On Simplicity of Semirings and Complete Semirings

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Abstract

In this paper, among other results, there are described (complete) simple – simultaneously ideal- and congruence-simple – endomorphism semirings of (complete) idempotent commutative monoids; it is shown that the concepts of simplicity, congruence-simplicity and ideal-simplicity for (complete) endomorphism semirings of projective semilattices (projective complete lattices) in the category of semilattices coincide if those semilattices are finite distributive lattices; there are described congruence-simple complete hemirings and left artinian congruence-simple complete hemirings. Considering the relationship between the concepts of ‘Morita equivalence’ and ‘simplicity’ in the semiring setting, we have obtained the following results: The ideal-simplicity, congruence-simplicity and simplicity of semirings are Morita invariant properties; A complete description of simple semirings containing the infinite element; The representation theorem – “Double Centralizer Property” – for simple semirings; A complete description of simple semirings containing a projective minimal one-sided ideal; A characterization of ideal-simple semirings having either infinite elements or a projective minimal one-sided ideal; A confirmation of Conjecture of \cite{18} and solving Problem 3.9 of \cite{17} in the classes of simple semirings containing either infinite elements or projective minimal left (right) ideals, showing, respectively, that semirings of those classes are not perfect and the concepts of ‘mono-flatness’ and ‘flatness’ for semimodules over semirings of those classes are the same. Finally, we give a complete description of ideal-simple, artinian additively idempotent chain semirings, as well as of congruence-simple, lattice-ordered semirings.

**Keywords:** ideal-simple semirings, congruence-simple semirings, simple semirings, distributive lattices, lattice-ordered semirings, complete semirings.

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1. Introduction

As is well known, structure theories for algebras of classes/varieties of algebras constitute an important “classical” area of the sustained interest in algebraic research. In such theories, so-called simple algebras, i.e., algebras possessing only two trivial congruences – the identity and universal ones – play a very important role of “building blocks”. In contrast to the varieties of groups and rings, the research on simple semirings has been just recently started, and therefore not much on the subject is known. Investigating semirings and their representations, one should undoubtedly use methods and techniques of both ring and lattice theory as well as diverse techniques and methods of categorical and universal algebra. Thus, the wide variety of the algebraic techniques involved in studying semirings, and their representations/semimodules, perhaps explains why the research on simple semirings is still behind of that for rings and groups (for some recent activity and results on this subject one may consult [26], [1], [27], [2], [14], [31], [13], and [20]).

At any rate, this paper concerns the ideal- and congruence-simpleness – in a semiring setting, these two notions of simpleness are not the same (see Examples 3.7 below) and should be differed – for some classes of semirings, as well as the ideal- and congruence-simpleness for complete semirings. The complete semirings originally appeared on the “arena” while considering both the generalization of classical formal languages to formal power series with coefficients in semirings and automata theory with multiplicities in semirings and, therefore, requiring the more general concepts of ‘infinite summability’ for semiring elements (see, for example, [5], [30], [10], [15], and [9]).

The paper is organized as follows. In Section 2, for the reader’s convenience, there are included all subsequently necessary notions and facts on semirings and semimodules. Section 3, among other results, contains two main results of the paper – Theorem 3.3, describing simple (i.e., simultaneously ideal- and congruence-simple) endomorphism semirings of idempotent commutative monoids, and its “complete” analog – Theorem 3.6. We also show (Corollaries 3.8 and 3.9) that the concepts of simpleness, congruence-simpleness and ideal-simpleness for endomorphism semirings (complete endomorphism semirings) of projective semilattices (projective complete lattices) in the category of semilattices with zero coincide iff those semilattices are finite distributive lattices. Theorem 4.6 and Corollary 4.7 of Section 4, describing all congruence-simple complete hemirings and left artinian congruence-simple complete hemirings, respectively, are among the main results of the paper, too.

In Section 5, considering the relationship between the concepts of ‘Morita equivalence’ and ‘simpleness’ in the semiring setting, there have been established the following main results of the paper: The ideal-simpleness, congruence-simpleness and simpleness of semirings are Morita invariant properties (The-
orem 5.6); A complete description of simple semirings containing the infinite element (Theorem 5.7); The representation theorem – “Double Centralizer Property” – for simple semirings (Theorem 5.10); A complete description of simple semirings containing a projective minimal one-sided ideal (Theorem 5.11); A characterization of ideal-simple semirings having either infinite elements or a projective minimal one-sided ideal (Theorem 5.12 and Corollary 5.13); A confirmation of Conjecture of [18] and solving Problem 3.9 of [17] in the classes of simple semirings containing either infinite elements or projective minimal left (right) ideals, showing, respectively, that semirings of those classes are not perfect (Theorem 5.15 and Corollary 5.16) and the concepts of ‘mono-flatness’ and ‘flatness’ for semimodules over semirings of those classes coincide (Theorem 5.17).

Section 6 contains two more central results of the paper: A complete description of ideal-simple, artinian additively idempotent chain semirings (Theorem 6.4 and Remark 6.6); And a complete description of congruence-simple, lattice-ordered semirings (Theorem 6.7 and Corollary 6.8).

Finally, in the course of the paper, there have been stated several, in our view interesting and promising, problems; also, all notions and facts of categorical algebra, used here without any comments, can be found in [25], and for notions and facts from semiring theory, universal algebra and lattice theory, we refer to [6], [8] and [3], respectively.

2. Preliminaries

Recall [6] that a hemiring is an algebra \((R, +, \cdot, 0)\) such that the following conditions are satisfied:

(1) \((R, +, 0)\) is a commutative monoid with identity element 0;
(2) \((R, \cdot)\) is a semigroup;
(3) Multiplication distributes over addition from either side;
(4) \(0r = 0 = r0\) for all \(r \in R\).

Then, a hemiring \((R, +, \cdot, 0)\) is called a semiring if its multiplicative reduct \((R, \cdot)\) is actually a monoid \((R, \cdot, 1)\) with the identity 1. A proper hemiring is a hemiring that is not a ring.

As usual, a left \(R\)-semimodule over a hemiring \(R\) is a commutative monoid \((M, +, 0_M)\) together with a scalar multiplication \((r, m) \mapsto rm\) from \(R \times M\) to \(M\) which satisfies the following identities for all \(r, r' \in R\) and \(m, m' \in M\):

(1) \((rr')m = r(r'm)\);
(2) \(r(m + m') = rm + rm';\)
(3) \((r + r')m = rm + r'm;\)
(4) \(r0_M = 0_M = 0m\).

Right semimodules over a hemiring \(R\) and homomorphisms between semimodules are defined in the standard manner. Considering semimodules over a semiring \(R\), we always presume that they are unital, i.e., \(1m = m\) for all
m \in M$. And, from now on, let $\mathcal{M}_R$ and $\mathcal{R}M$ denote the categories of right and left semimodules, respectively, over a hemiring $R$. As usual (see, for example, [6, Chapter 17]), in the category $\mathcal{R}M$, a free (left) semimodule $\sum_{i \in I} R_i$, $R_i \cong rR$, $i \in I$, with a basis set $I$ is a direct sum (a coproduct) of $|I|$ copies of the regular semimodule $rR$. And a projective left semimodule in $\mathcal{R}M$ is just a retract of a free semimodule. A semimodule $\mathcal{R}M$ is finitely generated if and only if it is a homomorphic image of a free semimodule with a finite basis set.

We need to recall natural extensions of the well known for rings and modules notions to a context of semirings and semimodules. Thus, the notion of a superfluous (or small) subsemimodule: A subsemimodule $S \subseteq M$ is superfluous (written $S \subseteq_s M$) if $S + N = M \Rightarrow N = M$, for any subsemimodule $N \subseteq M$. Then, taking into consideration Proposition 5 and Definition 4 of [28], for any semimodule $R \in |\mathcal{R}M|$, we have the subsemimodule $\text{Rad}(A)$ of $R$, the radical of $R$, defined by $\text{Rad}(A) := \{S \mid S \subseteq_s A \} = \bigcap \{M \mid M$ is a maximal subsemimodule of $A\}$. One can also extend the notions (as well as results involving them) of Descending Chain Condition and artinian module of the theory of modules over rings to a context of semimodules over semirings in an obvious fashion (see, e.g., [19]).

Following [6] (see also [6]), a left semimodule $M$ over a hemiring $R$ is called complete if and only if for every index set $\Omega$ and for every family $\{m_i \mid i \in \Omega\}$ of elements of $M$ we can define an element $\sum_{i \in \Omega} m_i$ of $M$ such that the following conditions are satisfied:

1. $\sum_{i \in \emptyset} m_i = 0_M$,
2. $\sum_{i \in \{1\}} m_i = m_1$,
3. $\sum_{i \in \{1, 2\}} m_i = m_1 + m_2$,
4. If $\Omega = \bigcup_{j \in \Lambda} \Omega_j$ is a partition of $\Omega$ into the disjoint union of nonempty subsets then $\sum_{i \in \Omega} m_i = \sum_{j \in \Lambda} \left( \sum_{i \in \Omega_j} m_i \right)$,
5. If $r \in R$ and $\{m_i \mid i \in \Omega\} \subseteq M$, then $r(\sum_{i \in \Omega} m_i) = \sum_{i \in \Omega} rm_i$.

As an immediate consequence of this definition, one gets that if $\pi$ is a permutation of $\Omega$ then

$$\sum_{i \in \Omega} m_i = \sum_{j \in \Omega} \left( \sum_{i \in \pi(j)} m_i \right) = \sum_{j \in \Omega} m_{\pi(j)}.$$

Complete right $R$-semimodules are defined analogously. For a complete left (right) $R$-semimodule $M$, we always have $\sum_{i \in \Omega} 0_M = \sum_{i \in \Omega} 0_R 0_M = 0_R (\sum_{i \in \Omega} 0_M) = 0_M$.

A hemiring is complete if and only if it is complete as a left and right semimodule over itself. The Boolean semifield $B = \{0, 1\}$ – an idempotent two element semiring – is a complete semiring if we define $\sum_{i \in \Omega} r_i = 0$ iff $r_i = 0$ for all $i \in \Omega$, and to be 1 otherwise.

If $M$ and $N$ are complete left $R$-semimodules then a $R$-homomorphism $\alpha : M \rightarrow N$ is complete if and only if it satisfies the condition that
\[ \alpha(\sum_{i \in \Omega} m_i) = \sum_{i \in \Omega} \alpha(m_i) \] for every index set \( \Omega \). Complete homomorphism of complete hemirings (semirings) are defined analogously. We will denote the set of all complete \( R \)-homomorphisms from \( M \) to \( N \) by \( \text{CHom}_R(M, N) \). Similarly, we denote the set of all complete \( R \)-endomorphisms of a complete left \( R \)-semimodule \( M \) by \( \text{CEnd}_R(M) \). Then \( \text{CEnd}_R(M) \) can be made a complete semiring with the infinite summation to be defined by \( (\sum_{i \in \Omega} \alpha)(m) = \sum_{i \in \Omega} \alpha_i(m) \) for every set \( \{\alpha_i | i \in \Omega\} \subseteq \text{CEnd}_R(M) \) and \( m \in M \).

Following [1], a hemiring \( R \) is congruence-simple if the diagonal, \( \Delta_R \), and universal, \( R^2 \), congruences are the only congruences on \( R \); and \( R \) is ideal-simple if \( 0 \) and \( R \) are its only ideals. A hemiring \( R \) is said to be simple if it is simultaneously congruence- and ideal-simple.

