Worldsheet Instanton Corrections to 5\textsuperscript{2}\textsubscript{2}-brane Geometry

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Abstract:
We study worldsheet instanton corrections to the exotic 5\textsuperscript{2}\textsubscript{2}-brane geometry in type II string theory. The BPS vortices in the \( \mathcal{N} = (4,4) \) gauged linear sigma model modify the geometry of the 5\textsuperscript{2}\textsubscript{2}-brane. We find that the modification of the geometry is understood by the localization in the T-dualized winding direction.
1 Introduction

In string theory, the Buscher rule of T-duality [1] exhibits the mixing of the metric and the NSNS B-field of spacetime geometries. Performing T-duality transformations along several directions, one obtains a non-conventional object whose metric is no longer single-valued. Due to the pathological nature, these kind of objects in string theory have been ignored for a long time. Recently, the non-conventional geometries again appear in string compactification scenarios with non-vanishing background fields. One of the most familiar example is known as the T-fold [2]. This contains information of the general coordinate transformation and the T-duality transformation. The feature of the T-fold comes from the winding modes as well as the momentum modes of the string coupled to the B-field on the space. Since the contribution of the winding modes cannot be represented in the conventional geometry, they are refereed to as the “non-geometric” backgrounds. A typical example of the T-fold is the exotic $5^2_2$-brane [3]. One finds this less familiar object from NS5-brane in type II string theory via T-duality along two transverse directions. So far, this has been well investigated in the supergravity viewpoint [3, 4, 5].

A salient feature of stringy corrections to the geometry comes from the worldsheet quantum effects. The gauged linear sigma model (GLSM) is a powerful tool to study phases of vacua [6] and quantum corrections to string worldsheet theory [7]. The IR limit of the GLSM is the non-linear sigma model (NLSM) whose target space is solutions to supergravities. The GLSM is known to be useful because of the following reasons: (i) the gauge multiplets in the UV regime connects various phases of vacua in the system, and (ii) the soliton solutions provided by the gauge fields can be mapped to the worldsheet instantons in the IR regime. In the previous paper [8], we have constructed the $\mathcal{N} = (4,4)$ GLSM for five-branes of co-dimension three which gives rise to the non-linear sigma model whose target space is the exotic $5^2_2$-brane. The model is obtained from the GLSMs for the NS5-branes and for the Kaluza-Klein (KK) monopoles via T-dualities. In [9, 10, 11], the instanton corrections to the NS5-branes and KK-monopoles are studied through the vortices in the GLSMs. In the present paper, based on the GLSM constructed in [8], we study the instanton corrections to the geometry of the exotic $5^2_2$-brane.

The organization of this paper is as follows: In Section 2 we briefly review the T-duality chains among the NS5-branes, the KK-monopoles and the exotic $5^2_2$-brane in the viewpoint of supergravity. We also discuss the instanton corrections and their geometrical meaning to the former two branes. In Section 3 we study the worldsheet instanton corrections to the exotic $5^2_2$-brane from the two distinct perspectives. One is from the T-dualized description of the KK-monopoles involving the instanton corrections. The other is from the direct calculation of instantons in the GLSM. We find that the instanton corrections to the geometry of the co-dimension three five-branes are carried over to that of the co-dimension two exotic $5^2_2$-brane. Although the two results are obtained from the different routes, we find that they are completely consistent with each other. Section 4 is devoted to conclusion and discussions.
2 T-duality chains among NS5-branes, KK-monopoles and exotic $5^2_2$-brane

In this section, we briefly explain the T-duality chains among the NS5-branes, the KK-monopoles and the exotic $5^2_2$-brane in type II string theory. These objects are realized as solutions to the equations of motion in ten-dimensional supergravities. Throughout this paper, we consider the configurations of the objects depicted in Table 1. These geometries are realized as the target spaces of the string NLSM. The action of the general supersymmetric NLSM is given by

\[
S = \frac{1}{2\pi} \int d^2x \left[ -\frac{1}{2} g_{\mu\nu} \partial_m X^\mu \partial^m X^\nu + \frac{1}{2} B_{\mu\nu} \varepsilon^{mn} \partial_m X^\mu \partial_n X^\nu + \frac{i}{2} g_{\mu\nu} \Omega_{\mu}^+ \Omega_{\nu}^+ + \frac{i}{2} g_{\mu\nu} \Omega_{\mu}^- \Omega_{\nu}^- + \frac{1}{4} R_{\mu\nu\rho\sigma} \Omega_{\mu}^+ \Omega_{\nu}^+ \Omega_{\rho}^- \Omega_{\sigma}^- \right],
\]

(2.1)

where $D_\pm = D_0^\pm \pm D_1^\pm$, $\varepsilon^{01} = 1$ and the worldsheet metric is $\eta_{mn} = \text{diag}(-1,1)$. The covariant derivative $D_m$ is defined with the positive and negative torsion. The functions $g_{\mu\nu}(X)$ and $B_{\mu\nu}(X)$ are the target space metric and the anti-symmetric NSNS B-field. The function $R_{\mu\nu\rho\sigma}$ is the Riemann tensor of the target space with the positive torsion. The real fermions $\Omega_\pm^\mu$ are supersymmetric partners of the string coordinates $X^\mu$.

