ON THE MINIMAL SPEED AND ASYMPTOTICS OF THE
WAVE SOLUTIONS FOR THE LOTKA VOLTERRA SYSTEM

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ABSTRACT. We study the minimal wave speed and the asymptotics of the traveling wave solutions of a competitive Lotka Volterra system. The existence of the traveling wave solutions is derived by monotone iteration. The asymptotic behaviors of the wave solutions are derived by comparison argument and the exponential dichotomy, which seems to be the key to understand the geometry and the stability of the wave solutions. Also the uniqueness and the monotonicity of the waves are investigated via a generalized sliding domain method.

1. Introduction and the Main result

We study the minimal wave speed and the asymptotic behaviors of the traveling wave solutions of the following classical Lotka-Volterra competition system

\[
\begin{align*}
    u_t &= u_{xx} + u(1 - u - a_1 v), \\
    v_t &= v_{xx} + rv(1 - a_2 u - v)
\end{align*}
\]

where \( u = u(x,t), v = v(x,t) \) and \( a_1, a_2, r \) are positive constants.

In the wave coordinates \( \xi = x + ct \), (1.1) is changed into

\[
\begin{align*}
    u_{\xi\xi} - cu_\xi + u(1 - u - a_1 v) &= 0, \\
    v_{\xi\xi} - cv_\xi + rv(1 - a_2 u - v) &= 0
\end{align*}
\]

Fei and Carr [3] investigated the traveling wave solutions and their minimal wave speed of system (1.2) under the assumptions:

[H1]. \( 0 < a_1 < 1 < a_2 \),

[H2]. \( 1 - a_1 \leq r(a_2 - 1) \).

Requiring further \( r(a_2-1) \leq 1 \), they showed that for each speed \( c \geq 2 \sqrt{r(a_2-1)} \) system (1.2) admits monotonic traveling waves \( (u(\xi), v(\xi))^T \) satisfying the following boundary conditions:

\[
\begin{pmatrix} u \\ v \end{pmatrix} (-\infty) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} u \\ v \end{pmatrix} (+\infty) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]
Under additional assumptions \( r = 1, a_1 + a_2 = 2 \) or \( r(a_2 - 1) = 1 - a_1 \) they also showed that system (1.2) has monotonic traveling wave solutions satisfying (1.3) for \( c \geq 2\sqrt{1 - a_1} \). However, the question of the minimal wave speed for the wave solutions of (1.2)-(1.3) remains unanswered.

We will prove that the minimal wave speed for (1.2) is indeed \( 2\sqrt{1 - a_1} \) if the following additional assumption

[H3]. \[ r(a_2 - 1) < (1 - a_1)(2 - a_1 + r) \]

is imposed. Noting that if \( r = 1 \) the condition [H3] includes Fei and Carr’s additional condition.

System (1.2) has three non-negative equilibria \((0, 0)\), \((0, 1)\) and \((1, 0)\), with \((0, 0)\) and \((0, 1)\) unstable and \((1, 0)\) stable (3). For the convenience of later use, we introduce the transformation \( \hat{v} = 1 - v \) to change system (1.2) into local monotone. Upon dropping the hat on the function \( t \) \( v \) system (1.2) is changed into

(1.4) \[
\begin{align*}
\beta u \xi - cu \xi + u(1 - a_1 - u + a_1 v) &= 0, \\
v \xi - cv \xi + r(1 - v)(a_2 u - v) &= 0
\end{align*}
\]

and the boundary conditions (1.3) are changed into

(1.5) \[
\begin{align*}
\left(\begin{array}{c}
u \\
v
\end{array}\right)(-\infty) &= \left(\begin{array}{c}0 \\
0
\end{array}\right), \\
\left(\begin{array}{c}
u \\
v
\end{array}\right)(+\infty) &= \left(\begin{array}{c}1 \\
1
\end{array}\right).
\end{align*}
\]

We have,

**Theorem.** Assuming conditions [H1]-[H3], then \( c^* = 2\sqrt{1 - a_1} \) is the minimal wave speed for system (1.4)-(1.5), namely, corresponding to each fixed \( c \geq c^* \) (1.4)-(1.5) has a unique traveling wave solution \((u(\xi), v(\xi))^T\); while for \( 0 < c < c^* \), (1.4)-(1.5) does not have any monotonic traveling wave solutions. Furthermore, the traveling wave solution has the following asymptotic behaviors:

1. Corresponding to each wave speed \( c > 2\sqrt{1 - a_1} \), the traveling wave solution \((u(\xi), v(\xi))^T\) satisfies, as \( \xi \rightarrow -\infty \);

(1.6) \[
\begin{align*}
\left(\begin{array}{c}u(\xi) \\
v(\xi)
\end{array}\right) &= \begin{pmatrix} A_1 \\
A_2
\end{pmatrix} e^{-\sqrt{c^2 - 4(1-a_1)} \xi} + o(e^{-\sqrt{c^2 - 4(1-a_1)} \xi}),
\end{align*}
\]

