Nonhomogeneous analytic families of trees

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Abstract

We consider a dichotomy for analytic families of trees stating that either there is a colouring of the nodes for which all but finitely many levels of every tree are nonhomogeneous, or else the family contains an uncountable antichain. This dichotomy implies that every nontrivial Souslin poset satisfying the countable chain condition adds a splitting real.

We then reduce the dichotomy to a conjecture of Sperner Theory. This conjecture is concerning the asymptotic behaviour of the product of the sizes of the $m$-shades of pairs of cross-$t$-intersecting families.

1 Introduction

The results in this paper are directed at the general question of just how fundamental the Cohen and random forcing notions are. More specifically we are interested in the following question from [She94].

**Question 1** (Shelah). Does $\{\text{Cohen, random}\}$ form a basis for the collection of all nontrivial Souslin ccc posets?

What came as a surprise, was a deep connection between this fundamental question in the theory of set theoretic forcing and extremal combinatorics.

Recall that a poset is *Souslin* if it can be represented a pair $(P, \leq)$ where $P \subseteq \mathbb{R}$ is analytic, and the partial ordering $\leq$ and the incompatibility relation are both analytic subsets of $\mathbb{R} \times \mathbb{R}$. Clearly the Cohen and random posets are both Souslin. By *nontrivial* we mean that the poset adds a new generic object to the ground model; in other words, the poset has a condition with no atoms below it. A *basis* for a class $\mathcal{C}$ of posets, is a subclass $\mathcal{B} \subseteq \mathcal{C}$ such that every member $Q$ of $\mathcal{C}$ has a member $P$ in $\mathcal{B}$ which *embeds* into it, i.e. there is a map $e : P \rightarrow Q^*$, where $Q^*$ is the completion of $Q$ (i.e. the complete Boolean algebra of regular open subsets of $Q$), such that $p \leq q \rightarrow e(p) \leq e(q)$ and $e$ preserves...
maximal antichains; equivalently, adding a generic object for \( Q \) adds a generic for \( P \).

The easier half of this question has been answered in [She94]:

**Theorem 1** (Shelah, 1994). *Every Souslin ccc poset which adds an unbounded real also adds a Cohen real.*

Thus the Cohen poset embeds into every Souslin ccc poset which is not weakly distributive, and thus it remains (for a positive answer) to show that every nontrivial weakly distributive Souslin poset with the ccc adds a random real. However, the remaining part of Question \( \dagger \) seems to be considerably more difficult. Indeed, many consider it a dubious conjecture that the collection \{Cohen, random\} does in fact form a basis. Veličković has suggested some test questions for this conjecture, including the following one from [Vel02].

**Question 2** (Veličković). *Does every nontrivial Souslin ccc poset add a splitting real?*

Recall that a real \( s \subseteq \mathbb{N} \) is *splitting* over some model \( M \), if both \( s \) and its complement intersect every infinite \( E \subseteq \mathbb{N} \) in \( M \), i.e. \( s \cap E \neq \emptyset \) and \( s^c \cap E \neq \emptyset \). Hence adding a splitting real means adding a splitting real over the ground model. This is a good test question because Cohen and random reals are both also splitting reals. One can see this by noting that \( \{ x \subseteq \mathbb{N} : x \subseteq y \} \) and \( \{ x \subseteq \mathbb{N} : x \cap y = \emptyset \} \) are both null (for the Haar measure on \( \mathcal{P}(\mathbb{N}) \cong \{0, 1\}^\mathbb{N} \)) and meager for every infinite \( y \subseteq \mathbb{N} \). But this fact also follows immediately from the more general corollary \( \dagger \) below.

We shall formulate a general dichotomy of descriptive set theory for analytic families of trees (dichotomy \( \dagger \)), which has a positive answer to question \( \dagger \) as a straightforward consequence. Whether or not this dichotomy is true remains unsolved. The main result of this paper, however, is the surprising discovery of a deep connection between this dichotomy and Sperner Theory. In particular, the famous Erdős–Ko–Rado Theorem (cf. [EKR61], but actually proved in 1938) on the maximum size of \( t \)-intersecting families is relevant. More specifically, in the paper [Hir08a], we arrived at a conjecture on the asymptotic behaviour of the maximum of the product of the size of the \( m \)-shades of a pair of cross-\( t \)-intersecting families. The main result of this paper is a substantial reduction of the dichotomy to this reasonable conjecture of extremal combinatorics. \( \dagger \)

There have been some relevant developments since the original version of this paper was written. First of all, question \( \dagger \) was solved by Veličković in [Vel05], where he moreover proved the following result:

**Theorem 2** (Veličković, 2003). \((*)_\mathbb{C} \) implies that every nontrivial weakly distributive ccc poset adds a splitting real.

\( ^1 \)A previous version of this paper ([Hir08b]) made a stronger combinatorial conjecture, and established the dichotomy as a consequence. However, that conjecture was disproved in [Hir08a].
is the standard $P$-ideal dichotomy on sets of size continuum (see e.g. [Tod00]).

However, Veličković’s theorem does not seem to say much on whether our dichotomy is true. In fact, even if Shelah’s question has a positive answer, it still does not seem to entail our dichotomy. On the other hand, our approach of extracting a statement of descriptive set theory is the approach used by Shelah in proving theorem 1; this has advantages because the descriptive set theoretic statement gives an actual description of the splitting real (specifically, in equation (4) below) as opposed to merely asserting its existence, and the descriptive set theory is of interest in its own right (e.g. the descriptive set theory extracted from the proof of theorem 1 is developed further in [Kam98] and [VCI02]).

The other developments have been on the measure theory front. In [BJP05] an old problem of von Neumann was settled. From this one could deduce that a positive answer to the famous Maharam’s Conjecture of abstract measure theory would imply a positive answer to Shelah’s question (question 1). The most astounding development however was Talagrand’s recent negative solution to Maharam’s Conjecture [Tal06]. While Maharam’s Conjecture was considered by some to be the most significant problem in abstract measure theory, Shelah’s question is now the natural place to look for even more challenging measure theoretic problems in a similar vein.

2 A dichotomy for analytic families of trees

We write $\mathbb{N} = \{0, 1, \ldots\}$ for the nonnegative integers. For a parameter $f \in \mathbb{N}^\mathbb{N}$ we let $T(f)$ denote the family of all subtrees of the tree

\[(1) \quad f^{< \mathbb{N}} = \bigcup_{n=0}^{\infty} \prod_{i=0}^{n-1} \{0, \ldots, f(i) - 1\}\]

of all functions $t$ with domain $\{0, \ldots, n-1\}$ for some $n \in \mathbb{N}$, with $t(i) < f(i)$ for all $i < n$, ordered by inclusion. To avoid trivialities we assume that $f \geq 2$, i.e.

\[(2) \quad f(n) \geq 2 \quad \text{for all } n.\]

Its topology is of course obtained by identifying $T(f)$ with the product $\{0, 1\}^{f^{< \mathbb{N}}}$. Note that $2^{< \mathbb{N}}$ is the binary tree and is more usually denoted by $\{0, 1\}^{< \mathbb{N}}$. Indeed, this is the primary case and one will not lose much by taking $f = 2$ throughout.

A branch through a tree $T$ refers to an infinite downwards closed chain of $T$. Recall that by König’s classical tree lemma, every infinite $T \in T(f)$ has an infinite branch. We say that $T$ is infinitely branching if $T$ has infinitely many branches. More familiar to set theorists are perfect trees $T$ where every node has an extension to a splitting node (i.e. a node with at least two immediate successors); thus perfect trees are in particular infinitely branching.
For a tree $T \in \mathcal{T}(f)$, we write $T(n) \subseteq \prod_{i=0}^{n-1} \{0, \ldots, f(i) - 1\}$ for the $n^{th}$ level of $T$, which consists of sequences of length $n$, and with $T(0)$ the singleton consisting of the empty sequence whenever $T \neq \emptyset$. And we write $T \upharpoonright n$ for the subtree consisting of the first $n$ levels $T(0), \ldots, T(n-1)$ of $T$. For $\epsilon = 0, 1$, a subset $A \subseteq f^\mathbb{N}$ is called $\epsilon$-homogeneous for some colouring $c : f^\mathbb{N} \to \{0, 1\}$, if $c(s) = \epsilon$ for all $s \in A$. We say that $A$ is homogeneous for $c$ if $c$ is constant on $A$, i.e. $A$ is nonhomogeneous for $c$ if it is not homogeneous.

