Gravitational excitons from extra dimensions

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Abstract

Inhomogeneous multidimensional cosmological models with a higher dimensional space-time manifold
\[ M = M_0 \times \prod_{i=1}^{n} M_i \quad (n \geq 1) \]
are investigated under dimensional reduction to \( D_0 \)-dimensional effective models. In the Einstein conformal frame, small excitations of the scale factors of the internal spaces near minima of an effective potential have a form of massive scalar fields in the external space-time. Parameters of models which ensure minima of the effective potentials are obtained for particular cases and masses of gravitational excitons are estimated.

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1 Introduction

The large scale dynamics of the observable part of our present time universe is well described by the Friedmann model with 4-dimensional Friedmann - Robertson - Walker (FRW) metric. However, it is possible that space-time at short (Planck) distances might have a dimensionality of more than four and possess a rather complex topology. String theory and its recent generalizations — p-brane, M- and F-theory widely use this concept and give it a new foundation. The most consistent formulations of these theories are possible in space-times with critical dimensions \( D_c > 4 \), for example, in string theory there are \( D_c = 26 \) or 10 for the bosonic and supersymmetric version, respectively. Usually it is supposed that a \( D \)-dimensional manifold \( M \) undergoes a "spontaneous compactification": \( M \rightarrow M^4 \times B^{D-4} \), where \( M^4 \) is the 4-dimensional external space-time and \( B^{D-4} \) is a compact internal space. So it is natural to consider cosmological consequences of such compactifications. With this in mind we shall investigate multidimensional cosmological models (MCM) with topology

\[ M = M_0 \times M_1 \times \ldots \times M_n , \]

where \( M_0 \) denotes the \( D_0 \)-dimensional (usually \( D_0 = 4 \)) external space-time and \( M_i \quad (i = 1, \ldots, n) \) are \( D_i \)-dimensional internal spaces. To make the internal dimensions unobservable at present time these internal...
spaces have to be compact and reduced to scales near Planck length $L_{Pl} \sim 10^{-33} \text{cm}$, i.e. scale factors $a_i$ of the internal spaces should be of order of $L_{Pl}$. In this case we cannot move in extra-dimensions and our space-time is apparently 4-dimensional. There is no problem to construct compact spaces with a positive curvature \cite{9,10}. (For example, every Einstein manifold with constant positive curvature is necessarily compact \cite{11}.) However, Ricci-flat spaces and negative curvature spaces can be compact also. This can be achieved by appropriate periodicity conditions for the coordinates \cite{12}-\cite{16} or, equivalently, through the action of discrete groups $\Gamma$ of isometries related to face pairings and to the manifold’s topology. For example, 3-dimensional spaces of constant negative curvature are isometric to the open, simply connected, infinite hyperbolic (Lobachevsky) space $H^3$ \cite{9,10}. But there exist also an infinite number of compact, multiply connected, hyperbolic quotient manifolds $H^3/\Gamma$, which can be used for the construction of FRW metrics with negative curvature \cite{12}-\cite{14}. These manifolds are built from a fundamental polyhedron (FP) in $H^3$ with faces pairwise identified. The FP determines a tessellation of $H^3$ into cells which are replicas of the FP, through the action of the discrete group $\Gamma$ of isometries \cite{14}. The simplest example of Ricci-flat compact spaces is given by $D$ - dimensional tori $T^D = \mathbb{R}^D/\Gamma$. Thus, internal spaces may have nontrivial global topology, being compact (i.e. closed and bounded) for any sign of spatial curvature.

In the cosmological context, internal spaces can be called compactified, when they are obtained by a compactification \cite{17} or factorization ("wrapping") in the usual mathematical understanding (e.g. by replacements of the type $\mathbb{R}^D \to S^D$, $\mathbb{R}^D \to \mathbb{R}^D/\Gamma$ or $H^D \to H^D/\Gamma$) with additional contraction of the sizes to Planck scale. The physical constants that appear in the effective 4-dimensional theory after dimensional reduction of an originally higher-dimensional model are the result of integration over the extra dimensions. If the volumes of the internal spaces would change, so would the observed constants. Because of limitation on the variability of these constants \cite{18,19} the internal spaces are static or at least slowly variable since the time of primordial nucleosynthesis and as we mentioned above their sizes are of the order of the Planck length. Obviously, such compactifications have to be stable against small fluctuations of the sizes (the scale factors $a_i$) of the internal spaces. This means that the effective potential of the model obtained under dimensional reduction to a 4-dimensional effective theory should have minima at $a_i \sim L_{Pl}$ ($i = 1, \ldots, n$). Because of its crucial role the problem of stable compactification of extra dimensions was intensively studied in a large number of papers \cite{20}-\cite{36}. As result certain conditions were obtained which ensure the stability of these compactifications. However, position of a system at a minimum of an effective potential means not necessarily that extra-dimensions are unobservable. As we shall show below, small excitations of a system near a minimum can be observed as massive scalar fields in the external space-time. In solid state physics, excitations of electron subsystems in crystals are called excitons. In our case the internal spaces are an analog of the electronic subsystem and their excitations can be called gravitational excitons. If masses of these excitations are much less than Planck mass $M_{Pl} \sim 10^{-5} \text{g}$, they should be observable confirming the existence of extra-dimensions. In the opposite case of very heavy excitons with masses $m \sim M_{Pl}$ it is impossible to excite them at present time and extra-dimensions are unobservable by this way.

The paper is organized as follows. In Sec.II we describe our model and obtain an effective theory in Brans-Dicke and Einstein conformal frames. In Sec.III it is shown that small excitations of the scale factors of the internal spaces near minima of an effective potential in the Einstein frame have a form of massive scalar fields in the external space-time. The masses of such scalar fields are evaluated for particular classes of effective potentials with minima in the case of one-internal-space models (Sec.IV) and two-internal-space models (Sec.V). In Sec.VI we show that conditions for the existence of stable configurations may be quite
different for these two types of models.

2 The model

We consider a cosmological model with metric

\[ g = g^{(0)} + \sum_{i=1}^{n} e^{2\beta(x)} g^{(i)}, \]

(2.1)

which is defined on manifold \([1,1]\) where \(x\) are some coordinates of the \(D_0\) - dimensional manifold \(M_0\) and

\[ g^{(0)} = g^{(0)}_{\mu\nu}(x) dx^\mu \otimes dx^\nu. \]

(2.2)

Let manifolds \(M_i\) be \(D_i\) - dimensional Einstein spaces with metric \(g^{(i)}\), i.e.

\[ R_{mn}[g^{(i)}] = \lambda^i g^{(i)}_{mn}, \quad m, n = 1, \ldots, D_i \]

(2.3)

and

\[ R[g^{(i)}] = \lambda^i R^i \equiv R^i. \]

(2.4)

In the case of constant curvature spaces parameters \(\lambda^i\) are normalized as \(\lambda^i = k_i (D_i - 1)\) with \(k_i = \pm 1, 0\). We note that each of the spaces \(M_i\) can be split into a product of Einstein spaces: \(M_i \rightarrow \prod_{k=1}^{n_i} M_k^{(i)}\). Here \(M_k^{(i)}\) are Einstein spaces of dimensions \(D_k^{(i)}\) with metric \(g^{(i)}_{(k)}\): \(R_{mn}[g^{(i)}_{(k)}] = \lambda_k^{(i)} g^{(i)}_{mn}(m, n = 1, \ldots, D_k^{(i)})\) and \(R[g^{(i)}_{(k)}] = \lambda_k^{(i)} D_k^{(i)}\). Such a splitting procedure is well defined provided \(M_k^{(i)}\) are not Ricci - flat \([37, 38]\). If \(M_i\) is a split space, then for curvature and dimension we have respectively \([37]\): \(R[g^{(i)}] = \sum_{k=1}^{n_i} R[g^{(i)}_{(k)}]\) and \(D_i = \sum_{k=1}^{n_i} D_k^{(i)}\). Later on we shall not specify the structure of the spaces \(M_i\). We require only \(M_i\) to be compact spaces with arbitrary sign of curvature.

With total dimension \(D = \sum_{i=0}^{n} D_i\), \(\kappa^2\) a \(D\) - dimensional gravitational constant, \(\Lambda\) - a \(D\) - dimensional cosmological constant and \(S_{YGH}\) the standard York - Gibbons - Hawking boundary term \([39, 40]\), we consider an action of the form

\[ S = \frac{1}{2\kappa^2} \int_M d^D x \sqrt{|g|} \{ R[g] - 2\Lambda \} + S_{add} + S_{YGH}. \]

(2.5)

The additional potential term

\[ S_{add} = - \int_M d^D x \sqrt{|g|} \rho(x) \]

(2.6)

is not specified and left in its general form, taking into account the Casimir effect \([22]\), the Freund - Rubin monopole ansatz \([1]\), a perfect fluid \([21, 22]\) or other hypothetical potentials \([34, 36]\). In all these cases \(\rho\) depends on the external coordinates through the scale factors \(a_i(x) = e^{\beta_i(x)} (i = 1, \ldots, n)\) of the internal spaces. We did not include into the action \((2.5)\) a minimally coupled scalar field with potential \(U(\psi)\), because in this case there exist no solutions with static internal spaces for scalar fields \(\psi\) depending on the external coordinates \([6]\).

