On the Interaction of Electrons, Magnetic Monopoles, and Photons

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We study quantum systems of interacting electrons, magnetic monopoles, and electromagnetic field. We formulate a convenient field theory, in which the electron-photon, monopole-photon, and electron-monopole interactions take simple forms.

I. INTRODUCTION

As well-known from the early times, inclusion of both electric and magnetic charges restores the symmetry between electricity and magnetism in the Maxwell equations\cite{1}. Several years after the birth of quantum mechanics, Dirac pointed out\cite{2} that magnetic monopole, if exists, cannot carry arbitrary amount of magnetic charges. In fact, the minimal magnetic charge \( g \) and the minimal electrical charge \( e \) must satisfy a quantum condition, \( eg = 2\pi n \) \( (n \) is an integer\), which is known as the Dirac quantization condition. Magnetic monopole has generated enduring interests in many fields of physics\cite{3–6}. For instance, it has played a crucial role in theoretical developments related to electromagnetic duality\cite{7–10}.

In Dirac’s original theory of monopoles, he used the auxiliary concept of “Dirac string”. In a work\cite{11} on classical action in 1948, Dirac introduced dynamic variables for the string. As a result, the entire theory became very complicated. A much more transparent formulation was put forward by Wu and Yang\cite{12, 13}, who borrowed the idea of fiber bundle from mathematics. In this language, systems with monopoles are identified with nontrivial \( U(1) \) bundles. To describe them quantitatively, overlapping patches of coordinates were used, and various quantities (such as gauge potential) follow prescribed transformation rules upon changing from one patch to another. In this formulation the troublesome Dirac string is absent.

Based on the language of fiber bundle, an elegant classical Lagrangian for monopoles was proposed in Ref.\cite{14}. A notable feature in this theory is that the action becomes multi-valued. The natural next step is to develop this theory into a form that describes quantum systems of interacting electrons, monopoles, and photons, however, this is not a straightforward problem\cite{15}. Without invoking this Lagrangian formulation, in a prescient work Tu, Wu, and Yang developed a Hamiltonian formalism\cite{16}, in which inspired guesswork was needed. Several other efforts on the quantum field theory of electron-monopole-photon systems can be found in Ref.\cite{17} and the references therein.

Aiming at a more convenient formulation for interacting electrons, monopoles, and photons, in the present paper we reinvestigate the classical Lagrangian of Wu and Yang\cite{14}. With this classical Lagrangian as a hint, we formulate a simple quantum field theory for electron-monopole-photon systems. This is done in the path integral approach. Simple forms for electron-photon, monopole-photon, and electron-monopole interactions are obtained in a natural manner.

The remainder of this paper is organized as follows. In Sec.II we study the interaction of monopoles and photons (but no electron). We work on this case first because it illustrates the way monopole-photon interaction emerges in this approach. In Sec.III we proceed to formulate the interactions of electrons, magnetic monopoles, and photons. The complete Lagrangian is also contained in Sec.III. In Sec.IV we shall study both classical and quantum equations of motion, followed by Sec.V in which we present a dual description. In Sec.VI we shall study the effective monopole-monopole interaction in both the original and the dual descriptions, which can be regarded as a consistency check of our formulation.

II. INTERACTION OF MAGNETIC MONOPOLES AND PHOTONS

In this section we formulate a field theory for interacting magnetic monopoles and photons (without electrons). The magnetic charge is treated as a topological charge, thus it is unnecessary to introduce a minimal coupling for monopoles. The basic idea dates back to Dirac\cite{11} (see also Wu and Yang\cite{14}), however, therein the monopole was treated as a classical particle instead of a quantum field, furthermore, the interaction between monopole and electromagnetic field was implicit in the Lagrangian. Later developments of quantum field theory of magnetic monopoles\cite{17} were often complicated by the Dirac string. In the present approach, the Dirac string is absent because of a convenient separation of electromagnetic field into two parts, the first part being dynamics, and the second part being kinematic.

Let us start from the Maxwell equations\cite{11}

\[
\begin{align*}
\nabla \cdot \mathbf{B} &= g \rho_m, \quad (a) \\
\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= -g \mathbf{j}_m, \quad (b) \\
\n\nabla \cdot \mathbf{E} &= e \rho_e, \quad (c) \\
\n\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} &= e \mathbf{j}_e, \quad (d)
\end{align*}
\]

where \((\rho_e, \mathbf{j}_e)\) are the density and currents of electrons\cite{18}, \((\rho_m, \mathbf{j}_m)\) are the density and currents of magnetic monopoles, and \(\mathbf{E}, \mathbf{B}\) are electric and magnetic fields. In the absence of electrons, we can just take \(\rho_e = \mathbf{j}_e = 0\). We would like to emphasize that, in classical physics\cite{14}, two of the Maxwell equations [(a) and (b)] should be regarded as constraints (kinematic equations), which are not derived from variation of action. The other two Maxwell equations, (c) and (d), are de-
derived from the action principle [19]. In the quantum mechanical formulation, we are going to regard (a) and (b) as kinematic equations and exploit their consequences. Let us begin with the first equation, namely, \( \nabla \cdot \mathbf{B} = g \rho_m \). We would like to separate out a part of \( \mathbf{B} \), such that the rest has vanishing divergence. In mathematical form, we have

\[
\mathbf{B} = \mathbf{b} + \mathbf{b}'
\]  

in which

\[
\mathbf{b}'(x, t) = g \int d \mathbf{y} \mathbf{G}(x, \mathbf{y}) \rho_m(\mathbf{y}, t),
\]

with the shorthand notation

\[
\mathbf{G}(x, \mathbf{y}) = -\frac{1}{4\pi|x - \mathbf{y}|} \frac{\mathbf{x} - \mathbf{y}}{4\pi|x - \mathbf{y}|^3}
\]

It is readily seen that \( \mathbf{b}'(x, t) \) satisfies \( \nabla \cdot \mathbf{b}'(x, t) = g \rho_m(x, t) \), therefore, we have \( \nabla \cdot \mathbf{b}(x, t) = 0 \), as we required. It is therefore possible to introduce a vector potential \( \mathbf{a} \) such that

\[
\mathbf{b} = \nabla \times \mathbf{a}
\]

Next we would like to do the similar trick for the electric field. This step is less straightforward. Here we would like to make use of the second kinematic equation, namely (b). We split the electric field as \( \mathbf{E} = \mathbf{e} + \mathbf{e}' \), such that the kinematic part \( \mathbf{e}' \) satisfies

\[
\nabla \times \mathbf{e}' + \partial_t \mathbf{b}' = -g \mathbf{j}_m
\]

Together with Maxwell equation (b), it also implies

\[
\nabla \times \mathbf{e} + \partial_t \mathbf{b} = 0
\]

Since the explicit formula for \( \mathbf{b}' \) has already been given, we can use Eq. (5) to determine \( \mathbf{e}' \), for which we indeed find a solution

\[
\mathbf{e}'(x, t) = g \int d \mathbf{y} \mathbf{G}(x, \mathbf{y}) \times \mathbf{j}_m(\mathbf{y}, t)
\]

To avoid distractions by too many details, we have left the calculations for this to Appendix A.

One of the merit of splitting \( \mathbf{E} \) as \( \mathbf{e} + \mathbf{e}' \) is as follows. Inserting Eq. (4) into Eq. (6), we have \( \nabla \times (\mathbf{e} + \partial_t \mathbf{a}) = 0 \), which suggests us to write down \( \mathbf{e} + \partial_t \mathbf{a} = -\nabla \mathbf{a}_0 \), or equivalently, \( \mathbf{e} = -\partial_t \mathbf{a} - \nabla \mathbf{a}_0 \). Summarizing these equations, we have the following mathematical expressions for the electric and magnetic fields in terms of gauge potential

\[
\mathbf{E} = -\partial_t \mathbf{a} - \nabla \mathbf{a}_0 + \mathbf{e}'; \quad \mathbf{B} = \nabla \times \mathbf{a} + \mathbf{b}'
\]

If magnetic monopole is absent, then \( \mathbf{b}' = \mathbf{e}' = 0 \), and these formulas are reduced to the usual ones. Roughly speaking, in Eq. (8) we have separated the electromagnetic field into two parts, the dynamic part described by \( \mathbf{e} \) and \( \mathbf{b} \), and the kinematic part described by \( \mathbf{e}' \) and \( \mathbf{b}' \). The latter is fully determined by the monopole density and current. It is worth emphasizing that the two equations obtained in Eq. (8) are natural consequences of (a) and (b) of the Maxwell equations.

