Generalized Resilience and Robust Statistics

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Abstract

Robust statistics traditionally focuses on outliers, or perturbations in total variation distance. However, a dataset could be corrupted in many other ways, such as systematic measurement errors and missing covariates. We generalize the robust statistics approach to consider perturbations under any Wasserstein distance, and show that robust estimation is possible whenever a distribution’s population statistics are robust under a certain family of friendly perturbations. This generalizes a property called resilience previously employed in the special case of mean estimation with outliers. We justify the generalized resilience property by showing that it holds under moment or hypercontractive conditions. Even in the total variation case, these subsume conditions in the literature for mean estimation, regression, and covariance estimation; the resulting analysis simplifies and sometimes improves these known results in both population limit and finite-sample rate. Our robust estimators are based on minimum distance (MD) functionals (Donoho and Liu, 1988), which project onto a set of distributions under a discrepancy related to the perturbation. We present two approaches for designing MD estimators with good finite-sample rates: weakening the discrepancy and expanding the set of distributions. We also present connections to Gao et al. (2019)’s recent analysis of generative adversarial networks for robust estimation.

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1 Introduction

We study the problem of statistical inference from corrupted data. For instance, consider the clean dataset and different corrupted datasets depicted below:

Figure 1: Clean dataset (left) and three different types of corruptions.

Figure 1 depicts possible corruptions: *process error* that affects the outputs, *measurement error* that affects the covariates, or some fraction of arbitrary *outliers*. We seek to provide a framework for studying robustness that encompasses these and other types of corruptions, study minimal assumptions needed to ensure robust estimation is possible with infinite samples given particular types of corruptions, and construct estimators with provably good performances in finite samples.

We will model corruptions in a *worst-case* or *adversarial* framework. This frees us from the need to model the precise distribution of errors, motivated by cases where the corruptions might be systematic rather than easily modeled as random noise. For instance, in genomics it is common for measurement errors to occur as “batch effects” that affect many data points at once in a structured way (Rosenberg et al., 2002; Li et al., 2008). More generally, data might suffer from outliers (Hodge and Austin, 2004; Aggarwal, 2015), human error in labeling or generating data (Huber, 2012, Chapter 2.4.3), or falsified records inserted by malicious actors (Chen et al., 2017).

Our framework for modeling corruptions considers some true population distribution $p^*$ (assumed to lie in some family $\mathcal{G}$), together with a corrupted distribution $p$ such that $D(p^*, p) \leq \epsilon$ under some discrepancy $D$. We then observe corrupted samples $X_1, \ldots, X_n$ and wish to output an estimate $\hat{\theta}(X_1, \ldots, X_n)$ such that some cost $L(p^*, \hat{\theta})$ is low. This is summarized in Figure 2; note that while samples come from a corrupted distribution, our goal is to estimate parameters of the original, uncorrupted distribution $p^*$. As a result, even as $n \to \infty$ we typically incur some non-vanishing error that depends on $\epsilon$.

![Framework for statistical inference from corrupted data.](image)

Figure 2: Framework for statistical inference from corrupted data.

Figure 2 illustrates an *oblivious* adversary that applies perturbations to the population distri-
distribution $p^*$. We also consider a more powerful adaptive adversary that can perturb the samples $X_1, \ldots, X_n$ directly. Defining adaptive adversaries correctly requires some technical care and is done formally in Section 2.2.

The worst-case estimation model illustrated in Figure 2 has been widely studied in robust statistics, but only for discrepancies $D$ that are closely related to the total variation distance $D = TV$. For instance, the Huber contamination model (Huber, 1973; Chen et al., 2018) allows the adversary to replace $p^*$ with $p = (1 - \epsilon)p^* + \epsilon q$ for arbitrary $q$ (adding points); this corresponds to the discrepancy $D_{\text{Huber}}(p^*, p) \equiv \inf \{ \epsilon \mid p(A) \geq (1 - \epsilon)p^*(A) \text{ for all events } A \}$, which satisfies $D_{\text{Huber}} \geq TV$ and hence allows weaker perturbations. Other functions such as Kolmogorov–Smirnov, Kuiper, Lévy, Prohorov, and Hellinger have also been studied (Huber, 2011; Donoho and Liu, 1988), while a significant fraction of robust statistics papers consider the TV distance itself (Adrover and Yohai, 2002; Lai et al., 2016; Diakonikolas et al., 2016; Charikar et al., 2017; Gao et al., 2018; Prasad et al., 2018; Klivans et al., 2018).

Corruptions under TV or Huber only allow an $\epsilon$-fraction of outliers or deletions. In many applications we might instead believe that all of the data have been slightly corrupted. We can model this by letting $D(p, q) = W_1(p, q)$ be the (standard) Wasserstein distance between $p$ and $q$, defined as the minimum cost in $\ell_2$-norm needed to move the points in $p$ to the points in $q$ (Definition 2.1). Other Wasserstein distances such as $W_\alpha$ (for $\alpha < 1$) allow more heavy-tailed outliers and interpolate towards the TV case. Little work has been done to analyze robustness under such corruptions.

Our aim is to generalize the theory of robust statistics to analyze richer families of perturbations. We will show that robust statistics can be applied to other perturbations beyond TV, such as Wasserstein distances, by constructing estimators with favorable worst-case bounds. A particular focus will be the setting where the dimension $d$ is large, which is a key difficulty in the TV case that has motivated considerable work (Maronna, 1976; Donoho, 1982; Donoho and Liu, 1988; Diakonikolas et al., 2016).

**Interpreting the problem parameters.** Our setting has several key elements: the cost function $L(p, \theta)$, the discrepancy $D$ and corruption level $\epsilon$, and the assumed distribution family $\mathcal{G}$. To build intuition we will briefly discuss each below.

The cost function $L(p^*, \theta)$ specifies our estimation goal. For mean estimation we take $L(p^*, \theta) = \|\mathbb{E}_{p^*}[X] - \theta\|_2$ to be the distance from $\theta$ to the true mean of $p^*$. For linear regression, where $p^*$ is a distribution over pairs $(x, y) \in \mathbb{R}^d \times \mathbb{R}$, our cost function could either be the predictive loss $L(p^*, \theta) = \mathbb{E}_{(X,Y) \sim p^*}[\{(Y - \langle \theta, X \rangle)^2\}]$ or the excess predictive loss $L(p^*, \theta) = \mathbb{E}_{(X,Y) \sim p^*}[\{(Y - \langle \theta, X \rangle)^2\}] - \inf_{\theta \in \mathbb{R}^d} \mathbb{E}_{(X,Y) \sim p^*}[\{(Y - \langle \theta, X \rangle)^2\}]$. Note that due to the adversarial nature of perturbations, we should expect the excess loss to be non-zero even when $n = \infty$, as long as $\epsilon \neq 0$.

As discussed above, the discrepancy $D$ specifies the family of allowed perturbations. Taking $D = TV$ allows deletions as well as arbitrary outliers, while taking $D = W_1$ allows all points to be perturbed so long as the average $\ell_2$-distance of the perturbation is small. One might also consider more exotic distances such as $W_{\ell_0,1}$, which measures the average $\ell_0$-distance between original and corrupted points and is a good model for missing or corrupted covariates.

Finally, the distribution family $\mathcal{G}$ encodes our assumptions about the true distribution $p^*$. Restricting $p^*$ to some non-trivial family $\mathcal{G}$ is generally necessary in the robust setting. Take for instance robust mean estimation with $D = TV$; then two distributions could be nearby in TV but have arbitrarily different means due to a small amount of mass at $\infty$. Without assumptions to rule one of these distributions out we cannot hope for any bound on the estimation error. Similar triviality happens when we consider robust covariance estimation with $D = W_1$.

In the literature on robust mean estimation, $\mathcal{G}$ is usually taken to be symmetric or elliptical.
distributions (Huber, 1973; Donoho and Liu, 1988; Adrover and Yohai, 2002; Gao et al., 2019), Gaussian distributions (Diakonikolas et al., 2016), sub-Gaussian distributions (Diakonikolas et al., 2017), or distributions with bounded moments (Lai et al., 2016; Diakonikolas et al., 2017). Steinhardt et al. (2018) recently introduced a condition called resilience that is weaker than the moment type conditions mentioned above and showed it is sufficient to obtain robust recovery guarantees.

Goals. Given cost function $L(p^*, \theta)$ and discrepancy $D(p^*, p)$, we wish to design assumptions $\mathcal{G}$ and estimators $\hat{\theta}$ to achieve the following goals:

- **Large $\mathcal{G}$**: We would like $\mathcal{G}$ to be large enough to contain most distributions that may arise; in other words, we do not want our assumptions to be too strong. Indeed, if $\mathcal{G}$ is too small and in real applications $p^* \notin \mathcal{G}$, we may fail to output a solution or incorrectly believe that we have achieved good robust recovery. Hence we view parametric assumptions such as Gaussianity as too strong, and seek non-parametric assumptions that can subsume sub-Gaussianity or bounded moments.

- **Population limit**: We want the set $\mathcal{G}$ to be small enough that at least when $n = \infty$ (population), there exists an estimator $\hat{\theta}$ achieving low error. Indeed, if $\mathcal{G}$ is too large then we may do too little to resist corruptions or be concerned about corruptions that are implausible, leading to weak error bounds.

- **Finite-sample properties**: We would like our estimators to possess fast statistical convergence rates in finite samples.

We will first focus on the population limit (Sections 3 and 4) and later study statistical rates (Sections 5 and 6). We seek to accomplish the three tasks above for a large variety of cost functions $L$ and discrepancies $D$, including Wasserstein distances.

Finally, we are not trying to design computationally efficient algorithms—our estimators are purely information theoretic in nature, although they suggest avenues for constructing efficient algorithms in some cases.

### 1.1 Typical Results

To give a flavor for what our techniques can provide, we informally state several typical results in concrete settings. The main text contains many other similar results as well as general techniques for establishing them.

We have four types of results: improving and simplifying population-limit results for $\text{TV}$ (Section 3), introducing robust estimators for $W_1$ and other Wasserstein distances under natural non-parametric assumptions (Section 4), improving finite-sample analysis for $\text{TV}$ (Section 5), and providing finite-sample analysis for $W_1$ (Section 6). We will give one example of each below.

**Generalizing TV estimators to large $\mathcal{G}$ (Section 3)**. Even in infinite samples robust estimation is non-trivial, and a key question is how the worst-case cost $L$ grows as a function of $\epsilon$ and other problem parameters, and what assumptions are needed to obtain a favorable dependence on $\epsilon$. We design estimators and identify assumptions that unify and improve existing assumptions in the literature. For instance, for linear regression we can prove the following:

**Proposition (Example 3.2)**. For $(X, Y) \sim p^*$, let $L(p, \theta) = \mathbb{E}_p[(Y - X^T \theta)^2 - (Y - X^T \theta^*(p))^2]$ be the excess predictive loss for linear regression. Let $Z = Y - X^T \theta^*(p)$ denote the residual error.
Suppose that
\[ E_{p^*}[(v^\top X)^4] \leq \kappa^4 E_{p^*}[(v^\top X)^2]^2 \text{ for all } v \in \mathbb{R}^d, \quad (1) \]
\[ E_{p^*}[Z^4] \leq \sigma^4. \quad (2) \]

Then we can design an estimator with excess loss \( L(p^*, \hat{\theta}(p)) \leq O(\kappa^2 \sigma^2 \varepsilon) \) when TV\((p^*, p) \leq \varepsilon).\)

This says that as long as the fourth moments of the error are bounded, and the covariates satisfy a hypercontractivity condition between the 2nd and 4th moments, we achieve error \( O(\varepsilon) \), which is information theoretically optimal. In contrast, Klivans et al. (2018) achieve a weaker error of \( O(\sqrt{\varepsilon}) \) under the same assumptions.

Given higher moment bounds we can obtain error approaching \( O(\varepsilon^2) \). In Section 3.2 we present this more general result as well as results for second moment estimation and joint mean and covariance estimation.

Robust estimators for \( W_1 \) (Section 4). For distances such as \( W_1 \) that admit large corruptions but are not closely connected to TV, and settings where naive estimators lead to infinite error, we are not aware of prior work providing good performance under natural assumptions. We present a general recipe for constructing estimator that are robust to general Wasserstein perturbations \( W_{c,k} \) (Definition 2.1), and apply them in the special case of \( W_1 \). For linear regression under \( W_1 \) perturbations we obtain:

Proposition (Example 4.2). For \((X, Y) \sim P^*\), let \( L(p, \theta) = E_p[(Y - X^\top \theta)^2] \) be the predictive loss for linear regression and let \( Z = Y - X^\top \theta^*(p) \) denote the residual error, where \( \theta^*(p) = \arg\min_{\theta \in \mathbb{R}^d} L(p, \theta). \) Let \( X' = [X, Z] \in \mathbb{R}^{d+1} \) be the concatenation of \( X \) and \( Z \). Assume that \( P^* \) satisfies the following condition:
\[ \sup_{v \in \mathbb{R}^{d+1}, \|v\|_2 \leq 1} E_{p^*}[(v^\top X')^3] \leq \sigma^3. \quad (3) \]

Then we can design an estimator with loss \( L(p^*, \hat{\theta}(p)) \leq \sigma^2 + O((R^2 + 1)^{3/4} \sigma^{3/2} \varepsilon^{1/2}) \) when \( W_1(p^*, p) \leq \varepsilon \).

For technical reasons we bound the loss rather than the excess loss, though if the parameter \( \sigma^2 \) is known then we achieve excess loss \( O((R^2 + 1)^{3/4} \sigma^{3/2} \varepsilon^{1/2}) \). We also show that it is necessary to assume \( \theta \) to be bounded. Stronger results under higher moments, as well as results for moment estimation, appear in Section 4.

Finite sample analysis for TV (Section 5). We also design versions of our estimators with good finite-sample bounds, and improve finite-sample analysis of existing algorithms. A core difficulty in finite-sample analysis is that the TV distance between \( p \) and the empirical distribution \( p_n \) converges to zero very slowly or not at all. We present two analysis techniques; the first replaces TV with a weaker distance TV that converges at a parametric rate, while the second analyzes projection under TV to a certain set \( \mathcal{M} \) which applies to many existing algorithms in the literature.

Using the first technique, we can generally replace \( \varepsilon \) with \( \varepsilon + \sqrt{d/n} \) in all of our bounds. Thus for linear regression we obtain the following finite-sample rate:

Proposition (Theorem 5.2). For \((X, Y) \sim P^*\), let \( L(p, \theta) = E_p[(Y - X^\top \theta)^2 - (Y - X^\top \theta^*(p))^2] \) be the excess loss and let \( Z = Y - X^\top \theta^*(p) \) denote the residual error. Suppose that
\[ E_{p^*}[(v^\top X)^4] \leq \kappa^4 E_{p^*}[(v^\top X)^2]^2 \text{ for all } v \in \mathbb{R}^d, \quad (4) \]
\[ E_{p^*}[Z^4] \leq \sigma^4. \quad (5) \]
Assume $TV(p^*, p) \leq \epsilon$ and let $\hat{p}_n$ denote the corrupted empirical distribution given $n$ samples. Then we can design a robust estimator with excess loss $L(p^*, \hat{\theta}(\hat{p}_n)) = O(\sigma^2 n^2 (\epsilon + \sqrt{(d + \log(1/\delta))/n}))$ with probability at least $1 - \delta$.

This matches the previous infinite-sample proposition with $\epsilon$ replaced by $\epsilon + \sqrt{(d + \log(1/\delta))/n}$. Thus when $n \gg d/\epsilon^2$ our performance matches the infinite-data limit up to constants. In contrast, Klivans et al. (2018) achieve a weaker error of $O(\sqrt{\epsilon})$ under the same assumptions when $n \gg \text{poly}(d^4, 1/\epsilon)$. Diakonikolas et al. (2019c) also proposed when $X$ is an isotropic Gaussian and $Z$ has bounded second moment, which guarantees $\|\hat{\theta} - \theta^*\|_2 \leq \epsilon \log(1/\epsilon)$ when $n \gg d/\epsilon^2$. We present further results for linear regression and second moment estimation in Section 5.1, and joint mean and covariance estimation in Appendix E.3.

For the second technique, a typical result concerns mean estimation for distributions with bounded $k$-th moment, where one projects the corrupted empirical distribution under $TV$ to a set containing bounded $k$-th moment distributions. The best general analysis yields a suboptimal complexity of $d^{1.5}$ (Steinhardt et al., 2018) while ad hoc analyses are often complicated and sometimes yield worse bounds like $d^{k/2}$ (Kothari and Steurer, 2017). We present a general analysis with nearly tight dependence on the dimension. A concrete result is:

**Proposition** (Theorem 5.6). Suppose that $p^*$ has mean $\mu$, identity covariance, and bounded $k$-th moment: $\sup_{\|v\|_2 \leq 1} E_{p^*}[|v^T (X - \mu)|^4] \leq \sigma^4$. Then, given $n$ samples, projecting the corrupted distribution $\hat{p}_n$ to the set of distributions with bounded covariances leads to an estimator with $\ell_2$ error $O(\sigma(\epsilon^{-1/k} + \sqrt{d \log d/n}))$ with high probability.

It achieves the optimal infinite sample error $\epsilon^{1-1/k}$ and near-linear sample complexity, and is potentially computationally efficient, since projecting to the set of bounded covariance distributions is generally efficiently solvable (Diakonikolas et al., 2017).

**Finite sample analysis for $W_1$ (Section 6).** Finally, we present the first finite sample analysis for $W_1$ perturbations. We employ a relaxation $\tilde{W}_1$ of $W_1$ similar to the TV case. Before we could replace $\epsilon$ with $\epsilon + \sqrt{d/n}$, but here due to the heavy-tailed nature of $W_1$ perturbations we get a worse dependence $\epsilon + \sqrt{d/n} + n^{-1/2+\delta}$, where $\delta$ depends on the behavior of $p^*$. For instance, for second moment estimation we obtain the following:

**Proposition** (Theorem 6.1). Suppose that $p^*$ has bounded 4th moment: $\sup_{\|v\|_2 \leq 1} E_{p^*}[|v^T (X - \mu)|^4] \leq \sigma^4$, and let $M = E_{p^*}[XX^\top]$ be the second moment matrix of $p^*$. Assume that $W_1(p^*, p) \leq \epsilon$ and let $\hat{p}_n$ be the corrupted empirical distribution given $n$ samples. Then we can design an estimator such that $\|M - \hat{M}(\hat{p}_n)\|_2 \leq O(\sigma^{1/3}(\epsilon + \sigma \sqrt{d/n}) + \sigma/n^{3/8})^{2/3}$ with high probability.

Note that now $\epsilon$ has the same units as $X$, hence $\sigma$ appearing in front of the dimensionless $\sqrt{d/n}$ makes the units match. We achieve error of $\sigma^{4/3} \epsilon^{2/3}$ when $n \gg d/\sigma^2 + 1/\epsilon^{2/3}$. We think the $\epsilon^{-8/3}$ term is likely unnecessary, and in general we consider finite sample analysis of $W_1$ to be largely open.

### 1.2 Outline of Techniques

Our algorithms for infinite samples are based on the **minimum distance (MD) functional** estimator (Donoho and Liu, 1988), which computes

$$\hat{\theta}(p) = \theta^*(q), \quad \text{where } q = \arg\min_{q \in \Theta} D(q, p)$$

(6)

and $\theta^*(q) = \arg\min_{\theta \in \Theta} L(q, \theta)$. In other words, this estimator finds the closest distribution $q$ to $p$ that lies in $\Theta$, then outputs the optimal parameters for $q$. 

8
When $D$ is a pseudometric, the worst-case loss of the MD functional $\hat{\theta}(p)$ is upper bounded by the modulus of continuity between $D$ and $L$ (Donoho and Liu, 1988), defined as

$$\mathfrak{m}(\mathcal{G}, 2\epsilon) = \sup_{p_1, p_2 \in \mathcal{G} : D(p_1, p_2) \leq 2\epsilon} L(p_1, \theta^*(p_2)).$$

While the adversary can choose distributions outside of $\mathcal{G}$, the modulus $\mathfrak{m}$ only involves pairs of distributions that lie within $\mathcal{G}$, making it amenable to analysis.

As a concrete example, if $D = TV$, $L(p^*, \theta) = ||\theta - \mathbb{E}_{p^*}[X]||_2$, and $\mathcal{G} = \mathcal{G}_{gauss}$ is the family of isotropic Gaussian distributions, then the modulus is linear in $\epsilon$ and independent of the dimension: $\mathfrak{m}(\mathcal{G}_{gauss}, 2\epsilon) \preceq \epsilon$. This achieves the goal of a small population limit, but $\mathcal{G}_{gauss}$ is much too small. We will see below how to design larger $\mathcal{G}$ that still have small modulus.

**Resilience for $D = TV$ (Section 3).** As a starting point we consider the case of robust mean estimation, where $D = TV$ and $L(p^*, \theta) = ||\theta - \mathbb{E}_{p^*}[X]||_2$. We will design $\mathcal{G}$ based on the notion of resilient distributions introduced in Steinhardt et al. (2018). A distribution $p$ over $\mathbb{R}^d$ is $(\rho, \epsilon)$-resilient if

$$||\mathbb{E}_p[X] - \mathbb{E}_{p^*}[X]||_2 \leq \rho$$

for all events $E$ with $\mathbb{P}_p(E) \geq 1 - \epsilon$. (8)

This says that all “$\epsilon$-deletions” of $p$ must have a mean similar to the mean of $p$. This property holds, for instance, for sub-Gaussians (with $\rho \preceq \epsilon \sqrt{\log(1/\epsilon)}$) as well as distributions with bounded covariance (with $\rho \preceq \sqrt{\epsilon}$). Importantly, $\rho$ usually has no explicit dependence on the dimension $d$.

Let $\mathcal{G}_{\text{TV}}(\rho, \epsilon)$ denote the family of $(\rho, \epsilon)$-resilient distributions. We show that the modulus $\mathfrak{m}(\mathcal{G}_{\text{TV}}(\rho, \epsilon), \epsilon) \preceq 2\rho$. The reason for this is a midpoint property of TV distance: if $TV(p_1, p_2) \leq \epsilon$ then there is a midpoint $r$ that can be obtained from either of the $p_i$ by conditioning on an event of probability $1 - \epsilon$. Thus $\mathbb{E}_r[X]$ is close to both $\mathbb{E}_{p_1}[X]$ and $\mathbb{E}_{p_2}[X]$ by resilience, and so $\mathbb{E}_{p_1}[X]$ and $\mathbb{E}_{p_2}[X]$ are close by the triangle inequality.

A general intuition is that $p \in \mathcal{G}_{\text{TV}}(\rho, \epsilon)$ formalizes the property that deleting the probability mass beyond the $\epsilon$-quantiles of $p$ will not move the mean by too much. In addition to sub-Gaussians and bounded covariance distributions discussed above, this also includes sub-exponential distributions ($\mathfrak{m}(\mathcal{G}, \epsilon) \preceq \epsilon \log(1/\epsilon)$) and distributions with bounded $k$-th moments ($\mathfrak{m}(\mathcal{G}, \epsilon) \preceq \epsilon^{1/\epsilon}$). We discuss these results in detail in Section 3.1.

Next suppose that $D = TV$ is still total variation distance but the loss $L$ is arbitrary. We generalize Steinhardt et al.’s definition of resilience to yield a family for any $L$ that has bounded modulus. Before, we used the closeness of mean between the distribution $p$ and the midpoint $r$. Since distance in mean is symmetric and satisfies the triangle inequality, we only needed the single condition (8). For general loss $L$ we will need two conditions: the first condition asks that the optimal parameters for $p$ do well on the midpoint $r$, while the second asks that if a parameter does well on $r$ then it also does well on $p$. These are stated formally below, where we use the notation $r \preceq \frac{p}{1 - \epsilon}$ to specify that $r$ can be obtained from $p$ by conditioning on an event of probability $1 - \epsilon$:

$$L(r, \theta^*(p)) \leq \rho_1 \text{ whenever } r \leq \frac{p}{1 - \epsilon},$$

$$L(p, \theta) \leq \rho_2 \text{ whenever } L(r, \theta) \leq \rho_1 \text{ and } r \leq \frac{p}{1 - \epsilon}.$$  

Let $\mathcal{G}_l(\rho_1, \epsilon)$ and $\mathcal{G}_r(\rho_1, \rho_2, \epsilon)$ be the distributions satisfying (↓) and (↑), respectively. We define the family of $(\rho_1, \rho_2, \epsilon)$-resilient distributions to be the intersection $\mathcal{G}_{TV} = \mathcal{G}_l(\rho_1, \epsilon) \cap \mathcal{G}_r(\rho_1, \rho_2, \epsilon)$. We show (Section 3.2) that this family has modulus bounded by $\rho_2$, and that it specializes to the family $\mathcal{G}_{TV}^{\text{mean}}(\rho, \epsilon)$ for mean estimation by taking $\rho_1 = \rho$, $\rho_2 = 2\rho$. [9]
While the conditions (↓) and (↑) may appear abstract, they yield sensible conditions in concrete cases. For instance, they yield the result in Section 1.1 on linear regression under bounded moments and hypercontractivity, and imply or improve most known infinite-sample results in the literature.

Resilience for Wasserstein distances. (Section 4) We next show how to extend the idea of resilience to Wasserstein distances $W_{c,k}$ (see Section 2.1 for a definition of Wasserstein distance). For TV distance, we showed that resilient sets have bounded modulus $m$; this crucially relied on the midpoint property that any $p_1, p_2$ have a midpoint $r$ obtained via deletions of $p_1$ or $p_2$. In other words, we used the fact that any TV perturbation can be decomposed into a “friendly” operation (deletion) and its opposite (addition). We think of deletion as friendlier than addition, as the latter can move the mean arbitrarily far by adding probability mass at infinity.

To extend this to other Wasserstein distances, we need to identify a similar way of decomposing a Wasserstein perturbation into a friendly perturbation and its inverse. Unfortunately, deletion is closely tied to the TV distance in particular. To get around this, we use the following re-interpretation: Deletion is equivalent to movement towards the mean under TV. More precisely:

$\hat{\mu}$ is a possible mean of an $\epsilon$-deletion of $p$ if and only if some $r$ with mean $\hat{\mu}$ can be obtained from $p$ by moving points towards $\hat{\mu}$ with TV distance at most $\epsilon$.

This is more easily seen in the following diagram:

Here we can equivalently either delete the left tail of $p$ or shift all of its mass to $\mu_r$; both yield a modified distribution with the same mean $\mu_r$. This motivates the following informal definition:

A distribution $r$ is an $\epsilon$-friendly perturbation of $p$ for a function $f(x)$ and distance $W_{c,k}$ if and only if one can transport $X \sim p$ to $Y \sim r$ with cost no more than $\epsilon$ while ensuring $f(Y)$ is between $f(X)$ and $\mathbb{E}_r[f(Y)]$ almost surely.

Note that friendliness is defined only in terms of one-dimensional functions $f : \mathcal{X} \to \mathbb{R}$; we will see how to handle higher-dimensional objects later. Intuitively, a friendly perturbation is a distribution $r$ for which there exists a coupling that ‘squeezes’ $p$ to $\mu_r$.

The key property of friendliness is the midpoint lemma (formal version Lemma 4.1), which states that every pair of nearby distributions has a friendly midpoint, analogously to the TV case:

**Lemma (informal).** Given any $p, q$ with $W_{c,k}(p,q) \leq \epsilon$ and any $f$, there exists an $r$ that is an $\epsilon$-friendly perturbation of both $p$ and $q$ for the function $f$.

We generalize resilience to Wasserstein distances by saying that a distribution is resilient if $\mathbb{E}_r[f(X)]$ is close to $\mathbb{E}_p[f(X)]$ for every $\eta$-friendly perturbation $r$ and every function $f$ lying within
some appropriate family $\mathcal{F}$. For instance, for second moment estimation we would consider functions $f_v(x) = \langle x, v \rangle^2$ with $\|v\|_2 = 1$. For more general losses $L(p, \theta)$, we show how to obtain an appropriate family $\mathcal{F}$ via the Fenchel-Moreau representation (Borwein and Lewis, 2010) of $L$, as long as $L$ is convex in $p$ for fixed $\theta$. Convexity in $p$ is a mild condition that often holds, e.g. any loss of the form $L(p, \theta) = E_{X \sim p}[\ell(\theta; X)]$ is linear (and hence convex) in $p$.

This construction is again somewhat abstract. The payoff is that it yields robust estimators in concrete cases, such as the $W_1$ linear regression example in Section 1.1. In Section 4 we will formally define friendly perturbations and resilience for $W_{c,k}$-perturbations; we will also present estimators and general sufficient conditions for robust second moment estimation and linear regression under $W_1$-perturbations. These are the first $W_1$-robust estimators we are aware of for both problems.

**Finite sample algorithm and analysis (Section 5).** All of the estimators discussed above use the minimum distance functional, which projects onto $\mathcal{G}$ under the distance $D$. In finite samples, the projection

$$q = \arg \min_{q \in \mathcal{G}} D(q, \hat{p}_n)$$

presents issues when $\hat{p}_n$ is the empirical distribution. For instance, when $D = TV$ the projection doesn’t make sense when $\mathcal{G}$ consists of continuous distributions, since the TV distance between any empirical distribution and any continuous distribution is 1. We present two approaches to resolving this, based on weakening the distance $D$ or expanding the set $\mathcal{G}$.

*First approach: weaken the distance.* Intuitively, the issue is that the TV distance is too fine—it reports a large distance even between a population distribution $p$ and the finite-sample distribution $\hat{p}_n$. A solution to this is to relax the distance. We will replace TV with the smaller distance

$$\tilde{TV}_H(p, q) \triangleq \sup_{f \in H, t \in \mathbb{R}} |\mathbb{P}_p(f(X) \geq t) - \mathbb{P}_q(f(X) \geq t)|. \tag{10}$$

We then apply the minimum distance functional under this smaller distance. Note that $\tilde{TV}$ is smaller than TV because it takes a supremum over only the events defined by threshold functions in $H$, while TV takes the same supremum over all measurable events.

To show that projection under $\tilde{TV}$ works, we need to check two properties:

- The modulus $\mathfrak{m}(\mathcal{G}, \epsilon)$ is still bounded when replacing TV with $\tilde{TV}_H$.
- The distance $\tilde{TV}_H(p, \hat{p}_n)$ between $p$ and its empirical distribution is small.

In fact, as long as the two properties hold, the $\tilde{TV}$ projection algorithm works under $\tilde{TV}$ perturbations, which can be much larger than TV perturbations. We establish the first property via a mean crossing lemma (Lemma 5.2) showing that any pair of resilient distributions that are close in TV have $\epsilon$-deletions whose means cross each other. This enables us to show that the modulus is small under $\tilde{TV}$ for most of the $\mathcal{G}^{TV}$ we designed. The second property is an instance of the VC inequality (Vapnik and Chervonenkis, 2015; Dudley, 1978), which implies that $\tilde{TV}_H(p, \hat{p}_n) = O(\sqrt{d/n})$ with high probability when the family of sets $\{\{x \mid f(x) \geq t\} \mid f \in H, t \in \mathbb{R}\}$ has VC-dimension $d$.

As an example, for mean estimation we take $H$ to consist of functions of the form $f(x) = v^\top x$, while for linear regression we take functions of the form $f(x) = (v_1^\top x)^2 - (v_2^\top x)^2$. In both cases the VC dimension is $O(d)$, yielding results such as the finite-sample bound for linear regression in Section 1.1. We discuss these in detail in Section 5.1.
In Appendix E.5 we discuss optimization aspects of $\tilde{TV}_H$, showing that it can be smoothed to produce a new distance that it closely related to generative adversarial networks (Goodfellow et al., 2014), extending observations in Gao et al. (2019) on connections between robustness and GANs.

**Second approach: expand the set.** An alternative approach to saving the minimum distance functional in (9) is to expand the destination set $G$ to some larger set $M$ (Lai et al., 2016; Diakonikolas et al., 2019a, 2018a; Steinhardt et al., 2018; Steinhardt, 2018). The idea is that even though $p^*$ is not close to the empirical distribution $\hat{p}_n$, some element of $M$ is close to $\hat{p}_n$. For this analysis strategy we need to show three things:

- $\mathcal{M}$ is large enough: $\hat{p}_n^* \in \mathcal{M}$ with high probability (here $\hat{p}_n^*$ is the uncorrupted empirical distribution).
- The modulus is still bounded: $\mathfrak{m}(\mathcal{M}, \epsilon)$ is small.
- The empirical loss $L(\hat{p}_n^*, \theta)$ is a good approximation to the population loss $L(p^*, \theta)$.

We actually need slightly weaker properties than the three above, as we discuss in Section 5.2; for instance we only need to bound a certain generalized modulus that is smaller than $\mathfrak{m}(\mathcal{M}, \epsilon)$.

A core challenge to this analysis strategy is that the empirical distribution $\hat{p}_n$ may not inherit good properties from $p^*$ in a reasonable number of samples. Typical ways of measuring tail behavior (such as the empirical moments) require more than $d$ samples to converge: for instance, the $k$-th moments require at least $d^{k/2}$ samples to converge (Appendix E.6.3).

To address this, we instead show that certain truncated moments of $\hat{p}_n^*$ are bounded with high probability given only $n = d$ samples. For instance, if $p^*$ has 4-th moments bounded by $\sigma^4$ then we will bound $\sup_{\|v\| \leq 1} \mathbb{E}_{\hat{p}_n}[\psi(|v^T X|)]$, where $\psi(x)$ is the smallest convex function on $[0, \infty)$ that coincides with $x^4$ when $0 \leq x \leq 4\sigma$. Bounding these truncated moments allows us to correct a weakness in a previous analysis (Steinhardt et al., 2018) that required $d^{3/2}$ samples for convergence.

Another technique shows that one may construct another distribution $\hat{p}'$ that deletes a small fraction of heavy-tailed “bad” events as long as any $\theta$ that performs well on $\hat{p}'$ also works well on $p^*$ (Steinhardt et al., 2017a; Diakonikolas et al., 2018a,c; Cheng et al., 2019a,b). With this modification, one can show that $\hat{p}'$ has covariance bounded by $O(\sigma)$ with $d \log d$ samples. We combine this technique with moment truncation to design a potentially efficient algorithm for $k$-th moment bounded isotropic distributions (Theorem 5.6), which guarantees error $O(\epsilon^{1-1/k})$ given $O(d \log(d)/\epsilon^2 - 2/k)$ samples.

**Finite sample algorithm for $W_1$ perturbation** Finally, we produce finite sample algorithms under $W_1$ perturbation by weakening $W_1$ to a distance $\tilde{W}_1$, similarly to the $TV \to \tilde{TV}$ weakening discussed above. The Kantorovich-Rubinstein duality yields the representation $W_1(p, q) = \sup \{ \mathbb{E}_p[f(X)] - \mathbb{E}_q[f(X)] \mid f \text{ is $1$-Lipschitz} \}$, so to weaken the $W_1$ distance we want to pick out a subset of 1-Lipschitz functions.

As in the TV case, the key step is to establish a mean cross lemma showing that if $\tilde{W}_1(p, q) \leq \epsilon$ then for a function $f$ there are $\epsilon$-friendly perturbations $r_p$, $r_q$ of $p$ and $q$ such that $\mathbb{E}_{r_p}[f(X)] \leq \mathbb{E}_{r_q}[f(X)]$. While before we proved this for all functions, here we only prove it for convex functions. The main idea is to use the integral representation of convex functions (Lukeš et al., 2009), which states that any convex function can be decomposed into a weighted integral of linear or rectified linear 1-Lipschitz functions. We then establish the mean cross lemma for this family of functions, which are a subset of all Lipschitz functions and hence provide a weakening $\tilde{W}_1$ of the $W_1$ distance. This is presented in Section 6. However, our bounds are likely loose and finite-sample analysis for $W_1$ and other non-TV discrepancies is largely open.
1.3 Open Problems

We hope our work has opened more doors than it closes. Except for some of the analyses in this paper that can be tightened, there are three major directions of future work.

**Design of $\mathcal{G}$ for other discrepancy $D$.** Robust estimation under TV perturbation is well-understood, while here we provide a principled way of designing $\mathcal{G}$ when $D = TV$ and extend it to $D = W_{c,k}$. What is the maximal $\mathcal{G}$ that can control population limit and guarantee the existence of good finite sample algorithm? How can we go beyond $W_{c,k}$? Can we exploit other geometric structure such as $f$-diverges, Bregman diverges, or general norms? At least some generalization should be possible, since we also give robust estimators for $\tilde{TV}_H$, which cannot be expressed as $W_{c,k}$.

**Finite sample algorithms for $W_{c,k}$ perturbation.** We provide two approaches for finite sample algorithm design under TV perturbation: weakening the distance and expanding the destination set. Can we do the same for $W_{c,k}$? General $W_{c,k}$ loses a unique property of TV: deleting $p_X$ under TV is equivalent to deleting $p_{f(X)}$ under TV for any $f$. Can we overcome this barrier by exploiting other properties? More generally, what is the optimal finite sample rate for the robust inference problem under $W_{c,k}$ perturbation?

**Design of computationally efficient algorithms.** We provide a principled way of designing statistically near-optimal algorithms; we would like to have a similarly principled method for designing computationally efficient algorithms. What minimum distance functionals are efficiently solvable? How can we design $\mathcal{G}$ with computational efficiency in mind? How large can $\mathcal{G}$ be in this case? Hopkins and Li (2019) showed that the requirement of computational efficiency may necessitate stronger assumptions and hence smaller $\mathcal{G}$.

1.4 Related Work

Robust statistics has a long history. A large variety of concrete inference problems are studied, such as robust mean estimation (Huber, 1973, 2011; Diakonikolas et al., 2017; Steinhardt et al., 2017a; Gao et al., 2018), robust classification (Huber, 2011; Klivans et al., 2009; Biggio et al., 2012; Awasthi et al., 2014; Steinhardt et al., 2017b; Diakonikolas et al., 2018c, 2019b), robust regression (Huber, 2011; Diakonikolas et al., 2018b; Rousseeuw and Leroy, 1987; Huber, 1973; Gao, 2017; Liu et al., 2018) and robust clustering (Guha et al., 2000; Davé and Krishnapuram, 1997). Below we discuss the literature that is most directly related to our paper.

**Minimum distance (MD) functional.** The minimum distance functional approach (Lemma 2.2) was developed by Donoho and Liu (1988), with precursors in Beran et al. (1977); Millar (1981). It was known that $M$-estimators in general cannot achieve high breakdown point in high dimensions (Maronna, 1976; Donoho, 1982), while minimum distance functionals are “automatically robust” (Donoho and Liu, 1988; Davies et al., 1992). Our key contributions beyond Donoho and Liu (1988) are to construct explicit nonparametric sets and compute the modulus for both TV and $W_{c,k}$ (Section 3 and 4), construct new MD functionals and improve the existing analysis, and connect the abstract MD functional with computational efficiency (Section 5 and 6).
**Design of \( \mathcal{G} \).** Our design of \( \mathcal{G} \) for TV perturbation is most related to Steinhardt et al. (2018); Steinhardt (2018). To our knowledge, a systematic study of robust estimation under Wasserstein perturbations did not exist prior to our work. The midpoint lemma (Lemma 4.1) for Wasserstein perturbation is related to but more complex than the Wasserstein interpolants (Ambrosio et al., 2008, Section 7.2), where the middle point is defined by transporting not from \( x \) to \( y \) in the original coupling, but from \( x \) to \((1 - t)x + ty \) for some \( t \in [0, 1] \).

**Finite sample algorithm and analysis.** The approach of weakening the distance was mainly studied in the statistics literature (Donoho and Liu, 1988; Chen et al., 2018; Gao et al., 2018, 2019), while the approach of expanding the set was mainly studied in the theoretical computer science literature and led to provably computationally efficient estimators for robust mean/covariance estimation (Diakonikolas et al., 2019a), classification (Klivans et al., 2009; Awasthi et al., 2014; Diakonikolas et al., 2018c), and regression (Klivans et al., 2018; Diakonikolas et al., 2019c). Here we unify the two approaches and provide systematic finite sample analysis for different estimation scenarios, and use the two approaches as a guide to design new algorithms.

The term “generalized Kolmogorov–Smirnov distance” was used in Devroye et al. (2013) to denote \( \sup_{A \in \mathcal{A}} |p(A) - q(A)| \) for any \( \mathcal{A} \). Our \( \overline{TV}_H \) is closer to the original KS distance since it is based on distances between cumulative distribution functions.

Our work on using GANs for robust inference (Appendix E.5) is inspired by Gao et al. (2019).

**Connection to distributionally robust optimization.** Distributionally robust optimization (DRO) solves the min-max problem

\[
\inf_{\theta} \sup_{q: D(q,p) \leq \epsilon} L(q; \theta). \tag{11}
\]

This is similar to our setting but omits the assumptions \( \mathcal{G} \) (or equivalently, takes \( \mathcal{G} \) to be all probability distributions). Consequently, (11) is in many cases not defined. For instance, when \( D = TV \) and \( L \) is any unbounded loss, the supremum is infinite for all \( \theta \); similarly for \( D = W_1 \) and \( L \) the linear regression loss. As a consequence, DRO typically considers bounded loss functions (Duchi and Namkoong, 2018), takes \( D \) to be some \( f \) divergences that only allow a small family of perturbations (Delage and Ye, 2010; Namkoong and Duchi, 2016), or takes \( L \) to be Lipschitz and \( D = W_2 \) (Volpi et al., 2018).

More conceptually, the optimal \( q \) in (11) will typically push outlying points to be even more outlying, and thus magnifies the influence of outliers, which is counter to our goal of resisting the effects of corruptions. On the other hand, if we restrict \( q \) in (11) to lie in \( \mathcal{G} \), we do resist outliers and in fact this modified DRO is the min-max optimal estimator in infinite samples under our framework, underscoring the importance of the assumptions \( \mathcal{G} \). We analyze a projection estimator below rather than DRO as it more readily admits bounds especially in finite samples, but study a DRO-based estimator in Appendix G.

**Connection to agnostic distribution learning.** Agnostic distribution learning (Yatracos, 1985; Devroye and Lugosi, 2012; Chan et al., 2014; Acharya et al., 2017; Zhu et al., 2019) concerns finding the distribution \( p^* \in \mathcal{G} \) that is closest to distribution \( p \) when only empirical samples from \( p \) are observed. The “Yatracos” method in agnostic learning constructs a distance \( \overline{TV} \leq TV \) such that \( \overline{TV}(p,q) \approx TV(p,q) \) for all \( p, q \in \mathcal{G} \) and then projects the empirical distribution \( \hat{p}_n \) to \( \mathcal{G} \) under \( \overline{TV} \). Zhu et al. (2019) show how to use agnostic learning algorithms to achieve robust inference when \( D = L = TV \). A key difference from our work is that agnostic distribution learning always
tak es $D = L$. For large sets $G$ such as the resilient set (8), there does not exist some $\widetilde{TV}$ weaker than $TV$ such that $\widetilde{TV}(p,q) = TV(p,q)$ for all $p,q \in G$. However, one can in fact apply the Yatracos method on top of our $TV_H$, by taking the loss in agnostic learning to be $\widetilde{TV}_H$.

## 2 Preliminaries

In this section, we provide definitions for frequently used terms throughout the paper, introduce the two corruption models considered in this paper, and discuss the corresponding information theoretic limits.

### 2.1 Some Definitions

We present here some definitions and notation that will be used throughout the paper. The rest are collected in the beginning of Appendix A. First, a pseudometric is a function $d: X \times X \to \mathbb{R}_+$ satisfying the following three properties: $d(x,x) = 0$, $d(x,y) = d(y,x)$ (symmetry), and $d(x,z) \leq d(x,y) + d(y,z)$ (triangle inequality). Unlike a metric, one may have $d(x,y) = 0$ for $x \neq y$. Similarly, a function $\parallel \cdot \parallel : X \to \mathbb{R}_+$ is a pseudonorm if $\parallel 0 \parallel = 0$, $\parallel cx \parallel = |c| \parallel x \parallel$ (homogeneity), and $\parallel x + y \parallel \leq \parallel x \parallel + \parallel y \parallel$ (triangle inequality).

For a pseudometric $c$, the Wasserstein distance is the minimum-cost matching between $p$ and $q$ according to $c$:

**Definition 2.1** ($W_{c,k}$ distance (Villani, 2003, Theorem 7.3)). Suppose $c(x,y)$ is a pseudometric. The Wasserstein-$k$ transportation distance for $k \geq 0$ is defined as

$$ W_{c,k}(p,q) = \begin{cases} \inf_{\pi \in \Pi(p,q)} \left( \int c^k(x,y)d\pi_{p,q}(x,y) \right)^{1/k} & k \geq 1 \\ \inf_{\pi \in \Pi(p,q)} \left( \int c^k(x,y)d\pi_{p,q}(x,y) \right) & k \in [0,1) \end{cases}, \quad (12) $$

where $\Pi(p,q)$ denotes the set of all couplings between $p$ and $q$.

If $c$ is a pseudometric, then so is $W_{c,k}$ (Villani, 2003, Page 209). If $c(x,y)$ is the Euclidean distance $\|x - y\|_2$, we usually omit the subscript $c$ and write $W_k$.

We will also want a way of measuring the tails of a distribution. We do so using Orlicz norms:

**Definition 2.2** (Orlicz function and norm). A function $\psi : [0, +\infty) \mapsto [0, +\infty)$ is called an Orlicz function if $\psi$ is convex, non-decreasing, and satisfies $\psi(0) = 0$, $\psi(x) \to \infty$ as $x \to \infty$. For a given Orlicz function $\psi$, the Orlicz norm of a random variable $X$ is defined as

$$ \|X\|_\psi \triangleq \inf \{ t > 0 : \mathbb{E}_p[\psi(\|X\|/t)] \leq 1 \}. \quad (13) $$

As special cases, we say that a random variable $X \sim p$ is sub-Gaussian with parameter $\sigma$ if $\|X - \mathbb{E}_p[X], v\|_{\psi_2} \leq \sigma$ whenever $\|v\|_2 \leq 1$, where $\psi_2(x) = e^{x^2} - 1$. We define a sub-exponential random variable similarly for the function $\psi_1(x) = e^{x^2} - 1$.

We define the generalized inverse of a non-decreasing function $\psi$ as

$$ \psi^{-1}(y) = \inf \{ x \mid \psi(x) > y \}, \quad (14) $$

which differs from the typical definition in the literature (Embretichs and Hofert, 2013) by replacing $\geq$ with $>$, since we want to ensure that $\psi(x) \leq y \Rightarrow x \leq \psi^{-1}(y)$.

Finally, for univariate random variables $X$ and $Y$, we say that $Y$ stochastically dominates $X$ (in first order) if $\mathbb{P}(X \leq t) \geq \mathbb{P}(Y \leq t)$ for all $t \in \mathbb{R}$ (Marshall et al., 1979, Definition B.19.a, B.19.b). We denote this as $X \leq_{sd} Y$ or $P_X \leq_{sd} P_Y$. It implies that $\mathbb{E}[X] \leq \mathbb{E}[Y]$. 


2.2 Corruption Models

We model our distribution as being adversarially corrupted within some $\epsilon$-ball according to a discrepancy $D$. There are two ways to formalize this: corruptions could occur either to the population distribution (with samples drawn from the corrupted distribution), or to the empirical distribution on $n$ samples (with samples originally drawn from the true distribution $p^*$ and then perturbed). Intuitively, the latter defines a more powerful adversary that is allowed to make decisions after seeing the random draw generating the data. It is related to the finite sample robustness model of (Donoho and Liu, 1988) and the full adversary setup in Diakonikolas et al. (2019a). We accordingly call the first model the oblivious model and the second the adaptive model:

**Definition 2.3** (Oblivious Corruption). Denote the true distribution as $p^*$. The oblivious adversary with level $\epsilon$ under $D$ is allowed to do the following. It first perturbs $p^*$ to $p$ such that $D(p^*, p) \leq \epsilon$, then takes $n$ i.i.d. samples from $p$ to get $(X_1, X_2, \ldots, X_n)$. We denote the empirical distribution of the observations $(X_1, X_2, \ldots, X_n)$ as $\hat{p}_n$.

We illustrate this in Figure 3.

![Oblivious corruption model](image-url)

Figure 3: Oblivious corruption model.

For adaptive corruption, the adversary can perturb the observed samples after observing the empirical samples from the real distribution $p^*$. A naïve attempt to define this model would constrain $D(\hat{p}_n^*, \hat{p}_n) \leq \epsilon$, where $\hat{p}_n^*$ is the empirical distribution of $n$ samples from $p^*$. However, consider the case $D = \text{TV}$, where changing an $\epsilon$ fraction of $p^*$ will affect a binomially-distributed $\text{Bin}(n, \epsilon)$ number of samples from the empirical distribution. The naïve definition would have the adaptive adversary change no more than exactly $\epsilon n$ samples, leading to an adversary that is neither stronger nor weaker than the oblivious adversary.

Instead, we define adaptive corruption in terms of requiring that the amount of corruption in empirical samples measured by $D$, which is $D(\hat{p}_n^*, \hat{p}_n)$, is stochastically dominated by that incurred by some oblivious perturbation with level $\epsilon$. It ensures that the adaptive corruption model is a stronger model than oblivious corruption with the same level $\epsilon$. In the special case of $D = \text{TV}$, it recovers the full adversary setup in Diakonikolas et al. (2019a).

**Definition 2.4** (Adaptive Corruption). Denote the true distribution as $p^*$. The adaptive adversary with level $\epsilon$ under $D$ is allowed to do the following. We take $n$ i.i.d. samples from $p^*$ to get the empirical distribution $\hat{p}_n^*$. The adversary inspects $\hat{p}_n^*$ and produces another empirical distribution supported on $n$ points, denoted as $\hat{p}_n$. The conditional distribution $\hat{p}_n \mid \hat{p}_n^*$ is allowed if there exists some coupling $\pi_{X,Y}$ satisfying $\pi_X = p^*, D(\pi_X, \pi_Y) \leq \epsilon$ such that $D(\hat{p}_n^*, \hat{p}_n)$ is stochastically dominated by $D\left(\frac{1}{n} \sum_{i=1}^n \delta_{X_i}, \frac{1}{n} \sum_{i=1}^n \delta_{Y_i}\right)$, where $\{(X_i, Y_i)\}_{i=1}^n$ denotes $n$ i.i.d. samples from $\pi_{X,Y}$.

We illustrate the adaptive corruption model in Figure 4.

![Adaptive corruption model](image-url)

Figure 4: Adaptive corruption model.

The following lemma is useful for controlling the behavior of adaptive corruptions:
Lemma 2.1. Suppose $\hat{p}_n | \hat{p}^*_n$ is an allowed adaptive corruption with level $\epsilon$ under $D = W_{c,k}$, $k \geq 1$ (Definition 2.1). Then the perturbed and true empirical distribution are close in expectation:

$$\mathbb{E}[W^k_{c,k}(\hat{p}_n, \hat{p}^*_n)] \leq \epsilon^k.$$

(15)

If additionally $0 \leq c(x, y) \leq C$ for all $x, y$, then with probability at least $1 - \delta$,

$$W_{c,k}(\hat{p}_n, \hat{p}^*_n) \lesssim \left( \epsilon^k + \frac{C^k \log(1/\delta)}{n} \right)^{1/k}.$$

(16)

The proof and a tighter bound when $D = \text{TV}$ are given in Appendix B.1.

2.3 Population (Infinite-Sample) Limit

It will be helpful to consider the infinite-sample regime where $\hat{p}^*_n = p^*$ and $\hat{p}_n = p$; in this case both corruption models defined above become identical. We will be interested in understanding the population limit, or the best (worst-case) error achievable by any estimator $\theta(p)$:

**Definition 2.5** (Population limit with $(\mathcal{G}, D, L, \epsilon)$). Given distribution family $\mathcal{G}$, discrepancy $D$, loss $L$, and perturbation level $\epsilon$, the population limit for the robust inference problem is

$$\inf_{\theta(p)} \sup_{p^*, p \in \mathcal{G}} L(p^*, \theta(p)).$$

(17)

This population limit is defined in Donoho and Liu (1988, Prop. 4.2).

**Modulus of continuity bounds population limit.** The following bound on the population limit will guide our design of $\mathcal{G}$ in the sequel:

**Lemma 2.2.** (Donoho and Liu, 1988) Suppose $D$ is a pseudometric. Then the population limit in (17) is at most the maximum loss between any pair of distributions in $\mathcal{G}$ of distance at most $2\epsilon$:

$$m(\mathcal{G}, 2\epsilon, D, L) \equiv \sup_{p_1, p_2 \in \mathcal{G} : D(p_1, p_2) \leq 2\epsilon} L(p_1, \theta^*(p_2)).$$

(18)

The upper bound is achieved by the projection (minimum distance functional) algorithm $q = \arg \min_{q \in \mathcal{G}} D(q, \cdot)$, where we output $\theta = \theta^*(q) = \arg \min_{\theta \in \Theta} L(q, \theta)$.

The quantity $m$ is called the gauge function or modulus of continuity (Donoho and Liu, 1988; Chen et al., 2018). Later we omit $D, L$ in the parameters if they are obvious from context. The projection procedure in Lemma 2.2 is oblivious to the cost $L$ as well as the perturbation level $\epsilon$, which are desirable features in practice. When we know the perturbation level $\epsilon$, then the upper bound also holds if output $\theta^*(q)$ for any $q \in \mathcal{G}$ such that $D(p, q) \leq \epsilon$. We show in Lemma B.2 that the projection algorithm is optimal up to constants if $L(p, \theta^*(q))$ is a pseudometric over $(p, q)$, and the modulus of continuity is a nearly tight upper bound if we further assume $D(p, q)$ is a pseudonorm such as TV or $W_1$.

3 Resilience: Design of $\mathcal{G}$ when $D = \text{TV}$

Among the parameters $(\mathcal{G}, D, L, \epsilon)$ for the population limit in Definition 2.5, the perturbation discrepancy $D$ and recovery cost $L$ specify our recovery goals, while $\mathcal{G}$ encodes knowledge about the true distribution $p^*$. We would like to design $\mathcal{G}$ that satisfies the following properties:
1. *Not too big:* the population limit in (17) is small;

2. *Not too small:* \( G \) is a superset of most of the known assumed distributions in the literature on robust statistics;

3. *Near-optimal finite sample algorithm:* we can design a general finite-sample algorithm that guarantees robust inference for the \( G \) we designed, and it is near-optimal for known cases.

We design \( G \) to meet the first two requirements for \( D = \text{TV} \) in this section, and generalize to Wasserstein distances in Section 4. We discuss finite sample algorithms in Sections 5 and 6.

We saw in Lemma 2.2 that the population limit of any set \( G \) is at most its modulus of continuity \( m \). Thus to show the set is *not too big*, we will upper bound the modulus of continuity. To show \( G \) is *not too small*, we study \( G \) in concrete cases such as mean estimation, linear regression and joint mean and covariance estimation, and show that in all the cases, our \( G \) is a superset of known distributional assumptions in the literature.

### 3.1 Warm-up: Mean Estimation when \( D = \text{TV} \)

As a warm-up, we consider the special case where \( D = \text{TV} \) and \( L(p, \theta) = \|\theta - E_p[X]\| \) for some norm \( \| \cdot \| \); this setting is called mean estimation under TV perturbations (Huber, 1973; Diakonikolas et al., 2017; Gao, 2017; Steinhardt et al., 2018; Donoho and Gasko, 1992).

According to Lemma 2.2, we want to construct a \( G \) such that any \( p, q \in G \) that are close in TV also have similar means. Steinhardt et al. (2018) introduced the set of resilient distributions which exactly satisfy this property and will be our starting point for more general settings. A distribution is called resilient if every \((1 - \eta)\)-subset of the distribution has a similar mean. More formally, the set of resilient distributions with parameters \((\rho, \eta)\) are

\[
G_{\text{mean}}^{\text{TV}}(\rho, \eta) \triangleq \{ p : \|E_r[X] - E_p[X]\| \leq \rho \text{ for all } r \leq \frac{p}{1 - \eta} \}, \tag{19}
\]

where \( \| \cdot \| \) is an arbitrary norm. The inequality \( r \leq \frac{p}{1 - \eta} \) can be formally understood as \( \frac{dr}{dp} \leq \frac{1}{1 - \eta} \), where \( \frac{dr}{dp} \) is the Radon-Nikodym derivative; an equivalent characterization is that \( r \) can be obtained from \( p \) by conditioning on an event \( E \) of probability at least \( 1 - \eta \), which is shown in Lemma C.1. Thus \( r \) can be thought of as an “\( \eta \)-deletion” of \( p \).

**Not too big.** We first bound the modulus of continuity \( m \) of \( G_{\text{mean}}^{\text{TV}} \). The key idea is that any two distributions that are close in TV have a midpoint distribution that is a deletion of both distributions (Lemma C.1). The midpoint distribution has mean close to both distributions via resilience, and it follows from the triangle inequality that the modulus of continuity \( m \) is bounded. We illustrate this idea in Figure 5 and make it precise in the following lemma:

**Lemma 3.1.** The modulus of continuity \( m \) in (18) for \( G_{\text{mean}}^{\text{TV}}(\rho, \eta) \) satisfies the bound \( m(\rho, \eta), 2\epsilon) \leq 2\rho \) for any \( 2\epsilon \leq \eta < 1 \).

By Lemma 2.2, this implies the population limit for \( G_{\text{mean}}^{\text{TV}}(\rho, \eta) \) is at most \( 2\rho \) when \( 2\epsilon \leq \eta < 1 \).

**Proof of Lemma 3.1.** Denote \( \mu_{p_1} = E_{p_1}[X], \mu_{p_2} = E_{p_2}[X] \). Recall that

\[
m(\rho, \eta), 2\epsilon) = \sup_{(p_1, p_2) : \text{TV}(p_1, p_2) \leq 2\epsilon, p_1, p_2 \in G_{\text{mean}}^{\text{TV}}(\rho, \eta)} \|\mu_{p_1} - \mu_{p_2}\|. \tag{20}
\]
for a formal proof). It then follows from (21) that the population limit is $\Theta(\sigma \epsilon \psi^{-1}(1/\epsilon))$ for $\epsilon < \frac{1}{4}$.

The proof is given in Appendix C.3. The first property shows that bounded Orlicz norm implies resilience, while the second shows that the bound implied by resilience (Lemma 3.1) is tight.

As a special case, by taking $\psi = \exp(x^2) - 1$, we see that when $p$ is sub-Gaussian with parameter $\sigma$ then $p \in G_T(\Theta(\sigma \epsilon \sqrt{\log(1/\eta)}), \eta)$ for $\eta \leq \frac{1}{2}$, and the population limit is $\Theta(\sigma \epsilon \sqrt{\log(1/\epsilon)})$ for $\epsilon \leq \frac{1}{4}$. By taking $\psi = \exp(x) - 1$, we similarly see that when $p$ is sub-exponential with parameter $\sigma$ then the population limit is $\Theta(\sigma \epsilon \log(1/\epsilon))$. Taking taking $\psi = x^k$ for $k \geq 1$, we see that when $p$ has its $k$-th central moment bounded by $\sigma^k$, the population limit is $\Theta(\sigma \epsilon^{1-1/k})$ for $\epsilon \leq \frac{1}{4}$.

### 3.2 General Resilient Set Design for $D = TV$

We saw above that robust mean estimation is possible for the family $G_{\text{mean}}^T$ of resilient distributions; the two key ingredients were the existence of a midpoint distribution and the triangle inequality for $L(p, \theta^*(q)) = ||\mu_p - \mu_q||$. We now extend the definition of resilience to arbitrary cost functions $L(p, \theta)$ that may not satisfy the triangle inequality. The general definition below imposes two conditions: (1) the parameter $\theta^*(p)$ should do well on all distributions $r \leq \frac{p}{1-\eta}$, and (2) any parameter that
does well on some \( r \leq \frac{p}{1-\eta} \) also does well on \( p \). We measure performance on \( r \) with a bridge function \( B(r, \theta) \), which is often the same as the loss \( L \) but need not be.

**Definition 3.1** \( G_{TV}(\rho_1, \rho_2, \eta) \). Given an arbitrary loss function \( L(p, \theta) \), we define \( G_{TV}(\rho_1, \rho_2, \eta) = G_\downarrow TV(\rho_1, \eta) \cap G_\uparrow TV(\rho_1, \rho_2, \eta) \), where:

\[
G_\downarrow TV(\rho_1, \eta) \triangleq \{ p \mid \sup_{r \leq \frac{p}{1-\eta}} B(r, \theta^*(p)) \leq \rho_1 \}, \tag{22}
\]

\[
G_\uparrow TV(\rho_1, \rho_2, \eta) \triangleq \{ p \mid \text{for all } \theta \in \Theta, r \leq \frac{p}{1-\eta}, (B(r, \theta) \leq \rho_1 \Rightarrow L(p, \theta) \leq \rho_2) \}, \tag{23}
\]

The function \( B(p, \theta) \) is an arbitrary cost function that serves the purpose of bridging. Here \( \theta^*(p) \in \arg \min_{\theta \in \Theta} B(p, \theta) \).

If we do not specify \( \Theta \), then it is clear from the context. The added flexibility in choosing \( B \) is important, for instance, in second estimation under Frobenius norm (Example 3.4), we choose \( B \) to be different from \( L \) due to finite sample consideration.

If we take \( B(p, \theta) = L(p, \theta) = ||E[p] - E[d]||, \rho_2 = 2\rho_1 \), then our design exactly reduces to the resilient set \( G_{mean}(\rho_1, \eta) \) for mean estimation discussed in Section 3.1.

As before, we will show that \( G_{TV} \) is not too big by bounding its modulus of continuity, and that it is not too small by showing that it subsumes various assumptions imposed in the literature.

**Not too big.** We show that the designed \( G_{TV}(\rho_1, \rho_2, \eta) \) has small modulus of continuity (and thus population limit by Lemma 2.2) in the following theorem, which extends Lemma 3.1.

**Theorem 3.1.** For \( G_{TV}(\rho_1, \rho_2, \eta) \) in Definition 3.1, if \( 2\epsilon \leq \eta < 1 \), we have

\[ m(G_{TV}(\rho_1, \rho_2, \eta), 2\epsilon) \leq \rho_2. \tag{24} \]

**Proof.** As illustrated in Figure 6, we still rely on the midpoint distribution \( r \) to bridge the modulus. Consider any \( p_1, p_2 \) satisfying \( TV(p_1, p_2) \leq 2\epsilon \leq \eta \). From Lemma C.1, there is a midpoint \( r \) such that \( r \leq \frac{p_1}{1-\eta} \) and \( r \leq \frac{p_2}{1-\eta} \). From the fact that \( p_1 \in G_{TV}(\rho_1, \rho_2, \eta) \subset G_\downarrow TV(\rho_1, \eta) \), we have \( B(r, \theta^*(p_1)) \leq \rho_1 \). From this and the fact that \( p_2 \in G_{TV}(\rho_1, \rho_2, \eta) \subset G_\uparrow TV(\rho_1, \rho_2, \eta) \), we then have \( L(p_2, \theta^*(p_1)) \leq \rho_2 \). Since \( p_1 \) and \( p_2 \) are arbitrary, this bounds the modulus of continuity by \( \rho_2 \). \( \square \)

\[
\begin{align*}
& \quad \quad \quad \quad p_1 \xrightarrow{TV(p_1, p_2) \leq \eta} p_2 \\
& p_1 \xrightarrow{\sup_{r \leq \frac{p_1}{1-\eta}} B(r, \theta^*(p)) \leq \rho_1} \xrightarrow{\rho_2 \leq \frac{p_2}{1-\eta}} \xrightarrow{\rho_2 \leq \frac{\min(p_1, p_2)}{1-\epsilon}} \xrightarrow{\rho_2} \xrightarrow{L(p_2, \theta^*(p_1)) \leq \rho_2} \\
& p_1 \in G_\downarrow TV \quad \Rightarrow \quad B(r, \theta^*(p_1)) \leq \rho_1 \\
& p_2 \in G_\uparrow TV \\
& L(p_2, \theta^*(p_1)) \leq \rho_2
\end{align*}
\]

Figure 6: Midpoint distribution helps bridge the modulus for \( G_{TV} \).

\[ ^1 \text{To see the reduction, note that } G_{mean} \text{ is equivalent to } G_\downarrow TV \text{ in Equation (22). Thus we only need to show that } G_\uparrow TV \text{ is a subset of } G_\downarrow TV. \text{ By our choice of } B, L \text{ and } \rho_2, \text{ the implication condition in } G_\uparrow TV \text{ follows from the triangle inequality.} \]
Not too small: Concrete examples for $G^{TV}$. Since the definition of $G^{TV}$ is somewhat abstract, we provide below sufficient conditions for inclusion in $G^{TV}$ in the case of linear regression, joint estimation of mean and covariance, and second moment estimation. In these cases $G^{TV}$ subsumes many known distribution families in the robust statistics literature, while at the same time Theorem 3.1 provides a bound on the population limit that matches or improves on known results.

Throughout, we will let $\psi$ denote an Orlicz function as in Example 3.1. Our first example gives concrete conditions for resilience in linear regression:

**Example 3.2 (Linear Regression).** Let $(X,Y) \sim p^*$ and take $B(p,\theta) = L(p,\theta) = \mathbb{E}_p[(Y - X^T \theta)^2 - (Y - X^T \theta^*(p))^2]$. Let $Z = Y - X^T \theta^*(p)$ denote the residual error. Assume the second moments of $X$ and $Z$ exist and satisfy the following conditions:

\[
\mathbb{E}_{p^*}[\psi(\frac{X^T Y}{\sigma^2_p})] \leq 1 \text{ for all } v \in \mathbb{R}^d, \text{ and } \mathbb{E}_{p^*}[\psi((Z/\sigma^2)^2)] \leq 1. \tag{25}
\]
\[
\[\mathbb{E}_{p^*}[\psi(\frac{X^T X}{\sigma^2_p})] \leq 1 \text{ for all } v \in \mathbb{R}^d, \text{ and } \mathbb{E}_{p^*}[\psi((Z/\sigma^2)^2)] \leq 1. \tag{26}
\]

Then $p^* \in G^{TV}(\rho,8\rho,\eta)$ for $\rho = 2(\sigma_1^2 e_{\psi^{-1}(1/\eta)}^2)^{1/2}$, for all $\eta$ satisfying $\frac{\sigma_1^2 e_{\psi^{-1}(1/\eta)}}{1-\eta} < \frac{1}{2}$. The population limit for the set satisfying (25) and (26) is $\Theta((\sigma_1 e_{\psi^{-1}(1/\rho)}^2)^2)$ when the perturbation level $\epsilon$ satisfies $\epsilon < 1/4$ and $\frac{\sigma_1^2 e_{\psi^{-1}(1/\rho)}}{1-2\epsilon} < 1/8$.

The proof is deferred to Appendix C.4. The condition (26) bounds the tails of the error $Z$, while (25) bounds the tails of $X$ in every direction relative to the second moment.

**Interpretation.** We interpret the dependence on $\epsilon$ in special cases. Klivans et al. (2018) shows that the following two conditions lead to $\epsilon^{1-2/k}$ error:

1. $k$-hyper-contractivity condition: $\mathbb{E}_{p^*}[|v^T X|^k] \leq (k\mathbb{E}_{p^*}[|v^T X|^2])^{k/2}$.
2. Bounded $k$-th moment of $Y - X^T \theta^*(p)$: $\mathbb{E}_{p^*}[|Y - X^T \theta^*(p)|^k] \leq \sigma^k$.

Taking $\psi(x) = x^{1/2}$, we recover these conditions and our result yields an improved bound of $\epsilon^{2-4/k}$. When $X$ and $Z$ are independent, we can improve this further to $\epsilon^{2-2/k}$, as we show in the proof.

When $X$ and $Z$ are both Gaussian, Gao (2017) shows that the information-theoretic limit is $\epsilon^2$. Taking $\psi(x) = \exp(x) - 1$, we see that when $\mathbb{E}_{p^*}[X X^T]^{-1/2} X$ and the noise $Y - X^T \theta^*(p)$ are both sub-Gaussian, the population limit is $O(\epsilon^2 \log^2(1/\epsilon))$. This also matches the rate in Diakonikolas et al. (2019c), where it is assumed that $X$ is isotropic Gaussian and the random noise $Z$ has bounded second moment.

**Lower bounds.** One might wonder whether a simpler condition such as sub-Gaussianity of $X$ and $Z$ would also guarantee a finite population limit. Even if $Z \equiv 0$, sub-Gaussianity of $X$ is not sufficient. In Appendix C.5 we exhibit a univariate sub-Gaussian $X$ for which an adversary can perturb $X$ to be zero almost surely, thereby destroying all information between $X$ and $Y$.

