REGULARITY CRITERION FOR THE THREE-DIMENSIONAL BOUSSINESQ EQUATIONS

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ABSTRACT. We prove that a solution \((u, \theta)\) to the three-dimensional Boussinesq equations does not blow-up at time \(T\) if \(\|u_{\leq Q}\|_{B^{1,\infty}}\) is integrable on \((0, T)\), where \(u_{\leq Q}\) represents the low modes of Littlewood-Paley projection of the velocity \(u\).

1. INTRODUCTION

Consider the three-dimensional incompressible Boussinesq equations,
\begin{align*}
\partial_t u + (u \cdot \nabla) u &= -\nabla p + \nu \Delta u + \theta e_3, \\
\partial_t \theta + (u \cdot \nabla) \theta &= \kappa \Delta \theta, \\
\nabla \cdot u &= 0,
\end{align*}
with initial data
\begin{align*}
u(x, 0) &= u_0(x), \\
\theta(x, 0) &= \theta_0(x),
\end{align*}
where \(x \in \mathbb{R}^3, t \geq 0, u = u(x, t)\) is the velocity vector with divergence-free initial data, \(p = p(x, t)\) is the pressure scalar, and \(\theta = \theta(x, t)\) is the temperature scalar. The fluid kinematic viscosity is \(\nu \geq 0\), the thermal diffusivity is \(\kappa \geq 0\), and \(e_3 = (0, 0, 1)^T\). When \(\theta\) vanishes, the system reduces to the incompressible Navier-Stokes equations, which can be further reduced to the incompressible Euler equations by setting \(\nu = 0\).

The Boussinesq equations are important in the study of atmospheric sciences and they yield a wealth of interesting and difficult problems from a mathematical perspective. The regularity of (1.1)-(1.3) has been studied throughly on its own and in relation to the regularity of other equations, such as the Navier-Stokes equations, Euler equations, and magneto-hydrodynamics (MHD) equations.

Regularity criteria for the Boussinesq and related equations mentioned above can be split into different classes, one of which is Beale-Kato-Majda-like (abbreviated “BKM-like”) criteria. The original result by Beale, Kato, and Majda [2] placed a condition on the vorticity in the Navier-Stokes equations. They proved if
\begin{equation}
\int_0^T \|\nabla \times u\|_{L^\infty} \ dt < \infty,
\end{equation}

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then a smooth solution to the Navier-Stokes equations on \((0, T)\) does not blow up at time \(T\). This condition was weakened for the Euler equations by Planchon \cite{10} and improved for the three-dimensional Navier-Stokes equations by Cheskidov and Shvydkoy \cite{21}. In \cite{7}, Cheskidov and Dai developed BKM-like, but weaker, regularity criterion for the three-dimensional MHD equations.

A related family of regularity criteria fall into the Ladyzhenskaya-Prodi-Serrin category. For the Navier-Stokes equations, the condition is

\[
\|u\|_{L^p(0, T; L^r)} \leq Q, \quad \text{for } 2 \frac{1}{p} + \frac{3}{r} = 1, \quad r \in (3, \infty].
\]  

Various extensions and improvements of this type of criteria have been developed since then, such as the extension to the case for \(r = 3\) by Escauriaza, Seregin, and Šverák \cite{15} and extensions to Besov spaces.

