ON THE STRUCTURE OF LIPSCHITZ-FREE SPACES

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Abstract. In this note we study the structure of Lipschitz-free Banach spaces. We show that every Lipschitz-free Banach space over an infinite metric space contains a complemented copy of $\ell_1$. This result has many consequences for the structure of Lipschitz-free Banach spaces. Moreover, we give an example of a countable compact metric space $K$ such that $\mathcal{F}(K)$ is not isomorphic to a subspace of $L_1$ and we show that whenever $M$ is a subset of $\mathbb{R}^n$, then $\mathcal{F}(M)$ is weakly sequentially complete; in particular, $c_0$ does not embed into $\mathcal{F}(M)$.

1. Introduction

Given a metric space $M$, it is possible to construct a Banach space $\mathcal{F}(M)$ in such a way that the Lipschitz structure of $M$ corresponds to the linear structure of $\mathcal{F}(M)$. This space $\mathcal{F}(M)$ is sometimes called “Lipschitz-free space”. We refer to the next section for some more details concerning the construction and basic properties of these spaces. Although Lipschitz-free spaces over separable metric spaces are easy to define, their structure is poorly understood to this day. The study of the linear structure of Lipschitz-free spaces over metric spaces has become an active field of study; see e.g. [8,9,11,13,14,18,20]. In the first part of this paper we prove the following general result.

Theorem 1.1. Let $M$ be an infinite metric space. For the Banach space $X = \mathcal{F}(M)$, we have

(i) $\ell_1 \hookrightarrow X$, i.e., there is a complemented subspace of $X$ isomorphic to $\ell_1$.

From this we get

(ii) $X \not\hookrightarrow C(K)$, i.e., $X$ is not isomorphic to a complemented subspace of a $C(K)$ space.

(iii) $X^*$ is not weakly sequentially complete; in particular, $X$ is not isomorphic to $L^1$-predual.

(iv) $X$ is not isomorphic to the Gurariĭ space.

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(v) $X$ is a projectively universal separable Banach space, i.e., for any separable Banach space $Y$ there exists a bounded linear operator from $X$ onto $Y$.

It often happens that the Lipschitz-free space over a “small enough” space is isomorphic to $\ell_1$. For example, if $M \subset \mathbb{R}$ is a set of measure zero or if $M$ is a separable ultrametric space, then $F(M)$ is isomorphic to $\ell_1$; see [13] and [7]. By the result of A. Dalet [8], $F(K)$ is a dual space with MAP whenever $K$ is a countable compact metric space. Hence, one could conjecture that in this case $F(K)$ is isomorphic to $\ell_1$. We give an example which shows that this is not the case.

**Theorem 1.2.** There is a countable compact metric space $K$ such that $F(K) \not\cong L_1$, i.e., $F(K)$ is not linearly isomorphic to a subspace of $L_1$. Moreover, $K$ is a convergent sequence, i.e., it has only one accumulation point.

If $M$ contains a bi-Lipschitz copy of $c_0$, then $F(M)$ is an isomorphically universal separable Banach space; i.e., $F(M)$ contains an isomorphic copy of every separable Banach space (for more details we refer to Section 5). Y. Dutrieux and V. Ferenczi in [9] asked for the converse.

**Question 1.** Let $X$ be a Banach space. Is $F(X)$ universal if and only if $X$ contains a bi-Lipschitz copy of $c_0$?

For a general metric space $X$ the above question has a negative answer. It follows from the result of P. Kaufmann [18] Corollary 3.3 that $F(c_0)$ is isomorphic to $F(B_{c_0})$ (thus, it is a universal) and of course, since $B_{c_0}$ is bounded, $B_{c_0}$ does not contain a bi-Lipschitz copy of $c_0$.

**Theorem 1.3.** Let $M \subset \mathbb{R}^n$ be an arbitrary set. Then $F(M)$ is weakly sequentially complete. Consequently, $c_0 \not\cong F(M)$, i.e., $c_0$ is not linearly isomorphic to a subspace of $F(M)$.

To the best of our knowledge, it was not even known whether there could be a Lipschitz-free space which neither embeds into $L_1$ nor is universal. The example given in Theorem 1.2 is one such example (because, by the result of A. Dalet [8], $F(K)$ is a separable dual space and so it does not contain $c_0$). Another one is the space $F([0,1]^{n})$; see Theorems 1.3 and 1.4.

In the last section of this note we mention some open problems related to the structure of isomorphically universal Lipschitz-free Banach spaces.

The notation and terminology we use are relatively standard. If $X$ and $Y$ are Banach spaces, the symbol $Y \hookrightarrow X$ (resp. $Y \not\hookrightarrow X$) means that $Y$ is (resp. is not) linearly isomorphic to a subspace of $X$. If $(M,d)$ is a metric space, $x \in M$ and $r \geq 0$, we use $U(x,r)$ and $B(x,r)$ to denote respectively the open and closed ball, i.e., the sets $\{y \in M : d(x,y) < r\}$ and $\{y \in M : d(x,y) \leq r\}$.

2. Basic facts about Lipschitz-free spaces

Let $(M,d)$ be a metric space with a distinguished point denoted by $0$. Consider the space $\text{Lip}_0(M)$ of all real-valued Lipschitz functions that map $0 \in M$ to $0 \in \mathbb{R}$. It has a vector space structure and one can define a norm $\| \cdot \|_{\text{Lip}}$ on $\text{Lip}_0(M)$, where for $f \in \text{Lip}_0(M)$, $\| f \|_{\text{Lip}}$ is the minimal Lipschitz constant, i.e., $\sup\{ \frac{|f(x)-f(y)|}{d(x,y)} : x \neq y \in M \}$. Then $(\text{Lip}_0(M), \| \cdot \|_{\text{Lip}})$ is a Banach space.
For any \( x \in M \), denote by \( \delta_x \in \text{Lip}_0(M)^* \) the evaluation functional, i.e., \( \delta_x(f) = f(x) \) for every \( f \in \text{Lip}_0(M) \). Denote by \( \mathcal{F}(M) \) the closure of the linear span of \( \{ \delta_x : x \in M \} \) with the dual space norm denoted simply by \( || \cdot || \). Observe that for any \( x, y \in M \), we have \( ||\delta_x - \delta_y|| = d(x, y) \).

This space is usually called Lipschitz-free Banach space (also Arens-Eells space) and it is uniquely characterized by the following universal property.

