Cauchy-characteristic matching for a family of cylindrical solutions possessing both gravitational degrees of freedom

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Abstract. This paper is part of a long term program to Cauchy-characteristic matching (CCM) codes as investigative tools in numerical relativity. The approach has two distinct features: (i) it dispenses with an outer boundary condition and replaces this with matching conditions at an interface between the Cauchy and characteristic regions, and (ii) by employing a compactified coordinate, it proves possible to generate global solutions. In this paper CCM is applied to an exact two-parameter family of cylindrically symmetric vacuum solutions possessing both gravitational degrees of freedom due to Piran, Safier and Katz. This requires a modification of the previously constructed CCM cylindrical code because, even after using Geroch decomposition to factor out the z-direction, the family is not asymptotically flat. The key equations in the characteristic regime turn out to be regular singular in nature.

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1. Introduction

This paper is part of a long term program to Cauchy-characteristic matching (CCM) codes as investigative tools in numerical tools in numerical relativity. For a review of CCM see the article of Winicour[1]. The approach has two distinct features: (i) it dispenses with an outer boundary condition and replaces this with matching conditions at an interface between the Cauchy and characteristic regions, and (ii) by employing a compactified coordinate, it proves possible to generate global solutions. This paper is continuation of work investigating CCM in systems with vacuum cylindrical symmetry (as a prototype system containing only one spatial degree of freedom). In [2] we developed the necessary machinery and in [3] we applied the approach to the exact solution of Weber and Wheeler [4] which has one degree of polarization and so only possesses one gravitational degree of freedom. An agreement with the exact solution was found with a maximum error better then 1 part in $10^3$. In the same paper we also investigated the propagation of Gaussian wave packets possessing two gravitational
degrees of freedom, but this work did not involve a check against any exact solution. Such a test is likely to be a more rigorous one because, unlike the Weber-Wheeler case, it would involve passing derivative information across the interface.

One of the major problems in numerical relativity is that there are very few exact solutions known which can be used to test numerical codes. In this paper CCM is applied to an exact two-parameter family of cylindrically symmetric vacuum solutions possessing both gravitational degrees of freedom due to Piran, Saffer and Katz [5]. This family is somewhat unphysical because the rotational degree of freedom diverges at future null infinity. Nonetheless, it can be used to test the CCM cylindrical code more rigorously because it involves passing derivative information across the interface. The previous CCM cylindrical code was constructed specifically for solutions which (after using Geroch decomposition to factor out the z-direction) are asymptotically flat and therefore can not be used for processing the Piran et al family. However, by working directly in terms of the Geroch potential it proves possible to develop a modified version of the code which is able to process the family. In section 3 we present the family of solutions and briefly discuss its properties. In section 4 we review the derivation of the field equations in the characteristic region for solutions which, after factoring out the z-direction, are asymptotically flat. In section 5 we discuss the asymptotic limit of the Piran et al family and derive the modified field equations and the interface equations. An appendix discusses the regular singular nature of two of the resulting equations. In section 6 we briefly discuss the results of the modified code.

2. The two-parameter family of solutions

Piran et al derive their metric by an indirect method starting from the Kerr line element in Boyer-Lindquist coordinates \((\tilde{t}, \tilde{r}, \tilde{\theta}, \tilde{\phi})\), namely

\[
\begin{align*}
 ds^2 &= -\frac{\Delta}{\rho^2}(d\tilde{t} - \tilde{a} \sin^2 \tilde{\theta} d\tilde{\phi})^2 + \frac{\sin^2 \tilde{\theta}}{\rho^2} [(\tilde{r}^2 + \tilde{a}^2)d\tilde{\phi} - \tilde{a} d\tilde{t}]^2 + \frac{\rho^2}{\Delta} d\tilde{r}^2 + \rho^2 d\tilde{\theta}^2, \tag{1}
\end{align*}
\]

where

\[
\begin{align*}
 \rho^2 &= \tilde{r}^2 + \tilde{a}^2 \cos^2 \tilde{\theta}, \tag{2}
 \Delta &= \tilde{r}^2 - 2m\tilde{r} + \tilde{a}^2. \tag{3}
\end{align*}
\]

They first transform to a new isotropic radial coordinate \(R\) where

\[
\tilde{r} = m + R + \frac{m^2 - \tilde{a}^2}{4R}, \tag{4}
\]

and then to cylindrical coordinates \((\tilde{t}, \tilde{\rho}, \tilde{Z}, \tilde{\phi})\) where

\[
\begin{align*}
 \tilde{\rho} &= R \sin \tilde{\theta}, \tag{5}
 \tilde{Z} &= R \cos \tilde{\theta}. \tag{6}
\end{align*}
\]

