Emergent Finite Frequency Criticality of Driven-Dissipative Correlated Lattice Bosons

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Critical points and phase transitions are characterized by diverging susceptibilities, reflecting the tendency of the system toward spontaneous symmetry breaking. Equilibrium statistical mechanics bounds these instabilities to occur at zero frequency, giving rise to static order parameters. In this work we introduce a new class of dynamical transitions in a quantum many body system far from thermal equilibrium, characterized by a susceptibility diverging at a finite non-zero frequency, an emerging scale set by interactions and non-equilibrium effects. In the broken-symmetry phase the corresponding macroscopic order parameter becomes non-stationary and oscillates in time without damping, thus breaking continuous time-translational symmetry. Our results, obtained for a paradigmatic model of bosons interacting on lattice in presence of drive and dissipation, are relevant for the upcoming generation of circuit QED arrays experiments and outline a generic framework to study time-domain instabilities in non-equilibrium quantum systems, including Floquet time crystals and quantum synchronization.
Classical and quantum phase transitions in systems at thermal equilibrium are characterized, according to the Landau paradigm, by the emergence of a static order parameter which spontaneously breaks a symmetry of the system, such as spin rotational invariance for magnetism, or spatial translational invariance for crystals \[1, 2\]. The resulting criticality is described in terms of a zero frequency instability of the normal symmetric phase, characterized by a singularity of a certain static susceptibility. For classical systems far away from thermal equilibrium, such as in presence of external forcing and dissipation, the variety of instabilities can be much richer, with both finite momentum and finite frequency modes going unstable and resulting in a rich phenomenology, including formation of patterns, propagating fronts, spatio-temporal chaos, synchronization or other oscillatory behaviors \[3, 4\].

In the quantum world, the question of whether finite frequency modes can become critical, giving rise to genuine time-domain instabilities of the quantum dynamics and to an associated breaking of time-translational invariance, is much less explored. Experimental breakthroughs at the interface between atomic physics, quantum optics and solid state physics, have brought forth a number of platforms which naturally probe non-equilibrium quantum many body dynamics, ranging from ultra cold atoms \[6\], trapped ions \[7\] and arrays of non-linear circuit QED cavities \[8, 11\], thus making the question of actual experimental relevance. In this respect, quantum many body systems in presence of both driving and dissipation mechanisms \[11, 12\], representing unavoidable particle losses, dephasing and decoherence processes leading to nonequilibrium stationary states, represent natural platforms to understand and explore such time domain dynamical instabilities at the quantum level. Theoretical investigations on non-equilibrium quantum criticality have typically focused on stationary instabilities \[13, 14\]. Only recently the attention has shifted toward the possibility of having limit cycles \[15–20\], i.e. non-stationary solutions of the equations of motion for a macroscopic order parameter. While several examples have been provided and discussed, yet, a general approach to address time-domain instabilities in quantum many body systems far from equilibrium from the point of view of critical phenomena is currently lacking.

In this work we introduce the framework of finite-frequency criticality for non equilibrium quantum many body systems. We argue that dynamical AC susceptibilities, which in thermal equilibrium are bound to remain finite and small since non-zero frequency modes are typically damped by interactions, can display genuine singularities at finite frequencies in driven-dissipative stationary states, as result of strong interactions and non-equilibrium effects. The critical frequency is not fixed a priori, but rather an emerging scale set by the microscopic parameters. As a result, the system undergoes a novel class of dissipative dynamical criticality where the order parameter in the broken symmetry phase becomes non-stationary and oscillates in time without damping, thus breaking the continuous time-translational symmetry. This stationary-state phase transition comes with unique dynamical features that appear in the dissipative time evolution, including a diverging relaxation time and a mode frequency softening. We write down the effective non-equilibrium field theory describing this transition and discuss the role of quantum fluctuations and the universality class of the finite frequency criticality. While our work focuses on a paradigmatic model of driven-dissipative quantum many body system describing bosons interacting on a lattice in presence of incoherent pump and losses, which is relevant for the upcoming generation of circuit QED arrays experiments \[21, 22\], it also outlines a generic framework to study dynamical instabilities in non-equilibrium quantum systems. Such a framework has the potential to be applied in a wide range of contexts, including for example driven and isolated Floquet systems, where breaking of discrete time-translational symmetry (so called time-crystal behavior) has been predicted \[22, 23\] and observed \[24, 27\], and quantum systems undergoing various forms of synchronization \[28, 30\]. In addition to providing an explicit microscopic quantum model showing breaking of time translation invariance, our work establish a conceptual bridge between phenomenological descriptions of classical dynamical instabilities and full quantum many body dynamics.

Results

System. We start introducing the Bose-Hubbard (BH) Hamiltonian

\[
H = -J \sum_{\langle ij \rangle} (a_i^\dagger a_j + h.c.) + \sum_i \left( \delta \omega_i n_i + \frac{U}{2} n_i^2 \right)
\]

modelling a lattice of circuit QED resonators \[21, 31, 32\] hosting a single bosonic cavity mode on each site, \( a_i, a_i^\dagger \), with frequency \( \delta \omega_i \) and repulsive interaction \( U \) proportional to the square of the local occupation, \( n_i^2 = (a_i^\dagger a_i)^2 \). Bosons can move around the lattice at a rate given by the hopping integral \( J \). We supplement the model with pump and losses, necessary to drive the system into a non-trivial steady state. Specifically, we describe the driven-dissipative dynamics in terms of a master equation for the system density matrix

\[
i\partial_t \rho = -i[H, \rho] + \mathcal{D}[\rho]
\]

