A family of singular ordinary differential equations of third order with an integral boundary condition

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Abstract

We establish in this paper the equivalence between a Volterra integral equation of second kind and a singular ordinary differential equation of third order with two initial conditions and an integral boundary condition, with a real parameter. This equivalence allow us to obtain the solution to some problems for nonclassical heat equation, the continuous dependence of the solution with respect to the parameter and the corresponding explicit solution to the considered problem.

Keywords : Singular ordinary differential equation of third order, integral boundary condition, Volterra integral equation, explicit solution, nonclassical heat equation.

2010 Mathematics Subject Classification : 34A05, 34B10, 34B16, 35C15, 35K05, 35K20, 45D05, 45E10.

1 Introduction

We consider the following family of singular ordinary differential equations of third order with an integral boundary condition, indexed by a parameter \( \lambda \in \mathbb{R} \) given by

\[
\begin{align*}
\left\{
\begin{array}{l}
y^{(3)}(t) - \lambda^2 y(t) = \frac{\lambda}{2\sqrt{\pi} t^{3/2}}, \quad t > 0, \\
y(0) = 1, \quad y'(0) = 0, \quad y''(1) = -\frac{\lambda}{\sqrt{\pi}} + \lambda^2 \int_0^1 y(t) dt.
\end{array}
\right.
\end{align*}
\]

(1.1)

where \( y^{(n)} \) denotes the \( n \)-derivative of the function \( y \).

Singular boundary value problems arise very frequently in fluid mechanics and in other branches of applied mathematics. There are results on the existence and asymptotic estimates of solutions for third order ordinary differential equations with singularly perturbed boundary value problems, which depend on a small positive parameter see for example [16, 19, 27], on third order ordinary differential equations with singularly perturbed boundary value problems and with nonlinear coefficients or boundary conditions see for example [3, 12, 29, 50], on third order ordinary differential equations with nonlinear boundary value problems see for example [18, 28], on existence results for third order ordinary differential equations see for example [17, 24], and particularly third order ordinary differential equations with integral boundary conditions see for example [2, 4, 7, 20, 39, 32, 37, 39].

In the last years there are several papers which consider integral or nonlocal boundary conditions on different branches of applications, e.g. for the heat equations see for example

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for the wave equations \cite{37}, for the second order ordinary differential equations see for example \cite{5, 31, 33, 44, 52, 53, 54}, for the fourth order ordinary differential equations see for example \cite{41, 51}, for higher order ordinary differential equations see for example \cite{25}, for fractional differential equations see for example \cite{23, 32, 46}.

Our goal is to prove in Section 2 that the system (1.1) is equivalent to the following Volterra integral equation of second kind

\[
y(t) = 1 - \frac{2\lambda}{\sqrt{\pi}} \int_0^t y(\tau) \sqrt{t - \tau} d\tau, \quad t > 0, \quad (\lambda \in \mathbb{R})
\]  

(1.2)

which allows us to obtain the solution to some problems for nonclassical heat equation for any real parameter $\lambda$ (see \cite{4, 8, 9, 11, 40, 43, 45}).

In Section 3, we establish the dependence of the family of singular ordinary differential equations of third order (1.1) with respect to the parameter $\lambda \in \mathbb{R}$ by using the equivalence with the Volterra integral equation (1.2).

\section{Equivalence and existence results}

Preliminary, we give some results useful in the next sections.

