A CONSTRUCTION OF $C^r$ CONFORMING FINITE ELEMENT SPACES IN ANY DIMENSION

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Abstract. This paper proposes a construction of $C^r$ conforming finite element spaces with arbitrary $r$ in any dimension. It is shown that if $k \geq 2^d r + 1$ the space $P_k$ of polynomials of degree $\leq k$ can be taken as the shape function space of $C^r$ finite element spaces in $d$ dimensions. This is the first work on constructing such $C^r$ conforming finite elements in any dimension in a unified way.

1. Introduction

This paper is to provide an $H^{r+1}$ conforming finite element method of $2(r + 1)$-th order elliptic problems on simplicial triangulations in $\mathbb{R}^d$. The conforming finite element method is to seek for piecewise polynomial function spaces with global $C^r$ continuity. The commonly used $H^1$ conforming element is the celebrated $C^0$ Lagrange element on simplicial triangulations in $d$ dimensions. While the Hermite element is still $C^0$ conforming but cannot admit higher global continuity when $d > 1$. The case $d = 1$ is an exception in the sense that the one-dimensional Hermite element is $H^2$ conforming. The construction of $C^r$ conforming finite elements on simplicial triangulations in $d$ dimensions is a long-standing open problem [26]. One main difficulty is the choice of the shape function space. It is commonly conjectured that the shape function space can be chosen as the space $P_k$ of polynomials of degree not greater than $k$ with $k \geq 2^d r + 1$. However, no successful construction can be found in the literature. Another main difficulty is the design of degrees of freedom. Indeed, the traditional bubble function technique does not work anymore in general.

Many efforts have been made for this problem and partial results can be found in Bramble and Zlámal [7], where two dimensional $C^r$ elements on triangular grids were constructed for any $r \geq 0$ with $k = 4r + 1$. For the case $r = 1$, it recovers the $P_5 - C^1$ Argyris element [4]. For the three-dimensional case, the first $P_9 - C^1$ element on tetrahedral grids was constructed by Ženíšek in [27]. Later, a family of $P_{9r+1} - C^r$ elements on tetrahedral grids was constructed for any $r \geq 0$ in Lai and Schumaker [19], which recovers the $P_9 - C^1$ Ženíšek element for the case $r = 1$. Zhang [28] also extended the Ženíšek element into a $P_{9k} - C^1$ element family for $k \geq 9$. A family of $P_5 - C^2$ elements on tetrahedral grids and a family of $P_k - C^1$ elements on 4-simplices with $k \geq 17$ were proposed in Zhang [29], where the bubble function spaces were tailored.

On the contrary, the construction of $C^r$ conforming finite elements on $d$-cube grids is much easier. In fact, a family of $C^r$ conforming finite elements was designed in Hu and Zhang [16] on macro-$d$-cube grids by using the space $Q_{r+1}$ consisting of polynomials of degree $\leq r + 1$ in each variable, see also Hu, Huang and Zhang [13] for $C^1$ conforming finite elements on macro-$d$-cube grids.

Given the difficulty of constructing $C^r$ conforming elements in $d$ dimensions, an alternative way is to weaken the continuity and to construct $H^{r+1}$ nonconforming elements. In this direction, the first and very elegant construction in any dimension is from Wang and Xu [23], where nonconforming finite elements on simplicial triangulations were proposed and analyzed for $d \geq r + 1$ by using the space $P_{r+1}$ as the shape function space. For the case $r = 1$, it recovers a very famous nonconforming element, namely, the Morley element, of fourth order problems [20, 21]. That family was later extended to the case $r = d$ by enriching the full polynomial space $P_{r+1}$ with higher order bubbles in Wu and Xu [25]. Recently, a family of $H^{r+1}$ nonconforming finite elements was established by employing an interior penalty technique for the case $r + 1 > d$, using the space $P_{r+1}$ as the shape function space of $C^r$ finite element spaces in $d$ dimensions. This is the first work on constructing such $C^r$ conforming finite elements in any dimension in a unified way.
function space, see Wu and Xu [24]. While in Hu and Zhang [17], a family of two dimensional $H^{r+1}$ nonconforming elements was constructed on triangular grids, using the space $P_{2r-1}$ as the shape function space when $r > 2$.

If non-polynomials are considered as the shape functions, the virtual element method [5, 6] can be used to design both conforming and nonconforming approximations of the space $H^{r+1}$ in any dimension. The interested readers can refer to Antonietti, Manzini and Verani [3], Chen and Huang [9], Huang [18] for relevant virtual element methods.

The construction generalizes all the conforming elements on simplicial triangulations introduced in (1.4) where the bubble space dimension. The interested readers can refer to Antonietti, Manzini and Verani [3], Chen and Huang [9], Huang [18] for relevant virtual element methods.

The neural network is a new method for discretization on partial differential equations. In Xu [26], the finite neuron method was proposed, utilizing a generalized ReLU neural network architecture to propose a conforming approximation of the space $H^{r+1}$ for any $r$ in $d$ dimensions.

A relevant topic of $C^r$ finite element methods is spline interpolations or supersplines [1, 11]. In Chui and Lai [11], the authors first constructed a family of vertex $C^r$ splines on simplicial triangulations in two dimensions by using piecewise polynomials of degrees not greater than $k$ with $k \geq 4r+1$, which is in fact a variant of the Bramble–Zlámal family [7]. Then they formally extended their approach to constructing vertex $C^r$ splines on simplicial triangulations in any dimension by using piecewise polynomials of degrees not more than $k$ with $k \geq 2d r + 1$. In Alfeld, Schumaker, and Sirvent [2], it was proved that when the polynomial of degrees $k \geq 2d r + 1$ there exists a structure of the spline spaces that allow the construction of a minimally supported basis. However, no degrees of freedom were proposed therein, which is a key of finite element methods. The reference [1] is a following-up paper of [2], whose main result is, by using Berstein–Bézier techniques, to show the existence of the local basis of spline spaces on simplicial triangulations for $d \leq 3$ with $k \geq 2d r + 1$. The construction of the supersplines on the Alfeld or Powell–Sabin split in two and three dimensions can be found in [19].

1.1. Main result. In this work, a family of $C^r$ conforming finite element spaces on simplicial triangulations is proposed, using the piecewise polynomials of degree not greater than $k$, with $k \geq 2d r + 1$. The construction generalizes all the conforming elements on simplicial triangulations introduced above, except the $P_{2r+1} - C^r$ family introduced in [19]. The main result is summarized in the following theorem.

**Theorem 1.1.** Given $u \in P_r(K)$, the space of polynomials of degree not greater than $k \geq 2d r + 1$ over $d$-dimensional simplex $K$, for any $(d-s)$-dimensional subsimplex $F$ of the simplicial triangulation $\mathcal{T}$ $(0 \leq s \leq d)$, let

$$D^\theta := \frac{\partial^n}{\partial n_1^{\theta_1} \cdots \partial n_s^{\theta_s}}$$

with $n := \sum_{i=1}^s \theta_i$

represent an $n$-th order normal derivative of $u$ on $F$ when $n > 0$, where $n_1, \cdots, n_s$ are orthonormal unit normal vector(s) of $F$. Define the following weighted moments

$$\frac{1}{|F|} \int_F (D^\theta u) \cdot v, \quad \forall v \in B_{F,n,k}, \text{ if } F \text{ is not a vertex,}$$

$$D^\theta u(F) \cdot v(F), \quad \forall v \in B_{F,n,k}, \text{ if } F \text{ is a vertex,}$$

where the bubble space

$$B_{F,n,k} := \text{span} \left\{ \prod_{i=s}^d \lambda_{F,i}^{\sigma_i} : (\sigma_s, \cdots, \sigma_d) \text{ satisfies (1.4) and (1.5)} \right\}$$

(1.3)

$$= \text{span} \{ \lambda_{F,s}^{\sigma_s} \cdots \lambda_{F,d}^{\sigma_d} : (\sigma_s, \cdots, \sigma_d) \text{ satisfies (1.4) and (1.5)} \}$$

with $\lambda_{F,i}, i = s, \cdots, d$, the barycenter coordinates of $F$, and the multi-indices $(\sigma_s, \cdots, \sigma_d)$ satisfy that

$$\sum_{i=s}^d \sigma_i = k - n$$

(1.4)
and
\[ \sigma_1 + \cdots + \sigma_l > 2l + s - 1 r - n, \quad \forall \{i_1, \ldots, i_l\} \subset \{s, \ldots, d\}, \]
with \( l = 1, \ldots, d - s, \) and \( 0 \leq n \leq 2^{s-1}r. \) When \( s = 0, \) let \( n = 0. \) Henceforth, the production \( \prod_{i=1}^{d} \lambda_{F,i}^{r} \) will be simply shortened as \( \lambda_{F,s}^{r} \cdots \lambda_{F,d}^{r} \) for convenience.

Then this set of degrees of freedom is unisolvent for the shape function space \( \mathcal{P}_k(K), \) and the resulting finite element space is of \( C^{r} \) continuity.

**Remark 1.2.** Note that the degrees of freedom defined in (1.1) and (1.2) are conventional and convenient for presentation. The more precise statement of the degrees of freedom of (1.1) should be \( \frac{1}{|T|} \int_T (D^0 u) : v_1, \) where \( v_1, i = 1, 2, \ldots, \dim B_{F,n,k}, \) form a basis of \( B_{F,n,k}. \)

The proof is based on an intrinsic decomposition of the associated set of multi-indices, which is similar to that in some spline construction such as [11]. However, as it will be seen below, this intrinsic decomposition will be used in a completely different way herein. To this end, a refinement of such a decomposition will be proposed, together with some basic properties. This will be discussed in Section 2. Based on the decomposition, two sets of degrees of freedom are constructed in Section 3. To help the readers get familiar with the notation and the main result, two-dimensional and three-dimensional examples are displayed in Section 4. The proof of unisolvency and continuity is given in Section 5.

The rest of the paper discusses some possible generalizations. In Section 6.1 some discontinuous finite element spaces are constructed. Section 6.2 shows that the constructed finite element spaces can be used to establish some new two-dimensional finite element Stokes complex. While in Section 6.3, another finite element smooth de Rham complex is built via constructing a generalized Stenberg element.

### 1.2. Notation

Some conventional notation is summarized here: \( I_d \) denotes the set \( \{0,1,\ldots,d\}, \) and \( T = T(\Omega) \) denotes a \( d \) dimensional simplicial triangulation (a conforming triangulation) of domain \( \Omega \) which can be exactly covered by simplices, \( x \) denotes a vertex of \( T, \) \( K \) denotes an element of \( T \) with vertices \( x_0, \ldots, x_d, \) and \( \lambda_i \) denotes the barycenter coordinate associated to vertex \( x_i \) of \( K, i = 0, \ldots, d. \) When \( d = 0, \) i.e. \( K \) is a vertex \( x, \) define \( \lambda_x = 1. \)

Given a subset \( I \) of \( I_d, \) let \( \langle I \rangle \) denote the simplex taking vertices \( \{x_i : i \in I\} \) as its vertices. Equivalently, \( \langle I \rangle \) is the convex hull of \( \{x_i : i \in I\}. \) Clearly, the mapping \( I \mapsto \langle I \rangle = \text{conv}(\{x_i : i \in I\}) \) defines a bijection between all sub-simplices of \( K \) and all nonempty subsets of \( I_d. \)

### 1.3. Argyris element

To gain some intuition and make the illustration smoother in the following, it is helpful to recall the triangular Argyris element [4] here. Given \( u \in \mathcal{P}_5(K), \) the degrees of freedom in notation of Theorem 1.1 are given as follows:

- The function value, first and second order derivatives of \( u \) at each vertex \( x \) of element \( K. \)

For this set of degrees of freedom, the integer \( n \) in Theorem 1.1 takes 1, 2, and 3. The corresponding bubble function spaces from Theorem 1.1 are \( B_{x,0,5} = \text{span}\{\lambda_x^5\}, B_{x,1,5} = \text{span}\{\lambda_x^2\}, B_{x,2,5} = \text{span}\{\lambda_x^3\}, \) respectively, and will be checked below. Recall that here \( \lambda_x = 1 \) is a function which is defined only at vertex \( x. \)

Take \( B_{x,0,5} \) as an example. Since vertex \( x \) is of codimension \( s = 2, \) by definition (1.3), the bubble space is spanned by \( \lambda_x^p \) for some nonnegative integer \( p. \) It follows from (1.4) that \( p = 5 - 0, \) while the second condition (1.5) vacuously holds since \( d = s = 2. \) Therefore, the bubble space \( B_{x,0,5} = \text{span}\{\lambda_x^5\}. \) Note again that \( \lambda_x \) is only defined at the vertex with value 1, and hence one can regard \( B_{x,0,k} \) as the constant function space \( \mathbb{R} \) for \( k = 3, 4, 5. \)

- The weighted moment
\[ \frac{1}{|e|} \int_e \left( \frac{\partial u}{\partial n} \right) \cdot (\lambda_{e,1}^2 \lambda_{e,2}^2) \]
for each edge \( e \) of element \( K. \) Here \( \lambda_{e,1} \) and \( \lambda_{e,2} \) are the barycenter coordinates of edge \( e. \)

For this set of degrees of freedom, note that \( n \leq 1. \) It suffices to show that \( B_{x,0,5} \) is an empty set, and \( B_{x,1,5} = \text{span}\{\lambda_x^2, \lambda_x^3\} \) is a one-dimensional polynomial space.
In fact, for $B_{e,0.5}$, by definition it is spanned by $\lambda^p_{e,1}\lambda^q_{e,2}$, and the conditions (1.4) and (1.5) are specified as $p + q = 5 - 0 = 5$ and $p, q > 2^{1+1-1} - 0 = 2$, respectively. However, for integers $p$ and $q$, these two conditions contradict each other, which implies that the bubble space $B_{e,0.5}$ is an empty set.

