Pseudo-spectral construction of non-uniform black string solutions in five and six spacetime dimensions

Michael Kalisch\(^1\) and Marcus Ansorg

Theoretisch-Physikalisches Institut, Friedrich-Schiller-Universität Jena, Max-Wien-Platz 1, D-07743 Jena, Germany

E-mail: michael.kalisch@uni-jena.de and marcus.ansorg@uni-jena.de

Received 12 July 2016, revised 5 September 2016
Accepted for publication 9 September 2016
Published 11 October 2016

Abstract
In this paper, we describe in detail a scheme for the construction of highly accurate numerical solutions to Einstein’s field equations in five and six spacetime dimensions, corresponding to non-uniform black strings. The scheme consists of a sophisticatedly adapted multi-domain pseudo-spectral method which incorporates a detailed understanding of the solution’s behavior at the domain boundaries and at critical points. In particular, the five-dimensional case is exceedingly demanding, as logarithmic terms appear which need to be treated with special care. Our scheme resolves these issues, and permits the investigation of unprecedentedly strong deformations of the black string horizon. As a consequence, we are able to study in detail the critical regime in phase diagrams displaying characteristic thermodynamic quantities such as mass and entropy. Our results show typical spiral curves in such diagrams which provide strong support for previous numerical works.

Keywords: pseudo-spectral methods, black holes, higher dimensions

(Some figures may appear in colour only in the online journal)

1. Introduction

The major motivation for the construction of stationary black hole solutions in higher spacetime dimensions \(D > 4\) originates from considerations of string theory and AdS/CFT correspondence. While analytic solutions are rare in this context, much effort has been expended to solve Einstein’s equations on the computer (see the recent review [1]).

\(^1\) Author to whom any correspondence should be addressed.
Naturally, numerical calculations are accompanied by errors, leading to only an approximation of a solution to the problem in question. Due to finite computational resources, errors cannot always be pushed below a reasonable value. In particular, when the underlying mathematical structure is more involved (e.g., if a strongly pronounced peak appears), then an accurate numerical solution becomes more challenging, and at some critical point the method may fail. Unfortunately, this typically happens when specifically interesting branch points or phase transitions are encountered. Consequently, in a situation where standard algorithms reach their limitations, it becomes necessary to design a method specifically adapted to the mathematical circumstances. With the help of a corresponding code developed along these lines, the critical regime can be explored, and hitherto unrevealed properties may become manifest.

A situation of this kind arises in the context of non-uniform black strings (henceforth ‘NBS’) in the limit of maximal deformation of the horizon. NBS emanate from the Gregory-Laflamme (henceforth ‘GL’) instability [2] of a uniform black string (henceforth ‘UBS’), which can be described as the product of a Schwarzschild-Tangherlini solution with a circle. This instability occurs when the mass of the string is sufficiently small compared to the size of the circle that it leads to a deformation of the string’s horizon along the compact dimension, see [3, 4] for reviews.

Static NBS solutions were first obtained approximatively with the help of perturbation theory in five spacetime dimensions \( D = 5 \). In the corresponding article [5] an appropriate measure of non-uniformity was introduced:

\[
\lambda = \frac{1}{2} \left( \frac{R_{\text{max}}}{R_{\text{min}}} - 1 \right)
\]

Here, \( R_{\text{max}} \) and \( R_{\text{min}} \) present maximal and minimal radii of the black string along the compact dimension. Naturally, UBS are characterised by \( R_{\text{max}} = R_{\text{min}} \), i.e. \( \lambda = 0 \), while for NBS solutions we have \( \lambda > 0 \).

The procedure described in [5] was later applied to higher dimensions [6, 7]. Going beyond the scope of perturbation theory, which considers small horizon deformations (\( \lambda \ll 1 \)), numerical constructions of NBS in different spacetime dimensions were performed in a number of works [6, 8–12]. Interestingly, the results suggest that along the NBS branch the ratio \( R_{\text{min}}/R_{\text{max}} \) shrinks gradually down to zero, i.e. NBS solutions exist for all \( \lambda \in (0, \infty) \). In the limit \( \lambda \to \infty \) a curvature singularity is encountered at which the horizon pinches off in precisely the region described above, where the numerics reaches its limitations. On the other hand, this limit is of particular interest as it describes, according to a conjecture presented in [13], a phase transition to another branch of solutions, known as localized or caged black hole solutions. This designation arises from the fact that the horizons of the corresponding objects do not extend to the entire compact dimension (see [14]). Numerical results provide evidence in favor of this phase transition [10, 15–18]. Moreover, the local geometry of the transit solution at the pinch-off point is conjectured to be that of a double-cone [13] (see [19] for further plausible arguments). Additional numerical evidence supporting this conjecture is presented in [9, 11, 20]. Very readable reviews regarding the black hole/black string phase transition can be found in [21–23].

In this paper we concentrate on the NBS branch. Motivated by contradictory results in previous works, we investigate in particular the critical regime of strong horizon deformations \( \lambda \gg 1 \). To be more precise, the discrepancy raises the question of whether, in five, and six spacetime dimensions, the mass in the NBS phase reaches a maximum for some finite \( \lambda \) [8, 9, 11]. Our results show agreement for both dimensions, with the findings reported in [8];
i.e., we detect a clearly pronounced maximum. Moreover, we are able to identify two further turning points in the mass curve if \( \lambda \) is further increased. Since other thermodynamic quantities show a similar behavior, we observe the beginning of a spiral curve in the black string phase diagram.

In the case \( D = 6 \), the aforementioned results have been outlined in [12]. Here we provide an elaborated description of the numerical algorithm and include the case \( D = 5 \), which requires a specific treatment (to be depicted in section 3.2) in order to deal with the logarithmic behavior of the solution.

Our numerical scheme is based on pseudo-spectral methods, and includes a number of sophisticated adaptations in order to resolve the critical regime satisfactorily. The techniques encompass several appropriate coordinate mappings (which provide, in particular, a compactification of infinity), the introduction of multiple domains, and the split of each metric function into two parts (near infinity). In addition, we need high resolution near the critical spatial point at which the pinch-off occurs for \( \lambda \to \infty \). We emphasize that the scheme is highly specialized to the construction of strongly deformed NBS. In principle, NBS solutions for \( D > 6 \) should be constructible in the same manner as described here (with some slight modifications). Some of the individual strategies, however, may also be applicable to other problems.

The paper is organized as follows. In section 2 we provide the form of the metric together with the system of equations to be solved in the sequel. Furthermore, for later convenience we quote relevant formulas and relations involving specific geometric and thermodynamic quantities. The numerical scheme is presented in section 3. In particular, differences in the construction of five and six-dimensional NBS are pointed out. The results are reported in section 4, including plots of characteristic geometric and thermodynamic variables. Also, we provide a discussion of the accuracies obtained. Finally, we conclude this article in section 5. More details on perturbation theory (see section A) and the numerical scheme (see section B) are provided in the appendix.

2. Physical setup

We consider the static NBS metric in \( D \) dimensions and with the background \( \mathbb{R}^{D-2,1} \times S^1 \) in the form

\[
\text{d}s^2 = -e^{2A(r,z)} \text{d}t^2 + e^{2B(r,z)} \left( \frac{\text{d}r^2}{f(r)} + \text{d}z^2 \right) + r^2 e^{2C(r,z)} \text{d}\Omega_{D-3}^2,
\]

where \( \text{d}\Omega_{D-3}^2 \) is the line element of a unit \((D-3)\)-sphere. The three unknown metric functions \( A, B, \) and \( C \) depend on the radial coordinate \( r \in [r_0, \infty) \) and the periodic coordinate \( z \in [0, L] \) varying along \( S^1 \). With

\[
f(r) = 1 - \left( \frac{r_0}{r} \right)^{D-4},
\]

the horizon resides at \( r = r_0 \).

It becomes apparent that \( A \equiv B \equiv C \equiv 0 \) corresponds to the UBS metric (which can be seen as a product of a Schwarzschild-Tangherlini solution in \( D - 1 \) dimensions with a circle of size \( L_1 \)). Holding \( r_0 \) fixed, the UBS is subject to the GL instability if \( L \) is larger than a critical value \( L_{GL} \), though it is stable if \( L < L_{GL} \). This instability breaks the translation invariance along the \( z \)-direction, thus leading to the NBS branch which can be described by non-vanishing potentials \( A, B, \) and \( C \).


2.1. Field equations

The NBS are described by specific vacuum solutions to Einstein’s field equations. Einstein tensor components $G^\mu_\nu$ satisfy $G^t_0 = 0$, $G^r_r + G^z_z = 0$, and $G^\theta_\theta = 0$ ($\theta$ is an angle of the $(D - 3)$-sphere), from which we get the following system of equations [8] (we use the notations $'$:= $\partial/\partial r$ and $\cdot$:= $\partial/\partial z$):

$$0 = A'' + \frac{\ddot{A}}{f} + A' + (D - 3) \left( \frac{A'C'}{f} + \frac{\dot{A}C'}{f} + \frac{A'}{r} + \frac{f'C'}{2f} \right) + \frac{3f'A'}{2f}, \quad (4a)$$

$$0 = B'' - (D - 3) \left( \frac{A'C'}{f} + \frac{\dot{A}C'}{f} + \frac{A'}{r} + \frac{f'C'}{2f} \right) - \frac{(D - 3)(D - 4)}{2r^2} \left( 1 - \frac{e^{2B - 2C}}{f} \right) + r^2C'' + 2rC' + \frac{r^2C^2}{f}, \quad (4b)$$

$$0 = C'' + \dot{C} + A'C' + \frac{\dot{A}C'}{f} + \frac{A'}{r} + \frac{f'C'}{f} + \frac{(D - 4)}{r^2} \left( 1 - \frac{e^{2B - 2C}}{f} \right) + (D - 3) \left( \frac{2C'}{r} + \frac{\dot{C}^2}{f} \right). \quad (4c)$$

Note that in our case five independent equations follow from Einstein’s field equations. The missing two elements, $G^r_0 = 0$ and $-G^z_z = 0$, can be regarded as constraints satisfied globally by solutions of (4), if appropriate boundary conditions are imposed as per Wiseman’s ‘constraint rule’ [6] (see [8] for the choice of coordinates used in (2)).