After dimensional reduction the action reads

\[ S = \frac{1}{2\kappa^2} \int_{M_0} d^{D_0} x \sqrt{|g^{(0)}|} \prod_{i=1}^{n} e^{D_i \beta^i} \left\{ R[g^{(0)}] - G_{ij} g^{(0)\mu\nu} \partial_{\mu} \beta^i \partial_{\nu} \beta^j + \sum_{i=1}^{n} R[g^{(i)}] e^{-2\beta^i} - 2\Lambda - 2\kappa^2 \rho \right\}, \]

(2.7)
where \( \kappa_0^2 = \kappa^2 / \mu \) is the \( D_0 \)-dimensional gravitational constant, \( \mu = \prod_{i=1}^{n} \mu_i = \prod_{i=1}^{n} \int d^{D_0} y \sqrt{|g^{(0)}|} \) and \( G_{ij} = D_i \delta_{ij} - D_i D_j \ (i, j = 1, \ldots, n) \) is the midisuperspace metric \([33, 44]\). Here the scale factors \( \beta^i \) of the internal spaces play the role of scalar fields. Comparing this action with the tree-level effective action for a bosonic string it can be easily seen that the volume of the internal spaces \( e^{-2\Phi} = \prod_{i=1}^{n} e^{D_i \beta^i} \) plays the role of the dilaton field \([37, 44, 45]\). We note that sometimes all scalar fields associated with \( \beta^i \) are called dilatons. Action \((2.7)\) is written in the Brans - Dicke frame. Conformal transformation to the Einstein frame the action of a self-gravitating \( \sigma \) yields

\[
S = \frac{1}{2\kappa_0^2} \int_{M_0} d^{D_0} x \sqrt{|g^{(0)}|} \left\{ \tilde{R} \left[ \tilde{g}^{(0)} \right] - \tilde{G}_{ij} \tilde{g}^{(0)\mu\nu} \partial_\mu \beta^i \partial_\nu \beta^j - 2U_{\text{eff}} \right\} .
\]

The tensor components of the midisuperspace metric (target space metric on \( \mathbb{R}_T^n \)) \( \tilde{G}_{ij} \) \((i, j = 1, \ldots, n)\), its inverse metric \( \tilde{G}^{ij} \) and the effective potential are respectively

\[
\tilde{G}_{ij} = D_i \delta_{ij} + \frac{1}{D_0 - 2} D_i D_j ,
\]

\[
\tilde{G}^{ij} = \frac{\delta^{ij}}{D_i} + \frac{1}{2 - D_i}
\]

and

\[
U_{\text{eff}} = \left( \prod_{i=1}^{n} e^{D_i \beta^i} \right)^{-2 \frac{\kappa_0^2}{D_0 - 2}} \left[ - \frac{1}{2} \sum_{i=1}^{n} R_i e^{-2\beta^i} + \Lambda + \kappa^2 \rho \right] .
\]

We remind that \( \rho \) depends on the scale factors of the internal spaces: \( \rho = \rho \left( \beta^1, \ldots, \beta^n \right) \). Thus, we are led to the action of a self-gravitating \( \sigma \)-model with flat target space \((\mathbb{R}_T^n, \tilde{G}) \) \((2.10)\) and self-interaction described by the potential \((2.13)\).

Let us first consider the case of one internal space: \( n = 1 \). Redefining the dilaton field as

\[
\varphi \equiv \pm \sqrt{\frac{D_1 (D_1 - 2)}{D_0 - 2}} \beta^1
\]

we get for action and effective potential respectively

\[
S = \frac{1}{2\kappa_0^2} \int d^{D_0} x \sqrt{|g^{(0)}|} \left\{ \tilde{R} \left[ \tilde{g}^{(0)} \right] - \tilde{g}^{(0)\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - 2U_{\text{eff}} \right\}
\]

and

\[
U_{\text{eff}} = e^{2\varphi \left( \frac{D_1}{(D_1 - 2)(D_0 - 2)} \right)^{1/2}} \left[ - \frac{1}{2} R_1 e^{2\varphi \left( \frac{D_1}{D_0 - 2} \right)^{1/2}} + \Lambda + \kappa^2 \rho(\varphi) \right],
\]

where in the latter expression we use for definiteness sign minus.

Coming back to the general case \( n > 1 \) we bring midisuperspace metric (target space metric) \((2.10)\) by a regular coordinate transformation

\[
\varphi = Q \beta , \quad \beta = Q^{-1} \varphi
\]

to a pure Euclidean form

\[
\tilde{G}_{ij} d\beta^i \otimes d\beta^j = \sigma_{ij} d\varphi^i \otimes d\varphi^j = \sum_{i=1}^{n} d\varphi^i \otimes d\varphi^i ,
\]

\[
\tilde{G} = Q' Q , \quad \sigma = \text{diag} \left( +1 + 1, \ldots, +1 \right).
\]
The prime denotes the transposition.) An appropriate transformation $Q : \beta^i \rightarrow \varphi^i = Q^i_j \beta^j$ is given e.g. by

\[
\varphi^i = -A \sum_{i=1}^{n} D_i \beta^i,
\]
\[
\varphi^i = [D_{i-1}/\Sigma_{i-1}\Sigma_i]^{1/2} \sum_{j=1}^{n} D_j \left( \beta^j - \beta^{j-1} \right), \quad i = 2, \ldots, n, \quad (2.18)
\]

where $\Sigma_i = \sum_{j=i}^{n} D_j$.

\[
A = \pm \left[ \frac{1}{D^2} \frac{D - 2}{D_0 - 2} \right]^{1/2}, \quad (2.19)
\]

and $D' = \sum_{i=1}^{n} D_i$. So we can write action (2.9) as

\[
S = \frac{1}{2\kappa^2} \int_{M_0} d^nx \sqrt{|\hat{g}(0)|} \left\{ \hat{R} \left[ \hat{g}(0) \right] - \sigma_{ik} \hat{g}^{(0)\mu\nu} \partial_{\mu} \varphi^i \partial_{\nu} \varphi^k - 2U_{\text{eff}} \right\} \quad (2.20)
\]

with effective potential

\[
U_{\text{eff}} = e^{A(2\kappa_0-2)^2} \left( -\frac{1}{2} \sum_{i=1}^{n} R_i e^{-2(Q^{-1})^i_k \kappa^k} + \Lambda + \kappa^2 \rho \right). \quad (2.21)
\]

### 3 Gravitational excitons

Let us suppose that the effective potential (2.12) has minima at points $\vec{\beta}_c = (\beta^1_c, \ldots, \beta^n_c)$

\[
\frac{\partial U_{\text{eff}}}{\partial \beta^c} \bigg|_{\vec{\beta}_c} = 0, \quad c = 1, \ldots, m \quad (3.1)
\]

and that its Hessian

\[
a_{(c)ik} := \left. \frac{\partial^2 U_{\text{eff}}}{\partial \beta^i \partial \beta^k} \right|_{\vec{\beta}_c} \quad (3.2)
\]

does not identically vanish at these points. For small fluctuations $\eta^i := \beta^i - \beta^i_c$ we have then up to second order in the Taylor expansion

\[
U_{\text{eff}} = U_{\text{eff}} \left( \vec{\beta}_c \right) + \frac{1}{2} \sum_{i,k=1}^{n} a_{(c)ik} \eta^i \eta^k \quad (3.3)
\]

As sufficient condition for the existence of minima at $\vec{\beta}_c$ we choose in this paper the strong condition consisting in the positivity of the quadratic form

\[
\eta'^T A_c \eta \equiv \sum_{i,k=1}^{n} a_{(c)ik} \eta^i \eta^k > 0, \quad \forall \eta^1, \ldots, \eta^n \quad (3.4)
\]

with exception of the point $\eta^1 = \eta^2 = \ldots = \eta^n = 0$. It is clear that for higher order expansions of the effective potential inequality (3.4) can be weaken to a nonnegativity condition $\eta'^T A_c \eta \geq 0$ with additional requirements on the multilinearforms occurring in this case. We note that according to the Sylvester criterion positivity of quadratic forms is assured by the positivity of the principal minors of the corresponding matrix,
in our case of the matrix $A_c$:

$$a_{(c)11} > 0, \begin{vmatrix} a_{(c)11} & a_{(c)12} \\ a_{(c)21} & a_{(c)22} \end{vmatrix} > 0, \ldots$$

$$\begin{vmatrix} a_{(c)1} & \cdots & a_{(c)1n} \\ \vdots & & \vdots \\ a_{(c)n1} & \cdots & a_{(c)nn} \end{vmatrix} = \det A_c > 0.$$

Equation (3.1) and Hessian (3.2) are affected by midisuperspace coordinate transformation (2.16) as follows:

$$\frac{\partial U_{\text{eff}}}{\partial \varphi^i} \bigg|_{\varphi_0} = \frac{\partial U_{\text{eff}}}{\partial \beta^k} \bigg|_{\beta_0} (Q^{-1})^k_i = 0, \quad \varphi_0 = Q\beta_0,$$

$$a_{(c)ik} = \frac{\partial^2 U_{\text{eff}}}{\partial \beta^i \partial \beta^k} \bigg|_{\beta_0} = \frac{\partial^2 U_{\text{eff}}}{\partial \varphi^i \partial \varphi^j} \bigg|_{\varphi_0} = \frac{\partial \varphi^j}{\partial \beta^k} = Q^t a_{(c)ij} Q_k.$$

This means that matrices $A_c$ and $\bar{A}_c$ are congruent matrices $[46]$ $A_c = Q'\bar{A}_c Q$, and hence their rank and signature coincide.

