A variational formulation for dissipative fluids in inhomogeneous temperature

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Abstract

We propose a formalization for dissipative fluids in inhomogeneous temperature from the viewpoint of a variational principle. The dynamics of the fluids satisfy the conservation laws of mass, momentum, angular momentum, and energy, and obey the second law of thermodynamics. By Noether’s theorem, these conservation laws are associated with the corresponding symmetries. On the other hand, the equation for entropy gives a nonholonomic constraint. Then Lagrangians and this nonholonomic constraint of the fluids satisfy these symmetries. The dynamics of fluid are given as a weak solution, minimizing an action subject to this constraint. For the existence of the weak solution, all the surface terms appearing in the variational calculus have to vanish with the aid of the nonholonomic constraint. In this study, we show that the equation for entropy is determined to be consist to “the second law of thermodynamics”, “the symmetries”, and “the necessary condition of the existence of the weak solution”. Our method gives a general scheme to derive the equation of motion for fluids in inhomogeneous temperature, and gives an explanation of some nontrivial thermal effects on the rotation of liquid crystal, vaporization of a one component fluid, and dissolution of a two component fluid.

1 Introduction

The dynamics of a fluid can be divided into a kinetic part and a thermodynamics part. The kinetic part is characterized by the equations of the conservation laws for mass, energy, momentum, and angular momentum. On the other hand, the thermodynamics is described by the equation of entropy. However, it is sometimes difficult to obtain these equations. On the assumption of constant temperature and low kinetic energy, various phase field models based on free energies are proposed for the dynamics\textsuperscript{[1,2,3].}

Under a constant temperature, the total Helmholtz free energy of a closed isolated system tends to decrease, because of the conservation law of total energy and the second law of thermodynamics, which can be stated as: \textit{the entropy of an isolated system cannot decrease\textsuperscript{[4].}} These models are widely used in the fields of soft matter such as liquid crystals (LCs) and polymer solutions. There can be important phenomena involved in inhomogeneous temperature and heat transfer.

The dynamics of cholesteric liquid crystals in a temperature gradient has attracted many researchers’ attention. In 1900, Lehmann observed the continu-
uous rotation of cholesteric droplets driven by a heat current along the cholesteric helix, which is called the Lehmann effect[5]. In 1968, F. Leslie predicted that in chiral LCs whose mirror symmetry is broken, any transport current could directly cause unidirectional molecular rotation[6]. The concept of the Lehmann effect was extended by de Gennes in 1973 to the electric and chemical cases, with a gradient of electric or chemical potential being formally equivalent to a temperature gradient[7]. Madhusudana and Pratibha found similar dynamics in cholesteric domains induced by an ionic current along the helix[8]. Lehmann’s old experiment has been well reproduced by Oswald and his coworkers [9, 10, 11, 12]. In Leslie’s phenomenological theory, the constant unidirectional rotation of the director is with the helical axis parallel to the heat current. Recently, Yoshioka et al. found the rotation of the director caused by a concave-convex surface. The striped droplets of a chiral LC, whose helical axis is perpendicular to the gradient temperature, rotate at a constant angular velocity[13]. This experiment suggests that another type of heat-driven unidirectional motion occurs through an essentially different mechanism than the Lehmann effect. To obtain a definitive understanding, we have to take into account not only the thermomechanical coupling terms, but also the fluid velocity and the inhomogeneous heat distribution inside.

Vaporization and dissolution are important processes associate with heat transfer, and frequently discussed in industry as well as in science. Latent heat caused by liquid-gas phase transition in a single-component fluid offers an efficient method to dissipate large heat fluxes in a compact device, such as a heat spreader or a heat pipe. Heat of dissolution in a multi-component fluid releases energy continuously on the process of the dissolution, and has a significant effect on the process of the dissolution. Dissolution of supercritical CO\(_2\) in interstitial water is one of the most crucial research topics in CO\(_2\) capture and storage (CCS), which stores the CO\(_2\) into deep underground geological rock formations[14]. The heat transfer during dissolution across the interface is also considered important for the dynamics of CO\(_2\) gas near its critical point because of its large thermal conductivity[15].

In these processes, interface energy plays the important role in the kinematics and thermodynamics. The high interface energy restrains the vaporization and dissolution, and the moving interface involves the heat of these process. For example, suppose that a drop of water shrinks by evaporation, and then its surface area also decreases. The entropy of the liquid area transfers to the gas area across the interface. This motion involves energy transfer, because the increasing and decreasing of entropy causes the internal energy to change. In the microscopic view, the high energy liquid molecules can go into the gas phase only through the surface of the drop and they take energy out of the surrounding molecules. Thus vaporization takes latent heat from the liquid phase to the gas phase. The energy required to free a molecule from the liquid is equivalent to the energy needed to break the molecular bonds and keep molecules away from surface. Therefore, the intermolecular forces that determine the interface internal energy of a substance are the same as those that determine its latent heat and boiling point. As might be expected, substances such as metals with high boiling points usually have high interface internal energy as noted in[16]. In the case of the multi-component fluid, the dynamics is more complicated. The diffusion flux can be induced by a temperature gradient, which is known as the Soret effect or thermal diffusion[17].

In previous works[6, 7], heuristic methods have been proposed to combine thermodynamics with the phase field models discussed above. Recently, Onuki discussed the liquid-gas phase transition of a single-component fluid in nonstationary, inhomogeneous temperature, and found a non-trivial entropy flux from this standpoint. He determined the stress tensor of the fluid by a consistency condition with the second law of thermodynamics with the aid of balance equations of momentum and energy describing their respective conservation laws. This model is called “The Dynamic van der Waals model”[18, 19]. In this theory, he assumed that the entropy density includes the gradient contributions. However, there remains difficulty in finding the proper equations of conservation laws and entropy. We also have a question whether if the derived entropy flux gives the proper condition of the moving interface between liquid and
gas.

In this study, we propose a new method based on the variational principle to derive the governing equations for these fluids above in an inhomogeneous temperature field. Our method properly enables us to formulate the dynamics of the fluid even if the thermodynamics of the fluid is complicated as we show in Sec. 3.

This article is organized as follows. We show the concept of our method in Sec. 2; the equation of entropy is determined to satisfy the symmetries associated with the conservation laws, and the dynamics is given by the stationary condition of an action subject to the equation of entropy. In Sec. 3 we explain the details of the calculation by using the models of various fluid as the following. We discuss the variational formulations of a Newtonian fluid and a liquid crystal, respectively in Secs. 3.1 and 3.2. Symmetries are the key-points for deriving the equations of motion. Next, we study about the fluid with interface energy in Sec. 3.3. The interface energies induce the entropy flux, which are related to the latent heat, and dissolution heat. Our method can be applied to the fluids where inhomogeneous temperature fields play important role. In Sec. 4, we discuss and summarize the formulation proposed in this study.

2 Analytical dynamics with thermodynamics

Let \( q \) be state variables except entropy \( s \). The dynamics of a system is described by the time development of \( q \) and \( s \). The equation of entropy is given in the form as

\[
\frac{ds}{dt} = \Theta.
\]  (1)

Here \( \Theta > 0 \) is a dissipative function describing entropy production rate per time. The equation (1) plays an important role in connecting the kinematics and thermodynamics. The low degree approximation of the dissipative function \( \Theta \) is given in the quadratic form of the time derivative of the state variables,

\[
\Theta(\dot{q}) = \dot{q}^T M \dot{q} \geq 0
\]  (2)

where \( \dot{q} \) denotes \( dq/dt \), and \( M \) is coefficient matrix. We can assume \( M \) as symmetric from the outset without loss of generality. In term of the variational principle, the conservation laws have corresponding symmetries by Noether’s theorem, which states that each of the conservation laws is associated with each corresponding symmetry. Then the conservation laws are automatically satisfied if the Lagrangian and constraints have the corresponding symmetries. For example, the conservation laws for mass, energy, momentum, and angular momentum are related to the Galilean invariance, the translation symmetries in time and space, and rotational symmetry, respectively. Thus to satisfy these conservation laws, (1) has to be consist with these symmetries. Our method is simple. We just give the Lagrangian by the kinetic energy minus the internal energy. The exact form of Eq. (1) is determined in the way above. The equations of motion are derived from the variational principle with the aid of Eq. (1).

2.1 Nonholonomic system

According to the variational principle in analytical mechanics, the realized motion of a system gives the least value of the action among the perturbation of the possible motion. It describes dynamics simply. However, in terms of control theory, all the system treated in analytical mechanics are categorized into holonomic systems. To deal the system with entropy, we have to generalize the variational principle in terms of optimized control theory\[20\]. In this viewpoint, physical systems are often considered as the system with control, and then the variational principle in analytical mechanics is regard as the optimization problem to find the optimized control minimizing a value functional (an action) with the condition of the initial and final states are fixed, and the equations of the motion is given as the result equation of the optimized control.

In order to introduce the basic concepts of the control theory it is convenient to start from a system described by the two state variables \((q, s)\). Its dynamics is described as a trajectory in the space of generalized coordinates \((q, s)\) and time \(t\). Suppose, further, that \(q\) and \(s\) are not independent variables. In other words,
the dynamics is restricted by a constraint given in the differential form,

\[ Tds + fdq + Qdt = 0, \tag{3} \]

where \( T, f, \) and \( Q \) are coefficient. If a constraint can be written in the form

\[ U(q, s, t) = 0, \tag{4} \]

it is called a holonomic constraint. If not, it is called a nonholonomic constraint.

We define the degree of the freedom as the number of the state variables which is needed to describe the dynamics of a system at least. For example, if we give the value of \( q \) at time \( t \) in Eq. (4), we can determine \( s \) as a function of \( q \) and time \( t \). On the other hand, a nonholonomic constraint does not reduce the degree of the freedom because it is impossible to know \( s \) from the set of \( q \) and \( s \), although \( s \) depends on the trajectory of \( q \). A system whose state variables are subject to a nonholonomic constraint is called a nonholonomic system\[21\]. Posture control during free falling, which is sometimes called the falling cat problem, is an example of the nonholonomic system. A free-falling cat can turn itself right-side-up as it falls without violating the law of conservation of angular momentum. Another simpler example of a nonholonomic system is the motion of a unicyclist\[22\]. These nonholonomic systems have history dependences.

### 2.1.1 Lagrangian

The time evolution of the state variables given as a function of \( q \) and \( u \),

\[ \frac{dq}{dt} = F(q, u), \quad q(t_{\text{init}}) = q_{\text{init}}, \quad u(t) \in \mathcal{U} \tag{5} \]

where \( \mathcal{U} \) is the set of admissible parameters. In terms of control theory, \( u \) is called control\[20\].

