Integrable and superintegrable Hamiltonian systems with four dimensional real Lie algebras as symmetry of the systems

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Abstract

We construct integrable and superintegrable Hamiltonian systems by using the realizations of four dimensional real Lie algebras as a symmetry of the system with the phase space $\mathbb{R}^4$ and $\mathbb{R}^6$. Furthermore, we construct some integrable and superintegrable Hamiltonian systems for which the symmetry Lie group is also the phase space of the system.

keywords: Integrable Hamiltonian systems, Superintegrable Hamiltonian systems, Lie algebra.

1 Introduction

A Hamiltonian system with $N$ degrees of freedom is integrable from the Liouville sense if it has $N$ invariants in involution (globally defined and functionally independent);[1] and is superintegrable if it has additional independent invariants up to $2N - 1$. Superintegrability forces analytic and algebraic solvability. The modern theory of superintegrability was pioneered by Smorodinsky, Winternitz and collaborators[2] (see for recent review [3]).

In this work, we construct new integrable and superintegrable Hamiltonian systems by using the realizations of four dimensional real Lie algebras [4] as a symmetry of the system with the phase space $\mathbb{R}^4$ and $\mathbb{R}^6$. Furthermore by use of these realizations we construct integrable and superintegrable Hamiltonian systems on symmetry Lie groups as phase space. Note that previously in [5] some integrable Hamiltonian systems were constructed on low dimensional real Lie algebra with their coalgebra as phase space. In that work, the invariants of the systems were not specified as a function of phase space variable.

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2 Integrable systems with phase space $\mathbb{R}^4$ and $\mathbb{R}^6$

Here, we use the classification of four dimensional real Lie algebra ($A_4$) which has been presented in [6], and construct integrable Hamiltonian systems with the phase space $\mathbb{R}^4$ or $\mathbb{R}^6$ such that the Casimir invariants of these Lie algebras are Hamiltonians of the systems. For this proposes, we consider the function $Q_i$ ($i = 1, \ldots, \text{dimension phase space}$) of the phase space ($\mathbb{R}^4$ or $\mathbb{R}^6$) variables ($x_a, p_a$) such that they satisfy the following Poisson brackets:

$$\{Q_i, Q_j\} = f^k_{ij} Q_k ,$$  \hspace{1cm} (1)

where $f^k_{ij}$ are the structure constants of the symmetry Lie algebra. Then one can consider the Casimir of the Lie algebra as Hamiltonian of the system where the dynamical observable $Q_i$’s replaced with the generators of the Lie algebra in the Casimir. For obtaining the functions of $Q_i$ we use the differential realization of the Lie algebras $A_4$ [4] such that in these realizations we replace the $\partial_{x_i}$ with the momentum $p_i$.

Now, let us consider an example; for Lie algebra $A_{4,1}$ according to [4] we have the following commutators and realization on $\mathbb{R}^6$:

$$X_1 = \partial_1 , \quad X_2 = \partial_2 , \quad X_3 = \partial_3 , \quad X_4 = x_2 \partial_1 + x_3 \partial_2 ,$$  \hspace{1cm} (2)

$$[X_2, X_4] = X_1 , \quad [X_3, X_4] = X_2 ,$$  \hspace{1cm} (3)

where $x_i$ are coordinates of $\mathbb{R}^6$ and $\partial_i \equiv \frac{\partial}{\partial x_i}$. Then, we construct the following $Q_i$’s, $i = 1, 2, 3, 4$ as a function of $(x_1, x_2, x_3, p_1, p_2, p_3)$ variables of $\mathbb{R}^6$ phase space from the above realization such that they have the following forms and Poisson brackets:

$$Q_1 = -p_1 , \quad Q_2 = -p_2 , \quad Q_3 = -p_3 , \quad Q_4 = -x_2 p_1 - x_3 p_2 ,$$  \hspace{1cm} (4)

$$\{Q_i, Q_j\} = f^k_{ij} Q_k ,$$  \hspace{1cm} (5)

where $f^k_{ij}$ is the structure constants [4] of the Lie algebra $A_{4,1}$. Now, with the above form for $Q_i$’s the Casimir of Lie algebra $A_{4,1}$ [6] as a Hamiltonian of the system has the following form:

$$H = Q_2^2 - 2Q_1Q_3 = p_2^2 - 2p_1p_2 .$$  \hspace{1cm} (6)

In this way, we construct a superintegrable system with Hamiltonian [6] and invariants ($H, Q_1, Q_2, Q_3$) on the phase space $\mathbb{R}^6$. The results for other four dimensional real Lie algebras are summarized in the table 1 and 2. In table 1 we summarized the integrable and superintegrable systems with phase space $\mathbb{R}^4$ and their symmetry Lie algebras. The result of above work with phase space $\mathbb{R}^6$ are summarized in table 2.
Table 1: Integrable and superintegrable systems with the phase space \( \mathbb{R}^4 \).