It is known that the string quantum corrections modify the target space geometries [6]. Especially, we focus on the worldsheet instanton corrections to the geometries. The worldsheet instantons are best examined in the language of the GLSM [6, 7] whose IR limit describes the NLSM (2.1). As we will discuss in the following section, instanton corrections to the four-point function of fermions in the GLSM in the UV regime result in the corrections to the Riemann tensor in the low-energy effective action (2.1). The instanton effects on the geometries are extracted from the Riemann tensor $R_{\mu\nu\rho\sigma}$. In the following subsections we introduce the worldsheet instanton corrections to the geometries of the H- and KK-monopoles.

|                  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|------------------|---|---|---|---|---|---|---|---|---|---|
| NS5-branes       | • | • | • | • | • | • | • | • | • | • |
| KK-monopoles     | • | • | • | • | • | • | • | • | • | ○ |
| $5^2_2$-brane     | • | • | • | • | • | • | • | • | • | ○ |

Table 1: The black dot • stands for the world-volume directions of each object. The others are transverse directions. Especially, starting from the NS5-branes, the compact directions where the T-duality transformations are performed are shown by the symbol ○.
2.1 NS5-branes and H-monopoles

Our T-duality chains start from the NS5-branes in type II string theory. The NS5-branes are known to be the co-dimension four solitonic solution to ten-dimensional supergravity equations of motion [13]. The explicit form of the solution is given by

\[ ds_{\text{NS5}}^2 = dx_{034567}^2 + H(\vec{R}) \, dx_{1289}^2, \quad e^{2\phi} = H(\vec{R}), \]

\[ H_{\mu\nu\rho} = \epsilon_{\mu\nu\rho\lambda} \partial_\lambda \log H(\vec{R}), \quad H(\vec{R}) = \frac{1}{g^2} + \frac{Q}{|\vec{R}|^2}, \]

where \( \phi \) is the dilaton and \( H_{\mu\nu\rho} \) (\( \mu, \nu, \rho = 1, 2, 3, 8 \)) is the field strength of the NSNS B-field. The constant \( g \) is a dimensionless parameter and \( Q \) is a charge with appropriate mass dimension. The vector \( \vec{R} = (X^1, X^2, X^8, X^9) \) represents the coordinates in the \( \mathbb{R}^4 \) plane. We used the notation such as \( dx_{034567}^2 = -(dx^0)^2 + (dx^3)^2 + \cdots + (dx^7)^2 \) and \( dx_{1289}^2 = (dx^1)^2 + (dx^2)^2 + (dx^8)^2 + (dx^9)^2 \).

Now we compactify the \( X^9 \)-direction on \( S^1 \) with radius \( R_9 \). Then the harmonic function \( H \) becomes

\[ H(r, \vartheta) = \frac{1}{g^2} + \sum_{n=-\infty}^{\infty} \frac{Q}{r^2 + (\vartheta - 2\pi R_9 n)^2}, \]

where \( r = (X^1, X^2, X^8), \quad r = |\vec{r}| \) and \( \vartheta = X^9 \). The summation in (2.3) is performed in the small \( R_9 \) limit. In this limit, the discrete summation over \( n \) is approximated by the continuous integral. The result is independent of \( \vartheta \):

\[ H(r, \vartheta) = \frac{1}{g^2} + \int_{-\infty}^{\infty} dn \frac{Q}{r^2 + (\vartheta - 2\pi R_9 n)^2} = \frac{1}{g^2} + \frac{Q/R_9}{2r}. \]

The solution (2.2) with the harmonic function (2.4) is called the H-monopoles or the smeared NS5-branes, i.e., the H-monopoles are obtained by compactifying a transverse direction in the NS5-brane solution (2.2). The H-monopole solution possesses a \( U(1) \) isometry along the \( X^9 \)-direction. We note that the asymptotic radius of the \( S^1 \) direction is given by \( 1/g \).

On the other hand, when the radius \( R_9 \) is kept finite in the summation (2.3), the result becomes

\[ H(r, \vartheta) = \frac{1}{g^2} + \frac{Q/R_9}{2r} \frac{\sinh(r R_9^{-1})}{\cosh(r R_9^{-1}) - \cos(\vartheta R_9^{-1})}. \]

The solution (2.2) with the harmonic function (2.4) is called the localized H-monopoles [14]. In this case, the isometry along the \( X^9 \)-direction is broken. Although the relation between the solutions with (2.4) and (2.5) is obvious in the viewpoint of supergravity, it is interesting to observe these results in the worldsheet perspective. The expansion of the function

\[ \frac{\sinh r}{\cosh r - \cos \vartheta} = 1 + \sum_{n=1}^{\infty} (e^{-nr+i\vartheta} + e^{-nr-i\vartheta}) \]

strongly suggests contributions to the harmonic function (2.4) from topological sectors labeled by the integer \( n \) behind the effect of the finite radius \( R_9 \). Indeed, it is pointed out in [9] that the integer
n is nothing but the topological winding number stemming from the string worldsheet instantons. The worldsheet instantons are interpreted as the BPS vortices in the \( \mathcal{N} = (4, 4) \) GLSM whose IR limit describes the H-monopoles. It is demonstrated that all the \( n \) instantons contribute to the four-point function of fermions and modify the geometry of the H-monopoles. The calculations are performed in the large asymptotic radius limit \( 1/g \to \infty \). It is explicitly shown that the instanton corrections break the isometry along \( X^9 \) and precisely reproduce the harmonic function (2.5) of the localized H-monopoles. From now on we refer to this kind of instantons as the \( X^9 \)-instantons.