While as \( \xi \rightarrow +\infty \) we have two cases to deal with: if \( r(a_2 - 1) \leq 1 \), then

(1.7) \[
\begin{align*}
\left(\begin{array}{c}u(\xi) \\
v(\xi)
\end{array}\right) &= \begin{pmatrix} 1 \\
1
\end{pmatrix} - \begin{pmatrix} \hat{A}_1 \\
\hat{A}_2
\end{pmatrix} e^{-\sqrt{c^2 - 4(1-a_1)} \xi} + o(e^{-\sqrt{c^2 - 4(1-a_1)} \xi}),
\end{align*}
\]

and if \( r(a_2 - 1) > 1 \), then

(1.8) \[
\begin{align*}
\left(\begin{array}{c}u(\xi) \\
v(\xi)
\end{array}\right) &= \begin{pmatrix} 1 \\
1
\end{pmatrix} - \begin{pmatrix} \hat{A}_1 e^{-\sqrt{c^2 - 4(1-a_1)} \xi} + \hat{A}_2 e^{-\sqrt{c^2 - 4(1-a_1)} \xi} \\
\hat{A}_3 e^{-\sqrt{c^2 - 4(1-a_1)} \xi}
\end{pmatrix} + \begin{pmatrix} 0(e^{-\sqrt{c^2 - 4(1-a_1)} \xi}) \\
0(e^{-\sqrt{c^2 - 4(1-a_1)} \xi})
\end{pmatrix},
\end{align*}
\]

where \( A_1, A_2, \hat{A}_1, \hat{A}_2, \hat{A}_3 \) are positive constants, and \( \hat{A}_2 \) is a real number.
2. For the traveling wave with the critical speed \( c^* = 2\sqrt{1-a_1} \), \((u(\xi), v(\xi))^T\) satisfies

\[
\begin{pmatrix}
  u(\xi) \\
  v(\xi)
\end{pmatrix} = \begin{pmatrix}
  A_{11c} + A_{12c}c^* \\
  A_{21c} + A_{22c}
\end{pmatrix} e^{\sqrt{1-a_1} \xi} + o(e^{\sqrt{1-a_1} \xi})
\]

as \( \xi \to -\infty \), and if \( r(a_2 - 1) \leq 1 \), we have

\[
\begin{pmatrix}
  u(\xi) \\
  v(\xi)
\end{pmatrix} = \begin{pmatrix}
  1 \\
  1
\end{pmatrix} - \begin{pmatrix}
  \tilde{A}_{11} e^{(\sqrt{1-a_1} - (1-a_1 + r(a_2 - 1))\xi)} \\
  \tilde{A}_{22}
\end{pmatrix} + o(e^{(\sqrt{1-a_1} - (1-a_1 + r(a_2 - 1))\xi})
\]

while if \( r(a_2 - 1) > 1 \), we have

\[
\begin{pmatrix}
  u(\xi) \\
  v(\xi)
\end{pmatrix} = \begin{pmatrix}
  1 \\
  1
\end{pmatrix} - \begin{pmatrix}
  \tilde{A}_{11} e^{(\sqrt{1-a_1} - (1-a_1 + r(a_2 - 1))\xi)} + \hat{A}_{12} e^{(\sqrt{1-a_1} - (1-a_1 + r(a_2 - 1))\xi)} \\
  \tilde{A}_{22} e^{(\sqrt{1-a_1} - (1-a_1 + r(a_2 - 1))\xi)}
\end{pmatrix} + \begin{pmatrix}
  o(e^{(\sqrt{1-a_1} - (1-a_1 + r(a_2 - 1))\xi)}) \\
  o(e^{(\sqrt{1-a_1} - (1-a_1 + r(a_2 - 1))\xi)})
\end{pmatrix}
\]

as \( \xi \to +\infty \), where \( A_{12c}, A_{22c} < 0 \), \( A_{11c}, A_{21c} \in \mathbb{R} \) and \( \tilde{A}_{11}, \tilde{A}_{22}, \hat{A}_{12}, \hat{A}_{22} > 0 \), \( \tilde{A}_{11} \) is a real number.

In the next section we prove the theorem. The proof uses monotone iteration of a pair of upper and lower solutions, which is different from that of [FeiCarr]. In fact, we fully explore properties of the wave solutions of the classical K.P.P (9) equation and the monotonic structure of system (1.4). For the existence of the traveling wave solutions to the Lotka Volterra systems with different assumptions on parameters, we refer to [4, 5, 6, 8, 10] and the references therein. Noting in the above mentioned results little attention has been paid to the asymptotics of the wave solutions. However, such information is the key to the understanding of the other properties of the traveling wave solutions such as the strict monotonicity, the uniqueness as well as the stability. As a final remark we point out that the existence and stability of the traveling wave solutions for (1.2) is investigated in [9] under conditions H1 and H2 (with the inequality reversed).

