Asymptotic normality of estimates in flexible seasonal time series model with weak dependent error terms

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**Abstract**

In this article, we consider flexible seasonal time series models which consist of a common trend function over periods and additive individual trend (seasonal effect) functions. The consistency and asymptotic normality of the local linear estimators were obtained under the $\alpha$-mixing conditions and without specifying the error distribution. We develop these results to consistency and asymptotic normality of local linear estimates by using central limit theorems for flexible seasonal time series model, which error terms are $k$-weak dependent and $\lambda$-weak dependent random variables.

**Keywords:** Flexible seasonal time series model; Local linear estimate; Consistency and asymptotic normality, Weak dependent random variables

**JEL classification:** C13; C14

### 1 Introduction and previous research

Let $y_{t1}, y_{t2}, \cdots, t = 1, 2, \cdots$ are seasonal time series. The flexible model is as follows.

\begin{equation}
y_{tj} = T_t + S_{tj} + e_{tj},
\end{equation}

where $T_t$ is the common trend same to different periods within a season, and $S_{tj}$ is the seasonal effect, satisfying $\sum_{j=1}^{d} S_{tj} = 0$. Semi-parametric seasonal time series model is as follows.

\begin{equation}
y_{tj} = \alpha(t) + \beta(t) + e_{tj}, \ i = 1, \cdots, n, \ j = 1, \cdots, d,
\end{equation}

where $r_j$ is seasonal factors. Hence the overall seasonal effect changes over periods in accordance with the modulating function $\beta(t)$. Implicity, model (1.2) assumes that the seasonal effect curves have the same shape (up to a multiplicative constant) for all seasons. We consider a more general flexible seasonal effect model having the following form:

\begin{equation}
y_{ij} = \alpha(t_i) + \beta_j(t_i) + e_{ij}, \ i = 1, \cdots, n, \ j = 1, \cdots, d,
\end{equation}
where \( t_i = \frac{i}{n}, \alpha(\cdot) \) is smooth trend function in \([0,1]\), \( \{\beta_j(\cdot), j = 1, \cdots, d\} \) are smooth seasonal effect functions in \([0,1]\), either fixed or random, subject to a set of constraints, and the error term \( e_{ij} \) is assumed to be stationary and weak dependent random variables. As in model (1.2), the following constraints are needed for fixed seasonal effects:

\[
\sum_{j=1}^{d} \beta_j(t) = 0, \forall t
\]

reflecting the fact that the sum of all seasons should be zero for the seasonal factor. In previous researches a local linear technique has been used to estimate the trend and seasonal functions, and the asymptotic properties of the resulting estimators have been studied assuming that error terms were \(\alpha\)-mixing random variables \([1]\). Also asymptotic properties of nonparametric estimators for various time series models has been studied by local linear method \([2, 3, 4, 8]\).

Weak dependence and Problems

In model (1.3), statistical properties of weighted least square estimators are depended conclusively on statistical structure of dependent error terms. Many authors have used the two type of dependence: one is, mixing properties introduced by Rosenblatt(1956); another is, martingales approximations or mixingales, following the works of Gordin(1969, 1973) and Mc Leisch(1974, 1975). Concerning strongly mixing sequences, very deep and elegant results have been established by Rio(2000) and Bradley(2002). However, many classes of time series do not satisfy any mixing condition, conversely most of such time series enter the scope of mixingales but limit theorems and moment inequalities are more difficult to obtain in this general setting, so between those directions Bickel and Bühlmann(1999) and seperatively Doukhan and Louhichi(1999) introduced a new idea of weak dependence. Their concept of weak dependence makes explicit the asymptotic independence between ‘past’ and ‘future’: this means that the ‘past’ is progressively forgotten. Roughly speaking, for convenient functions \( f \) and \( g \), they assumed that

\[
\text{Cov}(f(‘past’), g(‘future’))
\]

is small when the distance between the ‘past’ and the ‘future’ is sufficiently large. The main advantage is that such a kind of dependence contains lots of pertinent examples and can be used in various situations. Therefore the central limit theorems for weak dependent variables has been studied in recent years \([5, 6, 7]\). In this article, we are going to derive consistency and asymptotic normality of the weighted least square estimators with a local linear method, assuming that error terms are \(k\)-weak dependent and \(\lambda\)-weak dependent random variables.

