Research Article

Logarithmic Coefficient Bounds and Coefficient Conjectures for Classes Associated with Convex Functions

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It is well-known that the logarithmic coefficients play an important role in the development of the theory of univalent functions. If \( \mathcal{A} \) denotes the class of functions \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) analytic and univalent in the open unit disk \( \mathbb{U} \), then the logarithmic coefficients \( \gamma_n(f) \) of the function \( f \in \mathcal{A} \) are defined by \( \log \left( \frac{f(z)}{z} \right) = 2 \sum_{n=1}^{\infty} \gamma_n(f) z^n \). In the current paper, the bounds for the logarithmic coefficients \( \gamma_n \) for some well-known classes like \( \mathcal{B}(1 + az) \) for \( a \in (0, 1) \) and \( \mathcal{B}^{\text{pul}}(1/2) \) were estimated. Further, conjectures for the logarithmic coefficients \( \gamma_n \) for functions \( f \) belonging to these classes are stated. For example, it is forecasted that if the function \( f \in \mathcal{B}(1 + az) \), then the logarithmic coefficients of \( f \) satisfy the inequalities \( |\gamma_n| \leq a(2n(n+1)), n \in \mathbb{N} \). Equality is attained for the function \( L_{\alpha,n} \), that is, \( \log \left( L_{\alpha,n}(z) \right) / z = 2 \sum_{n=1}^{\infty} \gamma_n(L_{\alpha,n}) z^n = (a/n(n+1)) z^n + \cdots, z \in \mathbb{U} \).

1. Introduction

Let \( \mathbb{U} = \{ z \in \mathbb{C} : |z| < 1 \} \) denote the open unit disk in the complex plane \( \mathbb{C} \). Let \( \mathcal{A} \) be the category of analytic functions \( f \) in \( \mathbb{U} \) for which \( f \) has the following representation:

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{U}.
\] (1)

Also, let \( \mathcal{D} \) be the subclass of \( \mathcal{A} \) consisting of all univalent functions in \( \mathbb{U} \). Then, the logarithmic coefficients \( \gamma_n \) of the function \( f \in \mathcal{D} \) are defined with the aid of the following series expansion:

\[
\log \left( \frac{f(z)}{z} \right) = 2 \sum_{n=1}^{\infty} \gamma_n(f) z^n, \quad z \in \mathbb{U}.
\] (2)

These coefficients play an important role for different estimates in the theory of univalent functions, and note that we use \( \gamma_n \) instead of \( \gamma_n(f) \). Kayumov [1] solved Brennan’s conjecture for conformal mappings with the help of studying the logarithmic coefficients. The significance of the logarithmic coefficients follows from Lebedev-Milin inequalities ([2], chapter 2; see also [3, 4]), where estimates of the logarithmic coefficients were applied to obtain bounds on the coefficients of \( f \). Milin [2] conjectured the inequality

\[
\sum_{n=1}^{m} \sum_{k=1}^{m} \left( k|\gamma_k|^2 - \frac{1}{k} \right) \leq 0, \quad n = 1, 2, 3, \cdots,
\] (3)

that implies Robertson’s conjecture [5] and hence Bieberbach’s conjecture [6], which was the well-known coefficient problem in the theory of univalent functions. De Branges...
[7] proved Bieberbach’s conjecture by establishing Milin’s conjecture. Recall that we can rewrite (2) in the power series form as follows:

\[
2 \sum_{n=1}^{\infty} \gamma_n z^n = a_2 z + a_3 z^2 + a_4 z^3 + \cdots - \frac{1}{2} (a_2 z + a_3 z^2 + a_4 z^3 + \cdots)^2
\]
\[
+ \frac{1}{3} (a_2 z + a_3 z^2 + a_4 z^3 + \cdots)^3 + \cdots, \quad z \in \mathbb{U},
\]

and equating the coefficients of \(z^n\) for \(n = 1, 2, 3\), it follows that

\[
\begin{align*}
2\gamma_1 &= a_2, \\
2\gamma_2 &= a_3 - \frac{1}{2} a_2^2, \\
2\gamma_3 &= a_4 - a_2 a_3 + \frac{1}{3} a_2^3.
\end{align*}
\]

(5)

