The equivalence between doubly nonnegative relaxation and semidefinite relaxation for binary quadratic programming problems✩

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Abstract

It has recently been shown (Burer, Math. Program Ser. A 120:479-495, 2009) that a large class of NP-hard nonconvex quadratic programming problems can be modeled as so called completely positive programming problems, which are convex but still NP-hard in general. A basic tractable relaxation is gotten by doubly nonnegative relaxation, resulting in a doubly nonnegative programming. In this paper, we prove that doubly nonnegative relaxation for binary quadratic programming (BQP) problem is equivalent to a tighter semidifinite relaxation for it. When problem (BQP) reduces to max-cut (MC) problem, doubly nonnegative relaxation for it is equivalent to the standard semidifinite relaxation. Furthermore, some compared numerical results are reported.

Keywords: binary quadratic programming, semidefinite relaxation, completely positive programming, doubly nonnegative relaxation, max-cut problem

2000 MSC: 90C10, 90C26, 49M20

✩This work is supported by National Natural Science Foundation of China (Grant No. 11071158)

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1. Introduction

In this paper, we consider the following binary quadratic programming problem

\[
\begin{align*}
\text{(BQP)} \quad & \min x^T Q x + 2c^T x \\
& \text{s.t. } a_i^T x = b_i, \ i = 1, 2, \ldots, m, \\
& x \in \{-1, 1\}^n,
\end{align*}
\]

where \(x \in R^n\) is the variable, \(Q \in R^{n \times n}\), \(c \in R^n\), \(a_i \in R^n\) and \(b_i \in R\) for all \(i \in I := \{1, 2, \ldots, m\}\) are the data. Without loss of generality, \(Q\) is symmetric, and we assume \(Q\) is not positive semidefinite, which implies generally that problem (BQP) is nonconvex and NP-hard \[1\].

Problem (BQP) arises in many applications, such as financial analysis \[2\], molecular conformation problem \[3\] and cellular radio channel assignment \[4\]. Many combinatorial optimization problems are special cases of problem (BQP), such as max-cut problem \[5\]. For solving this type of problem, a systematic survey of the solution methods can be found in Chapter 10 in \[6\] and the references therein.

It is well-known that semidefinite relaxation (SDR) is a powerful, computationally efficient approximation technique for a host of very difficult optimization problems, for instance, max-cut problem \[5\], Boolean quadratic program \[7\]. It also has been at the center of some of the very exciting developments in the area of signal processing and communications \[8, 9\]. The standard SDR for problem (BQP) is as follows:

\[
\begin{align*}
\text{(SDR)} \quad & \min X \cdot Q + 2c^T x \\
& \text{s.t. } a_i^T x = b_i, \ \forall i \in I, \\
& a_i^T X a_i = b_i^2, \ \forall i \in I, \\
& X_{ii} = 1, \ \forall i = 1, 2, \ldots, n, \\
& X \succeq 0,
\end{align*}
\]

where symbol \(\cdot\) denotes the trace for any two conformal matrices. It is obviously that problem (SDR) is convex and gives a lower bound for problem (BQP) if the feasible set of problem (BQP) is nonempty. Moreover, if the optimal solution \((x^*, X^*)\) for problem (SDR) satisfy \(X^* = x^*(x^*)^T\), then we can conclude that \(x^*\) is an optimal solution for problem (BQP).

Recently, Burer \[10\] proves that a large class of NP-hard nonconvex quadratic programs with a mix of binary and continuous variables can be modeled as so called completely positive programs, which are convex but
still NP-hard in general. In order to solve such convex programs efficiently, a computable relaxed problem is obtained by approximation the completely positive matrices with doubly nonnegative matrices, resulting in a doubly nonnegative programming [11], which can be efficiently solved by some popular packages. For more details and developments of this technique, one may refer to [10, 11, 12, 13] and the references therein.

In this paper, a tighter SDR problem and a doubly nonnegative relaxation (DNNR) problem for problem (BQP) are established, respectively, according to the features of the constraints in problem (BQP) and the techniques of DNNP. And, we prove that doubly nonnegative relaxation for problem (BQP) is equivalent to the tighter semidefinite relaxation for it. Applying this result to max-cut (MC) problem, it is shown that doubly nonnegative relaxation for problem (MC) is equivalent to the standard semidefinite relaxation for it. Moreover, some compared numerical results are reported to illustrate the features of doubly nonnegative relaxation and semidefinite relaxation, respectively.

