Visibility of Cartesian products of Cantor sets

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Abstract

Let $K_\lambda$ be the attractor of the following IFS
\[ \{ f_1(x) = \lambda x, f_2(x) = \lambda x + 1 - \lambda \}, \quad 0 < \lambda < 1/2. \]

Given $\alpha \geq 0$, we say the line $y = \alpha x$ is visible through $K_\lambda \times K_\lambda$ if
\[ \{(x, \alpha x) : x \in \mathbb{R} \setminus \{0\} \} \cap ((K_\lambda \times K_\lambda)) = \emptyset. \]

Let $V = \{ \alpha \geq 0 : y = \alpha x$ is visible through $K_\lambda \times K_\lambda \}$. In this paper, we give a completed description of $V$, e.g., its Hausdorff dimension and its topological property. Moreover, we also discuss another type of visible problem which is related to the slicing problems.

1 Introduction

Projections, sections, geodesic curves and visibility are the main problems in geometry measure theory. It is related to many aspects of fractal geometry, for instance, the arithmetic sum of two self-similar sets is indeed the projectional problem [9] [14]: sections of some fractal sets are connected to the multiple representations of real numbers [12]; geodesic curves on fractal sets are distinct from the classical differential manifolds [6]. For more results on these problems see [19] [20] [3] [22] [21] [23] [18] and references therein. In this paper, we shall consider the visibility of the Cartesian products of some Cantor sets.

Given $\alpha \geq 0$ and some subset $F \subset \mathbb{R}^2$, we say the line $y = \alpha x$ is visible through $F$ if
\[ \{(x, \alpha x) : x \in \mathbb{R} \setminus \{0\} \} \cap F = \emptyset. \]

The concept of “visibility” was investigated by many scholars. Nikodym [13] constructed a subset $F$ of $\mathbb{R}^2$ such that every point of $F$ is visible from two diametrically opposite directions. In convex geometry, Krasnosel [2] offered a beautiful criterion which enables us to check whether the entire boundary of a compact set of $\mathbb{R}^2$ is visible from an interior point. Falconer and Fraser [8] proved that for a class of plane self-similar sets when the attractor $F$ has Hausdorff dimension greater than 1 then the Hausdorff dimension of the visible subset is 1. The readers can find more related results in [15] [10] [1] [5].

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In this paper, we shall analyze the following self-similar set. Let $K_\lambda$ be the attractor with the IFS
\[ \{ f_1(x) = \lambda x, f_2(x) = \lambda x + 1 - \lambda \}, \ 0 < \lambda < 1/2, \]
i.e.,
\[ K_\lambda = f_1(K_\lambda) \cup f_2(K_\lambda). \tag{1} \]
Let
\[ V = \{ \alpha \geq 0 : y = \alpha x \text{ is visible through } K_\lambda \times K_\lambda \}. \]
It is easy to verify that the line $y = \alpha x$ is visible through $K_\lambda \times K_\lambda$ if and only if
\[ \alpha \notin \frac{K_\lambda}{K_\lambda \setminus \{0\}} := \left\{ \frac{x}{y} : x, y \in K_\lambda, y \neq 0 \right\}. \]
Thus,
\[ V = [0, +\infty) \setminus \frac{K_\lambda}{K_\lambda \setminus \{0\}}. \tag{2} \]
By $A^o$ we denote the set of interior points of $A$, by $m(A)$ we denote the Lebesgue measure of $A$. In this paper, we obtain the following results.

**Theorem 1.1.** Let $K_\lambda$ be given by (1). Then

1. When $\frac{3 - \sqrt{5}}{2} \leq \lambda < 1/2$, $V = \emptyset$;
2. When $\frac{1}{3} \leq \lambda < \frac{3 - \sqrt{5}}{2}$,
\[ V = (0, +\infty) \setminus \bigcup_{k=-\infty}^{\infty} \lambda^k \left[ 1 - \lambda, \frac{1}{1-\lambda} \right]; \]
3. When $0 < \lambda < \frac{1}{3}$, $V^o \neq \emptyset$. In particular, when $\frac{1}{4} < \lambda < \frac{1}{3}$, $([0, +\infty) \setminus V)^o \neq \emptyset$;
   when $0 < \lambda \leq \frac{1}{4}$, $m([0, +\infty) \setminus V) = 0$ and $\dim_H([0, +\infty) \setminus V) = \frac{\log 4}{-\log \lambda}$.

