A review on the decoy-state method for practical quantum key
distribution

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Abstract

We present a review on the historic development of the decoy state method, including the background, principles, methods, results and development. We also clarify some delicate concepts. Given an imperfect source and a very lossy channel, the photon-number-splitting (PNS) attack can make the quantum key distribution (QKD) in practice totally insecure. Given the result of ILM-GLLP, one knows how to distill the secure final key if he knows the fraction of tagged bits. The purpose of decoy state method is to do a tight verification of the the fraction of tagged bits. The main idea of decoy-state method is changing the intensities of source light and one can verify the fraction of tagged bits of certain intensity by watching the the counting rates of pulses of different intensities. Since the counting rates are small quantities, the effect of statistical fluctuation is very important. It has been shown that 3-state decoy-state method in practice can work even with the fluctuations and other errors.
I. INTRODUCTION AND BACKGROUND

Although many standard quantum key distribution (QKD) protocols such as BB84 [1] have been proven to be unconditionally secure [2, 3, 4], this does not guarantee the security of QKD in practice, due to various types of imperfections in a practical set-up. In practical QKD, the source is often imperfect. Say, it may produce multi-photon pulses with a small probability. Normally weak coherent state is used in practical QKD. The probability of multi-photon pulses is around 10% among all non-vacuum pulses. On the other hand, the channel can be very lossy. For example, if we want to do QKD over a distance longer than 100kms, the overall transmittance can be in the magnitude order of $10^{-3}$ or even $10^{-4}$. This opens a door for the Eavesdropper (Eve) by the so called photon-number-splitting (PNS) attack [5].

It was then shown by Inamori, Lütkenhaus and Mayers (ILM) [7] and by Gottesman et al (GLLP) [8] on how to distill the secure final key even with an imperfect source, provided that we have a way to verify the upper bound of the fraction of tagged bits (counts caused by multiphoton pulses from the source) or equivalently, the lower bound of untagged bits (counts caused by single-photon pulses from the source). However, the ILM-GLLP [7, 8] result does not tell us how to make the verification itself. What it has presented is how to make the final key given the verified results of fraction of tagged bits or fraction of untagged bits. Therefore, the only difficulty remained for secure QKD in practice is the verification. Non-trivial verification is the central issue of the decoy-state method.

Before going into the decoy-state method, let’s first recall some concepts and results of ILM-GLLP [7, 8].

A. Tagged bits

Suppose in a QKD protocol, Alice is the sender and Bob is the receiver. In the standard protocols such as BB84 with perfect single photon source, there is no tagged bits because an Eavesdropper (Eve) in principle will cause errors to Bob’s bits if she wants to know some of the bit values of Bob. However, if Alice uses an imperfect source, things will be different. Suppose sometimes Alice sends a multi-photon pulse. All the photons in the pulse have the same state. Eve can keep one photon from the pulse and sends other photons of the
pulse to Bob. This action will not cause errors to the bit value to bob but Eve may have full information about Bob’s bit: After the measurement basis is announced by Alice or Bob, Eve will be always able to measure the photon she has kept in the correct basis. *If Eve can know some of bit values without causing any errors, these bits are defined as tagged bits.* Given an imperfect source, whenever Alice sends out a multi-photon pulse and Bob’s detector counts, we assume that a tagged bit has been produced. We don’t care how many photons the pulse may contain after it is transmitted to Bob’s side.

*Remark.* Bob cannot verify the tagged bits at his side by measuring the photon number in each coming pulses. Say, suppose he finds certain pulse contains only one photon. The bit caused by that pulse could be still a tagged bit because the pulse could have contained two photons at Alice’s side.

### B. Final key distillation with a fraction of tagged bits.

In the standard BB84 protocol with perfect single photon source, there is no tagged bits. To distill the final key, we need the information of bit-flip error rate $t_b$ and *phase-flip* rate $t_p$. Based on this information we can in principle have a CSS code to correct all bit-flip errors (error correction) and to compress any third party’s information to almost zero (privacy amplification). In particular, we shall use a CSS code that consumes $n_r H(t_b) = n_r[-t_b \log_2 t_b - (1 - t_b) \log_2 (1 - t_b)]$ raw bits to correct bit-flip errors and $n_r H(t_p)$ raw bits for the privacy amplification. Here $n_r$ is the number of raw bits. It was shown by ILM-GLLP that we can also distill the secure final key by a CSS code even with an imperfect source, if we know the bit-flip rate, phase-flip rate and upper bound value of the fraction of tagged bits, $\Delta$. In particular we shall use a CSS code that consumes $n_t H(t_b)$ raw bits for error correction and $n_r [\Delta + (1 - \Delta)H(\frac{t_p}{1 - \Delta})]$ for privacy amplification. This is to say, in a protocol with perfect single-photon source, we shall know how to distill the final key if we know the bit-flip rate and phase-flip rate; in a protocol with an imperfect source, we shall also know how to distill the final key if we know the bit-flip rate, phase-flip rate and $\Delta$ value, the upper bound of the tagged bits. Equivalently, we can also use $\Delta_1$, the lower bound of untagged bits. It is easy to verify the bit-flip rate and phase-flip rate: Alice and Bob simply take some samples and announce the bit values. Asymptotically, the error rate
rate of the remained raw bits is equal to the error rate of those samples. However, to know a
tight bound value of $\Delta$ or $\Delta_1$ is not so straightforward. The central task for the decoy-state
method is to make a tight verification of $\Delta$ or $\Delta_1$.

