Simplicity and finite primitive level of indecomposable set-theoretic solutions of the Yang-Baxter equation

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Abstract

This paper aims to deepen the theory of bijective non-degenerate set-theoretic solutions of the Yang-Baxter equation, not necessarily involutive, by means of q-cycle sets. We entirely focus on the finite indecomposable ones among which we especially study two classes of current interest: the simple solutions and those having finite primitive level. In particular, we provide two group-theoretic characterizations of these solutions, involving their permutation groups. Finally, we deal with some open questions.

Keywords: set-theoretic solution, Yang-Baxter equation, q-cycle set, cycle set, dynamical extension

2020 MSC: 16T25, 81R50, 20N02, 20E22

Introduction

A \textit{set-theoretic solution of the Yang-Baxter equation} on a non-empty set $X$ is a pair $(X, r)$, where $r : X \times X \to X \times X$ is a map such that the relation

$$(r \times \text{id}_X) (\text{id}_X \times r) (r \times \text{id}_X) = (\text{id}_X \times r) (r \times \text{id}_X) (\text{id}_X \times r)$$

is satisfied. The paper by Drinfel’d \cite{12}, a milestone in quantum group theory, moved the interest of several researchers for finding solutions of this equation in the last thirty years. Writing a solution $(X, r)$ as $r(x, y) = (\lambda_x(y) \rho_y(x))$, with $\lambda_x, \rho_x$ maps from $X$ into itself, for every $x \in X$, we say that $(X, r)$ is \textit{left non-degenerate} if $\lambda_x \in \text{Sym}_X$, \textit{right non-degenerate} if $\rho_x \in \text{Sym}_X$, \textit{non-degenerate} if it is both left and right non-degenerate. Moreover, $(X, r)$ is \textit{involutive} if $r^2 = \text{id}_{X \times X}$.

\*This work was partially supported by the Dipartimento di Matematica e Fisica “Ennio De Giorgi” - Università del Salento. The second author was partially supported by the ACROSS project ARS01_00702. The authors are members of GNSAGA (INdAM).

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Over the years, the involutive non-degenerate solutions have been widely studied starting from the seminal papers by Gateva-Ivanova and Van den Bergh [17], Gateva-Ivanova and Majid [16], and Etingov, Schedler, and Soloviev [13]. At the same time, Lu, Yan, and Zhu [22] and Soloviev [32] dealt with bijective non-degenerate solutions, not necessarily involutive.

A useful strategy for determining all the involutive solutions arises in [13] and consists in finding those that cannot be deconstructed into other ones, the so-called indecomposable solutions. In this context, several authors introduced useful tools for determining and classifying this type of solutions (see, for example, [2, 6, 18, 28, 29, 30, 31]). Later, in [9, 14], the notion of indecomposability was extended to solutions that are not necessarily involutive. Technically, a bijective non-degenerate solution \((X, r)\) is said to be decomposable if there exists a partition \(\{X_1, X_2\}\) of \(X\) such that \(r(X_i \times X_j) = X_j \times X_i\), for all \(i, j \in \{1, 2\}\); otherwise \((X, r)\) is called indecomposable. Etingof, Soloviev, and Guralnick [14, Lemma 2.1] exclusively characterized indecomposable solutions in group-theoretic terms. In this paper, we make explicit that a solution \((X, r)\) is indecomposable if and only if its permutation group, i.e., the group

\[
\mathcal{G}(X, r) = \langle \lambda_x, \eta_x \mid x \in X \rangle,
\]

acts transitively on \(X\), with \(\eta_x(y) := \rho_{\lambda_x^{-1}(x)}(y)\), for all \(x, y \in X\). In general, although the significant results obtained until now, finding and classifying all the indecomposable solutions is rather difficult.

Among involutive solutions, Vendramin first approached to the class of finite simple ones in [33]. More recently, Cedó and Okniński [8] have been introduced an equivalent definition in the indecomposable finite case; namely, an involutive solution \((X, r)\) is said to be simple if \(|X| > 1\) and, for every epimorphism of solutions \(f : (X, r) \to (Y, s)\), either \(f\) is an isomorphism or \(|Y| = 1\). In [8, Sections 4 and 5], several examples of involutive simple solutions can be found. This study was mainly motivated by the fact that the unique finite simple solutions known until then were two instances of order 4 (see [33, Example 2.11] and [2, Example 9]) and the primitive ones, i.e., solutions \((X, r)\) for which \(\mathcal{G}(X, r)\) acts on \(X\) as a primitive group. In this context, motivated by a question posed by Ballester-Bolinches in [20], Cedó, Jespers, and Okniński have been recently classified the finite primitive involutive solutions in [7, Theorem 3.1] and, in particular, they showed that they are all of prime order.

The main aim of this paper is to study finite indecomposable bijective non-degenerate solutions, not necessarily involutive.

**Convention.** Hereinafter, we briefly call solution any finite bijective non-degenerate set-theoretic solution of the Yang-Baxter equation.

To our purposes, we fully exploit the existing one-to-one correspondence between solutions and regular \(q\)-cycle sets, algebraic structures introduced by Rump [27]. Specifically, a non-empty set \(X\) endowed with two binary operations \(\cdot\) and \(:\) is said to be a \(q\)-cycle set if the map \(\sigma_x : X \to X, y \mapsto x \cdot y\) is bijective, for every \(x \in X\), and the
following conditions

\[(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z) \quad (q1)\]
\[(x : y) : (x : z) = (y : x) : (y : z) \quad (q2)\]
\[(x \cdot y) : (x : z) = (y \cdot x) \cdot (y : z) \quad (q3)\]

hold, for all \(x, y, z \in X\). Besides, \(X\) is regular if the map \(\delta_x : X \to X, y \mapsto x : y\) is bijective, for every \(x \in X\); non-degenerate if \(X\) is regular and the squaring maps, i.e., the maps \(q\) and \(q'\) from \(X\) into itself given by \(q(x) := x \cdot x\) and \(q'(x) := x : x\), for every \(x \in X\), are bijective. In [5, Theorem 5], it is proved that every finite regular q-cycle set \(X\) is non-degenerate. Thus, if \(X\) is a finite and regular q-cycle set, then \((X, r)\) is a solution, where \(r : X \times X \to X \times X\) is the map given by

\[r(x, y) = \left(\sigma_x^{-1}(y), \delta_{\sigma_x^{-1}(y)}(x)\right),\]

for all \(x, y \in X\). Vice versa, if \((X, r)\) is a solution, set

\[x \cdot y := \lambda_x^{-1}(y) \quad \text{and} \quad x : y := \rho_{\lambda_x^{-1}(y)}(y),\]

for all \(x, y \in X\), then \(X\) is a regular q-cycle set (cf. [23, Proposition 1]). Evidently, if \(X\) is a q-cycle set such that \(\cdot\) and \(:\) coincide, then \(X\) is a cycle set. Cycle sets were introduced by Rump in [22] and rather investigated (see, for instance, [2, 24, 26, 33]) for their one-to-one correspondence with left non-degenerate involutive solutions.

To study indecomposable solutions we look at indecomposable regular q-cycle sets. In light of our convention, throughout our treatise, it is clear that every q-cycle set will be finite even if it will not be specified. As one can expect, a regular q-cycle set \(X\) is indecomposable if and only if its permutation group \(\mathcal{G}(X) = \langle \sigma_x, \delta \mid x \in X \rangle\) acts transitively on \(X\).

As a first result, we show that any indecomposable q-cycle set \(X\) with \(|X| > 1\) and having regular permutation group \(\mathcal{G}(X)\) is retractable. In addition, if \(\mathcal{G}(X)\) is also abelian, then \(X\) is multipermutational. In this regard, given a regular q-cycle set \(X\), it is retractable if \(|X| = 1\) or there exist two distinct elements \(x, y \in X\) such that \(\sigma_x = \sigma_y\) and \(\delta_x = \delta_y\). Besides, it is possible to consider a congruence on \(X\), the retract relation \(\sim\) (see [3, Definition 1]), which ensures that the quotient \(\text{Ret}(X) := X/\sim\) is a q-cycle set. Moreover, for a multipermutational q-cycle set \(X\) of level \(n\), we mean that \(n\) is the minimal non-negative integer such that \(|\text{Ret}^n(X)| = 1\), where \(\text{Ret}^0(X) := X\) and \(\text{Ret}^i(X) := \text{Ret}(\text{Ret}^{i-1}(X)), \) for every \(i > 0\).

The core of this work is a description of indecomposable q-cycle sets in terms of dynamical extensions, a method to construct new families of q-cycle sets, already developed in [5]. Using this tool, we construct examples of indecomposable solutions that are not involutive. As another application, we prove that the permutation group \(\mathcal{G}(X)\) associated to any indecomposable retractable q-cycle set \(X\) always acts imprimitively on \(X\), whenever \(X\) has not prime size. Furthermore, motivated by a recent paper of Gateva-Ivanova [15], we focus our attention on indecomposable square-free q-cycle sets. We recall that a
non-degenerate q-cycle set is square-free if \( q = q' = \text{id}_X \). At first, we provide a structure theorem of indecomposable square-free q-cycle sets. Moreover, referring to [13, Question 9.6(7)], we exhibit a family of these q-cycle sets that are not self-distributive.

Another aspect we deal with is that of the simplicity of q-cycle sets, whose definition is consistent with that given for involutive solutions. Initially, we show that any finite simple q-cycle set \( X \) with \(|X| > 1\) is indecomposable. Furthermore, we describe in group-theoretic terms all the finite simple regular q-cycle set \( X \), involving a special group, namely, the displacement group of \( X \), given by

\[
\text{Dis}(X) = \langle \sigma_x^{-1}\sigma_y, \delta_x^{-1}\delta_y \mid x, y \in X \rangle.
\]

Note that, in the finite case, our results on the displacement group include those given in [1], where the authors first considered the previous group as a tool for studying latin cycle-sets (not necessarily finite).