Taking into account that transformation (2.16) holds also for small fluctuations near the minima

$$\xi = Q\eta, \quad \eta = Q^{-1}\xi, \quad \xi^i := \varphi^i - \varphi_0^i$$

we conclude that the quadratic form (3.4) is invariant under this transformation

$$\eta' A_c \eta = (Q^{-1}\xi)' Q'\bar{A}_c Q (Q^{-1}\xi) = \xi' \bar{A}_c \xi.$$

Together with the coinciding rank and signature of the congruent matrices $\bar{A}_c$ and $A_c$ this implies that the positivity of (3.4) remains preserved and minima of $U_{\text{eff}}$ in $\beta$-representation correspond to minima of $U_{\text{eff}}$ in $\varphi$-representation.

To get masses of excitations we need to diagonalize the matrices $\bar{A}_c$, keeping at the same time the kinetic term $g^{(0)\mu\nu} \sum_{i=1}^n \varphi^{i\mu} \varphi^{i\nu}$ in its diagonal form. One immediately checks that appropriate $SO(n)$-rotations $S_c$: $S'_c = S_c^{-1}$ fulfill these requirements

$$\bar{A}_c = S'_c M_c^2 S_c, \quad M_c^2 = \text{diag} (m_{(c)1}^2, m_{(c)2}^2, \ldots, m_{(c)n}^2)$$

and

$$g^{(0)\mu\nu} \sum_{i=1}^n \varphi^{i\mu} \varphi^{i\nu} = \tilde{g}^{(0)\mu\nu} \sum_{i=1}^n \phi^{i\mu} \phi^{i\nu},$$

where $\phi = S_c \varphi$. Introducing corresponding transformed fluctuation fields $\psi = S_c \xi$ we also verify that

$$\eta' A_c \eta = \xi' \bar{A}_c \xi = \psi' M_c^2 \psi.$$

It is clear from the Sylvester criterion that all diagonal elements of the matrix $M_c^2$ should be positive. From relations (2.17), (3.7), (3.10) follows that they are eigenvalues of matrix $\bar{A}_c$ as well as matrix $\tilde{G}^{-1} A_c$.

So explicit calculations of the matrices $S_c$ and $M_c^2$ go along standard lines $[46]$ and give e.g. in the case of two internal spaces ($n = 2$)

$$S_c = \begin{pmatrix} \cos \alpha_c & -\sin \alpha_c \\ \sin \alpha_c & \cos \alpha_c \end{pmatrix}$$

(3.13)
with the angle of rotation
\[ \tan 2\alpha_c = \frac{2\bar{a}_{(c)12}}{\bar{a}_{(c)22} - \bar{a}_{(c)11}} \] (3.14)
and
\[ m_{(c)1,2}^2 = \frac{1}{2} \left[ Tr(B_c) \pm \sqrt{Tr^2(B_c) - 4 \det(B_c)} \right] , \] (3.15)
where
\[ B_c = \bar{A}_c \quad \text{or} \quad B_c = \bar{G}^{-1}A_c . \] (3.16)
It can be easily seen that \( m_{(c)1}^2, m_{(c)2}^2 \) are positive because \( \bar{a}_{(c)11}, \bar{a}_{(c)22} > 0 \) and \( \bar{a}_{(c)11}\bar{a}_{(c)22} > \bar{a}^2_{(c)12} \).

So, the action functional \( (2.20) \) is equivalent to a family of action functionals for small fluctuations of the scale factors of internal spaces in the vicinity of the minima of the effective potential
\[
S = \frac{1}{2\kappa_0^2} \int \frac{d^Dx}{M_0} \sqrt{|\bar{g}(0)|} \left\{ \bar{R} \left[ \bar{g}(0) \right] - 2\Lambda_{(c)\text{eff}} \right\} + \\
+ \sum_{i=1}^n \frac{1}{2} \int \frac{d^Dx}{M_0} \sqrt{|g^{(0)}|} \left\{ -\bar{g}^{(0)}_{\mu\nu} \psi^i_{\mu} \psi^i_{\nu} - m_{(c)\psi}^2 \psi^i \psi^i \right\} , \quad c = 1, \ldots, m ,
\] (3.17)
where \( \Lambda_{(c)\text{eff}} := U_{\text{eff}}(\tilde{\phi}_c) \) and the factor \( \sqrt{\mu/k^2} \) has been included into \( \psi \) for convenience: \( \sqrt{\mu/k^2}\psi \rightarrow \psi. \)

Thus, conformal excitations of the metric of the internal spaces behave as massive scalar fields developing on the background of the external space-time. By analogy with excitons in solid state physics where they are excitations of the electronic subsystem of a crystal, the excitations of the internal spaces may be called gravitational excitons.

In the conclusion of this section we want to make a few remarks concerning the form of the effective potential. From the physical viewpoint it is clear that the effective potential should provide following conditions:

\[
\begin{align*}
(i) \quad a_{(c)i} &= e^{\beta_i} \geq L_{Pl} , \\
(ii) \quad m_{(c)i} &\leq M_{Pl} , \\
(iii) \quad \Lambda_{(c)\text{eff}} &\to 0 .
\end{align*}
\] (3.18)

The first condition expresses the fact that the internal spaces should be unobservable at the present time and stable against quantum gravitational fluctuations. This condition ensures the applicability of the classical gravitational equations near positions of minima of the effective potential. The second condition means that the curvature of the effective potential should be less than Planckian one. Of course, gravitational excitons can be excited at the present time if \( m_i \ll M_{Pl} \). The third condition reflects the fact that the cosmological constant at the present time is very small: \( \Lambda \leq 10^{-54}\text{cm}^{-2} \approx 10^{-120}L_{Pl} \). Thus, for simplicity, we can demand \( \Lambda_{\text{eff}} = U_{\text{eff}}(\tilde{\phi}_c) = 0 \). (We used the abbreviation \( \Lambda_{\text{eff}} := \Lambda_{(c)\text{eff}} \).)

Strictly speaking, in the multi-minimum case (\( c > 1 \)) we can demand \( a_{(c)i} \sim L_{Pl} \) and \( \Lambda_{(c)\text{eff}} = 0 \) only for one of the minima, namely the minimum which corresponds to the state of the present universe. For all other minima it may be \( a_{(c)i} \gg L_{Pl} \) and \( |\Lambda_{(c)\text{eff}}| \gg 0 \).

It can be easily seen that the conditions \( \Lambda_{\text{eff}} = 0 \) and \( \rho \equiv 0 \) are incompatible. In fact, the necessary extremum condition for the potential \( (2.21) \) reads
\[
\bar{B}^{-1} \frac{\partial U_{\text{eff}}}{\partial \phi^1} = \sum_{j=1}^n r_j(Q^{-1})^j_1 + \frac{\partial \rho}{\partial \phi^1} + q_i \bar{B}^{-1}U_{\text{eff}} = 0 ,
\]
\[
\bar{B}^{-1} \frac{\partial U_{\text{eff}}}{\partial \phi^i} = \sum_{j=1}^n r_j(Q^{-1})^j_i + \frac{\partial \rho}{\partial \phi^i} = 0 , \quad i = 2, \ldots, n ,
\] (3.19)
where \( r_i := R_i \exp\left(-2(Q^{-1})^i\varphi^k\right) \), \( \bar{B} := \exp\left(q_1\varphi^1\right) \) and \( q_1 = 2A(D_0 - 2) \). For \( U_{\text{eff}}|_{\min} = 0 \) and \( \rho \equiv 0 \) this system has a nontrivial solution iff \( Q = 0 \). But transformation (2.14) is regular. Thus, there are no solutions for \( U_{\text{eff}}|_{\min} = 0 \) and \( \rho \equiv 0 \) unless all internal spaces are Ricci-flat. Moreover, as follows from potential (2.12), the conditions \( U_{\text{eff}}|_{\min} = 0 \) and \( \frac{\partial U_{\text{eff}}}{\partial \delta^i}|_{\min} = 0 \) are compatible iff

\[
\sum_{i=1}^{n} R_i e^{-2\beta_i} = 2 \left( \Lambda + \kappa^2 \rho \left( \tilde{\beta}_c \right) \right) \tag{3.20}
\]

and

\[
R_i e^{-2\beta_i} = -\kappa^2 \frac{\partial \rho}{\partial \beta^i}\bigg|_{\tilde{\beta}_c}, \quad i = 1, \ldots, n. \tag{3.21}
\]

If all internal spaces are Ricci-flat \( (R_i \equiv 0, \quad i = 1, \ldots, n) \) and \( \rho \equiv 0 \), there are no extrema at all.

