Learning Mathematics from the Master: A Collection of Euler-based Primary Source Projects for Today’s Students, Part II

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Learning Mathematics from the Master: A Collection of Euler-based Primary Source Projects for Today’s Students, Part II

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Abstract

This article and its prequel together highlight a set of nine classroom-ready projects that draw on the remarkable writing of Leonhard Euler (1707–1783) as a means to help students develop an understanding of standard topics from today’s undergraduate mathematics curriculum. Part of a larger collection of primary source projects intended for use in a wide range of undergraduate mathematics courses, these projects are freely available to students and their instructors. We provide a general description of the pedagogical design underlying these projects, more detailed descriptions of the individual projects themselves, and instructions for obtaining downloadable copies for classroom use.

Keywords: Euler, active learning, using history to teach mathematics

1 Introduction

In their 2013 Convergence article “Teaching and Research with Original Sources from the Euler Archive” [15], Dominic Klyve, Lee Stemkoski, and Erik Tou wrote:

If one decides to study original sources in mathematics, one can hardly do better than to read the words of Leonhard Euler, one of the greatest didactic writers in the history of mathematics. More than perhaps any other mathematician, Euler wrote to be understood. His works brim with examples, computations, and even dead-ends in his thinking process—the kind of digression academics are usually taught to avoid in their publications.

In this article and its prequel, we highlight a set of primary source projects that draw on Euler’s remarkable writing as a means to help students develop
an understanding of standard topics from today’s undergraduate mathematics curriculum. We begin with a brief general description of the pedagogical design underlying these projects, which are part of a larger collection of freely available student-ready materials based on primary sources for use in a wide range of undergraduate mathematics courses.

The key design feature of a primary source project is a “guided reading” of select excerpts from a primary source. These excerpts are surrounded by a series of student tasks that prompt students to actively interpret the mathematics being developed by the source author. The tasks deliberately interrupt the reading of the source material to offer students various opportunities to engage in activities that model how mathematicians work: asking questions, grappling with uncertainties, making conjectures and testing them, verifying results, and proving theorems. Secondary commentary supplied by the project author ties the whole project together for students by discussing the historical context and mathematical significance of the source material, providing guidance in its interpretation, and connecting its mathematical content to contemporary terminology, notation, and standards.1

Importantly, primary source projects are intended to replace the standard textbook treatment of core topics in the undergraduate mathematics curriculum. The guided reading approach is also especially well suited to small-group discussions and other student-centered instructional strategies as an alternative to lecture. Each project is limited in scope, usually focusing on one particular topic (e.g., the derivative of the sine function) or a few related concepts that are unified within one particular source (e.g., “Observationes de theoremate quodam Fermatiano alisque ad numeros primos spectantibus” [6]). Classroom implementation of a project may last anywhere from one to two class periods (for shorter “mini” projects) to several weeks (for the longest of the “full-length” projects). The “Notes to Instructors” section provided with each project offers a sample schedule for its full implementation, as well as options for modifying the project to fit within other timeframes and further information about its goals and design. \LaTeX{} code of individual projects is also available from its author upon request to facilitate modifications that instructors may wish to make to better suit their goals for a particular course.

In our concluding section, we provide additional information about the development of this general pedagogical approach, along with details on where to obtain a downloadable copy of a project for classroom use. We first present more detailed descriptions of five particular Euler-based projects designed for use in courses on discrete mathematics/graph theory, number theory, and real analysis.

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1The genesis of this particular guided reading approach to learning via primary sources is described in [3]. It and other approaches to using primary sources in mathematics teaching, and the motivation behind each, are compared in [11] and [12].
2 Euler Circuits and The Königsberg Bridge Problem: A Project for Discrete Mathematics, Graph Theory or Transition to Proof

The three-day project “Early Writings in Graph Theory: Euler Circuits and The Königsberg Bridge Problem” [2] is based on excerpts from “Solutio problematis ad geometriam situs pertinentis” (E53) [7]. One of his most famous papers, Euler began his investigation by situating it within the branch of mathematics that later became known as topology.

