Cluster Properties in Relativistic Quantum Mechanics of N-Particle Systems

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Abstract
A general technique is presented for constructing a quantum theory of a finite number of interacting particles satisfying Poincaré invariance, cluster separability, and the spectral condition. Irreducible representations and Clebsch-Gordan coefficients of the Poincaré group are the central elements of the construction. A different realization of the dynamics is obtained for each basis of an irreducible representation of the Poincaré group. Unitary operators that relate the different realizations of the dynamics are constructed. This technique is distinguished from other solutions of this problem because it does not depend on the kinematic subgroups of Dirac’s forms of dynamics. Special basis choices lead to kinematic subgroups.
1 Introduction

This article illustrates a general method for constructing a relativistic quantum theory of $N$-interacting particles. The theory has a dynamical unitary representation of the Poincaré group, satisfies cluster separability, and has a four-momentum operator with spectrum in the future-pointing cone. These are the minimal elements of any physically motivated axioms of relativistic quantum theory.

Relativistic quantum theory of particles falls between non-relativistic quantum theory and local relativistic quantum field theory. It is interesting because it provides a mathematically well-defined framework for realizing the symmetry of special relativity in quantum theories. This makes it useful for applications to systems of a few strongly interacting particles.

The relativistic quantum theory constructed in this paper has many properties of local relativistic quantum field theory\[4\]. Both are quantum theories satisfying Poincaré invariance, cluster separability, and the spectral condition. The most significant distinction between the two theories is that local relativistic quantum field theory satisfies a microscopic locality constraint, which requires an infinite number of degrees of freedom.

The absence of theories that are simultaneously consistent with the axioms of local quantum field theory and applicable to realistic systems suggests that mathematically well behaved alternatives might be well suited to applications involving strongly interacting particles.

The essential features of quantum theory of particles are:

1. The model Hilbert space is the finite tensor product of single-particle
Hilbert spaces. This defines the degrees of freedom of the model.

2. There is a unitary representation of the Poincaré group \( \hat{U}(\Lambda, Y) \) on the model Hilbert space. This ensures that the quantum probabilities are independent of inertial frame. This representation necessarily contains the dynamics.

3. The four-momentum operators, which are the infinitesimal generators of the space-time translation subgroup of \( \hat{U}(\Lambda, Y) \), have a spectrum in the future-pointing light cone. This ensures the stability of the theory.

4. The operator \( \hat{U}(\Lambda, y) \) can be approximated by a tensor product of \( \hat{U}_i(\Lambda, y) \)'s on vectors describing subsets of particles in asymptotically separated regions. This justifies experiments on isolated sub-systems and provides the relation between few- and many-body systems.

5. The scattering operator is unitary and Poincaré invariant.

While relativistic quantum theory of particles is useful, independent of a relation to local quantum field theory, any local field theory should be well approximated by a quantum theory of particles when it is applied to reactions involving a finite number of particles. Because the defining requirements of relativistic quantum theory of particles are a subset of the axioms of local relativistic quantum field theory, the consequences of these requirements on the structure of the models of interacting particles are the same in both theories.

The Poincaré symmetry makes the problem of constructing a dynamical theory difficult. Poincaré covariance of the dynamics involves non-linear
constraints. The requirement that these constraints are preserved when the
system is separated into isolated subsystems introduces additional non-linear
constraints. These difficulties were recognized by Dirac [3] and have been
pointed out in a recent text by Weinberg [5].

The essential role of unitary representations of the Poincaré group in rela-
tivistic quantum theory was first emphasized by Wigner [6] in 1939. Most ap-
plications to finite systems of interacting particles cite Dirac’s 1949 paper,
which identified the essential difficulty and introduced kinematic subgroups
associated with different “forms of dynamics”. These subgroups, which re-
duce the number of constraints on the interactions, have played a role in all
subsequent theoretical development.

The problem of constructing interacting unitary representations of the
Poincaré group was first solved for the two-particle system by Bakamjian and
Thomas [7] in 1953. A three-particle solution satisfying $S$-matrix clustering
was given by Coester in 1965 [8]. The first complete solution of the problem
for $N$ particles was given by Sokolov in 1977 [9]. A general solution in all of
Dirac’s forms of dynamics appears in [2][9].

Relativistic quantum theory of particles is a practical framework for appli-
cations to few-hadron [10][11][12][13][14][15][16][17] and few-quark systems
[18][19][20][21][22]. All of these application are formulated in one of Dirac’s
forms of dynamics; they are limited to systems where cluster properties can
be trivially realized.

The construction in this paper is directly motivated by Wigner’s 1939
paper and makes essential use of irreducible representations of the Poincaré
group. It generalizes the two-body construction of [23] and leads to a rela-
tivistic N-body dynamics satisfying cluster properties and the spectral condition. Groups of unitary transformations that preserve the S-matrix and cluster properties are constructed. In the general construction all of the Poincaré generators may be interaction dependent. The kinematic subgroup symmetries can be implemented by imposing additional constraints on the general construction.

The resulting dynamics has interactions in between three and ten of the Poincaré generators. Unitary operators that preserve the S-matrix and cluster properties redistribute the interactions in ways that may be advantageous for different applications. These unitary operators are elements of a $C^*$ algebra of asymptotic constants, which is relevant for identifying physically equivalent theories.

This paper is organized as follows. Section two contains a brief account of Wigner’s formulation of relativistic quantum mechanics, which is central to the construction in this paper. Sections three to six summarize the group theory that is needed to construct the required representations. These sections discuss inhomogeneous $SL(2, C)$ ($ISL(2, C)$), which is the covering group of the Poincaré group, irreducible representations of $ISL(2, C)$, and Clebsch-Gordan coefficients of $ISL(2, C)$. Section seven provides an introduction to relativistic scattering theory, which is used in the general construction. This formulation of scattering theory does not assume the existence of a kinematic subgroup. Section eight introduces the cluster separability condition. Section nine introduces the $C^*$ algebra of asymptotic constants and its unitary elements, which are called scattering equivalences. This algebra provides a functional calculus of non-commuting operators that is used to establish
cluster properties. Section ten introduces the Möbius and Zeta function of the lattice of partitions. These combinatoric tools, which generalize standard Ursell cumulant expansions, are used extensively in the construction of the N-body dynamics. Section eleven contains the general solution of the two-body problem, which is the starting point of the recursive construction, and section twelve contains the recursive N-body construction. Section thirteen constructs scattering and cluster equivalences that relate dynamical models that utilize different bases. Section fourteen has conclusions. Technical aspects of the construction are included in the four appendices.

2 Relativity in Quantum Mechanics

In 1939 Wigner [6] showed that the relativistic invariance of all quantum probabilities

\[ P_{\psi\phi} := |\langle \psi | \phi \rangle|^2 \]  

is equivalent to the existence of a unitary representation of the Poincaré group. This was refined by Bargmann in 1954 [24] who observed that the dynamics could be realized by a single valued unitary representation of the covering group, ISL(2, C), of the Poincaré group. The central problem of relativistic quantum mechanics is to construct a unitary representation \( \hat{U}[\Lambda, Y] \) of ISL(2, C) which implements the dynamics.
3 Inhomogeneous $SL(2, C)$

In this section $ISL(2, C)$ is defined and related to the Poincaré group. Elements of $ISL(2, C)$ consist of ordered pairs of complex $2 \times 2$ matrices $(\Lambda, Y)$, where $\Lambda$ has determinant 1 and $Y$ is Hermitian. The group product is

$$(\Lambda_2, Y_2)(\Lambda_1, Y_1) = (\Lambda_2\Lambda_1, \Lambda_2Y_1\Lambda_1^\dagger + Y_2).$$  (2)

The relation to four-dimensional Poincaré transformations follows by representing four vectors $x^\mu$ by $2 \times 2$ Hermitian matrices $X$:

$$X := x^\mu\sigma_\mu, \quad x^\mu = \frac{1}{2}\text{Tr}(X\sigma_\mu)$$ (3)

where $\sigma_0$ is the identity and $\sigma_i$ are the Pauli matrices. In this matrix representation $ISL(2, C)$ transformations are affine transformations of the form

$$X' = \Lambda X\Lambda^\dagger + Y.$$ (4)

Any Poincaré transformation continuously connected to the identity can be represented in the form (4).

Elements of $ISL(2, C)$ can be parameterized by three components of a rotation vector $\vec{\theta}$, three components of a rapidity vector $\vec{\rho}$, and a space-time translation four vector $y^\mu$:

$$\Lambda(\vec{\theta}, \vec{\rho}) = e^{-\frac{i}{2}(\vec{\theta}+i\vec{\rho})\cdot\vec{\sigma}}, \quad Y(y) := y^\mu \sigma_\mu. (5)$$

Thus, the relativistic quantum dynamics, $\hat{U}[\Lambda, Y]$, satisfies:

$$\hat{U}^\dagger[\Lambda, Y] = \hat{U}^{-1}[\Lambda, Y] = \hat{U}[\Lambda^{-1}, -\Lambda^{-1}Y(\Lambda^{-1})^\dagger]$$ (6)

and

$$\hat{U}[\Lambda_2, Y_2]\hat{U}[\Lambda_1, Y_1] = \hat{U}[\Lambda_2\Lambda_1, \Lambda_2Y_1\Lambda_1^\dagger + Y_2].$$ (7)
4 \textit{ISL}(2, C') Generators

The infinitesimal generators of \( \hat{U}[\Lambda, Y] \) are defined. These operators are used to identify a maximal set of commuting self-adjoint operators. For structureless particles the eigenvalues of these commuting operators label the state of the particle. The spectrum of these operators is determined by the eigenvalues of the invariant mass and spin operators, which define an irreducible subspace, and group theoretic considerations. The single-particle Hilbert space is the space of square integrable functions of these eigenvalues.

The ten parameters \( y^\mu, \vec{\theta}, \vec{\rho} \) have the property that if any nine of them are set to zero, the group becomes a one-parameter unitary group with respect to the remaining parameter. These unitary one-parameter groups necessarily have the form \( \hat{U}(\lambda) = e^{-i\lambda \hat{G}} \) for a self-adjoint operator \( \hat{G} \). Thus a unitary representation \( \hat{U}[\Lambda, Y] \) of \textit{ISL}(2, C) can be parameterized as:

\[
\hat{U}[\Lambda(\vec{\theta}, \vec{\rho}), I] = e^{-i(\vec{\theta} \cdot \hat{J} + \vec{\rho} \cdot \hat{K})} \tag{8}
\]

\[
\hat{U}[I, Y(y)] = e^{iy \vec{P} - y^0 \hat{H}} \tag{9}
\]

with self-adjoint generators \( \hat{H}, \hat{P}, \hat{J} \) and \( \hat{K} \).

The commutation relations of the generators follow from the group representation property (7) and the definition (8) (9) of the generators. The commutation relations are consistent with \( \hat{P}^\mu := (\hat{H}, \hat{P}) \) transforming as a four-vector operator

\[
\hat{U}[\Lambda, 0]\hat{P}^\mu\hat{U}^\dagger[\Lambda, 0] = \hat{P}^\mu \Lambda_\nu^\mu \tag{10}
\]
and

\[ \hat{j}^{\mu \nu} := \begin{pmatrix} 0 & \hat{K}^1 & \hat{K}^2 & \hat{K}^3 \\ -\hat{K}^1 & 0 & \hat{J}^3 & -\hat{J}^2 \\ -\hat{K}^2 & -\hat{J}^3 & 0 & \hat{J}^1 \\ -\hat{K}^3 & \hat{J}^2 & -\hat{J}^1 & 0 \end{pmatrix} \]  \hspace{1cm} (11)

transforming as a rank-two antisymmetric tensor operator:

\[ \hat{U}[\Lambda, 0] \hat{j}^{\mu \nu} \hat{U}^\dagger[\Lambda, 0] = \hat{J}^{\alpha \beta} \Lambda^\alpha_\mu \Lambda^\beta_\nu. \]  \hspace{1cm} (12)

The Pauli-Lubanski vector \( \hat{W}^\mu \) is a four-vector valued function of \( \hat{P}_\mu \) and \( \hat{j}^{\mu \nu} \):

\[ \hat{W}^\mu := \frac{1}{2} \epsilon^{\mu \alpha \beta \gamma} \hat{P}_\alpha \hat{J}_{\beta \gamma} \]  \hspace{1cm} (13)

satisfying

\[ [\hat{j}^j, \hat{W}^k]_\mu = i \epsilon^{jkl} \hat{W}^l \] \hspace{1cm} \[ [\hat{J}^j, \hat{W}^0]_\mu = 0 \]  \hspace{1cm} (14)

\[ [\hat{K}^j, \hat{W}^k]_\mu = -i \delta^{jk} \hat{W}^0 \] \hspace{1cm} \[ [\hat{K}^j, \hat{W}^0]_\mu = -i \hat{W}^j \]  \hspace{1cm} (15)

\[ [\hat{P}_\mu, \hat{W}^\nu]_\mu = 0 \]  \hspace{1cm} (16)

\[ [\hat{W}_\mu, \hat{W}_\nu]_\mu = i \epsilon^{\mu \nu \rho \eta} \hat{W}_\rho \hat{P}_\eta \] \hspace{1cm} \[ \hat{W}_\mu \hat{P}_\mu = 0. \]  \hspace{1cm} (17)

The scalar operators

\[ \hat{M}^2 = -\hat{P}_\mu \hat{P}_\mu \]  \hspace{1cm} (18)

and

\[ \hat{W}^2 = \hat{W}_\mu \hat{W}_\mu \]  \hspace{1cm} (19)

are the two independent invariant polynomial functions of the generators of \( ISL(2, C) \).
When the spectrum of the mass operator is positive, the spin-squared operator is defined by

\[ j^2 := \frac{\hat{W}^2}{M^2}. \]  

(20)

5 Irreducible Representations of ISL(2, C)

The Hilbert space for an N-particle system is the tensor product of single particle Hilbert spaces. Single particle Hilbert spaces are irreducible representation spaces of ISL(2, C). The irreducible representations are labeled by the mass and spin of a particle. Eigenvalues of additional commuting self-adjoint functions of the ISL(2, C) generators are needed to specify the state of the particle. Simultaneous eigenstates of the commuting self-adjoint operators define a basis in the irreducible representation space. The single particle Hilbert space is the space of square integrable functions of the eigenvalues.

The irreducible representations of the ISL(2, C) were classified by Wigner [6] [27] [28] [5]. The displacement \( x_\mu^a - x_\mu^b \) between events \( a \) and \( b \) can be classified into six invariant classes depending on whether this displacement is zero, lightlike positive time, lightlike negative time, spacelike, timelike positive time, timelike negative time.

The irreducible representations corresponding to massive particles are the timelike positive-time representations. These irreducible representations of ISL(2, C) are labeled by the invariant eigenvalues of the mass (18) and spin operators (20). For a particle the mass eigenvalue \( m \) is discrete and the spin operator has the eigenvalue \( j(j + 1) \) where \( j \) is the spin of the particle.
The state of a structureless particle of mass $m$ and spin $j$ is determined by specifying the eigenvalues of a maximal set of commuting self-adjoint operators. These operators are the invariant mass $\hat{M}$, the spin $\hat{j}^2$ and four independent functions, $\hat{F}^i = F^i(\hat{P}^\mu, \hat{J}^\mu)$, of the $ISL(2, C)$ generators. The operators $\hat{F}^j$ cannot be invariant. They are arbitrary independent functions of the $ISL(2, C)$ generators subject to the constraints:

$$\hat{F}^i = (\hat{F}^i)\dagger \quad [\hat{F}^i, \hat{F}^j] = 0 \quad (21)$$

$$[\hat{F}^i, \hat{M}] = [\hat{F}^i, \hat{j}^2] = 0. \quad (22)$$

For particles with structure, additional invariant degeneracy operators are needed to get a maximal set of commuting operators.

The traditional choice for the operators $\hat{F}^i$ are the three components of the linear momentum $\hat{P}$ and the $z$-component of the canonical spin $\hat{z} \cdot \hat{J}_z$. In some applications it is advantageous to use the four velocity, the light-front components of the four momentum, or their conjugate variables. The helicity or light-front spin is sometimes used instead of the canonical spin. Any of the spin observables could be replaced by a component of the Pauli-Lubanski operator. These special cases are treated in Appendix I. Each choice of $\hat{F}^i$ corresponds to a single particle basis. In this paper the operators $\hat{F}^i$ are assumed to have a spectrum independent of the mass eigenvalue. This condition is not very restrictive and holds for all conventional choices.

The Hilbert space for a particle of mass $m$ and spin $j$ can be represented as the space of square integrable functions of the eigenvalues of the operators $\hat{F}^i$:

$$\mathcal{H}_{mj} = \left\{ \langle f|\psi\rangle | \int d\mu(f) |\langle f|\psi\rangle|^2 \right\} < \infty \quad (23)$$
where $f = \{f^1 \cdots f^4\}$ and $\int d\mu(f)$ indicates a sum over the discrete eigenvalues and an integral over the continuous eigenvalues of $\hat{F}^i$.

Basis vectors have the form
\[
|f\rangle := |f(m, j)\rangle := |f^1, f^2, f^3, f^4; m, j\rangle. \quad (24)
\]

The normalization convention is
\[
\langle f|f'\rangle = \delta[f, f'] \quad (25)
\]
where $\delta[f, f']$ is the product of Dirac or Kronecker delta functions in the variables $f^i$.

Irreducibility requires the transformation property:
\[
\hat{U}[^{\Lambda} Y]|f; m, j\rangle = \int |f'; m, j\rangle d\mu(f') \mathcal{D}^{m,j}_{f', f}[\Lambda, Y] \quad (26)
\]
where
\[
\mathcal{D}^{m,j}_{f', f}[\Lambda, Y]\delta_{m'm}\delta_{j'j} := \langle f'; m, j'|\hat{U}[\Lambda, Y]|f; m, j\rangle \quad (27)
\]
is the mass $m$, spin $j$ irreducible representation of $ISL(2, C)$ in the basis $\hat{F}^j$.

