Spin Chern-Simons and Spin TQFTs

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Abstract

In [14] we constructed classical spin Chern-Simons for any compact Lie group $G$: a gauge theory whose action depends on the spin structure of the 3-manifold. Here we apply geometric quantization to the classical Hamiltonian theory and investigate the formal properties of the partition function in the Lagrangian theory, all in the case $G = SO_3$. We find that the quantum theory for $SO_3$ spin Chern-Simons corresponds to the spin TQFT constructed by Blanchet and Masbaum [9] in the same way that the quantum theory for standard $SU_2$ Chern-Simons corresponds to the TQFT constructed by Reshetikhin and Turaev [19] or the TQFT constructed by Blanchet, Habegger, Masbaum, and Vogel [8].

1 Knot invariants and physics

The correlation between knot invariants and quantum Chern-Simons was first laid out in the seminal paper by Witten [21]. The crux of Witten's argument is that the quantum theory is a topological quantum field theory, or TQFT. In particular, he applies the formal properties of the quantum partition function to show that the theory satisfies the defining axioms of a TQFT. Together with certain results from conformal field theory, Witten uses these axioms to show that the quantum partition function associated to a compact oriented 3-manifold is the Jones polynomial of the corresponding knot. For a complete definition of a TQFT and an explanation of Witten's results we suggest [5].

Mathematicians -- in particular, Reshetikhin and Turaev [19] -- took a different approach to knot and 3-manifold invariants by constructing their own TQFT from the representation theory of quantum groups. So that their invariants matched Witten's, their TQFT was necessarily isomorphic to quantum Chern-Simons.

The quartet of Blanchet, Habegger, Masbaum, and Vogel (BHMV) come at TQFTs from the other direction; their starting point is a generalized version of the Reshetikhin and Turaev invariant. They apply algebro-categorical techniques to the Kauffman bracket and an oriented bordism category [8]. As desired, the BHMV TQFT nicely matches Witten's. We elaborate in section 3.1.
Our interest here, though, is in the work by Blanchet and Masbaum that followed the BHMV results. In the same spirit, their starting point is a refinement of the Reshetikhin and Turaev invariant – one that now depends on the spin structure of the 3-manifold. They apply the same algebro-categorical techniques, but now to a refined version of the Kauffman bracket and a spin bordism category. In this bordism category the manifolds and their boundaries must have compatible spin structures. The resulting construction satisfies the properties of what Blanchet and Masbaum dub a “spin TQFT”, where the axioms that define a spin TQFT are essentially refined versions of the axioms that define the “unspun” TQFTs mentioned above.

As we mentioned above, the BHMV TQFT corresponds well with quantum Chern-Simons. Upon learning of the Blanchet-Masbaum (BM) spin TQFT, one might naturally ask the following:

Question. Is there a quantum field theory that corresponds to the BM spin TQFT just as quantum Chern-Simons corresponds to the BHMV TQFT; and if so, what is it?

The answer, we claim, is yes; and the corresponding quantum field theory is the topic of this paper. We refer to this field theory as spin Chern-Simons.

In the first section of this paper we review the relevant aspects of classical spin Chern-Simons. These were worked out by the author in [14] and we refer to that paper for details and proofs. In the second part we review the BHMV TQFT and the BM spin TQFT. Here we carefully point out the correspondence between the BHMV TQFT and quantum Chern-Simons so that we can see how quantum spin Chern-Simons might correspond to the BM spin TQFT. We then address this hoped for correspondence on two fronts. The first is not necessarily rigorous. We use the formal properties of the quantum partition function to show that the 3-manifold invariants for both TQFTs display the same behavior. On the second front we compute the dimensions of the Hilbert spaces associated to closed, compact 2-manifolds in quantum spin Chern-Simons. We see that our dimension formulas are the same as those of the BM spin TQFT and that the Hilbert spaces of spin Chern-Simons are refinements of the Hilbert spaces of standard Chern-Simons. This in lock step with the way in which the Hilbert spaces of the BM spin TQFT refine the Hilbert spaces of the BHMV TQFT.

2 Classical spin Chern-Simons

To define the classical theory we choose a compact Lie group $G$ and an orthogonal, rank zero virtual representation $\rho \in RO(G)$. In practice, we often have an actual orthogonal representation $\rho_0 : G \to O(V)$ and take

$$\rho = \rho_0 - \dim V.$$ 

That is, $\rho$ is often the difference of an actual representation and the trivial representation of the same dimension.

We consider both the Lagrangian field theory over compact, spun 3-manifolds, and the Hamiltonian theory over closed, compact, spun 2-manifolds. In either
case, the fields over a manifold $X$ are principal $G$-bundles with connection over $X$. Technically, this “space” of fields is a category $\mathcal{C}_G(X)$ — a groupoid, in fact. The morphisms $\mathcal{G}_G(X)$ for this category are the $G$-bundle isomorphisms that cover the identity on $X$; in other words, the gauge transformations. Two objects in $\mathcal{C}_G(X)$ are considered physically equivalent if there is a gauge transformation taking one to the other.

2.1 Classical Lagrangian theory

Here, $X$ is a closed, compact, spun 3-manifold. The main object of interest in the Lagrangian field theory is the action. In our case, this is a $\mathbb{T}$-valued function on the space of fields $\mathcal{C}_G(X)$, where $\mathbb{T} \subset \mathbb{C}$ are the unit modulus complex numbers. To define the action we place a Riemannian structure on $X$ and so obtain a Dirac operator acting on sections of the spinor bundle

$$D_X : \Gamma(S_X) \to \Gamma(S_X).$$

From the pair $(P, A) \in \mathcal{C}_G(X)$ — where $P$ is a principal $G$-bundle over $X$ and $A$ is a connection on $P$ — we obtain the associated virtual vector bundle with connection $(\rho P, \rho A)$. We can couple $\rho A$ to the Dirac operator to obtain a twisted Dirac operator

$$D_X \otimes \rho A : \Gamma(S_X \otimes \rho P) \to \Gamma(S_X \otimes \rho P).$$

This operator is elliptic, self-adjoint and quaternionic so that it has a discrete, real spectrum of eigenvalues with even valued degeneracies. This allows us to define our action to be the assignment

$$\mathcal{C}_G(X) \to \mathbb{T}$$

$$A \mapsto \tau^{1/2}(D_X \otimes \rho A)$$

(2.1)

where $\tau^{1/2}(D)$ is a spectral invariant defined for any elliptic, self-adjoint, quaternionic operator $D$. In fact, it is (a square root of) the exponentiated boundary term in the Atiyah-Patodi-Singer index theorem \[3\] and we use this fact to our advantage in determining some of the action’s properties. In particular, if all of the data (the bundle, connection, metric, and spin structure) bounds a principal $G$-bundle $P_M$ with connection $A_M$ over a spun 4-manifold $M$ we can write

$$\tau^{1/2}(D_X \otimes \rho A) = \exp \frac{\pi i}{8\pi^2} \text{Tr} (\rho (\eta_1) \rho (\eta_2))$$

(2.2)

where $\Omega^M$ is the curvature of the connection $A_M$ and

$$\langle \eta_1, \eta_2 \rangle_\rho = \frac{1}{8\pi^2} \text{Tr} (\rho (\eta_1) \rho (\eta_2))$$

(2.3)

is a bilinear Ad-invariant form on the Lie algebra $\mathfrak{g} = \text{Lie}(G)$. Those familiar with standard Chern-Simons will note that the 4-dimensional integral is equal the action of that theory at the “level” determined by $\frac{1}{2}\langle \cdot, \cdot \rangle$. 

3
We summarize other properties in the following theorems whose proofs can be found throughout section (1) of [14]. First we state how the action depends on the smooth parameters.

**Theorem 2.1.** Assume that \( \dim \rho = 0 \) and that \( \langle \cdot, \cdot \rangle_\rho \) is a non-degenerate pairing. Then \( \frac{1}{2}(D_X \otimes \rho A) \) is independent of the metric and invariant under gauge transformations. With respect to the connection \( A \), the critical points are exactly the flat connections. In other words, \( d\frac{1}{2}(D_X \otimes \rho A) = 0 \) if and only if \( \Omega^A = 0 \).

Though we must choose a metric to define the action, the choice is ultimately irrelevant and the physical theory does not depend on it. Also, we see that the classical fields, i.e. the critical points of the action, are flat connections and that the action descends to the quotient \( C_G(X)/G \). In these three respects, spin Chern-Simons is the same as standard Chern-Simons which is implicitly independent of the metric and whose gauge invariant action also has flat connections as its critical points.

The next two theorems, in which we spell out the action’s dependence on the discrete parameters, each require some preemptive explanation.

One (of the two) discrete parameters is the spin structure \( \sigma \) on \( X \). As is well known, the equivalence classes of spin structures are affine over \( H^1(X; \mathbb{Z}/2\mathbb{Z}) \). Thus to track how the action changes with respect to the shift \( \sigma \rightarrow \sigma + \ell \) for some \( \ell \in H^1(X; \mathbb{Z}/2\mathbb{Z}) \) we consider the ratio \( q_{\sigma}(\rho P, \ell) = \frac{\tau^{1/2}(D_{\sigma+\ell} \otimes \rho A)}{\tau^{1/2}(D_\sigma \otimes \rho A)} \).

**Theorem 2.2.** If \( \sigma, \ell, \) and \( \rho \) are as above and \( w_1(\rho P) = 0 \) then

\[
q_{\sigma}(\rho P, \ell) = (-1)^w_2(\rho P) \cdot \ell
\]

Thus a change in spin structure could cause the action to change by a sign.

The other parameter is the virtual representation \( \rho \in RSO(G) \). In standard Chern-Simons the analogue to \( \rho \) is an Ad-invariant bilinear form \( \langle \cdot, \cdot \rangle \) on \( g \). Furthermore, in that theory, it is well known that the action’s dependence on \( \langle \cdot, \cdot \rangle \) factors through the Chern-Weil map

\[
\text{Sym}^2 g^* \rightarrow H^4(BG; \mathbb{R}).
\]

Similarly, in spin Chern-Simons, we have the following.

**Theorem 2.3.** The action’s dependence on \( \rho \) factors through a homomorphism

\[
\lambda_G : RSO(G) \rightarrow E^4(BG)
\]

where \( E^\bullet(\cdot) \) is a generalized cohomology.

The generalized cohomology in question has a corresponding spectrum of spaces \( \{E^j\} \) such that \( E^j(\cdot) = [\cdot, E^j] \). In particular, the space \( E^4 \) is (up to
homotopy) the bottom two rungs of the Postnikov tower for $BSO$. We will not say more here about $E^4(BG)$ in general but defer to the appendix in [14].

However, because this paper focuses on the cases $G = SU_2$ and $G = SO_3$, we do consider the following examples. Let $\rho_0 : SU_2 \to SO_4$ denote the realization of the standard action on $\mathbb{C}^2 = \mathbb{R}^4$ and let $id_{SO_3} : SO_3 \to SO_3$ denote the standard action on $\mathbb{R}^3$. With some foresight we define

$$1' = \lambda_{SU_2}(\rho_0 - 4) \quad \text{and} \quad 1 = \lambda_{SO_3}(id_{SO_3} - 3).$$

Then it turns out that

$$E^4(BSU_2) = \mathbb{Z} \cdot 1' \quad \text{and} \quad E^4(BSO_3) = \mathbb{Z} \cdot 1,$$

and that the covering homomorphism $SU_2 \to SO_3$ induces the homomorphism

$$E^4(BSO_3) \to E^4(BSU_2) \quad \quad n \cdot 1 \mapsto 2n \cdot 1'. \quad (2.4)$$

This fact plays a crucial role in determining the formal properties of the quantum partition function and in relating the Hamiltonian theories for $SU_2$ and $SO_3$.

