1 Introduction

Let $M$ be a smooth manifold of dimension $n \geq 2$. The Ricci flow

$$\frac{\partial g}{\partial t} = -2Ric$$

starting at a given Riemannian metric $g_0$ on $M$ deforms the Riemannian metric $g = g(t)$ in the opposite direction of its Ricci curvature tensor $Ric = Ric_{g(t)}$. It turns out that the Ricci flow becomes a nonlinear evolution equation of the parabolic type after a suitable gauge fixing is performed, see e.g. [CK]. In other words, we can view the Ricci flow as a nonlinear heat equation for Riemannian metrics. Indeed, the evolution equations associated with the Ricci flow for various curvature quantities such as the scalar curvature, the Ricci curvature tensor and the Riemann curvature tensor are all nonlinear heat equations. For this reason it is natural to look for geometric quantities associated with the Ricci flow which can be motivated by quantities in thermodynamics and related theories such as statistical mechanics and information.
theory. A central concept in thermodynamics is entropy. Entropy, from the Greek μετατροπή, meaning “transformation”, is a measure of the unavailability of a system’s energy to do the work. In terms of statistical mechanics, the entropy (more precisely, the Boltzmann entropy) describes the number of the possible microscopic configurations of a given system. The law of entropy, or the second law of thermodynamics, states that spontaneous changes in isolated systems occur with an increase in entropy.

So far two different kinds of entropy functional have been introduced into the theory of the Ricci flow. Both are motivated by concepts of entropy in thermodynamics-statistical mechanics-information theory. One is R. Hamilton’s entropy, the other is G. Perelman’s entropy. A major difference between these two concepts is this. In Hamilton’s entropy, the scalar curvature $R$ of the metric is viewed as the leading quantity of the system and plays the role of a probability density, while in Perelman’s entropy the leading quantity describing the system is the metric itself. We’ll explain below the motivation of Hamilton’s entropy in terms of Shannon entropy. In [P1], Perelman provided an interpretation of his entropy in terms of concepts in statistical mechanics including partition function, energy and entropy. We’ll briefly explain this interpretation, and provide a different, very natural motivation for Perelman’s entropy in terms of the logarithmic Sobolev inequality on the euclidean space $\mathbb{R}^n$.

Hamilton established the monotonicity of his entropy along the volume-normalized Ricci flow on the 2-sphere $S^2$ [H1]. He and B. Chow used this monotonicity and its generalization to study the Ricci flow on $S^2$ [H1] [Ch]. In [P1], Perelman established the monotonicity of his entropy along the Ricci flow in all dimensions. As an important application of this entropy monotonicity Perelman derived in [P1] the κ-noncollapsing property of the Ricci flow relative to upper bounds for $|Rm|$, the norm of the Riemann curvature tensor, under a finite upper bound for the time. Later, Perelman and the present author improved this result independently by replacing the upper bounds for $|Rm|$ with upper bounds for the scalar curvature, see [KL] and [Y1]. Note that the κ-noncollapsaping property is a crucial ingredient in Perelman’s work on the Ricci flow and the geometrization conjecture and the Poincaré conjecture.

Recently, Perelman’s entropy theory for the Ricci flow was further developed in [Y3], [Y4],[Y5], [Y6], [Y7] and [Y8]. In [Y3], the logarithmic Sobolev inequality along the Ricci flow was established. In general, this inequality depends on a finite upper bound for the time. But no such bound is required if the first eigenvalue $\lambda_0(g_0)$ of the operator $-\Delta + \frac{R}{4}$ for the initial metric is positive. $W^{1,2}$ Sobolev inequalities and κ-noncollapsing estimates along the Ricci flow were then derived in [Y3] as consequences of the logarithmic Sobolev inequality. In [Y4] and [Y5], the results of [Y3] were extended to the dimension $n = 2$ and the case $\lambda_0(g_0) = 0$. In [Y6], the log entropy functional was introduced and its monotonicity along the Ricci flow was established, based on Perelman’s entropy monotonicity. As a consequence, it was shown that the logarithmic Sobolev inequality improves along the Ricci flow. In [Y7], $W^{1,p}$ and $W^{2,p}$ Sobolev inequalities along the Ricci flow were obtained for $p \neq 2$. Several methods
were used in [Y7], including some tools from harmonic analysis and potential theory such as Bessel potentials and Riesz transforms.

Finally, Sobolev inequalities and κ-noncollapsing estimates were established for the Ricci flow with surgeries in [Y8]. The key construction in Perelman’s work on the Ricci flow and the geometrization conjecture and the Poincaré conjecture is the Ricci flow with surgeries [P2], which extends Hamilton’s earlier work on surgeries of the Ricci flow in a substantial way. The results in [Y8] can be used to replace the rather complicated arguments in [P2] for preserving the κ-noncollapsing property after surgeries. Moreover, the κ-noncollapsing property is established in [Y8] independent of the other properties, making the choice of the surgery parameters much simpler. It also becomes easier to establish e.g. the canonical neighborhood property.

We’ll review below these recent results, providing helpful clues whenever appropriate. We hope that our accounts can provide an integrated picture of the whole theory, offer some new perspectives, and explain the main ideas without getting into the technical details of the proofs.

\section{Hamilton’s entropy functional}

Let $M$ be a closed manifold of dimension $n = 2$ diffeomorphic to the 2-sphere $S^2$. R. Hamilton introduced the following entropy functional [H1]

$$S_0(g) = \int_M R \ln R \, dvol$$

(2.1)

for metrics $g$ of positive scalar curvature. It can be extended straightforwardly to metrics of nonnegative scalar curvature because $x \ln x \to 0$ as $x \to 0^+$. This formula of entropy can be motivated by Shannon entropy in information theory, which in turn is motivated by Gibbs entropy in statistical mechanics.

Consider a discret thermodynamic system $X$. The Boltzmann entropy of $X$ is defined to be

$$S_B = k_B \ln W,$$

(2.2)

where $k_B$ is the Boltzmann constant and $W$ denotes the number of microstates in $X$. The Gibbs entropy of $X$ is defined to be

$$S_G = -k_B \sum_i p_i \ln p_i,$$

(2.3)

where $p_i$ denotes the probability of the $i$-th microstate. If the microstates are equiprobable, then $S_B$ is reduced to $S_G$. In general, $S_G$ can be derived from $S_B$ as a limit when the number of microstates is very large, see e.g. [CC]. Next let $\mathcal{M}$
be a discrete message space with a probability measure. Then the Shannon entropy of $\mathcal{M}$ is defined to be

$$S = - \sum_i p_i \ln p_i,$$

(2.4)

where $p_i$ is the probability of the message $m_i$ taken from the message space $\mathcal{M}$. There is an integral version of Shannon entropy for a continuous message space $\mathcal{M}$, called the differential entropy, or differential Shannon entropy, which is defined as follows

$$S_D = - \int_{\mathcal{M}} f \ln f \, d\mu,$$

(2.5)

where $f$ denotes the probability density, and $\mu$ the background measure. (A word of caution: In passing from $S$ to $S_D$ in terms of a limit of Riemann sums, one has to throw out an infinite quantity.) It is the differential Shannon entropy which can be used to motivate Hamilton’s entropy.