Any hemiring \( R \) clearly can be considered as a subsemiring of the endomorphism semiring \( E_R \) of its additive reduct \( (R,+,0) \), and, by [31, Proposition 3.1], the congruence-simpleness of a proper hemiring \( R \) implies that the reduct \( (R,+,0) \) is an idempotent monoid, i.e., \( (R,+,0) \) is, in fact, an upper semilattice. In light of these observations, it is reasonable (and we will do that in the next section) first to consider the concepts of simpleness introduced above for the endomorphism semiring \( E_M \) of a semilattice – an idempotent commutative monoid – \( (M,+,0) \). Recall (see [31, Definition 1.6]) that a subhemiring \( S \subseteq E_M \) of the endomorphism semiring \( E_M \) is dense if it contains for all \( a,b \in M \) the endomorphisms \( e_{a,b} \in E_M \) defined by

\[
e_{a,b}(x) := \begin{cases} 0 & \text{if } x + a = a, \\ b & \text{otherwise,} \end{cases} \quad \text{for any } x \in M .
\]

We will need the following results.

**Lemma 2.1** ([31, Lemma 2.2]). For any \( a,b,c,d \in M \) and \( f \in E_M \) we have \( f \circ e_{a,b} = e_{a,f(b)} \) and

\[
e_{c,d} \circ f \circ e_{a,b} := \begin{cases} 0 & \text{if } f(b) \leq c, \\ e_{a,d} & \text{otherwise.} \end{cases}
\]

**Theorem 2.2** ([31, Theorem 1.7]). A proper finite hemiring \( R \) is congruence-simple iff \( |R| \leq 2 \), or \( R \) is isomorphic to a dense subhemiring \( S \subseteq E_M \) of the endomorphism semiring \( E_M \) of a finite semilattice \( (M,+,0) \).

### 3. Simpleness of Endomorphism Hemirings

Now let \( E_M \) be the endomorphism hemiring of an idempotent commutative monoid (a semilattice with zero) \( (M,+,0) \), \( F_M := \{e_{a,b} | a,b \in M\} \subseteq E_M \) the submonoid of \( (E_M,+,0) \) generated by all endomorphisms \( e_{a,b} \), \( a,b \in M \), and \( F_{rM} := \{f \in E_M | |f(M)| < \infty\} \subseteq E_M \) the submonoid of \( (E_M,+) \) consisting of all endomorphisms of the semilattice \( M \) having finite ranges. It is obvious that \( F_{rM} \) is an ideal of \( E_M \), and \( F_M \subseteq F_{rM} \); moreover, the following observations are true.
Lemma 3.1. Let $(M,+,0)$ be an idempotent commutative monoid such that $|\{x \in M \mid f(x) \leq a\}| < \infty$ for any $a \in M$ and $f \in E_M$. Then, $F_M$ is an ideal of $E_M$. In particular, if $M$ is finite, $F_M$ is an ideal of $E_M$.

Proof. By Lemma 2.1, it is enough to show that $F_M$ is a right ideal of $E_M$. Indeed, for any $a,b,x \in M$ and $f \in E_M$, $e_{a,b}f(x) := \begin{cases} 0 \text{ if } f(x) \leq a, \\ b \text{ otherwise} \end{cases}$.

Then, considering the set $\{x_1,x_2,\ldots,x_n\} := \{x \in M \mid f(x) \leq a\}$ and the element $c := x_1 + x_2 + \ldots + x_n$, one can easily see that $e_{a,b}f = e_{c,b}$. □

Lemma 3.2. For a subsemiring $S \subseteq E_M$ of the endomorphism semiring $E_M$ of an idempotent commutative monoid $(M,+,0)$, there holds:

(i) If $S$ is ideal-simple and $S \cap Fr_M \neq 0$, then $M$ is finite;
(ii) If $S$ is ideal-simple and $S \cap F_M \neq 0$, then $M$ is a finite distributive lattice.

Proof. (i). Consider the ideal $\langle f \rangle$ of $S$ generated by a nonzero endomorphism $f \in S \cap Fr_M$. Since $S$ is ideal-simple and $1_{E_M} = id_M \in S \subseteq E_M$, there are some endomorphisms $g_1,h_1,g_2,h_2,\ldots,g_n,h_n \in S$ such that $id_M = g_1fh_1 + g_2fh_2 + \ldots + g_nf h_n$. Whence and since $f \in Fr_M$, one has $id_M \in Fr_M$ and $|M| < \infty$.

(ii). As $F_M \subseteq Fr_M$, from (i) follows that $|M| < \infty$. Again, using the ideal-simpleness of $S$, one has that the ideal $\langle f \rangle$ of $S$ generated by a nonzero endomorphism $f \in S \cap F_M$ contains $id_M$, and therefore, by Lemma 3.1, $F_M = E_M$. Then, using [31, Proposition 4.9 and Remark 4.10], one concludes that the semilattice $M$ is, in fact, a finite distributive lattice. □

The next result describes the simple endomorphism semirings of idempotent commutative monoids.

Theorem 3.3. The following conditions for the endomorphism semiring $E_M$ of an idempotent commutative monoid $(M,+,0)$ are equivalent:

(i) $E_M$ is simple;
(ii) $E_M$ is ideal-simple;
(iii) The semilattice $M$ is a finite distributive lattice.

Proof. (i) $\implies$ (ii). It is obvious.

(ii) $\implies$ (iii) follows from Lemma 3.2.

(iii) $\implies$ (i). For $M$ is a finite distributive lattice, by [31, Proposition 4.9 and Remark 4.10], $F_M = E_M$, and therefore, by Theorem 2.2, $E_M$ is a congruence-simple semiring. So, we need only to show that $E_M$ is also ideal-simple. Indeed, let $I \subseteq E_M$ be a nonzero ideal of $E_M$, $0 \neq f \in I$, and $f(m) \neq 0$ for some $m \in M$. Then, by Lemma 2.1, $e_{a,b} = e_{0,b}f e_{a,m} \in I$ for any $a,b \in M$, and hence, $E_M = F_M \subseteq I$, i.e., $E_M = I$. □
Since the simpleness of additively idempotent semirings is one of our central interests in this paper, the following facts about additively idempotent semirings should be mentioned: By [7, Proposition 3.4] (or [6, Proposition 23.5]), an additively idempotent semiring \( R \) can be embedded in a finitary complete semiring \( S \) which additive reduct \((S, +, 0)\) is a complete semimodule over the Boolean semifield \( B = \{0, 1\} \). i.e., with the semilattice \((S, +, 0)\) being a complete lattice (see, for example, [6, Proposition 23.2]); And there exists the natural complete semiring injection from \( S \) to the complete endomorphism semiring \( \text{CEnd}_B(S) \). Thus, it is clear that an additively idempotent semiring \( R \) can be considered as a subsemiring of a complete endomorphism semiring of a complete lattice; and therefore, it is quite natural that our next results concern the simpleness of the complete endomorphism semiring \( \text{CEnd}_B(M) \) of a complete lattice \( M \).

So, let \( M \) be a complete lattice, which by [6, Proposition 23.2], for example, can be treated as a complete semimodule over the Boolean semifield \( B = \{0, 1\} \). The following observations are proved to be useful.

**Lemma 3.4.** Let \( M \) be a complete lattice. Then the following statements hold:

(i) \( e_{a, b} \in \text{CEnd}_B(M) \) for all \( a, b \in M \);
(ii) for any \( a, b \in M \) and \( f \in \text{CEnd}_B(M) \), there exists an element \( c \in M \) such that \( e_{a, b} f = e_{c, b} \).

**Proof.** (i). Let \( a, b \in M \) and \( \{m_i \mid i \in \Omega\} \) be a family of elements of \( M \). It is easy to see that

\[
\bigvee_{i \in \Omega} m_i \leq a \iff \forall i \in \Omega : m_i \leq a .
\]

Then, \( e_{a, b}(\bigvee_{i \in \Omega} m_i) = \bigvee_{i \in \Omega} e_{a, b}(m_i) \) and, hence, \( e_{a, b} \in \text{CEnd}_B(M) \).

(ii). For any \( a, b \in M \) and \( f \in \text{CEnd}_B(M) \), we have

\[
e_{a, b} f(x) := \begin{cases} 0 & \text{if } f(x) \leq a , \\ b & \text{otherwise} . \end{cases}
\]

Let \( X := \{x \in M \mid f(x) \leq a \} \) and \( c = \bigvee_{x \in X} x \). Then, \( f(c) = f(\bigvee_{x \in X} x) = \bigvee_{x \in X} f(x) \leq a \); hence, \( c \in X \) and \( x \leq c \iff f(x) \leq a \), and it is clear that \( e_{a, b} f = e_{c, b} \).

**Corollary 3.5.** For any complete lattice \( M \), the following statements hold:

(i) \( \mathbf{F}_M \) is an ideal in \( \text{CEnd}_B(M) \);
(ii) Any dense subhemiring of \( \text{CEnd}_B(M) \) is congruence-simple. In particular, \( \text{CEnd}_B(M) \) is a congruence-simple semiring.

**Proof.** (i). Since \( \text{CEnd}_B(M) \subseteq \mathbf{E}_M \), the statement immediately follows from Lemmas 2.1 and 3.4 (ii).

(ii). For any dense subhemiring of \( \text{CEnd}_B(M) \) is also a dense subhemiring of \( \mathbf{E}_M \) and the complete lattice \( M \) obviously contains the join-absorbing element \( \infty := \bigvee_{x \in M} x \), the statement right away follows from [31, Proposition 2.3].
The following result is a “complete” analog of Theorem 3.3 and describes the simple complete endomorphism semirings of complete lattices.

**Theorem 3.6.** For any complete lattice $M$, the following are equivalent:

(i) $\text{CEnd}_B(M)$ is simple;
(ii) $\text{CEnd}_B(M)$ is ideal-simple;
(iii) $M$ is a finite distributive lattice.

**Proof.** (i) $\implies$ (ii). It is obvious.

(ii) $\implies$ (iii). For $\text{CEnd}_B(M)$ is a subsemiring of $E_M$, by Corollary 3.5 (i) (or Lemma 3.4 (i)) $F_M \subseteq \text{CEnd}_B(M)$, and hence, by Lemma 3.2 (ii) $M$ is a finite distributive lattice.

(iii) $\implies$ (i). If $M$ is a finite nonzero distributive lattice, by Theorem 3.3 $E_M$ is a simple semiring. Since $0 \neq e_{0,m} \in F_M$ for all nonzero $m \in M$, by Lemma 3.1 $F_M$ is a nonzero ideal of $E_M$ and, hence, $E_M = F_M$. Then, using Corollary 3.5 (i) and $\text{CEnd}_B(M) \subseteq E_M = F_M \subseteq \text{CEnd}_B(M)$ one obtains that $\text{CEnd}_B(M) = E_M$ is simple. □

**Examples 3.7.** Congruence-simpleness of semirings does not imply their ideal-simpleness. Indeed, the following examples of infinite and finite semirings illustrate this situation:

a) Let $P(X)$ be the distributive complete lattice of all subsets of an infinite set $X$, and $\text{CEnd}_B(P(X))$ the complete endomorphism semiring of the complete lattice $(P(X), \cup)$. Then, by Theorem 3.6 the infinite semiring $\text{CEnd}_B(P(X))$ is not ideal-simple, however, by Corollary 3.5 (2) $\text{CEnd}_B(P(X))$ is congruence-simple.

b) Let $E_M$ be the endomorphism semiring of a finite semilattice $(M, +, 0)$ that is not a distributive lattice. Then, by Theorems 2.2 and 3.3, the finite semiring $E_M$ is an example of a congruence-simple, but not ideal-simple, finite semiring.

c) Obviously, the semifield $\mathbb{R}^+$ of all nonnegative real numbers is an example of an ideal-simple, but not congruence-simple (all positive real numbers constitute a congruence class of the nontrivial congruence), semiring.

One can clearly observe that the category of semilattices with zero coincides with the category $B_M$ of semimodules over the Boolean semifield $B$. Then, in light of this observation and Examples 3.7, the following result, describing projective $B$-semimodules with simple endomorphism semirings, is of interest.

