Polar alignment of a protoplanetary disc around an eccentric binary III: Effect of disc mass

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ABSTRACT
Martin & Lubow (2017) found that an initially sufficiently misaligned low mass protoplanetary disc around an eccentric binary undergoes damped nodal oscillations of tilt angle and longitude of ascending node. Dissipation causes evolution towards polar alignment in which the disc lies perpendicular to the binary orbital plane with angular momentum aligned to the eccentricity vector of the binary. We use hydrodynamic simulations and analytic methods to investigate how the mass of the disc affects this process. A more massive disc settles into a generalised polar state at somewhat lower levels of misalignment with respect to the binary orbital plane, in agreement with the analytic model. We obtain analytic criteria for polar alignment of a circumbinary ring with mass that broadly agree with the simulation results. The required level of initial misalignment angle for evolution to the polar state increases with disc-to-binary angular momentum, $j_0$, for the moderate levels adopted in the simulations. The analytic model suggests than this angle decreases at higher $j_0$. Very broad misaligned discs undergo breaking, but the inner regions at least may still evolve to a polar state. The long term evolution of the disc depends on the evolution of the binary eccentricity that we find tends to decrease. Although the range of parameters required for polar alignment decreases somewhat with disc mass, such alignment appears possible for a broad set of initial conditions expected in protostellar circumbinary discs.

Key words: accretion, accretion discs – binaries: general – hydrodynamics – planets and satellites: formation

1 INTRODUCTION
During the star formation process, chaotic accretion leads to the formation of misaligned discs around binary stars (e.g. McKee & Ostriker 2007; Bate et al. 2003; Monin et al. 2007; Bate et al. 2010; Bate 2018). Observations of circumbinary discs suggest misalignments may be common (e.g. Winn et al. 2004; Chiang & Murray-Clay 2004; Capelo et al. 2012; Brinch et al. 2016; Kennedy et al. 2012; Aly et al. 2018). The planet formation process in these discs will be altered by the torque from the binary that is not present in the single star case (e.g. Nelson 2000; Mayer et al. 2005; Boss 2006; Martin et al. 2014; Fu et al. 2015a,b, 2017; Franchini et al. 2019). Furthermore, giant planets that form in a misaligned disc may no longer remain coplanar to the disc (Picogna & Marzari 2015; Lubow & Martin 2016; Martin et al. 2016). In order to understand the observed properties of exoplanets, we first need to explain the disc evolution in misaligned systems.

A misaligned circumbinary disc around a circular orbit binary undergoes uniform nodal precession with constant tilt. The angular momentum vector of the disc precesses about the binary angular momentum vector. For a sufficiently warm and compact disc, the disc precesses as a solid body (e.g., Larwood & Papaloizou 1997). Dissipation within the disc leads to alignment with the binary orbital plane (Papaloizou & Terquem 1995; Lubow & Ogilvie 2000; Nixon et al. 2011; Nixon 2012; Facchini et al. 2013; Lodato & Facchini 2013; Foucart & Lai 2013, 2014).

Massless (test) particles that orbit around an eccentric orbit binary can undergo nodal libration oscillations of the tilt angle and the longitude of the ascending node, if the particle’s orbital plane is sufficiently misaligned with the binary’s orbital plane (e.g. Verrier & Evans 2009; Farago & Laskar 2010; Doolin & Blundell 2011). Rather than precessing about the angular momentum vector of the binary, such particles instead precess about the eccentricity vector of the binary.

Recently, we found that a low mass warm protostellar circumbinary disc around an eccentric orbit binary can evolve towards polar (perpendicular) alignment with respect to the binary orbital plane for sufficiently high initial inclination (Martin & Lubow 2017). The tilt evolution occurs due to damping of the libration oscillations by dissipation in the
disc and the disc angular momentum aligns to the eccentricity vector of the binary. Aly et al. (2015) found that cool discs that orbit binary black hole systems can also undergo such oscillations and polar alignment. This mechanism operates for sufficiently large misalignment angle (Aly et al. 2015; Lubow & Martin 2018; Zanazzi & Lai 2018).

In Lubow & Martin (2018) and Martin et al. (2014), we consider a model (Lubow & Martin 2007) that has been used extensively for simulations of misaligned accretion discs (e.g. Lodato & Pringle 2007; Ferrer & Osacar 1994; Miranda & Lai 2015; Price et al. 2012; Nixon et al. 2013; Martin et al. 2014; Fu et al. 2015a).

2.1 Simulation set-up

Table 1 summarises the parameters and some results for all of the simulations that we describe in this section. The binary has components with masses $M_1 = M_2 = 0.5 \ M_\odot$, where the total mass is $M = M_1 + M_2$. The binary orbits with semi-major axis $a_b$ and eccentricity vector $\mathbf{e}_b = (e_{1b}, e_{2b}, e_{3b})$. The binary orbit is initially in the $x-y$ plane with eccentricity vector $\mathbf{e}_b = (1, 0, 0)$. This $x-y$ plane serves as a reference plane for the orbital elements described below. The binary begins at apastron separation.

Initially the circumbinary disc is misaligned to the binary orbital plane by inclination angle $i$. We ignore the effects of self-gravity in our calculations. The surface density is initially distributed by a power law $\Sigma \propto R^{-3/2}$ between the initial inner radius $R_{in} = 2 a_b$ up to the initial outer radius $R_{out}$. Typically we take $R_{out} = 5 a_b$, but we do consider some larger values also. The initial inner disc truncation radius is chosen to be that of a tidally truncated coplanar disc (Artymowicz & Lubow 1994). However, the disc spreads both inwards and outwards during the simulation. As described in Lubow & Martin (2018), the inner edge of the disc extends closer to the binary because the usual gap-opening Lindblad resonances are much weaker on a polar disc around an eccentric binary than in a coplanar disc around a circular or eccentric orbit binary (see also Lubow et al. 2015; Nixon & Lubow 2015; Miranda & Lai 2015). We take the Shakhura & Sunyaev (1973) $\alpha$ parameter to be 0.01 in our simulations. The disc viscosity is implemented in the usual manner by adapting the SPH artificial viscosity according to Lodato & Price (2010). The disc is locally isothermal with sound speed $c_s \propto R^{-3/4}$ and the disc aspect ratio varies with radius as $H/R \propto R^{-1/4}$. Hence, $a$ and the smoothing length $\langle h \rangle / H$ are constant over the radial extent of the disc (Lodato & Pringle 2007). We take $H/R = 0.1$ at $R_{in} = 2 a_b$. We examined the effects of these two parameters in Lubow & Martin (2018). Particles in the simulation are removed if they pass inside the accretion radius for each component of the binary at $0.25 \ a_b$.

