On the universal $R$-matrix for the Izergin–Korepin model

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Received 2 May 2011, in final form 6 July 2011
Published 9 August 2011
Online at stacks.iop.org/JPhysA/44/355202

Abstract

We continue our exercises with the universal $R$-matrix based on the Khoroshkin and Tolstoy formula. Here we present our results for the case of the twisted affine Kac–Moody Lie algebra of type $A^{(2)}_2$. Our interest in this case is inspired by the fact that the Tzitzéica equation is associated with $A^{(2)}_2$ in a similar way as the sine-Gordon equation is related to $A^{(1)}_1$. The fundamental spin-chain Hamiltonian is constructed systematically as the logarithmic derivative of the transfer matrix. $L$-operators of two types are obtained by using $q$-deformed oscillators.

PACS numbers: 02.10.Hh, 02.20.Tw, 02.20.Uw

1. Introduction

The Izergin–Korepin model [14] was introduced as a quantum integrable model related to a classical integrable system described by the nonlinear differential equation

$$\partial_x \partial_x F = -m^2[\exp(-2F) - \exp(F)]$$

for a function $F$ of two independent variables. Although it was introduced for the first time by Tzitzéica within the framework of differential geometry, this equation is mostly known as the Dodd–Bullough–Mikhailov or Jiber–Mikhailov–Shabat model, since it was investigated later in the context of the theory of solitons and general aspects of classical integrability. It is also known that this equation is a particular case of the Toda equations associated with twisted loop
Here the affine group of type $A_2^{(2)}$ plays the role of the underlying symmetry group, which for the sine-Gordon equation takes the simplest affine group of type $A_1^{(1)}$.

To a certain extent, the model under consideration has features reminiscent of both the so-called $sl_3$ and $sl_2$ cases: on one hand, it is related to a three-dimensional matrix representation, while, on the other hand, the infinite-dimensional part of the whole representation should correspond to only one scalar field entering the integrable equation written above, and therefore we expect that a single copy of the $q$-deformed oscillator algebra will be sufficient to describe $L$- and $Q$-operators.

We work in the spirit and use the notations of [5], continuing our exercises with the universal $R$-matrix based on the remarkable formula presented by Khoroshkin and Tolstoy [18, 19, 30]. After reproducing the $R$-matrix of the Izergin–Korepin model, we construct two types of the $L$-operators using a $q$-deformed oscillator algebra. In the case under consideration, these $L$-operators turn out to be related by a similarity transformation. A similar relation holds for the $R$-matrix as well. We also discuss various (anti)automorphisms generating new $L$-operators from given ones. The $L$-operators can then be used for the subsequent construction of $Q$-operators encoding the information about the spectrum of the quantum integrable system under consideration.

The problem of the calculation of the eigenvalues of the transfer matrix for the Izergin–Korepin model was studied in [26]. An algebraic Bethe ansatz solution of this model was constructed in [27, 29]. This consideration was extended in [23, 22] to the case of open boundary conditions.

2. Generators and roots

Let $A = (a_{ij})_{i,j=0,1}$ be the generalized Cartan matrix of type $A_2^{(2)}$ having the explicit form

$$A = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}.$$  

The matrix $A$ is symmetrizable and we have $d_i a_{ij} = d_j a_{ji}$ for $d_0 = 2$ and $d_1 = 1/2$. We denote $A^s = (a^s_{ij}) = (d_i a_{ij}) = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$.

Before proceeding to the universal $R$-matrix, let us first describe the structure of the Kac–Moody algebra $g'(A)$, enveloping algebra $U(g'(A))$ and its quantum deformation $U_h(g'(A))$. In accordance with the general construction, see, for example, the books [17, 10], the twisted affine Lie algebra $g'(A)$ is generated by the elements $h_i, e_i, f_i, i = 0, 1$, with the defining relations

$$[h_i, h_j] = 0,$$  

$$[h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j,$$  

$$[e_i, f_j] = \delta_{ij} h_i,$$  

satisfied for all $i$ and $j$, and the Serre relations

$$(\text{ad } e_i)^{1-a_{ij}}(e_j) = 0, \quad (\text{ad } f_i)^{1-a_{ij}}(f_j) = 0,$$  

satisfied for all distinct $i$ and $j$.

We denote the linear span of the generators $h_i$ by $h'(A)$ and its dual space by $h'^*(A)$. The vector space $h'(A)$ is the Cartan subalgebra of $g'(A)$, and we have the decomposition

$$g'(A) = h'(A) \oplus \bigoplus_{\gamma \in \Delta(A)} g'(A)_\gamma,$$
where for any $\gamma \in \mathfrak{h}^* (A)$ we denote
\[
g'(A)_\gamma = \{ x \in g'(A) \mid [h, x] = \gamma (x) x \text{ for any } h \in \mathfrak{h}' (A) \}
\]
and
\[
\Delta (A) = \{ \gamma \in \mathfrak{h}^* (A) \mid \gamma \neq 0, g'(A)_\gamma \neq \{0\} \}.
\]
The elements of $\Delta (A)$ are roots, and $g'(A)_\gamma$ is the root space of $\gamma$ whose nonzero elements are root vectors.

The generators $e_i$ are evidently root vectors. We denote the corresponding roots by $a_i$. These are simple roots. Any other root is a linear combination of simple roots with integer coefficients all of which are either non-negative or non-positive. In the former case, we have a positive root and in the latter, a negative one. In particular, the generators $f_i$ are root vectors corresponding to the negative roots $-a_i$. One has $\Delta (A) = \Delta_+ (A) \sqcup \Delta_- (A)$, where $\Delta_+ (A)$ is the set of positive roots and $\Delta_- (A) = -\Delta_+ (A)$.

The symmetrized Cartan matrix $A^S$ determines a symmetric bilinear form on $\mathfrak{h}^* (A)$ by the equality
\[
(\alpha, \alpha) = a_{ij}^S.
\]
Explicitly we have
\[
(\alpha_0, \alpha_0) = 4, \quad (\alpha_0, \alpha_1) = -2, \quad (\alpha_1, \alpha_0) = -2, \quad (\alpha_1, \alpha_1) = 1.
\]
It is convenient to denote\(^5\)
\[
\delta = \alpha_0 + 2 \alpha_1, \quad \alpha = \alpha_1,
\]
so that
\[
(\delta, \delta) = (\delta, \alpha) = (\alpha, \delta) = 0, \quad (\alpha, \alpha) = 1.
\]

It can be shown that the set of the positive roots is
\[
\Delta_+ (A) = \{ \alpha + m \delta \mid m \in \mathbb{Z}_{\geq 0} \} \cup \{ 2 \alpha + (2m + 1) \delta \mid m \in \mathbb{Z}_{\geq 0} \}
\]
\[
\cup \{ m \delta \mid m \in \mathbb{Z}_{\geq 0} \} \cup \{ \delta - 2 \alpha + 2m \delta \mid m \in \mathbb{Z}_{\geq 0} \} \cup \{ \delta - \alpha + m \delta \mid m \in \mathbb{Z}_{\geq 0} \},
\]
see, for example, the book [10]. All root spaces corresponding to positive and negative roots are one dimensional. Choosing a root vector for each root and adding the Cartan generators $h_i$, we obtain a Cartan basis of $g'(A)$.

The enveloping algebra $U(g'(A))$ is defined as the unital associative algebra with generators $h_i, e_i, f_i$ and with the same relations (2.1)–(2.4) as $g'(A)$. Here we can rewrite the Serre relations (2.4) as
\[
\sum_{k=0}^{1-a_{ij}} (-1)^k \left( \begin{array}{c} 1 - a_{ij} \\ k \end{array} \right) (e_i)^{1-a_{ij}-k} e_j (e_i)^k = 0, \quad (2.5)
\]
\[
\sum_{k=0}^{1-a_{ij}} (-1)^k \left( \begin{array}{c} 1 - a_{ij} \\ k \end{array} \right) (f_j)^{1-a_{ij}-k} f_i (f_j)^k = 0. \quad (2.6)
\]
The Lie algebra $g'(A)$ can be naturally considered as a subspace of $U(g'(A))$, and a Cartan basis of $g'(A)$ as any of its bases generates a Poincaré–Birkhoff–Witt basis of $U(g'(A))$.

Let $h$ be an indeterminate and $q = \exp \ h$. In accordance with the general definition, see, for example, the book [11], the quantum deformation of $U(g'(A))$, the quantum group

\(^5\) In fact, $\delta$ is the minimal positive imaginary root [17, 10].
The quantum group $U_h(g'(A))$, is the topological $\mathbb{C}[[h]]$-algebra generated by six elements $h_i, e_i, f_i, i = 0, 1$, with the defining relations

$$[h_i, h_j] = 0, \quad \quad \quad (2.7)$$

$$[h_i, e_j] = a_{ij}^e e_j, \quad \quad \quad [h_i, f_j] = -a_{ij}^f f_j, \quad \quad \quad (2.8)$$

$$[e_i, f_j] = \delta_{ij} q^{d_{ih}} - q^{-d_{ih}}, \quad \quad \quad (2.9)$$

satisfied for all $i$ and $j$, and the Serre relations

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \begin{array}{c} 1-a_{ij} \\ n \\ \end{array} \right]_{q^e} (e_j)^{1-a_{ij}} e_j (e_i)^k = 0, \quad \quad \quad \quad \quad (2.10)$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \begin{array}{c} 1-a_{ij} \\ n \\ \end{array} \right]_{q^f} (f_j)^{1-a_{ij}} f_j (f_i)^k = 0, \quad \quad \quad \quad \quad (2.11)$$

satisfied for all distinct $i$ and $j$. Here we have introduced the $q$-binomial coefficients

$$\left[ \begin{array}{c} n \\ m \\ \end{array} \right]_q = \frac{[n]_q! [m]_q!}{[n-m]_q!}, \quad \quad [n]_q! = [n]_q [n-1]_q \cdots [1]_q, \quad \quad \quad [n]_q = \frac{q^n - q^{-n}}{q-q^{-1}}.$$

It is clear that relations (2.7)–(2.9) are deformations of relations (2.1)–(2.3), and the quantum Serre relations (2.10) and (2.11) are deformations of the Serre relations (2.5) and (2.6). Khoroshkin and Tolstoy use a slightly different definition of a quantum group [30, 19]. We come to this definition using the rescaling of the generators

$$h_i \rightarrow a_i^{-1} h_i, \quad e_i \rightarrow \left( \frac{q - q^{-1}}{q^d - q^{-d}} \right)^{1/2} e_i, \quad f_i \rightarrow \left( \frac{q - q^{-1}}{q^d - q^{-d}} \right)^{1/2} f_i.$$ 

After this, the defining relations (2.7)–(2.9) take the form

$$[h_i, h_j] = 0, \quad \quad \quad \quad \quad \quad (2.12)$$

$$[h_i, e_j] = a_{ij}^e e_j, \quad \quad \quad [h_i, f_j] = -a_{ij}^f f_j, \quad \quad \quad (2.13)$$

$$[e_i, f_j] = \delta_{ij} q^{d_{ih}} - q^{-d_{ih}}, \quad \quad \quad (2.14)$$

while the Serre relations (2.10) and (2.11) preserve their form. Below we work in terms of the rescaled generators. Note that the element $c = h_0 + 2h_1$ belongs to the center of $U_h(g'(A))$. It is convenient to assume that the definition of $U_h(g'(A))$ includes an additional relation

$$c = h_0 + 2h_1 = 0.$$ 

This allows us to use for the case of the quantum groups $U_h(g'(A))$ the Khoroshkin–Tolstoy formula for the universal $R$-matrix valid for quantum groups $U_h(g(A))$ just putting there $c = 0$.

