CONFIGURATIONS OF EXTREMAL TYPE II CODES

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Abstract. We prove configuration results for extremal Type II codes, analogous to the configuration results of Ozeki and of the second author for extremal Type II lattices. Specifically, we show that for \( n \in \{8, 24, 32, 48, 56, 72, 96\} \) every extremal Type II code of length \( n \) is generated by its codewords of minimal weight. Where Ozeki and Kominers used spherical harmonics and weighted theta functions, we use discrete harmonic polynomials and harmonic weight enumerators. Along we way we introduce \( t \)-designs as a discrete analog of Venkov’s spherical designs of the same name.

1. Introduction

We denote by \( \mathbb{F}_2 \) the two-element field \( \mathbb{Z}/2\mathbb{Z} \). By a “code” we mean a binary linear code of length \( n \), that is, a linear subspace of \( \mathbb{F}_2^n \). For such a code \( C \), and any integer \( w \), we define
\[
C_w := \{ c \in C : \text{wt}(c) = w \},
\]
where \( \text{wt}(c) := |\{ i : c_i = 1 \}| \) is the Hamming weight. Recall that the dual code of \( C \), denoted \( C^\perp \), is defined by
\[
C^\perp := \{ c' \in \mathbb{F}_2^n : (c, c') = 0 \text{ for all } c \in C \},
\]
where \( (\cdot, \cdot) \) is the usual bilinear pairing \( (x, y) = \sum_{i=1}^n x_i y_i \) on \( \mathbb{F}_2^n \). Then \( C^\perp \) is also linear, with \( \dim(C) + \dim(C^\perp) = n \). A code \( C \) is said to be self-dual if \( C = C^\perp \). Such a code must have \( \dim(C) = n/2 \); in particular 2 | \( n \). Because \( (c, c) \equiv \text{wt}(c) \mod 2 \), it follows that a self-dual code \( C \) is even, that is, has 2 | \( \text{wt}(c) \) for every word \( c \in C \); equivalently, \( C_w = \emptyset \) unless 2 | \( w \). A code \( C \) is said to be doubly even, or of Type II, if 4 | \( \text{wt}(c) \) for all \( c \in C \); equivalently, if \( C_w = \emptyset \) unless 4 | \( w \).

Mallows and Sloane [MS73] showed that a Type II code \( C \) of length \( n \) must contain nonzero codewords of weight at most \( 4 \lfloor n/24 \rfloor + 4 \) (see also [CS99, p. 194]). If \( C_w = \emptyset \) for all positive \( w < 4 \lfloor n/24 \rfloor + 4 \), then \( C \) is said to be extremal, because it is known that such a code has the largest minimal distance among all Type II codes of its length.

In this paper, we prove configuration results for extremal Type II codes. Specifically, we show that if \( C \) is an extremal Type II code of length \( n = 8, 24, 32, 48, 56, 72, \) or 96, then \( C \) is generated by its minimal-weight codewords. Our approach uses the machinery of harmonic weight enumerators introduced by Bachoc [Bac99] and developed further in [EK13], following...
the approach used to prove analogous results for lattices in the works of Venkov [Ven84],
Ozeki [Oze86a, Oze86b], and the second author [Kom09a].

2. Designs, Extremal Codes, and Discrete Harmonic Polynomials

Fix a positive integer \( n \). For each nonnegative integer \( w \leq n \), denote by \( \Omega_w \) the Hamming sphere of radius \( w \) about the origin of \( \mathbb{F}_2^n \). Thus \( \Omega_w \) consists of the \( \binom{n}{w} \) binary words of length \( n \) and weight \( w \). To such a word \( c \) we associate its support \( \Sigma_c \), which is the \( w \)-element set \( \{ i : 1 \leq i \leq n, c_i = 1 \} \).

We use the following definition of a \( t \)-design in \( \Omega_w \), which neither assumes that \( w \geq t \), nor that the design is nonempty.