\textbf{Lemma 2.1.} We have the following properties

\[ \int_0^t \left( \int_0^\tau y(\xi) d\xi \right) d\tau = \int_0^t y(\tau)(t - \tau) d\tau \]  

(2.1)

\[ \int_0^t \left( \int_0^{\xi} y(\tau)(t - \tau) d\tau \right) d\xi = \int_0^t y(\tau)(t - \tau)^2 d\tau \]  

(2.2)

\[ \int_0^t \frac{y(\tau)}{\sqrt{t - \tau}} d\tau = 2\sqrt{t} + 2 \int_0^t y'(\tau) \sqrt{t - \tau} d\tau \]  

(2.3)

\[ \int_\sigma^t \frac{d\tau}{\sqrt{t - \tau} \sqrt{t - \sigma}} = \pi. \]  

(2.5)

\textit{Proof.} The first three properties (2.1)-(2.3) follow from the simple integration process. To prove (2.4) we use the change of variable $\tau = \sigma + (t - \sigma)\xi$ then we obtain

\[
\int_\sigma^t \frac{\sqrt{\tau - \sigma}}{\sqrt{t - \tau}} d\tau = (t - \sigma) \int_0^1 \sqrt{\frac{\xi}{1 - \xi}} d\xi = (t - \sigma) \int_0^1 \xi^{\frac{3}{2} - 1}(1 - \xi)^{\frac{1}{2} - 1} d\xi
= (t - \sigma) B\left(\frac{3}{2}, \frac{1}{2}\right) = (t - \sigma) \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)} = \frac{\pi}{2} (t - \sigma),
\]
where $B$ and $\Gamma$ are the known Beta and Gamma functions defined by

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad x > 0, \quad y > 0,$$

$$\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}dt, \quad x > 0,$$

with the well known relations

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \Gamma(x+1) = x\Gamma(x) \quad \forall x > 0, \quad \Gamma(\frac{1}{2}) = \sqrt{\pi}, \quad \Gamma(n+1) = n! \quad \forall n \in \mathbb{N}.$$

To prove (2.5) we use the same change of variable, so we obtain

$$\int_\sigma^t \frac{d\tau}{\sqrt{t-\tau}\sqrt{\tau-\sigma}} = \int_0^1 \frac{d\xi}{\sqrt{\xi(1-\xi)}} = B\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} = \pi.$$

\[\Box\]

**Theorem 2.2.** $y$ is a solution to the singular ordinary differential equation (1.1) if and only if $y$ is a solution to the Volterra integral equation (1.2).

**Proof.** Firstly, we consider that $y$ is a solution to the singular ordinary differential equation (1.1). Then, by using an integration in variable $t$ we obtain

$$y^{(2)}(t) = y^{(2)}(0) + \lambda^2 \int_0^t y(\tau)d\tau - \frac{\lambda}{\sqrt{\pi}t}, \quad t > 0. \quad (2.6)$$

thus

$$y^{(2)}(1) = y^{(2)}(0) + \lambda^2 \int_0^1 y(\tau)d\tau - \frac{\lambda}{\sqrt{\pi}}.$$

And using the integral boundary condition

$$y^{(2)}(1) = -\frac{\lambda}{\sqrt{\pi}} + \lambda^2 \int_0^1 y(t)dt$$

so $y^{(2)}(0) = 0$. Thus taking this new condition into account, from (2.6) by using an integration in variable $t$, the condition $y'(0) = 0$ and (2.1) we get

$$y'(t) = \lambda^2 \left( \int_0^t y(\tau)d\tau \right) d\tau - \frac{2\lambda\sqrt{t}}{\sqrt{\pi}}$$

$$= \lambda^2 \int_0^t y(\tau)(t-\tau)d\tau - \frac{2\lambda\sqrt{t}}{\sqrt{\pi}}, \quad t > 0. \quad (2.7)$$

Finally, from (2.7) by using another integration in variable $t$, and the condition $y(0) = 1$, we obtain

$$y(t) = 1 + \lambda^2 \left( \int_0^t y(\tau)(t-\tau)d\tau \right) d\tau - \frac{4\lambda^3}{3\sqrt{\pi}t^{3/2}}$$

$$= 1 + \lambda^2 \int_0^t y(\tau)(t-\tau)^2d\tau - \frac{4\lambda}{3\sqrt{\pi}t^{3/2}}, \quad t > 0. \quad (2.8)$$

3
We can not arrive directly to the Volterra equation (1.2), but we can define the auxiliary function

\[ \varphi(t) = y(t) - 1 + \frac{2\lambda}{\sqrt{\pi}} \int_0^t y(\tau)\sqrt{t-\tau} \, d\tau \]  

(2.9)

and now our goal is to prove that \( \varphi = 0 \). We have \( \varphi(0) = 0 \), by using the boundary \( y(0) = 1 \).