where the dissipator takes the form \( \mathcal{D}[\rho] = \mathcal{D}_{\text{loss}}[\rho] + \mathcal{D}_{\text{pump}}[\rho] \). The first term describes single particle losses with rate \( \kappa \), \( \mathcal{D}_{\text{loss}}[\rho] = \kappa \sum_i (a_i \rho a_i^\dagger - \frac{1}{2} \{a_i^\dagger a_i, \rho\}) \), while the second term, \( \mathcal{D}_{\text{pump}}[\rho] \), accounts for incoherent pumping with a finite bandwidth \( \sigma \) and amplitude \( f \), which injects particles in each cavity without a well defined phase. Throughout this work we consider two different microscopic models of pumping: a random classical drive.
we plot the Hamiltonian (cavities incoherently preserves the problem, namely the intimate connection between the above symmetry, of particle number associated to the above symmetry, and narrower as the critical value \( J_c \) is approached. (b) At the critical point (top inset) the two peaks turn into a genuine finite frequency singularity, \( \text{Re} \chi^R(q = 0, \omega) \sim 1/(\omega - \Omega_\ast) \). (b) The Imaginary part of the susceptibility (bottom inset) shows also a narrow and sharp peak which evolves into a pole at the critical point. Parameters: Drive amplitude \( f = 0.0625U \), bandwidth \( \sigma = 1.5U \), loss rate \( \kappa = 0.0128U \), resonator frequency \( \delta \omega_0 = 0.0 \).

modulated with a coherent tone or an incoherent drive obtained through an ensemble of driven two-level emitters embedded in each cavity and undergoing a population inversion, as recently proposed. In the following we will focus on the robust and universal features of the resulting finite-frequency phase transition, while discussing elsewhere those aspects which depend on the specific driving protocol (see Supplemental Information. [? ]). We notice that driving the cavities incoherently preserves the \( U(1) \) symmetry of the Hamiltonian, \( a_i \rightarrow e^{i\theta}a_i \), as opposed to the cases of coherently driven cavities where this symmetry is explicitly broken by the drive. This leaves open the possibility for a non-equilibrium, driven-dissipative phase transition with spontaneous breaking of the \( U(1) \) symmetry, analog to the well known equilibrium quantum criticality of correlated lattice bosons. We notice however that, different from equilibrium systems, in the present case there is no conservation of particle number associated to the above symmetry, a signature of open quantum systems. Here we highlight another crucial non-equilibrium aspect of the problem, namely the intimate connection between the \( U(1) \) instability and the breaking of time-translational symmetry in the problem.

Interacting Normal Phase and its Finite-Frequency Instability. To gain some insight on the problem, we start considering the limit \( J = 0 \), where the lattice problem reduces to a collection of interacting, driven-dissipative decoupled single sites. In this limit \( \langle a_i \rangle = 0 \) and the balance between drive, dissipation and interaction results in a finite density of bosons per site (see Supplemental Information. [? ]). We expect this incoherent normal phase to remain stable even at non zero \( J \), at least below some critical hopping strenght \( J_c \). The focus of this work is exactly this strongly interacting, finite-hopping, non-equilibrium phase and its potential instabilities toward breaking of the \( U(1) \) symmetry, rather than the driven-dissipative superfluid regime at large hopping, well described by phenomenological Gross-Pitaevski weakly interacting theories. To unveil the properties of the normal phase we compute the susceptibility of the order parameter \( \psi_\ast = \langle a_i \rangle \) to a weak coherent drive, to obtain (see Methods)

\[
\chi^R(q, \omega) = \frac{1}{J_q \chi^R_\ast(q) - G^R_\text{loc}(\omega)}
\]

where in the above expression \( J_q = -2J \sum_{\alpha=1}^d \cos q_\alpha \) is the bare lattice dispersion while \( G^R_\text{loc}(\omega) \) defined as

\[
G^R_\text{loc}(\omega) = -i \int_0^\infty dt e^{i\omega t} \langle [a(t), a^\dagger(0)] \rangle_\text{loc}
\]

is the exact single-site retarded Green’s function in presence of interaction, drive and dissipation, which contain crucial information about the spectrum of the bosonic mode. For interacting lattice bosons in thermal equilibrium, the \( U(1) \) susceptibility is well known to show a zero-frequency singularity at a critical value of the hopping, at which the Mott insulating phase becomes unstable towards superfluidity. As we are now going to show the behavior of the same quantity in a non-equilibrium driven dissipative stationary state is remarkably different.

In figure we plot the \( q = 0 \) susceptibility, probing the instability of the homogeneous normal phase, for different values of the hopping strenght \( J \). We find a well defined resonance structure which gets sharper and narrower as the hopping is increased, and eventually turns into a genuine finite frequency pole at \( \omega = \Omega_\ast \), when a critical value \( J_c \) is reached (see Supplemental Information. [? ]). Right at \( J_c \) the susceptibility diverges as a power law around \( \Omega_\ast \), \( \chi^R(q = 0, \omega) = \chi_0 / (\omega - \Omega_\ast)^\alpha \), with \( \alpha = 1 \). The appearance of a singularity at finite frequency is a remarkable result with no counterpart in systems in thermal equilibrium, and ultimately arises from the interplay of interaction and non-equilibrium effects. To appreciate the origin of this result it is instructive to look more in detail into the structure of the retarded local Green’s function of the single site problem, \( G^R_\text{loc}(\omega) \) which enters the susceptibility through Eq. (3).

