TRAVELING WAVE SOLUTIONS TO THE MULTILAYER FREE BOUNDARY
INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

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Abstract. For a natural number $m \geq 2$, we study $m$ layers of finite depth, horizontally infinite, viscous, and incompressible fluid bounded below by a flat rigid bottom. Adjacent layers meet at free interface regions, and the top layer is bounded above by a free boundary as well. A uniform gravitational field, normal to the rigid bottom, acts on the fluid. We assume that the fluid mass densities are strictly decreasing from bottom to top and consider the cases with and without surface tension acting on the free surfaces. In addition to these gravity-capillary effects, we allow a force to act on the bulk and external stress tensors to act on the free interface regions. Both of these additional forces are posited to be in traveling wave form: time-independent when viewed in a coordinate system moving at a constant, nontrivial velocity parallel to the lower rigid boundary. Without surface tension in the case of two dimensional fluids and with all positive surface tensions in the higher dimensional cases, we prove that for each sufficiently small force and stress tuple there exists a traveling wave solution. The existence of traveling wave solutions to the one layer configuration ($m = 1$) was recently established and, to the best of our knowledge, this paper is the first construction of traveling wave solutions to the incompressible Navier-Stokes equations in the $m$-layer arrangement.

1. Introduction

1.1. Eulerian coordinate formulation. In this paper we study traveling wave solutions to the viscous surface-internal wave problem, which describes the evolution of a finite number of layers of incompressible and viscous fluid. We posit that the fluid layers contiguously occupy horizontally-infinite, finite-depth, and time-evolving slabs sitting atop a rigid hyperplane in ambient Euclidean space of dimension $n \in \mathbb{N} \setminus \{0, 1\}$ (the physically relevant dimensions are 2 and 3, but our analysis works more generally). Within each layer the fluid dynamics are described by the incompressible Navier-Stokes equations, and jump conditions couple the dynamics between layers. The multiple layers serve as a model of stratified fluids. These occur, for example, when salinity or temperature change rapidly with respect to depth.

In order to properly state the PDEs considered in this analysis, we first set the necessary notation. Fix the number of layers of fluid $m \in \mathbb{N} \setminus \{0, 1\}$. Let $\alpha = \{a_\ell\}_{\ell=1}^m$ be a strictly increasing sequence of positive real numbers, i.e. $0 < a_1 < \cdots < a_m$. We refer to $\alpha$ as the depth parameters. We associate to $\alpha$ the collection of admissible graph interfaces, which is the subset of $m$-tuples of continuous and bounded functions

$$\mathcal{A}(\alpha) = \{(\eta_\ell)_{\ell=1}^m \subset C_b^0(\mathbb{R}^{n-1}) : 0 < a_1 + \eta_1 \ldots < a_m + \eta_m \text{ on } \mathbb{R}^{n-1}\}. \quad (1.1)$$

If $\eta = (\eta_\ell)_{\ell=1}^m \in \mathcal{A}(\alpha)$, then for $\ell \in \{1, \ldots, m\}$ we define the slab-like domain

$$\Omega_{\ell}[\eta] = \begin{cases} (x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} : \eta_{\ell-1}(x) + a_{\ell-1} < y < \eta_\ell(x) + a_\ell \quad &\text{when } 2 \leq \ell \leq m \\ (x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} : 0 < y < \eta_1(x) + a_1 \quad &\text{when } \ell = 1 \end{cases} \quad (1.2)$$

and the free boundaries

$$\Sigma_\ell[\eta] = \{(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} : y = a_\ell + \eta_\ell(x)\}. \quad (1.3)$$

We also define the union of the slabs, entire domain, and rigid lower boundary, respectively, as

$$\Omega[\eta] = \Omega_1[\eta] \cup \cdots \cup \Omega_m[\eta], \quad \Omega^\circ[\eta] = \{(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} : 0 < y < a_m + \eta_m(x)\},$$

and $\Sigma_0 = \{(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} : y = 0\}. \quad (1.4)$
Observe that $(\overline{\Omega[n]})^0 = \Omega^n$. We will often need to distinguish between derivatives that are parallel to $\Sigma_0$ and vertical derivatives, so we write $\nabla = (\nabla_\parallel, \partial_n)$. Note that the operator $\nabla_\parallel$ is the full spatial gradient for the free surface functions, which have the spatial domain $\mathbb{R}^{n-1}$.

Suppose that $\eta = (\eta_t)_{\ell=1}^m \in \mathcal{A}(\alpha)$ is given and each $\eta_t$ is Lipschitz. Then for $X \in H^1(\Omega^n; \mathbb{R}^d)$, for some $d \in \mathbb{N}^+$, the restriction of $X$ to each $\Omega_\ell$ belongs to $H^1(\Omega^n; \mathbb{R}^d)$, and so from standard trace theory we have trace operators onto the upper and lower boundaries, $\Sigma_\ell$ and $\Sigma_{\ell-1}$, which we denote by $\text{Tr}^\uparrow_{\Sigma_\ell}[X]$ and $\text{Tr}^\downarrow_{\Sigma_{\ell-1}}[X]$, respectively. In turn, for $\ell \in \{1, \ldots, m\}$ this allows us to define the interfacial jumps via

$$[X]|_{\ell} = \begin{cases} \text{Tr}^\uparrow_{\Sigma_\ell}[X] - \text{Tr}^\downarrow_{\Sigma_{\ell-1}}[X] & \text{when } 1 \leq \ell \leq m - 1 \\ -\text{Tr}^\downarrow_{\Sigma_m}[X] & \text{when } \ell = m. \end{cases} \quad (1.5)$$

Note that $[X]|_m$ is not really a jump, but we will employ this notation for brevity in writing PDEs throughout the paper. If $u$ is a weakly differentiable vector field we define its symmetrized gradient as the matrix field $D u = \nabla u + \nabla u^t$, where the superscript ‘$t$’ denotes the matrix transpose. If $\mu = \{\mu_\ell\}_{\ell=1}^m \subset \mathbb{R}^+$ is a sequence of positive fluid viscosity parameters, we define the associated stress tensor as the mapping

$$S^\mu : \Omega^n \times H^1(\Omega^n; \mathbb{R}^n) \to \mathbb{R}^{n \times n}$$

by

$$S^\mu(p, u) = \sum_{\ell=1}^m \mathbb{1}_{\Omega_\ell^n} (p I_{n \times n} - \mu_\ell D u), \quad (1.6)$$

where $\mathbb{R}^{n \times n}$ denotes the set of $n \times n$ real symmetric matrices.

With the notation established, we are now equipped to describe the model and the equations of motion in time. At equilibrium, we posit that the layers of fluid occupy the domain $\Omega[0]$ with the $\ell$th layer occupying the region $\Omega_\ell[0]$; furthermore, when perturbed from equilibrium there are free surface functions $\zeta(t, \cdot) = (\zeta_\ell(t, \cdot))_{\ell=1}^m \in \mathcal{A}(\alpha)$ for time $t \geq 0$ describing the evolution of the fluid layer domains in such a way that $\Omega[\zeta(t, \cdot)]$ is the region occupied by the union of all the layers, and the $\ell$th layer occupies $\Omega_\ell[\zeta(t, \cdot)]$. The fluid velocity and pressure are described by the functions $w(t, \cdot) : \Omega[\zeta(t, \cdot)] \to \mathbb{R}^n$, $r(t, \cdot) : \Omega[\zeta(t, \cdot)] \to \mathbb{R}$. The density of the fluid occupying the region $\Omega_\ell[\zeta(t, \cdot)]$ is the constant $\rho_\ell \subset \mathbb{R}^+$, and the viscosity of this portion of the fluid is the constant $\mu_\ell \subset \mathbb{R}^+$. We assume that the fluid is acted upon by the following forces. The bulk (the region of fluid occupying $\Omega[\zeta(t, \cdot)]$) is acted on by: a uniform external gravitational field $-g e_n \subset \mathbb{R}^n$, for acceleration of gravity $g \subset \mathbb{R}^+$; and by a generic force $F(t, \cdot) : \Omega[\zeta(t, \cdot)] \to \mathbb{R}^n$. The $\ell$th free surface is acted upon by: a force generated by an externally applied stress tensor $T_\ell : \Sigma_\ell[\zeta(t, \cdot)] \to \mathbb{R}^{n \times n}$; and a force generated by the surface itself, which is modeled in the standard way by $-\sigma_\ell H(\zeta_\ell(t, \cdot))$ for $\sigma_\ell \geq 0$ a surface tension coefficient and

$$H(\eta_\ell) = \nabla_\parallel \cdot (\nabla_\parallel \eta_\ell (1 + |\nabla_\parallel \eta_\ell|^2)^{-1/2}) \quad (1.7)$$

the mean curvature operator. In addition, the upper surface (the $m$th one) is acted on by a constant external pressure $P_{\text{ext}} \subset \mathbb{R}$.

The equations of motion are:

$$\begin{align*}
\rho_\ell (\partial_t w + \nabla \cdot w) + \nabla \cdot S^\mu(w, r) &= -g \rho_\ell e_n + F \quad &\text{ in } \Omega(\zeta(t, \cdot)), \ &\ell \in \{1, \ldots, m\} \\
\nabla \cdot w &= 0 \quad &\text{ in } \Omega[\zeta(t, \cdot)] \\
\n P_{\text{ext}} \nu_m - S^\mu(w, r) \nu_m - \sigma_m H(\zeta_m) \nu_m &= T_m \nu_m \quad &\text{ on } \Sigma_m[\zeta(t, \cdot)] \\
\|S^\mu(w, r)\|_\ell \nu_\ell - \sigma_\ell H(\zeta_\ell) \nu_\ell &= T_\ell \nu_\ell \quad &\text{ on } \Sigma_\ell[\zeta(t, \cdot)], \ &\ell \in \{1, \ldots, m-1\} \\
\partial_t \zeta_\ell + w \cdot (\nabla_\parallel \zeta_\ell, 0) &= w \cdot e_n \quad &\text{ on } \Sigma_\ell[\zeta(t, \cdot)], \ &\ell \in \{1, \ldots, m\} \\
\|w\|_\ell &= 0 \quad &\text{ on } \Sigma_\ell[\zeta(t, \cdot)], \ &\ell \in \{1, \ldots, m-1\} \\
w &= 0 \quad &\text{ on } \Sigma_0.
\end{align*}$$

(1.8)

Here the upward pointing unit normal to the surface $\Sigma_\ell[\zeta(t, \cdot)]$ is

$$\nu_\ell = (1 + |\nabla_\parallel \zeta_\ell|^2)^{-1/2} (-\nabla_\parallel \zeta_\ell, 1), \quad (1.9)$$

and we write $\nabla \cdot S^\mu(w, r)$ to mean the $n$–vector with $i$th component equal to the divergence of the $i$th row of $S^\mu(w, r)$.

We briefly comment on the physics of the above system of PDEs. The first two equations of (1.8) are the incompressible Navier-Stokes equations. The first asserts a Newtonian balance of forces, while the second enforces that the associated flow is locally volume preserving and hence, because the density is constant in
the slab domains, mass is conserved. The third and fourth equations are the dynamic boundary conditions, which are understood as force balance on the interfaces, and the fifth equation is the kinematic boundary condition, which dictates the surfaces’ motion with the fluid. The final two equations are the no-slip conditions: the Eulerian velocity vanishes on the lower rigid boundary and is continuous across the free interface regions. For a more physical description of these equations and boundary conditions we refer to Wehausen-Laitone [WL60].

In this paper we construct traveling wave solutions to the system (1.8). These are solutions that are time-independent when viewed in an inertial coordinate system obtained from the above Eulerian coordinates through a Galilean transformation. In order for the stationary condition to hold, the moving coordinate system must be traveling at a constant velocity parallel to \( \Sigma_0 \). Up to a rigid rotation fixing the vector \( e_n \), we may assume that the traveling coordinate system is moving at a constant velocity \( \gamma e_1 \) for a signed wave speed \( \gamma \in \mathbb{R} \setminus \{0\} \).

In the new coordinates the stationary free surface functions are described by the unknowns \( \eta = (\eta_\ell)_{\ell=1}^m \in \mathcal{A}(\alpha) \); these are related to \( \zeta \) via \( \eta (x - \gamma e_1 t) = \zeta (t, x) \). Next we posit that \( v (x - \gamma e_1 t, y) = w (t, x, y) \),

\[
q (x - t \gamma e_1, y) = r (t, x, y) - P_{\text{ext}} - g \sum_{\ell=1}^m 1_{(a_{\ell-1},a_\ell)} (y) \left[ \rho (a_\ell - y) + \sum_{k=\ell+1}^m p_k (a_k - a_{k-1}) \right], \tag{1.10}
\]

\( \mathcal{F} (x - t \gamma e_1, y) = F (t, x, y) \), and \( \mathcal{T}_\ell (x - t \gamma e_1) = T_\ell (t, x, a_\ell + \zeta(t, x)) \), for \( t \geq 0 \) and \( (x, y) \in \mathbb{R}^{n-1} \times (0, a_m) \), where \( v : \Omega[0] \to \mathbb{R}^n \), \( q : \Omega[0] \to \mathbb{R} \), \( F : \Omega[0] \to \mathbb{R}^n \), and \( T_\ell : \mathbb{R}^{n-1} \to \mathbb{R}_{\text{sym}}^n \) are the stationary velocity field, renormalized pressure, external force, and external stresses, respectively. In the traveling coordinate system the PDE satisfied by the unknowns \( (q, v, (\eta_\ell)_{\ell=1}^m) \) with forcing \( (F, (T_\ell)_{\ell=1}^m) \) is the following system:

\[
\begin{cases}
\rho (v - \gamma e_1) \cdot \nabla v + \nabla \cdot S^\ell (q, v) = \mathcal{F} & \text{in } \Omega[0] \setminus \{0\} \\
\nabla \cdot v = 0 & \text{in } \Omega[0] \\
[S^\ell (q, v)]_\ell N_\ell = (g [\rho]_\ell \eta_\ell + \sigma e H (\eta_\ell)) N_\ell + T_\ell N_\ell & \text{on } \Sigma[0] \setminus \{0\} \\
-\gamma \partial_t \eta_\ell + v \cdot (\nabla \cdot \eta_\ell, 0) = v \cdot e_n & \text{on } \Sigma[0] \setminus \{0\} \\
\|v\|_{\ell, \ell} = 0 & \text{on } \Sigma[0] \setminus \{0\} \\
u = 0 & \text{on } \Sigma[0].
\end{cases}
\tag{1.11}
\]

In the above we write \( N_\ell = (-\nabla \cdot \eta_\ell, 1) \) and \( \rho = \sum_{\ell=1}^m 1_{\Omega[0]} \rho_\ell. \) Note that renormalizing the pressure in this way has the effect of shifting the gravitational term from the bulk to the interfaces.

We conclude our discussion of the model with a comment about the role of the forcing and interfacial stresses, \( (\mathcal{F}, (T_\ell)_{\ell=1}^m) \), appearing in (1.11). The simplest configuration occurs when \( \mathcal{F} = 0 \) and \( T_\ell = 0 \) for \( 1 \leq \ell \leq m - 1 \), but \( T_m = -\sigma I_{n,n} \) for a given scalar function \( \sigma : \mathbb{R}^{n-1} \to \mathbb{R} \). In this configuration, \( \sigma \) can be viewed as a spatially localized source of pressure moving with velocity \( \gamma e_1 \) above the fluid. We have chosen to study the more general framework with \( (\mathcal{F}, (T_\ell)_{\ell=1}^m) \) in order to allow for more sources of external force and stress.

### 1.2. Remarks on previous work.

Traveling wave solutions to the equations of fluid dynamics have been a subject of intense mathematical study for more than a century, so a complete review of the literature is well beyond the scope of this paper. The vast majority of this work has focused on inviscid models, in which the Navier-Stokes equations in (1.8) are replaced by the Euler equations. For a thorough review of the inviscid literature, we refer to the works of Toland [To06], Groves [Gr04], and Strauss [Str10].

In the viscous literature there are various results on stationary solutions to the free boundary problems, which correspond to traveling waves with vanishing velocity, \( \gamma = 0 \). For works on stationary solutions in layer geometries, we refer to Jean [Je80], Pileckas [P183, P184, P102], Gellrich [Ge93], Nazarov-Pileckas [NP99a, NP99b], Pileckas-Zalaškis [PZ03], and Bae-Choe [BC00]. Traveling wave solutions without a free boundary were constructed by Chae-Dubovskii [CD96] in full space and Kagei-Nishida [KN10] as bifurcations from Poiseuille flow in rigid channels.

To the best of our knowledge, the first construction of traveling wave solutions to the free boundary incompressible Navier-Stokes equations (system (1.8) for \( m = 1 \)) was only accomplished recently in the work of Leoni-Tice in [LT15], and there are no known results involving multiple layers. The multilayer problem is an important variant that appears in the study of internal waves in stratified fluids. This stratification can occur, for instance, due to changes in salinity or temperature.
As mentioned above, the system (1.11) can be used to model a source of spatially localized pressure translating above the fluid. This configuration has been studied in recent experiments with a tube of air, translating uniformly above a wave tank, blowing onto a single layer of viscous fluid and resulting in traveling waves. For details, we refer to the works of Akylas-Cho-Diorio-Duncan [CDAD11, CDAD11, Masnadi-Duncan [MD17], and Park-Cho [PC16, PC18].

1.3. Reformulation in an independent domain. The domain itself is one of the unknowns in the system (1.11), which presents a fundamental difficulty in producing solutions. Following the strategy of the single-layer case from [LT19], we overcome this obstacle with another change of coordinates and unknowns. We flatten to a domain that is independent of both time and the free surface functions, which comes at the expense of worsening the nonlinearities of the system.

We begin by defining the following family of flattening maps. Set \( a_0 = 0 \) and \( \eta_0 = 0 \). For \( \ell \in \{1, \ldots, m\} \) we define the mapping \( \mathcal{F}_\ell : \Omega_\ell[0] \to \Omega_\ell[\eta] \) with the assignment

\[
\mathcal{F}_\ell(x, y) = \left( x, \frac{a_\ell-y}{a_\ell-a_{\ell-1}} (a_{\ell-1} + \eta_{\ell-1}(x)) + \frac{y-a_{\ell-1}}{a_\ell-a_{\ell-1}} (a_\ell + \eta_\ell(x)) \right)
\]

(1.12)

for \( (x, y) \in \mathbb{R}^{n-1} \times [a_{\ell-1}, a_\ell] = \Omega_\ell[0] \). First we observe that each \( \mathcal{F}_\ell \) is bijective with inverse given via

\[
\mathcal{F}_\ell^{-1}(x, y) = \left( x, \frac{a_\ell+y(\eta_{\ell-1}(x)-a_{\ell-1}-\eta_{\ell-1}(x))}{a_\ell-a_{\ell-1}} a_{\ell-1} + \frac{y-a_{\ell-1}}{a_\ell-a_{\ell-1}} \eta_{\ell-1}(x) a_\ell \right)
\]

(1.13)

for \( (x, y) \in \Omega_\ell[\eta] \), whenever \( a_\ell - a_{\ell-1} \neq \eta_{\ell-1} - \eta_\ell \) pointwise. If this inequality holds, then \( \mathcal{F}_\ell \) is a homeomorphism inheriting the regularity of the tuple \( \eta \). We propose to paste these functions together to build our sought-after flattening map. That is, we define \( \mathcal{F} : \Omega_\ell[\eta] \to \Omega_\ell'[\eta] \) via \( \mathcal{F} = \mathcal{F}_\ell \) on \( \Omega_\ell[0] \). This assignment defines a homeomorphism because \( \mathcal{F}_\ell = \mathcal{F}_{\ell-1} \) on \( \Sigma_{\ell-1}[0] \) for \( \ell \in \{2, \ldots, m\} \).

Provided that \( \eta \) is differentiable, for \( (x, y) \in \Omega_\ell[0] \) we can compute the gradient

\[
\nabla \mathcal{F}_\ell(x, y) = \left( \frac{a_\ell-y}{a_\ell-a_{\ell-1}} \nabla \eta_{\ell-1}(x) + \frac{y-a_{\ell-1}}{a_\ell-a_{\ell-1}} \nabla \eta_\ell(x), \frac{0_{(n-1) \times 1}}{(n-1) \times 1} \right),
\]

(1.14)

the Jacobian

\[
J_\ell(x, y) = \det \nabla \mathcal{F}_\ell(x, y) = \frac{a_\ell + \eta_\ell(x) - a_{\ell-1} - \eta_{\ell-1}(x)}{a_\ell - a_{\ell-1}},
\]

(1.15)

and the geometry matrices

\[
\mathcal{A}_\ell(x, y) = \nabla \mathcal{F}_\ell(x, y)^{-1} = \begin{pmatrix}
I_{(n-1) \times (n-1)} & -\frac{a_\ell-y}{a_\ell-a_{\ell-1}} \nabla \eta_{\ell-1}(x) - \frac{y-a_{\ell-1}}{a_\ell-a_{\ell-1}} \nabla \eta_\ell(x) \\
0_{1 \times (n-1)} & 0_{(n-1) \times 1}
\end{pmatrix}.
\]

(1.16)

We then set \( \mathcal{A} : \Omega[0] \to \mathbb{R}^{n \times n} \) via \( \mathcal{A} = \mathcal{A}_\ell \) in \( \Omega_\ell[0] \), and \( J : \Omega[0] \to \mathbb{R} \) via \( J = J_\ell \) in \( \Omega_\ell[0] \): We may now reformulate (1.11) as a quasilinear system in the fixed domain \( \Omega[0] \).

\[
\begin{align*}
\rho_\ell ([u - \gamma v_\ell] \cdot \mathcal{A} \nabla |u + (\mathcal{A} \nabla) \cdot S^n_A(p, u)) = f & \quad \text{in } \Omega_\ell[0], \quad \ell \in \{1, \ldots, m\} \\
(\mathcal{A} \nabla) \cdot u = 0 & \quad \text{in } \Omega[0] \\
\left[ S^n_A(p, u) \right]_{\ell} \cdot \mathcal{N}_\ell = (g[\rho]_\ell \eta_\ell + \sigma_\ell \mathcal{H}(\eta_\ell)) \mathcal{N}_\ell + T_\ell \mathcal{N}_\ell & \quad \text{on } \Sigma_\ell[0], \quad \ell \in \{1, \ldots, m\} \\
\gamma \partial_1 \eta_\ell + u \cdot \mathcal{N}_\ell = 0 & \quad \text{on } \Sigma_\ell[0], \quad \ell \in \{1, \ldots, m-1\} \\
[u]_{\ell} = 0 & \quad \text{on } \Sigma_\ell[0], \\
u = 0 & \quad \text{on } \Sigma_\ell[0],
\end{align*}
\]

(1.17)

for the flattened velocity field and pressure \( u = v \circ \mathcal{F} \) and \( p = q \circ \mathcal{F} \). In the above we have also set \( f = \mathcal{F} \circ \mathcal{F} \), allowed \( \mathcal{A} \) to act on the ‘vector’ \( \nabla \) by standard matrix multiplication, and introduced the operator

\[
S^n_A(p, u) = \sum_{i=1}^m 1_{\Omega_i[0]} \left[ p I_{n \times n} - \mu_\ell (\nabla u) \mathcal{A}^i - \mu_\ell \mathcal{A}(\nabla u^i) \right].
\]

(1.18)

The \( n \)-vector \( (\mathcal{A} \nabla) \cdot S^n_A(p, u) \) has \( i \)-th component equal to the \( \mathcal{A} \)-divergence of the \( i \)-th row of \( S^n_A(p, u) \).
1.4. Statement of main results and discussion. We now give the two main results obtained from the analysis in this paper. We provide somewhat informal and abbreviated statements in order to avoid the need to introduce here some nonstandard function spaces we employ in our analysis. The proper statements are found later in the paper at the indicated theorems. The definitions of the function sets \( C^k \), \( C^k_b \), and \( C^k_0 \) can be found in Section 1.6.

Our first result regards the solvability of the flattened problem in (1.17): it tells us that if the strict Rayleigh-Taylor condition, \( 0 < \rho_m < \cdots < \rho_1 \), is satisfied along with certain conditions on the dimension \( n \) and the surface tensions \( \{ \sigma_i \}_{i=1}^n \), then the multilayer flattened free boundary problem (1.17) is well-posed for all nontrivial wave speeds and small forcing and applied stresses.

**Theorem 1** (Proved in Theorem 5.5). Suppose that either \( n = 2 \) and \( \{ \sigma_i \}_{i=1}^m = 0 \) or else \( n \geq 2 \) and \( \{ \sigma_i \}_{i=1}^m \subset \mathbb{R}^+ \). Let \( \mathbb{R} \ni s > n/2 \), \( 0 < \rho_m < \cdots < \rho_1 \), and \( \mathbb{N} \ni r < s - n/2 \). Then there exist Banach spaces

\[
X^s \hookrightarrow C_b^{1+r}(\Omega[0]) \times [C^0_b(\Omega^s[0]; \mathbb{R}^n) \cap C_b^{2+r}(\Omega[0]; \mathbb{R}^n)] \times (C_0^{3+r}(\mathbb{R}^{n-1}))^m
\]

and

\[
Z^s \hookrightarrow \mathbb{R} \times (C_0^{1+r}(\mathbb{R}^{n-1}; \mathbb{R}^{n \times n}))^m \times C^r_0(\Omega[0]; \mathbb{R}^n)
\]

and open sets \( V_s \subset X^s \) and \( U_s \subset Z^s \) such that the following hold.

1. \( (0, 0, (0)_{\ell=1}^m) \in V_s \) and \( (\mathbb{R} \setminus \{0\}) \times \{(0)_{\ell=1}^m\} \times \{0\} \subset U_s \).

2. For each \( (\gamma, (T_\ell)_{\ell=1}^m, f) \in U_s \) there exists a unique \( (p, u, (\eta_\ell)_{\ell=1}^m) \in V_s \) that is a classical solution to (1.17) with the former tuple as data. Moreover, the free surface functions obey the bound

\[
\max\{\|\eta_1\|_{C^0}, \ldots, \|\eta_n\|_{C^0}\} \leq \frac{1}{r} \min \{a_1, a_2 - a_1, \ldots, a_m - a_{m-1}\}.
\]

3. The mapping \( U_s \ni (\gamma, (T_\ell)_{\ell=1}^m, f) \mapsto (p, u, (\eta_\ell)_{\ell=1}^m) \in V_s \) is smooth.

Next, we take the solutions constructed by the previous theorem and build their associated inverse flattening maps. This process results in traveling wave solutions to the Eulerian formulation of the free boundary problem (1.11).

**Theorem 2** (Proved in Proposition 5.6 and Theorem 5.7). Let \( \mathbb{N} \ni k > n/2 \), \( 0 < \rho_m < \cdots < \rho_1 \), and \( \mathbb{N} \ni r < k - n/2 \). Suppose the dimension \( n \in \mathbb{N} \setminus \{0, 1\} \) and the surface tension coefficients \( \{ \sigma_i \}_{i=1}^m \) be related as in Theorem 1 and let \( U_k \) be as in the following. Then for each \( (\gamma, (T_\ell)_{\ell=1}^m, f) \in U_k \) the solution \( (p, u, (\eta_\ell)_{\ell=1}^m) \in V_k \) to (1.17) provided by Theorem 1 satisfies the following.

1. When defining the flattening map \( \tilde{\mathcal{F}} \) from the tuple \( \eta = (\eta_\ell)_{\ell=1}^m \) as in Section 1.3, the result is a bi-Lipschitz homeomorphism \( \tilde{\mathcal{F}} : \Omega^m[0] \rightarrow \Omega^m[\eta] \) that is a \( C^{3+r} \)-diffeomorphism on the \( m \) slab domains. In other words, \( \tilde{\mathcal{F}} \) and \( \tilde{\mathcal{F}}^{-1} \) are Lipschitz and satisfy the inclusions

\[
\tilde{\mathcal{F}} \in C^{3+r}(\Omega[0]; \Omega[\eta]) \quad \text{and} \quad \tilde{\mathcal{F}}^{-1} \in C^{3+r}(\Omega[\eta]; \Omega[0]).
\]

2. Setting

\[
(q, u, (\eta_\ell)_{\ell=1}^m) = (p \circ \tilde{\mathcal{F}}^{-1}, u \circ \tilde{\mathcal{F}}^{-1}, (\eta_\ell)_{\ell=1}^m)
\]

\[
\in C_b^{1+r}(\Omega^m[\eta]) \times [C^0_b(\Omega[\eta]; \mathbb{R}^n) \cap C_b^{2+r}(\Omega[\eta]; \mathbb{R}^n)] \times (C_0^{3+r}(\mathbb{R}^{n-1}))^m
\]

gives a classical solution to the free boundary problem (1.11) with signed wave speed \( \gamma \in \mathbb{R} \setminus \{0\} \), applied surface stresses \( (T_\ell)_{\ell=1}^m \subset C_b^{1+r}(\mathbb{R}^{n-1}; \mathbb{R}^{n \times n}) \), and external forcing \( F = f \circ \tilde{\mathcal{F}}^{-1} \in C^r_0(\Omega[\eta]; \mathbb{R}^n) \).

Following the lead of the single layer analysis in [11.19], our strategy for proving Theorems 1 and 2 can be succinctly described as follows: we find appropriate Banach spaces such that the locally defined mapping associated to the flattened problem in system (1.17) is well-defined, smooth, and satisfies the hypotheses of the implicit function theorem around the zero solution. This grants us the small data solution operator described in the first theorem. From these solutions to the flattened problem, we use the free surface functions to build the flattening map and its inverse to undo the reformulation described in Section 1.3. This then yields the second theorem.

The only serious difficulties in progressing from Theorem 1 to Theorem 2 lie in verifying that the flattening map \( \tilde{\mathcal{F}} \) and its inverse preserve not only the standard Sobolev spaces, but the specialized ones we employ in our analysis. Fortunately, these difficulties were already overcome in the single layer analysis.
of \textit{LT19}, and the solution is readily ported to the multilayer context of the present paper. As such, the main thrust of this paper is proving Theorem 1, which presents a number of nontrivial difficulties not encountered in the single layer analysis. The remainder of this discussion describes the path to this theorem in greater detail.

To invoke the implicit function theorem, we are led to study the linearization of system (1.17), which is recorded in (4.1). Even though this is a linear PDE, there are several obstacles that make solving it both an interesting and nontrivial endeavor. The first of these is the selection of appropriate Banach spaces for data and solutions for the linearized flattened problem. These spaces need be chosen so that: 1) the nonlinear operator associated to the flattened system is locally well-defined near the zero solution and is at least continuously differentiable, 2) they embed within subspaces of the classical scales measuring differentiability, and 3) the linearized problem induces a Banach isomorphism. The first and last point ensure that the hypothesis of the implicit function theorem are satisfied, and the second point guarantees that our notion of solution to the nonlinear flattened problem will be the classical one.

Unfortunately, for data belonging to subspaces of standard \(L^2\)-based Sobolev spaces, the natural a priori estimates associated to the linearized PDE (4.1) are too weak to force the solution tuple \((p,u, (\eta^m)_{\ell=1}^m)\) to belong to standard Sobolev spaces. The same problem was encountered in the single layer problem; to circumvent the issue, in Section 5 of \textit{LT19} novel scales of specialized Sobolev spaces were introduced, which satisfy the three requirements mentioned above. Fortunately for us, we find that the appropriate Banach spaces for the multilayer problem are natural modifications of the single-layer problem’s spaces: see Definitions 4.2 and 4.4. It is worth pointing out that, while these spaces arise naturally as the spaces that contain the solutions to (4.1), they have rather odd properties. For instance, in the most physically important case of \(n=3\) the space for the free surface functions is strongly anisotropic in the sense that it is not closed under composition with rotations (see Remark 5.4 in \textit{LT19}).