Consider a positive solution $u$ of the heat equation

$$\frac{\partial u}{\partial t} = \Delta u$$

(2.6)

on $M$, with $\Delta = \Delta_g$ for a given Riemannian metric $g$ on $M$. In the framework of thermodynamics $u$ represents the temperature of a physical system with the background given by the Riemannian manifold $(M, g)$. But we view $M$ as a message space with the evolving probability density function $u$ with respect to the background measure given by the volume measure of $g$. Since the average of $u$ is preserved by the heat flow, we can normalize $u$ such that $u \, d\text{vol}$ is a probability measure, i.e. $\int_M u \, d\text{vol} = 1$. The differential Shannon entropy of $u$ is then

$$S_D(u) = - \int_M u \ln u \, d\text{vol}.$$

(2.7)

There holds

$$\frac{d}{dt} S_D(u) = \int_M \frac{|\nabla u|^2}{u} \, d\text{vol},$$

(2.8)

whence $S_D(u)$ is increasing along the heat flow (2). Note that this leads to the following inequality

$$\int_M u \ln u \, d\text{vol} \geq \int_M \bar{u} \ln \bar{u} \, d\text{vol}$$

(2.9)

for any nonnegative function $u$ on $M$, where $\bar{u}$ denotes the average of $u$. Now the scalar curvature $R$ satisfies a nonlinear heat equation along the Ricci flow or the volume-normalized Ricci flow, hence it is analogous to the above $u$ of the heat flow. Obviously,
Hamilton’s entropy is simply the negative multiple of the differential Shannon entropy of \( R \), where \( R \) is viewed as the probability density function. (Note that \( \int_M R \text{dvol} \) is a constant along the volume-normalized Ricci flow by Gauss-Bonnet theorem.) The formula for Hamilton’s entropy \( S_0 \) analogous to (2.8) is

\[
\frac{dS_0}{dt} = - \int_M \frac{\| \nabla R \|^2}{R} \text{dvol} + \int_M R^2 \text{dvol}
\]  

(2.10)

along a smooth solution of the Ricci flow with positive scalar curvature. Indeed, since \( \frac{\partial}{\partial t} R = \Delta R + R^2 \) and \( \frac{\partial}{\partial t} \text{dvol} = -R \text{dvol} \) we have

\[
\frac{dS_0}{dt} = \int_M (\Delta R + R^2) \ln R \text{dvol} + \int_M (\Delta R + R^2) \text{dvol} - \int_M R^2 \ln R \text{dvol}
\]

\[
= -\int_M \frac{\| \nabla R \|^2}{R} \text{dvol} + \int_M R^2 \text{dvol}.
\]

Let \( f \) be Hamilton’s potential function which is defined by \( \Delta f = R - r \) with \( \int_M f \text{dvol} = 0 \), where \( r \) denotes the average of \( R \). Then (2.10) implies

\[
\frac{dS_0}{dt} = -\int_M \frac{\nabla R + R \nabla f}{R} \text{dvol} - 2 \int_M |H|^2 \text{dvol} + 8\pi r,
\]

(2.11)

where \( H \) is the trace-free part of the Hessian of \( f \). (This follows from (2.10), the formulas in the proof of Proposition 5.39 in [CK], and Gauss-Bonnet theorem.) The term \( 8\pi r \) can be given explicitly

\[
8\pi r = \frac{64\pi^2}{V_0 - 8\pi t} = -8\pi \frac{d}{dt} \ln(V_0 - 8\pi t),
\]

(2.12)

where \( V_0 \) is the volume of the initial metric. Hence we obtain the following monotonicity formula.

**Theorem 2.1** There holds along a smooth solution of the Ricci flow on \( M \)

\[
\frac{d}{dt} [S_0 + 8\pi \ln(V_0 - 8\pi t)] = -\int_M \frac{\nabla R + R \nabla f}{R} \text{dvol} - 2 \int_M |H|^2 \text{dvol}.
\]

(2.13)

In [H1], a simpler monotonicity formula is given for \( S_0 \) along a smooth solution of the volume-normalized Ricci flow. Indeed, there holds, similar to (2.10),

\[
\frac{dS_0}{dt} = -\int_M \frac{\| \nabla R \|^2}{R} \text{dvol} + \int_M (R - r)^2 \text{dvol}
\]

(2.14)

along a smooth solution \( g = g(t) \) of the volume-normalized Ricci flow with positive scalar curvature, which leads to the monotonicity formula of Hamilton

\[
\frac{dS_0}{dt} = -\int_M \frac{\nabla R + R \nabla f}{R} \text{dvol} - 2 \int_M |H|^2 \text{dvol},
\]

(2.15)
where $f$ and $H$ are defined in the same way as the $f$ and $H$ in (2.11). This monotonicity formula immediately yields an upper bound for $S_0$, which is combined in [H1] with the differential Harnack inequality for $R$ to produce an upper bound for $R$. This is the key step in [H1] for establishing the convergence of the volume-normalized Ricci flow in the case that the initial metric has positive scalar curvature. (For another approach to the Ricci flow on the 2-sphere based on the parabolic moving plane method in [Y1] we refer to [BSY].)

Next we consider a more general entropy quantity, the adjusted entropy $S_a$

$$S_a(g) = \int_M (R - a) \ln(R - a) \, d\text{vol} \quad (2.16)$$

for a given parameter $a$, assuming $R \geq a$. For a smooth solution $g = g(t)$ of the volume-normalized Ricci flow we can choose $a = a(t)$ to be the solution of the ODE

$$\frac{da}{dt} = a(a - r) \quad (2.17)$$

and consider $S_{a(t)}(g(t))$. This ODE corresponds to the evolution equation for the scalar curvature

$$\frac{\partial R}{\partial t} = \Delta R + R(R - r) \quad (2.18)$$

associated with the Ricci flow. The quantity $S_{a(t)}(g(t))$ is precisely the modified entropy introduced by B. Chow in [Ch]. (In the case of the Ricci flow, we can choose $a = a(t)$ to be the solution of the ODE $\frac{da}{dt} = a^2$, which corresponds to the evolution equation $\frac{\partial R}{\partial t} = \Delta R + R^2$.) Although $S_{a(t)}(g(t))$ is not necessarily monotone, Chow is able to obtain an upper bound for it. He uses this upper bound together with other tools to show that the scalar curvature always becomes positive in finite time regardless what initial metric is given, see [Ch] or [CK] for details.

Hamilton’s entropy $S_0$ can obviously be defined on manifolds of dimensions $n \geq 3$, but no monotonicity formula for it has been found there. Note that part of the analogy of the differential Shannon entropy is lost there, because in general the integral of the scalar curvature no longer stays a constant along the volume-normalized Ricci flow. We think that a deeper understanding of Hamilton’s entropy should be pursued.

### 3 Perelman’s entropy functional

G. Perelman introduced in [P1] the entropy functional $\mathcal{W}(g, f, \tau)$ and established its monotonicity along the Ricci flow. Let $M$ be a compact manifold of dimension $n \geq 2$. The definition of $\mathcal{W}(g, f, \tau)$ is as follows

$$\mathcal{W}(g, f, \tau) = \int_M [\tau(R + |\nabla f|^2) + f - n] \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}} \, d\text{vol}, \quad (3.1)$$

$^1$In [Y1], we neglected to mention Th. Aubin’s important contribution to the Yamabe problem.
where \( \tau \) is a positive number, \( g \) is a Riemannian metric on \( M \), and \( f \in C^\infty(M) \) satisfies
\[
\int_M \frac{e^{-f}}{(4\pi \tau)^{\frac{n}{2}}} d\text{vol} = 1. \tag{3.2}
\]
All geometric quantities in (3.1) and (3.2) are associated with \( g \).