**Corollary 3.8.** For a projective $B$-semimodule $M \in \mathcal{B}_M$, the following conditions are equivalent:

(i) $E_M$ is ideal-simple;
(ii) $E_M$ is simple;
(iii) $E_M$ is congruence-simple;
(iv) the semilattice $M$ is a finite distributive lattice.
Proof. First, as was shown in [11, Theorem 5.3], a semilattice \( M \in |B_M| \) is projective if and only if \( M \) is a distributive lattice such that \(|\{x \in M \mid x \leq a\}| < \infty \) for any \( a \in M \). Using this fact and Theorem 3.3, we need only to demonstrate the implication (iii) \( \implies \) (iv).

Indeed, consider the congruence \( \tau \) on \( E_M \) defined for \( f, g \in E_M \) by

\[
f \tau g \iff \exists a \in M \forall x \in M : f(x) + a = g(x) + a.
\]

If \( |M| \geq 2 \), one easily sees that there exists \( m \in M \) such that \( 0 \neq e_0, m \in E_M \) and \( (e_0, m) \in \tau \). From the latter and the congruence-simplesness of the semiring \( E_M \), one has that \( \tau = E_M \times E_M \), and hence, \( \text{id}_M \tau \neq 0 \), i.e., there exists an element \( a \in M \) such that \( \text{id}_M(x) + a = a \) for any \( x \in M \), i.e., \( x \leq a \) for any \( x \in M \). For \( M \) is a projective \( B \)-semimodule, from the fact mentioned above one gets that \( |M| < \infty \).

Taking into consideration Lemma 3.4 (i) and repeating the proof of Corollary 3.8 verbatim, we immediately get the following “complete” analog of the latter.

Corollary 3.9. For a complete, projective \( B \)-semimodule \( M \in |B_M| \), the following conditions are equivalent:

(i) \( \text{CEnd}_B(M) \) is ideal-simple;
(ii) \( \text{CEnd}_B(M) \) is simple;
(iii) \( \text{CEnd}_B(M) \) is congruence-simple;
(iv) \( M \) is a finite distributive lattice.

In light of Corollaries 3.8 and 3.9, we conclude this section with the following interesting open

Problem 1. Describe all \( B \)-semimodules \( M \in |B_M| \) for which the conditions (i), (ii), and (iii) of Corollaries 3.8 and 3.9 are equivalent.

4. Congruence-simplesness of complete hemirings

Considering a complete left \( R \)-semimodule \( (M, +, \Sigma) \) over a complete hemiring \( (R, +, \cdot, 0, \Sigma) \) with \( \Sigma \) symbolizing the ’summation’ in complete semimodules and hemirings, it is natural to call \( M \) a \( dR \)-semimodule if \( (\sum_{i \in \Omega} r_i)m = \sum_{i \in \Omega} r_i m \) for all \( m \in M \) and every family \( \{r_i \mid i \in \Omega\} \subseteq R \) of elements of \( R \).

Also, a left \( dR \)-semimodule \( M \) is called s-simple if \( RM \neq 0 \) and 0 is the only proper subsemimodule of \( M \); a subsemimodule \( N \) of a complete left \( R \)-semimodule \( M \) is complete if \( \sum_{i \in \Omega} m_i \in N \) for every family \( \{m_i \mid i \in \Omega\} \) of elements of \( N \). The following observation is almost obvious.

Lemma 4.1. For a left \( dR \)-semimodule \( M \) over a complete hemiring \( R \), the following statements are equivalent:

(i) \( M \) is s-simple;
(ii) \( M \neq 0 \), and \( M = Rm \) for any nonzero \( m \in M \);
(iii) \( R M \neq 0 \), and 0 is the only proper complete subsemimodule of \( M \).

Proof. (i) \( \implies \) (ii). It is clear that \( M \neq 0 \). Consider the subsemimodule \( N := \{ m \in M \mid Rm = 0 \} \) of \( M \). It is obvious that \( N \neq M \), and hence, because \( M \) is s-simple, we have \( N = 0 \), i.e., \( Rm \neq 0 \) for any nonzero \( m \in M \); again because \( M \) is s-simple, it follows that \( M = Rm \) for any nonzero \( m \in M \).

(ii) \( \implies \) (iii). Taking into consideration that \( M \) is a \( dR \)-semimodule, this implication can be established in the same fashion as the previous one.

The implications (ii) \( \implies \) (i) and (ii) \( \implies \) (iii) are obvious. \( \Box \)

A congruence \( \rho \) on a complete left \( R \)-semimodule \( M \) is called complete iff for any two families \( \{ m_i \mid i \in \Omega \} \) and \( \{ m'_i \mid i \in \Omega \} \) of elements of \( M \), the following implication

\[ \forall i \in \Omega : m_i \rho m'_i \implies (\sum_{i \in \Omega} m_i) \rho (\sum_{i \in \Omega} m'_i) \]

is true. For any complete left \( R \)-semimodule \( M \), the diagonal, \( \Delta_M \), and universal, \( M^2 \), congruences are obviously complete congruences on \( M \); and a complete left \( R \)-semimodule \( M \) is called complete congruence-simple (cc-simple) iff \( \Delta_M \) and \( M^2 \) are the only complete congruences on \( M \). Of course, the right-sided analog of the cc-simpleness is defined similarly. A left (right) \( dR \)-semimodule \( M \) is called simple iff it is both s-simple and congruence-simple, i.e., \( \Delta_M \) and \( M^2 \) are the only congruences on the left (right) \( R \)-semimodule \( M \) (and in this case, \( M \) is obviously cc-simple, too). The following fact, illustrating natural simple complete semimodules, deserves mentioning.

**Proposition 4.2.** Let \( M \) be a nonzero complete \( B \)-semimodule. Then, \( M \) is a simple complete left (right) \( R \)-semimodule over any dense complete subhemiring \( R \) of the endomorphism semiring \( E_M \). In particular, any complete lattice \( M \) is a simple complete left (right) \( \text{CEnd}_B(M) \)-semimodule.

**Proof.** For a dense complete subhemiring \( R \) of the dense, by Lemma 3.4, subsemiring \( \text{CEnd}_B(M) \) of the endomorphism semiring \( E_M \), the scalar multiplication from \( R \times M \) to \( M \) is defined by \( fm := f(m) \) for any \( f \in R \) and \( m \in M \). Then, it easy to see that \( M \) is a complete left \( R \)-semimodule and, since \( m' = e_0, m'(m) = e_0, m' m \in Rm \) for any nonzero \( m \in M \) and \( m' \in M \), we have that \( Rm = M \). Hence, by Lemma 4.1, \( M \) is an s-simple complete left \( dR \)-semimodule.

Now suppose that \( \rho \neq \Delta_M \) is a congruence on the left \( R \)-semimodule \( M \) and there exist elements \( x, y \in M \) such that \( x \neq y \) and \( (x, y) \in \rho \); moreover, since \( \rho \) is a congruence, without loss of generality we may even assume that \( y < x \). Then, \( (x, 0) = (e_{y,x}(x), e_{y,x}(y)) \in \rho \) and \( (m, 0) = (e_{0,m}(x), e_{0,m}(0)) = (e_{0,m} x, e_{0,m} 0) \in \rho \) for any \( m \in M \), and therefore, \( \rho = M^2 \). \( \Box \)

For any element \( x \in R \) of a complete hemiring \( R \) and any cardinal number \( n \), there exists the “additive \( n \)-power” \( nx \) of \( x \) defined by \( nx := \sum_{i \in I} x \) with \( I \) to be a set of the cardinality \( n \), i.e., \( |I| = n \). From the definition of the summation \( \Sigma \) right away follow the following facts:
(1) \((n_1n_2)x = n_1(n_2x)\) and \((n_1 + n_2)x = n_1x + n_2x\) for any cardinal numbers \(n_1\) and \(n_2\) and \(x \in R\);
(2) \(n(x + y) = nx + ny\) and \(n(xy) = (nx)y = x(ny)\) for any cardinal number \(n\) and \(x, y \in R\).

A complete left (right) congruence on a complete hemiring \(R\) is a complete congruence on the complete left (right) regular \(R\)-semimodule \(R\); a complete congruence on a complete hemiring \(R\) is a congruence that is simultaneously a complete left and right congruence on \(R\); a complete hemiring \(R\) is called complete congruence-simple (cc-simple) if \(\triangle_R\) and \(R^2\) are the only complete congruences on \(R\).

**Proposition 4.3.** For a cc-simple hemiring \(R\), the following statements hold:

(i) \(nx = x\) for any cardinal number \(n\) and \(x \in R\);
(ii) \(R\) is a complete lattice with respect to \(\bigvee_{i \in I} m_i := \sum_{i \in I} m_i\) for any family \(\{m_i \mid i \in I\}\) of elements of \(R\);
(iii) \(\inf R \neq 0\) and \(\infty := \sum_{x \in R} x\), then \(\infty^2 = \infty\).

**Proof.** (i). For \(a, b \in R\), we write \(a \preceq b\) iff there exist a cardinal number \(n\) and \(x \in R\) such that \(nb = x + a\); and we shall show that the relation \(\sim\) on \(R\), defined by
\[a \sim b \iff a \preceq b \text{ and } b \preceq a,\]
is a complete congruence on \(R\). Indeed, \(a \preceq a\) because \(1a = a = 0 + a\); if \(a \preceq b\) and \(b \preceq c\), then for some cardinal numbers \(m\) and \(n\) and \(x, y \in R\), we have \(nb = x + a\) and \(mc = y + b\), and, hence, \((nm)c = n(mc) = n(y + b) = ny + nb = ny + x + a\), and therefore, \(a \preceq c\). From this one may easily see that the relation \(\sim\) is an equivalence relation on \(R\).

To show the completeness of the relation \(\sim\), consider two families \((a_i)_{i \in I}\) and \((b_i)_{i \in I}\) of elements of \(R\) such that \(a_i \preceq b_i\) for each \(i \in I\). Then, there are cardinal numbers \(n_i\) and elements \(x_i \in R\), \(i \in I\), such that \(n_ib_i = x_i + a_i\). Appealing to the obvious natural sense of the cardinal arithmetic and assuming \(n := \sum_{i \in I} n_i\), one has \(nb_i = (n - n_i)b_i + n_i b_i = (n - n_i)b_i + x_i + a_i = x'_i + a_i\), where \(x'_i := (n - n_i)b_i + x_i\), \(i \in I\), and therefore, \(n\left(\sum_{i \in I} b_i\right) = \sum_{i \in I} \left(nb_i\right) = \sum_{i \in I} \left(x'_i + a_i\right) = \sum_{i \in I} x'_i + \sum_{i \in I} a_i\). The latter implies \(\sum_{i \in I} a_i \leq \sum_{i \in I} b_i\) and, hence, the completeness of \(\sim\).

To see that \(\sim\) is a congruence on \(R\), suppose that \(a \preceq b\) and \(c \preceq d\). Then there are cardinal numbers \(m\) and \(n\) and elements \(x, y \in R\) such that \(nb = x + a\) and \(md = y + c\), and therefore, \((mn)(bd) = m(nb)d = m(x + a)d = (x + a)md = (x + a)(y + c) = xy + x + ay + ac\). For the latter implies \(ac \preceq bd\), the relation \(\sim\) is a congruence on \(R\).

Thus \(\sim = \triangle_R\), or \(\sim = R^2\). In the second case, \(a \sim 0\) for every \(a \in R\) and, therefore, there exists a cardinal number \(n\) and an element \(x \in R\) such that \(0 = n0 = x + a\). From this, taking into consideration the hemiring variation of [3, Proposition 22.28] by which the complete hemiring \(R\) is zerosumfree, one gets \(R = 0\).
If \( \sim = \Delta_R \), for any \( a \in R \) and nonzero cardinal number \( n \) consider \( b := na \). Since \( na = 0 + b \) and \( 1b = b = na = (n - 1)a + a \), one gets that \( a \sim b \), and hence, \( a = b = na \).