**Definition 2.1.** We say that a subset \( D \subseteq \Omega_w \) is a \( t \)-predesign for an integer \( t \geq 0 \) if there exists an integer \( N = N_t(D) \) such that every subset \( I \subseteq \{ 1, 2, \ldots, n \} \) of cardinality at most \( t \) is contained in exactly \( N \) of the sets \( \Sigma_c \) with \( c \in D \). Then a subset \( D \subseteq \Omega_w \) is called a \( t \)-design if \( D \) is a \( t' \)-predesign for each positive integer \( t' \leq t \).

**Remarks.** It is well known that if \( w \geq t \), then \( D \) is a \( t \)-design if and only if

\[
(1) \quad \binom{n}{w} \sum_{c \in D} f(c) = |D| \binom{w}{t} \sum_{c \in \Omega_w} f(c)
\]

for any function \( f : \mathbb{F}_2^n \to \mathbb{C} \) that depends on at most \( t \) of the \( n \) coordinates, and that \( N_t(D) \) is given by the formula

\[
(2) \quad \binom{n}{w} N_t(D) = \binom{w}{t} |D|
\]

because both sides of \( 2 \) count ordered pairs \((I, c)\) such that \(|I| = t\), \( c \in D \), and \( I \subseteq \Sigma_c \). In this case a \( t \)-predesign \( D \) is automatically a \( t \)-design, but if \( w < t \) then every subset of \( \Omega_w \) is a \( t \)-predesign (with \( N_t(D) = 0 \), still in accordance with \( 2 \)), so we need the “predesign” property also for \( t' < t \) to assure that a \( t \)-design is also a \( t' \)-design for \( t' < t \). Moreover, the only \( w \)-predesigns in \( \Omega_w \) are \( \Omega_w \) itself and \( \emptyset \), so once \( t \geq w \) it follows that the only \( t \)-designs in \( \Omega_w \) are \( \Omega_w \) itself and \( \emptyset \). It follows that \( 1 \) still holds for \( t \geq w \).

Extremal codes yield designs by the following important special case of the Assmus-Mattson theorem. For \( n \equiv 0 \mod 8 \), we define \( t(n) \) by

\[
(3) \quad t(n) := \begin{cases} 
5 & n \equiv 0 \mod 24, \\
3 & n \equiv 8 \mod 24, \\
1 & n \equiv 16 \mod 24.
\end{cases}
\]

**Theorem 2.2** (AM69). If \( C \) is an extremal Type II code of length \( n \), then \( C_w \) is a \( t(n) \)-design for each \( w \).

In \( [EK13] \) Thm. 7.4] we gave a new proof of Theorem 2.2 using the discrete harmonic polynomials \( Q : \mathbb{F}_2^n \to \mathbb{C} \) introduced by Delsarte [Del78], via his characterization of \( t \)-designs:

**Theorem 2.3** (Del78 Thm. 7], [EK13 Prop. 7.1]). A set \( D \subseteq \Omega_w \) is a \( t \)-design if and only if

\[
(4) \quad \sum_{v \in D} Q(v) = 0
\]

for all nonconstant discrete harmonic polynomials \( Q \) with \( \deg Q \leq t \).

We note two important corollaries of Theorem 2.3. The first reorganizes \( 1 \):

\[
\begin{align*}
\text{Theorem 2.2} & \quad \text{If } C \text{ is an extremal Type II code of length } n, \text{ then } C_w \text{ is a } t(n)-\text{design for each } w. \\
\text{Theorem 2.3} & \quad \text{A set } D \subseteq \Omega_w \text{ is a } t \text{-design if and only if } \\
& \sum_{v \in D} Q(v) = 0 \\
& \text{for all nonconstant discrete harmonic polynomials } Q \text{ with } \deg Q \leq t. \\
\end{align*}
\]
Corollary 2.4 ([Del78 Thm. 6], [EK13 Cor. 7.3]). A set $D \subseteq \Omega_w$ is a $t$-design if and only if
\begin{equation}
\sum_{v \in D} Q(v) = \frac{|D|}{|\Omega_w|} \sum_{v \in \Omega_w} Q(v)
\end{equation}
for all discrete harmonic polynomials $Q$ with $\deg Q \leq t$.

Note that $\sum_{v \in \Omega_w} Q(v)$, and thus also $\sum_{v \in D} Q(v)$, vanishes unless $\deg Q = 0$.