**Theorem 3.1 (Perelman)** Let \( g = g(t) \) be a smooth solution of the Ricci flow on \( M \times I \) for some interval \( I \). Let \( \tau = \tau(t) \) be a scalar function on \( I \) with \( \tau' = -1 \), and \( f = f(t) \) a smooth solution of the equation
\[
\frac{\partial f}{\partial t} = -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau} \tag{3.3}
\]
associated with \( g = g(t) \) on \( M \times I \). Then there holds
\[
\frac{dW}{dt} = 2\tau \int_M |\text{Ric} + \nabla^2 f - \frac{1}{2\tau} g|^2 \frac{e^{-f}}{(4\pi \tau)^{\frac{n}{2}}} d\text{vol} \geq 0 \tag{3.4}
\]
in \( I \), where \( W = W(g(t), f(t), \tau(t)) \).

For a detailed proof of this theorem we refer to [KL]. In [P1], Perelman provided an interpretation of his entropy in terms of statistical thermodynamics, which we briefly explain here. Consider the thermodynamic system (in the sense of analogy) described by the pair \((g, f)\) with \( g \) denoting a Riemannian metric on \( M \) and \( f \) a smooth function on \( M \). By well-known formulas in statistical mechanics the entropy of the system is given by
\[
S = \beta < E > + \ln Z, \tag{3.5}
\]
where \( \beta \) is the inverse of the temperature, \(< E >\) the expectation value of energy, and \( Z \) the partition function. The formulas for \( Z \) and \( E \) are
\[
Z = \int e^{-\beta E} d\mu(E), \quad < E > = -\frac{\partial}{\partial \beta} \ln Z, \tag{3.6}
\]
with \( \mu \) denoting the “density of states” measure. Perelman states that the partition function is given by
\[
Z = \int_M \left(-f + \frac{n}{2}\right) dm, \tag{3.7}
\]
where
\[
dm = \frac{e^{-f}}{(4\pi \tau)^{\frac{n}{2}}} d\text{vol}. \tag{3.8}
\]
We do not attempt to explain this result here. Now let the pair \((g, f)\) evolve by the Ricci flow coupled with (3.3), with \(\tau\) as given in Theorem 3.12 playing the role of time. Then a standard computation yields

\[
S = -\mathcal{W}(g, f, \tau).
\]

(3.9)

We would also like to mention the following observation in [KL]

\[
\mathcal{W}(g, f, \tau) = \frac{d}{d\tau} \left( \tau \int_M (f - \frac{n}{2}) dm \right).
\]

(3.10)

Next we present a natural motivation of Perelman’s entropy functional in terms of the logarithmic Sobolev inequality on the euclidean space \(\mathbb{R}^n\). The logarithmic Sobolev inequality of L. Gross states

\[
\int u^2 \ln u^2 d\mu \leq 2 \int |\nabla u|^2 d\mu,
\]

(3.11)

assuming \(u \in W^{1,2}_{loc}(\mathbb{R}^n)\) and \(\int u^2 d\mu = 1\), where \(d\mu = (2\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2}} dx\). (We omit the integration domain which is the entire \(\mathbb{R}^n\).) This can be proved by employing the ordinary Sobolev inequality with the optimal constant and the product structure of the Euclidean spaces. Setting \(u = (2\pi)^{-\frac{n}{4}} e^{-f/2}\) we can convert (3.11) into the following formulation due to Perelman [P1]

\[
\int \left( \frac{1}{2} |\nabla f|^2 + f - n \right) \frac{e^{-f}}{(2\pi)^{n/2}} dx \geq 0,
\]

(3.12)

provided that \(f \in W^{1,2}_{loc}(\mathbb{R}^n)\), \(\int e^{-f}(2\pi)^{-n/2} dx = 1\) and \(\int |\nabla f|^2 e^{-f} dx < \infty\). Obviously, the left hand side of (3.12) is precisely \(\mathcal{W}(g_{\text{euc}}, f, \tau)\), where \(g_{\text{euc}}\) is the euclidean metric. To see how the general value of the parameter \(\tau\) enters into the play, we can consider the log gradient version of the logarithmic Sobolev inequality

\[
\int u^2 \ln u^2 dx \leq \frac{n}{2} \ln \left[ \frac{2}{\pi ne} \int |\nabla u|^2 dx \right]
\]

(3.13)

for \(u \in W^{1,2}(\mathbb{R}^n)\) with \(\int u^2 dx = 1\). Since \(\ln s \leq \sigma s - \ln \sigma - 1\) for all \(s > 0\) and \(\sigma > 0\) we deduce from (3.13)

\[
\int u^2 \ln u^2 dx \leq \frac{n}{2} \ln \frac{2}{\pi ne} + \frac{n\sigma}{2} \int |\nabla u|^2 dx - \frac{n}{2} \ln \sigma - \frac{n}{2}.
\]

(3.14)

To simplify the formula we replace \(\sigma\) by \(\frac{2}{n} \sigma\) and obtain

\[
\int u^2 \ln u^2 dx \leq \sigma \int |\nabla u|^2 dx - \frac{n}{2} \ln \sigma - \frac{n}{2} \ln \pi.
\]

(3.15)
This form of the logarithmic Sobolev inequality has been used extensively in the theory of ultracontractivity of symmetric Markov processes. We can absorb the terms $-\frac{n}{2}\ln\sigma-\frac{n}{2}\ln\pi$ if we replace $u$ by $(\pi\sigma)^{-n/4}u$. We obtain

$$\int u^2\ln u^2\frac{1}{(\pi\sigma)^{\frac{n}{2}}}dx \leq \sigma\int |\nabla u|^2\frac{1}{(\pi\sigma)^{\frac{n}{2}}}dx - n,$$

(3.16)

where $u$ satisfies

$$\int \frac{u^2}{(\pi\sigma)^{\frac{n}{2}}}dx = 1.$$  

(3.17)

Finally, we set $u = e^{-f/2}$ i.e. $f = -\ln u^2$ and $\sigma = 4\tau$ to deduce, assuming $u > 0$

$$-\int f\frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}}dx \leq \tau\int |\nabla u|^2\frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}}dx - n,$$  

(3.18)

provided that $\int e^{-f}(4\pi\tau)^{-n/2}dx = 1$. We summarize the above computations in a theorem.

**Theorem 3.2** Let $f \in W^{1,2}_{\text{loc}}(\mathbb{R}^n)$ and $\tau > 0$ such that

$$\int \frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}}dx = 1$$  

(3.19)

and

$$\int |\nabla f|^2e^{-f}dx < \infty.$$  

(3.20)

Then there holds

$$\int [\tau|\nabla f|^2 + f - n]\frac{e^{-f}}{(4\pi\tau)^{\frac{n}{2}}}dx \geq 0.$$  

(3.21)

The formula in (3.21) obviously suggests the general formula of $\mathcal{W}(g,f,\tau)$ on an arbitrary manifold. Of course, one still needs to come up with the idea of replacing $|\nabla f|^2$ by $|\nabla f|^2 + R$ in the general formula. The appearance of $R$ in the formula is quite natural in view of the classical total scalar curvature functional which plays an important role for the study of Einstein metrics. The fact that $R$ should be put together with $|\nabla f|^2$ is also natural in view of the common scaling weight of $R$ and $|\nabla f|^2$. 


