Notes on Lepton Gyromagnetic Ratios

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A compendium for outsiders.

I. ORIGINS

Dirac’s quantum theory of the electron tells us what an electron is [1]: a particle that carries half a quantum \((\hbar/2)\) of spin angular momentum, \(-1\) unit of electric charge, and magnetic moment \(\mu_e\) of minus one Bohr magneton, \(-\mu_B \equiv -\hbar e/2m_e\). Here \(\hbar\) is the quantum of action, \(-e\) the electron charge, and \(m_e\) the electron mass. The Bohr magneton is defined as the magnitude of the magnetic dipole moment of a point electron orbiting an atom with one unit \((\hbar)\) of orbital angular momentum. [The numerical value is \(\mu_B = 5.788\,381\,806(17) \times 10^{-11}\) MeV T\(^{-1}\) [2].] On the classical level, an orbiting point particle with electric charge \(e\) and mass \(m\) exhibits a magnetic dipole moment given by

\[
\vec{\mu}_L = \frac{e\hbar}{2m} \vec{L}.
\]  

(1)

Dirac’s prediction for the electron magnetic moment is thus twice the value that would arise for a half unit of orbital angular momentum, if that were possible. The ratio \(g_e \equiv \mu_e/(-\frac{1}{2}\mu_B)\) is called the gyromagnetic ratio of the electron; the Dirac equation predicts \(g_e = 2\).

These properties are precisely what is required to account for what Dirac calls “duplexity phenomena,” the observed number of quantum states for an electron in an atom being twice the number given by the quantum theory of a spinless point particle. A spin-\(\frac{1}{2}\) electron with a magnetic moment of one Bohr magneton matches precisely what Uhlenbeck and Goudsmit [3] inferred from their study of atomic spectra.

Dirac writes, The question remains as to why Nature should have chosen this particular model for the electron instead of being satisfied with the point-charge. One would like to find some incompleteness in the previous methods of applying quantum mechanics to the point-charge electron such that, when removed, the whole of the duplexity phenomena follow without arbitrary assumptions. In the present paper it is shown that this is the case, the incompleteness of the previous theories lying in their disagreement with relativity, or, alternatively, with the general transformation theory of quantum mechanics. It appears that the simplest Hamiltonian for a point-charge electron satisfying the requirements of both relativity and the general transformation theory leads to an explanation of all duplexity phenomena without further assumption [4].

II. MAGNETIC MOMENT AS A DIAGNOSTIC

A. Generalities

Within quantum electrodynamics (QED), a renormalizable local relativistic quantum field theory of photons and electrons, the gyromagnetic ratio of a lepton, \(g_\ell\), emerges unambiguously from the perturbation expansion. An extensive and interesting literature explores what makes \(g_\ell = 2\) the “natural value” for a structureless point particle of spin-\(\frac{1}{2}\) [5]. The requirement of good high-energy behavior implies that \(g_\ell - 2\) must vanish at tree level for any well-behaved theory [6].

Quantum corrections induce a deviation from the Dirac moment that is traditionally expressed as the magnetic moment anomaly,

\[
a_\ell \equiv \frac{g_\ell - 2}{2}.
\]

(2)

The predicted value of \(a_\ell\) can be confronted by experiment. The model dependence of the magnetic anomaly makes it an incisive test of QED and a sensitive probe for new-physics contributions [7]. The interplay between theory and experiment played a decisive role in the development and validation of quantum electrodynamics [8].

Beyond its role in challenging QED and probing for the virtual influence of undiscovered particles and forces, the anomalous magnetic moment provides stringent constraints on lepton substructure [9]. For that purpose, we may examine the magnetic anomaly defect, defined as the difference between theoretical prediction and experimental determination,

\[
\delta a_\ell \equiv a_\ell^{\text{th}} - a_\ell^{\text{exp}}.
\]

(3)

A very simple working hypothesis is that no cancellations or suppression factors due to symmetries in the underlying dynamics account for the small mass \(m_\ell\) of the composite lepton itself. Alternatively, so-called chiral models provide more modest constraints. In either case, it is informative to relate the compositeness scale \(M_\ell\) and the
radius $R_\ell$ of the lepton to the magnetic anomaly defect,

$$|\delta a_e| = \frac{m_\ell}{M_\ell} = m_\ell R_\ell \no\text{suppression, or}$$

(4a)

$$|\delta a_e| = \left(\frac{m_\ell}{M_\ell}\right)^2 = m_\ell^2 R_\ell^2 \text{ chiral model.} \quad (4b)$$

Searches for quark and lepton compositeness in high-energy collisions, reviewed in §92 of Ref. [2], reach above 10 TeV, again assuming no dynamical conspiracies.

### B. The Electron

The electron was the focus of Dirac’s theory and the test case for the developing theory of quantum electrodynamics in the late nineteen-forties. It is stable on the time scale of any conceivable experiment; the current bound on the electron lifetime, $\tau_e > 6.6 \times 10^{28}$ yr at 90% C.L. [10], greatly exceeds the age of the universe. The electron mass is $m_e = (0.5109989461 \pm 0.0000000031)$ MeV. CPT invariance requires that the gyromagnetic ratios of electron and positron be identical: $g_e^- = g_e^+$.

The anomalous magnetic moment of the electron was discovered in 1947 by Kusch and Foley [11], who inferred

$$a_e^{[KF]} = 0.001 \pm 5(5)$$

from their study of hyperfine structure in gallium atoms. Julian Schwinger showed that the one-loop $(O(e^2))$ quantum correction to the electron’s magnetic moment contributes [12]

$$a_e^{[2]} = \frac{\alpha}{2\pi} \approx 0.001162,$$  \quad (6)

where the numerical value reflects the value of the fine-structure constant as then known, $\alpha = 1/137$.