Have written \( \mathbf{E} \) and \( \mathbf{B} \) in convenient forms, we are now ready to present the path integral formulation. The Lagrangian density of the monopole-photon system is given as

\[
L = L_f + L_e + L_i
\]

where the first part is

\[
L_f = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2),
\]

\( \mathbf{E} \) and \( \mathbf{B} \) being given by Eq. (8). The second part of \( L \), namely \( L_e \), is the Lagrangian density of monopoles. If we take the monopoles as Dirac particles (though our results are not limited to Dirac particles), we have

\[
L_e = \psi^\dagger (i\partial_t + \frac{1}{2} \sum_{i=1}^3 \alpha_i \partial_i - \beta M) \psi
\]

where \( \alpha_i \) and \( \beta \) are Dirac matrices, and \( M \) is the (bare) mass of monopoles. The monopole density is \( \rho_m = \psi^\dagger \psi \), and the monopole current is \( j_m = \psi^\dagger \mathbf{a} \psi \) (or \( j_m^\dagger = \psi^\dagger \mathbf{a} \psi \)).

As the last part of the Lagrangian density, \( L_i \) is a gauge fixing term [20], which is added to ensure that the photon propagator is nonsingular. The nature of this term in our formulation is the same as that in usual quantum electrodynamics without monopole[21], therefore, we shall not discuss it in more detail. For simplicity, we take \( L_i \) to be

\[
L_i = \frac{1}{2\lambda}(\nabla \cdot \mathbf{a})^2
\]

though other choices are also allowed.

Now the full quantum theory is based on the path integral[22]

\[
\int \mathcal{D}a \mathcal{D}a_0 \mathcal{D}\psi \mathcal{D}\psi^\dagger \exp[i \int dt d^3x L(x, t)]
\]

To achieve a convenient field theory of interacting magnetic monopoles and photons, we have to investigate the Lagrangian density in more depth. A peculiar feature is worth noting. At first sight, it seems that the partial derivative \( \partial_t \) in \( L_e \) should be replaced by a covariant derivative, describing the monopole-photon interaction. Somewhat surprisingly, this is unnecessary and the partial derivative suffices. In fact, the monopole-photon interaction has already been included in \( L_f \). To see this fact, we can expand \( L_f \) as

\[
L_f = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2)
\]

\[
= \frac{1}{2}[(-\partial_t \mathbf{a} - \nabla \mathbf{a}_0 + \mathbf{e}')^2 - (\nabla \times \mathbf{a} + \mathbf{b}')^2]
\]

\[
= \frac{1}{2}[(-\partial_t \mathbf{a} - \nabla \mathbf{a}_0)^2 - (\nabla \times \mathbf{a})^2 + (\mathbf{e}')^2 - (\mathbf{b}')^2 - (\partial_t \mathbf{a} + \nabla \mathbf{a}_0) \cdot \mathbf{e}' - (\nabla \times \mathbf{a}) \cdot \mathbf{b}']
\]

After integration by parts and discarding total derivatives[23], we have

\[
L_f = \frac{1}{2}[(-\partial_t \mathbf{a} - \nabla \mathbf{a}_0)^2 - (\nabla \times \mathbf{a})^2 + (\mathbf{e}')^2 - (\mathbf{b}')^2 - (\partial_t \mathbf{a}) \cdot \mathbf{e}']
\]
Now the complete Lagrangian density of monopole-photon system is

\[
L_f + L_k + L_g = \frac{1}{2}(-\partial_t \mathbf{a} - \nabla a_0)^2 - \frac{1}{2} (\nabla \times \mathbf{a})^2 + \frac{1}{2\lambda} (\nabla \cdot \mathbf{a})^2 + \psi(\partial_i \mathbf{a} - i \sum_{i=1}^{3} \alpha_i \partial_i - \beta M)\psi
\]

\[+ \frac{1}{2} (\mathbf{e}')^2 - \frac{1}{2} (\mathbf{b}')^2 - (\partial_i \mathbf{a}) \cdot \mathbf{e}' \quad (16)
\]

Let us figure out the physical meanings of these terms. The first three terms of Eq.(16) give rise to the familiar bare photon propagator (propagator of \((\alpha_0, \mathbf{a})\)), which can be found in textbooks. We list them as \[24]\[25]\]

\[D^0(q) = \frac{\lambda g_0^2 q^2}{q^4}, \quad D^0(q) = \frac{\lambda g_0^2 q^2}{q^4}, \quad D^0(q) = \frac{\lambda g_0^2 q^2}{q^4} \quad (17)
\]

where \(q = (q_0, \mathbf{q})\) and \(q^2 = q_0^2 - \mathbf{q}^2\). These formulas simplify in the \(\lambda \to 0\) limit, wherein \(D^0 = 0\), thus \(\mathbf{a}\) and \(a_0\) are decoupled, and the longitudinal modes of \(\mathbf{a}\) are eliminated\[23].

What are the meanings of the last three terms of Eq.(16)? Let us first simplify the \(\frac{1}{2} (\mathbf{b}')^2\) term. For notational simplicity, we define

\[D(x, y) \equiv \frac{1}{4\pi|x - y|} \quad (18)
\]

which satisfies \[26\] \(G(x, y) = -\nabla D(x, y)\). With the input from Eq.(16), we can readily obtain that (See Appendix B for a derivation)

\[\frac{1}{2} \int dx (\mathbf{b}'(x))^2 = \frac{g^2}{2} \int dxdy \rho_m(x, y) D(x, y) \rho_m(y) \quad (19)
\]

which is just the magnetic Coulomb potential among monopoles. The \(\frac{1}{2} (\mathbf{e}')^2\) term in Eq.(16) can be simplified to current-current interactions (also see Appendix B)

\[\frac{1}{2} \int dx (\mathbf{e}'(x, t))^2 = \frac{g^2}{2} \int dxdy D(x, y) \mathbf{j}_m^T(x, t) \cdot \mathbf{j}_m^T(y, t) \quad (20)
\]

where \(\mathbf{j}_m^T\) denotes the transverse part of \(\mathbf{j}_m\).

The last term of Eq.(16) is more interesting. It can be recast as (see Appendix B)

\[- \int dx (\partial_i \mathbf{a}) \cdot \mathbf{e}' = g \int dxdy D(x, y)] [\nabla \times \partial_t \mathbf{a}(x, t)] \cdot \mathbf{j}_m(y, t) = \int dxdy D(x, y) \psi'(y, t) \mathbf{e}'(y, t) \mathbf{e}'(y, t) \quad (21)
\]

which describes a nonlocal interaction between monopoles and electromagnetic field: \(\mathbf{a}(x, t)\) is coupled to \(\mathbf{j}_m(y, t)\) with a decaying factor \(D(x, y)\). It is apparently invariant under a gauge transformation of \(\mathbf{a}\) \[27\]. In momentum-frequency space, Eq.(21) gives rise to the monopole-photon interaction

\[\int dxdy \frac{q_0}{q^2} (\mathbf{q} \times \mathbf{a}_q) \cdot \mathbf{j}_m(-q) = g \int dxdy \frac{q_0}{q^2} \psi_k(q) \alpha' \cdot (\mathbf{q} \times \mathbf{a}_q) \psi_k \quad (22)
\]

\(\mathbf{a}_q\), \(\mathbf{j}_m(q)\), \(\psi_k\) and \(\psi_k\) being the Fourier transformations of the corresponding quantities in spacetime. The \(\mathbf{q} \times \mathbf{a}_q\) factor in Eq.(22) indicates that \(\mathbf{a}\) suffers a \(\pi/2\) rotation around \(\mathbf{q}\) before being coupled to monopoles. This is physically intuitive, because \(\mathbf{E}\) and \(\mathbf{B}\) in a propagating electromagnetic wave are related by a \(\pi/2\) rotation around \(\mathbf{q}\).