The displacement group turns out to be also an essential tool to characterize indecomposable q-cycle sets having finite primitive level. In that regard, we exactly compute the primitive level of those having abelian permutation group. We say that a finite indecomposable q-cycle set \( X \) has primitive level \( k \) if \( k \) is the biggest positive integer such that there exist q-cycle sets \( X_1 = X, X_2, \ldots, X_k \) and an epimorphism \( p_{i+1} : X_i \to X_{i+1} \) with \(|X_i| > |X_{i+1}| > 1\), for every \( 1 \leq i \leq k - 1 \), and \( X_k \) is primitive. We specify that this notion was originally introduced in [8] in the involutive case and it establishes how far a solution is far from being a primitive solution. As a main application, referring to [8, Question 3.2], we characterize all the involutive solutions of primitive level equal to 2 and, among these, we completely classify the ones having abelian permutation group.

Finally, we show that the class of multipermutational cycle sets is contained in that of the cycle sets having finite primitive level. Consequently, we pose particular attention to some results and questions arisen in [8, 22, 31].

1. General results

This section is devoted to introducing some definitions and results involving the permutation group of a solution, already considered in [27] and [3]. Moreover, we give some notions on the algebraic structure of q-cycle set, that will be useful throughout the paper. Finally, we extend some results on the retractability of solutions given for the involutive case in [3] and [23].

**Definition 1.1.** Let \((X, r)\) be a solution and consider the permutation \( \eta_x \) on \( X \) given by \( \eta_x(y) := \rho_{\lambda_x^{-1}(a)}(y) \), for all \( x, y \in X \). Then, we name the group

\[
G(X, r) := \langle \lambda_x, \eta_x \mid x \in X \rangle
\]

the permutation group associated to \((X, r)\).
Note that if \((X, r)\) is involutive, then \(G(X, r) = \langle \lambda_x \mid x \in X \rangle\), cf. [13].

The following theorem characterizes indecomposable solutions in the finite case. We highlight that it is implicitly contained in the paper by Etingof, Soloviev, and Guralnick [14, Lemma 2.1]; however, to make our exposition self-contained, we give a proof of this result by using our terminology.

**Theorem 1.2.** Let \((X, r)\) be a solution. Then \((X, r)\) is indecomposable if and only if \(G(X, r)\) acts transitively on \(X\).

**Proof.** To get the claim, we show that \(x, y \in X\) are in the same orbit with respect to the action of the group \(G(X, r)\) if and only if they are in the same orbit with respect to the action of the group

\[ F(X, r) := \langle \lambda_x, \rho_x \mid x \in X \rangle. \]

Indeed, in [9, Proposition 6.6], it is proved that \((X, r)\) is indecomposable if and only if \(F(X, r)\) acts transitively on \(X\).

Now, suppose that \(x\) and \(y\) are in the same orbit with respect to the action of \(F(X, r)\). Then, there exist \(n \in \mathbb{N}\) and \(x_1, \ldots, x_n \in X\), such that \(g_{x_1} \cdots g_{x_n}(y) = y\), where \(g_{x_i} \in \{\lambda_x, \rho_x\}\), for every \(1 \leq i \leq n\). We show that there exist \(z_1, \ldots, z_n \in X\) and \(h_{z_1}, \ldots, h_{z_n} \in G(X, r)\), with \(h_{z_i} \in \{\lambda_{z_i}, \eta_{z_i}\}\), for every \(1 \leq i \leq n\), such that \(h_{z_1} \cdots h_{z_n}(x) = y\), by proceeding by induction on \(n\).

If \(n = 1\) we distinguish two cases. If \(g_{x_1} = \lambda_{x_1}\), then we set \(z_1 := x_1\) and \(h_{z_1} := \lambda_{z_1}\), while if \(g_{x_1} = \rho_{x_1}\), we set \(z_1 := \lambda_{x_1}(x_1)\) and \(h_{z_1} := \eta_{z_1}\).

Now, we assume that the claim holds for \(n - 1\). Then, there exist \(z_2, \ldots, z_n \in X\), \(h_{z_2}, \ldots, h_{z_n} \in G(X, r)\) such that \(h_{z_i} \in \{\lambda_{z_i}, \eta_{z_i}\}\), for every \(i \in \{2, \ldots, n\}\), and

\[ h_{z_2} \cdots h_{z_n}(x) = g_{x_2} \cdots g_{x_n}(x). \]

If \(g_{x_1} = \lambda_{x_1}\), we set \(z_1 := x_1\) and \(h_{z_1} := \lambda_{z_1}\). In this way, we trivially obtain that \(h_{z_1} \cdots h_{z_n}(x) = g_{x_1} \cdots g_{x_n}(x) = y\). Finally, if \(g_{x_1} = \rho_{x_1}\), then we set \(z_1 := \lambda_{g_{x_2} \cdots g_{x_n}(x)}(x_1)\) and \(h_{z_1} := \eta_{z_1}\). Hence,

\[ h_{z_1}h_{z_2} \cdots h_{z_n}(x) = h_{z_1}g_{x_2} \cdots g_{x_n}(x) = \rho_{\lambda_{g_{x_2} \cdots g_{x_n}(x)}(z_1)}(g_{x_2} \cdots g_{x_n}(x)) = y, \]

by completing the inductive step. In a similar way, one can show the converse implication.

**Remark 1.3.** In general, even if the orbits of the actions given by \(G(X, r)\) and \(F(X, r)\) coincide, these two groups are different. For instance, if \(X := \{1, 2, 3, 4\}\) and we consider the following permutations in \(\text{Sym}_X\)

\[ \lambda_1 := (2 \ 3) \quad \lambda_2 := (1 \ 4) \quad \lambda_3 := (1 \ 2 \ 4 \ 3) \quad \lambda_4 := (1 \ 3 \ 4 \ 2), \]

and we put \(\rho_y(x) := \lambda_{\lambda_y^{-1}(x)}\), for all \(x, y \in X\), then \((X, r)\) is an involutive solution, cf. [21, Example 8.2.14]. In this case, the group \(G(X, r)\) coincides with \(\langle \lambda_x \mid x \in X \rangle\) and it is different from \(F(X, r)\). Indeed, for example, \(\rho_1 = (2 \ 4)\) does not belong to \(G(X, r)\).
As a consequence of Theorem 1.2, we obtain that the indecomposability of a solution \((X, r)\) is linked to that of its derived solution, i.e., the pair \((X, r')\) where \(r'\) is the map defined by \(r'(x, y) = \left( y, \lambda_y \rho_{X^{-1}(y)}(x) \right)\), for all \(x, y \in X\) (see [32, p. 579]). Clearly, \(G(X, r')\) is a subgroup of \(G(X, r)\).

**Corollary 1.4.** Let \((X, r)\) be a solution and suppose that its derived solution \((X, r')\) is indecomposable. Then, \((X, r)\) is indecomposable.

**Proof.** By Theorem 1.2, \(G(X, r')\) acts transitively on \(X\). Since \(G(X, r') \leq G(X, r)\), it follows that \(G(X, r)\) also acts transitively on \(X\). Thus, by Theorem 1.2 the claim follows.

Now, in light of the existing correspondence between solutions and regular \(q\)-cycle sets given in [27, Proposition 1], let us introduce the analogous notion of indecomposable \(q\)-cycle set. To this end, according to [27, p. 143], we denote by \(G(X) = \langle \sigma_x, \delta_x \mid x \in X \rangle\) the permutation group associated to a regular \(q\)-cycle set \(X\). It is straightforward to check that the group \(G(X, r)\) coincides with the group \(G(X)\).

**Definition 1.5.** A regular \(q\)-cycle set \(X\) is said to be indecomposable if \(G(X)\) acts transitively on \(X\).

In view of Theorem 1.2, a solution \((X, r)\) is indecomposable if and only if the associated regular \(q\)-cycle set \(X\) is indecomposable.

Our aim is to show the retractability of any finite indecomposable \(q\)-cycle set \(X\) with regular permutation group \(G(X)\). In addition, if \(G(X)\) is abelian, \(X\) is multipermutational. To this purpose, we recall the relation of retraction of a regular \(q\)-cycle set \(X\), contained in [3, Definition 1]. Specifically, the relation \(\sim\) on \(X\) given by

\[ x \sim y \iff \sigma_x = \sigma_y \quad \text{and} \quad \delta_x = \delta_y, \]

for all \(x, y \in X\), is called the retract relation of \(X\). Such a relation is a congruence of \(q\)-cycle sets and ensures that the quotient \(\text{Ret}(X) := X/\sim\) can be endowed with the two operations induced by \(X\), which makes it into a \(q\)-cycle set structure. We call such a \(q\)-cycle set the retraction of \(X\). Analogously, given a solution \((X, r)\), one can define the retract relation of \((X, r)\), by using the maps \(\lambda_x\) and \(\rho_x\) (see [19, p. 3595]). As one can expect, the retraction of a solution corresponds to the retraction of a non-degenerate \(q\)-cycle set.

**Definition 1.6.** A regular \(q\)-cycle set \(X\) is said to be retractable if \(|X| = 1\) or if there exist two distinct elements \(x, y\) of \(X\) such that \(\sigma_x = \sigma_y\) and \(\delta_x = \delta_y\), otherwise \(X\) is irretractable. Moreover, \(X\) is multipermutational of level \(n\) if \(n\) is the minimal non-negative integer such that \(|\text{Ret}^n(X)| = 1\), where

\[ \text{Ret}^0(X) := X \quad \text{and} \quad \text{Ret}^i(X) := \text{Ret} \left( \text{Ret}^{i-1}(X) \right), \quad \text{for} \ i > 0. \]
In this case, we write mpl(\(X\)) = \(n\).

Clearly, a multipermutational q-cycle set of level \(n\) is retractable, but the converse is not necessarily true.

**Lemma 1.7.** Let \(X\) be an indecomposable q-cycle set such that \(\mathcal{G}(X)\) is regular. Then, \(q\) and \(q'\) coincide if and only if \(\cdot\) and \(:\) coincide.

*Proof.* Indeed, if \(q\) and \(q'\) coincide, then \(\sigma_x(x) = \delta_x(x)\), for every \(x \in X\). Moreover, since \(\mathcal{G}(X)\) acts regularly, \(\sigma_x = \delta_x\), i.e., the operations \(\cdot\) and \(:\) coincide. The converse implication is trivial. 

Let us observe that the following result is consistent with those contained in [6, Proposition 1] and [23, Corollary 3.3] in the context of cycle sets.

