A variant of the Hadwiger–Debrunner \((p, q)\)-problem in the plane

Sathish Govindarajan∗ Gabriel Nivasch†

Abstract

Let \(X\) be a convex curve in the plane (say, the unit circle), and let \(S\) be a family of planar convex bodies, such that every two of them meet at a point of \(X\). Then \(S\) has a transversal \(N \subset \mathbb{R}^2\) of size at most \(1.75 \cdot 10^9\).

Suppose instead that \(S\) only satisfies the following \(\ldots\)-condition: Among every \(p\) elements of \(S\) there are two that meet at a common point of \(X\). Then \(S\) has a transversal of size \(O(p^8)\). For comparison, the best known bound for the Hadwiger–Debrunner \((p, q)\)-problem in the plane, with \(q = 3\), is \(O(p^6)\).

Our result generalizes appropriately for \(\mathbb{R}^d\) if \(X \subset \mathbb{R}^d\) is, for example, the moment curve.

1 Introduction

Let \(S\) be a family of convex bodies in \(\mathbb{R}^d\). We say that \(S\) satisfies the \((p, q)\)-condition, for positive integers \(p \geq q\), if among every \(p\) elements of \(S\) there are \(q\) that meet at a common point. Hadwiger and Debrunner [12], in their celebrated problem, asked whether a family \(S\) that satisfies the \((p, q)\)-condition, for \(p \geq q \geq d + 1\), has a transversal of size bounded by a constant \(\text{HD}_d(p, q)\) that depends only on \(d, p,\) and \(q\). (A transversal for \(S\) is a set \(N \subset \mathbb{R}^d\) that intersects every element of \(S\).)

This problem is a generalization of Helly’s theorem [13]: Helly’s theorem states that, if every \(d+1\) elements of \(S\) intersect, then they all intersect; or, in other words, \(\text{HD}_d(d+1, d+1) = 1\).

It is clear that \(q\) cannot be smaller than \(d + 1\), since a family of \(n\) hyperplanes in general position provides a counterexample: Every \(d\) hyperplanes intersect, and yet a transversal must contain at least \(n/d\) points.

Hadwiger and Debrunner [12] showed that, for \(q > 1 + (d - 1)p/d\), one has \(\text{HD}_d(p, q) = p - q + 1\).

Alon and Kleitman [4] settled the general question in the affirmative, by tackling the hardest case \(q = d + 1\). Their proof uses an impressive array of tools from discrete geometry, including the fractional Helly theorem, linear-programming duality, and weak epsilon-nets. (Alon and Kleitman later published a more elementary proof in [5].)

∗gsat@cse.iisc.ernet.in. Indian Institute of Science, Bangalore, India.
†gabrieln@ariel.ac.il. Ariel University, Ariel, Israel.
†Throughout this paper we allow \(S\) to be a multi-set; meaning, the elements of \(S\) need not be pairwise distinct.
Fractional Helly. The fractional Helly theorem \cite{14, 15} (see also \cite{17, pp. 195}) states that, if $S$ is a family of $n$ convex bodies in $\mathbb{R}^d$ such that at least an $\alpha$-fraction of the $\binom{n}{d+1}$ $(d+1)$-tuples intersect, then there exists a point $z \in \mathbb{R}^d$ contained in at least $\beta n$ bodies, for some $\beta > 0$ that depends only on $d$ and $\alpha$. The bound $\beta \geq \alpha/(d+1)$ is asymptotically optimal for small $\alpha$.

Weak epsilon nets. Given a finite point set $P \subset \mathbb{R}^d$ and a parameter $0 < \varepsilon < 1$, a weak $\varepsilon$-net for $P$ (with respect to convex sets) is a set $N \subset \mathbb{R}^d$ that intersects every convex set that contains at least an $\varepsilon$-fraction of the points of $P$. Alon et al. \cite{2} showed that $P$ always has a weak $\varepsilon$-net of size bounded only by $d$ and $\varepsilon$. The best known bounds for the size of weak $\varepsilon$-nets are $f_2(\varepsilon) = O(\varepsilon^{-2})$ in the plane \cite{2, 7}, and $f_d(\varepsilon) = O(\varepsilon^{-d \text{polylog}(1/\varepsilon)})$ for dimension $d \geq 3 \cite{7, 19}$.