Both kinds of regularity criteria were developed for the three-dimensional incompressible Boussinesq equations. In \cite{18} and \cite{19}, Qiu, Du, and Yao developed Serrin-type blow-up criteria for the Boussinesq equations. Particularly in \cite{18}, the authors showed a smooth solution to \((1.1)-(1.3)\) on time interval \([0, T]\) will remain smooth at time \(T\) if \(u \in L^q(0, T; B_{p,\infty}^{0}(\mathbb{R}^3))\) for \(2 \frac{1}{q} + \frac{3}{p} = 1 + s, \quad \frac{3}{p+1} < p \leq \infty, \quad -1 < s \leq 1, \) and \((p, s) \neq (\infty, 1)\). Ishimura and Morimoto \cite{14} proved the Beale-Kato-Majda-like regularity criterion \(\nabla u \in L^1(0, T; L^\infty(\mathbb{R}^3))\). Later Fan and Zhou \cite{10} studied the Boussinesq equations with partial viscosity and proved BKM-like regularity criteria in terms of the vorticity: \(\nabla \times u \in L^1(0, T; \dot{B}^{0}_{\infty, \infty}(\mathbb{R}^3))\). More regularity conditions in the three-dimensional case were developed in the following years, some of which can be found in \cite{17}, \cite{21}, \cite{22}, \cite{24}, \cite{25}, and \cite{26}.

A great deal of literature has also been produced for the two-dimensional case. We will not discuss these results, but rather refer the reader a few publications on that topic: \cite{1}, \cite{3}, \cite{4}, \cite{5}, \cite{11}, \cite{12}, \cite{13}, \cite{20}, and \cite{23}.

The main theorem discussed here also falls into the two main kinds of regularity criteria discussed above. We will show:

**Theorem 1.1.** Let \((u, \theta)\) be a weak solution to \((1.1)-(1.3)\) on \([0, T]\), assume \((u, \theta)\) is regular on \((0, T)\), and

\[
\|u_{\leq Q}\|_{B_{\infty, \infty}^1} \in L^1(0, T).
\]  

Then \((u(t), \theta(t))\) is regular on \((0, T]\).

**Remark 1.2.** We note that the above BKM-like regularity criterion also recovers the previous known Prodi-Serrin-type regularity, in particular we improve upon the results in \cite{18}, by recovering the whole range, including the endpoint \((p, s) = (\infty, 1)\). Further, the criterion in Theorem 1.1 improves previous BKM-like criterion for the three-dimensional Boussinesq equations since we only impose a condition on a finite amount of modes of the projection of the velocity \(u\).

**Remark 1.3.** The notation \(u_{\leq Q}\) denotes the low modes of \(u\) and \(B_{\infty, \infty}^1\) is a Besov space. More precise definitions are presented in the next section (see (2.3) and Definition 2.2).

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**2. Background**

2.1. **Littlewood-Paley Decomposition.** We utilize Littlewood-Paley decomposition in our proof. Denote wave numbers as \(\lambda_q = 2^q\) (in some wave units). For
\[ \psi \in C^\infty(\mathbb{R}^3), \text{ define} \]
\[ \psi(\xi) = \begin{cases} 1, & \text{for } |\xi| \leq \frac{3}{4} \\ 0, & \text{for } |\xi| \geq 1. \end{cases} \]

Next define \( \phi(\xi) = \psi(\xi/\lambda_1) - \psi(\xi) \) and \[ \phi_q(\xi) = \begin{cases} \phi(\xi/\lambda_q), & \text{for } q \geq 0 \\ \psi(\xi), & \text{for } q = -1. \end{cases} \]

The Littlewood-Paley projection operator \( \Delta_q \) is defined as
\[ \Delta_q u = \int_{\mathbb{R}^3} u(x-y) \mathcal{F}^{-1}(\phi_q(y)) \, dy, \]
where \( \mathcal{F}^{-1} \) is the inverse Fourier transform. Primarily, we will denote \( \Delta_q u \), the \( q \)th Littlewood-Paley piece of \( u \), as \( u_q \) instead. In the sense of distributions, one has
\[ u = \sum_{q=-1}^{\infty} u_q. \]

We also define
\[ u_{\leq Q} = \sum_{q=-1}^{Q} u_q, \quad u_{\geq Q} = \sum_{q=Q}^{\infty} u_q, \]
and
\[ \tilde{u}_q = u_{q-1} + u_q + u_{q+1}, \]
which will be useful notation for the proof.