The paper is organized as follows. In Section 2, a new tighter semidefinite relaxation for problem (BQP) is proposed in Section 2.1. Problem (BQP) is relaxed to a doubly nonnegative programming problem in Section 2.2. Section 3 and Section 4 are devoted to show the equivalence of two relaxation problems for problem (BQP) and problem (MC), respectively. Some conclusions are given in Section 5.

2. New relaxation for problem (BQP)

2.1. New tighter SDR for problem (BQP)

First, note that problem (BQP) also can be relaxed to the following problem by SDR

\[
\begin{align*}
\min_{(SDR)} & \quad X \bullet Q + 2c^T x \\
\text{s.t.} & \quad a_i^T x = b_i, \forall i \in I, \\
\end{align*}
\]

If the optimal solution \((x^*, X^*)\) for problem\((SDR)\) satisfy \(X^* = x^*(x^*)^T\), it holds that \(x^*\) also is an optimal solution for problem (BQP). On one hand, it is worth noting that

\[
X - xx^T \succeq 0 \implies X \succeq 0 \quad (1)
\]
holds always, which further implies that any feasible solution of problem $\tilde{\text{SDR}}$ is also feasible for problem (SDR). It follows that $\text{Opt}(\text{SDR}) \leq \text{Opt}(\tilde{\text{SDR}})$ since the two problems have the same objective functions, where $\text{Opt}(\ast)$ denotes the optimal value for problem $(\ast)$. Therefore, we can conclude that problem $\tilde{\text{SDR}}$ is a tighter SDR problem for problem (BQP) than problem (SDR).

On the other hand, we can easily verify that the constraint $X - xx^T \succeq 0$ is nonconvex, since the quadratic term $-xx^T$ is nonconvex. Thus, problem $\tilde{\text{SDR}}$ is nonconvex and not solved by some popular packages for solving convex programs. In order to establish the convex representation for problem (SDR), a crucial theorem is given below and the details of its proof can be seen in Appendix A.5.5 Schur complement in [14].

**Theorem 2.1.** Let matrix $M \in S^n$ is partitioned as

$$M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}.$$  

If $\det A \neq 0$, the matrix $H = C - B^T A^{-1} B$ is called the Schur complement of $A$ in $M$. Then, we have the following relations:

(i) $M \succ 0$ if and only if $A \succ 0$ and $H \succ 0$.

(ii) If $A \succ 0$, then $M \succeq 0$ if and only if $H \succeq 0$.

According to Theorem 2.1(ii) and (1), it holds immediately that

$$\begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \succeq 0 \iff X - xx^T \succeq 0 \implies X \succeq 0,$$

i.e., the constraint $X - xx^T \succeq 0$ can be equivalently reformulated as $\begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \succeq 0$, which is not only convex, but also computable. So, problem (SDR) is equivalently reformulated as follows:

$$\begin{align*}
\text{min} & \quad X \cdot Q + 2c^T x \\
\text{s.t.} & \quad a_i^T x = b_i, \, \forall i \in I, \\
& \quad a_i^T X a_i = b_i^2, \, \forall i \in I, \\
& \quad X_{ii} = 1, \, \forall i = 1, 2, \ldots, n, \\
& \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \succeq 0,
\end{align*}$$

(SDR1)
which is not only convex in form, also can be efficiently solved by some popular packages for solving convex programs.

Here, some examples are given to show that problem (SDR1) is a tighter relaxation problem compared to problem (SDR), and the corresponding numerical results further show that problem (SDR1) is more efficient than problem (SDR). These examples are solved by CVX, a package for specifying and solving convex programs [15].

**Example 2.1.** This is a two dimensional nonconvex problem with one linear equality constraint, the corresponding coefficients are selected as follows:

\[
Q = \begin{bmatrix} 0 & -3 \\ -3 & -20 \end{bmatrix}, \quad c = \begin{bmatrix} -8 \\ 9 \end{bmatrix}, \quad A = [10, -10], \quad b = 0,
\]

where \( A = [a_1, a_2, \ldots, a_m]^T \), \( b = [b_1, b_2, \ldots, b_m]^T \).

