Algorithm for Finding the Maximum Clique Based on Continuous Time Quantum Walk

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In this work, we consider the application of continuous time quantum walking (CTQW) to the Maximum Clique (MC) Problem. Performing CTQW on graphs will generate distinct periodic probability amplitude for different vertices. We will show that the intensity of the probability amplitude at frequency indeed implies the clique structure of some special kinds of graph. And recursive algorithms with time complexity $O(N^5)$ in classical computers for finding the maximum clique are proposed. We have experimented on random graphs where each edge exists with probabilities 0.3, 0.5 and 0.7. Although counter examples are not found for random graphs, whether these algorithms are universal is beyond this work.

I. INTRODUCTION

The problem of finding the maximum clique (complete subgraph) is one of the NP-complete problems\cite{1}. The optimal complexity of exact algorithms is $O(2^{0.249N})$\cite{2}. Designing a polynomial algorithm for NP-complete problems on classical computer is notoriously difficult. As the performance of quantum algorithm has been proved better than the classical algorithm in a lot of situations\cite{3, 4, 5}, many scientists turned to consider quantum algorithm for NP-complete problem\cite{6, 7, 8}. With the advantage of quantum states, all possible solutions (combinations of vertices of a given graph) are encoded in an initial superposition state, and the optimal solution is searched by a process of quantum evolution in the previous quantum algorithms. The quantum algorithm asymptotically requires the square root of the number of operations that the classical algorithm requires\cite{3, 6}. Apparently the structure of graphs is not adequately considered in these algorithms. Noga Alon, Michael Krivelevich and Benny Sudakov’s work shows that the second eigenvector is related to the MC of random graphs\cite{11}. Generally, how to connect the structure of graphs with the NP-complete problem is unclear. In this work, the structure of the MC in a graph refers specifically to whether a vertex belongs to the MC. We mainly focus on the structure of center graphs because all kinds of graphs can be transformed to center graphs. A graph is called center graph if there exists one so-called center vertex adjacent to all other vertices. We will reveal how the structure of clique impact the continuous time quantum walks (CTQW) of several special kinds of center graph, and algorithms for the maximum clique problem will be proposed. In Farhi and Goldstone’s work, CTQW is defined as an evolution of a quantum system which is driven by the Laplacian matrix of a given graph\cite{12}. With some other physical models\cite{13, 14}, the Hamiltonian of CTQW is defined as the adjacency matrix of the corresponding graph in this work. Then the state of CTQW is determined by $\varphi(t) = e^{iAt}\varphi(0)$, where $A$ is adjacency matrix of the given graph $G$ and $e^{iAt}$ is evolution unitary operator. This operation has a series form $e^{iAt} = \sum_{s=0}^{\infty} \frac{(it)^s}{s!} A^s$. As $A^s$ is characterised by the number of walks in graphs, the CTQW does reflect the clique structure of several kinds of center graphs. The evolution can be estimated when eigenvectors and eigenvalues of a given adjacency matrix are obtained by numerical computation in $O(N^3)$ time on classical computers. The probability amplitude of CTQW is chosen as the critical feature to infer whether a vertex is a member of the maximum clique.

This paper is organized as follows: the second section presents CTQW on center graphs. In the third section, an algorithm, named algorithm A, with $O(N^5)$ time complexity based on CTQW for finding the maximum clique is introduced. In the fourth section, we study the probable error of algorithm A and present the approach for constructing a graph which is not valid for algorithm A. In the fifth section, we give another two algorithms, named algorithm B and algorithm C, to fix algorithm A. The last section is the conclusion.

II. THE CLIQUE STRUCTURE AND THE CTQW ON CENTER GRAPH

Generally, a graph is denoted as $G(V,E)$, consisting of a vertex set $V$ and an edge set $E$. The set $E$ is a subset of $V \times V$, which implies the connection relationship between any pair of vertices in $V$. Let the number of $V$ equals to $N$, the adjacency matrix of $G$ is an $N \times N$ real symmetric matrix $A$, where $A_{ij} = 1$ if vertex $v_j$ and $v_i$ are connected otherwise $A_{ij} = 0$.

