Mathematical representation analysis of students in solving mathematics problems

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Abstract. Representation of ideas in mathematics is fundamental to how students can understand and apply these ideas. This study aims to describe students’ mathematical representations in solving mathematics problems that include pictorial representations, symbol representations, and verbal-written representations. This study used descriptive qualitative which involved 20 respondents of seventh grade students of Asiyiyah junior high school Bengkulu. The instruments were sheet of problems and interview guides. Three respondents were selected based on the completeness of the answers then interviewed to clarify the answers. The completeness was based on the use of mathematical representations that include pictorial representations, symbol representations, and verbal-written representations. The results of the study showed that students can use various forms of mathematical representations in solving the given mathematics problems, and even students can use three forms of mathematical representations, and some student only used one or two forms of mathematical representations.

1. Introduction

Mathematics is a field of study that underlines and has an important role in the development of modern technology. Ignoring mathematics is the same as removing some of the foundations of all knowledge, if do not know about mathematics then we will not be able to know the other sciences [1]. Therefore, mathematics has a complete concept of theory. Mathematics can be used as a means for growth and development of logical, meticulous, critical, and creative way of thinking [2]. With regard to the importance of mathematics, every child should have the ability to master mathematics.

One form of ability that can be developed is the ability to think. Learning mathematics can stimulate the development of students' thinking ability. The teacher is one of the supporting factors that can encourage students to have math ability. One of the elements that the teacher must possess is the ability to master mathematical ideas presented to the students [3]. The steps of the idea to be conveyed by the teacher must arrive at the goal of acquiring a knowledge for the student, with clear steps and having a unified sequence. There was recommendation that teachers have to pay attention to everything needed in the learning process and the development of students' mathematical ideas [4].

The teacher's weakness in mastering mathematical ideas affects the level of student knowledge [5]. It means that teacher should represent well in the delivery, so students can understand math easily. In accordance with the results of research, the teacher's conception of representation as a process and practice of mathematics seem to be less developed [6]. The results of this research indicated that representation is viewed as a topic of study rather than as a general process and a goal for learning. Meanwhile, the results of research conducted showed that the involvement of students in learning
activities was very low and the results of learning after the test was in the low category [7]. Therefore, the mastery of mathematical ideas for students is very important in which the student can be able to communicate the information and understand that has been owned so that students’ involvement in learning will be more visible.

In recent years the mathematics education community has increasingly shown that representation is a useful tool for communicating information and understanding [8]. The awareness that this representation is important in mathematics is reflected in efforts to improve education and research studies that have been widely practiced. This is explained at the National Council of Teachers of Mathematics in the national standard document lifting representation is one of five process standards, namely problem solving, reasoning and verification, communication, connection, and representation. Representations can be used effectively to teach mathematics problem solving [9]. Moreover, representation is a practical strategy in problem solving [10].

Representation of ideas in mathematics is fundamental to determine how students can understand and apply these ideas. It is recommended that all levels of school to use representation to organize, record, and communicate mathematical ideas to model and interpret the problem using multiple representations. To understand the mathematical concepts being studied, it is better not only to be based on the new concept but also related to the concept previously studied, so that requires the retrieval of information that has been stored in the memory and it can be used to obtain new information which is the form of concepts mathematics [11]. The process of obtaining that memory is closely related to the encoding of the past experience, which is a mental activity that can be called as an internal representation.

Internal representation cannot be observed visually so it cannot be assessed and even analysed, because it is in the mind (minds on) no one knows but themselves. However, internal representations can be seen in words (spoken) or written-form statements, symbols, notations, graphics, images, and other forms in mathematics. The representation is a mental picture of learning process that can be understood through the mental development that is present in a person and is reflected as visualized in a verbal form, image, or concrete objects [12]. This shows that the process of drawing or mining something happens in a person's mind and the results of his thoughts are expressed in the form of statements, visuals, or notations. Thus, the embodiment of the internal representation is called an external representation.

