EXISTENCE AND REGULARITY OF SOLUTIONS OF FRACTIONAL DIFFERENTIAL EQUATIONS INVOLVING HILFER FRACTIONAL DERIVATIVE OF ORDER $1 < \alpha < 2$ AND TYPE $0 \leq \beta \leq 1$

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Abstract. In this paper we investigate the regularity of a solution of a linear problem involving Hilfer fractional derivative. We define the mild solution of an abstract Cauchy problem and obtain conditions under which a mild solution becomes a strong solution. We also study a semi-linear fractional evolution equation and give a suitable definition of a mild solution and establish some existence results for a mild solution.

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1. Introduction

In the last few decades, fractional calculus has been emerged as an efficient tool for analyzing various real problems due to its nonlocal behavior and linearity, this include fractional oscillator, viscoelastic models, diffusion in porous media, signal analysis, complex systems, medical imaging, pollution control and population growth etc. Several definitions of fractional derivatives and integrals are present in literature. The mostly used are Riemann-Liouville, Caputo and Grunwald-Letnikov fractional derivative etc. Later, Hilfer introduced a new notion of fractional derivative named as “Generalized fractional derivative of order $\alpha$ and type $\beta \in [0, 1]$”, in theoretical modeling of broadband dielectric relaxation spectroscopy for glasses, which generalizes both Riemann-Liouville and Caputo fractional derivative. It contains both Riemann-Liouville and Caputo fractional derivative as a special case for $\beta = 0$ and $\beta = 1$ respectively. Some important research papers containing Hilfer fractional derivative are [12], [13], [14], [15], [16], [17], [18], [19], [20]. In most of the cases $0 < \alpha < 1$.

In [2], Mei et. al studied the properties of Mittag-Leffler function $E_\alpha(at^\alpha)$ for $1 < \alpha < 2$ and defined a function named $\alpha$-order cosine function having the similar properties as $E_\alpha(at^\alpha)$ and proved that it is associated with a solution operator of an $\alpha$-order abstract Cauchy problem. The $\alpha$-order Cauchy problem is well-posed iff the linear operator generates an $\alpha$-order cosine function. In [5], Mei et. al studied $\alpha$-order Cauchy problem with Riemann-Liouville fractional derivative, $1 < \alpha < 2$, by defining a new family of bounded linear operators called as “Riemann-Liouville
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Shu et al. [4] investigated a fractional semilinear integro-differential equation of order $1 < \alpha < 2$

$$D_0^\alpha u(t) = Au(t) + f(t, u(t)) + \int_0^t q(t-s) g(s, u(s)) ds, \; t \in [0, T],$$  \hspace{1cm} (1.1)

$$u(0) + m(u) = u_0 \in X, \; u'(0) + n(u) = u_1 \in X$$  \hspace{1cm} (1.2)

in a Banach space $X$, where $A : D(A) \subset X \to X$ is a sectorial operator of type $(M, \theta, \alpha, \mu)$. They defined some families of operators to give a suitable definition of a mild solution of the problem and established the conditions for the existence of a mild solution by using Krasnoselskii’s fixed point theorem and contraction mapping principle. Similar type of problems has been discussed in [6]-[3]. In [6], Wang et al. investigated the existence of positive mild solutions by using Schauder’s fixed point theorem and Krasnoselskii’s fixed point theorem. In [3], Zhu et al. studied the existence of local and global mild solutions by using the solution operator, measure of non-compactness and some fixed theorems with compact and non-compact semigroups.

Mei et al. [1] considered a general fractional differential equations of order $1 < \alpha < 2$ and type $\beta \in [0, 1]$ described as follows

$$D_0^{\alpha,\beta} u(t) = Au(t), \; t > 0$$  \hspace{1cm} (1.3)

$$(g^{(1-\beta)(2-\alpha)} * u)(0) = 0, \; (g^{(1-\beta)(2-\alpha)} * u)'(0) = y,$$  \hspace{1cm} (1.4)

where $A : D(A) \subset X \to X$ is a closed densely defined linear operator on a Banach space $X$ and $D_0^{\alpha,\beta} u(t)$ is Hilfer fractional derivative of order $\alpha$ and type $\beta$. They defined a new family of bounded linear operators, named general fractional sine function of order $\alpha \in (1, 2)$ and type $\beta \in [0, 1]$ on Banach space $X$. They showed that they are essentially equivalent to a general fractional resolvent and used such theory to study the well-posedness of the above general fractional differential equations.

Motivated by the above articles, we consider the following linear

$$D_0^{\alpha,\beta} u(t) = Au(t),$$  \hspace{1cm} (1.5)

$$(g^{(1-\beta)(2-\alpha)} * u)(0) = u_1, \; (g^{(1-\beta)(2-\alpha)} * u)'(0) = u_2,$$  \hspace{1cm} (1.6)

and semilinear fractional differential equation

$$D_0^{\alpha,\beta} u(t) = Au(t) + f(t, u(t))$$  \hspace{1cm} (1.7)

$$(g^{(1-\beta)(2-\alpha)} * u)(0) = u_1, \; (g^{(1-\beta)(2-\alpha)} * u)'(0) = u_2,$$  \hspace{1cm} (1.8)

in the Banach space $X$, where $A$ is a densely defined closed linear operator on $X$. In this paper, we firstly consider a linear fractional differential equation in $\mathbb{R}$ and discuss about the existence of solution. Later we will generalize the properties of $t^{\alpha+\beta(2-\alpha)-2} \mathcal{E}_{\alpha,\alpha+\beta(2-\alpha)-1}(ct^\alpha)$ to give definition of a family of bounded linear operators, that will be used to study the problems (1.5), (1.6) and (1.7), (1.8).
2. Preliminaries

In this section, we give some basic definitions related to fractional derivatives and integrals and some results of measure of non-compactness.

**Definition 1.** Let \( \alpha > 0 \). The \( \alpha \)th order Riemann-Liouville fractional integral of a function \( u \) is defined and denoted by

\[
J_0^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)ds, \quad t > 0.
\]

**Proposition 1.** \([8]\) If \( W \subset C(I, X) \) is bounded and equicontinuous, then \( t \to \psi(W(t)) \) is continuous on \( I \), and

\[
\psi\left( \int_0^t W(s)ds \right) \leq \int_0^t \psi(W(s))ds \quad \text{for} \quad t \in I, \quad \psi(W) = \max_{t \in I} \psi(W(t)).
\]

**Proposition 2.** \([8]\) For a sequence of Bochner integrable functions \( \{u_n : I \to X\} \) with \( \|u_n(t)\| \leq m(t) \) for almost all \( t \in I \) and every \( n \geq 1 \), where \( m \in L^1(I, \mathbb{R}^+) \),
Proposition 3. \cite{11} If $W$ is bounded, then for $\epsilon > 0$, there is a sequence $\{w_n\}_{n=1}^\infty \subset W$, such that

$$\psi(W) \leq 2\psi(\{w_n\}_{n=1}^\infty) + \epsilon.$$ 

3. Linear Problem

In this section, we consider the following linear fractional differential problem

$$D_0^{\alpha,\beta} u(t) = cu(t) + f(t), \quad t > 0$$

(3.1)

where $c$ is a constant, $f : [0, T] \to \mathbb{R}$ and $g_\gamma(t) = \frac{t^\gamma}{\Gamma(\gamma)}, \gamma > 0$.

Lemma 1. If $f \in C^1([0, T], \mathbb{R})$, then $u(t) = \frac{t^{\alpha+\beta(2-\alpha)} - 2 E_{\alpha,\alpha+\beta(2-\alpha)-1}(ct^\alpha)x + t^{\alpha+\beta(2-\alpha)-1} E_{\alpha,\alpha+\beta(2-\alpha)-1}(ct^\alpha)y + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(c(t-s)^\alpha) f(s) ds}{\alpha,\alpha}$

is solution of the problem (3.1)-(3.2).

Proof. Let $u_1(t) = \frac{t^{\alpha+\beta(2-\alpha)} - 2 E_{\alpha,\alpha+\beta(2-\alpha)-1}(ct^\alpha)x + t^{\alpha+\beta(2-\alpha)-1} E_{\alpha,\alpha+\beta(2-\alpha)-1}(ct^\alpha)y}{\alpha,\alpha}$

and $u_2(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(c(t-s)^\alpha) f(s) ds$. Then

$$J_t^{(1-\beta)(2-\alpha)} u_2(t) = \frac{1}{\Gamma((1-\beta)(2-\alpha))} \int_0^t (t-\tau)^{1-\beta(2-\alpha)-1} \int_0^\tau (\tau-s)^{\alpha-1} E_{\alpha,\alpha}(c(\tau-s)^\alpha) f(s) ds d\tau$$

$$= \frac{1}{\Gamma((1-\beta)(2-\alpha))} \int_0^t f(s) \int_s^t (t-\tau)^{1-\beta(2-\alpha)-1} (\tau-s)^{\alpha-1} E_{\alpha,\alpha}(c(\tau-s)^\alpha) d\tau ds$$

$$= \int_0^t (t-s)^{1-\beta(2-\alpha)} E_{\alpha,2-\beta(2-\alpha)}(c(t-s)^\alpha) f(s) ds.$$ 

Hence

$$\frac{d}{dt} J_t^{(1-\beta)(2-\alpha)} u_2(t) = \frac{d}{dt} \int_0^t (t-s)^{1-\beta(2-\alpha)} E_{\alpha,2-\beta(2-\alpha)}(c(t-s)^\alpha) f(s) ds$$

$$= \int_0^t (t-s)^{-\beta(2-\alpha)} E_{\alpha,1-\beta(2-\alpha)}(c(t-s)^\alpha) f(s) ds$$

$$= \int_0^t u^{-\beta(2-\alpha)} E_{\alpha,1-\beta(2-\alpha)}(cu^\alpha) f(t-u) du.$$ 

(3.3)

