Abstract—This paper technically explores the secrecy outage probability (SOP) $\Lambda$ and a minimisation problem over it as $\min \mathbb{P}(\Lambda \geq \lambda)$. We consider a Riemannian manifold for it and we mathematically define a volume for it as $\mathbb{V}_\mathbb{R} \mathbb{M} \mathbb{A}$. Through achieving a new upper-bound for the Riemannian manifold and its volume, we subsequently relate it to the number of eigen-values existing in the relative probabilistic closure. We prove in-between some novel lemmas with the aid of some useful inequalities such as the Finsler’s lemma, the generalised Young’s inequality, the generalised Brunn-Minkowski inequality, the Talagrand’s concentration inequality.

Index Terms—NP-hard, Alice, Bob, Eve, Finsler’s lemma, generalised Brunn-Minkowski inequality, generalised Young’s inequality, Keyhole contour, matrix operands, Talagrand’s concentration inequality.

I. INTRODUCTION

Physical-layer security inevitably plays a vital role in 5G/6G and beyond. This widely supported concept [1], [2], [3], [4], [5], [6], [7], [8], [9] is emerged in parallel with traditional cryptography techniques while information-theoretic perspectives are promising.

In order to simultaneously enhance the fairness and the quality of service among all the users, the physical characteristics of the wireless channel are of an absolutely inconsistent nature, which originally comes from the channel’s broadcast behaviour — something that should be essentially managed.

The concept of secrecy outage probability (SOP) in telecommunication still shows up an open research field in the literature. This concept is useful e.g. for: reflecting intelligent surfaces [1], cognitive networks [2], cooperative communications [3], power-line communications [4], the internet of Things [5], terrestrial networks [6], mobile edge computing networks [7], molecular communications [8] and under-water networks [9].

In [1], [2], [3], [4], [5], [6], [7], [8], [9] and in totally various types of system models, some novel and closed-form mathematical expressions have been newly derived and proposed – some of them are optimisation based, some of them are statistical oriented and some of them are even jointly theoretical-practical.

A. Motivations and contributions

In this paper, we are interested in responding to the following question: How can we guarantee highly adequate relaxations over the principle of SOP? With regard to the non-complete version of the literature, the expressed question strongly motivate us to find an interesting solution, according to which our contributions are fundamentally described as follows.

• (i) We theoretically find a totally novel interpretation over the SOP minimisation problem. We consider a Riemannian mani-fold for the SOP and we mathematically define a volume for it for which we derive a new upper-bound. We use some insightful principles such as Keyhole contour.

• (ii) We subsequently relate the Riemannian mani-fold and its upper-bounded volume expressed above to the number of eigen-values. We use in-between some useful lemmas and inequalities such as the Finsler’s lemma, the generalised Young’s inequality, the generalised Brunn-Minkowski inequality, the Talagrand’s concentration inequality.

B. General notation

The notations widely used throughout the paper is given in Table I.

C. Organisation

The rest of the paper is organised as follows. The system set-up and our main results are given in Sections II and III. Subsequently, the evaluation of the framework and conclusions are given in Sections IV and V.

II. SYSTEM MODEL AND PROBLEM FORMULATION

A. System description

A traditional communication scenario includes a transmitter named Alice and a legitimate receiver named Bob and an
un-authorised one as an eavesdropper named Eve. The information capacity of the communication system is theoretically expressed by the general formula from Shannon. The secrecy capacity is interpreted as an upper-bound of the security performance of the communication system. We now have the following inequality

\[ C_s := \max_{f_k(x)} \left( I(X, A) - I(X, B) \right) \]

while \( X, A \) and \( B \) are random states relating to respectively Alice, Bob and Eve.

**B. Main problem**

**III. MAIN RESULTS**

In this section, our main results are theoretically provided in details.

**Definition 1.** Let us assign a random variable \( \Lambda(t) := \{\lambda_1, \cdots, \lambda_n\} \) for the SOP.

**Lemma 1:** For the random variable \( \Lambda(t) := \{\lambda_1(t), \cdots, \lambda_n(t)\} \) over the time horizon – for which the term \( t \) is neglected hereinafter for the ease of notation –, the expression

\[ \mathbb{E}(\Lambda) \approx \frac{\mathbb{E}(e^{\ell\Lambda}) - 1}{t}, \]

is satisfied.