The forms of representation are very diverse because there have been many theories that propose about the forms of representation. The representation can be categorized in 3 stages namely (1) through action (enactive) (2) through visual images (iconic) (3) through words and languages (symbolic) [13]. Based on those three categories, it is expected that students can go through these stages so that they can solve mathematics problems easily and skillfully. Furthermore, five forms of representation were expressed by Clement [14] namely images, manipulations, written symbols, spoken language and relevant situations, as in the following picture [14]:

![Figure 1. Form of Clements’ Mathematical Representation](image)
There are five forms of a mathematical representation [14]. The first form is picture, which means constructing an image of itself to be a good learning experience for students. The second form is manipulation, which means representing problems by using a real object and capable of being touched, and shifted, while the third form is spoken language. In the third form students can tell their answers and express their reasoning by using oral. Meanwhile, the fourth form is written symbols, which refers to the writing of mathematical symbols and knowledge related to the problem. The last but not least form is relevant situations. It is the use of mathematical ideas to the problems faced precisely, to solve problems more quickly.

Furthermore, Villaega [15] revealed the ability of mathematical representation into 3 forms. Firstly, verbal representation is a matter of words expressed in written or oral. Secondly, pictorial representation consists of drawings, diagrams or graphs and all actions related to the drawing. Thirdly, the symbolic representation consists of numbers, operations and dashes, algebraic symbols, and all sorts of actions referring to symbol.

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**Figure 2. Mathematical Representation Form of Villaega**

Based on some of the above explanations about forms of representation, the researchers chose to use representational forms, because they are often used as the representations in solving mathematics problems and are included in the flat matter. These three forms of representation are more abstract and higher levels of representation to solve mathematics problems. Mathematics is basically abstraction-based and deductive [16]. The ideas of mathematics such as the theorems are constructed from a deductive definition consisting of abstract symbols. Then the abstract symbols are formed in concrete objects such as image and spoken in verbal form of writing or oral.

2. **Research Methods**

This descriptive qualitative research was conducted by describing the analysis of mathematical representation of students in solving math problems. The respondents consisted of eighth grade students of Aisyiyah Junior High School Bengkulu as many as 20 students. Researchers chose three research respondents consisting of respondents 1 (S1), respondent 2 (S2), and respondent 3 (S3).

Respondents were asked to solve the mathematics problems given on the two-dimensional shape material. After the respondent was asked to solve the mathematics problem, the result of the respondent's answer was assessed and analysed. Furthermore, the results obtained was described on how the representation of students in solving mathematics problems with visual representation, symbols, and verbal writing. In solving the mathematics problems in the two-dimensional figure material, researchers could assess and analyse the data in accordance with the indicators that were adapted to the mathematical representation below.
Table 1. Indicators in mathematical representation data analysis

| The mathematical concept used | Mathematics Problems | Forms of Representation | Mathematics Problem Solving Analysis |
|-------------------------------|----------------------|-------------------------|-------------------------------------|
| The widespread concept of two-dimensional figures involving square areas, triangles, and circles | Look at the image below. A large square with an area of 144 square meters is divided into four small squares. Calculate the shaded area? | 1. Pictorial (in 1. image form) Make a drawing to clarify the problem and facilitate the settlement and can change the image according to the student's understanding of the given mathematics problem. This stage is included in the form of pictorial mathematical representation. |
|                              | 2. Symbol            | 2. Design and start to solve mathematics problems using the widescreen-wide formulas or with number operations. The second stage uses more representation form of mathematical symbols. |
|                              | 3. Verbal writing    | 3. Communicate and summarize the results of step 1 and 2, the third stage is more to mathematical representation form of verbal writing. |

In accordance with the table, indicators were presented for assessing and analysing mathematical representation data on given mathematics problems. In this stage of analysis, the forms of representation used by the students will be vary. Some students may be able to use the three forms of mathematical representation, or students only use 2 or 1 forms of mathematical representation. Therefore, the researchers does not limit it, because the mathematical problem given will bring up a different settlement process but has the same answer results.