Now using (3.3), we have

$$\frac{d^2}{dt^2} J_t^{(1-\beta)(2-\alpha)} u_2(t) = \frac{d}{dt} \int_0^t u^{-\beta(2-\alpha)} E_{\alpha,1-\beta(2-\alpha)}(cu^\alpha) f(t-u) du$$

$$= \int_0^t u^{-\beta(2-\alpha)} E_{\alpha,1-\beta(2-\alpha)}(cu^\alpha) f(t-u) du + t^{-\beta(2-\alpha)} E_{\alpha,1-\beta(2-\alpha)}(ct^\alpha) f(0)$$

$$= \int_0^t (t-u)^{-\beta(2-\alpha)} E_{\alpha,1-\beta(2-\alpha)}(c(t-u)^\alpha) f(t-u) du + t^{-\beta(2-\alpha)} E_{\alpha,1-\beta(2-\alpha)}(ct^\alpha) f(0).$$
Then
\[ D_0^{\alpha,\beta}u_2(t) = J_t^{\beta(2-\alpha)} \frac{d^2}{dt^2} J_t^{1-\beta}(2-\alpha)u_2(t) \]
\[ = f(t) - E_{\alpha}(ct^\alpha)f(0) + c \int_0^t (t-u)^{\alpha-1} E_{\alpha,\alpha}(c(t-u)^{\alpha})f(u)du + E_{\alpha}(ct^\alpha)f(0) \]
\[ = f(t) + c \int_0^t (t-u)^{\alpha-1} E_{\alpha,\alpha}(c(t-u)^{\alpha})f(u)du. \]

Hence,
\[ D_0^{\alpha,\beta}u(t) = c(t^{\alpha+\beta(2-\alpha)-2}E_{\alpha,\alpha+\beta(2-\alpha)-1}(ct^\alpha)x + t^{\alpha+\beta(2-\alpha)-1}E_{\alpha,\alpha+\beta(2-\alpha)}(ct^\alpha)y \]
\[ + c \int_0^t (t-u)^{\alpha-1} E_{\alpha,\alpha}(c(t-u)^{\alpha})f(u)du + f(t) = cu(t) + f(t), t \in (0, T]. \]

Clearly,
\[ J_t^{1-\beta}(2-\alpha)u(t) = E_{\alpha,1}(ct^\alpha)x + tE_{\alpha,2}(ct^\alpha)y + \int_0^t (t-s)E_{\alpha,1-\beta}(2-\alpha)(ct^\alpha)f(s)ds \to x \text{ and } \frac{d}{dt}J_t^{1-\beta}(2-\alpha)u(t) = ct^{\alpha-1}E_{\alpha,\alpha}(ct^\alpha)x + E_{\alpha,1}(ct^\alpha)y + \int_0^t (t-s)^{-\beta(2-\alpha)}E_{\alpha,1-\beta}(2-\alpha)(ct^\alpha)f(s)ds \to y, \text{ as } t \to 0, \text{ by dominated convergence theorem. Hence } u(t) = t^{\alpha+\beta(2-\alpha)-2}E_{\alpha,\alpha+\beta(2-\alpha)-1}(ct^\alpha)x + t^{\alpha+\beta(2-\alpha)-1}E_{\alpha,\alpha+\beta(2-\alpha)}(ct^\alpha)y + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(c(t-s)^{\alpha})f(s)ds \text{ is the solution of the problem } (3.1)-(3.2). \]

**Lemma 2.** If \( f \in J_t^{\beta(2-\alpha)}(L^1) \), then \( u(t) = t^{\alpha+\beta(2-\alpha)-2}E_{\alpha,\alpha+\beta(2-\alpha)-1}(ct^\alpha)x + t^{\alpha+\beta(2-\alpha)-1}E_{\alpha,\alpha+\beta(2-\alpha)}(ct^\alpha)y + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(c(t-s)^{\alpha})f(s)ds \) is solution of the problem \((3.1)-(3.2)\).

**Proof.** Similarly as in Lemma 1 define \( u_1(t) = t^{\alpha+\beta(2-\alpha)-2}E_{\alpha,\alpha+\beta(2-\alpha)-1}(ct^\alpha)x + t^{\alpha+\beta(2-\alpha)-1}E_{\alpha,\alpha+\beta(2-\alpha)}(ct^\alpha)y \) and \( u_2(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(c(t-s)^{\alpha})f(s)ds \). Then using [3.3], we have
\[ D_t^{\alpha,\beta}u_2(t) = J_t^{\beta(2-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\beta(2-\alpha)} E_{\alpha,1-\beta}(2-\alpha)(c(t-s)^{\alpha})f(s)ds \]
\[ = J_t^{\beta(2-\alpha)} \frac{d}{dt} I. \]

Using Theorem 5.1 of [21], we have
\[ I = c \int_0^t (t-s)^{\alpha-\beta(2-\alpha)} E_{\alpha,\alpha-\beta(2-\alpha)+1}(c(t-s)^{\alpha})f(s)ds + J_t^{1-\beta(2-\alpha)}f(t). \]

Since \( f \in J_t^{\beta(2-\alpha)}(L^1) \), there exist a function \( \phi \in L^1(0, T) \) such that \( f(t) = J_t^{\beta(2-\alpha)}\phi(t) \). Now
\[ \frac{dI}{dt} = c \int_0^t (t-s)^{\alpha-\beta(2-\alpha)-1} E_{\alpha,\alpha-\beta(2-\alpha)}(c(t-s)^{\alpha})f(s)ds + \frac{d}{dt}J_t^{1-\beta(2-\alpha)}f(t) \]
\[ = c \int_0^t (t-s)^{\alpha-\beta(2-\alpha)-1} E_{\alpha,\alpha-\beta(2-\alpha)}(c(t-s)^{\alpha})f(s)ds + \frac{d}{dt}J_t^{1-\beta(2-\alpha)}J_t^{\beta(2-\alpha)}\phi(t) \]
\[ = c \int_0^t (t-s)^{\alpha-\beta(2-\alpha)-1} E_{\alpha,\alpha-\beta(2-\alpha)}(c(t-s)^{\alpha})f(s)ds + \phi(t). \]
Using (3.4), we have
\[
D_0^{\alpha,\beta}u_2(t) = c \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(c(t-s)^{\alpha})f(s)ds + J_t^\beta(2-\alpha)\phi(t)
\]
\[
= c \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(c(t-s)^{\alpha})f(s)ds + f(t).
\]
Hence
\[
D_0^{\alpha,\beta}u(t) = f(t) + cu(t).
\]
\[
\square
\]

4. Linear Abstract Hilfer Fractional Cauchy Problem

In this section, we consider the following Cauchy problems
\[
D_0^{\alpha,\beta}u(t) = Au(t)
\]
\[
(g(1-\beta)(2-\alpha) * u)(0) = x, \quad (g(1-\beta)(2-\alpha) * u)'(0) = 0
\]
in a Banach space \(X\), where \(A : D(A) \subseteq X\) is a linear densely defined operator on \(X\).

**Definition 3.** We define general fractional resolvent operator \(\{C_{\alpha,\beta}\}_{t > 0}\) of order \(\alpha \in (1,2)\) and type \(0 \leq \beta \leq 1\) as a family of bounded linear operators such that following are satisfied

(i) \(C_{\alpha,\beta}(t)\) is strongly continuous i.e. for any \(x \in X, t \rightarrow C_{\alpha,\beta}(t)x\) is continuous over \((0, \infty)\):

(ii) \(\lim_{t \rightarrow 0^+} \frac{C_{\alpha,\beta}(t)x}{x + t^{\alpha+\beta(2-\alpha) - 1}} = \frac{x}{t^{\alpha+\beta(2-\alpha) - 1}}\);

(iii) \(C_{\alpha,\beta}(t)C_{\alpha,\beta}(s) = C_{\alpha,\beta}(s)C_{\alpha,\beta}(t)\) for all \(t, s > 0\).

(iv) \(C_{\alpha,\beta}(s)J_t^\alpha C_{\alpha,\beta}(t) - J_s^\alpha C_{\alpha,\beta}(s)C_{\alpha,\beta}(t) = \frac{s^{\alpha+\beta(2-\alpha) - 2}}{\Gamma(\alpha + \beta(2 - \alpha) - 1)} J_t^\alpha C_{\alpha,\beta}(t) - \frac{t^{\alpha+\beta(2-\alpha) - 2}}{\Gamma(\alpha + \beta(2 - \alpha) - 1)} J_s^\alpha C_{\alpha,\beta}(s)\) for all \(t, s > 0\).

We define the generator of general fractional resolvent operator as follows,

**Definition 4.** Let \(\{C_{\alpha,\beta}(t)\}_{t > 0}\) be a general fractional resolvent operator of order \(\alpha\) and type \(\beta\) on Banach space \(X\). We define the domain of the operator \(A\) as

\[
D(A) = \left\{ x \in X : \lim_{t \rightarrow 0^+} \left( \frac{C_{\alpha,\beta}(t)x - \frac{t^{\alpha+\beta(2-\alpha) - 2}}{\Gamma(\alpha + \beta(2 - \alpha) - 1)} x}{t^{\alpha+\beta(2-\alpha) - 2}} \right) \text{ exists} \right\}.
\]

Then, the operator \(A : D(A) \rightarrow X\) is defined by

\[
Ax = \Gamma(2\alpha + \beta(2 - \alpha) - 1) \lim_{t \rightarrow 0^+} \left( \frac{C_{\alpha,\beta}(t)x - \frac{t^{\alpha+\beta(2-\alpha) - 2}}{\Gamma(\alpha + \beta(2 - \alpha) - 1)} x}{t^{\alpha+\beta(2-\alpha) - 2}} \right).
\]

**Proposition 4.** Let \(\{C_{\alpha,\beta}(t)\}_{t > 0}\) be a general fractional resolvent operator of order \(\alpha\) and type \(\beta\) with generator \(A\), then

a) \(C_{\alpha,\beta}(t)D(A) \subseteq D(A)\) and \(C_{\alpha,\beta}(t)Ax = AC_{\alpha,\beta}(t)x \quad \forall \ x \in D(A)\);

b) \(\forall x \in X, t > 0 J_t^\alpha C_{\alpha,\beta}(t)x \in D(A)\) and

\[
C_{\alpha,\beta}(t)x = \frac{t^{\alpha+\beta(2-\alpha) - 2}}{\Gamma(\alpha + \beta(2 - \alpha) - 1)} x + AJ_t^\alpha C_{\alpha,\beta}(t)x;
\]
c) For all $x \in D(A)$
\[
C_{\alpha,\beta}(t)x = \frac{t^{\alpha+\beta(2-\alpha)-2}}{\Gamma(\alpha + \beta(2-\alpha) - 1)} x + J_0^\alpha C_{\alpha,\beta}(t)Ax;
\]
\]

d) $A$ is closed and densely defined.
\]
e) $A$ admits at most one general fractional resolvent operator of order $\alpha$ and type $\beta$.
\]
f) $C_{\alpha,\beta}(.)x \in C^1((0, \infty); X)$ for $x \in D(A)$.
\]
Proof.  
\]
a) Let $x \in D(A)$. For all $t, s > 0$, we have
\[
C_{\alpha,\beta}(s)C_{\alpha,\beta}(t)x - \frac{s^{\alpha+\beta(2-\alpha)-2}}{\Gamma(\alpha + \beta(2-\alpha) - 1)} C_{\alpha,\beta}(t)x = \frac{C_{\alpha,\beta}(t)C_{\alpha,\beta}(s)x - s^{\alpha+\beta(2-\alpha)-2}}{\Gamma(\alpha + \beta(2-\alpha) - 1)} C_{\alpha,\beta}(t)x.
\]
Since $C_{\alpha,\beta}(t)$ is bounded linear operator, we have
\[
\Gamma(\alpha + \beta(2-\alpha) - 1) \lim_{s \to 0^+} \frac{C_{\alpha,\beta}(s)C_{\alpha,\beta}(t)x - s^{\alpha+\beta(2-\alpha)-2}}{\Gamma(\alpha + \beta(2-\alpha) - 1)} C_{\alpha,\beta}(t)x = C_{\alpha,\beta}(t)Ax.
\]
Hence $C_{\alpha,\beta}(t)x \in D(A)$ and $AC_{\alpha,\beta}(t)x = C_{\alpha,\beta}(t)Ax$.