We now turn to the question of how to solve the problem (4.1), which is not a standard elliptic boundary value system (i.e. not in the form studied in the classic paper of Agmon,Douglas, and Nirenberg \cite{ADN64}) due to the fact that some of the unknowns, namely \((\eta^m)_{\ell=1}^m\), appear only on the boundary. Building on the strategy of \textit{LT19}, we attack this problem with the help of the normal stress to normal Dirichlet map (see Definition 2.14), which is \((\psi^m)_{\ell=1}^m \mapsto \nu_\gamma(\psi^m)_{\ell=1}^m = (\text{Tr}_{\Sigma_\ell} v \cdot e_n)_{\ell=1}^m\), where \((q,v)\) solve (2.44). Then a solution \((p,u, (\eta^m)_{\ell=1}^m)\) will take the form \(p = -g \sum_{\ell=1}^m [p]_\ell \eta^m (0,a_\ell) + q + r\) and \(u = v + w\) for \((q,v)\) solving (2.44) with data \((\psi^m)_{\ell=1}^m = (\sigma_\ell \Delta \eta^m)_{\ell=1}^m\) and \((r,w)\) solving

\[
\begin{align*}
\nabla \cdot S^\mu (r,w) - \gamma \rho_\ell \partial_1 w &= f + g \sum_{\ell=1}^m [p]_\ell \nabla \eta^m (0,a_\ell) & \text{in } \Omega_\ell, \ell \in \{1,\ldots,m\} \\
\nabla \cdot w &= g & \text{in } \Omega \\
[S^\mu (r,w) e_n]_\ell &= k_\ell & \text{on } \Sigma_\ell, \ell \in \{1,\ldots,m\} \\
w \cdot e_n &= h_\ell - \gamma \partial_1 \eta^m - [\nu_\gamma (\sigma_\ell \Delta \eta^m)]_\ell & \text{on } \Sigma_\ell, \ell \in \{1,\ldots,m-1\} \\
w|_\ell &= 0 & \text{on } \Sigma_0.
\end{align*}
\]

At first glance this seems no better than (4.1), but the advantage of this form is that even for given \(\eta = (\eta^m)_{\ell=1}^m\) belonging to the specialized Sobolev spaces, the right sides of this system belong to standard Sobolev spaces (see Proposition 4.3). However, if we think of \(\eta\) as given, then this system is overdetermined in the sense that \(n+1\) scalar boundary conditions are specified at each \(\Sigma_\ell\) rather than the \(n\) needed to uniquely determine solutions. This leads us to study the overdetermined problem (3.1).

The problem (3.1) cannot be solved for arbitrary data tuples \((g,f,(k_\ell)_{\ell=1}^m, (h_\ell)_{\ell=1}^m)\). Indeed, data for which a solution exists must satisfy certain compatibility conditions, which we identify in Section 3.1. Remarkably, this then yields a mechanism for solving (1.23) for general data \((g,f,(k_\ell)_{\ell=1}^m, (h_\ell)_{\ell=1}^m)\): we solve for \(\eta\) so that the modified data tuple

\[
(g, f + g \sum_{\ell=1}^m [p]_\ell \nabla \eta^m (0,a_\ell), (k_\ell)_{\ell=1}^m, (h_\ell - \gamma \partial_1 \eta^m)_{\ell=1}^m - \nu_\gamma (\sigma_\ell \Delta \eta^m)_{\ell=1}^m)
\]

satisfies the compatibility conditions, and then we solve for \((r,w)\) using the solvability theory for the overdetermined problem (3.1), which we also develop in Section 3.1.
In following this strategy for determining \( \eta \) in terms of the data, we uncover another remarkable fact: after horizontal Fourier transformation, the bulk term \( \mathbf{g} \sum_{\ell=1}^{m} [\mathbf{p}]_{\ell} \nabla \eta_{I_{\ell}}(0, a_{I}) \) in the compatibility condition shifts back to a boundary term involving the symbol of the pseudodifferential operator (ΨDO) \( \nu_{-\gamma} \), which allows us to show that the compatibility condition for the modified data tuple \([1.24] \) is equivalent to a system of pseudodifferential equations (ΨDEs) on \( \mathbb{R}^{n-1} \). More precisely, we show in Proposition 2.16 that \( \nu_{\gamma} \) has associated symbol \( \mathbf{n}_{\gamma} : \mathbb{R}^{n-1} \to \mathbb{C}^{m \times m} \), and we prove that the compatibility conditions are equivalent to the ΨDEs

\[
\mathbf{p}_{\gamma}(\xi) \mathcal{F}[\eta](\xi) = \mathcal{F}[\varphi](\xi) \quad \text{for} \quad \xi \in \mathbb{R}^{n-1},
\]

(1.25)

where \( \mathcal{F} \) denotes the Fourier transform on \( \mathbb{R}^{n-1} \) (acting on each component of the tuple \( \eta \) in the obvious way),

\[
\mathbf{p}_{\gamma}(\xi) = \mathbf{n}_{-\gamma}(\xi) \operatorname{diag}(-\mathbf{g} [\rho]_{1} + 4\pi^{2} |\xi|^{2} \sigma_{1}, \ldots, -\mathbf{g} [\rho]_{m} + 4\pi^{2} |\xi|^{2} \sigma_{m}) - 2\pi i \gamma \xi_{1} I_{m \times m} \in \mathbb{C}^{m \times m},
\]

(1.26)

and \( \varphi : \mathbb{R}^{n-1} \to \mathbb{C}^{m} \) is a particular tuple depending on the data \((g, f, (k_{\ell})_{\ell=1}^{m}, (h_{\ell})_{\ell=1}^{m})\). Note that the symbol \( \mathbf{p}_{\gamma} \) is a synthesis of the symbols for the differential operator \( \gamma \partial_{1} \), the normal stress to normal Dirichlet operator \( \nu_{-\gamma} \), and the elliptic capillary operators \( \mathbf{g} [\rho]_{\ell} + \sigma_{\ell} \Delta \).

Provided that \( \mathbf{p}_{\gamma} \) is almost everywhere invertible, we then have the determination \( \eta = \mathcal{F}^{-1}[\mathbf{p}_{\gamma}^{-1} \mathcal{F}[\varphi]] \). However, given the complicated form of \( \mathbf{p}_{\gamma} \), it is far from obvious that this holds or that, if it is true, the resulting formula for \( \eta \) produces free surfaces that are both physically sensible and mathematically useful in our implicit function theorem scheme. In order to prove these, we need to know two crucial pieces of information: detailed facts about the regularity of \( \varphi \), and precise asymptotic developments of \( \mathbf{n}_{\gamma}(\xi) \) as \(|\xi| \to 0 \) and \(|\xi| \to \infty \).

It is here where the present paper seriously deviates from the strategy employed for a single layer in [7,19], which involved brute forcing the asymptotics of the symbol from an explicit expression given by the solution to the ODE system resulting from applying \( \mathcal{F} \) to (2.44). Due to essential singularities in the symbol at \(|\xi| = \infty \), this approach is rather delicate and involves numerous tedious calculations for which computer algebra systems are of little assistance. If we were to attempt to port this brute force approach to the \( m \)-layer problem, the number of these tedious asymptotic developments that we would need to compute by hand would be on the order of \( m^{2} \), which is already disagreeable when \( m = 2 \) and is outright impossible in the general case \( m \geq 2 \).

In the present paper we thus abandon the brute force strategy and develop a more elegant and flexible method for deriving the asymptotic developments of the symbol \( \mathbf{n}_{\gamma} \). Our technique is based on a synthesis of novel energy estimates for solutions of the multilayer traveling Stokes system (2.1), a duality-based formulation of the compatibility conditions for (3.1), and estimates for solutions to certain prescribed divergence equations. The key observation is the energy equivalence of Theorem 2.15 for solutions to the applied normal stress problem (2.44), which characterizes the data space for solenoidal weak formulations (only employing solenoidal test functions to avoid introducing the pressure), and may thus be of independent interest in the study of the Stokes system.

The symbol \( \mathbf{p}_{\gamma}^{-1} \), together with the properties of \( \varphi \) (see Section 3.2), ultimately determine the nonstandard Sobolev spaces employed in our analysis. Thus, by employing this strategy, we can indeed solve for the free surface functions and then solve the linearized problem (1.1). This leads us to the isomorphism theorems Theorems 4.9 and 4.11 which then form the backbone of the implicit function scheme discussed above.

1.5. **Outline of paper.** We begin our linear analysis in Section 2 where we study the multilayer traveling Stokes equations subject to stress boundary conditions, as well as the specified divergence and normal trace problem. These are systems (2.1) and (2.2), respectively. The analysis of the latter PDE in Section 2.1 explores a necessary and sufficient compatibility condition for the data. The result is a solution operator and important technical estimates.

Section 2.2 dives into the analysis of the system (2.1). This system is elliptic, and the well-posedness theory is straightforward. However, the solution operator for this system plays a foundational role in our subsequent analysis, as they allow us to define the normal stress to normal Dirichlet ΨDO, \( \nu_{\gamma} \), as well as build more complicated solution operators to other PDE systems.
Section 2.3 next studies the normal stress to normal Dirichlet operator. The asymptotic developments of its symbol are computed using the energy structure of the multilayer traveling Stokes system and estimates from the specified divergence and normal trace problem.

In Section 3 we analyze the overdetermined variant of the multilayer traveling Stokes system. In Section 3.1 we characterize spaces of compatible data for which this PDE admits solutions. Then, in Section 3.2 we examine more closely what it means for data to be compatible and develop a particular measurement of compatibility for general data, which leads us to the tuple \( \varphi \) appearing in the \( \PsiDEs \) (1.25). We prove estimates for \( \varphi \) in frequency space that aid in the solving of these \( \PsiDEs \).

Section 4 synthesizes the previous two sections and draws from the specialized Sobolev space analysis of [LT19] to build the Banach isomorphism solution operator associated with linearized flattened problem. Section 4.1 proves that the proposed solution operator is well-defined and injective. Sections 4.2 and 4.3 prove surjectivity in the cases \( n \geq 2 \) and strictly positive surface tensions and \( n = 2 \) and vanishing surface tensions, respectively.

Section 5 contains the nonlinear analysis and the proofs of the main theorems. We combine the linear analysis from Section 3 with more results on specialized Sobolev spaces in order to satisfy the hypothesis of the implicit function theorem. Theorems 1 and 2 then follow.

Finally, in Appendix A we record some useful facts from analysis used throughout the paper. These include notions of real valued tempered distributions, (anti-)duality and the Lax-Milgram lemma, tangential Fourier multipliers, and Korn’s inequality.

1.6. Conventions of notation. The standard Lebesgue measure on the Euclidean space \( \mathbb{R}^d \) is \( \mathcal{L}^d \). The symbol \( \mathbb{K} \) will be used in situations in which both \( \mathbb{K} = \mathbb{R} \) and \( \mathbb{K} = \mathbb{C} \) are valid.

Whenever the expression \( a \lesssim b \) appears in a proof of a result, it means that there is a constant \( C \in \mathbb{R}^+ \), depending only on the parameters implicitly and explicitly quantified in the statement of the result, such that \( a \leq Cb \). We also write \( a \asymp b \) to mean \( a \lesssim b \) and \( b \lesssim a \).

Given complex vector spaces \( X, Y \), and \( Z \) we say that a mapping \( B : X \times Y \to Z \) is sesquilinear if it is linear in the left argument and antilinear in the right argument. The dot product \( \cdot \) denotes the standard sesquilinear Euclidean inner product on \( \mathbb{C}^d \), and we write \( : \) for the sesquilinear Frobenius inner product on \( \mathbb{C}^{d \times d} \). We denote the divergence and tangential divergence operators with

\[
\nabla \cdot f = \sum_{j=1}^n \partial_j (f \cdot e_j) \quad \text{and} \quad (\nabla_\| \cdot 0) \cdot f = \sum_{k=1}^{n-1} \partial_k (f \cdot e_k)
\]

(1.27)

for appropriate \( \mathbb{C}^n \)-valued functions \( f \). Note that this does not violate our sesquilinearity rule because the arguments of \( \nabla \cdot f \) and \( (\nabla_\| \cdot 0) \cdot f \) are outside of the domain of the dot product.

If \( \mathcal{H} \) is a complex Hilbert space, then \( \mathcal{H}^\ast \) denotes the set of continuous and antilinear functionals on \( \mathcal{H} \), i.e. the antidual. Sometimes we will need to simultaneously refer to complex and real Hilbert spaces. When doing so we will use \( \mathcal{H}^\ast \) to refer to the usual dual space when the base field is \( \mathbb{R} \) and the antidual when it’s \( \mathbb{C} \). Given a complex Hilbert \( \mathcal{H} \), the antidual pairing is the sesquilinear form \( \langle \cdot , \cdot \rangle_{\mathcal{H}^\ast, \mathcal{H}} : \mathcal{H}^\ast \times \mathcal{H} \to \mathbb{C} \) defined via \( \langle f, v \rangle_{\mathcal{H}^\ast, \mathcal{H}} = F(v) \). The Fourier transform is denoted \( \mathcal{F}[\cdot] \).

Finally, we set the following function space notation. If \( U \subset \mathbb{R}^{d_1} \) and \( V \subset \mathbb{R}^{d_2} \) are open subsets of Euclidean space and \( r \in \mathbb{N} \) we define

\[
\begin{align*}
C^r(U; V) & = \{ f : U \to V \mid f \text{ is continuous along with its derivatives of order } k, \forall \ k \in \mathbb{N}^+, \ k \leq r \} \\
C_c^r(U; V) & = \{ f \in C^r(U; V) \mid \max_{0 \leq k \leq r} \sup_{z \in U} |D^k f(z)| < \infty \} \\
C_0^r(\mathbb{R}^{d_1}; \mathbb{R}^{d_2}) & = \{ f \in C_0^r(\mathbb{R}^{d_1}; \mathbb{R}^{d_2}) \mid \lim_{|z| \to \infty} \max_{0 \leq k \leq r} |D^k f(z)| = 0 \}.
\end{align*}
\]

(1.28)

Let \( \eta \in \mathscr{A}(\alpha), \ s \geq 0, \) and \( \mathbb{K} \in \{ \mathbb{R}, \mathbb{C} \} \). If \( t, a \in \mathbb{R} \) we identify

\[
H^t(\mathbb{R}^{n-1} \times \{a\}; \mathbb{K}^d) \simeq H^t(\mathbb{R}^{n-1}; \mathbb{K}^d)
\]

(1.29)

in the obvious way. We say that a vector field is solenoidal if its distributional divergence vanishes. The closed subspace of \( H^1 \) consisting of solenoidal fields on \( \Omega^e[\eta] \) vanishing identically on the lower boundary is denoted

\[
_0H^1_e(\Omega[\eta]; \mathbb{K}^n) = \{ f \in H^1(\Omega^e[\eta]; \mathbb{K}^n) : \nabla \cdot f = 0 \text{ and } \Tr_{\Sigma_0} f = 0 \}.
\]

(1.30)
Note that functions in this space are required to be in $H^1$ on the entire domain $\Omega^e[j]$ (see \(1.4\)). For $s \in \mathbb{R}^+ \cup \{0\}$ we also define
\[
\mathcal{H}^{1+s}(\Omega^e[j]; \mathbb{K}^d) = \{ f \in H^1(\Omega^e[j]; \mathbb{K}^d) : f \restriction \Omega^e[j] \in H^s(\Omega^e[j]; \mathbb{K}^d) \text{ and } \text{Tr}_{\Sigma_0} f = 0 \}. \tag{1.31}
\]
Note that functions in this space are required to be in $H^1$ of the entire domain but only $H^{1+s}$ on each subdomain $\Omega^e[j]$. A norm that makes the above vector space Banach is given by
\[
\|f\|_{\mathcal{H}^{1+s}(\Omega^e[j])}^2 = \sum_{\ell=1}^m \|f \restriction \Omega^e[j]\|_{H^{1+s}(\Omega^e[j])}^2. \tag{1.32}
\]
Observe in particular that taking $s = 0$ implies that we will also denote $\mathcal{H}^1(\Omega^e[j]; \mathbb{K}^d)$ with $\mathcal{H}^1(\Omega^e[j]; \mathbb{K}^d)$.

2. Multilayer traveling Stokes with stress boundary and jump conditions

In this section and the two succeeding we analyze linear systems of PDEs in the fixed domains $\Omega^e[0]$ with boundary conditions prescribed on $\Sigma_j[0]$ for $\ell \in \{1, \ldots, m\}$ and $j \in \{0, 1, \ldots, m\}$. In the interest of concision we make the following change of notation: $\Omega^e[0] \mapsto \Omega^e$, $\Sigma_j[0] \mapsto \Sigma_j$, and $\Omega[0] \mapsto \Omega$.

Specific to this section of the paper is analysis of the following system of PDEs:
\[
\begin{cases}
\nabla \cdot S^u (p, u) - \gamma \rho \partial \eta u = f & \text{in } \Omega^e, \quad \ell \in \{1, \ldots, m\} \\
\nabla \cdot u = g & \text{in } \Omega \\
[S^u (p, u) e_n]_\ell = k_\ell & \text{on } \Sigma^e, \quad \ell \in \{1, \ldots, m\} \\
u = 0 & \text{on } \Sigma_0.
\end{cases} \tag{2.1}
\]
with unknown velocity $u$ and pressure $p$, and with prescribed data $f$, $g$, and $(k_\ell)_{\ell=1}^m$. The viscosity parameters are $\mu = \{\mu_\ell\}_{\ell=1}^m \subset \mathbb{R}^+$, $\{\rho_\ell\}_{\ell=1}^m \subset \mathbb{R}^+$ are the density parameters, and $\gamma \in \mathbb{R}$ is the signed wave speed. Out of necessity, in this section we will work with real and complex valued solutions. We recall from Section \ref{section1} that $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and in the complex case the symbols $\cdot$ and $:$ are sesquilinear, which allows us to suppress writing complex conjugates in many expressions.

2.1. Specified divergence and multi-normal trace problem. Before we dive into the analysis of \ref{2.1}, we first develop a few auxiliary results concerning the following multi-normal trace-divergence problem. That is, given a collection of normal traces $(g_\ell)_{\ell=1}^m$, with $g_\ell$ defined on $\Sigma^e$, and given $f : \Omega \to \mathbb{K}$ one asks for $u : \Omega \to \mathbb{K}^n$ satisfying:
\[
\begin{cases}
\nabla \cdot u = f & \text{in } \Omega \\
u = 0 & \text{on } \Sigma_0.
\end{cases} \tag{2.2}
\]
In general this problem is overdetermined in the sense that if $u$ is a solution belonging to a appropriate function space then a nontrivial compatibility condition must hold among the data $f$ and $(g_\ell)_{\ell=1}^m$. We codify this precisely in the following result.

**Proposition 2.1** (Divergence compatibility estimate). Let $u \in \mathcal{H}^1(\Omega; \mathbb{K}^n)$ and set $f = \nabla \cdot u \in L^2(\Omega; \mathbb{K})$ and $g_\ell = \text{Tr}_{\Sigma_\ell} u \cdot e_n \in H^{1/2}(\Sigma^e; \mathbb{K})$. Then for each $\ell \in \{1, \ldots, m\}$ we have the inclusion
\[
g_\ell(\cdot, a) - \int_{(0, a_\ell)} f(\cdot, y) \, dy \in \dot{H}^{-1}(\mathbb{R}^{n-1}; \mathbb{K}). \tag{2.3}
\]
Moreover, we have the bound
\[
\sum_{\ell=1}^m \left[ g_\ell - \int_{(0, a_\ell)} f \right]_{\dot{H}^{-1}} \leq 2\pi \left( \sum_{\ell=1}^m \sqrt{a_\ell} \right) \|u\|_{L^2}. \tag{2.4}
\]

**Proof.** As justified by the absolute continuity on lines characterization of $\mathcal{H}^1(\Omega; \mathbb{K}^n)$, we may integrate the equation $\nabla \cdot u = f$ in the vertical variable over $(0, a_\ell)$ and employ the second fundamental theorem of calculus. This results in the identity
\[
\int_{(0, a_\ell)} f = g_\ell + (\nabla ||, 0) \cdot \int_{(0, a_\ell)} u. \tag{2.5}
\]
Therefore, by Hölder’s inequality and Tonelli’s theorem,
\[ (g - \int_{(0,a)} f)_{H^{-1}} \leq 4\pi^2 \int_{\mathbb{R}^n} |u| \leq 4\pi^2 a \|u\|_{L^2}^2. \] (2.6)

The stated estimate follows.

The remainder of this subsection is devoted to the converse of the previous lemma: the satisfaction of this compatibility condition is also sufficient in guaranteeing the solvability of the PDE (2.2). The first ingredient we require is some right inverse to the divergence operator that enforces the vanishing trace on the lower boundary Σ₀.

**Lemma 2.2** (A right inverse to the divergence). Let \( a, b \in \mathbb{R} \) with \( a < b \) and set \( U = \mathbb{R}^{n-1} \times (a, b) \). There exists a linear and continuous mapping \( \Pi_U : L^2(U; \mathbb{K}) \to H^1(U; \mathbb{K}^n) \) such that \( \nabla \cdot \Pi_U f = f \) for all \( f \in L^2(U; \mathbb{K}) \).

**Proof.** The existence of such an operator in the case \( \mathbb{K} = \mathbb{R} \) is well-known. See, for instance, Proposition 2.1 in \[LT19\]. In the instance that \( \mathbb{K} = \mathbb{C} \) one may simply take the real valued operator to act on real and imaginary parts of the data separately in the obvious way.

Next, we may explicitly construct a solution operator to (2.2) in the special case of \( f = 0 \) and \( m = 1 \).

**Lemma 2.3** (Solenoidal extension operator). Let \( W = \mathbb{R}^{n-1} \times (a, b) \) and \( \Sigma = \mathbb{R}^{n-1} \times \{ b \} \) for \( a, b \in \mathbb{R}, a < b \). There exists a bounded linear operator \( P_W : H^1/2(\Sigma; \mathbb{K}) \cap H^{-1}(\Sigma) \to H^1(W; \mathbb{K}^n) \) such that \( \nabla \cdot P_W g = 0 \) and \( \text{Tr}_2 P_W g \cdot e_n = g \) for all \( g \in H^1/2(\Sigma; \mathbb{K}) \cap H^{-1}(\Sigma; \mathbb{K}) \).

**Proof.** It is sufficient to consider the case that \( a = 0 \) and \( b \in \mathbb{R}^+ \). We explicitly construct the solution operator with the horizontal Fourier transform. Given \( g \in H^{-1}(\Sigma; \mathbb{K}) \cap H^1/2(\Sigma; \mathbb{K}) \) we define the auxiliary functions \( v : \mathbb{R}^{n-1} \times (0, b) \to \mathbb{C}^{n-1} \) and \( w : \mathbb{R}^{n-1} \times (0, b) \to \mathbb{C} \) via
\[ v(\xi, t) = \hat{g}(\xi) \frac{i\xi \sinh (t|\xi|)}{2\pi |\xi| (\cosh (b|\xi|) - 1)} \quad \text{and} \quad w(\xi, t) = \hat{g}(\xi) \frac{\cosh (t|\xi|) - 1}{\cosh (b|\xi|) - 1}. \] (2.7)

We propose that setting \( P_W g = \mathcal{F}^{-1}(v, w) \) gives the desired solution operator. In order to check that this is well-defined and continuous, it is sufficient to use Parseval’s and Tonelli’s theorems and observe the following four computations: First:

\[ \|\mathcal{F}^{-1} v\|_{L^2 H^{-1}}^2 = \int_{(0,b)} \int_{\mathbb{R}^{n-1}} |\xi|^2 |\hat{g}(\xi)|^2 \sinh (t|\xi|)^2 \frac{1}{(\cosh (b|\xi|) - 1)^2} \, d\xi \, dt \]
\[ = \frac{1}{4} \int_{\mathbb{R}^{n-1}} \max\{|\xi|, |\xi|^2\} |\hat{g}(\xi)|^2 \min\{1, |\xi|^3\} \frac{2b|\xi| + \sinh (2b|\xi|)}{(\cosh (b|\xi|) - 1)^2} \, d\xi \leq c_0(b) \|g\|_{H^{-1}\cap H^1/2}^2. \] (2.8)

Second:

\[ \|\mathcal{F}^{-1} v\|_{L^2 H^1/2}^2 = \frac{1}{4\pi^2} \int_{(0,b)} \int_{\mathbb{R}^{n-1}} |\xi|^2 |\hat{g}(\xi)|^2 \cosh (t|\xi|)^2 \frac{1}{(\cosh (b|\xi|) - 1)^2} \, d\xi \, dt \]
\[ = \frac{1}{16\pi^2} \int_{\mathbb{R}^{n-1}} \max\{|\xi|, |\xi|^2\} |\hat{g}(\xi)|^2 \min\{1, |\xi|^3\} \frac{2b|\xi| + \sinh (2b|\xi|)}{(\cosh (b|\xi|) - 1)^2} \, d\xi \leq c_1(b) \|g\|_{H^{-1}\cap H^1/2}^2. \] (2.9)

Third:

\[ \|\mathcal{F}^{-1} w\|_{L^2 H^1}^2 = 4\pi^2 \int_{(0,b)} \int_{\mathbb{R}^{n-1}} |\xi|^2 |\hat{g}(\xi)|^2 \cosh (t|\xi|) - 1)^2 \frac{1}{(\cosh (b|\xi|) - 1)^2} \, d\xi \, dt \]
\[ = \pi^2 \int_{\mathbb{R}^{n-1}} \max\{|\xi|, |\xi|^2\} |\hat{g}(\xi)|^2 \min\{1, |\xi|^3\} \frac{6b|\xi| - 8 \sinh (b|\xi|) + \sinh (b|\xi|)}{(\cosh (b|\xi|) - 1)^2} \, d\xi \leq c_2(b) \|g\|_{H^{-1}\cap H^1/2}^2. \] (2.10)

Fourth:

\[ \|\mathcal{F}^{-1} w\|_{L^2 H^1/2}^2 = \int_{(0,b)} \int_{\mathbb{R}^{n-1}} |\xi|^2 |\hat{g}(\xi)|^2 \sinh (t|\xi|)^2 \frac{1}{(\cosh (b|\xi|) - 1)^2} \, d\xi \, dt \leq c_0(b) \|g\|_{H^{-1}\cap H^1/2}^2. \] (2.11)
It is straightforward to check that \( \nabla \cdot P_W g = 0 \), \( \text{Tr}_\Sigma P_W g \cdot e_n = g \), and \( \text{Tr}_{\mathbb{R}^{n-1} \times \{0\}} P_W g = 0 \). By Proposition \[A.2\] and Remark \[A.3\] it is also ensured that \( P_W g \) is real-valued whenever \( g \) is real-valued. This completes the proof. \( \square \)

We may piece together the operators from Lemmas \[2.3\] and \[2.2\] to solve problem \[2.2\] in the single prescribed normal trace case, \( m = 1 \).

**Proposition 2.4** (Solution operator to \[2.2\]: single layer case). Let \( a, b \in \mathbb{R} \) with \( a < b \). Define the Hilbert space

\[
\mathcal{W}(a,b) = \{(f,g) \in L^2(\mathbb{R}^{n-1} \times (a,b); \mathbb{K}) \times H^{1/2}(\mathbb{R}^{n-1} \times \{b\}; \mathbb{K}) : \| (f,g) \|_{\mathcal{W}} < \infty \}
\]

for the norm

\[
\| (f,g) \|_{\mathcal{W}}^2 = \| f \|_{L^2}^2 + \| \ell \|_{H^{1/2}}^2 + \left[ g - \int_{(a,b)} f \right]_{H^{-1}}^2.
\]

There exists a bounded linear \( Q^{a,b} : \mathcal{W}(a,b) \to H^1(\mathbb{R}^{n-1} \times (a,b); \mathbb{K}^n) \) such that \( \nabla \cdot Q^{a,b}(f,g) = f \) and \( \text{Tr}_{\mathbb{R}^{n-1} \times \{b\}} Q^{a,b}(f,g) \cdot e_n = g \).

**Proof.** Set \( W = \mathbb{R}^{n-1} \times (a,b) \) and \( \Sigma = \mathbb{R}^{n-1} \times \{b\} \). We propose that the assignment

\[
Q^{a,b}(f,g) = \Pi_W f + P_W [g - \text{Tr}_\Sigma \Pi_W f \cdot e_n]
\]

for \( (f,g) \in \mathcal{W}(a,b) \) has the desired properties. Well-definedness and continuity of \( Q^{a,b} \) are assured as soon as one observes the bound

\[
[g - e_n \cdot \text{Tr}_\Sigma \Pi_W f |_{H^{-1}}] \leq \left[ g - \int_{(a,b)} f \right]_{H^{-1}} + \left[ e_n \cdot \text{Tr}_\Sigma \Pi_W f - \int_{(a,b)} f \right]_{H^{-1}} \leq \| (f,g) \|_{\mathcal{W}} + 2\pi \sqrt{b-a} \| \Pi_W f \|_{L^2} \lesssim \| (f,g) \|_{\mathcal{W}}.
\]

Note that in the second to last inequality above we have employed the divergence compatibility estimate from Proposition \[2.1\]. \( \square \)

We need one final lemma before we solve the general case of problem \[2.2\].

**Lemma 2.5.** Let \( a, b \in \mathbb{R} \) with \( 0 < a < b \). There exists a bounded linear extension operator

\[
E^{a,b} : 0H^1(\mathbb{R}^{n-1} \times (0,a); \mathbb{K}^n) \to H^1_0(\mathbb{R}^{n-1} \times (0,b); \mathbb{K}^n).
\]

That is, \( E^{a,b}f = f \) on \( \mathbb{R}^{n-1} \times (0,a) \) for all \( f \in 0H^1(\mathbb{R}^{n-1}; \mathbb{K}^n) \).

**Proof.** We construct \( E^{a,b} \) via a simple reflection. Given \( f \in 0H^1(\mathbb{R}^{n-1} \times (0,a); \mathbb{K}^n) \) we define

\[
E^{a,b}f(x,y) = \begin{cases} f(x,y) & \text{when } (x,y) \in \mathbb{R}^{n-1} \times (0,a) \\ f(x,y) = f(x,a - a(y - a)/(b - a)) & \text{when } (x,y) \in \mathbb{R}^{n-1} \times (a,b). \end{cases}
\]

Thanks to the absolute continuity on lines characterization of \( H^1 \), we observe that \( E^{a,b} \) takes values within the claimed target. This extension is also continuous since bi-Lipschitz change of coordinates boundedly preserve \( H^1 \) inclusion. \( \square \)

We now have tools that are sufficient in solving the general case of problem \[2.2\].

**Theorem 2.6** (Solution operator to \[2.2\]: multilayer case). We define the appropriate Hilbert space for data in problem \[2.2\]. For \( \alpha = \{(a_i)_{i=1}^m \subset \mathbb{R}^+ \) with \( 0 < a_1 < \cdots < a_m \) we define

\[
\mathcal{X}^m(\alpha) = \{(f,\{g_{i,\ell}\}_{\ell=1}^m) \in L^2(\Omega; \mathbb{K}) \times \prod_{\ell=1}^m H^{1/2}(\Sigma_{\ell}; \mathbb{K}) : \| (f,\{g_{i,\ell}\}_{\ell=1}^m) \|_{\mathcal{X}^m(\alpha)} < \infty \}
\]

for the norm

\[
\| (f,\{g_{i,\ell}\}_{\ell=1}^m) \|_{\mathcal{X}^m(\alpha)}^2 = \| f \|_{L^2}^2 + \sum_{\ell=1}^m \left( \| g_{\ell} \|_{H^{1/2}}^2 + \left[ g_{\ell} - \int_{(0,a_{i,\ell})} f \right]_{H^{-1}}^2 \right).
\]

There exists a linear and continuous mapping \( Q_m : \mathcal{X}^m(\alpha) \to 0H^1(\Omega; \mathbb{K}^n) \) such that \( \nabla \cdot Q_m(f,\{g_{i,\ell}\}_{\ell=1}^m) = f \) and for each \( \ell \in \{1,\ldots,m\} \) one has \( \text{Tr}_{\Sigma_\ell} Q_m(f,\{g_{i,\ell}\}_{\ell=1}^m) \cdot e_n = g_\ell \) for all data \( (f,\{g_{i,\ell}\}_{\ell=1}^m) \in \mathcal{X}^m(\alpha) \).
Proof. We construct the desired solution operator by way of mathematical induction on the number of specified normal traces. The precise statement to be proved, which we denote by statement\((m)\) for \(m \in \mathbb{N}^+\), is as follows: for all strictly increasing sequences \(\alpha = \{a_\ell\}_{\ell=1}^m \subset \mathbb{R}^+\) there exists a bounded linear mapping \(Q_m : X^m(\alpha) \to \mathfrak{a}H^1(\Omega; \mathbb{K}^n)\) that is a solution operator to problem (2.2).