2 Main results and proof of theorems

Combination of (1.3) and (1.4) in a matrix expression leads to \(\theta\).

\[
Y_i = A\theta(t_i) + e_i
\]
Main results: for any time point $t$ and $\bar{z}$ bounded, and that $K(u/(n\lambda_n)) = R(u)$ as $n \to \infty$, which controls the amount of smoothing used in the estimation. By minimizing (2.2) with respect to $a$ and $b$, we obtain the local linear estimate $\hat{\theta}(t) = \hat{a}, \hat{\theta}'(t) = \hat{b}'$.

**Assumptions:**

A1. Assume that the kernel $K(u)$ is symmetric and satisfies the Lipschitz condition and $aK(u)$ is bounded, and that $\alpha(\cdot)$ and $\beta_j(\cdot)$ have continuous second derivatives in $[0, 1]$.  

A2. For each $n$, $\{e_n, \ldots, e_{n}\}$ have the same joint distribution as $\{\xi_1, \xi_2, \ldots, \xi_n\}$, where $\xi_i, t = \cdots, -1, 0, 1, \ldots$ is a strictly stationary time series with the covariance matrix $R(k-l) = \text{cov}(\xi_k, \xi_l)$. Assume that the time series $\{\xi_i\}$ is sequence of $k$-weak dependent random vectors with the finite $(2+\zeta)$th moment for some $\zeta > 0$ (i.e. $E\|\xi_k\|^{2+\zeta} < \infty$) and $k$-weak dependent coefficient $K_e(r)$ satisfying $K_e(r) = 0(h^{-2}r^{-k})$, where $k > 2 + 1/\zeta$. 

A2. Assume that the time series $\{\xi_i\}$ is sequence of $\lambda$-weak dependent random vectors satisfying the assumption A2 and $\lambda_e(r) = 0(h^{-2}r^{-\lambda})$, where $\lambda > 4 + 2/\zeta$.

**Main results:**
Lemma 2.1 Let a sequence of random vectors \( \{e_k\} \) is stationary with mean 0 and \( k \)-weak dependent (\( \lambda \)-weak dependent) and \( \{z_k\} \) is a sequence of stationary random variables defined as follows:

\[
z_k = hK_h(t_i - t)d^i e_k,
\]

then \( \{z_k\} \) are also \( k \)-weak dependent (\( \lambda \)-weak dependent) sequence and the following equality holds.

\[
|h|^2 K_e(r) = K_z(r), \quad |h|^2 \lambda_e(r) = \lambda_z(r),
\]

where \( K_e(r), K_z(r) \) and \( \lambda_e(r), \lambda_z(r) \) are \( k \)-weak dependent and \( \lambda \)-weak dependent coefficients respectively of \( \{e_k\}, \{z_k\} \).

Proof.

\[
K_z(r) = \sup_{u,v} \sup_{(i,j) \in \Gamma(u,v,r)} \sup_{f \in \mathcal{F}(u,v)} \frac{\text{cov}(f(z_{i1}, \cdots, z_{iu}), g(z_{j1}, \cdots, z_{ju}))}{\psi(f,g)}
\]

where in case of \( k \)-weak dependence \( \mathcal{F}_u(\mathcal{F}_u = J_u) \) is the wider set of functions from \( \chi^u \) to \( \mathbb{R} \), which are Lipschitz with respect to the distance \( \delta_1 \) on \( \chi^u \) defined by

\[
\delta_1(x,y) = \sum_{i=1}^u \delta(x_i, y_i),
\]

but which are not necessarily bounded. In this case

\[
\psi(f,g) = d_f d_g \text{Lip}(f) \text{Lip}(g)
\]

and in case of \( \lambda \)-weak dependence \( \mathcal{F}_u(\mathcal{F}_u = J_u) \) is the set of bounded functions from \( \chi^u \) to \( \mathbb{R} \), which are Lipschitz with respect to the distance \( \delta_1 \) on \( \chi^u \) defined by same method,

\[
\psi(f,g) = d_f \|g\|_{\infty} \text{Lip}(f) + d_g \|f\|_{\infty} \text{Lip}(g) + d_f d_g \text{Lip}(f) \text{Lip}(g).
\]