2.2 KK-monopoles

The KK-monopoles are known to be T-dual of the H-monopoles. For the H-monopoles, the solution has an isometry along the \( X^9 \)-direction. Therefore we can perform the T-duality transformation along this direction. Applying the Buscher rule [1] to the geometry (2.2) with (2.4), we have the KK-monopole geometry,

\[
\begin{align*}
\text{d}s^2_{\text{KKM}} &= \text{d}x_{034567}^2 + H(r) \text{d}x_{128}^2 + H^{-1}(r) \left( \text{d}x_9^2 + \frac{1}{2} \omega_i \text{d}x^i \right)^2, \\
\text{d}\omega &= *_3 \text{d}H, \quad \phi = 0, \quad H(r) = \frac{1}{g^2} + \frac{\hat{R}_9}{2r}, \quad \hat{R}_9 = Q/R_9,
\end{align*}
\]

where \( \hat{X}^9 \) is the dual coordinate of \( X^9 \) in the H-monopoles. The Hodge dual \(*_3\) is defined in \( \mathbb{R}^3 \) spanned by \((X^1, X^2, X^8)\). Then the vector \( \hat{\omega} \) satisfies the Dirac monopole equation, \( \text{rot} \hat{\omega} = -2\nabla H \). The transverse directions to the KK-monopoles are the Taub-NUT space which possesses a \( U(1) \) isometry. In the asymptotic regime \( r \to \infty \), the transverse space is the non-trivial fibration of \( S^1 \) over \( \mathbb{R}^3 \). The asymptotic radius of the \( S^1 \) direction is given by \( g \), which is just the T-dual circle. It is natural to consider the worldsheet instanton corrections to the KK-monopole geometry (2.7). In [10], the instanton effects in the GLSM for the KK-monopoles are studied. As in the same way in the H-monopoles, the BPS vortices in the GLSM modify the target space geometry (2.7). The instanton corrections are interpreted as the modification of the harmonic function \( H(r) \) in (2.7). The authors in [10] found that the modified function is nothing but the one in (2.5) obtained in the localized H-monopoles. Then the harmonic function depends on the “T-dual coordinate” \( \vartheta \), i.e., the KK-monopole is localized along the winding direction. Therefore the solution (2.7) with the harmonic function (2.5) is called the localized KK-monopoles.

A few comments are in order. First, the harmonic function depending on the winding coordinate \( \vartheta \) rather than the physical coordinate \( \tilde{X}^9 \) indicates the non-geometric structure of the solution. This kind of non-geometric nature is studied in the doubled formalism [15, 16]. Second, the configuration (2.7) with the modified harmonic function (2.5) is no longer the solution to the BPS equations in supergravity. This is because the calculation is performed in the small radius limit \( g \to 0 \) of the T-dual circle [10]. In the small radius limit, the KK modes become massive while the string winding

\(^1\)We call this also the \( X^9 \)-instantons not \( \tilde{X}^9 \)-instantons.
modes become lighter. We therefore need to incorporate the light winding modes into supergravities. Then the supergravity approximation of the string massless spectrum is lost. Finally, we note that even though the localized KK-monopoles show the non-geometric nature, they also preserve a $U(1)$ isometry along the $X^8$-direction when we further compactify this direction on $S^1$ (see Table 1). This fact enables us to T-dualize the KK-monopoles with the $X^9$-instanton corrections into the $5^2_2$-brane geometry. We will discuss this issue in Section 3.

2.3 Exotic $5^2_2$-brane

We next proceed to the exotic $5^2_2$-brane. The geometry of the $5^2_2$-brane is obtained by performing the T-duality transformation on the KK-monopoles. In order to utilize the Buscher rule, we compactify the $X^8$-direction on $S^1$ with radius $R^8$ in the KK-monopole geometry (2.7). The harmonic function becomes

$$H = \frac{1}{g^2} + \frac{1}{2} \sum_{l=-\infty}^{\infty} \frac{\tilde{R}_8}{\sqrt{g^2 + (X^8 - 2\pi R^8 l)^2}}, \quad g^2 = (X^1)^2 + (X^2)^2. \quad (2.8)$$

We approximate the discrete summation over $l$ by the continuous integration in the small $R^8$ [3]. Since the integration over $l$ diverges, we introduce the cutoff $\Lambda$ to regularize it. The result is

$$H = h_0 + \frac{\tilde{R}_8}{2\pi R^8} \log \frac{\mu}{\varrho}, \quad (2.9)$$

where $\mu > 0$ is the renormalization scale and $h_0 = \frac{1}{g^2} + \frac{\tilde{R}_8}{2\pi R^8} \log \frac{4\pi \Lambda}{\mu}$ is a constant which diverges in the limit $\Lambda \to \infty$. It is obvious that the harmonic function is ill-defined for $\varrho < \mu$. Therefore the metric is well-defined only in the region $\varrho < \mu$. The physical meaning of this renormalization scale is discussed in [4]. The vector $\vec{\omega} = (\omega_1, \omega_2, \omega_8)$ is obtained by solving the Dirac monopole equation with the harmonic function (2.9). Using the Buscher rule on the geometry (2.7), we find the $5^2_2$-brane solution,

$$\begin{align*}
\text{d}s_{5^2_2}^2 &= \text{d}x_{534567}^2 + H \text{d}x_{12}^2 + HK^{-1}((\text{d}x^8)^2 + (\text{d}x^9)^2), \quad (2.10a) \\
B_{89} &= \frac{1}{2} K^{-1} \omega_8, \quad K = H^2 + \frac{1}{4} \omega_8^2, \quad 2\phi = HK^{-1}, \quad \omega_8 = \frac{\tilde{R}_8 R^8}{\alpha'} \arctan \left( \frac{X^2}{X^1} \right), \quad (2.10b)
\end{align*}$$

where we have employed the gauge where $\omega_1 = \omega_2 = 0$ and $\tilde{R}_8 = \alpha'/R^8$ is the dual radius of $R^8$. The harmonic function is given by (2.9). We note that the asymptotic radius of the T-dual circle is not well-defined because of the artificial bound $\varrho < \mu$. The geometry (2.10) shows some exotic nature. For example, due to the function $\arctan(X^2/X^1)$, the metric is not single valued. Therefore it has non-trivial monodromy around the origin in the $(X^1, X^2)$-plane. Indeed, this is a common property of exotic objects [3].