| symmetry Lie algebra (nonzero commutation relations) | \( Q_1 \) | \( H \) | invariants |
|------------------------------------------------------|------------|------------|------------|
| \( A_{4,1} \) [\( e_2, e_4 \) = 0] | \( Q_1 = -p_1 \) | \( H = Q_1 = -p_1 \) | \( H, Q_2, Q_3 \) |
| \( e_3, e_4 \) = 0 | \( Q_2 = -2p_1 \) | \( Q_3 = -\frac{2}{3} p_1 \) | \( Q_4 = p_2 \) |
| \( A_{4,2} \) [\( e_1, e_4 \) = 0] | \( Q_1 = -p_1 \) | \( H = \frac{Q_1 Q_2}{2} = \frac{Q_2}{2} p_1 \) or \( H = Q_2 \exp(-\frac{Q_1}{Q_2}) = -x_2^2 p_1 \) | \( H, Q_1, Q_2, Q_3 \) |
| \( e_2, e_4 \) = 0 | \( Q_2 = -x_2 p_1 \) | \( Q_3 = -\frac{1}{2} (\ln |x_2|) p_1 \) | \( Q_4 = x_1 p_1 - x_2 p_2 \) |
| \( A_{4,3} \) [\( e_1, e_4 \) = 0] | \( Q_1 = -p_1 \) | \( H = Q_1 \exp(-\frac{Q_2}{Q_3}) = -x_2 p_1 \) | \( H, Q_1, Q_2, Q_3 \) |
| \( e_2, e_4 \) = 0 | \( Q_2 = -x_2 p_1 \) | \( Q_3 = x_2 (\ln |x_2|) p_1 \) | \( Q_4 = x_1 p_1 - x_2 p_2 \) |
| \( A_{4,4} \) [\( e_1, e_4 \) = 0] | \( Q_1 = -p_1 \) | \( H = -\exp(x_2) p_1 \) | \( H, Q_1, Q_2, Q_3 \) |
| \( e_2, e_4 \) = 0 | \( Q_2 = -\exp(x_2) p_1 \) | \( Q_3 = -x_2 p_1 + x_2 p_2 \) | \( Q_4 = x_1 p_1 + p_2 \) |
| \( A_{4,5} \) [\( e_1, e_4 \) = 0] | \( Q_1 = -p_1 \) | \( H = \frac{Q_1}{Q_2} = \frac{e^{(b-1)} p_1}{e^{(b-1)} p_2} \) or \( H = \frac{Q_1}{Q_2} = \frac{e^{(b-1)} p_1}{e^{(b-1)} p_2} \) | \( H, Q_1, Q_2, Q_3 \) |
| \( e_2, e_4 \) = 0 | \( Q_2 = -\exp(x_2) p_1 \) | \( Q_3 = -x_2 p_1 + x_2 p_2 \) | \( Q_4 = x_1 p_1 + p_2 \) |
| \( A_{4,6} \) [\( e_1, e_4 \) = 0] | \( Q_1 = -p_1 \) | \( H = -\exp(x_2) p_1 \) | \( H, Q_1, Q_2, Q_3 \) |
| \( e_2, e_4 \) = 0 | \( Q_2 = -\exp(x_2) p_1 \) | \( Q_3 = -x_2 p_1 + x_2 p_2 \) | \( Q_4 = x_1 p_1 + p_2 \) |
| \( A_{4,7} \) [\( e_1, e_4 \) = 0] | \( Q_1 = -p_1 \) | \( H = \frac{Q_1}{Q_2 + Q_3} = \frac{e^{(b-1)} p_1}{e^{(b-1)} p_2} \) | \( H, Q_1, Q_2, Q_3 \) |
| \( e_2, e_4 \) = 0 | \( Q_2 = -\exp(x_2) p_1 \) | \( Q_3 = -x_2 p_1 + x_2 p_2 \) | \( Q_4 = x_1 p_1 + p_2 \) |
| \( A_{4,8} \) [\( e_1, e_4 \) = 0] | \( Q_1 = -p_1 \) | \( H = \frac{Q_1}{Q_2 + Q_3} = \frac{e^{(b-1)} p_1}{e^{(b-1)} p_2} \) | \( H, Q_1, Q_2, Q_3 \) |
| \( e_2, e_4 \) = 0 | \( Q_2 = -\exp(x_2) p_1 \) | \( Q_3 = -x_2 p_1 + x_2 p_2 \) | \( Q_4 = x_1 p_1 + p_2 \) |
| \( A_{4,9} \) [\( e_1, e_4 \) = 0] | \( Q_1 = -p_1 \) | \( H = \frac{Q_1}{Q_2 + Q_3} = \frac{e^{(b-1)} p_1}{e^{(b-1)} p_2} \) | \( H, Q_1, Q_2, Q_3 \) |
| \( e_2, e_4 \) = 0 | \( Q_2 = -\exp(x_2) p_1 \) | \( Q_3 = -x_2 p_1 + x_2 p_2 \) | \( Q_4 = x_1 p_1 + p_2 \) |
| \( A_{4,10} \) [\( e_1, e_4 \) = 0] | \( Q_1 = -p_1 \) | \( H = \frac{Q_1}{Q_2 + Q_3} = \frac{e^{(b-1)} p_1}{e^{(b-1)} p_2} \) | \( H, Q_1, Q_2, Q_3 \) |
| \( e_2, e_4 \) = 0 | \( Q_2 = -\exp(x_2) p_1 \) | \( Q_3 = -x_2 p_1 + x_2 p_2 \) | \( Q_4 = x_1 p_1 + p_2 \) |
| \( A_{4,11} \) [\( e_1, e_4 \) = 0] | \( Q_1 = -p_1 \) | \( H = \frac{Q_1}{Q_2 + Q_3} = \frac{e^{(b-1)} p_1}{e^{(b-1)} p_2} \) | \( H, Q_1, Q_2, Q_3 \) |
| \( e_2, e_4 \) = 0 | \( Q_2 = -\exp(x_2) p_1 \) | \( Q_3 = -x_2 p_1 + x_2 p_2 \) | \( Q_4 = x_1 p_1 + p_2 \) |
| \( A_{4,12} \) [\( e_1, e_4 \) = 0] | \( Q_1 = -p_1 \) | \( H = \frac{Q_1}{Q_2 + Q_3} = \frac{e^{(b-1)} p_1}{e^{(b-1)} p_2} \) | \( H, Q_1, Q_2, Q_3 \) |
| \( e_2, e_4 \) = 0 | \( Q_2 = -\exp(x_2) p_1 \) | \( Q_3 = -x_2 p_1 + x_2 p_2 \) | \( Q_4 = x_1 p_1 + p_2 \) |
Table 2: Integrable and superintegrable systems with the phase space $\mathbb{R}^6$.

