Mobile Robot Navigation in Complex Polygonal Workspaces Using Conformal Navigation Transformations

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Abstract—This work proposes a novel transformation termed the conformal navigation transformation to achieve collision-free navigation of a robot in a workspace populated with arbitrary polygonal obstacles. The properties of the conformal navigation transformation in the polygonal workspace are investigated in this work as well as its capability to provide a solution to the navigation problem. The definition of the navigation function is generalized to accommodate non-smooth obstacle boundaries. Based on the proposed transformation and the generalized navigation function, a provably correct feedback controller is derived for the automatic guidance and motion control of the kinematic mobile robot. Moreover, an iterative method is proposed to construct the conformal navigation transformation in a multi-connected polygonal workspace, which transforms the multi-connected problem into multiple single-connected problems to achieve fast convergence. In addition to the analytic guarantees, the simulation study verifies the effectiveness of the proposed methodology in a workspace with non-trivial polygonal obstacles.

I. INTRODUCTION

The autonomous navigation of robots in cluttered environments is an actively studied topic of robotics research. Control laws based on artificial potential fields (APFs) constitute a widely researched tool for solving the robot navigation problem due to their intuitive design and ability to simultaneously solve path planning and motion planning subproblems [1], [2]. On the other hand, this control method suffers from the presence of local minima and the inability to cope well with non-convex obstacles.

The navigation function [3–5] method proposed by Rimon and Koditschek is an efficient method for constructing a class of local minimal free APFs in the sphere world, which can be extended to environments with star-shaped obstacles by constructing a diffeomorphism. The negative gradient of the navigation function can be served as a control input to guarantee collision-free motion of the kinematic mobile robot and convergence to the goal point from almost all initial points. A large number of successful applications have appeared in the literature, such as multi-robot systems [6–8], highly dynamic or partially known environments [9], [10], uncertain dynamic systems [11], and constrained stabilization problems [12]. In a recent work [13], the construction of navigation functions was extended to non-spherical convex obstacles; however, the construction of navigation functions in geometrically complex spaces is still a challenging problem.

Transforming geometrically complex spaces to geometrically simple spaces is an important and elegant approach to implementing navigation functions in complex workspaces. Techniques based on diffeomorphism and navigation transformation are a few of the promising methods found in the literature to achieve the above objective. In [4], a family of analytic diffeomorphisms was constructed for mapping any star world to an appropriate sphere world, requiring appropriate tuning of a certain parameter. To cope with the complexity of the workspace, the authors in [14] proposed a diffeomorphism based on the harmonic map, which maps the workspace to a punctured disc by solving multiple boundary problems. By applying a two-step navigation transformation, the author in [15] achieved a tuning-free solution by pulling back a trivial solution in the point world to the initial star world. In a continuous work [16], the authors propose a single-step navigation transformation candidate that directly maps a known star world to a point world.

The transformation candidates in most of the aforementioned works are only suitable for smooth star worlds rather than non-smooth workspaces, which hinders the application of navigation functions in realistic environments. In this work, we address the problem of navigating a kinematic mobile robot system in a workspace with arbitrary polygonal obstacles and connectedness by constructing a novel transformation termed the conformal navigation transformation. Unlike the transformations mentioned above, the proposed transformation requires neither tuning a parameter nor decomposing the workspace into trees of stars, and it is unique in a given workspace. Furthermore, the class of navigation functions defined in smooth workspaces is extended to accommodate polygonal obstacles. Then, the proposed transformation can pull back the extended navigation function in the sphere world to the initial workspace, as well as the trivial solution in the sphere world determined by the vector field, so that the navigation problem in the polygonal workspace can be
solved elegantly. Finally, we present a fast iterative method for constructing the proposed transformations in any complex workspace. To the best of the authors’ knowledge, this is the first work to provide a tuning-free transformation candidate that maps any workspace with non-star-shaped polygonal obstacles to a sphere world.

The rest of this paper is organized as follows. Section II introduces the necessary preliminaries. Section III analyzes the properties of the conformal navigation transformation. Section IV presents a mobile robot controller designed using the proposed transformation and the navigation function. Section V proposes the construction method of the conformal navigation transformation. Section VI provides the simulation study to demonstrate the effectiveness of the proposed control scheme. Section VII concludes this paper.

II. PRELIMINARIES

This section gives the necessary terminology and definitions for the development of the methodology.

