The sum of a maximal monotone operator of type (FPV) and a maximal monotone operator with full domain is maximal monotone

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Abstract

The most important open problem in Monotone Operator Theory concerns the maximal monotonicity of the sum of two maximal monotone operators provided that Rockafellar’s constraint qualification holds.

In this paper, we prove the maximal monotonicity of $A + B$ provided that $A$ and $B$ are maximal monotone operators such that $\text{dom } A \cap \text{int dom } B \neq \emptyset$, $A + N_{\text{dom } B}$ is of type (FPV), and $\text{dom } A \cap \text{dom } B \subseteq \text{dom } B$. The proof utilizes the Fitzpatrick function in an essential way.

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1 Introduction

Throughout this paper, we assume that $X$ is a real Banach space with norm $\| \cdot \|$, that $X^*$ is the continuous dual of $X$, and that $X$ and $X^*$ are paired by $\langle \cdot, \cdot \rangle$. Let $A : X \rightrightarrows X^*$ be
a set-valued operator (also known as multifunction) from $X$ to $X^*$, i.e., for every $x \in X$, $Ax \subseteq X^*$, and let $\text{gra } A = \{(x, x^*) \in X \times X^* \mid x^* \in Ax\}$ be the graph of $A$. Recall that $A$ is monotone if

$$\langle x - y, x^* - y^* \rangle \geq 0, \quad \forall (x, x^*) \in \text{gra } A \forall (y, y^*) \in \text{gra } A,$$

and maximal monotone if $A$ is monotone and $A$ has no proper monotone extension (in the sense of graph inclusion). Let $A : X \rightrightarrows X^*$ be monotone and $(x, x^*) \in X \times X^*$. We say $(x, x^*)$ is monotonically related to $\text{gra } A$ if

$$\langle x - y, x^* - y^* \rangle \geq 0, \quad \forall (y, y^*) \in \text{gra } A.$$

Let $A : X \rightrightarrows X^*$ be maximal monotone. We say $A$ is of type (FPV) if for every open convex set $U \subseteq X$ such that $U \cap \text{dom } A \neq \emptyset$, the implication

$$x \in U \text{ and } (x, x^*) \text{ is monotonically related to } \text{gra } A \cap U \times X^* \Rightarrow (x, x^*) \in \text{gra } A$$

holds. We say $A$ is a linear relation if $\text{gra } A$ is a linear subspace. Monotone operators have proven to be a key class of objects in modern Optimization and Analysis; see, e.g., the books [6, 7, 8, 10, 17, 18, 15, 24] and the references therein. We adopt standard notation used in these books: $\text{dom } A = \{x \in X \mid Ax \neq \emptyset\}$ is the domain of $A$. Given a subset $C$ of $X$, $\text{int } C$ is the interior of $C$, and $\overline{C}$ is the norm closure of $C$. The indicator function of $C$, written as $\iota_C$, is defined at $x \in X$ by

$$\iota_C(x) = \begin{cases} 0, & \text{if } x \in C; \\ \infty, & \text{otherwise.} \end{cases}$$

We set $\text{dist}(x, C) = \inf_{c \in C} \|x - c\|$, for $x \in X$. If $D \subseteq X$, we set $C - D = \{x - y \mid x \in C, y \in D\}$. For every $x \in X$, the normal cone operator of $C$ at $x$ is defined by $N_C(x) = \{x^* \in X^* \mid \sup_{c \in C} \langle c - x, x^* \rangle \leq 0\}$, if $x \in C$; and $N_C(x) = \emptyset$, if $x \notin C$. For $x, y \in X$, we set $[x, y] = \{tx + (1-t)y \mid 0 \leq t \leq 1\}$. Given $f : X \to [-\infty, +\infty]$, we set $\text{dom } f = f^{-1}(\mathbb{R})$ and $f^* : X^* \to [-\infty, +\infty] : x^* \mapsto \sup_{x \in X} \langle (x, x^*) - f(x) \rangle$ is the Fenchel conjugate of $f$. If $f$ is convex and $\text{dom } f \neq \emptyset$, then $\partial f : X \rightrightarrows X^* : x \mapsto \{x^* \in X^* \mid (\forall y \in X) (y - x, x^*) + f(x) \leq f(y)\}$ is the subdifferential operator of $f$. We also set $P_X : X \times X^* \to X : (x, x^*) \mapsto x$. Finally, the open unit ball in $X$ is denoted by $\mathbb{B}_X = \{x \in X \mid \|x\| < 1\}$, and $\mathbb{N} = \{1, 2, 3, \ldots\}$.

Let $A$ and $B$ be maximal monotone operators from $X$ to $X^*$. Clearly, the sum operator $A + B : X \rightrightarrows X^* : x \mapsto Ax + Bx = \{a^* + b^* \mid a^* \in Ax \text{ and } b^* \in Bx\}$ is monotone. Rockafellar’s [14] Theorem 1 guarantees maximal monotonicity of $A + B$ under Rockafellar’s constraint qualification $\text{dom } A \cap \text{int } \text{dom } B \neq \emptyset$ when $X$ is reflexive — this result is often referred to as “the sum theorem”. The most famous open problem concerns the maximal monotonicity of $A + B$ in nonreflexive Banach spaces when Rockafellar’s constraint qualification holds.
See Simons’ monograph [18] and [4, 5, 23] for a comprehensive account of some recent developments.

