Ion losses induced by Alfvén ion-cyclotron instability in mirror machine with skew neutral beam injection

Ivan S. Chernoshtanov
Budker Institute of Nuclear Physics SB RAS, Novosibirsk, Russia
E-mail: I.S.Chernoshtanov@inp.nsk.su

Abstract.
The dynamic of ions in an axisymmetric mirror cell with a perturbed field is investigated. The spatial distribution of the perturbation was found in the framework of an analysis of the linear stage of the Alfvén ion cyclotron instability in a mirror trap with skew neutral beam injection. It is demonstrated that unstable perturbations do not cause losses of ions with maximal energy driving the instability. Main anomalous losses are longitudinal losses of ions with low transversal energy due to stochastic diffusion by the magnetic moment.

1. Introduction
The neutral beam injection (NBI) is one of the main methods of heating and maintaining of plasma in mirror machines. It was used in TMX, TMX-U, 2XIIB machines, now NBI is used in Gas Dynamic Trap (GDT). The powerful NBI is planned to be used in the projected GDT-based source of thermonuclear neutrons [1] and may be used in proposed thermonuclear reactor based on a mirror trap with diamagnetic confinement [2].

NBI leads to the formation of population of fast ions with non-Maxwellian distribution function. In combination with finite \( \beta \) (which is the ratio of plasma pressure to magnetic field pressure) it can provoke the excitation of an electromagnetic instability, namely an Alfvén ion-cyclotron (AIC) instability [3, 4]. It is an instability of circularly-polarized waves propagating along the external magnetic field with frequency \( \omega \leq \Omega_{ci} \), here \( \Omega_{ci} \) is the ion cyclotron frequency. The mechanism of the instability in mirror machines is the inverse Landau damping by resonant ions (ions with velocity satisfying the ion cyclotron resonance condition \( \omega - \Omega_{ci} = k_{\parallel} v_{\parallel} \), here \( k_{\parallel} \) and \( v_{\parallel} \) are components of wavevector and velocity along external magnetic field) [5].

The unstable fluctuations results in collisionless scattering of ions so excitation of the AIC instability can drive anomalous losses of energy and particles. Anomalous transport driven by the AIC instability was observed in a number of mirror machines in particular in the TMX [6], Tara [7], GAMMA-10 [8], GDT [9]. The anomalous losses observed in TMX and GAMMA-10 restrict essentially plasma parameters in particular the particle lifetime. On the contrary, anomalous losses of fast ions observed in GDT are less than classical losses (which are determined by electron drag). This difference may be due to difference in the methods of plasma heating (normal NBI in TMX, ICR-heating in GAMMA-10 and skew NBI in GDT) which leads to different distribution functions of ions. It should be noted that negligibly low level of anomalous losses were observed on TMX-U device where skew NBI was used also.
Figure 1. An example of geometry of injection (arrows) and magnetic field lines (solid lines) of an axisymmetric mirror machine.

The skew NBI is planned will be used as main method of heating and maintaining of plasma in projected GDT-based neutron source [1]. So the AIC instability driven anomalous losses can be expected (in the source) and an investigation of threshold of the AIC instability and details of collisionless scattering of fast ions is essential for optimization of parameters of the source.

This article is aimed at the investigation of the dynamics of fast ions in an axisymmetric mirror trap with skew NBI and excited AIC instability. The spatial distribution of perturbed fields found in the linear approximation is used. The article is organized as follows. Method of linear stability analysis and spatial structure of unstable fluctuations are discussed briefly in second section. In the third section the Hamiltonian of ions is reduced to a form convenient for analysis. The instability driven magnetic moment diffusion of fast ions is investigated in the fourth section. The diffusion by radial coordinate is studied in the fifth section. The results is discussed in the Conclusion.

2. Spatial structure of perturbed fields

Experimental observations demonstrates that the excitation of the AIC instability results in formation of some circularly-polarized non-linear wave [6, 8, 10]. We find spatial distribution of this wave analytically in the linear approximation assuming that the nonlinear effects do not change it substantially.