In order to present a large number of simulations in this work we choose to use 300,000 particles in most of our following simulations. We found this number to provide sufficient resolution in Martin & Lubow (2018) for a time of about 1000 $P_{orb}$, where $P_{orb}$ is the orbital period of the binary. The disc is resolved with shell-averaged smoothing length per scale height $\langle h \rangle / H \approx 0.25$ for $R_{out} \approx 5 a_b$. For simulations with larger initial disc outer radii $R_{out} = 10 a_b$ and $R_{out} = 20 a_b$, we use 600,000 particles initially and the disc is initially resolved with $\langle h \rangle / H \approx 0.26$ and $\langle h \rangle / H \approx 0.31$, respectively.

In order to describe the evolution of the system, we compute the inclination of the disc relative to the instantaneous binary angular momentum as

$$i_{nd} = \cos^{-1}(\mathbf{l}_b \cdot \mathbf{l}_d). \tag{1}$$

where $\mathbf{l}_b = (l_{1b}, l_{2b}, l_{3b})$ is the unit vector in the direction of the binary angular momentum and $\mathbf{l}_d = (l_{1d}, l_{2d}, l_{3d})$ is a unit vector in the direction of the disc angular momentum vector.
The longitude of ascending node phase angle for the disc is

\[
\phi_a = \tan^{-1}\left(\frac{\ell_a}{i_a}\right) + \frac{\pi}{2}.
\]

(2)

We also determine the phase angle of the eccentricity vector of the binary projected onto the reference plane. We define this phase angle as

\[
\phi_b = \tan^{-1}\left(\frac{\ell_b}{i_b}\right) + \frac{\pi}{2}.
\]

(3)

This phase is plotted as red lines in the figures that we describe later. The inclination of the binary relative to the reference plane varies in time and is defined as

\[
i_b = \cos^{-1}\left(\frac{\ell_b}{i_b}\right).
\]

(4)

This angle is plotted as blue lines in the figures that we describe later.

### 2.2 Effect of the disc mass on the disc alignment

We first consider the effect of the disc mass on the standard disc model parameters shown in run1 of Table 1 which is the same model presented in Martin & Lubow (2017). The binary is equal mass with an initial orbital eccentricity of 0.5. The disc is initially inclined by 60° to the binary orbital plane. We calculate disc properties by dividing the disc into 100 bins in spherical radius. Within each bin, we calculate the mean properties of the particles, such as the surface density, inclination, longitude of ascending node, eccentricity.

The top left panel of Fig. 1 shows the evolution of the disc with our standard parameters in run1. We plot the evolution at a disc radius of \(r = 3a_b\) (solid lines) and \(r = 5a_b\) (dashed lines). The disc acts like a solid body since these lines nearly overlap. As described in Martin & Lubow (2017), the disc undergoes nodal libration in which the tilt and longitude of the ascending node oscillate. Dissipation causes the disc to evolve towards polar alignment where \(i_{bd} \approx 90°\). The disc angular momentum vector aligns with the eccentricity vector of the binary and the disc approaches a nonprecessing state. For this low mass disc, there is little evolution of the binary separation, eccentricity vector (as shown by the red line), or inclination (as shown by the blue line).

The other panels in Fig. 1 show the disc evolution with a higher initial mass of \(M_d = 0.01M\) (top right, run2), \(M_d = 0.02M\) (bottom left, run3), and \(M_d = 0.05M\) (bottom right, run4). Now the effect of the disc on the binary is no longer negligible. The binary undergoes apsidal precession (as seen by the red line in the phase angle plot), the inclination changes (see the blue lines), and the magnitude of the eccentricity of the binary oscillates and decays.

For all four disc masses in Fig. 1, the binary and disc phase angles are nearly the equal. The phase difference undergoes a small amplitude oscillation. Over this time, the disc is then nodally librating with respect to the binary, rather than circulating. The libration indicates that the system is in a state where it lies above the critical level of misalignment for polar-like behavior. This suggests that the system is undergoing evolution towards a polar-like state as found in the low mass disc case. There is a small reduction in binary semi-major axis that becomes larger with disc mass. In addition, the binary eccentricity has declined somewhat but the eccentricity decreases overall with disc mass. The reduction of binary eccentricity suggests that over longer timescales the disc might eventually become coplanar with

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**Table 1.** Parameters of the initial circumbinary disc set up for binary with total mass \(M\) and separation \(a\). The disc may be in a circulating (C) or librating (L) state.

| Name | Figure | \(M_d\) | \(i/°\) | \(\epsilon_b\) | \(R_{\text{run}}/a_b\) | C/L | Number of particles | Broken |
|------|--------|--------|--------|--------|-----------------|-----|-------------------|--------|
| run1 | 1      | 0.001  | 60     | 0.5    | 5 L             | 300,000 | No                |
| run2 | 1      | 0.01   | 60     | 0.5    | 5 L             | 300,000 | No                |
| run3 | 1      | 0.02   | 60     | 0.5    | 5 L             | 300,000 | No                |
| run4 | 1      | 0.05   | 60     | 0.5    | 5 L             | 300,000 | No                |
| run5 | 3      | 0.05   | 0     | 0.5    | 5 -             | 300,000 | No                |
| run6 | 3      | 0.01   | 0     | 0.5    | 5 -             | 300,000 | No                |
| run7 | 3      | 0.001  | 0     | 0.5    | 5 -             | 300,000 | No                |
| run8 | 4      | 0.05   | 20    | 0.5    | 5 C             | 300,000 | No                |
| run9 | 4      | 0.05   | 40    | 0.5    | 5 C             | 300,000 | No                |
| run10| 4      | 0.05   | 50    | 0.5    | 5 C             | 300,000 | No                |
| run11| 4      | 0.05   | 80    | 0.5    | 5 L             | 300,000 | No                |
| run12| 5      | 0.05   | 20    | 0.8    | 5 C             | 300,000 | No                |
| run13| 5      | 0.05   | 30    | 0.8    | 5 C             | 300,000 | No                |
| run14| 5      | 0.05   | 40    | 0.8    | 5 L             | 300,000 | No                |
| run15| 5      | 0.05   | 60    | 0.8    | 5 L             | 300,000 | No                |
| run16| 5      | 0.05   | 80    | 0.8    | 5 L             | 300,000 | No                |
| run17| 6      | 0.001  | 60    | 0.5    | 10 L            | 600,000 | No                |
| run18| 6      | 0.01   | 60    | 0.5    | 10 L            | 600,000 | No                |
| run19| 6      | 0.02   | 60    | 0.5    | 10 L            | 600,000 | No                |
| run20| 6      | 0.05   | 60    | 0.5    | 10 L            | 600,000 | No                |
| run21| 7      | 0.05   | 60    | 0.5    | 20 L            | 600,000 | Yes               |