(ii). By (i), \( R \) is additively idempotent, and we shall show that the order relation \( x \leq y \) if and only if \( x + y = y \) turns \( R \) into a complete lattice. Let \( (x_i)_{i \in I} \) be a family of elements in \( R \) and \( z := \sum_{i \in I} x_i \). If \( I = \emptyset \), then \( z \) is obviously the neutral and, hence, it is the supremum of the elements \( x_i, i \in I \), so assume that \( I \neq \emptyset \). First, note that for all \( j \in I \), we have \( x_j \leq z \) because \( x_j + \sum_{i \in I \setminus \{j\}} x_i = \sum_{i \in I} x_i \). Now, suppose that \( y \in S \) and satisfies \( x_j \leq y \) for all \( j \in I \), then, for \( \sum_{i \in I} x_i + y = \sum_{i \in I} x_i + \sum_{i \in I} y = \sum_{i \in I} (x_i + y) = \sum_{i \in I} y = y \), one gets that \( z \leq y \), and hence, \( z \) is the supremum again, and, therefore, \( R \) is a complete lattice.

(iii). Let \( a := \infty^2 \) and \( A := (a) := \{ x \in R \mid x \leq a \} \). First, note that if \( x, y \in R \), then \( xy \leq x \infty \leq \infty \infty = a \), so that \( xy \in A \); and, hence, applying (ii), one has that \( A \) is a complete subsemimodule of the complete regular left \( R \)-semimodule \( R \). Also, if \( x + y \in A \) and \( y \in A \), then \( x \leq x + y \leq x + a \leq a \), and \( x \in A \), too, i.e., \( A \) is a subtractive ideal of \( R \).

Next, consider the Bourne congruence (see, for example, [6, p. 78]) on \( R \) defined by the ideal \( A : x \equiv y \) iff there exist elements \( a, b \in A \) such that \( x + a = y + b \). For \( A \) is a complete subsemimodule of the complete left \( R \)-semimodule \( R \), the congruence \( \equiv_A \) is complete. Thus \( \equiv_A = \Delta_R \), or \( \equiv_A = R^2 \).

Let \( R \) be an additively idempotent complete hemiring, and \( R^* = \text{CHom}_R(R, B) \). Then \( R^* \) is a left \( dR \)-semimodule with the scalar multiplication defined by \( r \phi(x) := \phi(rx) \) for all \( r, x \in R \) and \( \phi \in R^* \). As it is clear that any homomorphism \( \phi \in R^* \) is uniquely characterized by the set \( A := \phi^{-1}(0) \), or by the element \( a := \sum_{x \in A} x \in A = (a) := \{ x \in R \mid x \leq a \} \), we will equally use \( \phi_A \) and \( \phi_a \) for \( \phi \). Using these notations, introduce the cyclic left \( R \)-subsemimodule \( R\phi_0 := \{ r \phi_0 \mid r \in R \} \subseteq R^* \) of \( R^* \) generated by the homomorphism \( \phi_0 \in R^* \), and note that for any \( r \phi_0 \in R\phi_0 \) and any \( x \in R \)

\[
    r \phi_0(x) = \phi_0(rx) = \begin{cases} 0 & \text{if } rx = 0, \\ 1 & \text{otherwise.} \end{cases}
\]

The following observations will prove to be useful.

**Lemma 4.4.** For an additively idempotent complete hemiring \( R \), the following statements are true:

(i) \( A = \phi^{-1}(0) \) is a left ideal of \( R \) for any \( \phi \in R\phi_0 \);
(ii) if \( R \) is also cc-simple, \( \psi \in R\phi_0 \) and \( \phi_a = \phi_A \in R\phi_0 \), then \( I := (a\psi)^{-1}(0) \) is an ideal of \( R \) and \( a\psi = 0 \) or \( a\psi = \phi_0 \).
Proof. (i). If $\phi = r\phi_0$, then $ar = 0$ for any $a \in A$, and hence, $xar = 0$ for any $x \in R$, and $xa \in A$ too.

(ii). By (i), to show that $I$ is an ideal, we need only to show that it is a right ideal. Since $a = \sum_{r \in A} r$ it holds $x \in I$ iff $(\sum_{r \in A} r) \psi(x) = 0$ iff $\psi(xr) = 0$ for every $r \in A$. From this and noting that, by (i), $A$ is a left ideal, for any $x \in I$, $r \in A$, one has $0 = \psi(xs) = \psi((xs)r)$ and, hence, $xs \in I$.

Next, it is also easy to see that $I$ is a subtractive complete ideal, and the Bourne congruence $\equiv_I$ given on $R$ by $x \equiv_I y$ iff there exist elements $a, b \in I$ such that $x + a = y + b$, is a complete one, and hence, $\equiv_I$ is $\Delta_R$ or $R^2$. If $\equiv_I = \Delta_R$, then $I = \{0\}$, and therefore, $a\psi = \phi_0$. In the second case, we have that $x \equiv_I 0$ for any $x \in R$, and the subtractiveness of $I$ implies that $I = R$, and therefore, $a\psi = 0$.

The following observation is important and will prove to be useful.

**Proposition 4.5.** For any cc-simple hemiring $R$ with nonzero multiplication, the left $R$-semimodule $R\phi_0$ is a simple complete semimodule in which $R\phi_0$ is a complete lattice with respect to $\bigvee_{i \in I} \phi_i := \sum_{i \in I} \phi_i$ for any family $\{\phi_i \mid i \in I\}$ of homomorphisms of $R\phi_0$.

Proof. It is clear that $R\phi_0$ is a left $dR$-semimodule as well as a complete lattice with $\bigvee_{i \in I} \phi_i := \sum_{i \in I} \phi_i$ for any family $\{\phi_i \mid i \in I\}$ of elements of $R\phi_0$. Now, let $\infty := \sum_{r \in R} r$; then, applying Proposition 4.3(iii), one gets that $\infty \phi_0(\infty) = \phi_0(\infty^2) = \phi_0(\infty) = 1$, and therefore, $RR\phi_0 \neq 0$.

Next, it is clear that $\phi_{\infty} \in R\phi_0$; and since $\infty \psi(\infty) = \psi(\infty^2) = \psi(\infty) \neq 0$ for any nonzero $\psi \in R\phi_0$, by Lemma 4.4, one gets that $\infty \psi = \phi_0$ and, hence, $R\psi = R\phi_0$ for any nonzero $\psi \in R\phi_0$; and therefore, by Lemma 4.1, $R\phi_0$ is s-simple.

Thus, we have only to show that $R\phi_0$ is a cc-simple $R$-semimodule; and, in fact, we shall prove that $R$-semimodule $R\phi_0$ is even congruence-simple. But first notice the following general and obvious fact: If a complete semilattice $(M, +, 0_M)$ is a left complete $dR$-semimodule over a cc-simple hemiring $R$, then any nonzero complete hemiring homomorphism from $R$ to the complete endomorphism semiring $\text{CEnd}_R(M)$ is injective, i.e., a hemiring embedding; in particular, if $RM \neq 0$, the natural homomorphism from $R$ to $\text{CEnd}_R(M)$ is an embedding.

Now, let $\rho \neq \Delta_R \phi_0$ be a congruence on the left $dR$-semimodule $R\phi_0$. Then, there exist homomorphisms $\psi, \phi \in R\phi_0$ and $r_0 \in R$ such that $(\psi, \phi) \in \rho$ and $\psi(r_0) \neq \phi(r_0)$; and without loss of generality, we may assume that $\psi(r_0) = 1 \in B$ and $\phi(r_0) = 0 \in B$. Then using Lemma 4.4, for $A := \phi^{-1}(0)$ and $a := \sum_{x \in A} x \in A$ and any $r \in R$, we have $a\phi(r) = \phi(ra) = 0$, i.e., $a\phi = 0$.

However, $a\psi \neq 0$. Indeed, supposing $a\psi = 0$, one gets $(\infty a)\psi = 0$ and $(\infty r_0)\psi = 0$ since $r_0 \leq a$, and hence, by Proposition 4.3(iii), $0 = (\infty r_0)\psi(\infty) = \psi((\infty^2 r_0) = \psi(\infty r_0)$. For $\phi_{\infty} \in R\phi_0$, by Lemma 4.4, we have $(\infty r)\phi_0 = \infty((rx)\phi_0) \geq (rx)\phi_0 = r(x\phi_0)$ for any $r, x \in R$, and therefore, taking into consideration the fact mentioned above together with the observation that the two multiplications on the left by $\infty r$ and $r \in R$, respectively, define
the corresponding natural complete endomorphism in $\text{CEnd}_B(R\phi_0)$, one immediately gets $\infty r \geq r$ for any $r \in R$. From the latter, we have $\psi(\infty r_0) \geq \psi(r_0) = 1$ which is a contradiction to $\psi(\infty r_0) = 0$. So, $a\psi \neq 0$, and therefore, by Lemma 4.4, $a\psi = \phi_0$.

Thus, the inclusion $(\psi, \phi) \in \rho$ and Lemma 4.4 imply that $(r(a\psi), r(a\phi)) = (r\phi_0, 0) \in \rho$ for any $r \in R$, what, in turn, implies that $\rho$ is the universal congruence on $R\phi_0$, i.e., $\rho = (R\phi_0)^2$.

In light of the proof of Proposition 4.5, it is natural to state the following, in our view interesting, problems.

**Problem 2.** Does there exist an s-simple, but not congruence-simple, complete left semimodule over a cc-simple hemiring?

**Problem 3.** For a cc-simple hemiring $R$, find and/or describe up to isomorphism all complete simple $R$-semimodules.

Of course, it is an important and natural question whether for complete hemirings the congruence-simpleness and cc-simpleness are the same concepts. Our next result not only has positively answered this question, but also describes all such hemirings.

**Theorem 4.6.** For a complete hemiring $R$, the following are equivalent:

(i) $R$ is congruence-simple;

(ii) $R$ is cc-simple;

(iii) $|R| \leq 2$, or $R$ is isomorphic to a dense complete subhemiring $S \subseteq \text{CEnd}_B(M)$ of the complete endomorphism hemiring $\text{CEnd}_B(M)$ of a nonzero complete lattice $M$.

**Proof.** (i) $\implies$ (ii). This is obvious.

(ii) $\implies$ (iii). Let $R$ be a cc-simple hemiring and, hence, by Proposition 4.3(i), $R$ is additively idempotent, too. For the hemiring $R$, there are only the following two possibilities: $R$ with nonzero multiplication, i.e., $RR \neq 0$, or $RR = 0$. In the second case, considering for a nonzero hemiring $R$ the complete hemiring homomorphism from $R$ onto the additively idempotent two element hemiring $2 = \{0, 1\}$ with the zero multiplication defined by $f(0) = 0$ and $f(x) = 1$ for all nonzero $x \in R$, we immediately obtain the isomorphism $R \cong 2$.

So, now let $RR \neq 0$, and $M$ be a simple complete left $dR$-semimodule in which $M$ is a complete lattice with respect to $\bigvee_{i \in I} m_i := \sum_{i \in I} m_i$ for any family $\{m_i \mid i \in I\}$ of elements of $M$; an existence of such an $R$-semimodule $M$ is guaranteed by Proposition 4.5. By the fact mentioned in the proof of Proposition 4.5, there exists the natural complete injection of the hemiring $R$ into the complete endomorphism semiring $\text{CEnd}_B(M)$, i.e., we can consider $R$ as a natural complete subhemiring of $\text{CEnd}_B(M)$.

