Maximal CP Violation in Flavor Neutrino Masses

Teruyuki Kitabayashi and Masaki Yasue

Department of Physics, Tokai University,
4-1-1 Kitakaname, Hiratsuka, Kanagawa 259-1292, Japan
teruyuki@tokai-u.jp, yasue@keyaki.cc.u-tokai.ac.jp

Since flavor neutrino masses $M_{\mu\mu,\tau\tau,\mu\tau}$ can be expressed in terms of $M_{ee,e\mu,e\tau}$, mutual dependence among $M_{\mu\mu,\tau\tau,\mu\tau}$ is derived by imposing some constraints on $M_{ee,e\mu,e\tau}$. For appropriately imposed constraints on $M_{ee,e\mu,e\tau}$ giving rise to both maximal CP violation and the maximal atmospheric neutrino mixing, we show various specific textures of neutrino mass matrices including the texture with $M_{\tau\tau} = M_{\mu\mu}^*$ derived as the simplest solution to the constraint of $M_{\tau\tau} - M_{\mu\mu} =$ imaginary, which is required by the constraint of $M_{e\mu} \cos \theta_{23} - M_{e\tau} \sin \theta_{23} =$ real for $\cos 2\theta_{23} = 0$. It is found that Majorana CP violation depends on the phase of $M_{ee}$.

Keywords: CP violation; atmospheric neutrino mixing; flavor neutrino masses.

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1. Introduction

The observed data of the neutrino oscillations\cite{1-25} have been accumulated to suggest the allowed range of the CP-violating Dirac phase $\delta_{CP}$ near $3\pi/2$ inducing the maximal CP violation and that of the atmospheric neutrino mixing angle $\theta_{23}$ near $45^\circ$ indicating the maximal atmospheric neutrino mixing.\cite{26,27} These maximal effects arouse theoretical interest that both the Dirac CP-violation and the atmospheric neutrino mixing are necessarily maximal because of a certain constraints imposed on flavor neutrino masses.

It has been discussed that these two maximal effects are correlated.\cite{28-46} Denoting the CP-violating Dirac phase by $\delta_{CP}$, the reactor neutrino mixing angle by $\theta_{13}$ and flavor neutrino masses by $M_{ij}$ for $i,j=e,\mu,\tau$, we can derive the following relation:\cite{35-37}

\begin{equation}
M_{\tau\tau} - M_{\mu\mu} \frac{\sin 2\theta_{23} - M_{\mu\tau} \cos 2\theta_{23}}{2} = \tan \theta_{13} \left( M_{e\mu} \cos \theta_{23} - M_{e\tau} \sin \theta_{23} \right) e^{-i\delta_{CP}},
\end{equation}

provided that the Pentecorvo-Maki-Nakagawa-Sakata mixing matrix $U_{PMNS}$\cite{47,48} takes the standard parameterization defined by the Particle Data Group (PDG).\cite{49,50}
to be $U^{PDG}_\nu = U^{PDG}_\nu K^{PDG}$:

$$U^{PDG}_\nu = \begin{pmatrix} c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i \delta_{CP}} \\ -s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i \delta_{CP}} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i \delta_{CP}} & -s_{12} s_{23} + c_{12} c_{23} s_{13} e^{i \delta_{CP}} \\ s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i \delta_{CP}} & -c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i \delta_{CP}} & c_{12} s_{23} + s_{12} c_{23} s_{13} e^{i \delta_{CP}} \end{pmatrix},$$

$$K^{PDG} = \begin{pmatrix} e^{i \phi_1/2} & 0 & 0 \\ 0 & e^{i \phi_2/2} & 0 \\ 0 & 0 & e^{i \phi_3/2} \end{pmatrix},$$

for $c_{ij} = \cos \theta_{ij}$ and $s_{ij} = \sin \theta_{ij}$ ($i, j = 1, 2, 3$), where $\theta_{12}$ is the solar neutrino mixing angle and $\phi_{1,2,3}$ stand for the Majorana phases, from which two independent combinations become the CP-violating Majorana phases. It is obvious that, from Eq.(1), the maximal atmospheric mixing giving $\cos 2 \theta_{23} = 0$ induces the maximal CP violation giving $\cos \delta_{CP} = 0$ if

$$M_{\tau \tau} - M_{\mu \mu} = \text{imaginary}, \quad M_{e \mu} \cos \theta_{23} - M_{e \tau} \sin \theta_{23} \left(= \frac{M_{e \mu} - \sigma M_{e \tau}}{\sqrt{2}} \right) = \text{real},$$

where $\sigma = \pm 1$ takes care of the sign of $\sin \theta_{23}$. The reactor neutrino mixing angle must satisfy $\sin \theta_{13} \neq 0$. It is obvious that the simplest solution to satisfy in Eq.(3) is that

$$M_{\tau \tau} = M^*_{\mu \mu}, \quad M_{e \tau} = -\sigma M^*_{e \mu},$$

which reproduce the following known flavor neutrino mass matrix $M_\nu$ consisting of $M_{ij}$:

$$M_{\nu} = \begin{pmatrix} M_{ee} & M_{e\mu} & -\sigma M^*_{e\mu} \\ M_{e\mu} & M_{\mu\mu} & M_{\mu\tau} \\ -\sigma M^*_{e\mu} & M_{\mu\tau} & M^*_{\mu\mu} \end{pmatrix}.$$  