For a family of trees $A \subseteq \mathcal{T}(f)$ and some tree $T \in \mathcal{T}(f)$, we denote $A_T = \{ S \in A : S \subseteq T \}$.

Note 1. It is important to note that we use the set theoretic terminology for antichains of a poset. I.e. if $(P, \leq)$ is a poset, then $A \subseteq P$ is an antichain iff $A$ is incompatible with $b$, written $a \perp b$, for all $a \neq b$ in $A$, where $a \perp b$ means that there is no $p \in P$ satisfying both $p \leq a$ and $p \leq b$.

We emphasize this because in extremal combinatorics (e.g. [And02]) an antichain refers instead to the weaker concept of a pairwise incomparable subset. These are also called Sperner families, and we shall use the latter terminology.

Dichotomy 1. Every analytic family $A$ of infinitely branching subtrees of $f^\mathbb{N}$ satisfies at least one of the following:

(a) There exists a colouring $c : f^\mathbb{N} \to \{0, 1\}$ and a $T \in A$ such that $S(n)$ is homogeneous for $c$ for at most finitely many $n \in \mathbb{N}$, for every $S \in A_T$.

(b) The poset $(A, \subseteq)$ has an uncountable antichain.

Remark 2. In a previous version ([Hir08b]) we had a stronger dichotomy where there was no restriction to $T$ in the first alternative (a), i.e. $c$ is nonhomogeneous for every $T \in A$. We think that the latter stronger dichotomy is entailed by the former; however, we wish to avoid the additional complications here.

Remark 3. We have not yet considered what happens when we remove the restriction that the subtrees be infinitely branching. It is obvious that dichotomy [\[1]] is false if we allow subtrees with only one branch, and we conjecture that it is still false even if we do not allow this triviality.

Let us see how this yields a splitting real.

Lemma 4. Dichotomy [\[1]] implies that every nontrivial Souslin poset with the ccc adds a splitting real.

Proof. Let $\mathcal{P}$ be a nontrivial Souslin poset with the ccc. It is known that $\mathcal{P}$ must add a new real; this is proved in [She94], and also follows from the fact that this statement is absolute, since one can construct a Souslin tree from any nontrivial poset with the ccc which does not add a real (see [She01]). Thus we can find a $\mathcal{P}$-name $\dot{r}$ for a new real in $\{0, 1\}^\mathbb{N}$. For each $p \in \mathcal{P}$, we let $T_p \in \mathcal{T}(2)$ be the tree of possibilities for $\dot{r}$ which is defined by

\[ T_p = \{ t \in \{0, 1\}^\mathbb{N} : q \models \check{t} \subseteq \check{r} \text{ for some } q \leq p \}. \]
The family \( \mathcal{A} = \{ T_p : p \in \mathcal{P} \} \) is an analytic subset of \( \{0, 1\}^{<\mathbb{N}} \), because \( \mathcal{P} \) is Souslin and since the ccc property allows for maximal antichains to be described as reals (see [She94], or [Kam98]). Since \( \check{r} \) names a new real, each \( T_p \) is a perfect tree, and thus in particular has infinitely many branches. And since \( q \leq p \) implies \( T_q \subseteq T_p \), if \( T_p \) and \( T_q \) are incompatible in the poset \( (\mathcal{A}, \subseteq) \), then \( p \) and \( q \) are incompatible in \( \mathcal{P} \). Since the second condition (b) entails the existence of an uncountable antichain in \( (\mathcal{A}, \subseteq) \), by the ccc dichotomy (with \( f = 2 \)) yields a colouring \( c : \{0, 1\}^{<\mathbb{N}} \rightarrow \{0, 1\} \) as in condition (a).

Define a \( \mathcal{P} \)-name \( \dot{s} \) for a real in \( \mathcal{P}(\mathbb{N}) \) by

\[
(4) \quad n \in \dot{s} \iff c(\check{r}↾ n) = 1.
\]

Then \( \dot{s} \) names a splitting real, because for every \( p \in \mathcal{P} \) and every infinite \( E \subseteq \mathbb{N} \), we can find an \( n \in E \) such that \( T_p(n) \) is nonhomogeneous for \( c \), and thus there are \( t_0 \) and \( t_1 \) in \( T_p(n) \) such that

\[
(5) \quad c(t_0) = 0 \quad \text{and} \quad c(t_1) = 1.
\]

And there are \( q_0, q_1 \leq p \) forcing that \( \check{r}↾ n = t_0 \) and \( \check{r}↾ n = t_1 \), respectively. Thus \( q_0 \models (\mathbb{N} \setminus \dot{s}) \cap E \neq \emptyset \) and \( q_1 \models \dot{s} \cap E \neq \emptyset \). \( \square \)

### 2.1 Generic splitting reals

Every \( f \in \mathbb{N}^\mathbb{N} \) determines a space of reals

\[
(6) \quad \mathbb{R}(f) = \prod_{n=0}^{\infty} \{0, \ldots, f(n) - 1\}.
\]

Note that for \( T \in \mathcal{T}(f) \), \([T] \subseteq \mathbb{R}(f) \) is a closed subset in the product topology, where \([T] \) is the set of all reals corresponding to a branch through \( T \) (i.e. \( x \upharpoonright n \in T \) for all \( n \)). By a perfect subset of a topological space, we mean a nonempty closed set with no isolated points. Thus \([T] \) is perfect whenever \( T \) is a perfect tree.

The natural measure on \( \mathbb{R}(f) \) is the Haar probability measure \( \mu_f \) determined by the group operation on \( \mathbb{R}(f) \) of coordinatewise addition modulo \( f(n) \). Notice that

\[
(7) \quad \mu_f([T]) = \lim_{n \to \infty} \nu_{fT}(n)
\]

where \( \nu_{fT}(0) \geq \nu_{fT}(1) \geq \cdots \) is the sequence

\[
(8) \quad \nu_{fT}(n) = \frac{|T(n)|}{\prod_{i=0}^{n-1} f(i)}.
\]

**Lemma 5.** For all \( f \in \mathbb{N}^\mathbb{N} \) \( (f \geq 2) \), there exists a colouring \( c : f^{<\mathbb{N}} \rightarrow \{0, 1\} \) such that for every \( T \in \mathcal{T}(f) \) at least one of the following holds.

(a) There are only finitely many \( n \) for which \( T(n) \) is homogeneous for \( c \).
(b) \( \mu_f([T]) = 0 \).

**Proof.** Define a colouring \( c : f^{<\omega} \to \{0, 1\} \) by

\[
(9) \quad c(s) = 0 \iff s(|s| - 1) < \left\lfloor \frac{f(|s| - 1)}{2} \right\rfloor.
\]

where \( |x| \) is the largest integer \( \leq x \). Observe that if \( T(n) \) is homogeneous for \( c \) (in either colour) for all \( n \) in some finite set \( F \), then setting \( n = \max(F) + 1 \),

\[
\mu_f([T]) \leq \frac{|T(n)|}{\prod_{i=0}^{f(n)-1} f(i)} \leq \frac{\prod_{i \in F} f(i) \cdot \prod_{i \not\in F} f(i)}{\prod_{i=0}^{f(n)-1} f(i)} \leq \left( \frac{2}{3} \right)^{|F|}.
\]

We say that a subfamily \( A \subseteq \mathbb{T}(f) \) separates points if every \( T \in A \) and every \( x, y \in [T] \) has a \( U \subseteq T \) in \( A \) such that at most one of \( x \) and \( y \) is in \( [U] \). It is clear that if \( A \) is a nonempty family of subtrees of \( f^{<\omega} \) that separates points, then the poset \( (A, \subseteq) \) forces a generic real in \( R(f) \).