b) Let $x \in X$,
\[
\Gamma(\alpha + \beta(2-\alpha) - 1) \lim_{s \to 0^+} \frac{C_{\alpha,\beta}(s)C_{\alpha,\beta}(t)x - s^{\alpha+\beta(2-\alpha)-2}}{\Gamma(\alpha + \beta(2-\alpha) - 1)} C_{\alpha,\beta}(t)x
\]
\[
= \Gamma(\alpha + \beta(2-\alpha) - 1) \lim_{s \to 0^+} \frac{J_0^\alpha C_{\alpha,\beta}(s)(C_{\alpha,\beta}(t)x - \frac{s^{\alpha+\beta(2-\alpha)-2}}{\Gamma(\alpha + \beta(2-\alpha) - 1)} C_{\alpha,\beta}(t)x)}{s^{\alpha+\beta(2-\alpha)-2}}
\]
\]
Now we claim that
\]
\[
\left| \Gamma(\alpha + \beta(2-\alpha) - 1) \frac{J_0^\alpha C_{\alpha,\beta}(s)x}{s^{\alpha+\beta(2-\alpha)-2}} - x \right|
\]
\[
= \left| \frac{\Gamma(\alpha)}{\Gamma(\alpha + \beta(2-\alpha) - 1)} \int_0^1 s^{\alpha-2-\beta(2-\alpha)}(s - \tau)^{\alpha-1} C_{\alpha,\beta}(\tau)x d\tau - x \right|
\]
\[
= \left| \frac{\Gamma(\alpha)}{\Gamma(\alpha + \beta(2-\alpha) - 1)} \int_0^1 s^{\alpha-\beta(2-\alpha)}(1 - \tau)^{\alpha-1} C_{\alpha,\beta}(s\tau)x d\tau - x \right|
\]
\[
= \left| \frac{\Gamma(\alpha + \beta(2-\alpha) - 1)}{\Gamma(\alpha)} \int_0^1 (1 - \tau)^{\alpha-1} \tau^{\alpha+\beta(2-\alpha)-2} \Gamma(\alpha + \beta(2-\alpha) - 1) \frac{C_{\alpha,\beta}(s\tau)}{(s\tau)^{\alpha+\beta(2-\alpha)-2}} x d\tau - x \right|
\]
\[
= \left| \frac{\Gamma(\alpha + \beta(2-\alpha) - 1)}{\Gamma(\alpha)} \int_0^1 (1 - \tau)^{\alpha-1} \tau^{\alpha+\beta(2-\alpha)-2} d\tau \sup_{\tau \in [0, 1]} \left| \Gamma(\alpha + \beta(2-\alpha) - 1) \frac{C_{\alpha,\beta}(s\tau)}{(s\tau)^{\alpha+\beta(2-\alpha)-2}} x - x \right| \right|
\]
\]
This implies
\]
\[
\Gamma(\alpha + \beta(2-\alpha) - 1) \left( \lim_{s \to 0^+} \frac{J_0^\alpha C_{\alpha,\beta}(s)x}{s^{\alpha+\beta(2-\alpha)-2}} - x \right) = x.
\]
(4.5)
c) Let $x \in D(A)$

\[
C_{\alpha,\beta}(t)x - \frac{t^{\alpha+\beta(2-\alpha)-2}}{\Gamma(\alpha + \beta(2 - \alpha) - 1)} x = AJ_0^x C_{\alpha,\beta}(t)x
\]

\[
= \frac{\Gamma(2\alpha + \beta(2 - \alpha) - 1)}{\Gamma(\alpha)} \lim_{s \to 0^+} \left( C_{\alpha,\beta}(s) - \frac{s^{\alpha+\beta(2-\alpha)-2}}{\Gamma(\alpha+\beta(2-\alpha)-1)} \right) \int_0^t (t-\tau)^{\alpha-1} C_{\alpha,\beta}(\tau) x d\tau
\]

\[
= \frac{\Gamma(2\alpha + \beta(2 - \alpha) - 1)}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} C_{\alpha,\beta}(\tau) \lim_{s \to 0^+} \left( C_{\alpha,\beta}(s) - \frac{s^{\alpha+\beta(2-\alpha)-2}}{\Gamma(\alpha+\beta(2-\alpha)-1)} \right) x d\tau
\]

\[
= \frac{\Gamma(2\alpha + \beta(2 - \alpha) - 1)}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} C_{\alpha,\beta}(\tau) \lim_{s \to 0^+} \left( C_{\alpha,\beta}(s) - \frac{s^{\alpha+\beta(2-\alpha)-2}}{\Gamma(\alpha+\beta(2-\alpha)-1)} \right) x d\tau
\]

\[
= J_1^x C_{\alpha,\beta}(t)Ax.
\]

Let $x_n \in D(A), x_n \to x$ and $Ax_n \to y$ as $n \to \infty$.

\[
C_{\alpha,\beta}(t)x - \frac{t^{\alpha+\beta(2-\alpha)-2}}{\Gamma(\alpha + \beta(2 - \alpha) - 1)} x = \lim_{n \to \infty} \left( C_{\alpha,\beta}(t)x_n - \frac{t^{\alpha+\beta(2-\alpha)-2}}{\Gamma(\alpha + \beta(2 - \alpha) - 1)} x_n \right)
\]

\[
= \lim_{n \to \infty} J_0^x C_{\alpha,\beta}(t)Ax_n
\]

\[
= \lim_{n \to \infty} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} C_{\alpha,\beta}(s) Ax_n ds
\]

\[
= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} C_{\alpha,\beta}(s) y ds
\]

\[
= J_0^x C_{\alpha,\beta}(t)y. \tag{4.6}
\]

Using definition of $D(A)$ and (4.5), we have

\[
Ax = \Gamma(2\alpha + \beta(2 - \alpha) - 1) \lim_{t \to 0^+} \left( C_{\alpha,\beta}(t)x - \frac{t^{\alpha+\beta(2-\alpha)-2}}{\Gamma(\alpha+\beta(2-\alpha)-1)} x \right)
\]

\[
= \Gamma(2\alpha + \beta(2 - \alpha) - 1) \lim_{t \to 0^+} J_0^x C_{\alpha,\beta}(t)y = y. \tag{4.7}
\]

Hence $A$ is closed operator.

For $x \in X$, set $x_t = J_t^0 C_{\alpha,\beta}(t)x$. Then $x_t \in D(A)$ from $(b)$ and by (4.5) we have $\Gamma(2\alpha + \beta(2 - \alpha) - 1) \lim_{t \to 0^+} J_0^x C_{\alpha,\beta}(t)x = x$. Thus $D(A) = X$.

e) Follows from $(c), (d)$ and Titchmarsh’s Theorem.

f) Next claim $C_{\alpha,\beta}(t)x \in C^1((0, \infty); X)$ for any $x \in D(A)$.

\[
\frac{d}{dt} C_{\alpha,\beta}(t)x = \frac{d}{dt} \left( \frac{t^{\alpha+\beta(2-\alpha)-2}}{\Gamma(\alpha + \beta(2 - \alpha) - 1)} x + J_0^x C_{\alpha,\beta}(t)Ax \right)
\]

\[
= \frac{\alpha + \beta(2 - \alpha) - 2}{\Gamma(\alpha + \beta(2 - \alpha) - 1)} \frac{t^{\alpha+\beta(2-\alpha)-3}}{\Gamma(\alpha + \beta(2 - \alpha) - 1)} x + J_1^x C_{\alpha,\beta}(t)Ax
\]

\[
= \frac{\alpha + \beta(2 - \alpha) - 2}{\Gamma(\alpha + \beta(2 - \alpha) - 1)} \frac{t^{\alpha+\beta(2-\alpha)-3}}{\Gamma(\alpha + \beta(2 - \alpha) - 1)} x + \frac{1}{\Gamma(\alpha - 1)} \int_0^t (t-\tau)^{\alpha-2} C_{\alpha,\beta}(\tau) Ax d\tau
\]

\[
\frac{\alpha + \beta(2 - \alpha) - 2}{\Gamma(\alpha + \beta(2 - \alpha) - 1)} \frac{t^{\alpha+\beta(2-\alpha)-3}}{\Gamma(\alpha + \beta(2 - \alpha) - 1)} x + \frac{e-1}{\Gamma(\alpha - 1)} \int_0^1 (1-\tau)^{\alpha-2} C_{\alpha,\beta}(\tau) Ax d\tau.
\]

The dominated convergence theorem gives $C_{\alpha,\beta}(t)x \in C^1((0, \infty); X)$. 

\[\square\]
Definition 5. We define the solution operator \{T(t)\}_{t > 0} of problem 4.7-4.8 as a family of bounded linear operators on \(X\) such that

1. for any \(x \in X\), \(T(.)x \in C([0, \infty), X)\) and \(\Gamma(\alpha + \beta(2-\alpha) - 1)\frac{T(t)}{t^{\alpha+\beta(2-\alpha)-1}}x = x \forall x \in X\);
2. \(T(t)D(A) \subset D(A)\), \(AT(t)x = T(t)Ax \forall x \in D(A)\);
3. For all \(x \in D(A)\)

\[ T(t)x = \frac{t^{\alpha+\beta(2-\alpha)-2}}{\Gamma(\alpha + \beta(2-\alpha) - 1)}x + J_t^{\alpha}T(t)Ax. \]

Definition 6. A function \(u \in C((0, \infty), X)\) is called a mild solution if \(J_t^{\alpha}u(t) \in D(A)\) and \(u(t) = \frac{t^{\alpha+\beta(2-\alpha)-2}}{\Gamma(\alpha + \beta(2-\alpha) - 1)}x + AJ_t^{\alpha}u(t), \ t \in (0, \infty)\).

