Quasi-conformal functions of quaternion and octonion variables, their integral transformations.

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12 March 2007

Abstract

The article is devoted to holomorphic and meromorphic functions of quaternion and octonion variables. New classes of quasi-conformal and quasi-meromorphic mappings are defined and investigated. Properties of such functions such as their residues and argument principle are studied. It is proved, that the family of all quasi-conformal diffeomorphisms of a domain form a topological group relative to composition of mappings. Cases when it is a finite-dimensional Lie group over $\mathbb{R}$ are studied. Relations between quasi-conformal functions and integral transformations of functions over quaternions and octonions are established. For this, in particular, noncommutative analogs of the Laplace and Mellin transformations are studied and used. Examples of such functions are given. Applications to problems of complex analysis are demonstrated.

1 Introduction

Complex analysis is one of the corner-stones of mathematics. On the other hand, natural generalizations of complex numbers obtained by subsequent doubling procedures with the help of generators were introduced in the second half of the 19-th century. The most important among them are quaternions invented by W.R. Hamilton and their generalizations such as octonions and Cayley-Dickson numbers are known [1, 12]. The problem of developing analysis over quaternions $\mathbb{H} = \mathbb{A}_2$ and octonions $\mathbb{O} = \mathbb{A}_3$ was posed by Hamilton and Yang and Mills for the needs of celestial mechanics and quantum field theory [11, 9, 30].

Quaternions and octonions were used in quantum mechanics and quantum field theory and even by J.C. Maxwell, but mainly algebraically, because theory of functions of quaternion and octonion variables was little developed [7, 9, 16]. Noncommutative analysis is being developed in recent years for the needs of mathematics and theoretical physics [2, 4, 5, 6, 14, 28], but it remains very little promoted in comparison with classical analysis, especially its non super-commutative part. Derivations of abstract algebras are widely used and a work with functions on algebras is frequently related with their representations by words and phrases [3, 29].

In preceding works of the author super-differentiable (in another words holomorphic) functions of Cayley-Dickson variables were investigated [17, 18, 19, 20] such that they generalize the theory of complex holomorphic functions. In the particular case of complex functions the notion of super-differentiability reduces to the usual complex differentiability. In these publications super-differentiability was defined as derivation of an algebra and
taking into account specific features of the Cayley-Dickson algebra. In view of the non-commutativity of the Cayley-Dickson algebra $\mathcal{A}_r$ with $2 \leq r$, the theory of functions over them is not only the usual theory of functions, but it also bears the algebraic structure and certainly is related with representations of functions with the help of words and phrases over Cayley-Dickson variables.

It is necessary to note, that there are natural embeddings $\theta^r_k$ of $\mathcal{A}_r$ into $\mathcal{A}_k$ for each $1 \leq r < k \in \mathbb{N}$ associated with the subsequent doubling procedure, but besides them there are others algebraic embeddings. The algebra $\mathcal{A}_\infty$ obtained by completion of the strict inductive limit $\text{str} - \text{ind}\{\mathcal{A}_r, \theta^r_k, \mathbb{N}\}$ relative to the $l_2$ norm has no any internal anti-automorphism $z \mapsto z^*$, since it is external, where $zz^* = |z|^2$. Therefore, it is natural to consider holomorphic functions of $\mathcal{A}_r$ variables with $2 \leq r < \infty$ as restrictions of functions of $\mathcal{A}_\infty$ variables on the corresponding domains. Though the hypercomplex Cayley-Dickson algebra $\mathcal{A}_\infty$ is noncommutative and non-associative, but with respect to the absence of the internal anti-automorphism $z \mapsto z^*$ it resembles by such property the complex field $\mathbb{C}$. Then operator theory over Cayley-Dickson algebras on the base of this function theory was studied in [22, 21]. Super-differentiable functions are locally analytic by their Cayley-Dickson variables, but series for them are more complicated in comparison with the complex case due to noncommutativity for $r \geq 2$ or non-associativity for $r \geq 3$ of $\mathcal{A}_r$. Then the noncommutative analog of the Laplace transformation was studied in [26, 27]. In particular, pseudo-conformal mappings over quaternions and octonions having properties closer to that of complex holomorphic functions were defined and studied in [23, 24]. Pseudo-conformal mappings over the quaternion skew field $\mathbb{H}$ or over the octonion algebra $\mathbb{O}$ are analogous to complex conformal functions, but in the noncommutative setting, so generally pseudo-conformal functions may be non-isometric (see Definition 2.1).

This work continues these investigations using preceding results. Professor Fred Van Oystaeyen has formulated in 2002 the problem of developing analysis over quaternions and octonions for the needs of mathematics and theoretical physics, particularly, of noncommutative geometry and their potential applications, as well as for problems of complex analysis such as the Riemann’s hypothesis [17]. Natural extensions of complex holomorphic functions are introduced over quaternions and octonions such that a notion of quasi-conformal mappings is defined. They form a different class than that of pseudo-conformal mappings. Quasi-conformal mappings on domains $W$ in $\mathcal{A}_b$ are formed from pseudo-conformal functions on domains $W$ in the Cayley-Dickson subalgebra $\mathcal{A}_r, 1 \leq r < b \leq 3$, with the help of operators to which rotations of the real shadow $\mathbb{R}^2$ correspond.

In the second section new classes of quasi-conformal and quasi-meromorphic mappings are defined and investigated. Properties of such functions such as their residues and argument principle are studied. It is proved, that the family of all quasi-conformal diffeomorphisms of a domain form a topological group relative to composition of mappings. Cases when it is a finite-dimensional Lie group over $\mathbb{R}$ are studied.

In the third section relations between quasi-conformal functions and integral transformations of functions over quaternions and octonions are established. For this, in particular, noncommutative analogs of the Laplace and Mellin transformations are studied and used. Examples of such functions are given. An effectiveness of analysis over quaternions and octonions is demonstrated for problems of complex analysis.

Many results of this paper are obtained for the first time.
2 Quasi-conformal functions

1. Definitions and Notation. Let $\mathcal{A}_r$ denote the Cayley-Dickson algebra of dimension $2^r$ over $\mathbb{R}$, where in particular $\mathbb{C} = \mathcal{A}_1$ is the field of complex numbers, $\mathbb{H} = \mathcal{A}_2$ is the skew field of quaternions, $\mathbb{O} = \mathcal{A}_3$ is the algebra of octonions. Suppose that $U$ is an open subset in $\mathcal{A}_r$, $2 \leq r \leq 3$. A function $f$ on $U$ we call pseudo-conformal at a point $\xi$ in $U$, if $f$ is holomorphic (super-differentiable) in a neighborhood of $\xi$ and satisfies Conditions (P1–3):

(P1) $\partial f(z)/\partial \bar{z} = 0$ for $z = \xi$;
(P2) $Re\{(|f(z)/\partial z|.h_1)||f(\bar{z})/\partial \bar{z}||h_2\} = 0$ for each $h_1, h_2 \in \mathcal{A}_r$;
(P3) $\partial f(z)/\partial z = 0$ for each $h \neq 0$ in $\mathcal{A}_r$, where $z = z^*$ denotes the adjoint number of $z \in \mathcal{A}_r$ such that $z\bar{z} = |z|^2$, $Re(z) := (z + \bar{z})/2$; for $f$ it is used either the shortest phrase compatible with these conditions or in the underlying real space (shadow) $\mathbb{R}^4$ or $\mathbb{R}^8$ non-proper rotations $[f']$ associated with $f'$ are excluded. That is, $[f'] \in SO(2^b, \mathbb{R})$, where $b = 2$ or $b = 3$, $SO(n, \mathbb{R})$ denotes the special orthogonal group of $\mathbb{R}^n$, $[f']$ denotes the operator in the real shadow corresponding to the super-derivative $f'$ over $\mathcal{A}_r$. For short $f(z, \bar{z})$ is written as $f(z)$ due to the bijectivity between $z \in \mathcal{A}_r$ and $\bar{z}$.

If $f$ is pseudo-conformal at each point $\xi \in U$, then it is called pseudo-conformal in the domain $U$.

For mappings of complex numbers, $r = 1$, each holomorphic function satisfying Condition (P3) fits to this definition, so we can include this case also.

We say that a function $\phi$ at $\zeta$ or on $V$ is $p$-pseudo-conformal, if $\phi(z) = f(z^p)$ and $f$ is pseudo-conformal at $\zeta$ or on $U$, where $\zeta^p = \xi$ or $U = \{z^p : z \in V\}$, $p \in \mathbb{N}$. A function of several variables $z, \cdots, n$ is called pseudo-conformal or $(p_1, \cdots, p_n)$-pseudo-conformal, if it is pseudo-conformal or $p_j$-pseudo-conformal by $jz$ for each $j = 1, \cdots, n$.

Let now $U$ be open in $\mathcal{A}_b$ and $W = U \cap \mathcal{A}_r \neq \emptyset$ be open in $\mathcal{A}_r$ and non-void, where $1 \leq r < b \leq 3$. Consider the natural embedding of $\mathcal{A}_r$ into $\mathcal{A}_b$ associated with the standard doubling procedure.

Suppose that $f$ is a holomorphic function on $U$ with values in $\mathcal{A}_b$ satisfying the following conditions:

(Q1) the function $g(z) := f(y_0 + z)$ has the $p$-pseudo-conformal restriction $g|_{W-y_0}$ on $W - y_0 := \{z : z = x - y_0, x \in W\}$ for some marked point $y_0 \in W$ and $g(W - y_0) \subset \mathcal{A}_r$;
(Q2) there exists a family of automorphisms $\tilde{R}_{z,x} = \tilde{R}_{z,x} : \mathcal{A}_b \to \mathcal{A}_b$ for each $z \in U - y_0$ and $x \in W - y_0$ with $Re(z) = Re(x)$ such that to each $\tilde{R}_{z,x}$ a proper rotation $T = [\tilde{R}_{z,x}]$ in $SO(2^b, \mathbb{R})$ of the real shadow $\mathbb{R}^{2^b}$ corresponds such that for each $z \in U - y_0$ there exists $x \in W - y_0$ for which $z = \tilde{R}_{z,x}x$, where $SO(n, \mathbb{R})$ denotes the special orthogonal group of the Euclidean space $\mathbb{R}^n$;
(Q3) $\tilde{R}_{z,x}|_{\mathbb{R}} = id_{\mathbb{R}}$ for each $z \in U - y_0$ and every $x \in W - y_0$, that is, $T = [\tilde{R}_{z,x}] \in SO_{\mathbb{R}}(2^b, \mathbb{R})$, where $SO_{\mathbb{R}}(n, \mathbb{R}) := \{T : T \in SO(n, \mathbb{R}); T|_{\mathbb{R}} = I\}$,
(Q4) $\tilde{R}_{z,x} = id$ for each $z \in W - y_0$ and every $x \in W - y_0$,
(Q5) $\tilde{R}_{z,x}$ depends $\mathcal{A}_b$ holomorphically on $z \in U - y_0$ and $\mathcal{A}_r$ holomorphically on $x \in W - y_0$ in a suitable $(z, x)$-representation,
(Q6) $g(z) = \tilde{R}_{z,x}g(x)$ for each $x \in W - y_0$ and every $z \in U - y_0$ such that $Re(z) = Re(x)$ and $z = \tilde{R}_{z,x}x$,
(Q7) $g'(\tilde{R}_{z,y}y).\tilde{R}_{z,y}h := g'(\eta).w|_{(\eta = \tilde{R}_{z,y}y, w = \tilde{R}_{z,y}h)} = \tilde{R}_{z,y}[g'(y).h]$ for each $z \in U - y_0$ and $y \in W - y_0$ such that $Re(z) = Re(y)$ and $z = \tilde{R}_{z,y}y$ and every $h \in \mathcal{A}_r$, where $g'(z)$ is the (super)derivative operator over $\mathcal{A}_b$. 3
We call such function \((p, r, b)\)-quasi-conformal. If a function \(f\) is \(A_b\) holomorphic on \(U\) and satisfies \((Q1 - Q6)\) on \(U\) and \(f\) is \((p, r, b)\)-quasi-conformal on \(U \setminus S_A\), where \(S_A := \bigcup \{ z + y_0 : z = R_{z,x}x ; x \in A - y_0, z \in U - y_0, Re(z) = Re(x) \}\), while \(A := A_f := \{ y + y_0 : y \in W - y_0, g'(y) = 0 \}\) is a discrete subset in \(W\) consisting of isolated points such that for each \(y_1 \in A\) there exists \(\delta > 0\) for which \(\inf_{y \in A, y \neq y_1} |y - y_1| \geq \delta\), then we call \(f\) the \((p, r, b)\)-quasi-regular function on \(U\). In the particular latter case of \(U = A_b\) we call \(f\) the \((p, r, b)\)-quasi-integral function.

If \(f\) is a \((p, r, b)\)-quasi-regular function on \(U \setminus S_C\), where \(C\) is a discrete set of isolated points in \(W\) at which \(f\) has poles (of finite orders), then we call \(f\) the \((p, r, b)\)-quasi-meromorphic function on \(U\).

For \(p = 1\) to shorten the notation we shall write that \(f\) is \((r, b)\)-quasi-conformal or \((r, b)\)-quasi-regular on \(U\) or \((r, b)\)-quasi-integral correspondingly. A function of several variables \(z_1, ..., n z\) is called \((p_1, ..., p_n; r, b)\)-quasi-conformal or \((p_1, ..., p_n; r, b)\)-quasi-regular on \(U\) open in \(A_b^p\) or \((p_1, ..., p_n; r, b)\)-quasi-integral, if it is \((p_j, r, b)\)-quasi-conformal or \((p_j, r, b)\)-quasi-regular or \((p_j, r, b)\)-quasi-integral by \(jz\) for each \(j = 1, ..., n\). If \(M\) is canonically closed, \(M\) is the closure of \(U\), then \(f\) is quasi-conformal or quasi-regular if it is such on \(U\) and \(f(z)|_{\partial M} = f'(z)|_{\partial M}\) are the continuous limits of \(f\) and \(f'\) in \(U\), where \(\partial M\) is the boundary of \(M\) such that \(M \cap A_b^p\) is a pseudo-conformal manifold.

2. Examples. Let a function \(g\) be \(A\), holomorphic and hence locally analytic. It has a local series expansion of \(g_{|W-y_0}\) with coefficients in \(A\), and the variable \(x \in W - y_0\) such that this series converges on an open ball \(B(A_r, \xi, R^-) \triangleq \{ x \in A : |x - \xi| < R \}\) for each \(\xi \in W - y_0\), where \(0 < R = R(\xi) \leq \infty\). Then the operator \(R_{z,x}\) acts on \(g(x)\) throughout a local series expansion of \(g_{|W-y_0}\) with coefficients in \(A\), and the variable \(x \in W - y_0\), since \(R_{z,x}\) is the automorphism of the Cayley-Dickson algebra \(A_b\). Therefore, \(g\) has the \(A_b\) extensions by the variable \(z \in B(A_r, \xi, R^-)\) such that \(U = \bigcup_{\xi \in W - y_0} B(\mathbb{A}_b, \xi, R^-)\). Though this extension satisfies Conditions \((Q1 - Q6)\), it need not be satisfying \((Q7)\) in general.

Each \(z \in A_b\) has the polar decomposition \(z = |z| \exp(\Arg(z))\), where \(\Arg(z) \in \mathcal{I}_b \triangleq \{ y \in \mathbb{A}_b : Re(y) = 0 \}\) (see Section 3 in [17, 18, 19]). Fix a branch of \(\Arg(z)\) choosing one definite branch of \(\text{Ln}\) over \(A_b\) such that \(\Arg(z) = \text{Ln}(z/|z|)\) and \(\Arg(z) = 0\) for each real \(z \geq 0\). Then

\[
(A) \quad \Arg(z) = M\phi = M(z)\phi(z), \quad M \in \mathcal{I}_b, \quad |M| = 1, \quad \phi \in \mathbb{R}, \quad |\phi| = |\Arg(z)|, \quad M = M(z), \quad \phi = \phi(z).
\]

Take without loss of generality \(y_0 = 0\). For the pair \((C, A_b)\) with \(2 \leq b \leq 3\) using the polar decomposition \(z - Re(z) = |z - Re(z)| \exp(M\psi)\) for \(z \in A_b\), where \(Re(M) = 0\), \(M = M(z - Re(z)) \in A_b, \quad |M| = 1, \quad \psi = \psi(z) = \phi(z - Re(z)) \in \mathbb{R}\), gives a family of automorphisms \(R_{z,x}\) for each \(z \in A_b\) and every \(x \in C\) satisfying the equality

\[
(1) \quad R_{z,x}(i_1) = R_{z,x}(i_1) = M \quad \text{with} \quad M = M(z - x - Re(z - x)) \quad \text{and} \quad \phi = \phi(z - x - Re(z - x))
\]

given by Equation \((A)\) and \(R_{z,x}(u) = R_{z,x}(u) = u\) for each \(u \in \mathbb{R}\) and \(R_{z,y}y = y\) for each \(z \in C\) and every \(y \in C\), hence

\[
R_{z,x}(x) = |x| \exp(M\phi) \quad \text{for each} \quad x = |x| \exp(i\phi) \in C,
\]

where \(\phi = \psi(x) \in \mathbb{R}, \quad z \in A_b \setminus C\). Indeed, the algebra isomorphic with \(A_b\) can be constructed by the subsequent doubling procedure starting from \(M\) as well instead of \(i = i_1\) choosing a generator \(M_2\) instead of \(i_2\) orthogonal to \(M\) and taking \(R_{z,x}(i_2) = M_2, \ R_{z,x}(i_3) = MM_2\), where \(M_2\) depends holomorphically on \(z\) and \(x\) (see also Proposition 3.2 and Corollary 3.5 [18, 19]), where \(\{ i_0, i_1, ..., i_{2r-1} \}\) are generators of \(A\), such that \(i_0 = 1, \ i_0^2 = -1\) and \(i_j i_k = -i_k i_j\) for each \(1 \leq j \neq k \leq 2^r - 1\). Then for \(b = 3\) take the doubling generator \(L \in \mathbb{I}_b\) orthogonal to \(M, M_2\) and \(MM_2\) such that \(L\) depends holomorphically on \(z\) and \(x\) and put \(R_{z,x}(i_4) = L\) (see in details below). Write \(z \in A_b\) in the form
(2) $z = \sum_{s \in b} w_s s$, where $w_s \in R$ for each $s \in b := \{1, i_1, ..., i_{2^b - 1}\}$, $b$ is the basis of generators of $A_b$, put $b := b \setminus \{1\}$, hence

(3) $z^* = (2^b - 2)^{-1}\{-z + \sum_{s \in b} s(z^s)\}$. Therefore,

(4) $|z| = (z^*z)^{1/2} = |z(2^b - 2)^{-1}\{-z + \sum_{s \in b} s(z^s)\}|^{1/2}$ and

(5) $Re(z) = (z + z^*)/2 = \{1 - (2^b - 2)^{-1}\}z + (2^b - 2)^{-1}\sum_{s \in b} s(z^s)$ are the holomorphic functions on $A_b \setminus \{0\}$ in these $z$-representations (4,5).

(6) $M(z)\phi(z) = Ln(z/|z|)$ for $z \neq 0$ and for $\phi(z) > 0$ with $z \in A_b \setminus R$ we have

(7) $M(z) = Ln(z/|z|)/Ln(z/|z|)$ is implied to be written in the $z$-representation with the help of Formula (4), putting $\phi(z) = 0$ for each real non-negative $z$. In view of Condition (Q2) it is sufficient to consider $\phi(z) > 0$ in the half-space of $A_b \setminus R$. The logarithmic function $Ln(z)$ is holomorphic on $A_b \setminus \{0\}$ with the noncommutative non-associative analog of the Riemann surface described in Section 3.7 [17, 18, 19]. In view of Formulas (4,5) the automorphism $\hat{R}_{z,x}$ given by Equations (1) becomes holomorphic by $z \in A_b$ and by $x \in C$ in the $(z,x)$-representation.

For the pair $(A_q, A_{q+1})$, where $1 \leq q \in N$, using the iterated exponent

(8) $exp(3M\phi(\xi)) = exp\{2M\phi_1(\xi) exp(3M\phi_2(\xi)) exp(-2M\phi_3(\xi))\},$

where $\xi = |z| \exp(3M\phi(\xi))$, $3M = MN \in I_{q+1}$, $\phi_1(\xi), \phi_2(\xi), \phi_3(\xi) \in R$, $z = z_1 + i_2 z_2$, $z \in A_{q+1}$, $z_1, z_2 \in A_q$, $|3M| = |2M| = |N| = 1$, $\xi = z - x - Re(z - x)$, $N = N(\xi) \perp 2M = 2M(\xi)$, that is, $Re(2MN) = 0$; $N$ and $2M \in I_{q+1}$, $l = i_2$. Consider (8) for $q = 0$ and then for $q = 2$. This gives the family of automorphisms $\hat{R}_{z,x}$ for each $z \in O = A_q$ and every $x \in H = A_q$ such that

(9) $\hat{R}_{z,x}(u) = R_{z,x}(u) = u$ for each $u \in R$, $\hat{R}_{z,x}(i_1) = R_{z,x}(i_1) = 2M$ and $\hat{R}_{z,x}(i_2) = R_{z,x}(i_2) = 2MN$, $R_{z,x}y = y$ for each $y \in H$ and every $x \in H$.

Indeed, the algebra isomorphic with $A_3$ can be constructed starting with $M, N, MN$ instead of $i_1, i_2, i_3$ and using the doubling procedure and choosing $L \perp R \oplus R \oplus \mathbb{R} \oplus MN$, $|L| = 1$ (see also Note 2.4 [27, 18]).

Since $e^M = \cos |M| + M(\sin |M|)/|M|$ for each $M \in I_b \setminus \{0\}$, $e^0 = 1$, then Equation (8) gives

$$3M\phi(\xi) = 2M\phi_1 \cos \phi_2 + N\phi_1 \sin \phi_2 \sin \phi_3 + N 2M\phi_1 \sin \phi_2 \cos \phi_3,$$

dependent on $z$ and $A_2$-holomorphically on $z$ and $A_2$-holomorphically on $x$ with the help of Equations (3 – 10). For example, take $2M \in i_2 R \oplus i_2 R \oplus i_7 R$, $N \in i_3 R \oplus i_8 R \oplus i_R$. Then choose the doubling generator $L \in I_b$ orthogonal to $2M$ and $N$ and $2MN$ such that $L$ depends holomorphically on $z$ and $x$. In view of Formulas (2 – 7) the automorphism $\hat{R}_{z,x}$ is holomorphic by $z \in O$ and by $x \in H$ in the $(z, x)$-representation.

This is possible for each $(r, b)$ pair using the sequence of embeddings $A_r \hookrightarrow A_{r+1} \hookrightarrow ... \hookrightarrow A_b$ and considering with the help of (2 – 7) subsequent holomorphic solutions of (8) for $A_q \hookrightarrow A_{q+1}$ in the corresponding $(z, x)$-representation for each $q = r, ..., b - 1$. If $R_{z,x}(i_2)$ are specified for $q = 0, 1, ..., b - 1$, then their multiplication in $A_b$ gives $R_{z,x}(i_2)$ for each $1 \leq j \leq 2^b - 1$ (see also [12]). This is evident, since $A_b = \{z \in A_b : \exists x \in A_{b-1}$ and $\exists T \in SO_R(2^b, R)$ such that $[z] = T[x]\}$ and $SO_R(2^b, R)$ is the real analytic Lie group isomorphic with $SO(2^b - 1, R)$, where $[x] \in R^{2^b}$, $[x] = (x_0, x_1, ..., x_{2^b-1, 0, 0, ...})$, $x = x_0 i_0 + x_1 i_1 + ... + x_{2^b-1} i_{2^b-1}, [z] = (z_0, ..., z_{2^b-1}), x_j, z_j \in R$ for each $j$, $2 \leq b \in N$.

From the construction of $R_{z,x}$ it follows that for the $(C, H)$ and $(C, O)$ pairs, that is, $r = 1$ and $b = 2, 3$, there exists $R_{z,x}$ satisfying conditions:
(11) $R_{vz,wy} = R_{z,y}$ for each $v$ and $w \in \mathbb{R} \setminus \{0\}$ such that $vw > 0$ and
(12) $R_{z,y} = R_{a,x}$ for each $Im(z) = Im(a)$ and $Im(y) = Im(x)$, where $Im(z) := z - Re(z)$. Therefore, if $R_{z,y}y = z$, then $R_{z,y} = \frac{z}{y}$.

Henceforth, up to an $\mathcal{A}_b$-pseudo-conformal diffeomorphism $\xi$ of a domain $U$ such construction of the family $R_{z,x}$ will be implied for $1 \leq r < b \leq 3$, $R_{z,x} \mapsto R_{\xi(z),\xi(x)}$, where $\xi(U) = U$, $\xi(W) = W$, $\xi(\mathbb{R} \cap W) = \mathbb{R} \cap W$, $R_{z,x}$ is the family of this example given by Equations (1 - 10).

Each $\mathcal{A}_r$-pseudo-conformal (particularly, complex holomorphic) function with real expansion coefficients of a power series converging by $x \in W - y_0$ evidently has an $(r, b)$-quasi-conformal extension due to Condition (Q3).

2.1. Definition. For each $p \in \mathbb{H} = \mathcal{A}_2$ let
(1) $E_2(p) := E(p) := p_0 + p_1i_1 \exp(-p_3i_3 \exp(-p_3i_1))$, while for each $p \in \mathbb{O} = \mathcal{A}_3$ put
(2) $E_3(p) := E(p) := p_0 + p_1i_1 \exp(-p_3i_3 \exp(-p_3i_1) \exp(-p_3i_3 \exp(-p_3i_1) \exp(-p_3i_3 \exp(-p_3i_1)))$,
where $p = p_0 + p_1i_1 + \ldots + p_{2^{b-1}}i_2^{b-1}, p \in \mathcal{A}_b, p_0, \ldots, p_{2^{b-1}} \in \mathbb{R}$. $L \xi \in \mathcal{A}_b$, $p \in \mathcal{A}_b$.

If $f^*$ is an $\mathcal{A}_b$-holomorphic function on a domain $V$ and $V = E^{-1}(U)$, where $U$ is a domain in $\mathcal{A}_b$, $f = f^* \circ E^{-1}$ is $(p, 1, b)$-quasi-conformal or quasi-regular (or quasi-integral) or quasi-meromorphic on $U$, then we call $f^*$ the $(p, 1, b)$-quasi-conformal or quasi-regular (or quasi-integral for $V = \mathcal{A}_b$) or quasi-meromorphic function in spherical $\mathcal{A}_b$-coordinates on $V$ respectively.

Certainly, in Formulas (1, 2) other choice of basic generators or some other order in the iterated exponent can be, but these formulas provide canonical spherical $\mathcal{A}_b$-coordinates.

3. Theorem. Let $U \subset \mathcal{A}_b^n$ be an open subset, let also $F = (f_1, \ldots, f_m) : U \to \mathcal{A}_b^n$ be a holomorphic mapping, where either $2 \leq r \in \mathbb{N}$ or $r = L$, $\text{card}(\Lambda) \geq 0$, $1 \leq m \leq n \in \mathbb{N}$. If $z_0 \in U$, $F(z_0) = 0$ and the operator $(\partial k/\partial jz)_{1 \leq j,k \leq m}$ is invertible at $z_0$, where $z = (z, \ldots, z) \in \mathcal{A}_b$, for each $j = 1, \ldots, n$, then there exist an open neighborhood $W$ of a point $x_0$ in $\mathcal{A}_b^n$ and a neighborhood $V$ of a point $y_0 \in \mathcal{A}_b^n - \{0\}$ with $W \times \mathbb{C} \subset V \subset U$ and a holomorphic function $G = (g_1, \ldots, g_m) : V \to \mathcal{A}_b^n$ such that $W \cap \{z \in U : F(z) = 0\} = \{z = (G(y), y) : y \in V\}$ and $g(x_0) = y_0$, where $z_0 = (x_0, y_0)$.

Proof. Consider the mapping $H = (1f, \ldots, m \bar{f}, m+1z, \ldots, n \bar{z}) : U \to \mathcal{A}_b^n$. Then the operator $L(z) := (\partial k/\partial jz)_{1 \leq j,k \leq n}$ is invertible at $z_0$, hence it is invertible in a neighborhood $U_0$ of $z_0$, since $L(z)$ is super-differentiable, where $(1h, \ldots, nh) = H$. Therefore, $L^{-1}(z)$ is super-differentiable in $U_0$, since $L^{-1}(z)L(z) = L(z)L^{-1}(z) = I$ for each $z \in U_0$, where $I$ is the unit operator. Then the operators $A(z) := (\partial k/\partial jz)_{1 \leq j,k \leq m}$ and $A^{-1}(z)$ are locally analytic in a neighborhood of $z_0$. Consider the mapping $g_0(x) := x - A^{-1}(0)F(x,y)$ in a neighborhood of $z_0$, where $(x, y) = z$, $x = (z, \ldots, z), y = (m+1 \bar{z}, \ldots, n \bar{z})$.