On one hand, we use CVX to solve problem (SDR), then we obtain \( \text{Opt}(SDR) = -\infty \), since problem (SDR) is unbounded below. On the other hand, when problem (SDR1) is solved, it follows that \( \text{Opt}(SDR1) = -28 \) with \( X = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \) and \( x = [-1, -1]^T \). Note that the relationship \( X = xx^T \) holds, thus we can conclude that \( x = [-1, -1]^T \) also is an optimal solution for problem (BQP). The results show that problem (SDR1) is more tighter and efficient than problem (SDR) for this problem.

**Example 2.2.** This problem is five dimensions with three linear equality constraints, the corresponding coefficients are chosen as follows:

\[
Q = \begin{bmatrix} -52 & 31 & 49 & -7 & 4 \\ 31 & -16 & -50 & -13 & -49 \\ 49 & -50 & 8 & 44 & -30 \\ -7 & -13 & 44 & 36 & 12 \\ 4 & -49 & -30 & 12 & 56 \end{bmatrix}, \quad c = \begin{bmatrix} -20 \\ 37 \\ 43 \\ 25 \\ -6 \end{bmatrix},
\]

\[
A = \begin{bmatrix} 4 & 10 & 29 & 14 & -36 \\ 38 & 9 & 1 & -17 & 23 \\ 48 & 39 & 5 & -17 & -13 \end{bmatrix}, \quad b = \begin{bmatrix} 11 \\ -50 \\ -36 \end{bmatrix}.
\]

If this problem is solved by CVX with problem (SDR), then it returns \( \text{Opt}(SDR) = -\infty \) since problem (SDR) is unbounded below. When we use problem (SDR1) to solve this problem, we have \( \text{Opt}(SDR1) = -307.548 \), however, the relationship \( X = xx^T \) is not holds for this problem. Therefore, we obtain a
tighter lower bound $-307.548$ for original problem. These results also show that problem (SDR1) is more effective than problem (SDR).

In fact, the constraint $x \in \{-1, +1\}^n$ in problem (BQP) further imply that the following relationship

$$(1 - x_i)(1 - x_j) \geq 0 \Rightarrow 1 - x_i - x_j + X_{ij} \geq 0, \forall 1 \leq i \leq j \leq n$$

(3)

always hold. Combing with (3) in problem (SDR1), we get the following new semidefinite relaxation problem

$$\begin{align*}
\min & \quad X \cdot Q + 2c^T x \\
\text{s.t.} & \quad a_i^T x = b_i, \forall i \in I, \\
& \quad a_i^T X a_i = b_i^2, \forall i \in I, \\
& \quad X_{ii} = 1, \forall i = 1, 2, \ldots, n, \\
& \quad 1 - x_i - x_j + X_{ij} \geq 0, \forall 1 \leq i \leq j \leq n,
\end{align*}$$

(SDR2)

The above semidefinite relaxation problem (SDR2) is more tighter than problem (SDR1) in form, since $\frac{n(n+1)}{2}$ inequality constraints are added into corresponding problem (SDR2). Furthermore, we will prove that problem (SDR2) is equivalent to another convex relaxation problem for problem (BQP) in Section 3.

Now, we test some problems to show that problem (SDR2) is tighter than problem (SDR1) from the computational point of view. These problems are of one of two types:

RdnBQP. We generate 50 instances of problem (BQP) by MATLAB function `randn(·)`. The symmetric matrix $Q$ is generated by $\text{tril}(\text{randn}(·), -1)+\text{triu}(\text{randn}(·), 0)$, and all instances are nonconvex.

RdiBQP. 50 instances of problem (BQP) are generated by MATLAB function `randi([-10, 10], ·)`. The symmetric matrix $Q$ is generated by

| Type    | Instances | $n$ | $m$ | Function   |
|---------|-----------|-----|-----|------------|
| RdnBQP  | 50        | 50  | 20  | randn(·)  |
| RdiBQP  | 50        | 50  | 20  | randi([-10, 10], ·) |
\texttt{randn(·)+randn(·)'.} Each element in the data coefficients is a random integer number in the range $[-10, 10]$. All instances are nonconvex binary quadratic programming problems.