The monopole-photon interaction obtained in Eq.(22) is apparently different from that of Ref.\[16\], wherein the interaction vertex does not contain a \(q_0\) factor. In addition, the direct current-current interaction, which is given by Eq.(20), is absent in Ref.\[16\]. As far as we can check for various physical processes, the two approaches lead to the same results, though the present approach is more convenient in many cases. In the Appendix D, we will return to a comparison of these two approaches.

Based on above calculations, we present a more explicit expression for Eq.(16) as

\[L_f + L_k + L_g = \frac{1}{2}(-\partial_t \mathbf{a} - \nabla a_0)^2 - \frac{1}{2} (\nabla \times \mathbf{a})^2 + \frac{1}{2\lambda} (\nabla \cdot \mathbf{a})^2 \]

\[+ \psi(\partial_i \mathbf{a} - i \sum_{i=1}^{3} \alpha_i \partial_i - \beta M)\psi
\]

\[- \frac{g^2}{2} \int dxdy \rho_m(x, y) D(x, y) \rho_m(y, t)
\]

\[+ \frac{g^2}{2} \int dxdy D(x, y) \mathbf{j}_m^T(x, t) \cdot \mathbf{j}_m^T(y, t)
\]

\[+ g \int dxdy D(x, y) [\nabla \times \partial_t \mathbf{a}(x, t)] \cdot \mathbf{j}_m(y, t) \quad (23)
\]

Compared with the Hamiltonian approach\[16\], in which inspired guesswork ( about the monopole-photon interaction, etc. ) is required, the present approach is more automatic.

Monopole-photon systems can also be described by minimally coupling monopoles to a (dual) gauge potential, the resultant theory being equivalent to the usual quantum electrodynamics. Therefore, the formulation in this section can be regarded as another version of the quantum electrodynamics\[28\]. The merit of our formalism will manifest in its application to electron-monopole-photon systems, to be investigated in the next section.

III. INTERACTIONS OF ELECTRONS, MAGNETIC MONOPOLES, AND PHOTONS

Having addressed the problem of monopole-photon interaction in the previous section, we shall formulate in this section the path integral quantization of electron-monopole-photon systems in the framework of fiber bundles, avoiding the troublesome “string singularities.”
In addition to the monopole–photon interaction found in the previous section, for electron-monopole-photon systems we have to write down the electron-photon and electron-monopole interactions. To this end, we kinematize the part of magnetic field as \( \mathbf{b}' = \nabla \times \mathbf{a}' \), where \( \mathbf{a}' \) is given by

\[
a'(x, t) = g \int dy A(x, y) \rho_m(y, t) \tag{24}
\]

in which we have introduced a function \( A(x, y) \), which satisfies \( \nabla \times A(x, y) = G(x, y) \). We can see that \( A(x, y) \) cannot be single-valued, otherwise we would have \( \nabla \cdot (\nabla \times A) = \int dy \nabla \cdot (\nabla \times A(x, y)) \rho_m(y, t) = 0 \). In fact, the most natural language for this problem is the fiber bundle\([13, 14]\). Following Ref.\([13, 14]\), we divide the space into two overlapping patches, \( R_a \) and \( R_b \). It is convenient to express \( x - y \) in the spherical coordinate \((r, \theta, \phi)\) (with \( \theta \in [0, \pi] \)), namely, \( x - y = (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta) \). The first patch \( R_a \) is defined by \( \theta < \theta_0 \), while the second patch \( R_b \) is defined by \( \theta > \pi - \theta_0, \theta_0 \) being a constant in \((\pi/2, \pi)\). We define\([13]\)

\[
A_b(x, y) = -\frac{1 - \cos \theta}{4\pi r^2 \sin^2 \theta} \hat{z} \times (x - y) \tag{25}
\]

It is readily found that \( A_b(x, y) - A_a(x, y) = 2\nu \phi \).

Let us define the full gauge potential \([23]\) \( A(x, t) = a(x, t) + a'(x, t) \) and \( A_0(x, t) = a_0(x, t) + a'_0(x, t) \), while the mathematical formula for \( a'_0 \) is to be determined shortly. The coupling of electromagnetic field to electrons reads

\[
e^{-eA_0(x, t)} \rho_e(x, t) + eA(x, t) \cdot j_e(x, t) = -e(a_0 + a'_0) \rho_e + e(a + a') \cdot j_e. \tag{26}
\]

Including this electromagnetic coupling, we have the following Lagrangian density for Dirac electrons

\[
L_e = c^\dagger (iD_t + \frac{3}{2} \sum_{i=1}^3 \alpha_i D_i - \beta \lambda)c - \frac{1}{2} \int dx L \psi \psi^\dagger \tag{27}
\]

where \( \psi \) is the position of monopole. To ensure the independence of \( L_e \) on patch choices, the Grassmann numbers \( c^\dagger, c \) have to be patch-dependent, and follow a prescribed transformation \( c^\dagger = e^{-\nu \gamma} c^\dagger(c) \), \( c = e^{\nu} c \), with \( \nu \) suitably chosen.

This is readily understood in the language of fiber bundle. To keep \( c^\dagger \) independent on the patch choice, we find [using Eq.\((28)\)] that \( \gamma \) satisfies

\[
\exp(i\nu) = e^{-\nu \int \frac{1}{2\pi} \theta} \tag{29}
\]

To ensure that \( e^{\nu} \) is single-valued, we must have the Dirac quantization condition \( e\nu/2\pi \) integer. In other words, the Dirac quantization condition is necessary for \( L_e \) to be independent of the patch choice. Moreover, to preserve the patch-independence of \( c^\dagger (iD_t)c \), we find that

\[
a^\dagger_0(x, t) - a^\dagger_0(b)(x, t) = -2\nu \phi(x, y, t) = 2\nu \psi^\dagger(y) \cdot \nabla \phi(x, y, t) \tag{30}
\]

To be consistent with this transformation rule, we have to define

\[
a_0(x, t) = g \int dy A(x, y) \cdot j_m(y, t) \tag{31}
\]

With this definition we find that the equation \( \mathbf{E}(x, t) = -\nabla \nu \theta_0(x, t) - \beta \mathbf{A}(x, t) \) is equivalent to \( \mathbf{E}(x, t) = -\nabla \nu \theta_0(x, t) + \mathbf{e}(x, t) \), which is a consistency check.

The full quantum theory is based on the path integral

\[
\int Da Da_0 Dc^\dagger Dc D\phi^\dagger D\phi \exp[i \int dt d^3L(x, t)] \tag{32}
\]

where the Lagrangian density is

\[
L = L_f + L_e + L_\theta + L_\lambda \tag{27}
\]

\[
= \frac{1}{2} (-\partial_t \mathbf{a} - \nabla \alpha_0)^2 - \frac{1}{2} (\nabla \times \mathbf{a})^2 + \frac{1}{2\lambda} (\nabla \cdot \mathbf{a})^2 + \psi^\dagger (i\partial_t + i \sum_{i=1}^3 \alpha_i \partial_i - \beta M) \psi + c^\dagger (iD_t + \frac{3}{2} \sum_{i=1}^3 \alpha_i D_i - \beta \lambda)c
\]

\[
+ \frac{1}{2} (\mathbf{e}^\dagger)^2 - \frac{1}{2} (\mathbf{b}^\dagger)^2 - (\partial_t \mathbf{a}) \cdot \mathbf{e} \tag{33}
\]

Eq.\((33)\) is a central equation of the present paper. Now the dynamical part ("photon") of electromagnetic field, described by \( \mathbf{a} \), is neatly separated out, while \( \mathbf{e}' \) and \( \mathbf{b}' \) are kinematic fields fully determined by electrons and monopoles[see Eq.\((2)\) and Eq.\((7)\)]. Various interactions are described by the covariant derivatives and the last three terms of Eq.\((33)\). The monopole-photon interaction is automatically included in \( L_f \), without the need of being put in by hand.

It is worth emphasizing that the partial derivative \( \partial_i \) appears in \( L_e \), while the covariant derivative \( D_i \) appears in \( L_f \). This feature resonates with the classical Lagrangian theory\([1,4]\).