2.2. Notation, Spaces, and Solutions. We will use the symbol \( \lesssim \) (or \( \gtrsim \)) to mean that an inequality holds up to an absolute constant, we will denote \( L^p \)-norms as \( \| \cdot \|_p \), and \( (\cdot, \cdot) \) will denote the \( L^2 \) inner product.

We use Littlewood-Paley decomposition to define some useful norms. Regularity (see Definition 2.4) is defined via Sobolev norms.

Definition 2.1. The homogeneous Sobolev space \( \dot{H}^s \) contains functions \( u \) such that the associated norm
\[ \| u \|_{\dot{H}^s} = \left( \sum_{q=-1}^{\infty} \lambda_q^{2s} \| u_q \|_2^2 \right)^{\frac{1}{2}}, \]
for \( s \in \mathbb{R} \), is finite. Note that \( \| u \|_{\dot{H}^s} \sim \| u \|_{\dot{H}^s} \), which we will keep in mind throughout the work below.

The regularity criterion is defined in terms of the following Besov norm:

Definition 2.2. The norm of \( u \) in Besov space \( B^1_{\infty, \infty} \) is defined as
\[ \| u(t) \|_{B^1_{\infty, \infty}} = \sup_{q \geq -1} \lambda_q \| u_q(t) \|_{\infty}. \]

The Besov space \( B^1_{\infty, \infty} \) is the space of tempered distributions \( u \) such that \( \| u(t) \|_{B^1_{\infty, \infty}} \) is finite.

We work in the class of weak solutions:
Definition 2.3. A weak solution of (1.1)-(1.3) on \([0,T]\) is a pair of functions \((u, \theta)\), \(u\) divergence free, in the class 
\[ u, \theta \in C_w([0,T]; L^2(\mathbb{R}^3)) \cap L^2(0,T; H^1(\mathbb{R}^3)) \]
such that 
\[ (u(t), \phi(t)) - (u_0, \phi(0)) \]
\[ = \int_0^t \left( (u(s), \partial_s \phi(s)) + \nu(u(s), \Delta \phi(s)) + (u(s) \cdot \nabla \phi(s), u(s)) + (\theta(s)e_3, \phi(s)) \right) \, ds \]
and 
\[ (\theta(t), \phi(t)) - (\theta_0, \phi(0)) \]
\[ = \int_0^t \left( (\theta(s), \partial_s \phi(s)) + \kappa(\theta(s), \Delta \phi(s)) + (u(s) \cdot \nabla \phi(s), \theta(s)) \right) \, ds, \]
for all divergence free test functions \(\phi \in C_\infty^0([0,T] \times \mathbb{R}^3)\).

Definition 2.4. A Leray-Hopf weak solution of (1.1)-(1.5) is regular on time interval \(I\) if the Sobolev norm \(\|u\|_{H^s}\) is continuous for \(s \geq \frac{1}{2}\) on \(I\).

Remark 2.5. One can apply a standard bootstrap argument to show if a solution is regular, then \(u\) and \(\theta\) are smooth.

2.3. The Dissipation Wave Number. The development of our regularity criterion is linked to Kolmogorov’s theory of turbulence and the dissipation wave number. The dissipation wave number is a time-dependent function that separates the low frequency inertial range, where the nonlinear term dominates the dynamics, from the high frequency dissipative range, where viscous forces takeover. In [9], Cheskidov and Shvydkoy defined the dissipation wave number and proved that if it belongs to \(L^{5/2}(0,T)\), then the solution of the Navier-Stokes equations is regular up to time \(T\). They also showed that the dissipation wave number belongs to \(L^1(0,T)\) for every Leray-Hopf solution.

We define the dissipation wave number \(\Lambda(t)\) for the Boussinesq equations in an analogous manner:
\[ Q(t) = \min \{ q : \lambda_p^{-1} \|u_p\|_\infty < c \min \{ \nu, \kappa \}, \forall p > q, q \geq 0 \}, \]
\[ \Lambda(t) = \lambda_{Q(t)}, \]
for absolute constant \(c > 0\).