II. HWANG’S IDEA AND PROTOCOL

The first idea of decoy-state method and the first protocol is given by Hwang[9]. Hwang
proposed to do the non-trivial verification by changing the intensity of pulses. In Hwang’s
first protocol, two intensities are used. The intensity of signal pulses is set to be around
0.3 and the intensity of decoy-state pulses is set to be 1. By watching the counting rate of
decoy pulses, one can deduce the upper bound of the fraction of tagged bits among all those
signal pulses.

For clarity, we first demonstrate why a tight verification is non-trivial. Knowing the
photon-number distribution of the source and the channel transmittance to one coherent
state is insufficient for a tight verification. Consider a coherent state with intensity $\mu$.
With the phase totally randomized, the state is a probabilistic mixture of different photon
numbers:

$$\rho_{\mu} = e^{-\mu} \sum_{n=0}^{\infty} \frac{\mu^n}{n!} |n\rangle\langle n|$$  \hspace{1cm} (1)

and $|n\rangle\langle n|$ is the Fock state of $n$ photons. Since Alice does not know the photon number
in each individual pulse, their knowledge about the channel transmittance is the averaged
transmittance to the whole mixed state. However, in principle, the channel (Eve’s channel)
transmittance can be selective: it can be more transparent given a multi-photon pulse.
Suppose the channel transmittance is $\eta$. (Since Eve may also control Bob’s detector, here $\eta$
is the overall transmittance for channel and detector.) It is possible that the transmittance
for single-photon pulses is actually zero. If we use the worst-case estimation, we require
$\mu < \eta$ in order to obtain a meaningful result for the verification. In practice, if the distance
is longer than 100kms, it is possible that $\eta$ value is in the magnitude order of $10^{-3}$ or $10^{-4}$,
then the worst-case verification doesn’t work.

Earlier, the PNS attack has been investigated where Alice and Bob monitor only how
many non-vacuum signals arise, and how many errors happen. However, it was then shown[6]
that the simple-minded method does not guarantee the final security. It is shown[6] that
in a typical parameter regime nothing changes if one starts to monitor the photon number statistics as Eve can adapt her strategy to reshape the photon number distribution such that it becomes Poissonian again.

Remark: Here PNS attack allows any method for Eve to split the multi-photon pulses, provided that it does not violate laws of the nature. Although some types of specific PNS attack, e.g., the beam-splitter attack could be detected by simple method such as tomography at Bob’s side, a method to manage whatever type of PNS attack is strongly non-trivial.

However, as it was first shown by Hwang[9], by changing the intensity of pulses, one can make the verification unconditionally and more efficiently. We now start from the classical statistical principle.

Principle 1. Given a large number of independent pulses, the averaged value of any physical quantity per pulse in a randomly chosen subset of pulses must be (almost) equal to that of the remained pulses, if the number of pulses in the subset and the number of remained pulses is large.

In a standard QKD protocols with perfect single-photon source, this principle is used for the error test: They check the error rate of a random subset, and use this as the error rate of the remained bits. Also, this principle can be used for estimation of other quantities, such as the counting rate. In the protocol, Alice sends pulses to Bob. Given a loss channel, whenever a pulse is sent out from Alice, Bob’s detector may click may not click. If his detector clicks, a raw bit is generated. Counting rate is the ratio of the number of Bob’s click and the number of pulses sent out from Alice. More specifically, if source $x$ sends out $N$ pulses and Bob’s detector clicks $n_x$ times meanwhile, the counting rate for pulses from source $x$ is $S_x = \frac{n_x}{N}$.

In the QKD protocol, Alice controls the source. Consider a case of a mixed source of X and source Y. When we say a mixed source of X and Y, we mean that each maybe from X or Y randomly. If these two sources produce the same state and each pulses are independent and the number pulses from each source is sufficiently large, then the counting rate of source X must be equal to that of source Y, since X and Y can be regarded as one source and pulses from X can be regarded as samples for testing and pulses from source Y can be regarded as the remained pulses.