Definition 7. A function \(u \in C((0, \infty), X)\) is called a strong solution if \(u(t) \in D(A)\) for all \(t > 0\) and \(D_t^{\alpha,\beta}u(t)\) is continuous on \((0, \infty)\) and \(4.7-4.8\) holds.

Theorem 1. Let \(A\) be the generator of general fractional resolvent operator of order \(\alpha\) and type \(\beta\) i.e. \(C_{\alpha,\beta}(t)\), then \(C_{\alpha,\beta}(t)x\) is strong solution for \(x \in D(A)\).

Proof. Let \(x \in D(A)\), then

\[ C_{\alpha,\beta}(t)x = \frac{t^{\alpha+\beta(2-\alpha)-2}}{\Gamma(\alpha + \beta(2-\alpha) - 1)}x + J_t^{\alpha}C_{\alpha,\beta}(t)Ax \]

and

\[ J_t^{(1-\beta)(2-\alpha)}C_{\alpha,\beta}(t)x = x + J_t^{2-\beta(2-\alpha)}C_{\alpha,\beta}(t)Ax. \]

Hence

\[ \lim_{t \to 0^+} J_t^{(1-\beta)(2-\alpha)}C_{\alpha,\beta}(t)x = x + \lim_{t \to 0^+} J_t^{2-\beta(2-\alpha)}C_{\alpha,\beta}(t)Ax \]

\[ = x + \lim_{t \to 0^+} \frac{1}{\Gamma(2 - \beta(2-\alpha))} \int_0^t (t-s)^{1-\beta(2-\alpha)}C_{\alpha,\beta}(s)Axds \]

\[ = x + \lim_{t \to 0^+} \frac{1}{\Gamma(2 - \beta(2-\alpha))} \int_0^1 t^{2-\beta(2-\alpha)}(1-u)^{1-\beta(2-\alpha)}C_{\alpha,\beta}(tu)Axdu \]

\[ = x + \lim_{t \to 0^+} \frac{1}{\Gamma(2 - \beta(2-\alpha))} \int_0^1 u^{\alpha+\beta(2-\alpha)-2}(1-u)^{1-\beta(2-\alpha)}(tu)^{2-\alpha-\beta(2-\alpha)}C_{\alpha,\beta}(tu)Axdu. \]

\[ = x, \]

by Lebesgue’s dominated convergence theorem. Similarly

\[ \frac{d}{dt}J_t^{(1-\beta)(2-\alpha)}C_{\alpha,\beta}(t)x = J_t^{1-\beta(2-\alpha)}C_{\alpha,\beta}(t)Ax \]

\[ = \frac{1}{\Gamma(1-\beta(2-\alpha))} \int_0^t (t-s)^{-\beta(2-\alpha)}C_{\alpha,\beta}(s)Axds \]

\[ = \frac{1}{\Gamma(1-\beta(2-\alpha))} \int_0^1 t^{1-\beta(2-\alpha)}(1-u)^{-\beta(2-\alpha)}T(tu)Axdu \]

\[ = \frac{1}{\Gamma(1-\beta(2-\alpha))} \int_0^1 u^{\alpha+\beta(2-\alpha)-2}(1-u)^{-\beta(2-\alpha)}(tu)^{2-\alpha-\beta(2-\alpha)}C_{\alpha,\beta}(tu)Axdu \]

\[ \to 0 \text{ as } t \to 0^+. \]
Then
\[ \frac{d^2}{dt^2} J_t^{(1-\beta)(2-\alpha)} C_{\alpha,\beta}(t)x = \frac{d}{dt} J_t^{1-\beta(2-\alpha)} C_{\alpha,\beta}(t)Ax = D^\beta_{0}(2-\alpha)C_{\alpha,\beta}(t)Ax. \]

Hence
\[ D_{0}^\beta C_{\alpha,\beta}(t)x = J_t^{\beta(2-\alpha)} D_{0}^\beta C_{\alpha,\beta}(t)Ax = C_{\alpha,\beta}(t)Ax = AC_{\alpha,\beta}(t)x. \]

**Lemma 3.** Let \( A \) be the generator of general fractional resolvent operator of order \( \alpha \) and type \( \beta \). Then there exist a solution operator \( S_\alpha(t) \) for the following fractional differential equation
\[
D_{\alpha}^\alpha [u(t) - x] = Au(t), \quad t > 0, \quad (4.8)
\]
\[
u(0) = x, \quad u'(0) = 0. \quad (4.9)
\]

**Proof.** Define \( S_\alpha(t) = J_t^{(1-\beta)(2-\alpha)} C_{\alpha,\beta}(t), t > 0 \).

For any \( x \in X \),
\[
\lim_{t \to 0^+} S_\alpha(t)x = \lim_{t \to 0^+} J_t^{(1-\beta)(2-\alpha)} C_{\alpha,\beta}(t)x = \frac{1}{\Gamma((1-\beta)(2-\alpha))} \lim_{t \to 0^+} \int_0^1 u^{\alpha+\beta(2-\alpha)-2}(1-u)^{(2-\alpha)(1-\beta)-1}(tu)^{2-\alpha-\beta(2-\alpha)}T(tu)udu = x,
\]
by dominated convergence theorem.

This gives, \( S_\alpha(0) = I \). Then, \( \{S_\alpha(t)\}_{t \geq 0} \) is strongly continuous with \( S_\alpha(0) = I \).

Now using property \([4]\) for all \( x \in D(A) \), we have
\[
S_\alpha(t) = J_t^{(1-\beta)(2-\alpha)} C_{\alpha,\beta}(t)x = x + J_t^{\alpha} J_t^{(1-\beta)(2-\alpha)} C_{\alpha,\beta}(t)Ax = x + J_t^{\alpha} S_\alpha(t)Ax.
\]

Therefore \( \{S_\alpha(t)\}_{t \geq 0} \) is a solution operator for the given fractional differential equation \([4.8]-[4.9]\). \( \square \)

**Lemma 4.** If \( A \) generates a general fractional resolvent operator of order \( \alpha \) and type \( \beta \), then it generates general fractional sine function \( \{S_{\alpha,\beta}(t)\}_{t \geq 0} \) (see \([1]\)) of order \( \alpha \) and type \( \beta \) also.

**Proof.** Define \( S_{\alpha,\beta}(t) = \int_0^t C_{\alpha,\beta}(s)ds, t > 0 \), then for \( x \in X \), we have
\[
S_{\alpha,\beta}(t)x = \frac{t^{\alpha+\beta(2-\alpha)-1}}{\Gamma(\alpha+\beta(2-\alpha))} + A J_t^\alpha S_{\alpha,\beta}(t)x. \quad (4.10)
\]

Now
\[
\Gamma(\alpha+\beta(2-\alpha)) \lim_{t \to 0^+} \frac{S(t)y}{t^{\alpha+\beta(2-\alpha)-1}} = \Gamma(\alpha+\beta(2-\alpha)) \lim_{t \to 0^+} \frac{\int_0^t C_{\alpha,\beta}(s)yds}{t^{\alpha+\beta(2-\alpha)-1}}
\]
\[
= \Gamma(\alpha+\beta(2-\alpha)) \lim_{t \to 0^+} t^{2-\alpha-\beta(2-\alpha)} \int_0^t C_{\alpha,\beta}(ts)yds
\]
\[
= \Gamma(\alpha+\beta(2-\alpha)) \lim_{t \to 0^+} \int_0^1 \hat{s}^{\alpha+\beta(2-\alpha)-2}(s)^{2-\alpha-\beta(2-\alpha)}C_{\alpha,\beta}(ts)yds = y,
\]
by dominated convergence theorem.

The commutativity of \( \{C_{\alpha,\beta}(t)\}_{t \geq 0} \) implies the commutativity of \( \{S_{\alpha,\beta}(t)\}_{t \geq 0} \).

Using \([4.10]\) and closedness of operator \( A \), we can get
\[
S_{\alpha,\beta}(s) J_s^\alpha S_{\alpha,\beta}(t)x - J_s^\alpha S_{\alpha,\beta}(s) S_{\alpha,\beta}(t)x = \frac{s^{\alpha+\beta(2-\alpha)-1}}{\Gamma(\alpha+\beta(2-\alpha))} J_s^\alpha S_{\alpha,\beta}(t)x - \frac{t^{\alpha+\beta(2-\alpha)-1}}{\Gamma(\alpha+\beta(2-\alpha))} J_s^\alpha S_{\alpha,\beta}(s)x; t, s > 0.
\]
Therefore, \( \{S_{\alpha,\beta}(t)\}_{t \geq 0} \) is a general fractional sine function of order \( \alpha \) and type \( \beta \). \( \square \)

**Proposition 5.** Define \( R(\lambda) = \int_{0}^{\infty} e^{-\lambda t} C_{\alpha,\beta}(t) dt \). Let \( R(\lambda) \) exist for some \( \lambda = \lambda_0 \). Then

\[
\lambda^{-\beta(2-\alpha)+1} R(\mu) - \mu^{-\beta(2-\alpha)+1} R(\lambda) = (\lambda^\alpha - \mu^\alpha) R(\mu) R(\lambda) \quad \text{for} \quad \lambda, \mu \geq \lambda_0.
\]

**Proof.** Taking Laplace transform w.r.t \( t \) and \( s \) of L.H.S. of (4.3), we have

\[
\int_{0}^{\infty} e^{-\lambda s}[C_{\alpha,\beta}(s)\lambda^{-\alpha} R(\lambda) - J_s^\alpha C_{\alpha,\beta}(s) R(\lambda)] ds = \lambda^{-\alpha} R(\mu) R(\lambda) - \mu^{-\alpha} R(\mu) R(\lambda).
\]

(4.11)

Taking Laplace transform of R.H.S of (4.3) w.r.t \( t \), we have

\[
\lambda^{-\alpha} \frac{s^{\alpha+\beta(2-\alpha)-2}}{\Gamma(\alpha + \beta(2-\alpha) - 1)} R(\lambda) - \lambda^{-\alpha-\beta(2-\alpha)+1} J_s^\alpha C_{\alpha,\beta}(s).
\]