Let $I_0 := \{r \in R \mid rx = 0\}$ be a complete left ideal of $R$ defined for any $x \in M$. For any family $\{m_i \mid i \in \Omega\}$ of elements of the lattice $M$ and $r \in R$,
it is clear that \( rm_i \leq \sum_{i \in \Omega} (rm_i) \) for any \( i \in \Omega \) and, hence, \( r(\sum_{i \in \Omega} m_i) = \sum_{i \in \Omega} (rm_i) = 0 \) if \( rm_i = 0 \) for each \( i \in \Omega \); and therefore, \( l_R(\sum_{i \in \Omega} m_i) = \cap_{i \in \Omega} l_R(m_i) \) for any family \( \{m_i | i \in \Omega\} \subseteq R \). As it is obvious that from \( l_R(m) = l_R(m') \) follows \( l_R(rm) = l_R(rm') \) for all \( r \in R \), defining the relation \( x \sim y \) iff \( l_R(x) = l_R(y) \) for \( x, y \in M \), one gets a complete congruence on \( M \) which, since \( M \) is simple, coincides with the identity one \( \Delta_M \). In particular, from the latter one gets that for all \( x, y \in M \),

\[
x \leq y \iff x + y = y \iff l_R(y) = l_R(x + y) = l_R(x) \cap l_R(y) \iff l_R(y) \subseteq l_R(x) .
\]

Now let \( \infty := \sum_{m \in M} m \), and \( a \in M \) such that \( a \neq \infty \). If \( x \in M \) and \( x \notin a \), then \( l_R(a) \not\subseteq l_R(x) \), and therefore, there is the obvious, nonzero, \( R \)-semimodule homomorphism \( l_R(a) \rightarrow M \) defined by \( r \mapsto rx \). For \( M \) is simple, this homomorphism is surjective, and, hence, there exists \( r_x \in l_R(a) \) such that \( r_x x = \infty \). Letting \( s := \sum_{x \notin a} r_x \in l_R(a) \), for any \( x \in M \) we have

\[
sx = \begin{cases} 
  s(x + a) = sx + sa = sa = 0 & \text{if } x \leq a \text{ or } (x + a = a) , \\
  \infty & \text{otherwise} .
\end{cases}
\]

Since, by Lemma 4.1, \( R \infty = M \), there exists \( r \in R \) such that \( r \infty = b \) for any \( b \in M \). From the latter, for any \( x \in M \), we have \( (rs)x = r(sx) = r0 = 0 \) if \( x \leq a \), and \( (rs)x = r(sx) = r \infty = b \) if \( x \notin a \), i.e.,

\[
(rs)x = \begin{cases} 
  r(sx) = r0 = 0 & \text{if } x \leq a , \\
  r(sx) = r \infty = b & \text{otherwise} .
\end{cases}
\]

Thus, for any \( a, b \in M \), there exists \( t \in R \) such that \( tx = e_{a,b}(x) \) for all \( x \in M \), and therefore, \( R \) is a dense complete subhemiring of \( \text{CEnd}_B(M) \).

(iii) \( \implies \) (i). This follows from Corollary 3.5.

As a corollary of Theorem 4.6, we obtain the following description of left artinian congruence-simple complete semirings.

**Corollary 4.7.** A left artinian complete hemiring \( R \) is congruence-simple iff \( |R| \leq 2 \), or it is isomorphic to a dense complete subhemiring \( S \subseteq \text{CEnd}_B(M) \) of the complete endomorphism hemiring \( \text{CEnd}_B(M) \) of a nonzero complete noetherian lattice \( M \).

Proof. For a nonzero complete lattice \( M \), the statement immediately follows from Theorem 4.6 and the following observation: an increasing \( m_1 < m_2 < \cdots < m_k < \cdots \) implies \( l_R(m_1) \supset l_R(m_2) \supset \cdots \supset l_R(m_k) \supset \cdots \) for any elements \( m_1, m_2, \ldots, m_k, \ldots \) in \( M \), for \( e_{m_n,m_n}(m_n) = 0 \) and \( e_{m_n,m_n}(m_{n+1}) = m_n \neq 0 \) for any \( n = 1, 2, 3, \ldots, \) i.e., \( e_{m_n,m_n} \in l_R(m_n) \) but \( e_{m_n,m_n} \notin l_R(m_{n+1}) \).
Remark 4.8. If $R$ is a finite additively idempotent hemiring, then it is a complete hemiring. Indeed, $R$ is a partially ordered semiring with the unique partial order on $R$ defined by $r \leq r'$ iff $r + r' = r'$ (see, for example, [15] and [22, Exercise 5.4] for details). With respect to this order, $R$ is a complete lattice and hemiring with $r \lor r' := r + r'$ and $r \land r' := \sum_{s \leq r, s \leq r'} s$ for all $r, r' \in R$.

Taking into consideration this remark, we finish this section with another corollary of Theorem 4.6, presenting an alternative proof of Theorem 2.2.

Corollary 4.9 ([31, Theorem 1.7]). A proper finite hemiring $R$ is congruence-simple iff $|R| \leq 2$, or $R$ is isomorphic to a dense subhemiring $S \subseteq E_M$ of the endomorphism hemiring $E_M$ of a finite semilattice $(M, +, 0)$.

Proof. $\Rightarrow$. By [31, Proposition 3.1], $R$ is additively idempotent, therefore, from Theorem 4.6 and Remark 4.8, one has that $|R| \leq 2$, or $R$ is isomorphic to a dense complete subsemiring $S \subseteq \text{CEnd}_B(M)$ of the complete endomorphism semiring $\text{CEnd}_B(M)$ of a nonzero complete lattice $M$. For, by Proposition 4.2, $M$ is a simple complete left $R$-semimodule, for any nonzero element $m \in M$, by Lemma 4.1, $M = Rm$, and hence, $M$ is a finite lattice. Since $\text{CEnd}_B(M)$ is a subhemiring of $E_M$, $S$ is a dense subhemiring of $E_M$, too.

$\Leftarrow$. This follows from Theorem 4.6.

5. Simpleness and Morita equivalence of semirings

As the next observation shows, the subclass of simple semirings plays a very special role in the class of ideal-simple semirings. But first recall [6, p. 122] that a surjective homomorphism of semirings $f : R \rightarrow S$ is a semiisomorphism iff $\text{Ker}(f) = \{0\}$; and the semiisomorphism $f$ is a strong semiisomorphism if for any proper ideal $I$ of $R$, the ideal $f(I)$ of $S$ is also proper.

Proposition 5.1. A semiring $R$ is ideal-simple iff $R$ is a simple ring, or there exists a strong semiisomorphism from $R$ onto an additively idempotent simple semiring $S$.

Proof. $\Rightarrow$. Let $\rho$ be a maximal congruence on $R$, which by Zorn’s lemma, of course, always exists and does not contain the pair $(1, 0)$. Then, the factor semiring $R/\rho$ is congruence-simple and, since $R$ is ideal-simple, ideal-simple as well, i.e., it is simple. For the natural surjection $\pi : R \rightarrow R/\rho$, since $\pi(1_R) \notin \text{Ker}(\pi)$, the ideal-simpleness of $R$ implies that $\text{Ker}(\pi) = \{0\}$; and from [31, Proposition 3.1] and the simpleness of $R/\rho$, we have that $R/\rho$ is either a simple ring or an additively idempotent simple semiring.

If $R/\rho$ is a simple ring, $\pi$ is an isomorphism. Indeed, from the equation $\pi(a) = \pi(b)$ for some $a, b \in R$, it follows that there exists $c \in R$ such that $\pi(c) = -\pi(b)$ and $\pi(a + c) = \pi(a) + \pi(c) = \pi(b) - \pi(b) = 0$, and hence, $a + c = 0$ since $\text{Ker}(\pi) = \{0\}$; so, for $c + b \in \text{Ker}(\pi)$ and hence $c + b = 0$ too, one has that $b = (a + c) + b = a + (c + b) = a$. From the latter, we obtain that $\pi$ is injective and $R$ is isomorphic to the simple ring $R/\rho$. 

16
In the case when $R/\rho$ is an additively idempotent simple semiring, it is obvious that the semiisomorphism $\pi$ is a strong one.

$\Leftarrow$. Taking into consideration that semiisomorphisms preserve ideals, this implication becomes obvious.

Recall (see [18] and [21]) that two semirings $R$ and $S$ are said to be Morita equivalent if the semimodule categories $R\mathcal{M}$ and $S\mathcal{M}$ are equivalent categories; i.e., there exist two (additive) functors $F : R\mathcal{M} \rightarrow S\mathcal{M}$ and $G : S\mathcal{M} \rightarrow R\mathcal{M}$, and natural isomorphisms $\eta : GF \rightarrow \text{Id}_{R\mathcal{M}}$ and $\xi : FG \rightarrow \text{Id}_{S\mathcal{M}}$. By [21, Theorem 4.12], two semirings $R$ and $S$ are Morita equivalent if and only if the semimodule categories $\mathcal{M}_{R}$ and $\mathcal{M}_{S}$ are equivalent categories, too. A left semimodule $R P \in |R\mathcal{M}|$ is said to be a generator for the category of left semimodules $R\mathcal{M}$ if the regular semimodule $R R \in |R\mathcal{M}|$ is a retract of a finite direct sum $\bigoplus P$ of the semimodule $R P$; and a left semimodule $R P \in |R\mathcal{M}|$ is said to be a progenerator for the category of left semimodules $R\mathcal{M}$ if it is a finitely generated projective generator for $R\mathcal{M}$. By [21, Proposition 3.9], a left semimodule $R P \in |R\mathcal{M}|$ is a generator for the category of left semimodules $R\mathcal{M}$ iff the trace ideal $\text{tr}(P) := \sum_{f \in R\mathcal{M}(R P, R R)} f(P) = R$. Also by [21, Theorem 4.12], two semirings $R$ and $S$ are Morita equivalent if and only if there exists a progenerator $P \in |R\mathcal{M}|$ such that the semirings $S$ and the endomorphism semiring $\text{End}(R P)$ of the semimodule $R P$ are isomorphic. Moreover, the following observation is true.

**Proposition 5.2** (cf. [24, Proposition 18.33]). For semirings $R$ and $S$, the following statements are equivalent:

(i) $R$ is Morita equivalent to $S$;

(ii) $S \cong e M_n(R)e$ for some idempotent $e$ in a matrix semiring $M_n(R)$ such that $M_n(R)e M_n(R) = M_n(R)$.

**Proof.** (i) $\Rightarrow$ (ii). Assume $R$ is Morita equivalent to $S$. By [21, Definition 4.1 and Theorem 4.12], there exists a progenerator $P \in |R\mathcal{M}|$ for $R\mathcal{M}$ such that $S \cong \text{End}(R P)$ as semirings. Applying [21, Proposition 3.1] and without loss of generality, we can assume that the semimodule $R P$ is a subsemimodule of a free semimodule $R R^n$, and there exists an endomorphism $e \in \text{End}(R R^n)$ such that $e^2 = e$, $P = e(R^n)$ and $e|_P = \text{id}_P$. Since $e \in \text{End}(R R^n) \cong M_n(R)$, one can consider the action of $e$ on $R R^n$ as a right multiplication by some idempotent matrix $(a_{ij}) \in M_n(R)$. In the same fashion as it has been done in the case of the modules over rings (see, for example, [24, Remark 18.10(D) and Exercise 2.18]), one may also show that $\text{tr}(P) = \sum R a_{ij} R$ and $r E_{ij} e E_{kl} r' = r a_{jk} r' E_{il}$, where $\{E_{ij}\}$ are the matrix units in $M_n(R)$ and $r, r' \in R$, and obtain that $M_n(R)e M_n(R) = M_n(\text{tr}(P))$, in the semimodule setting. Since $R P$ is a progenerator of the category of semimodules $R\mathcal{M}$ and [21, Proposition 3.9], $\text{tr}(P) = R$, and hence $M_n(R)e M_n(R) = M_n(R)$. We complete the proof by noting that the semiring homomorphism

$$\theta : \text{End}(R P) \rightarrow e \text{End}(R R^n)e \cong e M_n(R)e,$$
defined for all \( f \in \text{End}(R) \) by \( \theta(f) = eife \) with \( i : P \to R^n \) to be the natural embedding, is a semiring isomorphism.

