Dynamic mode decomposition analysis of coherent structures in rotating plane Couette flow

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Abstract. Large scale structures have been observed in many turbulent wall bounded flows, such as pipe, Couette or square duct flows. Many efforts have been made in order to capture such structures to understand and model them. However, commonly used methods have their limitations, such as arbitrariness in parameter choice or specificity to certain setups. In this manuscript we attempt to overcome these limitations by using two variants of Dynamic Mode Decomposition (DMD). We apply these methods to (rotating) Plane Couette flow, and verify that DMD-based methods are adequate to detect the coherent structures and to extract the distinct properties arising from different control parameters. In particular, these DMD variants are able to capture the influence of rotation on large-scale structures by coupling velocity components. We also show how high-order DMD methods are able to capture some complex temporal dynamics of the large-scale structures. These results show that DMD-based methods are a promising way of filtering and analysing wall bounded flows.

1. Introduction

Turbulent flows are generally characterized by chaotic motion and abundant mixing. But within certain canonical systems, experiments and simulations have detected coherent large scale structures \cite{1, 2}. The prime example of a turbulent system where such coherent flows are present is plane Couette (PC) flow, the shear-driven flow between two parallel plates. Direct numerical simulations (DNS) of PC flow have found extremely long structures that are far larger than the inter-plate distance \cite{3, 4, 5, 6}. To accurately capture these structures, we require computational boxes that are large enough to fit them. Large-scale structures have some important practical impact, having the general role of redistributing friction and heat flux along the walls and across the flow \cite{2}. Small computational boxes that fail to capture these large-scale structures can considerably alter the flow behaviour and statistics \cite{3}.

In the last decades, improvements in computation have made possible the use of numerical simulations as a tool for observing and exploring phenomena in many different systems. In this context, substantial effort has been made to characterize the origin, pinning and mathematical modelling of the coherent structures in turbulent flows. Methods for isolating these structures are still under development. Commonly used methods to extract structures include moving...
ensemble averages [4,7], two-point correlations [8] or Fourier analysis [9]. The main disadvantage of these methods is that they often include the relatively arbitrary choice of a cut-off to isolate the structures, be it in the shape of a cut-off frequency in Fourier space [2], or in the shape of a threshold value for a certain attribute [4,8]. Furthermore, these methods are usually tuned for one particular setup of the system; but as the control parameters of the flow changes, so do the shape of the coherent structure. Because of this, conventional methods can completely fail to track structures across certain dimensions of parameter space.

Our main aim is the development of a low-order model for the analysis of the evolution of large-scale structures in PC flow across parameter space. But to do this, we must be able to robustly capture the structures everywhere in parameter space, and separate them from the background turbulence. This filter must remove arbitrariness as far as possible. In this manuscript, we will assess the merits of Dynamic Mode Decomposition [10] (DMD) in fulfilling this tasks. The DMD method is fully detailed below, but for now we highlight that this method is a data–driven technique that algorithmically is a regression of data onto locally linear dynamics. This makes it a promising candidate to extract the large-scale structures in wall-bounded turbulent flow, as it is known that these structures behave in a quasi-inviscid, quasi-linear manner [11].

We will focus on rotating plane Couette (RPC) system, i.e. PC flow with an added solid-body rotation in the spanwise direction. A spanwise rotation coincides with the direction of the underlying streamwise vorticity, and it is known to significantly alter the shape and dynamics of the large-scale structures. Anti-cyclonic rotation induces a strong secondary flow perpendicular to the main flow direction, and pins the structures in the spanwise direction [12]. By comparing how DMD perform for both PC flow with no rotation and RPC flow with anti-cyclonic rotation, we can assess the merits of DMD in capturing our quantities of interest.

This article is organized as follows. In section 2 we briefly recall the general idea of the standard DMD method and of the variants we have used for our analysis. In section 3 we introduce the numerical details of the simulations. In section 4 we report the analysis of the method applied on both PC and RPC, with a particular care on the obtained large scale structures. The final section 5 contains a summary of the work done and the conclusions.

2. Dynamic mode decomposition

Dynamic mode decomposition (DMD) is a method that provides a spatio-temporal decomposition of data into a set of relevant dynamical modes from a sequence of snapshots of an evolving system. The method was firstly developed by Schmidt [10] to provide a linear approximation of non-linear dynamics, and soon after it was closely linked to Koopman operator analysis [13, 14, 15]. The Koopman operator is an infinite dimensional linear operator that represents the non-linear dynamic of a physical system. Because the underlying system is non-linear, and the Koopman operator does not linearize the system, the operator becomes infinite-dimensional. The DMD method can be interpreted as a way of calculating a finite approximation of the Koopman operator.

