Averaged dynamics of time-periodic advection diffusion equations in the limit of small diffusivity

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Abstract

We study the effect of advection and small diffusion on passive tracers. The advecting velocity field is assumed to have mean zero and to possess time-periodic stream lines. Using a canonical transform to action-angle variables followed by a Lie-transform, we derive an averaged equation describing the effective motion of the tracers. An estimate for the time validity of the first-order approximation is established. For particular cases of a regularized vortical flow we present explicit formulas for the coefficients of the averaged equation both at first and at second order. Numerical simulations indicate that the validity of the above first-order estimate extends to the second order.

Key words: Advection-diffusion, Lie-averaging, persistent patterns
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1 Introduction

The characterization of the behavior of a passive but diffusing scalar advected by a prescribed, smooth velocity field has been the subject of intensive research going back at least as far as Batchelor [3]. Above and beyond the obvious practical importance in applications ranging from micro-mixers to global climate dynamics, ‘scalar turbulence’ as exhibited by solutions of the linear advection diffusion equation also provides an avenue for insight into the structure of the Navier-Stokes equations [20].

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Despite the linearity of the governing equation, complete characterization of scalar solutions, especially the asymptotic decay of such solutions for vanishing diffusivity, continues to pose considerable difficulties even when restricted to the case of planar flows. Here we are concerned with the so-called Batchelor regime, where the spatial scale of the velocity ($l_v$) is assumed to be much larger than the diffusive length scale ($l_κ$). For the scalar turbulence problem with periodic velocity fluctuations in time and space, rigorous homogenization techniques \[4, 15, 17\] can be applied to compute effective, renormalized diffusivities on large time and space ($L \gg l_v$) scales assuming that the initial distribution of the scalar field satisfies the scale separation $l_s \ll l_v$. On the other hand, the situation for scalar fields with variations commensurate with both the velocity field and the domain size, ($l_s \sim l_v \sim L \gg l_κ$) requires different techniques.

The present paper is motivated in part by the phenomenon of persistent patterns, termed ‘strange eigenmodes’ \[18\], that occur under the action of periodic stirring. Such patterns, characterized by exponential decay of the scalar variance and self-similar evolution of scalar density functions, have been observed both numerically and experimentally \[22\], \[5\]. Theoretical predictions of the decay rates of the scalar and the connection between the observed eigenmodes and the phase space of the underlying advection dynamics have been investigated \[1, 7, 10, 19, 21\], most often in the context of non-linear maps.

In the strange eigenmode regime, the decay of the scalar contrast can be studied via Floquet theory. While well established for ODE’s, the existing theory for parabolic PDE’s requires that the PDE satisfies a restrictive spectral gap condition which the advection diffusion equation fails for vanishing diffusivity (see \[14\]). An alternate approach, shown by Chow et al. \[6\] for one dimensional parabolic equations, is to prove the existence of an inertial manifold for the system and then apply ordinary Floquet theory to the inertial form. Our goal here is less ambitious, but a potential first step in this direction. Following Krol \[13\], we propose a formal averaging procedure for the advection-diffusion equation when the velocity field is has zero mean and possesses time-periodic stream lines. The approach is perturbative, making explicit use of the disparity of time-scales between the advective and diffusive operators. Advection fields of the form $u = u(\xi, t) = \bar{u}(\xi) f(t)$ guarantee that, in the case of vanishing diffusion, the time-dependent system can be solved using action-angle variables, and that tracer trajectories will be time-periodic. The explicit solution in action-angle coordinates allows the original equation to be written in a form suitable for averaging. By applying Lie transform techniques, we derive an approximate averaged equation. The fact that this resulting equation - in contrast to the original problem - has time-independent coefficients, facilitates the theoretical and numerical analysis tremendously. The use of the Lie transform also allows relatively straight-forward computation of higher order corrections.
The form of the paper is as follows. In section 2 we consider the transformation which places the advection-diffusion equation in a form suitable for averaging. Lie transform techniques are used to average the equation in section 3. A proof of the convergence of solutions of the averaged equations to those of the original time dependent problem is given in section 4. An application to a specific flow field, a periodically modulated, regularized vortex, along with numerical comparisons of the solutions are given in sections 5 and 6.

2 Action-angle variables

We consider the advection diffusion equation in the following form

\[ c_t + (u \cdot \nabla) c - \kappa \nabla^2 c = 0. \]  

(1)

All functions depend on the spatial variable \( \xi = (x, y)^T \) and the time \( t \). We look at (1) as an initial value problem, assuming that

\[ c(0, \xi) = c^{(s)}(\xi) \]  

(2)

is a known function. Incompressibility \( \nabla \cdot u = 0 \) implies that the given velocity field \( u \) is derived from a stream function \( \Psi \) such that

\[ u(\xi, t) = \nabla^\perp \Psi(\xi, t) \]  

(3)

where \( \nabla^\perp \equiv (\partial_y, -\partial_x) \).