Consider the CTQW on a given graph, one can associate every vertex $v_j$ of the graph with a basis vector $|j\rangle$
in a N-dimensional vector space. The Hamiltonian of the system is

\[ H = A, \] (1)

If \( v_j \) is the initial state of system, and the transition probability amplitude from \( v_j \) to \( v_l \) is \( \alpha_{l,j}(t) \) or short format \( \alpha_{l,j} \), then:

\[ \alpha_{l,j}(t) = \langle l \mid e^{i\gamma At} \mid j \rangle, \] (2)

The probability \( \pi_{l,j}(t) \) or short format \( \pi_{l,j} \) can be written as

\[ \pi_{l,j}(t) = \left| \langle l \mid e^{i\gamma At} \mid j \rangle \right|^2. \] (3)

The eigenvalues of the \( A \) are denoted as \( \lambda_n (n = 1, 2, \ldots, N) \), the eigenvalues are arranged in descending order, namely \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \). The eigenvector belongs to \( \lambda_n \) is denoted as \( |\lambda_n\rangle \), producing:

\[ \alpha_{l,j}(t) = \sum_n e^{i\lambda_n t} \langle l \mid \lambda_n \rangle \langle \lambda_n \mid j \rangle, \] (4)

and

\[ \pi_{l,j}(t) = \left| \sum_n e^{i\lambda_n t} \langle l \mid \lambda_n \rangle \langle \lambda_n \mid j \rangle \right|^2. \] (5)

The real part of the amplitude \( \alpha_{l,j} \) can be represented as

\[ R(\alpha_{l,j}) = \sum_n p_n \cos(\lambda_n t) \] (6)

where \( p_n = \langle l \mid \lambda_n \rangle \langle \lambda_n \mid j \rangle \). This implies that the real part of amplitude of CTQW is a periodic function with \( N \) frequency components, the values of frequency are the eigenvalues of adjacency matrix and the intensity of the frequency \( \lambda_n \) is \( p_n \).

The amplitude can also be represented as a form of sums, i.e.,

\[ \alpha_{l,j}(t) = \sum_{s=0}^{\infty} \frac{(it)^s(A^*)_{l,j}}{s!}. \] (7)

Where \( A^*_{l,j} \) denotes the number of walks of length \( s \) [15]. Therefore, the CTQW is determined by the number of walks in graphs.

Consider a graph \( G \) and one of its vertices \( v_j \), the neighbors of \( v_j \) denoted by \( N(j) \). The induced subgraph of vertex \( v_j \bigcup N(j) \) is denoted as \( G_j \). We call \( G_j \) the center subgraph of vertex \( v_j \) and the vertex \( v_j \) the center vertex of \( G_j \). Note that the concept of center graph is not completely same with the concept of reference [16].

Two natural approaches can be used to transform a graph to a center graph or set of center graphs. The first way is to add a new vertex then connect it to every vertex of the original graph. The second approach is to induce a set of center graphs \( \{G_1, \ldots, G_N\} \) of the original graph \( G \).

A center graph \( G_j \) is called the first kind of ideal center graph, if there are two cliques in \( G_j \) and there has no edge connecting any pair of vertices \( \{v_j, v_l\} \) when \( v_j \) and \( v_l \) are members of distinct cliques. One of this kind of center graph is shown in Fig. 1.

![Fig. 1. The center graph \( G_5 \) of vertex 5. There exists two cliques, one is the subgraph induced by \( \{1, 2, 3, 4, 5\} \), another is the subgraph induced by \( \{5, 6, 7, 8\} \).](image)

It is hard to obtain the analytical solutions of eigenvalues and eigenvectors even for the first kind of ideal center graph. The method of counting the number of walks is used for solving the CTQW. Let \( W_s \) also denote the number of closed walks of center vertex, \( F_s \) denote the number of walks starting from the center vertex \( j \) ending with one of vertices in the MC, and \( H_s \) denote number of walks starting from the center vertex \( j \) ending with one of vertices not in the MC. It provides that

\[
\begin{align*}
W_{s+1} &= (m_1 - 1) F_s + (m_2 - 1) H_s \\
F_{s+1} &= W_s + (m_1 - 2) F_s \\
H_{s+1} &= W_s + (m_2 - 2) H_s
\end{align*}
\] (8)