The necessary condition of excitation of the AIC instability is an inverse population (positive sign of derivative of distribution function $f$ along perturbed trajectory $(\omega - k_\parallel v_{\parallel}) \partial_{v_{\parallel}} f + k_\parallel v_{\parallel} \partial_{v_{\parallel}} f$) of trajectories of resonant ions (ions with velocity satisfying the cyclotron resonance condition $v_\parallel = v_r \equiv (\Omega_{ci} - \omega)/k_\parallel$) [5]. In case of mirror trap with NBI the inverse population of resonant ions can arise only if the longitudinal velocity of resonant particles $v_r$ is close to the longitudinal velocity of injected atoms $v_{\parallel inj}$, i.e. $\Omega_{ci} - \omega \approx k_\parallel v_{\parallel inj}$ [4].

In a mirror trap the unstable perturbation can propagate from center behind the mirrors and thereby taking out the energy extracted from resonant ions. The AIC instability excites only if part of the expended energy returns to the trap center. This is due to the interaction between waves with different dependencies of the frequency $\omega$ on longitudinal component of wavevector $k_\parallel$. The interaction can occur in points where the phase velocities of waves coincide (so-called turning points) [11]. In the simplest case there are two waves: the usual Alfvén wave with $k_{\parallel A} \approx (\omega_{\parallel p}/c)\omega/\sqrt{\Omega_{ci} - \omega}$ and so-called resonant wave with $k_{\parallel r} \approx (\Omega_{ci} - \omega)/v_r$, here $v_A$ and $v_r$ are the Alfvén velocity and velocity of resonant ions, $\omega_{\parallel p}$ is the ion plasma frequency. The resonant wave propagates in the direction opposite to $v_r$ and extracts energy from the resonant ions and transmits it to the Alfvén wave in the left turning point. The Alfvén wave propagates in the direction of velocity of resonant particles and partially transmits energy to the resonant wave in the right turning point [4]. Because the distribution function of ions is symmetric in longitudinal velocity, there are two more waves with wavevectors $-k_{\parallel A}$ and $-k_{\parallel r}$. The spatial structure of excited wave is follows: the standing wave in the trap center and Alfvén waves...
propagating from the center to mirrors.

The longitudinal distribution of perturbed fields in the Wentzel-Kramer-Brillouin (quasiclassical) approximation is proportional to \( \exp(i k (z) dz - i \omega t) \), here \( k(z) \) is the solution of the local dispersion relation. The relation is the solvability condition for the system

\[
\left( \begin{array}{c}
D_{(ll)}^{(l)} \\
D_{(ll)}^{(rr)}
\end{array} \right)
\left( \begin{array}{c}
E_+ \\
E_-
\end{array} \right) = 0,
\]

Here \( E_+ \) and \( E_- \) are the circular components of perturbed electric field rotating in the direction of ion and electron gyro-rotation, \( D^{(a)} = s \omega^2 c^2 / \omega^2 - k^2 c^2 / (2 \omega^2) + \varepsilon^{(a)} \), \( s_{rt} = s_{tr} = 0 \), \( \varepsilon^{(a)} \) are linear combinations of components of tensor of dielectric permeability [11, 4]. The condition \( \omega_{pe} \gg \Omega_{ci} \geq \omega \) is assumed to be satisfied. The requirement of uniqueness of the solution allows the spatial distribution and frequency of perturbed field to be found [11].