There are mainly two types of visible problem [8]. Now, we shall consider another one. First, we introduce some definitions. Let $l_\theta$ denote the line going through the origin in direction $\theta \in (0, \pi/2)$, that is,
\[ l_\theta = \{(x, (\tan \theta)x) : x \in \mathbb{R}\}. \]
Given $\theta \in (0, \pi/2)$. The visible part of $K_\lambda \times K_\lambda$ is defined as follows:
\[ V_\theta(K_\lambda \times K_\lambda) = \{(x, y) \in K_\lambda \times K_\lambda : ((x, y) + l_\theta) \cap (K_\lambda \times K_\lambda) = \{(x, y)\}\}. \]
Let $(x, y) \in K_\lambda \times K_\lambda$. Define
\[ \text{Proj}_\theta(x, y) = y - x \tan \theta. \]
In other words, we project a point \((x, y)\) to the \(y\)-axis in direction \(\theta\). Moreover, we also define the following sets.

\[
\text{Proj}_\theta(K_\lambda \times K_\lambda) = \{y - x \tan \theta : (x, y) \in K_\lambda \times K_\lambda\},
\]

\[
\text{Proj}_\theta(V_\theta(K_\lambda \times K_\lambda)) = \{y - x \tan \theta : (x, y) \in V_\theta(K_\lambda \times K_\lambda)\}.
\]

Generally, \(\text{Proj}_\theta(K_\lambda \times K_\lambda)\) is not an interval. In what follows, we always assume that \(E = \text{Proj}_\theta(K_\lambda \times K_\lambda) = [−\tan \theta, 1]\), which is a natural assumption \([3]\). Clearly, \(E\) is the attractor of the following IFS,

\[
g_1(x) = \lambda x - (1 - \lambda) \tan \theta \\
g_2(x) = \lambda x + (1 - \lambda)(1 - \tan \theta) \\
g_3(x) = \lambda x \\
g_4(x) = \lambda x + 1 - \lambda.
\]

For the IFS of \(E\), i.e. \(\{g_i\}_{i=1}^4\), define \(T_j(x) := g_j^{-1}(x)\) for \(x \in g_j(E)\) and \(1 \leq j \leq 4\). We denote the concatenation \(T_{i_n} \circ \ldots \circ T_{i_1}(x)\) by \(T_{i_1 \ldots i_n}(x)\). Let

\[
H_i = g_i(E) \cap g_{i+1}(E), 1 \leq i \leq 3,
\]

i.e. we define \(H = H_1 \cup H_2 \cup H_3 = [a_1, b_1] \cup [a_2, b_2] \cup [a_3, b_3]\). The following two propositions are motivated by the results in open dynamical systems.

**Proposition 1.1.** Suppose that \(\text{Proj}_\theta(K_\lambda \times K_\lambda) = [−\tan \theta, 1]\). For any \([a_i, b_i], 1 \leq i \leq 3\), if there are some \(i_1 \cdots i_v, j_1 j_2 \cdots j_w\) such that

\[
T_{i_1 \cdots i_v}(a_i) \in H, T_{j_1 j_2 \cdots j_w}(b_i) \in H,
\]

then

\[
\text{Proj}_\theta(V_\theta(K_\lambda \times K_\lambda)) = \{a \in [−\tan \theta, 1] : \sharp\{(x, (\tan \theta)x + a) \cap (K_\lambda \times K_\lambda)\} = 1\}
\]

is a graph-directed self-similar sets with the strong separation condition, where \(\sharp(\cdot)\) denotes the cardinality.

Analogously, we have the following result.

**Proposition 1.2.** \(\text{Proj}_\theta(K_\lambda \times K_\lambda) = [−\tan \theta, 1]\). For any \([a_i, b_i], 1 \leq i \leq 3\), if all the possible orbits of \(a_i\) and \(b_i\) hit finitely many points, then apart from a countable set

\[
\text{Proj}_\theta(V_\theta(K_\lambda \times K_\lambda)) = \{a \in [−\tan \theta, 1] : \sharp\{(x, (\tan \theta)x + a) \cap (K_\lambda \times K_\lambda)\} = 1\}
\]

is a graph-directed self-similar sets with the open set condition.

Propositions [11] and [12] give a sufficient condition which allows us to calculate the dimension of the slicing set \(\{a \in [−\tan \theta, 1] : \sharp\{(x, (\tan \theta)x + a) \cap (K_\lambda \times K_\lambda)\} = 1\}.

The paper is arranged as follows. In section 2, we give the proof of Theorem [11]. In section 3, we give the proofs of Propositions [11] and [12]. Finally, we give some remarks.
2 Proofs of Main results

Before we prove Theorem 1.1, we give some definitions, and prove a useful lemma. Let $E = [0, 1]$. For any $(i_1, \ldots, i_n) \in \{1, 2\}^n$, we call $f_{i_1, \ldots, i_n}([0, 1]) = (f_{i_1} \circ \cdots \circ f_{i_n})([0, 1])$ a basic interval of rank $n$, which has length $\lambda^n$. Denote by $E_n$ the collection of all these basic intervals of rank $n$. Suppose $A$ and $B$ are the left and right endpoints of some basic intervals in $E_k$ for some $k \geq 1$, respectively. Denote by $G_n(\subset E_n)$ the union of all the basic intervals of rank $n$ which are contained in $[A, B]$. Let $I$ be a basic interval with rank $n$. Define $\tilde{I} = f_1(I) \cup f_2(I)$.

