Block-diagonalization of matrices over local rings II.

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Abstract. Consider rectangular matrices over a local ring $R$. In the previous work we have obtained criteria for block-diagonalization of such matrices, i.e. $UAV = A_1 \oplus A_2$, where $U, V$ are invertible matrices over $R$. In this short note we extend the criteria to the decomposability of quiver representations over $R$.

1. Introduction

This work is the continuation of [Kerner-Vinnikov].

1.1. Setup. Let $(R, \mathfrak{m})$ be a local (commutative, associative) ring over a field $\mathbb{k}$ of characteristic 0. As the simplest examples one can consider regular rings, e.g. formal power series, $\mathbb{k}[[x_1, \ldots, x_p]]$, rational functions that are regular at the origin, $\mathbb{k}[x_1, \ldots, x_p]_{(\mathfrak{m})}$, convergent power series, $k\{x_1, \ldots, x_p\}$, when $\mathbb{k}$ is a normed field. (If $\mathbb{k} = \mathbb{R}$ or $\mathbb{k} = \mathbb{C}$, one can consider the rings of germs of continuous or smooth functions as well.) Usually we assume the ring to be non-Artinian, i.e. of positive Krull dimension (though $R$ can be not pure dimensional).

Denote by $\text{Mat}(m, n; R)$ the set of matrices with entries in $R$. In this paper we always assume: $1 < m \leq n$. Usually we assume that the matrices ”vanish at the origin”, $A|_0 = 0$, i.e. $A \in \text{Mat}(m, n; m)$. Various matrix equivalences are important.

- In commutative algebra matrices are considered up to the left-right equivalence, $A \sim_{\text{lr}} UAV$, where $(U, V) \in G_{\text{lr}} := \text{GL}(m, R) \times \text{GL}(n, R)$.
- In representation theory (of algebras/groups) the matrices are considered up to the conjugation, $A \sim_{\text{conj}} UAU^{-1}$, $U \in \text{GL}(m, R)$.
- For the study of bilinear/quadratic/skew-symmetric forms one considers the congruence, $A \sim_{\text{cong}} UAU^T$, $U \in \text{GL}(n, R)$. (Note that we consider the non-primitive forms, i.e. $A$ vanishes mod $\mathfrak{m}$.)
- More generally, one studies the matrix problems/representations of quivers. Each such representation consists of a collection of (rectangular) matrices and a prescribed transformation equivalence.

Unlike the case of classical linear algebra (over a field), the matrices over a ring cannot be diagonalized or brought to some nice/simple/canonical form. For a given group action $G \otimes \text{Mat}(m, n; R)$ the natural weaker question is the decomposability:

\begin{equation}
\text{(1)} \quad \text{Which matrices are block-diagonalizable, i.e. } A \sim G \left( \begin{array}{c|c}
A_1 & 0 \\
\hline
0 & A_2
\end{array} \right) ?
\end{equation}

In [Kerner-Vinnikov] we have addressed this question for $G_{\text{lr}}$-equivalence. Recall that the Fitting ideals (the ideals of $j \times j$ minors, $\{I_j(A)\}$) are invariant under $G_{\text{lr}}$-equivalence, [Eisenbud-book, §20]. Thus the ideal of maximal minors of a block-diagonalizable matrix necessarily factorizes: $I_m(A) = I_{m_1}(A_1)I_{m_2}(A_2)$. We have obtained the following necessary and sufficient conditions for block-diagonalizability.

Theorem 2.1. [Kerner-Vinnikov, Theorem 2] (the case of square matrices)
Let $A \in \text{Mat}(m, m; m)$, $m > 1$, with $\det(A) = f_1f_2$. Suppose each $f_i \in R$ is neither invertible nor a zero divisor and $f_1, f_2$ are relatively prime, i.e. $(f_1) \cap (f_2) = (f_1f_2)$. Then $A \sim_{G_{\text{lr}}} A_1 \oplus A_2$, with
det(A_i) = f_i \text{ iff } I_{m-1}(A) \subseteq (f_1) + (f_2) \subseteq R.

