Robust Instability Analysis with Application to Neuronal Dynamics

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Abstract—This paper is concerned with robust instability analysis for guaranteed persistence of oscillating phenomena in dynamical systems. We first formalize the robust instability analysis problem for single-input-single-output linear time invariant systems by introducing a notion of the robust instability radius (RIR). We provide a method of finding the exact RIR for a certain class of systems and show that it works well for a class of second-order systems. This result is applied to the FitzHugh-Nagumo model for neuronal dynamics, and the effectiveness is confirmed by numerical simulations, where we properly take account of the change of the equilibrium point.

I. INTRODUCTION

There are a lot of interesting nonlinear oscillating phenomena in neuroscience and biology. For most cases, the phenomena occur when a certain equilibrium point is unstable, and it is desired to characterize the robustness of the instability against uncertainties. However, in contrast with robust stability analysis, robust instability analysis is not easy because instability may be sustained even when some of the nominally unstable poles become stable due to a certain perturbation. In other words, we need to keep track of the behavior of all nominally unstable modes. Consequently, the traditional small-gain type argument based on the \( \infty \)-norm may not work in general, meaning that the \( L_\infty \)-norm condition provides only sufficient conditions for the robust instability as seen in e.g., [3], [4]. Moreover, such conditions could be very conservative.

There is another interesting point on the issue. The problem is equivalent to strong stabilization by a minimum-norm, stable controller. The condition for strong stabilizability has been known, but the order of a strongly stabilizing controller is unknown [5]. Mimimization of a norm on some closed-loop transfer functions has been considered in the literature, but only partial solutions have been obtained due to the difficulty in enforcing the stability constraint on the controller, e.g. [6]. Therefore, it is required to develop a new fundamental theory for robust instability analysis.

We first formalize the robust instability problem for single-input-single-output linear time invariant (LTI) systems by introducing a notion of the robust instability radius (RIR) as the maximum allowable \( \infty \)-norm of the stable perturbations that maintain the instability. After showing the lower and upper bounds of RIR, we present two numerical examples of second order systems, which illustrate various situations by changing one real parameter. They prove useful to understand the reason why computing the exact RIR is hard in general. We then provide a method of finding the exact RIR for a certain class of systems. The idea is to use a first order all-pass function as a critical perturbation, and it is proven that the idea works well for a class of second-order systems. This result can be applied to the FitzHugh-Nagumo model [1], which represents neuronal dynamics. The effectiveness of the theoretical result is confirmed by numerical simulations, where we properly care the change of the equilibrium point.

II. ROBUST INSTABILITY RADIUS (RIR)

Given an unstable LTI positive feedback system with loop transfer function \( h(s) \), we consider robust instability against a class of multiplicative perturbations, i.e., the perturbed loop transfer function is represented by

\[
\hat{h}(s) = (1 + \delta(s))h(s), \text{ where } \delta(s) \text{ is in } \Delta := \{ \delta(s) \in \text{RH}_\infty \mid \| \delta \|_\infty < 1 \}. 
\]

The framework is the same as one for the standard robust stability analysis (see e.g., [7] and the references therein). Similarly to the robust stability analysis for multiplicative perturbations, the characteristic polynomial of the perturbed system represented by

\[
1 - \delta(s)g(s) = 0
\]

plays an important role, where

\[
g(s) := h(s)/(1 - h(s))
\]

is called the complementary sensitivity function.

Assuming nominal hyperbolic instability of \( g(s) \), we are interested in determining how much perturbation is allowed before the closed-loop system becomes stable. That is, the objective is to find the smallest norm stable perturbation \( \delta(s) \) such that \( \Delta \) has all roots in the open left half plane (OLHP).