Lemma 2.1. Let $F : U \to \mathbb{R}$ be a continuous function, where $U \subset \mathbb{R}^2$ is a non-empty open set. Suppose $A$ and $B$ are the left and right endpoints of some basic intervals in $G_{k_0}$ for some $k_0 \geq 1$ respectively such that $[A, B] \times [A, B] \subset U$. Then $K \cap [A, B] = \cap_{n=k_0}^\infty G_n$. Moreover, if for any $n \geq k_0$ and any two basic intervals $I, J \subset G_n$, such that

$$F(I, J) = F(\tilde{I}, \tilde{J}),$$

then $F(K \cap [A, B], K \cap [A, B]) = F(G_{k_0}, G_{k_0})$.

Proof. By the construction of $G_n$, i.e. $G_{n+1} \subset G_n$ for any $n \geq k_0$, it follows that

$$K \cap [A, B] = \cap_{n=k_0}^\infty G_n.$$

The continuity of $F$ yields that

$$F(K \cap [A, B], K \cap [A, B]) = \cap_{n=k_0}^\infty F(G_n, G_n).$$

Without loss of generality, we may assume that

$$G_n = \bigcup_{1 \leq i \leq t_n} I_{n,i} \text{ for some } t_n \geq 1,$$

where $I_{n,i}$ is a basic interval in $G_n$. By the condition in lemma, i.e. for any $n \geq k_0$ and any two basic intervals $I, J \subset G_n$, such that

$$F(I, J) = F(\tilde{I}, \tilde{J}),$$

it follows that

$$F(G_n, G_n) = \bigcup_{1 \leq i \leq t_n} \bigcup_{1 \leq j \leq t_n} F(I_{n,i}, I_{n,j})$$

$$= \bigcup_{1 \leq i \leq t_n} \bigcup_{1 \leq j \leq t_n} F(\tilde{I}_{n,i}, \tilde{I}_{n,j})$$

$$= F(\bigcup_{1 \leq i \leq t_n} I_{n,i}, \bigcup_{1 \leq j \leq t_n} I_{n,j})$$

$$= F(G_{n+1}, G_{n+1}).$$

Therefore, $F(K \cap [A, B], K \cap [A, B]) = F(G_{k_0}, G_{k_0})$. □

Lemma 2.2. Let $f(x, y) = \frac{x}{y}$, and $I = [a, a + t], J = [b, b + t]$ be two basic intervals. If $1/3 \leq \lambda < 1/2$, and $b \geq a \geq 1 - \lambda$, then $f(\tilde{I}, \tilde{J}) = f(I, J)$. 


Proof. Note that
\[ \tilde{I} = [a, a + \lambda t] \cup [a + t - \lambda t, a + t], \ 
\tilde{J} = [b, b + \lambda t] \cup [b + t - \lambda t, b + t]. \]
Therefore,
\[ f(\tilde{I}, \tilde{J}) = J_1 \cup J_2 \cup J_3 \cup J_4, \]
where
\[ J_1 = \left[ \frac{a}{b + t}, \frac{a + \lambda t}{b + t - \lambda t} \right] =: [r_1, s_1], \]
\[ J_2 = \left[ \frac{a}{b + \lambda t}, \frac{a + \lambda t}{b} \right] =: [r_2, s_2], \]
\[ J_3 = \left[ \frac{a + t - \lambda t}{b + t}, \frac{a + t}{b + t - \lambda t} \right] =: [r_3, s_3], \]
\[ J_4 = \left[ \frac{a + t - \lambda t}{b + \lambda t}, \frac{a + t}{b} \right] =: [r_4, s_4]. \]
Note that \( f(I, J) = [r_1, s_4] \). In the following, we verify that \( f(I, J) = J_1 \cup J_2 \cup J_3 \cup J_4 \)
Since \( b \geq a \geq 1 - \lambda \) and \( \lambda \geq \frac{1}{3} \), we have
\[ r_3 - r_2 = \frac{a + t - \lambda t}{b + t} - \frac{a}{b + \lambda t} = \frac{(1 - \lambda)(b - a + t\lambda)}{(b + t\lambda)(b + t)} \geq 0. \]
Now it suffices to check that
\[ s_1 - r_2 \geq 0, \quad s_2 - r_3 \geq 0 \quad \text{and} \quad s_3 - r_4 \geq 0. \]
We have
\[ s_1 - r_2 = \frac{a + \lambda t}{b + t - \lambda t} - \frac{a}{b + \lambda t} = \frac{t(2a\lambda - a + b\lambda + t\lambda^2)}{(b + t - \lambda t)(b + \lambda t)} \geq \frac{t(a(3\lambda - 1) + t\lambda^2)}{(b + t - \lambda t)(b + \lambda t)} \geq 0, \]
and
\[ s_2 - r_3 = \frac{a + \lambda t}{b} - \frac{a + t - \lambda t}{b + t} = \frac{t(a + (2\lambda - 1)b + t\lambda)}{b(b + t)} \geq \frac{t(b(1 - \lambda) + (2\lambda - 1)b + t\lambda)}{b(b + t)} \geq 0. \]
Finally,
\[ s_3 - r_4 = \frac{a + t}{b + t - \lambda t} - \frac{a + t - \lambda t}{b + \lambda t} = \frac{t(-a - t + 2a\lambda + b\lambda + 3t\lambda - t\lambda^2)}{(b + t - \lambda t)(b + \lambda t)}. \]
If \( b \neq a \), then \( b > a + t \). Therefore, we have
\[ -a - t + 2a\lambda + b\lambda + 3t\lambda - t\lambda^2 \geq -a + 2a\lambda + (a + t)\lambda + t(3\lambda - 1 - \lambda^2) \geq a(3\lambda - 1) + t(4\lambda - 1 - \lambda^2) \geq 0. \]
which leads to \( s_3 - r_4 \geq 0 \). However, if \( a = b \), then
\[ s_2 - r_4 = \frac{a + \lambda t}{a} - \frac{a + t - \lambda t}{a + \lambda t} = \frac{\lambda^2 t^2 + at(3\lambda - 1)}{a(a + \lambda t)} \geq 0. \]
Thus, we finish checking that \( f(I, J) = J_1 \cup J_2 \cup J_3 \cup J_4 = [r_1, s_4] = \left[ \frac{a}{b + t}, \frac{a + t}{b} \right]. \)
Lemma 2.3. We have

