Position and momentum bases on the sphere for the monochromatic Maxwell fish-eye

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Abstract. The sphere is well understood manifold, as is the basis of spherical harmonics for functions thereof. A stereographic projection of the sphere realizes the Maxwell fish-eye, an optical medium whose refractive index distribution is such that light rays travel in circles. Definite angular momentum on the sphere corresponds with monochromatic light in the fish-eye. A contraction of these manifolds (as the radius of the sphere grows without bound) results in a plane medium of wave functions subject to the Helmholtz equation. In the latter, there is the continuous basis of ‘momenta’ (plane waves) and a denumerably infinite basis of ‘positions’ and ‘normal derivatives’ (Bessel $\sim J_0(x)$ and $\sim J_1(x)/x$ functions) for the Hilbert space of these wavefields. We present the pre-contraction of these bases to the finite-dimensional systems of fish-eye medium and the sphere. The ‘momentum’ basis is a subset of Sherman-Volobuyev functions; in this paper we propose new ‘position’ and ‘normal derivative’ bases for this finite system. The bases are not orthogonal, so their measure is non-local, and here we find their dual bases.

1. The harmonic basis on the sphere
Phase space is a concept created in classical mechanics that applies as well in geometric optics [1]. In quantum mechanics and in wave optics, there are several approaches to phase space through the definition of the Wigner [2] and Wigner-like [3] quasidistribution functions. Those that follow Wigner’s original formulation hinge on the Heisenberg-Weyl algebra of noncommuting operators of position and momentum; other formulations also rely on a supporting Lie algebra [4].

In this work we start on a different path, based on the monochromatic Maxwell fish-eye wave-optical system, presenting what appear to be the natural wavefields of definite momentum and position. Since this system is a stereographic projection of the sphere on a plane or higher-dimensional flat manifold [5, 6, 7, 8, 9, 10, 11], [12, Sect. 6.4], the analysis remits us to the sphere, treated as in angular momentum theory through the well-known spherical harmonics, whose relevant properties are recalled in Sect. 2, while the link to the fish-eye model is summarized in Sect. 3. In our case we have not a continuous, but a finite system, i.e., the Hilbert space of wavefields for which we provide the bases is finite-dimensional.

The basis of functions of ‘most definite momentum’ are known as the Sherman-Volobuyev functions, of which we need only a finite number for fixed angular momentum in Sect. 4, because they correspond to monochromatic wavefields in the optical model. The basis of ‘most definite position’ presented in Sect. 5 actually involves two sub-bases, of positions and of normal
derivatives, which were proposed in Ref. [13]. Here we also examine their dual bases in Sect. 6. From that reference, in Sect. 7 we summarize the contraction of the algebra, sphere and optical fish-eye system, to a homogeneous medium subject to the Helmholtz equation, where we find that the limit wavefields match those of a related development addressed some time ago [14], where the inner product of the latter is non-local.

As we underline in the concluding Sect. 8, we are building a finite system which is a deformed (i.e. pre-contracted) model of a wave-optical system that has had important applications in waveguide and radar theory, with the intent of setting up an appropriate phase space grounded on finite function bases, rather than on a Lie algebra of operators.

2. The harmonic basis on the sphere

The quantum (or classical) angular momentum operators \( \vec{L} = -i \hat{\mathbf{s}} \times \hat{\nabla}_s \) (or Poisson operators \( \{\hat{L}_i, \hat{L}_j\} \), \( \hat{\mathbf{s}} \equiv (x, y, z) \in \mathbb{R}^3 \) generate the Lie algebra \( \mathfrak{so}(3) \) of rotations, and obey the commutation (or Poisson bracket) relations \( [L_i, L_j] = i L_k \), with \( i, j, k \) a cyclic permutation of \( x, y, z \). Free rotation by \( \omega \) about a unit axis \( \hat{n} \) is generated by \( \exp(-i \omega t \cdot \hat{n} \cdot \vec{L}) \); these rotations commute with the monochromatic wave equation \( L^2 F(s) = \ell(\ell + 1) F(s) \), where \( L^2 = \sum_i L_i^2 \) is the invariant Casimir operator of \( \mathfrak{so}(3) \), reduced to the sphere \( |\hat{s}| = \rho \), for fixed nonnegative \( \ell \). Spherical harmonics \( Y_{\ell,m}(\theta, \phi) \) are the eigenfunctions of \( L^2 \) and \( L_z \); their explicit expression is too well known to be reproduced here [15, Eqs. (3.141–153)]; the index \( m \) can take the \( 2\ell + 1 \) integer-spaced values \( m = -\ell, \ldots, \ell \). In particular, we shall use specifically the extreme \( (m = \ell) \) and the middle \( (m = 0) \) spherical harmonics:

\[
Y_{\ell,\ell}(\theta, \phi) = \frac{(-1)^\ell}{2\ell! \sqrt{4\pi}} (\sin \theta e^{i\phi})^\ell, \quad Y_{\ell,0}(\theta, \phi) = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos \theta). \tag{1}
\]

