Dependent Dirichlet Process Rating Model (DDP-RM)

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Abstract

Typical IRT rating-scale models assume that the rating category threshold parameters are the same over examinees. However, it can be argued that many rating data sets violate this assumption. To address this practical psychometric problem, we introduce a novel, Bayesian nonparametric IRT model for rating scale items. The model is an infinite-mixture of Rasch partial credit models, based on a localized Dependent Dirichlet process (DDP). The model treats the rating thresholds as the random parameters that are subject to the mixture, and has (stick-breaking) mixture weights that are covariate-dependent. Thus, the novel model allows the rating category thresholds to vary flexibly across items and examinees, and allows the distribution of the category thresholds to vary flexibly as a function of covariates. We illustrate the new model through the analysis of a simulated data set, and through the analysis of a real rating data set that is well-known in the psychometric literature. The model is shown to have better predictive-fit performance, compared to other commonly used IRT rating models.

KEYWORDS: Rating Scale Analysis, Bayesian Nonparametrics, Bayesian Inference
RUNNING TITLE: Dependent Dirichlet Process Rating Model.
1 Introduction

In social science research, it is often of interest to analyze examinee ratings to items of a test. An IRT rating model, and its parameters estimated from the given rating scale data set, provides useful information about various psychometric qualities. They include the difficulty parameter of each test item, the thresholds parameters of the rating categories, and the test ability (latent trait) parameter of each examinee. Typical IRT rating models include the Rasch rating scale model (Andrich, 1978), partial credit models (PCM) (Masters, 1982; Muraki, 1992), and the family of graded response models (Samejima, 1969, 1972), all of which have seen many successful applications in a wide range of research settings.

Nevertheless, these IRT models have their limitations. Typical IRT rating models assume that the rating category threshold parameters apply to all examinees. However, this assumption is violated when differential rating category usage occurs across the examinees. Differential rating category usage may be caused by differential item functioning (DIF); that is, when different clusters (groups) of examinees give rise to different threshold estimates for the rating categories after controlling for the level of examinee ability. The different clusters may either refer to unknown latent groups or known examinee groups (e.g., male and female). Differential category usage across examinees may also arise from non-systematic random error, such as when unclear labels that are assigned to the rating categories. Regardless, if a typical IRT model is used to analyze data which violates its assumption of no differential category usage, then the model may poorly fit the data and produce misleading results. The results would wrongly indicate that, for each test item, a single set of rating category threshold estimates applies for all examinees. In turn, this could lead to misleading examinee ability estimates. With traditional models, item fit statistics are often relied upon to identify items that misfit the model. However, fit statistics have low power in identifying DIF items (Seol, 1999; Smith & Suh, 2003). Moreover, even when an item fit statistic identifies an item as problematic, it does not explain why the item is misfitting.

Multiple-group IRT models (e.g., Lord, 1980; Wright & Masters, 1982) are more appropriate when the differential category usage is a result of DIF. These models specify interaction covariates between person and item characteristics (e.g., overall item difficulty and category thresholds). The regression coefficients of these interaction terms indicate whether DIF is present in an item, and provide some explanation about how rating category usage varies as a function of examinee characteristics. This modeling approach, however, is still limited because it assumes that the model contains all the covariates that could be associated in explaining DIF. As previously mentioned, latent or unknown examinee characteristics may also contribute to differential rating category usage, and/or random error may be present in the rating thresholds.

It then seems preferable to specify a discrete-mixture IRT rating model that can identify and account for differential rating category usage in the items, which may either result from multiple latent clusters (groups) of examinees, and/or result from known examinee characteristics (covariates). For each item, and conditioned on any other known covariates, the model would specify a (mixture) distribution for the rating category thresholds over all examinees, while assigning a distinct set of rating category threshold parameters to each latent cluster of examinees. If all examinees use (e.g., interpret) an item’s rating categories in the same manner, then the model’s threshold distribution becomes unimodal with near-
zero variance. Such a distribution would indicate a single cluster of examinees in terms of the rating thresholds, as in typical IRT rating models which assume no differential category usage. When an item exhibits differential rating category usage over examinees, then the model’s threshold distribution will have noticeable variance, with possible skewness and/or multimodality. A unimodal distribution with noticeable variance and/or skewness may either indicate uncertainty in the rating category usage of the item or DIF. A multimodal distribution would indicates DIF, with the multiple modes indicating multiple latent clusters of examinees. Finally, if an item’s threshold distribution is shown to depend on one or more known covariates that describe examinee background characteristics (e.g., gender, income), after controlling for examinee ability, then there is DIF due to known examinee groupings (as in multiple-group IRT).

A discrete mixture model has the general form (e.g., McLachlan & Peel, 2000):

\[ f_{G_{\mathbf{x}}}(y|\mathbf{x}) = \int f(y|x; \xi, \Psi(x)) dG_{\mathbf{x}}(\Psi) = \sum_{h=1}^{H} f(y|x; \xi, \Psi_h(x)) \omega_h(x), \]

given a mixing distribution \( G_{\mathbf{x}} \) that is possibly covariate \((\mathbf{x})\) dependent; component indices \( h = 1, \ldots, H \), kernel (component) densities \( f(y|x; \xi, \Psi_h(x)) \) \((h = 1, \ldots, H)\) with fixed parameters \( \xi \) and random parameters \( \Psi_h(x) \) that are subject to the mixture; and given mixing weights \( (\omega_h(x))_{h=1}^{H} \) which sum to 1 at every \( \mathbf{x} \in \mathcal{X} \). Mixture IRT models treat \( y \in \{k = 0, 1, \ldots, m\} \) as a scored item response (e.g., a rating), and specify each of the kernel densities \( f(y|x; \xi, \Psi_h(x)) \) by an ordinary IRT model, such as a 2-parameter logistic model, or a Rasch rating scale model.