3. Results and Discussion
Based on math problem solving in general, respondents could solve mathematical problems. However, eleven students were in the process of making a mistake. Such errors could be wrong in understanding mathematics problems, not knowing formulas two-dimensional figures, as well as errors in the concept of two-dimensional figures area while other nine students can solve math problems well. Therefore, the researchers chose three research respondents that solved math problems using various representations consisting of respondent 1 (S1), respondent 2 (S2), and respondent 3 (S3). In accordance with the exposure to the introduction of forms of representation used include pictorial (images), symbols, and verbal writing.
Figure 3. Step 1 of S1’s Answers (Pictorial Mathematical Representation)

Based on the answer, S1 could re-communicate what was known in the problem that was the large square area of 144 cm. Then, S1 re-created a large rectangular image with 4 small rectangles with the shaded area according to the problems and then manipulated the image by moving and combining the shaded area so as to form the new shaded area that fits 2 small rectangles while 2 other small rectangles were without shade. After the interview, S1 understood that the shaded area could form two small rectangles to facilitate S1 to solve math problems without having to use the area formula. This means that S1 could use pictorial representation (image) with image manipulation well, as suggested by McCoy [17] that by trying to make pictorial representations, it will train students to be able to create in other forms of representation.

Figure 4. Step 2 of S1’s Answers (Mathematical Representation Symbols and Written Verbal)

Furthermore, S1 continued the problem resolution in the form of symbols. It was proved by S1 by dividing the large square area of $\frac{144}{2}$ so that the area of two small rectangular or shaded area was 72 cm. Then S1 could deduce from completion in written verbal form) that the shaded area or area of two small squares was 72 cm. Through the process of solving problems that have been done by S1 coherently, S1 got the right solution. Problem solving done by S1 was a short solution with efficient time and appropriate results, so that S1 was the only respondent who could use the three forms of mathematical representation correctly.

Figure 5. Step 1 of S2’s answers (Mathematical Symbols Representation)

The next respondent, S2, used a more mathematical representation of mathematical symbols. In accordance with the problem, large squares had shaded regions on all small rectangles that were parts of large rectangles. S2 divided the shaded area into 3 shapes. In the shape 1, triangle has base of 12 cm and height of 6cm. After being interviewed, S2 knew the base and height of triangle by searching length of big square side. It was known that big square area of 144 cm and square area formula was the quadrat
of side thereby side is $\sqrt{\text{square area}}$ so that $\sqrt{144} = 12$, the length of the large square side is 12 cm. Furthermore, S2 obtained a triangular base = square side length of 12 cm. S2 said that each square of side length has the same side length, thus the height of the triangle is half the length of the large square side of 6 cm, because the height of the triangle was at the midpoint on a large square. Then, S2 went to know the area of the triangle $1/2 (a \times t)$ thus the area of shape 1 was $36 \text{ cm}^2$.

Figure 6. Step 2 of S2’s answers (Symbolic Mathematical Representation)

Shape 2 is a shaped area of $\frac{1}{4}$ circle, thus S2 looked for area of $\frac{1}{4}$ circle in which the radius of circle is $\frac{1}{2}$ of the side length of big square that is 6 cm, so the wide of $\frac{1}{4}$ circle is $\frac{1}{4} (\pi r^2)$ and area of shape 2 is $28.2 \text{ cm}^2$. In interview, S2 saw shape 2 as part of the circle in which the area of the circle is $\frac{1}{4}$ of a complete circle. Thus, S2 could solve the problem on shape 2 properly.

Figure 7. Step 3 of S2’s answers (Symbolic Mathematical Representation)

In this section, S2 calculated shape 3, which is the shaded area in which S2 gets the area by reducing the area of a small square with an area of $\frac{1}{4}$ circle. At the time of the interview, S2 said that shape 3 is part of the area that is not highlighted in shape 2. It is caused S2 to reduce the area of a small square with the area of the shape area 2, which the small square area is $\frac{1}{4}$ of the large square of 36 cm2, thus area shape 3 is $36 - 28.2 = 7.8 \text{ cm}^2$.

Figure 8. Step 4 of S2’s answers (Symbolic and Verbal Mathematical Representation)
As the completion, S2 was summing up all the buildings 1, 2 and 3 by using written verbal representation. Then, S2 got the shaded area of 72 cm². S2 could solve mathematics problem properly and correctly. Moreover, S2 did not use pictorial representation (picture) but the settlement used is less efficient in terms of time than the work done by S1. S2 used more formula of the triangle area and circle while S1 only used the operation of numbers.