\[
\frac{K_\lambda}{K_\lambda \setminus \{0\}} = \begin{cases} 
[0, \infty), & \text{when } \frac{3 - \sqrt{5}}{2} \leq \lambda < \frac{1}{2} \\
\bigcup_{k = -\infty}^{+\infty} \lambda^k \left[1 - \lambda, \frac{1}{1-\lambda}\right] \cup \{0\} & \text{when } 1/3 \leq \lambda < \frac{3 - \sqrt{5}}{2}.
\end{cases}
\] (3)

Proof. From Lemmas 2.1 and 2.2 it follows that if \( \lambda \geq \frac{1}{3} \)

\[
f_2(K_\lambda) = \left[1 - \lambda, \frac{1}{1-\lambda}\right] .
\]

Each \( x \in K_\lambda \) can be uniquely represented as

\[
x = \sum_{n=1}^{\infty} x_n \lambda^n \text{ with } x_n \in \{0, 1 - \lambda\}.
\]

Note that \( x \in f_2(K_\lambda) \) if and only if \( x_1 = 1 - \lambda \). Thus each \( x \in K_\lambda \setminus \{0\} \) is of form

\[
x = \lambda^m x^* \text{ with } m \in \{0, 1, 2, \ldots\}, x^* \in f_2(K_\lambda).
\]

Thus for any two \( x = \lambda^m x^*, y = \lambda^n y^* \in K_\lambda \setminus \{0\} \) with \( x^*, y^* \in f_2(K_\lambda) \) one has

\[
\frac{x}{y} = \lambda^{m-n} \cdot \frac{x^*}{y^*} \in \lambda^{m-n} \left[1 - \lambda, \frac{1}{1-\lambda}\right].
\]

Thus

\[
\frac{K_\lambda}{K_\lambda \setminus \{0\}} = \{0\} \cup \bigcup_{k = -\infty}^{\infty} \lambda^k \left[1 - \lambda, \frac{1}{1-\lambda}\right].
\]

It is easy to check that \( \{0\} \cup \bigcup_{k = -\infty}^{\infty} \lambda^k \left[1 - \lambda, \frac{1}{1-\lambda}\right] = [0, +\infty) \) when \( \frac{3 - \sqrt{5}}{2} \leq \lambda < 1/2 \), and intervals \( \lambda^k \left[1 - \lambda, \frac{1}{1-\lambda}\right] \) are pairwise disjoint when \( 1/3 \leq \lambda < \frac{3 - \sqrt{5}}{2} \). \( \square \)

Pourbarat [16], making use of the thickness of the Cantor sets, proved the following result.

Theorem 2.5. If \( \frac{\lambda^2}{(1 - 2\lambda)^2} > \lambda \), then \( \frac{K_\lambda}{K_\lambda \setminus \{0\}} \) contains an interior point.