The domain of integration is $\mathcal{G} = \{ (r, z) : r_0 \leq r < \infty, \quad 0 \leq z \leq L/2 \}$. Here we have incorporated reflection symmetry with respect to the coordinate line $z = L/2$, which is inherent to our problem. Hence it suffices to restrict the $z$-coordinate accordingly.

The following boundary conditions arise:

$$0 = A = B = C \quad \text{at} \quad r \to \infty, \quad (5)$$

$$0 = \dot{A} = \dot{B} = \dot{C} \quad \text{at} \quad z = 0 \quad \text{and} \quad z = L/2. \quad (6)$$

The conditions (5) imply asymptotic ‘flatness’, while the equations (6) result from the periodicity and reflection symmetry in $z^2$. On the horizon $r = r_0$ we can impose a ‘constant temperature condition’ (cf section 2.3) by

$$0 = A - B \quad \text{at} \quad r = r_0. \quad (7)$$

Apparently, condition (7) contains an undetermined constant of integration. We introduce the freely specifiable value $\beta_c$ by requiring that

$$0 = e^{-2B} - \beta_c \quad \text{at} \quad (r, z) = (r_0, L/2), \quad (8)$$

through which $B$ is prescribed at one point on the horizon and the constant in question is fixed. It turns out that $\beta_c$ (having no significant physical meaning) is a useful parameter to control the transition of our solutions along the NBS branch towards the critical regime. Beginning with $\beta_c = 1$ for a UBS, it monotonically decreases and reaches zero for the limiting pinch-off solution.

2 Here, ‘flat’ means Minkowski space times a circle.
There are two additional conditions which result from the field equations (4) in the degenerate limit \( r \to r_0 \). Instead of using these conditions, we follow [8] and introduce a modified radial coordinate \( \varphi \) via \( r/r_0 = \sqrt{\varphi^2 + 1} \) and obtain, as regularity requirements, that the derivatives with respect to \( \varphi \) vanish on the horizon. Our additional two boundary conditions are therefore:

\[
0 = A_\varphi = C_\varphi \quad \text{at} \quad r = r_0 \quad (\text{or} \ \varphi = 0).
\] (9)

Conditions (5) to (9) are in accordance with the aforementioned constraint rule [6]. Note that also \( B_\varphi = 0 \) holds on the horizon, but instead we employ equation (7). As a consequence of the validity of the constraints it then follows that \( B_\varphi \) vanishes for \( r = r_0 \).

Finally we note that a unique solution to equations (4)–(9) is obtained by not only scaling physical quantities in terms of appropriate powers of \( r_0 \), and fixing \( L/r_0 = L_{00}/r_0 \), cf (42), but also by prescribing a value for \( \beta_\epsilon \in [0, 1] \).

Note that although not all of the constraint conditions enter our numerical scheme (which solves the system (4)–(9)), they can be used \textit{a posteriori} to analyze the consistency and accuracy of the solution.

2.2. Asymptotics

Here, we proceed with a more elaborate discussion of the asymptotics of the metric functions. The equation (5) can be refined as follows [14]:

\[
\lim_{r \to \infty} r^{D-4}A = A_\infty r_0^{D-4},
\] (10a)

\[
\lim_{r \to \infty} r^{D-4}B = B_\infty r_0^{D-4},
\] (10b)

\[
\lim_{r \to \infty} g_D(r) C = C_\infty r_0 \quad \text{with} \quad g_D(r) = \begin{cases} r(\log r)^{-1} & \text{if } D = 5, \\ r & \text{if } D \geq 6, \end{cases}
\] (10c)

and constants \( A_\infty, B_\infty \), and \( C_\infty \). Note the logarithmic term in the asymptotics of function \( C \) in five spacetime dimensions, which calls for meticulous treatment if one wishes to produce highly accurate numerical results\(^3\).

From the asymptotic values we can extract two charges \([14, 24]\), namely the black string mass

\[
M = M_0 \left( 1 - 2A_\infty - \frac{2}{D - 3}B_\infty \right) \quad \text{with} \quad M_0 = \frac{(D - 3)r_0^{D-4}L\Omega_{D-3}}{16\pi G}
\] (11)

and the relative tension

\[
n = n_0 \frac{1 - 2A_\infty - 2B_\infty(D - 3)}{1 - 2A_\infty - 2B_\infty/(D - 3)} \quad \text{with} \quad n_0 = \frac{1}{D - 3}.
\] (12)

Here, \( \Omega_{D-3} \) is the surface area of a unit \((D - 3)\)-sphere and \( G \) is Newton’s constant of gravitation. The quantities \( M_0 \) and \( n_0 \) describe the corresponding values obtained for the UBS.

\(^3\) One might think of the simple coordinate transformation \( s = 1/r \) which compactifies the infinite region. In terms of \( s \), the fall-off in \( C \) reads: \( C \sim \log r/r = -s \log s \). Obviously, no derivatives (with respect to \( s \)) of this term exist at infinity \( (s = 0) \). This leads to a poor convergence of any spectral approximation of this function. We provide an alternative approach in section 3.2.1.
2.3. Horizon quantities

In this section we define relevant geometric as well as thermodynamic quantities which characterize the black string horizon. From the line element (2) we read off a $z$-dependent circumferential horizon radius:

$$R(z) = r_0 e^{C(r_0 z)}, \quad (13)$$

which is referred to as the ‘horizon areal radius’ [9]. The parameters $R_{\text{max}}$ and $R_{\text{min}}$ introduced in equation (1) are given by the corresponding extremal values:

$$R_{\text{min}} = \min_{0 \leq z \leq L/2} R(z), \quad R_{\text{max}} = \max_{0 \leq z \leq L/2} R(z). \quad (14)$$

Next, the proper length of the compact dimension on the horizon is defined by:

$$L_H = \int_0^L e^{B(r_0 z)} \, dz. \quad (15)$$

Apart from this, we introduce in the usual manner the following thermodynamic quantities: (i) the black string temperature $T$, which is proportional to the constant surface gravity on the horizon$^4$:

$$T = T_0 e^{A(r_0 z) - B(r_0 z)} \quad \text{with} \quad T_0 = \frac{D - 4}{4\pi r_0}, \quad (16)$$

and (ii) its entropy, which is proportional to the horizon area:

$$S = \frac{S_0}{L} \int_0^L e^{B(r_0 z) + (D-3)C(r_0 z)} \, dz \quad \text{with} \quad S_0 = \frac{r_0^{D-3} L \Omega_{D-3}}{4G}. \quad (17)$$

As above, $T_0$ and $S_0$ denote corresponding values of the UBS.

Together with mass and relative tension, temperature and entropy obey Smarr’s relation [14, 24]:

$$TS = \frac{D - 3 - n}{D - 2} M, \quad (18)$$

as well as the first law of black hole thermodynamics:

$$dM = T dS + \frac{nM}{L} dL. \quad (19)$$

Since these formulae relate asymptotic charges with horizon values, they can be employed as non-trivial tests of the consistency and accuracy of the numerical scheme, cf. section 4.3. In the following section, we fix $L/r_0$ by $L_{\text{GL}}/r_0$ (cf (42)), which means that the first law reduces to the form $dM = T dS$.

3. Numerical implementation

The basis of our numerical scheme is the expansion of any function $g : [a, b] \to \mathbb{R}$ in terms of Chebyshev polynomials $T_k(y) = \cos[k \arccos(y)]$, where $y \in [-1, 1]$ and

$^4$ Note that a $z$-independent temperature means that the difference between $A$ and $B$ is constant on the horizon, which is satisfied by our boundary condition (7).
Considering the values of the function on Lobatto grid points (which include the boundaries \(a\) and \(b\)):

\[
x_k = (b - a) \sin^2 \left[ \frac{\pi k}{2(N - 1)} \right] + a \quad \text{with} \quad k = 0, 1, \ldots, N - 1,
\]

we can calculate Chebyshev coefficients \(c_k\), as well as approximate derivatives of \(g\), by means of standard pseudo-spectral schemes. For a given expansion order \(N\), the accuracy of the approximation (20) depends crucially on the fall-off of the coefficients \(c_k\) which are governed (roughly speaking) by the smoothness properties of the underlying function \(g\). Thus, our strategy consists of a reformulation of the problem in order to obtain rapidly decaying coefficients of the solution, when considered on appropriate domains. We provide a detailed discussion of this treatment in sections 3.1 and 3.2.

Before moving on, it should be noted that we apply the Newton-Raphson method for solving a discretized system, describing the collection of non-linear partial differential equations and boundary conditions (4)–(9). In the several iterative steps of this scheme, a linear system involving a Jacobian matrix has to be solved. This is done by means of the so-called BiCGSTAB method [25], which we endow with a preconditioner that utilizes a finite difference representation of the Jacobian. We finally note that the linear system arising within the preconditional step is solved efficiently with the help of a band matrix decomposition algorithm [26].