Figure 9. Step 1 of S3’s answer (Using Pictorial Representation and Symbols)

The next discussion is respondent 3 (S3). In this section, S3 re-communicated with a symbolic representation of large square 144 cm² and then found the length of the large side of the square by looking for the square root of two of the large square area which was 12 cm. Next step was finding the length of the side of the small square by dividing the two sides of the large square to get 6 cm. After finding the small rectangular side, S3 continued by looking at the shaded area of the triangular shape. After knowing the longest side of the big square, S3 set the length of the big square side equal to the length of the base of triangle S3 which was 12 cm. The height of the triangle is equal to the length of the smallest square side of 6 cm. Likewise to the \( \frac{1}{4} \) circle shaded area, S3 considered the diameter of circle 6 cm and radius of circle 3 cm. At this stage, S3 made mistake in the concept diameter and radius of circle.

At the interview, the researchers asked about diameter and radius. S3 explained that the diameter was the length of the circumference of the circle passing through the centre of the circle and the radius was half of the diameter of the circle. Soon after S3 said it, S3 just realized making an error on the picture made by S3. S3 was disturbed by the shaded area, which was only \( \frac{1}{4} \) circle so that there was a mistake of deciding the diameter and radius of the circle. In general, at the beginning of this section S3 described each what needs to be known before the calculation and the deeper settlement were done by calculating the length of the shape side (square, triangle, and circle) associated with the shaded area in the drawing. Through interview, S3 wanted to know what can be known from the problem given at this early stage so that S3 thought that at a later stage it would make easier to process the settlement by incorporating into the area formula of two-dimensional figures. In the next section, S3 re-described a large square with its shaded area and places what has been found in the previous section. At this stage, S3 used a pictorial representation of the image, S3 understood that by redrawing, it would make easier for S3 to go further and find the shaded area.
Furthermore, S3 calculated the shaded area that was like a triangle. Calculating the area of the triangle with the formula \( \frac{1}{2} (\text{base} \times \text{height}) \), by entering the length of the base and the height that was known in the initial stage. It was found that the area of triangle was 36 cm\(^2\). Next the \( \frac{1}{4} \) circle-shape area was found by the area formula of \( \frac{1}{4} (\pi r^2) \), then the result was 7.065 cm\(^2\). The last step was reducing the area of \( \frac{1}{4} \) circle of 7.74 cm\(^2\), from the smallest square. At this stage, S3 had the same steps with S2 but S3 made an error on finding the area of \( \frac{1}{4} \) circle due to the diameter and radius of the circle.

Therefore, S3 added up the total area that had been found previously in the next stage. S3 was less precise in solving math problems due to mistakes in the early stages of defining the radius of the circle.

Although the students can use the three forms of representation, there was a mistake in the process of completion especially when the students have not understood the concept of the material or mistaken in understanding the problem given. Consequently, the completion of the results might be different.

4. Conclusion

Based on the results and discussion, mathematical representation on the three respondents of research is very diverse. Some research respondents used the three forms of representation i.e. pictorial, symbol, and verbal writing, but others only used one or two mathematical representations. Research respondents who can use all forms of mathematical representation does not guarantee the correctness of problem solving. The proof is that the Respondent 3 (S3) can use all forms of mathematical representation but made mistakes on the concept of two-dimensional figures.
Therefore, understanding mathematical concepts is the most important part in solving mathematics problems. Students at least understand the problems of a mathematics problems posed, so students can plan settlements, perform calculations appropriately, and be able to check or review what has been processed correctly. The aspect of mathematical representation is only a part of helping and facilitating students in the settlement. It can be in the form of providing an overview, translation, disclosure, reappraisal, mine, ideas, mathematical concepts, and relationships contained in a particular configuration, construction, or problem situation displayed by students in various forms in an attempt to gain clarity of meaning, to show their understanding, or to find solutions to their problems.

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