We assume that the stream function \( \Psi \) is of the particular form

\[ \Psi(\xi, t) = \bar{\Psi}(\xi)f(t) \]  

(4)

where the function \( f \) is periodic in time with period \( T \). For consistency in the averaging which follows, we also require that \( \langle f \rangle = \frac{1}{T} \int_0^T f(t)dt = 0 \).

A standard non-dimensionalization of (1) with velocity, length and time scales given respectively by \( (U, L, T) \) gives

\[ \frac{1}{St} (u \cdot \nabla) c - \epsilon \nabla^2 c = 0 \]  

(5)

where the dimensionless groups are the Strouhal number, \( St = UT/L \), the ratio of the forcing period to the advective time-scale, and \( \epsilon = T\kappa/L^2 \), the ratio of the forcing period to the diffusive time-scale. Throughout, we assume \( St = O(1) \) and \( \epsilon \ll 1 \). To clarify notation, we take \( u = u/St \) throughout.

Our aim is to derive an equation of the form

\[ \bar{c}_t + \mathcal{L} \bar{c} = 0 \]  

(6)
with a time-independent local linear operator $L$ that can be used to construct approximative solutions to the initial value problem given by (2) or (5), in the limit of small diffusivity $\kappa \ll 1$. Due to the fact that the leading order evolution is given by the periodically varying advection operator, we cannot apply averaging techniques directly. Instead, we seek a transformation to the Lagrangian frame which results in a new equation where the coefficients of both the advective and diffusive operators are zero-mean periodic functions of time. The transformed equation is then in a form suitable for averaging.

For the restricted class of flow fields considered, the proper transformation is simply to action-angle coordinates of the underlying, conservative advection equation. Introduce the function $F$ as

$$F(t) = \int_0^t f(t') \, dt'$$

and write the tracer coordinate $\xi$ as a function of $F$. We then obtain the autonomous Hamiltonian system

$$\frac{d\xi}{dF} = \nabla_\perp \bar{\Psi}(\xi).$$

Since this system integrable, there exists a canonical transformation [2]

$$C : (x, y) \to (J, \theta)$$

such that the advection-diffusion equation (1) becomes

$$c_t - f(t)\omega(J)c_\theta - \epsilon (\Gamma : \nabla \nabla + \delta \cdot \nabla) c = 0$$

with a matrix $\Gamma = \Gamma(\theta, J)$ and a vector $\delta = \delta(\theta, J)$, that are solely determined by the canonical transformation $C$. The advantage of the representation (10) lies in the fact that the evolution of the unperturbed problem is linear and given by

$$J = J_0, \quad \theta = \theta_0 - \omega(J)F(t).$$

Therefore, we can now use these stream lines as coordinates via the transformation

$$c(t, J, \theta) = v \left( t, J, \bar{\theta} = \theta - \omega(J)F(t) \right)$$

and with the transformation rules

$$c_J = -\omega' F v_\bar{\theta} + v_J,$$
$$c_{JJ} = (\omega')^2 F^2 v_{\bar{\theta} \bar{\theta}} - 2\omega' F v_{\bar{\theta} J} - \omega'' F v_\bar{\theta} + v_{JJ}$$

and the rescaling of time as $\tau = \epsilon t$ the equation for $v$ takes the form

$$v_\tau = (\tilde{\Gamma} : \nabla \nabla + \bar{\delta} \cdot \nabla) v.$$
Since the only explicit time dependence in the coefficients $\tilde{\Gamma}$ and $\tilde{\delta}$ is given in terms of the $T$-periodic function $F$, the equation (13) is now suitable for averaging.

3 Lie transform averaging

In order to average (13), we use a technique based on Lie transforms first developed in the finite-dimensional context [16] and then applied to cases involving an infinite number of degrees of freedom [9,12]. The basic idea of a Lie transform is to use a near identity transform of the type

$$v = \exp(\phi \cdot \nabla_L) V .$$

(14)

The linear operator, $\phi \cdot \nabla_L$, is chosen to eliminate the explicit time dependence of the coefficient of an equation

$$v_\tau = X(v,\tau)$$

(15)
in order to obtain an equation with time-independent coefficients of the form

$$V_\tau = Y(V) .$$

(16)

Since the functionals $X$ and $Y$ depend on $v$ and all its spatial derivatives, the operator $\phi \cdot \nabla_L$ will be defined in our case as

$$\phi \cdot \nabla_L = \sum_{n,m} \phi_{nx,my} \frac{\partial^{(n+m)}}{\partial V_{nx,my}}$$

(17)

where $\phi_{nx,my} = \partial^{(n+m)} \phi / \partial x^n \partial y^m$ and $V_{nx,my} = \partial^{(n+m)} V / \partial x^n \partial y^m$ respectively.

