A Generalization of Griffiths’ Theorem on Rational Integrals

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Abstract. We generalize Griffiths’ theorem on the Hodge filtration of the primitive cohomology of a smooth projective hypersurface, using the local Bernstein-Sato polynomials, the $V$-filtration of Kashiwara and Malgrange along the hypersurface and the Brieskorn module of the global defining equation of the hypersurface.

Introduction

Let $Y$ be a complex hypersurface of degree $d$ in $X := P^n$ with $n \geq 2$. Let $f$ be a defining equation of $Y$, which is a homogeneous polynomial of degree $d$, and is assumed to be reduced. Let $R$ be the Jacobian ring of $f$, and $R_j$ be its degree $j$ part. If $Y$ is smooth, Griffiths’ theorem [19] says that

$$R_{qd+d−n−1} = H^{p,q}_{\text{prim}}(Y, \mathbb{C}),$$

where $p + q = n − 1$ and the right-hand side is the $(p,q)$-part of the primitive cohomology, see also [18]. This follows from Griffiths’ theorem saying that the Hodge filtration is given by the pole order filtration in the smooth case, see [19]. The latter was generalized to the normal crossing case by Deligne ([8], [9]), and the comparison between the Hodge filtration and the pole order filtration was further studied in [10] and [30] inspired by a letter of P. Deligne which treated a certain special case, see Remark 4.6 in [30].

Let $F$ be the Hodge filtration on $\mathcal{O}_X(*Y) := \bigcup_{i>0} \mathcal{O}_X(iY)$ as defined in [29]. It induces the Hodge filtration on the cohomology of the complement $U$ of $Y$ (see [9]) by using the induced filtration of the de Rham complex, even if $Y$ is not a divisor with normal crossings. Let $P$ denote the pole order filtration on $\mathcal{O}_X(*Y)$ defined by $P_i = \mathcal{O}_X((i+1)Y)$ if $i \geq 0$ and $P_i = 0$ otherwise. Then we have $F_i \subset P_i$ in general, and $F_i = P_i$ for $i \leq \alpha_{Y,y}−1$ on a neighborhood of $y \in \text{Sing } Y$ if $−\alpha_{Y,y}$ is the maximal root of the Bernstein-Sato polynomial divided by $s + 1$ for a local defining equation of $Y$ at $y$, see [10], [30]. Set $\alpha_Y = \min_{y \in \text{Sing } Y} \{\alpha_{Y,y}\}$. We will denote also by $F$, $P$ the induced filtration on $\omega_X(*Y) = \omega_X \otimes \mathcal{O}_X(*Y)$ (where $\omega_X = \Omega_X^n$). Note that $F$, $P$ define the induced filtrations on the de Rham complex (see (2.1.1)) and on the de Rham cohomology of $U$.

Let $\Omega^*$ denote the de Rham complex of global algebraic differential forms on $\mathbb{A}^{n+1}$. Let $\mathcal{H}_f$ denote the free part of the algebraic Brieskorn module $\mathcal{H}_f := \Omega^{n+1}/df \wedge d\Omega^{n−1}$.
see [4]. It is known that $\mathcal{H}_f$ is a free graded $\mathbb{C}[t]$-module of finite rank according to an algebraic version of [20] (see also [1]), where the action of $t$ is given by the multiplication by $f$ and the degrees of $x_i$ and $dx_i$ are 1. (Here the $x_i$ are the coordinates of $\mathbb{A}^{n+1}$.) The action of the logarithmic Gauss-Manin connection $\nabla_{t\partial/\partial t}$ on the degree $k$ part $\mathcal{H}_{f,k}$ is the multiplication by $\frac{k}{d} - 1$, and we have isomorphisms $t: \mathcal{H}_{f,k} \cong \mathcal{H}_{f,k+d}$ for $k \geq nd$, see Remark (1.7) (iii). Let $\eta_0 = \frac{1}{d} \sum_{i=0}^{n} (-1)^i x_i dx_0 \wedge \cdots \wedge dx_i \cdots \wedge dx_n$. For $q \in \mathbb{Z}$, there are canonical morphisms

$$H^n(U, \mathbb{C}) \leftarrow \Gamma(X, P_q(\omega_X(*Y))) \rightarrow \mathcal{H}_{f,(q+1)d},$$

where the first morphism is defined by using the de Rham cohomology and the second morphism sends $f^{-q-1} g \eta_0$ to $g dx_0 \wedge \cdots \wedge dx_n$ for $g \in \mathbb{C}[x]_{(q+1)d-n-1}$ and is surjective. Note that $P_q(\omega_X(*Y))$ is related to $P^{n-q} := P_{q-n}$ on $H^n(U, \mathbb{C})$ by definition of the filtration on the de Rham complex, see (2.1.1).

**Theorem 1.** (i) For any $q \in \mathbb{Z}$, the above morphisms in (0.1) induce an isomorphism

$$P^{n-q}H^n(U, \mathbb{C}) = \mathcal{H}_{f,(q+1)d},$$

such that the inclusion $P^{n-q+1}H^n(U, \mathbb{C}) \hookrightarrow P^{n-q}H^n(U, \mathbb{C})$ is identified with the morphism $t: \mathcal{H}_{f,qd} \hookrightarrow \mathcal{H}_{f,(q+1)d}$ and we have $P^{n-q}H^n(U, \mathbb{C}) = H^n(U, \mathbb{C})$ for $q \geq n-1$.

(ii) The Hodge filtration $F^{n-q}H^n(U, \mathbb{C})$ is identified with the image of

$$\Gamma(X, F_q(\omega_X(*Y))) \subset \Gamma(X, P_q(\omega_X(*Y)))$$

by the last morphism of (0.1) for any $q \in \mathbb{Z}$. In particular,

$$F^{n-q}H^n(U, \mathbb{C}) = \mathcal{H}_{f,(q+1)d} \quad \text{for} \quad q \leq \alpha_Y - 1.$$

If $Y$ is a $\mathbb{Q}$-homology manifold, then $F^{n-q}H^n(U, \mathbb{C}) = F^{n-1-q}H^{n-1}(Y, \mathbb{C})$, and $H^j(Y, \mathbb{C})$ is naturally isomorphic to the intersection cohomology $IH^j(Y, \mathbb{C})$ (see [2], [15]). If furthermore $Y$ is smooth, then $\alpha_Y = +\infty$, and Theorem 1 is well known in the theory of hypersurface isolated singularities, see for instance [33], [34], [35], [39]. The essential part of the proof of Theorem 1 is showing the higher acyclicity of the Hodge filtration on each component of the de Rham complex, see (2.2). Note that it is not clear whether $F^{n-q} \neq P^{n-q}$ globally on $H^n(U, \mathbb{C})$ even if $F_q \neq P_q$ locally on $O_X(*Y)$, see (2.5). Theorem 1 gives a refinement of the above mentioned theorem of Griffiths by (1.6). We can also generalize a description of the Kodaira-Spencer map which is due to Griffiths in the smooth case, see (4.5).

If $Y$ is not smooth, $H^j(Y, \mathbb{C})$ is identified, up to the non primitive part, with the local cohomology $H^j_Y(X, \mathbb{C})$ or the homology $H^j_{2n-j-1}(Y, \mathbb{C})$. For $q \geq \lfloor \alpha_Y \rfloor$, it is not easy to describe $F^{n-q}H^n(U, \mathbb{C})$ in general. Consider first the case $q = 0$. Let $V$ denote the filtration on $O_X$ induced by the $V$-filtration of Kashiwara [22] and Malgrange [27] on $\mathcal{B}_f = O_X \otimes_{\mathbb{C}} \mathbb{C}[\partial_t]$ along $Y$. This coincides essentially with the filtration by the multiplier ideals [24], see [5]. It induces a filtration $V$ on $\omega_X(Y) = \omega_X \otimes_{O_X} O_X(Y)$ such that
of degree \(\alpha\) with singularities. Defined by a function \(F\) linear differential operators with the filtration on the singularities of \(\omega\) and \(\omega_Y = \omega_X(Y)/\omega_X\). Let \(\tilde{V}^>1O_X\) be the inverse image of \(V^>0B_f\) by the composition of the inclusion \(O_X \to B_f\) with \(\partial_i : B_f \to B_f\). This gives \(\tilde{V}^>0(\omega_X(Y)) := \tilde{V}^>1\omega_X \otimes O_X(Y)\) and \(\tilde{V}^>0\omega_Y := \tilde{V}^>0(\omega_X(Y))/\omega_X\) in the same way as above.

**Theorem 2.** There are canonical isomorphisms

\[
H^0(X, V^0(\omega_X(Y))) = H^0(Y, V^0\omega_Y) = F^nH^n(U, C),
\]

\[
H^0(X, \tilde{V}^>0(\omega_X(Y))) = H^0(Y, \tilde{V}^>0\omega_Y) = F^{n-1}H^{n-1}(Y, C),
\]

and the last term is canonically isomorphic to \(F^{n-1}H^{n-1}(Y, C)\). Furthermore, \(\tilde{V}^>1O_X\) coincides with the adjoint ideal ([14], [28], [38]), and \(\tilde{V}^>0\omega_Y\) is naturally isomorphic to the direct image of the dualizing sheaf \(\omega_X^\vee\) of a resolution of singularities \(\tilde{Y}\) of \(Y\). In particular, \(F^nH^n(U, C) = F^{n-1}H^{n-1}(Y, C)\) in case \(Y\) has only rational singularities.