We next consider joint estimation of the mean and covariance of a distribution $p$. As in Kothari and Steurer (2017), we use the recovery metric

\[
L(p,(\mu,\Sigma)) = \max (\|\Sigma_p^{-1/2}(\mu_p - \mu)\|_2^2/\eta, \|I_d - \Sigma_p^{-1/2}\Sigma_p^{-1/2}\Sigma_p^{-1/2}\|_2), \tag{27}
\]

where $\mu_p$ and $\Sigma_p$ are the mean and covariance of $p$. Making $L$ small requires that the estimated covariance $\Sigma$ is close to $\Sigma_p$, and also that $\mu$ is close to $\mu_p$ under the norm induced by $\Sigma_p$. 21
Due to the non-linear dependence of $\Sigma_p$ on $p$, $L$ turns out to be unsuitable as a bridge function. For bridge function $B$ we need to use $\Sigma$ and $\mu$ rather than $\Sigma_p$ and $\mu_p$ so that $B$ is convex as a function of $p$,\(^2\) thus we instead define

$$B(p, (\mu, \Sigma)) = \max \left( \|\Sigma^{-1/2}(\mu_p - \mu)\|_2^2/\eta, \|I_d - \Sigma^{-1/2} \Sigma_p[(X - \mu)(X - \mu)^\top] \Sigma^{-1/2}\|_2 \right).$$

(28)

**Example 3.3** (Joint mean and covariance estimation). Consider $B$ and $L$ defined in (28) and (27). For $X \sim p$ if we have

$$\sup_{v \in \mathbb{R}^d, \|v\|_2 = 1} \mathbb{E}_p \left[ \psi \left( \frac{(v^\top (X - \mu_p))^2}{\kappa \mathbb{E}_p[(v^\top (X - \mu_p))^2]} \right) \right] \leq 1,$n

(29)

then $p \in \mathcal{G}^{TV}(\rho, \delta\rho, \eta)$ with $\rho = 4\kappa^2 \eta \psi^{-1}(1/\eta)$, assuming $\eta \leq \frac{1}{2}$ and $(1 + \eta)\rho < \frac{1}{2}$.

Thus when the perturbation level is $\epsilon \leq \frac{\eta}{2}$, we can recover $\mu, \Sigma$ such that $\|\Sigma_p^{-1/2}(\mu_p - \mu)\|_2 = O(\kappa \sqrt{\psi^{-1}(1/\epsilon)})$ and $\|I_d - \Sigma_p^{-1/2} \Sigma_p^{-1/2}\|_2 = O(\kappa^2 \epsilon \psi^{-1}(1/\epsilon))$.

The proof is deferred to Appendix C.7. Similarly to Example 3.2, we need the tails of $X$ to be small relative to the covariance; equivalently, we need $\Sigma_p^{-1/2}X$ to have thin tails.

**Interpretation.** First take $\psi(x) = x^k$; then when $\Sigma_p^{-1/2}X$ has $2k$-th central moment bounded by $\kappa 2^k$, we can estimate the mean and covariance with respective errors $O(\kappa \epsilon^{1 - 1/(2k)})$ and $O(\kappa^2 \epsilon^{1 - 1/k})$. This matches results in Kothari and Steurer (2017).

Next take $\psi(x) = e^x - 1$; then when $\Sigma_p^{-1/2}X$ is sub-Gaussian with parameter $\kappa$, we can estimate the mean and covariance with respective errors $O(\kappa \epsilon \sqrt{\log(1/\epsilon)})$ and $O(\kappa^2 \epsilon \log(1/\epsilon))$.

Our final example considers estimation of the second moment matrix $M_p = \mathbb{E}_p[XX^\top]$ under the loss $L(p, M) = \|I - M_p^{-1/2}MM_p^{-1/2}\|_F$, which was studied in Diakonikolas et al. (2017). We use the bridge function $B(p, M) = \|I - M_p^{-1/2}MM_p^{-1/2}\|_F$, which avoids non-convexity in $p$ similarly to Example 3.3. We can estimate the second moment if all quadratic forms in $X$ have bounded tails:

**Example 3.4** (Second moment estimation). Take $B$ and $L$ defined above. For a random variable $X \sim p$ with second moment $M_p$, let $Y = M_p^{-1/2}X$. Assume that $Y$ satisfies

$$\sup_{B \in \mathbb{R}^{d \times d}, \|B\|_F = 1} \mathbb{E}_p \left[ \psi \left( \frac{Y^\top BY - \mathbb{E}_p[Y^\top BY]}{\kappa} \right) \right] \leq 1.$$

(30)

Then $p \in \mathcal{G}^{TV}(\rho, 2\rho(1 + \rho), \eta)$ for $\rho = 4\kappa \eta \psi^{-1}(1/\eta)$ when $\eta \in [0, \frac{1}{2}]$ and $\rho(1 + \rho) < \frac{1}{2}$. Thus the population limit for the set is $O(\kappa \psi^{-1}(\frac{1}{\epsilon}))$ when the perturbation level is $\epsilon \leq \frac{\eta}{2}$.

The proof is deferred to Appendix C.8. When $\psi(x) = x^2$ condition (30) asks that $YY^\top$ has bounded covariance when considered as a $d^2$-dimensional random variable.

**Interpretation.** The population limit for zero mean Gaussian distributions is $\Theta(\epsilon)$ (Diakonikolas et al., 2019a). By taking $\psi(x) = \exp(x) - 1$, we generalize the Gaussian assumption to a larger non-parametric family with rate-optimal population limit $O(\epsilon \log(1/\epsilon))$, which matches the result in Diakonikolas et al. (2017).

Besides all the examples above, we show in Appendix C.9 that with different choices of $B$ and $L$, we are able to derive different $\mathcal{G}^{TV}$ and corresponding sufficient conditions for robust classification.\(^2\) This is only important for finite sample analysis in Section 5 (Details are deferred to Appendix E.3).
4 \textit{W}_{c,k}\text{-Resilience: Design of } G

In this section, we show how to extend the idea of resilience to Wasserstein distances \( \text{W}_{c,k} \) (Definition 2.1). We present a general recipe for constructing estimator that are robust to general Wasserstein perturbations \( \text{W}_{c,k} \), and apply them in the special case of \( \text{W}_1 \).

For \text{TV} distance, we showed that resilient sets have bounded modulus \( m \); this crucially relied on the midpoint property that any \( p_1, p_2 \) have a midpoint \( r \) obtained via \textit{deletions} of \( p_1 \) or \( p_2 \).

In other words, we used the fact that any \text{TV} perturbation can be decomposed into a “friendly” operation (deletion) and its opposite (addition). We think of deletion as friendlier than addition, as the latter can move the mean arbitrarily far by adding probability mass at infinity.

To extend this to other Wasserstein distances, we need to identify a similar way of decomposing a Wasserstein perturbation into a friendly perturbation and its inverse. Unfortunately, deletion is closely tied to the \text{TV} distance in particular. To get around this, we use the following reinterpretation: \textit{Deletion is equivalent to movement towards the mean under} \text{TV}. More precisely:

\[ \hat{\mu} \text{ is a possible mean of an } \epsilon\text{-deletion of } p \text{ if and only if some } r \text{ with mean } \hat{\mu} \text{ can be obtained from } p \text{ by moving points } \text{towards} \hat{\mu} \text{ with } \text{TV} \text{ distance at most } \epsilon. \]

This is more easily seen in the following diagram:

Here we can equivalently either delete the left tail of \( p \) or shift all of its mass to \( \mu_r \); both yield a modified distribution with the same mean \( \mu_r \). Thus we can more generally say that a perturbation is friendly if it only moves probability mass towards the mean. This motivates the following definition:

\textbf{Definition 4.1 (Friendly perturbation).} For a distribution \( p \) over \( \mathcal{X} \), fix a function \( f : \mathcal{X} \to \mathbb{R} \). A distribution \( r \) is an \( \eta \)-friendly perturbation of \( p \) for \( f \) under \( \text{W}_{c,k} \), denoted as \( r \in F(p, \eta, \text{W}_{c,k}, f) \), if there is a coupling \( \pi_{X,Y} \) between \( X \sim p \) and \( Y \sim r \) such that:

- The cost \( (\mathbb{E}[c^k(X,Y)])^{1/k} \) is at most \( \eta \).
- All points move towards the mean of \( r \): \( f(Y) \) is between \( f(X) \) and \( \mathbb{E}_r[f(Y)] \) almost surely.

Note that friendliness is defined only in terms of one-dimensional functions \( f : \mathcal{X} \to \mathbb{R} \); we will see how to handle higher-dimensional objects later. Intuitively, a friendly perturbation is a distribution \( r \) for which there exists a coupling that ‘squeezes’ \( p \) to \( \mu_r \).

Another key property of deletion in the \text{TV} case is the existence of \textit{midpoint}: for any two distributions that are within \( \epsilon \) distance in \text{TV}, one can find another distribution that is an \( \epsilon \)-deletion of both distributions. We provide an analogous lemma for friendly perturbation in \( \text{W}_{c,k} \) perturbation. We would like to show that given any \( p, q \) with \( \text{W}_{c,k}(p, q) \leq \epsilon \) and any \( f \), there exists an \( r \) that is an \( \epsilon \)-friendly perturbation of both \( p \) and \( q \) for the function \( f \).
To show the existence of a midpoint, we rely on the intuition that any coupling between two one-dimensional distributions can be separated into two stages: in one stage all the mass only moves towards some point, in the other stage all the mass moves away from that point. This is illustrated in Figure 7.

Figure 7: Illustration of midpoint lemma. For any distributions $p_1, p_2$ that are close under $W_{c,k}$, the coupling between $p_1$ and $p_2$ can be split into couplings $\pi_{p_1,r}, \pi_{p_2,r}$ such that $p_1, p_2$ only move towards $\mu_r$ under the couplings. We do this by “stopping” the movement from $p_1$ to $p_2$ at $\mu_r$.

Now we formally state the midpoint lemma for friendly perturbations under $W_{c,k}$. We make the following topological assumptions regarding $c$ and $f$.

**Assumption 4.1** (Intermediate value property). Given $W_{c,k}$, we assume that for all $x$ and $y$ and all $u$ with $f(x) < u < f(y)$, there is some $z$ satisfying $f(z) = u$ and $\max(c(x,z), c(z,y)) \leq c(x,y)$.

This holds for all of our examples and for many other $f$ and $c$, e.g. when $c$ is a path metric (Gromov, 2007, Definition 1.7) and $f$ is continuous under the topology induced by the metric.

**Lemma 4.1** (Midpoint lemma for $W_{c,k}$ perturbation). Suppose Assumption 4.1 holds. Then for any $p_1$ and $p_2$ such that $W_{c,k}(p_1, p_2) < \eta$ and any $f$, there exists a distribution $r$ such that

$$r \in F(p_1, r, W_{c,k}, f) \cap F(p_2, r, W_{c,k}, f).$$

(31)

In other words, $r$ is an $\eta$-friendly perturbation of both $p_1$ and $p_2$ for $f$ under $W_{c,k}$.

See Appendix D.4 for a formal proof. With this lemma in hand, we generalize resilience to Wasserstein distances by saying that a distribution is resilient if $\mathbb{E}_r[f(X)]$ is close to $\mathbb{E}_p[f(X)]$ for every $\eta$-friendly perturbation $r$ and every function $f$ lying within some appropriate family $\mathcal{F}$. For instance, for second moment estimation we would consider functions $f_v(x) = \langle x, v \rangle^2$ with $\|v\|_2 = 1$.

We discuss this in more detail below.

### 4.1 Warm-up: Second Moment Estimation under $W_1$ Perturbation

Consider estimation of the second moment ($L(p, M) = \|M - \mathbb{E}_p[XX^\top]\|_2$ where $M \in \mathbb{R}^{d \times d}$) under $W_1$ (Definition 2.1). We do not consider mean estimation since it is trivial under $W_1$ perturbation (outputting the mean of $p$ incurs error $\epsilon$, which is optimal).

Recall that in Section 3.1, we designed $\mathcal{G}_{\text{mean}}^{TV}$ by asking that friendly perturbations (in that case deletions) did not move the mean by too much. With our definition of friendly perturbation for $W_{c,k}$ in hand, we similarly define $\mathcal{G}_{\text{sec}}^{W_1}$ for second moment estimation as the following:

$$\mathcal{G}_{\text{sec}}^{W_1}(\rho, \eta) = \{p | \sup_{\|v\|_2 = 1, r \in F(p, r, W_1, \|v\|_2)} \|\mathbb{E}_p[|v^\top X|^2] - \mathbb{E}_r[|v^\top X|^2]\| \leq \rho\},$$

(32)
This asks that in all unit directions \( v \), friendly perturbations under \( |v^\top x|^2 \) cannot move the second moment by more than \( \rho \). As before we will show that the set \( G^W_1 \) is not too big (has bounded modulus) and not too small (contains natural nonparametric distribution families).

**Not too big.** As in the TV perturbation case, we show that \( G^W_1 \) has controllable population limit by upper bounding its modulus of continuity.

**Theorem 4.2.** The modulus of continuity \( \mu \) in (18) for \( G^W_1 (\rho, \eta) \) is bounded above as \( \mu (G^W_1 (\rho, \eta), 2\epsilon) \leq 2\rho \) for any \( 2\epsilon \leq \eta \).

**Proof.** Denote \( M_p = \mathbb{E}_p [XX^\top] \). The modulus is defined as

\[
\sup_{p_1,p_2 \in G^W_1 (\rho, \eta), W_1(p_1,p_2) \leq 2\epsilon} \|M_{p_1} - M_{p_2}\|_2. \tag{33}
\]

Since Assumption 4.1 holds, by Lemma 4.1, for any unit vector \( v \) we can find \( r \) such that \( W_1(p_1, r) \leq 2\epsilon, W_1(p_2, r) \leq 2\epsilon \), and \( r \in \mathbb{F}(p_1, 2\epsilon, W_1, |v^\top X|^2) \) is a friendly perturbation for both \( p_1 \) and \( p_2 \). Now take some \( v^* \) with \( \|v^*\|_2 = 1 \) such that

\[
|(v^*)^\top (M_{p_1} - M_{p_2})v^*| = \|M_{p_1} - M_{p_2}\|_2. \tag{34}
\]

By symmetry of \( p_1,p_2 \) we may assume that the term inside the absolute value is positive. From \( p_1,p_2 \in G^W_1 (\rho, \eta) \), we know that for any \( 2\epsilon \leq \eta \),

\[
\mathbb{E}_{p_1}[(v^*^\top X)^2] - \mathbb{E}_{p_2}[(v^*^\top X)^2] \leq \rho, \tag{35}
\]

\[
\mathbb{E}_{p_2}[(v^*^\top X)^2] - \mathbb{E}_{p_2}[(v^*^\top X)^2] \leq \rho. \tag{36}
\]

Combining the two equations together gives us

\[
\mathbb{E}_{p_2}[(v^*^\top X)^2] - \mathbb{E}_{p_2}[(v^*^\top X)^2] \leq 2\rho. \tag{37}
\]

This shows that \( \|M_{p_1} - M_{p_2}\|_2 \leq 2\rho. \)

**Not too small.** We show that the set \( G^W_1 \) is a superset of Orlicz-norm bounded distributions:

**Example 4.1** (Second moment estimation under \( W_1 \)). Let \( \psi \) be an Orlicz function (Definition 2.2) that further satisfies \( \psi(x) \geq x \) for all \( x \geq 1 \), and define \( \tilde{\psi}(x) = x\psi(2x) \). Suppose that

\[
\sup_{v \in \mathbb{R}^d, \|v\|_2 = 1} \mathbb{E}_p \left[ \psi \left( \frac{|v^\top X|}{\sigma} \right) \right] \leq 1. \tag{38}
\]

Then, \( p \in G^W_1 (\rho, \eta) \) for \( \rho = \sigma \eta \psi^{-1}(\frac{2\sigma}{\eta}) \) whenever \( \eta < 2\sigma/\psi(8) \). Thus the population limit when the perturbation level is \( \epsilon \) is upper bounded by \( 2\sigma \epsilon \psi^{-1}(\frac{\epsilon}{4}) \) assuming \( \epsilon < \sigma/\psi(8) \).

The proof is deferred to Appendix D.3, where we show a more general statement for \( k \)-th moment estimation. We also extend the design of \( G^W_1 \) to arbitrary \( D = W_{c,k} \) and \( L = W_F \) in Appendix D.2.

Taking \( \psi(x) = x^m \), we know that when the \( (m+1) \)-th moment of \( X \) is bounded by \( 2(\frac{\sigma}{\eta})^{m+1} \), the population limit is \( O(\sigma^{1+1/m} \epsilon^{1-1/m}) \), while the same condition in TV perturbation would give error of \( O(\sigma \epsilon^{1-1/(m+1)}) \).

The function \( f \) in the friendly perturbation requirement \( (r \in \mathbb{F}(p, \eta, W_1, f)) \) can be chosen differently without changing the conclusions above. For example, we may replace the constraint \( r \in \mathbb{F}(p, \eta, W_1, |v^\top X|^k) \) with \( r \in \mathbb{F}(p, \eta, W_1, |v^\top X|) \), and the sufficient condition and modulus of continuity remains the same. This is discussed in Appendix D.3 and is used in Section 6 for finite sample algorithm design.
4.2 General Design of $\mathcal{G}^{W_{c,k}}$

Recall that for the general $\mathcal{G}^{TV}$ (Definition 3.1), we said a distribution is resilient if (1) the parameter $\theta^*(p)$ does well on all deletion $r \leq \frac{p}{1-\eta}$, and (2) any parameter that does well on some deletion $r \leq \frac{p}{1-\eta}$ also does well on $p$. We used a bridge function $B$ to measure performance on $r$.

Inspired by this argument, we extend the definition of resilience for second moment estimation $\mathcal{G}_{sec}^{W_1}$ to other losses and general $W_{c,k}$ perturbation. Since friendly perturbation for $W_{c,k}$ is only defined for a one-dimensional random variable $f(X)$, we apply the Fenchel-Moreau representation (Borwein and Lewis, 2010) of $B$ to decompose $B$ to the expectation of one dimensional functions $f \in \mathcal{F}$, as long as $B$ is lower semi-continuous and convex in $p$ for fixed $\theta$:

$$B(p, \theta) = \sup_{f \in \mathcal{F}_\theta} \mathbb{E}_p[f(X)] - B^*(f, \theta).$$

(39)

Here $B^*(f, \theta)$ is the convex conjugate of $B$, and $\mathcal{F}_\theta = \{ f \mid B^*(f, \theta) < \infty \}$. Convexity in $p$ is a mild condition that often holds, e.g. any loss of the form $B(p, \theta) = \mathbb{E}_{X \sim p}[\ell(\theta; X)]$ is linear (and hence convex) in $p$.

We thus define the resilient set in the same way as TV: we say a distribution is resilient if (1) $\theta^*(p)$ does well on all friendly perturbation perturbation $r$ and every function $f \in \mathcal{F}_{\theta^*(p)}$, and (2) for any parameter $\theta$, if all $f \in \mathcal{F}_\theta$, $\theta$ have a friendly perturbation $r \in B(p, \eta, W_{c,k}, f)$ where $\theta$ does well, then $\theta$ also does well on $p$ under $L$. We formally define the set $\mathcal{G}^{W_{c,k}}$ below.

**Definition 4.2 ($\mathcal{G}^{W_{c,k}}$).** We define

$$\mathcal{G}^{W_{c,k}}(\rho_1, \rho_2, \eta) = \mathcal{G}^{W_{c,k}}_\downarrow(\rho_1, \eta) \cap \mathcal{G}^{W_{c,k}}_\uparrow(\rho_1, \rho_2, \eta),$$

(40)

where

$$\mathcal{G}^{W_{c,k}}_\downarrow(\rho_1, \eta) = \{ p \mid \sup_{f \in \mathcal{F}_{\theta^*(p), r \in B(p, \eta, W_{c,k}, f)}} \mathbb{E}_r[f(X)] - B^*(f, \theta^*(p)) \leq \rho_1 \},$$

(41)

$$\mathcal{G}^{W_{c,k}}_\uparrow(\rho_1, \rho_2, \eta) = \left\{ p \mid \text{for all } \theta \in \Theta, \left( \sup_{f \in \mathcal{F}_{\theta}, r \in B(p, \eta, W_{c,k}, f)} \mathbb{E}_r[f(X)] - B^*(f, \theta) \leq \rho_1 \Rightarrow L(p, \theta) \leq \rho_2 \right) \right\}.$$  

(42)

The construction of $\mathcal{G}^{W_{c,k}}$ generalizes the idea in $\mathcal{G}^{TV}$ (Definition 3.1) and $\mathcal{G}^{W_{c,k}}_{sec}$ (Equation (32)). However, not all $\mathcal{G}^{W_{c,k}}$ can reduce to $\mathcal{G}^{TV}$. The difference between $\mathcal{G}^{TV}$ and $\mathcal{G}^{W_{c,k}}$ comes from the definition of friendly perturbation: the deletion operation is independent of $f$, while friendly perturbation for $p$ depends on the choice of function $f$. Thus in $\mathcal{G}^{TV}$, we look at $\theta$ that do well on the same deletion $r$ for all $f \in \mathcal{F}_\theta$, while in $\mathcal{G}^{W_{c,k}}$ we only require that for each $f$ there exists a friendly perturbation $r$ that $\theta$ does well on.

Now we show that the design of general $\mathcal{G}^{W_{c,k}}$ is also not too big and not too small, and that it enables us to conduct robust linear regression under $W_1$ perturbation.

**Not too big.** Assume that the choice of $B, W_{c,k}$ makes Assumption 4.1 hold for all $f \in \mathcal{F}_\theta, \theta \in \Theta$. We control the population limit of $\mathcal{G}^{W_{c,k}}$ by bounding its modulus of continuity.

**Theorem 4.3.** The modulus of continuity $m$ in (18) for $\mathcal{G}^{W_{c,k}}(\rho_1, \rho_2, \eta)$ is bounded above by $m(\mathcal{G}^{W_{c,k}}(\rho_1, \rho_2, \eta), 2\epsilon) \leq \rho_2$ for any $2\epsilon \leq \eta$.

The proof is deferred to Appendix D.5 and follows the same lines as Theorem 3.1, using Lemma 4.1 to produce the required midpoint distribution $r$.

---

4When $B = L = W_{c,k}$, $\mathcal{G}^{W_{c,k}}$ is equivalent to $\mathcal{G}^{TV}$; see Appendix D.2.
Not too small. By taking $B$ and $L$ as the cost of second moment estimation, we can recover the definition of $G_{W_1}^{W_2}$. We further study linear regression under $W_1$ perturbation below and provide sufficient conditions for distributions to be inside the $W_1$-resilient set. Here we measure the cost in $W_1$ as $c((x_0, y_0), (x_1, y_1)) = \sqrt{\|x_0 - x_1\|^2 + \|y_0 - y_1\|^2}$.

Example 4.2 (Linear regression under $W_1$ perturbation). Take $B(p, \theta) = L(p, \theta) = \mathbb{E}_p[(Y - X^\top \theta)^2]$. Denote by $X' = [X, Z]$ the $d + 1$ dimensional vector that concatenates $X$ with the noise $Z = Y - X^\top \theta(p^*)$, where $\theta^*(p^*) \triangleq \arg \min_{\theta \in \Theta} B(p^*, \theta), \Theta = \{\theta \mid \|\theta\|_2 \leq R\}$. Given an Orlicz function $\psi$ that further satisfies $\psi(x) \geq x$ for all $x \geq 1$, denote $\hat{\psi}(x) = x\psi(2x)$. Assume $p^*$ satisfies:

$$
\sup_{v \in \mathbb{R}^{d+1}, \|v\|_2 = 1} \mathbb{E}_{p^*}[\hat{\psi}\left(\frac{|v^\top X'|}{\sigma}\right)] \leq 1. \tag{43}
$$

Then for $\eta \sqrt{R^2/2 + 1} < 2\sigma/\psi(8)$, we have $p^* \in G_{W_1}(\sigma^2 + \Delta, \sigma^2 + (2R + 2)\Delta, \eta)$, where $\Delta = \sigma\eta\sqrt{R^2/2 + 4\psi^{-1}(2\sigma/(\eta\sqrt{R^2/2 + 1}))}$. The population limit for this set when the perturbation level is $\epsilon$ is upper bounded by $\sigma^2 + (2R + 2)\Delta$ where $\Delta = 2\epsilon\sqrt{2R^2 + 4\psi^{-1}(2\sigma/(2\epsilon\sqrt{R^2/2 + 1}))}$ assuming $\epsilon\sqrt{R^2/2 + 1} < \sigma/\psi(8)$.

The proof is deferred to Appendix D.7. Similar to second moment estimation, the function $|y - x^\top \theta|^2$ in the specification of $F$ can be replaced with $|y - x^\top \theta|$ without changing the conclusion.

Taking $\psi(x) = x^2$, we see that if the vector $(X, Z)$ has its 3-rd moment bounded by $\sigma^3$, the population limit is upper bounded by $\sigma^2 + O((\sigma^2(R^2 + 1))^{3/4}\sqrt{\epsilon})$. Thus if we know the original distribution’s optimal prediction error is bounded by $\sigma^2$, then our estimator’s risk approaches $\sigma^2$ as $\epsilon$ goes to 0. However, it is an open problem how to guarantee vanishing error for the excess predictive loss $L(p, \theta) = \mathbb{E}_p[(Y - X^\top \theta)^2 - (Y - X^\top \theta^*(p))^2]$. We show in Appendix D.8 that for $W_1$ linear regression, it is essential to have the compactness assumption that $\Theta = \{\theta \mid \|\theta\|_2 \leq R\}$. Otherwise, the population limit can be infinite.

5 Finite Sample Algorithms for TV

In Section 2.3 we saw that the minimum distance functional defined as

$$
q = \arg \min_{\tilde{\psi}} D(q, p) \tag{44}
$$

$$
\hat{\theta} = \theta^*(q) = \arg \min_{\theta \in \Theta} L(q, \theta). \tag{45}
$$

yields good bounds whenever the modulus of continuity $m(\mathcal{G}, \epsilon)$ is small. In Sections 3 and 4 we saw how to construct resilient sets such that the modulus $m$ is small, and thus such that the minimum distance functional performs well in infinite samples. We refer to (44) as the projection algorithm because it projects $p$ onto the destination set $\mathcal{G}$.

Ideally, we would like to turn this infinite-sample algorithm into a finite-sample algorithm by substituting $p$ with the empirical distribution $\hat{p}_n$ in (44). In other words, we would obtain $q$ via

$$
q = \arg \min_{\tilde{\psi}} D(q, \hat{p}_n). \tag{46}
$$

However, in many cases this method fails completely. For example, if $\mathcal{G}$ consists of continuous distributions and $D = TV$, then $TV(q, \hat{p}_n) = 1$ for all $q \in \mathcal{G}$, rendering the projection algorithm meaningless. In this section, we provide two ways to fix the finite sample projection algorithm:
1. **Weaken the discrepancy** $D$: one may choose to project under a different function $\tilde{D}(q, \hat{p}_n)$;

2. **Expand the destination set** $\mathcal{G}$: one may choose to project to a different set $\mathcal{M}$.

The intuition behind weakening the discrepancy $D$ is that while a discrepancy such as $D = \text{TV}$ is too strong such that $\text{TV}(p^*, \hat{p}_n)$ is very large, perhaps a weaker related discrepancy $\tilde{\text{TV}}$ would allow $\tilde{\text{TV}}(p^*, \hat{p}_n)$ to be small. Meanwhile, expanding the destination set $\mathcal{G}$ seeks to guarantee that there is some distribution $q \in \mathcal{G}$ (perhaps not $p^*$ itself) such that $D(q, \hat{p}_n)$ is small for the original discrepancy. The general projection algorithm is presented below.

**Algorithm 1 Projection algorithm $\Pi(\hat{p}_n; \tilde{D}, \mathcal{M})$ or $\Pi(\hat{p}_n; \tilde{D}, \mathcal{M}, \tilde{\epsilon})$**

**Input** observed distribution $\hat{p}_n$, discrepancy $\tilde{D}$, destination set $\mathcal{M}$, optional parameter $\tilde{\epsilon}$

if $\tilde{\epsilon}$ is given then
    find any $q \in \mathcal{M}$ such that $\tilde{D}(q, \hat{p}_n) \leq \tilde{\epsilon}$.
else
    find $q = \arg\min_{q \in \mathcal{M}} \tilde{D}(q, \hat{p}_n)$.
end if

**Output** $q$.

Given cost function $L(p, \theta)$, after obtaining $q$ from Algorithm 1, we output $\hat{\theta} = \theta^*(q) \triangleq \arg\min_{\theta \in \Theta} L(q, \theta)$ as the estimated parameter.

This algorithm is a generalization of the minimum distance functional (Donoho and Liu, 1988), where we have replaced $D$ with $\tilde{D}$ for the projection function, and $\mathcal{G}$ with $\mathcal{M}$ for the destination set, and we allow the algorithm to output any $q \in \mathcal{M}$ satisfying $\tilde{D}(q, \hat{p}_n) \leq \tilde{\epsilon}$ if the parameter $\tilde{\epsilon}$ is given. We will explore the effects of $\tilde{D}$ in Section 5.1 and the effects of $\mathcal{M}$ in Section 5.2.

### 5.1 Weaken Discrepancy $D$

In this section we focus on the approach of weakening the discrepancy $D$ when $D = \text{TV}$ and keeping the destination set intact ($\mathcal{M} = \mathcal{G}^{\text{TV}}$).

As discussed before, if the real distribution $p^*$ lies in a continuous family such as the family $\mathcal{G} = \{\mathcal{N}(\mu, I_d) \mid \mu \in \mathbb{R}^d\}$ of isotropic Gaussian distributions, then the naïve projection algorithm

$$q = \arg\min_{q \in \mathcal{G}} \text{TV}(q, \hat{p}_n)$$

fails completely since for any $q \in \mathcal{G}$, $\text{TV}(q, \hat{p}_n) = 1$. We define a family of pseudonorms that is weaker than $\text{TV}$, called *generalized Kolmogorov–Smirnov distance* $\tilde{\text{TV}}_{\mathcal{H}}$, to avoid the issue:

$$\tilde{\text{TV}}_{\mathcal{H}}(p, q) \triangleq \sup_{f \in \mathcal{H}, t \in \mathbb{R}} |\mathbb{P}_p[f(X) \geq t] - \mathbb{P}_q[f(X) \geq t]|. \quad (48)$$

When $X$ is a one-dimensional random variable and $\mathcal{H}$ is the singleton $\{\text{Id} : x \mapsto x\}$, we recover the Kolmogorov-Smirnov distance (Massey Jr, 1951). When $\mathcal{H}$ contains all functions, $\tilde{\text{TV}}_{\mathcal{H}} = \text{TV}$; otherwise $\tilde{\text{TV}}_{\mathcal{H}}$ is weaker than TV distance: $\tilde{\text{TV}}_{\mathcal{H}} \leq \text{TV}$. For any $\mathcal{H}$, $\tilde{\text{TV}}_{\mathcal{H}}$ is a pseudometric.

Assume $p^* \in \mathcal{G}^{\text{TV}}$ and $\text{TV}(p^*, p) \leq \epsilon$ (oblivious corruption model). The following result generalizes Donoho and Liu (1988) and guarantees good performance of the projection algorithm as long as we can bound $\tilde{\text{TV}}_{\mathcal{H}}(p, \hat{p}_n)$ along with the modulus of continuity under $\tilde{\text{TV}}_{\mathcal{H}}$.

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4All the argument can be extended to the case of $\mathcal{M} = \mathcal{G}^{\text{TV}} \supset \mathcal{G}^{\text{TV}}$. Here for simplicity we keep $\mathcal{M} = \mathcal{G}^{\text{TV}}$. 28
Proposition 5.1. The projection algorithm \( q = \Pi(\hat{p}_n; \tilde{TV}_H, \mathcal{G}^{TV}) \) satisfies
\[
L(p^*, \theta^*(q)) \leq m(\mathcal{G}^{TV}, \tilde{\epsilon}, \tilde{TV}_H, L) = \sup_{p_1, p_2 \in \mathcal{G}^{TV} : \tilde{TV}_H(p_1, p_2) \leq \tilde{\epsilon}} L(p_2, \theta^*(p_1)),
\]
where \( \tilde{\epsilon} = 2\epsilon + 2\tilde{TV}_H(p, \hat{p}_n) \).

As its proof below shows, the conclusions in Proposition 5.1 remain unchanged if we change the corruption model to allow \( \tilde{TV}_H \) perturbations instead of \( TV \). In other words, Proposition 5.1 analyzes the original algorithm under larger corruptions, and we will obtain good finite sample error bounds throughout this section for this more difficult problem.

Proof. Since \( \tilde{TV}_H \) satisfies triangle inequality, we have
\[
\tilde{TV}_H(q, p^*) \leq \tilde{TV}_H(q, \hat{p}_n) + \tilde{TV}_H(\hat{p}_n, p^*) \leq 2\tilde{TV}_H(\hat{p}_n, p^*),
\]
where the second step is because \( \tilde{TV}_H(q, \hat{p}_n) \) is minimizes \( \tilde{TV}_H(\cdot, \hat{p}_n) \) over \( \mathcal{G} \). Applying the triangle inequality again and \( \tilde{TV}_H(p^*, p) \leq \epsilon \), we obtain
\[
2\tilde{TV}_H(\hat{p}_n, p^*) \leq 2(\tilde{TV}_H(\hat{p}_n, p) + \tilde{TV}_H(p, p^*)) \leq 2\epsilon + 2\tilde{TV}_H(p, \hat{p}_n).
\]
Thus from \( p^*, q \in \mathcal{G}^{TV} \) and Lemma 2.2, we know that the error is bounded by \( m(\mathcal{G}^{TV}, \epsilon, \tilde{TV}_H, L) \). \( \Box \)

For any \( H \) with finite VC dimension, we can bound \( \tilde{TV}_H(p, \hat{p}_n) \) using the following lemma:

Lemma 5.1. Let \( \hat{p}_n \) be the empirical distribution of \( n \) i.i.d. samples from \( p \) and let \( \text{vc}(H) \) be the VC dimension of the collection of sets \( \{ \{ x \mid f(x) \geq t \} \mid f \in H, t \in \mathbb{R} \} \). Then, each of the following holds with probability at least \( 1 - \delta \):
\[
\tilde{TV}_H(p, \hat{p}_n) \leq C_{\text{vc}} \cdot \sqrt{\frac{\text{vc}(H) + \log(1/\delta)}{n}} \quad \text{for some universal constant } C_{\text{vc}},
\]
\[
\tilde{TV}_H(p, \hat{p}_n) \leq \sqrt{\frac{\ln(2|H|/\delta)}{2n}}, \quad \text{where } |H| \text{ denotes the cardinality of } H.
\]

Proof. The first statement follows from the VC inequality (Devroye and Lugosi, 2012, Chap 2, Chapter 4.3). Now we prove the second statement. Fix \( f \in H \) and denote \( M = |H| \). By the Dvoretzky-Kiefer-Wolfowitz inequality (Dvoretzky et al., 1956), with probability \( 1 - 2 \exp(-2n\epsilon^2) \) we have \( |\mathbb{P}_{\hat{p}_n} [f(x) \geq t] - \mathbb{P}_p [f(x) \geq t]| \leq \epsilon \) for all \( t \in \mathbb{R} \). Union bounding over \( f \in H \), we have that \( \tilde{TV}_H(\hat{p}_n, p) \leq \epsilon \) with probability at least \( 1 - 2M \exp(-2n\epsilon^2) \). Solving for \( \delta \), we obtain \( \epsilon = \sqrt{\log(2M/\delta)/2n} \), which proves the lemma. \( \Box \)

Since Lemma 5.1 bounds \( \tilde{TV}_H(p, \hat{p}_n) \), it remains to bound the modulus \( m(\mathcal{G}^{TV}, \epsilon, \tilde{TV}_H) \) for some \( H \) with small VC dimension. We show how to do this for different concrete cases below.

\( \tilde{TV}_H \) projection for mean estimation. For mean estimation in Example 3.1, we choose \( H \) as
\[
H = \{ v^\top x \mid v \in \mathbb{R}^d \}.
\]
This particular \( \tilde{TV}_H \) is also used in Donoho (1982); Donoho and Liu (1988). Intuitively, the reason for choosing this \( H \) is that linear projections of our data contain all information needed to recover the mean, so perhaps it is enough for distributions to be close only under these projections.
In Section 3, we controlled the modulus of continuity by using the existence of a midpoint distribution when two distributions are close in $\text{TV}$ (Lemma 3.1). Here we only know that distributions are close under $\widetilde{\text{TV}}_{\mathcal{H}}$ and cannot show the existence of midpoint. However, we can show that for two 1-dimensional distributions that are close under $\widetilde{\text{TV}}_{\mathcal{H}}$, we can delete a small fraction of probability mass to make their means cross. This is formally proved in the following lemma.

Figure 8: Illustration of mean cross lemma. For any distributions $p_1, p_2$ that are close under $\widetilde{\text{TV}}$, we can truncate the $\epsilon$-tails of each distribution to make their means cross.

**Lemma 5.2** (Mean crossing). Suppose two distributions $p, q$ on the real line satisfy

$$\sup_{t \in \mathbb{R}} |P_p(X \geq t) - P_q(Y \geq t)| \leq \epsilon.$$  \hspace{1cm} (55)

Then one can find some $r_p \leq p_1 - \epsilon$ and $r_q \leq q_1 - \epsilon$ such that $r_p$ is stochastically dominated by $r_q$, which implies that $E_{r_p}[X] \leq E_{r_q}[Y]$.

**Proof.** The idea of the proof is illustrated in Figure 8. Suppose $X \sim p, Y \sim q$. Starting from $p, q$, we delete $\epsilon$ probability mass corresponding to the largest points of $X$ in $p$ to get $r_p$, and delete $\epsilon$ probability mass corresponding to the smallest points $Y$ in $q$ to get $r_q$. Equation (55) implies that $P_{r_p}(X \geq t) \leq P_{r_q}(X \geq t)$ holds for all $t \in \mathbb{R}$. Hence, $r_q$ stochastically dominates $r_p$ and $E_{r_p}[X] \leq E_{r_q}[Y]$.

With this one-dimensional mean cross lemma, the key idea to show modulus of continuity in high dimension is to identify the optimal projected direction and apply the one-dimensional mean cross lemma in that direction. For mean estimation, we use this to obtain the following:

**Theorem 5.1.** Denote $\hat{\epsilon} = 2 \epsilon + 2C^{\text{vc}} \sqrt{\frac{d+1+\log(1/\delta)}{n}}$, where $C^{\text{vc}}$ is from Lemma 5.1. Assume $p^* \in \mathcal{G}^{\text{TVmean}}(\rho(\hat{\epsilon}), \hat{\epsilon})$. For $\mathcal{H}$ defined in (54), let $q$ denote the output of the projection algorithm $\Pi(\hat{p}_n; \widetilde{\text{TV}}_{\mathcal{H}}, \mathcal{G}^{\text{TV}})$. Then, with probability at least $1 - \delta$,

$$\|E_{p^*}[X] - E_{q}[X]\| \leq 2\rho(\hat{\epsilon}) = 2\rho \left( 2\epsilon + 2C^{\text{vc}} \sqrt{\frac{d+1+\log(1/\delta)}{n}} \right).$$  \hspace{1cm} (56)

**Proof.** By Proposition 5.1 it suffices to bound $\widetilde{\text{TV}}_{\mathcal{H}}(p, \hat{p}_n)$ and to bound the modulus of continuity for $\mathcal{G}^{\text{TV}}$ under $\widetilde{\text{TV}}_{\mathcal{H}}$.

Since the VC dimension of hyper-planes in $\mathbb{R}^d$ is $d + 1$, it follows from Lemma 5.1 that $\widetilde{\text{TV}}_{\mathcal{H}} \leq C^{\text{vc}} \sqrt{\frac{d+1+\log(1/\delta)}{n}}$ with probability at least $1 - \delta$. Now we upper bound the modulus, which equals

$$\sup_{p_1, p_2 \in \mathcal{G}^{\text{TVmean}}(\rho(\hat{\epsilon}), \hat{\epsilon}); \widetilde{\text{TV}}_{\mathcal{H}}(p_1, p_2) \leq \hat{\epsilon}} \|E_{p_1}[X] - E_{p_2}[X]\|.$$  \hspace{1cm} (57)
The condition that $\hat{TV}_H(p_1, p_2) \leq \bar{\epsilon}$ implies that for any $v \in R^d$, $\|v\|_* = 1$, where $\|\cdot\|_*$ is the dual norm of $\|\cdot\|$, 

$$\sup_{t \in R} |P_{p_1}[v^T X \geq t] - P_{p_2}[v^T X \geq t]| \leq \bar{\epsilon}. \quad (58)$$

It follows from Lemma 5.2 that there exist $r_{p_1} \leq \frac{p_1}{\bar{\epsilon}^2}, r_{p_2} \leq \frac{p_2}{\bar{\epsilon}^2}$ such that 

$$E_{r_{p_1}}[v^T X] \leq E_{r_{p_2}}[v^T X], \text{ or } E_{r_{p_1}}[v^T X] - E_{r_{p_2}}[v^T X] \leq 0. \quad (59)$$

Furthermore, from $p_1, p_2 \in G_{TV cerco}^{\rho_{mean}}(\bar{\rho}, \bar{\epsilon})$, we know the mean of $p_1$ is close to its deletion, and the same holds for $p_2$, thus for $\|v\|_* = 1$, 

$$E_{p_1}[v^T X] - E_{r_{p_1}}[v^T X] \leq \rho(\bar{\epsilon}), E_{r_{p_2}}[v^T X] - E_{p_2}[v^T X] \leq \rho(\bar{\epsilon}). \quad (60)$$

Adding the three inequalities together yields $E_{p_1}[v^T X] - E_{p_2}[v^T X] \leq 2\rho(\bar{\epsilon})$, or $v^T (\mu_{p_1} - \mu_{p_2}) \leq 2\rho(\bar{\epsilon}).$ Taking the maximum over $\|v\|_* = 1$ yields $\|\mu_{p_1} - \mu_{p_2}\| \leq 2\rho(\bar{\epsilon})$, where $\|\cdot\|_*$ is the dual norm of $\|\cdot\|$. which shows the modulus is small. The final conclusion follows from Proposition 5.1. \hfill \Box

**Interpretation and Comparison** For the $\hat{TV}_H$ projection algorithm, Theorem 5.1 bounds the error as 

$$2\rho(\bar{\epsilon}) = 2\rho \left( 2\epsilon + 2C^{\rho_{e}} \sqrt{\frac{d + 1 + \log(1/\delta)}{n}} \right). \quad (61)$$

If $p^*$ is sub-Gaussian, then $\rho(x) = \Theta(x \sqrt{\log(1/\epsilon)})$ and (61) implies an error $O(\epsilon \sqrt{\log(1/\epsilon)})$ when $n \geq \frac{d + \log(1/\delta)}{\epsilon^2}$. If $p^*$ has bounded covariance, then $\rho(x) = \Theta(\sqrt{x})$ and (61) implies an error of order $O(\sqrt{\epsilon})$ when $n \geq \frac{d + \log(1/\delta)}{\epsilon^2}$.

In both cases above, the dependence on $\epsilon$ is suboptimal; we improve the sample complexity by another analysis method in Section 5.2, and defer the detailed discussion to Section 5.3.

**Remark 5.1.** As comparison, we study the performance of the Tukey median in mean estimation in Appendix E.4, which can also be viewed as a minimum distance functional, and achieves good performance under a set of assumptions different from resilience. The techniques we use there are summarized in Section 7; they generalize Proposition 5.1.

**$\hat{TV}_H$ projection for linear regression.** For the linear regression problem in Example 3.2, we design the corresponding $H$ as 

$$H = \{(Y - X^T \theta_1)^2 - (Y - X^T \theta_2)^2 \mid \theta_1, \theta_2 \in R^d\}. \quad (62)$$

For linear regression, we need to slightly shrink the resilient set from 

$$G^TV \rho_1(\bar{\epsilon}, \bar{\epsilon}) \cap G^TV \rho_2(\rho(\bar{\epsilon}), \rho(\bar{\epsilon}), \bar{\epsilon}) \quad (63)$$

to 

$$G^TV \rho_1(\rho(\bar{\epsilon}), \rho(\bar{\epsilon})) \cap G^TV (2\rho(\bar{\epsilon}), \rho(\bar{\epsilon}), \bar{\epsilon}). \quad (64)$$

The conditions from Example 3.2 still imply that $p^*$ is inside this smaller set with appropriate parameters. The shrinkage comes from the following consideration: when using the mean cross lemma, we can only cross the mean of the same function $f(X)$ for $p$ and $q$ while the excess predictive loss in the original $G^TV$ requires the mean cross for two different functions.

The following theorem characterizes the performance of $\hat{TV}_H$ projection for linear regression.
Theorem 5.2. Denote \( \bar{\epsilon} = 2\epsilon + 2C^{vc}\sqrt{\frac{10d + \log(1/\delta)}{n}} \). Assume \( p^* \in \mathcal{G}^\text{TV}_\delta(\rho_1(\bar{\epsilon}), \bar{\epsilon}) \cap \mathcal{G}^\text{TV}_\delta(2\rho_1(\bar{\epsilon}), 2\bar{\epsilon}) \) in Example 3.2. For \( \mathcal{H} \) designed in (62), let \( q \) denote the output of the projection algorithm \( \Pi(\hat{p}_n; \mathcal{H}, \mathcal{G}^\text{TV}) \). Then, with probability at least \( 1 - \delta \),

\[
\mathbb{E}_{p^*}[(Y - X^\top \theta^*(q))^2 - (Y - X^\top \theta^*(p^*))^2] \leq \rho_2(\bar{\epsilon}) = \rho_2(\epsilon) + 2\epsilon + 2C^{vc}\sqrt{\frac{10d + \log(1/\delta)}{n}}. \tag{65}
\]

Proof. By Proposition 5.1 it suffices to bound \( \mathcal{TV}_\mathcal{H}(p, \hat{p}_n) \) and to bound the modulus of continuity for \( \mathcal{G}^\text{TV} \) under \( \mathcal{TV}_\mathcal{H} \).

Denote \( \bar{X} = (X, Y) \), then \( \mathcal{TV}_\mathcal{H}(p, q) \) can be upper bounded in the following way:

\[
\mathcal{TV}_\mathcal{H}(p, q) \leq \sup_{v_1, v_2 \in \mathbb{R}^{d+1}, t \in \mathbb{R}} \left| \mathbb{P}_p((v_1^\top X)^2 - (v_2^\top X)^2 \geq t) - \mathbb{P}_q((v_1^\top \bar{X})^2 - (v_2^\top \bar{X})^2 \geq t) \right|, \tag{66}
\]

From (Anthony and Bartlett, 2009, Theorem 8.3) we know that the VC dimension of the collection of sets \( \{\{x \in \mathbb{R}^{d+1} | (v_1^\top x)^2 - (v_2^\top x)^2 \geq t\} | v_1, v_2 \in \mathbb{R}^{d+1}, t \in \mathbb{R}\} \) is at most \( 10d \). Thus from Lemma 5.1 we have \( \mathcal{TV}_\mathcal{H}(p, q) \leq C^{vc}\sqrt{\frac{10d + \log(1/\delta)}{n}} \).

We now show the modulus of continuity is upper bounded by \( \rho_2(\bar{\epsilon}) \). We still apply mean cross lemma on the function \( f = \ell(\theta^*(p_2), X) - \ell(\theta^*(p_1), X) \). Define \( \ell(\theta, X) = (\theta^\top X - Y)^2 \), then the bridge function is \( B(p, \theta) = \mathbb{E}_p[\ell(\theta, X) - \ell(\theta^*(p), X)] \).

From Lemma 5.2, we know that for any \( f \in \mathcal{H} \), there exists \( r_{p_1} \leq \frac{p_2}{1 - \epsilon} \) and \( r_{p_2} \leq \frac{p_2}{1 - \epsilon} \) such that the mean under \( f \) of \( r_{p_1} \) and \( r_{p_2} \) can cross. Taking \( f = \ell(\theta^*(p_2), X) - \ell(\theta^*(p_1), X) \), we have

\[
\mathbb{E}_{r_{p_1}}[\ell(\theta^*(p_2), X) - \ell(\theta^*(p_1), X)] \leq \mathbb{E}_{r_{p_2}}[\ell(\theta^*(p_2), X) - \ell(\theta^*(p_1), X)] \leq \mathbb{E}_{r_{p_2}}[\ell(\theta^*(p_2), X) - \ell(\theta^*(r_{p_2}), X)] = B(r_{p_2}, \theta^*(r_{p_2})) \leq r_1(\epsilon). \tag{67}
\]

The last inequality comes from the fact that \( p_2 \in \mathcal{G}^\text{TV}_\delta(\rho_1(\epsilon), \epsilon) \). Combining the above equation with the fact that \( p_1 \in \mathcal{G}^\text{TV}_\delta(\rho_1(\epsilon), \epsilon) \), we know

\[
\mathbb{E}_{r_{p_1}}[\ell(\theta^*(p_2), X) - \ell(\theta^*(r_{p_1}), X)] = \mathbb{E}_{r_{p_1}}[\ell(\theta^*(p_2), X) - \ell(\theta^*(p_1), X) + \ell(\theta^*(p_1), X) - \ell(\theta^*(r_{p_1}), X)] \leq 2r_1(\epsilon). \tag{68}
\]

From \( p \in \mathcal{G}^\text{TV}_\delta(2\rho_1(\epsilon), \rho_2, \epsilon) \), this implies that \( L(p_1, \theta^*(p_2)) \leq r_2(\epsilon) \), which implies the final conclusion once we take \( B(p, \theta) = L(p, \theta) = \mathbb{E}_p[(Y - X^\top \theta)^2 - (Y - X^\top \theta(p))^2] \). This proof actually works for any \( \ell \) and \( B(p, \theta) = \mathbb{E}_p[\ell(\theta, X) - \ell(\theta^*(p), X)] \) as excess predictive loss.

**Interpretation and comparison** Combining the above result with the interpretation of Example 3.2, under \( k \)-hyper-contractivity condition of \( X \) and bounded \( k \)-th moment condition of \( Y - X^\top \theta^*(p) \), our projection algorithm guarantees the excess predictive loss to be \( O(\epsilon^{2-4/k}) \) given \( O(d/\epsilon^2) \) samples, while Klivans et al. (2018) gives \( O(\epsilon^{1-2/k}) \) assuming \( O(\text{poly}(d^k, 1/\epsilon)) \) samples.

When \( X \) and \( Z \) are both sub-Gaussian, our dependence on \( \epsilon \) is the same as the Gaussian example in Gao (2017) up to a log factor while the sample complexity matches Gao (2017) exactly.

Diakonikolas et al. (2019c) guarantee parameter error \( \|\theta - \theta^*\|_2 \leq (\epsilon \log(1/\epsilon))^2 \) given \( O(d/\epsilon^2) \) samples when \( X \) is isotropic Gaussian and \( Z \) has bounded second moment.
\( \tilde{\text{TV}}_{\mathcal{H}} \) projection for second moment estimation. Let \( M_p = \mathbb{E}_p[XX^T] \). For second moment estimation under the loss \( L(p, M) = \| I - M_p^{-1/2}MM_p^{-1/2} \|_F \) in Example 3.4, we take \( \mathcal{H} \) to be
\[
\mathcal{H} = \{ \text{Tr}(A(I - M^{-1/2}XX^T M^{-1/2})) \mid \| A \|_F \leq 1, M \in \mathbb{R}^{d \times d} \},
\]
motivated by the dual representation of \( B(p, M) = \| I - M^{-1/2}M_pM^{-1/2} \|_F \) as the Frobenius norm.

Directly applying mean cross lemma to \( f \in \mathcal{H} \) cannot control the modulus since \( B(p, M) \) does not satisfy triangle inequality. Instead, we apply minimax theorem to control the modulus of continuity.

**Theorem 5.3.** Denote \( \epsilon = 2\epsilon + 2C^\omega \sqrt{\frac{d^2 + 1 + \log(1/\delta)}{n}} \). Assume \( p^* \in G_{TV}(\rho(\epsilon), \rho_2(\epsilon), \tilde{\epsilon}) \) in Example 3.4. For \( \mathcal{H} \) designed in (69), let \( q \) denote the output of the projection algorithm \( \Pi(\tilde{p}_n; \tilde{\text{TV}}_{\mathcal{H}}, G_{TV}) \). Then, with probability at least \( 1 - \delta \),
\[
\| I - M_q^{-1/2}M_p^*M_q^{-1/2} \|_F \leq \rho_2(\epsilon) = \rho_2 \left( 2\epsilon + 2C^\omega \sqrt{\frac{d^2 + 1 + \log(1/\delta)}{n}} \right).
\]

**Proof.** By Proposition 5.1 it suffices to bound \( \tilde{\text{TV}}_{\mathcal{H}}(p, \tilde{p}_n) \) and to bound the modulus of continuity for \( G_{TV} \) under \( \tilde{\text{TV}}_{\mathcal{H}} \).

From Dudley’s lemma (Dudley, 1978), for every \( f \in \mathcal{H} \), there exists a matrix \( B \in \mathbb{R}^{d \times d}, a \in \mathbb{R} \) such that \( \{ x \mid f(x) \geq t \} = \{ x \mid x^\top Bx \geq a \} \). Thus the VC dimension is bounded by \( d^2 + 1 \). From Lemma 5.1 we have \( \tilde{\text{TV}}_{\mathcal{H}}(p, q) \leq C^\omega \sqrt{\frac{d^2 + 1 + \log(1/\delta)}{n}} \).

Next we bound the modulus of continuity. The mean cross lemma (Lemma 5.2) gives us that for any given \( f \in \mathcal{H} \), we can find some \( r_{p_1}, r_{p_2} \) that makes the mean under \( f \) cross. However, we want to find some \( r \) such that \( B(r, M) \) small due to the implication in \( G_1 \). Since the dual representation of \( B \) is taking supremum over all \( f \), we want the existence of a single \( r \) that controls the mean under any \( f \). To convert the maxmin to minimax form, we apply minimax theorem here.

From \( p_1 \in G_{TV}^+(\rho_1(\epsilon), \rho_2(\epsilon), \tilde{\epsilon}) \), and \( B(p, M) = \| I - M^{-1/2}M_pM^{-1/2} \|_F \), we know that it suffices to show that there exists some \( r_{p_1} \), such that \( \| I - M_{p_2}^{-1/2}M_{r_{p_1}}M_{p_2}^{-1/2} \|_F \leq \rho_1(\epsilon) \). From Lemma 5.2, we know that for any fixed \( A \) that satisfies \( \| A \|_F \leq 1 \), we have \( r_{p_1} \leq \frac{p_1}{\rho} \), \( r_{p_2} \leq \frac{\rho p}{\rho_2} \), such that
\[
\text{Tr}(A(I - M_{p_2}^{-1/2}M_{r_{p_1}}M_{p_2}^{-1/2})) \leq \text{Tr}(A(I - M_{p_2}^{-1/2}M_{r_{p_2}}M_{p_2}^{-1/2})) \leq \rho_1(\epsilon).
\]
The last inequality is from the fact that \( p_2 \in G_{TV}^+(\rho(\epsilon), \tilde{\epsilon}) \). Thus we have
\[
\sup_{\| A \|_F \leq 1} \inf_{r_{p_1} \leq \frac{p_1}{\rho}} \text{Tr}(A(I - M_{p_2}^{-1/2}M_{r_{p_1}}M_{p_2}^{-1/2})) \leq \rho_1(\epsilon).
\]

Now the key is to check that the minimax theorem holds, i.e.
\[
\min_{r_{p_1} \leq \frac{p_1}{\rho}} \sup_{\| A \|_F \leq 1} \text{Tr}(A(I - M_{p_2}^{-1/2}M_{r_{p_1}}M_{p_2}^{-1/2})) = \sup_{\| A \|_F \leq 1} \min_{r_{p_1} \leq \frac{p_1}{\rho}} \text{Tr}(A(I - M_{p_2}^{-1/2}M_{r_{p_1}}M_{p_2}^{-1/2})).
\]

Note that the set \( \| A \|_F \leq 1 \) is a compact set in \( \mathbb{R}^{d \times d} \), and the function is upper semicontinuous and linear in \( A \) and linear in \( r_{p_1} \). Thus we know the minimax theorem holds by (Fan, 1953, Theorem 2). Thus there exists some \( r_{p_1} \leq \frac{p_1}{\rho} \), such that
\[
\| I - M_{p_2}^{-1/2}M_{r_{p_1}}M_{p_2}^{-1/2} \|_F \leq \rho_1(\epsilon).
\]
From \( p_1 \in G_{TV}^+(\rho_1(\epsilon), \rho_2(\epsilon), \tilde{\epsilon}) \) we can derive the conclusion. \( \square \)
Interpretation and Comparison. Diakonikolas et al. (2017) constructed an algorithm that achieves error $\epsilon \log(1/\epsilon)$ with constant probability given sample size $n \gtrsim \frac{d^2}{\epsilon^2} \log(d/\epsilon)$ when the real distribution is Gaussian. We achieve the same error for all resilient distributions given sample size $n \gtrsim \frac{d^2}{\epsilon^2}$.

Not only the minimizer works in MD functional We show in Proposition 7.1 that instead of looking for the exact projection $q = \Pi(\hat{p}_n; \mathcal{TV}_H, \mathcal{G}^{TV})$, all the above results also hold if we simply find some distribution $q = \Pi(\hat{p}_n; \mathcal{TV}_H, \mathcal{G}^{TV}, \epsilon/2)$ in Algorithm 1, i.e. it suffices to find some $q \in \mathcal{G}^{TV}$ that is within $\epsilon + f(n, \delta, \mathcal{H})$ TV-distance of $\hat{p}_n$, where $f(n, \delta, \mathcal{H})$ is the $1 - \delta$ quantile of $\mathcal{TV}(p, \mathcal{H})$, which usually takes the form $C^{\text{vc}} \cdot \sqrt{\frac{\text{vc}(\mathcal{H}) + \log(1/\delta)}{n}}$ (Lemma 5.1).

We can also use $\tilde{\mathcal{TV}}_H$ projection for joint mean and covariance estimation; the detailed discussion is deferred to Appendix E.3, where we adopt a different method other than mean cross lemma to show the modulus, and discuss the recovery under two different choices of the loss $L$. For multiplicative cost (Theorem E.2), our results get the same population limit as Kothari and Steurer (2017) while improving the sample complexity’s dependence on dimension from $O(d^k)$ to $O(d)$. For cost under 2-norm (Theorem E.3), our results generalize the result in Gao et al. (2019) from Gaussians to a non-parametric set with a necessary sacrifice of a log factor.

5.2 Expand Destination Set $\mathcal{M}$

Besides weakening the distance $D = \mathcal{TV}$, an alternative way to rescue the projection algorithm is to ensure that the destination set $\mathcal{M}$ is large enough. In this section, we assume the corruption model is either oblivious corruption (Definition 2.3) or adaptive corruption (Definition 2.4) of level $\epsilon$, and denote $\hat{p}_n$ as the corrupted empirical distribution. Note that here our analysis can deal with both corruption models, while the analysis in Section 5.1 can only deal with oblivious corruption.

We will focus on mean estimation because it is well-studied, but our analysis strategy applies more generally. Interestingly, all the results in this section apply for both $\mathcal{TV}$ and $\tilde{\mathcal{TV}}$ projection. For $\mathcal{TV}$ projection, our results cover the existing filtering and convex programming approaches; for $\tilde{\mathcal{TV}}_H$ projection, our results provide new bounds for the algorithm in Section 5.1. We summarize the idea of analyzing $\mathcal{TV}$ projection in Figure 9. Intuitively, for projection onto $\mathcal{M}$ to work well, we need three properties to hold:

- $\mathcal{M}$ is large enough: $\hat{p}_n^* \in \mathcal{M}$ with high probability.
- The empirical loss $L(\hat{p}_n^*, \theta)$ is a good approximation to the population loss $L(p^*, \theta)$.
- The modulus is still bounded: $\min_{p,q \in \mathcal{M}: \mathcal{TV}(p,q) \leq 2\epsilon} L(p, \theta^*(q))$ is small.

In fact, we only need $\hat{p}_n^*$ to be near $\mathcal{M}$ with high probability, as long as there exists some $\tilde{p}' \in \mathcal{M}$ such that $\mathcal{TV}(\tilde{p}', \hat{p}_n^*)$ is small and $L(\tilde{p}', \theta)$ is a good approximation to $L(p^*, \theta)$. Moreover, we get stronger bounds when this $\tilde{p}'$ (or $\hat{p}_n^*$ itself) lies in some smaller set $\mathcal{G}' \subset \mathcal{M}$. This is formalized in the following proposition:

**Proposition 5.2.** For a set $\mathcal{G}' \subset \mathcal{M}$, define the generalized modulus of continuity as

$$m(\mathcal{G}', \mathcal{M}, \epsilon) \triangleq \min_{p \in \mathcal{G}', q \in \mathcal{M}: \mathcal{TV}(p,q) \leq \epsilon} L(p, \theta^*(q)).$$

Assume $\mathcal{TV}(\tilde{p}', \hat{p}_n^*) \leq \epsilon_1$ with probability at least $1 - \delta$ and $\tilde{p}' \in \mathcal{G}'$ with probability at least $1 - \delta$. Then the minimum distance functional projecting under $\mathcal{TV}$ onto $\mathcal{M}$ has empirical error $L(\tilde{p}', \theta)$ at most $m(\mathcal{G}', \mathcal{M}, \tilde{\epsilon})$ with probability at least $1 - 3\delta$, where $\tilde{\epsilon} = 2(\sqrt{\epsilon} + \sqrt{\frac{\log(1/\delta)}{2n}})^2 + 2\epsilon_1$. 

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Proof. From Lemma B.1, we know that with probability at least $1 - \delta$, $\text{TV}(\tilde{p}_n^*, \tilde{p}_n) \leq (\sqrt{\epsilon} + \sqrt{\log(1/\delta) / 2n})^2$. Thus by triangle inequality, $\text{TV}(\tilde{p}_n, \tilde{p}') \leq \epsilon_1 + (\sqrt{\epsilon} + \sqrt{\log(1/\delta) / 2n})^2 = \epsilon / 2$ with probability at least $1 - 2\delta$. If $\tilde{p}'$ lies in $\mathcal{G}'$, then since $\mathcal{G}' \subseteq \mathcal{M}$ we know that $\tilde{p}_n$ has distance at most $\epsilon / 2$ from $\mathcal{M}$, and so the projected distribution $q$ satisfies $\text{TV}(q, \tilde{p}_n) \leq \epsilon / 2$ and hence $\text{TV}(q, \tilde{p}') \leq \epsilon$. It follows from the definition that $L(\tilde{p}', \hat{\theta}) = L(\tilde{p}', \theta^*(q)) \leq \mathbf{m}(\mathcal{G}', \mathcal{M}, \hat{\epsilon})$.

To employ Proposition 5.2, we construct $\mathcal{M}$ and $\mathcal{G}'$ such that the generalized modulus is small, then exhibit some $\tilde{p}' \in \mathcal{G}'$ that is close to $\tilde{p}_n^*$. Often $\tilde{p}'$ will just be $\tilde{p}_n^*$, but sometimes it is a perturbed version of $\tilde{p}_n^*$ that deletes heavy-tailed “bad” events.

The key difficulty in analysis is that the empirical distribution $\tilde{p}_n^*$ may not inherit the good properties of $p^*$, e.g. the empirical distribution of an isotropic Gaussian distribution does not have constant $k$-th moment unless the sample size $n \gg d^{k/2}$ for constant $k$. Additionally, when $k = 2$, even if the original distribution has bounded Euclidean norm $\sqrt{d}$, its empirical distribution with $d$ samples is not resilient with the right scale of $\rho = O(\sqrt{n})$. We discussed this in Appendix E.6.3.

Since we cannot hope to establish properties like bounded $k$th moments for $\tilde{p}_n^*$, we instead rely on three main techniques to control some other properties of $\tilde{p}_n^*$ or $\tilde{p}'$: union bound, moment truncation, and perturbing $\tilde{p}_n^*$ to $\tilde{p}'$.

**Union bound.** Our first observation is that even if $\mathcal{G}$ is small, we can take $\mathcal{M}$ to be the family of resilient distributions while maintaining small modulus. Thus we only need $\tilde{p}_n^*$ to be resilient, which is easier to satisfy than e.g. bounded moments or sub-Gaussianity. For distributions with moment generating functions, a union bound leads to the following typical result (Lemma E.11): if $p^*$ is sub-Gaussian with parameter $\sigma$, then for any fixed $\eta$, the empirical distribution $\tilde{p}_n^*$ is $(\rho, \eta)$-resilient with probability $1 - \delta$, for $\rho = O(\sigma(\sqrt{\eta d + \log(1/\delta)} / n) + \eta \sqrt{\log(1/\eta)})$, which gives tighter bound for

![Figure 9: Illustration of projection algorithm. By perturbing $\tilde{p}_n^*$ to $\tilde{p}'$ a small amount ($TV(\tilde{p}_n^*, \tilde{p}') \lesssim \epsilon + 1/n$) under $TV$, we can ensure $\tilde{p}' \in \mathcal{G}'$ and control $TV(\tilde{p}_n, \tilde{p}')$ simultaneously. Thus projecting $\tilde{p}_n$ onto a larger set $\mathcal{M}$ can guarantee $TV_H(q, \tilde{p}_n) \leq TV(q, \tilde{p}_n) \lesssim \epsilon + 1/n$. In contrast, projecting onto the original $\mathcal{G}$ (as Section 5.1) can only guarantee $TV_H(q, \tilde{p}_n) \lesssim \epsilon + \sqrt{\text{vc}(\mathcal{H})/n}$, which gives sub-optimal sample complexity in most cases.](image-url)
resilience parameter than Diakonikolas et al. (2019a, Lemma 4.4). We thus obtain:

**Theorem 5.4 (Sub-Gaussian).** Denote $\bar{\epsilon} = 2(\sqrt{\epsilon} + \sqrt{\frac{\log(1/\delta)}{2n}})^2$. There exist some constants $C_1, C_2$ such that the following statement is true. Take $G$ as family of sub-Gaussian with parameter $\sigma$ and $M$ as resilient set, i.e.

$$G = \left\{ p \mid \sup_{v \in \mathbb{R}^d, ||v||_2 = 1} \mathbb{E}_p \left[ \exp \left( \frac{|v^\top (X - \mathbb{E}_p[X])|^2}{\sigma^2} \right) \right] \leq 2 \right\},$$

$$M = G_{\text{mean}}^{TV} \left( C_1 \sigma \cdot \left( \epsilon \sqrt{\log(1/\epsilon)} + \sqrt{\frac{d + \log(1/\delta)}{n}} \right), \bar{\epsilon} \right),$$

where $G_{\text{mean}}^{TV}$ is defined in (19). If $p^* \in G$ and $\bar{\epsilon} \leq 1/2$, then the projection $q = \Pi(\hat{p}_n; TV/\tilde{TV}_H, M)$ of $\hat{p}_n$ onto $M$ satisfies:

$$||\mathbb{E}_{p^*}[X] - \mathbb{E}_q[X]||_2 \leq C_2 \sigma \cdot \left( \epsilon \sqrt{\log(1/\epsilon)} + \sqrt{\frac{d + \log(1/\delta)}{n}} \right)$$

with probability at least $1 - 3\delta$. Moreover, this bound holds for any $q \in M$ within TV (or $\tilde{TV}$) distance $\hat{\epsilon}/2$ of $\hat{p}_n$.

The proof is deferred to Appendix E.7.

**Truncated moment.** The above technique only applies to the case when the population distribution has a moment generating function, because it requires union bounding over exponentially many events. If instead we only have bounded $k$th moments, we need a different approach. Although $\hat{p}_n^*$ does not have small $k$-th moment with less than $d^{k/2}$ samples, our key insight is that we can bound a certain truncated $k$-th moment with only $\Theta(d)$ samples, which is sufficient to ensure resilience (see Lemma E.10 for a rigorous statement and extension to any Orlicz norm). For instance, if $p^*$ has 4th moments bounded by $\sigma^4$ then we will bound $\sup_{||v||_2 \leq 1} \mathbb{E}_{p^*}[\psi(|v^\top X|)]$, where $\psi(x)$ is the smallest convex function on $[0, \infty)$ that coincides with $x^4$ when $0 \leq x \leq 4\sigma$.