Now we focus on the case when $A$ and $B$ satisfy the following three conditions: $\text{dom } A \cap \text{int dom } B \neq \emptyset$, $A + N_{\text{dom } B}$ is of type (FPV), and $\text{dom } A \cap \text{dom } B \subseteq \text{dom } B$. We show that the sum $A + B$ is maximal monotone in this setting. We note in passing that in [20, Corollary 2.9(a)], Verona and Verona derived the same conclusion when $A$ is the subdifferential operator of a proper lower semicontinuous convex function, and $B$ is maximal monotone with full domain. In [2, Theorem 3.1], it was recently shown that the sum theorem is true when $A$ is a linear relation and $B$ is the normal cone operator of a closed convex set. In [22], Voisei confirmed [17, Theorem 41.5] that the sum theorem is also true when $A$ is type (FPV) with convex domain, and $B$ is the normal cone operator of a closed convex set. Our main result, Theorem 3.4, generalizes all the above results and it also contains a result due to Heisler [11, Remark, page 17] on the sum theorem for two operators with full domain.

The remainder of this paper is organized as follows. In Section 2 we collect auxiliary results for future reference and for the reader’s convenience. The main result (Theorem 3.4) is proved in Section 3.

2 Auxiliary Results

Fact 2.1 (Rockafellar) (See [13, Theorem 3(b)], [18, Theorem 18.1], or [24, Theorem 2.8.7(iii)].) Let $f, g : X \to [\text{--}\infty, +\infty]$ be proper convex functions. Assume that there exists a point $x_0 \in \text{dom } f \cap \text{dom } g$ such that $g$ is continuous at $x_0$. Then $\partial(f + g) = \partial f + \partial g$.

Fact 2.2 (See [10, Theorem 2.28].) Let $A : X \rightrightarrows X^*$ be monotone with $\text{int dom } A \neq \emptyset$. Then $A$ is locally bounded at $x \in \text{int dom } A$, i.e., there exist $\delta > 0$ and $K > 0$ such that

$$\sup_{y^* \in Ay} \|y^*\| \leq K, \quad \forall y \in (x + \delta B_X) \cap \text{dom } A.$$

Fact 2.3 (Fitzpatrick) (See [9, Corollary 3.9].) Let $A : X \rightrightarrows X^*$ be maximal monotone, and set

$$(3) \quad F_A : X \times X^* \to [\text{--}\infty, +\infty] : (x, x^*) \mapsto \sup_{(a, a^*) \in \text{gra } A} \left( \langle x, a^* \rangle + \langle a, x^* \rangle - \langle a, a^* \rangle \right),$$

the Fitzpatrick function associated with $A$. Then for every $(x, x^*) \in X \times X^*$, the inequality $\langle x, x^* \rangle \leq F_A(x, x^*)$ is true, and the equality holds if and only if $(x, x^*) \in \text{gra } A$. 

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Fact 2.4 (See [21, Theorem 3.4 and Corollary 5.6], or [18, Theorem 24.1(b)].) Let $A, B : X \rightrightarrows X^*$ be maximal monotone operators. Assume $\bigcup_{\lambda>0} \lambda [P_X(\text{dom } F_A) - P_X(\text{dom } F_B)]$ is a closed subspace. If

$$(4) \quad F_{A+B} \geq \langle \cdot, \cdot \rangle \text{ on } X \times X^*,$$

then $A + B$ is maximal monotone.

Fact 2.5 (Simons) (See [18, Theorem 27.1 and Theorem 27.3].) Let $A : X \rightrightarrows X^*$ be maximal monotone with $\text{int dom } A \neq \emptyset$. Then $\text{int dom } A = \text{int } [P_X \text{dom } F_A]$, and $\text{dom } A$ is convex.

Now we cite some results on maximal monotone operators of type (FPV).

Fact 2.6 (Simons) (See [18, Theorem 48.4(d)].) Let $f : X \to ]-\infty, +\infty[$ be proper, lower semicontinuous, and convex. Then $\partial f$ is of type (FPV).

Fact 2.7 (Simons) (See [18, Theorem 46.1].) Let $A : X \rightrightarrows X^*$ be a maximal monotone linear relation. Then $A$ is of type (FPV).

Fact 2.8 (Simons and Verona-Verona) (See [18, Theorem 44.1] or [19].) Let $A : X \rightrightarrows X^*$ be a maximal monotone. Suppose that for every closed convex subset $C$ of $X$ with $\text{dom } A \cap \text{int } C \neq \emptyset$, the operator $A + N_C$ is maximal monotone. Then $A$ is of type (FPV).

The following statement first appeared in [17, Theorem 41.5]. However, on [18, page 199], concerns were raised about the validity of the proof of [17, Theorem 41.5]. In [22], Voisei recently provided a result that generalizes and confirms [17, Theorem 41.5] and hence the following fact.

