Universal optimization efficiency and bounds of Carnot-like heat engines and refrigerators under shortcuts to isothermality

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Based on a quantum thermodynamic protocol for shortcut to isothermality that smoothly modify the system-reservoir interaction can significantly speed up an isothermal process while keeping the overall dissipation constant [Phys. Rev. X. 10, 031015 (2020)], we extend the study of optimization performance of Carnot-like heat engines and refrigerators in a straightforward and unified way. We derive the universal optimization efficiency of heat engines and the optimization coefficient of performance of refrigerators under two unified optimization criterions, i.e., $\chi$ criterion and $\Omega$ criterion. We also derived the universal lower and upper bounds for heat engines and refrigerators, and found that these bounds can be reached under extremely asymmetric cases.

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1. INTRODUCTION

The standard Carnot cycle consists of two isothermal processes and two adiabatic processes operating between two reservoirs at temperatures $T_h$ and $T_c$ ($T_h > T_c$). The clockwise cycle is a heat engine that converts heat into work, and the counterclockwise cycle is a refrigerator that cools a cold reservoir. According to thermodynamics, the Carnot efficiency $\eta_c = 1 - T_c / T_h$ for heat engines and the Carnot coefficient of performance (COP) $\varepsilon_c = T_c / (T_h - T_c)$ for refrigerators are attainable only in reversible limit, i.e., all processes in cycles are quasistatic. This means that infinite long operation time, zero entropy production and zero power output or zero cooling power. Therefore, the thermodynamic processes of Carnot cycles should be speeded up to deliver finite power at finite time.

In recent years, the strategy of speeding up thermodynamic processes by external control has attracted much attention [1-3]. A technique derived from quantum control is called shortcuts to adiabaticity, which reproduces the same final state as the infinitely slow adiabatic process in a finite time [4,5]. This technique has been successfully determined both theoretically [1, 6-8] and experimentally [9-11], and has also been applied to enhance the performance of classical and quantum thermal machines (heat engines and refrigerators) [12-19]. Another method has been proposed as a finite time driving strategy to steer the system from an equilibrium state to another one at the same temperature, which is named the shortcuts to isothermality [20]. This method has been used experimentally to achieve fast equilibration of a Brownian particle in isothermal processes at finite-rates [21,22], and theoretically to efficiently evaluate free energy difference [23]. Shortcuts to isothermality can speed up significantly the thermodynamic cycles, which provides an effective way to optimize the finite-time performance of thermal machines [24-27].

Accelerating the thermodynamic cycles in a finite time can increase the power output of the heat engines and cooling power of refrigerators. However, any accelerated thermodynamic cycles will inevitably be accompanied by additional energy dissipation,
which will lead to irreversible entropy production and reduce their efficiency. Therefore, optimizing the energy dissipation of the accelerated thermodynamic processes plays a central role in improving the performance of thermodynamic cycles [21,24,28]. Recently, Pancotti et al. proposed an accelerated protocol, which can significantly accelerate the isothermal process by smoothly modifying the system-reservoir interaction, while keeping the overall dissipation constant. As a result, this proposal can increase the power of a Carnot-like heat engines and refrigerators without compromising their efficiency [24]. In this paper, based on the protocol of shortcut to isothermality proposed by Ref. [24], we extend the study of optimization performance of Carnot-like heat engines and refrigerators in a straightforward and unified way. The objective of this paper is twofold. First, we explicitly evaluate optimization efficiency of heat engines and the optimization coefficient of performance of refrigerators under two unified optimization criterions, i.e., \( \chi \) criterion [29] and \( \Omega \) criterion [30]. Second, we derived the universal lower and upper bounds for heat engines and refrigerators in the limits of extremely asymmetric condition.

The paper is organized as follows: In Sec. II, we derive the universal optimization efficiency and its bounds of Carnot-like heat engines under \( \chi \) criterion. In Sec. III, we derive the analytical expressions of the optimization coefficient of performance and its bounds for Carnot-like refrigerators under \( \chi \) criterion. In Sec. IV, we study the optimization performance of Carnot-like heat engines and refrigerators under \( \Omega \) criterion, respectively. We summarize our results in Sec. V.