For point sets $P$ that satisfy additional constraints, better bounds are known. For example, if $P \subset X$ for some convex curve $X \subset \mathbb{R}^2$, then $P$ has a weak $\varepsilon$-net of size $O((1/\varepsilon) \log d)$, where $\alpha(n)$ denotes the very slow-growing inverse-Ackermann function (Alon et al. \cite{3}).

Regarding lower bounds, Bukh et al. \cite{6} constructed, for every $d$ and $\varepsilon$, a point set $P \subset \mathbb{R}^d$ for which every weak $\varepsilon$-net has size $\Omega((1/\varepsilon) \log d)$.

Back to the Hadwiger–Debrunner problem. The argument of Alon and Kleitman \cite{4} yields $\text{HD}_d(p, d+1) \leq f_d(c_d p^{-d+1})$, where $f_d$ is the upper bound for weak epsilon-nets, and $c_d > 0$ is some constant. Thus, for the planar case we obtain $\text{HD}_2(p, 3) = O(p^2)$.

The lower bound $\text{HD}_d(p, d+1) = \Omega(p \log^{d-1} p)$ follows from the lower bound for weak epsilon-nets: Let $P \subset \mathbb{R}^d$ be a point set realizing the lower bound for weak $\varepsilon$-nets. Let $S$ be the set of all convex hulls of at least an $\varepsilon$-fraction of the points of $P$. Then $S$ satisfies the $(p, d+1)$-condition for $p = 1 + d/\varepsilon$; and every transversal for $S$ is a weak $\varepsilon$-net for $P$.

Related work. Many variants of the $(p, q)$-problem have been studied; see for example the survey \cite{10}. Regarding the case $q = 2$, Danzer \cite{8} (answering a question of Gallai) showed that any family of pairwise intersecting disks in the plane (i.e., satisfying the $(2, 2)$-condition) has a transversal of size 4, and that this bound is optimal. More generally, Grünbaum \cite{11} showed that any family of pairwise intersecting homothets of a fixed convex body in $\mathbb{R}^d$ has a transversal bounded in terms only of $d$.

Kim et al. \cite{16}, together with Dumitrescu and Jiang \cite{9}, showed that for homothets of a convex body in $\mathbb{R}^d$ having the $(p, 2)$-property, the transversal number is at most $c_d p$ for some constants $c_d$.

1.1 Our variant of the problem

We asked ourselves the following question: Can we obtain smaller transversals for $S$ if we impose an additional constraint in $S$, analogous to the convex-curve constraint for weak epsilon-nets?

In this spirit, we raised the following problem: Let $X$ be a convex curve in the plane (say, $X$ could be the unit circle $\{(x, y) \mid x^2 + y^2 = 1\}$). Let $S$ be a family of planar convex bodies as before. We now strengthen the $(p, q)$-condition by requiring that, among every $p$ elements
of $S$, at least $q$ meet at a point of $X$. What can we say then about the minimum size of a transversal for $S$?

**Problem 1.** Let $X$ be a convex curve in the plane, and let $S$ be a family of planar convex bodies, such that among every $p$ elements of $S$, three of them meet at a point of $X$. Then we know that $S$ has a transversal of size $\text{HD}_2(p,3) = O(p^6)$. Does $S$ have a smaller transversal?

Since the counterexample that required $q \geq 3$ does not hold in this new setting, we can ask what happens when $q = 2$.

**Problem 2.** Now suppose that among every $p$ elements of $S$, two of them meet at a point of $X$. Does $S$ then have a transversal of size depending only on $p$?

We do not know the answer to the first question, but we answer the second question in the affirmative:

**Theorem 1.** Let $X$ be a convex curve in the plane, and let $S$ be a family of planar convex bodies. Then:

(a) If every pair of elements of $S$ meet at a point of $X$, then there exists a point $z \in \mathbb{R}^2$ that intersects at least a $1/15800$-fraction of the elements of $S$, and $S$ has a transversal of size at most $175 \cdot 10^9$.  

(b) If among every $p$ elements of $S$, two of them meet at a point of $X$, then there exists a point $z \in \mathbb{R}^2$ that intersects a $\Omega(p^{-4})$-fraction of the elements of $S$, and $S$ has a transversal of size $O(p^8)$.

A generalization of Theorem 1 for $\mathbb{R}^d$ is discussed in Section 3.