Again taking Laplace transform w.r.t \( s \), we have

\[
\lambda^{-\alpha} \mu^{-\alpha-\beta(2-\alpha)+1} R(\lambda) - \mu^{-\alpha} \lambda^{-\alpha-\beta(2-\alpha)+1} R(\mu).
\]

(4.12)

Equating the both sides (4.11) and (4.12), we get the result. \( \square \)

**Theorem 2.** Let \( A \) generates a general fractional resolvent operator of order \( \alpha \) and type \( \beta \), and Laplace transform of \( C_{\alpha,\beta}(t) \) exists, then

\[
R(\lambda^\alpha, A) x = \lambda^{\beta(2-\alpha)-1} \int_{0}^{\infty} e^{-\lambda t} C_{\alpha,\beta}(t) x dt.
\]

**Proof.** For all \( x \in D(A) \), we have

\[
C_{\alpha,\beta}(t) x = \frac{t^{\alpha+\beta(2-\alpha)-2}}{\Gamma(\alpha + \beta(2-\alpha) - 1)} x + J_t^\alpha C_{\alpha,\beta}(t) A x.
\]

(4.13)

Taking Laplace transform on both sides of (4.13), we get

\[
R(\lambda)x = \lambda^{-\alpha-\beta(2-\alpha)+1} x + \lambda^{-\alpha} R(\lambda) A x, \forall x \in D(A).
\]

Since \( D(A) \) is dense in \( X \), we have

\[
\lambda^{\beta(2-\alpha)-1} R(\lambda)x = \lambda^{-\alpha} x + \lambda^{\beta(2-\alpha)-1} \lambda^{-\alpha} R(\lambda) A x \quad \text{on} \quad X
\]

Hence

\[
\lambda^{\beta(2-\alpha)-1} R(\lambda)(\lambda^\alpha - A)x = x
\]

and

\[
(\lambda^{-\alpha} - A) R(\lambda)x = \lambda^{-\alpha-\beta(2-\alpha)+1} x.
\]

This gives

\[
(\lambda^\alpha I - A) \lambda^{\beta(2-\alpha)-1} R(\lambda) x = x.
\]

Thus \( R(\lambda^\alpha, A)x = \lambda^{\beta(2-\alpha)-1} \int_{0}^{\infty} e^{-\lambda t} C_{\alpha,\beta}(t) x dt. \) \( \square \)

**Remark 1.** A family \( \{C_{\alpha,\beta}(t)\}_{t \geq 0} \) of bounded linear operators is a solution operator of (1.1) and (1.2) if and only if it is general fractional resolvent operator of order \( \alpha \) and type \( \beta \).
Consider the following problem
\[ D_0^{\alpha,\beta}u(t) = Au(t), \quad t > 0, \quad (4.14) \]
\[ (g(1-\beta)(2-\alpha) \ast u)(0) = x, \quad (g(1-\beta)(2-\alpha) \ast u)'(0) = y, \quad (4.15) \]
in a Banach space $X$ where $A$ is closed linear operator on $X$.

**Definition 8.** A function $u \in C((0, \infty), X)$ is called a mild solution of the problem \((4.14)-(4.15)\) if $J_t^\alpha u(t) \in D(A)$ and
\[
u(t) = \frac{\Gamma(\alpha+\beta(2-\alpha)-2)}{\Gamma(\alpha+\beta(2-\alpha)-1)} x + \frac{\Gamma(\alpha+\beta(2-\alpha)-1)}{\Gamma(\alpha+\beta(2-\alpha))} y + AJ_t^\alpha u(t), \quad t \in (0, \infty).\]

**Theorem 3.** Let $A$ generates a general fractional resolvent operator of order $\alpha$ and type $\beta$. Then for every $x, y \in X$, $C_{\alpha,\beta}(t)x + S_{\alpha,\beta}(t)y$ is the mild solution of the problem \((4.14)-(4.15)\).

### 5. SEMILINEAR ABSTRACT HILFER CAUCHY PROBLEM

In this section, we study the following in-homogeneous problem
\[ D_0^{\alpha,\beta}u(t) = Au(t) + f(t, u(t)), \quad t \in (0, T], \quad (5.1) \]
\[ (g(1-\beta)(2-\alpha) \ast u)(0) = u_1 \in X, \quad (g(1-\beta)(2-\alpha) \ast u)'(0) = u_2 \in X, \quad (5.2) \]
where $A$ generates a general fractional resolvent operator $C_{\alpha,\beta}(t)$ of order $1 < \alpha < 2$ and type $0 \leq \beta \leq 1$, such that Laplace transform of $C_{\alpha,\beta}(t)$ exist.

Taking Laplace transform on both sides of \((5.1)\), we get
\[ \hat{u}(s) = s^{1-\beta(2-\alpha)} R(s^{\alpha}, A)u_1 + s^{-\beta(2-\alpha)} R(s^{\alpha}, A)u_2 + R(s^{\alpha}, A)f(s), \]

Now taking inverse Laplace transform, we have
\[ u(t) = C_{\alpha,\beta}(t)u_1 + S_{\alpha,\beta}(t)u_2 + \int_0^t P_{\alpha,\beta}(t-s)f(s, u(s))ds, \]
where, $P_{\alpha,\beta}(t) = J_t^{1-\beta(2-\alpha)} C_{\alpha,\beta}(t)$.

Thus, we define the mild solution of the problem \((5.1)-(5.2)\) as $u \in C((0, T); X)$ such that $u$ satisfies
\[ u(t) = C_{\alpha,\beta}(t)u_1 + S_{\alpha,\beta}(t)u_2 + \int_0^t P_{\alpha,\beta}(t-s)f(s, u(s))ds, \quad t \in (0, T]. \]

Let $I = [0, T]$ and $I' = (0, T]$, we define a space $Y$ as
\[ Y = \{ u \in C(I', X) : \lim_{t \to 0^+} t^{(2-\alpha-\beta(2-\alpha))}u(t) \text{ exists and is finite} \} \]
with norm $\|u\|_Y = \sup_{t \in I'} \{t^{(1-\beta)(2-\alpha)}|u(t)| \}$.

Let $y : I \to X$ and $u(t) = t^{(2-\alpha)(\beta-1)}y(t) \in I'$, then $u \in Y$ iff $y \in C(I, X)$ and $\|u\|_Y = \|y\|_X$. Let $B_r(I) = \{ y \in C(I, X) : \|y\| \leq r \}$ and $B_r(Y) = \{ u \in Y : \|u\|_Y \leq r \}$.

**Theorem 4.** $u(t) = C_{\alpha,\beta}(t)x + S_{\alpha,\beta}(t)y + \int_0^t P_{\alpha,\beta}(t-s)f(s)ds$ is strong solution of the problem \((5.1)-(5.2)\) for $f(t, u(t)) = f(t)$ if $x, y \in D(A)$ and there exist a function $\phi \in L^1((0, T), X)$ such that $f(t) = J_t^{\beta(2-\alpha)} \phi(t), \phi(t) \in D(A)$ and $A\phi(t) \in L^1$. 
Proof. We can write,
\[ \int_0^t P_{\alpha, \beta}(t - s)f(s)ds = P_{\alpha, \beta}(t) \ast f(t) = J_t^{1-\beta(2-\alpha)}C_{\alpha, \beta}(t) \ast f(t). \]
\[ = g_{t-\beta(2-\alpha)}(t) \ast (C_{\alpha, \beta}(t) \ast f(t)). \]
\[ J_t^{1-\beta(2-\alpha)}(P_{\alpha, \beta}(t) \ast f(t)) = J_t^{1+(2-\alpha)(1-2\beta)}C_{\alpha, \beta}(t) \ast f(t) \]
\[ = J_t^{1+(2-\alpha)(1-2\beta)} \left[ \frac{\Gamma(\alpha + \beta(2-\alpha) - 1)}{\Gamma(\alpha + \beta(2-\alpha))}(1-2\beta) - 1 \right] \ast f(t) + \frac{\Gamma(\alpha + \beta(2-\alpha) - 1)}{\Gamma(\alpha + \beta(2-\alpha))}(1-2\beta) \ast f(t) \]
\[ = g_{1+(2-\alpha)(1-2\beta)}(t) \ast \left( \frac{\Gamma(\alpha + \beta(2-\alpha) - 1)}{\Gamma(\alpha + \beta(2-\alpha))}(1-2\beta) \ast f(t) \right) \]
\[ = J_t^{2-\beta(2-\alpha)}f(t) + J_t^{3-2\beta(2-\alpha)}C_{\alpha, \beta}(t) \ast Af(t) \]
\[ \frac{d}{dt} J_t^{1-\beta(2-\alpha)}(P_{\alpha, \beta}(t) \ast f(t)) = J_t^{1-\beta(2-\alpha)}f(t) + J_t^{2-2\beta(2-\alpha)}C_{\alpha, \beta}(t) \ast Af(t) \]
\[ = J_t^\alpha(\alpha, \beta) - J_t^\alpha(\alpha, \beta) \frac{\Gamma(\alpha + \beta(2-\alpha) - 1)}{\Gamma(\alpha + \beta(2-\alpha))}(1-2\beta) \ast f(t) \]
\[ = J_t^\alpha(\alpha, \beta) - J_t^\alpha(\alpha, \beta) \frac{\Gamma(\alpha + \beta(2-\alpha) - 1)}{\Gamma(\alpha + \beta(2-\alpha))}(1-2\beta) \ast f(t) \]
\[ = J_t^\alpha(\alpha, \beta) \frac{d^2}{dt^2} J_t^{1-\beta(2-\alpha)}(P_{\alpha, \beta}(t) \ast f(t)) \]
\[ = f(t) + A(P_{\alpha, \beta}(t) \ast f(t)). \]
Thus \( D_{0}^{\alpha, \beta} u(t) = Au(t) + f(t). \)

To find the conditions for the existence and uniqueness of a mild solution of the problem (H1)-(H2), we assume the following hypothesis:

**H1:** There exists a constant \( M > 0 \) such that \( \|C_{\alpha, \beta}(t)\| \leq Mt^{\alpha + \beta(2-\alpha) - 2} \) for \( t > 0 \);

