Non-Riemannian Description of Robinson-Trautman Spacetimes in Brans-Dicke Theory Of Gravity

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Abstract
The variational field equations of Brans-Dicke scalar-tensor theory of gravitation are given in a non-Riemannian setting in the language of exterior differential forms over 4-dimensional spacetimes. A conformally re-scaled Robinson-Trautman metric together with the Brans-Dicke scalar field are used to characterise algebraically special Robinson-Trautman spacetimes. All the relevant tensors are worked out in a complex null basis and given explicitly in an appendix for future reference. Some special families of solutions are also given and discussed.

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1 Introduction

Robinson-Trautman spacetimes \[1\,2\] were first presented in the early 60’s. Since then, they became one of the most important solutions of general relativity in our understanding of gravitational waves. These physically important solutions to the Einstein field equations are found by considering the fact that they should admit a null, geodesic and congruence without shear or twist. In order to illustrate the physical meaning of these solutions of the empty space Einstein field equations, it is found useful to study the asymptotic invariants of the Weyl tensor\[3\]. Mathematical studies on the global structure of Robinson-Trautman spacetimes are presented vividly in the literature\[4\], including cases of radiative space-times with cosmological constant\[5,\,6\] using analytical methods. One may consult the relevant chapters in Ref.\[7\] for details. More recent works\[8\] on the algebraic classification of Robinson-Trautman spacetimes provide non-trivial explicit examples in which the focus is on specific algebraic properties of the Robinson-Trautman spacetimes with a free scalar field. We note that the Robinson-Trautman spacetimes are frequently used to describe radially propagating spherical gravitational waves. Such studies are specifically realistic to depict imploding spherical stars and/or coalescing binary black holes.

Brans-Dicke scalar-tensor theory of gravity also dates from the early 1960’s and it has been recognised very early as the most viable alternative to Einstein’s theory of general relativity. It has found many applications since then to problems in gravitation and cosmology. String theorists have also pointed out similarities between string models and scalar-tensor theories, prompting further investigation of the latter\[9\]. Actually, the gravitational coupling constant has a characteristic aspect in modern cosmology: It may depend on the cosmic time. This idea is initially due to Dirac\[10\] who noticed that the analysis of constants of cosmological nature and the fundamental physical constants will become inter-related if these so-called constants are allowed to vary slowly over cosmological time scales. The work of Brans and Dicke\[11\,\,12\,\,13\] accomplished to detail this idea where a scalar field is regarded as a local gravitational coupling and carries gravity together with the metric tensor of general relativity. On the one hand, Gürsey\[14\] argued that Einstein’s theory of general relativity as it stands incorporates Mach’s Principle and detailed the reformulation of the metric as a product of a scalar density $\phi^2$ and a tensor density. On the other hand, conformal re-scaling properties of scalar-tensor gravity theories have been further elaborated, for instance, by Deser\[15\] and Anderson\[16\]. Here, we are generalising this model by considering a non-Riemannian description of spacetime where the spacetime
torsion is determined by the gradient of a scalar field\cite{17,18}. We are going to include in our discussion an arbitrary conformal scale factor of the metric as well in order to analyse the solutions in different "frames" (pictures). Robinson-Trautman solutions in the Einstein "frame" (picture) have been studied much in detail while the corresponding solutions in Brans-Dicke-Jordan "frame" (picture) or in the string "frame" (picture) are physically equivalent to those in the Einstein "frame" (picture) in the sense that these pictures could be related to each other by certain re-scalings of the basic fields. However, since a non-Riemannian description of scalar-tensor theories in the latter two pictures are possible\cite{17}, the equations of motion of non-spinning test masses in such spacetimes may differ in general from the geodesic equations of motion in a Riemannian setting\cite{19,20,21,22}. Motivated by these considerations, we discuss in this paper the Robinson-Trautman solutions of the Brans-Dicke field equations in a non-Riemannian setting. We are going to use extensively the language of exterior differential forms in a complex null basis.

The paper is organised as follows. In Section 2, the Brans-Dicke field equations will be derived by a first order variational principle using the language of exterior differential forms. The role played by the space-time torsion 2-forms that are proportional to the gradient of the Brans-Dicke scalar field will be discussed. The non-Riemannian space-time geometry expressed in terms of complex null differential forms and related formulas are given separately in the Appendix A. In Section 3, a conformally re-scaled Robinson-Trautman metric ansatz and a corresponding scalar field assumption are given. All the relevant geometrical expressions are explicitly worked out in Appendix B. We reduce the Brans-Dicke field equations in Robinson-Trautman spacetimes to a system of coupled non-linear differential equations and check their vacuum Einstein solutions for consistency. Furthermore a family of static, spherically symmetric solutions with non-vanishing scalar charge (the Janis-Newman-Winicour solutions) of the coupled Einstein-massless scalar field equations is written down. Moreover a new family of propagating solutions is constructed. It corresponds to a special case of recently found Robinson-Trautman type solutions of the Einstein massless scalar field equations\cite{25,26}. Section 4 is devoted to concluding remarks.
There are strong reasons motivated by recent astrophysical and cosmological observations that Einstein’s general relativity theory may require the inclusion of certain yet undetected scalar fields of either gravitational or matter origin. Furthermore low energy string field theories involve many unobserved scalar degrees of freedom and the most unified theories of strong and electroweak interactions predict scalar fields with both astrophysical and cosmological implications. In fact, Brans and Dicke suggested in 1961 a modification of Einsteinian gravity by introducing a real scalar field with particular couplings to matter via the space-time metric. This is arguably the simplest modification of general relativity. Brans-Dicke theory is a scalar-tensor theory of gravity that is still regarded as the most viable alternative to Einstein’s theory. A real parameter $\omega$ characterises the gravitational effects of the scalar field $\phi$ so that predictions of the Brans-Dicke theory are indistinguishable from those of Einstein’s theory for large values of $\omega$. Newton’s universal gravitational coupling constant $G$ is usually related in this theory to solutions which have a constant value of the scalar field, i.e. $<\phi> \sim \frac{1}{8\pi G}$. This modified theory can also be invoked in the context of conformal re-scalings of the space-time metric. A locally scale invariant theory of gravity occurs for the specific case for $\omega = -\frac{3}{2}$. The field equations of the Brans-Dicke theory are derived by a variational principle from an action $I = \int_M \mathcal{L}$, where the action density 4-form

$$\mathcal{L} = \frac{\phi}{2} R_{ab} \wedge * (e^a \wedge e^b) - \frac{\omega}{2\phi} d\phi \wedge *d\phi + \Lambda \phi^2 * 1.$$  \hspace{1cm} (1)$$