For $B_{e,1.5}$, the conditions (1.4) and (1.5) are specified as $p + q = 5 - 1 = 4$, and $p, q > 2^{1+1-1}-1 = 1$, respectively. Only the pair $(p, q) = (2, 2)$ meets the requirements. Therefore, $B_{e,1.5} = \text{span}\{\lambda^2_{e,1}\lambda^2_{e,2}\}$.

- In each element $K$, the bubble space is spanned by $\lambda^p_{F,0}\lambda^q_{F,1}\lambda^r_{F,2}$ where $\sigma_0 + \sigma_1 + \sigma_2 = 5$ and $\sigma_k > 2^{1+0-1} - 1 = 1$ for $k = 0, 1, 2$. Since no integers $\sigma_0, \sigma_1$, and $\sigma_2$ meet these requirements, the bubble space $B_{K,0.5}$ is an empty set.

![Figure 1](image.png)

**Figure 1.** The illustration of degrees of freedom of the Argyris element.

Figure 1 illustrates the degrees of freedom of the Argyris element. It is well known that the Argyris element has $C^1$ continuity, see [4, 8]. It is stressed that the above degrees of freedom are modified from those of the original Argyris element [4], as on each edge $e$ of $K$, the value of $\frac{\partial u}{\partial n}$ at the midpoint of $e$ is replaced by a weighted norm, as mentioned above.

2. INTRINSIC DECOMPOSITION

Given a positive integer $k$, this section constructs a decomposition, called an intrinsic decomposition, of the set of multi-indices

\[
\Sigma(I_d, k) := \{(\alpha_0, \alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^{d+1} : \sum_{i=0}^{d} \alpha_i = k\},
\]

which builds a relationship between a set of the geometric components of simplex $K$ and the set of all the multi-indices of degree $k$.

2.1. Definition and Assumption. Given a nonempty index set $I := \{i_0, i_1, \ldots, i_d\} \subseteq \mathbb{N}_0$ and an integer $k \geq 0$, define

\[
\Sigma(I, k) := \{(\alpha_{i_0}, \alpha_{i_1}, \ldots, \alpha_{i_d}) \in \mathbb{N}_0^{d+1} : \sum_{i \in I} \alpha_i = k\}.
\]

The decomposition is based on a continuity vector $r := (r_1, \ldots, r_d)$ with $r_i$ a nonnegative integer, $i = 1, 2, \ldots, d$. The vector specifies the continuity of piecewise polynomial finite element functions across the internal subsimplices of the given conforming simplicial triangulation $\mathcal{T}(\Omega)$. For example, in the standard $C^r$ Bramble–Zlámal element [7], the finite element functions of piecewise polynomials are of $C^2$ continuity across the vertices (0-simplices), and are of $C^r$ continuity across the internal edges (1-simplices) of the simplicial triangulation $\mathcal{T}(\Omega)$. For the more general case in two dimensions, the continuity vector is taken as $r = (r_1, r_2)$. For $d$ dimensions, the component $r_s$ of the continuity vector represents the continuity of the finite element functions of piecewise polynomials when crossing $(d - s)$-dimensional simplices (equivalently, with the codimension $s$). For example, for the standard triangular Argyris element, the continuity vector is chosen as $r = (r_1, r_2) = (1, 2)$, since it admits $C^1$ continuity when crossing internal edges while $C^2$ continuity when crossing vertices, by the choice of degrees of freedom. While for the three dimensional Ženěšek element, the continuity vector is chosen as $r = (r_1, r_2, r_3) = (1, 2, 4)$.

Throughout this paper, the following assumption is required for the continuity vector $r$, as well as the polynomial degree $k$, unless otherwise specified. Note that the assumption is a sufficient
condition for the construction, which seems to appear naturally in the existing attempts in two and three dimensions, while it is still not clear whether it is a necessary condition for the existence of \( C^\infty \) conforming finite element spaces, even for the two-dimensional case.

**Assumption A.** For the continuity vector \( \mathbf{r} = (r_1, \ldots, r_d) \) with nonnegative integers \( r_1, \ldots, r_d \) and the polynomial degree \( k \), it holds

\[
  r_d \geq 2r_{d-1} \geq 4r_{d-2} \geq \cdots \geq 2^{d-1}r_1
\]

and

\[
  k \geq 2r_d + 1.
\]

Given a continuity vector \( \mathbf{r} \) and a polynomial degree \( k \), the intrinsic decomposition is defined recursively as follows. For some technical reason, \( r_0 = 0 \) is always additionally assumed.

**Definition 2.1** (An intrinsic decomposition for \( \Sigma(I_d, k) \)). Given a continuity vector \( \mathbf{r} = (r_1, \ldots, r_d) \) and a polynomial degree \( k \) satisfying Assumption A, a decomposition of \( \Sigma(I_d, k) \) is defined inductively as follows:

\[
  \Sigma_d(I_d, k) := \{(\alpha_0, \ldots, \alpha_d) \in \Sigma(I_d, k) : \text{ There exists a subset } N_d \subseteq I_d, \text{ such that Card}(N_d) = d \text{ and } \sum_{i \in N_d} \alpha_i \leq r_d\},
\]

and then,

\[
  \Sigma_s(I_d, k) := \{(\alpha_0, \ldots, \alpha_d) \in \Sigma(I_d, k) : \text{ There exists a subset } N_s \subseteq I_d, \text{ such that Card}(N_s) = s \text{ and } \sum_{i \in N_s} \alpha_i \leq r_s \} \setminus \bigcup_{d' = s+1}^d \Sigma_{d'}(I_d, k),
\]

for \( s = d - 1, \ldots, 1 \) sequentially. Finally, set

\[
  \Sigma_0(I_d, k) := \Sigma(I_d, k) \setminus \bigcup_{s' = 1}^d \Sigma_{s'}(I_d, k).
\]

It follows from the definition of \( \Sigma_s(I_d, k), s = 0, 1, \ldots, d \), that

\[
  \Sigma(I_d, k) = \Sigma_0(I_d, k) \cup \Sigma_1(I_d, k) \cup \cdots \cup \Sigma_d(I_d, k),
\]

and that any two \( \Sigma_s(I_d, k) \) and \( \Sigma_s'(I_d, k) \) are disjoint if \( s \neq s' \). In fact, as can be seen below, a further refined decomposition is needed for the construction and analysis of \( C^\infty \) conforming finite element methods.

It is worth noting that in the definition of (2.4), the latter set on the right hand side in general is not a subset of the former set. To see this, consider a concrete two-dimensional example as follows, (here \( \mathbf{r} = (r_1, r_2) \))

\[
  \Sigma_2(I_2, k) := \{(\alpha_0, \alpha_1, \alpha_2) \in \Sigma(I_2, k) : \alpha_1 + \alpha_2 \leq r_2 \text{ or } \alpha_2 + \alpha_0 \leq r_2 \text{ or } \alpha_0 + \alpha_1 \leq r_2\},
\]

\[
  \Sigma_1(I_2, k) := \{(\alpha_0, \alpha_1, \alpha_2) \in \Sigma(I_2, k) : \alpha_0 \leq r_1 \text{ or } \alpha_1 \leq r_1 \text{ or } \alpha_2 \leq r_1\} \setminus \Sigma_2(I_2, k),
\]

\[
  \Sigma_0(I_2, k) := \Sigma(I_2, k) \setminus \{\Sigma_2(I_2, k) \cup \Sigma_1(I_2, k)\}.
\]

For example, with the continuity vector \( \mathbf{r} = (r_1, r_2) = (1, 4) \) and \( k = 9 \), the multi-index \((2, 2, 5)\) is in the set \( \Sigma_2(I_2, 9) \), but none of the components is less than 1, that is, it does not belong to the set

\[
  \{(\alpha_0, \alpha_1, \alpha_2) \in \Sigma(I_2, k) : \alpha_0 \leq r_1 \text{ or } \alpha_1 \leq r_1 \text{ or } \alpha_2 \leq r_1\}.
\]

**Remark 2.2.** Given a continuity vector \( \mathbf{q} = (q_1, \ldots, q_d) \) and a polynomial degree \( k \) satisfying Assumption A, for a given index set \( I = \{i_0, i_1, \ldots, i_d\} \subseteq N_0 \), the decomposition of the set \( \Sigma(I, k) \) defined in (2.2) can be defined in a similar way as Definition 2.1.

**Remark 2.3.** The last set \( \Sigma_0(I_d, k) \) of the intrinsic decomposition can be characterized as:

\[
  \Sigma_0(I_d, k) = \{(\alpha_0, \ldots, \alpha_d) \in \Sigma(I_d, k) : \alpha_{i_1} + \cdots + \alpha_{i_s} > r_s, \forall \{i_1, \ldots, i_s\} \subseteq I_d, s = 1, 2, \ldots, d\}.
\]
Example 2.4. For the Argyris element, the continuity vector \( r = (r_1, r_2) = (1, 2) \), the polynomial degree \( k = 5 \), and the intrinsic decomposition defined above reads
\[
\Sigma_0(\mathbb{I}_2, 5) = \emptyset,
\]
\[
\Sigma_1(\mathbb{I}_2, 5) = \{(1, 2, 2), (2, 1, 2), (2, 2, 1)\},
\]
and
\[
\Sigma_2(\mathbb{I}_2, 5) = \Sigma(\mathbb{I}_2, 5) \setminus \left( \Sigma_0(\mathbb{I}_2, 5) \cup \Sigma_1(\mathbb{I}_2, 5) \right).
\]

Note that the set \( \Sigma_0(\mathbb{I}_2, 5) \) is used to define the degrees of freedom inside each element \( K \) (thus no degrees of freedom are assigned inside element \( K \)), while the set \( \Sigma_1(\mathbb{I}_2, 5) \) indicates the degrees of freedom on edges (thus three degrees of freedom are assigned on edges in total).

2.2. A refined intrinsic decomposition. Recall the main result in Theorem 1.1, the bubble function space inside the \( d \)-dimensional simplex element \( K \) is defined by using the index set \( \Sigma_0(\mathbb{I}_d, k) \), with respect to the continuity vector \( r = (r_1, \ldots, r_d) = (r, 2r, \ldots, 2^{d-1}r) \). This subsection is to define the (in)complete bubble function spaces inside the \((d - s)\)-dimensional simplices of \( K \) by using the corresponding subsets \( \Sigma_s(\mathbb{I}_d, k) \) of \( \Sigma(\mathbb{I}_d, k) \) for \( 1 \leq s \leq d \). To this end, a refined intrinsic decomposition of \( \Sigma_s(\mathbb{I}_d, k) \) will be further introduced. As a result, each subset of the refined intrinsic decomposition is assigned to a \((d - s)\)-dimensional simplex of \( K \), indexed as a pair \((\mathcal{N}, n)\).

Definition 2.5. Let a continuity vector \( r = (r_1, \ldots, r_d) \) and a polynomial degree \( k \) satisfying Assumption A be given. For a given multi-index \( \alpha \in \Sigma_s(\mathbb{I}_d, k) \) from Definition 2.1, let \( \mathcal{N}(\alpha) \) denote the subset defined in (2.4), that is
\[
\mathcal{N}_s \subseteq \mathbb{I}_d \text{ such that } \sum_{i \in \mathcal{N}_s} \alpha_i \leq r_s \text{ and Card}(\mathcal{N}_s) = s,
\]
choose one if there are multiple possible choices. Notice that here the subscript \( s \) is only used to emphasize the cardinality of the set. Then let
\[
\Delta(\alpha) := \mathbb{I}_d \setminus \mathcal{N}(\alpha)
\]
and
\[
n(\alpha) := \sum_{i \in \mathcal{N}(\alpha)} \alpha_i.
\]

Particularly, when \( \mathcal{N}(\alpha) = \emptyset \), let \( n(\alpha) = 0 \). To adopt this, let \( r_0 = 0 \) in this section for convenience. Notice that this definition does not conflict with Assumption A since only \( r_1, \ldots, r_d \) are restricted therein.

Remark 2.6. As it will be seen below, \( \mathcal{N}(\alpha), \Delta(\alpha) \) and \( n(\alpha) \) will be used to define the degree of freedom associated with multi-index \( \alpha \). In particular, the set \( \mathcal{N}(\alpha) \) will be used to define the normal vector(s) involved in the degree of freedom \( \varphi_\alpha(\cdot) \) of (3.1) below, while the set \( \Delta(\alpha) \) will be used to locate the associated subsimplex of \( \varphi_\alpha(\cdot) \). At last, \( n(\alpha) \) is the order of the normal derivative taken in \( \varphi_\alpha(\cdot) \).

Remark 2.7. If \( \alpha \in \Sigma_s(\mathbb{I}_d, k) \), under Assumption A, then it holds that \( n(\alpha) := \sum_{i \in \mathcal{N}(\alpha)} \alpha_i \leq r_s \) and \( \alpha_j > r_{s+1} - n(\alpha) \geq r_s \) for any \( j \in \Delta(\alpha) \).