**H2:** \( P_{\alpha, \beta}(t)(t > 0) \) is continuous in the uniform operator topology for \( t > 0 \).

**H3:** for each \( t \in I' \), the function \( f(t, .) : X \to X \) is continuous and for each \( x \in X \), the function \( f(., x) : I' \to X \) is strongly measurable;

**H4:** there exists a function \( m \in L(I', \mathbb{R}^+) \) such that \( \|f(t, x)\| \leq m(t) \) for all \( x \in B_Y \) and almost all \( t \in [0, T] \);

**H5:** there exist a constant \( r > 0 \) such that

\[ M \|u_1\| + \frac{M}{\alpha + \beta(2-\alpha) - 1} \|u_2\| + \frac{MT^\alpha(\alpha + \beta(2-\alpha)) - 1}{\Gamma(\alpha)} \|m\|_{L^1} = r \]

**Lemma 5.** Assume that (H1) holds, then \( \|S_{\alpha, \beta}(t)\| \leq \frac{M}{\alpha + \beta(2-\alpha) - 1}t^{\alpha + \beta(2-\alpha) - 1} \) and \( \|P_{\alpha, \beta}(t)\| \leq \frac{MT^\alpha(\alpha + \beta(2-\alpha) - 1)}{\Gamma(\alpha)}t^{\alpha - 1} \) for \( t > 0 \).
Proof. Since $S_{\alpha,\beta}(t) = \int_0^t C_{\alpha,\beta}(s)\,ds$ and $P_{\alpha,\beta}(t) = J_t^{1-\beta(2-\alpha)}C_{\alpha,\beta}$, using Hypothesis (H1) we have

$$
\|S_{\alpha,\beta}(t)\| \leq M \int_0^t s^{\alpha+\beta(2-\alpha)-2}\,ds \leq \frac{M\Gamma(\alpha+\beta(2-\alpha)-1)}{\alpha+\beta(2-\alpha)-1}
$$

and

$$
\|P_{\alpha,\beta}(t)\| = \left\| \frac{1}{\Gamma(1-\beta(2-\alpha))} \int_0^t (t-s)^{-\beta(2-\alpha)}C_{\alpha,\beta}(s)\,ds \right\| \\
\leq \frac{M}{\Gamma(1-\beta(2-\alpha))} \int_0^t (t-s)^{-\beta(2-\alpha)\alpha+\beta(2-\alpha)-2}\,ds \\
= \frac{M\Gamma(\alpha+\beta(2-\alpha)-1)}{\Gamma(\alpha)} t^{\alpha-1}.
$$

For any $u \in B^y_t$, we define an operator $S$ as follows

$$(Su)(t) = (S_1u)(t) + (S_2u)(t)$$

where

$$(S_1u)(t) = C_{\alpha,\beta}(t)u_1 + S_{\alpha,\beta}(t)u_2, \text{ for } t \in (0, T],$$

$$(S_2u)(t) = \int_0^t P_{\alpha,\beta}(t-s)f(s, u(s))\,ds, \text{ for } t \in (0, T].$$

We can see easily that $\lim_{t \to 0^+} t^{2-\alpha-\beta(2-\alpha)}(S_1u)(t) = \frac{u_1}{\Gamma(\alpha+\beta(2-\alpha)-1)}$ and $\lim_{t \to 0^+} t^{2-\alpha-\beta(2-\alpha)}(S_2u)(t) = 0$. For $y \in B_r(I)$, set $u(t) = t^{\alpha+\beta(2-\alpha)-2}y(t)$ for $t \in (0, T]$. Then $u \in B^y_t$.

We define an operator $\mathcal{S}$ as follows

$$(\mathcal{S}y)(t) = (\mathcal{S}_1y)(t) + (\mathcal{S}_2y)(t),$$

where

$$(\mathcal{S}_1y)(t) = \begin{cases} 
\frac{t^{2-\alpha-\beta(2-\alpha)}(S_1u)(t)}{\Gamma(\alpha+\beta(2-\alpha)-1)} 
& \text{for } t \in (0, T] \\
\frac{u_1}{\Gamma(\alpha+\beta(2-\alpha)-1)} & \text{for } t = 0,
\end{cases}$$

and

$$(\mathcal{S}_2y)(t) = \begin{cases} 
\frac{t^{2-\alpha-\beta(2-\alpha)}(S_2u)(t)}{\Gamma(\alpha+\beta(2-\alpha)-1)} 
& \text{for } t \in (0, T] \\
0 & \text{for } t = 0.
\end{cases}$$

Lemma 6. For $t > 0$, $S_{\alpha,\beta}(t)$ and $P_{\alpha,\beta}(t)$ are strongly continuous.

Proof. For $x \in X$ and $0 < t_1 < t_2 \leq T$, we have

$$
\|S_{\alpha,\beta}(t_2)x - S_{\alpha,\beta}(t_1)x\| = \left\| \int_{t_1}^{t_2} C_{\alpha,\beta}(s)x\,ds \right\| \leq M \int_{t_1}^{t_2} s^{\alpha+\beta(2-\alpha)-2}\|x\|\,ds \\
= \frac{M}{\Gamma(\alpha+\beta(2-\alpha)-1)} (t_2^{\alpha+\beta(2-\alpha)-1} - t_1^{\alpha+\beta(2-\alpha)-1}) \to 0 \text{ as } t_2 \to t_1,
$$

$$
\|P_{\alpha,\beta}(t_2)x - P_{\alpha,\beta}(t_1)x\| = \\
\left\| \frac{1}{\Gamma(1-\beta(2-\alpha))} \int_{t_1}^{t_2} (t_2-s)^{-\beta(2-\alpha)}C_{\alpha,\beta}(s)x\,ds - \frac{1}{\Gamma(1-\beta(2-\alpha))} \int_{0}^{t_1} (t_1-s)^{-\beta(2-\alpha)}C_{\alpha,\beta}(s)x\,ds \right\| \\
\leq \left\| \frac{1}{\Gamma(1-\beta(2-\alpha))} \int_{t_1}^{t_2} (t_2-s)^{-\beta(2-\alpha)}C_{\alpha,\beta}(s)x\,ds \right\| + \left\| \frac{1}{\Gamma(1-\beta(2-\alpha))} \int_{0}^{t_1} (t_1-s)^{-\beta(2-\alpha)}C_{\alpha,\beta}(s)x\,ds \right\|. 
$$
\[
\left\| \frac{1}{\Gamma(1 - \beta(2 - \alpha))} \int_0^{t_1} ((t_2 - s)^{-\beta(2 - \alpha)} - (t_1 - s)^{-\beta(2 - \alpha)})C_{\alpha,\beta}(s)xds \right\| := I_1 + I_2
\]

\[
I_1 \leq \frac{M}{\Gamma(1 - \beta(2 - \alpha))\Gamma(\alpha + \beta(2 - \alpha) - 1)} \int_{t_1}^{t_2} (t_2 - s)^{-\beta(2 - \alpha)}s^{\alpha + \beta(2 - \alpha) - 2}\|x\|ds
\]

\[
\leq \frac{Mt_1^{\alpha + \beta(2 - \alpha) - 2}}{\Gamma(1 - \beta(2 - \alpha))\Gamma(\alpha + \beta(2 - \alpha) - 1)} \int_{t_1}^{t_2} (t_2 - s)^{-\beta(2 - \alpha)}\|x\|ds
\]

\[
= \frac{Mt_1^{\alpha + \beta(2 - \alpha) - 2}}{\Gamma(1 - \beta(2 - \alpha))\Gamma(\alpha + \beta(2 - \alpha) - 1)} \frac{(t_2 - t_1)^{1 - \beta(2 - \alpha)}}{1 - \beta(2 - \alpha)} \to 0 \text{ as } t_2 \to t_1.
\]

\[
I_2 \leq \frac{M}{\Gamma(1 - \beta(2 - \alpha))\Gamma(\alpha + \beta(2 - \alpha) - 1)} \int_0^{t_1} \| (t_2 - s)^{-\beta(2 - \alpha)} - (t_1 - s)^{-\beta(2 - \alpha)} \| s^{\alpha + \beta(2 - \alpha) - 2} \|x\|ds
\]

Noting that \( |(t_2 - s)^{-\beta(2 - \alpha)} - (t_1 - s)^{-\beta(2 - \alpha)}|s^{\alpha + \beta(2 - \alpha) - 2} \leq 2(t_1 - s)^{-\beta(2 - \alpha)}s^{\alpha + \beta(2 - \alpha) - 2} \)
and \( \int_0^{t_1} (t_1 - s)^{-\beta(2 - \alpha)}s^{\alpha + \beta(2 - \alpha) - 2}ds \) exists, then by Lebesgue’s dominated convergence theorem, we have \( I_2 \to 0 \) as \( t_2 \to t_1 \).

\]

**Lemma 7.** Assume that (H1)-(H4) hold, then \( \{ \mathcal{G}y : y \in B_r(J) \} \) is equicontinuous.

**Proof.** Let \( t_1 = 0 \) and \( 0 < t_2 \leq T \),

\[
\| \mathcal{G}y(t_2) - \mathcal{G}y(0) \| \leq \| \mathcal{G}_1y(t_2) - \mathcal{G}_1y(0) \| + \| \mathcal{G}_2y(t_2) - \mathcal{G}_2y(0) \|
\]

\[
\to 0 \text{ as } t_2 \to t_1
\]

Now, for \( 0 < t_1 < t_2 \leq T \), we have

\[
\| \mathcal{G}y(t_2) - \mathcal{G}y(t_1) \| \leq \| t_2^{2 - \alpha - \beta(2 - \alpha)}C_{\alpha,\beta}(t_2)u_1 - t_1^{2 - \alpha - \beta(2 - \alpha)}C_{\alpha,\beta}(t_1)u_1 \|
\]

\[
+ \| t_2^{2 - \alpha - \beta(2 - \alpha)}S_{\alpha,\beta}(t_2)u_2 - t_1^{2 - \alpha - \beta(2 - \alpha)}S_{\alpha,\beta}(t_1)u_2 \|
\]

\[
+ \| t_2^{2 - \alpha - \beta(2 - \alpha)} \int_0^{t_2} P_{\alpha,\beta}(t_2 - s)f(s, u(s))ds - t_1^{2 - \alpha - \beta(2 - \alpha)} \int_0^{t_1} P_{\alpha,\beta}(t_1 - s)f(s, u(s))ds \|
\]

\[
:= I_1 + I_2 + I_3.
\]

From strong continuity of \( C_{\alpha,\beta}(t) \) and \( S_{\alpha,\beta}(t) \), we get \( I_1, I_2 \to 0 \) as \( t_2 \to t_1 \).