Fact 2.9 (Voisei) Let $A : X \rightrightarrows X^*$ be maximal monotone of type (FPV) with convex domain, let $C$ be a nonempty closed convex subset of $X$, and suppose that $\text{dom } A \cap \text{int } C \neq \emptyset$. Then $A + N_C$ is maximal monotone.

Corollary 2.10 Let $A : X \rightrightarrows X^*$ be maximal monotone of type (FPV) with convex domain, let $C$ be a nonempty closed convex subset of $X$, and suppose that $\text{dom } A \cap \text{int } C \neq \emptyset$. Then $A + N_C$ is of type (FPV).

Proof. By Fact 2.9, $A + N_C$ is maximal monotone. Let $D$ be a nonempty closed convex subset of $X$, and suppose that $\text{dom } (A + N_C) \cap D \neq \emptyset$. Let $x_1 \in \text{dom } A \cap \text{int } C$ and $x_2 \in \text{dom } (A + N_C) \cap \text{int } D$. Thus, there exists $\delta > 0$ such that $x_1 + \delta B_X \subseteq C$ and $x_2 + \delta B_X \subseteq D$. Then for small enough $\lambda \in ]0, 1[$, we have $x_2 + \lambda (x_1 - x_2) + \frac{\lambda}{2} \delta B_X \subseteq D$. Clearly, $x_2 + \lambda (x_1 - x_2) + \lambda \delta B_X \subseteq C$. Thus $x_2 + \lambda (x_1 - x_2) + \frac{\lambda}{2} \delta B_X \subseteq C \cap D$. Since $\text{dom } A$
is convex, $x_2 + \lambda(x_1 - x_2) \in \text{dom} A$ and $x_2 + \lambda(x_1 - x_2) \in \text{dom} A \cap \text{int}(C \cap D)$. By Fact 2.1, $A + N_C + N_D = A + N_{C \cap D}$. Then, by Fact 2.3 (applied to $A$ and $C \cap D$), $A + N_C + N_D = A + N_{C \cap D}$ is maximal monotone. By Fact 2.8, $A + N_C$ is of type $(FPV)$. ■

Corollary 2.11 Let $A : X \rightrightarrows X^*$ be a maximal monotone linear relation, let $C$ be a nonempty closed convex subset of $X$, and suppose that $\text{dom} A \cap \text{int} C \neq \emptyset$. Then $A + N_C$ is of type $(FPV)$.

Proof. Apply Fact 2.7 and Corollary 2.10 ■

3 Main Result

The following result plays a key role in the proof of Theorem 3.4. The first half of its proof follows along the lines of the proof of [13, Theorem 44.2].

Proposition 3.1 Let $A, B : X \rightrightarrows X^*$ be maximal monotone with $\text{dom} A \cap \text{int} \text{dom} B \neq \emptyset$. Assume that $A + N_{\text{dom} B}$ is maximal monotone of type $(FPV)$, and $\text{dom} A \cap \text{dom} B \subseteq \text{dom} B$. Then $P_X[\text{dom} F_{A+B}] = \text{dom} A \cap \text{dom} B$.

Proof. By [9, Theorem 3.4], $\overline{\text{dom} A \cap \text{dom} B} = \overline{\text{dom}(A + B)} \subseteq P_X[\text{dom} F_{A+B}]$. It suffices to show that

$$P_X[\text{dom} F_{A+B}] \subseteq \overline{\text{dom} A \cap \text{dom} B}. \quad (5)$$

After translating the graphs if necessary, we can and do assume that $0 \in \text{dom} A \cap \text{int} \text{dom} B$ and that $(0,0) \in \text{gra} B$.

To show (5), we take $z \in P_X[\text{dom} F_{A+B}]$ and we assume to the contrary that

$$z \notin \overline{\text{dom} A \cap \text{dom} B}. \quad (6)$$

Thus $\alpha = \text{dist}(z, \overline{\text{dom} A \cap \overline{\text{dom} B}}) > 0$. Now take $y_0^* \in X^*$ such that

$$\|y_0^*\| = 1 \quad \text{and} \quad \langle z, y_0^* \rangle \geq \frac{2}{3}\|z\|. \quad (7)$$

Set

$$U_n = [0, z] + \frac{\alpha}{4n}B_X, \quad \forall n \in \mathbb{N}. \quad (8)$$
Since \(0 \in \text{dom} B\), \(B = A + N_{\text{dom} B}\). Since \(B\) is maximal monotone and \(B + N_{\text{dom} B}\) is a monotone extension of \(B\), we must have \(B = B + N_{\text{dom} B}\). Thus
\[
A + B = A + N_{\text{dom} B} + B.
\]
(9)

Since \(\text{dom} A \cap \overline{\text{dom} B} \subseteq \text{dom} B\) by assumption, we obtain
\[
\text{dom} A \cap \text{dom} B \subseteq \text{dom}(A + N_{\text{dom} B}) = \text{dom} A \cap \overline{\text{dom} B} \subseteq \text{dom} A \cap \text{dom} B.
\]
Hence
\[
\text{dom} A \cap \text{dom} B = \text{dom}(A + N_{\text{dom} B}).
\]
(10)