Let \( X \) be a Banach space and suppose \( L : M \to X \) is a Lipschitz map such that \( L(0) = 0 \). Then there exists a unique linear map \( \hat{L} : \mathcal{F}(M) \to X \) extending \( L \), i.e.,

\[
\begin{array}{ccc}
M & \xrightarrow{L} & X \\
\delta_M & \downarrow & \downarrow \text{id}_X \\
\mathcal{F}(M) & \xrightarrow{\hat{L}} & X,
\end{array}
\]

and \( ||\hat{L}|| = ||L||_{\text{Lip}} \) where \( || \cdot ||_{\text{Lip}} \) denotes the Lipschitz norm of \( L \).

This fact is usually referred to as folklore. The proof is so simple that we include it here.

Fix a Banach space \( X \) and a Lipschitz map \( L : M \to X \) mapping \( 0 \) to \( 0 \). Extend linearly \( L \) from \( M \) onto \( \text{span}\{\delta_x : x \in M\} \) and denote this extension by \( \hat{L} \). We only need to check that \( ||\hat{L}||_{\text{Lip}} = ||L||_{\text{Lip}} \). Pick some \( a \in \text{span}\{\delta_x : x \in M\} \). Then \( ||\hat{L}(a)||_X = f(\hat{L}(a)) \) for some \( f \in B_X \). However, \( f \circ L \) then belongs to \( \text{Lip}_0(M) \) and \( ||f \circ L||_{\text{Lip}} \leq ||L||_{\text{Lip}} \). It follows that \( ||a||_X \leq ||\hat{L}(a)||_X \) which proves the claim. Then we can extend \( \hat{L} \) to \( \mathcal{F}(M) \), the closure of \( \text{span}\{\delta_x : x \in M\} \).

Using this universal property of \( \mathcal{F}(M) \), it is immediate that \( \mathcal{F}(M)^* = \text{Lip}_0(M) \). Indeed, it is enough to consider \( X = \mathbb{R} \) in the universal property mentioned above.

Further, it is useful to observe that whenever \( N \) is a subspace of a metric space \((M, d)\), then \( \mathcal{F}(N) \) is linearly isometric to a subspace of \( \mathcal{F}(M) \). Indeed, the isometry is determined by sending \( \delta_x \in \mathcal{F}(N) \) to \( \delta_x \in \mathcal{F}(M) \); in order to see it is an isometry it is enough to use the well-known fact that any \( f \in \text{Lip}_0(N) \) can be extended to \( F \in \text{Lip}_0(M) \) with \( ||f||_{\text{Lip}} = ||F||_{\text{Lip}} \), e.g., by putting \( F(x) = \inf\{f(n) + ||f||_{\text{Lip}}d(n, x) : n \in N\} \), \( x \in M \); see e.g. [16] Lemma 7.39. Using this observation together with the universal property of \( \mathcal{F}(M) \) we see that the Lipschitz structure of \( M \) corresponds to the linear structure of \( \mathcal{F}(M) \). For example, if \( N \) is bi-Lipschitz equivalent (resp. isometric) to a subset of \( M \), then \( \mathcal{F}(N) \) is linearly isomorphic (resp. linearly isometric) to a subspace of \( \mathcal{F}(M) \), etc.

The last basic fact we would like to mention here is that it is possible to give an ‘internal’ definition of the norm on \( \mathcal{F}(M) \), i.e., by a formula which refers only to the metric on the metric space \( M \). This is in contrast to the ‘external’ definition given above which refers to the space \( \text{Lip}_0(M) \) in the computation of the norm. This is described e.g. in [27]. The proof is not difficult and so we include it here as well.

Let us consider another norm, denoted by \( || \cdot ||_{KR} \), on \( \text{span}\{\delta_x : x \in M\} \) which is a variant of the so-called Kantorovich-Rubinstein metric, a concept that penetrates many areas of mathematics and computer science. Let us identify \( \delta_0 \)
with $0 \in \mathcal{F}(M)$. For $a \in \operatorname{span}\{\delta_x \colon x \in M \setminus \{0\}\}$, set
\[
\|a\|_{KR} = \inf\{|\alpha_1| \cdot d(y_1, z_1) + \ldots + |\alpha_n| \cdot d(y_n, z_n) : \\
a = \alpha_1(\delta_{y_1} - \delta_{z_1}) + \ldots + \alpha_n(\delta_{y_n} - \delta_{z_n})\}.
\]
It is straightforward to check that $\| \cdot \|_{KR}$ is a seminorm. Moreover, it is the largest seminorm $\| \cdot \|$ on $\operatorname{span}\{\delta_x : x \in M\}$ satisfying $\|\delta_x - \delta_y\|' \leq d(x, y)$ for every $x, y \in M$. Indeed, any seminorm $\| \cdot \|$ with that property must satisfy the inequality $\|x\|' \leq |\alpha_1|\|\delta_{y_1} - \delta_{z_1}\|' + \ldots + |\alpha_n|\|\delta_{y_n} - \delta_{z_n}\|'$ when $x = \alpha_1(\delta_{y_1} - \delta_{z_1}) + \ldots + \alpha_n(\delta_{y_n} - \delta_{z_n})$ which shows that $\|x\|' \leq \|x\|_{KR}$. Since the standard norm $\| \cdot \|$ on $\mathcal{F}(M)$ satisfies the condition, we get that $\| \cdot \| \leq \| \cdot \|_{KR}$ which implies that $\| \cdot \|_{KR}$ is actually a norm and that $\|\delta_x - \delta_y\|_{KR} = d(x, y)$ for every $x, y \in M$.

Consider now the identity mapping $L : M \to \operatorname{span}\{\delta_x : x \in M\}$ sending $x$ to $\delta_x$. It is an isometric embedding. By the universality property of $\mathcal{F}(M)$, $L$ extends to $\hat{L} : \mathcal{F}(M) \to \operatorname{span}\{\delta_x : x \in M\}$ which is still 1-Lipschitz. It follows that $\| \cdot \|_{KR} \leq \| \cdot \|$, so the norms $\| \cdot \|$ and $\| \cdot \|_{KR}$ are one and the same. This fact is often referred to as the Kantorovich duality.

3. Embedding of $\ell_1$

The purpose of this section is to prove Theorem [11]. It will be deduced from the fact that $\ell_\infty$ embeds into the dual of a Lipschitz-free space, i.e., into the space of Lipschitz functions. Let us note that we do not know whether $\ell_\infty$ embeds isometrically into $\operatorname{Lip}_0(M)$ for every infinite metric space $M$. The natural way of embedding $\ell_\infty$ into the space of Lipschitz functions is described in the lemma below.

**Lemma 3.1.** Let $(M, d)$ be a metric space, let $K > 0$, and let $(x_n, y_n)_{n \in \mathbb{N}}$ be a sequence of pairs of points from $M$ satisfying the following three conditions:

(i) For every $n \in \mathbb{N}$, we have $x_n \neq y_n$.