The realized dynamics of the nonholonomic system is obtained by the optimal control \( u^* \), which minimizes an action (a value functional), \( \int_{t_{\text{init}}}^{t_{\text{fin}}} dt \, L(q, u, s) \) subject to Eq. (3) with the condition of fixing the initial and final value of \( q \),

\[ \delta q(t_{\text{init}}) = \delta q(t_{\text{fin}}) = 0 \tag{6} \]

Here, \( L(q, u, s) \) is Lagrangian. The equations of motion is given as the result equation of the optimized control. The equation (3) shows that the virtual displacement \( \delta q \) at any given instant in time are not independent,

\[ T\delta s + f\delta q = 0. \tag{7} \]

Note that we put \( \delta t = 0 \) because the time \( t \) is fixed in the variational calculus. Using the method of undetermined multiplier, the action to be minimized is given by

\[ I[q, p, u, s] = \int_{t_{\text{init}}}^{t_{\text{fin}}} dt \, \tilde{L}(q, p, u, s, \dot{q}), \tag{8} \]

where \( p \) is an undetermined multiplier, and \( \tilde{L}(q, p, u, s) \) is defined as

\[ \tilde{L}(q, p, u, s, \dot{q}) = L(q, u, s) + p \left( \frac{d}{dt} q - F(q, u) \right) \]

\[ = H(q, p, u, s) + p \frac{d}{dt} q. \tag{9} \]

In control theory, \( p \) is called costate. Subtracting Eq. (7) from the variation of Eq. (9) yields

\[ - \left( \frac{dp}{dt} + \frac{\partial \tilde{H}}{\partial q} + f \right) \delta q + \left( \frac{dq}{dt} - \frac{\partial \tilde{H}}{\partial p} \right) \delta p + \frac{\partial \tilde{H}}{\partial u} \delta u + \left( \frac{\partial \tilde{H}}{\partial s} - T \right) \delta s + \frac{d}{dt} (p\delta q), \tag{10} \]

We can define

\[ T = - \frac{\partial L}{\partial s} = \frac{\partial \tilde{H}}{\partial s} \tag{11} \]

without loss of generality. Then the stationary condition of Eq. (8) subject to Eq. (7) is given by

\[ 0 \approx \frac{\partial \tilde{H}}{\partial u}, \tag{12} \]

\[ \frac{dq}{dt} \approx \frac{\partial \tilde{H}}{\partial p}, \tag{13} \]

\[ \frac{dp}{dt} \approx - \frac{\partial \tilde{H}}{\partial q} - f. \tag{14} \]
In this study, we consider only infinitesimal transformations. Such transformations may be continuous or discrete. In a holonomic system, the number of both the state variables, costate variables, and the control variables are same. However, it is not always true. For example, in the physical system with entropy, which is considered as a nonholonomic system, the former is larger than the latter. In other words, although the time development of entropy is required in describing the dynamics, we don’t have the costate of entropy, and the number of the costate variables is the same as the number of the control variables.

### 2.1.2 Noether’s theorem with the constraint

Symmetries play a central role throughout this study. Thus, we generalize the Noether’s theorem to the system with constraint. To simplify the notation, we write \( r \) for \( r \equiv (q, p, u, s) \). Transformations \( (r, t) \to (r', t') \) define symmetries of a system if they leave the equations of motion invariant. In terms of the variational principle, transformations move an optimized trajectory \( (r^*, t) \) to other optimized trajectories \( (r'^*, t') \), i.e.,

\[
\delta I[r^*(t')] = \delta I[r'^*(t)] = 0. \tag{15}
\]

Such transformations may be continuous or discrete. In this study, we consider only infinitesimal transformations given by the form,

\[
\begin{align*}
    r'(t') &= r(t) + \alpha g(r(t), t), \\
    t' &= t + \alpha g_t(r(t), t),
\end{align*}
\]

where \( \alpha \) is an infinitesimal constant, and \( g \equiv (g_q, g_p, g_u, g_s) \) and \( g_t \) are functions. When an infinitesimal transformation, given by Eqs. (16) and (17) satisfies Eq. (15), we define it as a canonical transformation. By this definition, a canonical transformation keep the Hamilton’s equations (11)–(14), valid on the new variables \( (r', t') \). The original Noether’s theorem for a holonomic system states that there is a canonical transformation, then there be a corresponding conservation law. Let us generalize it to the nonholonomic system as the followings. Suppose the transformation given by Eqs. (16) and (17) keeps the Lagrangian \( L \) invariant, i.e.,

\[
\tilde{L}(r'(t'), \dot{q}(t'), t') = \tilde{L}(r(t), \dot{q}(t), t) - \alpha \frac{dW}{dt}, \tag{18}
\]

where the function \( W \) is called a generator, and satisfies Eq. (3), i.e.,

\[
T g_s + f g_q + Q g_t = 0. \tag{19}
\]

The transformation given by Eqs. (16) and (17) is a canonical transformation, because it satisfies Eq. (15). We can rewrite Eq. (18) into

\[
\tilde{H}(r'(t'), t') - p' \frac{dq'}{dt} = \tilde{H}(r(t), t) - p \frac{dq}{dt} - \alpha \frac{dW}{dt}. \tag{20}
\]

Then, substituting Eqs. (16) and (17) into lhs of Eq. (20) yields

\[
\begin{align*}
    \left( \frac{\partial \tilde{H}}{\partial q} + \frac{dp}{dt} \right) g_q + \left( \frac{\partial \tilde{H}}{\partial p} - \frac{dq}{dt} \right) g_p + \frac{\partial \tilde{H}}{\partial u} g_u - \frac{d}{dt} (pg_q) \right), \tag{21}
\end{align*}
\]

From Eqs. (19)–(21), we have

\[
0 = - \left( \frac{dp}{dt} + \frac{\partial \tilde{H}}{\partial q} + f \right) g_q + \left( \frac{dq}{dt} - \frac{\partial \tilde{H}}{\partial p} \right) g_p - \frac{\partial \tilde{H}}{\partial u} g_u + \left( \frac{\partial \tilde{H}}{\partial s} - T \right) g_s - \frac{d}{dt} (W - pg_q) - Q g_t. \tag{22}
\]

If \( r^*(t) \) satisfies Eqs. (11)–(13), the first and second lines on the rhs of (22) vanishes, and we have the equation,

\[
- \frac{d}{dt} (W - pg_q) - Q g_t = 0 \tag{23}
\]

Conversely in order to satisfy Eq. (23) associated with the infinitesimal translation given by Eqs. (16) and (17), the Lagrangian \( L \) and the nonholonomic constraint \( g \) have to satisfy Eqs. (18) and (19). Thus
we have a generalization on Noether’s theorem. If a constraint is given in a holonomic form (19), we can discuss the generalized Noether’s theorem more simply. The condition (19) means
\[ 0 = U(r'(t'), t') = U(r(t), t). \] (24)
The equation (23) is considered as a generalization of conservation law. In a physical system, the several conservation laws related to the symmetries are required. Thus the nonholonomic constraint (3) satisfies the symmetries of the systems, i.e., Lagrangians as Eq. (19). Actually, symmetries depend on the explicit form of the Lagrangian, i.e., on which system is considered, but in all the cases of physics it is usually assumed to satisfy a number of generic symmetries such as the translation symmetries in time and space, rotational symmetries, and Galilean symmetry, which are related to the conservation laws of energy, momentum, angular momentum, and mass, respectively. We give the details in App. A. Other symmetries, so-called internal (global and local or gauge) symmetries, under which a given system is assumed to be invariant, further specify the structure of the Lagrangian.

2.2 A simple nonholonomic system
A damped oscillator in a heat bath is an example of nonholonomic system. We assume that the heat bath is always at the equilibrium. The dynamics of this system is expressed by the time development of state variables: \( q, p, u, \) and \( s \) denoting the position, momentum, velocity of oscillator, and the entropy of the heat bath, respectively. The frictional force between oscillator and the heat bath dissipates kinetic energy of the oscillator, and release it to the heat bath as heat. The nonholonomic constraint (3) expresses the change of the entropy \( s \) caused by the motion of the oscillator. For example, the Lagrangian of a damped harmonic oscillator is given by
\[ \tilde{L}(q, \dot{q}, p, u, s) = \frac{1}{2}mu^2 - \left( \frac{1}{2}kq^2 + \epsilon(s) \right) + p \left( \frac{dq}{dt} - u \right). \] (25)
where \( m \) and \( k \) are the mass of the oscillator and the spring constant, respectively. The internal energy of the heat bath, \( \epsilon \), is a function of its entropy \( s \). The Lagrangian has the time translation symmetry. Then we require
\[ T\frac{ds}{dt} + f \frac{dq}{dt} + Q = 0, \] (26)
By Noether’s theorem discussed in Sec. 2.1.2, we have
\[ \frac{dH}{dt} - Q = 0, \] (27)
where \( H = \frac{p^2}{2m} + \left( \frac{kq^2}{2m} + \epsilon(s) \right) \) is the total energy. The details is given in App. A.1. To satisfy the conservation law of total energy, we put \( Q = 0 \). Using it and Eq. (26), we have the equation of the entropy,
\[ \frac{ds}{dt} = -\frac{f}{T} \frac{dq}{dt} \geq 0, \] (28)
with the aid of the second law of thermodynamics, \( ds/dt \geq 0 \). Here, \( f \) is considered as the frictional force exerted on the oscillator, and then \( f \) is proportional to \(-dq/dt\) in the linear approximation.

3 Field theory for fluids
Let us extend the discussion in Sec. 2 to the field theory. We define the Lagrangian density as the kinetic energy density minus the internal energy density, and the action as the integral over space and time. The realized dynamics of fluids, such as Newtonian fluid, viscoelastic fluid and liquid crystals, minimizes the action with the aid of the equation for entropy[23, 24, 25, 26, 27, 28]. In stead of Eq. (1), the equation of entropy is given in the form as
\[ \frac{\partial}{\partial t}(\rho s) = \Theta - \nabla \cdot J. \] (29)
where \( \rho \) is total mass density, \( s \) is specific entropy, \( J \) is entropy flux, and \( \Theta > 0 \) is a dissipative function describing entropy production rate per time. The equation (29) plays an important role in connecting the kinematics and thermodynamics. If we don’t know the exact form of the dissipative function \( \Theta \), we can fix it by considering the symmetries, which is associated with the conservation laws, and the second law of thermodynamics. On the other hand, the entropy flux \( J \) is determined to erase surface terms with free
boundary conditions appearing in the variational calculus. The entropy flux is related to the interface energy, yielding the surface terms. In the case that an action to be minimized has surface term without fixing boundary condition in the variational calculus, the entropy flux \( J \) is determined to erase it for the existence of the weak solution.