### 2.2 Classical Hamiltonian theory

Here $Y$ is a closed, compact, spun 2-manifold. The main object of interest in the Hamiltonian field theory is the prequantum line bundle over the classical phase space. To define it, we place a Riemannian structure on $Y$ and so obtain a chiral Dirac operator

$$D_Y : \Gamma(S_Y^+) \to \Gamma(S_Y^-).$$

Much as before, to any pair $(P, A) \in \mathcal{C}_G(Y)$ we associate a twisted chiral Dirac operator

$$D_Y \otimes \rho A : \Gamma(S_Y^+ \otimes \rho P) \to \Gamma(S_Y^- \otimes \rho P).$$

This operator is elliptic and skew-symmetric so that we can define the assignment

$$A \mapsto \text{Pfaff}^{-1}(D_Y \otimes \rho A) = \bigwedge^{\text{top}} \ker(D_Y \otimes \rho A) \quad (2.5)$$

where $\text{Pfaff}^{-1}(D)$ is the inverse Pfaffian line associated to a skew-symmetric operator $D$. If we fix a $G$-bundle $P \to Y$, then these lines fit together to form a smooth line bundle

$$\mathcal{L}^\rho(P) \to \mathcal{C}(P)$$

where $\mathcal{C}(P)$ denotes the (affine) space of connections on $P$. In fact, this line bundle has a natural hermitian structure and compatible connection [10].

We mention two important aspects of this inverse Pfaffian line bundle. The first is in regards to the holonomy associated to the natural connection. If
\( \gamma : S^1 \to \mathcal{C}(P) \) is a closed path of \( G \)-connections over \( Y \), it induces a \( G \)-connection \( A_\gamma \) over the 3-manifold \( S^1 \times Y \). Then the holonomy around \( \gamma \) is given by \( \text{hol}_\gamma = \tau^{1/2}(D_{S^1 \times Y} \otimes \rho A_\gamma) \). (2.6)

The second aspect is in regards to how \( G \)-connection automorphisms lift to the bundle. If \( \phi \) is a gauge transformation that preserves the \( G \)-connection \( A \), then \( \phi \) induces an automorphism of the line Pfaff\(^{-1}(D_Y \otimes \rho A) \). To compute that automorphism, we note that \( \phi \) and \( A \) induce another \( G \)-connection \( A_\phi \) over \( S^1 \times Y \). Then the induced automorphism is given by \( \text{aut}_\phi = \tau^{1/2}(D_{S^1 \times Y} \otimes \rho A_\phi) \). (2.7)

In both cases the spin structure on \( S^1 \times Y \) is the product spin structure induced by the given spin structure on \( Y \) and the bounding spin structure on \( S^1 \). This is the spin structure that extends to the disc.

We now address the fact that we had to choose a metric on \( Y \) to define the inverse Pfaffian bundle. The line bundle is independent of the metric on \( Y \) in the sense that, given a different metric, the two line bundles are canonically isomorphic. This follows immediately from the metric independence of the action and \( \text{hol}_\gamma \). Thus, in this sense, the Hamiltonian theory for spin Chern-Simons is independent of the metric much like it implicitly is for standard Chern-Simons.

Just as it is on compact 3-manifolds, the space of classical solutions is still the category of flat \( G \)-connections over \( Y \). Also, we still consider two \( G \)-connections to be equivalent if they lie in the same \( \mathcal{G}_G(Y) \)-orbit. Given \( \text{hol}_\gamma \) it is clear that any gauge transformation between two objects of \( \mathcal{C}_G(Y) \) induces a natural isomorphism between their corresponding inverse Pfaffian lines. Altogether this gives us the prequantum line bundle

\[ \mathcal{L}^\rho(Y) \to \mathcal{M}_G(Y) \]

where \( \mathcal{M}_G(Y) \) is the moduli stack of flat \( G \)-bundles over \( Y \). The claim is that, over the flat \( G \)-connections, the inverse Pfaffian bundle described above, along with all of its geometry, descends to the quotient under gauge transformations. This is proven in \([14]\). Here we have denoted the descendant line bundle by \( \mathcal{L}^\rho(Y) \). As mentioned above, the prequantum line bundle over \( \mathcal{M}_G(Y) \) is the main object of interest in the classical Hamiltonian theory. It also plays a large role in the quantum theory, as we see in section \( \ref{5} \).

Before we end this review of classical spin Chern-Simons we point out one final property of the prequantum line bundle. The fibers of the bundle \( \mathcal{L}^\rho(Y) \) each have a natural \( \mathbb{Z}/2\mathbb{Z} \) grading given by

\[ |\mathcal{L}^\rho(Y)_A| = \dim \ker(D_Y \otimes \rho A) \pmod{2}. \]

\(^1\)Generally, the holonomy is given by taking an adiabatic limit over metrics on \( S^1 \). In this case, metric independence eliminates the need to take this limit.
In other words, the grading is given by the mod-2 index of the skew-adjoint operator \( D_Y \otimes \rho A \). As this is a topological invariant the grading is locally constant over the moduli stack. This grading plays an important role in the quantum theory.

This ends our review of classical spin Chern-Simons.

3 The BHMV TQFT and BM spin TQFT

To better understand the correspondence between quantum spin Chern-Simons and the BM spin TQFT we first review the correspondence between the BHMV TQFTs and quantum Chern-Simons. We also review the relevant aspects of the BM spin TQFT. See [8] and [2] respectively for details of the constructions and more details regarding the results.

3.1 The BHMV TQFT versus the Witten TQFT

The BHMV TQFTs are constructed using combinatorial-topological techniques and categorical machinery in conjunction with the knot-theoretic Kauffman bracket. The result is a family of functors \( V_p : \mathcal{B} \to \mathcal{V} \) indexed by positive integers \( p \in \mathbb{Z}^>0 \). The objects of the domain category \( \mathcal{B} \) are closed, oriented 2-manifolds and its morphisms are 3-manifolds with boundary. Thus if \( Y_j, j = 1, 2 \), are two objects of \( \mathcal{B} \) and \( X \) is a 3-manifold such that \( \partial X = -Y_1 \cup Y_2 \), then \( X \) is a morphism \( Y_1 \to Y_2 \). The objects of the codomain category \( \mathcal{V} \) are finite dimensional, complex vector spaces and its morphisms are complex linear maps. Thus, in the example above, \( V_p(X) \) is a morphism \( V_p(Y_1) \to V_p(Y_2) \).

To the empty 2-manifold the functors assign \( V_p(\emptyset) = \mathbb{C} \). If, for example, \( \partial X = Y \) then \( X \) is a morphism \( \emptyset \to Y \), so that \( V_p(X) \) is a morphism \( \mathbb{C} \to V_p(Y) \); or what is the same, \( V_p(X) \) is a vector in \( V_p(Y) \). If, instead, \( \partial X = \emptyset \) then \( X \) is a morphism \( \emptyset \to \emptyset \) so that \( V_p(X) \) is a morphism \( \mathbb{C} \to \mathbb{C} \); or what is the same, \( V_p(X) \) is an element of \( \mathbb{C} \). For any object \( Y \) of \( \mathcal{B} \), the vector space \( V_p(Y) \) has a non-degenerate hermitian inner product \( \langle , \rangle_Y \) such that if \( \partial X_1 = \partial X_2 = Y \), then

\[
\langle V_p(X_1), V_p(X_2) \rangle_Y = V_p(X_1 \cup_Y (-X_2)) \in \mathbb{C}.
\]

On top of that, there are natural isomorphisms \( V_p(-Y) \to V_p(Y) \) and \( V_p(Y_1) \otimes V(Y_2) \to V_p(Y_1 \sqcup Y_2) \). Thus the functors \( V_p \) satisfy the axioms of a 2-dimensional topological quantum field theory [5].

As is well known, physicists believe that classical \( SU_2 \) Chern-Simons, in conjunction with the Feynmann path integral, defines a similar family of functors \( Z_k : \mathcal{B} \to \mathcal{V} \) [21]. The indexing set consists of positive elements \( k \in \mathbb{Z}^>0 \).

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\( ^2 \)The domain category considered in [8] is really that of 2-manifolds with \( p_1 \) structure, which is a central extension of the category \( \mathcal{B} \). For simplicity we demur the issue of \( p_1 \) structures.

\( ^3 \)The codomain category really considered in [8] are modules over an abstract cyclotomic field \( k_p \). We have taken the liberty of choosing a particular extension of \( k_p \) to \( \mathbb{C} \), one for which the 3-manifold invariants of the Witten and BHMV theories agree.
The formal properties of the Feynmann path integral offer an easy “proof” that the functors \( Z_k \) each define a 2-dimensional TQFT. This is assuming, of course, the one accepts the path integral. Taking this for granted, we can make the following observation.

**Observation 3.1.** If one compares the functors \( Z_k \) and \( V_p \) then, whenever \( p = 2(k + 2) \), one sees that

- For any object \( Y \) of \( \mathcal{B} \) the hermitian vector spaces \( V_{2(k+2)}(Y) \) and \( Z_k(Y) \) have the same dimension. In particular, if \( Y \) is a genus \( g \) 2-manifold then

\[
\dim V_{2(k+2)}(Y) = \dim Z_k(Y) = \left( \frac{k + 2}{2} \right)^{g-1} \sum_{j=1}^{k+1} \left( \sin \frac{\pi j}{k + 2} \right)^{2-2g},
\]

which is the famous Verlinde formula [20].

- For many examples of closed, oriented 3-manifolds \( X \), it has been shown that \( V_{2(k+2)}(X) = Z_k(X) \) (cf. [11], [21], and [15]).

- So that we can point out one last feature, we must first recall a fact about the algebraic topology of closed compact 2-manifolds. For any object \( Y \) of \( \mathcal{B} \) there is a central extension of \( H^1(Y;\mathbb{Z}/2\mathbb{Z}) \) by \( \mathbb{Z}/4\mathbb{Z} \); or what is the same, there exists a long exact sequence

\[
0 \to \mathbb{Z}/4\mathbb{Z} \to \Gamma(Y) \to H^1(Y;\mathbb{Z}/2\mathbb{Z}) \to 0.
\]

\( \Gamma(Y) \) is a quotient of the usual Heisenberg group \( H(Y) \) which itself fits into the short exact sequence

\[
0 \to \mathbb{Z} \to H(Y) \to H^1(Y;\mathbb{Z}) \to 0.
\]

The last feature we wish to point out it that the vector spaces \( V_{2(k+2)}(Y) \) and \( Z_k(Y) \) each support a natural \( \Gamma(Y) \) action and are equivalent as representations of \( \Gamma(Y) \) [11].

