Charged particles with spin in a gravitational wave and a uniform magnetic field

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Abstract

We study the motion of a pseudo-classical charged particle with spin in the space-time of a gravitational pp wave in the presence of a uniform magnetic field.

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1 Introduction

The theory of (pseudo)-classical charged spinning particles in curved space-time in the presence of electromagnetic fields has been a topic of interest in recent decades. The equations governing the motion of these particles was obtained by Dixon [1, 2] and Souriau [3] by generalizing the equations first obtained by Papapetrou [4] to describe spinning test particles. These consist of a set of equations of motion for the four-momentum and spin tensor of the particle, and a third set of equations called the supplementary equations needed to make the equations of motion complete. An extension of these equations to the case where torsion fields are also included was obtained in Refs. [5] and [6]. An alternative set of equations was obtained by van Holten [7]. The Dixon-Souriau equations reduce to the van Holten equations whenever the particle’s four-momentum and four-velocity become co-linear. It has also been shown that the Dixon-Souriau equations reduce to the well known Bargmann-Michel-Telegdi equations in the limit of the weak and homogeneous external field [8].

The Dixon-Souriau equations was used in Ref. [9] to study the motion of charged spinning particles in a Kerr-Newman background in the presence of electromagnetic fields. These equations was also applied to the case of motion in a Reissner-Nordstrom space-time in Ref. [10] (and also in Ref. [11] with a different supplementary equation). A linearized version of the Dixon-Souriau equations was used in Ref. [12] to study charged spinning particles interacting with a gravitational wave in the presence of a uniform magnetic field (see also [13]). The equations of Ref. [7] was used in Ref. [14] to study the motion of a charged spinning particle in the Schwarzschild space-time, in Ref. [15] to study the motion in the Reissner-Nordstrom space-time, in Ref. [16] to study the motion in the Taub-Nut space-time, and in Ref. [17] to study the motion in a uniform magnetic field on a static space-time.

The aim of the present work is to study the motion of a charged spinning test particles, in the space-time of a gravitational pp wave in the presence of a uniform magnetic field. This is motivated by the current interests in the dynamics of spinning particles in curved space-times and its possible application to the problem of the detection of gravitational waves [18]. We take the the electromagnetic field to be a uniform magnetic field, both for calculational reasons and for its possible astrophysical interests.

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In our study of charged spinning particles, we use the equations given in Ref. [7]. The advantage of using these equations instead of the Dixon-Souriau equations is their relative simplicity compared with the latter ones. As we will see, we will be able to find analytic solutions to the equations of motion without resort to linear approximation. Since the external fields considered here have very small amplitudes, we expect the results to have the same physical content as the results of the Dixon-Souriau equations.

2 The Equations of Motion

The equations describing the motion of a charged spinning particle are as follows

\[
\frac{D\dot{x}^\mu}{D\tau} = -\frac{1}{2m} R^\mu_{\nu\lambda\rho} S^\lambda\rho \dot{x}^\nu + \frac{q}{m} F^\mu_{\beta} \dot{x}^\beta + \frac{q}{2m^2} S^\nu\rho D^\mu F_{\nu\rho},
\]

(1)

\[
\frac{DS^{\mu\nu}}{D\tau} = -\frac{q}{m} (S^\nu\kappa F^\kappa_{\nu} - S^\nu\kappa F^\kappa_{\mu}),
\]

(2)

\[
\dot{x}_\mu S^{\mu\nu} = 0,
\]

(3)

where Greek indices run over the space-time dimensions, \(\tau\) is an affine parameter across the particle’s word-line, \(\dot{x}^\mu\) is the 4-velocity of the particle, \(S^{\mu\nu}\) are the components of the spin tensor of the particle, \(F^{\mu\nu}\) is the electromagnetic tensor, \(q\) and \(m\) represent the charge and the mass of the particle respectively, and

\[
\frac{D\dot{x}^\mu}{D\tau} = \frac{dx^\mu}{d\tau} + \Gamma^\mu_{\lambda\nu} \dot{x}^\lambda \dot{x}^\nu,
\]

\[
\frac{dS^{\mu\nu}}{d\tau} = \frac{dS^{\mu\nu}}{d\tau} + \Gamma^\nu_{\lambda\rho} \dot{x}^\lambda S^{\mu\rho} + \Gamma^\nu_{\lambda\rho} \dot{x}^\lambda S^{\mu\rho},
\]

\[
R^\nu_{\kappa\lambda\mu} = \partial_\lambda \Gamma^\nu_{\kappa\mu} - \partial_\mu \Gamma^\nu_{\kappa\lambda} + \Gamma^\nu_{\lambda\rho} \Gamma^\rho_{\kappa\mu} - \Gamma^\nu_{\mu\rho} \Gamma^\rho_{\kappa\lambda}.
\]

