Approximated Particle Flow for Continuous Filter, Predictor, and Smoother

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Abstract

The filtering and smoothing techniques for nonlinear continuous-discrete model which is applicable to physical state estimation problems are explained. The multiple distribution estimation and distribution selection are applied to design the non-Gaussian filter. After the execution of the filter, the non-Gaussian smoother is applied to improve the filtered state estimates further. The filtering and smoothing problem for the spacecraft re-entry simulation model is provided and the performance comparison of the proposed approach is compared with the other non-Gaussian filters and smoothers.

1 Introduction

The filter and smoother design problems for nonlinear continuous-discrete model have a long history, which was roughly started from 1960s, and many state-of-the-art techniques have been proposed. Since the model well suits to the real-world, physical state estimation problems, researchers have been pursued the effective and efficient filters and smoothers from both the engineering and mathematical interests. Since 2000s, the techniques originally developed to address the discrete nonlinear filtering problems have been started to be applied to the continuous-discrete models and thanks to the significant advancement of the computational power, the non-Gaussian filtering and smoothing now become practical.

In this paper, a new non-Gaussian filter for the continuous-discrete model is explained. The proposed filter is based on the techniques called the multiple distribution estimation (MDE)[1] and distribution selection (DS)[1] that were both originally devised for tackling the nonlinear discrete filtering problems. These techniques show the robustness against the particle degeneracy that terribly hurts the state estimation accuracy of the particle filter (PF) family. The non-Gaussian smoother performed in this paper is the ensemble Kalman smoother (EnKS) which is an approximation algorithm for the linear optimal smoother[2]. The accuracy improvement in the filtered state estimates by the EnKFs is also investigated.

As an numerical example, the filtering and smoothing problems for the spacecraft re-entry simulation model is provided and the performance comparison of the proposed approach against the other non-Gaussian filters is shown. The rest of this paper is organized as follows. Section 2 describes the problem statement and Section 3 explains the non-Gaussian filter based on the MDE and DS. In Section 4, the EnKS is explained and the numerical example is provided in Section 5. Finally, Section 6 concludes this paper.

2 Problem Statement

The state estimation problem for the following nonlinear continuous-discrete are addressed:

\[ dx(t) = f(x(t), t)dt + dw(t), \quad 0 \leq t \leq T \]
\[ y_k = h_k(x_k) + v_k, \quad k = 1, \ldots, N \]

Here, \( x(t) \) is the \( n \)-dimensional state vector at \( t \) and \( y_k \) is the \( p \)-dimensional observation vector at the discrete time \( t_k \). \( x(t_k) \) is also denoted as \( x_k \). \( w(t) \) is the \( m \)-dimensional Brownian motion vector whose spectral density is \( Q(t) \). \( v_k \) is the white Gaussian observation noise whose covariance matrix is \( R_k \). \( w(t) \) and \( v_k \) are independent for each other for \( 0 \leq t \leq T \) and \( k = 1, \ldots, N \), and they are also independent from \( x(0) \).

The filtering problem is summarized as the estimation of \( p(x_k | Y_k) \) where \( Y_k = \{y_1, \ldots, y_k\} \), while the smoothing problem is summarized as the estimation of \( p(x(t) | Y_N) \). These estimates are called the filtered and smoothed state probability density functions (PDFs), respectively. For Eqs. (1) and (2), \( p(x(t) | Y_k) \) and \( p(x(t) | Y_N) \) are generally non-Gaussian and to pursue their estimation accuracy, non-Gaussian filter and smoother are required.