We make some remarks about this last observation as it has bearing on the \( SO_3 \) spin-Chern-Simons theory. At the positive \( SU_2 \) levels \( k \) the vector spaces \( V_{2(k+2)}(Y) \) and \( Z_k(Y) \) decompose as representations of \( \Gamma(Y) \). In the cases \( k \equiv 0 \pmod{4} \) there is a natural one-one correspondence between irreducible components and \( \mathbb{Z}/2\mathbb{Z} \)-bundles on \( Y \); and in the cases \( k \equiv 2 \pmod{4} \) there is a natural one-one correspondence between irreducible components and spin structures on \( Y \). The latter correspondence will appear again when we consider the spin TQFTs of Blanchet and Masbaum in the next subsection and then yet again in the quantum spin-Chern-Simons theory for \( SO_3 \). In fact, we believe that the former and latter correspondences are part of a larger \( SO_3 \) gauge theory whose consideration we reserve for another time. In the cases \( k \equiv 1 \) or \( 3 \pmod{4} \) the vector spaces decompose but the irreducible components do not correspond to any topological structures.
As a clue to why the 0 (mod 2)-valued $SU_2$ levels should have anything to do with an $SO_3$ spin-Chern-Simons theory, we recall that levels of the later are elements of $E^4(BO_3)$. At the end of section 2.1 we stated that $E^4(BO_3) = \mathbb{Z} \cdot 1$ and that $E^4(BSU_2) = \mathbb{Z} \cdot 1'$. Furthermore we stated that the 2 : 1 covering map $\beta : SU_2 \to SO_3$ induces a homomorphism

$$\beta^* : E^4(BO_3) \longrightarrow E^4(BSU_2)$$

$$1 \longmapsto 2 \cdot 1'$$

Thus, the 0 (mod 2)-valued $SU_2$ levels correspond to $SO_3$ levels. We will have much to say about this is the sections to follow. For now we collate all of these correspondences in the following table:

| Corresponding Representations of $\Gamma(Y)$ |
|---------------------------------------------|
| BHMV level (mod 8) | 0 | 4 | 2 | 6 |
| $SU_2$ level (mod 4) | 2 | 0 | 1 | 3 |
| $SO_3$ level (mod 2) | 1 | 0 | 1 | 1 |
| Topological Structure | spin structure | $\mathbb{Z}/2\mathbb{Z}$-bundle |

Before moving on, we mention that our considerations throughout the rest of this paper focus on the corresponding representations in first column; i.e. those representations that correspond to spin structures on the 2-manifold.

### 3.2 The BM spin TQFT

The BM spin TQFTs are constructed using the same combinatorial-topological techniques and categorical machinery as was used in constructing the BHMV TQFTs; but they are used in conjunction with a knot-theoretic invariant that is sensitive to spin structures. The result is a family of functors $V^s_p : \mathcal{B}^s \to \mathcal{V}^s$ indexed by positive 0 (mod 8)-valued integers $p \in 8\mathbb{Z}^>0$. The objects of the domain category $\mathcal{B}^s$ are closed, spin 2-manifolds and its morphisms are spin 3-manifolds with boundary. 4 Thus if $(Y_j, \sigma_j), \ j = 1, 2$, are two objects of $\mathcal{B}^s$ and $(X, \Sigma)$ is a spin 3-manifold such that $\partial(X, \Sigma) = \{ -Y_1 \sqcup Y_2, -\sigma_1 \sqcup \sigma_2 \}$, then $(X, \Sigma)$ is a morphism $(Y_1, \sigma_1) \to (Y_2, \sigma_2)$. The objects of the codomain category $\mathcal{V}^s$ are finite dimensional, $\mathbb{Z}/2\mathbb{Z}$-graded complex vector spaces and its morphisms are complex linear maps that preserve the grading. 5

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4 Again, the domain category considered in [9] is really that of spin 2-manifolds with $p_1$ structure; or what is the same, 2-manifolds with string structure.

5 Again, the codomain category really considered in [9] is that of $\mathbb{Z}/2\mathbb{Z}$-graded modules over an abstract cyclotomic field $k_p$. We take the same liberties of we took for the BHMV TQFT in extending $k_p$ to $\mathbb{C}$. 9
All the axioms satisfied by the BHMV TQFTs are also satisfied by the BM spin TQFTs but with one caveat. For any object \((Y, \sigma)\) of \(\mathbb{B}^s\) we can write

\[ V_p^s(Y, \sigma) = V_{p,0}^s(Y, \sigma) \oplus V_{p,1}^s(Y, \sigma) \]

where \(V_{p,0}^s(Y, \sigma), V_{p,1}^s(Y, \sigma)\) are respectively the even and odd components of \(V_p^s(Y, \sigma)\). Then the caveat is that there are natural isomorphisms

\[ V_{p,0}^s(Y_1 \sqcup Y_2, \sigma_1 \sqcup \sigma_2) \rightarrow V_{p,0}^s(Y_1, \sigma_1) \otimes V_{p,0}^s(Y_2, \sigma_2) \oplus V_{p,1}^s(Y_1, \sigma_1) \otimes V_{p,1}^s(Y_2, \sigma_2) \]  

and

\[ V_{p,0}^s(Y_1 \sqcup Y_2, \sigma_1 \sqcup \sigma_2) \rightarrow V_{p,0}^s(Y_1, \sigma_1) \otimes V_{p,1}^s(Y_2, \sigma_2) \oplus V_{p,1}^s(Y_1, \sigma_1) \otimes V_{p,0}^s(Y_2, \sigma_2). \]

This is expressed more succinctly in terms of the graded tensor product “\(\hat{\otimes}\)”. Indeed, in the language of \(\mathbb{Z}/2\mathbb{Z}\)-graded vector spaces, (3.3) and (3.4) amount to saying that there is a natural isomorphism

\[ V_p^s(Y_1 \sqcup Y_2, \sigma_1 \sqcup \sigma_2) \rightarrow V_p^s(Y_1, \sigma_1) \hat{\otimes} V_p^s(Y_2, \sigma_2). \]

We enumerate some of the more interesting features of the BM spin TQFTs:

1. For a closed 3-manifold \(X\) the BM invariants \(V_p^s(X, \sigma)\) are refinements of the BHMV invariant \(V_p^s(X)\) in the sense that \(\sum_{\sigma} V_p^s(X, \sigma) = V_p^s(X)\).

2a. For a closed, genus \(g\), spin 2-manifold \((Y, \sigma)\)

\[
\dim V_{p,0}^s(Y, \sigma) = \frac{1}{2^{2g}} (\dim V_p(Y) + \left(\frac{p}{4}\right)^{g-1}((-1)^{\epsilon(\sigma)}2^g - 1))
\]

(3.5)

where \(\epsilon(\sigma)\) is the Arf invariant of the spin structure.

2b. For the same spin 2-manifold

\[
\dim V_{p,1}^s(Y, \sigma) = \frac{1}{2^{2g}} (\dim V_p'(Y) - \left(\frac{p}{4}\right)^{g-1}((-1)^{\epsilon(\sigma)}2^g - 1))
\]

(3.6)

where \(V_p'(Y)\) is a particular vector space associated to \(Y\) by the BHMV TQFT (see Remark 5.11 of [8]). At any rate its dimension is given by

\[
\dim V_p'(Y) = \left(\frac{p}{4}\right)^{g-1} \frac{p}{2^g} \sum_{j=1}^{\frac{p}{2}+1} (-1)^{j+1} \left(\sin \frac{2\pi j}{p}\right)^{2-2g}.
\]

(3.7)

3. There exists a canonical isomorphism

\[ V_p(Y) \rightarrow \bigoplus_{\sigma} V_{p,0}^s(Y, \sigma). \]
3.3 A conjecture and the Main Theorem

The points of Observation 3.1 amount to evidence toward a conjecture; the conjecture being that the quantum $SU_2$ Chern-Simons theory is, in fact, a TQFT and that the $SU_2$ TQFT at level $k$ is isomorphic to the BHMV TQFT at level $p = 2(k + 2)$. That this is only a conjecture comes from the fact that the Feynmann path integral is, at this point, not a well-defined mathematical object. In fact, it is very likely a conjecture that will not be proven any time soon. Despite that, the evidence offered in Observation 3.1 does make for a rather convincing empirical argument.

If the conjecture is to be believed it seems that the knot-theoretic BHMV TQFTs have a gauge-theoretic correspondence in the quantum $SU_2$ Chern-Simons theory. Upon discovering the knot-theoretic BM spin TQFTs one is naturally lead to ask if they have their own gauge-theoretic correspondence. This is the question that motivates our investigation of the quantum $SO_3$ spin-Chern-Simons theory. Indeed, we offer the following conjecture.

**Conjecture.** The quantum $SO_3$ spin-Chern-Simons theory is, in fact, a TQFT and at the $1 \mod 2$-valued $SO_3$ level $2m - 1$, it is isomorphic to the BHMV TQFT at level $p = 8m$.

Physicists would like to believe that the quantum partition function and canonical quantization can generate a spin TQFT from the classical theory described above. While it is (at this point) impossible to rigorously do so we can, nonetheless, exploit the features of this quantization map to compare the conjectured spin TQFT and the BM spin TQFT. In particular, we appeal to the formal properties of the quantum partition function and the procedure of geometric quantization.

Now consider spin Chern-Simons, as defined above, for a compact group $G$ and level $k$. Let $Z_G(X, \sigma, k)$ denote the quantum partition function associated to a closed spin 3-manifold $(X, \sigma)$, which we consider only formally. And let $H_G(Y, \sigma, k)$ denote the Hilbert space associated to a closed spin 2-manifold $(Y, \sigma)$ via geometric quantization. As evidence toward the conjecture above we provide the following theorem.

**Main Theorem.** For odd-valued $SO_3$-levels $k = 2m - 1$ we have the following:

1. $Z_{SO_3, 2m-1}(X^3, \sigma)$ has the same formal properties as $V^s_{8m}(X^3, \sigma)$ in the sense that
   $$Z_{SU_2, 4m-2} = \sum_{\sigma} Z_{SO_3, 2m-1}(X^3, \sigma).$$

2. $H_{SO_3, 2m-1}(Y^2, \sigma)$ is $\mathbb{Z}/2\mathbb{Z}$ graded. That is,
   $$H_{SO_3, 2m-1}(Y^2, \sigma) = H^0_{SO_3, 2m-1}(Y^2, \sigma) \oplus H^1_{SO_3, 2m-1}(Y^2, \sigma).$$

3. $\dim H^0_{SO_3, 2m-1}(Y^2, \sigma) = \dim V^s_{8m, 0}(Y^2, \sigma)$ and
   $$\dim H^1_{SO_3, k}(Y^2, \sigma) = \dim V^s_{8m, 1}(Y^2, \sigma).$$

4. $H_{SU_2, 4m-2}(Y^2) = \bigoplus_{\sigma} H^0_{SO_3, 2m-1}(Y^2, \sigma)$. 

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4 Formal Properties of the Three-Manifold Invariants

4.1 Spin Chern-Simons versus Chern-Simons

In this short section we show that if we consider the quantum partition function as a formal object then we can show that the spin 3-manifold invariants for the $SO_3$ spin-Chern-Simons theory at $SO_3$ level $k = 2m - 1$ are refinements of the 3-manifold invariants for the Chern-Simons theory at $SU_2$ level $k' = 4m - 2$.

To begin we prove the following proposition.