**Theorem 1.8.** If \(X\) is an indecomposable q-cycle set such that \(|X| > 1\) and \(\mathcal{G}(X)\) is regular, then \(X\) is retractable. Moreover, if \(\mathcal{G}(X)\) is abelian, then \(X\) is multipermutational.

*Proof.* If \(q\) and \(q'\) coincide, by Lemma 1.7, \(X\) is a cycle set and it is retractable by [23, Corollary 3.3]. If \(q\) and \(q'\) are different, there exists \(x \in X\) such that \(q(x) \neq q'(x)\). Moreover, by (q1), we have that \(\sigma_{q(x)}q = \sigma_{q'(x)}q\), hence, since \(\mathcal{G}(X)\) acts regularly on \(X\), it follows that \(\sigma_{q(x)} = \sigma_{q'(x)}\). With similar arguments, by (q2), we obtain that \(\delta_{q(x)} = \delta_{q'(x)}\). Thus, \(X\) is retractable.

Now, we suppose that \(\mathcal{G}(X)\) is abelian and we show the remaining part of the claim by induction on \(|X|\). If \(|X| = 2\), the claim trivially holds. Thus, we assume the claim for every indecomposable q-cycle set having order less than \(|X|\) and abelian permutation group. Clearly, by the first part of the proof \(X\) is retractable and \(|\text{Ret}(X)| < |X|\). Furthermore, \(\text{Ret}(X)\) is an indecomposable q-cycle set and \(\mathcal{G}(\text{Ret}(X))\) is again abelian. Therefore, by the inductive hypothesis, \(\text{Ret}(X)\) is multipermutational, which completes our claim.

In the last part of this section, we introduce the following subgroups of the permutation group of a q-cycle set. We underline that they will be essential tools for studying some classes of indecomposable q-cycle sets throughout our paper.

**Definition 1.9.** Let \(X\) be a regular q-cycle set not necessarily finite. Then, the groups

\[
\text{Dis}^+(X) := \langle \sigma_x \sigma_y^{-1}, \delta_x \delta_y^{-1} \mid x, y \in X \rangle \quad \text{Dis}^-(X) := \langle \sigma_x^{-1} \sigma_y, \delta_x^{-1} \delta_y \mid x, y \in X \rangle
\]

are called the *positive displacement group* and the *negative displacement group* of \(X\), respectively.

Let us observe that if \(X\) is a regular q-cycle set (not necessarily finite), then it holds \(\text{Dis}^+(X) \leq \text{Dis}^-(X)\). Indeed, by (q1) and (q2) we have that

\[
\sigma_x \sigma_y^{-1} = \sigma_{xy} \sigma_{yx} \quad \text{and} \quad \delta_x \delta_y^{-1} = \delta_{xy} \delta_{yx}.
\]
for all $x, y \in X$. In particular, if $X$ is a q-cycle set in which $\cdot$ and $:\colon$ coincide, then $\text{Dis}^+(X)$ and $\text{Dis}^-(X)$ are exactly the displacement groups introduced by Bonatto, Kinyon, Stanovský, and Vojtěchovský in [1, Section 2] in the context of cycle sets. The following results are consistent with Proposition 2.13 and Lemma 2.15 in [1] in the finite case; we underline that, by [5, Theorem 5], any finite regular q-cycle set is non-degenerate.

Lemma 1.10. If $X$ is a regular q-cycle set, the map $\Delta_{\cdot, :}\colon X \times X \to X \times X$ given by

$$\Delta_{\cdot, :}(x, y) = (x \cdot y, y : x),$$

for every $(x, y) \in X \times X$, is bijective.

Proof. Since $X$ is finite, we only show that $\Delta_{\cdot, :}$ is injective. If $(x, y), (x', y') \in X \times X$ are such that $\Delta_{\cdot, :}(x, y) = \Delta_{\cdot, :}(x', y')$, then

$$x \cdot y = x' \cdot y' \implies a(y : x) = a(x' : y')$$

$$= (y : x) \cdot (y : x) = (y', y' : x').$$

By (q1)

$$\implies (y : x) \cdot (y : x) = (y : x'),$$

since $y : x = y' : x'$. Therefore, the claim follows.

Proposition 1.11. Let $X$ be a regular q-cycle set. Then, $\text{Dis}^+(X) = \text{Dis}^-(X)$.

Proof. Initially, we know that $\text{Dis}^+(X) \leq \text{Dis}^-(X)$. Moreover, by Lemma 1.10 for every $(x, y) \in X \times X$ there exists a unique pair $(a, b) \in X \times X$ such that $(x, y) = (a \cdot b, b : a)$. Hence, by (q1) and (q2) we obtain that $\sigma_x^{-1} \sigma_y = \sigma_a \sigma_b^{-1}$ and $\delta_x^{-1} \delta_y = \delta_a \delta_b^{-1}$. Therefore, the claim follows.

In light of Proposition 1.11 for any regular q-cycle set $X$ we set

$$\text{Dis}(X) := \text{Dis}^+(X) = \text{Dis}^-(X)$$

and we call it the displacement group of $X$.

2. Dynamical extensions of indecomposable q-cycle sets and applications

In this section, we describe indecomposable q-cycle sets in terms of dynamical extensions, and we focus on some applications. At first, we show that the permutation group associated to any indecomposable retractable q-cycle set $X$ always acts imprimitively on $X$. Moreover, motivated by [15, Question 9.6(4)], we give a structure theorem of
indecomposable square-free q-cycle sets. Finally, referring to \[15, \text{Question } 9.6(7)\], we provide a family of this kind of q-cycle sets that are not self-distributive.

Initially, we recall the notion of dynamical pair contained in \([3]\), that is a useful tool to construct new q-cycle sets. Specifically, given a q-cycle set \(X\), a set \(S\), two maps \(\alpha : X \times X \times S \rightarrow \text{Sym}_S\) and \(\alpha' : X \times X \rightarrow S^S\), where \(S^S\) is the set of all the maps from \(S\) into itself, the pair \((\alpha, \alpha')\) is called a \textit{dynamical pair} if the following equalities
\[
\begin{align*}
\alpha_{(x,y),(x,z)}(s,t) \cdot \alpha_{(x,z)}(s,u) &= \alpha_{(y,z),(y,z)}(\alpha'_{(y,x)}(t,s), \alpha_{(y,z)}(t,u)) \\
\alpha'_{(x,y),(x,z)}(s,t) \cdot \alpha'_{(x,z)}(s,u) &= \alpha'_{(y,z),(y,z)}(\alpha_{(y,x)}(t,s), \alpha'_{(y,z)}(t,u)) \\
\alpha'_{(x,y),(x,z)}(s,t) \cdot \alpha_{(x,z)}(s,u) &= \alpha'_{(y,z),(y,z)}(\alpha_{(y,x)}(t,s), \alpha_{(y,z)}(t,u))
\end{align*}
\]
hold, for all \(x, y, z \in X\) and \(s, t, u \in S\).
As shown in \([3, \text{Theorem } 16]\), the triple \((X \times S, \cdot, ;)\) where
\[
(x, s) \cdot (y, t) := (x \cdot y, \alpha_{(x,y)}(s,t)) \quad (x, s) : (y, t) := (x : y, \alpha'_{(x,y)}(s,t)),
\]
for all \(x, y \in X\) and \(s, t \in S\), is a q-cycle set. If \(X\) is regular and \(\alpha'_{(x,y)}(s, -) \in \text{Sym}_S\), for all \(x, y \in X\) and \(s \in S\), then \((X \times S, \cdot, ;)\) is regular. Moreover, the converse is true if \(X\) and \(S\) have finite order. The q-cycle set \(X \times_{\alpha, \alpha'} S := (X \times S, \cdot, ;)\) is said to be a \textit{dynamical extension} of \(X\) by \(S\).

Any q-cycle set can be regarded as a particular dynamical extension by using covering maps. In detail, given two q-cycle sets \(X\) and \(Y\), a homomorphism \(p : X \rightarrow Y\) is called \textit{covering map} if it is surjective and all the fibers \(p^{-1}(y) := \{x \mid x \in X, p(x) = y\}\) have the same cardinality. For the ease of the reader, we recall such a result below.

**Theorem 2.1** (\(\text{Theorem } 18, [3]\)). Let \(X\) and \(Y\) be q-cycle sets and \(p : X \rightarrow Y\) a covering map. Then, there exist a set \(S\) and a dynamical pair \((\alpha, \alpha')\) such that \(X \cong X \times_{\alpha, \alpha'} S\).

The next two results are consistent with those given for cycle sets in \([2, \text{Section } 3]\).

**Lemma 2.2.** Every epimorphism \(p : X \rightarrow Y\) from an indecomposable q-cycle set \(X\) to a q-cycle set \(Y\) is a covering map.

**Proof.** To prove our claim, we show that
\[
\sigma_i(p^{-1}(p(x))) = p^{-1}(p\sigma_i(x)) \quad \delta_i(p^{-1}(p(x))) = p^{-1}(p\delta_i(x)),
\]
for all \(i, x \in X\). Note that, if \(j \in p^{-1}(p(x))\), then we get
\[
p(i \cdot j) = p(i) \cdot p(j) = p(i) \cdot p(x) = p(i \cdot x).
\]
Hence, \(\sigma_i(p^{-1}(p(x))) \subseteq p^{-1}(p\sigma_i(x))\). Conversely, if \(j \in p^{-1}(p\sigma_i(x))\), then there exists \(k \in X\) such that \(j = \sigma_i(k)\). Thus,
\[
p(i) \cdot p(x) = p(i \cdot x) = p(j) = p(i \cdot k) = p(i) \cdot p(k),
\]
and so \( p(x) = p(k) \). Then, they follow that \( k \in p^{-1}(p(x)) \) and \( j \in \sigma_s(p^{-1}(p(x))) \). In a similar way, one can check the second equality of \( 2 \).

Now, if \( x_1, x_2 \in X \), by the indecomposability of \( X \), there exists a set \( \pi \). By the equalities in \( 2 \), since \( X \) is finite we get \( \pi(p(x_1)) = p^{-1}(p(x_2)) \).