Typical IRT mixture models assume finite mixtures (i.e., \( H < \infty \)) (Rost, 1991; Smit, Kelderman, & van der Flier, 2003; Von Davier & Yamamoto, 2004; Frick, Strobl, Leisch, & Zeileis, 2012), which limits their ability to adequately describe many rating scale data sets. We could achieve greater modeling flexibility by turning to a fully nonparametric framework, through the specification of an infinite-mixture model (i.e., \( H = \infty \)). Such a model has infinitely-many parameters, and avoids the restrictive assumption of parametric IRT models, namely, that the distribution of item response data can be fully-described by finitely-many parameters. Along these lines, infinite-mixture IRT models have been developed. They include models based on the Dirichlet process (DP) mixture of the item parameters of a 3-parameter logistic model (Miyazaki & Hoshino, 2009), models based on a DP mixture of ability parameters in a Rasch model (San Martin et al., 2011), and a Dependent Dirichlet process (DDP) mixture model for the link function of the 2-parameter IRT model (Duncan & MacEachern, 2008). Karabatsos and Walker (2013 to appear) review the DP and DDP mixture models for IRT. However, none of the available mixture IRT models provide clustering of examinees in terms of the rating category threshold parameters. This is because they do not treat the rating category threshold parameters as the random parameters (i.e., \( \Psi_h(x) \)) that are subject to the mixture.

To address the limitations of the existing IRT models, we introduce a novel Bayesian nonparametric IRT rating model, which we call the DDP Rating Model (DDP-RM). This model is an infinite-mixture of Rasch partial credit models, with rating category threshold parameters subject to the mixture, and with covariate-dependent stick-breaking weights. The random parameters and the mixture weights are modeled by a Dependent Dirichlet
process (DDP) (MacEachern, 1999; 2000; 2001), which is defined by a novel modification of the local Dirichlet process (lDP) (Chung & Dunson, 2011).

In Section 2, we introduce our DDP Rating Model (DDP-RM). In Section 3, we illustrate our model on simulated data, in order to demonstrate the model’s ability to identify DIF as a result of latent (unknown) examinee characteristics (covariates). In Section 4, we illustrate our model on a real data set of rating scale items, which has been extensively studied in the psychometric modeling literature (De Boeck & Wilson, 2004). In this illustration, we also compare the goodness of predictive fit between our DDP-RM and other IRT rating scale model of common usage. In Section 5, we conclude by discussing possible future extensions of our model. Throughout, we denote \( n(\cdot|\cdot, \cdot) \), \( n_p(\cdot|\cdot, \cdot) \), \( ga(\cdot|\cdot, \cdot) \), \( ig(\cdot|\cdot, \cdot) \), \( beta(\cdot|\cdot, \cdot) \), and \( un(\cdot|\cdot, \cdot) \) as the probability density functions for the univariate normal, \( p \)-variate Normal, gamma, inverse gamma, beta, and uniform distributions, respectively. The gamma and inverse gamma distributions are parameterized by shape and rate parameters.

2 The Dependent Dirichlet Process Rating Model (DDP-RM)

Our rating model, the DDP-RM, is defined by an infinite mixture of IRT rating model. Specifically, this mixture model assumes that the probability of a rating \( Y = y \) is defined by:

\[
P(Y = y|x; \theta, \upsilon, \gamma, \psi) = \int f(y|\theta, \tau_h) \, dG_x(\tau) = \sum_{h=1}^{\infty} f(y|\theta, \tau_h) \omega_h(x^T\gamma; \upsilon, \gamma, \psi),
\]

where the kernel probability densities \( f(y|\theta, \tau_h) \) are specified by the partial credit model (PCM),

\[
f(y|\theta, \tau_h) = P(Y = y|\theta, \tau_h) = \frac{\exp(\theta y - \sum_{l=0}^{y} \tau_{lh})}{\sum_{k=0}^{m} \exp(k\theta - \sum_{l=0}^{m} \tau_{kh})}, \quad h = 1, 2, \ldots,
\]

where the mixture distribution \( G_x \) is covariate \((x)\) dependent and defined by:

\[
G_x(\cdot) = \sum_{h=1}^{\infty} \omega_h(x^T\gamma) \delta_{\tau_h(x^T\gamma)}(\cdot),
\]

and where \( \delta_{\tau}(\cdot) \) denotes the degenerate distribution which assigns probability 1 to the value \( \tau \). Additionally, \( \theta_t \) denotes the ability parameter of a given examinee \( t \), for a sample of examinees indexed by \( t = 1, 2, \ldots, N \); and for the \( m+1 \) rating categories indexed by \( k = 0, 1, \ldots, m \), the vector \( \tau_h = (\tau_{hl}, \ldots, \tau_{mh})^T \) gives the set of rating category threshold parameters for the \( h \)th mixture component, while assuming the constraint \( \tau_{0h} \equiv 0 \). The mixture distribution \( G_x(\tau) \) for the thresholds, and the corresponding covariate \((x)\)-dependent mixture weights \( \{\omega_h(x^T\gamma)\}_{h=1,2,\ldots} \) and atoms \( \{\tau_h(x^T\gamma)\}_{h=1,2,\ldots} \) are modeled by a modified local Dirichlet Process (lDP) prior. Therefore, the mixture weights have a stick-breaking form (see Sethuraman, 1994); later, we provide more details about the lDP and these weights. In general, \( x \) can be a vector of any \( p \) covariates, \( x = (x_1, \ldots, x_p) \), and they respectively correspond to
(positive-valued) linear regression coefficients $\gamma = (\gamma_1, \ldots, \gamma_p)^T$. For example, the covariates may be dummy (0-1) test item indicators, describe examinee characteristics (e.g., gender, race, and/or social economic status), and/or describe other test characteristics (e.g., time at which item was administered, item type, etc.).

As shown in equation (3), the DDP-RM is based on an infinite-mixture distribution $G_x(\tau)$ for the rating category thresholds $\tau$. Therefore, conditionally on $x$, the model can account for virtually all distributions of the rating category thresholds ($\tau_h$). These distributions include unimodal distributions with small-variance, indicating an item is free of DIF; unimodal distributions with larger-variance and/or skewness, indicating an item with more uncertainty in rating category usage, and possibly DIF; and multimodal distributions, which indicate the presence of multiple latent clusters of examinees (i.e., DIF). Also, the shape and location of the mixture distribution $G_x$ can change flexibly as a function of the covariates ($x$). Therefore, at one extreme, the mixture distribution $G_x$ may be unimodal with small variance for one value of the covariates $x$, while for the other extreme, the mixture distribution $G_{x'}$ may be highly skewed and multimodal for a different value of the covariates $x'$.