Where \( m_1 \) is the clique number and \( m_2 \) is the size of the remaining clique. Finding the solutions of Eq. (8) is equivalent to do eigen decomposition of adjacency matrix \( A \). But the exact numerical solutions are complex and helpless for inferring whether a vertex belongs to the maximum clique. We only need the relations between the probability amplitudes of different vertices. By using the result of literature [15, 17], we have

\[ W_s = \sum_{n=1}^{N} a_n \lambda_n^s. \] (9)

Where \( \sum_{n=1}^{N} a_n = 1 \). Taking Eq. (9) into the second and third terms of Eq. (8), we have

\[ F_{s+1} = \sum_{n=1}^{N} a_n \lambda_n^s + (m_1 - 2) F_s, \] (10)
and

\[ H_{s+1} = \sum_{n=1}^{N} a_n \lambda_n^s + (m_1 - 2) H_s. \]  
(11)

Solving Eq. (10) and Eq. (11), we have

\[ F_s = \sum_{n=1}^{N} a_n \frac{(m_1 - 2)^s - \lambda_n^s}{m_1 - 2 - \lambda_n} \]  
(12)

and

\[ H_s = \sum_{n=1}^{N} a_n \frac{(m_2 - 2)^s - \lambda_n^s}{m_2 - 2 - \lambda_n} \]  
(13)

Hence the probability amplitude of \( v_i \) which is a vertex of MC is

\[ \alpha_{l,j} (t) = \sum_{n=1}^{N} a_n \frac{e^{i(m_1 - 2)t} - e^{i\lambda_n t}}{m_1 - 2 - \lambda_n}, \]  
(14)

and for \( v_k \) which isn’t a member of the maximum clique, the probability amplitude is

\[ \alpha_{l,j} = \sum_{n=1}^{N} a_n \frac{e^{i(m_2 - 2)t} - e^{i\lambda_n t}}{m_2 - 2 - \lambda_n}. \]  
(15)

As \( m_j - 2 \) is not a eigenvalue for \( j = 1, 2 \), compare Eq. (4) with Eq. (15) to obtained that

\[ \sum_{n=1}^{N} a_n e^{i(m_j - 2)t} = 0. \]  
(16)

Let \( p_{l,n} \) denote the coefficient of \( \alpha_{l,j} \) at eigenvalue \( \lambda_n \). Then we have the follow theorem

**Theorem 1.** For the first kind of ideal center graph \( G_j \), \( v_j, v_1, v_k \in V(G_j) \), \( v_j \) is the center vertex, \( v_1 \) is a member of the maximum clique of \( G_j \), \( v_k \) is not a member of the maximum clique. Then

\[ p_{l,1} > p_{k,1}, \]

i.e.,

\[ \frac{1}{m_1 - 2 - \lambda_1} > \frac{1}{m_2 - 2 - \lambda_1}. \]

**Theorem 1** is obvious since \( \lambda_1 > m_1 - 2 > m_2 - 2 \). Therefore, it is easy to pick up the vertex that belongs to the maximum clique from the first kind of ideal center graph by using CTQW.

The second kind of ideal center graph is graphs derived from the first kind of ideal center graph via replacing the second large clique with a complete multi-partite graph, and keeping the maximum clique unchanged. An example graph of this type is shown in Fig. 2.

![Fig. 2. The second kind of ideal center graph. There has no edge that connect the vertices in the MC and the vertices not in the MC. The induced subgraph of blue vertices is a complete multi-partite graph.](image)

The maximum clique of graph shown in Fig. 2 is 5. Since the blue vertices constitute a complete multi-partite graph, adding edges between blue vertices will generate cliques with size not less than 5.