| symmetry Lie algebra (nonzero commutation relations) | N | $Q_i$ | $H$ | invariants |
|-----------------------------------------------------|---|-------|-----|------------|
| $A_{4,1}$ $\{e_2, e_4\} = e_1$ $\{e_3, e_4\} = e_2$ | 1 | $Q_1 = -p_1$ | $H = Q_1^2 - 2Q_1Q_2$ | $H, Q_1, Q_2, Q_3$ |
|                                                      |   | $Q_2 = -p_2$ |                               |                         |
|                                                      |   | $Q_3 = -p_3$ |                               |                         |
|                                                      |   | $Q_4 = -2x_2p_1 - x_3p_2$ | $= p_2^2 - p_1p_3$ |                         |
|                                                      | 2 | $Q_1 = -p_1$ | $H = Q_2^2 - 2Q_1Q_3$ | $H, Q_1, Q_2, Q_3$ |
|                                                      |   | $Q_2 = -p_2$ |                               |                         |
|                                                      |   | $Q_3 = -p_3$ |                               |                         |
|                                                      |   | $Q_4 = -x_2p_1 + p_3$ | $= p_2^2 + x_2^2 - x_3p_1p_2$ |                         |
|                                                      | 3 | $Q_1 = -p_1$ | $H = Q_2^2 - 2Q_1Q_3$ | $H, Q_1, Q_2, Q_3$ |
|                                                      |   | $Q_2 = -x_2p_1$ |                               |                         |
|                                                      |   | $Q_3 = -x_3p_1$ |                               |                         |
|                                                      |   | $Q_4 = p_2 + x_2p_1$ | $= (p_2^2 - 2x_3)p_2^2$ |                         |
| $A_{4,2}$ $\{e_1, e_4\} = b e_1$ $\{e_2, e_4\} = e_2$ $\{e_3, e_4\} = e_2 + e_3$ | 1 | $Q_1 = -p_1$ | $H = Q_2 \exp(-\frac{x_1}{Q_2})$ | $H, Q_1, Q_2, Q_3$ |
|                                                      |   | $Q_2 = -p_2$ |                               |                         |
|                                                      |   | $Q_3 = -p_3$ |                               |                         |
|                                                      |   | $Q_4 = -x_1p_1 - (x_2 + x_3)p_2 - x_3p_3$ | $= -p_2 \exp(-\frac{x_1}{Q_2})$ |                         |
|                                                      | 2 | $Q_1 = -p_1$ | $H = Q_2 \exp(-\frac{x_1}{Q_2})$ | $H, Q_1, Q_2, Q_3$ |
|                                                      |   | $Q_2 = -x_2p_1$ |                               |                         |
|                                                      |   | $Q_3 = -x_3p_1$ |                               |                         |
|                                                      |   | $Q_4 = -x_1p_1 - x_2p_2 + p_3$ | $= -p_2 \exp(-\frac{x_1}{Q_2})$ |                         |
|                                                      | 3 | $Q_1 = -p_1$ | $H = Q_2 \exp(-\frac{x_1}{Q_2})$ | $H, Q_1, Q_2, Q_3$ |
|                                                      |   | $Q_2 = -x_2p_1$ |                               |                         |
|                                                      |   | $Q_3 = -x_3p_1$ |                               |                         |
|                                                      |   | $Q_4 = -x_1p_1 - x_2p_2 - (x_3 - x_2)p_1$ | $= -p_2 \exp(-\frac{x_1}{Q_2})$ |                         |
| $A_{4,3}$ $\{e_1, e_4\} = e_1$ $\{e_3, e_4\} = e_2$ | 1 | $Q_1 = -p_1$ | $H = Q_4 \exp(-\frac{x_1}{Q_4})$ | $H, Q_1, Q_2, Q_3$ |
|                                                      |   | $Q_2 = -p_2$ |                               |                         |
|                                                      |   | $Q_3 = -p_3$ |                               |                         |
|                                                      |   | $Q_4 = -x_1p_1 - x_3p_2$ | $= -p_1 \exp(-\frac{x_1}{Q_4})$ |                         |
|                                                      | 2 | $Q_1 = -p_1$ | $H = Q_4 \exp(-\frac{x_1}{Q_4})$ | $H, Q_1, Q_2, Q_3$ |
|                                                      |   | $Q_2 = -x_2p_1$ |                               |                         |
|                                                      |   | $Q_3 = -x_3p_1$ |                               |                         |
|                                                      |   | $Q_4 = -(x_1 + x_2)p_1 - (x_2 + x_3)p_2 - x_3p_3$ | $= -p_1 \exp(-\frac{x_1}{Q_4})$ |                         |
|                                                      | 3 | $Q_1 = -p_1$ | $H = Q_4 \exp(-\frac{x_1}{Q_4})$ | $H, Q_1, Q_2, Q_3$ |
|                                                      |   | $Q_2 = -x_2p_1$ |                               |                         |
|                                                      |   | $Q_3 = -x_3p_1$ |                               |                         |
|                                                      |   | $Q_4 = -x_1p_1 - x_2p_2 - (x_3 - x_2)p_1$ | $= -p_1 \exp(-\frac{x_1}{Q_4})$ |                         |
| $A_{4,4}$ $\{e_1, e_4\} = e_1$ $\{e_2, e_4\} = e_1 + e_2$ $\{e_3, e_4\} = e_2 + e_3$ | 1 | $Q_1 = -p_1$ | $H = Q_2 \exp(-\frac{x_1}{Q_2})$ | $H, Q_1, Q_2, Q_3$ |
|                                                      |   | $Q_2 = -p_2$ |                               |                         |
|                                                      |   | $Q_3 = -p_3$ |                               |                         |
|                                                      |   | $Q_4 = -(x_1 + x_2)p_1 - (x_2 + x_3)p_2 - x_3p_3$ | $= -p_1 \exp(-\frac{x_1}{Q_2})$ |                         |
|                                                      | 2 | $Q_1 = -p_1$ | $H = Q_2 \exp(-\frac{x_1}{Q_2})$ | $H, Q_1, Q_2, Q_3$ |
|                                                      |   | $Q_2 = -p_2$ |                               |                         |
|                                                      |   | $Q_3 = -p_3$ |                               |                         |
|                                                      |   | $Q_4 = 2x_1p_2 - x_3p_2$ | $= -p_1 \exp(-\frac{x_1}{Q_2})$ |                         |
|                                                      | 3 | $Q_1 = -p_1$ | $H = Q_2 \exp(-\frac{x_1}{Q_2})$ | $H, Q_1, Q_2, Q_3$ |
|                                                      |   | $Q_2 = -x_2p_1$ |                               |                         |
|                                                      |   | $Q_3 = -x_3p_1$ |                               |                         |
|                                                      |   | $Q_4 = -x_1p_1 + p_2 + x_2p_3$ | $= -p_1 \exp(-\frac{x_1}{Q_2})$ |                         |
### Table 2: Integrable and superintegrable systems with the phase space $\mathbb{R}^6$ (continue).