To compare the performance of two relaxation problems for problem (BQP), by using problem (SDR1) and problem (SDR2), respectively, we use performance profiles as described in Dolan and Moré's paper \cite{16}. Our profiles are based on optimal values for problems (SDR1) and (SDR2). These problems are solved by \texttt{CVX}, and the results of performance are shown in Figure 1. From Figure 1, it is obviously that the lower bound which got from problem (SDR2) is much greater than that one of from problem (SDR1), for test problems RdnBQP and RdiBQP, respectively. Moreover, we find that optimal value of problem (SDR2) is strictly greater than that of problem (SDR1) for test problems in the experiment. Thus, the performance of problem (SDR2) is much better than problem (SDR1) for solving problem (BQP) in some sense.

2.2. Doubly nonnegative relaxation for problem (BQP)

Recently, Burer \cite{10} has shown that a large class of NP-hard nonconvex quadratic problems with binary constraints can be modeled as so-called completely positive programs (CPP), i.e., the minimization of a linear function over the convex cone of completely positive matrices subject to linear constraints. Motivated by the ideas, we first establish the CPP representation for problem (BQP), and then give its doubly nonnegative relaxation (DNNR)
formulation. Subsequently, some compared numerical results are presented in this section.

Let \( z = \frac{1}{2}(e - x) \) in problem (BQP), it follows that \( z \in \{0, 1\}^n \), and then problem (BQP) can be equivalently reformulated as follows:

\[
\begin{align*}
\min_{(BQP)} & \quad 4z^TQz - 4z^T(Qe + c) + e^TQe + 2c^Te \\
\text{s.t.} & \quad 2a_i^Tz = a_i^T e - b_i, \forall i \in I, \\
& \quad z \in \{0, 1\}^n,
\end{align*}
\]

where \( e \) denote the vector of ones with appropriate dimension. According to Theorem 2.6 in [10] and similar to the analysis in it, problem \((BQP)\) can be further equivalently transformed into the following CPP problem

\[
\begin{align*}
\min_{(CPP)} & \quad 4Q \cdot Z - 4z^T(Qe + c) + e^TQe + 2c^Te \\
\text{s.t.} & \quad 2a_i^Tz = a_i^T e - b_i, \forall i \in I, \\
& \quad 4a_i^TZa_i = (a_i^T e - b_i)^2, \forall i \in I, \\
& \quad Z_{ii} = z_i, \forall i = 1, 2, \ldots, n, \\
& \quad \begin{bmatrix} 1 & z^T \\ z & Z \end{bmatrix} \in C_{1+n},
\end{align*}
\]

where \( C_{1+n} \) is defined as follows:

\[
C_{1+n} := \left\{ X \in S^{1+n} : X = \sum_{k \in K} z^k (z^k)^T \right\} \cup \{0\},
\]

and for some finite \( \{z^k\}_{k \in K} \subset R^{1+n}_+ \setminus \{0\} \).

In view of the definition of convex cone in [14], \( C_{1+n} \) is a closed convex cone, and is called the completely positive matrices cone. Thus, problem \((CPP)\) is a convex problem. However, problem \((CPP)\) is NP-hard, since checking whether or not a given matrix belongs to \( C_{1+n} \) is NP-hard, which has been shown by Dickinson and Gijen in [17]. Thus, it has to be replaced by some computable cones, which can efficiently approximate cone \( C_{1+n} \). Note that the convex cone \((S_n)^+\) is self-dual, and so is the convex cone \( S^+_n \), where \((S_n)^+\) and \( S^+_n \) denotes the cone of \( n \times n \) nonnegative symmetric matrices and the cone of \( n \times n \) positive semidefinite matrices, respectively. Hence, Diananda’s decomposition theorem [18] can be reformulated as follows.

**Theorem 2.2.** \( C_n \subseteq S^+_n \cap (S_n)^+ \) holds for all \( n \). If \( n \leq 4 \), then \( C_n = S^+_n \cap (S_n)^+ \).
By the way, the matrices in $S_n^+ \cap (S_n)^+$ sometimes are called “doubly nonnegative”. Of course, in dimension $n \geq 5$ there are matrices which are doubly nonnegative but not completely positive, the counterexample can be seen in [19].