For the second kind of ideal center graph, the number of walks can be determined by the follow recursion equations:

\[
\begin{align*}
W_{s+1} &= (m_1 - 1) F_s + z (m_2 - 1) H_s \\
F_{s+1} &= W_s + (m_1 - 2) F_s \\
H_{s+1} &= W_s + z (m_2 - 2) H_s 
\end{align*}
\]  
(17)

Where \( z \) is the number of vertex in each independent set. Compare Eqs. (17) to Eqs. (8), one can find that the solutions format of Eqs. (17) are similar with the solutions of Eqs. (8), just replacing \( (m_2 - 2) \) by \( z (m_2 - 2) \) in Eq. (15). Therefore, for a vertex \( v_l \) which belongs to the maximum clique in the graph shown in Fig. 2, it provides that

\[ \alpha_{l,j} (t) = \sum_{n=1}^{N} a_n \frac{e^{i(m_1 - 2)t} - e^{i\lambda_n t}}{m_1 - 2 - \lambda_n}, \]  
(18)

and for a vertex which doesn’t belong to the maximum clique, it provides

\[ \alpha_{l,j} (t) = \sum_{n=1}^{N} a_n \frac{e^{i(z(m_2 - 2)t} - e^{i\lambda_n t}}{z (m_2 - 2) - \lambda_n}, \]  
(19)

Similar to **Theorem 1**, the following theorem is provided:

**Theorem 2.** For the second kind of ideal center graph \( G_j \), \( v_j, v_1, v_k \in V(G_j) \), \( v_j \) is the center vertex, \( v_1 \) is a member of the maximum clique of \( G_j \), \( v_k \) is not a member of the maximum clique. Then

\[ p_{l,1} < p_{k,1}, \]

i.e.,

\[ \frac{1}{m_1 - 2 - \lambda_1} < \frac{1}{z (m_2 - 2) - \lambda_1}. \]
\[
\text{if } m_1 - 2 < z (m_2 - 2) .
\]

In the previous instances, there has no edge connecting vertices in different cliques. For general cases, there exist multiple edges between vertices from distinct cliques. One of such instances is exhibited in Fig. 3.

![Fig. 3. An example graph. In this configuration, the maximum clique and the non-maximum clique connect by edge (1,10),(2,8),(3,21)](image)

For general graphs, the method of counting the number of walks to determine the CTQW is as hard as the eigen-decomposition of the adjacency matrix. For general graphs, even such the eigen-decomposition is obtained, to directly deduce that whether a vertex is a member of the maximum clique is still not known to us. An intuitive idea is to generate center subgraph of the original graph and find the maximum clique in that center subgraphs. For instance, the maximum clique attached to vertex 3 of the graph shown in Fig. 3 can be easily found. The center graph of vertex 3 is exhibited in Fig. 4. Note that the original center vertex 5 is not included in \( G_3 \).

![Fig. 4. The center subgraph of vertex 3. The graph is induced by vertex 3 and its neighbors except center vertex, namely \{1,2,3,4,21\}](image)

The result graph in Fig. 4 is the first kind of ideal center graph, and its maximum clique can be determined by Theorem. This inspires us that performing CTQW on center graphs in an orderly way can help us to reveal the maximum clique. In this procedure, we need repeatedly choose vertices and construct associated center subgraphs. In the following section, \( C(v_j) \) is used to denote the center graph of \( v_j \) or the procedure of constructing the center graph of \( v_j \). The procedure of deleting vertex \( v_j \) in graph \( G \) is frequently used and is denoted as \( D(v_j, G) \). The method about how to successively select a vertex to construct a center graph is presented in the next section.

### III. A Recursive Algorithm for Finding the Maximum Clique by CTQW

An algorithm, named algorithm A, based on eigenvectors of adjacency matrix of graph for finding the maximum clique is proposed in this section. The intensities of the real part(or imaginary part) of the probability amplitude are used as the critical feature for selecting probable vertices. The procedure of algorithm A is illustrated by a tree-like diagram in Fig. 5.