4 The log entropy functional and the log Sobolev constant

Motivated by the log gradient version of the logarithmic Sobolev inequality, the concept of log entropy functional was introduced in [Y6]. Let \( M \) be a closed manifold of dimension \( n \geq 2 \). We define the log entropy functional as follows

\[
\mathcal{Y}_0(g, u) = -\int_M u^2 \ln u^2 \, dvol + \frac{n}{2} \ln \left( \int_M (|\nabla u|^2 + \frac{R}{4} u^2) \, dvol \right),
\]  

(4.1)

where \( g \) is a smooth metric on \( M \) and \( u \in W^{1,2}(M) \) satisfies

\[
\int_M (|\nabla u|^2 + \frac{R}{4} u^2) \, dvol > 0.
\]  

(4.2)

Here, all geometric quantities are associated with \( g \). More generally, we define the log entropy functional with remainder \( a \) as follows

\[
\mathcal{Y}_a(g, u) = -\int_M u^2 \ln u^2 \, dvol + \frac{n}{2} \ln \left( \int_M (|\nabla u|^2 + \frac{R}{4} u^2) \, dvol + a \right).
\]  

(4.3)

Finally, we define the adjusted log entropy with remainder \( a \) as follows

\[
\mathcal{Y}_a(g, u, t) = -\int_M u^2 \ln u^2 \, dvol + \frac{n}{2} \ln \left( \int_M (|\nabla u|^2 + \frac{R}{4} u^2) \, dvol + a \right) + 4at.
\]  

(4.4)

Obviously, \( \mathcal{Y}_a(g, u) = \mathcal{Y}_a(g, u, 0) \).

Now we consider a smooth solution \( g = g(t) \) of the Ricci flow on \( M \times [\alpha, T) \) for some \( \alpha < T \), where \( \alpha \) is finite. Let \( u = u(t) \) be a smooth positive solution of the backward evolution equation

\[
\frac{\partial u}{\partial t} = -\Delta u + \frac{|\nabla u|^2}{u} + \frac{R}{2} u
\]  

(4.5)

which is derived from the equation (3.3) upon setting \( u = e^{-f/2} \). Let \( \lambda_0 \) denote the first eigenvalue of the operator \( -\Delta + \frac{R}{4} \). The following monotonicity result was obtained in [Y6].

**Theorem 4.1** Assume that \( a > -\lambda_0(g(\alpha)) \). Then \( \mathcal{Y}_a(t) \equiv \mathcal{Y}_a(g(t), u(t), t) \) is nondecreasing. Indeed, we have

\[
\frac{d}{dt} \mathcal{Y}_a \geq \frac{n}{4\omega} \int_M \left| \text{Ric} - 2 \frac{\nabla^2 u}{u} + 2 \frac{\nabla u \otimes \nabla u}{u^2} - \frac{g}{16\omega} |u^2| \right| \, dvol,
\]  

(4.6)
where
\[
\omega = \omega(t) = a + \int_M (|\nabla u|^2 + \frac{R}{4} u^2) dv |_t,
\]
which is positive.

This theorem was proved by combining Perelman’s entropy monotonicity formula with a minimizing procedure. To see its consequence on the behavior of the logarithmic Sobolev inequality along the Ricci flow, we define for each \(a > -\lambda_0(g)\) the logarithmic Sobolev constant with the \(a\)-adjusted scalar curvature potential
\[
C_{S, \log, a}(M, g) = \inf \left\{ -\int_M u^2 \ln u^2 dv + \frac{n}{2} \ln \left( \int_M (|\nabla u|^2 + (R + a) u^2) dv \right) : u \in W^{1,2}(M), \int_M u^2 dv = 1 \right\}.
\]
(4.8)

In other words, \(C_{S, \log, a}(M, g)\) is the optimal constant, i.e. the maximal possible constant, such that the logarithmic Sobolev inequality
\[
\int_M u^2 \ln u^2 dv \leq \frac{n}{2} \ln \left( \int_M (|\nabla u|^2 + \frac{R}{4} u^2) dv + a \right) - C_{S, \log, a}(M, g)
\]
holds true for all \(u \in W^{1,2}(M)\) with \(\int_M u^2 dv = 1\). The following monotonicity results were established in [Y6] as applications of Theorem 4.1.

**Theorem 4.2** The adjusted logarithmic Sobolev inequality improves along the Ricci flow. More precisely, \(C_{S, \log, a}(M, g(t)) + 4at\) is nondecreasing along an arbitrary smooth solution \(g(t)\) of the Ricci flow on \(M\), provided that \(a\) is greater than the negative multiple of the \(\lambda_0\) of the initial metric. In particular, the logarithmic Sobolev inequality improves along the Ricci flow, i.e. \(C_{S, \log, 0}(M, g(t))\) is nondecreasing along the Ricci flow, provided that \(\lambda_0 > 0\) at the start.

**Theorem 4.3** Assume \(\lambda_0 = 0\) at the start. Then either \(g = g(t)\) is a gradient soliton, in which case the logarithmic Sobolev constant \(C_{S, \log, a}(M, g(t))\) is independent of \(t\) for any given \(a > 0\); or \(C_{S, \log, 0}(M, g(t))\) is nondecreasing on \([\epsilon, T)\) for each \(\epsilon > 0\).

5 The logarithmic Sobolev and Sobolev inequalities along the Ricci flow

The following results on the logarithmic Sobolev inequality along the Ricci flow were obtained in [Y3]. Consider a compact manifold \(M\) of dimension \(n \geq 3\). Let \(g = g(t)\) be a smooth solution of the Ricci flow on \(M \times [0, T)\) for some (finite or infinite) \(T > 0\) with a given initial metric \(g(0) = g_0\).
Theorem 5.1 For each $\sigma > 0$ and each $t \in [0, T)$ there holds
\[
\int_M u^2 \ln u^2 dvol \leq \sigma \int_M (|\nabla u|^2 + \frac{R}{4} u^2) dvol - \frac{n}{2} \ln \sigma + A_1(t + \frac{\sigma}{4}) + A_2
\] (5.1)
for all $u \in W^{1,2}(M)$ with $\int_M u^2 dvol = 1$, where
\[
A_1 = \frac{4}{\tilde{C}_S(M, g_0)^2 \text{vol}_{g_0}(M) \frac{2}{n}} - \min R_{g_0},
\]
\[
A_2 = n \ln \tilde{C}_S(M, g_0) + \frac{n}{2} (\ln n - 1),
\]
and all geometric quantities are associated with the metric $g(t)$ (e.g. the volume form $dvol$ and the scalar curvature $R$), except the scalar curvature $R_{g_0}$, the modified Sobolev constant $\tilde{C}_S(M, g_0)$ (see Section 2 for its definition) and the volume $\text{vol}_{g_0}(M)$ which are those of the initial metric $g_0$.