**Proposition 4.1.** Let $\rho$ be a real rank zero virtual representation of a connected, simply connected, compact Lie group $G$ and let $\langle , \rangle_\rho$ denote the symmetric pairing defined by (2.3). If $A$ is a $G$-connection over a closed, spin 3-manifold $X$, we let $\exp 2\pi i S_X(A)$ denote the $T$-valued Chern-Simons invariant determined by the pairing $\frac{1}{2}\langle , \rangle_\rho$, as defined in [10]. Then

$$\tau^{1/2}_X(D_\rho A) = \exp 2\pi i S_X(A).$$

**Proof.** The proof relies on the fact that the cobordism group $\Omega^{\text{spin}}_3(BG) = 0$ whenever $G$ is compact and simply connected. In that case there exists a spin 4-manifold $M$ such that $\partial M = X$ as a spin manifold and there exists an extension $A'$ of $A$ over $M$. On the one hand it is well-know that

$$\exp 2\pi i S_X(\rho A) = \exp 2\pi i \int_M \frac{1}{2}\langle \Omega^{A'}, \Omega^{A'} \rangle_\rho.$$

On the other hand, since $\rho$ has rank zero the APS index theorem implies

$$\tau^{1/2}_X(D_\rho A) = \exp \pi i \int_M \langle \Omega^{A'}, \Omega^{A'} \rangle_\rho,$$

and this proves the proposition. \qed

To make the equality of these theories even stronger we point out the correspondence between their respective levels. Recall that for Chern-Simons the levels are elements of $H^4(BG)$ while for spin-Chern-Simons the level are elements of $E^4(BG)$. However, for $G$ simply connected there is a natural isomorphism $i : H^4(BG) \to E^4(BG)$ so that Chern-Simons theory at level $\alpha \in H^4(BG)$ is isomorphic to spin Chern-Simons at level $i(\alpha) \in E^4(BG)$. In particular, $SU_2$ Chern-Simons at $SU_2$ level $k \in H^4(BSU_2) \cong \mathbb{Z}$ is isomorphic to $SU_2$ spin-Chern-Simons at level $k \in E^4(BSU_2) \cong \mathbb{Z}$.

4.2 $SU_2$ and $SO_3$-connections

Let $X$ be a closed spin 3-manifold. The 2nd Steifel-Whitney class provides a one-one correspondence between isomorphism classes of $SO_3$ bundles on $X$ and elements of $H^2(X; \mathbb{Z}/2\mathbb{Z})$. Recall that $CG(X)$ denotes the quotient space
of all $G$-connections on $X$ with respect to the gauge group $G_G(X)$. Then the components of the quotient space $C_{SO_3}(X)$ are indexed by elements of $H^2(X; \mathbb{Z}/2\mathbb{Z})$ and we let $\tilde{C}_{SO_3,0}(X)$ denote the category of $SO_3$-connections $(P, A)$ such that $w_2(P) = b$.

All $SU_2$ bundles over $X$ have a section so that the quotient space $C_{SU_2}(X)$ has only one component. The standard $2:1$ covering homomorphism induces a functor $\beta: C_{SU_2}(X) \to C_{SO_3,0}(X)$. A simple argument shows that if two $SU_2$-connections map to the same $SO_3$-connection then they differ by a unique $\mathbb{Z}/2\mathbb{Z}$ bundle (up to a global choice of sign); and if two $SU_2$-connections differ by a $\mathbb{Z}/2\mathbb{Z}$ bundle then they map to the same $SO_3$-connection. Thus $C_{SO_3,0}(X) = C_{SU_2}(X)/C_{SO_3,0}(X)$.

From the discussion above it would seem that, at least formally, $\int_{C_{SU_2}(X)} \mu_{SU_2}(X) = (4.1)$ $\int_{C_{SO_3,0}(X)} \mu_{SO_3,0}(X) \cdot \mu_{\mathbb{Z}/2\mathbb{Z}}(X)$

where $\mu(G)$ is meant to be some sort of “measure” on $C_G(X)$. We point out that, while integrating over $C_{SU_2}(X)$ or $C_{SO_3}(X)$ does not make sense mathematically, integrating over the finite fibers of $C_{SU_2}(X) \to C_{SO_3,0}(X)$ involves a finite sum that does make sense.

### 4.3 Some formal manipulations

As an example of a formal manipulation using $\beta$, imagine that $f: \tilde{C}_{SO_3}(X) \to \mathbb{C}$ is a measurable function. Then, according to (4.1)

$$\int_{C_{SU_2}(X)} f \circ \beta \mu_{SU_2}(X) = (4.2)$$

$$\int_{C_{SO_3,0}(X)} \int_{C_{SU_2}(X)/C_{SO_3,0}(X)} f \mu_{SO_3,0}(X) \cdot \mu_{\mathbb{Z}/2\mathbb{Z}}(X) = \#(H^1(X; \mathbb{Z}/2\mathbb{Z})) \int_{C_{SO_3,0}(X)} f \mu_{SO_3,0}(X)$$

This is exactly the situation we encounter below.

On the one hand we consider $SO_3$ spin Chern-Simons at level $2m - 1$. The action is given by

$$A \mapsto f_\sigma(A) = \tau_{X,\sigma}^{1/2}(D_\rho A)^{2m-1}.$$ 

where $\rho = \text{id} - 3$ is the standard representation minus the 3-dimensional trivial representation and $\sigma$ is the spin structure on $X$. On the other hand we consider $SU_2$ Chern-Simons at level $4m - 2$ in which case a simple argument shows the action is given by

$$A' \mapsto f'(A') = f_\sigma \circ \beta(A').$$

(4.3)
We consider the formal integrals
\[ Z_{SO_3}(X, \sigma, 2m - 1) = \int_{C_{SO_3}(X)} f_\sigma \mu_{SO_3}(X) \quad \text{and} \]
\[ Z_{SU_2}(X, 4m - 2) = \int_{C_{SU_2}(X)} f' \mu_{SU_2}(X), \]
and put forth the following proposition.

**Proposition 4.2.** The formal spin 3-manifold invariants \( Z_{SO_3}(X, \sigma, 2m - 1) \) are refinements of the formal 3-manifold invariant \( Z_{SU_2}(X, 4m - 2) \) in the sense that
\[ Z_{SU_2}(X, 4m - 2) = \sum_{\sigma} Z_{SO_3}(X, \sigma, 2m - 1). \]

**Proof.** We begin with a more down-to-earth computation. Let \((P, A)\) be an \( SO_3 \)-connection such that \( w_2(P) = b \). Then we consider the sum
\[ \sum_{\sigma} \tau^{1/2}_{X,\sigma}(D_\rho A)^{2m-1} = \sum_{\ell} \tau^{1/2}_{X,\sigma+\ell}(D_\rho A)^{2m-1} = \left( \sum_{\ell} (-1)^{b-\ell} \right) \tau^{1/2}_{X,\sigma}(D_\rho A)^{2m-1}. \]
The second and third sums are over \( \ell \in H^1(X; \mathbb{Z}/2\mathbb{Z}) \). The second equality follows from Theorem 2.2. It is easy to see that the third sum is equal to zero if \( b \neq 0 \) and is equal to \( \# H^1(X; \mathbb{Z}/2\mathbb{Z}) \) if \( b = 0 \). The upshot of this computation is that, in summing over the spin structures, the \( SO_3 \)-connections with non-trivial \( w_2 \) contribute nothing and all non-zero contributions come from the \( SO_3 \)-connections with trivial \( w_2 \). In particular,
\[ \sum_{\sigma} Z_{SO_3}(X, \sigma, 2m - 1) = \quad (4.4) \]
\[ \#(H^1(X; \mathbb{Z}/2\mathbb{Z})) \int_{C_{SO_3,0}(X)} f_\sigma \mu_{SO_3,0}(X). \]
Notice that over \( C_{SO_3,0}(X) \) the function \( f_\sigma \) is independent of \( \sigma \). Now if we consider
\[ Z_{SU_2}(X, 4m - 2) = \int_{C_{SU_2}(X)} f' \mu_{SU_2}(X) \]
then (4.2), (4.3), and (4.4) imply the proposition. \( \square \)

5 The Quantum Hamiltonian Theory

5.1 Geometric quantization with Kähler polarization

In this section we review the procedure of geometric quantization when the symplectic manifold has a Kähler structure. Let \((M, \omega)\) be a smooth complex
manifold with a positive Kähler form and let \( \mathcal{L} \to M \) be a hermitian line bundle with a unitary connection \( \nabla \). Furthermore, we require that \( \nabla \circ \nabla = \Omega = -2\pi i \omega \). To conform with the literature (e.g. [22], [12]) we call \( \mathcal{L} \) the “pre-quantum line bundle”. Notice that, since \( \omega \) represents \( c_1(\mathcal{L}) \) in deRham cohomology, it takes integer values when integrated over smooth 2-cycles. This is already a “quantization” condition of sorts.

The connection \( \nabla \) determines a unique holomorphic structure \( \bar{\partial}_L \) on the pre-quantum line bundle. This follows from the fact that \( \Omega \) is a \((1,1)\) differential form in the bi-grading determined by the complex structure on \( M \). Thus, one way to obtain a Hilbert space from this system would be to take the holomorphic sections \( \mathcal{H}' = H^0(M; \mathcal{L}) \). The natural hermitian form for \( \mathcal{H} \) is

\[
\langle s_1, s_2 \rangle_\mathcal{H} = \int_M \langle s_1, s_2 \rangle_L \cdot \omega^{\dim M}.
\]

However, certain examples (cf. [12]) tell us that this is often not the “right” Hilbert space. A slight correction must be made and we explain this next.

In the most favorable cases, we can include the \textit{metaplectic correction} in the geometric quantization procedure. Let \( K = \bigwedge^{\text{top}}(T^{(1,0)}M)^* \) denote the canonical line bundle of \( M \). When the real tangent bundle of \( M \) has trivial 2nd Stiefel-Whitney class – or what is the same, the holomorphic tangent bundle has 1st Chern class that is trivial under mod-2 reduction – there exists a holomorphic line bundle \( K^{1/2} \to M \) such that \( (K^{1/2})^\otimes 2 = K \). In fact, given the existence of a \( K^{1/2} \), the set of equivalence classes of such line bundles is affine over \( H^1(M, \mathbb{Z}/2\mathbb{Z}) \). In the literature \( K^{1/2} \) is called the “bundle of half-forms”. In many cases, the “right” Hilbert space is \( \mathcal{H} = H^0(M; \mathcal{L} \otimes K^{1/2}) \).

On an affine symplectic vector space, for instance, the inclusion of the metaplectic correction is critical to identifying Hilbert spaces associated to different affine Kähler structures. More recently, [13], it has been discovered that in certain instances the metaplectic correction is required to identify the respective Hilbert spaces associated with a Kähler structure and a \textit{real polarization}. We feel that these cases provide strong motivation for including the correction, whenever possible, in our quantization procedure.

### 5.2 Geometric structures on the moduli stack

Let \( Y \) be a closed, 2-manifold with spin structure \( \sigma \) and let \( G \) be a compact Lie group. For simplicity we further require that \( G \) be connected. The moduli stack \( \mathcal{M}_G(Y) \) is, of course, independent any Riemannian or spin structure on \( Y \) but given such structures we obtain certain geometric structures on the moduli space. We discuss this next.

Fix a Riemannian structure on \( Y \) and let \( Q \to Y \) be a principal \( G \) bundle. Any Riemannian 2-manifold has a unique complex structure determined by its Hodge star operator. It is trivially integrable. According to a classical theorem of Narasimhan and Seshadri [17] (see also [2]), a complex structure on \( Y \) induces a complex structure on the moduli space \( \mathcal{M}(Q) \) of flat connections in the
following way. We let $G_C$ denote the complexification of $G$ and we let $Q_C$ denote the natural extension of $Q$ to a $G_C$ bundle. There is an identification between $\mathcal{M}(Q)$ and the moduli space $\mathcal{M}(Q_C) = \mathcal{C}(Q_C)^\ast / \mathcal{G}(Q_C)$ of semi-stable holomorphic structures on $Q_C$. The space $\mathcal{M}(Q_C)$ has a natural complex structure. Indeed, over a smooth point $B$, the holomorphic tangent space is modeled on $H^1(Y; \bar{\partial}_B)$, the first cohomology group of the complex

$$0 \to \Omega^0(\text{ad}Q_C) \xrightarrow{\bar{\partial}_B} \Omega^{0,1}(\text{ad}Q_C) \to 0. \quad (5.1)$$

Here the superscript $(0, 1)$ implies that these are the anti-holomorphic 1-forms with respect to the complex structure on $Y$ and $\bar{\partial}_B$ is the unique holomorphic structure on $Q_C$ determined by the flat connection $B$.

