Entangled Husimi distribution and Complex Wavelet transformation*

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Abstract

Based on the proceeding Letter [Int. J. Theor. Phys. 48, 1539 (2009)], we expand the relation between wavelet transformation and Husimi distribution function to the entangled case. We find that the optical complex wavelet transformation can be used to study the entangled Husimi distribution function in phase space theory of quantum optics. We prove that the entangled Husimi distribution function of a two-mode quantum state $|\psi\rangle$ is just the modulus square of the complex wavelet transform of $e^{-|\eta|^2/2}$ with $\psi(\eta)$ being the mother wavelet up to a Gaussian function.

Keywords: complex wavelet transformation, entangled Husimi distribution, IWOP technique

1 Introduction

Studying distribution functions of density operator $\rho$ in phase space has been a major topic in quantum statistical physics. Phase space technique has proved very effective in various branches of physics. Among various phase space distributions the Wigner function $F_w(q,p)\textsuperscript{[1, 2, 3, 4]}$ is the most popularly used. But the Wigner distribution function itself is not a probability distribution due to being both positive and negative. To overcome this inconvenience, the Husimi distribution function $F_h(q',p')$ is introduced $\textsuperscript{[5]}$, which is defined in a manner that guarantees it to be nonnegative. On the other hand, the optical wavelet transformations have been developed which can overcome some shortcomings of the classical Fourier analysis and the therefore has been widely used in Fourier optics and information science since 1980s $\textsuperscript{[6, 7, 8, 9]}$. In the previous Letter $\textsuperscript{[10]}$, we have employed the optical wavelet transformation to study the Husimi distribution function for single-mode case, and proved that the Husimi distribution function of a quantum state $|\psi\rangle$ is just the modulus square of the wavelet transform of $e^{-x^2/2}$ with $\psi(x)$ being the mother wavelet up to a Gaussian function, i.e.,

$$\langle\psi| \Delta_h (q, p, \kappa) |\psi\rangle = \int d\xi \psi^* (\frac{x - s}{\mu}) e^{-\xi^2/2},$$

where $s = \frac{\kappa q + ip}{\sqrt{\kappa}}$, $\mu = \sqrt{\kappa}$, and $\langle\psi| \Delta_h (q, p) |\psi\rangle$ is the Husimi distribution function,

$$\langle\psi| \Delta_h (q, p, \kappa) |\psi\rangle = 2 \int dq' dp' F_w(q', p') \exp \left[ -\kappa (q' - q)^2 - \frac{(p' - p)^2}{\kappa} \right],$$

as well as $\Delta_h (q, p, \kappa)$ is the Husimi operator,

$$\Delta_h (q, p, \kappa) = \frac{2\sqrt{\kappa}}{1 + \kappa} \exp \left\{ \frac{-\kappa (q - Q)^2}{1 + \kappa} - \frac{(p - P)^2}{1 + \kappa} \right\}.$$
here, \( Q = (a + a^\dagger)/\sqrt{2} \) and \( P = (a - a^\dagger)/(\sqrt{2}i) \) are the coordinate and the momentum operator, and \( a_1, a_1^\dagger \) the Bose annihilation and creation operators, \([a, a^\dagger] = 1, a \langle 0 \rangle = 0\). Thus a convenient approach for calculating various Husimi distribution functions of miscellaneous quantum states is presented.

Recalling that in Ref.\[11\], Fan and Guo have introduced the entangled Husimi operator \( \Delta_h (\sigma, \gamma, \kappa) \) which is endowed with definite physical meaning, and find that there corresponds a special two-mode squeezed coherent state representation such that \( \Delta_h (\sigma, \gamma, \kappa) = |\sigma, \gamma, \kappa \rangle \langle \sigma, \gamma, \kappa |. \) The entangled Husimi operator \( \Delta_h (\sigma, \gamma, \kappa) \) and the entangled Husimi distribution \( F_h (\sigma, \gamma, \kappa) \) of quantum state \( |\psi \rangle \) are given by