The $\mathcal{D}$-function includes $\delta$-functions that eliminate the integrals over the continuous spectrum in (26). Unitarity of the group representation property requires:
\[
\mathcal{D}^{m,j}_{f', f}[\Lambda, Y] = \left(\mathcal{D}^{m,j}_{f', f}[\Lambda^{-1}, -\Lambda Y \Lambda^\dagger]\right)^* \quad (28)
\]
and
\[
\int \mathcal{D}^{m,j}_{f', f''}[\Lambda_2, Y_2] d\mu(f'') \mathcal{D}^{m,j}_{f'', f}[\Lambda_1, Y_1] = \mathcal{D}^{m,j}_{f', f}[\Lambda_2 \Lambda_1, \Lambda_2 Y_1 \Lambda_2^\dagger + Y_2]. \quad (29)
\]
The restriction on the spectrum of $\hat{F}^i$ implies that range of values of $f$ in $\mathcal{D}^{m,j}_{f', f}[\Lambda, Y]$ is independent of $m$. 

12
Explicit representations for the \( ISL(2, C) \) Wigner \( D \)-functions corresponding to different \( \hat{F}^i \) are given in Appendix I. The form of the \( D \)-functions is basis dependent.

Irreducible representations in a basis of simultaneous eigenstates of a different set of commuting self-adjoint functions, \( \hat{G}^i \), of the generators are related to the representations in the basis \( \hat{F}^i \) by:

\[
D_{g,g'}^{m,j}[\Lambda, Y] = \int \langle g | f \rangle d\mu(f) D_{f,f'}^{m,j}[\Lambda, Y] d\mu(f') \langle f' | g' \rangle
\]  
(30)

where

\[
\langle f | g \rangle \delta_{m m'} \delta_{j j'} := \langle f ; m, j | g ; m', j' \rangle.
\]  
(31)

The coefficient functions \( \langle f | g \rangle \) can depend parametrically on the mass or spin. This parametric dependence on the mass is responsible for the dynamical differences that arise with different basis choices.

## 6 Clebsch-Gordan Coefficients

In this section Clebsch-Gordan coefficients \[8][27][28][29] and Racah coefficients of \( ISL(2, C) \) are defined. These are used to expand tensor products of irreducible representation as linear superpositions of irreducible representations and to transform between irreducible bases with different degeneracy quantum numbers.

The tensor product of irreducible representations of \( ISL(2, C) \) is reducible. The \( ISL(2, C) \) generators for a tensor product of two irreducible representations are

\[
\hat{P}^\mu = \hat{P}_1^\mu \otimes \hat{I}_2 + \hat{I}_1 \otimes \hat{P}_2^\mu.
\]  
(32)
\[ \hat{J}_{\mu\nu} = \hat{J}_{2\mu\nu} \otimes \hat{I}_1 + \hat{I}_1 \otimes \hat{J}_{1\mu\nu} . \]  

These operators act on the space

\[ \mathcal{H} = \mathcal{H}_{m_1j_1} \otimes \mathcal{H}_{m_2j_2} . \]  

The operators \( \hat{F}^i = F^i(\hat{P}^\mu, \hat{J}^\mu) \), \( \hat{M} = M(\hat{P}^\mu, \hat{J}^\mu) \), and \( \hat{j}^2 = j^2(\hat{P}^\mu, \hat{J}^\mu) \) are commuting self-adjoint operators on \( \mathcal{H} \). Because the tensor product is reducible, these operators do not define a maximal set of commuting self-adjoint operators. There are additional \( ISL(2, C) \) invariant degeneracy operators \( \hat{D}^i \) that distinguish multiple copies of the same irreducible representation. The degeneracy operators \( \hat{D}^i \) normally include the invariant operators \( \hat{M}_1, \hat{j}_1, \hat{M}_2, \hat{j}_2 \) of the factors of the tensor product and additional operators, \( \hat{R}_{12} \), that distinguish multiple copies of the \( m, j \) representation in the tensor product of the \( m_1, j_1 \) and \( m_2, j_2 \) representations.

The operators \( \hat{D}^i \) are invariant, self-adjoint functions of the single particle generators. The operators \( \hat{M}, \hat{j}^2, \hat{F}^1 \cdots \hat{F}^4, \hat{D}^1 \cdots \hat{D}^6 \) form a maximal set of commuting self-adjoint operators on \( \mathcal{H} \). Examples are given in the Appendix II.

The \((f, d)\) basis is the \( ISL(2, C) \)-irreducible basis for the tensor product space defined in terms of simultaneous eigenstates, \(|f, d; m, j\rangle\) of

\[ \{ \hat{F}^i, \hat{D}^k(\hat{M}, \hat{j}^2) \} . \]  

It follows that

\[ \hat{U}_1[\Lambda, Y] \otimes \hat{U}_2[\Lambda, Y]|f, d; m, j\rangle = \int |f', d; m, j\rangle d\mu(f') D_{f', j}^{m, j}[\Lambda, Y] \]  

where \( D_{f', j}^{m, j}[\Lambda, Y] \) is the irreducible representation matrix for a single particle of mass \( m \) spin \( j \). The \( D \)-function is independent of the invariant degeneracy parameters, \( d \).
The coefficients
\[ \langle f_1; m_1, j_1 : f_2; m_2, j_2 | f, d; m, j \rangle \]  \tag{37} \]
are Clebsch-Gordan coefficients of the Poincaré group in the \((f, d)\) basis. They are the kernel of the unitary transformation that relate tensor products of \(ISL(2, C)\) irreducible representations to direct integrals of irreducible representations. The \(ISL(2, C)\) Clebsch-Gordan coefficients have similar properties to \(SU(2)\) Clebsch-Gordan coefficients:
\[
\int \mathcal{D}^{m_1,j_1}[\Lambda,Y] \mathcal{D}^{m_2,j_2}[\Lambda,Y] d\mu(f'_1f'_2) \times \\
\langle f'_1; m_1, j_1 : f'_2; m_2, j_2 | f, d; m, j \rangle = \\
\int \langle f_1; m_1, j_1 : f_2; m_2, j_2 | f', d; m, j \rangle d\mu(f') \mathcal{D}^{m,j}_{f';f}[\Lambda,Y]. \]  \tag{38} \]
The new feature is that the irreducible representations are labeled by two Casimir operators and the mass operator has a continuous spectrum.

It is sometimes useful to replace the mass operator \(\hat{M}\) of the tensor product of two irreducible representations by the invariant relative momentum \(\hat{q}^2\), which has absolutely continuous spectrum, \([0, \infty)\):
\[
\hat{q}^2 = q^2(\hat{M}^2, \hat{M}_1^2, \hat{M}_2^2) := \\
\frac{\hat{M}^4 + \hat{M}_1^4 + \hat{M}_2^4 - 2\hat{M}_1^2\hat{M}_2^2 - 2\hat{M}_1^2\hat{M}_2^2 - 2\hat{M}^2\hat{M}_1^2 - 2\hat{M}^2\hat{M}_2^2}{4\hat{M}^2}. \]  \tag{39} \]

The Clebsch-Gordan coefficients have different forms in different bases. If \((f, d) \rightarrow (g, k)\) then the Clebsch-Gordan coefficients in the \((f, d)\) basis are related to the Clebsch-Gordan coefficients in the \((g, k)\) basis by
\[
\langle g_1; m_1, j_1 : g_2; m_2, j_2 | g; k; m, j \rangle = \
\]
\[ \int \langle g_1 | f_1' \rangle \langle g_2 | f_2' \rangle d\mu(f_1') d\mu(f_2') \times \]
\[ \langle f_1'; m_1, j_1 : f_2'; m_2, j_2 | f', d'; m, j \rangle d\mu(f', d') \langle f', d' | g, k \rangle \]  \hspace{1cm} (40)

where
\[ \langle g_i | f_i' \rangle \delta_{j_i j'_i} \delta_{m_i m'_i} := \langle g_i; m_i, j_i | f_i'; m'_i, j'_i \rangle \]  \hspace{1cm} (41)

and
\[ \delta_{jj'} \delta(m - m') \langle f, d | g', k' \rangle := \langle f, d; m, j | g', k'; m', j' \rangle. \]  \hspace{1cm} (42)

The Hilbert space for a system of \( N \)-particles is the \( N \)-fold tensor product of single particle Hilbert spaces:
\[ \mathcal{H} = \mathcal{H}_{m_1 j_1} \otimes \cdots \otimes \mathcal{H}_{m_N j_N}. \]  \hspace{1cm} (43)

The non-interacting representation of \( ISL(2,\mathbb{C}) \) on \( \mathcal{H} \) is defined by
\[ \hat{U}_0[\Lambda, Y] := \hat{U}_1[\Lambda, Y] \otimes \cdots \otimes \hat{U}_N[\Lambda, Y] \]  \hspace{1cm} (44)

where the 0 subscript is used to denote the non-interacting system. It follows that
\[ \hat{U}_0[\Lambda, Y]|f_1; m_1, j_1 \cdots f_N; m_N, j_N\rangle = \]
\[ \int |f_1'; m_1, j_1 : \cdots f_N'; m_N, j_N \rangle d\mu(f_1' \cdots f_N') \prod_{i=1}^{N} \mathcal{D}_{f_i'; f_i}^{m_i; j_i}[\Lambda, Y]. \]  \hspace{1cm} (45)

As in the case of \( SU(2) \), the tensor product of \( N \) irreducible representation spaces can be decomposed into a direct integral of irreducible representation spaces using successive pairwise coupling. The invariant degeneracy operators depend on the order of the coupling. It is also possible to use a simultaneous coupling scheme based on Mackey’s [30] theory of induced representations [31] which leads to a symmetric coupling.
Successive pairwise coupling is illustrated for the three-particle system:

\[ |f, d_{(12)3}; m, j) = \]

\[ \int |f_1; m_1, j_1 : f_2; m_2, j_2 : f_3; m_3, j_3) d\mu(f_1) d\mu(f_2) \times \]

\[ \langle f_1; m_1, j_1 : f_2; m_2, j_2 | f_{12}, d_{12}(m_{12}, j_{12}) | d\mu(f_12) d\mu(f_3) d\mu(m_{12}, j_{12}) \times \]

\[ \langle f_{12}; m_{12}, j_{12}: f_3; m_3, j_3 | f, d_{12,3}; m, j \rangle \]

(46)

where the invariant degeneracy parameters are

\[ d_{12,3} = \{ d_{12}, m_{12}, j_{12}, m_3, j_3, r_{12,3} \} \]

(47)

with

\[ d_{12} = \{ m_1, j_1, m_2, j_2, r_{12} \}. \]

(48)

Changing the ordering of the coupling from ((12)3) to ((23)1) changes the degeneracy parameters from \{ \{r_{12}, j_{12}, m_{12}, r_{12,3} \} \} to \{ \{r_{23}, j_{23}, m_{23}, r_{23,1} \} \}, leaving the operators \( \hat{M}, \hat{j}^2 \) and \( \hat{F}_i \) unchanged. The overlap coefficients have the general form

\[ \langle f, d_{ab,c}(m, j)| f', d'_{e,f,g}(m', j') \rangle = \]

\[ \delta[f, f'] \delta_{jj'} \delta(m - m') R^{m,j}_{d_{a,b,c},d'_{e,f,g}}. \]

(49)

The invariant quantities \( R^{m,j}_{d_{a,b,c},d'_{e,f,g}} \) are Racah coefficients for \( ISL(2, C) \). They are the kernel of the unitary transformation that changes the choice of degeneracy labels in subspaces corresponding the same mass, spin, and vector labels \( f \). They are independent of \( f \).

The Racah coefficients are important for performing computations because, as in the case of rotations, some operators have a simple form when the couplings are done in a specific order. Since many of the operators are
defined in specific representations, the Racah coefficients are needed for the evaluation of the abstract operator expressions.

The term Racah coefficient is used to indicate any change of irreducible basis with matrix elements of the form (49). Examples of Racah coefficients in representative bases are given in Appendix II.

7 Relativistic Scattering Theory

Relativistic scattering theory is formulated in this section. A kinematic subgroup is not assumed. The two-Hilbert space formulation [2][8][31] is used to treat multichannel scattering theory. The notation of this section follows [2]. Conditions on the interactions that are sufficient for a sensible relativistic scattering theory are discussed. Relativistic two-Hilbert space wave operators are essential elements of the general construction.

In this section the dynamical representation $\hat{U}[\Lambda, Y]$ of $ISL(2, C)$ is assumed to be given. The construction of $\hat{U}[\Lambda, Y]$ is the main topic of the remainder of this paper.

The first step in formulating relativistic scattering theory is to determine the bound states of $\hat{U}[\Lambda, Y]$; subsystem bound states are needed to formulate the asymptotic conditions in multi-channel scattering.

Bound states are associated with point eigenvalues of the mass and spin. For each bound-state channel $\alpha_b$ there is an irreducible subspace of $\mathcal{H}$. Vectors in the bound state subspace can be expressed as linear superpositions of simultaneous eigenstates of $\hat{M}, \hat{j}^2, \hat{F}^i$:

$$|\phi_{\alpha_b}\rangle = \int |f; m_\alpha, j_\alpha\rangle d\mu(f) \langle f | \chi\rangle \quad (50)$$
where in this expression $\hat{F}^i = F^i(\hat{P}^\mu, \hat{j}^{\mu\nu})$ are functions of the generators of $\hat{U}[\Lambda, Y]$.

The channel eigenstate $|f; m_{\alpha b}, j_{\alpha b}\rangle$ can be considered as a mapping, $\hat{\Phi}_{\alpha b}$, from the channel Hilbert space $\mathcal{H}_{\alpha b}$:

$$\mathcal{H}_{\alpha b} = \{ \langle f | \chi_{\alpha} \rangle | \int |\langle f | \chi_{\alpha} \rangle|^2 d\mu(f) < \infty \}$$  \hspace{1cm} (51)

to the invariant bound-state subspace of the Hilbert space $\mathcal{H}$:

$$\hat{\Phi}_{\alpha b} |\chi_{\alpha}\rangle := |\phi_{\alpha b}\rangle = \int |f; m_{\alpha b}, j_{\alpha b}\rangle d\mu(f) \langle f | \chi_{\alpha} \rangle.$$  \hspace{1cm} (52)

For each bound channel $\alpha b$ there is a channel injection operator $\hat{\Phi}_{\alpha b}$ and a channel Hilbert space $\mathcal{H}_{\alpha b}$. Since the bound channel spaces are irreducible representation spaces with respect to $\hat{U}[\Lambda, Y]$, the channel eigenstates transform irreducibly

$$\hat{U}[\Lambda, Y]|f; m_{\alpha b}, j_{\alpha b}\rangle = \int |f'; m_{\alpha b}, j_{\alpha b}\rangle d\mu(f') D_{f'f}^{m_{\alpha b}, j_{\alpha b}}[\Lambda, Y].$$  \hspace{1cm} (53)

Equation (53) can be expressed in terms of the channel injection operator as

$$\hat{U}[\Lambda, Y] \hat{\Phi}_{\alpha b} = \hat{\Phi}_{\alpha b} \hat{U}[\Lambda, Y].$$  \hspace{1cm} (54)

Scattering states are solutions of the time-dependent Schrödinger equation that look like mutually non-interacting bound or elementary subsystems in the asymptotic past or future. To formulate the asymptotic condition let $a$ denote a partition of $N$ particles into $n_a$ disjoint non-empty clusters. Denote the $i$-th cluster by $a_i$ and the number of particles in the $i$-th cluster by $n_{a_i}$.

For any partition $a$, the $N$-particle Hilbert space can be factored into a tensor product of subsystem Hilbert spaces $\mathcal{H}_{a_i}$:

$$\mathcal{H} = \bigotimes_{i=1}^{n_a} \mathcal{H}_{a_i}$$  \hspace{1cm} (55)
\[ \mathcal{H}_{a_i} = \bigotimes_{l \in a_i} \mathcal{H}_{m_{ij}}. \]  

(56)

A partition \( a \) has a scattering channel \( \alpha \) if the subsystem dynamics

\[ \hat{U}_{a_i} [\Lambda, Y] : \mathcal{H}_{a_i} \rightarrow \mathcal{H}_{a_i} \]  

(57)

associated with each cluster of \( a \) is either a one particle cluster or has a bound state.

For each bound subsystem channel, \( \alpha_i \), there is an injection operator, an asymptotic Hilbert space:

\[ \hat{\Phi}_{\alpha_i} : \mathcal{H}_{\alpha_i} \rightarrow \mathcal{H}_{a_i} \]  

(58)

and an irreducible asymptotic representation \( \hat{U}_{\alpha_i} [\Lambda, Y] \) of \( ISL(2, C) \) on \( \mathcal{H}_{a_i} \) satisfying:

\[ \hat{U}_{\alpha_i} [\Lambda, Y] \hat{\Phi}_{\alpha_i} = \hat{\Phi}_{\alpha_i} \hat{U}_{\alpha_i} [\Lambda, Y],. \]  

(59)

These relations hold trivially for the one particle clusters. The asymptotic Hilbert space for the scattering channel \( \alpha \) is defined as the tensor product of the bound channel subspaces for the subsystems:

\[ \mathcal{H}_{a} = \bigotimes_{i=1}^{n_a} \mathcal{H}_{\alpha_i}. \]  

(60)

The channel injection operator

\[ \hat{\Phi}_\alpha : \mathcal{H}_\alpha \rightarrow \mathcal{H} \]  

(61)

is defined by

\[ \hat{\Phi}_\alpha := \bigotimes_{i=1}^{n_a} \hat{\Phi}_{\alpha_i}. \]  

(62)

It follows from (59) that \( \hat{\Phi}_\alpha \) satisfies the intertwining relation

\[ \hat{U}_\alpha [\Lambda, Y] \hat{\Phi}_\alpha = \hat{\Phi}_\alpha \hat{U}_\alpha [\Lambda, Y] \]  

(63)
where
\[ \hat{U}_a[\Lambda, Y] := \otimes_{i=1}^{n_a} \hat{U}_{a_i}[\Lambda, Y] \] (64)

and
\[ \hat{U}_\alpha[\Lambda, Y] := \otimes_{i=1}^{n_a} \hat{U}_{\alpha_i}[\Lambda, Y]. \] (65)

In this notation a scattering state is a solution
\[ |\psi_{\pm}^\alpha(t)\rangle = \hat{U}[I, T]|\psi_{\pm}^\alpha\rangle \quad T := t\sigma_0 \] (66)
of the time-dependent Schrödinger equation satisfying the asymptotic condition
\[ \lim_{t \to \pm \infty} ||| \psi_{\pm}^\alpha(t) \rangle - \hat{U}_a[I, T] |\hat{\Phi}_\alpha \rangle |\chi_\alpha\rangle || \] (67)
for \( |\chi_\alpha\rangle \in H_\alpha \).