The robust instability radius (RIR), denoted by \( \rho_* \), with respect to real rational dynamic perturbation \( \delta \in \text{RH}_\infty \), is defined as the smallest magnitude of the perturbation that stabilizes the system:

\[
\rho_* := \inf_{\delta \in \mathbb{S}(g)} \| \delta \|_{\text{H}_\infty},
\]

where \( \mathbb{S}(g) \) is the set of real-rational, proper, stable transfer functions stabilizing \( g(s) \):

\[
\mathbb{S}(g) := \{ \delta \in \text{RH}_\infty : \delta(s)g(s) = 1 \Rightarrow \Re(s) < 0 \}.
\]
When the perturbation is parametric, the real and complex RIRs are defined by

\[ \rho_r := \inf_{\delta \in \mathcal{S}_c(g)} |\delta|, \quad \rho_c := \inf_{\delta \in \mathcal{S}_c(g)} |\delta|, \]

(5)

where \( \mathcal{S}_c(g) \subset \mathbb{R} \) and \( \mathcal{S}_c(g) \subset \mathbb{C} \) are defined as in [4] by replacing \( RH_{\infty} \) with \( \mathbb{R} \) and \( \mathbb{C} \), respectively. It is clear by definition that \( \rho_* \leq \rho_r \) and \( \rho_* \leq \rho_c \). However, it turns out that the relationship between \( \rho_* \) and \( \rho_c \) depends on \( g(s) \) as we will see later by examples. This is in stark contrast with the robust stability analysis in which the robust stability radius with respect to the dynamic uncertainty coincides with that for the complex parametric uncertainty. The fact that \( \rho_* \neq \rho_c \) makes the robust instability analysis for dynamic uncertainty extremely difficult.

The RIRs for the parametric uncertainty case can readily be calculated. For example, the stability region \( \mathcal{S}_c(g) \) can be fully characterized within the framework of the generalized frequency variable [2], and it is easy to compute \( \rho_r \) and \( \rho_c \) by gridding the frequency at least.

In contrast, the RIR for the dynamic uncertainty case is not easy to compute. It is noticed that our problem of robust instability has a clear correspondence with the so called strong stabilization, i.e., \( g(s) \) and \( \delta(s) \) respectively correspond to the unstable plant and the stabilizing controller, which is required to be stable. Recalling a classical result on strong stabilization by Youla et al. [5], we have the following proposition.

**Proposition 1:** Let an unstable transfer function \( g(s) \) be given. The following statements are equivalent.

(i) The robust instability radius \( \rho_* \) is finite.

(ii) Parity Interlacing Property (PIP) is satisfied, i.e., the number of unstable real poles of \( g(s) \) between any pair of real zeros in the closed right half complex plane (including zero at \( \infty \)) is even.

### III. Preliminaries

#### A. Lower and Upper Bounds of RIR

We here provide several results on lower and upper bounds of RIR as a preliminary. First note by the definition of \( \rho_* \) that there exists \( \delta(s) \) such that \( \|\delta\|_{H_{\infty}} = \rho_* \) and all the roots of \( 1 - \delta(s)g(s) = 0 \) satisfy \( R(s) \leq 0 \) with at least one \( s \) on the imaginary axis, say \( s = j\omega_c \), because otherwise it is impossible to stabilize \( g(s) \) by stable \( \delta(s) \) of norm arbitrarily close to \( \rho_* \) due to continuity of the characteristic roots. Thus the critical perturbation \( \delta(s) \) has to satisfy

\[ \delta(j\omega_c) = 1/g(j\omega_c), \quad |\delta(j\omega_c)| \leq \rho_* \]

Noting that \( |g(j\omega_c)| \leq \|g\|_{L_{\infty}}, \) a lower bound on \( \rho_* \) can be obtained in terms of the \( L_{\infty} \) norm of \( g(s) \), which was noted in [3], as follows.

**Proposition 2:** Define

\[ g_o := 1/\|g\|_{L_{\infty}}, \quad \|g\|_{L_{\infty}} := \sup_{\omega \in \mathbb{R}} |g(j\omega)|. \]

Then, we have

\[ g_o \leq \rho_* \leq \rho_r. \]

We can readily see that \( g_o \) is also a lower bound on \( \rho_* \), i.e., \( \rho_* \geq g_o \), since \( \rho_o = 1/|g(j\omega)| \) for some \( \omega \) as explained above. Graphically, we see that \( \rho_o = \rho_o \) holds when the projection of the origin onto the Nyquist plot of \( 1/g(s) \) is on the boundary of \( \mathcal{S}_c(g) \).

We now provide a special case where we can get a better lower bound of \( \rho_* \) than \( \rho_o \) for those \( g(s) \) with the peak gain strictly larger than the static gain.

