Robust Spectral Filtering
and Anomaly Detection

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Abstract

We consider a setting, where the output of a linear dynamical system (LDS) is, with an unknown but fixed probability, replaced by noise. There, we present a robust method for the prediction of the outputs of the LDS and identification of the samples of noise, and prove guarantees on its statistical performance. One application lies in anomaly detection: the samples of noise, unlikely to have been generated by our estimate of the unknown dynamics, can be flagged to operators of the system for further study.

1 Introduction

Across mathematics, statistics [29], artificial intelligence [25], and engineering [23, 36], much attention has been devoted to the identification of linear dynamical systems (LDS):

\[
\begin{align*}
    h_k &= Ah_{k-1} + Bx_k + \eta_k \\
    y_k &= Ch_k + Dx_k + \zeta_k,
\end{align*}
\]

where \(x_k \in \mathbb{R}^n\) are inputs, \(y_k \in \mathbb{R}^m\) are outputs, \(h_k \in \mathbb{R}^d\) is a hidden (latent) state, \(A, B, C, D\) are compatible matrices, and \(\eta_k, \zeta_k\) are compatible noise vectors with \(\sum_{k=1}^T \|\eta_k\|^2 + \|\zeta_k\|^2 < L\). In improper learning of such an LDS (which we refer to as an identification problem), one wishes to estimate \(\hat{y}_k\) such that \(\hat{y}_k\) are close to the best estimates \(y_k^*\) of \(y_k\) possible at time \(k\). When there is no hidden state, the identification problem is convex and a variety of methods work well. When there is a hidden state, the problem is non-convex and only rather recently spectral filtering [19, 18] has been used to obtain identification procedures with regret bounded by \(\tilde{O}(\log^7 \sqrt{k})\) at time \(k\), where \(\tilde{O}(\cdot)\) hides terms that depend polynomially on the dimension of the system and norms of the inputs and outputs and the noise.
We consider a Huber-like setting, where with a fixed probability $p > 0$, which may be known or unknown, the observations $y_k$ are replaced by noise. That is, we have:

$$h_k = Ah_{k-1} + Bx_k + \eta_k$$

$$y_k = \begin{cases} 
\xi_k & \text{with probability } p \\
Ch_k + Dx_k + \zeta_k & \text{otherwise}
\end{cases}$$

under assumptions described out in the next section. Our goals are two-fold: first, to predict $\hat{y}_{k+1}$ of $Ch_{k+1} + Dx_{k+1} + \zeta_{k+1}$. Our second goal is to identify when $\xi_k$ corrupts the observations such that it can be flagged for further study by operators of the system in the spirit of anomaly detection. We stress that this Huber-like model differs from the settings for both additive and non-additive changes surveyed in [6], where the additive changes are the changes in the mean of the distribution of the observed signals and non-additive changes are related to changes in variance, correlations, spectral characteristics, or dynamics of the signal or system.

Overall, our contributions are as follows:

- We present a novel Huber-like model for anomaly detection.
- We present algorithms combining spectral filtering and additive-decrease multiplicative-increasing (ADMI) for the related robust identification problem.
- We present conditions that allow for the identification of the corrupting noise and the underlying LDS.

Notice that our analytical results for the Huber-like model are stronger than those surveyed in [6] in three ways. First, we suggest what is the absolute value of the difference between samples of $\xi_k$ and the non-corrupted observation $Ch_k + Dx_k + \zeta_k$ sufficient to detect that an anomaly occurred at time $k$. Second, we provide guarantees on the regret of our estimate of the subsequent observation of $Ch_k + Dx_k + \zeta_k$, under the conditions, where anomalies are detectable. Third, in contrast to the usual assumption of a Gaussian process noise $\eta_k$ and measurement noise $\zeta_k$, we allow for arbitrarily-distributed, but bounded noise $\sum_{k=1}^{T} \|\eta_k\|^2 + \|\zeta_k\|^2 < L$, or equivalently, a bounded amount of adversarial perturbations to the system. We hence believe that our model and the results make for a valuable addition to the literature on anomaly detection based on non-additive changes.