Taking advantage of Eq.\((19)\), Eq.\((20)\), and Eq.\((21)\), we have the following more explicit expression for Eq.\((33)\)

\[
\int dx L(x, t) = \frac{1}{2} \int dx (-\partial_t \mathbf{a} - \nabla \alpha_0)^2 + \frac{1}{2} \int dx (\nabla \times \mathbf{a})^2
\]
+ \frac{1}{2\lambda} \int d^3x (\nabla \cdot a)^2 + g \int d^3xdy D(x,y)[(\nabla \times \partial_t a(x,t)) \cdot \mathbf{j}_m(y,t)
+ \int d^3x \psi^\dagger (i\partial_t + i \sum_{i=1}^3 \alpha_i \partial_i - \beta M)\psi
+ \int d^3x c^i (iD_t + i \sum_{i=1}^3 \alpha_i \partial_i - \beta M)\psi
+ \frac{e^2}{2} \int d^3xdy \rho_e(x,t)D(x-y \cdot \rho_e(y,t)
- \frac{g^2}{2} \int d^3xdy \rho_m(x,t)D(x-y \cdot \rho_m(y,t)
+ \frac{g^2}{2} \int d^3xdy D(x,y)\mathbf{j}_m^T(x,t) \cdot \mathbf{j}_m^T(y,t)
(34)

In the Coulomb gauge (the \( \lambda = 0 \) gauge), \( a_0 \) can be straightforwardly integrated out, yielding the electrical Coulomb energy. In this gauge the full Lagrangian can be recast as

\int d^3x L(x,t) = \frac{1}{2} \int d^3x (\partial_t a)^2 - \frac{1}{2} \int d^3x (\nabla \times a)^2
+ g \int d^3xdy D(x,y)[(\nabla \times a) \cdot \mathbf{j}_m(y,t)
+ \int d^3x c^i [i\partial_t + i e\alpha_0^i + i \sum_{i=1}^3 \alpha_i D_i - \beta M]c
- \frac{e^2}{2} \int d^3xdy \rho_e(x,t)D(x-y) \cdot \rho_e(y,t)
- \frac{g^2}{2} \int d^3xdy \rho_m(x,t)D(x-y) \cdot \rho_m(y,t)
+ \frac{g^2}{2} \int d^3xdy D(x,y)\mathbf{j}_m^T(x,t) \cdot \mathbf{j}_m^T(y,t)
(35)

where \( a^T \) denotes the transverse part of \( a \), and the longitudinal part of \( a \) is absent. In the momentum space, the last three terms of Eq. (35) read

- \sum_q \frac{1}{2q^2} [e^2 \rho_e(q,t) \partial_q(-q,t) + g^2 \rho_m(q,t) \rho_m(-q,t)]
+ \sum_q \frac{g^2}{2q^2} (\delta_{ij} - \frac{q_i q_j}{q^2}) f_m(q,t) f_m^*(-q,t)
(36)

IV. EQUATIONS OF MOTION

We will first discuss equations of motion as extremal conditions in the variation method, then we proceed to promote them to quantum equations of motion, which is readily done in the path integral formulation.

We take the Lagrangian of electron-monopole-photon systems, namely Eq. (34), as our starting point. We now show that the Maxwell equations, and the Dirac equations and Lorentz equations for both electrons and monopoles can be obtained from variation method (We have to discard the gauge fixing term \( L_4 \) in taking variation). Let us study the Maxwell equations first. The Maxwell equations (a) and (b) are automatically satisfied by our formulation, thus we only need to establish (c) and (d).

The action is defined as the integral of Lagrangian density, namely \( S = \int dt d^3x L(x,t) \). The variational equation \( \delta S / \delta a_0 = 0 \) leads to

- \nabla \cdot (\partial_t a + \nabla a_0) - e\partial \psi = 0
(37)

Because of \( \nabla \cdot e' = 0 \), this equation is equivalent to

\nabla \cdot (\partial_t a + \nabla a_0 + e') = e\partial \psi
(38)

which is simply the Maxwell equation (c).

The variational equation \( \delta S / \delta a = 0 \) leads to

- \partial_t (\partial_t a + \nabla a_0) - \nabla \times (\nabla \times a) - \partial_t e' + e\mathbf{j}_e = 0
(39)

in which the first two terms are obvious, the \( \partial_t e' \) term comes from the fourth term of Eq. (34), and the \( e\mathbf{j}_e \) term comes from the sixth term of Eq. (34). Eq. (39) can be rewritten as

- \partial_t (\partial_t a + \nabla a_0 + e') + \nabla \times (\mathbf{b} + \mathbf{b}') = e\mathbf{j}_e
(40)

where we have used \( \nabla \times \mathbf{b}' = 0 \). Eq. (40) is simply the Maxwell equation (d).

Now let us take variation of the action with respect to \( e' \). Because only the sixth term in Eq. (34) contributes, the result is simply

(\partial_t + i e\alpha \cdot \partial - \beta M)\psi(x,t)
- eg \int d^3y \mathbf{A}(x,y) \cdot \psi(x,t)\rho_e(y,t)
+ eg \int d^3y \mathbf{A}(x,y) \cdot \mathbf{j}_e(y,t) \psi(x,t)
+ g \int d^3y D(x,y)[\nabla \times a \cdot \mathbf{A}(x,t)] \cdot \alpha \psi(x,t)
+ g^2 \int d^3y D(x,y)\mathbf{j}_m^T(y,t) \cdot \alpha \psi(x,t)
- g^2 \int d^3y D(x,y)\rho_m(y,t) \cdot \alpha \psi(x,t)
= 0
(42)

which can be recast as

(\partial_t + i e\alpha \cdot \partial - \beta M)\psi = 0
(43)

where \( \partial_t = \partial_t + ig\Lambda_0 \) and \( \partial_{\Lambda} = \partial_t - ig\Lambda_t \), the potentials \( \Lambda_0 \) and \( \Lambda \) being given by

\( \Lambda_0(x,t) = -e \int d^3y \mathbf{A}(x,y) \cdot \mathbf{j}_e(y,t) + g \int d^3y D(x,y)\rho_m(y,t), \)

\( \Lambda(x,t) = -e \int d^3y \mathbf{A}(x,y)\rho_e(y,t) + \int d^3y D(x,y)\nabla \times a(y,t)
+ g \int d^3y D(x,y)\mathbf{j}_m(y,t)
(44)
In the path integral formalism, the classical equations of motion can be readily promoted to quantum equations of motion for the correlation functions of operators in the Heisenberg picture\(^2\). Let us take the Dirac equation for electrons as an example. By the invariance of the path integral (with insertions of field variables at arbitrary spacetime points) and the correlation functions of operators in Heisenberg picture (see Ref.\(^2\)), we have

\[
\langle \Omega | T (iD_t + i \sum_{i=1}^{3} \alpha_i D_i - \beta m) c_H \psi (x, t) | \Omega \rangle = -i \delta (x - x_0) \delta (t - t_1) \tag{47}
\]

where \(T\) denotes time ordering, \(|\Omega\rangle\) denotes the vacuum state (or ground state), and the subscript “\(H\)” refers to the Heisenberg picture. The simple equation \(\delta S / \delta c^\dagger (x, t) = (iD_t + i \sum_{i=1}^{3} \alpha_i D_i - \beta m) c(x, t)\) has been used in deriving Eq.(47). More concisely, we can write down the operator equation

\[
(iD_t + i \sum_{i=1}^{3} \alpha_i D_i - \beta m) c_H (x, t) = 0 \tag{48}
\]

In this way classical equations of motion can be translated into operator equations. Similarly, the Dirac equation for monopoles, and the Maxwell equations can be translated into operator equations.

Now let us study the fate of Lorentz equations. In classical electrodynamics, an electron feels Lorentz force in an electromagnetic field, such that its momentum satisfies \(p = e(E + v \times B)\), where \(v\) is the velocity of the electron. Similarly, a monopole satisfies a dual Lorentz equation \(\tilde{p} = g(B - v \times E)\), in which \(v\) is the velocity of the monopole. We would like to find the counterparts of classical Lorentz equations in our formulation.