The case \(m = 1\) is handled by Proposition 2.4. Now suppose that \(m \in \mathbb{N}^+\) is such that statement\((m)\) holds true. We will prove that statement\((m+1)\) is true.

Let \(\beta = \{b_\ell\}_{\ell=1}^{m+1} \subset \mathbb{R}^+\) be any sequence such that \(0 < b_1 < \cdots < b_m < b_{m+1}\) and set \(\alpha = \{b_\ell\}_{\ell=1}^m\). By hypothesis, there is a solution operator to problem (2.2), \(Q_m\), for the domain \(\Omega_m = \mathbb{R}^{n-1} \times (0, b_m)\) and boundary regions \(\Sigma_m = \mathbb{R}^{n-1} \times \{b_\ell\}\), for \(\ell \in \{1, \ldots, m\}\). Set \(\Omega_{m+1} = \mathbb{R}^{n-1} \times (0, b_{m+1})\) and \(\Sigma_{m+1} = \mathbb{R}^{n-1} \times \{b_{m+1}\}\).

We propose to define for \((f, (g_\ell)_{\ell=1}^{m+1}) \in X^{m+1}(\beta)\)

\[
Q_{m+1}(f, (g_\ell)_{\ell=1}^{m+1}) = E^{b_m,b_{m+1}}Q_m(f 1_{\Omega_m}, (g_\ell)_{\ell=1}^m) + Q_{b_m,b_{m+1}}(f 1_{\Omega_{m+1}\setminus\Omega_m} - \nabla \cdot E^{b_m,b_{m+1}}Q_m(f 1_{\Omega_m}, (g_\ell)_{\ell=1}^m))_{g_{m+1}},
\]

where \(E^{b_m,b_{m+1}}\) is the extension operator from Lemma 2.5 and \(Q_{b_m,b_{m+1}}\) is the solution operator from the single layer problem from Lemma 2.4. First, we check that this assignment is well-defined. It is clear that \((f 1_{\Omega_m}, (g_\ell)_{\ell=1}^m) \in X^m(\alpha)\) with \(\|f 1_{\Omega_m}, (g_\ell)_{\ell=1}^m\|_{X^m} \leq \|f, (g_\ell)_{\ell=1}^{m+1}\|_{X^{m+1}}\). Hence the first term appearing on the right hand side of the equality in (2.20) is a well-defined and continuous assignment. To check that these same properties hold for the second term too, we observe the following compatibility estimate:

\[
\begin{align*}
\left[g_{m+1} - \int_{(b_m,b_{m+1})} f + \int_{(b_m,b_{m+1})} \nabla \cdot E^{b_m,b_{m+1}}Q_m(f 1_{\Omega_m}, (g_\ell)_{\ell=1}^m)\right]_{H^{-1}} &\leq \left[g_{m+1} - g_m - \int_{(b_m,b_{m+1})} f\right]_{H^{-1}} \\
+ \left[g_m + \int_{(b_m,b_{m+1})} \nabla \cdot E^{b_m,b_{m+1}}Q_m(f 1_{\Omega_m}, (g_\ell)_{\ell=1}^m)\right]_{H^{-1}} &\leq 2\|f, (g_\ell)_{\ell=1}^{m+1}\|_{X^{m+1}} \\
&\quad + 2\pi \sqrt{b_{m+1} - b_m}\|E^{b_m,b_{m+1}}Q_m(f 1_{\Omega_m}, (g_\ell)_{\ell=1}^m)\|_{L^2}.
\end{align*}
\]

In the above we have employed the divergence compatibility estimate from Proposition 2.14 and the boundedness of the extension operator \(E^{b_m,b_{m+1}}\) from Lemma 2.5. Hence \(Q_{m+1}\) is well-defined and continuous. In the set \(\Omega_m\) we have the equality \(Q_m = Q_{m-1}\). The second term in the definition of \(Q_{m+1}\), equation (2.20), vanishes on \(\Sigma_m\) and the first term in the definition vanishes on \(\Sigma_{m+1}\); therefore, \(Q_{m+1}\) is a solution operator to the problem (2.2) in the \(m+1\)-prescribed normal trace case. This completes the induction. □

2.2. Isomorphism associated to multilayer traveling Stokes. In this subsection we construct a solution operator to the multilayer traveling Stokes problem with stress boundary conditions in equation (2.2). The validity of this section’s results over the fields \(\mathbb{R}\) and \(\mathbb{C}\) is integral to the proof of Theorem 2.19 in the next subsection. We remind the reader that the Euclidean inner product is sesquilinear with the left argument the linear one and that essential information regarding anti-duality can be found in Appendix A.2.

We begin by studying the weak formulation and showing the existence of weak solutions. First we focus on existence of a pressure with the following result.

Lemma 2.7 (Image of the gradient is the annihilator of solenoidal fields). Suppose that \(F \in (\mathfrak{a}H^1(\Omega; \mathbb{K}^n))^\mathfrak{u}\) vanishes on solenoidal fields. Then, there exists \(p \in L^2(\Omega; \mathbb{K})\) such that for all \(u \in \mathfrak{a}H^1(\Omega; \mathbb{K}^n)\)

\[
\langle F, u \rangle_{(\mathfrak{a}H^1)^\mathfrak{u}, \mathfrak{a}H^1} = \int_\Omega p \cdot (\nabla \cdot u).
\]

Proof. The case \(\mathbb{K} = \mathbb{R}\) is handled by Corollary 2.3 in [LT19]. We will show this is sufficient to justify the case \(\mathbb{K} = \mathbb{C}\) as well. Given an antilinear functional \(F \in (\mathfrak{a}H^1(\Omega; \mathbb{C}^n))^\mathfrak{u}\) we define the \(\mathbb{R}\)-linear functionals \(F_{\text{Re}}, F_{\text{Im}} \in (\mathfrak{a}H^1(\Omega; \mathbb{K}^n))^\mathfrak{u}\) via

\[
\langle F_{\text{Re}}, v \rangle = \text{Re}\left[\langle F, v \rangle\right], \quad \langle F_{\text{Im}}, v \rangle = \text{Re}\left[\langle F, iv \rangle\right] \quad \text{for } v \in \mathfrak{a}H^1(\Omega; \mathbb{R}^n).
\]

(2.23)
Observe that if $F$ annihilates solenoidal fields then $F_{\text{Re}}$ and $F_{\text{Im}}$ satisfy the hypothesis of the lemma for the $\mathbb{R}$-valued case. Therefore there are $q, r \in L^2(\Omega; \mathbb{R})$ such that for all $v, w \in H^1(\Omega; \mathbb{R}^n)$

$$\text{Re} \left[ (F, v + iw) \right] = \langle F_{\text{Re}}, v \rangle + \langle F_{\text{Im}}, w \rangle = \int_\Omega q \nabla \cdot v + r \nabla \cdot w = \text{Re} \left[ \int_\Omega (q + ir) \cdot (\nabla \cdot (v + iw)) \right].$$

(2.24)

This suggests that we set $p \in L^2(\Omega; \mathbb{C})$ via $p = q + ir$. It remains to check that $G \in (H^1(\Omega; \mathbb{C}^n))^\top$ defined via $(G, u) = (F, u) - \int_\Omega p \cdot (\nabla \cdot u) \in \mathbb{C}$ vanishes identically. The above computation shows that $\text{Re} \left[ (G, u) \right] = 0$ for all $u$. By antilinearity, the real part of $G$ determines $G$ entirely; i.e. for all $u \in H^1(\Omega; \mathbb{C}^n)$ it holds $\langle G, u \rangle = \text{Re} \left[ (G, u) + i\text{Re} \left[ (G, u) \right] \right]$. Thus $G = 0$ and the proof is complete. \hfill \Box

The truth of the two subsequent results in the $\mathbb{R}$-valued case is a consequence of Theorem 2.4 in [LT19]. We include a proof here in the $\mathbb{K}$-valued case for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

**Lemma 2.8.** For $\gamma \in \mathbb{R}^+$ we define the sesquilinear (bilinear if $\mathbb{K} = \mathbb{R}$) mapping $B_\gamma : H^1(\Omega; \mathbb{K}^n) \times H^1(\Omega; \mathbb{K}^n) \to \mathbb{C}$ via

$$B_\gamma(w, v) = \sum_{\ell=1}^m \int_{\Omega_\ell} \frac{\mu_\ell}{2} \nabla w : \nabla v - \gamma \rho_\ell \partial_1 w \cdot v.$$  

Then we have the identity

$$\text{Re} \left[ B_\gamma(w, w) \right] = \sum_{\ell=1}^m \int_{\Omega_\ell} \frac{\mu_\ell}{2} |\nabla w|^2, \text{ for all } w \in H^1(\Omega; \mathbb{K}^n).$$

(2.26)

In particular, $B_\gamma$ is coercive over the $H^1$-norm.

**Proof.** We observe that

$$- \gamma \sum_{\ell=1}^m \int_{\Omega_\ell} \rho_\ell \partial_1 w \cdot w = \gamma \sum_{\ell=1}^m \int_{\Omega_\ell} \rho_\ell w \cdot \partial_1 w = \sum_{\ell=1}^m \int_{\Omega_\ell} \rho_\ell \partial_1 w \cdot w \Rightarrow \gamma \sum_{\ell=1}^m \int_{\Omega_\ell} \rho_\ell \partial_1 w \cdot w = i\mathbb{R}. \quad (2.27)$$

Equation (2.26) follows. We now deduce that $B_\gamma$ is $H^1$-coercive by Korn’s inequality (see Appendix A.4). \hfill \Box

We now use the preceding lemmas to construct weak solutions to (2.1).

**Proposition 2.9 (Existence and uniqueness of weak solutions to (2.1)).** For $\gamma \in \mathbb{R}$ we define the mapping

$$\chi_\gamma : L^2(\Omega; \mathbb{K}) \times H^1(\Omega; \mathbb{K}^n) \to L^2(\Omega; \mathbb{K}) \times (H^1(\Omega; \mathbb{K}^n))^\top$$

via $\chi_\gamma(p, u) = (\nabla \cdot u, \mathcal{E}_\gamma(p, u)),$

(2.28)

where the antilinear functional $\mathcal{E}_\gamma(p, u) \in (H^1(\Omega; \mathbb{K}^n))^\top$ is defined via

$$\langle \mathcal{E}_\gamma(p, u), v \rangle_{(H^1)^\top} = - \int_\Omega p \cdot (\nabla \cdot v) + \sum_{\ell=1}^m \int_{\Omega_\ell} \frac{\mu_\ell}{2} \nabla u : \nabla v - \gamma \rho_\ell \partial_1 u \cdot v = - \int_\Omega S^u(p, u) : \nabla v - \gamma \sum_{\ell=1}^m \int_{\Omega_\ell} \rho_\ell \partial_1 u \cdot v \quad (2.29)$$

for $v \in H^1(\Omega; \mathbb{K}^n)$. Then $\chi_\gamma$ is a Hilbert isomorphism for all $\gamma \in \mathbb{R}$.

**Proof.** We begin by showing that $\chi_\gamma$ is a surjection. Thanks to the observation that the sesquilinear (bilinear when $\mathbb{K} = \mathbb{R}$) form $B_\gamma$ is bounded paired with the coercive estimate of Lemma 2.8, we are free to invoke the Lax-Milgram lemma (see Proposition A.5) for $B_\gamma$ on any closed subspace of $H^1(\Omega; \mathbb{K}^n)$.

Let $(g, F) \in L^2(\Omega; \mathbb{K}) \times (H^1(\Omega; \mathbb{K}^n))^\top$ be any data pair. Then Lax-Milgram implies that there exists a unique $w \in H^1(\Omega; \mathbb{K}^n)$ such that for all $v \in H^1(\Omega; \mathbb{K}^n)$

$$B_\gamma(w, v) = -B_\gamma(\Pi \Omega g, v) + \langle F, v \rangle_{(H^1)^\top}, \quad (2.30)$$

where $\Pi \Omega$ is the bounded right inverse to the divergence granted by Lemma 2.2. Next, we apply Lemma 2.7 to find that there is a pressure $p \in L^2(\Omega; \mathbb{K})$ such that for all $v \in H^1(\Omega; \mathbb{K}^n)$

$$B_\gamma(w, v) = -B_\gamma(\Pi \Omega g, v) + \langle F, v \rangle_{(H^1)^\top} + \int_\Omega p \cdot (\nabla \cdot v). \quad (2.31)$$

We may now conclude that $\chi_\gamma(p, w + \Pi \Omega g) = (g, F)$, showing this mapping to be a surjection.
On the other hand, suppose \((p, u) \in L^2(\Omega; \mathbb{K}) \times H^1(\Omega; \mathbb{K}^n)\) satisfy \(\chi_\gamma (p, u) = (g, F)\). Then we decompose \(u = w + \Pi_\Omega g\) and take \(v = w \in H^1(\Omega; \mathbb{K}^n)\) in identity (2.29) to see that
\[
\langle F, w \rangle_{(aH^1)^\tau, aH^1} = \langle \gamma_\gamma (p, u), w \rangle_{(aH^1)^\tau, aH^1} = B_\gamma (u, w) = B_\gamma (u, u) - B_\gamma (u, \Pi_\Omega g).
\] (2.32)

Again employing Lemma 2.2, we deduce from this that
\[
\|u\|_{aH^1}^2 \lesssim B_\gamma (u, u) \lesssim (\|g\|_{L^2} + \|F\|_{(aH^1)^\tau}) \|u\|_{aH^1}, \text{ and so } \|u\|_{aH^1} \lesssim \|g\|_{L^2} + \|F\|_{(aH^1)^\tau}.
\] (2.33)

We then take \(v = \Pi_\Omega p\) in (2.29) and use (2.33) to deduce that
\[
\|p\|_{L^2} \lesssim \|u\|_{aH^1} + \|F\|_{(aH^1)^\tau} \lesssim \|g\|_{L^2} + \|F\|_{(aH^1)^\tau}.
\] (2.34)

Estimates (2.33) and (2.34) show that \(\chi_\gamma^{-1}\) commutes with tangential multipliers (see Appendix A.3).

**Lemma 2.10.** Let \(\gamma \in \mathbb{R}\) and \(\omega \in L^\infty(\mathbb{R}^{n-1}; \mathbb{C}_+)_\mathbb{C}\), and consider the tangential multiplier \(M_\omega\) as defined in Definitions A.11 and A.13. If \((g, F) \in L^2(\Omega; \mathbb{C}) \times (0H^1(\Omega; \mathbb{C}^n))^\tau\) and \((p, u) = \chi_\gamma^{-1}(g, F)\), then \((M_\omega p, M_\omega u) = \chi_\gamma^{-1}(M_\omega g, M_\omega F)\).

**Proof.** We simply check that \(M_\omega g = M_\omega \nabla \cdot u = \nabla \cdot M_\omega u\) and note that if \(v \in (0H^1(\Omega; \mathbb{C}^n))^*_\mathbb{C}\), then
\[
\langle M_\omega F, v \rangle_{(aH^1)^\tau, aH^1} = \langle F, M_\omega v \rangle_{(aH^1)^\tau, aH^1} = \int_\Omega S^u (M_\omega p, M_\omega u) : \nabla v - \gamma \sum_{\ell=1}^m \int_\Omega \rho_\ell \partial_\ell M_\omega u \cdot v.
\] (2.35)

Therefore, \(\chi_\gamma (M_\omega p, M_\omega u) = (M_\omega g, M_\omega F)\). □

Next, we examine the regularity of weak solutions. We make the following notation.

**Definition 2.11.** For \(s \in \mathbb{R}_+ \cup \{0\}\) we define the continuous and linear maps
\[
\mathcal{G} : \prod_{\ell=1}^m H^{-1/2}(\Sigma_\ell; \mathbb{K}^n) \to \big(0H^1(\Omega; \mathbb{K}^n)\big)^\tau, \quad \mathcal{P} : H^s(\Omega; \mathbb{K}) \times \prod_{\ell=1}^m H^{1/2+s}(\Sigma_\ell; \mathbb{K}^n) \to \big(0H^1(\Omega; \mathbb{K}^n)\big)^\tau
\] (2.36)
with actions on \(v \in 0H^1(\mathbb{R}^n; \mathbb{K}^n)\) given by
\[
\langle \mathcal{G}(\phi_\ell)_{\ell=1}^m, v \rangle_{aH^1}^\tau = \sum_{\ell=1}^m \langle \phi_\ell, \text{Tr}(\Sigma_\ell v) \rangle_{H^{-1/2}, H^{1/2}}
\] (2.37)
for \((\phi_\ell)_{\ell=1}^m \in \prod_{\ell=1}^m H^{-1/2}(\Sigma_\ell; \mathbb{K}^n)\), and
\[
\langle \mathcal{P}(f, (k_\ell)_{\ell=1}^m), v \rangle_{(aH^1)^\tau} = \int_\Omega f \cdot v + \sum_{\ell=1}^m \int_{\Sigma_\ell} k_\ell \cdot v,
\] (2.38)
for \(f \in H^s(\Omega; \mathbb{K}^n)\) and \((k_\ell)_{\ell=1}^m \in \prod_{\ell=1}^m H^{1/2+s}(\Sigma_\ell; \mathbb{K}^n)\).

We can now state our regularity result.

**Proposition 2.12** (Regularity of weak solutions to (2.1)). Let \(s \in \mathbb{R}_+ \cup \{0\}\) and
\[
(g, f, (k_\ell)_{\ell=1}^m) \in H^{1+s}(\Omega; \mathbb{K}) \times H^s(\Omega; \mathbb{K}^n) \times \prod_{\ell=1}^m H^{1/2+s}(\Sigma_\ell; \mathbb{K}^n).
\] (2.39)

Suppose that \((p, u) \in L^2(\Omega; \mathbb{K}) \times 0H^1(\Omega; \mathbb{K}^n)\) are such that \(\chi_\gamma (p, u) = (g, \mathcal{P}(f, (k_\ell)_{\ell=1}^m))\). Then, in fact, we also have the inclusions \(p \in H^{1+s}(\Omega; \mathbb{K})\) and \(u \in H^{2+s}(\Omega; \mathbb{K}^n)\), as well as the universal estimate
\[
\sum_{\ell=1}^m \left[ \|p\|_{H^{1+s}(\Omega_\ell)} + \|u\|_{H^{2+s}(\Omega_\ell)} \right] \lesssim \sum_{\ell=1}^m \left[ \|g\|_{H^{1+s}(\Omega_\ell)} + \|f\|_{H^s(\Omega_\ell)} + \|k_\ell\|_{H^{1/2+s}} \right].
\] (2.40)

Finally the pair \((p, u)\) are a strong solution to the multilayer traveling stokes problem with stress boundary conditions in equation (2.1) with data tuple \((g, f, (k_\ell)_{\ell=1}^m)\).

**Proof.** This is a standard induction and interpolation argument based on applying horizontal difference quotients to derive control of horizontal derivatives and then exploiting the elliptic structure of the Stokes operator to control the vertical derivatives. For a sketch we refer the reader to the real valued one layer case in Theorem 2.5 of [LT19]. □
Theorem 2.13 (Existence and uniqueness of strong and classical solutions to (2.1)). Let \( s \in \mathbb{R}^+ \cup \{0\} \) and \( \gamma \in \mathbb{R} \). Define the bounded linear operator

\[
\Phi : H^{1+s}(\Omega; \mathbb{K}) \times H^s(\Omega; \mathbb{K}^n) \to H^{1+s}(\Omega; \mathbb{K}) \times H^s(\Omega; \mathbb{K}^n) \times \prod_{\ell=1}^m H^{1/2+s}(\Sigma_\ell; \mathbb{K}^n)
\]

with the assignment

\[
\Phi(p, u) = (\nabla \cdot u, \sum_{\ell=1}^m \sum_{j=1}^\infty [\nabla \cdot S^m(p, u) - \gamma \rho e \partial_1 u], (\nabla S^m(p, u) \cdot e_n)_{\ell=1}^m).
\]

Then \( \Phi \) is a Hilbert isomorphism.

Proof. Proposition 2.9 shows \( \Phi \) to be injective. Propositions 2.9 and 2.12 show that \( \Phi \) is a surjection. \( \square \)

2.3. Analysis of the normal stress to normal Dirichlet pseudodifferential operator. In this subsection we study a \( \Psi \)DO built from \( \Phi \). We make the following definition.

Definition 2.14 (Normal stress to normal Dirichlet \( \Psi \)DO). Let \( \gamma \in \mathbb{R} \) and \( s \in \mathbb{R}^+ \cup \{-1, 0\} \). We define the normal stress to normal Dirichlet pseudodifferential operator to be the bounded linear mapping \( \nu : \prod_{\ell=1}^m H^{1/2+s}(\Sigma_\ell; \mathbb{K}) \to \prod_{\ell=1}^m H^{3/2+s}(\Sigma_\ell; \mathbb{K}) \) given by

\[
\nu(\varphi e_n) = (\text{Tr}_{\Sigma_\ell} u \cdot e_n)_{\ell=1}^m,
\]

for \( (p, u) \in H^{1+s}(\Omega; \mathbb{K}) \times H^s(\Omega; \mathbb{K}^n) \) the unique solution to the normal stress problem:

\[
\begin{cases}
\nabla \cdot S^m(p, u) + \gamma \rho e \partial_1 u = 0 & \text{in } \Omega, \ell \in \{1, \ldots, m\} \\
\n\nabla \cdot u = 0 & \text{in } \Omega \\
\n[u]_{\ell} = \psi e_n & \text{on } \Sigma_\ell, \ell \in \{1, \ldots, m\} \\
\n[u]_{\ell} = 0 & \text{on } \Sigma_0.
\end{cases}
\]

In other words, \( (p, u) = \chi_{-\gamma}^{-1}(0, \Theta(\psi e_n))_{\ell=1}^m ) \) for the operators \( \Theta \) and \( \chi_{-\gamma}^{-1} \) from Definition 2.11 and Proposition 2.9, respectively (note the minus sign preceding \( \gamma \)).

Remark 2.15. The boundedness of \( \nu \) is a consequence of the boundedness of \( \Theta \), Proposition 2.9, and Theorem 2.13. The restriction \( s \in \mathbb{R}^+ \cup \{-1, 0\} \) is not important. By interpolation theory we are free to take any \( s \in [-1, \infty) \) in the previous definition statement.

We begin by proving that the Fourier transform diagonalizes \( \nu \). This gives us a representation of this linear mapping as a frequency space multiplication operator.

Proposition 2.16 (Diagonalization of \( \nu \)). There exists a bounded and measurable matrix field \( \mathbf{n} : \mathbb{R}^n \to \mathbb{C}^{m \times m} \) such that \( \mathbf{n}(\xi) = \nu(\xi) \) for a.e. \( \xi \in \mathbb{R}^n \) and for all \( s \in [-1, \infty) \) and all \( (\varphi e_n)_{\ell=1}^m \in \prod_{\ell=1}^m H^{1/2+s}(\Sigma_\ell; \mathbb{K}) \) we have the equality

\[
\mathcal{F}[\nu(\varphi e_n)]_{\ell=1}^m(\xi) = \mathbf{n}(\xi) \mathcal{F}[\varphi e_n](\xi) \text{ for a.e. } \xi \in \mathbb{R}^n. \tag{2.45}
\]

Moreover, there exists a constant \( c \in \mathbb{R}^+ \), depending only on the physical parameters, such that for a.e. \( \xi \in \mathbb{R}^n \) we have \( \mathbf{n}(\xi) \leq c(1 + |\xi|^2)^{-1/2} \).

Proof. Let \( j, k \in \{1, \ldots, m\} \). Define \( \nu_{jk}^{\delta,k} : H^{1/2}(\Sigma_j; \mathbb{K}) \to H^{1/2}(\Sigma_j; \mathbb{K}) \) via \( \nu_{jk}^{\delta,k} = \text{Tr}_{\Sigma_j} u \cdot e_n \) for \( (p, u) \) a solution pair to the normal stress PDE (2.44) with normal stress \( \psi \) on the surface \( \Sigma_k \). In other words,

\[
(p, u) = \chi_{-\gamma}^{-1}(0, \Theta(\psi e_n))_{\ell=1}^m, \quad \text{for } \delta \text{ the Kronecker delta.} \tag{2.46}
\]

It is clear that this assignment is bounded and, by Lemma 2.10, translation invariant. We are in a position to apply Proposition A.10 to obtain a measurable function \( \mathbf{n}^{\delta,k} : \mathbb{R}^n \to \mathbb{C} \) such that \( \mathbf{n}^{\delta,k}(\xi) = \mathbf{n}^{\delta,k}(\xi) \) for a.e. \( \xi \in \mathbb{R}^n \), that obeys the estimate

\[
\text{esssup}(1 + |\xi|^2)^{-1/2} |\mathbf{n}^{\delta,k}(\xi)| : \xi \in \mathbb{R}^n \leq 2 \|\nu_{jk}^{\delta,k}\|_{\mathcal{L}(H^{1/2},H^{1/2})}, \tag{2.47}
\]

and satisfies the identity \( \mathcal{F}[\nu_{jk}^{\delta,k}] = \mathbf{n}^{\delta,k} \mathcal{F}[\psi] \) for all \( \psi \in H^{1/2}(\Sigma_k; \mathbb{K}) \).
We set \( n_\gamma : \mathbb{R}^{n-1} \to \mathbb{C}^{m \times m} \) via \( n_\gamma(\xi)_{j,k} = n_j^{\gamma,k}(\xi) \) for \( \xi \in \mathbb{R}^{n-1} \) and \( j,k \in \{1, \ldots, m\} \). The following computation verifies that this matrix field is the sought after spectral representation:

\[
\mathcal{F}[\nu_\gamma(\psi_\ell)_\ell=1] = \sum_{j,k=1}^m (\mathcal{F}[\nu_\gamma(\psi_\ell\delta_{\ell,k})_\ell=1] \cdot e_j) e_j = \sum_{j,k=1}^m (\mathcal{F}[\nu_\gamma^{j,k} \psi_\ell] \cdot e_j) e_j = \sum_{j,k=1}^m (n^{j,k}_\gamma \mathcal{F}[\psi_\ell] \cdot e_j) e_j = n_\gamma \mathcal{F}[(\psi_\ell)_\ell=1].
\] (2.48)

We next observe how the multiplier \( n_\gamma \) changes under the map \( \gamma \mapsto -\gamma \).

**Proposition 2.17 (Adjoint of the normal stress to normal Dirichlet multiplier).** If \( \gamma \in \mathbb{R}^+ \) then for a.e. \( \xi \in \mathbb{R}^{n-1} \) we have the adjoint identity \( n_\gamma(\xi)^* = n_{-\gamma}(\xi) \).

**Proof.** Let \( (\psi_\ell)_\ell=1, (\phi_\ell)_\ell=1 \in \prod_{\ell=1}^m H^{-1/2}(\Sigma_\ell; \mathbb{C}) \) and denote \( (p, u) = \chi_{-\gamma}^{-1}(0, \mathcal{O}(\psi_\ell e_n)_\ell=1) \), \( (q, v) = \chi_{\gamma}^{-1}(0, \mathcal{O}(\psi_\ell e_n)_\ell=1) \), where we recall \( \chi_\gamma \) and \( \chi_{-\gamma} \) are defined in Proposition 2.9 and \( \mathcal{O} \) is from Definition 2.11. By testing in the weak formulation for \( (p, u) \), recalling that \( \nabla \cdot v = \nabla \cdot u = 0 \), and integrating by parts, we obtain the identities

\[
\langle (\psi_\ell)_\ell=1, (\operatorname{Tr}_\Sigma v \cdot e_n)_\ell=1 \rangle_{H^{-1/2}, H^{1/2}} = \sum_{\ell=1}^m \int_{\Omega} \frac{\mu_{\ell}}{2} \nabla u \cdot \nabla \psi_\ell + \gamma \rho_\ell \partial_1 u \cdot v = \sum_{\ell=1}^m \int_{\Omega} \frac{\mu_{\ell}}{2} \nabla u \cdot \nabla \psi_\ell - \gamma \rho_\ell \partial_1 v \cdot u = \langle (\phi_\ell)_\ell=1, (\operatorname{Tr}_\Sigma u \cdot e_n)_\ell=1 \rangle_{H^{1/2}, H^{-1/2}} = \langle (\phi_\ell)_\ell=1, (\psi_\ell)_\ell=1 \rangle_{H^{-1/2}, H^{1/2}}.
\] (2.49)

Hence, we may apply Propositions 2.6 and 2.16 along with Definition 2.14 to deduce that

\[
\int_{\mathbb{R}^{n-1}} \mathcal{F}[(\psi_\ell)_\ell=1] \cdot n_{-\gamma} \mathcal{F}[(\phi_\ell)_\ell=1] = \int_{\mathbb{R}^{n-1}} n_\gamma \mathcal{F}[(\psi_\ell)_\ell=1] \cdot \mathcal{F}[(\phi_\ell)_\ell=1].
\] (2.50)

Let \( a, b \in \mathbb{C}^m \) and \( \psi, \phi \in L^2(\mathbb{R}^{n-1}; \mathbb{C}) \). In (2.50) we are free to take \( (\psi_\ell)_\ell=1 = (a_\ell \psi)_\ell=1 \) and \( (\phi_\ell)_\ell=1 = (b_\ell \phi)_\ell=1 \) to see that

\[
\int_{\mathbb{R}^{n-1}} (a \cdot n_{-\gamma} b - n_a \cdot b) \mathcal{F}[\psi] \cdot \mathcal{F}[\phi] = 0.
\] (2.51)

As \( \phi \) and \( \psi \) are arbitrary, we deduce that, up to a null set depending on \( a \) and \( b \), we may equate \( a \cdot n_{-\gamma} b = n_a \cdot b \). By letting \( a \) and \( b \) range over the members of the standard basis of \( \mathbb{C}^m \) and recalling that a countable union of null sets is, again, a null set we conclude that \( n_\gamma(\xi)^* = n_{-\gamma}(\xi) \) for a.e. \( \xi \in \mathbb{R}^{n-1} \).

The remainder of this subsection is devoted to a more precise asymptotic development of the matrix field \( n_\gamma \) as \( \xi \to 0 \) and \( \xi \to \infty \), which we achieve with energy estimates. Recall that from Proposition 2.9 and trace theory we have the equivalence

\[
\| (\psi_\ell)_\ell=1 \|_{H^{-1/2}} \asymp \| u \|_{qH^1} + \| p \|_{L^2}
\] (2.52)

for \( (\psi_\ell)_\ell=1 \in \prod_{\ell=1}^m H^{-1/2}(\Sigma_\ell; \mathbb{K}) \) and \( (p, u) = \chi_{-\gamma}^{-1}(0, \mathcal{O}(\psi_\ell e_n)_\ell=1) \). Our next result shows that if we weaken the control of \( (\psi_\ell)_\ell=1 \) at low frequencies on the Fourier side, then we can remove \( p \) from the right. The resulting equivalence will play a key role in our asymptotic developments of \( n_\gamma \).