A. Sphere Worlds and Workspaces

The $n$-dimensional sphere world $\mathcal{M}$ as defined in [3] is a compact connected subset of $\mathbb{R}^n$ whose boundary is formed by the disjoint union of an external $(n-1)$-sphere and $M \in \mathbb{N}$ internal $(n-1)$-spheres. Each internal sphere obstacle $\hat{O}_i$ is the interior of the $(n-1)$-sphere, which is implicitly defined as $\hat{O}_i = \{ q \in \mathbb{R}^n : \| q - q_i \|^2 < \rho_i^2 \}$, $i \in \{1,\ldots,M\}$. For simplicity, the exterior of the external $(n-1)$-sphere is referred to as the zeroth sphere obstacle, denoted as $\check{O}_0$. In this regard, the $n$-dimensional sphere world with $M$ internal obstacles $\mathcal{M}$ is represented as:

$$\mathcal{M} = \{ q \in \mathbb{R}^n : -\| q - q_0 \|^2 \geq \rho_0^2, \| q - q_i \|^2 \geq \rho_i^2, \ldots, \| q - q_M \|^2 \geq \rho_M^2 \}. \tag{1}$$

Clearly, the sphere world is geometrically simple and can be used as a topological model of geometrically complex spaces. Formally, the geometrically complex robot workspace is defined as follows:

Definition 1: The $n$-dimensional robot workspace $\mathcal{W}$ is a connected and compact $n$-dimensional manifold with boundary such that $\mathcal{W}$ is homeomorphic to $\mathcal{M}$.

For a finite $M \in \mathbb{N}$, the internal obstacle $O_i$ of $\mathcal{W}$ is the interior of a connected and compact $n$-dimensional subset of $\mathbb{R}^n$ such that $O_i \cap O_j = \emptyset$, $i \neq j$, $i, j \in \{1,\ldots,M\}$. It is convenient to regard the unbounded component of the workspace’s complement as the zeroth obstacle. Each obstacle can be represented by an obstacle function $\beta_i : \mathbb{R}^n \rightarrow \mathbb{R}$ in the following form:

$$O_i = \{ q \in \mathbb{R}^n : \beta_i(q) < 0 \}, \quad i \in \{0,\ldots,M\}. \tag{2}$$

For the obstacle function, zero is not a critical value and the zero level set of $\beta_i$ defines the obstacle’s boundary: $\partial O_i = \{ q \in \mathbb{R}^n : \beta_i(q) = 0 \}, \quad i \in \{0,\ldots,M\}$. Moreover, according to the implicit function theorem, the boundary of an obstacle is an $(n-1)$-dimensional submanifold of $\mathcal{W}$. Then, the $n$-dimensional workspace with $M$ internal obstacles $\mathcal{W}$ is represented as:

$$\mathcal{W} = \{ q \in \mathbb{R}^n : \beta_0(q) \geq 0, \ldots, \beta_M(q) \geq 0 \}. \tag{3}$$

B. Polyhedral Obstacles and Polyhedral Workspaces

In realistic environments, obstacles may have “vertices” even in the simplest scenarios. Here, we will restrict our attention to the polyhedral obstacle and give a formal definition. Let $H_i$ be a open half-space in $\mathbb{R}^n$, $q_i$ be any point on $H_i$, and $n_i$ be the unit normal vector of $H_i$. The direction of $n_i$ is assumed to be outward pointing with respect to the polyhedron. Each half-space can be represented by a linear inequality in the following form:

$$H_i = \{ q \in \mathbb{R}^n : (q - q_i) \cdot n_i < 0 \}. \tag{4}$$

A set $O_i \subset \mathbb{R}^n$ is polyhedral if it can be expressed as a finite Boolean combination (via the set operations $\cup, \cap, -, \cap$) of half-spaces. Since set union and difference can be reduced to intersection and complement, polyhedrons can be generated from a finite number of half-spaces by intersection and complement operations. In particular, convex polyhedrons can be defined as the intersection of a finite number of half-spaces. Formally, the polyhedral obstacle is defined as follows:

Definition 2: A polyhedral obstacle, $O_i$, is a polyhedral set in $\mathbb{R}^n$ comprised of a finite intersection and complement of half-spaces, $H_{ij}$ for $j \in \{1,\ldots,M\}$, such that:

1) $O_i$ is the interior of a connected and compact $n$-dimensional manifold with boundary;

2) the intersection of the boundaries of any two intersecting half-space, $\partial H_{ij} \cap \partial H_{ik}$, is an $(n-2)$-dimensional subset of $\mathbb{R}^n$.

If all the obstacles of the workspace are polyhedral obstacles, then the workspace is called a polyhedral workspace, denoted by $\mathcal{P}$. In particular, we refer to the polyhedral obstacles and polyhedral workspaces in $\mathbb{R}^2$ as polygonal obstacles and polygonal workspaces, respectively.

The intersection of $\partial H_{ij}$ and $\partial O_i$ is an $(n-1)$-dimensional subset of $\mathbb{R}^n$, called the facet of the polyhedral obstacle, denoted by $\mathcal{F}_{ij}$. The boundary of all the facets in $\partial O_i$ is:

$$\mathcal{V}_i = \partial O_i - \bigcup_{j=1}^M \mathcal{F}_{ij},$$

is the set of vertices. The obstacle function $\beta_i$ for the polyhedral obstacle is an analytical function away from the vertices of $O_i$. In general, $\beta_i$ for the polyhedral obstacle can be constructed automatically based on the function $[4]$ for each half-space and the Boolean combination used to construct $O_i$.