If the functions \(f\) and \(g\) are analytic in \(\mathbb{U}\), the function \(f\) is called to be subordinate to the function \(g\), written \(f(z) < g(z)\), if there exists a function \(w\) analytic in \(\mathbb{U}\) with \(|w(z)| < 1, z \in \mathbb{U}\), and \(w(0) = 0\), such that \(f = g \circ w\). In particular, if \(g\) is univalent in \(\mathbb{U}\), then the following equivalence relationship holds true:

\[
f(z) < g(z) \iff f(0) = g(0), \quad f(\mathbb{U}) \subset g(\mathbb{U}).
\]

(6)

Using the principle of subordination, Ma and Minda [8] introduced the classes \(\delta^*(\varphi)\) and \(\mathcal{C}(\varphi)\), where we make here the weaker assumptions that the function \(\varphi\) is analytic in the open unit disk \(\mathbb{U}\) and satisfies \(\varphi(0) = 1\), such that it has a series expansion of the form

\[
\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots, z \in \mathbb{U}, \quad \text{with} \quad B_1 \neq 0.
\]

(7)

They considered the abovementioned classes as follows:

\[
\delta^*(\varphi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} < \varphi(z) \right\},
\]

\[
\mathcal{C}(\varphi) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} < \varphi(z) \right\},
\]

(8)

Some special subclasses of the class \(\delta^*(\varphi)\) and \(\mathcal{C}(\varphi)\) play a significant role in the Geometric Function Theory because of their geometric properties.

For example, taking \(\varphi(z) = (1 + Az)/(1 + Bz)\) where \(A \in \mathbb{C}, -1 \leq B \leq 0, A \neq B, \) we get the classes \(\delta^*[A, B]\) and \(\mathcal{C}[A, B]\), respectively (see also [9, 10]). The mentioned classes with the restriction \(-1 \leq B < A \leq 1\) reduce to the popular Janowski starlike and Janowski convex functions, respectively. By replacing \(A = 1 - 2\alpha\) and \(B = -1\), where \(0 \leq \alpha < 1\), we obtain the classes \(\delta^*(\alpha)\) and \(\mathcal{C}(\alpha)\) of the starlike functions of order \(\alpha\) and convex functions of order \(\alpha\), respectively. In particular, \(\delta^* = \delta^*(0)\) and \(\mathcal{C} = \mathcal{C}(0)\) are the class of starlike functions and of convex functions in the open unit disk \(\mathbb{U}\), respectively. Further, by altering \(A = \alpha\) and \(B = 0, \) where \(0 \leq \alpha < 1\), we get the classes \(\delta^*(1 + \alpha z)\) and \(\mathcal{C}(1 + \alpha z)\), which are the extensions of the classes \(\delta^*(1 + z)\) and \(\mathcal{C}(1 + z)\), respectively (see [11]), that is,

\[
\delta^*(1 + \alpha z) := \left\{ f \in \mathcal{A} : \left| \frac{zf'(z)}{f(z)} - 1 \right| < \alpha \right\},
\]

(9)

\[
\mathcal{C}(1 + \alpha z) := \left\{ f \in \mathcal{A} : \left| \frac{zf''(z)}{f'(z)} \right| < \alpha \right\},
\]

where \(0 < \alpha \leq 1\).

Supposing that \(\Psi_{\alpha,n} \in \delta^*(1 + az)\) is such that

\[
\frac{z^n \Psi_{\alpha,n}(z)}{\Psi_{\alpha,n}(\alpha)} = 1 + az^n, \quad n \in \mathbb{N},
\]

(10)

each function \(\Psi_{\alpha,n}\) is of the form

\[
\Psi_{\alpha,n}(z) = z \exp \left( \int_0^z 1 + az^n - 1 \frac{dt}{t} \right) = z + \frac{\alpha}{n(n + 1)} z^{n+1} + \cdots, \quad z \in \mathbb{U},
\]

(11)

and is the extremal function for various problems in \(\delta^*(1 + az)\). Also, suppose that \(L_{\alpha,n} \in \mathcal{C}(1 + az)\) is such that

\[
1 + \frac{z^n L_{\alpha,n}(z)'}{L_{\alpha,n}(\alpha)} = 1 + az^n, \quad n \in \mathbb{N}.
\]

(12)

Then, each function \(L_{\alpha,n}\) is of the form

\[
L_{\alpha,n}(z) = \int_0^z \exp \left( \int_0^x 1 + ax^n - 1 \frac{dt}{t} \right) dx.
\]

(13)

and plays as extremal function for some extremal problems in the set \(\mathcal{C}(1 + az)\).