The subscript at $\nabla_L$ distinguishes this operator from the usual $\nabla$. The generating function $\phi$ also depends on $V$ and all its derivatives. The idea is that the explicit time dependence will be kept in $\phi$ rather than in the equation for $V$, hence $\phi$ will also depend periodically on $\tau$. The general transformation rule under which (15) transforms to (16) using (14) is [11]

$$Y \cdot \nabla + \left( \frac{\partial}{\partial \tau} e^{\phi \cdot \nabla_L} \right) e^{-\phi \cdot \nabla} = e^{\phi \cdot \nabla_L} (X \cdot \nabla_L) e^{-\phi \cdot \nabla_L}$$

(18)

Both terms can be conveniently expanded using the Campbell-Baker-Hausdorff formulae

$$\left( \frac{\partial}{\partial \tau} e^{\phi \cdot \nabla_L} \right) = \left( \phi_t + \frac{1}{2!}[\phi,\phi_t]_L + \frac{1}{3!}[\phi,[\phi,\phi_t]_L]_L + ... \right) \cdot \nabla_L$$

(19)

$$e^{\phi \cdot \nabla_L} (X \cdot \nabla_L) e^{-\phi \cdot \nabla_L} = \left( X + [\phi,X]_L + \frac{1}{2!}[\phi,[\phi,X]_L]_L + ... \right) \cdot \nabla_L$$

(20)
where the Lie commutator is defined through \([A, B]_L = (A \cdot \nabla_L)B - (B \cdot \nabla_L)A\) and again the subscript distinguishes the Lie commutator from the usual commutator. We now expand both \(Y\) and \(\phi\) in a series in the small parameter \(\epsilon\) as

\[
Y = Y_0 + Y_1 + ..., \quad \phi = \phi_1 + \phi_2 + ...
\]  

(21)

where \(O(Y_n) = O(\phi_n) = \epsilon^n\) and differentiation by \(\tau\) lowers the order of \(\phi_n\) by one

\[
O(\partial \phi_n / \partial \tau) = \epsilon^{n-1}.
\]  

(22)

The equation for \(Y\) can then be solved order by order. At the leading order, we find

\[
Y_0 + \frac{\partial \phi_1}{\partial \tau} = X
\]  

(23)

and averaging this equation yields directly \(Y_0 = \langle X \rangle\) due to the periodicity of \(\phi_1\).

The transformed advection-diffusion equation (13) can be written in the form

\[
v_\tau = \tilde{L}v, \quad \tilde{L} = \tilde{\Gamma} : \nabla \nabla + \tilde{\delta} \cdot \nabla.
\]  

(24)

Averaging this equation immediately yields at the leading order

\[
V_\tau = \langle \tilde{L} \rangle V.
\]  

(25)

Here, and in what follows, \(\langle ... \rangle\) denotes averaging over one period. The averaged equation is then

\[
V_\tau = \left( \langle \tilde{\Gamma} \rangle : \nabla \nabla + \langle \tilde{\delta} \rangle \cdot \nabla \right) V.
\]  

(26)

Thus, the time-dependent coefficient are simply replaced by their time averages. The leading order of the generating function \(\phi_1\) is found as

\[
\phi_1 = L_1 V \equiv \left( \int_0^\tau \tilde{L} - \langle \tilde{L} \rangle \right) V.
\]  

(27)

Introducing \(\Gamma_1\) and \(\delta_1\) as

\[
\frac{d\Gamma_1}{d\tau} = \tilde{\Gamma} - \langle \tilde{\Gamma} \rangle, \quad \frac{d\delta_1}{d\tau} = \tilde{\delta} - \langle \tilde{\delta} \rangle
\]  

(28)

we can write \(L_1\) explicitly as

\[
L_1 = \Gamma_1 : \nabla \nabla + \delta_1 \cdot \nabla.
\]  

(29)

Higher order corrections can be calculated in an elegant way using the Campbell-Baker-Hausdorff formulae. For the second order term in the expansion of \(Y\), for example, we find
\[ Y_1 = \frac{1}{2} \langle [L_1 V, L_1 V]_L \rangle + \langle L_1 V, \langle \tilde{L} \rangle V \rangle_2 = \left( \langle \tilde{L} L_1 \rangle - \langle L_1 \langle \tilde{L} \rangle \rangle \right) V \] (30)

where the last equality follows after using the definition of the Lie commutator and integration by parts. Collecting first and second order, we obtain as averaged equation for \( V \)

\[ V_\tau = \left( \langle \tilde{L} \rangle + \epsilon \left( \langle \tilde{L} L_1 \rangle - \langle L_1 \langle \tilde{L} \rangle \rangle \right) \right) V \] (31)