\[
\Delta_h (\sigma, \gamma, \kappa) = 4 \int d^2 \sigma' d^2 \gamma' \Delta_w (\sigma', \gamma') \exp \left\{ -\kappa |\sigma' - \sigma|^2 - \frac{1}{\kappa} |\gamma' - \gamma|^2 \right\},
\]

and

\[
F_h (\sigma, \gamma, \kappa) = 4 \int d^2 \sigma' d^2 \gamma' F_w (\sigma', \gamma') \exp \left\{ -\kappa |\sigma' - \sigma|^2 - \frac{1}{\kappa} |\gamma' - \gamma|^2 \right\},
\]

respectively, where \( F_w (\sigma', \gamma') = \langle \psi | \Delta_w (\sigma', \gamma') |\psi \rangle \) with \( \Delta_w (\sigma', \gamma') \) being two-mode Wigner operator and \( F_w (\sigma', \gamma') \) being two-mode Wigner function. Thus we are naturally led to studying the entangled Husimi distribution function from the viewpoint of wavelet transformation.

In this paper, we shall expand the relation between wavelet transformation and Wigner-Husimi distribution function to the entangled case, that is to say, we employ the complex wavelet transformation (CWT) to investigate the entangled Husimi distribution function (EHDF) by bridging the relation between CWT and EHDF. We prove that the entangled Husimi distribution function of a two-mode quantum state \( |\psi \rangle \) is just the modulus square of the complex wavelet transform of \( e^{-|\eta|^2/2} \) with \( \psi (\eta) \) being the mother wavelet up to a Gaussian function. Thus we present a convenient approach for calculating various entangled Husimi distribution functions of miscellaneous two-mode quantum states.

2 Complex wavelet transform and its quantum mechanical version

In Ref.\[12\], Fan and Lu have proposed the complex wavelet transform (CWT), i.e., the CWT of a signal function \( g (\eta) \) by \( \psi \) is defined by

\[
W_\psi g (\mu, z) = \frac{1}{\mu} \int \frac{d^2 \eta}{\pi} g (\eta) \psi^* \left( \frac{\eta - z}{\mu} \right),
\]

whose admissibility condition for mother wavelets, \( \int \frac{d^2 \eta}{\pi} \psi (\eta) = 0 \), is examined in the entangled state representations \( \langle \eta | \) and a family of new mother wavelets (named the Laguerre–Gaussian wavelets) are found to match the CWT \[13\]\[14\]. In fact, by introducing the bipartite entangled state representation \( |\eta \rangle = \eta_1 + i \eta_2, |00 \rangle \)

\[
|\eta \rangle = \exp \left\{ -\frac{1}{2} |\eta|^2 + \eta a_1^\dagger - \eta^* a_2^\dagger + a_1^* a_2^\dagger \right\} |00 \rangle,
\]

which is the common eigenvector of relative coordinate \( Q_1 - Q_2 \) and the total momentum \( P_1 + P_2 \),

\[
(Q_1 - Q_2) |\eta \rangle = \sqrt{2} \eta_1 |\eta \rangle, \ (P_1 + P_2) |\eta \rangle = \sqrt{2} \eta_2 |\eta \rangle,
\]

where \( Q_j \) and \( P_j \) are the coordinate and the momentum operator, related to the Bose operators \( \{a_j, a_j^\dagger\}, \{a_j, a_j^\dagger\} = \delta_{ij} \) by \( Q_j = (a_j + a_j^\dagger)/\sqrt{2} \) and \( P_j = (a_j - a_j^\dagger)/(\sqrt{2}i) \) \((j = 1, 2)\), we can treat

\[
W_\psi g (\mu, z) = \frac{1}{\mu} \int \frac{d^2 \eta}{\pi} \langle \psi \left| \frac{\eta - z}{\mu} \right\rangle \langle \eta | g \rangle = \langle \psi | U_2 (\mu, z) |g \rangle,
\]
where \( z = z_1 + iz_2 \in C \), \( 0 < \mu \in R \), \( g(\eta) \equiv \langle \eta | g \rangle \), and \( \psi(\eta) = \langle \eta | \psi \rangle \) are the wavefunction of state vector \( |g\rangle \) and the mother wavelet state vector \( |\psi\rangle \) in \( \langle \eta \rangle \) representation, respectively, and

