SYMPLECTIC VIRTUAL LOCALIZATION OF
GROMOV-WITTEN INVARIANTS

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Abstract. We show that moduli spaces of stable maps admit virtual orbifold structure. The symplectic version of virtual localization formula is obtained.

Given a compact closed symplectic manifold \((M^{2n}, \omega)\) and an \(\omega\)-tamed almost complex structure \(J\), one can define the celebrated Gromov-Witten invariants using the moduli spaces of \(J\)-holomorphic curves. Such invariants were first discovered by Ruan-Tian on monotone manifolds\([18]\), then later defined on general manifolds independently by several different groups Fukaya-Ono\([8]\), Li-Tian\([13]\), Liu-Tian\([15]\), Ruan\([19]\) and etc. The breakthrough tool for their works is now well-known as virtual techniques. On the other hand, the algebraic version of the theory was first given by Li-Tian\([12]\).

Since the theory of Gromov-Witten invariants is set up, the computation of invariants has been one of the main issues of this area. One of the main tools of the computation is the localization technique. If the symplectic manifold admits a torus action, the action can be induced on the moduli spaces of \(J\)-holomorphic curves. Since the invariants are obtained via "integration" on the moduli spaces, Kontsevich observed that one may apply the Atiyah-Bott localization formula for computation\([10]\). To fulfill such an idea, we need to combine the Atiyah-Bott localization formula with virtual techniques. We call such a combination as the virtual localization. This has been done for algebraic varieties\([9]\). But such a formula has not been set-up in symplectic category. Our main goal of this paper is to prove a virtual localization formula on general symplectic manifolds. We remark that in this paper the group can act on the symplectic manifold as symplectomorphisms other than just Hamiltonian ones.

The ingredients of proving such a virtual localization formula are: (1), a modified gluing theory which provides smooth structures on moduli spaces, (2), virtual manifolds/orbifolds and (equivariant) integration theory on them. The abstract theory of virtual manifold/orbifolds has been established in \([6]\). In this paper, we mainly explain how to obtain smooth structures on moduli spaces via the gluing theory, and then generalize it to virtual moduli spaces accordingly.

The paper is organized as following: in Part I, we introduce some preliminary materials that is needed to understand the moduli spaces; in Part II, we describe the moduli spaces of the stable maps; in Part III we explain the full package of the gluing theory that provides a smooth structure on the moduli spaces; in the part IV, we develop the virtual theory on the moduli spaces and localization formula, at the end, as an application, we compute an example.

Acknowledge. The idea of this paper and that of \([6]\) was emerged 4 years ago. The drafts of papers have been written for quite a while, by some reason, they have not been completed until recently. First of all, special thanks to Y. Ruan and G.
Tian for their long time support on this project. During this long term preparation of papers, we would like to thank many people’s encouragement and discussion. The list includes G. Liu, K. Liu, M. Liu, W. Zhang, G. Zhao, Q. Zheng and etc. The material of the paper was explained as lectures in University of Wisconsin-Madison, Peking University. We would like to thank their hospitality. We would also like to W. Li, Y. Long and J. Robbin for their interests in the lectures.

Part I. Preliminary

1. Complex structures on \( \mathbb{R}^2 \)

1.1. Complex structures on \( \mathbb{R}^2 \). A complex structure \( j \) on \( \mathbb{R}^2 \) is a linear automorphism of \( \mathbb{R}^2 \) with \( j^2 = -1 \). It induces an orientation \( o(j) \) on \( \mathbb{R}^2 \) given by \( v \wedge jv \) for any \( 0 \neq v \in \mathbb{R}^2 \). Now fix an orientation \( o \) on \( \mathbb{R}^2 \). Set

\[
J_o(\mathbb{R}^2) = \{ j | j^2 = -1, o(j) = o \}.
\]

Fix a complex structure \( j_o \in J_o(\mathbb{R}^2) \). With a proper chosen basis, we may write \( j_o \) in terms of matrix as

\[
j_o = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

\((\mathbb{R}^2, j_o)\) can be identified with a complex plane \((\mathbb{C}, z)\) via

\[
\phi_o : \mathbb{R}^2 \to \mathbb{C}; \\
z = \phi_o(x, y) = x + y\sqrt{-1}.
\]

Let \( GL^+(2, \mathbb{R}) < GL(2, \mathbb{R}) \) be the subgroup that preserves \( o \). \( GL^+(2, \mathbb{R}) \) acts transitively on \( J_o(\mathbb{R}^2) \) via

\[
g \cdot j = gjg^{-1}.
\]

Via identification \( \phi_o \), \( GL(1, \mathbb{C}) \) is embedded in \( GL^+(2, \mathbb{R}) \) as a subgroup. Then

**Lemma 1.1.** \( J_o(\mathbb{R}^2) \cong GL^+(2, \mathbb{R})/GL(1, \mathbb{C}) \).

**Proof.** Since the isotropic group of the \( GL^+(2, \mathbb{R}) \) at \( j_o \) is \( GL(1, \mathbb{C}) \), the lemma follows. q.e.d.

1.2. Beltrami coefficients. Let \( f : (\mathbb{C}, z) \to (\mathbb{C}, w) \) be a linear isomorphism between two complex planes. Suppose

\[
w = f(z) = \alpha z + \beta \bar{z}.
\]

Then

\[
\mu_f = \alpha^{-1} \beta.
\]

is called the Beltrami coefficient of \( f \) with respect to coordinates \( z \) and \( w \).

Suppose that we change the coordinate of \( w \)-plane to \( \tilde{w} \)-plane by \( \tilde{w} = \gamma w, \gamma \in \mathbb{C} \). \( f \) is transformed to

\[
\tilde{f} : (\mathbb{C}, z) \overset{f}{\to} (\mathbb{C}, w) \overset{\gamma}{\to} (\mathbb{C}, \tilde{w}).
\]

We find

\[
\mu_{\tilde{f}} = \mu_f.
\]

This says that \( \mu_f \) is independent of the coordinate choice of \( w \)-plane.
Now suppose that we change the coordinate of \( z \)-plane to \( \tilde{z} \)-plane by \( \tilde{z} = \gamma^{-1}z, \gamma \in \mathbb{C} \). Then \( f \) is transformed to

\[
\tilde{f} : (\mathbb{C}, \tilde{z}) \to (\mathbb{C}, z) \to (\mathbb{C}, w).
\]

We have

\[
\mu_{\tilde{f}} = \mu f - \bar{\gamma}.
\]

This implies that

\[
(1.2) \quad \omega_f = \mu f \frac{d\tilde{z}}{dz}
\]

is invariant on the first plane. \( \omega_f \) is an \((-1,1)\)-form on \( z \)-plane. We call it the Beltrami form of \( f \).

Given a complex structure \( j \in J_{or}(\mathbb{R}^2) \) and an identification

\[
\phi : (\mathbb{R}^2, j) \to (\mathbb{C}, w),
\]

The identity map on \( \mathbb{R}^2 \) induces a map \( A_j \) via the diagram

\[
\begin{array}{ccc}
(\mathbb{R}^2, j_0) & \xrightarrow{id} & (\mathbb{R}^2, j) \\
\downarrow{\phi_0} & & \downarrow{\phi} \\
(\mathbb{C}, z) & \xrightarrow{A_j} & (\mathbb{C}, w).
\end{array}
\]

We define

\[
\mu : J_{or}(\mathbb{R}^2) \to \mathbb{C};
\]

\[
\mu(j) = \mu_{A_j}.
\]

This map is well defined since \( \mu(j) \) depends on \( \phi_0 \), but not on \( \phi \).

**Proposition 1.2.** \( \mu \) is injective and \( \text{Image}(\mu) = D \), the unit disk in \( \mathbb{C} \).

**Proof.** Since \( id \) (or \( A_j \)) is orientation preserving map, one can check that \( |\mu(j)| < 1 \). Hence \( \text{Image}(\mu) \subset D \).

\( \mu \) is injective: Suppose \( \mu(j_1) = \mu(j_2) \). We have diagram

\[
A_{j_2} : (\mathbb{C}, z) \xrightarrow{A_{j_1}} (\mathbb{C}, w_1) \xrightarrow{f_{12}} (\mathbb{C}, w_2),
\]

where \( f_{12} \) is defined by the equation. Suppose

\[
w_2 = f_{12}(w_1) = \alpha w_1 + \beta \bar{w}_1.
\]

Then one can check directly that

\[
\mu(j_1) = \mu(j_2) \iff \beta = 0.
\]

This says that \( f_{12} \) is holomorphic, and so \( j_1 = j_2 \).

\( \text{Image}(\mu) = D \): let \( \gamma \) be any complex number in \( D \), we solve \( j \) such that \( \mu(j) = \gamma \).

Suppose \( j = g^{-1}j_0g \). Then we have \( A_{j_0}^{-1} \) defined by

\[
\begin{array}{ccc}
(\mathbb{R}^2, j) & \xrightarrow{g} & (\mathbb{R}^2, j_0) \\
\downarrow{\phi} & & \downarrow{\phi_0} \\
(\mathbb{C}, w) & \xrightarrow{A_{j_0}} & (\mathbb{C}, z).
\end{array}
\]

By the definition of \( g \), \( A_{j_0}^{-1} \) is holomorphic. Furthermore

\[
\phi_0 g \phi_0^{-1} = A_g \circ A_{j_0}^{-1} : (\mathbb{C}, z) \to (\mathbb{C}, z).
\]
Hence, 
\[ \mu_{\phi_o \phi_o^{-1}} = \mu \lambda. \]
Suppose 
\[ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]
and \( \gamma = \alpha + \beta \sqrt{-1} \). Then 
\[ \phi_o g \phi_o^{-1}(z) = \left( \frac{a + d}{2} + \frac{c - b}{2} \sqrt{-1} \right) z + \left( \frac{a - d}{2} + \frac{b + c}{2} \sqrt{-1} \right) \bar{z}. \]

Now set 
\[ a = 1 + \alpha, d = 1 - \alpha, b = c = \beta. \]
We see that \( \det(g) = 1 - \alpha^2 - \beta^2 > 0 \), which says that \( g \in GL^+(2, \mathbb{R}) \), and 
\[ \mu_{\phi_o \phi_o^{-1}} = \gamma. \]
This solves \( \mu(j) = \gamma \). q.e.d.

1.3. A Universal family of \( J_{or}(\mathbb{R}^2) \). We combine the result of previous two subsections:
\[ GL^+(2, \mathbb{R})/GL(1, \mathbb{C}) \cong J_{or}(\mathbb{R}^2) \cong D. \]
The second isomorphism is given by \( \mu \) and \( \mu(j_o) = 0 \). The next proposition says that there exists a canonical section \( \sigma \) (with respect to \( j_o \)) for the principle bundle
\[ GL(1, \mathbb{C}) \rightarrow GL^+(2, \mathbb{R}^2) \]
\[ \downarrow \]
\[ D. \]

**Proposition 1.3.** \( \sigma(\gamma) = 1 - jj_o \) for \( \mu(j) = \gamma \).

**Proof.** Clearly \( j \sigma(\gamma) = \sigma(\gamma) j_o \). It remains to show that \( \sigma(\gamma) \in GL^+(2, \mathbb{R}) \).
Suppose \( j = g j_o g^{-1} \), where
\[ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \]
Without loss of generality, we assume that \( \det(g) = 1 \). Then
\[ j = \begin{pmatrix} ac + bd & -a^2 - b^2 \\ c^2 + d^2 & -ac - bd \end{pmatrix}, \]
and
\[ 1 - jj_o = \begin{pmatrix} 1 + a^2 + b^2 & ac + bd \\ ac + bd & 1 + c^2 + d^2 \end{pmatrix}. \]
Using the fact \( ad - bc = 1 \), we have \( \det(1 - jj_o) > 0 \). q.e.d.

Let \( \mathcal{R} \) be a tautological family of \( \mathbb{R}^2 \) with complex structure parameterized by \( D(\cong J_{or}(\mathbb{R}^2)) \): namely, we have
\[ r : \mathcal{R} = (D \times \mathbb{R}^2, \mathcal{J}) \rightarrow D, \]
where \( \mathcal{J} \) is the fiber-wise complex structure such that
\[ \mathcal{J}|_{r^{-1} \gamma} = \mu^{-1}(\gamma). \]

**Proposition 1.4.** There is a canonical trivialization with respect to \( j_o \)
\[ \Phi_o : \mathcal{R}_o := D \times (\mathbb{R}^2, j_o) \rightarrow \mathcal{R}. \]
Proof. We set $\Phi_0$ fiber-wise as
\[
\sigma(\gamma): \gamma \times (\mathbb{R}^2, j_0) \to \mathbb{R}^{-1}(\gamma).
\]

q.e.d.

2. Teichmüller spaces

2.1. Complex structures on $\Sigma_g$. Let $\Sigma$ be an oriented genus-$g$ surface with orientation $o(g)$. A complex structure $j$ on $\Sigma$ is a family of complex structures on $\mathbb{R}^2 \cong T_x \Sigma$ parameterized by $x \in \Sigma$. A complex structure $j$ induces an orientation $o(j)$ on $\Sigma$. Set
\[
J(\Sigma) = \{ j | o(j) = o(g) \}
\]
to be the set of complex structures on $\Sigma$ that is compatible with the given orientation $o(g)$.

Now fix a point $j_0 \in J(\Sigma)$. Let $\Omega^{-1,1}(\Sigma, j_0)$ denote the space of $(-1,1)$-forms on Riemann surface $(\Sigma, j_0)$. A $(-1,1)$-form is locally expressed in the form $f(z) d\bar{z}/dz$ in terms of local complex coordinate $(\mathbb{C}, z)$. For any $j \in J(\Sigma)$, it yields Beltrami coefficients $\mu(j(x))$ point-wisely, which gives a $(-1,1)$-form, denoted by $\omega_j$, on $\Sigma$. Set
\[
B\Omega^{-1,1}(\Sigma, j_0) = \{ \omega \in \Omega^{-1,1}(\Sigma, j_0) | \| \omega \| < 1 \}.
\]
Here $\| \omega \| = \max_x |f(x)|$ for $\omega = f(x) d\bar{z}/dz$. Then by proposition 1.2, we have

Proposition 2.1. $j \mapsto \omega_j$ gives an isomorphism $J(\Sigma) \cong B\Omega^{-1,1}(\Sigma, j_0)$. Moreover $\omega_{j_0} = 0$.

Conversely, given a form $\omega \in B\Omega^{-1,1}(\Sigma, j_0)$, we denote the corresponding complex structure by $j_{\omega}$.

2.2. Teichmüller spaces $T_g$. Here we give an informal review of Teichmüller spaces $T_g$.

$T_0$ consists of only one element, i.e, the standard sphere $S^2 = \mathbb{C} \cup \{ \infty \}$.

$T_1 \cong \mathbb{H}$, the upper half plane of $\mathbb{C}$. Given $\lambda \in \mathbb{H}$, we define a lattice
\[
L_\lambda = \{ m + n\lambda | m, n \in \mathbb{Z} \};
\]
then the corresponding torus is
\[
T_\lambda = \frac{\mathbb{C}}{L_\lambda}.
\]

For $g \geq 2$, define
\[
T_g = \frac{J(\Sigma)}{Diff^+_f(\Sigma)} = \frac{B\Omega^{-1,1}(\Sigma, j_0)}{Diff^+_f(\Sigma)}.
\]
Here $Diff^+_f(\Sigma)$ is the component of $o(g)$-preserving-diffeomorphism group $Diff(\Sigma)$ that contains 1. A classical theory on Teichmüller spaces says that the quotient
\[
\frac{B\Omega^{-1,1}(\Sigma, j_0)}{Diff^+_f(\Sigma)}
\]
has a global slice. Let $H^{-1,1}(\Sigma, j_0) \subset \Omega^{-1,1}(\Sigma, j_0)$ denote the space of holomorphic forms and set
\[
BH(j_0) = B\Omega^{-1,1}(\Sigma, j_0) \cap H^{-1,1}(\Sigma, j_0).
\]
Then
Theorem 2.2.
\[
\frac{\partial_1 \Omega^{-1,1}(\Sigma, j_0)}{Diff^+_0(\Sigma)} = BH(j_0).
\]

This theorem says that $BH(j_0)$ is a global slice of the quotient. By the Riemann-Roch theorem, we know that $BH(j_0)$ is a $6g - 6$ dimensional ball.

Corollary 2.3. $\dim T_g = 6g - 6, g \geq 2$.

We can also consider $T_{g,m}$, the Teichmüller space of genus $g$ Riemann surfaces with $m$ marked points. We give a complete list.

\[
\begin{align*}
T_{0,1} &= \{(S^2, \infty)\}; \\
T_{0,2} &= \{(S^2, 0, \infty)\}; \\
T_{0,3} &= \{(S^2, 0, 1, \infty)\}; \\
T_{0,m} &= T_{0,3} \times ((S^2 - \{0, 1, \infty\})^{m-3} - \Delta), m > 3; \\
T_{1,1} &= T_1 \times \{[0]\}, \text{ here } 0 \in \mathbb{C} \text{ and } [0] \text{ denote the point in tori}; \\
T_{1,m} &= T_{1,1} \times ((T_i - [0])^{m-1} - \Delta), m > 1 \text{ (refer this to } \text{uni}_1 \text{ in } \text{ §2.3}); \\
T_{g,m} &= T_g \times (\Sigma^m - \Delta).
\end{align*}
\]

In the first three terms, $S^2$ is identified with $\mathbb{C} \cup \{\infty\}$; $\Delta$ is the big diagonal of the product $X^m$.

2.3. Universal curves. By a universal curve, we mean a fibration

\[
\pi_{g,m} : \text{uni}_{g,m} \to T_{g,m}
\]

such that for $j \in T_{g,m}$ the fiber $\pi_{g,m}^{-1}(j)$ is the marked curve $j$. We show the existence of $\text{uni}_{g,m}$.

Case 1, $g = 0$. Then

\[
\begin{align*}
\text{uni}_0 &= \text{uni}_{0,0} = S^2; \text{uni}_{0,1} = (S^2, \infty); \\
\text{uni}_{0,2} &= (S^2, 0, \infty); \text{uni}_{0,3} = (S^2, 0, 1, \infty)
\end{align*}
\]

and

\[
\text{uni}_{0,m} = \text{uni}_{0,3} \times ((S^2 - \{0, 1, \infty\})^{m-3} - \Delta), m > 3.
\]

Case 2, $g = 1$. We first construct $\text{uni}_1$. Define an $\mathbb{Z} \times \mathbb{Z}$ action on $\mathbb{H} \times \mathbb{C}$ by

\[
(m, n) \cdot (\lambda, z) = (\lambda, z + m + n\lambda).
\]

Then

\[
\text{uni}_1 = \mathbb{H} \times \mathbb{C} \xrightarrow{\mathbb{Z} \times \mathbb{Z}} \mathbb{H}
\]

is the universal curve.

$\text{uni}_1$ can be topological trivialized to be $\mathbb{H} \times T_i, i = \sqrt{-1}$: define an $\mathbb{Z} \times \mathbb{Z}$ action on $\mathbb{H} \times \mathbb{C}$ by

\[
(m, n) \cdot (\lambda, z) = (\lambda, z + m + n\sqrt{-1}).
\]

Define the map

\[
\Phi : \mathbb{H} \times \mathbb{C} \to \mathbb{H} \times \mathbb{C} \\
\Phi(\lambda, z) = (\lambda, \phi_\lambda(z)),
\]

where $\phi_\lambda : \mathbb{C} \to \mathbb{C}$ is the linear map determined by $\phi_\lambda(1) = 1, \phi_\lambda(\sqrt{-1}) = \lambda$. Then the map is $\mathbb{Z} \times \mathbb{Z}$-equivariant. So $\Phi$ induces an isomorphism

\[
\Phi : \mathbb{H} \times T_i \to \text{uni}_1.
\]
In particular $\Phi(\lambda,[0]) = (\lambda,[0])$. We always assume $\text{uni}_1 = \mathbb{H} \times T_1$ from now on. Then

\[
\begin{align*}
\text{uni}_{1,1} &= \text{uni}_1 \times \{[0]\}; \\
\text{uni}_{1,m} &= \text{uni}_{1,1} \times ((T_1 - \{[0]\})^{m-1} - \Delta), m > 1.
\end{align*}
\]

**Case 3, $g \geq 2$.** We set $\text{uni}_g$ topologically to be

\[
\text{uni}_g = T_g \times \Sigma_g = BH \times \Sigma_g
\]

and the complex structure on the fiber over $\omega \in BH$ to be $j_\omega$. This gives $\text{uni}_g, g \geq 2$. Then set

\[
\text{uni}_{g,m} = \text{uni}_g \times (\Sigma^m - \Delta).
\]

We conclude that

**Theorem 2.4.** The universal curve $\pi : \text{uni}_{g,m} \to T_{g,m}$ exists.

3. **Moduli space** $M_{g,m}$

3.1. **Definitions.** By definition,

\[
\begin{align*}
T_g &= \frac{J(\Sigma_g)}{\text{Diff}^+ \,(\Sigma_g)}; \\
M_g &= \frac{J(\Sigma_g)}{\text{Diff}^+ \,(\Sigma_g)}.
\end{align*}
\]

The group $\Gamma_g = \text{Diff}^+ \,(\Sigma_g)/\text{Diff}^+ \,(\Sigma_g)$ is called the mapping class group. Then

\[
M_g = T_g/\Gamma_g.
\]

Similarly, one can define $M_{g,m}$, the moduli space of genus-$g$ curve with $m$-marked points.

When $g \leq 1$, $M_{g,m} = T_{g,m}$. When $g \geq 2$, $M_{g,m}$ are orbifolds. We give local descriptions for these orbifolds.

We recall some notions of orbifold. Let $X$ be an orbifold. For any $x \in X$, there exists a neighborhood $U_x$ of $x$ such that it is homeomorphic to $\mathbb{R}^n/G_x$ for some finite group $G_x$. The formal notion for these data is the so-called uniformization system $(V,G_x,\phi)$: here $V$ is a smooth manifold (usually is diffeomorphic to $\mathbb{R}^n$), $G_x$ acts smoothly on $V$ and

\[
\phi : V \xrightarrow{\pi} V/G_x \cong U_x.
\]

$G_x$ is called the isotropic group of $x$. Clearly, such a uniformization system describes the local of $x$.

Let

\[
(3.1) \quad \text{jo} = (\Sigma, j_\omega, x_{o1}, \ldots, x_{om}) \in M_{g,m}.
\]

We give a uniformization system for $j_\omega$. The isotropic group at $j_\omega$ is $\text{Aut}(j_\omega)$, the automorphism group of $j_\omega$. By an automorphism of $j_\omega$, we mean a bi-holomorphic maps of $(\Sigma, j_\omega)$ preserving $x_{o\ast}$’s. Recall that

\[
T_{g,m} = BH(j_\omega) \times (\Sigma^m - \Delta).
\]

$\text{Aut}(j_\omega)$ acts on this space naturally as

\[
\sigma \cdot (\omega, y_1, \ldots, y_m) = (\sigma_\ast(\omega), \sigma(y_1), \ldots, \sigma(y_m)).
\]
Then there exists a small Aut$(j_o)$-invariant neighborhood $\tilde{O}$ of $j_o$ such that $(\tilde{O}, \text{Aut}(j_o), \pi)$ is a uniformization system for $j_o$.

Moreover, Aut$(j_o)$ acts on uni$_{g,m}$ similarly. Hence

\begin{equation}
\text{uni}_{g,m}|_{\tilde{O}} \quad \text{Aut}(j_o)
\end{equation}

gives a universal curve for the neighborhood of $j_o$ in $M_{g,m}$.

3.2. Hyperbolic metrics. A curve in $M_{g,m}$ is called stable if and only if $2g + m \geq 3$. Let $j_o \in \mathcal{T}_{g,m}$ be a stable curve given as (3.1). We may assign it a hyperbolic metric. This is done as following: since $2g + m \geq 3$, the punctured surface $(\Sigma - \{x_{o1}, \ldots, x_{om}\}, j_o)$ admits a universal covering $\mathbb{H}$ such that the punctured surface is $\mathbb{H}/\Gamma$ for some Fuchsian group $\Gamma$. The hyperbolic metric on $\mathbb{H}$ induces a hyperbolic metric on this punctured Riemann surface.

Furthermore the neighborhood of $x_{oi}$ in $\Sigma$ has a nice description. Let $H_{\geq 1} = \{x + y\sqrt{-1} \in \mathbb{H} | y \geq 1\}$.

Then there exists a neighborhood of $x_{oi}$ (punctured at $x_{oi}$) that is identified with $H_{\geq 1}$ \langle $z \rightarrow z + 1$ \rangle.

We call the area the horocycle at $x_{oi}$. This area can be identified with the punctured disk $B^*(1)$ up to a rotation induced by:

$f(z) = c \exp(2\pi i z), c = \exp(2\pi)$.

From now on, by horocycle, we always refer it as $B^*(1)$. We note that Aut$(j_o)$ acts on $B^*(1)$ as rotations.