2. The proof

The proof of the Theorem is divided into several parts.

2.1. The existence. We show the existence of the traveling wave solutions by monotone iteration method given by [11]. Such method reduces the existence of the wave solutions to the finding of an ordered pair of upper and lower solutions. The construction of the upper and lower solutions seems to be new, see also [9].
Definition 1. A continuous and essentially bounded function \((\bar{u}(\xi), \bar{v}(\xi))\), \(\xi \in \mathbb{R}\) is an upper solution of (1.4) if it satisfies
\[
\begin{cases}
  u'' - cu' + u(1 - u - a_1 + a_1v) \leq 0, \\
  v'' - cv' + r(1 - v)(a_1u - v) \leq 0,
\end{cases}
\]
for \(\xi \in \mathbb{R}/\{y_1, y_2, \ldots, y_m\}\) and the boundary conditions
\[
\begin{pmatrix}
  u \\
  v
\end{pmatrix}
\begin{pmatrix}
  -\infty \\
  0
\end{pmatrix} \geq \begin{pmatrix}
  0 \\
  1
\end{pmatrix},
\]
while at \(y_i, \ i = 1, 2, \ldots, m, \ m \in \mathbb{N}\),
\[
\begin{pmatrix}
  \bar{u}'(y_i - 0), \bar{v}'(y_i - 0)
\end{pmatrix}^T \geq \begin{pmatrix}
  \bar{u}'(y_i + 0), \bar{v}'(y_i + 0)
\end{pmatrix}^T.
\]

A lower solution of (2.1) is defined similarly by reversing the above inequalities in (2.2) and (2.3).

First recall the following classical result ([9]) on the traveling wave solutions of K.P.P equation
\[
\begin{cases}
  w'' - cw' + f(w) = 0, \\
  w(-\infty) = 0, \quad w(+\infty) = b.
\end{cases}
\]
where \(f \in C^2([0, b])\) and \(f > 0\) on the open interval \((0, b)\) with \(f(0) = f(b) = 0\), \(f'(0) = d_1 > 0\) and \(f'(b) = -d_2 < 0\). The Lemma below describes the properties of the wave solution of (2.3).

Lemma 2. Corresponding to every fixed wave speed \(c \geq 2\sqrt{d_1}\), system (2.3) has a unique (up to a translation of the origin) strictly monotonically increasing traveling wave solution \(w(\xi)\) for \(\xi \in \mathbb{R}\). The traveling wave solution \(w\) has the following asymptotic behaviors:

For the wave solution with non-critical speed \(c > 2\sqrt{d_1}\), we have
\[
\begin{align*}
  w(\xi) &= a_w e^{-\sqrt{d_1}\xi} + o(e^{-\sqrt{d_1}\xi}) \quad \text{as } \xi \to -\infty, \\
  w(\xi) &= b - b_w e^{-\sqrt{d_1}\xi} + o(e^{-\sqrt{d_1}\xi}) \quad \text{as } \xi \to +\infty,
\end{align*}
\]
where \(a_w\) and \(b_w\) are positive constants.

For the wave with critical speed \(c = 2\sqrt{d_1}\), we have
\[
\begin{align*}
  w(\xi) &= (a_c + d_c\xi)e^{\sqrt{d_1}\xi} + o(\xi e^{\sqrt{d_1}\xi}) \quad \text{as } \xi \to -\infty, \\
  w(\xi) &= b - b_c e^{\sqrt{d_1}\xi} + o(\xi e^{\sqrt{d_1}\xi}) \quad \text{as } \xi \to +\infty,
\end{align*}
\]
where the constants \(d_c\) is negative, \(b_c\) is positive and \(a_c \in \mathbb{R}\).

According to Lemma 2 we let \(c \geq 2\sqrt{1 - a_1}\) be fixed and \(w(\xi)\), \(\xi \in \mathbb{R}\) be a solution of the following form of K.P.P equation
\[
\begin{cases}
  g''(\xi) - g'(\xi) + (1 - a_1)g(\xi)(1 - g(\xi)) = 0, \\
  g(-\infty) = 0, \ g(+\infty) = 1,
\end{cases}
\]
and for the same $c$ let $l$ be a number such that $\frac{r(a_2-1)-(1-a_1)}{a_1} \leq l < 1 - a_1$ and $\bar{u}(\xi), \xi \in \mathbb{R}$ be a solution of a K.P.P equation with the form

$$
\begin{align*}
\tag{2.10}
& h''(\xi) - ch'(\xi) + (1 - a_1)h(\xi)(1 - \frac{1-a_1-l}{1-a_1}h(\xi)) = 0, \\
& h(-\infty) = 0, \quad h(\infty) = \frac{1-a_1}{1-a_1}, \\
& \xi \in \mathbb{R}.
\end{align*}
$$

