Perturbative Subtraction Methods

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The effects of an automated, tenth order in $\kappa$ subtraction scheme on the noise variance of various Wilson QCD disconnected matrix elements are examined. It is found that there is a dramatic reduction in the variance of the lattice point-split electromagnetic currents and that this reduction persists at small quark mass.

1. Introduction

Many types of QCD matrix calculations require the global estimate of quark “disconnected” amplitudes. Examples for the nucleon include the pi-N sigma term, quark spin content and the strangeness form factors. There is a continuing need for new methods which can extract these matrix elements more efficiently.

Noise theory methods are based upon projection of the signal using random noise vectors as input. That is, given

$$Mx = \eta,$$

where $M$ is the $N \times N$ quark matrix, $x$ is the solution vector and $\eta$ is the noise vector, with

$$< \eta_i > = 0, < \eta_i \eta_j > = \delta_{ij},$$

where one is averaging over the noise vectors, any inverse matrix element, $M_{ij}^{-1}$, can then be obtained from

$$< \eta_j x_i > = \sum_k M_{ik}^{-1} < \eta_j \eta_k > = M_{ij}^{-1}. \quad (3)$$

Subtraction methods can be of great assistance in reducing the noise variance\(^2\). The key is using a perturbative expansion of the quark matrix as the subtraction matrices. The method described is completely iterative so that going to higher orders in perturbative subtraction is very easy and the overhead is also extremely low.

2. The Method

Let us review the results of noise theory for the expectation value and variance of matrices with various types of noises. Define\(^3\)

$$X_{mn} = \frac{1}{L} \sum_{l=1}^{L} \eta_{ml} \eta_{ml}^*, \quad (4)$$

$(m, n = 1, \ldots, N; l = 1, \ldots, L)$. We have

$$X_{mn} = X_{nm}^*, \quad (5)$$

and the expectation value,

$$< X_{mn} > = \delta_{mn}. \quad (6)$$

The expectation value of $Tr\{QX\}$ is $Tr\{Q\}$ and its noise variance is

$$V[Tr\{QX\}] \equiv < \sum_{m,n} q_{mn} X_{nm} - Tr\{Q\} >. \quad (7)$$

This results in

$$V[Tr\{QX\}] = \sum_{m \neq n} (< |X_{nm}|^2 > |q_{mn}|^2 +$$

$$< (X_{mn})^2 > q_{mn} q_{mn}^* ) + \sum_n < |X_{nn} - 1|^2 > |q_{nn}|^2. \quad (8)$$

Consider three cases. First, general real noise:

$$< |X_{mn}|^2 > = < (X_{mn})^2 > = \frac{1}{L}, \quad (9)$$

for $m \neq n$ so that

$$V[Tr\{QX_{\text{real}}\}] = \frac{1}{L} \sum_{m \neq n} (|q_{mn}|^2 + q_{mn} q_{mn}^*) +$$

$$+ \sum_n < |X_{nn} - 1|^2 > |q_{nn}|^2. \quad (10)$$

Compare this with the $Z(2)$ case which also has Eq.\(^3\) for $m \neq n$, but also

$$< |X_{nn} - 1|^2 > = 0. \quad (11)$$
This shows that
\[ V[Tr\{QX_Z(2)\}] \leq V[Tr\{QX_{\text{real}}\}] . \]  
(12)

For the \( Z(N) \) (\( N \geq 3 \)) case one has instead
\[ < |X_{mn}|^2 > = \frac{1}{L} , \]  
(13)
\[ < (X_{mn})^2 > = 0 , \]  
(14)
for \( m \neq n \), but again
\[ < |X_{nn} - 1|^2 > = 0 . \]  
(15)

Thus
\[ V[Tr\{QX_Z(N)\}] = \frac{1}{L} \sum_{m \neq n} |q_{mn}|^2 , \]  
(16)
and the variance relationship of \( Z(2) \) and \( Z(N) \) is not fixed in general for arbitrary \( Q \). However, if the phases of \( q_{mn} \) and \( q_{nm} \) are uncorrelated, then \( V[Tr\{QX_Z(2)\}] \approx V[Tr\{QX_Z(N)\}] (N \geq 3) \), which is apparently the case for the operators studied here and in Ref. [4].

