Deterministic Completion of Rectangular Matrices Using Asymmetric Ramanujan Graphs

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Abstract

In this paper we study the matrix completion problem: Suppose $X \in \mathbb{R}^{n_r \times n_c}$ is unknown except for an upper bound $r$ on its rank. By measuring a small number $m \ll n_r n_c$ of elements of $X$, is it possible to recover $X$ exactly, or at least, to construct a reasonable approximation of $X$? There are two approaches to choosing the sample set, namely probabilistic and deterministic. At present there are very few deterministic methods, and they apply only to square matrices. The focus in the present paper is on deterministic methods that work for rectangular as well as square matrices. The elements to be sampled are chosen as the edge set of an asymmetric Ramanujan graph. For such a measurement matrix, we derive bounds on the error between a scaled version of the sampled matrix and unknown matrix, and show that, under suitable conditions, the unknown matrix can be recovered exactly. Even for the case of square matrices, these bounds are an improvement on known results. Of course they are entirely new for rectangular matrices.

This raises the question of how such asymmetric Ramanujan graphs might be constructed. While some techniques exist for constructing Ramanujan bipartite graphs with equal numbers of vertices on both sides, until now no methods exist for constructing Ramanujan bipartite graphs with unequal numbers of vertices on the two sides. We provide a method for the construction of an infinite family of asymmetric biregular Ramanujan graphs with $q^2$ left vertices and $lq$ right vertices, where $q$ is any prime number and $l$ is any integer between 2 and $q$. The left degree is $l$ and the right degree is $q$. So far as the authors are aware, this is the first explicit construction of an infinite family of asymmetric Ramanujan graphs.

1 Introduction

1.1 General Statement

Compressed sensing refers to the recovery of high-dimensional but low-complexity objects from a small number of linear measurements. Recovery of sparse (or nearly sparse) vectors, and recovery of high-dimensional but low-rank matrices are the two most popular applications of compressed sensing. The object of study in the present paper is the matrix completion problem, which is a special case of low-rank matrix recovery. Matrix completion has been getting a lot of attention because of its application to different areas such as image processing, sketching, quantum tomography, and recommendation systems (e.g., the Netflix problem). An excellent survey of the matrix completion problem can be found in [1].

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1.2 Problem Definition

The matrix completion problem can be stated formally as follows: Suppose $X \in \mathbb{R}^{n_r \times n_c}$ is an unknown matrix that we wish to recover whose rank is known to be bounded by a known integer $r$. Let $[n]$ denote the set $\{1, \ldots, n\}$ for each integer $n$. In the matrix completion problem, a set $\Omega \subseteq [n_r] \times [n_c]$ is specified, known as the sample set. The measurements consist of $X_{i,j}$ for all $(i, j) \in \Omega$. Let us define the matrix $E_{i,j}$ to be the binary matrix with an element of 1 in the location $(i, j)$ and zeros elsewhere, and define

$$E_\Omega = \sum_{(i,j) \in \Omega} E_{i,j}.$$ 

To be specific, suppose $\Omega = \{(i_1,j_1), \ldots, (i_m,j_m)\}$, where $|\Omega| = m$ is the total number of samples. We are able to observe the values of the unknown matrix $X$ at the locations in the set $\Omega$. Then the measurement can be expressed as the Hadamard product $E_\Omega X$ where $E_\Omega \in \{0,1\}^{n_r \times n_c}$ is defined by

$$(E_\Omega)_{ij} = \begin{cases} 1 & \text{if } (i, j) \in \Omega, \\ 0 & \text{if } (i, j) \notin \Omega. \end{cases}$$

From these measurements, and the information that $\text{rank}(X) \leq r$, we aim to construct $X$ uniquely, or at least to construct a good approximation of $X$.

One possible approach to the matrix completion problem is to set

$$\hat{X} = \arg\min_{Z \in \mathbb{R}^{n_r \times n_c}} \text{rank}(Z) \text{ s.t. } E_\Omega Z = E_\Omega X. \quad (1)$$

The above problem is a special case of minimizing the rank of an unknown matrix subject to linear constraints, and is therefore NP-hard [2]. Since the problem is NP-hard, a logical approach is to replace the rank function by its convex relaxation, which is the nuclear norm, or the sum of the singular values of a matrix, as shown in [3]. Therefore the convex relaxation of (1) is

$$\hat{X} := \arg\min_{Z \in \mathbb{R}^{n_r \times n_c}} \|Z\|_N \text{ s.t. } E_\Omega Z = E_\Omega X. \quad (2)$$

It can be shown that, under suitable conditions, the unique solution to (2) is the true but unknown matrix $X$. Such results are reviewed in Section 2.

Another emerging trend is to use the so-called “max-norm” introduced in [4]. To define this norm, we begin by recalling that, if $U \in \mathbb{R}^{k \times l}$, then an induced matrix norm is given by

$$\|U\|_{2 \to \infty} := \max_{\|x\|_2 \leq 1} \|Ux\|_\infty = \max_{i \in [k]} \|u^i\|_2,$$

where $u^i$ denotes the $i$-th row of the matrix $U$. The max-norm of a matrix $X$ is defined as

$$\|X\|_m = \min_{X = UV^T} \|U\|_{2 \to \infty} \cdot \|V\|_{2 \to \infty}. \quad (3)$$

With this definition, an alternate approach to matrix completion is

$$\hat{X} := \arg\min_{Z \in \mathbb{R}^{n_r \times n_c}} \|Z\|_m \text{ s.t. } E_\Omega Z = E_\Omega X. \quad (4)$$

\(^1\)Recall that the Hadamard product $C$ of two matrices $A, B$ of equal dimensions is defined by $c_{ij} = a_{ij}b_{ij}$ for all $i, j$.  

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1.3 Contributions of the Present Paper

In the literature to date, most of the papers assume that the sample set $\Omega$ is chosen at random from $[n_r] \times [n_c]$, either without replacement as in [5], or with replacement [6]. The authors are aware of only two papers [7, 8] in which a deterministic procedure is suggested for choosing the sample set $\Omega$ as the edge set of a Ramanujan graph. (This concept is defined below).

In case $\Omega$ is chosen at random, it makes little difference whether the unknown matrix is square or rectangular. However, if $\Omega$ is to be chosen in a deterministic fashion, then the approach suggested in [7, 8] requires that the unknown matrix be square.

The reason for this is that, while it is possible to define the notion of a Ramanujan bigraph, until now there is not a single explicit construction of such a graph, only some abstract formulas that are not explicitly computable [9, 10]. One of the main contributions of the present paper is to present an infinite family of Ramanujan bigraphs; this is the first such explicit construction. Using this construction, we prove explicit deterministic procedures for choosing the sample set $\Omega$ to recover an unknown rectangular matrix, and prove bounds on the recovery error. These bounds are an improvement on the available bounds in two different ways. First, these bounds are applicable for rectangular matrices, while existing deterministic methods do not apply to this case. Second, even in the case of square matrices, our bounds improve currently available bounds. These improvements are achieved through modifying the so-called “expander mixing lemma” for bipartite graphs, which is a result that is possibly of independent interest.

In addition to developing the theory, we also study the “phase transition behavior” of nuclear norm minimization as a recovery technique, which show that the currently available sufficient conditions for matrix completion are quite far from being necessary.

2 Literature Review

In [5], the authors point out that the formulations (1) or (2) do not always recover an unknown matrix. They illustrate this by taking $X$ as the matrix with a 1 in the $(1, 1)$ position and zeros elsewhere. In this case, unless $(1, 1) \in \Omega$, the solution to both (1) and (2) is the zero matrix, which does not equal $X$. The difficulty in this case is that the matrix has high “coherence,” as defined next.

Definition 1. Suppose $X \in \mathbb{R}^{n_r \times n_c}$ has rank $r$ and the reduced singular value decomposition $X = U \Sigma V^T$, where $U \in \mathbb{R}^{n_r \times r}$, $V \in \mathbb{R}^{n_c \times r}$, and $\Sigma \in \mathbb{R}^{r \times r}$ is the diagonal matrix of the nonzero singular values of $X$. Let $P_U = U U^T \in \mathbb{R}^{n_r \times n_r}$ denote the orthogonal projection of $\mathbb{R}^{n_r}$ onto $U \mathbb{R}^{n_r}$. Finally, let $e_i \in \mathbb{R}^{n_r}$ denote the $i$-th canonical basis vector. Then we define

$$\mu_0(U) := \frac{n_r}{r} \max_{i \in [n_r]} \|P_U e_i\|_2^2 = \max_{i \in [n_r]} \|u_i\|_2^2,$$

where $u_i$ is the $i$-th row of $U$. The quantity $\mu_0(V)$ is defined analogously, and

$$\mu_0(X) := \max\{\mu_0(U), \mu_0(V)\}. \quad (6)$$

Next, define

$$\mu_1(X) := \sqrt{\frac{n_r n_c}{r}} \|UV^T\|_\infty,$$

(7)

\footnote{Though the paper [7] uses the notation $X \in \mathbb{R}^{n_r \times n_c}$, in the theorems it is assumed that $n_r = n_c$.}
It is shown in [5] that $1 \leq \mu_0(U) \leq \frac{2}{\sqrt{n}}$. The upper bound is achieved if any canonical basis vector is a column of $U$. (This is what happens with the matrix with all but one element equalling zero.) The lower bound is achieved if every element of $U$ has the same magnitude of $1/\sqrt{n}$, that is, a Walsh-Hadamard matrix.

To facilitate the statement of some known results in matrix completion, we reproduce from the literature two standard coherence assumptions on the unknown matrix $X = U\Sigma V^\top$.

