DYNAMICS ON SHIFT SPACES WITH A HOLE

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ABSTRACT. This paper examines the relationship between the escape rate and the minimal period of the hole. We consider the shift map on a subshift of finite type with union of cylinders based at words of identical length as the hole. The escape rate relates to the asymptotic behavior of the number of words of fixed length that do not contain a fixed set of (forbidden) words of identical length as subwords. We explore the relationship between the escape rate and a rational function \( r(z) \), evaluated at the alphabet size \( q \). In particular, we prove that the escape rate is faster for the hole with smaller \( r(q) \). Further we consider holes corresponding to a union of cylinders based at two collections with same number of words, each of equal length, having zero cross-correlation, and prove that the larger is the minimal period of the collection, the faster is the escape rate. However when the cross-correlations are non-zero, we give examples to prove that this result fails to hold. Our results are more general than the existing ones known for maps conjugate to a full shift with a single cylinder as the hole. The existing results arise as a special case of our results.

1. INTRODUCTION

Bunimovich and Yurchenko [3] considered maps conjugate to a full shift with Markov holes corresponding to a single cylinder. This work investigated the dependence of survival probability on the position and the size of the hole, and proved that the escape is fastest through the hole whose minimal period is maximum. The results in [3] are applicable only for maps conjugate to a full shift with a single cylinder as the hole. Similar results are not known to exist even for simple systems such as expansive Markov maps. Froyland and Stancevic [10], Haritha and Agarwal [15] have provided exploratory numerical examples. However these papers do not provide general results.

In this paper, we consider a general class of maps, the ones conjugate to a subshift of finite type with the union of cylinders based at words of identical length as holes. We extend and generalize results of [3,10,15] by considering maps conjugate to a subshift of finite type with the hole corresponding to a union of cylinders based at words of identical length. The results in these works follow as a special case of the results in our paper.

The existing results have extensively studied the problem of finding \( f(n) \), the number of words of length \( n \), which do not contain any of the words from a given finite set of words with symbols from a fixed set of alphabets. The problem has applications to comma-free codes, games, pattern matching, and several problems in probability theory, including finding the number...

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of events which avoid appearance of a given set of events as sub-events. We refer to [12] and [16] for an extensive account of several applications. Since \( f(n) \) does not have a simple explicit formula, it is often convenient to study its generating function \( F(z) \). The function \( F(z) \) is rational and its special form is helpful to understand the asymptotic behavior of \( f(n) \). The expression for \( F(z) \) when one word is forbidden was first described in [19]. A similar formula corresponding to a collection of words with some specific patterns is given in [9]. The generating function \( F(z) \) is described using the correlation between two words, which is a polynomial function representation of overlapping of one word onto another. The correlation of a word with itself is known as the auto-correlation. The question of characterizing and enumerating the words having a given auto-correlation is considered in [13] and [18], respectively.

Open dynamical systems or dynamical systems with hole are interesting because of their dynamical properties and also their applications, we refer to [1,2,3,8,11,14,20]. The problem of counting \( f(n) \) is also useful in computing the escape rate into a hole of a map conjugate to the shift map on a shift space, see [3] and [15]. Escape rate represents the average rate at which the orbits escape into the hole. Larger the escape rate, faster the orbits terminate.

The hole considered in this paper corresponds to a finite union of cylinders based at words of finite length. The words describing these cylinders are known as forbidden words. Hence the escape rate is related to the number \( f(n) \).

Guibas and Odlyzko in [11] and [12] gave an explicit combinatorial method (see Theorem 2.4) to compute \( F(z) \) using a system of linear equations involving the alphabet size and the correlation polynomials. This result was used by Bunimovich and Yurchenko in [3] to compare the escape rates of the doubling map into Markov holes, which correspond to cylinders based at words of identical length. Their results are applicable to all maps conjugate to any full shift. In [3], the calculation of \( f(n) \) using the generating function method was simple since the hole considered corresponded to a single cylinder, and thus there was only one forbidden word.

On the other hand, when the hole corresponds to a union of cylinders, in other words, when there is more than one forbidden word, the cross-correlations between each pair of words appear in the generating function, which makes the analysis harder. In any case, we show that the escape rate depends on the real positive root, say \( \lambda \), which is largest in modulus among all the roots of the denominator (polynomial function) of the generating function. In fact, we show in Theorem 3.3 that in case of full shift with holes being the union of two cylinders, escape rate is equal to \( \ln(q/\lambda) \), where \( q \) is the alphabet size of the underlying shift space. Further in Theorem 3.2, we show that \( \lambda \) increases monotonically with \( r(q) \), the corresponding rational function obtained from Theorem 2.4 evaluated at \( q \), and hence escape rate is smaller for the hole with larger value of \( r(q) \). These results are generalized in later sections. In Theorems 5.4 and 5.6 we will consider two collections with same number of words, each of equal length, having zero cross-correlation, and prove that the collection with larger minimal period
has the larger escape rate. We give counter-examples in Remark 4.10 discussing the violation of this result when the cross-correlations are non-zero.

1.1. Organization of the paper. In Section 2, we present some preliminaries on subshifts of finite type, relationship between the escape rate into a hole (which is a union of cylinders) and the topological entropy, and the form of the corresponding generating function in terms of the alphabet size $q$ and the associated correlation polynomials. We state the main results of this paper in Section 3, the proofs of which are presented in the future sections. In Section 4, we consider a full shift as the underlying space and compare the escape rate into two holes which are unions of two cylinders based at words of equal length. Theorems 3.1 and 3.2 give the relationship between the escape rate and $r(q)$ with certain assumptions on $q$ and the length $p$ of forbidden words. We prove Theorem 3.2 through a series of lemmas. Further, in Section 4.2, we discuss the relationship between the minimum period of the hole and the escape rate. In Section 5, we generalize the results obtained in Section 4 when holes are union of more than two cylinders. Theorems 5.3 and 5.4 are generalizations of Theorems 3.2 and 3.3 respectively. In Section 6, we discuss applications of the results obtained in Sections 4 and 5 to the case when the underlying space is a subshift of finite type. Finally, we give concluding remarks in Section 7.

2. Preliminaries

Let $q \geq 2$ and $\Lambda = \{0, 1, \ldots, q - 1\}$ be the set of symbols. We will denote by $\Sigma_q^+ = \Lambda^N$, the set of all one-sided sequences with symbols from $\Lambda$. Consider a finite collection $F$ of (forbidden) words of equal length say $p$ (this can always be assumed without loss of generality) with symbols from $\Lambda$. Let $\Sigma_F$ denote the collection of all sequences in $\Sigma_q^+$ which do not contain any of the words from $F$ as subwords. The collection $\Sigma_F$ is called a one-sided $(p - 1)$-step shift of finite type. Let $\sigma : \Sigma_F \to \Sigma_F$ be the left shift map.

Let $\Lambda^{p - 1}$ be the collection of all $q^{p - 1}$ words of length $p - 1$ with symbols from $\Lambda$. We will now treat each word in $\Lambda^{p - 1}$ as a symbol, and hence there are a total of $q^{p - 1}$ symbols. Define

$$\Sigma^{p - 1} = \{(u_i) \in (\Lambda^{p - 1})^N : a^i_{l+1} = a^i_{l+1}, \text{ for every } 1 \leq l \leq p - 2, i \in \mathbb{N}\},$$

where $u_i = a^i_1a^i_2\ldots a^i_{p-1}$.

Each word $w = a_1a_2\ldots a_p$ of length $p$ with symbols from $\Lambda$ has a corresponding word $(a_1a_2\ldots a_{p-1})(a_2a_3\ldots a_p)$ of length two (appearing as a subword in a sequence in $\Sigma^{p - 1}$), where $a_1a_2\ldots a_{p-1}$ and $a_2a_3\ldots a_p$ are symbols from $\Lambda^{p - 1}$. Through this correspondence, we obtain a collection $F'$ of words of length two with symbols from $\Lambda^{p - 1}$, using $F$. Note that both $F$ and $F'$ have same cardinality. Let $\Sigma^{p - 1}_{F'}$ denote the collection of all sequences in $\Sigma^{p - 1}$ which do not contain any of the words from $F'$ as subwords. The map given by $\varphi : \Sigma_F \to \Sigma^{p - 1}_{F'}$ as

$$\varphi(a_1a_2a_3\ldots) = (a_1a_2\ldots a_{p-1})(a_2a_3\ldots a_p)(a_3a_4\ldots a_{p+1})\ldots,$$

is a conjugacy. That is, $\sigma \circ \varphi = \varphi \circ \sigma$, where $\sigma$ is the left shift map on the corresponding sequence space. Therefore, to understand the properties of
the shift map $\sigma$ which are invariant under conjugacy, it is enough to study
the one step shifts (that is, $p = 2$).

For $p = 2$ (that is, forbidden words are of length two), define the adjacency
matrix $A = (a_{ij})_{q \times q}$ as $a_{ij} = 0$ if and only if the word $ij \in \mathcal{F}$ and $a_{ij} = 1$
otherwise (the rows and columns of $A$ are labeled from 0 to $q - 1$). We
assume that the adjacency matrix corresponding to the shift space $\Sigma_{\mathcal{F}}$ is irreducible.

If $f(k)$ denotes the number of words of length $k$ which appear as subwords in
sequences in $\Sigma_{\mathcal{F}}$, then the topological entropy (we refer \cite[Proposition
3.5]{17}) is given by

$$h_{\text{top}}(\Sigma_{\mathcal{F}}) = \lim_{k \to \infty} \frac{\ln(f(k))}{k} = \ln(\theta),$$

where $\theta$ is the largest (in modulus) eigenvalue of $A$ (which is real and positive
by Perron-Frobenius theorem), known as the Perron value.

Let $w = i_1i_2 \ldots i_k$ be an allowed word in $\Sigma_{\mathcal{F}}$ (that is, it does not contain
any word from the collection $\mathcal{F}$) and let $C_w = \{x_1x_2 \ldots \in \Sigma_{\mathcal{F}} \mid x_1 = i_1, x_2 = i_2, \ldots, x_k = i_k\}$,

the cylinder based at the word $w$. Then we obtain a probability
measure space with set $\Sigma_{\mathcal{F}}$, $\sigma$-algebra generated by cylinders based at all
allowed words of finite length, and the measure $\mu$ which is defined as follows:
for every allowed word $w = i_1i_2 \ldots i_k$, $\mu(C_w) = \frac{u_{i_1}v_{i_k}}{\theta^{k-1}}a_{i_1i_2}a_{i_2i_3} \ldots a_{i_{k-1}i_k}$,

where $\theta$ is the Perron value of $A$, $v = (v_0, \ldots, v_{q-1})^T$, and $u = (u_0, \ldots, u_{q-1})$,

are the normalized right and left eigenvectors with respect to $\theta$ such that $uv = 1$. The measure $\mu$ is called the Parry measure. In the case of full
shift ($\mathcal{F} = \emptyset$), $\mu(C_w) = 1/\theta^k$, where $w$ is any word of length $k$. If $\mathcal{F} \neq \emptyset$, it is immediate from the definition of $\mu$ that two cylinders based at words of identical length need not have the same measure.