Theorem 5.2 Assume that $\lambda_0(g_0) > 0$. For each $t \in [0, T)$ and each $\sigma > 0$ there holds
\[
\int_M u^2 \ln u^2 dvol \leq \sigma \int_M (|\nabla u|^2 + \frac{R}{4} u^2) dvol - \frac{n}{2} \ln \sigma + C
\] (5.2)
for all $u \in W^{1,2}(M)$ with $\int_M u^2 dvol = 1$, where $C$ depends only on the dimension $n$, a positive lower bound for $\text{vol}_{g_0}(M)$, a nonpositive lower bound for $R_{g_0}$, an upper bound for $C_S(M, g_0)$, and a positive lower bound for $\lambda_0(g_0)$.

The 2-dimensional case and the case $\lambda_0(g_0) = 0$ are treated in [Y4] and [Y5]. The class of Riemannian manifolds $(M, g_0)$ with $\lambda_0(g_0) \geq 0$ is a very large one and particularly significant from a geometric point of view. On the other hand, we would like to point out that in general the assumption $\lambda_0(g_0) \geq 0$ indispensable in (5.2). In other words, a uniform logarithmic Sobolev inequality like (5.2) without the assumption $\lambda_0(g_0) \geq 0$ is false in general. Indeed, by [HI] there are smooth solutions of the Ricci flow on torus bundles over the circle which exist for all time, have bounded curvature, and collapse as $t \to \infty$. In view of the $\kappa$-noncollapsing estimate implied by (5.2) (see Section 6) a uniform logarithmic Sobolev inequality like (5.2) fails to hold along these solutions.

Quantitative improvements of the above logarithmic Sobolev inequalities follow easily from the monotonicity theorems on the log Sobolev constant in the last section. Although the log gradient version of the logarithmic Sobolev inequality appears to be stronger than the above $\sigma$-version of the logarithmic Sobolev inequality, they are actually equivalent. On the other hand, the $\sigma$-version is often more convenient for applications. In particular, the ordinary Sobolev inequalities can be derived from it, which we address next.
It has been known for some time that there exist close relations between the logarithmic Sobolev inequality, the $W^{1,2}$ Sobolev inequality and the so-called ultracontractivity of the heat semigroup of the associated Schrödinger operator. Indeed, they are equivalent. This theory is presented e.g. in [D] in the general and abstract set-up of symmetric Markov processes. Besides PDE arguments, basic spectral analysis of self-adjoint operators and some basic theorems of harmonic analysis such as the Riesz-Thorin interpolation theorem and the Marcinkiewicz interpolation theorem play a crucial role. For the purpose of precise geometric estimates, transparent presentation and additional implications, the theory in [D] is adapted in [Y3] to the geometric set-up there and worked out in complete and self-contained details. (The paper [Z] provided help for us to find the reference [D].) Based on this theory, in [Y3] the following results on the Sobolev inequalities along the Ricci flow are derived from the above logarithmic Sobolev inequalities. Let $g = g(t)$ be a smooth solution of the Ricci flow on $M \times [0, T)$ with a given initial metric $g_0$ as before.

**Theorem 5.3** Assume that $\lambda_0(g_0) > 0$. There is a positive constant $A$ depending only on the dimension $n$, a nonpositive lower bound for $R_{g_0}$, a positive lower bound for $\text{vol}_{g_0}(M)$, an upper bound for $C_S(M, g_0)$, and a positive lower bound for $\lambda_0(g_0)$, such that for each $t \in [0, T)$ and all $u \in W^{1,2}(M)$ there holds

$$\left( \int_M |u|^{\frac{2n}{n-2}} \text{dvol} \right)^{\frac{n-2}{n}} \leq A \int_M (|\nabla u|^2 + \frac{R}{4} u^2) \text{dvol},$$

(5.3)

where all geometric quantities except $A$ are associated with $g(t)$.

**Theorem 5.4** Assume $T < \infty$. There are positive constants $A$ and $B$ depending only on the dimension $n$, a nonpositive lower bound for $R_{g_0}$, a positive lower bound for $\text{vol}_{g_0}(M)$, an upper bound for $C_S(M, g_0)$, and an upper bound for $T$, such that for each $t \in [0, T)$ and all $u \in W^{1,2}(M)$ there holds

$$\left( \int_M |u|^{\frac{2n}{n-2}} \text{dvol} \right)^{\frac{n-2}{n}} \leq A \int_M (|\nabla u|^2 + \frac{R}{4} u^2) \text{dvol} + B \int_M u^2 \text{dvol},$$

(5.4)

where all geometric quantities except $A$ and $B$ are associated with $g(t)$.

6 The $\kappa$-noncollapsing estimates

We first recall the following definitions [P1] [Y2].

**Definition 1** Let $g$ be a Riemannian metric on a manifold $M$ of dimension $n$. Let $\kappa$ and $\rho$ be positive numbers. We say that $g$ is $\kappa$-noncollapsed on the scale $\rho$, if $g$ satisfies $\text{vol}(B(x, r)) \geq \kappa r^n$ for all $x \in M$ and $r > 0$ with the properties $r < \rho$ and
sup\{|Rm|(x) : x \in B(x,r)\} \leq r^{-2}. \text{ We say that a family of Riemannian metrics} 
\text{g = g(t) is } \kappa\text{-noncollapsed on the scale } \rho, \text{ if } g(t) \text{ is } \kappa\text{-noncollapsed on the scale } \rho \text{ for each } t \text{ (in the given domain).}

\textbf{Definition 2} Let } g \text{ be a Riemannian metric on a manifold } M \text{ of dimension } n. \text{ Let } \kappa \text{ and } \rho \text{ be positive numbers. We say that } g \text{ is } \kappa\text{-noncollapsed on the scale } \rho \text{ relative to upper bounds of the scalar curvature, if } g \text{ satisfies } \text{vol}(B(x,r)) \geq \kappa r^n \text{ for all } x \in M \text{ and } r > 0 \text{ satisfying } r < \rho \text{ and } \text{sup}\{R(x) : x \in B(x,r)\} \leq r^{-2}.

The } \kappa\text{-noncollapsing property of the Ricci flow on a compact manifold can be derived directly from Perelman’s entropy monotonicity, as done in [P1] and [Y2], see also [KL]. But the Sobolev inequalities along the Ricci flow obtained in [Y3][Y4] and [Y5] and as presented in the last section lead to better results in several ways. First, we obtain explicit } \kappa\text{-noncollapsing estimates which have clear and rudimentary geometric dependences on the initial metric. Second, the estimates are uniform up to } t = 0, \text{ which is not immediately clear from the direct arguments. Third, the estimates hold true uniformly for all time (up to infinity), provided that } \lambda_0(g_0) \geq 0. \text{ We present in this section only the results from [Y3]. First we have the following general result on the } \kappa\text{-noncollapsing estimate implied by the Sobolev inequality.}

\textbf{Theorem 6.1} Consider the Riemannian manifold \((M, g)\) for a given metric } g, \text{ such that for some } A > 0 \text{ and } B > 0 \text{ the Sobolev inequality}
\begin{equation}
\left( \int_M |u|^{2n/(n-2)} \text{dvol} \right)^{n-2/n} \leq A \int_M (|\nabla u|^2 + R/4u^2) \text{dvol} + B \int_M u^2 \text{dvol} \tag{6.1}
\end{equation}
holds true for all } u \in W^{1,2}(M). \text{ Let } L > 0. \text{ Assume } R \leq \frac{1}{r^2} \text{ on a geodesic ball } B(x,r) \text{ with } 0 < r \leq L. \text{ Then there holds}
\begin{equation}
\text{vol}(B(x,r)) \geq \left( \frac{1}{2n+3A + 2BL^2} \right)^{\frac{n}{2}} r^n. \tag{6.2}
\end{equation}