First we study the Lorentz equation for electrons. The momentum \(p_t\) in the classical Lorentz equation is replaced by the local operator \(c_H^\dagger (-iD_t) c_H (x, t)\), and we have the operator equation

\[
\frac{d}{dt} [ c_H^\dagger (-iD_t) c_H ] = \langle c_H^\dagger (-iD_t) c_H | e_\psi (x, t) \rangle \tag{49}
\]

or equivalently,

\[
\frac{d}{dt} [ c_H^\dagger (-iD_t) c_H ] = e \rho_e E + e j_e \times B \tag{50}
\]

where \(E\) and \(B\) are also understood as operators in the Heisenberg picture. This is the “Lorentz equation” in our formulation. It is a local operator equation. Similarly, we can obtain an Lorentz equation for monopoles, which reads

\[
\frac{d}{dt} [ \psi^\dagger (-D) \psi_H ] = g \rho_m E + g j_m \times B \tag{51}
\]

where \(\tilde{E} = - \partial_\alpha \tilde{A} - \nabla \tilde{A}_0\) and \(\tilde{B} = \nabla \times \tilde{A}\). In the Appendix C we show that

\[
\tilde{B} (x, t) = -E (x, t) \tag{52}
\]

and

\[
\tilde{E} (x, t) = B (x, t) \tag{53}
\]

therefore, Eq.(51) can be recast as

\[
\frac{d}{dt} [ \psi^\dagger (-D) \psi_H ] = g \rho_m B - g j_m \times E \tag{54}
\]

To summarize this section, we have established that all Maxwell equations and Lorentz equations hold in our formulation as operator equations. The efficiency of promoting classical equations of motion to operator equations in the path integral formalism is notable.

\[\text{V. DUAL FORMULATION}\]

In Sec.[V] we have implicitly touched the dual description, where the dual electromagnetic field \(\tilde{B}\) and \(\tilde{E}\) were used. To discuss the dual formulation in a transparent way, we define the dual quantities

\[
\tilde{e} = g, \quad \tilde{g} = -e, \tag{55}
\]

\[
\tilde{\rho}_e = \tilde{e} \tilde{\psi}^\dagger \tilde{\psi} = \rho_m, \quad \tilde{\rho}_m = c^\dagger c = \rho_e, \tag{56}
\]

\[
\tilde{j}_e = \psi^\dagger \alpha \tilde{\psi} = j_m, \quad \tilde{j}_m = c^\dagger \alpha c = j_e, \tag{57}
\]

and the covariant derivatives

\[
\tilde{D}_t = \partial_t + i \tilde{e} \tilde{A}_0 = \partial_t - i \tilde{e} (\tilde{a}_0 + \tilde{a}_t), \tag{59}
\]

\[
\tilde{D}_t = \partial_t - i \tilde{e} \tilde{A}_0 = \partial_t - i \tilde{e} (\tilde{a}_t + \tilde{a}_0), \tag{59}
\]

The dual equations of Eq.(8) read

\[
\tilde{E} = -\partial_\alpha \tilde{a} - \nabla \tilde{a}_0 + \tilde{e}'; \quad \tilde{B} = \nabla \times \tilde{a} + \tilde{B}' \tag{60}
\]

in which \(\tilde{e}'\) is defined as

\[
\tilde{e}' (x, t) = \tilde{g} \int dy G (x, y) \times \tilde{j}_m (y, t) \tag{61}
\]

\[\int dy G (x, y) \times \tilde{j}_m (y, t) \tag{61}\]
which is dual to Eq. (7). Similarly, \( \vec{B}' \) is defined as the dual equation of Eq. (2). Together with the relation \( \vec{B} = -E \), Eq. (60) implies

\[
\vec{a}(x, t) = \int dyD(x, y)\nabla_y \times \partial_t a(y, t) + g \int dyD(x, y)j^T_\alpha(y)\]

which is also suggested by Eq. (44). Similarly, we also have [30]

\[
a(x, t) = -\int dyD(x, y)\nabla_y \times \partial_t \vec{a}(y, t) + e \int dyD(x, y)j^T_\alpha(y)\]

As a consistency check, we can see that Eq. (63) can be obtained from Eq. (62) by adding an overbar to each variable and using \( \bar{a} = -a \).

In the remainder of this section, we would like to show that a dual description can be obtained by changing the variables of path integral from \( (a_0, \vec{a}) \) to \( (\bar{a}_0, \bar{\vec{a}}) \). Furthermore, we show that the dual description is equivalent to the original description. For the purpose of this section, it is convenient to use a more compact but equivalent expression for the Lagrangian density, which reads

\[
L = \frac{1}{2}(E^2 - B^2) + \rho \cdot \partial_t \vec{a} + \vec{a} \cdot \partial_t \vec{a} + \vec{a} \cdot \vec{B} + \vec{E} \cdot \vec{D} + \lambda, 
\]

in which \( E \) and \( B \) is given by Eq. (63). It is readily seen that the Lagrangian density given in Eq. (64) is equal to the one given in Eq. (63). For simplicity we take the \( \lambda \to 0 \) limit (the Coulomb gauge) in this section, such that only transverse modes of \( a \) remain.

In the dual description, we use the dual gauge potentials \( \bar{a} \) and \( \bar{\vec{a}} \) as the fundamental variables in path integral. The dual Lagrangian \( \bar{L} \) is obtained from \( L \) by simply adding an overbar to each electromagnetic quantity. It is given as

\[
\bar{L} = \frac{1}{2}(\bar{E}^2 - \bar{B}^2) + \rho \cdot \partial_t \bar{\vec{a}} + \bar{\vec{a}} \cdot \partial_t \bar{\vec{a}} + \bar{\vec{a}} \cdot \bar{\vec{B}} + \bar{\vec{E}} \cdot \bar{\vec{D}} + \lambda, 
\]

in which the gauge fixing term in Eq. (65) is given as \( \lambda = \frac{1}{2\lambda}(\nabla \cdot \bar{a})^2 \).

It is a straightforward exercise to expand the dual Lagrangian \( \bar{L} \) as

\[
\bar{L} = \frac{1}{2}(\partial_t \bar{a} - \nabla \bar{\vec{a}}) - \frac{1}{2}(\nabla \times \bar{\vec{a}})^2 + \frac{1}{2\lambda}(\nabla \cdot \bar{\vec{a}})^2 + \rho \cdot \partial_t \bar{\vec{a}} + \bar{\vec{a}} \cdot \partial_t \bar{\vec{a}} + \bar{\vec{a}} \cdot \bar{\vec{B}} + \bar{\vec{E}} \cdot \bar{\vec{D}} + \lambda, 
\]

Compared to Eq. (63), the covariant derivative in Eq. (66) is associated with monopoles instead of electrons.

Now we would like to show that the difference \( L - \bar{L} \), without inclusion of the gauge fixing terms \( L_1 \) and \( \bar{L}_1 \), is actually a total derivative, therefore, the two Lagrangian \( \int dxL(x, t) \) and \( \int d\bar{x}\bar{L}(x, t) \) are equivalent. In fact, with the input of Eq. (58), we have

\[
\int dxL - \int d\bar{x}\bar{L} = \int dx[E^2 - \rho \cdot \partial_t a + e \cdot j_e + \bar{\vec{a}} \cdot \partial_t \bar{\vec{a}} - \bar{\vec{a}} \cdot \bar{\vec{E}} - \bar{\vec{D}} \cdot \bar{\vec{a}}] + \lambda, 
\]

in which we have excluded the gauge fixing terms \( L_1 \) and \( \lambda_1 \). It is not difficult to check that the last four terms in the parenthesis vanish. Moreover, the Maxwell equations can be used to rewrite \( e \int dx(-\rho \cdot \partial_t a + e \cdot j_e) \) as

\[
e \int dx(-\rho \cdot \partial_t a + e \cdot j_e) = \int dx [\partial_t (\bar{\vec{E}} - \bar{\vec{B}}) + \bar{\vec{a}} \cdot \partial_t \bar{\vec{a}} - \bar{\vec{E}} \cdot \bar{\vec{B}}] + \partial_t (\ldots) 
\]

in which \( \partial_t (\ldots) \) denotes total derivatives with respect to \( t \). It follows that

\[
\int dxL - \int d\bar{x}\bar{L} = \int dx[\bar{\vec{E}} \cdot \partial_t \bar{\vec{a}} - \bar{\vec{E}} \cdot \bar{\vec{B}} + \bar{\vec{a}} \cdot \partial_t \bar{\vec{a}} - \bar{\vec{E}} \cdot \bar{\vec{B}}] + \lambda. 
\]

In the Coulomb gauge (\( \lambda = 0 \)) in use, we have \( \bar{\vec{E}} \cdot \partial_t \bar{\vec{a}} = \int dx \bar{\vec{a}} \cdot \partial_t \bar{\vec{a}} = \int dx \bar{\vec{a}} \cdot \bar{\vec{a}} = \int dx \bar{\vec{a}} \cdot \partial_t \bar{\vec{a}} \) and \( \partial_t \bar{\vec{a}} \) remain.