As we see in figure the imaginary part of this local Green’s function, describing the spectral function of
the bosonic mode, has two sharp and well separated peaks, the distance between them being set essentially by the interaction $U$. While resembling the spectrum of an equilibrium zero temperature Bose Hubbard single site, if not for the broadening of the two peaks, a crucial feature appears in the driven-dissipative case which contrasts strikingly with the common expectation: the spectral function vanishes and changes its sign at a finite frequency $\Omega_\ast \neq 0$. This feature is not only absent in thermal equilibrium, where under general circumstances one expect bosonic spectral functions to change sign at zero frequency, but also for a driven-dissipative yet non-interacting system for which the imaginary part would be completely set by the external loss/pump rates, giving rise to a single Lorentzian feature with constant sign. The frequency $\Omega_\ast$ is therefore an emergent scale, not fixed a priori by any drive frequency (since each cavity is incoherently driven), but rather arising out of the non-interacting system for which the imaginary part changes its sign at a finite frequency $\Omega_\ast$. This emergent frequency is not fixed a priori but rather fully tunable and depending from the amplitude and bandwidth of the drive. Parameters: Drive amplitude $f = 0.0625U$, bandwidth $\sigma = 1.5U$, loss rate $\kappa = 0.0128U$, resonator frequency $\delta\omega_0 = 0.0$.

By expanding the local Green’s function around $\Omega^\ast$, $G_{\text{loc}}^{R}(\omega) = G_{\text{loc}}^{R}(\Omega) + K_1 (\omega - \Omega^\ast) + K_2 (\omega - \Omega^\ast)^2$, where $K_1 = \partial_\omega G_{\text{loc}}^{R}(\Omega^\ast)$ are in general complex numbers, and assuming for the time being to have both $K_1, K_2 \neq 0$, we find two branches of excitations $\delta\omega_\pm(q) = \omega_\pm(q) - \Omega^\ast$, which at low momentum reads

$$\delta\omega_\pm(q) \simeq (\omega_\mp^+(0) + i\omega_\pm^+(0)) + (D_\mp^+ + iD_\pm^+) q^2$$

with a mass/spectral gap $\omega_\pm^+(0) + i\omega_\pm^+(0)$ and a $z = 2$ dynamical critical exponent ($\omega \sim q^z$) corresponding to diffusive dynamics. Near the critical point $J_c$, one of the two branches at $q = 0$ becomes soft, with both the real and imaginary part of the spectral gap going to zero as $\omega_\pm^+(0)(q = 0) \sim |J - J_c|$. The instability of the normal incoherent phase towards a $U(1)$ symmetry breaking is therefore characterized by two energy scales vanishing, a fact that will have important dynamical consequences as we are now going to show.

**Dynamical Signatures of Finite Frequency Criticality.**

We now discuss the dynamical consequences of the finite-frequency, stationary-state instability we have presented so far and investigate the dissipative dynamics of the lattice problem for different values of the hopping strength $J$. (Top Panel) Normal phase, $J = 0.8J_c$, exponential decay of the absolute value of the order parameter toward an incoherent stationary state $|\psi(t)\rangle \sim e^{-t/\tau(J)}$, with oscillations at frequency $\Omega(J)$ (see inset) due to the phase linearly growing in time. (Bottom Panel, log scale) Approaching the critical point $J_c$, there is clear evidence of a critical slowing down and a growing relaxation time suggesting a power-law decay right at the transition, as we show analytically in the main text. Parameters: Drive amplitude $f = 0.0625U$, bandwidth $\sigma = 1.5U$, loss rate $\kappa = 0.0128U$, resonator frequency $\delta\omega_0 = 0.0$. 

**Figure 2. Local Spectral Function and the Emergent Frequency $\Omega^\ast$.** Retarded Green’s function, see equation (3) of the interacting single-site problem in Eq. (3) with drive and dissipation. The imaginary part, describing the spectral function vanishes and changes its sign at a finite frequency $\Omega^\ast$. This emergent frequency is not fixed a priori but rather fully tunable and depending from the amplitude and bandwidth of the drive. Parameters: Drive amplitude $f = 0.0625U$, bandwidth $\sigma = 1.5U$, loss rate $\kappa = 0.0128U$, resonator frequency $\delta\omega_0 = 0.0$.

**Figure 3. Dissipative Dynamical Transition.** Time evolution of the order parameter $\psi(t) = |\psi(t)|e^{i\sigma(t)}$ for different values of the hopping strenght $J$. (Top Panel) Normal phase, $J = 0.8J_c$, exponential decay of the absolute value of the order parameter toward an incoherent stationary state $|\psi(t)\rangle \sim e^{-t/\tau(J)}$, with oscillations at frequency $\Omega(J)$ (see inset) due to the phase linearly growing in time. (Bottom Panel, log scale) Approaching the critical point $J_c$, there is clear evidence of a critical slowing down and a growing relaxation time suggesting a power-law decay right at the transition, as we show analytically in the main text. Parameters: Drive amplitude $f = 0.0625U$, bandwidth $\sigma = 1.5U$, loss rate $\kappa = 0.0128U$, resonator frequency $\delta\omega_0 = 0.0$. 

We are now going to show.
that the absolute value of the order parameter shows an exponential relaxation toward zero, \(|\psi(t)| \sim e^{-t/\tau(J)}\), indicating an incoherent stationary state, while the phase grows linearly in time with a finite angular velocity \(\Omega\), \(\theta(t) = \Omega(J)t + \theta_0\). A closer inspection reveals that the characteristic frequency \(\Omega(J)\) differs from the value \(\Omega_s\), previously identified by an amount \(\delta \Omega(J) = |\Omega(J) - \Omega_s|\) which strongly depends on the hopping rate \(J\) and vanishes at the critical point \(J = J_c\) with a characteristic power law, \(\delta \Omega \approx (J_c - J)^{\alpha}\), as shown in the bottom left panel of Fig. 4. Similarly the relaxation time diverges upon approaching the critical hopping \(J_c\), \(\tau \sim 1/(J_c - J)\) and the order parameter shows a characteristic critical slowing down, as shown in the bottom right panel of figure 4. The behavior of the two energy scales \(\delta \Omega, \tau\) approaching the transition is the dynamical counterpart of the vanishing of real and imaginary part of the spectral gap at \(q = 0\) that we previously discussed, as it will become manifest in the following.