The two-loop contribution to the gyromagnetic ratio of the electron is also known analytically [13]. It is

$$a_e^{[4]} = \frac{\alpha^2}{\pi^2} \left(\frac{197}{144} + \frac{\pi^2}{12} + \frac{3}{4} \zeta(3) - \frac{\pi^2}{2} \ln 2\right) \quad (7a)$$

$$= -0.328 \left(\frac{\alpha}{\pi}\right)^2 \approx -0.00000177. \quad (7b)$$

Here $\zeta(3) = \sum_{i=1}^\infty \frac{i^{-3}}{i^3} = 1.202056903 \ldots$ is the Riemann zeta function of 3. The sum of Eqs. (6) and (7) yields the theoretical prediction ca. 1957 (for a theory of photons and electrons only),

$$a_e \approx a_e^{[2]} + a_e^{[4]} = 0.0011596. \quad (8)$$

The contributions of heavier fermion ($f\bar{f}$) bubbles are suppressed by mass ratios. The leading contribution when the mass ratio $(m_e/m_f)$ is small is [14]

$$a_e^{[4:]} \approx \frac{1}{45} \frac{\alpha^2}{\pi^2} \left(\frac{m_e}{m_f}\right)^2,$$  \quad (9)

so that $a_e^{[4:]} \approx 0.00000052 (\alpha^2/\pi^2)$. A form useful for evaluating fermion bubbles for all values of the mass ratio is given in Ref. [15].

Through heroic work over many decades, the calculation of the electron’s magnetic anomaly has been carried out through five loops in QED. The three-loop $O(\alpha^3)$ contribution is given in closed form in [16]; evaluated numerically, it gives

$$a_e^{[3]} = (1.181241456 \ldots) (\frac{\alpha}{\pi})^3. \quad (10)$$

Analytical calculations, reinforced by numerical evaluations, exist through four loops, i.e., up to $O(\alpha^4/\pi^4)$ or eighth order in the electron charge $e$. The coefficient of $(\alpha/\pi)^5$ is known from numerical integrations. In addition, the contributions of weak and hadronic interactions have been estimated. The predicted value as of 2019 is [17]

$$a_e^{[\text{th}[Cs]} = 115.965218160.6(11)/(12)(229) \times 10^{-14}, \quad (11)$$

where the first two uncertainties are from the tenth-order QED term and the hadronic term, respectively. The third and largest uncertainty comes from the value of the fine structure constant obtained from atom-interferometry measurements of the Cs atom, $\alpha^{-1}(Cs) = 137.035999946(27)$ [18]. A more recent determination using matter-wave interferometry to measure the recoil velocity of a rubidium atom that absorbs a photon, leads to $\alpha^{-1}(Rb) = 137.035999206(11)$ [19]. These two highly precise values differ by approximately 5.4 standard deviations [20].

To compare theory and experiment, it is efficient to combine the theoretical uncertainties of Eqs. (11), thus:

$$a_e^{[\text{th}[Cs]} = 115.965218161(023) \times 10^{-14}. \quad (12)$$

This represents a prediction at the level of 0.23 parts per trillion (ppt) for $g_e$. Adopting instead the 2020 value of $\alpha^{-1}(Rb) [19]$, the standard-model prediction is

$$a_e^{[\text{th}[Rb20]} = 115.965218025.2(95) \times 10^{-14}, \quad (13)$$

which carries an uncertainty of 0.1 ppt for $g_e$ and lies 5.3σ below $a_e^{[Cs]}$. A smaller value of $\alpha^{-1}$, which is to say a larger value of $\alpha$, implies a larger calculated value of $a_e$. The difference is very closely given by the Schwinger contribution,

$$\alpha(Cs) - \alpha(Rb20) \approx 0.136(25) \times 10^{-11}, \quad (14)$$

a shift to which modern measurements of $g_e$ are sensitive.

An independent numerical evaluation of the $O(e^{10})$ contribution of diagrams without lepton loops has been carried out by Volkov [21], with a result that differs slightly from the result of Ref. [17]. Although the contending values differ by 4.8σ, the implications for $a_e$ are not significant for current comparisons of theory and experiment; a resolution will be needed in the near future.
Today, experimental determinations of $g_e$ have attained sub-ppt precision—a stunning achievement. The evolution of experimental technique up to 1972 is reviewed in [22], which also contains a thorough historical summary of theoretical developments. That chronology recounts the landmark experiments at the University of Michigan that directly observed the spin precession of polarized electrons in a region of static magnetic field. This technique culminated in a 3.5 parts-per-billion (ppb) determination of $g_e$ [24].

$$a^{[\text{Mich}]}_e = 1159657.7(3.5) \times 10^{-9}. \quad \text{(15)}$$

Measurements on trapped single electrons led, over time, to a 4.3-pppb determination of $g_e$ at the University of Washington (UW) [24]. The result for the anomalous moment is

$$a^{[\text{UW}]}_e = 1159652.188.4(4.3) \times 10^{-12}. \quad \text{(16)}$$

The same experiment yields a positron anomaly of

$$a^{[\text{UW}]}_e = 1159652.187.9(4.3) \times 10^{-12}, \quad \text{(17)}$$

validating the prediction of CPT symmetry to a remarkable degree:

$$g_{e^-}/g_{e^+} = 1 + (0.5 \pm 2.1) \times 10^{-12}. \quad \text{(18)}$$

These stood as the definitive measurements for nearly two decades.

The most precise published result, from the Harvard University team, is obtained by resolving the quantum cyclotron and spin levels of a single electron suspended for months in a cylindrical Penning trap. This work reaches an uncertainty of 0.28 ppt for $g_{e^-}$ [25]. Written in the same form as the UW results, it is

$$a_{e^-}^{[\text{H08}]} = 1159652.180.73(0.28) \times 10^{-12}. \quad \text{(19)}$$

Comparing the prediction Eqs. (11) and measurement Eqn. (19), we find that the magnetic anomaly defect of the electron is

$$\delta a^{[\text{Cs}]}_e = a^{[\text{th}]}_e^{[\text{Cs}]} - a^{[\text{H08}]}_e = (88 \pm 37) \times 10^{-14}, \quad \text{(20)}$$

with the measurement $2.4\sigma$ below the prediction. The difference between Eqn. (13) and Eqn. (19) yields

$$\delta a^{[\text{Rb20}]}_e = a^{[\text{th}]}_e^{[\text{Rb20}]} - a^{[\text{H08}]}_e = (-48 \pm 30) \times 10^{-14}, \quad \text{(21)}$$

with the measurement $1.6\sigma$ above the prediction. Taking $|\delta a_e| \lesssim 10^{-12}$ as a rough measure of a potential offset between theory and experiment, we infer from Eqs. (14a) a compositeness scale $M_c \gtrsim 5 \times 10^9$ TeV, or equivalently a composite-electron radius $R_c \lesssim 4 \times 10^{-29}$ m. The chiral-invariant Ansatz gives, through Eqn. (11b), the more modest limits $M_c \gtrsim 500$ GeV and $R_c \lesssim 4 \times 10^{-19}$ m.