(ii) For every $n, m \in \mathbb{N}$, we have $x_n \notin U(y_n, K \cdot d(y_n, x_n))$.

(iii) For every $n \neq m$, we have $U(y_n, K \cdot d(y_n, x_n)) \cap U(y_m, K \cdot d(y_m, x_m)) = \emptyset$.

Then $\ell_\infty \hookrightarrow \operatorname{Lip}_0(M)$.

**Proof.** We may without loss of generality assume that $0 = x_1$ (because $\operatorname{Lip}_0(M) \ni f \mapsto f - f(x_1)$ is a linear isometry onto the space of Lipschitz functions $g$ with $g(x_1) = 0$). For every $n \in \mathbb{N}$, we define $f_n(x) := \max\{d(y_n, x) - \frac{d(y_n, x)}{K}, 0\}$, $x \in M$. Then $f_n \in \operatorname{Lip}(M)$. Moreover, it is easy to see that $\|f_n\|_{\operatorname{Lip}} \leq \frac{1}{K}$ and $\|f\|_\infty = d(y_n, x_n)$. By (ii), we have $K \cdot d(x_n, y_n) \leq d(x_1, y_1)$; hence, $f_n(0) = 0$.

Notice that condition (iii) implies that if $f_n(x) \neq 0$, then for every $m \neq n$, we have $f_m(x) = 0$. For every $x \in M$, we denote by $n(x)$ the unique $n \in \mathbb{N}$ with $f_n(x) \neq 0$ if it exists; otherwise, we put $n(x) := 1$. Finally, we define $T : \ell_\infty \to \operatorname{Lip}_0(M)$ by

$$T(\alpha)(x) := \alpha(n(x)) \cdot f_n(x)(x), \quad \alpha = (\alpha(n))_{n \in \mathbb{N}} \in \ell_\infty.$$ 

First, we will show that $T$ is linear and $\|T\| \leq \frac{2}{K}$. It is easy to see that $T$ is linear; hence, it suffices to show that for $\alpha = (\alpha(n))_{n \in \mathbb{N}} \in \ell_\infty$ with $\|\alpha\| = 1$, we have $\|T(\alpha)\|_{\operatorname{Lip}} \leq \frac{2}{K}$. Fix $x, y \in M$. We need to show that $|T(\alpha)(x) - T(\alpha)(y)| \leq \frac{2}{K}d(x, y)$. If $n(x) = n(y)$, this is easy because $f_n(x)$ is $\frac{1}{K}$-Lipschitz. Hence, we may
assume that \( n(y) \neq n(x) \). Thus, we have \( f_n(x)(y) = 0 = f_n(y)(x) \) and
\[
|T(\alpha)(x) - T(\alpha)(y)| \leq f_n(x)(x) + f_n(y)(y) = |f_n(x)(x) - f_n(x)(y)| + |f_n(y)(y) - f_n(y)(x)| \leq \frac{2}{k} d(x, y).
\]

In order to see that \( T \) is an isomorphism, we will use condition (ii). Fix \( \alpha = (\alpha(n))_{n \in \mathbb{N}} \in \ell_\infty \) and \( N \in \mathbb{N} \). By (ii), for every \( k \in \mathbb{N} \), we have \( f_k(x_N) = 0 \); hence, \( T(\alpha)(x_N) = 0 \) and we have
\[
|T(\alpha)(x_N) - T(\alpha)(y_N)| = |T(\alpha)(y_N)| = |\alpha(N)| f_N(y_N) = |\alpha(N)| d(x_N, y_N).
\]
Therefore, \( \|T(\alpha)\|_{\text{Lip}} \geq |\alpha(N)| \) and, since \( N \) was arbitrary, \( \|T(\alpha)\|_{\text{Lip}} \geq \|\alpha\|_\infty \). \( \square \)

The following result is the main step towards the proof of Theorem 3.1.

**Theorem 3.2.** Let \( M \) be an infinite metric space. Then \( \ell_\infty \hookrightarrow \text{Lip}_0(M) \).

**Proof.** First, note that we may without loss of generality assume that \( M \) is complete, because otherwise we take the completion \( N \) of \( M \) and use the obvious fact that \( \text{Lip}_0(\overline{M}) \) is linearly isometric to \( \text{Lip}_0(A) \) for every \( A \subset N \); in particular, \( \text{Lip}_0(N) \) is linearly isometric to \( \text{Lip}_0(M) \).

Now, we will prove the statement considering several cases. In each of them we will find a sequence of pairs of points from \( M \) satisfying the assumptions of Lemma 3.1.

**Case 1.** \( M \) is unbounded, i.e., for every \( K > 0 \), there are \( x, y \in M \) with \( d(x, y) > K \).

**Proof of Case 1**. Pick a sequence \((z_n)_{n=1}^\infty\) in \( M \) such that, for every \( n \in \mathbb{N} \), we have \( d(z_{n+1}, 0) > 2d(z_n, 0) \). Now, for each \( n \in \mathbb{N} \), put \( x_n := z_{2n-1} \) and \( y_n := z_{2n} \). We will show that the sequence \((x_n, y_n)_{n \in \mathbb{N}}\) satisfies the assumptions of Lemma 3.1 with \( K = \frac{1}{3} \).

Obviously, (i) is satisfied. Further, for \( n < m \), we have
\[
\begin{align*}
(3.1) & \quad d(z_n, z_m) \leq d(z_n, 0) + d(z_m, 0) < (1 + 2^{-(m-n)})d(z_m, 0), \\
(3.2) & \quad d(z_n, z_m) \geq d(z_m, 0) - d(z_n, 0) > (1 - 2^{-(m-n)})d(z_m, 0).
\end{align*}
\]

Let us show that (ii) holds. We need to show that, for \( m < n \), we have
\[
(3.3) \quad d(z_{2m-1}, z_{2n}) \geq \frac{1}{3} \cdot d(z_{2n-1}, z_{2n}).
\]
This is obvious if \( m = n \). If \( m < n \), then we have
\[
d(z_{2m-1}, z_{2n}) \geq (1 - 2^{-(2n-2m+1)})d(z_{2n}, 0) > (1 - 2^{-3})(1 + 2^{-1})^{-1}d(z_{2n}, z_{2n-1}).
\]
If \( n < m \), then we have
\[
d(z_{2m-1}, z_{2n}) \geq (1 - 2^{-1})d(z_{2m-1}, 0) > (1 - 2^{-1})d(z_{2n}, 0)
\]
\[
\geq (1 - 2^{-1})(1 + 2^{-1})^{-1}d(z_{2n}, z_{2n-1}).
\]
This proves (3.3); hence, condition (ii) from Lemma 3.1 is satisfied.