**Proposition 3:** Consider a real-rational, strictly proper transfer function \( g(s) \). Suppose \( g(s) \) has no pole at the origin and an odd number of unstable poles. Then

\[ \rho_* \geq g_o := 1/|g(0)| \]

holds. That is, \( g_o \) is a lower bound of RIR \( \rho_* \). Suppose further that \( g(s) \) is stabilizable by a static gain and \( g(j\omega) \) is real only at \( \omega = 0 \). Then the lower bound is tight, i.e., \( \rho_* = g_o \).

**Proof:** Let \( \delta(s) \) be a stable transfer function that stabilizes \( g(s) \). We will show \( \|\delta\|_{H_{\infty}} > g_o \). Let \( g(s) = n(s)/d(s) \) and \( \delta(s) = n(s)/d(s) \) be the polynomial coprime factorizations with monic polynomials \( d(s) \) and \( n(s) \). Define \( \Psi(s) \) as the characteristic polynomial, i.e., \( \Psi(s) := d(s)/d(s) - n(s)/n(s) \). Then, we have

\[ \Psi(s) = d(s)/d(s)[1 - g(s)/(\delta(s))] \]

which is monic because \( d(s) \) and \( n(s) \) are monic and \( g(s) \) is strictly proper. This leads to \( d(s) > 0 \), since \( \delta(s) \) stabilizes \( g(s) \). Also we have \( d(s) > 0 \), since \( \delta(s) \) is stable and \( d(s) \) is monic. Similarly, since \( d(s) \) is monic, the first supposition implies that \( d(0) = 0 \) and \( d(\infty) \) should have different signs, i.e., \( d(0) < 0 \). These conditions lead to \( 1 - g(0)/(\delta(0)) < 0 \), and hence \( |g(0)/(\delta(0))| > 1 \), implying \( \|\delta\|_{H_{\infty}} > g_o \). This proves (g).

Now suppose \( g(s) \) is stabilizable by a static gain and \( g(j\omega) \) is real only at \( \omega = 0 \). Let \( \delta_r \in \mathbb{R} \) be the perturbation of the smallest magnitude such that (a) the closed-loop system has all its poles in the closed left half plane and (b) \( \delta_r + \varepsilon \) with an arbitrarily small \( \varepsilon \) can stabilize the closed-loop system:

\[ \delta_r := \inf_{\delta \in \mathbb{R}} \{ |\delta| : 1 - g(s)/(\delta(s)) = R(s) \leq 0 \}. \]

Note that \( \rho_r := |\delta_r| \) is the real RIR and is an upper bound on the (dynamic) RIR; \( \rho_* \leq \rho_r \). Now, the characteristic equation \( 1 - g(s)/(\delta(s)) = R(s) \leq 0 \) has a root on the imaginary axis. If \( s = j\omega \) is a root, then \( g(j\omega) = -1/\delta_r \) is a real number. Hence, \( \omega \) must be zero by supposition. It then follows that \( \rho_* \leq \rho_r = |\delta_r| = |1/g(0)| = g_o \), proving tightness of the lower bound \( g_o \).

#### B. Illustrative Numerical Examples

We here provide two simple examples to gain theoretical insights. A variety of situations are captured by the examples to examine when the upper/lower bounds on the robust instability radius become tight.

\[ g(s) = \frac{s^2 + 2s + 1}{s^2 + 1}, \quad g_o = 1/|g(0)| = 1/2. \]

\[ g(s) = \frac{s^2 + 1}{s^2 + 0.1}, \quad g_o = 1/|g(0)| = 1/0.1. \]
• Example 1: Consider a class of 2nd-order feedback systems given by

$$h(s) = \frac{2(s - z)}{(s^2 + s - 2)}$$

with multiplicative uncertainty, i.e.,

$$g(s) := \frac{h(s)}{1 - h(s)} = \frac{1}{\phi(s) - 1}, \quad \phi(s) := \frac{1}{h(s)}$$

(11)

Here, \(z \in \mathbb{R}\) is a parameter, the unique real zero of \(h(s)\), and \(h(s)\) is minimum phase (resp. non-minimum phase) if \(z\) is negative (resp. non-negative). The characteristic polynomial for the nominal closed-loop system is given by \(1 - h(s) = 0\), or \(s^2 - s + 2(z - 1) = 0\). Hence, it is nominally unstable for any \(z\). What is nice in this quite simple example is that various situations appear by changing just one real parameter \(z\).