Basically, the Sobolev inequality implies a Faber-Krahn inequality for the first Dirichlet eigenvalue of subdomains in \(B(x,r)\) under the assumption } R \leq \frac{1}{r^2} \text{ on } B(x,r). \text{ By an elementary iteration argument in [Ca] we then arrive at the desired volume estimate.}

Combining Theorem 5.3 and Theorem 5.4 with Theorem 6.1 we arrive at the following } \kappa\text{-noncollapsing estimates in [Y3].}

\textbf{Theorem 6.2} Assume that } \lambda_0(g_0) > 0. \text{ Let } t \in [0,T). \text{ Consider the Riemannian manifold } (M, g) \text{ with } g = g(t). \text{ Assume } R \leq \frac{1}{r^2} \text{ on a geodesic ball } B(x,r) \text{ with } r > 0. \text{ Then there holds}
\begin{equation}
\text{vol}(B(x,r)) \geq \left( \frac{1}{2n+3A} \right)^{\frac{n}{2}} r^n, \tag{6.3}
\end{equation}
where \( A \) is from Theorem 5.3. In other words, the flow \( g = g(t), t \in [0, T) \) is \( \kappa \)-noncollapsed relative to upper bounds of the scalar curvature on all scales.

**Theorem 6.3** Assume that \( T < \infty \). Let \( L > 0 \) and \( t \in [0, T) \). Consider the Riemannian manifold \( (M, g) \) with \( g = g(t) \). Assume \( R \leq \frac{1}{r^2} \) on a geodesic ball \( B(x, r) \) with \( 0 < r \leq L \). Then there holds

\[
\text{vol}(B(x, r)) \geq \left( \frac{1}{2^{n+3}A + 2BL^2} \right)^n r^n, \tag{6.4}
\]

where \( A \) and \( B \) are from Theorem 5.4.

The importance of \( \kappa \)-noncollapsing estimates is that they yield estimates for the injectivity radius under the assumption of bounds for curvatures. In particular, they enable one to obtain smooth blow-up limits for the Ricci flow, which is crucial for blow-up analysis of singularities of the Ricci flow. The said estimates for the injectivity radius is implied by the result in [CGT] on estimates of the injectivity radius. We state here the local formulation of this result as given in [Y1].

**Theorem 6.4** Let \( (M, g) \) be a Riemannian manifold of dimension \( n \). Assume that the sectional curvatures \( K_g \) of \( g \) satisfies \( \kappa_1 \leq K_g \leq \kappa_2 \) on a geodesic ball \( B(p, r_0) \) in \( (M, g) \), such that \( r_0 \leq d(p, \partial M) \). (For the general definition of \( d(p, \partial M) \) see [Y2]. Note that \( d(p, \partial M) = \infty \) if \( M \) is closed.) Set \( r_1 = \frac{1}{4} \min\{r_0, \frac{\pi}{\sqrt{\kappa_2}}\} \). Then the injectivity radius \( i(q) \) at any \( q \in B(p, r_1) \) satisfies

\[
i(q) \geq r_2, \tag{B.1}
\]

where

\[
r_2 = \frac{r_1}{2} \left( 1 + \frac{V_{\kappa_1}(2r_1)^2}{\text{vol}_g(B(p, r_1))V_{\kappa_1}(r_1)} \right)^{-1}, \tag{B.2}
\]

\( r_1 = \frac{1}{4} \min\{r_0, \frac{\pi}{\sqrt{\kappa_2}}\} \), and for any \( r > 0 \), \( V_{\kappa_1}(r) \) denotes the volume of a geodesic ball of radius \( r \) in the \( n \)-dimensional model space (a simply connected complete Riemannian manifold) of sectional curvature \( \kappa_1 \).

### 7 The modified Ricci flows

By scaling invariance, the above results extend to the modified Ricci flow

\[
\frac{\partial g}{\partial t} = -2Ric + \lambda(g, t)g \tag{7.1}
\]

with a smooth scalar function \( \lambda(g, t) \) independent of \( x \in M \). The volume-normalized Ricci flow

\[
\frac{\partial g}{\partial t} = -2Ric + \frac{2}{n} \hat{R}g \tag{7.2}
\]
on a closed manifold, with \( \bar{R} \) denoting the average scalar curvature, is an example of the modified Ricci flow. The \( \lambda \)-normalized Ricci flow

\[
\frac{\partial g}{\partial t} = -2\text{Ric} + \lambda g
\]  

(7.3)

for a constant \( \lambda \) is another example. (Of course, it reduces to the Ricci flow when \( \lambda = 1 \).) The normalized Kähler-Ricci flow is a special case of it.

We have e.g. the following results from [Y3].

**Theorem 7.1** Theorem 5.3 and Theorem 6.2 extend to the modified Ricci flow.

Let \( g = g(t) \) be a smooth solution of the modified Ricci flow (7.1) on \( M \times [0, T) \) for some (finite or infinite) \( T > 0 \), with a given initial metric \( g_0 \). We set

\[
T^* = \int_0^T e^{-\int_0^s \lambda(g(s), s) ds} dt. 
\]  

(7.4)

**Theorem 7.2** Assume that \( T^* < \infty \).

1) There are positive constants \( A \) and \( B \) depending only on the dimension \( n \), a non-positive lower bound for \( R_{g_0} \), a positive lower bound for \( \text{vol}_{g_0}(M) \), an upper bound for \( C_S(M, g_0) \), and an upper bound for \( T^* \), such that for each \( t \in [0, T) \) and all \( u \in W^{1,2}(M) \) there holds

\[
\left( \int_M |u|^{\frac{2n}{n-2}} d\text{vol} \right)^{\frac{n-2}{n}} \leq A \int_M (|\nabla u|^2 + \frac{R}{4} u^2) d\text{vol} + B e^{-\int_0^t \lambda(g(s), s) ds} \int_M u^2 d\text{vol}. 
\]  

(7.5)

2) Let \( L > 0 \) and \( t \in [0, T) \). Consider the Riemannian manifold \( (M, g) \) with \( g = g(t) \). Assume \( R \leq \frac{1}{r^2} \) on a geodesic ball \( B(x, r) \) with \( 0 < r \leq L \). Then there holds

\[
\text{vol}(B(x, r)) \geq \left( \frac{1}{2^{n+3}A + 2Be^{-\int_0^t \lambda(g(s), s) ds} L^2} \right)^{\frac{n}{2}} r^n. 
\]  

(7.6)

Combining Theorems 7.1 and 7.2 with Perelman’s scalar curvature estimate [ST] we obtain the following result from [Y3].

**Theorem 7.3** Let \( g = g(t) \) be a smooth solution of the normalized Kähler-Ricci flow

\[
\frac{\partial g}{\partial t} = -2\text{Ric} + 2\gamma g
\]  

(7.7)

on \( M \times [0, \infty) \) with a positive first Chern class, where \( \gamma \) is the positive constant such that the Ricci class equals \( \gamma \) times the Kähler class. (We assume that \( M \) carries such
a Kähler structure.) Then the Sobolev inequality (7.8) holds true with \( \lambda(g(s), s) = 2\lambda \). Moreover, there is a positive constant \( L \) depending only on the initial metric \( g_0 = g(0) \) and the dimension \( n \) such that the inequality (7.6) holds true for all \( t \in [0, T) \) and \( 0 < r \leq L \).

