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Semianalyticity of isoperimetric profiles

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ABSTRACT: It is shown that, in dimensions $< 8$, isoperimetric profiles of compact real analytic Riemannian manifolds are semi-analytic.

RESUMÉ : On montre qu’en dimensions $< 8$, le profil isopérimétrique d’une variété riemannienne compacte est semi-analytique.

1 Introduction

1.1 The problem

Let $M$ be a compact real analytic Riemannian manifold. We are concerned with the regularity of the isoperimetric profile of $M$.

Given $0 < v < \text{vol}(M)$, consider all integral currents in $M$ with volume $v$. Define $I_M(v)$ as the least upper bound of the boundary volumes of such currents. In this way, one gets a function $I_M : (0, \text{vol}(M)) \to \mathbb{R}_+$ called the isoperimetric profile of $M$. In fact, for each $0 < v < \text{vol}(M)$, there exist currents in $M$ with volume $v$ and boundary volume $I_M(v)$. Such minimizing currents will be called bubbles, for short.

Here is a typical example. For $a > 0$, let $S$ denote the circle of length $2\pi$. Let $M = S \times S$. Then the isoperimetric profile of $M$ is easily computed to be

$$I_M(v) = \begin{cases} \sqrt{4\pi v} & \text{for } 0 < v \leq 4\pi, \\ 4\pi & \text{for } 4\pi \leq v \leq 4\pi(\pi - 1), \\ \sqrt{4\pi(4\pi^2 - v)} & \text{for } 4\pi(\pi - 1) \leq v < 4\pi^2. \end{cases}$$

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This is proven as follows. In 2 dimensions, the boundaries of these bubbles are smooth, they have constant geodesic curvature, therefore they lift to disjoint unions of circles of equal radii or lines in $\mathbb{R}^2 = \tilde{M}$. It follows that bubbles are either round disks or annuli bounded by parallel geodesics, or complements of such. There remains to minimize boundary length among these three families.

**Question 1** For general real analytic manifolds, is it true that bubbles fall into finitely many analytic families, and that the profile is piecewise analytic?

This has been proven in [7] in dimension 2 only.

### 1.2 The results

First, in a neighborhood of zero.

**Theorem 1** Let $M$ be a compact real analytic Riemannian manifold. There exists $\epsilon > 0$ such that $I_M$ is real analytic on $(0, \epsilon)$.

The isoperimetric profile of Euclidean space $\mathbb{R}^n$ is $I_{\mathbb{R}^n}(v) = n(\omega_n)^{1/n}v^{n-1/n}$, where $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$. In a curved manifold, $I_M(v)^{n-1} \sim n(\omega_n)^{1/n}v^{n-1/n}$ as $v$ tends to 0.

**Question 2** For a compact analytic Riemannian $n$-manifold, is $I_M(v)$ an analytic function of $v^{1/n}$ on $[0, \epsilon)$?
We have only a partial answer.

**Theorem 2** Let \( M \) be a compact real analytic Riemannian manifold. Assume that the absolute maxima of scalar curvature are nondegenerate critical points. Then there exists an analytic function \( f \) defined in a neighborhood of 0 such that \( I_M(v) = f(v^{1/n}) \) for \( v \) small enough.

Away from 0, our result also requires an extra assumption.

**Theorem 3** Let \( M \) be a compact real analytic Riemannian manifold. Let \( 0 < v_0 < \text{vol}(M) \). Assume that all bubbles of volume \( v_0 \) are smooth. Then \( I_M \) is semi-analytic on a neighborhood of \( v_0 \).

Since bubbles are known to be smooth in dimensions \(< 8, [1]\), it follows that

**Corollary 4** If the dimension of \( M \) is less than 8, \( I_M \) is semianalytic on \([0, \text{vol}(M)]\).

**Question 3** Our method of proof relies on the regularity of bubbles. Can this be circumvented?

### 2 Proof of Theorem 1

It relies on results from [5]. There, it is shown that small bubbles is a subset of a smooth finite dimensional family of domains called pseudo-balls. We merely need show that if the metric is real analytic, pseudo-balls form a compact, finite dimensional real analytic set, on which the volume and boundary volume functions are real analytic.

Pseudo-balls are solutions of a differential equations which is weaker than constancy of mean curvature, but to which the implicit function theorem can be applied. Specifically, for \( k \geq 0 \), consider the bundle \( \mathcal{F}^k \to M \) whose fiber at \( p \in M \) consists in \( C^{k,\alpha} \) functions on the unit sphere in the tangent space \( T_pM \). There is a smooth map \( \Phi : \mathbb{R} \times \mathcal{F}^2 \to \mathcal{F}^0 \) with the following properties.