(A1). There are known upper bounds $\mu_0, \mu_1$ on $\mu_0(X)$ and $\mu_1(X)$ respectively.

(A2). There is a constant $\delta_d$ such that

$$
\left\| \sum_{k \in S} \frac{n_r}{d_r} (U^k U^k) - I_r \right\|_S \leq \delta_d, \quad \forall S \subseteq [n_r], |S| = d_r,
$$

(8)

$$
\left\| \sum_{k \in S} \frac{n_c}{d_r} (V^k V^k) - I_r \right\|_S \leq \delta_d, \quad \forall S \subseteq [n_c], |S| = d_r,
$$

(9)

where $U^k$ is shorthand for $(U_k)^\top$.

2.1 Probabilistic Sampling

There are two approaches to choosing the sample set $\Omega$, namely probabilistic and deterministic. In the probabilistic approach the elements of $\Omega$ are chosen at random from $[n_r] \times [n_c]$. In this setting one can further distinguish between two distinct situations, namely sampling from $[n_r] \times [n_c]$ with replacement or without replacement. If one were to sample $m$ out of the $n_r n_c$ elements of the unknown matrix $X$ without replacement, then one is guaranteed that exactly $m$ distinct elements of $X$ are measured. However, the disadvantage is that the locations of the $m$ samples are not independent, which makes the analysis quite complex. This is the approach adopted in [5].

**Theorem 1.** (See [5, Theorem 1.1].) Draw

$$
m \geq C \max(n)^{5/4} r \log(n)
$$

(10)

samples from $[n_r] \times [n_c]$ without replacement. Then with probability at least $1 - \zeta$ where

$$
\zeta = cn^{-3}
$$

(11)

the recovered matrix $\hat{X}$ using (2) is the unique solution. Here $C, c$ are some universal constants that depend on $\mu$, and $n = \max(n_r, n_c)$.

An alternative is to sample the elements of $X$ with replacement. In this case the locations of the $m$ samples are indeed independent. However, the price to be paid is that, with some small probability, there would be duplicate samples, so that after $m$ random draws, the number of elements of $X$ that are measured could be smaller than $m$. This is the approach adopted in [6].

On balance, the approach of sampling with replacement is easier to analyze.

**Theorem 2.** (See [6, Theorem 2].) Assume without loss of generality that $n_r \leq n_c$. Choose some constant $\beta > 1$, and draw

$$
m \geq 32 \max\{\mu_1^2, \mu_0\} r (n_r + n_c) \beta \log^2(2n_c)
$$

(12)

samples from $[n_r] \times [n_c]$ with replacement. Define $\hat{X}$ as in (2). Then, with probability at least equal to $1 - \zeta$ where

$$
\zeta = 6 \log(n_c) (n_r + n_c)^{2 - 2\beta} + n_c^{2 - 2\beta} \sqrt{\beta},
$$

(13)

the true matrix $X$ is the unique solution to the optimization problem, so that $\hat{X} = X$. 

4
2.2 Basic Concepts from Graph Theory

In contrast with probabilistic sampling, known deterministic approaches to sampling make use of the concept of Ramanujan graphs. For this reason, we introduce a bare minimum of graph theory. Further details about Ramanujan graphs can be found in [11, 12].

Suppose $B \in \{0, 1\}^{n_r \times n_c}$. Then $B$ can be interpreted as the biadjacency matrix of a bipartite graph with $n_r$ vertices on one side and $n_c$ vertices on the other. If $n_r = n_c$, then the bipartite graph is said to be balanced, and is said to be unbalanced if $n_r \neq n_c$. The prevailing convention is to refer to the side with the larger ($n_c$) vertices as the “left” side and the other as the “right” side.

A bipartite graph is said to be left-regular with degree $d_c$ if every left vertex has degree $d_c$, and right-regular with degree $d_r$ if every right vertex has degree $d_r$. It is said to be ($d_r, d_c$)-biregular if it is both left- and right-regular with row-degree $d_r$ and column-degree $d_c$. Obviously, in this case we must have that $n_r d_r = n_c d_c$. It is convenient to say that a matrix $B \in \{0, 1\}^{n_r \times n_c}$ is “($d_r, d_c$)-biregular” to mean that the associated bipartite graph is ($d_r, d_c$)-biregular. The bipartite graph corresponding to $B$ is said to be an asymmetric Ramanujan graph if

$$|\sigma_2| \leq \sqrt{d_r - 1} + \sqrt{d_c - 1}. \quad (14)$$

2.3 Deterministic Sampling

The following result is claimed in [7].

**Theorem 3.** (See [7, Theorem 4.2].) Suppose Assumptions (A1) and (A2) hold. Choose $E_{t}$ to be the adjacency matrix of a $d$-regular graph such that $\sigma_2(E_{t}) \leq C\sqrt{d}$, and $\delta_d < 1/6$. Define $\hat{X}$ as in (2). With these assumptions, if

$$d \geq 36C^2\mu_0^2 r^2, \quad (15)$$

Then the true matrix $X$ is the unique solution to the optimization problem (2).

Theorems 1 and 2 pertain to nuclear norm minimization as in (2). An alternate set of bounds is obtained in [8] for max norm minimization as in (3). The matrix is assumed to be square, with $d_r = d_c = d$.

**Theorem 4.** (See [8, Theorem 2].) Suppose $E_{t}$ is the adjacency matrix of a $d$-regular graph with second largest (in magnitude) eigenvalue equal to $\lambda$. Define $\hat{X}$ as in (3). Then

$$\frac{1}{n^2}\|\hat{X} - X\|_F^2 \leq 8K_G \frac{\lambda}{d}\|X\|_m^2, \quad (16)$$

where $K_G$ is Grothendieck’s constant.

There is no closed-form formula for this constant, but it is known that

$$K_G \leq \frac{\pi}{2\log(1 + \sqrt{2})} \approx 1.78221.$$  

See [13] for this and other useful properties of Grothendieck’s constant.

Theorems 1 and 2 on the one hand, and Theorem 4 on the other hand, have complementary strengths and weaknesses. Theorems 1 and 2 ensure the exact recovery of the unknown matrix via nuclear norm minimization. However, the bounds involve the coherence of the unknown matrix as well as its rank. In contrast, the bound in Theorem 4 is “universal” in that it does not involve either the rank or the coherence of the unknown matrix $X$, just its max norm. Moreover, the bound is on the Frobenius norm of the difference $\hat{X} - X$, and thus provides an “element by element” bound. On the other hand, there are no known results under which max norm minimization exactly recovers the unknown matrix.
3 New Results

In this section we state without proof the principal new results in the paper. The proofs are given in subsequent sections.

3.1 Rationale of Using Ramanujan Bigraphs

We begin by giving a rationale of why biadjacency matrices of Ramanujan bigraphs are useful as measurement matrices. Suppose we could choose $E_{\Omega} = 1_{n_r \times n_c}$, the matrix of all ones. Then $E_{\Omega}X = X$, and we could recover $X$ exactly from the measurements. However, this choice of $E_{\Omega}$ corresponds to measuring every element of $X$, and there would be nothing “compressed” about this sensing. Now suppose that $E_{\Omega} = B$, the biadjacency matrix of a $(d_r, d_c)$-biregular graph. Then $\sigma_1 = \sqrt{d_r d_c}$ is the largest singular value of $B$, with corresponding row and column singular vectors $u_1 = (1/\sqrt{d_r})1_{n_r}$ and $v_1 = (1/\sqrt{d_c})1_{n_c}$. Let $\sigma_2$ denote the second largest singular value of $B$. Then

$$B = \sigma_1 u_1 v_1^\top + B_2,$$

where $\| \cdot \|_S$ denotes the spectral norm of a matrix (i.e., its largest singular value). Using the formulas for $u_1$ and $v_1$ and rescaling shows that

$$\sqrt{\frac{n_r n_c}{d_r d_c}} B = 1_{n_r \times n_c} + \sqrt{\frac{n_r n_c}{d_r d_c}} B_2.$$

This formula can be expressed more compactly by defining the constant $\alpha$, as

$$\alpha := \sqrt{\frac{d_r d_c}{n_r n_c}} = \frac{d_r}{n_r} = \frac{d_c}{n_c},$$

where the various equalities follow from the fact that $n_r d_r = n_c d_c$. One can think of $\alpha$ as the fraction of elements of the unknown matrix $X$ that are sampled. Since $1_{n_r \times n_c}X = X$, we see that

$$\frac{1}{\alpha} B.X = X + M.X,$$

where $M = (1/\alpha)B_2$. Therefore

$$\left\| \frac{1}{\alpha} B.X - X \right\|_S = \|M.X\|_S. \quad (17)$$

Now note that

$$\|M\|_S = \frac{\sigma_2}{\alpha} = \sigma_2 \cdot \sqrt{\frac{n_r n_c}{d_r d_c}} = \frac{\sigma_2}{\sigma_1} \sqrt{n_r n_c}.$$ 

Therefore, the smaller $\sigma_2$ is compared to $\sigma_1$, the better the approximation error is between $(1/\alpha)B.X$ and the unknown matrix $X$.\(^3\) Now, a Ramanujan graph is one for which this ratio is as small as possible. It is shown in [14] that, if $(d_r, d_c)$ are kept fixed while $(n_r, n_c)$ are increased, subject of course to the constraint that $n_r d_r = n_c d_c$, then (14) gives the best possible upper bound on $\sigma_2$.\(^3\)

\(^3\)Note that $n_r, n_c$ are the dimensions of the unknown matrix and are therefore fixed.
3.2 Error bounds using deterministic sampling

Theorem 5 below extends [7, Theorem 4.1] to rectangular matrices. It provides an upper bound on the error between a scaled version of the measurement matrix $E_{\Omega}.X$ an the true matrix $X$. Note that there is no optimization involved in applying this bound.