The second corollary is the special case of [1] when $Q$ is a discrete zonal harmonic polynomial, that is, a discrete harmonic polynomial such that $Q(v)$ depends only on the weights of $v$ and $v \cap \hat{v}$ for some fixed vector $\hat{v}$ (equivalently, $Q(v)$ depends only on $wt(v)$ and the distance between $v$ and $\hat{v}$). Given a degree $d$ and a fixed $\hat{v} \in \mathbb{F}_2^n$, we showed in [EK13 Sec. 6] that there is a one-dimensional space of discrete zonal harmonic polynomials, generated by
\begin{equation}
Q_{d,\hat{v}}(v) := \sum_{k=0}^{d} (-1)^k \left( \prod_{\ell=0}^{k-1} \frac{(n - wt(\hat{v})) - (d - \ell - 1)}{wt(\hat{v}) - \ell} \right) Q_{d,k;\hat{v}}(v),
\end{equation}
where
\begin{equation}
Q_{d,k;\hat{v}}(v) = \left( \sum_{i=0}^{k} (-1)^i \binom{wt(v \cap \hat{v})}{i} \binom{wt(\hat{v}) - wt(v \cap \hat{v})}{k-i} \right) \times \left( \sum_{i=0}^{d-k} \binom{wt(v) - wt(v \cap \hat{v})}{i} \frac{(n - wt(\hat{v})) - (wt(v) - wt(v \cap \hat{v}))}{d-k-i} \right).
\end{equation}

Corollary 2.5 ([EK13 Cor. 7.6]). If $D \subseteq \Omega_w$ is a $t$-design then
\begin{equation}
\sum_{v \in D} Q_{d,\hat{v}}(v) = 0
\end{equation}
for each positive $d \leq t$ and any $\hat{v} \in \mathbb{F}_2^n$.

The approach to Theorem 2.2 via discrete harmonic polynomials is motivated by the fruitful analogy between Type II codes and Type II lattices, which are even unimodular Euclidean lattices. Recall [Ser73 Ch. VII] that the rank of such a lattice $L$ must be a multiple of 8, and its theta function is a modular form for $\text{PSL}_2(\mathbb{Z})$. It follows via a theorem of Siegel [Sie69] that if $L$ has rank $n$ then its minimal nonzero norm is at least $2n/24 + 2$ (Mallows–Odlyzko–Sloane [MOS75]). If equality holds then $L$ is said to be extremal. In such a lattice the vectors of each given norm form a spherical $(2(n) + 1)$-design. As in Corollary 2.4 this means that the sum over those vectors of $P$ vanishes for any nonconstant harmonic polynomial $P$ of degree at most $2t(n) + 1$. The $(2(n) + 1)$-design property is proved by recognizing the sum as the coefficient of a modular form (a weighted theta function); our proof of Theorem 2.2 in [EK13] is analogous, using harmonic weight enumerators of Type II codes.

In the lattice setting, the modular-forms approach gives additional information on the configuration of lattice vectors of given norm, beyond the fact that the configuration is a $(2(n) + 1)$-design. Namely, while the sum of a spherical harmonic of degree $2t(n) + 2$ over lattice vectors of a given norm need not vanish (i.e., those vectors need not constitute a $(2(n) + 2)$-design), a spherical harmonic of degree $2t(n) + 4$ does sum to 0. (Odd harmonics sum to 0 automatically because the design is spherically symmetric.) Venkov [Ven84] calls such a spherical configuration a “$(2(n) + 1\frac{1}{2})$-design”. In [EK13 Prop. 7.5] we proved that for an extremal Type II code $C$ each $C_w$ satisfies an additional constraint, analogous to the $(2(n) + 1\frac{1}{2})$-design property of extremal lattices. We thus introduce parallel terminology in this setting. Recall (Theorem 2.3) that $D \subseteq \Omega_w$ is a $t$-design if and only if $\sum_{v \in D} Q(v) = 0$ for all nonconstant discrete harmonic polynomials $Q$ of degree at most $t$. 
Theorem 2.7 ([EK13, Prop. 7.5]). Let \( t = t(n) \). If \( C \) is an extremal Type II code of length \( n \), then \( C_w \) is a \( \frac{t}{2} \)-design for each \( w \). In particular, for each \( w \) and any \( \hat{v} \in \mathbb{F}_2^n \),

\[
\sum_{v \in C_w} Q_{d,\hat{v}}(v) = 0
\]

holds for positive \( d \leq t \) and also for \( d = t + 2 \).