The spherical harmonics form an orthonormal and complete basis for the Hilbert space \( \mathcal{L}^2(\mathbb{S}^2) \), whose inner product integrates over the sphere. We keep this inner product, but to be ‘monochromatic’ we fix \( \ell \), so \( \langle Y_{\ell,m}, Y_{\ell,m'} \rangle_{\mathbb{S}^2} = \delta_{m,m'} \) and \( \{Y_{\ell,m}\}_{m=-\ell}^{\ell} \) will be an orthonormal basis for a \((2\ell+1)\)-dimensional space that we indicate by \( \mathcal{M}_\ell \). Azimuthal and polar rotations...
yield the phases and linear combinations
\[
\exp(-i\alpha L_z) Y_{\ell,m}(\theta, \phi) = e^{-im\alpha} Y_{\ell,m}(\theta, \phi) = Y_{\ell,m}(\theta, \phi-\alpha),
\]
\[
\exp(-i\beta L_y) Y_{\ell,m}(\theta, \phi) = \sum_{m'=-\ell}^{\ell} Y_{\ell,m'}(\theta, \phi) d_{m',m}^{\ell}(\beta),
\]
where \(d_{m',m}^{\ell}(\beta)\) are the Wigner little-\(d\) functions [15, Eq. (3.72)], whose explicit form we also need not reproduce here in full. For later use we record the following relations [15, Eq. (3.141)]
\[
Y_{\ell,m}(\theta, \phi) = \sqrt{\frac{2\ell + 1}{4\pi}} d_{m,0}^{\ell}(\theta) e^{im\phi}, \quad d_{m,\ell}^{\ell}(\frac{1}{2}\pi) = \frac{1}{2\ell} \sqrt{\left(\frac{2\ell}{\ell + m}\right)},
\]
\[
3. \text{Stereographic projection on the fish-eye}
\]
The stereographic projection from the bottom pole of the sphere to a plane tangent to the top pole, is shown in geometric terms in Fig. 1. The map between polar coordinates and measures \((\theta, \phi)\) on the sphere (of radius \(\rho\)) and polar coordinates on the plane \((r, \phi)\), is
\[
\frac{\tan \frac{1}{2}\theta}{r/2\rho}, \quad d\cos\theta d\phi = \left(\frac{4\rho}{r^2 + 4\rho^2}\right)^2 r dr d\phi.
\]
Luneburg [5] investigated the transformation (5) and Buchdahl [7] recognized its symmetry of the fish-eye under rotations in one higher dimension. The stereographic map has been extended to a canonical map between the phase spaces of the sphere and plane in Ref. [9]. Since this map is point-to-point in the position spaces of Fig. 1, the map between momentum spaces will be comatic, i.e., if \((\mathbf{n}, \sigma)\) is the projection of the unit vector \(\hat{n} \in S^2\) in the \(\sigma\)-hemisphere \((\sigma = \pm 1)\) on the equatorial \(x-y\) plane, and \(\mathbf{r} \in \mathbb{R}^2\) is on the projected plane, then as detailed in [12, Sect. 6.4], when \(U_{\sigma}(|\mathbf{n}|)\) is any function on the sphere, then there is a canonical transformation to the projected plane,
\[
\mathbf{r} = U_{\sigma}(|\mathbf{n}|) \mathbf{n}, \quad \mathbf{p} = \frac{1}{U_{\sigma}(|\mathbf{n}|)} \mathbf{v} + W_{\sigma}(|\mathbf{n}|) \mathbf{n} \cdot \mathbf{v},
\]
where \(\mathbf{p}\) is the conjugate momentum in the projected plane, \(\mathbf{v}\) is the projection of momentum from the sphere on its \(x-y\) equatorial plane, and \(W_{\sigma}(|\mathbf{n}|)\) is found in terms of \(U_{\sigma}(|\mathbf{n}|)\) and its derivative \(U_{\sigma}'(|\mathbf{n}|)\). Since this map is canonical, their Poisson brackets satisfy \(\{n_i, v_j\} = \delta_{i,j} \Leftrightarrow \{r_i, p_j\} = \delta_{i,j}\). We know that classical and quantum mechanics follow each other under comatic maps such as (6), from Poisson brackets to commutators, so that canonicity becomes unitarity in wave-optical models. Since these transformations are invertible, they conserve the information contained in their corresponding states.

Note that two antipodal points on the sphere, \((\theta, \phi)\) and \((\pi-\theta, \phi+\pi)\) map onto conjugate points in the fish-eye plane, \((r, \phi)\) and \((\tilde{r}, \phi+\pi)\), with \(r^2 = (2\rho)^2\). The sphere rotates by \(\pi\) between these points along any of the maximal circles that join the pair; if the sphere rotates freely, on the fish-eye plane thus, all rays departing from one point will meet at the conjugate point in the same time, so the Maxwell fish-eye is a perfect focusing system in space and also in time. This optical system is referred to Maxwell [16] because he found that a refractive index
\[
n(r) = \frac{4n_o^2}{r^2 + 4\rho^2}, \quad n_o := n(0),
\]
would make light go in circles. Actually, only part of the fish-eye can be realized optically, because \(n(r) \geq 1\) is a physical constraint while \(n_o\) cannot be larger than 2.5 or so. But it is the half-fish-eye that receives technological application in ‘bubble-top’ radar installations using

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microwaves, where rubber foams serve to impress a refractive index in the concavity of a half-sphere antenna. A pulse at the back point of the antenna, seen as the source of circular arcs, will find the face of the half-sphere perpendicular to all of them, producing the same pulse in a beam of parallel lines. And conversely for detection of the target echo.

Finally, note that the measure in (5) is an obliquity factor for the map, and is proportional to the square of the refractive index (7). Hence, when functions $F$ on the sphere $\Omega \equiv (\theta, \phi)$ are corresponded with functions $f$ on the plane $r$ through

$$f(r) = \frac{4\rho}{r^2 + 4\rho^2} F(\Omega(r)), \quad F(\Omega) = \rho \sec^2 \frac{1}{2} \phi \ f(r(\Omega)).$$

The inner products in the two spaces will relate directly as

$$(F,G)_{S^2} := \int_{S^2} d^2\Omega \, F(\Omega)^* \, G(\Omega) = \int_{\mathbb{R}^2} d^2r \, f(r)^* \, g(r) =: (f,g)_{R^2}.$$ 

In particular, the basis of spherical harmonics of the first Section will map on a basis of wavefields on the fish-eye (FE) as

$$Y_{\ell,m}^{\text{FE}}(r) = \frac{4\rho}{r^2 + 4\rho^2} Y_{\ell,m}(\Omega(r)).$$

The set of functions $\{Y_{\ell,m}^{\text{FE}}(r)\}_{m=-\ell}^{\ell}$ will be orthonormal and complete in $\mathcal{M}_\ell$ under the inner product $(f,g)_{R^2}$ in (9). Figure 2 shows spherical harmonics on the sphere and their corresponding projections on the fish-eye medium.
The momentum basis

We singled out the extreme and middle spherical harmonics \( Y_{\ell,\ell} \) and \( Y_{\ell,0} \) in (1) because we intend these to be the ‘typical’ momentum and position functions to be found on the sphere: the extreme \( Y_{\ell,\ell}(\theta, \phi) \propto e^{i\ell\phi} \) exhibits an equatorial belt of oscillations which, on a sphere of radius \( \rho \), will accommodate \( \ell \) wavelengths \( \lambda = 2\pi \rho/\ell \); the middle \( Y_{\ell,0}(\theta, \phi) \propto P_\ell(\cos \theta) \) shows a maximum at the top pole \( \theta = 0 \), and a maximum (or minimum) at \( \theta = \pi \) for \( \ell \) even (or odd), and has the least second moment (with \( \cos^2 \theta \)) among all \( 2\ell+1 \) spherical harmonics of angular momentum \( \ell \) in \( \mathcal{M}_\ell \).