2. [Kerner-Vinnikov, Theorem 4] (the case of rectangular matrices)
Let \( A \in \text{Mat}(m, n; \mathbb{m}) \), \( m \leq n \). Suppose the ideal \( I_m(A) \) does not annihilate any non-zero element of \( R \), i.e. \( \text{ann}_R(I_m(A)) = \{0\} \). Suppose further that \( \text{ker}(A) \subseteq I_m(A)R^{\oplus n} \). Suppose \( I_m(A) = J_1J_2 \), where the (nontrivial) ideals \( J_1, J_2 \subseteq R \) are mutually prime, i.e. \( J_1 \cap J_2 = J_1J_2 \). Then \( A \cong A_1 \oplus A_2 \) with \( I_m(A_i) = J_i \) iff \( I_{m-1}(A) \subseteq (J_1 + J_2) \subseteq R \).

(In the second case, when speaking of \( \text{ker}(A) \), we consider \( A \) as the map of free modules \( R^\oplus n \to R^{\oplus m} \).

The condition \( \text{ker}(A) \subseteq I_m(A)R^{\oplus n} \) is the genericity assumption.)

In this short note we extend the decomposability criteria to other equivalences. More precisely, we reduce the decomposability of quiver representations to the decomposability of matrices under \( G_{tr} \)-equivalence. This reduction comes at the expense of enlarging the ring. However theorem 2 is "insensitive" to the dimension of the ring. Thus the theorem can be used effectively to obtain explicit decomposability criteria for various quivers/matrix problems.

## 2. Quiver representations over local rings

Given a quiver \( Q \) with the set of vertices \( I \). A representation of \( Q \) over a ring \( R \) is the collection of \( R \)-modules, \( \{M_i\}_{i \in I} \) and of their morphisms \( M_j \xrightarrow{A_{ij}} M_i \).

The following elementary observation is frequently used. Given a ring \( R \), consider the formal extension by some new variables, \( R[[\{x_i\}]] \). Given two matrices, \( A, B \) over \( R[[\{x_i\}]] \) whose entries are linear in \( \{x_i\} \), i.e. \( A = \sum_i x_iA_i \) for some matrices \( A_i \) over \( R \). If \( A = UBV \), where \( U, V \) are invertible, with entries in \( R[[\{x_i\}]] \), then there exist invertible matrices \( U, V \), over \( R \), satisfying \( A = UBV \). This leads to:

**Proposition 3.** Fix a subgroup \( G \subseteq G_{tr} \). Two tuples of matrices over \( R \) are simultaneously \( G \)-equivalent, \( (A_1, \ldots, A_N) \cong (B_1, \ldots, B_N) \), iff the corresponding matrices \( \sum_i A_ix_i \), \( \sum_i B_ix_i \) are \( G \)-equivalent (over \( R[[\{x_i\}]] \)).

We use this property for the groups \( G_{tr}, G_{conj}, G_{congr} \).

### 2.1. Replacing the quiver by a complete reduced quiver

Fix a quiver \( Q \) and its representation \( \{A_{ij}\} \).

- We can (and will) always assume that for any two vertices \( (i, j) \) of \( Q \) the quiver has arrows in both directions, \( i \Rightarrow j \). (Add all the missing arrows to the initial quiver and assume that the corresponding morphisms are zeros.)

- We can (and will) assume that there are precisely two arrows: \( i \to j \) and \( j \to i \). If there are more, e.g. there is a tuple of morphisms \( M_i \to (A_{ij}^{(1)}, \ldots, A_{ij}^{(N)}) \) \( M_j \), then we extend the ring to \( R[[\{x_i\}]] \), and replace this tuple by one morphism \( \sum_k A_{ij}^{(k)}x_k \). By proposition 3 we get an equivalent problem.

We call the so obtained quiver "a complete reduced" quiver.