1. Let \( r > 0, p \in M \) and \( x \in \mathcal{F}_p^2 \). If the graph, in polar coordinates, of \( r(1 + x) \) has constant mean curvature, then \( \Phi(r, p, x) = 0 \).
2. For all \( p \in M \), \( \Phi(0, p, 0) = 0 \).
3. The differential of \( \Phi \) restricted to the fibers is an isomorphism.

**Lemma 1** \( \Phi : \mathbb{R} \times \mathcal{F}^2 \to \mathcal{F}^0 \) is a real analytic map.
Theorem 2

For \((p, \rho) \in M \times \mathbb{R}_+\), let \(\beta(p, \rho)\) denote the pseudo-ball defined by

\[
\beta(p, \rho^*(\mathcal{N}^+(p, r))) = \mathcal{N}^+(p, r).
\]

Since \(\rho^*(\mathcal{N}^+(p, r)))^{1/n} \sim \omega_n^{1/n} r\) is a 1 to 1 analytic function of \(r\), the notation is unambiguous. Let

\[
f_\rho(p) = f(\rho^*(\beta(p, \rho))) = f_\rho(p).
\]

Then \(f\) is real analytic. Furthermore, among pseudo-balls of volume \(v = \rho^n\), bubbles are characterized as minima of \(f_\rho\). The following expansion

\[
f_\rho(p) = c_n \rho^{n-1}(1 - \frac{1}{2n(n+2)} \omega_n^{-2/n}Sc(p)\rho^2 + O(\rho^4))
\]

is computed in [5], Lemma 3.6, compare [9]. If the absolute maxima \(p_1, \ldots, p_k\) of the scalar curvature function \(Sc\) are non degenerate critical points, then each of them deforms into a critical point \(p_i(\rho)\) of \(f_\rho\) that depends analytically on \(\rho\). Therefore (Theorem 8 in [5]),

\[
I_M(\rho^n) = \min_{i=1, \ldots, k} f_\rho(p_i(\rho)).
\]

There exists \(\epsilon > 0\) and \(i\) such that the minimum is equal to \(f_\rho(p_i(\rho))\) for all \(\rho \in [0, \epsilon)\). Indeed, otherwise, some function \(f_\rho(p_i(\rho)) - f_\rho(p_j(\rho))\) would change sign infinitely many times near 0, contradiction. Thus the right hand side is analytic on \([0, \epsilon)\). This completes the proof of Theorem 2.
4 Proof of Theorem 3

We follow M. Tamm’s strategy, [8]. We aim at including bubbles in a parametrized analytic variety. We shall first do this in a neighborhood of a smooth bubble $B$ with volume $v_0$. Our first candidate is the set of domains whose boundary is a graph in normal exponential coordinates to $\partial B$ and has constant mean curvature. To decide whether this set is a submanifold in some function space, let us examine the mean curvature operator and its linearization.

4.1 Pseudo-bubbles

Let $B$ be a smooth bubble with volume $v_0$. Let $H_B : C^{2,\alpha}(\partial B) \to C^{0,\alpha}(\partial B)$ denote the operator which to a function $u$ on $\partial B$ associates the mean curvature of the graph of $u$ in normal exponential coordinates to $\partial B$. In particular, $H_B(0) = H(\partial B) = h_B$ is the constant mean curvature of $\partial B$. Let $L_B : C^{2,\alpha}(\partial B) \to C^{0,\alpha}(\partial B)$ denote its linearization at 0 (sometimes called the Jacobi operator).

Lemma 2 For all $v \in C^{2,\alpha}(\partial B)$

$$L_B(v) = -\Delta_{\partial B} v - (||II_{\partial B}||^2 + \text{Ric}(\nu)) v,$$

(1)

where $\Delta_{\partial B} v = \text{div}(\nabla v)$ is the Laplace operator on $\partial B$ (with negative spectrum when taken on the round sphere), $||II_{\partial B}||^2$ is the Hilbert-Schmidt squared norm ($\text{tr}(A^t A)$ for a square matrix $A$) of the second fundamental form of $\partial B$ and $\text{Ric}(\nu)$ is the Ricci curvature of the ambient manifold $M$ in the direction $\nu$ of the unit outward normal vector to $\partial B$ evaluated at a point of $\partial B$.