If \( \lambda_0(g_0) > 0 \), then the Sobolev inequality (8.1) holds true for \( g \). Moreover, there is a positive constant depending only on the initial metric \( g_0 \) and the dimension \( n \) such that the inequality (6.3) holds true for all \( t \in [0, T) \) and \( 0 < r \leq L \). Consequently, blow-up limits of \( g \) at the time infinity satisfy (6.3) for all \( r > 0 \) and the Sobolev inequality

\[
\left( \int_M |u|^{\frac{2n}{n-2}} dv_g \right)^{\frac{n-2}{n}} \leq A \int_M |\nabla u|^2 dv_g
\]

for all \( u \). (In particular, they must be noncompact.)

8 Further Sobolev inequalities

As is well-known, the case \( p = 2 \) of the \( W^{1,p}(M) \) Sobolev inequalities is used most often in analysis and geometry. However, it is of high interest to understand the situation \( 1 < p < 2 \) and \( 2 < p < n \), both from the point of view of a deeper understanding of the theory and the point of view of further applications. In [Y7], \( W^{1,p} \) and \( W^{2,p} \) Sobolev inequalities for general \( p \) along the Ricci flow were derived using several different methods, including Bessel potentials and Riesz transforms. Indeed, general results on deriving further Sobolev inequalities from a given Sobolev inequality were obtained in [Y7]. We state here part of the results from [Y7].

Theorem 8.1 Assume that \( \lambda_0(g_0) > 0 \). Let \( 2 < p < n \). Then there holds for each \( t \in [0, T) \) and all \( u \in W^{1,p}(M) \)

\[
\left( \int_M |u|^{\frac{2n}{n-2}} dv_g \right)^{\frac{n-2}{n}} \leq A \left( \max R^+ + 1 \right) \text{vol}(M)^{\frac{p}{2}} \int_M (|\nabla u|^p + |u|^p) dv_g,
\]

for all \( u \). (In particular, they must be noncompact.)

where all geometric quantities are associated with \( g(t) \), except the constant \( A \), which can be bounded from above in terms of the dimension \( n \), a nonpositive lower bound for \( R_{g_0} \), a positive lower bound for \( \text{vol}_{g_0}(M) \), an upper bound for \( C_S(M, g_0) \), a positive lower bound for \( \lambda_0(g_0) \), and an upper bound for \( \frac{1}{p} \). The quantity \( (\max R^+ + 1) \text{vol}(M)^{\frac{p}{2}} \) is at time \( t \) and the number \( m(p) \) is defined as follows: \( m(p) = 2^{k+1} \) for \( p \in (p_k, p_{k+1}], \ p_0 = 2, \) and \( p_{k+1} = \frac{n^2 p_k}{(n-p_k)^2 + np_k} \) for \( k \geq 0 \).
**Theorem 8.2** Assume $T < \infty$. Let $2 < p < n$. Then there holds for each $t \in [0, T)$ and all $u \in W^{1,2}(M)$

\[
\left( \int_M |u|^{\frac{np}{n-p}} \, d\text{vol} \right)^{\frac{n-p}{n}} \leq A \left[ 1 + (\max R^+ + 1) \text{vol}(M)^{\frac{n}{p}} \right]^{\frac{m(p)}{p}} \int_M (|\nabla u|^p + |u|^p) \, d\text{vol},
\]

(8.2)

where all geometric quantities are associated with $g(t)$, except the constant $A$, which can be bounded from above in terms of the dimension $n$, a nonpositive lower bound for $R_{g_0}$, a positive lower bound for $\text{vol}_{g_0}(M)$, an upper bound for $C_S(M, g_0)$, an upper bound for $T$, and an upper bound for $\frac{1}{n-p}$. The quantity $(\max R^+ + 1) \text{vol}(M)^{\frac{n}{p}}$ is at time $t$ and the number $m(p)$ is the same as in Theorem 8.1.

These two theorems are proved by employing an induction scheme based on the Hölder inequality. Note that this scheme cannot be extended to $1 < p < 2$. Next we have the results on nonlocal Sobolev inequalities in terms of the $(1, p)$-Bessel norm, which is defined as follows

\[
\|u\|_{B,1,p} = \|(-\Delta + 1)^{\frac{1}{2}} u\|_p
\]

(8.3)

for $u \in W^{1,p}(M)$.

**Theorem 8.3** Assume that $\lambda_0(g_0) > 0$. Let $1 < p < n$. There is a positive constant $C$ depending only on the dimension $n$, a positive lower bound for $\lambda_0(g_0)$, a positive lower bound for $\text{vol}_{g_0}(M)$, an upper bound for $C_S(M, g_0)$, an upper bound for $\frac{1}{p-1}$, and an upper bound for $\frac{1}{n-p}$, such that for each $t \in [0, T)$ and all $u \in W^{1,p}(M)$ there holds

\[
\|u\|_{\frac{np}{n-p}} \leq C (1 + R^+_{\text{max}})^{\frac{1}{2}} \|u\|_{B,1,p}.
\]

(8.4)

**Theorem 8.4** Assume $T < \infty$ and $1 < p < n$. There is a positive constant $C$ depending only on the dimension $n$, a nonpositive lower bound for $R_{g_0}$, a positive lower bound for $\text{vol}_{g_0}(M)$, an upper bound for $C_S(M, g_0)$, an upper bound for $T$, an upper bound for $\frac{1}{p-1}$, and an upper bound for $\frac{1}{n-p}$, such that for each $t \in [0, T)$ and all $u \in W^{1,p}(M)$ there holds

\[
\|u\|_{\frac{np}{n-p}} \leq C (1 + R^+_{\text{max}})^{\frac{1}{2}} \|u\|_{B,1,p}.
\]

(8.5)

To convert the above results into ordinary Sobolev inequalities, the method of Riesz transforms was used in [Y7]. Based on the $L^p$ estimates for the Riesz transform due to D. Bakry [B] the following results were obtained in [Y7].
Theorem 8.5 Assume $\lambda_0(g_0) > 0$. Let $1 < p < n$. There is a positive constant $C$ depending only on the dimension $n$, a positive lower bound for $\lambda_0(g_0)$, a positive lower bound for $\text{vol}_{g_0}(M)$, an upper bound for $C_s(M, g_0)$, an upper bound for $\frac{1}{p-1}$, and an upper bound for $\frac{1}{n-p}$, such that for each $t \in [0, T)$ and all $u \in W^{1,p}(M)$ there holds

$$
\|u\|_{W^{1,p}} \leq C(1 + R_{\text{max}}^{\frac{1}{2}})(\|\nabla u\|_p + (1 + \kappa)\|u\|_p). \tag{8.6}
$$

Theorem 8.6 Assume $T < \infty$ and $1 < p < n$. There is a positive constant $C$ depending only on the dimension $n$, a nonpositive lower bound for $R_{g_0}$, a positive lower bound for $\text{vol}_{g_0}(M)$, an upper bound for $C_s(M, g_0)$, an upper bound for $T$, an upper bound for $\frac{1}{p-1}$, and an upper bound for $\frac{1}{n-p}$, such that for each $t \in [0, T)$ and all $u \in W^{1,p}(M)$ there holds

$$
\|u\|_{W^{1,p}} \leq C(1 + R_{\text{max}}^{\frac{1}{2}})(\|\nabla u\|_p + (1 + \kappa)\|u\|_p). \tag{8.7}
$$

Similar results involving an integral norm of the Ricci curvature were also obtained in [Y7]. One should compare the above results with Gallot’s estimate of the isoperimetric constant [G] which implies an estimate for the Sobolev inequalities. In contrast to Gallot’s estimates, no upper bound for the diameter nor positive lower bound for the volume of $g(t)$ is assumed. Moreover, in (8.7) the lower bound for the Ricci curvature does not appear in front of $\|\nabla u\|_p$. (Gallot’s estimates lead to Sobolev inequalities in which the lower bound for the Ricci curvature gets involved with $\|\nabla u\|_p$.)

