THE $C^*$-ALGEBRA OF THE SEMI-DIRECT PRODUCT
$K \ltimes A$.

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Abstract. Let $G = K \ltimes A$ be the semi-direct product group of a compact group $K$ acting on an abelian locally compact group $A$. We describe the $C^*$-algebra $C^*(G)$ of $G$ in terms of an algebra of operator fields defined over the spectrum of $G$, generalizing previous results obtained for some special classes of such groups.

1. Introduction

It is well known that for a simply connected nilpotent Lie group and more generally for an exponential solvable Lie group $G = \exp(g)$, its dual space $\hat{G}$ is homeomorphic to the space of co-adjoint orbits $g^*/G$ through the Kirillov mapping (see [7]). If we consider semi-direct products $G = K \ltimes N$ of compact connected Lie groups $K$ acting on simply connected nilpotent Lie groups $N$, then again we have an orbit picture of the dual space of $G$ (see [8]) and one can imagine that the topology of $\hat{G}$ is linked to the topology of the admissible co-adjoint orbits.

By definition the $C^*$-algebra $C^*(G)$ of a locally compact group $G$ is the completion of the convolution algebra $L^1(G)$ with respect to the norm

$$\|f\|_{C^*(G)} := \sup_{\pi \in \hat{G}} \|\pi(f)\|_{\text{op}}.$$  

The unitary dual or spectrum $C^*(G)$ of $C^*(G)$ is in bijection with the dual space $\hat{G}$ of $G$. Define the Fourier transform $\mathcal{F}$ on $C^*(G)$ by

$$\mathcal{F}(c)(\pi) = \pi(c) \in \mathcal{B}(\mathcal{H}_\pi), \quad \text{for all } \pi \in \hat{G}, \ c \in C^*(G).$$  

Then $C^*(G)$ can be identified with the sub-algebra $\widehat{C^*(G)}$ of the big $C^*$-algebra $\ell^\infty(\hat{G})$ of bounded operator fields given by

$$\ell^\infty(\hat{G}) = \left\{ \phi : \hat{G} \to \bigcup_{\pi \in \hat{G}} \mathcal{B}(\mathcal{H}_\pi); \phi(\pi) \in \mathcal{B}(\mathcal{H}), \|\phi\|_\infty := \sup_{\pi \in \hat{G}} \|f(\pi)\|_{\text{op}} < \infty \right\}.$$  

Here $\mathcal{B}(\mathcal{H})$ denotes the space of bounded linear operators on the Hilbert space $\mathcal{H}$.

In order to understand the structure of the algebra $C^*(G)$, we must determine the special conditions which determine the operator fields $\phi \in \widehat{C^*(G)}$. 


For instance if $G = K$, then it is easy to see that

$$\hat{C}^\star(K) = \left\{ F : \hat{K} \to \bigcup_{\pi \in \hat{K}} B(\mathcal{H}_\pi) \mid F(\pi) \in B(\mathcal{H}_\pi), \lim_{\pi \to \infty} \|F(\pi)\|_{\text{op}} = 0 \right\}.$$  

Since $K$ is compact, its irreducible unitary representations are finite dimensional. So $\pi(c)$ is always trivially a compact operator. Furthermore, the spectrum of $K$ has the discrete topology. So there is no continuity condition for the operator fields.

If $G = A$ is abelian, then $\hat{A}$ is a locally compact Hausdorff space and

$$\hat{C}^\star(A) = \left\{ \phi : \hat{A} \to \mathbb{C}; \phi \text{ continuous and vanishes at infinity} \right\}.$$  

If $G$ is more generally a locally compact group of the form $K \rtimes A$, then its dual space $\hat{G}$ has a complicated structure determined by the $K$-orbits in $\hat{A}$ and the spectra of the stabilizer groups of the elements of $\hat{A}$. The topology of $\hat{G}$ has been described in the paper of W. Bagget [2], which will be used intensively here. For the motion groups of the form $G_n = SO(n) \times \mathbb{R}^n$ the conditions for $\hat{C}^\star(G_n)$ have been described explicitly in [3]. For some other groups only a detailed description of the topology of $\hat{G}$ is known (see for instance [4] and [9]).

In section 2, we recall the results of Bagget on the topology of $\hat{K} \rtimes A$, define the Fourier transform for $\hat{C}^\star(G)$ and discover the conditions which determine the algebra $\hat{C}^\star(G)$ inside the big algebra $\ell^\infty(\hat{G})$ (see Definition 2.20). To see what are the difficulties, consider a converging net $\Pi_M = (\pi_m = \pi_{\mu_m, \chi_m})_{m \in M}$ of irreducible representations of $G$. Here $(\chi_m)_{m \in M}$ is a net of unitary characters of the abelian group $A$, $K_m$ is the stabilizer of $\chi_m$ in $K$ and $\mu_m$ is an element of $K_m, m \in M$. Then $\pi_m = \text{ind}_{K_m \rtimes A}^{G} \mu_m \otimes \chi_m$. The problem is to understand for $a \in \hat{C}^\star(G)$ the behavior of the net of operators $(\pi_m(a))_{m \in M}$ acting on the different Hilbert spaces $\mathcal{H}_{\pi_m}$. We shall in fact construct for a certain converging subnet of the net $\Pi_M$ a common Hilbert space $\mathcal{H}_M$, such that $\mathcal{H}_M$ contains a copy $\mathcal{H}_m$ of the Hilbert space $\mathcal{H}_m$ of $\pi_m$ and a projection $P_m : \mathcal{H}_M \to \mathcal{H}_m$ for every $m$ in the subnet and we show that the essential condition for an operator field $\Phi$ to be an element of $\hat{C}^\star(G)$ is an operator norm convergence of the net $(\Phi(\pi_m))_{m \in M}$ to a limit operator $\sigma(\Phi_L) \in B(\mathcal{H}_M)$ which is determined by the restriction $\Phi_L$ of the operator field $\Phi$ to the limit set $L$ of the subnet.

In the last paragraph we present as example the group $G_{n,m} = SO(n) \times SO(m) \rtimes \mathbb{R}^{n+m}$, for $n, m \in \mathbb{N}$.

2. The $C^\star$-algebra of the group $G = K \rtimes A$.

2.1. Preliminaries. Let $K$ be locally compact group, $A$ be an abelian and suppose $\Psi$ is homomorphism of $K$ into the group of automorphisms of $A$
such that the mapping \( \Psi : K \times A \to A \) is continuous. For simplicity, we write the action of the automorphism \( \Psi(k) \) on an element \( a \) of \( A \) as \( k \cdot a \).

The semi-direct product \( G = K \rtimes A \) of the groups \( K \) and \( A \) is the following locally compact group. \( G \) is the topological product of \( K \) and \( A \) and \( G \) is equipped with the group law

\[
(k, a) \cdot (h, b) = (kh, h^{-1} \cdot a + b), \quad \forall (k, a), (h, b) \in G,
\]

where we write the multiplication in \( K \) with the symbol \( \cdot \) and multiplication in \( A \) additively as \( + \).