Nonequilibrium Field Theory of Finite Frequency Criticality. We now proceed to set up a non-equilibrium field theory for the finite-frequency dissipative transition, which will allow us to obtain a complete analytical picture of the mean field dynamics as well as to assess the role of quantum fluctuations beyond mean field and the universality class of the transition. The starting point is to notice that the dynamics of the order parameter is crucially controlled by the behavior of the inverse susceptibility in Eq. 3 around the critical frequency \(\Omega_s\). Upon expanding it for \(q \to 0\) and \(\omega \to \Omega_s\) and then moving to a rotating frame where the field is oscillating at frequency \(\Omega_s\), i.e. introducing the fields \(\psi_{c, \Omega}(x, t) = e^{-i\Omega_s t} \tilde{\psi}_{c, q}(x, t)\), we obtain

\[
\mathcal{S}_{eff} = \int dt dx \tilde{\psi}_c^* \left( -r + K_1 \partial_t \psi + \frac{K_2}{2} \partial_t^2 - K_3 \nabla^2 \right) \tilde{\psi}_q + \hbar c
\]

\[
+ \mathcal{S}_{\text{noise}} + \mathcal{S}_{\text{int}}
\]

where \(r = 1/zJ + \text{Re} \mathcal{G}_{loc}^R(\Omega_s) = (J_c - J)/J_c^2\) is the distance from the dissipative phase transition while \(K_3 = 1/zJ^2\) is the bare mass. The complex coefficients \(K_{1,2} = \partial_{\psi}^2 \mathcal{G}_{loc}^R(\Omega_s)\) play a crucial role for what concerns the universal properties of the transition encoded in the effective action. In thermal equilibrium whenever \(K_2 = 0\) one can drop the \(K_2\) coefficient and obtain a critical theory in the XY universality class. At the tip of the Mott lobes where \(K_1\) vanishes and a non-zero \(K_2\) gives rise to emergent particle-hole symmetry and a different universality class for the Mott to Superfluid transition [11]. In the dissipative case discussed here, the coefficient \(K_1\) is always different from zero along the phase boundary, from which we conclude that \(K_2\) can be dropped and the critical behavior of the dissipative transition belongs to a single universality class.

In Eq. 6 \(\mathcal{S}_{\text{noise}} = D \int dx dt \tilde{\psi}_c^* (x, t) \tilde{\psi}_q (x, t)\) represents the noise contribution in the equivalent stochastic (Langevin) dynamics with an effective diffusion coefficient \(D\) (see Supplemental Information). While \(\mathcal{S}_{\text{int}}\) accounts for the non-linearity and it is completely determined by the dynamical two-particle Green’s function of the driven-dissipative single site problem. If we restrict ourselves to terms which are linear in the quantum field, which is valid in high enough dimension according to canonical power counting [12, 42], we can write this term as \(\mathcal{S}_{\text{int}} = u \int dx dt \tilde{\psi}_c^* (x, t) \tilde{\psi}_q (x, t) \tilde{\psi}_q^* (x, t) + \hbar c\). We can now take the saddle point equation \(\delta \mathcal{S}/\delta \tilde{\psi}_c^* (x, t) = 0\) and obtain an equation of motion

\[
(iK_1 \partial_t - K_3 \nabla^2 - r) \psi_c + u |\psi|^2 \psi_c = 0
\]

which takes the form of a complex Ginzburg Landau equation, well known as phenomenological description of pattern formation in classical non-equilibrium systems [5, 6]. The spatially homogeneous solution \(\psi_c (t) \equiv |\psi_c (t)| e^{i\theta}\) can be obtained in closed form and reads

\[
|\psi_c(t)| = |\psi_c(0)| \sqrt{1 + \alpha \left( 1 - e^{-2r_t} \right)}
\]

\[
\theta(t) = -r_R t + \tilde{u}_R \int_0^t dt' |\psi_c(t')|
\]

with complex coefficients \(\tilde{r} = \tilde{r}_R + i\tilde{r}_I = r/K_1\) and \(\tilde{u} = \tilde{u}_R + i\tilde{u}_I = u/K_1\) as well as \(\alpha = |\psi_c(0)| |\tilde{u}_I|/\tilde{r}_I\). This solution describes, upon crossing the critical point \(J_c\) at which \(\tilde{r}_R, I \propto (J_c - J)\) change sign, a dynamical transition from a regime of damped oscillation toward