Define

$$
\begin{align*}
\tag{2.11}
& \begin{pmatrix}
\bar{u}(\xi) \\
\bar{v}(\xi)
\end{pmatrix} = \begin{pmatrix}
\min_{\xi \in \mathbb{R}} \{\bar{u}(\xi), 1\} \\
\min_{\xi \in \mathbb{R}} \{(1+l)\bar{u}(\xi), 1\}
\end{pmatrix}, \\
& \begin{pmatrix}
\underline{u}(\xi) \\
\underline{v}(\xi)
\end{pmatrix} = \begin{pmatrix}
\underline{u}(\xi) \\
\underline{v}(\xi)
\end{pmatrix}.
\end{align*}
$$

We have the following

**Lemma 3.** For every fixed $c \geq 2\sqrt{1-a_1}$, $(\bar{u}(\xi), \bar{v}(\xi))^T$ and $(\underline{u}(\xi), \underline{v}(\xi))^T$ in (2.11) define respectively an upper and lower solutions for $(1.4)-(1.5)$.

**Proof.** The verification for $(\underline{u}, \underline{v})$ being a lower solution for $(1.4)-(1.5)$ is straightforward, so we skip it. According to Lemma 2, $\bar{u}(\xi)$ is strictly monotonically increasing for $\xi \in \mathbb{R}$, there exist $N_1, N_2 \in \mathbb{R}$ such that $\bar{u}(N_1) = \frac{1}{1+l}$ and $\bar{u}(N_2) = 1$. We can therefore rewrite $(\bar{u}, \bar{v})$ as follows

$$
\begin{align*}
\tag{2.12}
& \begin{pmatrix}
\bar{u}(\xi) \\
\bar{v}(\xi)
\end{pmatrix} = \begin{cases}
\bar{u}(\xi), \bar{v}(\xi)^T, & \text{for } -\infty < \xi \leq N_1; \\
\bar{u}(\xi), 1)^T, & \text{for } N_1 < \xi \leq N_2; \\
(1,1)^T, & \text{for } N_2 < \xi < +\infty.
\end{cases}
\end{align*}
$$

For $N_2 < \xi < +\infty$, $(\bar{u}(\xi), \bar{v}(\xi))^T = (1,1)^T$ is obviously a solution of (1.4) and satisfying the inequality (2.3) on the boundary, we only need to verify that $(\bar{u}(\xi), \bar{v}(\xi))^T$ satisfies the inequality (2.3) on the intervals $(-\infty, N_1]$ and $(N_1, N_2)$ respectively. For the pair $(\bar{u}(\xi), \bar{v}(\xi))^T = (\bar{u}(\xi), (1+l)\bar{u}(\xi))^T$ on $-\infty < \xi \leq N_1$, we have

$$
\begin{align*}
& \bar{u}'' - c\bar{u}' + \bar{u}(1 - a_1 - \bar{u} + a_1 \bar{v}) \\
& = -(1 - a_1)\bar{u}(1 - \frac{1-a_1-l}{1-a_1}\bar{u}) + \bar{u}(1 - a_1 - \bar{u} + a_1(1 + l)\bar{u}) \\
& = -(1 - a_1)\bar{u}(1 - \frac{1-a_1-l}{1-a_1}\bar{u} - 1 + \frac{1-a_1(1+l)}{1-a_1}\bar{u}) \\
& = -(1 - a_1)l\bar{u}^2 \leq 0,
\end{align*}
$$
and
\[ \ddot{v}'' - c\ddot{v}' + r(1 - \ddot{v})(a_2\ddot{v} - \ddot{v}) \]
\[ = (1 + l)\dddot{u}'' - c\dddot{u}' + \frac{r}{1 + l}(1 - (1 + l)\dddot{u})(a_2\dddot{u} - (1 + l)\dddot{u}) \]
\[ = (1 + l)[\dddot{u}'' - c\dddot{u}' + (1 - a_1)\dddot{u}(1 - \dddot{u} + \frac{r}{1 - a_1} \dddot{u}) \]
\[ - (1 - a_1)\dddot{u}(1 - \dddot{u} + \frac{r}{1 - a_1} \dddot{u}) + \frac{r}{1 + l}(1 - (1 + l)\dddot{u})(a_2 - (1 + l)) ] \]
\[ = (1 + l)\dddot{u}[\frac{r}{1 + l}(1 - (1 + l)\dddot{u})(a_2 - (1 + l)) - (1 - a_1)(1 - \dddot{u} + \frac{r}{1 - a_1} \dddot{u})] \]
\[ = (1 + l)\dddot{u}(r\frac{a_2}{1 + l} - 1) - (1 - a_1) - \dddot{u}(r(a_2 - 1) - (1 - a_1 - l)] \]
\[ \leq 0. \]

The last inequality is true because of [H3] and the choice of \(l\).