At first glance, it seems that the choice of \( \mathcal{N}(\alpha) \) might be too arbitrary to make things in order. Nevertheless, it can be proved that the choice is always unique provided that Assumption A holds.

Proposition 2.8 (Uniqueness of \( \mathcal{N} \) and \( \Delta \)). Under Assumption A, for any multi-index \( \alpha \in \Sigma_s(\mathbb{I}_d, k) \subseteq \Sigma(\mathbb{I}_d, k) \), there exists a unique subset \( \mathcal{N}(\alpha) \) and consequently a unique \( \Delta(\alpha) := \mathbb{I}_d \setminus \mathcal{N}(\alpha) \) such that
\[
\sum_{i \in \mathcal{N}(\alpha)} \alpha_i \leq r_s \text{ and Card}(\mathcal{N}(\alpha)) = s.
\]
Proof. Suppose that, there are two different sets \( N_1 \) and \( N_2 \), with \( \text{Card}(N_1) = \text{Card}(N_2) = s \) and
\[
\sum_{i \in N_1} \alpha_i \leq r_s, \quad \sum_{i \in N_2} \alpha_i \leq r_s.
\]
If \( s = d \), it holds that \( \mathbb{I}_d = N_1 \cup N_2 \), hence
\[
k = \sum_{i \in \mathbb{I}_d} \alpha_i \leq \sum_{i \in N_1} \alpha_i + \sum_{i \in N_2} \alpha_i \leq r_d + r_d < k,
\]
which is a contradiction.

If \( s < d \), it holds that \( \text{Card}(N_1 \cup N_2) \geq s + 1 \), hence
\[
\sum_{i \in N_1 \cup N_2} \alpha_i \leq \sum_{i \in N_1} \alpha_i + \sum_{i \in N_2} \alpha_i \leq r_s + r_s \leq r_{\text{Card}(N_1 \cup N_2)},
\]
which contradicts with the choice of \( N_1 \) and \( s \). This completes the proof. \( \square \)

A refined version of the above intrinsic decomposition \( \Sigma_0(\mathbb{I}_d, k), \Sigma_1(\mathbb{I}_d, k), \cdots, \Sigma_d(\mathbb{I}_d, k) \) of the set \( \Sigma(\mathbb{I}_d, k) \) can be established as follows. Such a refined version is based on proper subsets \( N \) of \( \mathbb{I}_d \) and nonnegative integers \( n \in \mathbb{N}_0 \), which is a natural consequence of Remark 2.6.

**Definition 2.9 (A refined intrinsic decomposition of \( \Sigma(\mathbb{I}_d, k) \)).** Given a continuity vector \( r \) and a polynomial degree \( k \) with Assumption A, for a proper subset \( N \subseteq \mathbb{I}_d \) and an integer \( n \in \mathbb{N}_0 \), a refined intrinsic decomposition with respect to \( r \) and \( k \) is defined as
\[
\Sigma_{N,n}(\mathbb{I}_d, k) := \{ \alpha \in \Sigma(\mathbb{I}_d, k) : N(\alpha) = N \text{ and } n(\alpha) = n \}.
\]

Here, given \( \alpha \in \Sigma(\mathbb{I}_d, k) \), \( N(\alpha) \) and \( n(\alpha) \) are defined in Definition 2.5 above. Running over all \( (N,n) \) such that \( \Sigma_{N,n} \) is not empty leads to the following refined decomposition (a disjoint union)
\[
\Sigma(\mathbb{I}_d, k) = \bigcup_{N,n} \Sigma_{N,n}(\mathbb{I}_d, k).
\]

Given a nonempty proper subset \( N \subseteq \mathbb{I}_d \), let \( n \leq r_{\text{Card}(N)} \) and the multi-index \( \theta \in \Sigma(N,n) \) with the set \( \Sigma(N,n) \) defined in (2.2), define
\[
\Sigma_{N,n,\theta}(\mathbb{I}_d, k) = \{ \alpha \in \Sigma_{N,n}(\mathbb{I}_d, k) : \alpha_i = \theta_i, \ \forall i \in N \}.
\]

**Example 2.10.** Consider the refined intrinsic decomposition for the Argyris element, i.e., the refined intrinsic decomposition of \( \Sigma(\mathbb{I}_2, 5) \) with respect to the continuity vector \( r = (1,2) \). Here the focus is put in the classification of degrees of freedom at vertices and on edges, see Figure 2 for an illustration for the refined intrinsic decomposition of the set \( \Sigma(\mathbb{I}_2, 5) \).

- Consider \( x_0 \) as a typical vertex. In this case, \( N \) will be taken as \( \{1,2\} \), and \( n \) can be \( 0,1,2 = r_2 \). It follows that
\[
\Sigma_{\{1,2\},0}(\mathbb{I}_2, 5) = \{ (5,0,0) \}, \quad \Sigma_{\{1,2\},1}(\mathbb{I}_2, 5) = \{ (4,1,0), (4,0,1) \},
\]
and
\[
\Sigma_{\{1,2\},2}(\mathbb{I}_2, 5) = \{ (3,0,2), (3,1,1), (3,2,0) \},
\]
corresponding to the zeroth, first and second order derivatives at vertex \( x_0 \), respectively.

- Consider \( e_0 = (x_1, x_2) \) as a typical edge. In this case, \( N \) will be taken as \( \{0\} \), and the possible \( n \) can be \( 0,1 = r_1 \). It follows that
\[
\Sigma_{\{0\},0}(\mathbb{I}_2, 5) = \emptyset, \quad \Sigma_{\{0\},1}(\mathbb{I}_2, 5) = \{ (1,2,2) \},
\]
which is corresponding to the degrees of freedom on edge \( e_0 \). That is, no degree of freedom is for the function value, while one degree of freedom is for the first-order normal derivative on each edge.

From Definition 2.5 and Proposition 2.8, the following properties hold about this refined decomposition.

**Proposition 2.11.** Given a continuity vector \( r = (r_1, \cdots, r_d) \) and a polynomial degree \( k \) satisfying Assumption A, the refined intrinsic decomposition (2.8) possesses the following properties:
Figure 2. The refined intrinsic decomposition of $\Sigma(\mathbb{I}_2, 5)$.

(1) These sets $\Sigma_{N,n}(\mathbb{I}_d, k)$ are disjoint, i.e.,
$$\Sigma_{N,n}(\mathbb{I}_d, k) \cap \Sigma_{N',n'}(\mathbb{I}_d, k) = \emptyset \text{ if } (N, n) \neq (N', n').$$

(2) For any nonnegative integer $n$, it holds that
$$\Sigma_{N,n}(\mathbb{I}_d, k) \subseteq \Sigma_{\text{Card}(N)}(\mathbb{I}_d, k).$$

(3) For any pair $(N, n)$ such that $n > r_{\text{Card}(N)}$, it holds that
$$\Sigma_{N,n}(\mathbb{I}_d, k) = \emptyset.$$

From Proposition 2.11, the refined decomposition (2.8) of $\Sigma(\mathbb{I}_d, k)$ can be written in more details as
$$\Sigma(\mathbb{I}_d, k) = \bigcup_{N \subseteq \mathbb{I}_d} \Sigma_{N,n}(\mathbb{I}_d, k),$$
where the range of pairs $(N, n)$ is clarified.

Similar refined decompositions of $\Sigma(\mathbb{I}_d, k)$ were proposed in the literature, cf., [1, 11], which are used to determine the basis functions of the super spline space. However, no $C^r$ finite element methods were constructed so far.

Remark 2.12. In [11], the notion $N_{ji}$ therein is a subset of $\Sigma_{d-j}(\mathbb{I}_d, k)$ of this paper, which needs a further decomposition herein.

It now returns to the construction of degrees of freedom. Proposition 2.14 below, based on the refined intrinsic decomposition in $d$ dimensions, builds a bijection mapping. Such a mapping will be used to define the local degrees of freedom inside subsimplices of element $K$. Consequently, a unified definition of degrees of freedom can be carried out, which is a local version of degrees of freedom. The locality and unification provide many benefits when proving the unsolvability and continuity.

Before the introduction of the bijection mapping, an intrinsic decomposition based on a continuity vector $q = (q_1, \ldots, q_d)$ other than $r$ is needed for the set of multi-indices associated to the pair $(I, k')$ where
$$I := \{i_0, i_1, \ldots, i_{d'}\} \subseteq \mathbb{I}_d \text{ and } k' \in \mathbb{N}_0.$$ In particular, the last set of this decomposition, denoted as $\Sigma_0^{(q)}(I, k')$, reads
$$\Sigma_0^{(q)}(I, k') = \{(\alpha_{i_0}, \ldots, \alpha_{i_{d'}}) \in \Sigma(I, k') : \alpha_{j_1} + \cdots + \alpha_{j_s} > q_s, \forall \{j_1, \ldots, j_s\} \not\subseteq I, s = 1, 2, \ldots, d'\}.$$  

Remark 2.13. For the case $\text{Card}(I) = 1$, $q$ is an empty continuity vector, and $\Sigma_0^{(q)}(I, k') = \Sigma(I, k').$

Proposition 2.14. Given a continuity vector $r = (r_1, \ldots, r_d)$ and polynomial degree $k$ with Assumption $A$, for a nonempty proper subset $\Delta \subseteq \mathbb{I}_d$, let $N := \mathbb{I}_d \setminus \Delta$, $s := \text{Card}(N)$, $n \leq r_s$. Then it holds that the mapping
$$\mathcal{R}_{N,\Delta} : (\theta, \sigma) \mapsto \theta,$$
defined by
$$\theta_i = \sigma_i, i \in N \text{ and } \sigma_i = \alpha_i, i \in \Delta,$$
is a bijection between $\Sigma_{N,n}(\mathbb{I}_d, k)$ and $\Sigma(N, n) \times \Sigma_0^{(q)}(\Delta, k-n)$ with $q = q^{s,n} := (r_{s+1-n}, \ldots, r_d-n)$. 


Before proving it, several examples are introduced and discussed, in order to clarify the statement of Proposition 2.14 and its consequence in the main construction (will be shown in Section 3) of this paper.

**Example 2.15.** Set \( I_d = I_2 = \{0, 1, 2\} \), \( k \geq 5 \), and the continuity vector is chosen as \( r = (1, 2) \). For \( k = 5 \), it corresponds to the Argyris element, which has been discussed in Section 1.3. Here, the bubble spaces and the corresponding degrees of freedom are reinterpreted in the language of the refined intrinsic decomposition. Notice that \( N \) considered in Proposition 2.14 is a nonempty proper subset, all the possible cases are enumerated below.

- \( \text{Card}(N) = 1 \), and \( n = 0 \). Suppose that \( N = \{0\} \), then
  \[
  \Sigma_{N,n}(I_2,k) = \Sigma_{\{0\},n}(I_2,k) := \{(0,p,q) : p + q = k, p, q > r_2 = 2\},
  \]
  where the restriction of \( p, q \) comes from the definition of \( N \) (more precisely, the sum of any two indices is greater than \( r_2 = 2 \)). In this case, \( \Sigma(I,n) = \Sigma(\{0\}, 0) = \{(0)\} \), \( \Delta = \{1,2\} \), \( q = (2) \), and it follows from the definition (2.10) that
  \[
  \Sigma^0(q)(\Delta, k - n) = \Sigma^0(2)((1, 2), k - 0) := \{(p, q) : p + q = k, p, q > 2\}.
  \]
  In this case, it is easy to check the bijection holds. In particular, when \( k = 5 \), both \( \Sigma_{\{0\},n}(I_2, 5) \) and \( \Sigma^0(2)((1, 2), 5 - 0) \) are empty, which indicates that there are no degrees of freedom on edge with respect to the function value.

- \( \text{Card}(N) = 1 \), and \( n = 1 \). Suppose that \( N = \{0\} \), then
  \[
  \Sigma_{N,n}(I_2,k) = \Sigma_{\{0\},1}(I_2,k) := \{(1,p,q) : p + q = k - 1, p, q > r_2 - 1 = 1\}.
  \]
  In this case, \( \Sigma(N,n) = \Sigma(\{0\}, 1) = \{(1)\} \), \( \Delta = \{1,2\} \), and \( q = (1) \). It follows from (2.10) that
  \[
  \Sigma^0(q)(\Delta, k - n) = \Sigma^0(1)((1, 2), k - 1) := \{(p, q) : p + q = k - 1, p, q > 1\}.
  \]
  The bijection also holds. In particular, when \( k = 5 \), all the sets only consist of a single component.

- \( \text{Card}(N) = 2 \) and \( n \leq r_2 = 2 \). Suppose that \( N = \{1,2\} \) and \( \Delta = \{0\} \), then
  \[
  \Sigma_{N,n}(I_2,k) = \Sigma_{\{1,2\},n}(I_2,k) = \{(k - n, \alpha_1, \alpha_2) : \alpha_1 + \alpha_2 = n\},
  \]
  and \( q \) is an empty continuity vector. Thus by definition (2.10) \( \Sigma^0(q)(\Delta, k - n) = \Sigma(\{0\}, k - n) = \{(k - n)\}, \Sigma(\{1,2\}, n) = \{(\alpha_1, \alpha_2), \alpha_1 + \alpha_2 = n\} \). In this case, the bijection also holds.

The above example implies that the bijection relationship shown in Proposition 2.14 in two dimensions might be too simple to gain more information since either \( N \) or \( \Delta \) may be reduced to a set with only one element. To this end, consider the following three-dimensional example, showing some non-triviality of such a relationship. For simplicity, only the statement itself of the proposition is checked in the following, and the corresponding degrees of freedom will be not expanded. A detailed illustration of three-dimensional finite element spaces will be displayed in Section 4.2.