In Sec. 3.1 we derive the equation of entropy in a Newtonian fluid by considering symmetries and the second law of thermodynamics, and obtain the equation of motion for the fluid from the variational principle with the aid of the derived equation of entropy. We apply the method to LCs in Sec. 3.2. The symmetries are the key point for discussing the dynamics of the LCs. In Sec. 3.3 we discuss the entropy flux related to the interface energy. The entropy flux is determined to erase surface term appearing the variational calculus without fixing boundary condition. We give the details in the case of a one-component fluid in Sec. 3.3.1 and of a two-component fluid in Sec. 3.3.2.

### 3.1 Newtonian fluid

First of all, we discuss the variational formulation for a Newtonian fluid. In our previous work[23], the equation of entropy [29] is given beforehand. In this study, we show that it can be derived from the symmetries and the second law of thermodynamics as discussed in Sec. 2.2.

We assume the local thermal equilibration, and thus we can define the specific entropy \( s \) as the entropy density per mass. The internal energy density \( \epsilon \) per mass is given by the function of the mass density \( \rho \) and specific entropy \( s \). Then we have \( d\epsilon = -P \rho^{-1} + T ds \) in the thermodynamics. Temperature \( T \) and pressure \( P \) are respectively given by

\[
P = -\left( \frac{\partial \epsilon}{\partial \rho^{-1}} \right)_s = \rho^2 \left( \frac{\partial \epsilon}{\partial \rho} \right)_s \quad \text{and} \quad T = \left( \frac{\partial \epsilon}{\partial s} \right)_\rho,
\]

where the subscripts \( s \) and \( \rho \) indicate variables fixed in the respective partial differentiations. The velocity field \( \mathbf{v} \) draws the path line of a fluid particle. In Lagrangian description, we name fluid particles by its position \( \mathbf{A} = (A_1, A_2, A_3) \) at the initial time \( t_{\text{init}} \). The path line is expressed by the time development of the position of the fluid particle named \( \mathbf{A} \). On the other hand, in the Eulerian description, all the parameters of the fluid are given by the functions of the fixed point \( \mathbf{x} = (x_1, x_2, x_3) \) and time \( t \), and then the path line is expressed by the map from to the Eulerian coordinates \((t, \mathbf{A})\) to the Lagrangian coordinates \((t, \mathbf{A})\). The initial position of a fluid \( \mathbf{A} \) is sometimes called Clebsch potentials[29]. By the definition, the material derivative of \( A_i \) is zero,

\[
\frac{\partial A_i}{\partial t} + \mathbf{v} \cdot \nabla A_i = 0.
\]

Here and below, Roman indices run from 1 to 3. In terms of the control theory, \( \mathbf{v} \) control the state \( \mathbf{A} \) from Eq. (31). Below, we discuss the fluid dynamics in the Eulerian description.

#### 3.1.1 Lagrangian

The Lagrangian density is given by the kinetic energy minus inertial energy:

\[
\mathcal{L}(\rho, \mathbf{v}, s) \equiv \rho \left\{ \frac{1}{2} \mathbf{v}^2 - \epsilon(\rho, s) \right\},
\]

where \( \rho \mathbf{v}^2 / 2 \) is the kinetic energy density. The Lagrangian [32] has no explicit dependencies in time and space. Therefore it satisfies the infinitesimal translation symmetries in time and space. The Lagrangian [32] is also required to be invariant under a Galilean transformation. See the details in App. A.3. Then we have the mass conservation law:

\[
\rho(t, \mathbf{x}) - \rho_{\text{init}} J^{-1} = 0,
\]

where \( \rho_{\text{init}} \) is the initial mass density and \( J^{-1} \equiv \partial(A_1, A_2, A_3)/\partial(x_1, x_2, x_3) \) denotes the contraction percentage of a fluid particle. In fluid dynamics, we assume no singularity, i.e., \( J \neq 0 \). Thus the dynamics is described by the trajectory in the generalized coordinates and time \((\mathbf{A}(\mathbf{x}), s(\mathbf{x}), t)\). The time derivative of Eq. (33) yields

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,
\]
with the aid of Eq. (31). See the details in App. 6. The velocity \( v(t, x) \) is considered as the control determining the dynamics of the system. The time development of \( A \) is given by Eq. (31), and then entropy is determined by the time development of \( A \) because of Eq. (29). We write \( \delta \) for the infinitesimal changes of variables at a time \( t \). In the variational calculus, we fix the both ends of the path line

\[
\delta A(t_{\text{init}}, x) = \delta A(t_{\text{fin}}, x) = 0, \tag{35}
\]

and assume the path line at the boundary \( \partial V \) is fixed, i.e.,

\[
\delta A(t, x) = 0, \quad x \in \partial V. \tag{36}
\]

Then, we give the modified Lagrangian density by

\[
\tilde{\mathcal{L}} = \mathcal{L} + K(\rho - \rho_{\text{init}} J^{-1}) + \beta_i \left( \frac{\partial}{\partial t} A_i + v \cdot \nabla A_i \right), \tag{37}
\]

where \( K \) and \( \beta_i \) are undetermined multipliers. The equation of motion can be obtained by minimizing the integral of Eq. (37) over the considered space \( V \) and time interval \([t_{\text{init}}, t_{\text{fin}}]\),

\[
\int_{t_{\text{init}}}^{t_{\text{fin}}} dt \int_V d^3 x \tilde{\mathcal{L}}. \tag{38}
\]

subject to the constraint of entropy (29). In terms of control theory, the realized motion of the Newtonian fluid is given by the optimized control of \( v^* \), which minimizes the action (38) under the nonholonomic constraint (29). See the details in App. 5. In our previous works[23, 24, 25], the both of \( J \) and \( \Theta \) are given. Next, let us show that they are derived from the symmetries and the second law of thermodynamics.

### 3.1.2 Constraints for entropy

Conservation laws of energy, momentum and angular momentum are satisfied in the fluid dynamics. Thus the equations of motion derived from the stationary condition of Eq. (38) subject to the constraint for entropy (29) have to satisfy these conservation laws. By Noether’s theorem, Eq. (29) have to satisfy the corresponding symmetries. In this section, we determine Eq. (29) by using the same procedure in App. 2.1.2. Therefore the nonholonomic constraint for entropy is given in the differential form of \( ds, dA_i, \partial_i dA_j \), and \( dt \). Here we write \( \partial_i \) for the shorthand of \( \partial/\partial x_i \).

Note that \( \rho \) is given as a function of \( A \) from Eq. (33), and \( v \) is control to give the time development from Eq. (31). Without loss of the generality, we can give it as

\[
T \rho ds + M_i dX_i + \sigma_{ij} \partial_i (dX_j) + Q dt = 0. \tag{39}
\]

Here we define \( T \) as the temperature in Eq. (30), and write \( dX_j \) and \( \partial_i (dX_j) \) respectively for the short-hand of

\[
dX_j = -\frac{\partial x_j}{\partial A_i} dA_i \quad \text{and} \quad \partial_i dX_j = -\partial_i \left( \frac{\partial x_j}{\partial A_k} dA_k \right). \tag{40}
\]

The higher differentiation of \( X \) are not required in the Newtonian fluid. If we consider the dynamic of the one-component fluid with liquid-gas interface, these terms are required as we explain in Sec. 3.3.1.

The coefficients \( M_i, \sigma_{ij} \), and \( Q \) are determined to be consist with the symmetries associated with conservation laws, and entropy production law. First of all, let us discuss the space translation symmetry. Replacing \( d \) by \( \partial/\partial x_k \) in Eq. (39) yields

\[
T \rho \partial_j s - M_j = 0, \tag{41}
\]

with the aid of Eq. (40). Replacing \( d \) by \( \delta \) in Eq. (39) yields

\[
T \rho \delta s - T \rho \partial_j s \delta X_j + \sigma_{ij} (\partial_i \delta X_j) = 0. \tag{42}
\]

Note that \( \delta \) denotes the small variation at a fixed time point. Solving the stationary condition of Eq. (38) subject to Eq. (42) yields the equations of motion[24]. See the details in App. 6.

The coefficient \( Q \) in Eq. (39) are determined as the following. Using the generalized Noether’s theorem, we have

\[
\frac{\partial}{\partial t} \mathcal{H} + \partial_j \left[ \left( \mathcal{H} + P \right) \delta_{ij} + \sigma_{ij} \right] v_i + Q = 0. \tag{43}
\]

Here, \( \mathcal{H} = \rho \left( v^2 + c^2 \right) / 2 \) is the total energy density. To satisfies energy conservation law, \( Q \) should be given in the form,

\[
Q = \nabla \cdot J_q, \tag{44}
\]
where $J_q$ is considered as heat flux. Then we have have the nonholonomic constraint for entropy as

$$ T \rho ds = -(T \rho \nabla s) d\mathbf{X} + \sigma : \nabla d\mathbf{X} + \nabla \cdot J_q dt. \tag{45} $$

Replacing $d$ by $\partial / \partial t$ in Eq. \((45)\) yields

$$ \frac{\partial}{\partial t}(\rho s) = \frac{1}{T} \sigma_{ij} \partial_i v_j - \nabla \cdot (\rho s v) + \nabla \cdot J_q \frac{1}{T}, \tag{46} $$

with the aid of Eq. \((34)\). Here $\rho s v$ and $\sigma_{ij}$ are respectively considered as the entropy flux convected by mass flux, and viscous stress tensor.

