117(1988)353; E. Witten, *Introduction to cohomological field theories*, Int.J.Mod.Phys.A.6(1991)2273.