Employing the complex trick of sending

\[
\begin{align*}
 \tilde{t} \to i\tilde{Z}, \quad \tilde{\rho} \to \tilde{\rho}, \quad \tilde{Z} \to i\tilde{t}, \quad \tilde{\phi} \to \tilde{\phi}, \quad \tilde{a} \to ia, \tag{7}
\end{align*}
\]
they then introduce new cylindrical coordinates \((t, r, z, \phi)\) given by
\[
\begin{align*}
t &= \tilde{t} \left[ 1 + \frac{m^2 + a^2}{4(\tilde{\rho}^2 - \tilde{t}^2)} \right], \\
r &= \tilde{\rho} \left[ 1 - \frac{m^2 + a^2}{4(\tilde{\rho}^2 - \tilde{t}^2)} \right], \\
z &= Z - 2a^{-1}m(m + \sqrt{m^2 + a^2})\tilde{\phi}, \\
\phi &= \alpha \tilde{\phi},
\end{align*}
\]
where
\[
\alpha = \frac{\sqrt{m^2 + a^2}}{a},
\]
and the resulting line element is in the Jordan-Ehlers-Kompaneets form for a cylindrically symmetric spacetime \cite{7},
\[
ds^2 = -e^{2\gamma - 2\psi}(dt^2 - dr^2) + e^{2\psi}(dz + \omega d\phi)^2 + e^{-2\psi}r^2 d\phi^2.
\]
If we finally introduce coordinates
\[
\begin{align*}
u &= t - r, \\
v &= t + r,
\end{align*}
\]
then we can express the two-parameter family of solutions in the form
\[
ds^2 = -e^{2\gamma - 2\psi}dudv + e^{2\psi}(dz + \omega d\phi)^2 + \frac{e^{-2\psi}(v - u)^2}{4}d\phi^2,
\]
where explicitly
\[
\begin{align*}
e^{2\gamma} &= \frac{(\lambda_u + \lambda_v)^2 + \alpha^2(1 - \lambda_u\lambda_v)^2}{(1 + \lambda_u^2)(1 + \lambda_v^2)}, \\
e^{2\psi} &= \frac{\alpha^2(1 - \lambda_u\lambda_v)^2 + (\lambda_u + \lambda_v)^2}{\alpha^2\Xi^2 + (\lambda_u - \lambda_v)^2}, \\
\omega &= \frac{2a\sqrt{\alpha^2 - 1}}{\alpha}(\alpha + \sqrt{\alpha^2 - 1}) \\
&\quad - \frac{a\sqrt{\alpha^2 - 1}\Xi(\lambda_u + \lambda_v)^2}{\sqrt{\lambda_u\lambda_v}[\alpha^2(1 - \lambda_u\lambda_v)^2 + (\lambda_u + \lambda_v)^2]},
\end{align*}
\]
with
\[
\Xi = 1 + \lambda_u\lambda_v + 2\alpha^{-1}\sqrt{\alpha^2 - 1}\sqrt{\lambda_u\lambda_v},
\]
\[
\lambda_u = \left(\sqrt{a^2 + u^2} - u\right)/a, \\
\lambda_v = \left(\sqrt{a^2 + v^2} + v\right)/a.
\]
The coordinates \(u\) and \(v\) are null, \(\phi\) is the canonical azimuthal coordinate and \(z\) lies along the axis of symmetry. The two parameters are \(a\ (0 \leq a < \infty)\) which is a length scale and \(\alpha\ (1 \leq \alpha < \infty)\) which is a measure of the total energy of the system, with \(\alpha = 1\).
corresponding to flat space \([5]\). The solution is regular everywhere in the coordinate range

\[
-\infty < u < \infty, \\
-\infty < v < \infty, \\
0 \leq \phi < 2\pi, \\
-\infty < z < \infty.
\]

There is no conical singularity on the axis of symmetry and the solution reduces to Minkowski spacetime at past and future infinity. However, the solution is conical at spatial infinity and singular at both past and future null infinity. More precisely, at future null infinity (cf Eq. (10) in \([5]\))

\[
e^{2\psi} \to 1, \quad e^{2\gamma} \to \frac{1 + \alpha^2 \lambda^2}{1 + \lambda^2 u}, \quad \omega \to -\infty,
\]

as \(v \to \infty\), with analogous behaviour at past null infinity on interchanging \(u\) and \(v\). (Note that Eq. (10) in \([5]\) includes a typographical error and that \(u\) and \(v\) should be transposed).

