On the free product of ordered groups

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One of the fundamental questions of the theory of ordered groups is what abstract groups are orderable. E. P. Shimbireva [2] showed that a free group on any set of generators can be ordered. This leads to the following problem: under what conditions is it possible to order a free product of arbitrary groups?

Using the matrix presentation method for groups proposed by Malcev [1], in the present work we establish the orderability of a free product of arbitrary ordered groups.

Definition 1. An ordered group is a group endowed with a relation $>$, satisfying the following conditions:
1. For any elements $x$ and $y$ of the group either $x > y$, or $y > x$, or $x = y$.
2. If $x > y$ and $y > z$, then $x > z$.
3. If $x > y$, then $axb > ayb$ for any elements $a$ and $b$ of the group.

Definition 2. An ordered ring (field) is a ring (field) such that:
1. the additive group of the ring (field) is ordered, and
2. for any elements $a$, $x$, $y$ of the ring (field),
   $$(a > 0 \text{ and } x > y) \implies (ax > ay \text{ and } xa > ya).$$

Definition 3. The group algebra $\mathbb{k}\mathcal{G}$ of a group $\mathcal{G}$ over a field $\mathbb{k}$ is the algebra whose elements are formal finite linear combinations of elements of $\mathcal{G}$ with coefficients in $\mathbb{k}$. These sums are multiplied and added in the usual way. A group algebra has the obvious unit $1e$, where $e$ is the identity element of $\mathcal{G}$ and $1$ the unit of $\mathbb{k}$.

Lemma 1. If $\mathbb{k}$ is an ordered field and $\mathcal{G}$ an ordered group, then $\mathbb{k}\mathcal{G}$ is orderable.

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Proof. Let $A$ and $A'$ be elements of $k\mathcal{G}$ under the conditions of the lemma. Then they can be written as

$$A = \sum_{i=1}^{n} \alpha_i a_i, \quad A' = \sum_{i=1}^{n} \alpha'_i a_i,$$

where some of the $\alpha_i$ and $\alpha'_i$ might be zero, and $a_1 > \ldots > a_n$. We set $A > A'$ if for some $r \in \{1, \ldots, n\}$,

$$\alpha_1 = \alpha'_1, \quad \ldots, \quad \alpha_{r-1} = \alpha'_{r-1}, \quad \alpha_r > \alpha'_r.$$

It is easy to check that the conditions from Definition 2 hold.

We call a triangular matrix any matrix, finite or infinite, with zeroes under the main diagonal.

**Lemma 2.** The set of all triangular matrices with entries in an ordered unital ring, and with every element on the main diagonal positive and invertible, is an orderable group.

**Proof.** Triangular matrices of the form described in the statement clearly form a group. Let $X$ and $Y$ be such matrices. We will call preceding entries to a given entry $x_{ik}$, those $x_{nm}$ located to the right of or on the main diagonal, for which

$$n - m \leq k - i \quad \text{when} \quad m < i, \quad \text{and} \quad n - m < k - i \quad \text{when} \quad m \geq i.$$

Say that $X > Y$ if either of the following conditions holds:

- $x_{ii} = y_{ii}$ for $i = 1, \ldots, k - 1$, and $x_{kk} > y_{kk}$ for some $k$,
- $x_{ik} > y_{ik}$ for some $k > i$, and their preceding entries coincide.

One easily checks that the conditions of Definition 2 are satisfied.

**Lemma 3.** The direct product of two ordered groups is orderable.

**Proof.** Let $\mathfrak{A}$ and $\mathfrak{B}$ be ordered groups. Say that $(a, b) > (a', b')$ in $\mathfrak{A} \times \mathfrak{B}$ if either $a > a'$, or $a = a'$ and $b > b'$. It is easy to check that the conditions from Definition 1 hold.

We denote by $\mathfrak{M}$ the direct product of two ordered groups $\mathfrak{A}$ and $\mathfrak{B}$. A pair of the form $(a, e_1)$ where $e_1$ is the identity of $\mathfrak{B}$ will be denoted simply by $a$, and a pair of the form $(e, b)$ where $e$ is the identity of $\mathfrak{A}$ will be denoted by $b$.