Lately, Kanas et al. [12] introduced the categories \(\delta \mathcal{F}_{hpl}(s)\) and \(\mathcal{C}\mathcal{F}_{hpl}(s)\) by

\[
\delta \mathcal{F}_{hpl}(s) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} < q_s(z) := \frac{1}{(1-z)^s}, 0 < s \leq 1 \right\},
\]

(14)

\[
\mathcal{C}\mathcal{F}_{hpl}(s) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} < q_s(z) := \frac{1}{(1-z)^s}, 0 < s \leq 1 \right\},
\]

and obtained some geometric properties in these categories. Further, the functions
\[ \phi_{t,n}(z) = z \exp \left( \int_0^t \frac{q_n(t^n) - 1}{t} \, dt \right) = z + \frac{s}{n} z^{n+1} + \cdots, \quad z \in U, n \in \mathbb{N}, \]

\[ K_{t,n}(z) = \int_0^\infty \exp \left( \int_0^t \frac{q_n(t^n) - 1}{t} \, dt \right) \, dx = z + \frac{s}{n(n+1)} z^{n+1} + \cdots, \quad z \in \mathbb{D}, n \in \mathbb{N}, \quad (15) \]

play as extremal functions for some issues of the families \( \mathcal{S}_{\text{hpl}}(s) \) and \( \mathcal{C}_{\text{hpl}}(s) \), respectively.

Lately, several researchers have subsequently investigated same problems regarding the logarithmic coefficients and the coefficient problems [9, 13–23], to mention a few of them. For instance, the rotation of the Koebe function \( k(z) = z (1 - e^{\theta z})^{-2} \) for each \( \theta \in \mathbb{R} \) has the logarithmic coefficients \( \gamma_n = e^{\theta n}/n, n \geq 1 \). If \( f \in \mathcal{S} \), then applying the Bieberbach inequality for the first relation of (5), it follows that \( |\gamma_1| \leq 1 \), and using the Fekete-Szegő inequality for the second relation of (5) (see [24], Theorem 3.8) leads to

\[ |\gamma_2| = \frac{1}{2} (a_3 - \frac{1}{2} a_2^2) \leq \frac{1}{2} (1 + 2e^{-2}) = 0.635\ldots. \quad (16) \]

It was established in ([25], Theorem 4) that the logarithmic coefficients \( \gamma_n \) of \( f \in \mathcal{S} \) satisfy the inequality

\[ \sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{n^2}{6}, \quad (17) \]

and the equality is obtained for the Koebe function. For \( f \in \mathcal{S}^* \), the inequality \( |\gamma_1| \leq 1/n \) holds but is not true for the full class \( \mathcal{S} \), even in order of magnitude (see [24], Theorem 8.4).

In 2018, some first logarithmic coefficients \( \gamma_n \) were estimated for special subclasses of \textit{close-to-convex functions} in [15, 20]. However, the problem of the best upper bounds for the logarithmic coefficients of univalent functions for \( n \geq 3 \) is presumably still a concern. In [13], the authors obtained the bounds of logarithmic coefficients \( \gamma_n, n \in \mathbb{N} \), for the general class \( \mathcal{S}^*(\varphi) \), and the bounds of the logarithmic coefficients \( \gamma_n \) when \( n = 1, 2, 3 \) for the class \( \mathcal{C}(\varphi) \), while the estimated bounds would generalize many of the previous outcomes.

In the present study, which is motivated essentially by the recent works [13, 16], the bounds for the logarithmic coefficients \( \gamma_n, n \in \mathbb{N} \), of the class \( \mathcal{C}(1 + az) \) for \( a \in (0, 1] \) and \( \mathcal{S}_{\text{hpl}}(1/2) \) were estimated. Further, conjectures for the logarithmic coefficients \( \gamma_n \) for \( f \) belonging to these classes are stated.

### 2. Main Results

First, we will obtain the bounds for \( \gamma_n \) of the classes \( \mathcal{S}^*(1 + az) \) and \( \mathcal{C}(1 + az) \) for \( a \in (0, 1] \). In this regard, the following outcomes will be employed in the key results.