![Fig. 5. The recursive algorithm for finding the maximum clique. There are four sub-modules in algorithm A, and the fourth sub-module recursively calls the algorithm A on a smaller subgraph.](image)

As showed in Fig. 5 there are four cases(sub-modules) in algorithm A and the last is the recursive process. The first one is the sub-algorithm named Pick_max in which the strategy of picking the vertex with the largest intensity at frequency \( \lambda_1 \) in amplitude of CTQW was employed. The second one is the sub-algorithm named Delete_min in which the strategy of deleting the vertex with the weakest intensity at frequency \( \lambda_1 \) in amplitude of CTQW was employed. Then, we want to delete a vertex with the weakest intensity at frequency \( \lambda_1 \), this vertex is denoted as \( v_{\lambda_1,min} \). Since that vertex maybe a member of the maximum clique, the third sub-module Pick_max(\( C(v_{\lambda_1,min}) \)) is applied to find the maximum clique of \( v_{\lambda_1,min} \). The last sub-module is to recursively call the algorithm A. Note that the size of the graph de-
creases by one since a vertex is deleted in the fourth module. The maximum clique of \( C(v_s) \) is the largest clique found in these four cases. The sub-modules \( \text{Pick}_\text{max} \) and \( \text{Delete}_\text{min} \) are presented in the following charts.

TABLE I. The step of \( \text{Pick}_\text{max} \)

**Input:** Center graph \( G_s \), center vertex \( v_s \)
**Output:** Clique \( C \);  
1: Delete vertex \( v_s \) in \( G_s \) and for every other vertex \( v_i \), let \( C_i = \{ v_s, v_i \} \), find the central graph of \( v_i \), and denote as \( G_i, v_s \leftarrow v_i \);  
2: Do eigen-decomposition on adjacent matrix of \( G_i \), The largest eigenvalue is \( \lambda_1 \), corresponding eigenvector is \( x_1 \) and \( x_1^l \) denotes the \( l \)-th component of \( x_1 \);  
3: Calculate the intensity of amplitude of every vertex \( v_i \) at the maximum frequency \( \lambda_1 \), denoted as \( p_i, p_i = x_1^l x_1'^l \);  
4: Add vertex \( v_{1,\text{max}} \) which has the largest intensity at frequency \( \lambda_1 \) to \( C_i \); Delete \( v_i \) and construct center graph \( C(v_{1,\text{max}}) \), if size of \( C(v_{1,\text{max}}) \) is not 1, then \( v_i \leftarrow v_{1,\text{max}} \) and turn to step 2; else turn to step 5;  
5: Select the clique of max size in \( C_i \) as \( C \).

TABLE II. The step of \( \text{Delete}_\text{min} \)

**Input:** Center graph \( G_s \), center vertex \( v_s \)
**Output:** Clique \( C \);  
1: If \( G_s \) is a complete graph, then return all vertexes of \( G_s \) as clique \( C \);  
2: Do eigen-decomposition on adjacent matrix of \( G_s \), The largest eigenvalue is \( \lambda_1 \), corresponding eigenvector is \( x_1 \) and \( x_1^l \) denotes the \( l \)-th component of \( x_1 \);  
3: Delete vertex \( v_{1,\text{min}} \) which has the smallest intensity at frequency \( \lambda_1 \), turn to step 4.

Algorithm A has polynomial complexity. To observe this, we need solve the recursion of algorithm A. Let \( T(n) \) denote the complexity of algorithm A, and \( Y_1(n), Y_2(n), Y_3(n) \) denote the complexity of the corresponding sub-modules. Then the following recursion is satisfied.

\[
T(n) = Y_1(n) + Y_2(n) + Y_3(n) + T(n-1) \quad (20)
\]

As the complexity of \( Y_1(n), Y_2(n), Y_3(n) \) are all \( O(n^4) \), the Eq 20 can be reduced to

\[
T(n) = O(n^4) + T(n-1) \quad (21)
\]

Then from Eq 21 it provide that

\[
T(n) = O(n^5) \quad (22)
\]

We have experimented on random graphs with different edge connecting probability varying from 0.3 to 1 on classical computers. And in our experiments, counter example (the algorithm finds a clique which is not the maximum one) is not found yet. But counter examples can be elaborately constructed. We will illustrate an approach for designing such a graph in the next section.