This follows from [30]. Note that \(V^0\omega_Y = \omega_Y\) if and only if \(\alpha_Y \geq 1\), and \(\tilde{V}^>0\omega_Y = \omega_Y\) if and only if \(\alpha_Y > 1\) (i.e. \(Y\) has rational singularities). See also [25] for the case of isolated singularities.

For \(q \geq 1\), it is not easy to describe \(F_q(O_X(*Y))\) unless we impose some strong condition on the singularities of \(Y\). We now consider the case where \(Y\) has only isolated singularities which are locally semi-weighted-homogeneous. This means that \(Y\) is locally defined by a function \(h = \sum_{\alpha \geq 1} h_\alpha\), where the \(h_\alpha\) are weighted homogeneous polynomials of degree \(\alpha\) for some appropriate local coordinates \(y_1, \ldots, y_n\) with positive weights \(w_{y,1}, \ldots, w_{y,n}\) around \(y \in \text{Sing}Y\), and \(h^{-1}(0)\) (and hence \(Y\)) has an isolated singularity at \(y\). In this case, it is well known that \(\alpha_{Y,y} = \sum_i w_{y,i}\), see (3.5) (i). Let \(O_{X,y}^{\geq \beta}\) be the ideal of \(O_{X,y}\) generated by \(\prod_i y_i^{\nu_i}\) with \(\sum_i w_{y,i} \nu_i \geq \beta - \alpha_{Y,y}\). Let \(D_X\) be the sheaf of linear differential operators with the filtration \(F\) by the order of differential operators. Put \(k_0 = \lceil n - \alpha_{Y,y} \rceil - 1\). Let \(\mathcal{J}(q)\) be the ideal sheaf of \(O_X\) such that for \(y \in \text{Sing}Y\)

\[
\mathcal{J}_y(q)h^{-q-1} = \sum_{k=0}^{k_0} F_{q-k}D_{X,y}(O_{X,y}^{\geq k+1}h^{-k-1}),
\]

and \(\mathcal{J}(q)|_{X \setminus \text{Sing}Y} = O_{X \setminus \text{Sing}Y}\). For example, \(\mathcal{J}(0) = O_X^{\geq 1}\) if \(q = 0\). Let \(J(q)\) be the largest ideal of \(O (= O_{A^{n+1}})\) whose restriction to \(A^{n+1} \setminus \{0\}\) coincides with the pull-back of \(\mathcal{J}(q)\) by the projection \(A^{n+1} \setminus \{0\} \to \mathbb{P}^n\). Let \(\mathcal{J}(q)_f\) be the graded submodule of \(\mathcal{J}(q)\) defined by the image of \(J(q)dx_0 \wedge \cdots \wedge dx_n\). Then we have by [32] the following

**Theorem 3.** With the above assumption, there are canonical isomorphisms for \(q \in \mathbb{N}\)

\[
F_q(O_{X,y}(*Y)) = \mathcal{J}_y(q)h^{-q-1},
\]

\[
F^{n-q}H^n(U, C) = \mathcal{J}(q)_{f,(q+1)d}.
\]

For example, if \(q = 1\), \(w_{y,i} = 1/d_y\) for any \(i\), and \(\alpha_{Y,y} = n/d_y > 1\) for any \(y \in \text{Sing}Y\), then \(J(1) = O_X^{\geq 2}\). Note that if \(w_{y,i} = a_{y,i}^{-1}\) and the \(a_{y,i}\) are integers which are mutually
prime for any $y \in \text{Sing } Y$, then $Y$ is a $\mathbb{Q}$-homology manifold and the cohomology of $Y$ has good properties as remarked after Theorem 1. We are informed that a formula similar to Theorem 3 is obtained by L. Wotzlaw if $Y$ has only ordinary double points as singularities and $n = 3$. See [36] for a completely different approach to a similar problem.

In Sect. 1, we study meromorphic differential forms on projective spaces for the proof of Theorem 1. In Sect. 2, we prove the higher acyclicity of the Hodge filtration on each component of the de Rham complex, and complete the proof of Theorem 1. In Sect. 3, we recall some facts from the theory of $V$-filtration, and prove Theorems 2 and 3. In Sect. 4, we generalize a description of the Kodaira-Spencer map.

1. Rational differential forms on projective spaces

1.1. Let $f$ be a reduced homogeneous polynomial of degree $d$, and put $X = \mathbb{P}^n$, $Y = f^{-1}(0) \subset X$, and $U = X \setminus Y$. Let $x_0, \ldots, x_n$ be the coordinates of $\mathbb{A}^{n+1}$, and set

$$
\xi = \frac{1}{d} \sum_i x_i \partial_i,
$$

so that $\xi f = f$, where $\partial_i = \partial/\partial x_i$. Let $\iota_\xi$ and $L_\xi$ denote respectively the interior product and the Lie derivation. Then

$$(1.1.1) \ i_\xi(\frac{df}{f}) = 1, \quad \iota_\xi \circ \iota_\xi = 0, \quad \iota_\xi \circ d + d \circ \iota_\xi = L_\xi.
$$

Let $\Omega^*$ denote the de Rham complex of global differential forms on $\mathbb{A}^{n+1}$. This is isomorphic to the Koszul complex for the action of the vector fields $\partial_i$ on the polynomial ring $\mathbb{C}[x_0, \ldots, x_n]$. Let

$$
\mathcal{H}_f = \Omega^{n+1}/df \wedge d\Omega^{n-1},
$$

as in the introduction. It has a structure a graded module $\mathcal{H}_f = \bigoplus_{k \in \mathbb{Z}} \mathcal{H}_{f,k}$ where the degree of $x_i$ and $dx_i$ are 1. Let $\Omega^*[f^{-1}]_k$ denote the degree $k$ part of the localization $\Omega^*[f^{-1}]$ of $\Omega^*$. Then the restriction of $L_\xi$ to $\mathcal{H}_{f,k}$ and $\Omega^*[f^{-1}]_k$ is the multiplication by $k/d$. Let $\Omega^*[f^{-1}]_0(\xi)$ be the subcomplex of $\Omega^*[f^{-1}]_0$ defined by

$$
\Omega^j[f^{-1}]_0(\xi) = \text{Im}(\iota_\xi : \Omega^{j+1}[f^{-1}]_0 \to \Omega^j[f^{-1}]_0).
$$

1.2. Lemma. There is a canonical isomorphism of complexes

$$(1.2.1) \ \Gamma(U, \Omega_U) = \Omega^*[f^{-1}]_0(\xi).
$$

Proof. This follows by using the blow-up of $\mathbb{P}^{n+1}$ along the origin of $\mathbb{A}^{n+1} (\subset \mathbb{P}^{n+1})$ together with the pull-back by the projection to $\mathbb{P}^n = \mathbb{P}^{n+1} \setminus \mathbb{A}^{n+1}$. See also Proposition 6.1.16 in [12].
1.3. Lemma. The canonical morphism

\[ \frac{df}{f} \wedge : \Omega^*[f^{-1}]_0 \rightarrow \frac{df}{f} \wedge \Omega^*[f^{-1}]_0 \]

is an isomorphism of complexes, and its inverse is given by

\[ \iota_\xi : \frac{df}{f} \wedge \Omega^*[f^{-1}]_0 \xrightarrow{\sim} \Omega^*[f^{-1}]_0. \]

**Proof.** By (1.1.1), we have \( \iota_\xi \circ (\frac{df}{f} \wedge) = \text{id} \) on \( \Omega^*[f^{-1}]_0 \), and (1.3.1) is surjective by definition.

1.4. Lemma. The canonical morphism

\[ H^j(\frac{df}{f} \wedge \Omega^*[f^{-1}]_0) \rightarrow H^j(\frac{df}{f} \wedge \Omega^*[f^{-1}]_0) \]

is injective for any \( j \), and is bijective for \( j = n \).

**Proof.** Take \( f^{-i} \eta \in \Omega^{j-2}[f^{-1}]_0 \). Here we may assume \( i > 0 \), replacing \( \eta \) with \( f^k \eta \) and \( i \) with \( i + k \). Since the degree of \( \eta \) is \( di \), we have \( i\eta = L_\xi \eta = i_\xi(d\eta) + d(i_\xi \eta) \), and

\[ \frac{df}{f} \wedge d(i^{-i} \eta) = \frac{df}{f} \wedge d(i_\xi(f^{-i}d\eta)). \]

So the injectivity follows. The surjectivity follows from

\[ (\frac{df}{f} \wedge) \circ \iota_\xi = \text{id} \] on \( \Omega^{n+1}[f^{-1}] \).