Therefore, we get that \( |p^{-1}(p(x_1))| = |p^{-1}(p(x_2))| \) and so the map \( p \) is a covering map.

\[ \text{Theorem 2.3.} \]
If \( X \) is an indecomposable \( q \)-cycle set and \( p : X \rightarrow Y \) an epimorphism from \( X \) to a \( q \)-cycle set \( Y \), then there exist a set \( S \) and a dynamical pair \( (\alpha, \alpha') \) such that \( X \cong Y \times_{\alpha, \alpha'} S \).

\[ \text{Proof.} \]
Initially, by Lemma 2.2, the map \( p \) is a covering map. Therefore, by Theorem 2.3, there exist a set \( S \) and a dynamical pair \( (\alpha, \alpha') \) such that \( X \) is isomorphic to \( Y \times_{\alpha, \alpha'} S \), which is our claim.

As a first application of Theorem 2.3, we extend the results obtained in [2]. Remark 2.3 and Lemma 2.4, where the authors show that the permutation group \( G(X) \) of a finite retractable cycle set \( X \) acts imprimitively on \( X \), whenever \( X \) has not prime size. To this end, we recall some classical notions of group theory (see, for instance, [10, 11, 35]).

Given a finite transitive group \( G \) acting on a set \( X \), then \( G \) is said to be \emph{imprimitive} if there exists a subset \( \Delta \) of \( X \), \( \Delta \neq X \), with at least two elements, called \emph{block} for \( G \), such that, for each permutation \( g \) of \( G \), either \( g(\Delta) = \Delta \) or \( g(\Delta) \cap \Delta = \emptyset \). In this way, the set \( \{g(\Delta)\}_{g \in G} \) forms a partition of \( X \) which is said to be an \emph{imprimitive blocks system}.

\[ \text{Theorem 2.4.} \]
Let \( X \) be a retractable indecomposable \( q \)-cycle set such that \( |X| \) is not a prime number. Then, \( G(X) \) acts imprimitively on \( X \).

\[ \text{Proof.} \]
If \( \text{mpl}(X) = 1 \), then \( G(X) \) is an abelian group that acts transitively on \( X \). Since in this case the action of \( G(X) \) on \( X \) is equivalent to the action of \( G(X) \) on itself by left multiplication (see, for example, 1.6.7 in [24]), the statement follows from the fact that \( |X| \) is not a prime number.

If \( \text{mpl}(X) > 1 \), then \( |\text{Ret}(X)| > 1 \) and, by Lemma 2.3, the canonical epimorphism \( \sigma : X \rightarrow \text{Ret}(X) \) is a covering map. In this way, by Theorem 2.3, there exist a set \( S \) and a dynamical pair \( (\alpha, \alpha') \) such that \( X \) can be identified with \( \text{Ret}(X) \times_{\alpha, \alpha'} S \). Therefore, we obtain that the partition \( \{\{\sigma_x\} \times S\}_{\sigma_x \in \text{Ret}(X)} \) is an imprimitive blocks system for \( X \), hence the claim follows.

Below, we provide a characterization of indecomposable dynamical extensions of \( q \)-cycle sets that includes [2, Theorem 7] given in the context of cycle sets. To this purpose, given a finite \( q \)-cycle set \( X \), a set \( S \), and a dynamical pair \( (\alpha, \alpha') \), we denote by

\[ H_x := \left\{ h \mid h \in G(\{x\} \times_{\alpha} S), \ h(\{x\} \times S) = \{x\} \times S \right\} \]

the stabilizer of the set \( \{x\} \times S \), for every \( x \in X \).
Theorem 2.5. Let $X$ be a $q$-cycle set, $S$ a set, and $(\alpha, \alpha')$ a dynamical pair. Then, $X \ltimes_{\alpha, \alpha'} S$ is indecomposable if and only if $X$ is indecomposable and there exists $x \in X$ such that the stabilizer $H_x$ acts transitively on $\{x\} \times S$.

Proof. Clearly, if $X \ltimes_{\alpha, \alpha'} S$ is indecomposable, then $X$ is indecomposable. Moreover, if $x \in X$ and $s_1, s_2 \in S$, by the indecomposability of $X \ltimes_{\alpha, \alpha'} S$, there exists $g \in \mathcal{G}(X \ltimes_{\alpha, \alpha'} S)$ such that $g(x, s_1) = (x, s_2)$. Thus, $g \in H_x$ and the transitivity of $H_x$ on $\{x\} \times S$ follows. Conversely, suppose that $X$ is indecomposable and there exists $x \in X$ such that $H_x$ acts transitively on $\{x\} \times S$. By a standard calculation, one can prove that the subgroups $\{H_y\}_{y \in X}$ are all conjugate one to each other. Besides, the transitivity of $H_x$ on $\{x\} \times S$ implies the transitivity of $H_y$ on $\{y\} \times S$, for every $y \in X$. Since $X$ is indecomposable, if $(x_1, s_1), (x_2, s_2) \in X \times S$, there exist $y_1, \ldots, y_n \in X$ such that $\varphi_{y_1} \cdot \ldots \cdot \varphi_{y_n} (x_1) = x_2$, where $\varphi_{y_i} \in \{\sigma_y, \delta_y\}$, for every $i \in \{1, \ldots, n\}$. Furthermore, there exist $t_1, \ldots, t_n, s$ in $S$ such that $\varphi_{(y_1, t_1)} \cdot \ldots \cdot \varphi_{(y_n, t_n)} (x_1, s_1) = (x_2, s)$, where $\varphi_{(y_i, t_i)} \in \{\sigma_{(y_i, t_i)}, \delta_{(y_i, t_i)}\}$, for every $i \in \{1, \ldots, n\}$. Since $H_{x_2}$ acts transitively on $\{x_2\} \times S$, there exists $w \in H_{x_2}$ such that $w(x_2, s) = (x_2, s_2)$. Therefore, $w \varphi_{(x_1, t_1)} \cdot \ldots \cdot \varphi_{(x_n, t_n)} (x_1, s_1) = (x_2, s_2)$, hence the claim follows.

Remark 2.6. Note that, assuming that $\mathcal{G}(X \times S)$ acts on $X \times S$, there exists an induced action of $\mathcal{G}(X \times S)$ on $X$ given by $\sigma_{(x, s)} (y) = \sigma_x (y)$ and $\delta_{(x, s)} (y) = \delta_x (y)$, for all $x, y \in X$ and $s \in S$. Moreover, as a consequence of Theorem 2.5 if $X \ltimes_{\alpha, \alpha'} S$ is indecomposable, then the permutation group $\mathcal{G}(X \times S)$ imprimitively acts on $X \times S$ and the set $\{(x) \times S\}_{x \in X}$ is a blocks system, which we call the blocks system induced by $X$.

In the next, we construct examples of indecomposable $q$-cycle sets by using Theorem 2.5.

Example 1.

1. Let $X := \mathbb{Z}/4\mathbb{Z}$ be the $q$-cycle set given by $x \cdot y := y + 1$ and $x : y := y - 1$, for all $x, y \in X$, $S := \mathbb{Z}/2\mathbb{Z}$, and $\alpha, \alpha' : X \times X \times S \mapsto \text{Sym}_S$ the maps given by

$$
\alpha_{(x, y)} (s, t) = \alpha'_{(x, y)} (s, t) = \begin{cases} 
t & \text{if } x \in \{0, 2\} 
\t + 1 & \text{if } x \in \{1, 3\}
\end{cases}
$$

for all $y \in X$ and $s, t \in S$. Note that $X$ is indecomposable and $\langle \delta_{(0, 0)} \sigma_{(1, 0)} \rangle$ is a subgroup of $H_0$ that acts transitively on $\{0\} \times S$. Then, $X \ltimes_{\alpha, \alpha'} S$ is an indecomposable $q$-cycle set.

2. Let $k \in \mathbb{N}$, $X := \mathbb{Z}/2k\mathbb{Z}$ the $q$-cycle set given by $x \cdot y := x := y + 1$, for all $x, y \in X$, $S := \mathbb{Z}/2\mathbb{Z}$, and $\alpha, \alpha' : X \times X \times S \mapsto \text{Sym}_S$ the maps

$$
\alpha_{(x, y)} (s, t) = \begin{cases} 
t & \text{if } x \text{ is even} 
\t + 1 & \text{if } x \text{ is odd}
\end{cases}
\quad \alpha'_{(x, y)} (s, t) = \begin{cases} 
t + 1 & \text{if } x \text{ is even} 
\t & \text{if } x \text{ is odd}
\end{cases}
$$

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for all \( y \in X \) and \( s, t \in S \). Observe that \( X \) is indecomposable and \( \left\langle \sigma^{2k-1}_{(0,0)}\sigma_{(1,0)} \right\rangle \) is a subgroup of \( H_0 \) that acts transitively on \( \{0\} \times S \). Then, \( X \times_{\alpha, \alpha'} S \) is an indecomposable \( q \)-cycle set.

3. Let \( p \) be a prime number, \( X := \mathbb{Z}/p\mathbb{Z} \) the \( q \)-cycle set given by \( x \cdot y = x : y := y + 1 \) for all \( x, y \in X \), \( S := \mathbb{Z}/p\mathbb{Z} \), and \( \alpha, \alpha' : X \times X \times S \rightarrow \text{Sym}_S \) the maps given by

\[
\alpha_{(x,y)}(s,t) = t + x \quad \alpha'_{(x,y)}(s,t) = t + x + 1,
\]

for all \( x, y \in X \) and \( s, t \in S \). Observe that \( X \) is indecomposable and \( \left\langle \sigma^{p-1}_{(0,0)}\sigma_{(1,0)} \right\rangle \) is a subgroup of \( H_0 \) that acts transitively on \( \{0\} \times S \). Then, \( X \times_{\alpha, \alpha'} S \) is an indecomposable \( q \)-cycle set.

Inspired by some issues raised by Gateva-Ivanova in [15], we conclude this section focusing on indecomposable square-free \( q \)-cycle sets. In particular, the author asks for searching specific finite solutions that are indecomposable and square-free.

**Question 1** (Question 9.6(7), [15]). Find examples of indecomposable finite square-free solutions which are not left self-distributive.