By equating their formal (five-loop) expression for the electron anomaly to Eqn. (19), Aoyama et al. [17] take $a^{[\text{H08}]}_e$ as a measure of the fine structure constant,

$$\alpha^{-1}(a_{e^-}) = 137.035\,999\,149\,6(13)(14)(330). \quad \text{(22)}$$

\[ \text{The uncertainties are from the tenth-order QED term, hadronic term, and } a^{[\text{H08}]}_e, \text{ respectively. The inferred value, } \alpha^{-1}(a_e) = 137.035\,999\,150(33), \text{ lies } 0.104(43) \times 10^{-6} \text{ (2.4}\sigma) \text{ above } \alpha^{-1}(\text{Cs}) \text{ [18] and } 0.056(35) \times 10^{-6} \text{ (1.6}\sigma) \text{ below } \alpha^{-1}(\text{Rb}) \text{ [19], as we anticipate from Eqs. (20–21).} \]

Planned improvements in technique by the Gabrielse Research Group at Northwestern University (formerly Harvard) [26] aim at an order-of-magnitude improvement over the precision of $a^{[\text{H08}]}_e$ and a 150-fold improvement in the accuracy of $g_{e^-}/g_{e^+}$ compared with Eqn. (18).

Should further studies establish a discrepancy between the standard-model prediction and the experimental determination of $g_{e^+}$, it is of interest to ask whether a new light X boson, feebly coupled to the electron, might be responsible. The NA64 experiment at CERN, which directs a 100-GeV electron beam onto stationary nuclei has set new exclusion limits on a sub-GeV boson that decays predominantly into invisible final states [27].

\[ \text{C. The Muon} \]

The muon, the second-generation charged lepton, has a mass of $m_\mu = 105.658\,3745(24)$ MeV and a mean life $\tau_\mu = 2.196\,981.1(22) \times 10^{-6}$ s. The parity-violating decay $\mu^+ \rightarrow e^+\nu_e\bar{\nu}_\mu$ (and charge-conjugate) correlates the muon spin direction and the direction of the emitted positron (or electron), as first demonstrated by Garwin, Lederman, and Weinrich [28] at Columbia University’s Nevis Cyclotron. In leading approximation, the correlation is given by $\left(1 - \frac{1}{2} \cos \theta\right)$, where $\theta$ is the angle between the muon spin and the emitted electron momentum [29]. Garwin, et al. were able to exploit this fact in their discovery experiment to constrain the muon’s gyromagnetic ratio to lie within 5% of the $O(\alpha)$ prediction, $2(1 + \alpha/2\pi) \approx 2.001162$ (cf. Eqn. (4)). This was quickly followed by an improved measurement in Liverpool, which reported $g_\mu = 2.004 \pm 0.014$ and is notable for an early example of the “wiggle plot” that is a feature of later storage-ring experiments [30]. Subsequent measurements by Garwin and collaborators led by 1960 to the determination $a_\mu = 0.0013+0.00015$, in good agreement with the contemporaneous prediction, $a_{\mu,1960} = 0.001164(33)$.

Over nearly two decades, three elegant storage-ring experiments at CERN greatly advanced both technique and precision. They reported

$$a_{\mu,1965} = 116\,200.0(5000) \times 10^{-9}, \quad \text{(23a)}$$

$$a_{\mu,1972} = 116\,616.0(310) \times 10^{-9}, \quad \text{(23b)}$$

$$a_{\mu,1979} = 116\,592.4(8.5) \times 10^{-9}, \quad \text{(23c)}$$

The final entry averages the measured anomalous magnetic moments of positive and negative muons, $a_{\mu,\pm} = 116\,591.1(11) \times 10^{-9}$ and $a_{\mu,\pm} = 116\,593.7(12) \times 10^{-9}$.

The progression of Eqn. (23) represents a nearly 600-fold improvement through the series of experiments,
and the beginning of meaningful constraints on non-standard contributions to the anomalous moment. The CERN experiments provided a baseline for later efforts at Brookhaven and Fermilab.

The muon anomalous magnetic moment reported by Brookhaven experiment E821 value \[a_\mu^{\text{BNL}} = 116.592089 \times 10^{-11}\] set a new standard for precision: its uncertainty of 0.54 ppm represented a 14-fold improvement compared to the classic CERN measurements. The result was highly consequential, because it suggested a mismatch between theory and experiment. In the words of the Abstract,

While the QED processes account for most of the anomaly, the largest theoretical uncertainty, \(\approx 0.55\) ppm, is associated with first-order hadronic vacuum polarization. Present standard model evaluations, based on \(e^+e^-\) hadronic cross sections, lie 2.2–2.7 standard deviations below the experimental result.

The determination \((g_{\mu+} - g_{\mu-})/(g_{\mu}) = (-0.11 \pm 0.12) \times 10^{-8}\) is consistent with CPT invariance.

The E821 report stimulated not only a vigorous commerce in candidate interpretations of the putative break in the standard model, but also intensive effort to make the theoretical prediction more secure, largely by scrutinizing the expectations for the hadronic vacuum polarization and the contribution—confirmed to be small—of light-by-light scattering. Planning for an improved experiment to the development of Fermilab Muon g – 2 program using the relocated BNL muon storage ring.

A detailed review of theory and experiment through 2009 appears in Ref. \[14\]. At two-loop order, the contribution of a muon loop in the photon vacuum polarization, which enables them to predict \[16\] is negligible on the scale of current experimental capabilities.

The consensus relies on data-driven dispersion-relation evaluations of the hadronic vacuum polarization \[11\] \[42\]. Much current activity is devoted to \textit{a priori} evaluations using lattice techniques. The path toward a purely theoretical calculation was laid out by Blum \[43\] nearly two decades ago. An independent evaluation is of course desirable; in addition, the lattice calculation has advantages for the separation of QED effects from hadronic corrections and the treatment of isospin breaking. Perhaps half a dozen groups are pursuing this program, aiming to determine the hadronic vacuum polarization contribution with sub-percent precision. Their progress is reviewed in Refs. \[40\] \[44\].