II. HEAT ENGINES AND THE \( \chi \) CRITERION

A. Carnot-like heat engines model

We consider a finite-time Carnot-like cycle composed of two isothermal processes and two adiabatic processes. The heat engines cycle consists of four consecutive steps as follows:

(1) Isothermal expansion process: the working substance is in contact with the hot
reservoir at temperature $T_h$ and absorbs heat $Q_h$ from the hot reservoir. The time for completing this process is assumed to be $\tau_h$. Then the entropy production can be expressed as

$$\Delta S_h = \frac{Q_h}{T_h} + \Delta S_h^{ir}, \quad \text{(1)}$$

where $\Delta S_h^{ir} \geq 0$ is the irreversible entropy production comes from the dissipation of system-reservoir interaction.

(2) Adiabatic expansion process: this process is idealized as the system suddenly decouples from the hot reservoir and then contacts the cold reservoir instantaneously. There is no heat exchange and the entropy production is vanishing in this process, i.e., $Q_{ae} = 0$ and $\Delta S_{ae} = 0$.

(3) Isothermal compression process: the system is in contact with the cold reservoir at temperature $T_c$ and absorbs heat $Q_c$ from the cold reservoir during the time $\tau_c$. The entropy production is given by

$$\Delta S_c = \frac{Q_c}{T_c} + \Delta S_c^{ir}, \quad \text{(2)}$$

where $\Delta S_c^{ir}$ is the irreversible entropy production which is non-negative.

(4) Adiabatic compression process: this process is assumed to be instantaneous, both the heat exchange and the variation of entropy are vanishing, i.e., $Q_{ac} = 0$ and $\Delta S_{ac} = 0$.

Having undergone a whole cycle, the system returns to its initial state. There is no entropy production in the whole cycle. Therefore, the entropy productions in the two isothermal processes satisfy

$$\Delta S_h = -\Delta S_c \equiv \Delta S, \quad \text{(3)}$$

Then the heat between the system and each of the two reservoirs can be written as

$$Q_h = T_h \left( \Delta S - \Delta S_h^{ir} \right), \quad \text{(4)}$$
\[ Q_c = T_c \left( -\Delta S - \Delta S^c \right). \]  

The adiabatic processes are assumed to proceed in negligible time, such that the cycle time is given by \( \tau_{\text{cycle}} = \tau_h + \tau_c \), and the work extracted in one cycle is \( W = Q_h + Q_c \). So, the efficiency and the power output per cycle of Carnot-like heat engines are defined as

\[ \eta = \frac{W}{Q_h} = 1 + \frac{Q_c}{Q_h}, \]

\[ P = \frac{W}{\tau_{\text{cycle}}} = \frac{Q_h + Q_c}{\tau_h + \tau_c}. \]  

**B. Speed-ups to isothermality**

Recently, a quantum thermodynamic protocol called speed-ups to isothermality (also called shortcuts to isothermality), which smoothly modify the system-reservoir interaction can significantly speed up an isothermal process while keeping the overall dissipation constant, has been proposed. With fine-tuned external control, the dissipation for optimal speed-ups to isothermality can decays as \[ W_{\text{diss}} = \frac{\Sigma_i}{\tau_i^\gamma}. \]  

where \( i = h, c \) for two isothermal processes. For optimal speed-ups to isothermality, \( \gamma \geq 1 \) can be obtained by a suitable modification of the system-reservoir interaction, whereas \( \gamma = 1 \) corresponds to a standard isothermal process, which is called low-dissipation [31]. The coefficient \( \Sigma_i > 0 \) contains the details of the extra control protocol of speed-ups to isothermality [24].

For an isothermal processes, the irreversible entropy is proportional to the dissipation, i.e., \( \Delta S^i \propto W_{\text{diss}} \). Therefore, the heat of the two isothermal processes in Eqs. (4) and (5) can be rewritten as [24]

\[ Q_h = T_h \left( \Delta S - \frac{\Sigma_h}{\tau_h^\gamma} \right), \]
\[ Q_c = T_c \left( -\Delta S - \frac{\Sigma_c}{\tau_c} \right). \]  

(10)

In the shortcuts to isothermality regime, the efficiency and power of heat engines can be rewritten as

\[ \eta = 1 + \frac{T_c \left( -\Delta S - \Sigma_c \tau_c^{-\gamma} \right)}{T_h \left( \Delta S - \Sigma_h \tau_h^{-\gamma} \right)}, \]  