2 The Problem

As has been suggested in the previous section, we consider the problem of predicting $\hat{y}_{k+1}$ in the Huber-like extension of LDS (2), under several assumptions, starting with the identifiability of Hazan et al. [18].
Assumption 1. The outputs are generated by the stochastic difference equation (2), assuming:

1. Inputs and outputs are bounded: \( \|x_t\|_2 \leq R_x, \|y_t\|_2 \leq R_y \).

2. The system is Lyapunov stable, i.e., the largest singular value of \( A \) is at most 1: \( \rho(A) \leq 1 \).

3. \( A \) is diagonalizable by a matrix with small entries: \( A = \Psi \Lambda \Psi^{-1} \), with \( \|\Psi\|_F \|\Psi^{-1}\|_F \leq R_\Psi \).

4. \( B, C, D \) have bounded spectral norms: \( \|B\|_2, \|C\|_2, \|D\|_2 \leq R_\Theta \).

5. Let \( S = \{\alpha/|\alpha| : \alpha \text{ is an eigenvalue of } A\} \) be the set of phases of all eigenvalues of \( A \). There exists a monic polynomial \( p(x) \) of degree \( \tau \) such that \( p(\omega) = 0 \) for all \( \omega \in S \), the \( L^1 \) norm of its coefficients is at most \( R_1 \), and the \( L^\infty \) norm is at most \( R_\infty \).

While the Assumption 1 may seem restrictive, it essentially says that the system is identifiable \([16]\) and that eigenvectors of \( A \) corresponding to larger eigenvalues are not linearly dependent. Indeed, since Kalman \([21]\), it is understood that from input-output measurements, only the part of the system that is controllable and observable can be identified, while one can clearly achieve \([30]\) a near-perfect prediction \( \hat{y} \) of the output for an unstable system. Further, it is clear \([35]\) that the polynomial \( p(x) \) does exist, and we only introduce the notation for the norms of its coefficients.

We also make assumptions concerning the sparse noise, i.e., distribution of \( \xi_k \) and probability \( p \). Ideally, one would like to consider:

Assumption 2. Probability \( p < 1 \) is not known and the noise \( \xi_k \) is arbitrarily distributed.

We discuss Assumption 2 in Section 4. Notice, however, that some samples of the arbitrarily distributed noise may be indistinguishable from the output of the linear dynamical system. For a strong result on the identification of the LDS, we consider a separation condition:

Assumption 3. Probability \( p < 1 \) is not known. At time \( k \), the absolute value of the difference between the noise \( \xi_k \) and \( Cx_k + D_x_k + \zeta_k \) is greater than some instance- and algorithm-specific \( D_k \).

In Section 4, we prove the existence of \( D_k \), which goes to 0 in the large limit of \( k \). Under this assumption, we can also estimate the unknown \( p \).

3 The Algorithms

At a high-level, we suggest to use Algorithm 1 under Assumption 1. At the current time \( t \), the algorithm has the history of inputs, \( x_1 \ldots x_t \), and the outputs, \( y_1 \ldots y_{t-1} \), available. Based on the current input, \( x_t \), the algorithm produces a
forecast, \( \hat{y}_t \), after which the output, \( y_t \), of the (possibly corrupted) real system is observed. We then test whether the loss \( \| y_t - \hat{y}_t \|^2 \) is less than a certain threshold \( D_t \). If it is, we assume that \( y_t \) was generated by the LDS and use it in further predictions. Otherwise, we assume that the value is a sample of \( \xi_t \) and do not use it for further predictions. In the next section, Proposition 1 shows that there is a \( D_t \) of Assumption 3 decreasing to 0 in the large limit of \( t \) at a rate of \( \tilde{O}(t^{-1/2} \log^7(t)) \), which makes this schema meaningful.

In particular, we consider Algorithm 2 based on additive-decrease multiplicative-increase (ADMI), under Assumptions 1 and 3. There, we consider \( D_t \) of the form \( \text{mean}(L_k) + c \text{std}(L_k) \), where \( L_k \) are losses for predictions of values generated by the LDS, mean is the arithmetic mean, std is the standard deviation, and \( c \) is a coefficient greater or equal to 1.0. Subsequently, we update \( c \) using \( \alpha > 0, \beta > 1.0 \) as follows:

\[
    c \left\{ \begin{array}{ll}
        \beta c & \text{if } l_t > D_t \\
        c - \alpha & \text{otherwise}.
    \end{array} \right.
\]

That is: when we detect an anomaly, we raise the threshold for detecting anomalies relative to the losses observed so far. Otherwise, we decrease the threshold relative to the losses observed so far. Similar policies are widely used \cite{10} for congestion management in TCP/IP networking and distributed resource allocation. Again, we analyse this approach in the next section, specifically in Theorem 1.