### 2.2. Embedding the quiver representations into representation of the Kronecker quiver

Given a complete reduced quiver \( Q_R \), with its representation \( \{A_{ij}\} \), over \( R \). Associate to it the following square matrix:

\[
\{A_{ij}\}_{1 \leq i, j \leq m} \rightsquigarrow \mathbb{A}_Q := \begin{pmatrix}
x_{11}A_{11} + y_11 & x_{12}A_{12} & \cdots & x_{1m}A_{1m} \\
x_{21}A_{21} & x_{22}A_{22} + y_22 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
x_{m1}A_{m1} & \cdots & \cdots & x_{mm}A_{mm} + y_m1
\end{pmatrix}
\]
Here \( \{x_{ij}\} \) and \( \{y_i\} \) are some new formal variables (so that \( A_Q \) is linear in these variables), while \( \mathbb{I} \) denote the identity matrices of the appropriate sizes. We consider \( A_Q \) as a representation of the Kronecker quiver over \( R[[\{x_{ij}\}, \{y_i\}]] \).

**Proposition 5.** Two quiver representations \( \{A_{ij}\}, \{B_{ij}\} \) are equivalent (over \( R \)) iff the corresponding matrices \( A_Q \) and \( B_Q \) are \( G_i \)-equivalent (over \( R[[\{x_{ij}\}, \{y_i\}]] \)).

Apparently this embedding is well known but we know the reference only for the particular case of conjugation, when \( Q \) has just one vertex and one arrow, e.g. [Rao, §8].

**Proof.** \( \Rightarrow \) The equivalence of representations means \( A_{ij} = U_iB_{ij}U_j^{-1} \), for some matrices \( \{U_i \in GL(m_i, R)\} \). Therefore \( \begin{pmatrix} U_1 & \mathbb{O} & \cdots & \mathbb{O} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbb{O} & \cdots & U_m \end{pmatrix} A_Q \begin{pmatrix} U_1^{-1} & \mathbb{O} & \cdots & \mathbb{O} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbb{O} & \cdots & U_m^{-1} \end{pmatrix} = B_Q. \)

\( \Leftarrow \) Suppose \( B = \hat{U}A_Q\hat{V} \), where \( \hat{U}, \hat{V} \in GL(\sum m_i, R[[\{x_{ij}\}, \{y_i\}]]). \) Recall that \( B_Q, A_Q \) are linear in \( \{x_{ij}\}, \{y_i\} \). Thus, by proposition 3, we can choose invertible matrices \( U, V \) over \( R \) satisfying \( B_Q = U A_Q V \). In the later equality consider the \( y_i \) parts for each \( i \). In particular, put \( y_1 = \cdots = y_m \) to get \( UV = \mathbb{I} \). Now consider each \( y_i \) separately to get: \( U = \oplus_i U_i, V = \oplus_i V_i^{-1} \). Therefore \( (\oplus_i U_i) A_Q (\oplus_i U_i^{-1}) = B_Q. \) This implies \( U_i A_{ij} U_i^{-1} = B_{ij} \). \( \blacksquare \)

### 2.3. Decomposability of \( \{A_{ij}\} \) vs decomposability of \( A_Q \).

**Definition 6.** \( A_Q \) is quiver-block-diagonalizable if \( A_Q \overset{G_i}{\cong} A_1 \oplus A_2 \), where each of the determinants \( \det(A_k) \) contains a monomial \( \prod_{i=1}^{m} y_i^{l_i} \), where \( 0 < l_i < m_i \).

**Proposition 7.** The representation \( \{A_{ij}\} \) is decomposable iff the matrix \( A_Q \) is quiver-block-diagonalizable.

**Proof.** \( \Rightarrow \) Suppose \( \{A_{ij}\} \) is (non-trivially) decomposable, i.e. \( \{U_i A_{ij} U_j^{-1} = \begin{pmatrix} A_{ij}^{(1)} & \mathbb{O} \\ \mathbb{O} & A_{ij}^{(2)} \end{pmatrix} \} \). Denote \( U = \oplus U_i \). Then \( U A_Q U^{-1} \) consists of blocks, the \( ij \)'th block being \( \begin{pmatrix} A_{ij}^{(1)} + \delta_{ij} y_i \mathbb{I} & \mathbb{O} \\ \mathbb{O} & A_{ij}^{(2)} + \delta_{ij} y_i \mathbb{I} \end{pmatrix} \).

(Here \( \delta_{ij} = 1 \) if \( i = j \) and 0 if \( i \neq j \).) Note that each matrix \( y_i \mathbb{I} \) is of non-zero size.