Obviously we are identifying \( R(2) = \{0, 1\}^\omega \) with \( \mathcal{P}([\omega]) \) in corollary 6.

**Corollary 6.** If \( A \subseteq \mathbb{T}(2) \) is a nonempty family that separates points which has the property that \( \mu([T]) \neq 0 \) for all \( T \in A \), then the generic real of the poset \( (A, \subseteq) \) is a splitting real.

**Proof.** Let \( c : \{0, 1\}^{<\omega} \to \{0, 1\} \) be the colouring from (9). Notice that if \( \dot{r} \) is an \( A \)-name for the generic real then, the real \( \dot{s} \) defined in (4) satisfies

\[
(10) \quad n \in \dot{s} \iff \dot{r}(n-1) = 1.
\]

And by lemma 5 the second paragraph of the proof of lemma 4 shows that \( \dot{s} \) names a splitting real. It follows that \( \dot{r} \) which is a shift of \( \dot{s} \) is also a splitting real. \( \square \)

# 3 Decay of branching

The next dichotomy focuses on a lower bound for the size of the levels rather than nonhomogeneity.

**Lemma 7.** For every analytic \( A \subseteq \mathbb{T}(f) \) consisting of infinitely branching trees, at least one of the following holds:

(a) There is an \( h \in \omega^\omega \) with \( \lim_{n \to \infty} h(n) = \infty \) such that \( |T(n)| \geq h(n) \) for all but finitely many \( n \), for all \( T \in A \).

(b) There is a perfect subset \( B \subseteq A \) such that \( T \cap U \) has only finitely many branches for all \( T \not= U \) in \( B \).

Note that the condition (b) implies that the poset \( (A, \subseteq) \) has an antichain of size continuum. Lemma 7 is essentially proved in [She94]. It also follows immediately from the following closely related dichotomy, which is, essentially, implicit in [She94], and is explicit, with a different proof, in [Vc02]. The deduction of lemma 7 from lemma 8 is spelled out in [Hir08b]. Note that \( [\omega]^\omega \) is the...
family of all infinite subsets of \( \mathbb{N} \), and \( f \leq^* g \) if \( f(n) \leq g(n) \) for all but finitely many \( n \). For an infinite \( x \subseteq \mathbb{N} \) we write \( e_x \in \mathbb{N}^\mathbb{N} \) for the strictly increasing enumeration of \( x \) starting with \( e_x(0) = \min(x) \).

**Lemma 8.** For every analytic subset \( \mathcal{X} \subseteq [\mathbb{N}]^\infty \), at least one of the following holds:

(a) The family \( \{e_x : x \in \mathcal{X}\} \) is bounded in \((\mathbb{N}^\mathbb{N}, \leq^*)\).

(b) There is a perfect subset of \( \mathcal{X} \) consisting of pairwise almost disjoint sets.

Let us point out a corollary of lemma 8 which is interesting since, for example, every \( \Sigma^1_2 \) set is a union of \( \aleph_1 \) many analytic sets.

**Corollary 9.** If \( \mathcal{X} \subseteq [\mathbb{N}]^\infty \) is the union of less than \( b \) analytic sets, then either

(a) the family of enumerating functions of members of \( \mathcal{X} \) is bounded in \((\mathbb{N}^\mathbb{N}, \leq^*)\),

or

(b) there is a perfect almost disjoint subset of \( \mathcal{X} \).

Unfortunately, this dichotomy (lemma 8) is too weak for our needs. We need to bound the decay in branching, that is how much smaller the levels of \( S(n) \) are than \( T(n) \) for \( S \subseteq T \) in \( \mathcal{A} \), but first we must measure it.

### 3.1 Measuring the decay

Let \( \mathbb{N}_\infty^\mathbb{N} \subseteq \mathbb{N}^\mathbb{N} \) denote the subfamily of functions \( g \) such that \( \lim_{n \to \infty} g(n) = \infty \). For a function \( \tau : [0, \infty) \to [0, \infty) \) on the nonnegative reals, define a function \( \rho_\tau : \mathbb{N}_\infty^\mathbb{N} \times \mathbb{N}_\infty^\mathbb{N} \to [0, \infty] \) by

\[
\rho_\tau(g, h) = \liminf_{n \to \infty} \frac{h(n)}{\tau \circ g(n)}.
\]

Then define a relation \( \leq_\tau \) on \( \mathbb{N}_\infty^\mathbb{N} \) by

\[
g \leq_\tau h \quad \text{if} \quad \rho_\tau(g, h) > 0.
\]

For a family \( \mathcal{H} \) of such functions, we define a relation \( \leq_{\mathcal{H}} \) on \( \mathbb{N}_\infty^\mathbb{N} \) by

\[
g \leq_{\mathcal{H}} h \quad \text{if} \quad g \leq_{\tau} h \quad \text{for all} \ \tau \in \mathcal{H}.
\]

Typically we shall have

(i) \( \tau(x) = O(x) \) (i.e. \( \tau(x) \) is bounded by \( Cx \) for some \( C > 0 \)),

(ii) \( \lim_{x \to \infty} \tau(x) = \infty \),

(iii) \( \tau \) is concave
for all $\tau \in \mathcal{H}$. Condition (ii) ensures that

\[
g \leq h \quad \text{implies} \quad g \leq_{\tau} h,
\]

and in particular each $\leq_{\tau}$, and thus $\leq_{\mathcal{H}}$, is reflexive. Condition (iii) ensures that each $\leq_{\tau}$ is nontrivial. Concavity gives us

\[
\tau(cx) \geq c\tau(x) \quad \text{for all } c \leq 1.
\]

The reason we use a family of functions is to obtain transitivity. A family $\mathcal{H}$ with every $\tau \in \mathcal{H}$ satisfying (ii–iii), that is moreover closed under “taking roots”, i.e.

(iv) $\forall \tau \in \mathcal{H} \exists \sigma \in \mathcal{H} \quad \sigma \circ \sigma = \tau$,

is said to be suitable.

**Example 10.** The singleton $\mathcal{H} = \{\text{id}\}$ is suitable.

**Example 11.** $\mathcal{H} = \{x \mapsto x^\alpha : 0 < \alpha < 1\}$ is suitable since $x^{\sqrt{\alpha}} \circ x^{\sqrt{\alpha}} = x^\alpha$.

We could include $\alpha = 1$, but then $\leq_{\mathcal{H}}$ would be the same as $\leq_{\text{id}}$.

**Lemma 12.** Let $\mathcal{H}$ be suitable. Then $\leq_{\mathcal{H}}$ is a quasi ordering of $\mathbb{N}^\mathbb{N}$.

**Proof.** Suppose $g_0 \leq_{\mathcal{H}} g_1$ and $g_1 \leq_{\mathcal{H}} g_2$. Take $\tau \in \mathcal{H}$, and find $\sigma \in \mathcal{H}$ such that $\sigma \circ \sigma = \tau$. Choose $0 < \varepsilon_0 < 1$ so that $\varepsilon_0 \rho_\tau(g_0, g_1) \leq 1$ and let $0 < \varepsilon_1 < 1$ be arbitrary. Then there are $k_0$ and $k_1$ such that $g_1(n) > \varepsilon_0 \rho_\tau(g_0, g_1) \cdot \sigma \circ g_0(n)$ for all $n \geq k_0$ and $g_2(n) > \varepsilon_1 \rho_\tau(g_1, g_2) \cdot \sigma \circ g_1(n)$ for all $n \geq k_1$. Hence, using (15) and the fact that $\sigma$ must be nondecreasing, $g_2(n) > \varepsilon_1 \rho_\tau(g_1, g_2) \cdot \sigma(g_0(n))$ for all $n \geq \max(k_0, k_1)$, proving that $g_0 \leq_{\tau} g_2$.