Lemma 2.6. If \( \frac{1}{4} < \lambda < 1/3 \), then \( \frac{K_\lambda}{K_\lambda \setminus \{0\}} \) contains an interior point.

Proof. If \( \frac{1}{4} < \lambda < 1/3 \), then \( \frac{\lambda^2}{(1 - 2\lambda)^2} > \lambda \). Therefore, \( \frac{K_\lambda}{K_\lambda \setminus \{0\}} \) contains an interior point by Theorem 2.5.

Lemma 2.7. If \( 0 < \lambda < \frac{3 - \sqrt{5}}{2} \), then \( V \) has an interior point.
Proof. Note that \( f_2(K_\lambda) \subset [1 - \lambda, 1] \). Thus, by the argument in Lemma 2.3, we have

\[
\frac{K_\lambda}{K_\lambda \setminus \{0\}} \subseteq \{0\} \cup \bigcup_{k=-\infty}^{\infty} \lambda^k \left[ 1 - \lambda, \frac{1}{1 - \lambda} \right].
\]

Note that the intervals \( \left[ \lambda^k(1 - \lambda), \frac{\lambda^k}{1 - \lambda} \right] \) for \( k \in \mathbb{Z} \) are pairwise disjoint when \( 0 < \lambda < \frac{3 - \sqrt{5}}{2} \). Therefore, \( V \) has an interior point by (2). \( \square \)

**Lemma 2.8.** If \( 0 < \lambda < \frac{1}{4} \), then \( \frac{K_\lambda}{K_\lambda \setminus \{0\}} \) has Lebesgue measure zero.

**Proof.** If \( 0 < \lambda < \frac{1}{4} \), then

\[
\dim_H K_\lambda + \dim_H K_\lambda < 1.
\]

We note that for any \( X, Y \subseteq \mathbb{R} \), we have \( Y - X = \Pi_2(X \times Y) \), where \( \Pi_2(X \times Y) \) denotes the projection of \( X \times Y \) on the y axis along lines having \( \frac{\pi}{4} \) angle with the x axis. Therefore,

\[
\dim_H \frac{K_\lambda}{K_\lambda \setminus \{0\}} = \dim_H \frac{K_\lambda \setminus \{0\}}{K_\lambda \setminus \{0\}} = \dim_H (\ln(K_\lambda \setminus \{0\})) - \ln(K_\lambda \setminus \{0\})
\]

\[
\leq \dim_H (\ln(K_\lambda \setminus \{0\})) \times \ln(K_\lambda \setminus \{0\})
\]

\[
\leq \dim_H (\ln K_\lambda \setminus \{0\}) + \dim_P (\ln K_\lambda \setminus \{0\})
\]

\[
= 2 \dim_H (K_\lambda \setminus \{0\}) < 1,
\]

where we use the fact that \( \dim_H E = \dim_P \ln E \) and \( \dim_P E = \dim_P \ln E \). for a bounded set \( E \subset (0, +\infty) \). Thus, \( \frac{K_\lambda}{K_\lambda \setminus \{0\}} \) has Lebesgue measure zero. \( \square \)

We will use a result given by Simon and Solomyak [17].

**Theorem 2.10.** Let \( \Lambda \) be a self-similar 1-set in \( \mathbb{R}^2 \) with the open set condition, which is not on a line. Then

\[
m(P_{(0,0)}(\Lambda \setminus \{(0,0)\})) = 0,
\]

where

\[
P_{(0,0)} : \mathbb{R}^2 \setminus (0,0) \to S^1, P_{(0,0)}(\bar{x}) = \frac{\bar{x}}{|\bar{x}|}.
\]

**Lemma 2.9.** \( \frac{K_{1/4}}{K_{1/4} \setminus \{0\}} \) has Lebesgue measure zero.

**Proof.** Note that when \( \lambda = 1/4 \), \( \Lambda = K_\lambda \times K_\lambda \) is a self-similar set with the following IFS

\[
g_1(x, y) = \left( \frac{x}{4}, \frac{y}{4} \right), g_2(x, y) = \left( \frac{x + 3}{4}, \frac{y}{4} \right)
\]

\[
g_3(x, y) = \left( \frac{x}{4}, \frac{y + 3}{4} \right), g_4(x, y) = \left( \frac{x + 3}{4}, \frac{y + 3}{4} \right).
\]
Clearly, the above IFS satisfies the open set condition. Therefore, the Hausdorff dimension of $\Lambda$ is 1, and $0 < \mathcal{H}^1(\Lambda) < \infty$. Let

$$
\Gamma = \left\{ \frac{(x, y)}{\sqrt{x^2 + y^2}} : (x, y) \in K_\lambda \times K_\lambda \setminus \{(0,0)\} \right\} = P_{(0,0)}(\Lambda \setminus \{(0,0)\})
$$