IV. PROBABLE COUNTER EXAMPLES OF ALGORITHM A

We will present an approach for constructing probable counter examples of algorithm A in this section. Let \( W_s^{v_j} \) denote the number of walks of length \( s \) from the center vertex to the vertex \( v_j \). From the previous section,

\[
W_s^{v_j} = \sum_{k=1}^{N} a_k^{v_j} \lambda_k^s.
\]

Where, \( a_k^{v_j} = \langle s | \lambda_k | v_j \rangle \). Therefore, the amplitude of CTQW is

\[
p_{v_j} = \sum_{k=1}^{N} a_k^{v_j} e^{i \lambda_k t}.
\]

In algorithm A, \( a_k^{v_j} \) of different vertex at the largest eigenvalue \( \lambda_1 \) are compared. For a large enough \( s \), \( a_1^{v_j} \) and \( a_{v_h}^{v_j} \), \( a_1^{v_j} > a_{v_h}^{v_j} \) if and only if \( W_s^{v_j} > W_s^{v_h} \). This implies that if algorithm A is invalid for some graph \( G \), then every member of the maximum clique has some neighbor \( v_h \) that has the largest number of walks \( W_s^{v_h} \). In this case, no matter which vertex belonging to the maximum clique is chosen, \( v_h \) will be chosen in some layer of the recursion, and spontaneously the algorithm A is failed. For simplicity, assume all such subgraphs which are adjacent to every member of the maximum clique are the same, and it is a complete multi-partite graph with degree far larger than the size of MC. For clarity, we propose a kind of graph which is named as base graph. A schematic diagram of base graph is exhibited in Fig.6.

FIG. 6. Base graph. The graph contains a complete subgraph and a complete multi-partite subgraph. The number of independence set is not larger than the clique number, the number of vertices in the complete multi-partite subgraph is as great as possible.

The base graph consists of two elementary sub-graph, one is the maximum clique at the upper of Fig.6, the
other is a complete multi-partite graph containing all the light blue vertices in Fig.6. The vertex set in the same dashed box are not adjacent pairwise, i.e., it is an independent set. And vertices from different dashed box are fully adjacent. Edges can be added between the vertices of the maximum clique and the complete multi-partite graph. Firstly, a vertex, the red vertex in this instance, is chosen as the center vertex. Secondly, every combination of three vertices except the center vertex in the maximum clique are connected to \( q \cdot z \) \((qz > \omega(G) - 3, q + 4 < \omega(G))\) common vertices from at least two partite of the complete multi-partite graph, where \( q \) is the number of partite, \( z \) is the vertices in each independent set and \( \omega(G) \) is the clique number. After the two procedures of adding edges, every three vertices of the maximum clique have a common complete multipartite graph as neighbor. The condition \( qz > \omega(G) - 3 \) makes sure that a non-maximum clique vertex will be chosen in the procedure of the algorithm A, namely algorithm A failed in this situation.

Although algorithm A isn’t universal, it divides all graphs into two classes. The maximum clique of the first class of graphs can be found accurately by algorithm A but the second class of graphs couldn’t. Therefore, if there exists an algorithm can crack the second class of graphs, then the problem can be overcome. The algorithm B is designed to improve the performance of algorithm A for the second class of graphs. And algorithm C is derived from algorithm B by removing recursions. We will further describe them in the next section.

V. VARIATIONAL FREQUENCY SELECTION ALGORITHM FOR FINDING CLIQUE

In the algorithm B, we give greater attention to the contextual information. Supposing that \( v_s \) is a vertex with the smallest intensity at the largest frequency, and we want to find a maximum clique attached to \( v_s \). In the center graph of the original center vertex, \( v_s \) has the smallest intensity at the largest frequency, however, \( v_s \) has a considerable intensity at some frequency \( \lambda_j \), then the other members of the maximum clique are more likely to occur at frequency \( \lambda_j \). Then a vertex \( v_{ref} \) which has largest intensity at frequency \( \lambda_j \) except \( v_s \) is chosen, \( v_{ref} \) is called the reference vertex. The CTQW on the center graph of \( v_s \) and the location of vertex \( v_{ref} \) is used to determine the next reference vertex. The \( v_{ref} \) will become the new center graph and new reference vertex is used to find the next reference vertex. Such a way can apply to the second class of graph at some to some extent. To prove the algorithm B is totally valid for the second class of graphs is beyond this work. The procedure of algorithm B is illustrated in Fig.7.