3.1. Non-uniform black strings in 6 spacetime dimensions

We start with the description of the numerical scheme for the construction of solutions describing NBS in six spacetime dimensions. As mentioned above, these solutions are not plagued with logarithmic terms. The introduction of a modified radial coordinate \(\chi\) via

\[
\frac{r_0}{r} = 1 - (1 - \chi)^2 = \chi(2 - \chi),
\]

realizes on the one hand a compactification of the infinite domain, and yields on the other hand regularity conditions (9), which are realized by requiring that the derivatives with respect to \(\chi\) vanish on the horizon. A straightforward implementation of the equations (4) to (9) in terms of a pseudo-spectral method on a single compactified computational domain (as described above) does not provide us with satisfactory accuracy, even for solutions near the UBS with \(\lambda \ll 1\). In the following section, we describe a more sophisticated approach which consists of a special treatment of the solution’s behavior for \(r \to \infty\) (relevant for all \(\lambda \in (0, \infty)\), i.e. also for small horizon deformation), as well as a technique to handle the critical regime where the horizon deformation becomes very large (\(\lambda \gg 1\)).

3.1.1. Treatment of the asymptotics. A detailed analysis (see appendix A.1) of linear perturbations around the UBS leads us to the following ansatz for our metric potentials:

\[
A = A_0(r) \left( \frac{r_0}{r} \right)^2 + A_1(r, z) \cos \left( \frac{2\pi z}{L} \right) e^{-2\pi r/L} \left( \frac{r_0}{r} \right)^{3/2},
\]

\[
B = B_0(r) \left( \frac{r_0}{r} \right)^2 + B_1(r, z) \cos \left( \frac{2\pi z}{L} \right) e^{-2\pi r/L},
\]
\[ C = C_0(r) \left( \frac{r_0}{r} \right)^2 + C_1(r, z) \cos \left( \frac{2\pi}{L} z \right) e^{-2\pi r/L} \left( \frac{r_0}{r} \right). \]  

(23c)

This ansatz describes a decomposition into a part which depends only on the radial coordinate and is dominant near infinity, together with a remainder with specified leading terms in the radial direction. According to (10), we can directly read off the asymptotic charges \( M \) and \( n \) from the boundary values \( A_0\big|_{r \to \infty} = A_\infty \) and \( B_0\big|_{r \to \infty} = B_\infty \). Also, we have \( C_0\big|_{r \to \infty} = C_\infty \).

Note that there is a one-to-one map between \( \{ A_0, A_1 \} \) and \( A \) as they follow from:

\[ A_0(r) = \left( \frac{r_0}{r} \right)^2 A(r, z) \big|_{z = L/4}, \]

(24)

\[ A_1(r, z) = \frac{A(r, z) - A(r, z)\big|_{z = L/4}}{\cos \left( \frac{2\pi}{L} z \right)} e^{2\pi r/L} \left( \frac{r_0}{r} \right)^{-3/2}, \]

(25)

and similarly for \( B \) and \( C \).

We note the half-integer power of \( r \) in (23) which, expressed in terms of \( \chi \) (see (22)), is non-smooth: \( r^{-3/2} \sim \chi^{3/2} \). This consideration leads us, in the six-dimensional case, to the introduction of yet another radial coordinate \( \xi \in [0, 1] \) via:

\[ \frac{r_0}{r} = [1 - (1 - \xi^2)^2] = \xi^2 (2 - \xi^2). \]

(26)

Through the introduction of \( \xi \) we achieve (i) compactification of the infinite domain, (ii) vanishing \( \xi \)-derivatives on the horizon and (iii) regularization of half-integer powers of \( r_0/r \) near infinity. The coordinate values \( \xi = 0 \) and \( \xi = 1 \) correspond to infinity and to the horizon respectively. Note that the term \( e^{-2\pi r/L} \) is not analytic with respect to \( \xi \) at \( \xi = 0 \), because all of its \( \xi \)-derivatives vanish at this point (the same holds if it is expressed in terms of \( \chi \)).

The fall-off of spectral coefficients with respect to the angular direction \( z \) can be enhanced by introducing the new coordinate \( u \in [-1, 1] \):

\[ u = \cos \left( \frac{2\pi z}{L} \right), \]

(27)

where \( z = 0 \) corresponds to \( u = 1 \), \( z = L/4 \) to \( u = 0 \), and \( z = L/2 \) to \( u = -1 \). More specifically, in higher orders of perturbation theory, only powers of \( u \) appear. Hence we expect a rapid convergence of the spectral coefficients in this coordinate, at least in the vicinity of the UBS solution. We note that a Chebyshev expansion in \( u \) is basically equivalent to an even Fourier expansion in \( z \), cf equation (20).

Below, we present asymptotic boundary conditions for the auxiliary functions which constitute the metric components through (23). At first, we derive conditions for the \( z \)-independent functions \( A_0(\xi) \), \( B_0(\xi) \) and \( C_0(\xi) \) through the analysis of a power series in terms of \( \xi \) of the field equations (4) at \( u = 0 \) (i.e. \( z = L/4 \)) about \( \xi = 0 \). To zeroth order, we obtain \( A_{0,\xi} = 0 \) from (4a) and the condition \( C_{0,\xi} = 0 \) from both (4b) and (4c). Taking these findings into account for the first expansion order, we get:

\[ 0 = 3A_{0,\xi} - 2A_{0,\xi}C_\infty \]

(28)

from (4a) and:

\[ C_{0,\xi} = -4(2A_\infty + 4B_\infty + C_\infty^2) \]

(29)

We note that the ansatz (23) privileges the use of an odd number of grid points with respect to the \( u \)-direction, since then the central point \( u = 0 \) (corresponding to \( z = L/4 \)) is contained within the Lobatto grid (21).
again from both (4b) and (4c). At the second order we observe that, by virtue of the preceding relations, 
\( B_{0,\xi} = 0 \) also holds. A numerical scheme based on the boundary conditions of
vanishing \( \xi \)-derivatives for the three functions \( A_0, B_0 \) and \( C_0 \) at \( \xi = 0 \) works well, but yields
unsatisfactory accuracy. This is due to the fact that the condition \( B_{0,\xi} = 0 \) arises in the course
of the power law expansion only at second order, which affects the rounding error of internal
computations. Therefore, in order to restore high accuracy, we utilize the condition \( C_{0,\xi} = 0 \),
and perform another decomposition:

\[
C_0(\xi) = C_\infty + \xi^2 \, C_{01}(\xi). \tag{30}
\]

Then, reconsidering our power law expansion at second order, we also find, besides \( B_{0,\xi} = 0 \),
two further conditions, of which one reads:

\[
C_{0,\xi\xi} = 12(2A_\infty + 4B_\infty + C_\infty^2). \tag{31}
\]

Incorporating (30), we finally arrive at the following collection of boundary conditions for the
three functions \( A_0, B_0 \) and \( C_0 \) at \( \xi = 0 \):

\[
\begin{align*}
A_{0,\xi} &= 0, & B_{0,\xi} &= 0, \\
C_{01} + 2(2A_\infty + 4B_\infty + C_\infty^2) &= 0, & C_{01} + C_{01,\xi} &= 0. 
\end{align*} \tag{32}
\]

Here, the third and fourth conditions are equivalent to (29) and (31). Note that besides the
function \( C_{01} \), the value \( C_\infty \) appears as additional unknown in our numerical scheme, which
means that we have to take four (instead of three) conditions into account. Equation (28) is
neglected in the set of boundary conditions but emerges \emph{a posteriori} as a property of the final
solution.

For the \( z \)-dependent functions \( A_1, B_1 \), and \( C_1 \) the field equations at infinity, \( \xi = 0 \), give
the conditions:

\[
u(1 - u^2)X_{1,uu} - (2 - 3u^2)X_{1,u} - 2X_{1,u}|_{u=0} = 0, \tag{33}
\]

where \( X_i(u) = \{A_i(\xi, u), B_i(\xi, u), C_i(\xi, u)\} |_{\xi=0} \). The only regular solution of this ordinary
differential equation is \( X_{1,u}(u) = 0 \) and therefore we obtain the following boundary
conditions at \( \xi = 0 \):

\[
A_{1,u} = B_{1,u} = C_{1,u} = 0, \tag{34}
\]

In other words, these functions take \( z \)-independent constant values at infinity which are
unknown \emph{a priori}. This comes as no surprise, since the \( z \)-dependent modes of the functions
\( A_1, B_1 \), and \( C_1 \) again carry the factor \( e^{-2\pi r/L} \), and are therefore rapidly decreasing. This can be seen
explicitly through an expansion of the perturbation theory to higher orders.\footnote{By default the term \('rapidly decreasing'\) refers to functions which tend asymptotically to zero faster than any inverse power of \( r \).}

Accordingly, we can derive additional asymptotic conditions from a power law expansion
in terms of \( \xi \) of the corresponding equations at \( \xi = 0 \), thereby neglecting the \( z \)-dependent
modes of \( A_1, B_1 \), and \( C_1 \). The leading order yields:

\[
A_{1,\xi} = 0, \quad B_{1,\xi} = 0, \quad B_1 + 2\pi \frac{P_0}{L} C_1 = 0, \tag{35}
\]

which, together with (34), constitutes the set of asymptotic conditions for the functions \( A_1, B_1 \),
and \( C_1 \). In our numerical scheme, we require (34) at all grid points \((\xi, u) = (0, u)\) with
\( u \neq 0 \), while at \((\xi, u) = (0, 0)\) the conditions (35) are enforced.
3.1.2. Decomposition of the numerical domain. In the course of the development of the code it transpires that an appropriate splitting of the domain of integration is essential to render highly accurate solutions. With respect to the coordinates \((\xi, u)\) we denote the entire domain by \(\{ (\xi, u) : 0 \leq \xi \leq 1, \ -1 \leq u \leq 1 \} = [0, 1] \times [-1, 1]\). As a first step, we divide up the asymptotic region by \(\xi \leq \xi_j\) we express the functions \(A_1, B_1, C_1\) in terms of \(\xi, u\) (cf. (26) and (27)).