Without loss of generality using shifts we can consider $z_0 = 0$. Then $g_0(x) = x$ if and only if $F(x,y) = 0$. We have the identity: $\partial g_0(x)/\partial x = I - A^{-1}(0)(\partial F(x,y)/\partial x) = A^{-1}(0)(A(0) - \partial F(x,y)/\partial x)$. From the continuity of $\partial F(x,y)/\partial x$ it follows, that there exist $a > 0$ and $b > 0$ such that $\|\partial g_0(x)/\partial x\| \leq A^{-1}(0)\|\partial F(x,y)/\partial x\| < 1/2$ for each $z = (x, y)$ with $\|z\| < a$ and $\|y\| < b$.

Applying the fixed point theorem to this contracting mapping $g_0(x)$ we get a solution $G(y)$ in a neighborhood of 0 (see also the general implicit function theorem in §X.7 [33] and Theorems II.V.4.2, 5.1 and 6.1 [10]). Then the solution is locally analytic by $(z, \bar{z})$, since
f(z) and \( A^{-1}(z) \) and \( L(z) \) are locally analytic. Thus in a neighborhood of \((m+1)z_0, \ldots, n z_0)\) there are satisfied the identities \( k\big(\,1g, \ldots, m g, \, m+1z, \ldots, \, n z\big) = 0 \) for \( k = 1, \ldots, m \) and they are \((z, \zbar)\)-differentiable and the differentiation by \( \zbar \) gives:

\[
\sum_{i=1}^{n}(\partial k f/\partial i z)(\partial ig/\partial j z).h + \sum_{i=1}^{n}(\partial k f/\partial i z).\big((\partial ig/\partial j z)^* . h + (\partial k f/\partial j z)h\big) = 0
\]

for each \( h \in \mathcal{A}_r \), but \( \partial k f/\partial i z = \partial k f/\partial j z = 0 \), since \( f \) is \( \mathcal{A}_r \)-holomorphic and \((\partial k f/\partial j z)_{1 \leq i, k \leq m} \) is invertible by the condition of this theorem, where \( z^* = \zbar \) denotes the adjoint of \( z \) in the Cayley-Dickson algebra \( \mathcal{A}_r \). Therefore, \( \partial ig/\partial j z = 0 \) for each \( i = 1, \ldots, m \) and \( j = m + 1, \ldots, n \), consequently, \( G \) is holomorphic.

4. **Corollary.** Let \( U \) be an open subset in \( \mathcal{A}_p^m \), \( 1 \leq m \leq n \in \mathbb{N} \), \( F = (f, \ldots, m f) : U \to \mathcal{A}_p^m \) be a \((r, p)\)-quasi-conformal mapping, where \( 1 \leq r < p \leq 3 \). If \( z_0 \in U \), \( F(z_0) = 0 \) and the operator \((\partial k f/\partial j z)_{1 \leq i, k \leq m} \) is invertible at \( z_0 \), where \( z = (z_1, \ldots, z_n) \), \( j z \in \mathcal{A}_r \) for each \( j = 1, \ldots, n \), then there exist an open neighborhood \( W_p \) of a point \( x_0 \) in \( \mathcal{A}_p^m \) and a neighborhood \( V = V_p \) of a point \( y_0 \) in \( \mathcal{A}_p^{m-n} \) such that \((W_p \times V_p) \subset U \) and a holomorphic mapping \( G = (g_1, \ldots, m g) : \mathcal{A}_p^m \to \mathcal{A}_p^m \) such that \( W_p \cap \{z \in U : F(z) = 0\} = \{z = (G(y), y) : y \in V_p\} \) with \( g(x_0) = y_0 \).

**Proof.** Since \( F \) is \((r, p)\)-quasi-conformal, then it is holomorphic on \( U \) satisfying Conditions \((Q1 - Q7)\) with \( W = U \cap \mathcal{A}_p^m \subset U \), where \( \mathcal{A}_r \to \mathcal{A}_p^m \) is the natural embedding. In view of Theorem 3 there exists a holomorphic solution of this theorem.

5. **Corollary.** Let \( F \) satisfies conditions of Corollary 4 and \( n = 2 \) and \( m = 1 \). Then \( G \) is \((r, p)\)-quasi-conformal in a neighborhood of \( y_0 \) at each point \( y \in V = V_p \) such that \( F(G(y), y) = 0 \).

**Proof.** In view of Corollary 4 \( G \) is holomorphic, hence satisfies Condition \((P1)\). We have that

\[
(1) \ G'(y).h = \ -(F_x'(x, y))^{-1}.[(F_y'(x, y))_.h] \text{ for all } x = G(y) \text{ and each } h \in \mathcal{A}_p, \text{ when } F(G(y), y) = 0, \text{ since the quaternion skew field } \mathcal{H} = \mathcal{A}_2 \text{ is associative and the octonion algebra } \mathcal{O} = \mathcal{A}_3 \text{ is alternative.}
\]

The restriction of \( F \) on \( U \cap \mathcal{A}_p^2 \) is pseudo-conformal, hence \( F_x'(x, y) \) and \( F_y'(x, y) \) for \((x, y) \in U \cap \mathcal{A}_p^2 \) satisfy Conditions \((P2, P3)\). In view of Theorem 2.4 [23] \( (2) \ F_x'(x, y).h = a(x, y)bb(x, y) \) and \( F_y'(x, y).h = c(x, y)he(x, y) \) for each \( h \in \mathcal{H} \) and each \((x, y) \in U \cap \mathcal{A}_p^2 \), for \( r = 2 \), where \( a(x, y), b(x, y), c(x, y), e(x, y) \) are non-zero \( \mathcal{A}_r \)-holomorphic functions on \( U \cap \mathcal{A}_p^2 \). For \( r = 1 \), over \( \mathcal{C} \), evidently due to the commutativity of \( \mathcal{C} \) we take as usually \( b = 1 \) and \( e = 1 \).

Therefore, from Equation (1) it follows, that the restriction of \( G'(y) \) on \( V \cap \mathcal{A}_r \) satisfies Conditions \((P2, P3)\), since the quaternion skew field \( \mathcal{H} \) and the complex field \( \mathcal{C} \) are associative. But \( \tilde{R}_{x} \) are automorphisms of \( \mathcal{A}_r \) such that Conditions \((Q1 - Q6)\) are satisfied. For simplicity of the notation take the zero marked point. We have \( \cup_{q \in \mathcal{V}} \{\tilde{R}_{q,y}y : y \in V \cap \mathcal{A}_r\} = V \), hence \( \{q \in V : \exists y \in V \cap \mathcal{A}_r \text{ such that } \tilde{R}_{q,y}y = q\} = V \) (see also \((Q2)\)). Let \( q \in V \cap \mathcal{A}_r \) and \( y \in V \cap \mathcal{A}_r \) be such that \( \tilde{R}_{q,y}y = q \), then \( \tilde{R}_{q,y}F(x, y) = F(\tilde{R}_{q,y}x, q) \), but \( F(q, q) = 0 \) in \( W \) is equivalent to \( (q, \zbar) = (G(q), q) \) and \( q \in V \). Therefore, \( \tilde{R}_{q,y}F(G(y), y) = F(G(q), q) = 0 \) and \( \tilde{R}_{q,y}G'(y).h = G'(q).(\tilde{R}_{q,y}h) \) due to (1) for each \( h \in \mathcal{A}_r \) and every \( y \in V \cap \mathcal{A}_r \) and every \( q \in V \) such that \( \tilde{R}_{q,y}y = q \) and \( Re(y) = Re(q) \), consequently, \( \tilde{R}_{q,y}G(y) = G(q) \) for each \((q, y) \in V \times (V \cap \mathcal{A}_r) \) such that \( \tilde{R}_{q,y}y = q \) and \( Re(y) = Re(q) \), since \( F_x'(x, y) \) is invertible for \( x = G(y) \) and \( F \) is locally analytic and using expansion of \( F(x, y) \) by \( (x, y) \) with \( x = G(y) \).

Put \( H = G'(y).h \), then \( h = (G'(y))^{-1} H \), since \( F \) is pseudo-conformal and \( ker F_x'(x, y) = 0 \) and \( ker F_y'(x, y) = 0 \) for each \((x, y) \in U \cap \mathcal{A}_p^2 \), \( \mathcal{A}_p \) is the finite dimensional algebra over \( \mathbb{R} \). If the operator \( F_x'(x, y) : \mathcal{A}_r \to \mathcal{A}_r \) satisfies Condition \((Q7)\) and has the inverse, then its inverse also satisfies \((Q7)\), since the restriction \( F_x'(x, y)|_{\mathcal{A}_r} \) has Form (2) and each
non-zero number in $A_p$ is invertible, where $1 \leq p \leq 3$. Since the right hand side of Equation (1) satisfies Condition (Q7), then the left hand side of it satisfies (Q7) as well.

6. Corollary. If $f : U \to A_p$ is a $(r, p)$-quasi-conformal function, where $U$ is open in $A_p$ and $f(U) = V$ is open in $A_p$ and $f$ is bijective on $U$, then $f^{-1} : V \to A_p$ is $(r, p)$-quasi-conformal.

Proof. Take the function $F(x, y) = f(x) - y$, then it is $(r, p)$-quasi-conformal and satisfies Conditions of Lemma 5. Since $f(U) = V$ and $f : U \to V$ is bijective, then there exists $g = f^{-1} : V \to U$, which is $(r, p)$-quasi-conformal due to Lemma 5.

7. Theorem. Let $f$ and $g$ be $(p, r, b)$- and $(q, r, b)$-quasi-conformal mappings on neighborhoods $U$ of $z_0$ and $V$ of $y_0$ respectively such that $f(U) \supset V$ and $f(z_0) = y_0$, where $1 \leq r < b \leq 3$, $y_0$ and $z_0 \in A_r$, then their composition $g \circ f$ is $(pq, r, b)$-quasi-conformal on a neighborhood $W$ of $z_0$.

Proof. The composition of pseudo-conformal mappings is pseudo-conformal, so in accordance with Definition 1 it is sufficient to take the neighborhood $W$ of $z_0$ such that $W = f^{-1}(V)$ is open, since $f$ is continuous (see [23] and Theorem 2.6 [24]). Therefore, $g \circ f$ is $pq$-pseudo-conformal at each point in $W \cap A_r$, since $y_0$ and $z_0 \in A_r$. The composition of conformal mappings is conformal, the composition $T_1T_2$ of proper elements $T_1, T_2 \in SO(2^b, R)$ is the proper element $T_1T_2 \in SO(2^b, R)$, since $SO(2^b, R)$ is the special orthogonal group. If rotations $T_1$ and $T_2$ have a common axis, then their composition preserves this axis, hence $SO_R(n, R)$ is the subgroup of $SO(n, R)$. Take the families of automorphisms $R^g$ and $R^f$ for $g$ and $f$ correspondingly in accordance with Definition 1. Therefore, the composition $R^{g\circ f}_{z_0,x} := R^g_{(R^{f}_{x,0}(f(x+z_0)-y_0)),(f(x+z_0)-y_0)}$ is defined for each $x \in (W - z_0) \cap A_r$ and every $z \in W - z_0$ and it gives the restriction $R^{g\circ f}_{z_0,x} = id$ for each $x$ and $z \in (W - z_0) \cap A_r$, since $f(U \cap A_r) \subset A_r$. Thus the family of operators $R^{g\circ f}(z, x)$ satisfies Conditions (Q2 - Q5). Therefore, $g \circ f(z + z_0) = R^{g\circ f}_{z_0,x} h \circ f(z + z_0)$ for each $x \in (W - z_0) \cap A_r$ and every $z \in W - z_0$ with $Re(z) = Re(x)$ and $z = R^{g\circ f}_{z_0,x} x$, where $h(y) := g(y + y_0)$, consequently, $g \circ f$ satisfies (Q6). Since $(g \circ f)'(z + z_0).h = g'(f(z + z_0)) (f'(z + z_0).h)$ for each $z \in W - z_0$ and $h \in A_b$, while $g$ and $f$ satisfy (Q6, Q7) with $R^g$ and $R^f$ respectively, then $(g \circ f)'$ satisfies (Q7) with $R^{g\circ f}$ and inevitably $g \circ f$ is $(pq, r, b)$-quasi-conformal on $W$.

8. Corollary. Let $U$ be an open domain in $A_b$ with a marked point $y_0 \in U \cap A_r$, $1 \leq r < b \leq 3$, then the family of all $(r, b)$-quasi-conformal diffeomorphisms $f$ of $U$ onto $U$ preserving a marked point $y_0$ has the group structure.

Proof. In Accordance with Theorem 7 compositions of $(r, b)$-quasi-conformal mappings $g, f$ are $(r, b)$-quasi-conformal, since $f(y_0) = y_0$ and $g(y_0) = y_0$. In view of Corollary 6 the inverse mapping of $f$ is also $(r, b)$-quasi-conformal. Evidently, the identity mapping $id(x) = x$ for each $x \in U$ is pseudo-conformal, hence it is $(r, b)$-quasi-conformal. Since $f \circ id = id \circ f = f$ for each homeomorphism $f : U \to U$, then $id = e$ is the unit element of the family of $(r, b)$-quasi-conformal diffeomorphisms.

8.1. Remark. Topologize the family $H(M, P)$ of $A_b$ holomorphic mappings from a domain $M$ in $A_b^n$ into a domain $P$ in $A_b^n$ by the compact-open topology of locally analytic mappings as in the proof of Theorem 3.18 [24], where $n, k \in N$. This topology on $H(M, P)$ induces the topology on the group of $A_b$ holomorphic diffeomorphisms $DiffH(M)$ of $M$. For the family of $(r, b)$-quasi-conformal diffeomorphisms $f$ of $M$ suppose that $f(M \cap A_b^n) = M \cap A_b^n$.

8.2. Theorem. The family of all $(r, b)$-quasi-conformal diffeomorphisms $DiffQ(M)$ of a compact canonical closed domain $M$ in $A_b^n$ preserving a marked point $y_0$ (see 8.1), $1 \leq r < b \leq 3$, $n \in N$, form the topological metrizable group, which is complete relative to its metric and locally compact. The group $DiffQ(M)$ is the analytic Lie group over $R$. 

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Proof. In view of Theorems 3.24, 25 [24] the group $DifP(M \cap A^n)$ of all pseudo-conformal diffeomorphisms of $M \cap A^n$ is the topological metrizable locally compact analytic Lie group over $\mathbb{R}$. Consider $y_0 = 0$ without loss of generality. On the other hand, each $f \in DifQ(M)$ is obtained from the corresponding $q \in DifP(M \cap A^n)$ with the help of operators $R_{z,x}$ in accordance with Conditions $(Q1 - Q7)$ of Definition 1. In its turn each $R_{z,x}$ is the automorphism of $A_b$ depending $A_b$ and $A_r$ holomorphically on $z$ and $x$ respectively (see Example 2). Thus $f^{-1}$ is obtained from $q^{-1}$ with the help of $R_{z,y}$, $\zeta = f(z)$, $y = q(x)$, but $q^{-1}$ is pseudo-conformal, hence $f^{-1}$ is $(r, b)$-quasi-conformal due to Conditions $(Q6, Q7)$ for $f$, since $(f^{-1})' = (f')^{-1}$ and each $R_{z,x}$ is invertible. Since for each $f_1, f_2 \in DifQ(M)$ we have the corresponding $q_1, q_2 \in DifP(M \cap A^n)$ and $DifP(M \cap A^n)$ is the group, then $f_1 \circ f_2 \in DifQ(M)$ due to $(Q1 - Q7)$. The group $DifP(M \cap A_r)$ is the finite-dimensional locally compact analytic Lie group over $\mathbb{R}$ and the family $R_{z,x}$ also form the finite-dimensional analytic family over $\mathbb{R}$, hence $DifQ(M)$ is the finite-dimensional analytic Lie group over $\mathbb{R}$ and inevitably it is locally compact, metrizable and complete.

9.1. Proposition. If $q_1$ and $q_2$ are holomorphic functions on a domain $W$ in $\mathbb{C}$, $q_1$ and $q_2$ have $(1, b)$-quasi-conformal extensions $f_1$ and $f_2$ on $U$, $1 < b \leq 3$, with the same family $R_{z,x}$ and the same marked point $y_0 \in W$ for $f_1$ and $f_2$ (see $(Q1 - Q7)$ in Definition 1) and $(q_1, q_2)(x) \neq 0$ at each point $x \in W$, then their product $q_1 q_2$ has the $(1, b)$-quasi-conformal extension $f_1 f_2$ on $U$.

Proof. Conditions $(Q1 - Q6)$ are evidently satisfied for $f_1 f_2$, since $R_{z,x}$ is the automorphism of $A_b$ for each $z \in U - y_0$ and $x \in W - y_0$. On the other hand, the product of complex holomorphic functions is complex holomorphic, the product of $A_b$ holomorphic functions is $A_b$ holomorphic. Conditions $(Q6, Q7)$ are satisfied for $f_1$ and $f_2$, hence

$$(g_1 g_2)'(R_{z,y}) (R_{z,y}) = g_1(z) [g_2(R_{z,y})] + [(g_1)'(R_{z,y})] g_2(z)$$

for each $z \in U - y_0$ and $y \in W - y_0$ such that $Re(z) = Re(y)$ and $z = R_{z,y} y$ and every $h \in A_r$, where $g'(z)$ is the (super)derivative operator over $A_b$, consequently, $(Q7)$ is satisfied for $f_1 f_2$.

9.2. Corollary. Let $q_1$ and $q_2$ be holomorphic functions on a domain $W$ in $\mathbb{C}$ with isolated zeros of $q_1$ and $q_2$, $q_1$ and $q_2$ have $(1, b)$-quasi-regular extensions $f_1$ and $f_2$ on $U$, $1 < b \leq 3$, with the same family $R_{z,x}$ and the same marked point $y_0 \in W$ for $f_1$ and $f_2$ (see $(Q1 - Q7)$ in Definition 1), then their product $q_1 q_2$ has the $(1, b)$-quasi-regular extension $f_1 f_2$ on $U$.

Proof. In view of Theorem 9.1 $f_1 f_2$ is $(1, b)$-quasi-conformal on $U \setminus S_A$, where $S_A = \{z \in U - y_0, x \in W - y_0, q_1(x + y_0) = 0 \text{ or } q_2(x + y_0) = 0\}$, $S_A \cap W$ is the discrete subset consisting of isolated points in $W$. Conditions $(Q1 - Q6)$ are satisfied for $f_1 f_2$, since each $R_{z,x}$ is the automorphism of $A_b$. Thus $f_1 f_2$ is $(1, b)$-quasi-regular on $U$.

9.3. Remark. In general Theorem 9.1 and Corollary 9.2 may be not true for $((a_1 f_1)(a_2 f_2))$ instead of $f_1 f_2$, when $f_2$ are taken with constant non-real multipliers $a_j \in \mathbb{C} \setminus \mathbb{R}$ (see also Notes 13).

9.4. Theorem. Let $q_n$ be a sequence of complex holomorphic functions on an open connected convex domain $W$ in $\mathbb{C}$ such that the series $\sum_{n=1}^{\infty} q_n(y)$ converges uniformly on $W$ to a function $q(y)$ with $q'(y) \neq 0$ for each $y \in W$ (or $y \in W \setminus A$ with discrete subset $A$ consisting of isolated points in $W$) and $\sum_{n=1}^{\infty} q_n(y_0)$ converges at a marked point in $W$ to $q(y_0)$ while each $q_n$ has a $(1, b)$-quasi-conformal (or $(1, b)$-quasi-regular) extension $f_n$ on a domain $U \cap A_b$ with the same family $R_{z,x} : z \in U - y_0, x \in W - y_0$. Then the series $\sum_{n=1}^{\infty} q_n(y)$ converges on $W$ to a function $q(y)$ which has a $(1, b)$-quasi-conformal (or $(1, b)$-quasi-regular correspondingly) extension $f$ on $U$. 

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Proof. In view of Theorem XVI.3.4 [34] the series $\sum_{q=1}^\infty q_n(y)$ converges on $W$ to a function $q(y)$ and this convergence is uniform on compact subsets of $W$. Since $q'(y) = \sum_{n=1}^\infty q'_n(y)$ on $W$, then there exist $\partial q(y)/\partial y_1$ and $\partial q(y)/\partial y_2$, where $y = y_1 + iy_2$, $y_1, y_2 \in \mathbb{R}$, $i = (-1)^{1/2}$. Consequently, there exists $\partial q(y)/\partial \bar{y} = \sum_{n=1}^\infty \partial q_n(y)/\partial \bar{y} = 0$ on $W$, since each $q_n$ is holomorphic on $W$. Since $R_{s,x}$ is the automorphism of $A_b$ depending holomorphically on $z \in U - y_0$ and $x \in W - y_0$, then the series $\sum_{n=1}^\infty f_n(y)$ and $\sum_{n=1}^\infty f'_n(y)$ converge on $U$ to $f(y)$ and $f'(y)$ respectively and this convergence is uniform on $P$ and $P \times B$ for each compact subset $P$ in $U$, where $B = B(A_b,0,1) := \{z \in A_b : |z| \leq 1\}$. Consequently, there exists $\partial f(y)/\partial \bar{y} = 0$, since $\partial f_n(y)/\partial \bar{y} = 0$ for each $n \in \mathbb{N}$ and inevitably $f(y)$ is $A_b$ holomorphic on $U$. Conditions $(Q1 - Q6)$ are satisfied for each $f_n$ on $U$ and $(Q7)$ on $U \setminus S_{A_b}$, where $S_{A_b} = \emptyset$ in the $(1,b)$-quasi-conformal case, hence $(Q1 - Q6)$ are satisfied for $f(y)$ on $U$ and $(Q7)$ on $U \setminus S_A$, where $A = \emptyset$ in the $(1,b)$-quasi-conformal case, since the series $\sum_{n=1}^\infty f_n(y)$ and $\sum_{n=1}^\infty f'_n(y)$ converge on $U$ to $f(y)$ and $f'(y)$ correspondingly and $f'(y)h \neq 0$ for each $(y,h) \in (U \setminus S_A) \times (A_b \setminus \{0\})$.

9.5. Examples. 1. Let $q_n(y) = cn(y-y_0)^n$, coefficients be real $c_n \in \mathbb{R}$, $\sum_{n=1}^\infty |c_n|R^n < \infty$ for each $n \in \mathbb{N}$, where $0 < R < \infty$, $W = \{y \in \mathbb{C} : |y - y_0| < R\}$, $y_0 \in \mathbb{C}$ such that $\sum_{n=1}^\infty c_n(y-y_0)^n \neq 0$ on $W$, put $U := \{z \in A_b : |z - y_0| < R\}$ and take $\{R_{s,x}\}$ from Examples 2. Then conditions of Theorem 9.4 are satisfied, since $R_{s,x}c_n = c_n$ for each $n$. In particular, take $c_n = 1$ for each $n$, $0 < R < 1$, $y_0 = 0$, then $f(y) = 1/(1 - y)$ and $q'(y) = (1 - y)^{-2} \neq 0$ on $W$. If $q(y) = \sin(y)$ or $q(y) = \cos(y)$, then $q$ has the $(1,b)$-quasi-integral extension. Evidently, if $a = \text{const} \neq 0$, $a \in \mathbb{R} \setminus \{0\}$, $f(z) = (r,b)$-quasi-regular on $U$ or $(r,b)$-quasi-regular, then $q(z) = f(az)$ is $(r,b)$-quasi-regular on $U/a = \{z/a : z \in U\}$ or $(r,b)$-quasi-regular correspondingly.

2. Take $q_n(y) = cn\exp(cn(y-y_0))$, where $c_n \in \mathbb{R}$, $a_n \in \mathbb{R}$, $a_n \neq 0$ for each $n \geq 2$, $\sum_{n=1}^\infty c_n$ converges and $\sum_{n=1}^\infty |a_n|c_n \exp(|a_n|R) < \infty$ such that $\sum_{n=1}^\infty a_n c_n \exp(a_n(y-y_0)) \neq 0$ on $W := \{y \in \mathbb{C} : |y - y_0| < R\}$, $0 < R < \infty$, $y_0 \in \mathbb{C}$, $U := \{z \in A_b : |z - y_0| < R\}$ and $R_{s,x}$ is from Examples 2. Then Conditions of Theorem 9.4 are satisfied, since $R_{s,x}c_n = c_n$ and $R_{s,x}a_n = a_n$ for each $n$, while $\exp(a(y - y_0))$ is $A_b$-pseudo-conformal on $A_b$ for $a \neq 0$ in $A_b \setminus \{0\}$ (see [23] [24]).

In particular, for $cn = 1$ and $q_n(y) = n^{-y} = \exp(-y \ln n)$ for each $n \geq 1$, $y_0 = 0$, the series $\sum_{n=1}^\infty q_n(y)$ and $\sum_{n=1}^\infty q'_n(y)$ converge uniformly on $W_R := \{y \in \mathbb{C} : Re(y) > R\}$ for $1 < R < \infty$ to the holomorphic function $\zeta(y)$ and put $U_R := \{z \in A_b : Re(z) > R\}$. Therefore, take $W = \{y \in W_R : \zeta'(y) \neq 0, 1 < R < \infty\}$ and $U = U_1 \setminus \bigcup \{S'_y : y \in W_1 \setminus W\}$, where $S'_y := \{z : z = R_{s,x}y, z \in U_1, Re(z) = Re(y)\}$ for $y \in W_1 \setminus W$. Thus, $\zeta(y)$ has the $(1,b)$-quasi-conformal extension on $U$ from $W$ for $b = 2$ and for $b = 3$. Since the derivative $c'(y)$ is holomorphic on $W_1$ with isolated zeros, then $\zeta(y)$ has the $(1,b)$-quasi-regular extension on $U_1$ from $W_1$ for $b = 2$ and for $b = 3$.

If take

$$ (1) \quad q_n(y) = c_n \exp(v_n E(t_n(y-y_0))) $$

with $v_n, t_n \in \mathbb{R} \setminus \{0\}$, then these examples provide $(1,b)$-quasi-conformal or quasi-regular extensions in spherical $A_b$-coordinates (see Definition 2.1). For this choose the family $R_{E(t(p-y_0)),E(t(y-y_0))}$ for the $(C, A_b)$ pair in Example 2 independent from $t \in \mathbb{R} \setminus \{0\}$. This is possible due to additional Conditions 2.11,12, since $\exp(p_0 + pS) = \exp(p_0)(\cos(pS) + S\sin(pS))$ for each $p \in A_b$ and $\text{sign}(\sin(p_0t) \sin(p_0t)) = \text{sign}(p_0p_0)$ for each $t \in \mathbb{R} \setminus \{0\}$, where $p_0 = Re(p - y_0)$, $pS = Im(p - y_0) := p - y_0 - Re(p - y_0)$, $p_0, pS \in \mathbb{R}$, $S \in \mathbb{I}_b$, $|S| = 1$, $\text{sign}(t) = 1$ for $t > 0$, $\text{sign}(t) = -1$ for $t < 0$ and $\text{sign}(0) = 0$. Indeed, the group $SO_{2b, R}$ is isomorphic with $SO(2b-1, \mathbb{R})$ and $E(t(p-y_0)) \in \mathbb{R} \oplus SR$ for each $t \in \mathbb{R}$. Therefore,
(2) \( R_{E((p-y_0)),E((y-y_0))} = R_{E(p-y_0),E(y-y_0)} \)
for each \( p \in \mathcal{A}_b, y \in \mathcal{C} \) and each real nonzero \( t \in \mathbb{R} \setminus \{0\} \), where \( y_0 \in \mathcal{C} \) is the marked point. Then

\[
\begin{align*}
(3) \quad & R_{E((p-y_0)),E((y-y_0))} \sum_{n=1}^\infty q_n(y) = \\
& \sum_{n=1}^\infty R_{E(t_n(p-y_0)),E(t_n(y-y_0))} \exp(v_n E(t_n(y-y_0))) = \\
& \sum_{n=1}^\infty \exp(v_n E(t_n(p-y_0))) = f(p)
\end{align*}
\]
for each \( Re(E(p-y_0)) = Re(E(y-y_0)) \) with \( R_{E((p-y_0)),E((y-y_0))} E(y-y_0) = E(p-y_0) \), since \( R_{z,x}(tx) = t R_{z,x}x \) for each \( t \in \mathbb{R} \) and \( E(y) = y \) for each \( y \in \mathcal{C} \). Thus \( q(y) = \sum_{n=1}^\infty q_n(y) \) has the \((1, b)\)-quasi-conformal in spherical \( \mathcal{A}_b \)-coordinates extension \( f \) with \( q \) on \( V \) such that \( E(V) = W \) and \( f \) on \( P \) such that \( E(P) = U \) choosing the corresponding branches of \( L_n \).

Henceforth, it is supposed that Condition (2) is satisfied in spherical \( \mathcal{A}_b \)-coordinates.