| $A^{6,\pm}_{3,4}$, $abc \neq 0$ | $[e_1, e_2] = a e_1$ | $[e_2, e_3] = b e_2$ | $[e_3, e_4] = c e_3$ | $Q_1 = -p_1$ | $Q_2 = -p_2$ | $Q_3 = -p_3$ | $Q_4 = -ax_1 p_1 - bx_2 p_2 - cx_3 p_3$ | $H = \frac{Q^3_1}{Q^2_2} = \frac{(-p_1 b}{p_2^2}$ | $H, Q_1, Q_2, Q_3$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $a = b = 1$ | $c \neq 1$ | $a < 0$ | $b < 0$ if $a = -1$ | $Q_1 = -p_1$ | $Q_2 = -x_2 p_1$ | $Q_3 = -p_3$ | $Q_4 = -ax_1 p_1 - (a-b)x_2 p_2 - x_3 p_3$ | $H = \frac{Q^3_1}{Q^2_2} = \frac{(-p_1 b}{p_2^2}$ | $H, Q_1, Q_2, Q_3$ |
| $A^{6,\pm}_{3,4}$, $a > 0$ | $[e_1, e_4] = a e_1$ | $[e_2, e_4] = b e_2$ | $[e_3, e_4] = c e_3$ | $[e_2, e_3] = e_1$ | $[e_1, e_4] = (1 + b) e_1$ | $[e_2, e_4] = e_2$ | $[e_3, e_4] = b e_3$ | $[e_2, e_3] = e_1$ | $Q_1 = -p_1$ | $Q_2 = -p_2$ | $Q_3 = -x_2 p_1 - p_3$ | $Q_4 = (1 + b)x_1 p_1 - x_2 p_2 - bx_3 p_3$ | $H = Q_1 = p_1$ | $H, Q_2$ |
| $A^{6,\pm}_{3,4}$, $b > 0$ | $[e_1, e_4] = a e_1$ | $[e_2, e_4] = b e_2$ | $[e_3, e_4] = c e_3$ | $[e_2, e_3] = e_1$ | $[e_1, e_4] = (1 + b) e_1$ | $[e_2, e_4] = e_2$ | $[e_3, e_4] = b e_3$ | $[e_2, e_3] = e_1$ | $Q_1 = -p_1$ | $Q_2 = -p_2$ | $Q_3 = -x_2 p_1 - x_3 p_2$ | $Q_4 = (1 + b)x_1 p_1 - x_2 p_2 - p_3$ | $H = Q_1 = p_1$ | $H, Q_2$ |