The robust instability radius and its lower/upper bounds can be calculated analytically as follows.

**Facts on RIR for Example 1:** Consider \(h(s)\) in (10) with a parameter \(z \in \mathbb{R}\). Define the robust instability radius \(\rho_c\) by (8), its lower bounds \(\rho_l\) and \(\rho_u\) by (7) and (9), respectively, and its upper bound \(\rho_u\) by (5). We can see that \(\rho_c = \infty\) when \(0 \leq z < 1\) due to violation of the PIP condition. Otherwise, the result for various cases can be summarized as in Table I where “?” indicates unknown but finite numbers. “<” indicates that there is no \(\delta \in \mathbb{R}\) to stabilize \(g(s)\) in case of \(\rho_c\), or \(g(s)\) has an even number of unstable poles in case of \(\rho_u\). The lower bounds are determined by minimizing \(1/|g(j\omega)|^2\) over \(\omega\). If the optimizer is \(\omega = 0\), then \(\rho_l = \sigma_0 := 1/|g(0)| = |(z - 1)/z|\). Otherwise, the minimum \(\sigma_1\) is achieved at \(\omega_p = \omega_1\) where

$$\omega_1 := \sqrt{\eta(z) - z^2},$$

$$\sigma_1 := \frac{1}{|g(j\omega_1)|} = \frac{1}{2} \sqrt{2\eta(z) - 5 - 4c - 2z^2},$$

$$\eta(z) := (z - 1)(z + 2)(z^2 + 3z - 2).$$

The real zeros of

$$\mu(z) := 4z^3 - z^2 - 8z + 4$$

determine whether the optimizer is zero or non-zero, where \(b_1 = -1.51, b_2 = 0.54,\) and \(b_3 = 1.22\) are the roots of \(\mu(b_i) = 0\).

1 In the context of the multiplicative uncertainty, it is reasonable to consider the case where the magnitude of \(\delta(s)\) is less than one. However, here we consider an arbitrarily large perturbation for illustrative purposes; this is still reasonable if we see the uncertain system as the feedback connection of \(\delta(s)\) and \(g(s)\).

![Nyquist plots](image)

**Fig. 1. Nyquist plots of \(-\phi(s)\) for the six cases in Table II**

We now consider the five cases with different values of \(z\) that represent the columns of Table II. The numerical result is summarized in Table II. Figure 1 shows the Nyquist plots of \(-\phi(s)\), where the number of the characteristic roots in the ORHP is indicated in each region divided by the Nyquist plots. A circle is the projection of the critical point (star) onto the Nyquist plot of \(-\phi(s)\). The red vector from a circle to a star is \(\delta_p := \phi(j\omega_p) - 1\), and its length is \(\rho_p\).

**Table I**

| Case | \(z\) | \(\rho_l\) | \(\rho_u\) | \(\rho_c\) |
|------|-------|-------------|-------------|-----------|
| (a)  | \(z \leq b_0\) | \(\sigma_0\) | \(\sigma_0\) | \(\sigma_0\) |
| (b)  | \(b_0 < z \leq 0\) | ? | - | - |
| (c)  | \(1 \leq z \leq b_1\) | ? | ? | - |
| (d)  | \(b_1 < z \leq 2\) | ? | ? | - |
| (e)  | \(2 < z\) | ? | - | - |

**Table II**

| Case | \(z\) | \(\rho_u\) | \(\rho_c\) | \(\rho_l\) | \(\omega_p\) |
|------|-------|-------------|-------------|-----------|
| (a)  | -3 | 1.333 | 1.333 | 1.333 | 0 |
| (b)  | -0.5 | 3 | 3 | 3 | 1.725 | 1.57 |
| (c)  | 1.1 | ? | ? | - | 0.091 | 0 |
| (d)  | 1.5 | 0.258 | 0.258 | - | 0.80 | 7.76 |
| (e)  | 5 | 0.244 | 0.244 | - | 0 | 2.76 |
| (*) | -3 | ? | ? | 2 | 1.2 | 1.2 | 0 |

We can see in case (a) that the projection of \(-1\) onto the Nyquist plot \(-\phi(j\omega)\) takes it to the boundary of the stability region marked by “0” and that \(\rho_p = \rho_u\). This means that a slight extension of the projection will stabilize the system, i.e., \(\delta(s) = -(1 + \epsilon)\phi_p\) with small \(\epsilon > 0\) stabilizes, confirming tightness of the lower bound.
bound in Proposition 2. By this example, we are tempted to conjecture that (i) the lower bound is tight when the projection of $-1$ is on the boundary of the stability region and that (ii) the lower bound is not tight when the projection is not on the stability boundary.