The near-horizon region \(\xi \geq \xi_j\) is split up into the trapezoidal subdomain \(B\) and the triangular subdomain \(C\), which is divided up further, in order to resolve steep gradients at the critical point \((\xi, z) = (1, L/2)\), see figure 2. In \(B\) and \(C\), the numerical algorithm solves for the auxiliary functions \(\alpha, \beta\) and \(\gamma\) which are related through (36) to \(A, B\) and \(C\).

\[ A_0, B_0, C_0; A_1, B_1, C_1 \]

\[ \alpha, \beta, \gamma \]

Figure 1. Division of the integration domain into several subdomains for \(D = 6\). The asymptotic region \(\xi \leq \xi_j\) is split up into \(I\) rectangles (here \(I = 3\)) with different resolutions regarding the \(z\)-direction. In this region the numerical algorithm solves for the auxiliary functions \(A_0, B_0, C_0, A_1, B_1, C_1\) which constitute through (23) the metric functions \(A, B\) and \(C\). Note that the one-dimensional functions \(A_0, B_0\) and \(C_0\) are considered on the \(z = L/4\)-line (drawn more boldly, see also footnote 4). Also, for \(\xi \leq \xi_j\) we impose equality of the functions \(A, B, C\) at inner boundaries where two subdomains touch. Between domains close to the asymptotic region, with non-matching grid points at such inner boundaries, we use the Clenshaw recurrence algorithm to interpolate from one grid to the other. As the interpolations are only carried out at inner boundary grid points, the corresponding computational costs are not significant.
For the remaining subset $[\xi, 1] \times [-1, 1]$ we take into account that in the limit $\lambda \to \infty$ the functions $A$, $B$, and $C$ diverge at the critical point $(\xi, u) = (1, -1)$ (corresponding to $(r, z) = (r_0, L/2)$). In order to tackle this situation we introduce the auxiliary functions, $\alpha$, $\beta$ and $\gamma$ through:

$$
\alpha = e^{-2A}, \quad \beta = e^{-2B}, \quad \gamma = e^{2C}.
$$

In the limit $\lambda \to \infty$, the values of $\alpha$, $\beta$ and $\gamma$ tend to zero at the critical point. However, this point is plagued with steep gradients when considering large $\lambda$, in particular with respect to the $u$-direction. We recognize that in the critical regime the benefits of the coordinate $u$ are lost, and we return therefore nearer the horizon, back to the coordinate $z$ (elsewhere we can still profit from the properties of $u$).\footnote{Note that this is exactly the spatial grid point where the horizon is supposed to pinch-off in this limit.}

Nonetheless, in the coordinates $(\xi, z)$ we observe for large $\lambda$ clearly pronounced peaks of the functions $\alpha$, $\beta$ and $\gamma$ at the critical point. We resolve these peaks by splitting the domain $[\xi, 1] \times [0, L/2]$ into a trapezoidal subdomain $B$ and a triangular subdomain $C$, see figure 1. The latter is then further subdivided into $J$ subdomains $C_j$ ($j = 1, \ldots, J$) placed around the critical point (see figure 2). The corresponding coordinate transformations which provide the mappings of these subdomains onto rectangles are given in appendix B.

3.2. Non-uniform black strings in 5 spacetime dimensions

3.2.1. Treatment of the asymptotics. Again, a sophisticated analysis of linear perturbations around the UBS, to be conducted in appendix A.2, suggests the following ansatz near infinity in 5 spacetime dimensions:

$$
A = A_0(r) \frac{r_0}{r} + A_1(r, z) \cos \left( \frac{2\pi z}{L} \right) \left( \frac{r_0}{r} \right)^4,
$$

$$
B = B_0(r) \frac{r_0}{r} + B_1(r, z) \cos \left( \frac{2\pi z}{L} \right) \left( \frac{r_0}{r} \right)^4,
$$

In our numerical implementation we use dimensionless quantities, i.e. the calculations are carried out with respect to $z/L$.\footnote{In our numerical implementation we use dimensionless quantities, i.e. the calculations are carried out with respect to $z/L$.}
As for 6 spacetime dimensions (cf (23)), we split the functions into an asymptotically dominant part which only depends on $r$, and a secondary term depending on both coordinates, $r$ and $z$. The secondary terms are rapidly decreasing (cf footnote 5) asymptotically. However, in $D = 5$ spacetime dimension, we refrain from extracting the corresponding exponential factor from the secondary terms, see appendix A.2. Comparing (37) with (10) we again find $A_0 |_{r \to \infty} = A_\infty$, $B_0 |_{r \to \infty} = B_\infty$ and $C_0 |_{r \to \infty} = C_\infty$.

Similar to the case $D = 6$, we consider the ansatz (37) only in regions far away from the horizon. This means that the evidently incorrect condition $C(r_0, L/4) = 0$ (cf (37c)) does not appear in our numerical setup. Also, in these regions we again benefit from the coordinate $u$ defined in (27).

In the radial direction we work with the coordinate $\chi$ defined in (22). Note that in (37) no half-integer powers of $r_0/r$ appear, which eliminates the need to introduce the coordinate $\xi$ via (26). Such terms, as well as logarithmic expressions (like $\log^m(r_0/r)$) are suppressed by the rapid decay of functions $A_1$, $B_1$, and $C_1$. Consequently, we expect a sub-geometric (stronger than algebraic) convergence of the spectral coefficients of $A_1$, $B_1$, and $C_1$ with
respect to $\chi$. In contrast, the asymptotics of functions $A_0$, $B_0$ and $C_0$ contains logarithmic terms in $\chi$. In order to guarantee globally high accuracy, we consider these functions in terms of another radial coordinate $\eta \in [0, 1]$ given through:

$$\eta \in [0, 1], \chi \in [0, \chi_f]: \quad \eta = \frac{1}{1 - \log(\chi/\chi_f)} \Leftrightarrow \chi = \chi_f e^{1-\eta}, \quad (38)$$

where $0 \leq \chi_f \leq 1$. The asymptotic boundary $\chi = 0$ is obtained for $\eta \to 0$, whereas $\eta = 1$ corresponds to $\chi = \chi_f$, which, similarly to $D = 6$, separates the asymptotic from the near-horizon domain (see figure 3). Note that by means of (38), expressions of the kind $\chi^l \log^m \chi$ ($l, m > 0$) are differentiable (but not analytic) an infinite number of times with respect to $\eta$ at $\eta = 0$. In this way we assure sub-geometric convergence of the spectral coefficients of $A_0$, $B_0$ and $C_0$.

We summarize that functions $A_0$, $B_0$ and $C_0$ are considered at grid points with specifically prescribed values of the coordinate $\eta$ (cf (38)), while functions $A_1$, $B_1$, and $C_1$ are evaluated at specifically chosen coordinate values on the ($\chi$, $u$)-grid (see (22), (27)). In the numerical setup, we therefore need to apply spectral interpolation techniques (as before, we utilize the Clenshaw algorithm) in order to obtain a mapping between the different coordinate grids. Again, as this is done only at the coordinate line with fixed value $u = 0$ (i.e. $z = L/4$), the corresponding computational costs barely come into account. The two different radial grids are displayed in figure 3.

We can now provide the corresponding asymptotic boundary conditions. From the relevant equations for functions $A_0$, $B_0$ and $C_0$, derived from (4) in terms of the coordinate $\eta$ and considered at $u = 0$ (i.e. $z = L/4$), we obtain in the limit $\eta \to 0$:

$$A_{0,\eta} = 0, \quad B_{0,\eta} = 0, \quad A_{\infty} + 2B_{\infty} - C_{\infty} = 0. \quad (39)$$

This set of conditions turns out to work well, and yields satisfactory precision. Hence, an additional decomposition of the function $C_0$ (which was perfomed for $D = 6$) is not necessary. Finally, the rapid decay of functions $A_1$, $B_1$, and $C_1$ simply implies the conditions:

$$A_1 = B_1 = C_1 = 0, \quad (40)$$

to be satisfied asymptotiarcally at $\chi = 0$.

3.2.2. Decomposition of the numerical domain. Similarly to $D = 6$, we split up the integration domain $\mathcal{G} = \{(\chi, u) : 0 \leq \chi \leq 1, -1 \leq u \leq 1\} = [0, 1] \times [-1, 1]$ into several subdomains (see (22) and (27)). The region $[0, \chi_f] \times [-1, 1]$, in which we utilize the ansatz (37), is divided up into $I$ subdomains $\mathcal{A}_i = [\chi_{i-1}, \chi_i] \times [-1, 1]$, where $i = 1, 2, \ldots, I$ and $0 = \chi_0 < \chi_1 < \ldots < \chi_I$ for functions $A_1$, $B_1$, and $C_1$. Likewise, functions $A_0$, $B_0$ and $C_0$ are considered with respect to the radial coordinate $\eta$ (see (38)) on several intervals $[\eta_{i-1}, \eta_i]$ where $0 = \eta_0 < \eta_i < \ldots < \eta_I = 1$. The benefits of this approach are the same as for $D = 6$, as discussed in section 3.1.2.