With \( U_{\text{eff}}|_{\beta_c} = 0 \), \( \frac{\partial U_{\text{eff}}}{\partial \delta^i}|_{\beta_c} = 0 \) and eq. (3.21) the Hessian (3.2) of the potential (2.12) reads

\[
a_{(c)ik} = B\kappa^2 \left[ 2\delta_{ik} \frac{\partial \rho}{\partial \beta^i}\bigg|_{\tilde{\beta}_c} + \frac{\partial^2 \rho}{\partial \beta^i \partial \beta^k}\bigg|_{\tilde{\beta}_c} \right], \tag{3.22}
\]

where \( B := \exp\left[-\frac{2}{\kappa^2} \sum_{i=1}^{n} D_i \beta_c^i \right] \). The effective potential \( U_{\text{eff}} \) has minima at \( \tilde{\beta}_c \) if matrices \( a_{(c)ik} \) satisfy the Sylvester criterion (3.2). Because of \( B > 0 \), it is sufficient to check this criterion for the matrix elements

\[
\rho = \sum_{i=1}^{n} \left( \sum_{k=1}^{n} f_{ik} \beta^k \right), \tag{3.25}
\]

where \( A_n, f_{ik} \) are constants. This potential has very general form and includes, for example, Freund - Rubin "monopole" ansatz \( \bar{B} \), crude approximations of the Casimir effect due to non-trivial topology of the space-time \( [28] [29] \) and multicomponent perfect fluids \( [11] [12] \). In the former case ("monopole") the potential \( \rho \) reads \( [22] \)

\[
\rho = \sum_{i=1}^{n} (f_i)^2 = \sum_{i=1}^{n} \left( f_i \right)^2 e^{-2D_i \beta^i}, \tag{3.26}
\]

where \( f_i = \text{const} \). So, for the matrix \( f^i_k \) we have \( f^i_k = -2D_i \delta_{ik}, \quad i, k = 1, \ldots, n \). In the case of the multicomponent perfect fluid the energy density reads \( [11] [12] \)

\[
\rho = \sum_{a=1}^{m} \rho^a = \sum_{a=1}^{m} A_a \exp\left(-\sum_{k=1}^{n} \alpha^a_k D_k \beta^k \right), \tag{3.27}
\]

where \( A_a \) are constants. This formula describes the \( m \)-component perfect fluid with the equations of state \( P_{(i)}^{(a)} = (\alpha^{(a)}_1 - 1)^a \) in the internal space \( M_i \) \( (i = 1, \ldots, n) \). In the external space each component corresponds to vacuum: \( \alpha^a_0 = 0 \) \( (a = 1, \ldots, m) \). For this example \( f^a_k = -\alpha^a_k D_k \).

For the potential (3.2) equation (3.24) can be rewritten as

\[
r_k = -\kappa^2 \sum_{a=1}^{N} h_a f^a_k, \quad k = 1, \ldots, n, \tag{3.28}
\]

8
where \( r_k := R_k \exp \left( -2\beta_k^q \right) \) and \( h_n := A_n \exp \left( \sum_{k=1}^n f^a_k \beta_k^q \right) \). Now, the minimum - conditions (3.29) and (3.24) respectively read

\[
\sum_{a=1}^N h_a f^a_k (f^a_k + 2) > 0 , \quad k = 1, 2 \tag{3.29}
\]

and

\[
\prod_{k=1}^2 \left( \sum_{a=1}^N h_a f^a_k (f^a_k + 2) \right) > \left( \sum_{a=1}^N h_a f^a_1 f^a_2 \right)^2 . \tag{3.30}
\]

For example, for the "monopole" potential (3.26) we obtain the extremum - condition:

\[
R_k \exp \left( 2(D_k - 1)\beta^q_k \right) = 2D_k \kappa^2 (f_k)^2 , \quad k = 1, \ldots, n . \tag{3.31}
\]

It follows from this expression that there exists an extremum if \( \text{sign} \, R_k > 0 , \quad k = 1, \ldots, n \). Conditions (3.29), (3.31) show that this extremum is a minimum (for \( D_k > 1 \)).

### 4 One internal space

Here we consider the case of one internal space or, strictly speaking, the case where all internal spaces have one common scale factor. In the case under consideration the action and the effective potential are given by equations (2.14) and (2.15) respectively. To get masses of the gravitational excitons it is necessary to specify the potential \( \rho \). For this purpose we consider four particular examples:

a) pure geometrical potential: \( \rho \equiv 0 \).

The necessary condition for the existence of an extremum gives

\[
\frac{R_1}{D_1} e^{-2\beta_e} = \frac{2\Lambda}{D - 2} , \tag{4.1}
\]

where \( \beta := \beta^1 \). It follows from this expression that \( \text{sign} \, \Lambda = \text{sign} \, R_1 \). From the minimum-condition

\[
a_{11} = \frac{\partial^2 U_{\text{eff}}}{\partial \beta^2} \bigg|_{\beta_e} = -\frac{2(D - 2)}{D_0 - 2} R_1 \left( e^{-2\beta_e} \right)^{\frac{D_2 - 2}{D_0 - 2}} > 0 \tag{4.2}
\]

we see that bare cosmological constant and curvature of the internal space should be negative: \( \Lambda, R_1 < 0 \).

The effective cosmological constant is

\[
\Lambda_{\text{eff}} = \frac{D_0 - 2}{2D_1} R_1 \left( e^{-2\beta_e} \right)^{\frac{D_2 - 2}{D_0 - 2}} \tag{4.3}
\]

and negative for \( R_1 < 0 \). The mass squared of the exciton reads

\[
m^2 = -\frac{4\Lambda_{\text{eff}}}{D_0 - 2} = \frac{2R_1}{D_1} \left( e^{-2\beta_e} \right)^{\frac{D_2 - 2}{D_0 - 2}} . \tag{4.4}
\]

If we assume, for example, that for a space-time configuration \( M_0 \times M_1 \) with four-dimensional external space-time \( (D_0 = 4) \) and compact internal factor-space \( M_1 = H^{D_1}/\Gamma \) with constant negative curvature \( R_1 = -D_1(D_1 - 1) \) there exists a minimum of the effective potential at \( a_c = 10^2 L_{P1} \) then we get

\[
m^2 = 2(D_1 - 1)10^{-2(D_1 + 2)} M_{P1}^2 \quad \text{and} \quad \Lambda_{\text{eff}} = -10^{-2(D_1 + 2)}\Lambda_{P1} .
\]

Thus, according to observational data with \( |\Lambda_{\text{eff}}| \leq 10^{-120}\Lambda_{P1} \) there should be at least \( D_1 = 59 \) and the corresponding excitons would be extremely light particles with masses \( m \leq 10^{-60} M_{P1} \sim 10^{-55} \text{g} \). If one uses an reduction of the effective cosmological constant holding \( \Lambda = 2R_1 \) and \( R_1 \) fixed when \( D_1 \to \infty \) (this can be achieved by a conformal transformation \( g^{(1)} \to D_1^2 g^{(1)} \) with fixed \( \kappa^2 = \kappa^2/\mu \)) one gets \( a_c \to L_{P1} \) and \( \Lambda_{\text{eff}} \to 0 \). But at the same time the exciton
mass vanishes \((m \to 0)\) and the effective potential degenerates into a step function with infinite height: \(U_{\text{eff}} \to \infty\) for \(a < 1\) and \(U_{\text{eff}} = 0\) for \(a \geq 1\). Thus, in the limit \(D_1 \to \infty\) there is no minimum at all.

As it was shown in the previous section the effective cosmological constant is not equal to zero if \(\rho \equiv 0\). To satisfy this condition we should consider the case \(\rho \neq 0\).

b) Casimir potential: \(\rho = C e^{-D\beta}\).

Because of a nontrivial topology of the space - time, vacuum fluctuations of quantized fields result in a non - zero energy density of the form \(\rho = C e^{-D\beta}\) where \(C\) is a constant and its value depends strongly on the topology of the model. For example, for fluctuations of scalar fields the constant \(C\) was calculated to take the values: \(C = -8.047 \cdot 10^{-6}\) if \(M_0 = \mathbb{R} \times S^3\), \(M_1 = S^1\) (with \(e^{\vartheta_0}\) as scale factor of \(S^3\) and \(e^{\vartheta_0} \gg e^{\beta_1}\)) \(\gg 1\); \(C = -1.097\) if \(M_0 = \mathbb{R} \times \mathbb{R}^2\), \(M_1 = S^1\) \(\gg 1\) and \(C = 3.834 \cdot 10^{-6}\) if \(M_0 = \mathbb{R} \times S^3\), \(M_1 = S^3\) (with \(e^{\vartheta_0} \gg e^{\beta_1}\)) \(\gg 3\).

