Large Time Behavior on the Linear Self-Interacting Diffusion Driven by Sub-Fractional Brownian Motion With Hurst Index Large Than 0.5 I: Self-Repelling Case

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Let $S^H$ be a sub-fractional Brownian motion with index $\frac{1}{2} < H < 1$. In this paper, we consider the linear self-interacting diffusion driven by $S^H$, which is the solution to the equation

$$dX^H_t = dS^H_t - \theta \left( \int_0^t (X^H_t - X^H_s) ds \right) dt + \nu dt, \quad X^H_0 = 0,$$

where $\theta < 0$ and $\nu \in \mathbb{R}$ are two parameters. Such process $X^H$ is called self-repelling and it is an analogue of the linear self-attracting diffusion [Cranston and Le Jan, Math. Ann. 303 (1995), 87–93]. Our main aim is to study the large time behaviors. We show the solution $X^H$ diverges to infinity, as $t$ tends to infinity, and obtain the speed at which the process $X^H$ diverges to infinity as $t$ tends to infinity.

Keywords: the self-repelling diffusion, asymptotic distribution, convergence, sub-fractional Brownian motion, stochastic integral

1 INTRODUCTION

In 1995, Cranston and Le Jan [1] introduced a linear self-attracting diffusion

$$X_t = B_t - \theta \int_0^t \int_0^s (X_s - X_u) du ds + \nu t, \quad t \geq 0 \tag{1.1}$$

with $\theta > 0$ and $X_0 = 0$, where $B$ is a 1-dimensional standard Brownian motion. They showed that the process $X_t$ converges in $L^2$ and almost surely, as $t$ tends infinity. This is a special case of path dependent stochastic differential equations. Such path dependent stochastic differential equation was first developed by Durrett and Rogers [2] introduced in 1992 as a model for the shape of a growing polymer (Brownian polymer) as follows

$$X_t = X_0 + B_t + \int_0^t \int_0^s f(X_u - X_{u'}) du ds, \tag{1.2}$$

where $B$ is a $d$-dimensional standard Brownian motion and $f$ is Lipschitz continuous. $X_t$ corresponds to the location of the end of the polymer at time $t$. Under some conditions, they established asymptotic behavior of the solution of stochastic differential equation and gave some conjectures and questions. The model is a continuous analogue of the notion of edge (resp. vertex) self-interacting random walk. If $f(x) = g(x)/\|x\|$ and $g(x) \geq 0$, $X_t$ is a continuous analogue of a process introduced by
Diaconis and studied by Pemantle [3]. Let $\mathcal{X}(t, x)$ be the local time of the solution process $X$. Then, we have

$$X_t = X_0 + B_t + \int_0^t ds \int \mathbb{R} f(-x) \mathcal{X}(s, X_s + x) dx$$

for all $t \geq 0$. This formulation makes it clear how the process $X$ interacts with its own occupation density. We may call this solution a Brownian motion interacting with its own passed trajectory, i.e., a *self-interacting motion*. In general, the Eq. 1.2 defines a self-interacting diffusion without any assumption on $f$. If

$$x \cdot f(x) \geq 0 \quad (x \cdot f(x) \leq 0)$$

for all $x \in \mathbb{R}^d$, we call it self-repelling (resp. self-attracting). In 2002, Benaim et al [4] also introduced a self-interacting diffusion with dependence on the (convoluted) empirical measure. A great difference between these diffusions and Brownian polymers is that the drift term is divided by $t$. It is noteworthy that the interaction potential is attractive enough to compare the diffusion (a bit modified) to an Ornstein-Uhlenbeck process, in many case of $f$, which points out an access to its asymptotic behavior. More works can be found in Benaim et al. [5], Cranston and Mountford [6], Gauthier [7], Herrmann and Roynette [8], Herrmann and Scheutzow [9], Mountford and Tarr [10], Shen et al [11], Sun and Yan [12] and the references therein.

On the other hand, starting from the application of fractional Brownian motion in polymer modeling, Yan et al [13] considered an analogue of the linear self-interacting diffusion:

$$X^H_t = B^H_t - \theta \int_0^t \int_0^s (X^H_{t-u} - X^H_u) duds + \nu, \quad t \geq 0 \quad (1.3)$$

with $\theta > 0$ and $X^H_0 = 0$, where $B^H$ is a fractional Brownian motion (fBm, in short) with Hurst parameter $\frac{1}{2} \leq H < 1$. The solution of (1.3) is a Gaussian process. When $\theta > 0$, Yan et al [13] showed that the solution $X^H$ of (1.3) converges in $L^2$ and almost surely, to the random variable

$$X^H_\infty = \int_0^\infty h_0(s) dB^H_s + \nu \int_0^\infty h_0(s) ds$$

where the function is defined as follows

$$h_0(s) = 1 - \theta se^{\theta s^2} \int_s^\infty e^{-\theta u^2} du, \quad s \geq 0$$

with $\theta > 0$. Recently, Sun and Yan [14] considered the related parameter estimations with $\theta > 0$ and $\frac{1}{2} \leq H < 1$, and Gan and Yan [15] considered the parameter estimations with $\theta < 0$ and $\frac{1}{2} \leq H < 1$.