**Definition 2.1.** Consider $\Sigma_{\mathcal{F}}$. Let $\mathcal{F}_1$ be another finite collection of words
with symbols from $\Lambda$. We define the escape rate of the shift map on $\Sigma_{\mathcal{F}}$ into
the hole consisting of cylinders based at words in $\mathcal{F}_1$ by

$$\rho(\mathcal{F}_1; \Sigma_{\mathcal{F}}) := - \lim_{k \to \infty} \frac{1}{k} \ln \mu(\Sigma_{\mathcal{F}} \setminus \Omega_k(\mathcal{F}_1)),$$

where $\Sigma_{\mathcal{F}} \setminus \Omega_k(\mathcal{F}_1)$ is the collection of all sequences in $\Sigma_{\mathcal{F}}$ which do not contain words from $\mathcal{F}_1$ as subwords till their first $k$ positions. Here $\mu$ denotes
the Parry measure as defined above. In case of full shift ($\mathcal{F} = \emptyset$), we denote $\rho(\mathcal{F}_1)$ for the escape rate instead of $\rho(\mathcal{F}_1; \Sigma_{\mathcal{F}})$.

This limit may not exist for general holes, but in this case, it exists by Theorem \cite[2.2]{22}. Without loss of generality, assume that all the words in both $\mathcal{F}$ and $\mathcal{F}_1$ are of the same length, $p \geq 2$ and $\mathcal{F} \cap \mathcal{F}_1 = \emptyset$.

**Theorem 2.2.** \cite[Theorem 3.1]{15} The escape rate relates to the topological
entropy $h_{\text{top}}$ as follows:

$$\rho(\mathcal{F}_1; \Sigma_{\mathcal{F}}) = h_{\text{top}}(\Sigma_{\mathcal{F}}) - h_{\text{top}}(\Sigma_{\mathcal{F} \cup \mathcal{F}_1}).$$

\footnote{In case $A$ is not irreducible, we can define Markov measure.}
From Theorem 2.2 the escape rate is determined by the asymptotic behavior of the number of words of fixed length which do not contain a given collection of words. Thus the problem of computing the escape rate reduces to a purely combinatorial problem. Now we give some preliminaries from [12].

Let $F = \{w_1, w_2, \ldots, w_s\}$ be a reduced collection of words with symbols from $\Lambda$. Let $f(k)$ denote the number of words of length $k$ with symbols from $\Lambda$ which do not contain any of the words $w_1, w_2, \ldots, w_s$ from the collection $F$. Let

$$F(z) := \sum_{k=0}^{\infty} f(k)z^{-k}$$

be the generating function for $f(k)$.

**Definition 2.3.** Let $u$ and $w$ be two words of lengths $n_1$ and $n_2$, respectively. The *correlation polynomial* of $u$ and $w$ is defined as

$$(uw)_z = \sum_{\ell=1}^{n_1} b_\ell z^{n_1-\ell},$$

where $b_\ell = 0$, if $\sigma^{\ell-1}(C_u) \cap C_w = \emptyset$, and $b_\ell = 1$, otherwise, with $C_u$ and $C_w$ denote the cylinders based at $u$ and $w$, respectively.

When $u = w$, $(uu)_z$ is said to be the *auto-correlation polynomial* of $u$, and when $u \neq w$, $(uw)_z$ is said to be the *cross-correlation polynomial* of $u$ and $w$.

Note that if $\sigma^{\ell-1}(C_u) \cap C_w = \emptyset$, for all $1 \leq \ell \leq n_1$, then $(uw)_z = 0$. Also, if $u \neq w$, $b_1 = 0$ and if $u = w$, then $b_1 = 1$. Hence for $u \neq w$, $(uw)_z$ is a polynomial with degree at most $n_1 - 2$ and $(uu)_z$ is a monic polynomial with degree exactly equal to $n_1 - 1$.

**Theorem 2.4.** ([15, Theorem 3.3]) With the notations as above,

$$F(z) = \frac{z}{(z - q) + \frac{1}{r(z)}},$$

where $\frac{1}{r(z)}$ is the sum of entries of the matrix $M^{-1}$ with the correlation matrix $M$ given by

$$
\begin{bmatrix}
(w_1w_1)_z & (w_2w_1)_z & \cdots & (w_sw_1)_z \\
(w_1w_2)_z & (w_2w_2)_z & \cdots & (w_sw_2)_z \\
\vdots & \vdots & \ddots & \vdots \\
(w_1w_s)_z & (w_2w_s)_z & \cdots & (w_sw_s)_z
\end{bmatrix}.
$$

**Remarks 2.5.** The following are immediate from Theorem 2.4.

1. $1/r(z)$ is a rational function with the numerator being a polynomial of degree exactly $(s - 1)(p - 1)$, and denominator having degree exactly $s(p - 1)$.
2. The escape rate depends only on $q$ and the rational function $r(z)$.

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2For any $i \neq j$, $w_i$ is not a subword of $w_j$. 
(3) If \((w_i w_j)_z = 0\) for all \(i \neq j\), then
\[
\frac{1}{r(z)} = \sum_{i=1}^{s} \frac{1}{(w_i w_i)_z}.
\]

Definition 2.6. Let \(\mathcal{F} = \{u_1, \ldots, u_s\}\) be a collection of forbidden words of equal length \(p\), with symbols from \(\Lambda = \{0, 1, \ldots, q-1\}\). We define \(\tau_{\mathcal{F}}\) to be the minimum of the period of periodic points in the union of cylinders \(C = C_{u_1} \cup \cdots \cup C_{u_s}\) under the shift map. We call \(\tau_{\mathcal{F}}\) as the minimum period of the hole corresponding to the union \(C\) of cylinders.

Remarks 2.7. (1) Note that \(\tau_{\mathcal{F}} = \min\{\tau_{u_1}, \ldots, \tau_{u_s}\}\), where \(\tau_{u_i}\) denote the minimum of the period of periodic points in \(C_{u_i}\) under the shift map. For any finite word \(u\) of length \(p\), \(1 \leq \tau_u \leq p\). If \(1 \leq \tau_{u_i} \leq p-1\), then the second largest degree of \((u_i u_i)_z\) is \(z^{p-1-\tau_{u_i}}\). Moreover, \((u_i u_i)_z = z^{p-1}\) if \(\tau_{u_i} = p\).

(2) \(\tau_{\mathcal{F}}\) is the Poincaré recurrence time of \(C\).

3. Statement of main results

Here we state our main results. Proofs are presented in the later sections. The first two results consider a full shift \(\Sigma^+_q\) and compare the escape rate into two holes each corresponding to union of two cylinders based at words of equal length \(p\). The techniques used in the proofs are different when \(p = 2\) and when \(p \geq 3\). Hence we write these results separately. Let \(r_{\mathcal{F}_1}(z)\) and \(r_{\mathcal{F}_2}(z)\) denote the rational function corresponding to the collection \(\mathcal{F}_1\) and \(\mathcal{F}_2\), respectively, as described in Theorem 2.4.

Theorem 3.1. Let \(q \geq 2\) and \(\mathcal{F}_1, \mathcal{F}_2\) be the collections each having two words of length two. If \(r_{\mathcal{F}_1}(q) > r_{\mathcal{F}_2}(q)\), then \(\rho(\mathcal{F}_2) > \rho(\mathcal{F}_1)\).

Theorem 3.2. Let \(p \geq 3\), \(q \geq 7\), and \(\mathcal{F}_1, \mathcal{F}_2\) be the collections each having two words of length \(p\). If \(r_{\mathcal{F}_1}(q) > r_{\mathcal{F}_2}(q)\), then \(\rho(\mathcal{F}_2) > \rho(\mathcal{F}_1)\).

The following result is the key idea used to prove Theorem 3.2.

Theorem 3.3. For \(p \geq 3\), \(q \geq 7\), let \(\mathcal{F}\) be a collection of two words of length \(p\). Then the escape rate into the hole consisting of cylinders based at words from \(\mathcal{F}\) is given by
\[
\rho(\mathcal{F}) = -\ln(\lambda/q).
\]

where \(\lambda\) is the largest root of the denominator of the corresponding generating function \(F(z)\).

These results give the relationship between the escape rate and \(r(q)\), and are proved in Section 4.1.

The next set of results are for a full shift \(\Sigma^+_q\) and discuss the relationship between the minimum period of the hole and the escape rate when the holes correspond to a union of two cylinders based at words of length \(p\) with zero cross-correlations.

Theorem 3.4. Let \(q \geq 4\), \(p \geq 2\), and let \(\mathcal{F}_1 = \{u_1, u_2\}\) and \(\mathcal{F}_2 = \{w_1, w_2\}\), where \(u_1, u_2, w_1, w_2\) are words of equal length \(p\) with \((u_i u_j)_z = (w_i w_j)_z = 0\), for \(1 \leq i \neq j \leq 2\). If \(\tau_{\mathcal{F}_1} < \tau_{\mathcal{F}_2}\), then \(r_{\mathcal{F}_1}(q) > r_{\mathcal{F}_2}(q)\).
The claim in Theorem 3.4 holds true for $q = 3$ and $p \geq 2$ as well, see Theorem 4.17. We will also give an example to show that the result fails for $q = 2$. See Section 4.2.

Finally we generalize the results stated above for the case when holes are a union of more than two cylinders.

**Theorem 3.5.** Suppose $F_1$ and $F_2$ are finite collections of words each of length $p$ with symbols from $\Lambda$ and have the same number of words say $s$. Then there exist positive constants $C(s, r_{F_1}, r_{F_2})$ and $D(s, r_{F_1}, r_{F_2})$ such that for any $q \geq C$ and $p \geq D$, if $r_{F_1}(q) > r_{F_2}(q)$ then $\rho(F_2) > \rho(F_1)$.

**Theorem 3.6.** Suppose $\ell \geq 2$ and $q \geq 2$ be such that

$$\left((q-1)(\ell + 1) - \ell q \left(1 + \frac{1}{q}\right)^{\ell-1}\right) \geq 0 \quad (3)$$

Let $F_1 = \{u_1, u_2, \ldots, u_\ell\}$ and $F_2 = \{w_1, w_2, \ldots, w_\ell\}$ such that $u_i, w_i$ are words of equal length $p$ with $(u_i u_j)_z = (w_i w_j)_z = 0$ for $1 \leq i \neq j \leq \ell$. If $\tau_{F_1} < \tau_{F_2}$, then $r_{F_1}(q) > r_{F_2}(q)$. The inequality (3) holds true for $q \geq \ell (\ell^2 - 1)/2 + 1$.

These results are proved in Sections 5.1 and 5.2 respectively.

4. Full shift with union of two cylinders as hole

In this section, we consider the case of full shift $\Sigma = \Sigma_q^+$, that is, $F = \emptyset$. The hole that we consider here is a union of two cylinders based at words of equal length. In Section 4.1, we explore the relationship between the escape rate and $r(q)$ in Theorems 3.1 and 3.2. Further in Section 4.2, we discuss the relationship between the escape rate and the minimal period of the hole in Theorems 3.4 and 1.17. Our results easily extend to the case when the holes are a union of more than two cylinders, which will be discussed in Section 5.

4.1. Relationship between the escape rate and $r(q)$. For a finite collection $G$ of words with symbols from $\Lambda$, throughout this paper, as described earlier, we will use shorthand $\rho(G)$ to denote the escape rate of the shift map on full shift $\Sigma_q^+$ into the hole consisting of cylinders based at words in $G$. We recall the following result from the literature for the case of one forbidden word.

**Theorem 4.1.** [2] Let $i = 1, 2$, let $F_i = \{w_i\}$, where $w_1$ and $w_2$ are words of the same length. Then $(w_1 w_2)_q > (w_2 w_2)_q$ implies $\rho(F_2) > \rho(F_1)$, where $(ww)_q$ is the value of $(ww)_z$ evaluated at $z = q$.

Let us now consider the case of two forbidden words. Let $F_1 = \{u_1, w_1\}$ and $F_2 = \{u_2, w_2\}$ where $u_1 \neq w_1, u_2 \neq w_2$ are words of same length say $p$.

We would like to compare the escape rates $\rho(F_1)$ and $\rho(F_2)$ of the shift map on $\Sigma$ into the holes corresponding to the collections $F_1$ and $F_2$, respectively.