Next let us consider the rotational symmetry. The infinitesimal translations of the rotational symmetry for the scalar $\phi$ denoting $\rho, s,$ and $A_i$ is respectively given by

$$ \phi'(x) - \phi(x) = \phi(x + x \times e \theta) - \phi(x) = \frac{1}{2} \epsilon_{ijk} (x_j \partial_k - x_k \partial_j) \phi e_i \theta, \tag{47} $$

where unit vector $e$ indicates the direction of an axis of rotation, an angle $\theta$ describes the magnitude of the rotation about the axis, and $\epsilon_{ijk}$ is the Levi-Civita symbol. While the one of the vector $v$ is

$$ v'_a(x) = v_a(x) $$

$$ v(x + x \times e \theta) + \theta e \times v - v(x) = \frac{1}{2} \{ \epsilon_{ijk} (x_j \partial_k - x_k \partial_j) \delta_{\alpha \beta} - \epsilon_{\alpha \beta j} \} v_\beta e_i \theta. \tag{48} $$

With the aid of Eqs. \((47)\) and \((48)\), Eq. \((39)\) yields

$$ \epsilon_{ijk} \sigma_{kj} = 0. \tag{49} $$

The viscous tensor $\sigma_{ij}$ is divided into the symmetric tensor $\Sigma = (\sigma + \sigma_T)/2$ and asymmetric tensor $\Sigma^A = (\sigma - \sigma_T)/2$, i.e., $\sigma = \Sigma^S + \Sigma^A$. Then, Eq. \((49)\) can be rewritten into

$$ \epsilon_{ijk} \sigma_{kj}^A = 0. \tag{50} $$

With the aid of Eq. \((50)\), we can rewrite $\sigma_{ij} \partial_j v_i$ into

$$ \sigma^S_{ij} e_{ij} + \sigma^A_{ij} d_{ij} \tag{51} $$

where, $e_{ij}, d_{ij}$ are respectively defined as

$$ e_{ij} = \frac{1}{2} (\partial_j v_i + \partial_i v_j), \tag{52} $$

$$ d_{ij} = \frac{1}{2} (\partial_j v_i - \partial_i v_j). \tag{53} $$

The latter part $\sigma^A_{ij} d_{ij}$ of Eq. \((51)\) vanishes because of Eq. \((50)\). Thus we can rewrite Eq. \((46)\) in the form of Eq. \((29)\). The dissipative function $\Theta$ and entropy flux $J$ are respectively given by

$$ \Theta = \frac{1}{T} \sigma^S_{ij} e_{ij} + J_q \cdot \nabla \left( \frac{1}{T} \right) > 0, \tag{54} $$

$$ J = \rho s v + \frac{J_q}{T}. \tag{55} $$

The dissipative function $\Theta$ gives the entropy production rate, and should be positive because of the second law of thermodynamics. In the low approximation, the dissipative function is given in the quadric form, and then $\sigma^S$ and $J_q$ is given by linear combination of $e_{ij}$ and $\nabla T$.

Next, we assume that the heat flux $J_q$ is in proportion to only the gradient of temperature,

$$ J_q = -\lambda \nabla T, \tag{56} $$

where the thermal conductivity $\lambda > 0$ is positive. On the other hand, $\sigma^S_{ij}$ depends on only $e_{ij}$, when $J_q$ is given by Eq. \((50)\). To satisfy the rotational symmetry, $\sigma^S_{ij}$ have to be isotropic,

$$ \sigma_{ij} = 2 \zeta e_{ij} + (\zeta_b - 2 \zeta_3 / 3) \delta_{ij} e_{kk}, \tag{57} $$

where $\zeta_s$ and $\zeta_b$ are the coefficients of shear and bulk viscosities, respectively. If the both of $\zeta_s$ and $\zeta_b$ are positive, $\sigma_{ij} e_{ij}$ is also positive.

Note that we do not know the exact form of the equation of entropy \((29)\). It is obtained from the symmetries associated with conservation law, and the second law of thermodynamics.

### 3.2 Liquid crystal

Liquid crystals (LC) are composed of rod-like molecules. The long axes of the molecules in a fluid particle prefer to orient parallel to one another along a common axis. It is described by a dimensionless unit vector $n$, called director. The angular velocity $\dot{\omega}$ denotes the rotation of the director. The material time derivative of $n$ is given by

$$ \frac{\partial n}{\partial t} + (v \cdot \nabla) n = \omega, \tag{58} $$
where \( \omega \) is defined as \( \omega \equiv \hat{\omega} \times \mathbf{n} \). Note that the equation above expresses a kind of the rigid motion because the norm of \( \mathbf{n} \) is constant, \( |\mathbf{n}| = 1 \). We give the specific internal energy of nematic liquid crystal, \( \varepsilon \), as function of \( \rho, s, n_i, \) and \( \partial_t n_j \). The infinitesimal change of \( \delta \varepsilon \) yields \( \delta \varepsilon = -Pd\rho^{-1} + M\partial_t n_i + \pi \partial_t n_j \), where pressure \( P \) and temperature \( T \) are given as same as Eq. (30), and the new coefficients \( l_i \) and \( \pi_{ij} \) related with the director \( \mathbf{n} \) are defined as

\[
\begin{align*}
  l_i &\equiv \left( \frac{\partial \varepsilon}{\partial n_i} \right)_{\rho,s}, \\
  \pi_{ij} &\equiv \left( \frac{\partial \varepsilon}{\partial (\partial_t n_j)} \right)_{\rho,s}.
\end{align*}
\]  

(59)

We modify the nonholonomic constraint (39) as

\[
TpdS + M_j dX_j + \sigma_{ij}(\partial_t X_j) + g_i d\omega_i + Qdt = 0.
\]  

(60)

where the term \( g_i d\omega_i \) is added for the dependency of \( \mathbf{n} \). The coefficient \( M_j \) is \( M_j = T\rho\partial_x + \gamma g_i \partial_t n_i \) because of the space translation symmetry as in Eq. (41), and the coefficient \( N \) is given by Eq. (44) because of the energy conservation law. Using the rotational symmetry as in Eq. (40), we have

\[
\epsilon_{ijk} \sigma_{ij}^A + \epsilon_{ijk} \sigma_{ij}^B = 0.
\]  

(61)

Finally, we determine the coefficients \( \sigma \) and \( g \) in Eq. (60). Using the same procedure in Sec. 3.1.2, we can rewrite Eq. (60) in the form of Eq. (29). The entropy flux \( J \) and the dissipative function \( \Theta \) are respectively given by Eq. (55) and

\[
\begin{align*}
  \Theta &= \frac{1}{T}(\sigma_{ij}\partial_j v_i + g_i \omega_i) + J_q \cdot \nabla \left( \frac{1}{T} \right) \\
  &= \frac{1}{T}(\sigma_{ij}^s e_{ij} + \sigma_{ij}^d d_{ij} + g_i \omega_i) + J_q \cdot \nabla \left( \frac{1}{T} \right) \\
  &= \frac{1}{T}(\sigma_{ij}^s e_{ij} + g_i N_i) + J_q \cdot \nabla \left( \frac{1}{T} \right)
\end{align*}
\]  

(62)

where, \( e_{ij}, d_{ij} \) and \( N_i \) are respectively defined as Eqs. (52), (53), and

\[
N_i = \omega_i - d_{ij} n_j,
\]  

(63)

expressing the rotation of a director except the rotation of velocity. By the symmetry of the kinetic coefficient (30), we have

\[
\begin{align*}
  g &= \xi N + \eta \nabla T, \\
  J_q &= \eta TN - \lambda \nabla T.
\end{align*}
\]  

(64)

(65)

in the low degree approximation as seen in Eq. (2). Note that this symmetry yields Onsager reciprocal relations (17). Here, \( \xi, \eta, \) and \( \lambda \) are rotational viscosity, Lehman coefficient, heat conductivity. The Lehman coefficient \( \eta \neq 0 \) breaks chiral (mirror) symmetry of Eqs. (41) and (45). The temperature gradient \( \nabla T \) is a polar vector, while the torque \( \mathbf{N} \) is an axial vector. Then the Chiral transformation of them change the sign of the terms \( \xi N \) and \( \eta TN \).

The total kinetic energy destiny per mass is given by the sum of momentum energy \( \frac{1}{2} v^2 \), and angular momentum energy per mass, \( (\omega^T I \omega)/2 \). Here \( I_{ij} \) is the molecular’s moment of inertia, and promotional to \( \delta j \). The Lagrangian density is given by

\[
\mathcal{L}(\rho, n, v, \omega, s) \equiv \rho \left\{ \frac{1}{2} v^2 + \frac{1}{2} \omega^T I \omega - \epsilon(\rho, s, n) \right\}.
\]  

(66)

The Lagrangian density (66) satisfies the time and space translation symmetries, and the rotational symmetry. In the variational calculus, we use the boundary conditions, (35), (36), \( \delta n(t, x_{\text{init}}) = \delta n(t, x_{\text{final}}) = 0 \), and \( \delta n(t, x) = 0 \) where \( x \in \partial V \).

Then, the modified Lagrangian density is given by the sum of Eq. (67) and

\[
\lambda_i (\partial_t n_i + (v \cdot \nabla) n_i - \omega_i)
\]  

(67)

where \( \mathcal{L} \) is Eq. (66) and \( \lambda_i \) is an undetermined multiplier. The equation of motion can be obtained by minimizing its integral over the considered space \( V \) and time interval \([t_{\text{init}}, t_{\text{final}}]\) subject to the constraints (29).

Using the same procedure in Sec. 3.1, we obtain the equations of the motion. The angular velocity \( \omega \) satisfies

\[
\rho I_{ij}(\partial_t + v \cdot \nabla) \omega_j - \partial_j (\rho \pi_{ij}) + l_i - \rho g_i = 0.
\]  

(68)

To simplify the discussion, we assume that the velocity \( v \) is zero. In the steady state, the angular velocity \( \omega \) is constant, then we have

\[
\omega_i = -\frac{1}{\xi} \{ \partial_j (\rho \pi_{ij}) - \rho l_i - \eta \partial_i T \}.
\]  

(69)
The term $\eta \partial_i T$ induce the Lehmann effect in a chiral LC, namely the constant unidirectional rotation of the director with the helical axis parallel to the heat current. On the other hand, $\partial_j (\rho \pi_{ij})$ denotes that the surface geometry causes torque.

Recently, Yoshioka et al reported the rotation of the director caused by the surface concave-convex. The striped droplets of a chiral LC whose helical axis is perpendicular to the gradient temperature, rotate at a constant angular velocity\(^{[13]}\). This phenomenon can be expressed as the following. Suppose the internal energy $\epsilon$ has the mirror-symmetry breaking term, $k_{ch} \cdot \nabla \times n$, where $k_{ch}$ is a coefficient\(^{[31]}\). Then $\partial_j (\rho \pi_{ij}) - \rho l_i$ in Eq. (69) have the term, $\nabla \times (\rho k_{ch} n) - k_{ch} \rho \nabla \times n = \nabla (\rho k_{ch}) \times n$. (70)

Note that the directions of $\nabla (\rho k_{ch})$ in Eq. (70) is perpendicular to the helical axis, while the $\eta \partial_i T$ is parallel to the helical axis. If $k_{ch}$ depends on temperature $T$, the Chiral LC is rotated by the gradient temperature perpendicular to the helical axis. This may be the first explanation for the rotation reported in \(^{[13]}\).