**Lemma 1** (see [13], Theorem 1). Let \( f \in \mathcal{S}^*(\varphi) \). If \( \varphi \) is convex univalent, then the logarithmic coefficients of \( f \) satisfy the following inequalities:

\[ |\gamma_n| \leq \frac{|B_1|}{2n}, \quad n \in \mathbb{N}, \quad (18) \]

\[ \sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{4}{n^2}. \quad (19) \]

The inequalities in (18) and (19) are sharp, such that for any \( n \in \mathbb{N} \), there exist the function \( f_n \) given by \( z f_n(z)/f_n(z) = \varphi(z^n) \) and the function \( f \) given by \( z f'(z)/f(z) = \varphi(z) \), respectively, for those equalities we obtain.

**Lemma 2** (see [13], Theorem 2). Let \( f \in \mathcal{C}(\varphi) \). Then, the logarithmic coefficients of \( f \) satisfy the inequalities

\[ |\gamma_1| \leq \frac{|B_1|}{4}, \quad (20) \]

\[ |\gamma_2| \leq \begin{cases} \frac{|B_1|}{12}, & \text{if } |4B_2 + B_1^2| \leq 4|B_1|, \\ \frac{|4B_2 + B_1^2|}{48}, & \text{if } |4B_2 + B_1^2| > 4|B_1|, \end{cases} \quad (21) \]

and if \( B_1, B_2, \) and \( B_3 \) are real values, then

\[ |\gamma_3| \leq \frac{|B_1|}{24} H(q_2; q_2), \quad (22) \]

where \( H(q_2; q_2) \) is given in ([26], Lemma 2) or ([9], Lemma 5), \( q_1 = (B_1 + (4B_2/B_1))/2, \) and \( q_2 = (B_2 + (2B_3/B_1))/2 \). The bounds (20) and (21) are sharp.

**Lemma 3** (see [18], Theorem 30). If \( f \in \mathcal{C}_{\text{hpl}}(1/2) \), then

\[ |\gamma_1| \leq \frac{1}{8}, \quad |\gamma_2| \leq \frac{1}{24}, \quad |\gamma_3| \leq \frac{1}{48}. \quad (23) \]

The first two bounds are sharp for \( f = \mathcal{K}_{1/2,1} \) and \( f = \mathcal{K}_{1/2,2} \), respectively.

If we consider Lemma 1 with the function \( \varphi(z) = 1 + az \), then we immediately get the next result:

**Theorem 4.** If \( f \in \mathcal{S}^*(1 + az) \), then

\[ |\gamma_n| \leq \frac{a}{2n}, \quad n \in \mathbb{N}, \quad (24) \]

\[ \sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{a}{4}. \]

These inequalities are sharp for \( f = \Psi_{a,n} \) and \( f = \Psi_{a,1} \), respectively.
Corollary 5. Let \( f \in \mathcal{C}(1 + az) \). Then, the logarithmic coefficients of \( f \) satisfy the inequalities
\[
|y_1| \leq \frac{\alpha}{4},
\]
\[
|y_2| \leq \frac{\alpha}{12},
\]
\[
|y_3| \leq \frac{\alpha}{24}.
\]
(25)

Equalities in these inequalities are attained for the functions \( L_{n,n} \) for \( n = 1, 2, 3 \), respectively.

Proof. For \( \varphi(z) = 1 + az \), where \( B_1 = a, B_2 = B_3 = 0 \), in Theorem 6, we obtain the required result. Also, since
\[
\log \frac{L_{a,1}(z)}{z} = 2 \sum_{n=1}^{\infty} y_n(L_{a,1})z^n = \frac{\alpha}{2}z^2 + \mathcal{O}(z^3), \quad z \in \mathbb{U},
\]
\[
\log \frac{L_{a,2}(z)}{z} = 2 \sum_{n=1}^{\infty} y_n(L_{a,2})z^n = \frac{\alpha}{6}z^3 + \mathcal{O}(z^4), \quad z \in \mathbb{U},
\]
\[
\log \frac{L_{a,3}(z)}{z} = 2 \sum_{n=1}^{\infty} y_n(L_{a,3})z^n = \frac{\alpha}{12}z^4 + \mathcal{O}(z^5), \quad z \in \mathbb{U},
\]
(26)

it follows that these inequalities are attained for the functions \( L_{n,n} \) for \( n = 1, 2, 3 \), respectively.