The mixture distribution, $G_x$, of our model is formed according to our following novel modification of the local Dirichlet process (IDP) (Chung & Dunson, 2011), which is described as follows. First let

$$L_x = \{ h : d (x^T \gamma, h) \leq \psi(x) \} \subset \{ 1, 2, \ldots \}$$

denote the subset of mixture component indices $h \in \mathbb{Z}^+$ having fixed addresses $\{ \Gamma_h \equiv h \}$ that are within a $\psi(x)$-neighborhood around the linear predictor $x^T \gamma$, $\pi_l (x^T \gamma)$ is the $l^{th}$ ordered index in $L_x$, and $d (\cdot, \cdot)$ is a chosen distance measure (e.g., Euclidean). For example, if $x^T \gamma = 10$ and $\psi(x) = 2.5$, then the covariate ($x$)-dependent local subset becomes $L_x = \{8, 9, 10, 11, 12\}$, and $\pi_1 (x^T \gamma) = 8$, $\pi_2 (x^T \gamma) = 9$, ..., $\pi_{|L_x|} (x^T \gamma) = 12$, where $|L_x|$ is the cardinality of the set $L_x$. Under our formulation of the IDP, the local variables are defined by $v (x^T \gamma) = \{ v_h, h \in L_x \}$, in order to specify the mixture weights in (3) as having the covariate-dependent, stick-breaking form

$$\omega_l (x^T \gamma) = v_{\pi_l (x^T \gamma)} \prod_{r < l} (1 - v_{\pi_r (x^T \gamma)}) , \quad \text{(4)}$$

where the rating threshold atoms $\tau (x^T \gamma) = \{ \tau_h, h \in L_x \}$ are also covariate-dependent. We fix $v_{\max(L_x)} (x^T \gamma) \equiv 1$ to ensure that the mixture weights $\omega_l (x^T \gamma)$ sum to 1 for each $x$ (Chung & Dunson, 2011). In short, our IDP forms stick-breaking mixture weights by selecting the strict subset of stick-breaking parameters $\{ v_h \}$ and atoms $\{ \tau_h \}$ that are within the neighborhood centered around (a linearized) $x$. Then the mixture weights of Equation (4) gives rise to a covariate-dependent mixing distribution in equation (3), which can be rewritten as:

$$G_x (\cdot) = G_x (\cdot ; \tau, v, \gamma, \psi) = \sum_{l=1}^{|L_x|} \omega_l (x^T \gamma) \delta_{\tau_{\pi_l (x^T \gamma)}} (\cdot) , \quad \text{(5)}$$

where we denote $\tau = (\tau_h)_{h=1}^\infty$, $v = (v_h)_{h=1}^\infty$, and $\psi = (\psi(x))_{x \in \mathcal{X}}$. Based on this specification, for two covariates $x$ and $x'$, the level of similarity between $L_x$ and $L_{x'}$ determines the level of similarity between the two corresponding mixing distribution $G_x (\cdot)$ and $G_{x'} (\cdot)$, with the level of similarity controlled by the parameters $(\gamma, \psi)$. 
The DDP-RM is completed by the specification of the following prior distributions:

\[ \theta_t \sim n(0, \sigma^2), \ t = 1, 2, \ldots, N; \]
\[ \sigma^2 \sim ig(\sigma^2|a_{\sigma^2}, b_{\sigma^2}); \]
\[ \tau_h, \upsilon_h \sim n_m(x_0, \Sigma_\tau) \beta(a|1, \alpha), \ h = 1, 2, \ldots; \]
\[ \alpha, \gamma \sim ga(\alpha|a_\alpha, b_\alpha) \prod_{j=1}^p \text{un} (\gamma_j|a_\gamma, b_\gamma); \]
\[ \psi(x) \sim \text{un}(a_\psi, b_\psi), \ x \in \mathcal{X}. \]

If so desired, one may fix various model parameters to a particular constant by making specific extreme choices of prior. For example, we can fix \( \psi(x) \) to a constant \( c \) by setting \( a_\psi = b_\psi = c \) in the uniform prior. Additionally, we can fix \( \sigma^2 \) to 1, which is often done in many IRT models, by taking \( a_{\sigma^2} \to \infty \) and \( b_{\sigma^2} \to \infty \) in the inverse gamma prior. Similarly, we can fix \( \alpha \) to a fix value by appropriate choices of the gamma parameters.

### 2.1 Bayesian Posterior Inference of the DDP-RM

For notational convenience, we denote a sample set of rating data by \( D_n \), provided by \( N \) examinees \((t = 1, \ldots, N)\) on \( J \) test items \((j = 1, \ldots, J)\), and with \( n = NJ \) giving the total number of item responses in the data set. Each \( y_i \in D_n \) denotes a rating by a particular examinee on a particular item. Additionally, as before, we denote the parameters of our model by \( \ze = (\theta, \sigma^2, \tau, \upsilon, \alpha, \gamma, \psi) \), with \( \theta = (\theta_t)_{t=1}^N \), \( \tau = (\tau_h)_{h=1}^\infty \), \( \upsilon = (\upsilon_h)_{h=1}^\infty \), \( \gamma = (\gamma_h)_{h=1}^\infty \), and \( \psi = (\psi(x))_{x \in \mathcal{X}} \).

According to standard arguments of probability theory involving Bayes’ theorem, given a data set \( D_n \) having likelihood \( \prod_{i=1}^n f(y_i|x_i; \ze) \) under our model with parameters \( \ze \), with a proper prior density \( \pi(\ze) \) defined over the space \( \Omega_\ze \) of \( \ze \), the posterior density of \( \ze \) is proper and is given by:

\[
\pi(\ze|D_n) \propto \prod_{i=1}^n P(y_i|x_i; \ze) \pi(\ze)
\]

up to a proportionality constant. Then the posterior predictive density of \( Y \) for a chosen \( x \) is given by:

\[
f_n(y|x) = \int f(y|x; \ze) \pi(\ze|D_n) d\ze,
\]

with this density corresponding to posterior predictive mean (expectation) and variance (Var)

\[
E_n(Y|x) = \int y f_n(y|x) dy, \quad \text{Var}_n(Y|x) = \int (y - E(Y|x))^2 f_n(y|x) dy.
\]

Additionally, when investigating for DIF, it is of interest to infer functionals of the posterior predictive mean \( E_n[G_x(\cdot)] \) of the threshold mixture distribution \( G_x(\tau) \), such as its density. This posterior predictive mean is defined by

\[
E_n[G_x(\cdot)] = \int \int \int G_x(\cdot|\tau, \upsilon, \alpha, \psi) \pi(\tau, \upsilon, \alpha, \psi|D_n) d\tau d\upsilon d\alpha d\psi,
\]

given the marginal posterior density:

\[
\pi(\tau, \upsilon, \alpha, \psi|D_n) = \int \int \pi(\ze|D_n) d\theta d\sigma^2 d\alpha.
\]
In order to perform inference of functionals of the posterior density \( \pi(\zeta|D_n) \), including marginal posterior densities, posterior predictive densities \( f_n(y|x) \), the posterior mean mixing distribution \( E_n[G_x(\cdot)] \), we make use of standard MCMC sampling methods for Bayesian infinite-mixture models. These sampling methods are described by Kalli, Griffin, and Walker (2011). Appendix A provides more details about all the conditional posterior distribution of the model, which are sampled at each stage of the MCMC algorithm.

