ON THE BOMBIERI-LANG CONJECTURE OVER FINITELY GENERATED FIELDS

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ABSTRACT. The strong Bombieri-Lang conjecture postulates that, for every variety $X$ of general type over a field $k$ finitely generated over $\mathbb{Q}$, there exists an open subset $U \subset X$ such that $U(K)$ is finite for every finitely generated extension $K/k$. The weak Bombieri-Lang conjecture postulates that, for every positive dimensional variety $X$ of general type over a field $k$ finitely generated over $\mathbb{Q}$, the rational points $X(k)$ are not dense. Furthermore, Lang conjectured that every variety of general type $X$ over a field of characteristic 0 contains an open subset $U \subset X$ such that every subvariety of $U$ is of general type, this statement is usually called geometric Lang conjecture.

We reduce the strong Bombieri-Lang conjecture to the case $k = \mathbb{Q}$. Assuming the geometric Lang conjecture, we reduce the weak Bombieri-Lang conjecture to $k = \mathbb{Q}$, too.

In [Lan86], Lang famously stated a series of conjectures centered around the scarcity of rational points of a variety of general type over fields finitely generated over $\mathbb{Q}$. Independently, Bombieri stated part of these conjectures. We study their reduction to number fields.

In order to describe the conjectures, let us fix some conventions and recall some definitions.

Conventions. Fields are tacitly assumed to be of characteristic 0. For ease of reading, we will always use the letter $k$ for the base field, $h$ for finite extensions of $k$ and $K$ for finitely generated extensions of $k$. By convention, a possibly singular, non-proper variety is of general type if it is birational to a smooth, proper variety of general type.

Definition. A variety $X$ over $k$ is mordellic if $X(K)$ is finite for every finitely generated extension $K$ of $k$. We stress that $X(K)$ is required to be finite for all finitely generated extensions $K$ of $k$, not only finite ones. A variety is pseudo-mordellic if it has a non-empty open subset which is mordellic.

A variety $X$ over $k$ is geometrically mordellic, or GeM, if every subvariety of $X_k$ is of general type, and it is pseudo-GeM if it has a non-empty open subset which is GeM.

Following [AV96], we organize part of Lang’s program in four conjectures.

Geometric Lang conjecture: If $X$ is a variety of general type over a field $k$ of characteristic 0, then it is pseudo-GeM.

Lang conjecture for function fields: Let $K/k$ be a finitely generated extension of fields of characteristic 0 with $k$ algebraically closed in $K$, and $X/K$ a variety of general type. If $X(K)$ is Zariski dense in $X$, then $X$ is birational to a variety $X_0$ defined over $k$ and the "non-constant points" of $X(K) \setminus X_0(k)$ are not Zariski-dense in $X$.

Weak Bombieri-Lang conjecture: If $X$ is a positive dimensional variety of general type over a field $k$ finitely generated over $\mathbb{Q}$, the set of rational points $X(k)$ is not dense.

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**Strong Bombieri-Lang conjecture:** Every variety of general type over a field $k$ finitely generated over $\mathbb{Q}$ is pseudo-mordellic.

Faltings proved in [Fal94] that these conjectures hold for subvarieties of abelian varieties, but the general case remains widely open.

**Definition.** A field $k$ is weakly Lang if, for every positive dimensional variety $X$ of general type over $k$, $X(k)$ is not dense. A field $k$ is Lang if every positive dimensional variety $X$ of general type over $k$ is pseudo-mordellic. Clearly, Lang fields are weakly Lang, too.

Observe that in order to check whether a field $k$ is Lang one has to consider all finitely generated extensions of $k$, while to check whether $k$ is weakly Lang it is enough to work over $k$.

Using these definitions, the weak Bombieri-Lang conjecture states that finitely generated extensions of $\mathbb{Q}$ are weakly Lang, while the strong version states that they are Lang.

**Reduction to number fields.** In the last section of [Lan86], Lang discusses the problem of reducing his conjectures about finitely generated extensions of $\mathbb{Q}$ to number fields.