\[
I_3 \leq \| t_2^{2 - \alpha - \beta(2 - \alpha)} \int_0^{t_2} P_{\alpha,\beta}(t_2 - s)f(s, u(s))ds \|
\]

\[
+ \| t_2^{2 - \alpha - \beta(2 - \alpha)} \int_0^{t_2} P_{\alpha,\beta}(t_2 - s)f(s, u(s))ds \|
\]

\[
+ \| t_1^{2 - \alpha - \beta(2 - \alpha)} \int_0^{t_1} P_{\alpha,\beta}(t_1 - s)f(s, u(s))ds \|
\]

\[
:= I_{31} + I_{32} + I_{33}.
\]

Using the Hypothesis (H4), we have

\[
I_{31} \leq Mt_2^{2 - \alpha - \beta(2 - \alpha)} \int_0^{t_2} (t_2 - s)^{\alpha - 1}m(s)ds \leq Mt_2^{2 - \alpha - \beta(2 - \alpha)}(t_2 - t_1)^{\alpha - 1} \int_{t_1}^{t_2} m(s)ds
\]

\[
\to 0 \text{ as } t_2 \to t_1.
\]
Using Hypotheses (H2) and (H4), we have

\[ I_{32} \leq M|t_2^{2-\alpha} - t_1^{2-\alpha}| + t_1^{2-\alpha} - t_1^{2-\alpha} \int_0^{t_1} (t_2 - s)^{\alpha-1} m(s) ds \]

\[ \leq |t_2^{2-\alpha} - t_1^{2-\alpha}| + t_1^{2-\alpha} - t_1^{2-\alpha} \int_0^{t_1} m(s) ds \]

\[ \to 0 \text{ as } t_2 \to t_1. \]

Using Hypotheses (H2) and (H4), we have

\[ I_{33} \leq t_1^{2-\alpha} - t_1^{2-\alpha} \int_0^{t_1} \|P_{\alpha}(t_2 - s) - P_{\alpha}(t_1 - s)\| m(s) ds \]

\[ = t_1^{2-\alpha} - t_1^{2-\alpha} \int_0^{t_1} \|P_{\alpha}(t_2 - s) - P_{\alpha}(t_1 - s)\| m(s) ds + \int_0^{t_1} \|P_{\alpha}(t_2 - s) - P_{\alpha}(t_1 - s)\| m(s) ds \]

\[ \leq t_1^{2-\alpha} - t_1^{2-\alpha} \sup_{s \in [0, t_1 - \epsilon]} \|P_{\alpha}(t_2 - s) - P_{\alpha}(t_1 - s)\| \int_0^{t_1} m(s) ds + t_1^{2-\alpha} - t_1^{2-\alpha} C \int_0^{t_1} m(s) ds \]

\[ \to 0 \text{ as } t_2 \to t_1. \]

**Lemma 8.** Assume that (H1)-(H5) hold, then \( G \) maps \( B_r(I) \) into \( B_r(I) \), and \( G \) is continuous in \( B_r(I) \).

**Proof.** Let \( u \in B_r^Y \Rightarrow \|u\|_Y \leq r \), clearly \( Su(t) \in C(I', X) \) and

\[ t^{(2-\alpha)(1-\beta)} \|Su(t)\| \leq t^{(2-\alpha)(1-\beta)}(\|C_{\alpha, \beta}(t)u_1\| + \|S_{\alpha, \beta}(t)u_1\| + \int_0^t \|P_{\alpha}(t)\| m(s) ds) \]

\[ \leq M\|u_1\| + \frac{M}{\alpha + \beta(2-\alpha) - 1}\|u_2\| + \frac{M\Gamma(\alpha + \beta(2-\alpha) - 1) t^{(2-\alpha)(1-\beta)}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} m(s) ds \]

\[ \leq M\|u_1\| + \frac{M}{\alpha + \beta(2-\alpha) - 1}\|u_2\| + \frac{M\Gamma(\alpha + \beta(2-\alpha) - 1) t^{\alpha-1}}{\Gamma(\alpha)} \int_0^t m(s) ds \]

\[ \leq \frac{M}{\Gamma(\alpha + \beta(2-\alpha) - 1)}\|u_1\| + \frac{M}{\Gamma(\alpha + \beta(2-\alpha))}\|u_2\| + \frac{M\Gamma(\alpha + \beta(2-\alpha) - 1) T^{1-\beta(2-\alpha)}}{\Gamma(\alpha)} \|m\|_{L^1} = r \]

Hence \( G \) maps \( B_r(I) \) into \( B_r(I) \). Now, we prove that \( G \) is continuous in \( B_r(I) \).

Let \( y_m, y \in B_r(I) \) such that \( \lim_{m \to \infty} y_m = y(t) \) and \( \lim_{m \to \infty} u_m(t) = u(t) \), then we have \( \lim_{m \to \infty} y_m(t) = y(t) \) and \( \lim_{m \to \infty} u_m(t) = u(t) \), for \( t \in (0, T] \) where \( u_m(t) = t^{\alpha+\beta(2-\alpha)} y_m(t) \) and \( u(t) = t^{\alpha+\beta(2-\alpha)} y(t) \).

Now for \( t \in [0, T] \)

\[ \|(Gy_m)(t) - (Gy)(t)\| = t^{(2-\alpha)} \int_0^t P_{\alpha}(t-s) (f(s, u_m(s)) - f(s, u(s))) ds \]

\[ \leq t^{(2-\alpha)} M \int_0^t (t-s)^{\alpha-1} \|f(s, u_m(s)) - f(s, u(s))\| ds \]

\[ \leq t^{(2-\alpha)} M \int_0^t |f(s, u_m(s)) - f(s, u(s))| ds \leq 2T^{\alpha-1} m(s) \]

which is integrable. Hence by Lebesgue dominated convergence theorem, we have \( \|(Gy_m)(t) - (Gy)(t)\| \to 0 \). But Lemma ensures the continuity of \( G \).

**H6:** For bounded subset \( D \) of \( X, \psi(f(D)) \leq k\psi(D) \) for a.e. \( t \in [0, T] \).

**Theorem 5.** Assume that (H1)-(H6) hold, then the Cauchy problem \([5.1]-[5.2] \)

has at least one mild solution in \( B_r(I') \).
Proof. Let $B_0 \subset B_r(I)$
Define $\mathcal{S}^{(1)}(B_0) = \mathcal{S}(B_0), \mathcal{S}^{(n)}(B_0) = \mathcal{S}(\mathcal{S}^{(n-1)}(B_0)), n = 2, 3, \ldots$
Making use of proposition [13] we are able to get a sequence $\{y_n^{(1)}\}$ in $B_0$ for any $\epsilon > 0$ such that
\[
\psi(\mathcal{S}^{(1)}(B_0(t))) \leq 2\psi \left( f^{(1-\beta)(2-\alpha)} \int_0^t P_{\alpha,\beta}(t-s)f(s, \{s^{-(2-\alpha)(1-\beta)}y_n^{(1)}(s)\}_{n=1}^\infty)ds \right) + \epsilon
\]
\[
\leq 4M_{\alpha,\beta}[2t^{(2-\alpha)(1-\beta)} \psi(B_0)] \int_0^t (t-s)^{\alpha-1} \psi(f(s, \{s^{-(2-\alpha)(1-\beta)}y_n^{(1)}(s)\}_{n=1}^\infty))ds + \epsilon
\]
\[
\leq 4M_{\alpha,\beta}kt^{(2-\alpha)(1-\beta)} \psi(B_0) \int_0^t (t-s)^{\alpha-1}s^{(2-\alpha)(1-\beta)}ds + \epsilon
\]
\[
= 4M_{\alpha,\beta}kt^n \psi(B_0) \Gamma(\alpha)\Gamma((\alpha + \beta(2-\alpha) - 1) \Gamma(2\alpha + \beta(2-\alpha) - 1) + \epsilon.
\]
Since $\epsilon > 0$ is arbitrary, we have
\[
\psi(\mathcal{S}^{(1)}(B_0(t))) \leq 4M_{\alpha,\beta}kt^n \psi(B_0) \frac{\Gamma(\alpha)\Gamma((\alpha + \beta(2-\alpha) - 1)}{\Gamma(2\alpha + \beta(2-\alpha) - 1)}
\]
Again using proposition [13] we are able to get a sequence $\{y_n^{(2)}\}$ in $\mathcal{S}(B_0)$ for any $\epsilon > 0$ such that
\[
\psi(\mathcal{S}^{(2)}(B_0(t))) = \psi(\mathcal{S}(\mathcal{S}(\mathcal{S}^{(1)}(B_0(t)))))\]
\[
\leq 2\psi \left( f^{(2-\alpha)(1-\beta)} \int_0^t P_{\alpha,\beta}(t-s)f(s, \{s^{-(2-\alpha)(1-\beta)}y_n^{(2)}(s)\}_{n=1}^\infty)ds \right) + \epsilon
\]
\[
\leq 4M_{\alpha,\beta}[2t^{(2-\alpha)(1-\beta)} \psi(B_0)] \int_0^t (t-s)^{\alpha-1} \psi(f(s, \{s^{-(2-\alpha)(1-\beta)}y_n^{(2)}(s)\}_{n=1}^\infty))ds + \epsilon
\]
\[
\leq 4M_{\alpha,\beta}kt^{(2-\alpha)(1-\beta)} \psi(B_0) \int_0^t (t-s)^{\alpha-1}s^{(2-\alpha)(1-\beta)}ds + \epsilon
\]
\[
\leq 4M_{\alpha,\beta}kt^{(2-\alpha)(1-\beta)} \psi(B_0) \frac{(4M_{\alpha,\beta}k)^2t^{(2-\alpha)(1-\beta)}\Gamma(\alpha)\Gamma((\alpha + \beta(2-\alpha) - 1)}{\Gamma(2\alpha + \beta(2-\alpha) - 1)}
\]
\[
= 4M_{\alpha,\beta}kt^n \psi(B_0) + \epsilon.
\]
We can prove by mathematical induction that
\[
\psi(\mathcal{S}^{(n)}(B_0(t))) \leq \frac{(4kM_{\alpha,\beta})^n T^{n\alpha} \Gamma((\alpha + \beta(2-\alpha) - 1)}{\Gamma((n+1)\alpha + \beta(2-\alpha) - 1)} \psi(B_0), n \in \mathbb{N}.
\]
Since
\[
\lim_{n \to \infty} \frac{(4kM_{\alpha,\beta})^n T^{n\alpha} \Gamma((\alpha + \beta(2-\alpha) - 1)}{\Gamma((n+1)\alpha + \beta(2-\alpha) - 1)} = 0,
\]
there exists a constant $n_0$ such that
\[
\frac{(4kM_{\alpha,\beta})^n T^{n\alpha} \Gamma((\alpha + \beta(2-\alpha) - 1)}{\Gamma((n+1)\alpha + \beta(2-\alpha) - 1)} \leq \frac{(4kM_{\alpha,\beta})^n T^{n\alpha} \Gamma((\alpha + \beta(2-\alpha) - 1)}{\Gamma((n+1)\alpha + \beta(2-\alpha) - 1)} = p < 1.
\]
Assume that Theorem 6. 