The group \( A \) can be homeomorphically and isomorphically identified with the closed normal subgroup of \( G \) consisting of all pairs \((1_K, a)\), where \( a \) is an element of \( A \). Also, \( K \) is homeomorphic and isomorphic to the closed subgroup of \( G \) consisting of all pairs \((k, 0_A)\), where \( k \) is an element of \( K \). (We write \( 1_K \) for the multiplicative identity of \( K \), \( 0_A \) for the additive identity of \( A \), and \((1_K, 0_A)\) for the identity of \( G \)).

Now, let \( k \in K \) and \( \chi : A \to \mathbb{T} \in \hat{A} \) be a unitary character of \( A \), and let \( k \cdot \chi \) to be the character of \( A \) defined by

\[
k \cdot \chi(b) = \chi(k^{-1} \cdot b), \quad b \in A.
\]

Thus \( K \) acts on the left as a group of continuous transformations on \( \hat{A} \). If \( \chi \) is in \( \hat{A} \), the stability subgroup \( K_\chi \) of \( \chi \) is the closed subgroup

\[
K_\chi = \{ k \in K | k \cdot \chi = \chi \}.
\]

**Definition 2.1.**

1. We define the space \( \mathcal{K}(G) \) to be the collection of all closed subgroups of \( G \) equipped with the compact-open topology (see [5]).
2. Let \( \mathcal{A}(G) \) denote the set of all pairs \((C, T)\) where \( C \) is a closed subgroup of \( G \) and \( T \) is an irreducible unitary representation of \( C \).

**Proposition 2.2.** [see, [2] 2.1-D] Any closed subgroup of \( G \) which contains \( A \) is of the form \( J \ltimes A \), where \( J \) is a closed subgroup of \( K \). Further, a net \((J_n)\) of closed subgroups of \( K \) converges to \( J \) in \( \mathcal{K}(K) \) if and only if the net \((J_n \ltimes A)\) of subgroups of \( G \) converges to the subgroup \( J \ltimes A \) in \( \mathcal{K}(G) \).

Let now \( \chi \in \hat{A} \). Then the Hilbert space of the representation \( \delta = \text{ind}_{\{1_K \ltimes A\}}^G \chi \) can be identified with \( L^2(K) \), thus if \((k, a) \in G, f \in L^2(K)\) and \( h \in K \) we have

\[
\delta(k, a)(f)(h) = \delta((k, 0_A)(1_K, a))(f)(h) = \chi((h^{-1}k) \cdot a)f(k^{-1}h).
\]
2.2. The topology of the dual space of the group $G$.

The dual space or spectrum of $G$ has been described by G. Mackey (for details, see [10] and [11]).

For each character $\chi \in \hat{A}$ and any irreducible unitary representation $\mu$ of the stabilizer $K_{\chi}$ of $\chi$ in $K$, we have that

\begin{equation}
\sigma_{(\mu, \chi)} := \mu \otimes \chi
\end{equation}

is an irreducible unitary representation of

$G_{\chi} := K_{\chi} \ltimes A$,

whose restriction to $A$ is a multiple of $\chi$ (see [2] proposition 2 p.181) and the induced representation $\pi_{(\mu, \chi)} := \text{ind}^{G}_{G_{\chi}} \sigma_{(\mu, \chi)}$ is an irreducible representation of $G$.

On the other hand, every irreducible unitary representation $\tau_{\lambda}$ of $K$ extends to an irreducible representation (also denoted by $\tau_{\lambda}$) of the entire group $G$ defined by

$\tau_{\lambda}(k, a) := \tau_{\lambda}(k), \quad (k, a) \in G$.

The following Propositions give the relationship between $\hat{G}$ and the set of all elements $\pi_{(\mu, \chi)}$ (see [12])

Proposition 2.3. Let $\pi$ be an irreducible unitary representation of $G$. Then $\pi|_{A}$ is supported by a $K$-orbit $\theta$ in $\hat{A}$. Suppose $\chi$ is an element of $\theta$. Then $\pi$ is equivalent to a representation of the form $\pi_{(\mu, \chi)}$ where $\mu$ is an irreducible unitary representation of $K_{\chi}$.

Proposition 2.4. Let $\pi$ and $\theta$ be as in the above proposition. Assume $\chi$ and $\chi'$ are elements of $\theta$, i.e., $\chi' = k \cdot \chi$ for some element $k$ of $K$. Suppose further that $\pi$ is equivalent to $\pi_{(\mu, \chi)}$ and also is equivalent to $\pi_{(\mu', \chi')}$ for two elements $\mu \in K_{\chi}, \mu' \in K_{\chi'}$. Then

1. $G_{\chi'} = k \cdot G_{\chi} \cdot k^{-1}$.
2. The representation $h \mapsto \mu'(khk^{-1}), h \in K_{\chi}$, is equivalent to the representation $\mu$.

Definition 2.5. A cataloguing triple we mean a triple $(\chi, J, \mu)$, where $\chi$ is a character of $A$, $J$ is the stabilizer $K_{\chi}$ and $\mu$ is an irreducible unitary representation of $K_{\chi}$. We denote by $\pi_{(\mu, \chi)} := \pi_{(\mu, \chi)}$ the induced representation $\text{ind}^{G}_{J \ltimes A}(\mu \otimes \chi)$.

By Baggett in [2] (Proposition 2.4-D p.187), we have

Proposition 2.6. The mapping $(\chi, K_{\chi}, \mu) \mapsto \pi_{(\mu, \chi)}$ is onto $\hat{G}$.

For the proof of the following theorem, see §§4. 1-D of [2].

Theorem 2.7. The $C^{*}$-algebra $C^{*}(G)$ of the locally compact group $G = K \ltimes A$ is CCR.

Definition 2.8. Let $F$ be the set of all functions $f$ which satisfy:
Theorem 2.11. The topology of $\hat{G}$ may be described as follows: Let $B$ be a subset of $\hat{G}$ and $\pi$ an element of $\hat{G}$. $\pi$ is contained in the closure of $B$ if and only if there exist: a cataloguing triple $(\chi, (K, \mu))$ for $\pi$, an element $(H, S)$ of $A(K)$ and a net $(\chi_n, (K_n, \mu_n))_n$ of cataloguing triples, such that:

1. For each $n$, the element $\pi_{(\mu_n, \chi_n)}$ of $\hat{G}$ is an element of $B$.
2. The net $(\chi_n, (K_n, \mu_n))$ converges to $(\chi, (K', \mu'))$ in $\hat{A} \times A(K)$.
3. $K' \chi$ contains $K'$, and $\text{ind}_{K'}^{K'} \mu'$ contains $\mu$.

The following lemma is the key for Definition 2.20.

Lemma 2.12. Let $(\chi_n, (K_n, \mu_n))_n$ be a properly converging net with the limit $(\chi, (K', \mu'))$ (i.e. $K'$ is a closed subgroup of $K \chi$ and $\mu' \in \hat{K'}$). Then for some subnet we have that

$$\dim(\mu_k) = d_{\mu_k} = d_{\mu'} = \dim(\mu')$$

for all $k$ in the subnet.