Finally, in order to see that (iii) holds, it is sufficient to see that, for \( n \neq m \), we have
\[
(3.4) \quad d(z_m, z_n) > \frac{1}{3} \cdot (d(z_{2n-1}, z_{2n}) + d(z_{2m-1}, z_{2m})).
\]
We may assume that \( n < m \) and then we have
\[
d(z_{2n-1}, z_{2n}) + d(z_{2m-1}, z_{2m}) \leq (1 + 2^{-1})^2 d(z_{2m}, 0) \\
< (1 + 2^{-1})^2 (1 - 2^{-(2m-2n)})^{-1} d(z_{2m}, z_{2n}) \\
\leq (1 + 2^{-1})^2 (1 - 2^{-2})^{-1} d(z_{2m}, z_{2n}) = 3d(z_{2m}, z_{2n}).
\]
This proves \((3.4)\); hence, condition (iii) from Lemma 3.1 is satisfied. \( \square \)

**Case 2.** \( M \) is bounded and there is a closed infinite subset \( N \subset M \) such that each point \( n \in N \) is isolated in \( N \).

**Proof of Case 2** Fix \( N \) as above. Since \( M \) is bounded, there is \( D > 0 \) such that, for every \( x, y \in M \), we have \( d(x, y) \leq D \). Since \( N \) does not contain any nontrivial Cauchy sequence, there is an infinite subset \( P \subset N \) which is uniformly discrete, i.e., there is \( C > 0 \) such that, for every \( x, y \in P \) with \( x \neq y \), we have \( d(x, y) \geq C \) (this is a simple exercise using the classical Ramsey theorem; see e.g. [24, Exercise 5.5]). Fix a one-to-one sequence \((a_n)_{n=0}^{\infty}\) of points from \( P \) and, for every \( n \in \mathbb{N} \), put \( y_n = a_n \) and \( x_n = a_0 \). It remains to verify that the sequence \((x_n, y_n)_{n \in \mathbb{N}}\) satisfies the assumptions of Lemma 3.1 with \( K = \min\{\frac{C}{2D}, 1\} \). It is clear that conditions (i) and (ii) are satisfied, because \( K \leq 1 \). Moreover, for every \( n \in \mathbb{N} \), we have \( K \cdot d(x_n, y_n) \leq KD \leq C/2 \); hence, \( U(y_n, K \cdot d(x_n, y_n)) \subset U(y_n, C/2) \) and, since \((y_n)_{n \in \mathbb{N}}\) is \( C \)-discrete, the balls are pairwise disjoint. This verifies condition (iii) from Lemma 3.1. \( \square \)

**Case 3.** \( M \) is bounded and it contains infinitely many limit points.

**Proof of Case 3** First, let us assume there is a sequence \((y_n)_{n \in \mathbb{N}}\) consisting of limit points in \( M \) with \( y_n \to y \). Then, for each \( n \in \mathbb{N} \), put \( r_n := \text{dist}(y_n, \{y_m : m \neq n\}) > 0 \) and pick some \( x_n \in U(y_n, r_n/2) \) with \( x_n \neq y_n \). Then it is easy to see that the sequence \((x_n, y_n)_{n \in \mathbb{N}}\) satisfies the assumptions of Lemma 3.1 with \( K = 1 \).

Otherwise, the set \( N \) consisting of all the limit points in \( M \) satisfies the assumptions of Case 2. \( \square \)

Note that now it remains to handle the case when \( M \) is compact and it contains a nontrivial convergent sequence consisting of isolated points. Indeed, by the already proven Cases 1,3 we may assume \( M \) is bounded and contains only finitely many limit points. Then either \( M \) is compact, or there is an infinite closed set of isolated points in \( M \) and we may apply Case 2.

**Case 4.** \( M \) is compact and it contains a nontrivial convergent sequence consisting of isolated points.

**Proof of Case 4** Let \((a_n)_{n \in \mathbb{N}}\) be a nontrivial convergent sequence consisting of isolated points with the limit point \( a \). It is easy to construct by induction a sequence \((x_n, y_n)_{n \in \mathbb{N}}\) of pairs of points from \( M \) with
\[
\begin{align*}
(\text{a}) & \quad \forall n \in \mathbb{N} : \quad y_n \in \{a_n : n \in \mathbb{N} \} \text{ and } y_n \notin \{y_m : m < n\} \cup \{x_m : m < n\}, \\
(\text{b}) & \quad \forall n \in \mathbb{N} : \quad d(y_n, a) < \min\{d(y_n, y_m) : m < n\}, \text{ and} \\
(\text{c}) & \quad \text{for every } n \in \mathbb{N}, \text{ we pick } x_n \text{ to be any point with } d(x_n, y_n) = \text{dist}(y_n, M \setminus \{y_n\}).
\end{align*}
\]
Now, having such a sequence \((x_n, y_n)_{n \in \mathbb{N}}\), it remains to check that it satisfies the assumptions of Lemma 3.1 with \( K = 1 \). Obviously, (i) is satisfied. Moreover, we
have \(d(x_n, y_n) \leq d(y_n, x)\) for every \(x \in M \setminus \{y_n\}\) and so in order to verify (ii), it is enough to observe that, for \(n, m \in \mathbb{N}\), we have \(x_n \neq y_m\). This follows from (a) for \(n < m\), from (c) for \(n = m\); and from (b) for \(n > m\), because in the last case we have \(d(x_n, y_n) \leq d(y_n, a) < d(y_n, y_m)\).

It remains to verify (iii). But this is easy, because, for every \(n \in \mathbb{N}\), by the choice of \(x_n\) we have \(U(y_n, d(x_n, y_n)) = \{y_n\}\). □

Since the cases mentioned above cover all the possibilities, this completes the proof of Theorem 3.2.