In dimension 1, Lang cites two approaches to solve this reduction problem. The first approach is to use Mordell’s conjecture for function fields, which was proved first by Manin [Man63] and later, with different methods, by Grauert [Gra65]. The second approach uses a specialization argument, see the last paragraph of [Lan86]. These results were available years before the Mordell conjecture for number fields was proved by Faltings in 1983 [Fal83].

Let us now consider the higher dimensional case. The following easy lemmas are implicit in Lang’s writings.

**Lemma 1** (Lang). Assume that the Lang conjecture for function fields holds, and that finite extensions of $k$ are weakly Lang. Then finitely generated extensions of $k$ are weakly Lang.

*Proof.* Follows from the definitions. □

**Lemma 2** (Lang). Assume that the geometric Lang conjecture holds and that all finitely generated extensions of $k$ are weakly Lang. Then $k$ is Lang.

*Proof.* Let $X$ be a variety of general type over $k$. Since geometric Lang conjecture holds, there exists an open subset $U \subseteq X$ which is GeM. We claim that $U$ is mordellic. Let $K/k$ be a finitely generated extension and define $Z \subseteq U_{K}$ as the Zariski closure of $U_{K}(K)$. We have that $Z(K)$ is dense in $Z$ and all of its irreducible components are of general type since $U$ is GeM. Since $k$ is weakly Lang by hypothesis, then $Z$ has dimension 0 and thus $U(K) = Z(K)$ is finite. □

Combining these two lemmas, we see that the Lang conjecture for function fields (together with the geometric Lang conjecture which in dimension 1 is trivial) represents a generalization to higher dimensions of the first approach to the reduction problem. Moreover, in the very last sentence of [Lan86], Lang wishes for a generalization of the second approach (the specialization argument).

**Our results.** Up to our knowledge, both the approaches described above are still far from being complete already in dimension 2. We introduce a third approach based on studying the conjectures in all dimensions at the same time. This approach reduces the strong Bombieri-Lang conjecture to $\mathbb{Q}$ unconditionally. Moreover, assuming the geometric Lang conjecture, we show that the strong
Bombieri-Lang conjecture reduces to proving that \( \mathbb{Q} \) is weakly Lang. This removes the need, for arithmetic purposes, of Lang’s conjecture for function fields.

**Theorem A.** Let \( K/k \) be a finitely generated extension of fields of characteristic 0. Then \( k \) is Lang if and only if \( K \) is Lang. In particular, if \( \mathbb{Q} \) is Lang then the strong Bombieri-Lang conjecture holds.

**Theorem B.** Let \( k \) be a field of characteristic 0. Assume that the geometric Lang conjecture holds. Then \( k \) is Lang if and only if it is weakly Lang.

Compare Theorem B with Lemma 2: to conclude that \( k \) is Lang we only need \( k \) to be weakly Lang, not all of its finitely generated extensions.

Combining Theorems A and B, we immediately obtain the following.

**Theorem C.** If the geometric Lang conjecture holds and \( \mathbb{Q} \) is weakly Lang, then the strong Bombieri-Lang conjecture holds.

We stress that both Theorem A and Theorem B take into account the various conjectures for all dimensions at the same time. For instance, if we assume the strong Bombieri-Lang conjecture only for surfaces of general type over \( \mathbb{Q} \), we cannot apply Theorem A to conclude that the conjecture holds for surfaces of general type over finitely generated extensions of \( \mathbb{Q} \).

Our proofs are relatively easy, but we rely on two strong theorems: a particular case of the subadditivity of the Kodaira dimension by Viehweg [Vie82, Satz III] for Theorem A and a uniformity result by Caporaso-Harris-Mazur and Abramovich-Voloch [CHM97], [AV96, Theorem 1.7], [Abr97] for Theorem B.

Even though we are mainly interested in finitely generated extensions of \( \mathbb{Q} \), our arguments are general. Non-standard examples are given by finitely generated extensions of \( \mathbb{Q}(x_1, x_2, \ldots) \): it is easy to show that if the geometric Lang conjecture and the weak Bombieri-Lang conjecture hold, then \( \mathbb{Q}(x_1, x_2, \ldots) \) is weakly Lang. Theorems A and B then imply that all finitely generated extensions of \( \mathbb{Q}(x_1, x_2, \ldots) \) are Lang.