2.1. Standard DMD

In what follows, we briefly outline the key steps of DMD. The method is applied to spatio-temporal data organized in N-equispaced, M-dimensional realizations of the system:

\[ x_i \equiv x(t_i) \in \mathbb{R}^M \quad \text{where} \quad t_i = t_1 + (i-1)\Delta t \quad \text{for} \quad i = 1, \ldots, N. \]

The data are arranged in an \( M \times N \) matrix:

\[ X = [x_1, \ldots, x_N]. \]
The DMD method relies on the following Koopman assumption:

\[ x_{i+1} = Ax_i \quad \text{for} \quad i = 1, \ldots, N - 1, \]  

which can be written in matrix form as:

\[ X_N^2 = AX_1^{N-1}, \]  

where the snapshot matrices are generally defined as:

\[ X^k_j = [x_j, x_{j+1}, \ldots, x_k]. \]

In practice, when the dimension \( M \) is large, the matrix \( A \) may be intractable to analyze directly. DMD circumvents the eigendecomposition of \( A \) by considering the projection on its proper orthogonal decomposition (POD) modes, resulting in a rank-reduced matrix \( \tilde{A} \).

The DMD algorithm proceeds as follows \cite{16}:

1) Compute the truncated singular value decomposition (SVD) of the snapshot matrix \( X_1^{N-1} \):

\[ X_1^{N-1} = U \Sigma V^H \]  

where \( \cdot^H \) denotes the conjugate transpose of a matrix, \( \Sigma \in \mathbb{C}^{K \times K} \) is the diagonal matrix containing the retained SVD singular values sorted in decreasing order, and the (orthonormal) columns of the matrix \( U \in \mathbb{C}^{M \times K} \) and the matrix \( V \in \mathbb{C}^{N-1 \times K} \) are the spatial and temporal SVD-modes, respectively. We remark also that the left singular vectors of \( U \) correspond to POD modes.

2) Compute \( \tilde{A} \), the projection of the full matrix \( A \) onto \( U \):

\[ \tilde{A} = U^H AU = U^H Y V \Sigma^{-1} \]

3) Compute the eigenvalues \( \lambda_k \) and eigenvectors \( w_k \) of \( \tilde{A} \), arranged in the matrices \( \Lambda \) and \( W \):

\[ \tilde{A}W = W \Lambda \]

4) Reconstruct the eigendecomposition of \( A \) from \( W \) and \( \Lambda \):

\[ A \Phi = \Phi \Lambda, \quad \Phi = X_2^N V \Sigma^{-1} W \]

Note that the eigenvalues for \( A \) are equivalent to those of \( \tilde{A} \), and that the eigenvectors of \( A \) are given by the columns of \( \Phi \), and correspond to the DMD modes.

In this way, with the columns of the \( A \) and \( \Phi \) matrices, we arrive at the DMD representation of the snapshots:

\[ x_{DMD}(t) = \sum_{k=1}^{K} a_k \phi_k e^{(\delta_k + i \omega_k)t} \]

where, for convenience, we have written the eigenvalues as:

\[ \delta_k + i \omega_k = \frac{\log(\lambda_k)}{\Delta t}, \]

in order to distinguish in \cite{7} the growth rates \( \delta_k \), and frequencies \( \omega_k \) of the Fourier-like modes. The amplitudes \( a_k \) are computed via least squares fitting (as in optimized DMD \cite{17}) of the snapshots in expansion \cite{7}.

Standard DMD has several known limitations. The DMD method is based on an underlying SVD, whose well known weaknesses are the inability to efficiently handle invariances in the data, such as translational and/or rotational invariances of low-rank objects; and as other SVD–based methods, DMD is also unable to handle transient time phenomena (cf. chapter 1.5 in \cite{15}, or \cite{18} for examples). To address these problems, several modifications to the basic DMD method have been proposed. For this manuscript we will consider two of these methods.
2.2. Multi-resolution DMD (mrDMD)

The first method we will use is multi-resolution DMD (mrDMD). The method mrDMD is derived by the idea of separating slow and fast modes with techniques from foreground/background subtraction in video feeds [19]. It essentially combines features of the DMD decomposition with key concepts from wavelet theory and multi-resolution analysis [14, 20].