3. It is known, that the gamma function \( \Gamma(z) \) is holomorphic on \( \mathcal{C} \setminus \{0, -1, -2, -3, \ldots\} \) having poles of the first order at points \( 0, -1, -2, -3, \ldots \) with residues \( res_{z=-n} \Gamma(z) = (-1)^n/n! \), \( n = 0, 1, 2, \ldots \). Moreover, \( 1/\Gamma(z+1) = e^{\gamma z} \prod_{k=1}^\infty \left( 1 + \frac{z}{k} \right) e^{-z/k} \) and this product converges everywhere on \( \mathcal{C} \), where \( C = \lim_{n \to \infty} (\sum_{k=1}^n 1/k - \ln n) = 0.57721567 \ldots \) is the Euler constant (see [15]). In the product of Formula (1) all coefficients are real. It is possible to consider for this function different holomorphic extensions over \( \mathcal{A}_b \) (see Section 4 in [18]). Applying operators \( \hat{R}_{z,y} \) from Example 2 with \( y_0 = 0 \) and Proposition 9.1 and Theorem 9.4 to Equation (1) provides the \((1, b)\)-quasianmorphic extension of \( \Gamma(z) \), which is \((1, b)\)-quasi-conformal on \( \mathcal{A}_b \setminus \{ S_0, S_1, S_2, \ldots \} \) for \( 2 \leq b \leq 3 \), where \( S_n = S^T_n \). In particular, for \( y_0 = 0 \) we have \( S^T_n = \{-n\} \), since rotations are around the real axis, \( T = \hat{R}_{z,y} \in SO_R(2^b, \mathbb{R}) \). Moreover, \( 1/\Gamma(z) \) is (1, b)-quasi-integral, hence \( \Gamma(z) \) has no zeros in \( \mathcal{A}_b \).

10. **Definition.** Let \( a_1, \ldots, a_n, z \in \mathcal{A}_r \), put \( \text{Exp}_1(a_1; z) := \exp(a_1 z) \), \( \text{Exp}_n(a_1, \ldots, a_n; z) := \text{Exp}_{n-1}(a_1, \ldots, a_{n-1}; \text{Exp}_1(a_n; z)) \) for \( n > 1 \), where \( 2 \leq r \). For \( a_1 \neq 0, \ldots, a_n \neq 0, z \neq 0 \) put \( \text{Ln}_1(a_1; z) := a_1^{-1} \text{Ln}(z) \), \( \text{Ln}_{n-1} = \text{Ln}_{n-1}(a_1, \ldots, a_{n-1}; \text{Ln}_1(a_n; z)) \) for \( n > 1 \), where \( \text{Exp}_0(z) := id(z) = z \) and \( \text{Ln}_0(z) := id(z) = z \) for each \( z \in \mathcal{A}_r, 1 \leq r \). Here \( a_1, \ldots, a_{n-1} \) can be constants, but more generally \( \mathcal{A}_r \)-pseudo-conformal functions \( a_1(z) \neq 0, \ldots, a_{n-1}(z) \neq 0, a_n \neq 0 \) is a constant in \( \mathcal{A}_r, 2 \leq r \).

Suppose that \( \gamma(t) := z_0 + \rho \text{Exp}_1(a_1; \ldots, a_n; \xi(t)) \) is a curve in an open domain \( U \) in \( \mathcal{A}_r \) and \( f \) is a holomorphic function \( f : U \to \mathcal{A}_r, \) where \( a_1 = a_1(t) \neq 0, \ldots, a_{n-1} = a_{n-1}(t) \neq 0 \) are constants or pseudo-conformal functions with values in \( \mathcal{A}_r \) on an open domain \( V_a \supset [0, 1] \) or \( V_a \supset \mathbb{R} \) in \( \mathcal{A}_r, a_n \neq 0, \xi(t) \) is a rectifiable curve in \( \mathcal{A}_r, t \in [0, 1] \subset \mathbb{R}, f(z) \neq 0 \) for each \( z = \gamma(t), \) where \( 0 < \rho < \infty \). Then put

\[
\Delta \text{Arg}_n f := \Delta \text{Arg}_{a_1, \ldots, a_n} f := \int_{z \in \gamma} d \text{Ln}_n(a_1, \ldots, a_{n-1}, 1; f(z))
\]
with a chosen branch of \( \text{Ln}_n \).

10.1. **Note.** For \( n = 1 \) and \( \xi(t) = t \) with \( M \in \mathcal{A}_r, \) \( \text{Re}(M) = 0, |M| = 1, a_n = 2\pi M \), \( \{ \gamma(t) : t \in [0, 1] \} \) is the circle. If \( n = 1 \) and \( a_1 = 2\pi M, \) then \( \Delta \text{Arg}_{a_1} f = \Delta \text{Arg} f \) is the usual change of the argument of a function \( f \) along a curve \( \gamma \) (see also Section 3 in [17] [18] [19] and Theorem 2.23 [24]).

10.2. **Proposition.** The logarithmic function \( \text{Ln} \) on \( \mathcal{A}_r \setminus \{0\}, \) where \( 1 \leq r \leq \infty, \) has a countable number of branches.

**Proof.** For \( r = 1 \) we have \( \mathcal{A}_1 = \mathcal{C} \) and in this case the statement of this proposition is well-known. So consider \( 2 \leq r \leq \infty. \)

Each nonzero \( z \in \mathcal{A}_r \setminus \{0\} \) can be written in the polar form

\[
(1) \quad z = |z| \exp(M \phi + 2\pi n M),
\]
where \( M \in \mathcal{I}_r := \{ z \in \mathcal{A}_r : \text{Re}(z) = 0 \}, |M| = 1, \phi \in [0, 2\pi), n \in \mathbb{Z}, \text{Arg}(z) = M \phi + 2\pi n M \) (see Section 3 in [17] [18] [19]). If \( K \in \mathcal{I}_r, |K| = 1, K \) is not parallel to...
M, that is, \(|Re(MK^*)| < 1\), then M and K do not commute. When \(0 < \phi < \pi\), then \(\exp(M\phi + \pi Ks) \neq \exp(M\phi + \pi nM)\) for each \(s \neq 0, s \in \mathbb{Z} \setminus \{0\}\), and each \(n \in \mathbb{Z}\), since \(\exp(M\phi + \pi Ks) = \cos |M\phi + \pi Ks| + (M\phi + \pi Ks)(\sin |M\phi + \pi Ks|)/|M\phi + \pi Ks|\) while \(\exp(M\phi + \pi nM) = \cos |\phi + \pi n| + M(\phi + \pi n)(\sin |\phi + \pi n|)/|\phi + \pi n|\) and \(|M\phi + \pi Ks|^2 = \phi^2 + (\pi s)^2 + 2Re(MK^*)\phi s\) and \((\phi + \pi n) \notin \pi \mathbb{Z}\) and K is not parallel to M, where \(\mathbb{Z} := \{\ldots, -2, -1, 0, 1, 2, \ldots\}\). On the other hand, \(Im(z) := z - Re(z)\) is parallel to M in Equation (1), hence the only solutions of (1) are \(Arg(z) = M(\phi + 2\pi n)\), where \(\phi \in [0, 2\pi)\), \(n \in \mathbb{Z}\), M is parallel to \(Im(z)\). Therefore, \(L_n\) has only countable number of branches which can be enumerated by \(n \in \mathbb{Z}\).

In more details it is possible to construct the following noncommutative analog of the Riemann \(2^r\)-dimensional surface \(R\) of \(Ln\) such that \(Ln : \mathcal{A}_r \setminus \{0\} \rightarrow R\) is the univalent mapping, where \(2 \leq r \leq \infty\), \(2^\infty = \infty\). Consider copies \((\mathcal{A}_r, ni_1)\) of \(\mathcal{A}_r\) embedded into \(\mathcal{A}_r^2\), where \(ni_1 \in i_1 \mathbb{R}\) is in the second multiple \(\mathcal{A}_r\), \(n \in \mathbb{Z}\). Put \(P_j := \{z \in \mathcal{A}_r : z_0 < 0, z_j = 0, z_k \in \mathbb{R} \forall 0 < k \neq j\}\) and consider sections of \(\mathcal{A}_r\) (of the first multiple) by \(P_j\) for each \(1 \leq j \in \mathbb{Z}\), where \(z = \sum_{j=0}^{2^r-1} z_j i_j, z_j \in \mathbb{R}, i_j\) are standard generators of \(\mathcal{A}_r\). Then the set \(\{z \in \mathcal{A}_r : z_0 < 0\}\) is partitioned into the subsets \(S(k_1, k_2, \ldots)\) corresponding to definite combinations of signs of \(z_j\): either \(z_j \geq 0\) or \(z_j \leq 0\) with \(k_j = 1\) and \(k_j = -1\) respectively. For finite \(r\) the number of such parts is \(2^r\) with \(q = 2^r - 1\), since \(j = 1, \ldots, 2^r - 1\), for \(r = \infty\) their family is infinite and uncountable of the cardinality \(c = 2^{\aleph_0}\).

Then embed each partitioned copy of \((\mathcal{A}_r, ni_1)\) into \(\mathcal{A}_r^2\) and bend slightly each subset \(\{z \in \mathcal{A}_r : z_0 < 0\}\) \(ni_1\) in directions \(v_1, v_2, \ldots\) perpendicular to \((i_1, ni_1), (i_2, ni_1), \ldots\) using the imaginary part \(I_r\) of the second multiple such that after this procedure \(\{(z_1, z_2) \in (S(k_1, k_2, \ldots), ni_1) \cap (S(l_1, l_2, \ldots), ni_1) : z_0 < 0\}\) for each \(n \neq m\) with arbitrary \(k, l\) or \(n = m\) with \(k \neq l\), where \(l = (l_1, l_2, \ldots), z = (z_1, z_2) \in \mathcal{A}_r^2\), \(z_2 \in \mathcal{A}_r\). Then identify faces \(Q_j := P_j \setminus (U_{m,m \neq j} P_m)\) of two copies \(n + 1\) of \((S(k), ni_1)\) and \((S(k - 2e_j), (n + 1)i_1)\) by the corresponding straight rays of two copies of \((Q_j, ni_1)\) and \((Q_j, (n + 1)i_1)\), where \(k = (k_1, k_2, \ldots), k_j, k_{\bar{j}} \in \{1, -1\}, j \in \mathcal{J}, e_j = (0, 0, \ldots, 0, 1, 0, \ldots) \in \mathbb{R}^{2^r - 1}, 2^\infty - 1 = c\). Do this equivalence relation for all \(n \in \mathbb{Z}\), each \(1 \leq j \in \mathbb{Z}\) and each \(k\) with \(k_j = 1\). Consider after this identification that \(Q_j\) is the part of \((\mathcal{A}_r, (n + 1)i_1)\). Denote by \(L\) the \(2^r\)-dimensional surface in \(\mathcal{A}_r^2\) obtained by such procedure.

To each perpendicular transition through the face \(Q_j\) from \((S(k), ni_1)\) to \((S(k - 2e_j), (n + 1)i_1)\) attach the change \(2\pi i_j\) of the argument of the Cayley-Dickson number, where \(k_j = 1, 1 \leq j \in \mathbb{Z}\). To the perpendicular to \(Q_j\) transition in the opposite direction from \((S(k - 2e_j), (n + 1)i_1)\) to \((S(k), ni_1)\) with \(k_j = 1\) attach the opposite change of the argument \(-2\pi i_j\).

Consider the spherical coordinates \((a, \theta_1, \ldots, \theta_m)\) in the Euclidean space \(\mathbb{R}^{m+1}\) which are related with the Cartesian coordinates \(x_1, \ldots, x_{m+1} \in \mathbb{R}\) of a vector \(x = (x_1, \ldots, x_{m+1}) \in \mathbb{R}^{m+1}\) by the equations:

\[(2)\quad x_1 = a \cos(\theta_1), x_2 = a \sin(\theta_1) \cos(\theta_2), \ldots, x_m = a \sin(\theta_1) \ldots \sin(\theta_{m-1}) \cos(\theta_m), x_{m+1} = a \sin(\theta_1) \ldots \sin(\theta_m),\]

where \(0 \leq a = |x| < \infty, 0 \leq \theta_1 \leq 2\pi, 0 \leq \theta_2 \leq \pi, \ldots, 0 \leq \theta_m \leq \pi\) (see §XII.1 [34]). Then this gives the spherical coordinates in \(\mathcal{A}_r\) taking \(x_{j+1} = z_j\) for each \(j = 0, 1, 2, \ldots, 2^r - 1\) and \(m = 2^r - 1\), where \(z = \sum_{j=0}^{2^r-1} z_j i_j \in \mathcal{A}_r\). Comparing Equations (1) and (2) gives:

\[(3)\quad M = i_1 \cos(\theta_2) + i_2 \sin(\theta_2) \cos(\theta_3) + \ldots + i_{2^r-2} \sin(\theta_{2^r-2}) \cos(\theta_{2^r-1})+ i_{2^r-1} \sin(\theta_{2^r-2}) \ldots \sin(\theta_{2^r-1}) \text{ and } \theta_1 = \phi.\]

For \(\mathcal{A}_\infty\) the limit of (2) when \(r\) tends to the infinity gives spherical coordinates in \(\mathcal{A}_\infty\), since for each \(z \in \mathcal{A}_\infty\) the norm \(|z| := (\sum_{j=0}^{\infty} z_j^2)^{1/2} < \infty\) is finite. Therefore, each non-zero \(z = |z| \exp(M\phi z)\) is periodic (invariant) under substitutions \(\theta_j \mapsto \theta_j + 2\pi m_j\) for each \(j\), moreover, \(z\) is invariant relative to the pairwise substitutions: \(\theta_j \mapsto 2\pi m_j - \theta_j\) and \(\ldots\)
\[\theta_{j+1} \mapsto \theta_{j+1} + (2m_{j+1} + 1)\pi\] for each marked \(j\), where \(m_1, \ldots, m_{2r-1} \in \mathbb{Z}\).

To each spherical coordinates \((\theta_1 + 2\pi m_1, \theta_2 + 2\pi m_2, \ldots, \theta_{2r-1} + 2\pi m_{2r-1}) =: \psi\) attach two vectors \(m^+ = (m_1^+, m_2^+, \ldots)\) and \(m^- = (m_1^-, m_2^-, \ldots)\), where \(m_j^+ := \max(0, m_j)\), \(m_j^- := \min(0, m_j)\), \(|\psi|^2 := \sum_j \psi_j^2 < \infty\), the set \(\{j : m_j \neq 0\}\) is finite, only rectifiable curves in \(\mathcal{A}_r\) are considered. Then \(m = m^+ + m^-\) and put \(n^+ := \sum_j m_j^+\), \(n^- := \sum_j m_j^-\), hence \(0 \leq n^+ \in \mathbb{Z}\) and \(0 \geq n^- \in \mathbb{Z}\). Therefore, to each such \(\psi\) there corresponds a unique \(\text{Arg}(z)\) and \(z\) is uniquely characterized by two points \((y_1, y_2)\), where \(y_1 \in (\mathcal{L}, 1)\) belongs to \((\mathcal{A}_r, n^+)\) and \(y_2 \in (\mathcal{L}, 2)\) belongs to \((\mathcal{A}_r, n^-)\) whose spherical coordinates are \((|z|, \psi^+)\) and \((|z|, \psi^-)\) correspondingly, where \(\psi^+_i = \theta_1 + 2\pi m^+_i\), \(\psi^-_i = \theta_1 + 2\pi m^-_i\), and \(\psi^-_j = \theta_j + \pi m^-_j\) and \(\psi^+_j = \theta_j - \pi m^+_j\) for each \(j \geq 2\), \((\mathcal{L}, 1)\) and \((\mathcal{L}, 2)\) are two copies of \(\mathcal{L}\). Then embed \(\mathcal{R} := \{(y_1, y_2) : y_1 \in (\mathcal{L}, 1)\text{ with } n_1 \geq 0, y_2 \in (\mathcal{L}, 2)\text{ with } n_2 \leq 0\}\) into \(\mathcal{A}^2\), which is possible, since \(\mathcal{L} \subset \mathcal{A}_r \times \mathcal{I}_r\). Points \(y_1\) and \(y_2\) are equivalent if and only if \(n^+ = n^- = 0\). Then \(\mathcal{R}\) is the noncommutative for \(2 \leq r\) and non-associative for \(3 \leq r\) analog of the Riemann surface such that \(L_n : \mathcal{A}_r \setminus \{0\} \to \mathcal{R}\) is the univalent mapping and \(L_n\) has the countable number of branches such that \(L_n(z) = ln|z| + \text{Arg}(z)\), where \(ln : (0, +\infty) \to \mathcal{R}\) is the usual real natural logarithm.

11. **Lemma.** If \(U\) and \(V\) are open domains in \(\mathcal{A}_r\), \(2 \leq r \leq 3\), \(f : U \to \mathcal{A}_r\), \(\psi : V \to \mathcal{A}_r\), \(f\) is holomorphic on \(U\) and \(\psi\) is holomorphic diffeomorphism of \(V\) with \(\gamma\) is a rectifiable curve in \(U\), where \(\gamma(t) = z_0 + \text{Exp}_n(a_1, \ldots, a_n; t)\) for each \(t \in [0, 1]\), \(a_1, \ldots, a_n \in \mathcal{A}_r \setminus \{0\}\) are nonzero constants or \(\mathcal{A}_r\), pseudo-conformal functions, \(a_n = \text{const} \in \mathcal{A}_r \setminus \{0\}\), \(\text{Re}(a_n) = 0\), \(|a_n| = 2\pi\), \(f(\gamma(t)) \neq 0\) for each \(t \in [0, 1]\), then \(\Delta_\gamma \text{Arg}_n f = \Delta_\gamma \text{Arg}_n f \circ \psi\) and \(\Delta_\gamma \text{Arg}_n f\) is independent from \(a_1, \ldots, a_{n-1}\), when \(n \geq 2\), where \(\phi(z) := L_n(a_1, \ldots, a_n; \psi^{-1}(z))\) on \(U\) and \(\eta(t) := \text{Exp}_n(a_1, \ldots, a_n; \phi(\gamma(t)))\) for each \(t \in [0, 1]\).

**Proof.** Compositions of pseudo-conformal functions are pseudo-conformal, the inverse of a pseudo-conformal mapping is pseudo-conformal (see [23] and Theorem 2.6 [21]). Since \(\text{exp}\) and \(\text{Ln}\) are pseudo-conformal, the mappings \(z \mapsto az\) and \(z \mapsto za\) are pseudo-conformal for \(a \neq 0\), then \(\text{Exp}_n\) and \(\text{Ln}_n\) are also pseudo-conformal for \(a_1 \neq 0, \ldots, a_n \neq 0\). Choose a branch of the logarithmic function (see Proposition 10.2) and consider \(\phi(z) := L_n(a_1, \ldots, a_n; \psi^{-1}(z))\) and put \(\eta(t) := \text{Exp}_n(a_1, \ldots, a_n; \phi(\gamma(t)))\), hence \(\psi(\zeta) = \gamma(t) = z\) if and only if \(\zeta = \eta(t)\). On the other hand, \(\phi(z)\) is the holomorphic mapping as the composition of holomorphic mappings. Thus, \(\eta\) is the rectifiable curve, since \(\gamma\) is the rectifiable curve. A rectifiable curve \(\gamma\) is compact in \(\mathcal{A}_r\), hence it can be covered with a finite number of balls on each of which \(f\) has not zeros, since \(f\) is continuous and has not zeros on \(\gamma\). Since \(f^b \frac{dg(z)}{dz} = g(b) - g(a)\) for a holomorphic function on a ball \(W\) in \(\mathcal{A}_r\) (see Theorem 2.18 [24]), then

\[\Delta_\gamma \text{Arg}_n f = f_\gamma d\text{Ln}_n(a_1, \ldots, a_{n-1}, 1; f(z)) = f_\eta d\text{Ln}_n(a_1, \ldots, a_{n-1}, 1; f \circ \psi(\zeta)) = \Delta_\eta \text{Arg}_n f \circ \psi.\]

Since \(\text{Ln}_{n-1}(a_1, \ldots, a_{n-1}; z)\) is the inverse function of \(\text{Exp}_{n-1}(a_1, \ldots, a_{n-1}; y)\), then \(\Delta_\gamma \text{Arg}_n f\) is independent from \(a_1, \ldots, a_{n-1}\), when \(n \geq 2\). This is valid for each phrase \(\mu\) representing \(f\) and for each branch of the line integral, for example, specified with the help of the left or the right algorithm (see Lemma 2.16 and Theorems 2.17, 2.18 in [24] and [18]). Phrases corresponding to \(f\) are consistent for canonical (analytic) elements which are analytic extensions of each other in the domain due to the monodromy Theorem 2.1.5.4 [25] and 2.45 [24].

12. **Lemma.** Let \(f\) be a \((p, r, b)\)-quasi-conformal function on an open connected domain \(U\) in \(\mathcal{A}_b\) with a zero \(z_0 \in U \cap \mathcal{A}_r\), \(f(z_0) = 0\), where \(1 \leq r < b \leq 3\), \(0 < p \in \mathbb{Z}\). Then \(f\), if \(f\) has a connected surface \(S = S_{z_0}\), \(z_0 \in \mathcal{A}_b\), is also differentiable on \(\mathcal{A}_b\), zeros of \(f\) such that \(z_0 \in S\) and its dimension over \(\mathbb{R}\) is \(\text{dim}_RS = 2^b - 2^r\).

**Proof.** Since \(f(z + y_0) = R_{z,x}f(y_0 + x)\) for each \(x \in (U - y_0) \cap \mathcal{A}_r\) and \(z \in U - y_0\)
such that $Re(z) = Re(x)$ and $z = \hat{R}_{z,x}x$, where $y_0$ is a marked point in $U \cap A_r$, then $f(z) = \hat{R}_{z-y_0,z_0-y_0}f(z_0) = 0$ for each $z \in U$ such that $Re(z) = Re(z_0)$ and $z - y_0 = \hat{R}_{z-y_0,z_0-y_0}(z_0 - y_0)$, since $f(z) = \hat{R}_{z-y_0,z_0-y_0}f(z_0)$ is obtained from $f(z_0)$ with the help of the automorphism $\hat{R}_{z-y_0,z_0-y_0}$ of the Cayley-Dickson algebra $A_b$, which is the quaternion skew field $H$ for $b = 2$ or the octonion algebra $O$ for $b = 3$ (see Definition 1). The family of automorphisms $\hat{R}_{z,x}$ is holomorphic and satisfies Conditions $(Q2 - Q5)$ such that when $z$ tends to a point $\zeta$ in $(U - y_0) \cap A_r$, then $\hat{R}_{z,x}$ tends to the unit operator for a given $x \in (U - y_0) \cap A_r$, consequently, $S_{z_0} := S := \{z : z - y_0 = \hat{R}_{z-y_0,z_0-y_0}(z_0 - y_0), z \in U, Re(z) = Re(z_0)\}$ is connected. In particular, $\hat{R}_{z_0-y_0,z_0-y_0}(z_0 - y_0) = z_0 - y_0$, since $z_0 \in U \cap A_r$, hence $z_0 \in S$. Its dimension over $R$ is: $\dim_R S = 2^b - 2^r$, since $\dim_R A_b = 2^b$, $\dim_R A_b \cap A_r = 2^b - 2^r$ for $A_b \cap A_r$, considered as the $R$-linear space.

13. Notes. Generally the product of $(r, b)$-quasi-conformal functions (with a prescribed order of multiplication for $b = 3$) even for $r = 1$, where $1 \leq r < b \leq 3$ need not be $(r, b)$-quasi-conformal, since the derivative operator of the product is the sum of operators (see Definition 2.2(11) [17, 18, 19]). Indeed, the sum of pseudo-conformal or quasi-conformal functions may be non-pseudo-conformal or non-quasi-conformal respectively even when a derivative operator is non-zero, especially for $r = 2$, since there are projection operators $\pi_j$ from $A_b$ into $i_j R$ for each $j = 0, 1, \ldots, 2^b - 1$ and every $2 \leq b$, where $\pi_0(z) = (z + (2^b - 2)^{-1}\{z + \sum_{k=1}^{2^b-1} i_k(z_k^*)\})/2$, $\pi_j(z) = (-z + \sum_{k=1}^{j-1} i_k(z_k^*)^2)/2$ for each $j \in \{1, \ldots, 2^b - 1\}$, $\pi_j(z) = z_j$ for each $z \in A_b$, where $z = z_0 q_0 + \ldots + z_{2^b-1} q_{2^b-1}$. This is the effect of the noncommutativity of the Cayley-Dickson algebras for $2 \leq b$. Moreover, starting from complex constants $a = a_0 + ia_1$ with $a_0, a_1 \in R$ gives $\hat{R}_{z,y} a = a_0 + Ma_1$, where $M \in A_b$ depends on $z \in A_b$, $y \in A_r$, as it is described by Formulas 2(4, 6) in the $z$-representation, hence

1) $(\partial \hat{R}_{z,y} a/\partial z).h = ha_1$ for each $h \in \mathcal{I}_b, z \in A_b \setminus A_r$.

By the same reasoning the sum of pseudo-conformal or quasi-conformal mappings may be a non-pseudo-conformal or non-quasi-conformal mapping respectively for $2 \leq b$ even when its derivative is non-zero, especially for $r = 2$. Each complex holomorphic function $f$ on $C$ (integral function) can be decomposed in accordance with the Weierstrass Theorem V.72 [15] as $f(z) = z^n \exp[g(z)] \prod_{n=1}^\infty (1 - z/a_n) \exp(z/a_n + (z/a_n)^2/2 + \ldots + (z/a_n)^p_n/p_n)$, where $p_n$ is a sequence of natural numbers and $g(z)$ is an integral function, $m$ is the multiplicity of $z = 0$ as the zero of $f(z)$. But its extension with the help of automorphisms satisfying Conditions $(Q2 - Q5)$ may be non-quasi-conformal function in view of obstacles described above.

When a family $\hat{R}_{z,x}$ is given as $\hat{R}_{\xi(z),\xi(x)} = R_{\xi(z),\xi(x)}$ (see Example 2), then the phrases corresponding to canonical (analytic) elements of $f$ which are analytic extensions of each other in the domain are defined consistently due to the monodromy Theorem 2.1.5.4 [23] and 2.45 [24], where $\xi$ is a pseudo-conformal diffeomorphism of $U$. This is usually simpler, when $\xi = id$ is the identity mapping. If $f$ is $(1, b)$-quasi-regular, $2 \leq b \leq 3$, then $q = g|_{W-y_0}$ is the complex holomorphic function and phrases of analytic elements of $q$ are commutative over $C$.

14. Remarks and Definitions. 1. Zeros and poles of complex holomorphic functions are defined classically in the standard way. For an $A_r, p$-pseudo-conformal function $f$ on an open domain $V$ we call a point $z_0 \in V$ the zero of $f$, if $f(z_0) = 0$, where $2 \leq r \leq 3, 1 \leq p \in Z$. In view of Theorem 2.5 [24] and §1 its order is $p$. For an $A_r, p$-pseudo-conformal function $f$ on $V \setminus \{z_0\}$, where $2 \leq r \leq 3, V$ is open in $A_r$, and $z_0 \in A_r$ is a point, then we call $z_0$ the pole of $f$, if $g(z) := 1/f(z)$ is $A_r, p$-pseudo-conformal on $V \setminus \{z_0\}$ and $z_0$ is the zero of $g$. 14
Define \( f \) to be holomorphic in a neighborhood \( V \) of \( \infty \) or \((p, r)-\)pseudo-conformal or \((p, r, b)-\)quasi-conformal at \( \infty \) if and only if \( g(z) := f(1/z) \) is holomorphic in \( \{ z : 1/z \in V \} \) or \((p, r)-\)pseudo-conformal or \((p, r, b)-\)quasi-conformal at zero respectively. We say that \( y = \infty \) is a zero or a pole of \( f \) if and only if \( g(z) = f(1/z) \) has a zero or pole at \( z = 0 \) respectively.

2. Consider the following situation having the natural embedding of \( A \) into \( A_p \) associated with the standard doubling procedure, where \( 1 \leq r < b \leq 3 \). Suppose that \( U \) is an open connected domain in \( A \) and \( W = U \cap A \) is an open connected domain in \( A \) such that \( U \) is pseudo-conformally diffeomorphic with a domain \( V \), where \( V \) is obtained from \( W \) with the help of all rotations in all planes \( \mathbb{R}t \), \( v = 1, \ldots, 2^r - 1 \) and \( u = 2^r, \ldots, 2^b - 1 \) on angles \( \phi \in (0, 2\pi) \) with the real rotation axis, since each operator \( T \in SO(2^b, \mathbb{R}) \) can be presented as the finite product of one-parameter subgroups and here is considered its subgroup \( SO(2^b, \mathbb{R}) \) of operators restrictions of which on the real axis \( \mathbb{R} \) is the identity.

Let \( f \) be a \((p, r, b)-\)quasi-conformal or \((p, r, b)-\)quasi-regular mapping on \( U \) be besides a finite number of surfaces \( S \) such that \( \gamma \) is a finite point, then \( \gamma \) is a zero or pole of \( f \) and let also \( u \) with the help of all rotations in all planes \( \mathbb{R}t \), \( v = 1, \ldots, 2^r - 1 \) and \( u = 2^r, \ldots, 2^b - 1 \) on angles \( \phi \in (0, 2\pi) \) with the real rotation axis, since each operator \( T \in SO(2^b, \mathbb{R}) \) can be presented as the finite product of one-parameter subgroups and here is considered its subgroup \( SO(2^b, \mathbb{R}) \) of operators restrictions of which on the real axis \( \mathbb{R} \) is the identity.