And then \( \delta(x_i, y_i) \) is a distance on a space \( \chi \), in case of \( \{z_k\} \) we have \( \chi = \mathbb{R} \) and \( \delta(x_i, y_i) = |x_i - y_i| \). Now we define \( \bar{\delta}(e_i, e_j) \) on \( \mathbb{R}^d \) by

\[
\bar{\delta}(e_i, e_j) = \delta\left(K_h(t_i - t)e_i, K_h(t_j - t)e_j\right),
\]

where \( \delta \) is a usual distance defined by \( \| \cdot \| \) on \( \chi = \mathbb{R}^d \). We define

\[
F(e_{i1}, \cdots, e_{iu}) = f(z_{i1}, \cdots, z_{iu}).
\]

Then the following relations hold:

\[
F(e_{i1}, \cdots, e_{iu}) - F(e_{k1}, \cdots, e_{ku}) \leq \text{Lip}(f) \sum_{l=1}^u \|d^l(K_h(t_{il} - t)e_{il} - K_h(t_{kl} - t)e_{kl})\|
\]

\[
\leq \text{Lip}(f)\|d\|\|h\| \sum_{l=1}^u \|K_h(t_{il} - t)e_{il} - K_h(t_{kl} - t)e_{kl}\|
\]

\[
= \text{Lip}(f)\|d\|\|h\| \sum_{l=1}^u \delta(K_h(t_{il} - t)e_{il} - K_h(t_{kl} - t)e_{kl})
\]
Therefore, Lipschitz constant of $F$ is

$$\text{Lip}(F) = \text{Lip}(f)\|d\|\|h\| = \text{Lip}(f)|h|,$$

so

$$\psi(f, g) = |h|^{-2}\psi(F, G).$$

Hence

$$K_z(r) = \sup_{u, v} \sup_{i, j} \sup_{f \in \mathcal{I}, g \in \mathcal{J}} \frac{\text{cov}(f(z_{i_1}, \ldots, z_{i_1}), g(z_{j_1}, \ldots, z_{j_1}))}{\psi(f, g)} |h|^2$$

Finally convergence of two weak dependent coefficients are equivalent, hence $\{z_k\}$ are also $k$-weak dependent sequence. Correspondingly equivalence of $\lambda$-weak dependence is proved. □

**Lemma 2.2** Under assumptions of Lemma 2.1, we have

$$\lim_{n \to \infty} DB_{n0} = v_0 \Sigma_0, \quad B_{n1} \overset{P}{\to} 0,$$

where for $k = 0, 1$,

$$B_{nk} = (h/n)^{1/2} \sum_{i=1}^{n} (t_i - t) e_{ni} K_h(t_i - t), \quad k = 1, 2.$$

**Proof.** By the stationarity of $\{\xi_j\}$,

$$DB_{n0} = n^{-1}h \sum_{1 \leq k, l \leq n} R(k - l) K_h(t_k - t) K_h(t_l - t)$$

$$= n^{-1}h R(0) \sum_{k=1}^{n} K_h^2(t_k - t) + 2n^{-1}h \sum_{1 \leq l < k \leq n} R(k - l) K_h(t_k - t) K_h(t_l - t)$$