The quantum group $U_h(g'(A))$ is a topological Hopf $\mathbb{C}[[h]]$-algebra with correspondingly defined comultiplication, counit and antipode, see, for example, the book [11].

Associate now with each root from the infinite root system $\Delta(A)$ a corresponding root vector. We denote the root vector corresponding to a positive root $\gamma$ by $e_\gamma$, and the root vector corresponding to a negative root $-\gamma$ by $f_\gamma$. Let us start with the simple positive roots and put

$$e_{a_0} = e_{-2a_0} = e_0, \quad \quad e_{a_1} = e_a = e_1.$$
The higher root vectors corresponding to positive roots are defined recursively by the relations
\[ e_{\delta - \alpha} = ([2]_q)^{-1/2}[e_\alpha, e_{\delta - 2\alpha}]_q, \quad e_{\delta} = [e_\alpha, e_{\delta - \alpha}]_q, \]  
(2.15)
\[ e_{\alpha + m\delta} = ([3]_{q^{1/2}})^{-1}[e_{\alpha + (m-1)\delta}, e_{\delta}']_q, \quad e_{\delta - \alpha + m\delta} = ([3]_{q^{1/2}})^{-1}[e_{\delta}', e_{\delta - \alpha + (m-1)\delta}]_q, \]  
(2.16)
\[ e_{2\alpha + (2m+1)\delta} = ([2]_q)^{-1/2}[e_{\alpha + m\delta}, e_{\alpha + (m+1)\delta}]_q, \]  
(2.17)
\[ e_{2\alpha + 2(m+1)\delta} = ([2]_q)^{-1/2}[e_{\delta - \alpha + (m+1)\delta}, e_{\delta - \alpha + m\delta}]_q, \]  
(2.18)
\[ e_{\delta}'_{m\delta} = [e_{\alpha + (m-1)\delta}, e_{\delta - \alpha}]_q, \]  
(2.19)
where the \( q \)-deformed commutator is defined as
\[ [e_\gamma, e_{\gamma'}]_q = e_\gamma e_{\gamma'} - q^{[\gamma, \gamma']} e_{\gamma'} e_\gamma \]
for any two roots \( \gamma \) and \( \gamma' \) from the root system \( \Delta_+ (A) \). These relations allow us to construct the root vectors corresponding to all the roots from the set \( \Delta_+ (A) \). In fact, the expression for the universal \( R \)-matrix given by Khoroshkin and Tolstoy [30] contains the root vectors \( e_{m\delta} \) related to the root vectors \( e_{\delta}'_{m\delta} \) by the equality
\[ (q - q^{-1})e_\delta(x) = \log[1 + (q - q^{-1})e_\delta'](x), \]
where
\[ e_\delta'_{m\delta} = \sum_{m>0} e_{m\delta} x^{-m}, \quad e_\delta(x) = \sum_{m>0} e_{m\delta} x^{-m}. \]

To complete the set of root vectors, we need to construct root vectors for negative roots from the system \( \Delta_- (A) \). We do it using the rule
\[ f_\gamma = \omega(e_\gamma) = e_{-\gamma} \]
for any \( \gamma \in \Delta_+ (A) \). Here \( \omega \) is the Cartan anti-involution defined on the generators by the relations
\[ \omega(h_i) = h_i, \quad \omega(e_i) = f_i, \quad \omega(f_i) = e_i, \]  
(2.20)
supplied with the rule \( \omega(h) = -h \) implying that \( \omega(q) = q^{-1} \). Note that we use such normalization of the root vectors that for any \( \gamma \neq m\delta \), we have
\[ [e_\gamma, f_\gamma] = \frac{q^{h_\gamma} - q^{-h_\gamma}}{q - q^{-1}}, \]
where \( h_\gamma = \sum \alpha_i m_i h_i \) if \( \gamma = \sum m_i \alpha_i \). It appears that, as in the case of \( U(q' (A)) \), the root vectors corresponding to all roots from \( \Delta (A) \) together with the Cartan generators \( h_i \) generate a Poincaré–Birkhoff–Witt basis of \( U_h (q' (A)) \).

3. Universal \( R \)-matrix

We consider the case of a quasi-triangular Hopf algebra, see, for example, [11]. The corresponding universal \( R \)-matrix \( R \) satisfies the Yang–Baxter equation
\[ R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} \]  
(3.1)
and serves as a basic object for the construction and investigation of integrable models. In particular, let for any \( \zeta \in \mathbb{C}^\times \) a representation \( \varphi_\zeta \) of \( U_h (q' (A)) \) be given. Then the parameter-dependent \( R \)-matrix
\[ R(\zeta_1 | \zeta_2) = \varphi_{\zeta_1} \otimes \varphi_{\zeta_2} (R) \]

6 We use primed notation for the root vectors corresponding to the roots \( m\delta \) because we redefine them below.
satisfies the Yang–Baxter equation of the form
\[ R_{12}(\zeta_1 | \zeta_2) R_{13}(\zeta_1 | \zeta_3) R_{23}(\zeta_2 | \zeta_3) = R_{23}(\zeta_2 | \zeta_3) R_{13}(\zeta_1 | \zeta_3) R_{12}(\zeta_1 | \zeta_2). \]

Usually, one chooses the representation in such a way that \( R(\zeta_1 | \zeta_2) \) depends only on the combination \( \zeta_1 \zeta_2^{-1} \). This allows one to introduce the \( R \)-matrix \( R(z) \) depending on a single parameter, so that
\[ R(\zeta_1 | \zeta_2) = R(\zeta_1 \zeta_2^{-1}). \]

Now the Yang–Baxter equation takes its original form,
\[ R_{12}(\zeta_{12}) R_{13}(\zeta_{13}) R_{23}(\zeta_{23}) = R_{23}(\zeta_{23}) R_{13}(\zeta_{13}) R_{12}(\zeta_{12}). \] (3.2)

Here and below we use the notation \( \zeta_{ij} = \zeta_i \zeta_j^{-1} \).

An explicit construction of the universal \( R \)-matrix for quantum groups was proposed by Khoroshkin and Tolstoy [30]. For the case under consideration, it looks as follows [19].

First of all, we have to choose a normal ordering [21, 1] of roots from \( \Delta^+ (A) \). In fact there are only two possibilities. We use the ordering where the roots \( \alpha + m \delta \) and \( 2 \alpha + (2m + 1) \delta \) come first as
\[ \alpha, \; 2\alpha + \delta, \; \alpha + \delta, \; 2\alpha + 3\delta, \; \alpha + 2\delta, \; 2\alpha + 5\delta, \; \alpha + 3\delta, \; 2\alpha + 7\delta, \; \ldots; \]
then the roots \( m \delta \) come as
\[ \delta, \; 2\delta, \; 3\delta, \; 4\delta, \; 5\delta, \; 6\delta, \; 7\delta, \; \ldots; \]
and finally the roots \( \delta - \alpha + m \delta \) and \( \delta - 2\alpha + 2m \delta \) come in the order
\[ \ldots, \; \delta - 2\alpha + 6\delta, \; \delta - \alpha + 2\delta, \; \delta - 2\alpha + 4\delta, \; \delta - \alpha + \delta, \; \delta - 2\alpha. \]

Gathering, we can write
\[ \gamma + k\delta < m\delta < (\delta - \gamma) + \ell\delta, \]
where \( \gamma = \alpha, 2\alpha \). Another ordering is the reverse to this one.

Now we have all the ingredients needed to construct the universal \( R \)-matrix. According to Khoroshkin and Tolstoy, it has the form
\[ \mathcal{R} = \mathcal{R}_{-\delta} \mathcal{R}_{-\gamma} \mathcal{R}_{+\delta} \mathcal{R}_{+\gamma} \mathcal{K}. \]

The first factor is the product of the \( q \)-exponentials
\[ \mathcal{R}_{\gamma, m} = \exp_q ((q - q^{-1}) e_{\gamma + m\delta} \otimes f_{\gamma + m\delta}), \] (3.3)
where \( \gamma = \alpha, 2\alpha, m \in \mathbb{Z}_{\geq 0} \). Here we use the notation
\[ q^\gamma = q^{-(\gamma, \gamma)} \]
and understand the \( q \)-exponential as the series
\[ \exp_q(x) = 1 + x + \frac{x^2}{(2)_q!} + \ldots + \frac{x^n}{(n)_q!} + \cdots, \]
where
\[ (n)_q! = (n)_q (n - 1)_q \cdots (2)_q (1)_q, \quad (n)_q = \frac{q^n - 1}{q - 1}. \]
The order of the factors in $R_{\gamma,\delta}$ coincides with the chosen normal order of the roots $\gamma + m\delta$. The second factor is

$$R_{\gamma,\delta} = \exp \left( (q - q^{-1}) \sum_{m=0}^{\infty} b_m e_m \otimes f_m \right),$$

(3.4)

where

$$b_m = \frac{[m]_q}{m} (q^m - (-1)^m + q^{-m}).$$

(3.5)

Note here that

$$[e_{a+m\delta}, e_m h_q] = b_n e_{a+(m+n)\delta},$$

as it used to be for the untwisted case [19]. The factor $R_{\gamma,\delta}$ is the product of the $q$-exponentials

$$R_{\gamma,\delta,\mu} = \exp_q \left( (q - q^{-1}) e_{(\gamma-\mu)+m\delta} \otimes f_{(\delta-\gamma)+m\delta} \right),$$

(3.6)

where $\gamma = \alpha, 2\alpha$ and $m \in \mathbb{Z}_{\geq 0}$. The order of the factors in $R_{\gamma,\delta}$ coincides with the chosen normal order of the roots $(\delta - \gamma) + m\delta$. Finally, for the factor $K$, we have the expression

$$K = \exp(\hbar h_\theta \otimes h_\theta).$$

4. Finite-dimensional representation

Given $\xi \in \mathbb{C}^+$, define the three-dimensional representation $\varphi_\xi$ of the quantum group $U_h(g'(A))$ by the relations

$$\varphi_\xi(e_a) = \xi^a (E_{12} + E_{23}), \quad \varphi_\xi(e_{b-2a}) = \xi^{-a} ([2]_q)^{1/2} E_{13},$$

(4.1)

$$\varphi_\xi(h_a) = E_{11} - E_{33}, \quad \varphi_\xi(h_{b-2a}) = -2E_{11} + 2E_{33},$$

(4.2)

$$\varphi_\xi(f_a) = \xi^{-a} (E_{21} + E_{32}), \quad \varphi_\xi(f_{b-2a}) = \xi^{-a} ([2]_q)^{1/2} E_{13},$$

(4.3)

where $s_i, i = 0, 1$ are some integers, $i = 0, 1$, and the $3 \times 3$ matrix units $E_{ij}$ are defined as

$$(E_{ij})_{mn} = \delta_{im} \delta_{jn}.$$ 

The basic property following from this definition and used in what follows is given by the relation

$$E_{ij} E_{kl} = \delta_{jk} E_{il}.$$ 

Note that $\varphi_\xi(h_{b-2a}) + 2\varphi_\xi(h_a) = 0$. If we have an expression for $\varphi_\xi(a)$, where $a \in U_h(g'(A))$, in order to obtain the expression for $\varphi_\xi(g(a))$, we should simply take the transpose of $\varphi_\xi(a)$, also changing the deformation and spectral parameters as $q$ to $q^{-1}$ and $\xi$ to $\xi^{-1}$. We denote this operation by $\Omega_3$, where the index corresponds to the rank of the matrix it acts on. We will use similar operations for matrices of different ranks.