For \(N_1 < \xi < N_2\), we verify that \((\ddot{u}, 1)^T\) satisfies the inequality (2.1). We only verify for the first component since the one for \(v = 1\) is trivial.

\[ \dddot{u}'' - c\dddot{u}' + \dddot{u}(1 - a_1 - \dddot{u} + a_1 v) \]
\[ = \dddot{u}'' - c\dddot{u}' + \dddot{u}(1 - \dddot{u}) \]
\[ + (1 - a_1)\dddot{u}(1 - \frac{r}{1 - a_1} \dddot{u}) - (1 - a_1)\dddot{u}(1 - \frac{r}{1 - a_1} \dddot{u}) \]
\[ = \dddot{u}(1 - \dddot{u} - (1 - a_1) + (1 - a_1 - l) \dddot{u}) \]
\[ \leq 0. \]

Therefore, we have the conclusion of the Lemma.

To show the orderliness of the upper and lower solution pairs, we first introduce a sliding domain method which applies to a slightly more general system than (1.2). Noting that no monotonicity requirements are imposed on the upper and lower solutions.

**Lemma 4.** Let the \(C^2\) vector functions \(\tilde{U}(\xi) = (\tilde{u}_1(\xi), \tilde{u}_2(\xi), ..., \tilde{u}_n(\xi))^T\) and \(\tilde{U}(\xi) = (\tilde{u}_1(\xi), \tilde{u}_2(\xi), ..., \tilde{u}_n(\xi))^T\) be \(C^2\) and satisfy the following inequalities

\[ D\tilde{U}'' - c\tilde{U}' + F(U) \leq 0 \leq D\tilde{U}'' - c\tilde{U}' + F(U) \quad \text{for} \ \xi \in [-N, N] \]

and

\[ \tilde{U}(-N) < \tilde{U}(\xi) \quad \text{for} \ \xi \in (-N, N], \]

\[ \tilde{U}(\xi) < \tilde{U}(N) \quad \text{for} \ \xi \in [-N, N), \]

where \(D\) is a diagonal matrix with positive entries \(D_{ii}, i = 1, 2...n\), \(F(U) = (F_1(U), ..., F_n(U))^T\) is \(C^1\) with respect to its components and \(\frac{\partial F}{\partial u_j} \geq 0\) for \(i \neq j, i, j = 1, 2...n\), then
\begin{align}
U(\xi) &\leq \bar{U}(\xi), \quad \xi \in [-N, N]. 
\end{align}

**Proof.** We adapt the proof of [1]. Shift \( \bar{U}(\xi) \) to the left, for \( 0 \leq \mu \leq 2N \), consider \( \bar{U}^\mu(\xi) := \bar{U}(\xi + \mu) \) on the interval \( (-N - \mu, N - \mu) \). On both ends of the interval, by (2.14) and (2.15), we have

\begin{align}
U(\xi) < \bar{U}^\mu(\xi).
\end{align}

Starting from \( \mu = 2N \), decreasing \( \mu \), for every \( 0 < \mu < 2N \), the inequality (2.17) is true on the end points of the respective interval. On decreasing \( \mu \), suppose that there is a first \( \mu \) with \( 0 < \mu < 2N \) such that \( U(\xi) \leq \bar{U}^\mu(\xi) \) \( \xi \in (-N - \mu, N - \mu) \) and there is one component, for example the \( i \)-th, such that the equality holds on a point \( \xi_1 \) inside the interval. Let \( \bar{W}(\xi) = (w_1(\xi), \ldots, w_n(\xi))^T = \bar{U}_r(\xi) - \bar{U}(\xi) \), then \( w_i(\xi), i = 1, 2, \ldots, n \) satisfies

\begin{align*}
D_i w_i'' - c w_i' + \frac{\partial F_i}{\partial u_i} w_i \leq D_i w_i'' - c w_i' + \sum_{j=1}^n \frac{\partial F_j}{\partial u_i} w_j \leq 0,
\end{align*}

\( w_i(\xi_1) = 0, w_j(\xi) \geq 0 \) for \( \xi \in [-N - \mu, N - \mu] \), the Maximum principle further implies that \( w_i \equiv 0 \) for \( \xi \in [-N - \mu, N - \mu] \), but this is in contradiction with (2.17) on the boundary points \( \xi = -N - \mu \) and \( \xi = N - \mu \). So we can decrease \( \mu \) all the way to zero. This proves the Lemma. \( \square \)

**Lemma 5.** There exists a \( \nu \geq 0 \) such that \((\bar{u}, \bar{v})^T(\xi + \nu) \geq (u, v)^T(\xi) \) for \( \xi \in \mathbb{R} \).