**Theorem 5.** Suppose the sampling set $\Omega$ comes from a $(d_r,d_c)$-regular bipartite graph, and $\sigma_2 = \sigma_2(E_{\Omega})$ represents the magnitude of the second largest singular value (and of course $\sigma_1 = \sqrt{d_r d_c}$ is the largest singular value). Suppose $X \in \mathbb{R}^{n_r \times n_c}$ is a matrix of rank $r$ or less, and let $\mu_0$ denote its coherence as defined in (6). Then

$$ \left\| \frac{1}{\alpha} E_{\Omega}.X - X \right\|_S \leq \frac{\sigma_2}{\sigma_1} \mu_0 r \|X\|_S, $$

where $\| \cdot \|_S$ denotes the spectral norm (largest singular value) of a matrix.

**Remark:** Observe that the bound in (18) is a product of two terms: $\sigma_2/\sigma_1$ which depends on the measurement matrix $E_{\Omega}$, and $\mu_0 r \|X\|_S$ which depends on the unknown matrix $X$.

**Corollary 1.** Suppose the sampling set $\Omega$ comes from a $(d_r,d_c)$-regular asymmetric Ramanujan graph, Then

$$ \left\| \frac{1}{\alpha} E_{\Omega}.X - X \right\| \leq \mu_0 r \left( \frac{1}{\sqrt{d_r}} + \frac{1}{\sqrt{d_c}} \right) \|X\|_S. $$

Theorem 6 extends [8, Theorem 2] to rectangular matrices. Even for square matrices, the bound in Theorem 6 is smaller by a factor of two compared to that in [8, Theorem 2], stated here as Theorem 4. Note that, in contrast with Theorem 5, the bound in Theorem 6 does not involve the coherence of the unknown matrix, nor its rank. Moreover, the bound is on the Frobenious norm of the difference, and is therefore an “element by element” bound, unlike in Theorem 5.

**Theorem 6.** Suppose the sampling set $\Omega$ comes from a $(d_r,d_c)$-regular bipartite graph, and let $\sigma_2$ denote the second largest singular value of its biadjacency matrix. Suppose $\hat{X}$ is a solution of (3). Then

$$ \frac{1}{n_r n_c} \left\| X - \hat{X} \right\|_F^2 \leq 4K_G \frac{\sigma_2}{\sigma_1} \|X\|_m^2 $$

where $\| \cdot \|_F$ is the Frobenious norm, $\| \cdot \|_m$ is the max norm and $K_G$ is Grothendieck’s constant.

**Corollary 2.** Suppose the sampling set $\Omega$ comes from a $(d_r,d_c)$-regular asymmetric Ramanujan graph, Then

$$ \frac{1}{n_r n_c} \left\| X - \hat{X} \right\|_F^2 \leq 4K_G \left( \frac{1}{\sqrt{d_r}} + \frac{1}{\sqrt{d_c}} \right) \|X\|_m^2 $$

The next theorem presents a sufficient condition under which nuclear norm minimization as in (2) and sampling matrix from a Ramanujan bigraph leads to exact recovery of the unknown matrix. Note that [7, Theorem 4.2] claims to provide such a sufficient condition for square matrices. However, in the opinion of the authors, there is a gap in the proof, as discussed in the Conclusions section. Therefore Theorem 7 can be thought as the first one to prove exact recovery using nuclear norm minimization and a deterministic sampling matrix.

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4Note that biregularity implies that the largest singular value is $\sqrt{d_r d_c}$. 


Theorem 7. Suppose $X \in \mathbb{R}^{n_r \times n_c}$ is a matrix of rank $r$ or less, and satisfies the incoherence assumptions A1 and A2 with constants $\mu_0$ and $\delta_d$. Suppose $E_{\Omega} \in \{0, 1\}^{n_r \times n_c}$ a biadjacency matrix of a $(d_r, d_c)$ biregular graph $\Omega$, and let $\sigma_2$ denote the second largest singular value of matrix $E_{\Omega}$. Define
\[ \theta = \delta_d, \phi = \frac{\sigma_2}{\sigma_1} \mu_0 r, \] and suppose that
\[ \theta + \phi < 1/2, \] \[ \left(1 + \frac{4}{3\sqrt{2}}\right) \phi + \theta < 1. \] Then $X$ is the unique minimum of (2).

3.3 Construction of Asymmetric Ramanujan Graphs

There are very few explicit constructions of Ramanujan graphs. The first two explicit constructions are given in [15, 16] for some choices of $(n, d)$. Two recent papers [17, 18] prove the existence of bipartite Ramanujan graphs of all degrees $d$ and all vertex sizes $n$, but do not give readily computable procedures. The paper [19] gives a supposedly polynomial-time algorithm for implementing the recipes in these papers, but it is quite opaque and does not contain any pseudocode. Explicit construction of rectangular Ramanujan graphs are even fewer. The only constructions of which the authors are aware are in [9, 10], and these are very abstract and not explicitly computable.

We now present our construction of asymmetric Ramanujan graphs with $q^2$ vertices on one side and $lq$ vertices on the other side, for every prime number $q$ and every integer $l$ between 2 and $q$. As mentioned above, we believe this is the first explicit construction of an asymmetric Ramanujan graph. Note that when $l = q$ the bipartite graph is balanced, and our construction gives another class of Ramanujan graphs. This construction is inspired by so-called array codes from LDPC (low density parity check) codes [20, 21]. The construction is as follows: Let $P$ be a prime number, and let $P$ denote the “right shift” permutation. Thus $P_i = i - 1$ and the remaining elements are all zero, with $i - 1$ interpreted modulo $q$. Define
\[ B(q, l) = \begin{bmatrix} I_q & I_q & I_q & \cdots & I_q \\ I_q & P & P^2 & \cdots & P^{(q-1)} \\ I_q & P^2 & P^4 & \cdots & P^{2(q-1)} \\ I_q & P^3 & P^6 & \cdots & P^{3(q-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ I_q & P^{(l-1)} & P^{3(l-1)} & \cdots & P^{(q-1)(l-1)} \end{bmatrix}. \] (25)

It is easily seen that the bipartite graph defined by $B$ has $q^2$ left vertices, $lq$ right vertices, $d_r = q$, and $d_c = l$. Therefore the largest singular value of $B$ is $\sqrt{ql}$.

Theorem 8. The matrix $B(q, l)$ has a singular value of $\sqrt{ql}$, $l(q-1)$ singular values of $\sqrt{q}$, and $l - 1$ singular values of 0. Therefore, whenever $2 \leq q - 1$, $B(q, l)$ defines a Ramanujan bigraph. With $l = q$, $B(q, q)$ defines a balanced Ramanujan bipartite graph.

Note that, unlike [5, 6], we do not require the constant $\mu_1$. 

8
4 Proofs

In this section we give the proofs of various theorems in the previous section. We state a couple of lemmas that are used repeatedly in the sequel. Throughout we use the notation that if $A$ is a matrix, then $A^i, A_j$ denote the $i$-th row and $j$-th column of $A$ respectively. The $ij$-th element of $A$ is denoted by $A_{ij}$.

4.1 Some Preliminary Results

**Theorem 9.** Suppose $M \in \mathbb{R}^{n_r \times n_c}, A \in \mathbb{R}^{n_r \times r},$ and $B \in \mathbb{R}^{n_c \times r}$. Suppose further that $x \in \mathbb{R}^{n_r}$, $y \in \mathbb{R}^{n_c}$. Then

$$x^T (M(AB^T))y = \sum_{k \in [r]} (x.A_k)^T M(B_k.y).$$

(26)

**Proof.** The proof follows readily by expanding the triple product. Note that

$$(AB^T)_{ij} = \sum_{k \in [r]} A_{ik}B_{jk}.$$ 

Therefore

$$x^T (M(AB^T))y = \sum_{i \in [n_r]} \sum_{j \in [n_c]} x_i \left( \sum_{k \in [r]} M_{ij} A_{ik} B_{jk} \right) y_j$$

$$= \sum_{k \in [r]} \sum_{i \in [n_r]} \sum_{j \in [n_c]} x_i A_{ik} M_{ij} B_{jk} y_j$$

$$= \sum_{k \in [r]} (x.A_k)^T M(B_k.y),$$

as desired. \qed

**Theorem 10.** Suppose $M, A, B$ are as in Theorem 9. Suppose further that

$$\|A^i\|_2^2 \leq a^2, \|B^i\|_2^2 \leq b^2.$$ 

(27)

Then

$$\|M(AB^T)\|_S \leq ab\|M\|_S.$$ 

(28)