Now, using the recursive relations for the higher root vectors as given above and equations (4.1)–(4.3), we obtain

$$\varphi_\xi(e_{a+m\delta}) = q^{-m} \xi^{s_1+m} \left( (-1)^m E_{12} + q^{-m} E_{23} \right),$$

(4.4)

$$\varphi_\xi(f_{a+m\delta}) = q^m \xi^{-s_1-m} \left( (-1)^m E_{21} + q^m E_{32} \right),$$

(4.5)

$$\varphi_\xi(e_{b-2a+m\delta}) = -q^{-m} \xi^{s_1+m} \left( (-1)^{m+1} E_{21} + q^{-m-2} E_{32} \right),$$

(4.6)

7 Here and in what follows, we use instead of the integers $s_0$ and $s_1$ the integers $s = s_0 + 2s_1$ and $s_1$. 
This allows us to pass to unprimed quantities via the relation
\[ \varphi \zeta (x) = \varphi \zeta (e^\delta x) \]

where \( m = 0, 1, 2, \ldots \). For the primed positive imaginary root vectors, we have the following expressions:
\[ \varphi \zeta (e^m_\delta) = q^{-m+1} \zeta^{-m}(-1)^m E_{11} + [(1)^m q^{-m-1} - q^{-m}] E_{22} + q^{-m} E_{33}, \]
\[ \varphi \zeta (f^m_\delta) = q^{-m+1} \zeta^{-m}(-1)^m E_{11} + [(1)^m q - q^{-m}] E_{22} + q^{-m} E_{33}. \]
This allows us to pass to unprimed quantities via the relation
\[ (q - q^{-1}) \varphi \zeta (e^\delta x) = \varphi \zeta (\log[1 + (q - q^{-1}) e^\delta x]), \]

where
\[ \varphi \zeta (e^\delta_\delta) = \sum_{m=0}^\infty \varphi \zeta (e^{m_\delta}) x^{-m}, \quad \varphi \zeta (e^\delta x) = \sum_{m=0}^\infty \varphi \zeta (e^{m_\delta}) x^{-m}. \]

The corresponding expressions for the unprimed generators \( f^m_\delta \) can be found then by applying the Cartan anti-involution. After some calculations, we find
\[ \varphi \zeta (e^m_\delta) = \sum_{m=0}^\infty \varphi \zeta (e^{m_\delta}) x^{-m}, \quad \varphi \zeta (e^\delta x) = \sum_{m=0}^\infty \varphi \zeta (e^{m_\delta}) x^{-m}. \]

where \( m \) runs over the set of all positive integers.

5. R-matrix

In this section, we construct the R-matrix corresponding to the representation \( \varphi \zeta \) defined in the previous section. As usual, the most cumbersome part of the calculations is about the factor \( \varphi \zeta \otimes \varphi \zeta (R_{-\Lambda}) \). We obtain the following diagonal matrix:
\[ \varphi \zeta \otimes \varphi \zeta (R_{-\Lambda}) = e^{\lambda(\zeta)} - e^{-\lambda(\zeta)} \left[ E_{11} \otimes E_{11} + \frac{1 + q^{-1}_1\zeta_1}{1 - q^{-1}_1\zeta_1} E_{22} \otimes E_{22} + E_{33} \otimes E_{33} \right. \\
\left. \quad + \frac{1 - q^2_1\zeta_1}{1 - \zeta_1} (E_{11} \otimes E_{22} + E_{22} \otimes E_{33}) + \frac{1 - \zeta_1}{1 - q^{-2}_1\zeta_1} (E_{22} \otimes E_{11} + E_{33} \otimes E_{22}) \right. \\
\left. \quad + \frac{1 - q^2_1\zeta_1}{1 - \zeta_1} + \frac{1 + q^2_1\zeta_1}{1 + \zeta_1} E_{11} \otimes E_{33} + \frac{1 - \zeta_1}{1 - q^{-2}_1\zeta_1} + \frac{1 + q^{-1}_1\zeta_1}{1 + q\zeta_1} E_{33} \otimes E_{11} \right], \]
where
\[ \lambda(\zeta) = \sum_{m=0}^\infty \frac{1}{q^m - (-1)^m + q^{-m}} \zeta^m. \]
A useful property of the transcendental function \( \lambda \) is that
\[ \lambda(q\zeta) - \lambda(-\zeta) + \lambda(q^{-1}\zeta) = -\log(1 - \zeta). \]
Further, using equations (4.4), (4.5) and (4.8), (4.9), we see that \( \psi_{ci} \otimes \psi_{ci} (R_{\alpha,m}) \) commute with \( \psi_{ci} \otimes \psi_{ci} (R_{2\alpha,n}) \) for any \( m \) and \( n \), where \( R_{\alpha,m} \) and \( R_{2\alpha,n} \) are factors in \( R_{\alpha} \delta \) corresponding to the roots \( \alpha + m \delta \) and \( 2\alpha + (2n + 1)\delta \), respectively. Similarly, using equations (4.6), (4.7) and (4.10), (4.11), we see that \( \psi_{ci} \otimes \psi_{ci} (R_{\delta-2\alpha,m}) \) commute with \( \psi_{ci} \otimes \psi_{ci} (R_{\delta-a,n}) \) for any \( m \) and \( n \). Here \( R_{\delta-2\alpha,m} \) and \( R_{\delta-a,n} \) are factors present in \( R_{\alpha} \delta \) corresponding to the roots \( \delta - 2\alpha + 2m \delta \) and \( \delta - \alpha + m \delta \), respectively. Therefore, we can rearrange the factors entering \( \psi_{ci} \otimes \psi_{ci} (R_{\alpha} \delta) \) in the definition of the universal \( R \)-matrix, so that the factors corresponding to the roots \( \alpha + m \delta \) will come first, and then come the factors corresponding to the roots \( 2\alpha + (2n + 1)\delta \). Similarly, we can rearrange the factors entering \( \psi_{ci} \otimes \psi_{ci} (R_{\delta} \delta) \) in the definition of the universal \( R \)-matrix in such a way that the factors corresponding to the roots \( \delta - 2\alpha + 2m \delta \) come first, and only then come the factors corresponding to the roots \( \delta - \alpha + m \delta \). The same useful rearrangement turns out to be also valid for the matter of constructing the \( L \)-operators.

Hence, in the case under consideration, we can write

\[
\psi_{ci} \otimes \psi_{ci} (R_{\alpha}) = \psi_{ci} \otimes \psi_{ci} (R_{\alpha} R_{2\alpha}), \quad \psi_{ci} \otimes \psi_{ci} (R_{\delta} \delta) = \psi_{ci} \otimes \psi_{ci} (R_{\delta-2\alpha} R_{\delta-a}),
\]

where the factors entering this definition are given by the expressions

\[
\psi_{ci} \otimes \psi_{ci} (R_{\alpha}) = \prod_{m \geq 0} \exp_{q_{\alpha}} [(q - q^{-1}) \psi_{ci} (e_{\alpha + m \delta}) \otimes \psi_{ci} (f_{\alpha + m \delta})],
\]

\[
\psi_{ci} \otimes \psi_{ci} (R_{\delta-2\alpha}) = \prod_{m \geq 0} \exp_{q_{\alpha}} [(q - q^{-1}) \psi_{ci} (e_{\delta-2\alpha + 2m \delta}) \otimes \psi_{ci} (f_{\delta-2\alpha + 2m \delta})],
\]

and

\[
\psi_{ci} \otimes \psi_{ci} (R_{\delta-a}) = \prod_{m \geq 0} \exp_{q_{\alpha}} [(q - q^{-1}) \psi_{ci} (e_{\delta-a + m \delta}) \otimes \psi_{ci} (f_{\delta-a + m \delta})].
\]

Using the expressions for \( \psi_{ci} \otimes \psi_{ci} (R_{\alpha}), \psi_{ci} \otimes \psi_{ci} (R_{2\alpha}), \psi_{ci} \otimes \psi_{ci} (R_{\delta-2\alpha}), \psi_{ci} \otimes \psi_{ci} (R_{\delta-a}) \) presented in appendix A, we obtain by the respective matrix multiplication

\[
\psi_{ci} \otimes \psi_{ci} (R_{\alpha}) = 1 + \frac{q - q^{-1}}{1 - q^{2}z_{12}^{2i}} (E_{12} \otimes E_{23} + E_{23} \otimes E_{32})
\]

\[
+ \frac{q - q^{-1}}{1 + qz_{12}^{2i}} z_{12}^{2i} E_{12} \otimes E_{32} + \frac{q - q^{-1}}{1 + q^{-1}z_{12}^{2i}} z_{12}^{2i} E_{23} \otimes E_{21}
\]

\[
+ \frac{(q - q^{-1})(q - 1 + (q - q^{-1})q^{-1}z_{12}^{2i})}{(1 - z_{12}^{2i})(1 + q^{-1}z_{12}^{2i})} z_{12}^{2i} E_{13} \otimes E_{31}
\]

and

\[
\psi_{ci} \otimes \psi_{ci} (R_{\delta-2\alpha}) = 1 + \frac{(q - q^{-1})z_{12}^{2i-2n}}{1 - z_{12}^{2i}} (E_{21} \otimes E_{12} + E_{32} \otimes E_{23})
\]

\[
- \frac{(q - q^{-1})q^{2}z_{12}^{2i-2n}}{1 + qz_{12}^{2i}} E_{21} \otimes E_{23} - \frac{(q - q^{-1})q^{-2}z_{12}^{2i-2n}}{1 + q^{-1}z_{12}^{2i}} E_{32} \otimes E_{12}
\]

\[
+ \frac{(q - q^{-1})(q + q^{-1} + (q^{-1} - 1)q^{-1}z_{12}^{2i})}{(1 - z_{12}^{2i})(1 + q^{-1}z_{12}^{2i})} z_{12}^{2i} E_{31} \otimes E_{13}.
\]
For the simplest part of the calculations, the last factor in the definition of the universal \( R \)-matrix, we find the following diagonal matrix:

\[
\varphi_1 \otimes \varphi_1(\mathcal{K}) = qE_{11} \otimes E_{11} + E_{11} \otimes E_{22} + q^{-1}E_{11} \otimes E_{33} \\
+ E_{22} \otimes E_{11} + E_{22} \otimes E_{22} + E_{22} \otimes E_{33} \\
+ q^{-1}E_{33} \otimes E_{11} + E_{33} \otimes E_{22} + qE_{33} \otimes E_{33}.
\]

Now, multiplying all the factors in the given order, we finally obtain

\[
\varphi_1 \otimes \varphi_1(\mathcal{R}) = e^{\lambda(\varphi_1^{-1})-\lambda(\varphi_1)}R(\zeta_{12}),
\]

where

\[
R(\zeta) = qE_{11} \otimes E_{11} + \rho(\zeta)[E_{11} \otimes E_{22} + a(\zeta)E_{12} \otimes E_{21} \\
+ \Omega_9(a(\zeta)E_{12} \otimes E_{21}) + E_{12} \otimes E_{11}] \\
+ \sigma(\zeta)[E_{11} \otimes E_{33} + q\rho(\zeta)E_{12} \otimes E_{32} + c(\zeta)E_{13} \otimes E_{31} + b(\zeta)E_{23} \otimes E_{21} \\
+ \Omega_9(qb(\zeta)E_{12} \otimes E_{32} + c(\zeta)E_{13} \otimes E_{31} + b(\zeta)E_{23} \otimes E_{21}) \\
+ d(\zeta)E_{32} \otimes E_{22} + E_{33} \otimes E_{11} + \rho(\zeta)[E_{22} \otimes E_{33} + a(\zeta)E_{23} \otimes E_{32} \\
+ \Omega_9(a(\zeta)E_{23} \otimes E_{32}) + E_{33} \otimes E_{22}) + qE_{33} \otimes E_{33}.
\]