Theorem 6. Let \( f \in \mathcal{C}(1 + az) \). Then, the logarithmic coefficients of \( f \) satisfy the inequalities
\[
|y_n| \leq \frac{\alpha}{4n}, \quad n \in \mathbb{N}.
\]
(27)

This inequality is sharp for \( |y_1| \) for the function \( L_{a,1} \).

Proof. If \( f \in \mathcal{C}(1 + az) \), this is equivalent to \( f \in \mathcal{A} \) and
\[
1 + \frac{zf''(z)}{f'(z)} < 1 + az = \varphi_a(z).
\]
(28)

If we define \( p(z) = zf'(z)ff(z) \), then \( p(0) = 1 \), and the above subordination relation can be written as
\[
p(z) + \frac{zf''(z)}{p(z)} < \varphi_a(z).
\]
(29)

Supposing that the function \( \psi_a \) satisfies the differential equation
\[
\psi_a(z) + \frac{z\psi_a'(z)}{\psi_a(z)} = \varphi_a(z), \quad \psi_a(0) = 1,
\]
(30)

we will prove that \( \psi_a \) is a convex univalent function in \( \mathbb{U} \).

The function \( \varphi_a \) has positive real part in \( \mathbb{U} \) whenever \( a \in (0, 1] \). Therefore, using ([27], Theorem 1) for \( \beta = 1, \gamma = 0, \) and \( c = 1 \), it follows that the solution \( \psi_a \) of the differential equation (30) is analytic in \( \mathbb{U} \), with \( \psi_a(z) > 0 \) for all \( z \in \mathbb{U} \), and
\[
\psi_a(z) = H(z) \left( \int_0^z \frac{H(t)}{t} \, dt \right)^{-1} = \frac{az \exp(az)}{\exp((az) - 1)}
\]
(31)

where
\[
H(z) = z \exp \left( \int_0^1 \frac{\varphi_a(t) - 1}{t} \, dt \right) = z \exp(az),
\]
(32)

and all powers are considered at the principal branch, that is, \( \log 1 = 0 \).

Since \( \varphi_a \) is convex and \( \psi_a \) is analytic with \( \text{Re} \psi_a(z) > 0 \) for all \( z \in \mathbb{U} \), using [28] (Theorem 3.2i) for \( n = 1 \), we deduce that \( \psi_a \) is univalent in \( \mathbb{U} \). Moreover, from Figure 1 made with MAPLE software, we get
\[
\Psi(z) := \text{Re} \left( 1 + \frac{z\psi_a'(z)}{\psi_a(z)} \right) > 0, \quad z \in \mathbb{U},
\]
(33)

and \( \psi_a'(0) = a/2 \neq 0 \), so \( \psi_a \) is a convex function. Hence, it follows that \( \psi_a \) is a convex univalent function in \( \mathbb{U} \).

Therefore, according to [28] (Theorem 3.2i), the differential subordination (29) implies
\[
p(z) < \psi_a(z),
\]
(34)

for all \( 0 < a \leq 1 \), and \( \psi_a \) is the best dominant. Thus,
\[
\frac{zf'(z)}{f(z)} < \psi_a(z),
\]
(35)

for all \( 0 < a \leq 1 \). Hence,
\[
\mathcal{C}(1 + az) \subset \mathcal{S}(\psi_a).
\]
(36)

From the above relation, we get
\[
\sup \{ |\gamma_n(f)| : f \in \mathcal{C}(1 + az) \} \leq \sup \{ |\gamma_n(f)| : f \in \mathcal{S}(\psi_a) \}.
\]
(37)

Hence, from Lemma 1, we obtain
\[
\sup \{ |\gamma_n(f)| : f \in \mathcal{C}(1 + az) \} \leq \frac{\alpha}{4n}.
\]
(38)

Therefore, for \( f \in \mathcal{C}(1 + az) \) and for all \( n \in \mathbb{N} \), we conclude that
\[
|\gamma_n(f)| \leq \frac{\alpha}{4n},
\]
(39)
Remark 7. If we compare the results of Corollary 5 with those of Theorem 6, then we conclude that the results of Theorem 6 are not the best possible. We conjecture that if the function $f \in C(1 + \alpha z)$, then the logarithmic coefficients of $f$ satisfy the inequalities

$$|\gamma_n| \leq \frac{\alpha}{2n(n + 1)}, \quad n \in \mathbb{N}. \quad (40)$$

Equality is attained for the function $L_{\alpha,n}$, that is,

$$\log \frac{L_{\alpha,n}(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n \left(L_{\alpha,n}z^n = \frac{\alpha}{n(n + 1)} z^n + \cdots, \quad z \in \mathbb{U}. \quad (41)$$