Figure 4. Frequency Softening and Diverging Relaxation Time. Behavior of the two energy scales \(\delta \Omega\) and \(\tau\), extracted from the dynamics of the order parameter, as a function of the distance from the critical point, \(J_c - J\). These represent, respectively a slow frequency oscillation mode around the carrier frequency \(\Omega_s\) (top panel) and the relaxation time to reach a steady state (bottom panel). The dissipative dynamical transition is characterized by both energy scales becoming critical, in agreement with the behavior of the real/imaginary part of the spectral gap, \(\omega_{R,I}(q = 0)\). Parameters: Drive amplitude \(f = 0.0625U\), bandwidth \(\sigma = 1.5U\), loss rate \(\kappa = 0.0128U\), resonator frequency \(\delta \omega_0 = 0.0\).
zero, $\psi_c(t) \sim e^{-i\Omega t} e^{-i\tilde{\nu} R t}$ for $J < J_c$ to the broken symmetry phase for $J > J_c$, where amplification of oscillations and a long time saturation of the order parameter into a train of finite amplitude oscillations emerge, $\psi_c(t) \sim |\psi_c(\infty)| e^{-2i\tilde{\nu} R t}$, with $|\psi_c(\infty)| = \sqrt{|\tilde{\nu}||J|} \sim \sqrt{J-J_c}$. In the normal phase, where the non-linearity $u$ in Eq. (7) is essentially irrelevant, the transient dynamics shows harmonic oscillations while in the broken symmetry phase multiple frequencies are present, at least on intermediate time scales, as encoded in the phase dynamics (9). Right at the transition, for $J = J_c$ when $\tilde{\nu}_{R,I} \to 0$, the amplitude of the order parameter decays towards zero as a power-law $|\psi_c(t)| \sim 1/\sqrt{t}$ (13) (44), while the angular velocity vanishes and the phase growth in time only logarithmically, $\theta(t) \sim \log(1 + 2\tilde{\nu}_I|\psi_c(0)|t)$. We notice that while gaussian fluctuations resulting into Eq. (7) naturally rise to mean field exponents, the full effective action in Eq. (6) include the effect of non linearities, noise and quantum fluctuations, resulting in non trivial renormalizations of critical behavior and non-mean field exponents (12, 45). This raises the natural question of whether the finite-frequency criticality we have outlined here is a genuine feature of the stationary state once these fluctuations are included. While answering this question would require to develop a renormalization group treatment of the resulting finite-frequency criticality, along the lines discussed for the equilibrium Bose Hubbard model (46) as well as for weakly interacting non-equilibrium superfluids (42), the effective field theory we have derived already allows to obtain crucial insights on this subject. Indeed, provided the effective action (6) in the rotating frame admits a non-vanishing $U(1)$ order parameter, $\tilde{\psi}_c \neq 0$ which is expected in high enough dimensions, then the broken symmetry phase in the original frame will display undamped oscillations and breaking of time-translational invariance, $\psi_c \sim \psi_c e^{i\Omega t}$. An intriguing open question is whether such a phenomenon could be robustly observed in models of driven-dissipative systems with discrete broken symmetry phases (16, 18) or even in presence of a purely coherent drive, as for example in the context of optomechanical platforms (28, 47).

Discussion. In this work we have shown that a prototype model of correlated driven-dissipative lattice bosons, of direct experimental relevance for circuit QED arrays experiments (21, 22, 31) develops, for a critical value of the hopping rate, a diverging susceptibility at a non-zero frequency $\Omega$. The resulting finite-frequency criticality corresponds to the dissipative dynamics lacking of a stationary state and rather oscillating in time without damping, a signature of so called quantum time-crystal behavior (48). With respect to other proposals for robust breaking of time-translational invariance, which usually focus on thermally isolated systems subjected to periodic driving and to quenched random disorder (22, 25) or discuss it as a boundary (49) or prethermal (50) phenomenon in clean closed systems, our work takes advantage of the finite drive and dissipation rates to sustain the quantum time crystalline phase. Yet, it also provides a conceptual framework to analyze finite-frequency, time-domain instabilities of quantum evolution, encoded in dynamical non-equilibrium response functions whose investigation could open the door to a number of interesting new directions. For example analyzing discrete time-crystal behavior in periodically driven systems as well as quantum synchronization through the lens of the finite-frequency criticality discussed here, and connecting those phenomena to analogue instabilities of classical dynamical systems such as period doubling (51), appears an extremely interesting future direction.

Acknowledgements. We acknowledge discussions with A. Clerk, M. Goldstein, V. Savona. This work was supported by the CNRS through the PICS-USA-147504, by a grant "Investissements d’Avenir" from LabEx PALM (ANR-10-LABX-0039-PALM) and by a grant IRS-IQUPS of University Paris-Saclay.

Methods

Strong Coupling Approach to Driven-Dissipative Bose Hubbard. Here we introduce an elegant and powerful approach to study the instabilities of the normal incoherent phase of the Bose Hubbard model discussed in the main text, by generalizing to the driven-dissipative case the equilibrium strong-coupling approach of Ref (41). We start writing the Keldysh action associated to the many body quantum master equation (2), which allows to describe the non-equilibrium stationary state and the excitations on top of it. We have $Z = \int \prod_i d [\tilde{b}_i, \tilde{b}_i^\dagger] e^{iS_{loc}},$ where we can split $S$ into a local part and a hopping term: $S = \sum_i S_{loc} [\tilde{b}_i, \tilde{b}_i^\dagger] + \int_C dt \sum_{ij} \tilde{b}_i J_{ij} b_j,$ with $J_{ij} = J i$ if $(i,j)$ are nearest neighbours and $J_{ij} = 0$ otherwise; $\int_C$ means that the integration is performed on the Keldysh contour. The local part of the action, $S_{loc} [\tilde{b}_i, \tilde{b}_i^\dagger],$ includes hamiltonian and dissipator contributions, describing entirely the single-site problem, which is non-linear and driven-dissipative. In this application, we don’t need the explicit form of $S_{loc} [\tilde{b}_i, \tilde{b}_i^\dagger];$ nevertheless, this can be obtained both by starting from a full hamiltonian description of the problem, introducing the degrees of freedom of the environment and of the pump, and directly from the master equation in eq. (2) as shown for example in Ref. (12). The hopping term can be decoupled introducing the auxiliary fields $\{\psi_i\},$ by means of a Hubbard-Stratonovich transformation:

$$e^{iF_C dt \sum_{ij} \tilde{b}_i J_{ij} b_j} = \int \prod_i d [\tilde{\psi}_i, \psi_i] e^{-iF_C dt \sum_{ij} \tilde{\psi}_i J_{ij}^\dagger \psi_j - \sum_i \tilde{\psi}_i \tilde{b}_i + \tilde{b}_i^\dagger \psi_i]}$$ (10)