In addition, we can make use of Eq. (62) and rewrite the second term in Eq. (70) as

\[
\int dx \bar{\vec{E}} \cdot \partial_t \bar{\vec{a}} = -g \int dx dy \bar{\vec{E}} \cdot [\nabla \times \bar{\vec{D}}(x, y) j_m(x, t)], 
\]

in which the first term can be recast, according to Eq. (5), as

\[
\int dx \bar{\vec{E}} \cdot \partial_t \bar{\vec{a}} = g \int dx dy dz \bar{\vec{E}} \cdot [\nabla \times \bar{\vec{D}}(x, y) j_m(x, t)], 
\]

By summing Eq. (71) and Eq. (72), it is now straightforward to see that Eq. (70) reads

\[
\int dxL - \int dx\bar{L} = \partial_t (\ldots) 
\]

which is the central result of this section.

In the original description with Lagrangian \( L \) given in Eq. (59) or Eq. (64), electron-photon interaction is apparent
in the covariant derivative, while the monopole-photon interaction comes from \( L_f = \frac{1}{4}(E^2 - B^2) \). In the dual description with Lagrangian given in Eq. (65) or Eq. (66), monopole-photon interaction is apparent in the covariant derivative, while electron-photon interaction comes from \( \bar{\rho} \) total derivative terms, therefore, \( L_f = \frac{1}{4}E^2 \). It is reassuring to see in this section that \( L = L_f + \bar{\rho} \) describe the same physics, as they should do. The interested readers are also referred to Ref. [14] for the dual transformation of the classical Lagrangian.

VI. EFFECTIVE MONOPOLE-MONPOLE INTERACTION: A CONSISTENCY CHECK OF THE PROPOSED LAGRANGIAN

In our formulation monopoles are coupled to electromagnetic field in a unusual manner. For instance, there is a \( q_0 \) factor in the monopole-photon interaction found in Eq. (22). We would like to design nontrivial checks for it. We expect that different approaches lead to the same result for the same system, otherwise our theory would be in trouble.

For simplicity of notations, let us consider monopole-photon systems without the complication of electrons. We would like to calculate the effective action of monopoles after photons are integrated out. There are two methods to do this, as given in Sec VI A and Sec VI B below, which, by the internal consistency of our formulation, should lead to the same result.

A. Effective monopole-monopole interaction in the dual description

This method is the easier one. Because of electromagnetic duality, monopole-photon problem is equivalent to electron-photon problem. In other words, we can regard the monopole-photon problem as the dual of quantum electrodynamics. In this approach, magnetic charges are minimally coupled to the electromagnetic field as

\[
\bar{e}(\vec{a} \cdot \vec{j}_e - \bar{a}_0 \rho_e),
\]

in which \( \vec{j}_e \equiv \rho_m \vec{a}_0 \), and \( \bar{e} = g \). The quantities with overbar are the dual variables (see Sec V). It is straightforward to integrate out photons in the Coulomb gauge, yielding the effective action for monopoles as

\[
S_{eff} = \int dt dx \mathcal{L}_g - \frac{\bar{e}^2}{2} \sum_q \frac{1}{q^2}(\delta_{ij} - \frac{q_i q_j}{q^2})\bar{j}_m(q)\bar{j}_m(-q) - \frac{\bar{e}^2}{2} \sum_q \frac{1}{q^2}\bar{\rho}_m(q)\rho_m(-q),
\]

or equivalently,

\[
S_{eff} = \int dt dx \mathcal{L}_g - \frac{g^2}{2} \sum_q \frac{1}{q^2}(\delta_{ij} - \frac{q_i q_j}{q^2})j^e_m(q)j^e_m(-q) - \frac{g^2}{2} \sum_q \frac{1}{q^2}\rho_m(q)\rho_m(-q)
\]

where \( q = (q_0, \mathbf{q}) \), and \( q^2 \equiv q_0^2 - \mathbf{q}^2 \). The second term describes current-current interaction, and the last term describes the magnetic Coulomb energy.

B. Effective monopole-monopole interaction in the original description

In the original description, monopole-monopole interaction is described by the Lagrangian density given in Eq. (33). There are two contributions to the effective magnetic current-current interaction. The first part is mediated by \( a_e \), the interaction vertex being given by Eq. (22). Its contribution to the effective action of monopoles is found to be

\[
-\frac{g^2}{2} \sum_{q} \frac{1}{q^2} \left( \delta_{ij} - \frac{q_i q_j}{q^2} \right) j^e_m(q) j^e_m(-q),
\]

which looks quite different from the current-current interaction in Eq. (76), because of the awkward \( q_0^2/q^2 \) factor. Fortunately, there is a second contribution to the current-current interaction, namely the \( \frac{1}{2}(e')^2 \) term, which is simplified in Eq. (55). Adding these contributions together, we have the total current-current interaction

\[
-\frac{g^2}{2} \sum_q \frac{1}{q^2} \left( \delta_{ij} - \frac{q_i q_j}{q^2} \right) j^e_m(q) j^e_m(-q)
\]

This current-current interaction is the same as the one found in Eq. (76). The awkward \( q_0 \) factor turns out to be an indispensable part of the entire theory. The Coulomb energy is given by the \( \frac{1}{2}(b')^2 \) term in the Lagrangian density, and also equals to the last term of Eq. (76). Therefore, the complete effective action takes the same form as Eq. (76).

This exact match between two vastly different approaches reinforces our confidence in the validity of Eq. (53).

As a final remark to this section, we mention that the effective electron-monopole interaction can also be found in the original and dual descriptions, with matching results. Without going into details, we note that the effective electron-monopole interaction mediated by photon takes the form of \( egq_0 \mathbf{a} \cdot \vec{j}_m(q) \times j_m(-q)q_0^2 - |\mathbf{q}|^2 \). This part of the effective electron-monopole interaction can also be obtained using the Hamiltonian formalism [16], though the present approach is more convenient (e.g. The appearance of \( q_0 \) factor is less straightforward in the approach of Ref. [16]).
VII. FINAL REMARKS

In this paper we have formulated a method for the quantization of electron-monopole-photon systems through the path integral approach. In this formulation, the electron-photon, monopole-photon, and electron-monopole interactions emerge in a natural manner, for instance, the monopole-photon interaction is automatically generated from the Lagrangian of electromagnetic fields. In our formulation no Dirac string is involved, thanks to the language of fiber bundle.

Our formulation is applicable in both relativistic and non-relativistic cases. Since the Coulomb gauge is used, Lorentz invariance is not manifest in this formulation. In the present paper we have not addressed the problem of renormalization, which is left for future works.

On the one hand, the interaction of electrons, magnetic monopoles, and photons is a fundamental theoretical topic. On the other hand, magnetic monopoles have long been candidates of fundamental particles in high energy physics. Recently monopoles have also found renewed interests in condensed matter physics [31, 37]. We thus hope that our approach can be applied to a variety of systems.

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Appendix A: Derivation of Eq. (17)

In this appendix we show that Eq. (17) is a solution of Eq. (5). Taking the curl of Eq. (17), we have

\[ \nabla \times e'(x, t) = g \nabla \times \left[ \int dy G(x, y) \times j_m(y, t) \right] \]

\[ = g \int dy j_m(y, t) \nabla \cdot G(x, y) + g \int dy [j_m(y, t) \cdot \nabla] G(x, y) \]

\[ = -g \int dy j_m(y, t) \delta(x-y) - g \int dy [j_m(y, t) \cdot \nabla'] G(x, y) \]

\[ = -g j_m(x, t) + g \int dy [\nabla' \cdot j_m(y, t)] G(x, y) \]

\[ = -g j_m(x, t) - g \int dy \partial \rho_m(y, t) G(x, y) \]

\[ = -g (j_m(x, t) - g \partial \int dy \rho_m(y, t) G(x, y)) \]  

(A1)

where “\( \nabla' (\nabla') \)” refers to the gradient with respect to \( x (y) \). In this calculation we have used the law of conservation of magnetic charge. Combining Eq. (A1) with the definition of \( b' \), namely Eq. (2), we have

\[ \nabla \times e'(x, t) + \partial_t b'(x, t) = -g j_m(x, t) \]  

(A2)

which is just Eq. (7). We also note that \( \nabla \cdot e' = 0 \).