\[
U_2(\mu, z) \equiv \frac{1}{\mu} \int \frac{d^2\eta}{\pi} \frac{\eta - z}{\mu} \langle \eta | \mu = e^\lambda, \tag{10}
\]

is the two-mode squeezing-displacing operator \[15, 16, 17\]. Noticing that the two-mode squeezing operator has its natural expression in \( \langle \eta \rangle \) representation \[14\],

\[
S_2(\mu) = \exp \left[ (a_1^+ a_2^+ - a_1 a_2) \ln \mu \right] = \frac{1}{\mu} \int \frac{d^2\eta}{\pi} |\eta\rangle |\eta\rangle, \tag{11}
\]

which is different from the product of two single-mode squeezing (dilation) operators, and the two-mode squeezed state is simultaneously an entangled state, thus we can put Eq.(10) into the following form,

\[
U_2(\mu, z) = S_2(\mu) D(z), \tag{12}
\]

where \( D(z) \) is a two-mode displacement operator, \( D(z) |\eta\rangle = |\eta - z\rangle \) and

\[
D(z) = \int \frac{d^2\eta}{\pi} |\eta - z\rangle \langle \eta | = \exp \left[ iz_1 \frac{P_1 - P_2}{\sqrt{2}} - iz_2 \frac{Q_1 + Q_2}{\sqrt{2}} \right] = D_1(-z/2) D_2(z^*/2). \tag{13}
\]

It follows the quantum mechanical version of CWT is

\[
W_{\psi g}(\mu, \zeta) = \langle \psi | S_2(\mu) D(z) | g \rangle = \langle \psi | S_2(\mu) D_1(-z/2) D_2(z^*/2) | g \rangle. \tag{14}
\]

Eq.(14) indicates that the 2D CWT can be put into a matrix element in the \( \langle \eta \rangle \) representation of the two-mode displacing and the two-mode squeezing operators in Eq.(11) between the mother wavelet state vector \( |\psi\rangle \) and the state vector \( |g\rangle \) to be transformed. Thus the CWT differs from the direct product of two 1-dimensional wavelet transformations.

Once the state vector \( \langle \psi \rangle \) corresponding to mother wavelet is known, for any state \( |g\rangle \) the matrix element \( \langle \psi | U_2(\mu, z) | g \rangle \) is just the wavelet transform of \( g(\eta) \) with respect to \( \langle \psi \rangle \). Therefore, various quantum optical field states can then be analyzed by their wavelet transforms.

### 3 Relation between CWT and EHDF

In the following we shall show that the entangled Husimi distribution function (EHDF) of a quantum state \( |\psi\rangle \) can be obtained by making a complex wavelet transform of the Gaussian function \( e^{-|\eta|^2/2} \), i.e.,

\[
\langle \psi | \Delta_k(\sigma, \gamma, \kappa) | \psi \rangle = e^{-\frac{1}{2}|\gamma|^2} \int \frac{d^2\eta}{\sqrt{2\pi}} e^{-|\eta|^2/2} \psi^* \left( \frac{\eta - z}{\sqrt{\kappa}} \right)^2, \tag{15}
\]

where \( \mu = e^\lambda = \sqrt{\kappa}, z = z_1 + iz_2 \), and

\[
z_1 = \frac{\cosh \lambda}{1 + \kappa}[\gamma^* - \gamma - \kappa(\sigma^* + \sigma)], \tag{16}
\]

\[
z_2 = \frac{i \cosh \lambda}{1 + \kappa}[\gamma + \gamma^* + \kappa(\sigma - \sigma^*)], \tag{17}
\]
and $\langle \psi | \Delta_h (\sigma, \gamma, \kappa) | \psi \rangle$ is the Husimi distribution function as well as $\Delta_h (\sigma, \gamma, \kappa)$ is the Husimi operator,

$$
\Delta_h (\sigma, \gamma, \kappa) = \frac{4\kappa}{(1 + \kappa)^2} : \exp \left\{ \begin{array}{c}
- \frac{(a_1 + a_2^\dagger - \gamma)(a_1^\dagger + a_2 - \gamma^*)}{1 + \kappa} \\
- \kappa \left( a_1 - a_2^\dagger - \sigma \right) \left( a_1^\dagger - a_2 - \sigma^* \right) \end{array} \right\} : ,
$$

(18)

here $\cdot$ denotes normal ordering of operators.