In the near-horizon domain $(\chi, z) \in [\chi_f, 1] \times [0, L/2]$, the behavior of the functions is similar to that for $D = 6$. Therefore, we again introduce functions $\alpha$, $\beta$ and $\gamma$ according to (36), and utilize the original coordinate $z$. Also, we perform once more an additional domain decomposition into a trapezoidal subdomain $\mathcal{B}$ and a triangular subdomain $\mathcal{C}$. The entire setup is illustrated in figure 3. Again, further splitting of domain $\mathcal{C}$ turned out to be necessary, in order to enhance resolution near the critical point $(\chi, z) = (1, L/2)$ (see figure 2 and appendix B (in both cases, replace $\xi$ by $\chi$)).

9 We follow the terminology of Boyd [27].
4. Results

With the help of the numerical approach described in previous sections, we were able to obtain unprecedented horizon deformations characterized through large values of the parameter \( \lambda \). In particular, we were able to reach out to:

\[
\lambda \lesssim 340 \quad \text{for} \quad D = 5, \quad \lambda \lesssim 202 \quad \text{for} \quad D = 6.
\]  

(41)

Even for these large numerical limiting values, a high level of accuracy for the corresponding solutions was restored\(^{10}\). In the setup, we introduced eight subdomains in \( D = 5 \) (with \( I = 3 \) and \( J = 4 \), see figures 2 and 3) and seven subdomains in \( D = 6 \) (with \( I = 3 \) and \( J = 3 \), see figures 1 and 2). Although the spectral resolutions were of the order \( N \approx 50 \) (in all directions),\(^{11}\) the computational costs were kept at a moderate level, thus allowing us to establish a specific solution on a single computer within a few minutes\(^{12}\).

---

10 We note that in the large \( \lambda \) regime, calculations were performed using the C-programming language with ‘long double’ precision.

11 At least, such high resolutions are required for large \( \lambda \) in the vicinity of the horizon.

12 The most time consuming part of the calculation is the solution of the linear system inside the Newton-Raphson scheme.

Figure 4. Maximal and minimal horizon areal radii \( R_{\text{max}} \) and \( R_{\text{min}} \) (in units of \( r_0 \)) and the proper length of the compact dimension on the horizon \( L_\mathcal{H} \) (normalized by \( L \)) as functions of \( 1/(1 + \lambda) \). The insets (corresponding to the small boxes in the larger picture) show the region where \( R_{\text{max}} \) possesses three turning points. Lines of same type correspond to the same quantity in both plots.
At this point we want to emphasize that our results show qualitatively the same behavior for both spacetime dimensions considered, $D = 5$ and $D = 6$. Below, we present the findings obtained in these two cases, but we will not stress the qualitative agreement on every occasion.

Figure 5. Horizon areal radius $R$ (in units of $r_0$) for different $\lambda$ along the compact dimension. Proper distances $z_H(z) = \int_0^z e^{\kappa(z')} dz'$ are used (cf. (15)), and plots are centered around the origin. In the inset, the critical region (marked by a small box in the larger picture) is magnified. Also, the shape of the double-cone geometry is indicated.
4.1. Geometry

As depicted in appendix A, the value of \( L_{GL} \) (the critical length of the compact dimension, where the GL instability occurs) can be calculated based on linear perturbations in the vicinity of the UBS. The following values were obtained:

\[
\frac{L_{GL}}{r_0} = \begin{cases} 
7.1712728543704 & \text{for } D = 5, \\
4.9516154200735 & \text{for } D = 6.
\end{cases}
\]

In figure 4 we show the behavior of representative geometric quantities, defined on the horizon, as functions of \( \lambda \). In order to capture the regime of large \( \lambda \), we use \( 1/(1 + \lambda) \) as abscissas in the diagrams. In the range considered, the proper length of the compact dimensions \( L_H \) (see equation (15)) grow monotonically when \( \lambda \) is increased, while the minimal areal radius \( R_{min} \) decreases monotonically and approaches zero for \( \lambda \to \infty \) (note that \( R_{max} \) remains finite in this limit). Interestingly, \( R_{max} \) shows a non-monotonic behavior. In fact, on the way towards our maximally achieved horizon deformation, we encountered three turning points in the \( R_{max} \) curve.

A more qualitative picture, illustrating the growing deformation of the horizon as \( \lambda \) is increased, is given in figure 5. Here we plot the horizon areal radius against proper distances in the circle direction. We see a smooth behavior for moderate \( \lambda \), while for increasing \( \lambda \) the curve pinches off and develops a cusp in the limit \( \lambda \to \infty \).
More concretely, figure 5 shows that in the vicinity of the critical point the horizon approaches the double-cone geometry of [13] as $\lambda$ is increased. This supports the conjecture of a phase transition between black strings and localized black holes, with the double-cone metric as a local model of the transit solution at the point where the horizon pinches off [13].

4.2. Thermodynamic quantities

We now turn our focus to representative thermodynamic quantities. In figure 6 we display the behavior of mass, relative tension, temperature, and entropy as functions of $1/(1 + \lambda)$. Starting at $\lambda = 0$, mass and entropy increase with increasing $\lambda$ until a maximum is reached. Similarly, relative tension and temperature show a minimum. As $\lambda$ grows further, in each of these functions we observe two additional turning points, appearing on the way to the maximally achieved horizon deformation. These are shown in figure 7.

In [24] it was demonstrated that black string thermodynamics can be entirely derived from the curve displaying mass against relative tension. As an immediate consequence of the turning points of $M$ and $n$ along the NBS branch in figure 6, we see a spiral in the phase diagram in figure 8. In $D = 5$ as well as in $D = 6$ the spiral winds about one and half times (possibly a little more) before we reach our maximally numerically attainable value of $\lambda$. In figure 8 it becomes apparent that the extent of the spiral twists shrinks rapidly with each turn. In order to resolve these details, we plot the spiral once more in a logarithmically radially...

**Figure 7.** Magnification of the regions in figure 6 where the curves possess two further turning points (apart from their leading turning points).
rescaled version. To this end, we choose a point \((n_*, M_*)\) in the phase diagram (figure 8), located approximately at the position where the end-point of the spiral is expected. We center the coordinate system with respect to this point and introduce the radial distance \(d = (n_* - n/n_0)^2 + (M_* - M_0)^2\). Then we define the rescaled quantities:

\[
\tilde{M} = \frac{\log d}{d} \left( \frac{M}{M_0} - M_* \right) \quad \text{and} \quad \tilde{n} = \frac{\log d}{d} \left( \frac{n}{n_0} - n_* \right),
\]

(43)

to be plotted against one another in this diagram (see figure 9). Note that this procedure maps inner parts of the spiral to the outside and vice versa and, crucially, it would transform a spiral from the logarithmic to Archimedean type.

Although the spirals in figure 9 look rather bumpy, they are well resolved within our logarithmic rescaling. We conclude that in the original phase diagram (figure 8), the extent of the spiral twists shrinks exponentially with each turn, similar to the behavior of a logarithmic spiral. It is very tempting to conjecture that in the limit \(\lambda \to \infty\) the spirals wind an infinite number of times before reaching their endpoints (as with a logarithmic spiral).

Note that spiral curves are likewise exhibited in phase diagrams of other pairs of thermodynamic quantities. However, the \((M, S)\) diagram is an exception. This is due to the first law \(dM = TdS\), from which we follow that the turning points of mass and entropy coincide. This leads to cusps in the \((M, S)\) diagram rather than smooth twists.
4.3. Accuracy

The most interesting features in the figures presented in the previous section concern the regime of large $\lambda$-values, and are contained in tiny parts of the diagrams. In order to resolve these details, high accuracy of the solutions is needed, such that the uncertainties in the numerically determined values are much smaller than the magnitudes of the small features. In section 3, we described a numerical scheme capable of providing such solutions in an acceptable time-frame. Here, we want to show that, in the entire regime of $\lambda$-values considered, the results obtained using this scheme are very accurate. For this purpose we consider accuracy tests for the numerical scheme as well as consistency checks provided by the physics.

A commonly used method to measure the accuracy of a spectral algorithm is to compare a reference solution with high resolution with solutions of lower spectral resolution, obtained by the same procedure as the reference solution. The comparison is carried out by determining the function values of all solutions on a fine equidistant grid, using spectral interpolation techniques, then calculating the differences of reference solution to the solutions of lower resolution at each of these grid points. We call the largest magnitude of these differences the residue $R_N$, where $N$ indicates the resolution of the less resolved solution. As $N$ approaches the resolution of the reference solution, $R_N$ usually decreases and eventually saturates at a small value due to numerical limitations caused by finite machine precision and rounding errors. In figure 10 we display the convergence of the residue for the solutions with the largest horizon deformation obtained. It can be seen that the residue saturates at values of

![Figure 9. Logarithmic radial rescaling of figure 8. $\tilde{M}$ and $\tilde{n}$ are defined in equation (43).](image-url)
Apart from $R_N$, Figure 10 also shows the difference $D_{\text{Smarr}}$ between the right and left hand side of Smarr's formula (18). It converges similarly to the residue, thus Smarr's relation is satisfied very accurately. In addition, another verification of our solutions can be realized by checking the first law of black hole thermodynamics (see equation (19)). Parametrized with our control parameter $\beta_c$, the first law reads:

$$\frac{dM}{d\beta_c} = T \frac{dS}{d\beta_c}. \quad (44)$$

An accurate way to test equation (44) on a part of the NBS branch $\beta_c \leq \beta_c \leq \beta_c^2$ is to approximate $M(\beta_c), T(\beta_c)$ and $S(\beta_c)$ by means of a Chebychev expansion (20), where a solution to the field equations at each Lobatto grid point (21) is required. The derivatives in (44) can then be performed using standard spectral techniques. As one increases the number of Lobatto grid points, the deviation from the first law becomes smaller. Accordingly, one can plot a typical convergence plot, similar to figure 10. In this manner we confirmed that our solutions satisfy the first law on different parts of the NBS branch. The deviation always drops down to orders of $10^{-10}$ at least.