1 In addition to that branch of geometry which is concerned with magnitudes, and which has always received the greatest attention, there is another branch, previously almost unknown, which Leibniz first mentioned, calling it the geometry of position.² This branch is concerned only with the determination of position and its properties; it does not involve measurements, nor calculations made with them. It has not yet been satisfactorily determined what kind of problems are relevant to this geometry of position, or what methods should be used in solving them. Hence, when a problem was recently mentioned,³ which seemed geometrical but was so constructed that it did not require the measurement of distances, nor did calculation help at all, I had no doubt that it was concerned with the geometry of position especially as its solution involved only position, and no calculation was of any use. I have therefore decided to give here the method which I have found for solving this kind of problem, as an example of the geometry of position.

The method that Euler ultimately developed for solving the Königsberg Bridge Problem and others of its kind now serves as a key theorem in the study of Euler paths and Euler circuits in modern graph theory.⁴

²Gottfried W. Leibniz (1646–1716) described his thinking about the geometry of position in an 1670 letter to Christian Huygens (1629–1695) as follows:

I am not content with algebra, in that it yields neither the shortest proofs nor the most beautiful constructions of geometry. Consequently, in view of this, I consider that we need yet another kind of analysis, geometric or linear, which deals directly with position, as algebra deals with magnitude. (As quoted in [4, p. 30])

³See [10] for an interesting discussion of how the Königsberg bridge problem might have first come to Euler’s attention, along with other details about the contents of E 53.

⁴In fact, E53 is generally considered the first published work in both topology and graph theory.
Figure 1: Euler’s map representation of the Königsberg Bridge Problem. Image obtained from digitized copy of [7] at https://www.biodiversitylibrary.org/item/38573.

Euler stated (in Paragraph 2) the general problem he wished to solve as follows:

Whatever be the arrangement and division of the river into branches, and however many bridges there be, can one find out whether or not it is possible to cross each bridge exactly once?

He launched his search for an answer to this general question by analyzing the specific example of the seven bridges of Königsberg shown in Figure 1. He first briefly discussed (in Paragraph 3) the difficulties involved in attempting to find a solution by making “an exhaustive list of all possible routes, and then finding whether or not any route satisfies the conditions of the problem.” But Euler rejected that option in favor of looking for a simpler method “concerned only with the problem of whether or not the specified route could be found.” In fact, he found two such methods which he presented by carefully setting out his thinking at each step in a series of simplifications to the general problem, all leading up to a statement of his main results in the paper’s penultimate paragraph:

20 So whatever arrangement may be proposed, one can easily determine whether or not a journey can be made, crossing each bridge once, by the following rules:

   If there are more than two areas to which an odd number of bridges lead, then such a journey is impossible.

   If, however, the number of bridges is odd for exactly two areas, then the journey is possible if it starts in either of these areas.

   If, finally, there are no areas to which an odd number of bridges leads, then the required journey can be accomplished starting from any area.
Within the primary source project, students read E53 essentially in its entirety alongside definitions of modern terminology that are introduced through the secondary commentary that is interspersed between excerpts from Euler’s paper. While this provides them with an introduction to the basic theory of Euler paths and Euler circuits, the key theme of the project is an examination of the differences between an eighteenth-century proof and a modern treatment of the same result. This is accomplished via project tasks that ask students to compare Euler’s treatment of key results to the treatment of these same results in a modern textbook, with the objective of drawing students’ attention to current standards regarding formal proof. The project culminates with exercises that require students to ‘fill in the gaps’ in a modern proof of Euler’s main theorem.

The project is suitable for beginning-level discrete mathematics and graph theory courses and for a ‘transition to proof’ course. It assumes no prior background in graph theory, as it introduces modern graph theory terminology alongside Euler’s original writing. The project can thus be assigned prior to, concurrently with, or immediately following the introduction of basic graph theory concepts.