In this article, we would like to discuss the variation of Eq.(5) that gives the maximal CP violation and the maximal atmospheric neutrino mixing in the systematic way. In Sec.2 we derive $M_{\mu \mu, \tau \tau, \mu \tau}$ expressed in terms of $M_{ee, e\mu, e\tau}$, which are used in Sec.3 to find how $M_{\mu \mu, \tau \tau, \mu \tau}$ are correlated to each other for $\cos 2 \theta_{23} = 0$ and $\cos \delta_{CP} = 0$. To see the usefulness of our method, we show three new textures other than Eq.(5). In the Appendix A, redundant phases allowed in $U_{PMNS}$ are used to transform $M_\nu$ of Eq.(5) into a more general form of $M_\nu$ so that the associated $U_{PMNS}$ is described by three real numbers $u_{1,2,3}$ and three complex numbers $w_{1,2,3}$. In Sec.4 we discuss how these results are affected if $M_\nu$ is not compatible with the PDG convention. The final section Sec.5 is devoted to summary and discussions.

2. Neutrino Mixings and Flavor Neutrino Masses

To discuss the situation of the maximal CP violation, we utilize two more relations satisfied by $M_{ij}$ in addition to Eq.(1) and Eq.(3). The relations determine the remaining
two mixing angles, $\theta_{12}$ and $\theta_{13}$. After little calculus, we obtain that

\begin{align}
(\lambda_1 - \lambda_2) \sin 2\theta_{12} + 2 (c_{23} M_{\mu} - s_{23} M_{e}) \frac{\cos 2\theta_{12}}{c_{13}} = 0, \\
(M_{e} e^{-i\delta_{CP}} - \lambda_3 e^{i\delta_{CP}}) \sin 2\theta_{13} + 2 (s_{23} M_{\mu} + c_{23} M_{e}) \cos 2\theta_{13} = 0,
\end{align}

where $\lambda_{1,2,3}$ are defined by

\begin{align}
\lambda_1 &= \frac{c_{13}^2 M_{e} - s_{13}^2 e^{2i\delta_{CP}} \lambda_3}{\cos 2\theta_{13}}, \\
\lambda_2 &= c_{23}^2 M_{\mu} + s_{23}^2 M_{\tau} - M_{\mu \tau} \sin 2\theta_{23}, \\
\lambda_3 &= s_{23}^2 M_{\mu} + c_{23}^2 M_{\tau} + M_{\mu \tau} \sin 2\theta_{23}.
\end{align}

From these three relations, defining that

\begin{align}
M_{+} &= s_{23} M_{\mu} + c_{23} M_{e}, \\
M_{-} &= c_{23} M_{\mu} - s_{23} M_{e},
\end{align}

we can derive that

\begin{align}
M_{\mu \mu} &= \left( \frac{1}{c_{13} \tan 2\theta_{12}} - t_{13} e^{-i\delta_{CP}} \right) M_{-} + \left( \frac{e^{-i\delta_{CP}}}{\tan 2\theta_{13}} - \frac{1}{2} t_{13} e^{i\delta_{CP}} \right) M_{+} \\
&\quad + \frac{1 + e^{-2i\delta_{CP}}}{2} M_{e e} - M_{\mu \tau}^{(0)} \cos 2\theta_{23}, \\
M_{\tau \tau} &= \left( \frac{1}{c_{13} \tan 2\theta_{12}} + t_{13} e^{-i\delta_{CP}} \right) M_{-} + \left( \frac{e^{-i\delta_{CP}}}{\tan 2\theta_{13}} - \frac{1}{2} t_{13} e^{i\delta_{CP}} \right) M_{+} \\
&\quad + \frac{1 + e^{-2i\delta_{CP}}}{2} M_{e e} + M_{\mu \tau}^{(0)} \cos 2\theta_{23}, \\
M_{\mu \tau} &= M_{\mu \tau}^{(0)} \sin 2\theta_{23},
\end{align}

and

\begin{align}
M_{\mu \tau}^{(0)} &= -\left( \frac{1}{c_{13} \tan 2\theta_{12}} + t_{13} e^{-i\delta_{CP}} \right) M_{-} + \left( \frac{e^{-i\delta_{CP}}}{\tan 2\theta_{13}} + \frac{1}{2} t_{13} e^{i\delta_{CP}} \right) M_{+} \\
&\quad - \frac{1 - e^{-2i\delta_{CP}}}{2} M_{e e}.
\end{align}

where $t_{ij} = \tan \theta_{ij}$ ($i, j = 1, 2, 3$). In other words, the derived flavor masses of Eq.(10) are the solutions to Eqs.(1), (6) and (7). The relation of

\begin{equation}
M_{+} \propto \sin \theta_{13},
\end{equation}

holds in the limit of $\sin \theta_{13} \to 0$ as can be seen from Eq.(7).