Motivated by these results, as a natural extension one can consider the following stochastic differential equation:

$$X_t = G_t - \theta \int_0^t \int_0^s (X_s - X_u) duds + \nu, \quad t \geq 0 \quad (1.4)$$

with $\theta > 0$ and $X_0 = 0$, where $G = \{G_t, t \geq 0\}$ is a Gaussian process with some suitable conditions which includes fractional Brownian motion and some related processes. However, for a (general) abstract Gaussian process it is difficult to find some interesting fine estimates associated with the calculations. So, in this paper we consider the linear self-attracting diffusion driven by a sub-fractional Brownian motion (sub-fBm, in short). We choose this kind of Gaussian process because it is only the generalization of Brownian motion rather than the generalization of fractional Brownian motion. It only has some similar properties of fractional Brownian motion, such as long memory and self similarity, but it has no stationary increment. The so-called sub-fBm with index $H \in (0, 1)$ is a mean zero Gaussian process $S^H = \{S^H_t, t \geq 0\}$ with $S^H_0 = 0$ and the covariance

$$R_H(t, s) \equiv \mathbb{E}[S^H_t S^H_s] = s^{2H} + t^{2H} - \frac{1}{2} [(s + t)^{2H} + |t - s|^{2H}]$$

when $\theta > 0$. For $H = 1/2$, $S^H$ coincides with the standard Brownian motion $B$. $S^H$ is neither a semimartingale nor a Markov process unless $H = 1/2$, so many of the powerful techniques from stochastic analysis are not available when dealing with $S^H$. As a Gaussian process, it is possible to construct a stochastic calculus of variations with respect to $S^H$ (see, for example, Alós et al [16]). The sub-fBm has properties analogous to those of fBm and satisfies the following estimates:

$$[(2 - 2H^{-1}) \wedge 1](t - s)^{2H} \leq \mathbb{E}[ (S^H_t - S^H_s)^2 ] \leq [(2 - 2H^{-1}) \vee 1](t - s)^{2H} \quad (1.5)$$

More works for sub-fBm and related processes can be found in Bojdecki et al. [17–20], Li [21–24], Shen and Yan [25, 26], Sun and Yan [27], Tudor [28–31], Ciprian A. Tudor [32] Yan et al [33–35] and the references therein.

In this present paper, we consider the linear self-interacting diffusion

$$X^H_t = \xi^H_\infty - \theta \int_0^t \int_0^s (X^{H}_{t-u} - X^{H}_u) duds + \nu, \quad t \geq 0 \quad (1.7)$$

with $\theta < 0$ and $X^{H}_0 = 0$, where $\xi^H$ is a sub-fBm with Hurst parameter $\frac{1}{2} \leq H < 1$. Our main aim is to show that the solution of (1.7) diverges to infinity and obtain the speed diverging to infinity, as $t$ tends to infinity. The object of this paper is to expound and prove the following statements:

(I) For $\theta < 0$ and $\frac{1}{2} < H < 1$, the random variable

$$\xi^H_\infty = \int_0^\infty se^{\theta s^2} dS^H_s$$

exists as an element in $L^2$.

(II) For $\theta < 0$ and $\frac{1}{2} < H < 1$, as $t \to \infty$, we have

$$J^H_t(t, \theta, \nu) := te^{\theta s^2} X^H_s \to \xi^H_\infty - \frac{\nu}{\theta}$$

in $L^2$ and almost surely.

(III) For $\theta < 0$ and $\frac{1}{2} < H < 1$, define the times $f^H(n, \theta, \nu) = \{J^H_t(n, \theta, \nu), t \geq 0\}, n \geq 1$ by
Throughout this paper we assume that \( [16] \), Nualart \([36]\), and Tudor \([31]\) for a complete description of stochastic analysis are not available when dealing with \( \text{sub-fBm} \) unless \( \text{symmetrization with normality} \) implies that the sub-fBm \( \text{appears in} \text{Bojdecki} \text{et al} \text{[18]} \) in a high-density limit of occupation time of a system of independent particles moving in \( \text{with} \text{normality} \) and for all \( \theta \). We then have
\[
\frac{1}{t} \mathbb{E}
\begin{align*}
\int_{t}^{s} f(t, s) dS_{t}^{H} &= \int_{t}^{s} f(t, s) dS_{t}^{H} + \int_{t}^{s} S_{t}^{H} dt, \\
\end{align*}
\]
for all \( t \geq 0 \), provided \( u \) is of bounded \( q_{H} \)-variation on any finite interval with \( q_{H} > 1 \) and \( \frac{1}{p_{u}} \geq \frac{1}{q_{H}} > 1 \) (see, for examples, Bertoin \([37]\) and Föllmer \([38]\)).

Let \( \mathcal{H} \) be the completion of the linear space \( \mathcal{E} \) generated by the indicator functions \( 1_{[0, t]} \), \( t \in [0, T] \) with respect to the inner product
\[
\langle 1_{[0, t]}, 1_{[0, s]} \rangle_{\mathcal{H}} = R_{t}^{H}(s, t),
\]
for \( s, t \in [0, T] \). When \( \frac{1}{2} < H < 1 \), we can show that
\[
\| \phi \|_{\mathcal{H}}^2 = \int_{0}^{T} \int_{0}^{T} |\phi(t)\phi(s)R_{t}^{H}(t, s)|dtds
\]
for \( t, s \in [0, T] \). Define the linear mapping \( \mathcal{E} \ni \phi \mapsto S_{t}^{H}(\phi) \) by
\[
1_{[0, t]} \mapsto S_{t}^{H}(1_{[0, t]}) = \int_{0}^{T} 1_{[0, t]}(s) dS_{t}^{H} = S_{t}^{H},
\]
and it can be continuously extended to \( \mathcal{H} \) and we call the mapping \( \Phi \) is called the Wiener integral with respect to \( S_{t}^{H} \), denoted by
\[
S_{t}^{H}(\phi) = \int_{0}^{T} \phi(s) dS_{t}^{H}
\]
for any \( \phi \in \mathcal{H} \).