We deform the hyperbolic metric to be a local-flat metric such that it is flat in $B^*(5)$ and is rotational invariant in $B^*(1)$. So this new metric is still Aut$(j_o)$-invariant. We now assign the local-flat metrics fiber-wisely to uni$_{g,m}$. We denote the family metric to be $h$ and the metric on the fiber over $j$ as $h_j$. Then

**Proposition 3.1.** Let $j \in \mathcal{T}_{g,m}, \sigma \in \text{Aut}(j_o)$. Then

$h_{\sigma^*} = \sigma_* h_j$.

**Proof.** We may assign hyperbolic metrics $\hat{h}_j$ on each curve $\pi_{g,m}^{-1}(j)$ and get a family metric $\hat{h}$. By the property of hyperbolic metrics,

\begin{equation}
\hat{h}_{\sigma^*} = \sigma_* \hat{h}_j.
\end{equation}

Moreover, $\sigma$ preserves horocycles: it maps horocycle of $j$ to $\sigma_* j$ and behaves as rotations on $B^*(1)$. Hence, (3.3) is still true for $h$. q.e.d.

We always assume that uni$_{g,m}$ carries such a family of metric $h$ if $2g + m \geq 3$.

3.3. Local trivialization of universal curves. Recall that the local universal curve of $j_o \in M_{g,m}$ is given in the form (3.2). Topologically, the fiber is diffeomorphic to $(\Sigma, x_{o1}, \ldots, x_{om})$, or $\Sigma - \{x_{o1}, \ldots, x_{om}\}$ if we use the language of puncture curves. We now give a trivialization of uni$_{g,m}|_{\tilde{O}}$ when $m \geq 1$.

**Proposition 3.2.** There exists a smooth map

$\phi : \tilde{O} \times \Sigma \rightarrow \Sigma$

such that
(1) for any $j \in \tilde{O}$ with
\[\pi_{g,m}^{-1}(j) = (\Sigma, j, x_1, \ldots, x_m),\]
\[\phi(j, \ast) : \Sigma \to \Sigma \text{ is diffeomorphic and } \phi(x_{ok}) = x_k, 1 \leq k \leq m;\]
(2) $\phi(j, \ast)$ is holomorphic in small neighborhoods of $x_{ok}$ as a map
\[\phi(j, \ast) : (\Sigma, j_0) \to (\Sigma, j);\]
(3) $\phi$ is $\text{Aut}(j_0)$-equivariant in the sense
\[\phi(\sigma \cdot j, \sigma(z)) = \sigma(\phi(j, z)).\]

**Proof.** We explain for the case that $g \geq 2$. For $g = 0, 1$, the proof is simpler, we leave it to readers. For simplicity, we assume that $m = 1$. We have horocycles on $j_o$ and $j$ given by
\[\zeta_{j_o} : B^*(1) \to \Sigma - \{x_{o1}\},\]
and
\[\zeta_j : B^*(1) \to \Sigma - \{x_1\}.\]
As $x_1$ is close to $x_{o1}$, we may assume that $x_1 \in \zeta_{j_o}(B(75))$ and
\[\zeta_{j_o}(B(75)) \subset \zeta_j(B(1)).\]
Set $x'_1 = \zeta_{j_o}^{-1}(x_1)$. We define a map $\phi_j : B(75) \to B(75)$ by
\[\phi_j(z) = \begin{cases} z - x'_1, & \text{if } |z| \leq 0.25; \\ z, & \text{if } |z| \geq 0.5; \\ \eta(|\zeta_j(z)|)(z - x') + (1 - \eta(|\zeta_j(z)|))z, & \text{else} \end{cases}\]
Here $\eta(t)$ is a cut-off function that is 1 when $t \leq 0.25$ and is 0 when $t \geq 0.5$. We now set
\[\phi(j, \ast) = \zeta_j \phi_j \zeta_{j_o}^{-1}\]
on $\zeta_j(B(1))$ and extended it over $\Sigma$ by identity.

Conclusion (1) and (2) is obvious from the construction. It is well known that the horocycle $\zeta_j$ depends smoothly with respect to $j$, hence $\phi$ is smooth.

We now explain (3). It is clear that
\[\zeta_{\sigma \cdot j} = \sigma \circ \zeta_j.\]
Also note that $\sigma$ is a rotation in the unit disk. Then (3) can be verified directly.

q.e.d.

We can now use $\phi$ to trivialize the universal curve over the neighborhood of $j_o$. Define
\[\Phi_{j_o} : \tilde{O} \times j_o \to \text{uni}_{g,m} |_{\tilde{O}}\]
by
\[\Phi_{j_o}(j, \Sigma) = \phi(j, \Sigma).\]
In fact, claim (1) in the proposition already serves the purpose. Claim (3) implies that the trivialization is $\text{Aut}(j_0)$ equivariant. Claim (2) is an additional property that is needed later.
4. Deligne Mumford moduli space $\bar{M}_{g,m}$

4.1. Stable nodal curves. Let $\bar{M}_{g,m}$ be the Deligne-Mumford compactification of $M_{g,m}$. We call it Deligne-Mumford moduli space. This is a stratified space. The lower strata of $\bar{M}_{g,m}$ consists of equivalence classes of stable nodal curves. A nodal curve is a connected curve 

$$(\Sigma, j) = \bigcup_{k=1}^{c} (\Sigma_k, j_k), j = (j_1, \ldots, j_c),$$

with normal crossing singularities

$$\text{Sing}(\Sigma) = \{y_1, \ldots, y_s\}.$$ 

We call these $y_i, 1 \leq i \leq s$ the nodal points of $\Sigma$. A marked point on $\Sigma$ is a point $x \in \Sigma - \text{Sing}(\Sigma)$ Suppose we have $m$-marked points $\{x_1, \ldots, x_m\}$. Let 

$$j = (\Sigma, j, x_1, \ldots, x_m)$$

be a nodal curve with $m$-marked points.

Set 

$$D : \{1, \ldots, m\} \to \{1, \ldots, c\}$$

to be the map that assigns marked point $x_i$ to component $\Sigma_{D(i)}$. For each $y \in \text{Sing}(\Sigma)$, it is contained in two components $\Sigma_{c_1}$ and $\Sigma_{c_2}$. Here $c_1$ may equal to $c_2$.

We define the set $\text{comp}(y) = \{c_1, c_2\}$ (or, $\text{comp}(y) = \{c_1\}$ if $c_1 = c_2$).

Each component of this curve is

$$j_k = (\Sigma_k, j_k, \{x_i\}_{i \in D^{-1}(k)}, \text{Sing}(\Sigma) \cap \Sigma_k).$$

This is an one-component-curve maybe with nodal points. The nodal points are come from those singular $y$’s with $\text{comp}(y) = \{k\}$. Such a component admits a normalization $\mathcal{M}(j_k)$. Then the normalization of $j$ is defined to be the disjoint union

$$\mathcal{M}(j) := \coprod_{k=1}^{c} \mathcal{M}(j_k).$$

Recovering $j$ from $\mathcal{M}(j)$ is standard. It is given by a proper quotient map 

$$\pi : \mathcal{M}(j) \to \mathcal{M}(j)/\sim \cong j.$$ 

**Definition 4.1.** $j$ is stable if $\mathcal{M}(j_k)$ is stable for each $k$.

Two curves 

$$j = (\Sigma, x_1, \ldots, x_m) \text{ and } j' = (\Sigma', x'_1, \ldots, x'_m)$$

are equivalent if there exists a homeomorphism $\sigma : \Sigma \to \Sigma'$ such that $\sigma(x_i) = x'_i$ and the natural induced map $\mathcal{M}(\sigma) : \mathcal{M}(j) \to \mathcal{M}(j')$ is bi-holomorphic.

4.2. Data of stratum. Let $j$ be an $m$-marked stable nodal curve given by (4.1). We assign the following combinatoric data to this curve:

1. a (weighted) connected graph (with tails) $T$ (refer to item-2 for "weighted" and item-3 for "tail"): Let $V$ and $E$ be the set of vertices and edges of $T$ respectively, then each $k \in V$ stands for a component $\Sigma_k$ and $n \in E$ stands for a nodal point $y_n \in \text{Sing}(\Sigma)$;
(2) the genus $g_k$ of $\Sigma_k$ for each $k \in V$: $g_k$ is the weight of $k$ that is mentioned in item-1; the data of genus is denoted by $g = (g_1, \ldots, g_c)$.

Set

\begin{equation}
(4.3) \quad g = g(T) := \sum_{v=1}^{k} g_k + \text{rank} H_1(T);
\end{equation}

(3) a map

$D : \{1, \ldots, m\} \rightarrow \{1, \ldots, c\}$

mentioned in (4.2): for each $1 \leq j \leq m$ we assign it a tail, that is, for $D(j) = k$ we add $j$-th tail to vertex $k$.

We denote the data by $S = (T, D, g)$ and call it a stratum data in $\bar{M}_{g,m}$. Such a data is called stable if for each vertex $k$,

$$2g_k + \text{val}(k) \geq 3.$$ 

Here $\text{val}(k)$ is the valency of vertex $k$ (Tails are counted for valency). It is easy to check that

**Claim 4.2.** $j$ is stable if the data $S$ given by $j$ is stable.

On the other hand,

**Definition 4.3.** The genus of $j$ is defined to be $g = g(T)$. $j$ is called a stable $(g, m)$-curve.

We define $\bar{M}_{g,m}$ to be the set of equivalence classes of stable $(g, m)$-curves. This space admits a natural stratification given by data $S$’s: let $S = (T, D, g)$ be a stable data with $g = g(T)$, we define the stratum $M_S \subset \bar{M}_{g,m}$ to be the set of curves that give data $S$. The topology of $\bar{M}_{g,m}$ is not clear at the moment. However this is studied intensively ([?]). It is well known that $\bar{M}_{g,m}$ is a smooth orbifold of dimension $6g - 6 + 2m$ if $2g + m \geq 3$.

In the rest of the section. We describe the strata and their neighborhoods in $\bar{M}_{g,m}$ more carefully.

4.3. Some facts of data $S$. Let $S$ be a (stable) stratum data. There is an automorphism group $\text{Aut}(S)$ of $S$ defined as following

**Definition 4.4.** We say $\gamma \in \text{Aut}(S)$ if $\gamma : T \rightarrow T$ is a graph automorphism preserving weights and tails. Be precise, it induces isomorphisms $\gamma : V \rightarrow V$ and $\gamma : E \rightarrow E$ such that

$$\gamma(e(k_1, k_2)) = e(\gamma(k_1), \gamma(k_2))$$

and

$$g_{\gamma(k)} = g_k, \quad D(j) = \gamma(D(j)).$$

Let $D_{g,m}$ be the set of stable stratum data. It can be shown that

**Lemma 4.5.** $|D_{g,m}| < \infty$.

We skip the proof. The stability is crucial for the lemma.

For the set $D_{g,m}$ we can assign a partial order $\prec$. Let $S = (T, D, g)$ be a data. Let $e = e(v, w)$ be an edge of $T$. We can define a new data $S'$ by the following modifications on $S$: 
• a new graph $T'$ is obtained by (i) erasing edge $e$, (ii) identifying vertices $v$ and $w$ and denote the new vertex by $v'$;
• $g_{v'}$ is defined to be
  
  $$g_{v'} = g(T) - \sum_{k \neq v,w} g_k - \text{rank} H^1(T');$$

• $D'(i) = v'$ if $D(i) = v$ or $w$. The attaching vertices of tails are changed properly by new $D'$.

By this way, we say that $S'$ is a contraction of $S$ at edge $e$. We write $S' = S(e)$.
Similarly, we can define the contraction $S' = S(e_1, \ldots, e_l)$ of $S$ at edge $e_1, \ldots, e_l$.
Now, we say that $S \prec S'$ if $S'$ is a contraction of $S$. This induces a partial order on the strata of $\bar{\mathcal{M}}_{g,m}$. In fact, this is compatible with what we mean by "lower": $M_S$ is lower than $M_{S'}$ if and only if $S \prec S'$.

Let $T$ be the simplest graph that consists of 1 vertex and no edge. It defines $S_0$ and the stratum is just $\mathcal{M}_{S_0} = \bar{\mathcal{M}}_{g,m}$.

4.4. Strata $M_S$. Let $j \in M_S$. The notions for $j$ (cf. 4.1) and $S$ are same as before. Recall that we have normalizations $\mathcal{N}(j)$ and $\mathcal{N}(j_k)$, with

$$\pi : \mathcal{N}(j) \to j.$$ 

We write

$$\pi_S(\mathcal{N}(j)) = j.$$ 

These two maps are different! For $\pi$ the variable is a point on Riemann surface, while the variable for $\pi_S$ is a curve $\mathcal{N}(j)$. Suppose

$$(4.4) \quad \mathcal{N}(j_k) = (\Sigma_k, i_k, x_{k_1}, \ldots, x_{k_{m_k}}, \bar{y}_{k_1}, \ldots, \bar{y}_{k_{s_k}}).$$ 

Here $x$'s are marked points on $\Sigma_k$ and $\bar{y}$'s correspond to nodal points. Be precisely, we may further assign an edge $n = e(\bar{y}) \in E$ for $y$ that corresponds to the nodal point $\pi(\bar{y}) = y_n$.

Let

$$\mathcal{T}_S := \mathcal{T}_{g_1,m_1+s_1} \times \cdots \times \mathcal{T}_{g_c,m_c+s_c}.$$ 

This has a universal curve

$$\pi_S : \text{uni}_S = \text{uni}_{g_1,m_1+s_1} \times \cdots \times \text{uni}_{g_c,m_c+s_c} \to \mathcal{T}_S.$$ 

Since $\mathcal{N}(j) \in \mathcal{T}_S$, we represent it by $\pi_S^{-1}(\mathcal{N}(j))$.

We are now ready to describe the orbifold structure of $M_S$ at $j$. The isotropic group is $\text{Aut}(j)$: $\text{Aut}(j)$ is a fibration

$$\phi : \text{Aut}(j) \to \text{Aut}(S)$$ 

and acts on $\mathcal{T}_S$. We explain this. Suppose that $\mathcal{N}(j) \in \mathcal{T}_S$ is

$$\mathcal{N}(j) = (\mathcal{N}(j_1), \ldots, \mathcal{N}(j_c)).$$ 

Let $\gamma \in \text{Aut}(S)$. Then $\phi^{-1}(\gamma)$ is given by the following elements: define $\lambda_k : N(j_k) \to N(j_{\sigma(k)})$ such that

• the map
  
  $$\lambda_k : (\Sigma_k, i_k) \to (\Sigma_{\sigma(k)}, i_{\sigma(k)})$$ 

  is bi-holomorphic;
• $\lambda_k$ preserves marked points $x_i$'s;
By this way, we define $\text{Aut}(j)$. It acts naturally on the neighborhood of $\mathcal{N}(j)$ in $\mathcal{T}_S$. Let $\hat{O}$ be an $\text{Aut}(j)$-invariant neighborhood of $\mathcal{N}(j)$. Then $(\hat{O}, \text{Aut}(j), \phi)$ yields a uniformization system of $j$ in $M_S$ via $\pi_S$:

$$\hat{O} \xrightarrow{\phi} \hat{O}/\text{Aut}(j) \xrightarrow{\pi_S} M_S.$$ 

All these charts form the orbifold $M_S$.

As before, $\text{Aut}(j)$ also acts on universal curve $\text{uni}_S$. Hence it induces an universal curve over $\hat{O}/\text{Aut}(j)$:

$$\text{uni}_S|_{\hat{O}}.$$ 

On the other hand, by using the trivialization constructed in proposition 3.2, there exists a trivialization of the universal curve given by an $\text{Aut}(j)$-equivariant map

(4.5) $\Phi_j : \hat{O} \times \mathcal{N}(j) \rightarrow \text{uni}_S|_{\hat{O}}$.

4.5. **Smoothing nodal curves at nodal points.** Let $j \in M_S$ be a nodal curve. For any nodal point $y$ and a complex number $0 \neq \rho$ with small radius, we can smooth $j$ at $y$ and get a new curve $j_{y,\rho}$. This is what we mean by smoothing. We now explain this procedure.

Without loss of generality, we suppose that $\pi^{-1}(y) \subset \mathcal{N}(j)$ consists of $v_1 \in \Sigma_1$ and $v_2 \in \Sigma_2$. We treat $v_i$ as marked points on $\Sigma_i$. Since we are only concerned the local of $v_i$, we may assume that $\Sigma_i$ are smooth curves. The neighborhood of $v_i$ can be canonically identified with balls $B(1)$ up to rotations in $z_i$-planes for $i = 1, 2$: if $\Sigma_i$ is stable, we refer it as the horocycle at $v_i$; otherwise, $v_i$ is a special point $0$ or $\infty$, on $S^2$, we then refer the ball to be the semi-sphere containing the point. We write the balls $B_{v_i}(1)$. Furthermore we have

$$\phi_i : B_{v_i}^*(1) \rightarrow (-\infty, 0] \times S^1$$

by

$$\phi(re^{i\theta}) = (\log r, \theta).$$

We write the punctured surfaces as

(4.6) $\Sigma_i - \{v_i\} = \Sigma_{0i} \cup (-\infty, 0] \times S^1, i = 1, 2.$

The neighborhood of $y$ can be put in $\mathbb{C} \times \mathbb{C}$ as

(4.7) $\pi(B_{y_1}(1) \cup B_{y_2}(1)) = \{z_1z_2 = 0\} \cap B(1).$

For $\rho \in \mathbb{C}^*$ we deform (4.6) to

$$\{z_1z_2 = \rho\} \cap B(1).$$

The new curve is denoted by

$$j_{y,\rho} = (\Sigma_{\rho}, i_{\rho}, \ldots).$$

This smoothing procedure can be described explicitly. Set $\rho = r_0e^{i\theta_0}$. Let us focus on $B_{y_1}(1)$. The remainder of $\Sigma_{0i}$ remains unchanged in the whole process. We cut off the cylinder ends of two cylinders at $\{1.25 \log r_0\} \times S^1$, namely we get

$$[1.25 \log r_0, 0] \times S^1.$$
Then we glue two tubes along a sub-tube of length $-0.5 \log r_0$ with a twisted angle $\theta_0$. That is, we identify

$$(\log r_0 + t, \theta_1) \leftrightarrow (\log r_0 - t, \theta + \theta_0), \ t \in [0.25 \log r_0, -0.25 \log r_0].$$

The resultant curve is then $j_{y, \rho}$.

We note that the plane $C$ of $\rho$ can be treated as

$$T_v \Sigma_1 \otimes T_v \Sigma_2.$$ 

We denote the space as $C_y$. Then we just construct a map

$$(4.9) \quad g_{S_j} : B_\epsilon \subset C_y \to \bar{M}_{g,m}.$$ 

Here $\epsilon$ is any small constant less than 1.

We remark that $g_{S_j}$ is injective if and only if $j$ is stable.

### 4.6. Normal bundles of $M_S$ in $\bar{M}_{g,m}$

It is natural to ask the neighborhood of $j \in M_S$ in $\bar{M}_{g,m}$ when $S$ is stable. In particular, this is to ask what is the normal direction of $M_S$. We assume that $S$ is stable.

Suppose $S = (g, T, D)$. Given a point $j \in M_S$, we define a fiber

$$C_j^{\mid E} := \bigoplus_{y=1}^{\mid E \mid} C_{y,e}.$$ 

Here $E$ is the set of edges of $T$. By this, we define an orbifold bundle

$$L_S \to M_S$$

In fact, we have

$$\tilde{L}_S \to T_S.$$ 

with fiber $C_j$. $\text{Aut}(j)$ acts naturally on $\tilde{L}_S$. Locally, the quotient gives the uniformization system for $L_S$.

Let $O \subset M_S$ be a proper open subset. The gluing map described earlier defines a neighborhood of $O$ in $\bar{M}_{g,m}$:

$$(4.10) \quad g_{S,j} : L_{S,\epsilon}|O \to \bar{M}_{g,m},$$

where $g_{S,j}(\rho) = g_{S}(\rho)$. Here by $L_{S,\epsilon}$ we mean an $\epsilon$-neighborhood of 0 section. It is known that $g_{S,j}$ is injective and locally diffeomorphic when $S$ is stable.

Let $S' = (e_1, \ldots, e_l)$. We can similarly define a sub-bundle $L_{S,S'}$ of $L_S$ by requiring the fiber to be

$$L_{S,S'}|_j = \bigoplus_{y, e(y) = e_j, 1 \leq j \leq l} C_y.$$ 

We can then similarly define

$$(4.11) \quad g_{S,S'} : L_{S,S',\epsilon}|O \to M_{S'}.$$ 

Here, by $L_{S,S',\epsilon}$ we mean that the set of points whose all coordinates in fiber direction are not zero. For example, by $V_{\epsilon}^0$ for a vector space $V = \mathbb{C}^n$ we have

$$V_{\epsilon}^0 = \{(z_1, \ldots, z_n) | z_i \neq 0, |z_i| \leq \epsilon\}.$$ 

Finally, we remark that the above discussion can be generalized to unstable data $S$: $\tilde{L}_S$, $L_S$ and $g_S$ are still available. The only difference is that $g_S$ is neither injective nor locally diffeomorphic. But this is important when we consider stable maps.
Part II. Moduli spaces $\overline{M}_{g,m}(X, A)$

5. Moduli spaces of stable maps

We review the fundamental facts of stable maps in this section, such as notations of stable maps and related facts ([8]).

5.1. Stable maps without nodal points. Let $(X, \omega)$ be a $C^\infty$ compact, closed symplectic manifold of dimension $2n$. We choose an $\omega$-tamed almost complex structure $J$ on $X$, i.e., $\omega(\cdot, J \cdot)$ is positive. Define a $J$-compatible Riemannian metric $\langle \cdot, \cdot \rangle$ by

$$\langle V, W \rangle := \frac{1}{2} (\omega(V, JW) + \omega(W, JV)) .$$

Fix an element $A \in H_2(M, \mathbb{Z})$.

Let $j = (\Sigma, j)$ be a Riemann surface without nodal points. Consider the space of smooth maps

$$\text{Map}_j(X, A) = \{ u : \Sigma \to X | [u(\Sigma)] = A \} .$$

Here $[u(\Sigma)]$ represents the homology class of $u(\Sigma)$.

Define

$$\bar{\partial}_{J,i}(u) = \frac{1}{2} (du + J \cdot du \cdot j) .$$

We say that the map $u$ is $J$-holomorphic if

(5.1) $\bar{\partial}_{J,i}(u) = 0$.

Let $\tilde{M}_j(X, A) \subset \text{Map}_j(X, A)$ be the space of all $J$-holomorphic maps.

We explain $\bar{\partial}_{J,i}$. Given $u \in \text{Map}_j(X)$, we have a bundle $T^*\Sigma \otimes u^*TX$ over $\Sigma$. The space of sections of this bundle is denoted by

$$\text{End}(T^*\Sigma, u^*TX) = \Gamma(T^*\Sigma \otimes u^*TX) .$$

According to $j \otimes u^*J$, it has a decomposition

$$\text{End}(T^*\Sigma, u^*TX) = \text{End}^{1,0}(T^*\Sigma, u^*TX) \oplus \text{End}^{0,1}(T^*\Sigma, u^*TX) .$$

To be consistent with conventions, we set

$$\Omega_j^{0,1}(u^*TX) = \text{End}^{0,1}(T^*\Sigma, u^*TX) .$$

Now note that $du \in \text{End}(T\Sigma, u^*TX)$. Then $\bar{\partial}_{J,i}u \in \Omega_j^{0,1}(u^*TX)$ is the $(0,1)$-component of $du$.

We summarize that we have a bundle

$$\tilde{E}_j \to \text{Map}_j(X, A)$$

with fiber $\Omega_j^{0,1}(u^*TX)$ and $\bar{\partial}_{J,i}$ is a section of $\tilde{E}_j$. Then

$$\tilde{M}_j(X, A) = \text{zero section} \cap \bar{\partial}_{J,i} = \{ u | \bar{\partial}_{J,i}u = 0 \} .$$

We may replace $j = (\Sigma, j)$ by a marked Riemann surface $j = (\Sigma, j, x_1, \ldots, x_m)$.

Define

$$\tilde{M}_j(X, A) = \frac{\tilde{M}_j(X, A)}{\text{Aut}(j)} .$$

Let $\text{Aut}(u, j)$ be the stabilizer of the action for point $u \in \tilde{M}_j(X, A)$. $u$ is called stable if $|\text{Aut}(u, j)| \leq \infty$. 
Remark 5.1. \((u, j)\) is stable if \(j\) is stable. For this case, \(2g + m \geq 3\) and we call the map is pre-stable. Otherwise, it is called pre-unstable. We have four possibilities \((g, m) = (1, 0), (0, 0), (0, 1)\) and \((0, 2)\) for pre-unstable maps.

Proposition 5.2. \(u\) is stable if either \(j\) is stable or \(u\) is not constant.