(11)

\[ P = \frac{T_c \left( -\Delta S - \Sigma_c \tau_c^{-\gamma} \right) + T_h \left( \Delta S - \Sigma_h \tau_h^{-\gamma} \right)}{\tau_h + \tau_c}. \]  

(12)

C. Optimization efficiency and its bounds

Firstly, we consider a unified optimization criterion for both heat engines and refrigerators is called \(\chi\) criterion that is defined as the product of the converter efficiency \(z\) times the heat absorbed by the system \(Q_m\), divided by the time duration of cycle \(\tau_{cycle}\), i.e., \(\chi = zQ_m / \tau_{cycle}\) [29]. For a Carnot-like heat engine \(Q_m = Q_h\), and \(z = \eta\). Thus, the \(\chi\) criterion coincides with the power output of heat engine. Based on the Eq. (12), the maximum power is found by imposing the two conditions, i.e., \(\partial P / \partial \tau_h = 0\) and \(\partial P / \partial \tau_c = 0\). So, the unique physically acceptable solutions are given by

\[ \tau_c^{-\gamma} = \frac{\Delta S \left( T_h - T_c \right)}{(1 + \gamma) \left( \Sigma_c T_c H^{-\gamma} + \Sigma_c T_c \right)}, \]  

(13)

\[ \tau_h^{-\gamma} = \frac{\Delta S \left( T_h - T_c \right)}{(1 + \gamma) \left( \Sigma_c T_c H^{-\gamma} + \Sigma_h T_h \right)}. \]  

(14)

where \(H = \left( \Sigma_c T_c / \Sigma_h T_h \right)^{1/(1+\gamma)}\). Substituting Eqs. (13) and (14) into Eq. (11), the efficiency at maximum power is given by

\[ \eta_x = \frac{\eta_c \left[ \left( 1 - \eta_c \right) H^{-\gamma} \Sigma_c / \Sigma_h + 1 \right]}{(1 + \gamma) \left( 1 - \eta_c \right) H^{-\gamma} \Sigma_c / \Sigma_h + (1 - \eta_c) + \gamma}. \]  

(15)

In the case of \(\gamma = 1\) and symmetry condition \(\Sigma_c / \Sigma_h = 1\), the system degenerates to low-dissipation model for heat engines [31], and the efficiency recovers a standard
result, $\eta^m_{x}=1-\sqrt{1-\eta_c} \equiv \eta_{cA}$, which is reported by Curzon and Ahlborn with the so-called endoreversible model [32]. For $\gamma \to \infty$, the heat engine operates in the quasi-static case, and the efficiency $\eta_x = \eta_c$.

We now turn to the two main results derived from Eq. (15). First, the Eq. (15) can be expanded up to quadratic order in $\eta_c$ as

$$\eta_x = \frac{\gamma}{1+\gamma} \eta_c + \frac{1}{1+(\Sigma_c/\Sigma_h)^{1/(1+\gamma)}} \frac{\gamma^2}{(1+\gamma)^2} \eta_c^2 + O(\eta_c^3). \quad (16)$$

Second, in the limit $\Sigma_c/\Sigma_h \to 0$ and $\Sigma_c/\Sigma_h \to \infty$, the efficiency converges to the upper bound and to the lower bound, respectively

$$\frac{\gamma \eta_c}{1+\gamma} \leq \eta_x \leq \frac{\gamma \eta_c}{(1+\gamma)-\eta_c}. \quad (17)$$

It can be found that both the coefficients of Eq. (16) and the efficiency bounds in Eq. (17) depend on the parameter $\gamma$. For different parameters $\gamma$, the efficiency bounds and optimization regions are shown in Fig. 1. In the case of $\gamma = 1$, these results are recovered to the conclusion of low-dissipation model [31].

![Efficiency bounds and optimization regions of Carnot-like heat engines under $\chi$ criterion for different parameters $\gamma$.](image)
III. REFRIGERATORS AND THE $\chi$ CRITERION

A. Refrigerators model under speed-ups to isothermality

As a reverse cycle of the heat engines cycle, in the Carnot-like refrigerators model the adiabatic processes run instantaneously, while the isothermal processes under shortcuts to isothermality are carried out as follows. In the isothermal expansion process, the working substance is in contact with the cold reservoir at temperature $T_c$ during the time $\tau_c$. A certain amount of heat $Q_c$ is absorbed and the corresponding entropy charge can be expressed as

$$\Delta S_c = \frac{Q_c}{T_c} + \Delta S_c^{\text{ir}}$$  \hspace{1cm} (18)