**Example 2.16.** Let \( d = 3 \), \( k = 13 \), and the continuity vector \( r = (1, 3, 6) \). Consider the case where \( N = \{0, 1\} \) and \( \Delta = \{2, 3\} \). For \( n = 2 \) (hence \( q = (6 - 2 = 4) \)), it holds that \( \alpha_0 + \alpha_1 = 2 \), which implies \( (\alpha_0, \alpha_1) = (0, 2), (1,1) \) or \( (2,0) \). The restriction on \( (\alpha_2, \alpha_3) \) then are \( 2 + \alpha_2, 2 + \alpha_3 > r_3 = 6 \), and \( 2 + \alpha_2 + \alpha_3 = 13 \). A direct enumeration obtains that \( \Sigma_{\{0,1\},n}(I_3, k) = \Sigma_{\{0,1\},2}(I_3, 13) \) is the union of the following three sets

\[
\begin{align*}
(0, 2, \alpha_2, \alpha_3) : \alpha_2 + \alpha_3 = 11, \alpha_2, \alpha_3 > 4 & := \{(0,2,5,6), (0,2,6,5)\}, \\
(1, 1, \alpha_2, \alpha_3) : \alpha_2 + \alpha_3 = 11, \alpha_2, \alpha_3 > 4 & := \{(1,1,5,6), (1,1,6,5)\}, \\
(2, 0, \alpha_2, \alpha_3) : \alpha_2 + \alpha_3 = 11, \alpha_2, \alpha_3 > 4 & := \{(2,0,5,6), (2,0,6,5)\}.
\end{align*}
\]

As a result, the set \( \Sigma_{\{0,1\},2}(I_3, 13) \) can be decomposed as

\[
\Sigma_{\{0,1\},2}(I_3, 13) = \{(0, 2), (1,1), (2, 0)\} \times \{(5,6), (6,5)\}.
\]

This is exactly what Proposition 2.14 states, since

\[
\Sigma(N,n) = \Sigma(\{0,1\}, 2) = \{(0,2), (1,1), (2,0)\},
\]
and

\[ \Sigma_0^{(q)}(\Delta, k - n) = \Sigma_0^{(4)}(\{2, 3\}, 11) = \{(5, 6), (6, 5)\}. \]

Figure 3. The refined intrinsic decomposition of \(\Sigma(\emptyset_3, 13)\) with \(r = (1, 3, 6)\).

Finally, Proposition 2.14 is proved to close this section. The basic argument in the proof is similar to that of Example 2.16, with more technicality.

**Proof of Proposition 2.14.** For \(\alpha \in \Sigma_{N,n}(\emptyset_d, k)\) and \((\theta, \sigma) = R_{N,\Delta}(\alpha)\), it holds that

\[ \sum_{i \in N} \theta_i = \sum_{i \in N} \alpha_i = n, \quad \sum_{i \in \Delta} \sigma_i = \sum_{i \in \Delta} \alpha_i = k - n. \]

Moreover, for any nonempty subset \(N'\) of \(\Delta\), since \(N' \cap N = \emptyset\), it holds that

\[ \sum_{i \in N'} \sigma_i = \sum_{i \in N'} \alpha_i = \sum_{i \in N \cup N'} \alpha_i - \sum_{i \in N} \alpha_i > r_{\text{Card}(N')} + s - n = q_{\text{Card}(N')}\].

Hence \(\theta\) belongs to \(\Sigma(N, n)\) and \(\sigma\) belongs to \(\Sigma_0^{(q)}(\Delta, k - n)\).

It is straightforward to see that

\[ R_{N,\Delta} : \Sigma_{N,n}(\emptyset_d, k) \longrightarrow \Sigma(N, n) \times \Sigma_0^{(q)}(\Delta, k - n) \]

is an injection. Now, it suffices to show that \(R_{N,\Delta}\) is surjective. For \(\theta \in \Sigma(N, n)\) and \(\sigma \in \Sigma_0^{(q)}(\Delta, k - n)\), let \(\alpha \in \Sigma(\emptyset_d, k)\) such that \(\alpha_i = \theta_i\) for \(i \in N\) and \(\alpha_i = \sigma_i\) for \(i \in \Delta\). Then it holds that

\[ \sum_{i \in N} \alpha_i = \sum_{i \in N} \theta_i = n \leq r_s. \]
It remains to show for any proper subset $N'$ of $\mathbb{I}_d$ with $s+1 \leq \text{Card}(N') \leq d$, $\sum_{i \in N} \alpha_i > r_{\text{Card}(N')}$. If $N$ is a subset of $N'$, then it holds that $\text{Card}(N' \setminus N) = \text{Card}(N') - s$. It follows that

$$\sum_{i \in N'} \alpha_i = \sum_{i \in N' \setminus N} \sigma_i + \sum_{i \in N} \theta_i > q_{\text{Card}(N') - s} + n = r_{\text{Card}(N')} ,$$

where the first equation is from the construction of $\alpha$, and the last equation is from the definition of $q$. For the other case, if $N$ is not a subset of $N'$, then $\text{Card}(N' \setminus N) \geq \text{Card}(N') - s + 1$. It follows that

$$\sum_{i \in N'} \alpha_i \geq \sum_{i \in N' \setminus N} \sigma_i > q_{\text{Card}(N') - s + 1} = r_{\text{Card}(N')} + 1 - n \geq r_{\text{Card}(N')} .$$

In conclusion, it holds that $\alpha \in \Sigma_{N,n}(\mathbb{I}_d, k)$ and $R_{N,\Delta}(\theta, \sigma)$, which proves the surjection part.

In the following sections, Proposition 2.14 will be frequently used. Simply speaking, the components in the set $\Sigma(N, n)$ will correspond to certain (higher-order) derivatives that appear in the degrees of freedom, e.g., $u, \frac{\partial u}{\partial n}, \frac{\partial^2 u}{\partial n^2}$, etc. The components in the set $\Sigma_\theta(\Delta, k-n)$ will be used to define the bubble function spaces and the corresponding components of the refined intrinsic decomposition has been established.

3. TWO SETS OF DEGREES OF FREEDOM: LOCAL AND GLOBAL

This section introduces two sets of degrees of freedom, and shows their equivalence. The first set of degrees of freedom (3.1) will more intuitively portray the degrees of freedom on each subsimplex; while the second set of degrees of freedom (3.2) provides another perspective of (3.6), which makes sure that the proposed degrees of freedom can be used to design $C^r$ finite element spaces.

Given a subsimplex $F$, the averaging inner product $\overline{\int_F f \cdot g}$ is defined as $\frac{1}{|F|} \int_F f \cdot g$ if $F$ is not a vertex, as $f(F) \cdot g(F)$ if $F$ is a vertex. For the sake of clarity, the integral formulation $\frac{1}{|F|} \int_F f \cdot g$ will be also used when $F$ is a vertex, and should be understood as $f(F)g(F)$.

In what follows, the unit normal vectors of subsimplex $F$ of the element $K$ will be re-indexed for convenience. In particular, given a subsimplex $F := \langle \Delta \rangle := \text{conv}\{x_i : i \in \Delta\}$ of $K$, the orthonormal outer normal vectors will be denoted as $n_{F,i}, i \in N = \mathbb{I}_d \setminus \Delta$. A specific choice of these normal directions does not affect the construction and result, since the span of these normal vectors is the same space which is perpendicular to subsimplex $F$. However, the re-indexing can make the following proof more concise. The readers might recall the definition of $N(\alpha), n(\alpha), \Delta(\alpha)$ from Definition 2.5.

**Definition 3.1** (Two sets of degrees of freedom). Given a continuity vector $r = (r_1, \cdots, r_d)$ and a polynomial degree $k$ with Assumption A, two sets of degrees of freedom are introduced as follows. Note that unless otherwise specified, $\Delta$ and $N$ are dependent on $r$.

1. For $\alpha \in \Sigma(\mathbb{I}_d, k)$, let $\Delta = \Delta(\alpha)$ and $F := \langle \Delta \rangle$, let $N = N(\alpha)$ and $n = n(\alpha) := \sum_{i \in N} \alpha_i$. For $u \in P_k(K)$, define

$$\varphi_\alpha : u \mapsto \frac{1}{|F|} \left\langle \frac{\partial^n}{\prod_{i \in N} \partial n_{F,i}^{\alpha_i}} u |_{F}, \lambda^\Delta \alpha \right\rangle_F,$$

   where $\lambda^\Delta \alpha := \prod_{i \in \Delta} \lambda_i^{\alpha_i}$. There is no distinction among $\lambda_i$, $\lambda_{K,i}$ and $\lambda_{F,i}$, since $\lambda_{K,i}|_F = \lambda_{F,i}$ for $F \subseteq K$.

2. Given a nonempty proper subset $\Delta \subseteq \mathbb{I}_d$ and $F := \langle \Delta \rangle$, let $N := \mathbb{I}_d \setminus \Delta$, $s := \text{Card}(N)$, $n \leq r_s$, and $q = q^{r,s} := (r_{s+1} - n, \cdots, r_d - n)$. With $\theta \in \Sigma(N, n)$ and $\sigma \in \Sigma_\theta(\Delta, k-n)$ defined in (2.10), for $u \in P_k(K)$, define

$$\varphi_{\theta, \sigma} : u \mapsto \frac{1}{|F|} \left\langle \frac{\partial^n}{\prod_{i \in N} \partial n_{F,i}^{\theta_i}} u |_{F}, \lambda^\Delta \sigma \right\rangle_F,$$
where \( \lambda^\Delta \sigma := \prod_{i \in \Delta} \lambda^{\sigma_i}_i \). Moreover, for \( \Delta = \mathbb{I}_d \) and \( \sigma \in \Sigma_0(\mathbb{I}_d, k) \) (which implies that \( N(\sigma) \) is empty), define \( \varphi_{\varphi, \sigma} := \varphi_\sigma \) as in (3.1). Here, for \( r \) and \( k \) with Assumption \( A \), it is straightforward to show that both \( q^{s \cdot n} \) and \( k − n \) satisfy Assumption \( A \). Note that this is exactly the form in Theorem 3.3.

**Proposition 3.2.** Under Assumption \( A \), for any nonempty proper subset \( \Delta \subseteq \mathbb{I}_d \), let \( N := \mathbb{I}_d \setminus \Delta \), \( s := \text{Card}(\mathcal{N}) \), \( n \leq r_s \), and \( q = q^{s \cdot n} := (r_{s+1} - n, \ldots, r_d - n) \). Then the following two sets of degrees of freedom

\[
\{ \varphi_\alpha : \alpha \in \Sigma_{\mathcal{N}, n}(\mathbb{I}_d, k) \}
\]

and

\[
\{ \varphi_{\varphi, \sigma} : \sigma \in \Sigma(N, n), \sigma \in \Sigma_0^{(q)}(\Delta, k - n) \}
\]

coincide with each other. Moreover, the relationship can be written down explicitly, let \( (\theta, \sigma) = R_{N, \Delta}(\alpha) \), then it holds that \( \varphi_\alpha = \varphi_{\theta, \sigma} \).

**Proof.** From Proposition 2.14, the mapping \( R_{N, \Delta} : \alpha \mapsto (\theta, \sigma) \) is a bijection between \( \Sigma_{\mathcal{N}, n}(\mathbb{I}_d, k) \) and \( \Sigma(N, n) \times \Sigma_0^{(q)}(\Delta, k - n) \). The proposition can be immediately proved by this bijection. \( \square \)

In what follows, a global version of degrees of freedom will be proposed. Given a subsimplex \( F \) with codimension \( s \), it is clear that there are \( s \) pairwise orthonormal normal vectors of \( F \), denoted as \( n_{F,0}, \ldots, n_{F,s-1} \). Without loss of generality, set \( F = \langle x_s, \ldots, x_d \rangle := \text{conv}(\{x_s, \ldots, x_d\}) \) with dimension \( s \). Define a bubble function space on \( F \) (associated with the continuity vector \( r \) and polynomial degree \( k \)) by

\[
B_{F, n, k} := \text{span} \left\{ \lambda^{\sigma_1}_{F,1} : \sigma = (\sigma_s, \ldots, \sigma_d) \in \Sigma_{0}^{(q)}(\mathbb{I}_d \setminus \mathbb{I}_{s-1}, k - n) \right\}
\]

\[
\text{span}\{\lambda^{\sigma_1}_{F,s} \cdots \lambda^{\sigma_d}_{F,d} : \sigma = (\sigma_s, \ldots, \sigma_d) \in \Sigma_{0}^{(q)}(\mathbb{I}_d \setminus \mathbb{I}_{s-1}, k - n)\}.
\]

Here \( \lambda_{F,i} := \lambda_{K,i}|_F \), where \( \lambda_{K,i} \) is the barycenter coordinate associated with vertex \( x_i \) of \( K \), is also the barycenter coordinate associated to vertex \( x_i \) with respect to \( F \), \( i = s, \ldots, d \), and the continuity vector \( q = q^{s \cdot n} := (r_{s+1} - n, \ldots, r_d - n) \) for \( n = 0, 1, \ldots, r_s \).