$\{e^a\}$ are the co-frame 1-forms in terms of which the space-time metric is given by $g = \eta_{ab} e^a \otimes e^b$ with $\eta_{ab} = diag(-+++). *$ denotes the Hodge map referring to the space-time orientation $*1 = e^0 \wedge e^1 \wedge e^2 \wedge e^3$. $\phi$ is the Brans-Dicke scalar field. We also introduced a cosmological constant $\Lambda$. $\{\omega^a_b\}$ are the connection 1-forms that satisfy the Cartan structure equations

$$de^a + \omega^a_b \wedge e^b = T^a$$ \hspace{1cm} (2)$$

with the torsion 2-forms $T^a$ and

$$d\omega^a_b + \omega^a_c \wedge \omega^c_b = R^a_b$$ \hspace{1cm} (3)$$

are the curvature 2-forms of spacetime. Conventionally the Levi-Civita connection 1-forms $\{\tilde{\omega}^a_b\}$ are assumed which are fixed in a unique way by the metric tensor through the Cartan structure equations

$$de^a + \tilde{\omega}^a_b \wedge e^b = 0.$$ \hspace{1cm} (4)$$
Now the connection 1-forms admit a unique decomposition,
\[ \omega^a_b = \hat{\omega}^a_b + K^a_b \]  
(5)
in terms of the contortion 1-forms \( K^a_b \) that satisfy \( K^a_b \wedge e^b = T^a \). Then the field equations derived from the action above varied independently with respect to the co-frames and the scalar field, subject to the constraint that the connection is Levi-Civita, turn out to be (\( \omega \neq -\frac{3}{2} \))
\[ -\frac{\phi}{2} \tilde{R}^{bc} \wedge * (e_a \wedge e_b \wedge e_c) = \frac{\omega}{2\phi} (\iota_a \parb \phi \star d\phi + d\phi \wedge \iota_a \star d\phi) + \hat{D}(\iota_a \star d\phi) + \Lambda \phi^2 \star e_a, \]
\[ d \star d\phi = 0 \]  
(6)
where \( \hat{D} \) denotes the covariant exterior derivative with respect to the Levi-Civita connection. A first order variational derivation of these field equations is also possible with some interesting consequences. In that case, we set \( \phi = \alpha^2 \) for convenience. Then the Brans-Dicke action density 4-form becomes
\[ \mathcal{L} = \frac{\alpha^2}{2} R_{ab} \wedge *(e^a \wedge e^b) - \frac{c}{2} d\alpha \wedge *d\alpha + \Lambda \alpha^4 \star 1, \]
\[ (7) \]
where \( c \) is a real coupling constant. This action will be varied independently with respect to the co-frame 1-forms \( e^a \), connection 1-forms \( \omega^a_b \) and the scalar field \( \alpha \). The corresponding coupled field equations turn out to be \( (c \neq 0) \):
\[ -\frac{\alpha^2}{2} R^{bc} \wedge * (e_a \wedge e_b \wedge e_c) = c \tau_a[\alpha] + \Lambda \alpha^4 \star e_a, \]
\[ T^a = e^a \wedge \frac{d\alpha}{\alpha}, \]
\[ cd \star d\alpha^2 = 0; \]  
(8)
where
\[ \tau_a[\alpha] = \frac{1}{2} (\iota_a d\alpha \star d\alpha + d\alpha \wedge \iota_a \star d\alpha) \equiv T_{ab} \star e^b \]
\[ (9) \]
are the energy-momentum 3-forms of the scalar field. Therefore, taking the torsion 2-forms given by the above variational field equations and after some algebra, we arrive at the following decomposition for the Einstein 3-forms
\[ R^{bc} \wedge * (e_a \wedge e_b \wedge e_c) = \tilde{R}^{bc} \wedge * (e_a \wedge e_b \wedge e_c) - \frac{4}{\alpha^2} (\iota_a d\alpha \star d\alpha + d\alpha \wedge \iota_a \star d\alpha) - \frac{2}{\alpha^2} \iota_a (d\alpha \wedge *d\alpha) + \frac{4}{\alpha} \hat{D}(\star d\alpha). \]
\[ (10) \]
Substituting these expressions in the variational field equations we find
\[ -\frac{\alpha^2}{2} \hat{R}^{bc} \wedge * (e_a \wedge e_b \wedge e_c) = \frac{(c-6)}{2} (\iota_a d\alpha \ast d\alpha + d\alpha \wedge \iota_a \ast d\alpha) \]
\[ + \hat{D}(\iota_a \ast d\alpha^2) + \Lambda \alpha^4 \ast e_a, \]
\[ cd \ast d\alpha^2 = 0. \quad (11) \]
These coincide with the Brans-Dicke equations given before for \( \phi = \alpha^2 \)
provided we make the identification
\[ c = 4\omega + 6. \quad (12) \]
We also note the conformal re-scalings of the metric induced by the local transformations
\[ g \to e^{2\sigma(x)} g. \quad (13) \]
Then
\[ e^a \to e^a e^a, \quad \hat{\omega}^a_b \to \hat{\omega}^a_b - (\iota^a d\sigma)e_b + (\iota_b d\sigma)e^a. \quad (14) \]
If we now postulate the conformal re-scaling of the scalar field as
\[ \alpha \to e^{-\sigma} \alpha, \quad (15) \]
then it follows that
\[ K^a_b \to K^a_b + (\iota^a d\sigma)e_b - (\iota_b d\sigma)e^a. \quad (16) \]
Therefore we have \( \omega^a_b \to \omega^a_b \) and \( R^a_b \to R^a_b \), so that the Brans-Dicke action density 4-form is conformally scale invariant for the choice \( c = 0 \), that is, for \( \omega = -\frac{3}{2} \). In fact, Brans and Dicke postulated independently of their field equations that the non-spinning test masses should follow geodesic equations of motion given by
\[ \hat{\nabla}_c \hat{\nabla}^c \hat{C} = 0, \quad (17) \]
where \( \hat{\nabla} \) is the unique Levi-Civita connection of \( g \) and \( \hat{C} \) is a unit, time-like tangent vector field so that \( g(\hat{C}, \hat{C}) = -1 \). On the other hand here the autoparallel equations of motion relative to the non-Riemannian connection \( \nabla \) can be decomposed as
\[ \nabla_c \hat{C} = \hat{\nabla}_c \hat{C} + \frac{1}{\alpha} \iota_c (d\alpha \wedge \tilde{C}) = 0, \quad (18) \]
and would differ in general from the geodesic equations of motion.
Once the conformal re-scaling properties of the connection and curvature are set, a re-formulation of the Brans-Dicke action is possible if we adopt another orthonormal frame defined by $\tilde{e}_a = \frac{\alpha}{\alpha_0} e^a$ where $\alpha_0$ is a constant that fixes a scale. Note that since $e^a$’s are orthonormal with respect to the metric $g$, we must require $\tilde{e}^a$’s to be orthonormal with respect to a re-scaled metric

$$\tilde{g} = \left(\frac{\alpha}{\alpha_0}\right)^2 g.$$  \hspace{1cm} (19)