By (6) and (11), \(z \not\in \text{dom}(A + N_{\text{dom} B})\) and thus \((z, ny_0^*) \not\in \text{gra}(A + N_{\text{dom} B})\), \(\forall n \in \mathbb{N}\). For every \(n \in \mathbb{N}\), since \(z \in U_n\) and since \(A + N_{\text{dom} B}\) is of type (FPV) by assumption, we deduce the existence of \((z_n, z_n^*) \in \text{gra}(A + N_{\text{dom} B})\) such that \(z_n \in U_n\) and
\[
\langle z - z_n, z_n^* \rangle > n \langle z - z_n, y_0^* \rangle, \quad \forall n \in \mathbb{N}.
\]
(11)

Hence, using (8), there exists \(\lambda_n \in [0, 1]\) such that
\[
\|z - z_n - \lambda_n z\| = \|z_n - (1 - \lambda_n)z\| < \frac{1}{4} \alpha, \quad \forall n \in \mathbb{N}.
\]
(12)

By the triangle inequality, we have \(\|z - z_n\| < \lambda_n \|z\| + \frac{1}{4} \alpha\) for every \(n \in \mathbb{N}\). From the definition of \(\alpha\) and (10), it follows that \(\alpha \leq \|z - z_n\|\) and hence that \(\alpha < \lambda_n \|z\| + \frac{1}{4} \alpha\). Thus,
\[
\frac{3}{4} \alpha < \lambda_n \|z\|, \quad \forall n \in \mathbb{N}.
\]
(13)

By (12) and (7),
\[
\langle z - z_n - \lambda_n z, y_0^* \rangle \geq -\|z_n - (1 - \lambda_n)z\| > -\frac{1}{4} \alpha, \quad \forall n \in \mathbb{N}.
\]
(14)

By (14), (7) and (13),
\[
\langle z - z_n, y_0^* \rangle > \lambda_n \langle z, y_0^* \rangle - \frac{1}{4} \alpha > \frac{3}{4} \alpha - \frac{1}{4} \alpha = \frac{1}{4} \alpha, \quad \forall n \in \mathbb{N}.
\]
(15)

Then, by (11) and (15),
\[
\langle z - z_n, z_n^* \rangle > \frac{1}{4} n \alpha, \quad \forall n \in \mathbb{N}.
\]
(16)

By (8), there exist \(t_n \in [0, 1]\) and \(b_n \in \frac{\alpha}{n} B_X\) such that \(z_n = t_n z + b_n\). Since \(t_n \in [0, 1]\), there exists a convergent subsequence of \((t_n)_{n \in \mathbb{N}}\), which, for convenience, we still denote by \((t_n)_{n \in \mathbb{N}}\). Then \(t_n \to \beta\), where \(\beta \in [0, 1]\). Since \(b_n \to 0\), we have
\[
\frac{\alpha}{n} \to \beta \alpha.
\]
(17)
By (11), \( z_n \in \text{dom } A \cap \text{dom } B \); thus, \( \|z_n - z\| \geq \alpha \) and \( \beta \in [0, 1[ \). In view of (9) and (16), we have, for every \( z^* \in X^* \),

\[
F_{A+B}(z, z^*) = F_{A+N_{\text{dom } B} + B}(z, z^*) \\
\geq \sup_{\{n \in \mathbb{N}, y^* \in X^*\}} \left[ \langle z_n, z^* \rangle + \langle z - z_n, z_n^* \rangle + \langle z - z_n, y^* \rangle - \iota_{\text{gra } B}(z_n, y^*) \right]
\]

(18)

\[
\geq \sup_{\{n \in \mathbb{N}, y^* \in X^*\}} \left[ \langle z_n, z^* \rangle + \frac{1}{4} n \alpha + \langle z - z_n, y^* \rangle - \iota_{\text{gra } B}(z_n, y^*) \right].
\]

We now claim that

(19) \( F_{A+B}(z, z^*) = \infty \).

We consider two cases.

Case 1: \( \beta = 0 \).

By (17) and Fact 2.2 (applied to \( 0 \in \text{int } \text{dom } B \)), there exist \( N \in \mathbb{N} \) and \( K > 0 \) such that

(20) \( Bz_n \neq \emptyset \) and \( \sup_{y^* \in Bz_n} \|y^*\| \leq K, \forall n \geq N. \)

Then, by (18),

\[
F_{A+B}(z, z^*) \geq \sup_{\{n \geq N, y^* \in X^*\}} \left[ \langle z_n, z^* \rangle + \frac{1}{4} n \alpha + \langle z - z_n, y^* \rangle - \iota_{\text{gra } B}(z_n, y^*) \right]
\]

\[
\geq \sup_{\{n \geq N, y^* \in Bz_n\}} \left[ -\|z_n\| \cdot \|z^*\| + \frac{1}{4} n \alpha - \|z - z_n\| \cdot \|y^*\| \right]
\]

\[
\geq \sup_{\{n \geq N\}} \left[ -\|z_n\| \cdot \|z^*\| + \frac{1}{4} n \alpha - K \|z - z_n\| \right] \quad \text{(by (20))}
\]

\[
= \infty \quad \text{(by (17)).}
\]

Thus (19) holds.