3 Primes, Divisibility and Factoring: A Project for Number Theory

The five-day project “Primes, Divisibility and Factoring” [13] is based on Euler’s “Observationes de theoremate quodam Fermatiano aliisque ad numeros primos spectantibus” (E26) [6]. This, Euler’s first paper in number theory, contains a surprising number of new ideas in the theory of numbers. In a few short pages, he provided for the first time a factorization of $2^{25} + 1$ (believed by Fermat to be prime), discussed the factorization of $2^n - 1$ and $2^n + 1$, and began to develop the ideas that would later lead to the first proof of what we now call Fermat’s Little Theorem (or the Euler-Fermat Theorem). Other standard topics in a first number theory course touched on in this paper include Mersenne primes, perfect numbers, and the use of “modular arithmetic” (though the modern notation for this did not yet exist) to show that some particular forms of numbers must always be composite.

E26 is also eminently readable—it contains enough detail that it can be followed, but there is enough missing to make it an ideal paper for students to work through. In the project, students read the entire paper, pausing to work out details omitted by Euler along the way. The project begins, appropriately, with the beginning of “Observationes”:

It is known that the quantity $a^n + 1$ always has divisors whenever $n$ is an odd number or is divisible by an odd number aside from unity. Namely $a^{2m+1} + 1$ can be divided by $a + 1$ and $a^{p(2m+1)} + 1$ by $a^p + 1$, for whatever number is substituted in place of $a$. But on the other hand, if $n$ is a number which is divisible by no odd number aside from unity, which happens when $n$ is a power of two, no divisor of the number $a^n + 1$ can be assigned. So if there are
prime numbers of this form $a^n + 1$, they must all necessarily be included in the form $a^{2^n} + 1$. But it cannot however be concluded from this that $a^{2^n} + 1$ always exhibits a prime number for any $a$; for it is clear first that if $a$ is an odd number, this form will have the divisor 2.

There is a lot to unpack in this expansive opening paragraph, and the first eight tasks of the project walk students through this one sentence at a time, providing some scaffolding to guide them. Later tasks provide students the opportunity to make discoveries in the same way many number theorists do, by playing with patterns, making conjectures, and then developing proofs. As a bonus, students can also try their hand at proving some results that Euler believed were true, but that even he couldn’t prove when he wrote this paper! These include Euler’s early version of the “Fermat’s Little Theorem,” which is also used in an optional appendix that guides students through an exploration of Euler’s mysterious statement of the fact that 641 is a factor of $2^{25} + 1$.

Euler wrote “Observationes” expecting the reader to have a strong (elementary) algebra background, but not to know any number theory. It has been used successively in a lower-division Honors Seminar for first-year students, and indeed it could be used on the first day of class. The type of algebraic thinking Euler expected of his readers, though technically elementary, may be more sophisticated than we can expect of most college students. Instructors have achieved the most success with this project by making sure students are comfortable with modular arithmetic and basic number-theoretic reasoning, perhaps at the level of the first three weeks of a first course in the field.

4 Euler’s Rediscovery of $e$: A Project for Real Analysis

The two-day project “Euler’s Rediscovery of $e$” [17] is based on excerpts from one of Euler’s most influential works: the *Introductio in analysin infinitorum* (E101) [8]. Euler explicitly wrote this masterpiece as a precursor to his two later textbooks on differential and integral calculus. Unlike today’s precalculus texts, infinite series appear throughout the *Introductio* as a necessary skill for the later study of infinitesimal analysis. At the same time, because Euler introduced the general function concept as a new foundational concept of calculus in the *Introductio*, his writing about these ideas is more accessible to today’s students than that of his predecessors.