2 The proof

The first step (for case (b) only) is to apply Turán’s theorem [20] (see also [1]):

**Lemma 2.** Let $X$ be a convex curve in the plane, and let $S$ be a family of $n$ planar convex bodies, such that among every $p$ elements of $S$, two of them meet at a point of $X$. Then, the number of pairs of elements of $S$ that meet at a point of $X$ is at least $n^2/(2p)$.

**Proof.** Let $G$ be a graph containing a vertex for every element of $S$, and containing an edge for every pair of elements that do not meet at any point of $X$. Then our assumption on $S$ is equivalent to saying that $G$ contains no clique of size $p$. Therefore, by Turán’s theorem, $G$ contains at most $\left(1 - \frac{1}{p-1}\right) \frac{n^2}{2}$ edges, so it is missing more than $n^2/(2p)$ edges. □

The second, and main, step is to prove a fractional-Helly-type lemma:

**Lemma 3.** Let $X$ be a convex curve in the plane, and let $S$ be a family of $n$ planar convex bodies. Then:

(a) If every pair of elements in $S$ meet at a point of $X$, then there exists a point $z \in \mathbb{R}^2$ that is contained in at least $n/15800$ elements of $S$.
(b) If a $\gamma$-fraction of the $\binom{n}{2}$ pairs of elements of $S$ meet at a point of $X$, for some $0 < \gamma < 1$, then there exists a point $z \in \mathbb{R}^2$ that is contained in at least $\Omega(\gamma^4 n)$ elements of $S$.

Proof. Let $S_1, S_2, \ldots, S_n$ be the objects in $S$. We think of each set $S_i$ as “colored” with color $i$. For each pair $S_i, S_j$ that meet at a point of $X$, select a point $p_{ij} \in X \cap S_i \cap S_j$. In case (a) we have $N = \binom{n}{2}$ points $p_{ij}$, while in case (b) we have $N = \gamma \binom{n}{2}$ points. Note that these points are not necessarily pairwise distinct (in fact they could all be the same point); however, that would only make our problem easier.

Sort the points $p_{ij}$ in weakly circular order around $X$, and rename the sorted points $Q = (q_0, q_1, \ldots, q_{n-1})$. We treat $Q$ as a circular list, so after $q_{n-1}$ comes $q_0$. Each $q_a$ is colored with two distinct colors among $1, \ldots, n$ (corresponding to the two objects in $S$ that defined $q_a$), and each pair of colors occurs at most once (or exactly once in case (a)).

Let $Y = (y_0, \ldots, y_{n-1}) \subset X$ be a circular list of “separator” points, such that $y_i$ lies (weakly) between $q_{i-1}$ and $q_i$ for every $i$.

Note that each quadruple of separator points $y = (y_a, y_b, y_c, y_d)$ (listed in circular order) defines a partition of $Q$ into four intervals: $[q_a, q_{b-1}], [q_b, q_{c-1}], [q_c, q_{d-1}], [q_d, q_{a-1}]$. The quadruple $y$ is said to “pierce” color $i$ if each of these four intervals contains a point colored with color $i$.

We make use of the following observation, which was previously used in [3] and [7].

Observation 4. Let $y = (y_a, y_b, y_c, y_d)$ be a quadruple of separator points, and let $z \in \mathbb{R}^2$ be the point of intersection of segments $y_ay_b$ and $y_cy_d$. Then, if $y$ pierces color $i$, then $z \in S_i$ (see Figure 7 (left)).

Our strategy is to show that a randomly-chosen quadruple of separators pierces, in expectation, a constant fraction of the colors.

Define the distance between two points $q_a, q_b \in Q$ as $\min \{(b - a) \mod N, (a - b) \mod N\}$.

We now choose a parameter $\alpha < 1$ independent of $n$: For case (a) we set $\alpha = 0.027$, while for case (b) we set $\alpha = \gamma/300$. We call a color $i$ spread out if there exist four points $q_a, q_b, q_c, q_d \in Q$, colored with color $i$, such that all the pairwise distances between these four points are at least $\alpha N$.

Observation 5. A randomly-chosen quadruple $y = (y_a, y_b, y_c, y_d)$ has probability at least $24\alpha^3 (1 - 3\alpha)$ of piercing a given spread-out color.