For a subtree $T \subseteq f^{<\mathbb{N}}$ with infinitely many branches, clearly $h_T \in \mathbb{N}^\mathbb{N}$, were $h_T(n)$ is the size of the $n^{\text{th}}$ level:

\[
h_T(n) = |T(n)|.
\]

**Proposition 13.** Assume $\mathcal{H}$ is suitable. If $S \subseteq T$ then $h_S \leq h_T$, and hence $h_S \leq_{\mathcal{H}} h_T$.

**Definition 14.** Let $\mathcal{H}$ be a suitable family, and let $\mathcal{A} \subseteq \mathcal{T}(f)$ be a family of infinitely branching trees. We say that $\mathcal{A}$ has locally bounded branching decay with respect to $\mathcal{H}$ if there exists $T \in \mathcal{A}$ such that $h_T \leq_{\mathcal{H}} h_S$ for all $S \subseteq T$ in $\mathcal{A}$. We say that $\mathcal{A}$ has (globally) bounded branching decay with respect to $\mathcal{H}$ if the preceding statement holds for all $T \in \mathcal{A}$. For $\mathcal{H}$ as in example 11 we say that $\mathcal{A}$ has (locally) (globally) bounded branching decay for $x^{<1}$.

**Example 15.** We consider the case $\mathcal{H} = \{\text{id}\}$. Suppose that $\mathcal{A} \subseteq \mathcal{T}(f)$ with $f^{<\mathbb{N}} \in \mathcal{A}$. If $\mathcal{A}$ has globally bounded branching decay with respect to id, then by definition, $h_{f^{<\mathbb{N}}} \leq_{\mathcal{H}} h_T$ for all $T \in \mathcal{A}$. Observe however, that

\[
\rho_{\text{id}}(h_{f^{<\mathbb{N}}}, h_T) = \mu_f([T]) \quad \text{for all } T \in \mathcal{T}(f)
\]
(cf. equation (7)). Hence the global bounding property is equivalent to every member of \( \mathcal{A} \) having positive measure.

If \( \mathcal{A} \) has locally bounded branching decay with respect to \( \text{id} \), then there exists some \( T \in \mathcal{A} \) such that \( h_T \leq h_S \) for all \( S \subseteq T \) in \( \mathcal{A} \). Observe that \( \rho_{\text{id}}(h_T, \cdot) \) may define a finitely additive measure on \( \mathcal{A}_T \) (if \( T \) has “uniform” branching), even though we may have \( \mu_f([T]) = 0 \). Indeed, in this case we can define strictly positive finitely additive measure by \( \nu(S) = \lim_{n \to U} \frac{\h_S(n)}{\h_T(n)} \) for a nonprincipal ultrafilter \( U \). We remark, without proof, that it is possible to generalize the construction in equation (9) to obtain a nonhomogeneous colouring for \( \mathcal{A}_T \).

Now we consider a family \( \mathcal{A} \) that has locally bounded branching decay for \( x < 1 \), witnessed by some \( T \in \mathcal{A} \). First we notice that this is a weaker condition than having locally bounded branching decay for \( \text{id} \).

Let \( \mathcal{H} \) be the family from example 11.

**Proposition 16.** For all \( g, h \in \mathbb{N}_\infty, \) for all \( \tau \in \mathcal{H} \), \( \rho_{\tau}(g, h) > 0 \) implies \( \rho_{\tau}(g, h) = \infty \).

For reasons related to the central limit theorem (cf. \[Hir08a\]), we are especially interested in \( \alpha = \frac{1}{2} \).

**Proposition 17.** For all \( g, h \in \mathbb{N}_\infty \) and all \( 0 < \alpha < 1 \), \( g \leq h \) implies that \( \liminf_{n \to \infty} \frac{h(n)}{g(n)^\alpha} = \infty \).

It is easy to construct an analytic, indeed countable, family of trees that is not weakly distributive and does not have locally bounded branching decay for \( x < 1 \). For this reason, our Sperner Theoretic analysis does not apply to the non-weakly distributive case. An analytic family of trees that is equivalent as a forcing notion to Cohen forcing does have a nonhomogeneous colouring. However, the existence of nonhomogeneous colourings for arbitrary analytic non-weakly distributive ccc families of trees depends on the more difficult problem than question 1 of finding a complete characterization of Souslin ccc forcing notions, and not just a basis. Thus we restrict our attention to the weakly distributive case to obtain the following weakening of dichotomy 1.

**Dichotomy 2.** Every weakly distributive analytic family \( \mathcal{A} \) of infinitely branching subtrees of \( f^{< \mathbb{N}} \) satisfies at least one of the following:

(a) There exists a colouring \( c : f^{< \mathbb{N}} \to \{0, 1\} \) and a \( T \in \mathcal{A} \) such that \( S(n) \) is homogeneous for \( c \) for at most finitely many \( n \in \mathbb{N} \), for every \( S \in \mathcal{A}_T \).

(b) The poset \( (\mathcal{A}, \subseteq) \) has an uncountable antichain.

**Question 3.** Does every analytic family \( \mathcal{A} \subseteq \mathbb{T}(f) \) of infinitely branching trees, that both is weakly distributive and satisfies the ccc, have locally bounded branching decay for \( x < 1 \)?

We shall prove (theorem 4) that a positive answer to question 3 together with our Sperner Theoretic conjecture (conjecture 2) establishes dichotomy 2.
Let us point out that the Sperner Theory is making a substantial contribution towards the dichotomy. This is because if we ask instead for “locally bounded branching decay for id” in question 3, the answer is most likely negative. Indeed, in view of example [15] we expect that Talagrand’s construction ([Ta06]) of a counterexample to the Control Measure problem provides a counterexample to this question.

4 The coideals

For a subset $A \subseteq \mathbb{N}$ and $k \in \mathbb{N}$ we let

$$A - k = A \setminus \{0, \ldots, k\},$$

with $A - (-1) = A$. Please note that this disagrees with many authors’ usage!

For a function $F$ with domain $\mathbb{N}$ denote

$$\mathbb{N}(F) = \bigcup_{n=0}^{\infty} \prod_{n} F(n),$$

i.e. the collection of all pairs $(n, x)$ with $x \in F(n)$. And for a subset $S \subseteq \mathbb{N}(F)$ and $k \in \mathbb{N}$, denote

$$S - k = \{(n, x) \in S : n > k\}.$$

Throughout this article we assume that $F$ is a sequence of nonempty finite sets, and thus $\mathbb{N}(F)$ is countably infinite.

We recall that a set $A \subseteq \mathbb{N}$ diagonalizes a sequence $(A_k : k \in \mathbb{N})$ of subsets of $\mathbb{N}$ if $A - k \subseteq A_k$ for all $k \in A$. And a coideal $\mathcal{H}$ on $\mathbb{N}$ is selective if every descending sequence $A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$ of members of $\mathcal{H}$ has a diagonalization in $\mathcal{H}$. We next generalize the definition of selective to coideals on $\mathbb{N}(F)$.

**Definition 18.** $B \subseteq \mathbb{N}(F)$ diagonalizes a sequence $(B_k : k \in \mathbb{N})$ of subsets of $\mathbb{N}(F)$ if $B - k \subseteq B_k$ whenever $k \in \text{dom}(B)$. The definition of a selective coideal on $\mathbb{N}(F)$ is the same as for $\mathbb{N}$ but with this notion of diagonalization.