Then, by row/column permutations one can bring \( U A_Q U^{-1} \) to the form \( A_1 \oplus A_2 \), where each \( A_k \) consists of blocks \( A_{ij}^{(k)} + \delta_{ij} y_i \mathbb{I} \). As each block \( y_i \mathbb{I} \) is of non-zero size, \( \det(A_k) \) contains the monomial \( \prod_{i=1}^{m} y_i^{l_i} \) with \( 0 < l_i \).

\( \Leftarrow \) Consider the images of \( A_k \) under the projection \( R \overset{\phi}{\rightarrow} R/\{x_{ij}\} \). Note that \( \phi(A_Q) \) is diagonal and its entries are linear in \( \{y_i\} \), \( \phi(A) = \oplus y_i C_i \) and \( \sum \text{rank}(C_i) = \text{rank}(A_Q) \). Thus we can assume that the entries of each diagonal matrix \( \phi(A_k) \) are linear in \( \{y_i\} \) and the similar rank decomposition holds: \( \phi(A_k) = \oplus y_i C_i^{(k)} \). Now \( \phi(A_Q) \) and \( \phi(A_1) \oplus \phi(A_2) \) are related by row/column permutations. Fix the corresponding matrices \( U, V, \) over \( k \), such that \( U \phi(A_Q) V^{-1} = \begin{pmatrix} \phi(A_1) & \mathbb{O} \\ \mathbb{O} & \phi(A_2) \end{pmatrix} \).

Using the embedding \( k \hookrightarrow R \) consider \( U, V \) as matrices over \( R \). Consider the matrix \( U^{-1} \begin{pmatrix} A_1 & \mathbb{O} \\ \mathbb{O} & A_2 \end{pmatrix} V \).

It is of the form \( \begin{pmatrix} y_1 \mathbb{I} & \mathbb{O} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbb{O} & \cdots & y_m \mathbb{I} \end{pmatrix} + \mathbb{X}, \) where the block structure of \( \mathbb{X} \) is: \( \mathbb{X}_{ij} = \begin{pmatrix} X_{ij}^{(1)} & \mathbb{O} \\ \mathbb{O} & X_{ij}^{(2)} \end{pmatrix} \) with ...
size(\(X_{ij}^{(k)}\)) = size(\(C_{i}^{(k)}\)). Thus the matrix \(U^{-1} \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} V\) comes from a decomposable representation of the initial quiver. Now, invoke proposition 5 to get: \(\{A_{ij}\}\) is decomposable. □

Combining this result with theorem 2 we get:

**Corollary 8.** 1. If the representation \(\{A_{ij}\}\) is decomposable then \(\text{det}(A) \in R[[\{x_{ij}\}, \{y_{i}\}]]\) is reducible.

2. Suppose \(\text{det}(A) = f_1 f_2\), where each of \(f_1, f_2\) is a non-zero divisor and contains a monomial \(\prod y_i^k\). Suppose they are relatively prime, i.e. \(\langle f_1 \rangle \cap \langle f_2 \rangle = \langle f_1 f_2 \rangle\). Then the representation \(\{A_{ij}\}\) is decomposable iff \(I_{m-1}(A) \subset (f_1) + (f_2)\).

In many examples \(\text{det}(A)\) is square free, thus \(f_1, f_2\) are necessarily relatively prime. So, the corollary gives a very simple and effective decomposability criterion.

3. **Examples**

3.1. **Conjugation.** The conjugation, corresponds to the quiver with just one vertex (and several arrows). Explicitly, we are given a tuple of square matrices up to the simultaneous conjugation, \((A_1, \ldots, A_m) \rightarrow U(A_1, \ldots, A_m)U^{-1}\). Introduce the new variables, \(x_1, \ldots, x_m, y\). The associated matrix is \(A_Q = \sum_i x_i A_i + yI\). So that \(\{A_i\} \sim \{B_i\}\) iff \(A_Q \sim B_Q\). Proposition 7 reads:

\[(9)\]

the representation \(\{A_i\}\) is \(G_{\text{conj}}\)-decomposable iff \(A_Q = \sum_i x_i A_i + y I\) is \(G_{\text{tr}}\)-block-diagonalizable.

As the simplest case consider the \(2 \times 2\) matrices over \(R\).