Then it can be shown that our action density 4-form takes the form

$$L = \frac{\alpha_0^2}{2} R_{ab} \wedge \tilde{*}(\tilde{e}^a \wedge \tilde{e}^b) - \frac{c \alpha_0^2}{2 \alpha^2} d\alpha \wedge \tilde{*}d\alpha + \alpha_0^4 \Lambda \tilde{*}1.$$  \hspace{1cm} (20)

For $c \neq 0$, a further re-definition

$$\Phi = \ln \left(\frac{\alpha}{\alpha_0}\right),$$  \hspace{1cm} (21)

gives the action density 4-form in the Einstein picture:

$$L = \frac{\alpha_0^2}{2} R_{ab} \wedge \tilde{*}(\tilde{e}^a \wedge \tilde{e}^b) - \frac{c \alpha_0^2}{2} d\Phi \wedge \tilde{*}d\Phi + \alpha_0^4 \Lambda \tilde{*}1.$$  \hspace{1cm} (22)

The connection variations of this action yield the equation

$$\mathcal{D}_\omega \tilde{*}(\tilde{e}^a \wedge \tilde{e}^b) = 0$$  \hspace{1cm} (23)

whose unique solution implies that the connection should be the torsion-free Levi-Civita connection of the metric $\tilde{g}$. Thus although the torsion associated with the fields $(g, \alpha)$ in Brans-Dicke-Jordan picture need not be zero, the torsion of the re-defined fields in Einstein picture $(\tilde{g}, \Phi)$ should be zero. In particular, we wish to emphasise that the auto-parallel curves determined in terms of the connection with torsion in the Brans-Dicke-Jordan picture are equivalent to geodesic curves in the Einstein picture. This is essentially the observation of Dirac. On the other hand, Brans and Dicke argued in their paper that the scalar field $\phi$ should not couple to matter fields. This assumption in fact amounts to postulating geodesic equations of motion for spin-less test particles in the Brans-Dicke-Jordan picture that would be different in general from the auto-parallel equations of motion.
3 Robinson-Trautman Spacetimes

In order to solve the Brans-Dicke field equations we generalise (for convenience) the Robinson-Trautman ansatz as follows\[1\]:

\[
g = e^{2\sigma(u,r)} \left\{-C(u, r, \zeta, \bar{\zeta}) du^2 + 2dudr + \frac{4h^2(u, r)}{P^2(\zeta, \bar{\zeta})} d\zeta d\bar{\zeta}\right\},
\]

\[
\alpha = \alpha(u, r),
\]

(24)
in a coordinate chart \((u, r, \zeta, \bar{\zeta})\) where \(u\) is a null coordinate and \(\zeta, \bar{\zeta}\) are the stereographic projection coordinates on \(S^2\). We introduced a new function \(h(u, r)\) besides \(C(u, r, \zeta, \bar{\zeta})\) and an arbitrary scale factor \(\sigma(u, r)\), that all to be determined by the field equations.

Inserting the expressions for curvatures given in Appendix B in (70) and using (72) for the stress-energy-momentum tensor, the Einstein field equations to be solved reduce to the following system:

\[
\left(\frac{h_r C_r}{h} + \frac{h_r r C_r}{h} + \frac{4\sigma_r h_u}{h} + \frac{\alpha_r \alpha u}{\alpha} + \frac{\alpha_r C}{\alpha} + \frac{4\sigma_u h_r}{h} + \frac{C_r \alpha_r}{\alpha}\right) + \frac{4\sigma_r \sigma_u}{\alpha} + \frac{\alpha^2 C}{h^2} + \frac{2\alpha_r \alpha u}{\alpha} + \frac{h^2 C}{h^2} + \frac{4\alpha_r h_u}{h} + \frac{4\sigma_u h_r}{h} + \frac{4\sigma_r h_r C}{h}
\]

\[
+ \frac{4\sigma_r C \alpha_r}{\alpha} + \frac{P_{\zeta \bar{\zeta}}}{h^2} + \frac{P_{\zeta \bar{\zeta}}}{h^2} + 4\sigma_r \sigma_u + 2\sigma^2 C + \frac{2h_u r}{h} + \sigma_r C_r
\]

\[
+ \frac{2\alpha u_r}{\alpha} + 2\sigma u_r + \frac{4\alpha_r h_r C}{h}\alpha + \sigma_r C + \frac{2h_r h_u}{h^2} = 0 ,
\]

(25)

\[
\left(\frac{\sigma_r}{\alpha} - \frac{\alpha r_r}{\alpha} - \frac{h_r}{\alpha} + \frac{h^2}{\alpha} + \frac{2\sigma_r \alpha_r}{\alpha} + \frac{2\alpha_r^2}{\alpha^2} - \frac{1}{2} \frac{\alpha^2}{\alpha^2}\right) = 0 ,
\]

(26)

\[
\left(\frac{C_r \zeta}{\alpha} + \frac{C_{\zeta} \alpha_r}{\alpha} + \sigma_r C_\zeta\right) = 0 ,
\]