In the isothermal compression process, the working substance is in contact with the hot reservoir at temperature $T_h$ during the time $\tau_h$. A certain amount of heat $Q_h$ is absorbed from hot reservoir. Thus, the entropy charge can be expressed as

$$\Delta S_h = \frac{Q_h}{T_h} + \Delta S_h^{\text{ir}}$$  \hspace{1cm} (19)

Since after undergoing the full cycle, the system returns to its initial state, the entropy charge in the whole cycle is zero, i.e., $\Delta S_c + \Delta S_h = 0$. Therefore, we have $\Delta S_c = -\Delta S_h = \Delta S$. In the shortcuts to isothermality regime, the irreversible entropy production can be expressed as $\Delta S_c^{\text{ir}} = \Sigma_c/\tau_c^\gamma$, and $\Delta S_h^{\text{ir}} = \Sigma_h/\tau_h^\gamma$. As a result, the amount of heat per cycle can be written as

$$Q_c = T_c \left( \Delta S - \frac{\Sigma_c}{\tau_c^\gamma} \right),$$  \hspace{1cm} (20)

$$Q_h = T_h \left( -\Delta S - \frac{\Sigma_h}{\tau_h^\gamma} \right).$$  \hspace{1cm} (21)

The cooling power and COP of refrigerators is given by

$$P_c = \frac{Q_c}{\tau_h + \tau_c}.$$  \hspace{1cm} (22)
\[ \varepsilon = \frac{Q_h - Q_c}{Q_h - Q_c}. \]  

(23)

**B. Optimization COP and its bounds**

For refrigerators, the \( Q_{in} = Q_c \), and \( z = \varepsilon \). Thus, the \( \chi \) criterion is expressed as \( \chi = \varepsilon Q_c / \tau_{cycle} \) [29]. Next, for a fixed ratio \( \tau_h = R \tau_c \), the \( \chi \) criterion is transformed into

\[ \chi = \frac{\varepsilon Q_c}{\tau_{cycle}} = \frac{T_c^2 (\Delta S - \Sigma_c \tau_c^\gamma)}{(1 + R) \tau_c [T_h (\Delta S + \Sigma_h R^{-1} \tau_h^\gamma) - T_c (\Delta S - \Sigma_h \tau_c^\gamma)]}. \]  

(24)

Based on the Eq. (24), we optimize the \( \chi \) criterion with respect to \( \tau_c \) and can derive the optimized COP is

\[ \varepsilon_x = \frac{(\gamma - 1) M + (2\gamma - 1) R^\gamma \Sigma_c / \Sigma_h + N}{(\gamma + 1) M + (2\gamma + 1) R^\gamma \Sigma_c / \Sigma_h + N} \varepsilon_c, \]  

(25)

where the notations are expressed as

\[ M = 1 + (1 + R^\gamma \Sigma_c / \Sigma_h) \varepsilon_c, \]

\[ N = \sqrt{4(\gamma + 1) MR^\gamma \Sigma_c / \Sigma_h + [(\gamma - 1) M + (2\gamma + 1) R^\gamma \Sigma_c / \Sigma_h]^2}. \]

According to Eq. (25), we first consider the symmetric case, i.e., \( \Sigma_c / \Sigma_h = 1 \). In this case, we can recover the CA COP \( \varepsilon_{x}^{\text{sym}} = \sqrt{1 + \varepsilon_c^\gamma} - 1 = \varepsilon_{CA} [33] \) for \( \gamma = 1 \) and \( R = \sqrt{1 + \varepsilon_c} / \sqrt{1 + \varepsilon_c} \). In the extremely asymmetric case, the low and upper bounds of optimized COP are obtained when \( \Sigma_c / \Sigma_h \to 0 \) and \( \Sigma_c / \Sigma_h \to \infty \), respectively:

\[ \varepsilon_{x}^- = \frac{\gamma - 1}{\gamma} \varepsilon_c, \]  

(26)

\[ \varepsilon_{x}^+ = \left( \frac{2\gamma - 1}{2\gamma + 1} \right) \frac{(\gamma - 1) \varepsilon_c + \alpha}{(\gamma + 1) \varepsilon_c + \alpha}, \]  

(27)

where \( \alpha = \sqrt{4(\gamma + 1) \varepsilon_c + (1 - \varepsilon_c + 2\gamma + \gamma \varepsilon_c)^2} \). In Fig. 2, COP bounds and optimization
regions of Carnot-like refrigerators under $\chi$ criterion for different parameters $\gamma$ are shown as a function of the temperature ratio $T_h/T_c$. In the case of $\gamma = 1$, the Eqs. (26) and (27) revert to $\varepsilon^- = 0$ and $\varepsilon^+ = \left(\sqrt{9+8\varepsilon_c} - 3\right)/2$, which were first obtained in low-dissipation Carnot-like refrigerators [34].