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3See Remark 2.5 (3).
For \(i = 1, 2\), let \(f_i(k)\) denote the number of words of length \(k\) with symbols from \(\Lambda\), which do not contain any of the words \(u_i, w_i\), and let the corresponding generating function be given by \(F_{\mathcal{F}_i}(z)\), see [1]. By Theorem 2.4,

\[
F_{\mathcal{F}_i}(z) = \frac{zr_{\mathcal{F}_i}(z)}{1 + (z - q)r_{\mathcal{F}_i}(z)},
\]

where

\[
r_{\mathcal{F}_i}(z) = \frac{(u_iu_i)z(w_iw_i)z - (u_iw_i)z(w_iu_i)z}{(u_iu_i)z + (w_iw_i)z - (u_iw_i)z - (w_iu_i)z}.
\]

Table 1. Escape rate when \(p = 2\), where \(\mathcal{F}_1 = \{aa, bb\}\), or \(\{aa, ab\}\), or \(\{ab, ba\}\); \(\mathcal{F}_2 = \{ab, ca\}\), or \(\{ab, bc\}\); \(\mathcal{F}_3 = \{aa, bc\}\); and \(\mathcal{F}_4 = \{ab, cd\}\), or \(\{ab, ac\}\).

| \(q\) | \(r_{\mathcal{F}_1}(q)\) | \(\rho(\mathcal{F}_1)\) | \(r_{\mathcal{F}_2}(q)\) | \(\rho(\mathcal{F}_2)\) | \(r_{\mathcal{F}_3}(q)\) | \(\rho(\mathcal{F}_3)\) | \(r_{\mathcal{F}_4}(q)\) | \(\rho(\mathcal{F}_4)\) |
|---|---|---|---|---|---|---|---|---|
| 2  | 1.5 | 0.6931 | \cdots | \cdots | \cdots | \cdots | \cdots | \cdots |
| 3  | 2.0 | 0.2172 | 1.8 | 0.2550 | 1.71 | 0.2890 | 1.5 | 0.4055 |
| 4  | 2.5 | 0.1161 | 2.29 | 0.1302 | 2.22 | 0.1361 | 2.0 | 0.1583 |
| 5  | 3.0 | 0.0735 | 2.78 | 0.0804 | 2.73 | 0.0824 | 2.5 | 0.0918 |
| 6  | 3.5 | 0.0510 | 3.27 | 0.0550 | 3.23 | 0.0559 | 3.0 | 0.0609 |
| 7  | 4  | 0.0376 | 3.77 | 0.0401 | 3.73 | 0.0406 | 3.5 | 0.0436 |
| 8  | 4.5 | 0.0289 | 4.27 | 0.0306 | 4.23 | 0.0309 | 4  | 0.0328 |
| 9  | 5.0 | 0.023 | 4.76 | 0.0241 | 4.74 | 0.0243 | 4.5 | 0.0257 |
| 10 | 5.5 | 0.0187 | 5.26 | 0.0195 | 5.24 | 0.0196 | 5.0 | 0.0206 |

The values of \(r\) and \(\rho\) for the case of the holes corresponding to two forbidden words are given in Tables 1 and 2. These tables show the relationship between \(r_{\mathcal{F}}(q)\) and the escape rate for \(p = 2, 3\) and different values of \(q\), where \(a, b, c, d, e, f\) are distinct symbols from \(\Lambda\). These tables suggest that \(\rho(\mathcal{F})\) is monotonically decreasing as a function of \(r_{\mathcal{F}}(q)\). Indeed we have Theorems 3.1 and 3.2 dealing with the cases \(p = 2\) and \(p \geq 3\), respectively.

**Proof.** (Proof of Theorem 3.1) If the length \(p\) of words in the collections is two, there are four choices for \(r(z)\) namely

\[
r_1(z) = \frac{z + 1}{2}, \quad r_2(z) = \frac{z^2}{2z - 1}, \quad r_3(z) = \frac{z(z + 1)}{2z + 1}, \quad r_4(z) = \frac{z}{2},
\]

corresponding to various possible forms of the correlation matrix \(M\) as defined in Theorem 2.4.

Using the form of the generating function \(F_i(z)\) in terms of \(r_i(z)\), \(F_i(z) = \sum_{k=0}^{\infty} f_i(k)z^{-k}\); and comparing the coefficients, we get a recurrence relation. Solving this recurrence relation using the techniques described in [15], we see that if \(\mu_i\) is the largest real root of \(p_i(x)\), then for \(i = 1, \ldots, 4\),

\[
\lim_{k \to \infty} \frac{1}{k} \ln (f_i(k)) = \ln(\mu_i),
\]

where \(p_1(x) = x^2 - (q - 1)x - (q - 2), \quad p_2(x) = x^3 - qx^2 + 2x - 1, \quad p_3(x) = x^3 - (q - 1)x^2 - (q - 2)x + 1, \quad p_4(x) = x^2 - qx + 2\). Note that \(r_2, r_3, r_4\) are not possible for \(q = 2\). One can easily verify that, for \(q > 2\), \(r_1(q) > r_2(q) > r_3(q) > r_4(q)\).
Moreover, \( p_i(q-1) < 0 \) and \( p_i(q) > 0 \), for \( i = 1, \ldots, 4 \). Hence, there exists a real root \( \mu_i \) of \( p_i \) with \( q - 1 < \mu_i < q \), for \( i = 1, \ldots, 4 \). We claim that \( \mu_i \) is a simple root of \( p_i \) with the largest modulus, for \( i = 1, \ldots, 4 \). We will look at the polynomials separately.

1. \( \mu_1 \) is the largest root of \( p_1 \):
   Note that \( p_1(-1) > 0 \). Hence there exists a real root inside \((-1, q - 1)\). Hence as \( p_1 \) is a degree 2 polynomial and \( q \geq 2 \), \( \mu_1 \) is simple and has largest modulus.

2. \( \mu_2 \) is the largest root of \( p_2 \):
   Consider the polynomial \( p_2(z) = z^3 - qz^2 + 2z - 1 \). For \( q \geq 4 \), at \( |z| = q - 1 \), note that \( q|z|^2 > |z|^3 + 2|z| + 1 \) and hence \( p_2(z) \) has two roots inside the ball of radius \( q - 1 \) by Rouche’s theorem. Hence \( \mu_2 \) is the simple largest root of \( p_2 \). At \( q = 3 \), it is immediate that \( \mu_2 \) is the largest root of \( p_2(x) = x^3 - 3x^2 + 2x - 1 \).

3. \( \mu_3 \) is the largest root of \( p_3 \):
   It follows from the fact that \( p_3(-1) < 0 \), \( p_3(0) > 0 \), and \( p_3 \) is a degree 3 polynomial.

4. \( \mu_4 \) is the largest root of \( p_4 \):
   It follows from the fact that \( p_4(0) > 0 \) and \( p_4 \) is a degree 2 polynomial.

Further note that
\[
\mu_2 \mu_3 \mu_4 = \mu_2^3 - (q-1)\mu_2^2 - (q-2)\mu_2 = \mu_2^2 - q\mu_2 + 1 < 0,
\]
since \( \mu_2^2 - q\mu_2 + 1 = 1 - \mu_2 < 0 \). Hence \( p_1(\mu_2) < 0 \) which implies \( \mu_2 < \mu_1 \). Similarly \( p_3(\mu_2) > 0 \) implies \( \mu_3 < \mu_2 \), and \( p_3(\mu_4) < 0 \) implies \( \mu_4 < \mu_3 \), for \( q \geq 3 \). Thus we have the required result. \( \square \)

Next we prove Theorem 3.2 through a series of lemmas. Let us first fix a few notations. Let \( u, w \) be distinct words both of length \( p \) with symbols from \( \Lambda \), and let \( F = \{u, w\} \). Define
\[
f(z) := (uu)_z, \quad g(z) := (ww)_z, \quad h(z) := (uw)_z, \quad k(z) := (wu)_z,
\]
\[
A(z) := f(z)g(z) - h(z)k(z), \quad B(z) := f(z) + g(z) - h(z) - k(z).
\]
Then \( r(z) := r_F(z) = \frac{A(z)}{B(z)} \).
Observe that \( f, g, h, k \) are polynomials with coefficients either 0 or 1. The degree of both \( f(z) \) and \( g(z) \) is \( p - 1 \), whereas the degree of both \( h(z) \) and \( k(z) \) is at most \( p - 2 \). Hence \( A(z) \) and \( B(z) \) have degree \( 2(p-1) \) and \( p - 1 \), respectively. Moreover
\[
F(z) = \frac{zr(z)}{1 + (z-q)r(z)} = \frac{zA(z)}{B(z) + (z-q)A(z)}.
\]

Lemma 4.2. \( \square \) For \( |z| \geq 4 \),
\[
|A(z)| \geq \frac{1}{48}4^{2p}.
\]

\( ^4 \)This result is independent of \( q \).
Proof. Note that
\[ |f(z)| \geq |z|^{p-1} - \sum_{i=0}^{p-2} |z|^i = |z|^{p-1} - \frac{|z|^{p-1} - 1}{|z| - 1} \geq |z|^{p-1} \frac{|z| - 2}{|z| - 1}. \]
Similarly \(|g(z)| \geq |z|^{p-1} \frac{|z| - 2}{|z| - 1}.
Also
\[ |h(z)| \leq \sum_{i=0}^{p-2} |z|^i = \frac{|z|^{p-1} - 1}{|z| - 1} \leq \frac{|z|^{p-1}}{|z| - 1}. \]
Similarly \(|k(z)| \leq \frac{|z|^{p-1}}{|z| - 1}.
Hence
\[ |A(z)| \geq |f(z)||g(z)| - |h(z)||k(z)| \geq |z|^{2p-2} \left( \frac{|z| - 2}{|z| - 1} \right)^2 - \frac{|z|^{2p-2}}{(|z| - 1)^2} = |z|^{2p-2} \left( 1 - \frac{2}{|z| - 1} \right) \geq \frac{1}{48} 4^{2p}. \]
\[ \square \]

Lemma 4.3. For \( p \geq 3 \) and \( q \geq 5 \), the polynomial \((z - q)A(z) + B(z)\) has exactly one zero, say \( \lambda \), outside the disk \(|z| < 4\).

Proof. On the circle \(|z| = 4\),
\[ |B(z)| \leq |f(z)| + |g(z)| + |h(z)| + |k(z)| \leq 2 \sum_{i=0}^{p-1} |z|^i + 2 \sum_{i=0}^{p-2} |z|^i \leq \frac{2}{|z| - 1} |z|^{p-1} (|z| + 1) = \frac{5}{6} 4^p. \]
Thus from Lemma 12 on the circle \(|z| = 4\),
\[ \frac{|z - q||A(z)|}{|B(z)|} \geq \frac{|A(z)|}{|B(z)|} \geq \frac{1}{40} 4^p > 1, \tag{5} \]
since \( q \geq 5 \) and \( p \geq 3 \). Hence \( |z - q||A(z)| > |B(z)| \) on \(|z| = 4\).
By the Rouche’s theorem, both \((z - q)A(z)\) and \((z - q)A(z) + B(z)\), being of identical degrees, have same number of zeros in the region \(|z| \geq 4\).
Since by Lemma 12 \(|A(z)| > 0\), \( z = q \) is the only zero of \((z - q)A(z)\), for \(|z| \geq 4\), which proves this lemma. \square

Remark 4.4. Instead of using the properties of polynomials \(f(z), g(z), h(z)\) and \(k(z)\), if we simply use the fact that
\[ A(z) = z^{2(p-1)} + \sum_{i=0}^{2(p-1)-1} a_k z^k, \quad B(z) = 2z^{p-1} + \sum_{i=0}^{p-2} b_k z^k, \]
we obtain alternate estimates for \(A(z)\) and \(B(z)\).
If \( a = \max_k \{|a_k|, 1\} \) and \( b = \max_k \{|b_k|, 2\} \), then
\[ |A(z)| \geq |z|^{2(p-1)} \left( 1 - \frac{a}{|z| - 1} \right), \quad |B(z)| \leq b |z|^p. \]
Choose \( \alpha \in (a + 1, a + 2) \). Then for \( q > \alpha > a + 1 \) and \( |z| = \alpha \), we get
\[
\frac{|z - q||A(z)|}{|B(z)|} \geq \frac{1}{b}(q - \alpha)(\alpha - (a + 1))\alpha^{p-2},
\]
which can be made greater than 1 for large enough \( p \), since \( \alpha > 1 \).
These alternate estimates are worse than the ones obtained in Lemmas 4.2 and 4.3.