We now allow \(j\) varies. Let \(T_{g,m}\) and \(M_{g,m}\) be spaces of curves described in §3.1. Define

\[
\tilde{M}_{g,m}(X, A) = \coprod_{j \in T_{g,m}} \tilde{M}_j(X, A).
\]

Suppose that \((u, j)\) and \((u', j')\) are two maps. We say that they are equivalent if there exists an isomorphism \(\sigma : j \to j'\) such that \(u = u' \circ \sigma\). Define the moduli space to be

\[
\mathcal{M}_{g,m}(X, A) = \tilde{M}_{g,m}(X, A)/\sim.
\]

Roughly speaking, if the section \(\bar{\partial}_{J, i}\) transverses to the 0-section, \(\mathcal{M}_{g,m}(X, A)\) is an orbifold. We postpone the complete discussion to §?. Here we describe its local structure. Let \((u, j) \in \mathcal{M}_{g,m}(X, A)\). Recall that a neighborhood \(O_j\) of \(j \in M_{g,m}\) is given by

\[
O_j = \hat{O}_j / Aut(j).
\]

Define

\[
\tilde{M}_{\hat{O}}(X, A) = \coprod_{j' \in O_j} \tilde{M}_{j'}(X, A)
\]

and set

\[
\mathcal{M}_{\hat{O}}(M, A) = \frac{\tilde{M}_{\hat{O}}(X, A)}{Aut(j)}.
\]

This gives a neighborhood of \((u, j)\). Furthermore, we can choose a neighborhood \(\hat{U}\) of \((u, j)\) in the numerator that is invariant under the action of \(Aut(u, j)\). Then \((\hat{U}, Aut(u, j), \pi)\) is a uniformization system of

\[
U := \hat{U} / Aut(u, j).
\]

5.2. Stable maps with nodal points. Let \(j = (\Sigma, i, x_1, \ldots, x_m)\) be a nodal curve (may not be stable). Most of the discussion in §?? still works for unstable curves. We still have notions \(j_k\), \(\mathcal{N}(j)\) and etc. Similarly, we can define data \(S = (T, D, g)\). Suppose \(g = g(T)\).

Let \(u : \Sigma \to X\) be a continuous map such that \([u(\Sigma)] = A\). We say that it is holomorphic if each restriction \(u_k := u|_{\Sigma_k}\) lifts to a \((J, \mathcal{N}(j_k))-\text{holomorphic map}\)

\[
\tilde{u}_k : \tilde{\Sigma}_k \to X.
\]

Here \(\tilde{\Sigma}\) is the normalization of \(\Sigma\).

Definition 5.3. We say that \((u, j)\) is a \((g, m)\)-stable map if \((\tilde{u}_k, \mathcal{N}(j_k))\) is stable for any \(k\). The space of stable maps is denoted by \(\tilde{M}_{g,m}(X, A)\).

Two stable maps are equivalent, denoted by \((u, j) \sim (u', j')\), if there exists an isomorphism \(\sigma : j \to j'\) such that \(u = u' \circ \sigma\). Let

\[
\tilde{M}_{g,m}(X, A) = \tilde{M}_{g,m}(X, A)/\sim.
\]
The moduli space $\overline{M}_{g,m}(X,A)$ has a similar stratification as that of $\overline{M}_{g,m}$. Let $S = (T,D,g)$ be a stratum data of $\overline{M}_{g,m}$. We add an extra data $\mathfrak{A} = (A_1, \ldots, A_c)$ to it. Set $A = A_1 + \cdots + A_c$. We denote the new data to be $\mathfrak{S} = (T,D,g,A)$. Here $A_k$ represents the homology class of $u_k$. We then define the stratum $M_{\mathfrak{S}}(X,A) \subset M_{g,m}(X,A)$ to be the set that consists of equivalence classes of stable maps described above with the property $[u_k(\Sigma_k)] = A_k$.

Let $D_{g,m}^A$ be the set stratum data of stable map. As before Lemma 5.4. $|D_{g,m}^A| < \infty$.

We can also define a partial order $\prec$ on this set as in §4.3. The only extra information we should add is that $A_{v'} = A_v + A_w$. By this, we can define $\prec$ in $D_{g,m}^A$. So we can also say that the stratum $M_{\mathfrak{S}}(X,A)$ is lower than $M_{\mathfrak{S}'}(X,A)$ if $\mathfrak{S} \prec \mathfrak{S}'$.

With proper topology([?][?]), one has

**Theorem 5.5.** $\overline{M}_{g,m}(X,A)$ is compact. The closure of $M_{\mathfrak{S}}(X,A)$ is $\overline{M}_{\mathfrak{S}}(X,A) = M_{\mathfrak{S}}(X,A) \cup \bigsqcup_{\mathfrak{S}' \prec \mathfrak{S}} M_{\mathfrak{S}'}(X,A)$.

Using the local description of $M_{\mathfrak{S}}$, we would like to give a local description for $M_{\mathfrak{S}}(X,A)$ as well. Let $(u,j)$ be a $J$-holomorphic map in the stratum. Suppose $(O, Aut(j), \pi)$ is a uniformization system of a neighborhood $O$ of $j$ in $M_{\mathfrak{S}}$ (refer notations to §4.3). Let $\tilde{M}_O(X,A) = \prod_{j' \in \pi_S(\tilde{O})} \tilde{M}_{j'}(X,A)$. Then $M_O(X,A) = \frac{\tilde{M}_O(X,A)}{Aut(j)}$ is a neighborhood of $(u,j)$ in $M_{\mathfrak{S}}(X,A)$. As before, we can choose a neighborhood $\tilde{U}$ of $(u,j)$ in the numerator that is invariant under the action of $Aut(u,i)$. Then $(\tilde{U}, Aut(u,i), \pi)$ is a uniformization system of $U := \tilde{U}/Aut(u,i)$.

6. Analytic set-up

6.1. Analytic set-up for $M_{j}(X,A)$. Let $j \in M_{g,m}$. We first assume that $j$ is stable. Recall that $M_j(X,A)$ is viewed as zeros of section $\tilde{\partial}_{j,i}$ of bundle $\tilde{E}_j$. In order to show the smoothness of $\tilde{M}_j(M,A)$, we need put Sobolev norms on these spaces and apply the transversality theorem for Banach manifolds.

Let $p > 2$ be an even integer. We denote by $\chi_{j}^{1,p}(X,A)$ the space of continuous map $u : \Sigma \to M$ of class $W^{1,p}$ such that $[u(\Sigma)] = A$. We usually simplify the notation to be $\chi_{j}^{1,p}$. The space $\chi_{j}^{1,p}$ is an infinite dimensional Banach manifold.
For any map \( u \in \chi^1(p) \), its tangent space is the Banach space \( W^{1,p}(u^*TM) \) of \( W^{1,p} \)-vector fields \( \zeta \) along \( u \). The point-wise exponential map

\[
W^{1,p}(u^*TM) \to M : \zeta \to \exp_u \zeta
\]

identifies a neighborhood \( U_u \) of 0 in \( W^{1,p}(u^*TM) \) with a neighborhood of \( u \) in \( \chi^1(p) \). We have a coordinate chart \( (\exp_u U_u, \exp_u^{-1}) \). Without loss of generality, we assume that \( U_u \subset W^{1,p}(u^*TM) \) is a neighborhood of \( u \). The tangent bundle is

\[
T_{\chi^1(p)} \to \chi^1(p),
\]
a bundle with fiber \( W^{1,p}(u^*TM) \). We denote the bundle by \( \tilde{\mathcal{F}}_1 \).

Similarly, we consider bundle \( \tilde{\mathcal{E}}_j \) over \( \chi^1(p) \). We put \( L^p \) norm on the fiber. Hence the fiber over \( (u, i) \) is

\[
L^p(\Lambda^0,1(u^*TM)).
\]

Recall that \( \bar{\partial}_{J,i} \) is a section of this bundle.

Now fix \( u_o \in \chi^1(p)(X,A) \). We now trivialize \( \tilde{\mathcal{E}}_1 \) and \( \tilde{\mathcal{F}}_1 \) over a small neighborhood \( U_{uo} \) of \( u_o \). We trivialize \( \tilde{\mathcal{F}}_1 \) first. Let \( \zeta \in W^{1,p}(u_o^*TM) \) and \( u = \exp_{uo} \zeta \). Then the parallel transformation along path \( \exp_{uo} s \zeta \)

\[
P_\zeta : W^{1,p}(u_o^*TM) \to W^{1,p}(u_o^*TM)
\]
identifies two fibers. This defines a trivialization

\[
T_{\chi^1(p)}|_{U_{uo}} \cong U_{uo} \times W^{1,p}(u_o^*TM).
\]

Let \( \Pi^{0,1} \) be the projection

\[
\Pi^{0,1} : L^p(T^*\Sigma \otimes u_o^*TM) \to L^p(\Lambda^{0,1}(u_o^*TM)).
\]

Then

\[
\Pi^{0,1} \circ P_\zeta : L^p(\Lambda^{0,1}(u^*TM)) \to L^p(\Lambda^{0,1}(u_o^*TM))
\]
yields a trivialization of \( \tilde{\mathcal{E}}_1 \).

We summarize the data we have:
- \( u \)
- a base space \( U_{uo} \);
- a bundle \( \tilde{\mathcal{E}}_1 \cong U_u \times \Lambda^{0,1}(u_o^*TM) \);
- a section \( \bar{\partial} \) of the bundle \( \tilde{\mathcal{E}}_1 \);
- a tangent bundle \( \tilde{\mathcal{F}}_1 \).

Let \( (u, i) \in U_{uo} \). The linearization of \( \bar{\partial}_{J,i} \) at \( (u, i) \) is

\[
D_{u,i} : \tilde{\mathcal{F}}_1|(u, i) \to \tilde{\mathcal{E}}_1|(u, i).
\]

Be precise, we have

\[
D_{u,i} : W^{1,p}(u^*TM) \cong W^{1,p}(u^*TM) \to L^p(\Lambda^{0,1}(u^*TM)) \cong L^p(\Lambda^{0,1}(u_o^*TM)).
\]

Explicitly, by ignoring the identifications on the two ends given by trivialization

\[
D_{u,i}(\xi) = \frac{1}{2}(\nabla \xi + J(u) \nabla \xi) + \nabla \xi Jdu_i)
\]

**Proposition 6.1.** The index of \( D_{u,i} \) is \( n(2 - 2g) + 2\epsilon_1(A) \).

This follows from the Riemann-Roch theorem.

**Theorem 6.2.** If \( D_{u,i} \) is surjective for all \( (u, i) \in \tilde{\mathcal{M}}_1(X, A) \), \( \mathcal{M}_1(X, A) \) is a smooth orbifold of dimension \( n(2 - 2g) + 2\epsilon_1(A) \).
The proof of smoothness is standard ([1?]). We will give the proof in §\ref{6.2} using our terminology.

For $j$ is unstable, the treatment is similar.

6.2. **Analytic set-up for $\mathcal{M}_{g,m}(X,A)$**. For the analytic set-up, a general principle is to treat $\mathcal{M}_{g,m}$ as a parameter space. By this way, we can give a family version of set-up. However, there are some tedious issues.

Let us first assume $m = 0$. For this case, there is no essential change except that we replace $j$ by $\mathcal{M}_{g,m}$ (or $\tilde{O}$ if we emphasis the locality.) We summarize it:

- $\mathcal{U}_{uo}$ is replaced by $\tilde{O} \times \mathcal{U}_{uo}$;
- $\tilde{F}_j$ is replaced by $\tilde{O} \times \tilde{F}_j$;
- $\tilde{E}_j$ is replaced by the parameterized bundle $\tilde{E}_O$ which still can be trivialized as $\tilde{O} \times \tilde{E}_j$.

We explain the last statement. The fiber of $\tilde{E}_O$ over $(u, j')$ is $L^p(\Lambda_v^{0,1}(u^*TX))$. We explain the identification between $L^p(\Lambda_v^{0,1}(u^*TX)) \leftrightarrow L^p(\Lambda_1^{0,1}(u^*TX))$.

First the identification between $L^p(\Lambda_v^{0,1}(u^*TX)) \leftrightarrow L^p(\Lambda_1^{0,1}(u^*TX))$ is given by $\Pi^{0,1} \circ P_\zeta$ as before; secondly, the identification between $L^p(\Lambda_1^{0,1}(u^*TM)) \leftrightarrow L^p(\Lambda_1^{0,1}(u^*TM))$ is induced by proposition 1.4.

Next, we consider $m > 0$. This case is subtle. On the one hand, we can do the trivialization as what we do for $m = 0$ case. But on the other hand, we would like to trivialize bundles $\tilde{E}$ and $\tilde{F}$ in a different way. By using proposition 3.2, we may trivialize families $\tilde{E}_O$ and $\tilde{F}_O$ locally. This is necessary when we consider lower strata. However, these trivialization causes problems technically at the first sight: as it is pointed out in [16], the family with such trivialization are not smooth. Namely, we have trivialization for both families $\tilde{E}_O$ and $\tilde{F}_O$ locally. But two different trivialization $\tilde{E}_O$ and $\tilde{E}_O'$ do not patch smoothly. However this trouble can be solved by the following observation: first we note that trivialization is patched well by restricting on smooth objects; secondly, by the elliptic regularity property, all objects we are concerned are smooth. Hence we may always assume that $\tilde{E}_O$ and $\tilde{F}_O$ are trivialized and study the theory as if they are smooth families.

Hence,

**Theorem 6.3.** If $D_{a,j}$ is surjective for all $(u, j) \in \tilde{\mathcal{M}}_{g,m}(M, A)$, $\mathcal{M}_{g,m}(M, A)$ is a smooth orbifold of dimension $n(2 - 2g) + 2c_1(A) + 6g - 6 + 2m$.

**Proof.** Here $6g - 6 + 2m$ is the dimension of parameter space $M_{g,m}$. The theorem then follows from theorem 6.2.

6.3. **Analytic set-up for $\mathcal{M}_S(X, A)$**. Recall that

$$j_0 := (\Sigma, i_0, x_{o1}, \ldots, x_{om}).$$

and

$$\mathcal{R}(j_0) = (\mathcal{R}(j_{o1}), \ldots, \mathcal{R}(j_{oc})).$$
Let $\hat{\Sigma}_k$ be the surface for $\mathcal{R}(i_{o,k})$. Recall that
\[
\pi : \mathcal{R}(i_o) \to i_o.
\]
We define $\chi^1_{\mathcal{R}(i_o)}$ to be the set of elements
\[
\tilde{\nu} := (\tilde{u}_1, \ldots, \tilde{u}_c), \ u_k \in \chi^1_{\mathcal{R}(i_o)}
\]
such that it induces a continuous map $u : \Sigma \to X$, i.e., $u \circ \pi = \tilde{u}$. To avoid the complication of notations, we simply use $\chi^1_{i_o}$ for $\chi^1_{\mathcal{R}(i_o)}$ and $u$ for $\tilde{u}$. This simplification only causes a little ambiguity at nodal points. When this happens, we always refer to the normalization of curves.

With $i_o$ fixed, we still have $\tilde{F}_{i_o}$ and $\tilde{E}_{i_o}$. Their fibers are given by the followings. For $u_o \in \chi^1_{i_o}$, its tangent space is
\[
W^{1,p}(u_o^*TM) = \{ (\zeta_1, \ldots, \zeta_k) | \zeta_v \in W^{1,p}(u_v^*TM), \ \zeta_v(y) = \zeta_w(y) \text{ for } y \in \Sigma_v \cap \Sigma_w \}.
\]
This gives the fiber of $\tilde{F}_{i_o}$. Set
\[
L^p(\Lambda^{0,1}_{i_o}(u_o^*TM)) := \bigoplus_u L^p(\Lambda^{0,1}_{j_{o,v}}(u_v^*TM)).
\]
This gives the fiber of $\tilde{E}_{i_o}$. We have the linear operator
\[
D_{i_o,u_o} : W^{1,p}(u_o^*TM) \to L^p(\Lambda^{0,1}_{i_o}(u_o^*TM))
\]
\[
D_{i_o,u_o} := (D_{i_o,u_{o,1}}, \ldots, D_{i_o,u_{o,k}}).
\]

**Lemma 6.4.** $D_{i_o,u_o}$ is a Fredholm operator of index $2c_1(A) + 2n(1-g)$.

For the proof see [5].

To study $\mathcal{M}_S(X, A)$, we should allow that $j$ varies. Besides the similarities as above, there are parameters that record the nodal points on each component. This is reflected in the definition of $W^{1,p}(u_o^*TM)$. Therefore, we should use the trivialization method mentioned at the end of last subsection.

**Proposition 6.5.** The stratum $\mathcal{M}_S(X, A)$ is a smooth orbifold of dimension $n(2-2g) + 2c_1(A) + 6g - 6 + 2m - 2|\text{Sing}|$, if $D_{i_o,u}$ is surjective for any $(u,j) \in \mathcal{M}_S$.

**Proof.** We verify the claim of dimension. For a stable component, the moduli space of the component has dimension
\[
2c_1(A_k) + 2n(1-g_k) + 6g_k - 6 + 2m + 2s_k,
\]
where $m_k$ is the number of marked points and $s_k$ is the number of nodal points (on the normalized surface); for an unstable component, (only when $g = 0$ and $m_k + s_k \leq 2$), the dimension is
\[
2c_1(A_k) + 2n - 6 + 2m_k + 2s_k.
\]
Totally we have
\[
2c_1(A) + 2n(c - \sum g) + 6 \sum g - 6c + 2m + 4|\text{Sing}| - 2n|\text{Sing}|
\]
Note that
\[
g = \sum g_v + \text{rank}H^1(T); c - |\text{Sing}| = 1 - \text{rank}H^1(T).
\]
We have the formula of dimension.

For the smoothness, the proof is same as that of theorem 6.2. We omit it. q.e.d.
7. Coordinate charts for $\mathcal{M}_{g,m}(X,A)$

7.1. Data of coordinate charts. We consider $\mathcal{M}_j(X,A)$ with a fixed $j \in \mathcal{M}_{g,m}$. Let $u \in \chi^{1,p}_j$ and $U_u$ be a neighborhood of $u$. We may identify $U_u$ with an open set $W \subset W^{1,p}(u^*TM)$ via $\exp_u$. Set

$$L = L^p(\Lambda^1_{\mathcal{E},0}(u^*TM)), \quad M = \mathcal{M}_j(X,A) \cap W.$$

Definition 7.1. Let $W, L, M$ be as above. Suppose that we have

1. a smooth sub-manifold $U$ in $W$,
2. a small open ball $B_\delta \subset L$, a neighborhood $V$ of $u$ in $W$ and a diffeomorphism $\Phi : U \times B_\delta \to V$,
3. a smooth section $f : U \to B_\delta$

such that the map given by

$$F : U \xrightarrow{(1,f)} U \times B_\delta \xrightarrow{\Phi} V$$

maps $U$ onto $V \cap M$ and the map is diffeomorphic, we then call $(U, \phi, F)$ (or $(U, \Phi, f)$, if no confusion may be caused,) a data of coordinate chart.

Obviously, by the definition $(U, F)$ gives a coordinate chart for $M \cap V$.

Proposition 7.2. Any two coordinate charts given by two different data are $C^\infty$ compatible.

Proof. Suppose that we have two data of coordinate charts. Be precise: we have $u_i, i = 1, 2$ and $U_{u_i}$ which are identified with $W_i$; then furthermore, we have $(U_i, \Phi_i, F_i)$ which give coordinate charts $(U_i, F_i)$ for $V_i \cap M$. So we have a transition map

$$F_1^{-1}(F_1(U_1) \cap F_2(U_2)) \xrightarrow{F_1} F_1(U_1) \cap F_2(U_2) \xrightarrow{F_2^{-1}} F_2^{-1}(F_1(U_1) \cap F_2(U_2)).$$

This map is the composition of the following chain:

$$U_1 \xrightarrow{(1,f_1)} U_1 \times B_\delta \xrightarrow{\Phi_1} V_1 \xrightarrow{\Psi} V_2 \xrightarrow{\Phi_2} U_2 \times B_\delta \xrightarrow{\text{projection}} U_2$$

Here $\Psi = \exp_{u_2}^{-1} \exp_{u_1}$. Since each map in the chain is smooth, the transition map is smooth. q.e.d.

Note that in the proof, we use the fact that $\chi^{1,p}_j(X,A)$ is smooth. This is needed for the smoothness of the map $\Psi$. However, if we consider $\chi^{1,p}_{g,m}(X,A)$ and $\mathcal{M}_{g,m}(X,A)$, the fact is not true. The problem can be solved by a small modification:

Remark 7.3. We modify the definition by requiring that $U$ consists of smooth maps. Then we may repeat the argument of proposition 7.2 for the $\mathcal{M}_{g,m}(X,A)$. The only problem is $\Psi$. Although $\Psi$ is not smooth in general, it is smooth when restricted on smooth maps.
7.2. **Proof of theorem 6.3.** We only prove the smooth structure of $\mathcal{M}_j(X, A)$. The proof for that of $\mathcal{M}_{g,m}(X, A)$ is similar.

The goal is to construct a data of coordinate chart for each point $u \in \mathcal{M}_j(X, A)$. Set

$$W = W^{1,p}(u^*TM), L = L^p(A_j^{0,1}(u^*TM)).$$

By our assumption,

$$D_{u,j} : W \to L$$

is surjective. Hence we may construct a right inverse $Q_{u,j}$ to $D_{u,j}$ such that $Q_{u,j}$ is $\text{Aut}(u,j)$ equivariant: note that a right inverse gives a splitting

$$W = \ker D_{u,j} \oplus \text{range } Q_{u,j}$$

and vice versa. We choose $Q_{u,j}$ to be $(\ker D_{u,j})^\perp$ with respect to $L^2$-norm. Since $\ker D_{u,j}$ and $L^2$-norm are $\text{Aut}(u,j)$ invariant, the splitting is $\text{Aut}(u,j)$-equivariant.

Now we define

$$\Phi : \ker D_{u,j} \times L \to W;$$

$$\Phi(\xi, \eta) = \xi + Q_{u,j}\eta.$$

Then, there exists a small neighborhood $U$ of $0 \in \ker D_{u,j}$, a small ball $B_\delta \subset L$ and a neighborhood $V$ of $u$ in $W$ such that

$$\Phi : U \times B_\delta \to V$$

is diffeomorphic.

It remains to construct a section

$$f : U \to B_\delta$$

such that

$$\bar{\partial}_{J,i}(\Phi(\xi, f(\xi))) = 0$$

for any $\xi \in U$. For this purpose, we consider the map

$$H : U \times B_\delta \to U \times L;$$

$$H(\xi, \eta) = (\xi, \bar{\partial}_{J,i}(\Phi(\xi, \eta))).$$

Then

$$H(0, 0) = (0, 0); dH_{(0,0)} = \text{id}.$$

By the inverse function theorem, there is a smooth section solving (7.1). This completes the proof. q.e.d.

7.3. **Constructing data of coordinate charts.** Again, we only consider $\mathcal{M}_j(X, A)$. The situation is: let $U$ be a smooth sub-manifold of $\chi_1^{1,p}(X, A)$; fix a point $u_0 \in U$; set $W$ to be a small neighborhood of $u_0 \in W^{1,p}(u_o^*TM)$ and $L = L^p(A_j^{0,1}(u_o^*TM)$; let

$$Q = \{Q_{u,j} | u \in U\}$$

be a smooth family of right inverses for $u \in U$. Then we define

$$\Phi(u, \eta) = u + Q_{u,j}\eta.$$

Furthermore, we have the following assumption on $(U, Q)$:

**Assumption 7.4.** Let $(U, Q)$ be as above with properties

1. $\|\nabla u\|_{L^p} \leq C$ for any $u \in U$;
(2) for any $u \in U$  
\[ \| \bar{\partial}_{J,i} u \|_{L^p} \leq \epsilon; \]
(3) for any $\zeta \in T_u U$  
\[ \left\| \frac{d \bar{\partial}_{J,i} u}{d \zeta} \right\|_{L^p} \leq \epsilon \| \zeta \|; \]
(4) for right inverses  
\[ \| Q_{u,i} \| \leq C \]
and  
\[ \| Q_{u_1,i} - Q_{u_2,i} \| \leq C \| u_1 - u_2 \|_{L^1,p}. \]
Here $C$ is a constant and $\epsilon$ is a small constant such that $C \epsilon \ll 1$.

For any $(U, Q)$ satisfying the assumption, we explain that we may produce a data of coordinate chart from it for a neighborhood of $u_0$.

Applying the famous Taubes argument, we have

**Proposition 7.5.** There exists a smooth map  
\[ f : U \to B_\delta \]
such that $u + Q_u f(u)$ is holomorphic. Any holomorphic curve in the form $u + Q_u \xi, \xi \in B_\delta$ is given by $\xi = f(u)$. Here $\delta$ is a small number that depends only on $C$. Moreover

\[ \| f(u) \|_{L^p} \leq 2 \epsilon. \]

We remark that we may assume that $\epsilon \ll \delta \ll C$.

**Proof.** Composing with  
\[ \Phi : U \times L \to W, \]
we have a family of operators parameterized by $u \in U$:  
\[ \bar{\partial} : U \times L \xrightarrow{\Phi} W \xrightarrow{\bar{\partial}_{J,i}} L. \]
Be precise, for each $u$, we have  
\[ \bar{\partial}(u, \cdot) : L \to L; \bar{\partial}(u, \eta) = \bar{\partial}_{J,i}(u + Q_u \eta). \]
We now solve $\eta$ for the equation  
\[ \bar{\partial}_{J,i}(u + Q_u \eta) = 0. \]
Expand the equation we have  
\[ \bar{\partial}_{J,i}(u + Q_u \eta) = \bar{\partial}_{J,i} u + D_u Q_u \eta + N_u(Q_u \eta) = \cdots + \eta + \cdots. \]
Here $N_u(Q_u \eta)$ is a term with second or higher order. We use the fact  
\[ \| N_u(\xi_1) - N_u(\xi_2) \|_{L^p} \leq C_0 (\| \xi_1 \| + \| \xi_2 \|) (\| \xi_1 - \xi_2 \|). \]
Here $C_0$ depends only on $\| \nabla u \|_{L^p}$.