Recall from 2.2 that, given a real representation $\rho$ of rank zero, there is a hermitian line bundle $\mathcal{L}^\rho(Q) \to \mathcal{M}(Q)$ with a unitary connection $\nabla$. If $\Omega$ is the curvature of $\nabla$ then over smooth points of the moduli space, $\omega = i\Omega/2\pi$ defines a symplectic structure. With respect to the complex structure on $\mathcal{M}(Q)$ the symplectic form $\omega$ has bigrading $(1, 1)$. Thus $\mathcal{L}^\rho(Q)$ is a holomorphic line bundle and we are in the scenario described in the previous section.

As mentioned above, to obtain the “right” Hilbert space we require the canonical line bundle $K \to \mathcal{M}(Q)$ and then a square root of that bundle. We discuss this next. For the rest of this section we assume the genus of $Y$ is greater than one.

The zeroth cohomology $H^0(Y; \bar{\partial}_B)$ of the complex 5.1 is the Lie algebra of the stabilizing subgroup of $\bar{\partial}_B$. If $B$ represents a smooth point then the stabilizing subgroup consists of the center of $G$ so that $H^0(Y; \bar{\partial}_B) = 0$. Thus, at a smooth point $B$ we have the equivalence of lines

$$\text{Det}^{-1}_{\bar{\partial}_B} = \text{Det}H^0(Y; \bar{\partial}_B) \otimes \text{Det}^{-1}H^1(Y; \bar{\partial}_B) = \text{Det}^{-1}H^1(Y; \bar{\partial}_B). \quad (5.2)$$

Given that $H^1(Y; \bar{\partial}_B) = T^{1,0}\mathcal{M}(Q)$ we see that the far right hand side of (5.2) is the fiber of the canonical bundle $K$ at $B$. If we let $\text{Det}_{Q_C}^{-1} \to \mathcal{C}(Q_C)$ denote the inverse determinant line bundle whose fiber at $\bar{\partial}_B$ is the left hand side of (5.2), then $\text{Det}_{Q_C}^{-1}$ descends to $\mathcal{M}(Q)$. Over smooth points the descendant line bundle $\text{Det}_{Q_C}^{-1} \to \mathcal{M}(Q)$ is equivalent to the canonical bundle $K \to \mathcal{M}(Q)$. The inverse determinant line bundle definition of $\text{Det}_{Q_C}^{-1}$ is rather convenient in the setting of the Hamiltonian spin Chern-Simons theory with its Pfaffian line bundles. For this reason with work with $\text{Det}_{Q_C}^{-1}$ instead of $K$.

As a guide to finding a square root of $\text{Det}_{Q_C}^{-1}$ we point out that we may write

$$\bar{\partial} = D \otimes K_Y^{-1/2} \quad (5.3)$$

where the left hand side is the usual Dolbeault operator on $Y$ and the right hand side is the Dirac operator twisted by the square root of the canonical line bundle. Actually, $K_Y^{-1/2}$ is the spinor bundle $S_Y^-$ transposed to the holomorphic setting. The same follows when the two operators are twisted by any vector bundle with connection; in particular it follows when they are twisted by $\text{ad}Q_C$. We offer the following proposition. For simplicity we now assume $G$ is connected.
**Proposition 5.1.** Let $\rho_{ad} = ad - \dim G$ be the adjoint representation of $G$ minus the trivial representation of rank $\dim G$. Then there exists a connection preserving isometry between the two inverse determinant lines $\text{Det}_{Qc}^{-1} \to \mathcal{M}(Q)$ and $(\mathcal{L}^{\rho_{ad}}(Q))^{\otimes 2} \to \mathcal{M}(Q)$.

**Remark 5.2.** As will be clear from the proof, the above proposition applies more generally. Indeed, $\rho_{ad} Q$ may be replaced with any real oriented vector bundle and $K^{-1/2}_Y$ may be replaced with any fixed complex line bundle on $Y$. The upshot of this theorem is that we may model the canonical line bundle of $\mathcal{M}(Q)$ on the line bundle $(\mathcal{L}^{\rho_{ad}}(Q))^{\otimes 2}$ and so we may model the bundle of half-forms on the line bundle $(\mathcal{L}^{\rho_{ad}}(Q))$. Notice that a choice of square root for the canonical bundle $K_Y$ determines a choice of square root for the canonical bundle $K$ of the moduli space.

**Proof.** As the statement of the proposition suggests, we only prove existence and do not construct an isomorphism. To prove existence we show that the line bundle $L = \text{Det}_{Qc}^{-1} \otimes (\mathcal{L}^{\rho_{ad}}(Q))^{\otimes 2}$ has trivial holonomy. Then there exists a covariantly constant unitary section which is unique up to a factor in $T$.

Now we show that the holonomy around any closed path in $\mathcal{M}(Q)$ is the identity. From we know that the holonomy around any loop (in the case of inverse determinant lines) is given by the $\tau$-invariant of some twisted Dirac operator over $S^1 \times Y$ where the bounding spin structure is placed on $S^1$. What we show now is that

$$\tau_{S^1 \times Y}(D_E \otimes (K^{-1/2}_Y - 1)) = 1 \quad (5.4)$$

for any real oriented virtual vector bundle $E \to S^1 \times Y$ with orthogonal connection $\nabla^E$. We assume that the metric $S^1 \times Y$ is product and furthermore that the metric on $S^1$ is flat. This caveat is part of the hypothesis of the theorem which expresses the holonomy in terms of (adiabatic limits of) $\tau$-invariants.

We first claim that the right hand side of (5.4) in independent of $\nabla^E$. Indeed, the formula for the differential (of the log) of $\tau$ is given by the Atiyah-Patodi-Singer index theorem [3], and a straightforward computation of the local index shows that the differential is zero. From this we see that the right hand side of (5.4) only depends on the topological type of $E$. Because we are working over a 3-manifold, the topological type of $E$ is completely determined by $w_2(E)$. (See the appendix of [14] for a proof). The Kunneth formula implies

$$H^2(S^1 \times Y; \mathbb{Z}/2\mathbb{Z}) \cong H^2(Y; \mathbb{Z}/2\mathbb{Z}) \oplus H^1(S^1; \mathbb{Z}/2\mathbb{Z}) \otimes \mathbb{Z}/2\mathbb{Z} H^1(Y; \mathbb{Z}/2\mathbb{Z}).$$

Since both $w_2$ and $\tau$ depend linearly on $E$ we can assume, without loss of generality, that $w_2(E)$ lies in one of the above summands. Then we can show that, in each case, the right hand side of (5.4) is 1. First assume that $w_2(E) \in H^2(Y; \mathbb{Z}/2\mathbb{Z})$. In this case $E$ extends to a bundle $E' \to D^2 \times Y$ so that the
APS index theorem implies
\[
\tau_{S^1 \times Y}(D_E \otimes (K_Y^{-1/2} - 1)) = \\
\exp 2\pi i \left[ \int_{D^2 \times Y} \hat{A}(\Omega^{D^2 \times Y}) ch(\Omega^E) ch(K^{-1/2} - 1) \right]_{(0)}
\]
and another straightforward computation shows that the integral is zero. Now let \( \ell \) denote the non-trivial element of \( H^1(S^1; \mathbb{Z}/2\mathbb{Z}) \) and assume that \( w_2(E) = \ell \sim a \) for some \( a \in H^1(Y; \mathbb{Z}/2\mathbb{Z}) \). As we did in Section 1.1, identify \( a \) and \( \ell \) with the flat real line bundles that represent them. Then we can assume that \( E = a \oplus \ell \oplus \ell \oplus a \) so that
\[
\tau_{S^1 \times Y}(D_E \otimes (K_Y^{-1/2} - 1)) = \\
\tau_{S^1 \times Y}(D_a \otimes (K_Y^{-1/2} - 1)) \cdot \tau_{S^1 \times Y}(D_\ell \otimes (K_Y^{-1/2} - 1)) \cdot \tau_{S^1 \times Y}(D_{\ell \oplus a} \otimes (K_Y^{-1/2} - 1)).
\]
The first factor involves only vector bundles that can be extended over \( D^2 \times Y \) and can be shown to be equal to 1 using the same argument as in the case \( w_2(E) \in H^2(Y; \mathbb{Z}/2\mathbb{Z}) \). To deal with the last two factors we note that twisting by \( \ell \) is equivalent to placing the non-bounding spin structure on \( S^1 \), so that
\[
\tau_{S^1 \times Y}(D_E \otimes (K_Y^{-1/2} - 1)) = (-1)^{\text{index } D \otimes (K_Y^{-1/2} - 1)}(-1)^{\text{index } D_a \otimes (K_Y^{-1/2} - 1)}.
\]
Of course the index cannot detect the factor \( a \) so that the two indices above are equal and so cancel each other out (mod 2). Finally we see that the right hand side of (5.4) is 1 in this case, as well. \( \square \)

5.3 The moduli spaces for \( G = SU_2, \ G = SO_3 \)

We review some salient features of the moduli spaces \( M_{SU_2}(Y) \) and \( M_{SO_3}(Y) \). In particular, we restrict ourselves to the case of connected oriented 2-manifolds of genus 2 or higher.

For any connected group \( G \), the topological equivalence classes of principal \( G \)-bundles over \( Y \) are parametrized by the abelian group \( H^2(Y; \pi_1 G) \cong \pi_1 G \). Indeed, if we remove a disc \( D \) from \( Y \), then \( Y \setminus D \) is homotopic to its 1-skeleton over which any \( G \)-bundle is trivial. Thus, up to isomorphism, any \( G \)-bundle on \( Y \) is determined by the homotopy class of the clutching map \( g : \partial D \to G \) that glues the trivial bundle over \( Y \setminus D \) to the trivial bundle over \( D \). The elements of \( \pi_1 G \) are in one-one correspondence with the connected components of \( M_G(Y) \) since each topological type determines a component of the moduli stack. In particular, \( M_{SU_2}(Y) \) has one component while \( M_{SO_3}(Y) \) has two components: one for the each of the two possible 2nd Stiefel-Whitney classes. We denote these components by \( M_{SO_3, w_2}(Y) \) for \( w_2 = 0, 1 \).

For any compact Lie group \( G \) each connected component of the moduli space decomposes into a disjoint union of strata. The stratum of highest dimension is a smooth manifold and is a dense open subset of the component with respect to to quotient topology. For \( SU_2 \) the top stratum consists of the
irreducible flat connections. To explain what we mean by “irreducible” we remind the reader that a flat $G$-connection on $Y$ determines a representation $\pi_1 Y \to G$. It is this representation that is irreducible. The other strata of the $SU_2$ moduli space consist of reducible flat connections, in particular those that reduce to flat $T$ and $Z/2\mathbb{Z}$ connections. Recall that any maximal torus of $SU_2$ is isomorphic to $T$ and the center of $SU_2$ is isomorphic to $Z/2\mathbb{Z}$; this explains the appearance of $SU_2$-connections that reduce to these Lie groups.