We recall that a solution \((X, r)\) is left self-distributive if \( \rho_x = \text{id}_X \), for every \( x \in X \); let us note that such solutions correspond to \( q \)-cycle sets for which \( \delta_x = \text{id}_X \), for every \( x \in X \). For brevity, we call such \( q \)-cycle sets left self-distributive. Analogously, one can define the right version of the self-distributivity. For a self-distributive solution or a \( q \)-cycle set, we mean a solution or a \( q \)-cycle set that is left or right self-distributive.

Solutions of the type in Question [1] that are neither right nor left self-distributive, are unknown to date in the literature and finding instances is challenging. Below, we provide a family of such solutions, in terms of dynamical extensions of \( q \)-cycle sets, that are indecomposable by Theorem [2.5].

**Example 2.** Let \( S \) be a 2-elementary abelian group, \( X := \{1, 2, 3\} \) the indecomposable \( q \)-cycle set given by \( \sigma_1 := (2 \ 3), \sigma_2 := (1 \ 3), \sigma_3 := (1 \ 2) \) and \( \delta_x := \text{id}_X \), for every \( x \in X \), and \( \alpha, \alpha' : X \times X \times S \rightarrow \text{Sym}_S \) the maps defined by

\[
\alpha_{(x,y)}(s,t) = t \quad \text{and} \quad \alpha'_{(x,y)}(s,t) = \begin{cases} t & \text{if } x = y \\ s + t & \text{if } x \neq y \end{cases},
\]

for all \( x, y \in X \) and \( t, s \in S \). Then, \( X \times_{\alpha, \alpha'} S \) is an indecomposable square-free \( q \)-cycle set. To show this, we prove the equalities in [1]. Initially, note that the first equality is trivially satisfied. Moreover, if \( x, y, z \in X \) and \( s, t, u \in S \), then the third equality reduces to

\[
\alpha'_{(x,y)(x,z)}(t,u) = \alpha'_{(y,z)}(t,u)
\]
and since \( y = z \) if and only if \( x \cdot y = x \cdot z \), it is satisfied. Now, the second equality reduces to

\[
\alpha'_{(y, z)} \left( \alpha'_{(x, y)}(s, t), \alpha'_{(x, z)}(s, u) \right) = \alpha'_{(y, x, z)}(s, \alpha'_{(y, z)}(t, u))
\]

and, to prove it, we need to examine the following cases.

- If \( x = y = z \), both members in (3) are equal to \( u \).
- If \( x = y \) and \( x \neq z \), then since \( X \) is square-free, \( y \cdot x = x \) and so \( y \cdot x \neq z \). Therefore, the first side in (3) is equal to \( \alpha'_{(x, y)}(s, t) + \alpha'_{(x, z)}(s, u) = t + s + u \) and the second one is equal to \( s + \alpha'_{(y, z)}(t, u) = s + t + u \).
- If \( x \neq y \) and \( x = z \), then \( y \cdot x \neq x \). Hence, the first member in (3) is equal to \( \alpha'_{(x, y)}(s, t) + \alpha'_{(x, z)}(s, u) = s + t + u \) and the second one is equal to \( s + \alpha'_{(y, z)}(t, u) = s + t + u \).
- If \( y = z \) and \( x \neq y \), it follows that \( y \cdot x \neq z \), otherwise \( z \cdot x = y \cdot x = z \cdot z \) and so \( x = z \), a contradiction. Thus, the first side in (3) is equal to \( \alpha'_{(x, z)}(s, u) = s + u \) and the second one is equal to \( s + \alpha'_{(y, z)}(t, u) = s + u \).
- If \( x, y, z \) are all distinct, by the definition of the maps \( \sigma_t \), we have that \( y \cdot x = z \).

Hence, the first member in (3) is \( \alpha'_{(x, y)}(s, t) + \alpha'_{(x, z)}(s, u) = s + t + s + u = u + t \) and the second one is equal to \( s + \alpha'_{(y, z)}(t, u) = t + u \).

By [5, Theorem 16], \( X \times_{\alpha, \alpha'} S \) is a \( q \)-cycle set. To show that it is indecomposable, by Theorem [2,4.5] it is sufficient to prove that \( H_0 \) acts transitively on \( \{0\} \times S \). This fact is true, since we have that \( \delta_{(1, s)}(0, 0) = (1 : 0, \alpha'_{(1, 0)}(s, 0)) = (0, s) \), for every \( s \in S \). Moreover, it is easy to check that \( X \times_{\alpha, \alpha'} S \) is square-free. Finally, let us observe that, if \( s \in S \), with \( s \neq 0 \), then

\[
\sigma_{(1, s)}(2, 0) = (3, 0) \quad \text{and} \quad \delta_{(1, s)}(2, 0) = (2, s),
\]

hence \( \sigma_{(1, s)} \neq \text{id}_{X \times S} \) and \( \delta_{(1, s)} \neq \text{id}_{X \times S} \). Therefore, \( X \times_{\alpha, \alpha'} S \) is neither right nor left self-distributive.

Let us highlight that the smallest \( q \)-cycle set obtained by using the technique in Example 2 is of order 6. By computer calculations, we checked that there are not indecomposable square-free \( q \)-cycle sets of order 2, 3, and 4, that are not self-distributive. At present, we are not able to state if there exist \( q \)-cycle sets of order 5 of such a type.

Finally, we give a structure theorem for indecomposable square-free \( q \)-cycle sets in terms of dynamical extensions which partially goes in the direction of [15, Question 9.6(4)].

**Lemma 2.7.** Let \( X \) be an indecomposable square-free \( q \)-cycle set. Then, \( \text{Ret}^k(X) \) is a \( q \)-cycle set in which \( \cdot \) and \( : \) do not coincide, for every \( k \in \mathbb{N} \). In particular, \( X \) is not a multipermutational \( q \)-cycle set.
Proof. At first, observe that clearly Ret\(^k\) (X) is indecomposable, for every \(k \in \mathbb{N}\). Now, suppose that there exists \(\bar{k} \in \mathbb{N}\) such that Ret\(^{\bar{k}}\) (X) is a q-cycle set in which the operations \(\cdot\) and : are equal. Then, by [23, Theorem 1], Ret\(^{\bar{k}}\) (X) is decomposable, a contradiction. Moreover, if X is multipermutational of level \(h\), since X is square-free, then Ret\(^{h-1}\) (X) is the trivial decomposable cycle set given by \(x \cdot y = y\), for all \(x, y \in \text{Ret}^{h-1} (X)\), an absurd.

\[\]

**Theorem 2.8.** Let X be an indecomposable square-free q-cycle set. Then, there exist a set S, an irretractable square-free q-cycle set Y, and a dynamical pair \((\alpha, \alpha')\) such that \(X \cong Y \times_{\alpha, \alpha'} S\).

Proof. By Lemma 2.7 there exists \(k \in \mathbb{N}\) such that Ret\(^k\) (X) is an irretractable square-free q-cycle set. Therefore, by Theorem 2.3 there exist a set S and a dynamical pair \((\alpha, \alpha')\) such that X is isomorphic to Ret\(^k\) (X) \(\times_{\alpha, \alpha'} S\).

\[\]

3. Simple regular q-cycle sets

This section aims to give a group-theoretic characterization of simple regular q-cycle sets, of which we illustrate some applications. Moreover, we show that any simple regular q-cycle set is indecomposable.

The notion of simplicity for q-cycle sets has been already introduced in [5, Definition 3] and is consistent with that given by Vendramin for cycle sets in [33, Definition 2.9]. However, in the context of cycle sets, Cedó and Okniński gave another definition in [8, Definition 3.3], which coincides with that provided by Vendramin in the indecomposable case. For our purposes, we consider the next definition which includes that in [8].

**Definition 3.1.** Let X be a regular q-cycle set with \(|X| > 1\). Then, X is called simple if, for every epimorphism \(f : X \to Y\) of q-cycle sets, either \(f\) is an isomorphism or \(|Y| = 1\).

By Lemma 2.7 if X is an indecomposable q-cycle set, Definition 3.1 coincides with [5, Definition 3]. Let us note the fact that X is indecomposable is not surprising since this is the case of every simple q-cycle set, exactly as it happens for cycle sets, cf. [33, Lemma 4.1].

**Proposition 3.2.** Let X be a simple regular q-cycle set with \(|X| > 1\). Then, X is indecomposable.

Proof. Assume that X is decomposable. It follows that there exists a non-trivial orbit \(A\) with \(|A| < |X|\). Let us consider the trivial q-cycle set on the set \(Y := \{1, 2\}\), i.e., \(x \cdot y = x : y = y\), for all \(x, y \in Y\), and the surjective map \(f : X \to Y\) given by

\[
f(x) := \begin{cases} 
1 & \text{if } x \in A \\
2 & \text{if } x \in X \setminus A
\end{cases}
\]

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and we show that $f$ is a homomorphism of $q$-cycle sets. Let $x \in X$. If $y \in A$, since $A$ is an orbit we have that $x \cdot y \in A$, thus $f(x \cdot y) = 1 = f(y) = f(x) \cdot f(y)$. Now, if $y \in X \setminus A$, clearly $x \cdot y \in X \setminus A$, thus $f(x \cdot y) = 2 = f(y) = f(x) \cdot f(y)$. Similarly, $f(x) : f(y) = f(x : y)$, for all $x, y \in X$. Therefore, we obtain a contradiction and so the claim follows.

Now, we give the characterization of all simple regular $q$-cycle sets which we mentioned before. Using the terminology contained in [10, 11], given an imprimitive blocks system $B := \{g(\Delta)\}_{g \in G}$ for $G$, if $\Delta \in B$, the set-wise stabilizer of $\Delta$ in $G$ is the subgroup of $G$ given by

$$G_\Delta := \{g \mid g \in G, g(\Delta) = \Delta\}.$$  

Moreover, the fixer of $B$ is the subgroup of $G$ given by

$$\text{Fix}(B) = \bigcap_{\Delta \in B} G_\Delta.$$  

If $X$ is a $q$-cycle set, it is useful recalling that if $G(X)$ is imprimitive and $\{\Delta_x\}_{x \in X}$ is an imprimitive blocks system, then $g(\Delta_x) = \Delta_{g(x)}$, for all $g \in G(X)$ and $x \in X$.