Proof. The limit set $L$ of the net $(\pi_k)_k$ in $\hat{G}$ is according to [2] the set

$$L = \{\pi_{(\nu, \chi)}; \nu \in \hat{K'} \chi, \nu|_{K'} \ni \mu'\}.$$
We can as in [2] realize all these representations $\pi(\mu, \chi)$ on subspaces in the common Hilbert space $L^2(K)$. Take some $\mu \in \widehat{K}_\chi$ such that $\mu|_{K'}$ contains $\mu'$. Choose a $\rho \in \widehat{K}$, such that $\rho|_{K\chi}$ contains $\mu$. Let $L^2(K)^\rho$ be the minimal left and right translation $K$-invariant subspace of $L^2(K)$ containing $\mathcal{H}_\rho$ and thus also a copy of the Hilbert space $\mathcal{H}_\mu$ of the representation $\mu$ of $K\chi$. Then $L^2(K)^\rho$ has dimension $d_\rho$.

Then $L^2(K)^\rho$ also contains a copy $(\mathcal{H}_{\mu'}, \mu')$ of the irreducible representation $\mu'$ of $K'$. Let $\mathcal{H}^{\mu'}$ be the $\mu'$-isotopic component inside $\mathcal{H}_\rho$. We write

$$L^2(K)^\rho = \mathcal{H}^{\mu'} \oplus \mathcal{H}^{\mu}_{0},$$

where $\mathcal{H}^{\mu}_{0} := (\mathcal{H}^{\mu'})^\perp \subset L^2(K)^\rho$. There exists $l_{\mu} \in \mathbb{N}_{>0}$ such that $\mathcal{H}^{\mu'} \simeq l_{\mu}\mathcal{H}_{\mu'}$. Similarly we have $L^2(K)^\rho = \mathcal{H}^{\mu_{\nu}} \oplus \mathcal{H}^{\mu_{\nu}}_{0}$, $\mu|_{K\chi} \simeq l_{\mu_n}\mathcal{H}_{\mu_n}$ for some $l_{\mu} \in \mathbb{N}$. Of course $l_{\mu} \leq \dim(L^2(K)^\mu)$ and $l_{\mu}$ can be 0.

However, since the representations $(\pi(\mu_n, \chi_n))$ converge to the representation $(\pi(\mu, \chi))$, we can assume that for a subnet the subspaces $\mathcal{H}^{\mu_n}$ can be realized inside $L^2(K)^\rho$ for $n$ large enough (see [2], 4.2-D Theorem). This tells us also that $l_{\mu} > 0$ for $n$ large enough and hence, again for a subnet, we can suppose that $l_{\mu} = l$ is fixed for every $n$. Since the dimensions of the spaces $\mathcal{H}^{\mu_n}$ are smaller than the dimension of $L^2(K)^\rho$, we can also assume that all the dimensions $d_{\mu_n}$ are the same and equal to some common $d > 0$.

We choose for every $n$ an orthonormal basis $(\xi^n_j)_j$ of $L^2(K)^\rho$, which passes through the $l_{\mu}$ copies of $\mathcal{H}^{\mu_n}$ and through $\mathcal{H}^{\mu}_{0}$. We choose the $\xi^n_j$ such that $\dim(\mathcal{H}^{\mu_n})$ finite, we can assume (passing to a subnet) that $\lim_{n \to \infty} \xi^n_j = \xi_j$ exists in $L^2(K)$ for every $j \leq ld$. Let $c_n, n \in \mathbb{N}$, be the character of the irreducible representation $\mu_n$ of $K'$. Then $c_n * \xi_n = \xi_n$ for any $\xi_n \in \mathcal{H}^{\mu_n}$ and $c_n * \mathcal{H}^{\mu}_{0} = \{0\}$ for every $n$. According to Bagget, the pairs $(K\chi_n, c_n)$ converge in $\mathcal{A}(K)$ to the pair $(K', c_{\mu'})$, where $c_{\mu'}$ denotes the character of an irreducible representation $\nu$ (see [2], 7.1-B Lemma). This implies that (see [2], 1.4-A Proposition)

$$\xi_j := \lim_{n \to \infty} \xi^n_j = \lim_{n \to \infty} c_n * \xi^n_j = c_{\mu'} * \xi_j$$

for every $j$ and similarly

$$c_{\mu'} * \mathcal{H}^{\mu}_{0} = \{0\}.$$

This shows that $K'$ acts on $\mathcal{H}^{\mu'} = \lim_{n \to \infty} \mathcal{H}^{\mu_n}$ by a multiple of $\mu'$. Define for any $n$ and $i, j \in \{1, \cdots, d\}$ the function $c^n_{i,j}$ on $K$ by

$$c^n_{i,j}(k) := \frac{1}{d} \langle \lambda(k) \xi^n_i, \xi^n_j \rangle_{L^2(K)}, k \in K,$$
where $\lambda$ denotes the left regular representation of $K$ on $L^2(K)$. Since $\lim_{n \to \infty} \xi^n_i = \xi_i$ in $L^2(K)$ for every $i$, the functions $c^n_{i,j}$ converge uniformly on $K$ to the function $c_{i,j}(k) := \frac{1}{d} \langle \lambda(k)\xi_j, \xi_i \rangle_{L^2(K)}$.

Now the operator $\mu_n(c^n_{i,j}|_{K\chi_n})$, which acts on the Hilbert space $H_{\mu_n} := \text{span}\{\xi^n_j, 1 \leq j \leq d\}$ is the rank one operator $P_{\xi^n_i, \xi^n_j}$ which converges to the operator $P_{\xi_i, \xi_j}$ on the Hilbert space $H_{\mu'} := \lim_{n \to \infty} H_{\mu_n}$. This shows that the restriction of $\mu'$ to $H_{\mu'}$ is irreducible. Hence $d = \dim(H_{\mu'})$ and $l_{\mu'} = l$. □

2.3.

Remark 2.13. Identifying for every $n$ the Hilbert space $H_{\mu_n}$ with $\mathbb{C}^d$ via the basis given by the $\xi_j$'s, we see also that for every $f \in C(K)$ the operators $\mu_n(f|_{K\chi_n})$ converge strongly and hence in operator norm to the operator $\mu'(f|_{K'})$.

We have by §§4.5 of [5]:

Theorem 2.14. Let $A$ be a postliminal $C^*$-algebra. Then $A$ admits a composition net $(I_n)_{0 \leq n \leq \alpha}$ such that for any $n$, which is not an ordinal, the quotient $I_{n+1}/I_n$ is with continuous trace and such that for every ordinal $\beta \leq \alpha$ the relation $I_{\beta} = \bigcup_{n<\beta} I_n$ holds.

We take now as $C^*$-algebra our $A = C^*(G)$, which is CCR. Let

$$S_n := \{\pi \in \hat{G}|\pi(I_n) = \{0\}\}, 0 \leq n \leq \alpha.$$  

Then $S_0 = \hat{G}$ and $S_\alpha = \{\emptyset\}$. The subsets

$$\Gamma_n := S_n \setminus S_{n+1}, 0 \leq n \leq \alpha, n \text{ not an ordinal}$$

are locally compact and Hausdorff in their relative topologies, since $\Gamma_n$ is the spectrum of the algebra $I_{n+1}/I_n$, which is of continuous trace (see [5]). Let $S = \hat{G}$ be the spectrum of $G$. Then

$$S = \bigcup_{0 \leq n \leq \alpha} S_n = \bigcup_{0 \leq n \leq \alpha} \Gamma_n.$$

2.4. The Fourier transform. Let us first write down explicitly the representation $\pi_{(\mu,\chi)}$. Its Hilbert space $H_{(\mu,\chi)}$ can be identified with the space $L^2(G/K \ltimes A, \sigma_{(\mu,\chi)}) \simeq L^2(K/K_{\chi}, \mu) \subset L^2(K, \mathcal{H}_\mu)$. 