C. Navigation Function

The navigation function is a well-established technique for robot navigation in a smooth manifold with boundary [5]. However, the polyhedral workspace is not smooth because its tangent space at the vertices of the polyhedral obstacles is not well defined. In this case, the navigation function cannot be smooth over the whole workspace. This paper defines a
non-smooth version of the navigation function that relaxes the smooth property.

**Definition 3**: Let $\mathcal{P}^n \subset \mathbb{R}^n$ be a $n$-dimensional polyhedral workspace, $\mathcal{V} \subset \partial \mathcal{P}^n$ be the set of vertices of the polyhedral obstacles, and $x_d$ be the goal point in the interior of $\mathcal{P}^n$. A map $\phi: \mathcal{P}^n \to [0, 1]$, is a navigation function if it:

1) is continuous on $\mathcal{P}^n$ and analytic on $\mathcal{P}^n - \mathcal{V}$.
2) is admissible on $\mathcal{P}^n$: uniformly maximal on $\partial \mathcal{P}^n$.
3) is polar at $x_d$: has a unique minimum at $x_d$.
4) is more on $\mathcal{P}^n - \mathcal{V}$: has only non-degenerate critical points on $\mathcal{P}^n - \mathcal{V}$.
5) has a bounded gradient on $\mathcal{P}^n - \mathcal{V}$.

The Koditschek-Rimon navigation function $\phi_{kr} = \frac{1}{\|q-q_0\|^{\|q-q_0\|+r}}$, is a smooth navigation function constructed on the sphere world, so long as $k$ exceeds some threshold, where $\beta = \prod_{j=0}^{M} \beta_i$ is the aggregate obstacle function [5].

**D. Mobile Robot Model**

This work considers a mobile robot that navigates in a 2-dimensional polygonal workspace. The motion of the mobile robot is described by the trivial kinematic integrator model:

$$\dot{x}(t) = u(t),$$  \hspace{1cm} (5)

where $x \in \mathbb{R}^2$ is the robot position and $u \in \mathbb{R}^2$ is the corresponding control input vector. The initial position and goal position of the mobile robot are denoted as $x_0 \in \mathcal{P}^2$ and $x_d \in \mathcal{P}^2$, respectively.

**E. Conformal Navigation Transformation**

Let $U$ be a connected open set, and $z = x + iy$ be a complex variable. A complex-valued function $f(z) : U \subset \mathbb{C} \to \mathbb{C}$ is holomorphic on $U$ if it is complex differentiable at every point of $U$. Let $f(x, y) = (u(x, y), v(x, y)) : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be of class $C^1$ with non-vanishing Jacobian, then $f$ is a conformal map if and only if $f(z) = u(x, y) + iv(x, y) : U \subset \mathbb{C} \to \mathbb{C}$, as a function of $z$, is holomorphic [17].

The conformal navigation transformation defined below provides an efficient way to implement the navigation function in a geometrically complex polygonal workspace.

**Definition 4**: A conformal navigation transformation is a map $T : \mathcal{W}^2 \to \mathcal{M}^2$ which:

1) maps the interior of the workspace conformally to the interior of a 2-dimensional sphere world.
2) maps the external boundary of the workspace homeomorphically to the unit circle.
3) maps the boundaries of $M$ internal obstacles homeomorphically to $M$ small disjoint circles inside the unit circle.

**III. CONFORMAL NAVIGATION TRANSFORMATION ANALYSIS**

This section investigates the properties of the conformal navigation transformation on the polygonal workspace. These properties enable the proposed transformation to be applied to the navigation problem.

The following proposition establishes that the proposed transformation exists in any 2-dimensional workspace and provides the basis for constructing it on the polygonal space.

**Proposition 1** [18]: If $\mathcal{W}^2$ is a 2-dimensional workspace with $M$ internal obstacles, then there exists a 2-dimensional sphere world $\mathcal{M}^2$ with $M$ internal obstacles and a conformal navigation transformation $T$, such that $T(\mathcal{W}^2) = \mathcal{M}^2$.

The next proposition describes the uniqueness of the proposed transformation corresponding to a given workspace.

**Proposition 2** [18]: The conformal navigation transformation $T : \mathcal{W}^2 \to \mathcal{M}^2$ and $\mathcal{M}^2$ are uniquely determined by $\mathcal{W}^2$ if the images of the prescribed three points on the external boundary $\partial \mathcal{O}_0$ are fixed on the unit circle.

The uniqueness of the proposed transformation implies that no further parameter tuning is required after the construction of the conformal navigation transformation; that is, the resulting solution is correct-by-construction.

The above results are applicable to all 2-dimensional workspaces $\mathcal{W}^2$. However, the vertices of polygonal obstacles introduce a new technical difficulty: the boundaries of polygonal obstacles are not differential manifolds. To address the problem caused by the vertices, we continue to investigate the properties of $T$ on the polygonal workspace $\mathcal{P}^2$.