Given \( u \in \mathcal{P}_k(K) \), the degrees of freedom for the shape function space \( \mathcal{P}_k(K) \) are as follows:

\[
\frac{1}{|F|} (D^\theta u, v)_F \quad \forall v \in B_{F, n, k},
\]

for all the subsimplices \( F \) of \( d \)-dimensional simplex \( K \), and \( n = 0, 1, \ldots, r_s \) (when \( s = 0 \), let \( n = 0 \)). Here \( D^\theta u \) represents an \( n \)-th order normal derivative of \( u \) on \( F \) when \( n > 0 \) \( (D^0 u = u \) when \( n = 0 \), namely,

\[
D^\theta u := \frac{\partial^n}{\prod_{i=0}^{n-1} \partial n_{F,i}^{\theta_i}} u
\]

for some multi-index \( \theta \in \Sigma(\mathbb{I}_{s-1}, n) \), i.e., \( n = \sum_{i=0}^{s-1} \theta_i \). Then by the linearity of the space in (3.5), the degrees of freedom defined by (3.6) is equivalent to (3.2). Note that in fact there are \( \dim B_{F, n, k} = \text{Card}(\Sigma_{0}^{(q)}(\mathbb{I}_d \setminus \mathbb{I}_{s-1}, k - n)) \) degrees of freedom is defined by either (3.6) or (3.2).

The main result is the following unisolvency and \( C^\infty \) continuity of the constructed finite element spaces. The following theorem makes everything in Theorem 1.1 precise.

**Theorem 3.3.** Given a continuity vector \( r = (r_1, \ldots, r_d) \) and a polynomial degree \( k \) satisfying Assumption \( A \), the degrees of freedom defined in (3.6) are unisolvent for \( \mathcal{P}_k(K) \). Moreover, the global finite element space

\[
\{ u \in L^2(\Omega) : u|_K \in \mathcal{P}_k(K), u \text{ is single-valued for each degree of freedom on any subsimplex } F \text{ of dimension } \leq d-1 \}
\]

lies in \( C^{r_1}(\Omega) \).
Here \( u \) is single-valued for each degree of freedom \( \varphi_{\theta,s} \) on \( F \) means that, for any two \( d \)-dimensional simplices \( K^+ \) and \( K^- \), sharing common subsimplex \( F \), it holds that \( \frac{1}{|F|} \langle D^s u |_{K^+}, v \rangle_F = \frac{1}{|F|} \langle D^s u |_{K^-}, v \rangle_F \) for all \( |\theta| = n \) and \( v \in B_{F,n,k} \).

The theorem is the main result of this paper, but the proof is complicated since the definition of the intrinsic decomposition and \( B_{F,n,k} \) is not straightforward in \( d \) dimensions. Luckily, in two and three dimensions the characterization can be figured out, which is shown in Section 4. The theorem will be proved in Section 5 below.

4. Examples in Two and Three Dimensions

This section provides concrete examples in two and three dimensions. The following remark is also useful in the following, telling that when the bubble function spaces can be regarded as a complete polynomial bubble function space. Here completeness is a conventional mathematical notion. A polynomial bubble function space \( B \), defined on the subsimplex \( F \) (with codimension \( s \)), is complete if there exist non-negative integers \( r' \) and \( k' \) such that \( B = (\lambda_{F,s} \cdots \lambda_{F,d})^{r'} P_{k'}(F) \). On the contrary, \( B \) is incomplete if there do not exist such \( r' \) and \( k' \).

Remark 4.1. For the continuity vector \( r \), if \( r_s + 1 \leq s (r_1 + 1) \) holds for \( s = 1,\cdots,d \), then
\[
\Sigma_0(\|d,k) = \{(r_1 + 1 + \beta_0,\cdots,r_1 + 1 + \beta_d) : \beta \in \Sigma(1_d, k - (d + 1)(r_1 + 1))\}.
\]
As a result, it holds that
\[
B_{K,0,k} = \text{span}\{\lambda_0^{r_0} \cdots \lambda_d^{r_d} : \sigma = (\sigma_0,\cdots,\sigma_d) \in \Sigma_0(1_d, k)\} = (\lambda_0 \cdots \lambda_d)^{r_1 + 1} P_{k - (d + 1)(r_1 + 1)}.
\]

Under Assumption A, the condition \( r_s + 1 \leq s (r_1 + 1) \) cannot hold in general. In fact, if \( r_1 = 1 \), then both \( r_s \leq 2s \) and \( r_s \geq 2^{s-1} \) imply that \( d \leq 3 \). Therefore, only for lower dimensional cases, the polynomial bubble function spaces can be possible to be complete. Such a situation will become more complicated for higher dimensions and the case of higher continuity. This, in some sense, explains the challenge of the construction of \( C^r \) finite element spaces in any dimension.

4.1. \( C^r \) finite element spaces in two dimensions. First, recall the two-dimensional Bramble–Zlámáel element \([7]\), which possesses \( C^r \) continuity. The shape function space is taken as \( P = P_k(K) \) of the space of polynomials of degree \( \leq k \) for \( k \geq 4r + 1 \). Compared to the original paper \([7]\), a different but equivalent set of degrees of freedom is proposed here.

Given \( u \in P_k(K) \), the degrees of freedom in the notation of Theorem 1.1 are as follows:
- The function value, first, second, \( \cdots \), \( (2r) \)-th order derivatives of \( u \) at each vertex \( x \) of element \( K \), the corresponding bubble function spaces are
  \[
  B_{x,n,k} = \text{span}\{\lambda_x^{k-n} : n \leq 2r \},
  \]
  where \( \lambda_x \) is the associated barycenter coordinate of vertex \( x \). This set of degrees of freedom is in fact defined by the set of multi-indices \( \Sigma_2(1_d, k) \).
  - The weighted moment(s)
  \[
  \frac{1}{|e|} \int_e \left( \frac{\partial^m u}{\partial n^m} \right) \cdot v, \quad \forall v \in B_{e,n,k}
  \]
  on each edge \( e \) of element \( K \) for \( 0 \leq n \leq r \). Here \( n \) is the unit normal vector of edge \( e \), and the bubble function spaces \( B_{e,n,k} \) read
  \[
  B_{e,n,k} = \text{span}\{\lambda_{e,1}^{\sigma_1} \lambda_{e,2}^{\sigma_2} : (\lambda_{e,1} \lambda_{e,2})^{2r+1-n} P_{k+n-2(2r+1)}\},
  \]
  where \( \sigma_1 + \sigma_2 = k - n \) and \( \sigma_1, \sigma_2 \geq 2r + 1 - n \). This set of degrees of freedom is defined by the set of multi-indices \( \Sigma_1(1_d, k) \).
  - The weighted moment(s)
  \[
  \frac{1}{|K|} \int_K u \cdot v \quad \forall v \in B_{K,0,k} = \text{span}\{\lambda_x^{r_0} \lambda_1^{r_1} \lambda_2^{r_2} : \sigma \in \Sigma_0(1_d, k)\}.
  \]
This set of degrees of freedom is defined by the set of multi-indices \( \Sigma_0(1_d, k) \).
Figure 4 illustrates the cases when $r = 1, k = 6$ and $r = 2, k = 9$. The general case in two dimensions can be obtained by replacing $(r, 2r)$ by $(r_1, r_2)$, see Figure 5.

Figure 4. The illustration for the degrees of freedom in two dimensions, when $r = 1, k = 6$ and $r = 2, k = 9$.

Figure 5. The decomposition of $\Sigma(\mathbb{I}_2, 13)$ with the continuity vector $r = (1, 5)$ and polynomial degree $k = 13$, and corresponding degrees of freedom.

Remark 4.2. Two facts about the completeness are listed below without proof.
- The bubble spaces defined on a one-dimensional simplex (edge) are always complete. This holds for any dimension.
- In two dimensions, the bubble function space defined on a two-dimensional simplex (face) is complete if and only if $r_2 + 1 \leq 2(r_1 + 1)$, which implies $r = (r, 2r)$ or $r = (r, 2r + 1)$ for some nonnegative $r$.

4.2. $C^r$ finite element spaces in three dimensions. This subsection provides the $C^r$ finite element spaces, defined in (3.6) for three dimensions. For the case $r = 1$, it recovers the $P_9 - C^1$ Ženěšek element in Ženěšek [27]. The reader can also refer to [28, 29] for the $C^1$ and $C^2$ finite element methods in three dimensions, which extend the Ženěšek element, and can be derived from
the construction in this paper as well. However, the construction of a family of \( P_{sr+1} - C^r \) elements for \( r \geq 1 \), appearing in [19, Chapter 18.1], is different in nature from that given by this paper. In a very recent work [30], Zhang provided a set of explicit basis functions for several \( C^r \) finite element methods in three dimensions to verify the finite element methods constructed in this paper.

Given a continuity vector \( r = (r_1, r_2, r_3) \) and a polynomial degree \( k \) satisfying Assumption A, the degrees of freedom defined in (3.6) for \( u \in P_k(K) \) are as follows.

- The function value, first, second, \( \cdots \), \( r_3 \)-th order derivatives of \( u \) at each vertex \( x \) of element \( K \), the corresponding bubble function spaces are
  \[
  B_{x,n,k} = \text{span}\{\lambda_{k-n}^n\} \quad \forall n \leq r_3, 
  \]
  where \( \lambda_x \) is the associated barycenter coordinate of vertex \( x \). This set of degrees of freedom is in fact defined by the set of multi-indices \( \Sigma_3(\mathbb{I}_3, k) \).

- The weighted moments
  \[
  \frac{1}{|e|} \int_e \left( \frac{\partial^{p+q}}{\partial n_1^{q} \partial n_2^{p}} u \right) \cdot v \quad \forall v \in B_{e,p+q,k}
  \]
on each edge \( e \) of element \( K \), where \( p \geq 0, q \geq 0, p + q \leq r_2 \), and
  \[
  B_{e,p+q,k} = \text{span}\{\lambda_{\sigma_1}^\sigma_2 \lambda_{\sigma_3}^\sigma_3 \} = \{\lambda_{e,2} \lambda_{e,3}\}^{r_3-p-q} P_{k-2(r_3+1)+p+q},
  \]
  with \( \sigma_2 + \sigma_3 = k - p - q \) and \( \sigma_2, \sigma_3 \geq r_3 - p - q \), where \( \lambda_{e,2} \) and \( \lambda_{e,3} \) are the two barycenter coordinates with respect to edge \( e \). Here, \( \mathbf{n}_1, \mathbf{n}_2 \) are two linearly independent unit normal vectors of edge \( e \) and they are perpendicular to each other. This set of degrees of freedom is defined by the set of multi-indices \( \Sigma_2(\mathbb{I}_3, k) \).

- The weighted moment(s) inside element \( K \),
  \[
  \frac{1}{|F|} \int_F \left( \frac{\partial^n u}{\partial n} \right) \cdot v \quad \forall v \in B_{F,n,k},
  \]
on each face \( F, n = 0, 1, \cdots, r_1 \), where \( \mathbf{n} \) is the unit outer normal vector of face \( F \), and
  \[
  B_{F,n,k} = \text{span}\{\lambda_{\sigma_1}^\sigma_2 \lambda_{\sigma_3}^\sigma_3 \} : (\sigma_1, \sigma_2, \sigma_3) \in \Sigma_0(2-r_2-n, r_3-n, I, k - n) \},
  \]
  with \( I := \{1, 2, 3\} \), where \( \lambda_{F,1}, \lambda_{F,2} \) and \( \lambda_{F,3} \) are the three barycenter coordinates with respect to face \( F \). This set of degrees of freedom is defined by the set of multi-indices \( \Sigma_1(\mathbb{I}_3, k) \).

- The weighted moment(s) inside element \( K \),
  \[
  \frac{1}{|K|} \int_K u \cdot v \quad \forall v \in B_{K,0,k},
  \]
  \[
  B_{F,n,k} = \text{span}\{\lambda_{\sigma_0}^\sigma_1 \lambda_{\sigma_2}^\sigma_2 \lambda_{\sigma_3}^\sigma_3 \} : (\sigma_0, \sigma_1, \sigma_2, \sigma_3) \in \Sigma_0(\mathbb{I}_3, k) \},
  \]
  where \( \lambda_0, \lambda_1, \lambda_2 \) and \( \lambda_3 \) are the barycenter coordinates with respect to element \( K \). This set of degrees of freedom is defined by the set of multi-indices \( \Sigma_0(\mathbb{I}_3, k) \).

Consider a specific case \( r = (4, 8, 16) \) and \( k = 33 \). Given an element \( K \), the shape function space is \( P_k(K) \). The number of degrees of freedom defined above are as follows:

1. At each vertex, the number of degrees of freedom is \( \binom{16+3}{3} \). Hence, the total number of degrees of freedom at the four vertices of element \( K \) is \( 4 \times \binom{19}{3} = 3876 \), which is equal to \( \text{Card}(\Sigma_3(\mathbb{I}_3, 33)) \).

2. On each edge, the number of degrees of freedom is \( \sum_{\theta=0}^{8} \theta(\theta + 1) = 240 \). Hence, the total number of degrees of freedom on the six edges of element \( K \) is \( 6 \times 240 = 1440 \), which is equal to the number of components of the set \( \Sigma_2(\mathbb{I}_3, 33) \), namely, \( \text{Card}(\Sigma_2(\mathbb{I}_3, 33)) \).

