Generalized Stochastic Gauge Fixing

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Abstract

We propose a generalization of the stochastic gauge fixing procedure for the stochastic quantization of gauge theories where not only the drift term of the stochastic process is changed but also the Wiener process itself. All gauge invariant expectation values remain unchanged. As an explicit example we study the case of an abelian gauge field coupled with three bosonic matter fields in 0+1 dimensions. We nonperturbatively prove equivalence with the path integral formalism.

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Stochastic quantization was presented several years ago by Parisi and Wu [1] as a novel method for the quantization of field theories. It provides a remarkable connection between quantum field theory and statistical mechanics and has grown into a useful tool in several areas of quantum field theory, see refs. [2, 3] for comprehensive reviews and referencing.

One of the particularly interesting aspects of the stochastic quantization scheme lies in the quantization of gauge theories. Over the last years much hope has been put forward to gain new insights for a correct nonperturbative path integral formulation of gauge theories also from the stochastic quantization point of view. However, the fundamental question how stochastic quantization – if at all – compares with the conventional quantization schemes in the case of gauge theories so far remained unclear and no really compelling argument in favour for the stochastic quantization scheme has emerged.

In this paper we propose a generalized stochastic gauge fixing procedure which allows to extract the equilibrium Fokker-Planck probability distribution of a toy model in an appealing fashion. New hopes for related applications also for more complicated gauge models seem justified.

The crucial point of the Parisi-Wu approach for the gauge theory case [1] is to demand that the stochastic time evolution of the fields is given by a Langevin equation of the form

$$d\Phi^i(t, s) = -\frac{\delta S}{\delta \Phi^i(t)}_{\Phi(t) = \Phi(t, s)} ds + dW^i$$

(1)

Here we collectively denote by $\Phi^i(t, s), i = 1, ..., m$ the pure gauge as well as matter fields of the given gauge model. According to the stochastic quantization procedure these fields depend in addition to their usual coordinates – for shortness of notation denoted by just the single coordinate $t$ – on the stochastic time coordinate $s$ as well. $S$ denotes the original (Euclidean space-time) action of the given gauge model; it is the unmodified bare action without gauge symmetry breaking terms and without accompanying ghost field terms. The stochastic process (1) is defined in terms of the increments $dW^i$ of a $m$-dimensional Wiener process; it undergoes undamped diffusion and does not approach
an equilibrium distribution. Related to this fact is that a Fokker-Planck formulation for the $\Phi^k$ is not possible because the gauge invariance of the action leads to divergencies in the normalization condition of the Fokker-Planck density [1].

Zwanziger’s stochastic gauge fixing procedure [4] consists in adding an additional drift force to the Langevin equation (1) which acts tangential to the gauge orbits. This additional term generally can be expressed by the components $Z^i(t', t)$ of the generator of infinitesimal gauge transformations and an arbitrary function $\alpha$. The gauge generator is given by the vector field

$$Z_\xi(\Phi) = \int dt' dt \xi(t') Z^i(t', t) \frac{\delta}{\delta \Phi^i(t)}$$

where $\xi$ is an arbitrary element of the Lie algebra of the gauge group. Zwanziger’s modified Langevin equation reads as follows

$$d\Phi^i(t, s) = - \left[ \frac{\delta S}{\delta \Phi^i(t)} + \int dt' Z^i(t', t') \alpha(t') \right] ds + dW^i$$

One can prove that the expectation values of gauge invariant observables remain unchanged for any choice of the function $\alpha$ and that for specific choices of the – in principle – arbitrary function $\alpha$ the gauge modes’ diffusion is damped along the gauge orbits. As a consequence the Fokker-Planck density can be normalized [4].

We present now our generalization [5] of Zwanziger’s stochastic gauge fixing procedure by adding a specific drift term which not only has tangential components along the gauge orbits; in addition we modify the Wiener process itself. In this way we introduce more than just one function $\alpha$, in fact we add $m$ additional functions $\beta_i$ appearing in the drift term as well as in the Wiener process part of the Langevin equation.