4 An averaging theorem for parabolic differential equations

In the previous section, we applied a technique based on Lie transforms to average the equation (13), and we arrived at the equation (25). Here, we state and prove rigorously a theorem on averaging of parabolic partial differential equations which is due to Krol [13]. We assume that the differential operators in (13) are given by

\[ \tilde{\Gamma} = \epsilon \left( [a_{ij}(x, y, t)]^2 \right)_{i,j=1} \]

and

\[ \tilde{\delta} = \epsilon (b_1(t, x, y), b_2(t, x, y)), \]

where \( a_{ij}, b_i \in C^\infty(\mathbb{R}^2 \times [0, \infty)) \), and \( [a_{ij}] \) is symmetric and uniformly positive definite, i.e, there exists \( \theta > 0 \) such that for all \( \xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \) and \( (x, y, t) \in \mathbb{R}^2 \times [0, \infty) \) we have

\[ \sum_{i,j=1}^{2} a_{ij}(x, y, t)\xi_i\xi_j \geq \theta |\xi|^2. \]

In the following, let \( \tau_0 = \mathcal{O}(1/\epsilon) \), and let \( \| \cdot \|_\infty \) denote the usual supremum norm on either \( \mathbb{R}^2 \) or \( \mathbb{R}^2 \times [0, \tau_0] \), depending on the context.

**Theorem 4.1** Let \( v \) and \( V \) be solutions to the Cauchy problems \( v(0) = V(0) = v_0 \in C^\infty(\mathbb{R}^2) \) for the equations (13) and (25), respectively. Then

\[ \| v - V \|_\infty = \mathcal{O}(\epsilon). \]

**Proof:** First note that the existence and the uniqueness of bounded solutions \( v, V \) on \( C^2(\mathbb{R}^2 \times [0, \tau_0]) \) is well established (see [8]). Also, since (25) is autonomous, the derivatives \( V_\alpha \) of \( V \) also satisfy an autonomous parabolic differential equation with the same second order differential operator, however with different smooth and bounded first and zero order coefficients:

\[ V_\alpha_\tau = \left( \langle \tilde{\Gamma} \rangle : \nabla \nabla + \epsilon (b_1(\alpha)(x, y), b_2(\alpha)(x, y)) \cdot \nabla + \epsilon f(\alpha)(x, y) \right) V_\alpha. \]
The Phragmèn-Lindelöf principle for parabolic partial differential equations implies that
\[ \|V_\alpha\|_\infty \leq \|(v_0)_\alpha\|_\infty + \tau_0 \epsilon \|f(\alpha)\|_\infty = O(1). \]

Let us now define a near-identity transformation
\[ \hat{V}(x, y, \tau) = V(x, y, \tau) + \left[ \int_0^\tau (\tilde{L}(s) - \langle \tilde{L} \rangle) \, ds \right] V(x, y, \tau). \]

Since the integrand is \(T\)-periodic with zero average, the equation reads actually
\[ \hat{V}(x, y, \tau) = V(x, y, \tau) + \left[ \int_{\tau/T}^\tau (\tilde{L}(s) - \langle \tilde{L} \rangle) \, ds \right] V(x, y, \tau), \]
and it is an easy observation that \(\|\hat{V} - V\|_\infty = O(\epsilon).\)

On the other hand, one easily verifies that \(\hat{V}\) satisfies the equation
\[ \hat{V}_\tau = \tilde{L}(\tau) \hat{V} + \tilde{M}(\tau)V, \]
where \(\tilde{M}(\tau) = \int_0^\tau (\tilde{L}(s) - \langle \tilde{L} \rangle)(\tilde{L}(s) - \langle \tilde{L} \rangle) \, ds\), and the initial-value condition \(\hat{V}(0) = v_0\). Notice that \(\tilde{M}\) is a \(T\)-periodic fourth order operator with smooth bounded coefficients of order \(O(\epsilon^2)\). Consequently, the difference \(\hat{V} - v\) satisfies the equation
\[ (\hat{V} - v)_\tau = \tilde{L}(\tau)(\hat{V} - v) + \tilde{M}(\tau)V, \]
and the initial condition \((\hat{V} - v)(0) = 0\). By the Phragmèn-Lindelöf principle for parabolic partial differential equations, we conclude \(\|\hat{V} - v\|_\infty \leq \|\tilde{M}(\cdot)V\|_\infty \tau_0 = O(\epsilon)\). This concludes the proof.

We remark that similar techniques can be applied to solutions of the second order Lie-averaged equations leading to \(O(\epsilon^2)\) error estimates.