Using this result we obtain an effective action for the complex order parameter $\psi_i$, which takes the form

$$S_{eff} = \int_C dt \sum_{ij} \psi_i^\dagger J_{ij}^\dagger \psi_j + \sum_i \Gamma[\psi_i^\dagger, \psi_i]$$ (11)

where the second term represents the generating functional of the local bosonic Green’s functions, $\Gamma[\psi_i^\dagger, \psi_i] = \log(T e^{iF_C dt \psi_i^\dagger a_i^\dagger a_i^\dagger \psi_i})_{loc}$, where the average is taken over the single site problem, which includes interaction,
drive and dissipation. If we then perform a cumulant expansion in the fields ψ, ψ† we obtain, within a gaussian approximation, an effective action which in the classical/quantum basis ψ_{loc,q} = (ψ_{i+} ± ψ_{i−})/√2 takes the form \[ S_{eff} = \int \text{d}ω d^q \Psi(\Psi^\dagger) \tilde{G}^{-1}(q,\omega) \Psi(q,\omega) \] with \[ \Psi(q,\omega) = \tilde{\psi}^R(q,\omega) \tilde{\psi}^A(q,\omega) \] and \[ \tilde{G}^{-1}(q,\omega) = \begin{pmatrix} 0 & -G_{loc}^R(\omega) \\ -G_{loc}^A(\omega) & -G_{loc}^K(\omega) \end{pmatrix} \] In the above expression we have \( J_q = -2J \sum_{\alpha_1} \cos q_\alpha \), while \( G_{loc}^{R/A/K}(\omega) \) defined as \[ G_{loc}^{R/A/K}(\omega) = -i \int_0^\infty dt e^{i\omega t} \langle [a(t), a^\dagger(0)] \rangle_{loc} \] are the exact single-site retarded/advanced/Keldysh Green’s function in presence of interaction, drive and dissipation, which we compute using their representation in terms of exact eigenstates and eigenvalues of the single site Liouvillian. Then, the susceptibility \( \chi^R(q,\omega) \) given in Eq. (3) of the main text is given by the the retarded Green’s function of the order parameter field, \( \chi^R(x-x',t-t') = -i \langle \psi_i(x,t) \tilde{\psi}_i^R(x',t') \rangle \), after Fourier transform, which can be obtained directly from the effective action \( S_{eff} \), after a gaussian integration.

**Time-Dependent Gutzwiller.** To study the dissipative time evolution and the consequences of finite-frequency criticality we use a time-dependent Gutzwiller decoupling of the density matrix, i.e. ρ(t) = \( \prod \rho_i(t) \) that we further assume homogeneous in space, \( \rho_i(t) \equiv \rho_{loc}(t) \). This approximation results in an effective single-site problem \( \partial_t \rho_{loc}(t) = -i[H_{eff}(t),\rho_{loc}(t)] + D_{loc}[\rho_{loc}] \) where \( H_{eff}(t) = \delta \omega n + Un^2/2 + zJ \langle a^\dagger(t) + a(t) \rangle \) with \( z = d/2 \) the coordination number of the lattice, \( D_{loc} \) is the local dissipator including incoherent drive and losses, while \( \psi(t) = Tr \rho_{loc}(t) a \) is a self-consistent time dependent field.

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Supplementary Information for
“Emergent Finite Frequency Criticality of Driven-Dissipative Correlated Lattice Bosons”

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The results we have described in the main text concerning the finite frequency criticality are extremely robust with respect to the specific driving protocol, provided that bosons are injected in the lattice incoherently. There are however interesting differences in the nature of the normal phase which strongly depend on the nature of the drive, as we are going to discuss below. We start discussing more in detail the two driving protocols we focused on and then address the main qualitative differences in the results for the two cases.

I. MICROSCOPIC MODELS OF INCOHERENT PUMP AND LOSSES

As discussed in the main text we describe the driven-dissipative dynamics in terms of a master equation for the system density matrix

\[ i \partial_t \rho = -i[H, \rho] + \mathcal{D}[\rho] \] (S1)

where the dissipator \( \mathcal{D}[\rho] \) takes the form

\[ \mathcal{D}[\rho] = \mathcal{D}_{\text{loss}}[\rho] + \mathcal{D}_{\text{pump}}[\rho]. \] (S2)

The first term describes single particle losses with rate \( \kappa \), \( \mathcal{D}_{\text{loss}}[\rho] = \kappa \sum_i \left( a_i \rho a_i^\dagger - \frac{1}{2} \{ a_i^\dagger a_i, \rho \} \right) \), while the second term accounts for incoherent pumping. Throughout this work we consider two different microscopic models of incoherent pumping, that we now discuss in more detail. In the first scheme \( \text{(S1)} \), to which we will refer to as \( \text{hot drive} \) in the following, a random classical drive is modulated with a coherent tone as described by the time-dependent Hamiltonian

\[ H_{\text{pump}}^{\text{hot}}(t) = \sum_i \left( e^{i\omega_{\text{L}} t} a_i^\dagger \eta(t) + \text{hc} \right) \] (S3)

where \( \eta(t) \) is assumed to have gaussian statistics with zero average, \( \langle \eta(t) \rangle = 0 \), and correlations \( \langle \eta(t)\eta(t') \rangle = f C_{\sigma} (t - t') \). We assume the noise spectrum to be box-shaped with a finite bandwidth \( \sigma \), i.e. \( C_{\sigma}(\omega) = \theta(\sigma - |\omega|) \), and amplitude \( f \), although the results we obtain do not depend qualitatively from the exact shape of \( C_{\sigma}(\omega) \). A second scheme of incoherent pumping has been recently proposed \( \text{(S2-S4)} \), arising from an ensemble of \( N_{\text{at}} \) driven two-level emitters embedded in each cavity and having randomly distributed frequencies. In the following we will refer to this scheme as \( \text{cold drive} \), for reasons which will become clear in the next sections. The microscopic Hamiltonian for this driving scheme reads