**Proof.** We only prove for the wave speed \( c > 2\sqrt{1 - a_1} \) and \( r(a_2 - 1) \leq 1 \) since it is similar to show the other cases. We first derive the asymptotic behaviors of the upper- and lower-solutions at infinities. By Lemma 2 we have the following asymptotics for the upper and lower solutions

\begin{align}
\begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}(\xi) = \begin{pmatrix} A_1 \\ (1 + l)A_1 \end{pmatrix} e^{\frac{c - \sqrt{c^2 - 4(1 - a_1)}}{2} \xi} + o(e^{\frac{c - \sqrt{c^2 - 4(1 - a_1)}}{2} \xi})
\end{align}

and

\begin{align}
\begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}(\xi) = \begin{pmatrix} B_1 \\ B_1 \end{pmatrix} e^{\frac{c - \sqrt{c^2 - 4(1 - a_1)}}{2} \xi} + o(e^{\frac{c - \sqrt{c^2 - 4(1 - a_1)}}{2} \xi})
\end{align}

as \( \xi \to -\infty \);

\begin{align}
\begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}(\xi) \equiv \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\end{align}

and
as $\xi \to +\infty$, where $A_1, A_2, B_1, B_2$ are positive constants.

Since (2.7) is translation invariant, $\bar{v}(\xi) \equiv \bar{v}(\xi + \bar{r})$ is also a solution of (2.7) for any $\bar{r} \in \mathbb{R}$. It then follows that $(\bar{u}^\bar{r}, \bar{v}^\bar{r})^T(\xi)$ is also an upper-solution pair for system (1.4)-(1.5). For the asymptotic behavior of $(\bar{u}, \bar{v})^\bar{r}(\xi)$ at $-\infty$, we can simply replace $(A_1, (1 + l)A_1)$ by $(A_1, (1 + l)A_1)$ in (2.18). Now we choose $\bar{r} > 0$ large enough such that

$$e^{\frac{\sqrt{c^2-4(1-a_1)}}{2} \bar{r}} > 1.$$  

Then there exists a sufficiently large $N_1 > 0$ such that

(2.22)  

$$\begin{pmatrix} \bar{u}(\xi) \\ \bar{v}(\xi) \end{pmatrix} > \begin{pmatrix} \bar{u}(\xi) \\ \bar{v}(\xi) \end{pmatrix} \quad \text{for } \xi \in (-\infty, -N_1].$$

On the other hand, the boundary conditions of the upper- and lower-solutions at $+\infty$ also imply, on increasing $N_1$ if necessary, that

(2.23)  

$$\begin{pmatrix} \bar{u}(\xi) \\ \bar{v}(\xi) \end{pmatrix} > \begin{pmatrix} \bar{u}(\xi) \\ \bar{v}(\xi) \end{pmatrix} \quad \text{for } \xi \in \left[ N_1, +\infty \right).$$

On the interval $[-N_1, N_1]$, the strict monotonicity of the upper and lower solutions $(\bar{u}^\bar{r}, \bar{v}^\bar{r})^T$ and $(\bar{u}, \bar{v})^T$, and the inequalities (2.22)-(2.23) imply that

$$(\bar{u}, \bar{v})^T(-N_1) < (\bar{u}^\bar{r}, \bar{v}^\bar{r})^T(\xi) \quad \text{for } \xi \in (-N_1, N_1],$$

and

$$(\bar{u}, \bar{v})^T(\xi) < (\bar{u}^\bar{r}, \bar{v}^\bar{r})^T(N_1) \quad \text{for } \xi \in [-N_1, N_1).$$

Therefore, by Lemma 4 we have

(2.24)  

$$(\bar{u}, \bar{v})^T(\xi) \leq (\bar{u}^\bar{r}, \bar{v}^\bar{r})^T(\xi) \quad \text{for } \xi \in [-N_1, N_1].$$

Inequality (2.24) along with (2.22), (2.23) show the validity of the Lemma.  □

Proof of the Existence: We still use $(\bar{u}, \bar{v})^T(\xi)$ to denote the shifted upper-solution as given in lemma 4. Applying the monotone iteration method given in [11] to the upper and lower solutions defined in (2.11), we then have the existence of the traveling wave solutions for $c \geq 2\sqrt{a_1} - 1$. The boundary conditions that the upper and lower solutions satisfying lead to the boundary conditions (1.6) for traveling waves.
2.2. The asymptotics and the monotonicity. To derive the asymptotic decay rate of the traveling wave solution at \( \pm \infty \), we let \( c \geq 2\sqrt{1-a_1} \) and

\[
U(\xi) := (u(\xi), v(\xi))^T \quad \text{for} \quad -\infty < \xi < \infty
\]

be the corresponding traveling wave solution of (1.4)-(1.5) resulted from the monotone iteration. Lemma 2 implies that the upper- and the lower-solutions as derived in Lemma 3 have the same asymptotic rates at \(-\infty\). (1.6) and (1.8) then follow from Lemma 5. We differentiate (1.4) with respect to \( \xi \), and note that \((U(\xi))' := (w_1, w_2)^T(\xi)\) satisfies

\[
(w_1)_{\xi\xi} - c(w_1)_\xi + A_{11}(u, v)w_1 + A_{12}(u, v)w_2 = 0, \tag{2.26}
\]

\[
(w_2)_{\xi\xi} - c(w_2)_\xi + A_{21}(u, v)w_1 + A_{22}(u, v)w_2 = 0, \tag{2.27}
\]

where

\[
A_{11}(u, v) = 1 - a_1 - 2u + a_1v, \quad A_{12}(u, v) = a_1u \quad A_{21}(u, v) = a_2r(1 - v), \quad A_{22}(u, v) = -r(a_2u + 1 - 2v)
\]