The neutrino masses $m_{1,2,3}$ are calculated to be:

\begin{align}
m_1 e^{-i\phi_1} &= -\frac{t_{12}}{c_{13}} M_{-} - t_{13} e^{i\delta_{CP}} M_{+} + M_{e e}, \\
m_2 e^{-i\phi_2} &= \frac{1}{c_{13} t_{12}} M_{-} - t_{13} e^{i\delta_{CP}} M_{+} + M_{e e}, \\
m_3 e^{-i\phi_3} &= e^{-i\delta_{CP}} M_{+} + e^{-2i\delta_{CP}} M_{e e}.
\end{align}
The advantage of our method lies in the fact that the neutrino masses simply depend on the flavor masses of $M_{ee}$, $M_{e\mu}$, $M_{e\tau}$. It should be noted that somehow “natural” choice of $M_\pm$ and $M_{ee}$ is to take

$$M_-, M_{ee} = \text{real}, \quad \text{arg}(M_+) = -\delta_{CP}, \quad (14)$$

which induce

$$\phi_{1,2} = 0, \quad \phi_3 = 2\delta_{CP}. \quad (15)$$

As a result, the Majorana CP violation is characterized by the phase $2\delta_{CP}$. If textures belong to this type, no Majorana CP violation is induced for the maximal CP violation.

3. Maximal CP Violation and Maximal Atmospheric Neutrino Mixing

Using $e^{i\delta_{CP}} = \kappa i$ ($\kappa = \pm 1$) for the maximal CP violation and $\cos 2\theta_{23} = 0$ for the maximal atmospheric neutrino mixing, the main constraint Eq.(1) turns out to be the simplest one:

$$M_{\tau\tau} - M_{\mu\mu} = -2\kappa i \tan \theta_{13} (M_{e\mu} - \sigma M_{e\tau}) / \sqrt{2}. \quad (16)$$

The flavor neutrino masses of $M_{\mu\mu, \tau\tau, \mu\tau}$ are given by

$$M_{\mu\mu} = \left( \frac{1}{c_{13} \tan 2\theta_{12}} + i\kappa \sigma t_{13} \right) M_- - i\kappa \left( \frac{1}{\tan 2\theta_{13}} + \frac{1}{2} t_{13} \right) M_+, \quad$$

$$M_{\tau\tau} = \left( \frac{1}{c_{13} \tan 2\theta_{12}} - i\kappa \sigma t_{13} \right) M_- - i\kappa \left( \frac{1}{\tan 2\theta_{13}} + \frac{1}{2} t_{13} \right) M_+, \quad$$

$$\sigma M_{\mu\tau} = -\frac{1}{c_{13} \tan 2\theta_{12}} M_- - i\kappa \left( \frac{1}{\tan 2\theta_{13}} - \frac{1}{2} t_{13} \right) M_+ - M_{ee}, \quad (17)$$

where $M_\pm$ for the maximal atmospheric neutrino mixing are given by

$$M_+ = \frac{\sigma M_{e\mu} + M_{e\tau}}{\sqrt{2}}, \quad M_- = \frac{M_{e\mu} - \sigma M_{e\tau}}{\sqrt{2}}. \quad (18)$$

We define two real parameters, $x$ and $y$, and one complex parameter, $z$, to be:

$$x = \frac{1}{\tan 2\theta_{13}} + \frac{1}{2} t_{13}, \quad y = \frac{1}{\tan 2\theta_{13}} - \frac{1}{2} t_{13}, \quad z = \frac{1}{c_{13} \tan 2\theta_{12}} + i\kappa \sigma t_{13}, \quad (19)$$

leading to

$$M_{\mu\mu} = z M_- - i\kappa x M_+, \quad$$

$$M_{\tau\tau} = z^* M_- - i\kappa x M_+, \quad$$

$$\sigma M_{\mu\tau} = -\frac{z + z^*}{2} M_- - i\kappa y M_+ - M_{ee}. \quad (20)$$