We propose to use these to generate bases of ‘momentum’ and of ‘position’ for wavefunctions on the sphere, supported by their appearance in the contraction limit \( \rho, \ell \to \infty \), to be seen in Sect. 6, that maintains the ratio \( \rho/\ell = \lambda/2\pi = 1/k \) constant, so that the functions on the sphere and fish-eye plane come to be solutions of the Helmholtz equation of wavenumber \( k \), plane waves and localized Bessel \( J_0 \)'s, whose properties in this regard have been studied in [14] and [13].

By means of \( 2\ell+1 \) distinct rotations of the sphere by \( R_m, R_m' \in SO(3) \), and as long as the rotated functions \( R_m : Y_{\ell,\ell} \) and \( R_m' : Y_{\ell,0} \) are linearly independent, they can be used to expand any function \( F(\theta, \phi) \in \mathcal{M}_\ell \) of the same angular momentum. To obtain analytic results we should choose carefully these rotations \( R(\alpha_m, \beta_m, \gamma_m) \) that we characterize by the axis \( \hat{n}(\alpha_m, \beta_m) \) to which they send the top pole of the sphere.

For the momentum basis we choose \( 2\ell+1 \) equidistant points \( \alpha_m \) on the azimuthal circle \( \alpha_m := 2\pi m/(2\ell + 1) \), \( m|_{-\ell}^{\ell} \), to span \( \mathcal{M}_\ell \) with the extreme harmonics, defining the momentum functions as

\[
V_{\ell,m}(\theta, \phi) := \exp(-i\alpha_m L_z) \exp(-i\frac{\ell}{2} P \theta) Y_{\ell,\ell}(\theta, \phi)
\]

\[
= \sum_{m'=-\ell}^{\ell} \exp \left( \frac{-2\pi i m m'}{2\ell + 1} \right) d_{m', \ell}(\frac{1}{2} \pi) Y_{\ell,m'}(\theta, \phi)
\]

\[
= V_{\ell,0}(\theta, \phi - \alpha_m), \quad m|_{-\ell}^{\ell}.
\]

These we call the normalized Sherman-Volobuyev (nShV) functions over the sphere, since they have unit norm under the \( L^2(S^2) \) inner product, and whose ensemble \( \ell|_{0}^{\infty} \) has been studied by Sherman and Volobuyev [17, 18] as ‘Fourier analysis’ for arbitrary functions or distributions over the sphere.

The \( V_{\ell,m}(\theta, \phi) \)'s chosen in (11) exhibit their oscillation belts on meridian circles equally distributed around the azimuthal circle \( \alpha \) of directions, as shown in Fig. 3. These nShV functions...
Figure 4. Real part of the normalized Sherman-Volobuyev functions \( V_{6,m}^{\ell,0}(\theta, \phi) \), \( m = -4, 0, 4 \) in (11) on the sphere (top), and their projections on the fish-eye medium (bottom) with the minimal circle. The full basis is obtained through rotations around the z-axis by \( \alpha_m = 2\pi m/(2\ell+1) \), \( m \mid \ell - \ell \).

are not orthogonal, but

\[
(V_{\ell,m}, V_{\ell,m'})_{S^2} = d_{\ell,\ell}^0 (\alpha_{m'} - \alpha_m) = \left( \cos \frac{\pi (m' - m)}{2\ell+1} \right)^{2\ell}.
\] (12)

No pair of nShV functions is orthogonal because \( \frac{1}{2} (\alpha_{m'} - \alpha_m) = \frac{1}{2} \pi \) would imply \( m' - m = \ell + \frac{1}{2} \), but all \( m \)'s are integers. Also, one can verify that \( \det |d_{\ell,\ell}^0 (\alpha_{m'} - \alpha_m)| > 0 \), so the set of \( 2\ell + 1 \) nShV vectors form a complete basis for \( \mathcal{M}_\ell \).

Any monochromatic function \( F(\theta, \phi) \in L^2(S^2) \) can be thus expanded in the nShV basis,

\[
F(\theta, \phi) = \sum_{m=-\ell}^{\ell} \tilde{F}_m V_{\ell,m}(\theta, \phi).
\] (13)

Since the basis is not orthogonal, to recover the ‘momentum’ coefficients \( \tilde{F}_m, m \mid \ell - \ell \) we need the inner product in this basis, which is thus non-local:

\[
(F, G)_{S^2} = \sum_{m, m' = -\ell}^{\ell} \tilde{F}_m^* \left( \cos \frac{\pi (m - m')}{2\ell+1} \right)^{2\ell} G_{m'} =: (F, G)_{\mathcal{M}_\ell}
\] (14)

In Fig. 4 we show some of these functions over the sphere and their projection with the obliquity factor (8) which, following the convention used in (10) for bases on the fish-eye medium we denote by \( V_{FE,\ell,m}(r) \), but whose explicit expression we do not presently need.

5. The position and normal-derivative basis

For the proposed position basis we can use the \( Y_{\ell,0}(\theta, \phi) \)'s, rotated so that their axes point in \( 2\ell + 1 \) directions distributed on any azimuth circle parallel to the equator, or randomly over the sphere; these would provide acceptable bases provided they are linearly independent. For
Figure 5. Legendre ‘position’ functions over the sphere (left of each pair) $\Lambda_{4,m}(\theta, \phi)$, $m = -4, -2, 0, 2, 4$ in (15) which are even in azimuth $\phi$, and their projections $A_{4,m}(r)$ on the fish-eye medium (right of each pair), and the minimal circle.