Let $\xi$ be an element of $\mathcal{H}(\mu, \chi)$. For all $a \in A$ and $k, h \in K$ we use the same calculation as in (2.1) and we have that
\[
\pi(\mu, \chi)(k, a)\xi(h) = \chi(1_K, h^{-1} \cdot a)\xi(k^{-1}h) =: \chi(h^{-1} \cdot a)\xi(k^{-1}h).
\]

Let us compute for $f \in L^1(G)$ the operator $\pi(\mu, \chi)(f)$. We have for $h \in K$ and $\xi \in \mathcal{H}(\mu, \chi)$ that
\[
\pi(\mu, \chi)(f)\xi(h) = \int_{G} f(k, a)\pi(\mu, \chi)(k, a)\xi(h) \, da \, dk
\]
\[
= \int_{K} \int_{A} f(k, a)\chi(h^{-1} \cdot a)\xi(k^{-1}h) \, da \, dk
\]
\[
= \int_{K} \int_{A} f(hk^{-1}, a)\chi(h^{-1} \cdot a)\xi(k) \, da \, dk
\]
\[
= \int_{K/K} \int_{K} \left( \int_{A} f(hs^{-1}k^{-1}, h \cdot a)a^{-1}\chi(a)da \right)\mu(s^{-1})\xi(k) \, dsdk
\]
\[
= \int_{K/K} \int_{K} \hat{f}^2(hsk^{-1}, h \cdot \chi)\mu(s)\xi(k) \, dsdk
\]
\[
= \int_{K/K} f_{\mu, \chi}(h, k)\xi(k)dk,
\]
(2.2)

where
\[
f_{\mu, \chi} : K \times K \rightarrow \mathcal{B}(\mathcal{H}_\mu)
\]
(2.3)
\[
(h, k) \mapsto \int_{K} \hat{f}^2(hsk^{-1}, h \cdot \chi)\mu(s)ds.
\]

and
\[
\hat{f}^2(k, \mu) := \int_{A} \chi(a)f(k, a) \, da, k \in K, \chi \in \hat{A}.
\]

**Definition 2.15.** For each $f \in C^*(G)$, the Fourier transform $\mathcal{F}(f)$ of $f$ is the isometric homomorphism on $C^*(G)$ into $\ell^\infty(\hat{G})$ which is given by
\[
\mathcal{F}(f)(\mu, \chi) = \pi(\mu, \chi)(f) \in \mathcal{B}(\mathcal{H}(\mu, \chi)), \ (\chi, K, \mu) \text{ is a cataloguing triple.}
\]

Let now $L^1(G)_c$ be the dense subspace of $L^1(G)$ defined by
\[
L^1(G)_c := \left\{ f \in L^1(G); \text{ the function } \hat{f}^2 \text{ is in } C_c(K \times \hat{A}) \right\}.
\]

**Definition 2.16.** Let $L$ be a closed subgroup of the compact group $K$ and let $(\nu, \mathcal{H}_\nu)$ be an irreducible representation of $L$ with character $\chi_\nu$. We may identify the Hilbert space $\mathcal{H}_\nu$ with $\mathbb{C}^d$, $d = d_\nu = \dim(\nu)$. Let
\[
L^2(\nu) := L^2(K, \mathcal{H}_\nu) \cong L^2(K, \mathbb{C}^d).
\]
(1) Define for \( \chi \in \hat{A} \) such that \( L \subset K_{\chi} \) and for \( f \in L^1(G) \) the operator \( \tau_{\nu, \chi}(f) \) on \( L^2(K, \mathbb{C}^d) \) by
\[
\tau_{\nu, \chi}(f)(\xi)(x) := \int_K \int_L \left( \int_L (f^2(xy^{-1}, y \cdot \chi)) \nu(l) dl \right) (\xi)(y) dy.
\]

(2) Define for \( \chi \in \hat{A} \), for a closed subgroup \( L \subset K_{\chi} \) of \( K \), for \( \nu \in \hat{L} \), the linear projection \( P_{\nu} : L^2(K, \mathcal{H}_{\nu}) \rightarrow L^2(K/L, \nu) \) by
\[
P_{\nu}(\varphi)(x) := \int_L \nu(l)(\varphi(xl)) dl, \varphi \in C(K, \mathcal{H}_{\nu}), x \in K.
\]

**Proposition 2.17.**

(1) The linear operator \( P_{\nu} \) is an selfadjoint projection of the Hilbert space \( L^2(\nu) \).

(2) For any closed subgroup \( L \subset K_{\chi} \) of \( K \), \( \nu \in \hat{L} \), \( \chi \in \hat{A} \) and \( f \in L^1(G) \) we have
\[
\text{ind}_{L \times A}^G \chi \otimes \nu = \tau_{\nu, \chi}(f) = \nu_{\chi}(f), f \in L^1(G).
\]

**Proof.**

(1) We have for \( \varphi \in L^2(K, \mathcal{H}_\nu) \) that
\[
\|P_{\nu}(\varphi)\|^2 = \int_{K/L} \left\| \int_L \nu(l)(\varphi(kl)) dl \right\|^2_{\mathbb{C}^d} dk.
\]
\[
\leq \int_{K/L} \left( \int_L \|\varphi(kl)\|^2_{\mathbb{C}^d} dl \right) dk.
\]
\[
= \int_K \|\varphi(k)\|^2_{\mathbb{C}^d} dk.
\]
\[
= \|\varphi\|^2.
\]

Let \( \varphi \in \mathcal{H}_{\nu, \chi} = L^2(K/L, \nu) \subset L^2(K, \mathcal{H}_\nu) \). Then for \( k \in K \),
\[
P_{\nu}(\varphi)(k) = \int_L \nu(l)(\varphi(kl)) dl.
\]
\[
= \int_L \nu(l)\nu(l)^{-1}(\varphi(k)) dl.
\]
\[
= \int_L \varphi(k) dl.
\]
\[
= \varphi(k).
\]

Hence the operator \( P_{\nu} \) is the identity on \( \mathcal{H}_\nu \subset L^2(K, \mathcal{H}_\nu) \).