It can be found that the texture of Eq.(15) is one of the specific textures, where $M_+$ and $M_-$ are appropriately constrained to be $M_+ = \text{imaginary}$ and $M_- = \text{real}$. One may choose any types of constraints on $M_{ij}$ in place of this constraint to find favorite textures. Some of the examples including Eq.(5) are shown as follows:
(1) For $M_- = \text{real}(= R_0$ being a real number),

$$M_\nu = \begin{pmatrix}
\frac{M_{ee} R_0 + \sigma M_{\mu\tau}}{\sqrt{2}} & \frac{-\sigma R_0 - \sigma M_{\mu\tau}}{\sqrt{2}} \\
\frac{R_0 - \sigma M_{\mu\tau}}{\sqrt{2}} & \frac{-\sigma (R_0 - \sigma M_{\mu\tau})}{\sqrt{2}}
\end{pmatrix}, \quad (21)
$$

is obtained and $m_{1,2,3}$ are given by

$$m_1 e^{-i\phi_1} = -i \frac{t_{21} R_0 - \kappa t_{13} M_{\mu\tau}}{t_{13}} + M_{ee},
$$

$$m_2 e^{-i\phi_2} = i \left( \frac{t_{12}}{t_{13}} I_0 - \kappa t_{13} M_{\mu\tau} \right) + M_{ee},
$$

$$m_3 e^{-i\phi_3} = -i \left( \frac{\kappa}{t_{13}} M_{\mu\tau} + M_{ee} \right), \quad (22)
$$

where, if $M_+ = \text{imaginary}$ is further imposed, we reach Eq.(5) with $M_{e\tau} = -\sigma M_{\mu\mu}$ and $M_{\tau\tau} = M_{\mu\mu}^*$.

(2) For $M_- = \text{imaginary}(= iI_0$ with $I_0$ being a real number),

$$M_\nu = \begin{pmatrix}
\frac{M_{ee} (i I_0 - \sigma M_{\mu\tau})}{\sqrt{2}} & \frac{\sigma (I_0 + i M_{\mu\tau})}{\sqrt{2}} \\
\frac{I_0 - \sigma M_{\mu\tau}}{\sqrt{2}} & i \left( z I_0 - \kappa x M_{\mu\tau} \right)
\end{pmatrix}, \quad (23)
$$

is obtained and $m_{1,2,3}$ are given by

$$m_1 e^{-i\phi_1} = -i \left( \frac{t_{12} I_0 + \kappa t_{13} M_{\mu\tau}}{t_{13}} \right) + M_{ee},
$$

$$m_2 e^{-i\phi_2} = i \left( \frac{t_{12}}{t_{13}} I_0 - \kappa t_{13} M_{\mu\tau} \right) + M_{ee},
$$

$$m_3 e^{-i\phi_3} = -i \left( \frac{\kappa}{t_{13}} M_{\mu\tau} + M_{ee} \right), \quad (24)
$$

where, if $M_+ = \text{real}$ is further imposed, we observe that $M_{e\tau} = \sigma M_{\mu\mu}^*$ and $M_{\tau\tau} = -M_{\mu\mu}^*$ are satisfied in $M_\nu$:

$$M_\nu = \begin{pmatrix}
M_{ee} & M_{e\mu} & \sigma M_{e\mu}^* \\
M_{e\mu} & M_{\mu\mu} & M_{\mu\tau} \\
\sigma M_{e\mu}^* & M_{\mu\tau} & -M_{\mu\mu}
\end{pmatrix}, \quad (25)
$$

(3) For $M_+ = 0$,

$$M_\nu = \begin{pmatrix}
M_{ee} & -\sigma M_{e\mu} \\
M_{e\mu} & -\frac{\sigma M_{e\mu}}{\sqrt{2} z M_{e\mu}} \\
-\sigma M_{e\mu} & \frac{-\sigma M_{e\mu}}{\sqrt{2} z M_{e\mu}}
\end{pmatrix}, \quad (26)
$$

$$= \begin{pmatrix}
M_{ee} & \sqrt{2} z M_{e\mu} & -\sigma M_{e\mu} \\
M_{e\mu} & \frac{-\sigma M_{e\mu}}{\sqrt{2} z M_{e\mu}} & \frac{-\sigma M_{e\mu}}{\sqrt{2} z M_{e\mu}} \\
-\sigma M_{e\mu} & \frac{-\sigma M_{e\mu}}{\sqrt{2} z M_{e\mu}} & \frac{-\sigma M_{e\mu}}{\sqrt{2} z M_{e\mu}}
\end{pmatrix}.$$
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is obtained and \( m_{1,2,3} \) are given by

\[
m_1 e^{-i\phi_1} = -\sqrt{2} t_{12} \frac{M_{e\mu}}{c_{13}} + M_{ee},
\]

\[
m_2 e^{-i\phi_2} = \sqrt{2} t_{12} \frac{M_{e\mu}}{c_{13}} + M_{ee},
\]

\[
m_3 e^{-i\phi_3} = -M_{ee},
\]

(27)

where the mass hierarchy will be the degenerated one realized by \( |M_{ee}| \gg |M_{e\mu}| \);

(4) For \( M_{ee} = 0 \), Eq.(13) leads to \( m_1 = m_2 \), which is not allowed;

(5) For \( \sigma M_{\mu\tau} = -M_{ee} \) giving \( M_{ee} = i\kappa \tan 2\theta_{12} (c_{13}^2 - 2s_{13}^2) M_{+}/2s_{13}, \)

\[
M_{\nu} = \begin{pmatrix} M_{ee} & M_{e\mu} & M_{e\tau} \\ M_{e\mu} & -i\sqrt{2}\kappa t_{13} M_{e\tau} & -\sigma M_{ee} \\ M_{e\tau} & -\sigma M_{ee} & -i\sqrt{2}\kappa t_{13} M_{e\mu} \end{pmatrix},
\]