The simplest way to approximately account for longitudinal and transversal non-uniformity is the Pearlstein-Berk approximation [11]. Let’s look at a fully uniform plasma and consider perturbations with \( k_\perp = 0 \) (which are most unstable [11, 4]), so that the dispersion equation reduces to \( D^{(l)} = 0 \). Frequency \( \omega_0 \) and longitudinal component of wavevector \( k_{l0} \) satisfying the dispersion equation \( D^{(l)} = 0 \) and the condition for the merging of the roots \( \partial_{k_{l}} D^{(l)} = 0 \) can be found (case of \( \Im(\omega_0) > 0 \) corresponds to Briggs-Bers criterion for absolute instability in a homogenous medium). To take into account the spatial non-uniformity, the function \( D^{(l)} \) can be expanded in a Taylor series near \( \omega = \omega_0, k_{l} = k_{l0} \) and \( k_\perp = 0 \). After inverse Fourier transformation \( \vec{k} \rightarrow -i \nabla \) an equation (2) exists if frequency satisfies the condition \( \omega = \omega_0 - \kappa^2 D_{ll}^{(l)} / (2 D_{\omega}^{(l)}) - k_\perp^2 D_{kk_{l}}^{(l)} / (2 D_{\omega}^{(l)}) \), here \( \kappa^2 = \text{sign}(\text{Re}(\sqrt{D_{rr}/D_{kk_{l}}})) \sqrt{D_{rr}/D_{kk_{l}}} \) and \( k_{l0}^2 = \text{sign}(\text{Re}(\sqrt{D_{rr}/D_{kk_{l}}})) \sqrt{D_{rr}/D_{kk_{l}}} \). Typically \( 1 / k_{l0} \) is of the order of plasma radius and \( 1 / \kappa \) (which is the distance between wave turning points) is of the order of \( \sqrt{\rho_{nij} L} \), here \( \rho_{nij} \) is the Larmor radius of fast ions and \( L \) is the distance between mirrors. Finally, the distribution of perturbed electric field in the Pearlstein-Berk approximation can be written in the form

\[
E_+(z, r) = w B_0 \frac{\omega}{c k_{l0}} e^{-i \omega t} e^{-i k_{l0}^2 / 2} e^{-i k_{l0}^2 / 2} \sin(k_{l0}(z + \varphi_0)),
\]

here \( w \) is the ratio of amplitude of perturbed magnetic field to field of trap \( B_0 \) at \( z = 0 \) and \( r = 0 \). Typically, the amplitude of perturbations is very small (for example, microwave reflectometry on GAMMA-10 demonstrates \( w \sim 10^{-3} \) [12]). The analytical expression (2) is used below to analyze the dynamic of fast ions.

3. Hamiltonian

The paraxial approximation is used for describing the magnetic field of an axisymmetric mirror trap, \( A_{\varphi}^{(l)} = B(z) r / 2 \), and perturbation of vector potential is used in the form \( \delta A_r = -i A_{\varphi} = i (c / \omega) E_+(r, z) e^{-i \theta} \sqrt{2} \). The ion’s Hamilton function \( H_i(\vec{r}, \vec{p}) = (\vec{p} - e_i (\vec{A}^{(l)} + \delta \vec{A}) / c)^2 / (2 m_i) \) can be written in the form

\[
H(\mu, P_0, P_z; \Phi, \Theta, Z) = (\Omega_{ci}(Z) - \omega) \mu + \frac{P_0^2}{2 m_i} + \omega P_0 - \frac{w m_i \Omega_0}{k_{l0}} \sqrt{2 \Omega_{ci}(Z) / m_i} \text{Re} \left( e^{i \Phi - k_{l0}^2 / 2} e^{i \theta} e^{-k_{l0}^2 / 2} \sin(k_{l0}(Z + \varphi_0)) \right)
\]

(3)
after the canonical transformation

\[ \mu = \frac{p_r^2 + (p_\theta/r - m_i \Omega_{ci}r/2)^2}{2m_i \Omega_{ci}} + O(\frac{\rho B'}{B}), \quad \Phi = \arcsin \left( \frac{r - 2p_\theta}{m_i \Omega_{ci}r} \frac{1}{2\rho} \right) - \theta - \omega t + O(\frac{\rho B'}{B}), \]

\[ P_\theta = p_\theta + \mu, \quad \Theta = \arcsin \left( \frac{r + 2p_\theta}{m_i \Omega_{ci}r} \sqrt{\frac{m_i \Omega_{ci}}{8\mu + 8p_\theta}} \right) + \theta + \omega t + O(\frac{\rho B'}{B}), \]

\[ P_z = p_z + O(\frac{\rho B'}{B}), \quad Z = z + O(\frac{\rho B'}{B}), \quad (4) \]

where \( \Omega_{ci}(z) = e_iB/z / (m_i c) \) is the local cyclotron frequency, \( \Omega_0 = \Omega_{ci}(0) \), \( \vec{r} = (2(P_\theta + \mu - 2\sqrt{\mu B} \cos(\Phi + \Theta)) / (m_i \Omega_{ci})^{1/2} \) is the distance between trap axis and ion, \( \rho = \sqrt{2\mu / (m_i \Omega_{ci})} \) is the ion Larmor radius and \( B' = \partial_z B \). \( \mu \) and \( \Phi \) are the magnetic moment and conjugated phase. The distance between particle guiding center and trap axis is \( R = \sqrt{2\mu B / (m_i \Omega_{ci})} \). The terms of the order \( w^2, w \rho B'/B \) and \((\rho B'/B)^2 \) are neglected in the Hamiltonian (3).