**Proof.** Recall that, for any matrix $X$, we have that

$$\|X\|_S = \max_{\|x\|_2=1, \|y\|_2=1} x^T X y.$$ 

In particular

$$\|M(AB^T)\|_S = \max_{\|x\|_2=1, \|y\|_2=1} x^T (M(AB^T))y$$

$$= \max_{\|x\|_2=1, \|y\|_2=1} \sum_{k \in [r]} (x.A_k)^T M(B_k.y),$$

where the last step follows from Theorem 9. Now fix $x, y$ such that $\|x\|_2 = 1, \|y\|_2 = 1$. Then

$$x^T (M(AB^T))y \leq \|M\|_S \sum_{k \in [r]} \|x.A_k\|_2 \|B_k.y\|_2.$$
Therefore (28) is proved once it is established that, whenever $\|x\|_2 = 1$, $\|y\|_2 = 1$, it follows that
\[
\sum_{k \in [r]} \|x.A_k\|_2 \|B_k.y\|_2 \leq ab. \tag{29}
\]
To prove (29), apply Schwarz’ inequality to deduce that
\[
\sum_{k \in [r]} \|x.A_k\|_2 \|B_k.y\|_2 \leq \left( \sum_{k \in [r]} \|x.A_k\|_2^2 \right)^{1/2} \cdot \left( \sum_{k \in [r]} \|B_k.y\|_2^2 \right)^{1/2}. \tag{30}
\]
Now
\[
\sum_{k \in [r]} \|x.A_k\|_2^2 = \sum_{k \in [r]} \sum_{i \in [n_r]} (x_i A_{ik})^2
\]
\[
= \sum_{i \in [n_r]} x_i^2 \left( \sum_{k \in [r]} A_{ik}^2 \right)
\]
\[
= \sum_{i \in [n_r]} x_i^2 \|A^T\|_2^2
\]
\[
\leq a^2 \sum_{i \in [n_r]} x_i^2 = a^2.
\]
By entirely similar reasoning, we get
\[
\sum_{k \in [r]} \|B_k.y\|_2^2 \leq b^2.
\]
Substituting these two bounds into (30) establishes (29) and completes the proof. \qed

### 4.2 Proof of Theorem 5

**Proof.** As before, define
\[
M := (1/\alpha)E_\Omega - 1_{n_r \times n_c},
\]
and recall that
\[
M.X = (1/\alpha)E_\Omega.X - X, \|M\|_S = \frac{\sigma_2}{\alpha}.
\]
Now suppose $X = U\Phi V^\top$ is a singular value decomposition of $X$, so that $\Phi = \text{Diag}(\phi_1, \ldots, \phi_r)$. Define $A = U\Phi, B = V$. Then $X = AB^\top$. Moreover
\[
\sum_{k \in [r]} A_{ik}^2 = \sum_{k \in [r]} U_{ik}^2 \phi_k^2
\]
\[
\leq \phi_1^2 \sum_{k \in [r]} U_{ik}^2
\]
\[
\leq \|X\|_S^2 \frac{\mu_0 r}{n_r}.
\]
because $\|X\|_S = \phi_1$, and the definition of the coherence $\mu_0$. Similarly

$$\sum_{k \in [r]} B_{ik}^2 = \sum_{k \in [r]} B_{ik}^2 \leq \frac{\mu_0 r}{n_c}.$$  

Now apply Theorem 10 with

$$c = \|X\|_S \sqrt{\frac{\mu_0 r}{n_r}}, d = \sqrt{\frac{\mu_0 r}{n_c}},$$

and note that $\alpha \sqrt{n_r n_c} = \sqrt{d_r d_c} = \sigma_1$. Then (28) becomes

$$\|(1/\alpha)E_{\Omega}.X - X\|_S \leq \frac{\sigma_2}{\sigma_1} \mu_0 r \|X\|_S,$$

as desired.

4.3 Proof of Theorem 6

The proof of Theorem 6 is based on the following extension of the expander mixing lemma from [22] for rectangular expander graphs, which might be of independent interest.

**Lemma 1.** Let $E$ be the adjacency matrix of an asymmetric $(d_r, d_c)$ biregular graph with $(n_r, n_c)$ vertices so that $n_r d_r = n_c d_c$, and $\sigma_1 = \sqrt{d_r d_c}$ is the largest singular value of $E$. Let $\sigma_2$ denote the second largest singular value of $E$. Then for all $S \subseteq [n_r]$ and $T \subseteq [n_c]$, we have:

$$\left| \left| \begin{array}{c} |S| \quad |T| \\ n_r \\ n_c \end{array} - \frac{|E(S, T)|}{|E|} \right| \right| \leq \frac{\sigma_2}{\sigma_1} \sqrt{|S||T| \left(1 - \frac{|S|}{n_r}\right) \left(1 - \frac{|T|}{n_c}\right)}.$$

where $|E(S, T)|$ is the number of edges between the two vertex sets $S$ and $T$, and $|E| = n_r d_r = n_c d_c = \sqrt{n_r d_r n_c d_c}$ is the total number of edges in the graph.

**Remark:** First we explain why this result is called the “expander mixing lemma.” Note that $|S|/n_r$ is the fraction of rows that are in $S$, while $|T|/n_c$ is the fraction of columns that are in $T$. If the total number of edges $n_r d_r = n_c d_c$ were to be uniformly distributed, then the term on the left side of (31) would equal zero. Therefore the bound (31) estimates the extent to which the distribution of edges deviates from being uniform.

The above result extends [8, Theorem 8] which is adapted from [22, Lemma 2.5] to $(d_r, d_c)$ regular Ramanujan graphs. Moreover, the bound given here is tighter, because of the presence of the two square-root terms on the right side. As $|S|, |T|$ become larger, the square root terms tend to zero. No such term is present in [22, Lemma 2.5].

**Proof.** Let $1_S, 1_T$ denote the characteristic vectors of sets $S, T$ respectively. Then

$$|E(S, T)| = \sum_{u \in S, v \in T} E_{uv} = 1_S^\top E 1_T.$$  

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Write $E = \sum_{i=1}^r \sigma_i u_i v_i^\top$, and note that, due to the biregularity of $E$, we have that $u_1 = (1/\sqrt{n_r})1_{n_r}$, $v_1 = (1/\sqrt{n_c})1_{n_c}$, and $\sigma_1 = \sqrt{d_r d_c}$. Next, write $1_S = \sum_i \beta_i u_i + a$ and $1_T = \sum_j \gamma_j v_j + b$, where $a^\top, b$ belong to the row null space and column null space of $0$ respectively. Note that $\beta_1 = \langle 1_S, u_1 \rangle = |S|/\sqrt{n_r}$, and similarly $\gamma_1 = |T|/\sqrt{n_c}$. Then

$$|E(S, T)| = \left( \sum_i \beta_i u_i + a \right)^\top E \left( \sum_j \gamma_j v_j + b \right) = \frac{\sqrt{d_r d_c}}{\sqrt{n_r n_c}} |S||T| + \sum_{i=2}^r \sigma_i \beta_i \gamma_i.$$

Rearranging the above gives

$$\left| \frac{\sqrt{d_r d_c}}{\sqrt{n_r n_c}} |S||T| - |E(S, T)| \right| = \sum_{i=2}^r \sigma_i \beta_i \gamma_i \leq \sigma_2 \sum_{i=2}^r |\beta_i||\gamma_i|. \quad (32)$$

Next, by Schwarz' inequality, it follows that

$$\sum_{i=2}^r |\beta_i||\gamma_i| \leq \left( \sum_{i=2}^r \beta_i^2 \right)^{1/2} \left( \sum_{i=2}^r \gamma_i^2 \right)^{1/2}.$$

Now note that

$$\sum_{i=2}^r \beta_i^2 = \|1_S\|^2 - \alpha_1^2 = |S| - \frac{|S|^2}{n_r} = |S| \left( 1 - \frac{|S|}{n_r} \right),$$

and similarly

$$\sum_{i=2}^r \gamma_i^2 = |T| \left( 1 - \frac{|T|}{n_c} \right).$$

This implies that

$$\left| \sum_{i=2}^r \sigma_i \beta_i \gamma_i \right| \leq \sigma_2 \sqrt{|S| \cdot |T|} \left( 1 - \frac{|S|}{n_r} \right)^{1/2} \left( 1 - \frac{|T|}{n_c} \right)^{1/2}. \quad (33)$$

Substituting this into (32), dividing both sides by $\sqrt{d_r d_c n_r n_c} = |E|$, gives the first expression in (31). The second expression follows from $\sigma_1 = \sqrt{d_r d_c}$. \(\Box\)

**Theorem 11.** Suppose $R \in \mathbb{R}^{n_r \times n_c}$ and $\Omega$ is the edge set of an asymmetric $(d_r, d_c)$ biregular graph. Then

$$\left| \frac{1}{n_r n_c} \sum_{ij} r_{ij} - \frac{1}{|\Omega|} \sum_{(i,j) \in \Omega} r_{ij} \right| \leq \frac{\sigma_2}{\sigma_1} K_G \|R\|_m \quad (34)$$

where $K_G$ is Grothendieck's constant.

**Proof.** Let $M \in \mathbb{R}^{n_r \times n_c}$ be a rank 1 sign matrix with $\{1, -1\}$ entries, and define its corresponding binary matrix by $\bar{M} = 1/2(M + J)$, where $J$ is a matrix with all ones. Because $M$ is a rank 1 sign matrix, it can be expressed as $\beta \gamma^\top$, where $\beta \in \{-1, 1\}^{n_r}$ and $\gamma \in \{-1, 1\}^{n_c}$. Define $S := \{i \in [n_r] : \beta_i = 1\}, T := \{j \in [n_c] : \gamma_j = 1\}$. 