Equation (63) can be used to express the asymptotic condition as
\[ \lim_{t \to \pm \infty} ||| \psi_{\pm}^\alpha(t) \rangle - \hat{U}_a[I, -T] |\hat{\Phi}_\alpha \rangle \hat{U}_\alpha[I, T] |\chi_\alpha\rangle || = 0 \] (68)
which is identically satisfied by the bound-state channels.

Equation (68) can be expressed as
\[ |\psi_{\pm}^\alpha\rangle := \hat{\Omega}_{\alpha \pm} |\chi_\alpha\rangle \] (69)
where the channel wave operators
\[ \hat{\Omega}_{\alpha \pm} : H_\alpha \to H \] (70)
are defined by the strong limits
\[ \hat{\Omega}_{\alpha \pm} := \lim_{t \to \pm \infty} \hat{U}(I, -T) |\hat{\Phi}_\alpha \rangle \hat{U}_\alpha[I, T]. \] (71)
A sufficient condition for the existence of the channel wave operators is the Cook condition \[32]:

\[
\int_{t_c}^{\infty} \| \hat{V}_a \hat{U}_a (I, \pm T) | \chi \rangle \| dt < \infty
\] \hspace{1cm} (72)

where \( t_c \) is any constant and

\[
\hat{V}_a := \hat{H} \hat{\Phi}_a - \hat{\Phi}_a \hat{H}.
\] \hspace{1cm} (73)

The scattering operator for scattering from channel \( \alpha \) to channel \( \beta \) is the mapping from \( \mathcal{H}_\alpha \rightarrow \mathcal{H}_\beta \) defined by

\[
\hat{S}_{\beta \alpha} := \hat{\Omega}_\beta^\dagger \hat{\Omega}_\alpha.
\] \hspace{1cm} (74)

This is can be expressed compactly in a two-Hilbert space notation, where the asymptotic Hilbert space, \( \mathcal{H}_A \) is the orthogonal direct sum of all of the channel spaces, including the bound state channel spaces:

\[
\mathcal{H}_A = \bigoplus \mathcal{H}_\alpha.
\] \hspace{1cm} (75)

A two-Hilbert space injection operator \( \hat{\Phi}_A \):

\[
\hat{\Phi}_A : \mathcal{H}_A \rightarrow \mathcal{H}
\] \hspace{1cm} (76)

is defined as the sum of the channel injection operators

\[
\hat{\Phi}_A = \sum_\alpha \hat{\Phi}_\alpha
\] \hspace{1cm} (77)

where it is understood that each \( \hat{\Phi}_\alpha \) acts on the channel subspace \( \mathcal{H}_\alpha \) of \( \mathcal{H}_A \).

There is a natural unitary representation of \( ISL(2, C) \) on \( \mathcal{H}_A \) which transforms the particles or bound states as tensor products of irreducible representations:

\[
\hat{U}_A[\Lambda, Y] = \sum_\alpha \hat{U}_\alpha[\Lambda, Y]
\] \hspace{1cm} (78)

22
where \( \hat{U}_\alpha[\Lambda, Y] : \mathcal{H}_\alpha \to \mathcal{H}_\alpha \).

The bound state solutions and the scattering asymptotic conditions can be replaced by one two-Hilbert space equation:

\[
\Omega_\pm(\hat{H}, \hat{\Phi}_A, \hat{H}_A) = \lim_{t \to \pm \infty} \hat{U}[I, -T] \hat{\Phi}_A \hat{U}_A[I, T]
\]

where the limit is a strong limit. The wave operators \( \Omega_\pm(\hat{H}, \hat{\Phi}_A, \hat{H}_A) \) are mappings from \( \mathcal{H}_A \to \mathcal{H} \).

The scattering operator \( \hat{S} \) is a mapping from \( \mathcal{H}_A \to \mathcal{H}_A \) defined by

\[
\hat{S} := \Omega_+^\dagger(\hat{H}, \hat{\Phi}_A, \hat{H}_A) \Omega_-(\hat{H}, \hat{\Phi}_A, \hat{H}_A).
\]

(80)

The dynamics is asymptotically complete if the two-Hilbert space wave operators \( \Omega_\pm(\hat{H}, \hat{\Phi}_A, \hat{H}_A) \), which include all bound state channels, are unitary mappings from \( \mathcal{H}_A \to \mathcal{H} \). In all that follows the two-Hilbert space wave operators are assumed exist and to be unitary. These properties can be proved using the same methods used in non-relativistic scattering theory.

Fong and Sucher [33][2][34][35] showed that relativistic invariance of the scattering operator does not follow from the existence of \( \hat{U}[\Lambda, Y] \). This is because the \( ISL(2, C) \) transformations must commute with the limiting operations that are used to construct the scattering operator.

Invariance of \( \hat{S} \) is equivalent to the condition

\[
[\hat{U}_A[\Lambda, Y], \hat{S}]_- = 0.
\]

(81)

The following theorem provides a sufficient condition on \( \hat{U}[\Lambda, Y] \) for the \( ISL(2, C) \) invariance of the \( S \)-matrix:

**Theorem 1:** Let \( \Omega_\pm(\hat{H}, \hat{\Phi}_A, \hat{H}_A) \) be asymptotically complete two Hilbert space wave operators. A sufficient condition for \( \hat{S} \) to be \( ISL(2, C) \) invariant
is that for all $\Lambda$ and $Y$

$$\lim_{t \to \pm \infty} \left( \hat{\Phi}_A - \hat{U}^\dagger[I, Y] \hat{\Phi}_A \hat{U}_A[I, Y] \right) \hat{U}_A[I, T] = 0$$

(82)

and for any $Y$ of the form $Y = \vec{y} \cdot \vec{\sigma}$

$$\lim_{t \to \pm \infty} \left( \hat{\Phi}_A - \hat{U}^\dagger[I, Yt] \hat{\Phi}_A \hat{U}_A[I, Yt] \right) \hat{U}_A[I, T] = 0.$$

(83)

The limits above are strong limits. They must hold for both time directions.

Theorem 1 provides sufficient conditions on the interactions in the generators for a sensible relativistic scattering theory. The proof of this theorem is given in Appendix III.

The proof of Theorem 1 has a number of useful corollaries:

**Corollary 1** If the conditions of Theorem 1 are satisfied, then

$$\hat{U}[\Lambda, Y] \Omega_\pm(\hat{H}, \hat{\Phi}_A, \hat{H}_A) = \Omega_\pm(\hat{H}, \hat{\Phi}_A, \hat{H}_A) \hat{U}_A[\Lambda, Y].$$

(84)

This intertwining property ensures the $ISL(2, C)$ invariance of $S$.

**Corollary 2** If the conditions of Theorem 1 are satisfied, then

$$\Omega_\pm(\hat{H}, \hat{\Phi}_A, \hat{H}_A) = \Omega_\pm(\hat{P} \cdot \vec{y}, \hat{\Phi}_A, \hat{P}_A \cdot \vec{y})$$

(85)

where $y$ is any future-pointing time-like 4-vector.

This means that all future pointing time-like directions are equivalent for the purpose of formulating the asymptotic condition.

**Corollary 3** If the conditions of Theorem 1 are satisfied, then

$$\Omega_\pm(\hat{H}, \hat{\Phi}_A, \hat{H}_A) = \Omega_\pm(\hat{M}, \hat{\Phi}_A, \hat{M}_A)$$

(86)
This shows that in applications the Hamiltonian can be replaced by the mass operator in the wave operators. Both representations of the two-Hilbert space wave operators are used in the remainder of this paper.

Theorem 1 and its corollaries define conditions on the interactions that ensure that the dynamics is consistent with naive expectations for a relativistic scattering theory. In all that follows it is assumed that the two-Hilbert wave operators exist, are complete, and the dynamical operators satisfy (82) and (83).

8 Cluster Properties

Cluster properties provide the essential connection between the few and many-body problem. The cluster property requires that few-body interactions in the few-body problem are identical to the few-body interactions in the many-body problem. This establishes the justification for performing experiments on few-body systems.

The difficulty in satisfying cluster properties is that the interactions that appear in the $ISL(2,C)$ generators are uniquely determined by cluster properties up to an $N$-body interaction. Unfortunately, the $ISL(2,C)$ commutation relations put non-linear constraints on the $N$-body interactions which cannot be satisfied by setting these interactions to zero.

To formulate cluster properties let $a$ be a partition of the $N$ particle systems into $n_a$ disjoint clusters. Let $\hat{U}_{ai}[\Lambda,Y]$ be the subsystem representation of $ISL(2,C)$ for the particles in the $i$-th cluster of $a$. Define the cluster
translation operator $\hat{T}_a(Y_1, \cdots, Y_{n_a})$ on $\mathcal{H}$ by

$$\hat{T}_a[Y_1, \cdots, Y_{n_a}] := \otimes \hat{U}_a[I, Y_i].$$ (87)

The dynamical representation of the Poincaré group satisfies strong cluster properties if for all partitions $a$ and all $|\chi\rangle \in \mathcal{H}$

$$\lim_{\min(y_i-y_j)^2 \to +\infty} \langle \left( \hat{U} [\Lambda, Y] - \otimes_{i=1}^{n_a} \hat{U}_a[I, Y_i] \right) \hat{T}_a[Y_1, \cdots, Y_{n_a}] |\chi\rangle \| = 0.$$ (88)

Cluster properties will hold if (a)

$$\hat{U} [\Lambda, Y] \to U_a[\Lambda, Y] = \otimes_{i=1}^{n_a} \hat{U}_a[I, Y_i]$$ (89)

when the interactions involving particles in different clusters of $a$ are set to zero and (b) all of the interactions in each generator $\hat{G}$ satisfy:

$$\lim_{\min(y_i-y_j)^2 \to +\infty} \| \left( \hat{G} - \hat{G}_a \right) \hat{T}_a[Y_1, \cdots, Y_{n_a}] |\chi\rangle \| = 0$$ (90)

where $\hat{G}$ and $\hat{G}_a$ are the ISL(2, $C$) generators associated with $\hat{U} [\Lambda, Y]$ and $U_a[\Lambda, Y]$ respectively.

Condition (a) is called the algebraic cluster property [2]. It puts the non-linear constraints on the interactions of a relativistic quantum theory. It ensures that once the interactions between particles in different clusters are turned off the remainder is a tensor product. This condition is non-trivial because it must hold for every possible clustering.

The condition (b) is related to the range of the interaction. If the operators satisfy algebraic cluster properties the proof of the short range condition is similar to the non-relativistic proof [11] of cluster properties. In all that follows the interaction terms are assumed to satisfy condition (b).
When $\hat{U}[\Lambda, Y]$ does not satisfy algebraic cluster properties the limit (88) may not exist. A typical consequence is that the cluster limit eliminates interactions between particles in the same cluster [29].

The cluster condition (88) is a strong form of the cluster condition. It is also possible to formulate a weaker form of the cluster condition that applies only to the scattering matrix [2]. The stronger form is needed for the recursive construction in sections 12 and 13.

9 Scattering Equivalences

There is a large class of dynamical models with the same $S$-matrix. These models are called scattering equivalent models [36]. The freedom to transform between scattering equivalent models with different properties is an important tool for realizing cluster properties. What separates scattering equivalent models from unitary equivalent models is that scattering equivalent models do not change the description of free particles. They provide a parameterization of the freedom that is created by restricting the class of physical observables to asymptotic quantities ($t \to \pm \infty$).

While scattering equivalences necessarily preserve cluster properties of the $S$-matrix, they do not preserve cluster properties of the representation $\hat{U}[\Lambda, Y]$. Because of this property, scattering equivalences can be used to restore cluster properties of the dynamics.

The key to understanding scattering equivalences is to understand the algebra of operators that are asymptotically zero. A bounded operator $\hat{Z}$ on the $N$-particle Hilbert space is asymptotically zero if the following strong
limits vanish

\[ \lim_{t \to \pm \infty} \hat{Z} \hat{U}_0[I, T]|\psi\rangle = 0; \quad (91) \]

\[ \lim_{t \to \pm \infty} \hat{Z}^\dagger \hat{U}_0[I, T]|\psi\rangle = 0; \quad (92) \]

for both time limits, where

\[ T = t\sigma_0. \quad (93) \]

The subset of bounded operators that are asymptotically zero are denoted by \( \mathcal{Z} \). It is straightforward to show that for \( \hat{Z}_n \in \mathcal{Z} \) and \( \alpha \) complex that

\[ \alpha \hat{Z}_1 + \hat{Z}_2 \in \mathcal{Z} \quad (94) \]

\[ \hat{Z}_1 \hat{Z}_2 \in \mathcal{Z} \quad (95) \]

\[ \hat{Z}_1^\dagger \in \mathcal{Z} \quad (96) \]

\[ \| \hat{Z}_n - \hat{Z} \| \to 0 \Rightarrow \hat{Z} \in \mathcal{Z}. \quad (97) \]

Including the identity makes a \( C^* \) algebra, which we call the algebra of asymptotic constants, \( \mathcal{C} \).

A scattering equivalence \( \hat{A} \) is a unitary member of \( \mathcal{C} \) that is asymptotically equal to the identity \( \hat{I} \):

\[ \lim_{t \to \pm \infty} (\hat{A} - \hat{I}) \hat{U}_0[I, T]|\psi\rangle = 0; \quad (98) \]

\[ \lim_{t \to \pm \infty} (\hat{A}^\dagger - \hat{I}) \hat{U}_0[I, T]|\psi\rangle = 0; \quad (99) \]

The relation of these operators to scattering is through the following theorems:

**Theorem 2**: Let \( \hat{A} \) be a scattering equivalence. Let \( \Omega_{\pm}(\hat{H}, \hat{\Phi}_A, \hat{H}_A) \) be asymptotically complete two Hilbert space wave operators. Let \( \hat{H}' = \hat{A} \hat{H} \hat{A}^\dagger \)
and $\hat{\Phi}_A = \hat{A} \hat{\Phi}_A$. Then $\Omega_\pm(\hat{H}', \hat{\Phi}'_A, \hat{H}_A)$ exist, are asymptotically complete, and give the same $S$ matrix as $\Omega_\pm(\hat{H}, \hat{\Phi}_A, \hat{H}_A)$.

The proof follows from the identity

$$\hat{A} \Omega_\pm(\hat{H}, \hat{\Phi}_A, \hat{H}_A) = \Omega_\pm(\hat{H}', \hat{\Phi}'_A, \hat{H}_A).$$

(100)

While the structure of the injection operator $\hat{\Phi}_A$ depends on the representation of the subsystem bound states, it must become the identity in the scattering channel, ($\alpha = \alpha_0$), corresponding to $N$ free particles. Note that $\hat{\Phi}'_{\alpha_0} = \hat{A} \hat{I} = \hat{I} + \hat{Z} \neq \hat{I}$ where $\hat{Z}$ is asymptotically zero. This ensures that $\hat{\Phi}'_A$ can be replaced by another injection operator, $\hat{\Phi}''_A$, with $\hat{\Phi}''_{\alpha_0} = \hat{I}$:

$$\hat{\Phi}''_A = \hat{\Phi}'_A - \delta_{a\alpha_0} \hat{Z}.$$  

(101)

It follows that

$$\Omega_\pm(\hat{H}', \hat{\Phi}'_A, \hat{H}_{\alpha_0}) = \Omega_\pm(\hat{H}', \hat{\Phi}''_A, \hat{H}_{\alpha_0})$$

(102)

where $\hat{\Phi}''_{\alpha_0} = \hat{I}$.

Scattering equivalences are naturally constructed from pairs of wave operators that give the same $S$-matrix.

**Theorem 3:** Let $\hat{\Omega}_\pm := \Omega_\pm(\hat{H}, \hat{\Phi}_A, \hat{H}_A)$ and $\hat{\Omega}'_\pm := \Omega_\pm(\hat{H}', \hat{\Phi}'_A, \hat{H}_A)$ be asymptotically complete wave operators that give the same scattering matrix. Then there is a scattering equivalence $\hat{A}$ satisfying $\hat{H}' = \hat{A} \hat{H} \hat{A}^\dagger$.

To prove Theorem 3 note that the assumptions imply

$$S = \hat{\Omega}^\dagger_+ \hat{\Omega}_- = \hat{\Omega}'^\dagger_+ \hat{\Omega}'_-.$$  

(103)
Asymptotic completeness implies
\[ \hat{A} := \hat{\Omega}_+\hat{\Omega}_+^\dagger = \hat{\Omega}_-\hat{\Omega}_-^\dagger. \] (104)
This definition and the intertwining relations \[\text{for the Hamiltonian give} \]
\[ \hat{A}\hat{H}\hat{A}^\dagger = \hat{\Omega}_+\hat{\hat{H}}\hat{\hat{A}}\hat{\hat{H}}\hat{\hat{A}}^\dagger = \]
\[ \hat{H}'\hat{\Omega}_+\hat{\Omega}_+^\dagger\hat{A}^\dagger = \hat{H}'. \] (105)
Equations (104) and (105) imply
\[ \Omega_\pm(\hat{H}', \hat{\Phi}_A', \hat{H}_A) = \hat{A}\Omega_\pm(\hat{H}, \hat{\Phi}_A, \hat{H}_A) = \Omega_\pm(\hat{H}', \hat{\hat{A}}\hat{\Phi}_A, \hat{H}_A). \] (106)
The equality of the first and last terms gives the strong limit
\[ \lim_{t \to \pm\infty} (\hat{\Phi}_A' - \hat{A}\hat{\Phi}_A)\hat{U}_A[I, T] = 0. \] (107)
Unitarity of \(\hat{A}\) gives
\[ \lim_{t \to \pm\infty} (\hat{A}^\dagger\hat{\Phi}_A' - \hat{\Phi}_A)\hat{U}_A[I, T] = 0, \] (108)
restricting to the \(\alpha_0\) channel, using \(\hat{\Phi}_{\alpha_0} = \hat{\Phi}'_{\alpha_0} = \hat{I}\) and \(\hat{U}_{\alpha_0}[I, T] = \hat{U}_0[I, T]\) gives
\[ \lim_{t \to \pm\infty} (\hat{A} - \hat{I})\hat{U}_0[I, T] = 0 \] (109)
and
\[ \lim_{t \to \pm\infty} (\hat{A}^\dagger - \hat{I})\hat{U}_0[I, T] = 0 \] (110)
which establishes that \(\hat{A}\) is a scattering equivalence.