We suppose now that $\mathcal{U}$ is a nonprincipal selective ultrafilter on $\mathbb{N}$, and that $g \in \mathbb{N}^\mathbb{N}$ with $g(n)$ nonzero for all $n$ is a given parameter. For a subset $A \subseteq \mathbb{N}(F)$, write $A(n) = \{x : (n, x) \in A\}$. Define $\mathcal{H} = \mathcal{H}(\mathcal{U}, F, g)$ by

$$\mathcal{H} = \left\{ A \subseteq \mathbb{N}(F) : \lim_{n \to \mathcal{U}} \frac{|A(n)|}{g(n)} = \infty \right\}.$$

**Proposition 19.** If $\lim_{n \to \infty} \frac{g(n)}{|F(n)|} = 0$ then $\mathcal{H}$ is a nonempty selective coideal.

**Proof.** The assumptions that the limit is zero and that $\mathcal{U}$ is nonprincipal ensure that $\mathbb{N}(F) \in \mathcal{H}$. Obviously $\mathcal{H}$ is a coideal.
Suppose that \((A_k : k \in \mathbb{N})\) is a descending sequence of members of \(\mathcal{H}\). Then there is a descending sequence \((U_k : k \in \mathbb{N})\) of members of \(\mathcal{U}\) such that

\[(22) \quad \frac{|A_k(n)|}{g(n)} > k \quad \text{for all } n \in U_k,\]

for all \(k \in \mathbb{N}\). And then there is a \(U \in \mathcal{U}\) with \(U - k \subseteq U_k\) for all \(k \in U\). Now if we define

\[(23) \quad B = \bigcup_{n \in U} \left( \{n\} \times \bigcap_{k : n \in U_k} A_k(n) \right),\]

then \(B \in \mathcal{H}\) because \((A_k : k \in \mathbb{N})\) is descending. And for all \(k\), if \(k \in \text{dom}(B)\) then \(k \in U\) and thus \(B - k \subseteq \bigcup_{n \in U_k} \left( \{n\} \times B(n) \right) \subseteq A_k\). \(\square\)

### 5 The forcing notion

Define

\[(24) \quad \mathbb{F}(F) = \{ a \subseteq \mathbb{N}(F) : a \text{ is a finite partial function} \},\]

and for \(a \in \mathbb{F}(F)\), denote \(\hat{a} = \max(\text{dom}(a))\) with \(\hat{\emptyset} = -1\). For a subset \(S \subseteq \mathbb{N}(F)\) and \(a \in \mathbb{F}(F)\), we let

\[(25) \quad S - a = S - \hat{a}.\]

Suppose that \(\mathcal{H}\) is a nonempty selective coideal on \(\mathbb{N}(F)\). Let \(Q = \mathbb{Q}(\mathcal{H})\) be the poset of all pairs \((a, A)\) where \(a \in \mathbb{F}(F)\) and \(A \in \mathcal{H}\), ordered by \((b, B) \leq (a, A)\) if \(a \subseteq b\) (i.e. \(b\) end-extends \(a\)), \(b \setminus a \subseteq A\) and \(B \subseteq A\). An extension \((b, B) \leq (a, A)\) is called a **pure extension** of \((a, A)\) if \(b = a\). For convenience, we also insist that \(A - a = A\), i.e. \(\hat{a} < \min(\text{dom}(A))\).

Forcing with \(Q\) introduces a generic partial function \(\dot{c}\) on \(\mathbb{N}\) with \(\dot{c}(n) \in F(n)\), by defining

\[(26) \quad \dot{c} = \bigcup_{(a, A) \in \dot{G}} a,\]

where \(\dot{G}\) is a \(Q\)-name for the generic filter of \(Q\). We also write

\[(27) \quad \dot{D} = \text{dom}(\dot{c}).\]

**Lemma 20.** For every sentence \(\varphi\) in the forcing language of \(Q\), every condition in \(Q\) has a pure extension deciding whether or not \(\varphi\) is true.

**Proof.** Fix \((a, A) \in Q\) and a sentence \(\varphi\). For all \(b \in \mathbb{F}(F)\) and \(B \in \mathcal{H}\), we will say that \(B\) **accepts** \(b\) if \((b, B - b) \models \varphi\), \(B\) **rejects** \(b\) if there is no \(C \subseteq B\) in \(\mathcal{H}\) which accepts \(b\), and \(B\) **decides** \(b\) if it either accepts or rejects \(b\). The following facts are immediate from the definitions.
(28) For all $b \in \mathbb{F}(F)$ and $B \in \mathcal{H}$ there is a $C \subseteq B$ in $\mathcal{H}$ deciding $b$.

(29) If $B$ accepts/rejects $b$, then $C$ accepts/rejects $b$ for every $C \subseteq B$ in $\mathcal{H}$.

Claim 1. If $B \in \mathcal{H}$ accepts $b \cup \{s\}$ for all $s \in B - b$, then $B$ accepts $b$.

Proof. Assume that $B$ accepts $b \cup \{s\}$ for all $s \in B - b$. Supposing towards a contradiction that $B$ does not accept $b$, there exists $(c, C) \leq (b, B - b)$ such that

\[(30) \quad (c, C) \models \neg \varphi.\]

By extending $(c, C)$ if necessary, we can assume that $b$ is a proper initial segment of $c$. Then there is an $s \in B - b$ with $b \cup \{s\} \subseteq c$. However, by assumption, $(b \cup \{s\}, B - \{s\}) \models \varphi$, contradicting (30) because $(c, C) \leq (b \cup \{s\}, B - \{s\})$. \qed

Claim 2. There is a $B \subseteq A$ in $\mathcal{H}$ which decides $a \cup b$ for every finite partial function $b \subseteq B$.

Proof. Using (28), construct a descending sequence $A \supseteq B_0 \supseteq B_1 \supseteq \cdots$ of members of $\mathcal{H}$ such that

\[(31) \quad B_n \text{ decides } a \cup b \quad \text{for all partial functions } b \subseteq A \text{ with } \hat{b} \leq n,\]

for all $n$. Let $B \subseteq B_0$ in $\mathcal{H}$ diagonalize $(B_n : n \in \mathbb{N})$. Then $B$ decides $a$ by (29) and (31) with $n = 0$. Take $\emptyset \neq b \subseteq B$ in $\mathbb{F}(F)$. Then $B - (a \cup b) = B - \hat{b} \subseteq B_b$ and thus by (31), $B$ decides $a \cup b$. \qed

Letting $B$ be as in claim 2, in particular, $B$ decides $a$, and thus we need only concern ourselves when $B$ rejects $a$, in which case we make the following claim.

Claim 3. There is a $C \subseteq B$ in $\mathcal{H}$ which rejects $a \cup b$ for every finite partial function $b \subseteq C$.

Proof. Using claims 1 and 2 we can construct a descending sequence $B \supseteq C_0 \supseteq C_1 \supseteq \cdots$ in $\mathcal{H}$ so that

\[(32) \quad C_n \text{ rejects } a \cup b \cup \{s\} \quad \text{for all } s \in C_n - b, \]

for all $b \subseteq B$ in $\mathbb{F}(F)$ with $\hat{b} \leq n$ where $B$ rejects $a \cup b$.

Let $C \subseteq C_0$ in $\mathcal{H}$ diagonalize the sequence. It follows by induction that $C$ rejects $a \cup b$ for all $b \subseteq C$ in $\mathbb{F}(F)$, because if $b \subseteq C$, $s \in C - (a \cup b) = C - b \subseteq C_{b-1}$ (where $C_{-1} = C_0$) and $C$ rejects $a \cup b$, then since $B$ decides $a \cup b$, $B$ rejects $a \cup b$ by (29), and thus $C$ rejects $a \cup b \cup \{s\}$ by (32). \qed

Now if there were some $(b, D) \leq (a, C)$ forcing $\varphi$, then since $b \setminus a \subseteq C$, this would contradict that $C$ rejects $b$. Thus $(a, C) \not\models \neg \varphi$. \qed

Lemma 21. Let $\varphi(x)$ be a formula in the forcing language of $\mathcal{Q}$. Then for every $(a, A) \in \mathcal{Q}$, there exists $B \subseteq A$ in $\mathcal{H}$ such that $(a \cup b, B - b)$ decides $\varphi(\hat{a} \cup \hat{b})$ for every finite partial function $b \subseteq B$. 