\[ H_{\text{pump}}^{\text{cold}} = \sum_i N_{\text{at}} \sum_{n=1}^{N_{\text{at}}} \omega_{\text{at}}^{(n)} \sigma_i^{+(n)} \sigma_i^{-(n)} + g \sum_{i,n} \left( a_i^\dagger \sigma_i^{-(n)} + \text{hc} \right) \] (S4)

where the transition frequencies \( \omega_{\text{at}}^{(n)} \) of the two-level systems are assumed to be uniformly distributed over a finite range and each emitter is incoherently pumped in the excited state \( \text{(S3)} \). Treating the incoherent driving at the master-equation level we obtain, in both cases of Eqs. \( \text{(S3-S4)} \), a contribution to the dissipator which reads \( \text{\text{(S1, S3)}} \)

\[ \mathcal{D}_{\text{pump}}[\rho] = \sum_i f_{\text{in}} \tilde{\mathcal{D}}[a_i^\dagger, a_i; \rho] + f_{\text{out}} \tilde{\mathcal{D}}[a_i^\dagger, a_i; \rho] \] (S5)

where \( \tilde{\mathcal{D}}[X,Y] = X\rho Y + Y^\dagger \rho X^\dagger - X^\dagger Y^\dagger \rho - \rho Y X \) (S6)
FIG. S1. Local bosonic occupation in the interacting single-site driven dissipative problem, respectively as a function of the drive bandwidth $\sigma$ (left panel) showing a characteristic staircase structure and drive amplitude $f$ (right panel). The value of the local occupation in the plateaux depends strongly on the nature of the driving protocol and can in general be tuned continuously with the drive amplitude $f$. Parameters: loss rate $\kappa = 0.0128U$, resonator frequency $\delta \omega_0 = 0.0$, drive amplitude (left panel) $f = 0.0625U$.

while $\tilde{a}_\sigma^\dagger$ is the photon operator dressed by the finite bandwidth drive, $\tilde{a}_\sigma^\dagger = \sum_n \sqrt{n + 1} C_\sigma|n + 1\rangle\langle n|$, with $\Delta_{n+1} = Un + (\delta \omega_0 + U/2)$ the level spacing of an isolated single site in Eq. (1) of the main text. In the random noise case, Eq. (S3), we obtain $f_{in} = f_{out} \equiv f$, namely the drive acts both as a source and as a sink of particles, much like a finite-temperature bath. Instead, in the case of inverted random emitters we have $f_{in} = f$ and $f_{out} = 0$, namely there are no additional losses of particles associated to the drive.

II. ROLE OF DRIVING PROTOCOL

A. Local Bosonic Occupation

The balance between drive, dissipation and interaction results in a finite number of bosons per site. The boson number in the single-site problem as a function of pump bandwidth $\sigma$, plotted in figure S1 (left panel), shows a staircase structure characteristic of blockade physics [S1], with a value of $\Delta \sigma \sim U$ required to add extra bosons in the system. This can be understood naturally: to add a boson in an interacting site the drive has to provide extra energy, however since the system is ultimately open and boson number is not exactly conserved, the exact occupancy will be fixed by the ratio of pump and losses. While this resembles the ground state physics of a Bose-Hubbard interacting site, we stress that the average density in this open and dissipative implementation is never exactly integer, due to losses. In this respect, an important difference between the two driving protocols already appear, namely the cold drive is able to fix the occupancy to almost integer filling [S2], while the hot drive to half-integer filling, reflecting the fact that the stationary density matrix is almost pure in the cold drive case, while it has a box-shaped distribution of populations in the hot case. The amplitude of the steps is set by the interaction while the height can be tuned continuously by changing pump amplitude $f$, as we show in the right panel of figure S1.

B. Bosons Distribution Function and Effective Temperature

In the main text we have mostly focused on the spectral properties of system, in particular on the retarded susceptibility of the order parameter $\chi^R(q, \omega)$ (see Eq. (3) in the main text) which encodes information on the finite-frequency phase transition and the associated retarded Green’s function of the single-site problem, $G^R_{loc}$. In a full non-equilibrium setting like the one we are considering it is also interesting to investigate the distribution function, which contains crucial information on the occupation of bosonic modes. To this extent, we define a Keldysh component of the order parameter Green’s function, in the classical-quantum basis, as

$$\chi^K(x - x', t - t') = -i\langle \psi_c(x, t)\psi_c^\dagger(x', t') \rangle$$  (S7)
which can be obtained, in frequency and momentum space, directly from the gaussian effective action (see Eq. 11 in the main text) and gives

\[ \chi^K(q, \omega) = \frac{G^K_{loc}(\omega)}{(J_q^{-1} - G^K_{loc}(\omega)) (J_q^{-1} - G^A_{loc}(\omega))} \]  

(S8)

From this, we can define a bosonic distribution function in analogy with the thermal equilibrium case, i.e.

\[ F(\omega) = \frac{\chi^K(q, \omega)}{\chi^K(q, \omega) - \chi^A(q, \omega)} \]  

(S9)

Indeed in thermal equilibrium the fluctuation-dissipation theorem constrains the functional form of the distribution function to the canonical bosonic one, \( F_{eq}(\omega) = \coth \beta \omega/2 \), which at low frequency (or high temperature) becomes \( F_{eq}(\omega) \sim T/2\omega \). Here, in presence of drive and dissipation, such an identity does not hold and we use Eq. (S9) as operational definition of the distribution function. Using Eq. (S8-S9) as well as Eq.(3) of the main text we can easily obtain an expression for the distribution function which only depends on local single site Green’s functions, namely

\[ F(\omega) = \frac{G^K_{loc}(\omega)}{G^K_{loc}(\omega) - G^A_{loc}(\omega)} \]  