Taking $M$ to be the set of resilient distributions as before, we obtain:

**Theorem 5.5 (Bounded $k$-th moment).** Denote $\bar{\epsilon} = 2(\sqrt{\epsilon} + \sqrt{\frac{\log(1/\delta)}{2n}})^2$. There exist some constants $C_1, C_2$ such that the following statement is true. Take $G$ as bounded $k$-th moment set for $k \geq 2$ and $M$ as resilient set, i.e.

$$G = \left\{ p \mid \sup_{v \in \mathbb{R}^d, ||v||_2 = 1} \mathbb{E}_p [|v^\top (X - \mathbb{E}_p[X])|^k] \leq \sigma^k \right\}$$

$$M = G_{\text{mean}}^{TV} \left( C_1 \sigma \cdot \left( \frac{e^{1-1/k}}{\epsilon^{1/k}} + \frac{1}{\delta \sqrt{n}} \right), \bar{\epsilon} \right),$$

If $p^* \in G$ and $\bar{\epsilon} \leq 1/2$, then the projection $q = \Pi(\hat{p}_n; TV/\tilde{TV}_H, M)$ of $\hat{p}_n$ onto $M$ satisfies:

$$||\mathbb{E}_{p^*}[X] - \mathbb{E}_q[X]||_2 \leq C_2 \sigma \cdot \left( \frac{e^{1-1/k}}{\epsilon^{1/k}} + \frac{1}{\delta \sqrt{n}} \right)$$

with probability at least $1 - 4\delta$. Moreover, this bound holds for any $q \in M$ within TV (or $\tilde{TV}$) distance $\hat{\epsilon}/2$ of $\hat{p}_n$.

The proof is deferred to Appendix E.8. Steinhardt et al. (2018) presented an analysis for the same projection algorithm that requires $d^{3/2}$ samples, which our result improves to $d$. 

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Perturb \( \hat{p}_n^* \) to \( \hat{p}' \). If \( p^* \) has covariance operator norm bounded by \( \sigma \), then in general with even \( d \log d \) samples one cannot guarantee that \( \hat{p}_n^* \) has covariance with operator norm \( O(\sigma) \). However, it was realized in Steinhardt et al. (2017a) that one may construct another distribution \( \hat{p}' \) with 
\[
\TV(\hat{p}_n^*, \hat{p}') \lesssim \epsilon
\]
such that \( \hat{p}' \) has covariance bounded by \( O(\sigma) \) given \( d \log d \) samples. Thus instead of checking the empirical distribution \( \hat{p}_n^* \in G' \), we can construct some \( \hat{p} \in G' \) such that 
\[
\TV(\hat{p}_n^*, \hat{p}) \lesssim \epsilon.
\]

This allows us to take \( \mathcal{M} \) to be the set of bounded covariance distributions instead of all resilient distributions; the advantage of this is that there are computationally efficient algorithms that approximately solve the projection in some cases (Diakonikolas et al., 2017, 2019c).

This motivates us to consider mean estimation for bounded \( k \)-th moment distribution with identity covariance. The identity covariance assumption allows us to take \( \mathcal{M} \) to be the set of bounded covariance distributions, which we believe admits an efficient projection algorithm analogous to that of Diakonikolas et al. (2017) for isotropic sub-Gaussians. The statistical analysis requires the simultaneous application of the truncated moment and perturbation techniques described above, as well as a generalized modulus result (Lemma E.3). We obtain:

**Theorem 5.6** (Bounded \( k \)-th moment and identity covariance). Denote \( \tilde{\epsilon} = 4(\sqrt{\epsilon + \frac{\log(1/\delta)}{n}} + \sqrt{\frac{\log(1/\delta)}{2n}})^2 \). Take \( \mathcal{G} \) to be the set of isotropic distributions with bounded \( k \)-th moment, \( k \geq 2 \), and \( \mathcal{M} \) to be the set of bounded covariance distributions:

\[
\mathcal{G} = \{ p \mid \mathbb{E}_p[(X - \mu_p)(X - \mu_p)^\top] = I_d, \sup_{v \in \mathbb{R}^d, \|v\|_2=1} \mathbb{E}_p[|v^\top(X - \mu_p)|^k] \leq \sigma^k \}, \tag{82}
\]

\[
\mathcal{M} = \{ p \mid \|\mathbb{E}_p[(X - \mu_p)(X - \mu_p)^\top]\|_2 \leq 1 + f(n, d, \epsilon, \delta, \sigma) \}. \tag{83}
\]

If \( p^* \in \mathcal{G} \) and \( \tilde{\epsilon} < 1/2 \), then there exists some \( f(n, d, \epsilon, \delta, \sigma) \) such that the projection \( q = \Pi(\hat{p}_n; \TV/\tilde{\TV}_H, \mathcal{M}) \) of \( \hat{p}_n \) onto \( \mathcal{M} \) satisfies

\[
\|\mu_{p^*} - \mu_q\|_2 = O\left( \sigma \cdot \left( \frac{k \epsilon^{1-1/k}}{\delta^{1/k}} + \frac{k}{\delta} \sqrt{\frac{d \log(d)}{n}} \right) \right) \tag{84}
\]

with probability at least \( 1 - 8\delta \). Moreover, this bound holds for any \( q \in \mathcal{M} \) within \( \TV \) (or \( \tilde{\TV} \)) distance \( \tilde{\epsilon}/2 \) of \( \hat{p}_n \).

The proof is deferred to Appendix E.10. When \( \delta \) is constant and \( n \) goes to infinity, this bound recovers \( O(\epsilon^{1-1/k}) \), which is the population limit for bounded \( k \)-th moment distributions. It also guarantees sample complexity of \( O(d \log(d)/\epsilon^{2-2/k}) \).

The general framework of this analysis approach is summarized in Theorem 7.2, where we analyze the projection algorithm for general \( D, L \), and \( \mathcal{G} \).

**5.3 Interpretation and Comparison for Mean Estimation**

To contrast the approaches in Sections 5.1 and 5.2, here we compare their implied bounds for robust mean estimation (Table 1) and also discuss related literature.
| $\mathcal{G}$        | $\mathcal{M}$      | $D$                     | $||\mu_{p^*} - \mu_q||_2$ |
|----------------------|---------------------|-------------------------|-----------------------------|
| $\mathcal{G}^{TV}_{\text{mean}}$ | $\mathcal{G}^{TV}_{\text{mean}}$ | $\tilde{TV}$          | $\rho(2\epsilon + 2C^{\text{wc}} \sqrt{\frac{d+1+\log(1/\delta)}{n}})$ [Theorem 5.1] |
| sub-Gaussian         |                      | TV or $\tilde{TV}$     | $\epsilon \sqrt{\log(1/\epsilon)} + \sqrt{\frac{d+\log(1/\delta)}{n}}$ [Theorem 5.4] |
| bdd $k$-th moment ($k \geq 2$) |                      | TV or $\tilde{TV}$     | $\frac{\epsilon}{\delta/k} + \frac{1}{2} \sqrt{\frac{d}{n}}$ [Theorem 5.5] |
| bdd cov              | bdd cov             | TV or $\tilde{TV}$     | $\frac{\epsilon}{\delta/k} + \frac{1}{2} \sqrt{\log(d/\delta) \frac{d}{n}}$ [Theorem E.6] |
| bdd $k$-th moment ($k > 2$) + $I_d$ | bdd cov             | TV or $\tilde{TV}$     | $\frac{\epsilon}{\delta/k} + \frac{1}{2} \sqrt{\log(d/\delta) \frac{d}{n}}$ [Theorem 5.6] |

Table 1: Summary of results for generalized projection algorithm assuming $p^* \in \mathcal{G}$. Here ‘bdd’ is short for ‘bounded’.

**Interpretation and comparison for $\tilde{TV}_H$ projection algorithm**  
If $p^*$ is sub-Gaussian, analysis in Theorem 5.1 implies an error $O(\epsilon \sqrt{\log(1/\epsilon)})$ when $n \gtrsim \frac{d+\log(1/\delta)}{\epsilon^2 \log(1/\epsilon)}$, while analysis in Theorem 5.4 shows that the same algorithm actually only requires $n \gtrsim \frac{d+\log(1/\delta)}{\epsilon^2 \log(1/\epsilon)}$, which is optimal. Here Theorem 5.4 is not explicitly stated in the literature but should be well known among experts. We are also able to generalize it to all distributions with moment generating functions by Lemma E.11, which provides better bound than Diakonikolas et al. (2019a, Lemma 4.4) for sub-Gaussian case.

If $p^*$ has bounded covariance, then Theorem 5.1 implies an error of order $O(\sqrt{\epsilon})$ when $n \gtrsim \frac{d+\log(1/\delta)}{\epsilon^2 \log(1/\epsilon)}$. This dependence of $n$ on $\epsilon$ is sub-optimal; inspired by Prasad et al. (2019), our Theorem E.6 shows that the same $\tilde{TV}_H$ projection algorithm reaches $O(\sqrt{\epsilon})$ error when $n \gtrsim \frac{d \log(d/\delta)}{\epsilon}$, which has better dependence of $n$ on $\epsilon$ but worse dependence on $d$.

When $\epsilon = 0$, the current analysis in Theorem 5.1 for projecting under resilient set yields an error of $O(\sqrt{\frac{d+\log(1/\delta)}{n}})$ error, and our Theorem E.6 for projecting under bounded covariance set gives $O(\sqrt{\frac{d \log(d/\delta)}{n}})$, while median-of-means tournament method (Lugosi and Mendelson, 2019) can achieve $O((\frac{d+\log(1/\delta)}{n})^{1/2})$. It is an interesting problem to check whether one can tighten the analysis of those projection algorithms.

**Interpretation and comparison for TV projection algorithm**  
For $k$-th moment bounded distributions, the truncated moment approach (Lemma E.10) and Theorem 5.5 are new, and prior to our work Steinhardt (2018) shows that projection onto resilient set under TV distance works for $k$-th moment bounded distributions with $d^{3/2}$ samples. Prasad et al. (2019) used the approach of reducing high-dimensional mean estimation to one dimensional to achieve error $\epsilon^{1-1/k} + \sqrt{\frac{d \log(1/\delta)}{n}}$, while we generalize it to all distributions with bounded Orlicz norms (Corollary E.2). Theorem 5.6 presents the first guarantee for potentially efficient mean estimation algorithm of bounded $k$-th moment isotropic distributions with sample complexity $O(d \log d)$. It was shown in the literature that for sub-Gaussian distributions with identity covariance (Diakonikolas et al., 2018a,c; Cheng et al., 2019a,b), one can also choose $\mathcal{M}$ to be the bounded covariance set to achieve efficient computation and near-optimal statistical rate.

The flexibility of only needing to find some $q \in \mathcal{M}$ with $\text{TV}(q, \hat{p}_n) \leq \epsilon$ is useful for efficient algorithms. Indeed, many filtering and convex programming algorithms in the literature Diakonikolas et al. (2018a,c); Cheng et al. (2019a,b) can be interpreted as finding some distribution $q \in \mathcal{M}$, so that our analysis below leads to sample complexity bounds for those and similar algorithms. Usually these efficient algorithms are trying to find $q$ among deleted versions of $\hat{p}_n$, hence even after exhaustive search it is not guaranteed to find the $\hat{p}'$ (or $\hat{p}'_n$) $\in \mathcal{G}'$ since some of the samples might be already deleted. Hence, one additional step in analyzing the statistical properties...
of these efficient algorithms is to use our Lemma C.7 and Lemma C.5, which show that any Orlicz norm bounded set and resilient set are approximately closed under deletion. Thus as long as \( G' \) is of these types, any deleted version of \( \hat{p}' \) or \( \tilde{p}_n' \) would approximately share the same properties as \( \hat{p}' \) or \( \tilde{p}_n' \).

However, some filtering algorithms in the literature may not be cast as a projection algorithm; for instance in Diakonikolas et al. (2018a), one is required to follow a particular path of deleting \( \hat{p}_n \) to ensure that for each filter step one deletes way more bad points than good points, and the filter there only works under the Huber additive corruption model but not TV corruption.

6 Finite Sample Algorithm for \( W_1 \)

In this section, we design finite sample algorithms for robust estimation under \( W_1 \) perturbations. Throughout the section, we make the assumption that \( \hat{p}_n \) follows oblivious corruption model (Definition 2.3) of level \( \epsilon \) under \( W_1 \) perturbation.

Similar to the issue of TV\((p, \hat{p}_n)\) discussed in Section 5, in general \( W_{c,k}(p, \hat{p}_n) \) converges to zero slowly even when \( p \) is well behaved; for instance, \( \mathbb{E}[W_1(p, \hat{p}_n)] \geq n^{-1/d} \) for any measure \( p \) that is absolutely continuous with respect to the Lebesgue measure on \( \mathbb{R}^d \) (Dudley, 1969). Similarly to Section 5.1, we therefore weaken \( W_1 \) to a distance \( \tilde{W}_1 \) to achieve better finite sample performance.

6.1 Second Moment Estimation: Design of \( \tilde{W}_1 \)

We first introduce our design of \( \tilde{W}_1 \) for second moment estimation. Later we show this specific \( \tilde{W}_1 \) works for linear regression as well. Recall the dual representation of \( W_1 \):

\[
W_1(p, q) = \sup_{u \in \mathcal{U}} \mathbb{E}_p[u(X)] - \mathbb{E}_q[u(X)],
\]

where \( \mathcal{U} = \{ u : \mathbb{R}^d \rightarrow \mathbb{R} | \|u(x) - u(y)\| \leq \|x - y\| \} \) is the set of all 1-Lipschitz functions.

To design the weakened distance \( \tilde{W}_1(p, q) \), we take the supremaum over a smaller set. Let \( \mathcal{U}' \) be the set of all 1-Lipschitz linear or rectified linear functions:

\[
\mathcal{U}' = \{ \max(0, v^\top x - a) | v \in \mathbb{R}^d, \|v\| \leq 1, a \in \mathbb{R} \} \cup \{ v^\top x | v \in \mathbb{R}^d, \|v\| \leq 1 \}.
\]

Since all functions in \( \mathcal{U}' \) are Lipschitz-1, \( \mathcal{U}' \subset \mathcal{U} \), and we define \( \tilde{W}_1 \) as follows:

\[
\tilde{W}_1(p, q) = \sup_{u \in \mathcal{U}'} |\mathbb{E}_p[u(X)] - \mathbb{E}_q[u(X)]|.
\]

Similarly to Proposition 5.1 in Section 5.1, the key to analyzing \( \tilde{W}_1(p, q) \) is to control (i) \( \tilde{W}_1(p, \hat{p}_n) \) and (ii) the modulus of continuity for \( G^{W_1} \) under \( \tilde{W}_1 \).

We first use the following lemma to bound the statistical error \( \tilde{W}_1(p, \hat{p}_n) \):

**Lemma 6.1.** Assume \( W_1(p^*, p) \leq \epsilon \) and for some Orlicz function \( \psi \), \( p^* \) satisfies

\[
\sup_{v \in \mathbb{R}^d, \|v\|_2=1} \mathbb{E}_{p^*}[\psi(|v^\top (X - \mathbb{E}_{p^*}[X])|/\kappa)] \leq 1,
\]

\[
\|\mathbb{E}_{p^*}[(X - \mu_{p^*})(X - \mu_{p^*})^\top]\|_2 \leq \sigma^2.
\]

Then for \( \tilde{W}_1 \) defined in (87), we have

\[
\mathbb{E}_p[\tilde{W}_1(p, \hat{p}_n)] \leq 2\epsilon + 8\sigma \sqrt{\frac{d}{n}} + \frac{3\kappa \psi^{-1}(\sqrt{n})}{\sqrt{n}}.
\]

39
The proof is deferred to Appendix F.2. Compared to the lower bound $\mathbb{E}[W_1(p, \hat{p}_n)] \gtrsim n^{-1/d}$, the upper bound for $\tilde{W}_1$ reduces the sample complexity from exponential in $d$ to polynomial in $d$.

To bound the modulus of continuity, we establish a mean cross lemma showing that $p$ and $q$ have friendly perturbations $r_p$ and $r_q$ with $\mathbb{E}_{r_p}[f(X)] \leq \mathbb{E}_{r_q}[f(X)]$ (cf. Lemma 5.2). In Lemma 5.2 for $\tilde{W}_1$, we established the mean crossing property by showing that $r_p$ stochastically dominated $r_q$; for $\tilde{W}_1$ we will instead show that $r_p$ dominates $r_q$ in the convex order, which establishes mean crossing for convex functions $f$:

![Figure 10: Illustration of the mean cross lemma for $\tilde{W}_1$ in the special case that $p_1$ and $p_2$ have the same mean. If $\tilde{W}_1(\rho_1, \rho_2)$ is small, we can push the $\epsilon$-tails of $p_2$ to two points on the boundary of the new distribution to create a friendly perturbation $r_{p_2}$ that guarantees convex ordering.](image)

**Lemma 6.2.** Consider two distributions $p, q$ on the real line such that $\tilde{W}_1(p, q) \leq \epsilon$. Then, for $g(x) = x$ or $g(x) = |x|$, one can find $r_p \in \mathbb{F}(p, 7\epsilon, W_1, g(x))$ and $r_q \in \mathbb{F}(q, 7\epsilon, W_1, g(x))$ such that $r_p$ is less than $r_q$ in the convex order with respect to the random variable $g(X)$: $r_p \leq_{\text{conv}} r_q$. This is equivalent to saying that for all convex functions $f$ such that the expectations exist,

$$
\mathbb{E}_{r_p}[f(g(X))] \leq \mathbb{E}_{r_q}[f(g(X))].
$$

(91)

The proof is deferred to Appendix F.1. The idea is illustrated in Figure 10 in the special case that $p$ and $q$ have equal means. Intuitively, one can guarantee convex ordering by squeezing one of the distributions towards the mean. To apply the one-dimensional mean cross lemma, we consider all one-dimensional projections and identify the optimal direction.

The reason we only consider $g(x)$ instead of $f(x)$ in the friendly perturbation constraint is that the landscape of $f(x)$ can be complicated and it is harder to characterize the moving towards mean operation under $f(x)$. Thus for instance for second moment estimation, we consider friendly perturbations $r \in \mathbb{F}(p, \eta, W_1, |v^\top x|)$ instead of $r \in \mathbb{F}(p, \eta, W_1, |v^\top x|^2)$. As discussed in Example 4.1, the sufficient condition and modulus of continuity remain unchanged from this modification.

We combine these observations to guarantee finite-sample robustness for the projection algorithm $q = \Pi(\hat{p}_n; \tilde{W}_1, G_W)$, where $G_W$ is defined as

$$
G^W(p, \eta, \tilde{W}_1) = \left\{ p \mid \sup_{\|v\|_2=1, r \in \mathbb{F}(p, \eta, W_1, |v^\top x|)} |\mathbb{E}_p[|v^\top X|^2] - \mathbb{E}_r[|v^\top X|^2]| \leq \rho \right\}.
$$

(92)

**Theorem 6.1** (Second moment estimation, $\tilde{W}_1$ projection). Assume $p^*$ has bounded $k$-th moment for $k > 2$, i.e. $\sup_{v \in \mathbb{R}^d, \|v\|_2=1} \mathbb{E}_{p^*}[|v^\top X|^k] \leq \sigma^k$ for some $\sigma > 0$. Denote \( \tilde{\epsilon} = \frac{C_1}{\sigma} \left( \epsilon + \sigma \sqrt{\frac{2}{n} + \frac{\sigma^2}{\sqrt{n} n^{1-k}}} \right) \), where $C_1$ is some universal constant. Take the projection set as $G = G^W_{\text{sec}}(4\sigma^{1+1/(k-1)}(7\epsilon)^{1-1/(k-1)}, \tilde{\epsilon})$, where $G^W_{\text{sec}}$ is defined in Example 4.1. If $\tilde{\epsilon} < \sigma / 2^{2k-3}$, then the projection algorithm $q = \Pi(\hat{p}_n; \tilde{W}_1, G)$
or \( q = \Pi(\hat{p}_n; \hat{W}_1, \mathcal{G}, \epsilon/2) \) satisfies

\[
\| \mathbb{E}_q [XX^T] - \mathbb{E}_{p^*} [XX^T] \|_2 \leq C_2 \sigma \epsilon^{1+1/(k-1)} \left( \epsilon + \sigma \sqrt{d/n} + \sigma \sqrt{n^{-1/k}} \right)^{1-1/(k-1)}
\]  

(93)

with probability at least 1 - \( \delta \), where \( C_2 \) is some universal constant.

**Proof.** First, we show that the projected distribution \( q \) is close to \( p^* \) in \( \hat{W}_1 \). We know from Lemma 6.1 that under appropriate choice of \( C_1 \), with probability at least 1 - \( \delta \), we have

\[
\hat{W}_1(p^*, \hat{p}_n) \leq \frac{\epsilon}{2} = \frac{C_1}{\delta} \left( \epsilon + \sigma \sqrt{d/n} + \frac{\sigma}{\sqrt{n^{1-1/k}}} \right).
\]  

(94)

Note that \( \hat{W}_1 \) satisfies triangle inequality and \( \mathcal{U}' \subseteq \mathcal{U} \). Therefore

\[
\hat{W}_1(q, p^*) \leq \hat{W}_1(q, \hat{p}_n) + \hat{W}_1(\hat{p}_n, p^*) \leq \epsilon.
\]  

(95)

From Lemma 2.2 we know that the final result can be upper bounded by the modulus of continuity, thus it suffices to bound the term \( \sup_{p_1, p_2 \in \mathcal{G}, \hat{W}_1(p_1, p_2) \leq \epsilon} \| M_{p_1} - M_{p_2} \|_2 \). We apply Lemma 6.2 to show that the modulus of continuity can be bounded. By symmetry, without loss of generality we can take some \( v^* \in \mathbb{R}^d, \| v^* \|_2 = 1 \), such that

\[
v^*^T (M_{p_1} - M_{p_2}) v^* = \| M_{p_1} - M_{p_2} \|_2.
\]  

(96)

From \( p_1, p_2 \in \mathcal{G}_{\epsilon^2} \), \( \hat{W}_1(p_1, p_2) \leq \epsilon \) and Lemma 6.2, we know that there exist an \( r_{p_1} \in \mathcal{F}(p_1, \epsilon, W_1, [v^*^T X]) \) and an \( r_{p_2} \in \mathcal{F}(p_2, \epsilon, W_1, [v^*^T X]) \) such that

\[
\mathbb{E}_{r_{p_1}} [(v^*^T X)^2] \leq \mathbb{E}_{r_{p_2}} [(v^*^T X)^2].
\]  

(97)

From \( p_1, p_2 \in \mathcal{G}_{\epsilon^2} \), we know that

\[
\mathbb{E}_{p_1} [(v^*^T X)^2] - \mathbb{E}_{r_{p_1}} [(v^*^T X)^2] \leq 4 \sigma^{1+1/(k-1)} (\epsilon/2)^{1-1/(k-1)},
\]  

(98)

\[
\mathbb{E}_{r_{p_2}} [(v^*^T X)^2] - \mathbb{E}_{p_2} [(v^*^T X)^2] \leq 4 \sigma^{1+1/(k-1)} (\epsilon/2)^{1-1/(k-1)}.
\]  

(99)

for all friendly perturbations \( r_{p_1}, r_{p_2} \) of \( p_1, p_2 \). Thus we know that

\[
\| M_{p_1} - M_{p_2} \|_2 = v^*^T (M_{p_1} - M_{p_2}) v^*
\]

\[
= \mathbb{E}_{p_1} [(v^*^T X)^2] - \mathbb{E}_{p_2} [(v^*^T X)^2]
\]

\[
= \mathbb{E}_{p_1} [(v^*^T X)^2] - \mathbb{E}_{r_{p_1}} [(v^*^T X)^2] + \mathbb{E}_{r_{p_1}} [(v^*^T X)^2] - \mathbb{E}_{r_{p_2}} [(v^*^T X)^2]
\]

\[
+ \mathbb{E}_{r_{p_2}} [(v^*^T X)^2] - \mathbb{E}_{p_2} [(v^*^T X)^2]
\]

\[
\leq 8 \sigma^{1+1/(k-1)} (\epsilon/2)^{1-1/(k-1)}.
\]  

(100)

This gives the desired result.

**Comparison and Interpretation**  When the third moment of \( X \) is bounded by \( \sigma^3 \), the projection algorithm guarantees error \( O(\sigma^{3/2}/\epsilon) \) for \( \epsilon \) sufficiently small and \( n \geq \max(d/\epsilon^2, 1/\epsilon^4) \), while the same condition in \( TV \) perturbation would give error of \( O(\sigma \epsilon^{2/3}) \) for \( n \geq d/\epsilon^2 \) (Theorem E.3).
6.2 Application to Linear Regression

A close inspection of the proof of Theorem 6.1 shows that it can be generalized to any loss $L$ such that $\mathcal{F}_R$ contains only functions of the form $f(|p^\top x|)$, where all the $f$ are convex functions (see Proposition F.1 for proof). We then obtain the following result for robust linear regression under $W_1$ perturbation:

**Theorem 6.2** (Linear regression in Example 4.2, $\tilde{W}_1$ projection). Assume $\theta \in \Theta = \{\theta \in \mathbb{R}^d | \|\theta\|_2 \leq R\}$ and denote $\theta^*(p^\ast) = \arg\min_{\theta \in \Theta} \mathbb{E}_{p^\ast}(Y - X^\top \theta)^2$. Denote by $X' = [X, Y - X^\top \theta^*(p^\ast)]$ the $d + 1$ dimensional vector that concatenates both $X$ and the noise $Y - X^\top \theta^*(p^\ast)$. Assume $p^\ast$ satisfies:

$$\sup_{v \in \mathbb{R}^{d+1}, \|v\|_2 = 1} \mathbb{E}_{p^\ast}[|v^\top X'|^k] \leq \sigma^k$$

(101)

for $k > 2$. Denote $\bar{\epsilon} = C_1 \left(\epsilon + \sigma \sqrt{d/n + \sigma / \sqrt{n^{-1/2}}}\right)$, where $C_1$ is some universal constant. If $\bar{\epsilon} \sqrt{R^2/2} + 1 < \sigma / 8^{k-1}$, then for $\mathcal{G}$ in Example 4.2 with appropriate parameters, the projection algorithm $q = \Pi(p_n; \tilde{W}_1, \mathcal{G})$ or $q = \Pi(p_n; \tilde{W}_1, \mathcal{G}, \bar{\epsilon}/2)$ satisfies

$$\mathbb{E}_{p^\ast}[(Y - X^\top \theta^*(q))^2] \leq \sigma^2 + C_2 (R^2 + 1)^{1+1/(k-1)(k-1)\sigma^2}$$

(102)

with probability at least $1 - \delta$, where $C_2$ is some universal constant.

**Comparison and Interpretation.** When the third moment of $X'$ is bounded by $\sigma^3$, the projection algorithm guarantees predictive loss $\sigma^2 + O((R^2 + 1)^{3/4} \bar{\epsilon}^{3/2})$ for $\epsilon$ sufficiently small and $n \gtrsim \max(1/\epsilon^2, 1/\epsilon^3)$, while the same condition in TV perturbation would give excess predictive loss of $O(\sigma^2 \epsilon^{2/3})$ given $n \gtrsim d/\epsilon^2$ by using the $\tilde{TV}_R$ projection algorithm in Theorem 5.2.

7 Two Approaches for Finite Sample Analysis

We provide two approaches to analyze the performance of the general projection algorithms for general oblivious corruption (Definition 2.3) and adaptive corruption (Definition 2.4):

1. **Oblivious analysis:** this approach only applies to the oblivious corruption model in Definition 2.3. The oblivious analysis deals with the population distribution $p^\ast$ and its corrupted version $p$. Theorem 7.1 presents the general conditions under which this approach works.

2. **Coupling analysis:** this approach applies to both oblivious model in Definition 2.3 and adaptive corruption models in Definition 2.4. The key difference from oblivious analysis is that we focus on the properties of the empirical distribution sampled from the true distribution $\hat{p}_n$ and the empirical distribution of the corrupted data $\tilde{p}_n$. Theorem 7.2 presents the general conditions under which this approach works.

Among these two approaches there does not exist one analysis approach that strictly dominates the other, and in various cases one can apply both analysis approaches to obtain different bounds that are better in different parameter regimes.
7.1 Oblivious Analysis

In this section, we provide analysis of the projection algorithm that works for the oblivious corruption model, where the adversary perturbs the population distribution $p^*$ to $p$ under discrepancy $D$ and then we observe $n$ i.i.d. samples from $p$. We first introduce the sufficient condition for recovery under oblivious corruption model, which provides a general framework for us to check whether the projection algorithm guarantees decent finite sample error.

**Theorem 7.1** (Oblivious Analysis). Assume the oblivious corruption model of level $\epsilon$ under $D$. Denote the true distribution as $p^* \in \mathcal{G}$ and the perturbed population distribution as $p$ with $D(p^*, p) \leq \epsilon$. Denote the cost function as $L(p^*, \theta)$ and the empirical distribution of observed data as $\hat{p}_n$. Assume the following conditions:

1. **Robust to perturbation:** there exists a function $\overline{D}$ such that for any $p_2, p_3$,
   \[
   \sup_{p_1 \in \mathcal{M}} |\overline{D}(p_1, p_2) - \overline{D}(p_1, p_3)| \leq \overline{D}(p_2, p_3) \leq D(p_2, p_3). \tag{103}
   \]

2. **Generalized Modulus of Continuity:** $\mathcal{M} \supset \mathcal{G}$, and for $\overline{\epsilon} = 2\epsilon + 2\overline{D}(p, \hat{p}_n)$, we have
   \[
   \sup_{p_2^1, p_2^2 \in \mathcal{G}, \overline{D}(p_2^1, p_2^2) \leq \overline{\epsilon}} L(p_2^1, \theta^*(p_2^1)) \leq \rho(\overline{\epsilon}). \tag{104}
   \]

Then the projection algorithm $q = \Pi(\hat{p}_n; \overline{D}, \mathcal{M})$, $\theta^*(q) = \arg \min_{\theta \in \Theta} L(q, \theta)$ satisfies
\[
L(p^*, \theta^*(q)) \leq \rho(2\epsilon + 2\overline{D}(p, \hat{p}_n)). \tag{105}
\]

**Proof.** By the ‘robust to perturbation’ property of $D$, we have
\[
\overline{D}(q, p^*) - \overline{D}(p^*, p^*) \leq \overline{D}(q, p) + \overline{D}(p^*, p) - \overline{D}(p^*, p^*)
\leq \overline{D}(q, \hat{p}_n) + \overline{D}(p, \hat{p}_n) + \overline{D}(p^*, p) - \overline{D}(p^*, p^*)
\leq \overline{D}(p^*, p) + 2\overline{D}(p, \hat{p}_n) + \overline{D}(p^*, p) - \overline{D}(p^*, p^*)
\leq 2\overline{D}(p, \hat{p}_n) + 2\overline{D}(p^*, p)
\leq 2\overline{D}(p, \hat{p}_n) + 2\epsilon. \tag{106}
\]

We also know that $q \in \mathcal{M}$. Hence, by the generalized modulus of continuity property:
\[
L(p^*, \theta^*(q)) \leq \sup_{p_1^1 \in \mathcal{M}, p_2^1, p_2^2 \in \mathcal{G}, \overline{D}(p_1^1, p_2^1) - \overline{D}(p_2^1, p_2^2) \leq 2\epsilon + 2\overline{D}(p, \hat{p}_n)} L(p_2^1, \theta^*(p_2^1)) \leq \rho(2\epsilon + 2\overline{D}(p, \hat{p}_n)), \tag{107}
\]
we can derive the conclusion. \square

**Proposition 7.1** (Any $q$ suffices, not just the minimizer). Assume the conditions in Theorem 7.1 and further $\overline{D}(p, p) = 0$ for any $p$. Suppose that for any $p$ such that $D(p^*, p) \leq \epsilon$, $p^* \in \mathcal{G}$, we have $\overline{D}(p, \hat{p}_n) \leq d_n$ uniformly over $p$ with probability at least $1 - \delta$. Then, it follows from the proof of Theorem 7.1 that for any $q \in \mathcal{M}$ such that
\[
\overline{D}(q, \hat{p}_n) \leq \epsilon + d_n, \tag{108}
\]
we have,

\[ L(p^*, \theta^*(q)) \leq \rho(2\epsilon + 2d_n), \quad (109) \]

and the the existence of \( q \) satisfying (108) happens with probability at least \( 1 - \delta \).

**Proposition 7.2** (Solving robust inference under more general perturbations). It follows from the proof of Theorem 7.1 that the final finite sample error bound still holds if we allow more general perturbations: instead of allowing any \( p \) such that \( D(p^*, p) \leq \epsilon \), we allow any \( p \) such that \( \sup_{r \in M} |\bar{D}(r, p^*) - \bar{D}(r, p)| \leq \epsilon \). Hence, as long as the conditions in Theorem 7.1 are satisfied, the projection algorithm performs well with this bigger set of arbitrary perturbations.

**Corollary 7.1.** Consider the case of \( n = \infty \) in Theorem 7.1, we know that if \( q = \arg \min \{\bar{D}(q, p) \mid q \in M\}, \theta^*(q) = \arg \min_{\theta \in \Theta} L(q, \theta) \), then

\[ L(p^*, \theta^*(q)) \leq \rho(2\epsilon). \quad (110) \]

**Remark 7.1.** Setting \( \bar{D} = \bar{D} = TV \) and \( M = G_{TV}(\rho_1, \rho_2, \bar{\epsilon}) \) in Theorem 7.1 leads to the following bound (Theorem 3.1)

\[ L(p^*, \theta^*(q)) \leq \rho_2(\rho_1(\bar{\epsilon}), \bar{\epsilon}), \quad (111) \]

where \( \bar{\epsilon} = 2\epsilon + 2TV(p, \hat{p}_n) \). However, it would easily be a very loose bound if the contaminated distribution \( p \) is a continuous distribution since in this case \( TV(p, \hat{p}_n) = 1 \) almost surely. To fully utilize the power of Theorem 7.1, one needs to design \( \bar{D} \) and \( \bar{D} \) such that \( \bar{D}(p, \hat{p}_n) \) vanishes fast enough.

The oblivious analysis approach is summarized in the Figure 11.

![Figure 11: Framework for oblivious analysis.](image)

**7.2 Coupling Analysis**

We observe that in the paradigm of oblivious analysis in Figure 11, the empirical distribution \( \hat{p}_n \) from the true distribution \( p^* \) never shows up, and the analysis is crucially relying on the assumption that there is a perturbed population distribution \( p \) such that the observations can be viewed as \( n \) i.i.d. samples from it.
The coupling analysis discards \( p \), but operates on the coupling between the empirical distribution from the true distribution \( \hat{p}^n \) and the empirical distribution of the observations \( \hat{p}_n \). It can be applied to both oblivious and adaptive corruption models, and even in oblivious settings this analysis technique may produce better bounds than the oblivious analysis in certain parameter regimes.

One motivation for coupling analysis is, in practice, the discrepancy \( \mathcal{D}(p, \hat{p}_n) \) in equation (105) might converge very slowly while \( L(p, \hat{p}_n) \) still converges fast. For example, if we use TV projection as our algorithm, \( TV(p, \hat{p}_n) \) is always 1 for any continuous distribution \( p \) but \( \| \mathbb{E}_p[X] - \mathbb{E}_{\hat{p}_n}[X] \|_2 \) would converge to 0 as \( n \to \infty \) under mild tail conditions. Consequently, the oblivious analysis may not be used to produce tight statistical analysis for TV projection algorithms such as those proposed in (Diakonikolas et al., 2017).

**Theorem 7.2.** Assume either oblivious corruption or adaptive corruption model of level \( \epsilon \) under \( D \). Denote the true distribution as \( p^* \), \( \hat{p}^n \) as the empirical distribution sampled from \( p^* \) and \( \hat{p}_n \) as the empirical distribution of observed data. Denote the cost function as \( L(p^*, \theta) \). Assume the following conditions hold:

1. **Robust to perturbation:** there exists a function \( \mathcal{D} \) such that for any \( p_2, p_3 \),
   \[
   \sup_{p_1 \in \mathcal{M}} |\mathcal{D}(p_1, p_2) - \mathcal{D}(p_1, p_3)| \leq \mathcal{D}(p_2, p_3). \tag{112}
   \]

2. **Limited corruption:** \( \mathcal{D}(\hat{p}_n, \hat{p}^n) \leq \epsilon_2 \) with probability at least \( 1 - \delta \).

3. **Set for (perturbed) empirical distribution:** there exists a set \( \mathcal{G}' \subset \mathcal{M} \) such that there exists a distribution \( \hat{p}' \in \mathcal{G}' \) satisfying \( \mathcal{D}(\hat{p}_n^*, \hat{p}') \leq \epsilon_1 \) with probability at least \( 1 - \delta \).

4. **Generalized Modulus of Continuity:** \( \mathcal{G}' \subset \mathcal{M} \), and for \( \hat{\epsilon} = 2(\epsilon_1 + \epsilon_2) \), we have
   \[
   \sup_{p_1^* \in \mathcal{M}, p_2^* \in \mathcal{G}', \mathcal{D}(p_1^*, p_2^*)} L(p_2^*, \theta^*(p_1^*)) \leq \rho(\hat{\epsilon}), \tag{113}
   \]

5. **Generalization bound:** for any \( p^* \in \mathcal{G}, \theta \), there exists some constant \( C \) and some function \( g \) such that \( L(p^*, \theta) \leq C \cdot L(\hat{p}', \theta) + g(\hat{p}', p^*) \).

Then with probability at least \( 1 - 2\delta \), projection algorithm \( q = \Pi(\hat{p}_n; \mathcal{D}, \mathcal{M}) \), \( \theta^*(q) = \arg \min_{\theta} L(q, \theta) \) satisfies
\[
L(p^*, \theta^*(q)) \leq C \rho(2\epsilon_1 + 2\epsilon_2) + g(\hat{p}', p^*). \tag{114}
\]

**Remark 7.2.** In the mean estimation setting where \( L(p, \theta) = \| \mathbb{E}_p[X] - \theta \|_2 \), a generalization bound may be of the form \( C = 1 \),
\[
g(\hat{p}', p^*) = \| \mathbb{E}_{\hat{p}'}[X] - \mathbb{E}_{p^*}[X] \|_2, \tag{115}
\]
which can be shown using the triangle inequality.
Proof. It follows from the assumptions that with probability at least \( 1 - 2\delta \), there exists \( \hat{p}' \in G' \), \( \overline{D}(\hat{p}', \hat{p}_n^*) \leq \epsilon_1, \overline{D}(\hat{p}_n^*, \hat{p}_n) \leq \epsilon_2 \). Then,

\[
\tilde{D}(q, \hat{p}') - \tilde{D}(\hat{p}', \hat{p}_n) \leq \tilde{D}(q, \hat{p}_n) + \overline{D}(\hat{p}_n^*, \hat{p}_n) - \tilde{D}(\hat{p}', \hat{p}_n) \\
\leq D(q, \hat{p}_n) + \epsilon_1 - \tilde{D}(\hat{p}', \hat{p}_n) \\
\leq D(q, \hat{p}_n) + D(\hat{p}_n^*, \hat{p}_n) + \epsilon_1 - \tilde{D}(\hat{p}', \hat{p}_n) \\
\leq D(q, \hat{p}_n) + \epsilon_1 + \epsilon_2 - \tilde{D}(\hat{p}', \hat{p}_n) \\
\leq D(p', \hat{p}_n) + \epsilon_1 + 2\epsilon_2 - \tilde{D}(\hat{p}', \hat{p}_n) \\
\leq \tilde{D}(p', \hat{p}_n) + 2\epsilon_1 + 2\epsilon_2 - \tilde{D}(\hat{p}', \hat{p}_n) \\
= 2\epsilon_1 + 2\epsilon_2.
\] (116)

By the modulus of continuity condition and \( q \in M \), we know that with probability at least \( 1 - 2\delta \), we have

\[
L(p', \theta^*(q)) \leq \sup_{p_2^* \in M \mid D(p_2^*, p_2^*) \leq \tilde{D}(p_2^*, p_2^*)} L(p_2^*, p_2^*) \leq \rho(2\epsilon_1 + 2\epsilon_2). \quad (117)
\]

By the generalization bound condition, we have with probability at least \( 1 - 2\delta \),

\[
L(p^*, \theta^*(q)) \leq C\rho(2\epsilon_1 + 2\epsilon_2) + g(p', p^*). \quad (118)
\]

\[
\]

\[
\]

Proposition 7.3 (Any \( q \) suffices, not just the minimizer). Assume the conditions in Theorem 7.2 and further \( \overline{D}(p, p) = 0 \) for any \( p \). Then, it follows from the proof of Theorem 7.2 that for any \( q \in M \) such that

\[
\tilde{D}(q, \hat{p}_n) \leq \epsilon_1 + \epsilon_2, \quad (119)
\]

we have

\[
L(p^*, \theta^*(q)) \leq C\rho(2\epsilon_1 + 2\epsilon_2) + g(p', p^*), \quad (120)
\]

and the the existence of \( q \) satisfying (119) happens with probability at least \( 1 - 2\delta \).

The coupling analysis can be summarized in the Figure 12.
8 Acknowledgements

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References

Jayadev Acharya, Ilias Diakonikolas, Jerry Li, and Ludwig Schmidt. Sample-optimal density estimation in nearly-linear time. In Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms, pages 1278–1289. SIAM, 2017.

Jorge Adrover and Víctor Yohai. Projection estimates of multivariate location. The Annals of Statistics, 30(6):1760–1781, 2002.

Charu C Aggarwal. Outlier analysis. In Data mining, pages 237–263. Springer, 2015.

Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. Gradient flows: in metric spaces and in the space of probability measures. Springer Science & Business Media, 2008.

Martin Anthony and Peter L Bartlett. Neural network learning: Theoretical foundations. cambridge university press, 2009.

Pranjal Awasthi, Maria Florina Balcan, and Philip M Long. The power of localization for efficiently learning linear separators with noise. In Proceedings of the forty-sixth annual ACM symposium on Theory of computing, pages 449–458. ACM, 2014.

Rudolf Beran et al. Minimum hellinger distance estimates for parametric models. The annals of Statistics, 5(3):445–463, 1977.

Battista Biggio, Blaine Nelson, and Pavel Laskov. Poisoning attacks against support vector machines. arXiv preprint arXiv:1206.6389, 2012.

Jonathan Borwein and Adrian S Lewis. Convex analysis and nonlinear optimization: theory and examples. Springer Science & Business Media, 2010.

Stéphane Boucheron, Gábor Lugosi, and Pascal Massart. Concentration inequalities: A nonasymptotic theory of independence. Oxford university press, 2013.

Olivier Catoni and Ilaria Giulini. Dimension-free pac-bayesian bounds for matrices, vectors, and linear least squares regression. arXiv preprint arXiv:1712.02747, 2017.

Siu-On Chan, Ilias Diakonikolas, Rocco A Servedio, and Xiaorui Sun. Efficient density estimation via piecewise polynomial approximation. In Proceedings of the forty-sixth annual ACM symposium on Theory of computing, pages 604–613. ACM, 2014.

Moses Charikar, Jacob Steinhardt, and Gregory Valiant. Learning from untrusted data. In Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing, pages 47–60. ACM, 2017.

Mengjie Chen, Chao Gao, and Zhao Ren. Robust covariance and scatter matrix estimation under Hubers contamination model. The Annals of Statistics, 46(5):1932–1960, 2018.

Xinyun Chen, Chang Liu, Bo Li, Kimberly Lu, and Dawn Song. Targeted backdoor attacks on deep learning systems using data poisoning. arXiv preprint arXiv:1712.05526, 2017.

Yu Cheng, Ilias Diakonikolas, and Rong Ge. High-dimensional robust mean estimation in nearly-linear time. In Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms, pages 2755–2771. SIAM, 2019a.
Yu Cheng, Ilias Diakonikolas, Rong Ge, and David Woodruff. Faster algorithms for high-dimensional robust covariance estimation. *arXiv preprint arXiv:1906.04661*, 2019b.

Rajesh N Davé and Raghuram Krishnapuram. Robust clustering methods: a unified view. *IEEE Transactions on fuzzy systems*, 5(2):270–293, 1997.

Laurie Davies et al. The asymptotics of rousseeuw’s minimum volume ellipsoid estimator. *The Annals of Statistics*, 20(4):1828–1843, 1992.

Erick Delage and Yinyu Ye. Distributionally robust optimization under moment uncertainty with application to data-driven problems. *Operations research*, 58(3):595–612, 2010.

Luc Devroye and Gábor Lugosi. *Combinatorial methods in density estimation*. Springer Science & Business Media, 2012.

Luc Devroye, László Györfi, and Gábor Lugosi. *A probabilistic theory of pattern recognition*, volume 31. Springer Science & Business Media, 2013.

Ilias Diakonikolas, Gautam Kamath, Daniel M Kane, Jerry Li, Ankur Moitra, and Alistair Stewart. Robust estimators in high dimensions without the computational intractability. In *2016 IEEE 57th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 655–664. IEEE, 2016.

Ilias Diakonikolas, Gautam Kamath, Daniel M Kane, Jerry Li, Ankur Moitra, and Alistair Stewart. Being robust (in high dimensions) can be practical. In *Proceedings of the 34th International Conference on Machine Learning-Volume 70*, pages 999–1008. JMLR. org, 2017.

Ilias Diakonikolas, Gautam Kamath, Daniel M Kane, Jerry Li, Ankur Moitra, and Alistair Stewart. Robustly learning a gaussian: Getting optimal error, efficiently. In *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 2683–2702. Society for Industrial and Applied Mathematics, 2018a.

Ilias Diakonikolas, Gautam Kamath, Daniel M Kane, Jerry Li, Jacob Steinhardt, and Alistair Stewart. Sever: A robust meta-algorithm for stochastic optimization. *arXiv preprint arXiv:1803.02815*, 2018b.

Ilias Diakonikolas, Daniel M Kane, and Alistair Stewart. Learning geometric concepts with nasty noise. In *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing*, pages 1061–1073. ACM, 2018c.

Ilias Diakonikolas, Gautam Kamath, Daniel Kane, Jerry Li, Ankur Moitra, and Alistair Stewart. Robust estimators in high-dimensions without the computational intractability. *SIAM Journal on Computing*, 48(2):742–864, 2019a.

Ilias Diakonikolas, Daniel M Kane, and Pasin Manurangsi. Nearly tight bounds for robust proper learning of halfspaces with a margin. *arXiv preprint arXiv:1908.11335*, 2019b.

Ilias Diakonikolas, Weihao Kong, and Alistair Stewart. Efficient algorithms and lower bounds for robust linear regression. In *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 2745–2754. SIAM, 2019c.

David L Donoho. Breakdown properties of multivariate location estimators. Technical report, Technical report, Harvard University, Boston. URL http://www-stat. stanford , 1982.
David L Donoho and Miriam Gasko. Breakdown properties of location estimates based on halfspace depth and projected outlyingness. *The Annals of Statistics*, 20(4):1803–1827, 1992.

David L Donoho and Richard C Liu. The “automatic” robustness of minimum distance functionals. *The Annals of Statistics*, 16(2):552–586, 1988.

John Duchi and Hongseok Namkoong. Learning models with uniform performance via distributionally robust optimization. *arXiv preprint arXiv:1810.08750*, 2018.

Richard M Dudley. Central limit theorems for empirical measures. *The Annals of Probability*, pages 899–929, 1978.

Richard Mansfield Dudley. The speed of mean glivenko-cantelli convergence. *The Annals of Mathematical Statistics*, 40(1):40–50, 1969.

Aryeh Dvoretzky, Jack Kiefer, and Jacob Wolfowitz. Asymptotic minimax character of the sample distribution function and of the classical multinomial estimator. *The Annals of Mathematical Statistics*, 27(3):642–669, 1956.

Paul Embrechts and Marius Hofert. A note on generalized inverses. *Mathematical Methods of Operations Research*, 77(3):423–432, 2013.

Ky Fan. Minimax theorems. *Proceedings of the National Academy of Sciences of the United States of America*, 39(1):42, 1953.

Simon Foucart and Holger Rauhut. A mathematical introduction to compressive sensing. *Bull. Am. Math.*, 54:151–165, 2017.

Chao Gao. Robust regression via mutivariate regression depth. *arXiv preprint arXiv:1702.04656*, 2017.

Chao Gao, Jiyi Liu, Yuan Yao, and Weizhi Zhu. Robust estimation and generative adversarial nets. *arXiv preprint arXiv:1810.02030*, 2018.

Chao Gao, Yuan Yao, and Weizhi Zhu. Generative adversarial nets for robust scatter estimation: A proper scoring rule perspective. *arXiv preprint arXiv:1903.01944*, 2019.

Tilmann Gneiting and Adrian E Raftery. Strictly proper scoring rules, prediction, and estimation. *Journal of the American Statistical Association*, 102(477):359–378, 2007.

Ian Goodfellow, Jean Pouget-Abadie, Mehdi Mirza, Bing Xu, David Warde-Farley, Sherjil Ozair, Aaron Courville, and Yoshua Bengio. Generative adversarial nets. In *Advances in neural information processing systems*, pages 2672–2680, 2014.

Mikhail Gromov. *Metric structures for Riemannian and non-Riemannian spaces*. Springer Science & Business Media, 2007.

Sudipto Guha, Rajeev Rastogi, and Kyuseok Shim. Rock: A robust clustering algorithm for categorical attributes. *Information systems*, 25(5):345–366, 2000.

Uffe Haagerup. The best constants in the Khintchine inequality. *Studia Mathematica*, 70:231–283, 1981.
Victoria Hodge and Jim Austin. A survey of outlier detection methodologies. *Artificial intelligence review*, 22(2):85–126, 2004.

Samuel B Hopkins and Jerry Li. How hard is robust mean estimation? *arXiv preprint arXiv:1903.07870*, 2019.

Peter J Huber. Robust regression: asymptotics, conjectures and monte carlo. *The Annals of Statistics*, 1(5):799–821, 1973.

Peter J Huber. *Robust statistics*. Springer, 2011.

Peter J Huber. *Data analysis: what can be learned from the past 50 years*, volume 874. John Wiley & Sons, 2012.

Emilien Joly, Gábor Lugosi, Roberto Imbuzeiro Oliveira, et al. On the estimation of the mean of a random vector. *Electronic Journal of Statistics*, 11(1):440–451, 2017.

Adam Klivans, Pravesh K Kothari, and Raghu Meka. Efficient algorithms for outlier-robust regression. *arXiv preprint arXiv:1803.03241*, 2018.

Adam R Klivans, Philip M Long, and Rocco A Servedio. Learning halfspaces with malicious noise. *Journal of Machine Learning Research*, 10(Dec):2715–2740, 2009.

Pravesh K Kothari and David Steurer. Outlier-robust moment-estimation via sum-of-squares. *arXiv preprint arXiv:1711.11581*, 2017.

Kevin A Lai, Anup B Rao, and Santosh Vempala. Agnostic estimation of mean and covariance. In *2016 IEEE 57th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 665–674. IEEE, 2016.

Jun Z Li, Devin M Absher, Hua Tang, Audrey M Southwick, Amanda M Casto, Sohini Ramachandran, Howard M Cann, Gregory S Barsh, Marcus Feldman, and Luigi L Cavalli-Sforza. Worldwide human relationships inferred from genome-wide patterns of variation. *science*, 319(5866):1100–1104, 2008.

Liu Liu, Yanyao Shen, Tianyang Li, and Constantine Caramanis. High dimensional robust sparse regression. *arXiv preprint arXiv:1805.11643*, 2018.

Gábor Lugosi. Lectures on combinatorial statistics, 2017.

Gábor Lugosi and Shahar Mendelson. Sub-gaussian estimators of the mean of a random vector. *The Annals of Statistics*, 47(2):783–794, 2019.

Jaroslav Lukeš, Jan Malý, Ivan Netuka, and Jiří Spurný. *Integral representation theory: applications to convexity, Banach spaces and potential theory*, volume 35. Walter de Gruyter, 2009.

Ricardo Antonio Maronna. Robust m-estimators of multivariate location and scatter. *The annals of statistics*, pages 51–67, 1976.

Albert W Marshall, Ingram Olkin, and Barry C Arnold. *Inequalities: theory of majorization and its applications*, volume 143. Springer, 1979.

Pascal Massart. Concentration inequalities and model selection. 2007.
Frank J Massey Jr. The Kolmogorov-Smirnov test for goodness of fit. *Journal of the American statistical Association*, 46(253):68–78, 1951.

P Warwick Millar. Robust estimation via minimum distance methods. *Zeitschrift für Wahrscheinlichkeitsrechnung und verwandte Gebiete*, 55(1):73–89, 1981.

Michael Mitzenmacher and Eli Upfal. *Probability and computing: randomization and probabilistic techniques in algorithms and data analysis*. Cambridge university press, 2017.

Hongseok Namkoong and John C Duchi. Stochastic gradient methods for distributionally robust optimization with f-divergences. In *Advances in Neural Information Processing Systems*, pages 2208–2216, 2016.

Masashi Okamoto. Some inequalities relating to the partial sum of binomial probabilities. *Annals of the institute of Statistical Mathematics*, 10(1):29–35, 1959.

Adarsh Prasad, Arun Sai Suggala, Sivaraman Balakrishnan, and Pradeep Ravikumar. Robust estimation via robust gradient estimation. *arXiv preprint arXiv:1802.06485*, 2018.

Adarsh Prasad, Sivaraman Balakrishnan, and Pradeep Ravikumar. A unified approach to robust mean estimation. *arXiv preprint arXiv:1907.00927*, 2019.

Yao-Feng Ren and Han-Ying Liang. On the best constant in Marcinkiewicz-Zygmund inequality. *Statistics & probability letters*, 53(3):227–233, 2001.

Michael Rothschild and Joseph E Stiglitz. Increasing risk: I. a definition. In *Uncertainty in Economics*, pages 99–121. Elsevier, 1978.

Peter J Rousseeuw and Annick M Leroy. *Robust regression and outlier detection*, volume 1. Wiley Online Library, 1987.

Jacob Steinhardt. *Robust Learning: Information Theory and Algorithms*. PhD thesis, Stanford University, 2018.

Jacob Steinhardt, Moses Charikar, and Gregory Valiant. Resilience: A criterion for learning in the presence of arbitrary outliers. *arXiv preprint arXiv:1703.04940*, 2017a.

Jacob Steinhardt, Pang Wei W Koh, and Percy S Liang. Certified defenses for data poisoning attacks. In *Advances in neural information processing systems*, pages 3517–3529, 2017b.

Jacob Steinhardt, Moses Charikar, and Gregory Valiant. Resilience: A criterion for learning in the presence of arbitrary outliers. In *9th Innovations in Theoretical Computer Science Conference (ITCS 2018)*, volume 94, page 45. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2018.

Vladimir N Vapnik and A Ya Chervonenkis. On the uniform convergence of relative frequencies of events to their probabilities. In *Measures of complexity*, pages 11–30. Springer, 2015.

Roman Vershynin. Introduction to the non-asymptotic analysis of random matrices. *arXiv preprint arXiv:1011.3027*, 2010.
Roman Vershynin. *High-dimensional probability: An introduction with applications in data science*, volume 47. Cambridge University Press, 2018.

Cédric Villani. *Topics in optimal transportation*. Number 58. American Mathematical Soc., 2003.

Riccardo Volpi, Hongseok Namkoong, Ozan Sener, John C Duchi, Vittorio Murino, and Silvio Savarese. Generalizing to unseen domains via adversarial data augmentation. In *Advances in Neural Information Processing Systems*, pages 5334–5344, 2018.

Martin J Wainwright. *High-dimensional statistics: A non-asymptotic viewpoint*, volume 48. Cambridge University Press, 2019.

Yannis G Yatracos. Rates of convergence of minimum distance estimators and Kolmogorov’s entropy. *The Annals of Statistics*, pages 768–774, 1985.

William Henry Young. On classes of summable functions and their fourier series. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, 87(594):225–229, 1912.

Banghua Zhu, Jiantao Jiao, and David Tse. Deconstructing generative adversarial networks. *arXiv preprint arXiv:1901.09465*, 2019.
A General Lemmas and Facts

We first collect the notations used throughout this paper. We use capital letter \( X \) for random variable, lowercase letter \( p, q \) for population distribution, and the corresponding empirical distributions with \( n \) samples are denoted as \( \hat{p}_n, \hat{q}_n \). Blackbold letter \( \mathbb{P} \) is used for probability, e.g. \( \mathbb{P}_q(A) \) represents the probability of event \( A \) under distribution \( q \), and blackbold letter \( \mathbb{E} \) is used for expectation. We use bold lower case letters \( \mathbf{v} \) for vectors, bold upper case letters \( \mathbf{A} \) for matrices. We use \( \mathbf{A}^\top \) to denote the pseudo inverse of matrix \( \mathbf{A} \). We say \( \mathbf{A} \succeq 0 \) if \( \mathbf{A} \) is positive semidefinite, \( \mathbf{A} \succ 0 \) if \( \mathbf{A} \) is positive definite. We denote \( \mu_p = \mathbb{E}_p[X] \) and \( \Sigma_p = \mathbb{E}_p[(X - \mu_p)(X - \mu_p)^\top] \) as mean and covariance for distribution \( p \). We use \( \text{TV}(p, q) = \sup_A \mathbb{P}_p(A) - \mathbb{P}_q(A) \) to denote the total variation distance between \( p \) and \( q \). We use \( \triangleq \) to make definition. We denote \( \min\{a, b\} \) by \( a \land b \), and \( \max\{a, b\} \) by \( a \lor b \). For non-negative sequences \( a_\gamma, b_\gamma \), we use the notation \( a_\gamma \lesssim_\alpha b_\gamma \) to denote that there exists a constant \( C \) that only depends on \( \alpha \) such that \( \sup_\gamma \frac{a_\gamma}{b_\gamma} \leq C \), and \( a_\gamma \gtrsim_\alpha b_\gamma \) is equivalent to \( b_\gamma \lesssim_\alpha a_\gamma \).

When the constant \( C \) is universal we do not write subscripts for \( \lesssim \) and \( \gtrsim \). Notation \( a_\gamma \asymp b_\gamma \) is equivalent to \( a_\gamma \gtrsim b_\gamma \) and \( a_\gamma \lesssim b_\gamma \). Notation \( a_\gamma \gg b_\gamma \) means that \( \liminf_\gamma \frac{a_\gamma}{b_\gamma} = \infty \), and \( a_\gamma \ll b_\gamma \) is equivalent to \( b_\gamma \gg a_\gamma \). We write \( f(x) = O(g(x)) \) for \( x \to 0 \) if there exists some non-negative constant \( C_1, C_2 \) such that \( |f(x)| \leq C_1 |g(x)| \) for all \( x \in [0, C_2] \). We write \( f(x) = \Omega(g(x)) \) for \( x \to 0 \) if there exists some non-negative constant \( C_1, C_2 \) such that \( |f(x)| \geq C_1 |g(x)| \) for all \( x \in [0, C_2] \). We write \( f(x) = \Theta(g(x)) \) if \( f(x) = O(g(x)) \) and \( f(x) = \Omega(g(x)) \). For any norm \( \| \cdot \| \), we use \( \| \cdot \|_\alpha \) to denote its dual norm, which is defined as \( \|z\|_\alpha = \sup\{z^\top x \mid \|x\| \leq 1\} \).

In this section we present some general lemmas and facts that we use throughout the paper.

**Lemma A.1** (Non-decreasing property of function \( x \psi^{-1}(\sigma/x) \)). For any Orlicz function \( \psi \) as defined in Definition 2.2. The function \( x \psi^{-1}(\sigma/x) \) is non-decreasing for \( x \) for the region \( x \in [0, +\infty) \) for any constant \( \sigma > 0 \), where \( \psi^{-1} \) is the (generalized) inverse function of \( \psi \) defined in (14).

**Proof.** Denote \( t(x) = \psi^{-1}(\sigma/x) = \inf\{y \mid \psi(y) > \sigma/x\} \). Since \( \psi(x) \) is non-decreasing, we know that \( \psi^{-1}(\sigma/x) \) is a non-increasing function. Consider the function \( \frac{\psi(t)}{t} \). From the property of convex functions, we know that for any \( 0 < x_1 < x_2 \),

\[
\frac{\psi(x_1) - \psi(0)}{x_1} \leq \frac{\psi(x_2) - \psi(0)}{x_2}. \tag{121}
\]

Thus we know \( \frac{\psi(t)}{t} \) is an non-decreasing function. Thus the function \( f(t) = \frac{1}{\psi(t)} \) is an non-increasing function. Since the function \( \frac{1}{\psi} \psi^{-1}(\sigma/x) \) is composition \( (f \circ t)(x) \), we know that it is non-decreasing.

**Lemma A.2** (Generalized Holder’s Inequality). Define composition function as \( (\psi \circ f)(x) = \psi(f(x)) \). Given some Orlicz function \( \psi \) (Definition 2.2), for any two random variables \( X, Y \), any \( p, q > 0, 1/p + 1/q = 1 \),

\[
\|XY\|_\psi \leq \|X\|_{\psi \circ 1/p} \|Y\|_{\psi \circ 1/q}. \tag{122}
\]

**Proof.** Denote \( \|X\|_{\psi \circ 1/p} = \sigma_1 \), \( \|Y\|_{\psi \circ 1/q} = \sigma_2 \). It follows from Young’s inequality (Young, 1912) that

\[
|XY| \leq \frac{1}{p} |X|^p + \frac{1}{q} |Y|^q, \forall p, q > 0, \frac{1}{p} + \frac{1}{q} = 1. \tag{123}
\]
Thus

\[
\begin{align*}
\mathbb{E} \left[ \varphi \left( \frac{|XY|}{\sigma_1\sigma_2} \right) \right] & \leq \mathbb{E} \left[ \varphi \left( \frac{1}{p} \frac{|X|}{\sigma_1} + \frac{1}{q} \frac{|Y|}{\sigma_2} \right) \right] \\
& \leq \mathbb{E} \left[ \frac{1}{p} \varphi \left( \frac{|X|}{\sigma_1} \right) \right] + \mathbb{E} \left[ \frac{1}{q} \varphi \left( \frac{|X|}{\sigma_2} \right) \right] \\
& \leq \frac{1}{p} + \frac{1}{q} = 1.
\end{align*}
\] (124)

This shows that \( \|XY\|_{\psi} \leq \sigma_1\sigma_2 \).

The following lemma is a generalization of (Vershynin, 2018, Lemma 2.6.8) and shows that if a distribution has its non-centered Orlicz norm bounded, then its centered Orlicz norm is also bounded.

**Lemma A.3 (Centering).** For any Orlicz norm \( \psi \) in Definition 2.2, then if

\[
\sup_{f \in \mathcal{F}} \mathbb{E}_p \left[ \varphi \left( \frac{|f(X)|}{\sigma} \right) \right] \leq 1,
\] (125)

then

\[
\sup_{f \in \mathcal{F}} \mathbb{E}_p \left[ \varphi \left( \frac{|f(X) - \mathbb{E}_p[f(X)]|}{2\sigma} \right) \right] \leq 1.
\] (126)

**Proof.** For some fixed \( f \), note that \( \| \cdot \|_{\psi} \) satisfies the triangle inequality. Thus

\[
\|f(X) - \mathbb{E}_p[f(X)]\|_{\psi} \leq \|f(X)\|_{\psi} + \|\mathbb{E}_p[f(X)]\|_{\psi} \leq \sigma + \|\mathbb{E}_p[f(X)]\|_{\psi} \leq \sigma + |\mathbb{E}_p[f(X)]|/\psi^{-1}(1),
\] (127)

where \( \psi^{-1} \) is the (generalized) inverse function of \( \psi \) defined in (14). Now we show that \( |\mathbb{E}_p[f(X)]|/\psi^{-1}(1) \leq \|f(X)\|_{\psi} = \sigma \). By Jensen’s inequality, we have

\[
\mathbb{E}_p \left[ \varphi \left( \frac{\psi^{-1}(1)|f(X)|}{\mathbb{E}_p[f(X)]} \right) \right] \geq \varphi \left( \frac{\psi^{-1}(1)|\mathbb{E}_p[f(X)]|}{\mathbb{E}_p[f(X)]} \right) = \varphi(\psi^{-1}(1)) = 1.
\] (128)

This shows that \( |\mathbb{E}_p[f(X)]|/\psi^{-1}(1) \leq \|f(X)\|_{\psi} = \sigma \). Thus \( \|f(X) - \mathbb{E}_p[f(X)]\|_{\psi} \leq 2\sigma \). \( \Box \)

**Lemma A.4 (Convergence of mean under 2-norm for distribution with bounded second moment).** Assume distribution \( p \) has its second moment bounded, i.e.

\[
\sup_{v \in \mathbb{R}^d, \|v\|_2 = 1} \mathbb{E}_p \left[ |v^\top (X - \mathbb{E}_p[X])|^2 \right] \leq \sigma^2.
\] (129)

Then

\[
\mathbb{E}_p \left\| \frac{1}{n} \sum_{i=1}^n X_i - \mathbb{E}_p[X] \right\|_2 \leq \sigma \sqrt{\frac{d}{n}}.
\] (130)
Proof. By Jensen’s inequality,

\[ \mathbb{E}_p \left\| \frac{1}{n} \sum_{i=1}^{n} X_i - \mathbb{E}_p[X] \right\|_2 \leq \left( \mathbb{E}_p \left\| \frac{1}{n} \sum_{i=1}^{n} X_i - \mathbb{E}_p[X] \right\|_2^2 \right)^{1/2} \]

\[ = \left( \mathbb{E}_p \text{Tr} \left( \left( \frac{1}{n} \sum_{i=1}^{n} X_i - \mathbb{E}_p[X] \right) \left( \frac{1}{n} \sum_{i=1}^{n} X_i - \mathbb{E}_p[X] \right)^\top \right) \right)^{1/2} \]

\[ = \text{Tr} \left( \mathbb{E}_p \left[ \left( \frac{1}{n} \sum_{i=1}^{n} X_i - \mathbb{E}_p[X] \right) \left( \frac{1}{n} \sum_{i=1}^{n} X_i - \mathbb{E}_p[X] \right)^\top \right] \right)^{1/2} \]

\[ \leq \sqrt{d} \mathbb{E}_p \left[ \left( \frac{1}{n} \sum_{i=1}^{n} X_i - \mathbb{E}_p[X] \right) \left( \frac{1}{n} \sum_{i=1}^{n} X_i - \mathbb{E}_p[X] \right)^\top \right]^{1/2} \]

\[ = \sqrt{\frac{d}{n}} \left\| \mathbb{E}_p[(X - \mathbb{E}_p[X])(X - \mathbb{E}_p[X])^\top] \right\|_2^{1/2} \]

\[ \leq \sigma \sqrt{\frac{d}{n}}. \hspace{1cm} (131) \]

B  Related discussions and remaining proofs in Section 2

B.1 Proof of Lemma 2.1 and related discussions

Construct a coupling between \( \hat{p}_n^* = \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i} \) and \( \hat{p}_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{Y_i} \) as

\[ \lambda_{\hat{p}_n^*, \hat{p}_n} = \frac{1}{n} \sum_{i=1}^{n} \delta_{(x_i, y_i)}. \hspace{1cm} (132) \]

Then, it follows from the definition of \( W_{c,k}^*(\hat{p}_n^*, \hat{p}_n) \) that

\[ (W_{c,k}^*(\hat{p}_n^*, \hat{p}_n))^k \leq \frac{1}{n} \sum_{i=1}^{n} c^k(X_i, Y_i), \hspace{1cm} (133) \]

which proves the first two claims. Regarding the third part of the lemma, noting that \(|c^k(X, Y)| \leq C^k\) and \(\mathbb{E}_p[c^{2k}(X, Y)] \leq C^k\epsilon^k\) and applying Bernstein’s inequality, we have

\[ P \left( \frac{1}{n} \sum_{i=1}^{n} c^k(X_i, Y_i) - \mathbb{E}_p[c^k(X, Y)] \geq \epsilon^k \right) \leq \exp \left( -\frac{n^2 \epsilon^{2k}}{2 \left( nC^k\epsilon^k + \frac{C^k k!}{3} \right)} \right). \hspace{1cm} (134) \]

It implies that with probability at least \( 1 - \delta \),

\[ \frac{1}{n} \sum_{i=1}^{n} c^k(X_i, Y_i) \leq \epsilon^k + \max \left\{ \frac{4C^k \ln \left( \frac{1}{\delta} \right)}{3n}, \sqrt{\frac{4C^k c^k \ln \left( \frac{1}{\delta} \right)}{n}} \right\} \]

\[ \lesssim \epsilon^k + \frac{C^k \ln \left( \frac{1}{\delta} \right)}{n}. \hspace{1cm} (135) \]

Furthermore, when \( W_{c,k} = \text{TV} \), i.e. \( c(x, y) = 1(x \neq y), k = 1 \), we can provide a stronger bound.
Lemma B.1. For any \( \hat{p}_n \) generated by adaptive corruption model with level \( \epsilon \) under \( D = \text{TV} \), with probability at least \( 1 - \delta \), we have

\[
\text{TV}(\hat{p}_n, \hat{p}_n^*) \leq \begin{cases} 
(\sqrt{\epsilon} + \sqrt{\frac{\log(1/\delta)}{2n}})^2 & \text{for all } \epsilon \in [0,1], \delta \in (0,1], n \geq 1, \\
0 & \delta \geq 1 - (1 - \epsilon)^n
\end{cases},
\]

where \( \hat{p}_n^* \) is the empirical distribution of \( n \) i.i.d. samples from the true distribution \( p^* \).