5 Regularized vortical flow field

To illustrate the theory and to give a comparison to numerical simulations, we consider the particular case of a regularized vortical flow field whose stream function is given by
\[ \Psi(t, x, y) = \ln \left( \sqrt{a^2 + x^2 + y^2} \right) f(t). \]  

This flow represents perhaps the simplest example for studying the interplay between diffusion and nonlinear, time-periodic advection.
Since \( r^2 = x^2 + y^2 \) is a constant of motion, the unperturbed stream lines are given in Cartesian coordinates as

\[
x(t) = \cos(\omega(r) F(t)) x_0 + \sin(\omega(r) F(t)) y_0 \\
y(t) = -\sin(\omega(r) F(t)) x_0 + \cos(\omega(r) F(t)) y_0
\]

where

\[
\omega(r) = \frac{1}{a^2 + r^2}.
\]  

(33)

Obviously, we can take as a canonical transform to action-angle variables the usual transformation to polar coordinates \((x, y) \rightarrow (r, \theta)\) and in these coordinates, the advection-diffusion equation (1) is written for this particular flow field as

\[
\begin{align*}
ct - \omega(r) f(t) c_\theta &= \epsilon \left( \Delta c + \frac{1}{r} c_r + \frac{1}{r^2} c_{\theta \theta} \right) \\
\end{align*}
\]  

(34)

The transformed equation (13) becomes (using \( t \) instead of the rescaled \( \tau \))

\[
v_t = \epsilon \left( \Delta v + F \left( \left( \frac{\omega'}{r} + \omega'' \right) v_\theta + 2 \omega' v_\theta r \right) + F^2 (\omega')^2 v_\theta \theta \right)
\]  

(35)

Using the previous results, we obtain at leading order

\[
V_t = \epsilon \left( \Delta V + \langle F \rangle \left( \left( \frac{\omega'}{r} + \omega'' \right) V_\theta + 2 \omega' V_\theta r \right) + \langle F^2 \rangle (\omega')^2 V_\theta \theta \right)
\]  

(36)

As shown in the Appendix, the leading order contributions in Cartesian coordinates produce a time independent advection field with spatially dependent rotation and source-like terms. The averaged diffusivity tensor is usually full and symmetry breaking. The relative importance of the symmetry breaking terms depends upon the explicit form of the time dependence through the ratio of \( \langle F \rangle \) and \( \langle F^2 \rangle \).

For this flow-field, it is not difficult to compute corrections at second order as well. In order to make our notation more efficient, we introduce the two operators \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \)

\[
\mathcal{L}_1 \equiv \left( \frac{\omega'}{r} + \omega'' \right) \partial_\theta + 2 \omega' \partial_\theta \partial_r
\]  

(37)

\[
\mathcal{L}_2 \equiv (\omega')^2 \partial_\theta \partial_\theta
\]  

(38)

and introduce \( F_1 \equiv F \) and \( F_2 \equiv F^2 \). The first order averaged equation (36) becomes then

\[
V_t = \epsilon \left( \Delta V + \langle F_1 \rangle \mathcal{L}_1 V + \langle F_2 \rangle \mathcal{L}_2 V \right)
\]
Applying now (28), we introduce the functions $G_1$ and $G_2$ that can be found explicitly from $F_1$ and $F_2$ as

$$G_j(t_0) = \int_0^{t_0} (F_j(\tau) - \langle F_j \rangle) d\tau, \quad j = 1, 2.$$ (39)

At second order in $\epsilon$ the equation is

$$V_t = \epsilon (\Delta V + \langle F_1 \rangle \mathcal{L}_1 V + \langle F_2 \rangle \mathcal{L}_2 V) + \epsilon^2 ([\Delta, \langle G_1 \rangle \mathcal{L}_1] V + [\Delta, \langle G_2 \rangle \mathcal{L}_2] V$$

$$+ ((\langle F_1 G_1 \rangle - \langle F_1 \rangle \langle G_1 \rangle) \mathcal{L}_1^2 V + (\langle F_2 G_2 \rangle - \langle F_2 \rangle \langle G_2 \rangle) \mathcal{L}_2^2 V$$

$$+ ((\langle F_1 G_2 \rangle - \langle F_2 \rangle \langle G_1 \rangle) \mathcal{L}_1 \mathcal{L}_2 V + (\langle F_2 G_1 \rangle - \langle F_1 \rangle \langle G_2 \rangle) \mathcal{L}_2 \mathcal{L}_1 V)$$ (40)

where $[A, B] \equiv AB - BA$ denotes the usual commutator.

6 Numerical Simulations

We can now integrate both the original problem (1) and the averaged equation in first order (36) numerically and compare their solutions. In our numerical simulations, we use a standard Adams-Bashforth-Moulton Method in Fourier space, the Fourier transformations are done using FFTW. We work back in Cartesian space and for this purpose, we compare solutions at Poincaré-sections where $\mathcal{P}$ is zero, $\theta = \tilde{\theta}$ and (36) is written in Cartesian coordinates. The explicit form of (36) in Cartesian coordinates is given in the appendix.

Throughout, we consider a single initial condition of the form

$$c^{(s)}(x, y) = x \exp(-br^2)$$ (41)

where the constants $a$ and $b$ were chosen as $a = 1.0$ and $b = 4.0$.

As shown in (36), the form of the averaging depends explicitly on the nature of $f(t)$. For the particular case of $f(t) = \sin(t)$, we find $\langle F \rangle = 1$ and $\langle F^2 \rangle = 3/2$. Fig. 1 shows both the initial condition and the evolution after ten periods for this choice of $f(t)$.

