Some classes of Trivially Perfect Graphs

G. R. Gandal\textsuperscript{1}, R. Mary Jeya Jothi\textsuperscript{2}

\textsuperscript{1}Research Scholar, Department of Mathematics, Sathyabama Institute of Science and Technology, Chennai.
\textsuperscript{2}Associate Professor, Department of Mathematics, Sathyabama Institute of Science and Technology, Chennai.

e-mail id – ganesh.gandal

Abstract. A graph G is supposed to be trivially perfect if, in each induced subgraph H of G, the number of maximal cliques in H equivalents to the size of a maximum independent set in H. Trivially perfect graphs is subclasses of notable perfect graphs and its characterization have numerous continuous applications and it is adequate to research its subclasses. Along with this idea, in this paper, it is discussed trivially perfect graphs on the windmill graph and demonstrated a few outcomes on trivially perfect graphs.

Keywords: Trivially perfect graph, clique, independent set.

1. Introduction
Graphs are finite, connected, and simple, here. A path in G is an exchanging arrangement of vertices and edges. On the off chance that each vertex in a subset of vertex set is commonly adjacent, at that point, it is known as a clique. An independent set is a set of vertices of G, no two of which are adjacent. Each path contains a maximal clique of size two. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be any two graphs. The Cartesian product $G = G_1 \square G_2$ has the vertex set $V = V_1 \times V_2$ and there is an edge $A_i, A_j \in E(G)$ where $A_i = (u_i, v_i), A_j = (u_j, v_j)$ if and only if (i) $u_i u_j \in E_1$ in $G_1$ and $v_i = v_j$, or (ii) $u_i = u_j$ and $v_i v_j \in E_2$ in $G_2$. An independent set is supposed to be prevalent if for each induced subgraph $S$, each vertex of $S$ has a place with an independent set of $S$ meeting every maximal clique of $S$. Graph G is characterized to be a strongly perfect (SPG) if every induced subgraph $S$ has an independent set, meeting all maximal cliques of $S$. A SPG is supposed to be very strongly perfect (VSPG) if it contains a prevalent independent set. Cycle, tadpole, barbell, and friendship graphs are a portion of the subclasses of very strongly perfect graphs [1]. Perfect graphs manage the theoretical ideas of an independent set and clique.

A utilization of perfect graphs incorporates city life issues involving ideal heading findings of waste trucks. Additionally, It can be utilized to unravel calculation (polynomial time), utilizing ellipsoid strategy, for finding a stable set (maximal) and coloring (minimal) [4]. Because of the genuine uses of perfect graphs, it is important to research its subclasses as well.

Numerous subclasses of perfect graphs like strongly perfect graphs, super strongly perfect graphs have been explored [3]. As a trivially perfect graph is likewise one of the subclasses of perfect graphs, here it is explored a similar graph on windmill graph and demonstrated a few outcomes on trivially perfect graphs.
2. Trivially Perfect Graph (TPG)

A graph $G$ is supposed to be TPG if in each induced subgraph $H$ of $G$, the quantity of maximal cliques in $H$ counterparts to the size of a maximum independent set in $H$. Figure 1, represents the TPG.

Example 1

![Figure 1: Trivially Perfect Graph](image)

2. Windmill Graph

The windmill graph $W_d(r, n)$ is a graph, for $r \geq 2$ and $n \geq 2$ by joining $n$ duplicates of the complete graph $K_r$ at a common universal vertex. It has vertex connectivity 1 since its central vertex is an articulation point; be that as it may, similar to the complete graphs from which it is formed, it is $(k-1)$-edge-connected. For $r = 3$ windmill graph becomes a fan graph. Figure 2, demonstrates windmill graph $W_d(5, 4)$.

Example 2

![Figure 2: Windmill Graph $W_d(5, 4)$](image)

Here, $\{v\}$ is the universal vertex.
3.1. Theorem
If a graph $G$ is a windmill graph, then it is $TPG$.

**Proof:**
Consider $G = W_d(r, n)$ as a windmill graph, for $r \geq 2$ and $n \geq 2$ by joining $n$ duplicates of the complete graph $K_r$ at a common universal vertex 'v'. In this manner, the subgraph $G - v$ is disconnected, which induces 'n' connected component. Every one of the induced connected components is additionally complete subgraph of order $K_{r-1}$. Set an independent set that contains just a single vertex from every one of the connected components. Subsequently, $G$ is trivially perfect.

3.2. Lemma [2]
The following two conditions are equivalent:
(1) $G$ has no induced $P_4$;
(2) Every subset $B$ of $V; the subgraph $G_B$ or its complement $\overline{G_B}$ is disconnected.