For $M_{SO_3,0}(Y)$ and $M_{SO_3,1}(Y)$ moduli spaces the top strata also consist of irreducible flat connections. The other strata consist of reducible flat connections, in particular those that reduce to flat $O_2$, $SO_2$, $Z/2\mathbb{Z}$ and $Z/2\mathbb{Z} \times Z/2\mathbb{Z}$ connections. Recall that $SO_2$ embeds into $SO_3$ as rotations about a fixed axis and $O_2$ embeds as rotations about a fixed axis as well as such rotations composed with a $180^\circ$ rotation about some perpendicular axis. The group $Z/2\mathbb{Z}$ embeds as a subgroup of $SO_2$ and $Z/2\mathbb{Z} \times Z/2\mathbb{Z}$ embeds as compositions of $180^\circ$ rotations about three mutually orthogonal axes. This explains the appearance of $SO_3$-connections that reduce to these Lie groups. A trivial $SO_3$ bundle supports connections that reduce to each of these subgroups, however a non-trivial $SO_3$ bundle does not support connections that reduce to flat $SO_2$ or $Z/2\mathbb{Z}$ connections.

The complex structure of the $SU_2$ moduli space comes by way of extending the $SU_2$ bundles to $(SU_2)_C = SL_2(\mathbb{C})$ bundles and then identifying the moduli space of flat connections with the moduli space of semi-stable holomorphic structures. Similarly, we extend the $SO_3$ bundles to $(SO_3)_C = PL_2(\mathbb{C})$ bundles and use the identification of flat connections with holomorphic structures.

5.4 Flat $SO_3$ and Yang-Mills $U_2$ connections

We now address the apparent equality of dimensions for the top strata of the $SU_2$ and $SO_3$ moduli spaces. To do so we have to extend our consideration to Yang-Mills connections on principal $U_2$ bundles over $Y$ of which flat $SU_2$ connections are a subset. These connections were the main object of study in [2] and much of what we have to say about them comes from that reference.

Consider the short exact sequence of compact Lie groups

$$1 \to T \to U_2 \to PU_2 \to 1.$$  \hspace{1cm} (5.5)

The first map is multiplication by the $2 \times 2$ identity matrix and maps $T$ into the center $C \subset U_2$. The second map is projectivization. To connect to $SO_3$ we point out that $PU_2 \cong SO_3$. Indeed, the adjoint action of $U_2$ on the Lie algebra preserves the decomposition $u_2 = \mathfrak{c} \oplus \mathfrak{su}_2$; it acts trivially on $\mathfrak{c}$ and the non-trivial adjoint action on $\mathfrak{su}_2$ factors through $PU_2$. A choice of identification $\mathfrak{su}_2 \to \mathbb{R}^3$ induces a bijection $PU_2 \to SO_3$.

If $A$ is a $G$-connection over the oriented Riemannian 2-manifold $Y$, then it is Yang-Mills if $d_A \ast \Omega^4 = 0$, where $\ast$ denotes the Hodge star operator of the metric on $Y$. This implies that $\text{Tr}(i\Omega^4/2\pi)$ is the unique harmonic form that represents the first Chern class in deRham cohomology. The gauge group $G_G(Y)$ acts on the space of Yang-Mills connections and we can therefore talk
about the corresponding moduli space. The moduli space of Yang-Mills $U_2$
connections is a well-studied space, and we use the identification $PU_2 = SO_3$
to our advantage in quantizing our spin-Chern-Simons theory. We explain how
next.

Consider the short exact sequence of compact Lie groups

$$1 \to \mathbb{Z}/2\mathbb{Z} \to U_2 \to \mathbb{T} \times PU_2 \to 1. \quad (5.6)$$

The first map is multiplication of $\pm 1$ by the $2 \times 2$ identity matrix and the
second map is the cartesian product of the maps $\det : U_2 \to \mathbb{T}$ and $\mathbb{P} : U_2 \to \mathbb{P}U_2$. From this short exact sequence we see that a Yang-Mills $U_2$
connection $(P, A)$ is determined by the Yang-Mills connection $(\det P, \det A)$ and the flat
connection $(\mathbb{P}P, \mathbb{P}A)$ up to a $\mathbb{Z}/2\mathbb{Z}$-connection. To be precise, the category
$\mathcal{C}_{\mathbb{Z}/2\mathbb{Z}}(Y)$ of $\mathbb{Z}/2\mathbb{Z}$-connections acts on $\mathcal{C}_{U_2}(Y)$ by “tensor product”. Indeed,
for $Z \in \mathcal{C}_{\mathbb{Z}/2\mathbb{Z}}(Y)$ and $(P, A) \in \mathcal{C}_{U_2}(Y)$ we obtain a new $U_2$-connection:

$$(Z \otimes P, Z \otimes A) = (Z \times_Y (P, A))/\mathbb{Z}/2\mathbb{Z}$$

where the quotient is taken with respect to the diagonal $\mathbb{Z}/2\mathbb{Z}$ action on the
fiber product. This is the principal bundle analogue of tensoring a flat real
orthogonal line bundle with a hermitian rank 2 vector bundle, which justifies our
“tensor product” notation. We will use this notation throughout the rest of
this section. The important fact to observe is that tensoring $A$ with a $\mathbb{Z}/2\mathbb{Z}$-
connection preserves $\det A$ and $\mathbb{P}A$. This is what we mean when we say that
$\det A$ and $\mathbb{P}A$ determine $A$ up to a $\mathbb{Z}/2\mathbb{Z}$-connection.

If we fix a Yang-Mills $\mathbb{T}$-connection $(T, a)$ we can consider the category
$\mathcal{C}_{U_2}(T, a)$ of $U_2$ connections $(P, A)$ such that $(\det P, \det A) = (T, a)$. The
morphisms $\mathcal{G}_{U_2}(T, a)$ of this category are those elements $\phi$ of $\mathcal{G}_{U_2}(Y)$ such
that $\det \phi = \text{id}_T$; and so we may consider the moduli space $\mathcal{M}_{U_2}(T, a) =
\mathcal{C}_{U_2}(T, a)/\mathcal{G}_{U_2}(T, a)$. Now an element $A \in \mathcal{C}_{U_2}(T, a)$ is determined by $\mathbb{P}A$
up to a $\mathbb{Z}/2\mathbb{Z}$-connection. We will use this fact to identify the moduli space
$\mathcal{M}_{SO_3, w_2}(Y)$ as the quotient of some $\mathcal{M}_{U_2}(T, a)$ with respect to $\mathcal{M}_{\mathbb{Z}/2\mathbb{Z}}(Y)$. To
discern the correct choice of $(T, a)$ we point out that for a $U_2$ principal bundle
$P \to Y$, $w_2(\mathbb{P}P) = c_1(P) (\text{mod } 2)$ and that $c_1(P) = c_1(\det P)$. Thus we see
that if $(T, a)$ has even (resp. odd) degree then $\mathcal{M}_{SO_3, 0}(Y)$ (resp. $\mathcal{M}_{SO_3, 1}(Y)$)
is identified with $\mathcal{M}_{U_2}(T, a)/\mathcal{M}_{\mathbb{Z}/2\mathbb{Z}}(Y)$. In particular, if we fix $(T, a)$ to be
the trivial line bundle with trivial connection, then it is clear that $\mathcal{M}_{U_2}(T, a) =
\mathcal{M}_{SU_2}(Y)$ and we can identify the quotient of the latter with $\mathcal{M}_{SO_3, 0}(Y)$ as
well.

5.5 The lift of the $\mathcal{M}_{\mathbb{Z}/2\mathbb{Z}}(Y)$ action

Let $Y$ be a closed, oriented, genus-$g$ 2-manifold with fixed metric and spin
structure $\sigma$, and let $\beta : U_2 \to SO_3$ be the projective homomorphism composed
with some isomorphism $\mathbb{P}U_2 \to SO_3$. We also fix a real, rank zero virtual repre-
sentation $\rho$ of $SO_3$ and a degree-1 Yang-Mills $\mathbb{T}$-connection $(T, a)$. Throughout
this section we use the following shorthand. We let $\mathcal{C}_{\beta}(Y)$ stand in for either
\[C_{SU_2}(Y) \lor C_{U_2}(T,a); \text{ in particular } d = 0 \text{ for the category of } SU_2 \text{ connections and } d = 1 \text{ for the category of } U_2 \text{ connections whose determinants are equal to } (T,a). \text{ We let } C_{w_2}(Y) \text{ stand in for } C_{SO_3,w_2}(Y). \text{ We use a corresponding shorthand for the } SO_3 \text{ and } U_2 \text{ moduli stacks. To avoid typographical redundancy and emphasize its dependence of the spin structure, we drop the "Y" from the notation for the Pfaffian lines, writing } L^{(\rho)}(Y) \text{ for } L^{(\rho)}(Y) \text{ and } L^{\rho} \text{ for } L^{\rho}(Y). \]

In this section we take advantage of known results for the vector spaces \(H^0(M_d(Y); L^{\rho})\), \(d = 0,1\). In particular, the dimensions of these spaces are the well-known Verlinde formulae; the vector space for \(d = 1\) (and its corresponding Verlinde formula) is often referred to as the twisted case. We will use these previousy obtained Verlinde formulae to obtain formulae for the dimensions of \(H^0(M_{w_2}(Y); L^{\rho}(\sigma))\) for \(w_2 = d\). The formulas we obtain are not new and correspond to the formulas in [9] and [1]. The novelty here is our approach to the computation. Our vector spaces are derived from line bundles over the \(SO_3\) moduli stacks while the [1] computations are done in the context of \(U_2\) moduli stacks. In [1] the authors consider a central extension of \(M_{Z/2Z}(Y)\) and its action on \(H^0(M_d(Y); L^{\rho})\). They derive the vector spaces as the irreducible components of this action. See Section 3.1 or the cited reference for more details.

Our main concern here is the action of \(M_{Z/2Z}(Y) = H^1(Y;Z/2Z)\) on \(M_d(Y)\). This action really follows from an action of \(C_{Z/2Z}(Y)\) on \(C_d(Y)\) given by the tensor product action described above:

\[
C_{Z/2Z}(Y) \times C_d(Y) \longrightarrow C_d(Y)
(Z, (P,A)) \longmapsto (Z \otimes P, Z \otimes A)
\]

This categorical action descends to an action on the moduli stacks

\[
M_{Z/2Z}(Y) \times M_d(Y) \longrightarrow M_d(Y)
([Z], [P,A]) \longmapsto [(Z \otimes P, Z \otimes A)]
\]

In what follows we will make use of the identification

\[
M_d(Y)/M_{Z/2Z}(Y) \cong M_{w_2}(Y).
\]

Before we do anything more we characterize the fixed points of \([Z]\) in \(M_d(Y)\) by their images in \(M_{w_2}(Y)\). In particular we have the following proposition.

**Proposition 5.3 (Characterization of the Fixed Points).** The equivalence class \([P,A]\) is a fixed point of \([Z]\) if and only if \((\beta P, \beta A)\) can be reduced to an \(O_2\)-connection \((Q,B)\) such that \(\text{Det}Q \cong Z\).

Here \(\text{Det}Q\) denotes the orientation bundle of \(Q\) with its natural flat connection. This characterization is extremely useful in the forthcoming computation of \(\dim H^0(M_{w_2}(Y); L^{\rho}(\sigma))\).