The first term in the Hamiltonian (3) describes ions gyro-rotation with the frequency \( \Omega_{ci} - \omega \) (canonical transformation (4) implies a transition into rotating frame of reference), first and second terms describe bounce-oscillations along \( Z \) and last term describes scattering by \( \mu \) and \( P_\theta \) during excursions of trap center because of interaction with AIC perturbations.

The plasma radius \( R_p \) will be assumed to be large in the following sense: using the expression \( \exp(iz \sin \theta) = \sum_{n=-\infty}^{\infty} J_n(x) e^{i\theta} \) we can rewrite Hamiltonian (3) in the form

\[ H(\mu, P_\theta, P_z; \Phi, \theta, Z) = (\Omega_{ci}(Z) - \omega)\mu + \frac{P_z^2}{2m_i} + \omega P_\theta - \frac{m_i \Omega_0}{k_0^2 / m_i} \sqrt{2\Omega_{ci}(Z) / m_i} \times \]

\[ \times \sum_{n=-\infty}^{\infty} e^{-k_0^2 / 2m_i} I_n \left( \frac{2k_0^2 \sqrt{\mu B}}{m_i \Omega_{ci}} \right) R_0 \left( e^{-\kappa^2 Z^2 / 2 + i n \Theta + i (n+1) \Phi} \sin(k_0 Z + \varphi_0) \right), \quad (5) \]

Here \( I_n(x) \) are modified Bessel functions. The argument of this functions is assumed to be small \( 1/k_0^2 \gg \sqrt{\mu B} / (m_i \Omega_{ci}) \). This inequality can be written in the form \( R_p^2 \gg \rho R \).

The assumption that transversal non-uniformity of plasma is small enough allows us to eliminate terms with \( n > 1 \) in the Hamiltonian. It means that the influence of all cyclotron resonances \( \omega - (n + 1) \Omega_{ci} \pm k_0 \varphi = 0 \) with \( n > 1 \) is neglected. Because longitudinal velocity of majority of fast ions is less than the longitudinal velocity of injection (which is close to \( \Omega_{ci} - \omega \)) this simplification is reasonable.

4. Magnetic moment diffusion

In this section for simplicity the transversal non-uniformity is fully neglected, \( k_\perp = 0 \). So the moment \( \mu_\theta \) is a constant of motion and the particle motion is two-dimensional. It is convenient to use the Poincaré section of the plane \( Z = 0 \) for analysis of ion dynamic [13]. Let’s us to replace the Hamiltonian (3) with \( k_\perp = 0 \) by the equivalent twist mapping

\[ \mu_{n+1} = \mu_n + \text{Re} \left( i e^{i\Phi_n} (F_+ (\mu_{n+1}) - F_- (\mu_{n+1})) \right), \]

\[ \Phi_{n+1} = \Phi_n + 2\pi \alpha (\mu_{n+1}) - \text{Re} \left( e^{i\Phi_n} \frac{\partial}{\partial \mu} (F_+ (\mu_{n+1}) - F_- (\mu_{n+1})) \right), \quad (6) \]

where \( \mu_n \) and \( \Phi_n \) are particle magnetic moment and phase at \( z = 0 \) (in case when longitudinal velocity is positive), \( n \) is number of bounce oscillation, functions

\[ F_\pm = e^{i\alpha} (e^{\mp i\varphi_0} I_\pm (-z_1, z_2) - e^{\mp i\varphi_0 - i\pi \alpha} I_\pm (-z_1, 0) - e^{\mp i\varphi_0 - i\pi \alpha} I_\pm (0, z_1)), \]