reasons that will become clear after the contraction to the plane, we choose them to be on the upper half-meridian in the $x$-$z$ plane of the sphere, as was shown in Fig. 3. We thus define the Legendre harmonics as given through rotations of $Y_{\ell,0}$ by angles $\beta_m = \pi m/(2\ell + 1)$ around the $y$-axis,

$$A_{\ell,m}(\theta, \phi) := e^{-imL_y}Y_{\ell,0}(\theta, \phi) = \sum_{m' = -\ell}^{\ell} d_{\ell,0}^{m'}(\beta_m)Y_{\ell,m'}(\theta, \phi)$$

$$= \sum_{n=0}^{\ell} d_n^{\ell}(\beta_m)\frac{1}{2}\left(2-\delta_n,0\right)\left(Y_{\ell,n}(\theta, \phi) + Y_{\ell,-n}(\theta, \phi)^*\right)$$

$$= \sqrt{\frac{2\ell+1}{4\pi}} \left(d_{0,0}^{\ell}(\beta_m) d_{0,0}^{\ell}(\theta) + 2 \sum_{n=1}^{\ell} d_n^{\ell}(\beta_m) d_n^{\ell}(\theta) \cos n\phi\right)$$

$$= A_{\ell,m}(\theta, -\phi) = (-1)^\ell A_{\ell,m}(\theta \pm \pi, \phi),$$

where in the second line we used the symmetries $d_{m',0}^{\ell}(\beta) = (-1)^{m'}d_{-m',0}^{\ell}(\beta)$ and $Y_{\ell,-m'} = (-1)^m Y_{\ell,m'}^*$, reducing the sum to $n|_{-\ell}^{\ell}$. We show some Legendre functions $A_{4,m}(r)$ in Fig. 5, together with their projections with obliquity on the fish-eye plane, that following our convention we write $A_{4,m}^{FE}(r)$. On the sphere they are rotated versions of each other although on the fish-eye plane their appearance is different. The range $m|_{-\ell}^{\ell}$ for $A_{\ell,m}(\theta, \phi)$ is not warranted here, because all these functions are even under reflection across the $x$-$z$ plane, or in azimuth, $\phi \leftrightarrow -\phi$; Hence, only $\ell+1$ of these functions can be linearly independent.

We thus need a second set of $\ell$ functions with axes on the remaining points in the half-meridian that be odd under $\phi \leftrightarrow -\phi$. To propose them, we note that from angular momentum theory,

$$L_x Y_{\ell,0}(\theta, \phi) = -i\sqrt{\frac{2\ell+1}{4\pi}} P_{\ell}^1(\cos \theta) \sin \phi,$$
Figure 6. Legendre ‘normal derivative’ functions over the sphere (left of each pair) $A'_{\ell,m}(\theta, \phi)$, $m = -3, -1, 1, 3$ in (17) that are odd in azimuth $\phi$, and their projections $A'_{\ell,m}\,(\theta, \phi)$ on the fish-eye medium (right of each pair) with the minimal circle.

so we rotate these by $\beta_m = \pi m/(2\ell+1)$, to define the ‘normal derivative’ Legendre functions, using the same symmetries as in (15),

$$A'_{\ell,m}(\theta, \phi) := \sqrt{\frac{2}{\ell(\ell+1)}} e^{-i\beta_m} L_y \left(-i \right) L_x A_{\ell,0}(\theta, \phi)$$

$$= \sum_{n=1}^{2\ell+1} \left( d_{n,1}(\beta_m) + d_{n,-1}(\beta_m) \right) \frac{1}{2\pi} \left( Y_{\ell,n}(\theta, \phi) - Y_{\ell,-n}(\theta, \phi)^* \right)$$

$$= \sqrt{\frac{2\ell+1}{2\pi}} \sum_{n=1}^{2\ell+1} \left( d_{n,1}(\beta_m) + d_{n,-1}(\beta_m) \right) d_{n,0}(\theta) \sin n\phi$$

$$= -A'_{\ell,m}(\theta, -\phi) = (-1)^\ell A'_{\ell,m}(\theta \pm \pi, \phi),$$

which are odd under $\phi \leftrightarrow -\phi$ and shown in Fig. 6, together with their projections on the fish-eye plane $A'_{\ell,m}\,(\theta, \phi)$.

We choose the ranges of $m$ for the angles $\beta_m = \pi m/(2\ell+1)$ as follows:

for $A_{\ell,m}$, $m \in \mathcal{E}_\ell := \{-\ell, -\ell - 2, \ldots, 0, \ldots, \ell - 2, \ell\}$,

for $A'_{\ell,m}$, $m \in \mathcal{E}'_\ell := \{-\ell+1, -\ell + 3, \ldots, \ell - 1\}.

(18)

Since $|\beta_m| < \frac{1}{2}\pi$ the maximum of all these functions projects on the fish-eye plane inside the minimal circle $r = 2\rho$; their antipodes project to their conjugate points outside. To complement the set of Legendre functions, in Fig. 5 we show the normal-derivative Legendre functions with the $m$’s that were ‘absent’ in Fig. 6.

Although all $A_{\ell,m}$’s are orthogonal to all $A'_{\ell,m}$’s and each subset has been checked to be linearly independent, they are not orthogonal, but

$$(A_{\ell,m}, A_{\ell,m'})_{S^2} = d^\ell_{0,0}(\beta_m - \beta_m) = P_\ell(\cos(\beta_m - \beta_m)),

(19)$$

where the Legendre polynomials $P_\ell(\cos \beta)$ contain powers of $\cos^\nu \beta$, or equivalently of $\cos \nu \beta$, for $\nu \in \{\ell, \ell - 2, \ldots, 1\}$.

The inner products between ‘normal derivative’ Legendre functions are

$$(A'_{\ell,m}, A'_{\ell,m'})_{S^2} = \frac{1}{\ell} P_{\ell-1}^{(1,1)}(\cos(\beta_m - \beta_m)) =: Q_{\ell-1}(\cos(\beta_m - \beta_m)),

(20)$$
Thus we have the dual basis functions can be then written as the components of a vector \( S \), where the Jacobi polynomial \( P_{\ell-1}^{(1,1)} \equiv Q_{\ell-1} \) is of degree \( \nu \) in \( \cos' \beta \) or equivalently \( \cos \nu \beta \), for \( \nu \in \{ \ell-1, \ell-3, \ldots, 0 \} \), and they were checked to be also linearly independent.