Euler discussed logarithmic functions for various bases and their properties in Chapter VI of the *Introductio*. Part of his challenge in working with these functions was to find a logarithmic base for which infinite series expansions are convenient. With this goal in mind, Euler derived (in Chapter VII) $e$ as both the limiting value of the sequence $(1 + 1/j)^j$ and as the sum of the infinite series $1 + 1 + 1/2 + 1/3 + 1/2 \cdot 3 + \ldots$.

The primary source project uses excerpts from Chapter VII of E101 to present students the opportunity to see how $e$ appeared naturally in Euler’s development of exponential and logarithmic functions. Starting from the assumption that $a^\omega = 1 + kw$ for $\omega$ infinitely small (since $a^0 = 1$), Euler first provided examples
to show that "$k$ is a finite number which depends on the value of the base $a$." Using the binomial theorem to expand $a^{j\omega} = (1 + k\omega)^j$ and setting $j = \frac{z}{k}$ with $j$ an infinite number and $z$ finite, he then obtained the following (Section 115):

$$
\alpha^2 = (1 + kz/j)^j = 1 + \frac{1}{1} kz + \frac{1}{1 \cdot 2 \cdot j} k^2 z^2 + \frac{1}{1 \cdot 2 \cdot 3 \cdot j} k^3 z^3 + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4j} k^4 z^4 + \ldots.
$$

He continued in Section 116 by noting that $\frac{j-1}{j} = 1$, $\frac{j-2}{j} = 1$, etc. (since $j$ is infinitely large) so that equation (1) became

$$
\alpha^2 = 1 + \frac{1}{1} k z + \frac{1}{1 \cdot 2} k^2 z^2 + \frac{1}{1 \cdot 2 \cdot 3} k^3 z^3 + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} k^4 z^4 + \ldots
$$

This last equation then allowed Euler to express the relationship between the numbers $a$ and $k$ by taking $z = 1$:

$$
a = 1 + \frac{1}{1} k + \frac{1}{1 \cdot 2} k^2 + \frac{1}{1 \cdot 2 \cdot 3} k^3 + \ldots.
$$

Using this last equation, he concluded (in Section 122):

Since we are free to choose the base $a$ for the system of logarithms, we now choose $a$ in such a way that $k = 1$. Then the series found above in Section 116,

$$
1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \ldots
$$

is equal to $a$. If the terms are represented as decimal fractions and summed, we obtain the value $a = 2.71828182845904523536028 \ldots$. When this base is chosen, the logarithms are called natural or hyperbolic. The latter name is used since the quadrature of a hyperbola can be expressed through these logarithms. For the sake of brevity for this number $2.718281828459 \ldots$ we will use the symbol $e$, which will denote the base for the natural or hyperbolic logarithms.

A vital component of the project is giving a modern justification that $e = \lim_{j \to \infty} (1 + 1/j)^j$, using Euler’s ideas along with some modern theory. The approach using the Monotone Convergence Theorem outlined in the project tasks is a common approach in current analysis textbooks. Reading about it first in Euler’s own words gives context to the exercises and some appreciation of his dexterity with series, as well as the close connection with $e$ as a logarithm base to motivate the definition. Moreover, his series development of $e^x$ is an interesting alternative to the Taylor series approach students have seen in introductory calculus courses.

One question for instructors and students alike is how formally and thoroughly to treat Euler’s manipulations of infinitely large and small numbers, which are an aesthetic treat but a novel approach for a modern student. The
project author is of the opinion that students already have a personal sense of what these objects are and how they should work, having been through introductory calculus courses. Euler also made a good case for his development in the passages quoted in the project so students can follow his reasoning. Although a lengthy discussion of infinitesimal calculus is not appropriate in an introductory real analysis course (for which it was primarily designed), instructors who use the project in other courses (e.g., History of Mathematics) may want to spend more time on these issues.