![COP bounds and optimization regions of Carnot-like refrigerators under $\chi$ criterion for different parameters $\gamma$.](image)

**FIG. 2.** COP bounds and optimization regions of Carnot-like refrigerators under $\chi$ criterion for different parameters $\gamma$.

**IV. THE $\Omega$ CRITERION**

**A. Heat engines**

In this section, we consider another unified optimization criterion called $\Omega$ criterion, which is defined by considering a compromise between the useful energy and the lost energy [30]. For heat engines, the $\Omega$ criterion represented as a compromise between maximum power performed and minimum power lost, denoted as $\Omega = (2\eta - \eta_c) P/\eta$. Substitute Eq. (11) and (12), then the $\Omega$ criterion is expressed as

$$\Omega = \frac{T_h \left(\Delta S - \sum_{\tau_h^\gamma}\tau_h^\gamma\right) - T_c \left(\Delta S + 2\sum_{\tau_c^\gamma}\tau_c^\gamma + \sum_{\tau_h^\gamma}\tau_h^\gamma\right)}{\tau_h + \tau_c},$$

(28)

The Eq. (28) is optimized in two conditions $\partial\Omega/\partial\tau_h = 0$ and $\partial\Omega/\partial\tau_c = 0$, $\tau_h$ and $\tau_c$ are explicitly given by
\[ \tau^c = \frac{\Delta S(T_k - T_c)}{2\Sigma_c T_c (1 + \gamma + \gamma K) + \Sigma_h (T_h + T_c) K^{-\gamma}}, \quad (29) \]

\[ \tau^h = \frac{\Delta S(T_h - T_c)}{2\Sigma_c T_c (1 + \gamma + \gamma K) K^{-\gamma} + \Sigma_h (T_h + T_c)}. \quad (30) \]

where

\[ K = \left[ \frac{2\Sigma_c T_c}{\Sigma_h (T_h + T_c)} \right]^{1/(1+\gamma)}. \]

Substituting Eqs. (29) and (30) into Eq. (11), the efficiency at maximum \( \Omega \) condition is given by

\[ \eta_\Omega = \frac{1 + \left(1 + 2\gamma + 2\gamma K\right) K^{\gamma} \Sigma_c / \Sigma_h \eta_c}{2 + 2\left(1 + \gamma + \gamma K\right) K^{\gamma} \Sigma_c / \Sigma_h \eta_c}. \quad (31) \]

In the case of \( \gamma = 1 \) and \( \Sigma_c / \Sigma_h = 1 \), the efficiency recovers to

\[ \eta_\Omega^{\text{sym}} = 1 - \sqrt{(1 - \eta_c)(2 - \eta_c)/2}, \]

that first reported by Angulo et al. using the ecological criterion for endoreversible Carnot heat engines [35]. And it was verified again in the low-dissipation Carnot-like heat engines in the symmetric condition [36].

Expand Eq. (31) to the quadratic order of \( \eta_c \), it can be expressed as

\[ \eta_\Omega = \frac{1 + 2\gamma}{2 + 2\gamma} \eta_c + \frac{1}{1 + \left(\Sigma_c / \Sigma_h\right)^{1/(1+\gamma)}} \frac{\gamma}{4(1 + \gamma)^2} \eta_c^2 + O(\eta_c^3). \quad (32) \]

This expression depends strongly on the parameter \( \gamma \). For the case \( \gamma = 1 \), the universally of Eq. (32) has been demonstrated in Ref. [37].