It turns out, however, that neither of these conjectures is correct as shown by other cases due to subtle difference between static complex perturbation and real rational dynamic perturbation. Actually, case (b) is a counter example of the first conjecture. The red vector in Fig. I(b) clearly starts at one point on the boundary of the stability region, which means $\rho_p = \rho_c = 1.725$, but we can see from Proposition 2 that $\rho_p = \rho_o = 3$. This is also an example of $\rho_s \neq \rho_c$.

Other three cases, (c), (d), and (e), are related to the second conjecture. Note that the assumptions of Proposition 3 are not satisfied for those cases. The next section will show for cases (d) and (e) that we can find a stabilizing perturbation with slightly larger gain than $\rho_p$, meaning that $\rho_c = \rho_p$, although we were not able to find such a perturbation for case (c). This implies the second conjecture is not always true. The situations of the two cases (d) and (e) are slightly different. For case (e), we see that $\rho_p \leq \rho_s \leq \rho_r$. However, for case (d), there is no stabilizing static complex perturbation, i.e., $\rho_c (\leq \rho_r)$ is infinite.

• Example 2: We consider another example with $h(s) = (s-z)/(2s^2-s-1)$. The situation of this example is similar to that of Example 1 except for absence of type (e). Instead, a new case of interest ($z = -5$) gave a pattern not observed in the previous example. The result for this case is indicated by (s) in Table I and Fig. I. The projection of $-1$ onto the Nyquist plot of $-\psi(s)$ is real, and the corresponding real perturbation does not give marginal stability. However, $h(s)$ is stabilizable by a static real gain implying $\rho_r = 2$. This means that the RIR lies in the interval $1.2 \leq \rho_s \leq 2$ but otherwise remains unknown.

IV. EXACT RIR ANALYSIS

A. Idea: All-Pass Stabilization

The exact value of the RIR can be found if upper and lower bounds turn out to be the same value. The examples in the previous section have shown that the lower bounds in Propositions 2 and 3 are useful for a certain classes of $g(s)$, where we consider the stabilization by a static perturbation. However, the class of systems stabilized by a static gain is not large, and hence we need to consider the case of stabilization by a dynamic perturbation.

The aim of this section is to propose a simple way to get the exact RIR by a two step procedure; (Step 1) marginal stabilization by a 1st-order all-pass function, and (Step 2) slight modification of the function to get a stabilizing perturbation. The marginal stabilization here means that the closed-loop system consisting of $g(s)$ and $\delta(s)$ has a single complex conjugate pair (rather than multiple pairs) of (non-repeated) poles on the imaginary axis and the rest in the OLHP.

Our idea for Step 1 is as follows. Let $\delta(s)$ and $g(s)$ be given as

$$\delta(s) = b \cdot f(-s)/f(s), \quad g(s) = n(s)/d(s)$$

where $f(s)$ is a Hurwitz polynomial. The closed-loop system has a pair of poles $s = \pm j\omega_c$ and the rest of the poles are in the OLHP if and only if

$$b \cdot f(-s)n(s) - f(s)d(s) = (s^2 + \omega^2_c)p(s)$$

holds for some Hurwitz polynomial $p(s)$. Consider the case of the first order all-pass transfer function $f(s) = s + a$ with $a > 0$. Then we have

$$b(a-s)n(s) - (a+s)d(s) = (s^2 + \omega^2_c)p(s).$$

Equating the coefficients $s^i$ terms, we have the following characterization.