5 Euler’s Square Root Laws for Negative Numbers: A Project for Complex Variables

The short (one- or two-day) project “Euler’s Square Root Laws for Negative Numbers” [18] is based on excerpts from Part I of the two-volume textbook Vollständige Anleitung zur Algebra (E387) [9]. Euler most likely wrote his Algebra in 1765/66 during his Berlin period, although it first appeared in Russian translation in 1768/69 and then in its original German in 1770. Perhaps no other work shows the clarity of Euler’s writing to better advantage, with the number of editions and languages in which it appeared attesting to its success as an elementary textbook. Intended for self-study, the contents of the Algebra are laid out to lead a beginner from the basics of arithmetic on numbers through the elementary theory of functions (including exponential and logarithmic functions) to the study of determinate algebraic equations of low degree and an introduction to indeterminate equations, including a treatment of Fermat’s Last Theorem for degree 4.

The project itself focuses on what some mathematicians have argued was a mistake in Euler’s treatment of square roots of negative numbers in 387. Specifically, Euler wrote (in Paragraph 148):

Moreover, as \( \sqrt{a} \) multiplied by \( \sqrt{b} \) makes \( \sqrt{ab} \), we shall have \( \sqrt{6} \) for the value of \( \sqrt{-2} \) multiplied by \( \sqrt{-3} \).

The project brings up the controversy surrounding this claim via its first task as a hook to get students interested and perhaps give them a worthwhile sense of unease. It then follows the suggestion put forward by Martínez [16] that a careful reading of Euler’s notion of square roots reveals there may more to the story than a hasty generalization on Euler’s part of of the square root product law for positive numbers. In particular, the following passage from Paragraph 150 of Euler’s Algebra indicates that care is needed when using and interpreting the term “square root” and the radical symbol “\( \sqrt{} \)” as Euler used them.

The square root of any number has always two values, one positive and one negative; that \( \sqrt{a} \), for example, is both \( +2 \) and \( -2 \), and that, in general, we may take \( -\sqrt{a} \) as well as \( +\sqrt{a} \) for the square root of \( a \). This remark applies also to imaginary numbers; the square root of \( -a \) is both \( +\sqrt{-a} \) and \( -\sqrt{-a} \).
In light of this passage, Euler’s statement that “we shall have $\sqrt{6}$ for the value of $\sqrt{-2}$ multiplied by $\sqrt{-3}$” has multiple interpretations depending on whether we use $+\sqrt{-2}$ or $-\sqrt{-2}$ for $\sqrt{-2}$, and similarly for the values of $\sqrt{-3}$ and $\sqrt{6}$. Students examine these various options for both products and quotients of square roots through a series of tasks as a means to explore how Euler’s notion of square root could have allowed the square root product rule to actually be true with negative numbers as well.

Of course, square root laws for negative numbers are not in themselves of much interest today, and the project thus does not dwell on them. Instead, the issues with multiple roots are used to introduce the notion of a multivalued function and the difficulties that arise in trying to define a “normal” single-valued square root function. The project also brings in the important idea of multiple roots of a complex number, along with the related ideas of multivalued functions and branch cuts. It thus gives a peek at these ideas through the seemingly innocuous idea of square root laws for negative numbers. To extend the idea of multivalued functions beyond the square root, the project concludes with a brief discussion of the Euler-d’Alembert controversy about the logarithm of $-1$ (described in the next section of this article) and two related tasks which may create some (useful) confusion, but could be omitted in the interest of time.

The project is written with very few assumptions about student background beyond the basic sophistication developed in an introductory calculus course. For a complex variables course, it is designed to be used very early in the course, ideally at the first or second class meeting.

6 The Logarithm of $-1$: A Project for Complex Variables

The one-day project “The logarithm of $-1$” [14] is based on excerpts from letters (taken with some modification from [5]) in the correspondence between Euler and Jean Le Rond d’Alembert (1717–1783) in which they argued about the value of $\log(-1)$. One of the richest parts of the argument between these two mathematicians, from a pedagogical point of view, is that the definition of the logarithm function on the positive numbers could in principle be extended in several different ways to negative and complex values. There was, therefore, a decision to be made—a decision that is obfuscated by modern texts, which generally present multivalued logarithms as a fait accompli. The crux of the argument concerned the most basic question about negative logs, and this question gives the project its title: What is the logarithm of $-1$?