Proof. Suppose color $i$ is spread out. Consider four points $q_a, q_b, q_c, q_d \in Q$ in cyclic order, that prove that $i$ is spread out. Let the distances between them in cyclic order be $\beta_1 N, \beta_2 N, \beta_3 N, \beta_4 N$; so $\beta_1 + \cdots + \beta_4 = 1$. Then $y$ pierces color $i$ with probability at least $24 \beta_1 \beta_2 \beta_3 \beta_4$. Subject to the constraints $\beta_i \geq \alpha$ for $1 \leq i \leq 4$, this quantity is minimized when $\beta_1 = \beta_2 = \beta_3 = \alpha$ and $\beta_4 = 1 - 3\alpha$.

We now proceed to derive a lower bound on the number of spread-out colors.

First, we characterize when a color is spread out:

Observation 6. For each color $i$, exactly one of the following two options holds:

1. Color $i$ is spread out.
2. All the instances of color $i$ occur in at most three intervals of $Q$, each of length at most $\alpha N$.
Figure 1: Left: If the quadruple of separators pierces color $i$, then $z \in S_i$. Center: If color $i$ is not spread out, then it is contained in three small intervals. Right: A family that requires a transversal of size 3.

Proof. If the second condition is true then clearly color $i$ is not spread out, because we can at most choose $q_a$, $q_b$, and $q_c$ from three different intervals, and then we have no way of choosing $q_d$.

For the other direction, suppose color $i$ is not spread out. Let $I$ be the longest interval in $Q$ that is completely free of color $i$. We must certainly have $|I| > \alpha N$, since otherwise $Q$ would have $1/(2\alpha)$ points in cyclic order, with pairwise distances at least $\alpha N$, all colored with color $i$; and $1/(2\alpha) > 4$.

To the left and right of $I$ are points $q_a$ and $q_b$, respectively, colored with color $i$. Let $q_a'$ be the farthest point left of $q_a$, still within distance $\alpha N$ of $q_a$, that is colored with color $i$. Similarly, let $q_b'$ be the farthest point right of $q_b$, still within distance $\alpha N$ of $q_b$, that is colored with color $i$. Let $q_c$ be the first point left of $q_a'$ colored with color $i$; and let $q_d$ be the first point right of $q_b'$ colored with color $i$.

The distance between $q_c$ and $q_d$ must be less than $\alpha N$, since otherwise $q_c$, $q_a$, $q_b$, $q_d$ would prove that color $i$ is spread out. Thus, all instances of color $i$ are contained in the intervals $[q_d, q_c$, $[q_a', q_a]$, $[q_b, q_b']$. See Figure 1 (center).

We now derive an upper bound on the number of colors that are not spread out.

Lemma 7. In case (a) the number of colors that are not spread out is at most $3\sqrt{3an} + o(n)$. In case (b) this number is at most $(1 - \gamma/4)n + o(n)$.

Proof. We will use the following graph-theoretic observation:

Observation 8. Let $G = (V, E)$ be a graph, and for every $v \in V$ let $g(v) = \sum_{w \in N(v)} d(w)$ denote the sum of the degrees of the neighbors of $v$. Then, there exists a vertex $v \in V$ for which $g(v) \geq 4|E|^2/|V|^2$.

Proof. We have $\sum_{v \in V} g(v) = \sum_{v \in V} d^2(v)$, since each vertex $v$ contributes exactly $d(v)$ to $d(v)$ different terms of $\sum g(v)$. Therefore, the claim follows by the Cauchy–Schwarz inequality, noting that $\sum_{v \in V} d(v) = 2|E|$.
Let \( m = kn \) be the number of colors that are not spread out. In case \((b)\) we may assume that \( k > 1 - \gamma/2 \), since otherwise we are done.

Assume for simplicity that the non-spread-out colors are \( 1, \ldots, m \). For each color \( i, i \leq m \), let \( I_{ia}, I_{ib}, I_{ic} \) be the three intervals \( Q \), of length at most \( \alpha N \), on which color \( i \) occurs, according to Observation 6.

Let \( G = (V, E) \) be a graph with \( 3m \) vertices, labeled \( v_{ia} \) with \( 1 \leq i \leq m \) and \( 1 \leq a \leq 3 \), and with an edge connecting vertices \( v_{ia} \) and \( v_{jb} \) if and only there is a point \( p_{ij} \), having the pair of colors \( i \) and \( j \), lying on the intervals \( I_{ia} \) and \( I_{jb} \). In case \((a)\) we have \( |E| = \binom{m}{2} \).