As we can see from (8), the characteristic time of a mode is inversely proportional to \(|\log(\lambda_k)|\). It depends on the magnitude of both frequency and growth/decay rate. A mode can be considered “slow” if \(|\delta_k + i\omega_k| \approx 0\). This simply means that the mode changes somewhat slowly as the system evolves in time. To have an immediate representation of this behavior, when we plot a mode in a \(Re(|\log(\lambda_k)|) - Im(|\log(\lambda_k)|)\) space, a DMD mode with temporal frequencies near the origin in this space has both small \(\omega_k\) and \(\delta_k\) and therefore represents a slow mode. Such modes are interpreted as background (low-rank) portions of the given dynamics, while the terms with temporal frequencies far from the origin are their foreground (sparse) counterparts.

The mrDMD recursively removes low-frequency slowly-varying content from a given collection of snapshots. In this way at each step the slow-modes are removed first and the data is filtered for analysis of its higher frequency content. This recursive sampling structure is demonstrated to be effective in allowing for a reconstruction of a greater number of datasets with respect to classical DMD, since it overcomes the transitional and transient problems described above [19, 15, 21].

The mrDMD method is an iterative algorithm that can be briefly summarized as follow:

1) Compute DMD for available data.
2) Determine fast and slow modes.
3) Find the best DMD approximation to the available data constructed from the slow modes only.
4) Subtract off the slow-mode approximation from the available data.
5) Split the available data in half.
6) Repeat the procedure for the first half of data (including this step).
7) Repeat the procedure for the second half of data (including this step).

Mathematically, the mrDMD separates the DMD approximate solution (7) in the first pass as:

\[
x_{mrDMD}(t) = \sum_{k=1}^{K} a_k \phi_k^{(1)} e^{(\delta_k+i\omega_k)t} = \sum_{k=1}^{m_1} a_k \phi_k^{(1)} e^{(\delta_k+i\omega_k)t} + \sum_{k=m_1+1}^{K} a_k \phi_k^{(1)} e^{(\delta_k+i\omega_k)t},
\]

(9)

where we have ordered the modes from the slowest to the fastest. Note that the modes computed at this level are indicated with \(\phi_k^{(1)}\), and that we are separating the first \(m_1\) background slowest modes from the other ones. Since we have already removed the slowest modes in the first sum, in the second step we use the second sum to yield the fast scale data matrix:

\[
X_{K/2} = \sum_{k=m_1+1}^{K} a_k \phi_k^{(1)} e^{(\delta_k+i\omega_k)t}.
\]

(10)

The matrix \(X_{K/2}\) is now separated into two matrices:

\[
X_{K/2} = X_{K/2}^{(1)} + X_{K/2}^{(2)}.
\]

(11)
where each of the two matrices contain $K/2$ of the total $K$ snapshots. We then recompute DMD separately on both of the two matrices but we retain the slowest $m_2$ DMD modes, indicated as $\phi^{(2)}_k$, obtained from both the decomposition at this level.

The iterative process works by recursively removing slow frequency components until a desired multi-resolution decomposition has been achieved. The final DMD approximation reads as:

$$x_{m_{rDMD}}(t) = \sum_{k=1}^{m_1} a_k^{(1)} \phi_k^{(1)} e^{(\delta_k^{(1)}+i\omega_k^{(1)})t} + \sum_{k=1}^{m_2} a_k^{(2)} \phi_k^{(2)} e^{(\delta_k^{(2)}+i\omega_k^{(2)})t} + \sum_{k=1}^{m_3} a_k^{(3)} \phi_k^{(3)} e^{(\delta_k^{(3)}+i\omega_k^{(3)})t} + \ldots$$

To be precise, $\phi_k^{(i)}$ are the DMD modes obtained at $i$-th level of decomposition, and the $m_k$ are the number of slow modes retained at each level. In this way each successive level contains modes that are faster and faster, thereby providing a way to identify long-, medium-, and short-term trends in data. There is not a single set of modes that dominates the SVD and potentially obscures features at other time scales, but instead different DMD modes are used to represent key features at different time scales. We refer the reader to [20] for more detailed information on the method and its applications.

2.3. High order DMD

High order DMD (HODMD) is an extension of the classic DMD method which is appropriate to treat general periodic and quasi-periodic dynamics, and transients decaying to periodic and quasi-periodic attractors. This includes cases, not accessible to standard DMD, that show limited spatial complexity but a very large number of involved frequencies.