In the absence of the time-dependent advection field, the tracer field will obey the purely diffusive equation

$$c^{(vis)}_t = \epsilon \Delta c^{(vis)}, \quad c^{(vis)}(0, \xi) = c^{(s)}(\xi)$$ (42)

and the diffusion will simply spread the initial distribution out and preserve its symmetries. In the presence of the time-dependent vortical field, however, the particles will move back and forth within one period and the interplay of the time-dependent trajectories with the diffusion will give rise to a breaking of the symmetry of the initial distribution resulting in a “twist” that can be clearly
Fig. 1. Evolution of tracers in the time-periodically modulated vortical flow field. The figure on the left shows the initial condition, the figure on the right shows the result of the tracer motion after 10 periods.

seen in the right figure of Fig. 1. In order to determine how well the averaged equation (36) captures the differences between the purely diffusive case and the case with both time-dependent advection and diffusion, the difference of the purely viscous case and the solution of (1), hence $c - c^{(\text{vis})}$ and (b) the difference of the purely viscous case and the approximation $c^{(\text{av})}$ constructed using (36), hence $c^{(\text{av})} - c^{(\text{vis})}$ are plotted in Fig. 2. The first order approximation accurately captures the overall dynamics of the full, time-dependent problem.

Fig. 2. Comparison of the prediction of the averaged equation (36) to the evolution of the time-dependent equation (1). We plot the difference between the purely viscous solution and the solution incorporating the effects of the time-dependent advection field. The figure on the left shows $c - c^{\text{vis}}$ where $c$ is the solution of (1). The figure on the right shows $c^{(\text{av})} - c^{(\text{vis})}$, where $c^{(\text{av})}$ has been found using the averaged equation (36). The averaged equation is obviously able to capture the leading order impact of the time-dependent velocity field.

To quantify the accuracy of the approximation, we consider the time-evolution
of the canonical $L^2$-norm of the differences by defining

$$\|c - c^{(av)}\| = \left( \frac{\int_{\mathbb{R}^2} |c - c^{(av)}|^2 \, dx \, dy}{\int_{\mathbb{R}^2} |c|^2 \, dx \, dy} \right)^{1/2}$$  \hspace{1cm} (43)$$

Figure 3a shows that this error is small and approximately constant over the first 10 periods whereas the corresponding error between $c$ and $c^{(vis)}$ is comparatively large and growing exponentially in time. Figure 3b indicates that, at least for the case where $f(t) = \sin(t)$, the solutions of the averaged equation converge to those of the full equation faster than $\epsilon$.

![Figure 3a](image1)  \hspace{1cm} ![Figure 3b](image2)

Fig. 3. (a) Difference norms for the case $f(t) = \sin(t)$ and $\epsilon = 0.005$. (b) Average error between the first order averaged equation and the full solution versus $\epsilon$ for $f(t) = \sin(t)$. The error scales like $\epsilon^{1.8}$.

We examine the role of $f(t)$ in the averaged dynamic by setting $f(t) = \cos(t)$. This choice implies that $\langle F \rangle = 0$, leading to near degeneracy of the first order corrections. In this case, the time independent advection terms produced by
the averaging procedure do not contribute to ’twisting’ the scalar evolution. The symmetry breaking terms in the averaged diffusivity tensor are identically zero at first order. The effect of this near-degeneracy for $\langle F \rangle = 0$ is clearly seen in the comparison of panels (a) and (b) in Fig. 4. A comparison of the difference norms, shown in Fig. 5(a), indicates that the first order averaged equation is only a marginal improvement on the purely viscous solution at short times. However, as shown in Fig. 5(b), solutions to the first order averaged equations continue to converge to the true solution with decreasing $\epsilon$ although the convergence rate, $\sim \epsilon^{1/2}$, is considerably slower than that observed when $f(t) = \sin(t)$.

![Image](image-url)

Fig. 4. Solutions for the case $f(t) = \cos(t)$ and $\epsilon = 0.010$ after 10 periods. (a) Full solution, (b) first order averaged solution, (c) second order averaged solution, (d) difference between first and second order averaged.