**Proof of Theorem 1.1** This is a consequence of Theorem 3.2. Indeed, it is a classical result that, for every Banach space \(X\), \(\ell_\infty \hookrightarrow X^*\) if and only if \(\ell_1\) is isomorphic to a complemented subspace of \(X\) [1 Theorem 4]; hence, (i) follows. Since any complemented subspace of a \(C(K)\) space contains \(c_0\) (see e.g., [25 Theorem 5.1]), from (i) we get (ii) because \(c_0\) is not isomorphic to a subspace of \(\ell_1\). Since the dual space contains \(c_0\), it is not weakly sequentially complete and so it is not isomorphic to \(L^1(\mu)\) [25 Corollary III.C.14]. Therefore, \(X\) is not isomorphic to any \(L^1\)-predual, in particular, not to the Gurarii space [15]; see also [12 Theorem 2.17]. As it is well known that \(\ell_1\) is projectively universal, i.e., for any separable Banach space \(Y\) there exists a bounded linear operator from \(\ell_1\) onto \(Y\), the same is true for \(X\) since \(\ell_1\) is complemented there. □

**Remark 3.3.** During the review process of this paper we observed that from the assumptions of Lemma 3.1 it is possible not only to deduce that \(\ell_1\) is isomorphic to a complemented subspace of \(\mathcal{F}(M)\), but it is even possible to describe this subspace relatively easily. Let us assume that \((x_n, y_n)_{n \in \mathbb{N}}\) is as in Lemma 3.1. For each \(n \in \mathbb{N}\), put \(e_n := \frac{\delta_{y_n} - \delta_{x_n}}{d(y_n, x_n)} \in \mathcal{F}(M)\). Then, using a similar proof as in Lemma 3.1, we get that \((e_n)_{n \in \mathbb{N}}\) is \(2/K\)-equivalent to the \(\ell_1\) basis. Moreover, consider functions \((f_n)_{n \in \mathbb{N}}\) from the proof of Lemma 3.1 and define \(r : M \to \mathcal{F}(M)\) by \(r(x) := \sum_{n \in \mathbb{N}} f_n(x) e_n\), \(x \in M\). Then it is possible to verify that \(r\) is \(2/K\)-Lipschitz. Using the universal property of \(r\) we find \(P : \mathcal{F}(M) \to \mathcal{F}(M)\) with \(P \circ \delta = r\) and \(\|P\| \leq 2/K\). Finally, one can verify that \(P\) is actually a projection onto \(\mathrm{span}\{e_n : n \in \mathbb{N}\}\).

4. Embedding into \(L_1\)

The purpose of this section is to prove Theorem 1.2. In order to prove it, we will need the following result. The proof is just a modification of the arguments from [19].

**Theorem 4.1.** For any measure \(\mu\), \(\mathcal{F}([0,1]^2) \not\hookrightarrow L_1(\mu)\).

**Proof.** In order to shorten our notation, put \(I := [0,1]^2\). If there is a measure \(\mu\) with \(\mathcal{F}(I) \hookrightarrow L_1(\mu)\), then there is a continuous linear mapping from \(L_\infty(\mu)\) onto \(\text{Lip}_0(I)\). Since \(L_\infty(\mu)\) is a commutative \(C^*\)-algebra, there exists a compact Hausdorff space \(K\) such that \(L_\infty(\mu)\) is isometric to \(C(K)\). Hence, it suffices to show that there does not exist a bounded linear mapping \(T : C(K) \to \text{Lip}_0(I)\) which is onto. We only show that the “identity” mapping \(id : \text{Lip}_0(I) \to W^{1,1}(I)\) is absolutely summing. Then the rest can be proved by just copying line by line the arguments from [19 Theorem 3], where this statement is proved for the space \(C^1(I)\) instead of \(\text{Lip}_0(I)\) using the fact that “identity” mapping \(id : C^1(I) \to W^{1,1}(I)\) is
“absolutely summing” \( W^{1,1}(I) \) is the Sobolev space). So consider the “identity” mapping \( id : \text{Lip}_0(I) \rightarrow W^{1,1}(I) \). More precisely, having a Lipschitz function \( f \), we denote by \([f]\) the equivalence class containing all the functions which are equal to \( f \) almost everywhere. The “identity” mapping is the mapping \( f \mapsto [f] \). By the classical Rademacher’s theorem (see e.g. [23]), every Lipschitz function defined on \( I \) is almost everywhere differentiable and so it is possible to put \( ||[f]||_W := \int_{[0,1]^2} (|f(x,y)| + |\partial_1 f(x,y)| + |\partial_2 f(x,y)|) \, dx \, dy \). It is immediate that \( ||[f]||_W \leq 3||f||_{\text{Lip}} \) and it remains to show that the mapping \( f \mapsto [f] \) is absolutely summing; i.e., there is a constant \( C \) such that whenever \((f_i)_{i=1}^m \) are functions from \( \text{Lip}_0(I) \), then

\[
\sum_{i=1}^m ||[f_i]||_W \leq C \sup \{ \sum_{i=1}^m |x^*(f_i)| : x^* \in \text{Lip}_0(I)^*, ||x^*|| \leq 1 \}.
\]

Let us define \( \Phi : \text{Lip}_0([0,1]^2) \rightarrow L_\infty(I) \oplus_1 L_\infty(I) \oplus_1 L_\infty(I) \) by \( \text{Lip}_0(I) \ni f \mapsto \Phi(f) := (f, \partial_1 f, \partial_2 f) \). Note that \( \Phi \) is a linear bounded operator. Further, consider \( \Psi : L_\infty(I) \oplus_1 L_\infty(I) \oplus_1 L_\infty(I) \rightarrow L_1(I) \oplus_1 L_1(I) \oplus_1 L_1(I) \) defined as the identity. It is a standard fact (see e.g. [2] Remark 8.2.9]) that the identity operator from \( L_\infty(I) \) to \( L_1(I) \) is absolutely summing; hence, \( \Psi \) is absolutely summing. It is a classical fact that composition of a bounded operator with an absolutely summing one is absolutely summing; see e.g. [21] Proposition 8.2.5. Hence, \( id = \Psi \circ \Phi \) is absolutely summing.

**Remark 4.2.** The result that \( F(\mathbb{R}^2) \not\hookrightarrow L_1 \) is often mentioned as a result of A. Naor and G. Schechtmann [22]. The proof above shows that, using minor modifications, it actually follows already from [19].

The rest of this section is devoted to the proof of Theorem 1.2. First, we construct the countable compact space with one accumulation point and then in a series of claims we prove the statement.

For every \( n \geq 2 \), let \( (A_n, d_n) \) be the set \( \{(\frac{i}{n}, \frac{j}{n}) : 0 \leq i, j \leq n\} \) equipped with the Euclidean distance \( d_n \) inherited from \( \mathbb{R}^2 \). Denote by \( K \) the amalgamated metric sum of \( A_n \)'s over \( 0 \). That is, we take \( K \) to be the disjoint union \( \bigsqcup_n A_n \) with the zero element \((0,0)\) identified in all of them. The metric \( d \) on \( K \) is defined as follows. For \( a, b \in K \), we set

\[
d(a, b) = \begin{cases} d_n(a, b), & \exists n(a, b \in A_n), \\ d_n(a, 0) + d_m(b, 0), & a \in A_n, b \in A_m, n \neq m. \end{cases}
\]

It is easy to check that \( K \) is a countable compact metric space; in fact, it is a convergent sequence, i.e., it has only one accumulation point, the zero.