1.5. Proposition. There are canonical isomorphisms

\[ H^n(\Gamma(U, \Omega^*_U)) = H^n(\frac{df}{f} \wedge \Omega^*[f^{-1}]_0) = \lim_{k \rightarrow} \mathcal{H}_{f, kd}, \]

where the first isomorphism is induced by (1.2.1) and (1.3.1), and the limit is taken over \( k \in \mathbb{N} \) with transition morphisms \( \mathcal{H}_{f, kd} \rightarrow \mathcal{H}_{f,(k+1)d} \) given by the multiplication by \( f \).

**Proof.** The first isomorphism follows from Lemmas (1.2) and (1.3). The second isomorphism is defined by assigning \( \eta \) to \( \eta/f^k \in \Omega[f^{-1}]_0 \), because (1.4.2) implies

\[ \frac{df}{f} \wedge \Omega^n[f^{-1}]_0 = \Omega^{n+1}[f^{-1}]_0. \]

Since \( \eta/f^k = f \eta/f^{k+1} \), the isomorphism then follows from the definition of \( \mathcal{H}_f \) together with Lemma (1.4).

1.6. Proposition. We have a canonical isomorphism

\[ \mathcal{H}_f/f \mathcal{H}_f = \Omega^{n+1}/df \wedge \Omega^n (= \mathcal{O}/(\partial f) =: R), \]

where \((\partial f)\) is the Jacobian ideal generated by the \( f_i := \partial f/\partial x_i \) in \( \mathcal{O} = \mathbb{C}[x_0, \ldots, x_n] \).
Proof. The assertion is equivalent to
\[ df \wedge d\Omega^{n-1} + f\Omega^{n+1} = df \wedge \Omega^n = (\partial f)\Omega^{n+1}. \]
Since \( f = \sum_i \frac{1}{d} x_i \partial_i f / \partial x_i \), the inclusion \( \subset \) is clear. For the converse we use the Gauss-Manin connection \( \nabla_{\partial_t} \) on \( \mathcal{H}_f \). (We do not know a simple proof without using essentially the Gauss-Manin connection even in the isolated singularity case.) We have by definition
\[ \nabla_{\partial_t} \omega = d\eta \quad \text{with} \quad df \wedge \eta = \omega, \]
where \( \partial_t = \partial / \partial t \). Note that the inverse of the Gauss-Manin connection \( \nabla_{\partial_t}^{-1} \) is well-defined as a \( \mathbb{C} \)-linear endomorphism of \( \mathcal{H}_f \) by the de Rham lemma. It is well known that
\[ (1.6.2) \quad \nabla_{\partial_t}(f\omega) = (k/d)\omega \quad \text{for} \quad \omega \in \mathcal{H}_{f,k}. \]
Indeed, this follows from \( d(\xi \omega) = L_{\xi \omega} \) (see (1.1.1)) together with \( \frac{df}{f} \wedge (\xi \omega) = \omega \) by setting \( \eta = \xi \omega \).

We have to show that \( \omega \in f\mathcal{H}_f \) if \( \omega \) is represented by an element of \( df \wedge \Omega^n \). Here we may assume \( \omega \in \mathcal{H}_{f,k} \) (because \( f \) is homogeneous), and we have \( k > d \) by the definition of the grading on \( \Omega^{n+1} \) (because \( \deg f_i = d-1 \) and \( \deg dx_0 \wedge \cdots \wedge dx_n = n+1 > 1 \)). So the assertion follows from (1.6.2) together with \( \nabla_{\partial_t} t = t\nabla_{\partial_t} + id \).

1.7. Remarks. (i) Let \( \text{tor} \mathcal{H}_f \) denote the torsion part of \( \mathcal{H}_f \) as a \( \mathbb{C}[t] \)-module so that we have a short exact sequence
\[ 0 \to \text{tor} \mathcal{H}_f \to \mathcal{H}_f \to \mathcal{H}_f / \text{tor} \mathcal{H}_f \to 0. \]
It is well known after [20] that \( \mathcal{H}_f \) is finitely generated over \( \mathbb{C}[t] \) and \( \text{tor} \mathcal{H}_f \) is killed by a sufficiently high power of \( f \). (Indeed, this is easily proved by using a resolution of singularities, see e.g. [1], 2.3.) Then, by Proposition (1.6) together with the snake lemma applied to the multiplication by \( f \) on the above exact sequence, we get
\[ \text{tor} \mathcal{H}_{f,k} / f \text{tor} \mathcal{H}_{f,k-d} = \mathcal{H}_{f,k} / f\mathcal{H}_{f,k-d} = R_{k-n-1} \quad \text{for} \quad k \gg 0. \]
In particular, \( \text{tor} \mathcal{H}_f \) is not finitely generated over \( \mathbb{C}[t] \) unless \( Y \) is smooth. In the case where \( Y \) has only isolated singularities, then the dimension of \( R_{k-n-1} \) is closely related to the Tjurina numbers, see [7].

(ii) In [1] the Brieskorn module is defined by the cohomology \( H^j \mathcal{A}^*_f \) of the complex \( (\mathcal{A}^*_f, d) \) where \( \mathcal{A}^*_f = \text{Ker}(df \wedge : \Omega^j \to \Omega^{j+1}) \). For \( j = n+1 \), we have a canonical surjection
\[ H^{n+1} \mathcal{A}^*_f \to \mathcal{H}_f = \Omega^{n+1} / df \wedge d\Omega^{n-1}, \]
and its kernel is \( f \)-torsion by the acyclicity of \( (\Omega^*[f^{-1}], df \wedge) \). So the definition in [1] is compatible with the one in this paper as long as we take the free part. (However, it is not clear whether Proposition (1.6) holds with \( \mathcal{H}_f \) replaced by \( H^{n+1} \mathcal{A}^*_f \).)
(iii) For $k \geq nd$, we have isomorphisms $t: \overline{H}_{f,k} \sim \pi \overline{H}_{f,k+d}$. This follows for example from Remark (3.5) (ii) below because (3.5.1) implies the bijectivity of $t: \text{Gr}_V^n \overline{H}_f \to \text{Gr}_V^{n+1} \overline{H}_f$ for $\alpha > \beta_f - 1$ and we have $\beta_f - 1 < n$. We can also show the isomorphism by proving Theorem 1 with $q \geq n$ in the first assertion replaced by $q \gg n$, and then using the fact that the algebraic de Rham cohomology of $U$ is generated by meromorphic differential forms having poles of order $\leq n$ along $Y$, which follows from the relation between the Hodge and the pole order filtrations, see [10].

1.8. Milnor cohomology. With the notation of (1.1), let $S^i = \mathcal{O}_X(-i)$ and $E = \text{Spec}_X(\bigoplus_{i \in \mathbb{N}} S^i)$. Then $E$ is a line bundle over $X$, and the sheaf of local sections $\mathcal{O}_X(E)$ is naturally identified with $\mathcal{O}_X(1)$. Let $f$ be as in (1.1). Since it is identified with a section of $\mathcal{O}_X(d)$, it induces a shifted graded morphism $\rho_f: S^i \to S^{i-d}$ for $i \geq d$, which is compatible with the action of $u \in S^j$ defined by the multiplication $S^j \otimes S^i \to S^{i+j}$. So it determines an ideal $\mathcal{J}$ of $S^*: = \bigoplus_{i \in \mathbb{N}} S^i$ which is locally generated by $u - \rho_f(u)$ for local nonzero sections $u$ of $\mathcal{O}_X(-d)$. Let $\mathcal{S} = S^*/\mathcal{J}$, and $Z = \text{Spec}_X \mathcal{S}$ with the canonical projection $\pi: Z \to X$. Then $Z$ is a divisor on $E$, and we have as $\mathcal{O}_X$-modules

$$\pi_* \mathcal{O}_Z = \mathcal{S} = \bigoplus_{0 \leq i < d} S^i. \tag{1.8.1}$$

Let $F = f^{-1}(1) \subset \mathbb{A}^{n+1}$, and $\overline{F}$ be the closure of $F$ in the compactification $\mathbb{P}^{n+1}$ of $\mathbb{A}^{n+1}$. Then $Z$ can be identified with $\overline{F}$. Indeed, we have a projection $\pi': \overline{F} \to X$ induced by the projection $\mathbb{P}^{n+1} \setminus \{0\} \to X = \mathbb{P}^n$ (where $0 \in \mathbb{A}^{n+1} \subset \mathbb{P}^{n+1}$), and $\mathcal{F} = \pi'^{-1}(U)$. Furthermore, $\mathbb{A}^{n+1} \setminus \{0\}$ is identified with the total space of the $\mathbb{C}^*$-bundle associated to $\mathcal{O}_X(-1)$, and $\mathbb{P}^{n+1} \setminus \{0\}$ is identified with the line bundle corresponding to $\mathcal{O}_X(1)$ by exchanging the zero section and the $\infty$-section (using the involution of $\mathbb{C}^*$ sending $\lambda$ to $\lambda^{-1}$). So we get a canonical isomorphism

$$\overline{(F, F)} = (Z, \pi^{-1}(U)) \text{ over } U. \tag{1.8.2}$$

Let $G = \mathbb{Z}/d\mathbb{Z}$ be the covering transformation group of $F \to U$ and $Z \to X$ such that $F/G = U$ and $Z/G = X$. It has a generator $g$ which acts on $F$ by

$$g: (x_0, \ldots, x_n) \mapsto (\zeta x_0, \ldots, \zeta x_n),$$

where $\zeta = \exp(2\pi i/d)$. This coincides with the geometric Milnor monodromy. Furthermore, the subsheaf $S^i$ of $\pi_* \mathcal{O}_Z$ for $0 \leq i < d$ in (1.8.1) coincides with the eigenspace corresponding to the eigenvalue $\zeta^i$ for the action of $g^*$ because the above involution of $\mathbb{C}^*$ is used in the isomorphism (1.8.2). In particular, $S^i|_U$ is stable by the Gauss-Manin connection.