### 3.3 Interface energy

In this section, we derive the governing equations for fluids, where the interface energy in inhomogeneous temperature field plays an important role in the process of the vapor and dissolution. We discuss the dynamics of liquid-gas transition in a one-component fluid in Sec. \(^{3.3.1}\) and the dissolution in a two-component fluid in Sec. \(^{3.3.2}\).

The new point in this section is the determination of the entropy flux $J$. For the existence of the weak solution minimizing an action subject to this constraint, all the surface terms appearing in the variational calculus have to vanish with the aid of the nonholonomic constraint. We determine the entropy flux $J$ to erase the surface terms without fixing boundary conditions in the variational calculus.

#### 3.3.1 Liquid-gas interface of a one-component fluid

We assume that the resultant hydrodynamic equations contain the stress arising from the mass density gradient, $\nabla \rho$. The mass density $\rho$ jumps from high to low at liquid-gas interface. The interface energy is defined as an excessive internal energy per unit area induced by the density gradient in the interface zone. Introducing a gradient terms to describe the liquid-gas interface is originate from the study of van der Waals in 1893\(^{[32]}\). Let us redefine the internal energy as a function of $\rho$, $\nabla \rho$, and $s$,

$$\rho \epsilon + E. \quad (71)$$

Here, $E(\rho, \nabla \rho)$ is the interface internal energy of a substance, which is determined by the intermolecular forces, and then scarcely depends on the entropy\(^{[16]}\). We assume that $E(\rho, \nabla \rho)$ is isotropic, i.e.,

$$\frac{\partial E}{\partial \rho} \delta \rho = \frac{\partial E}{\partial \partial_j \rho} \delta \rho. \quad (72)$$

Using Eq. (71), we give the Lagrangian density by

$$\mathcal{L} = \rho \left( \frac{1}{2} (v^2) - \epsilon \right) - E. \quad (73)$$

The equation of motion can be obtained by minimizing the integral of Eq. (73) over the considered space and time subject to the constraints \(^{[31]}\), \(^{[43]}\) and the constraints for entropy \(^{[29]}\). The variation of the interface internal energy $E(\rho, \nabla \rho)$ is

$$\begin{align*}
- \frac{\partial E}{\partial \rho} \delta \rho - & \left( \frac{\partial E}{\partial \partial_j \rho} \partial_j \delta \rho \right) \\
= & \left\{ - \frac{\partial E}{\partial \rho} + \partial_j \left( \frac{\partial E}{\partial \partial_j \rho} \right) \right\} \delta \rho \\
- & \partial_j \left( \frac{\partial E}{\partial \partial_j \rho} \delta \rho \right). \quad (74)
\end{align*}$$

In terms of the variational calculus, the value of $\rho$ is not fixed at boundary $\partial V$, and the surface term in Eq. (74),

$$\partial_j \left( \frac{\partial E}{\partial \partial_j \rho} \delta \rho \right), \quad (75)$$
does not vanish in the variational calculus by itself. In the variational calculus, the values of state variables at the boundary are usually fixed, and then the resultant equations satisfy these boundary conditions. However, in the case of the fluids with interface, fixing the values of \( \rho \) at the boundary \( \partial V \) is an excessive boundary condition for the resultant equations, i.e., yields no resultant weak solution minimizing the action subject to the nonholonomic constraint [29]. Therefore, the equation of entropy (29) is given so as to erase the surface term (75). This is the necessary condition for the existence of the weak solution in the variational calculus.

Without loss of generality, the equation for entropy is given in the form of

\[
\rho D_t s = \frac{1}{T} (\Theta - \nabla \cdot J_q) - \nabla \cdot J_s. \tag{76}
\]

Here, \( D_t \equiv \partial / \partial t + v \cdot \nabla \) is the Lagrangian time derivative, \( \Theta \geq 0 \) is the dissipation function, and \( J_q \) is heat flux. The entropy flux \( J_s \) is determined to eliminate the surface term (75) and preserve conservation laws of energy and momentum. We find that if the entropy flux \( J_s \) is given by

\[
J_s = \frac{1}{T} \frac{\partial E}{\partial \rho} D_t \rho, \tag{77}
\]

the these three requirements above are satisfied. Let us make sure of it. First of all, we discuss the momentum conservation law associated with space translation symmetry. Replacing the time derivative \( \partial_t \) by \( d \) of \( T \nabla \cdot J_s \) where \( J_s \) is given by Eq. (77) yields

\[
T \partial_j \left\{ \frac{1}{T} \frac{\partial E}{\partial \rho} (d \rho + (\partial_k \rho) dX_k) \right\}, \tag{78}
\]

with the aid of Eq. (31). As we discussed in Eq. (31), replacing \( d \) by \( \partial_t \) make Eq. (78) vanish because this replacement of \( d \rho + (\partial_k \rho) dX_k \) yields \( (\partial_t - \delta_j \partial_j) \rho = 0 \) with the aid of Eq. (10). The term \( \nabla \cdot J_s \) turns into zero because \( D_t \rho \) in (77) vanishes in the replacement. The other terms in Eq. (76) also vanish as shown in Eq. (11). Thus Eq. (76) where \( J_s \) is given by Eq. (74) have the space translation symmetry. Next, let us confirm that Eq. (77) eliminates the surface term (75) in Eq. (71). Replacing the time derivative \( d \) by \( \delta \) of Eq. (78) yields

\[
\delta_j \left\{ \frac{\partial E}{\partial \rho} (\delta \rho + (\partial_k \rho) \delta X_k) \right\}
-(\partial_j T) \left\{ \frac{1}{T} \frac{\partial E}{\partial \rho} (\delta \rho + (\partial_k \rho) \delta X_k) \right\}. \tag{79}
\]

Then the first term in the bracket at the first line in (79) cancels the surface term (76). Finally, let us confirm the consistency of the energy conservation law and the second low. If \( \Theta \) and \( J_q \) are respectively given in the form of Eqs. (57) and (56), the constraint for entropy given by Eqs. (76) and (77) preserve the energy conservation law and satisfies the second low, as we discussed in 3.1.2. Thus we find out that the appropriate equation of entropy for the Lagrangian (73). If we define the interface internal energy as \( E = (K |V \rho|^2)/2 \), where \( K \) is a function of \( \rho \), Eq. (77) is equivalent to the theory which Omuki named “The Dynamic van der Waals model” [18, 19]. The properness of the Omuki’s nontrivial entropy flux (77) is confirmed in terms of the necessary condition of the existence of the weak solution.

The entropy flux (77) is related to the latent heat transfer across the liquid-gas interface. For example, suppose that a drop of water shrinks by evaporation, and then its surface area also decreases. The entropy of liquid area transfers to gas area across the interface. This motion involves the energy transfer, because the increasing and decreasing of entropy causes the ones of the internal energy. Remind that the entropy flux (77) is derived from the interface internal energy. In micro view, the high energy molecules liquid can go into gas phase only through surface and take energy out of surrounding molecules. Thus vaporization takes the latent heat from liquid to gas phase, i.e., entropy. The energy required to free an molecule from the liquid is equivalent to the energy needed to break the molecular bonds and keep molecules away from surface. Therefore, the intermolecular forces that determine the interface internal energy of a substance are the same as those that determine its latent heat and boiling point. As might be expected, substances such as metals with high boiling points usually high interface internal energy as refer-
eed in [16]. The entropy flux (77) clarifies the relation between interface internal energy and entropy flux.

### 3.3.2 Dissolution in a two-component fluid

We consider a two-component fluid composed of two substances: solute and solvent. The mass densities $\rho_{\text{solute}}$ and $\rho_{\text{solvent}}$ are convected by the mass average velocity $\mathbf{v}$, and the diffusion flux $\mathbf{j}_c$, as

$$\partial_t \rho_c + \nabla \cdot (\rho_c \mathbf{v} + \mathbf{j}_c) = 0, \quad (80)$$

where $c$ denotes solute or solvent. The diffusion flux $\mathbf{j}_c$ denotes the amount of the component transported by diffusion through unit area in unit time. When the diffusion occurs, besides the flux $\rho_c \mathbf{v}$ of the component in equation as it moves with the fluid, there is another flux which results in the transfer of the components even when the fluid as a whole is at rest. Note that the diffusion flux $\mathbf{j}_c$ satisfies $\mathbf{j}_\text{solute} = -\mathbf{j}_\text{solvent}$ because the sum of each mass conservation law (80) yields Eq. (81), where $\rho \equiv \rho_{\text{solute}} + \rho_{\text{solvent}}$ is the total mass density. Hereafter we write $\mathbf{j}$ for $\mathbf{j}_\text{solute}$. Therefore, the conservation laws (80) are rewritten into the set of Eq. (81) and the mass conservation law of the solute

$$\rho D_t \psi + \nabla \cdot \mathbf{j} = 0, \quad (81)$$

where $\psi \equiv \rho_a / \rho$ is the mass fraction of the solute. The diffusion flux $\mathbf{j}$ describes the relative motion of the solute and the solvent. Let $\mathbf{a}$ be the amount of the solute flowing through the unit interface orthogonal to the direction of $\mathbf{j}$, i.e.,

$$D_t \mathbf{a} - \mathbf{j} = 0. \quad (82)$$

We also fix the value of $\mathbf{a}$ at the boundary. The main purpose is to obtain the equations for $\mathbf{v}$ and $\mathbf{j}$ from the variational principle. We define the specific bulk internal energy $\epsilon$ as the function of $\rho$, $\psi$, and $s$. Here, $s$ is the specific entropy of the two-component fluid. Thus we have

$$d\epsilon = -P d\rho^{-1} + \mu d\psi + T ds, \quad (83)$$

in the thermodynamics. Pressure $P$ and temperature $T$ are defined as $P \equiv \rho^2 (\partial \epsilon / \partial \rho)_{s,\psi}$ and $T \equiv (\partial \epsilon / \partial s)_{\rho,\psi}$, where the subscripts $s$, $\rho$ and $\psi$ indicate variables fixed in the respective partial differentiations. The coefficient $\mu \equiv (\partial \epsilon / \partial \psi)_{s,\rho}$ is an appropriately defined chemical potential of mixture, $\mu = \mu_{\text{solute}}/m_{\text{solute}} - \mu_{\text{solvent}}/m_{\text{solvent}}$, where $\mu_{\text{solute}}$ and $\mu_{\text{solvent}}$ are the chemical potentials of the two substances, and $m_{\text{solute}}$ and $m_{\text{solvent}}$ are the masses of the two kinds of the particles as in Sec. 58 of Ref. [30].