According to Theorem 2.2, problem (CPP) can be relaxed to the following DNNP problem

$$
\begin{align*}
\text{min} & \quad 4Q \bullet Z - 4z^T(Qe + c) + e^TQe + 2e^Te \\
\text{subject to} & \quad 2a_i^Tz = a_i^Te - b_i, \quad \forall i \in I, \\
& \quad 4a_i^TZa_i = (a_i^Te - b_i)^2, \quad \forall i \in I, \\
& \quad Z_{ii} = z_i, \quad \forall i = 1, 2, \ldots, n, \\
& \quad \begin{bmatrix} 1 & Z^T \\ Z & 0 \end{bmatrix} \in S_{1+n}^+ \cap (S_{1+n})^+.
\end{align*}
$$

Up to now, the other convex relaxation problem for problem (BQP) is established, i.e., problem (DNNP), which is computable by some popular packages for solving convex programs, such as CVX, etc.

Note that problem (DNNP) has $n + \frac{n(n+1)}{2}$ equality constraints more than standard semidefinite relaxation problem (SDR1), and $n$ equality constraints more than problem (SDR2), respectively. Thus, the lower bound which get from problem (DNNP) is much greater than that one of by problem (SDR1) and problem (SDR2), respectively.

In the following, two types of problems are tested to show the performance of problem (SDR1), problem (SDR2) and problem (DNNP), respectively. The statistics of the test problems are chosen as follows:

| Type   | Instances | $n$ | $m$ | Function     |
|--------|-----------|-----|-----|--------------|
| RdBQP  | 50        | 50  | 25  | rand(·)     |
| RdsBQP | 50        | 50  | 25  | rands(·)    |

- **RdBQP**. For this type of problems, we generate 50 instances of problem (BQP) by using MATLAB function `rand(·)`. The symmetric matrix $Q$ is generated by `rand(·)+rand(·)'`, and all problems are nonconvex.
- **RdsBQP**. The coefficients of 50 instances of problem (BQP) are generated by using MATLAB function `rands(·)`, and the symmetric matrix $Q$ is generated by `rands(·)+rands(·)'`. All instances are nonconvex.
We use performance profiles \cite{16} to compare the performance of problem (SDR1), problem (SDR2) and problem (DNNP), for problem (BQP), respectively. The corresponding results of performance are shown in Figure 2. The profiles for Figure 2 are based on optimal values of problem (SDR1), problem (SDR2) and problem (DNNP), respectively, and these problems are solved by CVX. From Figure 2 it is obviously that the performances of problem (SDR2) and problem (DNNP) are almost the same, which are better than that one of problem (SDR1), for problems RdBQP and RdsBQP, respectively. Thus, we can conclude that it is more efficient to use problem (SDR2) and problem (DNNP) to solve problem (BQP), from the point of view of optimal values.

Furthermore, we will show the equivalence of the problems (DNNP) and (SDR2) in Section 3.

![Figure 2: Left figure is based on optimal values of problems RdBQP, right figure is based on optimal values of problems RdsBQP.](image)

3. Relationship between relaxation problems

In this section, we will investigate the relationship between two relaxation problems (SDR2) and (DNNP). First of all, the definition of the equivalence of two optimization problems is defined as follows.

**Definition 3.1.** We call two problems are equivalent if they satisfy the following two conditions:

(i) If from a solution of one problem, a solution of the other problem is readily found, and vice versa.

(ii) The two problems have the same optimal value.
Now, based on the above Definition 3.1, the main theorem is given below.

**Theorem 3.1.** Suppose that the feasible sets $\text{Feas}(\text{SDR2})$ and $\text{Feas}(\text{DNNP})$ are all nonempty. Then, two problems (SDR2) and (DNNP) are equivalent.

**Proof.** The proof can be divided into two parts. First of all, we will prove that $\text{Opt}(\text{SDR2}) \geq \text{Opt}(\text{DNNP})$.

Suppose that $(x^*, X^*)$ is an optimal solution of problem (SDR2), let $Z_{ij} = \frac{1}{4}(1 - x_i^* - x_j^* + X_{ij}^*)$ and $z_i = \frac{1}{2}(1 - x_i^*)$, $\forall 1 \leq i \leq j \leq n$, i.e.,

$$Z = \frac{1}{4}(ee^T - e(x^*)^T - x^*e^T + X^*), \quad z = \frac{1}{2}(e - x^*). \quad (4)$$

By $a_i^T x^* = b_i$ for all $i \in I$ and (4), we have

$$a_i^T x^* = a_i^T (e - 2z) = b_i \Rightarrow 2a_i^T z = a_i^T e - b_i, \forall i \in I. \quad (5)$$

From (4) and $a_i^T X^* a_i = b_i^2$ for all $i \in I$, it follows that

$$a_i^T X^* a_i = a_i^T (4Z - ee^T - x^*e^T + X^*)a_i = b_i^2 \Rightarrow 4a_i^T Z a_i = (a_i^T e - b_i)^2, \forall i \in I. \quad (6)$$

Again from (4), which imply that

$$Z_{ii} = \frac{1}{4}(1 - 2x_i^* + X_{ii}^*) = \frac{1}{2}(1 - x_i^*) = z_i, \forall i \in I, \quad (7)$$

since $X_{ii}^* = 1$.