### 2.2 Unique Features of the DDP-RM

As mentioned, one unique feature of the DDP-RM is that it flexibly allows the mixing distribution \( G_x \) to take on any shape, ranging from unimodal with small variance, to highly multimodal with large variance. Moreover, the mixing distribution \( G_x \) of the DDP-RM can flexibly change as a function of the covariates \( x \). This flexibility is enabled by a nonparametric specification of the mixing distribution \( G_x \) according to a flexible infinite mixture (involving an infinite number of parameters), with covariate-dependent mixture weights (i.e., \( \omega_h \)) and thresholds (i.e., \( \tau_h \)), as shown in equations (3) and (5). In other words, the model makes no finite-parametric assumptions about this mixing distribution, unlike traditional models, such as the assumption that the mixing distribution is normally distributed and can be described by a finite number of parameters (i.e., mean and variance). This assumption implies the empirically-falsifiable assumption that the mixing distribution is symmetric and unimodal. The DDP-RM, which is free from such limited assumptions about the mixture distribution \( G_x \), allows for accurate detection of rating scale category usage in the posterior distribution of \( G_x(\cdot) \) for covariates \( x \) of interest; for example, in the posterior means \( E_n[G_x(\cdot)] \). This could help reveal when subsets or all category labels are unclear, or when DIF is present.

Another unique feature of the DDP-RM is that it clusters item category thresholds based on the similarity in the mixing distribution. This similarity is captured through the neighborhood inducing parameter \( \gamma \). When two separate \( \gamma \)s have the same values, the mixture components are the same for the covariates associates with the two \( \gamma \)s. In the applications of our model for the simulated and real data sets in Sections 3 and 4, we specify the covariates \( x \) by dummy (0-1) test item indicators. Then, similar \( \gamma \)s would indicate that the items associated with the \( \gamma \)s have similar mixing distributions for the rating category thresholds.

### 2.3 Model Assessment of Predictive Performance

Given a set of data \( D_n \), one can use a a mean-squared predictive error criterion, namely the \( D(m) \) criterion (Gelfand & Ghosh, 1998), to compare the predictive performance among \( M \) different IRT rating models, with each model indexed by \( m = 1, ..., M \). For a given model \( m \in \{1, ..., M\} \) under comparison, the criterion is defined by:

\[
D(m) = \sum_{i=1}^{n} [y_i - E_n(Y_i|x_i, m)]^2 + \sum_{i=1}^{n} \text{Var}_n(Y_i|x_i, m) = \text{GF}(m) + \text{Pen}(m)
\]

In the right hand side of the equation above, the first term is a predictive bias measure that indicates the goodness-of-fit (\( \text{GF}(m) \)) of the model, to the sample data \( D_n \) at hand.
The second term is a penalty and is large when the model is either over-fitting or under-fitting the data set \( D_n \). For all other comparison models included in the present study, the \( E_n(Y_i|\mathbf{x}_i, \mu) \) and \( \text{Var}_n(Y_i|\mathbf{x}_i, \mu) \) are derived from marginal maximum or conditional maximum likelihood parameter estimates. For a non-Bayesian model having point estimate \( \hat{\zeta}_n = \hat{\zeta}(D_n) \), such as a maximum-likelihood estimate, the \( D(m) \) criterion is estimated via \( \hat{E}_n(Y_i|\mathbf{x}_i, \mu) = E(Y_i|\mathbf{x}_i, \mu, \hat{\zeta}_n) \) and \( \hat{\text{Var}}_n(Y_i|\mathbf{x}_i, \mu) = \text{Var}(Y_i|\mathbf{x}_i, \mu, \hat{\zeta}_n) \) \((i = 1, \ldots, n)\) (Gelfand & Ghosh, 1998).

3 Illustration of the DDP-RM on Simulated Data

In this section, we provide a simulation study in order to demonstrate the model’s ability to correctly identify DIF due to the presence of multiple latent examinee clusters, and to correctly identify the item free of DIF.

We generated item response data for 3000 examinees and 10 items, with each item scored on a 0-2 rating scale, yielding a total of \( n = 30,000 = 3000 \times 10 \) rating observations. These data were generated according to the parameters of a two-mixture Rasch logistic rating scale model, which are described as follows. Each simulated examinee was assigned an ability \( \theta \) parameter, according to an independent draw from a normal \( n(0,2.25) \) distribution. Additionally, each examinee was randomly assigned to one of two clusters, with equal probability. As a result, 1505 and 1496 examinees were assigned to the first and second cluster, respectively. Furthermore, each of the first nine items was specified as having no DIF in the rating category thresholds, with the second threshold parameter \( \tau_2 \) being 1 unit larger than the first threshold parameter \( \tau_1 \). For example, the fifth item was assigned thresholds \( \tau = (\tau_1 = -.5, \tau_2 = .5) \). Over all these nine items, the category thresholds had range \((-2.3, 2.3)\). In contrast, the tenth item was specified to have DIF for the threshold parameter \( \tau_2 \), but no DIF for the threshold \( \tau_1 \). Specifically, for this item, the first threshold was specified as \( \tau_1 = -1.25 \) for both examinee clusters. The second threshold parameter was specified as \( \tau_2 = 0 \) for the first examinee cluster, and specified as \( \tau_2 = 2 \) for the second examinee cluster.

To analyze the simulated rating data using the DDP-RM, we made the following model specifications for the purposes of demonstrating the model’s ability to differentiate between DIF and no-DIF items. First, we treated only two items as having random (mixed) threshold parameters. They included the fifth item, which was free of DIF, and the tenth item, which had DIF. For each of the remaining eight items, the thresholds were treated as fixed (non-mixed) parameters. Also, for the model, we specified covariates \( \mathbf{x} \) as 0-1 dummy indicators of the 10 items. Thus we can write the neighborhood size parameter as \( \psi(\mathbf{x}) = \psi_j \). Furthermore, we assigned proper prior distributions to the model’s parameters, namely \( \theta \sim iid n(0,\sigma^2), \sigma^2 \sim ig(1,1), \tau_h \sim iid n(\mathbf{0},2I_m), \nu_h \sim iid \text{beta}(1,\alpha), \alpha \sim \text{ga}(1,1), \gamma_j \sim iid \text{un}(1,745) \), while fixing \( \psi_j = 5 \) for all items we treated as random. For each of the eight items with fixed (non-mixed) threshold parameters, the thresholds were assigned prior \( \tau \sim n(\mathbf{0},10I_m) \). We believe that these prior distributions reflect priors that may be specified for typical real-data applications of the DDP-RM, where little prior information is available about the model parameters.