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1. **Proof of Theorem A**

**Lemma 3.** Let \( h/k \) be a finite extension of fields of characteristic 0. A variety \( X \) over \( k \) is pseudo-mordellic if and only if \( X_h \) is pseudo-mordellic.

**Proof.** If \( U \subseteq X \) is a mordellic open subset, then \( U_h \subseteq X_h \) is mordellic too. On the other hand, let \( U \subseteq X_h \) be a mordellic open subset, and let \( h' \) be a Galois closure of \( h/k \). Then \( \bigcap_{\sigma \in \text{Gal}(h'/k)} \sigma(U_{h'}) \) is Galois invariant, thus it descends to a mordellic open subset of \( X \). \( \square \)

**Lemma 4.** Let \( X_1, \ldots, X_n \) be varieties over a field \( k \), and assume that the product \( \prod_i X_i \) is pseudo-mordellic. Then each factor \( X_i \) is pseudo-mordellic.

**Proof.** Let us prove that \( X_1 \) is pseudo-mordellic. Let \( U \) be a non-empty mordellic open subset of \( \prod_i X_i \). Thanks to Lemma 3 we may assume that we have a rational point \( (x_i)_{i \in U(k)} \). Then \( U \cap (X_1 \times (x_2, \ldots, x_n)) \) is a non-empty mordellic open subset of \( X_1 \times (x_2, \ldots, x_n) = X_1 \). \( \square \)
If \( h/k \) is a finite extension and \( X \) is a variety over \( h \), recall that the Weil restriction \( R_{h/k}(X) \) is a variety over \( k \) representing the functor \( S \mapsto X(S_h) \) for schemes \( S \) over \( k \). The Weil restriction always exists for quasi-projective varieties, see [BLR90, §7.6].

If \( S \) is an \( h \)-scheme and \( \sigma : h \to h \) is a field automorphism, denote by \( \sigma^*S \) the \( h \)-scheme \( S \to \text{Spec } h \xrightarrow{\sigma^*} \text{Spec } h \), where \( \sigma^* \) is the morphism of schemes induced by \( \sigma \).

**Lemma 5.** Let \( h/k \) be a finite extension of fields of characteristic 0. Then \( k \) is Lang if and only if \( h \) is Lang.

**Proof.** If \( h \) is Lang, then \( k \) is Lang thanks to Lemma 3. Assume now that \( k \) is Lang, we want to prove that \( h \) is Lang. Thanks to the preceding case, we may assume that \( h/k \) is Galois.

Thanks to [Wei82, Theorem 1.3.2], we have that \( R_{h/k}(X)_h \simeq \prod_{\sigma \in \text{Gal}(h/k)} \sigma^*X \) is a product of varieties of general type, hence \( R_{h/k}(X)_h \) is of general type. Since we are assuming that \( k \) is Lang, \( R_{h/k}(X) \) is pseudo-mordellic, and thus \( R_{h/k}(X)_h = \prod_{\sigma} \sigma^*X \) is pseudo-mordellic too. By Lemma 4, it follows that \( X \) is pseudo-mordellic.

**Lemma 6.** Let \( k \) be a field of characteristic 0 and \( t \) an indeterminate. If \( k(t) \) is Lang, then \( k \) is Lang.

**Proof.** Let \( X \) be a positive dimensional variety of general type over \( k \), we want to show that it is pseudo-mordellic. By hypothesis, there exists an open, mordellic non-empty subset \( U \subseteq X_{k(t)} \). Let \( D \subseteq X \times \mathbb{A}^1 \) be the closure of \( X_{k(t)} \setminus U \). Since \( k \) is infinite and \( U \) is non-empty, there exists \( \lambda \in \mathbf{K} \) such that \( X \times \{ \lambda \} \) is not contained in \( D \) (otherwise, we would have \( D = X \times \mathbb{A}^1 \)). Write \( V = X \times \{ \lambda \} \setminus D \subseteq X \times \{ \lambda \} \cong X \), we claim that \( V \) is mordellic. Let \( K/k \) be a finitely generated extension and \( p \in V(K) \) a point, by definition of \( V \) we have that \( (p, \lambda) \in X \times \mathbb{A}^1 \) is not contained in \( D \). In particular, \( \{ p \} \times \mathbb{A}^1 \not\subseteq D \) and thus \( p_{K(t)} \in U(K(t)) \). Hence, we have that \( V(K) \subseteq U(K(t)) \) is finite since \( U \) is mordellic. 