Since Piran et al only derived this family of solutions by the indirect method described above, we tried to confirm that the family is vacuum by a direct computation of the Ricci tensor using both of the computer algebra systems SHEEP \([8]\) and GRTENSOR \([9]\). Unfortunately, neither system was able to complete the calculation because the metric involves nested radicals and algebra systems are notorious for the difficulties such quantities present. However, we were able to confirm the indirect derivation described above.

3. The standard field equations

The standard treatment of the Jordan-Ehlers-Kompaneets cylindrically symmetric line element \([13]\) in the Cauchy region

\[
t_0 \leq t \leq t_f, \\
0 \leq r \leq 1,
\]

is described in \([5]\). In particular, it is shown that the independent set of dynamical equations for the variables \(\psi, \omega, L^\phi_z\) and \(\tilde{L}\) are

\[
\psi_{,t} = \frac{1}{r} \tilde{L}, \\
\omega_{,t} = -2e^{-4\psi} L^\phi_z, \\
L^\phi_{,z,t} = \frac{1}{r} e^{4\psi} \left(\frac{1}{2} \omega_{,r} - \frac{1}{2} r \omega_{,rr} - 2r \psi_{,r} \omega_{,r}\right), \\
\tilde{L}_{,t} = \frac{1}{r} \left[ r^2 \psi_{,rr} + r \psi_{,r} - \frac{1}{2} e^{4\psi} \omega^2 + 2e^{-4\psi} (L^\phi_z)^2 \right],
\]
where \( L_\phi^z \) and \( \tilde{L} \) are defined in terms of the mixed components of the extrinsic curvature \( K^\mu_\nu \) by

\[
L_\phi^z = r^2 e^{-\psi} K^\phi_\phi, \\
\tilde{L} = r^2 e^{-\psi} (K^\phi_\phi - \omega K^\phi_\phi).
\]

These equations are augmented by the constraint equation

\[
\gamma, r = \frac{1}{4r} e^{4\psi} \omega, r^2 - r \psi, r^2 + \frac{1}{r} [\tilde{L}^2 + e^{-4\psi} (L_\phi^z)^2],
\]

which serves to determine \( \gamma \) once the main variables are known.

In the rest of this section we review the treatment in the characteristic region since this will need modification to cope with the Piran et al family of solutions. We introduce the compactified coordinate

\[
y = \frac{1}{\sqrt{r}} = \frac{1}{\sqrt{v - u}},
\]

in which case the line element becomes

\[
ds^2 = -e^{2(\gamma - \psi)} du^2 + \frac{4}{y^3} e^{2(\gamma - \psi)} du dy + e^{2\psi} (dz + \omega d\phi)^2 + \frac{e^{-2\psi}}{y^4} d\phi^2,
\]

where the metric functions \( \psi, \omega \) and \( \gamma \) are now to be regarded as functions of \( u \) and \( y \). The characteristic region then consists of

\[
u_0 \leq u \leq \nu_f, \\
0 \leq y \leq 1,
\]

where the interface is taken to be at \( r = y = 1 \) and future null infinity is given by \( y = 0 \). Instead of working directly with the metric functions \( \psi \) and \( \omega \), we use the related quantities \( m \) and \( w \) which are defined by

\[
m = e^{2\psi} - 1/y, \\
w = o/y = -\int_F \lambda^2 (\frac{1}{2} y^4 \omega, u + y \omega, u) du + \frac{1}{y} \int_F \lambda^2 y^2 \omega, y dy,
\]

where

\[
\lambda = e^{2\psi} = 1 + my,
\]

\( o \) is the Geroch potential \( \Box \), and the integration is along any path connecting a fixed point \( F \) on the interface (we choose \((u, y) = (\nu_0, 1)\)) to a general point \( P \) in the characteristic region (see Fig. \( \Box \)). With these definitions the vacuum equations can be written in the succinct form