**Corollary 10.** Given \(A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \text{Mat}(2, 2; R)\), suppose \(\text{tr}^2(A) \neq 4\text{det}(A)\). Then \(A\) is \(G_{\text{conj}}\)-decomposable iff \(\text{tr}^2(A) - 4\text{det}(A)\) is a full square in \(R\) and moreover the elements \(a_{12}, a_{21}, (a_{11} - a_{22})\) all belong to the ideal \((\sqrt{\text{tr}^2(A) - 4\text{det}(A)})\).

**Proof.** We want to check the \(G_{\text{tr}}\)-decomposability of \(x A + y I\). First of all we get that \(\text{det}(x A + y I) = y^2 + x y \cdot \text{tr}(A) + x^2 \cdot \text{det}(A)\) must factor over \(R[[y]]\). Consider this as a quadratic equation for \(y\). The roots are \(y_\pm = \frac{-\text{tr}(A) \pm \sqrt{\text{tr}^2(A) - 4\text{det}(A)}}{2}\). Thus \(\text{tr}^2(A) - 4\text{det}(A)\) must be a full square in \(R\).

To use theorem 2 we want the factors \((y - y_-), (y - y_+)\) of \(\text{det}(x A + y I)\) to be relatively prime. As \(y, x\) are independent variables, the factors are relatively prime iff \(y_+ \neq y_-\), i.e. \(\text{tr}^2(A) - 4\text{det}(A)\) ≠ 0.

Finally, if \(\text{tr}^2(A) - 4\text{det}(A)\) ≠ 0, but is a full square in \(R\), then the condition \(I_{1}(x A + y I) \subseteq J_{1} + J_{2}\) reads: \((x a_{12}, x a_{21}, 2y + x \text{tr}(A), x(a_{11} - a_{22})) \subseteq (2y + x \text{tr}(A), x \sqrt{\text{tr}^2(A) - 4\text{det}(A)})\). Which means \((a_{12}, a_{21}, a_{11} - a_{22}) \subseteq (\sqrt{\text{tr}^2(A) - 4\text{det}(A)})\). □

**Example 11.** Let \(A = \begin{pmatrix} a_{11} x_2 & a_{12} x_1 \\ a_{21} x_1 & a_{22} x_2 \end{pmatrix}\), where \(0 \neq a_{ij} \in \mathbb{k}\) and \(\mathbb{k} = \mathbb{k}\) and \(x_1, x_2\) are algebraically independent. Then \(\text{tr}^2(A) - 4\text{det}(A) = (a_{11} - a_{22})^2 x_2^2 + a_{12} a_{21} x_{1}^{k+l}\). If this expression is a full square then either \(a_{11} = a_{22}\) and \(k + l \in 2\mathbb{Z}\) or \(a_{12} a_{21} = 0\).

- Suppose \(a_{11} = a_{22}\) and \(k + l \in 2\mathbb{Z}\). To ensure \(\text{tr}^2(A) - 4\text{det}(A)\) ≠ 0 we assume \(a_{11} a_{22} \neq 0\). Then the condition \(\{a_{12} x_{1}^{k}, a_{21} x_{1}^{l}\} \subset \langle a_{12} a_{21} x_{1}^{k+l}\rangle\) means: \(k = l\).
- Suppose \(a_{12} a_{21} = 0\). Then the condition \(\{a_{12} x_{1}^{k}, a_{21} x_{1}^{l}, (a_{11} - a_{22}) x_{2}\} \subset ((a_{11} - a_{22}) x_{2})\) means: \(a_{12} = a_{21} = 0\), i.e. \(A\) is already diagonal.

Finally, if \(a_{11} = a_{22} = 0\), but \(a_{12} a_{21} \neq 0\) then the matrix is not \(G_{\text{conj}}\)-diagonalizable, e.g. by checking the characteristic polynomial.