We write $E$ for the interface energy density given as the function of $\rho$, $\psi$, and $\nabla \psi$, and assume that $E$ is isotropic, i.e.,

$$\frac{\partial E}{\partial \partial \psi} \partial \psi = \frac{\partial E}{\partial \partial \mathbf{j}} \partial \mathbf{j}. \quad (84)$$

as same as Eq. (72). The internal energy density is the sum of the bulk energy density and the interface energy density, i.e., Eq. (77). Here, $\epsilon$ is a function of $\rho$, $\psi$, and $s$. On the other hand, the kinetic energy density is given by

$$\sum_c \frac{1}{2} \rho_c \mathbf{v}_c^2, \quad (85)$$

where $\mathbf{v}_c$ is defined as $\mathbf{v}_c = (\rho_c \mathbf{v} + \mathbf{j}_c)/\rho$. We can rewrite Eq. (85) into

$$\frac{1}{2} \rho \mathbf{v}^2 + \frac{1}{2} \rho \left( \frac{1}{\psi} + \frac{1}{1 - \psi} \right) \mathbf{j}^2. \quad (86)$$

Thus the total kinetic energy density (86) is given by the sum of the kinetic energies associated to the mass average velocity and the diffusion fluxes of the solute and the solvent. The Lagrangian density $\mathcal{L}$ is given by $\frac{80}{80}$ minus Eq. (71), i.e.,

$$\mathcal{L} \equiv \rho \frac{1}{2} \mathbf{v}^2 + \frac{1}{2} \rho \left( \frac{1}{\psi} + \frac{1}{1 - \psi} \right) \mathbf{j}^2 - (\rho \mathbf{v} + E). \quad (87)$$

Next, let us discuss the thermodynamics. Considering the translation symmetries in time and space, which are respectively associated with the conservation laws of energy and momentum, the equation of the entropy is given in the form of

$$\rho D_t s = \frac{1}{T} (\sigma_{ij} \partial_i v_j - \nabla \cdot \mathbf{J}_q + \mathbf{v} \cdot D_t \mathbf{a}) - \nabla \cdot \mathbf{J}_s. \quad (88)$$


Here $\sigma$ and $\nu$ are coefficients, and $J_\phi$ is heat flux. Note that $v_j$ is the function of $\partial_i A_i$ and $\partial_j A_i$ from Eq. (31). We determine $J_s$ as

$$J_s = \frac{1}{T} \frac{\partial E}{\partial \psi} D_t \psi \quad (89)$$

to erase the surface term without the fixing boundary condition with respect to $\psi$,

$$\partial_j \left( \frac{\partial E}{\partial \psi} \delta \psi \right), \quad (90)$$

appearing in the variational calculus of the Lagrangian (87). Here $\partial E/\partial \psi$ takes large absolute value at the interface, and $D_t \psi$ expresses the moving of the interface. Thus Eq. (89) shows that entropy flux occurs with accompanying the moving interface, which is related to the heat of dissolution (16). The coefficient $\sigma$ is a symmetric tensor because of the rotational symmetry corresponding to the conservation law of angular momentum. We can rewrite Eq. (88) in the form of Eq. (29). Then $\Theta$ and $J$ are respectively given by

$$\Theta = \frac{\sigma_{ij} e_{ij}}{T} + \frac{\nu_j}{T} + J_q \cdot \nabla \left( \frac{1}{T} \right), \quad (91)$$

$$J = \rho s \nu + \frac{J_q}{T} + J_s. \quad (92)$$

Here $e_{ij} = (\partial_i v_j + \partial_j v_i)/2$ is the strain rate tensor. We determine $\sigma$, $\nu$, and $J_q$ to make Eq. (91) positive because of the second law of thermodynamics. In the low degree approximation, Eq. (91) is given by the quadratic form of $e_{ij}$, $J$, and $\nabla T$. Without loss of generality, we have

$$\nu = \xi j + \eta \nabla T, \quad (93)$$

$$J_q = -\eta T \nabla \xi j - \lambda \nabla T, \quad (94)$$

where $\xi$ is the coefficient of friction for the diffusion flux $j$, and $\lambda$ is the coefficient of thermal conductivity. The coefficient $\eta$ in Eq. (93) expresses the Soret effect describing the flow of the solute induced by a temperature gradient. On the other hand, the coefficient $\eta$ in Eq. (94) shows the Dufour effect describing the energy flux due to the diffusion flux $j$ occurring. The both of $\eta$ in Eqs. (93) and (94) expresses coupled effects of irreversible processes. The coefficients $\xi$, $\eta$, and $\lambda$ are determined to make Eq. (91) positive (17).

The action is given by the integral of Eq. (87) over the considered time and space. By solving the stationary condition of the action with subject to Eqs. (31), (33), (81), (82), and (88), we obtain the equations of motion for the mass average velocity $v$, and the diffusion flux $j$. The former is

$$\frac{\partial}{\partial t} (\rho v_j) + \partial_j (\rho v \nu_j + \Pi_{ij} + \sigma_{ij}) = 0, \quad (95)$$

where we use Eq. (84), and write $\Pi_{ij}$ for

$$\Pi_{ij} = \left( P + \rho \frac{\partial E}{\partial \rho} - E \right) \delta_{ij} + \frac{\partial E}{\partial \psi} \frac{\partial \psi}{\partial \psi} \psi. \quad (96)$$

The latter is

$$D_t \left\{ \frac{1}{\rho} \frac{1}{\psi} + \frac{1}{1 - \psi} \right\} j \right\} - \nabla \left\{ \mu^* + \frac{1}{2\rho^2} \left( \frac{1}{\psi^2} \psi_\psi - \frac{1}{(1 - \psi)^2} \right) \psi^2 \right\} - \nu, \quad (97)$$

where $\mu^*$ is the generalized chemical potential defined as

$$\mu^* \equiv \mu + \frac{1}{\rho} \frac{\partial E}{\partial \psi} - \frac{T}{\rho} \frac{\partial}{\partial \psi} \left( \frac{1}{T} \frac{\partial E}{\partial \psi} \right). \quad (98)$$

See the details in App. C. If the diffusion flux $j$ is static and small, we have

$$j = -\frac{1}{\xi} \nabla \mu^* - \frac{\eta}{\xi} \nabla T, \quad (99)$$

from Eqs. (93) and (97). The equation (99) shows that the diffusion flux $j$ occurs in response to the gradients of the generalized chemical potential $\mu^*$ and the temperature $T$. The third term in the right-hand side in Eq. (98) shows that the interface energy prevents the dissolution of the solute, when the temperature $T$ is low. For example, in a dilute solution, $\mu$ is given by $\mu = \mu^0 + k_BT \ln \psi$, where $\mu^0$ is a constant. Let the interface energy be

$$E = \rho \psi \frac{1}{2} (\nabla \psi)^2.$$

We assume that the total mass $\rho$ is homogeneous. In this simple model, the diffusion flux is governed by

$$j = -D \nabla \psi - \psi \nabla (\Delta \psi) - \frac{\eta}{\xi} \nabla T, \quad (100)$$

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where the diffusion coefficient $D$ is
\[
D = \frac{k_B T}{\xi \psi} - \frac{1}{\xi} \frac{1}{2} (\nabla \psi)^2 + \Delta \psi + \frac{\nabla T}{T}.
\]

The diffusion flux $j$ occurs in response to a concentration gradient. When the temperature $T$ is high, the diffusion coefficient is positive, and the solute moves from areas of high concentration to low concentration. On the other hand, when the temperature $T$ is low, $D$ is sensitive to the gradient terms of the concentration $\psi$ and the temperature $T$. This effect is different from the Soret effect. We obtain all the equations describing the whole dynamics of the two-component fluid. The interface energy plays the important role in thermodynamics and dissolution as shown in Eqs. (89) and (98), respectively. Previous theories based on the free energy [1, 2] assume a constant temperature and no heat flux in these theories, and cannot derive the entropy flux (89) and the generalized chemical potential (98).

4 Discussion

We propose a new theoretical method for fluids based on the variational principle. If we have a Lagrangian, we can find a proper equation for the entropy, and obtain the equation of motion by the variational principle. Our method shows that the dynamics of fluids are given by the two principles. The first one is the variational principle, stating that the realized motion minimizing an action. The second one is the second law of thermodynamics. The equations of motion for a fluid can be derived by the stationary condition of the action subject to the equation of entropy (29). In this way, the kinematics and thermodynamics are combined through Eq. (29) in the variational calculus.

The equation of entropy is given in a differential form, and determined to satisfy the following three requirements:

1. The symmetries associated with conservation laws,
2. The second law of thermodynamics,
3. The necessary condition for the existence of the weak solution in the variational calculus.

The first one is that the resultant equations have to satisfy the conservation laws. The second one denotes that the total entropy is increasing in the dynamics of the fluid. The third one means specifically erasing the surface terms without fixing boundary conditions in the variational calculus.

The details are given in the followings. By Noether’s theorem, conservation laws are related to the symmetries. Therefore the Lagrangian and the equation of entropy (29) have to satisfy these symmetries. As discussed in Sec. 3.1.2, the exact form of Eq. (29) is determined to satisfy these symmetries and the second law of thermodynamics. In Sec. 3.2, we applied this method to LCs, and clarifies the mechanism of the rotation of chiral LCs different reported in [13]. A mirror-symmetry breaking term in the internal energy $\epsilon$ yields the rotation of the chiral LC whose helical axis is perpendicular to the gradient mass and temperature as explain in Eq. (70). Note that this mechanism is different from the cross effect in Eqs. (64) and (65) known as the Lehmann effect.

In Sec. 3.3, we discussed the relation entropy flux $J$ and the interface energy. Interface energy involves heat associated phase transitions, such as liquid-gas transition, and dissolution. To obtain the governing equations, we have to erase the surface term without the fixing boundary conditions with respect $\rho$ or $\psi$, i.e., Eqs. (75) or (90).