The machinery of complex variables provides a convenient way to study $T$ on $\mathcal{P}^2$, which will be used in the following. In fact, $T(x)$ on $\mathcal{P}^2$ and $T(z)$ on the complex plane satisfy the relation: $T(x) = [\text{Re}(T(z)), \text{Im}(T(z))]$. The next proposition establishes the analyticity of $T$ on $\mathcal{P}^2 - \mathcal{V}$.

**Lemma 1**: The conformal navigation transformation $T : \mathcal{P}^2 \to \mathcal{M}^2$ is analytic on $\mathcal{P}^2 - \mathcal{V}$.

**Proof**: For the workspace with multiple obstacles, $T$ has the same boundary behavior as the workspace without internal obstacles [19]. Without loss of generality, let polygon $\partial \mathcal{O}_0$ be the only boundary of the $\mathcal{P}^2$ without internal obstacles and $\mathcal{V}_0$ be the set of vertices of $\mathcal{O}_0$. Since the rotation does not change the analyticity of $T$, we may as well assume that one of the edges of $\partial \mathcal{O}_0$ lies on the real axis, denoted as $\partial \mathcal{O}_0^1$. Notice that when $x \in \partial \mathcal{O}_0^1$ is a non-vertex point, there is a disk around $x$ such that the half disk in $\mathcal{P}^2$ does not contain the point $z_0$ with $T(z_0) = 0$. Then $\log T(z)$ has a single-valued branch when $z$ is in the half disk. Since $|T(z)|$ tends to 1 as $z$ approaches $\partial \mathcal{O}_0^1$, the real part of $\log T(z)$ tends to 0. By the Reflection Principle [19], $\log T(z)$ has an analytic extension to the whole disk. Taking exponential, we conclude that $T$ is an analytic function at $x \in \partial \mathcal{O}_0^1 - \mathcal{V}_0$. Rotating all other edges of $\partial \mathcal{O}_0$ to the real axis in turn, we have that $T$ is analytic on $\partial \mathcal{O}_0 - \mathcal{V}_0$. Since the analyticity of $T$ on $\partial \mathcal{P}^2 - \mathcal{V}$ is the same as that on $\partial \mathcal{O}_0 - \mathcal{V}_0$, $T$ is analytic on $\partial \mathcal{P}^2 - \mathcal{V}$. Hence, $T$ is analytic on $\mathcal{P}^2 - \mathcal{V}$.

Lemma 1 implies that $\phi \circ T$ is analytic on $\mathcal{P}^2 - \mathcal{V}$. The next lemma and propositions show that the Jacobian of $T$ is non-singular on $\mathcal{P}^2 - \mathcal{V}$.

**Lemma 2**: The conformal navigation transformation $T : \mathcal{P}^2 \to \mathcal{M}^2$ is injective on $\mathcal{P}^2$.

**Proof**: By definition, $T(z)$ is injective on $\partial \mathcal{P}^2$. Since $T(z)$ is a conformal equivalence from $\mathcal{P}^2$ onto $\mathcal{M}^2$, $T(z)$
is injective on $\mathbb{P}^2$. Hence, $T$ is injective on $\mathbb{P}^2$.

Based on the injectivity of $T$, the following proposition establishes that $T$ has a diffeomorphic extension to boundary points other than vertices.

**Proposition 3:** The conformal navigation transformation $T: \mathbb{P}^2 \to \mathcal{M}^2$ is an analytic diffeomorphism on $\mathbb{P}^2 - \mathcal{V}$.

**Proof:** Since an injective map cannot be constant, then according to the open mapping theorem, $T$ is an open map. Hence, the map $T^{-1}$ is continuous. From the Inverse function theorem, we know that $T^{-1}$ is analytic in the set of all $w = T(z)$ with $T'(z) \neq 0$. The set of all $w = T(z)$ with $T'(z) = 0$ is the image under $T$ of a discrete set in $\mathbb{P}^2 - \mathcal{V}$, hence discrete in $\mathcal{M}^2 - T(\mathcal{V})$. Then, by the Riemann removable singularity theorem, $T^{-1}$ is analytic on $\mathbb{P}^2 - \mathcal{V}$. Hence, $T$ is analytically diffeomorphic on $\mathbb{P}^2 - \mathcal{V}$. □

Since the Jacobian of $T$, $J_T$, is not well defined on $\mathcal{V}$, the boundedness of $\det(J_T)$ guaranteed by the compact set $\mathbb{P}^2$ and the continuous map $J_T$ needs to be re-verified. The next proposition shows that $\det(J_T)$ is bounded on $\mathbb{P}^2 - \mathcal{V}$.

**Proposition 4:** The Jacobian determinant of $T: \mathbb{P}^2 \to \mathcal{M}^2$ is bounded on $\mathbb{P}^2 - \mathcal{V}$.