From equations \((3.20)\) and \((3.21)\) (for \(n = 1\)), i.e. conditions \(\frac{\partial U_{\text{eff}}}{\partial \beta}|_{\beta_c} = 0\) and \(A_{\text{eff}} = 0\), we immediately derive

\[
R_1 e^{-2\beta_c} = \frac{2D}{D - 2} \Lambda
\]

and

\[
R_1 e^{(D - 2)\beta_c} = \kappa^2 \mathcal{C} D.
\]

An extremum exists if sign \(R_1 = \text{sign} \Lambda = \text{sign} C\). The expressions \((4.5)\) and \((4.7)\) provide fine tuning for the parameters of the model. Similar fine tuning was obtained by different methods in papers \(23\) (for one internal space) and \(34\) (for \(n\) identical internal spaces). The second derivative and mass squared reads respectively

\[
a_{11} = \frac{\partial^2 U_{\text{eff}}}{\partial \beta^2}|_{\beta_c} = (D - 2) R_1 (e^{-2\beta_c}) \frac{\partial R_1}{\partial \beta_c} e^{-2\beta_c},
\]

\[
m^2 = \frac{D_0 - 2}{D_1} R_1 (e^{-2\beta_c}) \frac{\partial R_1}{\partial \beta_c} e^{-2\beta_c}.
\]

Thus, the internal space should have positive curvature: \(R_1 > 0\) (or for split space \(M_1\) the sum of the curvatures of the constituent spaces \(M^i_1\) should be positive).

Let us consider a manifold \(M\) with topology \(M = \mathbb{R} \times S^3 \times S^3\) where \(e^{\vartheta_0} \gg e^{\beta_1}\). Then \(23\), \(C = 3.834 \cdot 10^{-6} > 0\). As \(C, R_1 > 0\), the effective potential has a minimum provided \(\Lambda > 0\). Normalizing \(\kappa_0^2\) to unity, we get \(\kappa^2 = \mu\) where \(\mu = 2 \pi^{(d + 1)/2} / \Gamma \left( \frac{1}{2} (d + 1) \right)\) for the \(d\) - dimensional sphere. For the model under consideration we obtain \(a_c \approx 1.5 \cdot 10^{-1} L_{\text{Pl}}\) and \(m \approx 2.12 \cdot 10^2 M_{\text{Pl}}\). Hence, the conditions (i) and (ii) are not satisfied for this topology. For other topologies this problem needs a separate investigation.

c) "monopole" potential: \(\rho = f^2 e^{-2D_1 \beta}\).

The "monopole" ansatz \(\boxed{1}\) consists in the proposal that an antisymmetric tensor field of rank \(D_1\) is not equal to zero only for components corresponding to the internal space \(M_1\). The energy density of this field reads \(\boxed{24, 25}\)

\[
\rho = f^2 e^{-2D_1 \beta},
\]

where \(f\) is an arbitrary constant (free parameter of the model). The equations \((3.20), (3.21)\) and \((3.31)\) yield following zero - extremum - conditions:

\[
\Lambda = \frac{D_1 - 1}{2D_1} R_1 e^{-2\beta_c}
\]
and
\[
\frac{R_1}{2D_1\kappa^2f^2} = e^{-2\beta c(D_1-1)},
\] (4.12)
which show that \( R_1, \Lambda > 0 \). The exciton mass squared reads
\[
m^2 = \frac{2(D_0 - 2)(D_1 - 1)}{D_1(D - 2)}R_1\left(e^{-2\beta c}\right)\frac{D - 2}{D_0 - 2}.
\] (4.13)
Condition (i) is satisfied if
\[
f^2 > R_1 / 2\kappa^2D_1.
\] (4.14)

Let \( M_1 \) be a 3-dimensional sphere, then \( R_1 = 6 \) and \( \kappa^2 = 2\pi^2 \). To get a minimum of the effective potential for a scale factor \( a_c = 10L_{Pl} \) we should take \( f^2 \approx 5 \cdot 10^2 \). For this value of \( a_c \) and for \( D_0 = 4 \) the mass squared is \( m^2 = \frac{16}{5} \cdot 10^{-5} \ll M_{Pl}^2 \). Thus, all three conditions (i) - (iii) are satisfied.

d) perfect-fluid potential:
\[
\rho = Ae^{-\alpha D_1\beta}.
\] (4.15)
The one-component perfect-fluid potential reads \[41, 42\]
\[
\rho = Ae^{-\alpha D_1\beta},
\] (4.15)
where \( A \) is an arbitrary positive constant. It describes vacuum in the external space and a perfect fluid with the equation of state \( P = (\alpha - 1)\rho \) in the internal space \( M_1 \). Physical values of \( \alpha \) are restricted to \( 0 \leq \alpha \leq 2 \).

(4.16)
It is easy to see that the case \( \alpha = 0 \) corresponds to the vacuum in the space \( M_1 \) and contributes to the bare cosmological constant \( \Lambda \). Therefore we shall not consider \( \alpha = 0 \) because in this case we come back to subsection a). The other limiting case with \( \alpha = 2 \) formally coincides here with the "monopole" potential (4.10).

For the perfect-fluid potential \[4.15\] a vanishing effective cosmological constant \( \Lambda_{eff} = 0 \) (eq. (3.20)) and extremum condition (3.21) yield
\[
R_1e^{(\alpha D_1-2)\beta c} = \kappa^2\alpha D_1A
\] (4.17)
and
\[
R_1e^{-2\beta c} = \frac{2\alpha D_1}{\alpha D_1 - 2}\Lambda.
\] (4.18)
For the second derivative of the effective potential in the minimum we obtain:
\[
a_{11} = \left. \frac{\partial^2 U_{eff}}{\partial \beta^2} \right|_{\beta_c} = (\alpha D_1 - 2)R_1\left(e^{-2\beta c}\right)\frac{D - 2}{D_0 - 2}.
\] (4.19)
Because of \( \alpha, A > 0 \), equation (4.17) shows that the internal space \( M_1 \) should have a positive curvature: \( R_1 > 0 \). From eq. (4.13) we see that there exists a minimum if \( \alpha > 2/D_1 \). The corresponding mass squared of the exciton is given as
\[
m^2 = \frac{(D_0 - 2)(\alpha D_1 - 2)}{D_1(D - 2)}R_1\left(e^{-2\beta c}\right)\frac{D - 2}{D_0 - 2}.
\] (4.20)
For the critical value of \( \alpha \) at \( \alpha = 2/D_1 \) the model becomes degenerated: \( U_{eff} \equiv 0 \).

As illustration, let \( M_1 \) be a 3-dimensional sphere and \( a_c = 10L_{Pl} \). This minimum can be achieved for \( A = (\alpha \pi^2)^{-1} \cdot 10^2 \). Thus, \( \frac{3}{\pi^2} < A \leq 5 \cdot 10^2 \) and \( 0 < m^2 \leq \frac{16}{5} \cdot 10^{-5} \) for \( 2/D_1 < \alpha \leq 2 \) and \( D_0 = 4 \). We see that all conditions (i) - (iii) are satisfied here.

In this section we considered four simple examples of the effective potential and showed that some of them satisfy conditions (i) - (iii).
5 Internal spaces with two scale factors

In this section we extend the consideration of possible excitons from effective potentials fulfilling conditions (3.18) to internal spaces with two scale factors. We analyze three potentials — the pure geometrical potential, the effective potential of a perfect fluid and the "monopole" potential. Stability considerations for Casimir-like potentials can be found in our paper [36].

a) pure geometrical potential $U_{eff,0} \equiv U_{eff}(\rho \equiv 0)$.

In this case the condition for the existence of an extremum $\partial U_{eff,0} / \partial \beta_k = 0$ implies a fine-tuning

$$R_k e^{-2\beta_k} = \frac{2A}{D-2}, \quad k = 1, 2 \quad \Rightarrow \quad e^{\beta_k} = \left[ \frac{R_k D_1}{R_1 D_k} \right]^{1/2} e^{\beta_c}$$

(5.1)

of the scale factors and sign $\Lambda = \text{sign } R_k$. From the Hessian

$$a_{(c)ik} \equiv \frac{\partial^2 U_{eff,0}}{\partial \beta_i \partial \beta_k} \bigg|_{\beta_c} = -\frac{4\Lambda_{eff}}{D_0-2} \left[ \frac{D_i D_k}{D_0-2} + \delta_{ik} D_k \right]$$

(5.2)

we see that according to the Sylvester criterion $a_{(c)11} > 0 , \quad a_{(c)22} > 0 , \quad a_{(c)11} a_{(c)22} > \left( a_{(c)12} \right)^2$ there exist massive excitons for this effective potential in the case of a negative cosmological constant $\Lambda < 0$ and negative scalar curvatures $R_k < 0$. The masses of the excitons are easy calculated as eigenvalues of the matrix $\hat{G}^{-1} A_c$ (3.15), (3.16). Because of

$$\hat{G}^{-1} A_c = \begin{pmatrix}
\frac{a_{(c)11} D_0}{D_2} & \frac{a_{(c)11} a_{(c)12}}{D_0-2} & \frac{a_{(c)11} a_{(c)12}}{D_0-2} \\
\frac{a_{(c)22} D_0}{D_2} & \frac{a_{(c)22} a_{(c)12}}{D_0-2} & \frac{a_{(c)22} a_{(c)12}}{D_0-2} \\
\frac{a_{(c)11} a_{(c)12}}{D_0-2} & \frac{a_{(c)22} a_{(c)12}}{D_0-2} & \frac{a_{(c)22} a_{(c)12}}{D_0-2}
\end{pmatrix} = -\frac{4\Lambda_{eff}}{D_0-2} \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}$$

(5.3)

they are given as

$$m_1^2 = m_2^2 = -\frac{4\Lambda_{eff}}{D_0-2} \exp \left[ \frac{-2}{D_0-2} \sum_{i=1}^{2} D_i \beta_i \right]$$

$$= 2 \left| \frac{R_k R_2}{D_1 D_2} \right|^{1/2} \exp \left[ \frac{2 D_0}{D_0-2} \right]^\frac{D_1}{D_0-2}$$

$$= 2 \left| \frac{R_k}{D_1} \right| \left( \frac{D_1}{R_1 D_2} \right)^\frac{D_1}{D_0-2}$$

(5.4)

where the last line follows immediately from the fine-tuning condition (5.1). From eq. (5.4) we see that the exciton masses $m_1 , \quad m_2$ of the two-scale-factor model are degenerated and related to the corresponding effective cosmological constant $\Lambda_{eff}$ in the same way as in the one-scale-factor case (3.14). As in the one-scale-factor model, for specific space configurations the two-scale-factor model allows the existence of excitons satisfying physical conditions (3.18).