Fermilab E989 recently reported its first results \[45\]:

\[a_\mu^{\text{FNAL}} = 116.592040(54) \times 10^{-11}\]

This 0.46-ppm determination of \(a_\mu\), within 0.6\(\sigma\) of the BNL value, is 3.3\(\sigma\) above the theory consensus. Combining the BNL and FNAL measurements gives a 2021 world average,

\[\langle a_\mu^{2021} \rangle = 116.592061(41) \times 10^{-11} \text{ (0.35 ppm)}\]

\(4.2\sigma\) above the theory consensus. The support for the BNL central value and the modest strengthening of the offset between theory and experiment is suggestive that new physics of some sort may be at play. Although measurements of \(a_\mu\) are less precise than those of \(a_e\), the muon’s heavier mass confers greater sensitivity to new-physics effects.

It is noteworthy that the observed offset,

\[\langle a_\mu^{2021} \rangle - a_\mu^{[\text{EW:2020}]} = 251(59) \times 10^{-11}\]

is not small. It is comparable in size to the electroweak contribution to the anomaly \[47\],

\[a_\mu^{[\text{EW}]} = 153.6(1.0) \times 10^{-11}\]

A good story needs a wrinkle, and Muon \(g - 2\) is no exception. The Budapest–Marlise–Wuppertal Collaboration has presented a new lattice-QCD evaluation of the hadronic vacuum polarization, which enables them to predict \[48\],

\[a_\mu^{[\text{BMW}]} = 116.591954(55) \times 10^{-11}\]

The difference from the consensus value Eqn. \[27\] is subtle but telling. The BMW value lies 144(70) \(\times 10^{-11}\) (2.1\(\sigma\)) above the consensus value and just 107(69) \(\times 10^{-11}\) (1.6\(\sigma\)) below the experimental world average.
Needless to say, theoretical and experimental work continues, with a particular focus on the hadronic vacuum polarization. The BABAR Collaboration [40], the CMD-3 [50] and SND [51] experiments at the Budker Institute in Novosibirsk, and the Belle-II experiment [52] recently commissioned at the KEK Super-B factory aim all at refining the $e^+e^- \to$ hadrons data sets. The MUonE experiment foreseen at CERN aims to gather information in the spacelike region by making precise measurements of $\mu e$ elastic scattering [53].

The (experimental average) magnetic anomaly defect with respect to the consensus prediction [41] is

$$\delta a_\mu = 251(59) \times 10^{-11} \quad (33)$$

which reflects the 4.2$\sigma$ mismatch between calculation and world average measurement. If we take $|\delta a_\mu| \lesssim 3 \times 10^{-9}$, we estimate $M^2_{\tau} \gtrsim 3.5 \times 10^4$ TeV and $R_\mu \lesssim 5.6 \times 10^{-24}$ m using Eqn. (4a) or $M^2_{\tau} \gtrsim 1.9$ TeV and $R_\mu \lesssim 10^{-19}$ m using Eqn. (5a).

The analysis reported in Ref. [45] is based on 6% of the planned E989 data sample. The current uncertainties are 434 ppb statistical and 157 ppb systematic. By the summer of 2022, the collaboration expects to report on their Run 2 and Run 3 data sets, increasing to four times the current sample (approximately 10 times the BNL E821 sample) and reducing the experimental error by a factor of two. At that point, they foresee a systematic uncertainty at the 100 ppb level. The ultimate goal is to record and analyze 20 times the BNL sample, leading to a further reduction of a factor of two in uncertainty.

At the Japan Proton Accelerator Research Complex in Tokai, J-PARC experiment E34 [54] will employ a very different technique, using a 300 MeV/c reaccelerated thermal muon beam with 50% polarization. The beam will be injected vertically into a solenoid storage ring with 1 ppm local magnetic field uniformity for the muon storage region with an orbit diameter of 66 cm. Compare the Brookhaven and Fermilab experiments, with muon momentum of 3.09 GeV/c and orbit diameter 14.224 m. The precision goal for $a_{\mu+}$ is a statistical uncertainty of 450 ppb, similar to the statistical weight of the BNL and FNAL-2021 samples, and a systematic uncertainty less than 70 ppb.

D. The Tau Lepton

The third charged lepton, $\tau$, has a mass $m_\tau = 1776.86$ (12) MeV, so its anomalous magnetic moment should have a heightened sensitivity to quantum corrections from heavy particles. In particular, $a_\tau$ is more sensitive than $a_\mu$ to heavy “new physics” by a ratio of $(m_\tau/m_\mu)^2 \approx 280$. By the same logic, $a_e$ should also be more sensitive to strong-interaction contributions.

In common with the muon, the tau lepton has parity-violation decays, here

$$\tau^- \to \mu^- \bar{\nu}_\mu \nu_\tau \quad (17.39 \pm 0.04)\% \quad (34a)$$
$$\tau^- \to e^- \bar{\nu}_e \nu_\tau \quad (17.82 \pm 0.04)\% \quad (34b)$$

that analyze its spin direction. However, experiments similar to those carried out for the muon are made challenging by the short lifetime of the $\tau$ lepton, $\tau_\tau = (290.3 \pm 0.5) \times 10^{-15}$ s. For that reason, conceiving a technique to measure $a_\tau$ demands original thinking [55].

A recent theoretical prediction, within the standard-model paradigm, is [42] [66]

$$a_{\tau}^{th} = 117717.1 \times 10^{-8}, \quad (35)$$

slightly greater than $a_e$ and $a_\mu$. What information we have from experiment is indirect, derived from limits on anomalous $\sigma_{\mu\nu}$ couplings of $\tau$ to electromagnetism. The Review of Particle Physics [2] takes as the best current constraint a limit from the DELPHI experiment at LEP [57] that derives from the measured cross section for the reaction $e^+e^- \to e^+e^- \tau^+\tau^-$, which is to say $\gamma\gamma \to \tau^+\tau^-$:

$$-0.052 < a_\tau < 0.013 \quad \text{at 95\% CL,} \quad (36)$$

or $a_\tau = -0.018 \pm 0.017$. While this result is consistent with the Dirac value, $g_\tau = 2$, it does not meaningfully test either the standard-model prediction or the presence of unexpected quantum corrections.