3 Design of Non-Gaussian Filter based on MDE and DS

The filtered state PDF at \( t_k \) is expressed as follows in the MDE approach.

\[ p(x_k | Y_k) = \int p(x_k, x_{k-1} | Y_k)dx_{k-1} \]
\[ = \int p(x_k | x_{k-1}, y_k)p(x_{k-1} | Y_k)dx_{k-1} \]
\[ = \frac{1}{D_k} \int p(x_k | x_{k-1}, y_k)p(y_k | x_{k-1}) \]
\[ \times p(x_{k-1} | Y_{k-1})dx_{k-1}, \quad (3) \]

where \( D_k = \int p(y_k | x_{k-1})p(x_{k-1} | Y_{k-1})dx_{k-1} \)
Then, suppose that the filtered state PDF at $t_{k-1}$ is obtained as

$$p(x_{k-1}|Y_{k-1}) \approx \frac{1}{M} \sum_{i=1}^{M} \delta(x_{k-1} - x_{k-1}^{(i)})$$  \hspace{1cm} (4)$$

Here, $\{x_{k-1}^{(i)}, i = 1, \ldots, M\}$ is the set of $M$ particles drawn from the filtered state estimate $x_{k-1}|Y_{k-1}$.

Then, Eq. (3) becomes

$$p(x_k|Y_k) \approx \frac{1}{D_k} \sum_{i=1}^{M} p(y_k|x_{k-1}^{(i)})p(x_k|x_{k-1}^{(i)}, y_k),$$  \hspace{1cm} (5)

where $D_k \approx \sum_{i=1}^{M} p(y_k|x_{k-1}^{(i)})$

Here, $p(y_k|x_{k-1}^{(i)})$ and $p(x_k|x_{k-1}^{(i)}, y_k)$ in Eq. (5) are

$$p(y_k|x_{k-1}^{(i)}) = \int p(y_k|x_k)p(x_k|x_{k-1}^{(i)})dx_k$$  \hspace{1cm} (6)

$$p(x_k|x_{k-1}^{(i)}, y_k) = \frac{p(y_k|x_k)p(x_k|x_{k-1}^{(i)})}{p(y_k|x_{k-1}^{(i)})}$$  \hspace{1cm} (7)

Equations (5) to (7) are the MDE for the filtered state PDF at $t_k$ given Eq. (4).

Then, suppose that $p(y_k|x_k)$ and $p(x_k|x_{k-1}^{(i)})$ in Eqs. (6) and (7) are both Gaussian and that $y_k$ is linear in $x_k$, $p(y_k|x_{k-1}^{(i)})$ and $p(x_k|x_{k-1}^{(i)}, y_k)$ in Eqs. (6) and (7) also become Gaussian, denoted as $N(\hat{x}_{k}^{(i)}, P_{k}^{(i)})$ and $N(y_k|\hat{y}_{k}^{(i)}, P_{y,k}^{(i)})$ respectively. Then, Eq. (5) becomes

$$p(x_k|Y_k) \approx \frac{1}{D_k} \sum_{i=1}^{M} w_{k}^{(i)} N(\hat{x}_{k}^{(i)}, P_{k}^{(i)}),$$  \hspace{1cm} (8)

where $w_{k}^{(i)} = N(y_k|\hat{y}_{k}^{(i)}, P_{y,k}^{(i)})$, $D_k = \sum_{i=1}^{M} w_{k}^{(i)}$

Here, $\hat{x}_{k}^{(i)}$, $P_{k}^{(i)}$, $\hat{y}_{k}^{(i)}$, and $P_{y,k}^{(i)}$ are obtainable by applying the discrete Gaussian filter\cite{3} such as the EKF, UKF, or CKF for the Gaussian-assumed $p(x_k|x_{k-1}^{(i)})$. Then, the problem left is the Gaussian estimation of $p(x_k|x_{k-1}^{(i)})$ for Eq. (1). Given Eq. (8), the filtered state estimate and the estimation error covariance matrix are

$$\hat{x}_k = \sum_{i=1}^{M} \frac{w_{k}^{(i)}}{D_k} \hat{x}_{k}^{(i)}$$  \hspace{1cm} (9)

$$P_k = \sum_{i=1}^{M} \frac{w_{k}^{(i)}}{D_k} (P_{k}^{(i)} + (\hat{x}_{k}^{(i)} - \hat{x}_k)(\hat{x}_{k}^{(i)} - \hat{x}_k)^T)$$  \hspace{1cm} (10)