Let us illustrate this situation with an extended version of the example of section (3.14). Suppose that $D_0 = 4 , \quad M_1 = H^D_1 / \Gamma_1 , \quad R_1 = -D_1 (D_1 - 1) , \quad D_1 = 2 , \quad a_{(c)11} = 10^5 L_{Pl} \quad ; \quad M_2 = H^D_2 / \Gamma_2 , \quad R_2 = -D_2 (D_2 - 1)$. Mass formula (5.4), effective cosmological constant and fine-tuning condition (5.1) read in this case:

$$m_1^2 = m_2^2 = 2 \cdot (D_2 - 1)^{-D_2/2} \cdot 10^{-2(D_2+4)} M_{Pl}^2 \,$$

$$\Lambda_{eff} = -(D_2 - 1)^{-D_2/2} \cdot 10^{-2(D_2+4)} L_{Pl} \,$$

$$\begin{align*}
a_{(c)1} &= (D_2 - 1)^{1/2} a_{(c)1} = (D_2 - 1)^{1/2} 10^2 L_{Pl} \\
a_{(c)2} &= (D_2 - 1)^{1/2} a_{(c)2} = (D_2 - 1)^{1/2} 10^2 L_{Pl}
\end{align*}$$

(5.5)
Thus, conditions (3.18) are fulfilled for internal spaces $M_2$ with dimensions $D_2 \geq D_{2,cr} = 40$. Indeed, in the case of $D_2 = 40$ we have $m_1^2 \simeq -2 \cdot 10^{-12} M_{Pl}^2$, $\Lambda_{eff} \simeq -10^{-12} \Lambda_{Pl}$, $a_{(c)} \simeq 6 \cdot 10^2 L_{Pl}$ and hence for $D_2 > 40$ there hold the relations $m_1 \ll M_{Pl}$, $|\Lambda_{eff}| < 10^{-12} \Lambda_{Pl}$, $a_{(c)} \gtrsim L_{Pl}$ as required in (3.18).

b) perfect fluid

For a multicomponent perfect fluid with energy density (3.27) the effective potential reads

$$U_{eff} = \left( \prod_{i=1}^{2} e^{D_i \beta_i} \right) \sum_{D_0} \left[ -\frac{1}{2} \sum_{i=1}^{2} R_i e^{-2\beta_i} + \Lambda + \kappa^2 \sum_{a=1}^{m} A_a \exp \left( -\frac{2}{\kappa} \alpha^{(a)}_k D_k \beta^k \right) \right].$$

(5.6)

Following the same scheme as in the previous considerations we calculate first extremum condition, Hessian and exciton masses in its general form and analyze then some concrete subclasses of potentials.

For shortness we introduce the abbreviations

$$u_k^{(a)} := \alpha_k^{(a)} + 2 \sum_{D_2} \alpha^{(a)}_k D_2, \quad v_k^{(a)} := \tilde{h}_a \alpha_k^{(a)}, \quad c_k := \frac{2 \Lambda D_k}{\kappa^2 - 2} ,$$

(5.7)

$$h_a := \alpha^2 \alpha^{(a)} e^{-\alpha^{(a)}_k D_1 \beta_1} e^{-\alpha^{(a)}_2 D_2 \beta_2} > 0 , \quad \tilde{h}_a := h_a \exp \left[ -\frac{2}{\kappa^2 - 2} \sum_{i=1}^{2} D_i \beta_i^2 \right].$$

Extremum condition and Hessian read then

$$\frac{\partial U_{eff}}{\partial \beta_k} = 0, \ k = 1, 2 \quad \Rightarrow \quad I_k := c_k + D_k \kappa^2 \sum_{a=1}^{m} A_a u_k^{(a)} e^{-\alpha^{(a)}_k D_1 \beta_1} e^{-\alpha^{(a)}_2 D_2 \beta_2} - R_k e^{-2\beta_k} = 0 , \quad k = 1, 2$$

(5.8)

$$a_{(c)k} = \frac{\partial^2 U_{eff}}{\partial \beta_i \partial \beta_k} \bigg|_{\beta_i} = -\frac{4 \Lambda e_{eff}}{D_0 - 2} \left[ \frac{D_i D_k}{D_0 - 2} + \delta_{i,k} D_k \right] + \sum_{a=1}^{m} \tilde{h}_a \alpha_k^{(a)} D_k \left( \alpha_i^{(a)} D_i - 2 \delta_{i,k} \right)$$

(5.9)

and from the auxiliary matrix

$$[G^{-1} A_c]_{ik} = -\frac{4 \Lambda e_{eff}}{D_0 - 2} \delta_{i,k} + J_{ik} , \quad J_{ik} = \sum_{a=1}^{m} v_k^{(a)} (D_k u_k^{(a)} - 2 \delta_{i,k})$$

(5.10)

we calculate the exciton masses squared as

$$m_{1,2}^2 = -\frac{4 \Lambda e_{eff}}{D_0 - 2} + \frac{1}{2} \left[ Tr(J) \pm \sqrt{Tr^2(J) - 4 \det(J)} \right].$$

(5.11)

From eq. (5.8) we see that the extremum condition has the form of a system of equations in variables $z_1 = e^{-\beta_1^2}$, $z_2 = e^{-\beta_2^2}$

$$I_k = c_k + D_k \kappa^2 \sum_{a=1}^{m} A_a u_k^{(a)} z_1^{\alpha_k^{(a)} D_1} z_2^{\alpha_k^{(a)} D_2} - R_k z_k^2 = 0 , \quad k = 1, 2$$

(5.12)

and for a given point $p = \{ \Lambda, R_1, R_2, A_1, \ldots, A_m, \alpha_1^{(a)}, \ldots, \alpha_m^{(a)} \}$ in parameter space $\mathbb{R}^{3(m+1)}$ positions of extrema should be found as solutions of this system. In the general case of $m > 1$ and $\alpha_1^{(a)}$ real ($\alpha_1^{(a)} \in \mathbb{R}$) this can be done most efficiently by numerical methods. Partially analytical methods can be applied, e.g. for $\alpha_1^{(a)}$ rational ($\alpha_1^{(a)} \in \mathbb{Q}$). In this case following representation holds $\alpha_1^{(a)} D_i = \frac{n_1^{(a)}}{d_1^{(a)}}$ with natural numerator $n_1^{(a)} \in \mathbb{N}$ and denominator $d_1^{(a)} \in \mathbb{N}^+$, where $n_1^{(a)}$, $d_1^{(a)}$ are relative prime, $\text{GCD}(n_1^{(a)}, d_1^{(a)}) = 1$. Introducing the least common multiple of the denominators $l = \text{LCM}(d_1^{(1)}, \ldots, d_1^{(m)})$ and the natural numbers $\tilde{g}_1^{(a)} := \frac{l}{d_1^{(a)} n_1^{(a)}}$ one has $\alpha_1^{(a)} D_i = \frac{\tilde{g}_1^{(a)}}{d_1^{(a)}}$. Eqs. (5.12) transform then to a system of polynomials

$$I_k = c_k + D_k \kappa^2 \sum_{a=1}^{m} A_a u_k^{(a)} \tilde{g}_1^{(a)} y_1^{\tilde{g}_1^{(a)}} y_2^{\tilde{g}_1^{(a)}} - R_k y_k^{2l} = 0 , \quad k = 1, 2$$

(5.13)
in the new variables \( y_k = z_k^{1/2} \), which can be analyzed by algebraic methods (resultant techniques [9], algebro-geometrical techniques [30]) and for rational parameters by methods of number theory [31]. So, for common roots of equations \( I_1 = 0 \), \( I_2 = 0 \) the resultants \( R_{y_1} [I_1, I_2] \), \( R_{y_2} [I_1, I_2] \) must necessarily vanish
\[
R_{y_1} [I_1, I_2] = w(y_2) = 0, \quad R_{y_2} [I_1, I_2] = w(y_1) = 0 \tag{5.14}
\]
and the analysis of (5.12) can be reduced to an analysis of the polynomials \( w(y_1) \), \( w(y_2) \) of degree
\[
\deg[w(y_1)], \deg[w(y_2)] \leq \left( \max \left( \alpha_1^{(a)} D_1 + \alpha_2^{(a)} D_2, 2 \right) \right)^2 \tag{5.15}
\]
in only one of the variables \( y_1 \) and \( y_2 \) respectively. For explicit considerations of extremum positions with the help of algebraic methods in the case of Casimir-like potentials we refer to [30].