**Theorem 2.18 (Normal stress and velocity energy equivalence).** For \( \gamma \in \mathbb{R} \) there exists a constant \( c \in \mathbb{R}^+ \), depending also on the remaining physical parameters, for which we have the equivalence

\[
c^{-1} \| u \|_{qH^1} \leq \left( \int_{\mathbb{R}^{n-1}} \min\{\|\xi\|^2, |\xi|^{-1}\} |\mathcal{F}[(\psi_\ell)_\ell=1]|(\xi) |^2 \; d\xi \right)^{1/2} \leq c \| u \|_{qH^1}
\] (2.53)

for all data \( (\psi_\ell)_\ell=1 \in \prod_{\ell=1}^m H^{-1/2}(\Sigma_\ell; \mathbb{K}) \), where \( (p, u) \in L^2(\Omega; \mathbb{K}) \times_0 H^1(\Omega; \mathbb{K}^n) \) is the uniquely determined weak solution to the normal stress problem in equation (2.44), i.e. \( (p, u) = \chi_{-\gamma}^{-1}(0, \mathcal{O}(\psi_\ell e_n)_\ell=1) \) for the operators \( \mathcal{O} \) and \( \chi_{-\gamma}^{-1} \) from Definition 2.11 and Proposition 2.9.
Proof. We begin by proving the left inequality of (2.53). By the definition of \((p,u) = \chi_{-\gamma}^{-1}(0, \mathcal{O}(\psi_\ell e_n)_{n=1}^m)\) we have that for all \(v \in H^1(\Omega; \mathbb{K}^n)\)
\[
\frac{1}{2} \int_{\Omega} (p \cdot \nabla v + \gamma \rho \partial_1 u \cdot v - \int_{\Omega} p \cdot (\nabla \cdot v) + \int_{\Omega} (\psi_\ell \cdot (\mathcal{T}_n u \cdot e_n))_{H^{-1/2},H^{1/2}}.
\]
Taking \(v = u\) in this equation and taking the real part (see Lemma 2.8) implies that
\[
\|u\|_{0,H^1} \lesssim \text{Re} \left[ \sum_{\ell=1}^m \langle \psi_\ell, \mathcal{T}_n u \cdot e_n \rangle_{H^{-1/2},H^{1/2}} \right] = \text{Re} \left[ \int_{\mathbb{R}^{n-1}} \mathcal{F} \left[ \psi_\ell \right]_{\ell=1}^m \cdot \mathcal{F} \left[ (\mathcal{T}_n u \cdot e_n)_{\ell=1}^m \right] \right]
\]
\[
\leq \left( \int_{\mathbb{R}^{n-1}} \min\{|\xi|^2,|\xi|\} \mathcal{F} \left[ (\mathcal{T}_n u \cdot e_n)_{\ell=1}^m \right] (\xi) \right)^{1/2} \left( \int_{\mathbb{R}^{n-1}} \max\{|\xi|^{-2},|\xi|\} \mathcal{F} \left[ (\mathcal{T}_n u \cdot e_n)_{\ell=1}^m \right] (\xi) \right)^{1/2},
\]
where we have used that \(u\) is solenoidal, Korn’s inequality (see Appendix A.4), and the (anti-)duality of Sobolev spaces (see Appendix A.2). Next we use the boundedness of traces and the divergence compatibility estimate of Proposition 2.1 to further bound
\[
\left( \int_{\mathbb{R}^{n-1}} \max\{|\xi|^{-2},|\xi|\} \mathcal{F} \left[ (\mathcal{T}_n u \cdot e_n)_{\ell=1}^m \right] (\xi) \right)^{1/2} \leq \|(\mathcal{T}_n u \cdot e_n)_{\ell=1}^m\|_{H^{1/2} \cap H^{-1}} \lesssim \|u\|_{0,H^1}.
\]
Combining estimates (2.55) and (2.56) gives the left hand inequality of (2.53).
We now prove the right inequality of (2.53). For \(\ell \in \{1, \ldots, m\}\) define \(\phi_\ell \in H^{1/2}(\Sigma_n; \mathbb{K}) \cap H^{-1}(\Sigma_n; \mathbb{K})\) via
\[
\mathcal{F} \left[ \phi_\ell \right] (\xi) = \min\{|\xi|^2,|\xi|^{-1}\} \mathcal{F} \left[ \psi_\ell \right] (\xi) \quad \text{for } \xi \in \mathbb{R}^{n-1}.
\]
We bound the norm of \(\phi_\ell\) as follows:
\[
\|\phi_\ell\|_{H^{1/2} \cap H^{-1}}^2 \leq 2 \int_{\mathbb{R}^{n-1}} \max\{|\xi|^2,|\xi|\} \min\{|\xi|^2,|\xi|^{-1}\} \mathcal{F} \left[ \psi_\ell \right] (\xi)^2 d\xi
\]
\[
= 2 \int_{\mathbb{R}^{n-1}} \min\{|\xi|^2,|\xi|^{-1}\} |\mathcal{F} \left[ \psi_\ell \right] (\xi)|^2 d\xi.
\]
We are in a position to apply Theorem 2.7 to obtain \(Q_m(0, (\phi_\ell)_{\ell=1}^m) \in H^1(\Omega; \mathbb{K}^n)\) with the estimate
\[
\|Q_m(0, (\phi_\ell)_{\ell=1}^m)\|_{0,H^1}^2 \leq \sum_{\ell=1}^m \|\phi_\ell\|_{H^{1/2} \cap H^{-1}}^2 \lesssim \int_{\mathbb{R}^{n-1}} \min\{|\xi|^2,|\xi|^{-1}\} |\mathcal{F} \left[ \psi_\ell \right]_{\ell=1}^m (\xi)|^2 d\xi.
\]
Testing \(v = Q_m(0, (\phi_\ell)_{\ell=1}^m)\) in the weak formulation of the normal stress PDE in equation (2.54) and using Proposition A.6 gives the identity
\[
\sum_{\ell=1}^m \int_{\mathbb{R}^{n-1}} \min\{|\xi|^2,|\xi|^{-1}\} |\mathcal{F} \left[ \psi_\ell \right]_{\ell=1}^m (\xi)|^2 d\xi = \sum_{\ell=1}^m \int_{\Omega} \frac{\mu}{2} Dv : DQ_m(0, (\phi_\ell)_{\ell=1}^m) + \gamma \rho \partial_1 u \cdot Q_m(0, (\phi_\ell)_{\ell=1}^m).
\]
The right inequality of (2.53) now follows by applying the Cauchy-Schwarz inequality to the right hand side of (2.60) and then utilizing estimate (2.59). \[\square\]

We are now in a position for finer asymptotic development of the symbol to the normal stress to normal Dirichlet \(\Psi\)DO.

Theorem 2.19 (Asymptotics of normal stress to normal Dirichlet multiplier). For each \(\gamma \in \mathbb{R}\) there exists a constant \(C \in \mathbb{R}^+\), depending only on the physical parameters, such that for a.e. \(\xi \in \mathbb{R}^{n-1}\) the following hold.

1. We have the estimate \(|n_\gamma(\xi)| \leq C \min\{|\xi|^2,|\xi|^{-1}\}|.
2. Letting \(\partial B_C(0,1)\) denote the unit sphere of \(\mathbb{C}^{n-1}\), we have the bound
\[
\min_{a \in \partial B_C(0,1)} \text{Re} \left[ n_\gamma(\xi) a \cdot a \right] \geq C^{-1} \min\{|\xi|^2,|\xi|^{-1}\}.
\]
3. The matrix \(n_\gamma(\xi)\) is invertible, and \(|n_\gamma^{-1}(\xi)| \leq C \max\{|\xi|^{-2},|\xi|\}|.
Proof. For item one, we first use the divergence compatibility estimates from Proposition 2.1. If \((\psi_{\ell})_{\ell=1}^{m} \in \prod_{\ell=1}^{m} H^{-1/2}(\Sigma_{\ell};\mathbb{C})\) are normal stress data then their associated velocity field \(u\) solving (2.44) is solenoidal and vanishing on \(\Sigma_{0}\). Hence, the normal traces on the hyperplanes \(\Sigma_{\ell}, \ell \in \{1, \ldots, m\}\) belong to \(H^{1/2} \cap H^{-1}\). In fact, we may bound with the divergence compatibility estimate and then the left inequality of Theorem 2.18 to obtain the bound

\[
\|\nabla (\psi_{\ell})_{\ell=1}^{m}\|_{H^{-1/2}} \leq \|u\|_{L^{2}}^{2} \leq \|u\|_{H^{1}}^{2} \leq \int_{\mathbb{R}^{n-1}} \min\{|\xi|^{2}, |\xi|^{-1}\} \big|\mathcal{F} [(\psi_{\ell})_{\ell=1}^{m}] (\xi)\big|^{2} \, d\xi
\]  

(2.62)

for all \((\psi_{\ell})_{\ell=1}^{m} \in \prod_{\ell=1}^{m} H^{-1/2}(\Sigma_{\ell};\mathbb{C})\).

Let \(b = (b_{1}, \ldots, b_{m}) \in (\mathbb{Q} + i\mathbb{Q})^{m}\) and \(\varphi \in L^{1}(\mathbb{R}^{n-1};\mathbb{R})\) such that \(\varphi(\xi) \geq 0\) for a.e. \(\xi \in \mathbb{R}^{n-1}\) and the support of \(\varphi\) is compact. Set \(\phi = \bigcap_{\ell \in \mathbb{N}} H^{1}(\mathbb{R}^{n-1};\mathbb{C}) \equiv \mathcal{F}^{-1} [\sqrt{\varphi}]\). We take \((\psi_{\ell})_{\ell=1}^{m} = (b_{\ell} \phi)_{\ell=1}^{m}\) in inequality (2.62) and use Proposition 2.16 to see that

\[
\int_{\mathbb{R}^{n-1}} |\xi|^{-2} |\nabla (\psi_{\ell})_{\ell=1}^{m} (\xi)|^{2} \, d\xi \leq c \int_{\mathbb{R}^{n-1}} \min\{|\xi|^{2}, |\xi|^{-1}\} |b|^{2} \varphi(\xi) \, d\xi.
\]

(2.63)

This inequality holds for all \(\varphi\) as above. Hence there exists \(E_{b} \subseteq B(0,1) \subset \mathbb{R}^{n-1}\) with \(\mathcal{L}^{n-1}(B(0,1) \setminus E_{b}) = 0\) and

\[
|\nabla (\psi_{\ell})_{\ell=1}^{m} (\xi)|^{2} \leq c |\xi|^{4} |b|^{2}, \quad \forall \xi \in E_{b}, \forall b \in (\mathbb{Q} + i\mathbb{Q})^{m}.
\]

(2.64)

Set \(E = \bigcap_{b \in (\mathbb{Q} + i\mathbb{Q})^{m}} E_{b}\) and note that since \((\mathbb{Q} + i\mathbb{Q})^{m}\) is countable, \(\mathcal{L}^{n-1}(B(0,1) \setminus E) = 0\). Then (2.64) implies that \(|\nabla (\psi_{\ell})_{\ell=1}^{m} (\xi)| \leq c |\xi|^{4} |b| \) for all \(\xi \in E\) and all \(b \in (\mathbb{Q} + i\mathbb{Q})^{m}\), but then by the density of \((\mathbb{Q} + i\mathbb{Q})^{m}\) in \(\mathbb{C}^{m}\) we find that this estimate continues to hold for all \(\xi \in E\) and \(b \in \mathbb{C}^{m}\). In turn, taking the supremum over \(b \in \mathbb{C}^{m}\) with \(|b| = 1\) and using the equivalence of the operator norm and Euclidean norm on \(\mathbb{C}^{m\times m}\), we deduce that there is a constant \(c > 0\), depending only on the physical parameters, such that \(|\nabla (\psi_{\ell})_{\ell=1}^{m} (\xi)| \leq c |\xi|^{2}\) for a.e. \(\xi \in B(0,1) \subset \mathbb{R}^{n-1}\). Combining this with the estimate from Proposition 2.16 then proves the first item.

We next prove the second item. Again we let \(b = (b_{1}, \ldots, b_{m}) \in (\mathbb{Q} + i\mathbb{Q})^{m}\) such that \(\varphi(\xi) \geq 0\) for a.e. \(\xi \in \mathbb{R}^{n-1}\) and the support of \(\varphi\) is compact. Set \(\phi = \bigcap_{\ell \in \mathbb{N}} H^{1}(\mathbb{R}^{n-1};\mathbb{C}) \equiv \mathcal{F}^{-1} [\sqrt{\varphi}]\). Notice that \((b_{\ell} \phi)_{\ell=1}^{m} \in \prod_{\ell=1}^{m} H^{-1/2}(\Sigma_{\ell};\mathbb{C})\). Thanks to Proposition 2.9 there are \((p, u) \in L^{2}(\mathbb{Q};\mathbb{C}) \times 0 H^{1}(\mathbb{Q};\mathbb{R}^{n})\), a weak solution to (2.44) with data \((b_{\ell} \phi)_{\ell=1}^{m}\), i.e. \((p, u) = \chi_{-\gamma^{-1}}(0, \mathcal{G}(b_{\ell} \phi)_{\ell=1}^{m})\) for the operators \(\mathcal{G}\) and \(\chi_{-\gamma^{-1}}\) from Definition 2.11 and Proposition 2.9 respectively. We test \(u\) itself in the weak formulation and use equation (2.26) to obtain the identity

\[
\mathcal{R}e[(b_{\ell} \phi)_{\ell=1}^{m}, \nabla (b_{\ell} \phi)_{\ell=1}^{m}]_{H^{-1/2},H^{1/2}} = \sum_{\ell=1}^{m} \int_{\Omega_{\ell}} \frac{\mu}{2} \|Du\|^{2}.
\]

(2.65)

Next we use the diagonalization of \(\nabla \varphi \) from Proposition 2.16 (anti-)duality (Proposition A.9), and finally the right inequality of Theorem 2.18 to obtain the estimate

\[
\int_{\mathbb{R}^{n-1}} \mathcal{R}e[b \cdot \nabla \varphi] \, d\xi = \mathcal{R}e[\sum_{\ell=1}^{m} (b_{\ell} \phi_{\ell} \cdot \mathcal{T}_{\Sigma_{\ell}} u \cdot e_{\ell})] = \sum_{\ell=1}^{m} \int_{\Omega_{\ell}} \mathcal{F} |\nabla u|^{2} \geq \tilde{c} \int_{\mathbb{R}^{n-1}} \min\{|\xi|^{2}, |\xi|^{-1}\} |b|^{2} \varphi(\xi) \, d\xi,
\]

(2.66)

where \(\tilde{c} \in \mathbb{R}^{+}\) depends only on the physical parameters. Therefore, there exists \(E_{b} \subseteq \mathbb{R}^{n-1}\) such that \(\mathcal{L}^{n-1}(\mathbb{R}^{n-1} \setminus E_{b}) = 0\) and

\[
\mathcal{R}e[b \cdot \nabla \varphi(\xi)] b \ni \mathcal{R}e[\min\{|\xi|^{2}, |\xi|^{-1}\} |b|^{2}, \forall \xi \in E_{b}, \forall b \in (\mathbb{Q} + i\mathbb{Q})^{m}.
\]

(2.67)

Set \(F = \bigcap_{b \in (\mathbb{Q} + i\mathbb{Q})^{m}} E_{b}\) and note \(\mathcal{L}^{n-1}(\mathbb{R}^{n-1} \setminus F) = 0\). If \(\xi \in F\) then by density of \((\mathbb{Q} + i\mathbb{Q})^{m}\) in \(\mathbb{C}^{m}\), we have

\[
\min_{b \in \partial B_{C}(0,1)} \mathcal{R}e[b \cdot \nabla \varphi(\xi)] b \ni \mathcal{R}e[\min\{|\xi|^{2}, |\xi|^{-1}\}]
\]

(2.68)

This proves the second item. In particular, this also shows that \(\nabla \varphi(\xi)\) has trivial kernel and thus is invertible. It remains only to estimate the inverse. If \(d \in \partial B_{C}(0,1)\) then there exists \(b \in \mathbb{C}^{m}\) such that \(\nabla \varphi(\xi) b = d\). Thus

\[
\tilde{c} \min\{|\xi|^{2}, |\xi|^{-1}\} |b|^{2} \leq \mathcal{R}e[b \cdot \nabla \varphi(\xi)] b = \mathcal{R}e[b \cdot d] \leq |b| \Rightarrow |\nabla \varphi(\xi) b| \leq \tilde{c}^{-1} \max\{|\xi|^{2}, |\xi|^{-1}\}.
\]

(2.69)
Taking the supremum over such \( d \) and again using the equivalence of the Euclidean and operator norms on \( \mathbb{C}^{m \times m} \), we complete the proof of the third item. \( \square \)

3. Overdetermined multilayer traveling Stokes

In this and the other remaining sections we exclusively study the \( \mathbb{R} \)-valued solvability for the PDE systems considered. We next turn our attention to the following variant of system (3.1):

\[
\begin{align*}
\nabla \cdot S^\mu (p, u) - \gamma \rho \partial_t u &= f & \text{in } \Omega_\ell, \ & \ell \in \{1, \ldots, m\} \\
\nabla \cdot u &= g & \text{in } \Omega \\
[S^\mu (p, u) e_n]_\ell &= k_\ell & \text{on } \Sigma_\ell, \ & \ell \in \{1, \ldots, m\} \\
u \cdot e_n &= h_\ell & \text{on } \Sigma_\ell, \ & \ell \in \{1, \ldots, m-1\} \\
u | = 0 & \text{on } \Sigma_0,
\end{align*}
\]

with unknown velocity \( u \), pressure \( p \), and prescribed data \( f, g, (k_\ell)_{\ell=1}^m, \) and \( (h_\ell)_{\ell=1}^m \). We recall that we are continuing to use the abbreviated notation for \( \Omega, \Omega_\ell, \) and \( \Sigma_\ell \) discussed at the start of Section 2 and that \( q = \{\mu_\ell\}_{\ell=1}^m \subset \mathbb{R}^+ \) are the viscosity parameters, \( \rho = \{\rho_\ell\}_{\ell=1}^m \subset \mathbb{R}^+ \) are the density parameters, and \( \gamma \in \mathbb{R} \) is the wave speed. Our analysis in the previous section shows that the data \( f, g, \) and \( (k_\ell)_{\ell=1}^m \) entirely determine the pressure and velocity field and hence the normal traces \( (h_\ell)_{\ell=1}^m \). In this sense problem (3.1) is overdetermined, so we cannot expect to solve it for general data.

3.1. Data compatibility and associated isomorphism. In this subsection we find the range of appropriate data for system (3.1). We begin introducing the following notation.

Definition 3.1. For \( \gamma \in \mathbb{R} \) we define the \( \mathbb{R} \)-bilinear mapping \( \mathcal{H}^\gamma : [L^2(\Omega) \times (0H^1(\Omega; \mathbb{R}^n))^* \times \prod_{\ell=1}^m H^{1/2}(\Sigma_\ell)] \times \prod_{\ell=1}^m H^{-1/2}(\Sigma_\ell) \to \mathbb{R} \)

via \( \mathcal{H}^\gamma[(g, F, (h_\ell)_{\ell=1}^m), (\psi_\ell)_{\ell=1}^m] = \langle F, v \rangle_{(0H^1)^*, 0H^1} - \int_\Omega gq - \sum_{\ell=1}^m \langle \psi_\ell, h_\ell \rangle_{H^{-1/2}, H^{1/2}}, \) (3.2)

for \( (q, v) \in L^2(\Omega) \times 0H^1(\Omega; \mathbb{R}^n) \) the unique weak solution to the normal stress problem in equation (2.44) with data \( (\psi_\ell)_{\ell=1}^m \), i.e. \( \sum_{\ell=1}^m (\psi_\ell, e_n) = 0 \) for the operators \( \mathcal{O} \) and \( \chi_{-\gamma}^{-1} \) from Definition 2.11 and Proposition 2.2, respectively. Thanks to boundedness of \( \chi_{-\gamma}^{-1} \) and \( \mathcal{O} \), we have that \( \mathcal{H}^\gamma \) is continuous.

The set of data for which \( \mathcal{H}^\gamma \) vanishes identically as a linear functional of its right argument will be denoted with

\[ \ker \mathcal{H}^\gamma = \{(g, F, (h_\ell)_{\ell=1}^m) : \mathcal{H}^\gamma[(g, F, (h_\ell)_{\ell=1}^m), (\psi_\ell)_{\ell=1}^m] = 0, \forall (\psi_\ell)_{\ell=1}^m \} \] (3.3)

and called the kernel of \( \mathcal{H}^\gamma \).

The following result characterizes the appropriate range of data for the system (3.1) as exactly the left kernel of \( \mathcal{H}^\gamma \). In what follows recall that \( \chi_{\gamma} \) and \( \chi_{-\gamma} \) are the mappings from Proposition 2.9.

Proposition 3.2 (Range of compatible data for (3.1)). For \( \gamma \in \mathbb{R} \) the mapping

\[ \tilde{\chi}_{\gamma} : L^2(\Omega) \times 0H^1(\Omega; \mathbb{R}^n) \to L^2(\Omega) \times (0H^1(\Omega; \mathbb{R}^n))^* \times \prod_{\ell=1}^m H^{1/2}(\Sigma_\ell) \]

with assignment \( \tilde{\chi}_{\gamma}(p, u) = (\chi_{\gamma}(p, u), (\text{Tr}_{\gamma, u} \cdot e_n)_{\ell=1}^m) \) (3.4)

is an injection with closed range \( \ker \mathcal{H}^\gamma \).

Proof. Proposition 2.9 tells us that \( \tilde{\chi}_{\gamma} \) is injective, and \( \ker \mathcal{H}^\gamma \) is closed by inspection. It remains only to show that the range of this mapping is the left kernel of \( \mathcal{H}^\gamma \).

Suppose first that \( (g, F, (h_\ell)_{\ell=1}^m) \in \ker \mathcal{H}^\gamma \) and define \( (p, u) \in L^2(\Omega) \times 0H^1(\Omega; \mathbb{R}^n) \) through \( (p, u) = \chi_{-\gamma}^{-1}(g, F) \). If \( (\psi_\ell)_{\ell=1}^m \in \prod_{\ell=1}^m H^{-1/2}(\Sigma_\ell) \) and \( (q, v) \in L^2(\Omega) \times 0H^1(\Omega; \mathbb{R}^n) \) are the associated solution to the normal stress PDE in equation (2.44), i.e. \( \chi_{-\gamma}^{-1}(0, \mathcal{O}(\psi(e_n))_{\ell=1}^m) \), then the identity

\[ \mathcal{H}^\gamma[(g, F, (h_\ell)_{\ell=1}^m), (\psi_\ell)_{\ell=1}^m] = \langle F, v \rangle_{(0H^1)^*, 0H^1} - \int_\Omega gq - \sum_{\ell=1}^m \langle \psi_\ell, h_\ell \rangle_{H^{-1/2}, H^{1/2}} = 0 \] (3.5)
implies that
\[
\sum_{\ell=1}^m (\psi_\ell, \text{Tr}_{\Sigma_\ell} u \cdot e_n)_{H^{1/2}, H^{1/2}} = \sum_{\ell=1}^m \int_{\Omega_\ell} \frac{\mu_\ell}{2} \nabla v : \nabla u - \gamma_\rho \partial_1 v \cdot u - \int_{\Omega} g g
\]
\[
\sum_{\ell=1}^m \int_{\Omega_\ell} \frac{\mu_\ell}{2} \nabla u : \nabla v + \gamma_\rho \partial_1 v \cdot u - \int_{\Omega} g g = (F, v)_{(0H^1)^*, aH^1} - \int_{\Omega} g g = \sum_{\ell=1}^m (\psi_\ell, h_\ell)_{H^{1/2}, H^{1/2}}.
\]
As \((\psi_\ell)_{\ell=1}^m\) is an arbitrary member of \(\prod_{\ell=1}^m H^{-1/2}(\Sigma_\ell)\) we learn that \(\text{Tr}_{\Sigma_\ell} u \cdot e_n = h_\ell\) for each \(\ell \in \{1, \ldots, m\}\) Therefore, \(\tilde{\chi}_\gamma (p, u) = (g, F, (h_\ell)_{\ell=1}^m)\).

On the other hand, if \((p, u) \in L^2(\Omega) \times \partial H^1(\Omega; \mathbb{R}^n)\) we let \((g, F, (h_\ell)_{\ell=1}^m) = \tilde{\chi}_\gamma (p, u)\) and \((g, v) = \chi_{-\gamma^{-1}}(0, \mathcal{O}(\psi e_n)_{\ell=1}^m)\) and compute, for \((\psi_\ell)_{\ell=1}^m \in \prod_{\ell=1}^m H^{-1/2}(\Sigma_\ell)\),
\[
\mathcal{H}^\gamma [(g, F, (h_\ell)_{\ell=1}^m), (\psi_\ell)_{\ell=1}^m] = (F, v)_{(0H^1)^*, aH^1} - \int_{\Omega} g g - \sum_{\ell=1}^m (\psi_\ell, h_\ell)_{H^{-1/2}, H^{1/2}} = (F, v)_{(0H^1)^*, aH^1} - \sum_{\ell=1}^m \int_{\Omega_\ell} \frac{\mu_\ell}{2} \nabla v : \nabla u + \gamma_\rho \partial_1 v \cdot u - \sum_{\ell=1}^m \int_{\Omega_\ell} \frac{\mu_\ell}{2} \nabla u : \nabla v - \gamma_\rho \partial_1 u \cdot v = 0.
\]
As this holds for all such \((\psi_\ell)_{\ell=1}^m\), we conclude that \(\tilde{\chi}_\gamma (p, u) \in \ker \mathcal{H}^\gamma\).

We now arrive at the isomorphism of Hilbert spaces associated to problem \((3.1)\).

**Theorem 3.3.** (Existence and uniqueness of solutions to \((3.1)\). Let \(\gamma \in \mathbb{R}\) and \(s \in \mathbb{R}^+ \cup \{0\}\). Consider the bounded linear injection
\[
\Psi_\gamma : H^{1+s}(\Omega) \times \partial H^{2+s}(\Omega; \mathbb{R}^n) \to H^{1+s}(\Omega) \times \partial H^s(\Omega; \mathbb{R}^n) \times \prod_{\ell=1}^m H^{1/2+s}(\Sigma_\ell; \mathbb{R}^n) \times \prod_{\ell=1}^m H^{3/2+s}(\Sigma_\ell) \quad (3.8)
\]
with assignment \(\Psi_\gamma (p, u) = (\Phi_\gamma (p, u), (\text{Tr}_{\Sigma_\ell} u \cdot e_n)_{\ell=1}^m)\), where \(\Phi_\gamma\) is from Theorem 2.13. The following are equivalent for \((g, f, (k_\ell)_{\ell=1}^m, (h_\ell)_{\ell=1}^m)\) belonging to the codomain of \(\Psi_\gamma\).

1. \((g, f, (k_\ell)_{\ell=1}^m, (h_\ell)_{\ell=1}^m)\) belongs to the range of \(\Psi_\gamma\).

2. The data tuple \((g, \mathcal{D}(f, (k_\ell)_{\ell=1}^m), (h_\ell)_{\ell=1}^m)\) belongs to \(\ker \mathcal{H}^\gamma\), where \(\mathcal{D}\) is from Definition 2.11.

**Proof.** Recall that \(\tilde{\chi}_\gamma\) is the mapping from Proposition 3.2. If the first item holds then \(\tilde{\chi}_\gamma (p, u) = (g, \mathcal{D}(f, (k_\ell)_{\ell=1}^m), (h_\ell)_{\ell=1}^m)\) for the unique \((p, u)\) belonging to the domain of \(\Psi_\gamma\) such that \(\Psi_\gamma (p, u) = (g, f, (k_\ell)_{\ell=1}^m, (h_\ell)_{\ell=1}^m)\). Thus, by Proposition 3.2 \((g, \mathcal{D}(f, (k_\ell)_{\ell=1}^m), (h_\ell)_{\ell=1}^m) \in \ker \mathcal{H}^\gamma\) and the second item follows.

If the second item holds then, by Proposition 3.2 again, we learn that there are \((p, u) \in L^2(\Omega) \times \partial H^1(\Omega; \mathbb{R}^n)\) such that \(\tilde{\chi}_\gamma (p, u) = (g, \mathcal{D}(f, (k_\ell)_{\ell=1}^m), (h_\ell)_{\ell=1}^m)\). In particular \(\chi_\gamma (p, u) = (g, \mathcal{D}(f, (k_\ell)_{\ell=1}^m))\) for \(\chi_\gamma\) from Proposition 2.9 and hence we may apply Proposition 2.12 to deduce that \((p, u)\) belongs to the domain of \(\Psi_\gamma\) and \(\Psi_\gamma (p, u) = (g, f, (k_\ell)_{\ell=1}^m, (h_\ell)_{\ell=1}^m)\) of \(\Psi_\gamma\). This shows the first item holds.

**3.2. Measuring data compatibility.** The previous subsection showed us that a nontrivial compatibility condition must be satisfied by the data in order for a solution to \((3.1)\) to exist. In this subsection we further explore this compatibility condition. We associate to each data tuple a tuple of functions that quantify how ‘close’ the data are to being compatible. We then study the dependence of the regularity and low Fourier mode behavior of the function tuple on the data.

The sense in which this association quantifies compatibility will be made clearer in the next section; however, the main idea is that the introduction of the free surface functions in the multilayer traveling Stokes with gravity-capillary boundary and jump conditions problem (equation \((4.1)\)) modify the data in a way that results in inclusion within the range of \(\Psi_\gamma\). This is achieved by the free surface functions solving certain \(\Psi\)DEs with forcing exactly this measurement of compatibility.

**Proposition 3.4.** For \(\gamma \in \mathbb{R}\) there is a bounded linear mapping
\[
\mathcal{G}^\gamma : L^2(\Omega) \times (0H^1(\Omega; \mathbb{R}^n))^s \times \prod_{\ell=1}^m H^{1/2}(\Sigma_\ell) \to \prod_{\ell=1}^m H^{1/2}(\Sigma_\ell) \quad (3.9)
\]
such that for all data tuples \((g, F, (h_\ell)_{\ell=1}^m) \in L^2(\Omega) \times (0H^1(\Omega; \mathbb{R}^n))^s \times \prod_{\ell=1}^m H^{1/2}(\Sigma_\ell)\) the following identity holds for all \((\psi_{\ell})_{\ell=1}^m \in \prod_{\ell=1}^m H^{-1/2}(\Sigma_\ell)\):

\[
\mathcal{H}_\gamma^[(g, F, (h_\ell)_{\ell=1}^m), (\psi_{\ell})_{\ell=1}^m] = \langle (\psi_{\ell})_{\ell=1}^m, \mathcal{J}_\gamma(g, F, (h_\ell)_{\ell=1}^m) \rangle_{H^{-1/2, H^{1/2}}}
= \sum_{k=1}^m \langle \psi_k, \mathcal{J}_\gamma(g, F, (h_\ell)_{\ell=1}^m) \cdot e_k \rangle_{H^{-1/2, H^{1/2}}},
\]

(3.10)

where \(\mathcal{H}_\gamma^\) is the bilinear mapping from Definition 3.7.