For simplicity, in this paper we assume that \( \frac{1}{2} < H < 1 \). Thus, if for every \( T > 0 \), the integral
\[
\int_{0}^{T} \phi(s) dS_{t}^{H}
\]
exists in \( L^{2} \) and
\[
\int_{0}^{\infty} \int_{0}^{\infty} \phi(t)\phi(s)\psi_{H}(t, s)dtds < \infty,
\]
we can define the integral
\[
\int_{0}^{\infty} \phi(s) dS_{t}^{H}
\]
for any \( \psi_{H} \in \mathcal{H} \).

Denote by \( \mathcal{S} \) the set of smooth functionals of the form
\[
F = f(S_{t}^{H}(\phi_{1}), S_{t}^{H}(\phi_{2}), \ldots, S_{t}^{H}(\phi_{n})),
\]
where \( f \in C_{b}^{\infty}(\mathbb{R}^{n}) \) and \( \phi_{i} \in \mathcal{H} \). The Malliavin derivative \( D \) of a functional \( F \) as above is given by
\[
D F = \sum_{i=1}^{n} f_{\phi_{i}}(S_{t}^{H}(\phi_{1}), S_{t}^{H}(\phi_{2}), \ldots, S_{t}^{H}(\phi_{n})) (D_{\phi_{i}} S_{t}^{H}(\phi_{1}), D_{\phi_{i}} S_{t}^{H}(\phi_{2}), \ldots, D_{\phi_{i}} S_{t}^{H}(\phi_{n})),
\]

\[
\begin{align*}
\int_{n}^{H} f_{n}(t, \theta, y) := \frac{\partial^{2}}{\partial \theta^{2}} \left( \int_{n}^{H} f_{n}(t, \theta, y) - (2n - 3)! \left( \frac{x_{\theta}}{y} \right) \right), \\
\end{align*}
\]
for all \( t \geq 0 \), \( \theta \in (0, H) \), \( y \in \mathbb{R} \), and \( n = 1, 2, \ldots \).
FIGURE 1 | A path of $X^H$ with $\theta = -1$ and $H = 0.7$.

FIGURE 2 | A path of $X^H$ with $\theta = -10$ and $H = 0.7$.

FIGURE 3 | A path of $X^H$ with $\theta = -100$ and $H = 0.7$.

FIGURE 4 | A path of $X^H$ with $\theta = -1$ and $H = 0.5$.

FIGURE 5 | A path of $X^H$ with $\theta = -10$ and $H = 0.5$.

FIGURE 6 | A path of $X^H$ with $\theta = -100$ and $H = 0.5$. 
The derivative operator $D$ is then a closable operator from $L^2(\Omega)$ into $L^2(\Omega; H)$. We denote by $\mathbb{D}^{1,2}$ the closure of $S$ with respect to the norm

$$
\|F\|_{1,2} := \sqrt{\mathbb{E}[|F|^2] + \mathbb{E}[DF]^2_{H}}.
$$

The divergence integral $d\delta$ is the adjoint of derivative operator $D^\dagger$. That is, we say that a random variable $u$ in $L^2(\Omega; H)$ belongs to the domain of the divergence operator $\delta$, denoted by $\text{Dom}(\delta)$, if

$$
\mathbb{E}[\langle DF, u \rangle_H] \leq c \mathbb{E}[\|F\|_{1,2}^2],
$$

for every $F \in \mathbb{D}^{1,2}$, where $c$ is a constant depending only on $u$. In this case $\delta(u)$ is defined by the duality relationship

$$
\mathbb{E}[F\delta(u)] = E[\langle DF, u \rangle_H].
$$ (2.3)
for an adapted process \( u \), and it is called Skorohod integral. Alós et al [16], we can obtain the relationship between the Skorohod and Young integral as follows

\[
\int_0^T u_s dS^H_s = \int_0^T u_s \delta S^H_s + \int_0^T \int_0^T D_u u_s \eta(t, s) ds dt,
\]

provided \( u \) has a bounded \( q \)-variation with \( 1 \leq q < \frac{1}{H} \) and \( u \in \mathbb{D}^{1,2}(\mathcal{H}) \) such that

\[
\int_0^T \int_0^T D_u u_s \eta(t, s) ds dt < \infty.
\]

**Theorem 2.1.** (Alós et al [16]). Let \( 0 < H < 1 \) and let \( f \in C^2(\mathbb{R}) \) such that

\[
\max(|f(x)|, |f'(x)|, |f''(x)|) \leq e^{|x|^2},
\]

where \( \kappa \) and \( \beta \) are two positive constants with \( \beta < \frac{1}{4} - 2H \). Then we have

\[
f(S^H_t) = f(0) + \int_0^t f(S^H_s) dS^H_s + H (2 - 2^{2H-1}) \int_0^t f(S^H_s) s^{2H-1} ds
\]

for all \( t \in [0, T] \).

### 3 SOME BASIC ESTIMATES

Throughout this paper we assume that \( 0 < \theta < \frac{1}{2} < H < 1 \). Recall that the linear self-interacting diffusion with sub-Bm \( S^H_t \) defined by the stochastic differential equation

\[
X^H_t = S^H_t - \frac{\theta}{2} \int_0^t S^H_s dS^H_s + \nu_t, \quad t \geq 0
\]

with \( \theta < 0 \). Define the kernel \((t, s)\rightarrow h_\theta(t, s)\) as follows

\[
h_\theta(t, s) = \begin{cases} 1 - \theta e^{\frac{1}{2}h_\theta}, & t \geq s, \\ \tfrac{1}{2} \theta^2 e^{-\frac{1}{2}h_\theta}, & 0 < t < s. \end{cases}
\]

for \( s, t \geq 0 \). By the variation of constants method (see, Cranston and Le Jan [1]) or Itô’s formula we may introduce the following representation:

\[
X^H_t = \int_0^t h_\theta(t, s) ds^H_s + \nu_0 + \int_0^t h_\theta(t, s) ds
\]

for \( t \geq 0 \).