Put for each \( z_0 \in S \) by the definition:

(1) \( (2\pi)^{-1} \int f(z)dz = res(z_0, f).N \) the residue operator of \( f \) at \( z_0 \), where \( N \in A \), \( Re(N) = 0 \), \( |N| = 1 \), \( \gamma(t) = z_0 + \rho \exp(2\pi t N) \in V \), \( t \in [0, 1] \), \( \rho > 0 \) is sufficiently small such that \( f|_{B(z_0)} \) is locally analytic and \( \gamma \) does not encompass another poles of \( f \) in the set \( \{ y_0 + z : z = \hat{R}_{y_0,y_0}(y - y_0), z \in U - y_0, Re(z) = Re(y - y_0) \} \) of poles \( y \in W \), which may only be points in \( W \), where \( 0 < \rho \in \mathbb{Z} \).

Suppose also that \( z_0 \in W \) is a zero or a pole of \( f \) and \( S \) be the surface corresponding to \( z_0 \) from Lemma 12, where \( W \) may contain only finite number of zeros or poles \( z_1 \), which may only be points. For a subset \( G \) in \( A \), let \( \pi_{s,q,t}(G) := \{ u : z \in G, z = \sum_{v \in b} w_v, u = w_s + w_v, w_v \in R \forall v \in b \} \) for each \( s \neq q \in b \), where \( t := \sum_{v \in b \setminus \{s, q\}} w_v \in A_{s,q,t} := \{ z \in A_r : z = \sum_{v \in b} w_v, w_s = w_q = 0, w_v \in R \forall v \} \). That is, geometrically \( \pi_{s,q,t}(G) \) is the projection on the complex plane \( \mathbb{C}_{s,q} \) of the intersection of \( G \) with the plane \( \pi_{s,q,t} \) and \( C_{s,q} := \{ as + bq : a, b \in \mathbb{R} \} \), since \( sq^* \in \hat{b} \), where \( b := \{ i_0, i_1, \ldots, i_{2^r - 1} \} \) is the family of the standard generators of the Cayley-Dickson algebra \( A_r \).

Suppose that \( \omega \) is a rectifiable loop, that is, a closed curve, \( \omega(0) = \omega(1) \), in an open sub-domain \( J \), \( z_0 \in J \subset W \) in \( A_r \) such that \( \omega \) encompasses \( z_0 \), where \( J \) does not contain any other zero or pole of \( f \), \( J \) is \((2^r - 1)\)-connected and \( \pi_{s,q,t}(J) \) is simply connected in \( C \) for each \( t \in A_{s,q,t} \) and \( u \in C_{s,q} \), \( s = i_{2k} \) and \( q = i_{2k+1} \), \( k = 0, 1, \ldots, 2^r - 1 \) for which there exists \( \zeta = u + t \in J \).

15. Theorem. Let \( f \) be a pseudo-conformal function on \( V \setminus \{ y \} \) with a pole at \( y \) in \( V \) and let also \( F \) be a univalent branch of its \((r, b)\)-quasi-conformal extension in \( W \setminus S \) relative to a marked point \( y_0 \), where \( W \) is an open subset in \( A \) such that \( W \cap A_r = V \). Then the residue operators \( res(y, f) \) and \( res(z, F) \) are such that \( res(z, F), M = \hat{R}_{z-y_0,y_0}[res(y, f), N] \) for each \( z \in S \cap W \) and every \( N \in A \), with \( Re(N) = 0 \), where \( M = \hat{R}_{z-y_0,y_0} N \). Moreover, \( res(z_0, f), N \) is \( \mathbb{R} \) homogeneous and \( A \) additive for \( f \).

Proof. If \( y \) is a finite point, then \( z \in S \) is a finite point and

(1) \( (2\pi)^{-1} \int f(z)dz = res(y, f), N \), where \( \gamma(t) = y + \rho \exp(2\pi t N) \) and \( \rho > 0 \) is sufficiently small such that \( \gamma \) does not encompass another poles of \( f \) in the set \( \{ y_0 + z : z = \hat{R}_{y_0,y_0}(y - y_0), z \in U - y_0, Re(z) = Re(y - y_0) \} \) for some \( 0 < \epsilon < \infty \) and some \( q \in A_b \) with \( Re(q) = Re(y - y_0) \).
such that \( N \in \hat{R}_{q,y-y_0}\mathcal{A}_r \). Using Conditions (Q6, Q7) the action of \( \hat{R}_{z-y_0,y-y_0} \) on both sides of Equation (1) gives

\[
(2) \quad \hat{R}_{z-y_0,y-y_0}[res(y, f).N] = (2\pi)^{-1} \int f F(s)ds \text{ for } \Re(z) = \Re(y),
\]

where \( \eta(t) = \hat{R}_{z-y_0,y-y_0}\gamma(t) \) for each \( t \).

Since each \( z \in S_y \) is the pole of \( F \) restricted on the corresponding subalgebra \( \hat{R}_{z-y_0,y-y_0}(\mathcal{A}_r) \) in \( \mathcal{A}_b \), then there is defined the \( \mathcal{A}_b \)-additive and \( \mathbf{R} \)-homogeneous operator \( res(z, F).M = (2\pi)^{-1} \int f F(s)ds \) for \( M \in \mathcal{A}_b \), \( \Re(M) = 0 \) (see Theorem 3.23 [17, 18, 19]).

Therefore, the first statement of this theorem follows from the first part of the proof. If \( y = \infty \), then consider \( g(z) = f(1/z) \) and \( g \) has the pole at \( z = 0 \), hence in this case the statement of this theorem follows from the first part of the proof.

**15.1. Example.** If a function \( f \) can be written in the form \( f(z) = (a(z)((b(z)1/(z - y))c(z)))e(z) \) in a neighborhood of \( y \in \mathcal{A}_r \), where \( a(z), b(z), c(z) \) and \( e(z) \) are \( \mathcal{A}_r \)-holomorphic and \( a(y) \neq 0, b(y) \neq 0, c(y) \neq 0 \) and \( e(y) \neq 0 \), \( 2 \leq r \leq 3 \). Then \( res(y, f).N = (2\pi)^{-1} \lim_{\rho \to 0} f_\gamma^{-1}(z)) e(z)dz = (a(y)(b(y))N)c(y) e(y) \). At the same time for \( \gamma \) from Definition 10 with \( a_n = 2\pi M, M \in \mathcal{A}_b, \Re(M) = 0, |M| = 1, z_0 = 0, \xi(t) = t \) for each \( t \), we have \( \Delta, \text{Arg}_n, \gamma = 2\pi M \).

**16. Theorem.** Let \( U \) be a proper open subset in \( \mathcal{A}_b \), let also \( f_1 \) and \( f_2 \) be two continuous functions from the closure \( \overline{U} := cl(U) \) of \( U \) into \( \mathcal{A}_b \) such that on a topological boundary \( Fr(U) \) of \( U \) they satisfy the inequalities \( \gamma(f_1) < |f_2| < \infty \) for each \( z \in Fr(U) \), where \( \mathcal{A}_b := \mathcal{A}_b \cup \{\infty\} \) is the one-point (Alexandroff) compactification of \( \mathcal{A}_b \). Suppose \( q_2 := f_2 \) and \( q_1 := f_1 + f_2 \) are \( \psi \)-quasi-meromorphic functions in \( U \) and zeros and poles of \( f_1|\mathcal{W} \) are isolated, where \( W = U \cap \mathcal{A}_r, 1 \leq r \leq 3, j = 1, 2 \). Let also \( q_j = (p, r, b) \)-quasi-conformal in a neighborhood \( U_{z_0} \) in \( U \) of each its zero \( z_0 \) and \( 1/q_j(z) \) be \( (p, r, b) \)-quasi-conformal in \( U_{z_0} \setminus \{z_0\} \) for each pole \( z_0 \), where \( p \in \mathbb{N} \) may depend on \( z_0 \), \( j = 1 \) and \( j = 2 \).

Suppose also that \( \gamma \) from Definition 10 is a loop, where \( \gamma(0) = \gamma(1) \) is a loop, does not cross any \( S_g(q_j) \) for any zero or pole \( y \) of \( q_j \) for \( j = 1 \) and \( j = 2 \), where \( \gamma \subset Fr(U), 1 \leq n \). Then \( \Delta, \text{Arg}_n, \gamma = 1 = \Delta, \text{Arg}_n, q_2 \).

**Proof.** Put without loss of generality \( z_0 = 0 \) and \( \rho = 1 \) for those of Definition 10. If \( n > 1 \) consider \( h_j := q_j \circ Exp_{-1}(a_1, \ldots, a_{n-1}; z) \) instead of \( q_j \) for \( j = 1 \) and \( j = 2 \), since the compositions of mappings are associative in the set theoretic sense. On the other hand, \( Exp_{-1}(a_1, \ldots, a_{n-1}; z) \) is the pseudo-conformal mapping for \( a_1 \neq 0, \ldots, a_{n-1} \neq 0 \). In view of Theorem 8.2 \( h_j \) satisfy suppositions of this theorem. Substituting \( q_j \) on \( h_j \) we can reduce the proof to the \( n = 1 \) case, since \( Ln_{n-1}(a_1, \ldots, a_{n-1}; \gamma(t)) = \xi(t) \) for a branch of \( Ln \) such that \( Ln(1) = 0 \). For example, it is possible to take \( a_n = 2\pi M \) and \( \xi(t) = t, t \in [0, 1], M \in \mathcal{A}_b \cap \mathcal{A}_r, |M| = 1, \Re(M) = 0 \) in Definition 10. Therefore, consider \( q_j \) for \( n = 1, \) where \( j = 1, 2 \).

The curve \( \gamma \) is rectifiable, hence compact. Zeros and poles of \( f_j|\mathcal{W} \) are isolated consequently, there exists a sequence \( \{\psi_m : m \in \mathbb{N}\} \) of rectifiable loops in \( U \) converging to \( \gamma \) uniformly when \( m \) tends to the infinity and such that each \( \psi_m \) does not cross any \( S_g(q_j) \) for \( j = 1 \) and \( j = 2 \) for zero or pole \( y \) of \( f_j \). Thus consider the integral \( \int \psi_m d\text{Ln}(q_j(z)) \) along \( \gamma \) as the limit of \( \int_{\psi_m} d\text{Ln}(q_j(z)) \) when \( m \) tends to the infinity, since \( f_j \) is continuous in a neighborhood \( V \) of \( Fr(U) \) in \( U \) and \( \psi_m \subset U \) for each \( m \) and \( j \), where \( V \) does not contain any zero or pole of \( q_1 \) and \( q_2 \). The latter \( V \) exists, since \( |f_1(z)| < |f_2(z)| < \infty \) for each
z ∈ Fr(U).

If z₀ is a pole of q₁ at z₀, then 1/q₁(z) has a zero at z₀. There are not any zeros or poles of q₁ and q₂ on Fr(U), since |f₁(z)| < |f₂(z)| < ∞ on Fr(U).

For each chosen branch of the logarithmic function there is the equality Ln(1/q₁) = −Ln(q₁) (see Proposition 10.2). Moreover, q₁ = f₁ + f₂ = f₂ + f₂[(1/f₂)f₁] = f₂(1 + (1/f₂)f₁), since A₃ = O is alternative, and |(1/f₂)f₁| < 1 on Fr(U). Consider the triangle formed by the vectors q₁(z), q₂(z) and q₁(z) − q₂(z) = f₁(z) for z ∈ Fr(U), then Arg q₁(z) = Arg q₂(z) + φ(z) such that |φ(z)| < π/2 for each z ∈ Fr(U) for a chosen branch of Ln, where Arg q₁ and φ(z) ∈ Θb. Therefore, Δξ Arg q₁ = Δξ Arg q₂ + Δξ φ = Δξ Arg q₂, since ξ(0) = ξ(1) and |φ(z)| < π/2 for each z ∈ ξ([0, 1]) ⊂ Fr(U) such that Δξ φ = 0. Indeed, the point w(z) = (1/f₂(z))f₁(z) is within the unit open ball B(A₆b, 0.1) := {w ∈ A₆b : |w| < 1}. Therefore, the vector v = 1+w can not rotate on 2π around zero. Thus the winding numbers of q₁ and q₂ around the zero are the same. From the relations of h_j with q_j for n > 1 the statement of this theorem follows for n > 1 as well: Δγ₁Argₙq₁ = Δγ₁Argₙq₂.

16.1. Theorem. Let suppositions of Remark 14.2 be satisfied, when W = B(A₄, y₀, R) \ A, where A := {y ∈ B(A₄, y₀, R) : f'y(y) = 0} consists of isolated points, 0 < R < ∞. Suppose also that W contains either zeros or poles of a (q, r, b)-quasi-regular function f, but not zeros and poles simultaneously, 1 ≤ r < b ≤ 3. Then for each rectifiable curve ω in J encompassing z₀ and each 2r ≤ n ≤ 2b − 1 there exists a family of rectifiable curves γ in U encompassing S = Sz₀ such that γ ∩ S_z₁ = ∅ for each zero or pole z₁ of f in W and such that γ is not contained in A₄ and

Δγ₁Argₙf = pKΔωArg₁f

for some K ∈ A₄, |K| = 1, Re(K) = 0, K = K(γ), 1 ≤ p ∈ Q.

Proof. The zero or pole z₀ of f is isolated in W, hence f(z) ≠ 0 in Y \ {z₀} for a sufficiently small neighborhood Y of z₀ in A₄. If z₀ is a pole of f, then z₀ is a zero of 1/f and vice versa. If z ∈ Y \ {z₀}, then f(z) ≠ 0 and apply automorphisms Rζ−y₀,z−y₀ to f(z) by all ζ ∈ U with Re(ζ) = Re(z) and (ζ − y₀) = Rζ−y₀,z−y₀(z − y₀). Take without loss of generality y₀ = 0, since z → z + y₀ is the bijective pseudo-conformal mapping from A₄ on A₆. This gives the closed surface Sf(z) analogous to S = Sz₀. In accordance with Lemma 12 dimRSz₀ = dimRSf(z) = 2b − 2r.

If z⁺ ∈ A₄ is obtained from z₁ ∈ A₄ with l ≠ k due to a rotation around the real axis in a plane πᵏ⁺l contained in A₄ ⊕ Aᵢ, which corresponds to the one-parameter over R subgroup of rotations in SOₙ(2h, R), then z⁺ and z₁ are both either zeros or poles due to the conditions of this theorem, since f is (q, r, b)-quasi-regular.

There are the following decompositions of algebras as the R-linear spaces due to the doubling procedure: H = C ⊕ i₂C, O = H ⊕ i₄H and O = C ⊕ i₂C ⊕ i₄C ⊕ i₆C corresponding to (r, b) pairs equal to (1, 2), (2, 3) and (1, 3) respectively, that gives the embedding of geometry in A₄ into geometry in A₆. Consider an intersection of the surface S with the plane π containing z₀ and perpendicular to the real axis R, π = z₀ + i₄R ⊕ i₄R, where 2r ≤ s < q ≤ 2b − 1. Then η := S ∩ π is a rectifiable loop containing z₀ and η has the winding number 1 for each internal point in the domain Pₙ enclosed by η in π with the boundary ∂Pₙ = η.

Consider a rectifiable loop γ consisting of the following parts: the loop γ⁺ outside Pₙ, the loop γ₋ inside Pₙ, ψ, where γ(t) = γ⁺(3t) for 0 ≤ t ≤ 1/3, γ(t) = ψ(6t − 2) for 1/3 < t < 1/2, γ(t) = γ₋(3t − 3/2) for 1/2 ≤ t ≤ 5/6, γ(t) = ψ(6t − 6t) for 5/6 < t ≤ 1 such that ψ joins γ⁺ with γ₋ such that ψ is gone along twice in one and the opposite direction, γ⁺ and γ₋ are in π for which |γ⁺(t)| > |η(t)| and |γ₋(1 − t)| < |η(t)| and |γ⁺(t) − η(t)| < δ and |γ₋(1 − t) − η(t)| < δ for each t ∈ [0, 1] and γ(t) ∈ U and γ(t) is not zero or pole.
of the function $f$ for each $t \in [0, 1]$, $\delta > 0$ is a sufficiently small constant such that $\gamma_-$ and $\eta$ encompass the same zeros and poles besides those belonging to $S_{z_0}$, $\gamma_+$ and $\gamma_-$ have opposite orientations (see also Theorem 16 and Equations (1, 2) below for more details). Since the set $A$ in $W$ consists of isolated points, then the loop $\gamma$ can be chosen such that $\gamma([0, 1]) \cap \{f_j : y \in A\} = \emptyset$. This encompassment is subordinated in $A_i$ to properties of $\Lambda_n$ (see Theorems 2.23 and 2.24 [21]). Take $0 < \rho_+ - \rho_- \text{ sufficiently small and use the approximation } f(z_0 + h) = f(z_0) + f'(z_0)h + O(h^2) \text{ and Properties 1.}(P_1 - P_3) \text{ in a neighborhood of a zero } z_0 \text{ of } f \text{ in } A_i \text{, and Properties 1.}(Q_1 - Q_7) \text{ in a neighborhood of } z_0 \text{ in } A_b, \text{ where } f(z_0) = 0 \text{ for a zero } z_0 \text{ of } f \text{ (see also Theorems 2.4 and 2.5 [21]).}

Choose $\psi$ in a plane $\pi_1$ containing $R$ and a point $\zeta' \in A_b \cap A_r$ such that $\psi$ does not intersect any $S_{z_1}$. Hence $\gamma$ does not intersect any $S_{z_1}$ and encompasses $S = S_{z_0}$. The direction of $\gamma$ is natural such that in the plane $\pi$ the loop $\gamma_+$ is gone counter-clock-wise and $\gamma_-$ is gone clock-wise as seen from the positive axis of $M_\pi R_\pi$, where $R_\pi := (0, \infty)$, $M_\pi \in \mathcal{I}_b$, $M_\pi \perp \pi$, $M_\pi$ corresponds to a vector being the vector product in the real shadow of basis vectors of the plane $\pi$. Though, instead of description of an orientation it is sufficient to write analytic formulas for curves, that is done below.

For $r = 1$ if $z_0$ is the zero or pole together with $z_0$, then $\gamma$ encompasses $z_0$ and $\bar{z}_0$ symmetrically, since $\bar{z}_0$ is obtained from $z_0$ by rotation on the angle $\pi$ around the real axis. For $r = 2$ if $z_j = \sum_{k=0}^{3} z_{j,k}i_k$ and $z_1 \in S_{z_0}$, then $z_j$ and $z_1$ are both either zeros or poles due to the condition imposed in Remark 14.2 and in this theorem, for example, when $z_{j,k} = -z_{j,k}$ for some $1 \leq k \leq 3$, where $z_{j,k} \in R$ for each $j, k$.

Using the iterated exponent choose $\gamma(t)$ up to an $A_b$-pseudo-conformal diffeomorphism of $U$ in the form

(1) $\gamma_+(t) = Re(z_0) + \rho_+ \exp_{n-1}(a_1, \ldots, a_{n-1}; \xi(t))$,

(2) $\gamma_-(t) = Re(z_0) + \rho_- \exp_{n-1}(a_1, \ldots, a_{n-1}; \xi(1-t))$,

where $0 < \rho_- < \rho_+ < R$, $1 \leq n \leq 2^b - 2^r$, $a_1, \ldots, a_{n-1} \in A_b$ are nonzero constants, $\xi([0, 1])$ is a loop, for example, $\xi(t) = \exp(2\pi M t)$, $t \in [0, 1]$, $M \in A_b \cap A_r$, $|M| = 1$, $Re(M) = 0$. Since $S$ is the smooth $C^\infty$ compact manifold having the $C^\infty$ real shadow, then it has $2^b - 2^r$ local coordinates. Moreover, $S$ is homeomorphic with the rotation surface such that $S$ can be parameterized with angles $\theta_1, \ldots, \theta_m$, where $0 \leq \theta_1 \leq 2\pi$ and $0 \leq \theta_j \leq \pi$ for $j = 2, \ldots, m$, $m = 2^b - 2^r$, such that $z \in S$ is the function $z = z(\theta_1, \ldots, \theta_m)$ (see Formulas 10.2(2)).

In view of Corollary 3.5 [18, 19] the sphere $S(A_b, y_0, R)$ in $A_b$ of radius $0 < R < \infty$ with the center at $y_0$ can be parameterized with the help of the iterated exponential functions. Let $\{i_0, i_1, i_2, i_3\}$ be the standard generators of the quaternion algebra $\mathcal{H}$, where $i_0 = 1$, $i_1^2 = i_2^2 = i_3^2 = -1$, $i_1i_2 = -i_2i_1 = i_3$, $i_2i_3 = -i_3i_2 = i_1$, $i_3i_1 = -i_1i_3 = i_2$, then

$$
\begin{align*}
(3) \exp(i_1(p_1t + \zeta_1) & \exp(-(i_3(p_2t + \zeta_2) \exp(-i_1(p_1t + \zeta_1) - i_2 \sin(p_3t + \zeta_3)))) = \\
& \exp(i_1(p_1t + \zeta_1)(\cos(p_2t + \zeta_2) - \sin(p_2t + \zeta_2) i_3 \cos(p_3t + \zeta_3) - i_2 \sin(p_3t + \zeta_3)))) = \\
& \exp(i_1(p_1t + \zeta_1)(i_1 \cos(p_2t + \zeta_2) + i_2 \sin(p_2t + \zeta_2) i_3 \cos(p_3t + \zeta_3) + i_3 \sin(p_2t + \zeta_2) \sin(p_3t + \zeta_3)))) = \\
& \cos(p_1t + \zeta_1) + i_1 \sin(p_1t + \zeta_1) \cos(p_2t + \zeta_2) + i_2 \sin(p_1t + \zeta_1) \sin(p_2t + \zeta_2) \cos(p_3t + \zeta_3) + i_3 \sin(p_1t + \zeta_1) \sin(p_2t + \zeta_2) \sin(p_3t + \zeta_3),
\end{align*}
$$

where $p_j, \zeta_j \in R$ for each $j$.

Further by induction the equality is accomplished:

$$
(4) \exp(qM(p, t; \zeta)) = \\
\exp(qM((i_1p_1 + \ldots + i_{2^r-1}p_{2^{r-1}}), t; (i_1\zeta_1 + \ldots + i_{2^r-1}\zeta_{2^r-1}) \exp(-i_1(2^{r+1}-1) (\sin(p_2t + \zeta_2) + i_2 \exp(-qM(i_1p_{2^{r-1}} + \ldots + i_{2^r-1}p_{2^{r-1}+1}, t; (i_1(2^r+1) + \ldots + i_{2^r-1}(2^{r+1}-2^r-1))))),
$$

where $i_j$ is the generator of the doubling of the algebra $A_{r+1}$ from the algebra $A_r$, such that $i_ji_{j+1} = i_{2^r}$ for each $j = 0, \ldots, 2^r - 1$, the function $M(p, t; \zeta)$ is written with the lower index $qM$ and it is given by the equation
(5) $M(p, t) = M(p, t; \zeta) = (p_1 t + \zeta_1)[i_1 \cos(p_2 t + \zeta_2) + i_2 \sin(p_2 t + \zeta_2)]$
\hspace{1cm} $\cos(p_3 t + \zeta_3) + \ldots + i_{2^n - 1} \sin(p_{2^n} t + \zeta_{2^n - 1}) \sin(p_{2^n - 1} t + \zeta_{2^n - 1})$

for the Cayley-Dickson algebra with $2 \leq q < \infty$, where $\zeta = \zeta_1 i_1 + \ldots + \zeta_{2^n - 1} i_{2^n - 1} \in A_q$ is the parameter of an initial phase, $\zeta_j \in \mathbb{R}$ for each $j = 0, 1, \ldots, 2^n - 1$. When $t = 1$ and $p_j$ are variables $p_1 \in [0, \pi]$ and $p_j \in [0, \pi)$ for each $j = 2, ..., 2^n - 1$, then the image of the iterated exponent given by Equation (4) for $2 \leq q < 3$ or, in particular, by Formula (3) is the unit sphere in $A_q$, where $\zeta_j$ is fixed for each $j = 1, ..., 2^n - 1$ and can also be taken particularly zero. This gives the one-sheeted covering of the unit sphere in $A_q$. If $p_j$ and $\zeta_j$, $j = 1, ..., 2^n - 1$ are fixed and $t$ is the variable, then Formulas (3) (4, 5) give the curve in $A_q$. It reduces to the loop, when $p_1 = 2\pi$ and $p_j = 0$ or $p_j = \pi$ for each $j = 2, ..., 2^n - 1$, $t \in (0, 1)$. Particularly, if $\zeta_j = 0$ for each $j > n$, $\zeta_1 \neq 0, \ldots, \zeta_n \neq 0$ and $p_k = 0$ for each $k \neq n$, then the iterated exponent in Formulas (3) or (4, 5) reduces to $Exp_p$.

Then $S$ is diffeomorphic with the intersection $S(A_0, y_0, R) \cap (i_2 R \oplus i_{2^2 - 1} R \oplus \ldots \oplus i_{2^n - 1} R)$. In accordance with the Riemann mapping Theorem 4.12.40 over $C$ [31] or Theorems 2.1.5.7 and 2.47 [21, 25] over $H$ and $O$ if $P$ is an open subset in $A_q$, $q = 1$ or $q = 2$ or $q = 3$, satisfying conditions of Remark 14 and with a boundary $\partial P$ consisting more, than one point, then $P$ is pseudo-conformally equivalent with the open unit ball in $A_q$.

Within each sub-domain $P$ of $U$ in $A_b$ satisfying conditions of Remark 14 is applicable the homotopy Theorem 2.11 [17, 18, 19] for the line integral over $A_b$. In view of Lemma 11 above we can consider a domain $U$ and hence a curve $\gamma$ in it up to a pseudo-conformal diffeomorphism. Therefore, the rest of the proof up to a pseudo-conformal diffeomorphism is with balls and spheres due to Conditions $(P1 - P3)$.

Therefore, there exist $n, a_1, \ldots, a_{n-1}$ and $\xi$ such that $\gamma_+ \text{ and } \gamma_-$ are given by Formulas (1, 2) and $\Delta \arg f \xi \neq 0$, consequently, $\Delta_+ \arg f \gamma \neq 0$, for example, $a_n = 2\pi M$, $M \in A_b \oplus A_r$, $|M| = 1$, $Re(M) = 0$. This is applicable both to $S_{z_0}$ and $S_{f(z)}$ obtained from $z_0$ and $f(z)$ by families of automorphisms. If we take $\chi = \pi \cap S_z \cap \chi$ with $Re(z) = Re(z_0), |Im(z)| > |Im(z_0)|$, $z \neq z_0$ such that $S_z$ does not contain any pole or zero of $f$, then each $\zeta \in (\pi - z_0 + f(z)) \cap S_{f(z)}$ has the form $\zeta = f(\chi(t))$ and the mapping $\{0, 2\pi\} \ni t \mapsto \zeta \in (\pi - z_0 + f(z)) \cap S_{f(z)}$ is bijective, hence $\Delta_+ \arg f \neq 0$. This is possible, since the set $A$ of poles or zeros of $f$ that consists of isolated points, $\forall z_0 \in A : \min\{|y - z_0| : y \in A \cap \{z_0\}\} > 0$. In view of Condition (Q7) for $a_n = 2\pi M$ and $\xi(t) = exp(2\pi M t)$ in Formulas (1, 2) we have $\Delta_+ \arg f = 2\pi u kM$, where $|M| = 1$, $Re(M) = 0$, $M \in A_b$, $k$ is the winding number of $\chi$, which we take equal to 1, $u$ is the sum of orders of all either poles or zeros from $W$ encompassed by $S_z$ (see also Theorem 16). For sufficiently small $|Im(z)| = |Im(z_0)| = \epsilon > 0$ the number $u$ is equal to the sum of orders of all either poles or zeros belonging to $S = S_{z_0}$ and to $B(A_r, Re(z_0), |Im(z_0)|) = \{y \in A_r : |y - Re(z_0)| \leq |Im(z_0)|\}$.

Then $u \neq 0$, since all $z_1$ belonging to $S$ are simultaneously zeros or poles together with $z_0$ (see above). It is possible to take $\gamma = \chi$, that gives $u = u_\chi$. If take $\gamma$ consisting of $\gamma_+$ and $\gamma_-$ and $\chi$ as above, then $u = u_+ - u_- > 0$, since $S_{z_0} \subset B(A_b, Re(z_0), \rho_+) \cap B(A_b, Re(z_0), \rho_-)$. On the other hand, $\Delta_\omega Ar f = 2\pi v N$, where $|N| = 1$, $Re(N) = 0$, $N \in A_r$, $v$ is the order of $z_0$, $v \geq 1$ for zero, $v \leq -1$ for pole. Thus $1 \leq p = u/v \in Q$, also $M = KN$ for $K = MN^*$ due to the alternativity of $A_r$ for $2 \leq r \leq 3$, since $u$ and $v$ are of the same sign and $|u| \geq |v|$, $|M|^2 = MM^* = M^*M = -M^2$ for purely imaginary $M \in A_b$, and inevitably the statement of this theorem follows.