Proof. Recall that \(\mathcal{H}_\gamma\) is continuous thanks to Proposition 2.9. Thus there is a constant \(c_0 \in \mathbb{R}^+\), depending only on the physical parameters, such that

\[
|\mathcal{H}_\gamma^[(g, F, (h_\ell)_{\ell=1}^m), (\psi_{\ell})_{\ell=1}^m]| \leq c_0 \left( \|g\|_{L^2} + \|F\|_{(0H^1)^s} + \sum_{\ell=1}^m \|h_\ell\|_{H^{1/2}} \right)
\]

for all \((g, F, (\psi_{\ell})_{\ell=1}^m)\) and \((\psi_{\ell})_{\ell=1}^m \in \prod_{\ell=1}^m H^{-1/2}(\Sigma_\ell)\). In particular, for fixed \((g, F, (h_\ell)_{\ell=1}^m)\) the assignment \((\psi_{\ell})_{\ell=1}^m \mapsto \mathcal{H}_\gamma^[(g, F, (h_\ell)_{\ell=1}^m), (\psi_{\ell})_{\ell=1}^m]\) is bounded and linear with operator norm at most \(c_0\). By duality (Proposition 2.6) there is a unique \(\mathcal{J}_\gamma(g, F, (h_\ell)_{\ell=1}^m) \in \prod_{\ell=1}^m H^{1/2}(\Sigma_\ell)\) such that (3.10) holds for all \((\psi_{\ell})_{\ell=1}^m \in \prod_{\ell=1}^m H^{-1/2}(\Sigma_\ell)\); moreover,

\[
\sum_{k=1}^m |\mathcal{J}_\gamma(g, F, (h_\ell)_{\ell=1}^m) \cdot e_k|_{H^{1/2}} \leq c_0 \left( \|g\|_{L^2} + \|F\|_{(0H^1)^s} + \sum_{\ell=1}^m \|h_\ell\|_{H^{1/2}} \right).
\]

(3.12)

It is also clear that \(\mathcal{J}_\gamma\) is linear. \(\square\)

We next show that \(\mathcal{J}_\gamma\) commutes with tangential Fourier multipliers, which are defined in Appendix A.3

**Lemma 3.5.** If \(\gamma \in \mathbb{R}, (g, F, (\psi_{\ell})_{\ell=1}^m) \in L^2(\Omega) \times (0H^1(\Omega; \mathbb{R}^n))^s \times \prod_{\ell=1}^m H^{1/2}(\Sigma_\ell), \) and \(\omega \in L^\infty(\mathbb{R}^{n-1}; \mathbb{C})\) satisfies \(\omega(-\xi) = \omega(\xi)\) for a.e. \(\xi \in \mathbb{R}^{n-1}\), then

\[
M_{\omega,\mathcal{J}_\gamma}(g, F, (h_\ell)_{\ell=1}^m) = \mathcal{J}_\gamma[M_\omega(g, F, (h_\ell)_{\ell=1}^m)] = \mathcal{J}_\gamma(M_\omega, g, F, (M_\omega h_\ell)_{\ell=1}^m),
\]

where the above is understood in the sense of Definitions A.11 and A.13 and is \(\mathbb{R}\)-valued by Proposition A.2.

**Proof.** Let \((\psi_{\ell})_{\ell=1}^m \in \prod_{\ell=1}^m H^{-1/2}(\Sigma_\ell)\) be normal stress data and denote the corresponding solution to (2.44) with \((g, v) \in L^2(\Omega) \times 0H^1(\Omega; \mathbb{R}^n)\), i.e. \((g, v) = \chi_\gamma(0, \theta(\psi_{\ell} e_\ell)_{\ell=1}^m)^{-1}\). Then by the definition of \(\mathcal{J}_\gamma\), Proposition 3.4 and then Lemma 2.10 we have that

\[
\langle (\psi_{\ell})_{\ell=1}^m, M_{\omega,\mathcal{J}_\gamma}(g, F, (h_\ell)_{\ell=1}^m) \rangle_{H^{-1/2, H^{1/2}}} = \langle (M_{\omega,\psi_{\ell}}(g, F, (h_\ell)_{\ell=1}^m))_{\ell=1}^m, \mathcal{J}_\gamma(g, F, (h_\ell)_{\ell=1}^m) \rangle_{H^{-1/2, H^{1/2}}}
= \mathcal{H}_\gamma[(g, F, (h_\ell)_{\ell=1}^m), (M_{\omega,\psi_{\ell}})_{\ell=1}^m] = \langle F, M_{\omega,\psi_{\ell}} \rangle_{(0H^1)^s, 0H^1} - \int_\Omega g(M_{\omega,\psi_{\ell}} h_\ell)_{H^{-1/2, H^{1/2}}} - \sum_{\ell=1}^m \langle M_{\omega,\psi_{\ell}} h_\ell \rangle_{H^{-1/2, H^{1/2}}} = \mathcal{H}_\gamma[(M_{\omega, g}, M_{\omega, F}, (M_\omega h_\ell)_{\ell=1}^m), (\psi_{\ell})_{\ell=1}^m] = \langle (\psi_{\ell})_{\ell=1}^m, \mathcal{J}_\gamma(M_{\omega, g}, M_{\omega, F}, (M_\omega h_\ell)_{\ell=1}^m) \rangle_{H^{-1/2, H^{1/2}}}.
\]

(3.14)

As this holds for all \((\psi_{\ell})_{\ell=1}^m \in \prod_{\ell=1}^m H^{-1/2}(\Sigma_\ell)\), the result follows. \(\square\)

The previous lemma allows us to deduce the regularity properties of \(\mathcal{J}_\gamma\). We record these now.

**Proposition 3.6.** If \(s \in \mathbb{R}^+ \cup \{0\}, \gamma \in \mathbb{R}, \) and

\[
(g, f, (k_\ell)_{\ell=1}^m, (h_\ell)_{\ell=1}^m) \in H^{1+s}(\Omega) \times H^s(\Omega; \mathbb{R}^n) \times \prod_{\ell=1}^m H^{1/2+s}(\Sigma_\ell; \mathbb{R}^n) \times \prod_{\ell=1}^m H^{3/2+s}(\Sigma_\ell)\)
\]

then \(\mathcal{J}_\gamma(g, \mathcal{P}(f, (k_\ell)_{\ell=1}^m), (h_\ell)_{\ell=1}^m) \in \prod_{\ell=1}^m H^{3/2+s}(\Sigma_\ell), \) where \(\mathcal{P}\) is described in Definition 2.11. Moreover, we have the universal estimate

\[
\sum_{k=1}^m \|\mathcal{J}_\gamma(g, F, (h_\ell)_{\ell=1}^m) \cdot e_k\|_{H^{3/2+s}} \lesssim \sum_{\ell=1}^m \left[ \|g\|_{H^{1+s}(\Omega)} + \|f\|_{H^s(\Omega)} + \|k_\ell\|_{H^{1/2+s}} + \|h_\ell\|_{H^{3/2+s}} \right].
\]

(3.16)
Proof. For \( k \in \mathbb{N}^+ \) we define the real valued radial function \( \omega_k \in L^\infty(\mathbb{R}^{n-1}; \mathbb{R}) \) with the assignment \( \omega_k(\xi) = 1_{B(0,k)}(\xi)(1 + |\xi|^2)^{-\frac{n+1}{2}} \). Using first Lemma 3.5 and then continuity of \( \mathcal{F}^\gamma \), we arrive at the estimate
\[
\|M_{\omega_k} \mathcal{F}^\gamma (g, F, (h_\ell)_{\ell=1}^m)\|_{H^{1/2}} = \|\mathcal{F}^\gamma (M_{\omega_k} g, M_{\omega_k} F, (M_{\omega_k} h_\ell)_{\ell=1}^m)\|_{H^{1/2}} 
\leq c_0 \left( \|M_{\omega_k} g\|_{L^2} + \|M_{\omega_k} F\|_{(0,H^1)^*} + \|(M_{\omega_k} h_\ell)_{\ell=1}^m\|_{H^{1/2}} \right),
\]
for \( c_0 \in \mathbb{R}^+ \) depending only on the physical parameters. By Proposition A.14 there is \( c_1 \in \mathbb{R}^+ \) depending only \( s \) and physical parameters such that
\[
\|M_{\omega_k} g\|_{L^2} + \|M_{\omega_k} F\|_{(0,H^1)^*} + \|(M_{\omega_k} h_\ell)_{\ell=1}^m\|_{H^{1/2}} 
\leq c_1 \sum_{\ell=1}^m \left[ \|g\|_{H^{1+s}(\Omega_\ell)} + \|f\|_{H^s(\Omega_\ell)} + \|k_\ell\|_{H^{1/2+s}} + \|h_\ell\|_{H^{3/2+s}} \right].
\]
Parrying equations (3.17) and (3.18) with Parseval’s theorem and Fatou’s lemma then yields the bound
\[
\|\mathcal{F}^\gamma (g, F, (h_\ell)_{\ell=1}^m)\|_{H^{3/2+s}} \leq \liminf_{k \to \infty} \|M_{\omega_k} \mathcal{F}^\gamma (g, F, (h_\ell)_{\ell=1}^m)\|_{H^{1/2}} 
\leq c_0 c_1 \sum_{\ell=1}^m \left[ \|g\|_{H^{1+s}(\Omega_\ell)} + \|f\|_{H^s(\Omega_\ell)} + \|k_\ell\|_{H^{1/2+s}} + \|h_\ell\|_{H^{3/2+s}} \right],
\]
which completes the proof. \( \square \)

For technical reasons that will become clear in the next section, we want to restrict to a smaller subspace of the domain of \( \mathcal{F}^\gamma \) that guarantees an image whose members’ low Fourier modes are more tame. We label this subspace as follows.

**Definition 3.7.** We define the Hilbert space
\[
\mathcal{Y}(\Omega) = \left\{ (g,F,(h_\ell)_{\ell=1}^m) \in L^2(\Omega) \times (0,H^1(\Omega;\mathbb{R}^n))^s \times \prod_{\ell=1}^m H^{1/2}(\Sigma_\ell) : \|(g,F,(h_\ell)_{\ell=1}^m)\|_\mathcal{Y} < \infty \right\}
\]
for the norm
\[
\|(g,F,(h_\ell)_{\ell=1}^m)\|_{\mathcal{Y}}^2 = \|g\|_{L^2}^2 + \|F\|_{(0,H^s)}^2 + \sum_{\ell=1}^m \left[ \|h_\ell\|_{H^{1/2}}^2 + \left( h_\ell - \int_{(0,a_\ell)} f \right)_H^2 \right].
\]

In our analysis of the action of \( \mathcal{F}^\gamma \) over \( \mathcal{Y}(\Omega) \) we utilize the following energy estimates of the normal stress problem with band limited data.

**Lemma 3.8.** Let \( \gamma \in \mathbb{R} \). There exists \( C \in \mathbb{R}^+ \), depending only on the physical parameters, such that for all tuples \((\psi_\ell)_{\ell=1}^m \in \prod_{\ell=1}^m H^{-1/2}(\Sigma_\ell)\) satisfying \( \text{supp} \mathcal{F} [((\psi_\ell)_{\ell=1}^m)] \subseteq B(0,1) \subseteq \mathbb{R}^{n-1} \) we may estimate
\[
\|v\|_{H^1}^2 \leq C^2 \int_{\mathbb{R}^{n-1}} \xi^2 |\mathcal{F} [((\psi_\ell)_{\ell=1}^m)](\xi)|^2 \, d\xi
\]
and
\[
\int_{\Omega} |q + \sum_{\ell=1}^m \psi_\ell 1_{(0,a_\ell)}| \leq C^2 \int_{\mathbb{R}^{n-1}} \xi^2 |\mathcal{F} [((\psi_\ell)_{\ell=1}^m)](\xi)|^2 \, d\xi,
\]
where for each \( \ell \in \{1, \ldots, m\} \) we have that \( \psi_\ell 1_{(0,a_\ell)} \in L^2(\Omega) \) is defined via \( \mathbb{R}^{n-1} \times (0,a_m) \ni (x,y) \mapsto \psi_\ell (x) 1_{(0,a_\ell)}(y) \in \mathbb{R} \), and \((q,v) = \chi_{-\gamma}^{-1} (0,\mathcal{O}(\psi_\ell e_\ell)_{\ell=1}^m) \in L^2(\Omega) \times 0H^1(\Omega;\mathbb{R}^n) \) are the solution to (2.44) with data \((\psi_\ell)_{\ell=1}^m\).

**Proof.** The band limited assumption on the data, paired with the left hand inequality in the energy estimate of Theorem 2.18 gives the first estimate.

For the second estimate we test \( v \in 0H^1(\Omega;\mathbb{R}^n) \) in the weak formulation of (2.44), write \( q = q + \sum_{\ell=1}^m \psi_\ell 1_{(0,a_\ell)} - \sum_{\ell=1}^m \psi_\ell 1_{(0,a_\ell)} \), and rearrange to arrive at the identity
\[
\int_{\Omega} \nabla \cdot w \left( q + \sum_{\ell=1}^m \psi_\ell 1_{(0,a_\ell)} \right) = \sum_{\ell=1}^m \int_{\Omega_\ell} \frac{M_{\xi}}{2} \mathbb{D} w - \gamma \rho_\ell \partial_1 v \cdot w - \int_{\mathbb{R}^{n-1}} \sum_{\ell=1}^m \psi_\ell \left[ \text{Tr}_{\Sigma_\ell} w \cdot e_\ell - \int_{(0,a_\ell)} \nabla \cdot w \right].
\]
Then by the first estimate and the divergence compatibility estimate from Proposition 2.11 we may bound
\[ \left| \int_\Omega \nabla \cdot w \left( q + \sum_{\ell=1}^m \psi_\ell \mathbb{1}_{(0,a_\ell]} \right) \right| \lesssim \left\| (\psi_\ell)_{\ell=1}^m \right\|_{H^1} \left[ \left\| w \right\|_{0,H^1} + \left\| w \right\|_{L^2} \right]. \] (3.25)

The second estimate now follows by taking \( w = \Pi_\Omega (q + \sum_{\ell=1}^m \psi_\ell \mathbb{1}_{(0,a_\ell]}), \) where \( \Pi_\Omega \) is the bounded right inverse of the divergence from Lemma 2.2.

We are now in a position to analyze the low frequency behavior of the image of \( \mathcal{Y}(\Omega) \) under \( \mathcal{S}^\gamma. \)

**Proposition 3.9.** If \((g, F, (h_\ell)_{\ell=1}^m) \in \mathcal{Y}(\Omega) \) and \( \gamma \in \mathbb{R}, \) then \( \mathcal{S}^\gamma (g, F, (h_\ell)_{\ell=1}^m) \in \prod_{\ell=1}^m \dot{H}^{-1}(\Sigma_\ell) \cap H^{1/2}(\Sigma_\ell) \) with the universal estimate \( \left\| \mathcal{S}^\gamma (g, F, (h_\ell)_{\ell=1}^m) \right\|_{H^{-1} \cap H^{1/2}} \lesssim \left\| (g, F, (h_\ell)_{\ell=1}^m) \right\|_{\mathcal{Y}}. \)

**Proof.** Recall that Lemma 3.5 guarantees that \( \mathcal{S}^\gamma \) commutes with tangential multipliers. We use this with the continuity of \( \mathcal{S}^\gamma \) and the boundedness of tangential multipliers (Proposition A.8 Definitions A.11 and A.13 and Lemma A.12) to estimate
\[ \left\| \mathcal{S}^\gamma (g, F, (h_\ell)_{\ell=1}^m) \right\|_{H^{-1} \cap H^{1/2}} \lesssim \left[ M_{1B(0,1)} \mathcal{S}^\gamma (g, F, (h_\ell)_{\ell=1}^m) \right]_{H^{-1}} + \sum_{\ell=1}^m \left\| M_{1B(0,1)} \mathcal{S}^\gamma (g, F, (h_\ell)_{\ell=1}^m) \right\|_{H^{-1/2}} \]
\[ + \left\| M_{1B(0,1)} \mathcal{S}^\gamma (g, F, (h_\ell)_{\ell=1}^m) \right\|_{H^{1/2}} + \left\| M_{1B(0,1)} \mathcal{S}^\gamma (g, F, (h_\ell)_{\ell=1}^m) \right\|_{H^{-1}} + \left\| F \right\|_{0,H^1} + \left\| (h_\ell)_{\ell=1}^m \right\|_{H^{1/2}}. \] (3.26)

Thus, it sufficient to show that \( M_{1B(0,1)} \mathcal{S}^\gamma (g, F, (h_\ell)_{\ell=1}^m) \in \prod_{\ell=1}^m \dot{H}^{-1}(\Sigma_\ell) \) with a bounded estimate. We do this via duality.

Let \((\psi_\ell)_{\ell=1}^m \in \prod_{\ell=1}^m H^{-1/2}(\Sigma_\ell) \) and denote the corresponding solution to the normal stress problem (2.4) via \((q, v) = \chi_{\gamma^{-1}}(0, \partial \psi_{\ell} e_\ell)_{\ell=1}^m \in L^2(\Omega) \times \dot{H}^1(\Omega; \mathbb{R}^n).\) We compute
\[ \int_{B(0,1)} \mathcal{F}[(\psi_\ell)_{\ell=1}^m] (\xi) \cdot \mathcal{F}[(\psi_\ell)_{\ell=1}^m] (\xi) \ d\xi \]
\[ = \langle (\psi_\ell)_{\ell=1}^m, M_{1B(0,1)} \mathcal{S}^\gamma (g, F, (h_\ell)_{\ell=1}^m) \rangle_{H^{-1/2}, H^{1/2}} = \mathcal{F}^\gamma \left[ \langle g, F, (h_\ell)_{\ell=1}^m \rangle, (M_{1B(0,1)} \psi_\ell)_{\ell=1}^m \right] \]
\[ = \langle F, M_{1B(0,1)} v \rangle_{0,H^1} - \int_\Omega g M_{1B(0,1)} q - \sum_{\ell=1}^m \langle M_{1B(0,1)} \psi_\ell, h_\ell \rangle_{H^{-1/2}, H^{1/2}} \]
\[ = \langle F, M_{1B(0,1)} v \rangle_{0,H^1} - \int_\Omega g M_{1B(0,1)} (q + \sum_{\ell=1}^m \psi_\ell \mathbb{1}_{(0,a_\ell]} \rangle + \int_{\mathbb{R}^n} \sum_{\ell=1}^m M_{1B(0,1)} \psi_\ell (-h_\ell + \int_{(0,a_\ell]} g) \rangle. \] (3.27)

Hence,
\[ \left| \int_{B(0,1)} \mathcal{F}[(\psi_\ell)_{\ell=1}^m] (\xi) \cdot \mathcal{F}[(\psi_\ell)_{\ell=1}^m] (\xi) \ d\xi \right| \leq \left\| F \right\|_{0,H^1} \left\| M_{1B(0,1)} v \right\|_{0,H^1} \]
\[ + \left\| g \right\|_{L^2} \left\| M_{2B(0,1)} q + \sum_{\ell=1}^m M_{1B(0,1)} \psi_\ell \mathbb{1}_{(0,a_\ell]} \right\|_{L^2} \leq \sum_{\ell=1}^m \left[ M_{1B(0,1)} \psi_\ell \right]_{H^1} \left[ h_\ell - \int_{(0,a_\ell]} g \right]_{H^{-1}}. \] (3.28)

Lemma 2.10 ensures us that \((M_{1B(0,1)} q, M_{2B(0,1)} v) = \chi_{\gamma^{-1}}(0, \partial (M_{1B(0,1)} \psi_{\ell} e_\ell)_{\ell=1}^m).\) As \((M_{1B(0,1)} \psi_\ell)_{\ell=1}^m \) is admissible band limited data, we may apply the second estimate of Lemma 2.28 to \((M_{1B(0,1)} \psi_\ell)_{\ell=1}^m \) to bound
\[ \left\| M_{2B(0,1)} q + \sum_{\ell=1}^m M_{1B(0,1)} \psi_\ell \mathbb{1}_{(0,a_\ell]} \right\|_{L^2} \lesssim \left[ (M_{1B(0,1)} \psi_\ell)_{\ell=1}^m \right]_{H^1} \leq \sum_{\ell=1}^m \left[ M_{2B(0,1)} \psi_\ell \right]_{H^1}. \] (3.29)

Therefore, by (3.28) and (3.29),
\[ \left| \int_{B(0,1)} \mathcal{F}[(\psi_\ell)_{\ell=1}^m] (\xi) \cdot \mathcal{F}[(\psi_\ell)_{\ell=1}^m] (\xi) \ d\xi \right| \lesssim \left\| (g, F, (h_\ell)_{\ell=1}^m) \right\|_{\mathcal{Y}} \sum_{\ell=1}^m \left[ M_{2B(0,1)} \psi_\ell \right]_{H^1}, \] (3.30)
and so we conclude that
\[ [M_{\mathbb{B}(0,1)} \mathcal{I}^\gamma (g, F, (h^m_{\ell=1}))]_{H^{-1}} \lesssim \sup \left\{ \left| \int_{\mathbb{B}(0,1)} \mathcal{F}[(\psi^m_{\ell=1}) (\xi) \cdot \mathcal{F}[\mathcal{I}^\gamma (g, F, (h^m_{\ell=1}))] (\xi) \, d\xi \right| : \sum_{\ell=1}^m [M_{\mathbb{B}(0,1)} \psi^m]_{H^1} \leq 1 \right\} \lesssim \| (g, F, (h^m_{\ell=1})) \|_{\mathcal{Y}}. \] (3.31)

This completes the proof. \( \square \)

**Remark 3.10.** For \( s \in \mathbb{R}^+ \cup \{0\} \) we may view
\[ H^{1+s} (\Omega) \times H^s (\Omega; \mathbb{R}^n) \times \prod_{\ell=1}^m H^{1/2+s} (\Sigma_\ell; \mathbb{R}^n) \times \prod_{\ell=1}^m H^{3/2+s} (\Sigma_\ell) \]
\[ \rightarrow L^2 (\Omega) \times (\Theta H^1 (\Omega; \mathbb{R}^n))^* \times \prod_{\ell=1}^m H^{1/2} (\Sigma_\ell) \] (3.32)
through the inclusion mapping \( (g, f, (k^m_{\ell=1}), (h^m_{\ell=1})) \mapsto (g, \mathcal{P}(f, (k^m_{\ell=1})), (h^m_{\ell=1})) \), for \( \mathcal{P} \) as in Definition 2.11.

We now synthesize the results of this subsection into a single result.

**Theorem 3.11.** Let \( \gamma \in \mathbb{R} \) and \( s \in \mathbb{R}^+ \cup \{0\} \). The linear mapping
\[ \mathcal{K}^\gamma : \mathcal{Y} (\Omega) \cap \left[ H^{1+s} (\Omega) \times H^s (\Omega; \mathbb{R}^n) \times \prod_{\ell=1}^m H^{1/2+s} (\Sigma_\ell; \mathbb{R}^n) \times \prod_{\ell=1}^m H^{3/2+s} (\Sigma_\ell) \right] \]
\[ \rightarrow \prod_{\ell=1}^m \tilde{H}^{-1} (\Sigma_\ell) \cap H^{3/2+s} (\Sigma_\ell) \] (3.33)
given by \( \mathcal{K}^\gamma (g, f, (k^m_{\ell=1}), (h^m_{\ell=1})) = \mathcal{I}^\gamma (g, \mathcal{P}(f, (k^m_{\ell=1})), (h^m_{\ell=1})) \) is both well-defined and continuous.

**Proof.** The result follows from Remark 3.10, Proposition 3.6, and Proposition 3.9. \( \square \)

4. **Multilayer traveling Stokes with gravity-capillary boundary and jump conditions**

Our linear analysis culminates in this section with the study of the linearized flattened free boundary problem (1.17). More precisely, we study the system

\[
\begin{aligned}
\nabla \cdot S^m (p, u) - \gamma \rho_\ell \partial_1 u &= f & \text{in } \Omega_\ell, \ell \in \{1, \ldots, m\} \\
\nabla \cdot u &= g & \text{in } \Omega \\
[S^m (p, u)]_n = k_\ell + (\rho [p])_\ell + \sigma_\ell \Delta u_n e_n &= \text{on } \Sigma_\ell, \ell \in \{1, \ldots, m\} \\
u \cdot e_n = h_\ell - \gamma \partial_1 u_\ell &= \text{on } \Sigma_\ell, \ell \in \{1, \ldots, m\} \\
[u]_n = 0 &= \text{on } \Sigma_\ell, \ell \in \{1, \ldots, m-1\} \\
u = 0 &= \text{on } \Sigma_0.
\end{aligned}
\] (4.1)

We remind the reader that we are still using the abbreviated notation for \( \Omega, \Omega_\ell, \) and \( \Sigma_\ell \) discussed at the start of Section 2 and that the unknown velocity is \( u \), the pressure is \( p \), and the free surface functions are in the tuple \( (\eta^m_{\ell=1}) \). The prescribed data are \( f, g, (k^m_{\ell=1}), \text{ and } (h^m_{\ell=1}) \). The viscosity parameters are \( \mu = \{\mu_\ell\}_{\ell=1}^m \subset \mathbb{R}^+ \), the fluid densities are \( \{\rho_\ell\}_{\ell=1}^m \subset \mathbb{R}^+ \), \( \sigma = \{\sigma_\ell\}_{\ell=1}^m \subset \mathbb{R}^+ \cup \{0\} \) are the surface tensions, \( \gamma \in \mathbb{R} \) is the signed wave speed, \( \rho \in \mathbb{R}^+ \) is the magnitude gravitational acceleration, and \( \rho = \sum_{\ell=1}^m \rho_\ell \mathbb{1}_{\Omega_\ell} \). Unlike the previous two sections, we now assume that the wave speed is non-trivial, i.e. \( \gamma \in \mathbb{R} \setminus \{0\} \) and that the density coefficients are strictly decreasing with layer number, i.e. \( 0 < \rho_m < \cdots < \rho_1 \) (this is consistent with the assumptions made in the introduction). Note that \( [p]_\ell = \rho_{\ell+1} - \rho_\ell < 0 \) for \( \ell \in \{1, \ldots, m-1\} \) and \( [p]_m = -\rho_m < 0 \).

Our goal in this section is to prove that the above system induces a linear isomorphism between an appropriate pair of Banach spaces. It turns out that the estimates obtained from (4.11) are too weak to guarantee that the free surface functions and the pressure belong to standard \( L^2 \)-based Sobolev spaces in dimensions three or higher. The resolution of this issue requires developing families of specialized Sobolev spaces to serve as the container spaces for the free surface functions and pressure. In this section and the next we establish and utilize variants of the specialized spaces developed in the single layer analysis of [LT19] that are appropriate for the multilayer context.
4.1. Specialized Sobolev space interlude, well-definedness, and injectivity. We first label the space of data for which we will solve (4.1).

**Definition 4.1.** For $s \in \mathbb{R}^+ \cup \{0\}$ we define the space

$$Y^s = \mathcal{Y}(\Omega) \cap \left[H^{1+s}(\Omega) \times H^s(\Omega; \mathbb{R}^n) \times \prod_{\ell=1}^m H^{1/2+s}(\Sigma_\ell; \mathbb{R}^n) \times \prod_{\ell=1}^m H^{3/2+s}(\Sigma_\ell)\right],$$

(4.2)

where $\mathcal{Y}(\Omega)$ is from Definition 3.17 and the intersection is understood in the sense of the inclusion from Remark 3.10. This space is Hilbert and as an equivalent norm we set

$$\|\mathbf{g}, f, (k_\ell)_{\ell=1}^m, (h_\ell)_{\ell=1}^m\|_{Y^s}^2 = \sum_{\ell=1}^m \left[\|g\|_{H^{1+s}(\Omega_\ell)}^2 + \|f\|_{H^s(\Omega_\ell)}^2 + \|k_\ell\|_{H^{1/2+s}(\Omega_\ell)}^2 + \|h_\ell\|_{H^{3/2+s}(\Omega_\ell)}^2 + \left|h_\ell - \int_{(0,a_\ell)} g \right|_{H^{-1}}^2\right].$$

(4.3)

Next we define the container space for the free surface functions, which is an anisotropic Sobolev space introduced in [LT19]. Note that for notational convenience we denote this space with a name different from the one used in [LT19].

**Definition 4.2.** For $s \in \mathbb{R}^+ \cup \{0\}$ we define the normed space

$$H^s(\mathbb{R}^{n-1}) = \{\zeta \in (\mathcal{S}(\mathbb{R}^{n-1}; \mathbb{R}))^* : \mathcal{F}[\zeta] \in L^1_{\text{loc}}(\mathbb{R}^{n-1}; \mathbb{C}) \text{ and } \|\zeta\|_{H^s} < \infty\}$$

for

$$\|\zeta\|_{H^s}^2 = \int_{B(0,1)} |\xi|^{-2}(\xi_1^2 + |\xi|^4) |\mathcal{F}[\zeta](\xi)|^2 \, d\xi + \int_{\mathbb{R}^{n-1} \setminus B(0,1)} |\xi|^{2s} |\mathcal{F}[\zeta](\xi)|^2 \, d\xi.$$

(4.4)

The following result summarizes the essential properties of this space.

**Proposition 4.3.** Let $s \in \mathbb{R}^+ \cup \{0\}$. Then the following hold for the space $H^s(\mathbb{R}^{n-1})$.

1. $H^s(\mathbb{R}^{n-1})$ is Hilbert.
2. If $k \in \mathbb{N}$ then $H^s(\mathbb{R}^{n-1}) \hookrightarrow C^k_0(\mathbb{R}^{n-1}) + H^s(\mathbb{R}^{n-1})$; in particular if $(n-1)/2 + k < s$ then we have the embedding $H^s(\mathbb{R}^{n-1}) \hookrightarrow C^k_0(\mathbb{R}^{n-1})$. We invite the reader to refer to Section 1.6.
3. If $\eta \in H^{5/2+s}(\mathbb{R}^{n-1})$ then $\partial_1 \eta \in H^{3/2+s}(\mathbb{R}^{n-1}) \cap H^{-1}(\mathbb{R}^{n-1})$ and $\Delta \eta \in H^{1/2+s}(\mathbb{R}^{n-1})$; moreover, these mappings are continuous.
4. (Fourier reconstruction) If $\vartheta \in L^1_{\text{loc}}(\mathbb{R}^{n-1}; \mathbb{C})$ satisfies $\vartheta(-\xi) = \overline{\vartheta(\xi)}$ for a.e. $\xi \in \mathbb{R}^{n-1}$ and

$$\int_{B(0,1)} |\xi|^{-2}(\xi_1^2 + |\xi|^4) |\vartheta(\xi)|^2 \, d\xi + \int_{\mathbb{R}^{n-1} \setminus B(0,1)} |\xi|^{2s} |\vartheta(\xi)|^2 \, d\xi < \infty$$

(4.5)

then there exists $\zeta_\vartheta \in H^s(\mathbb{R}^{n-1})$ with $\mathcal{F}[\zeta_\vartheta] = \vartheta$.
5. In the case $n = 2$ we have the equality of vector spaces with equivalence of norms: $H^s(\mathbb{R}^{n-1}) = H^s(\mathbb{R}^1)$.
6. If $\zeta \in H^s(\mathbb{R}^{n-1})$ satisfies $\text{supp}\mathcal{F}[\zeta] \subseteq \mathbb{R}^{n-1} \setminus B(0,\varepsilon)$, for some $\varepsilon \in \mathbb{R}^+$, then, in fact, we have the inclusion $\zeta \in H^s(\mathbb{R}^{n-1})$.

**Proof.** Items (1), (2), (3), and (5) follow from Proposition 5.3 and Theorems 5.6 and 5.7 in [LT19]. Item (4) follows from the definition of $H^s$ and completeness. Item (6) is clear given the definitions of the norms on $H^s$ and $H^s$. 

Now we are ready to define and study the container space for the pressure, which is a multilayer variant of the space introduced in [LT19], again given a different name for notational convenience. For the next definition and the subsequent proposition we switch back to the unabbreviated notation for multilayer domains as defined in Section 1.1.