Summarizing: if \(A\) is not diagonal then it is \(G_{\text{conj}}\)-diagonalizable iff \(k = l, a_{11} = a_{22} = 1\).
Example 12. Similarly consider $A = \begin{pmatrix} x_2 & x_1^k & 0 \\ 0 & x_2 & x_1^l \\ -x_1^{3n-k-l} & 0 & x_2 \end{pmatrix}$. Then $\det(A) = (x_2-x_1^n)(x_2^2 + x_2x_1^n + x_1^{2n})$, note that the two factors are mutually prime. Thus the $G_{tr}$-decomposability holds iff $k = l = n$. To check the $G_{com}$-decomposability we consider $A = \begin{pmatrix} y + x_2 & x_1^n & 0 \\ 0 & y + x_2 & x_1^n \\ -x_1^n & 0 & y + x_2 \end{pmatrix}$. Then $I_2(A) = (y + x_2)^2 + ((y + x_2)x_1^n) + (x_1^{2n}) \subseteq (x_2 - x_1^n) + (x_2^2 + x_2x_1^n + x_1^{2n})$. Thus $A$ is $G_{com}$-decomposable.

3.2. The "star" quiver. Consider the one-vertex quiver, with $l$ incoming arrows and $m$ outgoing arrows, $Q = A_{l+1} \swarrow \searrow A_k$. The corresponding matrix is

$$A_Q = \begin{pmatrix} x_{00}C + y_0E & 0 & \cdots & 0 & x_{0,l+1}A_{l+1} & \cdots & x_{0,k}A_k \\ x_{10}B_1 & y_1E & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{i0}B_i & 0 & \cdots & y_iE & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

(13)

Here $\det(A_Q) = \det(x_{00}C + y_0E) \prod_{i=1}^k y_i^{m_i}$, so any factorization $\det(A_Q) = f_1f_2$ is impossible with $f_1, f_2$ relatively prime. Theorem 2 does not produce any decomposability criterion here.

3.2.1. The string quiver. Consider the quiver $Q = \bullet \overset{A_1}{\cong} \cdots \overset{A_{m-1}}{\cong} \bullet$. The corresponding matrix is:

$$A_Q = \begin{pmatrix} y_1E & x_{12}A_1 & \cdots & 0 \\ x_{21}B_1 & y_2E & x_{23}A_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

(14)

The decomposability of a representation of $Q$ is controlled by $A_Q$.

Example 15. Consider the simplest case, $Q = \bullet \overset{A}{\cong} \bullet$, where $A, B \in \text{Mat}(2, 2; R)$. Then $\det(A_Q = B = y_1y_2^2 - y_1y_2x_{12}x_{21}\text{tr}(AB) + x_{12}^2x_{21}^2\det(AB))$. Thus the decomposability implies: $\text{tr}^2(AB) - 4\det(AB)$ is a full square in $R$. Further, if $A_Q$ is decomposable then the generators of $I_3(A_Q)$, i.e. the elements

$$y_1y_2^2, y_1^2y_2, \{y_1y_2x_{12}A_{ij}\}, \{y_1y_2x_{21}B_{ij}\}, x_{12}^2x_{21}^2\det(AB){A_{ij}}, x_{12}^2x_{21}^2\det(AB){A_{ij}}$$

belong to the ideal: $(2y_1y_2 - x_{12}x_{21}\text{tr}(AB)) + \left( x_{12}x_{21}\sqrt{\text{tr}^2(AB) - 4\det(AB)} \right)$. As $y_1, y_2, x_{12}, x_{21}$ are algebraically independent, only two cases are possible:

- $\text{tr}(AB) = 0$ and the elements $\det(A){B_{ij}}, \det(B){A_{ij}}$ belong to the ideal $(\sqrt{-\det(AB)})$. In this case the factors of $\det(A_Q$ are mutually prime iff $\det(AB) \neq 0$. If $\det(AB) \neq 0$ then theorem 2 ensures decomposability. If $\det(AB) = 0$, then the condition on $\det(A){B_{ij}}, \det(B){A_{ij}}$ forces: either $A = 0$ or $B = 0$ or $\det(A) = 0 = \det(B)$.

- $\det(AB) = a \cdot \text{tr}^2(AB)$, where $a - \frac{1}{4}$ is invertible in $R$. Then the factors of $\det(A_Q$ are mutually prime and together generate the ideal $(y_1y_2 + (x_{12}x_{21}\text{tr}(AB)))$. In this case the conditions give: both $\det(A){B_{ij}}$ $\det(B){A_{ij}}$ belong to the ideal $(\text{tr}(AB))$. 

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