The third module is named VFSA which is short for variational frequency selection algorithm for finding clique. The input of VFSA is \( C(v_{\lambda_1,min}) \) and \( v_{\ref} \). \( C(v_{\lambda_1,min}) \) is the center graph of vertex \( v_{\lambda_1,min} \) which denotes the vertex with the weakest intensity at the frequency \( \lambda_1 \). \( f_{ref} \) is the frequency where \( v_{\lambda_1,min} \) takes the largest intensity in \( C(v_s) \). And the vertex \( v_{ref} \) is the auxiliary adjacent vertex of \( v_{\lambda_1,min} \) and has the largest intensity except \( v_{\lambda_1,min} \) in frequency \( f_{ref} \). Vertex \( v_{ref} \) acts as the central vertex in the subsequent procedure. The steps of VFSA are listed in the following table.

![Algorithm B](image)

**FIG. 7.** The procedure of algorithm B. The algorithm has four sub-modules and only the third sub-module is different with the algorithm A.

| TABLE III. The step of VFSA |
|-------------------------------|
| **Input:** Center graph \( G \), center vertex \( v_s \), next center vertex \( v_{ref} \) |
| **Output:** Clique \( C(v_{ref}) \) |
| 1. If \( G \) is a complete graph, then return all vertexes of \( G \) as clique \( C \), if not, add \( v_s \) to clique \( C \), turn to step 2 |
| 2. Do eigen-decomposition on adjacent matrix of \( G \). The largest eigenvalue is \( \lambda_1 \), corresponding eigenvector is \( x_1 \) and \( x_1^t \) denotes the \( l \)-th component of \( x_1 \). Find a frequency \( f_{ref} \) such that vertex \( v_{ref} \) can take the largest intensity among all frequencies |
| 3. Find vertex \( v_{ref} \) which has the largest intensity at frequency \( f_{ref} \) in the neighbors of vertex \( v_{ref} \) |
| 4. Delete \( v_s \) and let \( v_s \leftarrow v_{ref} \), \( v_{ref} \leftarrow v_{ref} \), \( G_s \leftarrow C(v_{ref}) \), turn to step 1 |

In both algorithm A and algorithm B, the recursion is utilized to eliminate probable interference from low connectivity vertices. However, because a vertex will be removed in the recursion, this may lead not all of the maximum cliques are found when there are multiple maximum clique attached to one same vertex. For this case, we will transform algorithm B to an algorithm without recursion. The recursion-less algorithm, is called algorithm C, just calls VFSA for every vertex of the given graph, namely calls VFSA \( N - 1 \) times. The time complexity of algorithm C is \( O(N^5) \), as the complexity of NFSA is \( O(N^4) \) time.

VI. CONCLUSION

In this theme, we show that the clique structure of a graph is related with the CTQW. On some ideal graphs, the intensity of frequency of probability amplitude of CTQW is a good feature for direct speculating whether a vertex is a member of the maximum clique or not. As the frequencies of CTQW are the eigenvalues of the adja-
cency matrix, that is to say, the clique structure is related to the eigenvalues and the corresponding eigenvectors.

But for general graphs, this feature is not so obvious, and one can not find the maximum clique directly. To reveal the hidden maximum clique, we propose two recursive algorithms, the Algorithm A and algorithm B, with $N^5$ time complexity to find the maximum clique on graphs by using CTQW. We further simplify algorithm B to algorithm C which has $N^4$ time complexity. It seems that the algorithm A is valid for random graphs via numerical experiments. But counter examples can be elaborately constructed, we have shown how to construct such graphs. For such graphs whose maximum clique can not be found by algorithm A, we propose algorithm B to fix. To conform that whether the algorithm B can be used for all graphs or not is beyond this article. And it will be the future work to undertake.

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