Let \( \mu \in \hat{L} \) and let \( c_\mu \) its character. For \( \varphi \in L^2(K, \mathcal{H}_\nu) \) let
\[
\varphi_\mu(k) := \varphi \ast c_\mu(k) = \int_L \varphi(kl^{-1}) c_\mu(l) dl, k \in K.
\]
Then the mapping \( \varphi_\mu \) is also contained in \( L^2(\nu) \) and for another \( \omega \in \hat{L} \) we have that

\[
\langle \varphi_\mu, \varphi_\omega \rangle_{L^2(\nu)} = \int_K \langle \varphi_\mu(k), \varphi_\omega(k) \rangle_{H_\nu} dk
\]

\[
= \int_{K/L} \int_L \langle \varphi_\mu(ql), \varphi_\omega(ql) \rangle_{H_\nu} dldk
\]

\[
= \int_{K/L} \int_L \int_L \langle \varphi(ql^{-1}) c_\mu(l_1), \varphi(ql^{-1}) c_\omega(l_2) \rangle_{H_\nu} dldk
\]

\[
= \int_{K/L} \int_L \int_L \int_L \langle \varphi(ql^{-1}), \varphi(ql^{-1}) \rangle_{H_\nu} dldl_1 d_\nu d^{l_2}
\]

\[
= 0.
\]

It is easy to see now that

\[
L^2(\nu) = \sum_{\mu \in \hat{L}} L^2(\nu) * c_\mu,
\]

since \( L^2(K) = \sum_{\mu \in \hat{L}} L^2(K) * c_\mu \). Furthermore, for \( \mu \neq \nu \) it follows that

\[
P_\nu(\varphi_\mu)(k) = \int_L \nu(l)(\varphi_\mu(ql)) dl
\]

\[
= \int_L \nu(l) \left( \int_L \varphi_\mu(ql^{-1}) c_\mu(l_1) dl_1 \right) dl
\]

\[
= \int_L \nu(l) \left( \int_L \varphi_\mu(ql^{-1}) c_\mu(l_1) dl_1 \right) dl
\]

\[
= \int_L \nu(l) c_\mu(l_1) dl \left( \int_L \varphi_\mu(ql^{-1}) dl_1 \right)
\]

\[
= 0 \cdot \left( \int_L \varphi_\mu(ql^{-1}) dl_1 \right)
\]

\[
= 0.
\]

Hence \( L^2(\nu) = \sum_{\mu \neq \nu} L^2(\nu)_\mu \) and \( P_\nu \) is zero on \( L^2(\nu) \). This shows that

\[
P^*_\nu = P_\nu.
\]
(2) For \( f \in C_\r(G), \varphi \in L^2(K, \mathbb{C}^d), x \in K \), we have by (2.2) that

\[
\begin{align*}
\lim_{L \to A} \nu \otimes \chi(f)(P_\nu(\varphi))(x) &= \int_{K/L} \int_L \hat{f}^2(xlk^{-1}, k \cdot \chi) \nu(l) dl \, (P_\nu(\varphi)(k)) dk \\
&= \int_{K/L} \int_L \hat{f}^2(xlk^{-1}, k \cdot \chi) \nu(l) dl \, (P_\nu(\varphi)(k)) dk \\
&= \int_{K/L} \int_L \hat{f}^2(xlk^{-1}, k \cdot \chi) \nu(l) dl \, (P_\nu(\varphi)(k)) dk \\
&= \int_{K/L} \int_L \hat{f}^2(xlk^{-1}, k \cdot \chi) \nu(l) dl \, (P_\nu(\varphi)(k)) dk \\
&= \int_{K/L} \int_L \hat{f}^2(xlk^{-1}, k \cdot \chi) \nu(l) dl \, (P_\nu(\varphi)(k)) dk \\
&= \int_{K/L} \int_L \hat{f}^2(xlk^{-1}, k \cdot \chi) \nu(l) dl \, (P_\nu(\varphi)(k)) dk \\
&= \int_{K/L} \int_L \hat{f}^2(xlk^{-1}, k \cdot \chi) \nu(l) dl \, (P_\nu(\varphi)(k)) dk \\
&= \int_{K/L} \int_L \hat{f}^2(xlk^{-1}, k \cdot \chi) \nu(l) dl \, (P_\nu(\varphi)(k)) dk \\
&= \int_{K/L} \int_L \hat{f}^2(xlk^{-1}, k \cdot \chi) \nu(l) dl \, (P_\nu(\varphi)(k)) dk \\
&= \tau_{\nu, \chi}(f)(\varphi)(x).
\end{align*}
\]

Lemma 2.18. Let \( f \in C^*_r(G) \). Then we have:

(1) \( \lim_{(\mu, \chi) \to \infty} \| \mathcal{F}(f)(\mu, \chi) \|_{op} = 0 \).

(2) Let \( (\chi_m, L_m, \mu_m)_{m \in M} \) be a converging net in \( \hat{A} \times \mathcal{A}(K) \) with limit \( (\chi_\infty, L_\infty, \mu_\infty) \) such that \( \dim(\mu_m) = \dim(\mu_\infty) \) for any \( m \). Then, identifying the Hilbert spaces of the representations \( \mu_m \) with \( \mathbb{C}^d \), we have that \( \lim_{m \to \infty} \tau_{\mu_m, \chi_m}(f) = \tau_{\mu_\infty, \chi_\infty}(f) \) in operator norm for any \( f \in L^1(G) \).

Proof. (1) Let \( A \) be a \( C^* \)-algebra. According to ([3], chapter 3 §3.3) , if a net \( (\pi_k)_k \subset \hat{A} \) goes to infinity, i.e. this net has no converging subnet, then \( \lim_{k \to \infty} \| \pi_k(a) \|_{op} = 0 \) for any \( a \in A \). Now Bagget [2] has shown that for every net of cataloguing triples \( (\chi_k, K_{\chi_k}, \mu_k)_k \) we have that \( (\mu_k, \chi_k)_k \) goes to infinity, if and only if the net \( (\pi_k)_k \) goes to infinity in \( \mathcal{C}^*_r(G) \).

(2) Let first \( f \) be contained in \( L^1(G)_c \). Let

\[
f_m(x, y) := \int_{L_m} \hat{f}^2(xly^{-1}, y \cdot \chi_m) \mu_m(l) dl, x, y \in K, m \in M \cup \{ \infty \}.
\]

Then we see that \( f_m(x, y) \in \mathcal{B}(\mathbb{C}^n) \) and by Remark that (2.13)

\[
\lim_{m \to \infty} f_m(x, y) v = f_\infty(x, y) v, v \in \mathbb{C}^d,
\]

point wise in \( x, y \). Therefore also
Let \[ M_d(\mathbb{C}) \ni \lim_{m \to \infty} f_m(x, y) = f_\infty(x, y), \]
where \( M_d(\mathbb{C}) \) denotes the space of complex matrices of size \( d \).

Using Lebesgue, we see that
\[
\lim_{m \to \infty} \| \tau_{\mu_m, \chi_m}(f) - \tau_{\mu_\infty, \chi_\infty}(f) \|_{H.S}^2
= \lim_{m \to \infty} \int_{K \times K} \| f_m(x, y) - f_\infty(x, y) \|_{H.S}^2 dxdy
= 0.
\]
Hence
\[
\lim_{m \to \infty} \tau_{\mu_m, \chi_m}(f) = \tau_{\mu_\infty, \chi_\infty}(f).
\]
The lemma follows now from the density of \( L^1(G) \) in \( C^*(G) \).

2.5. A \( C^* \)-condition. Let \( G = K \times A \) be as before a semi-direct product of a compact group \( K \) with a locally compact abelian group \( A \).