(27)

\[
\left(\frac{P^2 C_{\zeta \bar{\zeta}}}{h^2} - \frac{4C \alpha_u}{\alpha} - \frac{2C_u \alpha_r}{\alpha} + \frac{4\sigma_u C \alpha_r}{\alpha} + \frac{4\sigma_r C \alpha_u}{\alpha} + \frac{2\sigma_r C^2 \alpha_r}{\alpha}\right)
\]

\[
- \frac{h_r r C^2}{h} - \frac{2\alpha^2 C^2}{\alpha^2} - \frac{\alpha r_r C^2}{\alpha} + \frac{h^2 C_r}{h} - 4\sigma_u u + \sigma^2 C^2 + 2\sigma_r C_r
\]

\[
- \sigma_r r C^2 + \frac{8\alpha_u C \alpha_r}{\alpha^2} + \frac{8\alpha_r^2}{\alpha^2} - \frac{4\alpha u u}{\alpha} + 4\alpha^2 + 4\sigma_u \sigma_r C + \frac{8\sigma_u \alpha_u}{\alpha}
\]

The separation of variables in the angular part is different than the standard one, e.g. that one may found in Ref.[3], motivated by some recent classical solutions in this class.
We also work out the scalar field equation to complete this system:

\[ \left( \frac{h_r C_r}{h} + \frac{h_{rr} C}{h} + \frac{2 \sigma r h_u}{h} - \frac{2 \sigma_r \alpha u}{\alpha} + \frac{2 \alpha r C_r}{\alpha} + \frac{2 \sigma_r h_r}{h} + \frac{2 C_r \alpha r}{\alpha} \right) = 0, \quad \text{(28)} \]

\[ \left( \frac{h_r C_r}{h} + \frac{h_{rr} C}{h} + \frac{2 \sigma r h_u}{h} - \frac{2 \sigma_r \alpha u}{\alpha} + \frac{2 \alpha r C_r}{\alpha} + \frac{2 \sigma_r h_r}{h} + \frac{2 C_r \alpha r}{\alpha} \right) \right) = 0. \quad \text{(29)} \]

We also work out the scalar field equation to complete this system:

\[ c \left( \frac{h_r C_r}{h} + \frac{h_{rr} C}{h} + \frac{2 \sigma r h_u}{h} - \frac{2 \sigma_r \alpha u}{\alpha} + \frac{2 \alpha r C_r}{\alpha} + \frac{2 \sigma_r h_r}{h} + \frac{2 C_r \alpha r}{\alpha} \right) = 0, \quad \text{(30)} \]

Thus we have five functions \( C(u, r, \zeta, \bar{\zeta}), P(\zeta, \bar{\zeta}), h(u, r), \sigma(u, r) \) and \( \alpha(u, r) \) to be determined by the coupled equations above.

### 3.1 Vacuum Einstein Solutions

For the sake of completeness and as a check on our field equations we first set \( \alpha = 1 \) and choose \( \sigma = 0 \) to retrieve the well known solutions of the vacuum Einstein field equations. Then we must take \( C(u, r) \) only \(^2\) and \( h(u, r) = re^{-\eta(u)} \) for some arbitrary function \( \eta(u) \). Then the above field equations reduce to the following coupled system of equations:

\[ \frac{1}{r^2} C + \frac{1}{r} C_r - \frac{1}{r^2} (PP_{\zeta \zeta} - P_{\zeta} P_{\zeta}) - \frac{4}{r} \eta' = 0, \]

\[ 4 \eta'' + \frac{4}{r} C + \frac{4}{r} C_r - 2 \eta' C_r - 2 \frac{2}{r} C_u = 0, \]

\(^2\) This case corresponds to subclass of only Petrov type D solutions as also confirmed by equation (33) which means that the 2-surfaces spanned by complex coordinates have constant curvature, so if they should be compact they are necessarily spheres. The choice of meromorphic functions in (34) does not provide any new solutions. The same applies to the line element (35) as well.
\[ C_{rr} + \frac{2}{r}C_r - \frac{4}{r} \eta' = 0. \] (31)

These equations are solved by

\[ C(r, u) = 2\eta'(u)r + \gamma_0 e^{2\eta(u)} + \frac{\gamma - 1}{r} e^{3\eta(u)} \] (32)

provided the function \( P(\zeta, \bar{\zeta}) \) satisfies

\[ PP_{\zeta} - P_{\zeta} P = \gamma_0. \] (33)

If we let \( P = e^B \), it reduces to the Liouville equation \( B_{\zeta\bar{\zeta}} = \gamma_0 e^{-2B} \). Hence a general solution can be written as

\[ P(\zeta, \bar{\zeta}) = \frac{1 + \gamma_0|\psi(\zeta)|^2}{|\psi(\zeta)|^2}, \] (34)

in terms of an arbitrary meromorphic function \( \psi(\zeta) \). In particular, for the choice \( \eta(u) = 0 \), the Schwarzschild metric is obtained with \( \psi(\zeta) = \zeta \) and \( \gamma_0 = 1, \gamma = -2M \) where \( M \) is the Schwarzschild mass.

A family of non-static solutions for \( \eta(u) \neq 0 \) is given by the metric

\[ g = - \left( 2\eta'(u)r + e^{2\eta(u)} - \frac{2M}{r} e^{3\eta(u)} \right) du^2 + 2dudr + \frac{4r^2 e^{-2\eta(u)}}{(1 + |\zeta|^2)^2} d\zeta d\bar{\zeta}. \] (35)

### 3.2 Brans-Dicke Solutions

In a milestone paper Janis, Newman and Winicour \[23, 24\] have constructed an explicit metric for the static spherically symmetric solution of the coupled Einstein-massless scalar field equations. They have taken the truncated Schwarzschild metric i.e. the exterior Schwarzschild solution as the physical solution with a non-zero scalar charge corresponding to a point mass. We found the corresponding solution of the Brans-Dicke field equations in the Robinson-Trautman form as given by the metric

\[ g = e^{2\sigma(r)} \left\{ - \left( \frac{r - M(\mu + 1)}{r + M(\mu - 1)} \right)^{\frac{1}{\mu}} du^2 + 2dudr + \frac{4h^2(r)}{(1 + |\zeta|^2)^2} d\zeta d\bar{\zeta} \right\}, \] (36)

and the scalar field

\[ \alpha = \alpha_0 \left( \frac{r - M(\mu + 1)}{r + M(\mu - 1)} \right)^{\frac{1}{2\mu}} \] (37)
where
\[ \sigma = \frac{A}{2\mu} \ln \left| \frac{r + M(\mu - 1)}{r - M(\mu + 1)} \right| \]
and
\[ h^2(r) = (r + M(\mu - 1))^{1 + \frac{1}{\mu}} (r - M(\mu + 1))^{1 - \frac{1}{\mu}} \]
provided
\[ (\mu - 1)(\mu + 1) = \frac{c}{2} A^2. \quad (38) \]
Here \( r \) stands for the standard Schwarzschild coordinate. The Schwarzschild mass \( M \) and the scalar charge \( A \) are identified in the weak field limit.\(^3\)