16.2. Remark. If suppositions of Theorem 16.1 are satisfied besides that $S_{z_0}$ contains both zeros and poles then it may happen, that $p = 0$ due to $N = P = 0$, where $N = \sum_k N_k$ is a number of zeros and $P = \sum_k P_k$ is a number of poles belonging to $S \cap A_r$, where each zero and pole is counted in accordance with its order $N_k$ and $P_k$ respectively.
17. Theorem. Let $f$ be a $(1,b)$-quasi-integral function such that $f(z) = \tilde{f}(z)$ for each $z \in \mathcal{A}_b$, also $f^*(p) = f^*(-p)$ for each $p \in \mathcal{A}_b$ with $\text{Re}(p) \neq 0$, where $f^*$ is a $(1,b)$-quasi-integral function in spherical $\mathcal{A}_b$-coordinates with $f = f^* \circ E^{-1}$ (see Definitions 2 and 2.1), $2 \leq b \leq 3$, $0 < q < \infty$, every zero $z_0$ of $f|_{\mathcal{C}}$ may be only in the strip $\{z \in \mathcal{C} : -q \leq \text{Re}(z) \leq q\}$, $f(z)$ has not any real zero. Then all zeros of the restriction $f|_{\mathcal{C}}$ of $f$ on the complex field $\mathcal{C}$ are complex and belong to the line $\text{Re}(z) = 0$.

Proof. Let $z_0$ be a complex zero of $f|_{\mathcal{C}}$, $f(z_0) = 0$, then $-q \leq \text{Re}(z_0) \leq q$ and $\text{Im}(z_0) \neq 0$ by the supposition of this theorem. Put $v_0 := \text{Re}(z_0)$. In the case $v_0 = 0$ there is nothing to prove. So suppose that $v_0 \neq 0$, then $f(-z_0) = 0$, $f(\tilde{z}_0) = 0$ and $f(-\tilde{z}_0) = 0$ due to the symmetry properties of $f$, since $f^*(p) = f^*(-p)$ for each $p \in \mathcal{A}_b$ with $\text{Re}(p) \neq 0$, $E(y) = y$ for each $y \in \mathcal{C}$, $f = f^* \circ E^{-1}$. Thus without loss of generality consider $0 < v_0 \leq q$. Hence zero surfaces of the $(1,b)$-quasi-conformal extension of $f(z)$ are: $S_{f_{z_0}}^f =: S_{z_0}$ and $S_{f_{\tilde{z}_0}}^f =: S_{\tilde{z}_0}$ (see Lemma 12), since $z_0, \tilde{z}_0 \in S_{z_0}^f$ and $-z_0, -\tilde{z}_0 \in S_{\tilde{z}_0}^f$. These surfaces $S_{z_0}$ and $S_{\tilde{z}_0}$ are symmetric relative to the hyperplane $\pi_0 := \{z \in \mathcal{A}_b : \text{Re}(z) = 0\}$. Without loss of generality put $i\text{Im}(z_0) > 0$, where $i\text{Im}(z_0) = i^1\text{Im}(z_0), \text{Im}(z_0) = z_0 - \text{Re}(z_0) = i\text{Im}(z_0)$.

Since the quaternion skew field $\mathcal{H} = \mathcal{A}_2$ has the natural embedding into the octonion algebra $\mathcal{O} = \mathcal{A}_3$, then it is sufficient to prove this theorem for $b = 2$. Mention, that $L_n$ has the countable number of branches with the noncommutative Riemann surface $\mathcal{R}$ given in §10.2. Therefore, $L_n(z_1, ..., z_{n-1}, 1; z_n)$ has the noncommutative Riemann surface embedded into $\mathcal{R}^n$, since $z \mapsto a^{-1}z$ is the pseudo-conformal mapping by $z$ for $a \neq 0$ in $\mathcal{A}_b$, $z \mapsto 1/z$ is also pseudo-conformal for $z \neq 0$ (see Corollary 2.7 [23]), where $z_j \neq 0$ and $z_j \in \mathcal{A}_b$ for each $j = 1, ..., n$, consequently, $E^{-1}$ has the noncommutative Riemann surface $\mathcal{R}_E$ of dimension $2^b$ over $\mathcal{R}$ embedded into $\mathcal{R}^n$ (see Equations 2.1(1, 2)).

Take $b = 2$ and consider the loops

1. $q_j(t) = v_j + \rho_j \exp(\pi K_j \exp(2\pi N_j t)/2)$

encompassing $S_{z_j}$ parameterized by $t \in [0, 1] \subset \mathcal{R}$, where $j = 0$ or $j = 1$, $v_1 = -v_0$, $K = i_1 = i$, $N = i_3$, $|N| = 1$, $N$ is a marked purely imaginary quaternion orthogonal to $K$, $\text{Re}(NK^*) = 0, 0 < \rho_j - |\text{Im}(z_0)|$ is sufficiently small, $\rho_0 = \rho_1$ (see Theorem 16.1). For $j = 0$ take $K_0 = K$ and $N_0 = N$ and for $j = 1$ take $K_1 = -K$ and $N_1 = -N$. Consider spheres $S(\mathcal{H}, v_0, |\text{Im}(z_0)|)$ and $S(\mathcal{H}, -v_0, |\text{Im}(z_0)|)$, where $S(\mathcal{A}_b, x, R) := \{y \in \mathcal{A}_b : |y - x| = R\}$, $0 < R < \infty$.

In accordance with Theorem I.20.2 [15] if $D$ is an open connected domain in $\mathcal{C}$ and functions $f_1$ and $f_2$ are holomorphic on $D$ such that $f_1(x_n) = f_2(x_n)$ for each $n \in \mathbb{N}$ and there exists a limit point $x \in D$ of a sequence $\{x_n : n \in \mathbb{N}\} \subset D$, then $f_1(x) = f_2(x)$ for each $y \in D$. The function $f$ is holomorphic on $\mathcal{C}$ and $(1,b)$-quasi-integral, hence by the latter theorem zeros of $f$ and $f'$ in $\mathcal{C}$ are isolated. Therefore, there exist $0 < \rho < \infty$ and $0 < \delta < |\text{Im}(z_0)|$ such that for each other complex zero $z_2 \notin \{z_0, -z_0, \tilde{z}_0, -\tilde{z}_0\}$ not belonging to $S(\mathcal{C}, v_0, |\text{Im}(z_0)|) \cup S(\mathcal{C}, -v_0, |\text{Im}(z_0)|)$, either $\{z_2, -z_2, \tilde{z}_2, -\tilde{z}_2\} \subset B(\mathcal{A}_b, v_0, |\text{Im}(z_0)| - \delta) \cup B(\mathcal{A}_b, -v_0, |\text{Im}(z_0)| - \delta)$ or $\{z_2, -z_2, \tilde{z}_2, -\tilde{z}_2\} \subset \mathcal{A}_b \setminus (B(\mathcal{A}_b, v_0, \rho + \delta) \cup B(\mathcal{A}_b, -v_0, \rho + \delta))$. Take $\rho_+ = \rho_0$ and $\rho_- = \rho - \delta$ in Formulas 16.1(1, 2) with $v_0 = \text{Re}(z_0)$ and $v_1 = -v_0 = \text{Re}(-z_0)$ as data in place of $\text{Re}(z_0)$ for $\gamma$ there and get two loops $\gamma_0$ and $\gamma_1$ corresponding to that of given by Equations (1, 2) for $\rho_+$ and $\rho_-$, denote them by $\gamma_{j+}$ and $\gamma_{j-}$ for $j = 0, 1$ respectively. Then $\gamma_0$ and $\gamma_1$ join by a rectifiable curve $\eta$ not containing any zero of $f$ such that $\eta$ is gone twice in one and the opposite direction.

There is the identity

$$K \exp(2\pi N t) = K \cos(2\pi t) + KN \sin(2\pi t) = \exp(\pi K \exp(2\pi N t)/2),$$

since $|K \cos(2\pi t) + KN \sin(2\pi t)| = 1$ and $e^M = \cos(|M|) + M \sin(|M|)/|M|$ for each $M \in \mathcal{I}_b \setminus \{0\}$, $\sin(\pi/2) = 1$, consequently, $q_j(t)$ is orthogonal to $\mathcal{R}$ in $\mathcal{H}$ relative to the
scalar product $(z, \xi) := \Re(z \xi^*)$, moreover, $q_0(0) = z_0 + (\rho_0 - \rho')i_1$, $q_1(0) = -z_0 - (\rho_0 - \rho')i_1$, since without loss of generality put \( \Im(z_0) = \rho' > 0 \), where $z_0 = v_0 + \im(z_0)i_1$. Consider also circles $q_{2+j}(t) = v_j + \rho_2 \exp(\pi K_j \exp(2\pi N_j t)/2)$. It is supposed that $\gamma$ is written in the $z$-representation with the help of formulas 2.2-(5).

Put $p_w(t) := v_0 + \rho_0i_1 - 2\pi i_2 t$, where $\rho_0 := \rho_+ + \rho_-$, $w = 0$ and $w = 2$, then $E_2(p_w(t)) = q_w(t)$ and $E_2(-p_w(t)) = q_{1+w}(t)$ for each $t \in [0,1]$ (see Definition 2.1 and Formula 2.1(1)), since $v_1 = -v_0 \neq 0$.

Therefore, 

\[
(2) \ f(q_{1+w}(t)) = f(q_{1+w}(t))
\]

for each $t \in [0,1]$ and for $w = 0, 2$, since $f^*(p) = f^*(-p)$ for each $p \in A_0$ with $Re(p) \neq 0$, $f = f^* \circ E_2^{-1}$ by the conditions of this theorem.

Consider the loop $\gamma$ consisting of $q_{1}(t)$ and twice gone paths joining them such that $\gamma$ is gone clock-wise by $q_1$ and $q_2$ and counter-clock-wise by $q_0$ and $q_3$ in the planes $v_1 + KR \oplus \pi KNR$ as seen from the negative axis of $((\infty, 0)N)$ perpendicular to the real axis: $\gamma(t) = \gamma_0(4t)$ for $0 \leq t < 1/4$, $\gamma(t) = \gamma_2(4(t-1/4))$ for $1/4 \leq t < 1/2$, $\gamma(t) = \gamma_1(4(t-1/2))$ for $1/2 \leq t < 3/4$ and $\gamma(t) = \gamma_2(1-4(t-3/4))$ for $3/4 \leq t < 1$, where $\gamma_0(t)$ and $\gamma_1(t)$ are composed from $q_0, q_2$ and $q_1, q_3$ for $S_{00}$ and $S_{00}$ respectively and the joining them paths gone twice in one and the opposite directions as in the proof of Theorem 16.1, the rectifiable curve $\{\gamma_2(t): 0 \leq t \leq 1\}$ joins $q_0(1)$ with $q_1(0)$ such that $\gamma((0,1)) \subset V := \{z \in H : -q \leq \Re(z) \leq q\}$, where $\gamma(0) = \gamma(1), \gamma_{0+} = q_0, \gamma_{1+} = q_1, \gamma_{0-} = q_2, \gamma_{1-} = q_3$, since $\bar{K}(KN) = -N$. Instead of talking about orientations it is sufficient to write analytic formulas for curves, that is done in this section.

Since $q$ and $|z|$ are finite, then the curve $\gamma$ can be chosen rectifiable. If $z_2$ is some other zero in the circles $S(C, v_0, |\im(z_0)|) \cup S(C, -v_0, |\im(z_0)|)$, then $S_{22}$ and $S_{-22}$ are encompassed by $\gamma$ as well in the sense of Theorem 16. Their additional value to $p$ of $\Delta_{\gamma} \ Arg z_2 f$ will be positive together with $z_0$ and $-z_0$ in accordance with Theorem 16.1. Let $p_0$ be that part of $p$, which corresponds to $z_2 \in S(C, v_0, |\im(z_0)|) \cup S(C, -v_0, |\im(z_0)|)$ with $Re(z_2) = 0$. Denote the set of such zeros of $f$ by $Z$, $Z := \{z \in S(C, v_0, |\im(z_0)|) \cup S(C, -v_0, |\im(z_0)|)): Re(z) = 0, f(z) = 0\}$. Then $Z$ is finite and may happen to be empty, since $Re(z_0) = v_0 \neq 0$ by the supposition made above. If $z \in C$ and $f(z) = 0$, then the imaginary part of $z$ is nonzero, $Im(z) \neq 0$, since $f$ has not any real zero by the supposition of this theorem.

There is not any zero of $f$ outside the band $-q \leq \Re(z) \leq q$ in $C$, hence there are not any chains of crossing spheres around zeros of $f$ of the type considered above besides may be pairs of spheres with centers at $v_0$ and $-v_0$ with $0 < v_0 \leq q$. Indeed, for $z_0 \in C$ with $|\im(z_0)| > q$ it may be only two such spheres with a given $z_0$, $Re(z_0) = v_0$. In the domain $V_q := \{x \in C : |Re(x)| \leq q$ and $|\im(x)| \leq q\}$ only finite number of zeros of $f$ may be and the consideration reduces to pairs of spheres if there exists a zero $z_0 \in V_q$ of $f$. If $z_3 \in C$ is a zero of $f$, then there exists an open neighborhood $W$ of $z_3$ such that $W$ can intersect no more, than a finite family of circles $S(C, v_j, |\im(z_0)|)$, where $v_0 = Re(z_0), v_1 = -v_0, j = 0$ or $j = 1, z_0$ is a zero of $f|C$ different from four zeros corresponding to $z_0, z_0 \notin \{z_2, z_3, z_4, z_5\}$. Therefore, the claimed loop $\gamma$ exists for each complex zero $z_0$ of $f$.

For each zero $z_2 \in S(C, v_0, |\im(z_0)|) \cup S(C, -v_0, |\im(z_0)|) \setminus Z$ if it exists and for the marked $z_0$ of $f$ the symmetry imposed on $f$ leads to the contradiction, when $Re(z_0) \neq 0$. To demonstrate this denote the family of such zeros $z_2$ and $z_0$ of $f$ by $Y$, $Y := \{z \in S(C, v_0, |\im(z_0)|) \cup S(C, -v_0, |\im(z_0)|): Re(z) \neq 0, f(z) = 0\}$. For $z \in Y$ let $k(z) \in N$ denotes its order. Then $Y$ is a finite set, since $f$ is the nontrivial integral function on $C$. If $z \in Y$, then $-z, \bar{z}, -\bar{z} \in Y$ also and $k(z) = k(-z) = k(\bar{z}) = k(-\bar{z})$. Then $p = p_0 + p_Y$, where $p_Y \geq 1$ and $p_0 \geq 0$ (if $p_0 \geq 1$ when $Z \neq \emptyset$) correspond to the sets $Y$ and $Z$ respectively.

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Applying Theorems 16 and 16.1 for \( r = 1 \) and \( b = 2 \) we get for \( \gamma \) that
\[
(3) \Delta_\gamma, \operatorname{Arg}_2 f = \Delta_\gamma, \operatorname{Arg}_2 f + \Delta_\gamma, \operatorname{Arg}_2 f = 0,
\]
since \( \Delta_\gamma, \operatorname{Arg}_2 f \) and \( \Delta_\gamma, \operatorname{Arg}_2 f \) are independent from \( a_1 = K \) and \( a_1 = -K \) respectively, but \( N_0 = -N_1 \). On the other hand,
\[
(4) \int_0^\infty \ln_2(\pi K_0/2, 1; f(z)) = \int_0^\infty \ln_2(\pi K_1/2, 1; f(z)),
\]
since \( \int_0^\infty \ln_2(\pi K_j/2, 1; f(z)) = \int_0^\infty \ln_2(\pi K_j/2, 1; f(g_j(t))) \) and the Equality (2) is satisfied. Therefore, the application of Theorems 16 and 16.1 to \( \gamma_0 \) only due to Equation (4) gives
\[
(5) |\Delta_\gamma, \operatorname{Arg}_2 f| = k\pi,
\]
where \( k = k_0 + k_\gamma \), \( k_0 \geq 0 \) corresponds to zeros from \( Z \), while \( k_\gamma = \sum_{z\in Y} k(z) \geq 4 \) corresponds to zeros of \( f \) from \( Y \). This gives the contradiction of Equation (3) with (5), consequently, all complex zeros of \( f \) may lie only on the line \( \Re(z) = 0 \).

### 17.1. Remarks

Examples of quasi-regular and quasi-integral functions can be provided with the help of Proposition 9.1, Corollary 9.2 and Theorem 9.4 and Section 9.5. Certainly each pseudo-conformal function on \( U \setminus A \) or \( A_b \setminus A \) besides a set \( A \) consisting of isolated points of zeros of its derivative \( A := \{ z \in U : f'(z) = 0 \} \) is at the same time quasi-regular on \( U \) or quasi-integral on \( A_b \) respectively (see about pseudo-conformal functions in [23, 24]).

On the other hand, if \( a > 0, q > 0 \), then put \( P(x) = (x - a - qi)(\bar{x} - a + qi)(x + a - qi)(\bar{x} + a + qi) \) for \( x \in \mathbb{C} \). The polynomial \( P \) satisfies the necessary symmetry properties on \( \mathbb{C} \), but it has not a quasi-regular extension on \( U \) open in \( A_b \) with \( W = U \cap \mathbb{C} \) open in \( \mathbb{C} \) for \( 2 \leq b \leq 3 \), since the left and the right sides of Formula (Q7) for \( \tilde{R}_{z,y} P \) differ on terms such as
\[
-q^2(Mv + vM)(z + a - qM)(\bar{z} + a + qM) - (z - a - qM)(\bar{z} - a + qM)q^2(Mv + vM) + q(zv - v\bar{z})(z + a - qM)(\bar{z} + a + qM) + q(z - a - qM)(\bar{z} - a + qM)(zv - v\bar{z}),
\]
where \( z \in \mathbb{A}_b \setminus \mathbb{C} \), \( z - \Re(z) := \Im(z) \neq 0 \), \( M = \Im(z)/|\Im(z)| \), \( v \neq 0 \), \( v||M, v \in I_b \setminus \mathbb{C} \), \( \Re(v) = 0 \), which follows from the \( z \)-representation with the help of Formulas 2.1(1–9). Then functions of the type \((f_1 P) f_2\) also generally need not satisfy conditions of Theorem 17 even when \( f_1 \) and \( f_2 \) are quasi-integral. Therefore, the class of functions satisfying conditions of Theorem 17 is rather narrow. The graph \( \{(z, f(z)) : z \in \mathbb{A}_b \} \subset \mathbb{A}_b^2 \) of \( f \) satisfying conditions of Theorem 17 has the natural interpretation. On the other hand, in view of Corollary 9.2 the power \( f^n \) of \( f \) satisfies conditions of Theorem 17 for each \( n = 2, 3, 4, \ldots \) if \( f \) satisfies them.

The existence problem of functions satisfying conditions of Theorem 17 is considered in the next section together with properties of their noncommutative integral transformations of the Laplace and Mellin types.

### 18. Theorem

Let \( f(z,t) \) be an \( A_b \) valued function on \( W := U \times [a, \infty) \) and there exists \((\partial f(z,t)/\partial z).h \) continuous on \( W \times B(A_b, 0, 1) \), where \( U = \{ z \in A_b : z_j \in [a_j, b_j], j = 0, 1, \ldots, 2^b - 1 \} \), \( z = z_0i_0 + \ldots + z_{2^b-1}i_{2^b-1} \), \( a_j < b_j \), \( B(A_b, y, R) := \{ z \in A_b : |z - y| \leq R \} \). Let \( x \in U \) be such that \( F(x) := f^\infty_a f(x, t)dt \) converges, while the improper integral depending on the parameter \( z \in U : G(z,h) := f^\infty_a (\partial f(z,t)/\partial z).h dt \) converges uniformly on \( U \times B(A_b, 0, 1) \). Then the improper integral \( F(z) := f^\infty_a f(z,t)dt \) depending on the parameter \( z \in U \) converges uniformly on \( U \) and for each \( z \in U \) there exists \( D_z F(z).h = f^\infty_a (\partial f(z,t)/\partial z).h dt = G(z,h) \) for each \( h \in B(A_b, 0, 1) \).

**Proof.** Write \((\partial f(z,t)/\partial z).h \) in the form \((\partial f(z,t)/\partial z).h = g_0i_0 + \ldots + g_{2^b-1}i_{2^b-1} \), where \( g : W \times A_b \to R \). Since \((\partial f(z,t)/\partial z).h \) is continuous on \( W \times B(A_b, 0, 1) \) and the improper integral \( G(z,h) := f^\infty_a (\partial f(z,t)/\partial z).h dt \) converges uniformly on \( U \times B(A_b, 0, 1) \), then
\[
f^\infty_a G(y,h)dy = \int^\infty_0 (f^\infty_a (\partial f(y,t)/\partial y).h)dt \]
and the improper integral on the right converges uniformly on \( U \times B(A_b, 0, 1) \), since \( t \) is the real parameter and \( R \) is the center of \( A_b \). Take \( h = w, |w| = 1 \), such that \( z = x + vw \), where
Then extend this operator by the formula $\int_{\gamma} G(y,w)\,dy$. Using the additivity of the integral we have $F(z) - F(\eta) = \int_{\gamma} G(y,h)\,dy$ for each $z, \eta \in U$, where $z - \eta = vh, v > 0$. Thus $\partial z(v)/\partial v = h$ and $\int_{\gamma} G(y,h)\,dy = \int_{\gamma} G(\eta + qh,h)\,dq$, where $q \in [0,v]$. Therefore, $D_{\gamma} F(z,h) = G(z,h)$ for each $z \in U$ and $h \in A_b$. Consequently, the improper integral $\int_{\gamma} G(y,h)\,dy$, where $\gamma$ is such branch of $L_{n-1}$. Then extend this operator by the formula $\text{res}_n(z_0;f)M := [\text{res}_n(z_0;f).M]\{M/|M|\}$ for $\gamma \in 0 \in I_b, \text{res}_n(z_0;f).0 := 0$.

For $n = 1$ this coincides with the usual definition of $\text{res}(z_0; f)$ [17] [18] [19].

20. Proposition. Let $g$ be a pseudo-conformal function in $V \setminus \{z_0\}$, where $V$ is an open ball with the center $z_0$ in $A_r$, $z_0 \in A_r$, $1 \leq r \leq 2$. Suppose that a $(r,b)$-quasi-conformal function $f$ in $U \setminus \{z_0\}$ is obtained from $g$ in accordance with Conditions $(Q1-Q7)$, where $U$ is an open ball with the center $z_0$ in $A_b$, $r < b < 3, 2 \leq n < 2^b$. Then $\text{res}_n(z_0;f)M = \text{res}(z_0;g)M$ for each $M \in I_r$.

Proof. Take $M \in I_b$ with $|M| = 1$. For the fixed branch of the logarithmic function consider the integral $\int_{\gamma} f(z_0 + L_{n-1}(a_1, \ldots, a_{n-1}; (z - z_0)))\,dL_{n-1}(a_1, \ldots, a_{n-1}; (z - z_0))$, where $\gamma_n(t) = z_0 + \rho \text{Exp}_n(a_1, \ldots, a_{n-1}; 2\pi M t)$, $M \in I_b, |M| = 1, 2 \leq b \leq 3, 0 < \rho < \infty, 0 \leq t \leq 1; 1 \leq a < 2^b, \text{Exp}_n(z) := id(z) = z, L_{n-1}(z) := id(z) = z, \text{Exp}_n$ and $L_{n-1}$ are given by Definition 10; $a_1, \ldots, a_{n-1} \in A_b$, $a_1 \neq 0, \ldots, a_{n-1} \neq 0, \gamma([0,1]) \subset U$, where $L_{n-1}(a_1, \ldots, a_{n-1}; \text{Exp}_n(a_1, \ldots, a_{n-1}; z)) = z$ is such branch of $L_{n-1}$. For each $t \in R$, when $a_1 \neq 0, \ldots, a_{n-1} \neq 0$. Choose $\rho > 0$ sufficiently small such that $\gamma([0,1]) \subset V$. Therefore, $\int_{\gamma} f(z_0 + L_{n-1}(a_1, \ldots, a_{n-1}; (z - z_0)))\,dL_{n-1}(a_1, \ldots, a_{n-1}; (z - z_0)) = f(z)\,dz$, where $\gamma(t) := z_0 + \rho \text{Exp}(Mt)$ for each $t \in R$. We have that $\gamma([0,1]) \subset A_r$ and $\text{res}_n(z_0;f).0 = 0$ and $\text{res}_n(z_0;g)0 = 0$. Since for $M \in I_r$ and such $\gamma$ there is the equality:

$$\int_{\gamma} f(z)\,dz = \int_{\gamma} g(z)\,dz$$

$$\int_{\gamma} g(z)\,dz = \int_{\gamma} f(z_0 + L_{n-1}(a_1, \ldots, a_{n-1}; (z - z_0)))\,dL_{n-1}(a_1, \ldots, a_{n-1}; (z - z_0))$$

consequently, $\text{res}_n(z_0;f)z_0 = \text{res}(z_0;g)z_0$, since $\gamma([0,1])$ is the loop gone around once and $\text{res}_n(z_0;f)M := [\text{res}_n(z_0;f).M]\{M/|M|\}$ for each $M \neq 0 \in I_b$.

Mention that $\text{res}_n(z_0;f)$ is independent of $a_1 \neq 0, \ldots, a_{n-1} \neq 0 \in A_b, \text{Ln}_{n-1}(a_1, \ldots, a_{n-1}; \text{Exp}_n(a_1, \ldots, a_{n-1}; z)) = z$ for each $z \in A_b$ for a branch of $L_{n}$ such that $L_{n}(1) = 0$. Using the homotopy theorem for $A_b$ line integrals we can consider more general $U$ than balls (see Theorems 2.11 and 3.9 in [17] [18] [19]).

3 Noncommutative integral transformations over $H$ and $O$.

1. Definitions. A function $f : R \rightarrow A_b$ we call function-original, where $A_b$ is the Cayley-Dickson algebra, which may be, in particular, $A_2 = H$ over the quaternion skew field or $A_3 = O$ the octonion algebra, if it satisfies the following conditions (1 - 3):

(1) $f(t)$ satisfies the Hölder condition: $|f(t + h) - f(t)| \leq A|h|^\alpha$ for each $|h| < \delta$ (where
$0 < \alpha \leq 1$, $A = \text{const} > 0$, $\delta > 0$ are constants for a given $t$) everywhere on $\mathbb{R}$ may be besides points of discontinuity of the first type. On each finite interval in $\mathbb{R}$ the function $f$ may have only the finite number of points of discontinuity of the first kind. Remind, that a point $t_0$ is called the point of discontinuity of the first type, if there exist finite left and right limits $\lim_{t \to t_0^{-}} f(t) =: f(t_0 - 0) \in \mathcal{A}_b$ and $\lim_{t \to t_0^{+}} f(t) =: f(t_0 + 0) \in \mathcal{A}_b$.

(2) $f(t) = 0$ for each $t < 0$.

(3) $f(t)$ increases not faster, than the exponential function, that is there exist constants $C = \text{const} > 0$, $s_0 = s_0(f) \in \mathbb{R}$ such that $|f(t)| < C \exp(s_0t)$ for each $t \in \mathbb{R}$.

If there exists an original

(4) $F(p; q) := F(p) := \int_{0}^{\infty} f(t)e^{-u(p,t;q)}dt$,

then $F(p)$ is called the Laplace transformation at a point $p \in \mathcal{A}_b$ of the function-originnal $f(t)$, where either $u(p,t; q) = pt + q$ or $u(p, t; q) = E(pt + q)$ (see Definition 2.2.1), $p \in \mathcal{A}_b$, $q \in \mathcal{A}_b$ is a parameter. It is supposed that the integral for $F(p; q)$ is written in the $(p, q)$-representation with the help of Formulas 2.2. (2 - 5). For $q = 0$ it can be omitted from $u$ putting $u(p, t; 0) = u(p, t)$. If $q$ is specified, then it can also be written shortly $F(p)$ instead of $F(p; q)$.

It can be taken the automorphism of the algebra $\mathcal{A}_b$ and instead of the standard generators $\{i_0, ..., i_{2^k-1}\}$ use new generators $\{N_0, ..., N_{2^k-1}\}$. Provide also $u(p, t; q) = u_N(p, t; q)$ relative to a new basic generators, where $2 \leq b \leq 3$. In this more general case an image we denote by $\mathcal{N}F(p)$ for the original $f(t)$ or in more details we denote it by $\mathcal{N}F_u(p; q)$ or $\mathcal{N}F_u(p; q)$.

Let $\gamma : (-\infty, \infty) \to \mathcal{A}_b$ be a path such that the restriction $\gamma_l := \gamma|_{[-l,l]}$ is rectifiable for each $0 < l \in \mathbb{R}$ and put by the definition

(5) $\int_{\gamma} f(z)dz = \lim_{l \to \infty} \int_{\gamma_{l}} f(z)dz$,

where $\mathcal{A}_b$ integrals by rectifiable paths were defined in §2.5 [17, 18, 19]. So they are defined along curves also, which may be classes of equivalence of paths relative to increasing piecewise smooth mappings $\tau : [a, b] \to [a_1, b_1]$ realizing reparametrization of paths. Then we shall talk, that an improper integral (5) converges.