9 The Ricci flow with surgeries

The key construction in Perelman’s work on the Ricci flow and geometrization of 3-manifolds is the Ricci flow with surgeries [P2], which extends Hamilton’s earlier work on surgeries of the Ricci flow in a substantial way. (For very nice accounts of Perelman’s work we refer to [MT] and [KL].) Perelman’s work provides a very clear picture of the behavior of the Ricci flow near blow-up singularities. Indeed, regions of large curvature are classified topologically and geometrically, and one can find nice portions of the manifold near blow-up singularities where surgeries can be performed.

Let $g = g(t)$ be a smooth solution of the Ricci flow on a closed 3-manifold $M$. (For technical reasons one needs to normalize the initial metric in a natural way.) Assume that $g(t)$ is smooth on $[0, T)$ for a finite $T$, but becomes singular somewhere as $t \to \infty$. (The situation of a Ricci flow with surgeries becoming singular upon approaching a finite time is similar.) By Perelman’s work, we can find a maximal region $\Omega \subset M$ such that $g(t)$ converges smoothly in $\Omega$ to a metric $g_T$. Then the curvature of $g_T$ blows up as one approaches the boundary of $\Omega$. For $\rho > 0$ we consider the region $\Omega_\rho = \{x \in \Omega : R < \rho^{-2}\}$ defined in terms of $g_T$. For sufficiently small $\rho$, $\partial \Omega_\rho$ consists
of boundaries of \( \epsilon \)-necks whose interiors lie outside of \( \Omega_\rho \). An \( \epsilon \)-neck is, after a suitable rescaling, \( \epsilon \)-close to the product \( S^2 \times [-\epsilon^{-1}, \epsilon^{-1}] \) in the \( C^{1/\epsilon} \) topology. Let \( Z_0 \) be one of such \( \epsilon \)-necks and \( Z \) the component of \( \Omega - \Omega_\rho \) containing \( Z_0 \). If \( Z \) is compact, then it is an \( \epsilon \)-cap, an \( \epsilon \)-neck or \( \epsilon \)-tube. In the last two cases, the both boundary components of \( Z \) are contained in \( \partial \Omega_\rho \). If \( Z \) is noncompact, then it is an \( \epsilon \)-horn consisting of \( \epsilon \)-necks of increasing magnitudes of curvature, with the curvature going to infinity at the end. We’ll call \( Z \) an attached \( \epsilon \)-horn for \( \Omega_\rho \).

The surgery will be performed on the attached \( \epsilon \)-horns for \( \Omega_\rho \). Indeed, for each attached \( \epsilon \)-horn \( Z \), we pick a suitable \( \epsilon \)-neck in \( Z \) and carry out the surgery on it. The surgery consists of cutting the \( \epsilon \)-neck and hence \( Z \) into two parts, throwing out the noncompact part of \( Z \), and gluing in a sufficiently long capped cylinder. The original metric \( g_T \) will be interpolated inside of the \( \epsilon \)-neck with a standard metric on the capped cylinder. After the surgery we restart the Ricci flow with the metric resulting from the surgery as the initial metric. This yields the first stage of a Ricci flow with surgeries. Repeating the process whenever we run into singularities at a finite time we then obtain a Ricci flow with surgeries on its maximal time interval.

The surgeries have to be done in a way such that the following three key properties of the Ricci flow are preserved for the extended solution of the Ricci flow after the surgery: 1) the Hamilton-Ivey pinching, 2) the \( \kappa \)-noncollapsing property, and 3) the canonical neighborhood property. The first property is the easiest to preserve. Indeed, a trick of Hamilton in [H2] can be applied without any difficulty. In [P2], Perelman uses the reduced length and the reduced volume to handle the second property. Because of the surgery, the behavior of the \( \mathcal{L} \)-geodesics are rather complicated, and the arguments are also forced to be rather complicated. Furthermore, the second and third properties are established in [P2] by rather involved combined arguments.

In [Y8], the arguments in [P2] for preserving the \( \kappa \)-noncollapsing property are replaced by a much simpler and more transparent argument. Indeed, the Sobolev inequality is first established for the metric resulting from the surgery. Then the Sobolev inequality will continue to hold by the results in [Y3], as presented in Section 5. Finally, we obtain a \( \kappa \)-noncollapsing estimate as a consequence of the Sobolev inequality, as in [Y3] and explained in Section 6. As an important feature of these arguments, the \( \kappa \)-noncollapsing property is established independent of the canonical neighborhood property, making the choice of the surgery parameters much simpler. It also becomes easier to establish the canonical neighborhood property.

The main results in [Y8] are as follows.

**Theorem 9.1** Let \( n = 3 \) and \( g = g(t) \) be a Ricci flow with surgeries as constructed in [P2] on its maximal time interval \( [0, T_{\text{max}}) \), with suitably chosen surgery parameters. Let \( g_0 = g(0) \). Let \( m(t) \) denote the number of surgeries which are performed up to the time \( t \in (0, T_{\text{max}}) \). Then there holds at each \( t \in [0, T_{\text{max}}) \)

\[
\left( \int_M |u|^6 \text{d}v\text{ol} \right)^{\frac{1}{3}} \leq A(t) \int_M (|\nabla u|^2 + \frac{R}{4} u^2) \text{d}v\text{ol} + B(t) \int_M u^2 \text{d}v\text{ol} \quad (9.1)
\]
for all \( u \in W^{1,2}(M) \), where \( A(t) \) and \( B(t) \) are bounded from above in terms of a nonpositive lower bound for \( R_{g_0} \), a positive lower bound for \( \text{vol}_{g_0}(M) \), an upper bound for \( C_S(M, g_0) \), and an upper bound for \( t \).

If \( \lambda_0(g_0) > 0 \), then there holds at each \( t \in [0, T_{\text{max}}) \)
\[
\left( \int_M |u|^6 d\text{vol} \right)^{\frac{1}{6}} \leq A(t) \int_M (|\nabla u|^2 + \frac{R}{4} u^2) d\text{vol}
\] (9.2)
for all \( u \in W^{1,2}(M) \), where \( A(t) \) is bounded from above in terms of a nonpositive lower bound for \( R_{g_0} \), a positive lower bound for \( \text{vol}_{g_0}(M) \), an upper bound for \( C_S(M, g_0) \), a positive lower bound for \( \lambda_0(g_0) \), and an upper bound for \( m(t) \).

\( \kappa \)-noncollapsing estimates follow from the above Sobolev inequalities.

This theorem is based on a general result on the Sobolev inequality on manifolds with surgeries and the specific patterns of surgeries for the Ricci flow in dimension 3 as discussed above.

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