Even though the $5^2_2$-brane is exotic, it is straightforward to realize the geometry (2.10) in the string non-linear sigma model. With observation of the instanton corrections to the H- and KK-
monopoles, we are interested in the quantum corrections to the $5_2^2$-brane geometry. In the next section, we examine the $X^9$-instanton corrections to the $5_2^2$-brane geometry.

3 Instanton corrections to $5_2^2$-brane geometry

In this section, we study the worldsheet instanton corrections to the exotic $5_2^2$-brane geometry in two independent ways. The first is through the localized KK-monopoles. We calculate the corrections to the $5_2^2$-brane geometry from the viewpoint of the $X^9$-instantons in the KK-monopole geometry. We then perform the T-duality transformation along the $X^8$-direction by using the Buscher rule. Notice that the $X^9$-instanton corrections to the KK-monopoles \[10\] preserve the isometry along the $X^8$-direction, whilst the one along the $X^9$-direction is broken. The second is through the multi-centered five-branes. In the previous paper, we have studied the $N = (4, 4)$ GLSM which gives rise to the NLSM with the target space geometry for co-dimension three multi-centered $k$ five-branes. In the following discussions we refer to them as $\hat{5}_2^2$-branes. The NLSM for the co-dimension two $5_2^2$-brane is obtained by the suitable choice of the Fayet-Iliopoulos (FI) parameters and the limit $k \to \infty$ \[8\]. We evaluate the instanton corrections to the four-point function of fermions in the GLSM. The instanton corrections modify the geometry of the $\hat{5}_2^2$-branes in the IR limit. The modified geometry is reduced to that of the co-dimension two $5_2^2$-brane by the above procedure. We will show that the two results coincide with each other even though those are obtained from the different ways.

3.1 KK-monopole perspective

We have discussed the $X^9$-instanton corrections to the geometry of the KK-monopoles in Section 2. We observed that the harmonic function of the KK-monopole is modified and is given in (2.5). Now we compactify the $X^8$-direction on $S^1$ with radius $R_8$. The localized KK-monopoles become those of the periodic array with an infinite number of mirror images. The harmonic function (2.5) becomes

\[
H = \frac{1}{g^2} + \frac{1}{2} \sum_{l=-\infty}^{\infty} \frac{\tilde{R}_9}{\sqrt{\vartheta^2 + (X^8 - 2\pi R_8 l)^2}} \left[ \frac{\sinh(\tilde{R}_9^{-1} \sqrt{\vartheta^2 + (X^8 - 2\pi R_8 l)^2})}{\cosh(\tilde{R}_9^{-1} \sqrt{\vartheta^2 + (X^8 - 2\pi R_8 l)^2}) - \cos(\tilde{R}_9^{-1} \vartheta)} \right].
\]

(3.1)

Since the summation over $l$ diverges, we need to regularize it. In order to find the regularization, we first decompose the harmonic function as the summation over the $n$ $X^9$-instantons:

\[
H = \frac{1}{g^2} + \sum_{l=-\infty}^{\infty} \frac{\tilde{R}_9}{\sqrt{\vartheta^2 + (X^8 - 2\pi R_8 l)^2}} \times \left[ 1 + \sum_{n=1}^{\infty} \left( e^{-n\tilde{R}_9^{-1} \sqrt{\vartheta^2 + (X^8 - 2\pi R_8 l)^2} + in\tilde{R}_9^{-1} \vartheta} + e^{-n\tilde{R}_9^{-1} \sqrt{\vartheta^2 + (X^8 - 2\pi R_8 l)^2} - in\tilde{R}_9^{-1} \vartheta} \right) \right].
\]

(3.2)
We approximate the discrete summation over $l$ by the continuous integration in the small $\mathcal{R}_8$, 

$$
H = \frac{1}{g^2} + \frac{\mathcal{R}_9}{2\pi \mathcal{R}_8} \int_0^\infty dl \frac{1}{g^2 + l^2} + \frac{\mathcal{R}_9}{2\pi \mathcal{R}_8} \sum_{n=1}^{\infty} \left( e^{In\mathcal{R}_9^{-1}\vartheta} + e^{-In\mathcal{R}_9^{-1}\vartheta} \right) \int_0^\infty dl \frac{1}{g^2 + l^2} e^{-n\sqrt{\mathcal{R}_9^2 + l^2}}.
$$

(3.3)

The second term diverges while the third term remains finite. The second term together with the first term has been already calculated in (2.9). The third term becomes

$$
\frac{\mathcal{R}_9}{2\pi \mathcal{R}_8} \sum_{n=1}^{\infty} \left( e^{In\mathcal{R}_9^{-1}\vartheta} + e^{-In\mathcal{R}_9^{-1}\vartheta} \right) K_0(n\mathcal{R}_9^{-1}\vartheta),
$$

(3.4)

where $K_0(x)$ is the modified Bessel function of the second kind. This harmonic function does not break the isometry along the $X^8$-direction. Therefore we can perform the T-duality transformation along this direction. Applying the Buscher rule, we obtain the $5^2_2$-brane geometry with the harmonic function given by

$$
H = h_0 + \frac{\mathcal{R}_9\mathcal{R}_8}{2\pi \alpha'} \log \frac{\mu}{\varrho} + \frac{\mathcal{R}_9\mathcal{R}_8}{2\pi \alpha'} \sum_{n \neq 0} e^{In\mathcal{R}_9^{-1}\vartheta} K_0(|n|\mathcal{R}_9^{-1}\vartheta).
$$

(3.5)

The first and second terms are just the harmonic function of the $5^2_2$-brane (2.9). The third term is corrections coming from the $X^9$-instantons in the KK-monopoles. It is worth pointing out that the function (3.5) also appears in the study of the D-instanton corrections to the moduli space of hypermultiplets in Calabi-Yau compactifications [17]. It is discussed that the singularity in the moduli space metric is smoothed out by the D-instanton effects. Indeed, the function (3.5) is regular at $\varrho = 0$ provided that $\vartheta$ is non-zero. The distinct origin of the functional form (3.5) provides the geometric intuition behind this fact. The $5^2_2$-brane is localized in the winding direction $\vartheta$, namely at the origin $\vartheta = 0$. The regularity of the metric at $\varrho = 0$ is interpreted due to the non-zero interval from the $5^2_2$-brane in the winding space. The singularity appears again when one approaches to the $5^2_2$-brane not only in the physical geometry $\varrho \to 0$ but also in the winding space $\vartheta \to 0$.