This shows that if two asymptotically complete wave operators give the same scattering matrix then the Hamiltonians are related by a scattering equivalence. Since \(\hat{A}\) is unitary it follows that
\[ \hat{U}'[\Lambda, Y] := \hat{A}\hat{U}[\Lambda, Y]\hat{A}^\dagger \] (111)
is a scattering equivalent representation of $ISL(2, C)$.

The important property of the scattering equivalences is that they are the unitary elements of the $C^*$ algebra of asymptotic constants. The $C^*$ algebra can be used to construct functions of the non-commuting scattering equivalences. When these functions are unitary and can be expressed as uniform limits of elements of this algebra, they are scattering equivalences. This provides the mechanism for constructing scattering equivalences with specialized properties.

10 Birkhoff Lattices:

The construction of operators satisfying cluster properties requires a significant amount of algebra involving cluster expansions of operators. The theory of Birkhoff lattices \cite{37,38,39,40,2} facilitates the required algebra. It provides closed-form expressions relating different standard cluster expansions of operators.

Let $\mathcal{P}$ denote the set of all possible partitions of N-particles into disjoint non-empty clusters. There is a natural partial ordering on $\mathcal{P}$ given by

$$a \supseteq b \quad (112)$$

if and only if every pair of particles in the same cluster of $b$ is in the same cluster of $a$. This means that $b$ can be obtained from $a$ by breaking up clusters.

The Zeta and Möbius functions \cite{38,40} for this partial ordering are integer
valued functions on $\mathcal{P} \times \mathcal{P}$ defined by
\[
\zeta(a \supseteq b) = \begin{cases} 
1 & a \supseteq b \\
0 & \text{otherwise}
\end{cases}
\] (113)
and
\[
\mu(a \supseteq b) = \zeta^{-1}(a \supseteq b) = \begin{cases} 
(-)^{n_a} \prod_{i=1}^{n_a} (-)^{n_{b_i}} (n_{b_i} - 1)! & a \supseteq b \\
0 & \text{otherwise}
\end{cases}
\] (114)
where $n_{b_i}$ are the number of clusters of $b$ in the $i$-th cluster of $a$. Note that both $\zeta(a \supseteq b)$ and $\mu(a \supseteq b)$ vanish unless $a \supseteq b$.

Intersections and unions, $a \cap b$ and $a \cup b$, of two partitions $a$ and $b$ are defined as the greatest lower bound and least upper bound with respect to this partial ordering.

It follows from the definitions that
\[
\zeta((a \cap b) \supseteq c) = \zeta(a \supseteq c)\zeta(b \supseteq c)
\] (115)
and
\[
\zeta(a \supseteq (b \cup c)) = \zeta(a \supseteq b)\zeta(a \supseteq c).
\] (116)

The set of partitions with the operations $\cup$ and $\cap$ form a semimodular lattice [37], called a partition or Birkhoff lattice. It provides a convenient means for keeping track of interactions. Let $\mathcal{O}$ be an operator that is a function of the physical $ISL(2, C)$ infinitesimal generators. Imagine putting a parameter $\lambda_i$ in front of each interaction that appears in the the physical $ISL(2, C)$ generators. The operator $\mathcal{O}_a$ is defined to be the result of turning off the interactions between particles in different clusters of $a$. In general the operator $\mathcal{O}_a$ will include the contributions of operators in $\mathcal{O}_b$ for all $a \supseteq b$. These can be recursively subtracted to construct truncated contributions $[\mathcal{O}]_b$. 

32
to $\mathcal{O}_a$. The truncated operators $[\mathcal{O}]_a$ vanish whenever interactions involving particles in any cluster of $a$ are turned off. The the M"obius function can be used to generate closed form expressions for the truncated operators in terms of the untruncated $\mathcal{O}_a$’s:

$$[\mathcal{O}]_a := \sum_b \mu(a \supseteq b) \mathcal{O}_b.$$  \hspace{1cm} (117)

This can be inverted using the Zeta function to get

$$\mathcal{O}_a := \sum_b \zeta(a \supseteq b) [\mathcal{O}]_b.$$  \hspace{1cm} (118)

If this is applied to the case where $a$ is the 1-cluster partition, this becomes

$$\mathcal{O} = \sum_b [\mathcal{O}]_b.$$  \hspace{1cm} (119)

While this generates the standard relations between ordinary multipoint functions and truncated multipoint functions based on cluster expansion methods, use of the lattice structure, and specifically the underlying partial ordering, has advantages that are useful in the recursive construction described in sections 12 and 13.

11 Two-Body Problem

The construction of two-body models follows [23]. The two-body Hilbert space is the tensor product of single particle spaces

$$\mathcal{H} = \mathcal{H}_{m_1,j_1} \otimes \mathcal{H}_{m_2,j_2}.$$  \hspace{1cm} (120)

Choose a basis $(f, d)$ and use the Clebsch-Gordan coefficient:

$$\langle f_1; m_1, j_1 : f_2; m_2, j_2 | f, d; m, j \rangle$$  \hspace{1cm} (121)
to construct an irreducible free-particle basis. The states
\[ |f, d; m, j) \] (122)
transform as mass \( m \) spin \( j \) irreducible representations of \( ISL(2, C) \) with respect to the non-interacting representation
\[ \hat{U}_0[\Lambda, Y] := \hat{U}_1[\Lambda, Y] \otimes \hat{U}_2[\Lambda, Y]. \] (123)

Using the \( ISL(2, C) \) transformation properties it is possible to construct operators \( \Delta \hat{F}_i^0 \) that change the value of \( f^i \), holding the values of \( f^j, (j \neq i) \) constant. If \( \hat{F}_i^0 \) has a continuous spectrum these operators are proportional to partial derivatives
\[ \Delta \hat{F}_i^0 = i \frac{\partial}{\partial f^i} \] (124)
holding \( f^k; k \neq j \) constant. If \( F_i^0 \) has discrete eigenvalues, a suitable \( \Delta \hat{F}_i^0 \) can typically be expressed in terms of a raising or lowering operators.

The operators \( \hat{M}_0, \hat{j}_0^2, \hat{F}_0^i, \Delta \hat{F}_0^i \) are functions of the free particle generators. Expression for the generators in terms of these operator can be constructed using the \( ISL(2, C) \) \( D \)-functions:
\[ \langle f, d; m, j| \tilde{K}_0| f', d'; m, s \rangle := \]
\[ i \frac{\partial}{\partial \rho} D_{f, f'}^{m, j}[\Lambda(\theta = 0, \rho), 0] \delta[d, d'] \delta(m - m') \delta_{jj'} \] (125)
\[ \langle f, d; m, j| \tilde{J}_0| f', d'; m, j \rangle := \]
\[ i \frac{\partial}{\partial \theta} D_{f, f'}^{m, j}[\Lambda(\theta, \rho = 0), 0] \delta[d, d'] \delta(m - m') \delta_{jj'} \] (126)
\[ \langle f, d; m, j| P_0^0| f', d'; m, j \rangle := \]
\[ -i \frac{\partial}{\partial y_\mu} D^{m,j}_{f,f'}[I,Y(y)] \delta[d,d'] \delta(m - m') \delta_{jj'} \]  

where all derivatives are computed at 0.

The chain rule gives explicit expressions for the ISL(2, C) generators in terms of the operators \( \hat{M}_0, \hat{j}_0^2, \hat{F}_0^i, \Delta \hat{F}_i^0 \):

\[ \hat{P}_0^\mu = \hat{P}^\mu(\hat{M}_0, \hat{j}_0^0, \hat{F}_0^i, \Delta \hat{F}_i^0) \]  

\[ \hat{j}_0^{\mu\nu} = \hat{j}^{\mu\nu}(\hat{M}_0, \hat{j}_0^0, \hat{F}_0^i, \Delta \hat{F}_i^0). \]

These expressions can be inverted to express \( \hat{M}_0, \hat{j}_0^2, \hat{F}_0^i, \Delta \hat{F}_i^0 \) in terms of the ISL(2, C) generators:

\[ \hat{M}_0 = M(\hat{P}_0^\mu, \hat{j}_0^{\mu\nu}) \]  

\[ \hat{j}_0^2 = j_0(\hat{P}_0^\mu, \hat{j}_0^{\mu\nu}) \]  

\[ \hat{F}_0^i = F^i(\hat{P}_0^\mu, \hat{j}_0^{\mu\nu}) \]  

\[ \Delta \hat{F}_i^0 = \Delta F^i(\hat{P}_0^\mu, \hat{j}_0^{\mu\nu}). \]

Examples of these operators for specific basis choices are computed in Appendix I to illustrate the general procedure.

Since \( \hat{M}_0^2 \) is a Casimir operator for ISL(2, C), it necessarily commutes with \( \hat{j}_0^2, \hat{F}_0^i, \) and \( \Delta \hat{F}_i^0 \). The ISL(2, C) commutation relations follow as consequences of the commutation relations of \( \hat{M}_0, \hat{j}_0^2, \hat{F}_0^i, \) and \( \Delta \hat{F}_i^0 \).

It follows that in order to construct a dynamical representation of ISL(2, C) it is enough to replace \( \hat{M}_0 \) by an operator \( \hat{M} = \hat{M}_0 + \hat{V} \) which also commutes with \( \hat{j}_0^2, \hat{F}_i^0, \) and \( \Delta \hat{F}_i^0 \). With this choice of interaction it follows that the operators

\[ \hat{P}_0^\mu \to \hat{P}^\mu = \hat{P}^\mu(\hat{M}, \hat{j}_0^2, \hat{F}_0^i, \Delta \hat{F}_i^0) \]  

35
\[ \hat{j}_0^{\mu\nu} \to \hat{j}^{\mu\nu} = \hat{j}^{\mu\nu}(\hat{M}, \hat{j}_0^2, \hat{F}_0^i, \Delta \hat{F}_0^i) \]  

(135)

automatically satisfy the ISL\((2,C)\) Lie algebra.

Cluster properties are satisfied for sufficiently short-range interactions. For the interaction to be non-trivial it should also satisfy

\[ [\hat{M}, \hat{M}_0] \neq 0 \]

(136)

and the spectral condition, \(\hat{M}_0 > \hat{V}\). In general the interaction can be treated as a perturbation of different functions of \(\hat{M}_0\), such as \(\hat{M}_0^2\). In all cases the interactions can be put in the form \(\hat{M} = \hat{M}_0 + \hat{V}\) by defining \(\hat{V} := \hat{M} - \hat{M}_0\), independent of how \(\hat{M}\) is constructed. The spectral condition constrains the interaction.

In the free particle irreducible basis an interaction \(\hat{V}\) commuting with \(\hat{j}_0^2, \hat{F}_0^i, \) and \(\Delta \hat{F}_0^i\) has a kernel with the structure:

\[ \langle f, d; m, j|\hat{V}|f', d'; m', j' \rangle = \]

\[ \delta[f, f']\delta_{jj'}\langle d, m||\hat{V}_j||d', m' \rangle. \]

(137)

The dynamical generators are given by (134) and (135) with \(\hat{M} = \hat{M}_0 + \hat{V}\). If the expression for a generator in (134) or (135) has an explicit mass dependence, the corresponding operator will be interaction dependent. Depending on the choice of basis \((f, d)\) between three and ten generators will have an explicit interaction dependence. Dirac’s forms of dynamics result from specific basis choices. A generic choice will not have a kinematic subgroup.

While it is straightforward to derive explicit expressions for the generators in terms of the \(\hat{F}^i\)'s, (see Appendix I) it is easier to directly solve for the dynamics in the free particle basis \(|f, d; m, j\rangle\).
In this basis $\hat{M}, \hat{F}_0, \hat{j}_0$ can be simultaneously diagonalized:

$$
\langle f', d'; m', j'| f; m, j \rangle = \delta[f, f'] \delta_{jj'} \phi_m^j(d', m')
$$

(138)

where $\phi_m^j(d', m')$ is the solution of the mass eigenvalue equation

$$
(m - m') \phi_m^j(d', m') = \sum \int dm'' dd'' \langle d', m'\| \hat{V}_j\| d'' m'' \rangle \phi_m^j(d'', m'').
$$

(139)

For suitable interactions $\hat{M}$ will be self-adjoint and the eigenstates $| f, d; m, j \rangle$ will define a complete set of simultaneous eigenstates of $\hat{M}, \hat{F}_i, \hat{j}_0$. Solving equation (139) is of comparable difficulty to solving the time-independent non-relativistic Schrödinger equation. It is assumed that the eigenstates include two-body bound states and scattering states satisfying incoming and outgoing wave asymptotic conditions.

Since the expressions (125-127) for the $ISL(2, C)$ generators were derived by evaluating the infinitesimal transformations in an irreducible basis, and $\{ \hat{M}_0, \hat{F}_i, \Delta \hat{F}_i, \hat{j}_0 \}$ and $\{ \hat{M}, \hat{F}_i, \Delta \hat{F}_i, \hat{j}_0 \}$ have the same commutation relations, the action of the dynamical representation of $ISL(2, C)$ on the eigenstates $| f; m, j \rangle$ has the same form as the free dynamics on $| f, d; m_0, j \rangle$, with the eigenvalue of $\hat{M}_0$ replaced by the eigenvalue of $\hat{M}$. It follows that

$$
\hat{U}[\Lambda, Y]| f; m, j \rangle = | f'; m, j \rangle d\mu(f') D_{f'j}^{mj}[\Lambda, Y].
$$

(140)

Since the states $| f; j, m \rangle$ are complete, this defines $\hat{U}[\Lambda, Y]$ on $\mathcal{H}$. Since $m$ is the eigenvalue of a dynamical operator, all of the mass dependent parts of $D_{f'j}^{mj}[\Lambda, Y]$ are interaction dependent.

This construction gives (1) an explicit expressions for the interaction dependent $ISL(2, C)$ Lie algebra, (2) a solution of the 2-body dynamics expressed as a direct integral of irreducible representations of $ISL(2, C)$, (3) and an explicit unitary representation of $ISL(2, C)$ on $\mathcal{H}$.  

37
This construction can be done in any irreducible basis. Consider the same construction in two bases \((f, d)\) and \((g, d)\) where, for simplicity, the degeneracy operators in both bases are assumed to have the same spectrum. In one model the interaction commutes with \(\hat{F}^i\) while in the other the interaction commutes with \(\hat{G}^i\). Because \(\hat{M}\) does not commute with \(\hat{M}_0\), if the relation between the \((f, d)\) and \((g, d)\) bases involves the mass, these two interactions cannot be the same.

Nevertheless, the form of the dynamical equation (139) is identical in both cases. Both will give the same bound state masses and scattering matrix elements. It follows, using Theorem 3, that the dynamical models constructed using the free particle bases

\[
(\hat{F}, \hat{d}, \hat{M}_1) \quad \text{and} \quad (\hat{G}, \hat{d}, \hat{M}_2)
\]

are scattering equivalent and are related by

\[
\hat{A} = \Omega_\pm(\hat{M}_1, \hat{\Phi}_1, \hat{M}_A)\Omega_\pm^\dagger(\hat{M}_2, \hat{\Phi}_2, \hat{M}_A).
\]

The transformation \(\hat{A}\) is not simply a change of basis; it is interaction dependent and changes the nature of the interactions. This illustrates the relation of the basis choice to the structure of the dynamics.

To understand the nature of the interaction dependence of \(\hat{A}\) note that both wave operators in (142) need to be computed in the same basis. This leads to an expression of the form

\[
\langle f | \hat{A} | f' \rangle = \int \langle f | \Omega_\pm(\hat{M}_f, \hat{\Phi}_f, \hat{M}_A) | f'' \rangle d\mu(f'') \langle f'' | g'' \rangle_{\hat{A}} d\mu(g'') \times
\]

\[
\langle g'' | \Omega_\pm^\dagger(\hat{M}_g, \hat{\Phi}_g, \hat{M}_A) | g' \rangle d\mu(g') \langle g' | f' \rangle.
\]

(143)
If the change of basis $f \leftrightarrow g$ involves the mass parametrically, then $\langle f|g \rangle_A$ will involve the physical mass eigenvalues while $\langle g|f \rangle$ involves the non-interacting masses. The interaction dependence is due to having the interacting mass in one of these expressions and the free mass in the inverse expression. In the limit that the interactions are turned off, this becomes the identity.

This completes the construction of the two-body dynamics. The construction provides a relativistic two-body model for any choice of basis and $ISL(2,C)$ Clebsch-Gordan coefficient.

To illustrate the structure of the dynamical equation in a familiar basis consider the case (see Appendix II) that $\hat{F} = \{\vec{P}, \hat{j}_{cz}\}$, corresponding to the linear momentum and $z$-component of the canonical spin, and $\hat{D}^i = \{j_1, m_1, j_2, m_2, \hat{l}, \hat{s}\}$ where $\hat{l}, \hat{s}$ are two-body orbital and spin angular momenta. The matrix elements of $\hat{V} = \hat{M} - \hat{M}_0$ have the form:

$$
\langle p, \mu, l, s; m, j | \hat{V} | p', \mu', l', s'; m', j' \rangle = \delta(p - p')\delta_{\mu\mu'}\delta_{jj'}\langle l, s, m | \hat{V}^j | l', s', m \rangle.
$$

(144)

If $m$ is replaced by the kinematic momentum $q$ defined by

$$
m = \sqrt{q^2 + m_1^2} + \sqrt{q^2 + m_2^2}
$$

(145)

the matrix element (144) has the same structure as the corresponding non-relativistic interaction. The eigenvalue equations (139) becomes:

$$
(m - \sqrt{q^2 + m_1^2} + \sqrt{q^2 + m_2^2})\phi_m^j(l, s, q) = 
\sum_{l' = 0}^{\infty} \sum_{s' = |j-l|}^{j+l} \int_0^\infty q'^2 dq' \langle l, s, q | \hat{V}^j | l', s', q' \rangle \phi_m^j(l', s', q').
$$

(146)
12 The N-Body Problem

The formulation of the N-body problem is by induction. The construction follows [1][2][9]. What is different is that the notion of “form of the dynamics” is replaced by a choice \((f,d)\) of basis for \(ISL(2,C)\) irreducible representation spaces and associated Clebsch-Gordan coefficients.

The construction of the N-body dynamics exploits the scattering equivalence of two representations of \(ISL(2,C)\). One representation satisfies algebraic cluster properties and the other has a kinematic spin, which is useful for the \(ISL(2,C)\) invariant addition of interactions.