In order to perform Bayesian posterior estimation of the DDP-RM parameters, we ran the
MCMC sampling algorithm for 200,000 MCMC sampling iterations. We discarded the first 100,000 MCMC samples (i.e., burn-in period), and saved every fifth sample thereafter. This resulted in a total of 20,000 MCMC samples that we saved and used for posterior inference. We then used standard procedures (Geyer, 2011) to evaluate the convergence of all MCMC samples to the posterior distribution of the model. Univariate trace plots of the MCMC samples of model parameters showed good mixing of the MCMC algorithm, in the sense that the MCMC samples of these parameters seemed to stabilize and explore the support of the posterior distribution with small auto-correlations. Also, we found that, for each model parameter, the 20,000 saved MCMC samples led to a rather small 95% Monte Carlo confidence interval (MCCI) for the parameter’s marginal posterior mean estimate, according to a consistent batch means estimator (Jones, et al. 2006). Over all model parameters, the size of the 95% MCCI half-width ranged between <.01 and .02. Hence, given all the results of the trace plots and 95% MCCIs, we generated a large-enough number of MCMC samples (200,000) to provide reasonably-accurate posterior estimates of the model’s parameters.

For the DDP-RM, the posterior mean estimates of the mixing distribution $G_x(\tau)$, given covariates $x$ (e.g. item indicators), reveal how examinees used the rating categories. For the fifth item, the top two panels of Figure 1 present the (marginal) posterior mean density estimates of the mixture distributions $G_x(\tau_1)$ and $G_x(\tau_2)$, which correspond to the two rating threshold parameters. As shown in the figure, for each of the two thresholds of this fifth item, the marginal posterior mean density estimate was unimodal with a very small variance. Thus, these estimates correctly shows that the item has no DIF, in the sense that a single common set of category thresholds applies to all examinees. That is, there is a single cluster of examinees in terms of these thresholds. Moreover, the posterior mean estimates of the thresholds were $\overline{\tau} = (\overline{\tau}_1 = -0.44, \overline{\tau}_2 = 0.43)^T$, and are thus very similar to the true data-generating values of $\tau = (\tau_1 = -0.5, \tau_2 = 0.5)^T$.

The bottom two panels of Figure 1 contain the estimated (marginal) posterior densities of $G_x(\tau_1)$ and of $G_x(\tau_2)$ for the two rating threshold parameters associated with the tenth item. For this item, the estimated marginal posterior density of the first threshold $G_x(\tau_1)$ is unimodal. Thus, this estimate correctly indicates the presence of non-DIF for threshold parameter $\tau_1$. The marginal posterior density estimate of the second threshold, however, is bimodal. Hence, this estimate correctly indicates that there is DIF for that item in that threshold. In other words, there are two latent clusters (modes) of examinees in terms of that threshold parameter. Furthermore, the first mode is slightly less than 0, and the second mode is approximately 2, and are thus very close to the true modes (0 and 2, respectively) that were used to simulate the rating data.

4 Illustration of the DDP-RM on Real Data

In this section, we illustrate the DDP-RM through the analysis of a real data set obtained from the verbal aggression study (see De Boeck & Wilson, 2004), which was based on the Verbal Aggression questionnaire. Moreover, we compare the predictive performance between the DDP-RM and several other IRT rating models. This data set has been frequently analyzed for the purposes of evaluating IRT models. Specifically, this data set contains ratings of 24 items that were made by each of 316 students (243 females and 73 males) who
attended a Dutch-speaking Belgian university. Each of the 24 items of the Verbal Aggression questionnaire represents a type of verbal aggression (e.g., “A bus fails to stop for me. I would want to curse.”), and can be categorized into a $2 \times 2 \times 3$ design: Behavior Mode (Want or Do) by Situation Type (Other-to-blame or Self-to-blame) by Behavior Type (Curse, Scold, or Shout). Each item was scored on a rating scale of $0 = \text{no}$, $1 = \text{perhaps}$, and $2 = \text{yes}$.

### 4.1 Model Specifications and MCMC Diagnostics

To analyze the verbal aggression rating data using the DDP-RM, we treated all items as having random (mixed) threshold parameters. Also, as before, we specified the covariates $\mathbf{x}$ as 0-1 dummy indicators for the 24 Verbal Aggression items. Hence, we may rewrite the neighborhood size parameter as $\psi(\mathbf{x}) = \psi_j$. Furthermore, we assigned priors $\theta_t \sim iid \, n(0, 1)$, $\boldsymbol{\tau}_h \sim iid \, n(0, 5 \mathbf{I}_m)$, $v_h \sim iid \, \text{beta}(1, \alpha)$, $\alpha \sim \text{ga}(1, 1)$, $\gamma_j \sim iid \, \text{un}(1, 745)$, and $\psi_j \sim iid \, \text{un}(5, 20)$, in our attempt to specify rather noninformative priors for the model parameters. Finally, as is done with other IRT models, we assumed that the item responses of the Verbal Aggression questionnaire are independent, conditionally on all model parameters. Since each of the 24 questionnaire items can be classified according to $2 \times 2 \times 3$ design in terms of item type, there may be a concern that the data violate this assumption. Though, if such a concern arises, then one can simply specify additional covariates in the DDP-RM that describe the levels of this design, so that it becomes more reasonable to assume conditional independence under the (expanded) model. However, for the interests of providing a simple illustration of the DDP-RM, we will analyze the data by specifying the covariates $\mathbf{x}$ as 0-1 dummy indicators of the 24 questionnaire items.