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Figure 1. Simulations of a circumbinary disc around an equal mass binary with initial binary eccentricity $e_b = 0.5$, initial disc inclination $i = 60^\circ$ and $H/R = 0.1$ at the disc inner edge of $R_{in} = 2 a_b$. The disc outer edge is initially at $R_{out} = 5 a_b$. In the upper two panels, the solid lines are for a radius of $R = 3 a_b$ and the dashed lines for $R = 5 a_b$. Top left: $M_d = 0.001 M$ (run1). Top right: $M_d = 0.01 M$ (run2). Bottom left: $M_d = 0.02 M$ (run3). Bottom right: $M_d = 0.05 M$ (run4). Upper panels: inclination of the disc angular momentum vector relative to the binary angular momentum vector, $i_{bd}$ (Equation (1)). The blue lines plot the inclination of angular momentum vector of the binary relative to the reference plane, $i_b$ (Equation (4)). Second panels: precession angles $\phi$. The black lines show the nodal precession angle for the disc $\phi_d$ (Equation 2). The red lines show the binary eccentricity vector phase angle $\phi_b$ (Equation (3)). Third panels: semi-major axis of the binary $a_b$. Lower panels: magnitude of the eccentricity of the binary, $e_b$.

The binary, since the polar disc mechanism requires a certain level binary eccentricity. Better resolution is required to study the longer term evolution.

Fig. 2 plots the evolution of the ratio of the angular momentum of the disc to that of the binary, $J_d/J_b$ for four different simulations that all have $M_d = 0.05 M$, including the case plotted in the lower right panel of Fig. 1 in the solid line. The ratios oscillate in time because the eccentricity of the binary oscillates. In all four cases, the disc angular momentum is quite significant with $J_d \approx 0.3 J_b$.

Unlike the very low mass disc case, a disc with significant mass evolves towards a highly misaligned nonprecessing state relative to the binary which is not perpendicular to the binary orbital plane. We define the stationary inclination angle that the disc is evolving towards as $i_{bd} = i_s$ where the disc precession rate relative to the binary vanishes, i.e., the disc phase angle is stationary (denoted by subscript $s$) relative to the binary phase angle. Only in the massless circumbinary disc case does the disc evolve to exactly polar alignment with $i_s = 90^\circ$.

As we discuss in Section 3.2, angle $i_s$ decreases with increasing particle angular momentum. Consequently, a narrow ring with the same orbital radius as the particle should also experience a decrease in the $i_s$ with increasing ring mass. Similar effects are expected for a disc. For the disc mass of $0.05 M$ (bottom right panel of Fig. 1), the binary-disc in-
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2.3 Binary orbital evolution

The binary orbital evolution is affected by a disc with significant mass. The evolution of the binary angular momentum is determined by both the accretion of angular momentum from the disc and the gravitational torques from the disc. The latter contributions to binary orbit changes involve the interaction of the disc with binary resonances that in turn depend on properties of the binary. In the coplanar binary-disc case for small binary eccentricity, the theory of resonant disc gravitational torques suggests that the binary eccentricity increases due to the dominant effects of a single resonance in the disc (Artymowicz et al. 1991). However, at higher binary eccentricities many resonances can lie within the disc, some of which cause binary eccentricity damping. For eccentricities $e_b \sim 0.5$ or greater, the binary eccentricity growth rate due to disc resonances may become very small or even become negative (Lubow & Artymowicz 1992).

Simulations by Armitage & Natarajan (2005) confirmed this increase in eccentricity along with a decreasing semi-major axis and suggested that this may solve the final parsec problem of merging massive black hole binaries, at least for extreme mass ratio binaries. More recently, Shi et al. (2012) performed the first 3D magnetohydrodynamic (MHD) simulations of a circumbinary disc around an equal mass circular binary. They found that the MHD stresses allowed accretion on to the binary resulting in the semi-major axis increasing slowly.

Because of the sensitivity of the binary evolution to system parameters, we consider here for comparison the evolution in our SPH models in the coplanar case. Fig. 3 shows the evolution of the binary in three coplanar disc simulations for varying disc mass. The eccentricity of the binary decreases in time while the semi-major axis also decreases. The eccentricity change is somewhat insensitive to the mass of the disc while the semi-major axis decreases more quickly for larger disc mass. For comparison, in the blue line in Fig. 3 we also show the binary orbit evolution in our standard inclination parameters of run1. A low mass inclined disc leads to very little binary orbital evolution over the timescale of our simulation.

2.4 Critical inclination for circulating and librating solutions

There is a critical inclination above which the disc is librating and below which it circulates. In Lubow & Martin (2018) we found that for a low mass disc, the critical inclination is close to that predicted for a test particle orbit. As we discuss in Section 3.3, the critical inclination for a massive third body depends upon the angular momentum of the body. Here we consider the critical inclination for two different binary eccentricities.