**Lemma 3.3.** Let $X$ be a regular $q$-cycle set. If $G(X)$ is imprimitive on $X$ and $\{\Delta_x\}_{x \in X}$ is an imprimitive blocks system, the relation defined by

$$\forall x, y \in X \quad x \approx y \iff \Delta_x = \Delta_y$$

is a left congruence of $X$.

**Proof.** Clearly, $\approx$ is an equivalence relation on $X$. Moreover, if $y, z \in X$ satisfy $y \approx z$, we get

$$y \approx z \implies \sigma_x(\Delta_y) = \sigma_x(\Delta_z) \quad \text{and} \quad \delta_x(\Delta_y) = \delta_x(\Delta_z)$$

$$\implies \Delta_{x \cdot y} = \Delta_{x \cdot z} \quad \text{and} \quad \Delta_{x : y} = \Delta_{x : z}$$

$$\implies x \cdot y \approx x \cdot z \quad \text{and} \quad x : y \approx x : z,$$

for every $x \in X$. 

In the following, given a regular $q$-cycle set $X$, if $\Delta$ is a block for $G(X)$, we denote by $\text{Dis}(X, \Delta)$ the subgroup of $\text{Dis}(X)$ given by

$$\text{Dis}(X, \Delta) := \langle \sigma_x^{-1} \sigma_y, \delta_x^{-1} \delta_y \mid x, y \in \Delta \rangle.$$  

**Theorem 3.4.** Let $X$ be a regular $q$-cycle set such that $|X| > 1$. Then, $X$ is simple if and only if $G(X)$ acts transitively on $X$ and, for every non-trivial imprimitive blocks system $\{\Delta_x\}_{x \in X}$, there exists $u \in X$ such that $\text{Dis}(X, \Delta_u) \not\leq \text{Fix}(\{\Delta_x\}_{x \in X})$. 

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Proof. Suppose that $G(X)$ acts transitively on $X$ and, for every non-trivial imprimitive blocks system $\{\Delta_x\}_{x \in X}$, there exists $\Delta \in \{\Delta_x\}_{x \in X}$ such that $\text{Dis}(X, \Delta) \not\subseteq \text{Fix}(\{\Delta_x\}_{x \in X})$. If $X$ is not simple, then there exist a q-cycle set $Y$ and a non trivial covering map $p : X \rightarrow Y$. Thus, if $x \in X$, set

$$\Delta_x := \{z \in X \mid p(x) = p(z)\},$$

we prove that $\{\Delta_x\}_{x \in X}$ is an imprimitive blocks system for $G(X)$. To this end, without loss of generality, we consider $y \in X$ and $\sigma_y \in G(X)$ such that $\sigma_y(\Delta_x) \cap \Delta_x \neq \emptyset$ and it is enough to prove that $\sigma_y(\Delta_x) \subseteq \Delta_x$. Thus, if $z \in \Delta_x$, since there exists $t \in \Delta_x$ such that $t = \sigma_y(a)$, for a certain $a \in \Delta_x$, we have that

$$p(\sigma_y(z)) = p(y \cdot z) = p(y \cdot a) = p(t) = p(x),$$

i.e., $\sigma_y(z) \in \Delta_x$. If $u \in X$, it follows that $\sigma_x(\Delta_u) = \Delta_{x \cdot u}$. Moreover, if $x' \in X$ is such that $\Delta_x = \Delta_{x'}$, since

$$p(x \cdot u) = p(x) \cdot p(u) = p(x') \cdot p(u) = p(x' \cdot u),$$

we obtain that $\sigma_x(\Delta_u) = \sigma_{x'}(\Delta_u)$. Similarly, one can show that $\delta_x(\Delta_u) = \delta_{x'}(\Delta_u)$. Clearly, these facts imply that $\text{Dis}(X, \Delta_x) \subseteq G(X)\Delta_u$, for every $u \in X$, that is an absurd.

Conversely, assume that $X$ is simple. Then, by Proposition 3.2, $X$ is indecomposable. By contradiction, suppose that $G(X)$ has a non-trivial imprimitive blocks system $\{\Delta_x\}_{x \in X}$ such that $\text{Dis}(X, \Delta_u) \subseteq \text{Fix}(\{\Delta_x\}_{x \in X})$, for every $u \in X$. To get the claim, since $X$ is finite, we show that the relation $\approx$ in (4) is a congruence. In this way, by Lemma 3.3, the canonical map $p : X \rightarrow X/\approx$ is an epimorphism of finite q-cycle sets and it is not trivial since $\{\Delta_x\}_{x \in X}$ is a non-trivial imprimitive blocks system, a contradiction. Thus, let $x, x' \in X$ such that $x \approx x'$. Then, $\Delta_x = \Delta_{x'}$ and, if $y \in X$, since $\text{Dis}(X, \Delta_x) \subseteq G(X)\Delta_y$, it follows that $\sigma_x^{-1}(\Delta_y) = \Delta_y$ and $\delta_x^{-1}(\delta_{x'}(\Delta_y)) = \Delta_y$. Hence, $\sigma_x(\Delta_y) = \sigma_{x'}(\Delta_y)$ and $\delta_{x'}(\Delta_y) = \delta_y(\Delta_y)$ and so $\Delta_{x \cdot y} = \Delta_{x' \cdot y}$ and $\Delta_{x \cdot y} = \Delta_{x' \cdot y}$. Therefore, $x \cdot y \approx x' \cdot y$ and $x : y \approx x' : y$, which is the desired conclusion. \hfill \Box

**Remark 3.5.** If $X$ is an indecomposable q-cycle set with $|X| > 1$ and $G(X)$ has a non-trivial imprimitive blocks system $\{\Delta_x\}_{x \in X}$ such that $\text{Dis}(X, \Delta_u) \not\subseteq \text{Fix}(\{\Delta_x\}_{x \in X})$, for every $u \in X$, then $\{\Delta_x\}_{x \in X}$ can be equipped with a q-cycle set structure. Indeed, by the proof of Theorem 3.4, one can observe that the quotient $X/\approx$ coincides with $\{\Delta_x\}_{x \in X}$.

Now, we close this section by showing how to use concretely Theorem 3.4 on specific q-cycle sets.

**Example 3.** (cf. Example 2.11) Let $X$ be the indecomposable q-cycle set on $\{1, 2, 3, 4\}$ given by

$$\sigma_1 = \delta_1 := (1 \ 4), \quad \sigma_2 = \delta_2 := (1 \ 3 \ 4 \ 2), \quad \sigma_3 = \delta_3 := (2 \ 3), \quad \sigma_4 = \delta_4 := (1 \ 2 \ 4 \ 3).$$
Then, by Theorem 3.4, \( X \) is simple. Indeed, the only imprimitive blocks system is given by \( \{ \Delta_1, \Delta_2 \} \), where \( \Delta_1 := \{1,4\} \) and \( \Delta_2 := \{2,3\} \), and, since \( \sigma_1\sigma_4^{-1}(\Delta_1) = \Delta_2 \), clearly \( \text{Dis}(X, \Delta_1) \not\subseteq \mathcal{G}(X)\Delta_1 \).

Example 4. (cf. \[8, Remark 4.11\]) Let \( X \) be the indecomposable q-cycle set on \( \{1,2,3,4,5,6,7,8,9\} \) given by

\[
\sigma_1 = \delta_1 := (1 3 8 4 5 2 9 7 6) \quad \sigma_2 = \delta_2 := (1 7 6 4 3 8 9 5 2) \\
\sigma_3 = \delta_3 := (1 7 8 4 3 2 9 5 6) \quad \sigma_4 = \delta_4 := (1 2 7 4 6 3 9 8 5) \\
\sigma_5 = \delta_5 := (1 8 5 4 2 7 9 6 3) \quad \sigma_6 = \delta_6 := (1 8 7 4 2 3 9 6 5) \\
\sigma_7 = \delta_7 := (1 9 4)(2 8 6) \quad \sigma_8 = \delta_8 := (1 9 4)(3 7 5) \\
\sigma_9 = \delta_9 := (2 8 6)(3 7 5).
\]

Then, by Theorem 3.4, \( X \) is simple. In fact, the only imprimitive blocks system is \( \{ \Delta_1, \Delta_2, \Delta_3 \} \), where \( \Delta_1 := \{1,4,9\} \), \( \Delta_2 := \{2,6,8\} \), and \( \Delta_3 := \{3,5,7\} \). Note that \( \text{Dis}(X, \Delta_1) \not\subseteq \mathcal{G}(X)\Delta_1 \), since \( \sigma_1\sigma_9^{-1}(\Delta_1) = \Delta_2 \).

Example 5. Let \( X := \{1,2,3,4,5,6\} \) be the indecomposable q-cycle set given by

\[
\sigma_1 := (2 4 5 3) \quad \sigma_2 := (1 3 6 4) \\
\sigma_3 := (1 5 6 2) \quad \sigma_4 := (1 2 6 5) \\
\sigma_5 := (1 4 6 3) \quad \sigma_6 := (2 3 5 4)
\]

and \( \delta_x := \text{id}_X \), for every \( x \in X \). The only imprimitive blocks system is \( \{ \Delta_1, \Delta_2, \Delta_3 \} \), where \( \Delta_1 := \{1,6\} \), \( \Delta_2 := \{2,5\} \), and \( \Delta_3 := \{3,4\} \). Note that \( \text{Dis}(X, \Delta_u) = \mathcal{G}(X)\Delta_v \), for all \( u, v \in X \). Hence, by Theorem 3.4, \( X \) is not simple.

4. Primitive level of indecomposable q-cycle sets

In this section, we introduce the notion of primitive level of q-cycle sets, which is consistent with the one given in \[8, p. 7\] for cycle sets. Moreover, we characterize all the indecomposable q-cycle sets having a finite primitive level in group-theoretic terms. In addition, we compute the primitive level of those having abelian permutation group.

Definition 4.1. A q-cycle set \( X \) is said to be primitive if \( \mathcal{G}(X) \) acts primitively on \( X \). Moreover, we say that a finite indecomposable q-cycle set \( X \) has primitive level \( k \) if \( k \) is the biggest positive integer such that

1. there exist q-cycle sets \( X_1 = X, X_2, \ldots, X_k \), with \( |X_i| > |X_{i+1}| > 1 \), for every \( 1 \leq i \leq k - 1 \);
there exists an epimorphism of q-cycle sets $p_{i+1} : X_i \to X_{i+1}$, for every $1 \leq i \leq k - 1$;

3. $X_k$ is primitive.