3.3. Theorem
If a graph $G$ with no induced $P_4$, then it is a $TPG$.

**Proof:**
Leave $G$ alone a graph with $n$ vertices.
We demonstrate the outcome by induction on the quantity of vertices of graph $G$.
Expect the outcome is valid for all the graphs of the order not as much as $n$ with no induced $P_4$ are trivially perfect. We have to show that a graph $G$ of order $n$ with no induced $P_4$ is additionally $TPG$.
By the Lemma 3.2, either $G$ or its compliment $\overline{G}$ is disconnected, without loss of all-inclusive statements we may expect that $G$ is disconnected with two connected segments state $G_1$ and $G_2$. By the induction theory, for $G_1$ there exist a maximum independent set $I_1$ with cardinality $m_1$, which has precisely $m_1$ maximal cliques. Correspondingly for $G_2$ there exist a maximum independent set $I_2$ with cardinality $m_2$, which has precisely $m_2$ maximal cliques. In this way, every independent set of graph $G$; which is of the structure $I = I_1 \cup I_2$; has the cardinality $m_1 + m_2$ with a similar number of maximal cliques. Thus, $G$ is $TPG$.

3.4. Theorem
If a graph $G$ is $TPG$, then it is a $VSPG$.

**Proof:**
Let $G$ be a $TPG$. Along these lines in each induced subgraph $H$ of $G$ the size of the biggest independent set equivalents to the number of maximal cliques. Let $I = \{u_1, u_2, u_3, \ldots, u_k\}$ be the biggest independent set with the cardinality $k$. Since $G$ is trivially perfect, it has precisely $k$ number of maximal cliques. In this manner each vertex in $I$ meets all the maximal cliques in $G$, therefore $I$ is a maximum independent set. Subsequently, every $TPG$ is $SPG$. Not conversely, for example, $C_5$ with two chords.

3.5. Theorem
Let $G_1$ and $G_2$ be connected graphs. Then $G_1 \square G_2$ is trivially perfect if and only if $V(G_1) = 1$ and $G_2$ is $TPG$.

**Proof:**
Since $G_2$ is trivially perfect, every induced subgraph $H_2$ of $G_2$, the number of maximal cliques in $H_2$ equals the size of the maximum independent set in $H_2$. But, $G_1$ is a $TPG$ (graph with only one vertex) therefore by definition of the Cartesian product, $G_1 \square G_2$ results in $G_2$ which is $TPG$.
3.6. Theorem
If $G_1$ has no induced subgraph $C_{2k+1}$ or $C_{2k+1} + e; k \geq 2$ and $G_2$ are $k_2$ then $G_1 \Box G_2$ is TPG.

Proof:
We prove the result by contradiction.
Case 1: if $G_1$ has $C_{2k+1}$ and $G_2$ is $k_2$ then by definition of Cartesian product $C_{2k+1} \Box k_2$ induces an even cycle of length four, contradicts $G_1 \Box G_2$ is TPG.
Case 2: if $G_1$ has $C_{2k+1} + e$ and $G_2$ is $k_2$ then by definition of Cartesian product $C_{2k+1} + e \Box k_2$ induces an even cycle of length four, contradicts $G_1 \Box G_2$ is TPG.

4. Conclusion
Here, it is investigated trivially perfect graphs on windmill graphs and demonstrated a few outcomes on trivially perfect graphs. The structural properties of very strongly perfect graphs, odd cycles, and strongly perfect graphs are discussed. We imagine that these techniques can be utilized to decide as extraordinary compared to other scientific models for a genuine circumstance, where one might want to pick an ideal set of pioneers from a given set of people groups. It is likewise utilized in group learning of AI models, which helps in choosing the ablest model for better after-effects of precision. In the future, this examination will be progressively relevant for the rest of the graph classes moreover.

References
[1] Ganesh R. Gandal and R. Mary Jeya Jothi, “Classes of Very Strongly Perfect Graphs”, International Journal of Pure and Applied Mathematics, Vol. 113, no.10, pp. 334 - 342, 2017.
[2] E. Mandrescu, “Strongly perfect product of Graphs”, Czechoslovak Math.J.41, Vol. 116, pp. 368-372, 1991.
[3] R. Mary Jeya Jothi, “A discussion on SSP Structure of Pan, Helm and Crown Graphs”, ARPN Journal of Engineering and Applied Sciences, Vol. 10, No. 9, pp. 4115 - 4121, 2015.
[4] M.Kwasnik and A. Szelecka, Strong Perfectness of the generalized Cartesian product of graphs, Discrete Mathematics, Vol. 164, pp. 213-220, 1997.