2.4.1 Initial binary eccentricity $e = 0.5$

Fig. 4 shows the effect of changing the initial inclination of the disc for an initially high disc mass of 0.05 $M$ and an initial binary eccentricity of 0.5. The simulations that have
initial inclination $20^\circ$ (top left, run8), $40^\circ$ (top right, run9) and $50^\circ$ (bottom left, run10) undergo nodal phase circulation of the disc relative to the binary. The disc and the binary are seen to precess in opposite directions. However, for initial inclination of $60^\circ$ (bottom right of Fig. 1, run4) and $80^\circ$ (bottom right of Fig. 4, run11), the disc is librating relative to the binary. The precession angles of the binary and the disc are nearly locked together. Thus, the critical angle between the two types of solution for these parameters is in the range $50^\circ$--$60^\circ$. A disc in this librating state is then in a polar-like orbit around the binary.

For the disc with the initial inclination of $80^\circ$, the tilt oscillations are in the opposite direction to the lower inclination discs. In other words, the inclination initially decreases and the eccentricity increases, vice versa for the lower inclination simulations. The disc is approaching its generalised polar angle $i_1$ from above.

2.4.2 Binary eccentricity $e = 0.8$

Fig. 5 shows the effect of changing the inclination of the disc around a binary with a higher eccentricity of $e_b = 0.8$. The disc varies from circulating phase at initial inclination of $20^\circ$ (top left panel, run12) to librating phase for initial inclination $40^\circ$ (top right panel, run14). Although we do not show a figure, we also ran a simulation with an initial inclination of $30^\circ$ and find that it is circulating (see run13 in Table 1). Thus, the critical angle is between $20^\circ$ and $30^\circ$. This angle is higher than the critical angle expected for a circulating disc at initial inclination of $30^\circ$ based on equation 2 of Doolin & Blundell (2011). The angular momentum evolution of the simulation that begins at $40^\circ$ (run14) is shown in the short–dashed line in Fig. 2.

2.5 Size of the disc

The size of the disc relative to the binary separation may take a wide range of values. Protoplanetary discs are thought to extend to around hundreds of au (e.g. Williams & Cieza 2011). For a close binary, this may be several hundred binary separations. However, for a wider binary this may be only a few times the binary separation. The simulations we have considered so far in this work have a moderate extent and are relevant to wider binaries. In Martin & Lubow (2018) we found that extending the outer disc radius, relative to the binary separation led to warped and even broken discs. If the sound crossing timescale over the disc is longer than the precession timescale, then the disc is unable to communicate fast enough to remain as a solid body.

2.5.1 Initial disc outer radius $R = 10a_b$

Fig. 6 shows the effect of increasing the initial size of the disc to $10a_b$ compared to $5a_b$ that we previously described. The figure shows the same four disc masses as shown in Fig. 1. The qualitative behaviour of the disc has not changed by increasing the initial disc radius. In each case, the disc is in a librating state. The two lines in the inclination and phase angle plots show the disc at a radius of $3a_b$ (solid lines) and $10a_b$ (dashed lines). There is a much more noticeable difference between these two radii now. That is, there is more warping in the larger disc. The warping is larger for the smaller disc mass because the tilt oscillations are larger. For high mass broader disc, the generalised polar (stationary) inclination is $i_1 \approx 60^\circ$ that is slightly lower than for the narrower disc. Thus, the disc begins very close to its stationary angle $i_1$ and so there is little inclination evolution. For the largest disc mass considered (run20), the evolution of the ratio of the disc angular momentum to the binary angular momentum is shown in the long–dashed line in Fig. 2.

2.5.2 Initial disc outer radius $R = 20a_b$

The left hand panel of Fig. 7 shows the high disc initial mass case of $M_2 = 0.05M$ with an even larger initial disc outer radius of $20a_b$ (run21). The two lines in the inclination and phase angle plots show the disc conditions at a radius of $3a_b$ (solid lines) and $20a_b$ (dashed lines). There is significant difference in properties between the two parts of the disc. Hence in the right hand panel we show the surface density, inclination and phase angle as a function of radius at three different times. There is a clear break in the disc at a radius of about $10a_b$. Circumbinary disc simulations around circular binaries have previously shown this behaviour (Nixon et al. 2012; Nixon & King 2012). The inner and the outer parts of the disc precess independently and show tilt oscillations on different timescales. The inner part of the broken disc at least can still achieve polar alignment. For this simulation (run21), the evolution of the ratio of the disc angular momentum to the binary angular momentum is shown in the dot–dashed line in Fig. 2.

3 GENERALISED POLAR ALIGNMENT OF A RING WITH MASS

The secular dynamics of a circumbinary particle are identical to those of a narrow circumbinary ring. In order to understand the stable polar alignment of a disc with significant mass, in this section we consider a three body problem for a circumbinary particle that takes into account the gravitational effects of the masses of all three bodies. We first determine the inclination at the centre of the librating region $i_c$, where the ring (particle) nodal phase is stationary with respect to the binary nodal phase. We then determine the conditions required for a circumbinary ring to evolve into a stationary (polar) configuration. The ring model provides insight into the effects gravitational interactions by the orbiting ring. But it does not include possible effects due to the radial extension of a disc or the advection of disc mass and angular momentum on to the binary.

3.1 Evolution equations

Farago & Laskar (2010) developed a secular theory for the motion of a circumbinary particle of nonzero mass. The principal approximation is that the binary potential is calculated in the quadrupole approximation. They utilize a Cartesian coordinate system that is defined relative to the binary orbit. The orbit changes in time due to gravitational interactions with the particle. The X-direction is along the instantaneous eccentricity vector of the binary, the z-direction is along the instantaneous binary angular momentum, and the
Figure 4. The effect of the initial inclination on the evolution of the high mass disc with initial mass $M_d = 0.05 M$ and $e_b = 0.5$ initially. Top left: initial inclination of $20^\circ$ (run8). Top right: initial inclination of $40^\circ$ (run9). Bottom left: initial inclination of $50^\circ$ (run10). Bottom right: initial inclination of $80^\circ$ (run11).

The $y$-direction is orthogonal to the $x$ and $z$ directions. The origin lies at the instantaneous center of mass of the binary. The equations of motion of the particle are expressed in terms of a unit vector that lies along the direction of the ring’s (particle’s) angular momentum that we denote by tilt vector $\mathbf{\ell} = (\ell_x, \ell_y, \ell_z)$ in this coordinate system.