### 3 Integrable and superintegrable Hamiltonian systems with the symmetry Lie group as phase space of the system

In this section, we construct the integrable Hamiltonian systems with the symmetry Lie group as a four dimensional phase space. For this propose, we consider those four dimensional real Lie groups such that they have symplectic structure. The list of symplectic four dimensional real Lie groups are classified in [8]. Here, we construct the models on those Lie groups separately as follows.
Lie group $A_{4,1}$:

According to [8], [9] and [10], non-degenerate Poisson $P^{\mu\nu}$ structure on this Lie group can be obtained in the following forms:

$$\{x_1, x_2\} = -\frac{c}{2} x_2^2, \quad \{x_1, x_3\} = cx_4, \quad \{x_1, x_4\} = -d, \quad \{x_2, x_3\} = -c,$$

where $c$ and $d$ are arbitrary real constants.

Now, one can find the following Darboux coordinates:

$$
\begin{align*}
y_1 &= \frac{y_3}{c} + \frac{(cx_4^2)}{8} + \frac{x_2^4}{(2d)}, \\
y_2 &= -x_1 + \frac{x_2^3}{c^2} + \frac{1}{4} cd x_2 x_4 - \frac{x_4 x_2^2}{4} + \frac{x_3 x_2^2}{cd} - \frac{3c^2 x_4^4}{64} + \frac{x_4^4}{4d^2} - \frac{c x_4^4}{8d}, \\
y_3 &= x_2 - \frac{2x_3 x_4}{cd} - \frac{x_3^3}{d^2} - \frac{c x_3^3}{4d}, \\
y_4 &= \frac{1}{d} x_4,
\end{align*}
$$

such that they satisfy the following standard Poisson brackets:

$$\{y_1, y_3\} = 1, \quad \{y_2, y_4\} = 1.$$  \hspace{1cm} (8)

In other words, the coordinate $y_i$ can be used as a coordinates for the phase space $\mathbb{R}^4$; such that the $y_1$ and $y_2$ are dynamical variables and $p_{y_1} = y_3$ and $p_{y_2} = y_4$ are their momentum conjugate. On the other hand, we can apply the realization of $A_{4,1}$ of table 1 with phase space $\mathbb{R}^4$ with coordinates $y_i$; in this respect, using (8) and after replacing $y_i$ in terms of $x_i$ in that realization we obtain the following realization for $Q_i$:

$$
\begin{align*}
Q_1 &= -x_2 + \frac{2 x_3 x_4}{cd} + \frac{x_2^3}{d^2} + \frac{c x_3^3}{4d}, \\
Q_2 &= (x_1 - \frac{x_2^3}{c^2} - \frac{1}{4} cd x_2 x_4 - \frac{x_3 x_2^2}{cd} + \frac{3c^2 x_4^4}{64} - \frac{x_4^4}{4d^2} + \frac{c x_4^4}{8d}) \\
&\quad \times \left( \frac{8 dx_3 x_4 + 4 x_3^4 + c^2 d x_4^2}{4 c d^2 + c^2 d x_4^2} \right)^2, \\
Q_3 &= -\frac{1}{2} (-x_1 + \frac{x_2^3}{c^2} + \frac{1}{4} cd x_2 x_4 - \frac{x_3 x_2^2}{cd} - \frac{3c^2 x_4^4}{64} + \frac{x_4^4}{4d^2} - \frac{c x_4^4}{8d})^2 \\
&\quad \times \left( \frac{8 dx_3 x_4 + 4 x_3^4 + c^2 d x_4^2}{4 c d^2 + c^2 d x_4^2} \right), \\
Q_4 &= \frac{1}{d} x_4,
\end{align*}
$$

such that they satisfy the following Poisson brackets by use of (7) as

$$\{Q_2, Q_4\} = Q_1, \quad \{Q_3, Q_4\} = Q_2.$$  \hspace{1cm} (9)