Throughout both Algorithms 1 and 2 we predict the next output \( \hat{y}_t \) of the system from inputs \( X_t \) until time \( t \) and outputs \( Y_{t-1} \) until \( t-1 \) in an online fashion. There, leading methods \cite{19, 18, 28, 17, 30} consider an overparametrisation, where the vector \( \tilde{X}_t \) is composed of the inputs to the system at all time-levels up to the current one, convolved with the eigenvectors of a certain Hankel matrix, as well as the outputs at the previous time level, and inputs at the current and previous time levels. Notice that the Hankel matrix is constant and its eigenvectors can be precomputed. See \cite{37, 26, 13, 22, 31, e.g.} for background, \cite{18} for the detailed derivation of the method we use, and Algorithm 2 for a sketch of our implementation. We note that the new hypothesis class \( \hat{H} \) arising from the over-parametrisation \cite{18} has been shown by \cite{18} to approximately contain the class of LDS satisfying Assumption 1, which makes it possible to derive regret bounds considering the convexification:

\[
    f(M) = \sum_{i=1}^{t} \left\| y_i - M \tilde{X}_i \right\| ^2, \quad \text{where } M \in \hat{H},
\]

instead of the non-convex problem at each point in time. In theory, one has to consider the convexifications growing with \( T \), but in practice, windowing works well. Furthermore, one can apply on-line optimisation techniques, such as a small number of iterations of a coordinate descent between two time levels, which benefit from the facts that the problem is strongly convex and that the optimizer of (4) changes only modestly between two time levels.
**Input:** time horizon \( T \)

1. Initialize \( M_t \) to suitable dimension.
2. for \( t = 1, \ldots, T \) do
   3. Form overparametrisation \( \tilde{X}_t \) from inputs \( X_t \) until time \( t \) and outputs \( Y_{t-1} \) until \( t-1 \)
   4. \( \hat{y}_t \leftarrow M_t \tilde{X}_t \)
   5. Observe \( y_t \) and compute \( l_t := \| y_t - \hat{y}_t \|^2 \)
   6. if \( l_t > D_t \) then
      7. Update \( M \) for next time-level
      8. Update \( D_{t+1} \), if needed
      9. Consider \( x_t, y_t \) for subsequent time levels
   else
      10. Ignore \( x_t, y_t \) for subsequent time levels
      11. Update \( D_{t+1} \), if needed
   12. end if
   13. end for
14. Return \( \hat{y}_T \)

**Algorithm 1:** A schema of an algorithm for the setting of Assumption 3

4 **An Analysis**

We could start with a result based on the work of Hazan et al. [18]:

**Proposition 1.** Under Assumption 1, Algorithm 1 makes it possible to consider \( D_k \) of Assumption 3 decreasing to 0 in the large limit of \( k \) at a rate of \( \tilde{O}(k^{-1/2} \log^2(k)) \), where \( O(\cdot) \) hides dependence on instance-specific constants. Furthermore, this choice of \( D_k \) allows for the perfect recovery of the probability \( p \) of (2). Furthermore, this allows for the perfect recovery of the non-corrupted entries.

**Proof.** We want to show that:

\[
\sum_k \| Ch_k + Dx_k + \zeta_k - \xi_k \| \leq \tilde{O} \left( R_1^3 R_2^2 R_3^4 R_4^2 R_5^2 d^{5/2} n \log^2 n \log^2 k \sqrt{k} \right) + O(R_\infty^{-3} R_\Theta^2 R_y^2 L),
\]

where the \( \tilde{O}(\cdot) \) suppresses factors polylogarithmic in \( n, m, d, R_\Theta, R_x, R_y \) allows for the perfect recovery of the non-corrupted entries. Recall that the parameters \( R_1, R_x, R_\Theta, R_y \), and \( R_y \) are those in Assumption 1, \( d \) is the dimension of the hidden state space, \( n \) is the dimension of the input space, and \( k \) is the number of time steps. This follows directly from Theorem 19 of Hazan et al. [18].