It is shown in [17] that a violation of the constraint equations would not cause any deviation of Smarr’s formula or the first law, as long as equations (4) are satisfied. As shown above our numerical approach relies on solving (4), while the constraint equations are only
indirectly satisfied by choosing the ‘right’ boundary conditions [6]. Therefore, an examination of the constraint equations by substituting the numerical solution is essential to check if the latter actually solves all of Einstein’s equations. The constraint violations for our solutions with maximal horizon deformations are of the order $10^{-8}$, at the point where the horizon is supposed to pinch off in the $\lambda \to \infty$ limit. Further from this point the violations are some orders of magnitude smaller, as they are for solutions with moderate horizon deformations.

5. Summary and conclusions

For the purpose of investigating the critical regime of large $\lambda$-values, corresponding to strong horizon deformations of the NBS, we have developed a well-adapted numerical scheme. Using a pseudo-spectral approach, we adjusted and extended our methods in a sophisticated manner, in order to obtain highly accurate solutions, in particular in the critical regime. We now discuss the crucial adaptions:

- The split of the functions near infinity:
  In the asymptotic region ($r \to \infty$) we separate each metric function into one part which only depends on $r$ and one which also depends on $z$ (see (23) and (37)). The reason for this decomposition is given by the fact that the $z$-dependent modes decay rapidly as $r \to \infty$. From the $z$-independent modes the asymptotic charges can be extracted very accurately, and we achieve greater precision in the overall solution. This procedure could be generally useful in problems where a compactified dimension is present, and the modes corresponding to the compact coordinate show a rapid decay at infinity.

- Multi-domain splitting and different resolutions in the two directions:
  We divide our integration domain in a sequence of straightly connected subdomains, see figure 1. Not only in the direction longitudinal to the sequence but also in the transversal direction, we use different resolutions, since the rapid decay of the $z$-dependent modes require, for globally highly accurate solutions, a substantially smaller resolution in the asymptotic regime. This leads to non-matching grid points at the common boundary of two neighboring domains.

- Coordinate transformations and multiple domains near the horizon:
  The density of numerical grid points near the critical point is tremendously increased by specific coordinate transformations (see appendix B). This enables us to resolve this regime with moderate resolutions very accurately. The corresponding coordinate transformations can be advantageous when strongly pronounced peaks appear in the solutions.

- Exponential coordinate transformation:
  With the help of a coordinate transformation (equation (38)) it is possible to avoid logarithmic behavior. As a consequence, the spectral convergence rate of the $z$-independent functions in $D = 5$ is changed from algebraic to sub-geometric, which significantly decreases the number of required grid points. This coordinate transformation could be useful in many situations, since logarithmic behavior is a frequently emerging problem in numerics.

In this work we constructed solutions in $D = 5$ and $D = 6$ spacetime dimensions. We expect that this scheme could easily be generalized for higher dimensions. As logarithmic terms are absent for $D \geq 6$, a modification of the asymptotic ansatz (23) to be incorporated into the routines for $D = 6$ should provide the corresponding numerical means.
The most interesting physical result is the appearance of a spiral curve in the black string’s phase diagram (see figures 8 and 9) in the case of both $D = 5$ and $D = 6$. Note that this was already conjectured in [8]. In both spacetime dimensions the spiral winds approximately one and half times before we reach our maximally numerically attainable deformation of the black string horizon. We conjecture, however, that the spiral winds up an infinite number of times in the critical limit $\lambda \to \infty$. Such behavior was observed in the context of hairy black holes in $AdS_5 \times S^5$ [28]. There are further examples of higher dimensional black objects where the beginning of a spiral curve in their phase diagram could be shown, for instance in the context of hairy black holes in global $AdS_5$ [29] or lumpy black holes [30, 31]. Note that all of these solutions have in common that their branch emanates from the zero-mode of an instability. In the case of the hairy black holes mentioned above, it is the superradiant instability of Reisner-Nordström black holes, while the ultraspinning instability of Myers-Perry black holes leads to their being 'lumpy'. As already stated, the NBS branch emanates from the GL instability of the UBS branch - hence the formation of a spiral curve in the phase diagram would seem to be a fairly generic feature for such situations13.

We conclude by considering some implications of the inspiral of the NBS branch. The conjecture regarding black hole/black string phase transition requires that these two branches meet at $\lambda \to \infty$ (from the NBS point of view). It seems very likely that the localized black hole branch will likewise pass through a spiral curve which joins to the NBS spiral with a common end-point. So far, one turning point in the phase diagram of localized black holes has been found [10]. Nevertheless, further extension of the localized black hole branch would be needed to see evidence of a possible spiral, in particular one which shows a continuous transition to the NBS spiral. Furthermore, as the turning point in the localized black hole branch goes along with the emergence of an unstable mode [10], we expect a similar property to be seen in the NBS phase diagram. In other words, if there are infinitely many twists before the two branches meet, then a growing number of unstable modes emerges as one approaches phase transition.

Acknowledgments

We thank Burkhard Kleihaus, Jutta Kunz, and Eugen Radu for drawing our attention to this problem and for numerous fruitful discussions. In particular, we thank Burkhard Kleihaus and Jutta Kunz for careful reading of the manuscript. Furthermore, we are grateful to Barak Kol for valuable discussions. This work was supported by the Deutsche Forschungsgemeinschaft (DFG) graduate school GRK 1523/2.

Appendix A. Perturbations around the UBS

Perturbation theory in the vicinity of the UBS was first developed in [5] for five dimensions and later applied to six dimensions in [6]. A generalization to arbitrary dimensions was carried out in [7]. We want to illustrate how the study of linear perturbations may prove useful for the development of an accurate and efficient numerical scheme to solve the full set of nonlinear equations (4). Furthermore we derive highly accurate values for $L_{GL}/r_0$ from this analysis.

13 We thank Oscar J C Dias for pointing this out.
For first order perturbations only the marginal GL mode appears and we can write:

\[ A = \varepsilon a(r)\cos \left( \frac{2\pi r}{L} \right), \]  
\[ B = \varepsilon b(r)\cos \left( \frac{2\pi r}{L} \right), \]  
\[ C = \varepsilon c(r)\cos \left( \frac{2\pi r}{L} \right), \]

for some small \( \varepsilon \). Having substituted this into the field equations, we take only linear orders of \( \varepsilon \) into account. Equation (4) then yields the following set of ordinary differential equations (see also [32]):

\[ 0 = f a'' + \left[ \frac{3}{2} f' + (D - 3) \frac{f}{r} \right] a' + \frac{1}{2} (D - 3) f' c' - \frac{4\pi^2}{L^2} a, \]  
\[ 0 = f b'' - (D - 3) \frac{f}{r} a' + \frac{1}{2} f' b' - \frac{1}{2} (D - 3) \left[ f' + 2(D - 4) \frac{f}{r} \right] c' + (D - 3)(D - 4) \frac{b - c}{r^2} - \frac{4\pi^2}{L^2} b, \]  
\[ 0 = f c'' + \frac{f}{r} a' + \left[ f' + 2(D - 3) \frac{f}{r} \right] c' - 2(D - 4) \frac{b - c}{r^2} - \frac{4\pi^2}{L^2} c. \]

A.1. 6D perturbations

An asymptotic analysis of (46) for \( D = 6 \) gives rise to the following ansatz:

\[ a = \tilde{a}(r) e^{-2\gamma r/L} \left( \frac{r_0}{r} \right)^{3/2}, \]  
\[ b = \tilde{b}(r) e^{-2\gamma r/L}, \]  
\[ c = \tilde{c}(r) e^{-2\gamma r/L} \left( \frac{r_0}{r} \right). \]

For the reasons explained in section 3.1.1, the introduction of the coordinate \( \xi \in [0, 1] \) is useful:

\[ \frac{r_0}{r} = [1 - (1 - \xi)^2] = \xi^2 (2 - \xi)^2. \]  

It is now straightforward to solve (46) for \( \tilde{a}, \tilde{b}, \) and \( \tilde{c} \) in terms of \( \xi \) numerically. This system is a set of homogeneous linear ordinary differential equations with non-trivial solutions. For a unique solution we need to impose a scaling condition, and we decided to choose \( \tilde{c} = 1 \) at the horizon, i.e. at \( \xi = 1 \). Analyzing the perturbation equations at the boundaries, we find that at infinity (\( \xi = 0 \)), the equations degenerate, and the resulting conditions correspond to (35). On the horizon (\( \xi = 1 \)), the equations yield only the two regularity requirements \( \tilde{a}_{,\xi} = 0 \) and \( \tilde{c}_{,\xi} = 0 \) (cf (9)).

Furthermore, one has to prescribe a value for the dimensionless quantity \( L/r_0 \) appearing in the equations. Interestingly, the additional regularity requirement \( \tilde{b}_{,\xi} = 0 \) to be imposed on the horizon is only satisfied for one specific value of \( L/r_0 \). We compute this value with high numerical precision by considering \( L/r_0 \) as further unknown and imposing \( \tilde{b}_{,\xi} = 0 \) as an
additional equation in our numerical scheme. This provides us with the value of $L_{GL}/r_0$ specified in (42).