$$=: D_1 + D_2.$$  

Clearly, by the Riemann sum approximation of an integral,

$$D_1 \approx R(0) h \int_0^{1} K_h^2(u - t) du \approx v_0 R(0).$$
Since \( nh \to \infty \), there exists \( c_n \to \infty \) such that \( c_n/(nh) \to 0 \). Let \( S_1 = \{(k,l) : 1 \leq k - l \leq c_n; 1 \leq l < k \leq n\} \) and \( S_2 = \{(k,l) : 1 \leq l < k \leq n\} \setminus S_1 \). Then, \( D_2 \) is split into two terms as \( \sum_{S_1} (\cdots) \), denoted by \( D_{21} \) and \( \sum_{S_2} (\cdots) \), denoted by \( D_{22} \). By assumptions of Lemma 2.1, we have
\[
|D_{22(jm)}| \leq Cn^{-1}h \sum_{S_2} |r_{jm}(k - l)|K_h(t_k - t)K_h(t_l - t)
\leq Cn^{-1}h \sum_{S_2} K(k - l)K_h(t_k - t)K_h(t_l - t)
\leq Cn^{-1} \sum_{k=1}^{n} K_h(t_k - t) \sum_{k_1 > A_n} K(k_1)
\leq C \sum_{k_1 > A_n} k_1^{-2 (1 + \zeta)}
\leq CA_n^{-1/\zeta} \sum_{k_1 > A_n} k_1^{-2}.
\]
Since \( A_n \to \infty \), the right side of above expression converges to zero. For any \((k,l) \in S_1\), by Assumption A1
\[
|K_h(t_k - t) - K_h(t_l - t)| \leq Ch^{-1} (t_k - t_l)/h \leq CA_n/(nh^2).
\]
From this inequality and the result of Lemma 4.2 in [7],
\[
|I| = \left| 2n^{-1}h \sum_{l=1}^{n-1} \sum_{1 \leq k - l \leq A_n} r_{jm}(k - l) \{K_h(t_k - t) - K_h(t_l - t)\} K_h(t_l - t) \right|
\leq CA_n n^{-2}h^{-1} \sum_{l=1}^{n-1} \sum_{1 \leq k - l \leq A_n} |r_{jm}(k - l)|K_h(t_l - t)
\leq CA_n n^{-2}h^{-1} \sum_{l=1}^{n-1} K_h(t_l - t) \sum_{k \geq 1} |r_{jm}(k)|
\leq CA_n/(nk) \to 0.
\]
Also the following result holds
\[
D_{21} = 2n^{-1}h \sum_{l=1}^{n-1} \sum_{1 \leq k - l \leq A_n} r_{jm}(k - l)K_h(t_k - t)K_h(t_l - t)
= 2n^{-1}h \sum_{l=1}^{n-1} K_h^2(t_l - t) \sum_{1 \leq k - l \leq A_n} r_{jm}(k - l) + I.
\]
Therefore
\[
\lim_{n \to \infty} D_{21} = 2 \pi_0 \sum_{k=1}^{\infty} r_{jm}(k),
\]
hence
\[ \lim_{n \to \infty} DB_{n0} = v_0 \left[ R_0 + 2 \sum_{k=1}^\infty R(k) \right] = v_0 \Sigma_0. \]

Otherwise, by the assumption A1, we get the following
\[ DB_{n1} = n^{-1}h \sum_{1 \leq k, l \leq n} R(k-l)(t_k - t)(t_l - t)K_h(t_k - t)K_h(t_l - t) \]
and
\[ Cn^{-1}h \sum_{1 \leq k, l \leq n} |R(k-l)| \leq Chn^{-1} \sum_{k=-\infty}^{\infty} |R(k)| \to 0. \quad \square \]

**Theorem 2.1** Under Assumptions A1 and A2, (or A1, \( \bar{A}2 \), we have
\[ \hat{\theta}(t) - \theta - \frac{h^2}{2} \mu_2 \theta^{(2)}(t) + o(h^2) = O_p \left( (nh)^{-1/2} \right). \]

**Proof.** Let \( \mu_k = \int u^k K(u)du \), \( v_k = \int u^k K^2(u)du \), then
\[ \lim_{n \to \infty} S_{n,k}(t) = h^k \mu_k \]
From Taylor explanation, we have
\[ \theta(t_i) = \theta + \theta'(t)(t_i - t) + \frac{\theta^{(2)}(t)}{2!}(t_i - t)^2 + o(h^2), \]
and hence it follows that
\[ n^{-1} \sum_{i=1}^{n} (t_i - t)^k \theta(t)K_h(t_i - t) = S_{n,k}(t)\theta(t) + S_{n,k+1}(t)\theta'(t) + \frac{1}{2}S_{n,k+2}(t)\theta^{(2)}(t) + o(h^2). \]
By the model (2.1)
\[ Y_i = A\theta(t_i) + e_i = A \left( \theta + \theta'(t)(t_i - t) + \frac{\theta^{(2)}(t)}{2!}(t_i - t)^2 + o(h^2) \right) + e_i \]
and applying the least square estimation result of [1]
\[ \hat{\theta}(t) = A^{-1} \sum_{i=1}^{n} S_i(t)Y_i = \theta(t) + \frac{1}{2} S_{n,2}(t) - S_{n,1}(t)S_{n,3}(t) \theta^{(2)}(t) \]
\[ + o(h^2) + A^{-1} \sum_{i=1}^{n} S_i(t)e_{ni}. \]
Then, by assumption $A1$ and using that $\mu_1 = 0, \mu_3 = 0, \mu_0 = 1$

\[
\hat{\theta}(t) - \theta(t) - \frac{h^2}{2} \mu_2 \theta^{(2)}(t) + o(h^2) = A^{-1} \sum_{i=1}^{n} S_i(t) e_{ni},
\]