Notice that this result is limited in two ways: First, it is not constructive, because the instance-dependent terms, based on the constants in Assumption 1 such as bounds on the spectral norms of matrices, are unknown a priori and non-trivial to estimate on-line. Second, one may wish for the width of the interval...
**Input:** time horizon $T$, data points $X_T, Y_{T-1}$, number $k$ of filters, pre-computed top $k$ eigenpairs $\{(\sigma_j, \phi_j)\}_{j=1}^k$ of a certain matrix $Z_T$

**Output:** prediction $\hat{y}_T, \hat{p}$

1: Initialize $M_t$ to suitable dimension, initialise $L_t$ to empty list, initialise $c$ to 1, $e_0$ to 0
2: for $t = 1, \ldots, T$ do
3: Form overparametrisation $\tilde{X}_t$ from inputs $X_t$ until time $t$, outputs $Y_{t-1}$ until $t-1$, and convolutions with pre-computed $\{(\sigma_j, \phi_j)\}_{j=1}^k$, as in [18]
4: $\hat{y}_t \leftarrow M_t \tilde{X}_t$
5: Observe $y_t$ and compute $l_t := \|y_t - \hat{y}_t\|^2$
6: Set $D_t := \text{mean}(L_t) + \text{cstd}(L_t)$
7: if $l_t > D_t$ or extra($l_t, D_t, \hat{p}_t$) then
8: Update $M$ for next time-level: $M_{t+1} \leftarrow \arg\min_M \sum_{i=1}^t \|y_i - M \tilde{X}_i\|^2$
9: Add $l_t$ to $L_t$
10: Update $c$ to $\beta c$
11: Set $e_t$ to $e_{t-1} + 1$
12: else
13: Update $c$ to $c - \alpha$
14: Set $e_t$ to $e_{t-1}$
15: end if
16: Estimate $\hat{p}_t$ as $e_t/T$
17: end for
18: Predict and return $\hat{y}_T$ and $\hat{p}$

**Algorithm 2:** An algorithm for the setting of Assumption 3 based on additive-decrease multiplicative-increase (ADMI).

to scale with (the square root of the second moment of the) losses obtained so far, because in many practical situations, the actual losses may be less than our analytical upper bound thereupon (and the second moment is important in the confidence estimates). To address these issues, we consider a policy, which dynamically adapts $D_k$ based on the additive-decrease multiplicative-increase (ADMI) updates:

**Theorem 1.** Under Assumption 4, Algorithm 3 even with extra($L_t, D_t, \hat{p}_t$) being a constant function returning False, makes it possible to compute $D_t$ of Assumption 3 scaling linearly with the mean of the losses observed so far. Furthermore, this choice of $D_t$ allows for our estimate $\hat{p}_t$ of the probability $p$ to converge in distribution as $t \rightarrow \infty$.

Theorem 1 says that there is a distribution such that, as $t \rightarrow \infty$, $\hat{p}_t$ follows this distribution. In other words, $\hat{p}_t$ will not be totally erratic, but it will be predictable in the sense that it will eventually resemble samples from a random variable with a fixed distribution. This legitimizes the use of simulations and studying the resulting sample distributions. Furthermore, under mild but technical conditions, the convergence occurs at a geometric rate [32, Theorem 1],
i.e., \( \mathbb{E}[\hat{p}_t - p] \) is \( O(r^n) \), where the growth rate \( r < 1 \) can be made explicit by a careful analysis of the drift function. Under further technical conditions, one could prove moment bounds [38].

**Proof sketch.** The process \( \hat{p}_t \) can be cast as a recurrent iterated function system (RIFS) on the normed space \((\mathbb{R}, \|\cdot\|_1)\) with a family of functions \( \{\omega_j \mid j \in \mathcal{K}\} \) that take one of the following two forms:

\[
\begin{align*}
\omega_1(c) &:= \beta c & \beta > 1.0 \\
\omega_2(c) &:= c - \alpha & \alpha > 0
\end{align*}
\]  

where clearly only \( \omega_2 \) is a contraction. We provide an overview of RIFS in Appendix A in the Supplementary material. By using a theorem of Barnsley et al. (restated as Theorem 3 in the Supplementary material, for convenience), we want to show that the system converges in distribution.