The equation to solve is  
\[ \eta = -\bar{\partial}_{J,i} u - N_u(Q_u \eta). \]
Let $H : B_\delta \to B_\delta$ be a map defined by  
\[ H \eta = -\bar{\partial}_{J,i} u - N_u(Q_u \eta). \]
By choosing proper $\delta$, $H$ is a contraction map. This follows by two simple estimates.

$$
\|H\eta\| \leq \|\hat{\partial}_{J,i}u\| + \|N_u(Q_u\eta)\|
\leq \epsilon + C_0C^2\|\eta\|^2
\leq \epsilon + C_0C^2\delta^2
\leq \epsilon + \delta/4 \leq \delta;
$$

here we require that $C_0C^2\delta < 1/4$ and $\epsilon \ll \delta$;

$$
\|H\eta_1 - H\eta_2\|_{L^p} = \|N_u(Q_u\eta_1) - N_u(Q_u\eta_2)\|
\leq C_0(\|Q_u\eta_1\| + \|Q_u\eta_2\|)\|Q_u(\eta_1 - \eta_2)\|
\leq 2C_0C^2\delta\|\eta_1 - \eta_2\|
\leq 0.5\|\eta_1 - \eta_2\|.
$$

We conclude that $H$ is a contraction map. On the other hand, we can also show that $H : B_{2\epsilon} \to B_{2\epsilon}$ is a contraction map. This implies the estimate for $f(u) = \eta$. q.e.d.

In this proposition, we essentially only use the property (1) in Assumption 7.4.

**Theorem 7.6.** There exists a small neighborhood $U' \subset U$ of $u_o$, $\delta_1 \leq \delta$ and $V \subset W$ such that

$$
\Phi : U' \times B_{\delta_1} \to V
$$

is diffeomorphic. Here $\delta_1$ depends only on $C$.

**Proof.** We may identify $W^{1,p}(u_o^*TM)$ with $\ker D_{u_o,i}L^p$ via

$$
\xi + Q_{u_o,i}\eta \leftrightarrow (\xi, \eta).
$$

We rewrite map $\Phi$ as

$$
\Phi : U \times B_{\delta} \to W^{1,p}(u_o^*TM) = \ker D_{u_o,i}L^p;
\Phi(u, \eta) = (\bar{u} + Q_u\eta - Q_{u_o}D_{u_o}(\bar{u} + Q_u\eta), D_{u_o}(\bar{u} + Q_u\eta)),
$$

Here $\bar{u} = u - u_o$. The tangent map of $\Phi$ at $u, \eta$ is

$$
D\Phi_{u,\eta}(\xi, \zeta) = \begin{pmatrix} \xi + I_{11} & I_{12} \\ I_{21} & \zeta + I_{22} \end{pmatrix},
$$

where

$$
I_{11} = \frac{dQ_u}{d\xi}\eta - Q_{u_o}D_{u_o}\xi - Q_{u_o}D_{u_o}\frac{dQ_u}{d\xi}\eta =: I_{111} + I_{112} + I_{113};
I_{12} = Q_u\zeta - Q_{u_o}D_{u_o}Q_u\zeta;
I_{21} = D_{u_o}(\xi + \frac{dQ_u}{d\xi}\eta);
I_{22} = D_{u_o}Q_{u_o}\zeta - \zeta.
$$

By direct estimates, we have that for proper chosen $U' \subset U, \delta' < \delta$ and $(u, \eta) \in U' \times B_{\delta'}$,

$$
\|I_{ij}\| \leq \frac{1}{100}\|\xi, \zeta\|
$$

Hence $D\Phi_{u,\eta}$ is invertible and

$$
\|D\Phi_{u,\eta}\| \leq 2
$$

for $(u, \eta) \in U' \times B_{\delta'}$. 

Finally, we show that $\Phi$ is injective. Suppose that
$$\Phi(u_1, \eta_1) = \Phi(u_2, \eta_2).$$
In general, we have
$$\Phi(u, \eta) = \Phi(u_0, 0) + D\Phi_{u_0, 0}(\bar{u}, \eta) + N(\bar{u}, \eta).$$
Here
$$N(\bar{u}, \eta) = (Q_u \eta - Q_{u_0} D_u Q_u \eta, D_{u_0} Q_u \eta - \eta).$$
It is not hard to get
\begin{equation}
\|N(\bar{u}_1, \eta_1) - N(\bar{u}_2, \eta_2)\| \leq C(\|\bar{u}_1, \eta_1\| + \|\bar{u}_2, \eta_2\|)(\|u_1 - u_2, \eta_1 - \eta_2\|).
\end{equation}
We have
$$D\Phi_{u_0, 0}((\bar{u}_1, \eta_1) - (\bar{u}_2, \eta_2)) = -(N(\bar{u}_1, \eta_1) - N(\bar{u}_2, \eta_2)).$$
Set $h = \|(u_1, \eta_1) - (u_2, \eta_2)\|$, then
$$\frac{h}{2} \leq C(\|\bar{u}_1, \eta_1\| + \|\bar{u}_2, \eta_2\|)h.$$
This is impossible if $\|(\bar{u}_v, \eta_v)\|, v = 1, 2$, are small. Here $\Phi$ is injective. q.e.d.
As a corollary, $(U, \Phi)$ yields a data of coordinate chart $(U', \Phi, F)$.

7.4. Estimates of $df/d\xi$. Finally, we discuss the derivative
$$\frac{df}{d\xi} \xi \in T_u U.$$
We show that
\begin{equation}
\|\frac{df}{d\xi}\| \leq C \epsilon \|\xi\|.
\end{equation}
\textbf{Proof.} The proof is rather long although it is straightforward.
Let $u_t$ be a path with $u_0 = u_0$ and representing $\xi \in T_u U$. We differentiate the equation
$$\bar{\partial}_{J,i} u_t + f(u_t) + N_{u_t}(Q_{u_t} f(u_t)) = 0$$
and get
\begin{equation*}
0 = \frac{d}{d\xi}\bar{\partial}_{J,i} u_t + \frac{df(u_t)}{d\xi} + \frac{d(N_{u_t} Q_{u_t} f(u_t))}{d\xi}
= \frac{d}{d\xi}\bar{\partial}_{J,i} u_t + \frac{df(u_t)}{d\xi}
+ \frac{dN_{u_t}(Q_{u_t} f(u_t))}{d\xi} + N_{u_0} \left(\frac{d(Q_{u_t} f(u_t))}{d\xi}\right)
= I_1 + I_2 + I_3 + I_4.
\end{equation*}
We have
$$\|I_1\| \leq \epsilon \|\xi\|$$
by property (2) in assumption.
To get the estimate for $I_4$ we consider
\[
N_{u_\circ}(Q_{u_\circ}f(u_t) - Q_{u_\circ}f(u_o))
\leq C_0(\|Q_{u_\circ}f(u_t)\| + \|Q_{u_\circ}f(u_o)\|)(\|Q_{u_\circ}f(u_t) - Q_{u_\circ}f(u_o)\|)
\leq 2C_0\|Q_{u_\circ}f(u_o)\|((\|u_t - u_o\|\|f(u_o)\| + C\|f(u_t) - f(u_o)\|)
\]
which says that
\[
I_4 \leq 2C_0C^2\|f(u_o)\|\|\xi\| + 2C_0C^2\|f(u_o)\|\|I_2\| \leq C\epsilon^2\|\xi\| + 0.5\|I_2\|.
\]
The estimate
\[
\|I_3\| \leq \epsilon\|\xi\|
\]
is given in the next lemma. Combine all these together, we have
\[
\|I_2\| \leq C\epsilon\|\xi\|.
\]
q.e.d.

**Proposition 7.8.** Let $u_t$, $N_u$ and $\xi$ be as above, then
\[
\|\frac{dN_u(\eta)}{d\xi}\| \leq C\|\xi\|_{L^1,p}(\epsilon + \|\eta\|_{L^1,p}).
\]

**Proof.** As we know
\[
\partial_{J,i}(u_t + \eta) = \partial_{J,i}u_t + D_{u_t}\eta + N_{u_t}\eta.
\]
on the other hand,
\[
\tilde{\partial}_{J,i}(u_t + \eta) = \tilde{\partial}_{J,i}u_o + D_{u_o}(\tilde{u}_t + \eta) + N_{u_o}(\tilde{u}_t + \eta),
\]
where $\tilde{u} = u - u_o$. Set two right hand sides equal. Then
\[
\frac{\partial_{J,i}u_t - \tilde{\partial}_{J,i}u_o}{t} + \frac{D_{u_t}\eta - D_{u_o}(\tilde{u}_t + \eta)}{t} + \frac{N_{u_t}(\eta) - N_{u_o}(\tilde{u}_t + \eta)}{t} = 0
\]
By taking $t \to 0$, we have
\[
\frac{\partial_{J,i}u_t - \tilde{\partial}_{J,i}u_o}{t} \to \frac{d}{d\xi}(\tilde{\partial}_{J,i}u);
\]
\[
\frac{D_{u_t}\eta - D_{u_o}(\tilde{u}_t + \eta)}{t} \to \frac{d}{d\xi}(D_u)\eta - D_{u_o}\xi;
\]
while for
\[
\frac{N_{u_t}(\eta) - N_{u_o}(\tilde{u}_t + \eta)}{t} = \frac{N_{u_t}(\eta) - N_{u_o}(\eta)}{t} + \frac{N_{u_o}(\eta) - N_{u_o}(\tilde{u}_t + \eta)}{t},
\]
its limit is
\[
(7.5) \quad \frac{d}{d\xi}N_{u_o}(\eta) + \lim_{t \to 0}\frac{N_{u_o}(\eta) - N_{u_o}(\tilde{u}_t + \eta)}{t}.
\]
Therefore
\[
\frac{d}{d\xi} N_{u_0}(\eta) = - \frac{d}{d\xi} (\bar{\partial}_{J,i} u) - \frac{d}{d\xi} D_u \eta + D_u \xi \lim_{t \to 0} N_{u_0}(\eta) - N_{u_0}(\bar{u}_t + \eta) \quad t = I_1 + I_2 + I_3 + I_4.
\]
For each term we have
\[
\|I_1\|_{L^p} \leq \epsilon \|\xi\|_{L^1,p},
\|I_2\|_{L^p} \leq C \|\xi\|_{L^1,p} \|\eta\|_{L^1,p},
\|I_3\|_{L^p} \leq \epsilon \|\xi\|_{L^1,p},
\|I_4\|_{L^p} \leq C \|\xi\|_{L^1,p} \|\eta\|_{L^1,p}.
\]
The estimate of \(I_4\) follows from lemma ??.

8. Balanced \(J\)-holomorphic curves

We consider the moduli space \(\mathcal{M}_{g,m}(X,A)\) with \(2g + m \leq 2\). There are 4 cases: \((g,m) = (0,0), (0,1), (0,2)\) and \((1,0)\). In this section, we focus on \((g,m) = (0,1)\) and \((0,2)\) since we need them when consider gluing.

Let \(j_m = (S^2, j, x_1, \ldots, x_m), 1 \leq m \leq 2\).

The moduli spaces are
\[
\mathcal{M}_{0,m}(X,A) = \frac{\tilde{\mathcal{M}}_{0,m}(X,A)}{\text{Aut}(j_m)},
\]
where \(\tilde{\mathcal{M}}_{0,m}(X,A)\) is defined below.

Since \(\text{Aut}(j_m)\) is a non-compact finite dimensional Lie group, it is useful to construct the slice for the quotient space, or reduce the quotient group to be compact. For this purpose, we introduce balanced holomorphic maps.

**Case 1**, \((g,m) = (0,1)\).

\(\mathcal{M}_{0,1}\) consists of only one element \(j_1 = (S^2, \infty)\). Here \(S^2 - \infty = \mathbb{C}\). We use \(\mathbb{C}\) in our discussion in this subsection. Let \(t = \mathbb{C}\) be the group of translations of \(\mathbb{C}\) and \(m = \mathbb{C}^*\) that acts on \(\mathbb{C}\) by multiplications. The semi-product \(\mathfrak{B} = t \ltimes m\) acts on \(\mathbb{C}\) as
\[
(t, m) \cdot z = m(z - t).
\]
It is well known that
\[
\text{Aut}(j_1) = \mathfrak{B}.
\]
Let
\[
\tilde{\mathcal{M}}_{0,1}(X,A) := \tilde{\mathcal{M}}_{0,0}(X,A) := \{u : S^2 \to X | \bar{\partial}_{J,i} u = 0, [u(S^2)] = A\}.
\]
Then
\[
\mathcal{M}_{0,1}(X,A) = \frac{\tilde{\mathcal{M}}_{0,1}(X,A)}{\mathfrak{B}}.
\]
For \(u \in \tilde{\mathcal{M}}_{0,0}(X,A)\) we usually call \(|du|^2\) the energy density. Note that the energy of \(u\) is \(\omega(A)\). Let \(h = \omega(A)/2\).

**Definition 8.1.** A \(J\)-curve \(u \in \tilde{\mathcal{M}}_{0,1}(X,A)\) is called balanced if
Let $\mathcal{M}_{0,1}^b(X, A)$ be the space of balanced $J$-curves.

We remark that for any $u \in \tilde{\mathcal{M}}_{0,1}(X, A)$ there is a canonical balanced curve $b_1(u)$ constructed
- by translating the energy center of $u$ to $0$;
- by proper dilation (i.e., multiplying a proper real number) such that the energy on the unit disk is $\hbar$.

It is then easy to see that

\[ \mathcal{M}_{0,1}(X, A) = \frac{\mathcal{M}_{0,1}^b(X, A)}{S^1}. \]  

Here $S^1$ acts on $\mathbb{C}$ by rotations and therefore has an induced action on $\mathcal{M}_{0,1}^b(X, A)$. When we consider $\mathcal{M}_{0,1}(X, A)$ we always use (8.1).

**Case 2,** $(g, m) = (0, 2)$.

This case is similar but easier. $\mathcal{M}_{0,2}$ consists only of an element $j_2 = (S^2, 0, \infty)$. Then

\[ \text{Aut}(j_2) = m = \mathbb{C}^*. \]

Set $\tilde{\mathcal{M}}_{0,2}(M, A) = \tilde{\mathcal{M}}_{0,0}(M, A)$.

**Definition 8.2.** A $J$-curve $u \in \tilde{\mathcal{M}}_{0,2}(M, A)$ is called balanced if the energy of $u$ on the unit disk is $\hbar$. Let $\mathcal{M}_{0,2}^b(X, A)$ be the space of balanced $J$-curves.

We also have

\[ \mathcal{M}_{0,2}(X, A) = \frac{\mathcal{M}_{0,2}^b(X, A)}{S^1}. \]  

**Part III. The Gluing Theory**

9. **Gluing maps**

In §12–§13 we discuss the basic case, i.e., the gluing theory for 1-nodal strata. Then we generalize it to general strata in §14.

9.1. **Pre-gluing.** Let $S = (g, \mathfrak{g}, T, D)$ be a data of stratum in $\overline{\mathcal{M}}_{g,m}(X, A)$. For simplicity, we assume $m = 0$. Here

\[ g = \{g_1, g_2\}, \mathfrak{g} = \{A_1, A_2\} \]

and $T$ consists of two vertices $v_1, v_2$ and one edge $e$. $D$ is trivial since $m = 0$.

Set $S = (g, T, D)$. Let $i_v \in M_S$ and $(u_v, i_v) \in \mathcal{M}_{i_v}(X, A)$. Suppose that $i_v$ consists of

\[ i_{ov} = (\Sigma_v, j_{ov}, y_{ov}), v = 1, 2. \]

By identifying $y_{o1}$ and $y_{o2}$, we get $i_o = (\Sigma, i_o)$. We write

\[ \Sigma = \Sigma_1 \cup_{y_{o1}=y_{o2}} \Sigma_2. \]

We denote the singular point by $y_o$. $u_o$ consists of $J$-holomorphic curves $u_{ov}: \Sigma_v \to M, [u_v(\Sigma_v)] = A_v$ with $u_{o1}(y_{o1}) = u_{o2}(y_{o2})$. 

Recall that we have an (orbi-)line bundle
\[ L_S \to M_S. \]
The forgetting-map map
\[ \mathcal{f} : M_S(X, A) \to M_S; \]
\[ \mathcal{f}(u, j) = j \]
induces an orbi-line bundle
\[ \mathcal{L}_S = \mathcal{f}^* L_S \to M_S(X, A). \]
Given a point \( p \in \mathcal{L}_S \), our goal is to construct a holomorphic map \( Gl(p) \in \mathcal{M}_{g,m}(X, A) \). Put in the local coordinate, we write \( p = (\rho, \beta), \rho = re^{i\theta} \), we construct \( Gl(u, \beta) \). The first step of the construction is pre-gluing, which gives an approximation holomorphic map \( \text{pgl}(u, \beta) \).

Recall that we have a gluing map for surfaces:
\[ \text{gs} : L_S \to M_{S_0}. \]
In local coordinates, we write
\[ j_{\rho} = \text{gs}(\rho, \beta) = (\Sigma_{\rho, y_{20}}, \rho_{\rho}). \]
Geometrically, \( \Sigma_{\rho, y_{20}} \) is obtained as the following. We use the holomorphic cylindrical coordinates \( (\log s_i, t_i) \) on \( \Sigma_i \) near \( y \), and write
\[ \Sigma_2 - \{ y_{20} \} = \Sigma_2 \bigcup \{ [0, \infty) \times S^1 \}, \]
\[ \Sigma_1 - \{ y_{20} \} = \Sigma_1 \bigcup \{ (-\infty, 0] \times S^1 \}. \]
We cut off the part of \( \Sigma_i \) with cylindrical coordinate glue the remainders by identifying the \( |\log r| \)-long ends of the cylinders with a twist of angle \( \theta \). The new curve is \( j_{\rho} \). \( \text{pgl}(u, \beta) \) is expected to be a map on \( j_{\rho} \).

More generally, we may replace holomorphic map \( u, b \) by \( u \in \chi_{1}^{1}(X, A) \). Write \( \phi = \text{pgl}(u, \beta) \) where \( u = (u_1, u_2) \). \( \phi \) is supposed to be a map on surface \( \Sigma_{\rho, y_{20}} \).

Define
\[ \phi(x) = \begin{cases} 
  u_1(x) & \text{if } x \in \Sigma_1 - D_{y_{20}}(2r^{1/4}) \\
  u_2(x) & \text{if } x \in \Sigma_2 - D_{y_{20}}(2r^{1/4}) \\
  p = u_1(y_{10}) = u_2(y_{20}) & \text{if } x \in D_{y_{20}}(r^{1/4}) - D_{y_{20}}(r^{3/4})
\end{cases} \]

To define the map in the rest part we fix a smooth cutoff function cutoff function \( \beta : \mathbb{R} \to [0, 1] \) such that
\[ \beta(s) = \begin{cases} 
  1 & \text{if } s \geq 2 \\
  0 & \text{if } s \leq 1
\end{cases} \]
and \( |\beta'(s)| \leq 2 \). We assume that \( r \) is small enough such that \( u_i \) maps the disk \( D_{y_i}(4r^{1/4}) \) into a normal coordinate domain of \( p \). We can define \( \phi \) by
\[ \phi(x) = \exp_p \left( \beta \left( \frac{x}{r^{1/4}} \right) \exp^{-1} u_1(x) + \beta \left( \frac{r^{1/4}}{z} \right) \exp^{-1} u_2 \left( \frac{z}{x} \right) \right). \]

**Lemma 9.1.** Suppose \( \phi = \text{pgl}(u, \beta) \), then
\[ \|\partial_{\beta_{j_{\rho}} \phi}\|_{L^p} \leq \|\partial_{\beta_{\rho}} u\|_{L^p} + C \| \rho \|^{1/4}, \]
where \( C \) is independent of \( \rho \). In particular,
\[ \|\partial_{\beta_{j_{\rho}} \phi}\|_{L^p} \leq C \| \rho \|^{1/4}. \]
if \( u \) is holomorphic.

The proof is given in \[10\].

For \( u = u_o \), set
\[ \phi_0 = \text{pgl}(u_o, J_o, \rho). \]

9.2. Right inverses. Let \( u \in \mathcal{A}_{1_{J_o}}^{1,p} \). We assume that \( D_{u, J_o} \) is surjective. Therefore, there is a right inverse
\[ Q_{u, J_o} : L^p(\Lambda^{0,1}_{\mathcal{A}_{1_{J_o}}} (u^*TM)) \to W^{1,p}(u^*TM), \]
with \( \| Q_{u, J_o} \| \leq C \). Let \( \phi = \text{pgl}(u, J_o, \rho) \). We construct the right inverse to \( D_{\phi, J_o} \).

We identify \( \Sigma''_v := \Sigma_v \cup \{(\log r, 0) \times S^1\}, v = 1, 2 \)
with \( \Sigma_\rho \) in an obvious way. We introduce two pairs \( \lambda_{v, \rho} \) and \( \gamma_{v, \rho}, v = 1, 2 \), of cut-off functions on them. We only describe these functions on the cylinder ends only since they are 1 on \( \Sigma_v \).

Let
\[ \lambda_{v, \rho}(t, \theta) = \begin{cases} 
1, & \text{if } t > \log r/2 + 1; \\
0, & \text{if } t < \log r/2 - 1,
\end{cases} \]
with \( \lambda_{1, \rho} + \lambda_{2, \rho} = 1 \). Let
\[ \gamma_{v, \rho}(t, \theta) = \begin{cases} 
1, & \text{if } t > \log r/2 - 1; \\
0, & \text{if } t < \log r,
\end{cases} \]
Note that \( \gamma_{v, \rho} \) is 1 on the support of \( \lambda_{v, \rho} \). Also
\[ |\nabla \gamma_s| \leq \frac{C}{|\log r|}. \]

Suppose that \( \eta \) is a function (or a form) on \( \Sigma_\rho \). We define
\[ \Lambda(\eta) = \lambda_{1, \rho} \eta + \lambda_{2, \rho} \eta \]
to be a function (or a form) on \( \Sigma \). Note that \( \lambda_{v, \rho} \eta, v = 1, 2 \) are functions (or forms) on \( \Sigma''_v \subset \Sigma \subset \Sigma_\rho \). By \( \oplus \), we mean the sum is taken over \( \Sigma \).

Conversely, suppose \( \sigma \) is a continuous function (or form) on \( \Sigma \). Define
\[ \sigma_1(x) = \gamma_{1, \rho}(\sigma(x) - \sigma(y)) + \sigma(y), x \in D_{y_1}(\sqrt{r}) \]
and equals to \( \sigma \) outside the disk. \( \sigma_1 \) is a function on \( \Sigma''_1 \subset \Sigma_\rho \). Similarly, we have a function \( \sigma_2 \) on \( \Sigma''_2 \subset \Sigma_\rho \). Define
\[ \Gamma(\sigma) = \sigma_1 \oplus \sigma_2 \]
to be a function on \( \Sigma_\rho \). Here by \( \oplus \), we mean the sum is taken over \( \Sigma_\rho \).

Lemma 9.2. For \( \eta \in L^p(\Lambda^{0,1}_{\mathcal{A}_{1_{J_o}}} \phi^*TM) \)
\[ \| D_{\phi, J_o} R \eta - \eta \|_{L^p} \leq \frac{C}{|\log r|} \| \eta \|_{L^p}, \]
where \( R = \Gamma Q_{u, J_o} \Lambda(\eta) \).

The proof is given in \[10\]. The lemma says that \( D_{\phi, J_o} R \) is invertible. Set
\[ Q_{\phi, J_o} = R(D_{\phi, J_o} R)^{-1}. \]
Proposition 9.3. $Q_{\phi, j_o, \rho}$ is a right inverse to $D_{\phi, j_o, \rho}$. Moreover

$$\|Q_{\phi, j_o, \rho}\| \leq C$$

where $C$ is independent of $\rho$.

In particular, for $\phi_o$ we construct the right inverse $Q_{\phi_o, j_o, \rho}$.

9.3. Gluing maps. With $\phi_o$ and $Q_{\phi_o, j_o, \rho}$, we can construct a holomorphic curve as in proposition 7.5.

We need the lemma

Lemma 9.4. Let $\phi = \text{pgl}(u, j_o, \rho)$.

$$\|N_\phi(\zeta_1) - N_\phi(\zeta_2)\|_{L^p} \leq C(\|\zeta_1\|_{L^{1,p}} + \|\zeta_2\|_{L^{1,p}})(\|\zeta_1 - \zeta_2\|_{L^{1,p}}),$$

where $C$ depends only on $\|u\|_{L^{1,p}}$.

Proof. By theorem ??, we have this inequality with some constant $C'$ depending on $\|\phi\|_{L^{1,p}}$. By the construction of $\phi$, we know that

$$\|\phi\|_{L^{1,p}} \leq C''\|u\|_{L^{1,p}}$$

So the claim follows. q.e.d.