How could we do better? A reasonable near-term goal might be to reach the level of the Schwinger contribution, Eqn. (6). A spin-precession experiment seems out of the question, so improved measurements of the rates for two-photon production of tau pairs, or of hard-photon emission accompanying tau-pair production, merit consideration.

An example of original thinking is a proposal to study $\gamma\gamma \to \tau^+\tau^-$ in ultraperipheral heavy-ion collisions at the Large Hadron Collider, aiming at a three-fold improvement on the DELPHI constraint [58]. The BELLE-II experiment [52] aims to record 45 billion tau pairs, which will enable new searches for a (P- and T-violating) electric dipole moment and constraints on $a_\tau$ [59]. What are the prospects for future electron–positron colliders: the International Linear Collider [60], FCC-ee [61], CEPC [62], CLIC [63], etc.?

III. $g = 2$ WITHOUT DIRAC?

Dirac’s statement of purpose and the elegant coherence of his results suggest that the gyromagnetic ratio $g = 2$ for the electron is a consequence of special relativity [64]. And yet, following an argument given by Feynman [65], modern quantum mechanics textbooks show that the
canonical result can be recovered by simple manipulations within the framework of nonrelativistic quantum mechanics, without explicitly invoking relativity [66].

Using the vector identity

$$\sigma \cdot X \sigma \cdot Y = X \cdot Y + i \sigma \cdot X \times Y,$$

(37)

where $X$ and $Y$ are arbitrary 3-vectors and $\sigma$ is the Pauli spin matrix, we may write the free-electron Schrödinger Hamiltonian as

$$\mathcal{H} = \frac{P^2}{2m_e} = \frac{(\sigma \cdot P)^2}{2m_e}$$

(38)

To include electromagnetic interactions we couple the vector potential $A$ as prescribed by nonrelativistic mechanics ($P \to \Pi = P - eA/c$) and expand

$$\frac{(\sigma \cdot \Pi)(\sigma \cdot \Pi)}{2m_e}$$

(39)

to obtain

$$\mathcal{H} = \frac{(P - eA/c)^2}{2m_e} + i\sigma \cdot \frac{(P - eA/c) \times (P - eA/c)}{2m_e}.$$  
(40)

Then with the substitution $P = -i\hbar \nabla$ and recognizing $\nabla \times A = B$ as the magnetic field, we find

$$\Pi \times \Pi = \frac{i\hbar}{c} B.$$  
(41)

The second term of Eqn. (40) becomes

$$- \frac{e\hbar}{2m_e c} \sigma \cdot B,$$  
(42)

so the gyromagnetic ratio $g = 2$ emerges from the Schrödinger equation, without specific mention of relativity. We did have to identify the electron as a spin-$\frac{1}{2}$ particle, whereas that emerges from Dirac’s construction, and we all learned in school that spin-$\frac{1}{2}$ particles correspond to particular representations of the Lorentz group, which seems to point to relativistic roots [57].

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[1] P. A. M. Dirac, “The quantum theory of the electron,” Proc. Roy. Soc. Lond. A 117, 610–624 (1928). See also *The Principles of Quantum Mechanics* (Oxford University Press, 1981) §70.
[2] P. A. Zyla et al. (Particle Data Group), “Review of Particle Physics,” Prog. Theor. Exp. Phys. 2020, 083C01 (2020). This is the source for parameters not otherwise attributed in this note.
[3] G. E. Uhlenbeck and S. Goudsmit, “Spinning electrons and the structure of spectra,” Nature 117, 264–265 (1926). Their first publication on electron spin is “Ersetzung der Hypothese vom unmechanischen Zwang durch eine Forderung bezüglich des inneren Verhaltens jedes einzelnen Elektrons,” Die Naturwissenschaften 13, 953–954 (1925). Spin is identified as the mysterious fourth quantum number posited by W. Pauli, “Über den Zusammenhang des Abschlusses der Elektroengruppen im Atom mit der Komplexstruktur der Spektren,” Zeitschrift für Physik 31, 765–783 (1925).
[4] What he understands by “transformation theory” is explained in P. A. M. Dirac, “The physical interpretation of the quantum dynamics,” Proceedings of the Royal Society of London A 113, 621–641 (1927). We might simply say, “quantum mechanics.”
[5] For a systematic presentation, see Barry R. Holstein, “How large is the ‘natural’ magnetic moment?” American Journal of Physics 74, 1104–1111 (2006). Other interesting considerations appear in H. Pfister and M. King, “The gyromagnetic factor in electrodynamics, quantum theory and general relativity,” Class. Quantum Grav. 20, 205 (2003); S. Ferrara, M. Porrati, and V. L. Telegdi, “$g = 2$ as the natural value of the tree-level gyromagnetic ratio of elementary particles,” Phys. Rev. D 46, 3529 (1992).
[6] Steven Weinberg, “Dynamic and algebraic symmetries,” in Lectures on Elementary Particles and Quantum Field Theory, Vol. 1, edited by S. Deser, M. Grisaru, and H. Pendleton (MIT Press, Cambridge, MA, 1970) 1970 Brandeis University Summer Institute.
[7] For a helpful introduction to the subject, see G. F. Giudice, P. Paradisi, and M. Passera, “Testing new physics with the electron $g - 2$,” JHEP 11, 113 (2012). arXiv:1208.6583 [hep-ph].
[8] Silvan S. Schweber, *QED and the Men Who Made It* (Princeton University Press, Princeton, 1994). See in particular Chapter 5, “The Lamb Shift and the Magnetic Moment of the Electron.”
[9] Stanley J. Brodsky and S. D. Drell, “The Anomalous Magnetic Moment and Limits on Fermion Substructure,” Phys. Rev. D 22, 2236 (1980).
[10] M. Agostini et al. (Borexino), “A test of electric charge conservation with Borexino,” Phys. Rev. Lett. 115, 231802 (2015) arXiv:1509.01223 [hep-ex]. Borexino is an exquisitely radiopure liquid scintillation detector located deep underground at the Gran Sasso Laboratory. The lifetime bound derives from a search for electron decay into a neutrino and a single monoenergetic photon.
[11] P. Kusch and H. M. Foley, “Precision Measurement of the Ratio of the Atomic ‘g’ Values in the $2P_3$ and $2P_1$ States of Gallium,” Phys. Rev. 72, 1256–1257 (1947). “The magnetic moment of the electron,” Phys. Rev. 74, 250–263 (1948).