**Lemma 4.5.** For all positive real numbers \( x \geq 2 \), we have the following inequalities.
\[
\begin{align*}
x^{p-1} & \leq f(x), g(x) \leq \frac{x^p - 1}{x - 1} \\
0 & \leq h(x), k(x) \leq \frac{x^{p-1} - 1}{x - 1} \\
(p - 1)x^{p-2} & \leq f'(x), g'(x) \leq (p - 1)\frac{x^{p-1}}{x - 1} \\
0 & \leq h'(x), k'(x) \leq (p - 2)\frac{x^{p-2}}{x - 1} \\
(p - 1)(p - 2)x^{p-3} & \leq f''(x), g''(x) \leq (p - 1)(p - 2)\frac{x^{p-2}}{x - 1} \\
0 & \leq h''(x), k''(x) \leq (p - 2)(p - 3)\frac{x^{p-3}}{x - 1} \\
\frac{x^{2p-1}(x - 2)}{(x - 1)^2} & \leq A(x) \leq \frac{(x^p - 1)^2}{(x - 1)^2} \\
2\frac{x^p - 2x^{p-1} + 1}{x - 1} & \leq B(x) \leq 2\frac{x^p - 1}{x - 1} \\
2\frac{(p - 1)x^{2p-2}(x - 2)}{(x - 1)^2} & \leq A'(x) \leq 2x^{2p-1} \frac{p - 1}{(x - 1)^2} \\
2\frac{(p - 1)x^{p-2}(x - 2)}{x - 1} & \leq B'(x) \leq 2(p - 1)x^{p-1}.
\end{align*}
\]

*Proof.* The proofs are straightforward using the forms of polynomials \( f, g, h, \) and \( k \). \( \square \)
Table 2. Escape rate when \( p = 3 \)

| \( \mathcal{F} \) | \( q = 3 \) | \( q = 4 \) | \( q = 5 \) | \( q = 6 \) | \( q = 7 \) | \( q = 8 \) |
|-----------------|-----------|-----------|-----------|-----------|-----------|-----------|
|                 | \( r_F(q) \) | \( \rho(F) \) | \( r_F(q) \) | \( \rho(F) \) | \( r_F(q) \) | \( \rho(F) \) | \( r_F(q) \) | \( \rho(F) \) |
| \{aaa, bbb\}    | \{aaa, aab\} | \{aaa, baa\} | \{aaa, aba\} | \{aaa, bcb\} | \{abc, bcb\} | \{abc, abb\} | \{abc, aab\} | \{abc, baa\} | \{abc, aba\} | \{abc, bcb\} | \{abc, abb\} | \{abc, aab\} | \{abc, baa\} | \{abc, aba\} | \{abc, bcb\} |
|                 |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |           |
|                 | 6.50      | 0.0579    | 10.50     | 0.0252    | 15.50     | 0.0133    | 21.50     | 0.0078    | 28.50     | 0.0050    | 36.50     | 0.0034    |           |           |           |           |           |
|                 | 6.14      | 0.0616    | 9.89      | 0.0268    | 14.64     | 0.0141    | 20.38     | 0.0083    | 27.13     | 0.00533   | 34.88     | 0.00361   |           |           |           |           |           |
|                 | 5.65      | 0.0686    | 9.39      | 0.0284    | 14.14     | 0.01464   | 19.89     | 0.00841   | 26.64     | 0.00537   | 34.384    | 0.00364   |           |           |           |           |           |
|                 | 5.62      | 0.0691    | 9.38      | 0.0285    | 14.13     | 0.01465   | 19.88     | 0.00842   | 26.630    | 0.005379  | 34.380    | 0.003642  |           |           |           |           |           |
|                 | 5.57      | 0.0700    | 9.33      | 0.0286    | 14.09     | 0.01469   | 19.85     | 0.00843   | 26.6      | 0.005385  | 34.35     | 0.003645  |           |           |           |           |           |
|                 | 5.40      | 0.0729    | 9.14      | 0.0293    | 13.89     | 0.0149    | 19.64     | 0.00852   | 26.38     | 0.005429  | 34.13     | 0.003669  |           |           |           |           |           |
|                 | 5.32      | 0.0746    | 9.08      | 0.0295    | 13.84     | 0.0150    | 19.59     | 0.00854   | 26.35     | 0.005436  | 34.10     | 0.003672  |           |           |           |           |           |
|                 | 5.00      | 0.0800    | 8.50      | 0.0317    | 13.00     | 0.0160    | 18.5      | 0.0090    | 25.00     | 0.0057    | 32.50     | 0.0038    |           |           |           |           |           |
|                 | 4.76      | 0.0857    | 8.26      | 0.0328    | 12.75     | 0.01632   | 18.25     | 0.009173  | 24.753    | 0.00578   | 32.252    | 0.003884  |           |           |           |           |           |
|                 | ...       | ...       | 8.24      | 0.0329    | 12.74     | 0.01634   | 18.24     | 0.009176  | 24.747    | 0.005789  | 32.248    | 0.003883  |           |           |           |           |           |
|                 | ...       | ...       | 8.00      | 0.0340    | 12.50     | 0.0167    | 18.00     | 0.0093    | 24.50     | 0.00585   | 32.00     | 0.00391   |           |           |           |           |           |
Lemma 4.6. For $p \geq 3$ and $q \geq 6$, $q - \lambda = O(q^{-p+2})$.

Proof. Since $\lambda$ is the single root of $(z - q)A(z) + B(z)$ with $|\lambda| \geq 4$, it has multiplicity one and is real.
Since $B(q) > 0$, at $z = q$, $(q - z)A(z) - B(z) = -B(q) < 0$.
For real $x \geq 2$,

$$(q - x)A(x) - B(x) = (q - x)(f(x)g(x) - h(x)k(x)) - (f(x) + g(x) - h(x) - k(x)) \geq (q - x)\left[ x^{2p-2} - \left( \frac{x^{p-1}}{x-1} \right)^2 \right] - 2 \left( \frac{x^p}{x-1} \right) \geq \frac{((q - x)(x^{2p} - 2x^{p-1} + 2x^{p-1} - 1) - 2x^p(x-1))}{(x-1)^2} \geq \frac{x^{p-1}}{(x-1)^2}((q - x)(x - 2)x^p - 2x^2 + 2q - (q - x)) \geq \frac{x^{p+1}}{(x-1)^2}((q - x)(x - 2)x^{p-2} - 2).$$

Moreover at $x = q - q^{-p+2}$,

$$(q - x)(x - 2)x^{p-2} - 2 = q^{-p+2}q^{p-2}(1 - q^{-p+1})p^{-2}(q - q^{-p+2} - 2) - 2 \geq (1 - (p - 2)q^{-p+1})(q - q^{-p+2} - 2) - 2 \geq 3 \left( 1 - \frac{p - 2}{q^{p-1}} \right) - 2 > 0,$$

since $(1 - x)^n \geq 1 - nx$, for $x > 0$ and $n \geq 1$, $p \geq 3$ and $q \geq 6$.
Hence $(q - x)A(x) - B(x) > 0$ for $x = q - q^{-p+2}$.
This implies that

$$q - q^{-p+2} < \lambda < q, \quad (6)$$

and thus the result follows.

Remark 4.7. As described in Remark 4.4, if we consider $A$ and $B$ as general polynomials, we get that for $p$ large enough, $(q - x)A(x) - B(x) < 0$ at $x = q$ and $(q - x)A(x) - B(x) > 0$ at $x = q - q^{-p+2}$ provided $q > a + b + 2$.

Lemma 4.8. For real $x > 2(2 + \sqrt{2})$, $r'(x) > 0$.

Proof. Note that

$$r'(x) = \frac{A'(x)B(x) - A(x)B'(x)}{B(x)^2}.$$
Using Lemma 4.5, observe that the numerator of $r'(x)$ is

\[
A'(x)B(x) - A(x)B'(x) \geq \left(2(p - 1)x^{2p-2}\frac{x - 2}{(x - 1)^2}\right) \left(2x^p - 2x^{p-1} + 1\right) - \left(\frac{x^p - 1}{x - 1}\right)^2 \left(2(p - 1)x^{p-1}\right) > 0
\]

\[
\Rightarrow 2(p - 1)x^{2p-3}(2x - 2 - x^2) = 2(p - 1)x^{2p-3}(2x - 2 - x^2)\frac{x - 2}{(x - 1)^3}
\]

Hence $r'(x) > 0$ if $2(x - 2)^2 - x^2 > 0$, which holds for all $x > 2(2 + \sqrt{2}) \sim 6.83$.

**Lemma 4.9.** The following holds for real $x > 2(2 + \sqrt{2})$:

\[
\frac{r'(x)}{r(x)} = O(px^{-1}), \text{ and } \frac{r''(x)}{r(x)} = O(p^2x^{-2}).
\]

**Proof.** Using Lemma 4.5,

\[
0 < \frac{r'(x)}{r(x)} \leq \frac{A'(x)}{A(x)} + \frac{B'(x)}{B(x)} \leq \frac{2p - 1}{x - 2} + \frac{p - 1}{x - 2} = 3\frac{p - 1}{x - 2}.
\]

which gives $\frac{r'(x)}{r(x)} = O(px^{-1})$.