The kernel function \((t, s)\rightarrow h_\theta(t, s)\) with \( \theta < 0 \) admits the following properties (these properties are proved partly in Sun and Yan [12]):

- For all \( s \geq 0 \), the limit

\[
\lim_{\theta \to 0} e^{\frac{1}{2}h_\theta}(t, s) = \frac{se^{2\theta^2}}{2^{2H-1}}
\]

for all \( s \geq 0 \).
• For all \( t \geq s \geq 0 \), we have
\[
1 \leq h_0(t, s) \leq e^{-\frac{1}{t-s}((t-s)^2)}.
\]

• For all \( t \geq s, r \geq 0 \), we have
\[
h_0(t,0) = h_0(t,t) = 1,
\]
\[
\int_0^t h_0(t,u)du = e^{\frac{1}{2}t^2} \int_0^t e^{\frac{1}{2}u^2}du.
\]

Lemma 3.1. Let \( \theta < 0 \) and define function
\[
I_\theta(t) = -\theta e^{\frac{1}{2}t^2} \int_0^t e^{\frac{1}{2}u^2}du - 1.
\]
We then have \( \lim_{t \to \infty} t^2 I_\theta(t) = \frac{1}{2} \) and
\[
\lim_{t \to \infty} t^2 \left( 1 + \theta e^{\frac{1}{2}t^2} \int_0^\infty e^{\frac{1}{2}u^2}du \right) = -\frac{1}{\theta}
\]

Proof. This is simple calculus exercise.

Lemma 3.2. (Sun and Yan [12]). Let \( \theta < 0 \) and define the functions \( t \to I_\theta(t,n), n = 1, 2, \ldots \) as follows
\[
I_\theta(t,1) = -\theta t^2 I_\theta(t), \quad I_\theta(t,n+1) = -\theta t^2 [I_\theta(t,n) - (2n-1)!!].
\]

Then we have
\[
\lim_{t \to \infty} I_\theta(t,n) = (2n-1)!!.
\] (3.5)
for every \( n \geq 0 \), where \((-1)!! = 1\).

Lemma 3.3. Let \( \theta < 0 \). Then the integral
\[
\Delta(H) = \int_0^\infty \int_0^\infty \frac{xy e^{\theta(x^2+y^2)}}{H(x,y)}dxdy
\] (3.6)
converges and as \( t \to \infty \),
\[
\lim_{t \to \infty} t^2 e^{\theta t^2} E(X_t^H)^2 = \Delta(H).
\]

Proof. An elementary may show that (3.6) converges for all \( \theta < 0 \).
It follows from L'Hôpital's rule that
\[
\lim_{t \to \infty} t^2 e^{\theta t^2} E(X_t^H)^2 = \lim_{t \to \infty} t^2 e^{\theta t^2} \int_0^t \int_0^t h_0(t,x)h_0(t,y)\psi_H(x,y)dxdy
\]
\[
= \lim_{t \to \infty} \frac{\theta^2}{t^2} \int_0^t dx \int_0^t \frac{xy e^{\theta(x^2+y^2)}}{H(x,y)}dxdy \int_0^t du \int_0^t e^{\frac{1}{2}u^2}du \int_0^t e^{\frac{1}{2}u^2}du
\]
\[
= 2 \lim_{t \to \infty} \frac{\theta^2}{t^2} \int_0^t dx \int_0^t \frac{xy e^{\theta(x^2+y^2)}}{H(x,y)}dxdy \int_0^t du \int_0^t e^{\frac{1}{2}u^2}du \int_0^t e^{\frac{1}{2}u^2}du
\]
\[
= \lim_{t \to \infty} \frac{-\theta_1}{t^2} \int_0^t dx \int_0^t e^{\frac{1}{2}u^2}du \int_0^t \frac{xy e^{\theta(x^2+y^2)}}{H(x,y)}dxdy
\]
\[
= \lim_{t \to \infty} \frac{-\theta_1}{t^2} \int_0^t dx \int_0^t e^{\frac{1}{2}u^2}du \int_0^t \frac{xy e^{\theta(x^2+y^2)}}{H(x,y)}dxdy
\]
\[
= \lim_{t \to \infty} \frac{-\theta_1}{t^2} \int_0^t dx \int_0^t e^{\frac{1}{2}u^2}du \int_0^t \frac{xy e^{\theta(x^2+y^2)}}{H(x,y)}dxdy
\]
\[
= \int_0^\infty dx \int_0^\infty \frac{xy e^{\theta(x^2+y^2)}}{H(x,y)}dxdy.
\]
where we have used the following fact:
\[
\lim_{t \to \infty} \frac{1}{t^2} \int_0^t e^{\frac{1}{2}u^2}du \int_0^t e^{\frac{1}{2}y^2}dy = \lim_{t \to \infty} \frac{1}{t^2} \int_0^t e^{\frac{1}{2}u^2}du \int_0^t e^{\frac{1}{2}y^2}dy
\]
\[
= \int_0^\infty xy e^{\theta(x^2+y^2)}\psi_H(x,y)dy + \int_0^\infty xy e^{\theta(x^2+y^2)}\psi_H(x,y)dy = 0.
\]

This completes the proof.

Lemma 3.4. Let \( \theta < 0 \). Then, convergence
\[
\lim_{t \to \infty} \frac{1}{t^2} e^{\frac{1}{2}t^2} \int_0^\infty \int_0^\infty xe^{\theta(x^2+y^2)}\psi_H(x,y)dsdr
\]
\[
= \frac{1}{4} (-\theta)^{-\frac{1}{2}} \Gamma (2H+1).
\] (3.7)
holds.