\[
M = m, u / \lambda, \\
W = w, u / \lambda, \\
M_y = f(y, m, m, y, m, yy, w, w, y, W), \\
W, y = g(y, m, m, y, w, w, y, w, yy, M),
\]
where $f$ and $g$ are explicit functions of their arguments $[3]$. Eqs. (14) and (15) serve to define $M$ and $W$ and (16) and (17) provide coupled propagation equations for $M$ and $W$ along the null rays ruling the hypersurfaces $u =$ constant. The initial data consists of specifying $m$ and $w$ on the initial hypersurface $\{u = u_0, \ 0 \leq y \leq 1\}$ and the interface $\{u_0 \leq u \leq u_f, \ y = 1\}$, together with $\gamma$ at their intersection $(u, y) = (u_0, 1)$. The essence of the iterative integration scheme is to use (16) and (17) to determine $M$ and $W$ on the interior null hypersurface $u = u_i$ and then (14) and (15) to determine $m, u$ and $w, u$ on $u = u_i$, which in turn determines $m$ and $w$ on the next neighbouring null slice. Finally $\gamma$ is determined from the constraint equation (Eq. (42) in [3]) which has the form

$$\gamma_{,y} = h(y, m, m_y, w, w_y),$$

(48)
where $h$ is an explicit function of its arguments. The CCM method requires the exchange of values of the metric functions and their derivatives at the interface at each iteration, the details of which are given in [3].

4. The modified field equations

The approach described in the last section breaks down for the Piran et al solution in the characteristic region because $\omega \to -\infty$ as $y \to 0$. Let us consider the behaviour of the various functions on the hypersurface $u = u_i$ in the asymptotic limit $y \to 0$. We have,

$$\lambda_u = (\sqrt{a^2 + u_i^2} - u_i)/a,$$

(49)

$$\lambda_v = \lambda_v/y^2,$$

(50)

$$\Xi = \Xi/y^2,$$

(51)

where we set

$$\bar{\lambda}_v \equiv \left(2 + u_i y^2 + \sqrt{4 + 4u_i y^2 + (a^2 + u_i^2)y^4}\right)/a = 4/a + O(y^2),$$

(52)
CCM for a family of cylindrical solutions

\[ \Xi \equiv \lambda_u \bar{\lambda}_v + 2y\sqrt{1 - \alpha^2}\lambda_u \lambda_v + y^2 = 4\lambda_u/a + O(y). \]  

(53)

Thus, asymptotically,

\[ e^{2\psi} = 1 + O(y), \]

(54)

\[ e^{2\gamma} = (1 + \alpha^2 \lambda_u^2)/(1 + \lambda_u^2) + O(y^2), \]

(55)

\[ \omega = -\left(\frac{2\sqrt{(\alpha^2 - 1)a\lambda_u}}{1 + \alpha^2 \lambda_u^2}\right)\frac{1}{y} + \frac{2a\sqrt{\alpha^2 - 1}}{\alpha} \left(\alpha + \sqrt{\alpha^2 - 1}\right) + O(y), \]

(56)

and

\[ m = \frac{2\alpha\sqrt{(\alpha^2 - 1)a\lambda_u^3}}{1 + \alpha^2 \lambda_u^2} + O(y), \]

(57)

\[ w = \frac{b}{y} + \frac{2\sqrt{(\alpha^2 - 1)a\lambda_u}}{1 + \alpha^2 \lambda_u^2} + O(y), \]

(58)

\[ o = b + \frac{2\sqrt{(\alpha^2 - 1)a\lambda_u}}{1 + \alpha^2 \lambda_u^2} y + O(y^2), \]

(59)

where

\[ b = o(u_i,0). \]

(60)

It is clear that \( \lambda_v, \Xi, \omega \) and \( w \) are all divergent as \( y \to 0 \). However, the ancillary quantities \( \bar{\lambda}_v \) and \( \bar{\Xi} \) as well as \( m \) and \( o \), the Geroch potential, are all regular. This suggests rewriting the system (44)–(48) in terms of these last two variables. We find explicitly that

\[ M = \frac{m_u}{1 + my}, \]

(61)

\[ O = \frac{o_u}{1 + my}, \]

(62)

\[ M_y = -\frac{o_y}{y(1 + my)}O + \frac{1}{4(1 + my)} \left[ -y(m + y^2 m_{yy} + 3ym_y) \right. \]

\[ + \frac{y^2}{1 + my} (m^2 + 2ym_{yn} + y^2 m_n - o^2_y) \right], \]

(63)

\[ O_y = \frac{1}{y}O + \frac{yo_y}{1 + my} M - \frac{y^2}{4(1 + my)} (yo_{yy} + o_y) \]

\[ + \frac{y^3}{2(1 + my)^2} (mo_y + ym_y o_y), \]

(64)

\[ \gamma_y = -\frac{y}{8(1 + my)^2} \left( (m + ym_{yn})^2 + o^2_y \right). \]

(65)