Further note that

\[
\frac{r''(x)}{r(x)} = \frac{A''(x)}{A(x)} - \left(\frac{A'(x)}{A(x)}\right)^2 - \frac{B''(x)}{B(x)} + \left(\frac{B'(x)}{B(x)}\right)^2 + \left(\frac{r'(x)}{r(x)}\right)^2.
\]

Using Lemma 4.5,

\[
A''(x) = f''(x)g(x) + f(x)g''(x) + 2f'(x)g'(x) - h''(x)k(x) - h(x)k''(x) - 2h'(x)k'(x)
\]

\[
\geq 2(p - 1)(p - 2)x^{2p-4} + 2(p - 1)^2x^{2p-4}
\]

\[
-2(p - 2)(p - 3)x^{2p-4}\frac{x - 2}{(x - 1)^2} - 2\left(p - 2\right)x^{p-2}\frac{x - 2}{(x - 1)^2} > 0,
\]

and similarly

\[
B''(x) = f''(x) + g''(x) - h''(x) - k''(x) \geq 2(p - 2)^2x^{p-3}\frac{x - 2}{(x - 1)^2} > 0.
\]

Hence

\[
\left|\frac{r''(x)}{r(x)}\right| \leq \frac{A''(x)}{A(x)} + \left(\frac{A'(x)}{A(x)}\right)^2 + \frac{B''(x)}{B(x)} + \left(\frac{B'(x)}{B(x)}\right)^2 + \left(\frac{r'(x)}{r(x)}\right)^2.
\]
For using Lemma 4.5 and the above inequalities, we obtain

\[
\left| \frac{A''(x)}{A(x)} \right| \leq \frac{f''(x)g(x) + f(x)g''(x) + 2f'(x)g'(x)}{A(x)} \leq \frac{2(p-1)(2p-3) \frac{x^{2p-2}}{(x-1)^2}}{A(x)} \
\leq \frac{2(p-1)(2p-3)}{x(x-2)} \leq \frac{2(p-1)(2p-3)}{(x-2)^2}.
\]

Similarly

\[
\left| \frac{B''(x)}{B(x)} \right| \leq \frac{(p-1)(p-2)}{(x-2)^2}.
\]

Using Lemma 4.5 and the above inequalities, we obtain

\[
\left| \frac{r''(x)}{r(x)} \right| \leq 19 \left( \frac{p-1}{x-2} \right)^2 = O(p^2x^{-2}).
\]

\[\Box\]

**Lemma 4.10.** For \( p \geq 3 \) and \( q \geq 7 \),

\[
\lambda = q - \frac{1}{r(q)} - \frac{r'(q)}{r(q)^3} + O \left( p^2q^{-3p+1} \right).
\]

**Proof.** First note that \( B(\lambda) \neq 0 \) since \( A(\lambda) \neq 0 \) by Lemma 4.2. Hence \((\lambda - q)A(\lambda) - B(\lambda) = 0\) implies

\[
r(\lambda) = \frac{A(\lambda)}{B(\lambda)} = \frac{1}{q-\lambda}.
\]

The Taylor’s formula gives

\[
\frac{1}{q-\lambda} = r(\lambda) = r(q) + (\lambda - q)r'(q) + \frac{1}{2}(\lambda - q)^2r''(u),
\]

for some \( u \) with \( \lambda \leq u \leq q \). Hence

\[
q - \lambda = \frac{1}{r(q)} \left( 1 + (\lambda - q) \frac{r'(q)}{r(q)} + \frac{1}{2}(\lambda - q)^2 \frac{r''(u)}{r(q)} + \ldots \right).
\]

For \( p \geq 3 \) and \( q \geq 7 \), using (\[\Box\]), we get

\[
\lambda > q - q^{-p+2} \geq 7 - 7^{-1} > 2(2 + \sqrt{2}).
\]

Thus by Lemma 4.8, \( r'(\lambda) > 0 \). Hence \( \frac{1}{q-\lambda} = r(\lambda) \leq r(q) \).

Also using Lemma 4.5

\[
\frac{r(u)}{r(q)} \leq \left( \frac{u}{q} \right)^{p+1} \frac{q-1}{q-2} \frac{q^2}{(u-1)(u-2)} \leq \frac{q-1}{q-2} \frac{q^2}{(u-1)(u-2)}.
\]

Now, using Lemma 4.5 since \( \frac{q}{2} \leq u \leq q \) and \( q \geq 7 \),

\[
\left| \frac{r''(u)}{r(q)} \right| = \left| \frac{r''(u)}{r(u)} \cdot \frac{r(u)}{r(q)} \right| \leq 19 \left( \frac{p-1}{u-2} \right)^2 \frac{q-1}{q-2} \frac{q^2}{(u-1)(u-2)} = O \left( p^2q^{-2} \right).
\]

Hence by Lemma 4.10 and Lemma 4.9

\[
(\lambda - q)^3 \frac{r''(u)}{r^2(q)} = O \left( p^2q^{-3p+1} \right), \quad (7)
\]
which gives
\[ q - \lambda = \frac{1}{r(q)} \left( 1 + (q - \lambda) \frac{r'(q)}{r(q)} \right) + O(p^2q^{-3p+4}). \]

Solving for \(\lambda\), we get
\[ \lambda = q - \frac{1}{r(q)} - \frac{r'(q)}{r(q)^3} + O(p^2q^{-3p+4}). \]

\[ \square \]

Remark 4.11. Using the bound on \(\frac{r'(q)}{r(q)}\), we get
\[ \lambda = q - \frac{1}{r(q)} + O(pq^{-2p+3}). \]  

(8)

Note that \(F(z)\) is a rational function with only one singularity \(\lambda\) in \(\{z : |z| \geq 4\}\), which is a simple pole. The residue of \(F(z)\) at \(\lambda\) is given by
\[ R_\lambda = \frac{\lambda A(\lambda)}{B'(\lambda) + (\lambda - q)A'(\lambda) + A(\lambda)} = \frac{\lambda}{1 - (q - \lambda)^2r'(\lambda)}. \]  

(9)

Consider
\[ E(z) := F(z) - \frac{R_\lambda}{z - \lambda}. \]  

(10)

Then \(E(z)\) is analytic on \(|z| \geq 4\), let
\[ E(z) = \sum_{k=0}^{\infty} e(k)z^{-k}. \]  

(11)

Lemma 4.12. For \(p \geq 3\) and \(q \geq 7\), \(e(k) = O(4^k)\).

Proof. For \(p \geq 3\), on \(|z| = 4\), using \([5]\), we get
\[ |1 + (z - q)r(z)| \geq |(z - q)r(z)| - 1 > \frac{1}{40}4^p - 1 \geq \frac{1}{120}4^p, \]

since \(\frac{4^p}{120} \geq 1\), and on \(|z| = 4\),
\[ |r(z)| \leq \frac{|f(z)||g(z)| + |h(z)||k(z)|}{|f(z) + g(z)| - |h(z)| - |k(z)|} \leq \frac{|z|^2 + 1}{2(|z| - 1)(|z| - 2)}|z|^{p-1} = c'4^{p-1}. \]

Hence on \(|z| = 4\), from \([4]\), \(|F(z)| \leq O(1)\). Also
\[ |R_\lambda| \leq \frac{\lambda}{1 - \left(\frac{q - \lambda}{r(\lambda)}\right)} = O(q), \]

and hence
\[ \frac{|R_\lambda|}{|z - \lambda|} = O(1), \]

on \(|z| = 4\). This implies that \(E(z) = O(1)\) on \(|z| = 4\).

Hence by the Cauchy’s integral formula,
\[ |e(k)| \leq \frac{1}{2\pi i} \int_{|z|=4} |z|^{k-1}|E(z)|dz = O(4^k). \]  

\[ \square \]
These lemmas lead to the following result which gives an explicit relationship between the escape rate $\rho(\mathcal{F})$, the number of symbols $q$, and the simple pole $\lambda$ obtained in Lemma 4.3. A similar result was obtained in [3] Lemma 4.3.1.

**Theorem 4.13.** For $p \geq 3$, $q \geq 7$, the escape rate into the hole consisting of cylinders based at words from $\mathcal{F}$ is given by

$$\rho(\mathcal{F}) = -\ln(\lambda/q).$$

**Proof.** Using (9), (10), and (11), we get

$$c_k \quad \text{Now we have}$$

$$\rho(\mathcal{F}) = -\ln(\lambda/q).$$

**Proof.** Using (9), (10), and (11), we get

$$f(k) = e(k) + R\lambda^{k-1} = e(k) + \frac{\lambda^k}{1 - (q - \lambda)^2r'(\lambda)}.$$ 

Hence Lemma 4.12 implies

$$f(k) = \frac{\lambda^k}{1 - (q - \lambda)^2r'(\lambda)} + O\left(4^k\right).$$

Thus there exists a positive constant $c$ such that

$$\frac{\lambda^k}{1 - (q - \lambda)^2r'(\lambda)} - c4^k \leq f(k) \leq \frac{\lambda^k}{1 - (q - \lambda)^2r'(\lambda)} + c4^k.$$ 

From Lemma 4.9 we have $(q - \lambda)^2r'(\lambda) \leq 3(q - \lambda)\frac{p-1}{\lambda-2} \leq 3\frac{p-1}{(\lambda-2)q^r} < 1$. Hence

$$\lambda^k - c4^k \leq f(k) \leq \frac{\lambda^k}{1 - 3\frac{p-1}{(\lambda-2)q^r}} + c4^k \leq c_2\lambda^k,$$

for some $c_2 > 0$, since $q \geq 7$ and $\frac{q}{2} \leq \lambda \leq q$. For the left-most inequality, choose $k_0 > 0$ such that $c\left(\frac{q}{2}\right)^k < 1/2$, for all $k \geq k_0$. Then $f(k) \geq \lambda^k\left(1 - c\left(\frac{q}{2}\right)^k\right) = c_3\lambda^k$, for some $c_3$ positive, for all $k \geq k_0$.

Now we have $c_1\lambda^k \leq f(k) \leq c_2\lambda^k$, for all $k \geq k_0$, and the result follows from Theorem 2.2.

**Proof.** (Proof of Theorem 4.2) Let us recall (8)

$$\lambda = q - \frac{1}{r(q)} + O\left(pq^{-2p+3}\right).$$

It is clear that $\lambda$ increases monotonically with $r(q)$. Hence if $r_{\mathcal{F}_1}(q) > r_{\mathcal{F}_2}(q)$, then $\lambda_1 > \lambda_2$. Hence by Theorem 4.6

$$\rho(\mathcal{F}_1) = -\ln(\lambda_1/q) < -\ln(\lambda_2/q) = \rho(\mathcal{F}_2),$$

which proves Theorem 3.2. 

The above results lead to the following straightforward corollary.

**Corollary 4.14.** For any $\epsilon > 0$ given, there exists a collection $\mathcal{F} = \{u, w\}$ of forbidden words of equal length such that $\rho(\mathcal{F}) < \epsilon$.

**Proof.** It follows from the fact that, by Lemma 4.6, for a given collection $\mathcal{F} = \{u, w\}$ with $|u| = |w| = p$, the largest root $\lambda$ root of $(z - q)A(z) + B(z)$ satisfies

$$q - q^{-p+2} \leq \lambda \leq q,$$

where $A(z)$ and $B(z)$ are as in (4). Hence $\lambda \rightarrow q$ as $p \rightarrow \infty$. 

□
4.2. Relationship between the escape rate and minimal period of the hole. We recall the next result from [3] which gives the relationship between the minimal period \(\tau_F\) and the escape rate \(\rho(F)\).

**Theorem 4.15.** [3, Theorem 4.0.8] For \(i = 1, 2\), let \(F_i = \{u_i\}\). Suppose that \(\tau_{F_1} < \tau_{F_2}\). Then \(\rho(F_1) < \rho(F_2)\).

This result is immediate from Theorem 4.11 since \(\tau_{F_1} < \tau_{F_2}\) implies \((u_1u_1)_q > (u_2u_2)_q\), which in turn implies \(\rho(F_1) < \rho(F_2)\). Theorem 3.4 generalizes this result in our setup, that is, when holes are union of two cylinders with an extra assumption that the cross-correlation polynomials between the forbidden words are zero. Later we prove that this assumption is necessary.

**Proof.** (Proof of Theorem 3.4) Let \((u_1u_i)_z = z^{p-1} + z^{p-1-\tau_{u_i}} + R_{u_i}(z)\), and \((w_iw_i)_z = z^{p-1} + z^{p-1-\tau_{w_i}} + R_{w_i}(z)\), where \(R_{u_i}, R_{w_i}\) are the remainder terms in the polynomial with degree less than \(p - 1 - \tau_{u_i}, p - 1 - \tau_{w_i}\), respectively, for \(i = 1, 2\). Since all the cross-correlation polynomials between the words are zero, we have

\[
\frac{1}{r_{F_1}(z)} = \frac{1}{(u_1u_1)_z} + \frac{1}{(u_2u_2)_z}, \quad \text{and} \quad \frac{1}{r_{F_2}(z)} = \frac{1}{(w_1w_1)_z} + \frac{1}{(w_2w_2)_z}.
\]

Without loss of generality, we assume that \(\tau_{u_1} \leq \tau_{u_2}\) and \(\tau_{w_1} \leq \tau_{w_2}\) and hence \(\tau_{F_0} = \tau_{u_1}\) and \(\tau_{F_2} = \tau_{w_1}\). Since \(\tau_{u_1} \leq \tau_{w_2}\), we have \(\frac{1}{(w_1w_1)_q} \leq \frac{1}{(u_2u_2)_q}\). Hence

\[
\frac{1}{r_{F_2}(q)} - \frac{1}{r_{F_1}(q)} \geq \frac{1}{(w_1w_1)_q} + \frac{1}{(w_2w_2)_q} - \frac{1}{(u_1u_1)_q} - \frac{1}{(u_2u_2)_q}.
\]

Since \(\tau_{u_1} < \tau_{w_1}\), we have \(1 \leq \tau_{u_1} \leq p - 1\). We will consider the following cases.