Then, the Hamiltonian of the superintegrable system with the $A_{4,1}$ as a phase space and symmetry group is obtained as follows:

$$H = Q_1 = -x_2 + \frac{2 x_3 x_4}{cd} + \frac{x_2^3}{d^2} + \frac{c x_3^3}{4d},$$

\footnote{Note that in [8] and [9] the symplectic structure $\omega_{ij}$ on Lie algebra have been given. For obtaining the symplectic structure $\omega_{\mu\nu} = e_{\mu}^i \omega_{ij} e_{\nu}^j$ on groups one can use the vierbein $e_{\mu}^i$ which have been obtained in [10] for four dimensional real Lie groups. Then, one can obtain the non-degenerate Poisson structure from $P^{\mu\nu} = (\omega_{\mu\nu})^i$.}
where the invariant of the system are \((H, Q_2, Q_3)\).

### Lie group \(A_{4,1}^{-1}\):

The non-degenerate Poisson structure on \(A_{4,1}^{-1}\) can be obtained as follows [8], [9], [10]:

\[
\{x_1, x_2\} = 2a, \quad \{x_1, x_3\} = -a, \quad \{x_2, x_4\} = b e^{-x_4},
\]

where \(a\) and \(b\) are arbitrary real constants. For this example, the Darboux coordinates has the following forms:

\[
y_1 = -\frac{e^{x_4}}{b} + x_3, \quad y_2 = -\frac{2ae^{x_4} - bx_1 + abx_2}{ab^2},
\]

\[
y_3 = \frac{2e^{x_4}}{b} + \frac{x_1}{a}, \quad y_4 = e^{x_4}.
\]

Then, after using the results of table 1, we have the following forms for the dynamical functions \(Q_i\):

\[
Q_1 = -\frac{2e^{x_4}}{b} - \frac{x_1}{a},
Q_2 = -\left(\frac{2ae^{x_4} - bx_1 + abx_2}{ab^2} + \frac{x_1}{a}\right),
Q_3 = -\frac{1}{2}\left(\frac{2ae^{x_4} - bx_1 + abx_2}{ab^2} + \frac{x_1}{a}\right) \ln\left|\frac{2ae^{x_4} - bx_1 + abx_2}{ab^2}\right|,
Q_4 = 2e^{x_4}\left(\frac{2ae^{x_4} - bx_1 + abx_2}{ab^2}\right) + \left(-\frac{e^{x_4}}{b} + x_3\right)\left(\frac{2e^{x_4}}{b} + \frac{x_1}{a}\right),
\]

such that they satisfy the following Poisson brackets by use of (13) as

\[
\{Q_1, Q_4\} = -Q_1, \quad \{Q_2, Q_4\} = Q_2, \quad \{Q_3, Q_4\} = Q_2 + Q_3.
\]

In this respect, the Hamiltonian of the maximal superintegrable system with the \(A_{4,1}^{-1}\) as a phase space and symmetry group is obtained as follows:

\[
H = \frac{1}{Q_1 Q_2} = \frac{1}{(\frac{2e^{x_4}}{b} + \frac{x_1}{a})^2\left(\frac{2ae^{x_4} - bx_1 + abx_2}{ab^2}\right)},
\]

where the invariant of the system are \((H, Q_1, Q_2, Q_3)\).

### Lie group \(A_{4,3}\):

From [8], [9] and [10], we have the following forms for the non-degenerate Poisson structure on \(A_{4,3}\):

\[
\{x_1, x_2\} = c x_4 e^{-x_4}, \quad \{x_1, x_3\} = d e^{-x_4},
\]

\[
\{x_1, x_4\} = h e^{-x_4}, \quad \{x_2, x_3\} = f,
\]

where \(c, d, h \) and \(f\) are arbitrary real constants.

Now, after finding of Darboux coordinates in the following forms:

Note that in the relation (10) and (12) and also the relations in the forthcoming models, one can choose the variables \(x_1\) and \(x_2\) as dynamical variables with momentum conjugates \(p_{x_1} = x_3\) and \(p_{x_2} = x_4\).
where

\begin{align}
y_1 &= \frac{dx_1}{f} + \frac{chx_3^2}{2df} - \frac{cx_3x_4}{f}, \\
y_2 &= \frac{x_1}{h} - \frac{de^{-x_4}x_2}{fh} - \frac{ce^{-x_4}x_2}{2df} + \frac{ce^{-x_4}x_3x_4}{fh}, \\
y_3 &= \frac{x_3}{d}, \\
y_4 &= e^{x_4}.
\end{align}

one can obtain the \( Q_i \) as follows:

\begin{align}
Q_1 &= -\frac{x_3}{d}, \\
Q_2 &= \frac{x_3}{d} - \frac{de^{-x_4}x_2}{fh} + \frac{ce^{-x_4}x_3x_4}{2dh} - \frac{ce^{-x_4}x_3x_4}{fh}, \\
Q_3 &= \frac{x_3}{d} - \frac{de^{-x_4}x_2}{fh} - \frac{ce^{-x_4}x_3x_4}{2dh} + \frac{ce^{-x_4}x_3x_4}{fh}(\ln(\frac{x_3}{h} - \frac{de^{-x_4}x_2}{fh} - \frac{ce^{-x_4}x_2}{2dh} + \frac{ce^{-x_4}x_3x_4}{fh}))
\end{align}

\begin{align}
Q_4 &= -e^{x_4}\frac{x_1}{h} + \frac{(d-hx_3)(2d^2x_2 + chx_3 - 2cdx_3x_4)}{2d^2hf}.
\end{align}

such that they satisfy the following Poisson brackets by use of (18) as

\begin{align}
\{Q_1, Q_4\} = Q_1, \quad \{Q_3, Q_4\} = Q_2.
\end{align}

Then the Hamiltonian of the maximal superintegrable system with the \( A_{4,3} \) as a phase space and symmetry group is obtained as

\begin{align}
H = Q_1exp(-\frac{Q_3}{Q_2}) = \frac{e^{-x_4}x_3}{2d^2fh}(chx_3^2 - 2d(fe^{x_4}x_1 - dx_2 + cx_3x_4)),
\end{align}

where the invariant of the system are \( (H, Q_1, Q_2, Q_3) \).

**Lie group \( A_{4,6} \):**

For this Lie group we have the following non-degenerate Poisson structure [8, 9, 10]:

\begin{align}
\{x_1, x_4\} = d e^{-ax_4}, \quad \{x_2, x_3\} = c, 
\end{align}

where \( c \) and \( d \) are arbitrary real constants. The Darboux coordinates for this structure are as follows:

\begin{align}
y_1 &= x_3, \quad y_2 = -\frac{e^{2ax_4}x_1}{ad}, \quad y_3 = -\frac{x_2}{c}, \quad y_4 = e^{-ax_4},
\end{align}

such that after the same calculation and using the results of table 1, the \( Q_i \) have the following forms:

\begin{align}
Q_1 &= \frac{x_2}{c}, \\
Q_2 &= \frac{e^{\frac{2ax_4}{d}}}{c} x_2 \cos(\frac{2ax_4}{ad}), \\
Q_3 &= \frac{e^{-\frac{2ax_4}{d}}}{c} x_2 \sin(\frac{2ax_4}{ad}), \\
Q_4 &= -e^{-ax_4} + \frac{a}{c}x_2x_3.
\end{align}

where they satisfy the following Poisson brackets by use of (23)

\begin{align}
\{Q_1, Q_4\} = a Q_1, \quad \{Q_2, Q_4\} = -Q_3, \quad \{Q_3, Q_4\} = Q_2.
\end{align}
The Hamiltonian of the *maximal superintegrable* system with the \( A_{4,0} \) as a phase space and symmetry group is obtained

\[
H = Q_2^2 + Q_3^2 = e^{-2x_4} \frac{x_2^2}{e^2},
\]

where the invariant of the system are \((H, Q_1, Q_2, Q_3)\).

**Lie group \( A_{4,7} \):**

The non-degenerate Poisson structure for this Lie group has the following form [8], [9], [10]:

\[
\{x_1, x_3\} = -2cx_3 e^{-2x_4}, \quad \{x_1, x_4\} = ce^{-2x_4}, \quad \{x_2, x_3\} = 2ce^{-2x_4},
\]

where \( c \) is arbitrary real constant. Furthermore, for this example one can find the following Darboux coordinates:

\[
y_1 = \frac{e^{2x_4}(x_2)}{2c}, \quad y_2 = \frac{-1 - e^{2x_4} + e^{4x_4}x_1 + e^{4x_4}x_2x_3}{2c},
\]

\[
y_3 = x_3, \quad y_4 = e^{-2x_4},
\]

such that the \( Q_i \) have the following forms:

\[
Q_1 = -x_3,
\]

\[
Q_2 = \frac{x_3(-1 - e^{2x_4} + e^{4x_4}x_1 + e^{4x_4}x_2x_3)}{2c},
\]

\[
Q_3 = e^{-2x_4},
\]

\[
Q_4 = x_3 \left( \frac{-e^{2x_4}(x_2)}{c} + \frac{(-1 - e^{2x_4} + e^{4x_4}x_1 + e^{4x_4}x_2x_3)^2}{8c^4} + \frac{e^{-2x_4}(-1 - e^{2x_4} + e^{4x_4}x_1 + e^{4x_4}x_2x_3)}{2c} \right),
\]

so that they satisfy the following Poisson brackets by use of (28) as

\[
\{Q_2, Q_3\} = Q_1, \quad \{Q_1, Q_4\} = 2Q_1, \quad \{Q_2, Q_4\} = Q_2,
\]

\[
\{Q_3, Q_4\} = Q_2 + Q_3.
\]

Then, the Hamiltonian of the *integrable* system with the \( A_{4,7} \) as a phase space and symmetry group is obtained

\[
H = Q_2 = \frac{x_3(-1 - e^{2x_4} + e^{4x_4}x_1 + e^{4x_4}x_2x_3)}{2c},
\]

where the invariant of the system are \((H, Q_1)\).