Principle 2. Asymptotically, given a mixed source of X and Y, Alice can verify the counting rate of source Y by watching counting rate of source X, if X and Y produce the same states and each pulses are independent.
Now we see how Hwang’s original protocol works. For mathematical simplicity, we give up Hwang’s original derivation which involves counting rates of each Fock states and complicated inequalities. We use the technique of density matrix convex and there are only a few parameters involved. We omit the dark count at the moment and assume that a vacuum pulse from Alice never causes counts at Bob’s side. Consider the source state in equation (1). The state can be re-written in the following equivalent convex form:

$$\rho_\mu = e^{-\mu}|0\rangle\langle 0| + \mu e^{-\mu}|1\rangle\langle 1| + cp_c$$  \hspace{1cm} (2)

and $c = 1 - e^{-\mu} - \mu e^{-\mu} > 0$,

$$\rho_c = \frac{1}{c} \sum_{n=2}^{\infty} P_n(\mu)|n\rangle\langle n|$$  \hspace{1cm} (3)

and $P_n(\mu) = \frac{e^{-\mu} \mu^n}{n!}$. This convex form shows that the source sends out 3 types of pulses: sometimes sends out vacuum, sometimes sends out $|1\rangle\langle 1|$, sometimes sends pulses of state $\rho_c$. Bob’s counts caused by $\rho_c$ from Alice are regarded as tagged bits. Since we know explicitly the probability of pulses $\rho_c$ for our source, we shall know the fraction of tagged bits if we know $s_c$, the counting rate of state $\rho_c$. The counting rate of any state $\rho$ is the probability that Bob’s detector counts whenever Alice sends $\rho$. If we have another source $A’$ which always produces state $\rho_c$, then we can combine source A, the coherent source $\rho_\mu$ and $A’$. Say, Alice uses a mixed source of A and A’. After Alice sends out all pulses, Bob announces which time is counted and which time is not. Then Alice knows the counting rate of source $A’$. The counting rate of source $A’$ is just the counting rate of all pulses $\rho_c$ from source A, asymptotically. Since Eve cannot treat the pulses from source $A’$ and the pulses of state $\rho_c$ from source A differently. This can be understood more easily in the following way: the above mixed source of A and $A’$ can be equivalently regarded as 4 sources since source A can be equivalently regarded as 3 sub-sources: sub-source $A_0$ containing all vacuum pulses from A, sub-source $A_1$ contains all single-photon pulses from A and sub-source $A_c$ contains all pulses of state $\rho_c$. Since the states of pulses from source $A’$ and source $A_c$ are identical, Eve can not treat them differently. Therefore the counting rate for pulses from source $A’$ must be equal to the that of source $A_c$. Since source $A_c$ contains all multi-photon pulses for source A, therefore the value $s_c$ for source A is verified by watching the counting rate of source $A’$. This is a natural consequence of Principle 2: Source $A’$ and sub-source $A_c$ makes a mixed source, they each produce the same identical pulses, therefore Alice can verify the counting rate of sub-source $A_c$ by watching that of source $A’$. 

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In the above toy model, we have used source A’ that produces state $\rho$ deterministically. In practice we don’t have such a source. But we have another coherent source with a different intensity $\mu' > \mu$. We now consider a realistic mixed source: source A is the coherent source with intensity $\mu$, source $A_{\mu'}$ is another coherent source with intensity $\mu'$. Since $\mu' > \mu$, the state for source $A_{\mu'}$ can be written in the convex form:

$$\rho_{\mu'} = e^{-\mu'}|0\rangle\langle 0| + \mu' e^{-\mu'}|1\rangle\langle 1| + \frac{\mu'^2 e^{-\mu'}}{\mu^2 e^{-\mu}} \rho_c + d \rho_d. \quad (4)$$

The source $A_{\mu'}$ can be equivalently regarded as a mixed source which contains the 4 sub-sources: $A_{\mu'0}$ which contains all vacuum pulses from $A_{\mu'}$, $A_{\mu'1}$ which contains all single-photon pulses from $A_{\mu'}$, $A_{\mu'c}$ that contains all $\rho_c$ pulses of source $A_{\mu'}$ and $A_{\mu'd}$ that contains all $\rho_d$ pulses of source $A_{\mu'}$. Therefore the mixed source of A, $A_{\mu'}$ now contains 7 sub-sources. Since state from sub-source $A_{\mu'c}$ is identical to that of sub-source $A_c$, the counting rate of source $A_{\mu'c}$ should be equal to that of $A_c$, i.e., the counting rate of state $\rho_c$ from source A. If Alice knew which pulses were from sub-source $A_{\mu'c}$, she would know the value $s_c$ exactly therefore know the value of fraction of tagged bits explicitly for source A, and then did key distillation based on raw bits from source A. However, Alice had no way to know which pulses are from sub-source $A_{\mu'c}$. But she know which pulses are from source $A_{\mu'}$. We shall show that she can know an upper bound of the fraction of tagged bits from source A by watching the counting rate of source $A_{\mu'}$.