We further look for a family of non-static solutions for which
\[ \alpha = e^{-\sigma}. \quad (39) \]
This assumption simplifies the field equations to a large extent. We take \( C(u, r) \) only and set \( PP_{\xi} = P_{\xi} = \gamma_0 \) as before. Then we are left with the following system of equations to solve:
\[ (hh_r C)_r + (2hh_r)_u = \gamma_0, \quad (40) \]
\[ 2 \frac{h_{rr}}{h} + c \left( \frac{\alpha_r}{\alpha} \right)^2 = 0, \quad (41) \]
\[ \frac{h_r}{h} C_u - \frac{h_u}{h} C_r + 2 \frac{h_{ru}}{h} C + 2 \frac{h_{uu}}{h} + c \left( \frac{\alpha_u}{\alpha} \right)^2 + c \frac{\alpha_u \alpha_r}{\alpha^2} = 0, \quad (42) \]
\[ \frac{1}{h^2} \left( h^2 C_r \right)_r + 4 \frac{h_{ur}}{h} + 2c \frac{\alpha_u \alpha_r}{\alpha^2} = 0, \quad (43) \]
\[ c \left\{ (h^2 C \frac{\alpha_r}{\alpha})_r + 2h^2 \left( \frac{\alpha_r}{\alpha} \right)_u + 2hh_r \frac{\alpha_u}{\alpha} + 2hh_u \frac{\alpha_r}{\alpha} \right\} = 0. \quad (44) \]
\(^3\)The scalar charge \( A \) is identified as the coefficient of the leading term in the expansion of the scalar field \( \alpha \) in powers of \( \frac{1}{r} \). In case the scalar charge \( A = 0 \) vanishes, the metric reduces to the standard Schwarzschild metric from which the mass \( M \) is identified.
A solution is found as follows:

\[ C(u, r) = \gamma_0 e^{-c_1 u} - c_1 r, \quad h(u, r) = \sqrt{e^{c_1 u} r^2 - e^{-c_1 u} c_0^2}, \]

\[ \alpha(u, r) = \alpha_0 \left( \frac{r + c_0 e^{-c_1 u}}{r - c_0 e^{-c_1 u}} \right)^{\frac{1}{\sqrt{2c}}}, \quad (45) \]

where \( \alpha_0, c_0, c_1 \) are arbitrary integration constants. This solution can be related with a special case of the Robinson-Trautman solutions of the Einstein-massless scalar field equations found recently by Tahamtan and Svitek\[25, 26\]. In particular in their set-up, one should consider \( k(x, y) = 0 \).

4 Concluding Remarks

We considered here a non-Riemannian description of the Brans-Dicke field equations. The fact that the field equations are expressed in a complex null basis facilitates the discussion of Robinson-Trautman space-times. All the relevant formulas are written down explicitly in an appendix for future reference.

We also provided a brief discussion on how a change of picture can be affected by local re-scalings of the metric and the scalar field. As a check on consistency, we presented the special case of vacuum Einstein solutions for \( \alpha = 1 \) and \( \sigma = 0 \). Similarly we presented a special family of static solutions in the Brans-Dicke-Jordan picture that corresponds to the well-known Janis-Newman-Winicour family which is customarily given in the Einstein picture.

Denoting the energy-momentum 3-forms of the scalar field and the space-time torsion 2-forms, we have presented the Brans-Dicke field equations obtained by a first order variation with respect to the co-frames and the scalar field. We generalized the ansatz by inserting a function \( h(u, r) \) and an arbitrary conformal factor \( \sigma(u, r) \). Solving the complex connection 1-forms from the Cartan structure equations, we calculated the complex null curvature 2-forms, Einstein 3-forms and Weyl curvature 2-forms. We checked the known source-free static and non-static solutions verifying the classical results of Robinson-Trautman. On the other hand, we presented the Brans-Dicke static solution of the massless scalar field derived by Janis-Newman-Winicour and a new class of non-static solutions. We noted at the end that more solutions can be obtained by choosing the relationship between the conformal factor \( \sigma(u, r) \) and the scalar field \( \alpha(u, r) \).
A propagating class of solutions is also found under the assumption $e^{-\sigma} = \alpha$. These correspond to a special case of Robinson-Trautman solutions of the Einstein-massless scalar field equations recently found by Tahamtan and Svitek\[25, 26\]. They have explicitly derived a Robinson-Trautman solution coupled to a scalar field and have shown that the solution has a singularity created by the divergence of the scalar field therein. The explicit solution given is asymptotically flat, contains a black hole and carries non-vanishing scalar charge. Compared to our metric, we can immediately see that the choice for the matter of convenience that they did lies in the solution of the field equations (26) and (27) with the scaling function taken as $\sigma(u, r) = -ln(\alpha(u, r))$. The only solution to (26) and (27) with the latter assumption is expressed by two functions $b_1$ and $b_2$ as $C(u, r, \zeta, \bar{\zeta}) = b_1(u, \zeta, \bar{\zeta}) + b_2(u, r)$.

For future work, we plan to find and investigate further new Robinson-Trautman type solutions of the Brans-Dicke gravity. There are some such solutions already in the literature\[27, 28, 29, 30, 31\]. Some solutions describing spherical gravitational waves in Brans-Dicke gravity are also known\[32, 33\]. However, a fresh physical outlook in both respects can be fruitful. In particular, Vaidya type of solutions\[34, 35\] in the Brans-Dicke-Jordan picture would be physically interesting in view of recent work on algebraically special solutions in AdS/CFT\[36, 37\].

In Brans-Dicke theory, as we have shown above, the gravitational field equations are modified by the scalar field, however, the motion of a (scalar) test mass under the influence of gravitation is still assumed to follow the space-time geodesics specified by the Levi-Civita connection of the metric tensor only. It was Dirac who first pointed out that it would be more natural to generate the motion of a (scalar) test mass from a locally scale invariant action. In such a case, the resulting equations of motion of a test mass differ in general from a Brans-Dicke geodesic. In the present framework, Dirac’s proposal amounts to the fact that the test masses under the influence of Brans-Dicke gravity should follow auto-parallels of the connection with torsion. It is a well known fact that geodesic curves and auto-parallel curves are distinct in general. In the absence of spinorial matter, no new physics of the fields may arise from re-formulation of the field equations as given above. On the other hand, the behaviour of scalar test masses in a space-time geometry with torsion would differ from their geodesic motion and hence observations must be analysed to decide between these two alternative approaches.