Observe that, by Lemma 2.2, if $X$ is an indecomposable q-cycle set of order $p_1^\alpha_1 \cdots p_n^\alpha_n$, for some primes $p_1, \ldots, p_n$, and with primitive level $k$, then $k$ is at most $\alpha_1 + \cdots + \alpha_n$. Clearly, q-cycle sets having primitive level 1 are exactly those for which $G(X)$ acts primitively on $X$. Cycle sets with primitive level 1 are completely classified (cf. [37, Theorem 3.1] and [13, Theorem 2.13]): they are, up to isomorphisms, the ones on $\mathbb{Z}$ and $\mathbb{Z}/p\mathbb{Z}$ (where $p$ is a prime number) given by $x \cdot y = y + 1$, for all $x, y \in \mathbb{Z}/p\mathbb{Z}$.

In general, there exist primitive q-cycle sets whose cardinality is not a prime number, as one can see in the following example.

**Example 6.** Let $X := \{1, 2, 3, 4\}$ be the q-cycle set given by

$$
\sigma_1 := (2 \ 4 \ 3) \quad \sigma_2 := (1 \ 3 \ 4) \quad \sigma_3 := (1 \ 4 \ 2) \quad \sigma_4 := (1 \ 2 \ 3),
$$

and $\delta_x := \text{id}_X$, for every $x \in X$. Then, $X$ is a primitive q-cycle set.

On the other hand, not all q-cycle sets have finite primitive level: for example, simple q-cycle sets with imprimitive permutation group have not finite primitive level. In the next, we give a characterization of all q-cycle sets having finite primitive level, in which the displacement group and its subgroups have a crucial role. We recall that if $G$ is a group that acts transitively on a finite set $X$, then a block $\Delta$ of $X$ is said to be maximal if, for every block $\Delta'$ such that $\Delta \subseteq \Delta' \subseteq X$, it follows that $\Delta = \Delta'$ or $\Delta' = X$ (cf. [35]). Moreover, an imprimitive blocks system $\mathcal{B}$ is said to be maximal if one (and hence all) of its blocks is maximal. Clearly, an imprimitive blocks system $\mathcal{B}$ is maximal if and only if the induced action of $G$ on $\mathcal{B}$ is primitive.

**Theorem 4.2.** Let $X$ be an indecomposable q-cycle set. Then, $X$ has finite primitive level if and only if there exists a maximal imprimitive blocks system $\{\Delta_x\}_{x \in X}$ such that $\text{Dis}(X, \Delta) \leq \text{Fix}(\{\Delta_x\}_{x \in X})$, for every $\Delta \in \{\Delta_x\}_{x \in X}$.

**Proof.** If $X$ has finite primitive level $k$, there exist q-cycle sets $X_1 = X, X_2, \ldots, X_k$, an epimorphism $p_{i+1} : X_i \to X_{i+1}$, with $|X_i| > |X_{i+1}| > 1$, for every $1 \leq i \leq k - 1$, and $X_k$ is primitive. Clearly, the composition $p_k p_{k-1} \cdots p_2$ is an epimorphism from $X$ to $X_k$, hence, by Theorem 2.1 there exist a set $S$ and a dynamical pair $(\alpha, \alpha')$ such that $X$ can be identified with $X_k \times_{\alpha, \alpha'} S$. Since $X_k$ is primitive, the action of $G(X_k \times_{\alpha, \alpha'} S)$ on $X_k$ (as in Remark 2.6) is primitive. It follows that the imprimitive blocks system $\{\{x\} \times S\}_{x \in X_k}$ induced by $X_k$ is maximal (see [35, p. 18]). Moreover, if $(x_1, s_1)$ and $(x_2, s_2)$ are in the same block, it follows that $x_1 = x_2$ and

$$
\sigma_{(x_1, s_1)}^{-1} \sigma_{(x_2, s_2)} (\{y\} \times S) = \{y\} \times S,
$$

for every $y \in X_k$. Analogously, one can see that $\delta_{(x_1, s_1)}^{-1} \delta_{(x_2, s_2)} (\{y\} \times S) = \{y\} \times S$. Hence, the necessary condition follows.
To prove the vice versa, it is sufficient to find an epimorphism from $X$ to a primitive q-cycle set. Thus, if $\{\Delta_x\}_{x \in X}$ is a maximal imprimitive blocks system such that $\text{Dis}(X, \Delta) \leq \text{Fix}(\{\Delta_x\}_{x \in X})$, for every $\Delta \in \{\Delta_x\}_{x \in X}$, then, by Theorem 3.3 $X$ is not simple. Furthermore, by Remark 3.5, there exist a set $S$ and a dynamical pair $(\alpha, \alpha')$ such that $X$ can be identified with $\{\Delta_x\}_{x \in X}$ and $\Delta_x$ is an epimorphism of q-cycle sets. Hence, by Theorem 2.1 there exist a set $S$ and a dynamical pair $(\alpha, \alpha')$ such that $X$ can be identified with $\{\Delta_x\}_{x \in X} \times \alpha, \alpha' S$. Moreover, since $\{\Delta_x\}_{x \in X}$ is a maximal imprimitive blocks system, we have that the action of $\mathcal{G}(X)$ on $\{\Delta_x\}_{x \in X}$ is primitive, and this clearly implies that $\mathcal{G}(\{\Delta_x\}_{x \in X})$ acts primitively on $\{\Delta_x\}_{x \in X}$. Therefore, $p$ is the requested epimorphism. \hfill \Box

Now, our aim is to show how to compute the finite primitive level of any indecomposable q-cycle set $X$ having $\mathcal{G}(X)$ as an abelian group. Let us recall that a transitive action of a finite abelian group $G$ on a finite set $X$ is equivalent to the left regular action of $G$ (for more details see, for instance, 1.6.7 in [24]). For this reason, without loss of generality, if $X$ is an indecomposable q-cycle set having abelian permutation group, we can assume that $X$ is an abelian group and, for every $x \in X$, there exist $h_x, h'_x \in X$ such that $\sigma_x$ and $\delta_x$ are the translations $h_x$ and $h'_x$, respectively, namely

$$\sigma_x(y) = h_x + y \quad \delta_x(y) = h'_x + y,$$

for every $y \in X$. Hereinafter, if $H$ is a subgroup of $G$, we denote by $\sim_H$ the usual equivalence relation with respect to $H$, i.e., $x \sim_H y$ if and only if $x - y \in H$, for all $x, y \in G$.

**Lemma 4.3.** Let $X$ be an indecomposable q-cycle and assume that $\mathcal{G}(X)$ is abelian. Then, there exists a non-trivial subgroup $H$ of $X$ such that $\text{Ret}(X) = X/H$. In addition, if $H'$ a subgroup of $X$ contained in $H$, the equivalence relation induced by $H'$ is a congruence of q-cycle sets.

**Proof.** Note that, by Theorem 1.8 $X$ is retractable. If $|\text{Ret}(X)| = 1$, put $H := X$, the first part of our claim follows. Otherwise, since $X$ has not prime order, by Theorem 2.3 the retraction induces an imprimitive blocks system $\mathcal{B}$. By a standard calculation, denoted by $\Delta_0 \in \mathcal{B}$ the block containing 0, one has that $H := \Delta_0$ is a subgroup of $X$. It follows that $\text{Ret}(X) = \{x + H\}_{x \in X}$.

Now, let $x, x', y, y' \in X$ such that $x \sim_{H'} x'$ and $y \sim_{H'} y'$. Note that, since $H' \subseteq H$, $\sigma_x = \sigma_{x'}$ and $\delta_x = \delta_{x'}$. Hence, there exists $h \in X$ such that $\sigma_x$ and $\sigma_{x'}$ are the translations by $h$. Thus, we get $y + h \sim_{H'} y' + h$ and so, $x \cdot y \sim_{H'} x' \cdot y'$. Similarly, one can show that $x : y \sim_{H'} x' : y'$. Therefore, the claim follows. \hfill \Box

**Theorem 4.4.** Let $X$ be an indecomposable q-cycle set with $|X| > 1$ and such that $\mathcal{G}(X)$ is abelian and has size $p_1^{\alpha_1} \cdots p_m^{\alpha_m}$, with $p_1, \ldots , p_m$ primes. Then, $X$ has primitive level equal to $\alpha_1 + \cdots + \alpha_m$. 

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Proof. We proceed by induction on $|X| = |\mathcal{G}(X)|$. If $|X| = 2$ the statement is trivial. Now, suppose that the claim is true for every $q$-cycle set $Y$ with $|Y| < |X|$ and having abelian permutation group. By Lemma 4.3 there exists a non-trivial subgroup $H$ of $X$ such that $\text{Ret}(X) = X/H$. Without loss of generality, we can suppose that $p_1 | |H|$.

Thus, let $H'$ be a subgroup of $H$ such that $|H'| = p_1$. Then, by Lemma 4.3 $X/H'$ is a $q$-cycle set and $p : X \to X/H'$ is an epimorphism of $q$-cycle sets. Finally, by the inductive hypothesis on $X/H'$, we have that the primitive level of $X/H'$ is $\alpha_1 + \cdots + \alpha_m - 1$. Therefore, the primitive level of $X$ is at least $\alpha_1 + \cdots + \alpha_m$ and hence the statement is proved.

As a consequence of Theorem 4.4, the class of multipermutational indecomposable $q$-cycle sets is contained in that of finite primitive level.

Corollary 4.5. Let $X$ be a multipermutational indecomposable $q$-cycle set. Then, $X$ has finite primitive level.

Proof. Assume that $X$ has multipermutational level $n$ and let $p : X \to \text{Ret}^{n-1}(X)$ be the canonical epimorphism. Then, $\text{Ret}^{n-1}(X)$ is a $q$-cycle set having abelian permutation group. Hence, if $\text{Ret}^{n-1}(X)$ has prime size, the claim is proved. Otherwise, by Theorem 4.3 there exists an epimorphism $\bar{p}$ from $\text{Ret}^{n-1}(X)$ to a primitive $q$-cycle set $Y$, with $|Y| < |\text{Ret}^{n-1}(X)|$. Therefore, $\bar{p}p$ is an epimorphism from $X$ to $Y$ and the statement follows.