3. On each face, the number of degrees of freedom is
  \[
  \sum_{\theta=0}^{4} \text{Card}(\Sigma_0(8-\theta, 16-\theta, \mathbb{I}_2, 33-\theta)) = 28 + 45 + 63 + 82 + 102 = 320.
  \]
  Hence, the total number of degrees of freedom on the four faces of element \( K \) is \( 4 \times 320 = 1280 \), which is equal to \( \text{Card}(\Sigma_1(\mathbb{I}_3, 33)) \).
(4) Inside $K$, the set of degrees of freedom is corresponding to the set $\Sigma_0(3, 3)$. The number of degrees of freedom inside element $K$ is 544.

The rest of this section considers the bubble function spaces $B_{F,n,k}$ from Theorem 1.1 inside the two dimensional faces $F$ of tetrahedron element $K$ for two lower order cases. Here let $I = \{1, 2, 3\}$.

In the first case, $r = (1, 2, 4)$ and $k = 9$. Then the corresponding bubble function spaces read

$$B_{F,0,9} = \text{span}\{\lambda_{F,1}^{r_1} \lambda_{F,2}^{r_2} \lambda_{F,3}^{r_3} : (\sigma_1, \sigma_2, \sigma_3) \in \Sigma_0^{(2,4)}(1, 9)\} = (\lambda_{F,1} \lambda_{F,2} \lambda_{F,3})^3 \mathcal{P}_0(F),$$

and

$$B_{F,1,9} = \text{span}\{\lambda_{F,1}^{r_1} \lambda_{F,2}^{r_2} \lambda_{F,3}^{r_3} : (\sigma_1, \sigma_2, \sigma_3) \in \Sigma_0^{(1,3)}(1, 8)\} = (\lambda_{F,1} \lambda_{F,2} \lambda_{F,3})^2 \mathcal{P}_2(F).$$

In the second case, $r = (2, 4, 8)$ and $k = 17$, the associated bubble function spaces as follows,

$$B_{F,0,17} = \text{span}\{\lambda_{F,1}^{r_1} \lambda_{F,2}^{r_2} \lambda_{F,3}^{r_3} : (\sigma_1, \sigma_2, \sigma_3) \in \Sigma_0^{(4,8)}(1, 17)\} = (\lambda_{F,1} \lambda_{F,2} \lambda_{F,3})^5 \mathcal{P}_2(F),$$

$$B_{F,1,17} = \text{span}\{\lambda_{F,1}^{r_1} \lambda_{F,2}^{r_2} \lambda_{F,3}^{r_3} : (\sigma_1, \sigma_2, \sigma_3) \in \Sigma_0^{(3,7)}(1, 16)\} = (\lambda_{F,1} \lambda_{F,2} \lambda_{F,3})^4 \mathcal{P}_4(F),$$

and

$$B_{F,2,17} = \text{span}\{\lambda_{F,1}^{r_1} \lambda_{F,2}^{r_2} \lambda_{F,3}^{r_3} : \Sigma_0^{(2,6)}(1, 15)\} = (\lambda_{F,1} \lambda_{F,2} \lambda_{F,3})^3 \left(\mathcal{P}_0(F) \setminus \text{span}\{\lambda_{F,0}^{r_1}, \lambda_{F,1}^{r_2}, \lambda_{F,2}^{r_3}\}\right).$$

Note that the last bubble function space $B_{F,2,17}$ cannot be regarded as a complete polynomial bubble function space.

**Remark 4.3.** The facts about the completeness of the bubble function spaces in three dimensions are listed below without proof:

- The bubble space defined on a two-dimensional simplex $F$ with respect to the $n$-th order normal derivative ($n \leq r_1$) is complete if and only if $r_3 = 2r_2 - n$ or $r_3 = 2r_2 - n + 1$, see Remark 4.2. Therefore, the bubble function space can never be complete when $n \geq 2$.
- In particular, when $n = 0$, the bubble function space on $F$ is complete if and only if $r_3 = 2r_2$ or $r_3 = 2r_2 + 1$. When $n = 1$, the bubble function space on $F$ is complete if and only if $r_3 = 2r_2$.
- The bubble function space defined on a three-dimensional simplex (cell) is complete if and only if $r = (0, 0, 0), (0, 0, 1), (0, 1, 2), (1, 2, 4), (1, 2, 5)$ or $(2, 4, 8)$. Hence, in almost all cases, the bubble function spaces are incomplete.

5. **Proof of Theorem 3.3: Unisolvency and Continuity**

This section proves Theorem 3.3. Recall the following refined decomposition of $\Sigma(I_d, k)$

$$\Sigma(I_d, k) = \bigcup_{N, n} \Sigma_{N,n}(I_d, k),$$

with the set $\Sigma_{N,n}(I_d, k)$ defined in Definition 2.9 above. Hence, a direct sum decomposition of the shape function space $\mathcal{P}_k(K)$ is as follows

$$(5.1) \quad \mathcal{P}_k(K) = \bigoplus_{N,n} \mathcal{P}_{N,n}.$$ 

Here,

$$\mathcal{P}_{N,n} := \text{span}\left\{\prod_{i \in I_d} \lambda_{\alpha_i}^{r_i} : \alpha \in \Sigma_{N,n}(I_d, k)\right\}.$$ 

This decomposition implies for any $u \in \mathcal{P}_k(K)$, there is a unique decomposition $u = \sum_{N,n} u_{N,n}$ with $u_{N,n} \in \mathcal{P}_{N,n}, N \subseteq I_d, n \leq r_{\text{card}(N)}$.

The proof of unisolvency is based on an induction argument. To this end, an order of the pairs $(N, n)$ will be defined below.
Definition 5.1 (Order of the pair). For all the pairs \((N,n), (N',n')\) \(\subseteq \mathbb{I}_d, n \in \mathbb{N}_0\), introduce the following order: Say \((N',n') \preceq (N,n)\) if
\[
N' \supseteq N
\]
or
\[
N' = N \quad \text{and} \quad n' \leq n.
\]
Say \((N',n') \prec (N,n)\) if \((N',n') \preceq (N,n)\) and \((N',n') \neq (N,n)\).

Again, consider the Argyris element as an example. For the three pairs \((\{1,2\},0), (\{1,2\},1)\) and \((\{1\},0)\), it holds that \((\{1,2\},0) \prec (\{1,2\},1) \prec (\{1\},0)\).

This order leads to the following key lemma, which in particular tells that \(\varphi_\alpha(\cdot)\) vanishes for \(P_{N,n}\) if the condition \((N,n) \preceq (N(\alpha),n(\alpha))\) does not hold.

Lemma 5.2 (Induction Lemma). Under Assumption A, for \(\alpha, \beta \in \Sigma(\mathbb{I}_d,k)\), let \(N := N(\alpha), n := n(\alpha)\). If the condition \((N(\beta),n(\beta)) \preceq (N,n)\) does not hold, then \(\varphi_\alpha(\lambda\beta) = 0\) with \(\varphi_\alpha(\cdot)\) defined in (3.3), where \(\lambda\beta := \prod_{i \in \mathbb{I}_d} \lambda_i^{\beta_i}\).

Proof. To start the proof, note that Definition 5.1 immediately implies that \(N\) cannot be empty. It follows from Definition 2.5 that \(n \leq r_{\text{Card}(N)}\). Then it can be asserted that \(\sum_{i \in \mathbb{I}_d} \beta_i > n\). Otherwise, suppose that \(\sum_{i \in \mathbb{I}_d} \beta_i \leq n\), it holds that \(\text{Card}(N(\beta)) \geq \text{Card}(N)\). Since the condition \((N(\beta),n(\beta)) \preceq (N,n)\) does not hold, it implies that \(N\) is not a subset of \(N(\beta)\). Therefore, it follows that
\[
\text{Card}(N(\beta) \cup N) \geq \text{Card}(N(\beta)) + 1
\]
and
\[
\sum_{i \in N(\beta) \cup N} \beta_i \leq \sum_{i \in N(\beta)} \beta_i + \sum_{i \in N} \beta_i
\]
\[
\leq r_{\text{Card}(N(\beta))} + r_{\text{Card}(N)}
\]
\[
\leq 2r_{\text{Card}(N(\beta))} \leq r_{\text{Card}(N(\beta) \cup N)},
\]
which contradicts with the definition of \(N(\beta)\), since \(N(\beta) \cup N\) satisfies the condition in Definition 2.1, while \(N(\beta)\) is the largest admissible choice. Here the second inequality is from \(\text{Card}(N(\beta)) \geq \text{Card}(N)\), while the last inequality is from \(\text{Card}(N(\beta) \cup N) \geq \text{Card}(N(\beta)) + 1\).

Let \(F := (\Delta) := \text{conv}\{x_i : i \in \Delta\}\) with the unit normal vectors \(n_{F,i}, i \in N\). Note that each barycenter coordinate \(\lambda_i\) associated with vertex \(x_i\) vanishes on \(F\) for \(i \in N\). With
\[
\lambda^\Delta \beta := \prod_{i \in \Delta} \lambda_i^{\beta_i}, \quad \lambda^N \beta := \prod_{i \in N} \lambda_i^{\beta_i},
\]
a direct calculation yields
\[
\frac{\partial^n \lambda^N \beta}{\prod_{i \in N} \partial n_{F,i}^{\theta_i'}} \bigg|_{F} = \frac{\partial^n \lambda^N \beta}{\prod_{i \in N} \partial n_{F,i}^{\theta_i'}} \left( \prod_{i \in N} \lambda_i^{\beta_i} \right) \bigg|_{F} = 0 \quad \text{for} \quad n' \leq n \quad \text{and} \quad \theta' \in \Sigma(N,n').
\]

It follows from (5.2) and the generalized Leibniz rule that
\[
\frac{\partial^n \lambda^N \beta}{\prod_{i \in N} \partial n_{F,i}^{\theta_i'}} \bigg|_{F} = \sum_{n'' \leq n, \theta' \in \Sigma_{N,n'',\theta}} \left( \frac{\partial^n \lambda^N \beta}{\prod_{i \in N} \partial n_{F,i}^{\theta_i'}} \right) \left( \frac{\partial^{n-n'} \lambda^N \beta}{\prod_{i \in N} \partial n_{F,i}^{\theta_i'}} \right) \bigg|_{F} = 0.
\]

Here \(\theta\) is the first component of the pair \((\theta, \sigma) = R_{N,\Delta} (\alpha)\) defined in Proposition 2.14, and the set \(\Theta_{N,n',\theta} := \{\theta' \in \Sigma(N,n') : \theta' \leq \theta_i, i \in N\}\). This implies that \(\varphi_\alpha(\lambda\beta) = 0\).

The following lemma is also crucial in the proof of unisolvency, indicating that for the subspace \(P_{N,n}\) of the shape function space, the degrees of freedom \(\{\varphi_\alpha(\cdot) : \alpha \in \Sigma_{N,n}(\mathbb{I}_d,k)\}\) are unisolvent.

Lemma 5.3 (Unisolvency for the subspace \(P_{N,n}\)). Under Assumption A, for any nonempty subset \(\Delta \subseteq \mathbb{I}_d\) and \(F := (\Delta) := \text{conv}\{x_i : i \in \Delta\}\), let \(N := \mathbb{I}_d \setminus \Delta, n \leq r_{\text{Card}(N)}\) and \(u_{N,n} \in P_{N,n}\). If \(\varphi_\alpha(u_{N,n}) = 0\) for all \(\alpha \in \Sigma_{N,n}(\mathbb{I}_d,k)\), then \(u_{N,n} = 0\).
Proof. First, consider the case \( \mathcal{N} \neq \emptyset \). A new basis of the space \( \mathcal{P}_{\mathcal{N}, n} \) corresponding to the set \( \{ n_{F,i} : i \in \mathcal{N} \} \) will be introduced. This new basis of the space \( \mathcal{P}_{\mathcal{N}, n} \) depends on a new basis \( \{ \nu_i : i \in \mathcal{N} \} \) of the space \( \text{span}\{ \lambda_i : i \in \mathcal{N} \} \), such that
\[
\text{grad} \, \nu_i = n_{F,i}, \quad \forall i \in \mathcal{N}.
\]
In fact, the new basis \( \{ \nu_i : i \in \mathcal{N} \} \) can be constructed as follows.

Suppose \( \mathbf{m}_i \) is the unique (outer) normal vector of codimension 1 subsimplex \( \langle I_d \setminus \{ I \} \rangle := \operatorname{conv}\{ x_i : i \in I_d \setminus \{ I \} \} \) for \( I \in I_d \). It follows from the definition that the set \( \{ n_{F,i} : i \in \mathcal{N} \} \) and the set \( \{ \mathbf{m}_i : i \in \mathcal{N} \} \) are two bases of the space perpendicular to subsimplex \( F \). Then there exist \( c_{ij} \in \mathbb{R} \), \( i,j \in \mathcal{N} \), such that
\[
n_{F,i} = \sum_{j \in \mathcal{N}} c_{ij} \mathbf{m}_j, \quad i \in \mathcal{N}.
\]
Clearly, there exists \( \xi_i \neq 0 \) such that
\[
\text{grad} \lambda_i = \xi_i \mathbf{m}_i, \quad i \in \mathcal{N}
\]
from the definition of \( \mathbf{m}_i \). Define
\[
\nu_i := \sum_{j \in \mathcal{N}} \xi_j \lambda_j, \quad i \in \mathcal{N},
\]
then it holds that
\[
\text{grad} \nu_i = n_{F,i}, \quad i \in \mathcal{N}.
\]
Since these vectors \( n_{F,i}, i \in \mathcal{N} \), are linearly independent, the functions \( \{ \nu_i : i \in \mathcal{N} \} \) form a basis of the space \( \text{span}\{ \lambda_i : i \in \mathcal{N} \} \). Thus, a new basis of \( \mathcal{P}_{\mathcal{N}, n} \) can be defined as follows.