Proof. It follows from L'Hôpital's rule that
\[
\lim_{t \to \infty} \frac{1}{t^2} e^{\frac{1}{2}t^2} \int_0^\infty \int_0^\infty xe^{\theta(x^2+y^2)}\psi_H(x,y)dsdr
\]
\[
= \frac{1}{t^2} \lim_{t \to \infty} \int_0^\infty \int_0^\infty xe^{\theta(x^2+y^2)}\psi_H(x,y)dsdr
\]
\[
= \frac{1}{2} (-\theta)^{-\frac{1}{2}} \Gamma (2H+1)
\]
for all \( \theta < 0 \) and \( \frac{1}{2} < H < 1 \). By making the change of variable \( \frac{1}{2} t^2 x^2 = x \), we see that
\[
\lim_{t \to \infty} \frac{1}{t^2} e^{\frac{1}{2}t^2} \int_0^\infty \int_0^\infty xe^{\theta(x^2+y^2)}\psi_H(x,y)dsdr
\]
\[
= \frac{1}{t^2} \lim_{t \to \infty} \int_0^\infty \int_0^\infty xe^{\theta(x^2+y^2)}\psi_H(x,y)dsdr
\]
\[
= \frac{1}{2} (-\theta)^{-\frac{1}{2}} \Gamma (2H+1)
\]
for all \( \theta < 0 \) and \( \frac{1}{2} < H < 1 \). This completes the proof.

Lemma 3.5. Let \( \theta < 0 \) and \( 0 \leq s < t \leq T \). We then have
\[
c(t-s)^{2H} \leq E \left[ (X_t^H - X_s^H)^2 \right] \leq C (t-s)^{2H}
\] (3.8)
Proof. Given \( 0 \leq s < t \leq T \) and denote
\[
\hat{X}_s^H = \int_0^t h_0(t,r)ds^H, \quad t \geq 0.
\]
It follows that
\[
E \left( \left( \int_t^H \theta \left( [h_0(t, r) - h_0(s, r)] dr \right) \right)^2 \right) = \theta \int_t^H e^{\frac{1}{2} \theta^2} dr \int_0^t e^{\frac{1}{2} \theta^2} dr \leq C_{H,T} \theta (t - s) \]
and
\[
\int_0^t [h_0(t, r) - h_0(s, r)] dr = \theta \int_t^H e^{\frac{1}{2} \theta^2} dr \int_0^t e^{\frac{1}{2} \theta^2} dr \leq C_{H,T} \theta (t - s) \]
for all \( \theta < 0 \) and \( 0 < s < t \leq T \). It follows that
\[
\left( \int_0^t [h_0(t, r) dr - \int_0^t h_0(s, r) dr] \right)^2 = \left( \int_0^t [h_0(t, r) - h_0(s, r)] dr \right)^2 + 2 \int_0^t h_0(t, r) dr \int_0^t [h_0(t, r) - h_0(s, r)] dr \leq C_{H,T} \theta (t - s)^2 \]
for all \( \theta < 0 \) and \( 0 < s < t \leq T \), which implies that
\[
E \left( \left( \int_t^H \theta \left( [h_0(t, r) - h_0(s, r)] dr \right) \right)^2 \right) \leq C_{H,T} \theta (t - s)^{2H} \]
for all \( \theta < 0 \) and \( 0 < s < t \leq T \). Noting that the above calculations are invertible for all \( \theta < 0 \) and \( 0 < s < t \leq T \), one can obtain the left hand side in (3.8) and the lemma follows.

4 CONVERGENCE

In this section, we obtain the large time behaviors associated with the solution \( X^H \) to Eq. 3.1. From Lemma 3.5 and Guassianess, we find that the self-repelling diffusion \( \{X^H_t, t \geq 0\} \) is H-\( \text{Hölder} \) continuous. So, the integral
\[
\int_0^t \theta X^H_t ds \leq C_{H,T} \theta (t - s)^{2H} \]
exists with \( t \geq 0 \) as a Young integral and
\[
t X^H_t = \int_0^t \theta X^H_s ds + \int_0^t X^H_s ds \]
for all \( t \geq 0 \). Define the process \( Y = \{Y_t, t \geq 0\} \) by
\[
Y_t = \int_0^t (X^H_t - X^H_s) ds = t X^H_t - \int_0^t X^H_s ds = \int_0^t \theta X^H_s ds + \frac{1}{2} \nu Y_t. \]
By the variation of constants method, one can prove
\[
Y_t = e^{\frac{1}{2} \theta^2} \int_0^t y e^{\frac{1}{2} \theta^2} ds \leq \frac{1}{\theta} (e^{\frac{1}{2} \theta^2} - 1) \]
for all \( t \geq 0 \). Define Gaussian process \( \xi^H = \{\xi^H_t, t \geq 0\} \) as follows
\[
\xi^H_t := \int_0^t \theta X^H_s ds, t \geq 0. \]
Lemma 4.1. Let $\theta < 0$ and $\frac{1}{2} < H < 1$. Then, the random variable
\[ \xi_{H}^{\infty} := \int_{0}^{\infty} xe^{\theta x} dS_{t}^{H} \]
exists as an element in $L^{2}$. Moreover, $\xi^{H}$ is $H$-Hölder continuous and $\xi_{t}^{H} \rightarrow \xi_{\infty}^{H}$ in $L^{2}$ and almost surely, as $t$ tends to infinity.