Case 2: \( \beta \neq 0 \).

Take \( v^*_n \in Bz_n \). We consider two subcases.

Subcase 2.1: \( (v^*_n)_{n \in \mathbb{N}} \) is bounded.

By (18),

\[
F_{A+B}(z, z^*) \geq \sup_{\{n \in \mathbb{N}\}} \left[ \langle z_n, z^* \rangle + \frac{1}{4} n \alpha + \langle z - z_n, v^*_n \rangle \right]
\]

\[
\geq \sup_{\{n \in \mathbb{N}\}} \left[ -\|z_n\| \cdot \|z^*\| + \frac{1}{4} n \alpha - \|z - z_n\| \cdot \|v^*_n\| \right]
\]

\[
= \infty \quad \text{(by (17) and the boundedness of \( (v^*_n)_{n \in \mathbb{N}} \)).}
\]
Hence (19) holds.

**Subcase 2.2**: \((v_n^*)_{n \in \mathbb{N}}\) is unbounded.

We first show

\[
\limsup_{n \to \infty} \langle z - z_n, v_n^* \rangle \geq 0. \tag{21}
\]

Since \((v_n^*)_{n \in \mathbb{N}}\) is unbounded and after passing to a subsequence if necessary, we assume that \(\|v_n^*\| \neq 0, \forall n \in \mathbb{N}\) and that \(\|v_n^*\| \to +\infty\). By \(0 \in \text{int dom } B\) and Fact 2.2 there exist \(\delta > 0\) and \(M > 0\) such that

\[
B_y \neq \emptyset \quad \text{and} \quad \sup_{y^* \in B_y} \|y^*\| \leq M, \quad \forall y \in \delta B_X. \tag{22}
\]

Then we have

\[
\begin{align*}
\langle z_n - y, v_n^* - y^* \rangle &\geq 0, \quad \forall y \in \delta B_X, y^* \in B_y, n \in \mathbb{N} \\
\Rightarrow \langle z_n, v_n^* \rangle - \langle y, v_n^* \rangle + \langle z_n - y, -y^* \rangle &\geq 0, \quad \forall y \in \delta B_X, y^* \in B_y, n \in \mathbb{N} \\
\Rightarrow \langle z_n, v_n^* \rangle - \langle y, v_n^* \rangle &\geq \langle z_n - y, y^* \rangle, \quad \forall y \in \delta B_X, y^* \in B_y, n \in \mathbb{N} \\
\Rightarrow \langle z_n, v_n^* \rangle - \langle y, v_n^* \rangle &\geq -\langle \|z_n\| + \delta \rangle M, \quad \forall y \in \delta B_X, n \in \mathbb{N} \quad \text{(by (22))} \\
\Rightarrow \langle z_n, v_n^* \rangle &\geq \langle y, v_n^* \rangle - (\|z_n\| + \delta)M, \quad \forall y \in \delta B_X, n \in \mathbb{N} \\
\Rightarrow \langle z_n, v_n^* \rangle &\geq \delta \|v_n^*\| - (\|z_n\| + \delta)M, \quad \forall n \in \mathbb{N} \\
\end{align*}
\]

\[
\Rightarrow \langle z_n, \frac{v_n^*}{\|v_n^*\|} \rangle \geq \delta - \frac{(\|z_n\| + \delta)M}{\|v_n^*\|}, \quad \forall n \in \mathbb{N}. \tag{23}
\]

By the Banach-Alaoglu Theorem (see [16, Theorem 3.15]), there exist a weak* convergent subnet \((v_{\gamma}^*)_{\gamma \in \Gamma}\) of \((v_n^*)_{n \in \mathbb{N}}\), say \(\frac{v_{\gamma}^*}{\|v_{\gamma}^*\|} \overset{w^*}{\rightharpoonup} w^* \in X^*\).

Using (17) and taking the limit in (23) along the subnet, we obtain

\[
\langle \beta z, w^* \rangle \geq \delta. \tag{25}
\]

Since \(\beta > 0\), we have

\[
\langle z, w^* \rangle \geq \frac{\delta}{\beta} > 0. \tag{26}
\]

Now we assume to the contrary that

\[
\limsup_{n \to \infty} \langle z - z_n, v_n^* \rangle < -\varepsilon,
\]