For \( \alpha \in \Sigma_{\mathcal{N}, n}(I_d, k) \), define \( \lambda_{\text{nor}} : \Sigma_{\mathcal{N}, n}(I_d, k) \to \mathcal{P}_{\mathcal{N}, n} \) such that
\[
\lambda_{\text{nor}} \alpha := \lambda^* \alpha \prod_{i \in \Delta} \nu_i^{\alpha_i}.
\]
It follows from Proposition 2.14 that \( \{ \lambda_{\text{nor}} \alpha : \alpha \in \Sigma_{\mathcal{N}, n}(I_d, k) \} \) is a basis of \( \mathcal{P}_{\mathcal{N}, n} \), where \( \lambda^* \alpha := \prod_{i \in \Delta} \lambda_i^{\alpha_i} \). Hence \( u_{\mathcal{N}, n} \) can be reexpressed as
\[
u_{\mathcal{N}, n} := \sum_{\alpha \in \Sigma_{\mathcal{N}, n}(I_d, k)} c_\alpha \lambda_{\text{nor}} \alpha
\]
for combination parameters \( c_\alpha, \alpha \in \Sigma_{\mathcal{N}, n}(I_d, k) \). Next, define
\[
u_{\mathcal{N}, n, \theta} := \sum_{\alpha \in \Sigma_{\mathcal{N}, n, o}(I_d, k)} c_\alpha \lambda^* \alpha
\]
for \( \theta \in \Sigma(\mathcal{N}, n) \). It follows from Proposition 2.14 that the mapping \( \mathcal{R}_{\mathcal{N}, \Delta} \) is a bijection between \( \Sigma_{\mathcal{N}, n}(I_d, k) \) and \( \Sigma(\mathcal{N}, n) \times \Sigma_0^q(\Delta, k - n) \). Hence, it holds that
\[
u_{\mathcal{N}, n} = \sum_{\alpha \in \Sigma_{\mathcal{N}, n}(I_d, k)} c_\alpha \lambda_{\text{nor}} \alpha
\]
and
\[
u_{\mathcal{N}, n, \theta} = \nu_{\mathcal{N}, n, \theta} \prod_{i \in \mathcal{N}} \nu_i^{\theta_i}.
\]
Since it holds that
\[
\frac{\partial \nu_i}{\partial n_{F,j}} = \delta_{ij}, \quad \forall i,j \in \mathcal{N},
\]
with \( \delta_{ij} \) being Kronecker’s delta, and that
\[
\frac{\partial \nu_i}{\partial \mathbf{m}} = 0, \quad \forall i \in \mathcal{N}
\]
for any vector $m$ such that $m \perp n_{F,j}$ for all $j \in N$, it follows that given $\theta \in \Sigma(N, n)$, for $\alpha \in \Sigma_{N,n,\theta}(I_d, k)$, and $\beta \in \Sigma_{N,n}(I_d, k)$,

\begin{equation}
(5.3) \quad \frac{\partial^n}{\prod_{i \in N} \partial n_{F,i}^{\alpha_i}} \lambda_{\text{nor} \beta} \big|_F = \begin{cases} 0, & \beta \not\in \Sigma_{N,n,\theta}(I_d, k), \\ \theta! \lambda^\Delta \beta, & \beta \in \Sigma_{N,n,\theta}(I_d, k), \end{cases}
\end{equation}

where $\Sigma_{N,n,\theta}(I_d, k)$ is defined in (2.9) above.

Now given $\theta \in \Sigma(N, n)$, for $\alpha \in \Sigma_{N,n,\theta}(I_d, k)$ and $\beta \in \Sigma_{N,n}(I_d, k)$, it follows from (5.3) that

\begin{equation}
\varphi_\alpha(\lambda_{\text{nor} \beta}) = \begin{cases} 0, & \beta \not\in \Sigma_{N,n,\theta}(I_d, k), \\ \theta! (\lambda^\Delta \beta, \lambda^\Delta \alpha)_F, & \beta \in \Sigma_{N,n,\theta}(I_d, k). \end{cases}
\end{equation}

Then the linearity of $\varphi_\alpha(\cdot)$ gives

\begin{align*}
\varphi_\alpha(u_{N,n}) &= \sum_{\beta \in \Sigma_{N,n,\theta}(I_d, k)} c_\beta \frac{\theta!}{|F|} (\lambda^\Delta \beta, \lambda^\Delta \alpha)_F \\
&= \frac{\theta!}{|F|} (v_{N,n,\theta}, \lambda^\Delta \alpha)_F.
\end{align*}

Notice that $\varphi_\alpha(u_{N,n}) = 0$ for $\alpha \in \Sigma_{N,n}(I_d, k)$. This leads to

\begin{equation}
(5.4) \quad 0 = \sum_{\alpha \in \Sigma_{N,n,\theta}(I_d, k)} c_\alpha \varphi_\alpha(u_{N,n}) = \frac{\theta!}{|F|} (v_{N,n,\theta}, u_{N,n})_F.
\end{equation}

It yields $v_{N,n,\theta} = 0$. Since $u_{N,n} = \sum_{\theta \in \Sigma(N,n)} (\theta) \sum_{i \in N} \theta_i |_i$, it concludes that $u_{N,n} = 0$.

Second, consider the case $N = \emptyset$ and $n = 0$. Then it holds that $\Sigma_{N,n}(I_d, k) = \Sigma_0(I_d, k)$. Thus, $u_{N,n}$ can be rewritten as $u_{N,n} = \sum_{\alpha \in \Sigma_0(I_d, k)} c_\alpha \lambda_\alpha$ for combination parameters $c_\alpha, \alpha \in \Sigma_0(I_d, k)$. Here

$$\lambda_\alpha := \prod_{i \in I_d} \lambda_i^\alpha.$$ 

Notice that $\varphi_\alpha(u_{N,n}) = 0$ holds for $\alpha \in \Sigma_{N,n}(I_d, k)$. This yields

\begin{align*}
0 &= \sum_{\alpha \in \Sigma_0(I_d, k)} c_\alpha \varphi_\alpha(u_{N,n}) \\
&= \sum_{\alpha \in \Sigma_0(I_d, k)} c_\alpha \frac{\theta!}{|F|} (u_{N,n}, \lambda_\alpha)_F \\
&= \frac{\theta!}{|F|} (u_{N,n}, u_{N,n})_F.
\end{align*}

Thus $u_{N,n} = 0$. \hfill \square

Now it is ready to show the unisolvency.

**Proposition 5.4 (Unisolvency).** Given an element $K \in T$, the set of degrees of freedom $\{\varphi_\alpha(\cdot) : \alpha \in \Sigma(I_d, k)\}$ defined in (3.1), is unisolvent for the shape function space $\mathcal{P}_k(K)$ if Assumption A is satisfied.

**Proof.** From the definition of the intrinsic decomposition of the set $\Sigma(I_d, k)$, the dimension of the shape function space and the number of degrees of freedom coincide. Hence it suffices to show if $u \in \mathcal{P}_k(K)$ vanishes on all degrees of freedom $\varphi_\alpha(\cdot)$ for $\alpha \in \Sigma(I_d, k)$, then $u = 0$. Suppose that $\varphi_\alpha(u) = 0$ for all $\alpha \in \Sigma(I_d, k)$ for some $u \in \mathcal{P}_k(K)$. Recall the decomposition (5.1) of $\mathcal{P}_k(K)$, namely $u = \sum_{N,n} u_{N,n}$ with $u_{N,n} \in \mathcal{P}_{N,n}$.

The proof is based on performing a mathematical induction. To begin with, consider the minimal element $(N, n)$ with respect to index order $\prec$, i.e., for all $(N', n')$, the condition $(N', n') \not\prec (N, n)$ does not hold. By Lemma 5.2, for any $\alpha \in \Sigma_{N,n}(I_d, k)$, it follows that

$$0 = \varphi_\alpha(u) = \sum_{(N', n') \not\prec (N, n)} \varphi_\alpha(u_{N', n'}) + \varphi_\alpha(u_{N,n}) = \varphi_\alpha(u_{N,n}).$$
Therefore by Lemma 5.3, it can be concluded that \( u_{N,n} = 0 \).

Suppose that for all \((N', n') \prec (N, n)\), it holds that \( u_{N', n'} = 0 \). Then, again, by Lemma 5.2, it follows that

\[
0 = \varphi_\alpha(u) = \sum_{(N', n') \not\prec (N, n)} \varphi_\alpha(u_{N', n'}) + \sum_{(N', n') \prec (N, n)} \varphi_\alpha(u_{N', n'}) + \varphi_\alpha(u_{N,n}) = \varphi_\alpha(u_{N,n}).
\]

Therefore by Lemma 5.3, it can be concluded that \( u_{N,n} = 0 \).

It then follows from the mathematical induction on the partial order sets that all \( u_{N,n} = 0 \).

Hence \( u = 0 \), which implies the unisolvency. \( \square \)

**Proposition 5.5** (Continuity). Let \( F \) be a \((d - 1)\)-dimensional simplex shared by two \( d\)-dimensional simplices \( K^+ \) and \( K^- \). Let functions \( u^\pm \) be defined on \( K^\pm \), respectively. Suppose that they are compatible on their common degrees of freedom. Then the piecewise polynomial \( u \) defined as \( u = u^+ \) on \( K^+ \) and \( u = u^- \) on \( K^- \) is of \( C^r \) on \( K^+ \cup K^- \).

**Proof.** It suffices to prove that: if all degrees of freedom \( \varphi_\alpha(\cdot) \) associated to \( F \) (including the degrees of freedom defined inside itself and/or any subsimplex of it, namely \( \langle \Delta(\alpha) \rangle \) is a subsimplex of \( F \)) vanish, then it holds that

\[
u = \frac{\partial u}{\partial n} = \frac{\partial^2 u}{\partial n^2} = \cdots = \frac{\partial^{r+1} u}{\partial n^{r+1}} = 0 \text{ on } F.
\]

Here \( n \) is the unit normal vector of \( F \). Without loss of generality, assume that \( F = \langle \mathbb{I}_d \setminus \{d\} \rangle = \langle \mathbb{I}_{d-1} \rangle \). By definition, \( \langle \Delta(\alpha) \rangle \subseteq F \) implies that \( d \in \mathbb{N}(\alpha) \).

For any nonempty subset \( \Delta \subseteq \mathbb{I}_{d-1} \) and \( f := \langle \Delta \rangle \), let \( N := \mathbb{I}_d \setminus \Delta \), \( s := \text{Card}(N) \). Consider \( \varphi_{\theta, \sigma} \) from (3.2), there holds that \( \varphi_{\theta, \sigma}(u) = 0 \) for \( \theta \in \Sigma(N, n) \) and \( \sigma \in \Sigma_0^q(\Delta, k - n) \) defined in (2.10), where \( q = (r_{s+1} - n, \ldots, r_d - n) \). Taking \( N' = N \setminus \{d\} \), \( 0 \leq l \leq r_1 \) and \( v = \frac{\partial^l}{\partial n^l} u \), it then follows that

\[
\frac{1}{|f|} \left\langle \frac{\partial^{n-l}}{\prod_{i \in N'} \partial n_{f,i}^{|q|}} v \right|_f, \lambda^\Delta \sigma \right\rangle = 0
\]

vanishes for \( \tilde{\theta} \in \Sigma(N', n - l) \) and \( \sigma \in \Sigma_0^q(\Delta, k - n) \), where \( \lambda^\Delta \sigma := \prod_{i \in \Delta} \lambda_i^{n_i} \). It is straightforward to see that the continuity vector \( p_l = (r_2 - l, r_3 - l, \cdots, r_d - l) \) and the polynomial degree \( k - l \) satisfies Assumption A. By the unisolvency with respect to the continuity vector \( p_l \) and the polynomial degree \( k - l \), it holds that \( v = 0 \), implying the continuity. \( \square \)

6. Generalizations

6.1. Discontinuous Elements. This section discusses a simple case where Assumption A is violated. This yields a construction of discontinuous elements. From now on, the following assumption is considered to replace Assumption A.

**Assumption B.** For the continuity vector \( r = (r_1, \ldots, r_d) \) such that \( r_1, \ldots, r_{t-1} = -1 \) for \( 1 < t < d \) and the polynomial degree \( k \), it holds that

\[
r_d \geq 2r_d - 1 \geq 4r_{d-2} \geq \cdots \geq 2^{d-t}r_t \geq k \geq 2r_d + 1.
\]

Under Assumption B, the corresponding intrinsic decomposition of \( \Sigma(\mathbb{I}_d, k) \) and \( \Sigma(I, k) \) can be defined in a similar way as Definition 2.1 and Remark 2.2. In particular, \( \Sigma_s(\mathbb{I}_d, k) = \emptyset \) for \( 1 \leq s < t \).