Note that the decay of the spectral coefficients of $\bar{a}$, $\bar{b}$ and $\bar{c}$, expressed in terms of the coordinate $\xi$, is geometric, while that of $a$, $b$, and $c$ only is sub-geometric. This is due to the exponential factor $e^{-2\pi\xi/L}$ which is non-analytic at infinity when expressed in terms of $\xi$.

Apart from an accurate value for $L_{GL}/r_0$ the solution obtained with this procedure yields an initial attempt at the solution of the non-linear equations (4) (an essential ingredient for the application of the Newton-Raphson scheme). However, we need to take care of some subtleties in the non-linear regime, which we want to describe now.

As we have seen above, linear perturbation in the vicinity of the UBS only yields the marginal GL mode (see (45)), i.e. the linear perturbation does not take the asymptotics (10) carried by the $z$-independent modes into account. It first occurs in the second order perturbation (see for example [6]). This leads us to an ansatz in which we split each of the functions $A$, $B$, and $C$ into two parts. One part is $z$-independent and its leading behavior at infinity is given by the asymptotics (10). Therefore the physical quantities which are encoded in the far field (mass and relative tension, cf (11) and (12)) follow from the $z$-independent part alone. The remaining part is $z$-dependent and its leading asymptotic behavior corresponds to that of $a$, $b$, and $c$. As explained above, we can achieve an enhancement of the decay of the spectral coefficients if we extract this behavior analogous to (47), although in the non-linear regime this yields a mere sub-geometric convergence, since the exponential factor appears in higher orders.

All of the above considerations lead us to the ansatz (23). Note that a sufficient initial guess close to the UBS is obtained by simply neglecting the $z$-independent functions and taking the solutions $\bar{a}$, $\bar{b}$ and $\bar{c}$ of the first order perturbation equations as seed for the $z$-dependent functions.

A.2. 5D perturbations

In the case $D = 5$, asymptotic analysis of (46) yields the ansatz:

$$a = \bar{a}(r) e^{-2\pi\xi/L}\left(\frac{n_0}{r}\right)^{1+\pi r_0/L}, \quad (48a)$$

$$b = \bar{b}(r) e^{-2\pi\xi/L}\left(\frac{n_0}{r}\right)^{2+\pi r_0/L}, \quad (48b)$$

$$c = \bar{c}(r) e^{-2\pi\xi/L}\left(\frac{n_0}{r}\right)^{1+\pi r_0/L}. \quad (48c)$$

The introduction of the coordinate $\chi \in [0, 1]$ via

$$\frac{n_0}{r} = 1 - (1 - \chi)^2 = \chi(2 - \chi), \quad (22 \text{ revisited})$$

means that for our metric functions, which are regular at the horizon, the derivatives with respect to $\chi$ vanish at $\chi = 1$. We can therefore solve the corresponding linear ordinary differential equations for $\bar{a}$, $\bar{b}$ and $\bar{c}$ numerically, in the same manner as explained in the previous section for $D = 6$. In particular, with the scaling condition $\bar{c} = 1$ and the regularity requirement $\bar{b}_{,\chi} = 0$, both imposed at the horizon, the equations are solved together with corresponding boundary conditions. As with $D = 6$, we simultaneously obtain an accurate value for $L_{GL}/r_0$ in $D = 5$, as specified in (42).
It would be tempting to proceed in a similar manner as for \( D = 6 \). Indeed, it is a good idea to split the functions \( A, B, \) and \( C \) into a \( z \)-independent and a \( z \)-dependent part in the non-linear regime. However, in contrast to \( D = 6 \), logarithmic behavior in the asymptotics (cf (10c)) appears in higher order perturbations.

Let us first concentrate on the \( z \)-dependent modes. According to (48) these modes always carry the exponential factor \( e^{-2\pi r/L} \). This factor suppresses any logarithmic behavior of \( r \) at infinity. For this reason we express the \( z \)-dependent modes in terms of \( \chi \) and refrain from extracting the exponential factor. This yields a sub-geometric convergence of the spectral coefficients of the \( z \)-dependent modes with respect to \( \chi \).

In order to deal with the logarithmic behavior of the \( z \)-independent modes we make use of the exponential coordinate transformation \( \chi = \chi(\eta) \) defined in (38). As explained in section 3.2.1, such terms now rapidly decrease with respect to \( \eta \) for \( \eta \to 0 \), and their spectral coefficients converge sub-geometrically. Note that it is essential to extract the leading asymptotics from the \( z \)-independent modes in order to get the values of \( A_{\infty}, B_{\infty} \) and \( C_{\infty} \).

With this considerations we are led to the ansatz:

\[
A = A_0(r) \frac{r_0}{r} + \tilde{A}_1(r, z) \cos \left( \frac{2\pi}{L} z \right), \quad (49a)
\]
\[
B = B_0(r) \frac{r_0}{r} + \tilde{B}_1(r, z) \cos \left( \frac{2\pi}{L} z \right), \quad (49b)
\]
\[
C = -C_0(r) \frac{r_0}{r} \log \frac{r_0}{r} + \tilde{C}_1(r, z) \cos \left( \frac{2\pi}{L} z \right), \quad (49c)
\]

where we regard the \( z \)-independent functions \( X_0 = \{A_0, B_0, C_0\} \) as functions of \( \eta \) and the \( z \)-dependent functions \( \tilde{X}_1 = \{\tilde{A}_1, \tilde{B}_1, \tilde{C}_1\} \) as functions of \( \chi \) and \( z \).

The equations for \( X_0 \) are obtained by substituting (49) into the field equation (4) and taking the specific coordinate value \( z = L/4 \). Having cancelled some prefactors in the resulting equations, their structure on the \( \eta \)-grid reads as follows:

\[
0 = F_0(X_0, \eta) + \left( \frac{r(\eta)}{r_0} \right)^4 F_1(\tilde{X}_1, \eta). \quad (50)
\]

Here, \( F_0 \) depends on the functions \( X_0 \) and their first and second derivatives with respect to \( \eta \), and \( F_1 \) depends on the functions \( \tilde{X}_1 \) and their first derivatives with respect to \( z \). Observe the factor \( (r/r_0)^4 \), which strongly blows up for small values of \( \eta \) due to exponential mapping (38). Mathematically, there is no problem, since the functions \( \tilde{X}_1 \) carry the exponential factor \( e^{-2\pi r/L} \). Hence, for small \( \eta \) this suppresses the \( (r/r_0)^4 \) behavior, and equation (50) is dominated by \( F_0 \). However, numerically, the evaluation of (50) for small \( \eta \) is highly problematic, since due to finite machine precision, one cannot guarantee that the functions \( \tilde{X}_1 \) are always small enough to compensate the \( (r/r_0)^4 \) behavior at each stage of the Newton-Raphson scheme. We tackle this technical problem by rewriting the functions \( \tilde{X}_1 \) as

\[
\tilde{X}_1 = \left( \frac{r_0}{r} \right)^4 X_1, \quad (51)
\]

which means that the problematic term \( (r/r_0)^4 \) in (50) is cancelled. Finally, by expressing \( (A, B, C) \) in terms of \( X_1 = \{A_1, B_1, C_1\} \) (as well as \( X_0 \)), we obtain the ansatz (37). We note that the fall-off of the spectral coefficients of the functions \( X_1 \) with respect to \( \chi \) is slower than that of \( \tilde{X}_1 \) but is still sub-geometric, and we thus achieve high accuracy with reasonable resolutions.
For constructing an initial guess for the Newton-Raphson scheme in the vicinity of the UBS, we proceed similarly to $D = 6$: We set the $z$-independent functions to zero and obtain the $z$-dependent functions from the solution of the first order perturbation equations.

Appendix B. Resolving the critical point

We want to describe the domain splitting and coordinate transformations which provide us with a high spatial resolution, particularly in vicinity of the critical point $(\xi, z) = (1, L/2)$.\footnote{The consideration in this section is made for the spacetime dimension $D = 6$ with the preferred coordinate $\xi$ in the vicinity of the horizon. For $D = 5$ one merely has to replace $\xi$ with the coordinate $\chi$ preferred in that case, e.g. the critical point in $D = 5$ is given by $(\chi, z) = (1, L/2)$.}

A first step is the decomposition of the region $\{(\xi, z) : \xi_l \leq \xi \leq 1, \ 0 \leq z \leq L/2\} = [\xi_l, 1] \times [0, L/2]$ into the trapezoidal domain $\mathcal{B}$ and the triangular domain $\mathcal{C}$, see figure 1. In terms of the new coordinate $\zeta \in [\xi_l, \xi_H]$ with:

$$\zeta(\xi, z) = \xi_l + \frac{(\xi_H - \xi_l)(\xi - \xi_l)}{(\xi_H - \xi_l) + (1 - \xi_H)(1 - 4 z/L)},$$

the domain $\mathcal{B}$ is mapped onto the rectangle $\{(\zeta, z) : \xi_l \leq \zeta \leq \xi_H, \ 0 \leq z \leq L/2\} = [\xi_l, \xi_H] \times [0, L/2]$. In particular, $\zeta = \xi_H$ corresponds to the hypotenuse of the triangular domain, which can be described by the equation $1 - 4 z/L = (\xi - \xi_H)/(1 - \xi_H)$, see figure 11.