for some \(\varepsilon > 0\).
Then, for all \( n \) sufficiently large,
\[
\langle z - z_n, v_n^* \rangle < -\frac{\varepsilon}{2},
\]
and so
\[
(27) \quad \langle z - z_n, \frac{v_n^*}{\|v_n^*\|} \rangle < -\frac{\varepsilon}{2\|v_n^*\|}.
\]
Then by (17) and (24), taking the limit in (27) along the subnet again, we see that
\[
\langle z - \beta z, w^* \rangle \leq 0.
\]
Since \( \beta < 1 \), we deduce \( \langle z, w^* \rangle \leq 0 \) which contradicts (26). Hence (21) holds. By (18),
\[
F_{A+B}(z, z^*) \geq \sup_{\{n \in \mathbb{N}\}} \left[ \langle z_n, z^* \rangle + \frac{1}{4} n \alpha + \langle z - z_n, v_n^* \rangle \right]
\]
\[
\geq \sup_{\{n \in \mathbb{N}\}} \left[ -\|z_n\| \cdot \|z^*\| + \frac{1}{4} n \alpha + \langle z - z_n, v_n^* \rangle \right]
\]
\[
\geq \limsup_{n \to \infty} \left[ -\|z_n\| \cdot \|z^*\| + \frac{1}{4} n \alpha + \langle z - z_n, v_n^* \rangle \right]
\]
\[
= \infty \quad (\text{by (17) and (21)}).
\]
Hence
\[
(28) \quad F_{A+B}(z, z^*) = \infty.
\]
Therefore, we have verified (19) in all cases. However, (19) contradicts our original choice that \( z \in P_X[\text{dom } F_{A+B}] \). Hence \( P_X[\text{dom } F_{A+B}] \subseteq \text{dom } A \cap \text{dom } B \) and thus (5) holds. Thus \( P_X[\text{dom } F_{A+B}] = \text{dom } A \cap \text{dom } B \).

**Corollary 3.2** Let \( A : X \rightrightarrows X^* \) be maximal monotone of type (FPV) with convex domain, and \( B : X \rightrightarrows X^* \) be maximal monotone with \( \text{dom } A \cap \text{int } \text{dom } B \neq \emptyset \). Assume that \( \text{dom } A \cap \text{dom } B \subseteq \text{dom } B \). Then \( P_X[\text{dom } F_{A+B}] = \text{dom } A \cap \text{dom } B \).

**Proof.** Combine Fact 2.5, Corollary 2.10 and Proposition 3.1.

**Corollary 3.3** Let \( A : X \rightrightarrows X^* \) be a maximal monotone linear relation, and let \( B : X \rightrightarrows X^* \) be maximal monotone with \( \text{dom } A \cap \text{int } \text{dom } B \neq \emptyset \). Assume that \( \text{dom } A \cap \text{dom } B \subseteq \text{dom } B \). Then \( P_X[\text{dom } F_{A+B}] = \text{dom } A \cap \text{dom } B \).

**Proof.** Combine Fact 2.5, Corollary 2.11 and Proposition 3.1. Alternatively, combine Fact 2.7 and Corollary 3.2.

We are now ready for our main result.
Theorem 3.4 (Main Result) Let $A, B : X \rightrightarrows X^*$ be maximal monotone with $\text{dom } A \cap \text{int } \text{dom } B \neq \emptyset$. Assume that $A + N_{\text{dom } B}$ is maximal monotone of type (FPV), and that $\text{dom } A \cap \text{dom } B \subseteq \text{dom } B$. Then $A + B$ is maximal monotone.

Proof. After translating the graphs if necessary, we can and do assume that $0 \in \text{dom } A \cap \text{int } \text{dom } B$ and that $(0, 0) \in \text{gra } A \cap \text{gra } B$. By Fact 2.3, $\text{dom } A \subseteq P_X(\text{dom } F_A)$ and $\text{dom } B \subseteq P_X(\text{dom } F_B)$. Hence,

$$
\bigcup_{\lambda > 0} \lambda (P_X(\text{dom } F_A) - P_X(\text{dom } F_B)) = X.
$$

Thus, by Fact 2.4 it suffices to show that

$$
F_{A+B}(z, z^*) \geq \langle z, z^* \rangle, \quad \forall (z, z^*) \in X \times X^*.
$$

Take $(z, z^*) \in X \times X^*$. Then

$$
F_{A+B}(z, z^*)
\geq \sup_{\{x, x^*, y^*\}} \left[ \langle x, z^* \rangle + \langle z, x^* \rangle - \langle x, x^* \rangle + \langle z - x, y^* \rangle - \iota_{\text{gra } A}(x, x^*) - \iota_{\text{gra } B}(x, y^*) \right].
$$

Assume to the contrary that

$$
F_{A+B}(z, z^*) < \langle z, z^* \rangle.
$$

Then $(z, z^*) \in \text{dom } F_{A+B}$ and, by Proposition 3.1

$$
z \in \overline{\text{dom } A \cap \text{dom } B} = P_X[\text{dom } F_{A+B}].
$$

Next, we show that

$$
F_{A+B}(\lambda z, \lambda z^*) \geq \lambda^2 \langle z, z^* \rangle, \quad \forall \lambda \in ]0, 1[.
$$

Let $\lambda \in ]0, 1[$. By 33 and Fact 2.5, $z \in P_X \text{dom } F_B$. By Fact 2.5 again and $0 \in \text{int } \text{dom } B$, $0 \in \text{int } P_X \text{dom } F_B$. Then, by 24 Theorem 1.1.2(ii), we have

$$
\lambda z \in \text{int } P_X \text{dom } F_B = \text{int } [P_X \text{dom } F_B].
$$

Combining (35) and Fact 2.5 we see that $\lambda z \in \text{int } \text{dom } B$.

We consider two cases.

Case 1: $\lambda z \in \text{dom } A$.