Proof: We recall here formula (6) of 3.3 of [6]

$$H(u) = -\text{div}(\nabla u) - \frac{\nabla_{g_u} u}{W_u} - \frac{1}{W^2_u} < \nabla_{g_u} u, \frac{\nabla_{g_u} u}{W_u} >_{g_u}$$

$$+ \frac{u^2}{W^3_u} \text{tr}^u_{\theta} \left( \nabla_{g_u} u, \nabla_{g_u} u \right)$$

$$- \frac{1}{W^2_u} H^u_{\theta}(u) + \frac{1}{W_u} < \nabla_{g_u} \left( \frac{1}{W_u} \right), \frac{\nabla_{g_u} u}{W_u} >_{g_u} .$$

Here $\theta$ denotes the gradient of the signed distance function to $\partial B$. For the meaning of the other terms involved in (2), see [6]. The operator $L_B$ satisfies then $H(tv) = H_{\theta B} + tL_B + O(t^2)$. At first observe that only the first and fourth term of (2) contribute to $L_B$. Denoting by $U(r) = \nabla \theta$ the shape operator of the equidistant hypersurfaces to $\partial B$ at distance $r$, we have $U(r) = U_0 + U_1 r + \cdots$, hence by using the Riccati equation satisfied by $U$ (see [4]) we can compute $U_1 = -U_0^2 - R$ where $R$ is the curvature tensor. Now taking traces we get $H^u_{\theta} = \text{tr}(-U) = H_{\partial B, \text{ext}} + (\text{tr}(U_0^2) + \text{Ric}) t + \cdots$ where $H_{\partial B, \text{ext}}$ is the outward mean curvature of the boundary of $B$. Finally (1) follows easily. □

$L_B$ is a selfadjoint elliptic operator, which has a discrete spectrum. Let $K_B = \text{kernel}(L_B), m(B) = \text{dim}(K_B)$. 

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If $L_B$ were invertible (i.e. $m(B) = 0$), the implicit function theorem would imply that nearby domains with constant mean curvature boundary come in one analytic family parametrized by the value of mean curvature.

Unfortunately, $L_B$ is not always invertible. Therefore, instead of solving $H(u) = h$, $h \in \mathbb{R}$, we shall solve

$$\Phi_B(u, h) = P_B(H_B(u) - h) = 0,$$

where $P_B$ is the orthogonal projection onto the $L^2$-orthogonal complement $K^\perp_B$ of $K_B$ in $C^{0,\alpha}(\partial B)$. Then

$$\Phi_B : C^{2,\alpha}(\partial B) \times \mathbb{R} \to K^\perp_B$$

is a real analytic map, whose linearization at 0 is $P_B \circ L_B$. By construction, it is onto. In fact, the restriction of $P_B \circ L_B$ to $K^\perp_B$ is an isomorphism (see for example [2], page 464). Note that $\Phi_B(0, h_B) = 0$. The following variant of the implicit function theorem provides us with an open neighborhood $U_B$ of $(0, h_B)$ in $C^{2,\alpha}(\partial B) \times \mathbb{R}$ in which the solutions of $\Phi_B(u, h) = 0$ form a real analytic submanifold. We shall call such solutions $B$-pseudo-bubbles.

\textbf{Lemma 3} Let $E$, $P$ and $F$ be real analytic Banach manifolds, let $e_0 \in E$, $f_0 \in B$, $p_0 \in P$ be such that $\Phi(e_0, p_0) = f_0$. Let $\Phi : E \times P \to B$ be a real analytic map. Assume that the differential $d\Phi$ of $\Phi$ at $(e_0, p_0)$ in the direction of $E$ has a finite dimensional kernel $K \subset T_{e_0}E$, which admits a closed complement $K^\perp$. Assume that the restriction of $d\Phi$ to $K^\perp$ is invertible. Then, in a neighborhood of $(e_0, p_0)$ the solutions of equation $\Phi(e, p) = f_0$ form a real analytic submanifold parametrized by a neighborhood of $(0, p_0)$ in $K \times P$.

\textbf{Proof:} Apply the implicit function theorem to $\Psi : E \times P \to F \times K$ defined by $\Psi(e, p) = (\Phi(e, p), \pi_K(e))$ where $\pi_K$ is a local submersion onto $K$. \hfill \blacksquare

\subsection{4.2 Compactness in $C^{2,\alpha}$-topology}

Let us define the $C^{2,\alpha}$-topology on the space of domains with smooth boundary as follows: as neighborhoods of a smooth domain $\beta$, take all domains $S$ whose boundary is the graph, in normal exponential coordinates, of a $C^{2,\alpha}$-small function on $\partial \beta$. Using a result from [6], we show that on smooth bubbles with volume close to $v_0$, the topologies induced by the $C^{2,\alpha}$-topology on smooth domains and the flat topology on currents coincide.