Instead of using the Koopman assumption (1), HODMD relies on the higher order Koopman assumption:

$$x_{i+d} = A_1 x_i + A_2 x_{i+1} + \cdots + A_d x_{i+d-1} \quad \text{for} \quad i = 1, \ldots, N - d,$$

(13)

for a tunable $d \geq 1$. Equation (13) can also be written as:

$$\hat{x}_{k+1} = \hat{A} \hat{x}_k,$$

(14)

where $\hat{x}_k$ and $\hat{A}$ are defined as:

$$\hat{x}_k = \begin{pmatrix} x_k \\ x_{k+1} \\ \vdots \\ x_{k+d-1} \end{pmatrix}, \quad \hat{A} = \begin{pmatrix} 0 & I & 0 & \ldots & 0 & 0 \\ 0 & 0 & I & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & I & 0 \\ A_1 & A_2 & A_3 & \ldots & A_{d-1} & A_d \end{pmatrix}.$$  

(15)

After applying a first dimension reduction on the general snapshot matrix $X_{N1}$, the HODMD algorithm proceeds the same way as the standard DMD described in section 2.1, and is applied on the reduced snapshots reorganized as in equation (14) (see [22] for more details).

Note that the main difference between equations (14) and (13) is that the latter relates each snapshot with not only the last but also the preceding $d-1$ snapshots. In fact, HODMD reduces to standard DMD if $d = 1$. For $d > 1$, instead, HODMD can be seen as a result of applying standard DMD to a set of enlarged snapshots (time-lagged snapshots) which also contains the delayed snapshots (sliding window).

The idea of using delayed snapshots to characterize chaotic dynamics comes from the time-delay embedding theory, and in particular from the delayed embedding theorem by Takens [23], who followed and formalized former seminal ideas by Packard et al. [24]. This theory
states that in some cases, it may be possible to reconstruct the entire attractor of a turbulent flow from a time series of a single point measurement, by enriching a single observable $x(t)$ with time–shifted copies of itself, known as delay coordinates or delay reconstruction, $x(t) = \{x(t), x(t-\tau), \ldots, x(t-(d-2)\tau), x(t-(d-1)\tau)\}$, where $\tau$ is the lag or delay time, and $d$ is the embedding dimension. Takens’ theory has been related not only to fluid attractors, but in general to nonlinear dynamics and data–driven methods [16, 25], as it can demarcate large coherent regions of phase space where the dynamics are approximately linear from those that are strongly nonlinear.

We note that in HODMD method two tunable tolerances $\epsilon_1$ and $\epsilon_2$ are defined; the first is linked to the first SVD truncation on $X^N_1$:

$$EE(M) = \frac{\sigma_{K+1}^2 + \cdots + \sigma_R^2}{\sigma_1^2 + \cdots + \sigma_R^2} \leq \epsilon_1,$$

with $R = \min\{N, M\}$ the rank of the full snapshot matrix. The second tolerance is selected to get rid of small amplitudes, i.e. a smaller number of $L$ modes is retained such that

$$\frac{a_{L+1}}{a_1} < \epsilon_2.$$

The possibility of tuning the parameters $d$, $\epsilon_1$ and $\epsilon_2$ makes the method more robust than standard DMD methods, since in this way it can get rid of noise or discretization/truncation errors and hence avoids capturing unphysical modes [22, 26].

3. Numerical setup

The flow data used for the analysis will be taken from direct numerical simulations (DNS) of rotating plane Couette flow (RPCF) in a three dimensional domain bounded by no-slip conditions at the walls in the wall normal ($y$) direction at $y = 0$ and $y = 2h$, and periodic in the streamwise ($x$) and spanwise ($z$) directions with periodicity lengths $L_x$ and $L_z$, respectively.

We have used an energy–conserving second–order centered finite–difference code for the spatial discretization, while the discretized system is integrated using a fractional–step method and advanced in time by a low–storage third order Runge–Kutta scheme. The simulation code used is based on the highly parallel FORTRAN-based AFiD (www.afid.eu) which has been used mainly for simulations of turbulent Rayleigh-Bénard convection and Taylor-Couette flow [27]. This code has been comprehensively validated. Detailed information regarding the code algorithms can be found in [27, 28].

All the simulations are performed in a reference frame such that the walls have opposite streamwise velocities $\pm U/2$, and the entire system rotates with angular velocity $\Omega = \Omega_{rf} e_z$ around the spanwise axis. In this frame the two control parameters are a shear Reynolds number $Re_s = U h / \nu$ and a Coriolis parameter $R_\Omega = 2 h \Omega_{rf} / U$.