1) When \(1 \leq \tau_{w_1}, \tau_{w_2} \leq p - 1\). Then \((u_1u_1)_q \geq q^{p-1} + q^{p-1-\tau_{u_1}}(u_2u_2)_q \geq q^{p-1} + q^{p-1-\tau_{u_2}}\), and \((w_1w_1)_q \leq q^{p-1} + q^{p-1-\tau_{w_1}} + q^{p-1-(\tau_{w_1}+1)} + \ldots + 1\). Hence

\[
\frac{1}{r_{F_2}(q)} - \frac{1}{r_{F_1}(q)} \geq \frac{2}{q^{p-1} + q^{p-1-\tau_{w_1}} + q^{p-1-(\tau_{w_1}+1)} + \ldots + 1} - \frac{1}{q^{p-1} + q^{p-1-\tau_{u_1}}} > 0,
\]

if and only if

\[
q^{2(p-1)-\tau_{u_1}}(1 + q^{-(\tau_{w_2} - \tau_{u_1})} + 2q^{-\tau_{u_1}}) > 0.
\]

Using the assumption that \(\tau_{u_1} \leq \tau_{w_1} - 1\), we get

\[
q^{2(p-1)-\tau_{u_1}} + q^{2(p-1)-(\tau_{w_1}+1)} + \ldots + q^{p-1} < \frac{q^{2(p-1)-(\tau_{u_1}-1)}}{q-1} \leq \frac{q^{2(p-1)-\tau_{u_1}}}{q-1}.
\]

Thus we have \(12\) if

\[
(q - 1)(1 + q^{-(\tau_{u_2} - \tau_{u_1})} + 2q^{-\tau_{u_1}}) > (2 + q^{-\tau_{u_1}} + q^{-\tau_{u_2}})
\]
and this is true for $q \geq 4$.

2) When $\tau_{u_2} = p$. Hence $(u_2u_2)_q = q^{p-1}$. Repeating the same calculations as in Case 1, we obtain

$$\frac{1}{r_{F_2}(q)} - \frac{1}{r_{F_1}(q)} \geq 0$$

if and only if $q - 1 \geq 2 + q^{-\tau_{u_1}} \geq 0$ and it is true for $q \geq 4$.

3) When $\tau_{u_1} = p$. In this case we have $(w_1w_1)_q = q^{p-1}$. Then it is immediate by repeating same calculations as in Case 1.

$\square$

**Remark 4.16.**

1) Using Theorems 3.2 and 3.4 for $q \geq 7$, under the hypothesis of Theorem 3.4, $\rho(F_1) < \rho(F_2)$.

2) Consider the collections $F_1 = \{u_1 = aaaa, u_2 = bbbb\}$ and $F_2 = \{w_1 = aaaa, w_2 = bcbbc\}$. Here $(u_1w_1)_z = (w_1u_1)_z = (u_2w_2)_z = 0$, $r_{F_1} = r_{F_2}$ but $r_{F_1}(q) > r_{F_2}(q)$ for any $q \geq 2$. Hence $\rho(F_1) < \rho(F_2)$ for $q \geq 7$.

3) Theorem 3.4 fails to hold when the cross-correlation polynomial between the words is non-zero. See Table 2 where the length of the words $p = 3$. For $F_1 = \{u_1 = abc, u_2 = bcd\}$ and $F_2 = \{w_1 = abc, w_2 = ddd\}$, we have $(u_1w_1)_z \neq 0$, $\tau_{F_1} = 3$ and $\tau_{F_2} = 1$ but $\rho(F_1) < \rho(F_2)$.

4) Theorem 3.4 does not hold for $q = 2$. For example, consider $F_1 = \{u_1 = 10111011, u_2 = 01001000\}$ and $F_2 = \{w_1 = 11100111, w_2 = 00011000\}$. Here $(u_1u_1)_z = (w_1w_1)_z = 0$ for $i \neq j$ and $\tau_{F_1} = 4 < 5 = \tau_{F_2}$. But $r_{F_1}(2) < r_{F_2}(2)$.

In the next result, we show that the conclusion of Theorem 3.4 holds true for $q = 3$ and $p \geq 2$. The proof given in Theorem 3.4 does not work for the case $q = 3$. Thus we give a separate proof for this case.

**Theorem 4.17.** The conclusion of Theorem 3.4 holds true for $q = 3$ and $p \geq 2$.

**Proof.** From the arguments presented in the proof of Theorem 3.4, it is enough to show that

$$\frac{2}{(w_1w_1)_3} - \frac{1}{(u_1u_1)_3} - \frac{1}{(u_2u_2)_3} > 0.$$ 

It is easy to see that the result holds true for the case when $1 \leq \tau_{u_1} \leq \tau_{u_2} \leq \tau_{w_1} \leq p$. Hence we assume that $1 \leq \tau_{u_1} < \tau_{w_1} \leq \tau_{u_2} \leq p$. Note that

$$\begin{align*}
(w_1w_1)_3 & \leq 3^{p-1} + 3^{p-1-\tau_{w_1}} + \ldots + 3^{p-1-k_w\tau_{w_1}} + 3^{p-1-(k_w\tau_{w_1}+1)} + \ldots + 1, \\
(u_1u_1)_3 & \geq 3^{p-1} + 3^{p-1-\tau_{u_1}} + 3^{p-1-2\tau_{u_1}} + \ldots + 3^{p-1-k_u\tau_{u_1}},
\end{align*}$$

where $k_w = \left\lfloor \frac{p-1}{\tau_{w}} \right\rfloor - 1, \left\lfloor \frac{p-1}{\tau_{w}} \right\rfloor$ is an integer, for $w = w_1, u_1, u_2$. Thus

$$\begin{align*}
\frac{2}{(w_1w_1)_3} - \frac{1}{(u_1u_1)_3} - \frac{1}{(u_2u_2)_3} & \geq f(p, \tau_{u_1}, \tau_{w_1}, \tau_{u_2}),
\end{align*}$$

where $f(p, \tau_{u_1}, \tau_{w_1}, \tau_{u_2})$ is a function...
where

\[ f(p, \tau_{u_1}, \tau_{w_1}, \tau_{u_2}) = \frac{2}{\sum_{j=0}^{k_{u_1}} 3^{p-1-j \tau_{w_1}} + \sum_{j=0}^{p-k_{u_1} \tau_{w_1}-2} 3^j} - \frac{1}{\sum_{j=0}^{k_{u_1}} 3^{p-1-j \tau_{w_1}} - \sum_{j=0}^{k_{u_2}} 3^{p-1-j \tau_{w_2}}} \]

Hence it is enough to show that \( f(p, \tau_{u_1}, \tau_{w_1}, \tau_{u_2}) > 0 \).

Note that \( f(p, \tau_{u_1}, \tau_{w_1}, \tau_{u_2}) > f(p, \tau_{u_1}, \tau_{u_1} + 1, \tau_{u_2}) \), and \( f(p, \tau_{u_1}, \tau_{u_1} + 1, \tau_{u_2}) \) is decreasing with respect to \( \tau_{u_2} \). Hence \( f(p, \tau_{u_1}, \tau_{u_1} + 1, \tau_{u_2}) > 0 \), if \( f(p, \tau_{u_1}, \tau_{u_1} + 1, p) > 0 \), for all \( 1 \leq \tau_{u_1} \leq p - 1 \).

Set \( t_1 = \tau_{u_1}, k_1 = k_{u_1}, k_2 = k_{w_1} \), where \( \tau_{w_1} = \tau_{u_1} + 1 \). We will consider the following cases.

1) When \( k_1, k_2 > 1 \),

\[ 3^{p-1} f(p, \tau_{u_1}, \tau_{u_1} + 1, p) > \frac{2}{1 - 3 - (k_2+1)(t_1+1)} - \frac{1 - 3^{-t_1}}{1 - 3^{t_1} - 1} > 0, \]

if

\[ (-3 + 3^{t_1+1}) + (3^{-k_1 t_1+1} + 3^{k_1 t_1+t_1+2}) + (3^{k_1 t_1+k_2 t_1+k_2+t_1} - 2.3^{(k_2+1)(t_1+1)}) \]

\[ + (3^{k_1 t_1+k_2 t_1+k_2+t_1} - 2.3^{(k_1+2)t_1+1}) + 4.3^{k_2 t_1+k_2} > 0, \]

which is true since \( k_1, k_2 > 1 \).

2) When \( k_2 = 1, k_1 > 1 \),

\[ 3^{p-1} f(p, \tau_{u_1}, \tau_{u_1} + 1, p) > \frac{4}{2 + 3^{-t_1}} - \frac{1 - 3^{-t_1}}{1 - 3^{t_1} - 1} > 0 \]

if \( 3^{k_1 t_1} - 2.3^{t_1} + 1 > 0 \) which is true since \( k_1 > 1 \).

3) When \( k_1 = 1 \), we have \( k_2 = 1 \) since \( 1 \leq k_2 \leq k_1 \). In this case,

\[ f(p, \tau_{u_1}, \tau_{u_1} + 1, p) > \frac{2}{3^{p-1} + 3^{p-1-t_1} - 1} - \frac{1}{3^{p-1} - 3^{p-1-t_1} + 1} > 0, \]

if \( 2.3^{2 t_1} + 3^{t_1} - 3^{p-1} > 0 \). This inequality holds true since \( k_1 = 1 \) implies \( 2t_1 > p - 1 \).

5. Full shift with union of more than two cylinders as hole

In this section, we generalize the results obtained in the previous section to the case of full shift when the holes are union of more than two cylinders (in other words, holes corresponding to more than two forbidden words). In Section 5.1, we explore the relationship between the escape rate and \( r(q) \), and the relationship between the escape rate and the minimal period of the hole is discussed in Section 5.2.

5.1. Relationship between the escape rate and \( r(q) \). In this section, we give a generalization of Theorem 4.1 in this setup. In view of Remark 2.5 (1), Remark 4.3 and Remark 4.7, with similar computations as in Section 4.1, we obtain Theorem 4.5.
Proof. (Sketch of Proof of Theorem 3.4) Let \( F \) be a given collection of forbidden words and \( r(z) = \frac{A(z)}{B(z)} \) be the corresponding rational function obtained from Theorem 2.4. Then by Remark 2.4, \( A(z) \) is a monic polynomial of degree \( s(p-1) \) and \( B(z) \) is a polynomial of degree \( (s-1)(p-1) \) with coefficient of \( z^{(s-1)(p-1)} \) equal to \( s \). Let \( a, b \) be as in Remark 4.4.

Step 1: Imitate Lemma 4.2, Lemma 4.3, and Lemma 4.6, using Remark 4.4, for some specific properties of associated polynomials.