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Let $1_S$ represent the characteristic vector of set $S$. Let $S \subseteq [n_r]$ and $T \subseteq [n_c]$, and let $S^c, T^c$ denote the complements of $S, T$ in the sets $[n_r], [n_c]$ respectively. Then $\beta = 1_S - 1_{S^c}$, $\gamma = 1_T - 1_{T^c}$, and $M = 1_S1_T^\top + 1_{S^c}1_{T^c}^\top$. Therefore

$$\begin{align*}
\left| \frac{1}{n_r n_c} \sum_{i,j} m_{ij} - \frac{1}{|\Omega|} \sum_{(i,j) \in \Omega} m_{ij} \right|
&= \left| \frac{1}{n_r n_c} \sum_{i,j} (2\bar{m}_{ij} - 1) - \frac{1}{|\Omega|} \sum_{(i,j) \in \Omega} (2\bar{m}_{ij} - 1) \right|
&= 2 \left| \frac{|S||T| + |S^c||T^c|}{n_r n_c} - \frac{E(S,T) + E(S^c,T^c)}{|\Omega|} \right|
&\leq 2 \left| \frac{|S||T|}{n_r n_c} - \frac{E(S,T)}{|\Omega|} \right| + 2 \left| \frac{|S^c||T^c|}{n_r n_c} - \frac{E(S^c,T^c)}{|\Omega|} \right|
&\leq (a) \frac{4\sigma_2}{\sigma_1} \left( \sqrt{\frac{|S||T|}{n_r n_c}} \sqrt{\frac{|S^c||T^c|}{n_r n_c}} \right)
&\leq (b) \frac{\sigma_2}{\sigma_1}.
\end{align*}$$

(35)

Here, the inequality (a) comes from Lemma 1 and the inequality (b) comes from $\sqrt{xy(1-x)(1-y)} \leq 1/4$, where equality holds when $x = y$ and $x = (1-x)$.

Any real matrix $R \in \mathbb{R}^{n_r \times n_c}$ can be expressed as a sum of rank-1 sign matrices in the form $R = \sum_i \nu_i M_i$. Define

$$\|R\|_\nu := \min \sum_i |\nu_i| \text{ s.t. } R = \sum_i \nu_i M_i,$$

where the number of terms in the summation is unspecified. As stated in [8, Theorem 7] the max-norm can be related to this new norm $\| \cdot \|_\nu$ via

$$\|X\|_m \leq \|X\|_\nu \leq K_G\|X\|_m.$$  

(36)
Therefore

\[
\left| \frac{1}{n_r n_c} \sum_{i,j} r_{ij} - \frac{1}{|\Omega|} \sum_{(i,j) \in \Omega} r_{ij} \right|
\]

\[
= \left| \sum_k \nu_k \left( \frac{1}{n_r n_c} \sum_{i,j} m_{kij} - \frac{1}{|\Omega|} \sum_{(i,j) \in \Omega} m_{kij} \right) \right|
\]

\[
\leq \sum_k |\nu_k| \left| \frac{1}{n_r n_c} \sum_{i,j} m_{kij} - \frac{1}{|\Omega|} \sum_{(i,j) \in \Omega} m_{kij} \right|
\]

\[
\leq \left( \frac{\sigma_2}{\sigma_1} \right) \sum_k |\nu_k|
\]

\[
\leq \left( \frac{\sigma_2}{\sigma_1} \right) K_G \|R\|_m
\]

where \( m_{kij} \) are the \((i,j)\)-th elements of \( M_k \) for all \( k \), (a) comes from (35), and (b) comes from (36).

\[
\text{Proof. (Of Theorem 6.) If } R = (X - \hat{X})(X - \hat{X}) \text{ then,}
\]

\[
\left| \frac{1}{n_r n_c} \sum_{i,j} (x_{ij} - \hat{x}_{ij})^2 - \frac{1}{|\Omega|} \sum_{(i,j) \in \Omega} (x_{ij} - \hat{x}_{ij})^2 \right|
\]

\[
= \left| \frac{1}{n_r n_c} \sum_{i,j} (x_{ij} - \hat{x}_{ij})^2 \right| = \|X - \hat{X}\|_F^2,
\]

because \( x_{ij} = \hat{x}_{ij} \) for all \((i,j) \in \Omega\). Since the max norm is multiplicative under Hadamard product, we have

\[
\|R\|_m \leq \|(X - \hat{X})\|_m^2 \leq (\|X\|_m + \|\hat{X}\|_m)^2 = 4\|X\|_m^2.
\]

Substituting both relationships into (34) gives the desired result. \(\square\)

4.4 Proof of Theorem 8

\[
\text{Proof. The proof is based on directly computing the eigenvalues of } BB^\top, \text{ where } B \text{ is a shorthand for } B(q,l). \text{ Note that because } P \text{ is the shift permutation, we have that } P^\top = P^{-1}. \text{ We continue to use the partition notation, which shows that the } (i,j)\text{-th block of the product } BB^\top \text{ is given by}
\]

\[
(BB^\top)_{ij} = \sum_{s=1}^q P^{(i-1)(s-1)}(P^\top)^{(s-1)(j-1)}
\]

\[
= \sum_{s=1}^q P^{(i-j)(s-1)} = \sum_{s=0}^{q-1} P^{(i-j)s}.
\]

It readily follows that

\[
(BB^\top)_{ii} = qI_q, \text{ } i = 1, \ldots, q.
\]
Now observe that, for any nonzero integer $k$, the set of numbers $ks$ modulo $q$ as $s$ varies over $\{0, \ldots, q-1\}$ equals $\{0, \ldots, q-1\}$. Therefore, whenever $i \neq j$, we have that

$$(BB^\top)_{ij} = \sum_{s=0}^{q-1} P^s = 1_{q \times q},$$

where $1_{q \times q}$ denotes the $q \times q$ matrix whose entries are all equal to one. Observe that the largest eigenvalue of $BB^\top$ is $ql$, with normalized eigenvector $(1/\sqrt{ql})1_{q \times 1}$. Therefore if we define $M = BB^\top - 1_{q \times q}$ and partition it commensurately with $B$, we see that the off-diagonal blocks of $M$ are all equal to zero, while the diagonal blocks are all identical and equal to $qI_q - 1_{q \times q}$. This is the Laplacian matrix of a fully connected graph with $q$ vertices, and thus has $q-1$ eigenvalues of $q$ and one eigenvalue of 0. Therefore $M = BB^\top - 1_{q \times q}$ has $l(q-1)$ eigenvalues of $q$ and $l$ eigenvalues of 0. Finally, we conclude that $BB^\top$ has a single eigenvalue of $ql$, $l(q-1)$ eigenvalues of $q$, and $l-1$ eigenvalues of 0. This is equivalent to the claim about singular values. Verifying that the bounds for the graph to be an asymmetric Ramanujan graph is elementary algebra.

## 5 Construction of High-Degree Ramanujan Graphs

There are very few explicit constructions of Ramanujan graphs. The first and simplest (to understand) construction is due to [15], with a later one due to [16]. Two recent papers [17, 18] prove the existence of bipartite Ramanujan graphs of all degrees $d$ and all vertex sizes $n$, but do not give readily computable procedures as in [15, 16]. Another paper [19] gives a supposedly polynomial-time algorithm for implementing the recipes in these papers, but it is quite opaque and does not contain any pseudocode.

The construction in [15], referred to hereafter as the LPS construction, begins with two distinct prime numbers $p, q$ both of which are congruent to 1 mod 4. Two different types of graphs arise from this construction, depending on the value of the Legendre symbol $(p|q)$. (See [23, Chapter 9] for an introduction to the Legendre symbol.) If the Legendre symbol $(p|q) = 1$, the LPS construction returns a graph with two connected components, each being a regular graph of degree $d = p+1$ with $n = (q(q^2-1))/2$ vertices; each component is a Ramanujan graph. If the Legendre symbol $(p|q) = -1$, the LPS construction returns a bipartite regular graph of degree $d = p+1$ with $n = (q(q^2-1))/2$ vertices on each side. Each biadjacency matrix represents a Ramanujan graph. Specifically, the method consists of generating $p+1$ matrices of dimension $2 \times 2$ in the projective general linear group $PGL(2,F_q)$, where $F_q$ is the Galois field with $q$ elements. In the original construction, $p < q$. Since $n \approx q^3/2$, the original LPS construction leads to $d < n^{1/3}$, which is not large enough to apply Theorems 5 and 6. However, a perusal of the proof in [15] shows that, if each of the $p+1$ generators in the LPS construction are distinct elements of $PGL(2,F_q)$, then the resulting construction will have only one edge between each pair of vertices, and will be a Ramanujan graph.\footnote{We thank Prof. Alex Lubotzky for confirming this in a personal communication to the authors.}

The authors have written Matlab routines for (i) determining, for a given pair $(p, q)$, whether the LPS construction leads to a Ramanujan graph or not, and (ii) implementing the LPS construction. For example, for $q = 13$ (leading to a graph with $n = 1,092$ vertices), if we choose $p = 157$, then every prime $< 157$ and equal to $1 \bmod 4$ leads to a Ramanujan graph. However, even higher values of $p$ are possible. For instance, with $q = 13$, the choices $p = 197, 229, 293$ lead to Ramanujan graphs. However, other values of $p$ (but larger than 157) do not necessarily lead to Ramanujan
multi-graphs\textsuperscript{7}. These routines can be shared upon request. In short, the LPS construction can be used to generate Ramanujan graphs where the degree $d/n$ is quite large. In turn such Ramanujan graphs can be used to complete matrices of fairly high rank.