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Proof. Using lemma 20, choose a descending sequence \( A \supseteq B_0 \supseteq B_1 \supseteq \cdots \) in \( \mathcal{H} \) such that
\[
(a \cup b, B_n) \text{ decides } \varphi(a \cup b) \quad \text{for all } b \subseteq A \text{ in } \mathbb{F}(F) \text{ with } \hat{b} \leq n.
\]
If \( B \subseteq B_0 \) in \( \mathcal{H} \) diagonalizes \( (B_n : n \in \mathbb{N}) \), then \( B \) satisfies the conclusion of the lemma. \( \square \)

6 Shades

One of the basic notions in Sperner theory is the shade (also called upper shadow) of a set or a family of sets (see e.g. [And02, Eng97]). For a subset \( x \) of a fixed set \( S \), the shade of \( x \) is
\[
\nabla(x) = \{ y \subseteq S : x \subseteq y \text{ and } |y| = |x| + 1 \},
\]
and the shade of a family \( X \) of subsets of \( S \) is
\[
\nabla(X) = \bigcup_{x \in X} \nabla(x).
\]
Recall that the \( m \)-shade (also called upper \( m \)-shadow or shade at the \( m \)th level) of \( x \) is
\[
\nabla_{\to m}(x) = \{ y \subseteq S : x \subseteq y \text{ and } |y| = m \},
\]
and \( \nabla_{\to m}(X) = \bigcup_{x \in X} \nabla_{\to m}(x) \). We follow the Sperner theoretic conventions of writing \([m, n]\) for the set \( \{m, m+1, \ldots, n\} \) and \([n]\) for the set \([1, n] = \{1, \ldots, n\}\).

We introduce the following notation for colouring sets with two colours. For a set \( S \), let \( \binom{S}{[m]} \) denote the collection of all colourings \( c : S \to \{0, 1\} \) with \( |c^{-1}(0)| = m \), i.e. \( c^{-1}(0) = \{ j \in S : c(j) = 0 \} \). This is related to shades, because for all \( c \in \binom{S}{[m]} \) and all \( x \subseteq S \),
\[
(37) \quad \text{if } x \text{ is homogeneous for } c \text{ iff } c^{-1}(0) \in \nabla_{\to m}(x) \text{ or } c^{-1}(1) \in \nabla_{\to |S|-m}(x).
\]

When a nonhomogeneous colouring is desired, it is most efficient to use colorings in \( \binom{S}{[m]} \) for \( |S| = 2m \). Equation (37) immediately gives us:

**Lemma 22.** Suppose \( X \) is a family of subsets of \([2m]\). Then
\[
\left| \left\{ c \in \binom{[2m]}{[m]} : \exists x \in X \text{ } x \text{ is homogeneous for } c \right\} \right| \leq 2|\nabla_{\to m}(X)|
\]
(the shades are with respect to \( S = [2m] \)).
6.1 Upper bounds

Recall that a family $\mathcal{A}$ of sets is $t$-intersecting if $|E \cap F| \geq t$ for all $E, F \in \mathcal{A}$; and a pair $(\mathcal{A}, \mathcal{B})$ of families of subsets of some fixed set are cross-$t$-intersecting if

$$|E \cap F| \geq t$$

for all $E \in \mathcal{A}$, $F \in \mathcal{B}$. Thus $\mathcal{A}$ is $t$-intersecting iff $(\mathcal{A}, \mathcal{A})$ is cross-$t$-intersecting.

We use the standard notation $\binom{[n]}{k}$ to denote the collection of all $k$-subsets of $S$, and hence $\binom{[n]}{k}$ denotes the collection of all subsets of $[n]$ of cardinality $k$. Let $I(n, k, t)$ denote the family of all $t$-intersecting subfamilies of $\binom{[n]}{k}$ (where $t \leq k \leq n$). Define the function

$$M(n, k, t) = \max_{\mathcal{A} \in I(n, k, t)} |\mathcal{A}|.$$

The investigation into the function $M$ and the structure of the maximal families was initiated by Erdős–Ko–Rado in 1938, but not published until [EKR61]. In this paper, they gave a complete solution for the case $t = 1$, and posed what became one of most famous open problems in this area. The following so called $4m$-conjecture for the case $t = 2$:

$$M(4m, 2m, 2) = \frac{1}{2} \left( \binom{4m}{2m} - \binom{2m}{m} \right)^2.$$

We briefly explain the significance of the right hand side expression. Define families

$$\mathcal{F}_i(n, k, t) = \left\{ F \in \binom{[n]}{k} : |F \cap [t + 2i]| \geq t + i \right\} \text{ for } 0 \leq i \leq \frac{n - t}{2}.$$

Clearly each $\mathcal{F}_i(n, k, t)$ is $t$-intersecting. It is not hard to compute that the cardinality of $\mathcal{F}_{m-1}(4m, 2m, 2)$ is equal to the right hand side of equation (40). Indeed, the $4m$-conjecture was generalized by Frankl in 1978 ([Fra78]) as follows: For all $1 \leq t \leq k \leq m \leq n$,

$$M(n, k, t) = \max_{0 \leq i \leq \frac{n - t}{2}} |\mathcal{F}_i(n, k, t)|.$$

In 1995, the general conjecture was proven true by Ahlswede–Khachatrian in [AK97].

Since we are interested in applying lemma 22, we define for $1 \leq t \leq k \leq m \leq n$,

$$M_0(n, m, k, t) = \max_{\mathcal{A} \in I(n, k, t)} |\nabla_m(\mathcal{A})|,$$

i.e. $M_0(n, m, k, t)$ is the maximum size of the $m$-shade of a $t$-intersecting family of $k$-subsets of $[n]$. We have that

$$M_0(n, m, k, t) \leq M(n, m, t),$$

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but this is not optimal. In [Hir08a], we argued that the following conjecture should be true.

**Conjecture 1.** \( M_0(n, m, k, t) = \max_{0 \leq i \leq \min(k-t, n-t)} |F_i(n, m, t)|. \)

We then deduced from conjecture [H] that whenever \( k \) and \( t \) are functions of \( m \) satisfying \( k(m) = o(m) \), i.e. \( \lim_{m \to \infty} k(m) / m = 0 \), \( \lim_{m \to \infty} k(m) = \infty \) and \( \lim_{m \to \infty} t(m) / \sqrt{k(m)} = \infty \),

\[
\lim_{m \to \infty} M_0(2m, m, k(m), t(m)) = 0.
\]

### 6.2 Cross-\( t \)-intersecting families

It turns out that for our application we need upper bounds on the size of shades of cross-\( t \)-intersecting families (cf. equation (38)). Let \( C(n, k, l, t) \) be the collection of all pairs \((A, B)\) of cross-\( t \)-intersecting families, where \( A \subseteq \binom{[n]}{k} \) and \( B \subseteq \binom{[n]}{l} \). Then the cross-\( t \)-intersecting function corresponding to \( M \) is defined by

\[
N(n, k, l, t) = \max_{(A, B) \in C(n, k, l, t)} |A| \cdot |B|.
\]

There are a number of results on cross-\( t \)-intersecting families in the literature; however, the state of knowledge seems very meager compared with \( t \)-intersecting families. The following theorem, proved in [MT89], is the strongest result of its kind that we were able to find.

**Theorem 3** (Matsumoto–Tokushige, 1989). \( N(n, k, l, 1) = \binom{n-1}{k-1} \binom{n-1}{l-1} \) whenever \( 2k, 2l \leq n \).

Generalizing \( M_0 \), we define the maximum size \( N_0(n, m_k, m_l, k, l, t) \) of the product of the \( m_k \)-shade with the \( m_l \)-shade of a pair of cross-\( t \)-intersecting families of \( k \)-subsets and \( l \)-subsets of \([n]\), respectively:

\[
N_0(n, m_k, m_l, k, l, t) = \max_{(A, B) \in C(n, k, l, t)} |\nabla_{-m_k}(A)| \cdot |\nabla_{-m_l}(B)|.
\]

For purposes of our dichotomy, we are exclusively interested in the numbers \( N_0(2m, m, k, k, t) \). Thus we define

\[
N_1(n, m, k, t) = N_0(n, m, m, k, k, t).
\]

In [Hir08a], we justify a conjecture for the value of \( N_0(n, m_k, m_l, k, l, t) \), corresponding to conjecture [H]. We then consider the asymptotic formula for cross intersecting pairs, corresponding to equation (45):

\[
\lim_{m \to \infty} \frac{\sqrt{N_1(2m, m_k, m_l, k, l, t)}}{\binom{2m}{m}} = 0.
\]
The main result of this paper is the following.