3.1 Approximated Particle Flow

For the Gaussian estimation of $p(x_k|x_{k-1}^{(i)})$, the following optimal predictor for Eq. (1) that provides the predicted state estimate and the estimation error covariance matrix given $Y_{k-1}$ is applied:

$$\frac{d\hat{x}(t)}{dt} = E[f(x(t), t)|Y_{k-1}]$$  \hspace{1cm} (11)

$$\frac{dP(t)}{dt} = \text{Cov}(x(t), f(x(t), t)|Y_{k-1}) + \text{Cov}(x(t), f(x(t), t)|Y_{k-1})^T + Q(t)$$  \hspace{1cm} (12)

Here, $t > t_{k-1}$. When $f(x(t), t)$ is linear in $x(t)$, Equations (11) and (12) become

$$\frac{d\hat{x}(t)}{dt} = f(\hat{x}(t), t)$$  \hspace{1cm} (13)

$$\frac{dP(t)}{dt} = P(t)F_t^T + F_tP(t) + Q(t)$$  \hspace{1cm} (14)

and Eqs. (13) and (14) reduce to the ordinary differential equations (ODEs). Here, $F_t$ is the system transition matrix at $t$. By integrating Eqs. (13) and (14) from $t_{k-1}$ to $t_k$ with the initial condition of $N(x_{k-1}^{(i)}, \phi)$, where $\phi$ is a null matrix, the Gaussian estimation of $p(x_k|Y_{k-1})$ is obtained. The numerical integration of Eqs. (13) and (14) based on the fourth-order Runge-Kutta (4RK) method is preferred from the accuracy viewpoint and the 4RK is thus performed by $M$ times.

3.2 Particle Re-sampling based on DS

Since Eq. (8) is Gaussian-sum, the particle resampling from the filtered state PDF is necessary to obtain the expression in Eq. (4) for the next time step. There are two choices for it and the first approach is sampling one particle from each $N(\hat{x}_k^{(i)}, P_k^{(i)})$ and re-sampling the $M$ sampled particles according to the weights $\{w_{k}^{(i)}/D_k, i = 1, \ldots, M\}$. However, this approach may result in the particle impoverishment after the re-sampling procedure. The second approach is selecting $M$ Gaussian components according to the weight $\{w_{k}^{(i)}/D_k, i = 1, \ldots, M\}$ and then, sampling one particle from each selected component. This approach called the DS always guarantees the particle diversity since the particle duplication never occurs. For example, when the $i$th Gaussian component $N(\hat{x}_k^{(i)}, P_k^{(i)})$ is selected three times, the three particles sampled from it are generally different since $w_{k}^{(i)}$ is not null.

4 Non-Gaussian Smoother by EnKS

Using the matrix notation as in [4], the EnKS for Eq. (1) is as follows:

$$\hat{X}_k = \hat{X}_k + [G_k^M \cdots G_k^M](\hat{X}_{k+1} - \hat{X}_k),$$  \hspace{1cm} (15)

where $G_k^M = \hat{X}_k W \hat{X}_k$.
filtered state estimate at $t_k$, the predicted state estimate at $t_{k+1}$, and the smoothed state estimate at $t_k$, respectively. $\hat{x}_k^M$, $\hat{x}_{k+1}^M$, and $\hat{x}_k^M$ are the corresponding ensemble means as follows:

\[
\hat{x}_k^M = [\hat{x}_k^{(1)} \cdots \hat{x}_k^{(M)}] \left[ \frac{1}{M} \cdots \frac{1}{M} \right]^T
\]

\[
\hat{x}_{k+1}^M = [\hat{x}_{k+1}^{(1)} \cdots \hat{x}_{k+1}^{(M)}] \left[ \frac{1}{M} \cdots \frac{1}{M} \right]^T
\]

\[
\hat{x}_k^M = [\hat{x}_k^{(1)} \cdots \hat{x}_k^{(M)}] \left[ \frac{1}{M} \cdots \frac{1}{M} \right]^T
\]