\textbf{Lemma 4} Let $B$ be a bubble of volume $v_0$. For all $\delta > 0$, there exist $\epsilon > 0$ such that if $\beta$ is a bubble of volume $\in [v_0 - \epsilon, v_0 + \epsilon]$ with $\text{vol}(\beta \Delta B) < \epsilon$, then there exists a smooth function $u$ on $\partial B$ with $\| u \|_{C^{2,\alpha}} < \delta$ such that $\partial \beta$ is the graph in normal exponential coordinates of $u$. Conversely, the graph of a $C^{2,\alpha}$-small function on $\partial B$ bounds a current which is close to $B$ volumewise.

\textbf{Proof:} By contradiction. Otherwise, there exists a sequence $\beta_j$ of bubbles with $\text{vol}(\beta_j) \to v_0$ and $\text{vol}(\beta_j \Delta B) \to 0$ such that $\partial \beta_j$ is not the normal exponential graph of a $C^{2,\alpha}$-small function on $\partial B$. \hfill \blacksquare
Theorem 1 of [6] asserts that for $j$ large enough, $\partial \beta_j$ is the graph in normal exponential coordinates of a function $u_j$ on $\partial B$ whose $C^{2,\alpha}$-norm tends to zero, contradiction. The last statement is obvious.

Lemma 5 Assume that all volume $v_0$ bubbles in $M$ are smooth. Then there exists $\epsilon > 0$ such that if $\beta$ is a bubble of volume $\in [v_0 - \epsilon, v_0 + \epsilon]$, then $\beta$ is smooth, and $(\beta, h_\beta)$ belongs to the open set $U(B)$ for some volume $v_0$ bubble $B$.

Proof: By contradiction. Otherwise, there exists a sequence $\beta_j$ of bubbles with $\operatorname{vol}(\beta_j) \to v_0$ that avoids all $U(B)$. By compactness of integral currents of bounded boundary volume, we can assume that $\beta_j$ converges in flat norm to some integral current $B$. By continuity of volume and semi-continuity of boundary volume, $B$ is a bubble of volume $v_0$. By assumption, $B$ is smooth. Theorem 1 of [6] asserts that for $j$ large enough, $\partial \beta_j$ is the graph in normal exponential coordinates of a $C^{2,\alpha}$-small smooth function $u_j$ on $\partial B$. Therefore $\beta_j$ is smooth and $(\beta_j, h_{\beta_j})$ belongs to $U(B)$, contradiction.

Lemma 6 Assume that all volume $v_0$ bubbles in $M$ are smooth. Then there exists $\epsilon > 0$ such that the set $B$ of pseudo-bubbles with volumes $\in [v_0 - \epsilon, v_0 + \epsilon]$ is contained in a finite union of compact semi-analytic pieces of finite dimensional real analytic manifolds, on which the volume and boundary volume functions are real analytic.

Proof: It was just proven that the set $B$ of bubbles with volumes $\in [v_0 - \epsilon, v_0 + \epsilon]$ is compact in flat topology and the set $\mathcal{BH} = \{(B, h_B) | B \in B\}$ is covered by the sets $U(B)$. According to Lemma 4, it is compact in $C^{2,\alpha}$-topology as well. Therefore, $\mathcal{BH}$ can be covered with finitely many open sets $U(B_1),...,U(B_N)$. The set $\Psi B_i$ of $B_i$-pseudo-bubbles in $U(B_i)$ is an analytic submanifold. There exist compact semi-analytic subsets $W_i \subset \Psi B_i$ which suffice to cover $\mathcal{BH}$.

The proof of Theorem 3 is completed in the same manner as the proof of Theorem 1. In dimensions less than 8, F. Almgren has shown that all bubbles are smooth. Therefore the profile is semi-analytic in a neighborhood of every point of the closed interval $[0, \operatorname{vol}(M)]$. It follows that it is semi-analytic on this interval. This proves Corollary 4.

References

[1] Frederick J. Almgren. Existence and regularity almost everywhere of solutions to elliptic variational problems with constraints. Number 165 in Memoirs. Amer. Math. Soc., Providence, R.I., 1976.

[2] Arthur L. Besse. Einstein Manifolds, volume 10 of Ergebnisse der Math. Grenz. Springer Verlag, 1987.

[3] Edward Bierstone and Pierre D. Milman. Semianalytic and subanalytic sets. Inst. Hautes Études Sci. Publ. Math., 67:5–42, 1988.
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