We next study the exponential decay rates of the traveling wave solution \( U(\xi) \) at \(+\infty\). The asymptotic system of (2.26) and (2.27) as \( \xi \to +\infty \) is

\[
\begin{cases}
(\psi_1)'' - c(\psi_1)' - \psi_1 + a_1\psi_2 = 0, \\
(\psi_2)'' - c(\psi_2)' - r(\psi_2 - 1)\psi_2 = 0.
\end{cases} \tag{2.28}
\]

It is easy to see that the system (2.28) admits exponential dichotomy. Since the traveling wave solution \((u(\xi), v(\xi))^T\) converges monotonically to a constant limit as \( \xi \to \pm \infty \), the derivative of the traveling wave solution satisfies \( (w_1(\pm \infty), w_2(\pm \infty)) = (0, 0) \) (11, p.658 Lemma 3.2). Hence we are only interested in finding exponentially decaying solutions of (2.28) at \(+\infty\).

One can write the the general solution of the second equation of (2.28) as

\[
\psi_2 = A^1e^{\frac{c-\sqrt{c^2+4r(a_2-1)}}{2}}\xi + B^1e^{\frac{-c-\sqrt{c^2+4r(a_2-1)}}{2}}\xi
\]

for some constants \( A^1 \) and \( B^1 \). Since \( w_2 \to 0 \) as \( \xi \to +\infty \), one immediately has \( A^1 = 0 \).

We then study the solution of the second equation of (2.28), rewriting the equation as

\[
(\psi_1)'' - c(\psi_1)' - \psi_1 = -a_1\psi_2, \tag{2.29}
\]

we have the following expression for the solution of (2.29),

\[
\psi_1 = B_1e^{\frac{-\sqrt{c^2+4r(a_2-1)}}{2}}\xi + B_2e^{\frac{-\sqrt{c^2+4r(a_2-1)}}{2}}\xi + B_3e^{\frac{c+\sqrt{c^2+4r(a_2-1)}}{2}}\xi. \tag{2.30}
\]

Since \( w_2(\xi) \to 0 \) as \( \xi \to +\infty \), then \( B_3 = 0 \). Also noticing that (2.29) is non-homogeneous, we have \( B_1 \neq 0 \). By roughness of the exponential dichotomy (levinson) and integration, we obtain the asymptotic decay rate of the traveling wave solutions at \(+\infty\) given in (1.10).

We next show the monotonicity of the traveling wave solutions. By the monotone iteration process (11), the traveling wave solution \( U(\xi) \) is increasing for \( \xi \in \mathbb{R} \), it then follows that \( W(\xi) = U''(\xi) \geq 0 \) satisfying (2.26) and (2.27) and
(2.31) \( w_1, w_2 \geq 0, (w_1, w_2)^T(\pm \infty) = 0. \)

The Maximum Principle implies that \( (w_1, w_2)^T(\xi) > 0 \) for \( \xi \in \mathbb{R} \). This concludes that the traveling wave solution is strictly increasing on \( \mathbb{R} \).

2.3. The Uniqueness. On the uniqueness of the traveling wave solution for every \( c \geq 2\sqrt{1-a_1} \), we only prove the conclusion for traveling wave solutions with asymptotic behaviors (1.6) and (1.7), since other case can be proved similarly. Let \( U_1(\xi) \) and \( U_2(\xi) \) be two traveling wave solutions of system (1.4)-(1.5) with the same speed \( c > 2\sqrt{1-a_1} \). There exist positive constants \( A_i, B_i, i = 1, 2, 3, 4 \) and a large number \( N > 0 \) such that for \( \xi < -N \),

\[
U_1(\xi) = \begin{pmatrix}
(A_1 + o(1))e^{\frac{c - \sqrt{c^2 - 4(1-a_1)}}{2}\xi} \\
(A_2 + o(1))e^{\frac{-c - \sqrt{c^2 - 4(1-a_1)}}{2}\xi}
\end{pmatrix}
\]

and for \( \xi > N \),

\[
U_2(\xi) = \begin{pmatrix}
(A_3 + o(1))e^{\frac{c - \sqrt{c^2 - 4(1-a_1)}}{2}\xi} \\
(A_4 + o(1))e^{\frac{-c - \sqrt{c^2 - 4(1-a_1)}}{2}\xi}
\end{pmatrix}
\]