**Proof:** From the boundary behavior of $T$, we know that the boundedness of $T$ on $\mathbb{P}^2 - \mathcal{V}$ is the same as that on $\partial \mathcal{V}_0$ [19]. Hence, it is sufficient to show that $T$ is bounded on $\partial \mathcal{V}_0 - \mathcal{V}_0$. Let the consecutive vertices be $z_1, ..., z_n$ in the counterclockwise direction. The angle at $z_k$ is given by the value of $\arg \frac{z_{k+1} - z_k}{z_k - z_{k-1}}$, denote by $\alpha_k \pi/2 < \alpha_k < \pi/2$. Let the circular sector $S_k$ be the intersection of a sufficiently small disk around $z_k$ with $\mathbb{P}^2$. For $S_k$, a single-valued branch of $\phi_{z_k}(\zeta) = (z - z_k)^{1/\alpha_k}$ maps $S_k$ onto a half disk $U_k^+$. A suitable branch of $\phi_{z_k}^{-1} = z_k + \zeta^{\alpha_k}$ maps $U_k^+$ onto $S_k$, and we may consider the function $G(\zeta) = T(z_k + \zeta^{\alpha_k})$ in $U_k^+$. According to the Reflection Principle [19], $G(\zeta)$ has an analytic continuation to the whole disk. Thus, $G(\zeta)$ is analytic at the origin and its Taylor expansion is given by $G(\zeta) = T(z_k + \zeta^{\alpha_k}) = T(z_k) + \sum_{m=1}^{\infty} a_m \zeta^m$, $a_0 \neq 0$.

Inverting the series in a neighborhood of $T(z_k)$ and on setting $w = T(z_k + \zeta^{\alpha_k})$, we have $\zeta = \sum_{m=1}^{\infty} b_m (w - w_k)^m$, $b_1 \neq 0$, where $w_k = e^{i\theta_k}$ is the image of vertex $z_k$ under $T$ in the unit circle $C$. Taking exponents, we obtain $T^{-1}(w) - z_k = (w - w_k)^{\alpha_k} E_k(w)$, where $E_k(w)$ is analytic and non-zero near $w_k$. Taking the derivative, we conclude that the product $F(w) = (T^{-1}(w)) \prod_{k=1}^{n} (w - w_k)^{-1}a_k$ is analytic and non-vanishing in the closed unit disk $\overline{D}$.

Next consider the boundedness of $|T^{-1}(w)|$ on $\overline{D} - T(\mathcal{V}_0)$. Since $T$ is analytic on $\mathbb{P}^2 - \mathcal{V}$ and homeomorphic on $\partial \mathbb{P}^2$, $T^{-1}$ smoothly maps the anticlockwise arc from $w_k$ to $w_{k+1}$ to the line segment from $z_k$ to $z_{k+1}$, with derivative non-vanishing. Thus on taking arguments

$$\arg \frac{d}{d\theta} (T^{-1}(e^{i\theta})) = \arg (z_{k+1} - z_k)$$

and thus by the chain rule

$$\arg (T^{-1})'(e^{i\theta}) = \arg (z_{k+1} - z_k) - (\theta + \pi/2).$$

For the factor $w - w_k$, we have that

$$w - w_k = 2 \sin \left(\frac{\theta - \theta_k}{2}\right) e^{i(\theta + \theta_k + \pi)},$$

which implies that its argument is $\theta$ plus a constant. Thus, we conclude that

$$\arg \prod_{k=1}^{n} (w - w_k)^{1-\alpha_k} = \frac{\theta}{2} \cdot \sum_{k=1}^{n} (1 - \alpha_k) + C_1,$$

where $C_1$ is a constant. From the fact that $\pi \cdot \sum_{k=1}^{n} (1 - \alpha_k)$ is the sum of the exterior angles of a polygon, we have that

$$\sum_{k=1}^{n} (1 - \alpha_k) = 2.$$

and

$$\arg F(w) = \arg (z_{k+1} - z_k) - (\theta + \pi/2) + \theta + C_1 = C_2,$$

where $C_2 = \arg (z_{k+1} - z_k) - \pi/2 + C_1$ is a constant. Hence, $\arg F(w)$ is constant between $w_k$ and $w_{k+1}$. Then, the function $\arg F(w)$ is constant on $w_k$ and $w_{k+1}$. Hence, for $T^{-1}$ on $\mathbb{P}^2 - T(\mathcal{V})$, we have that

$$(T^{-1})'(w) = C_3 \prod_{k=1}^{n} (w - w_k)^{\alpha_k - 1},$$

where $C_3$ is a constant. Since $w \neq w_k$ for $w \in \overline{D} - T(\mathcal{V}_0)$, $|(T^{-1})'(w)|$ is bounded.

Last consider the boundedness of $\det(J_T(x))$ on $\partial \mathcal{V}_0 - \mathcal{V}_0$. Since $|(T^{-1})'(w)|$ is bounded on $\overline{D} - T(\mathcal{V}_0)$, then according to inverse mapping theorem, $|T'(z)|$ is bounded on $\partial \mathcal{V}_0 - \mathcal{V}_0$. By the Cauchy-Riemann equations, we have that

$$\det(J_T(x)) = |T'(z)|^2.$$

Finally, since $|T'(z)|$ is bounded on $\partial \mathcal{V}_0 - \mathcal{V}_0$, the Jacobian determinant of $T$ is bounded on $\partial \mathcal{V}_0 - \mathcal{V}_0$. □

**IV. Controller Design Using The Conformal Navigation Transformation**

In this section, a provably correct feedback controller based on the conformal navigation transformation is presented and analyzed to illustrate the application of the proposed transformation to the robot navigation problem in complex 2D polygonal environments.