**Proof of Eq. (15).**

When the transformed $|g\rangle = |00\rangle$ (the two-mode vacuum state), noticing that $\langle \eta | 00 \rangle = e^{-|\eta|^2/2}$, thus we can express Eq. (15) as

$$
\frac{1}{\mu} \int \frac{d^2 \eta}{\pi} e^{-|\eta|^2/2} \psi^* \left( \frac{\eta - z}{\mu} \right) = \langle \psi | U_2 (\mu, z) | 00 \rangle .
$$

(19)

To combine the CWTs with transforms of quantum states more tightly and clearly, using the technique of integration within an ordered product (IWOP) of operators, we can directly perform the integral in Eq. (10) of [22] for the transformed two-mode squeezing operator $S_2 (\mu)$. From Eq. (20) it then follows that

$$
U_2 (\mu, z) | 00 \rangle = \text{sech} \lambda \exp \left\{ - \frac{(z_1 - iz_2)(z_1 + iz_2)}{2 (1 + \mu^2)} + a_1^\dagger a_2 + a_1^\dagger a_2^\dagger + 1 \right\} \left( a_1 - a_2^\dagger \right) \text{sech} \lambda

$$

(20)

where we have set $\mu = e^\lambda$, sech $\lambda = \frac{2}{e^{2\lambda} - 1}$, tanh $\lambda = \frac{e^{2\lambda} - 1}{e^{2\lambda} + 1}$, and we have used the operator identity $e^{a_1 a^\dagger} = : \exp \left( (e^g - 1) a^\dagger a \right) :$. In particular, when $z = 0$, it reduces to the usual normally ordered two-mode squeezing operator $S_2 (\mu)$.

Substituting Eqs. (16), (17) and tanh $\lambda = \frac{\kappa - 1}{\kappa + 1}$, cosh $\lambda = \frac{1}{\sqrt{\kappa}}$ into Eq. (21) yields

$$
e^{-\frac{1}{2\lambda} |\gamma|^2 - \frac{\kappa \gamma^* - \gamma\kappa^*}{2(\kappa + 1)}} U_2 (\mu, z_1, z_2) | 00 \rangle

$$

(22)

then the CWT of Eq. (19) can be further expressed as

$$
e^{-\frac{1}{2\lambda} |\gamma|^2 - \frac{\kappa \gamma^* - \gamma\kappa^*}{2(\kappa + 1)}} \int \frac{d^2 \eta}{\mu \pi} e^{-|\eta|^2/2} \psi^* \left( \frac{\eta - z_1 - iz_2}{\mu} \right) = \langle \psi | \sigma, \gamma \rangle_{\kappa} ,
$$

(23)
Using normally ordered form of the vacuum state projector \(|00\rangle \langle 00| = e^{-a_1^\dagger a_1 - a_2^\dagger a_2} , \) and the IWOP method as well as Eq.\( \text{(22)} \) we have

\[
|\sigma, \gamma\rangle_{\kappa\kappa} \langle \sigma, \gamma| = \frac{4\kappa}{(1 + \kappa)^2} \exp \left\{ \frac{-|\gamma|^2 + \kappa |\sigma|^2}{\kappa + 1} + \frac{\kappa \sigma + \gamma \gamma^* + \gamma^* - \kappa \sigma^*}{1 + \kappa}a_1^\dagger a_2^\dagger a_1 a_2 \right\} \]

\[
\rightarrow \frac{1}{2} \kappa \left( a_1 - a_2^\dagger - \sigma \right) \left( a_1^\dagger - a_2 - \sigma^* \right) \left( a_1^\dagger + a_2 - \gamma^* \right) \left( a_1 + a_2^\dagger - \gamma \right) \frac{1}{1 + \kappa} \]