**Proof.** We first prove sufficiency. Let \(Q \subset \beta P\) be an \(O_2\) subbundle such that \(\beta A\) preserves \(Q\) under parallel transport. We need to construct a morphism of pairs

\[
\Phi: (\text{Det}Q \otimes P, \text{Det}Q \otimes A) \longrightarrow (P,A).
\]
Let $R$ denote the rotation

$$
R = \begin{pmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{pmatrix} \in O_2 \subset SO_3
$$

and let $\tilde{R} \in SU_2$ be chosen so that $\beta(\tilde{R}) = R$. If $q, \beta p$ lie in the same fiber $Q_x$ over $x \in Y$ then we set

$$
\Phi(Det q \otimes p) = p \cdot \tilde{R} \cdot \frac{Det q}{Det \beta p}
$$

where

$$
\frac{Det q}{Det \beta p} \in \{\pm 1\}
$$

is the translation that carries $Det \beta p$ to $Det q$. We let the reader check that this is a well-defined morphism over for all such $q, p$. Equivariance now defines $\Phi$ at all other elements of $DetQ \otimes P$. Clearly $\Phi$ descends to an automorphism of $(\beta P, \beta A)$ so that it must send $DetQ \otimes A$ to $A$. This proves sufficiency.

To prove necessity we assume that there exists a morphism

$$
\Phi : (Z \otimes P, Z \otimes A) \rightarrow (P, A)
$$

If the stabilizing subgroup of $(P, A)$ is isomorphic to $U_1$ or $Z/2Z$ we can always chose $\Phi$ so that

$$
(id_Z \otimes \Phi) \circ \Phi = -id_P.
$$

If the stabilizing subgroup is isomorphic to $U_2$ then their is no non-trivial $Z$ that stabilizes $(P, A)$. Thus, in the non-trivial case, we can always chose $\Phi$ so that it descends, via $\beta$, to an automorphism

$$
\phi : (\beta P, \beta A) \rightarrow (\beta P, \beta A)
$$

such that $\phi \circ \phi = id_{\beta P}$. If $\phi = id_{\beta P}$ then its clear that $Z$ is trivializable. Assuming $\phi \neq id_{\beta P}$ it determines an $O_2$ sub-bundle

$$
Q = \{q \in \beta P | \phi(q) = q \cdot R\}
$$

which inherits a connection $B$ from $\beta A$ (since $\phi$ preserves $\beta A$). Our final task is to construct an isomorphism between $Z$ and $DetQ$. From the sufficiency argument we know that $\phi$ can be lifted to a morphism

$$
\Phi' : (DetQ \otimes P, DetQ \otimes A) \rightarrow (P, A)
$$

For $Det q \in DetQ_x$ and $z \in Z_x$ we define the map

$$
Det q \mapsto z \cdot \frac{\Phi'(Det q \otimes p)}{\Phi(z \otimes p)}
$$
where
\[
\frac{\Phi'(\text{Det} q \otimes p)}{\Phi(z \otimes p)} \in \{\pm 1\} \subset U_2
\]
is the translation that takes \( \Phi'(\text{Det} q \otimes p) \in P_x \) to \( \Phi(z \otimes p) \in P_x \). We let the reader check that the map is well-defined despite the choice of \( z \). This completes the proof.

We would like to see if and then how this lifts to the line bundles \( L^{\rho \alpha \beta} \). As before, to see the action we start on the categorical level. Indeed the natural isometry
\[
L^{\rho \alpha \beta}(P, A) \longrightarrow L^{\rho \alpha \beta}(Z \otimes P, Z \otimes A)
\]
is almost trivial thanks to the natural identification
\[
\beta(Z \otimes P) \longrightarrow \beta(P)
\]
\[
\beta(z \otimes p) \longrightarrow \beta(p)
\]
This naturally gives us the commutative diagram
\[
\begin{array}{ccc}
L^{\rho \alpha \beta} & \longrightarrow & L^\rho \\
\downarrow & & \downarrow \\
C_d(Y) & \longrightarrow & C_{w_2}(Y).
\end{array}
\]

**Remark 5.4.** This lift has a couple of nice properties. First of all, the lift induces a "group" action of \( C_{Z/2Z}(Y) \) on \( L^{\rho \alpha \beta} \). Thus \( L^{\rho \alpha \beta} \) is a \( C_{Z/2Z}(Y) \) equivariant line-bundle. Indeed, this is what is implied by (5.8). Second of all, the lift is covariant with respect to the natural connection on the bundles and commutes with the action of the bundle morphisms (after natural identifications). Thus the group action of \( C_{Z/2Z}(Y) \) not only lifts to the Pfaffian line bundle, it does so covariantly with respect to the connection over the moduli stack. Thus have the following commutative diagram:
\[
\begin{array}{ccc}
L^{\rho \alpha \beta} & \longrightarrow & L^\rho \\
\downarrow & & \downarrow \\
\mathcal{M}_d(Y) & \longrightarrow & \mathcal{M}_{w_2}(Y).
\end{array}
\]

Now we need only compute the actions over fixed points to know exactly how the actions lift over the whole bundle. With that in mind, we wish to see how this action lifts when we have a fixed point \( [Z \otimes P, Z \otimes A] = [P, A] \) on \( \mathcal{M}_d(Y) \). In this case there must be a morphism
\[
\Phi : (Z \otimes P, Z \otimes A) \longrightarrow (P, A)
\]
which, under the natural identification \( \beta(Z \otimes P) = \beta P \) descends to an automorphism
\[
\phi : (\beta(P), \beta(A)) \longrightarrow (\beta(P), \beta(A)).
\]
Notice that, under the natural identification $Z \otimes Z \otimes P = P$ we can compose $\Phi$ with the morphism

$$id_Z \otimes \Phi : (Z \otimes Z \otimes P, Z \otimes Z \otimes A) \mapsto (Z \otimes P, Z \otimes A)$$

so that we get an automorphism

$$(id_Z \otimes \Phi) \circ \Phi : (P, A) \mapsto (P, A)$$

which, via $\beta$, descends to the automorphism $\phi \circ \phi$.

We are finally in a good position to compute the action of $[Z] \in M_{Z/2Z}(Y)$ on a fixed point. The result is the following

**Proposition 5.5 (The Lift to the Pfaffian Line).** The lift of $[Z]$ to the line $\mathcal{L}^{\rho, \beta}[A]$ over a fixed point $[A]$ is multiplication by

$$q(\sigma, \rho(\beta P), Z) = (-1)^{w_2(\rho \circ \beta P) + w_2(\rho) (\text{ind}_2(D_\sigma + Z) - \text{ind}_2(D_\sigma))}$$

where $w_2(\rho \circ \beta P) \in \mathbb{Z}/2\mathbb{Z}$ denotes the invariant $w_2(\rho \circ \beta P) \sim [Y]$ and $w_2(\rho) \in \mathbb{Z}/2\mathbb{Z}$ is obtained from the map $w_2 : RO(SO_3) \to H^2(BSO_3; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.

**Proof.** By the proposition 5.3 we have that there is a sub-bundle $(Q, B) \subset (\beta P, \beta A)$ such that $\text{Det} Q \cong Z$. We also have that $\phi$ – the descendent (via $\beta$) of the morphism

$$\Phi : (Z \otimes P, Z \otimes A) \mapsto (P, A)$$

– restricts to $(-id_Q)$ on $Q$. The action on the line is multiplication by the $\tau^{1/2}$-invariant of

$$(Q, B) \times (-id_Q) S^1_b,$$

i.e. the bundle with connection gotten gluing the ends of $(Q, B) \times [0, 1]$ together with $\phi|Q = -id_Q$. Notice that in doing so we give $S^1$ the bounding spin structure. Recall that this is what is required to compute the trace (as opposed to the super-trace). Then, letting

$$i : O_2 \hookrightarrow SO_3$$

denote the standard inclusion homomorphism, we want to compute

$$\tau^{1/2}(\rho \circ i(Q, B) \times (-id_Q) S^1_b) = \tau^{1/2}(\rho \circ i(Q, B) \times \text{id}_Q S^1_{bh}) = (-1)^{\text{ind}_2(D_\sigma \otimes \rho \circ i(Q))}$$

Note that the mod-2 index is independent of $B$ (as it is a topological invariant). All that remains is to compute

$$\text{ind}_2(D_\sigma \otimes \rho \circ i(Q)) = w_2 \rho \circ i(Q) + w_2(\rho) (\text{ind}_2(D_\sigma + \text{Det} Q) - \text{ind}_2(D_\sigma)) = w_2(\rho(\beta P) + w_2(\rho) (\text{ind}_2(D_\sigma + Z) - \text{ind}_2(D_\sigma))$$

where the first equality follows from the $KO$-theoretic decomposition of $\rho \circ i(Q)$ and the second equality follows from $\text{Det} Q \cong Z$. \qed
Of course, this is not the only way in which one can lift the action of $[Z]$ to $\mathcal{L}^{\rho \beta}$. Indeed, we can also take the (rather natural) lift described above and multiply it by the scalar factor $q(\sigma, \rho \circ \beta(P), Z)$. This particular action, according to the previous theorem, lifts to a trivial action over the fixed points. The first action, which is more natural in the context of our spin Chern-Simons field theory, we denote by $[Z]_{CS,\sigma}$ while the second we denote by $[Z]_B$ in honor of A. Beauville who computed the trace of this particular action on the vector space $H^0(M_d(Y); \mathcal{L}^{\rho \beta})$ [6]. This trace was also independently computed by J. Andersen and G. Masbaum [1] and T. Pantev [18] in the non-trivial $SO_3$-bundle case.

**Theorem 5.6 (A. Beauville).** $\text{Tr}[Z]_B = (\lambda(\rho) + 1)^g - 1$ where $\lambda(\rho) \in \mathbb{Z}$ is obtained from the map $\lambda: RO(SO_3) \to E^4(BSO_3) \cong \mathbb{Z}$.

This computation makes use of the Lefschetz-Riemann-Roch fixed point formula [4] and algebro-geometric results characterizing the fixed points as certain abelian varieties [16]. In the case $w_2 \neq 0$ these are the Prym varieties, and in the case $w_2 = 0$ these are the Kummer varieties. The singularity of the fixed points in the case $w_2 = 0$ is dealt with by “transferring” the computation to the $w_2 \neq 0$ moduli space via the Hecke correspondence. For details we refer the reader to Beauville’s paper [6].

An easy corollary is then

**Corollary 5.7.** $\text{Tr}[Z]_{CS,\sigma} = q(\sigma, \rho(\beta P), Z)(\lambda(\rho) + 1)^g - 1$

We are finally in a good position to compute the dimension of $H^0(M_{w_2}(Y); \mathcal{L}^\sigma)$. Indeed, the commutative diagram (5.9) tells us that there is a one-one correspondence between $H^0(M_{w_2}(Y); \mathcal{L}^\sigma)$ and the $\mathcal{M}_{Z/2Z}(Y)$ invariant subspace of $H^0(M_d(Y); \mathcal{L}^{\rho \beta})$. The dimension of the latter is just the trace of the projection

$$P_\sigma = \frac{1}{2g^2} \sum_{[Z]} [Z]_{CS,\sigma}$$

where, unless there is notation to indicate otherwise, the sum is over all $[Z] \in \mathcal{M}_{Z/2Z}(Y)$. Based on this we have

**Proposition 5.8.** If $Y$ is a genus-$g$ 2-manifold then

$$\dim H^0(M_{w_2}(Y); \mathcal{L}^\sigma) = \frac{1}{2g^2}(\dim H^0(M_d(Y); \mathcal{L}^{\rho \beta}) + (-1)^{w_2(\rho \circ \beta P)}((-1)^{\epsilon(\sigma)}2^g - 1)(\lambda(\rho) + 1)^g - 1)$$

where $\epsilon(\sigma)$ is the Arf-invariant of $\sigma$.