The construction begins with the decomposition of the system into interacting subsystems, which are obtained by turning off the interactions between particles in different clusters of a partition \(a\). The tensor product of the subsystem representations define unitary representation of \(ISL(2,C)\) on the N-body Hilbert space. These representations are reducible and have interactions in both the N-body mass and spin operators. As \(a\) runs over all partitions these representation contain all interactions except the N-body interactions. Because the mass and spin operators for different decompositions into subsystems do not all commute, these tensor product representations are not suited to \(ISL(2,C)\) invariant addition of interactions.

In order to facilitate the invariant addition of interactions, scattering equivalences are introduced that transform each of the tensor product representations into scattering equivalent representations of \(ISL(2,C)\) where \(j^2\), \(F_j\) and \(\Delta F_j\) are free of interaction. In these representations all of the interactions are in the mass operators. Linear combinations of the mass operators for different decompositions into subsystems can be used to construct an over-
all N-body mass operator $\hat{M}$ that still commutes with the non-interacting operators $j_0^2$, $\hat{F}_0^j$, and $\Delta \hat{F}_0^j$.

The existence of the required scattering equivalences follows by induction from properties of the two-body solution. This is different than the solution presented in [2] where the kinematic subgroup and the $\vec{p} = 0$ condition played a central role in establishing the required scattering equivalences.

The properties of $\hat{M}$ guarantee that $ISL(2, C)$ generators expressed as functions of $\hat{M}, j_0^2, \hat{F}_0^j$, and $\Delta \hat{F}_0^j$ satisfy the $ISL(2, C)$ commutation relations. The associated unitary representation of $ISL(2, C)$, which is constructed using the same method used in the two-body construction, does not satisfy algebraic cluster properties for $N > 2$. Cluster properties are restored by constructing a suitable scattering equivalence, which introduces additional many-body interactions and introduces a non-trivial interaction dependence in the spin.

The induction begins with the two-body dynamics formulated in the previous section. The dynamical two-body representation, $\hat{U}[\Lambda, Y]$, of $ISL(2, C)$ satisfies:

- It becomes the tensor product of two one-body representations when the interaction is set to zero:

$$\hat{U}_{(12)}[\Lambda, Y] \rightarrow \hat{U}_1[\Lambda, Y] \otimes \hat{U}_2[\Lambda, Y].$$ \hspace{1cm} (147)

- The two-body mass operator commutes with the non-interacting $\hat{F}^j$, $\Delta \hat{F}^j$ and $j^2$: 
\[ [\hat{M}_{(12)}, \hat{F}_0] = [\hat{M}_{(12)}, \Delta \hat{F}_0] = [\hat{M}_{(12)}, \hat{j}_0^2] = 0. \quad (148) \]

These conditions cannot be simultaneously satisfied for systems of more than two particles. They are replaced by the following induction assumption, which reduces to the above condition when \( N = 2 \):

- For each proper subsystem \( s \) of the \( N \)-body system, there is a dynamical representation \( \hat{U}_s[\Lambda, Y] : \mathcal{H}_s \to \mathcal{H}_s \) with short-range interactions satisfying algebraic cluster properties. This means that if the interactions between particles in different clusters of the subsystem \( s \) are set to zero then

\[
\hat{U}_s[\Lambda, Y] \to \otimes_i \hat{U}_{a_i}[\Lambda, Y]. \quad (149)
\]

- For each proper subsystem there is a scattering equivalence \( \hat{C}_s \) satisfying

\[
\hat{C}_s \hat{U}_s[\Lambda, Y] \hat{C}_s^\dagger = \hat{U}_s[\Lambda, Y] \quad (150)
\]

with the property that the mass operator \( \hat{M}_s \) of the \( \hat{U}_s[\Lambda, Y] \) representation commutes with \( \hat{F}_s^i, \hat{j}_s^2, \Delta \hat{F}_s^i \) of the non-interacting subsystem, \( s \).

These conditions are trivially satisfied by the two-body construction of the previous section for \( \hat{C}_s = \hat{I} \) on each single particle Hilbert space.

First we show that if these conditions hold for all proper subsystems then they hold for any non-trivial partitioning of the \( N \)-body system.
The theorem below ensures the scattering equivalence of tensor products of subsystem representations that satisfy (150) to representations with a non-interacting $\hat{j}^2$, $\hat{F}^j$, and $\Delta \hat{F}^j$.

**Theorem 4:** Let $a$ be a partition of the $N$-particle system into $n_a$ disjoint mutually non-interacting subsystems, $a_i$. Assume that each subsystem has a dynamical representation $\hat{U}_{a_i}[\Lambda, Y]$ of $ISL(2, C)$ with an asymptotically complete scattering theory. Assume the each of the representations $\hat{U}_{a_i}[\Lambda, Y]$ is scattering equivalent to a representation that has $\hat{F}^j_{a_i} = \hat{F}^j_0$, $\Delta \hat{F}^j_{a_i} = \Delta \hat{F}^j_0$, $\hat{j}^2_{a_i} = \hat{j}^2_0$. Let

$$\hat{U}_{a}[\Lambda, Y] := \otimes_{i=1}^{n_a} \hat{U}_{a_i}[\Lambda, Y]$$

be the tensor product of subsystem representations of $ISL(2, C)$. Then $\hat{U}_{a}[\Lambda, Y]$ is scattering equivalent to a representation $\hat{U}_{a}[\Lambda, Y]$ that has $\hat{F}^j_a = \hat{F}^j_0$, $\Delta \hat{F}^j_a = \Delta \hat{F}^j_0$, $\hat{j}_a = \hat{j}_0$. Let

This states that if the subsystem mass operators are scattering equivalent to the subsystem mass operator with kinematic $\hat{F}^j_{a_i}$, $\Delta \hat{F}^j_{a_i}$, $\hat{j}_{a_i}$ then the tensor product of the subsystems has a mass operator that is scattering equivalent to a mass operator with kinematic $\hat{F}^j$, $\Delta \hat{F}^j$, and $\hat{j}$.

The induction assumptions (150) and (149) and the application of Theorem 4 imply that for every partition $a$ with at least two non-empty clusters there are representations $\hat{U}_{a}[\Lambda, Y]$, and $\hat{U}_{a}[\Lambda, Y]$, related by a scattering equivalence $\hat{B}_a$. The proof of Theorem 4 as well as the construction of $\hat{B}_a$ is given in Appendix IV.

To establish algebraic cluster properties let $\hat{X}$ be an operator valued function of the interactions. Assume that a coupling constant $\lambda_b$ is put in front
of all interactions involving particles in different clusters of a partition $b$. Let $(\hat{X})_b$ denote the operator obtained from $\hat{X}$ by setting $\lambda_b$ to 0.

Theorem 4 implies the following relation:

$$\bar{U}_a[\Lambda, Y] = \hat{B}_a \hat{U}_a[\Lambda, Y] \hat{B}_a^\dagger. \quad (152)$$

Turning off interactions between particles in different clusters of $b$ in (152) gives, using (149) and (151),

$$(\bar{U}_a[\Lambda, Y])_b = (\hat{B}_a)_b \hat{U}_{a \cap b}[\Lambda, Y] (\hat{B}_a)_b$$

when $b \cap a$ is a refinement of $a$.

Applying Theorem 4 directly to the partition $c = b \cap a$ gives

$$\bar{U}_{a \cap b}[\Lambda, A] = \hat{B}_{a \cap b} \hat{U}_{a \cap b}[\Lambda, Y] \hat{B}_{a \cap b}^\dagger. \quad (154)$$

This gives distinct scattering equivalences $\hat{B}_{a \cap b}$ and $(\hat{B}_a)_b$ relating $\bar{U}_{a \cap b}[\Lambda, Y]$ to different representations that commute with $\hat{F}_0^j$, $\Delta \hat{F}_0^j$, and $\hat{j}_0^2$. An illustration of this ambiguity in the four-body system occurs for $a = (123)(4)$, $b = (12)(34)$ and $c = (12)(3)(4)$.

It is desirable that the scattering equivalence obtained by turning off interactions agree with the scattering equivalence constructed directly by applying Theorem 4 to the tensor products. This can be achieved by recursively replacing the operators $\hat{B}_a$ of Theorem 4 with operators $\hat{A}_a$ that satisfy $(\hat{A}_a)_b = \hat{A}_{a \cap b}$. This replacement involves a redefinition of the $M_a$'s.

For $N - 1$ cluster partitions define

$$\hat{A}_a := \hat{B}_a. \quad (155)$$
Because $N - 1$ cluster interactions only have two-body interactions, both $\hat{A}_a$ and $\hat{B}_a$ become the identity when the interaction is turned off:

$$(\hat{A}_a)_b = \hat{A}_{a\cap b} = \hat{I}$$

(156)

In this case any non-trivial refinement of $a$ gives $N$ free particles.

Next consider a partition $a$ with $k$ clusters. By induction assume that scattering equivalences $\hat{A}_c$ have been defined for all partitions $c$ with more than $k$ clusters and that these operators satisfy $(\hat{A}_c)_d = \hat{A}_{c\cap d}$ for $n_c > k$.

Let $b$ be a partition such that $a \cap b$ has more than $k$ clusters. Note that

$$\hat{A}_{a\cap b}(\hat{B}_a)_b$$

is defined and commutes with $\hat{F}_0^j$, $\Delta \hat{F}_0^j$, and $\hat{j}_0$.

Define

$$\hat{A}_a := \left( \frac{\hat{I} - i\hat{\alpha}_a}{\hat{I} + i\hat{\alpha}_a} \right) \hat{B}_a$$

(158)

where

$$\hat{\alpha}_a := -\sum_{b \neq a} \mu(a \supset b)\hat{\alpha}_{a,b}$$

(159)

and

$$\hat{\alpha}_{a,b} := \frac{i\hat{I} - \hat{A}_b(\hat{B}_a)_b}{\hat{I} + A_b(\hat{B}_a)_b}$$

(160)

Note that $a \cap b = b$ was used in (160). These expressions utilize Cayley transforms to construct unitary functions of scattering equivalences. The resulting unitary operators will be scattering equivalences provided their Cayley transforms are in the algebra of asymptotic constants. This is not entirely trivial, because the algebra $\mathcal{C}$ is uniformly closed, but not strongly closed. $\hat{A}_a$ will be a scattering equivalence if the Cayley transforms $\hat{\alpha}_{a,b}$ are bounded. This will be assumed in all that follows.
The restriction \( b \neq a \) means that the \( b \)'s appearing in the sum are proper refinements of \( a \) and necessarily have more than \( k \) clusters. By induction the \( \hat{A}_b \)'s satisfy \((\hat{A}_b)_c = \hat{A}_{b \cap c}\). It follows for \( c \cap a \neq a \) that

\[
(\hat{\alpha}_{a,b})_c := \frac{i \hat{I} - \hat{A}_{b \cap c}(\hat{B}_a^\dagger)_{b \cap c}}{I + \hat{A}_{b \cap c}(\hat{B}_a^\dagger)_{b \cap c}} = \hat{\alpha}_{a,b \cap c}
\]

which gives

\[
(\hat{\alpha}_a)_c = -\sum_{b \neq a} \mu(a \supset b)\hat{\alpha}_{a,b \cap c} = -\sum_{b \neq a} \mu(a \supset b)\zeta((b \cap c) \supset d)\mu(d \supset e)\hat{\alpha}_{a,e}.
\]

(162)

Using (115) gives

\[
-\sum_{b \neq a} \mu(a \supset b)\zeta(\zeta(\mu(ad) - \delta_{ad} = \zeta(a \supset d) - \delta_{ad}. \text{ Using this in the above sum and observing that } \zeta(c \supset a) = 0, \text{ gives}
\]

\[
\sum_{d,e} \zeta(a \supset d)\zeta(\mu(d \supset e)\hat{\alpha}_{a,e} = \sum_{d,e} \zeta(a \cap c \supset d)\mu(d \supset e)\hat{\alpha}_{a,e} = \hat{\alpha}_{a,a \cap c}.
\]

(164)

It follows that

\[
(\hat{A}_a)_c = \frac{\hat{I} - i\hat{\alpha}_{a,c}}{I + i\hat{\alpha}_{a,c}}(\hat{B}_a)_c =
\]

\[
\hat{A}_{a \cap c}(\hat{B}_a^\dagger)_c(\hat{B}_a)_c = \hat{A}_{a \cap c}.
\]

(165)

This shows that if the result holds for more than \( k \) clusters, it holds for \( k \) clusters.

This process can be continued recursively until \( n_a = 2 \). The result is a set of scattering equivalences, \( \hat{A}_a \) and representations

\[
\hat{U}_a[\Lambda, Y], \bar{U}_a[\Lambda, Y]
\]

(166)

46
with the properties

\[ \hat{U}_a[\Lambda, Y] = \hat{A}_a \hat{U}_a[\Lambda, Y] \hat{A}_a^\dagger \]  
\[ \hat{U}_a[\Lambda, Y] = \otimes_{i=1}^{n_a} \hat{U}_{a_i}[\Lambda, Y] \]  
\[ \hat{A}_a \rightarrow \hat{A}_{a \cap b} \]

and

\[ \hat{F}_a^i = \hat{F}_0^i, \quad \Delta \hat{F}_a^i = \Delta \hat{F}_0^i, \quad \hat{J}_a^2 = \hat{J}_0^2. \]  

The final step is to complete the construction of the dynamics. For each partition \( a \) of the \( N \)-particle system with at least two clusters let \( \hat{M}_a \) be the mass operator for the tensor product representation \( \hat{U}_a[\Lambda, Y] \). Note that

\[ \hat{M}_a = \hat{A}_a \hat{M}_a \hat{A}_a^\dagger \]

is scattering equivalent to \( \hat{M}_a \) and commutes with \( \hat{F}_0^i \), \( \Delta \hat{F}_0^i \), and \( \hat{J}_0^2 \).

Define

\[ \bar{M} := - \sum_{a \neq 1} \mu(1 \supset a) \hat{M}_a + [\hat{M}]_N = - \sum_{a \neq 1} \mu(1 \supset a) \hat{A}_a \hat{M}_a \hat{A}_a^\dagger + [\hat{M}]_N \]

where \( [\hat{M}]_N \) is a possible additional \( N \)-body interaction that commutes \( \hat{F}_0^i \), \( \Delta \hat{F}_0^i \), and \( \hat{J}_0^2 \). By construction \( \bar{M} \) commutes with \( \hat{F}_0^i \), \( \Delta \hat{F}_0^i \), and \( \hat{J}_0^2 \). This expansion is equivalent to the cluster expansion of \( \bar{M} \). By the induction assumption, turning off the interactions between particles in different clusters of partition \( b \) gives

\[ (\bar{M})_b := - \sum_{a \neq 1} \mu(1 \supset a)(\bar{M}_a)_b = - \sum_{a \neq 1} \mu(1 \supset a) \hat{A}_a \hat{M}_a \hat{A}_a^\dagger_b = \]

\[ - \sum_{a \neq 1} \mu(1 \supset a) \zeta((a \cap b) \supset d) \mu(d \supset e) \hat{A}_e \hat{M}_e \hat{A}_e^\dagger = \]
\[-\sum_{a \neq 1} \mu(1 \supset a) \zeta(a \supset d) \zeta(b \supset d) \mu(d \supset e) \hat{A}_e \hat{M}_e \hat{A}_e^\dagger. \quad (173)\]

The sum gives \((1 - \delta_{1d}) \hat{I}\). Inserting this into (173) gives

\[\hat{M}_b = \hat{A}_b \hat{M}_b \hat{A}_b^\dagger \quad (174)\]

or

\[\hat{M}_b = \hat{M}_b. \quad (175)\]

This is not the mass operator \(\hat{M}_b\) corresponding to the tensor product of the subsystems associated with the clusters of \(b\). To correct this define the scattering equivalence

\[\hat{A} := \frac{\hat{I} + i \hat{\alpha}}{\hat{I} - i \hat{\alpha}} \quad (176)\]

with

\[\hat{\alpha} = -\sum_{a \neq 1} \mu(1 \supset a) \hat{\alpha}_a \quad (177)\]

\[\hat{\alpha}_a := \frac{i \hat{I} - \hat{A}_a}{\hat{I} + \hat{A}_a}. \quad (178)\]

Using the same algebra used to show that \((\hat{M})_b = \hat{M}_b\) it follows that

\[\hat{A}_b = \hat{A}_b. \quad (179)\]

Since \(\hat{A}\) is a scattering equivalence define

\[\hat{M} := \hat{A}^\dagger \hat{M} \hat{A}. \quad (180)\]

Since \(\hat{M}\) commutes with the kinematic operators \(\hat{F}_0^j\), \(\Delta \hat{F}_0^j\), and \(\hat{j}_0^2\), simultaneous eigenstates of \(\hat{M}\), \(\hat{F}_0^j\), and \(\hat{j}_0^2\) define a complete set of states that transform irreducibly. This can be used to construct a representation
\( \hat{U}[\Lambda, Y] \) of the \( ISL(2, C) \). The scattering equivalence \( \hat{A} \) defines a scattering equivalent representation

\[
\hat{U}[\Lambda, Y] := \hat{A} \hat{U}[\Lambda, Y] \hat{A} \tag{181}
\]

with the property that

\[
(\hat{U}[\Lambda, Y])_b := \hat{A}_b \hat{U}_b[\Lambda, Y] \hat{A}_b = \hat{U}_b[\Lambda, Y] = \bigotimes_{i=1}^{n_b} \hat{U}_b[i, Y]. \tag{182}
\]

The generators have the form

\[
\hat{P}^{\mu} = \hat{A}^{\dag}_i P^{\mu}(\hat{M}, \hat{j}_0^2, \hat{F}^i_0, \Delta \hat{F}^i_0) \hat{A}_i \tag{183}
\]

and

\[
\hat{J}^{\mu\nu} = \hat{A}^{\dag}_i J^{\mu\nu}(\hat{M}, \hat{j}_0^2, \hat{F}^i_0, \Delta \hat{F}^i_0) \hat{A}_i. \tag{184}
\]

This completes the proof of the induction.

The operator \( \hat{U}[\Lambda, Y] \) defined in (181) is the desired \( N \)-body representation of \( ISL(2, C) \) that is consistent with the dynamics and satisfies algebraic cluster separability. The effect of the transformation \( \hat{A} \) is to cancel the \( \hat{A}_a \)'s from the subsystems. It generates new many-body interactions that are necessary for the algebraic cluster properties of \( \hat{U}[\Lambda, Y] \).