We now turn to the consideration of some concrete subclasses of perfect fluids.
- b.1 \( m \)-component perfect fluid with \( \alpha_1^{(a)} = \alpha_2^{(a)} \)

In this case there exist no massive excitons for vanishing effective cosmological constants \( \Lambda_{\text{eff}} = 0 \). Indeed, \( m_{1,2}^2 > 0 \) and eq. (5.11) imply \( Tr(J) > 0 \), \( \det(J) > 0 \) which with
\[
J_{ik} = D_k W_1 - 2 \delta_{ik} W_2, \quad W_1 := \sum_{a=1}^{m} u(a)^{v(a)}, \quad W_2 := \sum_{a=1}^{m} v(a) \tag{5.16}
\]
read \( Tr(J) = D_k W_1 - 4W_2 > 0 \), \( \det(J) = 2W_2(2W_2 - D_k W_1) > 0 \). But because of \( v(a) = \bar{h}_a \alpha^{(a)} > 0 \) and hence \( W_2 > 0 \) this leads to a contradiction. Thus, for the existence of massive excitons \( m_{1,2}^2 > 0 \) the effective cosmological constant must be negative \( \Lambda_{\text{eff}} < 0 \).

- b.2 one-component perfect fluid with \( \alpha_1 \neq \alpha_2 \)

Again massive excitons are possible for negative effective cosmological constants \( \Lambda_{\text{eff}} < 0 \). Here at one hand we have \( \det(J) = -2v_1 v_2 \delta^{\frac{\beta_1 - \beta_2}{2}} > 0 \), \( \delta := D_1 \alpha_1 + D_2 \alpha_2 - 2 \) and hence \( \delta < 0 \). On the other hand from \( Tr(J) > 0 \) follows \( \delta(\alpha_1 + \alpha_2 - \frac{\beta_1 - \beta_2}{2}) > 0 \) and hence \( 0 > (D_0 - 2)(\alpha_1 + \alpha_2) + D_1 \alpha_1 + D_2 \alpha_2 \). Because of \( \alpha_k > 0 \) this is impossible.

- b.3 one-component perfect fluid with \( \alpha_1 = \alpha_2 = \alpha \)

For this subclass of b.1) extremum conditions (5.8) can be considerably simplified to yield
\[
h = \kappa^2 A e^{-\alpha(D_1 \beta_1 + D_2 \beta_2)} = \frac{1}{(D_0 - 2)\alpha + 2} \left( \frac{D - 2}{D_k} R_k e^{-2\beta_k} - 2\Lambda \right) \tag{5.17}
\]
and the same fine-tuning condition as in the case of a pure geometrical potential
\[
\hat{C} = \frac{R_1}{D_1} e^{-2\beta_1} = \frac{R_2}{D_2} e^{-2\beta_2} \tag{5.18}
\]
An explicit estimation of exciton masses and effective cosmological constant can be easily done. Using (5.7), (5.11), (5.16) we rewrite the exciton masses squared as
\[
\begin{pmatrix} m_1^2 \\ m_2^2 \end{pmatrix} = \frac{1}{D - 2} \left\{ -4\Lambda + h |(D_0 - 2)\alpha + 2| \left( \begin{array}{c} D' \alpha \\ 0 \end{array} \right) + 2 \right\} \exp \left[ -\frac{2}{D_0 - 2} \sum_{i=1}^{2} D_i \beta_i \right] \tag{5.19}
\]
and transform with (5.17) inequalities \( m_{1,2}^2 > 0 \), \( h > 0 \) to the following equivalent condition
\[
\frac{2}{D - 2}\Lambda < \hat{C} < 0 \tag{5.20}
\]
Hence stable space configurations with massive excitons are only possible for internal spaces with negative curvature \( R_k < 0 \). Reparametrizing \( \Lambda \) according to (5.20) as
\[
\Lambda = \frac{D - 2}{2} (\hat{C} - \tau) \tag{5.21}
\]
with $\tau > 0$ — a new parameter, we get for exciton masses squared and effective cosmological constant

$$
\left( \frac{m^2_1}{m^2_2} \right) = \begin{pmatrix} D \alpha \tau & 0 \\ 0 & 0 \end{pmatrix} - 2 \tilde{C} \exp \left[ -\frac{2}{D_0 - 2} \sum_{i=1}^{2} D_i \beta_i^e \right],
$$

(5.22)

$$
\Lambda_{eff} = -\frac{D_0 - 2}{2} \left[ \tau \left( \frac{(D-2)\alpha}{(D_0-2)\alpha + 2} \right) - \tilde{C} \right] \exp \left[ -\frac{2}{D_0 - 2} \sum_{i=1}^{2} D_i \beta_i^e \right].
$$

(5.23)

According to definition (5.21) and equations (5.17), (5.18) the parameter "monopole" potential formally coincides with the potential of a perfect fluid with parameters $\rho \sim \tau \equiv |\tilde{C}| \min(\frac{2}{D_0 - 2} \frac{(D_0-2)\alpha + 2}{(D-2)\alpha})$ we return to the pure geometrical potential considered in paragraph a). So physical conditions (3.18) are satisfied if $\tau > 0$, whereas in the two-scale-factor model they cannot occur. An explanation of this situation will be given in the next section.

- c) "monopole" potential: $\rho = \sum_{k=1}^{2} (f_k)^2 e^{-2D_k\beta_k}

For the "monopole" potential the extremum condition (3.1) leads in the case of vanishing effective cosmological constant $\Lambda_{eff} = 0$ to a fine-tuning of the scale factors

$$
\frac{R_k}{2D_k(J_k)^2} = e^{-2\beta_k(D_k-1)}
$$

(5.25)

and

$$
\Lambda = \frac{1}{2} \sum_{k=1}^{2} R_k e^{-2\beta_k} \frac{D_k - 1}{D_k}
$$

(5.26)

so that, as for the one-scale-factor model, extrema are only possible if $R_k > 0$, $\Lambda > 0$. Because the "monopole" potential formally coincides with the potential of a perfect fluid with parameters $a_k^{(a)} = 2\delta_{ak}$ the exciton masses are given by eq. (5.11)

$$
m^2_{1,2} = \frac{1}{2} \left[ \text{Tr}(J) \pm \sqrt{\text{Tr}^2(J) - 4 \det(J)} \right],
$$

(5.27)

where in terms of abbreviations (5.7) matrix $J$ reads

$$
J_{ik} = 4\tilde{h}_k(D_k - 1) \left[ \delta_{ik} - \frac{D_k}{D - 2} \right].
$$

(5.28)

One immediately verifies that $\text{Tr}(J) > 0$, $\det(J) > 0$, $\text{Tr}(J)^2 - 4 \det(J) \geq 0$ for dimensions $D_1 > 1$, $D_2 > 1$ and hence $0 < m^2_2 \leq \frac{1}{2} \text{Tr}(J) \leq m^2_1 < \text{Tr}(J)$. This means that physical conditions (3.18) are satisfied if $\text{Tr}(J) \leq M_{pi}^2$ and $e^{\beta_k} \geq L_{pi}$. Substituting

$$
\tilde{h}_k = \frac{R_k}{2D_k} e^{-2\beta_k} \exp \left[ -\frac{2}{D_0 - 2} \sum_{i=1}^{2} D_i \beta_i^e \right]
$$

(5.29)
Explicitly this projection operator can be constructed from the normalized base vector \( \bar{\sigma} \) transforms to \( \mathcal{K} \) masses of the reduced and unreduced two-scale-factor models is now easily established as

\[
\begin{align*}
\text{Tr}(J) & = \frac{2}{D-2} \left[ \sum_{k=1}^{E-1} \frac{(D_k-1)}{D_k} R_k (D-2-D_k) e^{-2\beta_k} \right] \exp \left[ -\frac{2}{D_0-2} \sum_{i=1}^{2} D_i \beta_i' \right].
\end{align*}
\] (5.30)

With this formula at hand we have e.g. for an internal space configuration \( M_1 \times M_2 : M_1 = S^3 \) , \( a_{(c)1} = 10L_{Pl} ; M_2 = S^5 \) , \( a_{(c)2} = 10^2 L_{Pl} \) the estimate \( \text{Tr}(J) \approx 56 \cdot 10^{-14} M_{Pl}^2 \ll M_{Pl}^2 \) and all conditions (i) - (iii) of (3.18) are satisfied.