Proof. Note that \( \frac{1}{n} \sum_{i=1}^{n} 1(X_i \neq Y_i) \) is being stochastically dominated by a binomial distribution \( \text{Bin}(n, \epsilon) \). The first result is a direct result of tail bound for binomial distribution (Okamoto, 1959).

To see the later result, note that the binomial distribution equals 0 with probability \( (1 - \epsilon)^n \). Thus when \( \delta \geq 1 - (1 - \epsilon)^n \), we have with probability at least \( 1 - \delta \), \( \text{TV}(\hat{p}_n^*, \hat{p}_n) = 0 \).

B.2 Proof of Lemma 2.2 and related discussions

It follows from the assumption \( D(p, p^*) \leq \epsilon \), and \( p^* \in \mathcal{G} \) that the projection algorithm can find some \( q \in \mathcal{G} \) such that

\[
D(p, q) \leq \epsilon.
\]

It follows from the triangle inequality of \( D \) that

\[
D(p^*, q) \leq D(p^*, p) + D(p, q) \leq 2\epsilon.
\]

Hence,

\[
L(p^*, \theta^*(q)) \leq \sup_{p_1 \in \mathcal{G}, p_2 \in \mathcal{G}, D(p_1, p_2) \leq 2\epsilon} L(p_1, \theta^*(p_2)). \tag{140}
\]

The following lemma provides lower bounds on the population limit.

Lemma B.2. (Donoho and Liu, 1988; Chen et al., 2018) Suppose that \( D(p, q) \) is a pseudometric and \( L(p, \theta^*(q)) \) is a pseudometric over \( (p, q) \). Then,

1. projection algorithm is near-optimal: for \( q = \arg \min_q D(q, p) \) and the observed corrupted distribution \( p \),

\[
\inf_{\theta(p)} \sup_{p^*: D(p^*, p) \leq \epsilon, p^* \in \mathcal{G}} L(p^*, \theta(p)) \geq \frac{1}{2} \sup_{p^* \in \mathcal{G}, D(p^*, p) \leq \epsilon} L(p^*, \theta^*(q)) \tag{141}
\]

2. if \( D(p, q) = \|p - q\| \), where \( \| \cdot \| \) is a pseudonorm, then the population limit in (17) is lower bounded by the modulus of continuity in (18) up to a factor of 2:

\[
\inf_{\theta(p)} \sup_{p^*, p^* \in \mathcal{G}, D(p^*, p) \leq \epsilon} L(p^*, \theta(p)) \geq \frac{1}{2} m(\mathcal{G}, 2\epsilon, D, L). \tag{142}
\]

It was discussed in detail in (Donoho and Liu, 1988) when the factor of 1/2 is tight.
Proof. Now we show the near-optimality of projection algorithm when \( L(p, \theta^*(q)) \) is a pseudometric for \((p, q)\). Indeed, for \( q = \arg \min_{\eta \in \cal G} D(q, p) \) and any estimator \( \theta(p) \), we have

\[
\sup_{p^* \in \mathcal{G}, D(p^*, p) \leq \epsilon} L(p^*, \theta^*(q)) \leq \sup_{p_1, p_2 \in \mathcal{G}, D(p_1, p_2) \leq \epsilon} L(p_1, \theta^*(p_2)) \leq \sup_{p_1, p_2 \in \mathcal{G}, D(p_1, p_2) \leq \epsilon} (L(p_1, \theta(p)) + L(p_2, \theta(p))) \leq 2 \sup_{p^* \in \mathcal{G}, D(p^*, p) \leq \epsilon, p^* \in \mathcal{G}} L(p^*, \theta(p)).
\]

Since the derivations above holds for any \( \theta(p) \), we know

\[
\sup_{p^* \in \mathcal{G}, D(p^*, p) \leq \epsilon} L(p^*, \theta^*(q)) \leq 2 \inf_{\theta(p)} \sup_{p^* \in \mathcal{G}, D(p^*, p) \leq \epsilon} L(p^*, \theta(p)).
\]

Last, we prove the near-optimality of modulus. For any \( p_1 \in \mathcal{G}, p_2 \in \mathcal{G}, D(p_1, p_2) \leq 2\epsilon \), since we assumed that \( D \) is generated by a pseudonorm, we have \( r = \frac{1}{2}(p_1 + p_2) \) such that

\[
D(p_1, r) \leq \epsilon, \\
D(p_2, r) \leq \epsilon,
\]

Hence, we get exactly the same observation \( r \) for two different true distributions \( p_1, p_2 \). Setting \( p = \frac{1}{2}(p_1 + p_2) \),

\[
\sup_{\theta} \inf_{p \in \mathcal{G}, D(p^*, p) \leq \epsilon} \sup_{p^* \in \mathcal{G}} L(p^*, \theta) \geq \inf_{\theta} \frac{1}{2} (L(p_1, \theta) + L(p_2, \theta)) \geq \frac{1}{2} L(p_1, \theta^*(p_2)).
\]

The last inequality comes from the assumption that \( L(p, \theta^*(q)) \) is a pseudometric for \( p, q \). Since this inequality holds for any \( p_1 \in \mathcal{G}, p_2 \in \mathcal{G}, D(p_1, p_2) \leq 2\epsilon \), the result follows.

\[\square\]

C Related discussions and remaining proofs in Section 3

C.1 Resilience for pseudonorm loss: generalization of mean estimation

We now present a straightforward generalization of the mean estimation example to the so-called \( W_{\mathcal{F}} \) (pseudo)norm \(^5\). The \( W_{\mathcal{F}}(p, q) \) pseudonorm between two probability distributions \( p, q \) is defined as

\[
W_{\mathcal{F}}(p, q) = \sup_{f \in \mathcal{F}} \mathbb{E}_p[f(X)] - \mathbb{E}_q[f(X)],
\]

where \( \mathcal{F} \) is symmetric, i.e. \( \forall f \in \mathcal{F} \), we have \(-f \in \mathcal{F}\). The corresponding resilient set is defined as

\[
\mathcal{G}_{W_{\mathcal{F}}}^{TV}(\rho, \eta) = \{ p \mid \sup_{r \leq \frac{1}{\rho}} W_{\mathcal{F}}(r, p) \leq \rho \}.
\]

With the same technique as mean estimation case, we can show

\[
m(\mathcal{G}_{W_{\mathcal{F}}}^{TV}(\rho, \eta), 2\epsilon) \leq 2\rho
\]

if \( 2\epsilon \leq \eta < 1 \).

\(^5\)It was shown in (Zhu et al., 2019, Lemma 1) that under some appropriate topology of distributions, any pseudonorm can be represented by \( W_{\mathcal{F}}\)-norm for some symmetric family \( \mathcal{F} \) satisfying that \(-\mathcal{F} = \mathcal{F}\).
C.2 Key Lemmas

C.2.1 General properties of $\mathcal{G}^{TV}$

For any two distributions $p, q$, a new distribution $r = \frac{\min(p, q)}{1 - TV(p, q)}$ is defined as follows. For any dominating measure $\nu$ satisfying $p \ll \nu, q \ll \nu$, we define $\frac{dr}{d\nu} = \min\left(\frac{dp}{d\nu}, \frac{dq}{d\nu}\right)/(1 - TV(p, q))$.

**Lemma C.1** (Properties of deletion). Denote by $\mathcal{P}$ the space of probability distributions. For any $\eta \in [0, 1)$, the following statements are true.

1. $\eta$-deletion belongs to $\eta$-TV perturbation: for any $r, p \in \mathcal{P}$,
   \[ r \leq \frac{p}{1 - \eta} \Rightarrow TV(r, p) \leq \eta \]  
   (154)

2. Existence of middle point: for any $p \in \mathcal{P}, q \in \mathcal{P}, TV(p, q) \leq \eta$, there exists some $r \in \mathcal{P}$ such that $r \leq \frac{p}{1 - \eta}, r \leq \frac{q}{1 - \eta}$.

3. Composition preserves being deletion: If $r \leq \frac{p}{1 - \eta}, r' \leq \frac{r}{1 - \eta}$, then $r' \leq \frac{p}{(1 - \eta)^2}$.

4. For any fixed $p \in \mathcal{P}$, the following three sets are equivalent:
   - $A_1 = \{ r \mid r \leq \frac{p}{1 - \eta}, r \in \mathcal{P} \}$,
   - $A_2 = \{ r \mid \mathbb{P}_r(X \in A) = \mathbb{P}_p[X \in A | Z = 0], Z \in \{0, 1\}, \mathbb{P}(Z = 0) \geq 1 - \eta \}$,
   - $A_3 = \{ \frac{\min(p, q)}{1 - TV(p, q)} \mid TV(p, q) \leq \eta, q \in \mathcal{P} \}$.

5. If $r \in \mathcal{P}, p \in \mathcal{P}$ as distributions of $X$ satisfy $r \leq \frac{p}{1 - \eta}$, then the induced distribution for $f(X)$ under both $r$ and $p$ satisfy the same relation for any measurable $f$.

**Proof.** The first claim can be shown via the following inequalities:

\[ r \leq \frac{p}{1 - \eta} \Rightarrow r - p \leq \eta r \]
\[ \Rightarrow TV(r, p) = \sup_A \mathbb{P}_r(A) - \mathbb{P}_p(A) \leq \sup_A \eta r(A) \leq \eta. \]  
   (155)

The second claim can be shown via taking $r = \frac{\min(p, q)}{1 - TV(p, q)}$. From $\int_{\{x : p(x) \geq q(x)\}} (p(x) - q(x))\nu(dx) = TV(p, q)$ we can see that $r$ is a probability distribution. Furthermore, from $TV(p, q) \leq \eta$, it is clear that

\[ r \leq \frac{p}{1 - \eta}, r \leq \frac{q}{1 - \eta}. \]  
   (156)

The third claim can be seen by

\[ r' \leq \frac{r}{1 - \eta} \leq \frac{p}{(1 - \eta)^2}. \]  
   (157)

Now we show the equivalence of three sets in the fourth claim. We first show that $A_1 \subseteq A_2$. For any $r \leq \frac{p}{1 - \eta}$, set distribution $q$ to satisfy that for any set $A$, $q(A) = \frac{p(A) - (1 - \eta)r(A)}{\eta}$. Then $q$ is a valid probability measure. We design the joint distribution of $X, Z$ such that

\[ X|Z = 0 \sim r, \]  
   (158)

\[ X|Z = 1 \sim q, \]  
   (159)

\[ \mathbb{P}_p(Z = 0) = 1 - \eta, \]  
   (160)

\[ \mathbb{P}_p(Z = 1) = \eta. \]  
   (161)
Then one can verify that $X \sim p$. We have found some $Z$ such that $\mathbb{P}_r[X \in A] = \mathbb{P}_p[X \in A | Z = 0], Z \in \{0,1\}, \mathbb{P}(Z = 0) \geq 1 - \eta$ for any measurable set $A$. This shows that $A_1 \subset A_2$.

We then show that $A_2 \subset A_3$. Given a distribution $r \in A_2$ and $p$, we choose a dominating measure $\nu$ such that $r \ll \nu, p \ll \nu$ and write the corresponding Radon–Nikodym derivatives as $r(x), p(x)$. Now the goal is to find some $q$ such that $r = \frac{\min(p,q)}{1 - \text{TV}(p,q)}$. We construct $q(x)$ as follows

$$q(x) = \begin{cases} 
\mathbb{P}(Z = 0 | X = x)p(x), & \mathbb{P}(Z = 0 | X = x) < 1 \\
C \cdot p(x), & \mathbb{P}(Z = 0 | X = x) = 1.
\end{cases} \quad (162)
$$

Here $C \geq 1$ is chosen such that $\int q(x) = 1$. Thus $\text{TV}(p,q)$ can be computed as

$$\text{TV}(p,q) = \int_{\{x: q(x) < p(x)\}} (p(x) - q(x)) \, dx$$
$$= \int_{\{x: \mathbb{P}(Z = 0 | X = x) < 1\}} (1 - \mathbb{P}(Z = 0 | X = x)) p(x) \, dx$$
$$= \int_{\{x: \mathbb{P}(Z = 0 | X = x) < 1\}} \mathbb{P}(Z = 1 | X = x) p(x) \, dx$$
$$= \int_X \mathbb{P}(Z = 1 | X = x) p(x) \, dx$$
$$= \mathbb{P}(Z = 1) \leq \eta. \quad (163)$$

One can check that

$$\frac{\min(p(x), q(x))}{1 - \text{TV}(p,q)} = p(x | Z = 0) = \begin{cases} 
\frac{\mathbb{P}(Z = 0 | X = x)p(x)}{\mathbb{P}(Z = 0)}, & \mathbb{P}(Z = 0 | X = x) < 1 \\
\frac{p(x)}{\mathbb{P}(Z = 0)}, & \mathbb{P}(Z = 0 | X = x) = 1
\end{cases} \quad (164)$$

which shows that $A_2 \subset A_3$.

Lastly, we show that $A_3 \subset A_1$. This can be seen by the construction in the second claim. From $\int_{\{x: p(x) > q(x)\}} (p(x) - q(x)) = \text{TV}(p,q)$ we can see that $r$ is a distribution. Furthermore, from $\text{TV}(p,q) \leq \eta$, it is clear that

$$r \leq \frac{p}{1 - \eta}, r \leq \frac{q}{1 - \eta}. \quad (165)$$

To show the fifth claim, if we know that for any measurable set $A$, $r_X(A) \leq \frac{p_X(A)}{1 - \eta}$, then for any measurable function $f(X)$, and any measurable set $A$, we have

$$r_{f(X)}(A) = r(f^{-1}(A)) \leq \frac{p(f^{-1}(A))}{1 - \eta} = \frac{p_f(X)(A)}{1 - \eta}, \quad (166)$$

where $f^{-1}(A) = \{x \mid f(x) \in A\}$.

Now, we show that if a distribution has bounded Orlicz norm, then it is inside some resilient set $G^\text{TV}_{p\sigma}$ defined in (152). This Lemma implies Example 3.1.

**Lemma C.2** (Bounded Orlicz norm implies resilience). *Given an Orlicz function $\psi$ defined in Definition 2.2, assume*

$$\sup_{f \in \mathcal{F}} \mathbb{E}_p \left[ \psi \left( \frac{|f(X) - \mathbb{E}_p[f(X)]|}{\sigma} \right) \right] \leq 1 \quad (167)$$
for some symmetric family \( \mathcal{F} \) and some \( \sigma > 0 \). For any \( \eta \in [0, 1) \), we have

\[
p \in \mathcal{G}_{W,\eta}^{TV} \left( \frac{\sigma \eta \psi^{-1}(1/\eta)}{1 - \eta} \wedge \sigma \psi^{-1} \left( \frac{1}{1 - \eta} \right), \eta \right), \tag{168}
\]

where \( \mathcal{G}_{W,\eta}^{TV} \) is defined in (152), \( \psi^{-1} \) is the (generalized) inverse function of \( \psi \) defined in (14).

**Proof.** The proof uses the property that for any \( r \leq \frac{p}{1 - \eta} \), there exists some event \( E \) such that \( \mathbb{P}_p(E) \geq 1 - \eta \) and \( \mathbb{E}_p[f(X)] = \mathbb{E}_p[f(X)|E] \) for any measurable \( f \) (Lemma C.1). For any event \( E \) with \( \mathbb{P}_p(E) \geq 1 - \eta \), denote its compliment as \( E^c \), by the definition of conditional expectation, we have

\[
\sup_{f \in \mathcal{F}} \mathbb{E}_p[f(X)|E] - \mathbb{E}_p[f(X)] = \sup_{f \in \mathcal{F}} \frac{\mathbb{P}_p(E^c)}{1 - \mathbb{P}_p(E^c)} \mathbb{E}_p[f(X) - \mathbb{E}_p[f(X)]|E^c] \tag{169}
\]

By the bounded condition and convexity of \( \psi \), one can see

\[
1 \geq \sup_{f \in \mathcal{F}} \mathbb{E}_p \left[ \psi \left( \frac{|f(X) - \mathbb{E}_p[f(X)]|}{\sigma} \right) \right] \geq \sup_{f \in \mathcal{F}} \mathbb{P}_p(E^c) \mathbb{E}_p \left[ \psi \left( \frac{|f(X) - \mathbb{E}_p[f(X)]|}{\sigma} \right) \right] |E^c] \geq \sup_{f \in \mathcal{F}} \mathbb{P}_p(E^c) \psi \left( \frac{|\mathbb{E}_p[f(X) - \mathbb{E}_p[f(X)]|E^c]|}{\sigma} \right). \tag{170}
\]

By definition of \( \psi^{-1} \) in (14), this gives us

\[
\sup_{f \in \mathcal{F}} \mathbb{E}_p[f(X)|E] - \mathbb{E}_p[f(X)] \leq \sigma \frac{\mathbb{P}_p(E^c)}{1 - \mathbb{P}_p(E^c)} \psi^{-1}(1/\mathbb{P}_p(E^c)) \leq \frac{\sigma \eta}{1 - \eta} \psi^{-1}(1/\eta). \tag{171}
\]

The last inequality uses the fact that \( x \psi^{-1}(1/x) \) is a non-decreasing function from Lemma A.1. Thus we have \( p \in \mathcal{G}_{W,\eta}^{TV} \left( \frac{\sigma \eta \psi^{-1}(1/\eta)}{1 - \eta}, \eta \right) \) for any \( \eta \in [0, 1) \).

Similarly, we have

\[
\sup_{f \in \mathcal{F}} \mathbb{E}_p[f(X)|E] - \mathbb{E}_p[f(X)] \leq \sigma \psi^{-1} \left( \frac{1}{\mathbb{P}_p(E)} \right) \leq \sigma \psi^{-1} \left( \frac{1}{1 - \eta} \right), \tag{172}
\]

since \( \mathbb{P}_p(E) \geq 1 - \eta \) and \( \psi^{-1}(1/x) \) is a non-increasing function of \( x \). It implies that \( p \in \mathcal{G}_{W,\eta}^{TV}(\sigma \psi^{-1} \left( \frac{1}{1 - \eta} \right), \eta) \) for any \( \eta \in [0, 1) \).

As shown below, the results in Lemma C.2 can be improved (usually by a constant) if each \( f(X) \) has moment generating function.

**Lemma C.3. (Massart, 2007, Lemma 2.3)** Let \( \psi \) be some convex and continuously differentiable function on \( [0, b) \) with \( 0 < b \leq \infty \), such that \( \psi(0) = \psi'(0) = 0 \). Assume that for \( \lambda \in (0, b) \),

\[
\sup_{f \in \mathcal{F}} \ln \left( \mathbb{E}_p[\exp(\lambda f(X)) \mathbb{E}_p[f(X)])] \right) \leq \psi(\lambda). \tag{174}
\]
Then, for any $\eta \in [0, 1)$,
\[
p \in \mathcal{G}_{W_\mathcal{F}}^{TV} \left( \frac{\eta}{1-\eta} \psi^{-1}(\ln(1/\eta)) \wedge \psi^{-1} \left( \ln \left( \frac{1}{1-\eta} \right) \right), \eta \right),
\]  
(175)
where $\mathcal{G}_{W_\mathcal{F}}^{TV}$ is defined in (152), $\psi^{-1}$ is the generalized inverse (14) of the Fenchel–Legendre dual of $\psi$:
\[
\psi^*(x) = \sup_{\lambda \in (0,b)} (\lambda x - \psi(\lambda)).
\]  
(176)
In particular, if $\psi(\lambda) = \frac{\lambda^2 \sigma^2}{2}$ for all $\lambda \in (0, \infty)$, then
\[
p \in \mathcal{G}_{W_\mathcal{F}}^{TV} \left( \frac{\sigma \eta}{1-\eta} \sqrt{2 \ln(1/\eta)} \wedge \sigma \sqrt{2 \ln(1/(1-\eta))}, \eta \right).
\]  
(177)

**Proof.** Fix $f \in \mathcal{F}$. It follows from (Massart, 2007, Lemma 2.3) that for any set $E$ we have
\[
\mathbb{E}_p[f(X)|E] - \mathbb{E}_p[f(X)] \leq \psi^{-1}(\ln(1/P_p(E))).
\]  
(178)
Hence, for any set $E$ such that $P_p(E) \geq 1 - \eta$, we have
\[
\mathbb{E}_p[f(X)|E] - \mathbb{E}_p[f(X)] \leq \psi^{-1}(\ln(1/(1-\eta))),
\]  
(179)
where we used the fact that $\psi^*$ is a non-negative convex and non-decreasing function on $\mathbb{R}_+$. Regarding the second bound, we first write
\[
\mathbb{E}_p[f(X)] - \mathbb{E}_p[f(X)|E] = \frac{P_p(E)}{1-P_p(E)}(\mathbb{E}_p[f(X)|E] - \mathbb{E}_p[f(X)]).
\]  
(180)
Thus,
\[
\mathbb{E}_p[f(X)] - \mathbb{E}_p[f(X)|E] \leq \frac{P_p(E)}{1-P_p(E)} \psi^{-1}(\ln(1/P_p(E)))
\]  
(181)
\[
\leq \frac{\eta}{1-\eta} \psi^{-1}(\ln(1/\eta)),
\]  
(182)
where in the last step we used the fact that $\frac{\text{\ensuremath{\eta}}}{1-x} \psi^{-1}(\ln(1/x))$ is a non-decreasing function on $(0,1)$. \hfill \Box

When $W_\mathcal{F}(p, q) = \|\mathbb{E}_p[XX^T] - \mathbb{E}_q[XX^T]\|_2 = \sup_{v \in \mathbb{R}^d,\|v\|_2 \leq 1, \xi \in \{\pm 1\}} (\mathbb{E}_p[\xi(v^T X)^2] - \mathbb{E}_q[\xi(v^T X)^2])$, the two lemmas above provide a simple proof for the upper bound of population limit of second moment estimation under operator norm, which matches the results of Gaussian case in (Gao et al., 2018) up to logarithmic factor. One can also design $\mathcal{TV}_H$ for this that achieves the same sample complexity as (Gao et al., 2018) up to logarithmic factor, which is shown in Theorem E.3. We further provide ways to give lower bound for the population limit in Lemma C.8.

The resilient set is closely related to tail bounds, which is described in the following lemma. Similar results are also shown in literature (Massart, 2007, Lemma 2.4), (Steinhardt, 2018, Example 2.7).

**Lemma C.4** (Resilience implies tail bounds). If $p \in \mathcal{G}_{W_\mathcal{F}}^{TV}(\rho, \eta)$, then for every $f \in \mathcal{F}$,
\[
P_p \left( f(X) - \mathbb{E}_p[f(X)] \geq \frac{(1-\eta)\rho}{\eta} \right) \leq \eta, \tag{183}
\]
\[
P_p \left( f(X) - \mathbb{E}_p[f(X)] \leq -\frac{(1-\eta)\rho}{\eta} \right) \leq \eta. \tag{184}
\]
Proof. We adopt a similar proof as (Massart, 2007, Lemma 2.4). Note that the set $G_{TV}^{W,F}(\rho, \eta)$ can be alternatively written as

$$G_{TV}^{W,F}(\rho, \eta) = \{p \mid \sup_{p(E)\geq 1-\eta, f \in F} \mathbb{E}_p[f(X)|E] - \mathbb{E}_p[f(X)] \leq \rho\}. \quad (185)$$

Note that for any event $E$, we have

$$\mathbb{P}_p(E)(\mathbb{E}_p[f(X)|E] - \mathbb{E}_p[f(X)]) + \mathbb{P}_p(E^c)(\mathbb{E}_p[f(X)|E^c] - \mathbb{E}_p[f(X)]) = 0. \quad (186)$$

It follows from the symmetry of $F$ that

$$\sup_{f \in F} \mathbb{P}_p(E)(\mathbb{E}_p[f(X)|E] - \mathbb{E}_p[f(X)]) = \sup_{f \in F} \mathbb{P}_p(E^c)(\mathbb{E}_p[f(X)|E^c] - \mathbb{E}_p[f(X)]). \quad (187)$$

Thus we have

$$\sup_{p(E)\leq \eta, f \in F} \mathbb{E}_p[f(X)] - \mathbb{E}_p[f(X)] = \frac{(1-\eta)\rho}{\eta}. \quad (188)$$

Taking $E = \{f(X) - \mathbb{E}_p[f(X)] \geq a\}$, by Markov’s inequality, we have

$$a \leq \mathbb{E}_p[f(X)] - \mathbb{E}_p[f(X)] \leq \frac{(1-\eta)\rho}{\mathbb{P}_p(E)}. \quad (189)$$

Thus

$$\mathbb{P}_p(f(X) - \mathbb{E}_p[f(X)] \geq a) \leq \frac{(1-\eta)\rho}{a}. \quad (190)$$

The other side holds analogously.

\[\square\]

C.2.2 Key Lemmas for lower bound

The below lemma shows that the population limit for resilient set is optimal up to constant under some topology assumption of $f$. We first show that if $p$ is inside some resilient set, then its deleted distribution $r$ is also inside some resilient set that has same population limit up to constant level in most cases.

Lemma C.5 (Resilience is approximately closed under deletion). Assume $p \in G_{TV}^{W,F}(\rho, \eta(2-\eta))$ defined in Equation (152). Then for any $r \leq \frac{p}{1-\eta}$, we have

$$r \in G_{TV}^{W,F}(2\rho, \eta). \quad (191)$$

Proof. From $r \leq \frac{p}{1-\eta}$, we know that for any $q \leq \frac{r}{1-\eta}$, we have

$$q \leq \frac{p}{(1-\eta)^2} = \frac{p}{1-2\eta+\eta^2}. \quad (192)$$

Thus from $p \in G_{TV}^{W,F}(\rho, \eta(2-\eta)), \eta(2-\eta) \geq \eta$, we have

$$W_F(q,p) \leq \rho, \quad (193)$$

$$W_F(r,p) \leq \rho. \quad (194)$$

Thus

$$\sup_{q \leq \frac{r}{1-\eta}} W_F(q,r) \leq \sup_{q \leq \frac{r}{1-\eta}} W_F(q,p) + W_F(p,r) \leq 2\rho. \quad (195)$$

\[\square\]
Lemma C.6 (Population limit for $G_{W_F}^{TV}(\rho, \eta)$ is optimal). Assume $\epsilon \in [0, 1)$ is the perturbation level, $\eta \geq \epsilon$, and there exist a distribution $p_1 \in G_{W_F}^{TV}(\rho/2, \epsilon(2-\epsilon))$ and some distribution $r_1 \leq \frac{p_1}{\epsilon}$ such that

$$W_F(p_1, r_1) \geq cp.$$  \hfill (196)

Then the population limit of the set $G_{W_F}^{TV}(\rho, \eta)$ under perturbation level $\epsilon$ is lower bounded by $cp/2$, i.e.,

$$\inf q(p) (p^*, p): TV(p^*, p) \leq \epsilon, p^* \in G_{W_F}^{TV}(\rho, \eta) \sup W_F(p^*, q) \geq \frac{1}{2} cp.$$  \hfill (197)

This matches the upper bound of population limit in (153) for $G_{W_F}^{TV}$ up to a constant.

Proof. From Lemma C.5 and $p_1 \in G_{W_F}^{TV}(\rho/2, \epsilon(2-\epsilon))$, we know that $r_1 \in G_{W_F}^{TV}(\rho, \epsilon)$. From the assumption we also know that $p_1$ is inside the same set. Assume the observed corrupted distribution is $p = p_1$. Then,

$$\inf q(p) (p^*, p): TV(p^*, p) \leq \epsilon, p^* \in G_{W_F}^{TV}(\rho, \eta) \sup W_F(p^*, q(p)) \geq \inf q(p^*, p_1) \leq \frac{1}{2} \inf (W_F(p_1, q) + W_F(q, r_1)) \geq \frac{1}{2} W_F(p_1, r_1) \geq \frac{cp}{2}.$$  \hfill (198)

Thus we know that the population limit of $G_{W_F}^{TV}(\rho, \epsilon)$ is lower bounded by $cp/2$. \hfill \qedsymbol

This lemma combined with Theorem 3.1 empowers us to show that under appropriate choice of $W_F$ and $\rho$, one can tightly bound the information-theoretic limit of $G_{W_F}^{TV}$ within universal constant factors.

The next two lemmas shows that we can also show similar results for Orlicz norm bounded set under some topological assumption of $f$.

Lemma C.7 (Orlicz norm bounded set is approximately closed under deletion). For some Orlicz function $\psi$ in Definition 2.2, define

$$G_{\psi}(\sigma) = \{ p \mid \sup_{f \in F} \mathbb{E}_p \left[ \psi \left( \frac{|f(X) - \mathbb{E}_p[f(X)]|}{\sigma} \right) \right] \leq 1 \}.$$  \hfill (199)

for some symmetric family $F$. Assume that there exist some distribution $p \in G_{\psi}(\sigma)$ and $\epsilon \leq 1/2$, then

$$r \in G_{(1-\epsilon)\psi}(5\sigma) = \left\{ p \mid \sup_{f \in F} \mathbb{E}_p \left[ (1-\epsilon)\psi \left( \frac{|f(X) - \mathbb{E}_p[f(X)]|}{5\sigma} \right) \right] \leq 1 \right\}.$$  \hfill (200)

Proof. We use $\| \cdot \|_{\psi,r}$ to represent the $\psi$--norm of $X \sim r$. Denote $\tilde{\psi} = (1-\epsilon)\psi$. By triangle inequality,

$$\sup_{f \in F} \| f(X) - \mathbb{E}_r[f(X)] \|_{\tilde{\psi},r} \leq \sup_{f \in F} \| f(X) - \mathbb{E}_p[f(X)] \|_{\psi,r} + \sup_{f \in F} \| \mathbb{E}_r[f(X)] - \mathbb{E}_r[f(X)] \|_{\tilde{\psi},r}.$$  \hfill (201)
Now we bound the first term of RHS, note that $p \in G_\psi(\sigma)$ implies that
\[
\sup_{f \in \mathcal{F}} \mathbb{E}_p \left[ \tilde{\psi} \left( \frac{|f(X) - \mathbb{E}_p[f(X)]|}{\sigma} \right) \right] = (1 - \epsilon) \sup_{f \in \mathcal{F}} \mathbb{E}_p \left[ \tilde{\psi} \left( \frac{|f(X) - \mathbb{E}_p[f(X)]|}{\sigma} \right) \right] \leq 1 - \epsilon. \tag{202}
\]
Thus from $r \leq \frac{p}{1 - \epsilon}$, we have
\[
\sup_{f \in \mathcal{F}} \mathbb{E}_r \left[ \tilde{\psi} \left( \frac{|f(X) - \mathbb{E}_r[f(X)]|}{\sigma} \right) \right] \leq \frac{1}{1 - \epsilon} \sup_{f \in \mathcal{F}} \mathbb{E}_p \left[ \tilde{\psi} \left( \frac{|f(X) - \mathbb{E}_p[f(X)]|}{\sigma} \right) \right] \leq 1. \tag{203}
\]

From Lemma C.2 and the definition of $\psi^{-1}$ in (14), the second term of RHS in Equation (201) is
\[
\sup_{f \in \mathcal{F}} \frac{|\mathbb{E}_p[f(X)] - \mathbb{E}_r[f(X)]|}{\psi^{-1}(1)} \leq \frac{2\sigma \epsilon \psi^{-1}(1/\epsilon)}{\psi^{-1}(1/\epsilon)}. \tag{204}
\]
Combining the upper bound of two terms together, we have
\[
\sup_{f \in \mathcal{F}} \|f(X) - \mathbb{E}_r[f(X)]\|_{\tilde{\psi},r} \leq \sigma \left( \frac{2\epsilon \psi^{-1}(1/\epsilon)}{\psi^{-1}(1/\epsilon)} + 1 \right). \tag{205}
\]
If we further assume that that $\epsilon \leq 1/2$, by Lemma A.1, we know $\epsilon \psi^{-1}(1/\epsilon) \leq (1 - \epsilon)\psi^{-1}(1/(1 - \epsilon))$. Thus
\[
\sup_{f \in \mathcal{F}} \|f(X) - \mathbb{E}_r[f(X)]\|_{\tilde{\psi},r} \leq \sigma \left( \frac{2(1/\epsilon)}{\psi^{-1}(1/\epsilon)} + 1 \right) \leq \sigma \left( \frac{2}{1 - \epsilon} + 1 \right) \leq 5 \sigma. \tag{206}
\]

Based on the above Lemma, we are able to show that under mild topological condition of the range of $f$, the upper bound we derive in Lemma C.2 is optimal up to a constant. With this Lemma, it is automatic that the upper bound in Example 3.1 is optimal.

**Lemma C.8.** [Population limit for Orlicz norm bounded set is optimal] For some Orlicz function $\psi$ in Definition 2.2, define
\[
G_\psi(\sigma) = \{ p \mid \sup_{f \in \mathcal{F}} \mathbb{E}_p \left[ \psi \left( \frac{|f(X) - \mathbb{E}_p[f(X)]|}{\sigma} \right) \right] \leq 1 \}. \tag{207}
\]
for some symmetric family $\mathcal{F}$. Assume that $\epsilon \leq 1/2$ and there exists some function $f \in \mathcal{F}$ that has range as a superset of $(-\infty, 0]$ or $[0, +\infty)$. Then the population limit of $G_\psi(\sigma)$ is lower bounded below:
\[
\inf_{q(p) \mid (p^*,q):TV(p^*,q) \leq \epsilon, p^* \in G_\psi(\sigma)} \sup_{p \in \mathcal{F}} W_\mathcal{F}(p^*,q) \geq \frac{\sigma \epsilon \psi^{-1}(1/\epsilon)}{20}, \tag{208}
\]
where $\psi^{-1}$ is the (generalized) inverse function of $\psi$ defined in (14). This matches the upper bound in Lemma C.2 up to a constant.
Proof. From Lemma C.7, we know that
\[ p_1 \in \mathcal{G}_{\psi/(1-\epsilon)}(\sigma/5) \Rightarrow \forall r_1 \leq \frac{p_1}{1-\epsilon}, p_1, r_1 \in \mathcal{G}_\psi(\sigma). \] (209)

Thus here we would like to show that there exist a distribution \( p_1 \in \mathcal{G}_{\psi/(1-\epsilon)}(\sigma/5) \) and some distribution \( r_1 \leq \frac{p_1}{1-\epsilon} \) such that
\[ W_F(p_1, r_1) \geq C_1 \sigma \epsilon \psi^{-1}(1/\epsilon), \] (210)

Assume the range of some \( f \in \mathcal{F} \) is a superset of \([0, +\infty)\). We construct the distribution \( p_1 \) as follows,
\[ \mathbb{P}_{p_1}(f(X) = t) = \begin{cases} \epsilon, & t = \sigma \psi^{-1}((1-\epsilon)/\epsilon)/5 \\ 1 - \epsilon, & t = 0 \\ 0, & \text{otherwise} \end{cases} \] (211)

Then we have
\[ \mathbb{E}_{p_1} \left[ \psi \left( \frac{f(X)}{\sigma/5} \right) \left/ (1 - \epsilon) \right. \right] = 1, \] (213)

which means that \( p_1 \in \mathcal{G}_{\psi/(1-\epsilon)}(\sigma/5) \), furthermore, we can design \( r \leq \frac{p_1}{1-\epsilon} \) by deleting the non-zero part of \( p_1 \), thus for \( \epsilon \leq 1/2 \), we have
\[ W_F(p_1, r_1) \geq \left| \mathbb{E}_{p_1}[f(X)] - \mathbb{E}_r[f(X)] \right| \geq \frac{\sigma}{5} \epsilon \psi^{-1}\left(\frac{1-\epsilon}{\epsilon}\right) \geq \frac{\sigma}{5} \epsilon \psi^{-1}\left(\frac{1}{2\epsilon}\right) \geq \frac{\sigma}{10} \epsilon \psi^{-1}\left(\frac{1}{\epsilon}\right) + \psi^{-1}(0) = \frac{\sigma}{10} \epsilon \psi^{-1}\left(\frac{1}{\epsilon}\right). \] (214)

The last two inequality is from Jensen’s inequality and the fact that \( \psi^{-1} \) is a concave function, \( \psi^{-1}(0) = 0 \). Thus if the observed corrupted population distribution \( p = p_1 \),
\[ \inf_{q(p')} \sup_{p'^*:TV(p'^*, p) \leq \epsilon, p'^* \in \mathcal{G}_\psi(\sigma)} W_F(p'^*, q) \geq \inf_{p_1} \sup_{p'^*:TV(p'^*, p_1) \leq \epsilon, p'^* \in \mathcal{G}_\psi(\sigma)} W_F(p'^*, q) \geq \frac{1}{2} \inf_q (W_F(p_1, q) + W_F(q, r_1)) \geq \frac{1}{2} W_F(p_1, r_1) \geq \frac{\sigma \epsilon \psi^{-1}(1/\epsilon)}{20}. \] (215)

\[ \square \]

C.3 Proof of Example 3.1

Note here that the conclusion \( p \in \mathcal{G}^\text{TV}_{\text{mean}}(\sigma \epsilon \psi^{-1}(1/\epsilon), \eta) \) for any \( \eta \in [0, 1) \) is a corollary of Lemma C.2. Thus by Lemma 3.1 the population limit when the perturbation level is \( \epsilon \) for some \( \epsilon < 1/4 \) is upper bounded by \( C \sigma \epsilon \psi^{-1}(1/\epsilon) \) for some universal constant \( C \). Furthermore, we show in Lemma C.8 that the population limit for Orlicz norm bounded set is lower bounded by \( C \sigma \epsilon \psi^{-1}(1/\epsilon) \) when \( \epsilon \leq \frac{1}{2} \).
C.4 Proof of Example 3.2

Upper bound. We first show the upper bound. Denote \( Z = Y - X^T \theta^*(p) \). Since the second moment of \( X \) and \( Z \) exist, we can denote \( \mathbb{E}_r[XZ] = \mu_r, \mathbb{E}_r[XX^T] = M_r, \mathbb{E}_{p^*}[XZ] = \mu_{p^*}, \mathbb{E}_{p^*}[XX^T] = M_{p^*} \). The optimal \( \theta \) in both cases can be written as a closed form solution:\(^6\)

\[
\theta^*(r) = \mathbb{E}_r[XX^T]^{-1}\mathbb{E}_r[XY] = \mathbb{E}_r[XX^T]^{-1}\mathbb{E}_r[XX^T \theta^*(p^*) + XZ] = \theta^*(p^*) + \mathbb{E}_r[XX^T]^{-1}\mathbb{E}_r[XZ] = \theta^*(p^*) + M_{r}^{-1}\mu_r. \tag{216}\]

Then \( p^* \in \mathcal{G}_{\psi}^{TV} \) implies that \( \forall r \leq \frac{p^*}{1-\eta}, \) we have

\[
\rho_1 \geq \mathbb{E}_r[\ell(\theta^*(p^*), X) - \ell(\theta^*(r), X)] = \mathbb{E}_r[(Y - X^T \theta^*(p^*))^2 - (Y - X^T \theta^*(r))^2] = \mathbb{E}_r[2YX^T(\theta^*(r) - \theta^*(p^*))^T X^T \theta^*(r) + \theta^*(p^*)^T XX^T \theta^*(p^*)] = \mathbb{E}_r[2(XX^T \theta^*(p^*) + Z)XX^T \theta^*(r) - \theta^*(p^*)^T XX^T \theta^*(r) + \theta^*(p^*)^T XX^T \theta^*(p^*)] = \mathbb{E}_r[2(XX^T \theta^*(p^*) + Z)XX^T \mu_r - \theta^*(p^*)^T XX^T \theta^*(p^*) + M_{r}^{-1}\mu_r + \theta^*(p^*)^T XX^T \theta^*(p^*)] = 2\theta^*(p^*)^T \mu_r + 2\mu_r^T M_{r}^{-1}\mu_r - 2\theta^*(p^*)^T \mu_r - \mu_r^T M_{r}^{-1}\mu_r - \theta^*(p^*)^T M_{r}\theta^*(p^*) + \theta^*(p^*)^T M_{r}\theta^*(p^*) = \mu_r^T M_{r}^{-1}\mu_r = \|M_{r}^{-1/2}\mu_r\|_2^2 \tag{217}\]

By similar calculation, we can see that \( \mathcal{G}_{\psi}^{TV} \) implies

\[
\forall \theta, \text{ if } \forall r \leq \frac{p^*}{1-\eta}, \|M_{r}^{-1/2}(\theta - \theta^*(p^*)) - M_{r}^{-1/2}\mu_r\|_2^2 \leq \rho_1 \Rightarrow \|M_{p^*}^{-1/2}(\theta - \theta^*(p^*))\|_2^2 \leq \rho_2. \tag{218}\]

We first check that \( p^* \in \mathcal{G}_{\psi}^{TV} \). Note that \( M_{r}^{-1/2}\mu_r = M_{r}^{-1/2}M_{p^*}^{-1/2}M_{p^*}^{-1/2}\mu_r \). We bound the term \( M_{r}^{-1/2}M_{p^*}^{-1/2} \) and \( M_{p^*}^{-1/2}\mu_r \) separately.

From the first condition, we know that the \( \psi \) norm of \( \frac{(v^T X)^2}{\mathbb{E}_{p^*}[(v^T X)^2]} \) is upper bounded by \(\sigma_1^2\), thus by Lemma A.3 we know the centered \( \psi \) norm is bounded by \(2\sigma_1^2\). By Lemma C.2, we have for any \( v \in \mathbb{R}^d \),

\[
\forall r \leq \frac{p^*}{1-\eta}, |\mathbb{E}_r[(v^T X)^2] - \mathbb{E}_{p^*}[(v^T X)^2]| \leq \frac{2\sigma_1^2 \eta \psi^{-1}(\frac{1}{\eta}) \mathbb{E}_{p^*}[(v^T X)^2]}{1-\eta}. \tag{219}\]

\(^6\)If either \( M_r \) or \( M_{p^*} \) is not invertible, we use \( M^{-1} = M^1 \) as its pseudoinverse.
Thus when \( \frac{2\sigma^2 \eta \psi^{-1}(\frac{1}{n})}{1-\eta} < \frac{1}{2} \), we have for any \( v \in \mathbb{R}^d \),
\[
v^\top M_r v \geq (1 - \frac{2\sigma^2 \eta \psi^{-1}(\frac{1}{n}) E_{\psi^*}(v^\top X)^2}{1-\eta}) v^\top M_{\psi^*} v
\]
\[
\Rightarrow M_r \geq \frac{1}{2} M_{\psi^*}
\]
\[
\Rightarrow M_r^{-1} \leq 2 M_{\psi^*}^{-1}
\]
\[
\Rightarrow M_r^{1/2} M_r^{-1/2} M_{\psi^*}^{1/2} \leq 2 I
\]
\[
\Rightarrow \|M_r^{1/2} M_r^{-1/2}\|_2 \leq \sqrt{2}.
\]

Equation (220) comes from the monotone property of matrix operator. Equation (221) comes from the fact that \( A \preceq B \) would lead to \( C^\top AC \preceq C^\top BC \). From the two conditions in (25) and (26) and Lemma A.2, we have
\[
\forall v \in \mathbb{R}^d, \|v^\top X Z\|_\psi \leq \sigma_1 \sigma_2 E_{\psi^*}[(v^\top X)^2]^{1/2}.
\]

(223)

Taking \( v = v' M_{\psi^*}^{-1/2} \), where \( \|v'\|_2 = 1 \), we can see that this is equivalent to
\[
\forall v \in \mathbb{R}^d, \|v\|_2 = 1, \|v^\top M_{\psi^*}^{-1/2} X Z\|_\psi \leq \sigma_1 \sigma_2.
\]

(224)

Note that \( E_{\psi^*}[(v^\top M_{\psi^*}^{-1/2} X Z] = 0 \). By Lemma C.2, this gives us that for any \( \eta \),
\[
\forall v \in \mathbb{R}^d, \|v\|_2 = 1, \forall r \leq \frac{p^*}{1-\eta}, E_r[v^\top M_{\psi^*}^{-1/2} X Z] \leq \frac{\sigma_1 \sigma_2 \eta \psi^{-1}(\frac{1}{n})}{1-\eta}.
\]

(225)

Thus we have \( \|M_{\psi^*}^{-1/2} M_r\|_2 \leq \frac{\sigma_1 \sigma_2 \eta \psi^{-1}(\frac{1}{n})}{1-\eta} \).

From the above results, we have
\[
\|M_r^{-1/2} M_r\|_2 = \|M_r^{-1/2} M_{\psi^*}^{1/2} M_r^{-1/2} M_r\|_2
\]
\[
\leq \|M_r^{-1/2} M_{\psi^*}^{1/2}\|_2 \|M_{\psi^*}^{-1/2} M_r\|_2
\]
\[
\leq \frac{\sqrt{2} \sigma_1 \sigma_2 \eta \psi^{-1}(\frac{1}{n})}{1-\eta}.
\]

(226)

Thus we can conclude that \( p^* \in \mathcal{G}_{TV}^\dagger(\rho, \eta) \) with \( \rho = 2(\frac{\sigma_1 \sigma_2 \eta \psi^{-1}(\frac{1}{n})}{1-\eta})^2 \) when \( \frac{2\sigma^2 \eta \psi^{-1}(\frac{1}{n})}{1-\eta} < \frac{1}{2} \). Then we check that \( p^* \in \mathcal{G}_{TV}^\dagger \). If \( \|M_r^{1/2}(\theta - \theta^*(p^*)) - M_r^{-1/2} M_r\|_2 \leq \rho \) holds, we have
\[
\|M_{\psi^*}^{1/2}(\theta^*(p^*) - \theta)\|_2 = \|M_{\psi^*}^{1/2} M_r^{-1/2} M_r^{1/2}(\theta^*(p^*) - \theta)\|_2
\]
\[
\leq \|M_{\psi^*}^{1/2} M_r^{-1/2}\|_2 \|M_r^{1/2}(\theta^*(p^*) - \theta)\|_2
\]
\[
\leq \|M_{\psi^*}^{1/2} M_r^{-1/2}\|_2 \left( \|M_r^{1/2}(\theta^*(p^*) - \theta) - M_r^{-1/2} M_r\|_2 + \|M_r^{-1/2} M_r\|_2 \right)
\]
\[
\leq 2\sqrt{2} \sqrt{\rho}.
\]

(227)

This gives that \( p^* \in \mathcal{G}_{TV}^\dagger(\rho, 8\rho, \eta) \).
Note that when \( \psi(x) = x^{k/2} \) and \( X, Z \) are independent, we can bound the term

\[
\sup_{v \in \mathbb{R}^k, \|v\|_2=1} \mathbb{E}_{P^*}[|v^\top M_p^{-1/2} X Z|^k] \leq \sup_{v \in \mathbb{R}^k, \|v\|_2=1} \mathbb{E}_{P^*}[|v^\top M_p^{-1/2} X|^k] \mathbb{E}_{P^*}[|Z|^k] \leq \sigma_1^k \sigma_2^k. \tag{228}
\]

Thus the final bound would be of the order \( O(\sigma_1^2 \sigma_2^2 \eta^{2-k}) \) when \( \eta \) is small enough. However, without independence assumption we can only bound the \( k/2 \)-th moment of \( v^\top M_p^{-1/2} X Z \), thus the result becomes \( O(\sigma_1^2 \sigma_2^2 \eta^{2-k}) \).

**Lower bound.** Then we show the lower bound for the population limit. Consider the set \( \tilde{G}_{\text{LinReg}} \) that is smaller than the original set in Example 3.2:

\[
\tilde{G}_{\text{LinReg}} = \{p \mid \mathbb{E}_p[X^2] = 1, \mathbb{E}_p[\psi(X^2)] \leq C, \mathbb{E}_p[\psi((Y - \theta^*(p) X)^2)] \leq C \}. \tag{229}
\]

Here \( C \) is some universal constant that may depend on \( \psi \). Then it suffices to show the population limit of the set \( \tilde{G}_{\text{LinReg}} \) is lower bounded by \( (\epsilon \psi^{-1}(1/\epsilon))^2 \), i.e. we need to show that for any estimator \( \theta(p) \),

\[
\inf_{\theta(p)} \sup_{(p^*, p) : p^* \in \tilde{G}_{\text{LinReg}}, \text{TV}(p^*, p) \leq \epsilon} \mathbb{E}_{P^*}[\psi(Y - X^\top \theta(p))^2 - (Y - X^\top \theta^*(p^*))^2] \geq C(\epsilon \psi^{-1}(1/\epsilon))^2. \tag{230}
\]

Consider the case of one-dimensional distribution \( X \). If \( \mathbb{E}_{P^*}[X^2] = \mathbb{E}_q[X^2] = 1 \), the cost \( L(p^*, \theta) \) can be written as

\[
L(p^*, \theta) = \mathbb{E}_{P^*}[((Y - X^\top \theta)^2 - (Y - X^\top \theta^*(p^*))^2]
= \mathbb{E}_{P^*}[(X\theta^*(p^*) + Z - X\theta)^2 - Z^2]
= \mathbb{E}_{P^*}[X^2]|(\theta - \theta^*(p^*))^2
= (\theta - \theta^*(p^*))^2
= (\mathbb{E}_{p^*}[XY] - \theta)^2. \tag{231}
\]

Here we use the fact that \( \theta^*(p^*) = \mathbb{E}_{p^*}[XX^\top]^{-1}\mathbb{E}_{p^*}[XY] = \mathbb{E}_{p^*}[XY] \).

Now we construct distribution \( p_1, p_2 \in \tilde{G}_{\text{LinReg}} \) with the same marginal distribution on \( X \):

\[
\mathbb{P}_{p_1}(X = t) = \mathbb{P}_{p_2}(X = t) = \begin{cases}
\frac{1-\epsilon}{2}, & t = 0 \\
\frac{1-\epsilon}{2}, & t = \sqrt{\frac{2(1-\epsilon \psi^{-1}(1/\epsilon))}{1-\epsilon}} \\
\epsilon, & t = \sqrt{\psi^{-1}(1/\epsilon)} \\
0, & \text{otherwise}
\end{cases} \tag{232}
\]

(233)

We have \( \mathbb{E}_{p_1}[X^2] = \mathbb{E}_{p_2}[X^2] = 1 \), and

\[
\mathbb{E}_{p_1}[\psi(X^2)] = \frac{1-\epsilon}{2} \cdot \psi\left(\frac{2(1-\epsilon \psi^{-1}(1/\epsilon))}{1-\epsilon}\right) + \epsilon \psi(\psi^{-1}(1/\epsilon)) \leq \frac{1}{2} \cdot \psi(2) + 1 \leq C, \tag{234}
\]

where \( C \) is some constant. Now we construct the conditional distribution \( Y|X \) under \( p_1 \) as follows.

\[
Y|X = t = \begin{cases}
0, & t \neq \sqrt{\psi^{-1}(1/\epsilon)} \\
X, & t = \sqrt{\psi^{-1}(1/\epsilon)}
\end{cases}. \tag{235}
\]
The conditional distribution $Y \mid X$ under $p_2$ is
\[
Y\mid X = t = \begin{cases} 
0, & t \neq \sqrt{\psi^{-1}(1/\epsilon)} \\
-X, & t = \sqrt{\psi^{-1}(1/\epsilon)}. 
\end{cases}
\tag{236}
\]

Then $\theta^*(p_1) = \mathbb{E}_{p_1}[XY] \in [0,1]$, $\theta^*(p_2) = \mathbb{E}_{p_2}[XY] \in [-1,0]$. For $Z = Y - \theta^*X$, in both cases we have $\mathbb{E}[\psi(Z^2)] \leq \mathbb{E}[\psi(X^2)] \leq C$. Furthermore, we have
\[
|\theta^*(p_1) - \theta^*(p_2)| = |\mathbb{E}_{p_1}[XY] - \mathbb{E}_{p_2}[XY]| = 2\epsilon\psi^{-1}(1/\epsilon).
\tag{237}
\]

Then the population limit of the set $\tilde{G}_{LinReg}$ is lower bounded by $\epsilon\psi^{-1}(1/\epsilon)$ once we assume the observed corrupted distribution $p = p_1$:
\[
\inf_{\theta(p)} \sup_{\mathbf{p}^* : TV(p^*,p) \leq \epsilon, p^* \in \tilde{G}_{LinReg}} L(p^*, \theta(p)) \geq \inf_{\theta} \sup_{\mathbf{p}^* : TV(p^*,p) \leq \epsilon, p^* \in \tilde{G}_{LinReg}} (\theta^*(p^*) - \theta)^2 \\
\geq \frac{1}{2} \inf_{\theta} ((\theta^*(p_1) - \theta)^2 + (\theta^*(p_2) - \theta)^2) \\
\geq \frac{1}{4} (\theta^*(p_1) - \theta^*(p_2))^2 \\
\geq (\epsilon\psi^{-1}(1/\epsilon))^2.
\tag{238}
\]

### C.5 Proof of Theorem C.1

**Theorem C.1.** Let
\[
G'_{LinReg} = \{ p \mid \sup_{v \in \mathbb{R}^d, \|v\|_2 = 1} \mathbb{E}_{p}[\exp((v^\top (X - \mathbb{E}_p[X]))^2/\sigma^2)] \leq 2, Y = X^\top \theta, \theta \in \mathbb{R}^d \}
\tag{239}
\]

be the family of distributions with sub-Gaussian $X$ and no noise in $Y$. For any estimator $\theta(p)$, there is a pair of distributions $(p, p^*)$ such that $p^* \in G'_{LinReg}$, $TV(p, p^*) \leq \epsilon$, and
\[
\mathbb{E}_{p^*}[(Y - X^\top \theta(p))^2 - (Y - X^\top \theta^*(p))^2] = \infty.
\tag{240}
\]

In other words, any estimator achieves arbitrarily large error in the worst case.

**Proof.** We show a stronger result than Theorem C.1 here: for any $f : \mathbb{R} \mapsto \mathbb{R}$ that is convex and non-negative, $f(0) = 0$, $f(x) \to \infty$ as $|x| \to \infty$ and any fixed estimator $\theta(p)$, there is a pair of distributions $(p, p^*)$ such that $p^* \in G'_{LinReg}$, $TV(p, p^*) \leq \epsilon$, and
\[
\mathbb{E}_{p^*}[f(Y - X^\top \theta(p)) - f(Y - X^\top \theta^*(p^*))] = \infty.
\tag{241}
\]

We consider the case that both $X$ and $Y$ are scalar random variables. We first construct the marginal distributions for two distributions $p_1^*, p_2^*$ as follows
\[
\mathbb{P}_{p_1^*}[X = x] = \begin{cases} 
\epsilon, & x = b(\sigma, \epsilon) \\
1 - \epsilon, & x = 0 \\
0, & \text{otherwise}
\end{cases}
\tag{242}
\]
\[
\mathbb{P}_{p_2^*}[X = x] = \begin{cases} 
\epsilon, & x = -b(\sigma, \epsilon) \\
1 - \epsilon, & x = 0 \\
0, & \text{otherwise}
\end{cases}
\tag{243}
\]

23
Here $b(\sigma, \epsilon)$ is the largest value such that $\|X\|_{\psi_2} \leq \sigma$, where $\psi_2$ is defined in Definition 2.2 as the Orlicz function for sub-Gaussian distributions. We design the joint distribution between $X, Y$ for $p_1^*$ as $Y = \theta^{(1)} X$, where $\theta^{(1)} = \frac{t}{b(\sigma, \epsilon)}$, and the joint distribution between $X, Y$ for $p_2^*$ as $Y = \theta^{(2)} X$, where $\theta^{(2)} = -\frac{t}{b(\sigma, \epsilon)}$. Here $t > 0$ is an arbitrary number that later will be taken to approach $\infty$.

Now we define the observed distribution $\tilde{p}$. Define

$$
\mathbb{P}_{\tilde{p}}[X = t] = \begin{cases} 
1, & t = 0 \\
0, & \text{otherwise} 
\end{cases} 
$$

(244)

The distribution of $Y$ is also 0 with probability 1. One can see that $\text{TV}(p_1^*, \tilde{p}) \leq \epsilon$, $\text{TV}(p_2^*, \tilde{p}) \leq \epsilon$, $p_1^* \in G_{\text{LinReg}}', p_2^* \in G_{\text{LinReg}}'$. So we have

$$
\inf_{\theta(\tilde{p})} \sup_{p^* \in G_{\text{LinReg}}'} \mathbb{E}_{p^*}[f(Y - X\theta) - f(Y - X\theta^*(p))] \\
\geq \inf_{\theta} \max_{p^* \in \{p_1^*, p_2^*\}} \mathbb{E}_{p^*}[f(Y - X\theta)] \\
\geq \frac{1}{2} \inf_{\theta} (\mathbb{E}_{p_1^*}[f(Y - X\theta)] + \mathbb{E}_{p_2^*}[f(Y - X\theta)]) \\
= \frac{1}{2} \inf_{\theta} \left( \epsilon f(b(\sigma, \epsilon)(\theta^{(1)} - \theta)) + \epsilon f(-b(\sigma, \epsilon)(\theta^{(2)} - \theta)) \right) \\
\geq \inf_{\theta} \epsilon f(b(\sigma, \epsilon)(\theta^{(1)} - \theta^*)/2) \\
= \epsilon f(t), 
$$

(245)

where the last inequality is due to Jensen’s inequality. Taking $t \to \infty$ finishes the proof. \hfill \Box

C.6 Robust gradient estimation implies robust regression (not necessarily optimally)

One approach for robust learning is through robust estimation of the gradient of the loss function (Diakonikolas et al., 2018b; Prasad et al., 2018). In this example, we show that if the gradient $\mathbb{E}_{p^*}[\nabla \ell(\theta, X)]$ can be estimated robustly, the distribution $p^*$ is inside $G^{TV}$ for both $B(p, \theta)$ and $L(p, \theta)$ being the excess predictive loss $\mathbb{E}_p[\ell(\theta, X) - \ell(\theta^*(p), X)]$, where $\theta^*(p) = \arg\min_{\theta} \mathbb{E}_p[\ell(\theta, X)]$. Note that linear regression in Example 3.2 is a special case of $G^{TV}$ for excess predictive loss. \footnote{We would like to point out it was shown in literature that any excess predictive loss function can be written as a Bregman divergence and any Bregman divergence can be represented as some excess predictive loss function from proper scoring function construction (Gneiting and Raftery, 2007). Thus we can define the set $G^{TV}$ for Bregman divergence and guarantee the population limit similarly. We omit the details here.}

Lemma C.9. For any $B(p, \theta) = L(p, \theta) = \mathbb{E}_p[\ell(\theta, X) - \ell(\theta^*(p), X)]$, and any $X \sim p^*$, suppose that the distribution of random variable $\nabla \ell(\theta, X)$ is inside $G_{\text{mean}}^{TV}(\rho, \eta)$, i.e. $\sup_{r \leq \rho} \|\mathbb{E}_r[\nabla \ell(\theta, X)] - \mathbb{E}_{p^*}[\nabla \ell(\theta, X)]\|_2 \leq \rho$. Then

1. If the radius of $\Theta$ is upper bounded by $R$, i.e. $\|\theta\|_2 \leq R$ for all $\theta \in \Theta$, then $p^* \in G^{TV}(2R\rho, 4R\rho, \eta)$. Thus the population limit is at most $4R\rho$ if $2\epsilon \leq \eta < 1$. 


2. If $\mathbb{E}_p[\ell(\theta, X)]$ is $\xi$-strongly convex in $\theta$, i.e.

$$\mathbb{E}_p[\ell(\theta_1, x)] - \mathbb{E}_p[\ell(\theta_2, x)] \geq \nabla \mathbb{E}_p[\ell(\theta_2, x)]^\top (\theta_1 - \theta_2) + \frac{\xi}{2} \|\theta_1 - \theta_2\|^2, \forall \theta_1, \theta_2 \in \Theta. \quad (246)$$

then $p^* \in G_{TV}^{\mathbb{E}^2 \left( \frac{3\rho^2}{\xi}, \eta \right)}$. Thus the population limit is at most $\frac{3\rho^2}{\xi}$ if $2\epsilon \leq \eta < 1$.

**Remark C.1.** Note that when we assume $\nabla \ell(\theta, X)$ has bounded covariance, i.e. $\text{Cov}_p[\nabla \ell(\theta, X)] \preceq \sigma^2 I$ for all $\theta \in \Theta$, we have $p^* \in G_{TV}^{\mathbb{E}_{TV}(\rho, \eta)}$ with $\rho = 2\sigma \sqrt{\eta}$ when $\eta < 1/2$ from Example 3.1. We know that bounded covariance set is a subset of $G_{mean}^{TV}$ from Lemma C.2, while $G_{mean}^{TV}$ is a subset of $G_{TV}^{TV}$ for regression from the above lemma. Thus it suffices to project onto a smaller set than $G_{TV}^{TV}$, which is bounded covariance set for the gradient. This motivates the algorithm design in (Diakonikolas et al., 2018b) in population level.

**Remark C.2.** Here we draw the connection between robust linear regression in Example 3.2 and Lemma C.9. Lemma C.9 requires the gradient to be inside resilient set, i.e. $\sup_{r \leq \frac{r^*}{1-\eta}} \|\mathbb{E}_r[XX^\top (\theta^*(p) - \theta) + XZ] - \mathbb{E}_p[XX^\top (\theta^*(p) - \theta) + XZ]\|_2 \leq \rho$. If we assume the same conditions in (25) and (26), and assume the radius of $\Theta$ is upper bounded by $R$, we can derive $\rho = 2\lambda_{\max}(\mathbb{E}_p[XX^\top])R\eta \psi^{-1}(1/\eta)$ if $\eta < 1/2$, here $\lambda_{\max}(A)$ is the largest eigenvalue for matrix $A$. If we further assume that $\lambda_{\min}(\mathbb{E}_p[XX^\top]) > 0$, then it follows from the second statement of Lemma C.9 that one can achieve population limit at most $\frac{C\lambda_{\max}(\mathbb{E}_p[XX^\top])R^2}{\lambda_{\min}(\mathbb{E}_p[XX^\top])} (\epsilon \psi^{-1}(1/\epsilon))^2$ for some constant $C$ when $\epsilon < 1/4$. In order to ensure this bound is of the same order as that in Example 3.2, one needs the additional assumptions of bounded radius and the matrix $\mathbb{E}_p[XX^\top]$ is near isotropic.

**Proof.** We first check all the distributions that satisfy the first condition are inside $G_{TV}^{TV} \cap G_{TV}^{TV}$. For $G_{TV}^{TV}$ and any $r \leq \frac{r^*}{1-\eta}$, we have

$$\mathbb{E}_r[\ell(\theta^*(p^*), X) - \ell(\theta^*(r), X)] = \mathbb{E}_r[\int_0^1 \nabla \ell(\theta^*(r) + t(\theta^*(p^*) - \theta^*(r)), X)^\top (\theta^*(p^*) - \theta^*(r))dt]$$

$$= \int_0^1 \mathbb{E}_r[\nabla \ell(\theta^*(r) + t(\theta^*(p^*) - \theta^*(r)), X)]^\top (\theta^*(p^*) - \theta^*(r))dt$$

$$\leq \int_0^1 (\mathbb{E}_r[\nabla \ell(\theta^*(r) + t(\theta^*(p^*) - \theta^*(r)), X)]^\top (\theta^*(p^*) - \theta^*(r))$$

$$+ \rho\|\theta^*(p^*) - \theta^*(r)\|_2)dt$$

$$\leq \mathbb{E}_r[\ell(\theta^*(p^*), X) - \ell(\theta^*(r), X)] + 2R\rho$$

$$\leq 2R\rho. \quad (247)$$

The last equation comes from $\mathbb{E}_p[\ell(\theta^*(p^*), X) - \ell(\theta^*(r), X)] \leq 0$. This shows that $p^* \in G_{TV}^{TV}(\rho, \eta)$.

Then we verify that the set is also inside $G_{TV}^{TV}$, which is defined such that for any $r \leq \frac{r^*}{1-\eta}$ and any $\theta$,

$$\mathbb{E}_r[\ell(\theta, X) - \ell(\theta^*(r), X)] \leq 2R\rho$$

$$\Rightarrow \mathbb{E}_p[\ell(\theta, X) - \ell(\theta^*(p^*), X)] \leq 4R\rho. \quad (248)$$
From $\|\mathbb{E}_r[\nabla \ell(\theta, X)] - \mathbb{E}_{p^*}[\nabla \ell(\theta, X)]\|_2 \leq \rho$, we have

$$
\mathbb{E}_{p^*}[\ell(\theta, X) - \ell(\theta^*(p^*), X)] = \mathbb{E}_{p^*}[\int_0^1 \nabla \ell(\theta^*(p^*) + t(\theta - \theta^*(p^*)), X)^\top (\theta - \theta^*(p^*))dt]
= \int_0^1 \mathbb{E}_{p^*}[\nabla \ell(\theta^*(p^*) + t(\theta - \theta^*(p^*)), X)]^\top (\theta - \theta^*(p^*))dt
\leq \int_0^1 (\mathbb{E}_r[\nabla \ell(\theta^*(p^*) + t(\theta - \theta^*(p^*)), X)]^\top (\theta - \theta^*(p^*)) + \rho \|\theta - \theta^*(p^*)\|_2)dt
\leq \mathbb{E}_r[\ell(\theta, X) - \ell(\theta^*(p), X)] + 2R\rho
\leq \mathbb{E}_r[\ell(\theta, X) - \ell(\theta^*(r), X)] + 2R\rho
\leq 4R\rho.
$$

(249)

This shows all distributions that satisfy condition 1 are inside $\mathcal{G}^{TV}(2R\rho, 4R\rho, \eta)$. Thus the population information-theoretic limit is $4R\rho$ if $2\epsilon \leq \eta < 1$.

Now we check that all the distributions that satisfy the second condition are inside $\mathcal{G}^{TV}(\frac{\epsilon^2}{2}, \frac{3\epsilon^2}{2}, \eta)$. For $\mathcal{G}^{TV}_\downarrow$, similar to the previous proof, we have

$$
\mathbb{E}_r[\ell(\theta^*(p^*), X) - \ell(\theta^*(r), X)] = \mathbb{E}_r[\int_0^1 \nabla \ell(\theta^*(r) + t(\theta^*(p^*) - \theta^*(r)), X)^\top (\theta^*(p^*) - \theta^*(r))dt]
= \int_0^1 \mathbb{E}_r[\nabla \ell(\theta^*(r) + t(\theta^*(p^*) - \theta^*(r)), X)]^\top (\theta^*(p^*) - \theta^*(r))dt
\leq \int_0^1 (\mathbb{E}_{p^*}[\nabla \ell(\theta^*(r) + t(\theta^*(p^*) - \theta^*(r)), X)]^\top (\theta^*(p^*) - \theta^*(r))
+ \rho \|\theta^*(p) - \theta^*(r)\|_2)dt
\leq \int_0^1 (\mathbb{E}_{p^*}[\nabla \ell(\theta^*(r) + t(\theta^*(p^*) - \theta^*(r)), X)]^\top (\theta^*(p^*) - \theta^*(r))dt
+ \rho \sqrt{\frac{2}{\xi}} \sqrt{\mathbb{E}_{p^*}[\ell(\theta^*(r), X) - \ell(\theta^*(p^*), X)]}
= -\mathbb{E}_{p^*}[\ell(\theta^*(r), X) - \ell(\theta^*(p^*), X)] + \rho \sqrt{\frac{2}{\xi}} \sqrt{\mathbb{E}_{p^*}[\ell(\theta^*(r), X) - \ell(\theta^*(p^*), X)]}
\leq \frac{\rho^2}{2\xi}.
$$

(250)

Here we use the property of strong convexity to get

$$
\mathbb{E}_{p^*}[\ell(\theta^*(r), X) - \ell(\theta^*(p^*), X)] \geq \nabla \mathbb{E}_{p^*}[\ell(\theta^*(p), X)](\theta^*(r) - \theta^*(p)) + \frac{\xi}{2} \|\theta^*(r) - \theta^*(p)\|_2^2
= \frac{\xi}{2} \|\theta^*(r) - \theta^*(p)\|_2^2.
$$

(251)
For $G_{r}^{TV}$, if $\mathbb{E}_{r}[\ell(\theta, X) - \ell(\theta^*(r), X)] \leq \frac{\rho^2}{2\xi}$, we have

$$
\mathbb{E}_{p^*}[\ell(\theta, X) - \ell(\theta^*(p^*), X)] = \mathbb{E}_{p^*}[\int_{0}^{1} \nabla \ell(\theta^*(p^*) + t(\theta - \theta^*(p^*)), X) \cdot (\theta - \theta^*(p^*))dt]
= \int_{0}^{1} \mathbb{E}_{p^*}[\nabla \ell(\theta^*(p^*) + t(\theta - \theta^*(p^*)), X)] \cdot (\theta - \theta^*(p^*))dt
\leq \int_{0}^{1} (\mathbb{E}_{r}[\nabla \ell(\theta^*(p^*) + t(\theta - \theta^*(p^*)), X)]^T \cdot (\theta - \theta^*(p^*)) + \rho(\epsilon)\|\theta - \theta^*(p^*)\|_2)dt
\leq \frac{\rho^2}{2\xi} + \rho \sqrt{\frac{2}{\xi} \mathbb{E}_{p^*}[\ell(\theta^*(r), X) - \ell(\theta^*(p^*), X)].}
$$

(252)

Solving this inequality, we have

$$
\mathbb{E}_{p^*}[\ell(\theta, X) - \ell(\theta^*(p^*), X)] \leq \frac{(\rho \sqrt{2/\xi} + 2\rho \sqrt{1/\xi})^2}{4}
\leq \frac{3\rho^2}{\xi}.
$$

(253)

This shows any distribution $p^*$ that satisfies condition 2 are inside $G_{r}^{TV}(\frac{\rho^2}{2\xi}, \frac{3\rho^2}{\xi}, \eta)$. Thus the population information-theoretic limit is $\frac{3\rho^2}{\xi}$ assuming $2\epsilon \leq \eta < 1$.