Consider now a function $f(z, y)$ defined for all $z$ from a domain $U$ and for each $y$ in a neighborhood $V$ of a curve $\gamma$ in $\mathcal{A}_b$. The integral $G(z) := \int_{\gamma} f(z, y)dy$ converges uniformly in a domain $U$, if for each $\epsilon > 0$ there exists $l_0 > 0$ such that

(6) $|\int_{\gamma} f(z, y)dy - \int_{\gamma_{l}} f(z, y)dy| < \epsilon$ for each $z \in U$ and $l > l_0$. Analogously is considered the case of unbounded $\gamma$ in one side with $[0, \infty)$ instead of $(-\infty, \infty)$.

2. Theorem. Let $V$ be a bounded neighborhood of a rectifiable curve $\gamma$ in $\mathcal{A}_r$, and a sequence of functions $f_n : V \to \mathcal{A}_r$ be uniformly convergent on $V$, where $2 \leq r < \infty$, then there exist the limit

(i) $\lim_{n \to \infty} \int_{\gamma} f_n(z)dz = \int_{\gamma} \lim_{n \to \infty} f_n(z)dz$.

Proof. For a given $\epsilon > 0$ in view of the uniform convergence of the sequence $f_n$ there exists $n_0 \in \mathbb{N}$ such that $|f_n(z) - f(z)| < \epsilon/l$ for each $n > n_0$, where $0 < l < \infty$ is the length of the rectifiable curve $\gamma$, $f(z) := \lim_{n \to \infty} f_n(z)$. In view of the Inequality 2.7(4) [17, 18, 19] there are only two positive constants $C_1 > 0$ and $C_2 > 0$ such that $|\int_{\gamma} f(z)dz - \int_{\gamma} f_n(z)dz| < (\epsilon/l)C_1 \exp(C_2R^2) = \epsilon C_1 \exp(C_2R^2)$, where $s = 2^2 + 2$, $0 < R < \infty$, such that $V$ is contained in the ball $B(\mathcal{A}_r, \gamma_{l_0}, 0)$ in $\mathcal{A}_r$ of the radius $R$ with the center at some point $z_0 \in \mathbb{K}$. This means the validity of Equality (i).

3. Theorem. If a function $f(z, y)$ holomorphic by $z$ is piecewise continuous by $y$ for each $z$ from a simply connected (open) domain $U$ in $\mathcal{A}_r$, with $2 \leq r < \infty$ and for each $y$ from a neighborhood $V$ of the path $\gamma$, where $\gamma_l$ is rectifiable for each $0 < l < \infty$, and the integral $G(z) := \int_{\gamma} f(z, y)dy$ converges uniformly in the domain $U$, then it is the holomorphic function in $U$. 24
Proof. For each $0 < l < \infty$ the function $f_{\gamma l} f(z, y)dy =: G_l(z)$ is continuous by $z$ in view of Theorem 2, that together with (1(6)) in view of the triangle inequality gives the continuous function $G(z)$ on $U$. In view of Theorems 2.16 and 3.10 [17, 18, 19] the integral holomorphy of the function $G(z)$ implies its holomorphy. But the integral holomorphy is sufficient to establish in the interior $\text{Int}(B(\mathcal{A}_r, z_0, R))$ of each ball $B(\mathcal{A}_r, z_0, R)$ contained in $U$. Let $\psi$ be a rectifiable path such that $\psi \subset \text{Int}(B(\mathcal{A}_r, z_0, R))$. Therefore, $f_\psi G(z)dz = f_\psi(f, f(z, y)dy)dz$. With the help of the proof of Theorem 2.7 [17, 18, 19] these integrals can be rewritten in the real coordinates and with the generators $A_n$ contained in $\mathcal{G}$ of the uniform convergence of integrals given above $h \in \mathbb{R}$ since $\text{Proposition 3.2 [17, 18, 19]}$, where $|p| = \text{Re}G(z)$ and the indicator of the growth of $f(t)$ is taken along the straight line $l$. On the other hand, along the straight line $l$ the function $G(z)$ is holomorphic by $\text{Definition 1}$, such that $\text{Proposition 3.2 [17, 18, 19]}$, where $|z|^2 = \text{Re}G(z)$, $\text{Im}G(z)$ in view of Corollary 3.3 [17, 18, 19], where $|z|^2 \in [0, \infty) \subset \mathbb{R}$, $M \in \mathcal{A}_r$, $\text{Re}(M) := (M + M)/2 = 0$. Therefore, 
$$[\partial(f_\psi f(t) \exp(-u(p, t; q))dt)/\partial\tilde{p}], h = 0$$
for each $h \in \mathcal{A}_r$, since $u(p, t; q)$ is written in the $(p, q)$-representation. In view of convergence of integrals given above $F(p)$ is (super)differentiable by $p$, moreover, $\partial F(p)/\partial \tilde{p} = 0$ in the considered p-representation, consequently, $F(p)$ is holomorphic by $p \in \mathcal{A}_b$ with $\text{Re}(p) > s_0$ due to Theorem 3.

5. Theorem. If a function $f(t)$ is an original (see Definition 1), such that $N \mathcal{F}_u(p; q) := \sum_{j=0}^{2^r-1} N_{\mathcal{F}_u,j}(p; q)N_j$ is its image, where the function $f$ is written in the form $f(t) = \sum_{j=0}^{2^r-1} f_j(t)N_j$, $f_j : \mathbb{R} \to \mathbb{R}$ for each $j = 0, 1, ..., 2^r - 1$, $f(\mathbb{R}) \subset \mathcal{A}_r$, $2 \leq r \leq 3$. Then at each point $t$, where $f(t)$ satisfies the Hölder condition there is accomplished the equality:

$$f(t) = (2\pi N_1)^{-1} \text{Re}(S_N) \sum_{j=0}^{2^r-1} (f_{a-j}^{<\infty} N_{\mathcal{F}_u,j}(p; q) \exp(u(p, t; q))dp)N_j,$$

where either $u(p, t; q) = pt + q_0$ with $S = N_1$, and $\text{Im}(q) = 0$, or $u(p, t; q) = E(pt + q)$, the integral is taken along the straight line $p(\tau) = a + S\tau \in \mathcal{A}_r$, $\tau \in \mathbb{R}$, $S \in \mathcal{A}_r$, $\text{Re}(S) = 0$, $|S| = 1$, $\text{Re}(S_N) \neq 0$ is non-zero, $\text{Re}(p) = a > s_0$ and the integral is understood in the sense of the principal value.

Proof. In view of the decomposition of a function $f$ in the form $f(t) = \sum_{j=0}^{2^r-1} f_j(t)N_j$ it is sufficient to consider the inverse transformation of the real valued function $f_j$, which we denote for simplicity by $f$. Since $t \in \mathbb{R}$, then $f_\infty f(t)dt$ is the Riemann integral. If $w$ is a holomorphic function of the Cayley-Dickson variable, then locally in a simply connected domain $U$ in each ball $B(\mathcal{A}_r, z_0, R)$ with the center at $z_0$ of radius $R > 0$ contained in the interior $\text{Int}(U)$ of the domain $U$ there is accomplished the equality

$$\partial f_\infty w(\zeta)d\zeta/\partial \zeta = w(z),$$

where the integral depends only on an initial $z_0$ and a final $z$ points of a rectifiable path in $B(\mathcal{A}_r, z_0, R)$. On the other hand, along the straight line $a + SR$ the restriction of the antiderivative has the form $f_{\theta_0} w(a + S\tau) d\tau$, since

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\[ f_x=a+\bar{S} \quad \text{where} \quad \partial f(z)/\partial \theta = (\partial f(z)/\partial z).S \text{ for super-differentiable by } z \in U \text{ function } f(z), \text{ moreover, the antiderivative is unique up to a constant from } \mathcal{A}_r \text{ with the given representation of the function and the branch of the noncommutative line integral (for example, specified with the help of the left or right algorithm) [17, 18, 19].} 

The integral \[ g_B(t) := f^{a+\bar{S}}_{a-\bar{S}} \quad \mathcal{F}_u(p; q) \exp(\mu(p; t; q))d\mu \text{ for each } 0 < B < \infty \text{ with the help of generators of the algebra } \mathcal{A}_r \text{ and the Fubini Theorem for real valued components of the function can be written in the form:} \]

\[ g_B(t) = (2\pi N_1)^{-1} \mathbb{R}(S N_1) f_0^{\infty} f(\tau) d\tau f^{a+\bar{S}}_{a-\bar{S}} \exp(\mu(p; t; q)) \exp(-\mu(p; \tau; q))d\mu, \]

since the integral \[ \int_0^{\infty} f(\tau) \exp(-u(p; \tau; q))d\tau \text{ is uniformly converging relative to } p \text{ in the half space } \mathbb{R}(p) > s_0 \text{ in } \mathcal{A}_r \text{ (see also Proposition 2.18 [26, 27]). In view of the alternativity of the algebra } \mathcal{A}_r \text{ use the automorphism } v \text{ from Lemma 2.17 [26, 27]. This gives the change of the basis of generators, hence instead of } \mathcal{F}_u(p; q) \text{ consider } \mathcal{F}_u(p; q) \text{, where } K_j = v(N_j) \text{ is the new basis of generators of } \mathcal{A}_r, j = 1, ..., 2^r - 1, N_0 = K_0 = 1. \text{ Then with such } v \text{ the function } u_k(p; t; q) = v(E(pt + q)) \text{ has the form given by the formulas:} \]

1. \( v(u_N(p; t; q)) = u_K(p; t; q) = (p_0 t + q_0) + (p_1 t + q_1) K, \)
2. \( K = [K_1 \cos(q_2') + K_2 \sin(q_3') \cos(q_4') + K_3 \sin(q_2') \sin(q_3')] \text{ for quaternionics;} \)
3. \( K = [K_1 \cos(q_2') + K_2 \sin(q_3') \cos(q_4') + ... + K_n \sin(q_2') \sin(q_3')] \text{ for octonionics, where } p_{0}, p_1, q_0, q_1, ..., q_{2r-1} \in \mathbb{R}, r \in \mathbb{R}, K_1, ..., K_{2r-1} \in \mathcal{A}_r \text{ are new generators with } \mathbb{R}(K_j) = 0 \text{ for each } j = 1, ..., 2^r - 1, \)

where \( K_0 = N_0 = 1, p_0 = p_0, q_0 = q_0, p = p_0 N_0 + p_1 N_1 + ... + p_{2r-1} N_{2r-1} = p_0 K_0 + p_1 K_1 + ... + p_{2r-1} K_{2r-1}, q = q_0 N_0 + q_1 N_1 + ... + q_{2r-1} N_{2r-1} = q_0 K_0 + q_1 K_1 + ... + q_{2r-1} K_{2r-1}, \) and, consequently, \( g(t) = t \text{ for each } t \in \mathbb{R}. \text{ Formula } (i) \text{ is satisfied if and only if it is accomplished after application of the automorphism } v \text{ to both parts of the Equality, since } v(z) = v(\zeta) \text{ for } z, \zeta \in \mathcal{A}_r \text{ is equivalent to that } z = \zeta. \)

Then up to an automorphism of the algebra \( \mathcal{A}_r \), the proof reduces to the case \( p = (p_0, p_1, 0, ..., 0), N = (N_0, N_1, N_2, ..., N_{2r-1}), \text{ where } N_0 = 1, \text{ since } \mathbb{R} \text{ is the center of the algebra } \mathcal{A}_r. \text{ But this gives } p_1 = p_1(t) = \mathbb{R}(S N_1) t \text{ for each } t \in \mathbb{R}. \text{ For } u(p; t; q) = pt + q_0 \text{ with } Im(q) = 0 \text{ take simply } S = N_1. \text{ Consider the particular case } c : = \mathbb{R}(S N_1) \neq 0, \text{ then the particular case } \mathbb{R}(S N_1) = 0 \text{ is obtained by taking the limit when } \mathbb{R}(S N_1) \neq 0 \text{ tends to zero. Thus,} \]

\[ g_B(t) = (2\pi N_1)^{-1} \int_0^{\infty} f(\tau) d\tau f^{a+\bar{S}}_{a-\bar{S}} \exp(-t + c(q_1 + t) K_1) \exp(-a t + c(q_1 + t) K_1) d\mu, \]

since \( q_0, a \in \mathbb{R}, \) where \( K_1 \) is either given by Formulas (2, 3) for \( u_N(p; t; q) = E(pt + q) \text{ or} \) \( K_1 = N_1 = S_1 \text{ can be taken for } u_N(p; t; q) = pt + q_0. \text{ Then} \]

\[ g_B(t) = (2\pi N_1)^{-1} \int_0^{\infty} f(\tau) e^{a(t - \tau)}[\sin(B(\tau) - \tau)](ct - ct)^{-1} d\tau, \]

where it can be used the substitution \( \tau - t = \zeta. \) Put \( w(t) := f(t) e^{-at}, \text{ where } w(t) = 0 \text{ for each } t < 0. \text{ Therefore,} \]

\[ g_B(t) = (\pi)^{-1} e^{at} \int_0^{\infty} d\tau [w(\zeta + t) - w(t)] \zeta^{-1} \sin(B\zeta) d\zeta, \]

The integral in the second term is known as the Euler integral: \( \int_0^{\infty} \zeta^{-1} \sin(B\zeta) d\zeta = \pi \text{ for each } B > 0, \text{ consequently, the second term is equal to } f(t). \text{ It remains to prove, that} \]

\[ \lim_{t \to \infty} \int_0^{\infty} d\tau [w(\zeta + t) - w(t)] \zeta^{-1} \sin(B\zeta) d\zeta = 0, \text{ that follows from the subsequent lemma.} \]

**6. Lemma.** If a function \( \psi(y) \) with values in the Cayley-Dickson algebra \( \mathcal{A}_r \) is integrable on the segment \([\alpha, \beta] \subset \mathbb{R}, \text{ then} \)

\[ \lim_{b \to \infty} \int_{\alpha}^{\beta} \psi(y) \sin(by) dy = 0. \]

**Proof.** If \( \psi \) is continuously differentiable on the segment \([\alpha, \beta], \text{ then the result of the integration by parts is:} \]

\[ \int_{\alpha}^{\beta} \psi(y) \sin(by) dy = -\psi(y) \cos(by) b^{-1}_y \mid_{\alpha}^{\beta} + \int_{\alpha}^{\beta} \psi'(y) \cos(by) b^{-1} dy, \]

and, consequently,
\[
\lim_{b \to \infty} \int_a^b \psi(y) \sin(by) \, dy = 0. \quad \text{If } \psi(y) \text{ is an arbitrary integrable function, then for each } \epsilon > 0 \text{ there exists a continuous differentiable function } \psi_t(y) \text{ such that } \int_a^b |\psi(y) - \psi_t(y)| \, dy < \epsilon/2. \]

Then \[
\int_a^b \psi(y) \sin(by) \, dy = \int_a^b [\psi(y) - \psi_t(y)] \sin(by) \, dy + \int_a^b \psi_t(y) \sin(by) \, dy,
\]
where \[
|\int_a^b [\psi(y) - \psi_t(y)] \sin(by) \, dy| < \epsilon/2 \text{ for each } b, \text{ since } |\sin(by)| \leq 1 \text{ and the second term tends to zero: } \lim_{b \to \infty} \int_a^b \psi_t(y) \sin(by) \, dy = 0 \text{ by the one proved above.}
\]

The final part of the Proof of Theorem 5. For a fixed \( \epsilon > 0 \) there is an equality:
\[
\lim_{B \to \infty} \int_{-\infty}^{\infty} \left[ \psi(w + t) - \psi(w) \right] y^{-1} \sin(By) \, dy = 0. \]

This theorem for the general function \( u_N(p,t;q) = E(pt + q) \) in the basis of generators \( \{ N_0, \ldots, N_{2r-1} \} \) follows also directly by the calculation of appearing integrals by real variables \( t \) and \( \tau \) using Lemma 6 with the help of integrals evaluated in [26, 27].

7. Theorem. An original \( f(t) \) with \( f(\mathbb{R}) \subset A_r \) for \( r = 2, 3 \) is completely defined by its image \( N F(p) \) up to values at points of discontinuity.

Proof. In view of Theorem 5 the value \( f(t) \) at each point \( t \) of continuity of \( f(t) \) is expressible throughout \( N F(p) \) by Formula 5(i). At the same time values of the original at points of discontinuity do not influence on the image \( N F(p) \), since on each bounded interval a number of points of discontinuity is finite.

8. Theorem. If a function \( N F_u(p) \) is analytic by the variable \( p \in A_r \) in the half space \( W := \{ p \in A_r : Re(p) > s_0 \} \), where \( 2 \leq r \leq 3 \), \( f(\mathbb{R}) \subset A_r \), either \( u(p,t) = pt \) or \( u(p,t) = E(pt) \), moreover, for each \( a > s_0 \) there exist constants \( C_a > 0 \) and \( \epsilon_a > 0 \) such that

\[
(i) \quad |N F_u(p)| \leq C_a \exp(-\epsilon_a |p|) \quad \text{for each } p \in A_r \text{ with } Re(p) \geq a, \text{ where } s_0 \text{ is fixed, the integral}
\]
\[
(ii) \quad \int_{a+\infty}^{b+\infty} N F_u(p) \, dp \quad \text{is absolutely converging, where } S \in I_r, |S| = 1. \quad \text{Then } N F_u(p) \text{ is}
\]
\[
(iii) \quad f(t) = (2\pi)^{-1} \hat{S} \int_{a+\infty}^{b+\infty} N F_u(p) \exp(u(p,t)) \, dp.
\]

Proof. The case \( u(p,t) = pt \) follows from \( u(p,t) := E(pt) \), when \( p = (p_0, p_1, 0, ..., 0) \), but the integral along the straight line \( a + St, t \in \mathbb{R} \), with such \( p \) in the basis of generators \( (N_0, ..., N_{2r-1}) \) can be obtained from the general integral by an automorphism \( v, z \mapsto v(z) \), of the algebra \( A_r \), \( 2 \leq r \leq 3 \). That is, as in the proof of Theorem 5 it is sufficient to prove the equality of the type \((iii)\) for \( K F_u(p) \) after the action of the automorphism \( v \).

Let \( Re(p) = a > s_0 \), then
\[
\int_{a+\infty}^{b+\infty} \exp(u(p,t)) \, N F_u(p) \, dp \leq \int_{a+\infty}^{b+\infty} |N F_u(a + S0)| \, dp.
\]

In view of the supposition of this theorem this integral converges uniformly relative to \( t \in \mathbb{R} \). For \( f(t) \) given by the Formula \((iii)\) for \( Re(\eta) := \eta_0 > s_0 \) and \( \eta - Re(\eta) := Im(\eta) \) parallel to \( S \), we get
\[
\int_{a+\infty}^{b+\infty} \exp(-\eta t) \, dt = (2\pi)^{-1} \hat{S} \sum_{j=0}^{2r-1} \int_{a+\infty}^{b+\infty} N F_u_j(p) \exp(u(p,t)) \, dp \exp(-\eta t) \, (dt) N_j,
\]
in which it is possible to change the order of the integration, since \( t \in \mathbb{R} \). Then
\[
\int_{a+\infty}^{b+\infty} \exp(-\eta t) \, dt = (2\pi)^{-1} \hat{S} \sum_{j=0}^{2r-1} \int_{a+\infty}^{b+\infty} \left( \int_{a+\infty}^{b+\infty} N F_u_j(p) \exp((p - \eta)t) \, dp \right) N_j,
\]
since \( e^v \in \mathbb{R} \) for each \( v \in \mathbb{R} \), \( e^{aM} e^{bM} = e^{(a+b)M} \) for each \( a, b \in \mathbb{R} \). In view of \( a < \eta_0 \) and
\[
\int_{a+\infty}^{b+\infty} e^{(a+b)M} \, dt = -(p - \eta)^{-1},
\]
then
\[ f_0^\infty f(t) \exp(-\eta t)dt = -(2\pi)^{-1} s \sum_{j=0}^{2n-1} (f_a^{a+s})_N F_{u,j}(p)(p-\eta)^{-1}dp. \]

To finish the proof it is necessary the following analog of the Jordan lemma.

9. Lemma. Let a function \( F \) of the variable \( p \) from the Cayley-Dickson algebra \( A_r \) with \( 2 \leq r \in \mathbb{N} \) satisfy Conditions (1–3):

1. the function \( F(p) \) is continuous by the variable \( p \in A_r \) in an open domain \( W \) of the half space \( \{ p \in A_r : Re(p) > s_0 \} \), moreover for each \( a > s_0 \) there exist constants \( C'_a > 0 \) and \( \epsilon_a > 0 \) such that

2. \[ |F(p)| \leq C'_a \exp(-\epsilon_a |p|) \] for each \( p \in S_{R_0}, S_R := \{ z \in A_r : |z| = R, Re(z) \geq a \}, \]

3. \[ \int_{a}^{a+s} F(p)dp \text{ is absolutely converging. Then} \]

4. \[ \lim_{n \to \infty} \int_{\gamma_n} F(p)\exp(-u(p,t,q))dp = 0 \]

for each \( t > 0 \) and each sequence of rectifiable curves \( \gamma_n \) contained in \( S_{R_0} \cap W \), moreover either \( F(p) \) is holomorphic in \( W \), which is \( (2^r-1) \)-connected open domain in \( A_r \) (see (32)), such that the projection \( \pi_{s,p,t}(W) \) is simply connected in \( sR \oplus \mathbb{R}^r \) for each \( s = i_{2k}, p = i_{2k+1}, k = 0, 1, \ldots, 2^{r-1} - 1 \) for each \( t \in A_{r,s,p} := A_r \oplus \mathbb{R} \oplus \mathbb{R} \) and \( u \in \mathbb{R} \oplus \mathbb{R} \), for which there exists \( z = t + u \in A_r \); or there exists a constant \( C'_r > 0 \) such that the variations (lengths) of curves are bounded \( V(\gamma_n) \leq C'_r R_n \) for each \( n \), where \( n \in \mathbb{N} \), either \( u(p,t,q) = pt + q \) or \( u(p,t,q) = E(pt + q) \).

Proof. If \( 0 < \epsilon \leq \min(a - s_0, \epsilon_a) \), then in view of Condition (2) there exists a constant \( C' > 0 \) such that

5. \[ |F(p)| \leq C'e^{-\epsilon|p|} \]

for each \( p \in A_r \) with \( Re(p) \geq a > s_0 \). If \( U \) is a domain in \( A_r \) of the diameter not greater than \( \rho \), then in view of (4) from the proof of Theorem 7 [17, 18, 19] there is accomplished the inequality:

\[ \sup_{p \in U} \| F(p) \| \leq \sup_{p \in U} |F(p)| C_1 \exp(C_2\rho^n), \]

where \( C_1 \) and \( C_2 \) are constants independent from \( F, n = 2^r + 2, 2 \leq r \in \mathbb{N} \). In particular, as \( U \) it is possible to take the interior of the parallelepiped with ribs of lengths not greater than \( \rho/2^{r/2} \). Then the path of integration can be covered by a finite number of such parallelepipeds. In the case of the circle of radius \( R \) a number of necessary parallelepipeds is not greater, than \( 2^{1+r/2}\pi R/\rho + 1 \). There exists \( R_0 > 0 \) such that for each \( R > R_0 \) there is accomplished the inequality \( 2^{1+r/2}\pi R/\rho < \exp(C_2\rho^{n-1}(R - \rho)) \). Therefore, in \( \rho \) neighborhood \( C'_r \) of the circle \( C'_r \) of radius \( R \) and with the center at zero with \( R > R_0 \) there is accomplished the inequality:

\[ \sup_{p \in C'_r \setminus Re(p) \geq a} \| F(p) \| \leq C'C_1 \exp((C_2\rho^{n-1} - \epsilon)R) \leq C'C_1 \exp(-\delta R) \]

in view of the condition imposed on \( F \), where \( C \) is the positive constant for the given \( F, \delta = \epsilon - C_2\rho^{n-1} > 0 \). With this the length of the path of integration does not exceed \( 2\pi R \) and \( \lim_{R \to \infty} C'C_1 2\pi R \exp(-\delta R) = 0 \). The function \( F(p) \) is continuous by \( p \), hence it is integrable along each rectifiable curve in the domain \( W \) of the half space \( \{ p \in A_r : Re(p) > s_0 \} \).

If \( F(p) \) is holomorphic, then in view of Theorem 2.11 [17, 18, 19]

\[ \int_{\gamma_n} F(p) \exp(-u(p,t,q))dp \]

is independent from the type of the curve and it is defined only by the initial and final its points. If \( V(\gamma_n) \leq C'_r R_n \) for each \( n \), then it is sufficient to prove
the statement of this Lemma for each subsequence $R_{n(k)}$ with $R_{n(k+1)} \geq R_{n(k)} + 1$ for each $k \in \mathbb{N}$. Denote for the simplicity such subsequence by $R_n$. Each rectifiable curve can be approximated by the converging sequence of rectifiable of polygonal type line composed of arcs of circles. If a curve is displayed on the sphere, then these circles can be taken with the common center with the sphere. Condition (2) in each plane $\mathbb{R} \oplus N \mathbb{R}$, where $N \in \mathcal{A}_r$, $\text{Re}(N) = 0$, $|N| = 1$, is accomplished, moreover, uniformly relative to a directrix $N$ and it can be accomplished a diffeomorphism $g$ in $\mathcal{A}_r$, such that $g(W) = W$, $g(S_{R_n}) = S_{R_n}$ for each $n \in \mathbb{N}$, and an image of a $C^1$ curve from $W$ is an arc of a circle, since $0 < R_n + 1 < R_{n+1}$ for each $n \in \mathbb{N}$ and $\lim_{n \to \infty} R_n = \infty$.

The function $(F, \gamma) \mapsto \int_S F(p) dp$ is continuous from $C^0(V, \mathcal{A}_r) \times \Gamma$ into $\mathcal{A}_r$, where $V$ is the compact domain in $\mathcal{A}_r$, $\Gamma$ is the family of rectifiable curves in $V$ with the metric $\rho(v, w) := \max(\sup_{z \in v} \inf_{\zeta \in w} |z - \zeta|, \sup_{z \in w} \inf_{\zeta \in v} |z - \zeta|)$ (see Theorem 2.7 [17] [18] [19]). The space $C^1$ of all continuously differentiable functions of the real variable is dense in the space of continuous functions $C^0$ in the compact-open topology in the case of a finite number of variables. In addition a rectifiable curve is an uniform limit of $C^1$ curves, since each rectifiable curve is continuous. Therefore, consider $\gamma_n = \psi_n \cap \{p \in \mathcal{A}_r : \text{Re}(p) > a\} \cap W$, where $\psi_n$ is a curve in $S_{R_n}$ corresponding to $\gamma_n$. Consequently,$$
 \lim_{n \to \infty} \int_S F(p) \exp(-u(p, t; q)) dp = 0,$$
 since this is accomplished for $\gamma_n = \pi_n \cap C_{R_n}$ and hence for general $\gamma_n$ with the same initial and final points, where $\pi_n$ are two dimensional over $\mathbb{R}$ planes in $\mathcal{A}_r$.

The continuation of the Proof of Theorem 8. In view of Lemma 9
\[
|\int_{\psi_R} F(p) (\rho - \eta)^{-1} dp| \leq u(R) \pi R /(R - |\eta|),
\]
where $0 < u(R) < \infty$ and there exists $\lim_{R \to \infty} u(R) = 0$, while $\psi_R$ is the arc of the circle $|p| = R$ in the plane $\mathbb{R} \oplus S \mathbb{R}$ with $\text{Re}(p) > a$, consequently,$$
\lim_{R \to \infty} \int_{\psi_R} F(p) (\rho - \eta)^{-1} dp = 0,
\]
since $u(R) \leq u_0 \exp(-\delta R)$ for each $R > R_0$, where $u_0 = \text{const} > 0$.

Then the straight line $a + S \theta$ with $\theta \in \mathbb{R}$ can be substituted by the closed contour $\phi_R$ composed from $\psi_R$ and the segment $[a + Sb, a - Sb]$ passed from the above to the bottom. Thus,$$
\int_0^\infty f(t) \exp(-\eta t) dt = (2\pi)^{-1} \tilde{S} \int_{\phi_R} F(p) (\rho - \eta)^{-1} dp,
\]
where the sign in front of the integral is changed due to the change of the pass direction of the loop $\phi_R$. Recall, that in the case of the Cayley-Dickson algebra $\mathcal{A}_r$ the residue of a function is the operator $\mathbb{R}$-homogeneous and $\mathcal{A}_r$-additive by the argument $L \in \mathcal{A}_r$ with $\text{Re}(L) = 0$, where the residue is naturally dependent on a function and a point. In the domain $\{p \in \mathcal{A}_r : \text{Re}(p) \geq a, |p| \leq R\}$ the analytic function $F(p)$ has only one point of singularity $p = \eta$, which is the pole of the first order with the residue $\text{res}(\eta; (p - \eta)^{-1}F(p)).L = LF(\eta)$ for each $L \in \mathcal{A}_r$, with $\text{Re}(L) = 0$, consequently,$$
\int_0^\infty f(t) \exp(-\eta t) dt = F(\eta),\text{ since } L = S \text{ in the given case and } \tilde{S} \tilde{S} = 1.
\]

For $t < 0$ in view of the aforementioned $\mathcal{A}_r$. Lemma 9 we get, that
\[
\lim_{R \to \infty} \int_{\phi_R} F(p) e^{u(p, t)} dp = 0,
\]
where $F(p)$ is analytic by $p$ together with $e^{u(p, t)}$ in the interior of the domain $\{p : p \in \mathcal{A}_r, |p| \leq R', \text{Re}(p) > \tau_0\}, a > \tau_0, 0 < R < R' \leq \infty$. Then the condition 2 for the original is accomplished. On the other hand,$$
|f(t)| \leq (2\pi)^{-1} \int_{\phi_R} F(p) e^{u(p, t)} dp = C e^{\eta t},
\]
where $C = (2\pi)^{-1} \int_{-\infty}^{\infty} |F(a + S\theta)| d\theta = Ce^{\eta t}$, consequently, Condition (3) is satisfied. As
well as \( f(t) \) is continuous, since the function \( F(p) \) in the integral is continuous satisfying Conditions (i, ii) and
\[
\lim_{R \to \infty} \int_{|s| \geq R} F(p) dp = 0.
\]
Moreover, the integral
\[
f_{a+S\infty} N F(p)(\partial \exp(u(p, t))/\partial t) dp
\]
converges due to Conditions (i, ii) and the proof above, consequently, the function \( f(t) \) is differentiable and hence satisfies the Lipschitz condition.