(ii) \( \implies \) (i). Let \( S \cong eM_n(R)e \) for some idempotent \( e \) in a matrix semiring \( M_n(R) \), and \( M_n(R)eM_n(R) = M_n(R) \). Applying the obvious semimodule modifications of the well-known results for modules over rings (see, for example, [24, Proposition 21.6 and Corollary 21.7]), we have \( S \cong eM_n(R)e \cong \text{End}(M_n(R)M_n(R)e) \). Then, using that by [21, Corollary 3.3] \( M_n(R)e \) is the projective left \( M_n(R) \)-semimodule generated by the idempotent \( e \) and the fact mentioned in (i) that \( \text{tr}(P) = \sum R_{ij} R \) for any finitely generated projective \( R \)-semimodule \( P \), we have that \( \text{tr}(M_n(R)M_n(R)e) = M_n(R)eM_n(R) = M_n(R) \), and, hence, the semimodule \( M_n(R)M_n(R)e \) is a progenerator of the category of semimodules \( M_n(R)\mathcal{M} \). From these observations and [21, Definition 4.1 and Theorem 4.12], it follows that the semirings \( S \) and \( M_n(R) \) are Morita equivalent. Finally, using the facts that by [18, Theorem 5.14] the semirings \( M_n(R) \) and \( R \) are also Morita equivalent, and by [21, Corollary 4.4] the Morita equivalence relation on the category of semirings is an equivalence relation, we conclude the proof. \( \square \)

This result, in particular, motivates us to consider more carefully the relationships between the ideal and congruence structures of semirings \( R \) and \( eRe \) corresponding to idempotents \( e \in R \). So, in the following observation, which will prove to be useful and is interesting on its own, we consider these relationships. Inheriting the ring terminology (see, for example, [23, p. 485]), we say that an idempotent \( e \in R \) of a semiring \( R \) is full if \( ReR = R \).

**Proposition 5.3.** Let \( e \) be an idempotent in the semiring \( R \).

(i) Let \( I \) be an ideal in the semiring \( eRe \). Then \( e(ReI)e = I \). In particular, \( I \mapsto RIR \) defines an injective (inclusion-preserving) map from ideals of \( eRe \) to those of \( R \). This map respects multiplication of ideals, and is surjective if \( e \) is a full idempotent.

(ii) Let \( \Theta \) be a congruence on the semiring \( eRe \). Then, the relation \( \Theta \) on \( R \) for all \( a,b \in R \) defined by

\[(a,b) \in \Theta \iff \forall r,s \in R : (\text{erase}, \text{erbse}) \in \Gamma,\]

is a congruence, and \((eRe)^2 \cap \Theta = \Gamma\). In particular, \( \Gamma \mapsto \Theta \) defines an injective (inclusion-preserving) map from congruences on \( eRe \) to those on \( R \). This map is surjective if the idempotent \( e \) is full.

**Proof.** (i). The proof given for rings in [24, Theorem 21.11] serves for our semiring setting as well; and just for the reader’s convenience, we briefly sketch it here. Namely, if \( I \) is an ideal in \( eRe \), then

\[e(ReI)e = eR(eI)eRe = (eRe)I(eRe) = I,\]

and, if \( I' \) is another ideal of \( eRe \), then

\[(RIR)(I'R) = RIRI'R = R(I'e)R(RI'e)R = RI'eR'I'R = R(I'I'R).\]
If the idempotent $e$ is full, then for any ideal $J$ in $R$ and the ideal $eJe$ in $eRe$, we have

$$R(eJe)R = Re(RJR)eR = (ReR)J(ReR) = RJR = J,$$

i.e., the correspondence $I \mapsto RIR$ is surjective in this case.

(ii). It is easy to see that the relation $\Theta$ on $R$ corresponding to a congruence $\Gamma$ on $eRe$ is, in fact, a congruence on $R$. And we shall show that $(eRe)^2 \cap \Theta = \Gamma$. Indeed, for any $(eae, ebe) \in \Gamma$, we have $(er(eae)ese, er(ebe)ese) \in \Gamma$ for all $r, s \in R$, and hence, $\Gamma \subseteq (eRe)^2 \cap \Theta$; and the opposite inclusion $(eRe)^2 \cap \Theta \subseteq \Gamma$ is obvious.

Now let an idempotent $e$ be full. Then there exist a natural number $n \geq 1$ and elements $\alpha_i, \beta_i$ in $R$ such that $\sum_{i=1}^{n} \alpha_i e \beta_i = 1$. Let $\Pi$ be a congruence on $R$ and $\Gamma := (eRe)^2 \cap \Pi$ the congruence on $eRe$; and $\Theta$ the congruence on $R$ corresponding to $\Gamma$ under the map $\Gamma \mapsto \Theta$. We shall show that $\Theta = \Pi$. Indeed, since $\Pi$ is a congruence on $R$, for any $(a, b) \in \Pi$ and $r, s \in R$, we always have $(eras, erbs) \in \Pi$, and hence, $(a, b) \in \Theta$, i.e., $\Pi \subseteq \Theta$. Conversely, for any $(a, b) \in \Theta$ and $r, s \in R$, we have $(eras, erbs) \in \Gamma$ and, therefore, $(eras, erbs) \in \Pi$ for all $r, s \in R$; in particular, for all $i, j = 1, 2, \ldots, n$, we have $(e\beta_ia \alpha_j e, e\beta_ia \alpha_j e) \in \Pi$ and, since $\Pi$ is a congruence on $R$, $(e\beta_ia \alpha_j e, e\beta_ia \alpha_j e) \in \Pi$, and therefore, $(a, b) = \sum_{j=1}^{n} \sum_{i=1}^{n} (e\beta_ia \alpha_j e, e\beta_ia \alpha_j e) \in \Pi$, i.e., $\Theta \subseteq \Pi$. Thus, for a full idempotent $e$, the map $\Gamma \mapsto \Theta$ is a surjective one.

From Proposition 5.3 immediately follows

**Corollary 5.4.** Let $e$ be a full idempotent in the semiring $R$. Then, $R$ is ideal-simple (congruence-simple) iff the semiring $eRe$ is ideal-simple (congruence-simple). In particular, $R$ is simple iff $eRe$ is simple.

We will use the following important, extending [2, Lemma 3.1], result:

**Proposition 5.5 ([20, Proposition 4.7]).** The matrix semirings $M_n(R)$, $n \geq 2$, over a semiring $R$ are congruence-simple (ideal-simple) iff $R$ is congruence-simple (ideal-simple). In particular, $M_n(R)$ is simple iff $R$ is simple.

Applying Propositions 5.2, 5.5 and Corollary 5.4, we immediately establish that ideal-simpleness, congruence-simpleness and simpleness are Morita invariants for semirings, namely:

**Theorem 5.6.** Let $R$ and $S$ be Morita equivalent semirings. Then, the semirings $R$ and $S$ are simultaneously congruence-simple (ideal-simple, simple).

In our next result, characterizing simple semirings with an infinite element and complete simple semirings, we also show that these classes of semirings are actually the same.

**Theorem 5.7.** For a semiring $R$, the following conditions are equivalent:

(i) $R$ is a simple semiring with an infinite element;
(ii) $R$ is Morita equivalent to the Boolean semiring $B$;

(iii) $R \cong E_M$, where $M$ is a nonzero finite distributive lattice;

(iv) $R$ is a complete simple semiring.

Proof. (i) $\implies$ (ii). Let $R$ be a simple semiring with the infinite element $\infty$. Because $\infty$ is the additively absorbing element of $R$, the additive reduct $(R, +, 0)$ is not a group and $R$ a proper semiring, and hence, by [31, Proposition 3.1], $R$ is an additively idempotent semiring. Then, one readily sees that the sets $A_r := \{ x \in R \mid Rx = 0 \}$ and $A_l := \{ x \in R \mid xR = 0 \}$ are ideals in $R$, and therefore, $A_r = A_l = 0$. The latter implies $\infty \cdot \infty \neq 0$ for any nonzero $x \in R$, in particular, $\infty^2 \neq 0$. Since the set $(\infty^2) := \{ x \in R \mid x \leq \infty^2 \}$ is obviously a nonzero ideal in $R$, one has $(\infty^2) = R$ and, hence, $\infty^2 = \infty$. Again, for, as it is easy to see, $(\infty x \infty) := \{ y \in R \mid y \leq \infty x \infty \}$ is an ideal in $R$, $(\infty x \infty) = R$ for any nonzero $x \in R$, and therefore, for any $x \in R$, we have $\infty x \infty = 0$ or $\infty x \infty = \infty$.

From this observations we conclude that $\infty R \infty = \{ 0, \infty \} \cong B$. Then, taking into consideration that $R \infty R$ is a nonzero ideal of $R$, one has $R \infty R = R$ and by Proposition 5.2 gets the implication.

(ii) $\implies$ (iii). Since the semiring $R$ is Morita equivalent to the Boolean semiring $B$, by [21, Theorem 4.12 and Definition 4.1] there exists a finitely generated projective (i.e., a progenerator) $B$-semimodule $B M \in |B M|$ such that the semirings $R$ and $\text{End}(B M) = E_M$ are isomorphic. Then, taking into consideration that $B M \in |B M|$ is a finitely generated semimodule and [11, Theorem 5.3] (see also [18, Fact 5.9]), one right away concludes that $M$ is a finite distributive lattice.

(iii) $\implies$ (iv). This implication immediately follows from Theorem 3.3 and Remark 4.8.

(iv) $\implies$ (i). It is obvious since by [6, Proposition 22.27] any complete semiring has an infinite element.

From this result we immediately obtain the following description of all finite simple semirings.

**Corollary 5.8.** Let $R$ be a finite semiring. Then one of the following holds:

(i) $R$ is isomorphic to a matrix semiring $M_n(F)$ for a finite field $F$, $n \geq 1$;

(ii) $R \cong E_M$, where $M$ is a nonzero finite distributive lattice.

**Proof.** Indeed, if $R$ is simple, then by [31, Proposition 3.1] $R$ is either a ring or an additively idempotent semiring. In the first case, one obtains $R \cong M_n(F)$ for some finite field $F$ by the well-known variation of the classical Wedderburn-Artin Theorem, characterizing simple artinian rings (see, for example, [24, Theorem 1.3.10]). In the case when $R$ is a finite additively idempotent semiring, $R$ has an infinite element and the result follows from Theorem 5.7.

In connection with this corollary, we wish to mention the following remarks and, in our view interesting, problem.
Remark 5.9. It is easy to see that a finite proper hemiring $R$ is simple iff $R \cong F_M$ for some finite lattice $M$. Indeed, if $R$ is simple, then, by [31, Theorem 1.7], for some finite lattice $M$, there exists a subhemiring $S$ of a semiring $E_M$ such that $R \cong S$ and $F_M \subseteq S \subseteq E_M$, and since, by Lemma 3.1, $F_M$ is an ideal of $E_M$, the ideal-simplicity of $S$ implies $S = F_M$. Inversely: If $R \cong F_M$ for some finite lattice $M$, then by the same [31, Theorem 1.7] $F_M$ is congruence-simple; and assuming that $I \subseteq F_M$ is a nonzero ideal and $f(m) \neq 0$ for some $f \in I$ and $m \in M$, we have $e_{a,b} = e_{0,b} f e_{a,m} \in I$ for all $a, b \in M$ and, hence, $F_M \subseteq I$, and therefore, $F_M = I$ and $F_M$ is ideal-simple too. Moreover, using [31, Proposition 2.3] in the same fashion one may easily see that for any lattice $M$ with an absorbing (infinite) element, $F_M$ is always a simple proper hemiring with an infinite element; however, the following open question is of interest.

Problem 4. Is it right that for any proper simple hemiring $R$ with the infinite element, there exists a lattice $M$ with the infinite element such that the hemirings $R$ and $F_M$ are isomorphic?

In our next observation we present a semiring analog of the well-known “Double Centralizer Property” of ideals of simple rings (see, e.g., [24, Theorem 1.3.11]).