**Claim 1.** \( F(K) \) is isometric to \( \bigoplus L_1 F(A_n) \).

**Proof.** This is easy and proved e.g. in [17] Proposition 5.1. \( \square \)

For every \( n \), consider the set \( nA_n := \{(\frac{i}{n}, \frac{j}{n}) : 0 \leq i, j \leq n\} \) again equipped with the Euclidean distance. Clearly, \( F(nA_n) \) is isometric to \( F(A_n) \). Indeed, since both spaces are finite-dimensional, it suffices to find an isometry of their duals; the mapping \( \phi : \text{Lip}_0(A_n) \rightarrow \text{Lip}_0(nA_n) \) defined by \( \phi(f)(x) := nf(x) \), \( x \in nA_n, f \in \text{Lip}_0(A_n) \), is such an isometry. As a consequence we get the following.

**Claim 2.** \( F(K) \) is linearly isometric to \( \bigoplus L_1 F(nA_n) \).
Since \( nA_n \), for each \( n \), is a subset of \([0, 1]^2\), we may and will consider \( F(nA_n) \) as a subspace of \( F([0, 1]^2) \). Notice that \( \bigcup_n nA_n \) is dense in \([0, 1]^2\). We need one more technical claim which says that finite-dimensional subspaces of \( F([0, 1]^2) \) can be approximated by finite-dimensional subspaces of \( F(nA_n) \) for large enough \( n \). In the following, by \( d_{BM} \) we denote the Banach-Mazur distance.

**Claim 3.** Let \( E \subseteq F([0, 1]^2) \) be a finite-dimensional subspace and let \( \varepsilon > 0 \) be arbitrary. Then there exist \( n \in \mathbb{N} \) and a finite-dimensional subspace \( E' \subseteq F(nA_n) \) such that \( d_{BM}(E, E') < 1 + \varepsilon \).

**Proof.** Let \( e_1, \ldots, e_m \) be a basis of \( E \). Since all norms on \( E \) are equivalent, there is \( D > 0 \) such that for all \( a \in \mathbb{R}^m \), we have \( \sum_{i=1}^m |a(i)| \leq D \sum_{i=1}^m a(i)e_i \). Fix \( \delta = \frac{\varepsilon}{2 + \varepsilon} \), i.e., such that \( \frac{1 + \delta}{1 - \delta} = 1 + \varepsilon \). Each \( e_i \) can be \((\delta/2mD)\)-approximated by some linear combination of elements from \( \text{span}\{\delta_y : y \in [0, 1]^2\} \). Without loss of generality, we may assume that for each \( i \leq m \), such a linear combination is of the same length. So for each \( i \leq m \), we choose some \( \alpha_i^1 \delta_{x_i^1} + \ldots + \alpha_i^m \delta_{x_i^m} \) ∈ \( \text{span}\{\delta_y : y \in [0, 1]^2\} \) such that \( \|e_i - (\alpha_i^1 \delta_{x_i^1} + \ldots + \alpha_i^m \delta_{x_i^m})\| < \delta/2mD \).

Now, since \( \bigcup_{n \in \mathbb{N}} nA_n \) is dense in \([0, 1]^2\), if we take \( n \) large enough, then for every \( i \leq m \) and \( j \leq l \), we can find \( \alpha_j^i \in nA_n \) such that \( \|\delta_{x_j^i} - \delta_{a_i^j}\| < \frac{\delta}{2m\lambda_dD} \), where \( \alpha = \max\{\|\alpha_j^i\| : j \leq l, i \leq m\} \). Consequently, for every \( i \leq m \), we get

\[
\|\alpha_j^i \delta_{x_j^i} + \ldots + \alpha_l^i \delta_{x_l^i} - (\alpha_1^i \delta_{a_1^i} + \ldots + \alpha_m^i \delta_{a_m^i})\| < \alpha \cdot \frac{\delta}{2m\lambda_dD} = \frac{\delta}{2mD}.
\]

Thus, if for every \( i \leq m \) we denote \( \alpha_i^1 \delta_{a_i^1} + \ldots + \alpha_i^m \delta_{a_i^m} \), by \( e_i' \), we have \( \|e_i - e_i'\| < \delta/mD \). Hence, for any \( a \in \mathbb{R}^m \), we have \( \|\sum_{i=1}^m a(i) e_i - \sum_{i=1}^m a(i) e_i'\| < \delta/D(\sum_{i=1}^m |a(i)|) \leq \delta \| \sum_{i=1}^m a(i)e_i \| \) and, consequently,

\[
\| \sum_{i=1}^m a(i) e_i \| < (1 + \delta) \| \sum_{i=1}^m a(i) e_i' \| \quad \text{and}
\]

\[
\| \sum_{i=1}^m a(i) e_i \| < \| \sum_{i=1}^m a(i) e_i' \| + \delta \| \sum_{i=1}^m a(i) e_i \| .
\]

Denote by \( E' \) the subspace \( \text{span}\{e_i' : i \leq m\} \subseteq F(nA_n) \). Using the above, the linear mapping determined by sending \( e_i \) to \( e_i' \), for \( i \leq m \), is a witness to the fact that \( d_{BM}(E, E') < \frac{1 + \delta}{1 - \delta} = 1 + \varepsilon \). \( \square \)

Let us now formulate a result of Lindenstrauss and Pelczyński that will help us finish the proof.

**Theorem 4.3** (Theorem 7.1 in [21]). Let \( X \) be a Banach space and fix \( \lambda \geq 1 \). If for every finite-dimensional subspace \( E \) of \( X \) there exists a finite-dimensional subspace \( E' \) of \( \ell_1 \) such that \( d_{BM}(E, E') < \lambda \), then there exists a measure \( \mu \) and a subspace \( Y \) of \( L_1(\mu) \) such that \( d_{BM}(X, Y) \leq \lambda \).

We are now ready to finish the proof of Theorem 1.2. By Theorem 4.1 we have that \( F([0, 1]^2) \) does not embed into \( L_1(\mu) \) for any measure \( \mu \). However, then by Theorem 4.3 we get that for every \( N \in \mathbb{N} \) there exists a finite-dimensional subspace \( E_N \) of \( F([0, 1]^2) \) such that for every finite-dimensional subspace \( E \) of \( \ell_1 \) we have \( d_{BM}(E_N, E) > N \). Using Claim 3 for each \( N \), we can find some \( n(N) \in \mathbb{N} \) and finite-dimensional subspace \( E_{n(N)} \) of \( F(n(N)A_n(N)) \) such that \( d_{BM}(E_N, E_{n(N)}) < 2 \).
Assume now that $\mathcal{F}(K)$ embeds into $L_1$ via some linear embedding of norm less than $N/8$ for some $N \in \mathbb{N}$. By Claim $2$, $\bigoplus_{\ell_1} \mathcal{F}(nA_n)$ embeds into $L_1$ via some linear embedding $T$ of norm less than $N/8$. Now, $T$ restricted on $E_{n(N)} \leq \mathcal{F}(nA_{n(N)})$ still has norm bounded by $N/8$. In particular, there is some finite-dimensional subspace $Y_{n(N)}$ of $L_1$ such that $d_{BM}(E_{n(N)}, Y_{n(N)}) \leq N/8$. Since $L_1$ is finitely representable in $\ell_1$ (see [2] Proposition 11.1.7), there exists a finite-dimensional subspace $Y_N$ of $\ell_1$ such that $d_{BM}(Y_N, Y_{n(N)}) < 2$.