From $1 - x_i^* - x_j^* + X_{ij}^* \geq 0, \forall 1 \leq i \leq j \leq n$, it holds that

$$Z_{ij} \geq 0, \forall 1 \leq i \leq j \leq n, \quad (8)$$

which combining with (7), further imply that

$$z_i \geq 0, \forall 1 \leq i \leq n. \quad (9)$$

By Theorem 2.1(ii) and (4), it follows that

$$Z - zz^T = \frac{1}{4}(ee^T - e(x^*)^T - x^*e^T + X^*) - \frac{1}{4}(e - x^*)(e - x^*)^T = \frac{1}{4}(X^* - x^*(x^*)^T) \succeq 0. \quad (10)$$

Combining (10) with (5), (6), (8) and (9), it follows that $(z, Z)$ defined by (4) is a feasible solution for problem (DNP).
Moreover, again from \((11)\), we have

\[
4Q \bullet Z - 4z^T(Qe + c) + e^TQe + 2c^Te
\]

\[
= Q \bullet (ee^T - e(x^*)^T - x^*e^T + X^*) - 2(e - x^*)^T(Qe + c) + e^TQe + 2c^Te
\]

\[
= Q \bullet X^* + 2c^T x^* = \text{Opt}(SDR2),
\]

which further imply that \(\text{Opt}(DNNP) \leq \text{Opt}(SDR2)\).

On the other hand, given an optimal solution \((z^*, Z^*)\) to problem \((DNNP)\), and let

\[
X_{ij} = 1 - 2z^*_i - 2z^*_j + 4Z^*_{ij}, \quad x_i = 1 - 2z^*_i, \quad \forall 1 \leq i \leq j \leq n,
\]

which imply that

\[
X_{ii} = 1 - 4z^*_i + 4Z^*_{ii} = 1
\]

since \(Z^*_{ii} = z^*_i, \forall i = 1, 2, \ldots, n\). Moreover,

\[
1 - x_i - x_j + X_{ij} = 1 - (1 - 2z^*_i) - (1 - 2z^*_j) + 1 - 2z^*_i - 2z^*_j + 4Z^*_{ij} = 4Z^*_i \geq 0, \quad \forall 1 \leq i \leq j \leq n.
\]

From \((11)\) and \(2a_i^T z^* = a_i^T e - b_i, \forall i \in I\), it follows that

\[
a_i^T x = a_i^T (e - 2z^*) = b_i, \quad \forall i \in I.
\]

Again from \((11)\) and \(4a_i^T Z^* a_i = (a_i^T e - b_i)^2, \forall i \in I\), we have

\[
a_i^T X a_i = a_i^T (ee^T - 2e(z^*)^T - 2z^*e^T + 4Z^*) a_i
\]

\[
= b_i^2, \quad \forall i \in I.
\]

From \((11)\) and Theorem \(2.1(ii)\), it holds that

\[
X - xx^T = ee^T - 2z^*e^T - 2e(z^*)^T + 4Z^* - (e - 2z^*)(e - 2z^*)^T
\]

\[
= 4(Z^* - z^*(z^*)^T) \succeq 0.
\]