Proof. (Proof of Theorem 3.6) Assume that \( 1 \leq \tau_{w_i}, \tau_{w_i} < p, i = 1, 2, \ldots, \ell \). Let \( (w_iu_i)_z = z^{p-1} + z^{p-1-\tau_{w_i}} + R_{w_i}(z), (w_iw_i)_z = z^{p-1} + z^{p-1-\tau_{w_i}} + R_{w_i}(z) \), where \( R_{w_i}, R_{w_i} \) are the reminder terms in the polynomial with degree less than \( \tau_{w_i} \).

Step 2: The following inequalities hold true:

\[
\begin{align*}
1 - \frac{a}{q-1} &\leq A(q) = \frac{a}{q^{s(p-1)}} \leq \left(1 + \frac{a}{q-1}\right), \\
(\frac{s-b}{q-1}) &\leq B(q) = \frac{b}{q^{s(p-1)(p-1)}} \leq \left(s + \frac{b}{q-1}\right), \\
1 - \frac{a}{q-1} &\leq \frac{A'(q)}{s(p-1)q^{s(p-1)-1}} \leq \left(1 + \frac{a}{q-1}\right), \\
\frac{s-b}{q-1} &\leq \frac{B'(q)}{(s-1)(p-1)q^{(s-1)(p-1)-1}} \leq \left(s + \frac{b}{q-1}\right), \\
1 - \frac{a}{q-1} &\leq \frac{A''(q)}{s(p-1)(s(p-1)-1)q^{(s-1)(p-1)-2}} \leq \left(1 + \frac{a}{q-1}\right), \\
\frac{s-b}{q-1} &\leq \frac{B''(q)}{(s-1)(p-1)((s-1)(p-1)-1)q^{(s-1)(p-1)-2}} \leq \left(s + \frac{b}{q-1}\right).
\end{align*}
\]

Step 3: Note that \( r(q) > 0 \) for \( q > a + b + 2 \). Also \( r'(q) > 0 \) if and only if \( A'(q)B(q) - A(q)B'(q) > 0 \) which is true when

\[
s(p-1) \left(1 - \frac{a}{q-1}\right) \left(s - \frac{b}{q-1}\right) \geq (s-1)(p-1) \left(1 + \frac{a}{q-1}\right) \left(s + \frac{b}{q-1}\right).
\]

Reducing this inequality, we get that \( r'(q) > 0 \) if \( q > \frac{1}{s}(sa+b)(2s-1) + 1 \). Define \( C := \frac{1}{s}(sa+b)(2s-1) + 1 \).

Step 4: Imitate the proofs of Lemmas 4.9, 4.10, 4.12, and Theorem 3.3, using the bounds in Step 2. \( \square \)

5.2. Relationship between the escape rate and minimal period of the hole. In this section, we generalize Theorem 3.4 when the collection of forbidden words can have more than two words, say \( \ell \geq 2 \). We will obtain a lower bound on \( q \) in terms of \( \ell \) which is worse than the one obtained in Theorem 3.3 for the case \( \ell = 2 \), since in Theorem 3.4 we were able to use some specific properties of associated polynomials.

Proof. (Proof of Theorem 5.6) Assume that \( 1 \leq \tau_{w_i}, \tau_{w_i} < p, i = 1, 2, \ldots, \ell \). Let \( (u_iu_i)_z = z^{p-1} + z^{p-1-\tau_{w_i}} + R_{w_i}(z), (w_iw_i)_z = z^{p-1} + z^{p-1-\tau_{w_i}} + R_{w_i}(z) \), where \( R_{w_i}, R_{w_i} \) are the reminder terms in the polynomial with degree less...
Hence inequality (13) holds if and only if
\[
\frac{1}{r_{F_1}(z)} = \frac{1}{(u_1 u_1)_z} + \frac{1}{(u_2 u_2)_z} + \cdots + \frac{1}{(u_\ell u_\ell)_z},
\]
\[
\frac{1}{r_{F_2}(z)} = \frac{1}{(w_1 w_1)_z} + \frac{1}{(w_2 w_2)_z} + \cdots + \frac{1}{(w_\ell w_\ell)_z}.
\]

Without loss of generality, we assume that \( \tau_{u_1} \leq \tau_{u_2} \leq \cdots \leq \tau_{u_\ell} \) and \( \tau_{w_1} \leq \tau_{w_2} \leq \cdots \leq \tau_{w_\ell} \). Since \( \tau_{F_1} < \tau_{F_2}, \tau_{u_1} < \tau_{u_1} \),
\[
\frac{1}{r_{F_2}(q)} - \frac{1}{r_{F_1}(q)} = \sum_{i=1}^{\ell} \frac{1}{(w_i w_i)_q} - \sum_{i=1}^{\ell} \frac{1}{(u_i u_i)_q} \geq \frac{\ell}{(w_1 w_1)_q} - \sum_{i=1}^{\ell} \frac{1}{q^{\tau_{u_i}}} \geq \frac{\ell}{q^{\tau_{w_1}} + \sum_{j=0}^{\ell-1} q} - \sum_{i=1}^{\ell} \frac{1}{q^{\tau_{u_i}}}.
\]
The above inequality is positive if and only if
\[
\ell \left( \prod_{j=1}^{\ell} \left( q^{\tau_{w_1}} + q^{\tau_{u_i}} \right) \right) > \left( q^{\tau_{w_1}} + \sum_{j=0}^{\ell-1} q \right) \left( \sum_{i=1}^{\ell} \left( \prod_{j \neq i} q^{\tau_{u_j}} \right) \right),
\]
if and only if
\[
\ell q^{(p-1)} \prod_{j=1}^{\ell} (1 + q^{-\tau_{u_j}}) > q^{(p-1)} \left( 1 + \frac{q^{-\tau_{u_1}}}{q-1} \right) \left( \sum_{i=1}^{\ell} \left( \prod_{j \neq i} (1 + q^{-\tau_{u_j}}) \right) \right),
\]
since \( \tau_{u_1} \leq \tau_{w_1} - 1 \). Now let
\[
\beta_1 = q^{-\tau_{u_1}} + q^{-\tau_{u_2}} + \cdots + q^{-\tau_{u_\ell}},
\]
\[
\beta_2 = q^{-\tau_{u_1}} q^{-\tau_{u_2}} + q^{-\tau_{u_1}} q^{-\tau_{u_3}} + \cdots + q^{-\tau_{u_{\ell-1}}} q^{-\tau_{u_\ell}},
\]
\[
\vdots
\]
\[
\beta_\ell = q^{-\tau_{u_1}} q^{-\tau_{u_2}} \cdots q^{-\tau_{u_\ell}}.
\]
Then \( \prod_{j=1}^{\ell} (1 + q^{-\tau_{u_j}}) = 1 + \beta_1 + \beta_2 + \cdots + \beta_\ell \) and \( \sum_{i=1}^{\ell} \left( \prod_{j \neq i} (1 + q^{-\tau_{u_j}}) \right) \) = \( \ell + (\ell - 1) \beta_1 + (\ell - 2) \beta_2 + \cdots + \beta_{\ell-1} \).

Hence inequality (13) holds if
\[
\ell (q-1) \left( 1 + \beta_1 + \beta_2 + \cdots + \beta_\ell \right)
- (q-1 + q^{-\tau_{u_1}}) (\ell + (\ell - 1) \beta_1 + (\ell - 2) \beta_2 + \cdots + \beta_{\ell-1})
= (q-1) (\beta_1 + 2 \beta_2 + \cdots + \ell \beta_\ell) - q^{-\tau_{u_1}} (\ell + (\ell - 1) \beta_1 + \cdots + \beta_{\ell-1})
\geq 0.
\]
Since \( \tau_{u_1} \leq \tau_{u_2} \leq \cdots \leq \tau_{u_\ell} \), we have

\[
\beta_1 < \beta_1 q^{\tau_{u_1}} = 1 + q^{-(\tau_{u_2} - \tau_{u_1})} + \cdots + q^{-(\tau_{u_\ell} - \tau_{u_1})} \leq \ell
\]

\[
\beta_2 < \beta_2 q^{\tau_{u_1}} = q^{-\tau_{u_2}} + \cdots + q^{-(\tau_{u_\ell} + \tau_{u_2} - \tau_{u_1})} + \cdots + q^{-(\tau_{u_\ell - 1} + \tau_{u_\ell} - \tau_{u_1})} \leq \left( \frac{\ell}{2} \right) \frac{1}{q}
\]

\[
\vdots
\]

\[
\beta_k < \beta_k q^{\tau_{u_1}} \leq \left( \frac{\ell}{k} \right) \frac{1}{q^{k-1}}
\]

for \( k = 1, 2, \ldots, \ell \). Now

\[
(q - 1)q^{\tau_{u_1}}(\beta_1 + 2\beta_2 + \cdots + \ell\beta_\ell) - (\ell + (\ell - 1)\beta_1 + \cdots + \beta_{\ell-1}) > q - 1 - \left( \ell + \sum_{k=1}^{\ell-1}(\ell - k)\frac{1}{q^{k-1}}\frac{\ell}{k} \right) = q - 1 - \ell - \ell q \left( 1 + \frac{1}{q} \ell - 1 \right) \geq 0,
\]

which is true by the assumption. Also, the inequality holds for \( q \geq 2\ell^{-1} + 1 \) since \( \frac{1}{q_{\tau_{u_1}}} \leq 1 \) for any \( 1 \leq k \leq \ell - 1 \).

For the case when \( \tau_{u_1} = p \), the calculations become easier and the result holds.

Now we consider the final case where \( 1 \leq \tau_{u_1}, \tau_{u_2}, \ldots, \tau_{u_s} < p \) and \( \tau_{u_{s+1}} = \cdots = \tau_{u_\ell} = p \) for some \( 1 \leq s \leq \ell \). Repeating the same calculations as before, the result holds if

\[
q - 1 - \left( \ell + \sum_{k=1}^{s}(\ell - k)\frac{1}{q^{k-1}}\frac{\ell}{k} \right) > q - 1 - \left( \ell + \sum_{k=1}^{\ell-1}(\ell - k)\frac{1}{q^{k-1}}\frac{\ell}{k} \right) \geq 0.
\]

\[ \square \]

Remark 5.1. Using Theorems 3.5 and 3.6 for \( q \geq C \) satisfying the inequality \( 3 \) and \( p \geq D \), under the hypothesis of Theorem 3.6 \( \rho(\mathcal{F}_1) < \rho(\mathcal{F}_2) \).