To summarize this construction; tensor products of subsystem dynamics are transformed to scattering equivalent representations where the operators \( \hat{F}^i, \Delta \hat{F}^i \), \( \hat{j} \) are free of interactions. The transformed mass operators are combined to construct a mass operator for a unitary representation of \( ISL(2, C) \) with kinematic \( \hat{F}^i, \Delta \hat{F}^i \), and \( \hat{j} \). This representation is transformed to a scattering equivalent representation satisfying cluster properties.
The construction, while complex, leads to a simple structure. All of the $ISL(2, C)$ generators can be expressed as sums of one, two, three, · · ·, $N$-body interactions. For any $ISL(2, C)$ generator, the $k$-body interaction in the $k$-body problem is identical to the $k$-body interaction in the many-body problem. At each stage of the construction the subsystem interactions remain unchanged. What is new is that cluster properties generate new many-body interactions. These do not change when they are imbedded in systems with more than $N$ particles. The spin, which is a non-linear function of these generators, is an interaction dependent quantity given by

$$\hat{j}^2 = \hat{A}^\dagger \hat{j}_0^2 \hat{A}. \quad (185)$$

The scattering equivalence $\hat{A}$ is an interaction dependent operator that becomes the identity when the interactions are switched off. While there is freedom to include many-body interactions, there is a class of many-body interactions that cannot be removed without violating cluster properties.

## 13 Cluster Equivalence

The dynamical unitary representation of $ISL(2, C)$ constructed in the previous section satisfies algebraic cluster properties. With suitable short ranged interactions it will satisfy cluster properties (SN) and the spectral condition. The choice of basis $(f, d)$ was an important element of this construction. In this section, this representation is shown to be scattering equivalent to a representation based on a different choice of basis, $(g, h)$. This representation also satisfies algebraic cluster properties.
This illustrates the existence of a subgroup of the group of scattering equivalences that relates the constructions based on different irreducible representation basis choices and preserves algebraic cluster properties. This subgroup will be called the group of cluster equivalences.

It follows that the choice of irreducible basis used in the construction has no fundamental physical significance. This generalizes the equivalence of choices of kinematic subgroups in two ways. First, it extends the result to the general setting of this paper where the form of dynamics is replaced by the basis choice \((f, d)\). Second, it shows that this equivalence respects cluster properties.

To illustrate the nature of the required scattering equivalence first let \(\hat{U}^f[\Lambda, Y]\) denote the representation constructed in the previous section using the \((f, d)\) basis. Turning off interactions between particles in different clusters of the partition \(a\) gives

\[
\hat{U}^f[\Lambda, Y] \rightarrow \hat{U}^f_a[\Lambda, Y] = \hat{A}_a^f \hat{U}^f[\Lambda, Y] \hat{A}_a^f
\]

where \(\hat{A}_a^f\) are the scattering equivalences constructed in the previous section. The superscript \(f\) indicates that the \((f, d)\) basis was used in the construction.

Algebraic cluster properties give the relations

\[
\hat{U}^f[\Lambda, Y] \rightarrow \hat{U}^f_a[\Lambda, Y] = \otimes_{i=1}^{n_a} \hat{U}^f_{a_i}[\Lambda, Y] =
\otimes_{i=1}^{n_a} \left( \hat{A}_{a_i}^f \hat{U}^f_{a_i}[\Lambda, Y] \hat{A}_{a_i}^f \right) = (\otimes_{i=1}^{n_a} \hat{A}_{a_i}^f)(\otimes_{i=1}^{n_a} \hat{U}^f_{a_i}[\Lambda, Y])(\otimes_{i=1}^{n_a} \hat{A}_{a_i}^f)
\]

where the \(\hat{A}_{a_i}^f\) are the \(\hat{A}_a^f\) operators for the subsystem consisting of the particles in the \(i-th\) cluster of \(a\).

It is useful to introduce the operators

\[
\tilde{U}^f_a[\Lambda, Y] := \otimes_{i=1}^{n_a} \hat{U}^f_{a_i}[\Lambda, Y]
\]
which are related to $\hat{U}^f_a[\Lambda, Y]$ by the scattering equivalence

$$
\hat{B}^f_a := \otimes_{i=1}^{n_a} \hat{A}^f_{a_i}.
$$

(189)

The construction of the previous section defined $\hat{U}^f_a[\Lambda, Y] := \tilde{U}^f_a[\Lambda, Y]$ for $n_a = N - 1$. All of the $\hat{U}^f_a[\Lambda, Y]$'s were recursively constructed from the $n_a = N - 1$ cluster representations.

Any of the representations $\bar{U}^f_a[\Lambda, Y]$ are scattering equivalent to a $\bar{U}^g_a[\Lambda, Y]$ representation. This scattering equivalence is realized by making the following replacements in the kernel of the barred mass operators:

$$
\langle f, d(m_0, j_0) | \bar{M}^f | f', d'(m'_0, j'_0) \rangle = \delta[f; f'] \delta_{j_0, j'_0} \langle m_0, d | \bar{M}^{j_0} | m'_0, d' \rangle
$$

(190)

by

$$
\langle g, h(m_0, j_0) | \bar{M}^g | g', h'(m'_0, j'_0) \rangle = \delta[g; g'] \delta_{j_0, j'_0} \langle m_0, h | \bar{M}^{j_0} | m'_0, h' \rangle
$$

(191)

where the reduced kernel $\langle m_0, h | \bar{M}^{j_0} | m'_0, h' \rangle$ is defined in terms of the reduced kernel $\langle m_0, d | \bar{M}^{j_0} | m'_0, d' \rangle$ by a variable change $d \rightarrow h$ implemented by kinematic $ISL(2, C)$-Racah coefficients. This means abstract reduced mass operators are identical. The operators $\bar{M}^g$ and $\bar{M}^f$ differ because of the delta functions in $f$ or $g$; but both operators manifestly give the same $S$ matrix elements and bound-state observables. The operators $\bar{M}^f$ and $\bar{M}^g$ define scattering equivalent representations of $ISL(2, C)$ with the non-interacting $\hat{F}^i$, $\Delta \hat{F}^i$ or $\hat{G}^i$, $\Delta \hat{G}^i$ respectively. The scattering equivalence is denoted by

$$
\hat{C}^{gf}: \hat{C}^{gf} \bar{U}^f_a[\Lambda, Y] \hat{C}^{gf \dagger} = \bar{U}^g_a[\Lambda, Y]
$$

(192)

Since this equivalence is valid for systems or subsystems, for each partition
the following representations are scattering equivalent:

\[ \hat{U}_a^f[\Lambda, Y], \bar{U}_a^f[\Lambda, Y], \check{U}_a^f[\Lambda, Y], \bar{U}_a^g[\Lambda, Y], \check{U}_a^g[\Lambda, Y]. \] (193)

These representations have the property that \( \hat{U}_a^f[\Lambda, Y] = \check{U}_a^f[\Lambda, Y] \) for \( N-1 \) cluster partitions and \( \hat{U}_a^f[\Lambda, Y] \) is scattering equivalent to \( \check{U}_a^f[\Lambda, Y] \) for the 1-cluster partition.

The goal is to find a \( \hat{U}_a^g[\Lambda, a] \) that is scattering equivalent to \( \bar{U}_a^g[\Lambda, Y] \) and \( \check{U}_a^f[\Lambda, Y] \) and also satisfies algebraic cluster properties, with \( \hat{U}_a^g[\Lambda, Y] = \check{U}_a^g[\Lambda, Y] \) for \( n_a = N - 1 \).

The first step is to define

\[ \check{U}_a^g[\Lambda, Y] = \bar{U}_a^g[\Lambda, Y] \] (194)

for \( n_a = N - 1 \). Following the construction of the previous section, this gives scattering equivalences \( \hat{A}_a^g \) relating \( \check{U}_a^g[\Lambda, Y] \) to \( \bar{U}_a^g[\Lambda, Y] \) for \( n_a = N - 1 \).

Next, assume by induction that \( \hat{U}_a^g[\Lambda, Y] \) has been defined for partitions with more than \( K \) clusters satisfying algebraic cluster properties and is scattering equivalent to \( \bar{U}_a^g[\Lambda, Y] \). The \( \bar{U}_a^g[\Lambda, Y] \) for \( K \)-cluster partitions is defined by (192). Its mass operator, \( \bar{M}_a^g \), is related to \( \check{M}_a^g \) by replacing delta functions in \( f \) by delta functions in \( g \). Since \( (\bar{M}_a^g)_b = \check{M}_a^g \cap_c b \) it follows that \( (\bar{M}_a^g)_b = \check{M}_a^g \cap_c b \) because the kernel of the two operators only differ by delta functions in the overall kinematic operators \( f \) or \( g \).

This means that \( \bar{M}_a^g \) differs from the cluster expansion

\[ \bar{M}_a^{g0} = -\sum_{b \neq a} \mu(a \supset b) \check{M}_b^g \] (195)

by at most an \( a \)-connected interaction term, \([\check{M}]_a^g\). In order to construct the
desired representation it is enough to define

\[
\hat{U}_g^a[\Lambda, Y] := \hat{A}_a^g \hat{U}_a^g[\Lambda, Y] \hat{A}_a^g
\]  
(196)

where

\[
\hat{A}_a^g = \frac{I + i\hat{\alpha}_a^g}{I - i\hat{\alpha}_a^g} \tag{197}
\]

\[
\hat{\alpha}_a^g := -\sum_{b\neq a} \mu(a, b)\hat{\alpha}_b^g \tag{198}
\]

\[
\hat{\alpha}_b^g = \frac{I + \hat{A}_b^g}{I - \hat{A}_b^g} \tag{199}
\]

Following the algebra used in (173) \( \hat{\alpha}_a^g \) has the property that

\[(\hat{\alpha}_a^g)_b = \hat{\alpha}_b^g \quad b \subset a \]  
(200)

and

\[(\hat{U}_g^a)_b[\Lambda, Y] := \hat{A}_{ar}^g \hat{U}_{ar}^g[\Lambda, Y] \hat{A}_{ar}^g = \hat{U}_{ar}^g[\Lambda, Y] \]  
(201)

This differs from the result of a direct construction in the \((g, h)\) basis because of the difference \([\bar{M}]^g_a\) between \(\bar{M}^g_a\) and \(\bar{M}^{g0}_a\). This introduces additional many-body interactions that are needed maintain the scattering equivalence at each stage of the recursion. Note that in this construction the factor \(\mu(a \supseteq b)\) ensures that only the \(b\) satisfying \(b \subset a\) appear in the sum. These partitions have more than \(K\)-clusters. This construction can be continued until \(K = 1\), where

\[
\hat{U}^g[\Lambda, Y] = \hat{U}^g_1[\Lambda, Y] = \hat{A}^g \hat{U}^g_1 \hat{A}^g \]  
(202)

is the desired representation based on the \((g, h)\) representation. The relevant scattering equivalence is

\[
\hat{U}^g[\Lambda, Y] = \hat{A}^g \hat{C}^gj \hat{A}^f \hat{U}^f[\Lambda, Y] \hat{A}^f \hat{C}^gfj \hat{A}^g. \quad \] (203)
It follows that $\hat{A}^g \hat{C}^g \hat{A}^f$ is the desired scattering equivalence connecting the construction of $\hat{U}[\Lambda, Y]$ using the $(f, d)$ representation to a dynamics satisfying cluster properties based on the $(g, h)$ representation.

It is important to emphasize that the $\hat{A}^g$ constructed in this manner are not identical to the corresponding operators that would have been constructed if one began with the $(g, h)$ basis. This is due to the presence of additional many-body interactions that are determined by the difference between the operators $\bar{M}_a^g$ and $\bar{M}_a^g$ for each $a$. These differences account for the dynamical differences that occur when the many-body dynamics is formulated with different basis choices, or using different forms of dynamics.

The cluster equivalences transform $ISL(2, C)$ generators in one representation to physically equivalent generators in another representation. In each representation the interactions are distributed differently among the generators. Specific representation have computational advantages.

14 Summary and Conclusion

This paper provides a general construction of a unitary representation $\hat{U}[\Lambda, Y]$ of $ISL(2, C)$ for a system of N-interacting particles based on the representation theory of $ISL(2, C)$. For suitable interactions the representation satisfies cluster properties and the spectral condition. The representation defines a non-trivial relativistic quantum theory of interacting particles. Unitary operators that preserve the S-matrix and cluster properties, called cluster equivalences, relate the different constructions.

Relativistic quantum theory of $N$-particles can be applied to model sys-
tems of strongly interacting particles. This framework has many features of non-relativistic quantum mechanics and local relativistic quantum field theory. Like non-relativistic quantum mechanics, it is a mathematically well behaved theory where exact numerical calculations are possible. Like quantum field theory, it is a quantum theory with an exact $ISL(2, C)$ symmetry that satisfies cluster properties and the spectral condition.

The generality of the construction suggests that any quantum theory dominated by a finite number of particle degrees of freedom which is consistent with Poincaré invariance, cluster properties, and the spectral condition will be related to a theory of the type discussed in this paper by a cluster equivalence.

The cluster equivalences introduced in section 13 relate physically equivalent representations of the same model. Cluster equivalent models have the same bound state observables and S-matrix elements. In each representation free particles are represented as tensor products of irreducible representations. The unitary representation of $ISL(2, C)$ that defines the dynamics clusters into tensor products of subsystems representations with the same properties. Cluster equivalence is a stronger condition than unitary equivalence or scattering equivalence. Scattering equivalences were shown to be unitary elements of the $C^*$ algebra of asymptotic constants. Cluster equivalences were shown to be a subgroup of the scattering equivalences.

The practical need to understand the relationship between different formulations of relativistic quantum theory suggests that it would be useful to have an abstract definition of a relativistic quantum theory of particles. The situation is different than the quantum field theory case, were there are sev-
eral sets of axioms [4] that are designed to define a suitable local field theory, with an absence of examples of realistic theories consistent with these axioms. In relativistic quantum theory there are many applications that claim to be relativistic quantum theories, with no universally accepted criteria of what it means to be a relativistic quantum theory of particles. The absence of an acceptable definition of what constitutes a relativistic quantum theory of particles makes comparison difficult. The construction of this paper, which focuses on mathematical formulation of observable physical properties, and how they can be realized in models, suggest minimal elements that need to be included in a set of axioms:

A1 : The Hilbert space $\mathcal{H}$ is the tensor product of irreducible representation spaces of $ISL(2,\mathbb{C})$ associated with the mass and spins of the constituent particles.

A2 : There is a unitary representation $\hat{U}[\Lambda, Y]$ of $ISL(2,\mathbb{C})$ on $\mathcal{H}$ with a positive mass and energy spectrum.

A3 : The Hilbert space can be factored into a tensor product of subsystem spaces, with each one supporting a subsystem unitary representation $\hat{U}_i[\Lambda, Y]$ of $ISL(2,\mathbb{C})$. For each partition $a$ into subsystems $a_i$ the operator $\hat{U}[\Lambda, Y]$ satisfies cluster property (88).

A4 The dynamics $\hat{U}[\Lambda, Y]$ has an asymptotically complete, $ISL(2,\mathbb{C})$ invariant $S$-matrix.

These requirements can be used to formulate a precise relationship between different formulations of relativistic quantum theory when they are applied to systems with finite energy and number of degrees of freedom.
The construction in section 12 points to some of the general features of relativistic quantum theory of particles. In the physical representations of $ISL(2, C)$ the scattering equivalence $\hat{A}$, which is an interaction dependent operator, normally generates interaction dependent terms in all of the operators using the relations:

$$\hat{F}^i = \hat{A}^{\dagger} \hat{F}_0^i \hat{A}$$  \hspace{1cm} (204)

$$\Delta \hat{F}^i = \hat{A}^{\dagger} \Delta \hat{F}_0^i \hat{A}$$  \hspace{1cm} (205)

$$j^2 = \hat{A}^{\dagger} j_0^2 \hat{A}$$  \hspace{1cm} (206)

$$P^\mu = P^\mu(M, j^2, \hat{F}^i, \Delta \hat{F}^i) = \hat{A}^{\dagger} P^\mu(M, j_0^2, \hat{F}_0^i, \Delta \hat{F}_0^i) \hat{A}$$  \hspace{1cm} (207)

$$J^{\mu\nu} = J^{\mu\nu}(M, j^2, \hat{F}^i, \Delta \hat{F}^i) = \hat{A}^{\dagger} J^{\mu\nu}(M, j_0^2, \hat{F}_0^i, \Delta \hat{F}_0^i) \hat{A}$$  \hspace{1cm} (208)

While the construction begins with representations having kinematic $j^2$, $\hat{F}^i$, and $\Delta \hat{F}^i$, all of these operators acquire an interaction dependence in the physical representation.

Tensor and spinor operator densities also play an important role in relativistic quantum mechanics. For example, the hadronic electroweak current operators provide the coupling of the hadronic dynamics to electroweak probes. In one-boson exchange approximations these current operators must transform as 4-vector densities with respect to $ISL(2, C)$

$$\hat{U}[\Lambda, Y]P^\mu[X]\hat{U}^{\dagger}[\Lambda, Y] = P^\nu[X\Lambda \Lambda^{\dagger} + Y]\Lambda_{\nu}^\mu.$$  \hspace{1cm} (209)
Because $\hat{U}[\Lambda,Y]$ is an interaction dependent operator, the covariance condition \((209)\) requires the existence of many-body contributions to the current. This is understood by considering covariance condition

\[
\langle f;m,j|\hat{I}^\mu[X]|f';m',j'\rangle = \\
\int d\mu(f'')d\mu(f''') \langle f''';m,j|\hat{I}^\nu[\Lambda X \Lambda^\dagger + Y]|f''';m',j'\rangle \times \\
\mathcal{D}^{m,j}_{\mu,f}[\Lambda,Y]\mathcal{D}^{m',j'}_{f'\nu,Y}[\Lambda,Y]\Lambda_{\nu}.
\]

In this expression the $m$ and $m'$ in the $\mathcal{D}$ functions are physical mass eigenvalues. This expression fixes a general matrix elements in terms of a set of independent current matrix elements and interaction ($m$) dependent coefficients. This is essentially the Wigner-Eckart theorem for $ISL(2,C)$. In this interpretation the interaction dependence arises because the Clebsch-Gordan coefficients depend on the physical mass eigenvalues. This means that the operators $\hat{I}^\mu(X)$ necessarily have interaction dependent terms that depend on the specific representation.