In the protocol, Alice can watch the (averaged) counting rate for source $A_{\mu'}$ and we suppose the value is $S_{\mu'}$. According to equation (1), we have the following equation:

$$S_{\mu'} = \mu' e^{-\mu'} s_1 + c \frac{\mu'^2 e^{-\mu'}}{\mu^2 e^{-\mu}} s_c + d s_d. \quad (5)$$

Here we have assumed no vacuum dark count and denoted $s_d$ for the counting rate for state $\rho_d$, i.e. for source $A_{\mu'd}$, $s_1$ for counting rate of single-photon pulses, i.e., for source $A_{\mu'}$. Alice does not know the value of $s_1$ or $s_d$ but she knows the fact $s_1 \geq 0$ and $s_d \geq 0$. Therefore we transform eq(5) into an inequality for the upper bound of $s_c$:

$$s_c \leq \frac{c \mu'^2 e^{-\mu'}}{\mu^2 e^{-\mu'}} S_{\mu'} \quad (6)$$

This bound is obtained based on the observation of source A. This is the bound value for counting rate of sub-source $A_{\mu'c}$. This is also the bound value for state $\rho_c$ from any source, including those $\rho_c$ pulses from source A, since Eve cannot treat pulses of the same state.
differently according to which source the pulse is from. Therefore we have the following upper bound for fraction of tagged bits of source A:

$$\Delta \leq \frac{\mu^2 e^{-\mu} S_{\mu'}}{\mu'^2 e^{-\mu'} S_{\mu}}$$  \hspace{1cm} (7)

and we have used

$$\Delta = \frac{S_c}{S_{\mu}}$$  \hspace{1cm} (8)

In the normal case that there is no Eve’s attack, Alice and Bob will find $S_{\mu'}/S_{\mu} = \frac{1-e^{-\mu'}}{1-e^{-\mu}} = \mu'/\mu$ in their protocol therefore they can verify $\Delta \leq \frac{\mu e^{-\mu}}{\mu' e^{-\mu'}}$, which is just eq.(13) of Hwang’s work.

The above is the main result of Hwang’s work. We have simplified the original derivation given by Hwang. In summary, Hwang’s protocol works in this way: Use the intensity $\mu' = 1$ for the decoy state. By watching the counting rate of decoy state, we can obtain $\Delta$ value for the signal state (intensity $\mu$).

Although Hwang’s result of verification has been much better than the trivial worst-case method, Hwang’s protocol should be further improved for immediate use in practice. Hwang’s verified result is still much larger than the true value. We want a tighter estimation.

Remark: The security of Hwang’s method is a direct consequence the separate prior art result of ILM-GLLP. ILM-GLLP have offered methods to distill the unconditionally secure final key from raw key if the upper bound of fraction of tagged bits is known, given whatever imperfect source and channel. Decoy-state method verifies such an upper bound for coherent-state source. We can consider an analog using the model of pure water distillation: Our task is to distill pure water by heating from raw water that may contain certain poison constitute. Surppose it is known that the poison constitute will be evaporated in the heating. We want to know how long the heating is needed to obtain the pure water for certain. If we blindly heat the raw water for too long, all raw water will be evaporated and we obtain nothing. If we heat the raw water for a too short period, the water could be still poisonous. “ILM-GLLP” finds an explicit formula for the heating time which is a function of the upper bound of the fraction of poison constitute. They have proven that we (almost) always obtain pure water if we use that formula for the heating time. However, the formula itself does not tell how to examine the fraction of poison constitute. “Decoy-state” method is a method to verify a tight upper bound of the fraction of the poison constitute. It
is guaranteed by the classical statistical principle that the verified upper bound by “decoy-state method” is (always) larger than the true value. Using this analog, the next question is how to obtain a tighter upper bound: if the verified value over estimates too much, it is secure but it is inefficient. We want a way to obtain a value that is only a bit larger than the true value in the normal case that there is no Eve (for efficiency), and it is (almost) always larger than the true in whatever case (for security). This can be achieved by the improved decoy-state method. Here the term “(almost) always” means “with a probability exponentially close to 1”.

III. IMPROVED DECOY-STATE METHOD

The improvement is possible because Hwang’s method has not sufficiently using the different intensities. Actually, in doing the verification, Alice has only used the counting rate of one intensity, the source $A_{\mu'}$. It should be interesting to consider the case that Alice uses more intensities.

After Hwang’s work, decoy-state method is further studied. Ref [11] reviewed the PNS attack and the elementary idea of decoy-state method with some shortly-stated suggestions for possible improvement, but there is no conclusive result. It suggests doing the verification by using two intensities, vacuum and very weak coherent state. However this idea seems to be inefficient in practice due to the possible fluctuation of dark count [12]. Latter, a protocol with infinite number of intensities is proposed [13], and the result in the infinity limit is given. The main result there [13] has been published in Ref [14].