For $\langle F \rangle = 0$, second order contributions are clearly important. Referring back to (40), this situation also leads to a relatively simple form for the second order expression. The coefficient in front of $\mathcal{L}_1$ vanishes, and in this particular case for $f(t) = \cos(t)$, (40) simplifies to

$$V_t = \epsilon \left( \Delta V + \frac{1}{2} \mathcal{L}_2 V \right) + \epsilon^2 \left( [\Delta, \mathcal{L}_1] V + \frac{1}{2} [\mathcal{L}_1, \mathcal{L}_2] V \right)$$

(44)
Evaluating this equation explicitly we find

\[
V_t = \epsilon \left( \Delta V + (\omega')^2 V_{\bar{\theta}\bar{\theta}} \right) + \epsilon^2 \left( \left( \frac{\omega'}{r^3} - \frac{\omega''}{r^2} + 2 \frac{\omega'''}{r} + \omega^{(iv)} \right) V_{\bar{\theta}} ight)
+ 4 \left( \frac{\omega''}{r} + \omega''' \right) V_{\bar{\theta}r} + \left( 4 \frac{\omega'}{r^3} - 2 (\omega')^2 \omega'' \right) V_{\bar{\theta}\bar{\theta}} + 4 \omega'' V_{\bar{\theta}rr} \right) \tag{45}
\]

As shown in the Appendix, this leads to \( O(\epsilon) \) symmetry breaking contributions to both the average advection terms and diffusivity tensor as well as contributions in the form of higher, third order, spatial derivatives. Numerically, such terms are easily computed spectrally. An example of a second order solution is shown in panel (c) of Fig. 4. Obviously, the second order solution is a clear improvement on the first order approximation and the difference between the two, shown in panel (d), demonstrates the restoration of advective twist at higher order. Figure 5 indicates both the increase in accuracy and the expected increase in convergence rate for the second order approximation.

7 Discussion

We have proposed a scheme for formally transforming the advection diffusion equation, in the limit of small diffusivity, into a form suitable for averaging. We have given an explicit means of averaging the transformed equation and proven the convergence of solutions of the averaged, time-independent approximation to the full dynamics. Throughout, however, we deal only with a restricted class of advecting fields, namely those which are zero-mean and possess time-periodic stream lines. While such fields are inherently integrable, and hence explicitly non-chaotic, the results presented are of interest in the study of ‘strange eigenmodes’ of the advection diffusion equation in the Batchelor regime. First, the emergence of strange eigenmodes is independent of the integrability or non-integrability of the underlying flow [5], and indeed, the simple example considered here produces non-trivial periodic patterns. Secondly, the results shown for even these extremely simple cases point to the delicate relationship between the non-linearity of the conservative advection operator and small diffusivity. The analysis points to fundamental differences between the dynamics of continuous time flows and discrete time maps. Indeed, for the periodically modulated vortex considered, the averaged dynamics depends strongly on the phase of the single frequency periodic modulation, a fact completely lost when considering the Poincaré map of the flow which is simply the identity.

In the case of a mean-free advection field with periodic stream lines, the trans-
Fig. 5. (a) Difference norms for the case $f(t) = \cos(t)$ and $\epsilon = 0.005$ for both first and second order averaging. (b) Average error between the averaged equation and the full solution versus $\epsilon$ for $f(t) = \cos(t)$. The first order solution is shown by the solid curve, the second order by the dashed curve. For comparison, the first order results for $f(t) = \sin(t)$ are shown in the light dot-dashed line.

form to action-angle variables of the Lagrangian flow was found to be the appropriate transformation for deriving accurate time-averaged dynamics in the limit of small diffusivity. Finding transformations with similar properties for other classes of advection fields will likely provide a means for understanding the interplay between advection and small diffusion.
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A The averaged equation in Cartesian coordinates

In order to convert (36) back to Cartesian coordinates, we note first that it follows directly from the transformation rules to polar coordinates that

\[ r \partial_r = x \partial_x + y \partial_y, \quad \partial_\theta = x \partial_y - y \partial_x \]

which, after straightforward calculations, yield the following transformation rules for the higher operators occurring in (36):

\[
\begin{align*}
\partial_{\theta} \partial_{\theta} &= -x \partial_x - y \partial_y \\
&+ y^2 \partial_x \partial_x - 2xy \partial_x \partial_y + x^2 \partial_y \partial_y \\
r \partial_{\theta} \partial_{r} &= x \partial_y - y \partial_x \\
&- xy \partial_x \partial_x + (x^2 - y^2) \partial_x \partial_y + xy \partial_y \partial_y
\end{align*}
\]

Rescaling time as \( \tau = \epsilon t \), we find for (36) then

\[ V_\tau + (\Xi \cdot \nabla)V = K : \nabla \nabla V \quad \text{(A.1)} \]

where the advection vector \( \Xi \) and is given as

\[
\Xi_1(x, y) = \langle F \rangle \left( \left( \frac{\omega'}{r} + \omega'' \right) + 2 \frac{\omega'}{r} \right) y + 2 \langle F^2 \rangle (\omega')^2 x \\
\Xi_2(x, y) = -\langle F \rangle \left( \left( \frac{\omega'}{r} + \omega'' \right) + 2 \frac{\omega'}{r} \right) x + 2 \langle F^2 \rangle (\omega')^2 y
\]