Equation (3) is the so called Pirani supplementary condition. A more widely used supplementary condition is the so called Tulczyjew supplementary condition which reads

\[
p_\mu S^{\mu\nu} = 0
\]

in which \(p^\mu\) is the particle’s four-momentum. For spinning particles, the four-momentum and the four-velocity are not co-linear in general, but their difference is gauge dependent and may be ignored in most practical cases [18]. By neglecting this difference the Tulczyjew condition reduces to the Pirani condition and the Dixon-Souriau equations of motion simplify to the equations (1) and (2). This simplification enables us to find analytic expressions for the trajectories of the particles and their spins without loss of physical content.

The above equations give the components of \(\dot{x}^\mu\) and \(S^{\mu\nu}\) and hence the trajectory of the particle. Also one may convert the spin tensor \(S^{\mu\nu}\) into the spin 4-vector \(S^\mu\) via the relation

\[
S^\mu = \frac{1}{2\sqrt{-g}} \epsilon^{\mu\nu\kappa\lambda} \dot{x}_\nu S_{\kappa\lambda}
\]

in which \(\epsilon^{\mu\nu\kappa\lambda}\) is the alternating symbol and we adopt \(\epsilon^{1234} = +1\). This sometimes gives a better insight into the physics of the problem.

3 The Space-Time

The space-time we are interested in, is a plane gravitational wave of general polarization and harmonic profile. We choose \((u = t - z, v = t + z, x, y)\) as our coordinate system and represent them by 1,2,3,4 as indices respectively. In these coordinates the metric is given by

\[
ds^2 = -du dv - K(u, x, y)du^2 + dx^2 + dy^2,
\]

(5)
where
\[ K(u, x, y) = f_+(u)(x^2 - y^2) + 2f_x(u)xy \tag{6} \]
and we take
\[ f_+(u) = h\omega^2 \sin(\omega u), \]
\[ f_x(u) = h\omega^2 \cos(\omega u), \]
h being the dimensionless wave amplitude. We assume that a constant magnetic field \( B \) is present in this space-time and neglect its effect on the space-time. We take the magnetic field parallel to the direction of the wave propagation, that is \( \mathbf{B} = B\mathbf{k} \) and so in terms of \( F^{\mu\nu} \), the only non-vanishing components are
\[ F_{34}^3 = -F_{43}^4 = B. \tag{7} \]

4 The Trajectories and Spins

We are interested in calculating the trajectory and spin evolution of a charged spinning particle in the space-time given by the metric \( g \) in the presence of a magnetic field given by \( h \). We first set \( \mu = 1 \) in equation \( 1 \) to obtain
\[ \frac{d^2 u(\tau)}{d\tau^2} = 0 \]
where we choose the following solution
\[ u = \tau \tag{8} \]
that is we take \( u \) to be the proper time along the particle’s world-line. Now by setting \( \mu = 1, \nu = 3, 4 \) in equation \( 2 \), we get
\[ \frac{dS_{13}^{13}(\tau)}{d\tau} = \omega_0 S_{14}^{14}(\tau), \]
\[ \frac{dS_{14}^{14}(\tau)}{d\tau} = -\omega_0 S_{13}^{13}(\tau). \]
where \( \omega_0 = \frac{qB}{m} \) is the cyclotron frequency. The solution to these equations reads
\[ S_{13}^{13}(\tau) = A \sin(\omega_0 \tau + \theta), \tag{9} \]
\[ S_{14}^{14}(\tau) = A \cos(\omega_0 \tau + \theta) \tag{10} \]
where \( A, \theta \) are constant. This means that these spin components depend only on the magnetic field and their initial values. Now by setting \( \mu = 3, 4 \) in equation \( 1 \) and using equations \( 9 \) and \( 10 \) we obtain
\[ \frac{d^2 x(\tau)}{d\tau^2} = -h\omega^2 \sin(\omega\tau)x(\tau) - h\omega^2 \cos(\omega\tau)y(\tau) + \frac{Ah\omega^2}{m}\cos((\omega_0 - \omega)\tau + \theta), \]
\[ \frac{d^2 y(\tau)}{d\tau^2} = h\omega^2 \sin(\omega\tau)y(\tau) - h\omega^2 \cos(\omega\tau)x(\tau) - \frac{Ah\omega^2}{m}\sin((\omega_0 - \omega)\tau + \theta). \]
These can be solved for \( \chi(\tau) = x(\tau) + iy(\tau) \) resulting in
\[ \chi(\tau) = e^{-i\omega_0\tau} (Pe^{i\omega_\tau} + Qe^{-i\omega_\tau}) + \frac{A}{m}e^\theta e^{-i\delta\omega_\tau} \tag{11} \]
which is valid for $|\delta - \frac{1}{2}| < 2h$, and