Although the equations (63) and (64) are coupled singular equations for \( M \) and \( O \), they are regular singular equations and the solutions remain regular as \( y \to 0 \) (see the Appendix). In fact, it is clear from the defining equations (61) and (62), together with (57) and (59), that both \( M \) and \( O \) are of order unity in the limit \( y \to 0 \). These are the modified equations on which the new code is based for investigating the Piran et al solutions.
We need to augment the equations with the interface equations which relate the two sets of variables in the two coordinate systems on the interface. The relationships required for \textit{injection}, i.e. going from the characteristic to the Cauchy region are,

\[ \psi = \frac{1}{2} \ln(1 + my), \]  
\[ \psi, r = -\frac{1}{2} y M - \frac{y^3}{4 \lambda} (ym)_y, \]  
\[ \psi, rr = \frac{1}{2} y M, u + \frac{1}{2} y^3 (ym)_y - \frac{y^6}{8 \lambda^2} [(ym)_y]^2 \]  
\[ + \frac{1}{8 \lambda} \left\{ 2(ym)_y + [y(ym)_y]_y \right\}, \]  
\[ \tilde{L} = \frac{1}{2} M y, \]  
\[ \omega, r = \frac{O}{y^2 \lambda}, \]  
\[ \omega, rr = -\frac{O}{y^2 \lambda} + \frac{MO}{y \lambda} + \frac{1}{2} \frac{yO (ym)_y}{\lambda} - \frac{1}{2} \frac{yO y}{\lambda}, \]  
\[ L^\phi = \frac{1}{2} \frac{\lambda O}{y^2} + \frac{1}{4} y o_y. \]

Similarly, the relationships required for \textit{extraction}, i.e. going from the Cauchy to the characteristic region are,

\[ \lambda = e^{2\psi}, \]  
\[ m = r^{1/2} (e^{2\psi} - 1), \]  
\[ m, y = -r^2 e^{2\psi} (4\psi, t + 4\psi, r + r^{-1}) + r, \]  
\[ m, yy = 2r^{7/2} e^{2\psi} (4\psi, tt + 8\psi, tr + 8\psi^2 t + 10r^{-1}\psi, t + 16\psi, t, r + 4\psi, rr \]  
\[ + 8\psi^2 r + 10r^{-1}\psi, r + r^{-2}) - 2r^{3/2}, \]  
\[ M = 2r^{-1/2} \tilde{L}, \]  
\[ o = \int_F r^{-1} e^{4\psi} \omega_y dt, \]  
\[ o, y = 4r^{1/2} L^\phi z - 2r^{1/2} e^{4\psi} \omega, r, \]  
\[ o, yy = 2r^2 e^{4\psi} (8\psi, t, \omega, r + 2\omega, tr + r^{-1}\omega, t + 8\psi, r, \omega, r + 2\omega, rr + r^{-1}\omega, t) \]  
\[ - 8r^2 (L^\phi z, t + L^\phi z, r), \]  
\[ O = r^{1/2} e^{2\psi} \omega, r. \]

where the integration in (78) is along the interface from the point \( F : (t, r) = (t_0, 1) \) to the point \( I : (t, r) = (t_i, 1) \) as shown in Fig. 1.

5. Results

In order to test the CCM code against the Piran et al solution we will first compare the metric quantity \( \psi \) in both the Cauchy and characteristic regions. Eq. (18) provides the exact solution in both cases when \( \lambda_u \) and \( \lambda_v \), Eqs. (21) and (22), are written in terms
of \((t, r)\) coordinates and \((u, y)\) coordinates for the Cauchy and characteristic regions respectively. It is not possible to use \(\omega\) for a similar comparison since from (56) we have seen that it diverges as \(y \to 0\). We use, instead, the Geroch potential, \(\sigma\), which is computed directly by the code in the characteristic region, but must be constructed in the Cauchy region using its definition (cf Eq. (24) of [2]),

\[
\sigma = \int r^{-1} e^{4\psi} \omega_r dt + \int r^{-1} e^{4\psi} \omega_t dr.
\]

(82)

It is straightforward to evaluate the integrals in the Cauchy region using finite-difference representations of \(\omega_r\) and \(\omega_t\). For the exact solution \(\omega_r\) and \(\omega_t\) can be evaluated by differentiation, however the integral must be evaluated numerically to find a semi-analytic \(\sigma\). A similar approach is required for the ‘exact’ value of \(\sigma\) in the characteristic region using Eq. (42).