3.2 $5^2_2$-branes perspective

In this subsection, we study the instanton corrections to the $5^2_2$-brane geometry from the direct calculations by the GLSM. In [8], we have studied the $\mathcal{N} = (4, 4)$ GLSM for the co-dimension three multi-centered $k$ $5^2_2$-branes. The bosonic sector of the GLSM Lagrangian is

$$
\mathcal{L} = \sum_{a=1}^{k} \frac{1}{2} \epsilon^a \left( \frac{1}{2} (F_{01,a})^2 - |\partial_m \sigma_a|^2 - 4|\partial_m M_c,a|^2 \right) - \frac{1}{2g^2} \left( (\partial_m r^1)^2 + (\partial_m r^3)^2 \right) - \frac{g^2}{2} \left( (\partial_m \gamma^2)^2 + (D_m \gamma^4)^2 \right) - \sum_{a=1}^{k} \left( |D_m q_a|^2 + |D_m \tilde{q}_a|^2 \right) - \sqrt{2} \epsilon^{mn} \sum_{a=1}^{k} \partial_m \left( (\varrho - t_{2,a}) A_{n,a} \right).
$$

8
The model exhibits $U(1)^k$ gauge symmetries with gauge fields $A_{m,a}$. The complex scalar fields $q_a$ and $\tilde{q}_a$ are components in the $\mathcal{N} = (4,4)$ charged hypermultiplets. The real scalar fields $r^1$, $r^2$, $r^3$ and $\gamma^4$ belong to the $\mathcal{N} = (4,4)$ neutral hypermultiplet. Under the duality transformations [18], the scalar fields $y^2$ and $\vartheta$ are related to $r^2$ and $\gamma^4$, respectively. The $\mathcal{N} = (4,4)$ gauge multiplets contain various scalar fields: $\sigma_a$ and $M_{c,a}$ are complex scalar fields. $A_{c,a}$, $\mathcal{A}_{c,a}$, $B_{c+b}$, $\mathcal{B}_{c+b}$ are auxiliary fields (for the details, see [8]). The gauge coupling constants $e_a$ are dimensionful while $g$ is dimensionless. The real FI parameters $t_{1,a}, s_{1,a}, s_{2,a}$ represent positions of the $k \delta_2^2$-branes in the $(r^1, r^2, r^3)$-directions while $t_{2,a}$ are shifts in the $\vartheta$-direction. The classical supersymmetric vacua of the GLSM in the $a$-th sector are found from the potential terms. From the positive definite parts, we find $\sigma_a = M_{c,a} = 0$. We take the branch where the last term in (3.6) vanishes. The fields $(r^1, r^2, r^3)$ and $(q_a, \tilde{q}_a)$ are the triplet and the doublet in the $SU(2)$ rotation. We employ a frame where $q_a \neq 0, \tilde{q}_a = 0$ and $r^1 = s_{1,a}, r^2 = s_{2,a}$ in the $a$-th sector. The vacuum expectation value of $r^3$ becomes a modulus of the $a$-th vacuum $r^3 = \zeta_a$. From these conditions, we have $|q_a| = \sqrt{2}(\zeta_a - t_{1,a})$.

We have also $y^2 = \gamma^4 = 0$, and $A_{m,a}$ stay at the pure gauge.

The IR limit of the GLSM (3.6) is obtained by taking $e_a \to \infty$. In this limit, the fields $A_{m,a}$, $\sigma_a$, $M_{c,a}$ are frozen and become auxiliary fields. From the potential terms, we obtain constraints among fields [8]. By using the constraints and integrating out all the auxiliary fields, we obtain the non-linear sigma model given by

$$
\mathcal{L} = -\frac{1}{2}H\left\{ (\partial_m r^1)^2 + (\partial_m r^2)^2 + (\partial_m r^3)^2 \right\} - \sqrt{2}e^{mn} \sum_{a=1}^{k} \partial_m((\vartheta - t_{2,a}) A_{n,a}) + \varepsilon^{mn}(\partial_m r^2)(\partial_n y^2) $$

$$
- \frac{1}{2}H^{-1}(\partial_m \vartheta)^2 - \frac{1}{2}(\omega_2)^2 H^{-1}(\partial_m r^2)^2 - \omega_2 H^{-1}(\partial_m \vartheta)(\partial_m r^2) $$

$$
- \frac{1}{2}(\omega_1)^2 H^{-1}(\partial_m r^1)^2 - \omega_1 \omega_2 H^{-1}(\partial_m r^1)(\partial_m r^2) - \omega_1 H^{-1}(\partial_m \vartheta)(\partial_m r^1) $$

(3.7)

Here the function $H$ is defined by

$$
H = \frac{1}{g^2} + \sum_{a=1}^{k} \frac{1}{\sqrt{2}R_a} ; \quad R_a = \sqrt{(r^1 - s_{1,a})^2 + (r^2 - s_{2,a})^2 + (r^3 - t_{1,a})^2} $$