Theorem 9.5. Suppose that $\phi$ is as above and let $C_0$ be the constant given in lemma 7.4. Suppose that

$$\|\bar{\partial}_{L^{1,0}, \phi}\|_{L^p} \leq \epsilon$$

for some $\epsilon \ll C_0^{-1}$. Then in the $\delta$-ball of $L^p(\Lambda_{i_o, \phi}^0, TM)$ with $\delta C_0 < 1/2$, there exists a unique element, denoted by $f(u, j_o, \rho)$, such that

$$\exp_\phi Q_{\phi, j_o, \rho} f(u, j_o, \rho)$$

is $J$-holomorphic and

$$\|f(u, j_o, \rho)\| \leq C\epsilon,$$

where $C$ can be any constant such that $CC_0 \epsilon < 1/2$.

The proof is a repeat of that in proposition 7.4.

Remark 9.6. Since we are working on some spaces with orbifold structure, we should require that the gluing maps are equivariant with respect to isotropic groups.

Let $(u_o, j_o) \in M_\delta(X, A)$, the local uniformization system for a neighborhood $O$ of $(u_o, j_o)$ in $M_\delta(X, A)$ and bundle $L_\delta|_O$ are in the form

$$(\bar{\partial}, \text{Aut}(u_o, j_o), \pi)$$

and

$$(\bar{L}_\delta|_O, \text{Aut}(u_o, j_o), \pi).$$

The gluing map is, at the moment, defined on $L_\delta|_O$ other than on $L|_O$. Then we note that

1. when $j_o$ is pre-stable, the gluing is $\text{Aut}(u_o, j_o)$-equivariant. Hence the gluing is defined on $L_\delta$;

2. when $j_o$ is not pre-stable, there is at least one non-pre-stable component $j_{ov}, v = 1, 2$. The component is of $g = 0, 1 \leq m \leq 2$. For this case, we have to use moduli spaces of balanced curves. Then it is easy to see that the gluing is well defined on $L_\delta$. 

Let $U \subset M(X, A)$ be any proper open subset. Define the gluing map to be

$$Gl : L^0_{g, \epsilon}|U \to M_{g, m}(X, A)$$

$$Gl(u, j, \rho) = pgl(u, j, \rho) + f(u, j, \rho).$$

Here $\epsilon$ depends only on $U$. To stress the process of gluing, we set

$$pert(u, j, \rho) = f(u, j, \rho).$$

Here pert stands for perturbation which is exactly what we are doing in the second step.

9.4. **Gluing maps for general strata.** Now suppose that $S = (g, \mathfrak{A}, T, D)$ is any stratum and $S = (g, T, D)$. For simplicity we assume that $M(X, A)$ is compact, otherwise we always restrict our discussion on a proper open subset in the stratum.

Recall that for any $S \prec S'$ (and correspondingly $S \prec S'$) there exists a gluing bundle $L_{S, S'}$. Repeat the process in §9.4, §9.3 we have a gluing map

$$Gl_{S, S'} : L^0_{S, S', \epsilon} \to M_{S'}(X, A).$$

Now consider a point $p \in L_S$

$$p = (u, j, \rho_1, \rho_2),$$

where $\rho_1$ denotes the coordinate corresponding to the fiber in $L_{S, S'}$ and $\rho_2$ is the rest. Then applying the gluing map $Gl_{S, S'}$ we have

$$(u, j, \rho_1, \rho_2) \to (Gl_{S, S'}(u, j, \rho_1), \rho_2) \in L_{S'}.$$

We denote this gluing map on the bundle level by $BGl$. It is clear that

**Lemma 9.7.** $Gl^*_{S, S'}(L_{S'}) = L_S$.

Suppose $S''$ is any stratum that is bigger than $S'$. Set $W = Gl_{S, S'}(L^0_{S, S', \epsilon})$. Had we proved that $Gl_{S, S'}$ is a homeomorphic, $Gl_{S, S'}$ would induce a gluing map

$$Gl'_{S, S''} : L^0_{S', S''} | W \to M_{S''}(X, A)$$

given by

$$Gl'_{S, S''} = Gl_{S, S'} \circ Gl^{-1}_{S, S'}.$$

The homeomorphism (in fact, diffeomorphism) of $Gl_{S, S'}$ will be proved in §12.

10. **Estimates**

10.1. **Estimates for pre-gluing maps.** We first prove lemma 9.1

Proof of lemma 9.1. Let $\Sigma_1' = \Sigma_1 - D_{y_{\epsilon_1}}(r^{1/2})$. We have

$$\|\tilde{\partial}_u \phi\|_{L^p(\Sigma_1')} \leq \|\tilde{\partial}_u \phi\|_{L^p(\Sigma_1)} + C \left( \int_{D_{y_{\epsilon_1}}(2r^{1/4})} |\nabla \beta(\frac{x}{r^{1/4}})(u - p)|^p \right)^{1/p}$$

$$+ C \left( \int_{D_{y_{\epsilon_1}}(2r^{1/4})} |\nabla J \cdot (u - p)|^p \right)^{1/p}.$$

Note that in $D_{y_{\epsilon_1}}(2r^{1/4})$

$$|\nabla \beta(\frac{x}{r^{1/4}})(u - p)| \leq C|u|_{C^1};$$

and

$$|\nabla J \cdot (u - p)| \leq C|J|_{C^1} r^{1/4}.$$
So on $\Sigma_1' \subset \Sigma_\rho$

$$\|\tilde{\partial}_{J_{1,\rho}} \phi\|_{L^p(\Sigma_1')} \leq \|\tilde{\partial}_{J_{1,\rho}} u\|_{L^p} + C r^{\frac{3}{2p}},$$

One can compute the other side on $\Sigma_2$ similarly, so the lemma follows. q.e.d.

We are also interested in the derivative of pre-gluing maps. Let $u_t = (u_{1t}, u_{2t}), t \in [0, \delta)$

be a path in $c^{1, p}$ with

$$\dot{u}_t|_{t=0} = \left. \frac{d}{dt} \right|_{t=0} u_t = \zeta := (\zeta_1, \zeta_2),$$

Let $\phi_t = \text{pgl}(u_t, J_0, \rho)$ we study $\phi_t|_{t=0}$. Similar to the computations for previous lemma, we have

**Lemma 10.1.** Let $\zeta = (\zeta_1, \zeta_2)$ be as above, Then

$$\left\| \left. \frac{d}{dt} \right|_{t=0} \phi_t \right\|_{L^1,p} \leq C \|\zeta\|_{L^1,p};$$

$$\left\| \left. \frac{d}{dt} \right|_{t=0} \tilde{\partial}_{J_{1,\rho}} \phi_t \right\|_{L^p} \leq \left\| \left. \frac{d}{dt} \right|_{t=0} \tilde{\partial}_{J_{1,\rho}} u_t \right\|_{L^p} + C r^{1/2p} \|\zeta\|_{C^1},$$

where $C$ is independent of $\rho$. In particular,

$$\left\| \left. \frac{d}{dt} \right|_{t=0} \tilde{\partial}_{J_{1,\rho}} \phi_t \right\|_{L^p} \leq C r^{1/2p} \|\zeta\|_{C^1},$$

if $u_t$ is a holomorphic path.

We leave the proof to readers. Note that the last term in the estimates is $\zeta$ with respect to $C^1$-norm rather than $L^1,p$-norm. Also for the last statement, it is clear that it holds as long as $\zeta \in \ker D_{J_0, u_0}$.

10.2. **Estimates for right inverses. Proof of lemma 9.2** Suppose $\sigma_1$ is constructed from $\sigma = Q_{u_0, J}(\Lambda \eta)$ as explained in §9.2. It is supported in $\Sigma_1''$. We compute

$$I := D_{\phi, J_0}(\sigma_1) = D_{\phi, J_0}(\sigma_1)(\gamma_{1, \rho}(\sigma - \sigma(y_{01})) + \sigma(y_{01})).$$

We find that

$$|I| \leq |\lambda_{1, \rho} \eta| + |\nabla \gamma_{1, \rho}(\sigma - \sigma(y_{01}))|$$

$$+ |J(\phi) \nabla \gamma_{1, \rho}(\sigma - \sigma(y_{01}))| + |(J(\phi) - J(u)) \gamma_{1, \rho} d\sigma|$$

$$+ |(\nabla J, \gamma_{1, \rho}) d(\phi - u)| + |(\nabla J, \sigma(y_{01}) - \gamma_{1, \rho}(\sigma(y_{01}))| d\phi|$$

$$=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6.$$

The difficult terms are $I_2$ and $I_3$. They behave similarly: for example,

$$\|I_2\|_{L^p} \leq \int_{D_{y_0}(r^{1/4})} \left( \frac{1}{\log r} \right)^{1/2} (r^{1/4})^{1-2/p} |\sigma|_{C^0} \right)^p = \frac{C}{|\log r|^p} |\sigma|^{p^p}_{C^0},$$

where $\alpha = 1 - 2/p$. So

$$\|I_2\|_{L^p} \leq \frac{C}{\log r} \|\eta\|_{L^p}.$$
Proposition 10.2. Let $Q_{t,1o}$ be the right inverse to $D_{t,1o}$ constructed as before. Then
\[
\|Q_{t,1o}\| \leq C; \\
\|\partial_\zeta Q_{t,1o}\| \leq C\|\zeta\|_{L^1,p},
\]
where $C$ are constants depending only on $u$.

Proof. All statements are standard except the last estimate. We explain this.

\[
\partial_\zeta Q_{t,1o} = (\partial_\zeta R)(D_{t,1o}R)^{-1} + R(\partial_\zeta R)^{-1}.
\]

For the first term it is sufficient to estimate
\[
\partial_\zeta R = \Gamma(\partial_\zeta Q_{1o})\Lambda.
\]

It is standard to have
\[
\|\partial_\zeta Q_{1o}\| \leq C\|\zeta\|_{L^1,p},
\]
and therefore
\[
\|\partial_\zeta R\| \leq C\|\zeta\|_{L^1,p}.
\]

For the second term, we use the identity
\[
\partial_\zeta (D_{t,1o}R)^{-1} = -(D_{t,1o}R)^{-1}\partial_\zeta (D_{t,1o}R)(D_{t,1o}R)^{-1}.
\]

Then the rest of estimates is standard. q.e.d.

10.3. Estimates of $f(u,\cdot,\rho)$. As a consequence of theorem 10.2 we have

Theorem 10.3. Let $\zeta \in \ker D_{u,1}$. Then
\[
\|\partial_\zeta f(u,1o,\rho)\|_{L^p} \leq C_{r,1/p}\|\zeta\|_{L^1,p},
\]
where $C$ depends only on $u$.

11. $C^0$-compatibility of gluing maps

11.1. Admissible gluing maps. As we have seen, gluing maps consist of two parts: pre-gluing and perturbation, i.e., map $\text{pgl}$ and pert described in §9. Hence, they depend on cut-off functions and right inverses used in the constructions. Since cut-off functions only depend on the coordinates of horocycles, we may assume that cut-off functions are fixed. This kills the ambiguities caused by cut-off functions.

On the other hand, there are more general gluing maps realized by the following data (again, we only explain for the 1-nodal stratum case): let $\Lambda = (V,Q)$ be a pair satisfying assumption §7.4. Suppose that it generates a data of coordinate chart $(V,\Phi,F)$ of a proper open subset $U$ of $\mathcal{M}_G(X,A)$. We may define a gluing map $\text{Gl}_\Lambda$ based on these data:

- for $p = (u,1o,\rho) \in \mathcal{L}_{1o}|U$ we define
  
  \[
  \text{pgl}_\Lambda(p) = \text{pgl}(F^{-1}(u),1o,\rho),
  \]

  set $\phi = \text{pgl}_\Lambda(p)$;

- construct right inverse for $Q_{t,1o}$ by using $Q_{F^{-1}(u),1o}$;

- construct pert $\Lambda$ by using $\phi$ and $Q_{\phi,1o}$.
More explicit, \( GL_\Lambda \) is the composition
\[
L_\Sigma|_V \to F^*L_\Sigma|_V \xrightarrow{GL_\Lambda} \mathcal{M}_{g,m}(X, A).
\]
We call a gluing map \( GL_\Lambda \) constructed as above is an admissible gluing map. Clearly, the original gluing maps are admissible.

**Definition 11.1.** \( GL_\Lambda \) is called type-1 if \( V \subset \mathcal{M}_S(X, A) \), otherwise, it is called type-2.

11.2. \( C^0\)-compatibility. Suppose that we have two different gluing maps \( GL_\Lambda \), \( \Gamma = (V, Q) \) and \( Gl \). Later, we will prove that both of them are compatible with the smooth structure of top stratum. How they compatible with each other? Note that all gluing maps are identity when \( \rho = 0 \). We want to understand how much difference between \( GL(u, j_\circ, \rho) \) and \( Gl_\Gamma(u, j_\circ, \rho) \) when \( \rho \to 0 \). The expected result should be

**Theorem 11.2.** \( \lim_{\rho \to 0} GL(u, j_\circ, \rho) = \lim_{\rho \to 0} Gl_\Gamma(u, j_\circ, \rho) \)

**Proof.** We show that
\[
||GL(u, j_\circ, \rho) - Gl_\Gamma(u, j_\circ, \rho)|| \leq C(\rho)
\]
where \( C(\rho) \to 0 \) for \( \rho \to 0 \).

For simplicity, we introduce notations. Let
\[
u_\circ = F^{-1}(u_\circ), \quad \eta_\circ = f(u'_\circ), \quad \sigma_\circ = u_\circ - u'_\circ = Q_{\nu_\circ} \eta_\circ.
\]
Let
\[
\phi'_\circ = pgl(u'_\circ, j_\circ, \rho), \quad \phi_\circ = pgl(u_\circ, j_\circ, \rho).
\]
Let \( \Lambda \) and \( R \) be those terms in §9.2. We compare
\[
\phi''_\circ := \phi'_\circ + Q_{\phi'_\circ, j_\circ}(\Lambda \eta_\circ)
\]
with \( \phi_\circ \). We claim that
\[
(11.1) \quad \|\phi''_\circ - \phi_\circ\| \leq C_1(\rho); \\
(11.2) \quad \|
\bar{\partial}_{J, j_\circ} \phi''_\circ \| \leq C_2(\rho),
\]
where \( C_j(\rho) \to 0, j = 1, 2 \), when \( \rho \to 0 \). These two equations imply this theorem by theorem 9.5. The proof of these two equations is given below. q.e.d.

**Proposition 11.3.** Equation (11.1) is true.

**Proof.** Step 1,
\[
(11.3) \quad \|Q_{\phi'_\circ}(\Lambda \eta_\circ) - R(\Lambda \eta_\circ)\| \leq \frac{C}{|\log \rho|} \|\eta_\circ\|.
\]
This follows directly by the definition of \( Q_{\phi'_\circ} \).

It remains to compare \( R(\Lambda \eta_\circ) \) with \( \beta \cdot \sigma_\circ \). By definition
\[
R(\Lambda \eta_\circ) = \Gamma Q_{\nu_\circ}(\Lambda \eta_\circ).
\]
Note that \( \sigma_\circ = Q_{\nu_\circ} \eta_\circ \). We can easily verify that
\[
\|\Gamma Q_{\nu_\circ}(\Lambda \eta_\circ) - \beta \cdot \sigma_\circ\| \leq C r^{1/2p}.
\]
Combine these, we get equation 11.1. q.e.d.

**Proposition 11.4.** Equation 11.2 is true.

**Proof.** We have that

\[ \| \bar{\partial} \phi \| \leq C r^{1/2} p. \]

and

\[ \| \bar{\partial} (\phi''_o - \phi_o) \|_{L^p} \leq C \| \nabla (\phi''_o - \phi_o) \|_{L^p} \leq C(\rho). \]

So 11.2 follows. q.e.d.

As a corollary, we have that

**Corollary 11.5.** $GL^{-1}GL_\Gamma$ and its inverse are continuous.

12. COORDINATE CHARTS FROM GLUING MAPS

We explain that how the differential structure on $\mathcal{M}_{g,m}(X, A)$ induced by gluing maps fits with the one given in §7.2.

We discuss these case by case:

- Case I: $2g_1 + m_1 \geq 3$ and $2g_2 + m - m_1 \geq 3$;
- Case II: $2g_1 + m_1 \geq 3$ and $2g_2 + m - m_1 < 3$;
- Case III: $2g_1 + m_1 < 3$ and $2g_2 + m - m_1 < 3$;

12.1. Case I. We study the gluing maps near $(u_o, j_o)$. By assumption $j_{ov}, v = 1, 2$ are stable, so is $j_o$. For simplicity, we will ignore finite groups $\Gamma_{i_o}, \Gamma_{j_{ov}}$ and etc. unless it is stressed.

Let $M_S$ be the stratum containing $j_o$. For simplicity, we assume that $M_S$ and $\mathcal{M}_S(X, A)$ are compact. It is known that

\[ g_S : L_{S,\epsilon}^0 \to M_{g,m} \]

is a local diffeomorphism.

Let $O$ be any neighborhood of $j_o$ in $M_S$. Let

\[ O = f^{-1}(O), \]

where $f$ is the forgetting-map map.

Set

\[ U = \text{pgl}(L_S|_O) \subset \chi_{g,m}(X, A). \]

For each $\phi = \text{pgl}(u, j, \rho) \in U$ we have right inverse $Q_{\phi, j, \rho}$. If we fix $(j, \rho)$, set

\[ U_{j, \rho} = \{ \text{pgl}(*, j, \rho) \}, \]

\[ Q_{j, \rho} = \{ Q_{\phi, j, \rho} | \phi \in U_{j, \rho} \}. \]

By estimates in §9.1 and §9.2 we have

**Theorem 12.1.** $(U_{j, \rho}, Q_{j, \rho})$ is a pair satisfying assumption 7.4. Hence the gluing map generates a coordinate chart of $\mathcal{M}_j(X, A)$. Since $(j, \rho)$ may be treated as parameters, $(U, Q)$ generates a coordinate chart of $\mathcal{M}_{g,m}(X, A)$ which is given by gluing maps.

In the other word, $GL_{i_o, \rho}$ is diffeomorphic automatically.
12.2. Gluing maps: case II. We now discuss the gluing for case II. That is: \( j_0 \) is stable and \( j_2 \) is unstable. In particular, we note that \( \Sigma_2 = S^2 \).

We will further divide case II into four subcases:

- IIa. \( m = m_1 \) and \( (\Sigma_1, j_1, x_1, \ldots, x_m) \) is stable;
- IIb. \( m_1 = m - 1 \) and \( (\Sigma_1, j_1, x_1, \ldots, x_m) \) is stable;
- IIC. \( m = m_1 \) and \( (\Sigma_1, j_1, x_1, \ldots, x_m) \) is unstable;
- IID. \( m_1 = m - 1 \) and \( (\Sigma_1, j_1, x_1, \ldots, x_m) \) is unstable.

We start with case IIa which is one of the most complicated cases. Before we proceed, let us remark what is new comparing with case 1. The problem is that \( g_s \) is no longer local diffeomorphic. Hence, we are not able to treat \( L_S \) as parameters.

**Case IIa.** We specify the notations for this case. \( j_o \) consists of

\[
j_0 = (\Sigma_1, j_{o1}, x_{o1}, \ldots, x_{om}, y_{o1})
\]

and

\[
j_2 = (S^2, y_{o2} = \infty).
\]

We describe \( M_S \). For simplicity, we assume \( m = 0 \) and \( (\Sigma_1, j) \) is stable. For \( j_1 = (\Sigma_1, j_1, y_1) \) set

\[
j'_1 = (\Sigma_1, j_1).
\]

Then

\[
M_S \cong M_{g_1} \times \Sigma_1 \times \{j_2\}
\]

where the isomorphism is given by

\[
(j_1, j_2) \leftrightarrow (j'_1, y_1, j_2).
\]

By the construction of

\[
gs : M_S \times \mathbb{C}^* \rightarrow \mathcal{M}_g
\]

we know it is an fibration with fiber

\[
\Sigma_1 \times \mathbb{C}^*.
\]

Geometrically, this says: with fixed surface

\[
j'_1 = (\Sigma_1, j_1),
\]

for any \( y \in \Sigma_1 \) and \( 0 \neq \rho \in \mathbb{C}^* \),

\[
gs(j_1, \rho) = j'_1.
\]

Let \( u = (u_1, u_2) \in M_S(X, A) \) be a map. We may assume that \( u_2 \) is balanced. Be precise, we define

\[
\mathcal{M}^S_{g}(X, A) = \{(u_1, u_2) | u_1 \in \mathcal{M}_{g, 1}(X, A_1), u_2 \in \mathcal{M}^0_{0, 1}(X, A_2), u_1(y) = u_2(y)\}.
\]

Then

\[
M_S(X, A) = \frac{\mathcal{M}^S_{g}(X, A)}{S^1}.
\]

By this exposition, we know that: \( y \) and \( \rho \) can not be treated as parameters, however \( j'_1 \) can be. So we will fixed \( j'_1 \) in the rest of argument for this subcase. This is equivalent to fixing \( j_1 \).
We summarize the notations again: \(j_o\) consists of

\[ j_{o1} = (\Sigma_1, j_{o1}, y_{o1}) \text{ and } j_{o2} = (S^2, \infty) \]

Set \(j'_{o1} = (\Sigma_1, j_{o1})\). Define

\[ M_{S, j'_{o1}} = \{(j_1 := (\Sigma_1, j_{o1}, y_1), j_{o2})\} \]

Clearly, \(M_{S, j'_{o2}} \cong \Sigma_1\). Then

\[ g_s : M_{S, j'_{o1}} \times C^*_\epsilon \to j'_{o1}. \]

Correspondingly, for moduli spaces, we have \(\mathcal{M}^b_{j}(X, A)\) and \(\mathcal{M}_j(X, A)\) for \(j \in M_{S, j'_{o1}}\). Set

\[ \mathcal{M}^b_{S, j'_{o1}}(X, A) = \prod_{j \in M_{S, j'_{o1}}} \mathcal{M}^b_j(X, A) \]

and

\[ \mathcal{M}_{S, j'_{o1}}(X, A) := \frac{\mathcal{M}^b_{S, j'_{o1}}(X, A)}{S^1}. \]

For any

\[ u \in \mathcal{M}^b_j(X, A) \subset \mathcal{M}^b_{S, j'_{o1}}(X, A), \]

we assume that \(D_{u, j}\) is surjective. Then we get a gluing map

\[ Gl : \mathcal{M}^b_{S, j'_{o1}}(X, A) \times_{S^1} C^*_\epsilon \to \mathcal{M}_{j_{o1}}(X, A). \]

The map is well defined: since it is easy to see that the gluing map defined on

\[ \mathcal{M}^b_{j_{o1}}(X, A) \times C^*_\epsilon \]

is \(S^1\)-equivariant. The balanced moduli spaces are necessary for the equivariance.

We move on to discuss the diffeomorphic issue.

Fix a map \(u_o = (u_{o1}, u_{o2}) \in \mathcal{M}^b_{S, j_{o1}}(X, A)\). Since \(D_{u_{o, j}}\) is surjective, \(\mathcal{M}^b_{S, j_{o1}, y_{o1}}(X, A)\) is a smooth manifold. Let \(N_o\) be a slice (with respect to the \(S^1\)-action) through \(u_o\). Let \(V\) be a neighborhood of \(y_{o1} \in \Sigma_1\). Then the neighborhood \(U_{u_o}\) of \(u_o \in \mathcal{M}_{S, j'_{o1}}(X, A)\) can be identified with

\[ U_{u_o} \cong V \times N_o. \]

Fix \(\rho_o = r_o\) and its neighborhood \(Glu(\rho_o) \subset C^*_\epsilon\). Then the gluing map is locally rewritten as

\[ (12.1) \quad Gl : V \times N \times Glu(\rho_o) \to \mathcal{M}_{j_{o1}}(X, A). \]

We want to show that this is local diffeomorphic. The new point is to compute differentiation with respect to new variables in \(V \times Glu(\rho_o)\). To treat them properly, we compare this map with another well-studied map, which has been shown to be diffeomorphic by case 1.

By adding two marked points \(\{0, 1\}\) to \(S^2\), we get a stable curve

\[ j'_{o2} = (S^2, 0, 1, \infty). \]

Let \(j_o = (j_{o1}, j_{o2})\). Set

\[ \tilde{y} = g_s(j_o, \rho_o) \in M_{g, 2}. \]

We regard \(u_o\) as an element in \(\mathcal{M}^b_{j_{o1}}(X, A)\) in an obvious way. Let \(\tilde{N}\) be a neighborhood of \(u_o\) in this moduli space. We have a gluing map

\[ Gl_{j_{o1}} : \tilde{N} \times \{\rho_o\} \to \mathcal{M}_{j}(X, A) \]
which is diffeomorphic according to case I. We rewrite the map as
\[ \text{Gl}_{i_{\alpha}, \rho_0} \times \tilde{N} \to \mathcal{M}_{Y_1}(X, A). \]
Since \( y_{01} \) and \( \rho_0 \) are fixed, \( \tilde{Y}' \) can be identified with \( \tilde{Y}'_{01} \) by forgetting the two extra marking points. This induces an isomorphism
\[ \mathcal{M}_Y(X, A) \leftrightarrow \mathcal{M}_Y'(X, A) \]
via forgetting-marked-point. So we have
\[ \text{Gl}_{i_{\alpha}, \rho_0} : \tilde{N} \to \mathcal{M}_{Y_1}(X, A). \]

Next, we explain that there is a natural isomorphism
\[ B : V \times N \times \text{Gl}(\rho_0) \to \tilde{N}. \]

Had \( \text{Gl} = \text{Gl}_{i_{\alpha}, \rho_0} \circ B \), we would prove that the former one is diffeomorphic. Though this is not case, they are rather close. This is what we do next.