Now we explain why \( \Delta_\kappa (\sigma, \gamma, \kappa) \) is the entangled Husimi operator. Using the formula for converting an operator \( A \) into its Weyl ordering form \[23\]

\[A = 4 \int \frac{d\alpha d\beta}{\pi} \langle -\alpha, -\beta | A | \alpha, \beta \rangle : \exp \left\{ 2 \left( \alpha^* a_1 - a_1^\dagger \alpha \right) + \beta^* a_2 - a_2^\dagger \beta - a_1^\dagger a_1 + a_2^\dagger a_2 \right\}. \]

where the symbol \( : \) denotes the Weyl ordering, \(| \beta \rangle \) is the usual coherent state, substituting Eq.\( \text{(24)} \) into Eq.\( \text{(25)} \) and performing the integration by virtue of the technique of integration within a Weyl ordered product of operators, we obtain

\[
|\sigma, \gamma\rangle_{\kappa\kappa} \langle \sigma, \gamma| = \frac{16\kappa}{(1 + \kappa)^2} \int \frac{d\alpha d\beta}{\pi} \langle -\alpha, -\beta | A | \alpha, \beta \rangle : \exp \left\{ -\frac{1}{\kappa} \left( a_1^\dagger + a_2 - \gamma \right) \left( a_1 + a_2^\dagger - \gamma^* \right) \right\} \]

\[
\times : \exp \left\{ 2 \left( \alpha^* a_1 - a_1^\dagger \alpha + \beta^* a_2 - a_2^\dagger \beta + a_1^\dagger a_1 + a_2^\dagger a_2 \right) \right\} \]

\[
= 4 \int \frac{d\alpha d\beta}{\pi} \langle -\alpha, -\beta | \Delta_\kappa (\sigma, \gamma, \kappa) | \alpha, \beta \rangle : \exp \left\{ -\kappa \left( a_1 - a_2^\dagger - \sigma \right) \left( a_1^\dagger - a_2 - \sigma^* \right) - \frac{1}{\kappa} \left( a_1^\dagger + a_2 - \gamma \right) \left( a_1 + a_2^\dagger - \gamma^* \right) \right\} \]

where we have used the integral formula

\[
\int \frac{d^2z}{\pi} \exp \left( \zeta |z|^2 + \xi z + \eta z^* \right) = \frac{1}{\zeta} e^{-\frac{\xi^2}{\zeta}} \text{Re} (\zeta) < 0. \]

This is the Weyl ordering form of \(|\sigma, \gamma\rangle_{\kappa\kappa} \langle \sigma, \gamma| \). Then according to Weyl quantization scheme \[24, 25\] we know the Weyl ordering form of two-mode Wigner operator is given by

\[
\Delta_w (\sigma, \gamma) = : \delta \left( a_1 - a_2^\dagger - \sigma \right) \delta \left( a_1^\dagger - a_2 - \sigma^* \right) \delta \left( a_1 + a_2^\dagger - \gamma \right) \delta \left( a_1^\dagger + a_2 - \gamma^* \right) : \]

thus the classical corresponding function of a Weyl ordered operator is obtained by just replacing \( a_1 - a_2^\dagger \rightarrow \sigma', a_1 + a_2^\dagger \rightarrow \gamma' \), i.e.,

\[
4 \int \frac{d\alpha d\beta}{\pi} \langle -\alpha, -\beta | \Delta (\sigma', \gamma') | \alpha, \beta \rangle : \exp \left\{ -\kappa \left( a_1 - a_2^\dagger - \sigma \right) \left( a_1^\dagger - a_2 - \sigma^* \right) - \frac{1}{\kappa} \left( a_1^\dagger + a_2 - \gamma \right) \left( a_1 + a_2^\dagger - \gamma^* \right) \right\} \]

\[
\rightarrow 4 \exp \left\{ -\kappa |\sigma' - \sigma|^2 - \frac{1}{\kappa} |\gamma' - \gamma|^2 \right\}. \]
we can further perform the integration in Eq. (4) and see