The conformal navigation transformation provides a geometric approach to transforming two navigation problems with different geometric details into the same problem, as described in the following theorem.

**Theorem 1:** Let $\mathcal{P} \subseteq \mathbb{R}^2$ be a polygonal workspace, and $V \subseteq \partial \mathcal{P}$ be the set of vertices. Let $\phi: \mathbb{M}^2 \to [0, 1]$ be a navigation function on $\mathcal{M}^2$. If $T: \mathbb{P}^2 \to \mathcal{M}^2$ is the unique conformal navigation transformation determined by $\mathcal{P}^2$, then $\phi = \phi \circ T$ is a navigation function on $\mathcal{P}^2$.
Proof: The continuity of \( \hat{\phi} \) on \( S^2 \) follows from the fact that both \( \phi \) and \( T \) have this property. From Lemma 1, we know that \( T \) is analytic on \( S^2 - V \). Then, \( \hat{\phi} \) is an analytic function on \( S^2 - V \) because it is a composition of two analytic functions.

Since \( T \) is an analytic diffeomorphism on \( S^2 - V \), the Jacobian of \( T \) is non-singular in \( S^2 - V \). Applying the chain rule yields \( \nabla \hat{\phi} = [J_T]^T \nabla (\phi \circ T) \). According to Proposition 4, the gradient \( \nabla \hat{\phi} \) is bounded on \( S^2 - V \).

As \( J_T \) is non-singular in \( S^2 - V \), \( T \) is a bijection from the set of critical points of \( \hat{\phi} \), \( \{x \} \), to the set of critical points of \( \phi \), \( \{x \} \). Furthermore, the Hessian matrix of \( \hat{\phi} \) at the critical points satisfies \( H(\hat{\phi}) = [J_T]^T H(\phi \circ T)[J_T] \). Since \( T \) is non-singular in \( S^2 - V \), \( H(\hat{\phi}) \) is non-singular at the critical points. Hence, \( \hat{\phi} \) satisfies the Morse function on \( S^2 - V \).

Next, consider the image of \( \partial \Theta_i \) under \( \hat{\phi} \). Since \( T \) is an analytic diffeomorphism on \( S^2 - V \) and a homeomorphism on each polygonal boundary \( \partial \Theta_i \), the image of \( \partial \Theta_i \) under \( T \) is exactly \( \partial \Theta_i \). Moreover, \( \hat{\phi} \) is uniformly maximal on \( \partial \Theta_i \). Hence, \( \hat{\phi} \) is admissible on \( S^2 \).

Finally, let us next verify the polar property of \( T \) in \( S^2 \). At the critical points, we have that: \( H(\hat{\phi}) = [J_T]^T H(\phi \circ T)[J_T] \). Since \( T \) is a bijection between \( \{x \} \) and \( \{x \} \), the Morse index of \( \hat{\phi} \) at each of the critical points. Moreover, \( \hat{\phi} \) has a unique minimum at \( x_\mu \) in \( S^2 - V \). Hence, \( \hat{\phi} \) is a polar function on \( S^2 - V \). Since the admissibility of \( T \) on \( S^2 \) means that \( \hat{\phi} \) is uniformly maximal on \( \partial S^2 \), any vertex in \( V \subset \partial S^2 \) is not a local minimum. It follows that \( \hat{\phi} \) satisfies the polar property on \( S^2 \).

Theorem 1 shows that the composition of the conformal navigation transformation \( T : S^2 \to M^2 \) with a navigation function on \( M^2 \) forms a navigation function on \( S^2 \). This property provides a method for designing feedback control laws to achieve convergence from almost all initial points to the goal point, as shown below.

**Theorem 2:** The mobile robot system \( \mathcal{S} \) under the feedback control law

\[
u(x) = -K(J_T(x))^T \nabla_{T(x)} \phi_{kr}(T(x)) = -K \|J_T(x)\| \nabla_{T(x)} \phi_{kr}(T(x)),
\]

where \( K \) is a positive gain, is globally asymptotically stable at the goal point \( x_\mu \) in \( S^2 \) from almost every initial point in \( S^2 - V \).

**Proof:** Since \( T \) is an analytic diffeomorphism on \( S^2 - V \), according to the Cauchy-Riemann equation, the Jacobian \( J_T(x) = \begin{bmatrix} u_x & -u_y \\ v_x & u_y \end{bmatrix} \) at \( x = (x_1, x_2) \) in \( S^2 - V \). Moreover, the Jacobian \( J_T(x) = (u_x^2 + v_x^2)^{\frac{1}{2}} R = \|J_T(x)\|^{\frac{1}{2}} R \), where \( R \) is a rotation matrix. Hence, for \( T \) on \( S^2 - V \), we have that: \( (J_T(x))^T = \|J_T(x)\|^{\frac{1}{2}} R^{-1} \).