The traveling wave solutions of system (1.4) (1.5) are translation invariant, thus for any \( \theta > 0 \), \( U^\theta_1(\xi) := U_1(\xi + \theta) \) is also a traveling wave solution of (1.4) and (1.5). By (2.32) and (2.34), the solution \( U_1(\xi + \theta) \) has the following asymptotic behaviors:

\[
U^\theta_1(\xi) = \begin{pmatrix}
(A_1 + o(1))e^{\frac{c - \sqrt{c^2 - 4(1-a_1)}}{2}\theta e^{\frac{c - \sqrt{c^2 - 4(1-a_1)}}{2}\xi}} \\
(A_2 + o(1))e^{\frac{-c - \sqrt{c^2 - 4(1-a_1)}}{2}\theta e^{\frac{-c - \sqrt{c^2 - 4(1-a_1)}}{2}\xi}}
\end{pmatrix}
\]

for \( \xi \leq -N \);

\[
U^\theta_1(\xi) = \begin{pmatrix}
1 - (B_1 + o(1))e^{\frac{c - \sqrt{c^2 - 4(1-a_1)}}{2}\theta e^{\frac{c - \sqrt{c^2 - 4(1-a_1)}}{2}\xi}} \\
1 - (B_2 + o(1))e^{\frac{-c - \sqrt{c^2 - 4(1-a_1)}}{2}\theta e^{\frac{-c - \sqrt{c^2 - 4(1-a_1)}}{2}\xi}}
\end{pmatrix}
\]

for \( \xi \geq N \).

It is clear that for \( \theta \) large enough, we have

\[
A_1 e^{\frac{-c - \sqrt{c^2 - 4(1-a_1)}}{2}\theta} > A_3,
\]

(2.38)
(2.39) \[ A_2 e^{-\left(\sqrt{c^2 + 4(a_2 - 1)}\right)\theta} > A_4, \]

(2.40) \[ B_1 e^{\left(\sqrt{c^2 + 4(a_2 - 1)}\right)\theta} < B_3, \]

(2.41) \[ B_2 e^{-\left(\sqrt{c^2 + 4(a_2 - 1)}\right)\theta} < B_4. \]

Inequalities (2.38) - (2.41) imply that for \( \theta \) large enough,

(2.42) \[ U_1^\theta(\xi) > U_2(\xi) \]

for \( \xi \in (-\infty, -N] \cup [N + \infty) \). We now consider \( U_1^\theta(\xi) \) and \( U_2(\xi) \) on \([-N, N] \).

On the interval \([-N_1, N_1]\), the strict monotonicity of \( U_1 \) and \( U_2 \) and the inequality (2.42) imply that

\[ U_2(-N_1) < U_1^\theta(\xi) \quad \text{for } \xi \in (-N_1, N_1], \]

and

\[ U_2(\xi) < U_1^\theta(N_1) \quad \text{for } \xi \in [-N_1, N_1). \]

Therefore, by Lemma 4 we have

(2.43) \[ U_1^\theta(\xi) \geq U_2(\xi) \quad \text{for } \xi \in [-N_1, N_1]. \]

Inequalities (2.43) along with (2.42) show the validity of the Lemma.

2.4. The range of the wave speed. The next Theorem shows that the lower bound \( 2\sqrt{1 - a_1} \) for the wave speed is optimal, hence \( c = 2\sqrt{\alpha} \) is the critical wave speed.

**Lemma 6.** There is no monotone traveling wave solution of \((1.4)-(1.5)\) for any \( 0 < c < 2\sqrt{\alpha} \).

**Proof.** Suppose there is a constant \( c \) with \( 0 < c < 2\sqrt{1 - a_1} \) and a solution \( V(\xi) = (v_1, v_2)^T(\xi) \) of \((1.4)-(1.5)\) corresponding to it. Similar to (2.2) the asymptotic behaviors of \( V(\xi) \) at \(-\infty\) are described by

\[
\begin{pmatrix}
v_1(\xi) \\
v_2(\xi)
\end{pmatrix}
= \begin{pmatrix}
A_s \\
B_s
\end{pmatrix}
e^{-\left(\sqrt{c^2 + 4(a_2 - 1)}\right)\xi} + \begin{pmatrix}
A_s \\
B_s
\end{pmatrix}
e^{+\sqrt{c^2 + 4(a_2 - 1)}\xi} + h.o.t,
\]

where \((A_s, B_s)^T\) and \((A_s, B_s)\) are not both zero. The condition \( 0 < c < 2\sqrt{1 - a_1} \) implies that \( V(\xi) \) is oscillating. This concludes that any solution of \((1.4)-(1.5)\) with \( c < 2\sqrt{1 - a_1} \) is not strictly monotonic. \( \Box \)

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