**Proof:** See Appendix A. \( \blacksquare \)

**Definition 2.** Let us, without loss of generality, consider the SOP as a Riemannian mani-fold for which the volume \( \mathbb{V}_{\mathcal{O}L}(\Lambda) \) is valid.

**Lemma 2:** Vitale’s random Brunn-Minkowski inequality\(^1\) – The expression

\[ \mathbb{V}_{\mathcal{O}L}\left(\mathbb{V}_{\mathcal{O}L}(\Lambda)\right) \geq \mathbb{V}_{\mathcal{O}L}\left(\Lambda\right), \]

holds.

**Lemma 3:** The expression

\[ \mathbb{V}_{\mathcal{O}L}\left(\frac{\mathbb{E}(e^{\ell\Lambda}) - 1}{t}\right) \geq \mathbb{V}_{\mathcal{O}L}\left(\Lambda\right), \]

holds.

**Proof:** The proof is easy to follow by an integration of Lemma 1 and Lemma 2. \( \blacksquare \)

\(^1\)Generalised Brunn-Minkowski inequality [10].

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**Table I: List of notations.**

| Notation | Definition | Notation | Definition |
|----------|-----------|----------|-----------|
| Min      | Minimisation | Max      | Maximisation |
| \( \mathbb{E} \) | Expected-value | \( I(\cdot) \) | Mutual-information |
| :=       | Is defined as | \( \approx \) | Is approximated to |
| \( \mathbb{V}_{\mathcal{O}L} \) | Volume | \( \mathbb{P} \) | Probability |
| \text{det}(\cdot) | Matrix determinant | \( \cdot^T \) | Transpose |
| \text{Tr}[\cdot] | Trace of matrix | \( \cdot^{-1} \) | Inverse of matrix |

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**Lemma 4:** The expression

\[ \mathbb{E}\{e^{\ell\Lambda}\} = e^{\ell\mathbb{P}(\Lambda \geq \lambda)}, \]

strongly holds.

**Proof:** See Appendix B. \( \blacksquare \)

**Lemma 5:** The problem

\[ \min_{(\cdot)} \mathbb{V}_{\mathcal{O}L}\left(e^{\ell\Lambda}\right) \mathbb{V}_{\mathcal{O}L}\left(\mathbb{P}(\Lambda \geq \lambda)\right), \]

is a dual one for the problem

\[ \min_{(\cdot)} \mathbb{E}\left\{\mathbb{V}_{\mathcal{O}L}\left(\Lambda\right)\right\}, \]

as its upperbound.

**Proof:** The proof is easy to follow with the aid of Yong’s inequality\(^2\) which says that

\[ f'(x)g'(x) \leq f(x) + g(x), \]

holds for the arbitrary functions \( f(\cdot) \) and \( g(\cdot) \), while \( (\cdot)' \) stands for the derivative. \( \blacksquare \)

**Lemma 7:** Finsler’s lemma\(^3\) – The problem

\[ \exists X, X^T A X = \xi, X^T B X \leq \xi \implies \exists z : B - zA < \xi, \]

holds for the arbitrary matrices \( A \) and \( B \) while \( \xi \) and \( T \) stand respectively for an arbitrary threshold and the transpose operand.

**Proposition 1:** Let us assume the descriptor system \((B, A)\), so, the characteristic polynomial is given as

\[ \mathbb{P}(z) = \text{det}(B - zA), \]

while \( \text{det}(\cdot) \) stands for the matrix determinant. The number of eigenvalues in the region associated with the polynomial

\(^2\)See e.g. [11] to understand what it is.

\(^3\)See e.g. [12] to understand what it is.
\( \mathcal{P}(z) \) over the Riemannian \( \mathcal{V} \) is related to \( \mathcal{B} = \mathcal{A}^{-1} \) and det(\( \mathcal{B} = \mathcal{A} \)) while \(-1\) stands for the inverse matrix.