The result is that the representation of tensor and spinor densities is related to the representation of the dynamics. Changing the representation of the dynamics by a cluster equivalence changes the representation of the interaction dependent parts of the tensor and spinor densities. This has important implications for modeling electromagnetic probes of hadronic systems.

Dirac’s forms of dynamics are obtained for special basis choices. Specifically, if the $ISL(2,C)$ Wigner $\mathcal{D}$ functions, $\mathcal{D}^{m,j}_{\mu,f}[\Lambda,Y]$, do not depend explicitly on $m$ for a subgroup $\mathcal{G}$ of $ISL(2,C)$, there are no interactions in the generators of the subgroup. This depends on the choice of commuting operators $\hat{F}^i$ that are used to label vectors in $ISL(2,C)$ irreducible subspaces.
Cluster equivalences can be used to relate a general model to an equivalent models in any of Dirac’s forms of dynamics.

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15 Appendix I

Examples of positive mass positive energy irreducible representations of $ISL(2, C)$ are constructed. The construction presented below is not as general as the abstract construction given in section 5, but it is general enough to include all of the representations that are commonly used in the literature.

Let $f^i(\vec{p}, m), \quad i = 1, 2, 3$ be three independent real valued functions of the three momentum and the mass. Since the $\hat{M}$ and $\hat{\vec{P}}$ commute, these three functions become commuting self-adjoint operators when $m$ and $\vec{p}$ are replaced by operators. Independence means that these functions can be uniquely inverted to express $\vec{p} = \hat{\vec{P}}(f, m)$ where $f$ denotes the three functions $f^i$. By the implicit function theorem this will be true provided the Jacobian matrix

$$\frac{\partial f^i}{\partial p^j}$$

is invertible for any $\vec{p}$ and any $m$ in the spectrum of $\hat{M}$.

Define the operators

$$\hat{F}^i = f^i(\hat{\vec{P}}, \hat{M})$$

for $i = 1$ to 3. Let $L(p)$ be an arbitrary but fixed $SL(2, C)$ valued function
of \( p = (\sqrt{m^2 + \vec{p}^2}, \vec{p}) \) with properties

\[
L(p)L^\dagger(p) = \frac{1}{m} \sigma_\mu p^\mu \tag{213}
\]

\[
L(p_0)L^\dagger(p_0) = \sigma_0 \quad p_0 := (m, 0, 0, 0). \tag{214}
\]

These equations mean that \( L(p) \) is an \( SL(2,C) \) representation of a Lorentz boost. In general it can differ from the canonical (rotationless boost) by a \( p \)-dependent rotation, \( R(p) \in SU(2) \):

\[
L(p) = L_c(p)R(p) \quad R(p_0) = I. \tag{215}
\]

Given the function \( L(p) \) it is possible to define the \( SL(2,C) \) valued matrix of operators \( L(\vec{P}) \) which is obtained by replacing \( p \) by the commuting operators \( (\hat{\vec{P}}, \hat{M}) \).

For a given \( L(p) \) define the \( l \)-spin by

\[
\hat{\vec{j}}_l := \frac{1}{2M} \text{Tr} \left[ \vec{\sigma}L(\vec{P})\hat{W}^\mu \sigma_\mu L^\dagger(\vec{P}) \right] \tag{216}
\]

where \( \hat{W}^\mu \) is the Pauli Lubanski vector. Since \( \hat{W}^\mu \) commutes with \( \hat{P}^\nu \), all components of \( \hat{\vec{j}}_l \) commute with \( \hat{F}^1, \hat{F}^2, \hat{F}^3 \). In addition, for any choice of \( L(p) \) the components of \( \hat{\vec{j}}_1 \) satisfy \( SU(2) \) commutation relations with \( \hat{\vec{j}}_1^2 = \hat{\vec{j}} = \hat{W}^2/M^2 \). Let \( \hat{\vec{F}}^4 = \hat{\vec{J}}_l \). The operators \( \hat{F}^1, \cdots \hat{F}^4, \hat{M}, \hat{J}^2 \) define a complete set of commuting self-adjoint operators.

Let \( f^1_0 = f^1(p_0), f^2_0 = f^2(p_0), f^3_0 = f^3(p_0) \). By construction \( f^1_0, f^2_0, f^3_0 \) is invariant under rotations, although \( f \) does not transform like an \( SO(3) \) vector. Let \( \bar{f} \) denote the eigenvalues of \( \hat{F}^1, \hat{F}^2 \) and \( \hat{F}^3 \) and \( \mu \) denote the eigenvalue of \( \hat{F}^4 \). Define \( f_\Lambda := f(\vec{p}_\Lambda, m) \), where \( p_\nu^\mu = \Lambda_\nu^\mu p^\mu \). For fixed \( \Lambda \), \( f_\Lambda \) is a function of \( f \) and \( m \).
Let $|f_0, \mu; j, m\rangle$ denote a rest eigenstate of $\hat{F}_i, \hat{M}, \hat{j}^2$ and let $R$ be a $SU(2)$ rotation. Define rotations and translations on the rest states by:

$$\hat{U}[R, 0]|f_0, \mu; j, m\rangle := \sum_{\nu=-j}^{j} |f_0, \nu; j, m\rangle D^j_{\nu\mu}(R)$$

$$\hat{U}[I, Y]|f_0, \mu; j, m\rangle := e^{-imy_0}|f_0, \mu; j, m\rangle.$$  \hfill (217)

Define states of arbitrary $\hat{F}$ by

$$|f, \mu; j, m\rangle := \hat{U}[L(f), 0]|f_0, \mu; j, m\rangle \sqrt{\left|\frac{\partial f_0}{\partial f}\right|}.$$  \hfill (219)

The expressions for $\hat{U}[R, 0]$ and $\hat{U}[I, Y]$ are manifestly unitary. The factor $\sqrt{\left|\frac{\partial f_0}{\partial f}\right|}$ assures that $\hat{U}[L(f), 0]$ unitarity for states with a delta-function normalization. These elementary relations determine a unitary representation $\hat{U}[\Lambda, Y]$ on any state by using the decomposition

$$\hat{U}[\Lambda, Y] = \hat{U}[I, Y]\hat{U}[L(f_\Lambda), 0]\hat{U}[R_{wl}(\Lambda, f), 0]\hat{U}[L^{-1}(f), 0]$$

where

$$R_{wl}(\Lambda, f) := L^{-1}(f_\Lambda)\Lambda L(f)$$

is the $l$-spin Wigner rotation and $L(f)$ is obtained from $L(p)$ by replacing $p$ by $p(f, m)$.

The irreducible representation in this basis follows as a consequence of the above relations:

$$\hat{U}[\Lambda, Y]|f, \nu; j, m\rangle = \sum_{\nu=-j}^{j} |f_\Lambda, \nu; j, m\rangle e^{ip(f_\Lambda, m)\cdot y} \left|\frac{\partial f_\Lambda}{\partial f}\right|^{1/2} D^j_{\nu\mu}[R_{wl}(\Lambda, f)].$$

Taking matrix elements give the $ISL(2, C)$ $D$-function

$$D^{mj}_{f', f}[\Lambda, Y] = e^{ip(f', m)\cdot y} D^j_{\mu'\mu}[R_{wl}(\Lambda, f)] \left|\frac{\partial f_\Lambda}{\partial f}\right|^{1/2} \delta^3(f' - f_\Lambda).$$
The infinitesimal generators of \( ISL(2,C) \) in this representation can be computed using (125-127). The results are:

\[
\hat{P}^\mu = p^\mu(\vec{f}, m) \tag{224}
\]

\[
\hat{j}^j = ie^{jkl} \frac{\partial f^m}{\partial p^k} \frac{\partial}{\partial f^m} \hat{p}^l + (\hat{c}_{1}^{jk}(p) + i\epsilon_{jlm}\hat{c}_{2}^{lk}(p)\hat{p}^m)\hat{j}^k \tag{225}
\]

\[
\hat{K}^j = -\frac{1}{2} \frac{\partial f^m}{\partial \hat{p}^k} [\Delta f^m, \hat{H}]_+ + i(\hat{c}_{1}^{jk}(p) - H\hat{c}_{2}^{jk}(p))\hat{j}^k \tag{226}
\]

where

\[
\hat{c}_{1}^{jk}(p) = \frac{1}{2} \text{Tr}(L^{-1}(\hat{p})\sigma_j L(\hat{p})\sigma_k) \tag{227}
\]

\[
\hat{c}_{2}^{jk}(p) = \text{Tr}(L^{-1}(\hat{p})\frac{\partial}{\partial \hat{p}^j} L(\hat{p})\sigma_k). \tag{228}
\]

These equations can be inverted to obtain explicit expressions (133) for \( \Delta \hat{f}^k \) in terms of the generators

\[
\Delta \hat{f}^k = -\frac{i}{2H} \frac{\partial \hat{H}}{\partial \hat{f}^k} - \frac{1}{H} \left[ \frac{\partial p^j}{\partial f^k} (\hat{K}^j - i(\hat{c}_{1}^{jm}(p) - H\hat{c}_{2}^{jm}(p))\hat{j}^m) \right] \tag{229}
\]

for \( k = 1, 2 \) or 3. This expression reduces [23] to the Newton-Wigner position operator when \( f^i = p^i \) and the \( l \)-spin is the canonical spin. The \( l \)-spin is given as a function of the infinitesimal generators by (216). The partial derivatives in this expression are computed with functions which are replaced by the appropriate operators after the differentiation is performed.

The \( \Delta f^4 \) for the spins are the raising and lowering operators

\[
\hat{j}_\pm := \hat{j}_x \pm i\hat{j}_y. \tag{230}
\]

This shows explicitly the equivalence between

\[
\{\hat{H}, \vec{P}, \vec{J}, \vec{K}\} \quad \text{and} \quad \{\hat{M}, \hat{j}^2, \vec{F}, \Delta \vec{F}\}. \tag{231}
\]
The basis choices illustrated above, while restrictive, include all of the basis choices that lead to Dirac’s forms of dynamics. The general construction yields a Dirac instant form of dynamics if $f^i$ are taken as the three components of the linear momentum and $L_l(p)$ is a canonical (rotationless) boost. Dirac’s point-form dynamics is obtained if $f^i$ is taken as the three components of the four velocity and $L_l(p)$ is the canonical boost. A front form dynamics is obtained if $f^i$ is taken as the three generators of translations tangent to a light front and $L_l$ is taken as corresponding the light front boost. Infinitely many other choices of $f^i$ and $L_l(p)$ are possible.

16 Appendix II

The Clebsch-Gordan coefficients for the representations in Appendix 1 can be computed from the tensor product representation using the same methods that were used to construct the single irreducible representations. The first step is to decompose the tensor product representation of the “rest state” into irreducible representation of $SU(2)$. This requires generalized Melosh rotations to ensure that all of the spins undergo the same rotations. The irreducible representation are then boosted with the appropriate $l$-boost. This generally leads to Wigner rotations. The general result is derived in [29]. The resulting Clebsch-Gordan coefficients for this basis are:

$$
\langle \tilde{f}, \mu; m, j, l, s \rangle = \\
\delta(f - \tilde{f}(f_1, f_2))\delta(m - m(f_1, f_2, m_1, m_2)) \left| \frac{\partial f(k)}{\partial f_1, f_2} \right|^{1/2} \frac{1}{k} \frac{\partial k}{\partial m} \times \\
D_{\mu_1, \mu_1'}^{j_1} [R_{wl}(p, k_1) R_{ml}(k_1)] D_{\mu_2, \mu_2'}^{j_2} [R_{wl}(p, k_2) R_{ml}(k_2)] \times \\
64
$$
where \( L_c(p) \) is the canonical boost and \( L_l(p) \) is a \( l \)-boost,

\[
k_i = \frac{1}{2} \text{Tr}(L_l^{-1}(p)p_i^\mu \cdot \sigma_\mu (L_l^{-1}(p))^\dagger)
\]

and

\[
R_{wl}(p, k_i) := L_l^{-1}(p_i)L_l(p)L_l(k_i) \tag{233}
\]

\[
R_{mlc}(k_i) := L_l^{-1}(k_i)L_c(k_i). \tag{234}
\]

These are the Wigner and Melosh rotations associated with the \( l \)-boost.

The Racah coefficient for this choice of basis can be computed in terms of four Clebsch-Gordan coefficients. It is simplest to compute the invariant part of this coefficient by choosing \( p = (m, 0, 0, 0) \) and integrating the result over \( SU(2) \). The Racah coefficients for the couplings ((12)(3)) \( \rightarrow \) ((23)(1)) become:

\[
\langle \vec{f}', \mu'; m', j', (12, 3) | \vec{f}, \mu; m, j, (23, 1) \rangle = \delta_{j'j} \delta_{\mu'\mu} \delta(\vec{f}' - \vec{f}) \delta(m - m') \frac{1}{2j + 1} \sum_{\mu_f} \times
\]

\[
\left[ \frac{8\pi^2 m_1 m_2 m_3 \omega_2 (q_3 + q_1)}{k_1' k_2' q_3' q_1 \omega_1 (k_1') \omega_2 (k_1') \omega_2 (k_2) \omega_3 (k_2) \omega_1 (q_3) \omega_2 (q_1)} \right] \times
\]

\[
\left[ \frac{\partial(f_{12}', f_3')}{\partial(f_{12}', f_3')} \left| \frac{\partial(f_{23}, k_2)}{\partial(f_{23}, k_2)} \right| \frac{\partial(f_{23}, f_1)}{\partial(f_{23}, f_1)} \right]^{1/2} \times
\]

\[
\left. \langle j, \mu_f | L', \mu_L', S', \mu_S' \rangle \langle S', \mu_S'| j_1', \mu_1', j_2', \mu_2' \rangle Y_{\mu_1'}^{L'}(k_1') \times
\right.
\]

\[
D_{\mu_1' \mu_2' \mu_1\mu_2}^{j_1} [R_{mlc}(-q_3)][Y_{\mu_2'}^{L'}(q_3)] \langle j_1', \mu_1 | L' \rangle \langle \mu_1 | \mu_1 \rangle \langle s' \rangle \langle s' | j_1', \mu_1' | j_1' \rangle \times
\]

\[
D_{\mu_1' \mu_1}^{j_2} [R_{mlc}(q_3') R_{wl}(-q_1, k_3)] D_{\mu_1 \mu_1}^{j_1} [R_{mlc}(k_1') R_{wl}^{-1}(-q_3', q_1') R_{mlc}(q_1)] \times
\]

\[
D_{\mu_2' \mu_2}^{j_2} [R_{mlc}(k_2') R_{wl}^{-1}(-q_3', q_2') R_{wl}(-q_1, k_2) R_{mlc}(k_2)]
\]

\[= 65\]
\[ Y^r_{\mu_l}(\hat{k}_2) \left< j_2 \mu_2 j_3 \mu_3 | s \mu_s \right> \left< l, \mu_l, s, \mu_s | j_3 \mu_3 \right> Y^L_{\mu_l}(\hat{\eta}_l) \times \]
\[ D^{j_23}_{\mu_23\mu_23} [R_{m}(\hat{-q}_1)] \left< j_2 \mu_2 j_1 \mu_1 | S \mu_S \right> \left< L, \mu_L, S, \mu_S | j \mu_f \right> \]

where \( m \) is the three body invariant mass, \( m_{ij} \) are the invariant masses of the \( ij \) and \( jk \) subsystems, \( w(k) \) are energies, and \( q_i \) are the operators

\[ \hat{q}_i := L^{-1}_i (\hat{p}) \hat{p}_i. \] (235)

17 Appendix III

To prove Theorem 1 first note that condition (82) implies

\[ \lim_{t \rightarrow \pm \infty} \| \hat{U} [I, -T] \left( \hat{\Phi}_A - \hat{U}^\dagger [\Lambda, Y] \hat{\Phi}_A \hat{U}_A [\Lambda, Y] \right) \hat{U}_A [I, T] \| \psi \| = 0 \] (236)

which is equivalent to

\[ \Omega_{\pm} (\hat{H}, \hat{\Phi}_A, \hat{H}_A) = \Omega_{\pm} (\hat{H}, \hat{U}^\dagger [\Lambda, Y] \hat{\Phi}_A \hat{U}_A [\Lambda, Y], \hat{H}_A). \] (237)

Since the Hamiltonian commutes with the linear and angular momentum operators, it follows that if \( (\Lambda, A) \) is a rotation or translation this becomes

\[ \Omega_{\pm} (\hat{H}, \hat{\Phi}_A, \hat{H}_A) = \hat{U}^\dagger [R, 0] \Omega_{\pm} (\hat{H}, \hat{\Phi}_A, \hat{H}_A) \hat{U}_A [R, 0] \] (238)

and

\[ \Omega_{\pm} (\hat{H}, \hat{\Phi}_A, \hat{H}_A) = \hat{U}^\dagger [I, Y] \Omega_{\pm} (\hat{H}, \hat{\Phi}_A, \hat{H}_A) \hat{U}_A [I, Y]. \] (239)

For the case of a rotationless Lorentz transformation condition (237) implies

\[ \Omega_{\pm} (\hat{H}, \hat{\Phi}_A, \hat{H}_A) = \Omega_{\pm} (\hat{H}, \hat{U}^\dagger [\Lambda, 0] \hat{\Phi}_A \hat{U}_A [\Lambda, 0], \hat{H}_A). \] (240)
The commutation relations imply
\[ \hat{U}^\dagger [\Lambda, 0] \hat{H} \hat{U} [\Lambda, 0] = \Lambda^0_\mu \hat{P}_\mu \] (241)
\[ \hat{U}_A^\dagger [\Lambda, 0] \hat{H}_A \hat{U}_A [\Lambda, 0] = \Lambda^0_\mu \hat{P}_\mu_A. \] (242)

It follows that
\[ \hat{U}^\dagger [\Lambda, 0] \Omega_\pm (\hat{H}, \hat{\Phi}_A, \hat{H}_A) \hat{U}_A [\Lambda, 0] = \Omega_\pm (\Lambda^0_\mu \hat{P}_\mu, \Lambda^0_\mu \hat{P}_\mu_A) \] (243)

which can be expressed as
\[ \Omega_\pm (\Lambda^0_\mu \hat{P}_\mu, \hat{\Phi}_A, \Lambda^0_\mu \hat{P}_\mu_A) = \lim_{t \to \pm \infty} e^{i \hat{H} \Lambda^0_0 t + i \Lambda^0_\mu \hat{P}_\mu t} \hat{\Phi}_A e^{-i \hat{H}_A \Lambda^0_0 t + i \Lambda^0_\mu \hat{P}_\mu_A t}. \] (244)

Since \( \Lambda^0_0 > 0 \) it is possible to redefine define \( t \to t' = \Lambda^0_0 t \) so the limit \( t \to \pm \infty \) is equivalent to the limit the \( t' \to \pm \infty \). This gives
\[ \lim_{t' \to \pm \infty} e^{i \hat{H} t'} \hat{U} [I, A t'] \hat{\Phi}_A \hat{U}_A [I, A t'] e^{-i \hat{H}_A t'} \] (245)
where
\[ A = \frac{\Lambda^0_0}{\Lambda^0_0 - \sigma_i}. \] (246)

Condition (83) then gives
\[ \lim_{t' \to \pm \infty} e^{i \hat{H} t'} \hat{U} [I, A t'] \hat{\Phi}_A \hat{U}_A [I, A t'] e^{-i \hat{H}_A t'} = \Omega_\pm (\hat{H}, \hat{\Phi}, \hat{H}_A). \] (247)

Combining (243) and (247) gives
\[ \Omega_\pm (\hat{H}, \hat{\Phi}_A, \hat{H}_A) = \hat{U}^\dagger [\Lambda, 0] \Omega_\pm (\hat{H}, \hat{\Phi}_A, \hat{H}_A) \hat{U}_A [\Lambda, 0]. \] (248)
To complete the proof of Theorem 1 note that (238), (248), (248) imply
\[
\hat{U}[A,Y] \Omega \pm (\hat{H}, \hat{\Phi}_A, \hat{H}_A) = \Omega \pm (\hat{H}, \hat{\Phi}_A, \hat{H}_A) \hat{U}_A[A,Y]
\]
which is the intertwining relation of corollary 1. Corollary 2 follows by identifying (241) and (247).