The Lebesgue measure of $\Gamma$ is 0 due to Theorem 2.10. Let

$$
\Gamma_1 = \left\{ \frac{(x, y)}{\sqrt{x^2 + y^2}} : (x, y) \in K_\lambda \times K_\lambda \setminus \{(0,0)\}, x \neq 0 \right\}.
$$

Clearly, $m(\Gamma_1) = m(\Gamma) = 0$. The metric on $\Gamma_1$, denoted by $d_1$, is the arc metric. It is well known that on $S^1$, the arc metric is equivalent to the Euclidean metric. Let

$$
\Gamma_2 = \left\{ \arctan \frac{y}{x} : (x, y) \in K_\lambda \times K_\lambda \setminus \{(0,0)\}, x \neq 0 \right\}.
$$

The metric on $\Gamma_2$ the Euclidean metric (we denote it by $d_2$). We define the the map

$$
\phi : \Gamma_1 \to \Gamma_2,
$$

by

$$
\phi \left( \frac{(x, y)}{\sqrt{x^2 + y^2}} \right) = \arctan \frac{y}{x}.
$$

The map $\phi$ is indeed mapping a point on $S^1$ into its associated polar angle in the polar coordinate system. Therefore, we may define $\phi$ in another way as follows: define

$$
\phi : \Gamma_1 \to \Gamma_2, \quad \phi(\vec{a}) = \theta_{\vec{a}}
$$

Clearly, $\phi$ is well-defined, and it is a bijection. Moreover, we shall prove that $\phi$ is a Lipschitz map, i.e. there exists some constant $L > 0$ such that

$$
d_2(\phi(\vec{a}), \phi(\vec{b})) \leq Ld_1(\vec{a}, \vec{b}).
$$

Note that $d_2(\phi(\vec{a}), \phi(\vec{b})) = d_2(\theta_{\vec{a}}, \theta_{\vec{b}})$, and that

$$
d_1(\vec{a}, \vec{b}) = d_2(\theta_{\vec{a}} \cdot 1, \theta_{\vec{b}} \cdot 1) = d_2(\theta_{\vec{a}}, \theta_{\vec{b}}).
$$

Now, $m(\Gamma_2) = 0$ follows from $\phi(\Gamma_1) = \Gamma_2$, $m(\Gamma_1) = 0$, and $\phi$ is Lipschitz. Therefore,

$$
m \left( \frac{K_{1/4}}{K_{1/4} \setminus \{0\}} \right) = 0.
$$

The following is from Bárany [3].

**Theorem 2.11.** Let $\Lambda$ be an arbitrary self-similar set in $\mathbb{R}^2$ not contain in any line. Suppose that $g : \mathbb{R}^2 \to \mathbb{R}$ is a $C^2$ map such that

$$
(g_x)^2 + (g_y)^2 \neq 0, (g_{xx}g_y - g_{xy}g_x)^2 + (g_{xy}g_y - g_{yy}g_x)^2 \neq 0
$$

for any $(x, y) \in \Lambda$. Then

$$
\dim_H g(\Lambda) = \min\{1, \dim_H(\Lambda)\}.
$$
Lemma 2.10. When $0 < \lambda \leq \frac{1}{4}$, $\dim_H([0, +\infty) \setminus V) = \frac{\log 4}{-\log \lambda}$.

Proof. By the argument in Lemma 2.3, we have

$$\frac{K_{\lambda}}{K_{\lambda} \setminus \{0\}} = \bigcup_{k=-\infty}^{\infty} \lambda^k \frac{f_2(K_{\lambda})}{f_2(K_{\lambda})} \cup \{0\}.$$

Thus

$$\dim_H \frac{K_{\lambda}}{K_{\lambda} \setminus \{0\}} = \dim_H \frac{f_2(K_{\lambda})}{f_2(K_{\lambda})}.$$

Clearly, $\Lambda = f_2(K_{\lambda}) \times f_2(K_{\lambda})$ is a two-dimensional self-similar set which is not contained in any line. Let $g(x, y) = \frac{x}{y}$, then

$$(g_x)^2 + (g_y)^2 \neq 0, (g_{xx}g_y - g_{xy}g_x)^2 + (g_{xy}g_y - g_{yy}g_x)^2 \neq 0$$

for any $(x, y) \in \Lambda$. Therefore, in terms of Theorem 2.11,

$$\dim g(\Lambda) = \dim_H \frac{f_2(K_{\lambda})}{f_2(K_{\lambda})} = \min\{\dim_H (f_2(K_{\lambda}) \times f_2(K_{\lambda})), 1\} = \min\{2 \dim_H (K_{\lambda}), 1\}.$$

Hence, if $0 < \lambda \leq 1/4$, then

$$\dim_H \frac{f_2(K_{\lambda})}{f_2(K_{\lambda})} = 2 \dim_H (K) = \frac{\log 4}{-\log \lambda}.$$ 

\[\square\]

Proof of Theorem 1.1. Theorem 1.1 (1) and (2) follows from Lemmas 2.3. Theorem 1.1 (3) follows from 2.6, 2.7, 2.8, 2.9 and 2.10.