**Definition 4.4.** Let $s \in \mathbb{R}^+ \cup \{0\}$ and $\zeta = \{\zeta_\ell\}_{\ell=1}^m \in (C^0_b(\mathbb{R}^{n-1}))^m$ be a tuple of continuous functions satisfying

$$\max\{\|\zeta_1\|_{C^0_b}, \ldots, \|\zeta_m\|_{C^0_b}\} \leq \frac{1}{4} \min\{a_1, a_2 - a_1, \ldots, a_m - a_{m-1}\}.$$  

(4.7)
We define the normed vector space
\[ \mathcal{P}^s(\Omega[\zeta]) = \{ q \in L^1_{\text{loc}}(\Omega[\zeta]) : \exists (p,(\eta_\ell)_{\ell=1}^m) \in H^s(\Omega[\zeta]) \times (\mathcal{H}^s(\mathbb{R}^{n-1}))^m \text{ such that } q = p - g \sum_{\ell=1}^m [\rho]_\ell \eta_\ell \mathbf{1}_{\Omega[\chi_{\zeta}]}, \]}
eq 0 \] equipped with the norm
\[ \|q\|_{\mathcal{P}^s} = \inf \left\{ \sum_{\ell=1}^m \left[ \|p\|_{H^s(\Omega[\zeta])} + \|\eta_\ell\|_{\mathcal{H}^s} : q = p - g \sum_{\ell=1}^m [\rho]_\ell \eta_\ell \mathbf{1}_{\Omega[\chi_{\zeta}]} \right] \right\}. \] (4.8)

When \( \zeta = 0 \) we will sometimes write \( \mathcal{P}^s(\Omega) \) in place of \( \mathcal{P}^s(\Omega[0]) \).

The following result records the essential properties of these spaces.

**Proposition 4.5.** The following properties hold for the scale of spaces \( \mathcal{P}^s(\Omega[\zeta]) \) for \( s \in \mathbb{R}^+ \cup \{0\} \) and \( \zeta \in (C^0_b(\mathbb{R}^{n-1}))^m \) satisfying (4.7).

1. \( \mathcal{P}^s(\Omega[\zeta]) \) is Banach.
2. If \( k \in \mathbb{N} \) then \( \mathcal{P}^s(\Omega[\zeta]) \hookrightarrow C^k_c(\Omega[\zeta]) + H^s(\Omega[\zeta]) \); in particular if \( n/2 + k < s \) then \( \mathcal{P}^s(\Omega[\zeta]) \hookrightarrow C^k_c(\Omega[\zeta]) \).
3. If \( p \in \mathcal{P}^{1+s}(\Omega[\zeta]) \) then \( \sum_{\ell=1}^m \mathbf{1}_{\Omega[\zeta]} \nabla p \in H^s(\Omega[\zeta];\mathbb{R}^n) \) and this map is continuous.
4. For \( \ell \in \{1, \ldots, m\} \) there are bounded trace operators: \( \text{Tr}^\ell_{\text{BD}}(0) : \mathcal{P}^{1+s}(\Omega[0]) \to \mathcal{H}^{1/2+s}(\mathbb{R}^{n-1}) \).
5. In the case \( n = 2 \) we have the equality of vector spaces with equivalence of norms: \( \mathcal{P}^s(\Omega[\zeta]) = H^s(\Omega[\zeta]) \).

**Proof.** The claims follow from simple multilayer adaptations of Theorems 5.9, 5.11, and 5.13 and Remark 5.10 of [LT19].

We have all the tools we need to label the spaces which hold the velocity, pressure, and free surface tuple.

**Definition 4.6.** For \( s \in \mathbb{R}^+ \cup \{0\} \) we define the Banach space
\[ \mathcal{X}^s = \{ (p,u,(\eta_\ell)_{\ell=1}^m) \in \mathcal{P}^{1+s}(\Omega) \times _a H^{2+s}(\Omega;\mathbb{R}^n) \times (\mathcal{H}^{5/2+s}(\mathbb{R}^{n-1}))^m : p + g \sum_{\ell=1}^m [\rho]_\ell \eta_\ell \mathbf{1}_{(0,a_\ell)} \in H^{1+s}(\Omega) \}, \] (4.10)

which we endow the norm
\[ \|(p,u,(\eta_\ell)_{\ell=1}^m)\|_{\mathcal{X}^s} = \|p\|_{\mathcal{P}^{1+s}} + \sum_{\ell=1}^m \left[ \|u\|_{H^{2+s}(\Omega)} + \|\eta_\ell\|_{\mathcal{H}^{5/2+s}} + \|p + g \sum_{k=1}^m [\rho]_k \eta_k \mathbf{1}_{(0,a_k)}\|_{H^{1+s}(\Omega)} \right]. \] (4.11)

Next we introduce the linear map that will turn out to be the Banach isomorphism solution operator associated to the problem (4.1).

**Proposition 4.7** (Uniqueness of solutions to (4.1)). For \( \gamma \in \mathbb{R} \setminus \{0\} \), \( \sigma = \{\sigma_\ell\}_{\ell=1}^m \subset \mathbb{R}^+ \cup \{0\} \), and \( s \in \mathbb{R}^+ \cup \{0\} \) the linear mapping \( \mathcal{Y}_{\gamma,\sigma} : \mathcal{X}^s \to \mathcal{X}^s \) with action
\[ \mathcal{Y}_{\gamma,\sigma}(p,u,(\eta_\ell)_{\ell=1}^m) = (\nabla \cdot u, \sum_{\ell=1}^m \mathbf{1}_{\Omega_\ell} [\nabla \cdot S^u(p,u) - \gamma \rho_\ell \partial_1 u], \] (4.12)

is well-defined, continuous, and injective.

**Proof.** We begin by checking that the mapping is well-defined and continuous. This is clear for the first component. The only possible point of contention in the second component is the expression with the pressure, \( \sum_{\ell=1}^m \mathbf{1}_{\Omega_\ell} \nabla p; \) however, we are in the clear thanks to item two of Proposition 4.5. For the third component we use that \( p + g \sum_{\ell=1}^m [\rho]_\ell \eta_\ell \mathbf{1}_{(0,a_\ell)} \in H^{1+s}(\Omega) \), paired with the usual trace theory and the jump calculation:
\[ \left[ \sum_{\ell=1}^m [\rho]_k \eta_k \mathbf{1}_{(0,a_k)} \right]_\ell = \sum_{j=\ell+1}^m [\rho]_j \eta_j - \sum_{k=\ell}^m [\rho]_k \eta_k = -[\rho]_\ell \eta_\ell \] (4.13)
to deduce the bounded inclusion
\[ [\rho]_\ell - g [\rho]_\ell \eta_\ell = [p + g \sum_{k=1}^m [\rho]_k \eta_k \mathbf{1}_{(0,a_k)}]_\ell \in H^{1/2+s}(\Sigma_\ell). \] (4.14)
where we recall that over

\[ \text{The above manipulations are justified by the fact that} \]

\[ \text{Item two of Proposition 4.3 tells us that} \]

\[ \text{Hence, Proposition 4.5, item four, and Proposition 4.3, item six, may be invoked to see that} \]

\[ \text{The normal stress PDE in equation (2.44). Note that we have introduced the band-limited approximation, in part, so that the above application of } \chi_\gamma^{-1}, \text{ as defined in Proposition 2.9 is well-defined.} \]

Since \( \gamma_{\gamma, \sigma}(p, u, (\eta^m)_{\ell=1}^m) = 0 \) we obtain the following string of identities by testing \( u \) in the weak formulation for \( q, v \) and integrating by parts (recall that \( v \) has vanishing divergence):

\[ \gamma \sum_{\ell=1}^m \int_{\Sigma_\ell} \psi_\ell \partial_1 \eta_\ell = \sum_{\ell=1}^m \int_{\Omega_\ell} \mu_\ell \nabla : \nabla u + \gamma \rho \partial_1 v \cdot u = \sum_{\ell=1}^m \int_{\Omega_\ell} \mu_\ell \frac{1}{2} \nabla : \nabla u - \gamma \rho \partial_1 u \cdot v \]

\[ = - \sum_{\ell=1}^m \int_{\Omega_\ell} \nabla \cdot \left( S^\mu \left( p + g \sum_{k=1}^m \|p_k \| \eta_k \mathbb{1}_{(0, a_k)} \right) \right) : \nabla v - \gamma \rho \partial_1 u \cdot v \]

\[ = \sum_{\ell=1}^m \int_{\Omega_\ell} \left[ \nabla \cdot S^\mu \left( p + g \sum_{k=1}^m \|p_k \| \eta_k \mathbb{1}_{(0, a_k)} \right) \right] \cdot v \]

\[ \text{The above manipulations are justified by the fact that} \]

\[ \text{Proposition 2.10 together with the fact that } (\psi_\ell)_{\ell=1}^m, \text{ defined in (4.16), is band limited yields the implications:} \]

\[ (\psi_\ell)_{\ell=1}^m = (M_{B(0, 2^r) \setminus B(0, 2^{-r})} \psi_\ell)_{\ell=1}^m \Rightarrow \text{Tr}_{\Sigma_\ell} v = B(0, 2^r) \setminus B(0, 2^{-r}) \Rightarrow \text{supp} \mathcal{F} [\text{Tr}_{\Sigma_\ell} v] \subseteq B(0, 2^r) \setminus B(0, 2^{-r}). \]
Parseval’s theorem:
\[
\int_{\Sigma_{\ell}} \left[ S^\mu \left( p + \sum_{k=1}^{m} [\rho]_k \eta k \mathbb{1}_{(0, a_k)}, u \right) e_n \right]_{\ell} \cdot v = \int_{\mathbb{R}^{n-1}} \mathcal{F} \left[ \left[ S^\mu \left( p + \sum_{k=1}^{m} [\rho]_k \eta k \mathbb{1}_{(0, a_k)}, u \right) e_n \right]_{\ell} \right] \cdot \mathcal{F} [\text{Tr}_{\Sigma_k} v]
\]
\[
= \int_{\mathbb{R}^{n-1}} \mathcal{F} \left[ \left[ S^\mu \left( p, u \right) e_n \right]_{\ell} \right] \cdot \mathcal{F} [\text{Tr}_{\Sigma_k} v] + \int_{\mathbb{R}^{n-1}} \mathcal{F} \left[ \left[ \sum_{k=1}^{m} [\rho]_k \eta k \mathbb{1}_{(0, a_k)} \right] \right]_{\ell} \cdot \mathcal{F} [\text{Tr}_{\Sigma_k} v \cdot e_n]
\]
\[
= \int_{\mathbb{R}^{n-1}} \mathcal{F} \left[ \left[ \sum_{k=1}^{m} [\rho]_k \eta k \mathbb{1}_{(0, a_k)} \right] \right]_{\ell} \cdot \mathcal{F} [\text{Tr}_{\Sigma_k} v \cdot e_n] - \mathcal{F} \left[ \mathcal{F} [\eta] \cdot \mathcal{F} [\text{Tr}_{\Sigma_k} v \cdot e_n] \right],
\]
where the decomposition into the two integrands in the middle line above is justified by (4.20).

In the final term in equation (4.21), we would like to use the vanishing divergence of and $\sigma_0$-trace of $v$ to simplify further. We first compute
\[
\mathcal{F} [\text{Tr}_{\Sigma_k} v \cdot e_n](\xi) = \mathcal{F} \left[ \int_{\{0,a\mu\}} \partial_n v(\cdot, y) \, dy \right](\xi)
\]
\[
= -\mathcal{F} \left[ \int_{\{0,a\mu\}} \sum_{j=1}^{n-1} \partial_j v(\cdot, y) \cdot e_j \, dy \right](\xi) = \int_{\{0,a\mu\}} \mathcal{F} [v(\cdot, y)](\xi) \cdot 2\pi i (\xi, 0) \, dy.
\]
Using (4.22), we may then rewrite
\[
- \mathcal{F} \left[ \int_{\{0,a\mu\}} \mathcal{F} [\eta] \cdot \mathcal{F} [\text{Tr}_{\Sigma_k} v \cdot e_n] = - \mathcal{F} \left[ \int_{\{0,a\mu\}} \mathcal{F} [\eta](\xi) \cdot \left[ \int_{\{0,a\mu\}} \mathcal{F} [v(\cdot, y)](\xi) \cdot 2\pi i (\xi, 0) \, dy \right] d\xi\right.
\]
\[
= - \mathcal{F} \left[ \int_{\{0,a\mu\}} \int_{\mathbb{R}^{n-1}} 2\pi i (\xi, 0) \, dy \right].
\]
Summing over $\ell \in \{1, \ldots, m\}$ in (4.21) and implementing (4.23) yields the identity
\[
\sum_{\ell=1}^{m} \int_{\Sigma_{\ell}} \left[ S^\mu \left( p + \sum_{k=1}^{m} [\rho]_k \eta k \mathbb{1}_{(0, a_k)}, u \right) \right]_{\ell} \cdot v = \sum_{\ell=1}^{m} \int_{\mathbb{R}^{n-1}} \mathcal{F} \left[ \left( [\rho]_{\ell} + \sigma_\ell \Delta_\parallel \right) \eta_k \cdot \mathcal{F} [\text{Tr}_{\Sigma_k} v \cdot e_n] - \mathcal{F} \left[ \mathcal{F} [\eta] \cdot \mathcal{F} [\text{Tr}_{\Sigma_k} v \cdot e_n] \right],
\]
and upon substituting (4.24) into (4.18) we deduce the equality
\[
- \gamma \sum_{\ell=1}^{m} \int_{\Sigma_{\ell}} \psi_k \partial_n \eta_k = \sum_{\ell=1}^{m} \int_{\mathbb{R}^{n-1}} \mathcal{F} \left[ \left( [\rho]_{\ell} + \sigma_\ell \Delta_\parallel \right) \eta_k \cdot \mathcal{F} [\text{Tr}_{\Sigma_k} v \cdot e_n] \right].
\]
Next we use the definition of $[\psi_k]^m_{\ell=1}$ on the left hand side and observe that
\[
- \gamma \sum_{\ell=1}^{m} \int_{\Sigma_{\ell}} \psi_k \partial_n \eta_k = - \gamma \sum_{\ell=1}^{m} \int_{\Sigma_{\ell}} [\rho]_{\ell} M_{B(0,2r) \backslash B(0,2r)} \eta_k \partial_k M_{B(0,2r) \backslash B(0,2r)} \eta_k
\]
\[
+ \gamma \sum_{\ell=1}^{m} \int_{\Sigma_{\ell}} \sigma_\ell \nabla M_{B(0,2r) \backslash B(0,2r)} \eta_k \cdot \partial_k \nabla M_{B(0,2r) \backslash B(0,2r)} \eta_k = 0.
\]
For the right hand side of (4.25) we use first Proposition 2.16 and then item two of Theorem 2.19 to bound
\[
0 = \sum_{\ell=1}^{m} \int_{\mathbb{R}^{n-1}} \mathcal{F} \left[ \left( [\rho]_{\ell} + \sigma_\ell \Delta_\parallel \right) \eta_k \cdot \mathcal{F} [\text{Tr}_{\Sigma_k} v \cdot e_n] \right] = \int_{\mathbb{R}^{n-1}} \text{Re} \left[ \mathcal{F} \left[ \left( [\rho]_{\ell} + \sigma_\ell \Delta_\parallel \right) \eta_k \right] \cdot n_{\ell} \mathcal{F} [\psi_k]^m_{\ell=1} \right]
\]
\[
\geq \int_{B(0,2r) \backslash B(0,2r)} \min \{|\xi|^2, |\xi|^{-1}\} \sum_{\ell=1}^{m} |\mathcal{F} \left[ \left( [\rho]_{\ell} + \sigma_\ell \Delta_\parallel \right) \eta_k \right](\xi)|^2 \, d\xi.
\]
Hence the right hand side above is zero for all $r \in \mathbb{N}^+$. This proves that for all $\ell \in \{1, \ldots, m\}$ we have $\eta_k = 0$. Thus, we have the inclusion $(p, u) \in H^{1+s}(\Omega) \times H^{2+s}(\Omega; \mathbb{R}^m)$. The space on the right is the domain for $\Phi_\gamma$. Since $[\eta_k]^m_{\ell=1} = 0$ and $\Phi_\gamma(p, u, (\eta_k)^m_{\ell=1}) = 0$ we have that $\Phi_\gamma(p, u) = 0$. In Theorem 2.13 we showed that $\Phi_\gamma$ is an isomorphism, so $u = 0$ and $p = 0$. Hence, $\gamma$ is an injection. □
4.2. Isomorphism in the case with surface tension. In this subsection we characterize the solvability of (4.11) for data belonging to the space $\mathcal{Y}^s$ and positive surface tensions, i.e. $\{\sigma_\ell\}_{\ell=1}^m \subset \mathbb{R}^+$. Before we state and prove the relevant isomorphism theorem, we show how the data determine the free surface functions.

**Lemma 4.8** (Determination of free surface functions: surface tension case). If $\gamma \in \mathbb{R} \setminus \{0\}$, $\{\sigma_\ell\}_{\ell=1}^m \subset \mathbb{R}^+$, and $(g, f, (k_\ell)_{\ell=1}^m, (h_\ell)_{\ell=1}^m) \in \mathcal{Y}^s$ for some $s \in \mathbb{R}^+ \cup \{0\}$ then there exists $(\eta_\ell)_{\ell=1}^m \in \left(H^{5/2+s}(\mathbb{R}^{n-1})\right)^m$ such that the modified data tuple

$$
(g, f + g \sum_{\ell=1}^m [\rho_\ell] \nabla \eta_\ell \mathbf{1}_{(a, \omega)}, (k_\ell + \sigma_\ell \Delta \|\eta_\ell\|_{L^2})_{\ell=1}^m, (h_\ell - \gamma \partial_1 \eta_\ell)_{\ell=1}^m)
$$

$$
in H^{1+s}(\Omega) \times H^s(\Omega; \mathbb{R}^n) \times \prod_{\ell=1}^m H^{1/2+s}(\Sigma_\ell; \mathbb{R}^n) \times \prod_{\ell=1}^m H^{3/2+s}(\Sigma_\ell)
$$

belongs to the range of $\Psi_\gamma$, where this latter operator is from Theorem 3.3. Moreover, we have the universal estimate:

$$
\|\eta_\ell\|_{L^{5/2+s}} \lesssim \|(g, f, (k_\ell)_{\ell=1}^m, (h_\ell)_{\ell=1}^m)\|_{\mathcal{Y}^s}.
$$

**Proof.** We divide the proof into three steps.

**Step 1: Establishing invertibility of a matrix field.** Let

$$
o_0 = -g \text{diag}([\rho_1], \ldots, [\rho_m]), \quad o_1 = \text{diag}(\sigma_1, \ldots, \sigma_m) \in \mathbb{R}^{m \times m},
$$

and for $\xi \in \mathbb{R}^{n-1}$ we set $o(\xi) = o_0 + 4\pi^2 |\xi|^2 o_1 \in \mathbb{R}^{m \times m}$ and

$$
p_\gamma(\xi) = n_\gamma(\xi)^* o(\xi) - 2\pi i \gamma \xi_1 I_{m \times m} = n_\gamma(\xi) o(\xi) - 2\pi i \gamma \xi_1 I_{m \times m} \in \mathbb{C}^{m \times m},
$$

where $n_\gamma(\xi)$ is as defined in Proposition 2.16 and satisfies $n_\gamma(\xi)^* = n_{-\gamma}(\xi)$ by Proposition 2.17. We claim that there exists a constant $C \in \mathbb{R}^+$, depending only on the physical parameters and $\gamma$, such that for all $b \in \mathbb{C}^m$ and a.e. $\xi \in \mathbb{R}^{n-1}$ we have that

$$
C^{-1}|p_\gamma(\xi)b|^2 \lesssim |(\xi_1^2 + |\xi|^4)1_{B(0,1)}(\xi) + |\xi|^2 1_{\mathbb{R}^{n-1}\setminus B(0,1)}(\xi)| |b|^2 \lesssim C|p_\gamma(\xi)b|^2.
$$

We begin the proof of the claim by recalling that there is a full measure set $E \subset \mathbb{R}^{n-1}$ such that if $\xi \in E$ then the estimates from Theorem 2.19 hold for the matrix $n_\gamma(\xi)$. Let $\xi \in E \setminus B(0,1)$ and $b \in \mathbb{C}^m$. Then from Theorem 2.19 we deduce that

$$
|p_\gamma(\xi)b|^2 \lesssim |\xi_1^2 + |\xi|^4| |b|^2 \lesssim |\xi|^2 |b|^2.
$$

Similarly, since $o(\xi)$ is self-adjoint, we have that $2\pi i \gamma \xi_1 b \cdot o(\xi)b$ is purely imaginary, so again Theorem 2.19 allows us to bound

$$
|\xi|^2 |p_\gamma(\xi)b| |b| \gtrsim \text{Re} |p_\gamma(\xi)b \cdot o(\xi)b| = \text{Re} |n_\gamma(\xi)^* o(\xi)b \cdot o(\xi)b| \gtrsim |\xi|^2 |o(\xi)b|^2 \lesssim |\xi|^4 |b|^2.
$$

Combining (4.33) and (4.34) gives estimate (4.32) for $\xi \in E \setminus B(0,1)$.

On the other hand, if $\xi \in B(0,1) \cap E$ and $b \in \mathbb{C}^m$, we once again appeal to Theorem 2.19 to arrive at the upper bound

$$
|p_\gamma(\xi)b|^2 \lesssim (\xi_1^2 + |\xi|^4) |b|^2.
$$

For the matching lower bound we combine the following estimates wherein we tacitly use: 1) for each $\ell \in \{1, \ldots, m\} - g \|\rho\|_\ell > 0$ and $|\gamma| > 0$, and 2) $n_\gamma o_\gamma^*$ is a self-adjoint matrix field. First:

$$
|\xi|^2 |p_\gamma(\xi)b| |b| \gtrsim |o(\xi)p_\gamma(\xi)b \cdot n_\gamma(\xi)^* o(\xi)b| = |n_\gamma(\xi)o(\xi)n_\gamma(\xi)^* o(\xi)b \cdot o(\xi)b - 2\pi i \gamma \xi_1 o(\xi)b \cdot n_\gamma(\xi)^* o(\xi)b| \gtrsim |\text{Im}[n_\gamma(\xi)o(\xi)n_\gamma(\xi)^* o(\xi)b \cdot o(\xi)b - 2\pi i \gamma \xi_1 o(\xi)b \cdot n_\gamma(\xi)^* o(\xi)b]| \gtrsim 2\pi |\gamma| |\xi_1^*| |\xi|^2 |o(\xi)b|^2 \gtrsim |\gamma| |\xi| |\xi|^2 |b|^2.
$$

Second:

$$
|p_\gamma(\xi)b| |b| \gtrsim \text{Re} |p_\gamma(\xi)b \cdot o(\xi)b| = \text{Re} |n_\gamma(\xi)^* o(\xi)b \cdot o(\xi)b| \gtrsim |\xi|^2 |o(\xi)b|^2 \gtrsim |\xi|^2 |b|^2.
$$

Estimates (4.35), (4.36), and (4.37) give (4.32) in the remaining cases.
Step 2: Construction of the free surface functions. Given \((g, f, (k_\ell)^m_{\ell=1}, (h_\ell)^m_{\ell=1}) \in \mathcal{Y}^s\) we propose to define, via item three of Proposition 4.3 (\(\eta_\ell^m\)) \(\in (\mathcal{H}^{5/2+s}(\mathbb{R}^n-1))^m\) through

\[
\mathcal{F}[(\eta_\ell^m)] = p_\gamma^{-1} \mathcal{H}^\gamma(g, f, (k_\ell)^m_{\ell=1}, (h_\ell)^m_{\ell=1}). \tag{4.38}
\]

Recall that \(\mathcal{H}^\gamma\) is the operator from Theorem 5.1. It is clear that since \(p_\gamma(-\xi) = \overline{p_\gamma(\xi)}\) (this realness assertion follows from that of \(n_\gamma\) - see Propositions 2.16 and A.2) for \(\xi\) then the above assignment will define a real-valued tempered distribution provided it defines a tempered distribution in the first place. For the latter to hold we need only observe that (note the use of inequality (4.32) and continuity of \(\mathcal{H}^\gamma\))

\[
\|\eta_\ell^m\|_{L^2(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n-1} \left|\xi\right|^{-2} \left(\xi^2 + |\xi|^4\right) 1_{B(0,1)}(\xi) + \left|\xi\right|^{5+2s} 1_{\mathbb{R}^n-1\setminus B(0,1)}(\xi) \cdot \mathcal{F}[(\eta_\ell^m)](\xi)^2 \, d\xi \leq \int_{\mathbb{R}^n-1} \max\{|\xi|^{-2}, |\xi|^{3+2s}\} \cdot \mathcal{F}[(\eta_\ell^m)](\xi)^2 \, d\xi \lesssim \|(g, f, (k_\ell)^m_{\ell=1}, (h_\ell)^m_{\ell=1})\|_{U^s}. \tag{4.39}
\]

This gives (4.29).

Step 3: Modification of the data. To show that the modified data tuple in equation (4.28) belongs to the range of \(\Psi_\gamma\) we use the second equivalence of Theorem 5.3 so let \(F \in (qH^1(\Omega; \mathbb{R}^n))^s\) be defined as \(F = \mathcal{P}(f, (k_\ell)^m_{\ell=1})\), for \(\mathcal{P}\) as in Definition 2.11. Using the definitions of \((\eta_\ell^m)\) and the mappings \(\mathcal{H}^\gamma, \mathcal{I}^\gamma, \) and \(\mathcal{H}^\gamma\) (Definition 3.1, Proposition 3.4 and Theorem 3.11 respectively), for any \((\psi_\ell^m)_{\ell=1}^m \in \prod_{\ell=1}^m H^{-1/2}(\Sigma_\ell)\) we may compute

\[
\mathcal{H}^\gamma[(g, F, (h_\ell)^m_{\ell=1}), (\psi_\ell^m)] = \langle (\psi_\ell^m)_{\ell=1}^m, \mathcal{H}^\gamma(g, F, (h_\ell)^m_{\ell=1}) \rangle_{H^{-1/2}, H^{1/2}}
\]

\[
= \langle (\psi_\ell^m)_{\ell=1}^m, 2\pi^2 |\xi|^2 \mathcal{O}_1 \mathcal{F}[(\eta_\ell^m)](\xi) \rangle_{\mathcal{H}^{-1/2}, \mathcal{H}^{1/2}} = \int_{\mathbb{R}^n-1} \mathcal{F}[(\psi_\ell^m)](\xi) \cdot \mathcal{F}[(\eta_\ell^m)](\xi) \, d\xi - \int_{\mathbb{R}^n-1} \mathcal{F}[(\psi_\ell^m)](\xi) \cdot 2\pi i \gamma_1 \xi_1 \mathcal{F}[(\eta_\ell^m)](\xi) \, d\xi + \int_{\mathbb{R}^n-1} \mathcal{F}[(\psi_\ell^m)](\xi) \cdot \mathcal{O}_0 \mathcal{F}[(\eta_\ell^m)](\xi) \, d\xi. \tag{4.40}
\]

For the first two terms after the last equality above we may use item two of Proposition 4.3 to justify the application of \(\mathcal{F}\)’s unitary properties; we also recall Proposition 2.16 which states that \(n_\gamma\) is a spectral representation of \(\gamma_\gamma\). Hence,

\[
\int_{\mathbb{R}^n-1} n_\gamma(\xi) \mathcal{F}[(\psi_\ell^m)](\xi) \cdot 2\pi^2 |\xi|^2 \mathcal{O}_1 \mathcal{F}[(\eta_\ell^m)](\xi) \, d\xi = -\sum_{\ell=1}^m \int_{\Sigma_\ell} v \cdot \sigma_\ell \Delta |\eta_\ell^m|, \tag{4.41}
\]

for \((q, v) = \chi_{-1}(0, \mathcal{O}_\ell \eta_\ell^m) \in L^2(\Omega) \times 0H^1(\Omega; \mathbb{R}^n)\) the solution to the applied stress PDE in (2.44) with data \((\psi_\ell^m)_{\ell=1}^m\). We also have the equality

\[
\int_{\mathbb{R}^n-1} \mathcal{F}[(\psi_\ell^m)](\xi) \cdot 2\pi i \gamma_1 \xi_1 \mathcal{F}[(\eta_\ell^m)](\xi) \, d\xi = \langle (\psi_\ell^m)_{\ell=1}^m, (\gamma^0 \eta_\ell^m)_{\ell=1}^m \rangle_{H^{-1/2}, H^{1/2}}. \tag{4.42}
\]

For the final term in (4.40) we cannot, in general, apply that \(\mathcal{F}\) is unitary directly since \((\eta_\ell^m)_{\ell=1}^m\) need not belong to \(\prod_{\ell=1}^m L^2(\Sigma_\ell)\). Instead we utilize the fact that \(v\) is solenoidal and vanishes on \(\Sigma_0\), which provides us identity (4.22). Hence,

\[
\int_{\mathbb{R}^n-1} n_\gamma \mathcal{F}[(\psi_\ell^m)](\xi) \cdot \mathcal{O}_0 \mathcal{F}[(\eta_\ell^m)](\xi) = -\sum_{\ell=1}^m \mathcal{O}_1 \mathcal{F}[(\eta_\ell^m)](\xi) \cdot \mathcal{F}[(\eta_\ell^m)](\xi) \, d\xi = -\sum_{\ell=1}^m \mathcal{O}_1 \mathcal{F}[(\eta_\ell^m)](\xi) \cdot \nabla \eta_\ell^m 1_{(0,a_\ell)} \cdot v. \tag{4.43}
\]
The last equality, which is an application of Plancherel’s and Fubini’s theorems, is justified by the third item of Proposition 4.5. Define $G \in (\mathcal{H}^1(\Omega; \mathbb{R}^n))^s$ through the assignment

$$
\langle G, w \rangle \equiv \langle F, w \rangle + \int_{\Omega} \left[ g \sum_{\ell=1}^{m} \left[ \rho \right]_{\ell} \nabla \eta \mathbb{1}_{(0, a\ell)} \right] \cdot w + \sum_{\ell=1}^{m} \int_{\Omega} \sigma \Delta \eta \mathbb{1}_{(0, a\ell)} \cdot w
$$

$$
= \int_{\Omega} f \cdot w + \sum_{\ell=1}^{m} \int_{\Sigma_{\ell}} k_{\ell} \cdot w + \int_{\Omega} \left[ g \sum_{\ell=1}^{m} \left[ \rho \right]_{\ell} \nabla \eta \mathbb{1}_{(0, a\ell)} \right] \cdot w + \sum_{\ell=1}^{m} \int_{\Omega} \sigma \Delta \eta \mathbb{1}_{(0, a\ell)} \cdot w, \quad w \in \mathcal{H}^1(\Omega; \mathbb{R}^n).
$$

We now synthesize identities (4.40), (4.41), (4.42), and (4.43):

$$
\mathcal{H}^\gamma[(g, F, (h_{\ell})_{\ell=1}^{m}), (\psi_{\ell})_{\ell=1}^{m}] = -\sum_{\ell=1}^{m} \int_{\Sigma_{\ell}} v \cdot \sigma \Delta \eta \mathbb{1}_{n} - \gamma \langle (\psi_{\ell})_{\ell=1}^{m}, (\partial_{1} \eta)_{\ell=1}^{m} \rangle_{H^{-1/2, H^{1/2}}}
$$

$$
- g \sum_{\ell=1}^{m} \left[ \rho \right]_{\ell} \int_{\Omega} \nabla \mathbb{1}_{(0, a\ell)} \cdot v = \mathcal{H}^\gamma[(0, F - G), (\gamma \partial_{1} \eta)_{\ell=1}^{m}), (\psi_{\ell})_{\ell=1}^{m}].
$$

Rearranging (4.45) and using that $\mathcal{H}^\gamma$ is bilinear shows that

$$
\mathcal{H}^\gamma[(g, G, (h_{\ell} - \gamma \partial_{1} \eta)_{\ell=1}^{m}), (\psi_{\ell})_{\ell=1}^{m}] = 0.
$$

As the above expression vanishes for all $(\psi_{\ell})_{\ell=1}^{m} \in \prod_{\ell=1}^{m} H^{-1/2}(\Sigma_{\ell})$, we conclude that the modified data tuple of equation (4.28) belongs to $\text{ker} \mathcal{H}^\gamma$, which Theorem 3.3 establishes is the range of $\Psi_{\gamma}$.