As shown by Farago & Laskar (2010), the circumbinary ring semi-major axis, the eccentricity (that we assume to be zero), and its angular momentum, $J_r$, are constants of motion. For the binary, the semi-major axis $a_b$ is a constant of motion, while its eccentricity, angular momentum $J_b$, and binary-ring mutual inclination $i$ are not constants of motion. However, the system angular momentum $J$ is a constant of motion. These properties imply that

$$J_b^2 + 2J_bJ_r \cos i = J^2 - J_i^2,$$

where the LHS is a constant of motion. In this equation, since $a_b$ is a constant of motion, binary angular momentum $J_b$ varies in time due to variations in binary eccentricity $e_b$ as inclination $i$ varies in time. This equation then determines a relationship between $e_b$ and $i$. In the limit that $J_b \gg J_r$, Equation (5) implies that $J_b$ and therefore $e_b$ are constants of motion, as applies for a low mass ring. In the opposite limit of a very massive ring $J_b \ll J_r$, we have that $J_b \cos i$ is a constant of motion. This condition holds because the $z$ component binary of binary angular momentum is conserved due to the static potential imposed by the massive stationary ring. The constant of motion in this case plays a key role in the study of Kozai-Lidov oscillations (Kozai 1962; Lidov 1962).

The equations of motion track the variations in time of the tilt vector $\mathbf{\ell}$ and the binary eccentricity $e_b$. We apply the secular evolution equations 3.15 - 3.18 of Farago & Laskar (2010). We make some changes in variables. We make use of
Figure 5. The effect of the initial inclination on the evolution of the high mass disc with initial mass $M_d = 0.05M$ and $e_b = 0.8$. Top left: initial inclination of $20^\circ$ (run12). Top right: initial inclination of $40^\circ$ (run14). Bottom left: initial inclination of $50^\circ$ (run15). Bottom right: initial inclination of $80^\circ$ (run16).

The ratio of the ring-to-binary angular momentum

$$j = \frac{J_r}{J_b}.$$  

(6)

The angular momentum ratio $j$ generally varies in time because $J_b$ varies in time, while $J_r$ does not change in time. We write the evolution equations as

$$\frac{d\ell_x}{d\tau} = (1 - e_b^2)\ell_y\ell_z + \gamma_r\sqrt{1 - e_b^2}\ell_y(2 - 5\ell_z^2),$$  

(7)

$$\frac{d\ell_y}{d\tau} = -(1 + 4e_b^2)\ell_x\ell_z$$

$$- \frac{\gamma_r\ell_x}{\sqrt{1 - e_b^2}} \left( (1 - e_b^2)(2 - 5\ell_z^2) + 5e_b^2\ell_x^2 \right),$$  

(8)

$$\frac{d\ell_z}{d\tau} = 5e_b^2\ell_x\ell_y$$

$$+ \frac{5\gamma_r e_b^2}{\sqrt{1 - e_b^2}}\ell_x\ell_y\ell_z,$$  

(9)

$$\frac{de_b}{d\tau} = 5\gamma_r e_b\sqrt{1 - e_b^2}\ell_x\ell_y.$$  

(10)

where we apply a scaled time equal to $\tau = a'\tau$ for time $t$ in taking the time derivatives above. Quantity $a'$ is constant in time and is defined by equation 3.9 of Farago & Laskar (2010). For our purposes of determining closed orbits, we do
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Figure 6. Same as Fig. 1 except the initial disc outer radius is 10a_b. The initial mass of the disc is 0.001 M (top left, run17), 0.01 M (top right, run18), 0.02 M (bottom left, run19) and 0.03 M (bottom right, run20). The solid lines show a radius of R = 3a_b and the dashed lines R = 10a_b.

not care about the actual time t and therefore do not need to know the value of α′. So we use τ as our time coordinate. Quantity γ_r is proportional to the ring angular momentum and is a constant of motion

\[ γ_r = \sqrt{1 - e_b^2} j_0. \]

(11)

For the purposes of numerically integrating these equations, it is convenient to set γ_r as

\[ γ_r = \sqrt{1 - e_b^2} j_0, \]

(12)

where e_b0 and j_0 are the initial eccentricity and ring-to-binary angular momentum ratio, respectively.

3.2 Stationary inclination

We are interested in determining the conditions for ℓ to be stationary in the ℓ_y = 0 plane, i.e., dℓ/dτ = 0 in Equations (7) - (10). For the test particle case, we know that this occurs when the particle orbit lies perpendicular to the binary orbital plane so that ℓ = (1, 0, 0). (It also occurs for ℓ = (-1, 0, 0) corresponding to an anti-alignment of particle angular momentum with binary eccentricity. But we omit discussion of that orientation.) In the present case, we take into account the nonzero ring mass. The stationary condition in the ℓ_y = 0 plane is given by

\[ ℓ_x = \sqrt{1 - ℓ_z^2}, \]

(13)

\[ ℓ_y = 0, \]

(14)

\[ ℓ_z = \frac{-\left(1 + 4e_b^2\right) + \sqrt{\left(1 + 4e_b^2\right)^2 + 60(1 - e_b^2)e_b^2}}{10j_0}. \]

(15)

as is consistent with Appendix 3.4 of Farago & Laskar (2010). In this stationary state, the binary eccentricity e_b = e_b0 and the ring-to-binary angular momentum ratio j = j_0.
are constant in time. From Equation (15) we can obtain the stationary tilt angle of the ring relative to the binary using the fact that

$$\cos \iota_s = \ell_z. \quad (16)$$

For small ring angular momentum, $j \ll 1$, we have that

$$\cos \iota_s \approx \frac{3j(1 - e_b^2)}{1 + 4e_b^2}. \quad (17)$$

A zero mass stationary ring is then perpendicular to the binary orbital plane, as expected. For arbitrary ring mass, in the limit of high eccentricity close to unity, we have that

$$\cos \iota_s \approx \frac{6j(1 - e_b)}{5}. \quad (18)$$

The stationary tilt angle $\iota_s$ then increases with binary eccentricity. The stationary angle is achieved at a near perpendicular orientation for sufficiently large binary eccentricity. In the limit of large ring angular momentum $j \gg 1$, the stationary inclination is

$$\cos \iota_s \approx \sqrt{\frac{3}{5}}(1 - e_b^2). \quad (19)$$

Note that in the case of circular binary orbit, the stationary angle is the critical angle for Kozai–Lidov oscillations of $39.2^\circ$ (Kozai 1962; Lidov 1962).