Work with the dissipation wave number and determining modes have provided key improvements to previous known regularity results for the surface quasi-geostrophic equations and the magneto-hydrodynamics equations (see [6], [7], and [8]), as well.

3. Proof of the Main Theorem

We prove Theorem 1.1 in this section.

Proof: We test (1.1) with \(\Delta_q^2 u\) and (1.2) with \(\Delta_q^2 \theta\), which yields
\[ \frac{1}{2} \frac{d}{dt} \|u_q\|^2 \leq -\nu \|\nabla u_q\|^2 + \int_{\mathbb{R}^3} \Delta_q (u \cdot \nabla u) \cdot u_q \, dx - \int_{\mathbb{R}^3} \Delta_q (\theta e_3) \cdot u_q \, dx, \]
\[ \frac{1}{2} \frac{d}{dt} \|\theta_q\|^2 \leq -\kappa \|\nabla \theta_q\|^2 + \int_{\mathbb{R}^3} \Delta_q (u \cdot \nabla \theta) \cdot \theta_q \, dx, \]
and then multiply (3.1) by $\lambda_q^{2s}$ and (3.2) by $\lambda_q^{2\sigma}$, add those two equations together, and sum over $q$ to arrive at

\begin{equation}
\frac{1}{2} \frac{d}{dt} \sum_{q=-1}^{\infty} \left( \lambda_q^{2s} \| u_q \|_2^2 + \lambda_q^{2\sigma} \| \theta_q \|_2^2 \right) \leq - \sum_{q=-1}^{\infty} \left( \lambda_q^{2s} \nu \| \nabla u_q \|_2^2 + \lambda_q^{2\sigma} \kappa \| \nabla \theta_q \|_2^2 \right) + I_1 + I_2 + I_3,
\end{equation}

where

\begin{align}
I_1 &= \sum_{q=-1}^{\infty} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (u \cdot \nabla u) \cdot u_q \, dx, \\
I_2 &= - \sum_{q=-1}^{\infty} \lambda_q^{2s} \int_{\mathbb{R}^3} \Delta_q (\theta e_q) \cdot u_q \, dx, \\
I_3 &= \sum_{q=-1}^{\infty} \lambda_q^{2\sigma} \int_{\mathbb{R}^3} \Delta_q (u \cdot \nabla \theta) \cdot \theta_q \, dx.
\end{align}

For (3.6), we use a similar method as in [7]. First, we decompose (3.6) into three parts:

\begin{align}
I_3 &= \sum_{q=-1}^{\infty} \sum_{|q-p| \leq 2} \lambda_q^{2\sigma} \int_{\mathbb{R}^3} \Delta_q (u_{\leq p-2} \cdot \nabla \theta_p) \theta_q \, dx \\
&\quad + \sum_{q=-1}^{\infty} \sum_{|q-p| \leq 2} \lambda_q^{2\sigma} \int_{\mathbb{R}^3} \Delta_q (u_p \cdot \nabla \theta_{\leq p-2}) \theta_q \, dx \\
&\quad + \sum_{q=-1}^{\infty} \sum_{p \geq q-2} \lambda_q^{2\sigma} \int_{\mathbb{R}^3} \Delta_q (u_p \cdot \nabla \theta_p) \theta_q \, dx \\
&= I_{3,1} + I_{3,2} + I_{3,3}.
\end{align}

We use Hölder’s inequality on $I_{3,2}$ to find

\begin{equation}
|I_{3,2}| \lesssim \sum_{q=-1}^{\infty} \lambda_q^{2\sigma} \| u_q \|_\infty \sum_{|q-p| \leq 2} \| \theta_p \|_2 \sum_{p' \leq p-2} \lambda_{p'} \| \theta_{p'} \|_2.
\end{equation}