\[ I_\pm(z_1, z_2) = w \frac{m_i \Omega_0}{2k_0^2} \int_{z_1}^{z_2} \sqrt{\frac{2\Omega_{ci}(Z)\mu}{m_i}} e^{-\kappa^2 Z^2 / 2} e^{i\phi + ik_0 Z} \frac{dz}{u(z)}. \quad (7) \]
describe the variation of the magnetic moment during a bounce-oscillation, \( \alpha \equiv (\Omega_m - \omega)/\Omega_b \) is so-called rotation number, \( \Omega_b \equiv \pi/(\int_{-z_l}^{z_l} dz/u(z)) \) is the bounce frequency, \( \langle \Omega_c \rangle \equiv 2\Omega_b \int_{-z_l}^{z_l} dz \Omega_c(z)/u(z) \) is the ion cyclotron frequency averaged over bounce oscillation, \( u(z) \equiv \sqrt{2(\varepsilon - \omega F_b)/m_i - 2(\Omega_c(z) - \omega)/m_i} \) describes dependence of unperturbed longitudinal velocity on longitudinal coordinate, \( \varepsilon \) is value of the Hamiltonian, \( z_l \) is positive solution of equation \( u(z) = 0, \phi(z) \equiv \int_0^z (\Omega_c(z') - \omega) dz'/u(z') \) describes dependence of unperturbed phase \( \Phi \) on longitudinal coordinate. The functions (7) are evaluated by integration of time derivative of magnetic moment \( -\partial H/\partial \Phi \) on trajectory of unperturbed motion, signs \( \pm \) correspond to interaction of particle with perturbations propagating in positive or negative direction (perturbations with spatial distribution proportional to \( e^{\pm ik_{||} z} \)).

The dynamics described by the twist mapping (7) are typical for two-dimensional weakly-perturbed integrable systems (see, for example, [14, 15]). For values of magnetic moment satisfying condition \( \alpha = n \) (n is integer) the resonant trajectory exists on which the change of phase \( \Phi_n \) is limited. The condition \( \alpha = n \) can be rewritten as \( \omega - \langle \Omega_c \rangle = n\Omega_b \). This condition means that the phase of cyclotron rotation in a rotating frame of reference changes an integer number of times during a bounce oscillation. The resonant trajectories are surrounded by separatrices and stochastic layers. At moderate values of \( \mu \) the resonances are separated by regular trajectories and change of magnetic moment in stochastic layers is limited. At small values of \( \mu \) the distance between resonances can become less than the width of separatrices. It leads to chaotic diffusion by magnetic moment of ions with small transversal energy [13]. The distance between neighboring resonances is approximately equal to \( (\partial_\phi \alpha)^{-1} \), the separatrix width is approximately \( \{I_+(z_l, z_l) - I_-(z_l, z_l)\}/(2\pi \partial_\phi \alpha)^{1/2} \), so the criterion of resonances overlapping [13] can be written in the form \( K \equiv (2\pi \partial_\phi \mu)I_+(z_l, z_l) - I_-(z_l, z_l) > 1 \). The diffusion coefficient can be estimated as squared change of magnetic moment during a bounce oscillation \( \Omega_b I_+(z_l, z_l) - I_-(z_l, z_l)^2 \) [16].

It should be noted that typically \( 1/k_{|| 0} \) and the bounce frequency are much less than the distance between mirrors and ion cyclotron frequency, respectively. So, the exponent index to be estimated analytically. Indeed, the exponents oscillate quickly except for the vicinity of the point where condition \( -\kappa^2 z + i\partial_\phi z - ik_{|| 0} = -\kappa^2 z + i(\Omega_c(z) - \omega)/u(z) - ik_{|| 0} = 0 \) is satisfied. This condition is simply cyclotron resonance condition complicated by longitudinal plasma non-uniformity. Further it is assumed that resonant interaction occurs near machine center so that all quantities can be expanded in a Taylor series for \( z = 0: \Omega_c(z) \approx \Omega_0(1 + z^2/L^2), \phi(z) \approx (z/V_0^l)(\Omega_0 - \omega)(1 + (1 - \omega/\Omega_0)V_0^l/V_0^\perp)^2/(6(1 - \omega/\Omega_0)(z^2/L^2)), \) here \( V_0^l = u(0) \) and \( V_0^\perp = \sqrt{2\Omega_0\mu/m_i} \) are components of unperturbed particle velocity for \( z = 0 \). The resonant condition is satisfied for \( z = \xi^{(+)} \), here