$$\chi(\tau) = P e^{i\alpha_1\omega\tau} + Q e^{i\alpha_2\omega\tau} + i\frac{A}{m} e^{i\theta} e^{-i\delta\omega\tau} + i\Delta_1 \mathcal{P} e^{-i(1+\alpha_1)\omega\tau} + i\Delta_2 Q e^{-i(1+\alpha_2)\omega\tau}$$

valid for other values of $\delta$ (we assume that $h < \frac{1}{4}$). Here, $P = P_1 + iP_2$, $Q = Q_1 + iQ_2$ are constants to be determined from $\chi(0), \dot{\chi}(0)$, and

$$\delta = \frac{\omega_0}{\omega},$$

$$\alpha = \frac{1}{2} \sqrt{2\delta - 2\delta^2 - 1 + 2\sqrt{\delta^4 - 2\delta^3 + \delta^2 + 4h^2}},$$

$$\alpha_1 = \frac{-1 + \sqrt{2\delta^2 - 2\delta + 1 + 2\sqrt{\delta^4 - 2\delta^3 + \delta^2 + 4h^2}}}{2},$$

$$\alpha_2 = \frac{-1 + \sqrt{2\delta^2 - 2\delta + 1 - 2\sqrt{\delta^4 - 2\delta^3 + \delta^2 + 4h^2}}}{2},$$

$$\Delta_1 = \frac{h}{(1 + \alpha_1)^2 - (1 + \alpha_1)\delta},$$

$$\Delta_2 = \frac{h}{(1 + \alpha_2)^2 - (1 + \alpha_2)\delta}.$$

The equation for $v(\tau)$ may be obtained either by setting $\mu = 2$ in equation (11) and solving the resulting equation for $v$, or more directly by integrating the following relation

$$\dot{v}(\tau) = 1 - K(u, x, y) + (\dot{x}(\tau))^2 + (\dot{y}(\tau))^2$$

which in turn is a consequence of the relation $\dot{x}\mu \dot{v} = -1$. Thus we have

$$v(\tau) = \tau + \int (\dot{x}^2(\tau) + \dot{y}^2(\tau) + K(u, x, y)) \, d\tau. \quad (13)$$

Let us now return to the equation (2), where for $\mu = 3, \nu = 4$ we have

$$\frac{dS^{34}(\tau)}{d\tau} = Ah\omega^2(x(\tau) \sin((\delta - 1)\omega\tau + \theta) - y(\tau) \cos((\delta - 1)\omega\tau + \theta)).$$

Now by substituting $x(\tau)$ and $y(\tau)$ into this equation, one may solve it for $S^{34}(\tau)$ to obtain

$$S^{34}(\tau) = \frac{-Ah\omega}{\alpha^2 + (\delta - \frac{1}{2})^2} \left( (P_1(\delta - \frac{1}{2}) + P_2\alpha)e^{\omega\tau} \cos((\delta - \frac{1}{2})\omega\tau + \theta) + (Q_1(\delta - \frac{1}{2}) - Q_2\alpha)e^{-\omega\tau} \cos((\delta - \frac{1}{2})\omega\tau + \theta) + (P_2(\delta - \frac{1}{2}) - P_1\alpha)e^{\omega\tau} \sin((\delta - \frac{1}{2})\omega\tau + \theta) + (Q_2(\delta - \frac{1}{2}) + Q_1\alpha)e^{-\omega\tau} \sin((\delta - \frac{1}{2})\omega\tau + \theta) \right)$$

$$- \frac{A^2h\omega}{m(1 - 2\delta)} \sin((1 - 2\delta)\omega\tau) + C \quad (14)$$

valid for $|\delta - \frac{1}{2}| < 2h$. For other values of $\delta$ we have

$$S^{34}(\tau) = Ah\omega \left( \frac{P_1}{\alpha_1 + 1 - \delta} \cos((\alpha_1 + 1 - \delta)\omega\tau - \theta) - \right.$$
The explicit form of these components is presented in the appendices for some values of $\delta$. If we set

$$A = 0, \quad B = 0,$$

from the absence of the last term of equation (1) in the subsequent equations for $\chi(\tau)$, this can be seen except for the interval $\frac{1}{2} - 2h < \delta < \frac{1}{2} + 2h$ which is narrow for small $h$. For the values of $\delta$ inside this interval, the oscillatory terms are modulated by slowly varying exponential terms $e^{\pm h\omega \tau}$. For $\delta = 1$ the coefficients $\alpha_1, \alpha_2$ become the same as those obtained in Ref. [25], see the appendices.