To perform Bayesian posterior estimation of the DDP-RM parameters, we ran the MCMC sampling algorithm for 200,000 MCMC sampling iterations. As before, we discarded the first 100,000 MCMC samples (i.e., burn-in period), and saved every fifth sample thereafter. This resulted in a total of 20,000 MCMC samples that we saved and used for posterior inference. Univariate trace plots of the MCMC samples of model parameters showed good mixing of the MCMC algorithm, in the sense that the MCMC samples of these parameters seemed to stabilize and explore the support of the posterior distribution with small auto-correlations. To provide more details, Figures 2 and 3 present the trace plots of the MCMC samples of the threshold parameters for three items, and of the ability parameters for six examinees. Also, we found that, for each model parameter, the 20,000 saved MCMC samples led to rather small 95% MCCIs for the marginal posterior mean estimates of various parameters. For example, the size of the 95% MC confidence interval half-width had range (.00, .03) for marginal posterior mean estimates of examinee ability parameters, and had range (.00, .03) for the marginal posterior standard deviation of these parameters. Also, over all 24 item of the Verbal Aggression questionnaire, the size of the 95% MC confidence interval half-width had range (.02, .93) for the posterior mean and .01 to .79 for the posterior standard deviation for the neighborhood location $\gamma$ and for the for the neighborhood size $\psi$.

Over all model parameters, the size of the MCCI half-width typically ranged between .00 and .03), with maximum value of .05. So given all the results of the trace plots and 95% MCCIs, it seems that we generated a large-enough number of MCMC samples (200,000) to provide reasonably-accurate posterior estimates of the model’s parameters.
4.2 Results

Table 1 presents posterior mean and standard deviation estimates of the category threshold parameters for each of the 24 items. As shown, the posterior means ranged from $-0.68$ to $3.32$. Similar to conclusions by others (e.g., De Boeck & Wilson, 2004), Item 21 was found to be the most difficult to endorse, as it attained the largest posterior means for the category thresholds. Item 4 was the easiest to endorse, as it had the smallest posterior means for the category thresholds.

For three of the Verbal Aggression questionnaire items, Figure 4 contains the marginal posterior mean density estimates of $G_x(\tau_1)$ and $G_x(\tau_2)$ for the two rating threshold parameters. As shown, Items 1 and 23 exhibit greater variability in their rating category thresholds compared to Item 2. For Item 1, the marginal posterior mean density estimate of the first threshold ($\tau_1$) and the second threshold ($\tau_2$) is tri-modal and bimodal, respectively. Thus, the item contains DIF, in the sense that there are three distinct latent clusters of examinees with respect to threshold $\tau_1$, and two distinct latent clusters of examinees with respect to threshold $\tau_2$. For Item 23, the marginal posterior mean density estimate is bimodal for threshold parameter $\tau_1$ and for threshold parameter $\tau_2$. Hence, this item also contains DIF. On the other hand, for Item 2, the marginal posterior mean density estimate for each threshold is unimodal with small variance. Thus, these estimates suggest no DIF, and indicate that there is a single cluster of examinees in terms of these threshold parameters.

For all 24 items of the Verbal Aggression questionnaire, Table 1 presents the marginal posterior mean, standard deviation, and modes of the threshold distributions $G_x(\tau_1)$ and $G_x(\tau_2)$. As shown, 21 of the 24 items are unimodal. The multimodal items, such as Items 21 and Item 23, may be referred to content experts on verbal aggression, so that they can provide further explanation as to why they are exhibiting DIF, and provide advice as to how to modify and improve the questionnaire for its future use. However, in the case the one must retain all possible data, such items do not pose problems for the DDP-RM itself because the model accounts for DIF, and therefore produces posterior parameter estimates (e.g., of examinee ability parameters) after controlling for any DIF. In contrast, for an IRT model that assumes no DIF in the rating threshold parameters, the presence of DIF in the data will lead to misleading parameter estimates. As mentioned in the Introduction section, such estimates would wrongly indicate that a single set of rating category threshold estimates applies for all examinees, for each test item. In turn, this may lead to misleading examinee ability estimates.

The DDP-RM also provides information about the similarities in mixing distributions over the 24 questionnaire items, through the neighborhood location and size parameters (i.e., $\gamma_j$ and $\psi_j$, respectively). Over all the 24 items, the marginal posterior mean estimates of the neighborhood location parameter $\gamma_j$ ranged from 6.0 to 255.6, whereas the marginal posterior mean estimates of neighborhood size parameter $\psi_j$ ranged from 7.5 to 19.8. In terms of the posterior posterior means, the items had noticeably different neighborhood locations and sizes, indicating that the items differed in terms of the mixing distribution $G_x(\tau)$. The box-plots in Figure 5 presents the marginal posterior median and interquartile range estimates for the neighborhood location and neighborhood size parameters, for each of the 24 questionnaire items.

Finally, over the 316 examinees (students), the marginal posterior mean estimate of
the ability parameter $\theta_t$ had range $(-2.37, 3.74)$, along with a mean $-0.02$ and standard deviation $1.01$.

### 4.3 Model Comparisons

In this section, for the Verbal Aggression data set, we compared the predictive performance between the DDP-RM, and other well-known IRT rating models. The other models include the partial credit model (PCM) (Masters, 1982), the generalized partial credit model (GPCM) (Muraki, 1992), the rating scale model (RSM) (Andrich, 1978), the graded response model (GRM) (Samejima, 1969), the nominal response model (NRM) (Bock, 1972), the mixture partial credit model (mix-PCM) (Rost, 1991), and a covariate-independent DP mixture PCM model that treated the category thresholds as random. All models except the latter two were fit using IRTPRO 2.1 (Cai, Thissen, & du Toit, 2011). The mix-PCM was fit in WINMIRA 2001 (von Davier, 2001). Among the one-, two-, three-, four-, and five-mixture PCMs, the 3-mixture PCM displayed the best predictive performance, according to the Akaike Information Criterion (Akaike, 1973), an index of model predictive fit. Thus, we report the predictive performance of the three-mixture PCM. The DP mixture PCM model was fit using code we wrote in MATLAB (2012, The MathWorks, Natick, MA). For this model, the baseline distribution for the set of $m$ thresholds was distributed as a multivariate normal distribution with density function $n(\tau|0, I_m)$, the examinee abilities were assigned a normal $n(0, 1)$ prior distribution, and the precision parameter $\alpha$ was fixed to 1. For the DDP-RM and the DP mixture PCM, we estimated the posterior distribution of parameters using 200,000 MCMC sampling iterations, as before. In each of these cases, the size of the 95% MCCI half width was generally less than 1, over all model parameters.