Functions on the sphere of angular momentum \( \ell \) are in \( \mathcal{M}_\ell \) and can be written as linear combinations of both sets of Legendre functions,

\[
F(\theta, \phi) = \sum_{m \in \mathcal{E}_\ell} F_m A_{\ell,m}(\theta, \phi) + \sum_{m \in \mathcal{E}'_\ell} F'_m A'_{\ell,m}(\theta, \phi),
\]

(21)

Since non-orthogonal bases imply non-local measures, as in (14), we now have

\[
(F, G)_{S^2} = \sum_{m,m' \in \mathcal{E}_\ell} F^*_m P_{\ell}(\cos \frac{\pi(m - m')}{2\ell + 1}) G_{m',m} + \sum_{m,m' \in \mathcal{E}'_\ell} F'^*_{m} Q_{\ell-1}(\cos \frac{\pi(m - m')}{2\ell + 1}) G'_{m',m} =: (F, G)_{\mathcal{M}_\ell}.
\]

(22)

6. The dual bases

We saw in the previous Section that neither the set of nShV nor Legendre functions on the sphere are orthogonal on \( \mathcal{M}_\ell \), although they have unit norm. It is natural to inquire on their duals, which will represent different wavefield bases on the sphere and fish-eye plane.

6.1. Dual of the nShV basis

The nShV functions \( V_{\ell,m}(\theta, \phi) \) in (11), for the \( 2\ell + 1 \) equally-spaced angles \( \alpha_m = 2\pi m/(2\ell + 1) \), \( m|\ell \), are complete in \( \mathcal{M}_\ell \). If we denote the column vector of spherical harmonics by \( \vec{Y}_\ell = \{ Y_{\ell,m}(\theta, \phi) \}_{m=-\ell}^{\ell} \) and the nShV column by \( \vec{V}_\ell = \{ V_{\ell,m}(\theta, \phi) \}_{m=-\ell}^{\ell} \), then (11) can be written with a matrix \( S^\ell = \{ S_{m,m'}^\ell \}_{m,m'=-\ell}^{\ell} \) as

\[
\vec{V}_\ell = S^\ell \vec{Y}_\ell, \quad S_{m,m'}^\ell := \exp\left( \frac{-2\pi i m'}{2\ell + 1} \right) d_{m,\ell}^{\ell}(\frac{1}{2}\pi).
\]

(23)

The dual basis functions can be then written as the components of a vector \( \vec{V}^\ell = \{ V_{\ell,m}(\theta, \phi) \}_{m=-\ell}^{\ell} \), and related to the orthonormal spherical harmonic vector by a matrix \( S^\ell = \{ S_{m,m'}^\ell \}_{m,m'=-\ell}^{\ell} \), so that

\[
(\vec{V}_{\ell,m}, V_{\ell,m'})_{S^2} = \sum_{m=-\ell}^{\ell} S_{m,m}^\ell S_{m,m'}^\ell = (S^\ell)_{m,m'} = \delta_{m,m'},
\]

(24)

where \( ^\dagger \) indicates adjunction. We must thus have \( S^\ell = (S^\ell)^\dagger \). Knowing finite Fourier analysis, the matrix elements of \( S^\ell \) can be found by inspection to be

\[
S_{m,m'}^\ell = \exp\left( \frac{2\pi i m m'}{2\ell + 1} \right) / (2\ell + 1) d_{m,\ell}^{\ell}(\frac{1}{2}\pi).
\]

(25)

Thus we have the dual nShV basis \( \vec{V}_\ell = S^\ell \vec{Y}_\ell \), whose normalized functions over the sphere are,

\[
V_{\ell,m}(\theta, \phi) = \frac{1}{2\ell + 1} \sum_{m'=-\ell}^{\ell} \exp\left( \frac{-2\pi i m m'}{2\ell + 1} \right) d_{m',\ell}^{\ell}(\frac{1}{2}\pi) Y_{\ell,m'}(\theta, \phi) = V_{\ell,0}(\theta, \phi-\alpha_m).
\]

(26)
Figure 7. Real part of the normalized dual Sherman-Volobuyev functions $V_{6,m}(\theta, \phi)$, $m = -4, 0, 4$ in (26) on the sphere (top), and their projections $V^\text{FE}_{6,m}(r)$ on the fish-eye medium (bottom) with the minimal circle. The full dual basis is obtained through rotations around the $z$-axis by $\alpha_m = 2\pi m/(2\ell+1)$, $m \mid \ell - \ell$.

We remark that it is unusual to see the Wigner little-$d$ function in a denominator, but this is never zero. In Fig. 7 we show some of the dual functions $V_{\ell,m}(\theta, \phi)$ over the sphere and $V^\text{FE}_{\ell,m}(r)$ on the fish-eye plane. Clearly, the dual functions on the sphere are rotated versions of each other; on the fish-eye plane they represent circular trajectories, all orthogonal to the line trajectories through the center that represent plane waves. They differ only by $2\ell + 1$ equidistant phases.

6.2. Dual of the Legendre bases

The duals of the Legendre bases of positions and normal derivatives in (15) and (17) can be found as in (24), but now we should be careful to handle separately the symmetric and antisymmetric sub-bases, of dimensions $\ell+1$ and $\ell$ respectively. We accommodate them into column vectors related by matrices appropriate to their dimensions,

$$\tilde{\Lambda}_\ell = \{A_{\ell,2n-\ell}\}_{n=0}^{\ell}, \quad \tilde{Y}_\ell^S = \{\text{Re} \ Y_{\ell,n}\}_{n=0}^{\ell}, \quad \tilde{\Lambda}_\ell = \mathbf{T}_\ell \tilde{Y}_\ell^S,$$

$$\tilde{\Lambda}'_\ell = \{A'_{\ell,2n-\ell-1}\}_{n=1}^{\ell}, \quad \tilde{Y}_\ell^A = \{\text{Im} \ Y_{\ell,n}\}_{n=1}^{\ell}, \quad \tilde{\Lambda}'_\ell = \mathbf{T}'_\ell \tilde{Y}_\ell^A.$$  

(27)

(28)