Proof. This is simple calculus exercise. In fact, we have
\[
E \left( \int_{0}^{\infty} xe^{\theta x} dS_{t}^{H} \right)^{2} = \int_{0}^{\infty} \int_{0}^{\infty} xe^{\theta (x+y)} \psi_{H}(x, y) dxdy \\
= 2 \int_{0}^{\infty} xe^{\theta x} dx \int_{0}^{\infty} ye^{\theta y} \psi_{H}(x, y) dy \\
= 2H(2H-1) \int_{0}^{\infty} xe^{\theta x} dx \\
\int_{0}^{\infty} xe^{\theta y} ((x-y)^{2H-2} - (x+y)^{2H-2}) dy \\
\leq 2H(2H-1) \int_{0}^{\infty} xe^{\theta x} dx \\
\int_{0}^{\infty} ((x-y)^{2H-2} - (x+y)^{2H-2}) y dy \\
= 2H(2H-1) \int_{0}^{\infty} xe^{\theta x} dS_{x} \\
= C_{0,H}(2H+2),
\]
for all $\theta < 0$ and $\frac{1}{2} < H < 1$, which shows that the random variable $\xi_{\infty}^{H}$ exists as an element in $L^{2}$.

Now, we show that the process $\xi^{a,b}$ is Hölder continuous. For all $0 < s < t$ by the inequality $e^{-x^{2}} x \leq C$ for all $x \geq 0$, we have
\[
E \left( \xi_{t}^{H} - \xi_{s}^{H} \right)^{2} = \int_{s}^{t} \int_{s}^{t} xe^{\theta (x+y)} \psi_{H}(x, y) dxdy \\
= 2 \int_{s}^{t} xe^{\theta x} dx \int_{s}^{t} ye^{\theta y} \psi_{H}(x, y) dy \\
= 2H(2H-1) \int_{s}^{t} xe^{\theta x} dx \int_{s}^{t} ye^{\theta y} ((x-y)^{2H-2} - (x+y)^{2H-2}) dy \\
\leq 2HC_{0}(2H-1) \int_{s}^{t} dx \int_{s}^{t} ((x-y)^{2H-2} - (x+y)^{2H-2}) dy \\
= C_{0,H}(t-s)^{2H}.
\]
Thus, the normality of $\xi^{H}$ implies that
\[
E \left( \xi_{t}^{H} - \xi_{s}^{H} \right)^{2n} \leq C_{0,H,n}(t-s)^{2nH}
\]
for all $0 \leq s < t$, $\frac{1}{2} < H < 1$ and integer numbers $n \geq 1$, and the Hölder continuity follows.

Nextly, we check the $\xi_{t}^{a,b}$ converges to $\xi_{\infty}^{H}$ in $L^{2}$. This follows from the next estimate:
\[
E \left( \xi_{t}^{H} - \xi_{\infty}^{H} \right)^{2} = \int_{t}^{\infty} \int_{t}^{\infty} xe^{\theta (x+y)} \psi_{H}(x, y) dxdy \\
= 2 \int_{t}^{\infty} \int_{t}^{\infty} xe^{\theta (x+y)} \psi_{H}(x, y) dxdy \\
\leq 2e^{\frac{\theta t}{2}} \int_{t}^{\infty} xe^{\theta x} dx \int_{t}^{\infty} ye^{\theta y} \psi_{H}(x, y) dy \\
\leq 2H(2H-1)e^{\frac{\theta t}{2}} \int_{t}^{\infty} xe^{\theta x} dx \int_{t}^{\infty} ye^{\theta y} ((x-y)^{2H-2} - (x+y)^{2H-2}) dy \\
\leq 2H(2H-1)e^{\frac{\theta t}{2}} \int_{t}^{\infty} xe^{\theta x} dx \int_{t}^{\infty} y((x-y)^{2H-2} - (x+y)^{2H-2}) dy \\
= 2H(2H-1) \left( \int_{t}^{\infty} \left( \int_{0}^{1} u(1-u)^{2H-2} du \right) e^{\theta u} \right) e^{\frac{\theta t}{2}} \int_{t}^{\infty} x^{2H+1} e^{\theta x} dx \\
\leq 0, \quad (4.1)
\]
as $t$ tends to infinity.

Finally, we check the $\xi_{t}^{a,b}$ converges to $\xi_{\infty}^{H}$ almost surely. By integration by parts we see that
\[
\xi_{t}^{H} - \xi_{\infty}^{H} = \int_{0}^{\infty} xe^{\theta x} dS_{t}^{H} = -te^{\frac{\theta t}{2}} S_{t}^{H} - \int_{0}^{\infty} (1+\theta s^{2})e^{\frac{\theta s^{2}}{2}} S_{t}^{H} ds
\]
for all $t \geq 0$. Elementary may check that the convergence
\[
\eta_{t}^{H} := \int_{0}^{\infty} (1+\theta s^{2})e^{\frac{\theta s^{2}}{2}} S_{t}^{H} ds \rightarrow 0
\]
holds almost surely, as $t$ tends to infinity. In fact, by inequality
\[
\int_{s}^{\infty} x^{a} e^{\frac{\theta x^{2}}{2}} dx \leq C_{a-1} x^{a-1} e^{\frac{\theta x^{2}}{2}}, \quad a > -1,
\]
with $t \geq 0$, we may show that
\[
E \left( \sup_{n \in \mathbb{N}, t \geq 1} \left| \eta_{t}^{H} \right| \right)^{2} \leq \int_{0}^{\infty} \int_{0}^{\infty} (1+\theta s^{2}) (1+\theta r^{2}) e^{\frac{\theta (s^{2}+r^{2})}{2}} \left| S_{t}^{H} \right| dS_{t}^{H} drds
\]
\[
\leq C \int_{0}^{\infty} s^{2H+1} e^{\frac{\theta s^{2}}{2}} ds
\]
for all integer numbers $n \geq 1$, and hence
\[
\sum_{n=0}^{\infty} P\left(\sup_{n,t|t+n\theta|} |\eta_t^{H,\theta}|^2 \geq \varepsilon\right) \leq C \varepsilon^{-2} \sum_{n=0}^{\infty} n^{-2H} e^{\Theta \varepsilon^2} < \infty.
\]

Thus, Borel-Cantelli’s lemma implies that \(\eta_t^{H,\theta}\) converges to zero almost surely as \(t\) tends to infinity, and the lemma follows from (4.2).