Remark 2.19. Let \((\pi(\mu_m, \chi_m))_{m \in M}\) be a net in \( \hat{G} \) which converges to \( \pi(\mu_\infty, \chi_\infty) \).
We can suppose that for a subnet (also denoted by \( M \) for simplicity of notation) that the triples \((\chi_m, K_{\mu_m}, \mu_m)\) converge to \((\chi_\infty, K_\infty, \mu_\infty)\) in \( \hat{A} \times A(K) \) and that the Hilbert spaces \( H_{\mu_m} \) and \( H_{\mu_\infty} \) are identified with \( \mathbb{C}^d \) for some \( d \in \mathbb{N}^* \) and that all these spaces \( H_{(\mu_m, \chi_m)}, m \in M \cup \{\infty\}, \) are subspaces of the common Hilbert space \( H_M := L^2(K, \mathbb{C}^d) \). The representation \( \tau_{\mu_\infty, \chi_\infty} = \text{ind}_{K_\infty \times A}^{G} \mu_\infty \otimes \chi_\infty \) can be disintegrated into an integral of irreducible representations supported by the limit set
\[
L = \left\{ \pi(\mu, \chi) \mid \mu \in \hat{K}_\infty, \mu|_{K_\infty} \text{ contains } \mu_\infty \right\}
\]
of the net \((\pi(\mu_m, \chi_m))_m\) (see Theorem 2.11). We denote by \( \sigma_{\mu_\infty, \chi_\infty} \) the corresponding representation of the algebra \( C^*(G)_L \) on the Hilbert space \( L^2(K/K_\infty, \mu_\infty) \subset L^2(K, \mathbb{C}^d) \). Let us observe that by the construction of \( \sigma_{\mu_\infty, \chi_\infty} \) we have that
\[
\sigma_{\mu_\infty, \chi_\infty}(a_{|L}) = \tau_{\chi_\infty, \mu_\infty}(a), a \in C^*(G).
\]

We can extend this representation \( \sigma_{\mu_\infty, \chi_\infty} \) to the larger \( C^* \)-algebra \( CB(L) \) consisting of all uniformly bounded operator fields \( F \) satisfying \( F(\pi) \in K(\mathcal{H}_\pi), \pi \in L, \) and we denote this extension also by \( \sigma_{\mu_\infty, \chi_\infty} \) (see [1]).

Definition 2.20. Let \( \mathcal{D}(G) = \mathcal{D} \) be the family consisting of all uniformly bounded operator fields \( F \) satisfying \( F(\pi) \in K(\mathcal{H}_\pi), \pi \in L, \) and we denote this extension also by \( \sigma_{\mu_\infty, \chi_\infty} \) (see [1]).

(1) \( F(\pi) \) is a compact operator on \( \mathcal{H}_\pi \) for every \( \pi \in \hat{G} \).
(2) \( \lim_{(\mu, \chi) \to (\infty, \infty)} \| F(\mu, \chi) \|_{op} = 0. \)
(3) Let \((\pi_{(\mu_m, \chi_m)})_{m \in M}\) be a properly converging net in \(\hat{G}\) with the properties and notations of the preceding Remark 2.19. Then
\[
\lim_{m \to \infty} \| F(\mu_m, \chi_m) \circ P_{\mu_m} - \sigma_{\mu_{\infty}, \chi_{\infty}}(F_L) \circ P_{\mu_{\infty}} \|_{\text{op}} = 0.
\]

**Proposition 2.21.** \(\mathcal{D}(G)\) is a \(C^*\)-algebra for the norm \(\| \cdot \|_{\text{op}}\) containing \(\overline{C^*(G)}\).

**Proof.** First we show that \(\mathcal{D}\) is a norm closed involutive subspace of \(l^\infty(\hat{G})\). It is clear that \(\mathcal{D}\) is a sub-space of \(\ell^\infty(\hat{G})\). The conditions (1), (2) are evidently true for every \(F\) in the closure \(\overline{\mathcal{D}}\) of \(\mathcal{D}\). For the condition (3), let \(F \in \overline{\mathcal{D}}\) and let \((F^k)_k \subset \mathcal{D}\) such that \(\lim_{k \to \infty} \| F^k - F \|_{\infty} = 0\). Then also \(\lim_{k \to \infty} \|(F^k)^* - F^*\|_{\infty} = 0\). Hence for any \(\varepsilon > 0\) there exists \(k_0\) such that such that \(\| F - F^k \|_{\infty} < \varepsilon\) for any \(k \geq k_0\). Therefore choosing some \(k > k_0\) we have for \(m\) large enough that
\[
\| F^k(\mu_m, \chi_m) \circ P_{\mu_m} - \sigma_{\mu_{\infty}, \chi_{\infty}}(F^k_L) \circ P_{\mu_{\infty}} \|_{\text{op}} \leq \varepsilon
\]
and so
\[
\| F(\mu_m, \chi_m) \circ P_{\mu_m} - \sigma_{\mu_{\infty}, \chi_{\infty}}(F_L) \circ P_{\mu_{\infty}} \|_{\text{op}} \\
\leq \| F(\mu_m, \chi_m) \circ P_{\mu_m} - F^k(\mu_m, \chi_m) \circ P_{\mu_m} \|_{\text{op}} \\
+ \| F^k(\mu_m, \chi_m) \circ P_{\mu_m} - \sigma_{\mu_{\infty}, \chi_{\infty}}(F^k_L) \circ P_{\mu_{\infty}} \|_{\text{op}} \\
+ \| \sigma_{\mu_{\infty}, \chi_{\infty}}(F^k_L) \circ P_{\mu_{\infty}} - \sigma_{\mu_{\infty}, \chi_{\infty}}(F_L) \circ P_{\mu_{\infty}} \|_{\text{op}} \\
\leq \varepsilon + \varepsilon + \| P_{\mu_{\infty}} \circ \sigma_{\mu_{\infty}, \chi_{\infty}}(F^k_L)^* - P_{\mu_{\infty}} \circ \sigma_{\mu_{\infty}, \chi_{\infty}}(F_L)^* \|_{\text{op}} \\
\leq 2\varepsilon + \| \sigma_{\mu_{\infty}, \chi_{\infty}}((F^k)^*_L - F^*|_L) \|_{\text{op}} \\
\leq 3\varepsilon.
\]
Hence \(F \in \mathcal{D}\). Since \(\sigma_{\mu_{\infty}, \chi_{\infty}}\) is a representation, it follows that \(\mathcal{D}\) is involutive and so \(\mathcal{D}\) is an involutive Banach space. Let us show that it is an algebra.

Let \(F, F' \in \mathcal{D}\). We must show that \(F \circ F'\) is in \(\mathcal{D}\) too.

The conditions (1), (2) are necessarily true for \(F \circ F'\).