Theorem 5.10 (cf. [24, Theorem 1.3.11]). Let $R$ be a simple semiring, and $I$ be a nonzero left ideal. Let $D = \text{End}(RI)$ (viewed as a semiring of right operators on $I$). Then the natural map $f : R \rightarrow \text{End}(ID)$ is a semiring isomorphism; the semimodule $I \in R\mathcal{M}$ is a generator in the category $R\mathcal{M}$, and the semimodule $I \in \mathcal{M}_D$ is a finitely generated projective right $D$-semimodule. Moreover, there is a nonzero idempotent $e$ of the matrix semiring $M_n(D)$, $n \geq 1$, such that the semirings $R$ and $eM_n(D)e$ are isomorphic, and the semiring $D$ is simple iff the left ideal $I$ is a finitely generated projective left $R$-semimodule.

Proof. Since the natural map $f : R \rightarrow \text{End}(ID)$ is defined by $r \mapsto f(r) \in \text{End}(ID)$, where $f(r)(i) := ri$ for any $i \in I$ and $r \in R$, it is clear that $f$ is a semiring homomorphism from $R$ to $\text{End}(ID)$, and $RI_D$ is an $R$-$D$-bimodule ([18]). For the semiring $R$ is congruence-simple, the map $f$ is injective and there should be only shown that it is surjective, too. And, for the ideal-simplesness of $R$, the latter can be established by repeating verbatim the scheme of the proof of Theorem 1.3.11 of [24]. Thus, $f : R \rightarrow \text{End}(ID)$ is an isomorphism and, hence, $R \cong \text{End}(ID)$.

Since $R$ is a simple semiring and $I$ is a nonzero left ideal in $R$, the trace ideal $\text{tr}(RI)$ coincides with $R$, i.e., $\text{tr}(RI) = R$, and by [21, Proposition 3.9] the semimodule $RI$ is a generator in the category of semimodule $R\mathcal{M}$. This fact implies that for some natural number $n \geq 1$ the regular semimodule $RI$ is a retract of the left $R$-semimodule $I^n$, i.e., there exist $R$-homomorphisms $\alpha : RI^n \rightarrow RI$ and $\beta : RI \rightarrow RI^n$ in the category $R\mathcal{M}$ such that $\alpha \beta = 1_{RI}$. Therefore, in the category $\mathcal{M}_D$, there are the obvious $D$-homomorphisms $\mathcal{M}_D(\alpha, 1_{RI})(\beta, 1_{RI}) : R\mathcal{M}(RI^n, RI) \rightarrow R\mathcal{M}(RI, RI)$ and $\mathcal{M}_D(\alpha, 1_{RI}) : R\mathcal{M}(RI, RI) \rightarrow R\mathcal{M}(RI^n, RI)$ such that
\[ M_D(\beta,1_R I)M_D(\alpha,1_R I) = 1_R M_D(R R, R I) = M_D(R R, R I) = D, \]

as well as \( R M(R I^p, R I) \cong \text{End}(R I)^n = D^n \) and \( R M(R R, R I) \cong I_D \). From these observations, one may easily see that the semimodule \( I \) is a retract of the right \( D \)-semimodule \( D^n \) and, therefore, \( I_D \) is a finitely generated projective right \( D \)-semimodule. Then, in the same manner as in the first part of the proof of Proposition 5.2, one may establish that \( R \cong e M_n(D)e \) for some nonzero idempotent \( e \) of the matrix semiring \( M_n(D) \).

Now assume that the left ideal \( I \) is a finitely generated projective left \( R \)-semimodule. Then, \( R I \) is a progenerator of the category \( R M \), and, hence, \( D \) is Morita equivalent to \( R \) and, by Theorem 5.6, \( D \) is a simple semiring, too.

Finally, suppose that \( D \) is a simple semiring. It is clear that each element \( x \in I \) produces the endomorphism \( \varphi(i) := i x \) for all \( i \in I \), and denote by \( \mathcal{T} := \{ \varphi \mid x \in I \} \subseteq D \) the set of all those endomorphisms. For any \( d \in D \), it is clear that \( \varphi d = \varphi d \) and, hence, \( \mathcal{T} \) is a right ideal of \( D \). For \( R \) is a simple semiring, the ideal \( l_R(I) := \{ r \in R \mid ra = 0 \text{ for all } a \in I \} = 0 \).

The latter implies that \( I^2 \neq 0 \) and, hence, \( \mathcal{T} \neq 0 \), and by the simplicity of \( D \) one has that \( \text{tr}(\mathcal{T}_D) = D \), what in turn, taking into consideration that the \( D \)-homomorphism \( \theta : I \to I_D \) defined by \( \theta(x) := \varphi \) for all \( x \in I \) is obviously a surjective one, gives that \( \text{tr}(I_D) = D \), and therefore, by [21, Proposition 3.9], the semimodule \( I_D \) is a generator in the category \( M_D \). Then, noting that \( R \cong \text{End}(I_D) \), in the similar way as it was done above for the semimodule \( R I \) only now substituting it with the semimodule \( I_D \), one obtains that \( R I \) is a finitely generated projective left \( R \)-semimodule.

As a corollary of Theorem 5.10, we immediately obtain the following description of all simple semirings having projective minimal left (right) ideals.

**Theorem 5.11.** For a semiring \( R \), the following statements are equivalent:

1. \( R \) is a simple semiring with a projective minimal left (right) ideal;
2. \( R \) is isomorphic either to a matrix semiring \( M_n(F) \), \( n \geq 1 \), over a division ring \( F \), or to an endomorphism semiring \( E_M \) of a nonzero finite distributive lattice \( M \).

**Proof.** (i) \( \implies \) (ii). Let \( R \) be a simple semiring with a projective minimal left ideal \( I \). Then, by Theorem 5.10, for \( D = \text{End}(R I) \) and some \( n \geq 1 \) and a nonzero idempotent \( e \) of the matrix semiring \( M_n(D) \), there exists a semiring isomorphism \( R \cong e M_n(D)e \). Actually, \( D \) is a division semiring: Indeed, for \( I \) is a minimal left ideal of the semiring \( R \), it is clear that \( f(I) = I \) for any nonzero element \( f \in D \), and, hence, any nonzero endomorphism \( f \in D \) is a surjection: from the latter and for \( R I \) is a projective left semimodule, it follows that there exists an injective endomorphism \( g \in D \) such that \( fg = 1_I \) for a nonzero endomorphism \( f \), and therefore, \( g \) and \( f \) are isomorphisms and \( D \) is a division, of course ideal-simple, semiring. From this and using Proposition 5.5, we get that the matrix semiring \( M_n(D) \) is ideal-simple, and, hence, \( M_n(D)e M_n(D) = M_n(D) \). Therefore, by Proposition 5.2, \( R \) is Morita equivalent to \( D \) and, hence, by Theorem 5.6, the division semiring \( D \) is simple, too, what, by [21, Theorem 4.5], implies that \( D \) is either a division ring or the Boolean semifield \( B \).
If $D$ is a division ring, then, since $R$ is Morita equivalent to $D$, it is easy to see that $R$ is a simple artinian ring, and therefore, $R \cong M_n(F)$ for some division ring $F$ and $n \geq 1$. In the case when $D$ is the Boolean semifield $B$, it is clear that $R$ is a finite additively idempotent simple semiring and, therefore, by Corollary 5.8, $R \cong E_M$ for some nonzero finite distributive lattice $M$.

(ii) $\implies$ (i). If $R \cong M_n(F)$ for some division ring $F$ and $n \geq 1$, the statement is a well-known classical result (see, for example, [24, Theorem 1.3.5]).

In the second case, from Theorem 3.3 it follows that $R$ is a finite additively idempotent simple semiring containing a minimal left ideal $I$. Let $D := \text{End}(R I)$ and show that $D \cong B$: Indeed, since $I$ a minimal left ideal of the finite semiring $R$, it is clear that for any nonzero element $f \in D$, we have $f(I) = I$, and therefore, $f$ is a surjection; also, for $I$ a finite left ideal, $f$ is an injection, too; then, since it is obvious that $D$ is a proper division finite semiring, from [9, Corollary 1.5.9] it follows that $D \cong B$. From the latter and Theorem 5.10, one concludes that $R I$ is a projective left $R$-semimodule.

The next result provides us with a characterization of ideal-simple semirings having either infinite elements or projective minimal left (right) ideals.

**Theorem 5.12.** For a proper semiring $R$ with either the infinite element or a projective minimal left (right) ideal, the following statements are equivalent:

(i) $R$ is ideal-simple;

(ii) $R$ is strongly semiisomorphic to the endomorphism semiring $E_M$ of a nonzero finite distributive lattice $M$.

**Proof.** (i) $\implies$ (ii). First consider the case of a semiring $R$ with the infinite element $\infty \in R$. By Proposition 5.1 there exists a strong semiisomorphism $\alpha$ from $R$ onto an additively idempotent simple semiring $S$. For $\alpha(\infty)$ is obviously the infinite element of $S$ and Theorem 5.7, we have $S \cong E_M$ for some nonzero finite distributive lattice $M$.

Now let $R$ be an ideal-simple semiring with a projective minimal left (right) ideal. Again, by Proposition 5.1, there exists a strong semiisomorphism $\alpha : R \to S$ from $R$ onto an additively idempotent simple semiring $S$. Let $R I$ be a projective minimal left ideal of $R$. Then, it is almost obvious that without loss of generality (also see, for instance, [12, Proposition 2.1 and Corollary 2.3]), one may assume that $I = Re$ for some idempotent $e \in R$; and $\alpha(I) = So(e)$ with $\alpha(e) = (\alpha(e))^2 \in S$ is also a minimal left ideal of $S$, which, again by [12, Corollary 2.3], is a projective left ideal of $S$. Therefore, applying Theorem 5.11, one has $S \cong E_M$ for some nonzero finite distributive lattice $M$.

(ii) $\implies$ (i). In the both cases, this follows from Theorem 3.3 and Proposition 5.1.

**Corollary 5.13.** A semiring $R$ possessing a projective minimal left (right) ideal is ideal-simple iff $R$ is either isomorphic to a matrix semiring $M_n(F)$, $n \geq 1$, over a division ring $F$, or strongly semiisomorphic to the endomorphism semiring $E_M$ of a nonzero finite distributive lattice $M$. 

23
Remark 5.14. As was shown in Examples 3.7, in general, even for finite semirings, the congruence-simpleness and ideal-simpleness of semirings are independent, different notions. However, since for any commutative proper semiring $R$ there exists the surjection $R \twoheadrightarrow B$ (see, for example, [18, Fact 5.5]), it is easy to see that for finite commutative semirings, the concepts of the ideal-simpleness and congruence-simpleness are always the same, i.e., coincide. In light of this observation and Theorem 5.12, the following two open questions, in our view, are of interest.

**Problem 5.** What is the class of all semirings (in particular, finite ones) for which the concepts of the ideal-simpleness and congruence-simpleness coincide? (In other words, to describe all semirings for which the concepts of the ideal-simpleness and congruence-simpleness coincide.)

**Problem 6.** Describe the class of all proper semirings (in particular, ones possessing either infinite elements or projective minimal one-sided ideals) which are strongly semiisomorphic to the endomorphism semirings $E_M$ of nonzero finite distributive lattices $M$.

We finish this section considering two more applications – to Conjecture and Problem 3.9 of [18] and [17], respectively – of Theorem 5.7. But first recall ([16, Definition 3.1] or [18]) that the tensor product bifunctor $- \otimes - : \mathcal{M}_R \times R \mathcal{M} \to \mathcal{M}$ of a right semimodule $A \in |\mathcal{M}_R|$ and a left semimodule $B \in |R \mathcal{M}|$ can be described as the factor monoid $F/\sigma$ of the free monoid $F \in |\mathcal{M}|$ generated by the cartesian product $A \times B$ and factorized with respect to the congruence $\sigma$ on $F$ generated by ordered pairs having the form

$$
\langle (a_1 + a_2, b), (a_1, b) + (a_2, b) \rangle, \langle (a, b_1 + b_2), (a, b_1) + (a, b_2) \rangle, \langle (ar, b), (a, rb) \rangle,
$$

with $a, a_1, a_2 \in A$, $b, b_1, b_2 \in B$ and $r \in R$.