**Remark 5.9.** The Arf-invariant has an index-theoretic formulation that is perhaps more appropriate considering the bent of our approach. Indeed, $\epsilon(\sigma) = \text{ind}_2(D_\sigma)$; that is the Arf-invariant is just the mod-2 index of the (uncoupled) Dirac operator associated to $\sigma$. For future reference we recall the well known fact that on a genus-$g$ surface there are $(2^{g-1} + 2^{g-1})$ spin structures $\sigma$ for which $\epsilon(\sigma) = 0$ and $(2^{2g-1} - 2^{g-1})$ spin structures $\sigma'$ for which $\epsilon(\sigma') = 1$. 

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Proof. From the discussion above we see that the dimension of $H^0(M_{w_2}(Y); \mathcal{L}^\rho)$ is given by the formula

$$\text{Tr}(\frac{1}{2^{2g}} \sum [Z]_{CS,\sigma}) = \frac{1}{2^{2g}} \sum [Z] \text{Tr}([Z]_{CS,\sigma})$$

For $[Z] \neq 0$ we have that

$$\text{Tr}[Z]_{CS,\sigma} = q(\sigma, \rho(\beta P), Z)(\lambda(\rho) + 1)^{g-1}$$

and, of course, if $[Z] = 0$ we have

$$\text{Tr}[Z]_{CS,\sigma} = \dim H^0(M_d(Y); \mathcal{L}^{\rho \circ \beta}).$$

Combining this with the formula above we have that the dimension of $H^0(M_{w_2}(Y); \mathcal{L}^\rho)$ is given by

$$\frac{1}{2^{2g}} \left( \dim H^0(M_d(Y); \mathcal{L}^{\rho \circ \beta}) + (\lambda(\rho) + 1)^{g-1} \sum_{[Z] \neq 0} q(\sigma, \rho(\beta P), Z) \right) \quad (5.12)$$

From what was said in the remark proceeding this proof we know that

$$\sum_{[Z]} q(\sigma, \rho(\beta P), Z) = (-1)^{w_2(\rho) + 1} \sum_{[Z]} (-1)^{\epsilon(\sigma + Z) - \epsilon(\sigma)}$$

$$= (-1)^{w_2(\rho \circ \beta P) + \epsilon(\sigma)} \sum_{\sigma'} (-1)^{\epsilon(\sigma')}$$

$$= (-1)^{w_2(\rho \circ \beta P) + \epsilon(\sigma)} ((2^{2g-1} + 2^{g-1} - (2^{2g-1} - 2^{g-1}))$$

$$= (-1)^{w_2(\rho \circ \beta P) + \epsilon(\sigma)} 2^g$$

so that

$$\sum_{[Z] \neq 0} q(\sigma, \rho(\beta P), Z) = (-1)^{w_2(\rho \circ \beta P)}((-1)^{\epsilon(\sigma)}2^g - 1).$$

Plugging this into (5.12) gives us the proposition. \qed

## 5.6 Application to spin Chern-Simons theory

In this section we use Proposition 5.8 to compute the dimensions of the Hilbert spaces of the spin Chern-Simons theory for closed, spin 2-manifolds of genera greater than one. The genus one case is considered in the following chapter. For the most part, our job is already done. However, some care must be taken to obtain the “correct” quantum theory. In particular this requires the proper choice of classical level. Indeed, to obtain the quantum theory which we conjecture to correspond with the spin-TQFT of Blanchet and Masbaum, we must consider only a certain subset of all possible levels. (Actually, as future work will show, there is a more comprehensive $SO_3$ theory that incorporates all possible levels, and the spin Chern-Simons we consider here is a subset of this theory.)
To discuss the levels consider the standard representation 3-dimensional representation \( \text{id}_{SO_3} \) and define

\[
1 = \lambda(\text{id}_{SO_3} - 3) \in E^4(BSO_3)
\]

then \( 1 \) generates the levels of \( SO_3 \) so that \( E^4(BSO_3) = \mathbb{Z} \cdot 1 \). We let the boldfaced \( k \) denote the level \( k \cdot 1 \) for any \( k \in \mathbb{Z} \). To obtain the spin-Chern-Simons that we want we consider only even valued levels. According to the prescription described in subsection 5.1 and the geometric equivalence determined by proposition 5.1, the Hilbert spaces we must consider is

\[
\mathcal{H}(Y, \sigma, k, w_2) = H^0(M_{w_2}(Y); \mathcal{L}^{k+1}(\sigma)). \tag{5.13}
\]

Having established the identity of our Hilbert space, the computation of its dimension is a trivial corollary of proposition 5.8. Recall from Section 2.2, that the Pfaffian line bundles \( \mathcal{L}^\rho(\sigma) \) are graded and that the grading is determined by the mod 2 index of the twisted Dirac operators. As this is a topological invariant the grading is obviously constant over connected components of the moduli stack. In particular the grading of the line bundles over \( M_{w_2}(Y) \) will depend on \( w_2 \) and the level. At level \( k + 1 \), \( k \) even, the line bundle has odd grading when \( w_2 = 1 \) and has even grading when \( w_2 = 0 \).

**Corollary 5.10.** Let \( Y \) be a closed, oriented genus-\( g \) 2-manifold with spin structure \( \sigma \). Then the dimension of the even Hilbert space of the quantum spin-Chern-Simons theory at level \( k \) is given by

\[
\dim \mathcal{H}(Y, \sigma, k, w_2 = 0) = \frac{1}{2g}(\dim H^0(M_{d=0}(Y); \mathcal{L}^{2k+2}) + (\varepsilon(\sigma)2^g - 1)(k + 2)^{g-1})
\]

and the dimension of the odd Hilbert space is given by

\[
\dim \mathcal{H}(Y, \sigma, k, w_2 = 1) = \frac{1}{2g}(\dim H^0(M_{d=1}(Y); \mathcal{L}^{2k+2}) - (\varepsilon(\sigma)2^g - 1)(k + 2)^{g-1}). \tag{5.15}
\]

This formula agrees with the one given in Theorem 19.1 in [9].

We consider, in particular, the even Hilbert spaces. The next proposition relates the Hilbert spaces of our \( SO_3 \) quantum theory to the Hilbert space of the well-known \( SU_2 \) quantum theory. To do so we must first say a few words about the \( SU_2 \) quantum theory. We start with the levels for the \( SU_2 \) theory.

To discuss the levels consider the realization of the standard \( \mathbb{C}^2 \) representation \( \rho : SU_2 \to SO(\mathbb{R}^4) \) and define

\[
1' = \lambda(\rho - 4) \in E^4(BSU_2)
\]

then \( 1' \) generates the levels of \( SU_2 \) so that \( E^4(BSU_2) = \mathbb{Z} \cdot 1' \). We let the boldfaced \( k' \) denote the level \( k \cdot 1' \) for any \( k \in \mathbb{Z} \). We point out that the
adjoint representation \( \text{ad} : SU_2 \to SO(\mathfrak{so}_3) \) represents the level \( \lambda(\text{ad}) = 2' \).

The homomorphism \( \beta : SU_2 \to SO_3 \) induces a homomorphism

\[
\beta^* : E^4(\text{BSO}_3) \longrightarrow E^4(\text{BSU}_2)
\]

\[
k \cdot 1 \longmapsto 2k \cdot 1'
\]

According to the prescription described in section 5.1 and the geometric equivalence determined by proposition 5.1, the Hilbert space we must consider is

\[
H'(Y, k') = H^0(M^\text{d}=0(Y); L^{k'+2'}(Y)).
\] (5.16)

It is easy to show that, over \( M^\text{d}=0(Y) \), \( L^{\rho \circ \beta} \) does not depend on the spin structure \( \sigma \) in the sense that for any two spin structures there exists a connection preserving isometry between the two corresponding line bundles which is unique up to a factor in \( T \). This is why \( \sigma \) does not appear in the denotation for the \( SU_2 \) Hilbert space.

The definitions (5.13) and (5.16) imply that we have inclusions

\[
i_\sigma : H(Y, \sigma, k, w_2 = 0) \hookrightarrow H'(Y, 2k')
\]

and that each these subspaces is the image of the (respective) projection

\[
P_\sigma = \frac{1}{2^g} \sum_{Z \in M_{Z/2Z}(Y)} [Z]_{CS, \sigma}.
\]

We offer the following proposition.

**Proposition 5.11.** Take any non-trivial element \( \ell \in H^1(Y; \mathbb{Z}/2\mathbb{Z}) \). Then

\[
P_{\sigma + \ell} \circ P_\sigma = 0.
\]

**Proof.** We first compute \( [Z]_{CS, \sigma + \ell} \) in terms of \( [Z]_{CS, \sigma} \). If we compose the former with the inverse of the latter we get a covariantly constant automorphism of \( L^{2k} \) which projects to the identity on \( M_{SU_2}(Y) \). This is just multiplication by some constant in \( T \) so that to compute it we only need to do so over a fixed point of \( [Z] \). Thus

\[
[Z]_{CS, \sigma + \ell} \circ [Z]_{CS, \sigma}^{-1} = q(\sigma + \ell, 0, Z) \cdot q(\sigma, 0, Z)^{-1}
\]

\[
= (-1)^{\text{ind}_2(D_{\sigma + \ell} + Z) - \text{ind}_2(D_{\sigma + \ell}) - \text{ind}_2(D_{\sigma + Z}) + \text{ind}_2(D_\sigma)}
\]

\[
= (-1)^{Z - \ell}.
\]

where the last equality follows from the fact that

\[
\text{ind}_2(D_{\sigma + Z}) - \text{ind}_2(D_\sigma)
\]
is quadratic with respect to $Z$ and the corresponding bilinear form on $H^1(Y; \mathbb{Z}/2\mathbb{Z})$ is the one determined by the cup product. Finally we compute

$$P_{\sigma + \ell} \circ P_\sigma = \sum_{Z'} \sum_Z [Z']_{CS,\sigma + \ell} \circ [Z]_{CS,\sigma}$$

$$= \sum_{Z'} \sum_Z (-1)^{Z' \cdot \ell} [Z']_{CS,\sigma} \circ [Z]_{CS,\sigma}$$

$$= \sum_{Z'} \sum_Z (-1)^{Z' \cdot \ell} [Z' + Z]_{CS,\sigma}$$

$$= \left( \sum_{Z'} (-1)^{Z' \cdot \ell} \right) P_\sigma$$

$$= 0$$

The upshot to this proposition is that the subspaces $\mathcal{H}(Y, \sigma, k, w_2 = 0) \subset \mathcal{H}'(Y, 2k')$ are disjoint for different spin structures. In fact, that the projections $P_\sigma$ are constructed out of isometries of the Hilbert space implies that the subspaces are orthogonal to each other. A straightforward dimension count shows that

$$\sum_\sigma \dim \mathcal{H}(Y, \sigma, k, w_2 = 0) = \dim \mathcal{H}'(Y, 2k').$$

We tie all of this together to conclude with the final proposition of this paper which is the Hamiltonian version of the Proposition 4.2.

**Proposition 5.12.** For $k$ even, we have the following orthogonal decomposition

$$\mathcal{H}'(Y, 2k') = \bigoplus_\sigma \mathcal{H}(Y, \sigma, k, w_2 = 0)$$

so that the Hilbert spaces for the $SO_3$ spin-Chern-Simons theory are refinements of the Hilbert space for the $SU_2$ theory.

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