(28)

is obtained and \( m_{1,2,3} \) are given by

\[
m_1 e^{-i\phi_1} = -i\kappa \left( t_{12} \tan 2\theta_{12} \frac{1 - 2t_{13}^2}{2t_{13}} + t_{13} \right) M_{+} + M_{ee},
\]

\[
m_2 e^{-i\phi_2} = i\kappa \left( t_{12} \tan 2\theta_{12} \frac{1 - 2t_{13}^2}{2t_{13}} - t_{13} \right) M_{+} + M_{ee},
\]

\[
m_3 e^{-i\phi_3} = - \left( i\kappa \frac{1}{t_{13}} M_{+} + M_{ee} \right),
\]

(29)

where the mass hierarchy will be also the degenerated one realized by \( |M_{ee}| \gg |M_{+}| \).

There are other textures based on other constraints on \( M_{\nu} \).

For the first texture with \( M_{e\tau} = -\sigma M_{e\mu}^* \) and \( M_{\mu\tau} = M_{\mu\mu}^* \), no Majorana CP violation is induced if \( M_{ee} \) is real as indicated by Eq.(22). It is noted in the Appendix A that \( U_{PMNS} \) including redundant phases can be parameterized by three real numbers \( u_{1,2,3} \) and three complex numbers \( w_{1,2,3} \) numbers\textsuperscript{28–34}

\[
U_{\text{Maximal}} = \begin{pmatrix} u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \\ w_1^* & w_2^* & w_3^* \end{pmatrix}.
\]

(30)

To reach \( U_{\text{Maximal}} \), \( M_{ee} \) is constrained to be real and no Majorana CP violation is induced. It is also noted in the Appendix A that the second texture with \( M_{e\tau} = \sigma M_{e\mu}^* \) and \( M_{\mu\tau} = -M_{\mu\mu}^* \) cannot be obtained from any phase transformation of the first texture with \( M_{e\tau} = -\sigma M_{e\mu}^* \) and \( M_{\mu\tau} = M_{\mu\mu}^* \) by adjusting redundant phases present in \( U_{PMNS} \). In this texture, no Majorana CP violation is induced if \( M_{ee} \) is imaginary as indicated by Eq.(24).
4. Flavor Neutrino Masses with Arbitrary Phases

For a general form of $M_\nu$, $M_{\mu\nu,\mu\tau,\tau\tau}$ does not satisfy Eq. (10). In fact, in terms of the redundant phases $\rho$, $\gamma$ and $\tau$ introduced in the Appendix A, yielding $\delta_{CP} = \delta + \rho + \tau$, $M_\nu$ should be replaced by Eq. (A.4), where $M_{ij}$ ($i,j = e, \mu, \tau$) for the PDG convention. Conversely, for a given general form of $M_\nu$, $M_\nu$ compatible with the PDG convention should be

$$M_\nu^{PDG} = \begin{pmatrix} e^{2i\rho} M_{ee} & e^{i(\rho + \gamma)} M_{e\mu} & e^{i(\rho - \gamma - \tau)} M_{e\tau} \\ e^{i(\rho + \gamma)} M_{\mu e} & e^{2i\gamma} M_{\mu\mu} & e^{-i\tau} M_{\mu\tau} \\ e^{i(\rho - \gamma)} M_{\tau e} & e^{-i\tau} M_{\tau\mu} & e^{-2i(\gamma + \tau)} M_{\tau\tau} \end{pmatrix}.$$  \hfill (31)

All constraints on $M_\nu$ including the key relation of Eq. (1) should be replaced by those on $M_\nu^{PDG}$. We can readily factor out $e^{-i\tau}$ as a common factor to have $\delta + \tau/2$, $\rho + \tau/2$ and $\gamma + \tau/2$, respectively, redefined to be $\delta$, $\rho$ and $\gamma$ resulting in $\delta_{CP} = \delta + \rho$.

The flavor neutrino masses $M_{\mu\mu,\mu\tau,\tau\tau}$ turn out to satisfy

$$e^{2i\gamma} M_{\mu\mu} = \left( \frac{1}{c_{13} \tan 2\theta_{12}} - \frac{t_{13} e^{-i\delta_{CP}}}{\sin 2\theta_{23}} \right) M_+ + \left( \frac{e^{-i\delta_{CP}}}{\tan 2\theta_{13}} - \frac{1}{2} t_{13} e^{i\delta_{CP}} \right) M_+$$

$$+ \frac{1 + e^{-2i\delta_{CP}}}{2} e^{2i\rho} M_{ee} - M_{\mu\mu}^{(0)} \cos 2\theta_{23},$$