For \( \alpha \in \Sigma_s(\mathbb{I}_d, k) \) with \( s = 0 \) and \( s \geq t \), the definitions of \( \mathbb{N}(\alpha) \), \( \Delta(\alpha) \) and \( \Delta(\alpha) \) are the same as those in Definition 2.5. The uniqueness of \( \mathbb{N}(\alpha) \) and \( \Delta(\alpha) \) can be proved by a similar argument in Proposition 2.8.
6.2. Stokes Complex in two dimensions. As an application, consider the following smoothing de Rham complex in two dimensions. Given a conforming triangular grid $\mathcal{T} = \mathcal{T}(\Omega)$ of the two-dimensional polygonal domain $\Omega$, denote the global finite element space defined in Theorem 3.3 with the continuity vector $r$ as $V_k(r)(\mathcal{T})$. Here and in the next subsection, the superscript $r$ will be used to emphasize the dependency of these finite element spaces on the continuity vector.

Note that the functions in $V_k(r)$ are of $C^{r_2}$ continuity across the vertices and of $C^{r_1}$ continuity for two dimensions.

Proposition 6.1. Let $r = (r_1, r_2)$, $r' = (r_1 - 1, r_2 - 1)$, $r'' = (r_1 - 2, r_2 - 2)$ with $r_1 \geq 1$. Suppose $r_2 \geq 2r_1$ and $k \geq 2r_1 + 1$. Then, it holds that the following sequence

$$
\mathbb{R} \xrightarrow{\text{curl}} V_k(r)(\mathcal{T}) \xrightarrow{\text{curl}} [V_{k-1}(\mathcal{T})]^2 \xrightarrow{\text{div}} V_{k-2}(\mathcal{T}) \rightarrow 0
$$

is a complex and exact, provided that the domain $\Omega$ is simply connected.

Remark 6.2. In Proposition 6.1, the continuity vector $r''$ might be $(-1, r''_2)$ for some nonnegative integer $r''_2$, which is a special case in Section 6.1. In particular, the case with $r_1 = 1, r_2 = 2$ in the above sequence (6.1) recovers the complex constructed in Falk and Neilan [12].

Proof. It is straightforward to see that (6.1) is a complex. To show the exactness of (6.1), it suffices to prove the discrete kernel of div is just the discrete image of curl, and to compute the dimensions of these finite element spaces in (6.1).

Notice that the dimension of $\mathcal{P}_k(r)(\mathcal{T})$ is just the total number of degrees of freedom defined at the vertices, on the edges and in the interior of element $K$. Denote the number of vertices, edges and faces by $V, E, F$, respectively. At each vertex, the number of degrees of freedom is $a_V = \binom{r_2 + 2}{2}$. On each edge, the number of degrees of freedom is

$$
a_E = \sum_{m=0}^{r_1} (k + m - 2r_2 - 1) = \frac{1}{2}(2k - 4r_2 - 2 + r_1)(r_1 + 1)
$$

$$
= (k - 2r_2 - 1)(r_1 + 1) + \binom{r_1 + 1}{2}.
$$

Inside each element, the number of degrees of freedom is $\binom{k+2}{2} - a_V - a_E$. As a summary, this gives

$$
dim(V_k(r)(\mathcal{T})) = \binom{r_2 + 2}{2}V + \sum_{m=0}^{r_1} (k + m - 2r_2 - 1)E + \left(\binom{k+2}{2} - a_V - a_E\right)F.
$$

Since

$$
\binom{n}{2} - 2\binom{n-1}{2} + \binom{n-2}{2} = 1
$$

and

$$
(k - 2r_2 - 1)(r_1 + 1) - 2(k - 2r_2)(r_1) + (k - 2r_2 + 1)(r_1 - 1) = -2,
$$

by Euler’s formula it follows that

$$
dim(V_k(r)(\mathcal{T})) - 2\dim(V_{k-1}(\mathcal{T})) + \dim(V_{k-2}(\mathcal{T})) = V - E + F = 1.
$$

It remains to show that the discrete kernel of div is just the discrete image of curl. Suppose that for some $v \in [V_{k-1}(\mathcal{T})]^2$ such that $\text{div} v = 0$, then by the exactness of the continuous Stokes complex, there exists $\phi \in H^2(\Omega)$ such that $\text{curl} \phi = v$. By Sobolev’s embedding, $\phi$ is continuous. Restricting this identity to each element $K$ immediately shows that $\phi$ is a polynomial of degree at most $k$ in each element $K$. It remains to show that $\phi$ satisfies the required continuity. Since $v = \text{curl} \phi$ is of $C^{r_2-1}$ continuity, it follows that $\phi$ is of $C^{r_2}$ continuity at each vertex. Similarly, it can be found that $v$ is of $C^{r_1}$ continuity across each edge.

Therefore, it holds that $\text{ker div} \subseteq \text{im curl}$ on the discrete kernel. The converse inclusion is from the definition of the complex. Hence it must hold that $\text{ker div} = \text{im curl}$ on the discrete level. Therefore, the complex (6.1) is exact. 

$\square$
6.3. $H(\text{div})$ Finite Element Space: A Generalized Stenberg Element. In this section, it is assumed that the continuity vector $r = (r_1, r_2)$ satisfies the conditions: $r_1 = l + 1/2$ for some integer $l \geq -1$, and $r_2 \geq 2r_1 + 1$ is a nonnegative integer. It is also assumed that the polynomial degree $k \geq 2r_2 + 1$. For edge $e$, denote by $n$ the (outer) normal vector and $t$ the tangential vector.

This section is motivated by the triangular Stenberg element [22], where $r$ is chosen as $(-\frac{1}{2}, 0)$. The shape function space for the Stenberg element is $[P_2(K)]^2$. Given $u \in [P_2(K)]^2$, the degrees of freedom are as follows:

- The value of $u(x)$ at each vertex $x$.
- $\int_e (u \cdot n)$ on each edge $e$.
- $\int_K u \cdot q$ for $q \in RT_0$, where $RT_0$ is the lowest order Raviart–Thomas element.

For the global Stenberg element space, there is another characterization, see [10]. Explicitly, the Stenberg element space $S$ admits the following decomposition

$$S(T) := [P_2(T)]^2 \oplus B_{\text{div}}(T).$$

Here $P_2(T)$ is the standard $H^1$ conforming quadratic Lagrange element space, while $B_{\text{div}}(T)$ is the elementwise $H(\text{div})$ bubble function space characterized by

$$B_{\text{div}}(K) := \{ u \in [P_2(K)]^2 : u \cdot n = 0 \text{ on each edge } e \}.$$

and the global $H(\text{div})$ bubble function space is defined as

$$B_{\text{div}}(T) := \{ u \in L^2(\Omega) : u|_K \in B_{\text{div}}(K), \forall K \in T \}.$$

It can be proved that, for each element $K$, the dimension of $B_{\text{div}}(K)$ is 3, and the last degrees of freedom can be modified as $\int_K u \cdot q$ for $q \in B_{\text{div}}(K)$. For further information, the interested readers can refer to [10] for the higher degree case and [14, 15] for some tensor generalization.

Given the continuity vector $r = (r_1, r_2)$, a generalized $H(\text{div})$ conforming finite element space $U$ is constructed in this section such that the following requirements are fulfilled: For $u \in U$, $u$ is of $C^{r_2}$ continuity across the vertex, and $u \cdot n$ is of $C^{r_1+1/2}$ continuity across the internal edges, while $u \cdot t$ is of $C^{r_1-1/2}$ continuity across the internal edges. To this end, it is natural to introduce the following generalized $H(\text{div})$ bubble function space.

A new generalized bubble function space $B^{(r)}_{\text{div}, k}(K)$ is defined as

$$B^{(r)}_{\text{div}, k}(K) := \{ u \in [P_k(K)]^2 : D^\alpha u(x) = 0 \text{ for } |\alpha| \leq r_2 \text{ at each vertex } x \in K, \\
(D^\beta)u \cdot n|_e = 0 \text{ for } |\beta| \leq r_1 + \frac{1}{2} \text{ on each edge } e \text{ of } K, \\
(D^\gamma)(u \cdot t)|_e = 0 \text{ for } |\gamma| \leq r_1 - \frac{1}{2} \text{ on each edge } e \text{ of } K \}.$$

Then the degrees of freedom are defined as follows.

**Definition 6.3.** Given $u \in [P_k(K)]^2$, the degrees of freedom are as follows

- $D^\alpha u(x)$ at vertex $x$ of $K$, for $|\alpha| \leq r_2$.
- $\int_e \frac{\partial}{\partial n}(u \cdot n)q$ for $q \in P_{k-2r_2-m}$, $m = 0, 1, \cdots, r_1 + \frac{1}{2}$ on each edge $e$ of $K$.
- $\int_e \frac{\partial}{\partial m}(u \cdot t)q$ for $q \in P_{k-2r_2-m}$, $m = 0, 1, \cdots, r_1 - \frac{1}{2}$ on each edge $e$ of $K$.
- $\int_K u \cdot q$ for $q \in B^{(r)}_{\text{div}, k}(K)$.

It is not easy to write down the explicit form of functions in $B^{(r)}_{\text{div}, k}(K)$. Nevertheless, it is possible to count the dimension of $B^{(r)}_{\text{div}, k}(K)$. Since $k_2 \geq 2(k_1 + \frac{1}{2})$, the degrees of freedom of the first three sets of degrees of freedom in Definition 6.3 are linearly independent. Therefore, the degrees of freedom defined in Definition 6.3 are unisolvent for the shape function space $[P_k(K)]^2$. Moreover, the resulting global finite element space admits the continuity in the following proposition.

**Proposition 6.4.** Let $r = (r_1 + 1/2, r_2 + 1)$, $r' = (r_1, r_2)$, $r'' = (r_1 - 1/2, r_2 - 1)$ with $r_1 \geq -1/2$. Suppose $r_2 \geq 2r_1 + 1$ and $k \geq 2r_2 + 1$. Then, it holds that the following sequence

$$\mathbb{R} \rightarrow V_{k-1}^{(r_1)}(T) \xrightarrow{\text{curl}} S_k^{(r')}(T) \xrightarrow{\text{div}} V_{k-1}^{(r'')}(T) \rightarrow 0$$

is a complex and exact, provided that the domain $\Omega$ is simply connected.
Proof. Recall from the proof in Proposition 6.1 that at each vertex, the sum of the numbers of degrees of freedom of $V^{(r_1)}_{k+1}(T)$ and $V^{(r_1)}_{k-1}(T)$ is $\left(\binom{r_2+3}{2}\right) + \left(\binom{r_2+1}{2}\right)$. On each edge, the sum of the numbers of degrees of freedom is

$$a_{E,1} + a_{E,3} = \sum_{m=0}^{r_1+1/2} (k + 1 + m - 2r_2 - 3) + \sum_{m=0}^{r_1-1/2} (k - 1 + m - 2r_2 + 1)$$

Now consider the degrees of freedom of the space $S^{(r')}_k(T)$ defined in Definition 6.3. At each vertex, the number of degrees of freedom is $\left(\binom{r_2+3}{2}\right)$. On each edge, the number of degrees of freedom is

$$a_{E,2} = \sum_{m=0}^{r_1+1/2} (k - 2r_2 - 1 + m) + \sum_{m=0}^{r_1-1/2} (k - 2r_2 - 1 + m).$$

Since

$$\left(\binom{n}{2}\right) - 2\left(\binom{n-1}{2}\right) + \left(\binom{n-2}{2}\right) = 1,$$

it holds that

$$a_{E,1} + a_{E,3} - a_{E,2} = \sum_{m=0}^{r_1+1/2} (-1) + \sum_{m=0}^{r_1-1/2} (1) = -1.$$

Denote by $a_{K,1}, a_{K,2}, a_{K,3}$ the degrees of freedom defined inside element $K$. Since

$$a_{K,1} = \dim P_{k+1} - \left(\binom{r_2+3}{2}\right) - a_{E,1},$$

$$a_{K,2} = \dim P^2_k - \left(\binom{r_2+2}{2}\right) - a_{E,2},$$

and

$$a_{K,3} = \dim P_{k+1} - \left(\binom{r_2+1}{2}\right) - a_{E,3},$$

it holds that

$$a_{K,1} - a_{K,2} + a_{K,3} = 1 - 1 + 1 = 1.$$

Consequently, by Euler’s formula, it holds that

$$\dim(V^{(r)}_{k+1}(T)) - \dim(S^{(r')}_k(T)) + \dim(V^{(r''})_{k-1}(T)) = V - E + F = 1.$$

It remains to show that the discrete kernel of div is just the discrete image of curl. Suppose that for some $v \in S^{(r')}_k(T)$ such that $\text{div} v = 0$, then by the exactness of the continuous Stokes complex, there exists $\phi \in H^1(\Omega)$ such that $\text{curl} \phi = v$. Restricting this identity to each element $K$ immediately tells that $\phi$ is a polynomial of degree at most $k+1$ in each element $K$. It remains to show that $\phi$ satisfies the required continuity. Since $\phi \in H^1(\Omega)$ is piecewise smoothing, it follows that $\phi$ is continuous.

At each vertex, it follows from the $C^{r_2-1}$ continuity of $v = \text{curl} \phi$ that $\phi$ is of $C^{r_2}$ continuity. On each edge, the assumption yields that

$$\frac{\partial}{\partial n^m}(\text{curl} \phi \cdot t) = \frac{\partial}{\partial n^{m+1}}(\phi)$$

is continuous (single-valued) across each internal edge $e$, for $m = 0, 1, \cdots, r_1 - 1/2$. Therefore, it holds that $\ker \text{div} \subseteq \text{im} \text{curl}$ on the discrete level. The converse inclusion is from the definition of the complex. Hence it must hold that $\ker \text{div} = \text{im} \text{curl}$ on the discrete level. Therefore, the complex (6.6) is exact. □
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