C.7 Proof of Example 3.3

With the choice of $B, L$, we have

$$
G_{r}^{TV}(\rho_1, \eta) = \{p \mid \forall r \leq \frac{p}{1 - \eta}, \max (\|\Sigma_p^{-1/2}(\mu_r - \mu_p)\|_2^2/\eta, \|I_d - \Sigma_p^{-1/2}\mathbb{E}_r[(X - \mu_p)(X - \mu_p)^T]\Sigma_p^{-1/2}\|_2) \leq \rho_1\},
$$

(254)

$$
G_{\uparrow}^{TV}(\rho_1, \rho_2, \eta) = \{p \mid \forall (\mu, \Sigma), \forall r \leq \frac{p}{1 - \eta},
\max (\|\Sigma^{-1/2}(\mu_r - \mu)\|_2^2/\eta, \|I_d - \Sigma^{-1/2}\mathbb{E}_r[(X - \mu)(X - \mu)^T]\Sigma^{-1/2}\|_2) \leq \rho_1
\Rightarrow \max (\|\Sigma_p^{-1/2}(\mu_p - \mu)\|_2^2/\eta, \|I_d - \Sigma_p^{-1/2}\Sigma\Sigma_p^{-1/2}\|_2) \leq \rho_2\}.
$$

(255)

We first show that with appropriate choices of $\rho_1$ and $\rho_2$, $G_{r}^{TV}(\rho_1, \eta)$ is a subset of $G_{\uparrow}^{TV}(\rho_1, \rho_2, \eta)$. It suffices to show that for any $p \in G_{r}^{TV}$, any $\mu, \Sigma, r \leq \frac{p}{1 - \eta}$ satisfying $\max (\|\Sigma^{-1/2}(\mu_r - \mu)\|_2^2/\eta, \|I_d - \Sigma^{-1/2}\mathbb{E}_r[(X - \mu)(X - \mu)^T]\Sigma^{-1/2}\|_2) \leq \rho_1$, we have

$$
\max (\|\Sigma_p^{-1/2}(\mu_p - \mu)\|_2^2/\eta, \|I_d - \Sigma_p^{-1/2}\Sigma\Sigma_p^{-1/2}\|_2) \leq \rho_2.
$$

(256)
We first note that
\[
\|I_d - \Sigma_r^{1/2} \Sigma_r^{-1/2}\|_2 = \|I_d - \Sigma_r^{1/2} \Sigma_r^{-1/2} \Sigma_r^{1/2}(X - \mu_r)(X - \mu_r)^\top \Sigma^{-1/2}\|_2 \\
= \|I_d - \Sigma_r^{1/2} \Sigma_r^{-1/2} \Sigma_r^{1/2}(X - \mu)(X - \mu)^\top + \Sigma^{-1/2}(\mu_r - \mu)(\mu_r - \mu)^\top \Sigma^{-1/2}\|_2 \\
\leq \|I_d - \Sigma_r^{1/2} \Sigma_r^{-1/2} \Sigma_r^{1/2}(X - \mu)(X - \mu)^\top \Sigma^{-1/2}\|_2 + \|\Sigma^{-1/2}(\mu_r - \mu)(\mu_r - \mu)^\top \Sigma^{-1/2}\|_2 \\
= \|I_d - \Sigma_r^{1/2} \Sigma_r^{-1/2} \Sigma_r^{1/2}(X - \mu)(X - \mu)^\top \Sigma^{-1/2}\|_2 + \|\Sigma^{-1/2}(\mu_r - \mu)\|_2^2 \\
\leq (1 + \eta)\rho_1. \quad (257)
\]
Thus we have
\[
(1 - (1 + \eta)\rho_1)I_d \preceq \Sigma_r^{1/2} \Sigma_r^{-1/2} \preceq (1 + (1 + \eta)\rho_1)I_d. \quad (258)
\]
From the fact that $A \preceq B$ leads to $C^\top AC \preceq C^\top BC$ and taking $C = \Sigma^{1/2}$, we have
\[
(1 - (1 + \eta)\rho_1)\Sigma_r \preceq (1 + (1 + \eta)\rho_1)\Sigma_r. \quad (259)
\]
Similarly, from $p \in G_\rho^T(\rho_1, \eta)$ we have
\[
\|I_d - \Sigma_p^{1/2} \Sigma_p^{-1/2}\|_2 = \|I_d - \Sigma_p^{1/2} \Sigma_p^{-1/2} \Sigma_p^{1/2}(X - \mu_p)(X - \mu_p)^\top \Sigma_p^{-1/2}\|_2 \\
= \|I_d - \Sigma_p^{1/2} \Sigma_p^{-1/2} \Sigma_p^{1/2}(X - \mu_p)(X - \mu_p)^\top \Sigma_p^{-1/2}\|_2 + \Sigma_p^{-1/2}(\mu_r - \mu_p)(\mu_r - \mu_p)^\top \Sigma_p^{-1/2}\|_2 \\
\leq \|I_d - \Sigma_p^{1/2} \Sigma_p^{-1/2} \Sigma_p^{1/2}(X - \mu_p)(X - \mu_p)^\top \Sigma_p^{-1/2}\|_2 + \|\Sigma_p^{-1/2}(\mu_r - \mu_p)(\mu_r - \mu_p)^\top \Sigma_p^{-1/2}\|_2 \\
= \|I_d - \Sigma_p^{1/2} \Sigma_p^{-1/2} \Sigma_p^{1/2}(X - \mu_p)(X - \mu_p)^\top \Sigma_p^{-1/2}\|_2 + \|\Sigma_p^{-1/2}(\mu_r - \mu_p)\|_2^2 \\
\leq (1 + \eta)\rho_1. \quad (260)
\]
Thus we have
\[
(1 - (1 + \eta)\rho_1)\Sigma_p \preceq \Sigma_r \preceq (1 + (1 + \eta)\rho_1)\Sigma_p. \quad (261)
\]
Combining Equation (259) and (261), we know that
\[
\frac{1 - (1 + \eta)\rho_1}{1 + (1 + \eta)\rho_1} \Sigma_p \preceq \Sigma \preceq \frac{1 + (1 + \eta)\rho_1}{1 - (1 + \eta)\rho_1} \Sigma_p \quad (262)
\]
When $(1 + \eta)\rho_1 \leq \frac{1}{3}$, we have
\[
(1 - 3(1 + \eta)\rho_1)\Sigma_p \preceq \Sigma \preceq (1 + 3(1 + \eta)\rho_1)\Sigma_p \quad (263)
\]
Thus
\[
\|I_d - \Sigma_p^{-1/2} \Sigma_p^{-1/2}\|_2 \leq 3(1 + \eta)\rho_1. \quad (264)
\]
Furthermore, we know that
\[ \|\Sigma_p^{-1/2}(\mu_r - \mu)\|_2 \leq \|\Sigma_p^{-1/2}\Sigma_1^{1/2}\|_2 \|\Sigma_p^{-1/2}(\mu_r - \mu)\|_2 \leq \sqrt{1 + 3(1 + \eta)\rho_1}\eta\rho_1. \] (265)

From \(\|\Sigma_p^{-1/2}(\mu_r - \mu_p)\|_2 \leq \sqrt{\eta\rho_1}\), by triangle inequality, we know that
\[ \|\Sigma_p^{-1/2}(\mu_p - \mu)\|_2 \leq \|\Sigma_p^{-1/2}(\mu_p - \mu_r)\|_2 + \|\Sigma_p^{-1/2}(\mu_r - \mu)\|_2 \]
\[ \leq (\sqrt{1 + 3(1 + \eta)\rho_1} + 1)\sqrt{\eta\rho_1}. \] (266)

Thus
\[ \|\Sigma_p^{-1/2}(\mu_p - \mu)\|_2 \leq \|\Sigma_p^{-1/2}(\mu_p - \mu_r)\|_2 + \|\Sigma_p^{-1/2}(\mu_r - \mu)\|_2 \]
\[ \leq (\sqrt{1 + 3(1 + \eta)\rho_1} + 1)\sqrt{\eta\rho_1}. \]

assuming \((1 + \eta)\rho_1 \leq \frac{1}{3}\). Therefore \(L(p, (\mu, \Sigma)) = \max (\|\Sigma_p^{-1/2}(\mu_p - \mu)\|_2, \|I_d - \Sigma_p^{-1/2}\Sigma\Sigma_p^{-1/2}\|_2) \leq 6\rho_1\) if \((1 + \eta)\rho_1 \leq \frac{1}{3}\). By taking \(\rho_2 = 6\rho_1\), we know that \(G_{TV}^\downarrow (\rho_1, \eta) \subset G_{TV}^\downarrow (\rho_1, \rho_2, \eta)\).

Now we only need to show that for any \(p\) that satisfies
\[ \sup_{v \in \mathbb{R}^d, ||v||_2 = 1} \mathbb{E}_p \left[ \psi \left( \frac{1}{\kappa^2} \left( \frac{v^\top \Sigma_p^{-1/2}(X - \mu_p)^2}{\mu^\top \Sigma_p^{-1/2}(X - \mu_p)^2} \right) \right) \right] \leq 1, \] (268)

we have \(p \in G_{TV}^\downarrow (\rho_1, \eta)\) for some \(\rho_1\). We view \(\Sigma_p^{-1/2}X\) as a random variable. Note that \(\psi \circ x^2\) is also an Orlicz function. From Lemma A.3, we know that bounded raw \(\psi\) norm can imply bounded central \(\psi\) norm. Thus from Lemma C.2 and centering Lemma A.3, for any \(\eta < 1/2,\)
\[ \|\Sigma_p^{-1/2}(\mu_r - \mu_p)\|_2 \leq 2\kappa\eta\sqrt{\psi^{-1}(1/\eta)}, \] (269)
\[ \|I_d - \Sigma_p^{-1/2}\Sigma_r\Sigma_p^{-1/2}\|_2 \leq 4\kappa^2\eta\psi^{-1}(1/\eta). \] (270)

We have shown that \(p \in G_{TV}^\downarrow (\rho_1, \eta)\) for \(\rho_1 = 4\kappa^2\eta\psi^{-1}(1/\eta)\). Thus \(p \in G_{TV}^\downarrow (\rho, 6\rho, \eta)\) for \(\rho = 4\kappa\eta\psi^{-1}(1/\eta)\) assuming \((1 + \eta)\rho \leq \frac{1}{3}\).

C.8 Proof of Example 3.4

We first show that \(p \in G_{TV}^\downarrow\). Note that the condition in \(G_{TV}^\downarrow\) can be written as
\[ \sup_{r \leq \frac{\sqrt{\eta}}{\kappa}} \|\mathbb{E}_p[M_p^{-1/2}XX^\top M_p^{-1/2}] - \mathbb{E}_r[M_p^{-1/2}XX^\top M_p^{-1/2}]\|_F \leq \rho. \] (271)

Recall that we define \(Y = M_p^{-1/2}X\). We flatten \(YY^T\) as a vector \(Y^{\otimes 2}\). Then \(p \in G_{TV}^\downarrow\) is equivalent to requiring that the distribution of \(Y^{\otimes 2}\) is inside set \(G_{mean}^{TV}\). By Lemma C.2, we know that from the given assumption in (30), the distribution of \(Y^{\otimes 2}\) is inside \(G_{mean}^{TV}(4\kappa\eta\psi^{-1}(\frac{1}{7}), \eta)\) for all \(\eta \in [0, 1/2]\), we know that the LHS of above equation can be controlled with \(\rho = 4\kappa\eta\psi^{-1}(\frac{1}{7})\).

Now we validate that \(G_{TV}^\downarrow (\rho, \eta) \subset G_{TV}^\downarrow (2\rho(1 + \rho), \eta)\). By \(p \in G_{TV}\) and the condition in \(G_{\uparrow}\), we have
\[ \|I - M_p^{-1/2}M_p^{-1/2}\|_F \leq \rho, \] (272)
\[ \|I - M^{-1/2}M^{-1/2}\|_F \leq \rho \] (273)
From the first bound, we know that \( \|M_p^{-1/2} M_r^{-1/2}\|_2^2 \leq 1 + \rho \). We bound the term:

\[
\begin{align*}
\|I - M_p^{-1/2} M M_p^{-1/2}\|_F &= \|M_p^{-1/2} M_r^{1/2}(M_r^{-1/2} M_p M_r^{-1/2} - M_r^{-1/2} M M_r^{-1/2}) M_r^{1/2} M_p^{-1/2}\|_F \\
&\leq \|M_p^{-1/2} M_r^{1/2}\|_2^2 \|M_r^{-1/2} M_p M_r^{-1/2} - M_r^{-1/2} M M_r^{-1/2}\|_F \\
&\leq \|M_p^{-1/2} M_r^{1/2}\|_2^2 \left( \|I - M_r^{-1/2} M_p M_r^{-1/2}\|_F + \|I - M_r^{-1/2} M M_r^{-1/2}\|_F \right) \\
&\leq 2\rho(1 + \rho). 
\end{align*}
\]

It can be seen from Theorem 3.1 that when the perturbation level is \( \epsilon \), the population limit for this set is upper bounded by \( 8\kappa \epsilon \psi^{-1}(\frac{1}{2\epsilon^2})(1 + 4\kappa \epsilon \psi^{-1}(\frac{1}{2\epsilon^2})) \).

C.9 Further discussion on robust classification

We study the sufficient conditions that can ensure a distribution is in \( \mathcal{G}^{TV} \) in Definition 3.1 where \( L(p, \theta) = \mathbb{E}_p[\mathbb{I}(Y X^\top \theta \leq 0)] \) is the zero-one loss function for linear classification, where \( (X, Y) \in \mathbb{R}^d \times \{-1, 1\} \). We always assume that we have augmented \( X \) by an additional dimension of constant 1 to avoid the non-zero offset term in the classifier.

C.9.1 Bridge function \( B(p, \theta) \) is zero-one loss and linearly separable case

We first consider the setting that \( B(p, \theta) = L(p, \theta) = \mathbb{E}_p[\mathbb{I}(Y X^\top \theta \leq 0)] \). Assume \( \theta \in \Theta = \{\theta \in \mathbb{R}^d : \|\theta\|_2 \leq 1\} \). We consider the special case of \( \rho_1 = \rho_2 = 0 \) in Definition 3.1:

\[
\mathcal{G}_i^{TV}(0, \eta) \triangleq \{p | \sup_{r \leq \frac{1}{2\eta}} \mathbb{E}_r[\mathbb{I}(Y X^\top \theta^*(p) \leq 0)] \leq 0\}, 
\]

\[
\mathcal{G}_i^{TV}(0, 0, \eta) \triangleq \{p | \forall \theta \in \Theta, \forall r \leq \frac{1}{\eta} \left( \mathbb{E}_r[\mathbb{I}(Y X^\top \theta \leq 0)] \leq 0 \Rightarrow \mathbb{E}_p[\mathbb{I}(Y X^\top \theta \leq 0)] \leq 0 \right) \}, 
\]

where \( \theta^*(p) = \arg \min_{\theta \in \Theta} \mathbb{E}_p[\mathbb{I}(Y X^\top \theta \leq 0)] \).

We investigate the sufficient conditions that imply \( p \in \mathcal{G}_i^{TV}(0, \eta) \cap \mathcal{G}_i^{TV}(0, 0, \eta) \).

**Proposition C.1.** Suppose distribution \( p \) of \( (X, Y) \in \mathbb{R}^d \times \{-1, 1\} \) satisfies the following properties:

- \( (X, Y) \) is linearly separable under \( p \);
- There does not exist \( \theta \in \Theta \) such that \( \mathbb{P}_p(Y X^\top \theta \leq 0) \in (0, \eta] \).

Then, \( p \in \mathcal{G}_i^{TV}(0, \eta) \cap \mathcal{G}_i^{TV}(0, 0, \eta) \) defined in (275) and (276).

**Proof.** If \( (X, Y) \) is linearly separable under \( p \), then

\[
\mathbb{E}_p[\mathbb{I}(Y X^\top \theta^*(p) \leq 0)] = 0. 
\]

Thus, for any \( r \leq \frac{1}{\eta} \),

\[
\mathbb{E}_r[\mathbb{I}(Y X^\top \theta^*(p) \leq 0)] \leq \frac{1}{\eta} \mathbb{E}_p[\mathbb{I}(Y X^\top \theta^*(p) \leq 0)] = 0,
\]

which shows that \( p \) being linearly separable implies that \( p \in \mathcal{G}^{TV}(0, \eta) \). It suffices to check that \( p \in \mathcal{G}^{TV}(0, \eta) \). We need to show that for any \( \theta \) and any \( r \leq \frac{1}{\eta} \), \( \mathbb{E}_r[\mathbb{I}(Y X^\top \theta \leq 0)] \leq 0 \) implies \( \mathbb{E}_p[\mathbb{I}(Y X^\top \theta \leq 0)] \leq 0 \).
For any $\theta \in \mathbb{R}^d$, $\|\theta\|_2 = 1$, by assumption only two situations will occur: $\mathbb{E}_p(1(YX^T \theta \leq 0)) = 0$ or $\mathbb{E}_p(1(YX^T \theta \leq 0)) > \eta$. If we observe $\mathbb{E}_r[1(YX^T \theta \leq 0)] = 0$, then there must be $\mathbb{E}_r[1(YX^T \theta \leq 0)] = 0$ since deletion would at most decrease the cost by $\eta$.

\[ \square \]

Intuitively, the second sufficient condition guarantees that deleting $\eta$ fraction of mass cannot decrease the loss from non-zero to zero. It is also necessary: if there exists some $\theta$ such that $\mathbb{E}_p(1(YX^T \theta \leq 0)) \in (0, \eta]$, then the adversary can delete all the mass with $1(YX^T \theta \leq 0)$ to construct a linearly separable distribution, then the implication in $\mathcal{G}_{\text{TV}}^+$ would fail.

However, the second sufficient condition is in general hard to be satisfied for continuous distributions in high dimensions: one can always rotate $\theta$ to satisfy $\mathbb{P}_p(YX^T \theta \leq 0) \in (0, \eta]$ since we know $\mathbb{P}_p(YX^T \theta^*(p) \leq 0) = 0$ and $\mathbb{P}_p(YX^T (-\theta^*(p)) \leq 0) = 1$. In next section, we change $B$ to hinge loss and show that the corresponding sufficient condition can be easier to satisfy.

### C.9.2 Bridge function $B(p, \theta)$ is hinge loss

Take $B(p, \theta) = \mathbb{E}_p[\max(0, 1 - YX^T \theta)]$, $L(p, \theta) = \mathbb{E}_p[1(YX^T \theta \leq 0)]$. Assume $\theta \in \Theta = \{\theta \in \mathbb{R}^d : \|\theta\|_2 \leq 1\}$. The set $\mathcal{G}^{TV}(\rho_1, \rho_2, \eta)$ is defined as $\mathcal{G}^{TV}(\rho_1, \eta) \cap \mathcal{G}^{TV}(\rho_1, \rho_2, \eta)$, where

\[
\mathcal{G}^{TV}(\rho_1, \eta) = \{p \mid \sup_{r \leq \frac{\rho_1}{1-\eta}} \mathbb{E}_r[\max(0, 1 - YX^T \theta^*(p))] \leq \rho_1 \}.
\]

\[
\mathcal{G}^{TV}(\rho_1, \rho_2, \eta) = \{p \mid \forall \theta \in \Theta, \forall r \leq \frac{\rho_1}{1-\eta}, \left( \mathbb{E}_r[\max(0, 1 - YX^T \theta)] \leq \rho_1 \Rightarrow \mathbb{P}_p(YX^T \theta \leq 0) \leq \rho_2 \right) \}.
\]

where $\theta^*(p) = \arg\min_{\theta \in \Theta} \mathbb{E}_p[\max(0, 1 - YX^T \theta)]$.

We investigate the sufficient conditions for a distribution to be inside $\mathcal{G}(\rho_1, \rho_2, \eta)$ as follows,

**Proposition C.2.** Suppose distribution $p$ of $(X, Y) \in \mathbb{R}^d \times \{-1, 1\}$ satisfies the following properties:

- $\mathbb{E}_p[\max(0, 1 - YX^T \theta^*(p))] \leq (1 - \eta)p_1$.
- $\forall \theta \in \Theta, \left( \mathbb{P}_p(YX^T \theta \leq \frac{1}{2}) \leq \eta + 2(1 - \eta)p_1 \Rightarrow \mathbb{P}_p(YX^T \theta \leq 0) \leq \rho_2 \right)$.

Then, $p \in \mathcal{G}^{TV}(\rho_1, \rho_2, \eta)$ defined in (279) and (280).

**Proof.** We first show that $p \in \mathcal{G}^{TV}(\rho_1, \eta)$. For any $r \leq \frac{\rho_1}{1-\eta}$, we have

\[
\mathbb{E}_r[\max(0, 1 - X^T \theta^*(p))] \leq \frac{\mathbb{E}_p[\max(0, 1 - X^T \theta^*(p))]}{1-\eta} \leq \frac{(1-\eta)p_1}{1-\eta} = \rho_1,
\]

which implies that $p \in \mathcal{G}^{TV}(\rho_1, \eta)$.

Next we show that $p \in \mathcal{G}^{TV}(\rho_1, \rho_2, \eta)$. Assume there exists some $\theta \in \Theta$ and $r \leq \frac{\rho_1}{1-\eta}$ such that $\mathbb{E}_r[\max(0, 1 - YX^T \theta)] \leq \rho_1$. Then we claim that $\mathbb{P}_p(YX^T \theta \leq \frac{1}{2}) \leq \eta + 2(1 - \eta)p_1$ must hold. If it does not hold, then $\mathbb{P}_p(YX^T \theta \leq \frac{1}{2}) > \eta + 2(1 - \eta)p_1$ would imply that $\mathbb{P}_r(YX^T \theta \leq \frac{1}{2}) > 2\rho_1$. Therefore $\mathbb{E}_r[\max(0, 1 - YX^T \theta)] > \rho_1$, which contradicts the assumption. \[ \square \]
Here the first condition is a standard assumption on the margin, and the second condition can be verified by the following condition: for any fixed $\theta \in \Theta$, denote $Z = YX^\top \theta$, and the left $\rho_2$ quantile of $Z$ as $q$, then as long as the following inequality holds, the second condition holds:

$$\mathbb{P}_p(Z \in [q, q + 1/2]) \geq \eta + 2(1 - \eta)\rho_1.$$  \hspace{1cm} (284)

When $XY$ is isotropic Gaussian distribution, the above conditions are satisfied for certain parameters.

In robust classification case, since the target loss function is zero-one loss, the robust error would be at most $\epsilon$ given $\epsilon$ perturbation in total variation. Thus one needs more stringent results for robustness to make the guarantee meaningful. Here our condition on concentration allows $\rho_2$ to be 0, which provides strong guarantee for the classification error.

**Remark C.3.** In the literature of robust classification (Klivans et al., 2009; Awasthi et al., 2014; Diakonikolas et al., 2018c), it is usually assumed that $P_X$ instead of $P_{X|Y}$ satisfies some nice concentration properties. However, one can easily create a toy example where $P_{X|Y=1}$ and $P_{X|Y=-1}$ are well separated and satisfy our sufficient conditions, but $P_X$ does not have good concentration property.

**C.9.3 Estimating Chow-parameters implies robust classification under polynomial threshold function**

As another example under our framework of $\mathcal{G}_{TV}$, it is proposed in (Diakonikolas et al., 2018c, Lemma 3.4) that with appropriate estimate of Chow-parameters, one can guarantee certain level of classification accuracy if the classification function is the sign of some degree-$d$ polynomial threshold functions $f(x)$. Thus the classification problem can be reduced to robustly estimating the mean of $\mathbb{E}_{p^*}[f(X)t(X)]$ where $t(x)$ is one of the polynomial functions with degree at most $d$. Thus the result can be incorporated into our framework when $B$ measures the loss in estimating the Chow parameters, and $L$ is the classification zero-one loss, and the conditions in (Diakonikolas et al., 2018c) serves as sufficient conditions for $p^*$ being inside $\mathcal{G}_{TV}$.

**D Related discussions and remaining proofs in Section 4**

**D.1 Key Lemmas**

The following lemma produces an upper bound on $|\mathbb{E}[g(X) - g(Y)]|$.

**Lemma D.1.** Assume we are given two distributions $p, q$, some function $g : \mathbb{R} \mapsto \mathbb{R}$, some non-negative cost function $c(x, y)$ and some Orlicz function $\psi$ in Definition 2.2. For any coupling $\pi_{p,q}$ between $p, q$ such that $X \sim p, Y \sim q$, and any $\sigma > 0$, we have

$$|\mathbb{E}[g(X) - g(Y)]| \leq \sigma \mathbb{E}_{\pi_{p,q}}[c(X, Y)]\psi^{-1}\left(\frac{\mathbb{E}_{\pi_{p,q}}[c(X, Y)\psi\left(\frac{|g(X) - g(Y)|}{\sigma c(X, Y)}\right)]}{\mathbb{E}_{\pi_{p,q}}[c(X, Y)]}\right).$$  \hspace{1cm} (285)

where $\psi^{-1}$ is the (generalized) inverse function of $\psi$ defined in (14).

We remark here that the left-hand side of the conclusion does not depend on the coupling $\pi_{p,q}$, so one can take the infimum over all couplings on the right-hand side.
Proof. We omit the $\pi$ in the subscript in $\mathbb{E}$. Applying Jensen’s inequality to the new measure $c(X,Y) \propto |c(X,Y)| d\pi_{\rho,q}$,

$$
\psi \left( \frac{\mathbb{E}[g(X) - g(Y)]}{\sigma \mathbb{E}[c(X,Y)]} \right) = \psi \left( \frac{\mathbb{E} \left[ \frac{c(X,Y)}{\mathbb{E}[c(X,Y)]} \cdot \frac{g(X) - g(Y)}{\sigma c(X,Y)} \right]}{\mathbb{E}[c(X,Y)]} \right) 
\leq \mathbb{E} \left[ \frac{c(X,Y)}{\mathbb{E}[c(X,Y)]} \psi \left( \frac{|g(X) - g(Y)|}{\sigma c(X,Y)} \right) \right] 
= \frac{\mathbb{E} \left[ c(X,Y) \psi \left( \frac{|g(X) - g(Y)|}{\sigma c(X,Y)} \right) \right]}{\mathbb{E}[c(X,Y)]}. 
\tag{286}
$$

It implies that

$$
|\mathbb{E}[g(X) - g(Y)]| \leq \sigma \mathbb{E}[c(X,Y)] \psi^{-1} \left( \frac{\mathbb{E} \left[ c(X,Y) \psi \left( \frac{|g(X) - g(Y)|}{\sigma c(X,Y)} \right) \right]}{\mathbb{E}[c(X,Y)]} \right). \tag{287}
$$

With this lemma, we show that bounded Orlicz norm implies resilience for $k$-th moment estimation under $W_1$ perturbation.

**D.2 $W_{c,k}$-Resilient Set Design for $L = W_F$**

Recall that in Section 3.1, we design $\mathcal{G}_{\text{mean}}^{TV}$ as that the means of any friendly perturbation distribution and the original distribution are close, and extend the idea to arbitrary $W_F$ pseudonorm in (152). Since we have already defined friendly perturbation for $W_{c,k}$, we similarly define $\mathcal{G}_{W_F}^{W_{c,k}}$ as that the $W_F$ pseudonorm between any friendly perturbation distribution and the original distribution is close. Concretely, given $L = W_F$ defined in (151), the set $\mathcal{G}_{W_F}^{W_{c,k}}$ can be defined as

**Definition D.1 ($\mathcal{G}_{W_F}^{W_{c,k}}(\rho, \eta)$).** Assume Assumption 4.1 holds. We define

$$
\mathcal{G}_{W_F}^{W_{c,k}}(\rho, \eta) = \{ p \mid \sup_{f \in \mathcal{F}, r \in F(p, \eta, W_{c,k}, f)} \mathbb{E}_r[f(X)] - \mathbb{E}_p[f(X)] \leq \rho \}. \tag{288}
$$

If $c(x, y) = 1(x \neq y)$ and $k = 1$, then $W_{c,k}$ reduces to TV, and one can show that the resilient set $\mathcal{G}_{W_F}^{W_{c,k}}(\rho, \eta)$ defined in (288) reduces to that in (152) in the TV case. The reason is that these two definitions of resilient sets share the same worst case perturbed $\mathbb{E}_r[f(X)]$; it is always to either delete the largest (or smallest) $\epsilon$ probability mass of $f(X)$, or to move the largest (or smallest) $\epsilon$ probability mass to the $\mathbb{E}_r[f(X)]$. Similar to TV case, we can show that the design of $\mathcal{G}_{W_F}^{W_{c,k}}$ is not too big such that its population limit can be controlled, and not too small such that some usual non-parametric assumptions such as bounded $k$-th moment implies being in the set.

**Not too big.** Similar to TV perturbation case, we show that $\mathcal{G}_{W_F}^{W_{c,k}}$ has controllable population limit by upper bounding its modulus of continuity.

**Theorem D.1.** Assume Assumption 4.1 holds. The modulus of continuity $m$ in (18) for $\mathcal{G}_{W_F}^{W_{c,k}}(\rho, \eta)$ is bounded above as $m(\mathcal{G}_{W_F}^{W_{c,k}}(\rho, \eta), 2\epsilon) \leq 2\rho$ for any $2\epsilon \leq \eta$. 

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Proof. The modulus is defined as

\[
\sup_{(p_1,p_2) : W_{c,k}(p_1, p_2) \leq 2\epsilon, p_1 \in \mathcal{G}_{W_{c,k}}^W (\rho,\eta), p_2 \in \mathcal{G}_{W_{c,k}}^W (\rho,\eta)} W_F(p_1, p_2).
\] (289)

By Lemma 4.1, for any \( f \in \mathcal{F} \), we are able to pick \( r \) such that \( W_{c,k}(p_1, r) \leq 2\epsilon \), \( W_{c,k}(p_2, r) \leq 2\epsilon \), and \( r \in \mathcal{F}(p_1, 2\epsilon, W_{c,k}, f) \cap \mathcal{F}(p_2, 2\epsilon, W_{c,k}, f) \) is a friendly perturbation for both \( p_1 \) and \( p_2 \). Take

\[
f^* \in \arg \max_{f \in \mathcal{F}} \mathbb{E}_{p_1}[f(X)] - \mathbb{E}_{p_2}[f(X)].
\] (290)

From \( p_1, p_2 \in \mathcal{G}_{W_{c,k}}^W (\rho,\eta) \) and the symmetricity of \( \mathcal{F} \), we know that for any \( 2\epsilon \leq \eta \),

\[
\mathbb{E}_{p_1}[f^*(X)] - \mathbb{E}_r[f^*(X)] \leq \rho,
\] (291)

\[
\mathbb{E}_r[f^*(X)] - \mathbb{E}_{p_2}[f^*(X)] \leq \rho.
\] (292)

Combining the two equations together gives us

\[
\mathbb{E}_{p_1}[f^*(X)] - \mathbb{E}_{p_2}[f^*(X)] \leq 2\rho.
\] (293)

This shows that \( W_F(p_1, p_2) \leq 2\rho \). 

\[\Box\]

Not too small. To see the generality of the design, we study a special case of estimating \( k \)-th moment under \( W_1 \) (Definition 2.1) perturbation.

Taking \( \mathcal{F} = \{ f \mid X \mapsto \xi|v^T X|^2, v \in \mathbb{R}^d, \|v\| = 1, \xi \in \{-1\} \} \), \( W_{c,k} = W_1 \) in Equation (288), one can check that Assumption 4.1 holds. The set \( \mathcal{G}_{W_{c,k}}^W \) becomes

\[
\mathcal{G}_{W_{c,k}}^W (\rho,\eta) \equiv \{ p \mid \sup_{v \in \mathbb{R}^d, \|v\| = 1, r \in \mathcal{F}(p,\rho,W_1,|v^T X|^k)} \mathbb{E}_r[|v^T X|^2] - \mathbb{E}_r[|r^T X|^2] \leq \rho \}. \] (294)

D.3 Proof of Example 4.1

We show a stronger statement of \( k \)-th moment estimation here.

Example D.1 (Bounded Orlicz norm implies resilience for \( k \)-th moment estimation under \( W_1 \) perturbation). For \( k > 1 \), taking \( \mathcal{F} = \{ f \mid X \mapsto \xi|v^T X|^k, v \in \mathbb{R}^d, \|v\| = 1, \xi \in \{-1\} \} \), \( W_{c,k} = W_1 \) in Equation (288), one can check that Assumption 4.1 holds. The set \( \mathcal{G}_{W_{c,k}}^W \) becomes

\[
\mathcal{G}_{W_{c,k}}^W (\rho,\eta) \equiv \{ p \mid \sup_{v \in \mathbb{R}^d, \|v\| = 1, r \in \mathcal{F}(p,\rho,W_1,|v^T X|^k)} \mathbb{E}_r[|v^T X|^k] - \mathbb{E}_r[|r^T X|^k] \leq \rho \}. \] (295)

Assume that there exists some Orlicz function \( \psi \) as is defined in Definition 2.2 and \( \psi \) satisfies \( \psi(x) \geq x, \forall x \geq 1 \). Denote \( \tilde{\psi}(x) = x\psi(kx^{-k-1}) \). We assume that

\[
\sup_{v \in \mathbb{R}^d, \|v\| = 1} \mathbb{E}_p \left[ \frac{\tilde{\psi} \left( \frac{|v^T X|}{\sigma} \right)}{\sigma} \right] \leq 1.
\] (296)

Then,

\[
p \in \mathcal{G}_{W_{c,k}}^W (\sigma^{k-1} \eta \psi^{-1}(\frac{2\sigma}{\eta}), \eta), \forall \eta < \min(\sigma/2k, 2\sigma/\psi(\max(k, 8))), \] (297)

where \( \psi^{-1} \) is the (generalized) inverse of \( \psi \) defined in (14).

\footnote{If the argmax is not achievable, we can take a sequence of \( f^*_i \) such that \( E_{p_1}[f^*_i(X)] - E_{p_2}[f^*_i(X)] \) goes to the maximum value as \( i \to +\infty \).}
Proof. From the fact that \( r \in \mathbb{F}(p, \eta, W_1, |v^\top X|^k) \), we know that for any coupling \( \pi_{p,r} \) that makes \( r \) friendly perturbation, we have

\[
\sup_{v \in \mathbb{R}^d, \|v\|_2 = 1} \mathbb{E}_{\pi_{p,r}}|v^\top (X - Y)| \leq \mathbb{E}_{\pi_{p,r}} \sup_{v \in \mathbb{R}^d, \|v\|_2 = 1} |v^\top (X - Y)| \leq \eta. \quad (298)
\]

For any fixed \( v \in \mathbb{R}^d, \|v\|_2 = 1 \), we claim that the worst perturbation only happens when for any \((x, y) \in \text{supp}(\pi_{p,r})\), \( |v^\top x|^k \geq |v^\top y|^k \) or for any \((x, y) \in \text{supp}(\pi_{p,r})\), \( |v^\top x|^k \leq |v^\top y|^k \). If it is not one of the two cases, we can always remove the movement from \( x \) to \( y \) that decreases or increases \( g \) to make \( |\mathbb{E}_{\pi}||v^\top X|^k - |v^\top Y|^k| \) larger without increasing \( \mathbb{E}_{\pi} \|X - Y\| \).

Thus we can assume for any \((x, y) \in \text{supp}(\pi_{p,r})\), \( |v^\top x|^k \geq |v^\top y|^k \) or for any \((x, y) \in \text{supp}(\pi_{p,r})\), \( |v^\top x|^k \leq |v^\top y|^k \). For the first case, by Lemma \( D.1 \), we bound the worst case perturbation as follows. For any \( v \in \mathbb{R}^d, \|v\|_2 = 1 \),

\[
|\mathbb{E}_{(X, Y) \sim \pi_{p,r}}[|v^\top X|^k - |v^\top Y|^k]| \leq \sigma^{k-1} \mathbb{E}_{\pi_{p,r}}[|v^\top (X - Y)|]^\psi^{-1} \frac{\mathbb{E}_{\pi_{p,r}}[|v^\top (X - Y)|]}{\mathbb{E}_{\pi_{p,r}}[|v^\top (X - Y)|]}
\]

\[
\leq \sigma^{k-1} \eta \psi^{-1} \frac{\mathbb{E}_{\pi_{p,r}}[|v^\top (X - Y)|]}{\eta} \quad (299)
\]

\[
\leq \sigma^{k-1} \eta \psi^{-1} \frac{\mathbb{E}_{\pi_{p,r}}[|v^\top (X - Y)|]}{\eta} \quad (300)
\]

\[
\leq \sigma^{k-1} \eta \psi^{-1} \frac{\mathbb{E}_{\pi_{p,r}}[|v^\top (X - Y)|]}{\eta} \quad (301)
\]

\[
= \sigma^{k-1} \eta \psi^{-1} \frac{\mathbb{E}_{\pi_{p,r}}[|2v^\top X|]}{\eta} \quad (302)
\]

\[
\leq \sigma^{k-1} \eta \psi^{-1} \frac{2\sigma}{\eta}. \quad (303)
\]

Here Equation (299) comes from the fact that \( x\psi^{-1}(C/x) \) is a non-decreasing function of \( x \) for the region \([0, +\infty)\) for any \( \sigma > 0 \) (Lemma \( A.1 \)). Equation (300) uses the fact that for any \((x, y) \in \text{supp}(\pi_{p,r}), |v^\top x| \geq |v^\top y| \). Equation (303) is from the assumption in (296).

Otherwise, for any \((x, y) \in \text{supp}(\pi_{p,r}), |v^\top x|^k \leq |v^\top y|^k \), by Lemma \( D.1 \), we bound the worst
case perturbation as follows:

$$
|E_{(X,Y) \sim \pi_{p,r}}[v^\top X^k - |v^\top Y|^k]| \leq \sigma^{k-1} \eta^{-1} \left( \frac{E_{\pi_{p,r}}[v^\top (X - Y)]}{\eta} \left( \frac{|v^\top X^k|}{\sigma^{k-1} v^\top (X - Y)} \right) \right)
\leq \sigma^{k-1} \eta^{-1} \left( \frac{E_{\pi_{p,r}}[v^\top (X - Y)]}{\eta} \left( \frac{|v^\top X^k|}{\sigma^{k-1} v^\top (X - Y)} \right) \right)$$

$$\leq \sigma^{k-1} \eta^{-1} \left( \frac{E_{\pi_{p,r}}[v^\top (X - Y)]}{\eta} \left( \frac{|v^\top X^k|}{\sigma^{k-1} v^\top (X - Y)} \right) \right).$$

(304)

For any \((x, y) \in \text{supp}(\pi_{p,r}), x \neq y\), we have \(|v^\top y|^k \leq E[|v^\top Y|^k] \) from the definition of friendly perturbation. Thus

$$E_{(X,Y) \sim \pi_{p,r}}[|v^\top Y|^k - |v^\top X|^k] \leq \sigma^{k-1} \eta^{-1} \left( \frac{E_{\pi_{p,r}}[v^\top (X - Y)]}{\eta} \left( \frac{k|v^\top Y|^{1-1/k}}{\sigma^{k-1}} \right) \right)$$

$$= \sigma^{k-1} \eta^{-1} \left( \frac{k|v^\top Y|^{1-1/k}}{\eta} \right)$$

$$= \eta \sigma^{-1} \left( \frac{k|v^\top Y|^{1-1/k}}{\eta} \right).$$

(305)

If \(|v^\top Y|^k \leq \sigma^k\), we have \(E_{(X,Y) \sim \pi_{p,r}}[|v^\top Y|^k - |v^\top X|^k] \leq k\sigma^{k-1} \eta\). Otherwise we further upper bound it to get

$$E_{(X,Y) \sim \pi_{p,r}}[|v^\top Y|^k - |v^\top X|^k] \leq k\sigma^{k-1} \eta E \left[ \frac{|v^\top Y|^k}{\sigma^k} \right].$$

(306)

Solving the inequality, we know that when \(\eta < \sigma/k\),

$$E[|v^\top Y|^k] \leq \frac{E[|v^\top X|^k]}{1 - k\eta/\sigma}.$$  

(307)

Now we bound the term \(E[|v^\top X|^k]\). When \(k|v^\top x|^{k-2}/\sigma^{k-1} \leq 1\), we have \(|v^\top x|^k \leq \sigma/k^{k/(k-1)}\). When \(k|v^\top x|^{k-2}/\sigma^{k-1} > 1\), by assumption \(\psi(x) \geq x, \forall x \geq 1\), we have \(|v^\top x|^k \leq \frac{\sigma}{k \eta^{-1}} \psi \left( \frac{k^{1-1/k}}{\sigma^{k-1}} \right)\).

$$E[|v^\top X|^k] \leq \frac{\sigma}{k^{k/(k-1)}} + \frac{\sigma^{k-1} E[|v^\top X|^{1/k}] \psi \left( \frac{k^{1-1/k}}{\sigma^{k-1}} \right)}{k}$$

$$\leq \frac{\sigma}{k^{k/(k-1)}} + \frac{\sigma^{k}}{k}$$

$$\leq \frac{2\sigma}{k}.$$  

(308)

Thus combining Equation (305) and (307), we have if \(\eta < \sigma/2k\),

$$E[|v^\top Y|^k] - E[|v^\top X|^k] \leq k\eta E[|v^\top Y|^k]^{1-1/k} \leq k\eta \left( \frac{E[|v^\top X|^k]}{1 - k\eta/\sigma} \right)^{1-1/k} \leq k\eta \left( \frac{2\sigma^{k}}{k(1 - k\eta/\sigma)} \right)^{1-1/k} \leq 8\sigma^{k-1}.$$

(309)
Combining the two cases, we know that the movement is upper bounded by \( \max(\sigma^{k-1} \eta \psi^{-1}(\frac{2\varepsilon}{\eta}), \max(k, 8) \sigma^{k-1} \eta) \) when \( \eta < \sigma/2k \). Thus for \( \eta < \min(\sigma/2k, 2\sigma/\psi(\max\{k, 8\})) \), we have \( p \in G^{W_{r,k}}_{\psi^{-1}(\frac{2\varepsilon}{\eta}), \eta} \).

We remark here that the above proof also applies to the case when we requires \( r \in F(p, \eta, W_{1,1}, |v^\top X|) \) instead of \( r \in F(p, \eta, W_{1,1}, |v^\top X|^k) \). The only difference to note is in above (305) where we need to apply Jensen’s inequality to derive \( |v^\top y|^k \leq (\mathbb{E}[|v^\top Y|])^k \leq \mathbb{E}[|v^\top Y|^k] \). This proves to be crucial in finite sample algorithm design in Section 6.

\[ \square \]

D.4 Proof of Lemma 4.1

Given any two points \( x \) and \( y \), without loss of generality we assume \( f(x) \leq f(y) \), define

\[
s_{xy}(u) = \begin{cases} 
  \min(f(x), f(y)), & u \leq \min(f(x), f(y)) \\
  u, & u \in [f(x), f(y)] \\
  \max(f(x), f(y)), & u \geq \max(f(x), f(y)).
\end{cases}
\] (310)

If we imagine \( u \) increasing from \( -\infty \) to \( +\infty \), we can think of \( s_{xy} \) as a “slider” that tries to be as close to \( u \) as possible while remaining between \( f(x) \) and \( f(y) \).

By Assumption 4.1, there must exist some point \( z \) such that \( \max(c^k(x, z), c^k(z, y)) \leq c^k(x, y) \) and \( f(z) = s_{xy}(u) \). Call this point \( z_{xy}(u) \).

Given a coupling \( \pi(x, y) \) from \( p_1 \) to \( p_2 \), if we map \( y \) to \( z_{xy}(u) \), we obtain a coupling \( \pi_1(x, z) \) to some distribution \( r(u) \), which by construction satisfies the monotonicity property, except that it is relative to \( u \) rather than the mean \( \mu(u) = \mathbb{E}_{X \sim r(u)}[f(X)] \). However, note that \( u - \mu(u) \) is a continuous, monotonically non-decreasing function (since \( u - s_{xy}(u) \) is non-decreasing) that ranges from \( -\infty \) to \( +\infty \). It follows that there is a point with \( \mu(u) = u \), in which case \( r(u) \) satisfies the monotonicity property with respect to \( \mu(u) \).

Moreover, \( \mathbb{E}_{(X,Z) \sim \pi_1}[c^k(X, Z)] \leq \mathbb{E}_{(X,Y) \sim \pi}[c^k(X, Y)] = W_{c,k}^k(p_1, p_2) \). The coupling \( \pi_1 \) therefore also has small enough cost, and so satisfies all of the properties required in Lemma 4.1.

To finish, we need to construct \( \pi_2 \); but this can be done by taking the reverse coupling from \( y \) to \( z_{xy}(u) \), which satisfies the required properties by an identical argument as above.

D.5 Proof of Theorem 4.3

Proof. It suffices to upper bound the modulus

\[
\sup_{(p_1, p_2): W_{c,k}(p_1, p_2) \leq 2\varepsilon, p_1 \in G^{W_{c,k}}(p_1, p_2), p_2 \in G^{W_{c,k}}(p_1, p_2)} \mathbb{E}_r[f(X)] - B^*(f, \theta^*(p_1)) \leq \rho_1.
\] (312)

Note that \( 2\varepsilon \leq \eta \). It follows from the condition that \( p_1 \in G^{W_{c,k}}(p_1, p_2) \subset G_{\psi^{-1}(\eta)}^{W_{c,k}}(p_1) \) that

\[
\sup_{f \in F_{\theta^*(p_1)}, r \in F(p_1, 2\varepsilon, W_{c,k}, f)} \mathbb{E}_r[f(X)] - B^*(f, \theta^*(p_1)) \leq \rho_1.
\] (313)

By Lemma 4.1, for any \( f \in F_{\theta^*(p_1)} \), we are able to pick \( r \) such that \( W_{c,k}(p_1, r) \leq 2\varepsilon, W_{c,k}(p_2, r) \leq 2\varepsilon \), and \( r \in F(p_1, 2\varepsilon, W_{c,k}, f) \cap F(p_2, 2\varepsilon, W_{c,k}, f) \) is a friendly perturbation for both \( p_1 \) and \( p_2 \), which implies that

\[
\sup_{f \in F_{\theta^*(p_1)}, r \in F(p_2, 2\varepsilon, W_{c,k}, f)} \mathbb{E}_r[f(X)] - B^*(f, \theta^*(p_1)) \leq \rho_1.
\] (313)
It then follows from $p_2 \in G_{W_{c,k}}(\rho_1, \rho_2) \subset G_{\uparrow}(\rho_1, \rho_2)$ that

$$L(p_2, \theta^*(p_1)) \leq \rho_2.$$  (314)

\[ \square \]

### D.6 Proof of reduction from $G_{W_{c,k}}$ to $G_{W_{F}}$

We prove here that by taking $B = L = W_{F}$, $\rho_1 = \rho$, $\rho_2 = 2\rho$ in Equation (40), we can recover the definition of $G_{W_{F}}$.

**Proof.** Here we identify $\theta$ as $q$, and and take $\mathcal{F}_{\theta} = \mathcal{F}$ for any $\theta$. Under the choices of $B, \rho_1$, we have $G_{W_{F}} = G_{\downarrow}^{W_{c,k}}$. We only need to show that under the choices of $\rho_1, \rho_2, B, L$, we have $G_{\downarrow}^{W_{c,k}}(\rho, \eta) \subset G_{\uparrow}(\rho, 2\rho, \eta)$. For any $p \in G_{\downarrow}^{W_{c,k}}$, we have

$$\sup_{f \in \mathcal{F}, r \in \mathcal{F}(p, \rho, W_{c,k}, f)} \mathbb{E}_r[f(X)] - \mathbb{E}_p[f(X)] \leq \rho. \quad (315)$$

Note that the condition $p \in G_{\downarrow}^{W_{c,k}}$ is equivalent to that for any $q$, we have

$$\left( \sup_{f \in \mathcal{F}} \inf_{r \in \mathcal{F}(p, \rho, W_{c,k}, f)} \mathbb{E}_r[f(X)] - \mathbb{E}_q[f(X)] \leq \rho \right) \Rightarrow \sup_{f \in \mathcal{F}} \mathbb{E}_p[f(X)] - \mathbb{E}_q[f(X)] \leq 2\rho. \quad (316)$$

Now it suffices to show that for any $p \in G_{\downarrow}^{W_{c,k}}(\rho, \eta)$, we have $p \in G_{\uparrow}(\rho, 2\rho, \eta)$.

Assume $p \in G_{\downarrow}^{W_{c,k}}(\rho, \eta)$. For any $q$ that satisfies the LHS condition in Equation (316). Then we know for any $f \in \mathcal{F}$, there exists some $r \in \mathcal{F}(p, \rho, W_{c,k}, f)$ with $\mathbb{E}_r[f(X)] - \mathbb{E}_q[f(X)] \leq \rho$, denote by $f^*$ the function that satisfies

$$\sup_{f \in \mathcal{F}} \mathbb{E}_p[f(X)] - \mathbb{E}_q[f(X)] = \mathbb{E}_p[f^*(X)] - \mathbb{E}_q[f^*(X)]. \quad (317)$$

Denote by $r^*$ the friendly perturbation in $\mathcal{F}(p, \rho, W_{c,k}, f^*)$. We have $\mathbb{E}_r[f^*(X)] - \mathbb{E}_q[f^*(X)] \leq \rho$. Then,

$$\sup_{f \in \mathcal{F}} \mathbb{E}_p[f(X)] - \mathbb{E}_q[f(X)] = \mathbb{E}_p[f^*(X)] - \mathbb{E}_q[f^*(X)]$$

$$\leq \mathbb{E}_p[f^*(X)] - \mathbb{E}_{r^*}[f^*(X)] + \mathbb{E}_{r^*}[f^*(X)] - \mathbb{E}_q[f^*(X)]$$

$$\leq 2\rho. \quad (320)$$

Hence, we have shown that $G_{\downarrow}^{W_{c,k}}(\rho, \eta) \subset G_{\uparrow}(\rho, 2\rho, \eta)$. Proof is completed. \[ \square \]

---

\[ ^9 \text{For any } a > 0, \text{ there always exists some } f^* \text{ such that } \mathbb{E}_p[f^*(X)] - \mathbb{E}_q[f^*(X)] \geq \sup_{f \in \mathcal{F}} \mathbb{E}_p[f(X)] - \mathbb{E}_q[f(X)] - a. \]

The following steps can go through using this limiting argument if the supremum is not attained.
D.7 Proof for Example 4.2

Proof. We know that $G^{W_1} = G^{W_1}_4(\rho_1, \eta) \cap G^{W_1}_7(\rho_2, \rho_2, \eta)$, where

$$G^{W_1}_4(\rho_1, \eta) = \{ p \mid \sup_{x \in \mathbb{F}(p, \eta, W_{c,k}, |X^\top \theta^*(p) - Y|)} \mathbb{E}_r [(X^\top \theta^*(p) - Y)^2] \leq \rho_1 \}, \quad (321)$$

$$G^{W_1}_7(\rho_2) = \left\{ p \mid \forall \tau \geq 0, \forall \theta \in \Theta, \forall r \in \mathbb{F}(p, \eta, W_{c,k}, |X^\top \theta - Y|), \right.$$ 

$$\left. \mathbb{E}_r[\mathbb{E}_r[(X^\top \theta - Y)^2] \leq \rho_1 \Rightarrow \mathbb{E}_p[(X^\top \theta - Y)^2] \leq \rho_2 \right\}. \quad (322)$$

We first show that $\mathbb{E}_{p^*}[|X| \psi(\frac{|2X|}{\sigma})] \leq 1$ implies $\mathbb{E}_{p^*}[X^2] \leq 2\sigma^2$. Note that $\mathbb{E}_{p^*}[|X| \psi(\frac{|2X|}{\sigma})] \leq 1$ is equivalent to

$$1 \geq \mathbb{P}_{p^*}(|X| \leq \sigma)\mathbb{E}_{p^*}[|X| \psi(\frac{|X|}{\sigma})] + \mathbb{P}_{p^*}(|X| > \sigma)\mathbb{E}_{p^*}[|X| \psi(\frac{|X|}{\sigma})] \geq \mathbb{P}_{p^*}(|X| > \sigma)\mathbb{E}_{p^*}[X^2/\sigma^2 | |X| > \sigma], \quad (323)$$

since $\psi(x) \geq x$ for $x \geq 1$. Thus we have

$$\mathbb{E}_{p^*}[X^2/\sigma^2] = \mathbb{P}_{p^*}(|X| \leq \sigma)\mathbb{E}_{p^*}[X^2/\sigma^2 | |X| \leq \sigma] + \mathbb{P}_{p^*}(|X| > \sigma)\mathbb{E}_{p^*}[X^2/\sigma^2 | |X| > \sigma] \leq 2. \quad (324)$$

Denote $Z = Y - X^\top \theta^*(p^*)$. Then $X' = [X, Z]$, and $\mathbb{E}_{p^*}[Z^2] \leq 2\sigma^2$. Furthermore, since $X' = [X, Y - X^\top \theta^*(p)]$, for any two distributions $p_1, p_2$ defined on $(X, Y)$ space with $W_1(p_1, p_2) \leq \eta$, converting them to $X'$ space to derive $\tilde{p}_1, \tilde{p}_2$ would give $W_1(\tilde{p}_1, \tilde{p}_2) \leq \eta \sqrt{R^2 + 2}$.

From the same proof as in second moment estimation (Example 4.1) we know that the condition in (43) implies that

$$\sup_{v \in \mathbb{R}^{d+1}, \|v\|_2 = 1, r \in \mathbb{F}(p^*, \epsilon, W, \epsilon, X^\top X')} |\mathbb{E}_{p^*}[(v^\top X')^2] - \mathbb{E}_r[(v^\top X')^2]| \leq \sigma \eta \sqrt{2R^2 + 4\psi^{-1}(2\sigma/\eta \sqrt{R^2/2 + 1})}, \quad (325)$$

assuming $\eta \sqrt{R^2/2 + 1} < \min(\sigma/4, 2\sigma/\psi(8))$. By setting the last element of $v$ as 1 and all others as 0, we have $p^* \in G^{W_{c,k}}_4(\rho_1, \eta)$ where $\rho_1 = \sigma^2 + \sigma \eta \sqrt{2R^2 + 4\psi^{-1}(2\sigma/\eta \sqrt{R^2/2 + 1})}$.

Now we show that $p^* \in G^{W_{c,k}}_7(\rho_2)$. Note that for any $\theta$, we know that

$$X^\top \theta - Y = X^\top (\theta - \theta^*(p^*)) + (X^\top \theta^*(p^*) - Y) = X^\top (\theta - \theta^*(p^*)) + Z = v^\top X' \cdot \sqrt{||\theta - \theta^*(p^*)||_2^2 + 1} \leq v^\top X' \cdot \sqrt{4R^2 + 1}. \quad (326)$$

where $v$ is the unit vector in the direction of $(\theta - \theta^*(p^*), 1)$. Since friendly perturbation is invariant to scaling, this gives us

$$\sup_{\theta \in \Theta, r \in \mathbb{F}(p^*, \eta, W_{c,k}, |X^\top \theta - Y|)} |\mathbb{E}_{p^*}[(X^\top \theta - Y)^2] - \mathbb{E}_r[(X^\top \theta - Y)^2]| \leq \sigma \eta \sqrt{(2R^2 + 4)(4R^2 + 1)\psi^{-1}(2\sigma/\eta \sqrt{R^2/2 + 1})}. \quad (327)$$
Thus we know that if \( E_r[(X^\top \theta - Y)^2] \leq r_1 \), we have
\[
E_{p^*}[(X^\top \theta - Y)^2] \leq \sigma^2 + \sigma \eta \sqrt{2R^2 + 4\psi(2\sigma/(\sqrt{R^2/2 + 1}))} + \sigma \eta \sqrt{(2R^2 + 4\psi(2\sigma/(\sqrt{R^2/2 + 1}))}
\]
assuming \( \eta \sqrt{R^2/2 + 1} < 2\sigma/\psi(8) \). Thus we have \( p^* \in G_{W_1}(\sigma^2 + \sigma \eta \sqrt{2R^2 + 4\psi(2\sigma/(\sqrt{R^2/2 + 1}))}, \sigma^2 + (2R + 2)\sigma \eta \sqrt{2R^2 + 4\psi(2\sigma/(\sqrt{R^2/2 + 1}))}, \eta) \).

\[\Box\]

D.8 Necessity of compactness assumption in Example 4.2

To show that the necessity of compactness assumption \( \|\theta\|_2 \leq R \), we provide a lower bound for \( W_1 \) linear regression question.

**Theorem D.2.** Taking \( B(p, \theta) = L(p, \theta) = E_p[(Y - X^\top \theta)^2] \) in (40). Denote \( G \) a set of two-dimensional distributions:
\[ G = \{p \mid (X, Y) \sim p, Y = X^\top \theta, |\theta| \leq R, E_p[|X|^2] \leq 1.\} \]

Then the population information theoretic limit is lower bounded:
\[ \sup_{p} \sup_{\theta(p)p^* \in G, W_1(p^*, p) \leq \epsilon} E_{p^*}[(Y - X^\top \theta(p))^2] \geq R^2 \epsilon^2/4. \] (330)

As \( R \to +\infty \), the limit goes to \( +\infty \).

**Proof.** Consider \( (X, Y) \sim p_1 \), with probability 1/2, \( (X, Y) = (\epsilon/2, R\epsilon/2) \), with probability 1/2, \( (X, Y) = (-\epsilon/2, -R\epsilon/2) \), and another \( (X, Y) \sim p_2 \), with probability 1/2, \( (X, Y) = (\epsilon/2, R\epsilon/2) \), with probability 1/2, \( (X, Y) = (-\epsilon/2, -R\epsilon/2) \). Then \( W_1(p_1, p_2) \leq \epsilon \). We have
\[
\sup_{p} \sup_{\theta(p)p^* \in G, W_1(p^*, p) \leq \epsilon} E_{p^*}[(Y - X^\top \theta(p))^2] \geq \inf_{\theta(p_1)p^* \in G, W_1(p^*, p_1) \leq \epsilon} E_{p^*}[(Y - X^\top \theta(p_1))^2]
\]
\[
\geq \inf_{\theta(p_1)} \frac{1}{2} (E_{p_1}[(Y - X^\top \theta(p_1))^2] + E_{p_2}[(Y - X^\top \theta(p_1))^2])
\]
\[
= \inf_{\theta} \frac{1}{2} ((\epsilon/2 - \theta\epsilon/2)^2 + (\epsilon/2 + \theta\epsilon/2)^2)
\]
\[
\geq R^2 \epsilon^2/4. \] (331)

\[\Box\]

E Related sample discussions and remaining proofs in Section 5

E.1 Finite sample analysis for \( \tilde{TV}_H \) projection to \( G_{TV}^{W_\mathcal{F}} \) in (152)

We now present a theorem for robust learning with loss function \( W_\mathcal{F} \).

**Theorem E.1.** Choose some symmetric \( H \subset \mathcal{F} \) such that \( W_H(p, q) \geq \frac{1}{2} W_\mathcal{F}(p, q), \forall p, q \in G_{TV}^{W_\mathcal{F}}(\rho(\tilde{\epsilon}), \tilde{\epsilon}). \)

Denote \( \tilde{\epsilon} = 2\epsilon + \sqrt{\frac{2\ln(2|H|/\delta)}{n}} \). Assume \( p^* \in G_{TV}^{W_\mathcal{F}}(\rho(\tilde{\epsilon}), \tilde{\epsilon}) \) and the oblivious corruption model of level \( \epsilon \) with \( TV \) perturbation. Denote the empirical distribution of observed data as \( \hat{p}_n \). Let \( q \) denote the output of the projection algorithm \( \Pi(\hat{p}_n; \tilde{TV}_H, G_{TV}) \) or \( \Pi(\hat{p}_n; \tilde{TV}_H, G_{TV}, \epsilon/2) \). Then, with probability at least \( 1 - \delta \),
\[
W_\mathcal{F}(p^*, q) \leq 4\rho \left( 2\epsilon + \sqrt{\frac{2\ln(2|H|/\delta)}{n}} \right). \] (332)
Proof. The \( \tilde{\epsilon} \) bound follows from Proposition 5.1 and Lemma 5.1. Now we upper bound the modulus:

\[
\sup_{p_1, p_2 \in \mathcal{G}^{TV}_{\mathcal{F}}(\rho(\tilde{\epsilon}), \tilde{\epsilon}) : \mathcal{T} \mathcal{V}(p_1, p_2) \leq \tilde{\epsilon}} W_{\mathcal{F}}(p_1, p_2) \leq 4\rho(\tilde{\epsilon}).
\] (333)

The condition that \( \mathcal{T} \mathcal{V}(p_1, p_2) \leq \tilde{\epsilon} \) implies that for any \( h \in \mathcal{H} \),

\[
\sup_{t \in \mathbb{R}} |P_{p_1}[h(X) \geq t] - P_{p_2}[h(X) \geq t]| \leq \tilde{\epsilon}.
\] (334)

Take \( h = \arg\max_{h \in \mathcal{H}} \mathbb{E}_{p_1}[h(X)] - \mathbb{E}_{p_2}[h(X)] \), hence \( \mathbb{E}_{p_1}[h(X)] - \mathbb{E}_{p_2}[h(X)] = W_{\mathcal{H}}(p_1, p_2) \). It follows from Lemma 5.2 that there exist some \( r_{p_1} \leq \frac{\rho}{1 - \tilde{\epsilon}}, r_{p_2} \leq \frac{\rho}{2 - \tilde{\epsilon}} \) such that

\[
\mathbb{E}_{r_{p_1}}[h(X)] \leq \mathbb{E}_{r_{p_2}}[h(X)].
\] (335)

Furthermore, from \( p_1, p_2 \in \mathcal{G}^{TV}_{\mathcal{F}}(\rho(\tilde{\epsilon}), \tilde{\epsilon}), \mathcal{H} \subset \mathcal{F} \), we have

\[
\mathbb{E}_{p_1}[h(X)] - \mathbb{E}_{r_{p_1}}[h(X)] \leq \rho(\tilde{\epsilon}),
\] (336)

\[
\mathbb{E}_{r_{p_2}}[h(X)] - \mathbb{E}_{p_2}[h(X)] \leq \rho(\tilde{\epsilon}).
\] (337)

Then,

\[
W_{\mathcal{H}}(p_1, p_2) = \mathbb{E}_{p_1}[h(X)] - \mathbb{E}_{p_2}[h(X)]
\] (338)

\[= \mathbb{E}_{p_1}[h(X)] - \mathbb{E}_{r_{p_1}}[h(X)] + \mathbb{E}_{r_{p_1}}[h(X)] - \mathbb{E}_{p_2}[h(X)] + \mathbb{E}_{r_{p_2}}[h(X)] - \mathbb{E}_{p_2}[h(X)]\] (339)

\[\leq 2\rho(\tilde{\epsilon}),\] (340)

which implies that \( W_{\mathcal{F}}(p_1, p_2) \leq 2W_{\mathcal{H}}(p_1, p_2) \leq 4\rho(\tilde{\epsilon}) \), which shows the modulus is small. The final conclusion follows from Proposition 5.1.

\[
\square
\]

E.2 Modulus bound on \( \mathcal{T} \mathcal{V}_{\mathcal{H}} \)

It follows from Proposition 5.1 that it suffices to check the following modulus

\[
\sup_{p_1, p_2 \in \mathcal{G}^{TV}_{\mathcal{F}} : \mathcal{T} \mathcal{V}(p_1, p_2) \leq \eta} L(p_2, \theta^*(p_1))
\] (341)

to guarantee the finite sample error of \( \mathcal{T} \mathcal{V}_{\mathcal{H}} \) projection algorithms. The following lemma generalizes the modulus computation in Theorem 5.3 via assuming \( B(p, \theta) \) is convex in \( p \) and minimax theorem. The lemma here also provides a principled way of choosing \( \mathcal{H} \) given \( B \), which applies to almost all the examples in Section 5.1.

**Lemma E.1.** Assume \( B(p, \theta) \) is convex in \( p \) for all \( \theta \) in \( \mathcal{G}^{TV}(p_1, p_2, \eta) \), consider the dual representation of \( B \):

\[
B(p, \theta) = \sup_{f \in \mathcal{F}_\theta} \mathbb{E}_{p}[f(X)] - B^*(f, \theta).
\] (342)

Here \( \mathcal{F}_\theta = \{ f \mid B^*(f, \theta) < \infty \} \). We take \( \mathcal{H} = \bigcup_{\theta \in \Theta} \mathcal{F}_\theta \) and assume that for any \( \theta \in \Theta \), the minimax theorem holds:

\[
\min_{\mathcal{F}_\theta} \sup_{f \in \mathcal{F}_\theta} \mathbb{E}_{r}[f(X)] - B^*(f, \theta) = \sup_{f \in \mathcal{F}_\theta} \min_{\mathcal{F}_\theta} \mathbb{E}_{r}[f(X)] - B^*(f, \theta).
\] (343)

Then, the modulus is being controlled by \( \rho_2 \):

\[
\sup_{p_1, p_2 \in \mathcal{G}^{TV}_{\mathcal{F}} : \mathcal{T} \mathcal{V}(p_1, p_2) \leq \eta} L(p_2, \theta^*(p_1)) \leq \rho_2.
\] (344)
Proof. Recall that
\[
\widetilde{TV}_\mathcal{H}(q,p) = \sup_{f \in \bigcup_{q \in \mathcal{G}} \mathcal{F}_q, t \in \mathbb{R}} |\mathbb{P}_p[f(X) \geq t] - \mathbb{P}_q[f(X) \geq t]|. \tag{345}
\]
From \( p, q \in \mathcal{G}_{TV} \),
\[
\forall r_q \leq \frac{q}{1 - \eta}, \quad \sup_{f \in \mathcal{F}_{\theta^*(q)}} \mathbb{E}_{r_q}[f(X)] - B^*(f, \theta^*(q)) \leq \rho_1 \tag{346}
\]
From Lemma 5.2, we know that for any \( f \in \mathcal{H} \), there exists \( r_p, r_q \) such that
\[
\mathbb{E}_{r_p}[f(X)] \leq \mathbb{E}_{r_q}[f(X)]. \tag{347}
\]
Thus for any \( f \in \mathcal{F}_{\theta^*(q)} \), we have
\[
\mathbb{E}_{r_p}[f(X)] - B^*(f, \theta^*(q)) \leq \mathbb{E}_{r_q}[f(X)] - B^*(f, \theta^*(q)) \leq \rho_1. \tag{348}
\]
Therefore we know
\[
\sup_{f \in \mathcal{F}_{\theta^*(q)} \cap \mathcal{F}_q, r_p \leq \frac{r_q}{1 - \eta}} \min_{r_q \leq \frac{q}{1 - \eta}} \mathbb{E}_{r_p}[f(X)] - B^*(f, \theta^*(q)) \leq \rho_1. \tag{349}
\]
Since we assumed minimax theorem holds, one has
\[
\min_{r_p \leq \frac{r_q}{1 - \eta}} \sup_{f \in \mathcal{F}_{\theta^*(q)}} \mathbb{E}_{r_p}[f(X)] - B^*(f, \theta^*(q)) = \sup_{f \in \mathcal{F}_{\theta^*(q)} \cap \mathcal{F}_q, r_p \leq \frac{r_q}{1 - \eta}} \min_{r_q \leq \frac{q}{1 - \eta}} \mathbb{E}_{r_p}[f(X)] - B^*(f, \theta^*(q)) \leq \rho_1. \tag{350}
\]
The LHS is exactly the condition in \( \mathcal{G}_{TV} \). Thus from \( p \in \mathcal{G}_{TV} \), we know that
\[
L(p, \theta^*(q)) \leq \rho_2, \tag{351}
\]
proof is complete.

\[ \square \]

E.3 Discussion and proof on \( \widetilde{TV} \) projection for joint mean and covariance estimation

Here we show the finite sample performance guarantee for \( \widetilde{TV}_\mathcal{H} \) projection algorithm for Example 3.3.