6 Exciton masses and scale factor constraints

In this section we derive a relation between the exciton masses \( m_{(c)1}, m_{(c)2} \) of a model with two independently varying scale factors \( \beta_1, \beta_2 \) and the effective mass \( m_{(c)0} \) of the exciton which occurs under scale factor reduction, i.e. when the scale factors of the model are connected by a constraint \( \beta = \beta_1 = \beta_2 \).

In order to simplify our calculation we introduce the projection operator \( P \) on the constraint subspace \( \mathbb{R}_\beta = \{ \beta = (\beta_1, \beta_2) \mid \beta_1 - \beta_2 = \bar{a} \cdot \beta = \bar{a} = (1,-1) \} \) of the 2-dimensional target space \( \mathbb{R}_T^2 \) of the \( \sigma \)-model

\[
P_{\mathbb{R}_T^2} = \mathbb{R}_\beta \subset \mathbb{R}_T^2.
\] (6.1)

Explicitly this projection operator can be constructed from the normalized base vector \( \bar{e} \) of the subspace \( \mathbb{R}_\beta ^\perp \). With \( \bar{e} = \frac{1}{\sqrt{2}} \left( \begin{array} {c} 1 \\ 1 \end{array} \right) \) we have

\[
P = \bar{e} \otimes \bar{e}' = \frac{1}{2} \left( \begin{array} {c} 1 & 1 \\ 1 & 1 \end{array} \right) \otimes \left( \begin{array} {c} 1 \\ 1 \end{array} \right) = \frac{1}{2} \left( \begin{array} {c} 1 & 1 \\ 1 & 1 \end{array} \right)
\] (6.2)

and \( P^2 = P \cdot P = 0 \).

Let us now calculate the exciton mass \( m_{(c)0} \) for the reduced model. For this purpose we introduce the exciton Lagrangian, written according to section [3] in terms of the fluctuation fields \( \bar{\eta} = (\eta^1, \eta^2) \), \( \eta^i = \beta^i - \beta^i_c \)

\[
\mathcal{L}_{\text{exc}} = - \left[ \bar{\eta} \hat{G} \bar{K} \bar{\eta} + \bar{\eta} A_{(c)} \bar{\eta} \right].
\] (6.3)

\( \hat{K} := \partial_{(\mu} \bar{g}^{(\nu)\mu\nu} \partial^\nu \) denotes the pure kinetic operator. Under scale factor reduction \( \bar{\eta} = (\eta, \eta) \) this Lagrangian transforms to

\[
\mathcal{L}_{\text{exc}} = - \left[ \gamma_1 \bar{\eta} \tilde{K} \bar{\eta} + \gamma_{(c)2} \bar{\eta} \right],
\] (6.4)

\[
\gamma_1 := 2 \bar{e}' \tilde{G} \bar{e} = \sum_{i,j} \tilde{G}_{ij} \quad , \quad \gamma_{(c)2} := 2 \bar{e}' A_{(c)} \bar{e} = \sum_{i,j} A_{(c)ij}
\] (6.5)

so that the substitution \( \eta = \gamma_1^{-1/2} \) yields the effective one-scale-factor Langrangian

\[
\mathcal{L}_{\text{exc}} = - \left[ \bar{\psi} \tilde{K} \bar{\psi} + \psi m_{(c)0} \bar{\psi} \right] \text{ with exciton mass } m_{(c)0} = \gamma_{(c)2}/\gamma_1 .
\]

Taking into account that \( \bar{e}' A_{(c)} \bar{e} = \text{Tr} \left[ PA_{(c)} \right] \), \( A_{(c)} = \bar{Q} \bar{S}_c M_{(c)2}^2 \bar{S}_c Q \) and \( M_{(c)} = \text{diag}(m_{(c)1}^2, m_{(c)2}^2) \) the needed relation between the exciton masses of the reduced and unreduced two-scale-factor models is now easily established as

\[
m_{(c)0}^2 = 2 \gamma_1^{-1} \text{Tr} \left[ Q P Q' S_c M_{(c)2}^2 S_c \right].
\] (6.6)

With the use of

\[
Q P Q' = \frac{1}{2} D' \frac{D - 2}{D_0 - 2} \left( \begin{array} {cc} 1 & 0 \\ 0 & 0 \end{array} \right), \quad \gamma_1 = D' \frac{D - 2}{D_0 - 2}
\] (6.7)
and the $SO(2)\, -$rotation matrix $S_b \,$ from (3.13), (3.14) this formula can be considerably simplified to give the final relation

$$m^2_{(c)0} = \cos^2(\alpha_c)m^2_{(c)1} + \sin^2(\alpha_c)m^2_{(c)2}. \quad (6.8)$$

In its compact form this mass formula implicitly reflects the behavior of the effective potential $U_{eff}$ in the vicinity $\Omega_{\beta_0} \subset \mathbb{R}^2_\rho$ of the extremum point $\beta_0$. So, the exciton masses squared $m^2_{(c)1}$, $m^2_{(c)2}$ describe the potential as function over the two-dimensional $\beta_0$ – vicinity $\Omega_{\beta_0}$, whereas $m^2_{(c)0}$ characterizes $U_{eff}$ as function over the line interval $\Omega_{\beta_0} \cap \mathbb{R}^1_\rho$ only. Comparison of the minimum conditions of the unreduced and reduced two-scale-factor model

$$m^2_{(c)1,2} > 0 : \quad a_{(c)11} > 0, \quad a_{(c)22} > 0, \quad a_{(c)11} \cdot a_{(c)22} > (a_{(c)12})^2 \quad (6.9)$$

and

$$m^2_{(c)0} > 0 : \quad (a_{(c)11} + a_{(c)22} + 2a_{(c)12}) > 0 \quad (6.10)$$

shows that stable configurations of reduced models with $m^2_{(c)0} > 0$ are not only possible for stable configurations of the unreduced model $m^2_{(c)1} > 0$, $m^2_{(c)2} > 0$, but even in cases when the potential $U_{eff}$ has a saddle point at $\beta_0$ and the unreduced model is unstable. For the masses we have in these cases $m^2_{(c)1} > 0$, $m^2_{(c)2} < 0$ or $m^2_{(c)1} < 0$, $m^2_{(c)2} > 0$ and massive excitons in the reduced model correspond to exciton – tachyon configurations in the unreduced model.

7 Conclusions

This paper was devoted to the problem of stable compactification of internal spaces. This is one of the most important problems in multidimensional cosmology, because via stable compactification of the internal dimensions near Planck length we can explain unobservability of extra-dimensions. With the help of dimensional reduction we obtained an effective four - dimensional theory in Brans - Dicke and Einstein frames. The Einstein frame was considered here as a physical one \[22\]. In this frame we derived an effective potential. It was shown that small excitations of the scale factors of internal spaces near minima of the effective potential have a form of massive scalar particles (gravitational excitons) developing in the external space - time. Detection of these excitations can prove the existence of extra - dimensions. Particular examples of effective potentials were investigated in the one - and two - internal - space cases. Parameters of the models which ensure a minimum were obtained and masses of the excitons were estimated. The solutions at the minima of the potential are stable against small perturbations of the scale factor(s) of the expanding external universe \[26\]. We would like to note, that the problem of stable compactification in MCM with more than one internal scale factor was considered first for pure geometrical models in papers \[28, 29\]. However, the analysis of the effective potential minima existence was not complete there.

Our analysis shows that conditions for the existence of stable configurations may be quite different for one- and two-scale-factor models. For example, in the case of a one-scale-factor model which is filled with a one-component perfect fluid stable compactifications are possible for vanishing effective cosmological constant $\Lambda_{eff} = 0$ and parameters $\alpha$ from the restricted interval $2/D_1 < \alpha \leq 2$ determining the equation of state in the internal space: $p_1 = (\alpha - 1)\rho$. In the case of two-scale-factor models stable compactifications can exist for negative effective cosmological constants $\Lambda_{eff} < 0$ only, but for values of the parameter $\alpha$ from the usual interval : $0 \leq \alpha \leq 2$ (here, $\alpha$ determines the equations of state in both internal spaces :
\( P_1 = (\alpha - 1)\rho, P_2 = (\alpha - 1)\rho \). At first sight the difference in the behavior of these two models looks a bit strange because the one-scale-factor model can be obtained by reduction of the two-scale-factor model with the help of the constraint \( \beta_1 = \beta_2 = \beta \). As it was shown in section 6, such a different behavior may take place because stable configurations of reduced models are not only possible for stable configurations of unreduced models, but even in cases when the effective potential \( U_{eff} \) of the unreduced model has a saddle point. In the case of our two-scale-factor model with one-component perfect fluid we get such a saddle point for configurations with \( \Lambda_{eff} = 0 \) and \( 2/(D_1 + D_2) < \alpha \leq 2 \).

In the present paper we did not consider the case of degenerated minima of the effective potential, for example, self-interaction-type potentials or Mexican-hat-type potentials. In the former case one obtains massless fields with self-interaction. In the latter case one gets massive fields together with massless ones. Here, massless particles can be understood as analog of Goldstone bosons. This type of the potential was described in [3].

Another possible generalization of our model consist in the proposal that the additional potential \( \rho \) may depend also on the scale factor of the external space. It would allow, for example, to consider a perfect fluid with arbitrary equation of state in the external space.

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