10. Note. In Theorem 8 Condition (i) can be replaced on
\[
\lim_{n \to \infty} \sup_{p \in S_{R_n}} \|\hat{F}(p)\| = 0
\]
by the sequence \( S_{R_n} := \{ z \in \mathcal{A}_r : |z| = R_n, \text{Re}(z) > s_0 \} \), where \( R_n < R_{n+1} \) for each \( n \), \( \lim_{n \to \infty} R_n = \infty \), since this leads to the accomplishment of the \( \mathcal{A}_r \) analog of the Jordan Lemma for each \( r \geq 2 \) (see also Note 36 [26, 27]). But in Theorem 8 itself it is essential the alterativity of the algebra, that is, in it in general are possible only \( r = 2 \) or \( r = 3 \) for \( f(\mathbb{R}) \subset \mathcal{A}_r \).

11. Definition. Consider function-originals, satisfying conditions (1 – 3) below:

1. \( f(t) \) satisfies the Hölder condition: \( |f(t + h) - f(t)| \leq A|h|^\alpha \) for each \( |h| < \delta \) (where \( 0 < \alpha \leq 1, A = \text{const} > 0, \delta > 0 \) are constants for a given \( t \)) everywhere on \( \mathbb{R} \) may be besides points of discontinuity of the first kind. On each finite interval in \( \mathbb{R} \) a function \( f \) may have only a finite number of points of discontinuity and of the first kind only.

2. \( |f(t)| < C_1 \exp(-s_1 t) \) for each \( t > 0, \) where \( C_1 = \text{const} > 0, s_1 = s_1(f) = \text{const} \in \mathbb{R} \).

3. \( |f(t)| < C_2 \exp(s_0 t) \) for each \( t \geq 0, \) that is, \( f(t) \) is growing not faster, than the exponential function, where \( C_2 = \text{const} > 0, s_0 = s_0(f) \in \mathbb{R} \).

The two-sided Laplace transformation over the Cayley-Dickson algebras \( \mathcal{A}_r \) with \( 2 \leq r \leq 3 \) is defined by the formula:

\[
F^*(f; p) := \int_{-\infty}^{\infty} f(t) \exp(-u(p, t; q)) dt
\]
for all numbers \( p, q \) in \( \mathcal{A}_r \), for which the integral exists, where \( q \in \mathcal{A}_r \) is a parameter, either \( u(p, t; q) = pt + q \) or \( u(p, t; q) = E(pt + q) \) (see Definition 2.2.1). Denote for short \( F^*(f; p) \) through \( F^*(p) \). For a basis of generators \( \{ N_0, \ldots, N_{2r-1} \} \) in \( \mathcal{A}_r \) we shall write in more details \( N \mathcal{F}^*(f; p) \) or \( N F^*(p) \) in the case of necessity.

12. Note. Naturally, that the two-sided Laplace integral can be considered as the sum of two one-sided integrals

\[
(1) \int_{-\infty}^{\infty} f(t) \exp(-u(p, t; q)) dt = \int_{-\infty}^{0} f(t) \exp(-u(p, t; q)) dt + \int_{0}^{\infty} f(t) \exp(-u(p, t; q)) dt
\]
\[
(2) \int_{-\infty}^{\infty} f(-t) \exp(-u(p, -t; q)) dt + \int_{0}^{\infty} f(-t) \exp(-u(p, -t; q)) dt.
\]

The second integral converges for \( \text{Re}(p) > s_0 \). Since \( u(p, -t; q) = u(-p, t; q) \), then the first integral converges for \( \text{Re}(-p) > -s_1 \), that is, for \( \text{Re}(p) < s_1 \). Then there is a region of convergence \( s_0 < \text{Re}(p) < s_1 \) of the two-sided Laplace integral. For \( s_1 = s_0 \) the region of convergence reduces to the vertical hyperplane in \( \mathcal{A}_r \) over \( \mathbb{R} \). For \( s_1 < s_0 \) there is no any common domain of convergence and \( f(t) \) can not be transformed with the help of the two-sided transformation 1(4).

13. Example. \( F^*(\exp(-\alpha t^2); p) = \int_{-\infty}^{\infty} \exp(-\alpha t^2 - pt) dt = (\pi/\alpha)^{1/2} \exp(p^2/(4\alpha)) \), where \( \alpha > 0 \), since \( \int_{-\infty}^{\infty} \exp(-t^2) dt = (\pi)^{1/2} \). For comparison the one-sided Laplace transformation gives:

\[
\mathcal{F}(\exp(-\alpha t^2) Ch_{[0, \infty]}; p) = \int_{0}^{\infty} \exp(-\alpha t^2 - pt) dt
\]
\[
= (\alpha)^{-1/2} \exp(p^2/(4\alpha)) \int_{p/(2\alpha)^{1/2}}^{\infty} \exp(-t^2) dt
\]
\[
= 2^{-1}(\pi/\alpha)^{1/2} \exp(p^2/(4\alpha))Erf(p/(2\alpha^{1/2}))
\]
(see also [26, 27], where numerous examples of calculations of noncommutative Laplace transformations of such and more general type over \( \mathcal{H} \) and \( \mathcal{O} \) and their applications to super-differential equation were given).

The application of Theorem 4 to \( \int_{0}^{\infty} f(-t) \exp(-u(-p, t; q)) dt \) and
\[ f_0^\infty f(t) \exp(-u(p; t; q))dt \] gives.

14. Theorem. If an original \( f(t) \) satisfies Conditions 11(1–3), and moreover, \( s_0 < s_1 \), then its image \( \mathcal{F}^s(f; p) \) is holomorphic by \( p \) in the domain \( \{ z \in \mathcal{A}_r : s_0 < \Re(z) < s_1 \} \), where \( 2 \leq r \leq 3 \).

15. Examples. 1. There may be cases, when a domain of convergence for a sum is greater, then for each additive. For example, \( \mathcal{F}^s((\exp(at) - 1)U(t); p) = (p - a)^{-1} \), also \( \mathcal{F}^s(\exp(at)U(t); p) = ap^{-1}(p - a)^{-1} \) for \( \Re(p) > a \) in both cases, when \( a \in \mathbb{R}, U(t) := 1 \) for \( t > 0, U(0) = 1/2 \), while \( U(t) = 0 \) for \( t < 0 \). But \( (p - a)^{-1} - ap^{-1}(p - a)^{-1} = p^{-1} \) and \( \mathcal{F}^s(U(t); p) = p^{-1} \) for each \( \Re(p) > 0 \).

It is necessary to note, that the two-sided Laplace transformation of the function \( t^n \) does not exist, but the one-sided transformation was elucidated in examples 2.30.1 and 2.33 [26, 27].

2. \( \mathcal{F}^s((e^t + 1)^{-1}; p) = f_0^\infty (e^t + 1)^{-1} \exp(-pt)dt \) in the domain \(-1 < \Re(p) < 0\). Make the substitution \( v = (e^t + 1)^{-1} \), then the integral reduces to the Euler integral of the first kind \( f_0^\infty v^p(1 - v)^{-p+1}dv = -\pi/\sin(\pi p) \) (see Proposition 4.6, Definition 4.14 and Theorem 4.17 in [18]).

16. Theorem. If a function \( f(t) \) is an original such that

\[ N^s\mathcal{F}^s(f; p; q) := \sum_{j=0}^{2r-1} N^s\mathcal{F}^s_{a,j}(p; q)N_j \] is its image, where a function \( f \) is written in the form

\[ f(t) = \sum_{j=0}^{2r-1} f_j(t)N_j, f_j : \mathbb{R} \to \mathbb{R} \] for each \( j = 0, 1, ..., 2r - 1, f(\mathbb{R}) \subset \mathcal{A}_r \) for \( 2 \leq r \leq 3 \),

\[ N^s\mathcal{F}^s_{a,j}(p; q) := f_\infty^\infty f_j(t) \exp(-u(p; t; q))dt \] then at each point \( t \), where \( f(t) \) satisfies the H"older condition there is true the equality:

(i) \( f(t) = (2\pi N_1)^{-1} \Re(\mathcal{S}N_1) \sum_{j=0}^{2r-1} (f_{a,j}(t; q) N_j \exp(u(p; t; q))dp)N_j \)

in the domain \( s_0(f) < \Re(p) < s_1(f) \), where either \( u(p; t; q) = pt + q \) with \( S = N_1 \) and \( \Im(q) = 0 \), or \( u(p; t; q) = E(p; q) \) and the integral is taken along the straight line \( p(\tau) = a + S\tau \in \mathcal{A}_r, \tau \in \mathbb{R} \), \( S \in \mathcal{A}_r, \Re(S) = 0, |S| = 1, \Re(S\mathcal{N}_1) \neq 0 \) is non-zero, while the integral is understood in the sense of the principal value.

Proof. The two-sided transformation in the basis of generators

\[ N = \{ N_0, N_1, ..., N_{2r-1} \} \] can be written in the form

\[ N\mathcal{F}^s(f; p; q) := \int_0^\infty f(t) \exp(-u(p; t; q))dt = \mathcal{F}^s(f(\mathcal{U}(t); p; q) + \mathcal{F}^s(f(1 - U(t); p; q)), \]

where the index \( N \) is omitted, \( U(t) = 1 \) for \( t > 0, U(0) = 1/2, U(t) = 0 \) for \( t < 0 \), also

\[ \mathcal{F}^s(f(1 - U(t); t; p; q) = \int_0^\infty f(t)U(t) \exp(-u(-p; t; q))dt \]

since \( u(-p; t; q) = u(-p; t; q) \), where \( |f(-t)| \leq C_1 \exp(s_1t) \) for each \( t > 0 \). The common domain of the existence

\[ \int_0^\infty f(-t)U(t) \exp(-u(-p; t; q))dt \] and \( \int_0^\infty f(t)U(t) \exp(-u(p; t; q))dt \)

is \( s_0(f) < \Re(p) < s_1(f) \), since the inequality \( \Re(-p) > -s_1(f) \) is equivalent to the inequality \( \Re(p) < s_1(f) \). Then the application of Theorem 5 twice to \( f(t)U(t) \) and to \( f(-t)U(t) \) gives the statement of this theorem.

17. Theorem. If a function \( N^s\mathcal{F}^s_u(p) \) is analytic by the variable \( p \in \mathcal{A}_r \) in the domain

\[ W := \{ p \in \mathcal{A}_r : s_0 < \Re(p) < s_1 \} \], where \( 2 \leq r \leq 3, f(\mathbb{R}) \subset \mathcal{A}_r, \) either \( u(p, t) = pt \) or \( u(p, t) := E(pt) \). Let also \( N^s\mathcal{F}^s_{u,0}(p) \) can be written in the form

\[ \mathcal{F}^s_{u,0}(p) = \mathcal{F}^s_{u,0,0}(p) + \mathcal{F}^s_{u,0,1}(p) \]

where \( \mathcal{F}^s_{u,0}(p) \) is holomorphic by \( p \) in the domain \( s_0 < \Re(p) \), also \( \mathcal{F}^s_{u,1}(p) \) is holomorphic by \( p \) in the domain \( \Re(p) < s_1, S \in \mathcal{I}_r, \) \( |S| = 1 \), moreover, for each \( a > s_0 \) and \( b < s_1 \) there exists constants \( C_a > 0, C_b > 0 \) and \( \epsilon_a > 0 \) and \( \epsilon_b > 0 \) such that

(i) \( |N^s\mathcal{F}^s_{u,0,0}(p)| \leq C_a \exp(-\epsilon_a|p|) \) for each \( p \in \mathcal{A}_r \) with \( \Re(p) \geq a \),

(ii) \( |N^s\mathcal{F}^s_{u,0,1}(p)| \leq C_b \exp(-\epsilon_b|p|) \) for each \( p \in \mathcal{A}_r \) with \( \Re(p) \leq b \), where \( s_0 \) and \( s_1 \) are fixed, also the integral.
converges absolutely for $k = 0$ and $k = 1$ for $s_0 < w < s_1$. Then $\tilde{N}_{F_u}^{S}(p)$ is the image of the function

$$(iv) \quad f(t) = (2\pi)^{-1} \tilde{S} \int_{-\infty}^{w-\infty} \hat{N}_{F_u}^{S}(p) \exp(u(p,t)) dp.$$  

**Proof.** For the function $\tilde{N}_{F_u}^{S}(p)$ we consider the substitution of the variable $p = -g$, $-s_1 < Re(g)$. In view of Theorem 8 there exist originals $f^0$ and $f^1$ for functions $\tilde{N}_{F_u}^{S}(p)$ and $\tilde{N}_{F_u}^{S}(p)$ while a choice of $w \in \mathbb{R}$ in the common domain $s_0 < Re(p) < s_1$, that is, $s_0 < w < s_1$. At the same time the supports of the functions $f^0$ and $f^1$ are contained in $[0, \infty)$ and $(-\infty, 0)$ respectively. Then $f = f^0 + f^1$ is the original for $\tilde{N}_{F_u}^{S}(p)$ while $q = 0$, since

$$f(t) = f^0(t) + f^1(t) = (2\pi)^{-1} \tilde{S} \int_{-\infty}^{w-\infty} \hat{N}_{F_u}^{S}(p) \exp(u(p,t)) dp + (2\pi)^{-1} \tilde{S} \int_{-\infty}^{w-\infty} \hat{N}_{F_u}^{S}(p) \exp(u(p,t)) dp = (2\pi)^{-1} \tilde{S} \int_{-\infty}^{w-\infty} \hat{N}_{F_u}^{S}(p) \exp(u(p,t)) dp$$

due to the distributivity of the multiplication in the algebra $\mathcal{A}_r$.

**18. Note.** While the definition of the one- and two-sided Laplace transformations over the Cayley-Dickson algebras above the Riemann integral of the real variable was used, while for the inverse transformation the noncommutative integral along paths over $\mathcal{A}_r$ from the works [17][18]. It can be considered also a generalization of the direct transformation with the Riemann-Stieltjes integral as the starting point. For a function $\alpha(t)$ with values in $\mathcal{A}_r$ of the variable $t \in \mathbb{R}$ such that $\alpha(t)$ has a bounded variation on each finite segment $[a, b] \subset \mathbb{R}$, we consider the Stieltjes integral

$$\int_{-\infty}^{\infty}(do(t)) \exp(-u(p, t; q)) := \lim_{u \to \infty} \int_{-\infty}^{\infty}(do(t)) \exp(-u(p, t; q)) + \lim_{b \to \infty} \int_{-\infty}^{\infty}(do(t)) \exp(-u(p, t; q)),$$

where

$$\int_{-\infty}^{\infty}(do(t)) \exp(-u(p, t; q)) = \exp\left(-u(p, t; q)\right) \exp\left(u(p, t; q)\right),$$

$i_0, i_1, \ldots, i_{2r-1}$ are generators of the Cayley-Dickson algebra $\mathcal{A}_r$, $f_v$ and $\alpha_v$ are real-valued functions such that $\alpha = \sum_v \alpha_v i_v$ and $f = \sum_v f_v i_v$, also $f_v(f_v(t) \alpha_v(t))$ is the usual Stieltjes integral over the field of real numbers on a finite segment $[a, b]$. Under imposing the condition $\alpha(t) \exp(-p_0 t) \int_{-\infty}^{\infty} = 0$ the integration by parts gives the relation

$$\int_{-\infty}^{\infty}(do(t)) \exp(-u(p, t; q)) = \int_{-\infty}^{\infty}(\alpha(t)) d[\exp(-u(p, t; q))].$$

In view of the associativity of $\mathcal{H}$ and the alternativity of $\mathcal{O}$ it gives:

- $\int_{-\infty}^{\infty}(do(t)) \exp(-p t) = F^s(\alpha(t); p; p).$
- For $u(p, t; q) = E(pt + q)$ there is the formula

$$\int_{-\infty}^{\infty}(do(t)) \exp(-u(p, t; q)) = p_0 F^s(\alpha(t); p; q) + p_1 F^s(\alpha(t); p; q - i_1 \pi/2) + \ldots + p_{2r-1} F^s(\alpha(t); p; q - i_{2r-1} \pi/2)$$

over $\mathcal{A}_r$ with $2 \leq r \leq 3$.

Thus, the Laplace transformation over $\mathcal{A}_r$ can be spread on a more general class of originals. For this it is used instead of an ordinary notion of convergence of improper integrals their convergence by Cesaro of order $p > 0$:

$$(C, p) \int_{-\infty}^{\infty} f(t) dt := \lim_{n \to \infty} \int_{-\infty}^{\infty} f(t) \left(1 - \frac{t}{b}\right)^p dt.$$  

If this integral converges by Cesaro for some $p > 0$, then it converges for each $q > p$, moreover:

$$(C, p) \int_{-\infty}^{\infty} f(t) dt = (C, q) \int_{-\infty}^{\infty} f(t) dt.$$  

That is, with the growth or the order $p$ a family of functions enlarges for which an improper integral converges. The limit case of the limit by Cesaro is the Cauchy limit:

$$(C) \int_{-\infty}^{\infty} f(t) dt := \lim_{n \to \infty} \int_{-\infty}^{\infty} f(t) \exp(-ct) dt.$$  

For two-sided integrals convergence of improper integrals by Cesaro of order $p$ is defined by the equality:

$$(C, p) \int_{-\infty}^{\infty} f(t) dt := \lim_{n \to \infty} \int_{-\infty}^{\infty} f(t) \left(1 - \frac{|t|}{b}\right)^p dt$$  

and by Cauchy:
(C) \( \int_{-\infty}^{\infty} f(t)dt := \lim_{\epsilon\to+0} \int_{-b}^{b} f(t) \exp(-\epsilon|t|)dt \), when these limits exist.

The noncommutative Mellin transformation is based on the two-sided transformation of the Laplace type over Cayley-Dickson algebras, which was presented above.

19. Remark. If \( f \) is an original function of the two-sided Laplace transformation over the Cayley-Dickson algebra \( A_r \), and \( g(\tau) = f(ln \tau) \) for each \( 0 < \tau < \infty \), then Conditions 11(1 – 3) for \( f \) are equivalent to the following Conditions M(1-3):

\( M(1) \) \( g(\tau) \) satisfies the Hölder condition: \( |g(\tau + h) - g(\tau)| \leq A|h|^\alpha \) for each \( |h| < \delta \) (where \( 0 < \alpha \leq 1, A = \text{const} > 0, \delta > 0 \) are constants for a given \( \tau \)) everywhere on \( \mathbb{R} \) may be besides points of discontinuity of the first kind. On each finite segment \([a, b]\) in \((0, \infty)\) a function \( g \) may have only a finite number of points of discontinuity and of the first kind only.

\( M(2) \) \( |g(\tau)| < C_1 \tau^\alpha \) for each \( 0 < \tau < 1 \), where \( C_1 = \text{const} > 0, s_0 = s_0(g) = \text{const} \in \mathbb{R} \).

\( M(3) \) \( |g(\tau)| < C_2 \tau^{-s_1} \) for each \( \tau \geq 1 \), that is, \( g(\tau) \) is growing not faster, than the power function, where \( C_2 = \text{const} > 0, s_1 = s_1(g) \in \mathbb{R} \).

This is because of the fact that the logarithmic function \( \ln : (0, \infty) \to (-\infty, \infty) \) is the diffeomorphism.

20. Definition. In the two-sided integral transformation of the Laplace type substitute variables \( p \) on \(-p\) and \( t \) on \( e^t \) and \( q \) on \(-q\), then the formula takes the form:

\( M(g; p; q) := \int_{-\infty}^{\infty} f(ln \tau) \exp(-u(p, \ln \tau; -q))\tau^{-1}d\tau, \)

where \( f \) is an original function, \( g(\tau) = f(ln \tau) \) for each \( 0 < \tau < \infty \) (see also Definition 11).

For a specified basis \( \{N_0, N_1, \ldots, N_{2^r-1}\} \) of generators of the Cayley-Dickson algebra \( A_r \),

we can write the notation in more details \( \mathcal{M} A(u; p; q) \) if necessary.

21. Theorem. If an original function \( g(\tau) \) satisfies Conditions 20.1.3, then its image \( \mathcal{M} A(g; p; q) \) is holomorphic by \( \tau \) in the domain \( \{z \in A_r : s_0 < \text{Re}(z) < s_1\} \),

where \( 2 \leq r \leq 3 \).

Proof. The application of Theorem 4 to \( g(\tau) = f(ln \tau) \) gives the statement of this theorem.

22. Theorem. Let \( g(\tau) \) be an original function such that

\( \mathcal{N} \mathcal{M} A(g; p; q) := \sum_{j=0}^{2^r-1} \mathcal{N} G_{u,j}(p; q)N_j \) be its image, where a function \( g \) is written in the form

\( g(\tau) = \sum_{j=0}^{2^r-1} g_j(\tau)N_j, \quad g_j : (0, \infty) \to \mathbb{R} \) for each \( j = 0, 1, \ldots, 2^r - 1 \), \( g((0, \infty)) \subset A_r \) for \( 2 \leq r \leq 3, \)

\( \mathcal{N} G_{u,j}(p; q) := \int_{-\infty}^{\infty} g_j(\tau) \exp(-u(p, \ln \tau; -q))\tau^{-1}d\tau. \)

Then at each point \( \tau \), where \( g(\tau) \) satisfies the Hölder condition the equality is accomplished:

(i) \( g(\tau) = (2\pi N_1)^{-1}\text{Re}(S N_1) \sum_{j=0}^{2^r-1}(\frac{a+j}{a+S})\mathcal{N} G_{u,j}(p; q) \exp(u(p, -p, \ln \tau; -q))dp)N_j \)

in the domain \( s_0(g) < \text{Re}(p) < s_1(g) \), where either \( u(p, t; q) = pt + q \) with \( S = N_1 \) for \( \text{Im}(q) = 0 \), or \( u(p, t; q) = E(pt + q) \) (see §2.2.1) and the integral is taken along the straight line \( p(\theta) = a + S\theta \in A_r, \theta \in \mathbb{R}, S \in A_r, \text{Re}(S) = 0, \) \( |S| = 1, \text{Re}(S N_1) \neq 0 \) is non-zero, while the integral is understood in the sense of the principal value.

Proof. Putting \( t = \ln \tau \) and substituting \( p \) on \(-p\) and applying Theorem 16 we get the statement of this theorem.

23. Theorem. If a function \( \mathcal{N} G_{u}(p) \) is analytic by the variable \( p \in A_r \) in the domain \( W := \{p \in A_r : s_0 < \text{Re}(p) < s_1\} \), where \( 2 \leq r \leq 3, g((0, \infty)) \subset A_r \), either \( u(p, t; q) = pt \) or \( u(p, t) := E(pt) \). Let also \( \mathcal{N} G_{u}(p) \) can be written in the form \( \mathcal{N} G_u(p) = \mathcal{N} G_{u_a}(p) + \mathcal{N} G_{u_b}(p) \), where \( \mathcal{N} G_{u_a}(p) \) is holomorphic by \( p \) in the domain \( s_0 < \text{Re}(p) \), also \( \mathcal{N} G_{u_b}(p) \) is holomorphic by \( p \) in the domain \( \text{Re}(p) < s_1 \), \( S \in I_r, |S| = 1, \) moreover, for each \( a > s_0 \) and \( b < s_1 \) there exists constants \( C_a > 0, C_b > 0 \) and \( \epsilon_a > 0 \) and \( \epsilon_b > 0 \) such that

\( (1) \ | \mathcal{N} G_{u_a}(p)| \leq C_a \exp(-\epsilon_a|p|) \) for each \( p \in A_r \) with \( \text{Re}(p) \geq a, \)
(2) \( |N G^1_u(p)| \leq C_b \exp(-\epsilon_b|p|) \) for each \( p \in \mathcal{A}_r \) with \( \text{Re}(p) \leq b \), where \( s_0 \) and \( s_1 \) are fixed, while the integral

\[
\int_{-\infty}^{S_\infty} N G^k_u(p) \, dp
\]

converges absolutely for \( k = 0 \) and \( k = 1 \) for \( s_0 < w < s_1 \), then \( N G_u(p) \) is the image of the function.

(4) \( g(\tau) = (2\pi)^{-1} \int_{-\infty}^{S_\infty} N G_u(p) \exp(u(-p, \ln \tau)) \, dp \).

Proof. The change of the variable \( p \) on \(-p\) and the substitution \( t = \ln \tau \) for \( \tau > 0 \) with the help of Theorem 17 gives the statement of this Theorem.

24. Theorem. Let \( f \) be an original function from Definition either 1 or 11 or 20. Suppose that \( F \) is its image function of the noncommutative either Laplace or two-sided Laplace transformation for either \( u(p,t) = pt \) or \( u(p,t) = E(pt) \) in the domain \( V := \{ z \in \mathcal{A}_b : s_0 < \text{Re}(z) < s_1 \} \), \( b = 2 \) or \( b = 3 \), where \( s_1 = \infty \) for the noncommutative one-sided Laplace transformation. Then \( F \) is either \( (1, b) \)-quasi-
regular or \( (1, b) \)-quasi-regular in spherical \( \mathcal{A}_b \)-coordinates respectively in \( V \) with \( y_0 = 0 \) if and only if its original is real \( f(t) \in \mathbb{R} \) for each continuity point \( t \) of \( f \) either in \([0, \infty)\) or \( \mathbb{R} \) or \((0, \infty)\) respectively.

Proof. Since the Mellin transformation is obtained from the two-sided Laplace transformation with the help of smooth change of real variables and the one-sided Laplace transformation is the particular case of that of two-sided, then it is sufficient to prove this theorem for the two-sided noncommutative Laplace transformation. Thus consider \( F(p) = \int_{-\infty}^{\infty} f(t) \exp(-u(p,t)) \, dt \), since \( q = 0 \), \( y_0 = 0 \) by the conditions of this theorem. We have that \( \hat{R}_{p,x} = R_{w(p),w(x)} \), \( w \) is a pseudo-conformal diffeomorphism of \( V \), \( R_{p,x} \) is given in Examples 2.2 and 2.9.5.2. To each automorphism \( \hat{R}_{p,x} \) the operator belonging to the Lie group \( \text{SO}(Q^2, \mathbb{R}) \) on the real shadow \( \mathbb{R}^{2b} \) corresponds. Therefore,

(1) \( \hat{R}_{p,x} y = \int_{-\infty}^{\infty} [R_{w(p),w(y)} f(t)] \exp(-(R_{w(p),w(y)} y) t) \, dt \) for \( u = pt \), for each \( p \in V \) and every \( y \in V \cap \mathbb{C} \) such that \( \text{Re}(p) = \text{Re}(y) \) and \( R_{w(p),w(y)} y = p \), since \( \hat{R}_{p,y} |_{\mathbb{R}} = \text{id} \).

(2) \( \hat{R}_{E(p),E(y)} y = \int_{-\infty}^{\infty} [R_{w(E(p)),w(E(y))} f(t)] \exp(-(R_{w(E(p)),w(E(y))} E(t)) \, dt \)

for \( u(p,t) = E(pt) \) respectively due to Formula 9.5.2(2) for each \( p \in V \) and every \( y \in V \cap \mathbb{C} \) such that \( \text{Re}(E(p)) = \text{Re}(E(y)) \) and \( R_{w(E(p)),w(E(y))} E(y) = E(p) \).

(3) \( F_u(p) = \hat{R}_{p,y} y = \int_{-\infty}^{\infty} f(t) \exp(-(pt) \, dt = \int_{-\infty}^{\infty} [\hat{R}_{p,y} f(t)] \exp(-pt) \, dt \)

for \( u = pt \), for each \( p \in V \) and every \( y \in V \cap \mathbb{C} \) such that \( \text{Re}(p) = \text{Re}(y) \) and \( R_{w(p),w(y)} y = p \).