We know that \( \tilde{N} = \mathfrak{V}_0N \). By \( \mathfrak{V}_0 \) we mean a neighborhood of identity \((0, 1) \in \mathfrak{V}\). So it is sufficient to define a map \( b : V \times \text{Glu}(\rho_0) \to \mathfrak{V}_0 \). This is given by
\[ b(y, \rho) = (r_0^{-1}(y - y_{0}), r_0^{-1}(\rho)). \]

So
\[ B(y, u, \rho) = b(y, \rho) \cdot u. \]

Set
\[ \text{Gl}_{i_{\alpha}, \rho_0} = \text{Gl} \circ B^{-1}. \]

And think of \( \text{Gl}_{i_{\alpha}, \rho_0} \) and \( \text{Gl}_{i_{\alpha}, \rho_0} \) are both maps from \( \tilde{N} = \mathfrak{V}_0N \) to \( \mathcal{M}_{Y_1}(M, A) \).

Let
\[ \phi_0 = \text{pgl}(u_{0}, i_{0}, \rho_0), \hat{u}_0 = \text{Gl}(u_{0}, i_{0}, \rho_0). \]

Set
\[ \phi = \text{pgl}(u_{\alpha}, i_{\alpha}, \rho_0), \hat{u}_0 = \text{Gl}(u_{\alpha}, i_{\alpha}, \rho_0). \]

We know that \( \text{Gl}_{i_{\alpha}, \rho_0} \) induces the following data of a coordinate chart:

1. \( O_{i_{\alpha}, \rho_0} = \text{pgl}(\mathfrak{V}_0N, i_{\alpha}, \rho_0); \)
2. \( \phi_{i_{\alpha}, \rho_0} : O_{i_{\alpha}, \rho_0} \times L \to W \) given by
\[ \Phi_{i_{\alpha}, \rho_0}(\phi, \eta) = \phi + Q_{\phi, i_{\alpha}} \eta; \]
3. \( \text{pert}_{i_{\alpha}, \rho_0} : O \to L \) the map that yields the gluing map \( \text{Gl}_{i_{\alpha}, \rho_0} \).

**Theorem 12.2.** The following is the data of a coordinate chart induced by \( \text{Gl}_{i_{\alpha}, \rho_0} : \)

1. \( O := \text{pgl}(V \times N \times \text{Glu}(\rho_0) \times \{i_{\alpha}\}) = \text{pgl}(B^{-1}(\mathfrak{V}_0N), i_{\alpha}); \)
2. \( \Phi : O \times L \to W \) given by
\[ \Psi(\phi, \eta) = \psi + Q_{\psi, i_{\alpha}} \eta; \]
3. \( \text{pert} : O \to L \) the map that yields the gluing map \( \text{Gl}. \)

**Proof.** We only need to verify that \( \psi \) is an isomorphism. Since \( O \cong \mathfrak{V}_0N \), it is equivalent to show that
\[ \Phi = \Phi_{i_{\alpha}, \rho_0} \circ \text{pgl} : \mathfrak{V}_0N \times L \to W \]
is an isomorphism.

On the other hand, by [12.1] we know that
\[ \Phi = \Phi_{i_{\alpha}, \rho_0} \circ \text{pgl} : \mathfrak{V}_0N \times L \to W \]
is an isomorphism. Now both maps \( \tilde{\Psi} \) and \( \tilde{\Phi} \) have same domain and range. We claim that when \( r_o \) is small,

\[
\| \tilde{\Psi} - \tilde{\Psi} \| \leq \epsilon,
\]

(12.5)

\[
\| d\tilde{\Psi} - d\tilde{\Psi} \| \leq \epsilon
\]

(12.6)

for some small \( \epsilon \). Then that the isomorphism of \( \tilde{\Psi} \) implies that of \( \tilde{\Phi} \). The proof of (12.5) and (12.6) is rather straightforward but tedious. The proof of them is explained below. q.e.d.

**Definition 12.3.** Let \( Gl_1 \) and \( Gl_2 \) be two gluing maps defined on same (local) domain. Let \( \Psi_1 \) and \( \Psi_2 \) are corresponding maps defined in the form as (12.4). We say

\[
Gl_1 \approx Gl_2
\]

if \( \Psi_1 - \Psi_2 \) satisfies (12.5) and (12.6).

To compare \( \tilde{\Psi} \) and \( \tilde{\Phi} \), we should go through the process of gluing and compare them in each step.

**Pre-gluing maps.** We first compare the pre-gluing maps for two different gluing processes.

Suppose \( (t, z) \in \mathcal{B} \) is given. Let \( (y, \rho) = b^{-1}(t, z) \). Let \( u = (u_1, u_2) \in N \) and \( \tilde{u} = (u_1, (t, z) \cdot u_2) =: (u_1, \tilde{u}_2) \).

Set

\[
j_y = ((\Sigma_1, j_o), y, \tilde{u}_2).
\]

The pre-gluing for \( Gl'_{j_o, \rho} \) is \( \text{pgl}(u, j_y, \rho) \) and that for \( Gl_{j_o, \rho} \) is \( \text{pgl}(\tilde{u}, \tilde{j}_o, \rho_o) \). We denote them by

\[
\text{pgl}_{j_o, \rho} : \mathcal{B}_0N \to W
\]

respectively. We have

**Proposition 12.4.** Let \( (t, z) = b(y, \rho), u \in N, \) i.e,

\[
\rho = r_o z, y = r_o t.
\]

Then

\[
\| \text{pgl}_{j_o, \rho}((t, z) \cdot u), \text{pgl}_{j_o, \tilde{j}_o}((t, z) \cdot u) \|_{L^1 \cdot \rho} \leq C|t|\sqrt{r_o}.
\]

Here \( C \) is a constant independent of \( r_o \).

**Proof.** Set

\[
\phi' = \text{pgl}_{j_o, \rho}((t, z) \cdot u); \phi = \text{pgl}_{j_o, \tilde{j}_o}((t, z) \cdot u).
\]

By the construction of pre-gluing, \( \phi' \) and \( \phi \) are maps on \( \Sigma_{y, \rho} \) and \( \Sigma_{y, \rho_o} \). We should identify them properly: in fact, both of them are identified with \( \Sigma_1 \) in a canonical way and so they are identified. In particular, we explain how two sphere components identified. We name the spheres \( S^2_y \) and \( S^2_{y_o} \). Let

\[
C_y = S^2_y - \{\infty\}; C_{y_o} = S^2_{y_o} - \{\infty\}.
\]

We write down the identification map

\[
w : C_y \to C_{y_o};
\]

\[
w(z) = (t_o^{-1}(y + \rho z^{-1}))^{-1},
\]
The inverse of $w$ is
\[ w^{-1}(z) = \left( \rho^{-1}(r_0 x^{-1} - y) \right)^{-1}. \]

Explicitly, we write down $\phi$ and $\phi'$ on $\Sigma_{y_o, \rho_o}$. We separate $\Sigma_{y_o, \rho_o}$ into three pieces:

- $P_1 := \Sigma_1 = \Sigma - D_{y_o}(2\sqrt{r_0})$;
- $P_2 := \Sigma_2 = \Sigma - D_{y_o}(2\sqrt{r_0})$;
- $P_3 = D_{y_o}(2\sqrt{r_0}) - D_{y_o}(\sqrt{r_0}/2) \subset \Sigma_1$.

On $P_3$,
\[
\phi = \phi' = u(y_o).
\]

On $P_1$,
\[
\phi(z) = u(y_o) + \beta(\frac{z}{\sqrt{r_0}})(u_1(z) - u(y_o)),
\]
\[
\phi'(z) = u(y_o) + \beta(\frac{z - y}{\sqrt{r}})(u_1(z) - u(y_o)).
\]

On $P_2$,
\[
\phi(z) = u(y_o) + \beta(\frac{z}{\sqrt{r_0}})(\tilde{u}_2(z) - u(y_o)),
\]
\[
\phi'(z) = u(y_o) + \beta(\frac{w^{-1}(z)}{\sqrt{r}})(\tilde{u}_2(z) - u(y_o)).
\]

Clearly, to prove the proposition, the computation of cut-off functions is involved. We need the results from appendix ??.

We explain the computation on $P_1$.
\[
\phi(z) - \phi'(z) = (\beta(\frac{z}{\sqrt{r_0}}) - \beta(\frac{z - y}{\sqrt{r}}))(u_1(z) - u(y_o)).
\]

This is supported in $D_{y_o}(3\sqrt{r_0}N) - D_{y_o}(\sqrt{r_0}N/2)$ And we have estimates in this area:
\[
|\beta(\frac{z}{\sqrt{r_0}}) - \beta(\frac{z - y}{\sqrt{r}})(u_1(z) - u(y_o))| \leq C|\frac{y}{\sqrt{r_o}}|\sqrt{r_0}^{1-2/p}
\]

and
\[
|\nabla \left( \beta(\frac{z}{\sqrt{r_0}}) - \beta(\frac{z - y}{\sqrt{r}})(u_1(z) - u(y_o)) \right) \right| \leq C|\frac{y}{\sqrt{r_o}}|\sqrt{r_0}^{1-2/p} + C|\frac{y}{\sqrt{r_o}}|
\]

Then their $L^p$-norms are bounded by
\[
C|y| + C|y|\sqrt{r_0}^{-1/2} + C|y|\sqrt{r_0}^{-p/2}.
\]

Plug in $y = r_o t$, we have
\[
\|\phi - \phi'\|_{L^1(P_1)} \leq C|t|\sqrt{r_o}.
\]

The computation on $P_2$ is same. Then the claim of proposition follows. q.e.d.

**Remark 12.5.** The key to the whole process is that we use coordinate $(t, z)$ rather than $(y, \rho)$: we note that the computation of cut-off functions with respect to $(y, \rho)$ does not perform friendly, while there is no problem when it is with respect to $(t, z)$. This is due to the factor $r_0$. On the other hand, we know that it is $(t, z) \in \mathcal{B}$ that is essential inspired by the map $\mathcal{B}_{t_o, \rho_o}$. So it is not surprise that the computation behaves well. We will skip the computations of the rest of these type results. It is just a matter of recycling the above computations and those in appendix.
Proposition 12.6. Given \( u \in \mathcal{N} \) and a path \((t(s), z(s)) \in \mathcal{B}, s \in [0, 1)\) with \((t, z) = (t(0), z(0))\) and
\[
(v_1, v_2) = \frac{\partial}{\partial s} \bigg|_{s=0} (t(s), z(s)),
\]
Then
\[
\| \frac{\partial}{\partial s} \bigg|_{s=0} \text{pgl}_{i_o}(t(s), z(s)) - \frac{\partial}{\partial s} \bigg|_{s=0} \text{pgl}_{i_o}'(t(s), z(s)) \cdot u \|_{L^1, p} \leq C \sqrt{\rho_0} \|(v_1, v_2)\|.
\]
In particular, this implies that at \((t, s) \cdot u\)
\[
\|d(\text{pgl}_{i_o} - d\text{pgl}_{i_o}')\| \leq C \sqrt{\rho_0}.
\]
when \((t, s)\) is bounded.

Right inverses. Recall that in the construction of right inverse \(Q\), we first define \(R = \Gamma Q_\phi \Lambda\) and then set \(Q_\phi = R(DR)^{-1}\). Here \(\Gamma\) and \(\Lambda\) involves cut-off functions. Hence we should deal with the derivatives of cut-off functions as well.

For gluing maps \(Gl_{i_o, \rho_o}\) and \(Gl_{i_o, \rho_o}'\) we have two families of right inverses \(Q\) and \(Q'\):
\[
Q = \{Q_{\text{pgl}_{i_o}(x), \rho_o} | x \in \mathcal{B}_0 N\};
\]
\[
Q' = \{Q_{\text{pgl}_{i_o}'(x), \rho_o} | x \in \mathcal{B}_0 N\}.
\]
We may treat them as maps
\[
Q, Q': \mathcal{B}_0 N \times L \to W.
\]
Then

Proposition 12.7. Let \(Q\) and \(Q'\) be as above.
\[
\|Q - Q'\| \leq C \sqrt{\rho_0};
\]
\[
\|dQ - dQ'\| \leq C \sqrt{\rho_0}.
\]

Combine these results, we prove theorem (12.5) and (12.6).

Remark 12.8. We explain the idea that guides us in the above proof. Let \((u_o, \rho_o) \in \tilde{M}_g(X, A) \times C_*\)
and \(U_{u_o} \times \text{Glu}(\rho_o)\) be a neighborhood of this point. We may be expecting a gluing map
\[
\tilde{Gl} : \tilde{U}_{u_o} \times \text{Glu}(\rho_o) \to \mathcal{M}_{g, m}(M, A).
\]
We may construct a gluing map defined on a proper chosen slice. This is essentially what \(\tilde{Gl}\) does. Another reasonable approach would be \(\tilde{Gl} \Rightarrow \tilde{Gl}_1 \Rightarrow \tilde{Gl}'\). Here
\[
\tilde{Gl}_1 : \tilde{U}_{u_o} \times \{\rho_o\} \to \mathcal{M}_{g, m}(M, A).
\]
We use \(\text{Glu}(\rho_o) \cong m\).

Set
\[
\tilde{i}_{o,y} = ((\Sigma_1, i_{o1}, y), i_o); i_o = \tilde{i}_{o,y}.
\]
Let
\[
\tilde{M}V(X, A) = \prod_{y \in V} \mathcal{M}_{i_{o,y}}(X, A).
\]
Suppose that
\[ \tilde{\mathcal{M}}_{j_0, y}(X, A) \cong \tilde{\mathcal{M}}_{j_0}(X, A) \]
and
\[ \tilde{\mathcal{M}}_V(X, A) \cong \tilde{\mathcal{M}}_{j_0}(X, A) \times V. \]
Set
\[ \tilde{U}_{u_0, y_0} = \tilde{U}_{u_0} \cap \tilde{\mathcal{M}}_{j_0}(X, A). \]
Then using a natural identification of \( V \) with \( t \) we reduce \( \tilde{\text{Gl}}_1 \) to \( \text{Gl}' : \tilde{U}_{u_0, y_0}(X, A) \times \{ \rho \} \to \mathcal{M}_{g, m}(M, A). \)

Elements in \( \tilde{U}_{u_0, y_0} \) are treated pre-stable by adding two marked points on \( j_0^2 \). Hence \( \text{Gl}' \) is exactly \( \text{Gl}_{j_0, \rho_0} \). So it is not surprise to have a natural comparison between \( \text{Gl} \) and \( \text{Gl}' \). Although the computation is tedious, it is quite straightforward.

From this, we also see that in this local comparison \( \text{Glu}(r_0) \) can always compare with \( m \). So we will always cancel \( \text{Glu}(r_0) \) with \( m \) when the similar issue occurs.

Case 2b and 2d. These two cases are simpler. The group \( \mathfrak{B} \) in case 2a is replaced by \( m < \mathfrak{B} \). We skip them.

Case 2c. This is a relatively new case. The point is that the resultant curves after gluing are pre-unstable. The treatment of this case is same as case 3. We discuss case 3 directly.

12.3. Gluing maps: case 3. Both
\[ (\Sigma_1, i_1, x_1, \ldots, x_{m_1}, y) \quad \text{and} \quad (\Sigma_2, i_2, x_{m_1+1}, \ldots, x_m, y) \]
are unstable, \( \Sigma_j = S^2, j = 1, 2 \) and \( m_1 \leq 1, m - m_1 \leq 1 \). We take the most complicated case: \( m = 0 \). To tell the difference between two components, we mark spheres by \( S_j^2 \). Namely
\[ \Sigma = (S_1^2, y = \infty_1) \cup (S_2^2, y = \infty_2). \]
Then
\[ \text{Aut}(\Sigma) = \mathfrak{B}_1 \times \mathfrak{B}_2. \]
We put the subscripts to tell the difference. We define a normal subgroup of \( \text{Aut}(\Sigma) \)
\[ \text{Aut}_y(\Sigma) = \{ (\psi_1, \psi_2) | \psi_i \in \text{Aut}(\Sigma_i), d\psi_1(y_1) \otimes d\psi_2(y_2)|_{T_{y_1}(\Sigma_1) \otimes T_{y_2}(\Sigma_2)} = 1 \}. \]
Set
\[ \Delta^*(m) = \{ (m_1, m_2) \in m_1 \times m_2 | m_1 m_2 = 1 \}. \]
By direct computation, we have
\[ \text{Lemma 12.9.} \quad \text{Aut}_y(\Sigma) = t_1 \times t_2 \times \Delta^*(m). \]
Then
\[ \frac{\text{Aut}(\Sigma)}{\text{Aut}_y(\sigma)} \cong m \cong m_2. \]
Now we consider the gluing. For each component, we use balanced curves, i.e,
\[ \mathcal{M}_{0,1}(X, A_j) = \frac{M_{0,1}^h(M, A_j)}{S_1}. \]
Therefore
\[ M_S(X, A) = \frac{M_S^b(X, A)}{S^1 \times S^1}, \]
where \( A = A_1 + A_2 \). For simplicity, we assume that the stratum is compact. The gluing is

(12.12) \[ Gl : M_S^b(X, A) \times S^1 \times S^1 \times C^*_\epsilon \to \tilde{M}_{0,0}(X, A). \]

or in a more precise form, the right hand side is treated as a subset
\[ M_S^b(X, A) \times S^1 \times S^1 \times C^*_\epsilon \subset M_{0,0}^b(M, A_1) \times S^1 \times C^*_\epsilon \times M_{0,0}^b(M, A_2). \]

Note that \[ M_{0,0}(X, A) = \frac{\tilde{M}_{0,0}(X, A)}{\text{Aut}(S^2)}. \]

In order to show that \( Gl \) defined in (12.12) induces a local-diffeomorphic gluing map
\[ M_S^b(X, A) \times S^1 \times S^1 \times C^*_\epsilon \to M_{0,0}(X, A). \]
we should conclude that

Theorem 12.10. The image of \( Gl \) represents a slice in \( \tilde{M}_{0,0}(X, A) \) with respect to the action of \( \text{Aut}(S^2) \).

We now explain the idea following the guide line given in remark 12.8 to speculate the proof. The key is proposition 12.11.

Locally, an expecting map is

(12.13) \[ \tilde{Gl} : \frac{U}{\mathcal{B}_1 \times \mathcal{B}_2} \times \text{Glu}(r_o) \to \frac{\tilde{M}_{0,0}(X, A)}{\text{Aut}(S^2)}. \]

Here \( U \subset \tilde{M}_S(X, A) \) is a small open subset in the stratum that is \( \mathcal{B}_1 \times \mathcal{B}_2 \) invariant. Again, \( U \) should be thought as a subset of
\[ \tilde{M}_{0,1}(X, A_1) \times \tilde{M}_{0,1}(X, A_2) \]
and
\[ \frac{U}{\mathcal{B}_1 \times \mathcal{B}_2} \subset \frac{\tilde{M}_{0,1}(X, A_1)}{\mathcal{B}_1} \times \frac{\tilde{M}_{0,1}(X, A_2)}{\mathcal{B}_2}. \]

The left hand side of (12.13) can be written as
\[ \left( \frac{U}{\text{Aut}_y(\Sigma)} \right) / m_1 \times \text{Glu}(r_o). \]
As before, locally \( m_1 \) is cancelled by \( \text{Glu}(\rho_0) \), and we have
\[ \tilde{Gl}_1 : \frac{U}{\text{Aut}_y(\Sigma)} \times \{ \rho_0 \} \to \frac{\tilde{M}_{0,0}(M, A)}{\text{Aut}(S^2)}. \]

Next we need an important fact for this kind of gluing.

Proposition 12.11. For any \( \rho \leq r_o \), there exists a neighborhood \( V \) of \( \text{id} \) in \( \text{Aut}_y(\Sigma) \), a neighborhood \( V' \) of \( \text{id} \) in \( \text{Aut}(S^2) \) and a diffeomorphism map
\[ gl : V \to V'. \]

Here \( gl \) is construct via gluing process.
We skip the proof. Using this fact:

\[ \text{Aut}_y(\Sigma) \cong \text{Aut}(S^2) \]

locally, we would show that the image of \( Gl \) is a slice.

**Sketch the proof of theorem 12.10.** First we introduce a slice of

\[ \frac{\tilde{M}_S(X, A)}{\text{Aut}_y(\Sigma)}. \]

We say an element \((u_1, u_2) \in \tilde{M}_S(X, A)\) is balanced with respect to \( \text{Aut}_y(\Sigma) \) if \( u_1 \) is balanced and \( u_2 \) is centered. Let \( \tilde{M}^b_S(X, A) \) denote the set of such elements. It is not hard to see that

\[ \frac{\tilde{M}_S(X, A)}{\text{Aut}_y(\Sigma)} = \frac{\tilde{M}^b_S(X, A)}{S^1_1 \times S^1_2}, \]

and

\[ \tilde{M}_S^b(X, A) = \frac{\tilde{M}_S^b(X, A)}{m_2}. \]

Set

\[ Gl_1 : \tilde{M}_S^b(X, A) \times \{\rho_0\} \to \tilde{M}_0(X, A) \]

to be a gluing map. Then locally

\[ Gl \approx Gl_1 \]

in the sense of definition 12.3. The problem is now translated to show that the image of \( Gl_1 \) is a slice in \( \tilde{M}_0(X, A) \).

Take a slice \( N \) in \( \tilde{M}_S^b(X, A) \). A neighborhood of \( N \) in \( \tilde{M}_S(X, A) \) is \( V \cdot N \). Define a map

\[ Gl_2 : V \cdot N \times \{\rho_0\} \to \tilde{M}_0(X, A) \]

by

\[ Gl_2(v, n) = gl(v) \cdot Gl(n). \]

We compare it with the original gluing map, extending \( Gl_1 \),

\[ Gl : V \cdot N \times \{\rho_0\} \to \tilde{M}_0(X, A). \]

By using the property of \( gl \), it is straightforward to show that

\[ Gl \approx Gl_2. \]

Since \( Gl \) is local diffeomorphic, we conclude that

\[ Gl_2(1 \cdot N) = Gl(1 \cdot N) \]

represents a slice in \( \tilde{M}_0(X, A) \). This proves the theorem.
12.4. On gluing maps for lower strata. We generalize our results from 1-nodal case to general strata.

**Corollary 12.12.** The gluing map

\[ Gl_{S,S'} : \mathcal{L}^0_{S,S',\epsilon} \to \mathcal{M}_\Sigma(X,A) \]

gives a coordinate chart for \( \mathcal{M}_{\Sigma'}(X,A) \).

Similarly,

**Corollary 12.13.** The isomorphism

\[ Gl^*_{S,S'}(\mathcal{L}_{\Sigma'}) = \mathcal{L}_{\Sigma} \]

is diffeomorphic.

Set \( W = Gl_{S,S'}(\mathcal{L}^0_{S,S',\epsilon}) \). Since \( Gl_{S,S'} \) is diffeomorphic, the gluing map

\[ Gl'_{S',S''} : \mathcal{L}_{S',S'',\epsilon}|_W \to \mathcal{M}_{\Sigma'}(X,A) \]

given by

\[ Gl'_{S',S''} = Gl_{S,S''} \circ Gl^{-1}_{S,S'} \]

is a diffeomorphism.

Moreover,

**Corollary 12.14.** \( Gl'_{S',S''} \) is admissible, so it is \( C^0 \)-compatible with \( Gl_{S',S''} \).

**Proof.** \( Gl'_{S',S''} \) is admissible by its construction and definitions. The second assertion follows from §11. q.e.d.

13. Smooth structures on \( \overline{\mathcal{M}}_{g,m}(X,A) \)

13.1. **Topology on** \( \overline{\mathcal{M}}_{g,m}(X,A) \). By far, \( \overline{\mathcal{M}}_{g,m}(X,A) \) is a union of strata, each of which is a smooth orbifold. We have not defined the topology on the whole set. This is provided by gluing maps.

Recall that for any \((u,i) \in \mathcal{M}_\Sigma(X,A)\), there exists a neighborhood \( U \subset \mathcal{M}_\Sigma(X,A) \) of \((u,i)\) and \( \epsilon \) such that the gluing map

\[ Gl_\Sigma : \mathcal{L}_{\Sigma,\epsilon}|_U \to \overline{\mathcal{M}}_{g,m}(X,A) \]

exists. We define the image of \( Gl_\Sigma \) to be a neighborhood of \((u,i) \in \overline{\mathcal{M}}_{g,m}(X,A)\). By this way, we may define a topological base at \((u,i)\): to see we form a topological base, we use the property of \( C^0 \)-compatibility between gluing maps, which says that any two such open sets are compatible. Therefore, we have a topology on \( \overline{\mathcal{M}}_{g,m}(X,A) \). In fact, we have

**Theorem 13.1.** \( \overline{\mathcal{M}}_{g,m}(X,A) \) is a topological orbifold.

**Proof.** For each point \((u,i) \in \mathcal{M}_\Sigma(X,A)\), a neighborhood described above has a coordinate chart:

\[ (\mathcal{L}_{\Sigma,\epsilon}|_U, Gl_\Sigma) \]

The transition maps between any two charts are \( C^0 \). Hence it is an orbifold. q.e.d.
13.2. Smooth structures on $\overline{\mathcal{M}}_{g,m}(X,A)$. In this subsection, we explain that there exists an atlas such that $\overline{\mathcal{M}}_{g,m}(X,A)$ is smooth. However, we do not show any two atlas are compatible.