The $(\ell+1) \times (\ell+1)$ matrix $\mathbf{T}_\ell \equiv \{T_{\ell,n'}\}_{n,n'=0}^{\ell}$ and the $\ell \times \ell$ matrix $\mathbf{T}'_\ell \equiv \{T'_{\ell,n'}\}_{n,n'=1}^{\ell}$ have their elements given by the second lines in (15) and (17),

$$T_{\ell,n'} = (2 - \delta_{n,0}) d_{\ell,0}(\beta_{2n' - \ell}),$$

$$T'_{\ell,n'} = d_{\ell,1}(\beta_{2n' - \ell - 1}) + d_{\ell,-1}(\beta_{2n' - \ell - 1}).$$

(29)

(30)

As in the previous subsection, we must find the inverse adjoint matrices

$$\mathbf{T}_\ell := (\mathbf{T}_\ell)^\dagger - 1, \quad \mathbf{T}'_\ell := (\mathbf{T}'_\ell)^\dagger - 1.$$  

(31)

to build the two dual sub-bases,

$$\bar{\mathcal{I}}_{\ell,2n-\ell}(\theta, \phi) = \sum_{n'=0}^{\ell} T_{\ell,n'}^{\ell} Y^S_{\ell,n'}(\theta, \phi), \quad n|\ell.$$  

(32)
Figure 8. Real part of the duals of the Legendre position and normal-derivative functions \( \Lambda_{6,m}(\theta, \phi) \), \( m = -4, 0, 4 \) from (32) and \( \Lambda'_{6,m}(\theta, \phi) \), \( m = -3, -1, 1, 3 \) from (33) on the sphere (top), and their fish-eye projections \( \Lambda_{FE,6,m}(r) \) and \( \Lambda'_{FE,6,m}(r) \) on the fish-eye medium (bottom) with the minimal circle.

The dual vectors have the same parity as their original counterparts, and are also rotated copies of each other as (15) and (17). Again, on the fish-eye plane they superficially appear to be rotated versions of each other, but are slightly different from the original basis and from each other.

7. Contraction to the Helmholtz medium

By contraction of so(3) to the Euclidean Lie algebra iso(2) we shall see that the assignments of momentum and position to the nShV and Legendre bases are consistent with what we know of bases for solutions of the Helmholtz equation, and thus confirm their roles. We note the following limits of spherical harmonics and Wigner d-functions to Bessel functions:

\[
\Lambda'_{\ell,2n-\ell-1}(\theta, \phi) = \sum_{n'=1}^{\ell} T_{\ell,n'}^{\ell} Y_{\ell,n'}(\theta, \phi), \quad n|\ell. \tag{33}
\]

We cannot provide a closed expression for the matrix elements of \( T^\ell \) and \( T'^\ell \), so we had to resort to numerical computation of the matrices in order to show in Fig. 8 some of the dual basis functions (32)–(33), together with their fish-eye projections, \( \Lambda_{FE,6,m}(r) \) and \( \Lambda'_{FE,6,m}(r) \). We checked their duality properties by numerical integration over the sphere for several low values of \( \ell \). The dual vectors have the same parity as their original counterparts, and are also rotated copies of each other as (15) and (17). Again, on the fish-eye plane they superficially appear to be rotated versions of each other, but are slightly different from the original basis and from each other.

By contraction of so(3) to the Euclidean Lie algebra iso(2) we shall see that the assignments of momentum and position to the nShV and Legendre bases are consistent with what we know of bases for solutions of the Helmholtz equation, and thus confirm their roles. We note the following limits of spherical harmonics and Wigner d-functions to Bessel functions:

\[
\frac{k}{\rho} \lim_{\rho \to \infty} Y_{\ell,m}(\frac{r}{\rho}, \phi) = (-1)^m \sqrt{\frac{k^2}{2\pi}} J_m(kr) e^{im\phi}, \quad \frac{k}{\rho} \lim_{\rho \to \infty} d_{\ell,n'}^{\ell}(\frac{r}{\rho}) = J_{m-n'}(kr). \tag{34}
\]
As we noted in Sect. 4, the limit \( \rho \to \infty \) of the radius of the sphere in Fig. 1 should maintain a constant wavelength \( \lambda \) where the sphere is tangent to the fish-eye plane, so we must consider increasing \( \ell \propto \rho \), related by the wavenumber as,

\[
k := 2\pi/\lambda = \ell/\rho, \quad \ell = k\rho, \tag{35}
\]

Also the polar angle \( \theta \) in (34), to map onto points at a finite radius \( r \) on the fish-eye plane (5), must be taken in the limit \( \rho \to \infty \) as \( \theta \approx r/\rho \to 0 \). In this limit the refractive index of the medium (7) becomes the constant \( n_o \); finally, the inner product (9) and the obliquity factor (10) require renormalizations of functions to keep them finite.

The spherical harmonic basis \( Y_{\ell,m}^{FE}(r) \) in (10) is given by (34). We thus obtain the multipole basis for non-exponential solutions of the Helmholtz medium, that we indicate by lower-case letters,

\[
y_{k,m}(r, \phi) := \lim_{\rho \to \infty} \frac{1}{\sqrt{\rho}} \frac{k}{2\pi} V_{\ell,m}(r/\rho, \phi) = (-1)^m \sqrt{k/2\pi} e^{im\phi}, \quad m|_{-\infty}^{\infty}, \tag{36}
\]

where \( m \) now ranges over all integers. The contraction of the Lie algebra generators yields

\[
L_x = -i(y\partial_z - z\partial_y) \to i\rho\partial_y, \\
L_y = -i(z\partial_x - x\partial_z) \to -i\rho\partial_x. \tag{37}
\]

while \( L_z \) remains invariant. The Casimir equation \( L^2 = \ell(\ell+1)I \) after normalization becomes the Helmholtz equation for the wavefields \( (\partial_x^2 + \partial_y^2) f(r) = k^2 f(r) \); and \( e^{-i\beta_m L_y} \to e^{-x_m\partial_x} \) for \( x_m = \rho\beta_m \) finite will apply to the angles of the Legendre basis.