In case \((b)\) we have \( |E| \geq N - (1 - k)n^2 \), since the number of points \( p_{ij} \) that have a spread-out color (meaning, that \( i > m \) or \( j > m \)) is at most \( n(n - m) = (1 - k)n^2 \). Thus, \( |E| \geq (k + \gamma/2 - 1)n^2 \) ignoring lower-order terms. Note that this quantity is positive, by our assumption on \( k \).

Denote by \( g(v) = \sum_{w \in N(v)} d(w) \) the sum of the degrees of the neighbors of vertex \( v \in V \).

By Observation 8, there exists a vertex \( v_{ia} \) for which \( g(v_{ia}) \geq 4|E|^2/|V|^2 \).

Consider the interval \( I_{ia} \) corresponding to this vertex \( v_{ia} \). Recall that \( I_{ia} \) has length at most \( \alpha N \). Let \( I' \) be an interval of \( Q \) of length \( 3\alpha N \) centered around \( I_{ia} \). All the intervals \( I_{jb} \) that correspond to neighboring vertices \( v_{jb} \in N(v_{ia}) \) lie in \( I' \). Each such \( I_{jb} \) contains \( d(v_{jb}) \) points colored with color \( j \). Thus, \( I' \) contains at least \( g(v_{ia}) \) “colorings” of points. But at most two “colorings” happen at each point, so \( |I'| \geq g(v_{ia})/2 \geq 2|E|^2/|V|^2 \).

Therefore, \( 3\alpha N \geq 2|E|^2/|V|^2 \). In case \((a)\) we substitute \( |V| = 3m \), \( |E| \approx m^2/2 \), and \( N \approx n^2/2 \) (ignoring lower-order terms); we obtain \( m \leq 3\sqrt{3\alpha n} + o(n) \), as claimed.

In case \((b)\) we substitute \( |V| = 3kn \), \( |E| \geq (k + \gamma/2 - 1)n^2 \), \( N = \gamma n^2/2 \), and \( \alpha = \gamma/300 \). Solving for \( k \), we obtain

\[
k \leq \frac{1 - \gamma/2}{1 - 3\gamma/20}.
\]

Since \( 0 < \gamma < 1 \), this quantity is at most \( 1 - \gamma/4 \), completing the proof.

Thus, the number of spread-out colors is at least \( (1 - 3\sqrt{3\alpha})n - o(n) \) in case \((a)\), and \( \Omega(\gamma n) \) in case \((b)\).

To conclude the proof of Lemma 3, we put together Observation 5 and Lemma 7. They give us a lower bound on the expected number of colors that are pierced by a randomly-chosen quadruple of separators. There must exist a quadruple \( y = (y_a, y_b, y_c, y_d) \) that achieves this expectation.

In case \((a)\), the expectation is \( 24\alpha^3(1 - 3\alpha)(1 - 3\sqrt{3\alpha})n - o(n) \). Since we chose \( \alpha = 0.027 \) (which is close to optimal), this is at least \( n/15800 \) for large enough \( n \).

For case \((b)\) we note that the bound in Observation 5 is \( \Omega(\alpha^2) \), which is \( \Omega(\gamma^3) \) by our choice of \( \gamma \). Hence, \( y \) pierces \( \Omega(\gamma^4 n) \) colors.

In both cases, by Observation 4, the point of intersection \( z = y_a y_c \cap y_b y_d \) is the desired point. This completes the proof of Lemma 3.

The final step is to apply the standard Alon–Kleitman machinery. We follow Matoušek’s presentation in 17:

**Proof of Theorem 7.** We recall some concepts. Given a finite family \( S \) of objects in \( \mathbb{R}^d \), a *fractional transversal* for \( S \) is a finite point set \( N \subset \mathbb{R}^d \), together with a weight function \( w : N \to [0, 1] \), such that \( \sum_{x \in N \cap S} w(x) \geq 1 \) for each \( S \in S \). (A regular transversal is then a
fractional transversal for which \(w(x) = 1\) for all \(x \in N\). The size of the fractional transversal is defined as \(\sum_{x \in N} w(x)\).