6 Phase transition

The bound in (18) is only a sufficient condition for matrix completion. In an attempt to determine how close it is to being necessary, we have carried out numerical experiments as follows: We chose $q = 13$ which led to $n = 1,092$, and varied $p$ from 5 to 157, which is the largest number such that every prime number equal to 1 mod 4 results in a Ramanujan graph. For each choice of $p$, we generated 100 random matrices of rank $r$ for each $r$. We increased $r$ and noted the maximum value of $r$, call it $\tilde{r}$, such that when $r \leq \tilde{r}$, the recovery percentage was 100%. The results are shown in Figure 1. It can be observed from this figure that $\tilde{r} \approx d/3 = (p + 1)/3$ over the entire range of $p$. Then we observed what happens when the rank is increased beyond this maximum value. Our experiments establish the following: There is a phase transition, whereby the success ratio of nuclear norm minimization changes abruptly from 100% when $r = \tilde{r}$ to 0% if $r = \tilde{r} + 2$ or $\tilde{r} + 3$. The maximum rank $\tilde{r}$ for which 100% recovery took place was the same, whether the sampling scheme was to use a Ramanujan graph or random selection. The phase transition of the success ratio going from 100% to 0% when $r = \tilde{r} + 3$ also held true in both sampling schemes. However, the average error across the 100 randomly generated unknown matrices was slightly higher with random sampling as opposed to Ramanujan sampling. Figures 2 through 5 illustrate these observations. The difference is perhaps not sufficient to justify the use of Ramanujan sampling. However, the authors believe that a theoretical analysis of the maximum recoverable rank $\tilde{r}$ as well as the phase transition behavior would be far easier with Ramanujan sampling.

In the case of $\ell_1$-norm minimization with randomly generated Gaussian measurement matrices, a similar phase transition is reported in [24]. A theoretical analysis is carried out in several papers, including [25, 26] and others. Yet another paper [27] reports purely numerical results which show that, when the random measurement matrix is replaced by any of several deterministic matrices, the phase transition behavior continues to manifest itself. Our conclusions for the matrix completion problem are somewhat similar.

7 Conclusions and Future Work

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\textsuperscript{7}Multi-graph are those graphs with multiple edges between any two vertices
Figure 1: Plot of maximum rank for 100% recovery versus the prime $p$.

Figure 2: Average recovery error $\|\hat{X} - X\|_F / \|X\|_F$ over 100 randomly chosen matrices of rank $r$ with $p = 149$. 
Figure 3: Average recovery error $\|\hat{X} - X\|_F / \|X\|_F$ over 100 randomly chosen matrices of rank $r$ with $p = 197$.

Figure 4: Average recovery error $\|\hat{X} - X\|_F / \|X\|_F$ over 100 randomly chosen matrices of rank $r$ with $p = 229$. 
Figure 5: Average recovery error $\|\hat{X} - X\|_F / \|X\|_F$ over 100 randomly chosen matrices of rank $r$ with $p = 293$. 

| $p = 61$ | $p = 149$ | $p = 197$ |
|---|---|---|
| $r$ | RD | RM | $r$ | RD | RM | $r$ | RD | RM |
| 10 | 100 | 100 | 40 | 100 | 100 | 60 | 100 | 100 |
| 11 | 100 | 100 | 41 | 100 | 100 | 61 | 100 | 100 |
| 12 | 100 | 100 | 42 | 100 | 100 | 62 | 100 | 100 |
| 13 | 100 | 100 | 43 | 100 | 100 | 63 | 99 | 98 |
| 14 | 100 | 100 | 44 | 98 | 100 | 64 | 54 | 80 |
| 15 | 97 | 98 | 45 | 84 | 88 | 65 | 4 | 9 |
| 16 | 38 | 23 | 46 | 32 | 34 | 66 | 0 | 0 |
| 17 | 1 | 0 | 47 | 1 | 1 | 67 | 0 | 0 |
| 18 | 0 | 0 | 48 | 0 | 0 | 68 | 0 | 0 |
| 19 | 0 | 0 | 49 | 0 | 0 | 69 | 0 | 0 |
| 20 | 0 | 0 | 50 | 0 | 0 | 70 | 0 | 0 |

| $p = 229$ | $p = 293$ |
|---|---|
| $r$ | RD | RM | $r$ | RD | RM |
| 70 | 100 | 100 | 100 | 100 | 100 |
| 71 | 100 | 100 | 101 | 100 | 100 |
| 72 | 100 | 100 | 102 | 100 | 100 |
| 73 | 100 | 100 | 103 | 99 | 100 |
| 74 | 100 | 100 | 104 | 88 | 95 |
| 75 | 100 | 100 | 105 | 34 | 48 |
| 76 | 89 | 98 | 106 | 4 | 5 |
| 77 | 43 | 55 | 107 | 0 | 0 |
| 78 | 15 | 20 | 108 | 0 | 0 |
| 79 | 0 | 1 | 109 | 0 | 0 |
| 80 | 0 | 0 | 110 | 0 | 0 |
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Appendix: Proof of Theorem 7

This appendix contains the proof of Theorem 7. Suppose $X = U\Sigma V^T$ is the unknown matrix of rank $r$ or less that is to be recovered, where $U \in \mathbb{R}^{n_r \times r}$, $V \in \mathbb{R}^{n_c \times r}$, and $\Sigma$ is diagonal of dimensions $r \times r$. Throughout this appendix, the symbols $U$ and $V$ denote only these matrices and nothing else. Define $\mathcal{T} \subseteq \mathbb{R}^{n_r \times n_c}$ to be the subspace spanned by all matrices of the form $UB^T$ and $CV^T$.

It is easy to show that the projection operator $\mathcal{P}_\mathcal{T}$ equals

$$\mathcal{P}_\mathcal{T} Z = UU^T Z + ZVV^T - UU^T ZV V^T$$

$$= UU^T Z + U_U U_U^T ZV V^T$$

$$= UU^T ZV V^T + ZV V^T,$$

where $U_U U_U^T = I_{n_r} - UU^T$ and $V_V V_V^T = I_{n_c} - VV^T$. Suppose now that $Z \in \mathcal{T}$. Then

$$Z = \mathcal{P}_\mathcal{T} Z = UU^T Z + U_U U_U^T ZV V^T.$$

Thus one can write $Z = UB^T + CV^T$, where

$$B^T = U^T Z, C = U_U U_U^T ZV.$$

(37)
Throughout, we define the symbols $B$ and $C$ as above.

The heart of the proof is roughly similar to [6, Theorem 2].

**Lemma 2.** Suppose there exists a $Y \in \mathbb{R}^{n_r \times n_c}$ such that

1. $Y$ belongs to the image of $E_\Omega$, that is $Y_{ij} = 0 \forall (i, j) \not\in \Omega$.

2. $Y$ satisfies
   \begin{align*}
   &\|P_T Y - U V^T\|_F \leq \sqrt{\frac{\alpha}{32}}, \quad \|P_T \perp (Y)\|_S < \frac{3}{4}.
   \end{align*}
   (38)

Suppose further that the operator norm of $(1/\alpha)P_T E_\Omega - I$ when restricted to the subspace $T$ is no larger than $1/2$. In other words

\begin{align*}
\| (1/\alpha)P_T E_\Omega Z - Z \|_F \leq (1/2)\|Z\|_F, \forall Z \in T.
\end{align*}
(39)

Under these assumptions, for any $\Delta \in \mathbb{R}^{n_r \times n_c}$ such that $E_\Omega \Delta = 0$, we have that

\begin{align*}
\| X + \Delta \|_N > \|X\|_N,
\end{align*}
(40)

so that $\hat{X} = X$ is the unique solution to (1).

**Proof.** Suppose $E_\Omega \Delta = 0$, so that $\|E_\Omega \Delta\|_F = 0$. Then

\begin{align*}
\|E_\Omega \cdot P_T \Delta\|_F^2 &= \langle E_\Omega \cdot P_T \Delta, P_T \Delta \rangle_F \\
&= \langle P_T E_\Omega \cdot P_T \Delta - \alpha P_T \Delta, P_T \Delta \rangle_F \\
&\quad + \alpha \langle P_T \Delta, P_T \Delta \rangle_F \\
&\geq (a) \alpha \|P_T \Delta\|_F^2 - \alpha/2 \|P_T \Delta\|_F^2 \\
&= \alpha/2 \|P_T \Delta\|_F^2,
\end{align*}

where $(a)$ follows from (39). Now, since $\|E_\Omega \Delta\|_F = 0$, we have $\|E_\Omega \cdot P_T \Delta\|_F = \|E_\Omega \cdot P_T \perp \Delta\|_F$. Therefore,

\begin{align*}
\|P_T \perp \Delta\|_N \geq \|P_T \perp \Delta\|_F \geq \|E_\Omega \cdot P_T \perp \Delta\|_F \geq \sqrt{\alpha/2} \|P_T \Delta\|_F
\end{align*}
(41)

Next, recall that for any matrix $M$, it is true that

\begin{align*}
\|M\|_N = \max_{U',V'} \langle U' V'^T, M \rangle_F
\end{align*}

over all matrices $U', V'$ with orthogonal columns. In particular, for a particular $\Delta$, it is possible to choose $U_\perp, V_\perp$ such that $[U \ U_\perp], [V \ V_\perp]$ have orthogonal columns, and

\begin{align*}
\langle U_\perp V_\perp^T, P_T \perp \Delta \rangle_F = \|P_T \perp \Delta\|_N.
\end{align*}
For such a choice, we have
\[
\|X + \Delta\|_N \geq (a) (UV^T + U_{\perp} V_{\perp}^T, X + \Delta)_F \\
= (b) \|X\|_N + (UV^T + U_{\perp} V_{\perp}^T, \Delta)_F \\
= (c) \|X\|_N + (UV^T + U_{\perp} V_{\perp}^T, \Delta)_F \\
- (Y, \Delta)_F \\
= \|X\|_N + (UV^T - P_T Y, P_T \Delta)_F \\
+ (U_{\perp} V_{\perp}^T - P_{\perp} Y, P_{\perp} \Delta)_F \\
\geq (d) \|X\|_N - \|UV^T - P_T Y\|_F \|P_T \Delta\|_F \\
+ \|P_{\perp} \Delta\|_N - \|P_{\perp} Y\|_F \|P_{\perp} \Delta\|_N \\
> (e) \|X\|_N - \sqrt{\alpha/32} \|P_T \Delta\|_F \\
+ 1/4 \|P_{\perp} \Delta\|_N \\
\geq \|X\|_N
\] (43)

where (a) follows from the characterization of the nuclear norm, (b) follows from \((UV^T, X)_F = 0\), (c) follows from \((Y, \Delta)_F = 0\), (d) follows from Hölder’s inequality, and (e) follows from (38). 