(3.8)

and the set $(\omega_1, \omega_2)$ is the solution to the Dirac monopole equation with the harmonic function (3.8). The target space geometry of (3.7) represents the multi-centered $k \delta_2^2$-branes of co-dimension three. In order to obtain the $\delta_2^2$-brane of co-dimension two, we compactify the $r^2$-direction on $S^1$
with radius \( R \). The FI parameters \( s_{1,a}, t_{1,a}, t_{2,a} \) should be independent of \( a \) in the compactification. In the following, we take \( s_{1,a} = t_{1,a} = t_{2,a} = 0 \) for simplicity. The positions of the \( S^2 \)-branes in the \( r^2 \)-direction become \( s_{2,a} = 2 \pi R a, \ a \in \mathbb{Z} \) and \( k \to \infty \). The summation over \( a \) becomes continuous integral in the small \( R \). After integrating out the field \( r^2 \), the result is the non-linear sigma model whose target space is the \( S^2 \)-brane geometry \((2.10)\) [8].

Now we are going to study the instanton corrections to the vacuum moduli space in the GLSM \((3.6)\). As discussed in [9, 10], there is no quantum moduli space of vacua by the Coleman-Mermin-Wagner theorem [19, 20]. However, we can study the quantum corrected moduli space of vacua in the IR regime by integrating out the high momentum modes. It is also discussed that there is no exact BPS instanton solution in the GLSM for the H- and KK-monopoles. The situation is the same even for the present case. Following [9, 10], we therefore consider constrained (point-like) instantons [21, 6] in the limit \( g \to 0 \) where an exact BPS solution exists. In the limit \( g \to 0 \), the fields \( r^1, r^3 \) are frozen and the kinetic terms of \( y^2, \gamma^4 \) vanish. In order to find the instantons in the \( g \to 0 \) limit, we consider the configuration where \( A_{m,a}, q_a \) are dynamical fields and the others stay at the vacua. Then we find the following truncated model:

\[
\mathcal{L} = \sum_{a=1}^{k} \left[ \frac{1}{2} \epsilon_a^2 (F_{01,a})^2 - \sum_{a=1}^{k} |D_m q_a|^2 - \sum_{a=1}^{k} \frac{\epsilon_a^2}{2} \left( |q_a|^2 - \sqrt{2}\zeta_a \right)^2 - \sqrt{2} \sum_{a=1}^{k} \partial F_{01,a} \right], \tag{3.9}
\]

where we have dealt with the field \( \vartheta \) as a constant. This is a conceivable assumption since the field \( \vartheta \) is no longer dynamical and appears only in the topological term in \((3.6)\). This is a reminiscent of the instanton calculus in the KK-monopoles [10, 11] where \( \vartheta \) is recognized as a constant parameter rather than the field on the worldsheet theory. The truncated model \((3.9)\) is just the \( k \) Abelian-Higgs model with the FI parameters \( \zeta_a \). After the Wick rotation to the Euclidean space \((x^2 = i x^0)\), we find that the Lagrangian becomes

\[
\mathcal{L}_E = \sum_{a=1}^{k} \left[ \frac{1}{2} \epsilon_a^2 (F_{12,a})^2 + |D_m q_a|^2 + \frac{\epsilon_a^2}{2} \left( |q_a|^2 - \sqrt{2}\zeta_a \right)^2 + i\sqrt{2} \sum_{a=1}^{k} \partial F_{12,a} \right]
\]

\[
= \sum_{a=1}^{k} \left[ \frac{1}{2} \epsilon_a^2 \left( F_{12,a} \pm \epsilon_a^2 (|q_a|^2 - \sqrt{2}\zeta_a) \right)^2 + \left( D_1 \pm i D_2 \right) q_a \right] \pm \sqrt{2}\zeta_a F_{12,a}
\]

\[
+ i\sqrt{2} \partial F_{12,a} \right]. \tag{3.10}
\]

From the Bogomol’nyi bound, we find the BPS equations:

\[
F_{12,a} \equiv \pm \epsilon_a^2 (|q_a|^2 - \sqrt{2}\zeta_a), \quad (D_1 \pm i D_2) q_a = 0 \quad \text{with} \quad a = 1, \cdots, k. \tag{3.11}
\]

These are the Abrikosov-Nielsen-Olesen (ANO) vortex equations [22]. Then the action \( S = \frac{i}{\hbar} \int d^2 x \mathcal{L}_E \) becomes

\[
S = \sqrt{2} \sum_{a=1}^{k} \zeta_a |n_a| - i \sqrt{2} \vartheta \sum_{a=1}^{k} n_a. \tag{3.12}
\]
Here we defined the instanton number in the $a$-th sector,

$$n_a = \frac{1}{2\pi} \int d^2 x \, F_{12,a}. \quad (3.13)$$

The analytic solution to the equations (3.11) is not known but these satisfy the half BPS condition, namely, the solution keeps $\mathcal{N} = (2, 2)$ worldsheet supersymmetries.