Lemma 1: The positive feedback system consisting of

$$\delta(s) = b \cdot \frac{a-s}{a+s}, \quad g(s) = \frac{\beta_0 + \beta_1 s + \cdots + \beta_n s^n}{\alpha_0 + \alpha_1 s + \cdots + \alpha_n s^n}$$

has all the closed-loop poles in the open left half plane except for a pair of poles at $s = \pm j\omega_c$ if and only if

$$\Omega_c \psi = -A_d \alpha + A_n \beta,$$

holds for some Hurwitz polynomial

$$p(s) := \psi_0 + \psi_1 s + \cdots + \psi_{n-1} s^{n-1},$$

where $\Omega_c \in \mathbb{R}^{(n+2)\times n}$, $A_d, A_n \in \mathbb{R}^{(n+2)\times (n+1)}$, $\psi \in \mathbb{R}^n$, and $\alpha, \beta \in \mathbb{R}^{n+1}$ are defined by

$$\Omega_c := \begin{bmatrix} \omega_c^2 & I_n \\ 0 & I_n \end{bmatrix}, \quad \psi := [\psi_0, \cdots, \psi_{n-1}]^T,$$

$$A_d := \begin{bmatrix} a_0 + a_1 s & \cdots & a_{n-1} s^{n-1} \\ 0 & \cdots & 0 \end{bmatrix}, \quad \alpha := [\alpha_0, \cdots, \alpha_n]^T,$$

$$A_n := \begin{bmatrix} a_0 + a_1 s & \cdots & a_{n-1} s^{n-1} \\ 0 & \cdots & 0 \end{bmatrix}, \quad \beta := [\beta_0, \cdots, \beta_n]^T.$$
(ii) $\rho_* = \rho_p := 1/\|g\|_{L_\infty} = 1/|g(j\omega_p)|$ holds if $p < 0$, $q > 0$, $r^2q^2 + 2q - p^2 > 0$, where $\omega_p^2 = q - p^2/2$ for $r = 0$ and otherwise
\[ \omega_p^2 = \left(\sqrt{r^2q^2 + 2q - p^2}r^2 + 1\right) - 1/\sqrt{r^2}. \] (16)

Proof: See Appendix A.

Let us now confirm the effectiveness of the theorem by using Example 1, where
\[ g(s) = 2z - \frac{s^2 - 1}{s^2 - s + 2(z - 1)}, \]
i.e., $r = 1/z$, $p = -1$, and $q = 2(z - 1)$. For cases (a) and (b), we can apply the first case with $q = 2(z - 1) < 0$ and see that $p + rq = -1 + 2(z - 1)/z = 1 - 2/z > 0$. Hence, we have $\rho_* = \rho_p = \rho_r = 1/|g(0)| = 1 - 1/z$.

There are two different situations based on the peak frequency when $q > 0$; $\omega_p = 0$ for case (c), and $\omega_p > 0$ for cases (d) and (e). Since condition (15) holds for cases (d) and (e), our proposed 1st-order all-pass function works, meaning that $\rho_p$ gives the exact RIR $\rho_*$. However, we have no answer for case (c).

V. APPLICATIONS TO NEURONAL DYNAMICS

This section is devoted to an application to an analysis of neuronal dynamics of excitable membranes for robustly generating action potentials. We use the FitzHugh-Nagumo (FHN) model [1], which is a second-order nonlinear system represented by
\[ c\dot{v} = \psi(v) - w, \]
\[ \tau \dot{w} = v + \alpha - \beta w, \] (17)
where $c$, $\tau$, $\alpha$, and $\beta$ are positive scalars, and
\[ \psi(v) = v - v^3/3 + i. \]

The variable $v(t)$ represents the membrane potential of the neuronal cell, $w(t)$ is the recovery variable that captures the net effect of the channel conductances, and $i(t)$ is the current injection input to the cell, which is assumed constant in the following development.

We consider the case where $\beta < 1$, which guarantees that the system has a unique equilibrium point. Based on the shape of the function $\psi(v)$, it can be shown that all the trajectories are ultimately bounded. Hence, an oscillation occurs when the equilibrium is hyperbolically unstable. Typically, the system has a stable limit cycle which is seen as a spike train. An example is shown in Fig. 2, where the parameter values are
\[ c = 1, \quad \tau = 10, \quad \alpha = 0.7, \quad \beta = 0.8, \quad i = 0.4. \] (18)

Assuming these values, we will examine robustness of the oscillation against unmodeled dynamics.