$$e^{-2i\gamma} M_{\mu\tau} = \left( \frac{1}{c_{13} \tan 2\theta_{12}} + \frac{t_{13} e^{-i\delta_{CP}}}{\sin 2\theta_{23}} \right) M_+ + \left( \frac{e^{-i\delta_{CP}}}{\tan 2\theta_{13}} - \frac{1}{2} t_{13} e^{i\delta_{CP}} \right) M_+$$

$$+ \frac{1 + e^{-2i\delta_{CP}}}{2} e^{2i\rho} M_{ee} + M_{\mu\mu}^{(0)} \cos 2\theta_{23},$$  \hfill (32)

with the same equation of $M_{\mu\tau} = M_{\mu\mu}^{(0)} \sin 2\theta_{23}$, where

$$M_+ = s_{23} e^{i(\rho + \gamma)} M_{\mu\mu} + c_{23} e^{i(\rho - \gamma)} M_{e\tau}, \quad M_- = c_{23} e^{i(\rho + \gamma)} M_{e\mu} - s_{23} e^{i(\rho - \gamma)} M_{e\tau},$$

$$M_{\mu\tau}^{(0)} = - \left( \frac{1}{c_{13} \tan 2\theta_{12}} + \frac{t_{13} e^{-i\delta_{CP}}}{\tan 2\theta_{23}} \right) M_+ + \left( \frac{e^{-i\delta_{CP}}}{\tan 2\theta_{13}} + \frac{1}{2} t_{13} e^{i\delta_{CP}} \right) M_+$$

$$- \frac{1 - e^{-2i\delta_{CP}}}{2} e^{2i\rho} M_{ee}.  \hfill (33)$$

Practically speaking, $\delta$, $\rho$ and $\gamma$ can be calculated from $M_{ij}$ $(i,j = e, \mu, \tau)$ for $M = M_1^1 M_\nu$. The phases $\rho$ and $\delta$ are given by

$$\rho = \arg(X), \quad \delta = - \arg(Y),  \hfill (34)$$

for

$$X = e^{i\gamma} c_{23} M_{e\mu} - e^{-i\gamma} s_{23} M_{e\tau}, \quad Y = e^{i\gamma} s_{23} M_{e\mu} + e^{-i\gamma} c_{23} M_{e\tau},$$  \hfill (35)

where $\gamma$ can be derived from

$$\cos 2\gamma \mathrm{Im} (M_{\mu\tau}) - \sin 2\gamma \mathrm{Re} (M_{\mu\tau}) = t_{13} \sin \delta_{CP} |X|,  \hfill (36)$$

with $|X| = \Delta m_{21}^2 \sin 2\theta_{12}/2$ ($\Delta m_{21}^2 = m_2^2 - m_1^2$). More details can be found in Refs. [55]-[58].
It is observed that the maximal atmospheric neutrino mixing and maximal CP violation are realized by the same constraint on $M_{\tau\tau}$: $M_{\tau\tau} = M_{\mu\mu}^*$ and by the modified one on $M_{e\tau}$:

$$M_{e\tau} = -\sigma e^{-2i\rho}M_{e\mu}^*. \quad (37)$$

The additional constraint that $M_{ee} = \text{real}$ is replaced by $e^{2i\rho}M_{ee} = \text{real}$.

5. Summary

We have demonstrated the usefulness of flavor neutrino masses expressed in terms of $M_{ee, e\mu, e\tau}$. Appropriate constraints on $M_{ee, e\mu, e\tau}$ reveal the necessary mutual dependence among $M_{\mu\mu, \tau\tau, \mu\tau}$ to have the maximal CP violation. For example, we have stressed the significant role of the relation: $(M_{\tau\tau} - M_{\mu\mu})\sin 2\theta_{23}/2 - M_{\mu\tau}\cos 2\theta_{23} = \tan \theta_{13} (M_{e\mu}\cos \theta_{23} - M_{e\tau}\sin \theta_{23})e^{-i\delta_{CP}}$. If we impose a constraint on $M_{e\mu, e\tau}$ such as the constraint of $M_{e\mu} - \sigma M_{e\tau} = \text{real}$ for the maximal atmospheric neutrino mixing signaled by $\cos 2\theta_{23} = 0$, we observe that the maximal CP violation signaled by $\cos \delta_{CP} = 0$ is induced by the constraint of $M_{\tau\tau} - M_{\mu\mu} = \text{imaginary}$. The simplest solution to satisfy both constraints consists of $M_{e\tau} = -\sigma M_{e\mu}^*$ and $M_{\tau\tau} = M_{\mu\mu}^*$, which provide the known texture. If $M_{\nu}$ is not associated with the PDG convention, $M_{e\tau} = -\sigma M_{e\mu}^*$ should be replaced by $M_{e\tau} = -\sigma e^{-2i\rho}M_{e\mu}^*$, where $\rho$ is the redundant Dirac phase associated with the 1-2 rotation. Other constraints on $M_{ij}$ ($i, j = e, \mu, \tau$) lead to new textures and we have shown three such examples. It can also be discussed how $M_{\nu}$ of Eq. (5) is modified by the inclusion of the effect from $\cos 2\theta_{23} \neq 0$ allowing an arbitrary atmospheric mixing angle. For example, at a first glance, we may choose the constraint of $M_{e\mu}\cos \theta_{23} - M_{\mu\tau}\sin \theta_{23} = \text{real}$ leading Eq. (4) for the maximal atmospheric neutrino mixing. This subject will be discussed elsewhere.