7.1. Contraction of the \( nShV \) basis

The azimuth angles \( \alpha_m = 2\pi m/(2\ell+1) \) of the \( nShV \) basis become dense on the \( \alpha \)-circle, as \( \Delta \alpha = 2\pi/(2\ell+1) \to 0 \). We write \( \alpha \) for \( \alpha_m \)'s on the circle and no longer count \( m \) as a label. Using (34), the Bessel generating function, other limits for the functions involved and renormalization, we obtain

\[
v_{k,\alpha}(r, \phi) := \lim_{\rho \to \infty} \left( \frac{k\pi}{\rho} \right)^{1/4} V_{\ell,m}(r/\rho, \phi) = \sqrt{\frac{k}{2\pi}} \exp(i\pi m \sin(\phi - \alpha)). \tag{38}
\]

These are plane waves of wavenumber \( k \) with wavefronts in the \( \phi = \alpha \) direction, also expressed in terms of the multipole wavefields in (36).

The inner product on the sphere in (14) was non-local in the coefficients of the \( nShV \) basis. In the \( \rho, \ell \to \infty \) limit, this kernel becomes a Dirac \( \delta \),

\[
\lim_{\rho \to \infty} \sqrt{\rho} \cos \left( \frac{\pi(m - m')}{2\ell + 1} \right)^{2\ell} = \frac{1}{\sqrt{\pi k}} \delta(\alpha - \alpha'), \tag{39}
\]

so the inner product for Helmholtz wavefields \( f, g \) in the plane-wave basis becomes local:

\[
(f,g)_{o} := \int_{-\pi}^{\pi} d\alpha \tilde{f}(\alpha)^* \tilde{g}(\alpha), \quad \tag{40}
\]
The two sets of functions of the Legendre bases, being rotated versions of each other, contract $M$ project the maximum of their derivatives on points $x, y$ (45), implies that their duals decrease as $\sim m$ extend to the finite basis on the pre-contracted sphere. Again, the dual Legendre bases vanish: $J$, where $\ell, m$.

The Legendre harmonics $\Lambda_{\ell,m}(r, \phi)$ in (15) point their ‘fingers’ at angles $\beta_m = \pi m/(2\ell+1)$ on an upper half-meridian, and which project on points $x_m$ in the $x$-axis of the fish-eye plane. We saw that for $m \in E$, these are a sub-basis in $M_\ell$ for even functions of azimuth $\phi \leftrightarrow -\phi$, and of $y \leftrightarrow -y$ on the fish-eye plane. Similarly, the sub-basis of normal derivatives $\Lambda_{\ell,m}^\prime(\theta, \phi)$ in (17) project the maximum of their derivatives on points $x_m$ for $m \in E$, and form the complementary sub-basis in $M_\ell$ for functions that are odd in $\phi \leftrightarrow -\phi$ and $y \leftrightarrow -y$.

In the $\ell, \rho \to \infty$ limit (35), the set of points $x_m$, will become the infinite set of equally-spaced points

$$x_m = 2\rho \tan \frac{1}{2} \beta_m \to x_m = \rho \beta_m = \frac{1}{2} \lambda m, \quad m \in E \text{ or } m \in E'.$$  

The two sets of functions of the Legendre bases, being rotated versions of each other, contract into translated copies, each set on points $x_m$ separated by $\frac{1}{2}$; the even position functions we place on the points $x_m$ with even $m \in E$, and the odd normal derivative functions on $x_m$ with odd $m \in E'$. We use (34) for finite $r = \rho \theta$, to give their limits on the Helmholtz plane and renormalized by $1/\sqrt{\rho}$. The limit of the Legendre bases functions are

$$\lambda_{k,m}(r) := \lim_{\rho \to \infty} \frac{1}{\sqrt{\rho}} \Lambda_{k,m}(r/\rho, \phi) = \sqrt{\frac{k}{2\pi}} J_0(k|r-x_m|),$$

$$\lambda_{k,m}^\prime(\theta, \phi) := \lim_{\rho \to \infty} \frac{-i}{\sqrt{\rho}} \Lambda_{k,m}^\prime(\theta, \phi) = \sqrt{\frac{k}{\pi}} J_1(k|r-x_m|)/|r-x_m|,$$

where $|r-x_m| = \sqrt{(r-x_m)^2 + y^2}$. The $J_0$’s in (44) are the narrowest Helmholtz wavefields [14], and $J_1(x)/x$ have the largest normal derivatives, so we confirm that (44) plays the role of a position and normal derivative sub-bases for Helmholtz wavefields. This identification we can extend to the finite basis on the pre-contracted sphere. Again, the dual Legendre bases vanish: the growth $\sim \rho^{1/2}$ of the original $\Lambda_{\ell,m}$’s and $\Lambda_{\ell,m}^\prime$’s that required renomalization in (44) and (45), implies that their duals decrease as $\sim \rho^{-1/2}$.

Expansions of the Legendre bases (44) and (45) in terms of the standard multipole basis $y_{k,m}(r)$ in (36) can be found from (15) and (17) also using (34),

$$\lambda_{k,m}(r) = \sum_{m' \in E} J_{m'}\left(\frac{1}{2} \pi m\right) y_{k,m'}(r), \quad m \in E \text{ (even)},$$

$$\lambda_{k,m}^\prime(r) = \sum_{m' \in E'} m' J_{m'}\left(\frac{1}{2} \pi m\right) y_{k,m'}(r), \quad m \in E' \text{ (odd)}.$$

with $\sigma = \cos \alpha$, and similarly for $g(r)$. This expression, termed the wave transform only requires us to know the values of the fields and normal derivatives on the $x$-axis. As understood by Sherman and Volobuyev [17, 18], we have a natural momentum basis on the sphere, which is reduced to a single wavenumber in the monochromatic Maxwell fish-eye.

The contraction limit of the dual basis $\{\langle V_{\ell,m}(\theta, \phi)\rangle\}_{m=-\ell}^{\ell}$ is the null vector; this is due to the factor $2\ell+1$ in the denominator of (26), and is also evident in (38) where the growth of $V_{\ell,m}(r/\rho, \phi)$ by $\sim \rho^{1/4}$ entails that its dual $\langle V_{\ell,m}(r/\rho, \phi)\rangle$ must vanish as $\sim \rho^{-1/4}$ to maintain the inner products $\langle V_{\ell,m}, V_{\ell,m}\rangle = 1$.