[12] Julian S. Schwinger, “On Quantum electrodynamics and the magnetic moment of the electron,” Phys. Rev. 73, 416–417 (1948). Schwinger’s grave marker in the Mount Auburn Cemetery in Cambridge, Massachusetts, bears the inscription. \(\frac{\alpha}{\pi}\).

[13] The first calculation to include a fermion loop was given by Robert Karpplus and Norman M. Kroll, “Fourth-Order Corrections in Quantum Electrodynamics and the Magnetic Moment of the Electron,” Phys. Rev. 77, 536–540 (1950) An important correction leading to the result displayed here is due to A. Petermann, “Fourth order magnetic moment of the electron,” Helv. Phys. Acta 30, 407–408 (1957) and to Charles M. Sommerfield, “Magnetic Dipole Moment of the Electron,” Phys. Rev. 107, 328–329 (1957). “The magnetic moment of the electron,” Annals of Physics 5, 26–57 (1958).

[14] Fred Jegerlehner and Andreas Nyffeler, “The Muon $g-2$,” Phys. Rept. 477, 1–110 (2009) arXiv:0902.3360 [hep-ph].

[15] M. Passera, “The Standard model prediction of the muon anomalous magnetic moment,” J. Phys. G 31, R75–R94 (2005) arXiv:hep-ph/0411168 [hep-ph].

[16] S. Laporta and E. Remiddi, “The Analytical value of the electron $(g-2)$ at order $\alpha^3$ in QED.” Phys. Lett. B 379, 283–291 (1996) arXiv:hep-ph/9602417 [hep-ph].

[17] Tatsumi Aoyama, Toichiro Kinoshita, and Makiko Nio, “Theory of the anomalous magnetic moment of the electron,” Atoms 7, 28 (2019).

[18] Richard H. Parker, Chenghui Yu, Weicheng Zhong, Brian Estey, and Holger Müller, “Measurement of the fine-structure constant as a test of the standard model,” Science 360, 191–195 (2018).

[19] Léo Morel, Zhibin Yao, Pierre Cladé, and Saïda Guellati-Khélifa, “Determination of the fine-structure constant with an accuracy of 81 parts per trillion,” Nature 588, 61–65 (2020). Saïda Guellati-Khélifa, “Measuring the fine-structure constant to refine the Standard Model predictions,” CERN EP Seminar, May 4, 2021. For an assessment, see Holger Müller, “Standard model of particle physics tested by the fine-structure constant,” Nature 588, 37–38 (2020).

[20] Ref. 17 quoted an earlier measurement of $\alpha^{-1}$ (Rb): Ryn Bouchendira, Pierre Cladé, Saïda Guellati-Khélifa, François Nez, and François Biraben, “New determination of the fine-structure constant and test of the quantum electrodynamics,” Phys. Rev. Lett. 106, 080801 (2011) arXiv:1012.3627 [physics.atom-ph].

[21] Sergey Volkov, “Calculating the five-loop QED contribution to the electron anomalous magnetic moment: Graphs without lepton loops,” Phys. Rev. D 100, 096004 (2019) arXiv:1909.08015 [hep-ph]. See §1 for the consequences for $\alpha$ and $\alpha^{-1}$.

[22] A. Rich and J. C. Wesley, “The current status of the lepton $g$ factors,” Rev. Mod. Phys. 44, 250–283 (1972).

[23] John C. Wesley and Arthur Rich, “High-field electron $g - 2$ measurement,” Phys. Rev. A 4, 1341–1363 (1971).

[24] Robert S. Van Dyck, Paul B. Schwinberg, and Hans G. Dehmelt, “New high-precision comparison of electron and positron $g$ factors,” Phys. Rev. Lett. 59, 26–29 (1987) See also Hans Dehmelt, “Experiments with an isolated subatomic particle at rest,” Rev. Mod. Phys. 62, 525–530 (1990) alternate source: 1989 Nobel Lecture Lowell S. Brown and Gerald Gabrielse, “Geonium Theory: Physics of a Single Electron or Ion in a Penning Trap,” Rev. Mod. Phys. 58, 233 (1986).

[25] D. Hanneke, S. Fogwell, and G. Gabrielse, “New Measurement of the Electron Magnetic Moment and the Fine Structure Constant,” Phys. Rev. Lett. 100, 120801 (2008) arXiv:0801.1134 [physics.atom-ph]. The value of $\alpha^{-1}$ inferred from the measurement, based on eighth-order theory, is superseded by the tenth-order result quoted in Eqn. (22).

[26] G. Gabrielse, S. E. Fayer, T. G. Myers, and X. Fan, “Towards an Improved Test of the Standard Model’s Most Precise Prediction,” Atoms 7, 45 (2019) arXiv:1904.06174 [quant-ph]. For general information, see the Gabrielse Lab web site. See also X. Fan and G. Gabrielse, “Driven one-particle quantum cyclotron,” Phys. Rev. A 103, 022821 (2021).

[27] Yu. M. Andreev et al. (NA64), “Constraints on New Physics in the Electron $g - 2$ from a Search for Invisible Decays of a Scalar, Pseudoscalar, Vector, and Axial Vector.” (2021), Phys. Rev. Lett. (to be published), arXiv:2102.01885 [hep-ex]. This article contains an extensive list of useful references.

[28] R. L. Garwin, L. M. Lederman, and Marcel Weinrich, “Observations of the Failure of Conservation of Parity and Charge Conjugation in Meson Decays: The Magnetic Moment of the Free Muon,” Phys. Rev. 105, 1415–1417 (1957) see also J. I. Friedman and V. L. Telegdi, “Nuclear Emulsion Evidence for Parity Nonconservation in the Decay Chain $\pi^+ \rightarrow \mu^+ \rightarrow e^+\nu$,” Phys. Rev. 106, 1290–1293 (1957).

[29] For a treatment of the muon spin asymmetry through $O(\alpha^2)$ in QED, see Fabrizio Caola, Andrzej Czarnecki, Yi Liang, Kirill Melnikov, and Robert Szafron, “Muon decay spin asymmetry,” Phys. Rev. D 90, 053004 (2014) arXiv:1403.3386 [hep-ph].