As mentioned previously, the two parameters \(a\) and \(\alpha\) of the Piran et al solution represent the length scale and strength of the gravitational wave respectively. For \(t < 0\) the wave moves in from infinity, reaches its highest concentration at \(t = 0\), where it rebounds from the \(r = 0\) axis, and is outgoing for \(t > 0\). CCM allows us to evolve initial data containing an ingoing gravitational wave, something which is difficult to do for a Cauchy only evolution. Since we have set the interface at \(r = 1\) the value of \(a\) can be chosen such that a substantial fraction of the the wave will move onto the Cauchy region and will be well resolved with the numbers of grid points we choose. Changing the interface position while keeping the ‘interface distance to \(a\)’ ratio the same results in an identical evolution on the Cauchy portion of the grid.

The primary variable we used for comparison are \(\psi\) and \(\sigma\) (since \(\gamma\) is a derived quantity). We considered grid resolution numbers of \(N = 301, 601\) and 1201, where the error is defined as

\[
\varepsilon(\psi) = \frac{|| \psi_E - \psi_C ||}{|| \psi_E ||},
\]

(83)

where \(\psi_E\) is the exact value, \(\psi_C\) is the code computed value and \(|| \cdot ||\) denotes the \(L_2\) norm. We ran the modified code for a whole range of values in the \(a, \alpha\) parameter space and found an error in \(\psi\) of no more than 0.01% and an error in \(\sigma\) of less than 0.2%. However, in this version of the code, although the convergence rate of the solution starts as second-order it deteriorates to first order after long time evolution. In some recent work, colleagues in the Southampton Numerical Relativity group have modified the code so that it is now fully second order and very long time stable [10]. The improvement was achieved by dispensing with variables which are related by exponential or logarithmic functions at the interface, using the Geroch potential both in the Cauchy and characteristic regimes and also using an implicit method at the interface. This version of the code is currently being applied successfully to modelling dynamic cosmic strings.

Surface plots are shown for evolutions of the metric quantity \(e^{2\psi}\) in Fig. 2, the Geroch potential \(\sigma\) in Fig. 3, and the radial derivative \(\gamma_r\) (which indicates the distribution of energy within the wave) in Fig. 4, where the parameters are \(a = 0.5\) and
Figure 2. Surface plot showing the time evolutions of the metric quantity $e^{2\psi}$. The initial data has the parameters $a = 0.5$ and $\alpha = 10.0$, representing a strong field spacetime. The interface is placed at $z = 1$ denoted by a gap in the surface plot. Note that for clarity the $z$-axis is reversed in this figure.

For comparison purposes figures 5, 6 and 7 shows the same quantities, but for parameter values $a = 0.5$ and $\alpha = 1 \times 10^{-1}$, which represents an almost flat spacetime. Again the wave reaches its maximum concentration on the $r = 0$ axis at $t = 0$, however the peak is much less dominant in this case.
Figure 3. Surface plot showing the time evolutions of the Geroch potential, \( o \).

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Appendix A. Regular-singular behaviour of modified propagation equations

Let

\[
F = \frac{o_y}{(1 + my)},
\]  

(A.1)

which by (57) and (59) is regular as \( y \to 0 \). The homogeneous part of the modified propagation equations (63) and (64) is

\[
M_{,y} + \frac{F}{y} O = 0,
\]  

(A.2)

\[
O_{,y} - yFM - \frac{1}{y} O = 0.
\]  

(A.3)
Differentiating (A.3) with respect to $y$ and using (A.3) to eliminate $M$ and (A.2) to eliminate $M_y$, we get

\[
O_{,yy} - \left( \frac{2}{y} + \frac{F_y}{F} \right) O_{,y} + \left( \frac{2}{y^2} + \frac{F_{yy}}{yF} + F^2 \right) O = 0,
\]

which has dominant singular part

\[
O_{,yy} - \frac{2}{y} O_{,y} + \frac{2}{y^2} O = 0.
\]

Substituting in the trial solution $y^k$ we obtain the auxiliary equation

\[
k^2 - 3k + 2 = 0
\]

which has roots $k = 1, 2$ and so gives rise to the regular independent solutions $y$ and $y^2$.

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Figure 5. Same as Fig. 2 but with parameters $a = 0.5$ and $\alpha = 1.01$ representing an almost flat spacetime.

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Figure 6. Same as Fig. 3 but with parameters $a = 0.5$ and $\alpha = 1.01$ representing an almost flat spacetime.
Figure 7. Same as Fig. 4, but with parameters $a = 0.5$ and $\alpha = 1.01$ representing an almost flat spacetime.