Appendix A. $U_{PMNS}$ for the maximal CP violation and the maximal atmospheric neutrino mixing

The general form of $U_{PMNS}$ contains seven phases: three phases of the Dirac type to be denoted by $\delta$, $\rho$ and $\tau$, three phases of the Majorana type to be denoted by $\varphi_{1,2,3}$ and the remaining phase to be denoted by $\gamma$. For $U_{PMNS} = U_{\nu}K$, we
This unitary matrix can be casted into parameterize $U_{\nu}$ and $K$ as follows:

\[
U_{\nu} = \begin{pmatrix}
1 & 0 & 0 \\
0 & e^{i\gamma} & 0 \\
0 & 0 & e^{-i\gamma}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta_{23} & \sin \theta_{23}e^{i\tau} \\
0 & -\sin \theta_{23}e^{-i\tau} & \cos \theta_{23}
\end{pmatrix}
\begin{pmatrix}
\cos \theta_{13} & 0 & \sin \theta_{13}e^{-i\delta} \\
0 & 1 & 0 \\
-\sin \theta_{13}e^{i\delta} & 0 & \cos \theta_{13}
\end{pmatrix}
\]

\[
K = \begin{pmatrix}
e^{i\varphi_{1}/2} & 0 & 0 \\
e^{i\varphi_{2}/2} & 0 & 0 \\
0 & 0 & e^{i\varphi_{3}/2}
\end{pmatrix}.
\]  

\[
M_{\nu} = \begin{pmatrix}
e^{-2i\rho}M_{ee} & e^{-i(\rho+\gamma)}M_{e\mu} & e^{-i(\rho-\gamma-\tau)}M_{e\tau} \\
-e^{-i(\rho+\gamma)}M_{\mu e} & e^{-2i\gamma}M_{\mu\mu} & e^{i\tau}M_{\mu\tau} \\
-e^{-i(\rho-\gamma-\tau)}M_{\tau e} & e^{i\tau}M_{\mu\tau} & e^{2i(\gamma+\tau)}M_{\tau\tau}
\end{pmatrix}.
\]

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\[
U_{PMNS} = \begin{pmatrix}
e^{i\rho} & 0 & 0 \\
0 & e^{i\gamma} & 0 \\
0 & 0 & e^{-i(\gamma+\tau)}
\end{pmatrix}U_{PDG},
\]  

where

\[
\delta_{CP} = \delta + \rho + \tau, \quad \phi_{1} = \varphi_{1} - 2\rho, \quad \phi_{2} = \varphi_{2}, \quad \phi_{3} = \varphi_{3} + 2\tau,
\]

in $U_{PDG}$. Accordingly, from $M_{\nu}$ for $U_{PDG}$ consisting of $M_{ij}$ ($i,j=e,\mu,\tau$), we reach $M_{\nu}$ for $U_{PMNS}$ of Eq. (A.1):

\[
M'_{\nu} = \begin{pmatrix}
e^{-2i\rho}M_{ee} & e^{-i(\rho+\gamma)}M_{e\mu} & -\sigma e^{-i(\rho-\gamma-\tau)}M_{e\tau} \\
e^{-i(\rho+\gamma)}M_{\mu e} & e^{-2i\gamma}M_{\mu\mu} & e^{i\tau}M_{\mu\tau} \\
-\sigma e^{-i(\rho-\gamma-\tau)}M_{\tau e} & e^{i\tau}M_{\mu\tau} & e^{2i(\gamma+\tau)}M_{\tau\tau}
\end{pmatrix}.
\]

For the maximal CP violation and the maximal atmospheric neutrino mixing, Eq. (5) provides

\[
M'_{\nu} = e^{-2i\rho} \begin{pmatrix}
M_{ee} & e^{i(\rho-\gamma)}M_{e\mu} & -\eta \sigma (e^{i(\rho-\gamma)}M_{e\mu})^* \\
-e^{i(\rho-\gamma)}M_{\mu e} & M_{\mu\mu} & \eta M_{\mu\tau} \\
-\sigma e^{i(\rho-\gamma)}M_{\tau e} & \eta M_{\mu\tau} & e^{2i(\gamma+\tau)}M_{\tau\tau}
\end{pmatrix}.
\]