7.2. Contraction of the Legendre bases

The Legendre harmonics $\Lambda_{\ell,m}(\theta, \phi)$ (15) point their ‘fingers’ at angles $\beta_m = \pi m/(2\ell+1)$ on an upper half-meridian, and which project on points $x_m$ in the $x$-axis of the fish-eye plane. We saw that for $m \in E$, these are a sub-basis in $M_\ell$ for even functions of azimuth $\phi \leftrightarrow -\phi$, and of $y \leftrightarrow -y$ on the fish-eye plane. Similarly, the sub-basis of normal derivatives $\Lambda_{\ell,m}^\prime(\theta, \phi)$ (17) project the maximum of their derivatives on points $x_m$ for $m \in E$, and form the complementary sub-basis in $M_\ell$ for functions that are odd in $\phi \leftrightarrow -\phi$ and $y \leftrightarrow -y$.

In the $\ell, \rho \to \infty$ limit (35), the set of points $x_m$, will become the infinite set of equally-spaced points

$$x_m = 2\rho \tan \frac{1}{2} \beta_m \to x_m = \rho \beta_m = \frac{1}{2} \lambda m, \quad m \in E \text{ or } m \in E'.$$  

The two sets of functions of the Legendre bases, being rotated versions of each other, contract into translated copies, each set on points $x_m$ separated by $\frac{1}{2}$; the even position functions we place on the points $x_m$ with even $m \in E$, and the odd normal derivative functions on $x_m$ with odd $m \in E'$. We use (34) for finite $r = \rho \theta$, to give their limits on the Helmholtz plane and renormalized by $1/\sqrt{\rho}$. The limit of the Legendre bases functions are

$$\lambda_{k,m}(r) := \lim_{\rho \to \infty} \frac{1}{\sqrt{\rho}} \Lambda_{k,m}(r/\rho, \phi) = \sqrt{\frac{k}{2\pi}} J_0(k|r-x_m|),$$

$$\lambda_{k,m}^\prime(\theta, \phi) := \lim_{\rho \to \infty} \frac{-i}{\sqrt{\rho}} \Lambda_{k,m}^\prime(\theta, \phi) = \sqrt{\frac{k}{\pi}} J_1(k|r-x_m|)/|r-x_m|,$$

where $|r-x_m| = \sqrt{(r-x_m)^2 + y^2}$. The $J_0$’s in (44) are the narrowest Helmholtz wavefields [14], and $J_1(x)/x$ have the largest normal derivatives, so we confirm that (44) plays the role of a position and normal derivative sub-bases for Helmholtz wavefields. This identification we can extend to the finite basis on the pre-contracted sphere. Again, the dual Legendre bases vanish: the growth $\sim \rho^{1/2}$ of the original $\Lambda_{\ell,m}$’s and $\Lambda_{\ell,m}^\prime$’s that required renomalization in (44) and (45), implies that their duals decrease as $\sim \rho^{-1/2}$.

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$$\lambda_{k,m}(r) = \sum_{m' \in E} J_{m'}\left(\frac{1}{2} \pi m\right) y_{k,m'}(r), \quad m \in E \text{ (even)},$$

$$\lambda_{k,m}^\prime(r) = \sum_{m' \in E'} m' J_{m'}\left(\frac{1}{2} \pi m\right) y_{k,m'}(r), \quad m \in E' \text{ (odd)}.$$
Figure 9. Left: The limit of the Legendre ‘position’ functions are the $J_0$-Bessel functions \( \lambda_{k,m}(r) \) in (44) centered on \( x_m, m \in \mathcal{E} \) even integers. Right: The limit of the ‘normal derivative’ Legendre functions \( \lambda'_{k,m}(r) \) in (45) centered on \( x_m, m \in \mathcal{E}' \) odd integers.

A generic non-exponential Helmholtz wavefield can thus be written in terms of the position and normal derivative bases (44) and (45) as

\[
f(r) = \sum_{m \in \mathcal{E}} f_m \lambda_{k,m}(r) + \sum_{m \in \mathcal{E}'} f'_m \lambda'_{k,m}(r). \tag{48}
\]

In terms of these coefficients the inner product between two Helmholtz fields \( f, g \), is the limit of the non-local inner product (22). We find that the limit inner product remains non-local in the discrete positions \( x_m \).

\[
(f,g)_Z := \sum_{m,m' \in \mathcal{E}} f_m J_0(\frac{1}{2} \pi (m'-m)) g_{m'} + \sum_{m,m' \in \mathcal{E}'} f'_m J_1(\frac{1}{2} \pi (m'-m)) g'_{m'}. \tag{49}
\]

This result is known from Refs. [19, 14, 20].

8. Conclusions

Phase space is a well-defined object in classical mechanical and optical systems; in quantum systems, the Wigner quasidistribution function provides a proper and working representation of states both in position and momentum, even though they stem from noncommuting operators in the Heisenberg-Weyl Lie algebra. A variety of other systems have been endowed with phase space descriptions, as has the classical sphere, which is a symplectic manifold [1].

Our approach here has been to start with functions on the manifold of the sphere with definite angular momentum, and propose two function bases —rotated \( Y_{\ell,\ell} \) and \( Y_{\ell,0}' \)'s— to be (not eigenfunctions of any two operators, but) the closest analogues of quantum momentum and position wavefunctions. The use of Sherman-Volbuyev functions as ‘Fourier’ bases for functions on the sphere is known, but those we called Legendre functions of position and normal derivative appear to be new in this context. Their contraction limit to the Helmholtz medium binds these bases to previously known ones. Then, the stereographic projection of the sphere, the fish-eye medium, provides a physical model where the corresponding wavefields acquire a degree of physicality, which may be used later to extend this concept to other related Luneburg systems in wave optics.

The use of fish-eye optics for waveguides and radar antennas by itself suggests that our inquiry is not vacuous. Their perfect focusing properties in space and in time open other questions, such as bases for a sphere whose lower hemisphere is screened, corresponding to the fish-eye region bounded by the minimal circle where a waveguide is bounded by vacuum. Finally, the half fish-eye used in microwave radar antennas, coupled to a homogeneous, Euclidean-invariant medium may require some interesting group-theoretical techniques.
Acknowledgments
We thank Guillermo Krötzsch for his assistance in the preparation of the figures. Support for this research has been provided by the Óptica Matemática projects by the Universidad Nacional Autónoma de México PAPIIT in101011, and by the Consejo Nacional de Ciencia y Tecnología project SEP-CONACYT CB220692.

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