The instanton contributions to the geometry are given by the sum of those in each $U(1)$ sector [11]. The calculations are the same in each sector $a$. For the time being, we concentrate on the instantons in the $a$-th sector. The path integral on the $n_a$-instantons background is reduced to the integral over the instanton moduli space $\mathcal{M}_{n_a}$. Since the instantons keep $\mathcal{N} = (2, 2)$ supersymmetries, the bosonic and fermionic non-zero modes of the solution are cancelled out. We only need the zero modes. The integral measure on the moduli space is discussed in [9]. We consider the four-point correlation function of $\psi_{a\pm}, \tilde{\psi}_{a\pm}$, which are supersymmetric partners of the charged scalars $q_a, \tilde{q}_a$, on the $n_a$-instantons background:

$$G_4^{(n_a)}(x_1, x_2, x_3, x_4) = \langle \tilde{\psi}_{a+}(x_1)\psi_{a-}(x_2)\tilde{\psi}_{a+}(x_3)\psi_{a-}(x_4) \rangle_{n_a\text{-instantons}}. \quad (3.14)$$

For our purpose, we only need the asymptotic solution to the BPS equations (3.11) [23]. The fermionic position moduli in the solutions $\psi_{a\pm}, \tilde{\psi}_{a\pm}$ saturate the associated Grassmann measure in the moduli space integration. Since the solution preserves $\mathcal{N} = (2, 2)$ supersymmetries in two dimensions, the moduli action has corresponding supersymmetries in $0 + 0$ dimensions. Then there are terms with four centered fermionic moduli in the moduli action [23] which saturate the Grassmann measure associated with the centered fermionic moduli. Therefore we conclude that all the $n_a$-instantons give the non-zero contributions in (3.14) even though the metric on the centered moduli space is not known for $n_a > 1$.

We note that the constraints among the scalar fields $q_a, \tilde{q}_a$ and the neutral ones $r^1, r^2, r^3$ induce those on the supersymmetric partners of the charged fields $\psi_{a\pm}, \tilde{\psi}_{a\pm}$ and the neutral fermions $\chi_{\pm}, \tilde{\chi}_{\pm}$ (for the fermionic field contents, see [8]),

$$\chi_{\pm} = 2 \sum_{a=1}^{k} (\tilde{q}_a \psi_{a\pm} + q_a \tilde{\psi}_{a\pm}), \quad \tilde{\chi}_{\pm} = 2i \sum_{a=1}^{k} (\tilde{q}_a \psi_{a\pm} - q_a \tilde{\psi}_{a\pm}). \quad (3.15)$$

The complex fermions $\chi_{\pm}, \tilde{\chi}_{\pm}$ are decomposed into the real fermions $\Omega_{\pm}^\mu$ [10]. Therefore the four-point function (3.14) is translated to the coefficient of the four-point interaction of $\Omega_{\pm}^\mu$ in the IR Lagrangian (2.1). Then we find the instanton corrections to the Riemann tensor of the target space geometry for the $\hat{A}_2$-branes. Since the calculation itself is the same discussed in [9, 10], we never repeat it here. The corrections to the Riemann tensor at the leading order in $1/R_a$ in the limit $g \to 0$ is found to be

$$\delta R_{1313}\big|_a = \frac{N}{4R_a} \sum_{n_a=1}^{\infty} n_a^2 e^{-n_a R_a X^{-1}} \left( e^{i n_a X^{-1} \varphi} + e^{-i n_a X^{-1} \varphi} \right), \quad (3.16)$$
\[ \delta R_{132} \bigg|_a = -\delta R_{2313} \bigg|_a = -\frac{\mathcal{N}}{4R_a} \sum_{n_a=1}^{\infty} n_a^2 e^{-n_a R_a R'} \left( e^{in_a R' \varphi} - e^{-in_a R' \varphi} \right), \quad (3.16b) \]

where the symbol \( \big|_a \) stands for the contributions from the \( a \)-th instanton sector. The corrections (3.16a), (3.16b) contain an overall factor \( \mathcal{N} \) stemming from the volume of the centered moduli space of the ANO vortices which no one knows \(^2\). The dimensionful coordinates in (3.16a), (3.16b) are normalized by a parameter \( R' \). The (anti)symmetric structure of the Riemann tensor suggests that (3.16a) comes from the corrections to the metric while (3.16b) from these to the torsion. We note that the off-diagonal part of the target space metric in (3.7) are dropped in the limit \( g \to 0 \) and the non-zero component of the metric is \( g_{ij} = H \delta_{ij} \) where \( i, j = 1, 2, 3 \). Assuming that the instanton corrections are only in the diagonal part, the result (3.16a) is reproduced by the following modification to the metric at the leading order in \( 1/R_a \) expansion \([9]\),

\[ \delta g_{11} \bigg|_a = \delta g_{22} \bigg|_a = \delta g_{33} \bigg|_a = \frac{\mathcal{N}}{2R_a} \sum_{n_a=1}^{\infty} n_a e^{-n_a R_a R'} \left( e^{in_a R' \varphi} + e^{-in_a R' \varphi} \right). \quad (3.17) \]

In addition, we have the corrections to the torsion from (3.16b) \([10]\),

\[ T_{128} \bigg|_a = -H_{128} \bigg|_a = -\frac{i\mathcal{N}}{2R_a} \sum_{n_a=1}^{\infty} n_a e^{-n_a R_a R'} \left( e^{in_a R' \varphi} - e^{-in_a R' \varphi} \right). \quad (3.18) \]

Therefore the instanton corrections to the \( k \) \( 5^2 \)-branes of co-dimension three (3.7) result in the following modified harmonic function,

\[ H = \frac{1}{g^2} + \frac{1}{2} \sum_{a=1}^{k} \sum_{n_a=1}^{\infty} \frac{\mathcal{N}}{R_a} \left( e^{-n_a R_a R' - in_a R' \varphi} + e^{-n_a R_a R' + in_a R' \varphi} \right) \]

\[ = \frac{1}{g^2} + \sum_{a=1}^{k} \frac{\mathcal{N}}{2R_a} \frac{\sinh(R_a R')}{\cosh(R_a R') - \cos(R' \varphi)}. \quad (3.19) \]