[30] J. M. Cassels et al., “Experiments with a polarized muon beam,” Proc. Phys. Soc. A 70, 543–546 (1957). The (anomalous) precession frequency of a muon moving through a magnetic field is imprinted on the time distribution of high-energy positrons emitted along the $\mu^+$ spin direction in the form of oscillations on the exponential decay curve characteristic of the (boosted) muon lifetime.

[31] T. Coffin, R. L. Garwin, S. Penman, L. M. Lederman, and A. M. Sachs, “Magnetic Moment of the Free Muon,” Phys. Rev. 109, 973–979 (1958) R. L. Garwin, D. P. Hutchinson, S. Penman, and G. Shapiro, “Accurate Determination of the $\mu^+$ Magnetic Moment,” Phys. Rev. 118, 271–283 (1960).

[32] G. Charpak, P. J. M. Farley, E. L. Garwin, T. Muller, J. C. Sens, and A. Zichichi, “The anomalous magnetic moment of the muon,” Nuovo Cim. 37, 1241–1363 (1965).

[33] J. Bailey, W. Bartl, G. von Bochmann, R. C. A. Brown, F. J. M. Farley, M. Giesch, H. Jostlein, S. van der Meer, E. Picasso, and R. W. Williams, “Precise Measurement of the Anomalous Magnetic Moment of the Muon,” Nuovo Cim. A 9, 369–432 (1972).

[34] J. Bailey et al. (CERN-Mainz-Daresbury), “Final Report on the CERN Muon Storage Ring Including the Anomalous Magnetic Moment and the Electric Dipole Moment...
of the Muon, and a Direct Test of Relativistic Time Dilatation," Nucl. Phys. B 150, 1–75 (1979).

35. G. W. Bennett et al., “Final Report of the Muon E821 Anomalous Magnetic Moment Measurement at BNL,” Phys. Rev. D 73, 072003 (2006) [arXiv:hep-ex/0602035]. Also see the E821 web site. The central value is adjusted to the latest value of $\mu_{\mu}/\mu_B$ by the Muon g – 2 Theory Initiative.

36. G. W. Bennett et al. (Muon (g – 2) Collaboration), “Measurement of the negative muon anomalous magnetic moment to 0.7 ppm,” Phys. Rev. Lett. 92, 161802 (2004).

37. R. M. Carey et al., “The New (g – 2) Experiment: A proposal to measure the muon anomalous magnetic moment to ±0.14 ppm precision,” FERMILAB-PROPOSAL-0989 (2009).

38. Hiroshi Suura and Eyyvid H. Wichmann, “Magnetic Moment of the Mu Meson,” Phys. Rev. 105, 1930–1931 (1957).

39. A. Petermann, “Magnetic moment of the mu meson,” Phys. Rev. 105, 1931 (1957); and Sommerfield, Ref. 13. For a complete evaluation of the electron loop, retaining $m_e/m_\mu$ terms, see H. H. Elen, “On the anomalous magnetic moment of the muon,” Physics Letters 20, 682–684 (1966) [erratum: ibid. 21, 720 (1966)].

40. Tatsushi Aoyama, Masashi Hayakawa, Toichiro Kinoshita, and Makiko Nio, “Complete Tenth-Order QED Contribution to the Muon g – 2,” Phys. Rev. Lett. 109, 111808 (2012) [arXiv:1205.5370 [hep-ph]]

41. T. Blum, “Lattice calculation of the lowest-order hadronic contribution to the muon g – 2,” Phys. Rev. D 74, 034503 (2001) [arXiv:hep-ph/0007350].

42. M. Davier, A. Hoecker, B. Malaceu, and Z. Zhang, “A new evaluation of the hadronic vacuum polarisation contributions to the muon anomalous magnetic moment and to $a(M_Z^2)$,” Eur. Phys. J. C 80, 241 (2020) [arXiv:1908.00921 [hep-ph]].

43. Alexander Keshavarzi, Daisuke Nomura, and Thomas Teubner, “$g – 2$ of charged leptons, $a(M_Z^2)$, and the hyperfine splitting of muonium,” Phys. Rev. D 101, 014029 (2020) [arXiv:1911.00367 [hep-ph]].

44. T. Blum, “Lattice calculation of the lowest-order hadronic contribution to the muon anomalous magnetic moment,” Phys. Rev. Lett. 91, 052001 (2003).

45. Harvey B. Meyer and Hartmut Wittig, “Lattice QCD and the anomalous magnetic moment of the muon,” Progress in Particle and Nuclear Physics 104, 46–96 (2019).

46. B. Abi et al. (Muon g – 2 Collaboration), “Measurement of the positive muon anomalous magnetic moment to 0.46 ppm,” Phys. Rev. Lett. 126, 141801 (2021). More information is available at the E939 web site.

47. It is an interesting challenge to exploit the greater precision of $a_\tau$ to test interpretations of the $a_\tau$ deviation. See F. Terranova and G. M. Tino, “Testing the $a_\tau$ anomaly in the electron sector through a precise measurement of $h/m_\tau$,” Phys. Rev. A 89, 052118 (2014).

48. Andrzej Czarnecki, William J. Marciano, and Arkady Vainshtein, “Refinements in electroweak contributions to the muon anomalous magnetic moment,” Phys. Rev. D 67, 073006 (2003) [Erratum: Phys. Rev.D 73, 119901 (2006)] [arXiv:hep-ph/0212229]. G. Gnendiger, D. Stöckinger, and H. Stöckinger-Kim, “The electroweak contributions to $(g – 2)_\mu$ after the Higgs boson mass measurement,” Phys. Rev. D 88, 053005 (2013) [arXiv:1306.5546 [hep-ph]]. The corresponding contribution for the electron is $a_\tau^{[\text{EW}]} = 3.053 (23) \times 10^{-14}$. Fred Jegerlehner, “Variations on Photon Vacuum Polarization,” EPJ Web Conf. 218, 01003 (2019) [arXiv:1711.06089 [hep-ph]].