**Definition 13.2.** A stratum-covering of $\overline{\mathcal{M}}_{g,m}(X,A)$ consists of $U_S, \epsilon_S$ for each stratum such that

- $U_S$ is a proper subset of $\mathcal{M}_S(X,A)$;
- there exists a $Gl_S$ on $L_{S,\epsilon_S}|U_S$;
- for $W_S = Gl_S(L_{S,\epsilon_S}|U_S)$, $W_S \cap W_{S'} \neq \emptyset$ if and only if $S \prec S'$ (or, $S' \prec S$);
- $\{W_S\}_{S \in p_{g,m}}$ is a covering of $\overline{\mathcal{M}}_{g,m}(X,A)$;

The following lemma shows that stratum-coverings are abundance.

**Lemma 13.3.** There are many stratum-coverings.

**Proof.** Set $D = D_{g,m}$. Let $S_0$ be the set of smallest strata $S \in D$. Choose $U_S = \mathcal{M}_S(X,A)$.

They are compact. By the gluing theory, there exists $\epsilon_S$ such that the gluing map exists on $L_{S,\epsilon_S}$. If we choose $\epsilon_S$ small, we may have $W_S \cap W_{S'} = \emptyset$.

Inductively, let $S_k$ be the set of smallest strata $S \in D - S_{k-1}$.

Suppose that $U_S, \epsilon_S$ are chosen for all $S \in S_i, i \leq k - 1$. Set $W_{S,S'} = Gl((L_{S,S'},\epsilon_S)|U_S)$.

For any $S \in S_k$ we choose a proper open set $U_S$ such that $\{W_{S',S}|S' \prec S\} \cup \{U_S\}$ covers $\mathcal{M}_S(X,A)$. Moreover, we choose $\epsilon_S$ such that there exists a gluing map $Gl_S$ defined on $L_{S,\epsilon_S}|U_S$ and $W_S$ is disjoint with other $W_{S'}$ unless $S' \prec S$. Inductively, this construct a stratum-covering. Since we are free to choose $\epsilon_S, U_S$ (except $S \in S_0$) and $Gl_S$, hence there are many choices of stratum-coverings. q.e.d.

Note that for a given stratum-covering, we have an atlas on $\overline{\mathcal{M}}_{g,m}(X,A)$ given by $(L_{S,\epsilon_S}|U_S, Gl_S)$.

Given such an atlas, we ask if the transition maps between any two charts $Gl_S \circ Gl^{-1}_{S'}$ are smooth. If so, we have shown the smoothness of $\overline{\mathcal{M}}_{g,m}(X,A)$. However, this may be too tedious and not true. Instead, we show that there exists certain $Gl_S$ for each $S$ such that

$Gl_S \circ Gl^{-1}_{S'}$

are smooth for any pair $(S \prec S')$.

The main idea is given by the following. Let $S_i \in D, i = 1, 2$, with $S_1 \prec S_2$. Let $U_{S_i}$ be proper open subsets of $\mathcal{M}_{S_i}, i = 1, 2$. Suppose that we have a gluing map $Gl_{S_1} : L_{S_1,\epsilon_1}|U_{S_1} \rightarrow \overline{\mathcal{M}}_{g,m}(X,A)$. 
Set
\[ W_{S_1, S_2} = Gl_{S_1}(L_{S_1, S_2, \epsilon}|U_{S_2}) \cap U_{S_2}; \]
\[ W'_{S_1, S_2} = Gl_{S_1}(L_{S_1, S_2, 0.5 \epsilon}|U_{S_2}) \cap U_{S_2}; \]
\[ W''_{S_1, S_2} = Gl_{S_1}(L_{S_1, S_2, 0.75 \epsilon}|U_{S_2}) \cap U_{S_2}; \]

As explained in \[\text{§9.4}\], \( Gl_{S_1} \) induces a gluing map \( Gl'_{S_2} \) on \( W_{S_1, S_2} \). We show that

**Proposition 13.4.** There exists \( \epsilon_2 \) and gluing map
\[ Gl_{S_2} : L_{S_2, \epsilon_2}|U_{S_2} \to \overline{M}_{g, m}(X, A) \]
such that \( Gl_{S_2} = Gl'_{S_2} \) on
\[ L_{S_2, \epsilon_2}|W'_{S_1, S_2}. \]

**Proof.** By the gluing theory, there exist \( \epsilon \) and a type-1 gluing map
\[ Gl'_{S_2} : L_{S_2, \epsilon}|U_{S_2} \to \overline{M}_{g, m}(X, A). \]

Note that \( Gl'_{S_2} \) is of type-2. We use a cut-off function on gluing parameter to patch these two gluing maps. To be precise, let us introduce coordinates: by local coordinates, a point in \( W_{S_1, S_2} \) is denoted by
\[ Gl(u, j, \rho), (u, j, \rho) \in L_{S_1, S_2}. \]

Let \( \beta \) be a cut-off function such that
\[ \beta(t) = \begin{cases} 
1, & t \leq 0.5 \epsilon \\
0, & t \geq 0.75 \epsilon.
\end{cases} \]

For an admissible gluing, we start with a coordinate data \((V, \Phi, F)\). Suppose this is the data used for \( Gl'_{S_2} \). Namely,
\[ V = pgls_{S_1, S_2}(L_{S_1, S_2, \epsilon_1}) \]
and
\[ F : V \to \mathcal{M}_{S_2}(X, A) \]
realizes the gluing map. In terms of formula, it says
\[ Gl_{S_1, S_2}(u, j, \rho) = pgls_{S_1, S_2}(u, j, \rho) + \text{pert}(pgls_{S_1, S_2}(u, j, \rho)). \]

Now we define \( V' \) to be
\[ V' = \{ pgls_{S_1, S_2}(u, j, \rho) + \beta(\rho)\text{pert}(pgls_{S_1, S_2}(u, j, \rho)) \}. \]

Start with \((V, Q)\), it is easy to generate a new pair \((V', Q')\). Therefore, we define a new admissible gluing map \( Gl_2 \) based on this coordinate data. Since the part of \( V' \) is in \( \mathcal{M}_{S_2}(X, A) \) when \( |\rho| \geq 0.75 \epsilon \), we may extend \( Gl_{S_2} \) over \( U_{S_2} \). q.e.d.

We remark that the cut-off function used to patch two gluing maps is a function on \( L_S \). We call the method to be patching gluing maps.

**Theorem 13.5.** There exists a stratum-covering \((U_S, \epsilon_S)\) and gluing maps \( Gl_S \) such that for any \( S \) \( Gl_S \) agrees with any gluing map \( Gl'_S \) induced from \( Gl_{S'}, S' \preceq S \), on the overlapping domain.
Proof. We use the same process as in lemma 13.3. For $S \in S_0$, no modification is needed. Suppose the construction is done for all $S \in S_l, l \leq k - 1$.

Let $S \in S_k$. For any $S' \prec S$, set $W_{S', S}$ as before. Let 

$$W_S = \cup_{S' \prec S} W_{S', S}.$$

For the moment, we denote $Gl_S(S')$ for the gluing map defined over $W_{S', S}$ induced by $Gl_{S'}$. We assert that $Gl_S(S') = Gl_S(S'')$ over $W_{S', S} \cap W_{S'', S}$. First of all, by the definition of stratum-covering, the intersection is non-empty if and only if $S'' \prec S'$. Since $Gl_{S'}(S'') = Gl_{S'}$ over $W_{S', S'} \cap U_{S'}$, hence they induce same gluing maps on stratum $M$. So totally, we have a gluing map $Gl_S'$ over $W_S'$ induced by all gluing maps from lower strata. For any gluing map $Gl_{S''}'$ defined over $U_{S''}$, we may apply proposition 13.4 and get a new gluing map $Gl_S$ that is a patching of $Gl_S'$ and $Gl_{S''}'$. Then by induction, we complete the construction. q.e.d.

As a corollary, we have

**Theorem 13.6.** $\overline{M}_{g,m}(X, A)$ admits smooth structure.

### Part IV. Virtual theory on $\overline{M}_{g,m}(X, A)$

In [6], we introduce a new concept "virtual manifolds/orbifolds". Furthermore, we develop the integration theory on them, which including the equivariant integration and localization formulae. The background of the concept is to define invariants on the moduli spaces from Fredholm systems. In this part, our goal is to construct a (smooth) virtual orbifold from $\overline{M}_{g,m}(X, A)$. Then all the theory on virtual orbifolds can be applied to this particular moduli space. Therefore the virtual localization formulae of Gromov-Witten invariants follow.

In §14, we review the material of virtual orbifolds in [6]. Then we construct the virtual orbifold structure on $\overline{M}_{g,m}(X, A)$ in §15-§17. An application is given in §18.

### 14. Virtual orbifolds

14.1. Basic concepts. Let $N = \{1, \ldots, n\}$ and $\mathcal{N} = 2^N$ be the set of all subsets of $N$. Let 

$$\mathcal{X} = \{X_I | I \in \mathcal{N}\}$$

be a collection of sets indexed by $\mathcal{N}$. For any $I \subset J$ there exist $X_{I,J} \subset X_I, X_{J,I} \subset X_J$ and a surjective map 

$$\phi_{I,J} : X_{I,J} \rightarrow X_{I,J}.$$

Set $\Phi = \{\phi_{I,J} | I \subset J\}$. We always assume that $X_\emptyset \neq \emptyset$.

**Definition 14.1.** A pair $(\mathcal{X}, \Phi)$ is called patchable if for any $I, J \in \mathcal{N}$ we have

P1. $X_{I \cup J, I \cup J} = X_{I \cup J, I} \cap X_{I \cup J, J}$;

P2. $X_{I \cap J, I \cap J} = X_{I \cap J, I} \cap X_{I \cap J, J}$;

P3. $\phi_{I \cup J, I \cup J} = \phi_{I, I \cap J} \circ \phi_{I \cup J, J} = \phi_{I, J \cap I} \circ \phi_{I \cup J, I}$;

P4. $\phi_{I \cup J, I \cap J} (X_{I \cup J, I \cap J}) = \phi_{I, I \cap J}^{-1} (X_{I \cap J, I \cup J})$;
P5. \( \phi_{I,J,I}(X_{I,J,I \cap J}) = \phi_{J,I,I}(X_{I \cap J,I \cup J}) \).

Set
\[
X_{I,J} = \phi_{I,J,I}(X_{I,J,I \cap J}) = \phi_{I,J,I}(X_{I \cap J,I \cup J}),
\]
\[
X_{J,I} = \phi_{I,J,J}(X_{I,J,I \cap J}) = \phi_{I,J,J}(X_{I \cap J,I \cup J}).
\]

There is an equivalence relation for points in \( \bigcup X_I \): For \( x \in X_I \) and \( y \in X_J \) we say that \( x \sim y \) if and only if there exists a \( K \subseteq I \cap J \) such that
\[
\phi_{I,K}(x) = \phi_{J,K}(y).
\]

We "patch" \( X_I \) together and get a set
\[
X = \bigcup_{I \in N} X_I / \sim.
\]

A virtual manifold is a patchable pair \((X, \Phi)\) with specified properties.

**Definition 14.2.** Let \((X, \Phi)\) be a patchable pair. Suppose that

- \( X_I \in X \) are smooth orbifolds;
- \( X_{I,J} \) and \( X_{J,I} \) are open suborbifolds in \( X_I \) and \( X_J \) respectively;
- \( \Phi_{J,I} : X_{J,I} \to X_{I,J} \) is an orbifold vector bundle.

Then \((X, \Phi)\) is called a virtual orbifold if for any \( I \) and \( J \),
\[
\phi_{I,J,I} : X_{I,J} \to X_{I \cap J,I \cup J},
\]
\[
\phi_{J,I,J} : X_{J,I} \to X_{I \cap J,I \cup J}
\]
are orbifold vector bundles and
\[
X_{I,J,I \cap J} = X_{I,J} \times_{X_{I \cap J,I \cup J}} X_{J,I}.
\]

We call
\[
X = \bigcup_{I \in N} X_I / \sim
\]
the virtual space of \((X, \Phi)\). We denote the projection map \( X_I \to X \) by \( \phi_I \).

Let \( d_I \) be the dimension of \( X_I \). We call \( d_0 \) the virtual dimension of \((X, \Phi)\).

One can also define virtual manifolds/orbifolds with boundary. From now on, for simplicity, we forget the orbifold singularities and focus on manifolds only.

The following example gives a typical method to construct virtual manifolds.

**Example 14.3.** Let \( X \) be a manifold. Let \( \{U_0, U_1, \ldots, U_n\} \) be an open cover of \( X \).
Let \( U_i^n = \overline{U_i}, i \geq 1 \). Here \( \overline{U_i} \) just means an open subset whose closure is in \( U_i \).

We use \( U_i^n \) to make the notations more suggestive.

Let \( N = \{1, \ldots, n\} \) and \( I, J, K \) be as before. Define
\[
X_0 = U_0 - \bigcup_{i=1}^n U_i^o,
\]
\[
X_I = \bigcap_{i \in I} U_i - \bigcup_{j \notin I} U_j^o.
\]

Let \( X = \{X_I | I \in N\} \). Define
\[
X_{I,J} = X_{J,I} = X_I \cap X_J.
\]

All possible \( \psi_{J,I} \) are taken to be identities and let \( \Phi = \{\phi_{J,I}\} \). Then \((X, \Phi)\) is a virtual manifold (cf. Proposition ??). Moreover, the virtual space \( X \) is \( X \).
We can define differential forms on virtual manifolds. There are two types. The first type is nature. Let \((\mathcal{X}, \Phi)\) be a virtual manifold.

**Definition 14.4.** A \(k\)-form on \((\mathcal{X}, \Phi)\) is
\[
\alpha = \{ \alpha_I \in \Omega^k(X_I) \mid I \in \mathcal{N} \}
\]
such that
\[
\alpha_J = \phi^*_{J,I} \alpha_I
\]
on \(X_{J,I}\).

This is called a pre-\(k\)-form in [6]. It, in fact, induces a \(k\)-form on the virtual manifold in the sense of [6].

In order to consider the second type of forms, we need Thom forms \(\Theta_{J,I}\) of the bundle \(\Psi_{J,I} : X_{J,I} \to X_{I,J}\). To avoid the unnecessary complication caused by the degree of forms, we always assume that the degree of \(\Theta_{J,I}\) is even.

**Definition 14.5.** A set of forms \(\Theta = \{ \Theta_{J,I} \}_{I \subseteq J}\) is called a transition data of \(\mathcal{X}\) if it satisfies the following compatibilities: for any \(I\) and \(J\),
\[
\Theta_{I \cup J, I \cap J} = \Psi^*_{I \cup J, I} \Theta_{I, I \cap J} \wedge \Psi^*_{I \cup J, J} \Theta_{J, I \cap J}
\]
on \(X_{I \cup J, I \cap J}\).

**Definition 14.6.** A virtual form on \((\mathcal{X}, \Phi)\) is
\[
\mathfrak{z} = \{ \mathfrak{z}_I \in \Omega^*(X_I) \mid I \in \mathcal{N} \}
\]
such that
\[
\mathfrak{z}_J = \phi^*_{J,I} \mathfrak{z}_I \wedge \Theta_{J,I}
\]
on \(X_{J,I}\) for some transition data \(\Theta\). \(\mathfrak{z}\) is called a \(\Theta\)-form on \(\mathcal{X}\).

For either forms or virtual forms, one can define close and compact supported forms. Let \(\mathfrak{z}\) be a compact supported \(\Theta\)-form, one can define integration
\[
\int_X \mathfrak{z}.
\]
The Stokes’ theorem holds for this type integration.

The discussion given above can be generalized to the equivariant case. Let \(G\) be a compact Lie group.

**Definition 14.7.** By a \(G\)-virtual manifold \((\mathcal{X}, \Phi)\), we mean that (a.) \((\mathcal{X}, \Phi)\) is a virtual manifold, (b.) each \(X_I\) is \(G\)-manifold and (c.) \(\Psi_{J,I} : X_{J,I} \to X_{I,J}\) are \(G\)-equivariant bundles for any \(I \subset J\).

To study the \(G\)-equivariant integration theory on \(\mathcal{X}\), we may consider \(G\)-equivariant transition data \(\Theta^G = \{ \Theta^G_{J,I} \}_{I \subseteq J}\). Then similarly, we may define: \(G\)-equivariant forms, \(G\)-equivariant \(\Theta^G\)-forms, and etc. For a compact supported \(\Theta^G\) form \(\zeta = (\zeta_I)\), we can define
\[
\int_X \zeta^G.
\]
The virtual localization formula is stated as
Theorem 14.8. Let $\mathcal{X}$ be a finite dimensional virtual manifold with $G = S^1$ action. Let $X$ be its virtual space. Let $\zeta \in \Omega_{\Theta_G,c}(\mathcal{X}^0)$ and $\alpha \in \Omega^*_G(X)$, then

$$\mu_{\zeta}(\alpha) = \int_{X^G} i^*_{X^G}(\alpha \wedge \zeta)/e_G(X^G).$$

We explain the notations. $\zeta$ is a compact supported $\Theta_G$ forms in the interior of $\mathcal{X}$;

$$\mu_{\zeta}(\alpha) := \int_X \zeta \wedge \alpha;$$

$X^G$ is the fix locus of the action, which itself is a virtual manifold; and $e_G(X^G)$ is the $G$-equivariant Euler class of the virtual normal bundle of $X^G$ in $X$.

14.2. From Fredholm systems to virtual manifolds. We start with the following set-up.

Definition 14.9. A Fredholm system consists of following data:

(B1) let $\pi : \mathcal{F} \to \mathcal{B}$ be a Banach orbifold bundle over a Banach orbifold $\mathcal{B}$;

(B2) let $S : \mathcal{B} \to \mathcal{F}$ be a proper smooth section. In particular, the properness implies that $M = S^{-1}(0)$ is compact;

(B3) for any $x \in M$, let $L_x$ be the linearization of $S$ at $x$

$$L_x : T_x \mathcal{B} \to \mathcal{F}_x.$$ We assume that $L_x$ is a Fredholm operator. Let $d$ be the index of the operator.

We refer the triple $(\mathcal{B}, \mathcal{F}, S)$ as a Fredholm system. $M$ is called the moduli space of the system.

A core topic in studying moduli problems is to define invariants on such a system. This is based on the study of $M$. It is well known that if $L_x$ is surjective for all $x \in M$, $M$ is a compact smooth orbifold. Then $M$ can be thought as a cycle in $H_d(\mathcal{B})$ representing the Euler class of bundle $\mathcal{F} \to \mathcal{B}$. Let $a \in H^d(\mathcal{B}, \mathbb{R})$, define

$$\Phi(a) = \int_M a.$$ The challenging problem is to define invariants when the surjectivity of $L_x$ fails. The virtual technique is introduced to deal with this situation. There are several different versions of this technique, however the main idea is the stabilization, which has become popular since 60’s. Our method follows [16] closely.

We recall stabilization for a Fredholm system. Let $U$ be an open subset of $\mathcal{B}$, let

$$a : \mathcal{O}_U \to U$$

be a rank-$k$ vector bundle, let

$$s : \mathcal{O}_U \to \mathcal{F}_U$$

be a bundle map. Define a map

$$\tilde{S} : \mathcal{O}_U \to \mathcal{F}_U; \tilde{S}(u, o) = (u, S(u) + s(o)),$$

where the expression is given in the form of local coordinates and $S(u) + s(o)$ is the sum on fibers. By abusing the notations, we usually use $S + s$ for $\tilde{S}$ to emphasis that $S$ is stabilized by $s$. 
Let \( \hat{L}_{(u,o)} \) be the linearization of \( \hat{S} \) as a map
\[
\hat{L}_{(u,o)} : T_{(u,o)} \mathcal{O}_U \to \mathcal{F}_U.
\]
We say that the pair \((\mathcal{O}_U, s)\) stabilizes the system \((\mathcal{B}, \mathcal{F}, S)\) at \(U\) if \(\hat{L}_{(u,o)}\) are surjective for all \((u,o) \in \mathcal{O}_U\). Set
\[
V_U = \hat{S}^{-1}(0) \subseteq \mathcal{O}_U.
\]
This is now a smooth manifold of dimension \(d+k\). Clearly, \(M \cap U \subseteq V_U\) and
\[
(u,o) \in M \iff o = 0.
\]
We now explain the existence of local stabilizations.

Suppose \(L_x \) is not surjective for some \(x \in M\). Let \(O^x\) be a finite dimensional subspace of \(\mathcal{F}_x\) such that
\[
\text{Image}(L_x) + O^x = \mathcal{F}_x.
\]
For example, we may take \(O^x\) to be the "cokernel" of \(L_x\).

Let \(U^x\) be a neighborhood of \(x\) in \(\mathcal{B}\). In order to make notations more suggestive, we assume that \(U^x = B_r(x)\) is the radius-\(r\) disk centered at \(x\) and \(cU^x = B_{cr}(x)\) for \(c \in \mathbb{R}^+\).

We can restate this construction by using the concept of Fredholm system. Let \(\sigma^* \mathcal{F} \to \mathcal{O}_U\) be the pull-back bundle over \(\mathcal{O}_U\). \(\hat{S}\) then gives a canonical section of this bundle in an obvious way. For simplicity, we still denote the section by \(\hat{S}\). Therefore, we have a Fredholm system \((\mathcal{O}_U, \sigma^* \mathcal{F}, \hat{S})\). If \((\mathcal{O}_U, s)\) stabilizes the system at \(U\), we say that \((\mathcal{O}_U, \sigma^* \mathcal{F}, \hat{S})\) stabilizes \((\mathcal{B}, \mathcal{F}, S)\) at \(U\). \(V_U \subseteq \mathcal{O}_U\) is the moduli space of the new system.

We may construct a canonical bundle \(\sigma^* \mathcal{O}_U \to V_U\), then there is a canonical section \(\sigma : V_U \to \sigma^* \mathcal{O}_U\) given by \((u,o) \to (u,o,o)\) with respect to the local coordinates. Then \(M \cap U = \sigma^{-1}(0)\). This reduces the infinite dimensional system \((\mathcal{U}, \mathcal{F}_U, S)\) to a finite dimensional system \((V_U, \sigma^* \mathcal{O}_U, \sigma)\). We call \((V_U, \sigma^* \mathcal{O}_U, \sigma)\), or simply \(V_U\), to be the virtual neighborhood of \(M\) at \(U\). Bundles \(\mathcal{O}_U\) and \(\sigma^* \mathcal{O}_U\) are called the obstruction bundles.

Suppose that \(\mathcal{F}_U^x\) is trivialized as \(\mathcal{F}_{U^x} = U^x \times \mathcal{F}_x\). We now describe the stabilization using the notations given above by setting \(U = U^x\):

1. \(\text{(C1)}\) the obstruction bundle is
   \[
   \mathcal{O}_{U^x} = U^x \times O^x;
   \]
2. \(\text{(C2')\, the bundle map \,} s = I^x : \mathcal{O}_{U^x} \to \mathcal{F}_{U^x}\) is the standard embedding via the trivialization of \(\mathcal{F}_{U^x}\) given above.

We may assume that the pair \((\mathcal{O}_{U^x}, I^x)\) stabilizes the system at \(U^x\) if \(U^x\) is chosen small. This explains the existence of local stabilization. The trivialization of \(\mathcal{F}_{U^x}\) prevents us to extend the construction outside \(U^x\). This is "taken care" by modifying the bundle map \(s\) as the following. Let \(\eta^x\) be a cut-off function on \(U^x\) such that \(\eta^x = 1 \text{ in } \frac{U^x}{3U^x}\) and \(= 0\) outside \(\frac{2U^x}{3U^x}\). \(\text{(C2')\, is then replaced by}\)

\(\text{(C2) \, the bundle map} \, \, s^x = \eta^x I^x\).

Clearly, \((\mathcal{O}_{U^x}, s^x)\) stabilizes the system at \(\frac{U^x}{3U^x}\). In this paper, we always use \(\text{(C2)}\) to construct virtual neighborhoods. It turns out that \(\text{(C2)}\) is the key towards the construction of virtual orbifolds from a Fredholm system.
Since $M$ is compact by our assumption, there exists finite points $\{x_i\}_{i=1}^n$ in $M$ such that

$$M \subseteq \bigcup_{i=1}^n \frac{1}{\beta} U^{x_i} =: U,$$

where $U^{x_i}$ are as above.