By (31),

$$
F_{A+B}(\lambda z, \lambda z^*)
\geq \sup_{\{x, x^*, y^*\}} \left[ \langle \lambda z, \lambda z^* \rangle + \langle \lambda z, x^* \rangle - \langle \lambda z, x^* \rangle + \langle \lambda z - \lambda z, y^* \rangle - \iota_{\text{gra } A}(\lambda z, x^*) - \iota_{\text{gra } B}(\lambda z, y^*) \right]
= \langle \lambda z, \lambda z^* \rangle.
$$
Hence (34) holds.

Case 2: \( \lambda z \notin \text{dom} A \).

Using \( 0 \in \text{dom} A \cap \text{dom} B \) and the convexity of \( \text{dom} A \cap \text{dom} B \) (which follows from (33)), we obtain \( \lambda z \in \text{dom} A \cap \text{dom} B \subseteq \text{dom} A \cap \text{dom} B \). Set

\[
U_n = \lambda z + \frac{1}{n} B_X, \quad \forall n \in \mathbb{N}.
\]

Then \( U_n \cap \text{dom}(A + N_{\text{dom} B}) \neq \emptyset \). Since \( (\lambda z, \lambda z^*) \notin \text{gra}(A + N_{\text{dom} B}) \), \( \lambda z \in U_n \), and \( A + N_{\text{dom} B} \) is of type (FPV), there exists \( (b_n, b_n^*) \in \text{gra}(A + N_{\text{dom} B}) \) such that \( b_n \in U_n \) and

\[
\langle \lambda z, b_n^* \rangle + \langle b_n, \lambda z^* \rangle - \langle b_n, b_n^* \rangle > \lambda^2 \langle z, z^* \rangle, \quad \forall n \in \mathbb{N}.
\]

Since \( \lambda z \in \text{int} \text{dom} B \) and \( b_n \to \lambda z \), by Fact 2.2 there exist \( N \in \mathbb{N} \) and \( M > 0 \) such that

\[
b_n \in \text{int} \text{dom} B \quad \text{and} \quad \sup_{v^* \in B_{B_n}} \|v^*\| \leq M, \quad \forall n \geq N.
\]

Hence \( N_{\text{dom} B}(b_n) = \{0\} \) and thus \( (b_n, b_n^*) \in \text{gra} A \) for every \( n \geq N \). Thus by (31), (37) and (38),

\[
F_{A+B}(\lambda z, \lambda z^*) \geq \sup_{\{b_n, \lambda z^*\}} \left[ \langle b_n, \lambda z^* \rangle + \langle \lambda z, b_n^* \rangle - \langle b_n, b_n^* \rangle + \langle \lambda z - b_n, v^* \rangle \right], \quad \forall n \geq N
\]

\[
\geq \sup_{\{v^* \in B_{B_n}\}} \left[ \lambda^2 \langle z, z^* \rangle + \langle \lambda z - b_n, v^* \rangle \right], \quad \forall n \geq N \quad \text{(by (37))}
\]

\[
\geq \sup \left[ \lambda^2 \langle z, z^* \rangle - M \|\lambda z - b_n\| \right], \quad \forall n \geq N \quad \text{(by (38))}
\]

\[
\geq \lambda^2 \langle z, z^* \rangle \quad \text{(by } b_n \to \lambda z)\).
\]

Hence \( F_{A+B}(\lambda z, \lambda z^*) \geq \lambda^2 \langle z, z^* \rangle \).

We have verified that (31) holds in both cases. Since \( (0, 0) \in \text{gra} A \cap \text{gra} B \), we obtain \( (\forall (x, x^*) \in \text{gra}(A + B)) \langle x, x^* \rangle \geq 0 \). Thus, \( F_{A+B}(0, 0) = 0 \). Now define

\[
f: [0, 1] \to \mathbb{R}: t \to F_{A+B}(tz, tz^*).
\]

Then \( f \) is continuous on \([0, 1]\) by [21] Proposition 2.1.6. From (31), we obtain

\[
F_{A+B}(z, z^*) = \lim_{\lambda \to 1^-} F_{A+B}(\lambda z, \lambda z^*) \geq \lim_{\lambda \to 1^-} \langle \lambda z, \lambda z^* \rangle = \langle z, z^* \rangle,
\]

which contradicts (32). Hence

\[
F_{A+B}(z, z^*) \geq \langle z, z^* \rangle.
\]

Therefore, (30) holds, and \( A + B \) is maximal monotone. \( \square \)

Theorem 3.4 allows us to deduce both new and previously known sum theorems.
Corollary 3.5 Let \( f : X \to ]-\infty, +\infty[ \) be proper, lower semicontinuous, convex, and let \( B : X \rightrightarrows X^* \) be maximal monotone with \( \text{dom } f \cap \text{int } \text{dom } B \neq \emptyset \). Assume that \( \text{dom } \partial f \cap \text{dom } B \subseteq \text{dom } B \). Then \( \partial f + B \) is maximal monotone.

**Proof.** By Fact 2.5 and Fact 2.1, \( \partial f + N_{\text{dom } B} \text{dom } B = \partial (f + \iota_{\text{dom } B}) \). Then by Fact 2.6, \( \partial f + N_{\text{dom } B} \) is type of (FPV). Now apply Theorem 3.4.