Using these we can generalize the first assertion of Theorem 1 to the Milnor cohomology. Indeed, Proposition (1.5) is generalized to the following assertion for $0 \leq i < d$:

$$H^n(\Gamma(U, (S^i|_U) \otimes \Omega_X^*)) = H^n(df^{-1} \wedge \Omega^*[f^{-1}]_{-i}) = \lim_k \mathcal{H}_{f,kd-i}. \tag{1.8.3}$$

For $0 \leq i < d$, we define the pole order filtration on $S_i \otimes \mathcal{O}_X(*Y)$ by

$$P_j(S_i \otimes \mathcal{O}_X(*Y)) = S_i \otimes \mathcal{O}_X((j+1)Y) \text{ if } j \geq 0,$$
and it is zero otherwise. We have the Hodge filtration $F$ on $\mathcal{S}_i \otimes \mathcal{O}_X(*Y)$ because (1.8.1) is the decomposition by the action of $g$. Furthermore, $F_j \subset P_j$, and they coincide at the smooth point of $D$.

For $q \geq n+1$, there are canonical isomorphisms $H^n(F, C)_{e(i/d)} = \overline{H}_{f, qd-i}$ compatible with $t : \overline{H}_{f, qd-i} \sim \overline{H}_{f, (q+1)d-i}$, where $H^n(F, C)_{e(i/d)}$ is the $e(i/d)$-eigenspace for the action of the monodromy, where $e(i/d) = \exp(2\pi \sqrt{-1}i/d)$. For $q \in \mathbb{Z}$, we have the surjective morphisms

\begin{equation}
\Gamma(X, P_q(S_i \otimes \omega_X(*Y))) \rightarrow \overline{H}_{f, (q+1)d-i},
\end{equation}

and, for $q \leq n$, $P^{n-q}H^n(F, C)_{e(i/d)}$ coincides with the image of

$$t^{n-q} : \overline{H}_{f, (q+1)d-i} \hookrightarrow \overline{H}_{f, d(n+1) - i} = H^n(F, C)_{e(i/d)}.$$

However, the remaining assertion of Theorem 1 cannot be generalized, because we have to use the shifted $b$-function $b_f(s + i/d)$ corresponding to $f^{i/d}f^s = f^{s+i/d}$, and consider the minimal root of $b_f(-s + i/d)$ instead of $b_f(-s + i/d)/(-s + i/d + 1)$.

1.9. Remark. Let $\omega_0 = dx_0 \wedge \cdots \wedge dx_n$, and consider the following Briançon-Skoda type property:

There is a positive integer $k$ such that $f^k\omega_0 \in df \wedge d\Omega^{n-1}$.

This property is clearly equivalent to $[\omega_0] = 0$ in $\overline{H}_{f, n+1}$. By the above discussion, we get the following corollary:

Assume $e(-(n+1)/d)$ is not an eigenvalue of the monodromy acting on $H^n(F, C)$. Then $f$ satisfies the above Briançon-Skoda type property.

The above hypothesis is satisfied, for example, if $f = x^3 + y^2z$, the equation of a cuspidal cubic plane curve, or $f = x^2z + y^3 + xyz$, the equation of a cubic surface such that $H^3(F) = 0$, see for details [11], Example 4.3. Higher dimensional examples can be constructed easily using the hypersurfaces introduced in [12], p. 148, Proposition (2.24).

2. Hodge and pole order filtrations

2.1 With the notation of (1.1), let $\mathcal{O}_X(*Y)$ be the localization of $\mathcal{O}_X$ by local defining equations of $Y$. Let $P$ and $F$ be respectively the pole order filtration and the Hodge filtration on the left $D_X$-module $\mathcal{O}_X(*Y)$ (see [10], [29]) as in the introduction (e.g. $P_i = \mathcal{O}_X((i+1)Y)$ if $i \geq 0$ and $P_i = 0$ otherwise). They induce the filtrations $F$ and $P$ on the de Rham complex $\text{DR}_X(\mathcal{O}_X(*Y)) = \Omega_X^*(*)Y$ by

\begin{equation}
F_i(\Omega^n_X(*Y)) = \Omega^n_X \otimes F_{i+j}(\mathcal{O}_X(*Y)),
\end{equation}

and similarly for $P$. Note that the filtrations $F, P$ on $\Omega^n_X$ are different from those on $\omega_X$ defined in the introduction (i.e. there is a shift by $n$).
If $i > 0$, it follows from Bott’s vanishing theorem that

\begin{equation}
H^k(X, \Omega_X^j(iY)) = 0 \quad \text{for} \quad k > 0,
\end{equation}

This implies the $\Gamma$-acyclicity of the components of $P_i\text{DR}_X(\mathcal{O}_X(*Y))$, i.e.

\begin{equation}
H^k(X, P_i(\Omega_X^j(*Y))) = 0 \quad \text{for} \quad k > 0.
\end{equation}

### 2.2. Proposition.

$H^k(X, F_i(\Omega_X^j(*Y)) \otimes_{\mathcal{O}_X} \mathcal{O}_X(r)) = 0$ for $k > 0, r \geq 0$ and $i, j \in \mathbb{Z}$.

**Proof.** We proceed by increasing induction on $n = \dim X \geq 0$ and increasing induction on $i \geq -n$. The assertion is trivial if $n = \dim X = 0$. Since $F_{-1}\mathcal{O}_X(*Y) = 0$, we have

$$F_{-n}(\Omega_X^*(*Y)) = \Omega_X^n \otimes F_0(\mathcal{O}_X(*Y))[-n].$$

Combining this with the vanishing of $H^j(U, C)$ for $j > n$, the assertion for $i = -n$ and $r = 0$ follows from the strictness of the Hodge filtration on the direct image of $(\mathcal{O}_X(*Y), F)$ by $X \to pt$. Indeed, the latter means the injectivity of

\begin{equation}
H^j(X, F_p(\Omega_X^*(*Y))) \to H^j(X, \Omega_X^*(*Y)) = H^j(U, C),
\end{equation}

(even if $Y$ is not a divisor with normal crossings), because the direct image by $X \to pt$ is defined by using the de Rham complex.

Let $X_0$ be a general hyperplane of $X = \mathbb{P}^n$, and set $Y_0 = Y \cap X_0$. Then $X_0$ is non characteristic to the $\mathcal{D}_X$-module $\mathcal{O}_X(*Y)$, and the vanishing cycle $\mathcal{D}$-module $\varphi_g \mathcal{O}_X(*Y)$ ($= \bigoplus_{0 \leq \alpha < 1} \text{Gr}_V^\alpha \mathcal{O}_X(*Y)$) vanishes, where $g$ is a local equation of $X_0 \subset X$. This implies that the filtration $V$ on $\mathcal{O}_X(*Y)$ along $X_0$ is given by the $g$-adic filtration (because $\text{Gr}_V^1 \mathcal{O}_X(*Y) = 0$ unless $\alpha$ is a positive integer). So the restriction of the filtered $(\mathcal{O}_X(*Y), F)$ by the inclusion $i_0 : X_0 \to X$ is given by the tensor product with $\mathcal{O}_{X_0}$ over $\mathcal{O}_X$ (because this gives $\text{Gr}_V^1$). Then, by the uniqueness of the direct image of a mixed Hodge module by the affine open morphism $X_0 \setminus Y_0 \to X_0$ (see [29], 2.11), we get

$$F_i(\mathcal{O}_{X_0}(*Y_0)) = F_i(\mathcal{O}_X(*Y)) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_0}.$$ 

Since $\Omega^n_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-n - 1)$, we get furthermore a short exact sequence

$$0 \to F_i(\Omega^n_X(*Y)) \to F_i(\Omega^n_X(*Y))(r + 1) \to F_i(\Omega^n_{X_0}(*Y_0))(r) \to 0,$$

where $M(r)$ denotes $M \otimes_{\mathcal{O}_X} \mathcal{O}_X(r)$ for an $\mathcal{O}_X$-module $M$. Using the inductive hypothesis for $X_0$, this implies

\begin{equation}
H^k(X, F_i(\Omega^n_X(*Y))(r + 1)) = 0 \quad \text{for any} \quad k > 0,
\end{equation}

if $H^k(X, F_i(\Omega^n_X(*Y))(r)) = 0$ for any $k > 0$.