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1 Translators’ note: we believe that there is a mistake here, $x_{nm}$ should probably be replaced with $x_{mn}$.
Consider now the following transcendental triangular matrix:

\[
X = \begin{pmatrix}
1 & x_{12} & x_{13} & x_{14} & \cdots & \cdot & \cdot & \cdot \\
& 1 & x_{23} & x_{24} & \cdots & \cdot & \cdot & \cdot \\
& & 1 & x_{34} & \cdots & \cdot & \cdot & \cdot \\
& & & 1 & \cdots & \cdot & \cdot & \cdot \\
& & & & \ddots & \cdots & \cdots & \cdot \\
& & & & & \ddots & \cdots & \cdot \\
& & & & & & \ddots & \cdots \\
& & & & & & & \ddots \\
\end{pmatrix}
\]

We denote by \( \mathfrak{G} \) the free abelian group generated by the entries \( x_{ij} \) of \( X \). This group is orderable (see [2] and references therein). By Lemma 1, the group algebra \( \mathbb{K} = \mathbb{Q} \mathfrak{G} \) is orderable, and thus has no zero divisors. The field of fractions \( \text{Frac}(\mathbb{K}) \) of this algebra is also orderable [3]. Consider the group algebra \( \mathcal{L} = \text{Frac}(\mathbb{K}) \mathcal{M} \), where \( \mathcal{M} = \mathfrak{A} \times \mathfrak{B} \) as above. According to Lemmas 1 and 3, the algebra \( \mathcal{L} \) is orderable.

**Lemma 4.** Consider the diagonal matrix

\[
A = \begin{pmatrix}
1 & & & & \\
& a & & & \\
& & 1 & & \\
& & & a & \\
& & & & \ddots & \\
& & & & & \ddots & \\
& & & & & & \ddots & \\
& & & & & & & \ddots \\
\end{pmatrix}
\]

where 1 is the unit of \( \mathcal{L} \) and \( a \in \mathcal{L} \) is neither 0 nor 1. Then every entry of the matrix \( B = X^{-1}AX \) located to the right of or on the main diagonal is non-zero.

**Proof.** Put \( X^{-1} = (y_{ik}) \) and \( B = (b_{ik}) \). Clearly

\[
y_{in} = -x_{in} + \sum_{i < \alpha_1 < n} x_{i\alpha_1} x_{\alpha_1 n} - \sum_{i < \alpha_1 < \alpha_2 < n} x_{i\alpha_1} x_{\alpha_1 \alpha_2} x_{\alpha_2 n} + \cdots + \left( -1 \right)^{n-i} x_{i, i+1} x_{i+1, i+2} \cdots x_{n-1, n}
\]

\( ^{2} \) Translators’ note: we corrected the last term of the formula given for \( y_{in} \). Note also that this formula holds only for \( i \neq n \), as \( y_{ii} = 1 \). As a result, the very last formula of this proof is slightly incorrect when \( i \) is odd, but the main point—that the coefficient of \( b_{ik} \) is not 0—seems to hold true after all.
and
\[ b_{ik} = 1(y_{i1}x_{1k} + y_{i3}x_{3k} + \cdots + y_{i,2l+1}x_{2l+1,k}) + \]
\[ a(y_{i2}x_{2k} + y_{i4}x_{4k} + \cdots + y_{i,2r}x_{2r,k}). \]

From this follows:
\[ y_{i1}x_{1k} + y_{i3}x_{3k} + \cdots + y_{i,2l+1}x_{2l+1,k} = \]
\[ -\sum x_{in}x_{nk} + \sum_{i<\alpha_1<n} x_{i\alpha_1}x_{\alpha_1n}x_{nk} - \sum_{i<\alpha_1<\alpha_2<n} x_{i\alpha_1}x_{\alpha_1\alpha_2}x_{\alpha_2n}x_{nk} + \cdots, \]
where the external sums are over all odd integers \( n \) between \( i \) and \( k \). This equality shows that the coefficient of 1 in \( b_{ik} \) is non-zero, and so \( b_{ik} \neq 0. \]

**Theorem.** The free product of two ordered groups can be endowed with a group order whose restriction to each factor is the original order.

**Proof.** Consider, together with the triangular matrix \( X \) introduced before, the following transcendental triangular matrices:

\[ Y = \begin{pmatrix} 1 & y_{12} & y_{13} & y_{14} & \cdots \\ & 1 & y_{23} & y_{24} & \cdots \\ & & 1 & y_{34} & \cdots \\ & & & \ddots & \vdots \\ & & & & 1 & \cdots \\ & & & & & \cdots \end{pmatrix}, \]

\[ U = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_3 \end{pmatrix}, \quad V = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_3 \end{pmatrix}. \]

Let \( \mathfrak{A} \) and \( \mathfrak{B} \) be ordered groups. As before, we construct an algebra \( \mathcal{L} = \text{Frac}(\mathbb{Q}\mathfrak{G})\mathfrak{M} \) with \( \mathfrak{M} = \mathfrak{A} \times \mathfrak{B} \), where now the free abelian group \( \mathfrak{G} \) is generated by the set of all formal entries not only of \( X \), but also of \( Y, U, \)
and $V$. To every $a = (a, e_1) \in \mathcal{M}$ we associate the diagonal matrix
\[ \overline{A}_a = \begin{pmatrix} 1 & \ & \ & \ \\ & a & \ & \\ & & 1 & \ \\ & & & a \end{pmatrix}, \]
and to every $b = (e, b) \in \mathcal{M}$ the diagonal matrix
\[ \overline{B}_b = \begin{pmatrix} 1 & \ & \ & \ \\ & b & \ & \\ & & 1 & \ \\ & & & b \end{pmatrix}. \]
Clearly the two sets of matrices $\overline{A} = \{ \overline{A}_a \mid a \in \mathcal{A} \}$ and $\overline{B} = \{ \overline{B}_b \mid b \in \mathcal{B} \}$ form groups naturally isomorphic to $\mathcal{A}$ and $\mathcal{B}$ respectively.