\[
\xi^{(\pm)} = -2i \frac{V_0^l \Delta \omega}{\kappa^2 V_0^\perp^2 \pm \sqrt{2\Delta \omega(\omega - \Omega_0)V_0^\perp^2/L^2 - 4\Omega_0 \Delta \omega V_0^\perp^2/L^2 + \kappa^4 V_0^\perp^4}}, \quad \Delta \omega = \omega - \Omega_0 + k_{||} V_0^l, \tag{8}
\]

but \( \xi^{(-)} \) is fictive solution (\( \xi^{(-)} \to \infty \) when \( \kappa L \to \infty \) or \( \Delta \omega/\Omega_0 \to 0 \)). Expanding exponent index in integrals (8) for \( z = \xi^{(+)} \), one can find

\[
I_+(z_1, z_2) \approx \frac{\Omega_0}{2k_{|| 0}} \sqrt{\frac{2m_i \Omega_c(z^{(+)}) \mu}{u(\xi^{(+)})}} e^{-((\kappa \xi^{(+)})^2/(3(1 + \xi^{(+)}/\xi^{(+)})}} \xi^{(+)} g \left( \frac{\kappa \xi^{(+)}}{2}, \frac{\xi^{(-)} - \xi^{(+)}}{\xi^{(-)} + \xi^{(+)}} \right), \tag{9}
\]

function \( g(x) = \sqrt{\pi} e^{x^2}(1 - \text{erf}(x))/(2x) \) if \( z_1 = 0 \) and \( z_2 = z_t, \) \( g(x) = \sqrt{\pi} e^{x^2}(1 + \text{erf}(x))/(2x) \) if \( z_1 = -z_t \) and \( z_2 = 0, \) and \( g(x) = \sqrt{\pi} e^{x^2}/x \) if \( z_1 = -z_t \) and \( z_2 = z_t, \) \text{erf}(x) \) is the error function.
4.1. Limit of large distance between mirrors
Typically $1/\kappa \sim L^{1/2}$, so product $\kappa L$ can be arbitrary large if $L \to \infty$. The analytical expressions for variation of magnetic moment during bounce-oscillation are essentially simplified if $\kappa L \gg 1$. It allows qualitative picture of ions motion to be clarified. Let’s look at a mirror trap with parabolic profile of magnetic field ($\Omega_{ci} = \Omega_0(1+z^2/L^2)$) and assume that unstable perturbations are localized near the minimum of the magnetic field, $\kappa L \gg 1$.

The expression (9) reduces to $I_+(-z_t, z_t) \approx w \sqrt{\pi/2} (m_i \Omega_0/k||0)(V_\parallel/V_\parallel') \exp(-\omega - \Omega_0 + k||0V_||^2/(2\kappa^2V^2))$ and the criterion of resonances overlapping $K > 1$ can be written in the form

$$V_\parallel^4 < \frac{\pi^3/2 L\Omega_0^3(3V_\parallel^2 + 4(1 - \omega/\Omega_0)V_\parallel^2)}{k||0\kappa V_\parallel} e^{-\Delta \omega^2/(2\kappa^2V^2)} \approx w \frac{3\pi^3/2 L\Omega_0^3 V_\parallel}{k||0\kappa} e^{-(\omega-\Omega_0+k||0V_||^2)/(2\kappa^2V^2)}$$

(10)

Ions with velocity satisfying condition (10) fall into the loss cone due to stochastic diffusion by magnetic moment. It should be noted that the margin of chaotic motion heavily depends of longitudinal velocity: ions are most strongly scattered, whose velocity is of the order of $(\Omega_0 - \omega)/k||0$. From the other hand, if the perturbation amplitude $w$ is small enough and transversal velocity of injected atoms $v_{\perp inj}$ exceeds margin value (10), the AIC instability excitation cannot lead to anomalous longitudinal losses of resonant ions driving the instability. In this case resonant ions may move chaotically, but variation of magnetic moment of resonant ions is restricted by the order of the resonance width (which is of the order of $\sqrt{w/(\kappa L)}m_i v^2_{\perp inj}/\Omega_0$ in the case $\kappa L \gg 1$). Such collisionless scattering of resonant ions is seems to play important role in nonlinear saturation of the instability.