**Theorem 4.** Assume conjecture 2 is true. Let \( A \subseteq \mathcal{T}(f) \) be a weakly distributive analytic family of infinitely branching trees satisfying the ccc that has locally bounded branching decay for \( x < 1 \) and a \( T \in A \) such that \( S(n) \) is nonhomogeneous for \( c \) for all but finitely many \( n \), for all \( S \in A_T \).

**Proof.** Let \( A \) be as hypothesized. Find \( T' \in A \) such that
\[
(50) \quad h_{T'} \leq_N h_S \quad \text{for all } S \in A_{T'}
\]
(cf. equation (16)), where \( \mathcal{H} = \{ x^\alpha : 0 < \alpha < 1 \} \). By lemma 5 we may assume that there exists \( T \subseteq T' \in A \) with \( \mu_f([T]) = 0 \). Then setting \( k(n) := h_T(n) \), we have
\[
(51) \quad |S(n)| \leq k(n) \quad \text{for all } n, \text{ all } S \in A_T,
\]
\[
(52) \quad \lim_{n \to \infty} \frac{k(n)}{\prod_{i=0}^{n-1} f(i)} = 0.
\]
Furthermore, by lemma 12, proposition 13 and equation (50), \( h_T \leq_N h_S \) for all \( S \in A_T \), and thus by proposition 17, we have
\[
(53) \quad \liminf_{n \to \infty} \frac{|S(n)|}{k(n)^\alpha} = \infty \quad \text{for all } S \in A_T, \text{ for all } 0 < \alpha < 1.
\]
Finally, we notice that by increasing the function \( f \) if necessary, we may assume that \( f(0) \) is even.

Now we are ready to specify the parameters for the coideal. Define \( m \in \mathbb{N}^\mathbb{N} \) by
\[
(54) \quad m(n) = \frac{\prod_{i=0}^{n-1} f(i)}{2},
\]
which makes sense, taking the empty product as zero, since \( f(0) \) is even. The function \( F \) on \( \mathbb{N} \) is given by
\[
(55) \quad F(n) = \left( \frac{2m(n)}{|m(n)|} \right) \quad \text{for all } n,
\]
where the “\( 2m(n) \)” is identified with the set
\[
(56) \quad \prod_{i=0}^{n-1} \{ 0, \ldots, f(i) - 1 \}
\]
of cardinality \( 2m(n) \). Fix \( \frac{1}{2} < \beta < 1 \), and define \( g \in \mathbb{N}^\mathbb{N} \) by
\[
(57) \quad g(n) = \left\lfloor \sqrt[\alpha]{N_1(2m(n), m(n), k(n), k(n)^\beta)} \right\rfloor.
\]
Note that by \(52\), \(k(n) = o(m(n))\), \(\lim_{n \to \infty} k(n) = \infty\) since \(T\) is infinitely branching and \(\lim_{n \to \infty} k(n)^\beta/\sqrt{k(n)} = \lim_{n \to \infty} k(n)^\beta/\sqrt{k(n)} = \infty\). Therefore, conjecture \(2\) applies, giving \(\lim_{n \to \infty} \frac{g(n)}{|F(n)|} = \lim_{n \to \infty} \frac{\sqrt{N(2m(n), m(n), k(n), k(n)^\beta)}}{(2m(n))} = 0\). Thus proposition \(13\) applies.

By going to a forcing extension, if necessary, we assume that there exists a selective nonprincipal ultrafilter \(\mathcal{U}\) on \(\mathbb{N}\). Note that this existence follows from CH, or more generally if the covering number of the meager ideal is equal to the continuum. We are forcing with \(Q = Q(\mathcal{H}(\mathcal{U}, F, g))\), and we claim that \(Q\) forces that for all \(S \in A_T\),

\[
(58) \quad S(n) \text{ is nonhomogeneous for } \dot{c} \text{ for all but finitely many } n \in \dot{D}.
\]

This will complete the proof, because the existence of \(D\) and \(c\) as in the first alternative is a \(\Sigma_1^2\) statement, and thus Schoenfield’s Absoluteness Theorem applies to give these two in the ground model. Once we have a colouring that is nonhomogeneous on some fixed infinite set, this trivially yields a colouring for which each \(S \in A_T\) is nonhomogeneous for all but finitely many levels.

Suppose towards a contradiction that this is false. Then there exists a \(Q\)-name \(\dot{S}\) for a member of \(A_T\) and a condition \((a, A) \in Q\) forcing that

\[
(59) \quad \dot{S}(e_D(n)) \text{ is homogeneous for } \dot{c} \text{ for infinitely many } n.
\]

Before continuing, we give a very rough idea of how a contradiction comes about. The definition of \(\mathcal{H}(\mathcal{U}, F, g)\) ensures that, in the limit as \(n \to \mathcal{U}\), the \(A(n)\’s\) contain a large number of colourings of the tree of height \(n\) in equation \(50\). From this and \(59\) we argue (in claim \(6\)) that along some “path” \(p\), where \(p \upharpoonright n\) decides \(\dot{S}(n) = t\), the uncertainty in \(\dot{S}(n)\) is large, i.e., the set \(R(p \upharpoonright n)\) of all finite trees \(t\) such that some extension of \((a, A)\) forces \(\dot{S}(t) \neq t\) is large; more precisely, we argue that the \(m\)-shade of \(R(p \upharpoonright n)\) is large. This entails that along two different paths \(p \neq q\), the product of the cardinalities of the pair \((R(p \upharpoonright n), R(q \upharpoonright n))\) will be too large to be \(\ell(n)\)-intersecting for a suitably large \(t(n)\). This in turn will entail that the two interpretations of \(\dot{S}\) along the paths \(p\) and \(q\) will have an intersection whose branching is too low to be a member of \(A\). Hence, from uncountably many paths we obtain an uncountable antichain.

Note that \((b, B)\) decides \(e_D(|b| - 1) = \dot{b}\) and not \(e_D(|b|)\).

**Claim 4.** The set of all conditions \((b, B) \in Q\) forcing that \(\dot{S}(e_D(|b|))\) is homogeneous for \(\dot{c}\) is dense below \((a, A)\).

**Proof.** Fix \((a_1, A_1) \leq (a, A)\). By lemma \(21\) there is a \(B \subseteq A_1\) in \(\mathcal{H}\) such that \((a_1 \cup b, B - b)\) decides whether or not \(\dot{S}(e_D(|a_1 \cup b|))\) is homogeneous for \(\dot{c}\) for every finite partial function \(b \subseteq B\). And by \(59\), there exists \((b, C) \leq (a_1, B)\) and \(n \geq |a_1|\) such that \((b, C)\) forces \(\dot{S}(e_D(n))\) is homogeneous for \(\dot{c}\). By going to an extension of \((b, C)\), we can assume that \(|b| \geq n\). Then letting \(d \subseteq b\) be the initial segment of length \(n\), \((d, B - d)\) decides whether \(\dot{S}(e_D(|d|))\) is
homogeneous for $\check{c}$ because $d \setminus a_1 \subseteq B$ and $a_1 \subseteq d$ as $|a_1| \leq n$. However, as $(b, C) \leq (d, B - d)$, it must decide this positively. Since $(d, B - d) \leq (a_1, A_1)$, the proof is complete.