Since $\rho$ and $\tau$ are redundant, we choose $\rho$ and $\tau$ to satisfy $e^{i(2\rho+\tau)} = \eta$ ($\eta = \pm 1$) yielding\[\text{[3],[5]}

\[
M'_{\nu} = e^{-2i\rho} \begin{pmatrix}
M_{ee} & e^{i(\rho-\gamma)}M_{e\mu} & -\eta \sigma (e^{i(\rho-\gamma)}M_{e\mu})^* \\
-e^{i(\rho-\gamma)}M_{\mu e} & M_{\mu\mu} & \eta M_{\mu\tau} \\
-\eta \sigma e^{i(\rho-\gamma)}M_{\tau e} & \eta M_{\mu\tau} & e^{2i(\gamma+\tau)}M_{\tau\tau}
\end{pmatrix}.
\]

The corresponding $U_{PMNS}$ is simply given by

\[
U_{PMNS} = e^{i\rho} \begin{pmatrix}
u_{1} e^{i\phi_{1}/2} & u_{2} e^{i\phi_{2}/2} & -i\kappa u_{3} e^{i\phi_{3}/2} \\
w_{1} e^{i\phi_{1}/2} & u_{2} e^{i\phi_{2}/2} & -i\kappa w_{3} e^{i\phi_{3}/2} \\
-i\sigma w_{1} e^{i\phi_{1}/2} & -i\sigma w_{2} e^{i\phi_{2}/2} & i\kappa w_{3} e^{i\phi_{3}/2}
\end{pmatrix}.
\]
where $u_{1,2,3}$ and $w_{1,2,3}$ are given by

$$u_1 = c_{12}c_{13}, \quad u_2 = s_{12}c_{13}, \quad u_3 = s_{13},$$

$$w_1 = -e^{i(\gamma - \rho)} \frac{s_{12} + i\kappa c_{12}s_{13}}{\sqrt{2}}, \quad w_2 = e^{i(\gamma - \rho)} \frac{c_{12} - i\kappa s_{12}s_{13}}{\sqrt{2}},$$

$$w_3 = e^{i(\gamma - \rho)} \frac{i\kappa c_{13}}{\sqrt{2}}.$$  \hfill (A.8)

The Majorana phases $\phi_{1,2,3}$ are so determined to satisfy Eq. (13). The global phase $\rho$ in Eqs. (A.6) and (A.7) are cancelled each other in $U_{PMNS}^T M'_\nu U_{PMNS}$. To reach $U_{Maximal}$ in Eq. (30), it is further required that Eq. (15) be satisfied; namely, $\phi_1 = \phi_2 = 0$ and $\phi_3 = 2\delta_{CP}(= \pm \pi)$. This choice subsequently requires that $M_{ee} = \text{real}$ (as indicated by Eq. (13)), which is nothing but the additional constraint imposed in Refs. [28, 31]. The resulting $U_{PMNS}$ of Eq. (A.2) turns out to be:

$$U_{PMNS} = e^{i\rho} \begin{pmatrix} u_1 & u_2 & u_3 \\ w_1 & w_2 & \sigma w_3 \\ -\eta w_1^* & -\eta w_2^* & -\eta w_3^* \end{pmatrix}. \hfill (A.9)$$

The obtained $U_{PMNS}$ is literally coincident with $U_{Maximal}$ of Eq. (30) for $\sigma = 1$ and $\eta = -1$. It is thus proved that the texture for $U_{Maximal}$ can be derived from the texture for $U_{PDG}$. No Majorana CP violation is induced because of $\phi_1 = \phi_2 = 0$ and $\phi_3 = \pm \pi$.

One may wonder if the texture of Eq. (23) with $M_{ee} = \sigma M_{\mu\mu}^*$ and $M_{\tau\tau} = -M_{\mu\mu}^*$ is obtained from Eq. (A.5) for the texture with $M_{ee} = -\sigma M_{\mu\mu}^*$ and $M_{\tau\tau} = M_{\mu\mu}^*$ by adjusting the redundant phases. Requiring that $M'_{\mu\mu} = M'^*_{\mu\mu}$ realized by $e^{i(2\rho + \gamma)} = i\eta$, we reach

$$M'_\nu = e^{-i\rho} \begin{pmatrix} M_{ee} & e^{i(\rho - \gamma)} M_{e\mu} - i\eta \sigma (e^{i(\rho - \gamma)} M_{e\mu})^* \\ e^{i(\rho - \gamma)} M_{e\mu} & M_{\mu\mu} & \eta M_{\mu\tau} \\ -i\eta \sigma (e^{i(\rho - \gamma)} M_{e\mu})^* & \eta M_{\mu\tau} & (e^{2i(\rho - \gamma)} M_{\mu\mu})^* \end{pmatrix}, \hfill (A.10)$$

indicating $M'_{e\tau} = -i\eta \sigma M'^*_{e\tau}$, which cannot be equivalent to $M_{e\tau} = \sigma M_{e\mu}^*$. The texture with $M_{ee} = \sigma M_{\mu\mu}^*$ and $M_{\tau\tau} = -M_{\mu\mu}^*$ is not connected to Eq. (A.5).

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