Here we use the notations

\[
a(\zeta) = \frac{(q - q^{-1})}{1 - \zeta^2}, \quad b(\zeta) = \frac{(q - q^{-1})}{1 + q^{-1}\zeta^2}, \\
c(\zeta) = \frac{(q - q^{-1})(q - 1 + (q + q^{-1})q^{-1}\zeta^s)}{(1 - \zeta^s)(1 + q^{-1}\zeta^s)}, \\
d(\zeta) = \frac{q + (q - 1)(q - q^{-1})q^{-1}\zeta^s - q^{-2}\zeta^2}{(1 - \zeta^s)(1 + q^{-1}\zeta^s)}, \\
\rho(\zeta) = \frac{1 - \zeta^s}{1 - q^{-1}\zeta^2}, \quad \sigma(\zeta) = \frac{q^{-1}(1 - \zeta^s)(1 + q^{-1}\zeta^s)}{(1 - q^{-2}\zeta^2)(1 + q^{-3}\zeta^2)}.
\]

Choosing \( s_0 = 1 \) and \( s_1 = 0 \), we recover the \( R \)-matrix presented in [19]. To compare with the original result of [14], one should apply certain similarity transformation and also adjust the respective parametrization, see, for example, \([20, 23]\). Here we have that

\[
\Omega_1(d(\zeta)) = d(\zeta), \quad \Omega_1(\rho(\zeta)) = q^{-2}\rho(\zeta), \quad \Omega_1(\sigma(\zeta)) = q^{-2}\sigma(\zeta),
\]

and we take into account that we have

\[
\Omega_9(E_{ij} \otimes E_{mn}) = E_{ij} \otimes E_{mn}.
\]

The explicit matrix form of the \( R \)-matrix for the considered representation is given in figure 1.

It is clear, in particular, that diagonalizing this matrix means an independent diagonalization of two of its \(2 \times 2\) sub-matrices with \( \rho \) and one \( 3 \times 3 \) block with \( \sigma \). Here we obtain three different eigenvalues \( \{q, q\frac{q^{-1}\zeta^t - 1}{q^{-1}\zeta^t}, q\frac{q^{-1}\zeta^t + 1}{q^{-1}\zeta^t}\} \), having multiplicities \( \{5, 3, 1\} \), respectively.

\[8 \text{ Compare with the untwisted case associated with } A_1^{(1)}, \text{ where one has only two different eigenvalues } \{1, \frac{q^{-1}\zeta^t - 1}{q^{-1}\zeta^t}\} \text{ with multiplicities } \{6, 3\}, \text{ respectively.} \]
The \( R \)-matrices corresponding to different values of \( s = s_0 + 2s_1 \) and \( s_1 \) are related by a change of the spectral parameter and a gauge transformation [9, 5]. In the case under consideration, we have the following relation\(^9\):

\[
R^{(s,s_1)}(\zeta_{12}) = [G(\zeta_1) \otimes G(\zeta_2)] R^{1,0}(\zeta_{12}) [G(\zeta_1) \otimes G(\zeta_2)]^{-1},
\]

where

\[
G(\zeta) = \begin{pmatrix} \zeta^{s_1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta^{-s_1} \end{pmatrix}
\]

and \( \zeta_{12} \) denotes the ratio \( \zeta_1 / \zeta_2 \).

Note finally that the expression for the \( R \)-matrix remarkably factorizes. Indeed, the expression for the universal \( R \)-matrix can be written as

\[
\mathcal{R} = (\mathcal{R}_{+\delta})(\mathcal{R}_{-\delta} \mathcal{K})(\mathcal{K}^{-1} \mathcal{R}_{-\delta} \mathcal{K}).
\]

Observing that the relation

\[
\psi_{\zeta_1} \otimes \psi_{\zeta_2} (\mathcal{K}^{-1} \mathcal{R}_{-\delta} \mathcal{K}) = \Omega_0 \circ \psi_{\zeta_1} \otimes \psi_{\zeta_2} (\mathcal{R}_{+\delta})
\]

holds in the case under consideration, we can thus write

\[
R = R_+ R_0 R_-,
\]

where we have denoted the upper-triangular, diagonal and lower-triangular factors as

\[
R_+ (\zeta_{12}) = \psi_{\zeta_1} \otimes \psi_{\zeta_2} (\mathcal{R}_{+\delta} \mathcal{R}_{2\delta}) = I + \frac{q - q^{-1}}{1 - \zeta_{12}^{s_1}} (E_{12} \otimes E_{21} + E_{23} \otimes E_{32})
\]

\[
+ \frac{q - q^{-1}}{1 + q \zeta_{12}^{s_1}} E_{12} \otimes E_{32} + \frac{q - q^{-1}}{1 + q^{-1} \zeta_{12}^{s_1}} E_{23} \otimes E_{21}
\]

\[
+ \frac{(q - q^{-1})(q - 1 + (q + q^{-1})q^{-1} \zeta_{12}^{s_1})}{(1 - \zeta_{12}^{s_1})(1 + q^{-1} \zeta_{12}^{s_1})} \xi_{12}^{s_1} E_{13} \otimes E_{31}.
\]

\(^9\) It is implied here that \( R(\zeta) = R^{(s,s_1)}(\zeta), R^{1,0}(\zeta_1) = R^{(s,0)}(\zeta) \), and a similar convention will be used for the \( L \)-operators.
$R_0(\xi_{12}) = e^{-i(q_1^{\xi_1}+i)q_1^{\xi_2}q_2^{\xi_2}} \varphi_{\xi_1} \otimes \varphi_{\xi_2}(R_{-\delta}K) = qE_{11} \otimes E_{11} + 1 - q^2 \xi_{12}^2 E_{11} \otimes E_{22} + \left(1 - q^2 \xi_{12}^2\right) q^{-1} E_{11} \otimes E_{33}
+ 1 - \xi_{12}^2 q^{-1} E_{22} \otimes E_{11} + 1 + q\xi_{12}^2 E_{22} \otimes E_{22} + 1 - q^2 \xi_{12}^2 E_{22} \otimes E_{33}
+ \left(1 - \xi_{12}^2\right) \left(1 + q^{-1} \xi_{12}^2\right) q^{-1} E_{33} \otimes E_{11}
+ 1 - \xi_{12}^2 q^{-1} E_{33} \otimes E_{22} + qE_{33} \otimes E_{33},

R_{-\delta}(\xi_{12}) = \varphi_{\xi_1} \otimes \varphi_{\xi_2}(K^{-1}R_{-\delta-\delta}R_{-\delta-\delta}K) = I + \frac{(q - q^{-1}) \xi_{12}^{\omega}}{1 - \xi_{12}^2} (E_{21} \otimes E_{12} + E_{32} \otimes E_{23}) - \frac{(q - q^{-1}) q^{\omega} \xi_{12}^{-1}}{1 + q\xi_{12}^2} E_{21} \otimes E_{23} - \frac{(q - q^{-1}) q^{-1} \xi_{12}^{-1}}{1 + q^{-1} \xi_{12}^2} E_{32} \otimes E_{12}
+ \frac{(q - q^{-1}) (q + q^{-1} + (q - 1) q^{-1} \xi_{12}^{-2}) \xi_{12}^{-1}}{(1 - \xi_{12}^2)(1 + q^{-1} \xi_{12}^2)} E_{31} \otimes E_{13},

respectively. Here we also have the Cartan anti-involution relations

$\Omega_0 (R_+ (\xi)) = R_- (\xi), \quad \Omega_0 (R_0 (\xi)) = q^{-2} R_0 (\xi),\n$

implying that

$\Omega_0 (R (\xi)) = q^{-2} R (\xi)$. 

6. The spin-chain Hamiltonian

The transfer matrix of the system based on $N$ sites is given by the relation

$T (\xi_1, \ldots, \xi_N) = tr_{0} (R_{01} (\xi / \xi_1) R_{02} (\xi / \xi_2) \cdots R_{0N} (\xi / \xi_N)),$

where the string of $R_{0i} (\xi / \xi_i)$ acts in $V_0 \otimes V_1 \otimes \cdots \otimes V_N$ and the trace is taken over $V_0$ in the given representation. The corresponding Hamiltonian is defined by the formula

$H = T^{-1} (\xi) \frac{dT (\xi)}{d\xi} \bigg|_{\xi=1},$

where $T (\xi) = T (\xi [1, \ldots, 1])$, and allows one to obtain the ground-state energy of the system, see, for example, [31, 32].

Direct calculations show that the Hamiltonian in the case under consideration is a sum of four terms,

$\tilde{H} = H_{(12)} + H_{(23)} + H_{(31)} + H_{(123)},$

where

$H_{(12)} = -\frac{1}{q - q^{-1}} \sum_{l=1}^{N} \left[ E_{12}^{(l)} E_{21}^{(l+1)} + E_{21}^{(l)} E_{12}^{(l+1)} - q^{-1} E_{11}^{(l)} E_{22}^{(l+1)} - q E_{22}^{(l)} E_{11}^{(l+1)} \right],

H_{(23)} = -\frac{1}{q - q^{-1}} \sum_{l=1}^{N} \left[ E_{23}^{(l)} E_{32}^{(l+1)} + E_{32}^{(l)} E_{23}^{(l+1)} - q^{-1} E_{22}^{(l)} E_{33}^{(l+1)} - q E_{33}^{(l)} E_{22}^{(l+1)} \right].$

Also a certain twist operator can be introduced in the definition of the transfer matrix corresponding to specific boundary conditions.
\[ H_{(31)} = -\frac{1}{q-q^{-1}} \left[ \frac{3}{2} \gamma_2^2 \sum_{l=1}^{N} \left( E_{33}^{(l)} E_{13}^{(l+1)} + E_{13}^{(l)} E_{31}^{(l+1)} \right) \right. \\
\left. - \left( q^{-1} + q(q-1) \right) E_{33}^{(l)} E_{11}^{(l+1)} - (q - q^{-2})(q-1)) E_{11}^{(l)} E_{33}^{(l+1)} \right] , \]

\[ H_{(123)} = -\frac{3}{2} \gamma_2 \sum_{l=1}^{N} \left[ q^{1/2} (q E_{12}^{(l)} E_{32}^{(l+1)} - q^{-1} E_{32}^{(l)} E_{12}^{(l+1)}) \right. \\
\left. + q^{-1/2} (q E_{23}^{(l)} E_{12}^{(l+1)} - q^{-1} E_{12}^{(l)} E_{23}^{(l+1)}) - (q^{1/2} - q^{-1/2}) (2\gamma_2 E_{22}^{(l)} E_{22}^{(l+1)}) \right] \]

in the standard notation, with \( s_j \) fixed as \( s_0 = 1, s_1 = 0 \).