5. Some applications to the indecomposable involutive solutions

In this section, we specialize some of the results previously obtained to indecomposable cycle sets. Into the specific, following [8, Questions 3.2 and 7.3], we characterize indecomposable cycle set having primitive level 2, and we give a sufficient condition to guarantee the simplicity of indecomposable cycle sets of size $p^2$. Moreover, we discuss some matters contained in [8, 28, 31].

The following is a characterization of indecomposable cycle sets having finite primitive level. Unlike Theorem 4.2 one can note that this result entirely involves the displacement group.

Theorem 5.1. Let $X$ be an indecomposable cycle set. Then, $X$ has finite primitive level if and only if there exists an imprimitive blocks system $\{\Delta_x\}_{x \in X}$ having prime size such that $\text{Dis}(X) \leq \text{Fix}(\{\Delta_x\}_{x \in X})$.

Proof. If $X$ has finite primitive level, with a similar argument in the proof of Theorem 4.2 we obtain that $X$ can be identified with a dynamical extension $X_k \times_{\alpha, \alpha'} S$, where $X_k$ is a primitive cycle set. By [2, Theorem 3.1], $|X_k| = p$, with $p$ a prime number, and $\sigma_x(y) = \gamma(y)$, for all $x, y \in X_k$, where $\gamma$ is a $p$-cycle. These facts imply that
has size $p$ and abelian permutation group, up to isomorphism.

Conversely, since $\text{Dis}(\Delta) \leq \text{Fix}((\Delta_x)_x \in X)$, for every $\Delta \in \{\Delta_x\}_{x \in X}$, and every imprimitive blocks system $\{\Delta_x\}_{x \in X}$ having prime size is maximal, the vice versa is a consequence of Theorem 5.2.

Among involutive solutions of finite primitive level, Cedó and Okniński asked the following.

**Question 2.** ([8, Question 3.2]) Describe involutive solutions of primitive level 2.

Using Theorem 5.1, we obtain a characterization of indecomposable cycle sets having primitive level 2. To this purpose, we recall that, given a group $G$ which acts transitively on a finite set $X$ and two imprimitive blocks systems $B, B'$ of $G$, then $B'$ is a refinement of $B$ if there exist $\Delta \in B$ and $\Delta' \in B'$ such that $\Delta' \subseteq \Delta$.

**Theorem 5.2.** Let $X$ be an indecomposable cycle set such that $|X|$ is not prime. Then, $X$ has primitive level 2 if and only if the following conditions hold:

1. there exists an imprimitive blocks system $\{\Delta_x\}_{x \in X}$ having prime size such that $\text{Dis}(X) \leq \text{Fix}((\Delta_x)_x \in X)$;
2. for every non-trivial refinement $\{\Delta'_x\}_{x \in X}$ of an imprimitive blocks system $\{\Delta_x\}_{x \in X}$ satisfying 1., there exists $\Delta' \in \{\Delta'_x\}_{x \in X}$ such that $\text{Dis}(X, \Delta') \not\leq \text{Fix}((\Delta'_x)_{x \in X})$.

**Proof.** If $X$ has primitive level 2 then, by Theorem 5.1 the condition 1. is satisfied. Thus, let $\{\Delta_x\}_{x \in X}$ be such an imprimitive blocks system having prime size and such that $\text{Dis}(X) \leq \text{Fix}((\Delta_x)_{x \in X})$ and consider a non-trivial refinement $\{\Delta'_x\}_{x \in X}$ such that $\text{Dis}(X, \Delta') \leq \text{Fix}((\Delta'_x)_{x \in X})$, for every $\Delta' \in \{\Delta'_x\}_{x \in X}$. It follows that the maps $p_1, p_2$ given by $p_1 : X \to \{\Delta_x\}_{x \in X}, x \mapsto \Delta_x$ and $p_2 : \{\Delta'_x\}_{x \in X} \to \{\Delta_x\}_{x \in X}, \Delta' \mapsto \Delta_x$ are epimorphisms of cycle sets, hence $X$ has primitive level at least 3, a contradiction.

Conversely, if conditions 1. and 2. hold, then, by Theorem 5.1 $X$ has finite primitive level. Moreover, if $X$ has primitive level greater than 2, there exist two indecomposable cycle sets $X_1$ and $X_2$, where $X_2$ is primitive, and two epimorphisms $p_1 : X \to X_1$ and $p_2 : X_1 \to X_2$. Hence, by Theorem 2.1 $X_1$ can be identified with a cycle set having $X_2 \times T$ as underlying set, for a suitable set $T$. In the same way, $X$ can be identified with a cycle set having $X_1 \times S = X_2 \times T \times S$ as underlying set, for a suitable set $S$. Therefore, the set $\{\{u\} \times T\}_{u \in X_1}$ is a refinement of $\{\{u\} \times T \times S\}_{u \in X_2}$. Moreover, by Theorem 5.1 $\text{Dis}(X, \{\{u\} \times S\}_{u \in X_1}) \leq \text{Fix}((\{x\} \times S)_{x \in X_1})$, for every $\{a\} \times S \in \{\{x\} \times S\}_{x \in X_1}$, a contradiction.

As a consequence of Theorem 5.3 we concretely describe cycle sets having primitive level 2 and abelian permutation group, up to isomorphism.
Corollary 5.3. All the indecomposable cycle sets with abelian permutation group and primitive level 2 have size $pq$ with $p, q$ not necessarily distinct primes. Hence, they are the ones provided in [6, Theorem 21].

Proof. It follows by Theorem 4.4 and [6, Theorem 21].

In general, by Lemma 2.2, the retractable indecomposable cycle sets having size $pq$, with $p, q$ not necessarily distinct primes, have primitive level equal to 2. In [8], one can find the next question:

Question 3 (Question 7.5, [8]). Does there exist a simple involutive solution $(X, r)$ such that $|X| = p_1 p_2 \cdots p_n$, for $n > 1$, with $p_1, p_2, \ldots, p_n$ distinct primes?

Inspecting all the indecomposable cycle sets having size 6 (using a GAP package in [34]), we note that they are all retractable. These facts clearly imply that it does not exist a simple indecomposable cycle set having size 6, answering in this specific case in the negative sense to the Question 3.

In [31, Corollary 6.6], the authors give a sufficient condition for the decomposability of a multipermutational finite cycle set. For the ease of the reader, we recall below such a result.

Corollary 5.4. [31, Corollary 6.6] Let $X$ be a multipermutational cycle set with $|X| > 1$ and suppose that, for every $x \in X$, there exists $y \in X$ such that $x \cdot y = y$ and $y \cdot x = x$. Then, $X$ is decomposable.

In [31, Question 6.7] Smoktunowicz and Smoktunowicz asked the following:

Question 4. Is Corollary 5.4 also true without the assumption that the multipermutational level of $(X, r)$ is finite?

In [28, Theorem 4], Rump showed that Corollary 5.4 in general, is not true if the cycle set is not of multipermutational type; nevertheless the hypotheses can be relaxed. In this context, we show that Corollary 5.4 is not unexpected: indeed, using dynamical extensions and Corollary 4.5, such cycle sets belong to a larger class of decomposable cycle sets.

Proposition 5.5. Let $X, Y$ be cycle sets, $p : X \rightarrow Y$ a covering map, and suppose that $Y$ is an indecomposable cycle set having prime size. Then, $x \cdot y \neq y$, for all $x, y \in X$.

Proof. If $x, y \in X$ are such that $x \cdot y = y$, then $p(x) \cdot p(y) = p(x \cdot y) = p(y)$, which contradicts [13, Theorem 2.13].

Below, we provide method, alternative to Corollary 5.4, to check that a cycle set is decomposable without the hypothesis of multipermutationality.

Corollary 5.6. Let $X, Y$ be cycle sets, with $|Y|$ a prime number. Suppose that $Y$ is an epimorphic image of $X$ and that there exist $x, y \in X$ such that $x \cdot y = y$. Then, $X$ is decomposable.
Proof. If by absurd $X$ is indecomposable, then so $Y$ is. Hence, by Lemma 2.2 and Proposition 5.5, we obtain a contradiction. Therefore, the statement is proved.

Now, let us focus on two applications of Corollary 5.6 in the first one, we give a direct simple proof of Corollary 5.4 which does not make use of the brace theory; in the second one, we extend [28, Theorem 4] in the case of finite cycle sets, giving more information on the cyclic structure of the left multiplications.

Proof of Corollary 5.4. If $X$ is indecomposable, by Corollary 5.5, it has finite primitive level. Hence, there exists an epimorphism $p$ from $X$ to an indecomposable cycle set $Y$ having prime size. Therefore the claim follows by Corollary 5.6.

The next result clearly implies [28, Theorem 4] in the finite case.

Corollary 5.7. Let $X$ be an indecomposable cycle set having finite primitive level. Then, $\sigma_x$ has no fixed point, for every $x \in X$.

Proof. Since $X$ has finite primitive level, there exists an epimorphism from $X$ to a cycle set $Y$ having prime size. If there exists $x \in X$ such that $\sigma_x$ has a fixed point, by Corollary 5.6, $X$ is decomposable, a contradiction.

Let us note that Corollary 5.7 allows for giving a sufficient condition for [8, Question 7.3], which we recall below.

Question 5. Let $p$ be a prime and $(X, r)$ an indecomposable and irretractable solution of cardinality $p^2$. Is $(X, r)$ simple?

Corollary 5.8. Let $X$ be an indecomposable cycle set having size $p^2$, for some prime $p$. If there exist $x, y \in X$ such that $x \cdot y = y$, then $X$ is simple.

Proof. Clearly, $X$ is simple or $X$ has primitive level 2. In the second case, by Corollary 5.7, it follows that $x \cdot y \neq y$, for all $x, y \in X$, against the hypothesis.

Finally, observe that the converse of Corollary 5.8 does not hold: indeed, in [8, p. 19] the authors give a simple indecomposable cycle set of size 9, where the left multiplications have no fixed points.
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