**Lie group \( A_{4,9} \):**

For this Lie group the non-degenerate Poisson structure has the following form [8], [9], [10]:

\[
\{x_1, x_3\} = 2cx_3 e^{-2x_4}, \quad \{x_1, x_4\} = -ce^{-2x_4}, \quad \{x_2, x_3\} = -2ce^{-2x_4},
\]
where $c$ is arbitrary real constant. On the other hand, after the same calculation one can find the Darboux coordinates as follows:

$$y_1 = -\frac{e^{2x_4}(x_2)}{2c}, \quad y_2 = -1 - e^{2x_4} + e^{4x_4}x_1 + e^{4x_4}x_2x_3,$$

$$y_3 = x_3, \quad y_4 = e^{-2x_4},$$

(34)

Then, according to the results of table 1, the $Q_i$ are obtained as follows:

$Q_1 = -x_3$,
$Q_2 = -e^{-2x_4}$,
$Q_3 = -x_3(-1 - e^{2x_4} + e^{4x_4}x_1 + e^{4x_4}x_2x_3)$,
$Q_4 = \frac{e^{2x_4}(x_2x_3)}{2c} - \frac{e^{-2x_4}(-1 - e^{2x_4} + e^{4x_4}x_1 + e^{4x_4}x_2x_3)}{2c},$

(35)

such that they satisfy the following Poisson brackets by use of (33) as

$$\{Q_2, Q_3\} = Q_1, \quad \{Q_1, Q_4\} = 2Q_1, \quad \{Q_2, Q_4\} = Q_2,$$

$$\{Q_3, Q_4\} = Q_3$$

(36)

In this way, the Hamiltonian of the integrable system with the $A_{4,9}^1$ as a phase space and symmetry group is obtained

$$H = Q_1 = -x_3,$$

(37)

where the invariant of the system are $(H, Q_2)$.

**Lie group $A_{4,12}$:**

Finally, for this Lie group we have the following non-degenerate Poisson structure [8, 9, 10]:

$$\{x_1, x_3\} = -c e^{-x_3}(a \cos(x_4) + b \sin(x_4)),$$
$$\{x_1, x_4\} = c e^{-x_3}(-b \cos(x_4) + a \sin(x_4)),$$
$$\{x_2, x_3\} = c e^{-x_3}(b \cos(x_4) - a \sin(x_4)),$$

$$\{x_2, x_4\} = -c e^{-x_3}(a \cos(x_4) + b \sin(x_4)),$$

(38)

where $c = \frac{1}{ax^2 + bx}$ and $a, b$ are arbitrary real constants. One can find the following Darboux coordinates for this structure:

$$y_1 = e^{2x_3}(ax_1\cos(x_4) - bx_2\cos(x_4) + bx_1\sin(x_4) + ax_2\sin(x_4)),$$
$$y_2 = -e^{2x_3}(bx_1\cos(x_4) + ax_2\cos(x_4) - ax_1\sin(x_4) + bx_2\sin(x_4)),$$
$$y_3 = e^{x_3},$$
$$y_4 = x_4.$$

(39)

Then, by use of table 1 one can obtain the $Q_i$ as follows:

$$Q_1 = -e^{-x_3},$$
$$Q_2 = bx_1\cos(x_4) + ax_2\cos(x_4) - ax_1\sin(x_4) + bx_2\sin(x_4),$$
$$Q_3 = -e^{x_3}(ax_1\cos(x_4) - bx_2\cos(x_4) + bx_1\sin(x_4) + ax_2\sin(x_4)),$$

(40)
\[ Q_4 = -e^{2x_3}(ax_1\cos(x_4) - bx_2\cos(x_4) + bx_1\sin(x_4) + ax_2\sin(x_4))(bx_1\cos(x_4) + ax_2\cos(x_4) - ax_1\sin(x_4) + bx_2\sin(x_4)) + x_4(1 - e^{2x_3}(bx_1\cos(x_4) + ax_2\cos(x_4) - ax_1\sin(x_4) + bx_2\sin(x_4))), \]

such that they satisfy the following Poisson brackets by use of (38) as

\[
\begin{align*}
\{Q_1, Q_3\} &= Q_1 , & \{Q_2, Q_3\} &= Q_2 , & \{Q_1, Q_4\} &= -Q_2, \\
\{Q_2, Q_4\} &= Q_1 .
\end{align*}
\]

(41)

Then, the Hamiltonian of the integrable system with the \(A_{4,12}\) as a phase space and symmetry group is obtained

\[ H = Q_1 = -e^{-2x_4} \]

(42)

where the invariant of the system are \((H, Q_2)\).

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