Appendix B: Simplification of Eq. (16)

We have recast the last three terms of Eq. (16) as the expressions presented in Eq. (19), Eq. (20), and Eq. (21). In this appendix we provide calculational details.

First let us provide a derivation for Eq. (19). By a somewhat tedious calculation [38], we have

\[ \frac{1}{2} \int dx [e'(x, t)]^2 \]

\[ = \frac{g^2}{2} \int dx dy dz [G(x, y) \times j_m(y, t)] \cdot [G(x, z) \times j_m(z, t)] \]

\[ = \frac{g^2}{2} \int dx dy dz [\nabla D(x, y) \times j_m(y, t)] \cdot [\nabla D(x, z) \times j_m(z, t)] \]

\[ = \frac{g^2}{2} \int dx dy dz \nabla \times [D(x, y) j_m(y, t)] \cdot [\nabla \times (D(x, z) j_m(z, t))] \]

\[ = \frac{g^2}{2} \int dx dy dz D(x, z) j_m(z, t) \cdot [\nabla \times [D(x, y) j_m(y, t)]] + \partial_t (x-y) j_m(y, t) \]

\[ = \frac{g^2}{2} \int dx dy D(x, y) j_m(x, t) \cdot j_m(y, t) \]

which is just Eq. (19).

Next we shall derive Eq. (20). By a somewhat tedious calculation [38], we have

\[ \frac{1}{2} \int dx [e'(x, t)]^2 \]

\[ = \frac{g^2}{2} \int dx dy dz [G(x, y) \times j_m(y, t)] \cdot [G(x, z) \times j_m(z, t)] \]

\[ = \frac{g^2}{2} \int dx dy dz [\nabla D(x, y) \times j_m(y, t)] \cdot [\nabla D(x, z) \times j_m(z, t)] \]

\[ = \frac{g^2}{2} \int dx dy dz \nabla \times [D(x, y) j_m(y, t)] \cdot [\nabla \times (D(x, z) j_m(z, t))] \]

\[ = \frac{g^2}{2} \int dx dy dz D(x, z) j_m(z, t) \cdot [\nabla \times [D(x, y) j_m(y, t)]] + \partial_t (x-y) j_m(y, t) \]

\[ = \frac{g^2}{2} \int dx dy D(x, y) j_m(x, t) \cdot j_m(y, t) \]

\[ + \frac{g^2}{2} \int dx dy dz D(x, z) j_m(z, t) \cdot [\nabla \times [D(x, y) j_m(y, t)]] \]

\[ = \frac{g^2}{2} \int dx dy D(x, y) j_m(x, t) \cdot j_m(y, t) \]

\[ + \frac{g^2}{2} \int dx dy dz D(x, z) j_m(z, t) \cdot [\nabla \times [D(x, y) j_m(y, t)]] \]

\[ = \frac{g^2}{2} \int dx dy D(x, y) j_m(x, t) \cdot j_m(y, t) \]

\[ + \frac{g^2}{2} \int dx dy dz [\nabla \times [D(x, z) j_m(z, t)]] \cdot [\nabla D(x, y) \times j_m(y, t)] \]

\[ = \frac{g^2}{2} \int dx dy D(x, y) j_m(x, t) \cdot j_m(y, t) \]
\[
\frac{g^2}{2} \int dx dy dz \{ \nabla D(x, y) \cdot j_m(y, t) \} \{ \nabla D(x, z) \cdot j_m(z, t) \}
\]

\[
= \frac{g^2}{2} \int dx dy D(x, y) j_m^\dagger(x, t) \cdot j_m(y, t)
\]  
(B2)

where \( \nabla \) denotes differential operators with respect to the coordinate \( x \). The expression \( j_m^\dagger(q) \) denotes the transverse part of \( j_m(q) \). In momentum space, \( j_m^\dagger \) is defined by

\[
j_m^\dagger(q) \equiv \frac{[q \times j_m(q)] \times q}{|q|^2} = j_m(q) - \frac{q \cdot j_m(q)}{|q|^2} q
\]  
(B3)

or

\[
j_m^\dagger(i)^j(q) \equiv (\delta_{ij} - \frac{q_i q_j}{|q|^2}) j_m^\dagger(q)
\]  
(B4)

where \( j_m^\dagger \) denotes the \( i \)-th component of \( j_m \), and similarly for \( j_m^\dagger \).

Eq. (B2) can be recast in momentum space as

\[
\frac{1}{2} \int dx [e'(x, t)]^2
= \frac{g^2}{2} \sum_q j_m^\dagger(q) DJ_{ij}(q) j_m^\dagger(-q, t)
= \frac{g^2}{2} \sum_q \frac{1}{|q|^2} (\delta_{ij} - \frac{q_i q_j}{|q|^2}) j_m^\dagger(q) j_m^\dagger(-q)
\]  
(B5)

We would like to mention that the last line of Eq. (B2) can also be written as \( \frac{1}{2} \int dx dy j_m^\dagger(q) D_{ij}^\dagger(q) j_m^\dagger(q) \), where we have defined the shorthand notation

\[
D_{ij}^\dagger(x, y) = D(x, y) \delta_{ij} - \int dz \frac{\partial}{\partial z_i} D(z, x) \frac{\partial}{\partial z_j} D(z, y)
\]  
(B6)

In momentum space we have

\[
D_{ij}^\dagger(q) = \frac{1}{|q|^2} (\delta_{ij} - \frac{q_i q_j}{|q|^2})
\]  
(B7)

The last term in Eq. (16) has been simplified to the expression given in Eq. (21). The detail of this calculation is given as follows

\[
- \int dx [\partial_i a(t)] \cdot e' = - \int dx dy [\partial_i a(x, t)] \cdot [G(x, y) \times j_m(y, t)]
= \int dx dy [\partial_i a(x, t)] \cdot [\nabla \times \{ D(x, y) \}]
= - \int dx dy \nabla \cdot \{ \partial_i a(x, t) \times j_m(y, t) \} D(x, y)
+ \int dx dy D(x, y) j_m(y, t) \cdot [\nabla \times \partial_i a(x, t)]
= \int dx dy D(x, y) j_m(y, t) \cdot \nabla \cdot \partial_i a(x, t)
= g \int dx dy D(x, y) \psi'(y, t) \alpha \cdot [\nabla \partial_i a(x, t)] \psi(y, t)
\]  
(B8)

Appendix C: Calculations of dual electric and magnetic fields

We would like to calculate the dual electromagnetic fields from the dual potential defined in Eq. (44). First let us calculate \( \mathbf{B} \equiv \nabla \times \mathbf{A} \). It is found to be

\[
\mathbf{B}(x, t) = -e \int dy G(x, y) \rho_e(y, t) - g \int dy G(x, y) \times j_m(y, t)
+ \partial_i a^T(x, t)
= e \nabla \cdot \{ \int dy D(x, y) \rho_e(y, t) \} - e' \mathbf{e}(x, t) + \partial_i a^T(x, t)
= - \nabla a_0(x, t) - e' \mathbf{e}(x, t) + \partial_i a^T(x, t)
= -e \mathbf{E}(x, t)
\]  
(C1)

where \( a^T \) is the transverse part of \( a \). Here we have chosen the Coulomb gauge \( (\lambda = 0 \) gauge), thus \( e \int dy D(x, y) \rho_e(y, t) = a_0(x, t) \). In the derivation of Eq. (C1), a useful intermediate step is to rewrite \( \int dy \nabla_y D(x, y) \times [\nabla_y \times \partial_i a(y, t)] \) according to the well known formula \( \nabla (X \cdot Y) = X \times (\nabla \times Y) + (\nabla \times X) \cdot Y \), while the \( Y \) is dual \( X \).