Then we split the sum into high and low modes. For the high modes we use the definition of $\Lambda(t)$ and for the low modes we use

\begin{equation}
f(t) = \| u_{\leq Q(t)}(t) \|_{B_{s,\infty}} \sup_{q \leq Q(t)} \lambda_q \| u_q(t) \|_\infty,
\end{equation}
(which will be used to define the regularity criterion later) to find

$$|I_{3,2}| \lesssim C_k \sum_{q=Q+1}^{\infty} \lambda_q^{2\sigma+1} \sum_{|q-p| \leq 2} \|\theta_p\|_2 \sum_{p' \leq p-2} \lambda_{p'} \|\theta_{p'}\|_2$$

$$+ f(t) \sum_{q=-1}^{Q} \lambda_q^{2\sigma-1} \sum_{|q-p| \leq 2} \|\theta_p\|_2 \sum_{p' \leq p-2} \lambda_{p'} \|\theta_{p'}\|_2$$

$$\lesssim C_k \sum_{q=Q+1}^{\infty} \lambda_q^{2\sigma+1} \|\theta_q\|_2 \sum_{p' \leq q} \lambda_{p'} \|\theta_{p'}\|_2$$

$$+ f(t) \sum_{q=-1}^{Q+2} \lambda_q^{2\sigma-1} \|\theta_q\|_2 \sum_{p' \leq q} \lambda_{p'} \|\theta_{p'}\|_2.$$  

After a rearrangement and an application of Jensen’s inequality, we arrive at

$$|I_{3,2}| \lesssim C_k \sum_{q=Q-1}^{\infty} \lambda_q^{2\sigma+1} \|\theta_q\|_2 \sum_{p' \leq q} \lambda_{p'} \|\theta_{p'}\|_2$$

$$+ f(t) \sum_{q=-1}^{Q+2} \lambda_q^{2\sigma-1} \|\theta_q\|_2 \sum_{p' \leq q} \lambda_{p'} \|\theta_{p'}\|_2$$

$$\lesssim C_k \sum_{q=-1}^{\infty} \lambda_q^{2\sigma+2} \|\theta_q\|_2^2 + f(t) \sum_{q=-1}^{Q+2} \lambda_q^{2\sigma} \|\theta_q\|_2^2,$$

for

$$\sigma < 0.$$  

By Bony’s paraproduct and commutator notation, which says

$$[\Delta_q, u_{\leq p-2} \cdot \nabla] \theta_p = \Delta_q(u_{\leq p-2} \cdot \nabla \theta_p) - u_{\leq p-2} \cdot \nabla \Delta_q \theta_p,$$

one may decompose $I_{3,1}$ as

$$I_{3,1} = \sum_{q=-1}^{\infty} \sum_{|q-p| \leq 2} \lambda_q^{2\sigma} \int_{R^3} [\Delta_q, u_{\leq p-2} \cdot \nabla] \theta_p \theta_q \ dx$$

$$+ \sum_{q=-1}^{\infty} \int_{R^3} u_{\leq q-2} \cdot \nabla \theta_q \theta_q \ dx$$

$$+ \sum_{q=-1}^{\infty} \sum_{|q-p| \leq 2} \lambda_q^{2\sigma} \int_{R^3} (u_{\leq p-2} - u_{\leq q-2}) \cdot \nabla \Delta_q \theta_p \theta_q \ dx$$

$$= I_{3,1,1} + I_{3,1,2} + I_{3,1,3}.$$  

In [7], they note that their term (equivalent to our $I_{3,1,1}$) can be estimated as

$$|I_{3,1,1}| \lesssim C_k \sum_{q=Q+2}^{\infty} \lambda_q^{2\sigma+2} \|\theta_q\|_2^2 + f(t) \sum_{q=-1}^{\infty} \lambda_q^{2\sigma} \|\theta_q\|_2^2.$$
The second term, $I_{3,1,2} = 0$ because of the divergence-free condition on $u$. We also refer the reader to [7], where one can find

$$|I_{3,1,3}| + |I_{3,3}| \lesssim c_k \sum_{q=-1}^{\infty} \lambda_q^{2\sigma+2} \|\theta_q\|_2^2 + f(t) \sum_{q=-1}^{Q+2} \lambda_q^{2\sigma} \|\theta_q\|_2^2.$$  