The triangular domain $\mathcal{C}$ is mapped by means of the following coordinate transformation:

$$\sigma(\xi, z) = 1 - \frac{1}{2}[1 - (1 - \xi) + (1 - \xi_H)(2 - 4 z/L)],$$

![Figure 11. Domain setup in the vicinity of the horizon, i.e. for $\xi \gg \xi_l$. The triangular domain is subdivided into $J$ layers surrounding the critical point (here $J = 3$). The following coordinate lines were drawn: $\zeta = \text{const. (violet)}, z = \text{const. (dark blue)}, \sigma = \text{const. (orange)}$ and $\varphi = \text{const. (light blue)}. 

For constructing an initial guess for the Newton-Raphson scheme in the vicinity of the UBS, we proceed similarly to $D = 6$: We set the $z$-independent functions to zero and obtain the $z$-dependent functions from the solution of the first order perturbation equations.
onto the rectangle $\{(\sigma, \varphi) : \xi_H \leq \sigma \leq 1, -1 \leq \varphi \leq 1\} = [\xi_H, 1] \times [-1, 1]$. Crucially, coordinate lines of constant $\varphi$-values converge towards the critical point, which can be seen from the inverted form of equations (53) and (54):

$$\xi = 1 - (1 - \sigma)(1 - \varphi),$$

$$1 - 4 \frac{z}{L} = \frac{(1 - \sigma)(1 + \varphi)}{1 - \xi_H} - 1.$$  

For all $\varphi \in [-1, 1]$ the coordinate value $\sigma = 1$ corresponds to the critical point $(\xi, z) = (1, L/2)$, which means that this single point in the $(\xi, z)$-chart is blown up to an edge in the $(\sigma, \varphi)$-chart. The remainder of the horizon $(\xi = 1, z < L/2)$ corresponds to $\varphi = 1, \sigma < 1$ and the hypotenuse of the triangular domain is obtained for $\sigma = \xi_H$. Finally, $z = L/2$ is associated with $\varphi = -1$.

The domain decomposition and coordinate mappings described above allow us to use Chebyshev expansions on the several rectangular domains and, moreover, to obtain, in our discretized numerical scheme, densely distributed grid points in the vicinity of the critical point. However, the steep gradients at that point still require a careful treatment which we address in particular through the two additional steps below.

Similar to the decomposition of the domain $\mathcal{A}$ in section 3.1.2, we subdivide the triangular domain $\mathcal{C} = \{(\sigma, \varphi) : \xi_H \leq \sigma \leq 1, -1 \leq \varphi \leq 1\} = [\xi_H, 1] \times [-1, 1]$ into $J$ further subdomains $\mathcal{C}_j = [\sigma_{j-1}, \sigma_j] \times [-1, 1]$, where $j = 1, 2, \ldots, J$ and $\xi_H = \sigma_0 < \sigma_1 < \ldots < \sigma_J = 1$. The benefits are similar to those described in the previous context. We illustrate this domain subdivision in figure 11.

Finally, we resolve steep gradients at the critical point through an analytic mesh-refinement carried out within the triangular subdomain $\mathcal{C}_f$ which contains the critical point. The analytic mesh-refinement is defined by the mapping:

$$\sigma(\tilde{\sigma}) = 1 - (1 - \sigma_{j-1}) \frac{\sinh \left(\frac{\kappa - \tilde{\sigma}}{1 - \sigma_{j-1}}\right)}{\sinh \kappa},$$

where the new coordinate $\tilde{\sigma}$ is located in $[\sigma_{j-1}, 1]$. Depending on the parameter $\kappa$, the grid points in the $\sigma$-chart, chosen according to (21), are densely distributed about $\sigma = 1$ in the $\sigma$-chart. On the other hand, the mesh is coarser at the opposite edge, $\sigma = \sigma_{j-1}$, see figure 12.
This coordinate transformation proved to be appropriate to resolve steep gradients as demonstrated in [33, 34] (see [35] for a recent application). The parameter $\kappa > 0$ has to be chosen such that the fall-off of the spectral coefficients of the solution is as rapid as possible (there is an optimal $\kappa$, mostly of $O(1)$). With this analytic mesh-refinement, the spectral coefficients according to $\tilde{\sigma}$ show a more rapid decay as compared to the fall-off obtained when using $\sigma$. Thus, the costs of the numerical scheme are reduced and, moreover, we observe a substantial increase of the accuracy of the results.

References

[1] Dias O J C, Santos J E and Way B 2016 Numerical methods for finding stationary gravitational solutions Class. Quant. Grav. 33 133001
[2] Gregory R and Laffamme R 1993 Black strings and p-branes are unstable Phys. Rev. Lett. 70 2837–40
[3] Harmark T, Niarchos V and Obers N A 2007 Instabilities of black strings and branes Class. Quant. Grav. 24 R1–90
[4] Gregory R 2012 The Gregory-Lafmanne instability, in Black holes in Higher Dimensions ed G T Horowitz (Cambridge: Cambridge University Press)
[5] Gubser S S 2002 On nonuniform black branes Class. Quant. Grav. 19 4825–44
[6] Wiseman T 2003 Static axisymmetric vacuum solutions and nonuniform black strings Class. Quant. Grav. 20 1137–76
[7] Sorkin E 2004 A Critical dimension in the black string phase transition Phys. Rev. Lett. 93 031601
[8] Kleihaus B, Kunz J and Radu E 2006 New nonuniform black string solutions J. High Energy Phys. JHEP06(2006)016
[9] Sorkin E 2006 Non-uniform black strings in various dimensions Phys. Rev. D 74 104027
[10] Headrick M, Kitchen S and Wiseman T 2010 A new approach to static numerical relativity, and its application to Kaluza–Klein black holes Class. Quant. Grav. 27 035002
[11] Figueras P, Murata K and Reall H S 2012 Stable non-uniform black strings below the critical dimension J. High Energy Phys. JHEP11(2012)071
[12] Kalisch M and Ansorg M 2015 Highly deformed non-uniform black strings in six dimensions 14th Marcel Grossmann Meeting on Recent Developments in Theoretical and Experimental General Relativity, Astrophysics, and Relativistic Field Theories (MG14) Rome, Italy (July 12–18, 2015) arXiv:1509.03083
[13] Kol B 2005 Topology change in general relativity, and the black hole black string transition J. High Energy Phys. JHEP10(2005)049
[14] Kol B, Sorkin E and Piran T 2004 Caged black holes: Black holes in compactified space-times: 1. Theory Phys. Rev. D 69 064031
[15] Wiseman T 2003 From black strings to black holes Class. Quant. Grav. 20 1177–86
[16] Sorkin E, Kol B and Piran T 2004 Caged black holes: Black holes in compactified space-times: 2. 5-d numerical implementation Phys. Rev. D 69 064032
[17] Kudoh H and Wiseman T 2004 Properties of Kaluza–Klein black holes Prog. Theor. Phys. 111 475–507
[18] Kudoh H and Wiseman T 2005 Connecting black holes and black strings Phys. Rev. Lett. 94 161102
[19] Asnin V, Kol B and Smolkin M 2006 Analytic evidence for continuous self similarity of the critical merger solution Class. Quant. Grav. 23 6805–27
[20] Kol B and Wiseman T 2003 Evidence that highly nonuniform black strings have a conical waist Class. Quant. Grav. 20 5493–504
[21] Kol B 2006 The phase transition between caged black holes and black strings, a review Phys. Rep. 422 119–65
[22] Harmark T and Obers N A 2005 Phases of Kaluza–Klein black holes: A brief review arXiv:hep-th/0503020
[23] Horowitz G T and Wiseman T 2012 General Black holes in Kaluza–Klein Theory, in Black Holes in Higher Dimensions ed G T Horowitz (Cambridge: Cambridge University Press)
[24] Harmark T and Obers N A 2004 New phase diagram for black holes and strings on cylinders Class. Quant. Grav. 21 1709
[25] Barrett R, Berry M, Chan T, Demmel J, Donato J, Dongarra J, Eijkhout V, Pozo R, Romine C and van der Vorst H 1994 Templates for the Solution of Linear Systems: Building Blocks for Iterative Methods 2nd edn (Philadelphia, PA: Society for Industrial and Applied Mathematics)

[26] Press W H, Teukolsky S A, Vetterling W T and Flannery B P 2007 Numerical Recipes 3rd Edition: The Art of Scientific Computing 3rd edn (New York: Cambridge University Press)

[27] Boyd J 2001 Chebyshev and Fourier Spectral Methods Dover Books on Mathematics 2nd edn (Mineola, NY: Dover Publications)

[28] Bhattacharyya S, Minwalla S and Papadodimas K 2011 Small hairy black holes in $AdS_5 \times S^5$ J. High Energy Phys. JHEP11(2011)035

[29] Dias O J C, Figueras P, Minwalla S, Mitra P, Monteiro R and Santos J E 2012 Hairy black holes and solitons in global $AdS_5$ J. High Energy Phys. JHEP08(2012)117

[30] Dias O J C, Santos J E and Way B 2014 Rings, ripples, and rotation: Connecting black holes to black rings J. High Energy Phys. JHEP07(2014)045

[31] Emparan R, Figueras P and Martinez M 2014 Bumpy black holes J. High Energy Phys. JHEP12(2014)072

[32] Kol B and Sorkin E 2004 On black-brane instability in an arbitrary dimension Class. Quant. Grav. 21 4793–804

[33] Meinel R, Ansorg M, Kleinwächter A, Neugebauer G and Petroff D 2012 Relativistic figures of Equilibrium (New York: Cambridge University Press)

[34] Macedo R P and Ansorg M 2014 Axisymmetric fully spectral code for hyperbolic equations J. Comput. Phys. 276 357–79

[35] Ammon M, Leiber J and Macedo R P 2016 Phase diagram of 4D field theories with chiral anomaly from holography J. High Energy Phys. JHEP03(2016)164