Here we are most interested in a protocol that is practically efficient. Obviously, there should be several criterion. 1. The protocol must be clearly stated. For example, there should be quantitative description about the intensities used and quantitative result about the verification. Because we need the explicit information of intensities in the implementation and the explicit value of $\Delta$ for key distillation. 2. The result of verified value $\Delta$ should be tight in the normal case when there is no Eve. This criterion is to guarantee a good final key rate. 3. It should only use a few different intensities. 4. It should be robust to possible statistical fluctuations. Say, in the non-asymptotic case, it only needs a reasonable number of pulses to make the verification. Note that the counting rates are very small parameters. The effects of possible statistical fluctuations can be very important. Concerning the
above criterion, a 3-intensity protocol is then proposed. The protocol uses 3 intensities: vacuum, $\mu$ and $\mu'$ for the verification. For convenience, we shall always assume

$$\mu' > \mu; \mu' e^{-\mu'} > \mu e^{-\mu} \quad (9)$$

in this paper. Since we randomly change the intensities among 3 values, we can regard it as the mixing of 3 sources. Source $A_0$ contains those vacuum pulses, $A$ contains those pulses of intensity $\mu$ and source $A_{\mu'}$ contains those pulses of intensity $\mu'$. States from source $A$ and $A_{\mu'}$ are given by eq.(2) and eq.(4), respectively. In the protocol, they can direct watch the counts of each source of $A_0, A, A_{\mu'}$. Suppose they find $S_0, S_{\mu}, S_{\mu'}$ for each of them. In the asymptotic case, we have the following equations:

$$S_{\mu} = e^{-\mu}S_0 + \mu e^{-\mu}s + cs_c \quad (10)$$

$$S_{\mu'} = e^{-\mu'}s_0 + \mu' e^{-\mu'}s_1 + c\frac{\mu'^2 e^{-\mu'}}{\mu^2 e^{-\mu}}s_c + ds_d \quad (11)$$

In the above we have used the same notations $S_0, s_1, s_c$ in both equations. This is because we have assumed that the counting rates of the same state from different sources are equal. $S_0$ is known, $s_1$ and $s_d$ are unknown, but they are never less than 0. Therefore setting $s_d, s_1$ to be zero we can obtain the following crude result by using eq.(11) alone.

$$cs_c \leq \frac{\mu'^2 e^{-\mu'_1}}{\mu'^2 e^{-\mu'}} \left( S_{\mu'} - e^{-\mu'}s_0 - \mu' e^{-\mu'}s_1 \right). \quad (12)$$

However, we can tighten the verification by using eq.(10). Having obtained the crude results above, we now show that the verification can be done more sophisticationally and one can further tighten the bound significantly. In the inequality (13), we have dropped terms $s_1$ and $s_d$, since we only have trivial knowledge about $s_1$ and $s_d$ there, i.e., $s_1 \geq 0$ and $s_d \geq 0$. Therefore, inequality(12) has no advantage at that moment. However, after we have obtained the crude upper bound of $s_c$, we can have a larger-than-0 lower bound for $s_1$, provided that our crude upper bound for $\Delta$ given by eq.(6) is not too large. From eq.(2) we have

$$e^{-\mu}s_0 + \mu e^{-\mu}s_1 + cs_c = S_{\mu}. \quad (13)$$

With the crude upper bound for $s_c$ given by eq.(6), we have the non-trivial lower bound for $s_1$ now:

$$s_1 \geq S_{\mu} - e^{-\mu}s_0 - cs_c > 0. \quad (14)$$
Therefore tight values for $s_c$ and $s_1$ can be obtained by solving the simultaneous constraints of equation (13) and inequality (12). We have the following final bound after solving them:

$$\Delta \leq \frac{\mu}{\mu' - \mu} \left( \frac{\mu e^{-\mu S_{\mu'}}}{\mu e^{-\mu} S_{\mu}} - 1 \right) + \frac{\mu e^{-\mu} s_0}{\mu' S_{\mu}}.$$ (15)

Here we have used eq.(8). In the case of $s_0 < \eta$, if there is no Eve., $S_{\mu'}/S_{\mu} = \mu'/\mu$. Alice and Bob must be able to verify

$$\Delta = \left. \frac{\mu (e^{\mu' - \mu} - 1)}{\mu' - \mu} \right|_{\mu' - \mu \to 0} = \mu$$ (16)

in the protocol. This is close to the real value of fraction of multi-photon counts: $1 - e^{-\mu}$, given that $\eta << 1$. In this 3-intensity protocol for the verification, both $\mu$ and $\mu'$ can be set in a reasonable range therefore both of them can be used for final key distillation. Of course, if we also want to use source $A_{\mu'}$ for key distillation, we need the value $\Delta'$, the upper bound of the fraction of tagged bits for source $A_{\mu'}$. Given $s_c$, we can calculate the lower bound of $s_1$ through eq.(14). Given $s_1$, we can also calculate the upper bound of $\Delta'$, the fraction of multi-photon count among all counts caused by pulses from source $A_{\mu'}$. Explicitly,

$$\Delta' \leq 1 - (1 - \Delta - \frac{e^{-\mu} s_0}{S_{\mu}}) e^{\mu' - \mu} - \frac{e^{-\mu'} s_0}{S_{\mu'}}.$$ (17)

The values of $\mu, \mu'$ should be chosen in a reasonable range, e.g., from 0.2 to 0.5.