Theorem 8. Let $f \in C^{\mathcal{V}_{-bpl}}(1/2)$. Then, the logarithmic coefficients of $f$ satisfy the inequalities

$$|\gamma_n| \leq \frac{1}{6n}, \quad n \in \mathbb{N}. \quad (42)$$

This inequality is sharp for $|\gamma_n|$ for the function $K_{1/2,1}$.

Proof. Letting $f \in C^{\mathcal{V}_{-bpl}}(1/2)$, it follows that

$$1 + \frac{zf''(z)}{f'(z)} < \frac{1}{\sqrt{1 - z}} = q_{1/2}(z). \quad (43)$$

Suppose that $p$ satisfies the differential equation

$$p(z) + \frac{zp'(z)}{p(z)} = \frac{1}{\sqrt{1 - z}}. \quad (44)$$

If we define $p(z) = zf'(z)/f(z)$, then the subordination (43) can be rewritten as

$$p(z) + \frac{zp'(z)}{p(z)} < q_{1/2}(z). \quad (45)$$

According to the inequality (20) of [12] (Theorem 2.3), the function $q_{1/2}$ is analytic with positive real part in $\mathbb{U}$. Therefore, using [27] (Theorem 1) for $\beta = 1$, $\gamma = 0$, and $c = 1$, it follows that the solution $p$ of the differential equation (44) is analytic in $\mathbb{U}$ with $\Re p(z) > 0$, $z \in \mathbb{U}$, and

$$p(z) = H(z) \left( \int_0^z \frac{H(t)}{t} \, dt \right)^{-1} = \frac{4z}{\left(1 + \sqrt{1 - z} \right)^2} \left[ \frac{1}{8} \left(1 + \sqrt{1 - z} \right) - 6 \ln \left(1 + \sqrt{1 - z} \right) + 4 + 8 \ln 2 \right]
= 1 + \frac{1}{4} \sqrt{z} + \cdots, \quad z \in \mathbb{U}. \quad (46)$$
and we obtain the result. This completes the proof. □

Remark 9. If we compare the results of Lemma 1 with those of Theorem 8, then we conclude that the results of Theorem 8 are not the best possible. We conjecture that if the function $f \in \mathcal{V}^{hpl}_{1/2}$, then the logarithmic coefficients $f$ satisfy the inequalities

$$|\gamma_n| \leq \frac{1}{4n(n+1)}, \quad n \in \mathbb{N}. \quad (52)$$

Equality is attained for the function $K_{1/2, n}$, that is,

$$\log \frac{K_{1/2, n}}{z} = 2 \sum_{n=1}^{\infty} \gamma_n(K_{1/2, n})z^n = \frac{1}{2n(n+1)}z^n + \cdots, \quad z \in \mathbb{U}. \quad (53)$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] I. R. Kayumov, “On Brennan’s conjecture for a special class of functions,” Mathematical Notes, vol. 78, no. 3-4, pp. 498–502, 2005.
[2] I. M. Milin, “Univalent functions and orthonormal systems,” Translations of Mathematical Monographs, vol. 49, 2008.
[3] I. M. Milin, “On a property of the logarithmic coefficients of univalent functions,” in Metric Questions in the Theory of Functions, pp. 86–90, Naukova Dumka, Kiev, 1980.
[4] I. M. Milin, “On a conjecture for the logarithmic coefficients of univalent functions,” Zap. Nauch. Semin. Leningr. Otd. Mat. Inst. Steklova, vol. 125, pp. 135–137, 1983.
[5] M. S. Robertson, “A remark on the odd schlicht functions,” Bulletin of the American Mathematical Society, vol. 42, no. 6, pp. 366–371, 1936.
[6] L. Bieberbach, “Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln,” Sitzungsberichte Preussische Akademie der Wissenschaften, vol. 138, pp. 940–955, 1916.
[7] L. De Branges, “A proof of the Bieberbach conjecture,” Acta Mathematica, vol. 154, no. 1-2, pp. 137–152, 1985.
[8] W. C. Ma and D. Minda, “A unified treatment of some special classes of univalent functions,” in Proceedings of the Conference on Complex Analysis (Tianjin, 1992), pp. 157–169, Cambridge, MA, USA, 1992.