The incompressible Navier-Stokes equations then become:

$$\nabla \cdot u = 0,$$

$$\frac{\partial u}{\partial t} + u \cdot \nabla u + R_\Omega e_z \times u = -\nabla p + Re_s^{-1} \nabla^2 u.$$

A uniform grid is used in the streamwise and spanwise directions, while a Chebychev-type clustering near the walls is used in the wall-normal direction. For this study we have fixed $Re_s = 3 \times 10^4$ and varied the Coriolis forcing, $R_\Omega = 0$ and $0.1$. The computational box used is of $4\pi h \times 2h \times 2\pi h$ in streamwise, wall-normal and spanwise directions respectively, the same size as in [29]. This has led to a numerical resolution of $n_x \times n_y \times n_z = 512 \times 384 \times 512$ in
the wall-normal, streamwise and spanwise directions, respectively. In order to achieve temporal convergence, the simulations are run until the difference between the time-averaged shear at both walls is less than 1%.

To perform the DMD analysis, we have used open source algorithms. For mrDMD we have used the python package described in [30]; while for HODMD we have used the MATLAB solver described in [31], that can be found in [31].

4. Results
Before introducing the results obtained with DMD methods, we report here a visualization of the velocity flow fields of 2D slices in the $x-z$ plane taken at fixed wall-normal distance: at the centerline $y = h$ and close to the wall at $y^+ \approx 15$. We have reported both the instantaneous and the temporal average of the flow, for both the rotationless case $R_\Omega = 0$ (figure 1) and the anti-cyclonic case $R_\Omega = 0.1$ (figure 2). We can see the complexity of the flow, composed by fast-evolving small structures, especially close to the wall, and large structures, whose shape is not perfectly defined in the $R_\Omega = 0$ case, and are approximately streamwise invariant, and pinned in the $R_\Omega = 0.1$ case.

**Figure 1.** From top to bottom: instantaneous flow near the wall at $y^+ \approx 15$, instantaneous flow at the centerline $y = h$, mean flow near the wall at $y^+ \approx 15$, mean flow at the centerline $y = h$. From left to right, the panel represent the $v_x$ (streamwise), $v_y$ (wall-normal) and $v_z$ (spanwise) velocity components, in the rotationless case $R_\Omega = 0$. 

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The figure shows the velocity fields for different cases, highlighting the complexity of the flow with fast-evolving small structures near the wall and large structures that are approximately streamwise invariant when $R_\Omega = 0$. The visualization is provided for both the rotationless ($R_\Omega = 0$) and the anti-cyclonic ($R_\Omega = 0.1$) cases, allowing for a detailed comparison of the flow dynamics under different rotational conditions.
4.1. mrDMD

We first applied the mrDMD method to a collection of 1000 snapshots of 2D slices in the $x-z$ plane taken at fixed wall-normal distance: at the centerline $y = h$ and close to the wall at $y^+ \approx 15$. The temporal spacing between each snapshot was fixed as $\Delta t U/h = 4 \times 10^{-2}$.

In what follows we analyze the components in the DMD decomposition (equation (12)), and the effect of the Coriolis force term $R_\Omega$ on this decomposition. We will particularly focus on the slowest modes, which should capture the dynamics of the large, coherent structures. As an additional test, we perform the analysis on a snapshot matrix built by considering the three velocity components simultaneously or separately, in order to capture the dominant modes of either the entire flow or of a single component.

4.1.1. No rotation

We first analyze the results obtained in the case with no rotation, $R_\Omega = 0$. We begin by reporting in figure 3 the stability of the modes. This is done through the eigenvalues as written in equation (8), in order to check both growth/decay rate and frequency of the modes. In particular we have highlighted the first level of the decomposition. As we have explained in subsection 2.2, mrDMD method recursively separates slow modes from fast modes; thus the first level of this decomposition is the first step in the recursive procedure, and corresponds to modes with both the lowest frequency and growth/decay rate. Moreover this level corresponds to the first summation addend of the decomposition written as in equation (12). If we look then
Figure 3. Growth/decay and frequency of the modes $\delta_k + i\omega_k = \frac{\log(\lambda_k)}{\Delta t}$ in the rotationless case $R_\Omega = 0$, near the wall at $y^+ \approx 15$ (top row) and at the centerline $y = h$ (central row). The panels correspond to the modes derived, from left to right, from the $v_x$ (streamwise), $v_y$ (wall-normal) and $v_z$ (spanwise) velocity component singularly. The last row corresponds to the the modes derived from all the velocities simultaneously, close to the wall (left) and at the centerline (right). In the panels the first three levels of iteration of equation (12) are highlighted with red (level 1), blue (level 2) and magenta (level 3) markers.