Recall that for the untwisted case, i.e. the system associated with \( A_{n-1}^{(1)} \), we have the Hamiltonian also expressed in terms of the respective matrix units,

\[ H_n = -\frac{1}{q-q^{-1}} \left[ \sum_{i,j=1}^{N} E_{ij}^{(l)} E_{ji}^{(l+1)} - q^{-1} \sum_{i<j} E_{ii}^{(l)} E_{jj}^{(l+1)} - q \sum_{i>j} E_{ij}^{(l)} E_{ji}^{(l+1)} \right] . \]

In the best studied cases, \( n = 2 \) and \( n = 3 \), the Hamiltonian can be represented by means of the generators of the first fundamental representation of the corresponding finite-dimensional algebras, \( A_1 \) and \( A_2 \), respectively,

\[ H_2 = -\frac{1}{q-q^{-1}} \left[ \sum_{l=1}^{N} E_{ii}^{(l)} F_{ii}^{(l+1)} + F_{ii}^{(l)} E_{ii}^{(l+1)} - q + q^{-1} \right] + \frac{1}{4} (H^{(1)} - H^{(N+1)}) - \frac{N q + q^{-1}}{4 q - q^{-1}} , \]

\[ H_3 = -\frac{1}{q-q^{-1}} \left[ \sum_{l=1}^{N} \left( E_{ii}^{(l)} F_{ii}^{(l+1)} + E_{ii}^{(l)} F_{ii}^{(l+1)} + E_{ii}^{(l)} F_{ii}^{(l+1)} + F_{ii}^{(l)} E_{ii}^{(l+1)} + F_{ii}^{(l)} E_{ii}^{(l+1)} \right. \right. \\
\left. \left. + F_{ii}^{(l)} E_{ii}^{(l+1)} + \frac{q + q^{-1}}{3} (H_{ii}^{(l)} H_{ii}^{(l+1)} - H_{ii}^{(l)} H_{ii}^{(l+1)}) + \frac{q}{3} H_{ii}^{(l)} H_{ii}^{(l+1)} \right) - \frac{1}{3} (H_{ii}^{(l)} - H_{ii}^{(N+1)} + H_{ii}^{(l)} - H_{ii}^{(N+1)}) + \frac{N q + q^{-1}}{3 q - q^{-1}} \right] . \]

Here we kept the boundary terms and the constant terms.

Now, comparing these expressions with what we have obtained for the twisted \( A_2^{(1)} \) case, we see that the Hamiltonian of the system under consideration is the sum of two pure \( A_1^{(1)} \)-type systems’ Hamiltonians \( H_{(12)} \) and \( H_{(23)} \), related to the indices (12) and (23), one more somewhat shifted \( A_1^{(1)} \)-type system’s Hamiltonian \( H_{(31)} \) and an essentially \( A_2^{(2)} \) addition presented explicitly by the part \( H_{(123)} \). Here, we cannot express the Hamiltonian \( \tilde{H} \) in terms of matrices representing the algebra generators, which is in contrast with the untwisted cases.

### 7. L-operators in q-oscillator representation

A useful object to investigate the properties of an integrable system is the corresponding \( Q \)-operator. According to the modern approach, it is constructed as the trace of some monodromy-type operator constructed, in turn, from an \( L \)-operator. Here the \( L \)-operator is obtained by taking one of the factors of the universal \( R \)-matrix in an infinite-dimensional representation. Usually, it is some \( q \)-oscillator representation [4, 2, 5].
In the case under consideration, $\mathcal{R}$ is an element of the tensor product of the Borel subalgebras of the quantum group $U_h(\mathfrak{g}(A))$, and thus we have

$$\mathcal{R} \in U_h(\mathfrak{b}_+^-(A)) \otimes U_h(\mathfrak{b}_-^+(A)) \subset U_h(\mathfrak{g}(A)) \otimes U_h(\mathfrak{g}(A)).$$

The two Borel subalgebras here are unital associative algebras generated by the elements $h_i, e_i$ and $h_i, f_i$, respectively. To construct an $L$-operator, it therefore suffices to have representations of the Borel subalgebras.

### 7.1. Resolving the Serre relations

In this section, we construct $L$-operators based on the $q$-deformed oscillator algebra defined as an associative algebra $\text{Osc}_h$ with generators $a, a^\dagger$ and $D$ subject to the relations\(^{11}\)

\[
[D, a] = -a, \quad [D, a^\dagger] = a^\dagger, \\
aa^\dagger = 1 - q^2q^{-D}, \quad a^\dagger a = 1 - q^{2D}.
\]

The transformations

\[
a \rightarrow \kappa a q^{-D}, \quad a^\dagger \rightarrow \kappa^{-1}q^{-5D}a^\dagger, \quad D \rightarrow D
\]

form a two-parametric group of automorphisms of $\text{Osc}_h$. One can use these transformations to obtain different $L$-operators; however, the trace used in the definition of $Q$-operators is invariant with respect to the action of this automorphism group [4, 7].

To construct $L$-operators, we have to consider the representations $\chi_\xi$ and $\psi_\xi$ of $U_h(\mathfrak{b}_+^-(A))$ and $U_h(\mathfrak{b}_-^+(A))$, respectively, in addition to the representation $\phi_\xi$ already described in section 3. Slightly more abstractly, we will use the homomorphisms $\chi_\xi$ and $\psi_\xi$ of $U_h(\mathfrak{b}_+^-(A))$ and $U_h(\mathfrak{b}_-^+(A))$ to $\text{Osc}_h$. It is easy to switch to representations, when necessary, using the well-known representations of $\text{Osc}_h$.

By an $L$-operator of type $\hat{L}$, we denote an element of $\text{End}(\mathbb{C}^3) \simeq \text{Mat}_3(\text{Osc}_h)$ defined as

$$\hat{L}(\xi; \omega) = \chi_\xi \otimes \psi_\xi(R).$$

Here, we have assumed that the homomorphisms $\chi_\xi$ and $\psi_\xi$ are such that they result in an $L$-operator depending only on $\xi; \omega$. It follows from the Yang–Baxter equation (3.1) that the $L$-operators of type $\hat{L}$ should satisfy the equation

$$R_{23}(\xi_1; \omega_1) \hat{L}_{12}(\xi_2; \omega_2) = \hat{L}_{12}(\xi_2; \omega_2) R_{23}(\xi_1; \omega_1).$$

As an $L$-operator of type $\hat{L}$, we define an element of $\text{End}(\mathbb{C}^3) \otimes \text{Osc}_h \simeq \text{Mat}_3(\text{Osc}_h)$,

$$\hat{L}(\xi_1; \omega_1) = \psi_\xi \otimes \psi_\xi(R),$$

assuming again that the homomorphisms $\psi_\xi$ and $\psi_\xi$ are such that they result in a dependence on the ratio $\xi_1/\xi_2$. Using the Yang–Baxter equation (3.1), we derive the following equation for the $L$-operators of type $\hat{L}$:

$$R_{12}(\xi_1; \omega_1) \hat{L}_{13}(\xi_2; \omega_2) = \hat{L}_{13}(\xi_2; \omega_2) R_{12}(\xi_1; \omega_1).$$

In terms of the matrices $\hat{R}$ and $\tilde{R}$ defined by means of the matrix of the permutation operator $P_{12}$ as

$$\hat{R}(\xi) = R(\xi)P, \quad \tilde{R}(\xi) = PR(\xi),$$

\(^{11}\) Instead of the naturally $q$-deformed oscillators, defined by the relations $[N', b] = -b, [N', b^\dagger] = b^\dagger, bb^\dagger = [N'+1]_q$ and $b^\dagger b = [N]_q$, we use slightly different objects used earlier in [6–8]. The latter are related to the former as $D = N'$, $a = -(q - q^{-1})bq^{N'}$ and $a^\dagger = b^\dagger$.\]
equations (7.2) and (7.3) take the forms
\[ \hat{R}(\zeta_{12})(\hat{L}(\zeta_1) \boxtimes \hat{L}(\zeta_2)) = (\hat{L}(\zeta_2) \boxtimes \hat{L}(\zeta_1))\hat{R}(\zeta_{12}) \]
and
\[ \hat{R}(\zeta_{12})(\hat{L}(\zeta_1) \boxtimes \hat{L}(\zeta_2)) = (\hat{L}(\zeta_2) \boxtimes \hat{L}(\zeta_1))\hat{R}(\zeta_{12}), \]
respectively, where \( \boxtimes \) means a generalization of the Kronecker product to the matrices with arbitrary algebra-valued entries [11, 5].

Remember that the Borel subalgebra \( U_b(\mathfrak{b}'_+(A)) \) is generated by the elements \( h_i \) and \( e_i \), while the dual Borel subalgebra \( U_b(\mathfrak{b}'_-(A)) \) is generated by \( h_i, f_i, i = 0, 1 \). Here, the corresponding Serre relations (2.10) and (2.11) are explicitly of the form
\[
e^a \varepsilon_{\zeta-2\alpha} - (q^2 + q^{-2})e_{\zeta-2\alpha}e_a + e_{\zeta-2\alpha}^2 e_a = 0,
\]
\[
e^a \varepsilon_{\zeta-2\alpha} - [5]_{q^{1/2}}^1 e_a^d e_{\zeta-2\alpha}^d + [4]_{q^{1/2}}^1 [5]_{q^{1/2}}^1 (e_a^3 e_{\zeta-2\alpha}^2 - e_a^2 e_{\zeta-2\alpha}^3 - e_a^2 e_{\zeta-2\alpha}^3 = 0,
\]
and we have the same equations for \( f_{\zeta-2\alpha} \) and \( f_a \). Besides, the Cartan generators have to satisfy equations (2.13) and the condition \( h_{\zeta-2\alpha} + 2h_\alpha = 0 \).

To fulfill the defining relations of \( U_b(\mathfrak{b}'_-(A)) \), including also the corresponding Serre relations, we use the homomorphism \( \chi \) defined by the equations
\[
\chi(h_{\zeta-2\alpha}) = 2D, \quad \chi(h_\alpha) = -D, \quad \chi(e_a) = \mu_0 a q^2 D, \quad \chi(f_a) = \mu_1 a q^D,
\]
with free parameters \( \mu_0, \mu_1, \) and \( v \). The \( \zeta \)-dependent homomorphism \( \chi_\zeta \) can be defined by the same procedure used earlier for the homomorphism \( \varphi_\zeta \). Here, changing the parameter \( \mu_0 \), we always change the coefficient at \( \zeta \). Note that the parameters \( \mu_0, \mu_1, v \) can be freely changed by the transformations (7.1).

Analogously, to satisfy the defining relations of \( U_b(\mathfrak{b}'_+(A)) \), meaning also the corresponding Serre relations, we use the homomorphism \( \psi \) defined by the equations
\[
\psi(h_{\zeta-2\alpha}) = -2D, \quad \psi(h_\alpha) = D, \quad \psi(e_a) = \mu_0 a q^{2D}, \quad \psi(f_a) = \mu_1 a q^D,
\]
where \( \mu_0, \mu_1, \) and \( v \) are again free parameters. The \( \zeta \)-dependent homomorphism \( \psi_\zeta \) can be defined by the same procedure used earlier for the homomorphism \( \varphi_\zeta \). Changing the parameter \( \mu_0 \), we always change the coefficient at \( \zeta \), and the parameters \( \mu_0, \mu_1, \) can be freely changed by the transformations (7.1). In both cases, the \( L \)-operators corresponding to the different values of \( \mu_0, \mu_1, \) and \( v \) are equivalent.

7.2. \( L \)-operators of type \( \hat{L} \)

Thus, to construct \( L \)-operators of type \( \hat{L} \), we use the homomorphism \( \chi_\zeta \) from \( U_b(\mathfrak{b}'_-(A)) \) to \( \text{Osc}_b \) defined by the following relations\(^{12}\):
\[
\chi_\zeta(h_{\zeta-2\alpha}) = 2D, \quad \chi_\zeta(h_\alpha) = -D, \quad \chi_\zeta(e_a) = \frac{1}{(q-1)\sqrt{12}} a q^{-2D} \zeta^{n_\alpha}, \quad \chi_\zeta(f_a) = \frac{1}{q-q^{-1}} a \zeta^{n_\alpha}.
\]

\(^{12}\) As above, also in what follows the more convenient combination of the integers \( s = s_0 + 2s_1 \) and \( s_1 \) will be used instead of the initial ones, \( s_0 \) and \( s_1 \).
This corresponds to the choice of the free parameters $\mu_0 = (q - 1)^{-1}(2[2])^{-1/2}$, $\mu_1 = 1/(q - q^{-1})$ and $\nu = 0$.