3 Visible sets, slicing sets and open dynamical systems

In this section, we give the proofs of Propositions 1.1 and 1.2. Define

$$P_{ij} = \text{Proj}_\theta(f_i([0, 1]) \cap f_j([0, 1])) = \{y - x \tan \theta : (x, y) \in f_i([0, 1]) \cap f_j([0, 1])\}, 1 \leq i, j \leq 2.$$ 

It is easy to see that the length of $P_{ij}$ is $\lambda(1 + \tan \theta)$. In what follows, we always assume that $E = \text{Proj}_\theta(K_{\lambda} \times K_{\lambda}) = [-\tan \theta, 1]$. Clearly, in terms of $P_{ij}$, $E$ is the attractor of the following IFS,

$$g_1(x) = \lambda x - (1 - \lambda) \tan \theta$$
$$g_2(x) = \lambda x + (1 - \lambda)(1 - \tan \theta)$$
$$g_3(x) = \lambda x$$
$$g_4(x) = \lambda x + 1 - \lambda.$$
Motivated by Lemma 3.1, we may define the orbits of the points of $K$.

**Definition 3.2.** Let $x \in K$. Then $(i_n)_{n=1}^\infty \in \{1, \ldots, m\}^\mathbb{N}$ is a coding for $x$ if and only if $T_{i_{n-i}}(x) \in K$ for all $n \in \mathbb{N}$.

It is easy to see that for different codings, the orbits of $x$ may be distinct.

The set $\text{Proj}_\theta(V_\theta(K_\lambda \times K_\lambda))$ can be viewed as a slicing set, i.e.

\[ \text{Proj}_\theta(V_\theta(K_\lambda \times K_\lambda)) = \{a \in [-\tan \theta, 1] : \#((x, (\tan \theta)x + a) \cap (K_\lambda \times K_\lambda)) = 1\}, \]

where $\#(\cdot)$ denotes the cardinality. The following lemma is trivial.

**Lemma 3.2.** Suppose that $\text{Proj}_\theta(K_\lambda \times K_\lambda) = [-\tan \theta, 1]$. Then

\[ \text{Proj}_\theta(V_\theta(K_\lambda \times K_\lambda)) \]

is exactly the univoque set of $E$ under the IFS $\{g_i\}_{i=1}^4$, i.e.

\[ \text{Proj}_\theta(V_\theta(K_\lambda \times K_\lambda)) = \{a \in [-\tan \theta, 1] : \#((x, (\tan \theta)x + a) \cap (K_\lambda \times K_\lambda)) = 1\} = U_1. \]

**Proof.** The proof is trivial. We leave it to readers. \(\square\)

If $\theta \in [\pi/4, \pi/2)$, let $H_i = g_i(E) \cap g_{i+1}(E), 1 \leq i \leq 3$, i.e.

\[ H_1 = [1 - \lambda - \tan \theta, \lambda - (1 - \lambda)\tan \theta], \]
\[ H_2 = [-\lambda \tan \theta, 1 - (1 - \lambda)\tan \theta], \]
\[ H_3 = [1 - \lambda - \lambda \tan \theta, \lambda]. \]

Let $H = H_1 \cup H_2 \cup H_3 = [a_1, b_1] \cup [a_2, b_2] \cup [a_3, b_3]$. Similarly, if $\theta \in (0, \pi/4)$, then we can also define the $H_i = [a_i, b_i], 1 \leq i \leq 3$. Now the following result is a corollary of the main result of [2].
Proposition 3.1. Suppose that \( \text{Proj}_\theta(K_\lambda \times K_\lambda) = [-\tan \theta, 1] \). For any \([a_i, b_i], 1 \leq i \leq 3\), if there are some \(i_1 \cdots i_v, j_1 j_2 \cdots j_w\) such that
\[
T_{i_1 \cdots i_v}(a_i) \in H, T_{j_1 j_2 \cdots j_w}(b_i) \in H,
\]
then
\[
\{ a \in [-\tan \theta, 1] : \# \{(x, (\tan \theta)x + a) \cap (K_\lambda \times K_\lambda) \} = 1 \}
\]
is a graph-directed self-similar sets with the strong separation condition.

The following is a corollary of the main result of [11].

Proposition 3.2. \( \text{Proj}_\theta(K_\lambda \times K_\lambda) = [-\tan \theta, 1] \). For any \([a_i, b_i], 1 \leq i \leq 3\), if all the possible orbits of \(a_i\) and \(b_i\) hit finitely many points, then apart from a countable set
\[
\{ a \in [-\tan \theta, 1] : \# \{(x, (\tan \theta)x + a) \cap (K_\lambda \times K_\lambda) \} = 1 \}
\]
is a graph-directed self-similar sets with the open set condition.