By \((13)\), \((14)\), \((15)\) and \((16)\), we can conclude that \((x, X)\) defined by \((11)\) is a feasible solution for problem \((SDR2)\). Furthermore, we have

\[
X \bullet Q + 2c^T x = (ee^T - 2e(z^*)^T - 2z^*e^T + 4Z^*) \bullet Q + 2c^T (e - 2z^*)
\]

\[
= 4Z^* \bullet Q - 4(z^*)^T(Qe + c) + e^TQe + 2c^Te
\]

\[= \text{Opt}(DNNP),
\]

which imply that \(\text{Opt}(SDR2) \leq \text{Opt}(DNNP)\). Summarizing the analysis above and according to Definition \(3.1\), we can conclude that problem \((DNNP)\) is equivalent to problem \((SDR2)\).
Although Opt(SDR2) = Opt(DNNP) in view of Theorem 3.1 and Definition 3.1, problem (DNNP) has \( n \) equality constraints more than problem (SDR2) in form. So, the amount of computation for solving problem (DNNP) may be much greater than that one of solving problem (SDR2). In order to illustrate this point of view, the compared performance results are shown in Figure 3 and Figure 4, respectively, which are based on the number of iterations and CPU time for solving problems RdBQP and RdsBQP. The results in Figure 3 show that the performance of problem (SDR2) is better than that one of problem (DNNP) for problems RdBQP, but the performance of problem (DNNP) is better than that one of problem (SDR2) for problems RdsBQP, in view of the points of the number of iterations. From the results of the performance of CPU time, it is obviously that problem (SDR2) is more efficient than problem (DNNP) for solving problems RdBQP and RdsBQP, respectively. Summarizing the analysis above, we can efficiently solving problem (BQP) by solving problem (SDR2) or problem (DNNP) in practice.

Figure 3: Left figure is based on the number of iterations of problems RdBQP, right figure is based on the number of iterations of problems RdsBQP.

4. An application to max-cut problem

The max-cut (MC) problem is a kind of important combinatorial optimization problem on undirected graphs with weights on the edges, and also is NP-hard [20]. Given such a graph, (MC) problem consists in finding a partition of the set of nodes into two parts so as to maximize the total weight of edges cut by the partition.
Figure 4: Left figure is based on CPU time of problems RdBQP, right figure is based on CPU time of problems RdsBQP.

Let $G$ be an $n$-node graph, vertex set $V := \{1, 2, \ldots, n\}$, $A(G)$ the adjacency matrix of graph $G$, $L$ the Laplacian matrix associated with the graph, i.e., $L := \text{Diag}(A(G)e) - A(G)$. Let the vector $u \in \{+1, -1\}^n$ represent any cut in the graph $G$ via the interpretation that the sets $\{i : u_i = +1\}$ and $\{i : u_i = -1\}$ form a partition of the node set of $G$, we can get the following formulation for (MC) problem

$$(MC) \max \frac{1}{4} u^T L u \quad \text{s.t.} \quad u \in \{+1, -1\}^n.$$ 

On one hand, by using the standard semidefinite relaxation technique to (MC) problem, we can get the following problem

$$(SDR) \max \frac{1}{4} L \bullet U \quad \text{s.t.} \quad U_{ii} = 1, \forall i = 1, 2, \ldots, n, \quad U \succeq 0.$$ 

Goemans and Williamson [5] have provided estimates for the quality of problem (SDR) bound for (MC) problem. By a randomly rounding a solution to problem (SDR), they propose a 0.878-approximation algorithm for solving problem (MC) based on problem (SDR), which is known to be the best approximation ration of polynomial-time algorithm for solving problem (MC).

On the other hand, according to the technique introduced in Section 2.2 problem (MC) can also be relaxed to the following doubly nonnegative
programming problem
\[
\begin{align*}
\text{(DNNP)} & \quad \max \ L \cdot X - x^T Le + \frac{1}{4}e^T Le \\
\text{s.t.} & \quad X_{ii} = x_i, \forall i = 1, 2, \ldots, n, \\
& \quad \begin{bmatrix}
1 & x^T \\
X
\end{bmatrix} \in S_{1+n}^+ \cap (S_{1+n})^+.
\end{align*}
\]

**Remark 4.1.** (i) Note that for two relaxation problems (SDR) and (DNNP), the feasible sets are all nonempty. It is obviously that the identity matrix $E$ is a feasible solution for problem (SDR), and $(x, X) = (0, 0)$ feasible for problem (DNNP).

(ii) Compared with problem (SDR), problem (DNNP) has not only $n + \frac{n(n+1)}{2}$ new inequality constraints, but also $n$ variables.

Thus, according to Theorem 3.1 and Remark 4.1(i), we have the following theorem.