Fig. 8 shows the stationary inclination as a function of the ratio of the ring angular momentum to the binary angular momentum for three different binary eccentricities (using equations (15) and (16)). The dashed lines show the corresponding limit of large particle angular momentum given in Equation (19). We compare this to the numerical hydrodynamical disc simulations in Section 3.4.1.

3.3 Conditions for polar evolution

We consider here the conditions required for a ring with nonzero mass to evolve towards a stationary noncoplanar (polar) orientation. For such evolution to occur, the ring needs to be in a state where its angular momentum direction $l$ undergoes libration oscillations about the stationary direction described in Section 3.2.

We determine the minimum inclination required for a librating orbit, given the binary eccentricity and a measure of the ring-to-binary angular momentum $j$. Since this ratio varies in time as $J_b$ varies, we select a value of $j = j_0$ where the line of ascending notes is equal to $\phi = 90^\circ$ and the inclination is smaller than the stationary value $\iota_s$. The latter
condition is applied because a librating orbit of \( l \) forms a closed loop that is double valued in \( \phi \) corresponding to two different values of inclination \( i \) (see points A and C in Figure 9). We select the \( j \) value at the smaller value of \( i (i < i_0) \) for the reference quantity \( j_0 \). Similarly binary eccentricity \( e_b \) varies in time and we apply the value of the reference binary eccentricity \( e_{b0} \) that is the value of eccentricity at this same phase \( \phi = 90^\circ \) and inclination.

For a given initial value of binary eccentricity \( e_{b0} \), angular momentum ratio \( j_0 \) and assumed initial binary-ring inclination \( i_0 < i_1 \), we integrate the evolution Equations (7) - (10) together with Equation (12). The initial conditions are given by

\[
(x, y, z, \epsilon_b) = (\sin i_0, 0, \cos i_0, \epsilon_{b0}).
\]  

(20)

For a fixed set of values of \( e_{b0} \) and \( j_0 \), we determine the minimum value of \( i_0 \) for which the orbit of \( l \) is librating, rather than circulating. This is done using a bisection method. We sometimes refer to that librating orbit as the critical orbit.

Figure 9 plots two critical librating orbits as heavy black lines in a phase portrait of \( i \cos \phi \) versus \( i \sin \phi \). The distance from the origin to a point on the plot is the mutual inclination \( i_0 \), while the angle from horizontal to the line from the origin to a point on the plot is equal to the longitude of ascending node \( \phi \). Both plots are for a system with \( \epsilon_{b0} = 0.5 \). The upper plot has \( j_0 = 0.1 \), while the lower plot has \( j_0 = 0.3 \). The grey lines plot the circulating orbits that result from slightly smaller values of \( i_0 \) than for the critical orbit. Both the minimum and maximum values of \( i \) along an orbit always occur where \( \phi = 90^\circ \) corresponding to points A and C in Figure 9. The minimum inclination along a librating orbit thus occurs at the initial time when \( i = i_0 \) (as described above in Equation (20)).

Figure 10 plots as a dotted line the numerically determined minimum tilt angles for critical librating orbits as a function of \( j_0 \) for three different values of binary eccentricity \( e_{b0} \). In addition, the solid blue line plots the stationary angle. Notice that the stationary angles lie above the minimum tilt angles, as expected (point D in Figure 9 lies above point A). The minimum tilt angle increases with \( j_0 \) for small values \( j_0 \). It flattens and then decreases for larger values for \( j_0 \). If a ring tilt lies above the minimum value given in Figure 10, it does not immediately follow that the ring is in a librating state. The full condition also involves the phase \( \phi \) as we see below. The condition on tilt is necessary for libration, but not sufficient.

### 3.3.1 Lower \( j \) / higher \( e_b \) branch

For sufficiently small values of \( j \) or large values of \( e_b \), it is possible to determine the libration conditions analytically. As seen in the upper panel of Figure 9 for \( j \) small, the critical orbit of \( l \) that separates libration from circulation has a cusp at \( \phi = 0^\circ \) and \( 180^\circ \) that corresponds to \( \ell_x = 0 \). The cusp involves \( d\ell_x/d\tau = 0 \) on the \( \ell_x = 0 \) plane (see also Appendix A3 of Farago & Laskar (2010)). Strictly speaking this is a stationary point. But, this stationary point is unstable, unlike the stationary point in the \( \ell_x = 0 \) plane corresponding to polar configuration discussed in Section 3.2. Since it is unstable, orbits that lie extremely close to it will diverge away from it, either as a librating or circulating orbit. The librating orbit that comes infinitesimally close to a stationary point with the same binary eccentricity is the critical librating orbit.

From Equation (7), we have that this \( \ell_x = 0 \) stationary
that has
time due the variations in its eccentricity only. The quantity
\( J \) is then independent of time. We want to express
\( t \) and \( \ell \) at this stationary point as a function of these con-
tant value infintesimally close to
\( \ell = 2 (\ell_x^2 - 2(1 - e_b^2)j(2j + \cos i) > 0. \)
which are the same as equations A12 and A15 of
Farago & Laskar (2010).

We consider a secular Hamiltonian based on equation
3.21 of Farago & Laskar (2010)
\[
H = \ell_s^2 + e_b^2 (2 - \ell_s^2 - 5 \ell_x^2),
\]
where we ignore an overall factor that is independent of \( t \)
and \( \ell \) that is irrelevant to our considerations below.