A fractional packing for \(S\) is a weight function \(\phi : S \to [0, 1]\), such that \(\sum_{S \in S} \phi(S) \leq 1\) for every point \(x \in \mathbb{R}^d\). The size of the fractional packing is defined as \(\sum_{S \in S} \phi(S)\).

Since \(S\) has a finite number of elements, they define a partition of \(\mathbb{R}^d\) into a finite number of regions. It does not matter which point we choose from each region, and therefore, there is only a finite number of points we have to consider.

The problems of minimizing the size of a fractional transversal of \(S\), and of maximizing the size of a fractional packing of \(S\), are both linear programs, and furthermore, they are duals of each another. Therefore, by LP duality, the size of their optimal solutions coincide (see also [13]). We denote by \(\tau^*(S)\) the optimal size of the linear programs.

Now consider the family \(S\) given in Theorem 1. Recall that \(S\) satisfies our strengthened \((p, 2)\)-condition: Among every \(p\) elements of \(S\), two meet at a point of \(X\) (with \(p = 2\) in case \((a)\)). We can assume that every element of \(S\) intersects \(X\), since otherwise, the remaining elements would satisfy the \((p - 1, 2)\)-condition.

Let \(\phi\) be a fractional packing for \(S\) achieving the optimal size \(\tau^* = \tau^*(S)\). We can assume that \(\phi(S)\) is rational for every \(S \in S\). Write \(\phi(S) = m(S)/D\), where \(m(S)\) and \(D\) are integers and \(D\) is a common denominator. Then \(\sum_{S \in S} m(S) = \tau^* D\), and

\[
\sum_{S \in S} m(S) \leq D \quad \text{for every point } x \in \mathbb{R}^d. \quad (1)
\]

Define a family of objects \(T\) obtained by repeating each \(S \in S\) \(m(S)\) times. Since \(S\) satisfies our strengthened \((p, 2)\)-condition, so does \(T\) (if among the \(p\) elements we select two copies of the same object, then they clearly meet in \(X\)). Thus, by Lemmas 2 and 3, there exists a point \(z \in \mathbb{R}^2\) contained in at least an \(\varepsilon\)-fraction of the \(\tau^* D\) objects in \(T\), where \(\varepsilon = 1/15800\) in case \((a)\) or \(\varepsilon = \Omega(p^{-4})\) in case \((b)\). On the other hand, equation (1) implies that \(z\) cannot intersect more than \(D\) objects of \(T\). Hence, \(\tau^* \leq 1/\varepsilon\).

By LP duality, this means that \(T\) has a fractional transversal \((N, w)\) of size at most \(1/\varepsilon\). As before, we can assume that all the weights in the fractional transversal are rational. We replace \(N\) by an unweighted point set \(N'\), in which each point of \(x \in N\) is replaced by a tiny cloud of size proportional to \(w(x)\). Then, each object in \(T\) (and thus, each object in \(S\)) contains at least an \(\varepsilon\)-fraction of the points of \(N'\).

Finally, we take a weak \(\varepsilon\)-net \(M\) for \(N'\). Since \(M\) intersects every convex set that contains an \(\varepsilon\)-fraction of the points of \(N'\), \(M\) is our desired transversal for \(S\). Its size is \(f_2(\varepsilon) = O(\varepsilon^{-2})\), which in case \((b)\) is \(O(p^8)\). For case \((a)\) we use the more explicit bound \(f_2(\varepsilon) \leq 7\varepsilon^{-2}\) of Alon et al. [2], and we get \(|M| \leq 1.75 \cdot 10^9\), as claimed.^2

3 Generalization to \(\mathbb{R}^d\)

Convex curves. A convex curve in \(\mathbb{R}^d\) is a curve that intersects every hyperplane at most \(d\) times [22, p. 314]. The most well known convex curve is the moment curve

\[\{(t, t^2, \ldots, t^d) \mid t \in \mathbb{R}\}.\]

^2The bound of Alon et al. can actually be improved to \(f_2(\varepsilon) \leq 6.37\varepsilon^{-2} + o(\varepsilon^{-2})\) by simply optimizing the parameter involved in the divide-and-conquer argument. This would lead to a modest improvement in our bound for \(|M|\).
If $d$ is even, then a convex curve in $\mathbb{R}^d$ can be open (like the moment curve) or closed, like the Carathéodory curve [21, p. 75]

$$\{(\sin t, \cos t, \sin 2t, \cos 2t, \ldots, \sin \frac{d}{2}t, \cos \frac{d}{2}t) \mid 0 \leq t < 2\pi\}.$$  

For $d$ even it is convenient to think of the curve as being always closed, by pretending, if necessary, that the curve’s two endpoints are joined together. In other words, for $d$ odd we consider the points on the curve to be linearly ordered, while for $d$ even we consider the points to be circularly ordered.