49. Sz. Borsanyi et al. (BMW Collaboration), “Leading hadronic contribution to the muon magnetic moment from lattice QCD,” Nature 593, 51–55 (2021) [arXiv:2002.12347 [hep-lat]]. See also Zoltan Fodor, “$(g – 2)_\mu$ from lattice QCD and experiments: 4.2σ?,” DESY Colloquium, April 13, 2021.

50. Vladimir Druzhinin (BaBar), “Recent results on hadronic cross sections measurements at BABAR for the $g – 2$ calculation,” PoS EPS-HEP2019, 535 (2020).

51. A. Ryzhennov et al., “Overview of the CMD-3 recent results,” Journal of Physics: Conference Series 1526, 012009 (2020).

52. M. N. Achasov et al. (SNF), “Measurement of the $e^+e^- \rightarrow \pi^+\pi^-$ process cross section with the SND detector at the VEPP-2000 collider in the energy region $0.525 < \sqrt{s} < 0.883$ GeV,” JHEP 01, 113 (2021) [arXiv:2004.00263 [hep-ex]].

53. E. Kon, F. Urquijo, W. Altmannohefer, et al., “The Belle II Physics Book,” PTEP 2019, 123C01 (2019) [arXiv:1808.10567 [hep-ex]].

54. C. M. Carbone, R. Patrignani, M. Passera, T. Tantacuo, and G. Venanzoni, “A new approach to evaluate the leading hadronic corrections to the muon g-2,” Phys. Lett. B 746, 325–329 (2015) [arXiv:1504.02228 [hep-ph]]. G. Abbiendi et al., “Measuring the leading hadronic contribution to the muon g-2 via $\mu e$ scattering,” Eur. Phys. J. C 77, 139 (2017) [arXiv:1609.08887 [hep-ex]]. For a recent update, see Giovanni Abbiendi, “Status of the MUnOe experiment,” PoS ICHEP2020, 223 (2021) [arXiv:2012.07016 [hep-ex]].

55. M. Abe et al., “A new approach for measuring the muon anomalous magnetic moment and electric dipole moment,” Progress of Theoretical and Experimental Physics 2019, 053C02 (2019). More information is available at the experiment’s web page.

56. See Problem 1.6 of Chris Quigg, Gauge Theories of the Strong, Weak, and Electromagnetic Interactions (Benjamin / Cummings, Reading, Massachusetts, 1983); second edition, (Princeton University Press, Princeton, 2013).

57. For an earlier estimate, see S. Eidelman and M. Passera, “Theory of the tau lepton anomalous magnetic moment,” Mod. Phys. Lett. A 22, 159–179 (2007) [arXiv:hep-ph/0701260].

58. J. Abdallah et al. (DELPHI), “Study of tau-pair production in photon-photon collisions at LEP and limits on the anomalous electromagnetic moments of the tau lepton,” Eur. Phys. J. C 35, 159–170 (2004) [arXiv:hep-ex/0406010]. Compare the value $a_\tau = 0.004 \pm 0.027 \pm 0.023$ from the rate of hard photons in $e^+e^- \rightarrow \tau^+\tau^-\gamma$ reported by M. Acciarri et al. (L3), “Measurement of the anomalous magnetic and electric dipole moments of the tau lepton,” Phys. Lett. B 434, 169–179 (1998) [see also K. Ackerstaff et al. (OPAL), “An upper limit on the anomalous magnetic moment of the tau lepton,” Phys. Lett. B 431, 188–198 (1998)] [arXiv:hep-ex/9803020] for the result $–0.068 < a_\tau < 0.063$.

59. Lydia Beresford and Jesse Liu, “New physics and tau $g – 2$ using LHC heavy ion collisions,” Phys. Rev. D 102, 113008 (2020) [arXiv:1908.05180 [hep-ph]].
but competitive—measurements are reported in CMS Physics Analysis Summary HIN-21-09, “Observation of $\tau$ lepton pair production in ultraperipheral nucleus–nucleus collisions,” and in ATLAS Collaboration, “Observation of the $\gamma\gamma \rightarrow \tau\tau$ process in Pb+Pb collisions and constraints on the $\tau$-lepton anomalous magnetic moment with the ATLAS detector,” (2022), arXiv:2204.13478 [hep-ex]

[59] Matteo Fael, Lorenzo Mercalli, and Massimo Passera, “Towards a determination of the tau lepton dipole moments,” Nucl. Phys. B Proc. Suppl. 253-255, 103–106 (2014), arXiv:1301.5302 [hep-ph]; S. Eidelman, D. Epifanov, M. Fael, L. Mercalli, and M. Passera, “$\tau$ dipole moments via radiative leptonic $\tau$ decays,” JHEP 03, 140 (2016), arXiv:1601.07987 [hep-ph]

[60] “The International Linear Collider,” web site.

[61] A. Abada et al. (FCC), “FCC Physics Opportunities: Future Circular Collider Conceptual Design Report Volume 1,” Eur. Phys. J. C 79, 474 (2019)

[62] Mingyi Dong et al. (CEPC Study Group), “CEPC Conceptual Design Report: Volume 2 - Physics & Detector,” (2018), arXiv:1811.10545 [hep-ex]

[63] “CLIC Detectors and Physics,” web site.

[64] This conviction is reinforced by Dirac’s debt to L. H. Thomas, “The motion of a spinning electron,” Nature 117, 514 (1926); “The kinematics of an electron with an axis,” Phil. Mag. Ser. 7 3, 1–21 (1927) For a reminiscence, see “Recollections of the discovery of the Thomas precessional frequency,” AIP Conf. Proc. 95, 4–12 (1983)

[65] Richard P. Feynman, Quantum Electrodynamics (W. A. Benjamin, 1961).

[66] J. J. Sakurai, Modern Quantum Mechanics (Addison-Wesley, 1967), pp. 78–79. R. Shankar, Principles of Quantum Mechanics (Springer US, 1994) Chapter 20, where Dirac’s derivation is recast in modern vector notation. See also the survey by Ronald J. Alder and Robert A. Martin, “The electron $g$ factor and factorization of the Pauli equation,” American Journal of Physics 60, 837–839 (1992)

[67] For an argument that spin emerges from (nonrelativistic) Galilean invariance, see Jean-Marc Lévy-Leblond, “Non-relativistic particles and wave equations,” Communications in Mathematical Physics 6, 286–311 (1967)