It follows that
\[
\hat{U}^\dagger_A[A,Y] \hat{S} \hat{U}_A[A,Y] =
\hat{U}^\dagger_A[A,Y] \Omega^\dagger_+ (\hat{H}, \hat{\Phi}_A, \hat{H}_A) \Omega_- (\hat{H}, \hat{\Phi}_A, \hat{H}_A) \hat{U}_A[A,Y] =
\Omega^\dagger_+ (\hat{H}, \hat{\Phi}_A, \hat{H}_A) \hat{U}^\dagger[A,Y] \hat{U}[A,Y] \Omega_- (\hat{H}, \hat{\Phi}_A, \hat{H}_A) = \hat{S}. \tag{250}
\]
This completes the proof of Theorem 1.

To prove corollary 3 note that equation (86) is equivalent to
\[
s - \lim_{s \to \pm \infty} \left[ e^{-i \hat{M}^s} \Omega \pm (\hat{H}, \hat{\Phi}, \hat{H}_A) - \hat{\Phi} e^{-i \hat{M}_A^s} \right] = 0 \tag{251}
\]
The intertwining properties that follow from Theorem 1 give the strong limit:
\[
s - \lim_{s \to \pm \infty} \left[ (\Omega \pm (\hat{H}, \hat{\Phi}, \hat{H}_A) - \hat{\Phi}) \right] e^{-i \hat{M}^s} = 0. \tag{252}
\]
The proof that this holds on the dense set of asymptotic states with bounded momentum follows the proof of theorem IX.23 of [41] (see also [42, 43]). The extension to the strong limit follows the argument in [2].

18 Appendix IV

To prove Theorem 4 let \( \hat{C}_{a_i} \) be the scattering equivalence that maps \( \hat{U}_{a_i}[A,Y] \) to the representation \( \hat{U}_{a_i}[A,Y] \) with kinematic \( \hat{F}^i_{a_j}, \Delta \hat{F}^i_{a_j}, \hat{j}_{a_j} \). Define
\[
\hat{C}_{a} := \otimes_{i=1}^{n_a} \hat{C}_{a_i} \tag{253}
\]
and
\[\tilde{U}_a[\Lambda, Y] := \hat{C}_a \tilde{U}_a[\Lambda, Y] \hat{C}_a^\dagger = \otimes_{i=1}^{n_a} (\hat{C}_{a_i} \tilde{U}_a[\Lambda, Y] \hat{C}_{a_i}^\dagger).\] (254)

By assumption, the representations \(\tilde{U}_a[\Lambda, A]\) and \(\tilde{U}_a[\Lambda, A]\) have the same scattering matrix elements, which are products of the single cluster scattering matrix elements. In addition, because
\[(\hat{I} - \hat{C}_a)\tilde{U}_0[I, T] = \otimes_{i=1}^{n_a} (I_{a_i} - \hat{C}_{a_i})\tilde{U}_{0a_i}[I, T]\] (255)
it follows that
\[\lim_{t \to \pm \infty} (\hat{I} - \hat{C}_a)\tilde{U}_0[I, T] = 0\] (256)
which shows that \(\hat{C}_a\) is a scattering equivalence on the N-body Hilbert space.

The representation \(\tilde{U}_a[\Lambda, Y]\) does not have kinematic \(\hat{F}_i, \Delta \hat{F}_i\) or \(\hat{j}\), even though each factor of the tensor product has this property. The advantage of the representation \(\tilde{U}_a[\Lambda, Y]\) is that it is scattering equivalent to a representation \(\tilde{U}_a[\Lambda, Y]\) that has a kinematic \(\hat{F}_i, \Delta \hat{F}_i\) and \(\hat{j}\).

To show this consider the structure of the single cluster \(\tilde{H}_a\) and \(\tilde{M}_a\). The Hamiltonian \(\tilde{H}_a\) of the representation \(\tilde{U}_a[\Lambda, Y]\) is
\[\tilde{H}_a := \sum_{i=1}^{n_a} \tilde{H}_{a_i} \otimes \hat{I}_i\] (257)
where \(\hat{I}_i\) is the identity on the remaining factors in the tensor product. The mass operator \(\tilde{M}_a\) is a function of the commuting operators \(\tilde{M}_{a_i} \otimes \hat{I}_i\) and \(\tilde{P}_{a_i} \otimes \hat{I}_i\). Corollary 3 of Theorem 1 give mild conditions on the interactions for \(\tilde{H}_a\) and \(\tilde{M}_a\) to lead to the same S-matrix.

The matrix elements of \(\tilde{M}_{a_i} \otimes \hat{I}_i\) in the tensor product of \(n_a\) free particle irreducible representations have the form
\[\langle \otimes_j (f_j, d_j; m_j, j_j) | \tilde{M}_{a_i} \otimes \hat{I}_i | \otimes_k (f'_k, d'_k; m'_k, j'_k) \rangle = \]
\[ \delta[f_i, f'_i]\delta_{j_i, j'_i}\langle d_i, m_i||\tilde{\mathcal{M}}^{j_i}_{a_i}||d'_i, m'_i\rangle \times \prod_{k \neq i} \delta[f_k, f'_k]\delta[d_k, d'_k]\delta_{j_k, j'_k}\delta(m_k - m'_k). \] (258)

An irreducible free particle basis for the \( N \)-body system can be constructed by successive use of the \( ISL(2, C) \) Clebsch-Gordan coefficients to decompose the basis \( \otimes_j (f_j, d_j(m_j, j_j)) \) into a direct integral of irreducible representations. What is relevant for the proof of this theorem is that the variables \( m_i, d_i \) and \( j_i \) that appear in the kernel \( \langle d_i, m_i||\tilde{\mathcal{M}}^{j_i}_{a_i}||d'_i, m'_i\rangle \) of \( \tilde{\mathcal{M}}_{a_j} \) are degeneracy parameters in this representation.

In order to be precise assume that the irreducible free particle basis is obtained by successively coupling clusters in the order \( \cdot \cdot \cdot (((12)3)4) \cdot \cdot \cdot n_a \). In addition, at each stage in the coupling define \( \hat{q}_i \) as the solution to

\[ \hat{M}_{0(\cdots(12)3)\cdots\cdot \cdot \cdot i(\cdot \cdot \cdot(12)\cdots n_a)\cdot \cdot \cdot i+1)} = \sqrt{\hat{q}_i^2 + \hat{M}_{0(i+1)}^2} + \sqrt{\hat{q}_i^2 + \hat{M}_{0(\cdots (12)3)\cdots i)}^2}. \] (259)

The operators \( \hat{q}_i \) are alternate labels for the kinematic invariant masses \( \hat{M}_{0(\cdots (12)3)\cdots i)} \).

Define the single cluster mass operators \( \bar{\mathcal{M}}_{a_i} \) in this irreducible representation

\[ \langle f, d; m, j||\bar{\mathcal{M}}_{a_i}||f', d'; m', j'\rangle = \delta[f, f']\delta[j, j']\delta_{j_k, j'_k}\langle d_i, m_i||\bar{\mathcal{M}}^{j_i}_{a_i}||d'_i, m'_i\rangle \times JJ' \prod_{k \neq i} \delta_{j_k, j'_k}\delta(m_k - m'_k) \prod_{l=1}^{n_a} \delta(q_l - q'_l) \delta_{r_i, r'_i}. \] (260)

where the \( q_l \)'s are considered functions of the kinematic invariant masses, the \( r_l \) are degeneracy parameters that result when particle \( l \) is coupled to the irreducible \( (1 \cdot \cdot \cdot l - 1) \) system, and \( J \) and \( J' \) are Jacobians

\[ J = |\frac{\partial(q_1 \cdot \cdot \cdot q_{n_a-1})}{\partial(m_{(12)} \cdot \cdot \cdot m_{0(\cdots (12)\cdots (n_a)})}|^{1/2}. \] (261)
The three important observations about this definition are

- The non-trivial part of this kernel is identical to the non-trivial part of the kernel of \( \tilde{M}_a \) in the tensor product representation \( (258) \).
- Each \( \bar{M}_{a_j} \) commutes with \( \hat{F}_i^0, \Delta \hat{F}_0^i, \hat{j}_0 \)
- \( [\bar{M}_{a_i}, \bar{M}_{a_j}] = 0 \)

The relations \( (259) \) can be inverted to express the free mass as a function of the free single cluster mass operators and the \( q_i \)’s:

\[
\hat{M}_0 = M(\hat{M}_{0_1}, \ldots, \hat{M}_{0_n}, \hat{q}_1, \ldots, \hat{q}_{n-1}) \tag{262}
\]

The commutation relations allow the definition:

\[
\bar{M}_a := M(\tilde{M}_{a_1}, \ldots, \tilde{M}_{a_n}, \hat{q}_1, \ldots, \hat{q}_{n-1}) \tag{263}
\]

where the \( \hat{q}_i \)’s in \( (263) \) are identical to the non-interacting \( \hat{q}_i \)’s in \( (262) \). By construction \( \bar{M}_a \) commutes with \( \hat{F}_0^i, \Delta \hat{F}_0^i, \hat{j}_0 \). Simultaneous eigenstates of \( \bar{M}_a, \hat{F}_0^i \) and \( j_0 \) transform as mass \( \bar{M}_a \) spin \( j_0 \) irreducible representations of \( ISL(2, \mathbb{C}) \). This defines the representation \( \bar{U}_a[\Lambda, Y] \).

In order to construct a scattering theory we need to define a suitable injection operator to the asymptotic Hilbert space for \( \bar{M}_a \). The channel injection operator for the representation \( \bar{U}_a[\Lambda, Y] \) is the tensor product of irreducible eigenstates

\[
\bar{\Phi}_\alpha = |f_1, \alpha_1, \ldots, f_{n_a}, \alpha_1\rangle. \tag{264}
\]

The corresponding channel injection operator for the representation \( \bar{U}_a[\Lambda, A] \) is defined as the simultaneous eigenstates of \( \bar{M}_a, \hat{j}_0^2, \hat{F}_0^i, \hat{q}_{i0}, \hat{r}_I \), and \( \tilde{M}_{ai} \).
corresponding the same bound states of the $\tilde{M}_a$:

$$\tilde{\Phi}_\alpha = |f, j_0, q_1, \cdots, q_{na-1}, r_1, \cdots, r_{na-1}, \alpha_1 \cdots \alpha_{na}\rangle. \quad (265)$$

These differ by the delta functions that multiply the cluster eigenfunctions.

With this definition it follows that

$$\bar{\Omega}_{a\pm} := \Omega_{\pm}(\bar{M}_a, \bar{\Phi}_{a\alpha}, H_{a\alpha}) \quad (266)$$

exist and are complete. The scattering operator

$$\bar{S}_a = \bar{\Omega}_a^\dagger \Omega_{a-} - \delta_{j_0j'_0} \delta[f, f'] \prod_{i=1}^{na-1} \delta(q_i - q'_i) \prod_i \delta_{j_i j'_i} \bar{S}_a \quad (267)$$

is identical to $\bar{S}_a$ if the Clebsch-Gordan coefficients are used to replace the irreducible spectator variables by the single cluster $f_i, j_i$’s. The equivalence follows because the $S$ matrix elements are determined by the single cluster mass operators, which have identical reduced kernels in representations $(258)$ and $(260)$.

This establishes that the representations $\bar{U}_a[\Lambda, Y]$ and $\bar{U}_a[\Lambda, Y]$ give the same scattering matrix elements. By Theorem 3 they are scattering equivalent. Let $\hat{D}_a$ be the scattering equivalence that relates these two representations:

$$\bar{U}_a[\Lambda, Y] = \hat{D}_a \hat{U}_a[\Lambda, Y] \hat{D}_a^\dagger. \quad (268)$$

It follows from $(254)$ and $(268)$ that

$$\bar{U}_a[\Lambda, Y] = \hat{B}_a \bar{U}_a[\Lambda, Y] \hat{B}_a^\dagger \quad (269)$$

where

$$\hat{B}_a := \hat{D}_a \hat{C}_a. \quad (270)$$

The operator $\hat{B}_a$ is a scattering equivalence since it is a product of scattering equivalences. This completes the proof of Theorem 4.
References

[1] S. N. Sokolov, Dokl. Akad. Nauk USSR 233(1977)575.

[2] F. Coester and W. N. Polyzou, Phys. Rev. D26(1982)1348.

[3] P.A.M. Dirac, Rev. Mod. Phys. 21(1949)392.

[4] Rudolf Haag, Local Quantum Physics, Springer Verlag, 1992.

[5] S. Weinberg, The Quantum Theory of Fields, I, Cambridge University Press, 1995, see page 169.

[6] E. P. Wigner, Ann. Math. 40(1939)149.

[7] B. Bakamjian and L. H. Thomas, Phys. Rev. 92(1953)1300.

[8] F. Coester, Helv. Phys. Acta 38(1965)7.

[9] W. Klink and W. Polyzou, Phys. Rev. C54(1996)1189.

[10] W. Glöckle, T. S-H Lee, F. Coester, Phys. Rev. C33(1986)709.

[11] P. L. Chung, W.N. Polyzou ,F. Coester, B.D. Keister, Phys. Rev. C37(1988)2000.

[12] By Z.J. Cao, B.D. Keister, Phys. Rev. C42(1990)2295.

[13] W.N. Polyzou, W. Glöckle, Phys. Rev. C53(1996)3111.

[14] M.G. Fuda, Few Body Syst. 23(1998)127.

[15] Y. Elmessiri, M. G. Fuda, Phys. Rev. C60(1999)044001.
[16] E. Pace, G. Salme, Nucl.Phys. A689(2001)441.

[17] T.W. Allen, W.H. Klink, W.N. Polyzou, Phys. Rev. C63(2001)034002.

[18] P.L. Chung, F. Coester, W.N. Polyzou, Phys. Lett. B205(1988)545.

[19] P.L. Chung, F. Coester, Phys. Rev. D44(1991)229.

[20] S. Capstick and B.D. Keister, Phys. Rev. D51(1995)3598.

[21] F. Cardarelli, I.L. Grach, I.M. Narodetsky, G. Salme, S. Simula, Few Body Syst. Suppl. 9(1995)267.

[22] L.Y. Glozman, M. Radici, R.F. Wagenbrunn, S. Boffi, W. Klink, W. Plessas, Phys.Lett. B516(2001)183.

[23] W. N. Polyzou, Ann. Phys. (N.Y.) 193(1989)367.

[24] V. Bargmann, Ann. Math. 59(1954)1.

[25] M. Reed and B. Simon, Functional Analysis, Academic Press, New York, 1972.

[26] M. Hammermesh, Group Theory and its Application to Physical Problems, Dover, Mineola N.Y., 1962.

[27] H. Joos, Fortsch. Phys. 10(1962)65.

[28] P. Moussa and R. Stora, in “Lectures in Theoretical Physics”, (Ed. W. E. Brittin, and A. O. Barut), Vol VIIA, Summer Institute for Theoretical Physics, The University of Colorado, Boulder, CO 1964, University of Colorado Press, 1965.
[29] B. D. Keister and W. N. Polyzou, Advances in Nuclear Physics, Vol. 20, Ed. J. W. Negele and E. Vogt, Plenum Press, New York, 1991.

[30] G. W. Mackey, *Induced Representations of Groups and Quantum Mechanics*, Benjamin, N.Y., 1966.

[31] C. Chandler and A. G. Gibson, J. Math. Phys. 30(1989)1533.

[32] J. Cook, J. Math. Phys. 36(1957)82.

[33] Fong and J. Sucher, J. Math. Phys. 5(1964)456.

[34] W. N. Polyzou, Few Body Systems, 27(1999)57.

[35] M. Fuda, Phys. Rev. C64(2001)027001.

[36] F. Coester, *Proceedings, Mathematical Methods and Applications of Scattering Theroy*, ed A. W. Sanez, W. W. Zachary, Washington DC, 1979.

[37] G. Birkhoff, *Lattice Theory*, A.M.S. Colloquium Publications V25, Providence, RI, 1995.

[38] *Gian Carlo-Rota on Combinatorics - Introductory Papers and Commentaries*, ed. J.P.S. Kung, Birkhäuser, Boston, Basel, Berlin, 1995.

[39] W. N. Polyzou, J. Math. Phys. 22(1981)798.

[40] K.L.Kowalski, W. N. Polyzou, E.F.Redish, J. Math. Phys. 22(1981)1965.
[41] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Volume III, Scattering Theory*, Academic Press, N.Y., 1979.

[42] T. Kato, *Perturbation Theory for Linear Operators*, Springer Verlag, N.Y 1964.

[43] C. Chandler and A. Gibson, *Indiana University Math. J.*, 25#5 (1976)443.