For simplicity, we set

$$U_i = U^{x_i}, \mathcal{O}_i = \mathcal{O}_{U^{x_i}}, s_i = s^{x_i}.$$

We call the data $\{(U_i, \mathcal{O}_i, s_i)\}$ a local stabilization system of $U$. From such a local stabilization system, one is able to construct a virtual manifold and other data that yield integrations on it. This is stated as

**Proposition 14.10.** Let $(\mathcal{B}, \mathcal{F}, S)$ be a Fredholm system.

1. there exists a local stabilization system $\{(U_i, \mathcal{O}_i, s_i)\}$.
2. Let $X$ be the natural virtual manifold for $\mathcal{B}$ generated by the covering $\{U_i\}$. Using the stabilization data given above, one is able to define a virtual manifold $\mathcal{W} = \{W_i\}$, where $(W_i, \mathcal{O}_i, \sigma)$ is a virtual neighborhood over $U_i$. Let $W$ be the virtual space of $\mathcal{W}$.
3. $\mathcal{O}$ is a virtual bundle over $\mathcal{W}$. $\sigma$ is a section of the bundle;
4. Let $\Theta_i$ be Thom form of $\mathcal{O}_i$. All Thom forms $\Theta_I$ of $\mathcal{O}_I$ restricting on $W_I$ form a $\Theta$-form. Denote the form by $\theta$. If the moduli space $M$ is compact, $\theta \in \Theta^*_\mathcal{O,c}(\mathcal{W})$. $\theta$ is an Euler class of $\mathcal{O}$.
5. For any $a \in \Omega^*(\mathcal{B})$, let $a_I = \pi_I^* a$ on $W_I$. Then $(a_I)_{I \subseteq N} \in \Omega^*(\mathcal{V})$. To abuse the notations, we still denote the form by $a$.

By the proposition, we have $\mu_\theta(a)$. Also we know that this is well defined not only on $\Omega^*(\mathcal{B})$, but also on $H^*(\mathcal{B})$. If a global stabilization as in §?? exists, it is easy to see that

$$\Phi(a) = \mu_\theta(a).$$

This leads to the following definition.

**Definition 14.11.** Let $(\mathcal{B}, \mathcal{F}, S)$ be a Fredholm system. Let $\{(U_i, \mathcal{O}_i, s_i)\}$ be a local stabilization system constructed in Proposition 14.10. Let $\mathcal{W}, \theta$ be the virtual manifold and obstruction form given above. For $a \in H^*(\mathcal{B})$, define the invariants $\Phi(a)$ to be $\mu_\theta(a)$.

One can prove that the invariants is independent of the choice of local stabilization systems.

One can further assume that the Fredholm system admits an $S^1$-action. Then we can construct a $G$-virtual manifold $\mathcal{V}$ from a local $G$-stabilization system. Then we replace $\Theta_i$ by equivariant Thom forms $\Theta_i^G$. So we have $\Theta^G_I$‘s and $\Theta^G_{j,I}$’s. Clearly, $\theta^G = \{\Theta^G_I\}_I$ is a $\Theta^G = \{\Theta^G_{j,I}\}$ form. For any $\alpha \in \Omega^*_G(\mathcal{B})$, define

$$\Phi_G(\alpha) = \mu_\mathcal{V},\theta^G_G(\alpha).$$

Now we can state the virtual localization formula for Fredholm systems. Again, let $G = S^1$. We consider the Fredholm system $(\mathcal{B}, \mathcal{F}, S)$ with $G$-action. Let $\mathcal{V}$ be the virtual orbifold for the moduli space $M$. Let $V$ denote the virtual space. Then $\mathcal{V}^G$ is the virtual orbifold for $M^G$ and its virtual space is $V^G$. We have
Theorem 14.12. Let \((B, F, S)\) be an \(S^1\)-Fredholm system. For \(\alpha \in \Omega_G^*(B)\),
\[
\Phi_G(\alpha) = \int_{V_G} i^*_G \alpha \wedge \theta_G / e_G(V_G) = \mu e_G(V_G)(i^*_G \alpha).
\]

15. Local Stabilizations

15.1. Neighborhoods in \(\bar{\chi}_{g,m}(X, A)\). Let \(u_o \in \bar{\chi}_{g,m}(X, A)\). For simplicity, we will drop \((X, A)\) and write \(\bar{\chi}_{g,m}\). We describe neighborhoods of \(u_o\). Suppose that \(u_o \in \chi_j \subset \chi^S_o\), where \(j_o \in M^S_o\). Within the stratum \(\chi^S_o\) the neighborhoods of \(u_o\) is well defined. Here we give an explicit construction of neighborhoods which may be generalized to \(\bar{\chi}_{g,m}\).

Neighborhoods of \(u_o\) within the stratum: let \(V\) be a neighborhood of \(j_o\) in \(M^S_o\), there is a trivialization of \(\chi_V := \bigcup_{j \in V} \chi_j\) given by \(\phi : \chi_{j_o} \times V \to \chi_V\).

Set \(u_j = \phi(u, j)\).

We now consider two cases with respect to whether \(S\) is stable or not. First, suppose that \(S\) is stable. Let \(U_j(u_j, \delta)\) be an \(\delta\)-neighborhood of \(u_j\) in \(\chi_j\). We define a neighborhood of \(u_o\) in the stratum to be
\[
U_{S_o}(u_o, \delta, V) = \bigcup_{j \in V} U_j(u_j, \delta).
\]

Now if \(S\) is unstable. \(u_o\) may have nontrivial isotropic group \(\text{Aut}(u_o)\). Set
\[
\Lambda_{j_o} = \text{Aut}(u_o) \cdot u_o.
\]

We define a normal bundle of \(\Lambda_{j_o}\): at \(u_o\) we use \(L^2\)-norm to define a normal tangent space \(N_{u_o}\) that is normal to \(\Lambda_{j_o}\), then define \(N_u = \alpha \cdot N_{u_o}\) for \(u = \alpha \cdot u_o\). This automatically define a normal bundle \(N\) over \(\Lambda_{j_o}\) with fiber \(N_u\). Then take a \(\delta\)-disk bundle \(N_\delta\) of \(N\) and use exp mapping it to \(\chi_{j_o}\) to get a neighborhood
\[
U_{i(o,u_o)}(u_o, \delta) = \exp N_{\Lambda_{j_o}} N_\delta.
\]

Similarly, we do this for all \(u_j\) in \(\chi_j\). We put them together and get \(U_{S_o}(u_o, \delta, V)\).

The method provide here is standard to treat nontrivial isotropic groups.

Next we consider “neighborhoods” of \(u_o\) in \(\tilde{\chi}_{g,m}\). Recall that there is a gluing bundle \(L_{S_o}\) over \(M_{S_o}\) and
\[
L_{\tilde{S}_o} \to \chi^S_o
\]
which is \(\pi^* L_{S_o}\) via projection \(\pi : \chi^S_o \to M_{S_o}\). Let \(L_{S_o, \epsilon_o}\) be the \(\epsilon_o\)-disk bundle of \(L_{S_o}\). There is a gluing-surface map
\[
gs : L_{S_o, \epsilon_o} \to \tilde{M}_{g,m}.
\]

Set
\[
j_\rho = gs(j, \rho).
\]

Case 1. We now assume that \(j_o\) is stable. Then \(gs\) is injective and \(j_\rho\) is stable. On \(j_\rho\) there is a map
\[
u_{j_\rho} = \text{pgl}(u_1, \rho).
\]
We define a neighborhood of \( u_o \) in \( \bar{\chi}_{g,m} \) given by

\[
U_{u_o}(\delta, \epsilon_o, V) = \bigcup |\rho| < \epsilon_o \bigcup_{j \in V} U_{j \rho}(u_{j \rho}, \delta).
\]

**Case 2.** Suppose that \( j_o \) is not stable but \( gs(j_o, \rho) \) is stable. The typical example is case (IIa) in §12.2. We take it as an example. To avoid too much complication caused by notations. We follow notations in §12.2. As it is explained, \( gs \) is no longer injective.

The neighborhood of \( j_o \) in \( M_{S_o} \) can be parameterized by

\[
V' \times D_{y_o1},
\]

where \( V' \) is a neighborhood of \( j'_o1 \) in its stratum, denoted by \( M'_{S_o} \), and \( D_{y_o1} \) is a neighborhood of \( y_o1 \) in \( \Sigma_1 \). For \( j = (j'_1, y) \) we write \( u_j \) to be \( u_{j'_1, y} \). Note that \( gs(j_o, \rho) = j'_1 \). Set

\[
u_{j'_1, y, \rho} = \text{pgl}(u_{j'_1, y}, \rho).
\]

Hence on \( \chi'_{j'_1} \) we get a slice

\[
\Lambda_{j'_1} = \{ u_{j'_1, y, \rho} | y \in D_{y_o1}, \rho < \epsilon_o \}.
\]

We then give a \( \delta \) neighborhood of this slice: as usual we use \( L^2 \)-norm to get its normal bundle in \( \chi'_{j'_1} \) and then using exp we map a \( \delta \)-disk on \( \chi'_{j'_1} \) to get a neighborhood. *But* there is a tricky point: the \( L^2 \)-norm is induced from the metrics on \( \Sigma_1 \) and \( X \), here we require the metric on \( \Sigma_1 \) varies as parameters \( y, \rho \) vary. In fact, for fixing \( y \) and \( \rho \) the metric we use is the metric on the connected sum

\[
\Sigma_1 \sharp y, \rho S^2.
\]

We can arrange the metric varies smoothly with respect to \( y \) and \( \rho \). By this way, we get a \( \delta \)-neighborhood

\[
U'_{j'_1}(\Lambda_{j'_1}, \delta)
\]

of \( \Lambda_{j'_1} \). Then the neighborhood of \( u_o \) is defined to be

\[
U_{u_o}(\delta, \epsilon_o, V) = U_{S_o}(u_o, \delta, V) \cup \bigcup_{j'_1 \in V'} U'_{j'_1}(\Lambda_{j'_1}, \delta).
\]

**Case 3.** We consider the case that \( j_o \) and \( gs(j_o, \rho) \) are both un-stable. The typical example is §12.3. The idea is a combination of case 2 and case 1 with non-trivial isotropic groups. We leave the construction to readers.

15.2. Cut-off functions. On neighborhoods \( U_{u_o}(\delta, \epsilon_o, V) \), we can construct smooth cut-off functions easily: let \( \beta_1 \) be a cut-off function such that \( \beta_1(t) = 1, t \leq \delta/4 \) and 0 when \( t \geq \delta/2 \); let \( \beta_2 \) be a cut-off function such that \( \beta_2(t) = 1, t \leq \epsilon_o/4 \) and 0 when \( t \geq \epsilon_o/2 \); let \( \beta_3 \) be a cut-off function on \( V \) which is supported in \( V/4 \), then we set a cut-off function \( \beta_{u_o} \) on \( U_{u_o}(\delta, \epsilon_o, V) \) as

\[
\beta_{u_o}(\exp_{u_{j \rho}} \zeta) = \beta_1(\|\zeta\|)\beta_2(|\rho|)\beta_3(j).
\]
15.3. **Obstruction bundles.** For any \( u \in \mathcal{M}_j(X, A) \subset \overline{\mathcal{M}}_{g,m}(X, A) \) we let \( \text{coker}_u \) be the cokernel of operator \( D_{u,j} \). Choose a cut-off function \( \beta \) on \( j \) such that it is support away from nodal points and \( \text{Aut}(j) \)-invariant. For a proper choice of \( \beta \), namely, if the support of \( \beta - 1 \) is near nodal points, then the space

\[ O_u := \beta \cdot \text{coker}_u \]

is complement to the image of \( D_{u,j} \). For any \( u \) and its neighborhood \( U_u(\delta, \epsilon, V) \) we define the local obstruction bundle

\[ O_u = U_u(\delta, \epsilon, V) \times O_u. \]

15.4. **Local stabilization.** We now can follow the argument in §14. Be equipped with \( O_u \) and cut-off functions \( \beta_u \) we can construct the local stabilization at \( U_u(\delta, \epsilon, V) \) if \( \delta, \epsilon \) and \( V \) are small. Be precise, for small \( \delta, \epsilon \) and \( V \), we can embed \( O_u \) into \( E_u \) properly for any \( u \in U_u(\delta, \epsilon, V) \). Then we define the stabilized equation over \( O_u \) by

\[ \hat{S}_u(u, \xi) = \bar{\partial}_{J,u} + \beta_u(u) \xi =: (\bar{\partial} + s_u)(u, \xi). \]

This finishes the construction of local stabilization for \( \overline{\mathcal{M}}_{g,m}(X, A) \).

16. **Virtual structures for \( \overline{\mathcal{M}}_{g,m}(X, A) \) and the Gromov-Witten invariants**

16.1. **Virtual orbifold structures on \( \overline{\mathcal{M}}_{g,m}(X, A) \).** As explained in §14 the existence of local stabilization and the compactness of \( \overline{\mathcal{M}}_{g,m}(X, A) \) imply that there is a virtual (topological) orbifold for \( \overline{\mathcal{M}}_{g,m}(X, A) \). We formulate notations.

Suppose that there are \( n \) points

\[ \Lambda = \{u_1, \ldots, u_n\} \subset \overline{\mathcal{M}}_{g,m}(X, A) \]

with neighborhoods \( U_{u_i}(2\delta_i, 2\epsilon_i, 2V_i) \) such that

\[ \bigcup_{i=1}^n U_{u_i}(\delta_i, \epsilon_i, V_i) \supset \overline{\mathcal{M}}_{g,m}(X, A). \]

Following the construction given in §14 we have a sequence of orbifolds, (which may not be smooth,)

\[ \{W_I\}_{I \subset \{1, \ldots, n\}}. \]

Hence we have a virtual orbifold \( W \) given by

\[ W_k = \{W_I\}_{|I|=k}. \]

The goal is to show that

**Theorem 16.1.** \( W \) admits a smooth structure. Hence it can be a smooth virtual orbifold.

**Proof.** Let

\[ p : W_I \to \overline{\mathcal{M}}_{g,m}(X, A) \]

be the projection. For each \( W_I \) set

\[ W_I(\Sigma) = p^{-1}(\chi_\Sigma) \cap W_I. \]
Then
\[ \mathcal{W}_S = \{ W_I(S) \} \]
forms a smooth virtual orbifold for each \( S \). This is due to the construction in §14 for a Fredholm system. Since we are working within a stratum, the smooth structure exists automatically.

Next we show that \( \mathcal{W} \) admits a smooth structure at \( \mathcal{W}_S \). For each \( W_I \) the smooth structure at \( W_I(S) \) is induced by gluing maps. Let \( \mathcal{L}_S \) be the gluing bundle over \( \chi_S \). It induces a bundle over each \( W_I(S) \), we denote the bundle by \( \mathcal{L}_{I,S} \). Then we have gluing maps
\[ Gl_{I,S} : \mathcal{L}_{I,S,e} \to W_I. \]

Note that \( \{ \mathcal{L}_{I,S} \} \) itself is a smooth virtual orbifold. If \( Gl_{I,S} \) is compatible with the overlapping maps, then the smooth structures induced on \( W_I \) are compatible with the virtual structure on \( W_I \). To be precise, this is what we mean: suppose we have \( I \subset J \) and \( x \in W_{I,J}, y \in W_{J,I} \) with \( x = \pi_{I,J}(y) \). We denote them by
\[ x = (u, o_1), y = (u, o_1, o_2). \]

For any gluing parameter \( \rho \) we want
\[ (16.1) \quad Gl_{J,S}(y, \rho) = Gl_{I,S}(x, o_2). \]

To make \( (16.1) \) available, we should require that the pre-gluing maps and right inverses \( Q_x, Q_y \) used for gluing map are same. There is no problem for the consistency of pre-gluing maps. For right inverses, this can be easily achieved as well: let \( Q_{I,S} \) be right inverses used for \( W_I(S) \), we can use partition of unity to reproduce a new group of right inverses \( Q'_{I,S} \) such that for any \( x \) and \( y \) as above
\[ Q_x \in Q'_{I,S}, Q_y \in Q'_{J,S} \]
are equal. This allows us to give a smooth structure of \( \mathcal{W} \) at \( \mathcal{W}_S \).

As before, since the smooth structures on \( \mathcal{W} \) induced by gluing maps from different strata may be different, we should apply the technique given in §13. Let \( \mathcal{S}_0 \) be the set of smallest strata. for any \( S \in \mathcal{S}_0 \) let
\[ \mathcal{W}_S = \{ W_I | p(W_I) \cap \chi_S \neq \emptyset \}. \]
It is a smooth virtual orbifold. We may assume that
\[ \mathcal{W}_S \cap \mathcal{W}_{S'} = \emptyset. \]
Hence,
\[ \mathcal{W}_0 := \bigcup_{S \in \mathcal{S}_0} \mathcal{W}_S \]
still form a smooth virtual orbifold.

Next we consider the set \( \mathcal{S}_1 \) of smallest strata next to those in \( \mathcal{S}_0 \). Then for \( S \in \mathcal{S}_1 \), \( \mathcal{W}_S \) is still a smooth virtual orbifold. However, on \( \mathcal{W}_S \cap \mathcal{W}_0 \), they may have two different smooth structures due to the discrepancy of gluing maps on different strata. We can then apply the argument in §13.2 to perturb the gluing maps on \( \mathcal{W}_S \) such that its smooth structure is compatible with that induced from \( \mathcal{W}_0 \). By this way, we have a modified smooth structure on
\[ \mathcal{W}_S = \bigcup_{S \in \mathcal{S}_1} \mathcal{W}_S \]
such that
\[ W_{S_0 \cup S_1} = W_{S_0} \cup W_{S_1} \]
forms a smooth virtual orbifold. We continue the process, then we have a smooth structure on \( W \). q.e.d.

16.2. The Gromov-Witten invariants. For the moduli space \( \overline{M}_{g,m}(X,A) \) we have constructed an associated virtual orbifold \( W \). As explained in §14, we have a transition data on \( W \)
\[ \Theta = \{ \Theta_{J,I} = \bigwedge_{j \in J-I} \Theta_j \} \]
and a \( \Theta \)-form \( \theta = (\Theta_I) \).

Suppose that the virtual dimension of \( \overline{M}_{g,m}(X,A) \) then for any degree \( d \) form \( \alpha \) on \( W \) we define the Gromov-Witten invariants to be
\[ \mu_\theta(\alpha). \]

In general, \( \alpha \) is induced from forms on \( X \) (by evaluation maps) or from forms on \( \overline{M}_{g,m} \). Moreover, the invariant is independent of the construction of \( W \).

17. Symplectic virtual localization

We now derive the symplectic virtual localization formula for Gromov-Witten invariants.

Let \( G = S^1 \) act on \( (X,\omega) \) symplectomorphically. It then induces an action on \( \overline{\chi}_{g,m}(X,A) \) and on \( \overline{M}_{g,m}(X,A) \). First we can modify the construction of virtual orbifold \( W \) such that it is an \( S^1 \)-virtual orbifold. The forms \( \Theta, \theta \) and \( \alpha \) are then replaced by equivariant forms \( \Theta^G, \theta^G \) and \( \alpha^G \).

Then applying the virtual localization formula for \( G \)-virtual orbifolds, we have

**Theorem 17.1.** Suppose that \( (X,\omega) \) admits an \( S^1 \) symplectomorphic action. Then the virtual localization formula for Gromov-Witten invariant \( \mu_\theta(\alpha) \) is given by
\[ \mu_{\theta^G}(\alpha^G) = \int_{W^G} i\omega_G(\alpha^G \wedge \theta^G) / e_G(W^G). \]

Here \( W^G \) is the virtual orbifold for \( \overline{\chi}_{g,m}^G(X,A) \) and \( e_G(W^G) \) is the equivariant Euler form of the normal bundle of \( W^G \) in \( W \).

18. An application of the virtual localization formula

18.1. Models \( W_k \) and their Gromov-Witten invariants. Let
\[ V_k = \{(u_1, u_2, u_3, u_4) | u_1^2 + u_2^2 + u_3^2 + u_4^2 = 0 \} \]
for \( k = 1, 2, \ldots \). \( V_k \) contains a singularity at 0. By blowing-up at 0, we have \( W_k \) with an exceptional line \( A = \mathbb{P}^1 \). \( W_k \) can be given by two coordinate patches \( (w, z_1, z_2) \) and \( (x, y_1, y_2) \) and by a transition map between the coordinate patches. Here \( A \) is given by \( \{ z_1 = z_2 = 0 \} = \{ y_1 = y_2 = 0 \} \). The transition map is given by
\[ \begin{cases} 
  z_1 = x^2 y_1 + x y_2^k \\
  z_2 = y_2 \\
  w = 1/x.
\end{cases} \]
The normal bundle of $A$ in $W_k$ is known as

$$
\begin{cases}
\mathcal{O}(-1) \oplus \mathcal{O}(-1), & \text{when } k = 1 \\
\mathcal{O} \oplus \mathcal{O}(-2), & \text{when } k \geq 2.
\end{cases}
$$

$W_k$ are Calabi-Yau threefolds. When $k = 1$, this is well-known conifold. Since $A$ is extremal ray, the moduli spaces

$$
\overline{M}_{g,0}(W_k, d[A]) = \overline{M}_{g,0}(A, d[A]).
$$

Hence, we are allowed to define (local) Gromov-Witten invariants. When $k = 1$, the invariants on $W_1$ is computed by Faber-Pandharipande in [7] by localization techniques. When $k > 2$, the invariants are computed by Bryan-Katz-Leung [2] by using deformation arguments. The results are given in the following theorem.

**Theorem 18.1** (Faber-Pandharipande, Bryan-Katz-Leung). Let $C_k(g, d)$ be the Gromov-Witten invariants for moduli spaces $\overline{M}_{g,0}(W_k, d[A])$. Then

$$
C_1(g, d) = \frac{|B_{2g}| d^{2g-3}}{2g \cdot (2g-2)!}
$$

and

$$
C_k(g, d) = kC_1(g, d).
$$

In this paper, we use the localization formula to verify (18.2). Such a model is closely related to the framework of Li-Ruan’s study on Gromov-Witten theory with respect to flops. Such a problem was first proposed and solved in [11], and then later reconsidered in [13]. In [13], a computation of (18.2) without using deformation is also asked. On the other hand, in orbifolds, there is a similar problem in this framework. It is known that the deformation technique can not be applied for orbifold case. Partial results have been considered in [3]. The localization technique would be a key to understand the orbifold Gromov-Witten invariants. These are the motivations for recompute (18.2) by using localization.

### 18.2. Localization set-up.

There is a $T^2$-action on $W_k$ given by

$$(t_1, t_2) \cdot (w, z_1, z_2) = (t_1^\lambda w, t_2^u z_2, t_1^{-\lambda} t_2^k u z_1).$$

The weights of the action is said to be $(\lambda, u, -\lambda + ku)$.

The moduli space is $\overline{M}_{g,0}(A, d[A])$. The fix loci of the action in this moduli space are associated with graphs([10], [9]). For each graph $\Gamma$, we denote the fixed loci by $M_\Gamma$. The Gromov-Witten invariant is given by

$$
\int_{V_k} 1 = \sum_\Gamma \int_{M_\Gamma} \frac{\theta}{e_T(N_{M_\Gamma})} = \sum_\Gamma \int_{M_\Gamma} \frac{1}{e_T^\Theta(N_{M_\Gamma})},
$$

where $V_k$ is a virtual neighborhood of the moduli space $\overline{M}_{g,0}(A, d[A])$ and $\theta$ is a $\Theta$-form constructed from cokernels. Unlike the well-known case $k = 1$, neither $V_k$ is the moduli space, nor $V_k = V_{k'}$ when $k \neq k'$. On $M_\Gamma$ we have is a $K$-bundle $\mathbb{H}^0 - \mathbb{H}^1$. The fiber of $H^i, i = 0, 1$ over $f : \Sigma \to W_k$ is given by $H^i(\Sigma, f^*TW_k)$. Then

$$
\theta|_{M_\Gamma} = e_T(\mathbb{H}^1), \quad N_{M_\Gamma} = \mathbb{H}^0.$$

18.3. Proof of Theorem 18.1. We follow the computation in [7]. We denote the left hand side of (18.3) by $J$, each term on right hand side by $I_{\Gamma}(\lambda, u)$ and the sum by $I(\lambda, u)$. Clearly, $J = I(\lambda, u)$ implies that $I$ is independent of choice of $u$. We will compute
\[
\lim_{u \to 0} I(1, u).
\]
By the same reason as the computation in [7], we know that
\[
\lim_{u \to 0} I_{\Gamma}(1, u) = 0
\]
unless $\Gamma$ is the graph that consists of a single edge. Now let $\Gamma$ be such a graph that consists of one edge. Its two ends are marked by $p_1$ and $p_2$, the fixed point on $A$ of the action. ($p_1$ is the point with $w = 0$ and $p_2$ is the other one.) Suppose the corresponding genus are $g_1, g_2$ with $g_1 + g_2 = g$. Such a graph is denoted by $\Gamma_{g_1, g_2}$. Then by a direct computation, we find that
\[
\lim_{u \to 0} I_{\Gamma_{g_1, g_2}}(1, u) = kd^{2g-3}b_{g_1}b_{g_2}.
\]
Therefore,
\[
\lim_{u \to 0} I(1, u) = kd^{2g-3}\sum_{g_1 + g_2 = g} b_{g_1}b_{g_2} = kC(g, d).
\]
The last equation is proved in [7]. This proves the theorem.

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