First, we justify the choice of \( \mathcal{H} \) from the dual representation of \( B \) in Example 3.3. From the general choice of \( \mathcal{H} \) in Lemma E.1, we first write down the dual representation of \( B \). Recall that in Example 3.3, we take \( B \) as
\[
B(p, (\mu, \Sigma)) = \max \left( \| \Sigma^{-1/2}(\mu_p - \mu) \|_2^2 / \eta, \| I_d - \Sigma^{-1/2} \mathbb{E}_p[(X - \mu)(X - \mu)^\top] \Sigma^{-1/2} \|_2 \right)
= \max( \sup_{v_1 \in \mathbb{R}^d, \|v_1\|_2 = 1} \langle v_1 \Sigma^{-1/2}(\mu_p - \mu) \rangle^2 / \eta, \sup_{v_2 \in \mathbb{R}^d, \|v_2\|_2 = 1} |1 - v_2 \Sigma^{-1/2} \mathbb{E}_p[(X - \mu)(X - \mu)^\top] \Sigma^{-1/2} v_2|)
= \max( \sup_{y \in \mathbb{R}, v_1 \in \mathbb{R}^d, \|v_1\|_2 = 1} y \langle v_1 \Sigma^{-1/2}(\mu_p - \mu) \rangle - y^2 / \eta, \sup_{v_2 \in \mathbb{R}^d, \|v_2\|_2 = 1} |1 - v_2 \Sigma^{-1/2} \mathbb{E}_p[(X - \mu)(X - \mu)^\top] \Sigma^{-1/2} v_2|)
= \sup_{f \in \mathcal{F}_1 \cup \mathcal{F}_2} \mathbb{E}_p[f(X)], \tag{352}
\]
where $\mathcal{F}_1 = \{(y(v_1^\top \Sigma^{-1/2}(x - \mu)) - y^2)/\eta \mid y \in \mathcal{R}, v_1 \in \mathcal{R}^d, \|v_1\|_2 = 1\}$, $\mathcal{F}_2 = \{\xi(1 - v_2\Sigma^{-1/2}(x - \mu)v_2) \mid \xi \in \{\pm 1\}, v_2 \in \mathcal{R}^d, \|v_2\|_2 = 1\}$. This can also be viewed as $B(p, q)$ where $\mu = \mu_q, \Sigma = \Sigma_q$.

We show that by taking $\mathcal{H} = \{v^\top x \mid v \in \mathcal{R}^d\}$, $\tilde{\mathcal{TV}}_\mathcal{H}$ small can imply $\tilde{\mathcal{TV}}_{\mathcal{F}_1 \cup \mathcal{F}_2}$ small.

$$
\tilde{\mathcal{TV}}_{\mathcal{F}_1}(p, q) = \sup_{y \in \mathcal{R}, v_1 \in \mathcal{R}^d, \|v_1\|_2 = 1} \left| \mathbb{P}_p((y(v_1^\top \Sigma^{-1/2}(x - \mu)) - y^2)/\eta \geq t) - \mathbb{P}_q((y(v_1^\top \Sigma^{-1/2}(x - \mu)) - y^2)/\eta \geq t) \right|
\leq \sup_{v \in \mathcal{R}^d, \|v\|_2 = 1, t \in \mathcal{R}} \left| \mathbb{P}_p(|v^\top \Sigma^{-1/2}(x - \mu)| \geq t) - \mathbb{P}_q(|v^\top \Sigma^{-1/2}(x - \mu)| \geq t) \right|
\leq \sup_{v \in \mathcal{R}^d, \|v\|_2 = 1, t \in \mathcal{R}} \left| \mathbb{P}_p(v^\top X \geq t) - \mathbb{P}_p(v^\top X \geq t) \right|
\leq \tilde{\mathcal{TV}}_\mathcal{H}(p, q). \quad (353)
$$

Furthermore, for $\tilde{\mathcal{TV}}_{\mathcal{F}_2}$, we have

$$
\tilde{\mathcal{TV}}_{\mathcal{F}_2}(p, q) = \sup_{\xi \in \{\pm 1\}, v \in \mathcal{R}^d, \|v\|_2 = 1, t \in \mathcal{R}} \left| \mathbb{P}_p(\xi v^\top \Sigma^{-1/2}(X - \mu)(X - \mu)^\top \Sigma^{-1/2}v \geq t) - \mathbb{P}_q(\xi v^\top \Sigma^{-1/2}(X - \mu)(X - \mu)^\top \Sigma^{-1/2}v \geq t) \right|
\leq \sup_{v \in \mathcal{R}^d, \|v\|_2 = 1, t \in \mathcal{R}} \left| \mathbb{P}_p(|v^\top \Sigma^{-1/2}(X - \mu)| \geq t) - \mathbb{P}_q(|v^\top \Sigma^{-1/2}(X - \mu)| \geq t) \right|
\leq \sup_{v \in \mathcal{R}^d, \|v\|_2 = 1, t > 0} \left| \mathbb{P}_p(v^\top X \geq t) - \mathbb{P}_p(v^\top X \geq t) \right|
\leq \sup_{v \in \mathcal{R}^d, \|v\|_2 = 1, t \in \mathcal{R}} \left| \mathbb{P}_p(v^\top X \geq t) - \mathbb{P}_p(v^\top X \geq t) \right|
\leq 2\tilde{\mathcal{TV}}_\mathcal{H}(p, q). \quad (354)
$$

Thus we have $\tilde{\mathcal{TV}}_{\mathcal{F}_1 \cup \mathcal{F}_2} \leq 2\tilde{\mathcal{TV}}_\mathcal{H}$.

Assume the oblivious corruption model of level $\epsilon$. Denote the empirical distribution of observed data as $\hat{p}_n$. We have the following theorem for Example 3.3:

**Theorem E.2.** Denote $\tilde{\epsilon} = 2\epsilon + 2C_n^{1/2}(\epsilon + \log(1/\delta))/n$, $\mathcal{G} = \bigcap_{\epsilon \in [0, 1/2]} \mathcal{G}_\epsilon^\mathcal{TV}(\rho(\epsilon), \epsilon)$, where $\mathcal{G}_\epsilon^\mathcal{TV}$ is the same as in Example 3.3. Assume $\tilde{\epsilon} < 1/2$, $p^* \in \mathcal{G}$. For $\mathcal{H} = \{v^\top x \mid v \in \mathcal{R}^d, \|v\|_2 = 1\}$, let $q$ denote the output of the projection algorithm $\Pi(\hat{p}_n; \tilde{\mathcal{TV}}_\mathcal{H}, \mathcal{G})$ or $\Pi(\hat{p}_n; \tilde{\mathcal{TV}}_\mathcal{H}, \mathcal{G}, \tilde{\epsilon}/2)$. Then there exist some $C_1$ such that when $\tilde{\epsilon} \leq C_1$, with probability at least $1 - \delta$,

$$
\|\Sigma^{-1/2}(\mu_p - \mu_q)\|_2 \lesssim \sqrt{\epsilon \rho(3\tilde{\epsilon})}, \quad (355)
\|I_d - \Sigma^{-1/2}\Sigma^{-1/2}p_{p^*}\Sigma^{-1/2}\|_2 \lesssim \rho(3\tilde{\epsilon}). \quad (356)
$$

**Proof.** The bound on $\tilde{\epsilon}$ is the same as in the proof of Theorem 5.1. It suffices to show the modulus of continuity.

We first show that when $\tilde{\mathcal{TV}}_\mathcal{H}(p_1, p_2) \leq \tilde{\epsilon}$, $p_1, p_2 \in \mathcal{G}$, we have $\|I_d - \Sigma^{-1/2}\Sigma^{-1/2}p_{p^*}\Sigma^{-1/2}\|_2$. Without loss of generality, assume $\Sigma_{p_1}$ is invertible. Consider any fixed direction $v \in \mathcal{R}^d, \|v\|_2 = 1$, from
We truncate \( p_1, p_2 \) by deleting all the mass the satisfies \(|v^\top \Sigma_{p_1}^{-1/2}(X - \mu_{p_1})| \geq \sqrt{\frac{\rho(\epsilon)}{\epsilon}}\) to get deleted distribution \( r_{p_1}, r_{p_2} \). Then we know that \( TV(p_1, r_{p_1}) \leq \tilde{\epsilon}, TV(p_2, r_{p_2}) \leq 3\tilde{\epsilon}. \) From \( p_1, p_2 \in \mathcal{G} \), we know that

\[
\|\Sigma_{p_1}^{-1/2}(\mu_{p_1} - r_{p_1})\|_2 \leq \sqrt{\epsilon\rho(\epsilon)}, \|I_d - \Sigma_{p_1}^{-1/2}E_{r_{p_1}}[(X - \mu_{p_1})(X - \mu_{p_1})^\top]\Sigma_{p_1}^{-1/2}\|_2 \leq \rho(\epsilon) \tag{359}
\]

\[
\|\Sigma_{p_2}^{-1/2}(\mu_{p_2} - r_{p_2})\|_2 \leq \sqrt{3\epsilon\rho(3\epsilon)}, \|I_d - \Sigma_{p_2}^{-1/2}E_{r_{p_2}}[(X - \mu_{p_2})(X - \mu_{p_2})^\top]\Sigma_{p_2}^{-1/2}\|_2 \leq \rho(3\epsilon). \tag{360}
\]

From the above inequality we also have

\[
\|I_d - \Sigma_{p_2}^{-1/2}E_{r_{p_2}}\Sigma_{r_{p_2}}^{-1/2}\|_2 \leq \|I_d - \Sigma_{p_2}^{-1/2}E_{r_{p_2}}[(X - \mu_{p_2})(X - \mu_{p_2})^\top]\Sigma_{p_2}^{-1/2}\|_2 + \|\Sigma_{p_2}^{-1/2}(\mu_{p_2} - r_{p_2})\|_2^2 \leq \rho(3\epsilon) + 3\epsilon\rho(3\epsilon). \tag{361}
\]

This is equivalent to

\[
(1 - \rho(3\epsilon) - 3\epsilon\rho(3\epsilon))\Sigma_{r_{p_2}} \leq \Sigma_{p_2} \leq (1 + \rho(3\epsilon) + 3\epsilon\rho(3\epsilon))\Sigma_{r_{p_2}}. \tag{362}
\]

Now we know that the random variable \( v^\top \Sigma_{p_1}^{-1/2}X \) under \( r_{p_1}, r_{p_2} \) has bounded support, and \( TV(r_{p_1}, r_{p_2}) \leq 5\epsilon, \) thus we have

\[
|v^\top \Sigma_{p_1}^{-1/2}(\mu_{r_{p_1}} - \mu_{r_{p_2}})| = |v^\top \Sigma_{p_1}^{-1/2}(\mu_{r_{p_1}} - \mu_{p_1} + \sqrt{\frac{\rho(\epsilon)}{\epsilon}} - (\mu_{r_{p_2}} - \mu_{p_1} + \sqrt{\frac{\rho(\epsilon)}{\epsilon}}))| = \int_{t=\mu_{p_1} - \sqrt{\frac{\rho(\epsilon)}{\epsilon}}}^{\mu_{p_1} + \sqrt{\frac{\rho(\epsilon)}{\epsilon}}} (\mathbb{P}_{r_{p_1}}(v^\top \Sigma_{p_1}^{-1/2}X \geq t) - \mathbb{P}_{r_{p_2}}(v^\top \Sigma_{p_1}^{-1/2}X \geq t))dt| \leq \int_{t=\mu_{p_1} - \sqrt{\frac{\rho(\epsilon)}{\epsilon}}}^{\mu_{p_1} + \sqrt{\frac{\rho(\epsilon)}{\epsilon}}} 5\epsilon dt \leq 10\sqrt{\epsilon\rho(\epsilon)}. \tag{363}
\]

and

\[
|\mathbb{E}_{r_{p_1}}[(v^\top \Sigma_{p_1}^{-1/2}(X - \mu_{r_{p_1}}))^2] - \mathbb{E}_{r_{p_2}}[(v^\top \Sigma_{p_1}^{-1/2}(X - \mu_{r_{p_1}}))^2]| = \int_{t=0}^{\sqrt{\frac{\rho(\epsilon)}{\epsilon}}} (t\mathbb{P}_{r_{p_1}}(v^\top \Sigma_{p_1}^{-1/2}(X - \mu_{r_{p_1}}) \geq t) - t\mathbb{P}_{r_{p_2}}(v^\top \Sigma_{p_1}^{-1/2}(X - \mu_{r_{p_1}}) \geq t))dt| \leq \int_{t=0}^{\sqrt{\frac{\rho(\epsilon)}{\epsilon}}} 10\epsilon dt = 5\rho(\epsilon). \tag{364}
\]
We have

\[
|1 - \mathbb{E}_{p_2}[\langle v^\top \Sigma_{p_1}^{-1/2}(X - \mu_{p_2}) \rangle^2]| \\
\leq |\mathbb{E}_{p_1}[\langle v^\top \Sigma_{p_1}^{-1/2}(X - \mu_{p_1}) \rangle^2] - \mathbb{E}_{p_2}[\langle v^\top \Sigma_{p_1}^{-1/2}(X - \mu_{p_2}) \rangle^2]| + \rho(\bar{\epsilon}) \\
\leq |\mathbb{E}_{p_1}[\langle v^\top \Sigma_{p_1}^{-1/2}(X - \mu_{p_1}) \rangle^2] - \mathbb{E}_{p_1}[\langle v^\top \Sigma_{p_1}^{-1/2}(X - \mu_{p_1}) \rangle^2]| \\
+ \mathbb{E}_{p_1}[\langle v^\top \Sigma_{p_1}^{-1/2}(X - \mu_{p_1}) \rangle^2] - \mathbb{E}_{p_2}[\langle v^\top \Sigma_{p_1}^{-1/2}(X - \mu_{p_2}) \rangle^2] \\
+ \mathbb{E}_{p_2}[\langle v^\top \Sigma_{p_1}^{-1/2}(X - \mu_{p_2}) \rangle^2] - \mathbb{E}_{p_2}[\langle v^\top \Sigma_{p_1}^{-1/2}(X - \mu_{p_2}) \rangle^2]| + \rho(\bar{\epsilon}) \\
\leq \langle v^\top \Sigma_{p_1}^{-1/2}(\mu_{p_2} - \mu_{p_1}) \rangle^2 + 5\rho(\bar{\epsilon}) + \langle v^\top \Sigma_{p_1}^{-1/2}(\mu_{p_1} - \mu_{p_2}) \rangle^2 + \rho(\bar{\epsilon}) \\
\leq 6\rho(\bar{\epsilon}) + 10\bar{\epsilon}\sqrt{\rho} \lesssim \rho(\bar{\epsilon}).
\]

(365)

On the other hand, we have

\[
|1 - \mathbb{E}_{p_2}[\langle v^\top \Sigma_{p_1}^{-1/2}(X - \mu_{p_2}) \rangle^2]| = \left| 1 - \frac{\mathbb{E}_{p_2}[\langle v^\top \Sigma_{p_1}^{-1/2}(X - \mu_{p_2}) \rangle^2]}{\mathbb{E}_{p_1}[\langle v^\top \Sigma_{p_1}^{-1/2}(X - \mu_{p_1}) \rangle^2]} \right|
\]

(366)

Denote \( v' = \frac{\Sigma_{p_1}^{-1/2}v}{\|\Sigma_{p_1}^{-1/2}v\|_2} \), from (362) we have

\[
(1 - \rho(3\bar{\epsilon}) - 3\bar{\epsilon}\rho(3\bar{\epsilon})) \frac{\mathbb{E}_{p_2}[\langle v'^\top (X - \mu_{p_2}) \rangle^2]}{\mathbb{E}_{p_1}[\langle v'^\top (X - \mu_{p_1}) \rangle^2]} \leq \frac{\mathbb{E}_{p_2}[\langle v'^\top (X - \mu_{p_2}) \rangle^2]}{\mathbb{E}_{p_1}[\langle v'^\top (X - \mu_{p_1}) \rangle^2]} \leq (1 + \rho(3\bar{\epsilon}) + 3\bar{\epsilon}\rho(3\bar{\epsilon})) \frac{\mathbb{E}_{p_2}[\langle v'^\top (X - \mu_{p_2}) \rangle^2]}{\mathbb{E}_{p_1}[\langle v'^\top (X - \mu_{p_1}) \rangle^2]}
\]

(367)

Thus

\[
\left| 1 - \frac{\mathbb{E}_{p_1}[\langle v'^\top (X - \mu_{p_1}) \rangle^2]}{\mathbb{E}_{p_1}[\langle v'^\top (X - \mu_{p_1}) \rangle^2]} \right| \lesssim \rho(3\bar{\epsilon}).
\]

(368)

Now we have shown for any \( v' \in \mathbb{R}^d \), the above inequality holds. Taking \( v' = \Sigma_{p_1}^{-1/2}v \), we have

\[
\|I_d - \Sigma_{p_1}^{-1/2}\Sigma_{p_2}\Sigma_{p_1}^{-1/2}\|_2 \lesssim \rho(3\bar{\epsilon}).
\]

(369)

This gives multiplicative bound for covariance. Now we only need to bound the difference between mean. From above proof, we already know that when \( \rho(3\bar{\epsilon}) \leq 1 \), for the fixed \( v \in \mathbb{R}^d, \|v\|_2 = 1 \),

\[
|v^\top \Sigma_{p_1}^{-1/2}(\mu_{p_1} - \mu_{p_2})| \lesssim 10\sqrt{\rho(\bar{\epsilon})},
\]

(370)

\[
|\Sigma_{p_1}^{-1/2}(\mu_{p_1} - \mu_{p_1})|_2 \leq \sqrt{\rho(\bar{\epsilon})},
\]

(371)

\[
|\Sigma_{p_1}^{-1/2}(\mu_{p_2} - \mu_{p_2})|_2 \lesssim \sqrt{\rho(3\bar{\epsilon})}.
\]

(372)

Thus we have

\[
|v^\top \Sigma_{p_1}^{-1/2}(\mu_{p_1} - \mu_{p_2})| \lesssim \sqrt{\rho(3\bar{\epsilon})}.
\]

(373)

This shows that

\[
|\Sigma_{p_1}^{-1/2}(\mu_{p_1} - \mu_{p_2})|_2 \lesssim \sqrt{\rho(3\bar{\epsilon})}.
\]

(374)
Also, we are able to guarantee robustness for joint mean and covariance estimation by choosing a different set of $B, L$ pairs in $G$:

$$B(p, (\mu, \Sigma)) = L(p, (\mu, \Sigma)) = \max (\|\mu_p - \mu\|^2 \eta, \|\Sigma - \mathbb{E}_p[(X - \mu)(X - \mu)^T]\|_2). \quad (375)$$

We have the following finite sample error bounds, which generalizes (Gao et al., 2019, Theorem 4.1) to nonparametric classes.

**Theorem E.3.** Denote $\tilde{\epsilon} = 2\epsilon + 2C\epsilon\sqrt{\frac{d + 1 + \log(1/\delta)}{n}}$, $G = G_{TV}^L(\rho(\tilde{\epsilon}), \tilde{\epsilon}) \cap G_{TV}^L(\rho(2\tilde{\epsilon}), 2\tilde{\epsilon})$ when $B, L$ are chosen as (375). Assume $\tilde{\epsilon} < 1/2$, $p^* \in G$. Assume the oblivious corruption model of level $\epsilon$. Denote the empirical distribution of observed data as $\hat{p}_n$. For $H = \{v^T x \mid v \in \mathbb{R}^d, \|v\|_2 = 1\}$, let $q$ denote the output of the projection algorithm $\Pi(\hat{p}_n; TV_H, G)$ or $\Pi(\hat{p}_n; TV_H, G, \tilde{\epsilon}/2)$. Then, with probability at least $1 - \delta$,

$$\|\mu_p - \mu_q\|_2 \leq 2\sqrt{\epsilon\rho(\tilde{\epsilon})}, \quad (376)$$
$$\|\Sigma_p - \Sigma_q\|_2 \leq 7\rho(2\tilde{\epsilon}), \quad (377)$$

**Proof.** The bound on $\tilde{\epsilon}$ is the same as in the proof of Theorem 5.1. It suffices to show the modulus of continuity.

Assume $TV_H(p_1, p_2) \leq \tilde{\epsilon}$, and $p_1, p_2 \in G_{TV}^L(\rho(\tilde{\epsilon}), \tilde{\epsilon})$. Following the same argument as mean estimation in Theorem 5.1, we know that

$$\|\mu_{p_1} - \mu_{p_2}\|_2 \leq 2\sqrt{\epsilon\rho(\tilde{\epsilon})}. \quad (378)$$

Thus it suffices to bound the modulus of continuity for covariance estimation. Note that

$$\sup_{v \in \mathbb{R}^d, \|v\|_2 = 1, t \in \mathbb{R}^d} |\mathbb{P}_{p_1}(v^T(X - \mu_{p_2})(X - \mu_{p_2})^Tv \geq t) - \mathbb{P}_{p_2}(v^T(X - \mu_{p_2})(X - \mu_{p_2})^Tv \geq t)|$$
$$= \sup_{v \in \mathbb{R}^d, \|v\|_2 = 1, t \in \mathbb{R}^d} |\mathbb{P}_{p_1}(|v^T(X - \mu_{p_2})| \geq t) - \mathbb{P}_{p_2}(|v^T(X - \mu_{p_2})| \geq t)|$$
$$\leq \sup_{v \in \mathbb{R}^d, \|v\|_2 = 1, t \in \mathbb{R}^d} |\mathbb{P}_{p_1}(v^T(X - \mu_{p_2}) \geq t) - \mathbb{P}_{p_2}(v^T(X - \mu_{p_2}) \geq t)|$$
$$+ |\mathbb{P}_{p_1}(v^T(X - \mu_{p_2}) \leq -t) - \mathbb{P}_{p_2}(v^T(X - \mu_{p_2}) \leq -t)|$$
$$\leq 2TV_H(p_1, p_2)$$
$$\leq 2\tilde{\epsilon}. \quad (379)$$

Without loss of generality, assume that there exists some $v^*$ such that

$$v^T(\Sigma_{p_1} - \Sigma_{p_2})v^* = \|\Sigma_{p_1} - \Sigma_{p_2}\|_2. \quad (380)$$

Thus from $TV_H(p_1, p_2) \leq \tilde{\epsilon}$ and Lemma 5.2, there exist some $r_{p_1} \leq \frac{p_1}{1-2\epsilon}$, $r_{p_2} \leq \frac{p_2}{1-2\epsilon}$, such that

$$v^T\mathbb{E}_{r_{p_1}}[(X - \mu_{p_2})(X - \mu_{p_2})^T]v^* \leq v^T\mathbb{E}_{r_{p_2}}[(X - \mu_{p_2})(X - \mu_{p_2})^T]v^*. \quad (381)$$

From $p_1, p_2 \in G_{TV}^L(\rho(\tilde{\epsilon}), \tilde{\epsilon})$, we know that

$$\sup_{r_{p_1} \leq \frac{p_1}{1-2\epsilon}} v^T(\Sigma_{p_1} - \mathbb{E}_{r_{p_1}}[(X - \mu_{p_2})(X - \mu_{p_2})^T])v^* \leq \rho(2\tilde{\epsilon}), \quad (382)$$
$$\sup_{r_{p_2} \leq \frac{p_2}{1-2\epsilon}} v^T(\mathbb{E}_{r_{p_2}}[(X - \mu_{p_2})(X - \mu_{p_2})^T] - \Sigma_{p_2})v^* \leq \rho(2\tilde{\epsilon}). \quad (383)$$
Thus overall, we have
\[ \|\Sigma_{p_1} - \Sigma_{p_2}\|_2 = v^*^\top (\Sigma_{p_1} - \Sigma_{p_2}) v^* \]
\[ = v^*^\top (\Sigma_{p_1} - \E_{r_{p_1}} [(X - \mu_{p_1}) (X - \mu_{p_1})^\top]) \]
\[ + \E_{r_{p_1}} [(X - \mu_{p_1}) (X - \mu_{p_1})^\top] - \E_{r_{p_1}} [(X - \mu_{r_{p_1}}) (X - \mu_{r_{p_1}})^\top] \]
\[ + \E_{r_{p_1}} [(X - \mu_{r_{p_1}}) (X - \mu_{r_{p_1}})^\top] - \E_{r_{p_1}} [(X - \mu_{p_2}) (X - \mu_{p_2})^\top] \]
\[ + \E_{r_{p_2}} [(X - \mu_{p_2}) (X - \mu_{p_2})^\top] - \Sigma_{p_2} v^* \]
\[ \leq \rho(2\hat{\epsilon}) + \|\mu_{p_1} - \mu_{r_{p_1}}\|_2^2 + \|\mu_{p_2} - \mu_{r_{p_1}}\|_2^2 + 0 + \rho(2\hat{\epsilon}) \]
\[ \leq 2\rho(2\hat{\epsilon}) + 10\hat{\epsilon}\rho(2\hat{\epsilon}) \]
\[ \leq 7\rho(2\hat{\epsilon}). \] (384)

\[ \square \]

E.4 Analysis of Tukey median

As another example of weakening the TV distance, we analyze Tukey median, a well-known algorithm for robust mean estimation (Huber, 1973; Donoho, 1982; Donoho and Gasko, 1992; Gao, 2017), which can be presented as the minimum distance functional

\[ q = \Pi(\hat{p}_n; D_{\text{Tukey}}, \mathcal{M}), \] (385)
\[ \hat{\mu}_{\text{Tukey}} = \E_q [X], \] (386)

where \( \Pi \) is the projection algorithm described in Algorithm 1,

\[ D_{\text{Tukey}}(q, p) = \sup_{v \in \mathbb{R}^d} \P_p \left( (X - \E_q [X])^\top v > t \right), \] (387)

and \( \hat{p}_n \) is the empirical distribution of the observed corrupted data. Here \( \mathcal{M} \) can be the set of all distributions since the \( D_{\text{Tukey}} \) only depends on \( q \) through its mean \( \mu_q \). Note that \( D_{\text{Tukey}}(q, p) \) is asymmetric, does not satisfy the triangle inequality, and \( D_{\text{Tukey}}(p, p) \) is in general non-zero.

Gao (2017) shows that the Tukey median achieves good finite sample error for mean estimation if the real distribution \( p^* \) belongs to the family of elliptical distributions. The theorem below analyzes the performance of Tukey median under more general distributional assumptions and TV_\( \mathcal{H} \) perturbation, where \( \mathcal{H} \) is defined in (54).

**Theorem E.4.** Denote the true distribution as \( p^* \) and assume that there exists a non-negative function non-decreasing function \( h(t) \) for \( t \geq 0 \) such that for all \( t \geq 0 \) and any \( v \in \mathbb{R}^d, \|v\|_\infty = 1, \)

\[ \P_p ((X - \E_p [X])^\top v > t) \leq 1/2 - h(t). \] (388)

Consider either the Oblivious (Definition 2.3) or Adaptive corruption (Definition 2.4) of level \( \epsilon \) under \( D = \text{TV}_\mathcal{H} \) where \( \mathcal{H} \) is defined in (54).

Denote the corrupted empirical distribution as \( \hat{p}_n \). Then with probability at least \( 1 - \delta \), there exists a universal constant \( C > 0 \) such that

\[ \|\hat{\mu}_{\text{Tukey}} - \E_{p^*}[X]\| \leq h^{-1} \left( 2\epsilon + C \cdot \sqrt{\frac{d + 1 + \log(1/\delta)}{n} - h(0)} \right), \] (389)

where \( \hat{\mu}_{\text{Tukey}} \) is the Tukey median in (386), and \( h^{-1} \) is the generalized inverse function of \( h \) defined in (14).
A few remarks are in order:

1. For Gaussian distribution with bounded covariance, we can find some \( h(t) \) that is linear when \( t \) is small and \( h(0) = 0 \), and Theorem E.4 implies that Tukey median achieves the the minimax rate \( \epsilon + \sqrt{d/n} \) for robust Gaussian mean estimation. Our condition is more general than the elliptical distribution assumption in (Gao, 2017).

2. It follows from the proof of Theorem E.4 that the technical reason why Tukey median works under the more general perturbation metric \( \tilde{TV}_H \) (where \( H \) is defined in (54)) in the oblivious corruption model is that for any \( p, q, r \), we have \( |D_{Tukey}(r, p) - D_{Tukey}(r, q)| \leq \tilde{TV}_H(p, q) \). Similar relationship also holds between \( \tilde{TV}_H \) for linear regression (Theorem 5.2) and regression depth in (Gao et al., 2018), and between \( \tilde{TV}_H \) for covariance estimation (Theorem E.3) and matrix depth in (Chen et al., 2018). For any projection algorithm \( q = \Pi(\hat{p}_n; D, \mathcal{M}) \), under mild conditions the algorithm works with new perturbation metric \( \sup_{r \in \mathcal{M}} |\hat{D}(r, p^*) - \hat{D}(r, p)| \leq \epsilon \) (Proposition 7.2).

3. If we strengthen the assumption and further assume that for any \( v \in \mathbb{R}^d \),

\[
\mathbb{P}(v^\top (X - \mathbb{E}_p[X]) > 0) = \frac{1}{2}, \tag{390}
\]

and denote the corresponding set of \( p^* \) as \( G_{Tukey} \), then the projection algorithm \( q = \Pi(\hat{p}_n; \tilde{TV}_H, G_{Tukey}) \) achieves the same performance guarantee in Theorem E.4. In some sense, Tukey median achieves robust estimation under a more general condition, and the algorithm has no explicit dependence on \( G_{Tukey} \); however, it cannot be shown to achieve decent finite sample error bounds for the resilient sets \( G_{TV} \) we designed in Section 3.

**Proof.** Let \( q \) denote the output of Tukey median \( q = \arg \min_q D_{Tukey}(q, \hat{p}_n) \). Then,

\[
D_{Tukey}(q, p^*) \leq D_{Tukey}(q, \hat{p}_n) + \tilde{TV}_H(\hat{p}_n, p^*) \tag{391}
\]

\[
\leq D_{Tukey}(p^*, \hat{p}_n) + \tilde{TV}_H(\hat{p}_n, p^*) \tag{392}
\]

\[
\leq D_{Tukey}(p^*, p^*) + 2\tilde{TV}_H(\hat{p}_n, p^*) \tag{393}
\]

\[
\leq D_{Tukey}(p^*, p^*) + 2\tilde{TV}_H(\hat{p}_n, \hat{p}_n^*) + 2\tilde{TV}_H(\hat{p}_n^*, p^*), \tag{394}
\]

where we repeated use the fact that for any \( p, q, r \), we have \( |D_{Tukey}(r, p) - D_{Tukey}(r, q)| \leq \tilde{TV}_H(p, q) \).

It follows from the definition of Adaptive corruption (Definition 2.4) that \( \tilde{TV}_H(\hat{p}_n, \hat{p}_n^*) \) is being stochastically dominated by \( \tilde{TV}_H(\frac{1}{n} \sum_{i=1}^n \delta Y_i, \frac{1}{n} \sum_{i=1}^n \delta X_i) \), where \((X_i, Y_i)_{i=1}^n\) are i.i.d. samples from some \( \pi_X, \pi_Y \) where \( \pi_X = p^*, \pi_Y = p, \tilde{TV}_H(\pi_X, \pi_Y) \leq \epsilon \).

We then upper bound

\[
\tilde{TV}_H(\frac{1}{n} \sum_{i=1}^n \delta Y_i, \frac{1}{n} \sum_{i=1}^n \delta X_i) \leq \tilde{TV}_H(\frac{1}{n} \sum_{i=1}^n \delta Y_i, p) + \tilde{TV}_H(p, p^*) + \tilde{TV}_H(p^*, \frac{1}{n} \sum_{i=1}^n \delta X_i) \tag{395}
\]

\[
\leq \epsilon + \tilde{TV}_H(\frac{1}{n} \sum_{i=1}^n \delta Y_i, p) + \tilde{TV}_H(p^*, \frac{1}{n} \sum_{i=1}^n \delta X_i). \tag{396}
\]

It follows from (52), the fact that the family of sets \( \{x \mid v^\top x \geq t \} \mid \|v\| = 1, t \in \mathbb{R}, v \in \mathbb{R}^d \} \) has VC dimension \( d + 1 \) that for any distribution \( r \), if \( \hat{r}_n \) denotes the empirical distribution from \( r \)
with $n$ i.i.d. samples, then with probability at least $1 - \delta$,

$$\tilde{\text{TV}}_{\mathcal{H}}(r, \tilde{r}_n) \leq C^{\text{vc}} \cdot \sqrt{\frac{d + 1 + \log(1/\delta)}{n}}. \tag{397}$$

Hence, with probability at least $1 - 3\delta$,

$$D_{\text{Tukey}}(q, p^*) - D_{\text{Tukey}}(p^*, p^*) \leq 2\epsilon + 6C^{\text{vc}} \cdot \sqrt{\frac{d + 1 + \log(1/\delta)}{n}}. \tag{398}$$

Let $\tilde{v} = \arg \max_{\|v\| \leq 1} v^\top (\mu_{p^*} - \mu_q)$, we have

$$D_{\text{Tukey}}(q, p^*) - D_{\text{Tukey}}(p^*, p^*) = \sup_{v \in \mathbb{R}^d} \mathbb{P}_{p^*}(v^\top (X - \mu_q) > 0) - \mathbb{P}_{p^*}(v^\top (X - \mu_{p^*}) > 0) \tag{399}$$

$$\geq \mathbb{P}_{p^*}(\tilde{v}^\top (X - \mu_q) > 0) + h(0) - 1/2 \tag{400}$$

$$\geq \mathbb{P}_{p^*}(\tilde{v}^\top (X - \mu_{p^*}) > \tilde{v}^\top (\mu_q - \mu_{p^*})) + h(0) - 1/2 \tag{401}$$

$$= \mathbb{P}_{p^*}(\tilde{v}^\top (X - \mu_{p^*}) > \|\mu_q - \mu_{p^*}\|) + h(0) - 1/2 \tag{402}$$

$$= h(0) + \frac{1}{2} - \mathbb{P}_{p^*}(\tilde{v}^\top (X - \mu_{p^*}) \leq -\|\mu_q - \mu_{p^*}\|). \tag{403}$$

Hence, we know that

$$\mathbb{P}_{p^*}(\tilde{v}^\top (X - \mu_{p^*}) \leq -\|\mu_q - \mu_{p^*}\|) \geq \frac{1}{2} + h(0) - 2\epsilon - 6C^{\text{vc}}(\sqrt{\frac{d + 1 + \log(1/\delta)}{n}}). \tag{404}$$

We show that it implies

$$\|\mu_q - \mu_{p^*}\| \leq h^{-1} \left(2\epsilon + 6C^{\text{vc}} \cdot \sqrt{\frac{d + 1 + \log(1/\delta)}{n}} - h(0) \right). \tag{405}$$

Indeed, for any $t$ such that $h(t) > 2\epsilon + 6C^{\text{vc}} \cdot \sqrt{\frac{d + 1 + \log(1/\delta)}{n}} - h(0)$, if $\|\mu_q - \mu_{p^*}\| > t$, we have

$$\mathbb{P}_{p^*}(\tilde{v}^\top (X - \mu_{p^*}) \leq -\|\mu_q - \mu_{p^*}\|) = \mathbb{P}_{p^*}((-\tilde{v})^\top (X - \mu_{p^*}) \geq \|\mu_q - \mu_{p^*}\|) \geq \mathbb{P}_{p^*}((-\tilde{v})^\top (X - \mu_{p^*}) > t) \tag{406}$$

$$\leq \mathbb{P}_{p^*}((-\tilde{v})^\top (X - \mu_{p^*}) > t) \tag{407}$$

$$\leq \frac{1}{2} - h(t) \tag{408}$$

$$< \frac{1}{2} + h(0) - 2\epsilon - 6C^{\text{vc}}(\sqrt{\frac{d + 1 + \log(1/\delta)}{n}}), \tag{409}$$

resulting in a contradiction. The proof is completed via noting

$$h^{-1}(t) \triangleq \inf\{x | x \geq 0, h(x) > t\}. \tag{410}$$
E.5 Generative Adversarial Networks and Robust Inference

Recall the definition of $\tilde{TV}_H$:
\[
\tilde{TV}_H(p, q) = \sup_{h \in H, t \in \mathbb{R}} |E_p[1(f(X) + t \geq 0)] - E_q[1(f(X) + t \geq 0)]|, \tag{411}
\]

However, it may be difficult to compute the $\tilde{TV}_H$ projection in practice due to the non-differentiable zero-one loss. We show below a large family of smooth loss function can also be used as projection discrepancies to achieve decent finite sample bounds.

The key observation is that the we have a generalization of the mean cross lemma (Lemma 5.2) that works for the smoothed generalized Kolmogorov-Smirnov distance defined as
\[
S(p, q) = \sup_{f \in F, t \in \mathbb{R}} |E_p[T(f(X) + t)] - E_q[T(f(X) + t)]|, \tag{412}
\]

Here $T(x)$ is the cumulative distribution function of some zero mean random variable $Z$, i.e. $T(-\infty) = 0, T(+\infty) = 1$ and $T$ is right-continuous (e.g. the sigmoid function $T(x) = 1/(1 + e^{-x})$). When $Z = 0$ almost surely, it is reduced to the case of generalized KS distance. The generalized mean cross lemma is presented below.

Lemma E.2 (Closeness in smoothed KS distance implies mean cross). Assume for two distribution $p, q$,
\[
\sup_{t \in \mathbb{R}} |E_p[T(X + t)] - E_q[T(X + t)]| \leq \epsilon, \tag{413}
\]

where $T(x)$ can be written as the cumulative distribution function of some zero mean random variable $Z$ (e.g. the sigmoid function $T(x) = 1/(1 + e^{-x})$). Assume the distribution of $Z$ is inside resilience family of $G_{\text{mean}}(\rho_Z, \epsilon)$. Then there exists $r_p \leq \frac{p}{1-\epsilon}, r_q \leq \frac{q}{1-\epsilon}$,
\[
E_{r_q}[X] - E_{r_p}[X] \leq 2\rho_Z. \tag{414}
\]

Clearly, for the generalized KS distance, $Z \equiv 0$ almost surely, and $\rho_Z = 0$, recovering part of the conclusion of Lemma 5.2.

Proof. For a fixed $x$, $T(x + t) = P(Z \leq x + t) = P(x - Z \geq -t)$. Thus we know
\[
\sup_{t \in \mathbb{R}} |P_q(X - Z \geq -t) - P_p(X - Z \geq -t)| \leq \epsilon, \tag{415}
\]

where $X$ and $Z$ are independent. Denote $\tilde{p}$ as the distribution of $X - Z$ when $X \sim p$, $\tilde{q}$ as the distribution of $X - Z$ when $X \sim q$. By Lemma 5.2, we know that there exist $\tilde{r}_p \leq \frac{p}{1-\epsilon}, \tilde{r}_q \leq \frac{q}{1-\epsilon}$, such that
\[
E_{r_q}[X - Z] \leq E_{\tilde{r}_p}[X - Z]. \tag{416}
\]

Denote the original distribution of $Z$ as $p_Z$. Since the marginal distribution of both $\tilde{r}_p, \tilde{r}_q$ is a deletion of $p_Z$, rearranging the equation, we get
\[
E_{r_q}[X] - E_{\tilde{r}_p}[X] \leq E_{r_q}[Z] - E_{\tilde{r}_p}[Z] \\
\leq |E_{r_q}[Z] - E_{p_Z}[Z]| + |E_{\tilde{r}_p}[Z] - E_{p_Z}[Z]|. \tag{417}
\]

□
We now show that given the mean cross lemma for smoothed KS distance, the smoothed distance can also be used to achieve decent robust statistical error for non-parametric robust estimation.

**Theorem E.5.** For any \( p^* \in \bigcap_{\eta \in [0,1]} \mathcal{G}_{TV}^{TV}(\rho(\eta), \eta) \) in Equation (152), denote \( p \) as the observed corrupted distribution such that \( TV(p^*, p) \leq \epsilon \), define

\[
A(p, q) = \sup_{(d_1, d_2) \in \mathcal{D}} \mathbb{E}_p[d_1(X)] + \mathbb{E}_q[d_2(X)].
\]  

(418)

Here \( \mathcal{D} \) is some family for discriminator function pairs \((d_1, d_2)\), \( T(x) \) is the cumulative distribution function of some zero mean random variable \( Z \) (e.g. the sigmoid function \( T(x) = 1/(1 + e^{-x}) \)). Assume the following conditions:

1. For all \((d_1, d_2) \in \mathcal{D}\), and all \( x \in \mathbb{R}\), we have \( |d_2(x)| \leq 1/2 \).

2. For any distribution pair \( p, q \in \bigcap_{\eta \in [0,1]} \mathcal{G}_{TV}^{TV}(\rho(\eta), \eta) \), we have

\[
A(q, p) - A(p, p) \leq \tilde{\epsilon} \Rightarrow \sup_{f \in \mathcal{F}, b \in \mathbb{R}} |\mathbb{E}_q[T(f(X) + b) - \mathbb{E}_p[T(f(X) + b)]| \leq C\tilde{\epsilon}
\]

(419)

for some constant \( C > 0 \) and \( \tilde{\epsilon} = 2\epsilon + 2\tilde{A}(p, \tilde{p}_n) \), where

\[
\tilde{A}(p, q) = \sup_{(d_1, d_2) \in \mathcal{D}} |\mathbb{E}_p[d_2(X)] - \mathbb{E}_q[d_2(X)]|.
\]  

(420)

3. the distribution of \( Z \) is inside resilient set \( \bigcap_{\eta \in [0,1]} \mathcal{G}_{TV}^{TV}(\rho_Z(\eta), \eta) \).

Then the projection algorithm \( q = \Pi(\tilde{p}_n; A, \bigcap_{\eta \in [0,1]} \mathcal{G}_{TV}^{TV}(\rho(\eta), \eta)) \) guarantees

\[
W_F(p^*, q) \leq 2\rho(C\tilde{\epsilon}) + 2\rho_Z(C\tilde{\epsilon}).
\]

(421)

Proof. It suffices to verify that the conditions in Theorem 7.1.

1. **Robustness to perturbation:** For any distribution \( p_1, p_2, p_3 \), we have

\[
|A(p_1, p_2) - A(p_1, p_3)| = \sup_{(d_1, d_2) \in \mathcal{D}} |\mathbb{E}_{p_1}[d_1(X)] + \mathbb{E}_{p_2}[d_2(X)] - \mathbb{E}_{p_1}[d_1(X)] + \mathbb{E}_{p_3}[d_2(X)]| \leq \tilde{A}(p_2, p_3)
\]

(422)

2. **Generalized modulus of continuity:** For any \( p, q \in \mathcal{G}_{TV}^{TV} \), from \( A(q, p) - A(p, p) \leq \tilde{\epsilon} \), we have

\[
\sup_{f \in \mathcal{F}, t \in \mathbb{R}} |\mathbb{E}_q[T(f(X) + t)] - \mathbb{E}_p[T(f(X) + t)]| \leq C\tilde{\epsilon}.
\]

(423)
From Lemma E.2, we know that for any fixed \( f^* \in \arg\max_{f \in \mathcal{F}} \mathbb{E}_p[f(X)] - \mathbb{E}_q[f(X)] \), there exist \( r_p \leq \frac{p}{1 - \varepsilon} \), \( r_q \leq \frac{q}{1 - \varepsilon} \), such that
\[
\mathbb{E}_{r_p} f^*(X) \leq \mathbb{E}_{r_q} f^*(X) + 2\rho_Z(C \varepsilon).
\] (424)

From \( q \in \bigcap \mathcal{G}_{W_p}^\text{TV}(\rho(\eta), \eta) \), we have
\[
\forall r \leq \frac{q}{1 - C \varepsilon}, W_p(r, q) \leq \rho(C \varepsilon).
\] (425)

Therefore
\[
W_p(p, q) = \mathbb{E}_p[f^*(X)] - \mathbb{E}_q[f^*(X)]
\leq \mathbb{E}_p[f^*(X)] - \mathbb{E}_{r_p} f^*(X) + \rho(C \varepsilon)
\leq \mathbb{E}_p[f^*(X)] - \mathbb{E}_{r_q} f^*(X) + 2\rho_Z(C \varepsilon) + \rho(C \varepsilon)
\leq 2\rho(C \varepsilon) + 2\rho_Z(C \varepsilon).
\] (426)

By Theorem 7.1 the conclusion is derived.

**Remark E.1.** The first condition on the magnitude of \( d_2 \) can be easily satisfied by setting the output of the neural network designed for \( d_2 \) to pass through a bounded activation function. In (Gao et al., 2019), it was shown that under mild conditions, one can produce valid \( (d_1, d_2) \) using proper scoring rules and appropriate neural network architectures to ensure condition (419). Our result, combined with (Gao et al., 2019), extends the results for mean and second moment estimation results in (Gao et al., 2019) to general resilient sets including sub-Gaussian and bounded \( k \)-th moments, while (Gao et al., 2019) gives guarantee for elliptical distributions as semi-parametric classes. It is an interesting open problem to provide provable guarantees for computational efficiencies of GANs.

**Remark E.2.** If \( T(x) = \text{sigmoid}(x) = \frac{1}{1 + e^{-x}} \), the corresponding random variable \( Z \) is sub-exponential and it incurs an additional error of \( \Theta(\varepsilon \log(1/\varepsilon)) \). If \( Z \) is the uniform distribution on \([-1, 1]\), then the additional error incurred is \( \Theta(\varepsilon) \).

### E.6 Key Lemmas

#### E.6.1 Generalized Modulus of Continuity

The following Lemma is essentially the same as (Diakonikolas et al., 2017, Corollary A.25). It shows that the generalized modulus of continuity between bounded covariance set and resilient set can be controlled. For completeness we present the proof here.

**Lemma E.3.** For some constant non-negative \( \rho_1, \rho_2, \tau, \) assume \( \tau \geq \varepsilon \), denote \( \mu_p = \mathbb{E}_p[X] \). Define
\[
\mathcal{G}_1 = \{ p : \forall r \leq \frac{p}{1 - \varepsilon}, \| \mu_r - \mu_p \|_2 \leq \rho_1, \lambda_{\min}(\mathbb{E}_r[(X - \mu_p)(X - \mu_p)^\top]) \geq 1 - \rho_2 \}
\] (427)
\[
\mathcal{G}_2 = \{ p : \| \mathbb{E}_p[(X - \mu_p)(X - \mu_p)^\top] \|_2 \leq 1 + \tau \}.
\] (428)

Here \( \lambda_{\min}(A) \) is the smallest eigenvalue of symmetric matrix \( A \). Then, for any \( \varepsilon \in [0, 1) \) we have
\[
\sup_{p \in \mathcal{G}_1, q \in \mathcal{G}_2, \text{TV}(p, q) \leq \varepsilon} \| \mathbb{E}_p[X] - \mathbb{E}_q[X] \|_2 \leq C(\sqrt{\tau \varepsilon} + \sqrt{\rho_2 \varepsilon} + \rho_1).
\] (429)

Here \( C \) is some universal constant.
Proof. Assume \( p \in \mathcal{G}_1, q \in \mathcal{G}_2, p \neq q \). Without loss of generality, we assume \( \mu_p = 0 \). From \( \text{TV}(p, q) = \epsilon_0 \leq \epsilon \), we construct distribution \( r = \frac{p}{1 - \epsilon_0} \). By Lemma C.1 we know that \( r \leq \frac{q}{1 - \epsilon_0} \). Denote \( \tilde{r} = (1 - \epsilon_0)r \). Consider measure \( p - \tilde{r}, q - \tilde{r} \). We have \( \mu_q = \mu_p - \mu_{p - \tilde{r}} + \mu_{q - \tilde{r}} = -\mu_{p - \tilde{r}} + \mu_{q - \tilde{r}} \). Note that \( \|\mu_p - \tilde{r}\|_2 = \|\mu_p - \mu_{\tilde{r}}\|_2 = \|\mu_{\tilde{r}}\|_2 \leq (1 - \epsilon_0)\rho_1 \leq \rho_1 \). For any \( v \in \mathbb{R}^d, \|v\|_2 = 1 \), we have

\[
v^\top \Sigma_q v^\top = v^\top (E_q[XX^\top] - \mu_q \mu_q^\top) v \\
= v^\top (E_{\tilde{r}}[XX^\top] + E_{q - \tilde{r}}[XX^\top] - (\mu_{q - \tilde{r}} - \mu_{p - \tilde{r}})(\mu_{q - \tilde{r}} - \mu_{p - \tilde{r}})^\top) v \\
\geq (1 - \rho_2)(1 - \epsilon_0) + E_{q - \tilde{r}}[(v^\top X)^2] - (v^\top \mu_{q - \tilde{r}})^2 + 2v^\top \mu_{q - \tilde{r}} v^\top \mu_{p - \tilde{r}} - (v^\top \mu_{p - \tilde{r}})^2 \\
\geq 1 - \rho_2 - \tau + E_{q - \tilde{r}}[(v^\top X)^2] - (v^\top \mu_{q - \tilde{r}})^2 - 2\|\mu_{q - \tilde{r}}\|_2\|\mu_{p - \tilde{r}}\|_2 - \|\mu_{p - \tilde{r}}\|^2 \\
\geq 1 - \rho_2 - \tau + E_{q - \tilde{r}}[(v^\top X)^2] - (v^\top \mu_{q - \tilde{r}})^2 - 2\rho_1\|\mu_{q - \tilde{r}}\|_2 - \rho_1^2. \tag{430}
\]

Here we use the fact that \( \epsilon_0 \leq \epsilon \leq \tau \). Denote \( b_q = \frac{q - \tilde{r}}{\epsilon_0} \). Then \( b_q \) is a distribution. If \( \mu_{b_q} = 0 \), then we already know that \( \|\mu_q - \mu_r\| \leq \epsilon_0\|\mu_{b_q}\|_2 = 0 \). Otherwise we take \( v = \frac{\mu_{b_q}}{\|\mu_{b_q}\|_2} \). Then we can see \( E_{q - \tilde{r}}[(v^\top X)^2] = \epsilon_0 E_{b_q}[(v^\top X)^2] \geq \epsilon_0\|\mu_{b_q}\|_2^2 \). From \( q \in \mathcal{G}_2 \), we know that \( v^\top \Sigma_q v \leq 1 + \tau \). Thus

\[
(\epsilon_0 - \epsilon_0^2)\|\mu_{b_q}\|_2^2 - 2\epsilon_0\rho_1\|\mu_{b_q}\|_2 \leq \rho_1^2 + \rho_2 + 2\tau. \tag{431}
\]

Solving the inequality, we derive that

\[
\|\mu_q - \mu_r\|_2 \leq \epsilon_0\|\mu_{b_q}\|_2 \leq C(\sqrt{\tau\epsilon_0} + \sqrt{\rho_2\epsilon_0} + \sqrt{\epsilon_0\rho_1}) \leq C(\sqrt{\tau\epsilon} + \sqrt{\rho_2\epsilon} + \sqrt{\epsilon\rho_1}). \tag{432}
\]

where \( C \) is some universal constant. Thus we can conclude

\[
\|\mu_p - \mu_q\|_2 \leq \|\mu_p - \mu_r\|_2 + \|\mu_r - \mu_q\|_2 \leq C(\sqrt{\tau\epsilon} + \sqrt{\rho_2\epsilon} + \rho_1). \tag{433}
\]

Furthermore, we show a stronger lemma that the generalized modulus of continuity for the same sets under \( \text{TV}_H \) distance is also bounded. The Lemma is critical in showing the generalized modulus of continuity for both bounded \( k \)-th moment distribution and sub-Gaussian distribution with identity covariance assumption under \( \text{TV}_H \) distance.

**Lemma E.4.** For some non-negative constant \( \rho_1, \rho_2, \tau \), assume \( \tau \geq \epsilon \). We denote \( \mu_p = E_p[X] \). Define

\[
\mathcal{G}_1 = \{ p : \forall r \leq \frac{p}{1 - 2\epsilon}, \|\mu_r - \mu_p\|_2 \leq \rho_1, \lambda_{\text{min}}(E_r[(X - \mu_p)(X - \mu_p)^\top]) \geq 1 - \rho_2 \}, \tag{434}
\]

\[
\mathcal{G}_2 = \{ p : \|E_p[(X - \mu_p)(X - \mu_p)^\top]\|_2 \leq 1 + \tau \}, \tag{435}
\]

\[
\text{TV}_H(q,p) = \sup_{v \in \mathbb{R}^d, \tau \in \mathbb{R}} |P_p(v^\top X \geq t) - P_q(v^\top X \geq t)|. \tag{436}
\]

Here \( \lambda_{\text{min}}(A) \) is the smallest eigenvalue of symmetric matrix \( A \). Assume \( \epsilon < 1/3 \), then there exists a universal constant \( C \) such that

\[
\sup_{q \in \mathcal{G}_2, p \in \mathcal{G}_1, \text{TV}_H(q,p) \leq \epsilon} \|\mu_p - \mu_q\|_2 \leq C(\sqrt{\tau\epsilon} + \sqrt{\rho_2\epsilon} + \rho_1), \tag{437}
\]

where \( C \) is some universal constant.
Proof. From Lemma C.2 and \( \mathcal{T} \), \( H(q, p) \leq \epsilon < 2\epsilon \), for \( v = \frac{\mu_q - \mu_p}{\|\mu_q - \mu_p\|_2} \), there exist \( r_p \leq \frac{p}{1 - 2\epsilon} \), and \( r_q \leq \frac{q}{1 - 2\epsilon} \),

\[
\mathbb{E}_{r_p}\langle (X - \mu_q, v) \rangle^2 \leq \mathbb{E}_{r_q}\langle (X - \mu_q, v) \rangle^2
\]

Thus we have

\[
\mathbb{E}_q[(X - \mu_q, v)^2] \geq (1 - 2\epsilon)\mathbb{E}_{r_q}[(X - \mu_q, v)^2]
\]

\[
\geq (1 - 2\epsilon)\mathbb{E}_{r_p}[(X - \mu_q, v)^2]
\]

\[
= (1 - 2\epsilon)\mathbb{E}_{r_p}[(X - \mu_p, v)^2 + \langle \mu_p - \mu_q, v \rangle^2 + 2\langle X - \mu_p, v \rangle \cdot \langle \mu_p - \mu_q, v \rangle]
\]

\[
\geq (1 - 2\epsilon)(1 - \rho_p + \|\mu_p - \mu_q\|^2 - 2\rho_p\|\mu_p - \mu_q\|_2)
\]

(439)

The last inequality comes from the fact that \( p \in \mathcal{G}_1 \). From \( q \in \mathcal{G}_2 \), we know

\[
\|\mathbb{E}_q[(X - \mu_q, v)^2]\|_2 \leq 1 + \tau
\]

\[
\Rightarrow \|\mu_p - \mu_q\|^2 - 2\rho_p\|\mu_p - \mu_q\|_2 \leq \frac{1 + \tau}{1 - 2\epsilon} - 1 + 2\rho_p < 2\rho_p + 9\tau.
\]

(440)

Here we use the assumption that \( \epsilon < 1/3 \). Solving the equation, we can derive

\[
\|\mu_p - \mu_q\|_2 \leq 5\rho_p + 4\sqrt{\rho_p + 9\tau}.
\]

(441)

Next we show that if \( \|\mu_p - \mu_q\|_2 \leq 11\rho_p + 9\sqrt{\rho_p + 9\tau}\), we have

\[
\|\mathbb{E}_q[(X - \mu_q, (X - \mu_q)^\top)]\|_2 > 1 + \tau.
\]

(442)

Consider the unit vector \( v = \frac{\mu_q - \mu_p}{\|\mu_q - \mu_p\|_2} \). For \( a \in [0, 1] \), \( b \in [0, 1] \), consider the random variable

\[
\frac{\rho_1}{a} \mathbb{P}_p(\frac{a\mathbb{V}^\top(X - \mu_p)}{\rho_1} + \frac{b(1 - (v^\top(X - \mu_p))^2)}{\rho_2} \geq t) - \mathbb{P}_q(\frac{a\mathbb{V}^\top(X - \mu_p)}{\rho_1} + \frac{b(1 - (v^\top(X - \mu_p))^2)}{\rho_2} \geq t) \leq 2\epsilon.
\]

(443)

To see this, indeed, for any given \( t \in \mathbb{R}^d \), when the equation \( \frac{\rho_1}{a} \mathbb{P}_p(\frac{a\mathbb{V}^\top(X - \mu_p)}{\rho_1} + \frac{b(1 - (v^\top(X - \mu_p))^2)}{\rho_2} \geq t) \) has two solutions (denoted as \( x_0, x_1 \)), we have

\[
\|\mathbb{P}_p(\frac{a\mathbb{V}^\top(X - \mu_p)}{\rho_1} + \frac{b(1 - (v^\top(X - \mu_p))^2)}{\rho_2} \geq t) - \mathbb{P}_q(\frac{a\mathbb{V}^\top(X - \mu_p)}{\rho_1} + \frac{b(1 - (v^\top(X - \mu_p))^2)}{\rho_2} \geq t) \|_2 \leq 2\epsilon.
\]

(444)

When the equation has one or zero solution and \( b \neq 0 \), the difference is 0. When the equation has one solution and \( b = 0 \), we know that the difference is bounded by \( \epsilon \). For any \( a, b \in [0, 1] \), from Lemma C.2, we know that there exists \( r_p \leq \frac{p}{1 - 2\epsilon} \), and \( r_q \leq \frac{q}{1 - 2\epsilon} \) such that

\[
\mathbb{E}_{r_p}(\frac{a\mathbb{V}^\top(X - \mu_p)}{\rho_1} + \frac{b(1 - (v^\top(X - \mu_p))^2)}{\rho_2}) \leq \mathbb{E}_{r_q}(\frac{a\mathbb{V}^\top(X - \mu_p)}{\rho_1} + \frac{b(1 - (v^\top(X - \mu_p))^2)}{\rho_2}) \leq a + b \leq 2.
\]

(445)
This is from that \( p \in G_1 \). Thus we have

\[
\max_{a \in [0,1], b \in [0,1]} \min_{r_q \leq \frac{q}{1-2\epsilon}} \mathbb{E}_{r_q} \left[ \frac{a v^\top (X - \mu_p)}{\rho_1} + \frac{b (1 - (v^\top (X - \mu_p))^2)}{\rho_2} \right] \leq 2. \tag{446}
\]

By minimax theorem, we can see that

\[
\min_{r_q \leq \frac{q}{1-2\epsilon}} \max_{a \in [0,1], b \in [0,1]} \mathbb{E}_{r_q} \left[ \frac{a v^\top (X - \mu_p)}{\rho_1} + \frac{b (1 - (v^\top (X - \mu_p))^2)}{\rho_2} \right] \leq 2. \tag{447}
\]

Thus there exists some \( r_q \leq \frac{q}{1-2\epsilon} \), such that for any \( a, b \in [0,1] \),

\[
\mathbb{E}_{r_q} \left[ \frac{a v^\top (X - \mu_p)}{\rho_1} + \frac{b (1 - (v^\top (X - \mu_p))^2)}{\rho_2} \right] \leq 2. \tag{448}
\]

By taking \( a = 0, b = 1 \) and \( a = 1, b = 0 \), we have

\[
\mathbb{E}_{r_q} [(v^\top (X - \mu_p))^2] \geq 1 - 2\rho_2, \tag{449}
\]

\[
\mathbb{E}_{r_q} [v^\top (X - \mu_p)] \leq 2\rho_1. \tag{450}
\]

Denote \( \tilde{r}_q = (1 - 2\epsilon) r_q \), then we have \( q \geq \tilde{r}_q \). To bound from below the maximum eigenvalue, it is sufficient to lower bound the term

\[
v^\top \mathbb{E}_{\tilde{r}_q} [(X - \mu_q)(X - \mu_q)^\top] v = v^\top \mathbb{E}_{\tilde{r}_q} [(X - \mu_q)(X - \mu_q)^\top] v + v^\top \mathbb{E}_{q-\tilde{r}_q} [(X - \mu_q)(X - \mu_q)^\top] v. \tag{451}
\]

Now we bound the two terms separately, first note that

\[
\mathbb{E}_{q-\tilde{r}_q} [(X - \mu_q, v)] \geq \mathbb{E}_{q-\tilde{r}_q} [(X - \mu_p, v)] - \int (q - \tilde{r}_q) \langle \mu_p - \mu_q, v \rangle \]

\[
= \mathbb{E}_q [(X - \mu_p, v)] - \mathbb{E}_{\tilde{r}_q} [(X - \mu_p, v)] - \int (q - \tilde{r}_q) \langle \mu_p - \mu_q, v \rangle \\
\geq ||\mu_q - \mu_p||_2 - \mathbb{E}_{\tilde{r}_q} [(X - \mu_p, v)] - 2\epsilon ||\mu_q - \mu_p||_2 \\
\geq (1 - 2\epsilon) ||\mu_q - \mu_p||_2 - (1 - 2\epsilon) 2\rho_1 \\
> (1 - 2\epsilon) (11\rho_1 + 9\sqrt{(\rho_2 + 9\tau)\epsilon - 2\rho_1}) \\
= (1 - 2\epsilon) (9\rho_1 + 9\sqrt{(\rho_2 + 9\tau)\epsilon}) \\
> \frac{1}{3} (9\rho_1 + 9\sqrt{(\rho_2 + 9\tau)\epsilon}) \\
> 3(\rho_1 + \sqrt{(\rho_2 + 9\tau)\epsilon}) \tag{452}
\]

By Cauchy-Schwarz inequality,

\[
\mathbb{E}_{q-\tilde{r}_q} [(X - \mu_q, v)]^2 \int (q - \tilde{r}_q) \geq \left( \mathbb{E}_{q-\tilde{r}_q} [(X - \mu_q, v)] \right)^2 > 9(\rho_1 + \sqrt{(\rho_2 + 9\tau)\epsilon})^2. \tag{453}
\]

Thus

\[
\mathbb{E}_{q-\tilde{r}_q} [(X - \mu_q, v)]^2 > \frac{9(\rho_1 + \sqrt{(\rho_2 + 9\tau)\epsilon})^2}{\epsilon}. \tag{454}
\]
Now we bound the term $v^\top \mathbb{E}_q[(X - \mu_q)(X - \mu_q)^\top]v$. We can see from Equation (449) and (450) that
\[
\mathbb{E}_q[(X - \mu_q, v)^2] = \mathbb{E}_q[(X - \mu_p, v)^2 + (\mu_p - \mu_q, v)^2 + 2(X - \mu_p, v)(\mu_p - \mu_q, v)] \\
\geq (1 - 2\epsilon)(1 - 2\rho_2 - 4\rho_1 \cdot (5\rho_1 + 4\sqrt{\rho_2 + 9\tau}) \\
> 1 - 2\epsilon - 2\rho_2 - 4\rho_1 \cdot (5\rho_1 + 4\sqrt{\rho_2 + 9\tau}) \\
> 1 - 2\tau - 2\rho_2 - 20\rho_1^2 - 16\rho_1\sqrt{\rho_2 + 9\tau}.
\] (455)

Combining two inequalities together, we see that when $\epsilon < 1/3$,
\[
v^\top \mathbb{E}_q[(X - \mu_q)(X - \mu_q)^\top]v = v^\top \mathbb{E}_{p_r}[((X - \mu_q)(X - \mu_q)^\top)v + v^\top \mathbb{E}_{r_+}[(X - \mu_q)(X - \mu_q)^\top]v \\
> 1 - 2\tau - 2\rho_2 - 20\rho_1^2 - 16\rho_1\sqrt{\rho_2 + 9\tau} + \frac{9(\rho_1 + \sqrt{\epsilon(\rho_2 + 9\tau)})^2}{\epsilon} \\
= 1 - 2\tau - 2\rho_2 - 20\rho_1^2 - 16\rho_1\sqrt{\rho_2 + 9\tau} + \frac{9\rho_1^2}{\epsilon} + 9(\rho_2 + 9\tau) + \frac{18\rho_1\sqrt{\rho_2 + 9\tau}}{\sqrt{\epsilon}} \\
> 1 + \tau,
\] (456)
which contradicts with the fact that $q \in G_2$. Thus we have
\[
\sup_{q \in G_2, p \in G_1, TV_H(q, p) \leq \epsilon} \| \mu_p - \mu_q \|_2 \leq 11\rho_1 + 9\sqrt{(\rho_2 + 9\tau)\epsilon}.
\] (457)

**E.6.2 General Convergence and Concentration results**

**Lemma E.5** (Convergence of mean for empirical distribution with bounded support and bounded second moment (Foucart and Rauhut, 2017, Corollary 8.45)). Given distribution $p$ satisfying the following conditions:
\[
\sup_{v \in \mathbb{R}^d, \|v\|_2 = 1} \mathbb{E}_p \left[ (v^\top (X - \mathbb{E}_p[X]))^2 \right] \leq \sigma^2, \tag{458}
\]
\[
\|X - \mathbb{E}_p[X]\|_2 \leq R \text{ a.s.}. \tag{459}
\]

Denote the empirical distribution of $n$ i.i.d. samples from $p$ as $\hat{p}_n$. Then with probability at least $1 - \delta$, there is some constant $C$ such that
\[
\|\mathbb{E}_p[X] - \mathbb{E}_{\hat{p}_n}[X]\|_2 \leq C \left( \sigma \sqrt{\frac{d}{n}} + \sigma \sqrt{\frac{\log(1/\delta)}{n}} + \frac{R \log(1/\delta)}{n} \right). \tag{460}
\]

**Lemma E.6** (Convergence of covariance for empirical distribution with bounded support (Vershynin, 2010, Thoerem 5.44)). Given distribution $p$, denote $\Sigma = \mathbb{E}_p[XX^\top]$. Assume that $\|X\|_2 \leq R$ almost surely. Denote $\Delta_1 = \sqrt{\frac{R^2 \log(d/\delta)}{n}}$. Then with probability at least $1 - \delta$, there exists some constant $C_1$ such that
\[
\left\| \frac{1}{n} \sum_{i=1}^n X_i X_i^\top - \Sigma \right\|_2 \leq C_1 \max(\|\Sigma\|_2^{1/2} \Delta_1, \Delta_1^2).
\] (461)
Denote $\Delta_2 = \sqrt{\frac{R^2 \log(d)}{n}}$. By integrating over $\delta$, we know that for some constant $C_2$

$$\mathbb{E}_p \left[ \left\| \frac{1}{n} \sum_{i=1}^{n} X_i X_i^\top - \mathbb{E}_p[XX^\top] \right\|_2 \right] \leq C_2 \max(\|\Sigma\|_2^{1/2} \Delta_2, \Delta_2^2). \quad (462)$$

The below lemma controls the tail of $\|X\|_2$ when $X$ has bounded $k$-th moment.