An analysis of the equations of motion of a test mass in Robinson-Trautman spacetimes we found would be important. The auto-parallel
trajectories in the non-Riemannian framework in the Brans-Dicke-Jordan picture differ in general from the geodesic trajectories postulated by Brans and Dicke themselves. Such a difference may give rise to profound effects on the black hole interpretation of the Robinson-Trautman type solutions [38, 39, 40].

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Appendix A: Spacetime Geometry with Torsion in a Complex Null Basis

Since we will be using the complex null co-frame in our calculations extensively, in the following we will summarize the basics. First of all, we are fixing the relations between the orthonormal and complex null co-frames by

\[ l = \frac{e^3 + e^0}{\sqrt{2}}, \quad n = \frac{e^3 - e^0}{\sqrt{2}}, \quad m = \frac{e^1 + ie^2}{\sqrt{2}}, \quad \bar{m} = \frac{e^1 - ie^2}{\sqrt{2}} \]  

(46)

and their inverse by

\[ e^0 = \frac{l - n}{\sqrt{2}}, \quad e^3 = \frac{l + n}{\sqrt{2}}, \quad e^1 = \frac{\bar{m} + m}{\sqrt{2}}, \quad e^2 = -i \frac{m - \bar{m}}{\sqrt{2}}. \]  

(47)

The metric becomes

\[ g = 2l \otimes n + 2m \otimes \bar{m} \]  

(48)

where the bar over a symbol denotes its the complex conjugate.

In terms of the orthonormal quantities, the relation \( \iota_a e^b = \delta^b_a \) is written in the complex null basis with the 1-forms as

\[ \iota_l n = \iota_n l = \iota_m \bar{m} = \iota_{\bar{m}} m = 1, \quad \iota_l l = \iota_m n = \iota_m \bar{m} = \iota_{\bar{m}} m = 0. \]  

(49)

The relations between the orthonormal and the null interior product is such that

\[ \iota_l = \frac{\iota_3 - \iota_0}{\sqrt{2}}, \quad \iota_n = \frac{\iota_3 + \iota_0}{\sqrt{2}}, \quad \iota_m = \frac{\iota_1 + i\iota_2}{\sqrt{2}}, \quad \iota_{\bar{m}} = \frac{\iota_1 - i\iota_2}{\sqrt{2}} \]  

(50)

and their inverse as follows

\[ \iota_0 = \frac{-\iota_l + \iota_n}{\sqrt{2}}, \quad \iota_3 = \frac{\iota_l + \iota_n}{\sqrt{2}}, \quad \iota_1 = \frac{\iota_m + i\iota_{\bar{m}}}{\sqrt{2}}, \quad \iota_2 = -i \frac{\iota_m - \iota_{\bar{m}}}{\sqrt{2}}. \]  

(51)

The Hodge dual defined in the orthonormal quantities is

\[ *e^{a_1\ldots a_p} = \frac{1}{(D - p)!} \epsilon^{a_1\ldots a_p a_{p+1}\ldots a_D} e^{a_{p+1}\ldots a_D}, \]  

(52)

where we consider a \( D \) dimensional manifold and \( \epsilon_{01\ldots D} = +1 \) is the totally antisymmetric Levi-Civita symbol in D-dimensions. In \( D = 4 \) dimensions, this linear map acting on the null basis 1-forms gives

\[ \*1 = il \wedge n \wedge m \wedge \bar{m}, \] 
\[ \*n = in \wedge m \wedge \bar{m}, \] 
\[ \*l = -il \wedge m \wedge \bar{m}, \] 
\[ \*m = -il \wedge n \wedge m, \] 
\[ \*(l \wedge n) = -im \wedge \bar{m}, \] 
\[ *\bar{m} = -il \wedge \bar{m}, \] 
\[ *\bar{m} = \bar{m} \wedge n \wedge m. \]  

(53)
Let us define the complex null connection 1-forms as

$$\omega^k = -\frac{1}{2}(i\omega^0_k + \frac{1}{2}\epsilon_{ijk}\omega^j)$$

(54)

where \(i, j, k, \ldots = 1, 2, 3\) and \(\epsilon_{ijk}\) is totally antisymmetric and satisfies the relation \(\epsilon_{123} = +1\). Let's write explicitly

$$\omega_1 = -\frac{1}{2}(i\omega^0_1 + \omega^2_3), \quad \omega_2 = -\frac{1}{2}(i\omega^0_2 - \omega^1_3), \quad \omega_3 = -\frac{1}{2}(i\omega^0_3 + i\omega^1_2)$$

(55)

and conversely

$$\omega^0_1 = i(\omega_1 - \bar{\omega}_1), \quad \omega^0_2 = i(\omega_2 - \bar{\omega}_2), \quad \omega^0_3 = i(\omega_3 - \bar{\omega}_3),
\omega^2_3 = -(\omega_1 + \bar{\omega}_1), \quad \omega^1_3 = (\omega_2 + \bar{\omega}_2), \quad \omega^1_2 = -(\omega_3 + \bar{\omega}_3).$$

(56)

At this point we adopt a more useful notation by setting

$$\omega_+ = \omega^1 + i\omega^2, \quad \omega_- = \omega^1 - i\omega^2, \quad \omega_0 = \omega^3$$

(57)

and conversely

$$\omega^1 = \frac{1}{2}(\omega_+ + \omega_-), \quad \omega^2 = -\frac{i}{2}(\omega_+ - \omega_-), \quad \omega^3 = \omega_0.$$  

(58)

Given the torsion 2-forms

$$T^a = d\epsilon^a + \omega^a_b \wedge e^b,$$

the first set of Cartan structure equations are written in terms of the complex quantities as

$$dl + i(\omega_0 - \bar{\omega}_0) \wedge l + i\omega_+ \wedge \bar{m} - i\bar{\omega}_+ \wedge m = \frac{T^3 + T^0}{\sqrt{2}},$$

$$dn - i(\omega_0 - \bar{\omega}_0) \wedge n + i\omega_- \wedge \bar{m} - i\bar{\omega}_- \wedge m = \frac{T^3 - T^0}{\sqrt{2}},$$

$$dm + i(\omega_0 + \bar{\omega}_0) \wedge m - i\omega_+ \wedge n - i\bar{\omega}_- \wedge l = \frac{T^1 + iT^2}{\sqrt{2}}.$$  