For $n \geq 1$, there is a short exact sequence

$$0 \to \mathcal{O}_X \to \bigoplus \mathcal{O}_X(1) \to \Theta_X \to 0,$$
where \( \Theta_X \) is the sheaf of vector fields. This induces a short exact sequence

\[
0 \to \wedge^{i-1} \Theta_X \to \wedge^{i}(\bigoplus \mathcal{O}_X(1)) \to \wedge^i \Theta_X \to 0.
\]

Since \( \Omega^j_X = \Omega^n_X \otimes \wedge^{n-j} \Theta_X \), it implies by decreasing induction on \( j < n \)

\[
(2.2.3) \quad H^k(X, F_{i+n-j}(\Omega^j_X(\ast Y))(r)) = 0 \quad \text{for any} \quad k > 0, r \geq 0,
\]

if \( H^k(X, F_i(\Omega^j_X(\ast Y))(r)) = 0 \) for any \( k > 0, r \geq 0 \).

Here the hypothesis is reduced to the case \( r = 0 \) by (2.2.2).

Consider now the spectral sequence

\[
(2.2.4) \quad E_1^{p,q} = H^q(X, F_i(\Omega^p_X(\ast Y))) \Rightarrow H^{p+q}(X, F_i(\Omega^\ast_X(\ast Y))).
\]

By inductive hypothesis for \( i \) together with (2.2.2) and (2.2.3), we have \( E_1^{p,q} = 0 \) for \( p < n, q > 0 \). On the other hand, \( H^k(X, F_i(\Omega^\ast_X(\ast Y))) = 0 \) for \( k > n \) by the strictness of the Hodge filtration. So \( E_1^{p,q} = H^q(X, F_i(\Omega^\ast_X(\ast Y))) = 0 \) for \( q > 0 \), and we can proceed by increasing induction on \( i \) using (2.2.2) and (2.2.3) (with \( n \) fixed). This completes the proof of Proposition (2.2).

2.3. Corollary. We have canonical isomorphisms for \( q \in \mathbb{Z} \)

\[
F^{n-q}H^n(U, \mathbb{C}) = \text{Im}(\Gamma(X, F_q(\Omega^r_X(\ast Y))) \to H^n(X, \Omega^r_X(\ast Y))).
\]

Proof. By the definition of the direct image of mixed Hodge modules [29], we have a canonical isomorphism

\[
F^{n-q}H^n(U, \mathbb{C}) = \text{Im}(H^n(X, F_{q-n}(\Omega^r_X(\ast Y))) \to H^n(X, \Omega^r_X(\ast Y))).
\]

So the assertion follows from Proposition (2.2).

2.4. Proof of Theorem 1. The assertion (i) follows from (1.5). We have \( F_q \mathcal{O}_X = P_q \mathcal{O}_X \) for \( q \leq \alpha_{Y,y} - 1 \) on a neighborhood of \( y \in \text{Sing} Y \) as in the introduction. Since \( \mathcal{H}_f \) is the quotient of \( \mathcal{H}_f \) by the \( f \)-torsion part, we see that \( \mathcal{H}_{f,j} \) is isomorphic to the image of \( \mathcal{H}_{f,j} \) in the inductive limit of \( \mathcal{H}_{f,j+kd} \) over \( k \in \mathbb{N} \) where the transition morphisms are given by the multiplication by \( f \). So the assertion (ii) follows from (2.3).

2.5. Remark. If \( \alpha_{Y,y} < 1 \), we have locally \( F_0 \neq P_0 \) on \( \mathcal{O}_{X,Y}(\ast Y) \). However, it is not clear whether \( F^n \neq P^n \) on \( H^n(U, \mathbb{C}) \) if \( d \leq 3 \). Indeed, if \( Y \) has only ordinary double points and \( n = 2 \), then \( \alpha_Y = 1 \) and hence \( F_0 = P_0 \). In case \( Y \) is a rational cubic curve with a cusp (i.e. \( f = x^3 + y^2z \)), we have \( \alpha_Y = 5/6 \) but \( H^2(U, \mathbb{C}) = 0 \) because \( Y \) is a \( \mathbb{Q} \)-rational manifold. However, for \( d = 4 \), we have \( F^2 \neq P^2 \) on \( H^2(U, \mathbb{C}) \) if \( Y \) is the union of a smooth cubic curve and a line which intersect only at one point, see [12], p. 186, Remark 1.33. We have also \( F^2 \neq P^2 \) if \( d = 4 \) and \( Y \) has two cusps \( O, O' \) so that its normalization is an elliptic curve \( E \), e.g. if \( f = x^2y^2 + xz^3 + yz^3 \). Indeed, let \( g_1, g_2 \) be linear functions of \( x, y, z \) such that \( g_i^{-1}(0) \) passes
through both \(O, O'\) and \(g_2^{-1}(0)\) passes through \(O\) but not \(O'\). Let \(\omega_1\) be the differential 2-form on \(U\) corresponding to \(f^{-1}g_1dx \wedge dy \wedge dz\) by Theorem 1, and \(\eta_1\) be the differential 1-form on \(Y \setminus \{O, O'\}\) obtained by taking the residue of \(\omega_1\). Let \(E\) be the normalization of \(Y\). Then \(\eta_1\) is extended to a nowhere vanishing 1-form on \(E\) by Theorem 3, and hence \(\eta_2\) has only a pole of order 2 at the point \(P'\) corresponding to \(O'\) because the pull-back of \(g_2/g_1\) to \(E\) has only such a pole. Since \(E' := E \setminus \{P'\}\) is affine, the cohomology of \(E'\) is calculated by the complex of algebraic differential forms on \(E'\), and the cohomology class of \(\eta_2\) modulo \(\mathbb{C}\eta_1\) does not vanish, i.e. it is not an exact form modulo \(\mathbb{C}\eta_1\), because there is no rational function on an elliptic curve having only one pole of order 2.

Note however that we have \(F = P\) on \(H^2(U, \mathbb{C})\) if any singular point of \(Y\) is a cusp or an ordinary double point and if the irreducible components of \(Y\) are rational curves. Indeed, we have \(F^2H^2(U, \mathbb{C}) = H^2(U, \mathbb{C})\) using the weight spectral sequence because the normalization of each irreducible component of \(Y\) is a smooth rational curve.

If \(\dim Y = n - 1 = 2\) and \(Y\) has only rational singularities, then we have locally \(F_1 \neq P_1\) on \(\mathcal{O}_X(*Y)\) by [32], but it is not easy to show that \(F^2 \neq P^2\) globally on \(H^3(U, \mathbb{C})\), or equivalently, \(F^1 \neq P^1\) on \(H^2(Y, \mathbb{C})\). In this case, \(Y\) is a \(\mathbb{Q}\)-homology manifold and if \(Y'\) is a smooth hypersurface of degree \(d\) in \(\mathbb{P}^3\), then we have \(h^{p-p-2}(Y') = h^{p-p-2}(Y)\) for \(p \neq 1\) and \(h^{1,1}(Y') - h^{1,1}(Y)\) is the sum of the Milnor numbers. More generally, if \(Y\) is a \(\mathbb{Q}\)-homology manifold, then \(h^{p,q}(Y)\) may be expressed in terms of the degree \(d\) and the list of singularities on \(Y\). For details, see [13].

### 3. \(V\)-Filtration

3.1. Let \(X\) be a smooth complex algebraic variety, and \(h\) be an algebraic function on \(X\). Set \(Y = h^{-1}(0)\). Let \(i_h : X \to X \times \mathbb{A}^1\) be the graph embedding. We denote by \(\mathcal{B}_h\) the direct image of \(\mathcal{O}_X\) by \(i_h\) as a \(\mathcal{D}\)-module, see [30] for more details. Let \(\mathcal{B}_h[t^{-1}]\) be the localization of \(\mathcal{B}_h\) by \(t\). This is identified with the direct image of \(\mathcal{O}_X(*Y)\) by \(i_h\) as a \(\mathcal{D}\)-module.