Choosing \( V(x) = \phi_{kr}(T(x)) \) as a candidate Lyapunov function. Differentiating \( V \) along the system’s trajectories yields: \( \dot{V}(x) = -K \|J_T(x)\| \nabla_{T(x)} \phi_{kr}(T(x)) \). Since \( J_T(x) \) is non-singular, the set for \( \dot{V}(x) = 0 \) consists only of the critical points of \( \phi_{kr}(T(x)) \). The critical points of \( \phi_{kr}(T(x)) \) include the unique minimum point \( x_\mu \) and the isolated saddle points. Lasalle’s Invariance Theorem dictates that the closed loop system will converge to the largest positive invariant set that consists only of the goal and the saddle points. Since a saddle point has a stable manifold of 1-dimension in \( S^2 \), all isolated saddle points have attractive basins of zero measure. Hence, \( \dot{V}(x) < 0 \) holds almost everywhere, which implies that the system is globally asymptotically stable at \( x_\mu \) from almost every initial point in \( S^2 - V \).

V. CONSTRUCTION OF THE CONFORMAL NAVIGATION TRANSFORMATION

In [18], a construction method for the conformal navigation transformation is proposed on a 2-dimensional workspace with smooth boundaries. Here, we will present a construction method for \( T \) that operates on a polygonal workspace. The first step is to construct \( T \) on a simply connected polygonal workspace. The second step is to use the method proposed in [18] to obtain \( T \) on a multiply connected polygonal workspace.

Let \( U^+ \) and \( U^- \) be a bounded and an unbounded simply connected workspace in the extended complex plane \( \mathbb{C} \cup \{\infty\} \), respectively. For \( U^+ \), assume that \( z_0 \) is a given point in \( U^+ \). For \( U^- \), assume that \( z_0 \) is a given point in the complement of \( U^- \) and \( \infty \notin U^- \). The boundary \( \Gamma \) is a closed polygonal curve parametrized by a \( 2\pi \)-periodic complex analytic function \( \gamma(s) \) except at \( n \) points

\[ s_k = (k - 1)\frac{2\pi}{n} \in [0, 2\pi], k \in \{1, 2, ..., n\}. \]

The orientation of \( \Gamma \) is counterclockwise for \( U^+ \) and clockwise for \( U^- \).

The construction method for \( T(z) \) on \( U^+ \) with a smooth boundary is given by the following results. Moreover, the described method can be straightforwardly applied to solve \( T(z) \) on \( U^- \). For more details, see [18].

For a bounded simply connected workspace \( U^+ \) with a smooth boundary, \( T(z) \) is constructed as follows:

\[ T(z) = \hat{T}(z)(z - z_0)e^{i(\gamma(z) - \gamma(z_0))}, \]

where \( f(z) \) is a holomorphic function on \( U^+ \). The boundary values of the function \( f(z) \) are given by:

\[ G(s)f(\gamma(s)) = \mu(s) + c(s) + i\nu(s), \]

where \( G(s) = \gamma(z) - z_0, \mu(s) = -\log |\gamma(s) - z_0|, c(s) = -\log (T(z)), v(s) = \theta(s) - \arg(\gamma(s) - z_0). \) The unknown functions \( c \) and \( \nu \) are uniquely determined by the known \( \mu \).

The following proposition describes the above conclusion in more detail. The various variables mentioned in the sequel are defined in [18].

**Proposition 5 [18]:** For the given real-valued function \( \mu \), there exists a unique constant \( c \) and a unique function \( v \), such that \( \mu \) are boundary values of a holomorphic function \( f \) in \( U^+ \). The function \( v \) is the unique solution of the equation

\[ \mu - R\mu = -Hv, \]

and \( c \) is given by

\[ c = [H\mu - (I - R)v]/2. \]
where \( t,s \) and an eigenfunction of \( H \) is the conformal navigation transformation on \( \Gamma \) can be rewritten as

\[ T(z) = e^{-z} \left( z - z_c \right) e^{(z - z_c)f(z)} \quad (12) \]

is the conformal navigation transformation on \( U^+ \).

For a bounded simply connected workspace \( U^+ \) with a polygonal boundary, the curve parametric equation \( \gamma(s) \) with \( \gamma(s) \neq 0 \), the holomorphic function

\[ T(z) = e^{-z} \left( z - z_c \right) e^{(z - z_c)f(z)} \quad (12) \]

is the conformal navigation transformation on \( U^+ \).

For a bounded simply connected workspace \( U^+ \) with a polygonal boundary, the curve parametric equation \( \gamma(s) \) is nonsmooth at \( s_k \), \( k \in \{1, 2, ..., n\} \). Therefore, before using the above results, it is necessary to rewrite equation \((10)\) and eliminate the singularity at \( s_k \).