Now, we compute the first derivative of \( \varphi \) using the property (2.3), we get

\[ \varphi'(t) = y'(t) + \frac{\lambda}{\sqrt{\pi}} \int_0^t \frac{y(\tau)}{\sqrt{t-\tau}} \, d\tau = y'(t) + \frac{\lambda}{\sqrt{\pi}} \left( 2\sqrt{t} + 2 \int_0^t y'(\tau)\sqrt{t-\tau} \, d\tau \right) \quad t > 0. \]  

(2.10)

From the other hand, by using (2.9), (2.7), (2.10) and the property (2.4) we obtain

\[ \int_0^t \frac{\varphi(\tau)}{\sqrt{t-\tau}} \, d\tau = \int_0^t \frac{y(\tau)}{\sqrt{t-\tau}} \, d\tau - 2\sqrt{t} + 2 \int_0^t y'(\tau)\sqrt{t-\tau} \, d\tau = \int_0^t \frac{y(\tau)}{\sqrt{t-\tau}} \, d\tau + \frac{\sqrt{\pi}}{\lambda} y'(t) = \frac{\sqrt{\pi}}{\lambda} \varphi'(t), \quad t > 0. \]

That is

\[ \varphi'(t) = \frac{\lambda}{\sqrt{\pi}} \int_0^t \frac{\varphi(\tau)}{\sqrt{t-\tau}} \, d\tau, \quad t > 0, \]  

(2.11)

thus \( \varphi'(0) = 0 \). Therefore, we have

\[ \varphi'(t) = \frac{\lambda}{\sqrt{\pi}} \int_0^t \frac{\varphi(t-\tau)}{\sqrt{\tau}} \, d\tau, \quad t > 0, \]  

(2.12)

and then we obtain

\[ \varphi^{(2)}(t) = \frac{\lambda}{\sqrt{\pi}} \int_0^t \frac{\varphi'(t-\tau)}{\sqrt{\tau}} \, d\tau = \frac{\lambda}{\sqrt{\pi}} \int_0^t \frac{\varphi'(\tau)}{\sqrt{\tau}} \, d\tau, \quad t > 0, \]  

(2.13)

thus \( \varphi^{(2)}(0) = 0 \), and so on we obtain \( \varphi^{(n)}(0) = 0 \) for all \( n \in \mathbb{N} \), then this part holds.

Secondly, we consider that \( y \) is a solution of the Volterra integral equation (1.2), then we have the condition \( y(0) = 1 \) which is automatically satisfied.

Then, by derivation of (1.2) and by using the property (2.4) we have
\[ y'(t) = -\frac{\lambda}{\sqrt{\pi}} \int_0^t \frac{y(\tau)}{\sqrt{t - \tau}} d\tau \]
\[ = -\frac{\lambda}{\sqrt{\pi}} \left( 2\sqrt{t} - \frac{2\lambda\sqrt{t}}{\sqrt{\pi}} \left( \int_0^t \frac{1}{\sqrt{t - \tau}} \left( \int_0^\tau y(\sigma) \sqrt{\tau - \sigma} \, d\sigma \right) d\tau \right) \right) \]
\[ = -\frac{2\lambda}{\sqrt{\pi}} \sqrt{t} + \frac{\lambda^2}{\sqrt{\pi}} \int_0^t \left( \int_0^\tau \frac{y(\sigma)}{\sqrt{t - \sigma}} d\sigma \right) d\tau \]
\[ = -\frac{2\lambda}{\sqrt{\pi}} \sqrt{t} + \lambda^2 \int_0^t y(\sigma)(t - \sigma) d\sigma, \quad t > 0, \quad (2.14) \]