Put $\overline{A} = U^{-1}X^{-1}A\overline{X}U$ and $\overline{B} = V^{-1}Y^{-1}\overline{B}\overline{Y}V$. We are going to show that the representations of $\mathcal{A}$ and $\mathcal{B}$ given by $a \mapsto \overline{A}_a$ and $b \mapsto \overline{B}_b$ induce a faithful representation of the free product $\mathcal{A} * \mathcal{B}$, that is, given elements of $\mathcal{A} * \mathcal{B}$ of type
\[ r_1 = \prod_{i=1}^n a_i b_i, \quad r_2 = \left( \prod_{i=1}^n a_i b_i \right) a_k, \quad r_3 = b_k \prod_{i=1}^n a_i b_i, \quad r_4 = \prod_{i=1}^n b_i a_i, \]
the corresponding matrices
\[ R_1 = \prod_{i=1}^n \overline{A}_i \overline{B}_i, \quad R_2 = \left( \prod_{i=1}^n \overline{A}_i \overline{B}_i \right) \overline{A}_k, \quad R_3 = \overline{B}_k \prod_{i=1}^n \overline{A}_i \overline{B}_i, \quad R_4 = \prod_{i=1}^n \overline{B}_i \overline{A}_i \]
are not the identity matrix. We will write down the proof for $R_1$ only, as the three remaining cases are similar.

Every entry $\overline{a}_{kl}^i$ of the matrix $\overline{A}_i$ is equal to $u_k^{-1}a_{kl}^{i'}u_l$, where $a_{kl}^{i'}$ is an entry of $A'_i = X^{-1}A_iX$, and $u_k^{-1}$ and $u_l$ are diagonal entries of the matrices $U^{-1}$ and $U$. Similarly, $\overline{b}_{kl}^i = v_k^{-1}b_{kl}^{i'}v_l$, where $b_{kl}^{i'}$ is an entry of $B'_i = X^{-1}B_iX$, and $v_k^{-1}$ and $v_l$ are diagonal entries of the matrices $V^{-1}$ and $V$.

By Lemma 4, every matrix in the groups $\mathcal{A}' = X^{-1}A\overline{X}$ and $\mathcal{B}' = Y^{-1}\overline{B}\overline{Y}$ different from the identity matrix has only non-zero entries to the right of or
on the main diagonal. The entries of the matrix $R_1$ are given by

$$ r_{ik} = \sum_{i \leq i_2 \leq i_3 \leq \ldots \leq i_{2n} \leq k} a_{i_2}^{(1)} b_{i_2}^{(1)} a_{i_3}^{(2)} b_{i_3}^{(2)} \ldots a_{i_{2n-1}}^{(n)} b_{i_{2n-1}}^{(n)} b_{i_{2n,k}}^{(n)} $$

Here $i \leq k$. This sum can be regarded as a polynomial in the diagonal entries of $U$, $V$ and of their inverses. The coefficients of this polynomial are products of entries of the matrices $A'_1, B'_1, A'_2, B'_2, \ldots$. Observe that no monomial occurs twice in the sum as it is given. Moreover, every coefficient is non-zero, since it is a product of non-zero elements of the algebra $L$, which has no zero divisors.

Therefore, we have a faithful representation of the free product $\mathfrak{A} \ast \mathfrak{B}$, given by

$$ r_i \mapsto R_i. $$

Every diagonal entry of $R_i$ is either the unit of $L$ or a positive invertible element of $L$ distinct from the unit. It follows then from Lemma 2 that all matrices of all four types $R_i$ together form an orderable group. Therefore, the free product $\mathfrak{A} \ast \mathfrak{B}$ is orderable.

The proof presented here for two factors obviously works for any number of factors.

References

[1] A. Malcev. On isomorphic matrix representations of infinite groups. Rec. Math. [Mat. Sbornik] N.S., 8 (50):405–422, 1940.

[2] H. Shimbireva. On the theory of partially ordered groups. Rec. Math. [Mat. Sbornik] N.S., 20(62):145–178, 1947.

[3] B. L. van der Waerden. Modern Algebra. Vol. I. M.–L., 1934.