Since the constant is an eigenfunction of \( R \) corresponding to \(-1 \) and an eigenfunction of \( H \) corresponding to \( 0 \), the equation \((10)\) can be rewritten as

\[ 2\mu - R(s, t)[\mu(t) - \mu(s)] = -H(s, t)[\nu(t) - \nu(s)], \quad (13) \]

where \( t, s \in [0, 2\pi], \ t \neq s \).

Now, define the function

\[ \sigma(s) = 2\pi \frac{[g(s)]^q}{[g(s)]^q + [g(2\pi - s)]^q}, \quad (14) \]

where

\[ g(s) = \left( \frac{1}{q} - \frac{1}{2} \right) \left( \frac{\pi - s}{\pi} \right)^3 + \frac{s - \pi}{q\pi} + \frac{1}{2}. \quad (15) \]

The function \( \sigma(s) \) is a smooth, strictly monotonically increasing and bijective function. The parameter \( q \) is an integer and \( p \geq 2 \).

Define the function

\[ \eta(s) = \frac{1}{n} \sigma \left( n(s - s_k) \right) + s_k, \ s \in [s_k, s_{k+1}]. \quad (16) \]

For the function \( \eta(s) \), we have that \( \dot{\eta}(s_k) = 0 \) and \( \dot{\eta}(s) \neq 0 \) for \( s \neq s_k \).

Finally, define

\[ \hat{\gamma}(\tau) = \eta(\eta(\gamma(\tau))), \ \tau \in [0, 2\pi]. \quad (17) \]

and consequently we obtain that

\[ \int_0^{2\pi} \hat{\gamma}(\tau)d\tau = \int_0^{2\pi} \gamma(s)ds. \quad (18) \]

Then, the function \( \hat{\gamma} \) is smooth on \( \Gamma \) and \( \hat{\gamma}(0) = \hat{\gamma}(2\pi) = 0 \). Hence, the singularity at \( s_k \) is eliminated. After rewriting Equation \((10)\) and eliminating the singularity at \( s_k \), the solution of \( T \) on \( U^+ \) can be obtained using the construction method of \( T \) on \( U^+ \) with a smooth boundary.

The construction method of \( T(z) \) on a multiply connected workspace proposed in \([18]\) is an efficient and fast iterative method, which is inspired by the conventional Koebe’s method on unbounded regions. Meanwhile, having obtained \( T \) on \( U^+ \) and \( U^- \), the method can be applied directly to solving the conformal transformation on a multiply connected polygonal workspace. See section V in \([18]\) for details.

**VI. SIMULATION RESULTS**

In order to validate the effectiveness of the proposed methodology, a numerical simulation was presented in this section. The simulation was carried out for controller \([6]\), and the Koditschek-Rimon navigation function \( \phi_{kr} \) was used in the sphere world. The parameter \( k \) in \( \phi_{kr} \) was set as \( k = 6 \). The gain \( K \) in \([6]\) was set as \( K = 1 \). For the conformal navigation transformation, we iterated until \( ||T_n - T_{n-1}|| < 1 \times 10^{-12} \).

In the simulation, the feedback control law \([6]\) provided in Theorem 2 is applied to the kinematic mobile robot \([5]\). The workspace is set up with three star-shaped internal
polygonal obstacles: a triangle, a rectangle, a hexagonal star dodecagon; and a non-star-shaped polygonal obstacle that cannot be handled well by the homeomorphic-based transformations. The goal point is set as $x_d = (0, 0)$, which the robot aims to converge to from 4 different initial points, namely $x_0 = (1.5, 1), (-1.5, 1), (-1, -1.5)$ and $(0.05, -1.5)$. Fig. 1 and Fig. 2 depict the level sets of the navigation function and robot trajectories in the initial workspace and the transformed sphere world, respectively. As we can observe, the designed feedback control law successfully performs the task of maintaining the robot in the initial workspace and the transformed sphere world, avoiding collisions and stabilizing it at the goal point. In this example, the number of iterations is 7 and $\|T_f - T_d\| = 1.5831 \times 10^{-13}$.

VII. CONCLUSION

This paper proposes a methodology for the navigation problem in complex polygonal workspaces based on a novel spatial transformation called the conformal navigation transformation. The class of navigation functions defined in smooth workspaces is extended to accommodate polygonal obstacles. With the application of the proposed transformation, the navigation function in the geometrically simple sphere world is pulled back to the initial workspace. The negative gradient of the pulled-back navigation function then serves as a control law that enables the dynamic robotic system to converge from almost all initial points to the goal point. It is shown that this method provides solutions for both path planning and motion planning subproblems. Furthermore, an iterative method for constructing the proposed transformation is presented, which ensures fast convergence and is suitable for non-star-shaped polygonal obstacles. Finally, simulation results support the theoretical results.

Future research directions include extending the solution to account for dynamic obstacles, multi-agent navigation problems, and more realistic scenarios with local sensing.

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