The proof of Theorem 7 consists of showing that, under the stated hypotheses, there exists a \(Y\) that satisfies the conditions of Lemma 2. This is achieved through some preliminary lemmas.

**Lemma 3.** Suppose \(E_\Omega\{0,1\}^{n_c \times n_c}\) is a \((d_r, d_c)\)-biregular sampling matrix, and let \(U, V, \delta_d\) be as before.

1. For arbitrary \(B \in \mathbb{R}^{n_c \times r}\), define

\[
F^\top := (1/\alpha) U^\top E_\Omega (UB^\top) - B^\top.
\] (44)

Then
\[
\|F\|_F \leq \delta_d \|B\|_F.
\] (45)

2. For arbitrary \(C \in \mathbb{R}^{n_r \times r}\), define

\[
G = (1/\alpha) E_\Omega (CV^\top) V - C.
\] (46)

Then
\[
\|G\|_F \leq \delta_d \|C\|_F.
\] (47)

**Proof.** Fix \(i \in [r], j \in [n_c]\). Then
\[
F_{ji} = (F^\top)_{ij} = e_i^\top F^\top e_j \\
= (1/\alpha) e_i^\top U^\top E_\Omega (UB^\top) e_j - B_{ji}.
\]

Let us focus on the first term after ignoring the factor of \(1/\alpha\). From Theorem 9, specifically (26), we get
\[
e_i^\top U^\top E_\Omega (UB^\top) e_j = U_i^\top E_\Omega (UB^\top) e_j \\
= \sum_{k \in [r]} (U_i U_k)^\top E_\Omega (B_k e_j).
\]
Now observe that $B_k.e_j = B_{jk}e_j$, so that $E\Omega(B_k.e_j) = (E\Omega)_jB_{jk}$. Therefore

$$e_i^T U^TE\Omega(UB^T)e_j = \sum_{k\in[r]} (U_iU_k)^T(E\Omega)_jB_{jk}.$$ 

For this fixed $j$, define

$$\mathcal{N}(j) = \{l \in [nr]: (E\Omega)_{lj} = 1\},$$

and note that $|\mathcal{N}(j)| = d_c$ due to regularity. Then, for fixed $k \in [r]$, we have

$$(U_iU_k)^T(E\Omega)_j = \sum_{l \in \mathcal{N}(j)} U_{il}U_{lk}.$$ 

Therefore

$$\left((F^T)_{ij}\right) = \frac{1}{\alpha}e_i^T U^TE\Omega(UB^T)e_j - B_{ji}$$

$$= \left(\frac{1}{\alpha} \sum_{k\in[r]} \left(\sum_{l \in \mathcal{N}(j)} U_{il}U_{lk}^T\right) B_{jk} - B_{ji}\right)$$

$$= \left((1/\alpha) \sum_{l \in \mathcal{N}(j)} U_{il}^TU^T - I_r\right) B^T_{ij}.$$ 

By (8), the matrix inside the square brackets has spectral norm $\leq \delta_d$. Therefore

$$\|(F^T)_{ij}\|_2 \leq \frac{\delta_d}{\alpha}\|(B^T)_{ij}\|_2, \forall j \in [nr].$$

Taking the norm squared and summing over all $j$ proves (45), after noting that a matrix and its transpose have the same Frobenius norm. This establishes Item (1).

To prove Item (2), we use Item (1). Note that $(X.Y)^T = X^TY^T$. So (46) is equivalent to

$$G^T = (1/\alpha)V^TE\Omega^T(VC^T) - C^T.$$ 

Now every column of $E\Omega^T$ (or every row of $E\Omega$) contains $d_r$ ones. Therefore (47) follows from (45).

**Lemma 4.** Suppose $E\Omega\{0,1\}^{nr\times nc}$ is a $(d_r, d_c)$-biregular sampling matrix, and that $Z \in \mathcal{T}$. Define, as before,

$$B^T = U^TZ, C = U_\perp U_\perp^T Z V,$$

so that $Z = UB^T + CV^T$. Next, define

$$\bar{Z} := (1/\alpha)P_T E\Omega.Z - Z,$$

$$\bar{B}^T = U^T\bar{Z}, \bar{C} = U_\perp U_\perp^T \bar{Z}.$$ 

Let $\mu_0, \delta_d, \sigma_1, \sigma_2$ be as before. Then

$$\|\bar{B}\|_F \leq \frac{\delta_d}{\alpha}\|B\|_F + \frac{\sigma_2}{\sigma_1}\mu_0r\|C\|_F,$$

$$\|\bar{C}\|_F \leq \frac{\sigma_2}{\sigma_1}\mu_0r\|B\|_F + \delta_d\|C\|_F.$$ 

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Remark: The above two relations can be expressed compactly as
\[
\begin{bmatrix}
\|\tilde{B}\|_F \\
\|\tilde{C}\|_F
\end{bmatrix} \leq \begin{bmatrix}
\theta & \phi \\
\phi & \theta
\end{bmatrix} \begin{bmatrix}
\|B\|_F \\
\|C\|_F
\end{bmatrix},
\]
(53)
where, as in (22), we have
\[
\theta = \delta_d, \phi = \frac{\sigma_2}{\sigma_1} \mu_0 r,
\]
(54)

Proof. We establish (51), and the proof of (52) is entirely similar.

The definition of \( P_T \) makes it clear that
\[
U^\top P_T Y = U^\top Y, U \perp U^\top \perp P_T Y = U \perp U^\top \perp Y, \forall Y \in \mathbb{R}^{n_r \times n_c}.
\]
Therefore
\[
\tilde{B}^\top = U^\top ((1/\alpha)E_{\Omega}(UB^\top) - UB^\top) + (1/\alpha)U^\top E_{\Omega}.(CV^\top),
\]
because \( U^\top C = 0 \). Define \( \tilde{B}^\top = \tilde{B}^\top_1 + \tilde{B}^\top_2 \), where
\[
\tilde{B}^\top_1 = U^\top ((1/\alpha)E_{\Omega}(UB^\top) - UB^\top)
\]
\[
= (1/\alpha)U^\top E_{\Omega}(UB^\top) - B^\top,
\]
\[
\tilde{B}^\top_2 = (1/\alpha)U^\top E_{\Omega}.(CV^\top).
\]
Then it follows from Lemma 3 that
\[
\|\tilde{B}^\top_1\|_F \leq \delta_d \|B\|_F.
\]
(55)

To estimate \( \|\tilde{B}^\top_2\|_F = \|\tilde{B}^\top_2\|_F \), we proceed as follows:
\[
(\tilde{B}^\top_2)^i = e_i^\top \tilde{B}^\top_2 = (1/\alpha)U_i^\top E_{\Omega}.(CV^\top),
\]
\[
\| (\tilde{B}^\top_2)^i \|_2^2 = \max_{y \in \mathbb{R}^{n_c}, \|y\|_2 = 1} (\tilde{B}^\top_2)^i y
\]
\[
= \max_{\|y\|_2 = 1} (1/\alpha)U_i^\top E_{\Omega}.(CV^\top)y.
\]

Fix a \( y \in \mathbb{R}^{n_c} \) such that \( \|y\|_2 = 1 \) but otherwise arbitrary. Then it follows by Theorem 9 that
\[
(1/\alpha)U_i^\top E_{\Omega}.(CV^\top)y = (1/\alpha) \sum_{k \in [r]} (U_i.C_k)^T E_{\Omega}(V_k.y).
\]

Now \( U_i \perp C_k \), so that \( U_i.C_k \perp \mathbf{1}_{n_r} \). Therefore
\[
(U_i.C_k)^T E_{\Omega}(V_k.y) \leq \sigma_2 \|U_i.C_k\|_2 \|V_k.y\|_2, \forall k \in [r],
\]
\[
(1/\alpha)U_i^\top E_{\Omega}.(CV^\top)y \leq \frac{\sigma_2}{\alpha} \sum_{k \in [r]} \|U_i.C_k\|_2 \|V_k.y\|_2
\]
\[
\leq \left( \sum_{k \in [r]} \|U_i.C_k\|_2 \right)^{1/2}
\]
\[
\cdot \left( \sum_{k \in [r]} \|V_k.y\|_2 \right)^{1/2},
\]
(56)
where we use Schwarz’ inequality in the last step.