Intended to be used in courses on complex variables, the project introduces students to the definition of the logarithm function on negative and complex numbers via this epistolary argument between Euler and d’Alembert. The biggest hurdle that it tries to help students clear is the concept of a “multivalued function,” which might seem a contradiction in terms based on their previous experience. In fact, it was this exact property of the logarithm that made extending the domain of this function to negative and natural numbers so difficult in the eighteenth century. Johann Bernoulli (1667–1748) and d’Alembert both
seemed unable to make the leap to multivalued functions, and even Euler (who 
eventually extended the domain of the log function in the way we think of it 
today) struggled for several years.

Euler’s discussion of this question actually looks quite modern:

I believe that I have shown [in another work, not yet published] that 
it [the value of log(−1)] is imaginary and that it is = π(1 ± 2n)√−1 
where π indicates the circumference of a circle whose diameter = 1, 
and n is any whole number⁵.

Because, I have shown just as every sine responds to an infinity of 
arc's of the circle, so the logarithm of every number have an infinite 
amount of different values, among which there is only one that is 
real when the number is positive, but when the number is negative 
all the numbers are imaginary.

D’Alembert, on the other hand, offered a counter-argument:

All difficulties reduce, it seems to me, to knowing the value of 
log(−1). Now why may we not prove it by the following reasoning? 
−1 = 1/ −1, so log(−1) = log 1 − log(−1). Thus 2 log(−1) = 
log 1 = 0. Thus log(−1) = 0.

His reasoning is flawed, but to detect the flaw, students need to embrace mul-
tivalued logarithms, not just for negative arguments, but for positive ones, too.

The project assumes very little formal background, but students need the 
mathematical maturity to work with reasonably complicated functions, and to 
know the basic laws of logarithms.

7 Conclusion

In Part 1 of this article, we described a set of four additional Euler-based primary 
source projects designed for use in first-year calculus and differential equations. 
These and the five projects described above are part of a larger collection of 
nearly 120 primary source projects intended for use in a wide range of undergraduate 
mathematics courses that have been developed and extensively site tested 
with support from the National Science Foundation via a series of three grants. 
Information about all three and digital access to the primary source projects de-
veloped with their support are available through the website of the most recent 
grant: a multi-year, multi-institution collaborative effort entitled TRansform-
ing Instruction in Undergraduate Mathematics via Primary Historical Sources 
(TRIUMPHS): https://blogs.ursinus.edu/triumphs.

Of course, given the range, nature, and quality of his writing, the works by 
Euler currently represented in this collection are but a small fraction of those 
which are accessible to undergraduate mathematics students (including first-
year students). In this regard, we propose that engaging with his writing during 
classroom implementation of a primary source projects may serve as an entice-
ment for students to explore Euler’s original works more deeply. For example,

⁵Euler would eventually use the symbol i to represent √−1, but not until 1777.
Klyve, Stemkoski, and Tou [15] described three possibilities for undergraduate research projects that make use of the Euler Archive [1]:

- translation projects for students with the appropriate foreign language skills;
- in-depth studies of the mathematical content contained in a particular text; and
- historical investigations that seek to understand “how Euler’s work fit into the general milieu of the times, and how ... that historical moment impact[ed] later understandings of the topic.”

Here, we add yet another possibility that combines pedagogy, mathematics, and history into a single research project for students in mathematics education:

- development of a guided-reading primary source project based on a particular Euler text.

_Euleriana_ readers (and their students) who wish to experience the rewards (and challenges!) of writing their own Euler-based primary source project are invited to reach out to us for advice on how to get started. We also encourage readers to send us their top picks for Euler sources that are especially ripe as the foundation of projects that would allow today’s students to learn even more mathematics directly from the master.

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