**Proof:** See Appendix D.

IV. NUMERICAL RESULTS

We have done our simulations w.r.t. the Bernoulli-distributed data-sets using GNU Octave of version 4.2.2 on Ubuntu 16.04.

Table II shows the SOP vs. \( \mathcal{I}(X, \mathcal{A}) \) while changing \( \rho \) – something that is perfect for the evaluation here.

V. CONCLUSION

A new interpretation over the SOP minimisation problem was explored. We considered a Riemannian manifold for the SOP and a volume for it. Towards such end, some highly professional and insightful principles such as Keyhole contour, Finsler’s lemma, the generalised Brunn-Minkowski inequality etc were used.

**Appendix A**

**Proof of Lemma 1**

The proof is performed according to the Taylor expansion of

\[ e^{\mathcal{A}t} \approx 1 + t\mathcal{A}. \]

Now, by applying an expected-value operand, we consequently reach out

\[ \mathbb{E}\{\mathcal{A}\} \approx \frac{\mathbb{E}\{e^{\mathcal{A}t}\} - 1}{t}. \]

The proof is now completed.

**Appendix B**

**Proof of Lemma 4**

The cumulative distribution function (CDF)

\[ \mathcal{F}_\mathcal{A}(\mathcal{A}) = \mathbb{P}(\mathcal{A} \geq \mathcal{A}) \]

\[ = 1 - \mathbb{P}(\mathcal{A} \geq \mathcal{A}) \]

\[ = 1 - e^{-t\mathcal{A}}\mu_\mathcal{A}(t) \]

holds while \( \mu_\mathcal{A}(t) \) is the moment-generating function (MGF), so, we have

\[ \mathbb{E}\{e^{t\mathcal{A}}\} = e^{t\mathcal{A}}\mathbb{P}(\mathcal{A} \geq \mathcal{A}), \]

holds.

The proof is now completed.

**Appendix C**

**Proof of Lemma 5**

Let us assume that we have the optimisation problem of \( \min \mathbb{E}\{\mathcal{V} \} \), something that is equivalent to the minimisation over its upper-bound as in

\[ \min_{\mathcal{V}} \mathcal{V} \mathcal{O} \left( \frac{\mathbb{E}\{e^{t\mathcal{A}}\} - 1}{t} \right), \]

or with the aid of Lemma 4, the problem

\[ \min_{\mathcal{V}} \mathcal{V} \mathcal{O}\left( \frac{e^{t\mathcal{A}}\mathcal{F}_\mathcal{A}(\mathcal{A} \geq \mathcal{A})}{t} \right), \]

or

\[ \min_{\mathcal{V}} \mathcal{V} \mathcal{O}\left( \frac{e^{t\mathcal{A}}\mathcal{F}_\mathcal{A}(\mathcal{A} \geq \mathcal{A})}{t} \right), \]

or finally

\[ \min_{\mathcal{V}} \mathcal{V} \mathcal{O}\left( \frac{e^{t\mathcal{A}}\mathcal{F}_\mathcal{A}(\mathcal{A} \geq \mathcal{A})}{t} \right). \]

The proof is now completed.

**Appendix D**

**Proof of Proposition 1**

The proof is provided here in terms of the following solution.

Where \( K \) is a constant scaling factor, one can re-write the polynomial as

\[ \mathcal{P}(z) = K \prod_{i=1}^{n} (z - \gamma_i), \]

while \( \gamma_i, i \in \{1, \cdots, n\} \) stands literally for the \( i \)-th eigenvalue.

Now, recall the term \( \mathcal{V} \mathcal{O}\left( \frac{\mathbb{P}(\mathcal{A} \geq \mathcal{A})}{t} \right) \) versus \( \mathcal{V} \mathcal{O}\left( \frac{\mathbb{P}(\mathcal{A} \geq \mathcal{A})}{t} \right) \). By differentiating \( \mathcal{P}(z) \) with respect to \( z \) as \( \mathcal{P}'(z) \), \( \mathcal{P}(z) \) is obtained as

\[ \frac{\mathcal{P}'(z)}{\mathcal{P}(z)} = \sum_{i=1}^{n} \frac{1}{z - \gamma_i}. \]