The above estimates on the pieces of (3.3) yield

$$(3.11) \quad |I_3| \lesssim c_k \sum_{q=-1}^{\infty} \lambda_q^{2\sigma+2} \|\theta_q\|_2^2 + f(t) \sum_{q=-1}^{\infty} \lambda_q^{2\sigma} \|\theta_q\|_2^2.$$  

For (3.4), we refer the reader to the estimates carried out in [9] on the Navier-Stokes equations (and the equivalent term in [7] on the MHD equations), where they show

$$(3.12) \quad |I_1| \lesssim c\nu \sum_{q=-1}^{\infty} \lambda_q^{2\sigma+2} \|u_q\|_2^2 + f(t) \sum_{q=-1}^{\infty} \lambda_q^{2\sigma} \|u_q\|_2^2,$$

where $f(t)$ is the same as in (3.8). The bound (3.12) holds for

$$(3.13) \quad \frac{1}{2} \leq s < 1.$$  

We use Young’s inequality to estimate (3.5) as

$$(3.14) \quad |I_2| = \left| \sum_{q=-1}^{\infty} \lambda_q^{2\sigma} \int_{\mathbb{R}^3} \Delta q \left( q e_3 \right) \cdot u_q \, dx \right| \lesssim \sum_{q=-1}^{\infty} \lambda_q^{2\sigma} \|u_q\|_2^2 + \|\theta_q\|_2^2.$$  

We may break up this sum as follows:

$$(3.15) \quad |I_2| \lesssim \sum_{q=-1}^{\infty} \lambda_q^{2\sigma} \|u_q\|_2^2 + \sum_{q=-1}^{N} \lambda_q^{2\sigma} \|\theta_q\|_2^2 + \sum_{q=N}^{\infty} \lambda_q^{2\sigma} \|\theta_q\|_2^2,$$

where $N$ is finite and will be determined later.

Estimates (3.12), (3.15), and (3.11) in (3.3) yield

$$\frac{1}{2} \frac{d}{dt} \sum_{q=-1}^{\infty} \left( \lambda_q^{2\sigma} \|u_q\|_2^2 + \lambda_q^{2\sigma} \|\theta_q\|_2^2 \right) \lesssim -\nu \sum_{q=-1}^{\infty} \lambda_q^{2\sigma+2} \|u_q\|_2^2 + c\nu \sum_{q=-1}^{\infty} \lambda_q^{2\sigma+2} \|u_q\|_2^2$$

$$- \kappa \sum_{q=-1}^{\infty} \lambda_q^{2\sigma+2} \|\theta_q\|_2^2 + c\kappa \sum_{q=-1}^{\infty} \lambda_q^{2\sigma+2} \|\theta_q\|_2^2$$

$$+ \sum_{q=-1}^{N} \lambda_q^{2\sigma} \|\theta_q\|_2^2 + \sum_{q=N}^{\infty} \lambda_q^{2\sigma} \|\theta_q\|_2^2$$

$$+ (f(t) + 1) \sum_{q=-1}^{\infty} \lambda_q^{2\sigma} \|u_q\|_2^2$$

$$+ f(t) \sum_{q=-1}^{\infty} \lambda_q^{2\sigma} \|\theta_q\|_2^2.$$  

For

$$(3.16) \quad 2s < 2\sigma + 2$$  

$$(3.17) \quad 2s < 2\sigma + 2$$
and absolute constants $C_1, C_2, C_3, C_4$ and $C_5$, we have