In particular, the probability of applying \( \omega_1 \) at iteration \( t + 1 \) is given by the probability of: \( l_k > D_k \) where \( l_k \) is the loss \( \|y_t - \hat{y}_t\|^2 \) and \( D_k \) is a threshold. Our goal is hence to prove that there exists a Markov chain, with \( K \) states and a transition probability matrix \( P \in [0, 1]^{K \times K} \), such that \( \text{prob}(i_{t+1} = j|i_t) = p_{i,j} \), i.e., the probability of applying a specific \( \omega_j \) depends on the last applied function \( \omega_i \). Clearly, this is not true in case of \( K = 2 \), but one can consider an arbitrary number of copies of the two functions, and hence a much larger \( K \).

Notice that even with the hidden state, the evolution of the underlying \( y_t \) in (1) can be easily modelled with a Markov chain. Our goal is hence to show that the evolution of \( \hat{y}_t \) can be modelled by a Markov chain. Although the solution of the convexification in Line 8 may seem non-linear, one should consider the fact that the existence of a Moore–Penrose pseudoinverse guarantees that there exists a linear representation of the evolution of \( y_t \). The existence of such a Markovian representation in turn guarantees that there exist instance-specific conditions such that the RIFS is contractive on average and by Theorem 3, we then conclude that \( \hat{p}_k \) converges in distribution.

Notice, however, that so far, we have not considered the setting of Assumption 2 and we have not made any use of the \( \hat{p}_t \). Let us now consider Assumptions 1 and 2 and a function extra\((L_t, D_t, \hat{p}_t)\), which would from some time level \( t_0 \) onwards perform two actions: First, consider the \( t_0 \) most recent entries in \( L_t, D_t \) and compute an estimate of a probability of an anomaly based on such time-window of length \( t_0 \). Second, estimate the probability that such an estimate is not drawn from a sample distribution of \( \hat{p}_t \). This is motivated by the intuition that in case \( l_t \) is less than \( D_t \), we may want to force a \( \hat{p}_t \) fraction of any time-window to be an anomaly. We conjecture that under Assumptions 1 and 2 Algorithm 2 with such an extra allows for the recovery of the non-corrupted entries, with high probability in the large limits of \( t, t_0 \). Intuitively, the proof may use the technique of the proof of Theorem 1 and a law of large numbers for non-identically distributed Bernoulli random variables, e.g., from Kolmogorov’s strong law. However, the proof would have to operate with a much larger state.
space of the Markov chain than the one used in the proof of Theorem 1, and the reasoning would be complicated by the fact that until \( t_0 \), we may corrupt our estimate of the underlying LDS by mis-interpreting some elements of noise \( \xi_t \) as outputs of the LDS, which Theorem 1 avoids by Assumption 3.

5 Empirical Results

To illustrate the performance of the algorithms, we chose the same single-input single-output (SISO) system as in [18], where:

\[
B^T = C = [1 \ 1], \quad D = 0, \quad A = \text{diag}([0.999, 0.5]),
\]

(8)
time horizon \( T = 100 \), and noise terms \( \eta \) and \( \xi \) are i.i.d. Gaussians. For comparison purposes, we consider the trivial last-value prediction, also known as persistence-based prediction, which uses the most recent value \( \hat{y}_{t+1} := y_t \), and the same thresholding of Algorithm 2.

First, we illustrate the performance on one sample run of the method. Figure 1a presents the true inputs \( x_t \) (in blue), true outputs \( y_t \) (in green), which have been corrupted in 10% of samples by noise \( U(0, 100) \) (as indicated by black vertical bars in the bottom third of the picture), and predictions \( \hat{y}_t \) of the output of our method (in red). Below, Figure 1b presents the thresholds (in dotted lines), the output of our method (in blue), the last-value prediction (in green, often overlapping with the red line), and the corresponding anomalies as semi-transparent vertical bars. In particular, the pink vertical bars in the top third of the plot correspond to anomalies detected by the last-value prediction and the pale blue bars in the middle third of the plot correspond to the anomalies detected by our method. In this one sample, with no statistical significance, the harmonic mean of precision and recall (F1 score) of our method is 0.93, while the last-value prediction results in F1 score of 0.56. Notice that the uniformly-distributed noise \( U(0, 100) \) violates Assumption 3 and indeed, the first detected anomaly does not differ from the previous output of the LDS by much.