**Weak epsilon-nets.**  The result by Alon et al. [3] on weak epsilon-nets mentioned in the introduction generalizes as follows: If $P$ is a finite point set that lies on a convex curve $X \subset \mathbb{R}^d$, then $P$ has a weak $\varepsilon$-net of size at most $(1/\varepsilon)^2\text{poly}(\alpha(1/\varepsilon))$.

Note that this bound is barely superlinear in $1/\varepsilon$, and it is much stronger than the general bound for weak $\varepsilon$-nets in $\mathbb{R}^d$.

### 3.1 Generalization of our result

Theorem 1(b) generalizes as follows:

**Theorem 9.** Let $X$ be a convex curve in $\mathbb{R}^d$, and let $S$ be a family of convex bodies in $\mathbb{R}^d$, with the property that among every $p$ elements of $S$, two meet at a point of $X$. Then, there exists a point $z \in \mathbb{R}^d$ intersecting a $\Omega(p^{-1})$-fraction of the elements of $S$, for some constant $j = d^2/2 + O(d)$.

As a result, $S$ has a transversal of size $O(p^{j'})$ for some constant $j' = d^2/2 + O(d^2)$.

For comparison, the bound for $\text{HD}_d(p,d+1)$ obtained by Alon and Kleitman is only $O(p^{j''})$ for $j'' = d^2 + O(d)$.

The proof of Theorem 9 proceeds like the proof of Theorem 1(b), with the following main changes:

Instead of Observation 4 we use the following Lemma:

**Lemma 10** (Alon et al. [3]). Let $X$ be a convex curve in $\mathbb{R}^d$, and define

$$j = \begin{cases} \left(\frac{d^2 + d + 2}{2}\right), & \text{d even;} \\ \left(\frac{d^2 + 1}{2}\right), & \text{d odd.} \end{cases} \tag{2}$$

Let $A$ be a set of $j$ points on $X$. Note that $A$ partitions $X$ into $j + 1$ intervals if $d$ is odd, or $j$ intervals if $d$ is even.

Then, there exists a point $p \in \text{conv}(A)$ with the following property: For every set $B \subset X$ that contains a point in each of the above-mentioned intervals, we have $p \in \text{conv}(B)$.

In our application of the Lemma, $A$ plays the role of the separator points, and $B$ plays the role of the points colored with color $i$. Hence, instead of quadruples of separator points, we consider $j$-tuples.

A color $i$ is now spread out if there exist $j + 1$ points for $d$ odd, or $j$ points for $d$ even, colored with color $i$, such that all the pairwise between these points are at least $\alpha N$. Then,

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3Alon et al. state this lemma specifically for the moment curve, but it is true for any convex curve.
exactly one of the following is true: Either color $i$ is spread out, or all instances of color $i$ occur in at most $j$ intervals for $d$ odd, or $j - 1$ intervals for $d$ even, each of length at most $\alpha N$.

The probability of a random $j$-tuple of separators piercing a spread-out color is now $\Omega(\alpha^d)$ for $d$ odd, and $\Omega(\alpha^{d-1})$ for $d$ even. Instead of setting $\alpha = \gamma/300$, we set $\alpha = c_d \gamma$ for a small enough positive constant $c_d$.

The remaining details are left to the reader.

4 Conclusion

Figure 1 (right) shows a family of seven convex sets, every pair of which meet at a point of the unit circle, that requires a transversal of size 3. The points $a, \ldots, g$ are uniformly spaced along the unit circle, except for $f$, which has been moved a bit towards $e$. The seven sets are the convex hulls of $abc$, $cde$, $efa$, $bdf$, $adg$, $beg$, $cfg$, respectively. If 2 points were enough to pierce all the triangles, then at least one point must intersect 4 triangles. There are three regions which are overlaps of 4 triangles (the darkest shades of gray in the figure). But in each case, there are three triangles left that cannot be intersected with a single point.

We believe that the true bound for this problem is less than 10.

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