(59)

The corresponding complex null curvature 2-forms are given by

$$R^k = -\frac{1}{2}(iR^0_k + \frac{1}{2}\epsilon_{ijk}R^j).$$

(60)

Again a more convenient notation can be introduced so that

$$R_+ = R^1 + iR^2, \quad R_- = R^1 - iR^2, \quad R_0 = R^3$$

(61)

and conversely

$$R^1 = \frac{1}{2}(R_+ + R_-), \quad R^2 = -\frac{i}{2}(R_+ - R_-), \quad R^3 = R_0.$$  

(62)
Thus the complex null curvature 2-forms are determined by the second set of Cartan structure equations
\[ R_0 = d\omega_0 - i\omega_- \wedge \omega_+ , \]
\[ R_+ = d\omega_+ + 2i\omega_0 \wedge \omega_+ , \]
\[ R_- = d\omega_- - 2i\omega_0 \wedge \omega_- . \] (63)
The Einstein 3-forms \( G_a := -\frac{1}{2} R^{bc} \wedge *e_{abc} \) can be stated now as such:
\[ G_3 + G_0 = \frac{1}{\sqrt{2}} ((R_0 + \bar{R}_0) \wedge n + R_+ \wedge m + \bar{R}_- \wedge \bar{m}) , \]
\[ G_3 - G_0 = -\frac{1}{\sqrt{2}} ((R_0 + \bar{R}_0) \wedge l - \bar{R}_+ \wedge m - R_- \wedge \bar{m}) , \]
\[ G_1 + iG_2 = -\frac{1}{\sqrt{2}} ((R_0 - \bar{R}_0) \wedge m - \bar{R}_- \wedge l + R_+ \wedge n) , \]
\[ G_1 - iG_2 = -\frac{1}{\sqrt{2}} ((R_0 - \bar{R}_0) \wedge \bar{m} - R_- \wedge l + \bar{R}_+ \wedge n) . \] (64)
We also introduce the Weyl curvature 2-forms as usual:
\[ C_{ab} = R_{ab} - \frac{1}{2} (e_a \wedge P_b - e_b \wedge P_a) + \frac{R}{6} e_{ab} \] (65)
where \( R_{ab}, P_a = \iota_b R^b_a \) and \( R \) are, respectively, the Riemann curvature 2-forms, Ricci 1-forms and the curvature scalar. In terms of the Schouten 1-forms given by
\[ 2S_a = P_a - \frac{R}{6} e_a \]
in \( D = 4 \) dimensions, we have
\[ R_{ab} = C_{ab} + (S_a \wedge e_b - S_b \wedge e_a) . \]
Then the Cotton curvature 2-forms are determined by \( Y_a = D S_a \). In the complex null basis, the components of the Weyl curvature 2-forms are given as follows:
\[ iC_+ = iR_+ + \frac{1}{2} (l \wedge (\frac{S_1 + iS_2}{\sqrt{2}}) - m \wedge (\frac{S_3 - S_0}{\sqrt{2}}) ) ; \]
\[ iC_0 = iR_0 + \frac{1}{4} (l \wedge (\frac{S_3 + S_0}{\sqrt{2}}) - n \wedge (\frac{S_4 - S_0}{\sqrt{2}}) + m \wedge (\frac{S_1 - iS_2}{\sqrt{2}}) ) ; \]
\[ iC_- = iR_- + \frac{1}{2} ( - n \wedge (\frac{S_1 - iS_2}{\sqrt{2}}) + \bar{m} \wedge (\frac{S_3 + S_0}{\sqrt{2}}) ) . \] (66)
Appendix B: Connections and Curvatures in Robinson-Trautman Spacetimes

From this point on the Appendix will be used to introduce the complex null co-frame formalism and to illustrate our results. For the Robinson-Trautman metric (24), the complex null basis 1-forms in terms of the coordinate 1-forms are given by

\[ l = e^\sigma du, \quad n = e^\sigma (dr - \frac{C}{2} du), \quad m = e^\sigma \sqrt{\frac{2h}{P}} d\zeta. \]  

(67)

The torsion 2-forms in complex null basis are found to be

\[ \frac{T^3 + T^0}{\sqrt{2}} = e^{-\sigma} \frac{\alpha_r}{\alpha} l \land n, \quad \frac{T^3 - T^0}{\sqrt{2}} = e^{-\sigma} \left( \frac{\alpha_u}{\alpha} + \frac{C\alpha_r}{2\alpha} \right) n \land l, \]

\[ \frac{T^1 + iT^2}{\sqrt{2}} = e^{-\sigma} \left( \frac{\alpha_u}{\alpha} + \frac{C\alpha_r}{2\alpha} \right) m \land l + e^{-\sigma} \frac{\alpha_r}{\alpha} m \land n. \]  

(68)

After some lengthy calculations we determine the complex null connection 1-forms with torsion as

\[ i\omega_+ = -e^{-\sigma} \left( \frac{h_r}{h} + \sigma_r + \frac{\alpha_r}{\alpha} \right) m, \]

\[ i\omega_0 = \frac{e^{-\sigma}}{2} \left( \frac{C_r}{2} + \sigma_u + \frac{\sigma_r C}{2} + \frac{\alpha_u}{\alpha} + \frac{\alpha_r C}{2\alpha} \right) l - \frac{e^{-\sigma}}{2} \left( \frac{\alpha_u}{\alpha} + \sigma_r \right) n \]

\[ -e^{-\sigma} \frac{P_\zeta}{2\sqrt{2h}} m + e^{-\sigma} \frac{P_{\bar{\zeta}}}{2\sqrt{2h}} \bar{m}, \]

\[ i\omega_- = e^{-\sigma} \left( \frac{h_u}{h} + \frac{h_r C}{2h} + \sigma_u + \frac{\sigma_r C}{2} + \frac{\alpha_u}{\alpha} + \frac{\alpha_r C}{2\alpha} \right) \bar{m} + e^{-\sigma} \frac{C_{\zeta} P}{2\sqrt{2h}} l. \]