Defining $w^M = [\frac{1}{M} \cdots \frac{1}{M}]^T$, $W$ in Eq. (15) is

\[
W = [I - [w^M \cdots w^M]] \text{diag} \left( \frac{1}{M} \cdots \frac{1}{M} \right) \times [I - [w^M \cdots w^M]]^T
\]

$[G_k^M \cdots G_k^M]$ in Eq. (15) is $m \times (m \times M)$ matrix.

The EnKS is an approximation algorithm for the linear optimal smoother which is optimal among the linear smoothers[2]. In Eq. (15), the ensembles for the smoothed state estimate are propagated backwards in time from $t_{k+1}$ to $t_k$.

5 Numerical Example

5.1 Model and Data

The spacecraft re-entry model is often used as a benchmark simulation problem for evaluating the performance of nonlinear filters[4]. The system model is formally expressed as follows:

\[
\begin{align*}
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4 \\
\dot{x}_3 &= D x_3 + G x_1 + w_1 \\
\dot{x}_4 &= D x_4 + G x_2 + w_2 \\
\dot{x}_5 &= w_3 \\
D &= b \cdot \exp \left( \frac{r_0 - r}{h_0} \right) v \\
b &= b_0 \exp(x_5) \\
r &= \sqrt{x_1^2 + x_2^2} \\
v &= \frac{\sqrt{x_1^2 + x_2^2}}{\sqrt{x_3^2 + x_4^2}} \\
G &= -\frac{\mu}{r^3} \\
r_0 &= 6374, \ b_0 = -0.59783 \\
h_0 &= 13.406, \ \mu = 3.9860 \times 10^5
\end{align*}
\]

The units are kilometers, kilograms, and seconds. Here, $D$ and $G$ are regarding the drag force and the gravity force from the earth ($\mu$ is the gravity parameter of the earth). $w_i, (i = 1, 2, 3)$ are zero-mean, delta-correlated Gaussian processes with joint spectral density $Q(t) = \text{diag}(2.4064 \times 10^{-4}, 2.4064 \times 10^{-4}, 0)$. $x_5$ is the aerodynamic property of the spacecraft. The true initial state is Gaussian whose expectation and covariance matrix are $x(0) = [6500.4, 349.14, -1.8093, -6.7967, 0.6932]^T$ and $P_0 = \text{diag}(10^{-6}, 10^{-6}, 10^{-6}, 10^{-6}, 0)$.

The observation radar is located at $(r_0, 0)$ and the observation model is

\[
\begin{align*}
r(t_k) &= \sqrt{(x_1(t_k) - r_0)^2 + x_2(t_k)^2} + v(t_k)_1 \\
\theta(t_k) &= \tan^{-1} \left( \frac{x_2(t_k)}{x_1(t_k) - r_0} \right) + v(t_k)_2
\end{align*}
\]

here, $v(t_k), (j = 1, 2)$ are zero-mean, white Gaussian noises with joint covariance matrix $R(t_k) = \text{diag}(10^{-6}, (0.01 \pi / 180)^2)$. $t_k, (k = 1, 2, \cdots)$ is set to $(1 \times k)(s)$ and the simulation duration is 250 seconds. To generate the data, the system model is re-expressed by the Itô’s SDE and discretized using the EM scheme with 10000 steps between each observation.

5.2 Filter Setting

The initial filtered state PDF is Gaussian, having $\hat{x}(0) = [6500.4, 349.14, -1.8093, -6.7967, 0]^T$ and $P_0 = \text{diag}(10^{-6}, 10^{-6}, 10^{-6}, 10^{-6}, 1)$. For the filter execution, the $w_i, (i = 1, 2, 3)$ is set to having $Q = \text{diag}(2.4064 \times 10^{-4}, 2.4064 \times 10^{-4}, 10^{-5})$ to simultaneously estimate the state and the unknown aerodynamic property. The observation noise filtered is exactly the same as that used for simulating the data.