Now let us turn to the dual electric field, namely, \( \mathbf{E} \equiv -\nabla A_0 - \partial_\mathbf{A} \). According to Eq. (44), we find that

\[
\mathbf{E}(x, t) = b'(x, t) - e \int dy G(x, y) \times j_m(y, t)
- \int dy D(x, y) \nabla_y \times \partial^2_i a^T(y, t) - g \int dy D(x, y) \partial_i j_m^\dagger(y, t)
\]  
(C2)

Now it becomes more convenient to translate this equation into momentum space, wherein it reads

\[
\mathbf{E}(q) = b'(q) + \frac{e q \times j_m(q)}{|q|^2} + \frac{ig_0 j_m^\dagger(q)}{|q|^2} + g \frac{q \times j_m(q)}{|q|^2}
\]  
(C3)

in which \( q = (q_0, \mathbf{q}) \). Taking advantage of the transverse part of the quantum equations of motion associated with the Maxwell equations \( (d) \), or more explicitly, Eq. (39), which in momentum space is given as

\[
(q_0^2 - |\mathbf{q}|^2) a^T - gq_0 \frac{q \times j_m(q)}{|q|^2} + e j_m^\dagger(q) = 0,
\]  
(C4)

we can obtain that

\[
\mathbf{E}(q) = i q \times a^T(q) + b'(q).
\]  
(C5)

which in spacetime is exactly

\[
\mathbf{E}(x, t) = \nabla \times a^T(x, t) + b'(x, t)
= \mathbf{B}(x, t)
\]  
(C6)

Appendix D: Comparison with the Hamiltonian formalism

The apparent differences between the present approach and Tu-Wu-Yang’s Hamiltonian approach[16], other than that ordinary numbers instead of noncommutative operators are used here, appear in the form of monopole-photon interaction and the presence of the last term in Eq. (44) or Eq. (45). In spite of these differences in formulation, we shall show that our approach is equivalent to the Tu-Wu-Yang approach[16].
Let us start from the present Lagrangian formulation. We shall work in the Coulomb gauge (the $\lambda = 0$ gauge), in which only transverse part of $\mathbf{a}$ appears. From Eq. (43) (or its promotion as an operator equation in path integral approach), we know that the effective vector potential to which monopoles are coupled is

$$\mathbf{A} = \bar{\mathbf{a}} + \bar{\mathbf{a}}^T,$$

in which

$$\bar{\mathbf{a}} = -e \int dy \mathbf{A}(x, y) \rho(y, t),$$

and

$$\bar{\mathbf{a}}^T(x, t) = \int dy \mathbf{D}(x, y) \nabla_y \times \partial_t \bar{\mathbf{a}}^T(y, t)$$

$$+ g \int dy \mathbf{D}(x, y) \mathbf{\bar{j}}_0^T(y, t).$$

Let us compare this with the gauge potential to which monopoles are coupled in Ref. [16], namely the Eq.(3.2) in Ref. [14]. Eq. (D2) can apparently be identified with the first term of Eq.(3.2) of Ref. [16]. Now let us investigate Eq. (D3). According to the formulas $\mathbf{B} = \nabla \times \bar{\mathbf{a}}^T + \bar{\mathbf{b}}^T$, $\mathbf{E} = -\partial_t \mathbf{a}^T - \nabla \phi_0 + \mathbf{e}$, and the dual relation $\mathbf{B} = -\mathbf{E}$, we have

$$\partial_t \mathbf{a}^T = \nabla \times \bar{\mathbf{a}}^T + \mathbf{e}$$

Eq. (D4) can also be obtained by directly calculating the curl of $\bar{\mathbf{a}}^T$. Similarly, we have

$$\partial_t \bar{\mathbf{a}}^T = -\nabla \times \bar{\mathbf{a}}^T + \mathbf{e}^T$$

(D5)

We emphasize that Eq. (D4) and Eq. (D5) can be regarded as operator equations, according to the correspondence between classical equations and operator equations in the path integral formalism.[40]

Now the connection to the Tu-Wu-Yang approach becomes clear. In the Tu-Wu-Yang formalism,[16] the electron-photon and monopole-photon interaction are given by their Eq.(3.2) and Eq.(3.3), in which the evolution of $\mathbf{A}^T$ and $\mathbf{B}^T$[41] is fully determined by their Eq.(4.6) and Eq.(4.7). It is evident that our Eq. (D4) and Eq. (D5) take the same forms as Eq.(4.6) and Eq.(4.7) in Ref. [16], which allows us to identify $\mathbf{a}^T$ and $\bar{\mathbf{a}}^T$ in our paper as $\mathbf{A}^T$ and $\mathbf{B}^T$ in Ref. [16] respectively. Therefore, the couplings of monopoles to electromagnetic field are essentially the same in these two approaches, though they are seemingly different.

To be more explicit, we can solve Eq.(4.6) of Ref. [16], and find that their $\mathbf{B}^T$ can be expressed in terms of $\mathbf{A}^T$ as

$$\mathbf{B}^T(x, t) = \int dy \mathbf{D}(x, y) \nabla_y \times \partial_t \mathbf{A}^T(y, t)$$

$$+ g \int dy \mathbf{D}(x, y) \mathbf{\bar{j}}_0^T(y, t),$$

(D6)

which takes the same form as our Eq. (D3), therefore, we see again that $\mathbf{B}^T(x, t)$ in Ref. [16] can be identified as $\bar{\mathbf{a}}^T$ in our paper (the identification of $\mathbf{A}^T$ in Ref. [16] as our $\mathbf{a}^T$ is obvious). Therefore, the monopole-photon interaction turns out to be essentially the same in Ref. [16] and in our paper.

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[18] In this paper “electron” refers to any particle with electric charge $e$ (we take the convention that $e > 0$). It can be regarded as the positron or the proton.
[19] Due to electromagnetic duality, we can also regard the last two equations as constraints, while the first two as derived from variational principle. This is the dual description.
[20] Different gauge fixing terms amount to different gauge choices (in the Faddeev-Popov approach[43]). See Ref. [21] or Ref. [24] for more details.
[21] M. E. Peskin and D. V. Schroeder, An introduction to quantum field theory (Westview, 1995).
[22] Here $\psi, \bar{\psi}$ are regarded as Grassmann numbers when monopoles are fermionic. Our formulation is also valid when monopoles are bosons, in which case $\psi, \bar{\psi}$ are replaced by ordinary number. Similarly for electrons.
[23] For instance, the last term can be simplified as $-(\nabla \times \mathbf{a}) \cdot \mathbf{b}' + \nabla \cdot (\mathbf{b}' \times \mathbf{a}) + \mathbf{a} \cdot (\nabla \times \mathbf{b}') = \nabla \cdot (\mathbf{b}' \times \mathbf{a})$, which is a total derivative. Similarly for electrons.
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[25] The $\lambda \to \infty$ limit is singular because $D^\mu$ diverges.
[26] In this paper, $\nabla \left( \nabla \cdot \right)$ denotes differential operator with respect to $\mathbf{x}$ (y).
A remark: In our formulation monopoles are inert to gauge transformation of $a$.

Electron-photon systems can also be described in the formalism presented here, provided that we take $(c)$ and $(d)$ of the Maxwell equations as kinematic equations.

Note that “$A(x, t)$” and “$A(x, y)$” refer to different objects.

For simplicity we take the Coulomb gauge ($\lambda = 0$) in this section, therefore, $a$ contains only transverse modes.

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In the above calculation we have used well known formulas in vector analysis, such as $\nabla \cdot (X \times Y) = Y \cdot (\nabla \times X) - X \cdot (\nabla \times Y)$, where $X, Y$ are continuous vector functions.

The calculation for Eq. (C1) is easier if we do it in the momentum space and then translate back into spacetime.

See Sec. IV of the present paper or Ref. [21] for the promotion of classical equations to operator equations in the path integral approach.

Do not confuse about the notational differences between our paper and Ref. [16]. In Ref. [16] both $A^T$ and $B^T$ refer to gauge potential. In our paper $B$ refers to the magnetic field, while $B^T$ never appear.

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