The value of the Hamiltonian at this stationary point
in the \( \ell_s = 0 \) plane based on Equations (24) and (25) is equal
to
\[
H = 2(1 - \gamma^2 - \gamma_s^2).
\]
By applying Equations (11) and (22), we then have that at
any point on the critical librating orbit that has a Hamilton-
ian value infintesimally close to \( H_0 \)
\[
H_\ell = 2 \left( e_b^2 - 2 (1 - e_b^2) j (j + \cos i) \right),
\]
where we use the fact that \( \ell_x = \cos i \). But this equation only
holds if \( e_b \) is real in Equation (24), which again through the
application of Equations (11) and (22), implies that
\[
\gamma = \frac{\sqrt{1 - e_b^2} J}{J_b}.
\]
Both \( \gamma \) and \( \gamma \) are constants of motion. The reason is that
the magnitude of the binary angular momentum varies
in
\[
\ell_{z_2} = \frac{2 \gamma_s}{\sqrt{1 - e_{b_s}^2}}
\]
at this stationary point \( x \). From Equations (5) and (11), we
have that total angular momentum conservation implies
\[
y^2 = \gamma^2 + 1 - e_b^2 + 2 \gamma \sqrt{1 - e_b^2} \ell_z.
\]
points at $\phi = 0^\circ$ and $180^\circ$ as expected. The critical librating orbit then covers a large $180^\circ$ range of $\phi$. The lower plot with $\chi < 0$ has cusps points that cover a much smaller range in $\phi$.

Libration requires that $H < H_{cr}$. We then obtain the condition

$$\Lambda_1 = -(1 - e_0^2)(2j + \cos\phi)^2 + 5e_0^2 \sin^2\phi > 0,$$

(30)

where we use $\ell_x = \sin\phi \sin i$ and $\ell_z = \cos i$ in evaluating $H$. For a massless ring we have that $j = 0$ and we recover equation 51 of Zanazzi & Lai (2018).

To find the minimum possible tilt $i$ for libration to occur given values of $e_0$ and $j_0$, we use Equation (30) with $\phi = 90^\circ$ and $\Lambda_1 = 0$ to obtain

$$\cos i_{\text{min}} = \frac{\sqrt{3}e_0 \sqrt{4e_0^2 + 2j_0(1 - e_0^2) + 1 - 2j_0(1 - e_0^2)}}{1 + 4e_0^2}.$$  

(31)

For small $j_0$, the above equation can be expanded as a series to linear order in $j_0$ to give

$$\sin i_{\text{min}} \approx \frac{1 - e_0^2}{1 + 4e_0^2} 1 + \frac{2\sqrt{3}e_0 j_0}{\sqrt{1 + 4e_0^2}}.$$  

(32)

For a massless ring we have that $j = 0$ and $e_0 = e_0$ (constant binary eccentricity), and we recover the minimum tilt angle given in equation (2) of Doolin & Blundell (2011). The term proportional to $j_0$ shows that the minimum angle for libration increases with ring angular momentum for small $j_0$.

For high binary eccentricity, $e_0$ close to unity, we have to lowest order in $1 - e_0$ that

$$\sin i_{\text{min}} \approx \sqrt{\frac{2}{5}(1 + 2j_0)} \sqrt{1 - e_0^2}. $$  

(33)

For large $j_0$ in this equation, libration can occur over a wide range of tilt angles provided that $e_0 \geq 1 - 5/(8j_0^2)$.

3.3.2 Higher $j$ / lower $e_b$ branch

The results in Section 3.3.1 are based on a critical librating orbit that passes through a stationary point (where $d\ell/dt = 0$) that has the property that $\ell_z = 0$. These results apply for $\chi \geq 0$ in Equation (29) which is based on the requirement that the binary eccentricity be a real quantity in Equation (24). For larger $j$ or lower $e_b$ values, where $\chi < 0$, there does not exist a stationary point with $\ell_z = 0$. Instead, as we show below, the critical librating orbit for such larger $j_0$ cases involves a stationary point with the property that $\ell_{bz} = 0$. The properties of this stationary point are also discussed in section A2 of Farago & Laskar (2010).

From Equation (22) with $e_b = 0$, we have that

$$\ell_{bz} = \frac{y^2 - y_1^2 - 1}{2y_1}. $$  

(34)

Using equation (26) for the Hamiltonian, we have that its value at this stationary point for the critical librating orbit is then

$$H_{cr} = \left(\frac{y^2 - y_1^2 - 1}{2y_1}\right)^2. $$  

(35)

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We proceed as in Section 3.3.1 through the use of Equations (11) and (22) in Equation (35) and require that $H < H_{cr}$ for a librating orbit to obtain the condition that

$$\Lambda_2 = e_0^2 + 4j(1 - e_0^2)\left(-\cos i + j(-2 + 5\sin^2\phi)\right) > 0, $$

(36)

which is valid for $\chi < 0$ in Equation (29). The condition $\chi < 0$ is satisfied for sufficiently large values of $j$ or small values of $e_b$.

To find the minimum possible tilt $i$ for libration to occur given values of $e_0$ and $j_0$, we use Equation (36) with $\phi = 90^\circ$ and $\Lambda_2 = 0$ to obtain

$$\cos i_{\text{min}} = \frac{\sqrt{1 - e_0^2 - (1 - e_0^2)j_0}}{10(1 - e_0^2). $$  

(37)

In the limit of large $j_0$, we have to first order

$$\cos i_{\text{min}} \approx \frac{3}{5} \sqrt{\frac{1}{10j_0}.$$

(38)

which approaches the critical angle for Kozai-Lidov oscillations as $j_0$ goes to infinity.

3.3.3 Summary of analytic conditions for polar alignment

Suppose we have a system with the following parameters at some instant in time: ring-to-binary angular momentum ratio $j = J_r/J_b$, mutual binary-ring inclination $i$, and longitude of ascending node for the ring $\phi$. If $\chi > 0$ in Equation (29), then Equation (30) determines whether the system undergoes libration. If $\chi < 0$, then Equation (36) determines whether the system undergoes libration. Libration in turn can lead to alignment to a stationary (polar) configuration.

The values of $i_{\text{min}}$ given by Equation (31) are plotted solid red lines in Figure 10 over the range of $j_0$ values where $\chi > 0$ in Equation (29). The values of $i_{\text{min}}$ given by Equation (37) are plotted solid green lines in Figure 10 over the range of $j_0$ values where $\chi < 0$. Notice that the red and green lines pass through the dotted lines, indicating excellent numerical agreement between the two independent methods (numerical and analytic) for determining minimum angles for both branches. Notice also that there is a change in behaviour of the minimum tilt angle near the largest value of $j_0$ plotted in red that occurs when $\chi = 0$. Figure 9 shows a major change in orbital behaviour for the librating orbits with a change in sign of $\chi$. Figure 10 shows that for $\chi < 0$ (beyond the red lines), the minimum tilt does not increase as rapidly with $j_0$ and decreases for sufficient large $j_0$, as is also indicated by Equation (35).