(S10)

We plot in figure S2 (left panel) the distribution function for a given value of interaction, drive and dissipation. While its shape shows strong departure from thermal equilibrium remarkably we find that around the critical frequency \( \Omega_* \) the system develops a singularity of the form \( F(\omega) \sim T_{eff}/(\omega - \Omega_*) \), ultimately arising from the fact that, as we discussed extensively in the main text, \( \text{Im} G^K_{loc}(\omega) \) has a zero at \( \Omega_* \) while the Keldysh component is finite around the same frequency range. This suggests an asymptotic thermalization of the critical modes around the frequency \( \Omega_* \), with an effective temperature \( T_{eff} \) and with the frequency \( \Omega_* \) playing now the role of an effective chemical potential. The dependence of \( T_{eff} \) from the drive bandwidth is plotted in the right panel of figure S2 and reveals a rather substantial difference in the two driving protocols for what concerns the effective heating properties of the system. Indeed upon increasing the drive bandwidth the effective temperature increases in the hot drive case while decreases in the cold case one. This offers an alternative perspective on the recent proposed scheme to engineer effective ground state phases of interacting photons through the use of non-markovian reservoirs S2 S3. Finally we conclude by noticing that the effective temperature \( T_{eff} \) also controls the noise properties of the nonequilibrium field theory we have introduced in the main text, see Eq.(6), since the diffusion coefficient \( D \sim G^K_{loc}(\Omega_*) \sim T_{eff} \).

C. Dependence of Critical Frequency \( \Omega_* \) from microscopic parameters

In the main text we have shown the emergence of a finite frequency \( \Omega_* \) at which the normal phase susceptibility diverges and the system becomes unstable toward the formation of a non-stationary order parameter. As we emphasized, such a frequency is not externally imposed by the driving scheme but rather emerges as a result of interactions
and non-equilibrium effects. It is therefore interesting to ask how it depends from the microscopic parameters, in particular from the details of the drive. The existence of such a frequency is a property already encoded in the solution of the single-site driven dissipative problem, in particular in its vanishing spectral function, which then translates into a genuine phase transition at the full lattice level. To study \( \Omega_\ast \) is therefore sufficient to look at the dependence of the zero of the single-site spectral function from the external parameters, i.e. defining \( \Omega_\ast \) such that \( \text{Im} G_{\text{loc}}^R(\Omega_\ast) = 0 \). In figure S3 we plot this quantity as a function of the drive bandwidth \( \sigma \) (left panel) and drive amplitude \( f \) (right panel).

![Graph](image)

**FIG. S3.** Dependence of the Critical Frequency \( \Omega_\ast \) at which the system develops an instability from the external parameters, in particular drive bandwidth \( \sigma \) and drive amplitude \( f \). Parameters: loss rate \( \kappa = 0.0128U \), resonator frequency \( \delta \omega_0 = 0.0 \), drive amplitude (left panel) \( f = 0.0625U \), drive bandwidth (right panel) \( \sigma = 0.625U \).

![Graph](image)

**FIG. S4.** Stationary State Phase diagram as a function of hopping strength \( J \) and drive bandwidth \( \sigma \), for the two driving protocols considered namely cold drive (left panel) and hot drive (right panel). Parameters: loss rate \( \kappa = 0.0128U \), resonator frequency \( \delta \omega_0 = 0.0 \), drive amplitude (left panel) \( f = 0.0625U \).

We see that \( \Omega_\ast \) depends continuously from the amplitude, growing linearly and then saturating for a large value of \( f \). Instead a staircase structure emerges in the dependence of the critical frequency from the bandwidth \( \sigma \), with clear steps whose size and height is controlled by the interaction \( U \), a behavior that is ultimately due to the box-shaped nature of the drive spectrum.

**D. Phase Diagram of Finite Frequency Criticality**

Finally we conclude presenting the stationary phase diagram of the finite-frequency phase transition for the two driving protocols we have discussed so far. In the main text we considered a specific value of the drive bandwidth \( \sigma \) (\( \sigma = 1.5U \)) while now we present the phase boundary in the \( \sigma, J \) plane. We notice that upon increasing \( \sigma \) the local occupation of the bosonic mode increases (see figure S1) so in a certain sense this can be interpreted as a parameter...
playing the role of effective chemical potential. For both driving protocols we generically find a similar behavior, namely a small hopping phase $J < J_c(\sigma)$ which has a stable stationary state fully incoherent and a large hopping regime $J > J_c(\sigma)$ where the stationary state becomes unstable toward an oscillating regime and the system develops a $U(1)$ order parameter at finite frequency. It is nevertheless quite interesting to discuss the different shape of the phase boundary, which instead rather strongly depends on the protocol. We notice that for the cold drive case (left panel) the boundary resembles the ground state one, with a lobe-like structure for different values of the local density and a critical hopping $J_c$ which decreases as the local filling increases. Viceversa, in the hot drive case we find a rather opposite effect, namely the critical hopping increases with $\sigma$ and the region of normal phase stability expands. We can understand this effect from the discussion on the occupation of the bosonic mode and the effective temperature: indeed in the hot drive case, increasing the bandwidth $\sigma$ (i.e. moving along the vertical axis in Figure S4) has the effect of both changing the local occupation (see Figure S1) and of increasing the effective temperature (see Figure S2), with the result of shrinking the broken symmetry region due to effectively increased thermal fluctuations.

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