Let us check point (3). It follows from property (3) for \(F\), using the involution * , that also
\[
\lim_{m \to \infty} \| P_{\mu_k} \circ F(\mu_m, \chi_m) - P_{\mu_{\infty}} \circ \sigma_{\mu_{\infty}, \chi_{\infty}}(F_L) \|_{\text{op}} = 0.
\]
We then have that, since \( P_\mu \circ F(\mu, \chi) \circ P_\mu = F(\mu, \chi) \circ P_\mu \) for every \( \chi \in \tilde{A}, \mu \in \tilde{K} \),
\[
\lim_{m \to \infty} \| F \circ F'(\mu_m, \chi_m) \circ P_{\mu_m} - \sigma_{\mu_\infty, \chi_\infty}(F \circ F'_{\mu_L}) \circ P_{\mu_\infty} \|_{\text{op}} \\
= \lim_{m \to \infty} \| P_{\mu_m} \circ F(\mu_m, \chi_m) \circ F'(\mu_m, \chi_m) \circ P_{\mu_m} - P_{\mu_\infty} \circ \sigma_{\mu_\infty, \chi_\infty}(F_{\mu_L}) \circ \sigma_{\mu_\infty, \chi_\infty}(F'_{\mu_L}) \circ P_{\mu_\infty} \|_{\text{op}} \\
\leq \lim_{m \to \infty} \| P_{\mu_m} \circ F(\mu_m, \chi_m) \circ F'(\mu_m, \chi_m) \circ P_{\mu_m} - P_{\mu_\infty} \circ \sigma_{\mu_\infty, \chi_\infty}(F_{\mu_L}) \circ F'(\mu_m, \chi_m) \circ P_{\mu_m} \|_{\text{op}} + \\
+ \lim_{m \to \infty} \| P_{\mu_\infty} \circ \sigma_{\mu_\infty, \chi_\infty}(F_{\mu_L}) \circ F'(\mu_m, \chi_m) \circ P_{\mu_m} - P_{\mu_\infty} \circ \sigma_{\mu_\infty, \chi_\infty}(F_{\mu_L}) \circ \sigma_{\mu_\infty, \chi_\infty}(F'_{\mu_L}) \circ P_{\mu_\infty} \|_{\text{op}} \\
\leq \lim_{m \to \infty} C \| P_{\mu_m} \circ F(\mu_m, \chi_m) - P_{\mu_\infty} \circ \sigma_{\mu_\infty, \chi_\infty}(F_{\mu_L}) \|_{\text{op}} + \\
+ \lim_{m \to \infty} C \| F'(\mu_m, \chi_m) \circ P_{\mu_m} - \sigma_{\mu_\infty, \chi_\infty}(F'_{\mu_L}) \circ P_{\mu_\infty} \|_{\text{op}} \\
= 0,
\]
where \( C = \max(\| F \|_{\infty}, \| F' \|_{\infty}) \).

Since \( \tilde{C}^*(G) \) satisfies all the conditions of \( D^*(G) \) it follows that \( \tilde{C}^*(G) \) is contained in \( D^*(G) \).

**Proposition 2.22.** The spectrum \( \tilde{D}(G) \) of the algebra \( D(G) \) can be identified with \( \tilde{G} \).

**Proof.** We have by \( \S \S \S 4.5 \) of [5]:

**Theorem 2.23.** Let \( A \) be a postliminal \( C^* \)-algebra. Then \( A \) admits a composition sequence \((I_n)_{0 \leq n \leq \alpha}\) such that the quotients \( I_{n+1}/I_n \) are \( C^* \)-algebras with continuous trace.

This theorem applies of course to our \( C^* \)-algebra \( C^*(G) \). Let now
\[
S_n := \{ \pi \in \tilde{G} \mid \pi(I_n) = \{0\} \}, 0 \leq n \leq \alpha.
\]
The subsets
\[
\Gamma_n := S_n \setminus S_{n+1}, 0 \leq n \leq \alpha,
\]
are locally compact and Hausdorff in their relative topologies, since \( \Gamma_n \) is the spectrum of the algebra \( I_{n+1}/I_n \), which is of continuous trace (see [5]). Then
\[
\tilde{G} = \bigcup_{0 \leq n \leq \alpha} S_n, \\
= \bigcup_{0 \leq n \leq \alpha} \Gamma_n, \\
S_{n-1} \supset S_n, 0 < n \leq \alpha \\
S_0 = \tilde{G}, \\
S_\alpha = \{0\}.
\]
Evidently \( \tilde{\mathcal{D}} \supset \tilde{G} \). Define:
\[
J_n := \{ F \in \mathcal{D} \mid F(\pi) = 0, \pi \in S_n \}, 0 \leq n \leq \alpha.
\]
Then the $J_n$’s are closed ideals of $\mathcal{D}$ and

\[ J_n \supset J_{n-1}, \quad 0 < n \in N, \quad n \text{ not an ordinal}, \]

\[ J_\alpha = \mathcal{D}, \]

\[ J_0 = \{0\}. \]

Let now $\pi \in \hat{D} \setminus \hat{G}$. Let

\[ n_\pi := \sup_{n \in N} \pi(J_n) = \{0\}. \]

If $n_\pi = \alpha$, then $\pi(J_n) = \{0\}$ for every $0 \leq n < \alpha$ and then $\pi(\mathcal{D}) = \pi(J_0) = \{0\}$, which is impossible. Hence

\[ n_\pi < \alpha. \]

We have now that $\pi(J_{n+1}) \neq \{0\}$, but $\pi(J_n) = \{0\}$.

This means in particular that $\pi$ is contained in the hull of the ideal $J_{n+1}$. Hence, there exists a net $(\pi_k)_k \subset \Gamma_{n+1}$, such that $\pi = \lim_{k \to \infty} \pi_k$ in $\hat{D}$. Hence, there exists for $\xi \in \mathcal{H}_\pi$ and for any $k$ an element $\xi_k \in \mathcal{H}_{\pi_k}$, such that for any $F \in \mathcal{D}$, we have that

\[ \lim_{k \to \infty} \langle F(\pi_k)(\xi_k), \xi_k \rangle = \langle \pi(F)\xi, \xi \rangle. \]

Then $\pi_k$ does not go to infinity in $\hat{G}$ because of condition (2). Hence, either for a subnet, the net $(\pi_k = \pi_{(\mu_k, \chi_k)})_k$ converges in $\Gamma_{n+1}$ to some $\pi_\infty = \pi_{(\mu_\infty, \chi_\infty)} \in \Gamma_{n+1}$, and then by condition (3)

\[ \|\sigma_{\mu_\infty, \chi_\infty} \circ P_{\mu_\infty} - F(\mu_k, \chi_k) \circ P_{\mu_k}\|_{\text{op}} = 0 \]

for any $F \in \mathcal{D}$. Now for any $k$ we have that

\[ \xi_k = P_{\mu_k}(\xi_k). \]

This shows that

\[ (2.4) \quad \lim_{k \to \infty} \langle \sigma_{\mu_\infty, \chi_\infty}(F)[L] \circ P_{\mu_\infty}(\xi_k), P_{\mu_\infty}(\xi_k) \rangle = \langle \pi(F)\xi, \xi \rangle, F \in \mathcal{D}. \]

Since in our case $L = \{\pi_\infty\}$, relation [2.4] implies that $\ker(\pi_\infty) \subset \ker(\pi)$. But then $\pi = \pi_\infty$, since $\pi_\infty$ is completely continuous (see [5], 4.1.11. Corollary).

The other possibility is that the net converges in $\hat{G}$ to a limit set $L$ contained in $S_{n+1}$.

Now for $F \in J_{n+1}$, it follows from condition (3), that

\[ \lim_{m \to \infty} \|F(\pi_k) \circ P_{\mu_k}\|_{\text{op}} = \lim_{m \to \infty} \|F(\pi_k) \circ P_{\mu_k} - \sigma_{\mu_\infty, \chi_\infty}(F)[L] \circ P_{\mu_\infty}\|_{\text{op}} = 0, \]

which shows that $\lim_{k \to \infty} \|F(\pi_k)\|_{\text{op}} = 0$. But this implies then by [2.4] that $\pi(F)(\xi) = 0$ for every $\xi$. Therefore $\pi$ is 0 on $J_{n+1}$. This contradiction shows that $\pi = \pi_\infty \in \hat{G}$. \[ \square \]
Theorem 2.24. Let $G = K \rtimes A$ be the semi-direct product of a compact group $K$ with an abelian locally compact group $A$. Then the $C^*$-algebra $\mathcal{D}(G)$ is isomorphic to the group $C^*$-algebra $C^*(G)$.