It is sometimes difficult to find the proper equations of conservation laws. In our method, the derived equations of motion satisfies these conservation laws by considering the corresponding symmetries. We find that the equation of entropy is systematically given by considering the symmetries, the second laws and the necessary condition of the existence of the weak solution.

Our proposed method can be applied to various more complicated fluids, and yields the governing equations consistent with the conservation laws and thermodynamics. What are required in our theory are the kinetic and the internal energy densities. The exact form of Eq. (29) is determined to satisfy the three requirements above in this section. The equa-
tions of motion are derived from the variational principle with the aid of Eq. (29).

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A Symmetries of Lagrangians

In this section, we give some examples of the Lagrangians which have symmetries, and the conservation laws associated with these symmetries.

A.1 Time translation symmetry

The most basic symmetry is the time translation symmetry, which is connected with energy conservation law. Let us define the time shifting of \( r \) in near future \( \alpha \) by

\[
\begin{align*}
    r'(t') &= r(t + \alpha) = r(t) + \alpha \frac{dr(t)}{dt} \quad (101) \\
    t' &= t + \alpha \quad (102)
\end{align*}
\]

If the Lagrangian is independent of the base time, it has the identity

\[
\tilde{L}(r'(t')) = \tilde{L}(r(t)) - \alpha \frac{d}{dt} \tilde{L}(r(t)), \quad (103)
\]

under the infinitesimal time translation, given by Eqs. (101) and (102). We also assume that the time translation satisfies Eq. (3), i.e.,

\[
\alpha \left\{ T \frac{ds}{dt} + \int \frac{dq}{st} + Q \right\} = 0, \quad (104)
\]

Using the same procedure in Sec. 21.2, we have

\[
\frac{dH}{dt} - Q = 0. \quad (105)
\]

Here we use \( H(q, p, s) \equiv \tilde{H}(q, p, u^*, s) = -\tilde{L} + pF = -\tilde{L} + p(q, s) \) on the condition that \( u^* \) satisfies (12).

A.2 Space translation symmetry

Let us consider the Lagrangian,

\[
\tilde{L}(q_1 - q_2, \dot{q}_1, \dot{q}_2, p_1, p_2, u_1, u_2, s).
\]

It is invariant, i.e., \( \tilde{L}(r') = \tilde{L}(r) \) under the transformation,

\[
q_i'(t) = q_i(t) + \alpha, \quad \text{where} \ i = 1, 2 \quad (106)
\]

and the rest \( (p_i', u_i', s', t') \) remain the same. If the coefficients of \( Tds + f dq + Qdt = 0 \) satisfies

\[
f_1 + f_2 = 0, \quad (107)
\]

we have the conservation law of momentum \( p_1 + p_2 \) = constant.

A.3 Galilean symmetry

Let us give the Lagrangian by

\[
\tilde{L}(q_1 - q_2, \dot{q}_1, \dot{q}_2, p_1, p_2, u_1^2, u_2^2, s),
\]

and consider the Galilean transformation,

\[
\begin{align*}
    q_i'(t) &= q_i(t) + \alpha t \quad (108) \\
    p_i'(t) &= p_i(t) + m_i \alpha \quad (109) \\
    u_i'(t) &= u_i(t) + \alpha \quad (110)
\end{align*}
\]

where \( i = 1, 2 \) and \( m_i = (\partial L / \partial u_i^2) \), and the rest \( (s', t') \) remain the same. We have

\[
\tilde{L}(r') = \tilde{L}(r) + \alpha m_i \left( u_i + \frac{\partial L}{\partial p_i} \right) = 0. \quad (111)
\]
In the case of \( \partial \tilde{L} / \partial p_i = \dot{q}_i - u_i \), and \( m \) is constant, the rhs of Eq. (111) becomes \( ad(m_i q_i)/dt \), and then the Lagrangian is invariant. If we have Eq. (107), we obtain the conservation conservation of center of momentum

\[
d \frac{d}{dt}(M_{CM} - Pt) = 0, \tag{112}
\]

where we define \( M \equiv m_1 + m_2 \), \( P \equiv p_1 + p_2 \), and \( q_{CM} \equiv (m_1 q_1 + m_2 q_2)/M \).

We can apply the calculus to a one-component fluid. The second term in rhs of Eq. (111) corresponds

\[
\rho v_i + \rho \left( \frac{\partial A_i}{\partial t} - v \cdot \nabla A_i \right). \tag{113}
\]

If the mass conservation law (34) is satisfied, Eq. (113) can be rewritten into

\[
\frac{\partial}{\partial t} + \nabla \cdot v \left( \rho x_i + A_i \right),
\]

and therefore the Lagrangian (32) satisfies Galilean symmetry.

## C Mass conservation law

Calculating by means of the cofactors yields

\[
\frac{\partial J^{-1}}{\partial A_i} = J^{-1} \frac{\partial x_j}{\partial A_i}, \tag{117}
\]

while some algebra yields

\[
\partial_j \left( J^{-1} \frac{\partial x_j}{\partial A_i} \right) = 0. \tag{118}
\]

Using Eqs. (117) and (118), we have

\[
\delta J^{-1} = J^{-1} \frac{\partial x_j}{\partial A_i} \frac{\partial \delta A_i}{\partial x_j}.
\]

Because the initial mass density \( \rho_{init} \) is the function of \( A_i \), then we have

\[
\delta \rho_{init} = \frac{\partial \rho_{init}}{\partial A_i} \delta A_i = \frac{\partial \rho_{init}}{\partial x_j} \frac{\partial \delta A_i}{\partial x_j}.
\]

We obtain

\[
\delta \left( \frac{\partial (A_1, A_2, A_3)}{\partial (x_1, x_2, x_3)} \right) = \frac{\partial \rho_{init}}{\partial A_i} \delta A_i J^{-1} + \rho_{init} \delta J^{-1} = \frac{\partial \rho_{init}}{\partial A_i} \delta A_i.
\]

Replacing \( \delta \) by \( \partial / \partial t \), Eq. (122) yields the well known mass conservation law (34) with the aid of Eq. (31). The change of the mass density \( \rho \) does not depend on the historical values of \( A \). We call the constraints like Eq. (33) holonomic constraints as we discussed in App. B. Multiplying Eq. (122) by the undermined multiplier \( K \), we obtain

\[
\int_v dx^3 \left\{ K \delta \rho + \rho \frac{\partial \delta A_i}{\partial x_j} \frac{\partial x_j}{\partial A_i} \right\} = 0. \tag{123}
\]
with the aid of the partial integral and the boundary condition.

D The Navier Stokes equations

Subtracting Eq. (12) from the variation of Eq. (38) yields

\[
\int_{t_{init}}^{t_{fin}} dt \int_V d^3x \left[ (\rho v + \beta_j \nabla A_j) \cdot \delta v \right. \\
+ \left( \frac{1}{2} \rho v^2 - h + K \right) \delta \rho + \left( \frac{\partial \beta_i}{\partial t} + \partial_j (\beta_i v_j) \right) \delta A_i \\
+ \left( \frac{\partial \beta_i}{\partial t} + \partial_j (\beta_i v_j) - \rho \partial_j K - \partial_k \sigma_{jk} + T \rho \partial_j \delta A_i \right) \delta A_i \\
+ \left( \rho - \rho_{init} \right) \delta K + (\partial_t A_i + v \cdot \nabla A_i) \delta \beta_i \\
+ \partial_k \left\{ \left( \sigma_{jk} \frac{\partial x_j}{\partial A_i} + \rho K \frac{\partial x_j}{\partial A_i} \right) \delta A_i \right\} + \frac{\partial}{\partial t} (\beta_i \delta A_i) \right] \]

The last line in Eq. (124) is surface term and vanishes because of Eqs. (35) and (36). Then, we obtain Eqs. (31), (33) and the followings,

\[
v + \frac{\beta_j}{\rho} \nabla A_j = 0, \tag{125}
\]

\[
\frac{1}{2} v^2 - h + K = 0, \tag{126}
\]

\[
\frac{\partial \beta_i}{\partial t} + \partial_j (\beta_i v_j) - (\rho \partial_j K - \partial_k \sigma_{jk} - \rho T \partial_j \delta A_i) \frac{\partial x_j}{\partial A_i} = 0 \tag{127}
\]

Multiplying (125) by \( L_v \equiv \{ \partial_i + \nabla (v \cdot) - v \times \nabla \times \} \) yields

\[
\partial_t v + v (\nabla \cdot v) - \nabla \times (v \times v) + \frac{1}{\rho} D_i \beta_i \nabla A_i = 0, \tag{128}
\]

with the aid of Eq. (31) \[24, 33\]. Here, \( L_v \) denotes the convected time derivative for the cotangent vector, and considered as the Lie derivative in mathematics \[33\]. Substituting Eq. (127) into Eq. (128), we obtain the equation of motion,

\[
\frac{\partial}{\partial t} (\rho v_j) + \partial_k (\rho v_j v_k + P \delta_{jk} + \sigma_{jk}) = 0. \tag{129}
\]

Using the same procedure in Sec. 2.1.2 we obtain

\[
\frac{\partial}{\partial t} (\beta_i \partial_t A_i - \mathcal{L}) + \partial_j \left\{ \beta_i v_j \partial_t A_i + \left( \rho K \delta_{jk} - \sigma_{jk} \right) \frac{\partial x_k}{\partial A_i} \right\} + Q = 0, \tag{130}
\]

with the aid of (124). Doing some calculations of Eq. (130) yields Eq. (43).