The FHN model captures the essential dynamics of action potential (spike) generation in the simplest way, ignoring the details of various channel conductances. Here we model the neglected dynamics by the multiplicative uncertainty $\delta(s)w$, where $\delta(s)$ is an uncertain stable transfer function. The uncertain FHN model is then given by
\[ \dot{v} = \psi(v) - (1 + \delta(s))w, \]
\[ \tau \dot{w} = v + \alpha - \beta w. \] (19)

Let $(\bar{v}, \bar{w})$ be an equilibrium point, characterized by
\[ \psi(\bar{v}) = (1 + e)\bar{w}, \quad \bar{v} = \beta \bar{w} - \alpha, \] (20)
where $e := \delta(0)$. It can be verified that the equilibrium is unique if $1 + e > \beta$. Linearizing the system around $(\bar{v}, \bar{w})$, the characteristic equation is given by
\[ 1 = \delta(s)g_e(s), \quad g_e(s) := -\frac{1}{\epsilon \tau s^2 + (\beta c - \tau \gamma)s + 1 - \beta \gamma}, \]
where $\gamma := \psi'(\bar{v}) = 1 - \bar{v}^2$. Note that $g_e(s)$ depends on $e$ through $\bar{v}$, and this dependence is indicated by the subscript. For the nominal parameters in (18), the linearization $g_o(s)$ (i.e. $g_e(s)$ with $e = 0$) is unstable since $\beta c < \tau \gamma$ holds. Thus we have nominal instability.

The question is: What is the smallest norm of $\delta(s)$ such that $1 = \delta(s)g_e(s)$ has all its roots in the left half plane?

From Theorem 1 the RIR for $g_o(s)$ can be found as $\rho_* = 1/\|g_o\|_{L_\infty} = 0.283$. This means that the equilibrium remains unstable under perturbations satisfying $\|\delta\|_{H_\infty} < \rho_*$, provided the equilibrium does not move by the perturbation, i.e., $\delta(0) = 0$. However, there may be a perturbation $\delta(s)$ with a nonzero static gain such that the equilibrium is moved by $\delta(0) = e$ and becomes stable. Indeed, for a stable perturbation $\delta(s)$ of norm $\|\delta\|_{H_\infty} = 0.2$, the new equilibrium $(\bar{v}, \bar{w})$ in (20) with $e = \delta(0) = 0.2$ is stable, and the simulated response converges to $(\bar{v}, \bar{w})$ as shown in Fig. 3 where $x$ is the state of $\delta(s)$.

A stabilizing perturbation $\delta(s)$ of a smaller norm can be found as follows. First, solve $|e| = 1/\|g_o\|_{L_\infty}$ for $e$ by a line search. The solution is found to be $e_o = -0.118$ and the peak gain of $g_{e_o}(j\omega)$ is attained at $\omega_p = 0.299$. Then by Theorem 1 the RIR for $g_{e_o}(s)$ is equal to $\|e_o\|_{H_\infty}$ and a stabilizing perturbation is given by $\delta(s) = (1 + \varepsilon)\delta_o(s)$ with a sufficiently small $\varepsilon > 0$, where
\[ \delta_o(s) = 0.118 \cdot \frac{s - 0.320}{s + 0.320} \]
is a stable first order all-pass transfer function uniquely determined by $\delta_o(j\omega_p) = 1/|g(j\omega_p)|$. For example, with $\varepsilon = 0.1$, the perturbation $\delta(s)$ has norm $\|\delta_o\|_{H_\infty} = 0.130$, and the new equilibrium $(\bar{v}, \bar{w})$ in (20) with $e = \delta(0) = -0.130$ is stable. Consequently, the simulated response converges to $(\bar{v}, \bar{w})$ as shown in Fig. 4.

For this particular example, the upper bound $\rho_p = 1/\|g_e\|_{L_\infty}$ on the RIR for $g_e(s)$ turned out to be a monotonically increasing function of $e$ in the neighborhood of $e = 0$. Since $|\delta_e|_{H_\infty} \geq |\delta(0)|$, a lower bound is given by $|e|$. Hence $e = e_o$, at which the lower and upper bounds become coincident, gives the norm $|e_o|$ of the smallest stabilizing perturbation. Indeed, when $e$ is negative with a small $|e|$, the perturbation
magnitude $\|\delta\|_{H_{\infty}}$ is less than $\epsilon_o$, and the equilibrium point remains unstable, resulting in a stable limit cycle as shown in Fig. 5 for the case $\epsilon = -0.1$.