Table 2 presents the $D(m)$ mean-squared error predictive criterion for each of the IRT models used to analyze the Verbal Aggression data set. As shown, the DDP-RM outperformed all comparison models, by at least 49 $D(m)$ units. Moreover, in terms of the $D(m)$, there was no overlap between any two of the models after accounting for the 95% MCCI of the $D(m)$ estimate. In all, the three mixture models outperformed the traditional, non-mixture models. This result suggests that more than one latent class is present in the data set. Nevertheless, the finite-mixture Rasch PCM model, while outperforming the traditional models, was still bested by the two infinite-mixture models. The DDP-RM outperforming the DP-mixture PCM suggests that all items do not share a common mixing distribution. As mentioned in the previous subsection, the items have noticeably different posterior mean estimates for the neighborhood location item parameters, and for the neighborhood size item parameters.

### 5 Conclusions

We have introduced a novel Bayesian nonparametric rating scale IRT model named the DDP-RM. This is an infinite-mixture model that is based on the local Dirichlet process formulation of the DDP. The model, through posterior mean estimates of the mixing distribution for the threshold parameters, describes how the examinees used the rating categories. Specifically, the posterior number of modes in the mixing distribution reveals the number of clusters...
(groups) of examinees in terms of an item category thresholds. Moreover, using a real data set that is well-known in the psychometric field, we demonstrated that the new model provides a substantially-better predictive fit of the rating data compared to other IRT models commonly used.

In future research, the DDP-RM can be extended by assigning a nonparametric prior for the ability distribution, such as a DP prior (San Martin, Jara, Rolin, & Mouchart, 2011). Also, it would be of interest to extend the model by specifying $G_x(\tau)$ as a more flexible, infinite mixture. For example, Karabatsos and Walker (2012) proposed novel mixture weights that are based on an infinite-ordered probits regression model, with covariate dependence in the mean and in the variance of the probits. Alternatively, the infinite number of mixture weights can be specified by a covariate-dependent version of normalized random measures (Regazzini, Lijoi & Prünster, 2003; Lijoi, Meña, & Prünster, 2005, 2007; James, Lijoi, & Prünster, 2009).

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APPENDIX A: MCMC Sampling Methods

We implement the MCMC sampling method of Kallii et al., (2011) to estimate our infinite-mixture IRT model. This MCMC sampling method involves introducing strategic latent variables in order to implement exact MCMC algorithms for the estimation of the model’s posterior distribution. That is, for our DDP-RM (Section 2), we introduce the latent variables \((u_i, z_i \in \mathbb{Z})\) and a decreasing function \(\xi_h = \exp(-h)\), so that the model’s data likelihood can be written as the joint distribution:

\[
\prod_{i=1}^{n} f(u_i, z_i, y_i | x; \zeta) = \prod_{i=1}^{n} \left\{ \mathbb{I} (0 < u_i < \xi_{z_i}) \xi_{z_i}^{-1} f \left( y_i | \theta_{t(i)}, \tau_{z_i} \right) \omega_{z_i} \right\},
\]

(6)

where \(\theta_{t(i)}\) denotes the ability of examinee \(t\) who provided the rating \(y_{t}\), and where \(\mathbb{I} (\cdot)\) is the indicator function. Marginalizing over each of the latent variables \((u_i, z_i)\) in Equation (6) for each \(i = 1, ... n\), returns the original likelihood,

\[
\prod_{i=1}^{n} \left\{ \sum_{h=1}^{\infty} f \left( y_i | \theta_{t(i)}, \tau_{h} \right) \omega_h \left( x^\top \gamma \right) \right\},
\]

of our infinite-dimensional IRT model. Thus, provided the latent variables, the model can be characterized as a finite-dimensional model, which in turn, permits the use of standard MCMC methods to sample the model’s full joint posterior distribution. Given all variables, save the \((z_i)_{i=1}^{n}\), the choice of each \(z_i\) has minimum 1 and maximum \(N_{\text{max}}\), where \(N_{\text{max}} = \max_i \mathbb{I} (u_i < \xi_{h}) h\).

Specifically, for each \(i = 1, ..., n\) and \(t = 1, ..., T\), each of the model parameters is sampled from its corresponding full conditional posterior distribution at each stage \(s (s = 1, ..., S)\) of the MCMC algorithm. We assume the prior form as in the empirical illustration of our model, in the analysis of the verbal aggression data set, as in Section 4. The full conditional posterior distribution for each block of model parameters are as follows:

1. \(\pi (u_i|...) = \text{un}(u_i|0, \xi_{z_i});\)
2. \(\pi (z_i = h|...) \propto \mathbb{I} (u_i < \xi_{h}) \xi_{h}^{-1} f \left( y_i | \theta_{t(i)}, \tau_{h} \right) \omega_h \left( x^\top \gamma \right), h = 1, ..., N_{\text{max}};\)
3. \(\pi (\theta_{t}|...) \propto n(\theta|0, \sigma^2) \prod_{i:t(i)=t} f \left( y_i | \theta_{t(i)}, \tau_{z_i} \right);\)
4. \(\pi (\sigma^2|...) = \text{ig}(\sigma^2|a_{\sigma^2} + N/2, b_{\sigma^2} + \frac{1}{2} \sum_{i=1}^{N} \theta_i^2);\)
5. \(\pi (\gamma|...) \propto \{ \prod_{j=1}^{p} \text{un}(\gamma_j|a_{\gamma}, b_{\gamma}) \} \prod_{i=1}^{n} v_{z_i} \prod_{l=1}^{z_i-1} (1 - u_l);\)
6. \(\pi (\psi(x)|...) \propto \text{un}(\psi(x)|a_{\psi}, b_{\psi}) \prod_{i=1}^{n} v_{z_i} \prod_{l=1}^{z_i-1} (1 - u_l);\)
7. \(\pi (\tau_h|...) \propto n_{m} (\tau_h|0, \Sigma_{\tau}) \prod_{i:h} f \left( y_i | \theta_{t(i)}, \tau_{z_i} \right), h = 1, ..., N_{\text{max}};\)
8. \(\pi (v_{h}|...) = \text{beta} \left( v_h | 1 + \sum_{i=1}^{n} \mathbb{I}(z_i = h \& z_i \neq \max \{L_x\}, \alpha + \sum_{i=1}^{n} \mathbb{I}(z_i > h) \right), h = 1, ..., N_{\text{max}};\)
9. \( \pi(\alpha|...) = \text{ga}(\alpha | a \alpha + n_{clus} - 1(u > \{O/(1 + O)\}), \{b \alpha - \log(\eta)\}^{-1}) \), given draws \( \eta \sim \text{beta}(\alpha + 1, n) \), \( u \sim \text{un}(0, 1) \), and \( O = (a \alpha + n_{clus} - 1)/(\{b \alpha - \log(\eta)\}n) \), where \( n_{clus} \) is the number of unique \( z_i \), over \( (i = 1, \ldots, n) \) (Escobar & West, 1995, p.584).