Now we can bound the second term as follows:

\[
\sum_{k \in [r]} \|V_k.y\|_2^2 = \sum_{k \in [r]} \sum_{l \in [n_c]} V_{lk}^2 y_l^2 \\
= \sum_{l \in [n_c]} y_l^2 \sum_{k \in [r]} V_{lk}^2 \\
\leq \frac{\mu_0 r}{n_c} \sum_{l \in [n_c]} y_l^2 = \frac{\mu_0 r}{n_c},
\]

where in the last step we use the definition of the coherence \(\mu_0\). Substituting this bound into (56) gives

\[
\|\bar{B}_2\|_F \leq \frac{\sigma_2^2}{\alpha^2} \frac{\mu_0 r}{n_r n_c} \|C\|_F^2.
\]

Now the last term can be bounded in a manner analogous to the above. We have that

\[
\sum_{i \in [r]} \sum_{k \in [r]} \|U_i.C_k\|_2^2 = \sum_{i \in [r]} \sum_{k \in [r]} \sum_{l \in [n_r]} U_{li}^2 C_{lk}^2 \\
= \sum_{k \in [r]} \sum_{l \in [n_r]} C_{lk}^2 \sum_{i \in [r]} U_{li}^2 \\
\leq \frac{\mu_0 r}{n_r} \sum_{k \in [r]} \sum_{l \in [n_r]} C_{lk}^2 \\
= \frac{\mu_0 r}{n_r} \|C\|_F^2.
\]

Substituting this bound in (57) gives

\[
\|\bar{B}_2\|_F^2 \leq \frac{\sigma_2^2}{\alpha^2} \left(\frac{\mu_0 r}{n_r n_c}\right) \|C\|_F^2 = \left(\frac{\sigma_2}{\sigma_1} \frac{\mu_0 r}{\alpha^2}\right)^2 \|C\|_F^2.
\]

Taking square roots of both sides gives

\[
\|\bar{B}_2\|_F \leq \frac{\sigma_2}{\sigma_1} \frac{\mu_0 r}{\alpha} \|C\|_F,
\]

\[
\|\bar{B}\|_F \leq \|\bar{B}_1\|_F + \|\bar{B}_2\|_F \leq \delta \|B\|_F + \frac{\sigma_2^2}{\sigma_1} \frac{\mu_0 r}{\alpha^2} \|C\|_F,
\]

which is (51). The proof of (52) is entirely similar.
Lemma 5. Suppose $E_\Omega \{0,1\}^{d_r \times d_c}$ is a $(d_r,d_c)$-biregular sampling matrix, that $Z \in \mathcal{T}$, and define

$$\tilde{Z} := (1/\alpha)P_T E_\Omega Z - Z,$$

Then

$$\|\tilde{Z}\|_F \leq (\theta + \phi)\|Z\|_F,$$

where $\theta, \phi$ are defined in (51) and (52) respectively.

Remark: The above lemma can be stated as follows: The map $Z \mapsto (1/\alpha)P_T E_\Omega Z - Z$, when restricted to $\mathcal{T}$, has an operator norm $\leq \theta + \phi$.

Proof. Define, as before,

$$B^T = U^T Z, C = U_{\perp} U_+^T Z V,$$

$$\bar{B}^T = U^T \bar{Z}, \bar{C} = U_{\perp} U_+^T \bar{Z},$$

so that $Z = U B^T + C V^T$, $\bar{Z} = U \bar{B}^T + \bar{C} V^T$. Note that

$$\langle UB^T, CV^T \rangle_F = \text{tr}(BU^T CV^T) = 0,$$

because $U^T C = 0$. Therefore

$$\|Z\|_F^2 = \|UB^T\|_F^2 + \|CV^T\|_F^2 + 2\langle UB^T, CV^T \rangle_F$$

$$= \|UB^T\|_F^2 + \|CV^T\|_F^2 = \|B\|_F^2 + \|C\|_F^2,$$

because left multiplication by $U$ and right multiplication by $V^T$ preserve the Frobenius norm. Similarly

$$\|\tilde{Z}\|_F^2 = \|\bar{B}\|_F^2 + \|\bar{C}\|_F^2.$$

Now it is easy to verify that the spectral norm of the matrix in (53) is $\theta + \phi$. Therefore

$$\|\tilde{Z}\|_F^2 = \|\bar{B}\|_F^2 + \|\bar{C}\|_F^2 \leq (\theta + \phi)^2 (\|B\|_F^2 + \|C\|_F^2)$$

$$= (\theta + \phi)^2 \|Z\|_F^2.$$

This is the desired conclusion. \qed

Proof. (Of Theorem 7.) At last we come to the proof of the theorem itself. Recall from Lemma 2 that $X$ is the unique solution of (2) provided the following conditions hold: First, there exists a $Y \in \mathbb{R}^{n_r \times n_c}$ that satisfies the following conditions:

1. $Y$ belongs to the image of $E_\Omega$, that is $Y_{ij} = 0 \forall (i,j) \not\in \Omega$.

2. $Y$ satisfies

$$\|P_T Y - UV^T\|_F \leq \sqrt{\frac{\alpha}{32}},$$

$$\|P_T (Y)\|_S \leq \frac{3}{4}.$$

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Second, the operator norm of \((1/\alpha)P_T E_{\Omega} - I\) when restricted to the subspace \(T\) is no larger than \(1/2\). Lemma 5 shows that the above operator norm is \(\leq \theta + \phi\). Therefore if (23) holds, then this condition is satisfied. So it remains to construct a suitable \(Y\).

We do this as follows: Define \(W_0 = UV^T\), and define \(W_i\) recursively as

\[
W_i = W_{i-1} - (1/\alpha)P_T E_{\Omega}W_{i-1},
\]

(62)

\[
Y_p = \sum_{i=0}^{p-1} (1/\alpha)E_{\Omega}W_i.
\]

(63)

Then it is obvious that each \(Y_p\) belongs to the image of \(E_{\Omega}\). So the proof is complete once it is shown that \(Y\) satisfies the two conditions (60) and (61). Towards this end, note that

\[
(1/\alpha)E_{\Omega}W_i = W_i - W_i + 1.
\]

So

\[
Y_p = \sum_{i=0}^{p-1} (W_i - W_i + 1) = W_0 - W_p.
\]

Therefore

\[
\|Y_p - W_0\|_F = \|W_p\|_F \leq (\theta + \phi)p\|W_0\|_F.
\]

Therefore, for sufficiently large \(p\) (which could be computed, but it is not necessary), we have that

\[
\|Y_p - W_0\|_F = \|Y - UV^T\|_F \leq \frac{\sqrt{\alpha}}{32},
\]

which is (60).

To establish (61) and complete the proof, we reason as follows:

\[
P_{T^\perp}(Y) = P_{T^\perp} \left[ \sum_{i=0}^{p-1} (1/\alpha)E_{\Omega}W_i \right]
\]

\[
= P_{T^\perp} \left[ \sum_{i=0}^{p-1} [(1/\alpha)E_{\Omega}W_i - W_i] \right],
\]

because \(W_i \in T\) and hence \(P_{T^\perp}W_i = 0\). Therefore

\[
\|P_{T^\perp}(Y)\|_S = \left\| P_{T^\perp} \left[ \sum_{i=0}^{p-1} [(1/\alpha)E_{\Omega}W_i - W_i] \right] \right\|_S
\]

\[
\leq (a) \left\| \sum_{i=0}^{p-1} [(1/\alpha)E_{\Omega}W_i - W_i] \right\|_S
\]

\[
\leq (b) \sum_{i=0}^{p-1} \| (1/\alpha)E_{\Omega}W_i - W_i \|_S
\]

\[
\leq (c) \phi \sum_{i=0}^{p-1} \| W_i \|_S
\]

\[
\leq (d) \phi \sum_{i=0}^{p-1} \| W_i \|_F.
\]
Here (a) follows because the spectral norm is submultiplicative and the spectral norm of $P_{T^\perp} = 1$, (b) follows from the triangle inequality, (c) is a consequence of Theorem 5 and in particular (18), and (d) follows from the fact that the spectral norm is no larger than the Frobenius norm. Now we apply the recursion bound from Lemma 4. It states that, if we define

$$B_i^T = U^T W_i, C_i = U_{\perp} U_{\perp}^T W_i V,$$

then

$$\begin{bmatrix} \|B_{i+1}\|_F \\ \|C_{i+1}\|_F \end{bmatrix} \leq \begin{bmatrix} \theta & \phi \\ \phi & \theta \end{bmatrix} \begin{bmatrix} \|B_i\|_F \\ \|C_i\|_F \end{bmatrix}. \tag{64}$$

Now at $i = 0$, we have that $W_0 = UV^T = UB_0^T + C_0 V^T$ with $B_0^T = (1/2)V^T$, $C_0 = (1/2)U$. Since the columns of $U$ and of $V$ are normalized, and there are $r$ columns in each matrix, we have that

$$\begin{bmatrix} \|B_0\|_F \\ \|C_0\|_F \end{bmatrix} = \frac{\sqrt{r}}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$ 

Now note that $[1 \ 1]^T$ is an eigenvector of the matrix in (64), with eigenvalue $\theta + \phi$. Thus applying (64) recursively leads to

$$\begin{bmatrix} \|B_i\|_F \\ \|C_i\|_F \end{bmatrix} \leq \frac{\sqrt{r}}{2} (\theta + \phi)^i \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

So

$$\|W_i\|_F = (\|B_i\|_F^2 + \|C_i\|_F^2)^{1/2} \leq \sqrt{r} (\theta + \phi)^i,$$

$$\phi \sum_{i=0}^{p-1} \|W_i\|_F \leq \phi \sum_{i=0}^{p-1} \sqrt{r} (\theta + \phi)^i \leq \phi \sum_{i=0}^{\infty} \sqrt{r} (\theta + \phi)^i = \phi \sqrt{\frac{\sqrt{r}}{2} \frac{1}{1 - (\theta + \phi)}}.$$

Now it is routine algebra to show that (24) can be rewritten as follows:

$$\left(1 + \frac{4}{3} \sqrt{\frac{r}{2}}\right) \phi + \theta < 1 \iff \phi \sqrt{\frac{r}{2} \frac{1}{1 - (\theta + \phi)}} < \frac{3}{4}.$$

Hence (61) also holds. This shows that $Y$ satisfies the requisite conditions, and as a consequence, $X$ is the unique solution to (2). \qed