Our generalization is done in such a way that expectation values of gauge invariant observables again remain unchanged for any choice of $\alpha$ and $\beta_i$. The main motivation behind our generalization is that for specific choices of these extra functions $\alpha$ and $\beta_i$ the fluctuation-dissipation theorem can be applied which leads to drastic simplifications of the stochastic process in the equilibrium limit; such a mechanism is not possible in the original approach of Zwanziger.
Our generalized Langevin equation reads
\[
d\Phi^i(t, s) = -\left[\frac{\delta S}{\delta \Phi^i(t)} + \int dt' Z^i(t, t')\alpha(t')\right] \bigg|_{\Phi(\cdot) = \Phi(\cdot, s)} ds
+ \int dt_1 dt_2 \frac{\delta Z^i(t, t_1)}{\delta \Phi^k(t_2)} \zeta^k(t_1, t_2) \bigg|_{\Phi(\cdot) = \Phi(\cdot, s)} dW^k(t_2, s)
\]
(4)

We introduced \(\zeta^k(t_1, t_2)\) as a shorthand notation of
\[
\zeta^k(t_1, t_2) = 2\delta^{k\ell} \beta^\ell(t_1, t_2) + \int dt_3 dt_4 Z^k(t_2, t_3) \beta^\ell(t_3, t_4) \delta^{\ell m} \beta^m(t_1, t_4).
\]
(5)

We see that the new drift term clearly is not acting tangential to the gauge orbit; its rather complicated structure is necessary for leaving unchanged gauge invariant expectation values; the straightforward proof is given in [5].

In order to proceed more explicitly we decided to study the so called Helix model, which describes the minimal coupling of an abelian gauge field with three bosonic matter fields in 0 + 1 dimensions. This model was originally proposed by deWit [6] and was investigated intensively within the Hamiltonian framework by Kuchař [7]. Recently the helix model came to new life again [8, 9] in the course of studies on problems with gauge fixing. The helix model is defined by the Lagrange density
\[
L(t) = \frac{1}{2}\left[\left(\dot{\varphi}^1 - A\varphi^2\right)^2 + \left(\dot{\varphi}^2 + A\varphi^1\right)^2 + \left(\dot{\varphi}^3 - A\right)^2\right]
- \frac{1}{2}\left[(\varphi^1)^2 + (\varphi^2)^2\right]
\]
(6)
where the dot denotes time derivation and the fields \((\varphi(t), \varphi^3(t)) = (\varphi^1(t), \varphi^2(t), \varphi^3(t))\) and \(A(t)\) are regarded as elements of the function spaces \(\mathcal{E} = C^\infty(\mathbb{R}, \mathbb{R}^3)\) and \(\mathcal{A} = C^\infty(\mathbb{R}, \mathbb{R})\), respectively. Hence the total number of gauge and matter fields is \(m = 4\) and \(\Phi = (\varphi, \varphi^3, A)\). Let \(\mathcal{G} = C^\infty(\mathbb{R}, \mathbb{R})\) denote the abelian group of gauge transformations and consider the following transformation on the configuration space
\[
(\varphi, \varphi^3, A) \rightarrow (R(g)\varphi, \varphi^3 - g, A - \dot{g})
\]
(7)
where \(g \in \mathcal{G}\) and
\[
R(g) = \begin{pmatrix}
\cos g & -\sin g \\
\sin g & \cos g
\end{pmatrix}.
\]
The Lagrange density \( L(t) \) is easily verified to be invariant under these transformations. The components \( Z^i(t', t) \) of the generator of infinitesimal gauge transformations can be read off the vector field

\[
Z_\xi(\Phi) = \int_{\mathbb{R}} dt (-\xi \frac{\delta}{\delta \varphi^1(t)} + \xi \frac{\delta}{\delta \varphi^2(t)} - \xi \frac{\delta}{\delta \varphi^3(t)} - \dot{\xi} \frac{\delta}{\delta A(t)}) \] (8)

We rewrite the stochastic process in terms of gauge invariant and gauge dependent fields (for the explicit geometrical structure see [5])

\[
\Psi = R(\varphi^3) \varphi \\
\Psi^3 = A - \dot{\varphi}^3 \\
\Psi^4 = -\varphi^3
\] (9)