Finally, making use of the relation (11), we obtain

$$S^1(\tau) = -S^{14}(\tau) + \dot{\chi}(\tau)S^{13}(\tau) - \dot{\chi}(\tau)S^{14}(\tau),$$

(19)

$$S^2(\tau) = 2S^{14}(\tau) + \dot{\chi}(\tau)S^{13}(\tau),$$

(20)

$$S^3(\tau) = S^{14}(\tau) + \dot{\chi}(\tau)S^{13}(\tau),$$

(21)

$$S^4(\tau) = -S^{13}(\tau) + \dot{\chi}(\tau)S^{14}(\tau).$$

(22)

The explicit form of these components is presented in the appendices for some values of $\delta$.

5 Discussion

If we set $B = 0$, that is if the magnetic field is absent, we have a spinning particle moving in a gravitational wave background. This has been studied extensively in Refs. [18]-[25]. For the case of the vanishing magnetic field, the equations obtained here reduce to those obtained in Ref. [25].

An interesting feature of motion of a charged spinning particle in the external fields considered here is that spin-electromagnetic interaction does not affect the particle’s trajectory directly, this can be seen from the absence of the last term of equation (11) in the subsequent equations for $\chi(\tau)$. However in the case that the spin has an initial transverse component, $A \neq 0$, this interaction affects the spin and this in turn affects the trajectory. If the particle has no initial transverse spin components, its spins remains unaffected. This can be seen by letting $A = 0$ in equations (11), (13), and (14), (15). Also in this case, if the particle is sitting initially at the origin of coordinates, it remains in this coordinate point at other times. Another interesting feature of these solutions is that they consist only of oscillatory terms, except for the interval $\frac{1}{h} - 2h < \delta < \frac{1}{h} + 2h$ which is narrow for small $h$. For the values of $\delta$ inside this interval, the oscillatory terms are modulated by slowly varying exponential terms $e^{\pm h\omega \tau}$. For $\delta = 1$ the coefficients $\alpha_1, \alpha_2$ become the same as those obtained in Ref. [25], see the appendices.
For an object like an electron (treated classically) moving in a uniform magnetic field of the same typical magnitude as that of the earth, and for the gravitational wave frequencies ranging from $1\text{Hz}$ up to $10^4\text{Hz}$, the parameter $\delta$ ranges from an order of magnitude of $10^7$ down to $10^3$, and so the effect of the gravitational wave is suppressed by the magnetic field. For gravitational waves with frequencies lower than $1\text{Hz}$, $\delta$ exceeds $10^7$. For weaker magnetic fields such as cosmological magnetic fields, which could be as weak as $10^{-12}T$, this parameter, again for an electron, is of an order of magnitude of $10^{-\sim10^4}$. Thus $\delta = 1$ corresponds to situations such as the case where an electron moves in a weak magnetic field (of say, $10^{-9}T$) and a gravitational wave of frequency of say $10^2\text{Hz}$. The calculation of relative acceleration of charged spinning particles in gravitational waves and magnetic fields will be reported elsewhere.

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**A The Case of $\delta = \frac{1}{2}$**

For $\delta = \frac{1}{2}$ we have

$$\alpha = \frac{1}{2} \sqrt{\frac{1}{2} + \frac{1}{2} \sqrt{1 + 64h^2}} \approx 2h$$

and so neglecting $h^2$ and higher order terms, we obtain

$$x(\tau) = (P_1 e^{2h\omega\tau} + Q_1 e^{-2h\omega\tau}) \cos \left( \frac{\omega \tau}{2} \right) + $$$$ \left( P_2 e^{2h\omega\tau} + Q_2 e^{-2h\omega\tau} \right) \sin \left( \frac{\omega \tau}{2} \right) + \frac{A}{m} \sin \left( \frac{\omega \tau}{2} - \theta \right), \quad (23)$$

$$y(\tau) = \left( P_2 e^{2h\omega\tau} + Q_2 e^{-2h\omega\tau} \right) \cos \left( \frac{\omega \tau}{2} \right) - $$$$$ \left( P_1 e^{2h\omega\tau} + Q_1 e^{-2h\omega\tau} \right) \sin \left( \frac{\omega \tau}{2} \right) + \frac{A}{m} \cos \left( \frac{\omega \tau}{2} - \theta \right), \quad (24)$$

$$S^{34}(\tau) = \frac{1}{2} A \omega \left( (P_1 \sin \theta - P_2 \cos \theta) e^{2h\omega\tau} + $$$$$ (Q_2 \cos \theta - Q_1 \sin \theta) e^{-2h\omega\tau} \right) - \frac{A^2 h^2}{m} \tau + C. \quad (25)$$