**Theorem 4.1.** Problem (SDR) is equivalent to problem (DNNP).

**Proof.** On one hand, suppose that $U^*$ is an optimal solution for problem (SDR), and let $X_{ij} = \frac{1}{4}(U^*_{ij} + 1)$ and $x_i = \frac{1}{2}, \forall 1 \leq i \leq j \leq n$, i.e.,

\[
X = \frac{1}{4}(U^* + ee^T), \quad x = \frac{1}{2}e, \quad (17)
\]

which imply that

\[
X_{ii} = \frac{1}{4}(U^*_{ii} + 1) = x_i = \frac{1}{2} > 0, \quad \forall 1 \leq i \leq n, \quad (18)
\]

since $U^*_{ii} = 1$. Then, from $U^* \succeq 0$, it follows that

\[
0 \leq U^*_{ii}U^*_{jj} - (U^*_{ij})^2 = 1 - (U^*_{ij})^2,
\]

i.e.,

\[
-1 \leq U^*_{ij} \leq 1, \quad \forall 1 \leq i < j \leq n,
\]

combining with (17), we have

\[
X_{ij} = \frac{1}{4}(U^*_{ij} + 1) \geq 0, \quad 1 \leq i < j \leq n. \quad (19)
\]
Moreover, from (17), it follows that
\[ X - xx^T = \frac{1}{4}(U^* + ee^T) - \frac{1}{4}ee^T = \frac{1}{4}U^* \succeq 0, \]
(20)
it followed by \( U^* \succeq 0. \) Combining (18), (19) and (20) as well as Theorem 2.1(ii), it holds that \((x, X)\) is a feasible solution for problem \((\hat{DNNP})\). Again from (17), we have
\[ L \cdot X - x^TLe + \frac{1}{4}e^TLe = \frac{1}{4}L \cdot U^* = \text{Opt}(\hat{SDR}), \]
which further imply that \( \text{Opt}(\hat{DNNP}) \geq \text{Opt}(\hat{SDR}). \)

On the other hand, suppose that \((x^*, X^*)\) is an optimal solution for problem \((\hat{DNNP})\), and let
\[ U = 4X^* - 2x^*e^T - 2e(x^*)^T + ee^T, \]
(21)
which imply that
\[ U_{ii} = 4X_{ii}^* - 2x_{ii}^* - 2x_{ii}^* + 1 = 1, \quad \forall 1 \leq i \leq n, \]
(22)
since \( X_{ii}^* = x_{ii}^* \), \( \forall 1 \leq i \leq n. \) From (21), it follows that
\begin{align*}
U &= 4X^* - 2e(x^*)^T - 2x^*e^T + ee^T \\
&= 4(X^* - x^*(x^*)^T) + (2x^* - e)(2x^* - e)^T \succeq 0.
\end{align*}
(23)
From (22) and (23), we can conclude that \( U \) defined by (21) is a feasible solution for problem \((\hat{SDP})\). Furthermore, again from (21), we have
\[ \frac{1}{4}L \cdot U = L \cdot X^* - (x^*)^TLe + \frac{1}{4}e^TLe = \text{Opt}(\hat{DNNP}), \]
which imply that \( \text{Opt}(\hat{SDR}) \geq \text{Opt}(\hat{DNNP}). \) The proof is completed. \( \square \)

By Theorem 4.1, we can obtain doubly nonnegative relaxation for problem \((MC)\) exactly equal to the standard semidefinite relaxation, i.e. problem \((\hat{DNNP})\) and problem \((\hat{SDR})\) are equivalent according to Definition 3.1 without the boundedness assumption of two feasible sets of two problems.

5. Conclusions

In this paper, a class of nonconvex binary quadratic programming problem is considered, which is NP-hard in general. In order to solve this problem
efficiently by some popular packages for solving convex programs, two convex representation methods are proposed. One of the methods is semidefinite relaxation, by the structure of the binary constraints of original problem, which results in a new tighter semidefinite relaxation problem (SDR2). The other method is doubly nonnegative relaxation. The original problem is equivalently transformed into a convex problem (CPP), which is also NP-hard in general. Then, by virtue of the features of constraints in this problem, a computable convex problem (DNNP) is obtained through doubly nonnegative relaxation. Moreover, the two convex relaxation problems are equivalent. These results are applied to (MC) problem, we can conclude that doubly nonnegative relaxation for problem (MC) is equivalent to the standard semidefinite relaxation for it. Furthermore, some compared numerical results are reported to show the performance of two relaxed problems.

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