At the first level of the decomposition performed on the single velocity components in figure 3 we can see that non-oscillatory modes –with zero imaginary component, are only present when analyzing the streamwise velocity. We also note that at the centerline the non-oscillatory mode is slightly decaying, while close to the wall, this mode is perfectly stable.

When we perform the analysis on the velocity fields simultaneously, we observe that the first level of the decomposition at the centerline has an imaginary component, probably meaning that the non-oscillatory, streamwise velocity is not strong enough to dominate the whole flow. We will return to this later.

To better understand the flow decomposition, we look at the mode amplitudes $\alpha_i$, reported in figure 4. Here we can clearly see that for the streamwise velocity, the first (slowest) mode has a large influence in the dynamics of the flow both close to the wall and in the centerline, as its amplitude is significantly larger than that of other modes. This does not happen for the other two velocity components, where fast and slow modes contribute with similar weight to the dynamics. When we perform the method on the velocity fields simultaneously, we can again see...
that the first (slowest) mode has a large influence in the dynamics of the flow both close to the wall and in the centerline.

Both these results hint at the fact that mrDMD is only capturing a large-scale mode in the streamwise velocity. Indeed, these behaviours are reflected when visualizing the modes themselves. In figure 5, we show the first (slowest) mode obtained with the mrDMD method applied to the three velocity components simultaneously. We observe the clear presence of a large-scale structure only for the streamwise velocity. In particular we have an evident large scale motion at the centerline, and close to the wall there is a structure pattern that is clearly related to the mid–gap one. Meanwhile, for the other velocity components, large-scale structures are completely absent.

4.1.2. Anti-cyclonic rotation. The situation is different when we inspect the results obtained in the case where anti-cyclonic rotation, $R\Omega = 0.1$, is imposed to the flow. Again we start by analyzing the stability of the slowest modes in figure 3, written as in equation (8). We have then analyzed the first level of the decomposition, that is, as we have stated in the previous subsection, the first summation addends of the decomposition written as in (12), which correspond to the the slowest modes of the flow. When we analyse the modes on the single components, we find
Figure 5. Real part of the slowest structure in the rotationless case $R_\Omega = 0$, near the wall at $y^+ \approx 15$ (left) and at the centerline $y = h$ (right). The panels correspond to the mode derived, from top to bottom, from the $v_x$ (streamwise), $v_y$ (wall-normal) and $v_z$ (spanwise) velocity components.

that the first level of the decomposition of every component is composed by a single mode, whose frequency is exactly zero. This happens both at $y^+ \approx 15$ and at the centerline for all the velocity fields, except for the wall-normal velocity close to the wall, whose first level of decomposition has no modes. Moreover all the non oscillating modes have eigenvalues close to the origin of the axis, which means that they are all either slightly decaying or perfectly stable modes.

If we now perform the method on the velocity fields simultaneously, we observe that, in contrast to the $R_\Omega = 0$ case, the first level of the decomposition both close to the wall and at the centerline has no imaginary component.

This more complex scenario is confirmed if we look at the amplitudes in figure 7. Again we have that for the streamwise velocity, the first mode is the one that has the largest influence in determining the evolution of the flow. However, this mode is now marked at the centerline for the wall-normal velocity and at the near-wall in the spanwise component. This configuration is confirmed when we perform the method on the velocity fields simultaneously, since both close to the wall and in the centerline the first mode is dominating the dynamics of the flow. These properties are also reflected when visualizing the slowest mode obtained with the analysis applied to the three velocity fields simultaneously in figure 8. In this case there is clear presence of the large structures in both streamwise and wall normal components at the centerline, and streamwise and spanwise direction close to the wall.