We then compactify the \( r^2 \)-direction with the small radius \( R \). This procedure is equivalent to take \( s_{1,a} = 2\pi R a \) with \( k \to \infty \) in the NLSM as we have discussed before. The summation over \( a \) now becomes the continuous integral. Again, the integral diverges and we introduce the cutoff \( \Lambda \) and the renormalization scale \( \mu \). Then the harmonic function of the \( 5^2 \)-brane of co-dimension two becomes

\[ H = h_0 + \frac{\mathcal{N}}{2\pi R} \log \frac{\mu}{g'} + \frac{\mathcal{N}}{2\pi R} \sum_{n \neq 0} e^{in R' \varphi} K_0(|n| R' \varphi'), \quad (3.20) \]

where \( g' = \sqrt{(r^1)^2 + (r^3)^2} \) and \( h_0 = \frac{1}{g} + \frac{\mathcal{N}}{2\pi R} \log \frac{4\pi^2 \Lambda}{\mu} \). Now we identify the fields in the NLSM and the coordinates of geometry \( (r^1, r^2, r^3) = (X^1, X^8, X^2) \) and \( g' = g \). Consequently we have the correspondences \( R = R_8, \mathcal{N} = R' = R_g, h_0 = h_0 \). Then the result (3.20) completely coincides with the one with the \( X^9 \)-instanton corrections elucidated in (3.5).

\(^2\) The factor \( \sqrt{2} \) in the harmonic function in (3.8) is also absorbed in this overall factor. This factor \( \sqrt{2} \) comes from the canonical normalization of the two-dimensional scalar fields \([8]\) which is different from the one in \([9]\).
4 Conclusion and discussions

In this paper we investigated the worldsheet instanton corrections to the exotic $5^2_2$-brane geometry. First we briefly reviewed the H-monopoles, the KK-monopoles and the $5^2_2$-brane from the spacetime viewpoint. These objects are related by the T-duality chains in type II string theories. When the T-duality circle has finite radius, the geometry of the H-monopoles is modified. This modification is interpreted as the worldsheet instanton corrections which is analyzed in the language of the GLSM [9]. We then mentioned the instanton corrections to the KK-monopoles which causes the dependence on the winding coordinate in the harmonic function [10]. This results in the notion of the localized KK-monopoles.

In the main part of the current paper, we investigated the instanton corrections to the geometry of the exotic $5^2_2$-brane from the two distinct perspectives. One is from the T-duality transformation of the localized KK-monopoles. Since the localized KK-monopoles possess the $U(1)$ isometry along the $X^8$-direction, we applied the Buscher rule to the geometry and obtained the $X^9$-instanton corrected $5^2_2$-brane geometry. We found that the correction to the harmonic function depends on the winding coordinate $\vartheta$. The other is from the direct calculation of the instantons in the GLSM for the multi-centered $\mathring{5}^2_2$-branes [8]. We found that the truncated model of the GLSM in the limit $g \to 0$ is just the multi-centered generalization of the one discussed in [9, 10]. The BPS solutions with the topological numbers $n_a = -\frac{1}{2\pi} \int d^2x \ F_{12,a}$ are the ANO vortices. We found that the instanton corrections to the co-dimension three $\mathring{5}^2_2$-branes are carried over to that of the $5^2_2$-brane geometry by the appropriate choice of the FI parameters and the limit $k \to \infty$. The resulting harmonic function completely coincides with the one obtained from the localized KK-monopoles. The relation between the two results are summarized in Figure 1.

![Figure 1: Instanton corrections to the $5^2_2$-brane. Here KKM stands for the Kaluza-Klein monopole.](image)

We note that the instanton corrections to the off-diagonal part of the metric and the vector $\omega_i$ are still intractable. This is because the full geometry is no longer a solution to conventional supergravities in the $g \to 0$ limit. This can be understood from the harmonic function (2.9).
though the asymptotic radius of the T-dual circle is ill-defined due to the renormalization scale $\mu$, we can go to the asymptotic region $g \sim \Lambda$ when the renormalization scale is large $\mu \sim \Lambda$. We then find that the approximate radius of the T-dual circle is given by $g$. Consequently, as we have mentioned in Section 2, the winding states become massless in the limit $g \to 0$ and the supergravity approximation lose its meaning. The full solution of the $5^2_2$-brane with the $X^9$-instanton corrections would be studied in the framework of the double field theory [24] where the KK and the winding massless modes are treated democratically.

There is a question on the quantum deformation to the $5^2_2$-brane geometry. We have studied the $X^9$-instanton effects which cause the localization of the $5^2_2$-brane in the $\vartheta$-direction. This $\vartheta$-dependence originates from the T-dual of the $X^9$-dependence in the KK-monopoles. The localization appears from the worldsheet instanton effects or from the finite radius effect of the $X^9$ circle. It is natural to consider the other quantum deformation caused by the finiteness of the $X^8$ radius. Indeed, when we perform the discrete summation over $l$ in (3.1) by keeping the radius $R_8$ finite, we have the harmonic function which depends not only on $X^9$ but also on $X^8$. Even though the discrete summation in (3.1) breaks the $U(1)$ isometry along the $X^8$-direction, we qualitatively expect that the harmonic function on the T-dualized $5^2_2$-brane geometry depends on the two winding coordinates $\vartheta$ and $\vartheta' = X^8$. The $\vartheta$-dependence is what we have investigated in this paper. The worldsheet origin of the $\vartheta'$-dependence is still an open question. The localization in the $\vartheta'$-direction is expected to be induced by the topological term $\vartheta' F_{01}$ which is missing in our model. It is interesting to find the GLSM which contains a topological term that enables one to localize the $5^2_2$-brane in the $\vartheta'$-direction. We will study these issues in the next paper [25].

Acknowledgements

The work of S. S is supported in part by Sasakawa Scientific Research Grant from The Japan Science Society and Kitasato University Research Grant for Young Researchers.

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