At last we are ready to state and prove an isomorphism of Banach spaces induced by the PDE (4.1).

**Theorem 4.9** (Existence and uniqueness of solutions to (4.1): surface tension case). For $\gamma \in \mathbb{R} \setminus \{0\}$, $\sigma = \{\sigma_{\ell}\}_{\ell=1}^{m} \subset \mathbb{R}^+$, and $s \in \mathbb{R}^+ \cup \{0\}$ the bounded linear mapping $\mathcal{Y}_{\gamma, s} : \mathcal{X}^s \to \mathcal{Y}^s$, with action given by (4.12), is an isomorphism.

**Proof.** Proposition 4.7 ensures that this mapping is well-defined and injective, so it remains only to prove surjectivity. Let $(g, f, (k_{\ell})_{\ell=1}^{m}, (h_{\ell})_{\ell=1}^{m}) \in \mathcal{Y}^s$, and define the associated tuple of free surface functions $(\eta_{\ell})_{\ell=1}^{m} \in (H^{5/2+s}(\mathbb{R}^{n-1}))^m$ via Lemma 4.8. Then the modified data tuple in (4.28) belongs to the range of $\Psi_{\gamma}$. Consequently, there exists $(q, u) \in H^{1+s}(\Omega) \times \mathcal{H}^{5/2+s}(\Omega; \mathbb{R}^n)$ such that

$$
\Psi_{\gamma}(q, u) = (g, f + g \sum_{\ell=1}^{m} \left[ \rho \right]_{\ell} \nabla \mathbb{1}_{(0, a\ell)}, (k_{\ell} + \sigma \Delta \eta \mathbb{1}_{n})_{\ell=1}^{m}, (h_{\ell} - \gamma \partial_{1} \eta)_{\ell=1}^{m}).
$$

Set $p \in \mathcal{P}^{1+s}(\Omega)$ via $p = q - g \sum_{\ell=1}^{m} \left[ \rho \right]_{\ell} \eta \mathbb{1}_{(0, a\ell)}$. As $p + g \sum_{\ell=1}^{m} \left[ \rho \right]_{\ell} \eta \mathbb{1}_{(0, a\ell)} = q \in H^{1+s}(\Omega)$, we have that $(p, u, (\eta_{\ell})_{\ell=1}^{m}) \in \mathcal{X}^s$. We then observe that

$$
\left[ S^h(p, u)e_{n}\right]_{\ell} = \left[ S^h(q, u)e_{n}\right]_{\ell} - g \sum_{k=1}^{m} \left[ \rho \right]_{k} \eta \mathbb{1}_{(0, a\ell)} = k_{\ell} + \sigma \Delta \eta \mathbb{1}_{n}.
$$

It is now straightforward to check $\mathcal{Y}_{\gamma, s}(p, u, (\eta_{\ell})_{\ell=1}^{m}) = (g, f, (k_{\ell})_{\ell=1}^{m}, (h_{\ell})_{\ell=1}^{m})$, which completes the proof that $\Psi_{\gamma, s}$ is a surjection.

**4.3. Isomorphism in the case without surface tension.** In this subsection we study (4.1) in the case of a two-dimensional fluid ($n = 2$) and vanishing surface tension ($\sigma = \{\sigma_{\ell}\}_{\ell=1}^{m} = 0$). Again, we first present how the free surface functions are determined from the data. In this instance the proof is simpler because item four of Proposition 4.3 tells us that the function spaces holding the tuple of free surface functions are familiar Sobolev spaces.

**Lemma 4.10** (Determination of free surface functions: case without surface tension). If $\gamma \in \mathbb{R} \setminus \{0\}$ and $(g, f, (k_{\ell})_{\ell=1}^{m}, (h_{\ell})_{\ell=1}^{m}) \in \mathcal{Y}^s$ for some $s \in \mathbb{R}^+ \cup \{0\}$, then there exists $(\eta_{\ell})_{\ell=1}^{m} \in (H^{5/2+s}(\mathbb{R}^{n-1}))^m = (H^{5/2+s}(\mathbb{R}^{n-1}))^m$ such that the modified data tuple

$$
(g, f, (k_{\ell} + g \left[ \rho \right]_{\ell} \eta \mathbb{1}_{(0, a\ell)}, (h_{\ell} - \gamma \partial_{1} \eta)_{\ell=1}^{m})
$$

$$
\in H^{1+s}(\Omega) \times H^{s}(\Omega; \mathbb{R}^n) \times \prod_{\ell=1}^{m} H^{1/2+s}(\Sigma_{\ell}; \mathbb{R}^n) \times \prod_{\ell=1}^{m} H^{3/2+s}(\Sigma_{\ell})
$$

belongs to the range of $\Psi_{\gamma}$, where this latter operator is from Theorem 3.3. Moreover, we have the universal estimate

$$
\| (\eta_{\ell})_{\ell=1}^{m} \|_{H^{5/2+s}} \lesssim \| (g, f, (k_{\ell})_{\ell=1}^{m}, (h_{\ell})_{\ell=1}^{m}) \|_{\mathcal{Y}^s}.
$$

(4.50)
\textbf{Proof.} We again proceed in three steps as in the proof of Lemma 4.8

\textbf{Step 1: Estimates and invertibility.} Again let \( \mathbf{o}_0 = \text{diag} (\sigma_1, \ldots, \sigma_m) \in \mathbb{R}^{m \times m} \). We claim that the matrix field

\[ \mathbf{p}_\gamma (\xi) = \mathbf{n}_\gamma (\xi) * \mathbf{o}_0 - 2 \pi i \gamma \mathbf{I}_{m \times m} = \mathbf{n}_{-\gamma} (\xi) \mathbf{o}_0 - 2 \pi i \gamma \mathbf{I}_{m \times m} \in \mathbb{C}^{m \times m}, \]

where \( \mathbf{n}_\gamma (\xi) \) is as defined in Proposition 2.16 and, satisfies the estimate

\[ C^{-1} |\mathbf{p}_\gamma (\xi)| b^2 \leq |\xi|^2 |b|^2 \leq C |\mathbf{p}_\gamma (\xi)| b^2 \]

for a.e. \( \xi \in \mathbb{R} \) and all \( b \in \mathbb{C}^m \), for a constant \( C \in \mathbb{R}^+ \) depending only on the physical parameters. Recall that since \( n = 2 \), \( \xi = \xi_1 \).

By the first item of Theorem 2.19 there is a universal constant \( C_0 \in \mathbb{R}^+ \) and a full measure set \( E \subset \mathbb{R} \) such that if \( \xi \in E \), then \( |\mathbf{n}_\gamma (\xi) * \mathbf{o}_0| \leq C_0 \min \{ |\xi|^2, |\xi|^{-1} \} \). Thus the left inequality in (4.52) follows from the triangle inequality. Also, as a consequence of this estimate on \( \mathbf{n}_\gamma * \mathbf{o}_0 \), we learn that there are radii (depending only on \( C_0 \) and \( |\gamma| \)) \( 0 < R_0 < 1 < R_1 \) such that if \( \xi \in \mathbb{R} \setminus \{(-R_1, -R_0) \cup (R_0, R_1)\} \), then

\[ 2 \pi |\gamma| |\xi| - C_0 \min \{ |\xi|^2, |\xi|^{-1} \} \geq \pi |\gamma| |\xi| \]

(4.53).

Estimate (4.53) gives the right inequality in (4.52) for \( \xi \in E \) with \( |\xi| \leq R_0 \) or \( |\xi| \geq R_1 \), by the reverse triangle inequality. For \( \xi \in E \cap (-R_1, R_1) \setminus (-R_0, R_0) \) we use the second item of Theorem 2.19. Let \( b \in \mathbb{C}^m \). As \( \mathbf{o}_0 \) is self-adjoint, we may estimate

\[ |\mathbf{p}_\gamma (\xi) b | b | \geq |\text{Re} \mathbf{p}_\gamma (\xi) b \cdot \mathbf{o}_0 b | = |\text{Re} \mathbf{b} \cdot \mathbf{n}_\gamma (\xi) \mathbf{o}_0 b | \geq \min \{ |\xi|^2, |\xi|^{-1} \} |b|^2 \geq \min \{ R_0^2, R_1^{-1} \} |b|^2 \].

Therefore (4.52) is shown.

\textbf{Step 2: Construction of the free surface functions.} Again using item three of Proposition 4.3 we define, given \( (g, f, (k_\ell)^{m}_{\ell = 1}, (h_\ell)^{m}_{\ell = 1}) \in \mathcal{Y}^s \), a corresponding tuple of functions \( (\eta_\ell)^{m}_{\ell = 1} \in (H^{5/2+s}(\Omega; \mathbb{R}^m)) \) via \( \mathcal{F} [(\eta_\ell)^{m}_{\ell = 1}] = \mathbf{p}_\gamma^{-1} \mathcal{F} [\mathcal{H}^\gamma (g, f, (k_\ell)^{m}_{\ell = 1}, (h_\ell)^{m}_{\ell = 1})] \). This is well-defined thanks to the estimate

\[ ||(\eta_\ell)^{m}_{\ell = 1}||^2_{H^{5/2+s}} \leq \int_{\mathbb{R}^n} \frac{1}{|\xi|^2} |\mathcal{F} [(\eta_\ell)^{m}_{\ell = 1}] (\xi)|^2 \, d\xi \]

\[ \lesssim \int_{\mathbb{R}^n} (1 + |\xi|^2) |\mathcal{F} [\mathcal{H}^\gamma (g, f, (k_\ell)^{m}_{\ell = 1}, (h_\ell)^{m}_{\ell = 1})] (\xi)|^2 \, d\xi \lesssim \| (g, f, (k_\ell)^{m}_{\ell = 1}, (h_\ell)^{m}_{\ell = 1}) \|_{\mathcal{Y}^s}. \]

(4.55)

Hence the estimate of (4.50) holds.

\textbf{Step 3: Data correction.} To show that the modified data tuple in (4.39) belongs to the range of \( \Psi_\gamma \), we appeal to the equivalence presented in Theorem 3.3. Again we define \( F \in \langle (0 H^1 (\Omega; \mathbb{R}^m))^s \) as \( F = \mathcal{F} (f, (k_\ell)^{m}_{\ell = 1}) \), for \( \mathcal{F} \) in Definition 2.11. Using the definition of \( \mathcal{H}^\gamma \), we compute

\[ \mathcal{H}^\gamma [(g, F, (h_\ell)^{m}_{\ell = 1}), (\psi_\ell)^{m}_{\ell = 1}] = \langle (\psi_\ell)^{m}_{\ell = 1}, \mathcal{F} (g, F, (h_\ell)^{m}_{\ell = 1}) \rangle_{H^{-1/2},H^{1/2}} = \int_{\mathbb{R}^n} \mathcal{F} [(\psi_\ell)^{m}_{\ell = 1}] \cdot \mathbf{p}_\gamma \mathcal{F} [(\eta_\ell)^{m}_{\ell = 1}] \]

\[ = \int_{\mathbb{R}^n} \mathbf{n}_\gamma \mathcal{F} [(\psi_\ell)^{m}_{\ell = 1}] \cdot \mathbf{o}_0 \mathcal{F} [(\eta_\ell)^{m}_{\ell = 1}] - \int_{\mathbb{R}^n} \mathcal{F} [(\psi_\ell)^{m}_{\ell = 1}] (\xi) \cdot 2 \pi i \xi \mathcal{F} [(\eta_\ell)^{m}_{\ell = 1}] (\xi) \, d\xi, \]

(4.56)

for \( (\psi_\ell)^{m}_{\ell = 1} \in \prod_{\ell = 1} H^{-1/2}(\Sigma) \). Denote the solution to the normal stress PDE in equation (2.41) with data \( (\psi_\ell)^{m}_{\ell = 1} \) as \( (q, v) = \chi^{-1}_{\gamma} (0, \mathcal{F} (\psi_\ell)^{m}_{\ell = 1}) \in L^2 (\Omega) \times 0 H^1 (\Omega; \mathbb{R}^m) \) (recall that \( \chi^{-1}_{\gamma} \) and \( \mathcal{F} \) are defined in Proposition 2.9 and Definition 2.11). We next use the fact that \( \mathcal{F} \) is unitary on \( L^2 \) to rewrite (4.56) as

\[ \mathcal{H}^\gamma [(g, F, (h_\ell)^{m}_{\ell = 1}), (\psi_\ell)^{m}_{\ell = 1}] = -\frac{m}{\Sigma} \sum_{\ell = 1} \mathbf{[p]} \int_{\Sigma} \eta \psi \cdot e_n - \langle (\psi_\ell)^{m}_{\ell = 1}, (\gamma \mathcal{F} (\eta_\ell)^{m}_{\ell = 1}) \rangle_{H^{-1/2},H^{1/2}}. \]

(4.57)

Set \( G \in \langle (0 H^1 (\Omega; \mathbb{R}^m))^s \) via

\[ \langle G, w \rangle_{(0 H^1)^s, 0 H^1} = \langle F, w \rangle_{(0 H^1)^s, 0 H^1} + \frac{m}{\Sigma} \sum_{\ell = 1} \mathbf{[p]} \int_{\Sigma} \eta w \cdot e_n \]

\[ = \int_{\Omega} f \cdot w + \sum_{\ell = 1} k_\ell \cdot w + \frac{m}{\Sigma} \sum_{\ell = 1} \mathbf{[p]} \int_{\Sigma} \eta w \cdot e_n \text{ for } w \in 0 H^1 (\Omega; \mathbb{R}^m). \]

(4.58)
Then (4.57) implies that \( \mathcal{M} \gamma \left[ (g, G, (h_\ell - \gamma \partial_1 \eta_\ell)_\ell=1^n, (\psi_\ell)_\ell=1^n) \right] = 0 \) for all \( (\psi_\ell)_\ell=1^n \), so we conclude, using Theorem 3.5, that the modified data tuple belongs to the range of \( \Psi_\gamma \).

Finally, we state and prove the analogue to Theorem 4.9.

**Theorem 4.11** (Existence and uniqueness of solutions to (4.1) case without surface tension). For \( \gamma \in \mathbb{R} \setminus \{0\} \), \( n = 2 \), and \( s \in \mathbb{R}^+ \cup \{0\} \) the bounded linear mapping \( \Upsilon_{\gamma,0} : \mathcal{X}^s \to \mathcal{Y}^s \), with action given by (4.12), is an isomorphism.

**Proof.** Proposition 4.7 ensures that this mapping is well-defined and injective, so only surjectivity remains. Let \( (g, f, (k_\ell)_\ell=1^n, (h_\ell)_\ell=1^n) \in \mathcal{Y}^s \) and define the associated tuple of free surface functions \( (\eta_\ell)_\ell=1^n \in (H^{5/2+s}(\mathbb{R}^{n-1}))^n \) via Lemma 4.10. Then the modified data tuple in equation (4.49) belongs to the range of \( \Psi_\gamma \). Consequently, there exists \( (p, u) \in H^{1+s}(\Omega) \times H^{2+s}(\Omega; \mathbb{R}^n) \) such that

\[
\Psi_\gamma (p, u) = (g, f, (k_\ell)_\ell=1^n, (h_\ell - \gamma \partial_1 \eta_\ell)_\ell=1^n).
\] (4.59)

By item four of both Propositions 4.3 and 4.5, we have that \( (p, u, (\eta_\ell)_\ell=1^n) \in \mathcal{X}^s \). It’s also clear that \( \Upsilon_{\gamma,0} (p, u, (\eta_\ell)_\ell=1^n) = (g, f, (k_\ell)_\ell=1^n, (h_\ell)_\ell=1^n) \). Hence, \( \Upsilon_{\gamma,0} \) is a surjection. □

5. Nonlinear analysis

We now use the Banach isomorphisms constructed in the previous section to solve the fully nonlinear problems (1.17) and (1.11) for small data by way of the implicit function theorem. The proofs of most of the results in this section essentially mirror those used in the one layer analysis of [LT19] (except that we use our new isomorphisms), so for the sake of brevity we will mostly sketch the details. For full details we refer to Section 8 of [LT19].

5.1. Preliminaries. This subsection is dedicated to showing that the nonlinear mapping associated to the flattened PDE (1.17) is both well-defined and smooth. We begin by examining the smoothness of the nonlinearities present. First we have a simple product estimate.

**Proposition 5.1.** Let \( \ell \in \{1, \ldots, m\} \) and \( s \in \mathbb{R}^+ \) with \( (n - 1)/2 < s \). If \( f \in H^s(\mathbb{R}^{n-1}) \) and \( g \in H^s(\Sigma_\ell) \), then the pointwise product satisfies the inclusion \( fg \in H^s(\Sigma_\ell) \). Moreover the bilinear mapping \( H^s \times H^s \ni (f, g) \mapsto fg \in H^s \) is continuous and hence smooth.

**Proof.** This is Theorem 5.8 in [LT19]. □

The more complicated nonlinearities present in system (1.17) are also smooth, as a consequence of the following result.

**Proposition 5.2.** Let \( s \in \mathbb{R}^+ \) with \( s > n/2 \), and \( m = 1 \) There exists a positive radius \( \delta (s) \in \mathbb{R}^+ \) such that the following hold.

1. If \( \eta \in H^s(\mathbb{R}^{n-1}) \) satisfies \( \|\eta\|_{H^s} < \delta (s) \) then \( \|\eta\|_{C^0} < \frac{1}{2} \).
2. By the first item for \( \eta \in B_{H^s}(0, \delta (s)) \), \( w \in H^s(\mathbb{R}^{n-1}) \), and \( v \in H^s(\Omega) \) we are free to define pointwise \( \Gamma_0 (\eta, w) = \frac{w}{1+\eta} \) and \( \Gamma_1 (\eta, v) = \frac{v}{1+\eta} \). Then \( \Gamma_0 (\eta, w) \in H^s(\mathbb{R}^{n-1}) \) and \( \Gamma_1 (\eta, v) = \frac{v}{1+\eta} \in H^s(\Omega) \), and the mappings \( \Gamma_0 : B_{H^s}(0, \delta (s)) \times H^s(\mathbb{R}^{n-1}) \to H^s(\mathbb{R}^{n-1}) \) and \( \Gamma_1 : B_{H^s}(0, \delta (s)) \times H^s(\Omega) \to H^s(\Omega) \) are smooth.

**Proof.** The existence of a \( \delta_1 (s) \in \mathbb{R}^+ \) for which the first item holds follows from the supercritical embedding within item 2 of Proposition 4.3.

Theorem 5.15 in [LT19] states that for some \( \varepsilon_0 \in \mathbb{R}^+ \) the mapping \( \Gamma_2 : B_{P^s}(0, \varepsilon_0) \times H^s(\Omega) \to H^s(\Omega) \), for \( B_{P^s}(0, \varepsilon_0) \) the open \( \varepsilon_0 \)-ball of the space \( P^s(\Omega) \) (Definition 4.4 for \( m = 1 \)), defined by \( \Gamma_2 (\zeta, u) = \frac{u}{1+\varepsilon_0} \) is smooth and well-defined. Denote the continuous (and hence smooth) inclusion mapping: \( \iota : H^s(\mathbb{R}^{n-1}) \to P^s(\Omega) \). There then exists \( \delta_2 (s) \in \mathbb{R}^+ \) such that \( \iota (B_{H^s}(0, \delta_2 (s))) \subseteq B_{P^s}(0, \varepsilon_0) \). Let \( \hat{\delta} (s) = \min \{ \delta_1 (s), \delta_2 (s) \} \). Then \( \Gamma_1 \) is smooth as \( \Gamma_1 = \Gamma_2 \circ (\iota, \id_{H^s(\Omega)}) \). By letting \( \Psi : H^s(\Omega) \to H^s(\mathbb{R}^{n-1}) \) denote the smooth projection onto this closed subspace and \( \iota : H^s(\mathbb{R}^{n-1}) \to H^s(\Omega) \) denote this smooth inclusion we deduce that \( \Gamma_0 = \Psi \circ \Gamma_2 \circ (\iota, \hat{\iota}) \) is also smooth. □
The data spaces $\mathcal{Y}^s$ for which we solve the linearized flattened problem enforce the divergence compatibility condition from Proposition 2.11. To ensure that the nonlinear mapping associated to the flattened problem has a target enforcing this condition, we require the following result.

**Proposition 5.3.** Suppose that $s \in \mathbb{R}^+$ satisfies $s > n/2$, $u \in H^{2+s}(\Omega[0];\mathbb{R}^n)$, and $(\eta_k)_{k=1}^m \subset B_{H^{5/2+s}}(0, \delta)$ for $\delta = \frac{1}{2} \min(a_1, a_2 - a_1, \ldots, a_m - a_{m-1})$ for $\delta (5/2 + s) \in \mathbb{R}^+$, where $\delta (5/2 + s) \in \mathbb{R}^+$ is as in Proposition 5.2. Then for each $\ell \in \{1, \ldots, m\}$ we have the identity
\[
\int_{(a_{k-1}, a_k)} J A \nabla \cdot u = u \cdot N_\ell(\cdot, a_\ell) + (\nabla \parallel, 0) \cdot \int_{(a_{k-1}, a_k)} J A^\top u,
\]
where $J$, $A$, and $N_\ell$ are functions of $(\eta_k)_{k=1}^m$ as defined in Section 1.3.

**Proof.** Let $k \in \{1, \ldots, \ell\}$. Arguing as in Proposition 8.2 in [LT19] we arrive at
\[
\int_{(a_{k-1}, a_k)} J_k A_k \nabla \cdot u = u \cdot N_\ell(\cdot, a_\ell) - u \cdot N_{\ell-1}(\cdot, a_{\ell-1}) + (\nabla \parallel, 0) \cdot \int_{(a_{k-1}, a_k)} J_k A_k^\top u,
\]
where we take $N_0 = e_n$. Summing over $k \leq \ell$ and using that $u$ vanishes on $\Sigma_0$ gives the result. \( \square \)

We now arrive at our final preliminary result, which states that the nonlinear mapping associated to the flattened problem (11.17) is well-defined and smooth.

**Theorem 5.4.** Let $s \in \mathbb{R}^+$ with $s > n/2$, $\sigma = \{\sigma_\ell\}_{\ell=1}^m \subset \mathbb{R}^+ \cup \{0\}$, and $\kappa \in \mathbb{R}^+$. Define the open set
\[
U_\kappa^s = \{ (p, u, (\eta_\ell)_{\ell=1}^m) \in \mathcal{X}^s : \eta_\ell \in B_{H^{5/2+s}}(0, \kappa) \text{ for } \ell \in \{1, \ldots, m\} \}
\]
and the mapping $\Sigma_\sigma : \mathbb{R} \times \prod_{\ell=1}^m H^{1/2+s}(\Sigma_\ell[0]; \mathbb{R}^{n \times n}) \times U_\kappa^s \to \mathcal{Y}^s$ with action given via
\[
\Sigma_\sigma(\gamma, (T_\ell)_{\ell=1}^m, p, u, (\eta_\ell)_{\ell=1}^m) = \left( J \nabla \cdot u, \sum_{\ell=1}^m 1_{\Omega[0]} \left[ \rho \nu (u - \gamma e_1) \cdot A \nabla \right] u + (A \nabla) \cdot S_a^m (p, u), \right)
\]
for $J$, $A$, and $N_\ell$ defined as functions of $(\eta_k)_{k=1}^m$ as in Sections 1.4 and 1.3. There exists $\kappa_0 \in \mathbb{R}^+$ such that for all $0 < \kappa \leq \kappa_0$ the mapping $\Sigma_\sigma$ is well-defined, i.e. maps into $\mathcal{Y}^s$, and is smooth.

**Proof.** Theorem A.10 of [LT19] asserts that there is $\delta_1 \in \mathbb{R}^+$ for which the mean curvature operator $H : B_{H^{5/2+s}}(0, \delta_1) \to H^{1/2+s}(\mathbb{R}^{n-1})$ is well-defined and smooth. Thus we set $\kappa_0$ to be the minimum of $\delta_1$ and $\delta$ for the radius from Proposition 5.3. By combining the analysis of the nonlinearities from Propositions 5.1 and 5.2 with the nonlinear divergence compatibility of Proposition 5.3 we may argue as in Theorem 8.3 in [LT19] to deduce well-definedness and smoothness. \( \square \)

5.2. Solvability of (11.17) and (11.11). To solve (11.17) we combine the smoothness result from Theorem 5.4 with the linear isomorphisms of Theorems 5.9 and 5.11.

**Theorem 5.5.** Suppose that $\sigma = \{\sigma_\ell\}_{\ell=1}^m \subset \mathbb{R}^+ \text{ and } n \geq 2$ or $\sigma = 0$ and $n = 2$. Assume that $\mathbb{R}^+ \ni s > n/2$. Then there exists open sets $\mathcal{V}_s \subset \mathcal{X}^s$ and $\mathcal{U}_s \subset \mathbb{R} \times \{0\} \times \prod_{\ell=1}^m H^{1/2+s}(\Sigma_\ell[0]; \mathbb{R}^{n \times n}) \times H^s(\Omega[0]; \mathbb{R}^n)$ such that the following hold.

1. $(0, 0, (0)_{\ell=1}^m) \subset \mathcal{V}_s$ and $(\mathbb{R} \times \{0\}) \times \{(0)_{\ell=1}^m \} \times \{0\} \subset \mathcal{U}_s$.
2. For each $(\gamma, (T_\ell)_{\ell=1}^m, f) \in \mathcal{U}_s$ there exists a unique $(p, u, (\eta_\ell)_{\ell=1}^m) \in \mathcal{V}_s$ solving (11.17) classically.
3. The mapping $\mathcal{U}_0 \ni (\gamma, (T_\ell)_{\ell=1}^m, f) \mapsto (p, u, (\eta_\ell)_{\ell=1}^m) \in \mathcal{V}_s$ is smooth.

**Proof.** We apply the implicit function theorem to $\Sigma_\sigma$ (see, for instance, Theorem 2.5.7 in [AMR88]). Denote the Hilbert space $\mathcal{E}^s = \mathbb{R} \times \prod_{\ell=1}^m H^{1/2+s}(\Sigma_\ell[0]; \mathbb{R}^{n \times n})$. Viewing the domain of $\Sigma_\sigma$ as the product $\mathcal{E}^s \times U_\kappa^s \subset \mathcal{E}^s \times \mathcal{X}$ we define the partial derivatives with respect to the first and second factors via
\[
D_1 \Sigma_\sigma : \mathcal{E}^s \times U_\kappa^s \to \mathcal{L}(\mathcal{E}^s, \mathcal{Y}^s) \quad \text{and} \quad D_2 \Sigma_\sigma : \mathcal{E}^s \times U_\kappa^s \to \mathcal{L}(\mathcal{X}^s, \mathcal{Y}^s).
\]

For any $\gamma \in \mathbb{R}$ we have $\Sigma_\sigma(\gamma, (0)_{\ell=1}^m, 0, (0)_{\ell=1}^m) = 0$ and $D_2 \Sigma_\sigma(\gamma, (0)_{\ell=1}^m, 0, (0)_{\ell=1}^m) = \mathcal{Y}_{\gamma, \sigma}$, for the latter operator as in Proposition 4.7 Theorems 5.3, 4.9 and 4.11 witness the satisfaction of the implicit function theorem’s hypotheses whenever $\gamma \in \mathbb{R} \setminus \{0\}$.
Therefore for each \( \gamma \in \mathbb{R} \setminus \{0\} \) there exists open sets \( \mathfrak{X}(\gamma) \subset \mathcal{E}^s \), \( \mathfrak{B}(\gamma) \subset U_{n_0}^s \), and \( \mathfrak{C}(\gamma) \subset \mathcal{Y}^s \) such that \( (\gamma, (0)^{m}_{l=1}) \in \mathfrak{X}(\gamma) \), \( (0, 0, (0)^{m}_{l=1}) \in \mathfrak{B}(\gamma) \), and \( (0, 0, (0)^{m}_{l=1}, (0)^{m}_{l=1}) \in \mathfrak{C}(\gamma) \) and a smooth mapping \( \omega_{\gamma} : \mathfrak{X}(\gamma) \times \mathfrak{C}(\gamma) \rightarrow \mathfrak{B}(\gamma) \) such that

\[
\Xi_\sigma(\gamma, (T)_{l=1}^m, \omega_{\gamma}(\gamma, (T)_{l=1}^m, g, f, (k_l)_{l=1}^m, (h_l)_{l=1}^m)) = (g, f, (k_l)_{l=1}^m, (h_l)_{l=1}^m) \tag{5.6}
\]

for all \( (g, f, (k_l)_{l=1}^m, (h_l)_{l=1}^m) \in \mathfrak{C}(\gamma) \) and all \( (\gamma, (T)_{l=1}^m) \in \mathfrak{X}(\gamma) \). Moreover the tuple \((p, u, (\eta)_{l=1}^m) = (\omega_{\gamma}, (\gamma, (T)_{l=1}^m, g, f, (k_l)_{l=1}^m, (h_l)_{l=1}^m) \in \mathfrak{B}(\gamma) \) is the unique solution to (5.6) in \( \mathfrak{B}(\gamma) \).

Define the open sets

\[
\mathcal{C}_1(\gamma) = \{ f : (0, f, (0)^{m}_{l=1}, (0)^{m}_{l=1}) \in \mathfrak{C}(\gamma) \} \subseteq H^s(\Omega[0]; \mathbb{R}^n),
\]

\[
\mathcal{U}_s = \bigcup_{\gamma \in \mathbb{R} \setminus \{0\}} \mathfrak{X}(\gamma) \times \mathcal{C}_1(\gamma) \subset \mathcal{E}^s \times \mathcal{X}^s(\Omega[0]; \mathbb{R}^n), \text{ and } \mathcal{V}_s = \bigcup_{\gamma \in \mathbb{R} \setminus \{0\}} \mathfrak{B}(\gamma) \subset U_{n_0}^s. \tag{5.7}
\]

Observe that the first item is satisfied with these open sets. Define \( \varphi : \mathcal{U}_s \rightarrow \mathcal{V}_s \) via \( \varphi(\gamma, (T)_{l=1}^m, f) = (\omega_{\gamma}, (\gamma, (T)_{l=1}^m, f) \) when \( (\gamma, (T)_{l=1}^m) \in \mathfrak{X}(\gamma) \) for some \( \gamma \in \mathbb{R} \setminus \{0\} \). The map \( \varphi \) is well-defined and smooth by the previous analysis.

Taking \((p, u, (\eta)_{l=1}^m) = (\omega_{\gamma}, (\gamma, (T)_{l=1}^m, f) \) and noting the embeddings of the specialized Sobolev spaces (see Propositions 4.3 and 4.5) completes the justification of the second and third items.

Next we examine system (1.11). Our first result gives some of the mapping properties of the flattening map \( \mathfrak{F} \) and its inverse from Section 1.3.

**Proposition 5.6.** Let \( n, k \in \mathbb{N} \) with \( 1 \leq n/2 < k \), let \( \eta = (\eta)_{l=1}^m \in (H^{5/2+k}(\mathbb{R}^{n-1}))^m \) be such that

\[
\max\{\|\eta_1\|_{C^0_{\gamma}}, \ldots, \|\eta_m\|_{C^0_{\gamma}}\} \leq \frac{1}{4} \min\{a_1, a_2 - a_1, \ldots, a_m - a_m - 1\}, \tag{5.8}
\]

and define \( \mathfrak{G} : \Omega[\eta] \rightarrow \Omega[0] \) via \( \mathfrak{G} = \mathfrak{F}_\ell^{-1} \) in the set \( \Omega[\ell] \) for each \( \ell \in \{1, \ldots, m\} \) as in (1.13). Then the following hold.