Using these in the Cartan structure equations (63), we calculate the corresponding complex null curvature 2-forms

\[ iR_+ = -e^{-2\sigma} \left( \frac{1}{2} \alpha_{rr} C + \frac{\sigma_r h_u}{h} + \frac{\alpha_r h_r C}{h\alpha} + \frac{\alpha_r u}{\alpha} + \frac{\alpha_r u}{\alpha} + \frac{1}{2} C_r \alpha_r \right. \]

\[ + \frac{1}{2} \frac{h_{rr} C}{h} + \frac{\sigma_r C \alpha_r}{\alpha} + \frac{\alpha_r h_u}{h\alpha} + \frac{\alpha_r h_r C}{h\alpha} + \frac{\alpha_r u}{\alpha} + \frac{h_{ur}}{h} \]

\[ + \frac{1}{2} \frac{\sigma_u h_r}{h} + \frac{1}{2} \frac{h_r C_r}{h} + \sigma_r \sigma_u + \frac{1}{2} \sigma_r^2 C + \frac{1}{2} \sigma_r C_r \) \quad l \land m. \]
\[ i R_0 = e^{-2\sigma} \left( \frac{2 \sigma_C}{\alpha} - \frac{\alpha_{rr}}{\alpha} - \sigma_{rr} + \sigma_C^2 - \frac{h_{rr}}{h} + \frac{2 \alpha_C^2}{\alpha^2} \right) n \wedge m, \]

\[ i R_+ = e^{-2\sigma} \left( - \frac{1}{2} \sigma_{\alpha C}^2 \right) l \wedge n + e^{-2\sigma} \left( - \frac{1}{4 \sqrt{2}} \frac{C_P \zeta \eta}{h} + \frac{1}{2 \sqrt{2}} \frac{P h_C \zeta}{h} \right) l \wedge m, \]

\[ i R_- = e^{-2\sigma} \left( - \frac{1}{2} \sigma_C \frac{h_u}{h} - \frac{1}{2} \sigma_{\alpha C} \frac{h_u}{h} - \frac{2 \alpha_{u \alpha u}}{\alpha} - \sigma_{u \sigma}, C - \frac{2 \alpha_{u \alpha C}}{\alpha^2} - \sigma_{u \alpha C} \frac{F}{\alpha} \right) l \wedge m, \]

\[ + e^{-2\sigma} \left( \frac{h_u}{h} + \frac{h_{uu}}{h} + \frac{h_{ur} C}{h} + \frac{h_{rr} C}{h} - \frac{1}{2} \sigma_{u \sigma}, C + \frac{1}{4} \frac{C_p \zeta}{h} \right) l \wedge m, \]

\[ + e^{-2\sigma} \left( \frac{h_u}{h} + \frac{h_{uu}}{h} + \frac{h_{ur} C}{h} + \frac{h_{rr} C}{h} - \frac{1}{2} \sigma_{u \sigma}, C + \frac{1}{4} \frac{C_p \zeta}{h} \right) n \wedge m, \]

\[ + e^{-2\sigma} \left( \frac{1}{2 \sqrt{2}} \frac{P h_C}{h} - \frac{1}{2 \sqrt{2}} \frac{P \sigma_C}{h} - \frac{1}{2 \sqrt{2}} \frac{C_P \zeta}{h} - \frac{1}{2 \sqrt{2}} \frac{C_{P \alpha}}{h} \right) l \wedge n, \]

\[ - e^{-2\sigma} \left( \frac{1}{2 \sqrt{2}} \frac{P \sigma_C}{h} + \frac{1}{2 \sqrt{2}} \frac{C_{P \alpha}}{h} \right) l \wedge m. \]
Finally, the Weyl curvature 2-forms are

\begin{align*}
C_+ &= -\frac{1}{6} \left( -\frac{P \zeta P}{h^2} + \frac{P \zeta P}{h^2} + \frac{h^2 C}{h^2} - \frac{h_r C}{h} - \frac{h_{rr} C}{h} - \frac{2 h_{ur}}{h} + \frac{2 h_u h_r}{h^2} \\
&\quad + \frac{C_{rr}}{2} \right) e^{-2\sigma} il \wedge m,
C_- &= \frac{P}{2h^2} \left( \frac{P \zeta C + PC\zeta}{2} \right) e^{-2\sigma} il \wedge m \\
&\quad + \frac{1}{6} \left( -\frac{P \zeta P}{h^2} + \frac{P \zeta P}{h^2} + \frac{h^2 C}{h^2} - \frac{h_r C}{h} - \frac{h_{rr} C}{h} - \frac{2 h_{ur}}{h} + \frac{2 h_u h_r}{h^2} \\
&\quad + \frac{C_{rr}}{2} \right) e^{-2\sigma} in \wedge \bar{m} - \frac{P}{2h} \left( \frac{h_r C}{h} - \frac{C_{r\zeta}}{2} \right) e^{-2\sigma} \frac{i}{\sqrt{2}} (l \wedge n + m \wedge \bar{m}),
C_0 &= -\frac{P}{2\sqrt{2}h} \left( \frac{h_r C}{h} - \frac{C_{r\zeta}}{2} \right) e^{-2\sigma} il \wedge m \\
&\quad + \frac{1}{3\sqrt{2}} \left( -\frac{P \zeta P}{h^2} + \frac{P \zeta P}{h^2} + \frac{h^2 C}{h^2} - \frac{h_r C}{h} - \frac{h_{rr} C}{h} - \frac{2 h_{ur}}{h} + \frac{2 h_u h_r}{h^2} \\
&\quad + \frac{C_{rr}}{2} \right) e^{-2\sigma} \frac{i}{\sqrt{2}} (l \wedge n + m \wedge \bar{m}).
\end{align*}

It turns out from the above expressions that the Weyl scalars \( \Psi_0 = \Psi_1 = 0 \) vanish so that our spacetime is of Petrov type II in general.

In order to determine the coupled equations to be solved we also write down the stress-energy-momentum tensor of the scalar field in the complex null basis as follows:

\begin{align*}
\tau_3 + \tau_0 &= -e^{-2\sigma} (\alpha_u + \frac{\alpha_r C}{2})^2 il \wedge m \wedge \bar{m}, \\
\tau_3 - \tau_0 &= e^{-2\sigma} \alpha_r^2 in \wedge m \wedge \bar{m},
\end{align*}

\begin{align*}
\tau_1 + i\tau_2 &= e^{-2\sigma} \alpha_r (\alpha_u + \frac{\alpha_r C}{2}) il \wedge n \wedge m.
\end{align*}
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