$$\frac{1}{2} \frac{d}{dt} \sum_{q=-1}^\infty (\lambda_q^{2s} \|u_q\|_2^2 + \lambda_q^{2\sigma} \|\theta_q\|_2^2) \leq -\nu \sum_{q=-1}^\infty \lambda_q^{2s+2} \|u_q\|_2^2 + C_1 \nu \sum_{q=-1}^\infty \lambda_q^{2s+2} \|u_q\|_2^2$$

$$- \kappa \sum_{q=-1}^\infty \lambda_q^{2\sigma+2} \|\theta_q\|_2^2 + C_2 \kappa \sum_{q=-1}^\infty \lambda_q^{2\sigma+2} \|\theta_q\|_2^2$$

$$+ C_3 \sum_{q=-1}^N \lambda_q^{2s} \|\theta_q\|_2^2 + C_4 \lambda_N^{2s-2\sigma-2} \sum_{q=-1}^\infty \lambda_q^{2\sigma+2} \|\theta_q\|_2^2$$

$$+ C_5 (f(t) + 1) \sum_{q=-1}^\infty (\lambda_q^{2s} \|u_q\|_2^2 + \lambda_q^{2\sigma} \|\theta_q\|_2^2).$$

(3.18)

For $N$ large enough, one may choose $N = N(\kappa)$ such that $C_4 \lambda_N^{2s-2\sigma-2} \leq \kappa/2$. In fact, one may solve for this $N$ explicitly. Since such a finite $N$ exists, then we can use the following two facts:

(3.19) $$\sum_{q=-1}^N \lambda_q^{2s} \|\theta_q\|_2^2 < \infty,$$

and

(3.20) $$C_4 \lambda_N^{2s-2\sigma-2} \sum_{q=N}^\infty \lambda_q^{2\sigma+2} \|\theta_q\|_2^2 \leq \frac{\kappa}{2} \sum_{q=-1}^\infty \lambda_q^{2\sigma+2} \|\theta_q\|_2^2.$$

Using (3.19) and (3.20) in (3.15), we arrive at the following differential inequality:

$$\frac{1}{2} \frac{d}{dt} \sum_{q=-1}^\infty (\lambda_q^{2s} \|u_q\|_2^2 + \lambda_q^{2\sigma} \|\theta_q\|_2^2) \leq (-\nu + C_1 \nu) \sum_{q=-1}^\infty \lambda_q^{2s+2} \|u_q\|_2^2$$

$$- \left(\frac{\kappa}{2} + C_2 \kappa\right) \sum_{q=-1}^\infty \lambda_q^{2\sigma+2} \|\theta_q\|_2^2$$

$$+ C_5 (f(t) + 1) \sum_{q=-1}^\infty (\lambda_q^{2s} \|u_q\|_2^2 + \lambda_q^{2\sigma} \|\theta_q\|_2^2)$$

$$+ C_3 \sum_{q=-1}^N \lambda_q^{2s} \|\theta_q\|_2^2$$

(3.21)

The choice $c = \min\{\frac{1}{c_1}, \frac{1}{2c_2}\}$ yields

(3.22) $$\frac{d}{dt} (\|u\|_{H^s}^2 + \|\theta\|_{H^\sigma}^2) \leq C(\nu, \kappa, s, \sigma) (f(t) + 1) (\|u\|_{H^s}^2 + \|\theta\|_{H^\sigma}^2)$$

$$+ C_6 \sum_{q=-1}^N \lambda_q^{2s} \|\theta_q\|_2^2.$$

By Grönwall’s inequality (noting (3.19)), we have that $\|u\|_{H^s}^2 + \|\theta\|_{H^\sigma}^2$ remains bounded on $(0, T)$ for $\frac{1}{2} \leq s < 1$ and $s - 1 < \sigma < 0$ if

(3.23) $$\|u\|_{L^\infty_0} \in L^1(0, T).$$

Thus, by Definition 2.4 we reach the desired conclusion.
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