The EKF is used for propagating particles in the proposed non-Gaussian filter. Non-Gaussian filters compared are the PF (Monte Carlo Filter), Gaussian PF (GPF), and the extended Kalman PF (EKPF). The Gaussian PF is almost the same as the PF instead of resampling particles from the Gaussian-assumed filtered state PDF whose first two moments are calculated by the particle averages. The EKFs for the proposed filter and the EKPF are integrated by the 4RK using 10 steps between each observation, while the SDEs for the PF and Gaussian PF are integrated by the EM method with 10 steps between each observation.

The number of particles is varied from 100 to 1000 to investigate the improvement in the state estimation accuracy.

5.3 Filtering and Smoothing Results

Table 1 shows the following root mean square errors (RMSEs) evaluated by 20 Monte Carlo runs:

\[
\text{RMSE1}(t_k) = \sqrt{E[(x_1(t_k) - \hat{x}_1(t_k))^2 + (x_2(t_k) - \hat{x}_2(t_k))^2]}
\]

\[
\text{RMSE2}(t_k) = \sqrt{E[(x_5(t_k) - \hat{x}_5(t_k))^2]}
\]

Here, $\hat{x}_s(s = 1, 2, 5)$ is the corresponding element of the filtered state estimate at $t_k$. The RMSEs further averaged over the simulation duration are shown in the table. From Table 1, we can clearly see that the proposed filter shows the best estimation accuracy among
Table 1: Simulation results. The first row is \( M \). For each filter, the first and second rows denote the averaged RMSE1 and RMSE2 over the simulation duration.

|       | \( 100 \) | \( 200 \) | \( 500 \) | \( 1000 \) |
|-------|--------|--------|--------|--------|
| PF    | 26.63  | 28.23  | 32.52  | 34.95  |
|       | 0.71   | 0.69   | 0.79   | 0.63   |
| GPF   | --     | --     | --     | --     |
|       | --     | --     | --     | --     |
| EKPF  | 27.54  | 32.86  | 46.46  | 42.97  |
|       | 1.0    | 1.1    | 1.46   | 1.45   |
| Proposed Filter | 0.014 | 0.013 | 0.012 | 0.012 |
|       | 0.50   | 0.49   | 0.15   | 0.11   |

the PF family compared. The GPFs were diverged for all \( M \) settings due to the loss of the positive definiteness for the filtered state estimation error covariance matrices. The results for the PF, GPF, and EKPF are significantly worse and this is because of the particle degeneracy. On the other hands, the proposed filter showed the robustness against the degeneracy problem and the estimation accuracy is improved as the \( M \) is increased.

Table 2 shows the RMSEs for the smoothing results for the proposed filter.

Table 2: Simulation results. The first row is \( M \). For the filter and smoother, the first and second rows denote the averaged RMSE1 and RMSE2.

|                  | \( 1000 \) | \( 2000 \) |
|------------------|----------|----------|
| Proposed Filter  | 0.012    | 0.012    |
|                  | 0.11     | 0.098    |
| Proposed Filter + EnKS | 0.008 | 0.008 |
|                  | 0.048    | 0.046    |

The averaged RMSEs of 0.008 and 0.046 indicate that the expected estimation error for the state position is about 8m and that for the aerodynamic property is around 0.046. The Gaussian-assumed Type I continuous smoother in [5] was also tested but the results were significantly worse than those for the EnKS.

6 Conclusions

The multiple distribution estimation and distribution selection techniques are applied to design the non-Gaussian filter for nonlinear continuous-discrete models. The filtering and smoothing problem for the spacecraft re-entry simulation model is provided as the numerical example and the proposed approach showed the robustness against the particle degeneracy inherent in the PF family. The effectiveness of the non-Gaussian smoother, EnKS, is also confirmed.

References

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