### IV. STATISTICAL FLUCTUATIONS

The results above are only for the asymptotic case. In practice, there are statistical fluctuations, i.e., Eve. has non-negligibly small probability to treat the pulses from different sources a little bit differently, even though the pulses have the same state. Mathematically, this can be stated by

$$s_{\rho}(\mu') = (1 + r_{\rho}) s_{\rho}(\mu)$$ (18)

and the real number $r_{\rho}$ is the relative statistical fluctuation for counting rate of state $\rho$ in different sources of pulses. It is insecure if we simply use the asymptotic result in practice. Since the actual values are actually different from what we have estimated from the observed data. Our task remained is to verify a tight upper bound of $\Delta$ and the probability that the real value of $\Delta$ breaks the verified upper bound is exponentially close to 0.
The counting rate of any state $\rho$ from different sources now can be slightly different from the counting rate of the same state $\rho$ from another sources, $A_\mu$, with non-negligible probability. We shall use the primed notation for the counting rate for any state from source $A_\mu'$ and the original notation for the counting rate for any state from source $A$. Explicitly, constraints (6,14) are now converted to

$$
\begin{cases}
    e^{-\mu} s_0 + \mu e^{-\mu} s_1 + c s_c = S_\mu, \\
    c s'_c \leq \frac{\mu^2 e^{-\mu}}{\mu'^2 e^{-\mu}} \left( S_{\mu'} - \mu e^{-\mu} s'_1 - e^{-\mu} s'_0 \right).
\end{cases}
$$

(19)

Setting $s'_x = (1 - r_x) s_x$ for $x = 1, c$ and $s'_0 = (1 + r_0) s_0$ with $r_x > 0$ we obtain

$$
\mu' e^\mu \left[ (1 - r_c) \frac{\mu'}{\mu} - 1 \right] \Delta \leq \mu e^{\mu'} S_{\mu'} / S_\mu - \mu' e^{\mu} + [(\mu' - \mu) s_0 + r_1 s_1 + r_0 s_0] / S_\mu.
$$

(20)

From this we can see, if $\mu$ and $\mu'$ are too close, $\Delta$ can be very large. The important question here is now whether there are reasonable values for $\mu', \mu$ so that our method has significant advantage to the previous method[9]. The answer is yes.

Given $N_1 + N_2$ copies of state $\rho$, suppose the counting rate for $N_1$ randomly chosen states is $s_\rho$ and the counting rate for the remained states is $s'_\rho$, the probability that $s_\rho - s'_\rho > \delta_\rho$ is less than $\exp \left( -\frac{1}{4} \delta_\rho^2 N_0 / s_\rho \right)$ and $N_0 = \min(N_1, N_2)$. Now we consider the difference of counting rates for the same state from different sources, $A$ and $A_{\mu'}$. To make a faithful estimation for exponentially sure, we require $\delta_\rho^2 N_0 / s_\rho = 100$. This causes a relative fluctuation

$$
 r_\rho = \frac{\delta_\rho}{s_\rho} \leq 10 \sqrt{\frac{1}{s_\rho N_0}}.
$$

(21)

The probability of violation is less than $e^{-25}$. To formulate the relative fluctuation $r_1, r_c$ by $s_c$ and $s_1$, we only need to check the number of pulses in $\rho_c$, $|1\rangle\langle 1|$ in each sources in the protocol. That is, using eq.(21), we can replace $r_1, r_c$ in eq.(19) $|1\rangle\langle 1|$ in each sources in the protocol. That is, using eq.(21), we can replace $r_1, r_c$ in eq.(19) by $10 e^{\mu/2} \sqrt{1 / \mu s_1 N}$, $10 \sqrt{1 / c s_c N}$, respectively and $N$ is the number of pulses in source $A$. Since we assume the case where vacuum-counting rate is much less than the counting rate of state $\rho_\mu$, we omit the effect of fluctuation in vacuum counting, i.e., we set $r_0 = 0$. With these inputs, eq.(19) can now be solved numerically. The results are listed in the following table. From this table we can see that good values of $\mu, \mu'$ indeed exist and our verified upper bounds are sufficiently tight to make QKD over very lossy channel. Note that so far this is the only non-asymptotic result among all existing works on decoy-state. From the table we can see that our non-asymptotic
TABLE I: The verified upper bound of the fraction of tagged pulses in QKD. $\Delta_H$ is the result from Hwang’s method. $\Delta_R$ is the true value of the fraction of multi-photon counts in case there is no Eve. $\Delta_H$ and $\Delta_R$ do not change with channel transmittance. $\Delta_W$ is bound for pulses from source $A$, given that $\eta = 10^{-3}$. $\Delta_W$ and $\Delta_{W'}$ are bound values for the pulses from source $A, A'$ respectively, given that $\eta = 10^{-4}$. We assume $s_0 = 10^{-6}$. The number of pulses is $10^{10}$ from source $A, A'$ in calculating $\Delta_W$ and $8 \times 10^{10}$ in calculating $\Delta_W, \Delta_{W'}$. (Our results will only increase by 0.03 even if we only use $10^{10}$ pulses. Actually, as we shall show it in Table 2 and 3, pretty good results can be obtained with only $10^{10}$ pulses[17].) $4 \times 10^9$ vacuum pulses is sufficient for source $A_0$. The bound values will change by less than 0.01 if the value of $s_0$ is 1.5 times larger. The numbers inside brackets are chosen values for $\mu'$. For example, in the column of $\mu = 0.25$, data 30.9%(0.41) means, if we choose $\mu = 0.25, \mu' = 0.41$, we can verify $\Delta \leq 30.9\%$ for source $A$.