**Corollary 4.1.** For all \(\gamma > 0\), we have
\[
t' \left( \psi_t^{H,\theta} - \psi_0^{H,\theta} \right) = t' \int_0^\infty r \text{sech}^2 r \, ds_h^H \rightarrow 0,
\]
in \(L^2\) and almost surely, as \(t\) tends to infinity.

**Lemma 4.2.** Let \(\theta < 0\) and \(\frac{1}{2} < H < 1\). Then, we have
\[
\Lambda_{\gamma}(t, \theta) := t' \int_0^t e^{\frac{1}{2} \bar{\Theta} r} \left( \xi^{ab}_{\infty} - \xi^{ab}_u \right) \, du \rightarrow 0
\]
in \(L^2\) and almost surely for every \(\gamma \geq 0\), as \(t\) tends to infinity.

**Proof.** Given \(0 < s \leq t\), \(\theta < 0\) and denote
\[
\Psi_0(s, t) := \int_{s}^{t} e^{-\frac{1}{2} \bar{\Theta} r} \, I_{r}^{\infty} r \text{sech}^2 r \, \psi_{H}(s, r) \, dr
\]
\[
= \int_{s}^{t} r \text{sech}^2 r \, \psi_{H}(s, r) \, dr + \int_{s}^{t} e^{-\frac{1}{2} \bar{\Theta} r} \, I_{r}^{\infty} r \text{sech}^2 r \, \psi_{H}(s, r) \, dr
\]
\[
\leq C \int_{s}^{t} r \psi_{H}(s, r) \, dr + C t e^{-\frac{1}{2} \bar{\Theta} t} \int_{s}^{t} r \text{sech}^2 r \, \psi_{H}(s, r) \, dr
\]
\[
\leq C \int_{s}^{t} r \psi_{H}(s, r) \, dr + (t - s)^{2H-2} t^{-1},
\]
where we have used the fact
\[
\int_{s}^{t} e^{-\frac{1}{2} \bar{\Theta} r} \, dr \leq C \frac{r}{x} e^{-\frac{1}{2} \bar{\Theta} x}, \quad \forall x \geq 0
\]
and estimates
\[
\int_{s}^{t} r \text{sech}^2 r \, \psi_{H}(s, r) \, dr = H(2H - 1) \int_{s}^{t} r (r - s)^{2H-2} e^{\Theta \varepsilon^2} \, dr
\]
\[
\leq H(2H - 1) \int_{s}^{t} r (r - s)^{2H-2} e^{\Theta \varepsilon^2} \, dr
\]
\[
\leq H(2H - 1) (t - s)^{2H-2} \int_{s}^{t} r \text{sech}^2 r \, dr
\]
\[
= \frac{H(2H - 1)}{-\theta} (t - s)^{2H-2} e^{\Theta \varepsilon^2}.
\]
It follows that
\[
E[|\Lambda_{\gamma}(t, \theta)|^2] = t'^2 \int_0^t e^{-\frac{1}{2} \bar{\Theta} r} \left( \xi^{ab}_{\infty} - \xi^{ab}_u \right) \, du
\]
\[
\leq t'^2 \int_0^t e^{-\frac{1}{2} \bar{\Theta} r} \left( \xi^{ab}_{\infty} - \xi^{ab}_u \right) \, du
\]
\[
\leq t'^2 \int_0^t \text{sech}^2 r \, \psi_{H}(s, r) \, dr
\]
\[
\leq t'^2 \int_0^t \text{sech}^2 r \, \psi_{H}(s, \theta) \, ds
\]
\[
\leq t'^2 \int_0^t \text{sech}^2 r \, \psi_{H}(s, \theta) \, ds
\]
\[
\leq t'^2 \int_0^t \text{sech}^2 r \, \psi_{H}(s, \theta) \, ds
\]
\[
\rightarrow 0 \quad (t \rightarrow \infty),
\]
which shows that \(\Lambda_{\gamma}(t, \theta)\) converges to zero in \(L^2\).

Now, we obtain the convergence with probability one. Noting that
\[
\psi_t^{H,\theta} - \psi_0^{H,\theta} = \int_{\infty}^{\psi_t^{H,\theta}} \text{sech}^2 r \, ds
\]
for all \(u \geq 0\), we get
\[
|\Lambda_{\gamma}(t, \theta)| \leq t'^2 \int_0^t e^{-\frac{1}{2} \bar{\Theta} r} \left( \xi^{ab}_{\infty} - \xi^{ab}_u \right) \, du
\]
\[
\leq t'^2 \int_0^t e^{-\frac{1}{2} \bar{\Theta} r} \left( \xi^{ab}_{\infty} - \xi^{ab}_u \right) \, du
\]
\[
\leq t'^2 \int_0^t \text{sech}^2 r \, \psi_{H}(s, \theta) \, ds
\]
\[
\leq t'^2 \int_0^t \text{sech}^2 r \, \psi_{H}(s, \theta) \, ds
\]
\[
\rightarrow 0 \quad (t \rightarrow \infty),
\]
almost surely for all \(\gamma \geq 0\), \(\theta < 0\) and \(\frac{1}{2} < H < 1\), as \(t\) tends to infinity. This completes the proof.
The objects of this paper are to prove the following theorems which give the long time behaviors for \(X_H^t\) with \(\frac{1}{2} < H < 1\).