and the boundary condition \( y'(0) = 0 \) holds. Therefore from (2.14) we have

\[ y^{(2)}(t) = -\frac{\lambda}{\sqrt{\pi} t} + \lambda^2 \int_0^t y(\tau) d\tau, \quad t > 0, \quad (2.15) \]

thus for \( t = 1 \) we get the integral boundary condition.

Finally, from (2.15) we have

\[ y^{(3)}(t) = \frac{\lambda}{\sqrt{\pi}} t^{-3/2} + \lambda^2 y(t), \quad t > 0, \quad (2.16) \]

so the singular ordinary differential equation (1.1) holds, thus the proof of the theorem is complete.

**Theorem 2.3.** The solution of the Volterra integral equation (1.2) is given by the following expression

\[ y(t) = I(t) - \sqrt{\frac{\pi}{3}} J(t), \quad t > 0, \quad (2.17) \]

with

\[ I(t) = \sum_{n=0}^{+\infty} \frac{(\lambda^{2/3} t)^{3n}}{(3n)!} \quad (2.18) \]

\[ J(t) = \sum_{n=0}^{+\infty} \frac{(2\lambda^{2/3} t)^{3(2n+1)}}{(3(2n+1))!!} \quad (2.19) \]

are series with infinite radii of convergence and we use the definition

\[ (2n + 1)!! = (2n + 1)(2n - 1)(2n - 3) \cdots 5 \cdot 3 \cdot 1. \]

for compactness expression.

**Proof.** By using the Adomian method [1, 48] we propose, for the solution of the Volterra integral equation (1.2), the following serie of expansion functions given by

\[ y(t) = \sum_{n=0}^{+\infty} y_n(t) \]
and we obtain the following recurrence expansions:

\[ y_0(t) = 1, \quad y_n(t) = \frac{2\lambda}{\sqrt{\pi}} \int_0^t y_{n-1}(\tau) \sqrt{t - \tau} \, d\tau, \quad \forall n \geq 1. \]

Then, following [9] we obtain (2.17) where \( I(t) \) and \( J(t) \) are given by (2.18) and (2.19) respectively, and the result holds.

The solution of the Volterra integral equation (1.2) is the key in order to obtain the solution of the following nonclassical heat conduction problem given by

\[
\begin{align*}
  u_t(x, t) - u_{xx}(x, t) &= -\lambda \int_0^t u_x(0, \tau) d\tau, \quad x > 0, \quad t > 0, \\
  u(0, t) &= 0, \quad t > 0, \\
  u(x, 0) &= h_0 > 0, \quad x > 0,
\end{align*}
\]

with a parameter \( \lambda \in \mathbb{R} \). Then the solution of the problem above is given by

\[
u(x, t) = h_0 \text{erf}\left(\frac{x}{2\sqrt{t}}\right) - \lambda \int_0^t \text{erf}\left(\frac{x}{2\sqrt{t - \tau}}\right) U(\tau) d\tau
\]

where \( U(t) \) is given by

\[
U(t) = \frac{h_0}{\sqrt{\pi}} \int_0^t \frac{g(\tau)}{\sqrt{t - \tau}} d\tau
\]

and \( g \) is the solution of the Volterra integral equation (1.2). Moreover, the heat flux on \( x = 0 \) is given by

\[
u_x(0, t) = U'(t) = \frac{h_0}{\sqrt{\pi t}} - h_0 \lambda \int_0^t g(\tau) d\tau, \quad t > 0.
\]

For the complete proof see [9].

\[ \Box \]

3 Dependence of the solution with respect to \( \lambda \)

From now on, we will consider that the solution to the singular ordinary differential equation of third order with an integral boundary condition (1.1) or equivalently the solution of the Volterra integral equation (1.2) depends also on the parameter \( \lambda \in \mathbb{R} \).