The higher root vectors are defined according to the recursive relations (2.15)–(2.19). Applying the homomorphism $\chi_\ell$, we subsequently obtain

$$\chi_\ell(e_m^\prime) = \frac{q^m \zeta^{ms}}{(q - 1)(q - q^{-1})} \left[(1 - q^{2m-1}) - (1 - q^{2m+1})q^{2D}y^{2(m-1)D}\right],$$

which gives

$$\chi_\ell(e_m^\prime) = -\frac{q^m \zeta^{ms}}{m(q - q^{-1})} \left(1 - \frac{m}{|m|_q}b_m q^m q^{-2mD}\right),$$

where the quantities $b_m$ are given explicitly by (3.5). Taking also the expression for $\varphi_\ell(f_{m\delta})$ from equation (4.15), we obtain for the image of the factor $R_{-\delta}$ the following expression:

$$\chi_\ell \otimes \varphi_\ell(R_{-\delta}) = e^{1(-q)^{R_{-\delta}}} \left((1 + q^2 q_2^{D})E_{11} + \frac{(1 + q^2 q_2^{D})}{1 + q^4 q_2^{D}}E_{22} + \frac{(1 + q^3 q_2^{D})}{1 + q^5 q_2^{D}}E_{33}\right).$$

The simplest part of the calculations is, as usual, given by the operator $K$. In our case, it is represented by a diagonal matrix of the form

$$\chi_\ell \otimes \varphi_\ell(K) = q^{-D}E_{11} + E_{22} + q^D E_{33}.$$

As we already did for the $R$-matrix in section 5, we can rearrange the factors entering $\chi_\ell \otimes \varphi_\ell(R_{-\delta})$ in the definition of the universal $R$-matrix in such a way that the factors corresponding to the roots $\alpha + m\delta$ come first, and only then come the factors corresponding to the roots $2\alpha + (2m + 1)\delta$. Similarly, concerning the factor $\chi_\ell \otimes \varphi_\ell(R_{+\delta})$, we rearrange the factors entering there in such a way that the ones corresponding to the roots $\delta - 2\alpha + 2m\delta$ come first, and finally come those objects corresponding to $\delta - \alpha + m\delta$.

Altogether, the relations derived in appendix B allow us to write down expressions corresponding to the factors $R_{-\delta} = R_{a} R_{2a}$ and $R_{+\delta} = R_{3-2a} R_{3-a}$. We obtain

$$\chi_\ell \otimes \varphi_\ell(R_{-\delta}) = I + \zeta_1^{\alpha} a(1 + q^2 q_2^{D})^{-1}E_{21} + \zeta_1^{\alpha} a(1 - q^3 q_2^{D})^{-1}E_{32} + \frac{q}{q + 1} \zeta_2^{2\alpha} a^2(1 + q^3 q_2^{D})^{-1}E_{31}$$

and

$$\chi_\ell \otimes \varphi_\ell(R_{+\delta}) = I - \frac{q - q^{-1}}{q - 1} q^2 \zeta_2^{\alpha} a(1 + q^4 q_2^{D})^{-1}E_{12} + \frac{q - q^{-1}}{q - 1} q^2 \zeta_2^{\alpha} a(1 - q^5 q_2^{D})^{-1}E_{23} + \frac{q - q^{-1}}{q - 1} \zeta_2^{2\alpha} a^2 q^{-2D}(1 + q^4 q_2^{D})^{-1}E_{13}.$$

Multiplying the above presented factors in the given order, $R_{-\delta} R_{+\delta} R_{+\delta} K$, we finally obtain the $L$-operator. According to our terminology, the corresponding object is called an $L$-operator of type $\hat{L}$, and therefore, we will use the respective notation. In matrix form, we have

$$\hat{L}(\xi) = e^\lambda(-q^{\xi}) \left(\begin{array}{ccc} q^{-D} + q^2 q^2 q^{-D} & -(q + 1)q^{\xi^{r-n}a^\dagger} & \frac{q + 1}{q} q^{\xi^{r-n}a} q^{2\alpha} q^{-D} \\ q^{\xi^{r-n}a} q^D & 1 - q^\xi a^\dagger & \frac{q + 1}{q} q^{\xi^{r-n}a^\dagger} q^{-D} \\ \frac{q}{q + 1} q^{2\alpha} a^2 q^{-D} & q^{\xi a} & q^D + q^\xi q^{-D} \end{array}\right).$$
We have the following relation between the $L$-operators with arbitrary values of the parameters and fixed ones:
\[
\hat{L}^{(r,s_1)}(\xi_{12}) = \gamma_r(G(\xi_2)\hat{L}^{(s_1)}(\xi_{12})G^{-1}(\xi_2)),
\]
where the matrix $G(\xi)$ was defined earlier by (5.2), while the mapping $\gamma_r$, $\xi \in \mathbb{C}^*$, is defined by the relations
\[
\gamma_r(a) = a\xi^{-s_1}, \quad \gamma_r(a^\dagger) = a^\dagger \xi^{s_1}, \quad \gamma_r(D) = D.
\]
It means, in particular, that the $Q$-operators based on $L$-operators with different values of $s$ and $s_1$ are related by a change of the spectral parameters and a similarity transformation.

Applying to $\hat{L}$ the automorphism $\sigma$ generated by the transformation
\[
a^\dagger \to a\xi^{-D}, \quad a \to -q^{-D}a^\dagger, \quad D \to -D - 1,
\]
we obtain another $\hat{L}$-operator of type $\hat{L}_\sigma$,
\[
\sigma : \hat{L} \to \hat{L}_\sigma,
\]
where
\[
\hat{L}_\sigma(\xi) = e^{\lambda(-q\xi^s)}\hat{L}_\sigma(\xi),
\]
and we have used the notation
\[
\hat{L}_\sigma(\xi) = \begin{pmatrix}
qD + \xi^sq^{-D} & -(q + 1)q\xi^{s-s_1}aq^{-D} & (q + 1)q\xi^{s-2s_1}a^2q^{-D} \\
-\xi^{s_1}a^\dagger & 1 - q\xi^s & (q + 1)\xi^{s-s_1}a^\daggerq^{-D} \\
q^{-1}q^{2s_1}a^\dagger q^{-D} & -q^{-1}\xi^{s_1}a^\dagger q^{-D} & q^{-1}(q^{-D} + q^2\xi^sD^s)
\end{pmatrix}.
\]

Note also that an object similar to $\hat{L}_\sigma$ was used in [12] in an attempt to describe so-called defects in affine Toda field theory.

7.3. Type $\hat{L}$ and some useful relations

The $\hat{L}$-operators of type $\hat{L}$ can be constructed by means of the homomorphisms $\psi_\xi$ and $\psi_\xi$. This requires us to map the Borel subalgebra $U_h(b'_+(A))$ to the finite-dimensional matrix representation, while realizing the Borel subalgebra $U_h(b'_-(A))$ in $\text{Osc}_h$. Here we consider the following homomorphism for the generators $f_{a_0}, f_{a_1}$ and $h_{a_0}, h_{a_1}$:
\[
\psi_\xi(f_{a+2}) = \frac{1}{(q - 1)\sqrt{[2]_q}}a^{2}q^{-2D}\xi^{-s_0}, \quad \psi_\xi(f_u) = \frac{1}{q - q^{-1}}a\xi^{-s_1}
\]
and
\[
\psi_\xi(h_{a-2}) = -2D, \quad \psi_\xi(h_u) = D.
\]

Using the Cartan anti-involution, we obtain from the recursive relations (2.15)–(2.19) recursive relations for the higher root vectors spanning the whole $U_h(b'_-(A))$. Performing the whole procedure in the same way as in the preceding subsection, we obtain the following matrix form of the $\hat{L}$-operator of type $\hat{L}$:
\[
\hat{L}(\xi) = e^{\lambda(-q\xi^s)}\begin{pmatrix}
qD + \xi^sq^{-D} & \xi^{s_1}a & q + 1\xi^{s-s_1}a^2q^{-D} \\
q^{-1}q^{2s_1}a^\dagger q^{-D} & 1 - q\xi^s & \xi^{s_1}aq^{-D} \\
n+1q^{-1}\xi^{s-2s_1}a^\dagger q^{-D} & -(q + 1)q\xi^{s-s_1}a^\dagger & q^{-D} + q^2\xi^sD^s
\end{pmatrix}.
\]

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Here, to bring the scalar factor to the given simple form, we have used the relation
\[
e^{-\lambda(q^4 \zeta)} \frac{1}{(1 + q^2 \zeta)(1 - q^3 \zeta)} = \hat{e}^{\lambda(-q \zeta^4)}
\]
(7.7)
following from the basic equation (5.1) for the function \( \lambda \). For the \( L \)-operators of type \( \hat{L} \), we have the following relation:
\[
\hat{L}^{(\sigma, s)}(\zeta_{12}) = G(\zeta_1)g(\zeta)(\hat{L}^{(1, 0)}(\zeta_{12}))G^{-1}(\zeta_1).
\]

Now, comparing the explicit forms of the \( L \)-operators of the type \( \hat{L} \) and \( \hat{L} \), we see that they are related as
\[
\hat{L}(\zeta) = JL(\zeta)J,
\]
where \( J \) is the \( 3 \times 3 \) skew-diagonal unit matrix\(^{13} \). Besides, for the matrices \( \hat{R} \) and \( \hat{R} \) introduced by equation (7.4), we obtain the following relation:
\[
\hat{R} = (J \otimes J)\hat{R}(J \otimes J).
\]

Also the inverse of the \( L \)-operator of type \( \hat{L} \) at \( \zeta^{-1} \) should be an \( L \)-operator of type \( \hat{L} \). Explicitly we have
\[
\hat{L}^{-1}(\zeta) = \frac{e^{-\lambda(-q \zeta^4)}}{(1 + q^2 \zeta^4)(1 - q^3 \zeta^4)} \hat{L}(\zeta),
\]
where we have denoted
\[
\hat{L}(\zeta) = \begin{pmatrix}
q^D - q^3 \zeta q^\frac{D}{q} & (q + 1)q^3 \zeta^{r+s} a q^{-D} a^1 & -(q + 1)q^3 \zeta^{r-2s} a q^{-D} a^{12} \\
-q^s a & 1 + q^4 \zeta^3 & -(q + 1)q^3 \zeta^{r-s} a^1 \\
q^{-1} q^2 \zeta q^{-D} a^2 & -q^s q^{-D} a & q^{-D} - q^5 \zeta q^D
\end{pmatrix}.
\]
Again, using equation (7.7), one can simplify the form of the scalar factor at \( \hat{L} \):

Note finally that, if \( L(\zeta) \) is an \( L \)-operator of type \( \hat{L} \), then \( \tau(\hat{L}(\zeta^{-1})) \) is an \( L \)-operator of type \( \hat{L} \), and conversely, if \( \hat{L}(\zeta) \) is an \( L \)-operator of type \( \hat{L} \), then \( \tau(\hat{L}(\zeta^{-1})) \) is an \( L \)-operator of type \( L \), where \( \tau \) means the anti-involution of the \( q \)-oscillator algebra \( \text{Osc}_h \) defined by the relations
\[
\tau(a) = a^\dagger, \quad \tau(a^\dagger) = a, \quad \tau(D) = D.
\]
Besides, applying to \( L \) the automorphism \( \sigma \) described in the preceding sub-section, one can obtain another \( L \)-operator of type \( \hat{L} \).