On the other hand,

(4) \( R_{w(E(p)),w(E(y))} \exp(E(ty)) = R_{w(E(p)),w(E(y))} \exp(E(ty)) = \exp(E(tp)) \)

for \( u(p,t) = E(pt) \), for each \( \text{Re}(E(p)) = \text{Re}(E(y)) \) with \( R_{w(E(p)),w(E(y))} E(y) = E(p) \), since \( R_{z,x} (tx) = tR_{z,x} x \) for each \( t \in \mathbb{R} \) and \( E(y) = y \) for each \( y \in \mathbb{C} \). Consequently,

(5) \( F_u(p) = \hat{R}_{E(p),E(y)} f_u(y) = \int_{-\infty}^{\infty} f(t) \exp(-E(tp)) \, dt = \int_{-\infty}^{\infty} [\hat{R}_{E(p),E(y)} f(t)] \exp(-E(tp)) \, dt \)

for \( u(p,t) = E(pt) \). Particularly, \( w = \text{id} \) can also be taken.

The two-sided Laplace transformation is injective such that \( \mathcal{F}^s(f_1;z) = \mathcal{F}^s(f_2;z) \) for each \( z \in V \) if and only if \( f_1(t) = f_2(t) \) at each point \( t \) in \( \mathbb{R} \) where \( f_1(t) \) and \( f_2(t) \) are continuous (see Theorems 5, 7, 8, 16, 17, 22 and 23). Thus due to Formulas (3, 4) \( F(z) \) is either \((1, b)\)-quasi-
regular or \((1, b)\)-quasi-regular in spherical \( \mathcal{A}_b \)-coordinates correspondingly if and only if either \( \hat{R}_{p,y} f(t) \) is \( f(t) \) or \( \hat{R}_{E(p),E(y)} f(t) \) is \( f(t) \) respectively for each \( t \in \mathbb{R} \) a point of continuity of \( f \) and each \( p \in V \) and every \( y \in V \cap \mathbb{C} \) such that \( \text{Re}(p) = \text{Re}(y) \) and \( \text{Re}(E(p)) = \text{Re}(E(y)) \) (see \( 2.1(Q2 - Q5) \)). Since \( f \) is continuous besides points of discontinuity of the first kind, then using limits from the left or from the right redefine \( f \) at points of discontinuity such that \( f \) will be real everywhere on \( \mathbb{R} \).
If $f$ is real-valued, then $F_u$ satisfies Conditions 2.1(Q1 − Q5) by the construction of $F_u$. Then $F_u$ satisfies 2.1(Q6, Q7) as well due to Theorem 2.18, since the function $e^{\exp p} = v(p)$ is pseudo-conformal for $a \neq 0$ on $A_p, p \in A_b$.

25. Theorem. Let suppositions of Theorem 24 be satisfied for the noncommutative two-sided Laplace or Mellin transformation. If $f$ is real-valued, then

(1) $F_u(p) = F_u(p)$ for $u(p, t) = pt$ or

(1') $F_u(p_0 - p_1 t_1 + p_2 t_2 + \ldots + p_{2l-1} t_{2l-1}) = F_u(p)$ for $u(p, t) = E(pt)$ respectively for each $p \in V$. Moreover, either $f(t) = f(-t)$ is even for each $t \in \mathbb{R}$ or $f(t) = f(1/t)$ for each $t > 0$ at each point of continuity of $f$, if and only if its noncommutative two-sided Laplace or Mellin transformation $F_u(p)$ for $u(p, t) = pt$ or $u(p, t) = E(pt)$ satisfies the condition:

(2) $F_u(-p) = F_u(p)$ for each $p \in V$ for both types of $u$.

Proof. If an original $f$ is real-valued, then

$$[\int_{-\infty}^{\infty} f(t) \exp(-u(p, t)) dt]^* = \int_{-\infty}^{\infty} [\exp(-u(p, t))]^*[f(t)]^* dt = \int_{-\infty}^{\infty} f(t) \exp(-u(p, t))^* dt,$$

but $[u(p, t)]^* = u^*(t)$ for $u = pt$ and $[u(p, t)]^* = u((p_0 - p_1 t_1 + p_2 t_2 + \ldots + p_{2l-1} t_{2l-1}), t)$ for $u(p, t) = E(pt)$ (see Formulas 2.2.1(1, 2) or 16.1(3, 5)), where $p = p_0 + p_1 t_1 + \ldots + p_{2l-1} t_{2l-1}, p_j \in \mathbb{R}$ for each $j = 0, \ldots, 2b - 1$. Therefore, either (1) or (1') respectively is satisfied.

An original $f$ is even on $\mathbb{R}$ if and only if

$$\int_{-\infty}^{\infty} f(t) \exp(-u(p, t)) dt = \int_{-\infty}^{\infty} f(-t) \exp(-u(p, t)) dt$$

for both variants $u(p, t) = pt$ and $u(p, t) = E(pt)$, since $u(p, -t) = u(-p, t)$ while the two-sided Laplace transformation is injective such that $F^*(f_1; z) = F^*(f_2; z)$ for each $z \in V$ if and only if $f_1(t) = f_2(t)$ at each point $t \in \mathbb{R}$ where $f_1(t)$ and $f_2(t)$ are continuous (see Theorems 16, 17). Consequently, Condition (2) is equivalent to $f(t) = f(-t)$ for each $t \in \mathbb{R}$ for the noncommutative two-sided Laplace transformation.

Substituting $t$ on $\ln(\tau)$ and $p$ on $-p$ gives that (2) is equivalent to $f(t) = f(1/t)$ for each $t > 0$ for the noncommutative Mellin transformation due to Theorems 22 and 23.

25.1. Proposition. Let $f$ be either a $(1, b)$-quasi-regular or $(1, b)$-quasi-regular in spherical $A_b$-coordinates function on a domain $V$, $f(z) \neq 0$ for each $z \in V$, where $2 \leq b \leq 3$. Then $1/f(z)$ is either a $(1, b)$ quasi-regular or $(1, b)$-quasi-regular in spherical $A_b$-coordinates function respectively on $V$.

Proof. Take without loss of generality $y_0 = 0$. Since $\hat{R}_{z, x}$ and $\hat{R}_{E(z), E(x)}$ are automorphisms of $A_b$, then $1/f$ or $1/f \circ E^{-1}$ respectively satisfies Conditions 2.1(Q1 − Q6) on $V$ (see also Definition 2.2.1). Since $f$ is $A_b$ holomorphic, then $1/f$ is also $A_b$ holomorphic, $(\partial(1/f(z))/\partial z).h = 0$ for each $h \in A_b$ and all $z \in V$ (see [17, 18, 19]). On the other hand, $f(z)[1/f(z)] = 1$ for each $z \in V$, hence

(1) $[\partial(1/f(z))/\partial x].h = -f(x)[f'(x).h](1/f(x))]$

for each $h \in A_b$ and every $x \in V$, since $O = A_3$ is alternative, $H = A_2$ is associative, where $f'(z).h = (\partial f(z)/\partial z).h$. Acting on both sides of Equation (1) by either $\hat{R}_{z, x}$ or $\hat{R}_{E(z), E(x)}$ gives (Q7) for $1/f(z)$ or $1/f \circ E^{-1}$ respectively, since $f(z)$ or $f \circ E^{-1}$ correspondingly satisfies (Q1 − Q7).

26. Examples. 1. Consider now the zeta function on $A_b$ (see Example 9.5.2). In view of Theorem 2.1 [33] the zeta function $\zeta(s)$ has the holomorphic extension in $\mathbb{C} \setminus \{1\}$ with the pole at $s = 1$ with residue 1, moreover, it satisfies the functional equation $\zeta(s) = 2^s \pi^{-s} \sin(s\pi/2) \Gamma(1 - s) \zeta(1 - s)$.

Construct for $\zeta(s)$ (1, b)-quasi-conformal in Examples 1 and 2 and (1, b)-quasi-conformal in spherical $A_b$-coordinates in Example 3 extensions in $A_b \setminus \{1\}$. For this put $z = x + iyM$, where $x, y \in \mathbb{R}, Re(M) = 0, |M| = 1, r = 1, y_0 = 0$. Then $z$ is obtained from $s = x + iy$ by
the automorphism $\hat{R}_{z,s}$ such that $\hat{R}_{z,s}(i) = M$, where $i = i_1$. Then $\mathbb{R} \oplus M =: \mathbb{C}$ is the subalgebra in $A_b$ isomorphic with $\mathbb{C}$. Let $a$ and $q$ be positive integers, $q > a$, $z \neq 1$, then

$$\sum_{n=a+1}^{q} z^{-n} = \frac{(q^{-1} - a^{-1})}{(1 - z) - z f_0^x(x - x - 1/2)x^{-1-1}dx + (q^{-1} - a^{-1})/2},$$

where $[x]$ denotes the greatest integer not exceeding $x$. For $Re(z) =: \sigma > 1$ and $a = 1$ consider $q \to \infty$, then from Formula (1) we get

$$\zeta(z) = z \int_{[x]}^{\infty} [x - x + 1/2]x^{-1-1}dx + 1/(z - 1) + 1/2.$$  

The function $[x] - x + 1/2$ is bounded, consequently, this integral converges for $\sigma > 0$ and uniformly converges in the domain $\sigma > \delta$ in $A_b$, where $\delta > 0$ is the constant. Therefore, this integral defines a holomorphic function of $z (1, b)$-quasi-regular for $\sigma > 0$, $z \neq 1$, due to Theorem 24. The right hand side of Equation (2) thus provides the $A_b$ holomorphic continuation of $\zeta(z)$ up to $\sigma = 0$, while there is a simple pole at $z = 1$ with residue 1.

For $0 < \sigma < 1$ Formula (2) may be written as

$$\zeta(z) = z \int_{0}^{\infty} ([x] - x)x^{-1-1}dx,$$

since $f_0^x([x] - x)x^{-1-1}dx = - f_0^1 x^{-1}dx = 1/(z - 1)$ and $z f_0^\infty x^{-1-1}dx/2 = 1/2$. Consider $f(x) = [x] - x + 1/2$, $f_1(x) = f_1^x f(y)dy$, then $f_1(y)$ is bounded, since $f_k(y)dy = 0$ for each integer $k$. Consequently, $f_k f_{x_k} f(x) x^{-1-1}dx = f_1(x) x^{-1-1}i_{x=1}^{k} + (z + 1)f_{x_1} f_1(x) x^{-2-2}dx,$ which tends to zero as $x_1 \to \infty$ and $x_2 \to \infty$, while $\sigma > -1$. Therefore, the integral in (2) is convergent for $\sigma > -1$, hence (2) gives the holomorphic continuation of $\zeta(z)$ for $\sigma > -1$.

Since $z f_0^1([x] - x + 1/2)x^{-1-1}dx = 1/(z - 1) + 1/2$ for $-1 < \sigma < 0$. In view of Proposition 2.9.1 and Theorem 24 and Formulas (2, 3) and using the continuous extension from $\{z \in A_b : -1 < Re(z) < 0 \text{ or } 0 < Re(z)\}$ the function $\zeta(z)$ is $(1, b)$-quasi-regular in the domain $\{z \in A_b : -1 < Re(z), z \neq 1\}$.

$$\lim_{R \to \infty} \sum_{n=1}^{\infty} n^{-1} \int_{R}^{\infty} \sin(2\pi nx)x^{-1-1}dx = 0$$

for each $\sigma$. Consequently, $\lim_{R \to \infty} \sum_{n=1}^{\infty} n^{-1} \int_{nR}^{\infty} \sin(2\pi nx)x^{-1-1}dx = 0$ for $-1 < \sigma < 0$. Since there is the Fourier series expansion: $x^{-1-1}dx = \sum_{n=1}^{\infty} \sin(2\pi nx)/(\pi n)^{-1}$ for non-integer real $x$, then integrating in (3) term by term series we obtain

$$\zeta(z) = \frac{z(\pi)}{\{\pi \}} \sum_{n=1}^{\infty} n^{-1} \int_{0}^{\infty} \sin(2\pi nx)x^{-1-1}dx = \frac{z(\pi)}{\{\pi \}} \sum_{n=1}^{\infty} \frac{n^{-1} \int_{0}^{\infty} \sin(y)y^{-1-1}dy}{\{\pi \}} \zeta(1 - z),$$

where $\Gamma(z)$ the $(1, b)$-quasi-conformal extension of Example 2.9.5.3 is used. Formula (4) is initially valid for $-1 < \sigma < 0$, but the right-hand side of (4) is true also for each $\sigma < 0$, where $\sigma = Re(z)$. Thus this provides the $(1, b)$-quasi-regular extension of $\zeta(z)$ on $A_b \setminus \{1\}$ and the following formula is satisfied:

$$\zeta(1 - z) = 2^{1-1} - \pi^{-1} \cos(z\pi/2)\Gamma(z)\zeta(z).$$

Equation (5) transforms into

$$\zeta(z) = \chi(z)\zeta(1 - z),$$

where

$$\chi(z) = 2^{1-1} \pi^{-1} \sin(\pi z/2)\Gamma(1 - z)$$

by changing $z$ into $1 - z$. Then $\chi(z) = \pi^{-1/2}\Gamma(1/2 - z/2)/\Gamma(z/2)$, hence $\chi(z)\chi(1 - z) = 1$. Then $\xi(z) = \xi(1 - z)$ for each $Re(z) \neq 1/2$, where $\xi(z) = z(z - 1)\pi^{-1/2}\Gamma(z/2)\zeta(z)/2$, consequently,

$$\Upsilon(z) = \Upsilon(-z)$$

for each $Re(z) \neq 0$,

$$\Upsilon(z) = \xi(z + 1/2).$$

Since $(2^z)^* = 2^{z^*}$, $(\pi^{-z})^* = \pi^{-z^*}$, $\sin(\pi z^*/2) = (\sin(\pi z/2))^*$, $\Gamma(1 - z^*) = (\Gamma(1 - z))^*$ for each $z \in A_b$, then

$$\Upsilon(z)^* = \Upsilon(z^*)$$

for each $z \in A_b$, where $z^* := \bar{z}$.  

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For $\sigma > 0$ we have $\int_0^\infty x^{z-1}e^{-nx}dx = n^{-z}\int_0^\infty y^{z-1}e^{-y}dy = n^{-z}\Gamma(z)$, since $n$ and $y$ are real and $\eta^{-1}$ is defined as $\text{Exp}((z-1)Ln(\eta))$ with the branch of the logarithm $\text{Ln}(R)$ real for $R > 0$ so that $n^{-z}$ and $y^{z-1}$ commute. For $\sigma > 1$ we have the convergent series

$$
\sum_{n=1}^\infty \int_0^\infty x^{z-1}e^{-nx}dx = \Gamma(\sigma)\zeta(\sigma). \quad \text{Therefore,} \quad \Gamma(z)\zeta(z) = \sum_{n=1}^\infty \int_0^\infty x^{z-1}e^{-nx}dx = \int_0^\infty x^{z-1}(e^x - 1)dx.
$$

Consider the integral $J(z) = \int_C \eta^{-1}(e^{(e^1)1})d\eta$, where the contour $C$ starts at infinity on the positive real axis, encircles the origin in the plane $\mathbb{R} \oplus M\mathbb{R}$ in the positive direction besides the points $2\pi Mk$, where $0 \neq k \in \mathbb{Z}$ and returns to positive infinity. Therefore, $\text{Arg}(\text{Ln}(\eta))$ varies from 0 to $2\pi M$ round the contour. So we take $C$ consisting of the real axis from infinity to $0 < R < 2\pi$, the circle $|z| = R$, and the real axis from $R$ to $\infty$. Thus on the circle $|z^{1/\sigma}| = \exp((z-1)Ln|\eta| - t \text{Arg}(\eta)) \leq |\eta|^{1/\sigma} \exp(2\pi |t|) \text{ and } |\text{Exp}(\eta) - 1| > A|\eta|$, where $\text{Arg}(\eta) = M^*\text{Arg}(\eta)$, $z = \sigma + tM$, $\sigma = \text{Re}(z)$, $t \in \mathbb{R}$. Consequently, the integral round the circle tends to zero while $R \to \infty$ for $\sigma > 1$. Taking the limit with $R \to \infty$ gives $J(z) = -\int_0^\infty x^{z-1}(\sigma+1)dx + \int_0^\infty (x\text{Exp}(2\pi M))^{z-1}(e^x - 1)dx = (\text{Exp}(2\pi Mz) - 1)\Gamma(z)\zeta(z) = 2\pi M\text{Exp}(\pi z M)(\Gamma(1 - z))^{-1}\zeta(z)$, hence

$$
(9) \quad \zeta(z) = \Gamma(1 - z)\text{Exp}(-\pi Mz)(\Gamma(1 - z))^{-1}\zeta(z).
$$

The latter formula has been proved for $\sigma > 1$. But the integral $J(z)$ is uniformly convergent in $G_M$ for any bounded region $G_M$ of the $\mathbb{R} \oplus M\mathbb{R}$ plane and uniformly by purely imaginary $M \in A_b$, $\text{Re}(M) = 0$, $|M| = 1$, where $G_M = \tilde{R}_{M,1}\mathbb{G}_1$. Thus Formula (9) defines the $(1,b)$-quasi-regular function on $A_b \setminus \{1\}$.

Formulas (4, 9) have been obtained by the same family $R_{z,x}$ of Example 2.2. If a pole of a complex meromorphic function is at the real axis, then for its quasi-conformal extension with a marked point $y_0 = 0$ its pole will remain the same real pole, since the rotation axis is $\mathbb{R}$. Thus the only possible singularities of $\zeta(z)$ may be poles of $\Gamma(1-z)$, $z = 1, 2, 3, ...$. In view of (4) $\zeta(z)$ is regular at $z = 2, 3, ...$, more exactly $J(z)$ vanishes at these points (see [33] and Theorem 2.11 [17, 18]). At $z = 1$ we have $J(1) = \int_C(\text{Exp}(z) - 1)^{-1}dz = 2\pi M$ and $\Gamma(1 - z) = -(z - 1)^{-1} + ...$, hence the residue at this pole is 1.

2. For the logarithmic derivative $\psi(1 + z) = d\text{Ln}\Gamma(1 + z)/dz$ of the gamma function there is the expression $\psi(1 + z) = -C - \sum_{k=1}^\infty ((z + k) - 1)$ (see Formula VII.89(9) in [15]). Hence it is valid for its $(1,b)$-quasi-meromorphic extension with the operators $\tilde{R}_{x,y}$ as in Example 2, where $y_0 = 0$, $2 \leq b \leq 3$. Take $-1 < a < 0$, then in view of the noncommutative $\mathbb{A}_b$ analog of the Jordan lemma 9 and Notes 10 above 2.47 [18] with $-W := \{z : \text{Re}(z) < s_0\}$ instead of $W$ and with $a < s_0 < 0$ and Theorem 3.9 about residues [18, 17, 19] we have

$$
(1) \quad \zeta(z) = \text{Exp}(\pi z M)(2\pi)^{-1}\text{M}^{\text{H}^a_{x-M}}(\psi(1 + \eta) - \text{Ln}(\eta))\eta^{-z}d\eta \text{ for each } \sigma > 1, \text{ where } M \in A_b, \text{Re}(M) = 0, |M| = 1, z \in A_b, \sigma = \text{Re}(z), z = \sigma + M\nu, \sigma, \nu \in \mathbb{R}, -1 < a < 0.
$$

The function $\{\psi(1 + \eta) - \text{Ln}(\eta)\}\eta^{-z}$ is real on $(0, \infty) = \{x \in \mathbb{R} : 0 < x\}$, where the branch of Ln is such that $\text{Ln}|_{\mathbb{R}} = \text{Ln} : (0, \infty) \to \mathbb{R}$. In view of the theorems about uniqueness and inversion of the noncommutative version of the Mellin transformation the $(1,b)$-quasi-regular extension of $\zeta(z)$ coincides with the noncommutative version of the Mellin transform (2), when $z \in A_b$ with $0 < \text{Re}(z) < 1$. Then $\tilde{R}_{x,y}g(y) = g(z)$ for each $y \in \mathbb{C}$ and $z \in A_b$ with $0 < \text{Re}(y) = \text{Re}(z) < 1$ such that $\tilde{R}_{x,y}y = z$, where $g(z) := \int_0^\infty \{\psi(1 + x) - \text{Ln}(x)\}x^{-z}dx, y_0 = 0$. In view of Theorem 2.18 there exists $(\partial \int_0^\infty \{\psi(1 + x) - \text{Ln}(x)\}x^{-z}dx)h$ for each $h \in \mathbb{R} \oplus M\mathbb{R}$ and
every $0 < \sigma < 1$, where $z = \sigma + Mv$. Thus $g(z)$ satisfies (Q1, Q6) and $g'(z)$ satisfies (Q7) when $g'(z) \neq 0$, consequently, $g(z)$ is the $(1, b)$-quasi-regular function.

In view of (1.6) there is the symmetry relation: $g(z) = -(\sin(\pi z))^{-1}\pi \zeta(z) = -(\sin(\pi(1 - z)))^{-1}\pi \zeta(1 - z)$, since $\sin(\pi - \phi) = \sin(\phi)$ for each $\phi \in \mathcal{A}_b$, where

$$\chi(z) := 2\pi^{-1}\sin(\pi z/2)\Gamma(1 - z), \quad \chi(z)\zeta(1 - z) = 1.$$  But $|2\sigma| = 2\sigma, |\pi^{-1}| = \pi^{-1},$ sin(\pi z/2) = 0 if and only if $z = 2k$ with $k \in \mathbb{Z},$ sin(\pi z/2) has not poles, $\Gamma(1 - z)$ has not zeros, $\Gamma(1 - z)$ has a pole at $z$ if and only if $z = 1, 2, 3, \ldots$, consequently, $\chi(z)$ has not any zero or pole in the domain $V := \{z \in A_b : 0 < Re(z) < 1\}$. At the same time the multiplier (sin(\pi z))^{-1}1_V has not any pole or zero in $V$.

3. Consider now new type of an extension in spherical $A_b$-coordinates. Let

$$(1) \psi(x) := \sum_{n=1}^{\infty} \exp(-n^2\pi x),$$

where $x > 0$, then

$$(2) \zeta(y) = \pi^{y/2}[\Gamma(1/2)]^{-1} \int_0^{\infty} x^{y/2-1} \psi(x)dx$$

for $\sigma = Re(y) > 1, y \in \mathbb{C}$. It is known that

$$(3) 2\psi(x) + 1 = [2\psi(1/x) + 1]/(x)^{1/2}$$

for each $x > 0$. Therefore, from (2, 3) it follows, that

$$(4) \pi^{-y/2}\Gamma(1/2)\zeta(y) = \int_0^1 x^{y/2-1}\psi(x)dx + \int_1^{\infty} x^{y/2-1}\psi(x)dx$$

$$= \int_0^1 \psi(1/x)(x)^{-1/2} + (1/2 - 2/1)dx + \int_1^{\infty} x^{y/2-1}\psi(x)dx$$

$$= 1/(y - 1) - 1/y + \int_1^{\infty} x^{y/2-3/2}\psi(x)dx + \int_1^{\infty} x^{y/2-1}\psi(x)dx$$

$$= 1/[y(y - 1)] + \int_1^{\infty} (x^{-y/2-1/2} + x^{y/2-1})\psi(x)dx.$$
Theorems 14, 24. Put \( \Omega(q) := \xi(q+1/2) \). Then \( \Omega(q) \) has the \((1, b)\)-quasi-integral in spherical \( A_b \)-coordinates extension \( \Omega^*(p) = \{w^*(p)\}^{-1}g^*(p) \). The function \( f(t) := -\exp(-|t|/2) + 2\exp(|t|/2)\psi(2|t|) \) is real-valued and even on \( \mathbb{R} \). In view of Theorems 24 and 25 \( \Omega^*(p) \) has the symmetry properties 25(1’), 2. This also can be seen from Equations (7, 8). The symmetry property 25(1’) for \( f^* \) implies 25(1) for \( f = f^* \circ E^{-1} \), since if \( z = E(p) \), then the adjoint number is \( \bar{z} = E(p_0 - p_1 i_1 + p_2 i_2 + \ldots + p_{2^r-1} i_{2^r-1}) \) in accordance with Formulas 2.2.1(1, 2).

It is known that \( \zeta(z) \) has no any poles in \( \mathbb{C} \) besides \( z = 1 \), that is, \( \zeta(z) \) contains only complex zeros in the domain \( 0 < Re(y) < 1 \) in \( \mathbb{C} \). It is well-known that all complex zeros of \( \zeta(z) \) are in the complex strip \( 0 < \sigma < 1 \) and they form a discrete set in \( \mathbb{C} \) without finite accumulation points [33]. Thus the function \( f = f^* \circ E^{-1} \) with \( f^*(p) = \Omega^*(p) \) satisfies conditions of Theorem 2.17, since \( \zeta(z) \) has not any real zeros and all its complex zeros are in the strip \( 0 \leq Re(z) \leq 1 \) (see page 30 [33]). On the other hand, \( \zeta(z) \) and \( \xi(z) \) have common all complex zeros and \( E(y)|_{C} = y \) for each \( y \in \mathbb{C} \). Thus in view of Theorem 2.17 it is proved the following.

\textbf{27. Theorem.} All complex zeros of the \( \zeta \) function lie only on the line \( Re(z) = 1/2 \).

\textbf{27.1. Remark.} This is not so surprising, since by Theorem 2.13 [33] each meromorphic function \( f(s) = G(s)/P(s) \), where \( G \) is an integral function of finite order and \( P \) is a polynomial on \( \mathbb{C} \), and \( f \) is satisfying the symmetry property 26.1(6) and having the series expansion \( f(s) = \sum_{n=1}^{\infty} a_n n^{-s} \) absolutely convergent for \( \sigma > 1 \), then \( f(s) \) is \( c \zeta(s) \), where \( c = \text{const, } a_n \in \mathbb{C} \) is a constant for each \( n \in \mathbb{N} \). On the other hand, the class of \((r, b)\)-quasi-conformal functions is more narrow and specific in comparison with the class of \( A_b \) holomorphic functions, where \( 1 \leq r < b \leq 3 \) (see also Notes 2.13 and 2.17.1). Moreover, the class of \((1, b)\)-quasi-integral functions is more narrow than that of \((1, b)\)-quasi-regular which in its turn is restricted by Theorem 24. Mention, that if \( z = z_0 + z_1 i_1 + z_2 i_2 + z_3 i_3 = E_2(p) \), then \( E_2(-p) = -z_0 - z_1 i_1 + z_2 i_2 - z_3 i_3 \) in accordance with Formulas 2.2.1(1) and 16.1(3), where \( z_0, z_1, z_2, z_3 \in \mathbb{R}, \ z, p \in \mathbb{H} \). Consequently, \( E_2 \) and \( E_6 \) are neither even nor odd functions. More narrow class is that of satisfying symmetry properties 25(1’), 2, which need to be met for using Theorem 2.17. For example, the Dirichlet function does not satisfy conditions of Theorem 2.17 (see §10.25 [33]).

Consider the identity \( \int_{-\infty}^{\infty} f(t) \exp(p_0 + i p_1 t) dt = g(p) \int_{-\infty}^{\infty} f(t) \exp((1-p_0)t- ip_1 t) dt \) for an original nonzero function \( f : \mathbb{R} \rightarrow \mathbb{R} \) and a meromorphic function \( g \) in \( \mathbb{C} \) such that \( g \) is holomorphic and without zeros in the band \( G := \{ p \in \mathbb{C} : s_0 < p_0 < s_1, s_0 < 1 - p_0 < s_1 \} \), where \( g \) may have only isolated poles in \( \mathbb{C} \), \( p_0, p_1 \in \mathbb{R}, \ p = p_0 + i p_1, \ 0 < s_0(f) < s_1(f) < 1, |f(t)| < C_1 \exp(-s_1 t) \) for each \( t < 0 \), \( f(t) < C_2 \exp(s_0 t) \) for each \( t \geq 0 \), \( s_0 = s_0(f), s_1 = s_1(f) \). Then \( g(p)g(1-p) = 1 \) and \( g(p) = g(\bar{p}) \) in \( G \) besides poles and \( g(1/2) = 1 \), and \( \bar{F}(p) = F(\bar{p}) \) in \( G \), where \( F(p) = \int_{-\infty}^{\infty} f(t) \exp(pt) dt \). In particular, \( g'(1/2 + i p_1) = g'(1/2 - i p_1) \) for each \( p_1 \mathbb{R} \) besides poles of \( g \). Since the two-sided Laplace transformation of \( f(t) \) is holomorphic in the band \( \{ p \in \mathbb{C} : s_0 < p_0 < s_1 \} \), then the differentiation of this identity by \( p \) in \( G \) gives:

\[ f_{-\infty}^{\infty} f(t) t \exp(pt) dt = g'(p)(\int_{-\infty}^{\infty} f(t) \exp((1-p)t) dt - g(p) \int_{-\infty}^{\infty} f(t) t \exp((1-p)t) dt \] (see also Theorem 2.18). Therefore, the class of such functions \( F(p) \) is narrow.

\textbf{Acknowledgement.} The author is sincerely grateful to Prof. Fred van Oystaeyen and Prof. Jan van Casteren for helpful discussions on noncommutative analysis and geometry over quaternions and octonions at Mathematical Department of Antwerpen University in 2002 and 2004 and for hospitality.
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