Now, putting all of these inequalities together we get

$$\frac{N}{8} \geq d_{BM}(E_{n(N)}, Y_{n(N)}) \geq \frac{d_{BM}(E_N, Y_N)}{d_{BM}(E_{n(N)}, Y_{n(N)})} > \frac{N}{4},$$

and that is a contradiction finishing the proof.

**Remark 4.4.** During the review process of this paper, the paper [17] was published (see [18]). However, in the published version the statement [17, Proposition 5.1], which we cite in the proof of Claim 1 above is missing. Thus, we would like to sketch the easy proof of it here.

Consider

$$\Phi : \bigoplus_{\ell_\infty} \text{Lip}_0(A_n) \to \text{Lip}_0(K),$$

defined by $\Phi((f_n))(x) = f_n(x)$, $(f_n) \in \bigoplus_{\ell_\infty} \text{Lip}_0(A_n)$, $x \in A_n$. Then it is easy to verify that $\Phi$ is an isometry onto and a $w^* - w^*$ homeomorphism. Hence, it is the adjoint of an isometry from $\mathcal{F}(K)$ onto $\bigoplus_{\ell_1} \mathcal{F}(A_n)$.

**Remark 4.5.** During the review process of this paper, it was observed by G. Lancien and A. Procházka that our method of proof actually gives that $K$ from the statement of Theorem [11] can be taken as a subset of $[0, 1]^2$ and that there does not exist a bi-Lipschitz embedding of $\mathcal{F}(K)$ into $L_1$. Let us sketch the argument here.

First, the only place where we used the metric of $K$ was to prove Claim 1. However, it is easy to see that taking a sequence $(k_n)_{n \in \mathbb{N}}$ increasing fast enough, we have that $\mathcal{F}(\bigcup_{n \in \mathbb{N}} A_{k_n})$ is linearly isomorphic to $\bigoplus_{\ell_1} \mathcal{F}(A_{k_n})$ (using the same mapping $\Phi$ as in Remark 4.3), which would be enough for the rest of the proof. Hence, we may have $K = \bigcup_{n \in \mathbb{N}} A_{k_n}$. Moreover, our proof gives that $\mathcal{F}(K)$ does not linearly embed into any Banach space finitely representable in $\ell_1$. If there was a bi-Lipschitz embedding of $\mathcal{F}(K)$ into $L_1$, then, by [2] Corollary 7.10, $\mathcal{F}(K)$ embeds linearly into $(L_1)^{**}$ which is by the principle of local reflexivity [2] Theorem 11.2.4 finitely represented in $L_1$ (in particular, $(L_1)^{**}$ is finitely represented in $\ell_1$ because $L_1$ is), a contradiction.

5. Embedding of $c_0$

Let $M$ be a separable metric space that contains a bi-Lipschitz copy of every separable metric space. By [14] Theorems 2.12 and 3.1, we have $X \hookrightarrow \mathcal{F}(X)$ for every separable Banach space $X$; therefore, $\mathcal{F}(M)$ is a universal separable Banach space, i.e., $\mathcal{F}(M)$ contains an isomorphic copy of every separable Banach space. Note that by the result of Aharoni [1] this is equivalent to the condition that $M$ contains a bi-Lipschitz copy of $c_0$. Y. Dutrieux and V. Ferenczi in [9] asked for the converse; see Question [1]. In this section we prove Theorem [1.3] making partial progress towards the answer to this question.
Let $M$ be either $[0, 1]^n$ or $\mathbb{R}^n$. By $C^1(M)$ we denote the space of functions $F : M \to \mathbb{R}$ whose derivatives of order $\leq 1$ are continuous on $M$. For $F \in C^1(M)$, we define $\|F\|_{\infty} := \max\{\|F\|_{\infty}, \|\partial_x F\|_{\infty} : i \leq n\}$. It is well known that the space $(C^1([0, 1]^n), \|\cdot\|_{\infty}^1)$ is a Banach space.

The following result was essentially proved by J. Bourgain [5], [6]. The result of J. Bourgain concerns the space of smooth functions over $n$-dimensional torus; however, the same proof works for the $n$-dimensional cube. We refer also to [28], where a more detailed proof of the result of J. Bourgain may be found (use Example III.D.30 and Theorem III.D.31 and conclude similarly as in the proof of Corollary III.C.14).

**Theorem 5.1.** For every $n \in \mathbb{N}$, the Banach space $(C^1([0, 1]^n))^*$ is weakly sequentially complete, i.e., weakly Cauchy sequences are weakly convergent.

**Lemma 5.2.** Let $A \subset \mathbb{R}^n$ be a finite set and let $f : A \to \mathbb{R}$ be a 1-Lipschitz function (on $\mathbb{R}^n$ we consider Euclidean norm). Then, for every $\varepsilon > 0$, there exists $g \in C^1(\mathbb{R}^n)$, an extension of $f$ (i.e., $g \supset f$), with $\|g\|_{\infty} < \max\{\|f\|_{\infty}, 1\} + \varepsilon$.

**Proof.** Find $\delta > 0$ such that the balls $\{B(a, 2\delta) : a \in A\}$ are pairwise disjoint. Fix some even Lipschitz function $\tau \in C^1(\mathbb{R})$ with $\tau(0) = 1$, $\|\tau\|_{\infty} \leq 1$ and $\{x : \tau(x) \neq 0\} \subset (-\delta, \delta)$; e.g.,

$$\tau(x) = \begin{cases} e^{-\frac{1}{2\pi x^2 + \delta^2}}, & |x| < \delta, \\ 0, & \text{otherwise.} \end{cases}$$

Let $K$ be such that $\tau$ is $K$-Lipschitz and $K > 1$.

We may assume that $0 \in A$ and $f(0) = 0$. First, we extend $f$ to a 1-Lipschitz function defined on $\mathbb{R}^n$; see e.g. [16, Lemma 7.39]. We again call this extension $f$. Now, we find a 1-Lipschitz $\tilde{g} \in C^1(\mathbb{R}^n)$ with $\|f - \tilde{g}\|_{\infty} < \varepsilon/2K$; e.g. using the standard integral convolution [16, Lemma 7.1].