| $\mu$ | 0.2  | 0.25 | 0.3  | 0.35 |
|-------|------|------|------|------|
| $\Delta_H$ | 44.5% | 52.9% | 60.4% | 67.0% |
| $\Delta_R$ | 18.3% | 22.2% | 25.9% | 29.5% |
| $\Delta_{W1}$ | 23.4%(0.34) | 28.9%(0.38) | 34.4%(0.43) | 39.9%(0.45) |
| $\Delta_{W2}$ | 25.6%(0.39) | 30.9%(0.41) | 36.2%(0.45) | 41.5%(0.47) |

| $\mu'$ | 0.39 | 0.41 | 0.45 | 0.47 |
|--------|------|------|------|------|
| $\Delta_H$ | 71.8% | 74.0% | 78.0% | 79.8% |
| $\Delta_R$ | 32.3% | 33.7% | 36.2% | 37.5% |
| $\Delta_{W1}'$ | 40.1% | 42.2% | 45.8% | 48.6% |

values are less than Hwang’s asymptotic values already. Our verified values are rather close to the true values. We have assumed the vacuum count rate $s_0 = 10^{-6}$ in the calculation. If $s_0$ is smaller, our results will be even better. Actually, the value of $s_0$ (dark count) can be even lower than the assumed value here[15, 16]. In the real set-up given by Gobby et al[15], the light losses a half over every 15km, the devices and detection loss is 4.5% and $s_0 \leq 8.5 \times 10^{-7}$. Given these parameters, we believe that our protocol works over a distance longer than 120km with with $\mu = 0.3, \mu' = 0.45$ and a reasonable number of total pulses. In the table, we have chosen both values of $\mu, \mu'$ in a reasonable range. Of course, our method and Eq.(15) also work for other values of $\mu, \mu'$ which are beyond the table. This shows that eq.(15) indeed gives a rather tight upper bound.
V. ROBUSTNESS TO OTHER SMALL ERRORS

We now study how robust our method is. In the protocol, we use different intensities. In practice, there are both statistical fluctuations and small operational errors in switching the intensity. We shall show that, by using the counting rates of 3 intensities, one can still verify tight bounds even we take all these errors and fluctuations into consideration.

There are small operational errors inevitably. Say, in setting the intensity of any light pulse, the actual intensity can be slightly different from the one we have assumed. More specifically, suppose the number of pulses from source $A_0, A, A_{\mu'}$ are $N_0, N_\mu, N_\mu'$, respectively.

Due to the small operational error, the intensity of light pulses in source $A_0$ could be slightly larger than 0. This doesn’t matter because a little bit overestimation on the vacuum count will only decrease the efficiency a little bit but not at all undermine the security. Therefore we don’t care about the operational error of this part. Say, given $n_0$ counts for all the pulses from source $A_0$, we then simply assume the tested vacuum counting rate is $s_0 = n_0/N_0$, though we know that the actual value of vacuum counting rate is less than this.