\[
\langle \sigma, \gamma \rangle_{\kappa} \langle \sigma, \gamma \rangle = 4 \int d^2\sigma' d^2\gamma' \delta \left( a_1 - a_2 - \sigma \right) \delta \left( a_1^+ - a_2 - \sigma^* \right) \delta \left( a_1 + a_2^+ - \gamma \right) \times \delta \left( a_1^+ + a_2 - \gamma^* \right) \exp \left\{ -\kappa |\sigma' - \sigma|^2 - \frac{1}{\kappa} |\gamma' - \gamma|^2 \right\} = 4 \int d^2\sigma' d^2\gamma' \Delta_w (\sigma', \gamma') \exp \left\{ -\kappa |\sigma' - \sigma|^2 - \frac{1}{\kappa} |\gamma' - \gamma|^2 \right\}.
\]

In reference to Eq. (5) in which the relation between the entangled Husimi function and the two-mode Wigner function is shown, we know that the right-hand side of Eq. (30) should be just the entangled Husimi distribution function of a two-mode quantum state, i.e.,

\[
\langle \sigma, \gamma \rangle_{\kappa} \langle \sigma, \gamma \rangle = 4 \int d^2\sigma' d^2\gamma' \Delta_w (\sigma', \gamma') \exp \left\{ -\kappa |\sigma' - \sigma|^2 - \frac{1}{\kappa} |\gamma' - \gamma|^2 \right\} = \Delta_h (\sigma, \gamma, \kappa),
\]

thus Eq. (15) is proved by combining Eqs. (31) and (23).

Motivated by the preceding Letter [10], we have further expanded the relation between wavelet transformation and Wigner-Husimi distribution function to the entangled case. That is to say, we prove that the entangled Husimi distribution function of a two-mode quantum state \( |\psi\rangle \) is just the modulus square of the complex wavelet transform of \( e^{-|\eta|^2/2} \) with \( \psi (\eta) \) being the mother wavelet up to a Gaussian function, i.e.,

\[
\langle \psi | \Delta_h (\sigma, \gamma, \kappa) |\psi\rangle = e^{-\frac{1}{2} |\gamma|^2} \int \frac{d^2\eta}{\sqrt{\pi\kappa}} e^{-|\eta|^2/2} |\psi (\eta + z) / \sqrt{\kappa}|^2.
\]

Thus we have a convenient approach for calculating various entangled Husimi distribution functions of miscellaneous quantum states. For more discussion about the wavelet transformation in the context of quantum optics, we refer to Refs. [26, 27].

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**Appendix**

We can check Eq. (31) by the following way.

Using the normally ordered form of the two-mode Wigner operator [11]

\[
\Delta_w (\sigma, \gamma) = \frac{1}{\pi^2} : \exp \left\{ - \left( a_1 - a_2 - \sigma \right) \left( a_1^+ - a_2 + \sigma^* \right) - \left( a_1 + a_2^+ - \gamma \right) \left( a_1^+ + a_2 - \gamma^* \right) \right\} :,
\]

we can further perform the integration in Eq. (4) and see

\[
\Delta_h (\sigma, \gamma, \kappa) = 4 \int d^2\sigma' d^2\gamma' \frac{1}{\pi^2} \exp \left\{ -\kappa |\sigma' - \sigma|^2 - \frac{1}{\kappa} |\gamma' - \gamma|^2 \right\} \times \exp \left\{ - \left( a_1 - a_2 + \sigma \right) \left( a_1^+ - a_2 + \sigma^* \right) - \left( a_1 + a_2^+ - \gamma \right) \left( a_1^+ + a_2 - \gamma^* \right) \right\} : = 4 \kappa \left( 1 + \kappa \right)^{-2} : \exp \left\{ - \frac{\left( a_1 + a_2^+ - \gamma \right) \left( a_1 + a_2 - \gamma^* \right) - \kappa \left( a_1 + a_2^+ - \sigma \right) \left( a_1^+ + a_2 - \sigma^* \right) \right\} : = \Delta_h (\sigma, \gamma, \kappa),
\]

which is the confirmation of Eq. (31).

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