If we choose $\chi(0) = \dot{\chi}(0) = 0$, we obtain

$$x(\tau) = \frac{A}{m} \sin \left( \frac{1}{2} \omega \tau - \theta \right) \left( 1 - \cosh(2h\omega\tau) \right), \quad (26)$$

$$x(\tau) = \frac{A}{m} \sin \left( \frac{1}{2} \omega \tau - \theta \right) \left( 1 - \cosh(2h\omega\tau) \right), \quad (27)$$

$$S^{34}(\tau) = \frac{A^2 \omega}{2m} (\sinh(2h\omega\tau) - 2h\omega\tau) + C, \quad (28)$$

and hence

$$S^4(\tau) = -\frac{A^2 \omega}{2m} (\sinh(2h\omega\tau) - 2h\omega\tau) - C, \quad (29)$$

$$S^2(\tau) = \frac{A^2 \omega}{2m} (\sinh(2h\omega\tau) - 2h\omega\tau) + C, \quad (30)$$

$$S^3(\tau) = A \cos(\omega \tau + \theta), \quad (31)$$

$$S^4(\tau) = -A \sin(\omega \tau + \theta). \quad (32)$$
We also have
\[ S^z(\tau) = \frac{1}{2} (S^2(\tau) - S^1(\tau)) \approx \frac{A^2 \omega}{2m} (\text{sinh}(2h\omega \tau) - 2h\omega \tau) + C. \] (33)

## B The Case of $\delta = 1$

For $\delta = 1$ we have
\[ \alpha_1 = -1 + \frac{\sqrt{1 + 4h}}{2}, \quad \alpha_2 = -1 + \frac{\sqrt{1 - 4h}}{2}, \quad \Delta_1 = -\Delta_2 = 1 \]
and hence
\[ \chi(\tau) = P e^{i\alpha_1 \omega \tau} + iPe^{-i(1+\alpha_1)\omega \tau} + Qe^{i\alpha_2 \omega \tau} \]
\[ - iQe^{-i(1+\alpha_2)\omega \tau} + i\frac{A}{m} e^{-i(\omega \tau - \theta)}. \] (34)

In addition, if we let $\chi(0) = \dot{\chi}(0) = 0$, we have
\[ \chi(\tau) = A \frac{h}{m} \left( \frac{\alpha_1}{1 + 2\alpha_1} e^{i\alpha_1 \omega \tau} - \frac{i(1 + \alpha_1)e^{i\theta}}{1 + 2\alpha_1} e^{-i(1+\alpha_1)\omega \tau} + (1 + \alpha_2)e^{i\theta} - i\alpha_2 e^{i\theta} \frac{1 + 2\alpha_2}{1 + 2\alpha_2} e^{-i(1+\alpha_2)\omega \tau} + 2i e^{-i(\omega \tau - \theta)} \right). \] (35)

For small $h$ we have
\[ x(\tau) = \frac{Ah}{m} (\cos \theta - \cos(\omega \tau + \theta)) - \frac{A}{m} \sin \theta \sin(h\omega \tau), \] (36)
\[ y(\tau) = \frac{Ah}{m} (- \sin \theta + \sin(\omega \tau + \theta)) - \frac{A}{m} \cos \theta \sin(h\omega \tau), \] (37)
\[ S^{34}(\tau) = C, \] (38)
and hence
\[ S^1(\tau) = -C + \frac{A^2 h \omega}{m} \sin \theta \left( \sin(\omega \tau + \theta) - \cos(\omega \tau + \theta) \right), \] (39)
\[ S^2(\tau) = C + \frac{A^2 h \omega}{m} \sin \theta \left( \sin(\omega \tau + \theta) - \cos(\omega \tau + \theta) \right), \] (40)
\[ S^3(\tau) = A \cos(\omega \tau + \theta) - \frac{hCA \omega}{m} (\sin(\omega \tau + \theta) - \sin \theta), \] (41)
\[ S^4(\tau) = -A \sin(\omega \tau + \theta) - \frac{hCA \omega}{m} (\cos(\omega \tau + \theta) - \cos \theta). \] (42)

We also obtain
\[ S^z(\tau) = \frac{1}{2} (S^2(\tau) - S^1(\tau)) = C. \] (43)

which can be used to determine constant $C$ in terms of the spin component along the direction in which the wave propagates.
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