Corollary 3.6 Let \( A : X \rightrightarrows X^* \) be maximal monotone of type (FPV), and let \( B : X \rightrightarrows X^* \) be maximal monotone with full domain. Then \( A + B \) is maximal monotone.

**Proof.** Since \( A + N_{\text{dom } B} = A + N_{X} = A \) and thus \( A + N_{\text{dom } B} \) is maximal monotone of type (FPV), the conclusion follows from Theorem 3.4.

Corollary 3.7 (Verona-Verona) (See [20, Corollary 2.9(a)] or [18, Theorem 53.1].) Let \( f : X \to ]-\infty, +\infty[ \) be proper, lower semicontinuous, and convex, and let \( B : X \rightrightarrows X^* \) be maximal monotone with full domain. Then \( \partial f + B \) is maximal monotone.

**Proof.** Clear from Corollary 3.5. Alternatively, combine Fact 2.6 and Corollary 3.6.

Corollary 3.8 (Heisler) (See [11, Remark, page 17].) Let \( A, B : X \rightrightarrows X^* \) be maximal monotone with full domain. Then \( A + B \) is maximal monotone.

**Proof.** Let \( C \) be a nonempty closed convex subset of \( X \). By Corollary 3.7, \( N_{C} + A \) is maximal monotone. Thus, \( A \) is of type (FPV) by Fact 2.8. The conclusion now follows from Corollary 3.6.

Corollary 3.9 Let \( A : X \rightrightarrows X^* \) be maximal monotone of type (FPV) with convex domain, and let \( B : X \rightrightarrows X^* \) be maximal monotone with \( \text{dom } A \cap \text{int } \text{dom } B \neq \emptyset \). Assume that \( \text{dom } A \cap \text{dom } B \subseteq \text{dom } B \). Then \( A + B \) is maximal monotone.

**Proof.** Combine Fact 2.5, Corollary 2.10 and Theorem 3.4.

Corollary 3.10 (Voisei) (See [22].) Let \( A : X \rightrightarrows X^* \) be maximal monotone of type (FPV) with convex domain, let \( C \) be a nonempty closed convex subset of \( X \), and suppose that \( \text{dom } A \cap \text{int } C \neq \emptyset \). Then \( A + N_{C} \) is maximal monotone.

**Proof.** Apply Corollary 3.9.

Corollary 3.11 Let \( A : X \rightrightarrows X^* \) be a maximal monotone linear relation, and let \( B : X \rightrightarrows X^* \) be maximal monotone with \( \text{dom } A \cap \text{int } \text{dom } B \neq \emptyset \). Assume that \( \text{dom } A \cap \text{dom } B \subseteq \text{dom } B \). Then \( A + B \) is maximal monotone.
Proof. Combine Fact 2.7 and Corollary 3.9.

**Corollary 3.12** (See [2, Theorem 3.1].) Let $A : X \rightrightarrows X^*$ be a maximal monotone linear relation, let $C$ be a nonempty closed convex subset of $X$, and suppose that $\text{dom} A \cap \text{int} C \neq \emptyset$. Then $A + N_C$ is maximal monotone.

**Proof.** Apply Corollary 3.11.

**Corollary 3.13** Let $A : X \rightrightarrows X^*$ be a maximal monotone linear relation, and let $B : X \rightrightarrows X^*$ be maximal monotone with full domain. Then $A + B$ is maximal monotone.

**Proof.** Apply Corollary 3.11.

**Example 3.14** Suppose that $X = L^1[0,1]$, let

$$D = \{ x \in X \mid x \text{ is absolutely continuous}, x(0) = 0, x' \in X^* \},$$

and set

$$A : X \rightrightarrows X^* : x \mapsto \begin{cases} \{ x' \}, & \text{if } x \in D; \\ \emptyset, & \text{otherwise}. \end{cases}$$

By Phelps and Simons’ [12, Example 4.3], $A$ is an at most single-valued maximal monotone linear relation with proper dense domain, and $A$ is neither symmetric nor skew. Now let $J$ be the duality mapping, i.e., $J = \partial \frac{1}{2} \| \cdot \|^2$. Then Corollary 3.13 implies that $A + J$ is maximal monotone. To the best of our knowledge, the maximal monotonicity of $A + J$ cannot be deduced from any previously known result.

**Remark 3.15** In [3], it was shown that the sum theorem is true when $A$ is a linear relation, $B$ is the subdifferential operator of a proper lower semicontinuous sublinear function, and Rockafellar’s constraint qualification holds. When the domain of the subdifferential operator is closed, then that result can be deduced from Theorem 3.4. However, it is possible that the domain of the subdifferential operator of a proper lower semicontinuous sublinear function does not have to be closed. For an example, see [11, Example 5.4]: Set $C = \{(x, y) \in \mathbb{R}^2 \mid 0 < 1/x \leq y \}$ and $f = \iota_C$. Then $f$ is not subdifferentiable at any point in the boundary of its domain, except at the origin. Thus, in the general case, we do not know whether or not it is possible to deduce the result in [3] from Theorem 3.4.

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