**Lemma E.7** (Tail bound for the norm of bounded $k$-th moment distribution). Assume distribution $p$ has its $k$-th moment bounded for $k \geq 2$, i.e.

$$\sup_{v \in \mathbb{R}^d, \|v\|_2 = 1} \mathbb{E}_p \left[ |v^\top X|^k \right] \leq \sigma^k. \quad (463)$$

Then,

$$\mathbb{P}_p (\|X\|_2 \geq t) \leq \left( \frac{\sigma \sqrt{d}}{t} \right)^k. \quad (464)$$

**Proof.** Since the $k$-th moment is bounded, by Chebyshev’s inequality and Khinchine’s inequality (Haagerup, 1981),

$$\mathbb{P}_p (\|X\|_2 \geq t) \leq \frac{\mathbb{E}_p \|X\|_2^k}{t^k} \leq \frac{\mathbb{E}_{X \sim p, \xi \sim \{\pm 1\}^d} |\xi^\top X|^k}{t^k}. \quad (465)$$

From the fact that $X$ has bounded $k$-th moment, we have

$$\mathbb{E}_{X \sim p, \xi \sim \{\pm 1\}^d} |\xi^\top X|^k \leq \sigma^k \mathbb{E}_{\xi \sim \{\pm 1\}^d} \|\xi\|_2^k = \sigma^k d^{k/2}. \quad (466)$$

Thus overall,

$$\mathbb{P}_p (\|X\|_2 \geq t) \leq \left( \frac{\sigma \sqrt{d}}{t} \right)^k. \quad (467)$$

**Remark E.3.** If we know the distribution $p$ is sub-Gaussian with parameter $\sigma$, then we know the $k$-th moment of $p$ is bounded by $(C \sigma \sqrt{k})^k$ for some constant $C$, i.e.

$$\sup_{v \in \mathbb{R}^d, \|v\|_2 = 1} \mathbb{E}_p [|v^\top X|^k] \leq (C \sigma \sqrt{k})^k. \quad (468)$$

Then we have a better bound from the Hanson-Wright inequality (Vershynin, 2018, Exercise 6.3.5),

$$\mathbb{P}(\|X\|_2 \geq C_1 \sigma \sqrt{d} + t) \leq \exp(-\frac{C_2 t^2}{\sigma^2}). \quad (469)$$

One can see that by taking $t = C_3 \sigma \sqrt{d}$, the above bound gives tail of $\exp(-Cd)$ while (464) only gives constant tail bound. It is an open problem whether we can do better given bounded $k$-th moment condition.
E.6.3 Some negative results on empirical distribution from bounded ψ-norm distribution

**Lemma E.8.** Suppose zero mean distribution $p$ on $\mathbb{R}^d$ satisfies

$$\mathbb{P}_p(\|X\| \geq C\sqrt{d}) \geq \frac{1}{2}, \quad (470)$$

where $C$ is some universal constant. Denote the empirical distribution of $n$ i.i.d. samples from $p$ as $\hat{p}_n$. Then, for any Orlicz function $\psi$ (Definition 2.2), the Orlicz norm of random vector $Y \sim \hat{p}_n$, defined as

$$\|Y\|_\psi \triangleq \sup_{v \in \mathbb{R}^d, \|v\|_2 = 1} \|v^\top Y\|_\psi, \quad (471)$$

satisfies

$$\mathbb{P}_p(\|Y\|_\psi \geq \frac{C\sqrt{d}}{\psi^{-1}(n)}) \geq \frac{1}{2}, \quad (472)$$

In particular, if we take $\psi(x) = x^k$, then Lemma E.8 shows that it requires at least $\Omega(d^{k/2})$ samples to ensure the empirical distribution $\hat{p}_n$ has constant $k$-th moment with probability $1/2$. Similarly, $\psi(x) = \exp(x^2) - 1$ corresponds to sub-Gaussian, which implies we would need at least $\exp(\Omega(d))$ number of samples to guarantee the empirical distribution has constant sub-Gaussian norm with probability $1/2$.

**Proof.** Denote the samples in $\hat{p}_n$ as $X_1, X_2, \ldots, X_n$. We have

$$\sup_{v \in \mathbb{R}^d, \|v\|_2 \leq 1} \mathbb{E}_{\hat{p}_n}[\psi(\|v^\top X\|/\sigma)] \geq \mathbb{E}_{\hat{p}_n}[\psi(\|X_1\|/\sigma)] \geq \frac{1}{n} \psi(\|X_1\|/\sigma), \quad (473)$$

where in the first inequality we have taken $v = \frac{X_1}{\|X_1\|}$. Hence, with probability at least $1/2$, we have

$$\sup_{v \in \mathbb{R}^d, \|v\|_2 \leq 1} \mathbb{E}_{\hat{p}_n}[\psi(\|v^\top X\|/\sigma)] \geq \frac{1}{n} \psi(C\sqrt{d}/\sigma). \quad (475)$$

If

$$\sigma = \frac{C\sqrt{d}}{\psi^{-1}(n)}, \quad (476)$$

then $\frac{1}{n} \psi(C\sqrt{d}/\sigma) \geq 1$. 

The next lemma shows that the lower bound in Lemma E.8 in the case of $\psi(x) = x^2$ (bounded second moment) is not tight: even if the distribution has bounded support ($\|x\| \leq \sqrt{d}$ almost surely), the sample size needed to ensure either resilience with the right parameter or bounded second moment is superlinear in $d$.

**Lemma E.9.** There exist a distribution $p$ on $\mathbb{R}^d$ with the following properties:

1. bounded support: for $X \sim p$, $\|X\| \leq \sqrt{d}$ almost surely;
2. identity population second moment:

$$E_p[XX^T] = I_d$$  \hspace{1cm} (477)

3. growing empirical second moment: let $\hat{p}_n$ denote the empirical distribution of $n$ i.i.d. samples $X_1, X_2, \ldots, X_n$ from $p$. Then, if $n = d$,

$$E_p[\| \frac{1}{n} \sum_{i=1}^n X_i X_i^\top \|] \asymp \frac{\ln d}{\ln \ln d};$$  \hspace{1cm} (478)

4. empirical distribution not resilient: if $n = d$, then there exists an absolute constant $C_1 > 0$ such that the following does not hold for any absolute constant $C_2 > 0$: $\hat{p}_n \in G_{TV}(C_2 \sqrt{\eta}, \eta)$ for $\eta = \frac{C_1 \ln n}{n \ln \ln n}$ with probability at least $1/2$.

Proof. Let $p$ be the distribution of $X$ that takes value $\sqrt{d} e_i$ with probability $1/d$, where $\{e_i\}_{i=1}^{d}$ is the standard basis in $\mathbb{R}^d$. Then, the third part follows from (Vershynin, 2018, Exercise 5.4.14).

Regarding the last statement, we have

$$\frac{1}{n} \sum_{i=1}^n X_i = \frac{\sqrt{d}}{n} \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_d \end{bmatrix},$$  \hspace{1cm} (479)

where $Z_i = \sum_{k=1}^n 1(X_k = \sqrt{d} e_i)$, and $(Z_1, Z_2, \ldots, Z_d) \sim \text{mult}(n, (1/d, 1/d, \ldots, 1/d))$. It follows from (Mitzenmacher and Upfal, 2017, Chapter 5) that if $n = d$ then $\max_{k \in [d]} Z_k$ tightly concentrates on $\frac{\ln n}{\ln \ln n}$.

Without loss of generality assume $Z_1 = \max_{k \in [d]} Z_k$. Consider the deletion operation that deletes all samples that contribute to the counts in $Z_1$. Hence the deletion fraction is $\frac{Z_1}{n}$. Since $Z_1$ tightly concentrates around $\frac{\ln n}{\ln \ln n} \gg 1$ as $n \to \infty$, there exist two absolute constants $c_1, c_2$ such that $\frac{c_1 \ln n}{\ln \ln n} \leq Z_1 \leq \frac{c_2 \ln n}{\ln \ln n}$ with probability at least $1/2$ for $n$ large enough. From now on we condition on this event. Denote the empirical distribution of remaining samples as $r$. Then, take $n = d$,

$$\|E_{\hat{p}_n}[X] - E_r[X]\| \geq |c_1^T (E_{\hat{p}_n}[X] - E_r[X])| = \frac{\sqrt{d}}{n} Z_1 = \frac{Z_1}{\sqrt{n}} \approx \frac{\ln n}{\sqrt{n} \ln \ln n} \gg \frac{Z_1}{n} \times \sqrt{\frac{\ln n}{n \ln \ln n}}.$$

\hspace{1cm} (480)

\begin{proof}

\end{proof}

\section*{E.6.4 Empirical distribution from bounded $\psi$-norm distributions is resilient for all $\eta$}

In Lemma E.9, it is shown that given $n = d$ samples from a distribution with bounded second moment and bounded support, the empirical distribution is not resilient for small $\eta$ with the right rate. However, we show in the lemma below that the empirical distribution is in fact resilient with the right rate for $\eta$ large enough. In particular when $\eta \geq \frac{cd}{n}$ for some $c > 0$ under bounded second moment and bounded support assumptions.

\textbf{Lemma E.10.} For a given Orlicz function $\psi$, we define

$$\mathcal{G} = \{ p \mid \sup_{f \in F} E_p \left[ \psi \left( \frac{|f(X) - E_p[f(X)]|}{\sigma} \right) \right] \leq 1 \}.$$  \hspace{1cm} (481)

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Assume \( p \in \mathcal{G} \), and the empirical distribution \( \hat{p}_n \) of \( n \) i.i.d. samples from \( p \) satisfies

\[
\mathbb{E}_p [W_F(p, \hat{p}_n)] \leq \xi_n. \tag{482}
\]

Assume the following equations have solutions, denoted as \( x_0, t \):

\[
\sigma x_0 \psi^{-1}(1/x_0) = \xi_n, \tag{483}
\]

\[
4\psi'(\psi^{-1}(\frac{t}{x_0})) x_0 \psi^{-1}(\frac{1}{x_0}) = t. \tag{484}
\]

We assume that for any \( \epsilon, t > 0 \), there exists some \( C_t \) that only depends on \( \psi \) and \( t \) such that

\[
\psi^{-1}(t/\epsilon) \leq C_t \psi^{-1}(1/\epsilon). \tag{485}
\]

Define

\[
\rho_\delta(\eta) = \frac{C_t + 2}{1 - \eta} \left( \sigma \eta \psi^{-1}(1/\delta \eta) + \frac{\xi_n}{\delta} \right). \tag{486}
\]

Then with probability at least \( 1 - 2\delta \),

\[
\hat{p}_n \in \bigcap_{\eta \in [0,1]} \mathcal{G}^\text{TV}_{\psi,F}(\rho_\delta(\eta), \eta) = \{ p \mid \forall \eta \in [0,1), \sup_{f \in F, \sigma \leq \frac{\xi_n}{\eta}} |\mathbb{E}_p[f(X)] - \mathbb{E}_r[f(X)]| \leq \rho_\delta(\eta) \}. \tag{487}
\]

Here the first term \( \sigma \eta \psi^{-1}(1/\sigma \eta) \) is close to the population limit in Lemma C.2, and the second term \( \xi_n/\delta \) is similar to the finite sample error bound without corruption.

**Proof.** We use the similar technique as Lemma C.2 to show that \( \hat{p}_n \in \mathcal{G}' \) with high probability. Note that \( x_0 > 0 \) is defined as the solution to the following equation:

\[
\sigma x \psi^{-1}(1/x) = \xi_n. \tag{488}
\]

We then define a convex function \( \tilde{\psi} \) for \( t > 0 \) as

\[
\tilde{\psi}(x) = \begin{cases} 
\psi(x), & 0 \leq x \leq \psi^{-1}(\frac{t}{x_0}), \\
\psi'(\psi^{-1}(\frac{1}{x_0}))(x - \psi^{-1}(\frac{1}{x_0})) + \frac{t}{x_0}, & x > \psi^{-1}(\frac{1}{x_0}).
\end{cases} \tag{489}
\]

One can see that \( \tilde{\psi} \) is convex, non-negative, non-decreasing and \( \tilde{\psi}(|x|) \leq \psi(|x|) \). Hence,

\[
\tilde{\psi}^{-1}(\frac{1}{x}) = \begin{cases} 
\frac{1}{\psi'(\psi^{-1}(t/x_0))} - \frac{t}{x_0 \psi'(\psi^{-1}(t/x_0))} + \psi^{-1}(\frac{t}{x_0}), & 0 \leq x \leq \frac{x_0}{t}, \\
\psi^{-1}(\frac{1}{x}), & x > \frac{x_0}{t}.
\end{cases} \tag{490}
\]

Note that from Lemma A.1, we know that \( x \tilde{\psi}^{-1}(1/x) \) is non-decreasing, and

\[
x \tilde{\psi}_t^{-1}(\frac{t}{x}) \leq \begin{cases} 
x_0 \psi^{-1}(\frac{t}{x_0}), & 0 \leq x \leq x_0, \\
x \psi^{-1}(\frac{1}{x}), & x > x_0.
\end{cases} \tag{491}
\]

Now we bound the term:

\[
\sup_{f \in F} \mathbb{E}_{\hat{p}_n} \left[ \tilde{\psi} \left( \frac{|f(X) - \mathbb{E}_p[f(X)]|}{\sigma} \right) \right] = \sup_{f \in F} \frac{1}{n} \sum_{i=1}^{n} \tilde{\psi} \left( \frac{|f(X_i) - \mathbb{E}_p[f(X)]|}{\sigma} \right) \tag{492}
\]

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By \( p \in \mathcal{G} \), \( \sup_{f \in \mathcal{F}} \mathbb{E}_p \left[ \psi \left( \frac{|f(X) - \mathbb{E}_p[f(X)]|}{\sigma} \right) \right] \leq 1 \), we have

\[
\mathbb{E}_p \left[ \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \psi \left( \frac{|f(X_i) - \mathbb{E}_p[f(X)]|}{\sigma} \right) \right] \\
\leq \mathbb{E}_p \left[ \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \psi \left( \frac{|f(X_i) - \mathbb{E}_p[f(X)]|}{\sigma} \right) \right] - \mathbb{E}_p \left[ \psi \left( \frac{|f(X) - \mathbb{E}_p[f(X)]|}{\sigma} \right) \right] \\
+ \sup_{f \in \mathcal{F}} \mathbb{E}_p[\psi \left( \frac{|f(X) - \mathbb{E}_p[f(X)]|}{\sigma} \right)] \tag{493}
\]

\[
\leq 2 \mathbb{E}_{p, \epsilon \sim \{\pm 1\}^n} \left[ \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \epsilon_i \psi \left( \frac{|f(X_i) - \mathbb{E}_p[f(X)]|}{\sigma} \right) \right] + 1 \tag{494}
\]

\[
\leq 2 \psi'(\psi^{-1}(\frac{t}{x_0})) \mathbb{E}_{p, \epsilon \sim \{\pm 1\}^n} \left[ \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} \left( \frac{f(X_i) - \mathbb{E}_p[f(X)]}{\sigma} \right) \right] + 1 \tag{495}
\]

\[
\leq 4 \psi'(\psi^{-1}(\frac{t}{x_0})) \mathbb{E}_p \left[ \frac{1}{n} \sum_{i=1}^{n} \left( \frac{f(X_i) - \mathbb{E}_p[f(X)]}{\sigma} \right) \right] + 1 \tag{496}
\]

\[
= 4 \psi'(\psi^{-1}(\frac{t}{x_0})) \mathbb{E}_p \left[ \frac{1}{n} \sum_{i=1}^{n} \frac{f(X_i) - \mathbb{E}_p[f(X)]}{\sigma} \right] + 1 \tag{497}
\]

\[
\leq 4 \psi'(\psi^{-1}(\frac{t}{x_0})) \mathbb{E}_p \left[ \frac{1}{n} \sum_{i=1}^{n} \frac{f(X_i) - \mathbb{E}_p[f(X)]}{\sigma} \right] + 1 \tag{498}
\]

Here Equation (494) is from triangle inequality of sup. Equation (495) and Equation (497) are from symmetrization inequality (Wainwright, 2019, Proposition 4.11). Equation (496) is from Talagrand contraction inequality (Vershynin, 2018, Exercise 6.7.7).

Now we apply a similar argument in Lemma C.2 to show that \( \hat{p}_n \) is in the resilient set induced by \( \tilde{\psi} \).

For any event \( E \), denote its compliment as \( E^c \), by the definition of conditional expectation and symmetry of \( \mathcal{F} \),

\[
\sup_{f \in \mathcal{F}} \mathbb{P}_{\hat{p}_n}(E)(\mathbb{E}_{\hat{p}_n}[f(X)|E] - \mathbb{E}_{\hat{p}_n}[f(X)]) = \sup_{f \in \mathcal{F}} \mathbb{P}_{\hat{p}_n}(E^c)(\mathbb{E}_{\hat{p}_n}[f(X) - \mathbb{E}_{\hat{p}_n}[f(X)|E^c]]) \tag{500}
\]

Thus we have

\[
\sup_{f \in \mathcal{F}} \mathbb{E}_{\hat{p}_n}[f(X)|E] - \mathbb{E}_{\hat{p}_n}[f(X)] = \sup_{f \in \mathcal{F}} \left\{ \frac{\mathbb{P}_{\hat{p}_n}(E^c)}{1 - \mathbb{P}_{\hat{p}_n}(E^c)} \mathbb{E}_{\hat{p}_n}[f(X) - \mathbb{E}_{\hat{p}_n}[f(X)] | E^c] \right\} \\
\leq \sup_{f \in \mathcal{F}} \left\{ \frac{\mathbb{P}_{\hat{p}_n}(E^c)}{1 - \mathbb{P}_{\hat{p}_n}(E^c)} \mathbb{E}_{\hat{p}_n}[f(X) - \mathbb{E}_p[f(X)] | E^c] \right\} \\
+ \sup_{f \in \mathcal{F}} \left\{ \frac{\mathbb{P}_{\hat{p}_n}(E^c)}{1 - \mathbb{P}_{\hat{p}_n}(E^c)} \mathbb{E}_{\hat{p}_n}[f(X)] - \mathbb{E}_p[f(X)] \right\} \\
= \sup_{f \in \mathcal{F}} \left\{ \frac{\mathbb{P}_{\hat{p}_n}(E^c)}{1 - \mathbb{P}_{\hat{p}_n}(E^c)} \mathbb{E}_{\hat{p}_n}[f(X) - \mathbb{E}_p[f(X)] | E^c] \right\} \\
+ \frac{\mathbb{P}_{\hat{p}_n}(E^c)}{1 - \mathbb{P}_{\hat{p}_n}(E^c)} \mathcal{W}(p, \hat{p}_n). \tag{501}
\]

We then control the first term in RHS. From Equation (497), by Markov’s inequality, we know that
for any \( \eta \in [0, 1) \), with probability at least \( 1 - \delta \)

\[
\frac{4\psi'(\psi^{-1}(\frac{t}{x_0})) \frac{\xi_n}{\eta} + 1}{\delta} \geq \sup_{f \in \mathcal{F}} \mathbb{E}_{\hat{p}_n} \left[ \tilde{\psi} \left( \frac{|f(X) - \mathbb{E}_p[f(X)]|}{\sigma} \right) \right]
\]

\[
\geq \sup_{\mathbb{P}_{\hat{p}_n}(E) \geq 1 - \eta, f \in \mathcal{F}} \mathbb{P}_{\hat{p}_n}(E^c) \mathbb{E}_{\hat{p}_n} \left[ \tilde{\psi} \left( \frac{|f(X) - \mathbb{E}_p[f(X)]|}{\sigma} \right) \right] | E^c
\]

\[
\geq \sup_{\mathbb{P}_{\hat{p}_n}(E) \geq 1 - \eta, f \in \mathcal{F}} \mathbb{P}_{\hat{p}_n}(E^c) \tilde{\psi} \left( \mathbb{E}_{\hat{p}_n} \left[ \frac{|f(X) - \mathbb{E}_p[f(X)]|}{\sigma} \right] | E^c \right). (502)
\]

This gives us with probability at least \( 1 - \delta \),

\[
\sup_{\mathbb{P}_{\hat{p}_n}(E) \geq 1 - \eta, f \in \mathcal{F}} \mathbb{E}_{\hat{p}_n} [f(X)|E] - \mathbb{E}_{\hat{p}_n} [f(X)]
\]

\[
\leq \sup_{\mathbb{P}_{\hat{p}_n}(E) \geq 1 - \eta} \frac{\sigma \mathbb{P}_{\hat{p}_n}(E^c)}{1 - \mathbb{P}_{\hat{p}_n}(E^c)} \tilde{\psi}^{-1} \left( \frac{4\psi'(\psi^{-1}(\frac{t}{x_0})) \frac{\xi_n}{\eta} + 1}{\delta \mathbb{P}_{\hat{p}_n}(E^c)} \right) + \frac{\eta}{1 - \eta} W_F(p, \hat{p}_n)
\]

\[
= \frac{\sigma \eta}{1 - \eta} \tilde{\psi}^{-1} \left( \frac{4\psi'(\psi^{-1}(\frac{t}{x_0})) x_0 \psi^{-1}(\frac{1}{x_0})}{\delta \eta} \right) + \frac{\sigma \eta}{1 - \eta} \tilde{\psi}^{-1} \left( \frac{4\psi'(\psi^{-1}(\frac{t}{x_0})) x_0 \psi^{-1}(\frac{1}{x_0})}{\delta \eta} \right) + \frac{\eta}{1 - \eta} W_F(p, \hat{p}_n). (503)
\]

Equation (503) uses the fact that \( x \psi^{-1}(b/x) \) is a non-decreasing function in \( [0, 1) \) for any \( b > 0 \) in Lemma A.1. Equation (504) is from the concave and non-negative property of \( \psi^{-1} \). By Markov’s inequality, we know that with probability at least \( 1 - \delta \), we have \( W_F(p, \hat{p}_n) \leq \frac{\xi_n}{\delta} \). By union bound we have for any \( \eta \in [0, 1) \), with probability at least \( 1 - 2\delta \),

\[
\sup_{\mathbb{P}_{\hat{p}_n}(E) \geq 1 - \eta, f \in \mathcal{F}} \mathbb{E}_{\hat{p}_n} [f(X)|E] - \mathbb{E}_{\hat{p}_n} [f(X)] \leq \frac{\sigma \eta}{1 - \eta} \tilde{\psi}^{-1} \left( \frac{4\psi'(\psi^{-1}(\frac{t}{x_0})) x_0 \psi^{-1}(\frac{1}{x_0})}{\delta \eta} \right) + \frac{\sigma \eta}{1 - \eta} \tilde{\psi}^{-1} \left( \frac{4\psi'(\psi^{-1}(\frac{t}{x_0})) x_0 \psi^{-1}(\frac{1}{x_0})}{\delta \eta} \right) + \frac{\eta}{1 - \eta} \frac{\xi_n}{\delta}. (505)
\]

Note that \( t \) is the solution to

\[
t = 4\psi'(\psi^{-1}(\frac{t}{x_0})) x_0 \psi^{-1}(\frac{1}{x_0}). (506)
\]

Denote \( \tilde{\rho}_\delta(\eta) \) as

\[
\tilde{\rho}_\delta(\eta) = \frac{1}{1 - \eta} (\sigma \eta \tilde{\psi}^{-1}(\frac{t}{x_0}) + \sigma \eta \tilde{\psi}^{-1}(\frac{1}{x_0}) + \frac{\eta \xi_n}{\delta}). (507)
\]

So far we have shown that \( \hat{p}_n \in \bigcap_{\eta \in [0, 1]} \mathcal{G}_{W_F}^{TV}(\tilde{\rho}_\delta(\eta), \eta) \) Now we show that \( \bigcap_{\eta \in [0, 1]} \mathcal{G}_{W_F}^{TV}(\hat{\rho}_\delta(\eta), \eta) \subset \bigcap_{\eta \in [0, 1]} \mathcal{G}_{W_F}^{TV}(\rho_\delta(\eta), \eta) \), where

\[
\rho_\delta(\eta) = \frac{C_1 + 2}{1 - \eta} \left( \sigma \eta \psi^{-1}(\frac{1}{\eta \delta}) + \frac{\xi_n}{\delta} \right). (508)
\]

From Equation (491),

\[
\frac{\eta}{1 - \eta} \tilde{\psi}^{-1}(\frac{t}{x_0}) \leq \left\{ \begin{array}{ll}
\frac{x_0}{\eta (1 - \eta \delta)} \psi^{-1}(\frac{t}{x_0}) & 0 \leq \eta \leq \frac{x_0}{\delta}, \\
\frac{\eta}{1 - \eta} \tilde{\psi}^{-1}(\frac{t}{x_0}) & \eta > \frac{x_0}{\delta}.
\end{array} \right. (509)
\]

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From Equation (485), we know that for any $\eta$,

$$\psi^{-1}\left(\frac{t}{\delta\eta}\right) \leq C_t\psi^{-1}\left(\frac{1}{\delta\eta}\right). \quad (510)$$

These two equations combined show that

$$\tilde{\rho}_\delta(\eta) \leq \frac{(C_t + 1)}{1 - \eta} \left(\frac{1}{\delta\eta}\right) + \frac{\eta\xi_n}{(1 - \eta)\delta} = \rho_\delta(\eta). \quad (511)$$

As a corollary, we can easily show that the empirical distribution for bounded $k$-th moment distribution is inside resilience family:

**Corollary E.1.** Suppose for $k \geq 2$,

$$G = \{ p \mid \sup_{v \in \mathbb{R}^d, \|v\|_2 = 1} \mathbb{E}_p \left[ |\langle X - \mathbb{E}_p[X], v \rangle|^k \right] \leq \sigma^k \}. \quad (512)$$

Define

$$\rho_\delta(\eta) = Ck\sigma \left(\frac{\eta^{1-1/k}}{\delta^{1/k}} + \frac{1}{\delta} \sqrt{\frac{d}{n}} \right). \quad (513)$$

Then with probability at least $1 - 2\delta$, the empirical distribution $\hat{p}_n$ of $n$ i.i.d. samples from $p$ satisfies

$$\hat{p}_n \in \bigcap_{\eta \in [0, 1]} G^{TV}_{\text{mean}}(\rho_\delta(\eta), \eta) = \{ p \mid \forall \eta \in [0, 1), \sup_{r \leq \eta} \|\mathbb{E}_p[X] - \mathbb{E}_r[X]\|_2 \leq \rho_\delta(\eta) \}. \quad (514)$$

**Proof.** We first check the conditions required in Lemma E.10. Here $\psi(x) = x^k$ for $k \geq 2$. From Lemma A.4 we have

$$\mathbb{E}_{p^*}[\|\mathbb{E}_{p^*}[X] - \mathbb{E}_{p_n}[X]\|_2] \leq \sigma \sqrt{\frac{d}{n}}. \quad (515)$$

One can solve $x_0 = \left(\frac{d}{n}\right)^{k/2(k-1)}, t = k^k$ from the following equations:

$$\sigma x_0 \psi^{-1}\left(\frac{1}{x_0}\right) = \xi_n, \quad (516)$$

$$4\psi'(\psi^{-1}\left(\frac{t}{x_0}\right))x_0 \psi^{-1}\left(\frac{1}{x_0}\right) = t. \quad (517)$$

Then for any $\eta \in [0, 1)$, we have

$$\psi^{-1}\left(\frac{t}{\eta}\right) \leq k\psi^{-1}\left(\frac{1}{\eta}\right) \quad (518)$$
for some universal constant $C$. Then when $\delta > x_0$, with probability at least $1 - 2\delta$,
$$
\hat{p}_n^* \in \mathcal{G}.
$$
(519)

Then by Lemma E.10, we know that with probability at least $1 - 2\delta$, for any $r \leq \hat{p}_n^*$,
$$
\|\mu_r - \mu_{\hat{p}_n^*}\|_2 \leq C k \sigma \left( \frac{\eta^{1-1/k}}{\delta^{1/k}} + \frac{1}{\delta} \sqrt{\frac{d}{n}} \right).
$$
(520)

\[ \Box \]

E.6.5 Empirical distribution from distributions with moment generating function is resilient with better rate for some $\eta$

The next lemma shows that the empirical distribution of i.i.d. samples from distributions with moment generating functions is resilient with a fixed $\eta$, whose dependence of the parameters on $\delta$ is better than that in Lemma E.10. It is a generalization of (Diakonikolas et al., 2019a, Lemma 4.4).

**Lemma E.11.** Let $\psi$ be some convex and continuously differentiable function on $[0, b]$ with $0 < b \leq \infty$, such that $\psi(0) = \psi'(0) = 0$. Assume that for $\lambda \in (0, b)$,
$$
\sup_{v \in \mathbb{R}^d, \|v\|_2 = 1} \ln(\mathbb{E}_p[\exp(\lambda(v^\top X - \mathbb{E}_p[v^\top X])]]) \leq \psi(\lambda).
$$
(521)

Denote by $\psi^*(x)$ the Fenchel–Legendre dual of $\psi$:
$$
\psi^*(x) = \sup_{\lambda \in (0,b)} (\lambda x - \psi(\lambda)).
$$
(522)

Fix $\eta \in [0, 1)$. Then, there exists an absolute constant $C > 0$ such that with probability at least $1 - \delta$,
$$
\hat{p}_n^* \in \{ p \mid \sup_{r \leq \frac{p}{1-\eta}} \|\mathbb{E}_p[X] - \mathbb{E}_r[X]\| \leq \rho \},
$$
(523)

where
$$
\rho = \max \left\{ \frac{4\eta}{1-\eta} \psi^{-1}\left( \frac{C d + \ln(2/\delta) + n h(\eta)}{n \eta} \right), \frac{4\eta}{1-\eta} \psi^{-1}\left( \frac{C d + \ln(2/\delta)}{n} \right) \right\},
$$
(524)

$h(p) = p \ln(1/p) + (1 - p) \ln(1/(1-p))$ is the binary entropy function, and $\psi^{-1}$ is the generalized inverse of $\psi^*$ (14). In particular, if $\psi(\lambda) = \frac{\lambda^2 \sigma^2}{2}$, and $\eta \in [0, 1/2]$, then one can take
$$
\rho = C \sigma \cdot \left( \sqrt{\eta} \sqrt{\frac{d + \ln(1/\delta)}{n}} + \eta \sqrt{\ln(1/\eta)} \right),
$$
(525)

where $C$ is some universal constant.

**Proof.** Throughout this proof the constant $C$ may be different from line by line, but is always an absolute constant. It follows from the Chernoff method (Boucheron et al., 2013, Page 24) that for any $v \in \mathbb{R}^d, \|v\|_2 = 1$, we have for any $t \geq 0$,
$$
\mathbb{P}\left( \frac{1}{n} \sum_{i=1}^n v^\top X_i - v^\top \mathbb{E}_p[X] \geq t \right) \leq \exp(-n\psi^*(t)).
$$
(526)
It follows from (Vershynin, 2018, Corollary 4.2.13) that we can take a net of vectors $\mathcal{N} \subset \{v \mid v \in \mathbb{R}^d, \|v\|_2 = 1\}$ such that $|\mathcal{N}| \leq C^d$ and that for any $x$, $\sup_{v \in \mathcal{N}} v^T x \geq \frac{1}{2}\|x\|_2$. Then it follows from the union bound that

$$
P \left( \sup_{v \in \mathbb{R}^d, \|v\|_2=1} \frac{1}{n} \sum_{i=1}^n v^T X_i - v^T \mathbb{E}_p[X] \geq t \right) \leq \P \left( \sup_{v \in \mathcal{N}} \frac{1}{n} \sum_{i=1}^n v^T X_i - v^T \mathbb{E}_p[X] \geq \frac{t}{2} \right) \leq \sum_{v \in \mathcal{N}} \P \left( \frac{1}{n} \sum_{i=1}^n v^T X_i - v^T \mathbb{E}_p[X] \geq \frac{t}{2} \right) \leq |\mathcal{N}| \exp(-n\psi^*(t/2)) \leq \exp(Cd - n\psi^*(t/2)).$$

(527)

(528)

(529)

(530)

Denote $\hat{\mu}_n = \frac{1}{n} \sum_{i=1}^n X_i$. From now on we assume $\eta n$ is an integer. Our goal is to find the parameter $\rho$ such that

$$
P \left( \sup_{J \subset [n], |J| \geq (1-\eta)n} \left\| \frac{1}{|J|} \sum_{i \in J} X_i - \hat{\mu}_n \right\| \geq \rho \right) \leq \delta. \quad (531)$$

It follows from a replacement argument that it suffices to consider only those $J$ such that $|J| = (1-\eta)n$. Noting that for any $J \subset [n], |J| = (1-\eta)n$,

$$
\frac{1}{|J|} \sum_{i \in J} X_i - \hat{\mu}_n = \frac{\eta}{1-\eta} \left( \hat{\mu}_n - \frac{1}{\eta n} \sum_{i \notin J} X_i \right). \quad (532)
$$

Hence,

$$
P \left( \sup_{J \subset [n], |J| \geq (1-\eta)n} \left\| \frac{1}{|J|} \sum_{i \in J} X_i - \hat{\mu}_n \right\| \geq \rho \right)
= P \left( \sup_{J \subset [n], |J| = (1-\eta)n} \left\| \frac{1}{|J|} \sum_{i \in J} X_i - \hat{\mu}_n \right\| \geq \rho \right) \leq \P \left( \frac{\eta}{1-\eta} \| \hat{\mu}_n - \mathbb{E}_p[X] \| \geq \rho/2 \right) + P \left( \sup_{J \subset [n], |J| = (1-\eta)n} \frac{\eta}{1-\eta} \left\| \frac{1}{\eta n} \sum_{i \notin J} X_i - \mathbb{E}_p[X] \right\| \geq \rho/2 \right) \quad (533)
$$

$$
\leq \exp(Cd - n\psi^*(\rho(1-\eta)/(4\eta))) + \exp(nh(\eta) + Cd - \eta n\psi^*(\rho(1-\eta)/(4\eta))), \quad (534)
$$

where in the last step we used the inequality $\binom{n}{j} \leq \exp(nh(k/n))$, where $h(p) = p\ln(1/p) + (1-p)\ln(1/(1-p))$ is the binary entropy function. It now suffices to guarantee that

$$
n\psi^*(\rho(1-\eta)/(4\eta)) - Cd \geq \ln(2/\delta) \quad (535)$$

$$
\eta \psi^*(\rho(1-\eta)/(4\eta)) - nh(\eta) - Cd \geq \ln(2/\delta). \quad (536)$$

$$
\eta \psi^*(\rho(1-\eta)/(4\eta)) - nh(\eta) - Cd \geq \ln(2/\delta). \quad (537)$$

$$
\eta \psi^*(\rho(1-\eta)/(4\eta)) - nh(\eta) - Cd \geq \ln(2/\delta). \quad (538)$$

$$
\eta \psi^*(\rho(1-\eta)/(4\eta)) - nh(\eta) - Cd \geq \ln(2/\delta). \quad (539)$$
It we choose \( \rho \) such that
\[
\rho \geq \frac{4n}{1 - \eta} \left( \frac{C + \ln(2/\delta) - n(\eta)}{n} \right),
\]
the two bounds above would be satisfied.

**E.6.6 Empirical distribution from distributions with bounded \( k \)-th moment and bounded support has bounded covariance and is resilient for all \( \eta \)**

The lemma below shows that if we delete the distribution with bounded \( k \)-th moment, we are able to guarantee good properties of the deleted distribution that is required in the modulus of continuity in Lemma E.3, which holds for all \( \eta \in [0,1/k] \).

**Lemma E.12 (Properties of deleted distribution of bounded \( k \)-th moment).** Assume the distribution \( p^* \) has identity covariance and has its \( k \)-th moment bounded by \( \sigma \) for \( k \geq 2 \), i.e.
\[
\sup_{v \in \mathbb{R}^d, ||v||_2=1} E_{p^*} [v^T (X - E_{p^*}[X])] \leq \sigma^k, E_{p^*} [(X - E_{p^*}[X])(X - E_{p^*}[X])^T] = I_d.
\]

For any fixed \( \eta \in [0,1/2] \), we define a new distribution \( p' \) satisfying that for any event \( A \),
\[
P_{p'}(X \in A) = P_p(X \in A | \|X - \mu_p\| \leq \sigma \sqrt{d/\eta^{1/k}}).
\]

Denote the empirical distribution of \( n \) i.i.d. samples from \( p' \) as \( \hat{p}' \). Denote \( \Delta_1 = \sqrt{\frac{\sigma^2 d \log(d/\delta)}{np^{2/k}}} \), \( \Delta_2 = \sqrt{\frac{\sigma^2 d \log(d)}{np^{2/k}}} \), and
\[
G_1(\eta) = \left\{ p | \forall r \leq \frac{p}{1 - \eta}, \|E_r[X] - E_p[X]\|_2 \leq C_1 k \sigma \left( \left( \frac{\eta^{1-1/k}}{\delta^{1/k}} + \frac{1}{\delta} \sqrt{\frac{d}{n}} \right) \right) \right\},
\]
\[
G_2(\eta) = \left\{ p | \|E_p[(X - \mu_p)(X - \mu_p)^T]\|_2 \leq \frac{1}{1 - \eta} + C_3 \max(\Delta_1, \Delta_2) \right\},
\]
\[
G_3(\eta) = \left\{ p | \forall r \leq \frac{p}{1 - \eta}, \|E_r[(X - E_p[X])(X - E_p[X])^T] - I_d\|_2 \leq C_2 k \left( \frac{\sigma_2 \eta^{1-1/k}}{\delta^{2/k}} + \max(\Delta_2, \Delta_2) \right) \right\},
\]

where \( C_1, C_2, C_3 \) are some universal constants. Then the following holds:

1. \( \text{TV}(p^*, p') \leq \eta, \)
2. With probability at least \( 1 - 2\delta \), \( p' \in G_1(\eta), \)
3. With probability at least \( 1 - \delta \), \( p' \in G_2(\eta), \)
4. If \( k > 2 \), with probability at least \( 1 - 6\delta \), \( p' \in G_3(\eta). \)

**Proof.** We show the four conclusions separately.
1. From Lemma E.7, we know that
\[
\mathbb{P}_{p^*}(\|X - \mathbb{E}_{p^*}[X]\|_2 \geq t) \leq \frac{\sigma^k d^{k/2}}{t^k}.
\] (545)
By taking \( t = \sigma \sqrt{d/\eta^{1/k}} \), we have
\[
\mathbb{P}_{p^*}(\|X - \mathbb{E}_{p^*}[X]\|_2 \geq \sigma \sqrt{d/\eta^{1/k}}) \leq \eta.
\] (546)
Thus we know \( \text{TV}(p^*, p') \leq \eta \).

2. From Lemma C.7, we know that Orlicz-norm bounded function is approximately closed under deletion. From now we condition on the event that \( \text{TV}(p', p^*) \leq \eta \). Since \( p' \) is a deletion of \( p^* \), by Lemma C.7 we have for any \( \eta \in [0, 1/2] \)
\[
\sup_{v \in \mathbb{R}^d, \|v\|_2 = 1} \mathbb{E}_{p'} \left[ \left( \frac{|v^\top (X - \mu_{p^*})|}{5\sigma} \right)^k \right] \leq \frac{1}{1 - \eta}.
\] (547)
From Lemma A.3 (Centering) and Corollary E.1, we know that with probability at least \( 1 - 2\delta \), for any \( \eta \in [0, 1/2] \) and any \( r \leq \frac{\rho'}{1 - \eta} \),
\[
\|\mu_r - \mu_{\rho'}\|_2 \leq C_1 k \sigma \left( \frac{1}{1 - \eta} \left( \frac{\eta^{1-1/k}}{d^{1/k}} + \frac{\sqrt{d}}{\sqrt{n}} \right) \right),
\] (548)
where \( C_1 \) is some universal constant. This shows that \( \rho' \in \mathcal{G}_1(\eta) \) with probability at least \( 1 - 2\delta \).

3. Since \( p^* \) is inside \( \mathcal{G}_{\text{mean}}^{TV} \) and \( p' \) is a \( \eta \) deletion, by definition of resiliense 19 and Lemma C.2 we have
\[
\|\mu_{p'} - \mu_{p^*}\|_2 \leq 2\sigma \eta^{1-1/k}.
\] (549)
Furthermore, by \( p' \) having bounded \( k \)-th moment and bounded support, from Lemma E.5, we have with probability at least \( 1 - \delta \),
\[
\|\mu_{p'} - \mu_{\rho'}\|_2 \leq C \left( \sqrt{\frac{d}{n}} + \sqrt{\frac{\log(1/\delta)}{n}} + \frac{\sqrt{d \log(1/\delta)}}{n \eta^{1/k}} \right).
\] (550)
Denote \( \Delta_1 = \sqrt{\frac{\sigma^2 d (\log(d) + \log(1/\delta))}{n \eta^{1/k}}} \). From Lemma E.6, we know that with probability at least \( 1 - \delta \),
\[
\|\mathbb{E}_{p'}[(X - \mu_{p^*})(X - \mu_{p^*})^\top] - \mathbb{E}_{p'}[(X - \mu_{p^*})(X - \mu_{p^*})^\top]\|_2 \leq C \max(\Delta_1, \Delta_1^2).
\] (551)
Thus we know that with probability at least \( 1 - \delta \),
\[
\|\mathbb{E}_{p'}[(X - \mu_{\rho'})(X - \mu_{\rho'})^\top]\|_2 \leq \|\mathbb{E}_{p'}[(X - \mu_{p^*})(X - \mu_{p^*})^\top]\|_2 \leq \|\mathbb{E}_{p'}[(X - \mu_{p^*})(X - \mu_{p^*})^\top]\|_2 + C \max(\Delta_1, \Delta_1^2)
\]
\[
\leq \frac{1}{1 - \eta} + C \max(\Delta_1, \Delta_1^2).
\] (552)
This shows that with probability at least \( 1 - \delta \), \( \rho' \in \mathcal{G}_2(\eta) \).

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4. When \( k > 2 \), denote \( \Delta_2 = \sqrt{\frac{\sigma^2 d \log(d)}{n \eta^2 / k}} \). From Lemma E.6, we also know that with probability at least \( 1 - \delta \),

\[
\mathbb{E}_{\rho'}[\|\mathbb{E}_{\rho'}[(X - \mu_{\rho'}) (X - \mu_{\rho'})^\top] - \mathbb{E}_{\rho'}[(X - \mu_{\rho'}) (X - \mu_{\rho'})^\top]\|_2] \leq C \max(\Delta_2, \Delta_2^2).
\] (553)

for some constant \( C \). By Lemma C.7 we know that the distribution of \((v^\top (X - \mathbb{E}_{\rho'}[X]))^2\) has its \((1 - \eta)x^{k/2}\)-norm bounded by \(5 \sigma\) under \( X \sim \rho'\). From centering lemma in Lemma A.3 we know that

\[
\sup_{v \in \mathbb{R}^d, \|v\|_2 = 1} (1 - \eta)\mathbb{E}_{\rho'} \left( (v^\top (X - \mathbb{E}_{\rho'}[X]))^2 - \mathbb{E}_{\rho}(v^\top (X - \mathbb{E}_{\rho}[X]))^2 \right)^{k/2} \leq (5 \sigma)^k.
\] (554)

This combined with (553) and Lemma E.10 gives that when \( k > 2 \), with probability at least \( 1 - 2\delta \), for any \( r \leq \frac{\rho'}{\tau' \eta} \),

\[
\|\mathbb{E}_r[(X - \mathbb{E}_{\rho'}[X]) (X - \mathbb{E}_{\rho'}[X])^\top] - \mathbb{E}_{\rho'}[(X - \mathbb{E}_{\rho'}[X]) (X - \mathbb{E}_{\rho'}[X])^\top]\|_2 \leq \frac{C_2 k}{1 - \eta} \left( \frac{\sigma^2 \eta^{1-2/k}}{\delta^{2/k}} + \max(\Delta_2, \Delta_2^2) \right).
\] (555)

We also have with probability at least \( 1 - \delta \),

\[
\|\mathbb{E}_{\rho'}[(X - \mu_{\rho'}) (X - \mu_{\rho'})^\top] - I_d\|_2 \leq \|\mathbb{E}_{\rho'}[(X - \mu_{\rho'}) (X - \mu_{\rho'})^\top] - \mathbb{E}_{\rho'}[(X - \mu_{\rho'}) (X - \mu_{\rho'})^\top]\|_2 + \|\mathbb{E}_{\rho'}[(X - \mu_{\rho'}) (X - \mu_{\rho'})^\top] - I_d\|_2 \\
\leq C(\max(\Delta_1, \Delta_1^2) + \sigma^2 \eta^{1-2/k}),
\] (556)

and from Equation (549) and (550), we have with probability at least \( 1 - 3\delta \),

\[
\|\mathbb{E}_r[(X - \mathbb{E}_{\rho'}[X]) (X - \mathbb{E}_{\rho'}[X])^\top] - \mathbb{E}_r[(X - \mathbb{E}_{\rho'}[X]) (X - \mathbb{E}_{\rho'}[X])^\top]\|_2 \\
\leq \|\mathbb{E}_r[(X - \mathbb{E}_{\rho'}[X]) (X - \mathbb{E}_{\rho'}[X])^\top] - \mathbb{E}_r[(X - \mathbb{E}_{\rho'}[X]) (X - \mathbb{E}_{\rho'}[X])^\top]\|_2 + \|\mathbb{E}_r[(X - \mathbb{E}_{\rho'}[X]) (X - \mathbb{E}_{\rho'}[X])^\top] - \mathbb{E}_r[(X - \mathbb{E}_{\rho'}[X]) (X - \mathbb{E}_{\rho'}[X])^\top]\|_2 \\
\leq \|\mu_{\rho'} - \mu_r\|_2^2 + \|\mu_{\rho'} - \mu_r\|_2^2 \\
\leq \frac{(\|\mu_{\rho'} - \mu_{\rho'}\|_2^2 + \|\mu_{\rho'} - \mu_{\rho'}\|_2^2 + \|\mu_{\rho'} - \mu_r\|_2^2)}{\frac{k_2^2 \sigma^2 \eta^{2-2/k}}{\delta^{2/k}} + \frac{d \log^2(1/\delta)}{n \delta^2}}.
\] (557)

Combining above three inequalities, we know that when \( \eta < 1/k \), with probability at least \( 1 - 6\delta \), there exists some constant \( C_2 \) such that

\[
\|\mathbb{E}_r[(X - \mathbb{E}_{\rho'}[X]) (X - \mathbb{E}_{\rho'}[X])^\top] - I_d\|_2 \leq \frac{C_2 k^2}{1 - \eta} \left( \frac{\sigma^2 \eta^{1-2/k}}{\delta^{2/k}} + \max(\Delta_2, \Delta_2^2) + \frac{k \sigma^2 d \log^2(1/\delta)}{n \delta^2} \right).
\] (558)

Then we know that with probability at least \( 1 - 6\delta \), \( \hat{\rho'} \in \mathcal{G}_3(\eta) \). We remark here in fact we have shown that \( \hat{\rho'} \in \mathcal{G}_1(\eta) \cap \mathcal{G}_2(\eta) \cap \mathcal{G}_3(\eta) \) with probability at least \( 1 - 6\delta \).
E.7 Proof of Theorem 5.4

Proof. We verify the five conditions in Theorem 7.2.

1. Robust to perturbation: True since $\text{TV}$ satisfies triangle inequality.

2. Limited Corruption: It follows from Lemma B.1 that with probability at least $1 - \delta$,

$$\text{TV}(\hat{p}_n, \bar{p}_n^*) \leq \left( \sqrt{\epsilon} + \sqrt{\frac{\log(1/\delta)}{2n}} \right)^2 = \frac{\epsilon}{2}. \quad (559)$$

3. Set for (perturbed) empirical distribution: It can be seen from Lemma E.11 that for some fixed $\eta \in [0, 1/2]$, with probability at least $1 - \delta$, there exists some constant $C$ such that

$$\hat{p}_n^* \in \mathcal{G}_{\text{mean}}^{\text{TV}} \left( C\sigma \cdot \left( \sqrt{\frac{d + \log(1/\delta)}{n}} + \eta\sqrt{\log(1/\eta)} \right), \eta \right). \quad (560)$$

4. Generalized Modulus of Continuity:

We construct $\mathcal{M} = \mathcal{G}' = \mathcal{G}_{\text{mean}}^{\text{TV}} \left( C\sigma \cdot \left( \sqrt{\frac{d + \log(1/\delta)}{n}} + \tilde{\epsilon}\sqrt{\log(1/\tilde{\epsilon})} \right), \tilde{\epsilon} \right)$. Thus it follows from the population limit of $\mathcal{G}_{\text{mean}}^{\text{TV}}$ in Lemma 3.1 that for some constant $C_1$, we have

$$\sup_{p_1^* \in \mathcal{M}, p_2^* \in \mathcal{G}', \text{TV}(p_1^*, p_2^*) \leq \tilde{\epsilon}} ||\mathbb{E}_{p_1^*}[X] - \mathbb{E}_{p_2^*}[X]||_2 \leq C_1\sigma \left( \sqrt{\frac{d + \log(1/\delta)}{n}} + \tilde{\epsilon}\sqrt{\log(1/\tilde{\epsilon})} \right). \quad (561)$$

Since $f(x) = x\sqrt{\log(1/x)}$ is a concave function, and $f(0) = 0$, we have

$$f(a + b) = \frac{a}{a + b}f(a) + \frac{b}{a + b}f(b) \leq f(a) + f(b). \quad (562)$$

From the assumption in theorem statement we know that $n \geq \log(1/\delta)$, we have

$$\tilde{\epsilon}\sqrt{\log(1/\tilde{\epsilon})} \leq \epsilon\sqrt{\log\frac{1}{\epsilon}} + \frac{\log(1/\delta)}{n}\sqrt{\log(1/\delta)} \leq \epsilon\sqrt{\log(1/\epsilon)} + \sqrt{\log(1/\epsilon)n}. \quad (563)$$

From Theorem 5.1, we know that the generalized modulus for $\tilde{\text{TV}}_{\mathcal{H}}$ is the same as $\text{TV}$ for resilient set.

5. Generalization bound: Since we take $\hat{p}' = \hat{p}_n^*$, we have

$$||\mathbb{E}_{p^*}[X] - q[X]||_2 \leq ||\mathbb{E}_{p^*}[X] - \hat{\mathbb{E}}_{p_n^*}[X]||_2 + ||\mathbb{E}_{\hat{p}_n^*}[X] - q[X]||_2. \quad (564)$$

Thus by with probability at least $1 - \delta$,

$$||\mathbb{E}_{p^*}[X] - q[X]||_2 \leq ||\mathbb{E}_{\hat{p}_n^*}[X] - q[X]||_2 + C\sigma\sqrt{\frac{d + \log(1/\delta)}{n}}. \quad (565)$$

The convergence of $||\mathbb{E}_{\hat{p}_n^*}[X] - p^*[X]||_2$ is from (Lugosi, 2017, Equation (5.5)).

Combining the five conditions, from Theorem 7.2, for projection algorithm $q = \arg\min\{\text{TV}(q, \hat{p}_n) \mid q \in \mathcal{M}\}$, we have with probability at least $1 - 3\delta$,

$$||\mathbb{E}_{p^*}[X] - q[X]||_2 \leq C\sigma \cdot \left( \sqrt{\frac{d + \log(1/\delta)}{n}} + \epsilon\sqrt{\log(1/\epsilon)} \right), \quad (566)$$

where $C$ is some universal constant. \qed
E.8 Proof of Theorem 5.5

Proof. Among the five conditions in Theorem 7.2, we only need to verify the set for (perturbed) empirical distribution, generalized modulus of continuity and generalization bound. Other two conditions are identical to the proof in Appendix E.7.

1. Set for (perturbed) empirical distribution: From Corollary E.1, we know that with probability at least $1 - 2\delta,$

$$\hat{p}_n^* \in \bigcap_{\eta \in (0,1)} G_{\text{mean}}^{TV} \left( \frac{Ck\sigma}{1 - \eta} \left( \frac{\eta^{1-1/k}}{\delta^{1/k}} + \frac{1}{\delta} \sqrt{\frac{d}{n}} \right), \eta \right) = G', \quad (567)$$

for some constant $C$.

2. Generalized modulus of continuity: Denote $\tilde{\epsilon} = \left( \sqrt{\epsilon + \sqrt{\frac{\log(1/\delta)}{2n}}} \right)^2,$ from assumption we know that $\tilde{\epsilon} < 1/4$. Since $f(x) = x^{1-1/k}$ is a concave function, following a similar analysis as (563), we know that

$$\epsilon^{1-1/k} \leq 2\epsilon^{1-1/k} + 2\sqrt{\frac{\log(1/\delta)}{n}}. \quad (568)$$

Thus with appropriate choice of $C_1$, we can make $G' \subset \mathcal{M} = G_{\text{mean}}^{TV} \left( C_1k\sigma \left( \frac{\epsilon^{1-1/k}}{\delta^{1/k}} + \frac{1}{\delta} \sqrt{\frac{d}{n}} \right), \left( \sqrt{\epsilon + \sqrt{\frac{\log(1/\delta)}{2n}}} \right) \right)$.

Therefore the generalized modulus of continuity for perturbation level $\tilde{\epsilon}$ is upper bounded by $C_1k\sigma \left( \frac{\epsilon^{1-1/k}}{\delta^{1/k}} + \frac{1}{\delta} \sqrt{\frac{d}{n}} \right)$. From Theorem 5.1, we know that the generalized modulus for $\tilde{TV}_{\mathcal{H}}$ is the same as $TV$ for resilient set.

3. Generalization bound: From Lemma A.4, we know that with probability at least $1 - \delta,$

$$\| \mathbb{E}_{p^*} [X] - \mathbb{E}_q [X] \|_2 \leq \| \mathbb{E}_{\hat{p}_n^*} [X] - \mathbb{E}_q [X] \|_2 + \| \mathbb{E}_{p^*} [X] - \mathbb{E}_{\hat{p}_n^*} [X] \|_2. \quad (569)$$

By Chebyshev’s inequality, we have

$$\mathbb{P}_{p^*} (\| \mathbb{E}_{p^*} [X] - \mathbb{E}_{\hat{p}_n^*} [X] \|_2 \geq t) \leq \frac{\mathbb{E}_{p^*} [\| \mathbb{E}_q [X] - \mathbb{E}_{\hat{p}_n^*} [X] \|_2]}{tk} \quad (570)$$

Since the $k$-th moment is bounded, by Khinchine’s inequality (Haagerup, 1981), there is

$$\mathbb{E}_{p^*} \left( \left\| \frac{1}{n} \sum_{i=1}^{n} X_i - \mathbb{E}_{p^*} [X] \right\|_2^k \right) \leq \mathbb{E}_{X_i \sim p^*, \xi \sim (\pm 1)^d} \left\| \frac{1}{n} \xi^\top \left( \sum_{i=1}^{n} X_i - \mathbb{E}_{p^*} [X] \right) \right\|_2. \quad (571)$$

By Marcinkiewicz-Zygmund inequality (Ren and Liang, 2001) there exists some $C_2$, such that for any $v \in \mathbb{R}^d$, 

$$\mathbb{E}_{X_i \sim p^*} \left| \frac{1}{n} v^\top \left( \sum_{i=1}^{n} X_i - \mathbb{E}_{p^*} [X] \right) \right|^k \leq \frac{(C_2\sigma \sqrt{k})^k}{n^{k/2}} \| v \|_2^k. \quad (572)$$
Therefore by first conditioning on $\xi$, we have

$$
\mathbb{E}_{X_i \sim p, \xi \sim \{\pm 1\}^d} \left| \frac{1}{n} \xi^\top \left( \sum_{i=1}^{n} X_i - \mathbb{E}_p[X] \right) \right|^k \leq \left( \frac{C_2 \sigma \sqrt{k}}{n^{1/2}} \right)^k \mathbb{E}_{\xi \sim \{\pm 1\}^d} \|\xi\|_2^k
$$

$$
= \left( C_2 \sigma \sqrt{k} \sqrt{\frac{d}{n}} \right)^k.
$$

(573)

Thus overall, we know that with probability at least $1 - \delta$, there exists some constant $C$, such that

$$
\left\| \frac{1}{n} \sum_{i=1}^{n} X_i - \mathbb{E}_p[X] \right\|_2 \leq C \sigma \sqrt{k} \sqrt{\frac{d}{\delta^1/k}} \sqrt{\frac{d}{n}}.
$$

(574)

We have with probability at least $1 - \delta$,

$$
\| \mathbb{E}_{p^*}[X] - \mathbb{E}_q[X] \|_2 \leq \| \mathbb{E}_{\hat{p}_n}[X] - \mathbb{E}_q[X] \|_2 + \frac{C \sigma \sqrt{k}}{\delta^1/k} \sqrt{\frac{d}{n}}.
$$

(575)

Combining the five conditions, from Theorem 7.2, for projection algorithm $q = \arg\min \{ \text{TV}(q, \hat{p}_n) \mid q \in \mathcal{M} \}$, we have with probability at least $1 - 4\delta$,

$$
\| \mu_{p^*} - \mu_q \|_2 \leq C k \sigma \cdot \left( \epsilon_1^{1/k} + \frac{1}{\delta} \sqrt{\frac{\log(1/\delta)}{2n}} \right).
$$

(576)

\[ \square \]

### E.9 Mean estimation via projecting to bounded covariance set

The below theorem is following a similar flow of proof and conclusion as in (Prasad et al., 2019). Here we use our framework to give a clean proof.

**Theorem E.6** (Bounded covariance, TV projection). Denote

$$
\epsilon_1 = \max \left( \frac{d \log(d/\delta)}{n}, \epsilon + \frac{\log(1/\delta)}{n} \right), \bar{\epsilon} = 4 \left( \sqrt{\epsilon_1} + \sqrt{\frac{\log(1/\delta)}{2n}} \right)^2.
$$

(577)

We take both $\mathcal{G}$ and $\mathcal{M}$ to be the set of bounded covariance set as below:

$$
\mathcal{G} = \{ p \mid \| \Sigma_p \|_2 \leq \sigma^2 \},
$$

(578)

$$
\mathcal{M} = \left\{ p \mid \| \Sigma_p \|_2 \leq 2 C_2^2 \left( 1 + \frac{d \log(d/\delta)}{n \epsilon_1} \right) \right\}.
$$

(579)

If $p^* \in \mathcal{G}$ and $\bar{\epsilon} < 1/2$, then the projection $q = \Pi(\hat{p}_n; \text{TV}/\text{TV}_{\mathcal{H}}, \mathcal{M})$ of $\hat{p}_n$ onto $\mathcal{M}$ satisfies:

$$
\| \mathbb{E}_{p^*}[X] - \mathbb{E}_q[X] \|_2 \leq C_3 \sigma \cdot \left( \sqrt{\bar{\epsilon}} + \sqrt{\frac{d \log(d/\delta)}{n}} \right).
$$

(580)

with probability at least $1 - 3\delta$. Moreover, this remains true for any $q \in \mathcal{M}$ within TV (or $\text{TV}$) distance $\bar{\epsilon}/2$ of $\hat{p}_n$. 

71
Proof. Among the five conditions in Theorem 7.2, the ‘robust to perturbation’ and ‘limited corruption’ conditions are identical to the proof in Appendix E.7. We only need to verify the other three conditions.

1. **Set for (perturbed) empirical distribution:**

   For any fixed $\epsilon$ as perturbation level, we show that there exists some distribution $\hat{p}'$ that has bounded covariance and $\text{TV}(\hat{p}', \hat{p}_n)$ is small.

   We truncate the distribution $p^*$ by removing all $X$ with $\|X - \mu_{p^*}\| \geq \sigma \sqrt{d}/\sqrt{\epsilon_1}$ to get a new distribution $p'$, where $\epsilon_1$ is some parameter to be specified later. Denote the empirical distribution of $p'$ with $n$ samples as $\hat{p}'$. From Lemma E.12, we know that $\text{TV}(\hat{p}^*, p') \leq \epsilon_1$. It follows from Lemma B.1 that with probability at least $1 - \delta$,

   \[
   \text{TV}(\hat{p}', \hat{p}_n^*) \leq \left( \sqrt{\epsilon_1} + \frac{\log(1/\delta)}{2n} \right)^2. \tag{581}
   \]

   Denote $\Delta_1 = \sqrt{\frac{\sigma^2 d \log(d/\delta)}{n \epsilon_1}}$ and

   \[
   G' = \{ p \mid \| E_p[(X - \mu_p)(X - \mu_p^\top)] \|_2 \leq \sigma^2 + C \max(\Delta_1, \Delta_1^2) \}, \tag{582}
   \]

   where $C$ is some universal constant to be specified. From Lemma E.6, we know that with probability at least $1 - \delta$,

   \[
   \| E_{\hat{p}'}[(X - \mu_{p^*})(X - \mu_{p^*})^\top] - E_{\hat{p}'}[(X - \mu_{p^*})(X - \mu_{p^*})^\top] \|_2 \leq C \max(\sigma \Delta_1, \Delta_1^2). \tag{583}
   \]

   Thus we know that with probability at least $1 - \delta$,

   \[
   \| E_{\hat{p}'}[(X - \mu_{p'})(X - \mu_{p'})^\top] \|_2 \leq \| E_{\hat{p}'}[(X - \mu_{p^*})(X - \mu_{p^*})^\top] \|_2 \leq \| E_{\hat{p}'}[(X - \mu_{p^*})(X - \mu_{p^*})^\top] \|_2 + C \max(\sigma \Delta_1, \Delta_1^2) \leq \frac{\sigma^2}{1 - \epsilon} + C \max(\sigma \Delta_1, \Delta_1^2). \tag{584}
   \]

   Here we use the fact that $p'$ is a deletion of $p^*$, thus $E_{\hat{p}'}[(v^\top (X - \mu_{p^*}))^2] \leq \frac{1}{1 - \epsilon} E_{\hat{p}'}[(v^\top (X - \mu_{p^*}))^2]$. Thus we have $\hat{p}' \in G'$ with probability at least $1 - \delta$.

2. **Generalized Modulus of Continuity:** Assume that $\epsilon + \log(1/\delta)/n \leq \epsilon_1$. Then the perturbation level for modulus of continuity is $2 \left( \sqrt{\epsilon_1} + \sqrt{\frac{\log(1/\delta)}{2n}} \right)^2 + 2 \left( \sqrt{\epsilon} + \sqrt{\frac{\log(1/\delta)}{2n}} \right)^2 \leq 4 \left( \sqrt{\epsilon_1} + \sqrt{\frac{\log(1/\delta)}{2n}} \right)^2$. Denote the right hand side as $\hat{\epsilon}$. From $\hat{p}' \in G'$, and that both $G'$ and $M$ guarantees the covariance to be upper bounded, we know from Lemma C.2 that for any $\hat{\epsilon} \in [0, 1/2]$, there exist constants $C_1, C_2$ such that

   \[
   \sup_{p \in G', q \in M, \text{TV}(p, q) \leq \hat{\epsilon}} \| E_p[X] - E_q[X] \|_2 \leq \sqrt{\frac{\sigma^2}{1 - \epsilon} + C_1 \max(\sigma \Delta_1, \Delta_1^2) \sqrt{\hat{\epsilon}}} \leq C_2 \sigma \left( 1 + \frac{d \log(d/\delta)}{n \epsilon_1} \right) \epsilon_1. \tag{585}
   \]

   From Theorem 5.1, we know that the generalized modulus for $\hat{\text{TV}}_H$ is the same as $\text{TV}$ for bounded covariance set.
3. Generalization bound: Note that \( p' \) is a \( \epsilon_1 \)-deletion of \( p^* \). By triangle inequality and the resilient condition for \( p^* \), we have

\[
\| \mathbb{E}_{p^*}[X] - \mathbb{E}_q[X] \|_2 \leq \| \mathbb{E}_{p^*}[X] - \mathbb{E}_{p'}[X] \|_2 + \| \mathbb{E}_{p'}[X] - \mathbb{E}_q[X] \|_2 + \| \mathbb{E}_{p'}[X] - \mathbb{E}_q[X] \|_2
\]

\[
\leq \sigma \sqrt{\frac{d}{n}} + \| \mathbb{E}_{p'}[X] - \mathbb{E}_q[X] \|_2 + \| \mathbb{E}_{p'}[X] - \mathbb{E}_q[X] \|_2.
\]

(586)

From Lemma E.5 and the assumption that \( \epsilon + \log(1/\delta)/n \leq \epsilon_1 \), we know that with probability at least \( 1 - \delta \), there exists some constant \( C_3 \) such that

\[
\| \mathbb{E}_{p'}[X] - \mathbb{E}_q[X] \|_2 \leq C_3 \left( \sigma \sqrt{\frac{d}{n}} + \frac{\log(1/\delta)}{n} + \frac{\sigma \sqrt{d \log(1/\delta)}}{n \sqrt{\epsilon + \log(1/\delta)/n}} \right)
\]

\[
\leq C_3 \left( \sigma \sqrt{\frac{d}{n}} + \frac{\log(1/\delta)}{n} + \frac{\sigma \sqrt{d \log(1/\delta)}}{n \sqrt{\log(1/\delta)/n}} \right)
\]

\[
= C_3 \left( \sigma \sqrt{\frac{d}{n}} + \frac{\log(1/\delta)}{n} + \sigma \sqrt{\frac{d \log(1/\delta)}{n}} \right)
\]

(587)

Thus with probability at least \( 1 - \delta \), there exists some constant \( C_4 \) such that

\[
\| \mathbb{E}_{p^*}[X] - \mathbb{E}_q[X] \|_2 \leq \| \mathbb{E}_{p'}[X] - \mathbb{E}_q[X] \|_2 + \sigma \sqrt{\epsilon_1} + C_4 \sigma \sqrt{\frac{d \log(1/\delta)}{n}}.
\]

(588)

Combining the five conditions, from Theorem 7.2, for projection algorithm \( q = \Pi(p_n; \equiv M, \epsilon/2) \), there exists some constant \( C \) such that with probability at least \( 1 - 3\delta \),

\[
\| \mathbb{E}_{p^*}[X] - \mathbb{E}_q[X] \|_2 \leq C \sigma \cdot (1 + \frac{d \log(d/\delta)}{n \epsilon_1}) \sqrt{\epsilon_1} + \sqrt{\frac{d \log(1/\delta)}{n}}.
\]

(589)

By taking \( \epsilon_1 = \max \left( \frac{d \log(d/\delta)}{n}, \epsilon + \frac{\log(1/\delta)}{n} \right) \), we can see that \( 1 + \frac{d \log(d/\delta)}{n \epsilon_1} \leq 2 \). Thus we can get the following bound:

\[
\| \mathbb{E}_{p^*}[X] - \mathbb{E}_q[X] \|_2 \leq C \sigma \cdot \left( \sqrt{\epsilon} + \sqrt{\frac{d \log(d/\delta)}{n}} \right).
\]

(590)

\[ \square \]

E.10 Proof of Theorem 5.6

Proof. Among the five conditions in Theorem 7.2, the robust to perturbation and limited corruption condition is identical to the proof in Appendix E.7. We only need to verify the other three conditions.

1. Set for (perturbed) empirical distribution:

For any fixed \( \epsilon \) as perturbation level, we show that there exists some distribution \( p' \) that satisfies conditions required in Lemma E.4 for modulus of continuity and TV(\( p', \hat{p}_n \)) is small.

We truncate the distribution \( p^* \) by removing all \( X \) with \( \| X - \mu_p \| \geq \sigma (kd)^{1/2}/\epsilon_1^{1/k} \) to get a new distribution \( p' \), where \( \epsilon_1 \) is some parameter to be specified later and we assume \( \epsilon_1 \geq \epsilon + \)
log(1/\delta)/n. Denote the empirical distribution of \( p' \) with \( n \) samples as \( \hat{p}' \). From Lemma E.12, we know that \( \text{TV}(p^*, p') \leq \epsilon_1 \). It follows from Lemma B.1 that with probability at least 1 - \( \delta \),

\[
\text{TV}(p', \hat{p}'_n) \leq \left( \sqrt{\epsilon_1} + \sqrt{\frac{\log(1/\delta)}{2n}} \right)^2.
\]

(591)

Then we apply the result in Lemma E.12. Denote \( \Delta_1 = \sqrt{\frac{\sigma^2 d \log(d/\delta)}{n \epsilon_1^3}}, \Delta_2 = \sqrt{\frac{\sigma^2 d \log(d)}{n \epsilon_1^2}}, \) and

\[
G' = \left\{ p : \forall r \leq \frac{p}{1 - \epsilon_1}, \|\mathbb{E}_r[X] - \mathbb{E}_p[X]\|_2 \leq \frac{C_1 k \sigma}{1 - \epsilon_1} \left( \frac{\epsilon_1^{1-1/k}}{\delta^{1/k}} + \frac{1}{\delta} \sqrt{\frac{d}{n}} \right), \right.
\]

\[
\left. \|\mathbb{E}_p[(X - \mu_p)(X - \mu_p)^\top]\|_2 \leq \frac{1}{1 - \epsilon_1} + C_3 \max(\Delta_1, \Delta_2^2), \right.
\]

\[
\|\mathbb{E}_r[(X - \mathbb{E}_p[X])(X - \mathbb{E}_p[X])^\top] - I_d\|_2 \leq \frac{C_2 k}{1 - \epsilon_1} \left( \frac{k \sigma^2 \epsilon_1^{1-2/k}}{\delta^{2/k}} + \frac{\max(\Delta_2, \Delta_2^2)}{\delta} + \frac{k \sigma^2 d}{n \delta^2} \right) \}
\]

(592)

where \( C_1, C_2, C_3 \) are some universal constants. Then when \( k > 2 \), from Lemma E.12, under appropriate choice of constants \( C_1, C_2, C_3 \), for any \( \epsilon_1 < 1 \), we have \( \hat{p}' \in G' \) with probability at least 1 - \( 6\delta \).