VI. CONCLUSION

We have formalized the robust instability problem by introducing a notion of robust instability radius (RIR). We provided a method of finding the exact RIR for a class of second-order systems, and the effectiveness has been confirmed by an application to the FitzHugh-Nagumo model. The future work includes the generalization of the RIR theory for applications in biology.

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APPENDIX

A. Proof of Theorem 1

We first compute $q_p$ and the critical frequency $\omega_p$. Noting that $G(\Omega) := |g(j\omega)|^2$, $\Omega := \omega^2$ is given by

$$G(\Omega) = \frac{r^2\Omega + 1}{\Omega^2 + (p^2 - 2q)\Omega + q^2},$$

$\omega_p^2$ is given by the maximizer of $G(\Omega)$ for $\Omega \geq 0$. The derivative of $G(\Omega)$ with respect to $\Omega$ is equal to zero is written as

$$F(\Omega) := -|r^2\Omega^2 + 2\Omega + (p^2 - 2q - r^2q^2)| = 0.$$

Hence, for $r = 0$, we have

$$\Omega_p = q - p^2/2, \quad 1/G(\omega_p) = (q - p^2/4)p^2;$$

$$\Omega_p = 0, \quad 1/G(\omega_p) = q^2;$$

$$\Omega_p = (\sqrt{(r^2q^2 + 2q - r^2p^2)^2/4} - 1)/r^2.$$

Otherwise, $\Omega_p = 0$.

The proof of (i) for the case of $q < 0$ and $\Omega_p = 0$ is easy. The characteristic equation for the closed-loop system with $\delta$ is given by $s^2 + (p - \delta r)s + q + \delta = 0$. With $\delta > 1/(g(0)) = -q$, the two roots are both negative, and we can show that $g(j\omega)$ is real only at $\omega = 0$. This completes the proof of (i).

To prove (ii), we here only consider the case of $r = 0$. We can do similarly for the case of $r \neq 0$ with complicated formulas, but we omit the proof due to the page limitation. Let us first define

$$F(\Omega) := 1/|g(j\omega)|^2 = \Omega^2 + (p^2 - 2q)\Omega + q^2;$$

where $\Omega := \omega^2 \geq 0$ to get the critical frequency $\omega_p$. Since $dF(\Omega)/d\Omega = 2\Omega - (2q - p^2)$, the square of the critical frequency $\Omega_p := \omega_p^2$ is given by $\Omega_p = q - p^2/2$. Then, a simple calculation yields $F(\Omega_p) = p^2(4q - p^2)/4 > 0$.

We now apply Lemma 2 by setting $\omega^2 = \omega_p^2$ and $b = -\sqrt{F(\Omega_p)} = p\sqrt{4q - p^2}/2 < 0$. Equation (12) implies

$$x = a + p > 0, \quad \omega_p^2 = ap - b + q > 0,$$

$$x\omega_p^2 = (a + p)(ap - b + q) = a(q + b) > 0.$$

These three relations lead to

$$a = (-p + \sqrt{4q - p^2})/2 > 0, \quad x = (\sqrt{4q - p^2} + p)/2 > 0,$$

which guarantee the marginal stabilization. Next, we show the existence a stabilizing perturbation by choosing a slightly larger perturbation

$$\delta_\varepsilon(s) = (1 + \varepsilon)b \cdot \frac{a - s}{a + s}, \quad \varepsilon > 0.$$ 

It is readily seen that the corresponding characteristic polynomial is given by $s^3 + d_2s^2 + d_1s + d_0 = 0$ with

$$d_2 := a + p = (\hat{q} + p)/2 > 0,$$

$$d_1 := ap - (1 + \varepsilon)b + q = p^2/2 - b\varepsilon,$$

$$d_0 := a\{q + (1 + \varepsilon)b\} = \hat{q}^2(\hat{q} + p)/4 + (\hat{q} - |\hat{q}|\varepsilon)/2,$$

where $\hat{q} := \sqrt{4q - p^2}$. We can see that $d_0$ and $d_1$ are both positive for sufficiently small $\varepsilon > 0$, and a simple calculation shows that $d_1d_2 - d_0 = -eb\sqrt{4q - p^2} > 0$, which guarantees the stability of the perturbed feedback system. This means that $\rho_* = q_p = |b| = |p|\sqrt{4q - p^2}/2$, and hence the proof is completed.