Standard MCMC Gibbs sampling methods can be used to sample the full conditionals in Steps 1, 2, 4, 8, and 9. The full conditionals in Steps 3, 5, 6, and 7 are each sampled using an adaptive random-walk Metropolis-Hastings algorithm (Roberts & Rosenthal, 2009). The above 9-step sampling algorithm is repeated a large number \( S \) of times to construct a discrete-time Harris ergodic Markov chain \( \{\zeta(s) = (\theta, \sigma^2, \tau, \nu, \alpha, \gamma, \psi)^{(s)}\}_{s=1}^{S} \), having a posterior distribution \( \Pi(\zeta|D_n) \) as its stationary distribution, provided that a proper prior is assigned to \( \zeta \).

The posterior predictive density \( f_n(y|x) \), and the posterior mean of the mixing distribution \( E_n[G_x(\cdot)] \), and the functionals thereof (e.g., a kernel density estimate), can each be estimated as by-products of the MCMC algorithm. In order to estimate the posterior predictive density \( f_n(y|x) \), a step is added to the MCMC algorithm, to sample from the full conditional posterior distribution \( f(y_i|\theta_{t(i)}, \tau_{z_i}) \), which is a multinomial distribution defined by the Rasch partial credit model. A MCMC sample of \( E_n[G_x(\cdot)] \) is given by \( \tau_{z_i} \).

We have written MATLAB (2012, The MathWorks, Natick, MA) code that implements the MCMC sampling algorithm. The analysis of the verbal aggression data took approximately 24 hours, using a Dell Precision T3600, 3.2 GHz 6-core, and 32 gigs of RAM.
| Item                  | $\tau_1$ Mean | $\tau_1$ SD | $\tau_2$ Mean | $\tau_2$ SD | Modes $\tau_1$ | Modes $\tau_2$ |
|-----------------------|---------------|-------------|---------------|-------------|----------------|----------------|
| 1: bus-want-curse     | -.42          | 1.27        | -.03          | 1.87        | -.1, -.9, 1.4  | .6, -.5        |
| 2: bus-want-scold     | .06           | .83         | .20           | .85         | .1             | .2             |
| 3: bus-want-shout     | .28           | .85         | 1.09          | 1.00        | .4             | 1.4            |
| 4: train-want-curse   | -.68          | 1.47        | .09           | 1.55        | -.3, -.9       | .6, -.1, 1.9   |
| 5: train-want-scold   | -.10          | .25         | .25           | .26         | -.2            | .2             |
| 6: train-want-shout   | .33           | 1.74        | .67           | 1.21        | -.2            | .6             |
| 7: grocery-want-curse | -.14          | .87         | 1.11          | 1.43        | -.4            | 1.5            |
| 8: grocery-want-scold | .82           | .29         | 2.01          | .42         | .8             | 2.0            |
| 9: grocery-want-shout | 1.52          | .52         | 2.75          | .70         | 1.6            | 2.8, 3.8       |
| 10: operator-want-curse | -.63       | .52         | .70           | .56         | -.8            | .7             |
| 11: operator-want-scold       | .63       | .47         | 1.29          | .59         | .7             | 1.4            |
| 12: operator-want-shout       | 1.28       | 1.05        | 1.70          | 1.16        | 1.6            | 2.0            |
| 13: bus-do-curse       | -.61          | .46         | .21           | .47         | -.6            | .2             |
| 14: bus-do-scold       | .14           | .72         | .63           | 1.2         | -.06           | .84            |
| 15: bus-do-shout       | 1.15          | .86         | 1.69          | 1.58        | 1.38, .22      | 2.23           |
| 16: train-do-curse     | -.25          | .92         | .20           | 1.24        | -.46           | .33            |
| 17: train-do-scold     | .48           | .77         | 1.04          | 1.35        | .46            | 1.29           |
| 18: train-do-shout     | 1.62          | 1.00        | 2.17          | 1.2         | 1.94           | 2.47           |
| 19: grocery-do-curse   | .89           | .64         | 2.12          | .93         | 1.02           | 2.25           |
| 20: grocery-do-scold   | .96           | .38         | 2.24          | .56         | 1.10           | 2.21           |
| 21: grocery-do-shout   | 2.87          | .52         | 3.31          | .77         | 2.92           | 3.22           |
| 22: operator-do-curse  | -.22          | .86         | .80           | 1.34        | -.48           | 1.06           |
| 23: operator-do-scold  | .61           | 1.83        | 1.01          | 1.27        | -.15, 2.67     | .64, 1.05      |
| 24: operator-do-shout  | 2.06          | .07         | 2.56          | .89         | 2.17           | 2.45           |

Table 1: For the DDP-RM, the posterior estimates of the ordered category threshold parameters, by item. For the posterior mean and SD estimates, the 95 percent MCCI half-width typically ranged between .00 to .03, with maximum .05.
Table 2: For various IRT models, the overall mean-squared predictive error (D), the goodness of fit (GF), and the penalty (Pen).

| Model (m)       | D(m) | GF(m) | Pen(m) |
|-----------------|------|-------|--------|
| DDP-RM          | 4984 | 2008  | 2976   |
| DP-PCM          | 5033 | 2077  | 2956   |
| 3-Mixture PCM   | 5163 | 2485  | 2679   |
| PCM             | 5716 | 2783  | 2934   |
| GPCM            | 5686 | 2774  | 2912   |
| RSM             | 5726 | 2791  | 2936   |
| NRM             | 5689 | 2774  | 2915   |
| GRM             | 5709 | 2783  | 2925   |
Figure Captions

**Figure 1.** For the simulated data, the marginal posterior mean density estimates of the rating category thresholds, for two items.

**Figure 2.** For three items of the Verbal Aggression questionnaire, trace plots of the MCMC posterior samples of the threshold parameters.

**Figure 3.** For six examinees of the Verbal Aggression questionnaire, trace plots of the MCMC posterior samples of the ability parameters.

**Figure 4.** For three items of the Verbal Aggression questionnaire, the marginal posterior mean density estimate of the rating category thresholds.

**Figure 5.** Median, interquartile, and 95-percentile range of the marginal posterior distribution of the neighborhood location ($\gamma$) and size ($\psi$), for each item of the Verbal Aggression questionnaire.