In Figure 2, we present the loss \( l_t := \|y_t - \hat{y}_t\|^2 \) of our method (in red) and the last-value prediction (in blue). In particular, we plot the mean (in a solid line) and standard deviation (shaded). While it is not possible to infer any generalisations from this one particular LDS, the F1 score of 0.88 of our method (averaged over the sample paths) improves considerably over the F1 score of 0.46 using the last-value predictions.

Next, we present perhaps an even more intriguing example, which is not supported by our theory. In particular, we consider a time-varying system, where we vary \( B_k \) from \( k = 50 \) such that each entry of \( B_k \) follows a sinusoid:

\[
\begin{cases} 
1 & \text{if } k < 50, \\
1.01 + \sin\left(\frac{\pi(k-50)}{180}\right) & \text{otherwise}.
\end{cases}
\]

(9)

Figures 3 and 4 clearly demonstrate that the performance of the last-value prediction does not change materially, but that the performance of our method

\[8\]
improves. We have been able to replicate this behaviour on a variety of small examples. This naturally opens the question as to whether one could prove regret bounds for spectral filtering in a time-varying system, and consequently guarantee the performance of the anomaly detection therein.

6 Related Work

In system identification, there are over 65 years of research [21], which we draw upon. Specifically, our spectral filtering follows the tradition of subspace methods for identification [37, 26, 13, 22, 31, e.g.]. There, one consider “wave filters”, which are based on convolving data with eigenvectors of a certain Hankel matrix $Z_T$. Notice that the eigendecomposition of the Hankel matrix $Z_T$ can be pre-computed, as the matrix does not depend on the input data. In particular, we consider the regularised version of Hazan et al. [19, 18]. Several other authors [28, 17, 30] have derived similarly important results at the same time.
Figure 3: Illustrations on the time-varying system (9). Left: Inputs, outputs, and predictions of the output by our method. Right: Thresholds and anomalies detected.

Figure 4: Mean and standard deviation of the loss $l_t$ on the time-varying system (9).

Subsequently, a number of authors [11, 1, 14, 15, 3, 7] have applied them to the (Linear-Quadratic, LQ) control of an unknown system, which underlies much of reinforcement learning.

In anomaly detection [9] and and closely related problems, there is a possibly even longer history of related work. For anomaly detection in general, and especially for anomaly detection in LDS, the book of Basseville and Nikiforov [6] is the standard reference. In particular, the closest to our work is Ting et al. [34], who employ Kalman filters [21] in anomaly detection, i.e., assume that $A, B$ are known. In contrast, we do not assume that $A, B$ are known. Further, we should like to point to the closely related problems of deviation [27], or (on-line, complex) event [10] detection, outlier analysis [2], detection [33], or pursuit [39], foreground detection [8] and the complementary background subtraction or background maintenance, or even dynamic anomalography [24].
7 Conclusions

While anomaly detection is notoriously hard to benchmark, due to the fact that each application has its own assumptions as to what is normal, we believe that the assumption of normal data being generated by an unknown linear dynamical system and anomalies replacing the observations arbitrarily in a Huber-like fashion may have a broad appeal, especially in conjunction with the methods with performance guarantees, which we have presented.

There is a considerable scope for further work, including the rates of convergence [32, cf. Theorem 1], moment bounds [38, cf.], and extensions of the results to time-varying systems on the theoretical side, and novel variants of the thresholding on the algorithmic side. In particular, we envision that exponential smoothing and upper confidence bounds may well be worth investigating.

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A Background on Iterated Function Systems

In a generalization of a Markov chain, known variously as iterated function system or iterated random functions, one has a state space $X$ with its metric $d$, a family $W$ of Lipschitz functions $W = \{ w_j : X \to X \mid j \in \mathcal{K} \}$, where $\mathcal{K}$ is some index set, which we assume to be finite or countably infinite, and a measure $\nu$ that makes $(\mathcal{K}, \cdot, \nu)$ a probability space.