It should be noted that the increase of the injection angle (decrease in the longitudinal and increase in the transverse components of velocity of injected atoms) facilitates the fulfillment of the condition (10) and increases the longitudinal losses of ions. Moreover, an increase of the injection angle facilitates the excitation of the AIC instability [3, 4] and the increases amplitude of perturbed fluctuations. Thus a decrease of the injection angle is an effective method of reducing anomalous losses driven by the AIC instability in mirror machines.

5. Radial diffusion
Accounting for the transversal non-uniformity ($k_{\perp 0} \neq 0$) allows the moment $P_\theta$ to change and can result in diffusion by particle guiding centers. To investigate the radial dynamic of ions in case of weak transversal non-uniformity the twist mapping (6) should be modified:

$$\mu_{n+1} = \mu_n + \text{Re} \left( i e^{i\Phi_n - k_{\perp 0}(P_\theta,n+1+\mu_{n+1})/(m_i \Omega_0)} I_0 \left( \frac{2k_{\perp 0}^2 \sqrt{\mu_{n+1} P_\theta,n+1}}{m_i \Omega_0} \right) \{ F_+ (\mu_{n+1}) - F_- (\mu_{n+1}) \} \right),$$

$$\Phi_{n+1} = \Phi_n + 2\pi \alpha (\mu_{n+1}) -$$

$$- \text{Re} \left( e^{i\Phi_n} \frac{\partial}{\partial \mu} e^{-k_{\perp 0}(P_\theta,n+1+\mu_{n+1})/(m_i \Omega_0)} I_0 \left( \frac{2k_{\perp 0}^2 \sqrt{\mu_{n+1} P_\theta,n+1}}{m_i \Omega_0} \right) \{ F_+ (\mu_{n+1}) - F_- (\mu_{n+1}) \} \right),$$

$$P_\theta,n+1 = P_\theta,n + \text{Re} \left( \sum_{n-1} e^{i(n+1)\Phi_n+i\Theta_n} \frac{\partial}{\partial P_\theta} \{ G^{(n)} (\mu_{n+1}, P_\theta,n+1) - G^{(n)} (\mu_{n+1}, P_\theta,n+1) \} \right),$$

$$\Theta_{n+1} = \Theta_n + \frac{2\pi \omega}{\Omega_0(\mu_{n+1}, P_\theta,n+1)} - \text{Re} \left( \sum_{n-1} e^{i(n+1)\Phi_n+i\Theta_n} \frac{\partial}{\partial P_\theta} \{ G^{(n)} (\mu_{n+1}, P_\theta,n+1) - G^{(n)} (\mu_{n+1}, P_\theta,n+1) \} \right),$$

(11)

Here

$$G^{(n)} (\mu, P_\theta) = ne^{i\pi n(\alpha+\omega/\Omega_b)+i\pi \alpha \times}$$
\begin{align*}
\times \left( e^{\pm i\varphi_0} J_\pm^{(n)}(-z_t, z_t) - e^{\mp i\varphi_0 + i\pi(n+1)} J_\pm^{(n)}(-z_t, 0) - e^{\mp i\varphi_0 - i\pi(n+1)} J_\pm^{(n)}(0, z_t) \right),
\end{align*}

\begin{align*}
J_\pm^{(n)}(z_1, z_2) = w \frac{m_i \Omega_i}{2k_{\|0}} \int_{z_1}^{z_2} \sqrt{\frac{2\Omega_{i\|}(z)}{m_i}} e^{-k_{\perp0}(\rho^2 + R^2)/2} I_n \left( k_{\perp0}^2 \rho R \right) e^{i(n+1)\phi + i\alpha\pm i\beta \pm i\gamma_{\|0}} z - \kappa^2 z^2/2 \, \frac{dz}{u(z)},
\end{align*}

where the function \( \tilde{\theta}(z) = \omega \int_0^z dz'/u(z') \) describes dependence unperturbed phase \( \Theta \) on longitudinal coordinate.