We shall only consider sources $A, A_{\mu'}$. First, there are statistical fluctuations to the states itself since the number of pulses are finite. Say, e.g., given $N_\mu$ pulses of intensity $\mu$, the number of vacuum, single-photon state and multi-photon state $\rho_c$ could be a bit different from the assumed values of $P_0(\mu), P_1(\mu), P_c(\mu)$, respectively. The similarly deviations also apply to the pulses of intensity $\mu'$. But the deviation should be small given that the number of pulses in each sources is not too small. Say, e.g., given $N_\mu \geq 10^9, \mu = 0.2$, the relative fluctuation for the probability of state $\rho_c$ is less than $10\sqrt{\frac{1}{N_\mu(1-e^{-\mu}-\mu e^{-\mu})}} < 0.2\%$. Besides this, there are operational errors. At any time Alice decides to set the intensity of the pulse to be $\mu$ or $\mu'$, the actual intensity could be $\mu_i$ or $\mu'_i$, which can be a bit different from $\mu$ or $\mu'$. Therefore we should replace $P_\mu(n)$ and $P_{\mu'}(n)$ by slightly different distributions of $\tilde{P}_\mu(n) = \frac{1}{N_\mu} \sum_{i=0}^{N_\mu} P_{\mu_i}(n) = P_\mu(n)(1 + \epsilon_n)$ and $\tilde{P}_{\mu'}(n) = \frac{1}{N_{\mu'}} \sum_{i=0}^{N_{\mu'}} P_{\mu'_i}(n) = P_{\mu'}(n)(1 + \epsilon_n')$. Base on these, we have the following new convex formula for the actual states of each source given both statistical errors of states and operational errors:

$$\tilde{\rho}_\mu = \tilde{P}_0|0\rangle\langle 0| + \tilde{P}_1|1\rangle\langle 1| + \tilde{c}\tilde{\rho}_c$$

and $\tilde{P}_{0,1} = \tilde{P}_\mu(0,1); \tilde{c} = 1 - \tilde{P}_0 - \tilde{P}_1$. Since the new distributions are only a bit different from the old ones, there must exist a positive number $\tilde{d}$ and a density operator $\tilde{\rho}_d$ so that
the state from source $A_{\mu'}$ is convexed by

$$\tilde{\rho}_{\mu'} = \tilde{P}_0'(0)\langle 0| + \tilde{P}_1'(1)\langle 1| + \tilde{c}'\tilde{\rho}_c + \tilde{d}'\tilde{\rho}_d.$$ (23)

This, together with the possible statistical fluctuation to counting rates gives the following simultaneous constraint:

$$\begin{cases}
P_{\mu}(0)(1 + \epsilon_0)s_0 + P_{\mu}(1)(1 + \epsilon_1)s_1 + c(1 + \epsilon_c)s_c = S_\mu, \\
P_{\mu'}(0)(1 + \epsilon'_0)s'_0 + P_{\mu'}(1)(1 + \epsilon'_1)s'_1 + \frac{\mu'^2e^{-\mu'}}{\mu e^{-\mu}}c(1 + \epsilon'_c)s'_c \leq S_{\mu'}
\end{cases}$$ (24)

and $c = 1 - P_{\mu}(0) - P_{\mu}(1); c' = \frac{\mu'^2e^{-\mu'}}{\mu e^{-\mu}}c$. Suppose we know the upper bounds of all values of $|\epsilon_x|, |\epsilon'_x|, x = 0, 1, c$, we can calculate the lower bound of $s_1$ and upper bound of $s_c$. Say, we try all possible values of $\epsilon_x, \epsilon'_x$ and take the worst case as the verified result. After calculation, we find the result does not change too much given small $|\epsilon_x|, |\epsilon'_x|$. For example, given that $|\epsilon_x| < 2\%, |\epsilon'_x| < 2\%$ and $N_\mu = N_{\mu'} = 10^{10}, N_0 = 4 \times 10^9$ and $S_0 = 10^{-6}$, the lower bound of single-photon counting rate can be verified to be larger than $0.95\tilde{s}_1$ and $\tilde{s}_1$ is the lower bound of single-photon counts given $\epsilon_x = \epsilon'_x = 0$.

VI. FURTHER STUDIES

After the major works presented in [9, 10, 14], the decoy-state method has been further studied. Harrington studied the effect of fluctuation of the state itself. Ref. [17] proposed a 4-state protocol: using 3 of them to make optimized verification and using the other one $\mu_s$ as the main signal pulses. This is because, if we want to optimize the verification of $\Delta$ value, $\mu, \mu'$ cannot be chosen freely. Therefore we use another intensity $\mu_s$ to optimize the final key rate. It is shown numerically on how to choose the intensity for the main signal pulses ($\mu_s$) and good key rates are obtained in a number of specific conditions. Ref. [18] further studied the 3-intensity protocol. In particular, statistical fluctuations to the bit error rates are also considered there and a type of stronger key rate formula is suggested there. An experiment was also done.

VII. SUMMARY

In summary, we have reviewed the historic development of the decoy state method, including the background, development and some delicate concepts. Given an imperfect source
and a very lossy channel, the PNS attack can make the QKD in practice totally insecure. Given the result of ILM-GLLP[7, 8], one knows how to distill the secure final key if he knows the fraction of tagged bits. The purpose of decoy state method is to do a tight verification of the the fraction of tagged bits. The main idea of decoy-state method is changing the intensities of source light and one can verify the fraction of tagged bits of certain intensity by watching the the counting rates of pulses of different intensities. Since the counting rates are small quantities, the effect of statistical fluctuation is very important. It has been shown that 3-state decoy-state method in practice can work even with the fluctuations and other errors.

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