8. Concluding remarks

We have constructed the \( R \)-matrix for the twisted Kac–Moody algebra of type \( A_2^{(2)} \), which is given by the universal \( R \)-matrix under the action of a twisted evaluation homomorphism. The latter allowed us to bring the dual Borel subalgebras to a finite-dimensional matrix representation. We have found an expression for the respective spin-chain Hamiltonian. We have also constructed \( L \)-operators realizing one of the Borel subalgebras in the \( q \)-deformed oscillator algebra.

We have seen, in particular, that the \( R \)-matrix and \( L \)-operators have two points of degeneracy in this twisted case in contrast to the earlier considered untwisted cases where we had only one such point.

\(^{13} \) This expression \( J \otimes J \) implies actually the superposition of two operations, where one has to take first the usual transposition of \( x \) with respect to its main diagonal, and then the transposition of the obtained matrix with respect to the skew diagonal.
Recall that one usually considers a useful decomposition of the $R$-matrix in the form

$$R(\zeta) = zR_0 - z^{-1}R_0^{-1},$$

where the non-degenerate matrix $R_0$ does not depend on the spectral parameter $z$. Instead, taking, for simplicity $s_0 = 1$, $s_1 = 0$, so that $s = 1$, we will have another relation,

$$R(\zeta) = \Lambda(\zeta) \left[ q^{1/2}(\zeta^{-1}r_1 - r_0) - \Omega_0 \left( q^{1/2}(\zeta^{-1}r_1 - r_0) \right) \right],$$

where the upper-triangular matrices $r_0$ and $r_1$ do not depend on $\zeta$ and the scalar pre-factor $\Lambda$ is a rational function subject to the condition

$$\Omega_1(\Lambda(\zeta)) = -q^{-2}\Lambda(\zeta).$$

Here we explicitly have

$$r_0 = E_{11} \otimes E_{11} + qE_{11} \otimes E_{22} + E_{11} \otimes E_{33}
+ q(E_{22} \otimes E_{11} + (1 - (q - q^{-1}))E_{22} \otimes E_{22} + E_{22} \otimes E_{33})
+ E_{33} \otimes E_{11} + qE_{33} \otimes E_{22} + E_{33} \otimes E_{33}
- (q - q^{-1})(q^{-2}E_{12} \otimes E_{21} - qE_{12} \otimes E_{32} - E_{23} \otimes E_{21})
+ (q + q^{-1})qE_{13} \otimes E_{31} + q^{-2}E_{23} \otimes E_{32}).$$

$$r_1 = q^2E_{11} \otimes E_{11} + qE_{11} \otimes E_{22} + E_{11} \otimes E_{33}
+ q(E_{22} \otimes E_{11} + E_{22} \otimes E_{22} + E_{22} \otimes E_{33})
+ E_{33} \otimes E_{11} + qE_{33} \otimes E_{22} + q^2E_{33} \otimes E_{33}
+ (q - q^{-1})qE_{12} \otimes E_{21} + qE_{12} \otimes E_{32} + E_{23} \otimes E_{21}
+ (q - 1)E_{13} \otimes E_{31} + qE_{23} \otimes E_{32})$$

and

$$\Lambda(\zeta) = \frac{q^{-3/2}\zeta}{(1 - q^{-2}\zeta)(1 + q^{-3}\zeta)}.$$

We have also constructed $L$-operators of two types, $\hat{L}$ and $\bar{L}$, in the $q$-oscillator representation. They satisfy Yang–Baxter equations with the $R$-matrices $R$ and $\bar{R}$, respectively. Here, the $L$-operators and $R$-matrices of different types are related by similarity transformations.

The $L$-operators allow for the decomposition of the usual form $\zeta L_+ - \zeta^{-1}L_-$, where the constituents $L_+$ and $L_-$ are presented by non-degenerate matrices. The latter differs from the untwisted case, where one of two such parts, $L_+$ or $L_-$, turned out to be degenerate [5]. However, to also find for the $L$-operators a decomposition similar to what we have for the $R$-matrix, one needs to define an analog of the Cartan anti-involution in the $q$-oscillator algebra.

An interesting and quite non-trivial question in a lattice model is about continuum analogs of any monodromy-type matrices. In a reasonable classical limit, one should anticipate that the corresponding objects, in terms of suitable phase space variables, would have certain fundamental Poisson brackets with the classical $r$-matrix. This problem, in a somewhat reversed form, was formulated in the pioneering papers on the quantum inverse scattering method, see, for example, [28, 15]. Also in our case, it will be the subject of further investigation.

Another special question concerns the quasi-classical limit. If we evaluate our $L$-operators naively at $\hbar \to 0$, where $q \to 1$ and $\zeta \to 1$, we obtain a rather meaningless degenerate matrix. Instead, we could follow an idea exploited e.g. in [3] and first renormalize the $L$-operators by
means of certain linear transformation. Let us consider \( \hat{L}(\zeta) \) of section 7.2 with fixed \( s = 2 \) and \( s_1 = 0 \) and change there \( \zeta^2 \) by \(-\zeta^2\) for convenience. Performing now the transformation

\[
\hat{L}(\zeta) \to \zeta^{-1} e^{-\lambda(q \zeta^2)} I_q \hat{L}(\zeta) I_q,
\]

where

\[
I_q = \begin{pmatrix} 1 & 1 \\ 1 & -1/(q - q^{-1}) \end{pmatrix}, \quad J_q = J_{I_q} J,
\]

we rewrite the transformed \( L \)-operator in terms of the naturally \( q \)-deformed oscillators (see footnote 11) \( N, b \) and \( b^\dagger \) and use the parametrization \( q = e^\hbar \) and \( \zeta = e^{\hbar z} \). Then, in the limit \( \hbar \to 0 \), we have

\[
\hat{L}_c(z) = \begin{pmatrix} z + N + 1 & 2b^\dagger & -2b^{2\dagger} \\ b & 2 & -2b^\dagger \\ b^2/2 & b & z - N \end{pmatrix}.
\]

One can treat the obtained matrix \( \hat{L}_c(z) \) as a local quantum Lax operator for the quasi-classical limit of the Izergin–Korepin model.

The objects constructed in this work will be used in future work to develop the method of functional relations including the transfer matrix and Baxter’s \( Q \)-operators for the quantum integrable system considered here. Note that to obtain functional relations, one can also use an approach based on the notion of fundamental modules over the Borel subalgebras of the quantum algebras [13, 16], allowing one to avoid explicit forms of the \( L \)-operators.

Acknowledgments

We are grateful to M Jimbo for communication. This work was supported in part by the Volkswagen Foundation. AVR was supported in part by the RFBR grants 09-01-93107 and 10-01-00300.

Appendix A. Calculating the \( R \)-matrix

The calculation of the factors \( \psi_{\zeta_i} \otimes \psi_{\zeta_j}(\mathcal{R}_{2\nu}) \) and \( \psi_{\zeta_i} \otimes \psi_{\zeta_j}(\mathcal{R}_{\delta - 2\nu}) \) belongs obviously to the simplest part of the work, since, due to the relation

\[
(E_{ij})^\ell = 0, \quad i \neq j, \quad \ell > 1,
\]

the infinite products in the corresponding expressions reduce to simple infinite geometric series. Indeed, we have

\[
\psi_{\zeta_i} \otimes \psi_{\zeta_j}(\mathcal{R}_{2\nu}) = \prod_{m \geq 0} (I + (q - q^{-1}) \psi_{\zeta_i} (e^{2\nu + (2m+1)\delta}) \otimes \psi_{\zeta_j} (f_{2\nu + (2m+1)\delta}))
\]

\[
= I + (q - q^{-1}) \sum_{m \geq 0} \psi_{\zeta_i} (e^{2\nu + (2m+1)\delta}) \otimes \psi_{\zeta_j} (f_{2\nu + (2m+1)\delta}),
\]

where \( I \) stands for the \( 9 \times 9 \) unit matrix. Then we easily perform the summation in the expression above and obtain

\[
\psi_{\zeta_i} \otimes \psi_{\zeta_j}(\mathcal{R}_{2\nu}) = I + (q - q^{-1}) [2] q^{\frac{\zeta + 2s_1}{2}} \frac{1}{1 - q^{2s} E_{12} \otimes E_{31}}.
\]
Similarly, for the factor $R_{\delta-2\alpha}$, we obtain

$$\psi_{\zeta_1} \otimes \psi_{\zeta_2} (R_{\delta-2\alpha}) = \prod_{m \geq 0} (I + (q - q^{-1})\psi_{\zeta_1} (e_{\delta-2\alpha+2m\delta}) \otimes \psi_{\zeta_2} (f_{\delta-2\alpha+2m\delta}))$$

$$= I + (q - q^{-1}) \sum_{m \geq 0} \psi_{\zeta_1} (e_{\delta-2\alpha+2m\delta}) \otimes \psi_{\zeta_2} (f_{\delta-2\alpha+2m\delta}).$$

This allows us to write down the expression

$$\psi_{\zeta_1} \otimes \psi_{\zeta_2} (R_{\delta-2\alpha}) = I + (q - q^{-1})[2\delta_0 \frac{\zeta_1^{r_{2r_1}}}{1 - \zeta_1^{2r_1}}] E_{31} \otimes E_{13}.$$

The most complicated part of our calculations concerns the factors $R_\alpha$ and $R_{\delta-\alpha}$. This is due to the fact that the images of the generators in the corresponding exponentials are now nilpotent of degree 3. Hence, we need to take into account terms quadratic in such generators,

$$\psi_{\zeta_1} \otimes \psi_{\zeta_2} (R_\alpha) = \prod_{m \geq 0} \exp_{q_0} (x_{\alpha,m}) = \prod_{m \geq 0} \left( 1 + x_{\alpha,m} + \frac{x_{\alpha,m}^2}{2 q_0} \right),$$

while all higher order terms vanish. Here, for convenience, we have used the notation

$$x_{\alpha,m} = (q - q^{-1}) \psi_{\zeta_1} (e_{\alpha+m\delta}) \otimes \psi_{\zeta_2} (f_{\alpha+m\delta}),$$

where $\psi_{\zeta_1} (e_{\alpha+m\delta})$ and $\psi_{\zeta_2} (f_{\alpha+m\delta})$ are explicitly given above. The infinite product in the expression for $\psi_{\zeta_1} \otimes \psi_{\zeta_2} (R_\alpha)$ can be rewritten as the following infinite sum:

$$\psi_{\zeta_1} \otimes \psi_{\zeta_2} (R_\alpha) = I + \sum_{m \geq 0} x_{\alpha,m} + \frac{q}{q + 1} \sum_{m \geq 0} x_{\alpha,m}^2 + \sum_{m \geq 0} \sum_{k \geq 0} x_{\alpha,m} x_{\alpha,k+m+1}.$$

Performing the corresponding summations

$$\sum_{m \geq 0} x_{\alpha,m} = \frac{q - q^{-1}}{1 - \zeta_1^{r_1}} (E_{12} \otimes E_{21} + E_{23} \otimes E_{32})$$

$$+ \frac{q - q^{-1}}{1 + q \zeta_1^{r_1}} E_{12} \otimes E_{32} + \frac{q - q^{-1}}{1 + q^{-1} \zeta_1^{r_1}} E_{23} \otimes E_{21},$$

the infinite product in the expression for $\psi_{\zeta_1} \otimes \psi_{\zeta_2} (R_\alpha)$ can be rewritten as the following infinite sum:

$$\psi_{\zeta_1} \otimes \psi_{\zeta_2} (R_{\delta-\alpha}) = \prod_{m \geq 0} \exp_{q_0} (y_{\alpha,m}) = \prod_{m \geq 0} \left( 1 + y_{\alpha,m} + \frac{y_{\alpha,m}^2}{2 q_0} \right),$$

we obtain

$$\psi_{\zeta_1} \otimes \psi_{\zeta_2} (R_{\delta-\alpha}) = I + \frac{q - q^{-1}}{1 - \zeta_1^{r_1}} (E_{12} \otimes E_{21} + E_{23} \otimes E_{32})$$

$$+ \frac{q - q^{-1}}{1 + q \zeta_1^{r_1}} E_{12} \otimes E_{32} + \frac{q - q^{-1}}{1 + q^{-1} \zeta_1^{r_1}} E_{23} \otimes E_{21}$$

$$+ \frac{(q - q^{-1})(q - 1)(1 - q^{-2} \zeta_1^{r_1})}{(1 - \zeta_1^{2r_1})(1 + q^{-1} \zeta_1^{r_1})} \sum_{m \geq 0} \sum_{k \geq 0} x_{\alpha,m}^2 x_{\alpha,k+m+1}.$$