2. Generalized Modulus of Continuity:

Under the assumption that \( \epsilon + \log(1/\delta)/n \leq \epsilon_1 \), the perturbation level for modulus of continuity is \( 2 \left( \sqrt{\epsilon_1} + \sqrt{\frac{\log(1/\delta)}{2n}} \right)^2 + 2 \left( \sqrt{\epsilon} + \sqrt{\frac{\log(1/\delta)}{2n}} \right)^2 \leq 4 \left( \sqrt{\epsilon_1} + \sqrt{\frac{\log(1/\delta)}{2n}} \right)^2 \). Denote the right hand side as \( \tilde{\epsilon} \). Then we also have \( \tilde{\epsilon} \leq \epsilon_1 \). Now we take \( \mathcal{M} \) as

\[
\mathcal{M} = \left\{ p : \mathbb{E}_p[(X - \mu_p)(X - \mu_p)] \leq 1 + \frac{C_2 k}{1 - \epsilon_1} \left( \frac{k \sigma^2 \epsilon_1^{1-2/k}}{\delta^{2/k}} + \frac{\max(\Delta_2, \Delta_2^2)}{\delta} + \frac{k \sigma^2 d}{n \delta^2} \right) \right\}.
\]

(593)

When \( k > 2 \), in Lemma E.3, take \( \rho_1 = \frac{C_1 k \sigma}{1 - \epsilon_1} \left( \frac{\epsilon_1^{1-1/k}}{\delta^{1/k}} + \frac{1}{\delta} \sqrt{\frac{d}{n}} \right), \rho_2 = \tau = \frac{C_2 k}{1 - \epsilon_1} \left( \frac{k \sigma^2 \epsilon_1^{1-2/k}}{\delta^{2/k}} + \frac{\max(\Delta_2, \Delta_2^2)}{\delta} + \frac{k \sigma^2 d}{n \delta^2} \right) \}

We have for any \( \tilde{\epsilon} \in [0, 1) \),

\[
\sup_{p \in G', q \in \mathcal{M}, \text{TV}(p, q) \leq \tilde{\epsilon}} \|\mathbb{E}_p[X] - \mathbb{E}_q[X]\|_2 \leq \frac{C_6}{1 - \epsilon_1} \left( \frac{k \sigma^2 \epsilon_1^{1-1/k}}{\delta^{1/k}} + \sqrt{\frac{k \max(\Delta_2, \Delta_2^2)}{\delta} \epsilon_1} + \frac{k \sigma}{\delta} \sqrt{\frac{d}{n}} \right).
\]

(594)

From Lemma E.4, we know that the generalized modulus for \( \widehat{TV}_H \) is the same as \( \text{TV} \) for \( G' \) and \( \mathcal{M} \).

3. Generalization bound:

From the same argument as (586), we know that with probability at least 1 - \( \delta \),

\[
\|\mathbb{E}_p[X] - \mathbb{E}_q[X]\|_2 \leq \|\mathbb{E}_{\hat{p}'}[X] - \mathbb{E}_q[X]\|_2 + C_7 \left( \frac{\sigma^2 \epsilon_1^{1-1/k}}{\delta^{1/k}} + \sigma \sqrt{\frac{d}{n}} + \sigma \sqrt{\frac{\log(1/\delta)}{n}} + \frac{\sigma \sqrt{d} \log(1/\delta)}{n \epsilon_1^k} \right).
\]

(595)
By taking \( \epsilon_1 = \epsilon + \frac{\log(1/\delta)}{2n} \), we can see that \( \hat{\epsilon} \asymp \epsilon_1 \asymp \epsilon + \frac{\log(1/\delta)}{2n} \). Combining the five conditions, from Theorem 7.2, for projection algorithm \( q = \Pi(\hat{p}_n; \text{TV}, \mathcal{M}, \hat{\epsilon}/2) \) where \( \hat{\epsilon} < 1/2 \), we have with probability at least \( 1 - 8\delta \),

\[
\|\mu_{p^*} - \mu_q\|_2 \leq \frac{k\sigma\epsilon^{1/k}}{\delta^{1/k}} + \sqrt{\frac{k \max(\Delta_2, \Delta_2^2) \epsilon}{\delta}} + \frac{k \sigma}{\delta} \sqrt{\frac{\bar{d}}{n}} + \frac{\sigma}{\delta} \sqrt{\frac{d \log(1/\delta)}{n}} + \frac{\sigma \sqrt{d \log(1/\delta)}}{n \epsilon^{1/k}}
\]

Thus we have with probability at least \( 1 - \delta \),

\[
\|\mu_{p^*} - \mu_q\|_2 \leq C_3 k\sigma \cdot \left( \frac{\epsilon^{1/k}}{\delta^{1/k}} + \frac{1}{\delta} \sqrt{\frac{d \log(d)}{n}} \right) \tag{596}
\]

for some constant \( C_3 \).

We remark here that by appropriate choice of constants, we can make the projection set \( \mathcal{M} \) smaller than the set \( \mathcal{M} \) in Theorem E.6, and we know that bounded \( k \)-th moment would imply bounded second moment. Thus the bound in Theorem E.6 also applies to the projection algorithm here, the final bound shall be the minimum of the two terms. \( \square \)

### E.11 Improving the sample complexity by reducing high dimensional mean estimation to one dimension

We show that by adopting a different algorithm and analysis technique, we can improve the sample complexity in bounded Orlicz norm distributions discussed in Section 5.2. We first that this results in good sample complexity in one-dimensional mean estimation, and show that high dimensional mean estimation can be reduced to a one-dimension problem.

**Theorem E.7** (One dimensional mean estimation for bounded Orlicz norm distribution under TV projection). Assume the corruption model is either oblivious corruption (Definition 2.3) or adaptive corruption model (Definition 2.4) of level \( \epsilon \). For some Orlicz function \( \psi \) (Definition 2.2) that satisfies \( \psi(x) \geq x \) when \( x \geq 1 \), we take both \( \mathcal{G} \) and \( \mathcal{M} \) to be the set of one-dimensional bounded Orlicz norm distributions as below:

\[
\mathcal{G} = \left\{ p \mid E_p \left[ \psi \left( \frac{(X - \mu_p)^2}{\sigma^2} \right) \right] \leq 1 \right\}, \tag{598}
\]

\[
\mathcal{M} = \left\{ p \mid E_p \left[ \psi \left( \frac{(X - \mu_p)^2}{4\sigma^2} \right) \right] \leq 4 \right\}. \tag{599}
\]
Denote
\[
\hat{\epsilon} = 4 \left( \sqrt{\epsilon + \frac{\log(1/\delta)}{n}} + \sqrt{\frac{\log(1/\delta)}{2n}} \right)^2.
\] (600)

If \( p^* \in G \) and \( \hat{\epsilon} < 1/2 \), then the projection algorithm \( q = \Pi(\hat{p}_n; TV, M) \) or \( q = \Pi(\hat{p}_n; TV, M, \hat{\epsilon}/2) \) satisfies the following with probability at least \( 1 - 3\delta \):
\[
\| \mathbb{E}_{p^*}[X] - \mathbb{E}_{q}[X] \|_2 \leq C\sigma \cdot \left( \epsilon \sqrt{\psi^{-1}(\epsilon)} + \sqrt{\frac{\log(1/\delta)}{n}} \right),
\] (601)
where \( C \) is some universal constant.

**Proof.** Among the five conditions in Theorem 7.2, the ‘robust to perturbation’ and ‘limited corruption’ conditions are identical to the proof in Appendix E.7. We only need to verify the other three conditions. Denote \( \epsilon_1 = \epsilon + \frac{\log(1/\delta)}{n} \).

1. **Set for (perturbed) empirical distribution:**

   For any fixed \( \epsilon \) as perturbation level, we show that there exists some distribution \( \tilde{p}' \) that has bounded Orlicz norm (inside \( M \)) and \( TV(\tilde{p}', \hat{p}_n) \) is small.

   We truncate the distribution \( p^* \) by removing all \( X \) with \( |X - \mu_{p^*}| \geq \sigma \sqrt{\psi^{-1}(1/\epsilon_1)} \) to get a new distribution \( p' \), where \( \epsilon_1 \) is some parameter to be specified later. Denote the empirical distribution of \( p' \) with \( n \) samples as \( \tilde{p}' \). By Markov’s inequality, we have for any \( t \geq 0 \),
\[
\mathbb{P}_{p^*}(|X - \mu_{p^*}| \geq t) = \mathbb{P}_{p^*} \left( \psi \left( \frac{(X - \mu_{p^*})^2}{\sigma^2} \right) \geq \frac{\psi(t^2/\sigma^2)}{\sigma^2} \right) \leq \frac{\mathbb{E}_{p^*}[\psi((X - \mu_{p^*})^2/\sigma^2)]}{\mathbb{E}_{p^*}[\psi(t^2/\sigma^2)]} \leq \frac{1}{\mathbb{E}_{p^*}[\psi(t^2/\sigma^2)]}.
\] (602)

   By taking \( t = \sigma \sqrt{\psi^{-1}(1/\epsilon_1)} \), we know that \( TV(p^*, p') \leq \epsilon_1 \). It follows from Lemma B.1 that with probability at least \( 1 - \delta \),
\[
TV(\tilde{p}', \hat{p}_n^*) \leq \left( \sqrt{\epsilon_1} + \sqrt{\frac{\log(1/\delta)}{2n}} \right)^2.
\] (603)

   Consider the random variable \( \psi \left( \frac{(X - \mu_{p^*})^2}{\sigma^2} \right) \), where \( X \sim p' \), we know that
\[
\mathbb{E}_{p'}[\psi \left( \frac{(X - \mu_{p^*})^2}{\sigma^2} \right)] \leq 1,
\] (605)
\[
\text{Var}(\psi \left( \frac{(X - \mu_{p^*})^2}{\sigma^2} \right)) = \mathbb{E}_{p'}[\psi^2 \left( \frac{(X - \mu_{p^*})^2}{\sigma^2} \right)] - \mathbb{E}_{p'}[\psi \left( \frac{(X - \mu_{p^*})^2}{\sigma^2} \right)]^2 \leq \frac{1}{\epsilon_1}.
\] (606)

   By Bernstein’s inequality, we know that
\[
\mathbb{P}_{p'} \left( \sum_{i=1}^{n} \psi \left( \frac{(X_i - \mu_{p^*})^2}{\sigma^2} \right) - \mathbb{E}_{p'}[\psi \left( \frac{(X - \mu_{p^*})^2}{\sigma^2} \right)] \geq t \right) \leq \exp \left( -\frac{t^2}{2n/\epsilon_1 + 2t/3\epsilon_1} \right).
\] (607)
Assume $\epsilon_1 \geq \epsilon + \log(1/\delta)/n$. Let RHS be $\delta$ and solving $t$, we know that with probability at least $1 - \delta$, 
\[
\mathbb{E}_{\hat{p}'}[\psi(\frac{(X - \mu_{p'})^2}{\sigma^2})] \leq \mathbb{E}_{\hat{p}'}[\psi(\frac{(X - \mu_{p'})^2}{\sigma^2})] + \left( \frac{2\log(1/\delta)}{n\epsilon_1} + \frac{2\log(1/\delta)}{3n\epsilon_1} \right) \leq 1 + \left( \frac{2\log(1/\delta)}{n\epsilon_1} + \frac{2\log(1/\delta)}{3n\epsilon_1} \right) < 4. \quad (608)
\]

From centering lemma in A.3 we know that with probability at least $1 - \delta$,
\[
\mathbb{E}_{\hat{p}'}[\psi(\frac{(X - \mu_{p'})^2}{4\sigma^2})] \leq 4. \quad (609)
\]
By taking $G' = M$, we know that $p' \in G'$ with probability at least $1 - \delta$.

2. **Generalized Modulus of Continuity:** Assume that $\epsilon + \log(1/\delta)/n \leq \epsilon_1$. Then the perturbation level for modulus of continuity is $2 \left( \sqrt{\epsilon_1 + \sqrt{\log(1/\delta) \frac{2n}{2n}}} \right) + 2 \left( \sqrt{\epsilon + \sqrt{\log(1/\delta) \frac{2n}{2n}}} \right) \leq 4 \left( \sqrt{\epsilon_1 + \sqrt{\log(1/\delta) \frac{2n}{2n}}} \right)^2$. Denote the right hand side as $\bar{\epsilon}$. Then we also know $\bar{\epsilon} \leq \epsilon_1$. We know from Lemma C.2 that for any $\bar{\epsilon} \in [0, 1/2]$, there is
\[
\sup_{p \in G', q \in M, TV(p, q) \leq \bar{\epsilon}} \|\mathbb{E}_p[X] - \mathbb{E}_q[X]\|_2 \leq \sigma \epsilon_1 \sqrt{\psi^{-1}(1/\epsilon_1)}. \quad (610)
\]

3. **Generalization bound:** We first show that $\mathbb{E}_{p'}[\psi(X^2/\sigma^2)] \leq 1$ implies $\mathbb{E}_{p'}[X^2] \leq 2\sigma^2$. Note that $\mathbb{E}_{p'}[\psi(X^2/\sigma^2)] \leq 1$ is equivalent to
\[
1 \geq \mathbb{P}_{p'}(|X| \leq \sigma)\mathbb{E}_{p'}[\psi(X^2/\sigma^2) \mid |X| \leq \sigma] + \mathbb{P}_{p'}(|X| > \sigma)\mathbb{E}_{p'}[\psi(X^2/\sigma^2) \mid |X| > \sigma]
\]
\[
\geq \mathbb{P}_{p'}(|X| > \sigma)\mathbb{E}_{p'}[X^2/\sigma^2 \mid |X| > \sigma], \quad (611)
\]

since $\psi(x) \geq x$ for $x \geq 1$. Thus we have
\[
\mathbb{E}_{p'}[X^2/\sigma^2] = \mathbb{P}_{p'}(|X| \leq \sigma)\mathbb{E}_{p'}[X^2/\sigma^2 \mid |X| \leq \sigma] + \mathbb{P}_{p'}(|X| > \sigma)\mathbb{E}_{p'}[X^2/\sigma^2 \mid |X| > \sigma]
\]
\[
\leq 2. \quad (612)
\]
Thus we know that $p'$ has its variance bounded. Note that $p'$ is a $\epsilon_1$-deletion of $p^*$. By triangle inequality and the resilient condition for $p^*$, we have
\[
|\mathbb{E}_{p'}[X] - \mathbb{E}_q[X]| \leq |\mathbb{E}_{p'}[X] - \mathbb{E}_{p'}[X]| + |\mathbb{E}_{p'}[X] - \mathbb{E}_{q}[X]| + |\mathbb{E}_{q}[X] - \mathbb{E}_q[X]|
\]
\[
\leq \sigma \epsilon_1 \sqrt{\psi^{-1}(1/\epsilon_1)} + |\mathbb{E}_{p'}[X] - \mathbb{E}_{q}[X]| + |\mathbb{E}_{q}[X] - \mathbb{E}_q[X]|. \quad (613)
\]
From Lemma E.5 and the assumption that $\epsilon + \log(1/\delta)/n \leq \epsilon_1$, we know that with probability at least $1 - \delta$, there exists some constant $C_3$ such that
\[
|\mathbb{E}_{p'}[X] - \mathbb{E}_{q}[X]| \leq C_3 \left( \sigma \sqrt{\frac{\log(1/\delta)}{n}} + \frac{\sigma \log(1/\delta)}{n \sqrt{\epsilon + \log(1/\delta)/n}} \right)
\]
\[
\leq C_3 \left( \sigma \sqrt{\frac{\log(1/\delta)}{n}} + \frac{\sigma \log(1/\delta)}{n \sqrt{\log(1/\delta)/n}} \right)
\]
\[
= 2C_3 \sigma \sqrt{\frac{\log(1/\delta)}{n}}. \quad (614)
\]
Thus with probability at least $1 - \delta$, there exists some constant $C_4$ such that

$$
|\mathbb{E}_{p^*}[X] - \mathbb{E}_q[X]| \leq |\mathbb{E}_{p^*}[X] - \mathbb{E}_q[X]| + \sigma \epsilon_1 \sqrt{\psi^{-1}(1/\epsilon_1)} + C_4 \sigma \sqrt{\log(1/\delta)/n}. \tag{615}
$$

Take $\epsilon_1 = \epsilon + \log(1/\delta)/n$. From the same argument as (563), we have

$$
\epsilon_1 \sqrt{\psi^{-1}(1/\epsilon_1)} \leq \epsilon \sqrt{\psi^{-1}(1/\epsilon)} + \frac{\log(1/\delta)}{n} \sqrt{\psi^{-1}(1/\epsilon) + \frac{\log(1/\delta)}{n}}. \tag{616}
$$

From assumption $\hat{\epsilon} < 1/2$ we know that $\frac{n}{\log(1/\delta)} > 1$. From $\psi^{-1}(x) \leq x$ for $x \geq 1$, we know that

$$
\frac{\log(1/\delta)}{n} \sqrt{\psi^{-1}(\frac{n}{\log(1/\delta)})} \leq \sqrt{\frac{\log(1/\delta)}{n}}. \tag{617}
$$

Thus overall, we have

$$
|\mathbb{E}_{p^*}[X] - \mathbb{E}_q[X]| \leq |\mathbb{E}_{p^*}[X] - \mathbb{E}_q[X]| + \sigma \epsilon \sqrt{\psi^{-1}(1/\epsilon)} + C_4 \sigma \sqrt{\log(1/\delta)/n}. \tag{618}
$$

Combining the five conditions, from Theorem 7.2 and taking , for projection algorithm $q = \Pi(\hat{p}_n; TV, \mathcal{M})$ or $q = \Pi(\hat{p}_n; TV, \mathcal{M}, \hat{\epsilon}/2)$, there exists some constant $C$ such that with probability at least $1 - 3\delta$,

$$
|\mathbb{E}_{p^*}[X] - \mathbb{E}_q[X]| \leq C \sigma \cdot \left( \epsilon \sqrt{\psi^{-1}(1/\epsilon)} + \sigma \sqrt{\log(1/\delta)/n} \right). \tag{619}
$$

Now we show that if we are able to get good mean estimator for one-dimensional random variable, we are guaranteed to get good mean estimator for high-dimensional random variable. Similar idea also appears in (Catoni and Giulini, 2017; Joly et al., 2017; Prasad et al., 2019).

**Lemma E.13.** Assume $X \sim p^*$ is a $d$-dimensional random variable, and there exists an estimator $\hat{\mu}_v$ such that for any fixed $v \in \mathbb{R}^d$, $\|v\|_2 = 1$, with probability at least $1 - \delta$,

$$
|\hat{\mu}_v - \mathbb{E}_{p^*}[v^\top X]| \leq f(\delta). \tag{620}
$$

Denote the minimum $1/2$-covering of the unit sphere $S^{d-1} = \{v \mid v \in \mathbb{R}^d, \|v\|_2 = 1\}$ as $\mathcal{N}(S^{d-1}, 1/2)$, i.e. $\mathcal{N}(S^{d-1}, 1/2)$ is the set with minimum elements that satisfies $\forall v \in S^{d-1}, \exists y \in \mathcal{N}(S^{d-1}, 1/2)$ such that $\|v - y\|_2 \leq 1/2$. Then the linear programming

$$
\hat{\mu} = \arg \min_{\mu} \sup_{v \in \mathcal{N}(S^{d-1}, 1/2)} |v^\top \mu - \hat{\mu}_v| \tag{620}
$$

satisfies

$$
\|\hat{\mu} - \mathbb{E}_{p^*}[X]\|_2 \leq 4 f(\delta/5^d) \tag{621}
$$

with probability at least $1 - \delta$.

**Proof.** We have

$$
\|\hat{\mu} - \mathbb{E}_{p^*}[X]\|_2 = \sup_{v \in S^{d-1}} |\mathbb{E}_{p^*}[v^\top (X - \hat{\mu})]| \leq \sup_{v \in \mathcal{N}(S^{d-1}, 1/2)} |\mathbb{E}_{p^*}[v^\top (X - \hat{\mu})]| + \frac{1}{2} \|\hat{\mu} - \mathbb{E}_{p^*}[X]\|_2. \tag{622}
$$
Thus we know that

$$
\|\hat{\mu} - E_p[X]\|_2 \leq 2 \sup_{v \in \mathcal{N}(S^{d-1}, 1/2)} E_p[|v^T (X - \hat{\mu})|] 
\leq 2 \left( \sup_{v \in \mathcal{N}(S^{d-1}, 1/2)} |E_p[v^T X] - \hat{\mu}_v| + \sup_{v \in \mathcal{N}(S^{d-1}, 1/2)} |v^T \hat{\mu} - \hat{\mu}_v| \right) 
\leq 4 \sup_{v \in \mathcal{N}(S^{d-1}, 1/2)} |E_p[v^T X] - \hat{\mu}_v|.
$$

Since $|\mathcal{N}(S^{d-1}, 1/2)| \leq 5^d$ (Wainwright, 2019), by taking the union bound over all the vectors in $\mathcal{N}(S^{d-1}, 1/2)$ in (619), we know that with probability at least $1 - 5^d \delta$, 

$$
\|\hat{\mu} - E_p[X]\|_2 \leq 4f(\delta).
$$

Substituting $\delta$ with $\tilde{\delta} = 5^d \delta$ gives the final result.

\[\square\]

**Remark E.4.** The above result also applies to general case of estimating under $L = \mathcal{W}_F(p, q) = \sup_{f \in \mathcal{F}} |E_p f(X) - E_q f(X)|$ by designing $W_{\mathcal{H}}$ where $\mathcal{H} \subset \mathcal{F}$ and $W_{\mathcal{H}}(p, q) \geq \frac{1}{2} W_F(p, q)$. Furthermore, it is necessary to estimate each $f(X)$ for $f \in \mathcal{F}$ very well. Recall that the modulus of continuity for $L = \mathcal{W}_F$ is a nearly tight upper bound for the population limit (Lemma B.2). The modulus can be written as

$$
\sup_{p_1, p_2 \in \mathcal{G}, \text{TV}(p_1, p_2) \leq \epsilon} \mathcal{W}_F(p_1, p_2) = \sup_{f \in \mathcal{F}} \sup_{p_1, p_2 \in \mathcal{G}, \text{TV}(p_1, p_2) \leq \epsilon} |E_{p_1}[f(X)] - E_{p_2}[f(X)]|,
$$

while $\sup_{p_1, p_2 \in \mathcal{G}, \text{TV}(p_1, p_2) \leq \epsilon} |E_{p_1}[f(X)] - E_{p_2}[f(X)]|$ is the modulus for estimating $E_{p}[f(X)]$. Hence, robust estimation under $\mathcal{W}_F$ is equivalent to robust estimation of each $f \in \mathcal{F}$.

Combining Theorem E.7 and Lemma E.13, we have the following corollary.

**Corollary E.2.** Assume the corruption model is either oblivious corruption (Definition 2.3) or adaptive corruption model (Definition 2.4) of level $\epsilon$. For some Orlicz function $\psi$ (Definition 2.2) that satisfies $\psi(x) \geq x$ when $x \geq 1$, we take $\mathcal{G}$ to be the set of $d$-dimensional bounded Orlicz norm distributions, and $\mathcal{M}$ to be the set of 1-dimensional bounded Orlicz norm distributions:

$$
\mathcal{G} = \left\{ p \mid \sup_{v \in \mathbb{R}^d, \|v\|_2 = 1} E_p \left[ \psi \left( \frac{(v^T (X - \mu_p))^2}{\sigma^2} \right) \right] \leq 1 \right\},
$$

$$
\mathcal{M} = \left\{ p \mid E_p \left[ \psi \left( \frac{(X - \mu_p)^2}{4\sigma^2} \right) \right] \leq 4 \right\}.
$$

Denote

$$
\tilde{\epsilon} = 4 \left( \sqrt{\epsilon + \frac{\log(1/\delta)}{n}} + \sqrt{\frac{\log(1/\delta)}{2n}} \right)^2.
$$

For some fixed $v \in S^{d-1}$, denote the projection of distribution $\hat{\mu}_n$ along direction $v$ as $\hat{\mu}_n^v$. Denote $q_v = \Pi(\hat{\mu}_n^v; \text{TV}, \mathcal{M}, \tilde{\epsilon}/2)$, and the algorithm outputs

$$
\hat{\mu} = \arg\min_{\mu} \sup_{v \in \mathcal{N}(S^{d-1}, 1/2)} |v^T \mu - \mu_{q_v}|.
$$
If \( p^* \in G \) and \( \tilde{\epsilon} < 1/2 \), then \( \hat{\mu} \) satisfies the following with probability at least \( 1 - 3\delta \):

\[
\| \mu_{p^*} - \hat{\mu} \|_2 \leq C\sigma \cdot \left( \epsilon \sqrt{\psi^{-1}(\epsilon)} + \sqrt{d \log(1/\delta)/n} \right),
\]

where \( C \) is some universal constant.

By taking \( \psi(x) = x \), we are assuming the true distribution has bounded covariance, and guarantee estimation error \( O(\sqrt{\epsilon} + \sqrt{d \log(1/\delta)/n}) \).

**F Related discussions and remaining proofs in Section 6**

**F.1 Proof of Lemma 6.2**

*Proof.* We first show the result when \( g(x) = x \). From (Rothschild and Stiglitz, 1978)(Marshall et al., 1979, Proposition B.19.c, remark 2), we know that it suffices to check the following two conditions to guarantee convex order of \( X \):

1. \( \mathbb{E}_{p}[X] = \mathbb{E}_{q}[X] \),
2. \( \forall z \in \mathbb{R}, \int_{-\infty}^{z} \mathbb{P}_{p}[X \leq t] dt \geq \int_{-\infty}^{z} \mathbb{P}_{q}[X \leq t] dt \).

Denote function \((x)_+ = \max(x,0), (x)_- = -\min(-x,0)\). Note that we have

\[
\mathbb{E}_{p}[(X - a)_+] = \int_{a}^{+\infty} \mathbb{P}_{p}(X \geq t) dt,
\]

which is also equivalent to the \( W_1 \) cost that moves all the mass left to \( a \) to the point \( a \). Similarly, the \( W_1 \) cost that moves all the mass right to \( a \) to the point \( a \) can be represented as

\[
\mathbb{E}_{p}[(X - a)_-] = \int_{-\infty}^{a} \mathbb{P}_{p}(X \leq t) dt.
\]

Given \( p,q \), we construct \( r_p \) and \( r_q \) as follows. We consider different cases of \( \mu_p, \mu_q \):

1. Assume \( \mu_p = \mu_q = \mu \) for some \( \mu \). We construct some coupling for \( p, r_p \) pair and \( q, r_q \) pair such that under the coupling, all the mass move towards \( \mu \) and the mean is unchanged. Since \( \mathbb{E}_{p}[(X - \mu)_+] - \mathbb{E}_{p}[(X - \mu)_-] = \mathbb{E}_{p}[X - \mu] = 0 \), we know that \( \mathbb{E}_{p}[(X - \mu)_+] = \mathbb{E}_{p}[(X - \mu)_-] = 0 \). Similarly we have \( \mathbb{E}_{q}[(X - \mu)_+] = \mathbb{E}_{q}[(X - \mu)_-] \).

If \( \mathbb{E}_{p}[(X - \mu)_+] = \mathbb{E}_{p}[(X - \mu)_-] \leq 2\epsilon \), from \( \tilde{W}_1(p,q) \leq \epsilon \), we know that

\[
\mathbb{E}_{q}[(X - \mu)_+] = \mathbb{E}_{q}[(X - \mu)_-] \leq 3\epsilon.
\]

Then we move all the mass of \( p,q \) to a single point \( \mu \) to get the new distribution \( r_p, r_q \), from the above condition we know that \( \tilde{W}_1(p, r_p) \leq 4\epsilon, \tilde{W}_1(q, r_q) \leq 6\epsilon \), and \( r_p, r_q \) satisfies the two conditions required since they are identically distributed.

Otherwise, we have

\[
\mathbb{E}_{p}[(X - \mu)_+] = \mathbb{E}_{p}[(X - \mu)_-] > 2\epsilon,
\]
and consequently,
\[ \mathbb{E}_q[(X - \mu)_+] = \mathbb{E}_q[(X - \mu)_-] > \epsilon. \] (638)

Then we are able to find some \( \tau_2 < \mu < \tau_1 \) such that
\[ \int_{-\infty}^{\tau_2} \mathbb{P}_q(v^\top X \leq t)dt = \epsilon, \] (639)
\[ \int_{\tau_1}^{+\infty} \mathbb{P}_q(v^\top X \geq t)dt = \epsilon. \] (640)

We move all the mass of \( q \) that is left to \( \tau_2 \) to \( \tau_2 \), and all the mass of \( q \) that is right to \( \tau_1 \) to \( \tau_1 \) to get \( r_q \), and keep \( r_p = p \). Then we have \( W_1(r_q, q) = 2\epsilon \), \( \mu_{r_q} = \mu_q = \mu_p = \mu_{r_p} \), and
\[ \forall z < \tau_2, \int_{-\infty}^{z} \mathbb{P}_{r_p}[X \leq t]dt = 0 = \int_{-\infty}^{z} \mathbb{P}_{r_q}[X \leq t]dt, \] (641)
\[ \forall z \in [\tau_2, \tau_1] \int_{-\infty}^{z} \mathbb{P}_{r_p}[X \leq t]dt = \int_{-\infty}^{z} \mathbb{P}_p[X \leq t]dt \]
\[ = \int_{-\infty}^{z} \mathbb{P}_p[X \leq t]dt - \int_{-\infty}^{z} \mathbb{P}_q[X \leq t]dt + \int_{-\infty}^{z} \mathbb{P}_q[X \leq t]dt \]
\[ - \int_{-\infty}^{z} \mathbb{P}_{r_q}[X \leq t]dt + \int_{-\infty}^{z} \mathbb{P}_{r_q}[X \leq t]dt \]
\[ \geq -\epsilon + \epsilon + \int_{-\infty}^{z} \mathbb{P}_{r_q}[X \leq t]dt \]
\[ = \int_{-\infty}^{z} \mathbb{P}_{r_q}[X \leq t]dt, \] (642)
\[ \forall z > \tau_2, \int_{-\infty}^{z} \mathbb{P}_{r_p}[X \leq t]dt = \int_{-\infty}^{z} \mathbb{P}_p[X \leq t]dt \]
\[ = \mathbb{E}_p((X - z)_-) \]
\[ = \mathbb{E}_p((X - z)_+) - \mathbb{E}_p[X - z] \]
\[ \geq \mathbb{E}_{r_q}((X - z)_+) - \mathbb{E}_{r_q}[X - z] \]
\[ = \mathbb{E}_{r_q}[(X - z)_-] \]
\[ = \int_{-\infty}^{z} \mathbb{P}_{r_q}[X \leq t]dt. \] (643)

Thus the two conditions to guarantee convex order are satisfied.

2. Assume \( \mu_p > \mu_q \). From \( W_1(p, q) \leq \epsilon \) we know that \( |\mu_p - \mu_q| \leq \epsilon \). Take \( \mu = \mu_p \).

If \( \mathbb{E}_q[(X - \mu)_-] \leq \epsilon \), then from \( \mathbb{E}_q[(X - \mu)_+] - \mathbb{E}_q[(X - \mu)_-] = \mathbb{E}_q[X - \mu] < 0 \), we know that \( \mathbb{E}_q[(X - \mu)_+] \leq \epsilon \). Thus from \( W_1(p, q) \leq \epsilon \), we know that \( \mathbb{E}_p[(X - \mu)_-] \leq 2\epsilon \), \( \mathbb{E}_p[(X - \mu)_+] \leq 2\epsilon \).

Thus we can move all the mass of \( p, q \) to a single point \( \mu \) to get the new distribution \( r_p, r_q \), from the above condition we know that \( W_1(p, r_p) \leq 4\epsilon \), \( W_1(q, r_q) \leq 2\epsilon \), and \( r_p, r_q \) satisfies the two conditions required since they are identically distributed.

Otherwise, we know \( \mathbb{E}_q[(X - \mu)_-] > \epsilon \). Then we first move the left most part of \( X \) under \( q \) to make \( \mathbb{E}_{r_q}[X] = \mu \). Thus we have \( \mathbb{E}_{r_q}[(X - z)_-] \leq \mathbb{E}_q[(X - z)_-] \) for any \( z \in \mathbb{R} \). Starting from \( p, r_q \), we know that their means are equal. Thus we repeat the first step to construct \( r_p, r_q \) that satisfies the two conditions. Overall we have \( W_1(q, r_q) \leq 7\epsilon \).
3. Assume $\mu_p < \mu_q$. Denote $\mu = \mu_p$. Then if $\mathbb{E}_q[(X - \mu)_-] \leq \epsilon$, we follow the same procedure as the case of $\mu_p > \mu_q$. Otherwise we first move the right most part of $X$ under $q$ to make $\mathbb{E}_{r_q}[X] = \mu_p$. Then we repeat the first step to construct $r_p, r_q$ that satisfies the two conditions. Overall we have $W_1(q, r_q) \leq 7\epsilon$.

When $g(x) = |x|$, note that the movement in above construction from $p, q$ to $r_p, r_q$ only includes deleting the left most or right most of $X$. By replacing $X$ with $|X|$, all above arguments go through without increasing the cost $W_1(p, q)$ and $W_1(q, r_q)$. This is because for any movement from $a$ to $b$ for $|X|$, where both $a$ and $b$ are non negative, one can map it back to movement in $x$ space from $a$ to $b$ or from $-a$ to $-b$ without increasing the cost. Thus the result also holds for $g(x) = |x|$.

\[ \Box \]

F.2 Proof of Lemma 6.1

Proof. To show the result, we first prove the following Lemma.

Lemma F.1. Consider any distribution $p$ and denote its empirical distribution of $n$ i.i.d. samples as $\hat{p}_n$. For any $M > 0$, define

$$\xi_1 = \mathbb{E}_p \left[ \frac{1}{n} \sum_{i=1}^{n} X_i - \mathbb{E}_p[X] \right]_2,$$

$$\xi_2(M) = \sup_{v \in \mathbb{R}^d, \|v\|_2 = 1} \mathbb{E}_p \left[ \max \left( 0, v^\top (X - \mathbb{E}_p[X]) - M \right) \right].$$

Then for any $M > 0$,

$$\mathbb{E}_p[\tilde{W}_1(p, \hat{p}_n)] \leq 8\xi_1 + \xi_2(M) + 2 \frac{M}{\sqrt{n}},$$

where $C$ is some universal constant.

Denote the contaminated population distribution as $p$ satisfying $W_1(p, p^*) \leq \epsilon$, and the empirical distribution of observed data as $\hat{p}_n$. Under the oblivious corruption model $\hat{p}_n$ represents $n$ i.i.d. samples from $p$. Define

$$\mathcal{U}_1 = \{v^\top x : v \in \mathbb{R}^d, \|v\|_2 \leq 1\},$$

$$\mathcal{U}_2 = \{\max(0, v^\top (x - a)) : a, v \in \mathbb{R}^d, \|v\|_2 \leq 1\},$$

$$\mathcal{U}_3 = \{-\max(0, v^\top (x - a)) : a, v \in \mathbb{R}^d, \|v\|_2 \leq 1\}.$$

Note that $\mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3$ is symmetric. We have

$$\tilde{W}_1(p, \hat{p}_n) = \sup_{u \in \mathcal{U}_1} \left[ \mathbb{E}_{\hat{p}_n}[u(X)] - \mathbb{E}_p[u(X)] \right]$$

$$= \sup_{u \in \mathcal{U}_1} \left[ \mathbb{E}_p[u(X)] - \frac{1}{n} \sum_{i=1}^{n} u(X_i) \right]$$

$$= \max \left\{ \sup_{u \in \mathcal{U}_1} \mathbb{E}_p[u(X)] - \frac{1}{n} \sum_{i=1}^{n} u(X_i) \right\}.$$

(650)
We have
\[
\sup_{u \in \mathcal{U}_1} \mathbb{E}^p[u(X)] - \frac{1}{n} \sum_{i=1}^n u(X_i) = \sup_{v \in \mathbb{R}^d, \|v\|_2 \leq 1} v^T \mathbb{E}^p[X] - \frac{1}{n} \sum_{i=1}^n v^T X_i
\]
(651)
\[
= \left\| \frac{1}{n} \sum_{i=1}^n X_i - \mathbb{E}^p[X] \right\|_2
\]
(652)

Now we bound the uniform law of large number for \( \mathcal{U}_2 \). Here we first shift \( X \to \bar{X} = X - \mathbb{E}^p[X] \). Since \( a \) is taken in \( \mathbb{R}, \tilde{W}_1(p, \hat{p}_n) \) wouldn’t change. With a bit of abuse of notation, we still use \( X \) to represent the mean-0 shifted distribution. We have
\[
\mathbb{E} \left[ \sup_{u \in \mathcal{U}'_2} \mathbb{E}^p[u(X)] - \frac{1}{n} \sum_{i=1}^n u(X_i) \right] = \mathbb{E} \sup_{v \in \mathbb{R}^d, \|v\|_2 = 1, a \in \mathbb{R}} \mathbb{E}^p \left[ \max(0, v^T X - a) \right] - \frac{1}{n} \sum_{i=1}^n \max(0, v^T X_i - a)
\]
(653)

We bound three cases separately, i.e. \(-M \leq a \leq M, a > M \) and \( a < -M \).

For \(-M \leq a \leq M \), from symmetrization inequality (Wainwright, 2019, Proposition 4.11), we have
\[
\mathbb{E}^p \sup_{v \in \mathbb{R}^d, \|v\|_2 = 1, a \in [-M, M]} \mathbb{E}^p \left[ \max(0, v^T X - a) \right] - \frac{1}{n} \sum_{i=1}^n \max(0, v^T X_i - a)
\]
(654)
\[
\leq 2 \mathbb{E}_{p, \epsilon \sim \{\pm 1\}^d} \sup_{v \in \mathbb{R}^d, \|v\|_2 = 1, a \in [-M, M]} \left[ \frac{1}{n} \sum_{i=1}^n \epsilon_i \max(0, v^T X_i - a) \right]
\]
(655)

From Talagrand contraction inequality (Vershynin, 2018, Exercise 6.7.7) as below, we have
\[
2 \mathbb{E}_{p, \epsilon \sim \{\pm 1\}^d} \sup_{v \in \mathbb{R}^d, \|v\|_2 = 1, a \in [-M, M]} \left[ \frac{1}{n} \sum_{i=1}^n \epsilon_i \max(0, v^T X_i - a) \right]
\]
(656)
\[
\leq 2 \mathbb{E}_{p, \epsilon \sim \{\pm 1\}^d} \sup_{v \in \mathbb{R}^d, \|v\|_2 = 1, a \in [-M, M]} \left[ \frac{1}{n} \sum_{i=1}^n \epsilon_i (v^T X_i - a) \right]
\]
(657)
\[
\leq 2 \mathbb{E}_{p, \epsilon \sim \{\pm 1\}^d} \sup_{v \in \mathbb{R}^d, \|v\|_2 = 1} \left[ \frac{1}{n} \sum_{i=1}^n \epsilon_i v^T X_i \right] + 2 \mathbb{E}_{\epsilon \sim \{\pm 1\}^d} \sup_{a \in [-M, M]} \left[ \frac{1}{n} \sum_{i=1}^n \epsilon_i a \right]
\]
(658)
\[
\leq 2 \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^n (X_i - \mathbb{E}^p[X]) \right]_2 + 2M \mathbb{E}_{\epsilon \sim \{\pm 1\}^d} \left[ \frac{1}{n} \sum_{i=1}^n \epsilon_i \right]
\]
(659)
\[
\leq 4 \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^n (X_i - \mathbb{E}^p[X]) \right]_2 + 2M \sqrt{\mathbb{E}_{\epsilon \sim \{\pm 1\}^d} \left( \frac{1}{n} \sum_{i=1}^n \epsilon_i \right)^2}
\]
(660)
\[
\leq 4 \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^n (X_i - \mathbb{E}^p[X]) \right]_2 + 2 \frac{M}{\sqrt{n}}
\]
(661)

Here Equation (660) comes from symmetrization inequality (Wainwright, 2019, Proposition 4.11).
For $a > M$, we have
\[
\mathbb{E}_p \left[ \sup_{v \in \mathbb{R}^d, \|v\|_2 = 1, a > M} \mathbb{E}_p \left[ \max(0, v^T X - a) \right] - \frac{1}{n} \sum_{i=1}^n \max(0, v^T X_i - a) \right] \quad (662)
\]
\[
\leq \mathbb{E}_p \left[ \sup_{v \in \mathbb{R}^d, \|v\|_2 = 1, a > M} \mathbb{E}_p \left[ \max(0, v^T X - a) \right] \right] \quad (663)
\]
\[
= \sup_{v \in \mathbb{R}^d, \|v\|_2 = 1, a > M} \mathbb{E}_p \left[ \max(0, v^T X - a) \right] \quad (664)
\]
\[
< \sup_{v \in \mathbb{R}^d, \|v\|_2 = 1} \mathbb{E}_p \left[ \max(0, v^T X - M) \right] \quad (665)
\]

For $a < -M$, we have
\[
\mathbb{E}_p \left[ \sup_{v \in \mathbb{R}^d, \|v\|_2 = 1, a < -M} \mathbb{E}_p \left[ \max(0, v^T X - a) \right] - \frac{1}{n} \sum_{i=1}^n \max(0, v^T X_i - a) \right] \quad (666)
\]
\[
= \mathbb{E}_p \left[ \sup_{v \in \mathbb{R}^d, \|v\|_2 = 1, a < -M} \mathbb{E}_p \left[ v^T X - a + \max(0, -v^T X + a) \right] - \frac{1}{n} \sum_{i=1}^n \left( v^T X_i - a + \max(0, -v^T X + a) \right) \right] \quad (667)
\]
\[
= \mathbb{E}_p \left[ \sup_{v \in \mathbb{R}^d, \|v\|_2 = 1, a < -M} \mathbb{E}_p \left[ v^T X - a \right] - \frac{1}{n} \sum_{i=1}^n \left( v^T X_i - a \right) + \left( \mathbb{E}_p \left[ \max(0, -v^T X + a) \right] - \frac{1}{n} \sum_{i=1}^n \max(0, -v^T X_i + a) \right) \right] \quad (668)
\]
\[
\leq \mathbb{E}_p \left[ \left\| \frac{1}{n} \sum_{i=1}^n X_i - \mathbb{E}_p[X] \right\|_2 \right] + \sup_{v \in \mathbb{R}^d, \|v\|_2 = 1, a < -M} \mathbb{E}_p \left[ \max(0, -v^T X + a) \right] \quad (669)
\]
\[
\leq \mathbb{E}_p \left[ \left\| \frac{1}{n} \sum_{i=1}^n X_i - \mathbb{E}_p[X] \right\|_2 \right] + \sup_{v \in \mathbb{R}^d, \|v\|_2 = 1} \mathbb{E}_p \left[ \max(0, -v^T X - M) \right] \quad (670)
\]
\[
= \mathbb{E}_p \left[ \left\| \frac{1}{n} \sum_{i=1}^n X_i - \mathbb{E}_p[X] \right\|_2 \right] + \sup_{v \in \mathbb{R}^d, \|v\|_2 = 1} \mathbb{E}_p \left[ \max(0, v^T X - M) \right] \quad (671)
\]

Thus we have
\[
\sup_{u \in U_2} \mathbb{E}_p[u(X)] - \frac{1}{n} \sum_{i=1}^n u(X_i) \leq 4 \mathbb{E}_p \left[ \left\| \frac{1}{n} \sum_{i=1}^n X_i - \mathbb{E}_p[X] \right\|_2 \right] + \frac{2M}{\sqrt{n}} + \sup_{v \in \mathbb{R}^d, \|v\|_2 = 1} \mathbb{E}_p \left[ \max(0, v^T X - M) \right] \quad (672)
\]

Following a similar argument, we have
\[
\sup_{u \in U_3} \mathbb{E}_p[u(X)] - \frac{1}{n} \sum_{i=1}^n u(X_i) \leq 4 \mathbb{E}_p \left[ \left\| \frac{1}{n} \sum_{i=1}^n X_i - \mathbb{E}_p[X] \right\|_2 \right] + \frac{2M}{\sqrt{n}} + \mathbb{E}_p \left[ \sup_{v \in \mathbb{R}^d, \|v\|_2 = 1} \frac{1}{n} \sum_{i=1}^n \max(0, v^T X_i - M) \right] \quad (673)
\]

To see the final results, we first show that
\[
\mathbb{E}_p \left[ \max(0, v^T X - M) \right] \leq \mathbb{E}_p \left[ \sup_{v \in \mathbb{R}^d, \|v\|_2 = 1} \frac{1}{n} \sum_{i=1}^n \max(0, v^T X_i - M) \right] \quad (674)
\]
This can be seen from that for any $v \in \mathbb{R}^d$, $\|v\|_2 = 1$,
\[
\frac{1}{n} \sum_{i=1}^n \max \left(0, v^\top X_i - M \right) \leq \sup_{v \in \mathbb{R}^d, \|v\|_2 = 1} \frac{1}{n} \sum_{i=1}^n \max \left(0, v^\top X_i - M \right). \tag{675}
\]

Taking expectation on both sides, we can see that for any $v \in \mathbb{R}^d$, $\|v\|_2 = 1$,
\[
\mathbb{E}_p[\max \left(0, v^\top X_i - M \right)] \leq \mathbb{E}_p[\sup_{v \in \mathbb{R}^d, \|v\|_2 = 1} \frac{1}{n} \sum_{i=1}^n \max \left(0, v^\top X_i - M \right)]. \tag{676}
\]

Thus we only need to bound the RHS of the above equation. Following the same approach of symmetrization and contraction inequality, we have
\[
\mathbb{E}_p \left[ \sup_{v \in \mathbb{R}^d, \|v\|_2 = 1} \frac{1}{n} \sum_{i=1}^n \max \left(0, v^\top X_i - M \right) \right]
\leq 2\mathbb{E}_{p, \epsilon \sim \{\pm 1\}^d} \left[ \sup_{v \in \mathbb{R}^d, \|v\|_2 = 1} \frac{1}{n} \sum_{i=1}^n \epsilon_i \max \left(0, v^\top X_i - M \right) \right] + \mathbb{E}_p[\max \left(0, v^\top X - M \right)]
\leq 2\mathbb{E}_{p, \epsilon \sim \{\pm 1\}^d} \left[ \sup_{v \in \mathbb{R}^d, \|v\|_2 = 1} \frac{1}{n} \sum_{i=1}^n \epsilon_i v^\top X_i \right] + \mathbb{E}_p[\max \left(0, v^\top X - M \right)]
\leq 4\mathbb{E}_p \left[ \sup_{v \in \mathbb{R}^d, \|v\|_2 = 1} \frac{1}{n} \sum_{i=1}^n v^\top (X_i - \mathbb{E}_p[X]) \right] + \mathbb{E}_p[\max \left(0, v^\top X - M \right)]
\leq 4\mathbb{E}_p \left[ \left\| \frac{1}{n} \sum_{i=1}^n v^\top X_i - \mathbb{E}_p[X] \right\|_2 \right] + \mathbb{E}_p[\max \left(0, v^\top X - M \right)]. \tag{677}
\]

Overall, we have
\[
\mathbb{E}_p[\tilde{W}_1(p, \hat{p}_n)] \leq 8\mathbb{E}_p \left[ \left\| \frac{1}{n} \sum_{i=1}^n X_i - \mathbb{E}_p[X] \right\|_2 \right] + \mathbb{E}_p[\max(0, v^\top X - M)] + 2\frac{M}{\sqrt{n}}. \tag{678}
\]

Combining all the results give the conclusion.

Now we are ready for the proof of main Lemma. Note that $\tilde{W}_1$ is a pseudometric. By triangle inequality, we have
\[
\tilde{W}_1(p, \hat{p}_n) \leq \tilde{W}_1(p, p^*) + \tilde{W}_1(p^*, \hat{p}_n^*) + \tilde{W}_1(\hat{p}_n^*, \hat{p}_n). \tag{679}
\]

Taking the expectation over the optimal coupling $\pi$ between $p, p^*$, by Lemma 2.1, we know that
\[
\mathbb{E}_p[\tilde{W}_1(p, \hat{p}_n)] \leq \tilde{W}_1(p, p^*) + \mathbb{E}_{p^*}[\tilde{W}_1(p^*, \hat{p}_n^*)] + \mathbb{E}_\pi[\tilde{W}_1(\hat{p}_n^*, \hat{p}_n)]
\leq \epsilon + \mathbb{E}_{p^*}[\tilde{W}_1(p^*, \hat{p}_n^*)] + \mathbb{E}_\pi[\tilde{W}_1(\hat{p}_n^*, \hat{p}_n)]
\leq 2\epsilon + \mathbb{E}_{p^*}[\tilde{W}_1(p^*, \hat{p}_n^*)]. \tag{680}
\]
Thus it suffices to bound the term $\mathbb{E}_{p^*}[\hat{W}_1(p^*, \tilde{p}_n^*)]$. By Lemma 6.1, we have for any $M > 0$,

$$
\mathbb{E}_{p^*}[\hat{W}_1(p^*, \tilde{p}_n^*)] \leq 8\xi_1 + \xi_2(M) + 2\frac{M}{\sqrt{n}}.
$$

(681)

where $\xi_1 = \mathbb{E}_{p^*}\left\| \frac{1}{n}\sum_{i=1}^{n} X_i - \mathbb{E}_{p^*}[X]\right\|_2$, $\xi_2(M) = \sup_{v \in \mathbb{R}^d, \|v\|_2 = 1} \mathbb{E}_{p^*}\left[ \max(0, v^\top(X - \mathbb{E}_{p^*}[X]) - M) \right]$. Now we bound the two terms $\xi_1, \xi_2$ separately. From Lemma A.4, we know that

$$
\xi_1 = \mathbb{E}_{p^*}\left\| \frac{1}{n}\sum_{i=1}^{n} X_i - \mathbb{E}_{p^*}[X]\right\|_2 \leq \sigma \sqrt{\frac{d}{n}}.
$$

(682)

Now we bound the term $\xi_2(M)$. From Lemma C.2, we know for some fixed $v \in \mathbb{R}^d, \|v\|_2 = 1$,

$$
\|\mathbb{E}_{p^*}[X | v^\top(X - \mathbb{E}_{p^*}[X]) \geq M] - \mathbb{E}_{p^*}[X]\|_2 \leq \kappa\psi^{-1}(1/(1 - \mathbb{P}_{p^*}(v^\top(X - \mathbb{E}_{p^*}[X]) \leq M)))
$$

(683)

By Markov’s inequality, we have

$$
\mathbb{P}_{p^*}(v^\top(X - \mathbb{E}_{p^*}[X]) \geq M) \leq \mathbb{P}_{p^*}(\psi(|v^\top(X - \mathbb{E}_{p^*}[X])/\kappa|) \geq \psi(M/\kappa))
$$

$$
= \mathbb{E}_{p^*}[\psi(|v^\top(X - \mathbb{E}_{p^*}[X])/\kappa|)] / \psi(M/\kappa)
$$

(684)

Thus we have

$$
\xi_2(M) = \sup_{v \in \mathbb{R}^d, \|v\|_2 = 1} \mathbb{E}_{p^*}\left[ \max(0, v^\top(X - \mathbb{E}_{p^*}[X]) - M) \right]
$$

$$
= \sup_{v \in \mathbb{R}^d, \|v\|_2 = 1} \mathbb{P}_{p^*}(v^\top(X - \mathbb{E}_{p^*}[X]) \geq M)\|\mathbb{E}_{p^*}[X - \mathbb{E}_{p^*}[X] | v^\top(X - \mathbb{E}_{p^*}[X]) \geq M]\|_2
$$

$$
\leq \sup_{v \in \mathbb{R}^d, \|v\|_2 = 1} \kappa\mathbb{P}_{p^*}(v^\top(X - \mathbb{E}_{p^*}[X]) \geq M)\psi^{-1}(1/\mathbb{P}_{p^*}(v^\top(X - \mathbb{E}_{p^*}[X]) \geq M))
$$

$$
\leq \frac{M}{\psi(M/\kappa)}.
$$

(685)

The last inequality uses the fact that $\epsilon\psi^{-1}(1/\epsilon)$ is nondecreasing from Lemma A.1. Now we balance the term $\xi_2(M) + 2\frac{M}{\sqrt{n}}$. By taking $M = \kappa\psi^{-1}(\sqrt{n})$, we have

$$
\xi_2(M) + 2\frac{M}{\sqrt{n}} \leq \frac{3\kappa\psi^{-1}(\sqrt{n})}{\sqrt{n}}.
$$

(686)

\[\square\]

### F.3 General proposition of $\hat{W}_1$ projection algorithm

We first slightly generalize $G^{W,c,k}$ in Definition 4.2 to $G^{W,c,k}(\rho_1, \rho_2, \eta) = G_{\downarrow}^{W,c,k}(\rho_1, \eta) \cap G_{\uparrow}^{W,c,k}(\rho_1, \rho_2, \eta)$, where

$$
G_{\downarrow}^{W,c,k}(\rho_1, \eta) = \left\{ p \mid \sup_{f \in \mathbb{F}_{\theta^*(p), g \in \mathbb{P}(f), r \in \mathbb{P}(\eta, W,c,k,g)}} \mathbb{E}_r[f(X)] - B^*(f, \theta^*(p)) \leq \rho_1 \right\},
$$

(687)

$$
G_{\uparrow}^{W,c,k}(\rho_1, \rho_2, \eta) = \left\{ p \mid \forall \theta, \left( \sup_{f \in \mathbb{F}_{\eta}, g \in \mathbb{P}(f), r \in \mathbb{P}(\eta, W,c,k,g)} \mathbb{E}_r[f(X)] - B^*(f, \theta) \leq \rho_1 \right) \Rightarrow \mathcal{L}(p, \theta) \leq \rho_2 \right\}.
$$

(688)
Here the only difference between Definition 4.2 is that we allow the friendly perturbation has different projection function $g$ rather than $f$, which is needed in Theorem 6.1. In second moment cases, we have $f(x) = (v^T x)^2$ while $g(x) = |v^T x|$. It follows the same proof as Theorem 4.3 that the modulus of continuity for the above set is bounded.

In this section, we make the assumption that $\mathcal{F}_\theta$ contains only the function of the form $f(|v^T x|)$, where all the $f$ are convex functions. We show the robustness guarantee for general projection algorithm $q = \Pi(\hat{p}_n; \tilde{W}_1, G^{W_1})$. The following proposition is a corollary of Theorem 7.1.

**Proposition F.1.** Under the oblivious corruption model of level $\epsilon$ with $\tilde{W}_1$ perturbation, where $\tilde{W}_1$ is defined in (87). Assume $p^*$ satisfies that

$$
\sup_{v \in \mathbb{R}^d, \|v\|_2 = 1} \mathbb{E}_{p^*}[\psi(|v^T X|/\kappa)] \leq 1,
$$

(689)

$$
\|\Sigma_{p^*}\|_2 \leq \sigma^2.
$$

(690)

Denote the empirical distribution of observed data as $\hat{p}_n$, and

$$
\tilde{\epsilon} = 2(1 + \frac{1}{\delta})\epsilon + \frac{8\sigma}{\delta}\sqrt{\frac{d}{n}} + \frac{3\kappa\psi^{-1}(\sqrt{n})}{\delta\sqrt{n}}.
$$

(691)

Suppose $g(x) = either v^T x or |v^T x|$. When we take the projection set $G = G^{W_1}(\rho_1(7\epsilon), \rho_2(7\epsilon), 7\epsilon)$ in Definition 4.2, where all functions in $\mathcal{F}_\theta$ are of the form of $f(g(x))$ for some convex function $f$, and $\mathcal{P}(f(g(x))) = \{g(x)\}$, we have with probability at least $1 - \delta$, the projection algorithm $\Pi(\hat{p}_n; \tilde{W}_1, G^{W_1})$ or $\Pi(\hat{p}_n; W_1, G^{W_1}, \tilde{\epsilon}/2)$ satisfies

$$
L(p^*, \theta^*(q)) \leq \rho_2(7\epsilon).
$$

(692)

**Proof.** The two conclusions are all corollaries of Theorem 7.1. We only need to verify the two conditions in Theorem 7.1.

1. **Robust to perturbation:** Note that $\tilde{W}_1$ satisfies triangle inequality and $U' \subset U$. For any $p_1, p_2, p_3$, we have

$$
|\tilde{W}_1(p_1, p_2) - \tilde{W}(p_1, p_3)| = \sup_{u \in U'} |\mathbb{E}_{p_1}[u(X)] - \mathbb{E}_{p_2}[u(X)]| - \sup_{u \in U'} |\mathbb{E}_{p_1}[u(X)] - \mathbb{E}_{p_3}[u(X)]| \\
\leq \sup_{u \in U'} |\mathbb{E}_{p_2}[u(X)] - \mathbb{E}_{p_3}[u(X)]| \\
= \tilde{W}_1(p_2, p_3) \\
\leq W_1(p_2, p_3).
$$

(693)

2. **Generalized Modulus of Continuity:** Assume $p_1, p_2 \in G^{W_1}(\rho_1(7\epsilon), \rho_2(7\epsilon), 7\epsilon)$ and $\tilde{W}_1(p_1, p_2) \leq \tilde{\epsilon}$. For any fixed $v$, from Lemma 6.2, we know that when $g(x)$ takes either $v^T x$ or $|v^T x|$, there exists an $r_{p_1} \in F(p_1, 7\epsilon, W_1, g(x))$ and an $r_{p_2} \in F(p_2, 7\epsilon, W_1, g(x))$ such that for convex $f(x)$, we have

$$
\mathbb{E}_{r_{p_1}}[f(g(X))] \leq \mathbb{E}_{r_{p_2}}[f(g(X))].
$$

(694)

From $p_2 \in G^{W_1}_+$, we know that

$$
\sup_{f \in F_{\rho_2}(p_2), g \in \mathcal{P}(f), r \in \mathcal{F}(p_2, 7\epsilon, W_{c, k}, g(x))} \mathbb{E}_r[f(X)] - B^*(f, \theta^*(p_2)) \leq \rho_1(7\epsilon).
$$

(695)

87
Thus we know for any given $f, g$, and any friendly perturbation $r_{p_2}$, the above inequality holds. Thus we also have

$$
\mathbb{E}_{r_1} [f(X)] - B^*(f, \theta^*(p_2)) \leq \mathbb{E}_{r_2} [f(X)] - B^*(f, \theta^*(p_2)) \leq \rho_1(7\tilde{\epsilon}).
$$

(696)

Rewriting the above statement, we know that

$$
\sup_{f \in \mathcal{F}, \theta \in \mathcal{G}(v, W_{c,k}(x)), g \in \mathcal{P}(f)} \inf_{r_1, \tilde{\epsilon}} \mathbb{E}_r [f(X)] - B^*(f, \theta^*(p_2)) \leq \rho_1(7\tilde{\epsilon}).
$$

(697)

Since we also know that $p_1 \in \mathcal{G}_d W$, we have

$$
L(p_1, \theta^*(p_2)) \leq \rho_2(7\tilde{\epsilon}).
$$

(698)

Thus from Theorem 7.1, we have with probability at least $1 - \delta$,

$$
L(p^*, \theta^*(q)) \leq \rho_2(7\tilde{\epsilon}).
$$

(699)

We provide ways to bound the statistical error term $\tilde{W}_1(\hat{p}_n, p)$ in Lemma 6.1. Combining these two lemmas gives the results.

\[\square\]

G Related discussions in Section 7: Connections with robust optimization

We provided two approaches to analyze the finite sample projection algorithm $q = \Pi(\hat{p}_n; \tilde{D}, \mathcal{M})$: oblivious analysis in Theorem 7.1 and adaptive analysis in Theorem 7.2. In this section, we build the connections between our projection algorithms and DRO.

We first show that under appropriate conditions (Theorem G.1), the projection algorithm is approximately solving the following distributionally robust optimization problem, and any approximate solution of the DRO below produces a decent robust estimate of $\theta$.

$$
\hat{\theta} = \arg \min_{\theta(\hat{p}_n)} \sup_{r : r \in \mathcal{G}, \tilde{D}(r, \hat{p}_n) \leq \epsilon} L(r, \hat{\theta}(\hat{p}_n))
$$

(700)

Note that here we use $r$ to denote the dummy variable in the DRO.

**Theorem G.1.** Assume the oblivious contamination model of level $\epsilon$ under $D$. Denote the true distribution as $p^* \in \mathcal{G}$ and the perturbed population distribution as $p$ with $\tilde{D}(p, p^*) \leq \epsilon$. Denote the cost function as $L(p^*, \theta)$ and the empirical distribution of observed data as $\hat{p}_n$. Assume the following conditions.

1. **Robust to perturbation:** $\tilde{D}(p, q)$ is a pseudometric that satisfies

   $$
   \tilde{D}(p, q) \leq D(p, q).
   $$

   (701)

2. **Generalized Modulus of Continuity:** there exists a set $\mathcal{M} \supset \mathcal{G}$ and for $\tilde{\epsilon} = 2\epsilon + 2\tilde{D}(p, \hat{p}_n)$, we have

   $$
   \sup_{p_1^* \in \mathcal{M}, p_2^* \in \mathcal{G}, \tilde{D}(p_1^*, p_2^*) \leq \tilde{\epsilon}} L(p_2^*, \theta^*(p_1^*)) \leq \rho(\tilde{\epsilon}).
   $$

   (702)
Then,

1. Any approximate solution of DRO suffices: for any \( \theta, \rho \), if

\[
\sup_{r : r \in G, \tilde{D}(r, \hat{p}_n) \leq \epsilon + \tilde{D}(p, \hat{p}_n)} L(r, \theta) \leq \rho,
\]

then

\[
L(p^*, \theta) \leq \rho.
\] (704)

2. Projection approximately solves DRO: for \( q = \Pi(\hat{p}_n; \tilde{D}, \mathcal{M}, \epsilon + \tilde{D}(p, \hat{p}_n)) \), \( \theta^*(q) = \arg \min_{\theta \in \Theta} L(q, \theta) \), there is

\[
\sup_{r : r \in G, \tilde{D}(r, \hat{p}_n) \leq \epsilon + \tilde{D}(p, \hat{p}_n)} L(r, \theta^*(q)) \leq \rho(2\epsilon + 2\tilde{D}(p, \hat{p}_n)).
\] (705)

Proof. Regarding the first claim, it suffices to show that \( \tilde{D}(p^*, \hat{p}_n) \leq \epsilon + \tilde{D}(p, \hat{p}_n) \). It is true since \( \tilde{D} \) is a pseudometric and \( \tilde{D} \leq D \):

\[
\tilde{D}(p^*, \hat{p}_n) \leq \tilde{D}(p^*, p) + \tilde{D}(p, \hat{p}_n) \leq D(p^*, p) + \tilde{D}(p, \hat{p}_n) \leq \epsilon + \tilde{D}(p, \hat{p}_n).
\]

Now we verify the second claim. In order to apply the Generalized Modulus of Continuity property, it suffices to show that for any \( r \) such that \( r \in G, \tilde{D}(r, \hat{p}_n) \leq \epsilon + \tilde{D}(p, \hat{p}_n) \) and \( q \in \mathcal{M} \supset G \) being the output of the projection algorithm, we have

\[
\tilde{D}(q, r) \leq 2\epsilon + 2\tilde{D}(p, \hat{p}_n).
\] (709)

Note that \( q \) is the output of the projection algorithm implies either \( \tilde{D}(q, \hat{p}_n) \leq \epsilon + \tilde{D}(p, \hat{p}_n) \) or \( \tilde{D}(q, \hat{p}_n) \leq \tilde{D}(r, \hat{p}_n) \leq \epsilon + \tilde{D}(p, \hat{p}_n) \). Hence,

\[
\tilde{D}(q, r) \leq \tilde{D}(q, \hat{p}_n) + \tilde{D}(r, \hat{p}_n) \leq 2(\epsilon + \tilde{D}(p, \hat{p}_n)).
\] (711)

The above theorem shows that solving DRO in Equation (700) will also provide robustness guarantee. When we specify \( G = \mathcal{G}^TV \), we remark that we can solve the following DRO:

\[
\hat{\theta} = \arg \min_{\hat{\theta}(\hat{p}_n)} \sup_{p^*: p^* \in \mathcal{G}^TV \uparrow, \tilde{D}(p^*, \hat{p}_n) \leq \epsilon + \tilde{D}(p, \hat{p}_n)} L(p^*, \hat{\theta}(\hat{p}_n))
\] (712)

It is interesting to note that when solving DRO, we only need to consider the \( p^* \) inside \( \mathcal{G}^TV \uparrow \). Meanwhile for projection algorithm, it suffices to project \( \hat{p}_n \) onto \( \mathcal{G}^TV \).

We now show that similar interpretations can also be made for the DRO formulation below.

\[
\hat{\theta} = \arg \min_{\hat{\theta}(\hat{p}_n)} \sup_{r : r \in G', \tilde{D}(r, \hat{p}_n) \leq \epsilon_1 + \epsilon_2} L(r, \hat{\theta}(\hat{p}_n))
\] (713)
Theorem G.2. Assume either oblivious contamination or adaptive contamination model of level $\epsilon$ under $D$. Denote the true distribution as $p^* \in \mathcal{G}$, $\hat{p}_n^*$ as the empirical distribution sampled from $p^*$ and $\hat{p}_n$ as the empirical distribution of observed data. Denote the cost function as $L(p^*, \theta)$. Assume the following conditions.

1. **Project via pseudometric:** $\tilde{D}(p, q)$ is a pseudometric.

2. **Limited contamination:** $\tilde{D}(\hat{p}_n, \hat{p}_n^*) \leq \epsilon_2$ with probability at least $1 - \delta$.

3. **Set for (perturbed) empirical distribution:** there exists a set $\mathcal{G}' \subset \mathcal{M}$ such that there exists a distribution $\hat{p}' \in \mathcal{G}'$ satisfying $\tilde{D}(\hat{p}_n^*, \hat{p}') \leq \epsilon_1$ with probability at least $1 - \delta$.

4. **Generalized Modulus of Continuity:** $\mathcal{G}' \subset \mathcal{M}$, and for $\tilde{\epsilon} = 2(\epsilon_1 + \epsilon_2)$, there is

$$\sup_{p_1^* \in \mathcal{M}, p_2^* \in \mathcal{G}', \tilde{D}(p_1^*, p_2^*) \leq \tilde{\epsilon}} L(p_2^*, \theta^*(p_1^*)) \leq \rho(\tilde{\epsilon}), \quad (714)$$

5. **Generalization bound:** for any $p^* \in \mathcal{G}$, $\theta \in \Theta$, there exists some constant $C$ and some function $g$ such that $L(p^*, \theta) \leq C \cdot L(\hat{p}', \theta) + g(\hat{p}', p^*)$.

Then,

1. Any approximate solution of DRO suffices: for any $\theta, \rho$, if

$$\sup_{r: r \in \mathcal{G}', \tilde{D}(r, \hat{p}_n) \leq \epsilon_1 + \epsilon_2} L(r, \theta) \leq \rho, \quad (715)$$

then, with probability at least $1 - 2\delta$,

$$L(p^*, \theta) \leq C\rho + g(\hat{p}', p^*). \quad (716)$$

2. Projection approximately solves DRO: for $q = \Pi(\hat{p}_n; \tilde{D}, \mathcal{M}, \epsilon_1 + \epsilon_2), \theta^*(q) = \arg\min_{\theta \in \Theta} L(q, \theta),

$$\sup_{r: r \in \mathcal{G}', \tilde{D}(r, \hat{p}_n) \leq \epsilon_1 + \epsilon_2} L(r, \theta^*(q)) \leq \rho(2\epsilon_1 + 2\epsilon_2). \quad (717)$$

Proof. We know that with probability at least $1 - 2\delta$, there exist $\hat{p}' \in \mathcal{G}'$, $\tilde{D}(\hat{p}', \hat{p}_n^*) \leq \epsilon_1$, $\tilde{D}(\hat{p}_n^*, \hat{p}_n) \leq \epsilon_2$. Hence

$$\tilde{D}(\hat{p}', \hat{p}_n) \leq \tilde{D}(\hat{p}', \hat{p}_n^*) + \tilde{D}(\hat{p}_n^*, \hat{p}_n) \leq \epsilon_1 + \epsilon_2. \quad (718)$$

Hence, we know $L(\hat{p}', \theta) \leq \rho$, and it follows from the generalization bound that

$$L(p^*, \theta) \leq C\rho + g(\hat{p}', p^*). \quad (720)$$

Now we prove the second claim using the Generalized Modulus of Continuity property. It suffices to show that for any $r \in \mathcal{G}'$, $\tilde{D}(r, \hat{p}_n) \leq \epsilon_1 + \epsilon_2$, $q \in \mathcal{M} \supset \mathcal{G}'$ being the output of the projection algorithm, we have

$$\tilde{D}(q, r) \leq 2\epsilon_1 + 2\epsilon_2. \quad (721)$$

Note that $q$ is the output of the projection algorithm implies either $\tilde{D}(q, \hat{p}_n) \leq \epsilon_1 + \epsilon_2$ or $\tilde{D}(q, \hat{p}_n) \leq \tilde{D}(r, \hat{p}_n) \leq \epsilon_1 + \epsilon_2$. Hence,

$$\tilde{D}(q, r) \leq \tilde{D}(q, \hat{p}_n) + \tilde{D}(\hat{p}_n, r) \leq 2(\epsilon_1 + \epsilon_2). \quad (722)$$

□