3.4 Comparison of the analytic criteria to the hydrodynamical simulations

3.4.1 Stationary inclination

Fig. 2 shows the ratio of the disc angular momentum to the binary angular momentum for some of the hydrodynamical simulations. The high mass disc simulations ($M_A = 0.05 M$) with binary eccentricity $e_b = 0.5$ have initially $J_d/J_b = 0.42$. In the analytic model in Fig. 8, this corresponds to a stationary inclination of $i_s = 69.7^\circ$. This is close to the inclination
that the hydrodynamic discs oscillate about (as shown the bottom right panel of Fig. 1 for initial inclination 60° and the bottom right panel of Fig. 4 for initial inclination of 80°).

For the larger eccentricity binary, $e_b = 0.8$, with the same disc parameters (Fig. 5), the system has angular momentum ratio initially $J_d/J_b = 0.61$ and in the analytic model in Fig. 8, this corresponds to a stationary inclination of $i_s = 80.6°$. This is in good agreement with the simulations shown in the top right and bottom left and right panels of Fig. 5 that are librating.

The simulation with the larger disc size (outer radius 10 au) in the bottom right panel of Fig. 6 (run20), has initially $J_d/J_b = 0.52$. In the analytic model this corresponds to a stationary inclination of $i_s = 67.1°$. The largest disc size we considered (outer radius 20 au) in Fig. 7 (run21), has initially $J_d/J_b = 0.68$ and this corresponds to $i_s = 64.0°$.

The stationary inclination for the disc is consistently slightly less than the value predicted for the ring in Fig. 2. However, the angular momentum of the disc and the binary evolve in the disc simulations. While the ratio oscillates because the angular momentum of the binary oscillates, the angular momentum of the disc generally decreases in time.

Furthermore, as described in the Introduction, the dynamics of a ring are somewhat different from that of an extended disc.

### 3.4.2 Condition for polar evolution

For the simulations described in Section 2.4.1 for binary eccentricity $e_b = 0.5$ and a high mass disc $M_d = 0.05 M_\odot$, the transition from librating to circulating solutions is in the range 50° – 60°. This system has angular momentum ratio $J_d/J_b = 0.42$. The critical inclination between librating and circulating solutions for this high disc angular momentum in the analytic model is 51° using Equation (37). For the same disc parameters except disc mass 0.001 $M_\odot$, we previously found that the critical angle was in the range 40° – 50° (Martin & Lubow 2018). This system has angular momentum ratio $J_d/J_b = 0.0084$. In the analytic model for low angular momentum ratio, the critical angle for this angular momentum ratio is 38.4° (Equation (37)). The analytic model slightly underestimates the critical angle.

### 4 DISCUSSION

The polar aligned disc observed by Kennedy et al. (2019) is within 4° of being perpendicular to the binary orbital plane. The binary has a semi-major axis of 1 au, eccentricity $e_b > 0.78$, and the circumbinary gas disc in carbon monoxide extends from about 1.6 au out to about 6.4 au. We assume that the disc has evolved to a stationary configuration and that the binary is equal mass. Then using the Equation (15) for a ring we have that $J_s \sim 0.2 J_b$. Assuming the disc density falls off inversely with radius, we then find that the disc mass is $M_d \lesssim 0.02 M_\odot$. This mass range is quite plausible for protostellar discs.

Solid bodies may form within a massive gaseous disc that reaches its stationary inclination $i_s < 90°$. Such bodies will likely remain within the gaseous disc due to gravitational coupling, unless they are massive enough to open gaps. As the gas disc dissipates, its tilt angle can increase until it reaches the polar state at 90° misalignment with respect to the binary. The orbits of the solid bodies will likely remain coplanar with the disc again due to gravitational coupling, however, once the gaseous disc mass becomes sufficiently small this coupling will break down and the solid bodies may decouple from the gas disc before it reaches its final value of 90° inclination. Once the solid bodies break free of the gas disc their libration speeds will no longer be coordinated and they will randomize relative to each other. The random velocities could effect the planet formation process. Just how this operates is beyond the scope of this paper.

### 5 CONCLUSIONS

In this work we have investigated the conditions under which the nodal libration mechanism can operate in a protostellar disc around an eccentric binary as first described by Martin & Lubow (2017). We apply both SPH simulations and analytic methods. Such discs undergo oscillations of the tilt and longitude of ascending node, similar to test particle orbits. However, for the case of a disc, dissipation leads to polar alignment of the disc. We have investigated the effect of a nonzero mass disc on the system evolution. The mass of the disc affects the outcome of the process because the binary evolution is affected. The disc affects the binary orbit gravitationally and through advection of mass and angular momentum. The binary eccentricity and tilt oscillate. The eventual alignment of disc with nonzero mass is at an angle less than 90°. This has significant implications for planet formation around eccentric binaries and for the detection properties of such discs.

We applied the secular equations of Farago & Laskar (2010) to determine conditions related to polar alignment for an arbitrary mass ring that orbits around an eccentric orbit binary. We determined the stationary misalignment angle, the generalised polar angle, between the ring and binary as a function of system parameters. In the presence of dissipation, the ring tilt could evolve to this angle. This angle, given analytically by Equations (9) and (16), decreases with increasing ratio of ring-to-binary angular momentum and decreasing binary eccentricity (see Figure 8). A very small mass ring lies perpendicular to the binary orbit plane. Departures from the perpendicular state provide constraints on the disc mass, as discussed in Section 4.

We determined analytic criteria required for a ring with mass to evolve to a polar configuration (see Section 3.3.3) and determined the minimum misalignment inclination angles (see Figure 10). For small values of the disc-to-binary angular momentum ratio $j_b$, the minimum tilt angle increases with with $j_b$. But for larger $j_b$, this angle decreases. The change in behaviour is understood in terms of a transition between different types of stationary points for marginally librating orbits. As discussed in Section 3.4, we found approximate agreement between the results of the SPH simulations and the analytic model.

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