[9] D. AliMohammadi, N. E. Cho, E. A. Adegani, and A. Motamednezhad, “Argument and coefficient estimates for certain analytic functions,” Mathematics, vol. 8, no. 1, p. 88, 2020.

[10] J. Sokół, "A certain class of starlike functions," Computers & Mathematics with Applications, vol. 62, no. 2, pp. 611–619, 2011.

[11] R. Singh, "On a class of star-like functions," Compositio Mathematica, vol. 19, pp. 78–82, 1967.

[12] S. Kanas, V. S. Mashh, and A. Ebadian, "Relations of a planar domains bounded by hyperbolas with families of holomorphic functions," Journal of Inequalities and Applications, vol. 246, 2019.

[13] E. A. Adegani, N. E. Cho, and M. Jafari, "Logarithmic coefficients for univalent functions defined by subordination," Mathematics, vol. 7, p. 408, 2019.

[14] E. A. Adegani, N. E. Cho, A. Motamednezhad, and M. Jafari, "Bi-univalent functions associated with Wright hypergeometric functions," Journal of Computational Analysis and Applications, vol. 28, pp. 261–271, 2020.

[15] M. F. Ali and A. Vasudevarao, "On logarithmic coefficients of some close-to-convex functions," Proceedings of the American Mathematical Society, vol. 146, pp. 1131–1142, 2018.

[16] D. AliMohammadi, E. A. Adegani, T. Bulboacă, and N. E. Cho, "Logarithmic coefficients for classes related to convex functions," Bulletin of the Malaysian Mathematical Sciences Society, vol. 44, pp. 2659–2673, 2021.

[17] N. E. Cho, B. Kowalczyk, O. S. Kwon, A. Lecko, and Y. J. Sim, "On the third logarithmic coefficient in some subclasses of close-to-convex functions," Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas, vol. 114, p. 54, 2020.

[18] A. Ebadian, T. Bulboacă, N. E. Cho, and E. A. Adegani, "Coefficient bounds and differential subordinations for analytic functions associated with starlike functions," Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas, vol. 114, p. 128, 2020.

[19] R. Kargar, "On logarithmic coefficients of certain starlike functions related to the vertical strip," Journal of Analysis, vol. 27, no. 4, pp. 985–995, 2019.

[20] U. P. Kumar and A. Vasudevarao, "Logarithmic coefficients for certain subclasses of close-to-convex functions," Monatshefte für Mathematik, vol. 187, no. 3, pp. 543–563, 2018.

[21] M. Obrodović, S. Ponnusamy, and K.-J. Wirths, "Logarithmic coefficients and a coefficient conjecture for univalent functions," Monatshefte für Mathematik, vol. 185, no. 3, pp. 489–501, 2018.

[22] S. Ponnusamy, N. L. Sharma, and K.-J. Wirths, "Logarithmic coefficients of the inverse of univalent functions," Results in Mathematics, vol. 73, no. 4, p. 160, 2018.

[23] S. Ponnusamy, N. L. Sharma, and K.-J. Wirths, "Logarithmic coefficients problems in families related to starlike and convex functions," Journal of the Australian Mathematical Society, vol. 109, no. 2, pp. 230–249, 2019.

[24] P. L. Duren, Univalent functions, Springer, Amsterdam, 1983.

[25] P. L. Duren and Y. J. Leung, "Logarithmic coefficients of univalent functions," Journal d’Analyse Mathématique, vol. 36, no. 1, pp. 36–43, 1979.

[26] D. V. Prokhorov and J. Szynal, "Inverse coefficients for \((\alpha, \beta)\) -convex functions," Annales Universitatis Mariae Curie-Skłodowska, vol. 35, pp. 125–143, 1981.

[27] S. S. Miller and P. T. Mocanu, "Univalent solutions of Briot-Bouquet differential equations," Journal of Differential Equations, vol. 56, no. 3, pp. 297–309, 1985.

[28] S. S. Miller and P. T. Mocanu, Differential subordinations: theory and applications, Marcel Dekker Inc., New York, NY, USA, 2000.