We have canonical isomorphisms
\[
\mathcal{B}_h = \mathcal{O}_X \otimes \mathbb{C}[\partial_t], \quad \mathcal{B}_h[t^{-1}] = \mathcal{O}_X(*Y) \otimes \mathbb{C}[\partial_t],
\]
where \(t\) is the coordinate of \(\mathbb{A}^1\) and \(\partial_t = \partial/\partial t\). These isomorphisms are compatible with the action of \(\mathcal{O}_X\) and \(\partial_t\). The actions of \(t, g \otimes \partial^i\) and of a vector field \(\xi\) on \(X\) are given by
\[
\begin{align*}
\xi(g \otimes \partial^i) &= (\xi g) \otimes \partial^i - (\xi h) g \otimes \partial^{i+1}, \\
t(g \otimes \partial^i) &= hg \otimes \partial^i - ig \otimes \partial^{i-1}.
\end{align*}
\]

We have the Hodge filtration \(F\) on \(\mathcal{B}_h, \mathcal{B}_h[t^{-1}]\) by
\[
F_p(\mathcal{B}_h[t^{-1}]) = \sum_{i \geq 0} F_{p-i}(\mathcal{O}_X(*Y)) \otimes \partial_t^i,
\]
and similarly for \(F_p \mathcal{B}_h\). (Here the shift of the index by 1 associated to the direct image by a closed embedding of codimension 1 is omitted to simplify the notation.)
Let $V$ be the filtration of Kashiwara [22] and Malgrange [27] on $B_h, B_h[t^{-1}]$ along $X \times \{0\}$ indexed by $Q$. This is an exhaustive decreasing filtration of coherent $D_X$-submodules, and satisfies the following conditions:

(i) $t(V^\alpha B_h[t^{-1}]) \subset V^{\alpha+1}B_h[t^{-1}], \partial_t(V^\alpha B_h[t^{-1}]) \subset V^{\alpha-1}B_h[t^{-1}]$ for $\alpha \in Q$,
(ii) $t(V^\alpha B_h[t^{-1}]) = V^{\alpha+1}B_h[t^{-1}]$ for $\alpha > 0$,
(iii) $\partial_t - \alpha$ is nilpotent on $Gr^\alpha_V B_h[t^{-1}]$ for $\alpha \in Q$,

and similarly for $B_h$. Here $Gr^\alpha_V = V^\alpha/V^{\alpha+1}$ with $V^{\alpha} = \bigcup_{\beta > \alpha} V^\beta$, and we assume $V$ is indexed discretely and satisfies $V^\alpha = V^{\alpha-\epsilon}$ for $\epsilon > 0$ sufficiently small. Note that the filtration $V$ on $B_h$ is induced by that of $B_h[t^{-1}]$, and

$$(3.1.3) \quad (V^\alpha B_h, F) = (V^\alpha B_h[t^{-1}], F) \quad \text{for} \quad \alpha > 0,$$

because $B_h[t^{-1}]/B_h$ is supported on $X \times \{0\}$ so that $Gr^\alpha_V(B_h[t^{-1}]/B_h) = 0$ unless $-\alpha \in N$. We will denote also by $V$ the filtration on $\mathcal{O}_X(\ast Y)$ induced by that on $B_h[t^{-1}]$.

3.2. Proposition. Let $j : X \times (A^1 \setminus \{0\}) \to X \times A^1$ denote the inclusion. Then

$$(3.2.1) \quad F_0(B_h[t^{-1}]) = V^0(B_h[t^{-1}]) \cap j_\ast j^\ast F_0 B_h,$$

$$(3.2.2) \quad F_0(\mathcal{O}_X(\ast Y)) = V^0(\mathcal{O}_X(\ast Y)) = V^0(\mathcal{O}_X(Y)).$$

Proof. The isomorphism (3.2.1) follows from the property of the Hodge filtration of a mixed Hodge module on which the action of $t$ is bijective (see [30], 4.2) because $\text{min}\{p \in \mathbb{Z} | F_p(B_h[t^{-1}]) \neq 0\} = 0$. Then (3.2.1) implies the first isomorphism of (3.2.2), because $F_0(B_h[t^{-1}])_{U'} = V^0(\mathcal{O}_X(\ast Y))_{U'} = \mathcal{O}_X(Y)_{U'}$ where $U' = X \setminus \text{Sing} Y$. The last isomorphism of (3.2.2) is equivalent to $V^k \mathcal{O}_X \subset h^{-1} \mathcal{O}_X$ for any positive integer $k$ because $gh^{-k} \in V^0(\mathcal{O}_X(\ast Y))$ with $g \in \mathcal{O}_X$ is equivalent to $g \in V^k \mathcal{O}_X$. So it is proved by restricting to the smooth points of $Y_{\text{red}}$ because $\mathcal{O}_{X,x}$ is a unique factorization domain.

3.3. Proof of Theorem 2. We have $	ilde{V}^{>0}(\omega_X(Y)) \supset \omega_X$ because $	ilde{V}^{>1} \mathcal{O}_X \supset V^{>1} \mathcal{O}_X = \mathcal{O}_X(\ast Y)$. So the first isomorphisms in the first and the second rows of Theorem 2 follow from the vanishing of $H^k(X, \omega_X)$ for $k = 0, 1$. Then the second isomorphism of the first row follows from the last assertion of Proposition 3.2. We now reduce the remaining assertions to

$$(3.3.1) \quad \tilde{V}^{>0} \omega_Y = \tilde{\omega}_Y := \rho_\ast \omega_Y, \quad \tilde{V}^{>1} \omega_X = \rho_\ast \omega_{\tilde{X}/X}(-E),$$

where $\rho : \tilde{X} \to X$ is an embedded resolution of singularities of $Y$ such that $\rho^{-1}(Y)$ and the exceptional divisor $E$ are divisors with normal crossings and the proper transform $\tilde{Y}$ of $Y$ is smooth. Here the multiplicities of $E$ is defined by $\rho^\ast(Y) = E + \tilde{Y}$, and the adjoint ideal is defined by $\rho_\ast \omega_{\tilde{X}/X}(-E)$. 

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Assuming (3.3.1) and using, for example, a cubic resolution, the last isomorphism in the second row is more or less well-known. The relation with the intersection cohomology is reduced to the assertion that \( \tilde{\omega}_Y \) gives the first nontrivial piece of the Hodge filtration of the mixed Hodge module corresponding to the intersection complex. Here the last assertion follows easily from the decomposition theorem in the category of mixed Hodge modules because \( \tilde{\omega}_Y \) is torsion-free.

We now show (3.3.1). By [17] we have the exactness of the short exact sequence

\[
0 \to \rho_* \omega_X \to \rho_* \omega_X(\tilde{Y}) \to \rho_* \omega_{\tilde{Y}} \to 0.
\]

So the two assertions in (3.3.1) are equivalent to each other, because the second assertion is equivalent to \( \tilde{V}^0 \omega_X(Y) = \rho_* \omega_X(Y) \), see also [5], [14], [38]. We will show the first assertion of (3.3.1).

Since the assertion is local and we have a local equation \( h \) of \( Y \), we may consider \( \tilde{V}^1 \omega_X \) instead of \( \tilde{V}^0 \omega_X(Y) \) by trivializing \( O_X(\tilde{Y}) \). By definition \( \tilde{V}^1 \omega_X \) contains \( V^1 \omega_X = \omega_X(-Y) \) and

\[
\tilde{V}^1 \omega_X/V^1 \omega_X \subset Gr_1^V \omega_X
\]

is identified with

\[
\text{Ker}(N : Gr_1^V B_h \to Gr_1^V B_h) \otimes \omega_X \cap Gr_1^V \omega_X,
\]

because \( t : Gr^0_1 B_h \to Gr^1_1 B_h \) is injective. (Here \( N = t \partial_t \) as usual, and the tensor product with \( \omega_X \) may be viewed as the transformation between left and right \( D \)-modules.) Then it is further identified with \( \tilde{\omega}_Y \subset Gr_1^V \omega_X \), see the proof of Th. 0.6 in [30]. So the assertion follows.

### 3.4. Proof of Theorem 3.

The first isomorphism is shown in [32], 0.9, where the acyclicity of the Koszul complex associated to the partial derivatives \( \partial f/\partial x_i \) is used in an essential way. The second isomorphism then follows from Theorem 1 by considering the image of \( \Gamma(X, F_q(\omega_X(Y))) \) in \( \Gamma(X, P_q(\omega_X(Y))) \) by the last morphism of (0.1) in the introduction.

### 3.5. Remarks.

(i) If \( h \) is semi-weighted-homogeneous with weight \( (w_1, \ldots, w_n) \) as in Introduction, it is well known that \( \alpha_{Y,y} = \sum w_i \). If \( h \) is weighted homogeneous, this is due to Kashiwara (unpublished), and follows also from [26], [34] together with [33], [39] (or we can use a calculation of the Gauss-Manin connection by Brieskorn together with [26]). If \( h \) is semi-weighted-homogeneous, the assertion is then reduced to the fact that \( \alpha_{Y,y} \) does not change by a \( \mu \)-constant deformation in the isolated singularity case. The last assertion follows, for example, from the fact that \( \alpha_{Y,y} \) coincides with the minimal spectral number [35] by [26], [33], [39], because the spectral numbers are constant under a \( \mu \)-constant deformation [40].