We consider that \( t \rightarrow g_\lambda(t) \) be the solution of the Volterra integral equation (1.2) for the parameter \( \lambda \). For \( \varepsilon \in (0, 1) \) be a fixed real number and \( T > 0 \), let consider the parameter \( \lambda \) such that

\[
|\lambda| \leq \lambda_{\varepsilon, T} = \frac{3\sqrt{\pi}}{4} \frac{\varepsilon}{T^{3/2}},
\]

and we define the norm

\[
\|g\|_T = \max_{0 \leq t \leq T} |g(t)|.
\]

Therefore, we obtain the following dependence results.
Theorem 3.1. We have the boundedness
\[ \|g_\lambda\|_T \leq \frac{1}{1 - \varepsilon}, \quad \forall \lambda : \ |\lambda| \leq \lambda_{\varepsilon,T}. \tag{3.2} \]

Moreover the application \( \lambda \mapsto g_\lambda(t) \) defined from \([-\lambda_{\varepsilon,T}, \lambda_{\varepsilon,T}]\), to \( C([0,T]) \) is Lipschitzian.

Proof. From the Volterra integral equation (1.2) we obtain
\[
|g_\lambda(t)| \leq 1 + 2|\lambda| \|g_\lambda\|_t \int_0^t \sqrt{t-\tau} \, d\tau \leq 1 + 4 \frac{\lambda_{\varepsilon,T} T^{3/2}}{3\sqrt{\pi}} \|g_\lambda\|_T
\]
and by using (3.1) follows (3.2). Moreover, consider \( g_i(t) \) the solution of the Volterra integral equation (1.2) for \( \lambda_i \) (\( i = 1, 2 \)), such that
\[
|\lambda_i| \leq \lambda_{\varepsilon,T}.
\]
Then, we have
\[
|g_2(t) - g_1(t)| \leq \frac{4}{3\sqrt{\pi}} |\lambda_2 - \lambda_1| \|g_1\|_T + |\lambda_2| \|g_2 - g_1\|_T.
\]
Therefore, we get
\[
\|g_2 - g_1\|_T \leq \frac{4}{3\sqrt{\pi}} \frac{T^{3/2}}{(1 - \varepsilon)^2} |\lambda_2 - \lambda_1|
\]
thus the result holds.

Now, we obtain the dependence of the solution to the nonclassical heat conduction problem (2.20)-(2.22) with respect to the parameter \( \lambda \). We consider that \( U_\lambda \) and \( u_\lambda \) are given respectively by
\[
U_\lambda(t) = \frac{h_0}{\sqrt{\pi}} \int_0^t \frac{g_\lambda(\tau)}{\sqrt{t-\tau}} \, d\tau
\]
and
\[
u_\lambda(x,t) = h_0 \text{erf} \left( \frac{x}{2\sqrt{t}} \right) - \lambda \int_0^t \text{erf} \left( \frac{x}{2\sqrt{t-\tau}} \right) U_\lambda(\tau) \, d\tau.
\]

Then, we obtain the following results:

Theorem 3.2. We have the boundedness
\[ \|U_\lambda\|_T \leq \frac{2h_0 T^{1/2}}{\sqrt{\pi} (1 - \varepsilon)}, \quad \forall \lambda : \ |\lambda| \leq \lambda_{\varepsilon,T}. \tag{3.6} \]
Moreover, the application $\lambda \mapsto U_\lambda(t)$, from $[-\lambda_cT, \lambda_cT]$ to $C([0,T])$ is Lipschitzian. We have also the following boundedness
\[ \|u_\lambda\|_{0, +\infty}[x\times[0,T]} \leq h_0 \left(1 + \frac{3\varepsilon}{2(1-\varepsilon)}\right) \quad \forall \lambda : \ |\lambda| \leq \lambda_cT, \quad (3.7) \]
the estimates
\[ \|u_\lambda - u_0\|_{0, +\infty}[x\times[0,T]} \leq \frac{2h_0 T^{3/2}}{\sqrt{\pi(1-\varepsilon)}} |\lambda|, \quad \forall \lambda : \ |\lambda| \leq \lambda_cT, \quad (3.8) \]
and that the application $\lambda \mapsto u_\lambda(x,t)$, from $[-\lambda_cT, \lambda_cT]$ to $C([0, +\infty\times[0,T])$ is Lipschitzian.
Proof. From (2.24) we have
\[ u_\lambda(x,t) = \lambda_cT \phi(x, \lambda_cT t) \quad (\lambda \mapsto \phi) \quad (3.6) \]
thus (3.6) holds. Consider now $U_i(t)$ given by (3.4), for $\lambda_i$ $(i = 1, 2)$ satisfying $|\lambda_i| \leq \lambda_cT$. We have
\[ |U_i(t)| \leq h_0 \frac{1}{\sqrt{\pi}} \left|\int_0^t \frac{d\tau}{\sqrt{t-\tau}}\right| \leq \frac{2h_0 T^{1/2}}{\sqrt{\pi(1-\varepsilon)}} \quad (3.8) \]
thus the application $\lambda \mapsto U_\lambda$ is Lipschitzian.
From (3.6) we have
\[ |u_\lambda(x,t)| \leq h_0 + t|\lambda||U_\lambda| \leq h_0 \left(1 + \frac{3\varepsilon}{2(1-\varepsilon)}\right), \quad \forall x \in [0, +\infty[, \quad (3.7) \]
thus (3.7) holds.
From (3.5) also, we have
\[ |u_\lambda(x,t) - u_0(x,t)| \leq t|\lambda||U_\lambda| \leq \frac{2h_0 T^{3/2}}{\sqrt{\pi(1-\varepsilon)}} |\lambda| \quad (3.9) \]
thus (3.9) holds.
Consider now $u_i(x,t)$ given by (3.5) for $\lambda_i$ $(i = 1, 2)$ satisfying $|\lambda_i| \leq \lambda_cT$. Then, we have
\[ |u_2(x,t) - u_1(x,t)| \leq t|\lambda_2 - \lambda_1||U_1| + t|\lambda_2||U_2 - U_1| \leq T|\lambda_2 - \lambda_1||U_1| + T|\lambda_2||U_2 - U_1| \leq \frac{2h_0 T^{1/2}}{\sqrt{\pi(1-\varepsilon)}} (T + \frac{\varepsilon\pi}{1-\varepsilon}) |\lambda_2 - \lambda_1| \quad (\forall x \in [0, +\infty[, \quad (3.10) \]
thus
\[ \|u_2 - u_1\|_{0, +\infty}[x\times[0,T]} \leq \frac{2h_0 T^{1/2}}{\sqrt{\pi(1-\varepsilon)}} \left(\frac{\varepsilon\pi}{1-\varepsilon} + T\right) |\lambda_2 - \lambda_1| \quad (3.11) \]
and the result holds.
Conclusion We have obtained the equivalence between a family of singular ordinary differential equations of third order with an integral boundary condition (1.1) and the Volterra integral equation (1.2) with a parameter $\lambda \in \mathbb{R}$. We have also given the explicit solution of these equations and then some nonclassical heat conduction problems can be solved explicitly, for any real parameter $\lambda$. Finally, we have established the dependence of the family of singular differential equations of third order with respect to the parameter $\lambda$.

Acknowledgements: This paper was partially sponsored by the Institut Camille Jordan St-Etienne University for first author, and the projects PIP # 0275 from CONICET-Austral (Rosario, Argentina) and Grant AFOSR-SOARD FA 9550-14-1-0122 for the second author.

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