**H7:** The function $u(t)$ is absolutely continuous and $u'(t)$ is bounded. Then by the Schauder fixed point theorem, 

$$
\psi(T^{m_0}(B_0(t))) = \max_{t \in [0,T]} \psi(T^{m_0}(B_0(t))).
$$

Hence,

$$
\psi(T^{m_0}(B_0)) \leq p\psi(B_0).
$$

Applying a similar method as in Theorem 4.2, we can get a nonempty, convex and compact subset $D$ in $B_r(I)$ such that $\mathcal{G}(\mathcal{D}) \subset \mathcal{D}$ and $\mathcal{G}(\mathcal{D})$ is compact. Then by the Schauder fixed point theorem, $\mathcal{G}$ has a fixed point $y^*$ in $B_r(I)$. Let $u^*(t) = t^{(2-\alpha)(\beta-1)}y^*(t)$. Then $u^*(t)$ is a mild solution of (5.1)-(5.2).

**H7:** There exist a constant $k > 0$ such that for any $u, v \in B_r^Y(I)$ and $t \in (0, T]$ we have

$$
\|f(t, u(t)) - f(t, v(t))\| \leq k\|u - v\|_Y.
$$

**Theorem 6.** Assume that (H1), (H3) – (H5) and (H7) holds. Then the problem (5.1)-(5.2) has a unique mild solution for every $u_1, u_2 \in X$, if $\frac{M}{T^{\beta-2(\alpha-1)}}k < 1$.

**Proof.** Let

$$
y_1(t) = t^{(2-\alpha)(\beta-1)(2-\alpha)}u(t) \quad \text{and} \quad y_2(t) = t^{(2-\alpha)(\beta-1)(2-\alpha)}v(t).
$$

Then

$$
\|\mathcal{G}(y_1) - \mathcal{G}(y_2)\| = t^{2-\alpha(2-\alpha)}\| \int_0^t P_{\alpha, \beta}(t-s)(f(s, u(s)) - f(s, v(s))) \| \\
\leq t^{2-\alpha(2-\alpha)} M \| f(s, u(s)) - f(s, v(s)) \| \\
\leq t^{2-\beta(2-\alpha)} M \| u - v \|_Y
$$

Hence unique mild solution exist by Banach contraction theorem.

6. **Example**

Let $X = L^2([0, \pi], \mathbb{R})$. Consider the following fractional partial differential equation with Hilfer fractional derivative

$$
D_t^{\alpha, \beta}u(x, t) = \frac{\partial^2}{\partial x^2}u(x, t) + f(t, u(x, t))
$$

$$(g^{(1-\beta)(2-\alpha)} * u(x, t))(t = 0) = u_1(x), \quad (g^{(1-\beta)(2-\alpha)} * u')'(t = 0) = u_2(x)
$$

where $D_t^{\alpha, \beta}$ is the Hilfer fractional derivative of order $\alpha \in (1, 2)$ and type $\beta \in [0, 1]$, $f$ is a given function.

We define an operator $A$ by $Av = v''$ with the domain $D(A) = \{v() \in X : v, v'' \text{ are absolutely continuous and } v'' \in X, v(0) = v(\pi) = 0\}.$

$A$ has eigen values $\{\lambda_n = -n^2\}$ with eigenfunctions $\{\sin nx, n = 1, 2, \ldots\}$, we define the family of operators $\{C_{\alpha, \beta}(t)\}, \{S_{\alpha, \beta}(t)\}$ as follows

$$
(C_{\alpha, \beta}(t)g)(x) = \sum_{n=1}^{\infty} I^{\alpha+\beta(2-\alpha)}_{-2}E_{\alpha, \alpha+\beta(2-\alpha)-1}(-n^2t^\alpha)g_n \sin nx,
$$

where $g_n$ are the eigenfunctions of $A$. The family of operators $\{C_{\alpha, \beta}(t)\}$ forms a semi group of contraction operators.
EXISTENCE AND REGULARITY OF SOLUTIONS OF FRACTIONAL DIFFERENTIAL EQUATIONS INVOLVING HILFER FRACTIONAL DERIVATIVE OF ORDER $1 < \alpha < 2$ AND TYPE $0 \leq \beta \leq 1$

$$(S_{\alpha,\beta}(t)g)(x) = \sum_{n=1}^{\infty} t^{\alpha+\beta(2-\alpha)-1} E_{\alpha,\alpha+\beta(2-\alpha)}(-n^2 t^\alpha) g_n \sin nx,$$

where $E_{\alpha,\gamma}$ is the Mittag-Leffler function defined by

$$E_{\alpha,\gamma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \gamma)}.$$ 

The operator $A = \triangle$ is a sectorial operator of type $(M, \theta, \alpha, \mu)$. We can easily calculate to get a constant $M > 0$ such that $\| C_{\alpha,\beta}(t) \| \leq M t^{\alpha+\beta(2-\alpha)-2}$. It is well known from [23] that $P_{\alpha,\beta}(t)$ is continuous in the uniform operator topology.

Consider $f(t,u) = t \sin(u)$. Then $r = M \left\| u_1 \right\| + M \frac{\left\| u_2 \right\|}{\alpha+\beta(2-\alpha)-1} \frac{M^{(\alpha+\beta(2-\alpha))}}{2 \Gamma(\alpha)}$.

Then Theorem 5 guarantees the existence of a mild solution.

REFERENCES

[1] Mei, Z.D., Peng, J.G., Gao, J.H.: General fractional differential equations of order $\alpha \in (1, 2)$ and type $\beta \in [0, 1]$ in Banach spaces. Semigroup Forum 94, 712–737 (2017).
[2] Mei, Z.D., Peng, J.G., Jia, J.X.: A new characteristic property of Mittag-Leffler functions and fractional cosine functions. Studia Mathematica 220 (2), 119–140 (2014).
[3] Zhu, B., Liu, L., Wu, Y., Local and global existence of mild solutions for a class of semilinear fractional integro-differential equations. Fract. Calc. Appl. Anal., Vol. 20, No 6 (2017), pp. 1338–1355.
[4] Shu, X. B., Wang, Q., The existence and uniqueness of mild solutions for fractional differential equations with nonlocal conditions of order $1 < \alpha < 2$. Computers and Mathematics with Applications 64(2012) 2100–2110.
[5] Mei, Z.D., Peng, J.G., Zhang, Y., An operator theoretical approach to Riemann-Liouville fractional Cauchy problem. Math. Nachr. 288, No. 7, 784–797 (2015).
[6] Wang, X., Shu, X., The existence of positive mild solutions for fractional differential evolution equations with nonlocal conditions of order $1 < \alpha < 2$. Advances in Difference Equations 159 (2015).
[7] Lizama, C., Poblete, F.: On a functional equation associated with $(a, k)$-regularized resolvent families. Abstr. Appl. Anal. Article ID 495487 (2012)
[8] Guo, D.J., Lakshmikantham, V., Liu, X.Z.: Nonlinear integral equations in abstract spaces. Kluwer Academic, Dordrecht, (1996).
[9] Monch, H.: Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces. Nonlinear Anal: TMA4, (1980).
[10] Zhou, Y.: Basic Theory of Fractional Differential Equations. World Scientific, Singapore, (2014).
[11] Bothe, D.: Multivalued perturbation of $m$-accretive differential inclusions. Israel. J. Math. 108, 109–138 (1998).
[12] Sandev, Trifce and Metzler, Ralf and Tomovski, Živorad:Fractional diffusion equation with a generalized Riemann–Liouville time fractional derivative. Journal of Physics A: Mathematical and Theoretical, 44(25),255203 (2011).
[13] Furati, Khaled M and Kassim, Mohammed D and others: Existence and uniqueness for a problem involving Hilfer fractional derivative. Computers & Mathematics with Applications, 64(6), 1616–1626 (2012).
[14] Kamocki, Rafal and Obczynski, Cezary:On fractional Cauchy-type problems containing Hilfer’s derivative. Electronic Journal of Qualitative Theory of Differential Equations, 50, 1–12,(2016).
[15] Mahmudov, NI and McKibben, MA: On the approximate controllability of fractional evolution equations with generalized Riemann-Liouville fractional derivative. Journal of Function Spaces, (2015).
[16] Ahmed, Hamey M and El-Borai, Mahmoud M and El-Owaidy, Hassan M and Ghanem, Ahmed S:Impulsive Hilfer fractional differential equations. Advances in Difference Equations, 1, 226, (2018).
[17] Gu, H. and Trujillo, J.J., 2015. Existence of mild solution for evolution equation with Hilfer fractional derivative. Applied Mathematics and Computation, 257, pp.344-354.
[18] Ahmed, H.M. and Okasha, A., 2018. Nonlocal Hilfer Fractional Neutral Integrodifferential Equations. International Journal of Mathematical Analysis, 12(6), pp.277-288.
[19] Mei, Z.D., Peng, J.G. and Gao, J.H., 2015. Existence and uniqueness of solutions for nonlinear general fractional differential equations in Banach spaces. Indagationes Mathematicae, 26(4), pp.669-678.
[20] Hilfer, R. (2002). Experimental evidence for fractional time evolution in glass forming materials. Chemical Physics, 284(1-2), 399–408.
[21] Hambold, H.J., Mathai, A.M. and Saxena, R.K., 2011. Mittag-Leffler functions and their applications. Journal of Applied Mathematics, 2011.
[22] Zhang, L., Zhou, Y.: Fractional Cauchy problems with almost sectorial operators. Applied Mathematics and Computation. 257,145–157 (2015).
[23] Lizama, C., Pereira A. and Ponce R.:On the compactness of fractional resolvent operator functions. Semigroup Forum (2016).