6. Subshift of finite type

In this section, we will extend the results presented in the earlier sections for the shift map on a subshift of finite type \( \Sigma_{\mathcal{F}} \). We first consider the case when \( \mathcal{F} = \{ w \} \), and compare escape rates for two holes in \( \Sigma_{\mathcal{F}} \) which correspond to the collections of forbidden words \( \mathcal{F}_1 = \{ w_1 \} \) and \( \mathcal{F}_2 = \{ w_2 \} \), with \( |w| = |w_1| = |w_2| = p \). It follows from Theorems 5.1 and 5.2 that if \( p = 2 \) and \( q \geq 2 \), or if \( p \geq 3 \) and \( q \geq 7 \), then \( r_{\mathcal{F}_1 \cup \mathcal{F}_2}(q) > r_{\mathcal{F}_1}(q) \) implies \( \rho(\mathcal{F}_2; \mathcal{F}) > \rho(\mathcal{F}_1; \mathcal{F}) \), where

\[
r_{\mathcal{F}_1 \cup \mathcal{F}_2}(z) = \frac{(ww)_z(w_1w_1)_z - (ww_1)_z(w_1w)_z}{(ww)_z + (w_1w)_z - (ww_1)_z - (w_1w)_z},
\]

\[ i = 1, 2. \] This easily follows from Theorem 2.2 since

\[
\rho(\mathcal{F}_2; \Sigma_{\mathcal{F}}) = h_{\text{top}}(\Sigma_{\mathcal{F}}) - h_{\text{top}}(\Sigma_{\mathcal{F} \cup \mathcal{F}_2}) = \rho(\mathcal{F} \cup \mathcal{F}_2) - \rho(\mathcal{F}) > \rho(\mathcal{F} \cup \mathcal{F}_1) - \rho(\mathcal{F}) = \rho(\mathcal{F}_2; \Sigma_{\mathcal{F}}).
\]
This can be seen using Tables 3, 4, 5, and 6. Note that the cases of collections with different $r$ values that we consider here are similar to cases considered in Tables 1 and 2 since escape rate depends only on the value of $r_{F_1 \cup F_2}(q)$. Interesting thing to note here is that the holes with same measure can have different escape rates and holes with different measures can have the same escape rate. For example, in Table 3, $q = 3$, $F = \{00\}$, $F_1 = \{w_1 = 11\}$, $F_2 = \{w_2 = 01\}$, and $F_3 = \{w_3 = 12\}$. Here $\mu(C_{w_1}) = \mu(C_{w_3})$, but $\rho(F_1; F) \neq \rho(F_3; F)$. Moreover $\mu(C_{w_1}) \neq \mu(C_{w_2})$, but $\rho(F_1; F) = \rho(F_2; F)$. Here $\mu$ denotes the Parry measure on $\Sigma_F$.

| $q$ | $r_{F_1 \cup F_2}(q)$ | $\rho(F_1; F)$ | $r_{F_1 \cup F_2}(q)$ | $\rho(F_2; F)$ |
|----|-------------------|-----------------|-------------------|-----------------|
| 2  | 1.5               | 0.4812          | ...              | ...             |
| 3  | 2.0               | 0.1237          | 1.71             | 0.1955          |
| 4  | 2.5               | 0.0625          | 2.22             | 0.0826          |
| 5  | 3.0               | 0.0386          | 2.73             | 0.0475          |
| 6  | 3.5               | 0.0264          | 3.23             | 0.0313          |
| 7  | 4                 | 0.0193          | 3.73             | 0.0222          |
| 8  | 4.5               | 0.0147          | 4.23             | 0.0167          |
| 9  | 5.0               | 0.0117          | 4.74             | 0.0130          |
| 10 | 5.5               | 0.0095          | 5.24             | 0.0104          |

Table 4. Escape rates for different values of $q$ with $F = \{aa\}; F_1 = \{bb\}, \{ab\}, \{ba\}; and F_2 = \{bc\}$.

| $q$ | $r_1$ | $\rho_1$ | $r_2$ | $\rho_2$ | $r_3$ | $\rho_3$ | $r_4$ | $\rho_4$ |
|----|-------|----------|-------|----------|-------|----------|-------|----------|
| 3  | 2.0   | 0.0810   | 1.8   | 0.1188   | 1.71  | 0.1528   | 1.5   | 0.2693   |
| 4  | 2.5   | 0.0468   | 2.29  | 0.0609   | 2.22  | 0.0668   | 2.0   | 0.0890   |
| 5  | 3.0   | 0.0308   | 2.78  | 0.0378   | 2.73  | 0.0398   | 2.5   | 0.0491   |
| 6  | 3.5   | 0.0220   | 3.27  | 0.0260   | 3.23  | 0.0269   | 3.0   | 0.0318   |
| 7  | 4     | 0.0165   | 3.77  | 0.0191   | 3.73  | 0.0195   | 3.5   | 0.0225   |
| 8  | 4.5   | 0.0129   | 4.27  | 0.0146   | 4.23  | 0.0149   | 4     | 0.0168   |
| 9  | 5.0   | 0.0104   | 4.76  | 0.0116   | 4.74  | 0.0117   | 4.5   | 0.0131   |
| 10 | 5.5   | 0.0085   | 5.26  | 0.0094   | 5.24  | 0.0095   | 5.0   | 0.0105   |

The next result is a straightforward consequence of the preceding discussion.

**Theorem 6.1.** Let $F = \{w = a_1a_2 \ldots a_p\}$ and $\mathcal{F}$ be the collection of all words $u$ of length $p$ such that $(wu)_z = (uw)_z = 0$. Let $a, b \in \Lambda$ be distinct symbols in $\Lambda \setminus \{a_1, a_p\}$. Let $F_u = \{u\}$, for all $u \in \mathcal{G}$. Then

$$\min_{u \in \mathcal{G}}(\rho(F_u; F)) = \rho(F_{u_0}; F),$$
Table 5. Escape rates for different values of $q$ with $F = \{aaa\}; F_1 = \{bbb\}, \{aab\}, \{baa\}; F_2 = \{aba\}; F_3 = \{bcb\}; F_4 = \{abb\}, \{bba\}, \{abc\}, \{bca\}; \text{ and } F_5 = \{bcd\}, r_i = r_{F \cup F_i}(q), \rho_i = \rho(F_i; F), i = 1, \ldots, 5.$

| $q$ | $r_1$ | $\rho_1$ | $r_2$ | $\rho_2$ | $r_3$ | $\rho_3$ |
|-----|-------|----------|-------|----------|-------|----------|
| 3   | 6.5   | 0.0308   | 6.14  | 0.0345   | 5.65  | 0.0414   |
| 4   | 10.5  | 0.0129   | 9.89  | 0.0145   | 9.39  | 0.0162   |
| 5   | 15.5  | 0.00675  | 14.64 | 0.00756  | 14.14 | 0.00809  |
| 6   | 21.5  | 0.00398  | 20.38 | 0.00834  | 19.89 | 0.00464  |
| 7   | 28.5  | 0.00255  | 27.13 | 0.00281  | 26.64 | 0.00291  |
| 8   | 36.5  | 0.00173  | 34.88 | 0.00189  | 34.38 | 0.00194  |
| 9   | 45.5  | 0.00097  | 43.38 | 0.00134  | 43.13 | 0.00136  |
| 10  | 55.5  | 0.000907 | 53.38 | 0.000979 | 52.88 | 0.000997 |

Table 6. Escape rates for different values of $q$ with $F = \{abc\}, r = r_{F \cup F_1}(q), \rho = \rho(F_1; F)$.

| $F_1$ | $q = 3$ | $q = 4$ | $q = 5$ |
|-------|---------|---------|---------|
| $\{bcb\}$ | 5.6250 | 0.0280 | 9.3793 | 0.0122 | 14.1304 | 0.00648 |
| $\{ccc\}/\{bca\}$ | 5.5714 | 0.029 | 9.3333 | 0.0123 | 14.099 | 0.00653 |
| $\{bcb\}/\{bcc\}$ | 5.4000 | 0.039 | 9.1429 | 0.0130 | 13.8889 | 0.00676 |
| $\{bcd\}/\{bcb\}$ | 5.3182 | 0.0336 | 9.0811 | 0.0132 | 13.8393 | 0.00681 |
| $\{cde\}/\{cba\}$ | 5.0000 | 0.039 | 8.5000 | 0.0154 | 13.000 | 0.0078 |
| $\{cda\}$ | 4.7647 | 0.0446 | 8.2581 | 0.0165 | 12.7551 | 0.00815 |
| $\{ded\}/\{cbb\}$ | 5.0000 | 0.039 | 8.0000 | 0.0177 | 12.5000 | 0.00852 |
| $\{def\}/\{dcb\}$ | 5.0000 | 0.039 | 8.0000 | 0.0177 | 12.5000 | 0.00852 |
| $\{dee\}$ | 8.0000 | 0.0177 | 12.5000 | 0.00852 |

where $u_0 = \overline{a}$, and

$$\max_{u \in \mathcal{G}} (\rho(F_u; F)) = \rho(F_u; F),$$

where $u_1 = ab$.

Proof. Substituting $(uu)_z = (wv)_z = 0$ in Equation $14$, we get $r(z) = \frac{1}{(wv)_z + (uv)_z}$. Note that $r(q)$ is maximum when $(uu)_z = z^{p-1} + z^{p-2} + \cdots +
1 and minimum when \((uu)z = z^{p-1}\). Since \(a, b \notin \{a_1, a_p\}\), \(u_0, u_1 \in G\), \((u_0u_0)z = z^{p-1} + z^{p-2} + \cdots + 1\), and \((u_1u_1)z = z^{p-1}\). \(\square\)

Remark 6.2. Since the escape rate is invariant under conjugacy, all the results stated in this paper can be applied to the maps that are conjugate to a subshift of finite type with holes corresponding to the union of cylinders. Examples for such maps are given in [15].

The next theorem extends Theorem 3.5 when underlying space is a subshift of finite type \(\Sigma_F\), for some collection \(F\) with words of length \(p\) with symbols from \(\Lambda\).

**Theorem 6.3.** Consider \(\Sigma_F\). Suppose \(F_1\) and \(F_2\) are finite collections of words each of length \(p\) with symbols from \(\Lambda\) and each having \(s\) words. Assume that \(F \cap F_i = \emptyset\), for \(i = 1, 2\). Then there exist positive constants \(C(s, r_{F_1}, r_{F_2})\) and \(D(s, r_{F_1}, r_{F_2})\) such that for any \(q \geq C\) and \(p \geq D\), if \(r_{F_1} > r_{F_2}(q)\), then \(\rho(F_2) > \rho(F_1)\).

Remark 6.4. Under the hypothesis of Theorem 6.3 for the collection of forbidden words \(F \cup F_i\), \(i = 1, 2\), using Theorems 3.6 and 6.3, for \(p \geq D\), and for \(q \geq C\) satisfying the inequality \([\text{?}]\), we obtain \(\rho(F_1) < \rho(F_2)\).

7. Concluding remarks and future directions

In this paper, we considered the shift map on a subshift of finite type with \(q\) symbols and compared the escape rates into holes which are unions of same number of cylinders based at words of identical length.

In Section 4, we considered the full shift space and the hole which is a union of two cylinders based at words of identical length \(p\). With certain assumptions on \(p\) and \(q\), we showed that the escape rate depends on the pole, say \(\lambda\) of \(F(z)\), which is real and largest in modulus. We proved that the escape rate is in fact equal to \(\ln(q/\lambda)\), and is the maximum for the hole with the minimum value of the corresponding rational function \(r(z)\) evaluated at \(z = q\). In Theorem 5.1 we proved that when the forbidden words have zero cross-correlation, then the hole with larger minimal period has the larger escape rate. It is interesting to note that Theorem 5.1 does not hold true if the words have non-zero cross-correlation. In Section 5 we give generalization of the results obtained in Section 4 to the full shift with holes corresponding to more than two forbidden words. In Section 6, we consider subshift of finite type and see that the corresponding results in this case can easily be derived from results discussed in Sections 4 and 5.

This work leads to several interesting problems. One could compare the escape rate into holes where the space has general Markov measure, and not necessarily the Parry measure, and explore the relationship between the escape rate and the topological entropy. We saw here that the escape rate into a hole is equal to \(\ln(q/\lambda)\), where \(\lambda \in (q - 1, q)\) is the simple pole of \(F(z)\) with largest modulus. One could attempt to see how the escape rate changes with \(q\). The numerics presented in the tables suggest that the escape rate decreases with increasing \(q\), which is expected since the size of the hole reduces. If this is true, we would like to know at what rate the escape rate decays. The numerics presented in the tables also suggest that
Theorem 3.2 holds true for all values of $p \geq 2$ and $q \geq 2$. Our proof is not helpful in this regard. Another interesting question is when the cross-correlation between the forbidden words is non-zero, what does the rational function $r(z)$ represent? It is certain from Remark 4.16 that it does not depend on the minimal period of the hole. A question of general interest is to explore other factors that influence the escape rate into the hole other than the length, number of the corresponding forbidden words, and the minimum period of the hole.

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