A semimodule $B \in |R \mathcal{M}|$ is said to be flat [17] iff the functor $- \otimes B : \mathcal{M}_R \to \mathcal{M}$ preserves finite limits or iff a semimodule $B$ is a filtered (directed) colim of finitely generated free (even projective) semimodules [17, Theorem 2.10]: a semiring $R$ is left (right) perfect [18] iff every flat left (right) $R$-semimodule is projective.

In [18, Corollary 5.12], it was shown that in the class of additively regular commutative semirings, perfect semirings are just perfect rings, and proposed the conjecture that the same situation would take place in the entire, not only commutative, class of additively regular semirings. Then this conjecture has been confirmed for additively regular semisimple semirings in [21, Theorem 5.2]. By using Theorem 5.7, we confirm the conjecture for the class of simple semirings with either infinite elements or projective minimal left (right) ideals.

**Theorem 5.15.** A proper simple semiring $R$ with either the infinite element or a projective minimal left (right) ideal is not left (right) perfect.

**Proof.** First consider the case of a semiring $R$ with the infinite element $\infty \in R$. By Theorem 5.7, $R$ is Morita equivalent to the Boolean semiring $B$. Therefore,
by [21, Theorem 4.12] the semimodule categories $R\mathcal{M}$ and $B\mathcal{M}$ are equivalent, i.e., there exist two (additive) functors $F : R\mathcal{M} \to B\mathcal{M}$ and $G : B\mathcal{M} \to R\mathcal{M}$, and natural isomorphisms $\eta : GF \to \text{Id}_{R\mathcal{M}}$ and $\xi : FG \to \text{Id}_{B\mathcal{M}}$. By [21, Lemma 4.10 and Proposition 5.12], the functors $F$ and $G$ establish the equivalences between the subcategories of projective and flat left semimodules of the categories $R\mathcal{M}$ and $B\mathcal{M}$, respectively. However, by [18, Theorem 5.11] (see also [20, Theorem 5.2]) $B$ is not a perfect semiring and, hence, $R$ is not perfect, too.

In the case of a semiring $R$ with a projective minimal left (right) ideal, from Theorem 5.11 follows that $R \cong E_M$ for some nonzero finite distributive lattice $M$ and, therefore, by Theorem 5.7, $R$ is Morita equivalent to $B$, and, as it was shown in the first case above, $R$ is not a perfect semiring.

Using Corollary 5.13 and taking into consideration, for example, [24, Theorems 3.10 and 23.10], one has

**Corollary 5.16.** A simple semiring with a projective minimal left (right) ideal $R$ is perfect iff it is an artinian simple ring.

A semimodule $G \in |_{R\mathcal{M}|}$ is said to be mono-flat [17] iff $\mu \otimes 1_G : F_1 \otimes G \to F \otimes G$ is a monomorphism in $\mathcal{M}$ for any monomorphism $\mu : F_1 \to F$ of right semimodules $F_1, F \in |\mathcal{M}_{R}|$. By [17, Proposition 2.1 and Theorem 2.10], every flat left semimodule is mono-flat, but the converse is not true [17, Example 3.7] (see also [21, Theorem 5.19]); and the question of describing and/or characterizing semirings such that the concepts of ‘mono-flatness’ and ‘flatness’ for semimodules over them coincide constitutes a natural and quite interesting problem [17, Problem 3.9]. For additively regular semisimple semirings this problem has been solved in [21, Theorem 5.19], and in our next result we positively resolve this problem for the class of simple semirings with either infinite elements or projective minimal left (right) ideals, namely:

**Theorem 5.17.** For left (right) $R$-semimodules over a simple semiring $R$ with either the infinite element or a projective minimal left (right) ideal, the concepts of ‘mono-flatness’ and ‘flatness’ are the same.

**Proof.** By Theorems 5.7 and 5.11, the semiring $R$ is an artinian simple ring or Morita equivalent to the semiring $B$. In the first case, the statement is obvious. If $R$ is Morita equivalent to $B$, in the notations introduced in the proof of Theorem 5.15 and applying [21, Proposition 5.12], we have that the functors $F$ and $G$ establish the equivalences between the subcategories of mono-flat and flat left semimodules of the categories $R\mathcal{M}$ and $B\mathcal{M}$, respectively. However, by [17, Theorem 3.2] the concepts of ‘mono-flatness’ and ‘flatness’ for $B$-semimodules coincide and, therefore, they are the same for left (right) $R$-semimodules, too.

6. Simplesness of Additively Idempotent Chain Semirings

Obviously, the additive reduct $(R, +, 0)$ of an additively idempotent semiring $R$ in fact forms an upper semilattice and there exists the partial ordering
≤ on \( R \) defined for any two elements \( x, y \in R \) by \( x \leq y \) iff \( x + y = y \). If any two elements \( x, y \in R \) of the poset \( (R, \leq) \) are comparable, i.e., either \( x \leq y \) or \( y \leq x \), the partial order relation \( \leq \) is said to be total, \( (R, \leq) \) forms a chain, and the semiring \( R \) is called an additively idempotent chain semiring or, in short, \(aic\)-semiring. In this final and short section, we conclude the paper considering the congruence- and ideal-simpleness concepts in the context of artinian simple \(aic\)-semirings and congruence-simple lattice-ordered semirings \[6, \text{Section 21}\]. In particular, we show that the only \((\max, \cdot)\)-division semirings over totally-ordered multiplicative groups \[6, \text{Example 4.28}\] are left (right) artinian ideal-simple \(aic\)-semirings. But first let us make some necessary observations.

**Lemma 6.1.** Any left (right) artinian semiring \( R \) is Dedekind-finite, i.e., \(ab = 1\) implies \(ba = 1\) for any \(a, b \in R\).

**Proof.** Let \( R \) be a left artinian semiring, \(ab = 1\) for some \(a, b \in R\), and \(Ra^n \supseteq Ra^{n+1}\) for any \(n \in \mathbb{N}\). Then, for some \(m \in \mathbb{N}, m \geq 1\), we have \(Ra^m = Ra^{m+1}\), and, hence, \(a^m = ca^{m+1}\) for some \(c \in R\). From \(ab = 1\) follows \(a^mb^m = 1\), and, therefore, \(1 = a^{m}b^{m} = ca^{m+1}b^{m} = ca\). The latter implies \(b = 1b = (ca)b = cb = \frac{1}{c}\), and, hence, \(ba = 1\).

**Lemma 6.2.** For an \(aic\)-semiring \( R \), the subset \(J := \{a \in R \mid \forall r \in R : ra \neq 1\} \subseteq R\) of the semiring \( R\) is the only maximal left ideal of \( R\); and therefore, \(J = \text{Rad}(R)\).

**Proof.** Obviously, we need only to show that \(J\) is a left ideal of \( R\). Indeed, if \(a+b \notin J\) for some \(a, b \in J\), then there exists \(r \in R\) such that \(1 = r(a+b) = ra + rb\) and, since \( R\) is an \(aic\)-semiring, \(ra = 1\) or \(rb = 1\), what contradicts \(a, b \in J\). It is clear that \(ra \in R\) for any \(a \in J\) and \(r \in R\).

In contrast to the ring case (see, e.g., \[24, \text{Corollary 2.4.2}\]), the radical \(\text{Rad}(R)\) of a semiring \( R\) in general is not an ideal of \( R\) \[29, \text{Remark, p.134}\]; however, for artinian \(aic\)-semirings it is not a case, namely:

**Lemma 6.3.** For a left (right) artinian \(aic\)-semiring \( R\), the radical \(\text{Rad}(R)\) is an ideal of \( R\), and \(\text{Rad}(R) = \text{Rad}(R)\).

**Proof.** Indeed, if \(ar \notin \text{Rad}(R)\) for some \(a \in \text{Rad}(R)\) and \(r \in R\), then by Lemma 6.2 there exists \(s \in R\) such that \(s(ar) = 1\), and hence, using Lemma 6.1, one has \((rs)a = 1\). From the latter, it follows that \(a \notin \text{Rad}(R)\), what contradicts to \(a \in \text{Rad}(R)\).

Now we are ready to describe all ideal-simple artinian \(aic\)-semirings.

**Theorem 6.4.** A left (right) artinian \(aic\)-semiring \( R\) is ideal-simple iff it is a division \(aic\)-semiring.

**Proof.** \(\Longrightarrow\). Using the ideal-simpleness of \( R\), the result immediately follows from Lemmas 6.3 and 6.2.

\(\Longleftarrow\). It is obvious.
For a division aic-semiring \( R \) is obviously zerosumfree, there always exists the surjection \( R \rightarrow B \), and therefore, from Theorem 6.4 we immediately obtain

**Corollary 6.5.** A left (right) artinian aic-semiring \( R \) is simple iff \( R \cong B \).

**Remark 6.6.** Let \( G \) be a totally-ordered multiplicative group and \( R := G \cup \{0\} \). Then, extending the order on \( G \) to \( R \) by setting \( 0 \leq g \) for any \( g \in G \), and defining \( 0g = g0 = 0 \) for all \( g \in G \), one has that \( (R, \max, \cdot) \) is a division aic-semiring. And Theorem 6.4 actually says that all left (right) artinian ideal-simple aic-semirings can be obtained in such a fashion for a suitable group \( G \).

Recall \([6, \text{Section 21}]\) that a semiring \( R \) is lattice-ordered if and only if there also exists a lattice structure \((R, \lor, \land)\) on \( R \) such that \( a + b \lor a \land b \leq a \land b \) for all \( a, b \in R \) with respect to the partial order naturally induced by the lattice operations. From the definition, it immediately follows that a lattice-ordered semiring is an ideal-idempotent semiring, and \( 1 = 1 + 0 = 0 + 1 \) for all \( a \in R \), i.e., 1 is the infinite element in \( R \). Form these observations and Theorem 5.7, one readily describes all congruence-simple lattice-ordered semirings.

**Theorem 6.7.** For a lattice-ordered semiring \( R \) the following statements are equivalent:

1. \( R \) is congruence-simple;
2. \( R \) is simple;
3. \( R \cong B \).

**Proof.** (i) \( \implies \) (ii). We need only to show that \( R \) is ideal-simple. Indeed, let \( I \) be a nonzero ideal of \( R \). Then, the Bourne relation on \( R \), defined by setting \( x \equiv_I y \) iff there exist elements \( a, b \in I \) such that \( x + a = y + b \), is obviously a congruence on \( R \), and therefore, \( x \equiv_I y \) for any \( x, y \in R \). In particular, we have \( 1 \equiv_I 0 \), i.e., there exist elements \( a, b \in I \) such that \( 1 = 1 + a = 0 + b \in I \), what implies that \( I = R \).

(ii) \( \implies \) (iii). By Theorem 5.7, \( R \cong E_M \) for some nonzero finite distributive lattice \( M \). From this and using the fact that 1 is the infinite element in \( R \), we get that \( 1_M : M \rightarrow M \) is the infinite element in \( E_M \). On the other hand, putting \( \infty_M := \bigvee_{m \in M} m \), it is easy to see that \( e_{0, \infty_M} \) is also the infinite element in \( E_M \). Hence, \( 1_M = e_{0, \infty_M} \); therefore, \( M = \{0, \infty_M\} \) and \( R \cong E_M \cong B \).

(iii) \( \implies \) (i). It is obvious.

**Corollary 6.8.** For a lattice-ordered aic-semiring \( R \) the following statements are equivalent:

1. \( R \) is ideal-simple;
2. \( R \) is congruence-simple;
3. \( R \) is simple;
4. \( R \cong B \).

**Proof.** (i) \( \implies \) (iv). It follows from the observation that the subset \( I := \{a \in R \mid a < 1\} \subset R \) is an ideal.

The implication (iv) \( \implies \) (i) is obvious.

The remaining implications follow from Theorem 6.7.
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