In these new coordinates, gauge transformations are given purely as translations, i.e. \( (\Psi, \Psi^3, \Psi^4) \rightarrow (\Psi, \Psi^3, \Psi^4 - g) \) where \( g \in G \). With respect to these variable changes we introduce the vielbeins \( E \) and their inverses \( e \)

\[
E^\mu_i(t, t') = \frac{\delta \Psi^\mu(t)}{\delta \Phi^i(t')}; \quad e^i_\mu(t, t') = \frac{\delta \Phi^i(t)}{\delta \Psi^\mu(t')},
\] (10)

as well as the induced inverse metric \( G^{\mu\nu} \)

\[
G^{\mu\nu}(t_1, t_2) = \int_{\mathbb{R}} dt_3 E^\mu_i(t_1, t_3) \delta^{ij} E^\nu_j(t_2, t_3).
\] (11)

We can choose such specific values for the functions \( \alpha \) and \( \beta_k \) that the gauge modes’ diffusion is damped along the gauge orbits (as a consequence the Fokker-Planck density can be normalized) and that the equilibrium limit of the stochastic process can explicitly be derived. We take

\[
\alpha(t) = \int_{\mathbb{R}} dt' \left[ G^{4\nu}(t, t') \frac{\delta S}{\delta \Psi^\nu(t')} - \gamma(t, t') \Psi^4(t') \right]
\] (12)

as well as

\[
\beta_k(t_1, t_2) = -E^4_k(t_1, t_2) + \delta_{kl} e^l_4(t_1, t_2)
\] (13)

and

\[
\gamma(t_1, t_2) = \int_{\mathbb{R}} dt_3 e^k_4(t_1, t_3) \delta_{kl} e^l_4(t_2, t_3)
\] (14)
The Langevin equations now read (according to the rules of Ito’s stochastic calculus)

\[
d\Psi^\mu(t, s) = \int_\mathbb{R} dt_1 \left\{ \left[ -\bar{G}^{\mu\nu}(t, t_1) \frac{\delta S_{\text{tot}}}{\delta \Psi^\nu(t_1)} + \frac{\delta \bar{G}^{\mu\nu}(t, t_1)}{\delta \Psi^\nu(t_1)} \right] ds + \bar{E}^\mu_\kappa(t, t_1) dW^\kappa(t_1, s) \right\} \bigg|_{\Psi(\cdot) = \Psi(\cdot, s)}
\]

(15)

Here we introduced the total action

\[
S_{\text{tot}} = S + \frac{1}{2} \int_\mathbb{R} dt (\Psi^4(t))^2,
\]

(16)

the matrix \(\bar{G}\)

\[
\bar{G}^{\bar{\mu}\bar{\nu}}(t_1, t_2) = G^{\bar{\mu}\bar{\nu}}(t_1, t_2) \quad \bar{\mu}, \bar{\nu} = 1, 2, 3
\]

\[
\bar{G}^{\bar{\mu}4}(t_1, t_2) = \bar{G}^{4\bar{\nu}}(t_1, t_2) = 0
\]

\[
\bar{G}^{44}(t_1, t_2) = \gamma(t_1, t_2)
\]

(17)

and the vielbein \(\bar{E}\)

\[
\bar{E}^\mu_\kappa(t_1, t_2) = E^\mu_\kappa(t_1, t_2) + \delta^\mu_4 [-E^4_\kappa(t_1, t_2) + \delta_{k\ell} e^\ell_4(t_1, t_2)]
\]

(18)

We remark that \(\bar{G}\) is explicitly decomposable as

\[
\bar{G}^{\mu\nu}(t_1, t_2) = \int_\mathbb{R} dt_3 \bar{E}^\mu_\kappa(t_1, t_3) \delta^{\kappa\ell} \bar{E}^\nu_\ell(t_2, t_3)
\]

(19)

and by construction is a positive matrix.

We now derive the equilibrium distribution of the stochastic process described by the above Langevin equation (15) by studying the corresponding Fokker-Planck equation. We remind that we restricted ourselves to a well converging stochastic processes, so that the Fokker-Planck probability distribution is normalizable. Most crucially we have that \(\bar{G}\) is positive and is appearing in the Fokker-Planck operator in factorized form. As a consequence (due to the fluctuation-dissipation theorem) the formal stationary limit of the Fokker-Planck probability distribution can be identified with the equilibrium limit and reads

\[
\rho[\Psi]_{\text{equil.}} = \frac{e^{-S_{\text{tot}}}}{\int D\Psi e^{-S_{\text{tot}}}}
\]

(20)

After integrating out \(\Psi^3\) and \(\Psi^4\) [5], our path integral density is agreeing nicely with the result of [8].
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