For the above equation, where \( j = \sqrt{-1} \) is the imaginary unit, \( \mathcal{D} \supseteq \mathcal{V} \mathcal{O}\left( \frac{\mathbb{P}(\mathcal{A} \geq \mathcal{A})}{t} \right) \) is a closed anti-clockwise curve on the complex plane, and \( \mathbb{C} \supseteq \mathcal{V} \mathcal{O}\left( \frac{\mathbb{P}(\mathcal{A} \geq \mathcal{A})}{t} \right) \) is the region enclosed by \( \mathcal{D} \), it is achieved as

\[ \oint_{\mathcal{D}} \frac{1}{z - \gamma_i} dz = \begin{cases} 2\pi j, & \text{if } \gamma_i \in \mathbb{C}, \\ 0, & \text{if } \gamma_i \notin \mathbb{C}, \end{cases} \]

accoring to which one can say that the number of the eigenvalues in the region \( \mathbb{C} \) is

\[ N = \frac{1}{2\pi j} \oint_{\mathcal{D}} \frac{\mathcal{P}'(z)}{\mathcal{P}(z)} dz \]

\[ = \frac{1}{2\pi j} \sum_{i=1}^{n} \oint_{\mathcal{D}} \frac{1}{z - \gamma_i} dz. \]

On the other hand, \( \mathcal{P}'(z) \) is obtained as \([15]\)

\[ \mathcal{P}'(z) = \text{det}(\mathcal{B} - \mathcal{A}) \text{Tr}\left[ (\mathcal{B} - \mathcal{A})^{-1} \frac{\partial}{\partial z} (\mathcal{B} - \mathcal{A}) \right], \]

\[ \text{See e.g. [13].} \]

\[ \text{See e.g. [14].} \]

\[ \text{Page 8, eqn. 46.} \]
while $Tr[\cdot]$ stands for the trace of the matrix, something that is equivalent to
\[
P'(z) = P(z)Tr \left( B - zA \right)^{-1}(-A) \right],
\]
according to which one can say
\[
N = \frac{1}{2\pi} \int_{\mathcal{L}} Tr \left( B - zA \right)^{-1}(-A) dz.
\]

The last integral, i.e., the equation appeared above can be efficiently solved by some digitised methods such as the Rayleigh-Ritz method \cite{16}, \cite{17}, \cite{18}.

In order to conclude the proof, let us ultimately go over the essential relevance between the number of eigen-values and $\forall \alpha \in \mathbb{R} \cup \{\infty\}$ in the context of the following lemma.

Lemma 8: The number of eigen-values discussed above relies fundamentally upon $Pr(A)$.

Proof. In relation to the term $\forall \alpha \in \mathbb{R} \cup \{\infty\}$, we get in hands
\[
Pr\left( \alpha \geq \lambda \right) \geq \rho \leq 1 - \frac{e^{-\rho^2}}{Pr(A)}
\]
according to the Talagrand’s Concentration inequality \cite{7}, while $\rho$ is an arbitrary threshold. This means that $\forall \alpha \in \mathbb{R} \cup \{\infty\}$, $\mathbb{C}$ and $\mathbb{L}$ are functions of $\left( \rho; Pr(A) \right)$—something that proves Lemma 8.

Remark 2. The accuracy of evaluating the eigen-values expressed here can be fully able to be controlled by $\rho$.

The proof is now completed.

TABLE II: Simulations: SOP vs. $I(X, A)$ while changing $\rho$.

| $I(X, A)$ | SOP | $I(X, A)$ | SOP | $I(X, A)$ | SOP |
|-----------|-----|-----------|-----|-----------|-----|
| 0         | 0.99 | 0.005     | 0.1 | 0.99      | 0.5 |
| 0.005, $\rho = 0.2$ | 0.5 | 0.99      | 0.1 | 0.99      | 0.2 |
| 0.0001, $\rho = 0.1$ | 1  | 0.99      | 0.1 | 0.99      | 0.02 |

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\[\text{See e.g. [19] to understand what it is: It says that the complement of the given random variable in a bounded probability closure is emphatically upperbounded.}\]