which implies that

\[
(2.4) \quad \sqrt{n} h \left\{ \hat{\theta}(t) - \theta(t) - \frac{h^2}{2} \mu_2 \theta^{(2)}(t) + o(h^2) \right\} = A^{-1} \frac{S_{n,2}(t) B_{n0} - S_{n,1}(t) B_{n1}}{S_{n,0}(t) S_{n,2}(t) - S_{n,1}(t)^2},
\]

where both $B_{n0}$ and $B_{n1}$ are defined in Lemma 2.2. From Lemma 2.2 and Eq. (2.3), we prove the theorem. □

**Theorem 2.2** Under Assumptions $A1$ and $A2$, (or $A1, \tilde{A}2$), we have

\[
\sqrt{n} h \left\{ \hat{\theta}(t) - \theta(t) - \frac{h^2}{2} \mu_2 \theta^{(2)}(t) + o(h^2) \right\} \to N(0, \Sigma_{\theta}),
\]

where $\Sigma_{\theta} = v_0 A^{-1} \Sigma_0 (A^{-1})'$. 

**Proof.** From Eq. (2.4), we get

\[
\sqrt{n} h \left\{ \hat{\theta}(t) - \theta(t) - \frac{h^2}{2} \mu_2 \theta^{(2)}(t) + o(h^2) \right\} = A^{-1} \frac{S_{n,2}(t)}{S_{n,0}(t) S_{n,2}(t) - S_{n,1}(t)^2} \left\{ B_{n0} - \frac{S_{n,1}(t)}{S_{n,2}(t)} B_{n1} \right\}.
\]

So

\[
\frac{S_{n,1}(t)}{S_{n,2}(t)} B_{n1} = \left\{ \frac{S_{n,1}(t) - \mu_1 h}{S_{n,2}(t)} + \frac{\mu_1 h}{S_{n,2}(t)} \right\} B_{n1},
\]

\[
\frac{S_{n,2}(t)}{S_{n,0}(t) S_{n,2}(t) - S_{n,1}(t)^2} = 1 + \frac{S_{n,1}(t) - S_{n,2}(t) S_{n,0}(t) + S_{n,2}(t)}{S_{n,0}(t) S_{n,2}(t) - S_{n,1}(t)^2}.
\]

By Lemma 2.2 and Assumption $A1$, we have

\[
\frac{S_{n,1}(t)}{S_{n,2}(t)} B_{n1} \xrightarrow{p} 0, \quad \frac{S_{n,2}(t)}{S_{n,0}(t) S_{n,2}(t) - S_{n,1}(t)^2} \to 1,
\]

which implies that to establish the asymptotic normality of $\hat{\theta}(t)$, we only need to consider the asymptotic normality for $B_{n0}$.

Hence it remains to prove the asymptotic normality of $d' B_{n0}$ for all $d \in \mathbb{R}^d (\|d\| = 1)$. Let $Z_{ni} = \sqrt{h} d' e_{ni} K_h(t, t)$, then clearly $d' B_{n0} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_{ni}$. Moreover

\[
(2.5) \quad D(d' B_{n0}) = v_0 d' \Sigma_0 d\{1 + o(1)\} = \theta_d^2 (1 + o(1))
\]
Since \( k \)-weak dependence and \( \lambda \)-weak dependence of \( \{Z_{ni}\} \) holds from Lemma 2.1, we can apply Theorem 7.1 and Theorem 7.2 of [7]. If we consider assumptions of this theorem, then the central limit theorem holds for \( Z_{ni} \), therefore \( dB_{n0} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_{ni} \) converges in distribution to \( N(0, \theta_{d}^2) \). Hence \( \sqrt{n}h \left\{ \hat{\theta}(t) - \theta t - \frac{h^2}{2} \mu_2 \theta^{(2)}(t) + o(h^2) \right\} \) converges in distribution to \( N(0, \Sigma_{\theta}) \), where covariance matrix \( \Sigma_{\theta} = v_0 A^{-1} \Sigma_0 (A^{-1})' \).

\( \square \)

3 Conclusions

In this work we derived a general seasonal time series model with \( k \)-dependent and \( \lambda \)-dependent errors, which are new concepts of dependence. In this model we derived the consistency and asymptotic normality of non-parametric estimates constructed by local linear method.

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