At each iteration $t$ of the iterated function system, $j$ is selected from $\mathcal{K}$ according to $\nu$ and $w_j$ is applied to the current state $x_k$ to obtain $x_{k+1}$. Formally:

$$\text{prob}(X_{k+1} \in A \mid X_k = x_k) \overset{\text{def}}{=} \sum_{\mathcal{K}} \mathds{1}_{\{ \{ j \mid w_j(x_k) \in A \} \}}(j) \nu(j),$$

i.e. the probability of $x_{k+1}$ ending up in a set $A$ is the probability of selecting an index $j$ such that $w_j(x_k)$ is in $A$ (the measure of the set of indices $j$ for which $w_j(x_k)$ is in $A$). Here the Markov property is clear: the distribution of the next state $X_{k+1}$ depends only on the current state $x_k$ and not any “older” states $x_{k-1}$ etc.

In this way, the IFS “jumps” around $X$. Unless we have a degenerate case such as all $w_j$ having the same fixed point, we can not expect the sequence $\{x_k\}$ to converge in a classical sense. Instead, we can establish conditions for convergence in distribution: that there is a distribution $\Pi$ on $X$ such that as $k \to \infty$, the set $\{x_0, x_1, \ldots, x_k\}$ will be distributed according to $\Pi$.

**Theorem 2** (E.g. [12] Thm. 1.1). Let $L_j$ denote the Lipschitz constant of $w_j$ and assume that the IFS is contractive on average, i.e.

$$\sum_{\mathcal{K}} \nu(j) \log(L_j) < 0. \quad (10)$$

Then, there is a distribution $\Pi$ on $X$ such that $\{x_0, x_1, \ldots, x_k\}$ is distributed according to $\Pi$ as $k \to \infty$.

If $W$ is a family of contractions, i.e. if $L_j < 1$ for all $j$, then (10) is trivially satisfied.

One can generalise this notion further to a recurrent iterated function system (RIFS), which is an IFS with an underlying Markov chain that modifies $\nu$ at each time step. More precisely, we have an IFS as described in the last section with a finite index set $\mathcal{K}$, say $\mathcal{K} = \{1, \ldots, K\}$. Additionally, there is a Markov Chain with $K$ states and transition probability matrix $P \in [0, 1]^{K \times K}$. The probability of applying $w_j$ at iteration $k+1$ is now given by $\text{prob}(i_{k+1} = j \mid i_k) = p_{i_k,j}$, i.e. the probability of applying a specific $w_j$ depends on what the last applied function $w_{ik}$ was! This is in contrast to the basic case above, where the probability to select a specific $w_j$ was always the same and given by $\nu(j)$. Notice that the way that $X_k$ jumps around in $X$ now is not a Markov process anymore — the distribution of $X_k$ not only depends on $X_{k-1}$, but also on $i_{k-1}$ — but the joint process of $(X_k, i_k)$ jumping around in $X \times \mathcal{K}$ is.

Results analogous to Theorem 2 can be stated for this case, see e.g. [3, 5]. We state

**Theorem 3** ([5]). Assume we have an RIFS as described above, and let $m : \{1, \ldots, K\} \to [0, 1]$ denote the stationary distribution of the underlying Markov chain (i.e. $m$ corresponds to the normalized Perron eigenvector of $P^T$). Then, if

$$\sum_{i=1}^K m(i) \log L_i = E_m \{ \log L_i \} < 0, \quad (11)$$

\footnote{A function $f$ on the metric space $(X, d)$ is Lipschitz with constant $s$, or “$s$-Lipschitz,” if for all $x, y \in X$, we have $d(f(x), f(y)) \leq sd(x, y)$.}
there is a unique stationary distribution $\nu$ of the Markov process $(X_k, i_k)$ and $X_k$ converges in distribution to $\nu$ with $\nu(B) = \nu(B \times K)$. Here, $L_i$ again denotes the Lipschitz constant of $w_i$, and $E_m$ denotes expected value with respect to $m$.

Proof. This is just a corollary (much weaker, but sufficient for our purposes) to \cite{5} Thm. 2.1 (ii), which follows by taking $n = 1$ and removing the specifics of the stationary distributions. \hfill \square

If (11) holds, we again say that the RIFS is average contractive or contractive on average.