Now consider \( \bar{Q} \) such that
\[ < Tr\{\bar{Q}\} > = 0 . \]  
(17)

Obviously,
\[ < Tr\{(Q - \bar{Q})X\} >= < Tr\{Q\} > . \]  
(18)

However,
\[ V[Tr\{(Q - \bar{Q})X\}] \neq V[Tr\{QX\}] . \]  
(19)

As we have seen for \( Z(N) \) (\( N \geq 2 \)), the variance originates exclusively from off diagonal entries. So the trick is to try to find matrices \( \bar{Q} \) which are traceless (or can be made so) but which mimic the off-diagonal part of \( Q \) as much as possible.

The natural choice is simply to choose as \( \bar{Q} \) the perturbative expansion of the quark matrix. This is given by (\( \{IJ\} \) are collective indices)
\[ (M_p^{-1})_{IJ} = \frac{1}{\delta_{IJ} - \kappa P_{IJ}} , \]  
(20)

where
\[ P_{IJ} = \sum_\mu [(1 + \gamma_\mu)U_\mu(x)\delta_{x,y-a_\mu} + (1 - \gamma_\mu)U_\mu^\dagger(x - a_\mu)\delta_{x,y+a_\mu}] . \]  
(21)

Expanding this in \( \kappa \) gives,
\[ M_p^{-1} = \delta + \kappa P + \kappa^2 P^2 + \kappa^3 P^3 + \cdots , \]  
(22)

One constructs \( < \eta_j(M_p^{-1})_{ik}\eta_k > \) and subtracts it from \( < \eta_j M_p^{-1}\eta_k > \), where \( \eta \) is the noise vector. This construction is an iterative process and so is easy to code and extend to higher powers on the computer. One can put coefficients in front of the various terms in Eq. (22) and vary them to find the minimum in the variance, but such coefficients take on their perturbative value except perhaps for low order expansions. Interestingly, significant variance improvement occurs in some operators even at 0th order in \( \kappa \).

For a given operator, \( \mathcal{O} \), the matrix \( OM_p^{-1} \) encountered in \( < \bar{\psi}\bar{\psi}\mathcal{O}\psi >_{\text{gauge}} = Tr(OM_p^{-1}) \) is not traceless in general. To correct for this one must re-add the perturbative part, subtracted earlier, to get the full, statistically unbiased answer. How does one calculate the perturbative part? Hard, exact way: explicitly construct all the gauge invariant paths (up to a given \( \kappa \) order) for a given operator. Easy, statistical way: subject the perturbative contribution to a separate Monte Carlo estimation, identical to the Monte Carlo applied to the nonperturbative part. This separate Monte Carlo is easy to do because one is simply constructing a matrix rather than inverting one. Local operators require perturbative corrections starting at 4th order and point-split ones have corrections starting at 3rd order. In the following I will carry this procedure to \( \kappa^{10} \).

3. Variance Ratio Results

I will show the ratio of unsubtracted noise variance to subtracted variance, \( V_{\text{unsub}}/V_{\text{sub}} \) for \( Z(2) \) noise. Since computer time is proportional to the operator variance, the ratio gives a measure of the decrease in the computer time needed to reach a given noise variance level. The lattices are Wilson 16\(^3\) \( \times \) 24, \( \beta = 6.0 \). Note that I am using a “one noise” inversion method, meaning that a global estimate of the operator is obtained after a single matrix inversion. The appropriate operators to apply this to are \( \bar{\psi}\psi \) and local and point split \( \bar{\psi}\gamma_\mu\psi \) (see Ref. [3]). Fig. 1 shows the effect of the level of subtraction on the point-split charge.
density operator at $\kappa = 0.148$. Fig. 2 shows the ratio of variances for the scalar (“S”), the four local vector (“L VEC 1-4”), and the four point-split vector operators (“P-S VEC 1-4”) after 10th order subtraction, also at $\kappa = 0.148$.

4. Conclusions and Acknowledgments

It has been demonstrated that a large reduction in the noise variance of certain lattice operators is obtained using perturbative subtraction methods. The method is effective for the scalar and local vector currents, but most effective for the point-split vector currents. The method can become less effective at lower quark masses, depending on the operator. The 10th order point-split vector, local vector, and scalar variance ratios change from $\sim 35$, $\sim 12$, and $\sim 10$ at $\kappa = 0.148$, to $\sim 25 \sim 10$, and $\sim 5$ at $\kappa = 0.152$, respectively. Similar methods can be devised for other operators (axial, pseudoscalar, tensor) by implementing this algorithm in the context of “12 noise” methods.

The operators considered here are all zero momentum. Of course for disconnected form factor evaluations one is more interested in nonzero momentum data. Although the results are not shown here I have found essentially identical results to the above for the momentum transformed data. These methods should be extremely useful in the lattice evaluations of strangeness form factors, which are of current experimental interest.

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