**Theorem 4.1.** Let \(\theta < 0\) and \(\frac{1}{2} < H < 1\). Then, as \(t \to \infty\), the convergence
\[
f_H^t(t; \theta, \nu) := t e^{i\theta^2} X_t^H \to \xi_H^\infty - \frac{\nu}{\theta}
\]
holds in \(L^2\) and almost surely.

**Proof.** Given \(t > 0\) and \(\theta < 0\). Simple calculations may prove
\[
j_H^t(t; \theta, \nu) = t e^{i\theta^2} X_t^H
\]
\[
= t e^{i\theta^2} \int_0^t h_\theta(t, s) ds + \nu t e^{i\theta^2} \int_0^t \int_0^s e^{i\theta^2 u} d\xi_n(t, s) ds
\]
\[
= t e^{i\theta^2} S_t^H - \theta t e^{i\theta^2} \int_0^t s e^{i\theta^2 u} \left( \int_0^u e^{i\theta^2 v} d\xi_n(t, s) \right) ds
\]
\[
+ \nu t e^{i\theta^2} \int_0^t e^{-i\theta^2 u} ds
\]
\[
= t e^{i\theta^2} S_t^H - \theta t e^{i\theta^2} \int_0^t e^{-i\theta^2 u} \xi_n(t, s) ds
\]
\[
+ \nu t e^{i\theta^2} \int_0^t e^{-i\theta^2 u} ds.
\]
(4.3)

It follows from Lemma 4.1, Corollary 4.1, and Lemma 4.2 that
\[
j_H^t(t; \theta, \nu) - \left( \xi_H^\infty - \frac{\nu}{\theta} \right)
\]
\[
= t e^{i\theta^2} S_t^H - \theta t e^{i\theta^2} \int_0^t e^{-i\theta^2 u} \left( \xi_n(t, s) - \xi_H^\infty \right) ds
\]
\[
+ \left( \xi_H^\infty - \frac{\nu}{\theta} \right) \left( -\theta t e^{i\theta^2} \int_0^t e^{-i\theta^2 u} du - 1 \right) \to 0 \quad (t \to \infty)
\]
(4.4)
in \(L^2\) and almost surely for all \(\theta < 0\) and \(\frac{1}{2} < H < 1\), as \(t\) tends to infinity.

**Theorem 4.2.** Define the processes \(J_H^t(n, \theta, \nu) = \{J_H^t(n, \theta, \nu), t \geq 0\}, n \geq 1\) by
\[
J_H^t(n, \theta, \nu) := \theta t \left( J_H^t(n-1, \theta, \nu) - (2n - 3)! \left( \xi_H^\infty - \frac{\nu}{\theta} \right) \right)
\]
for all \(n \geq 1\), where \((-1)! = 1\). Then, the convergence
\[
J_H^t(n, \theta, \nu) \to (2n - 1)! \left( \xi_H^\infty - \frac{\nu}{\theta} \right)
\]
holds in \(L^2\) and almost surely for every \(n \geq 1\), as \(t \to \infty\).

**Proof.** From the proof of Theorem 4.1, we find that the identities
\[
j_0^H(t; \theta, \nu) - \left( \xi_H^\infty - \frac{\nu}{\theta} \right) = t e^{i\theta^2} S_t^H + \theta t e^{i\theta^2} \int_0^t e^{i\theta^2 u} \left( \xi_H^\infty - \xi_n(t, s) \right) ds
\]
\[
+ \left( \xi_H^\infty - \frac{\nu}{\theta} \right) \left( \theta t e^{i\theta^2} \int_0^t e^{-i\theta^2 u} du - 1 \right),
\]
\[
j_n^H(t; \theta, \nu) = \left( \xi_H^\infty - \frac{\nu}{\theta} \right) I_n(t, \theta) + t \left( \theta^2 \right)^n e^{-i\theta^2} S_t^H
\]
\[
+ \theta t \left( \theta^2 \right)^n e^{i\theta^2} \left( \xi_n(t, s) - \xi_H^\infty \right) ds.
\]
holds for all \(t > 0\), \(n \geq 1\) and \(\theta < 0\), where \(I_n(t, \theta)\) is given in Lemma 3.2. Thus, the theorem follows from Lemma 4.1, Corollary 4.1, Lemma 4.2 and Theorem 4.1.

**5 SIMULATION**

We have applied our results to the following linear self-repelling diffusion driven by a sub-fBm \(S_t^H\) with \(\frac{1}{2} < H < 1\):
\[
dX_t^H = dS_t^H - \theta \left( \int_0^t X_s^H - X_t^H ds \right) dt + \nu dt, \quad X_0^H = 0,
\]
where \(\theta < 0\) and \(\nu \in \mathbb{R}\) are two parameters. We will simulate the process with \(\nu = 0\) in the following cases:

- \(H = 0.7\) and \(\theta = -1\), \(\theta = -10\), and \(\theta = -100\), respectively (see, Figure 1, Figure 2, Figure 3, and Table 1, Table 2, Table 3);
- \(H = 0.5\) and \(\theta = -1\), \(\theta = -10\), and \(\theta = -100\), respectively (see, Figure 4, Figure 5, Figure 6, and Table 4, Table 5, Table 6);

**Remark 1.** From the following numerical results, we can find that it is important to study the estimates of parameters \(\theta\) and \(\nu\).

**DATA AVAILABILITY STATEMENT**

The original contributions presented in the study are included in the article/Supplementary Material, further inquiries can be directed to the corresponding author.

**AUTHOR CONTRIBUTIONS**

All authors listed have made a substantial, direct, and intellectual contribution to the work and approved it for publication.

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