1. \( \mathfrak{G} \in C^{0,1}(\Omega[\eta], \Omega[0]) \) is a bi-Lipschitz homeomorphism with inverse given by \( \mathfrak{F} \in C^{0,1}(\Omega[0]; \Omega[\eta]) \), as defined in (1.12).
2. Set \( \mathfrak{G}_\ell = \mathfrak{G} | \Omega[\ell] \) for \( \ell \in \{1, \ldots, m\} \). Then \( \mathfrak{G}_\ell \in C^r(\Omega[\ell], \Omega[0]) \) is a diffeomorphism with inverse given by \( \mathfrak{F}_\ell = \mathfrak{F} | \Omega[\ell] \in C^r(\Omega[0], \Omega[\ell]) \), where \( N \geq r < 3 + k - n/2 \).
3. If \( g \in \mathfrak{G} H^1(\Omega[0]) \) then \( g \circ \mathfrak{G} \in \mathfrak{G} H^1(\Omega[\eta]) \). Moreover there is \( c \in \mathbb{R}^+ \), independent of \( g \), such that \( \|g \circ \mathfrak{G}\|_{0, H^1} \leq c \|g\|_{0, H^1} \).
4. For \( \mathbb{R}^+ \cup \{0\} \ni s \leq k + 2 \), if \( f \in H^s(\Omega[0]) \) then \( f \circ \mathfrak{G} \in H^s(\Omega[\eta]) \). Moreover there is \( \bar{c} \in \mathbb{R}^+ \), independent of \( f \), such that \( \|f \circ \mathfrak{G}\|_{H^s(\Omega[0])} \leq \bar{c} \|f\|_{H^s(\Omega[0])} \).

**Proof.** By inspection, \( \mathfrak{G} \) is a homeomorphism with weak derivative in \( \Omega[\eta] \) given by \( \nabla \mathfrak{G}(x, y) = \mathcal{A}^t \circ \mathfrak{G} \), for \( \mathcal{A}^t \) the geometry matrix field from Section 1.3. By the embedding of item two in Proposition 1.3 this weak gradient is essentially bounded. Hence \( \mathfrak{G} \) is Lipschitz. A similar argument shows that \( \mathfrak{F} = \mathfrak{G}^{-1} \) is also Lipschitz. Hence the first and third items are now shown. The second and fourth items are now shown by applying the arguments of Theorem 8.4 in [LT19] to the restrictions \( \mathfrak{G} | \Omega[\ell] \) for \( \ell \in \{1, \ldots, m\} \).

Finally, we prove the solvability of the free boundary problem (1.11).

**Theorem 5.7.** Let \( n, k \in \mathbb{N} \) with \( 1 \leq n/2 < k \). Suppose that \( \sigma = \{\sigma_l\}_{l=1}^m \subset \mathbb{R}^+ \) and \( n \geq 2 \) or \( \sigma = 0 \) and \( n = 2 \). For all \( \gamma \in \mathbb{R} \setminus \{0\} \), there exists \( \varepsilon \in \mathbb{R}^+ \) such that if \( (T)_{l=1}^m H^{1/2+k} (\sigma_l 0; \mathbb{R}^{n \times n}) \), \( f \in H^k(\Omega[0]; \mathbb{R}^n) \), and \( \sum_{l=1}^m \left[ \|T_l\|H^{1/2+k} + \|f\|_{H^k(\Omega[0])} \right] < \varepsilon \) then there exists a tuple of free surface functions \( \eta = (\eta)_{l=1}^m \in (H^{5/2+k}(\mathbb{R}^{n-1}))^m \) satisfying (5.8) such that the following hold.

1. If \( \mathfrak{G} \) is the diffeomorphism from Proposition 5.6, then we have the inclusion \( F := f \circ \mathfrak{G} \in H^k(\Omega[0]; \mathbb{R}^n) \).
2. There exists \( (q, v) \in \mathfrak{P}^{1+k}(\Omega[\eta]) \times \mathfrak{H}^{2+k}(\Omega[\eta]; \mathbb{R}^n) \) such that \((q, v, \eta)\) is a classical solution to system (1.11) with forcing \( F \) and applied surface stresses \((T)_{l=1}^m \).

**Proof.** We argue as in the proof of Theorem 1.3 in [LT19]. For small data we may solve the flattened problem via Theorem 5.5. Then we obtain the associated flattening mapping via Proposition 5.6. Finally, we pre-compose the solution to the flattened problem with the inverse of the flattening map to obtain the desired solution to the free boundary problem.

□
APPENDIX A. TOOLS FROM ANALYSIS

This appendix records various tools and results used throughout the paper.

A.1. Real-valued tempered distributions. Recall the notion of a real valued tempered distribution.

**Definition A.1** (Real valued tempered distributions). We say that \( F \in (\mathcal{S}(\mathbb{R}^d; \mathbb{C}))^* \) is \( \mathbb{R} \)-valued if \( F \) equals its complex conjugate \( \overline{F} \), where we define \( \overline{F} \in (\mathcal{S}^*(\mathbb{R}^d; \mathbb{C}))^* \) with action \( \langle \overline{F}, \varphi \rangle = \langle F, \overline{\varphi} \rangle \) for \( \varphi \in \mathcal{S}(\mathbb{R}^d; \mathbb{C}) \).

The following are useful characterizations of the \( \mathbb{R} \)-valued tempered distributions. Here we recall that the reflection operator \( \delta_{-1} \) acts on functions \( f : \mathbb{R}^d \to \mathbb{C}^k \) via \( \delta_{-1} f(x) = f(-x) \) and acts on \( F \in (\mathcal{S}(\mathbb{R}^d; \mathbb{C}))^* \) via \( \langle \delta_{-1} F, \varphi \rangle = \langle F, \delta_{-1} \varphi \rangle \).

**Proposition A.2** (Characterizations of real-valued tempered distributions). For \( F \in (\mathcal{S}(\mathbb{R}^d; \mathbb{C}))^* \) the following are equivalent.

1. \( F \) is \( \mathbb{R} \)-valued.
2. \( \overline{F} = \delta_{-1} F \).
3. \( F \in (\mathcal{S}(\mathbb{R}^d; \mathbb{R}))^* \) in the sense that \( \langle F, \varphi \rangle \in \mathbb{R} \) for all \( \varphi \in \mathcal{S}(\mathbb{R}^d; \mathbb{R}) \).

**Proof.** The equivalence of the first and second items is standard; see, for instance, Lemma A.1 of [LT19] for a proof. We prove that the first and third items are equivalent. Suppose first that (3) holds. If \( \varphi \in \mathcal{S}(\mathbb{R}^d; \mathbb{C}) \) then \( \text{Re}[\varphi], \text{Im}[\varphi] \in \mathcal{S}(\mathbb{R}^d; \mathbb{R}) \); hence we are free to equate

\[
\langle F, \varphi \rangle = \langle F, \text{Re}[\varphi] \rangle + i\langle F, \text{Im}[\varphi] \rangle = \langle F, \text{Re}[\varphi] \rangle + i\langle F, \text{Im}[\varphi] \rangle = \langle F, \varphi \rangle.
\]

Therefore (3) \( \Rightarrow \) (1). Next suppose that (1) holds. If \( \varphi \in \mathcal{S}(\mathbb{R}^d; \mathbb{R}) \) then \( \varphi = \overline{\varphi} \). Hence,

\[
\langle F, \varphi \rangle = \langle \overline{F}, \varphi \rangle = \langle \overline{F}, \overline{\varphi} \rangle = \langle F, \overline{\varphi} \rangle \Rightarrow \langle F, \varphi \rangle \in \mathbb{R}.
\]

Thus (1) \( \Rightarrow \) (3).

**Remark A.3.** By the previous proposition it is not an abuse of notation to denote the space of real valued tempered distributions with \((\mathcal{S}(\mathbb{R}^d; \mathbb{R}))^*\).

**Remark A.4.** If \( f \in (\mathcal{S}(\mathbb{R}^d; \mathbb{C}))^* \cap L^1_{\text{loc}}(\mathbb{R}^d; \mathbb{C}) \) then by the third item of Proposition A.2 \( f \) is a \( \mathbb{R} \)-valued tempered distribution if and only if \( f(x) \in \mathbb{R} \) for a.e. \( x \in \mathbb{R}^d \).

A.2. (Anti-)duality and Lax-Milgram. Recall notions of sesquilinearity and anti-duality as defined in Section 1.6 of the introduction. The following variant of the Lax-Milgram lemma is adapted to anti-duality. For the well-known \( \mathbb{R} \)-valued version of this result we refer, for instance, to Corollary 5.8 of [Bre11].

**Proposition A.5** (Lax-Milgram). Suppose that \( H \) is \( \mathbb{C} \)-Hilbert and \( B : H \times H \to \mathbb{C} \) is a continuous and sesquilinear mapping for which there exists \( c \in \mathbb{R}^+ \) such that for all \( u \in H \) one has the coercive estimate \( ||u||_H^2 \leq c\text{Re}[B(u, u)] \). Then, there exists a \( \mathbb{C} \)-linear continuous isomorphism \( \beta : \mathbb{H}^\mathbb{C} \to H \) satisfying

\[
B(\beta F, v) = \langle F, v \rangle_{\mathbb{H}^\mathbb{C}, H} \text{ for all } F \in \mathbb{H}^\mathbb{C} \text{ and } v \in H.
\]

**Proof.** Let \( K \) be the \( \mathbb{R} \)-Hilbert space with underlying vector space equal to that of \( H \) and equipped with inner product \( \langle \cdot, \cdot \rangle_K = \text{Re}[\langle \cdot, \cdot \rangle_H] \). The map \( \text{Re}[B(\cdot, \cdot)] : K \times K \to \mathbb{R} \) is then a bilinear form satisfying the hypotheses of the \( \mathbb{R} \)-valued Lax-Milgram lemma; in other words, \( \text{Re}[B(\cdot, \cdot)] \) is bounded and coercive. Thus there exists an \( \mathbb{R} \)-isomorphism \( \alpha_0 : K^* \to K \) such that for all \( v \in K \) and all \( G \in K^* \) we have \( \text{Re}[B(\alpha_0 G, v)] = \langle G, v \rangle_{K^*, K} \). Let \( \alpha_1 : H^\mathbb{C} \to K^* \) be the \( \mathbb{R} \)-linear mapping defined via \( \langle \alpha_1 F, v \rangle_{K^*, K} = \text{Re}[\langle F, v \rangle_{\mathbb{H}^\mathbb{C}, H}] \). Set \( \beta : H^\mathbb{C} \to H \) via \( \beta = \alpha_0 \alpha_1 \).

By definition of \( \beta \) for all \( F \in H^\mathbb{C} \) and \( v \in H \) we have \( \text{Re}[B(\beta F, v) - \langle F, v \rangle_{\mathbb{H}^\mathbb{C}, H}] = 0 \). By antilinearity:

\[
B(\beta F, v) - \langle F, v \rangle_{\mathbb{H}^\mathbb{C}, H} = \text{Re}[B(\beta F, v) - \langle F, v \rangle_{\mathbb{H}^\mathbb{C}, H}] + i\text{Re}[B(\beta F, iv) - \langle F, iv \rangle_{\mathbb{H}^\mathbb{C}, H}] = 0.
\]

Finally \( \beta \) is a \( \mathbb{C} \)-linear isomorphism as it is the inverse of the following \( \mathbb{C} \)-linear mapping: \( \alpha_2 : H \to H^\mathbb{C} \) with action \( \langle \alpha_2 v, w \rangle_{\mathbb{H}^\mathbb{C}, H} = B(v, w) \).

To conclude this subsection, we review some representation formulae of (anti)-dual spaces.
Proposition A.6 ((Anti-)duality representation of Sobolev spaces). Let \( s \in \mathbb{R} \) and \( K \in \{ \mathbb{R}, \mathbb{C} \} \). Recall that the \( K^k \)-valued \( L^2 \)-based Sobolev space on \( \mathbb{R}^d \) of order \( s \) is defined as

\[
H^s(\mathbb{R}^d; K^k) = \left\{ f \in ((\mathcal{S}(\mathbb{R}^d; K))^*)^k : \mathcal{F}[f] \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}^d; C^k) \right\}
\]

and \( \| f \|_{H^s}^2 = \int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\mathcal{F}[f](\xi)|^2 \, d\xi < \infty \). \hspace{1cm} (A.5)

We have the representation formula \((H^s(\mathbb{R}^d; K^k))^\mathbb{T} = H^{-s}(\mathbb{R}^n; K^k)\) where one may view the (anti-)dual pairing as the sesquilinear (bilinear when \( K = \mathbb{C} \)) \( L^2 \)-pairing of Fourier transforms. In other words, for all \( G \in (H^s(\mathbb{R}^d; K^k))^\mathbb{T} \) there exists a unique \( g \in H^{-s}(\mathbb{R}^d; K^k) \) such that for all \( f \in H^s(\mathbb{R}^d; K^k) \) one has the equality

\[
\langle G, f \rangle_{(H^s)^\mathbb{T},H^{-s}} = \int_{\mathbb{R}^d} \mathcal{F}[g] \cdot \mathcal{F}[f] =: \langle g, f \rangle_{H^{-s},H^s}. \hspace{1cm} (A.6)
\]

Conversely if \( g \in H^{-s}(\mathbb{R}^d; K^k) \) then \( f \mapsto \langle g, f \rangle_{H^{-s},H^s} \) defines a member of \((H^s(\mathbb{R}^d; K^k))^\mathbb{T}\).

Proof. The assertions for the case \( K = \mathbb{C} \) are a consequence of the discussion after Theorem 6.3 in \cite{Fo95}. Suppose that \( K = \mathbb{R} \) and that \( G \in (H^s(\mathbb{R}^d; K^k))^* \). We may define \( G_0 \in (H^s(\mathbb{R}^d; C^k))^\mathbb{T} \) via

\[
(G_0, f)_{(H^s)^\mathbb{T},H^s} = \langle G, \text{Re}[f] \rangle_{(H^s)^\mathbb{T},H^s} - i \langle G, \text{Im}[f] \rangle_{(H^s)^\mathbb{T},H^s}, \quad f \in H^s(\mathbb{R}^d; C^k). \hspace{1cm} (A.7)
\]

Applying the result for the \( \mathbb{C} \)-valued case gives us \( g \in H^{-s}(\mathbb{R}^d; C^k) \) such that \( \langle G_0, f \rangle_{(H^s)^\mathbb{T},H^s} = \langle g, f \rangle_{H^{-s},H^s} \) for all \( f \in H^s(\mathbb{R}^d; C^k) \). We next note that \( (\mathcal{S}(\mathbb{R}^d; \mathbb{R}))^* \) by Proposition A.2 that is, if \( f \in \mathcal{S}(\mathbb{R}^d; \mathbb{R}) \subset H^s(\mathbb{R}^d; C^k) \), then

\[
\langle g, f \rangle_{\mathcal{S}(\mathbb{R}^d; \mathbb{R})} = \langle g, f \rangle_{H^{-s},H^s} = \langle G, \text{Re}[f] \rangle_{(H^s)^\mathbb{T},H^s} - i \langle G, \text{Im}[f] \rangle_{(H^s)^\mathbb{T},H^s} = \langle G, f \rangle_{(H^s)^\mathbb{T},H^s} \in \mathbb{R}. \hspace{1cm} (A.8)
\]

Finally, \( g \) is uniquely determined by the following argument. Suppose that \( g_0 \in H^{-s}(\mathbb{R}^d; K^k) \) also satisfies \( \langle G, f \rangle_{(H^s)^\mathbb{T},H^s} = \langle g_0, f \rangle_{H^{-s},H^s} \) for all \( f \in H^s(\mathbb{R}^d; \mathbb{R}) \). Define \( f_0 \in H^s(\mathbb{R}^d; \mathbb{C}) \) via \( \mathcal{F}[f_0](\xi) = (1 + |\xi|^2)^{-s} \mathcal{F}[g - g_0](\xi), \xi \in \mathbb{R}^d \). Then

\[
0 = \langle g - g_0, f_0 \rangle_{H^{-s},H^s} = \int_{\mathbb{R}^d} (1 + |\xi|^2)^{-s} |\mathcal{F}[g - g_0]|^2 \, d\xi = \| g - g_0 \|^2_{H^{-s}}. \hspace{1cm} (A.9)
\]

\( \square \)

Remark A.7. In this paper we choose identify the functional \( G \) with the tempered distribution \( g \) and the (anti-)duality pairing \( \langle \cdot, \cdot \rangle_{(H^s)^\mathbb{T},H^s} \) with \( \langle \cdot, \cdot \rangle_{H^{-s},H^s} \).

A.3. Fourier multipliers. We begin this subsection recalling the characterization of essentially bounded Fourier multipliers as \( L^2 \)-bounded translation invariant linear mappings.

Proposition A.8. Let \( K \in \{ \mathbb{R}, \mathbb{C} \} \). The following are equivalent for a continuous linear mapping \( T \in \mathcal{L}(L^2(\mathbb{R}^d; \mathbb{K})) \).

1. \( T \) commutes with all translation operators.
2. There exists \( m \in L^\infty(\mathbb{R}^d; \mathbb{K}) \) such that \( T f = \mathcal{F}^{-1}[m \mathcal{F}[f]] \) for all \( f \in L^2(\mathbb{R}^d; \mathbb{K}) \).

In either case \( \| T \|_{\mathcal{L}(L^2)} \leq \| m \|_{L^\infty} \leq 2 \| T \|_{\mathcal{L}(L^2)} \), and if \( K = \mathbb{R} \) then \( m(\xi) = m(-\xi) \) for a.e. \( \xi \in \mathbb{R}^d \).

Proof. The case \( K = \mathbb{C} \) is handled in Theorem 2.5.10 of \cite{Gra14}. It remains to handle the case \( K = \mathbb{R} \). The implication (2) \( \Rightarrow \) (1) is clear. On the other hand, given a translation invariant \( T \in \mathcal{L}(L^2(\mathbb{R}^d; \mathbb{R})) \) define \( T_0 \in \mathcal{L}(L^2(\mathbb{R}^d; \mathbb{C})) \) via \( T_0 f = T \text{Re}[f] + it \text{Im}[f] \) for \( f \in L^2(\mathbb{R}^d; \mathbb{C}) \). \( T_0 \) is also translation invariant. Hence by the \( \mathbb{C} \)-valued case there is \( m \in L^\infty(\mathbb{R}^d; \mathbb{C}) \) such that the action of \( T_0 \) is given by multiplication of \( m \) in frequency space. Using Proposition A.2 and Remark A.4 we compute:

\[
\overline{m \mathcal{F}[f]} = \mathcal{F}[T \text{Re}[f]] - i \mathcal{F}[T \text{Im}[f]] = \delta_{-1} \mathcal{F}[T \text{Re}[f]] + i \delta_{-1} \mathcal{F}[T \text{Im}[f]] = \delta_{-1} m \mathcal{F}[f] = \delta_{-1} m \mathcal{F}[f]. \hspace{1cm} (A.10)
\]

The above equality holds for all \( f \in L^2(\mathbb{R}^n; \mathbb{C}) \) and so we deduce that \( \overline{m} = \delta_{-1} m \) almost everywhere. \( \square \)

We may also generalize the previous theorem to a characterization of continuous, linear, and translation invariant mappings between the \( L^2 \)-based fractional Sobolev spaces. First we recall the Bessel potential.
Definition A.9 (Bessel potential). For $s \in \mathbb{R}$ we define the Bessel Potential of order $s$ as the operator $J^s \in \mathcal{L}(\mathcal{S}(\mathbb{R}^d; \mathbb{C})^*)$ defined via $J^s F = \mathcal{F}^{-1}[(1+|\cdot|^2)^{s/2}F]$ for $F \in (\mathcal{S}(\mathbb{R}^d; \mathbb{C})^*)$. Thanks to Proposition A.12, $J^s F$ is $\mathbb{R}$-valued whenever this true of $F$. We also recall that for any $t \in \mathbb{R}$ and $k \in \{\mathbb{R}; \mathbb{C}\}$ $J^s \in \mathcal{L}(H^{1+s}(\mathbb{R}^d; k); H^t(\mathbb{R}^d; k))$ is an isometric isomorphism.

Proposition A.10. Let $k \in \{\mathbb{R}; \mathbb{C}\}, s, t \in \mathbb{R}$. The following are equivalent for a continuous linear mapping $T \in \mathcal{L}(H^s(\mathbb{R}^d; k); H^t(\mathbb{R}^d; k)).$

1. $T$ commutes with all translation operators.
2. There exists a measurable function $\mu : \mathbb{R}^d \to \mathbb{C}$ such that
   \[ m_\mu[r] := \text{esssup}\{(1 + |\xi|^2)^{r/2} |\mu(\xi)| : \xi \in \mathbb{R}^d\} \in [0, \infty] \] (A.11)
   is finite for $r = t - s$ and for all $f \in H^s(\mathbb{R}^d; k)$ one has $T f = \mathcal{F}^{-1}[\mu \mathcal{F}[f]].$

In either case $\|T\|_{\mathcal{L}(H^s,H^t)} \leq m_\mu[t-s] \leq 2 \|T\|_{\mathcal{L}(H^s,H^t)}$, and if $k = \mathbb{R}$ then $\delta_{-1} \mu = \mathcal{F}$.

Proof. Suppose that the first item holds. Using the Bessel potentials from the previous definition, we obtain the bounded and translation-invariant $L^2$-operator $T_0 := J^t J^{-s}$. Applying Proposition A.8 grants us $\omega \in L^\infty(\mathbb{R}^d; k)$ such that if $k = \mathbb{R}$ then $\delta_{-1} \omega = \mathcal{F}$ and $T_0 F = \mathcal{F}^{-1}[\omega \mathcal{F}[f]]$ for $F \in L^2(\mathbb{R}^d; k)$. Set $\mu(\xi) := (1 + |\xi|^2)^{(t-s)/2} \omega(\xi)$. We check that $\mu$ is the desired spectral representation of $T$:

\[ TF = J^{-t}T_0 J^s F = J^{-t}T_0 \mathcal{F}^{-1}[(1+|\cdot|^2)^{t/2}F] = \mathcal{F}^{-1}[(1+|\cdot|^2)^{(t-s)/2} \omega(\xi) (1+|\cdot|^2)^{s/2} \mathcal{F}[f]] = \mathcal{F}^{-1}[\mu \mathcal{F}[f]]. \] (A.12)

for $F \in H^s(\mathbb{R}^d; k)$. Using once more Proposition A.8 we arrive at the bounds

\[ m_\mu[t-s] = \|\omega\|_{L^\infty} \times \|T_0\|_{\mathcal{L}(L^2)} = \|T\|_{\mathcal{L}(H^s,H^t)}. \] (A.13)

Thus the forward direction is shown. The reverse implication is proved in a similar manner.

We now set notation for $L^2$-bounded translation invariant mappings, emphasizing that this space is parameterized by essentially bounded functions.

Definition A.11 (Tangential Fourier multipliers I). Let $k \in \{\mathbb{R}, \mathbb{C}\}$ and let $\omega \in L^\infty(\mathbb{R}^d; \mathbb{C}^{k \times k})$ be a multiplier such that if $k = \mathbb{R}$ then $\mathcal{F} = \delta_{-1} \omega$. We make the following definitions.

1. $M_\omega \in \mathcal{L}(L^2(\mathbb{R}^d; k))$ is defined via $M_\omega f = \mathcal{F}^{-1}[\omega \mathcal{F}[f]]$ for $f \in L^2(\mathbb{R}^d; k)$.
2. If $a < b$ are real numbers we set $U = \mathbb{R}^d \times (a,b)$ and extend $M_\omega$ to be a member of $L^2(U; k^k)$ via $M_\omega f(\cdot, y) = \mathcal{F}^{-1}[\omega \mathcal{F}[f(\cdot, y)]]$ for $y \in (a,b)$ and $f \in L^2(U; k^k)$.

We would like to further extend the definitions of $M_\omega$ to the spaces $H^s(U \times (a,b); k^k)$ for $s \in \mathbb{R}^+$ and $(aH^1(U \times (a,b); k^k))^\mathbb{R}$ and study their boundedness properties. To do this we need the following preliminary estimates.

Lemma A.12. Let $s \in \mathbb{R}^+ \cup \{0\}$, $k$, and $\omega$ be as in Definition A.11. $a < b$ be real, and $U = \mathbb{R}^d \times (a,b).$ Then the following hold.

1. If $f \in H^s(U; k^k)$ then $M_\omega f \in H^s(U; k^k)$ and $\|M_\omega f\|_{H^s} \leq c_0 \|\omega\|_{L^\infty} \|f\|_{H^s}$ for a constant $c_0 \in \mathbb{R}^+$ depending only on $a$, $b$, $d$, $s$. Moreover if $s > 1/2$ then for $z \in [a,b]$ we have
   \[ \mathcal{F}_{\mathbb{R}^d \times \{z\}} M_\omega f = M_\omega \mathcal{F}_{\mathbb{R}^d \times \{z\}} f. \] (A.14)
2. If $f \in H^s(U; k^k)$, then setting $J^s f(\cdot, y) := \mathcal{F}^{-1}[(1+|\cdot|^2)^{s/2} \mathcal{F}[f(\cdot, y)]]$ for $y \in (a,b)$ defines an $L^2$-function $J^s f \in L^2(U; k^k)$, and there is a constant $c_1 \in \mathbb{R}^+$, dependent only upon $a$, $b$, $d$, and $s$, for which $\|J^s f\|_{L^2} \leq c_1 \|\omega\|_{H^s}$.

Proof. The first item follows from interpolation, the fact that $M_\omega$ commutes with distributional derivatives, and Proposition A.8. The second item follows from Corollary A.6 of [LT19].

By the first item of the previous lemma, we may extend the definition of tangential Fourier multipliers in the following way.
Definition A.13 (Tangential Fourier multipliers II). Let \( K \) and \( \omega \) be as in Definition A.1, \( a < b \) be real, and \( U = \mathbb{R}^d \times (a, b) \). If \( F \in (0H^1(U; \mathbb{K}^k))^\mathbb{F} \) we define \( M_\omega F \in (0H^1(U; \mathbb{K}^k))^\mathbb{F} \) to be the (anti-)linear functional with action on \( \varphi \in 0H^1(U; \mathbb{K}^k) \) given by
\[
\langle M_\omega F, \varphi \rangle_{0H^1} = \langle F, M_\omega \varphi \rangle_{0H^1}.
\]

Thanks to the first item of Lemma A.12, \( M_\omega F \) is well-defined and \( \| M_\omega F \|_{(0H^1)^\mathbb{F}} \leq \| \omega \|_{L^\infty} \| F \|_{(0H^1)^\mathbb{F}} \).

Finally, we arrive at the principal result of this subsection.

Proposition A.14. Let \( s \in \mathbb{R}^+ \cup \{0\} \), \( \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\} \), and \( \omega \in L^\infty(\mathbb{R}^d; \mathbb{C}^{k \times k}) \) be such that if \( \mathbb{K} = \mathbb{R} \) then \( \overline{\omega} = \delta_{-1}\omega \). Let \( m \in \mathbb{N}^+ \) and \( \{a_\ell\}_{\ell=1}^m \subset \mathbb{R}^+ \) be such that \( a_0 = 0 < a_1 < \cdots < a_m \), and set \( U = \bigcup_{\ell=1}^m U_\ell \), and \( W = (U)^c \). Then there exists a constant \( c_2 \in \mathbb{R}^+ \), independent of \( \omega \), for which the following estimates hold, where \( m_\omega[\cdot] \) is as in (A.11).

1. If \( g \in H^{1+s}(U; \mathbb{K}^k) \), then \( \| M_\omega g \|_{L^2} \leq c_2 m_\omega[1 + s] \sum_{\ell=1}^m \| g \|_{H^{1+s}(U_\ell)} \).

2. If \( f \in H^s(U; \mathbb{K}^k) \), \( \{k_\ell\}_{\ell=1}^m \in \prod_{\ell=1}^m H^{1/2+s}(\mathbb{R}^d \times \{a_\ell\}; \mathbb{K}^k) \), and we define \( F \in (0H^1(W; \mathbb{K}^k))^\mathbb{F} \) via
\[
\langle F, \varphi \rangle_{0H^1} = \int_U f \cdot \varphi + \sum_{\ell=1}^m \int_{\mathbb{R}^d \times \{a_\ell\}} k_\ell \cdot \varphi, \text{ then } M_\omega F \in (0H^1(W; \mathbb{K}^k))^\mathbb{F} \text{ with }
\]
\[
\| M_\omega F \|_{(0H^1)^\mathbb{F}} \leq c_2 m_\omega[1 + s] \sum_{\ell=1}^m \| f \|_{H^s(U_\ell)} + \| k_\ell \|_{H^{1/2+s}}.
\]

**Proof.** For the first item we use the second assertion of Lemma A.12 to bound
\[
\| M_\omega g \|_{L^2} \leq m_\omega[1 + s]^2 \sum_{\ell=1}^m \| J^{1+s}g \|_{L^2(U_\ell)}^2 \leq c_1^2 m_\omega[1 + s]^2 \sum_{\ell=1}^m \| g \|_{H^{1+s}(U_\ell)}^2.
\]

We next prove the second item. Suppose that \( \varphi \in 0H^1(W; \mathbb{K}^k) \). If \( \ell \in \{1, \ldots, m\} \) then by trace theory and the first assertion of Lemma A.12
\[
\int_{\mathbb{R}^d \times \{a_\ell\}} k_\ell M_\omega \varphi = \langle (J^{1+s}k_\ell, J^{-1-s}M_\omega \varphi(\cdot, a_\ell))_{H^{-1/2, H^{1/2}}} \rangle_{H^{1/2+s}} \leq \| k_\ell \|_{H^{1/2+s}} \| M_\omega \varphi(\cdot, a_\ell) \|_{H^{1/2}}
\]
\[
\leq \bar{c} \| k_\ell \|_{H^{1/2+s}} \| M_\omega \varphi \|_{0H^1} \leq \bar{c} c_0 m_\omega[1 + s] \| k_\ell \|_{H^{1/2+s}} \| \varphi \|_{0H^1},
\]
for \( \bar{c} \in \mathbb{R}^+ \) a constant from trace theory and the auxiliary multiplier \( \omega(\xi) = (1 + |\xi|^2)^{-(s+1)/2} \omega(\xi), \xi \in \mathbb{R}^d \).

By again Lemma A.12 item one, we finally estimate
\[
\int_U f \cdot M_\omega \varphi = \int_U J^s f \cdot J^{-s}M_\omega \varphi \leq c_1 \sum_{\ell=1}^m \| f \|_{H^s(U_\ell)} \| M_\omega J^s \varphi \|_{L^2} \leq m_\omega[1 + s] \| \varphi \|_{0H^1} \sum_{\ell=1}^m \| f \|_{H^s(U_\ell)}.
\]
Combining (A.19) and (A.20) gives the second item. \( \Box \)

### A.4. Korn’s inequality.
We record a version of Korn’s inequality stating that the \( L^2 \)-norm of the symmetrized gradient controls the \( H^1 \) norm on the closed subspace of functions vanishing on the lower boundary.

**Proposition A.15.** Let \( a, b \in \mathbb{R} \) with \( a < b \). Then there exists a constant \( c \in \mathbb{R}^+ \), depending only on \( b - a \) and \( n \), such that for all \( f \in H^1(\mathbb{R}^{n-1} \times (a, b); \mathbb{R}^n) \) such that \( \mathcal{T}_{\mathbb{R}^{n-1} \times \{a\}} f = 0 \) we have the inequality:
\[
\| f \|_{L^2} + \| \nabla f \|_{L^2} \leq c \| \mathcal{D} f \|_{L^2}.
\]

**Proof.** We refer the reader to the proof of Lemma 2.7 of [Bea81]. \( \Box \)
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