For $a \in A$, define $\phi_a : \mathbb{R}^n \to \mathbb{R}$ by $\phi_a(x) = (f(a) - \tilde{g}(a))\tau(\|x - a\|)$, $x \in \mathbb{R}^n$. Then $h := \sum_{a \in A} \phi_a$ is a well-defined $\varepsilon/2$-Lipschitz function such that $\|h\|_{\infty} \leq \varepsilon/2K$ and $h(a) = f(a) - \tilde{g}(a)$ for every $a \in A$. Moreover, since on a Hilbert space the norm is smooth everywhere except 0 and since the function $\tau$ is even, it is easy to observe that $h \in C^1(\mathbb{R}^n)$. It remains to put $g := \tilde{g} + h$. Then we have $g \in C^1(\mathbb{R}^n)$, $\|g\|_{\infty} < \|f\|_{\infty} + \varepsilon$ and $g$ is $(1+\varepsilon/2)$-Lipschitz; hence, $\|g\|_{\infty} < \max\{\|f\|_{\infty}, 1\} + \varepsilon$. □

**Remark 5.3.** Note that it follows from Lemma 5.2 that whenever $A$ is a finite set in $[0, 1]^n$ and $f$ is a 1-Lipschitz function on $A$ with $f(0) = 0$, there is $g \in C^1(\mathbb{R}^n)$ with $\|g\|_{\infty}^1 \leq \sqrt{n} + 1$. Therefore, by [10, Theorem 1], there is a linear extension operator $T : \text{Lip}_0(A) \to \text{Lip}_0(\mathbb{R}^n)$ with norm depending only on $n$; hence, $T^*|_{F(\mathbb{R}^n)}$ is a projection from $F(\mathbb{R}^n)$ onto $F(A)$. Consequently, whenever we have $M \subset [0, 1]^n$ and $A \subset M$ a finite set, $F(A)$ is $C(n)$-complemented in $F(M)$, where the constant $C(n)$ depends only on the dimension $n$. This gives another proof of the fact that $F(M)$ has BAP whenever $M \subset [0, 1]^n$ [20, Proposition 2.3].

**Lemma 5.4.** For every $n \in \mathbb{N}$, there is an isomorphism of $F([0, 1]^n)$ into $(C^1([0, 1]^n))^*$.

**Proof.** Put $Y = \{f \in C^1([0, 1]^n) : f(0) = 0\}$. Then $Y$ is a closed subspace of codimension 1; hence, it is complemented and $Y^*$ is isomorphic to a subspace of $(C^1([0, 1]^n))^*$. For every $x \in [0, 1]^n$, we define $T(\delta_x) \in Y^*$ by $T(\delta_x)(f) := f(x)$,
$f \in Y$. Extend $T$ linearly to the set $\text{span}\{\delta_x : x \in [0,1]^n\}$. Now, it is enough to verify that $T$ is an isomorphism into $Y^\star$.

Fix an element $\mu \in \text{span}\{\delta_x : x \in [0,1]^n\}$. There are $k \in \mathbb{N}$, $\alpha \in \mathbb{R}^k$ and $x_1, \ldots, x_k \in [0,1]^n$ with $\mu = \sum_{i=1}^{k} \alpha(i)\delta_{x_i}$. We have to find constants $C > 0$ and $D > 0$ with

$$C \sup \left\{ \left| \sum_{i=1}^{k} \alpha(i)f(x_i) \right| : \|f\|_1^1 \leq 1, f(0) = 0 \right\} \leq \sup \left\{ \left| \sum_{i=1}^{k} \alpha(i)f(x_i) \right| : \|f\|_{\text{Lip}} \leq 1, f(0) = 0 \right\} \leq D \sup \left\{ \left| \sum_{i=1}^{k} \alpha(i)f(x_i) \right| : \|f\|_\infty^1 \leq 1, f(0) = 0 \right\}.$$ 

The existence of constant $C$ follows from the basic fact that every function with total differential bounded by $K$ is $K$-Lipschitz; see [26, Theorem 9.19]. Hence, we may put $C = 1/\sqrt{n}$. The existence of $D$ follows from Lemma 5.2, which gives $D = \sqrt{n}$. □

**Proof of Theorem 1.3** By [18, Corollary 3.3], $\mathcal{F}(\mathbb{R}^n)$ is isomorphic to $\mathcal{F}([0,1]^n)$. Hence, $\mathcal{F}(\mathbb{R}^n)$ is weakly sequentially complete by Theorem 5.1 and Lemma 5.4. Finally, using the fact mentioned in Section 2 that $\mathcal{F}(M)$ is isometric to a subspace of $\mathcal{F}(\mathbb{R}^n)$, we see that $\mathcal{F}(M)$ is weakly sequentially complete. Consequently, $c_0$ does not embed isomorphically into $\mathcal{F}(M)$ because, as is well known and easy to prove, $c_0$ is not weakly sequentially complete. □

6. OPEN PROBLEMS

As was mentioned in Section 5 if $M$ contains a bi-Lipschitz copy of $c_0$, then $\mathcal{F}(M)$ is a universal separable Banach space. Hence, we have quite a rich family of universal separable Banach spaces. By Theorem 1.1 they are all different from $\mathcal{C}(K)$ spaces and from the Gurari˘ı space. One example is Pelczyński’s universal basis space $\mathbb{P}$ (which is unique up to isomorphism). This space is isomorphic to $\mathcal{F}(\mathbb{P})$; see [14, p. 139]. Another example is the Holmes space, i.e., the Lipschitz-free space over the Urysohn universal metric space. By [11, Theorem 4.2], the Holmes space is not isomorphic to $\mathbb{P}$. By [9, Theorem 5], $\mathcal{F}(c_0)$ is isomorphic to each $\mathcal{F}(\mathcal{C}(K))$. It could be of some interest to find out what isomorphic types of universal Banach spaces we are able to get using the Lipschitz-free construction. For example, the following seems to be open.

**Question 2.** Is $\mathcal{F}(c_0)$ isomorphic to the Holmes space or to $\mathbb{P}$?

In light of Theorem 1.3 it is also natural to ask the following.

**Question 3.** Is it true that $c_0 \hookrightarrow \mathcal{F}(\ell_2)$?

Note that $c_0$ does not bi-Lipschitz embed into $\ell_p$ ($1 \leq p < \infty$); see e.g. [3, p. 169]. Hence, the negative answer to the above question would be partial progress towards the answer to Question 11. Similarly, we do not know the answer to the following question.

**Question 4.** Is it true that $c_0 \hookrightarrow \mathcal{F}(\ell_1)$?

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