With mrDMD we have confirmed nothing more than a clear detection of the organized, pinned
Figure 6. Growth/decay and frequency of the modes $\delta_k + i\omega_k = \frac{|\log(\lambda_k)|}{\Delta t}$ in the case with anti-cyclonic rotation $R\Omega = 0.1$, near the wall at $y^+ \approx 15$ (top row) and at the centerline $y = h$ (central row). The panels correspond to the mode derived, from left to right, from the $v_x$ (streamwise), $v_y$ (wall-normal) and $v_z$ (spanwise) velocity components singularly. The last row corresponds to the the modes derived from all the velocities simultaneously, close to the wall (left) and at the centerline (right). In the panels the first three levels of iteration of equation (12) are highlighted with colors as in figure 3.

structures of RPC flow as described in [32, 12]: at the centerline there is a strong correlation between the $v_x$ and $v_y$ velocity component due to the Coriolis force. On the other hand, close to the wall the $v_y$ velocity component has to be very close to zero, and owing to the existence of the roll large structure, there has to be a strong $v_z$ component. With this analysis we can say that without rotation, i.e., with $R\Omega = 0$, we do not see a clear coupling between the structures across velocity directions, while for the anti-cyclonic case, $R\Omega = 0.1$, the three components of velocity are strongly coupled.

This could be a first step to explain why the rolls are fixed when we add rotation, and they are not if this is missing: for $R\Omega = 0$ these large structures are just “eigenmodes” of the streamwise velocity, while for $R\Omega = 0.1$ the Coriolis term is modifying the eigenmodes by forcing the coupling between the velocity components, in particular of $v_x$ and $v_y$ in the centerline.
4.2. HODMD

In this section we want to test the reliability of the results obtained in the previous section by applying HODMD: a different variant of standard DMD which is appropriate to study periodic and quasi-periodic dynamics, as we have already pointed out in section 2, or in general to that dynamics that show limited spatial complexity but a very large number of involved frequencies. Moreover HODMD has the possibility of fine-tuning the order parameters \( d \), and the tolerances \( \epsilon_1 \) and \( \epsilon_2 \). In this way we gain both the ability of getting rid of noise or discretization/truncation errors and thus avoid capturing unphysical modes, and also the possibility get rid of irrelevant or spurious modes by eliminating small amplitude modes.

Due to these reasons, we applied HODMD on a collection of 100 2D snapshots on the three velocity fields simultaneously, but however focusing on the streamwise velocity at the centerline. For this method, we use a larger temporal step between snapshots of \( \Delta t U/h = 4 \times 10^{-1} \), in order to focus more on slower evolving structures, and compare the output obtained when varying \( \epsilon_1 = \epsilon_2 = 10^{-2}, 10^{-4}, 10^{-6} \), and changing \( d = 2, 5, 10, 50 \) for \( R_\Omega = 0 \) and \( d = 2, 10, 20, 50 \) for \( R_\Omega = 0.1 \).

As we can see from figure 8 fixing the order and reducing the tolerance value increases the number of retained modes. On one side these modes are located at higher values of...
Figure 8. Real part of the slowest structure in the case with anti-cyclonic rotation $R_\Omega = 0.1$, near the wall at $y^+ \approx 15$ (left) and at the centerline $y = h$ (right). The panels correspond to the mode derived, from top to bottom, from the $v_x$ (streamwise), $v_y$ (wall-normal) and $v_z$ (spanwise) velocity components.

frequencies, indicating that they are related with high frequency, small scale structures; on the other side there is an increasing number of spurious modes, which are high frequency modes with high amplitudes. This happens for all the values of order $d$ considered, and for both the rotation and the rotationless cases. Because of this, in figure [11] we fix the tolerances to $\epsilon_1 = \epsilon_2 = 10^{-2}$ and vary the order of the method $d$ in both rotation and rotationless cases. As we can see, we have that the order that better captures the complexity of the flow in its large scales without introducing spurious effects is $d = 10$ for both $R_\Omega = 0$ and $R_\Omega = 0.1$, both highlighted with the black markers. Moreover these two figures emphasize the greater number of eigenvalues/eigenmodes when anti-cyclonic rotation is present with respect to the rotationless case, and the less clear pattern in the influence of them in the evolution of the flow.

It is worth noting that, in contrast with mrDMD, the dominant mode captured by HODMD has zero frequency, while with mrDMD the rotationless case was exhibiting an oscillating behaviour. This could be a consequence to the fact that HODMD is more appropriate to detect periodic or quasi-periodic dynamics, thus it is isolating the streamwise, quasi–steady large scale mode; on the other side seems that mrDMD is keeping also the dynamics of the other two velocity fields, which are faster and oscillatory.