We have the same situation with the factor $R_{\delta-\alpha}$. Here we have to carry out the expression

$$\psi_{\zeta_1} \otimes \psi_{\zeta_2} (R_{\delta-\alpha}) = \prod_{m \geq 0} \exp_{q_0} (y_{\alpha,m}) = \prod_{m \geq 0} \left( 1 + y_{\alpha,m} + \frac{y_{\alpha,m}^2}{2 q_0} \right).$$
where now we have denoted

\[ y_{a,m} = (q - q^{-1}) \varphi_{\xi_1}(\epsilon_{\delta-a+m\delta}) \otimes \varphi_{\xi_2}(f_{\delta-a+m\delta}) , \]

with \( \varphi_{\xi_1}(\epsilon_{\delta-a+m\delta}) \) and \( \varphi_{\xi_2}(f_{\delta-a+m\delta}) \) given in the preceding section. Again, we rewrite the infinite product in the expression for \( \varphi_{\xi_1} \otimes \varphi_{\xi_2}(R_{\delta-a}) \) as an infinite sum,

\[ \varphi_{\xi_1} \otimes \varphi_{\xi_2}(R_{\delta-a}) = I + \sum_{m \geq 0} y_{a,m} + \frac{q}{q + 1} \sum_{m \geq 0} y_{a,m} + \sum_{m \geq 0, k \geq 0} y_{a,k+m+1}y_{a,m} . \]

After some calculations,

\[
\sum_{m \geq 0} y_{a,m} = \frac{(q - q^{-1}) \xi_{12}^{s_{r-s}^1}}{1 - \xi_{12}^s} (E_{21} \otimes E_{12} + E_{32} \otimes E_{23}) \\
= \frac{(q - q^{-1}) q^2 \xi_{12}^{s_{r-s}^1}}{1 + q \xi_{12}^s} E_{21} \otimes E_{23} - \frac{(q - q^{-1}) q^{-2} \xi_{12}^{s_{r-s}^1}}{1 + q^{-1} \xi_{12}^s} E_{32} \otimes E_{12},
\]

\[
q + \sum_{m \geq 0} y_{a,m}^2 = \frac{(q - q^{-1}) (q - 1) \xi_{12}^{2(s_{r-s}^1)}}{1 - \xi_{12}^s} E_{31} \otimes E_{13},
\]

we obtain

\[
\varphi_{\xi_1} \otimes \varphi_{\xi_2}(R_{\delta-a}) = I + \frac{(q - q^{-1}) \xi_{12}^{s_{r-s}^1}}{1 - \xi_{12}^s} (E_{21} \otimes E_{12} + E_{32} \otimes E_{23}) \\
- \frac{(q - q^{-1}) q^2 \xi_{12}^{s_{r-s}^1}}{1 + q \xi_{12}^s} E_{21} \otimes E_{23} - \frac{(q - q^{-1}) q^{-2} \xi_{12}^{s_{r-s}^1}}{1 + q^{-1} \xi_{12}^s} E_{32} \otimes E_{12} \\
+ \frac{(q - q^{-1}) (q - 1) \xi_{12}^{2(s_{r-s}^1)} (1 - \xi_{12}^s)}{(1 + q \xi_{12}^s)(1 + q^{-1} \xi_{12}^s)} E_{31} \otimes E_{13}.
\]

### Appendix B. Calculating the L-operators

With respect to the highest roots \( 2\alpha + (2m + 1)\delta \), we obtain the expression

\[
\chi_{\xi_1}(\epsilon_{2\alpha + (2m + 1)\delta}) = \frac{(q - 1)^{[2]} q^{[2]^{2\alpha + (2m + 1)\delta}}}{(q - q^{-1})(q^2 - q^2)} a^2 q^{2(2m + 1)D},
\]

which, in addition to (4.9) for \( \varphi_{\xi_1}(f_{2\alpha + (2m + 1)\delta}) \), leads to

\[
\chi_{\xi_1} \otimes \varphi_{\xi_2}(R_{2\alpha}) = I + \frac{q(q - 1)}{q - q^{-1}} \xi_{12}^{s_{r-s}^1} q^2 q^{2D} (1 + q^3 \xi_{12}^{2(s_{r-s}^1)} a^2 q^{D})^{-1} E_{31}.
\]

Related to the roots \( -2\alpha + 2m\delta \), we obtain the expression

\[
\chi_{\xi_1}(\epsilon_{-2\alpha + 2m\delta}) = \frac{q^{2m} \xi_{-2\alpha + 2m\delta}}{(q - 1)^{[2]^{2m}} a^2 q^{2(2m - 1)D}},
\]

and so, using also equation (4.11) for \( \varphi_{\xi_1}(f_{-2\alpha + 2m\delta}) \), we obtain

\[
\chi_{\xi_1} \otimes \varphi_{\xi_2}(R_{-2\alpha}) = I + \frac{q(q - 1)}{q - q^{-1}} \xi_{12}^{s_{r-s}^1} a^2 q^{2D} (1 + q^3 \xi_{12}^{2(s_{r-s}^1)} a^2 q^{2D})^{-1} E_{13}.
\]
Further, for the factor $R_a$, we obtain the equation
\[
\chi_{\ell_1} \otimes \varphi_{\zeta_1}(R_a) = I + \sum_{m \geq 0} x_{a,m} + \frac{q}{q + 1} \sum_{m \geq 0} x_{a,m}^2 + \sum_{m \geq 0} \sum_{k \geq 0} x_{a,m} x_{a,k+m+1},
\]
where now
\[
x_{a,m} = (q - q^{-1}) \chi_{\ell_1}(e_{a+m}) \otimes \varphi_{\zeta_1}(f_{a+m}).
\]

Using the expressions
\[
\chi_{\ell_1}(e_{a+m}) = \frac{q^m \zeta_1^m}{q - q^{-1}} a q^{2mD},
\]
and (4.5) for $\varphi_{\zeta_1}(f_{a+m})$, we obtain
\[
\sum_{m \geq 0} x_{a,m} = \zeta_1^m a \left( 1 + q^2 \zeta_1^2 q^{2D} \right)^{-1} E_{21} + \left( 1 - q^3 \zeta_1^2 q^{2D} \right)^{-1} E_{32},
\]
\[
\frac{q}{q + 1} \sum_{m \geq 0} x_{a,m}^2 = \frac{q^2 \zeta_1^2}{q + 1} a^2 \left( 1 + q^3 \zeta_1^2 q^{2D} \right)^{-1} E_{31},
\]
\[
\sum_{m \geq 0} \sum_{k \geq 0} x_{a,m} x_{a,k+m+1} = -q^2 \zeta_1^2 \zeta_1^{2m+1} a^q q^{2D} \left( 1 + q^3 \zeta_1^2 q^{4D} \right)^{-1} \left( 1 + q^3 \zeta_1^2 q^{2D} \right)^{-1} E_{31}.
\]

This leads us to the expression
\[
\chi_{\ell_1} \otimes \varphi_{\zeta_1}(R_a) = I + \zeta_1^m a \left( 1 + q^2 \zeta_1^2 q^{2D} \right)^{-1} E_{21} + \frac{q}{q + 1} \zeta_1^2 a^2 \left( 1 + q^3 \zeta_1^2 q^{4D} \right)^{-1} E_{31} - q^2 \zeta_1^2 \zeta_1^{2m+1} a^q q^{2D} \left( 1 + q^3 \zeta_1^2 q^{4D} \right)^{-1} \left( 1 + q^3 \zeta_1^2 q^{2D} \right)^{-1} E_{31}.
\]

For the factor $R_{\delta-a}$, we have the equation
\[
\chi_{\ell_1} \otimes \varphi_{\zeta_1}(R_{\delta-a}) = I + \sum_{m \geq 0} y_{a,m} + \frac{q}{q + 1} \sum_{m \geq 0} y_{a,m}^2 + \sum_{m \geq 0} \sum_{k \geq 0} y_{a,k+m+1} y_{a,m},
\]
where we have used the notation
\[
y_{a,m} = (q - q^{-1}) \chi_{\ell_1}(e_{\delta-a+m}) \otimes \varphi_{\zeta_1}(f_{\delta-a+m}).
\]

Using the expressions
\[
\chi_{\ell_1}(e_{\delta-a+m}) = -\frac{q^{3m+2} \zeta_1^{2m+1}}{q - 1} a^q q^{2mD},
\]
and (4.7) for $\varphi_{\zeta_1}(f_{\delta-a+m})$, we obtain
\[
\sum_{m \geq 0} y_{a,m} = -\frac{q - q^{-1}}{q - 1} q^{2m+1} \zeta_1^m a^q \left[ \left( 1 + q^2 \zeta_1^2 q^{2D} \right)^{-1} E_{12} - q^2 \left( 1 - q^3 \zeta_1^2 q^{2D} \right)^{-1} E_{23} \right],
\]
\[
\frac{q}{q + 1} \sum_{m \geq 0} y_{a,m}^2 = -\frac{q^2}{q + 1} \left( q - q^{-1} \right)^2 \zeta_1^2 \zeta_1^{2m+1} a^2 \left( 1 + q^4 \zeta_1^2 q^{4D} \right)^{-1} E_{13},
\]
\[
\sum_{m \geq 0} \sum_{k \geq 0} y_{a,k+m+1} y_{a,m} = \frac{(q - q^{-1})^2}{(q - 1)^2} q^{2m+1} \zeta_1^2 \zeta_1^{2m+1} a^q q^{2D} \left( 1 + q^4 \zeta_1^2 q^{4D} \right)^{-1} \left( 1 + q^6 \zeta_1^2 q^{2D} \right)^{-1} E_{13}.
\]
Summing up these terms, we obtain
\[
\chi_{\xi} \otimes \varphi_{\xi} (R_{\alpha-\alpha}) = I - \frac{q - q^{-1}}{q - 1} q^3 \xi_{12}^{-s_1} a^1(I + q^4 \xi_{12}^4 q^{-2D})^{-1} E_{12}
\]
\[
+ \frac{q - q^{-1}}{q - 1} q^3 \xi_{12}^{-s_1} a^1(I - q^5 \xi_{12}^4 q^{-2D})^{-1} E_{23}
\]
\[
+ \frac{q^7}{q + 1} \frac{(q - q^{-1})^2}{(q - 1)^2} \xi_{12}^{-2(s-s_1)} a^{12}(I + q^{11} \xi_{12}^4 q^{-4D})^{-1} E_{13}
\]
\[
+ \frac{(q - q^{-1})^2}{(q - 1)^2} q^{12} \xi_{12}^{-2s_1} a^{12} q^{-2D}(I + q^{11} \xi_{12}^4 q^{-4D})^{-1} (I + q^4 \xi_{12}^4 q^{-2D})^{-1} E_{13}.
\]

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