Proof. We know now that $\hat{\mathcal{D}} = \hat{G}$. It suffices to apply the theorem of Stone-Weierstrass to the $C^*$-algebra $\mathcal{D}$ and its subalgebra $\hat{C}^*(G)$.

3. Examples

Example 3.1. For all $n \in \mathbb{N}^*$, let $G_n$ the the semi-direct product of the compact Lie group $SO(n)$ with the abelian group $\mathbb{R}^n$. The $C^*$-algebra of this group is described by Abdelmoula, Elloumi and Ludwig in [3].

We can parameterize the dual space in the following way:

$$\Gamma_1 = \hat{SO(n-1)} \times \mathbb{R}^*_+$$

$$\Gamma_0 = \hat{SO(n)}.$$

For $\rho_{\mu} \in \hat{SO(n-1)}$, and $r \in \mathbb{R}^*_+$, we denote by $K_{\chi_r} = SO(n-1) \times \mathbb{R}^*_+$ the stabilizer of $\chi_r$. The projection $P_r : L^2(SO(n)) \to L^2(SO(n))$ are

$$P_r(\varphi)(x) := \int_{K_{\chi_r}} \rho_{\mu}(l)\varphi(xl)dl,$$

and for $r = 0$

$$P_0(\varphi)(x) := \int_{SO(n)} \tau_{\lambda}(l)\varphi(xl)dl, \quad \text{where } \tau_{\lambda} \in \hat{SO(n)}.$$

For any $f \in L^1(G_n)$ and $\xi \in L^2(SO(n))$ we have that

$$\tau_{\mu,r}(f)(x) := \int_{SO(n)} \left( \int_{K_{\chi_r}} (\hat{f}^2(xly^{-1}, y \cdot \chi_r)\rho_{\mu}(l)dl) \right) \xi(y)dy, \quad x \in SO(n).$$

$$\tau_{\mu,0}(f)(x) := \int_{SO(n)} \hat{f}^2(y,0)\xi(y^{-1}x)dy, \quad x \in SO(n).$$

Let $\mathcal{D}_n$ be the family consisting of all operator fields $F \in \ell^\infty(\hat{G}_n)$ satisfying the following conditions:

1. $F(\gamma)$ is a compact operator on $\mathcal{H}_n$ for every $\gamma \in \Gamma_1$,
2. $\lim_{\gamma \to \infty} \|F(\gamma)\|_{op} = 0$,
3. $\lim_{r \to 0} \|F(\mu, r) \circ P_r - F(\mu, 0) \circ P_0\|_{op} = 0$.

Then, the $C^*$-algebra of the group the group $G_n$ is isomorphic to $\mathcal{D}_n$ under the Fourier transform.
Example 3.2. Define the abelian group $A$ and the compact groups $L$ and $K$ by:

\[
A := \{(z_i)_{i \in \mathbb{N}} \in \mathbb{Z}^\infty | z_i \neq 0 \text{ for a finite number of indices}\}
\]
with the discrete topology.

\[
L := \{1, -1\}
\]

\[
K := L^\infty
\]
with the product topology.

Then, by [13], Ch. 4, 2, 3.1., we have that

\[
\hat{A} = T^\infty.
\]

Furthermore, the spectrum $\hat{K}$ of the abelian group $K$ is the set of all infinite products

\[
\chi = \chi_1 \times \chi_2 \times \cdots \in \{1, -1\} \times \{1, -1\} \times \cdots,
\]
where $\chi_j = 1$ almost everywhere.

Define for $m \in \mathbb{N}$ the subgroups $L^m$ and $L_m$ of $K$ by

\[
L_m := \{1\} \times \{1\} \times \cdots \times \{1\} \times \{1, -1\} \times \{1, -1\} \cdots
\]

and

\[
L^m := L \times L \times \cdots \times L \times \{1\} \times \{1\} \cdots
\]

Then $L^m$ is isomorphic to the quotient group $K/L_m$ and it has $2^m$ elements. The Haar measure $dl^m$ on $L^m$ is given by

\[
\int_{L^m} f(l_m) dl^m = \frac{1}{2^m} \left( \sum_{x \in L^m} f(x) \right), f \in C(L^m).
\]

A function $f : K \to \mathbb{C}$ is continuous, if and only if for every $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that

\[
\sup_{x \in K, l_m \in L_m} |f(x) - f(xl_m)| \leq \varepsilon.
\]

Then the Haar measure $dk$ on $K$ is given according to [13], Ch. 3, 3, example (vi) by

\[
\int_K f(k) dk = \lim_{m \to \infty} \int_{L^m} f(l_m) dl^m.
\]

For $\chi = (\chi_i)_{i \in \mathbb{N}} \in \hat{A}$, the stabilizer $K_\chi$ is the direct product

\[
K_\chi = \prod_{i \in \mathbb{N}} K_{\chi_i}
\]
where $K_{\chi_i} = \{1, -1\}$ if $\chi_i \in \{1, -1\}$ and $K_{\chi_i} = \{1\}$ if $\chi_i \not\in \{1, -1\}$. 
Let for $m \in \mathbb{N}$
\[
\chi_m = (\chi_j^m)_{j \in \mathbb{N}}
\]
where
\[
\chi_j^m = \begin{cases} 
1 & \text{if } j \leq m, \\
0 & \text{if } j > m.
\end{cases}
\]

Then
\[
\lim_{m \to \infty} \chi_m = 1
\]
in $\hat{A}$. The stabilizer $K_{\chi_m}$ in $K$ is the subgroup
\[
K_{\chi_m} = \{1\} \times \{1\} \times \cdots \times \{1\} \times \{1, -1\} \times \{1, -1\} \cdots
\]

Then
\[
\lim_{m \to \infty} K_{\chi_m} = 1 = \{1\} \times \{1\} \times \{1\} \times \cdots
\]

Choose for every $m \in \mathbb{N}$ the trivial character $\mu_m$ of $K_{\chi_m}$. Then, according to Definition 2.16, the dimensions $d_{\mu_m}$ are one and the projections $P_{\mu_m} : L^2(K) \to L^2(K)$ are given by
\[
P_{\mu_m}(\varphi)(x) := \int_{K_{\chi_m}} (\varphi(xl_m)) dl^m, \varphi \in C(K), x \in K.
\]

Therefore, by Remark 2.19 the limit set of the sequence $(\pi_{\mu_m, \chi_m})$ is the spectrum $\hat{K}$ itself and then for any $f \in L^1(G)$ and $\xi \in L^2(K)$ we have that
\[
\tau_{\mu_m, \chi_m}(f)\xi(x) := \int_K \left( \int_{K_{\chi_m}} \overline{f(xy^{-1}, y \cdot \chi_m)} dl \right) (\xi(y)) dy, x \in K,
\]
\[
\tau_{\mu_\infty, \chi_\infty}(f)\xi(x) := \int_K \overline{f(y, 1)} \xi(y^{-1} x) dy, x \in K.
\]

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