There is another interesting behavior that can be studied if we look now at the frequencies $\omega_k$ of the eigenvalues. If we look at figure [11], we can see a spectral decay of frequencies of the modes, indicating a period cascade in the periodicity lengthscale of the eigenmodes. In a better
Figure 9. Amplitude on frequency plot of the eigenvalues of the streamwise component of the velocity for a fixed order $d$ of HODMD and varying the tolerances between values $\epsilon_1 = \epsilon_2 = 10^{-2}$ (stars), $10^{-4}$ (pluses) and $10^{-6}$ (crosses). Left: $R_\Omega = 0$ case; right: $R_\Omega = 0.1$.

Figure 10. Amplitude on frequency plot of the eigenvalues of the streamwise component of the velocity for a fixed tolerance $\epsilon_1 = \epsilon_2 = 10^{-2}$ and varying the order of the method $d$ between values $d = 2$ (pluses), $d = 5$ (crosses), $d = 10$ (circles) and $d = 50$ (pentagons) for the $R_\Omega = 0$ case (left panel), $d = 2$ (pentagons), $d = 10$ (circles), $d = 20$ (pluses) and $d = 50$ (crosses) for the $R_\Omega = 0.1$ case (right panel).
Figure 11. Frequency of the eigenvalues (stars) and linear fit (solid line), for $R_\Omega = 0$ (left) and $R_\Omega = 0.1$ (right).

way, we can say that the largest scales of the streamwise velocity can be decomposed in a mean flow (zero frequency) and a single periodic wave that repeats itself in the higher modes doubling its frequency in space with the mode number.

This behavior is clearly visible if we plot the slowest in frequency modes, as we have done in figure 12. In both, rotation and rotationless, cases we capture the largest, zero-frequency structures that are almost identical to the ones found with mrDMD in figure 5 and 8. We also capture oscillating structures, whose frequency is faster with the increasing mode order. This behavior recalls a result found for Rayleigh-Bénard flow during the transition to turbulence in [33]. During the transition it was found a fundamental harmonic with all the subharmonics of that fundamental oscillation frequency that during the transition was also indicating a route to chaos following a period-doubling scenario, until the transition to turbulence is complete, and the frequency spectrum appears as a continuous curve. It is possible than that a similar behavior could be present also for PC and RPFC flow, and that even at a completely turbulent flow, the largest scales still preserve the fundamental frequency with all the subharmonics, as a footprints of the transition.

5. Summary and conclusion

In this manuscript, we applied two variants of standard DMD on PC flow and RPC flow. The aim of the work was to find a filter for large-scale structures that was robust across the parameter space and that was able to correctly detect the coherent structures arising in the turbulent flow.

We first used mrDMD on a large amount of 2D snapshots, and this method was able to capture the different behaviors of the structures in the flow. In particular, we showed that when there is no added rotation, the large structures develop predominantly in the streamwise component, and are decoupled from the other velocity components, that show no evident structures; this means that the wavyness in the streamwise structures has no effect on both wall-normal and spanwise directions. On the other side, when rotation is added on the system, large structures are detected both in streamwise and wall-normal components; this means that the Coriolis force is forcing the coupling among the different components of the flow.

We also applied HODMD on a smaller selection of snapshots with a greater spacing in time, as in this way the method is able to eliminate small scale components and spurious elements that arise elsewhere. This serves as a check on mrDMD’s accuracy. We found the same coherent structures for the two methods. Moreover, we detected an extra feature: large scales of both PC and RPC systems can be decomposed in mean flow (the zero frequency stable elongated structures) and a single periodic wave, or harmonic, that repeat itself in the smaller structures with a doubling period cascade. This feature can be linked to the period cascade in the route to turbulence in Rayleigh-Bénard flow.

With this work, we have demonstrated that DMD methods can be a robust and reliable
Figure 12. Real part of the largest structure in the rotationless case $R_\Omega = 0$ (left panels) and anti-cyclonic rotation $R_\Omega = 0.1$ (right panels) at the centerline $y = h$, for the $v_x$ (streamwise) velocity components. The panels represent, from top to bottom, the zero-frequency mode, the first, second and third harmonics.

way to capture the large scale motion and try to understand their origin and their pinning. However there are known issues due to the large amount of snapshots that DMD methods need to properly capture the motion. For this reason this analysis was limited to a small size domain and to 2D collection of snapshots instead of full 3D fields. Thus to develop a full three-dimensional model for the structures we need to reduce the computational load of the methods used in this manuscript, using for example a further reduction of the dimension of the snapshot matrix, or a parallel distributed version of the codes on multiple CPUs or GPUs.

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