(ii) Let \( \mathcal{G}_f^j \) be the algebraic Gauss-Manin systems associated to a homogeneous polynomial \( f : \mathbb{A}^{n+1} \to \mathbb{A}^1 \) as in the introduction. These are the direct image sheaves of \( \mathcal{B}_f \) by the projection \( \mathbb{A}^{n+1} \times \mathbb{A}^1 \to \mathbb{A}^1 \), and are defined by using the relative de Rham complex associated to the projection. We identify these with the corresponding modules over the Weyl algebra by taking the global section functor. They have the filtration \( V \), which
satisfy the conditions similar to those in (3.1), and are induced by the filtration \( V \) on \( \mathcal{B}_f \) by using the relative de Rham complex. Let \( \delta(t - f) \) denote the canonical generator \( 1 \otimes 1 \) of \( \mathcal{B}_f \). Let \( b_f(s) \) be the Bernstein-Sato polynomial of \( f \). Let \( \alpha_f \) and \( \beta_f \) be respectively the minimal and the maximal root of \( b_f(-s)/(-s + 1) \). Then \( \alpha_f > 0 \) by [21], \( \beta_f \leq n + 1 - \alpha_f \) by [31], and

\[
V > \beta_f - 1 \mathcal{G}_f^0 \subset \mathcal{H}_f \subset V^{\alpha_f} \mathcal{G}_f^0.
\]

This holds also for the analytic Brieskorn modules and the analytic Gauss-Manin systems associated to a germ of a holomorphic function \( f \). In the isolated singularity case, this is well-known, and follows from [26] together with [33], [35], [39] (see also [34] for the weighted homogeneous case). In general, the first inclusion of (3.5.1) is reduced to

\[
\mathcal{D}_X[s] \delta(t - f) \supset V > \beta_f - 1 \mathcal{B}_f,
\]

and the latter follows from the fact that \( b_f(s) \) is the minimal polynomial for the action of \( s = -\partial_t \) on

\[
\mathcal{D}_X[s] \delta(t - f) / t \mathcal{D}_X[s] \delta(t - f).
\]

The second inclusion of (3.5.1) follows from the fact that the action of \( \partial_t \) is bijective on \( \mathcal{G}_f^0 \) so that the filtration \( V \) can be replaced with the microlocal filtration in [31].

Note that the first inclusion of (3.5.1) is equivalent to

\[
\text{Gr}^V_{\mathcal{H}_f} = \text{Gr}^V_{\mathcal{G}_f^0} \quad \text{for any } \alpha > \beta_f - 1,
\]

and also to the isomorphisms

\[
\tau : \text{Gr}^V_{\mathcal{H}_f} \xrightarrow{\sim} \text{Gr}^{\alpha + 1}_{\mathcal{H}_f} \quad \text{for any } \alpha > \beta_f - 1.
\]

Note also that (1.6.2) is equivalent to

\[
\text{Gr}^V_{\mathcal{H}_f} = \mathcal{H}_{f,k} \quad \text{for } \alpha = k/d.
\]

### 4. Kodaira-Spencer map

#### 4.1. Deformation of smooth open varieties. Let \( \pi : \tilde{X} \rightarrow S \) be a proper smooth morphism of complex manifolds, and \( \tilde{Y} \) be a divisor with normal crossings which is flat over \( S \) and such that \( \tilde{Y}_s := \tilde{Y} \cap \pi^{-1}(s) \) is a divisor with normal crossings on \( \tilde{X}_s := \pi^{-1}(s) \) for any \( s \in S \). Put \( U = \tilde{X} \setminus \tilde{Y} \), and \( U_s = \tilde{X}_s \setminus \tilde{Y}_s \).

Fix \( s \in S \), and let \( \Theta_{\tilde{X}_s}(\log \tilde{Y}_s) \) be the sheaf of logarithmic vector fields on \( \tilde{X}_s \) along \( \tilde{Y}_s \) which is the dual of \( \Omega^1_{\tilde{X}_s}(\log \tilde{Y}_s) \) (i.e. \( \xi \in \Theta_{\tilde{X}_s}(\log \tilde{Y}_s) \) belongs to \( \Theta_{\tilde{X}_s}(\log \tilde{Y}_s) \) if and only if \( \xi \mathcal{I}_{\tilde{Y}_s} \subset \mathcal{I}_{\tilde{Y}_s} \) where \( \mathcal{I}_{\tilde{Y}_s} \) is the (reduced) ideal sheaf of \( \tilde{Y}_s \)). There is the Kodaira-Spencer map

\[
T_{S,s} \rightarrow H^1(\tilde{X}_s, \Theta_{\tilde{X}_s}(\log \tilde{Y}_s)),
\]
as in the classical case (where $\widetilde{Y} = \emptyset$), see e.g. [23], [37]. Furthermore, it is known that the $O_S$-linear part of the Gauss-Manin connection

\[(4.1.2) \quad \text{Gr}_{F} \nabla_\xi : \text{Gr}^p_F H^j(U_s, \mathbb{C}) \to \text{Gr}^{p-1}_F H^j(U_s, \mathbb{C})\]

coincides up to a sign with the action of the image of $\xi$ by the Kodaira-Spencer map, see loc. cit. In the classical case, this is due to Griffiths, see [18].

4.2. Case of the complement of hypersurfaces. Let $X_s = \mathbb{P}^n$, and let $\{Y_s\}_{s \in S}$ be an equisingular family of divisors defined by polynomials $f_s$ which depend algebraically on $s \in S$, where $S$ is assumed to be a smooth affine variety. Here equisingular means that $(X_s, Y_s)_{s \in S}$ admits a simultaneous embedded resolution $(\tilde{X}_s, \tilde{Y}_s) \to (X_s, Y_s)$ $(s \in S)$ which is induced by an embedded resolution $(\tilde{X}, \tilde{Y}) \to (X, Y)$ by restricting to the fibers at each $s \in S$, where $X = \mathbb{P}^n \times S$ and $Y = \bigcup_{s \in S} Y_s \times \{s\}$. This assumption implies that $\{U_s\}_{s \in S}$ is topologically locally trivial, and $\{H^j(U_s, \mathbb{C})\}_{s \in S}$ (or, more precisely, $R^j\pi_* \mathbb{C}_{U}$) is a local system on $S$ which underlies naturally a variation of mixed Hodge structure, see [29]. In particular, the dimension of the Hodge filtration $F^k H^n(U_s, \mathbb{C})$ for $s \in S$ is constant.

4.3. Remarks. (i) It is not clear whether the dimension of the pole order filtration $P_n^{n-q}$ on $H^n(U_s, \mathbb{C})$ is constant for $s \in S$. By Theorem 1 this is equivalent to that dim $H_{f_s, qd}^j$ is constant. Note that dim $(\partial f_s)_{q, d - n - 1}$ $(s \in S)$ is not necessarily constant, where $(\partial f_s)_j$ is the degree $j$ part of the Jacobian ideal $(\partial f_s)$. For example, consider a family of plane curves defined by $f_s = x^4z + y^3 + sx^2y^3 = 0$ for $s \in \mathbb{C}$. Here $\text{Sing}_s Y_s$ is one point, and the family is equisingular in the sense of (4.2) (indeed, the resolution of singularities in the case of irreducible plane curves depends only on the Puiseux pairs). Then the local Tjurina number jumps at $s = 0$, and dim $(\partial f_s)_k$ $(s \in S)$ is not constant at $s = 0$ for $k \gg 0$, see [7].

(ii) There is a theory of versal family for deformations of varieties with normal crossing divisors as in (4.1), see [23]. If $Y_s$ has only isolated singularities (where $s$ is a base point of $S$), then one can apply the above theory to a deformation of a resolution of singularities $(\tilde{Y}_s, E_s) \to (Y_s, \Sigma_s)$ $(where E_s is the exceptional divisor and \Sigma_s = \text{Sing}_s Y_s)$ instead of applying it to the embedded resolution $(\tilde{X}_s, \tilde{Y}_s) \to (X_s, Y_s)$ as in (4.2). In certain cases, it is possible to blow down a deformation $(\tilde{Y}_s', E_s')$ of $(\tilde{Y}_s, E_s)$ by [16], e.g. if $\dim Y_s = 2$ and the singularities of $Y_s$ are rational double points so that the exceptional divisor $E_s$ is a disjoint union of copies of $\mathbb{P}^1$. This has an advantage that the dimension of $\tilde{Y}_s$ is smaller than that of $\tilde{X}_s$. However, it it not clear whether the blow-down of an arbitrary deformation $(\tilde{Y}_s', E_s')$ of $(\tilde{Y}_s, E_s)$ is still a hypersurfaces of $\mathbb{P}^n$, and we have to determine the subset of the base space of the versal deformation of $\tilde{Y}_s$ consisting of the points corresponding to hypersurfaces of $\mathbb{P}^n$.

There is also a problem about the difference between the versal deformation of $(\tilde{Y}_s, E_s)$ and that of $\tilde{Y}_s$. Assume, for simplicity, $E_s$ is smooth, and let $N_{E_s}$ denote the normal bundle of $E_s$ in $\tilde{Y}_s$. Then there is a short exact sequence

\[(4.3.1) \quad 0 \to \Theta_{\tilde{Y}_s}(\log E_s) \to \Theta_{\tilde{Y}_s} \to N_{E_s} \to 0,\]

15
inducing a long exact sequence

$$
H^0(E_s, N_{E_s}) \to H^1(\tilde{Y}_s, \Theta_{\tilde{Y}_s}(\log E_s)) \to H^1(\tilde{Y}_s, \Theta_{\tilde{Y}_s})
$$

(4.3.2) $$
\cong H^1(E_s, N_{E_s}) \to H^2(\tilde{Y}_s, \Theta_{\tilde{Y}_s}(\log E_s)) \to H^2(\tilde{Y}_s, \Theta_{\tilde{Y}_s}).
$$

This may be used to study the difference between the versal deformation of $\tilde{Y}_s$ and that
$(\tilde{Y}_s, E_s)$ (the latter may be viewed as the versal equisingular deformation of $(Y_s, \Sigma_s)$ in the
sense of (4.2) if we can blow down as above). We have usually $H^0(E_s, N_{E_s}) = 0$ (e.g. if $N_{E_s}$ is negative). We have sometimes $H^1(E_s, N_{E_s}) = 0$ for example if $E_s = \mathbb{P}^{m-1}$ with $m := \dim \tilde{Y}_s > 2$ by the Bott vanishing theorem. However, the morphism $\gamma$ in (4.3.2) does not vanish in general. For example, if $\omega_{\tilde{Y}_s}$ is trivial, then $\gamma$ is identified with the restriction morphism

$$
H^1(\tilde{Y}_s, \Omega^{m-1}_{\tilde{Y}_s}) \to H^1(E_s, \Omega^{m-1}_{E_s}),
$$

and this morphism is surjective in the case $m = 2$. This may be related to [6].