and the diffusion matrix \( K \) becomes

\[
\begin{align*}
K_{11} &= -2 \frac{\omega'}{r} xy \langle F \rangle + \langle F^2 \rangle (\omega')^2 y^2 \\
K_{12} &= K_{21} = \frac{\omega'}{r} \langle F \rangle (x^2 - y^2) - \langle F^2 \rangle (\omega')^2 xy \\
K_{22} &= 2 \frac{\omega'}{r} xy \langle F \rangle + \langle F^2 \rangle (\omega')^2 x^2
\end{align*}
\]
This together with the formulas
\[
\omega' = \frac{-2r}{(a^2 + r^2)^2}, \quad \frac{\omega'}{r} + \omega'' = 4 \frac{r^2 - a^2}{(r^2 + a^2)^3}
\]
gives directly the coefficients of (36) in Cartesian coordinates. Note that all operators give a contribution to the averaged advection field in Cartesian coordinates.

For the second-order, the corresponding terms are more complicated. The explicit transformations of the operators are given by
\[
r^2 \partial_r \partial_r = 2xy \partial_y^2 + 2(x^2 - y^2) \partial_x \partial_y - 2xy \partial_x^2
\]
\[
+ xy \partial_y^2 + (x^2 - 2xy) \partial_x^2 \partial_y + (2x^2y - y^3) \partial_x \partial_y^3 - x^2y \partial_x^3
\]
\[
\partial_\theta^3 = -x \partial_y + y \partial_x
\]
\[
+ 3xy \partial_x^2 + 3(y^2 - x^2) \partial_x \partial_y - 3xy \partial_y^2
\]
\[
+ x^3 \partial_y^3 - 3x^2y \partial_x \partial_y^2 + 3xy \partial_x^2 \partial_y - y^3 \partial_x^3
\]
Note in particular that the first term in the transformation of \( \partial_\theta^3 \) corresponds to \(- \partial_\theta \), hence is one example of a term that will have a "twisting" effect on the solution and that can lead to symmetry breaking. We can now rewrite (45) in Cartesian coordinates (rescaling time \( \tau = \epsilon t \) as before) and obtain:

\[
V_\tau + (\Xi \cdot \nabla) V = K_{11} V_{xx} + 2K_{12} V_{xy} + K_{22} V_{yy}
\]
\[
+ M_{111} V_{xxx} + M_{112} V_{xxy} + M_{122} V_{xyy} + M_{222} V_{yyy}
\]

(A.2)

where the coefficients of the advection field \( \Xi \) are given by

\[
\Xi_1 = f_5 x + \epsilon \left( (f_1 - f_3)y + f_2 \frac{y}{r} \right)
\]
\[
\Xi_2 = f_5 y + \epsilon \left( (f_3 - f_1)x - f_2 \frac{x}{r} \right),
\]
and the diffusion matrix \( K \) results in

\[
K_{11} = 1 + f_3 y^2 + \epsilon \left( 3f_3 xy - f_2 \frac{xy}{r} - 2f_4 \frac{x y}{r^2} \right)
\]
\[
K_{22} = 1 + f_3 x^2 + \epsilon \left( -3f_3 xy + f_2 \frac{xy}{r} + 2f_4 \frac{x y}{r^2} \right)
\]
\[
2K_{12} = -2f_5 xy + \epsilon \left( f_2 \frac{x^2 - y^2}{r} + 3f_3 (y^2 - x^2) + 2f_4 \frac{x^2 - y^2}{r^2} \right).
\]

As for the tensor \( M \) yielding higher-order contributions, we find
\[ M_{111} = -f_3 y^3 - f_4 \frac{x^2 y}{r^2} \]
\[ M_{222} = f_3 x^3 + f_4 \frac{xy^2}{r^2} \]
\[ M_{112} = 3f_3 xy^2 + f_4 \frac{x^3 - 2xy^2}{r^2} \]
\[ M_{122} = -3f_3 x^2 y - f_4 \frac{y^3 - 2x^2 y}{r^2} \]

and the coefficients \( f_k \) are explicitly written as

\[
\begin{align*}
    f_1 &= \frac{\omega'}{r^3} - \frac{\omega''}{r^2} + 2 \frac{\omega'''}{r} + \omega^{(iv)} = 64 \frac{r^4 - 4a^2r^2 + a^4}{(r^2 + a^2)^5} \\
    f_2 &= 4 \left( \frac{\omega''}{r} + \omega''' \right) = -8 \frac{9r^4 - 14a^2r^2 + a^4}{(r^2 + a^2)^4} \\
    f_3 &= 4 \frac{\omega'}{r^3} - 2(\omega')^2 \omega'' = -8 \frac{16a^2r^2 - 3r^4}{r^2(r^2 + a^2)^2} \frac{2r^2}{(r^2 + a^2)^3} \\
    f_4 &= 4\omega'' = 4 \frac{2r^2}{(r^2 + a^2)^3} \\
    f_5 &= \frac{1}{2}(\omega')^2 = \frac{2r^2}{(a^2 + r^2)^4}.
\end{align*}
\]

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