As in the previous section, the magnitude of variation of moment \( P_0 \) due to bounce-oscillation can be estimated analytically. The resonance condition \( u(z) = \{(n + 1)\Omega_i - \omega + i\kappa^2 z\}/k_{\|0} \) can be satisfied only if \( n = -1 \) (because longitudinal velocity is restricted by inequality \( |u| < V_{\|\text{inj}} \approx (\Omega_i - \omega)/k_{\|0} \)). Assuming that resonant interaction occurs near \( z = 0 \) and expanding \( \tilde{\theta}(z) \approx (\omega z/V_{\|}) \left( 1 + (V_{\perp0}^2 + V_{\perp0}^2)(z^2/(6L^2)) \right) \), we find that the resonance condition \( i\omega/V(z) - ik_{\|0} - \kappa^2 z = 0 \) is satisfied for \( z \approx \xi_{\theta}^{(\pm)} \), where

\begin{align*}
\xi_{\theta}^{(\pm)} = \frac{2i}{\kappa^2 V_{\|}^2} \pm \frac{V_{\|}(\omega - k_{\|0} V_{\|})}{2\omega(\omega - k_{\|0} V_{\|})V_{\perp}^2/L^2 + \kappa^4 V_{\|}^4}.
\end{align*}

The integral \( J_{\theta}^{(-1)} \) in (12) giving the main contribution in variation of \( P_0 \) can be written in the form

\begin{align*}
J_{\theta}^{(-1)}(z_1, z_2) \approx w \frac{\Omega_i}{2k_{\|0}} \sqrt{\frac{2m_i \Omega_i}{u(z)}(\xi_{\theta}^{(\pm)})} e^{-k_{\perp0}(\rho^2 + R^2)/2} I_1(k_{\perp0}^2 \rho R) \times \\frac{e^{-i\xi_{\theta}^{(\pm)}z/2/(3(\pm\xi_{\theta}^{(-)}/\xi_{\theta}^{(+)})\xi_{\theta}^{(+)})} \xi_{\theta}^{(+)}}{\xi_{\theta}^{(+)}/2(\xi_{\theta}^{(-)}/\xi_{\theta}^{(+)}) (\xi_{\theta}^{(-)} + \xi_{\theta}^{(+)})}.
\end{align*}

Comparison of (9) and (14) shows that variation of \( P_0 \) during bounce-oscillation is less than variation of \( \mu \) in \( e^{k_{\perp0}^2(\rho^2 + R^2)/2} I_1(k_{\perp0}^2 \rho R) \approx 1/(k_{\perp0}^2 \rho R) \gg 1 \times \). So, anomalous losses due to radial diffusion seem to be unessential in comparison with anomalous longitudinal losses while plasma radius is not too small, \( R_p \gg \sqrt{\rho R} \).

6. Conclusion

An investigation of anomalous losses due to the AIC instability is connected with discussion about reactor perspectives of mirror traps. Unfortunately the stability margin and details of nonlinear dynamics depend on a very large number of parameters (details of distribution function and magnetic field geometry, plasma density and pressure etc.). Such a number of parameters restricts the capability of numerical simulation. Thus the qualitative understanding of mechanisms of losses and analytical estimation of losses rate seem important.

The dynamics of fast ions in a mirror machine under an excited AIC instability and mechanisms of anomalous losses were considered in this article. If the plasma radius exceeds several Larmor radii of the fast ions, the main mechanism is longitudinal losses due to magnetic moment diffusion. If the injection angle differs essentially from 90°, such diffusion influences only on ions with low transversal energy and the energy distribution of lost ions depends strongly on details of the ion distribution function at low transversal energy. The majority of ions do not change noticeably their velocity in this case. An increase the injection angle expands the area of chaotic motion and increases longitudinal losses.

Accounting for finite plasma radius leads to radial diffusion of ions with low transversal energy and Arnold diffusion. The first effect seems to be insignificant in comparison with diffusion by the magnetic moment. The Arnold diffusion seems to be slow in comparison with the diffusion due to the classical Coulomb collisions.
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