4 Some remarks

The main idea of this paper is to establish a connection between the visible problem and arithmetic on the fractal sets. Our idea can be implemented for other overlapping self-similar sets. Similar results can be obtained if we replace the line \(y = \alpha x\) in the definition of \(V\), by some parabolic curves or hyperbolic curves. Nevertheless, for these cases, the analysis can be difficult. We shall discuss these problems in another paper.

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References

[1] Ida Arhosalo, Esa Järvenpää, Maarit Järvenpää, Michał Rams, and Pablo Shmerkin. Visible parts of fractal percolation. Proc. Edinb. Math. Soc. (2), 55(2):311–331, 2012.

[2] Simon Baker, Karma Dajani, and Kan Jiang. On univoque points for self-similar sets. Fund. Math., 228(3):265–282, 2015.

[3] B. Bárány and M. Rams. Dimension of slices of Sierpinski-like carpets. J. Fractal Geom., 1(3): 273-294, 2014.

[4] Balázs Bárány. On some non-linear projections of self-similar sets in \( \mathbb{R}^3 \). Fund. Math., 237(1):83–100, 2017.
[5] M. Bond, I. Laba, and J. Zahl. Quantitative visibility estimates for unrectifiable sets in the plane. Trans. Amer. Math. Soc., 368(8):5475–5513, 2016.

[6] Yiming Li and Lifeng Xi. Manhattan property of geodesic paths on self-affine carpets. Arch. Math. (Basel), 111(3):279–285, 2018.

[7] Ludwig Danzer, Branko Grünbaum, and Victor Klee. Helly’s theorem and its relatives. In Proc. Sympos. Pure Math., Vol. VII, pages 101–180. Amer. Math. Soc., Providence, R.I., 1963.

[8] Kenneth J. Falconer and Jonathan M. Fraser. The visible part of plane self-similar sets. Proc. Amer. Math. Soc., 141(1):269–278, 2013.

[9] Michael Hochman and Pablo Shmerkin. Local entropy averages and projections of fractal measures. Ann. of Math. (2), 175(3):1001–1059, 2012.

[10] Esa Järvenpää, Maarit Järvenpää, Paul MacManus, and Toby C. O’Neil. Visible parts and dimensions. Nonlinearity, 16(3):803–818, 2003.

[11] K. Jiang and K. Dajani. Subshifts of finite type and self-similar sets. Nonlinearity, 30(2):659–686, 2017.

[12] Kan Jiang and Lifeng Xi. Arithmetic representations of real numbers in terms of self-similar sets. Annales Academi Scientiarum Fennic Mathematica, 44:1-19, 2019.

[13] O. Nikodym. Sur la mesure des ensembles plans dont tous les points sont rectilinéairement accessibles. Fund. Math., 10:116–168, 1927.

[14] Yuval Peres and Pablo Shmerkin. Resonance between Cantor sets. Ergodic Theory Dynam. Systems, 29(1):201–221, 2009.

[15] Toby C. O’Neil. The Hausdorff dimension of visible sets of planar continua. Trans. Amer. Math. Soc., 359(11):5141–5170, 2007.

[16] Mehdi Pourbarat. On the arithmetic difference of middle cantor sets. Discrete and Continuous Dynamical Systems., 38(9):4259-4278, 2018.

[17] Károly Simon and Boris Solomyak. Visibility for self-similar sets of dimension one in the plane. Real Anal. Exchange, 32(1):67–78, 2006/07.

[18] Songjing Wang, Zhouyu Yu, and Lifeng Xi. Average geodesic distance of Sierpinski gasket and Sierpinski networks. Fractals, 25(5):1750044, 8, 2017.

[19] Zhi-Ying Wen and Li-Feng Xi. On the dimensions of sections for the graph-directed sets. Ann. Acad. Sci. Fenn. Math., 35(2):515–535, 2010.

[20] Lifeng Xi, Wen Wu, and Ying Xiong. Dimension of slices through fractals with initial cubic pattern. Chin. Ann. Math. Ser. B, 38(5):1145–1178, 2017.

[21] Jinjin Yang, Songjing Wang, Lifeng Xi, and Yongchao Ye. Average geodesic distance of skeleton networks of Sierpinski tetrahedron. Phys. A, 495:269–277, 2018.

[22] Qianqian Ye, Long He, Qin Wang, and Lifeng Xi. Asymptotic formula of eccentric distance sum for Vicsek network. Fractals, 26(3):1850027, 8, 2018.
[23] Luming Zhao, Songjing Wang, and Lifeng Xi. Average geodesic distance of Sierpinski carpet. *Fractals*, 25(6):1750061, 8, 2017.

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