E The Ericksen-Leslie equations

The stationary condition of Eq. (67) subject to the constraint (60) regenerates Eqs. (31), (33), (58), and yields

\[
\delta v : \rho v + \beta_i \nabla A_j + \lambda_j \nabla n_j = 0, \tag{131}
\]

\[
\delta \omega : \rho I_{ij} \omega_j - \lambda_i = 0, \tag{132}
\]

\[
\delta \rho : \frac{1}{2} \rho v^2 + \frac{1}{2} \omega^t I \omega - h + K = 0, \tag{133}
\]

\[
\delta A_i : \frac{\partial \beta_i}{\partial t} + \partial_j (\beta_i v_j) - \rho \partial_j K - \partial_k \sigma_{jk} - \rho T \partial_j \delta A_i = 0, \tag{134}
\]

\[
\delta n_i : \partial_j (\rho \sigma_{ij}) - \left\{ \frac{\partial \lambda_i}{\partial t} + \partial_j (\lambda_i \nu_j) \right\} + g_i = 0. \tag{135}
\]

The equation (135) becomes

\[
\frac{\partial \beta_i}{\partial t} + \partial_j (\beta_i v_j) = \left\{ -\rho \partial_j K - \rho T \partial_j s - \partial_k \sigma_{jk} - g_k \partial_j n_k \right\} \frac{\partial x_j}{\partial A_i}, \tag{137}
\]

with aid of Eq. (133) and

\[
\partial_j h = \partial_j \left( \epsilon + \frac{P}{\rho} \right) = \frac{1}{\rho} \partial_j P + T \partial_j s + \pi_{lm} \partial_j \partial_l n_m,
\]

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The Lie derivative $L_v$ of Eq. (131) yields
\[ \rho L_v v + (L_v \beta_i) \nabla A_i + L_v (\lambda_i \nabla n_i) = 0, \] (138)
with the aid of Eqs. (31) and (68). Using Eqs. (132) and (136), we can rewrite the third terms of (138) into
\[ L_v (\lambda_i \nabla n_i) = (L_v \lambda_i) \nabla n_i + \lambda_i \nabla (L_v n_j) = (\partial_j (\rho \pi_{ij}) + g_i) \nabla n_i - \lambda_i \nabla \omega j \]
and (136), we can rewrite the third terms of (138) with the aid of Eqs. (31) and (58). Using Eqs. (132) and (125), and the followings
\[ L \lambda_i \nabla \omega j \]
\[ \partial \rho \delta v = -P \delta \omega j - \rho \pi_{ki} \partial \omega j n_k + \sigma_{ij}. \] (141)
The Lie derivative $L_v \equiv \partial_t + \nabla \cdot (v )$ of Eq. (132) yields the equation of the angular velocity of a fluid (68). The set of Eqs. (68) and (140) is exactly the Ericksen-Leslie equations.

Then we have the equation of motion,
\[ \frac{\partial}{\partial t} (\rho v_j) + \partial_k (\rho v_j v_k + \Pi_{jk} + \sigma_{jk}) = 0. \] (143)
Here we write $\Pi_{jk}$ for
\[ \left\{ P - \rho T \partial_i \left( \frac{1}{T} \frac{\partial E}{\partial \partial \rho} \right) + \rho \frac{\partial E}{\partial \rho} - E \right\} \delta_{jk} + \frac{\partial E}{\partial \partial \rho} \partial_k \rho \]
and use the identities, (122) and
\[ \left\{ \frac{\partial E}{\partial \rho} - \partial_j \left( \frac{\partial E}{\partial \partial \rho} \right) \right\} \partial_k \rho = \partial_j \left( E \delta_{jk} - \frac{\partial E}{\partial \partial \rho} \partial_k \rho \right). \]

\section{G The equation of motion for a two-component fluid}

We use the same procedure in Sec. 3.3.1. We give the Lagrangian to be minimized as the sum of Eq. (37) and
\[ \gamma (\rho D_t \psi + \nabla \cdot j) + b \cdot (D_t a - j), \] (144)
where $\gamma$ and $b$ are undetermined multipliers. Solving the stationary condition of (38) with subject to the nonholonomic condition (68) yields Eqs. (125), (51), (33), and the followings
\[ \delta v : \rho v + \beta_j \nabla A_j + \rho \gamma \nabla \psi + b_j \nabla a_j = 0, \] (145)
\[ \delta \rho : \frac{1}{2} v^2 - h - \frac{\partial E}{\partial \rho} + K \]
\[ + \gamma D_t \psi - \frac{1}{2} \rho \frac{1}{\psi} \left( \frac{1}{\psi} + \frac{1}{1 - \psi} \right) j^2 = 0, \] (146)
\[ \delta A_i : \frac{\partial \beta_i}{\partial t} + \partial_k (\beta_i v_k) + \rho \partial_j T \left( \frac{1}{T} \frac{\partial E}{\partial \partial \rho} \right) \partial_j \rho \partial_k \rho - \partial \sigma_{jk} \frac{\partial x_j}{\partial A_i} \partial A_i = 0. \] (147)
\[ \delta j : \frac{1}{\rho} \left( \frac{1}{\psi} + \frac{1}{1 - \psi} \right) j - \nabla \gamma - b = 0 \] (148)
\[ \delta \psi : \rho \mu + \frac{\partial E}{\partial \psi} - T \partial_k \left( \frac{1}{T} \frac{\partial E}{\partial k \psi} \right) \\
+ \rho D \gamma + \frac{1}{2 \rho} \left( \frac{1}{\psi^2} - \frac{1}{(1 - \psi)^2} \right) j^2 = 0 \quad (149) \]

\[ \delta \alpha : -D \partial_t b_j + \nu_j = 0 \quad (150) \]

From Eqs. (145), (146) and (147), then we have the equation of the motion,
\[ \frac{\partial}{\partial t} (\rho v_j) + \partial_k (\rho v_j v_k + \Pi_{jk} + \sigma_{jk}) = 0. \quad (151) \]

Here we write \( \Pi_{ij} \) for \[ \Pi_{jk} = \left\{ P + \rho \frac{\partial E}{\partial \rho} - E \right\} \delta_{jk} + \frac{\partial E}{\partial \partial_j \psi} \partial_k \psi \]
and use the identities, (83) and
\[ \frac{\partial E}{\partial \rho} \partial_k \rho + \left\{ \frac{\partial E}{\partial \rho} - \partial_j \left( \frac{\partial E}{\partial \partial_j \psi} \right) \right\} \partial_k \psi = \partial_j \left( E \delta_{jk} - \frac{\partial E}{\partial \partial_j \psi} \partial_k \psi \right). \]

References

[1] J.D. van der Waals. The thermodynamic theory of capillarity flow under the hypothesis of a continuous variation in density. Verhandel. Konink. Akad. Weten. Amsterdam, 1:1–56, 1893.

[2] John W Cahn and John E Hilliard. Free energy of a nonuniform system. i. interfacial free energy. The Journal of chemical physics, 28(2):258–267, 1958.

[3] Masao Doi. Gel dynamics. Journal of the Physical Society of Japan, 78(5):052001, 2009.

[4] Clement John Adkins. Equilibrium thermodynamics. Cambridge University Press, 1983.

[5] O. Lehmann. Structur, system und magnetisches verhalten flüssiger krystalle und deren mischbarkeit mit festen. Annalen der Physik, 307(8):649–705, 1900.

[6] F. M. Leslie. Some thermal effects in cholesteric liquid crystals. Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences, 307(1490):359–372, 1968.

[7] Pierre-Gilles De Gennes and Jacques Prost. The physics of liquid crystals, volume 23. Clarendon press Oxford, 1993.

[8] N. V. Madhusudana and R. Pratibha. An experimental investigation of electromechanical coupling in cholesteric liquid crystals. Liquid Crystals, 5(6):1827–1840, 1989.

[9] P. Oswald and A. Dequidt. Lehmann effect in chiral liquid crystals and langmuir monolayers: an experimental survey. Liquid Crystals, 36(10-11):1071–1084, 2009.

[10] Patrick Oswald and Alain Dequidt. Measurement of the continuous lehmann rotation of cholesteric droplets subjected to a temperature gradient. Phys. Rev. Lett., 100:217802, May 2008.

[11] P. Oswald. About the leslie explanation of the lehmann effect in cholesteric liquid crystals. EPL (Europhysics Letters), 97(3):36006, 2012.

[12] P. Oswald and A. Dequidt. Direct measurement of the thermomechanical lehmann coefficient in a compensated cholesteric liquid crystal. EPL (Europhysics Letters), 83(1):16005, 2008.

[13] Jun Yoshioka, Fumiya Ito, Yuto Suzuki, Hiroaki Takahashi, Hideaki Takizawa, and Yuka Tabe. Director/barycentric rotation in cholesteric droplets under temperature gradient. Soft Matter, 10:5869–5877, 2014.

[14] Demonstrating storage of \{CO2\} in geological reservoirs: The sleipner and \{SACS\}.

[15] Leslie A Guildner. The thermal conductivity of carbon dioxide in the region of the critical point. Proceedings of the National Academy of Sciences of the United States of America, 44(11):1149, 1958.
[16] Jacob N Israelachvili. *Intermolecular and surface forces: revised third edition*. Academic press, 2011.

[17] Lars Onsager. Reciprocal relations in irreversible processes. *i. Phys. Rev.*, 37:405–426, Feb 1931.

[18] R. Teshigawara and A. Onuki. Droplet evaporation in one-component fluids: Dynamic van der waals theory. *EPL (Europhysics Letters)*, 84(3):36003, 2008.

[19] Akira Onuki. Dynamic van der waals theory. *Phys. Rev. E*, 75:036304, Mar 2007.

[20] Lev Semenovich Pontryagin. *Mathematical theory of optimal processes*. CRC Press, 1987.

[21] Anthony M Bloch. *Nonholonomic mechanics and control*, volume 24. Springer, 2003.

[22] Toshihiro Iwai. A geometric setting for classical molecular dynamics. 1987.

[23] Hiroki Fukagawa and Youhei Fujitani. A variational principle for dissipative fluid dynamics. *Progress of Theoretical Physics*, 127(5):921–935, 2012.

[24] Hiroki Fukagawa and Youhei Fujitani. Clebsch potentials in the variational principle for a perfect fluid. *Progress of Theoretical Physics*, 124(3):517–531, 2010.

[25] Hiroki Fukagawa. *Improvements in the Variational Principle for Fluid Dynamics (In Japanese)*. PhD thesis, Keio University, 2012. In Japanese.

[26] Yunk Yong Hyon, Do Young Kwak, and Chun Liu. Energetic variational approach in complex fluids: maximum dissipation principle. *DCDS-A*, 24(4):1291–1304, 2010.

[27] Chun Liu. *An introduction of elastic complex fluids: an energetic variational approach*. World Scientific: Singapore, 2009.

[28] B. D. Jones. Navier-Stokes Hamiltonian. *ArXiv e-prints*, July 2014.

[29] R. L. Seliger and G. B. Whitham. Variational principles in continuum mechanics. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 305(1480):1–25, 1968.

[30] L. D. Landau, E. M. Lifshitz, J. B. Sykes, and W. H. Reid. *Fluid Mechanics*. Pergamon Press Oxford, England, 1959.

[31] S. Chandrasekhar. *Liquid crystals / S. Chandrasekhar*.

[32] J.S. Rowlinson. *Journal of Statistical Physics*, 20(2):197–200, 1979.

[33] Bernard F. Schutz. *Geometrical methods of mathematical physics*. Cambridge University Press, 1984.