Let us now prove Theorem A. Let \( K/k \) be a finitely generated extension. If \( K \) is Lang, by Lemma 5 and Lemma 6 plus an easy inductive argument on \( \text{trdeg}(K/k) \) we have that \( k \) is Lang.

Assume now that \( k \) is Lang, we want to prove that \( K \) is Lang. Thanks to Lemma 5 we may assume that \( k \) is algebraically closed in \( K \), let \( V/k \) be some variety such that \( k(V) = K \) and let \( \eta \in V \) be its generic point. Thanks again to Lemma 5, we may pass to finite extensions of \( K \) and assume that \( V \) is of general type.

Let \( X \) be a variety of general type over \( K = k(V) \), we want to prove that \( X \) is pseudo-mordellic. There exists a variety \( \overline{X} \) over \( k \) with a morphism \( \overline{X} \to V \) such that \( \overline{X}_\eta = X \). Since both \( V \) and the generic fiber of \( \overline{X} \to V \) are of general type, by [Vie82, Satz III] we have that \( \overline{X} \) is of general type. Since we are assuming that \( k \) is Lang, there exists a non-empty mordellic open subset \( U \subseteq \overline{X} \), i.e. \( U(K') \) is finite for every finitely generated extension \( K' \) of \( k \). In particular, \( U_\eta(K') \subseteq U(K') \) is finite for every finitely generated extension \( K' \) of \( k(V) \) and hence \( U_\eta \) is a non-empty mordellic open subset of \( \overline{X}_\eta = X \). This completes the proof of Theorem A.

2. Proof of Theorem B

Assume that the geometric Lang conjecture holds and that \( k \) is weakly Lang, we want to show that it is Lang. Since we are assuming the geometric Lang conjecture, it is enough to show that every GeM variety over \( k \) is mordellic.
Let $X/k$ be a GeM variety. We want to apply a theorem by Abramovich-Voloch [AV96, Theorem 1.7], [Abr97] (generalization of a result by Caporaso-Harris-Mazur [CHM97]) which implies that there exists an uniform bound on $|X(h)|$ for finite extensions $h/k$ of bounded degree, but to do so we need to make a small remark on their hypotheses. In fact, they assume that the base field $k$ is finitely generated over $\mathbb{Q}$ and that weak Bombieri-Lang conjecture holds, but a careful analysis of their proof shows that the only hypothesis on $k$ they actually use is that $k$ is weakly Lang.

Since we are assuming that $k$ is weakly Lang, we may thus apply the uniform bound theorem by Abramovich and Voloch. Let $K/k$ be a finitely generated extension, we want to show that $X(K)$ is finite. Let $V$ be an integral scheme of finite type over $k$ whose function field is $K$, choose any generically finite, dominant rational map $V \to \mathbb{P}^n$ and let $d$ be its degree. Since rational points are dense in $\mathbb{P}^n$, it follows that $V_d = \{p \in V(\bar{k}) \mid [k(p) : k] \leq d\}$ is dense in $V$. Thanks to the uniform bound theorem, there exists an $N$ such that $|X(h)| \leq N$ for every finite extension $h/k$ with $[h : k] \leq d$. Let us prove that $|X(K)| \leq N$.

If by contradiction we have $N + 1$ different sections Spec $K \to X$, up to shrinking $V$ we may assume that they extend to $N + 1$ morphisms $f_1, \ldots, f_{N+1} : V \to X$. Since $|X(h)| \leq N$ for every finite extension $h/k$ with $[h : k] \leq d$, we have that for every $p \in V_d$ there exists a pair of different indexes $i \neq j$ with $f_i(p) = f_j(p)$. Since $V_d$ is dense and the pairs of different indexes are finite, there exists a pair $i \neq j$ and a subset $S \subseteq V_d$ dense in $V$ such that $f_i(p) = f_j(p)$ for every $p \in S$, thus $f_i = f_j$ which gives a contradiction.

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