4.4. Gauss-Manin systems. Let $Z = \mathbb{A}^{n+1}$, and $f$ be a function on $Z \times S$ whose
restriction to $Z \times \{s\}$ is $f_s$. Let $\mathcal{G}_{(f, \text{pr})}$ be the direct image of $\mathcal{O}_{Z \times S}$ by $(f, \text{pr}): Z \times S \to \mathbb{A}^1 \times S$ as an algebraic $\mathcal{D}$-module. This is obtained by taking the global section functor of the relative de Rham complex $\mathcal{D}R_{Z \times S/S}(B_f)$. Here we may take the global section functor because $\mathbb{A}^1 \times S$ is affine. Recall that

$$
\mathcal{B}_f = \mathcal{O}_{Z \times S} \otimes_{\mathbb{C}} \mathbb{C}[\partial_t],
$$

and the actions of $\xi \in \Theta_{Z \times S}$ and $t$ are given by

$$
\xi(g \otimes \partial^i) = (\xi g) \otimes \partial^i - (\xi f) g \otimes \partial^{i+1},
$$

$$
t(g \otimes \partial^i) = f g \otimes \partial^i - i g \otimes \partial^{i-1},
$$

(4.4.1) where the actions of $\mathcal{O}_{Z \times S}$ and $\partial_t$ are natural ones.

Let $\mathcal{G}_{f_s}$ be the restriction of $\mathcal{G}_{(f, \text{pr})}$ to $\mathbb{A}^1 \times \{s\}$ which is by definition the tensor product of $\mathcal{G}_{(f, \text{pr})}$ with $\Gamma(\mathbb{A}^1 \times \{s\}, \mathcal{O})$ over $\Gamma(\mathbb{A}^1 \times S, \mathcal{O})$. By the assumption on the equisingularity, $\mathcal{G}_{f_s}$ is the direct image of $\mathcal{B}_f$, which is defined by replacing $f$ with $f_s$ and $S$ with $\{s\}$. There are decompositions

$$
\mathcal{G}_{(f, \text{pr})} = \bigoplus_{k \in \mathbb{Z}} \mathcal{G}_{(f, \text{pr}), k}, \quad \mathcal{G}_{f_s} = \bigoplus_{k \in \mathbb{Z}} \mathcal{G}_{f_s, k},
$$

such that the degrees of $x_i$ and $d x_i$ are 1, where the $x_i$ are the coordinates of $Z := \mathbb{A}^{n+1}$.

We see that $\mathcal{G}_{f_s, k}$ is the tensor product of $\mathcal{G}_{(f, \text{pr}), k}$ and $\Gamma(\mathbb{A}^1 \times \{s\}, \mathcal{O})$ over $\Gamma(\mathbb{A}^1 \times S, \mathcal{O})$, and the canonical morphism

$$
\mathcal{H}_{f_s} \to \mathcal{G}_{f_s}
$$

(4.4.2) is compatible with the decomposition. Note that the action of $\nabla_{t \partial/\partial t}$ on $\mathcal{G}_{f_s, k}$ is the multiplication by $k/d - 1$, and the image of (4.4.2) is $\overline{\mathcal{H}}_{f_s}$, see [1]. By the assumption
on the equisingularity, \( \dim \mathcal{G}_{f_s, k} \) is constant, and hence \( \mathcal{G}_{(f, pr), k} \) is a projective \( R \)-module, where \( R = \Gamma(S, \mathcal{O}_S) \). However, this is not clear for \( \overline{\mathcal{H}}_{f_s, k} \), see Remark (4.3)(i).

4.5. Theorem. Assume \( \dim P^j H^n(U_s, \mathbb{C}) \) (\( s \in S \)) is constant for any \( j \in \mathbb{Z} \). Then for \( \xi \in \Theta_S \), the action of the Gauss-Manin connection \( \nabla_\xi \) induces a well-defined morphism

\[
\text{Gr}_P \nabla_\xi : \text{Gr}_P^{n-q+1} H^n(U_s, \mathbb{C}) \to \text{Gr}_P^{n-q} H^n(U_s, \mathbb{C}),
\]

which corresponds to the multiplication by \(-q(\xi f)_s\) using the identification in Theorem 1. Furthermore there is a commutative diagram

\[
\begin{array}{ccc}
\text{Gr}_P^{n-q+1} H^n(U_s, \mathbb{C}) & \xrightarrow{\text{Gr}_P \nabla_\xi} & \text{Gr}_P^{n-q} H^n(U_s, \mathbb{C}) \\
\downarrow & & \downarrow \\
\text{Gr}_P^{n-q+1} H^n(U_s, \mathbb{C}) & \xrightarrow{\text{Gr}_P \nabla_\xi} & \text{Gr}_P^{n-q} H^n(U_s, \mathbb{C}) \\
\text{H}_{f,qd}/f \text{H}_{f,(q-1)d} & \xrightarrow{-q(\xi f)_s} & \text{H}_{f,(q+1)d}/f \text{H}_{f,qd} \\
\end{array}
\]

where the vertical morphisms are either the natural morphisms or the isomorphisms given by Theorem 1.

Proof. Let \( \text{H}_{(f, pr), k} \) be the image of the canonical morphism

\[
(C[x]_{k-\cdot-1} \otimes \mathbb{C} R \otimes 1) dx_0 \wedge \cdots \wedge dx_n \to \mathcal{G}_{(f, pr), k},
\]

where \( C[x]_j \) is the degree \( j \)-part of the polynomial ring \( C[x] \). Since the tensor product is right exact, the \( \text{H}_{f, k} \) are given by the image of the tensor product of the inclusion

\[
\text{H}_{(f, pr), k} \to \mathcal{G}_{(f, pr), k}
\]

with \( C_s := R/m_s \) over \( R \) where \( m_s \) is the maximal ideal at \( s \). Since \( \dim P^j H^n(U_s, \mathbb{C}) \) is constant, we see that \( \text{H}_{(f, pr), k} \) is a projective \( R \)-module. Let

\[
\text{H}_{(f, pr)} = \bigoplus_{k \in \mathbb{N}} \text{H}_{(f, pr), k} \subset \mathcal{G}_{(f, pr)}.
\]

Then \( \text{H}_{(f, pr)} \) is stable by the action of \( t \xi \) for \( \xi \in \Theta_S \). Indeed, we have by (4.4.1)

\[
t_\xi (g \otimes 1) = t((\xi g) \otimes 1 - (\xi f) g \otimes \partial_t)
\]

\[
= f(\xi g) \otimes 1 - t\partial_t((\xi f) g \otimes 1),
\]

and the action of \( t\partial_t \) on \((\xi g) g \otimes 1) dx_0 \wedge \cdots \wedge dx_n \) is the multiplication by \( k/d \) if \( g \) has pure degree \( k - n - 1 \). Calculating \( \xi (g f^{-q} dx_0 \wedge \cdots \wedge dx_n) \) we see that the action of \( \xi \) is compatible with the isomorphism in Theorem 1. Since the action of the Gauss-Manin connection is compatible with the structure of \( \mathcal{D} \)-modules, we get the well-definedness of \( \text{Gr}_P \nabla_\xi \) together with the above commutative diagram. This completes the proof of (4.5).

4.6. Remark. In general it is quite difficult to get a good condition for the constantness of \( \dim P^j H^n(U_s, \mathbb{C}) \) (\( s \in S \)) for any \( j \in \mathbb{Z} \). If \( F^j = P^j \) on \( H^n(U_s, \mathbb{C}) \) for \( j = k, k + 1 \) and \( \dim P^{k-1} H^n(U_s, \mathbb{C}) \) is constant (e.g. if \( \dim Y_s = 2, d = 4, k = 2 \) and \( P^1 H^2(U_s, \mathbb{C}) = H^2(U_s, \mathbb{C}) \)), then we can calculate by (4.5)

\[
\text{Gr}_F \nabla_\xi : \text{Gr}_F^k H^n(U_s, \mathbb{C}) \to \text{Gr}_F^{k-1} H^n(U_s, \mathbb{C}) \subset \text{Gr}_F^{k-1} H^n(U_s, \mathbb{C}).
\]
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