In the extreme asymmetric limit \( \Sigma_c / \Sigma_h \to 0 \), the Eq. (31) converges to the upper bound

\[ \eta_\Omega^+ = \frac{(1 + 2\gamma) - (1 + \gamma) \eta_c}{(2 + 2\gamma) - (2 + \gamma) \eta_c} \eta_c, \quad (33) \]

while for the case \( \Sigma_c / \Sigma_h \to \infty \), the Eq. (31) reduces to the lower bound

\[ \eta_\Omega^- = \frac{1 + 2\gamma}{2 + 2\gamma} \eta_c. \quad (34) \]
The bounds and optimization regions for different parameters \( \gamma \) are shown in Fig. 3.

In the case of \( \gamma = 1 \), the bounds are recovered to the conclusion of low-dissipation model [36].

![Fig. 3. Efficiency bounds and optimization regions of Carnot-like heat engines under \( \Omega \) criterion for different parameters \( \gamma \).]

**B. Refrigerators**

For refrigerators, the \( \Omega \) criterion represented as a trade-off between maximum cooling load and minimum lost cooling load, denoted as \( \Omega = (2\varepsilon - \varepsilon_c)P_c/\varepsilon \).

Substitute Eq. (22) and (23), then the \( \Omega \) criterion is expressed as

\[
\Omega = \frac{\Delta S(T_h - T_c) - \Sigma_c(2T_h - T_c)\tau_c^{-\gamma} - \Sigma_hT_h\tau_h^{-\gamma}}{\tau_h + \tau_c} \frac{T_c}{T_h - T_c},
\]

(35)

The optimized \( \tau_h \) and \( \tau_c \) are given by

\[
\tau_c^{-\gamma} = \frac{\Delta S(T_h - T_c)G^{1+\gamma}}{\Sigma_hT_h(G + \gamma + \gamma G) + \Sigma_c(2T_h - T_c)G^{1+\gamma}},
\]

(36)

\[
\tau_h^{-\gamma} = \frac{\Delta S(T_h - T_c)G}{\Sigma_hT_h(G + \gamma + \gamma G) + \Sigma_c(2T_h - T_c)G^{1+\gamma}},
\]

(37)

where

\[
G = \left[ \frac{\Sigma_c(2T_h - T_c)}{\Sigma_hT_h} \right]^{-1/(1+\gamma)}.
\]
Thus, the corresponding COP at maximum \( \Omega \) condition is given by

\[
\varepsilon_\Omega = \frac{(G + \gamma + \gamma G) + G^{1+\gamma} \Sigma_c/\Sigma_h}{(2G + \gamma + \gamma G) + 2G^{1+\gamma} \Sigma_c/\Sigma_h} \varepsilon_C. \tag{38}
\]

In the case of \( \gamma=1 \) and \( \Sigma_c/\Sigma_h=1 \), the COP reverts to

\[
\varepsilon_\Omega^{nm} = \varepsilon_C \left[ \sqrt{(1+\varepsilon_C)(2+\varepsilon_c)} - \varepsilon_c \right],
\]

this result first reported for Carnot-like refrigerators in the endoreversible model [30].

The lower and upper bounds can be obtained, respectively, in the asymmetric conditions \( \Sigma_c/\Sigma_h \to 0 \) and \( \Sigma_c/\Sigma_h \to \infty \):

\[
\varepsilon_\Omega^- = \frac{1+\gamma}{2+\gamma} \varepsilon_C, \tag{39}
\]

and

\[
\varepsilon_\Omega^+ = \frac{(1+2\gamma) + (1+\gamma)\varepsilon_c}{(2+2\gamma) + (2+\gamma)\varepsilon_c} \varepsilon_c, \tag{40}
\]

as shown in Fig. 4. In the case of \( \gamma=1 \), the COP bounds are recovered to the conclusion of low-dissipation model [36].

![FIG. 4. COP bounds and optimization regions of Carnot-like refrigerators under \( \Omega \) criterion for different parameters \( \gamma \).](image)

V. CONCLUSIONS
We have extended the study of optimization performance of Carnot-like heat engines and refrigerators in a straightforward and unified way based on an optimal protocol of shortcut to isothermality. The main results of this paper are to derive universal analytical expressions for the optimization efficiency of heat engines and optimization COP of refrigerators under two different optimization criteria, and to obtain their upper and lower bounds in the limits of extremely asymmetric condition. These results can be recovered to the conclusions of low-dissipation Carnot-like heat engines and refrigerators in the case of $\gamma=1$. The results obtained here provide universal optimization results for designing more efficient finite-time heat engines and refrigerators under shortcut to isothermality.

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