Let $M < H_{(2^{n_0})^+}$ be a countable elementary submodel containing $f$, $T$, $U$ and $\check{S}$. Let $\overline{M}$ be its transitive collapse, and let $\mathcal{D}_a (a \in \mathbb{N})$ enumerate all of the dense subsets of $Q^{\overline{M}} = Q \cap \overline{M}$ in $\overline{M}$. We construct a subtree $U$ of $(\mathcal{F}(F), \subseteq)$, conditions $(b_u, B_u)$ $(u \in U)$ in $Q^{\overline{M}}$, $R(u) \subseteq f^{\leq n_{\beta}} (u \in U)$ and $n_i \in \mathbb{N} (i \in \mathbb{N})$ by recursion on $|u|$ so that

(i) $(b_0, B_0) \leq (a, A)$,

(ii) $n_i \in \bigcap_{|u|=i} \text{dom}(B_u)$,

(iii) $B_u(n_i) > 2g(n_i)$ for all $|u| = i$,

(iv) $u \subseteq v$ implies $(b_u, B_u) \leq (b_v, B_v)$,

(v) $(b_u, B_u) \in \mathcal{D}_{|u|}$,

(vi) $(b_u, B_u)$ forces that $\check{S}(\check{e}_F([b_u]))$ is homogeneous for $\check{c}$,

(vii) $U(i + 1) = \{ u^\prec : u \in U(i), s = (n_i, x), x \in B_u(n_i) \}$,

(viii) $b_u^\prec s \subseteq b_{u^\prec s}$ for all $u^\prec s \in U(i + 1)$,

(ix) $(b_u^\prec s, B_u^\prec s) \models \check{S}(n_i) = R(u^\prec s)$ for all $|u| = i$.

It is possible to satisfy (i) and (iii) by the definition of $\mathcal{H}(U, F, g)$, using the fact that $U$ is a filter. Claim 4 is used for (vi).

**Claim 5.** $R(u^\prec (n_i, x))$ is homogeneous for $x$, for all $u^\prec (n_i, x) \in U(i + 1)$.

**Proof.** Set $s = (n_i, x)$. Since $(b_u^\prec s, B_u) \models \check{e}_F([b_u]) = n_i$, $(b_u^\prec s, B_u^\prec s) \models \check{S}(n_i)$ is homogeneous for $\check{c}$ by (vi) and (viii). And since $(b_u^\prec s, B_u) \models \check{c}(n_i) = x$, the claim follows from (ix).

**Claim 6.** For all $u \in U(i)$,

\[
|\nabla_{-m(n_i)} \{ R(v) : v \in \text{insucc}(u) \} | > \sqrt{N_1(2m(n_i), m(n_i), k(n_i), k(n_i)^3)}.
\]

**Proof.** By lemma 22, the number of colourings in $\binom{2m(n_i)}{m(n_i)}$ for which some $R(v)$ is homogeneous is at most

\[
2|\nabla_{-m(n_i)} \{ R(v) : v \in \text{insucc}(u) \} |.
\]

Suppose towards a contradiction that the value in (61) is smaller than $B_u(n_i)$. Then there exists $x \in B_u(n_i)$ for which no member of $\{ R(v) : v \in \text{insucc}(u) \}$ is homogeneous. Thus, putting $s = (n_i, x)$, $R(u^\prec s)$ is nonhomogeneous for $x$, contradicting claim 5. Therefore, by (iii), the value in (61) is as big as

\[
B_u(n_i) > 2\sqrt{N_1(2m(n_i), m(n_i), k(n_i), k(n_i)^3)}.
\]
Define a map $G$ on $\mathbb{N}$ where $G(i)$ is the set of all functions from $\{0, 1\}^i$ into

$$\prod_{u \in U(i)} \text{imsucc}(u)$$

for each $i$. Denote the space

$$\mathbb{R}(G) = \prod_{i=0}^{\infty} G(i)$$

and endow it with its product topology. Now let $c \in \mathbb{R}(G)$ be a Cohen real over the ground model, and set $d := \bigcup_{i=0}^{\infty} c(i)$; hence, the domain of $d$ is $\{0, 1\}^{<\mathbb{N}}$ and $d(s) = c(|s|)(s) \in \prod_{u \in U(|s|)} \text{imsucc}(u)$ for all $s \in \{0, 1\}^{<\mathbb{N}}$. In the Cohen extension, we define a mapping $\Phi : \{0, 1\}^{\mathbb{N}} \to T(f)$ as follows. For each $z \in \{0, 1\}^{\mathbb{N}}$, define by recursion on $i$, $b_0(z) = \emptyset$ and

$$b_{i+1}(z) = b_i(z) \ominus d(z \upharpoonright i)(b_i(z)) \in U(i+1).$$

Thus for every $z \in \{0, 1\}^{\mathbb{N}}$, $\bigcup_{i=0}^{\infty} b_i(z)$ is a branch through $U$. Then let

$$\Phi(z) = \bigcup_{i=0}^{\infty} R(b_i(z)).$$

Let $C$ be the poset directly giving the Cohen real $c$, i.e. each condition of $C$ is an element of $\prod_{i=0}^{n} G(i)$ for some $n \in \mathbb{N}$.

**Claim 7.** The range of $\Phi$ is a subset of $A_T$.

**Proof.** Fix $z \in \{0, 1\}^{\mathbb{N}}$, and set $x := \bigcup_{i=0}^{\infty} b_i(z)$. By 14, $\{b_i(z) : i \in \mathbb{N}\}$ is $Q_{\mathcal{M}}$-generic over $\mathcal{M}$. Thus, in $\mathcal{M}[\{b_i(z) : i \in \mathbb{N}\}] = \mathcal{M}[x]$, $\hat{S}[x] = \Phi(z)$ by 13, where $\hat{S}[x]$ denotes the generic interpretation of $\hat{S}$. Hence $\mathcal{M}[x] \models \Phi(z) \in A_T$. By analytic absoluteness between the transitive models $\mathcal{M}[x]$ and $\mathcal{V}[c]$, $\mathcal{V}[c] \models \Phi(z) \in A_T$.

**Claim 8.** Let $y$ and $z$ be distinct reals from the ground model. Then $\Phi(y)$ and $\Phi(z)$ are incompatible in $(A, \subseteq)$.

**Proof.** Fix $y \neq z$ in $\{0, 1\}^{\mathbb{N}}$ from the ground model. Suppose towards a contradiction that $\Phi(y)$ and $\Phi(z)$ are compatible. Then since there must be a member of $A_T$ contained in $\Phi(y) \cap \Phi(z)$, by equation 23, there some integer $j$ such that

$$\left| (\Phi(y) \cap \Phi(z))(n_i) \right| \geq k(n_i)^3 \quad \text{for all } i \geq j.$$

Let $r \in C$ be a condition forcing 67. By increasing $j$ if necessary, we assume that

$$y \upharpoonright j \neq z \upharpoonright j.$$
By extending $r$ if necessary, we assume that $|r| \geq j$. Set $i = |r|$. By claim \(6\)
\[
|\nabla_{m(n_i)} \{ R(u) : u \in \text{imsucc}(b_i(y)) \} | \cdot |\nabla_{m(n_i)} \{ R(v) : v \in \text{imsucc}(b_i(z)) \} |
\geq N_1(2m(n_i), m(n_i), k(n_i), k(n_i)\beta).
\]

Hence \(\{ R(u) : u \in \text{imsucc}(b_i(y)) \}, \{ R(v) : v \in \text{imsucc}(b_i(z)) \}\) cannot be a cross-$k(n_i)\beta$-intersecting pair. This means that there are $u \in \text{imsucc}(b_i(y))$ and $v \in \text{imsucc}(b_i(z))$ such that
\[
|R(u) \cap R(v)| < k(n_i)\beta.
\]

By \(68\), we can define an extension $r_0 = r^\triangleright g$ of $r$, where $g \in G(i)$, so that
\[
b_i(y)^\triangleright g(y \upharpoonright i)(b_i(y)) = u \quad \text{and} \quad b_i(z)^\triangleright g(z \upharpoonright i)(b_i(z)) = v.
\]

However, $r_0 \Vdash (\Phi(y) \cap \Phi(z))(n_i) = R(u) \cap R(v)$ which by \(70\) is in contradiction with \(67\).

Claims \(7\) and \(8\) entail that, in the Cohen extension, the image under $\Phi$ of the ground model reals is an antichain of $(\mathcal{A}, \subseteq)$ of size continuum. By absoluteness, this implies the existence of an uncountable antichain of $(\mathcal{A}, \subseteq)$ in the ground model, concluding the paper.

\[\Box\]  

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