3. Configuration Results

3.1. Preliminaries. For any \( \hat{v} \in \mathbb{F}_2^n \), any length-\( n \) binary linear code \( C \), and any \( j \) (\( 0 \leq j \leq n \)), we denote by \( N_j(C; \hat{v}) \) the value

\[
N_j(C; \hat{v}) := |\{ c \in C_{\text{min}(C)} : \text{wt}(c \cap \hat{v}) = j \} |.
\]

For \( c \in C^\perp \), we must have \( N_{2j'+1}(C; c) = 0 \) for all \( j' \) with \( 0 \leq j' \leq \lfloor n/2 \rfloor \).

Throughout the remainder of this section, \( C \) denotes a length-\( n \) extremal Type II code, and \( w_0 := \text{min}(C) \) denotes the minimal weight of codewords in \( C \).

Lemma 3.1. If \( \hat{c} \) is a minimal-weight representative of the class \([\hat{c}] \in C/C_{w_0}(C)\) and \( c \in C_{w_0} \), then

\[
\text{wt}(c \cap \hat{c}) \leq \frac{w_0}{2}.
\]

Proof. If \( \text{wt}(c \cap \hat{c}) > w_0/2 \), then \([\hat{c}]\) contains a codeword \( c + \hat{c} \) of weight

\[
\text{wt}(c + \hat{c}) = \text{wt}(c) + \text{wt}(\hat{c}) - 2\text{wt}(c \cap \hat{c}) < \text{wt}(\hat{c}).
\]

This contradicts the minimality of \( \hat{c} \) in \([\hat{c}]\). \( \square \)

3.2. Extremal Type II Codes of Lengths 48 and 72. We begin with a configuration result for Type II codes of lengths \( n = 48, 72 \).

Theorem 3.2. If \( C \) is an extremal Type II code of length \( n = 48 \) or 72, then

\[
C = C_{w_0}(C).
\]

Proof. We consider the equivalence classes of \( C/C_{w_0}(C) \) and assume for the sake of contradiction that there is some class \([\hat{c}] \in C/C_{w_0}(C)\) with minimal-weight representative \( \hat{c} \) with \( \text{wt}(\hat{c}) = s > w_0 \).

As \( C \) is self-dual, we have \( N_{2j'+1}(C; c) = 0 \) for all \( 0 \leq j' \leq \lfloor n/2 \rfloor \). Additionally, by Lemma 3.1, we must have \( N_{2j'}(C; \hat{c}) = 0 \) for \( j' > w_0/4 \). We now develop a system of equations in the variables \( N_0(C; \hat{c}), N_2(C; \hat{c}), \ldots, N_{w_0/2}(C; \hat{c}) \).

Combining the \( t(n) + 1 \) equations of Corollary 2.5 with the equation

\[
N_0(C; \hat{c}) + N_2(C; \hat{c}) + \cdots + N_{w_0/2}(C; \hat{c}) = |C_{w_0}|
\]

gives a system of

\[
t(n) + 2 > \frac{w_0}{4} + 1
\]

equations in the variables \( N_{2j'}(C; \hat{c}) \) (\( 0 \leq j' \leq w_0/4 \)).
For \( n = 48, 72 \), the (extended) determinants of these inhomogeneous systems are

\[
\begin{align*}
(10) & \quad 2^{26}3^55^27^111^223^243^147^1 \left( \frac{11s^3 - 396s^2 + 4906s - 20736}{(s - 3)(s - 2)^2(s - 1)^3s^3} \right), \\
(11) & \quad 2^{42}3^55^27^211^213^117^323^267^1 \left( \frac{39s^4 - 2600s^3 + 67410s^2 - 800440s + 3650496}{(s - 4)(s - 3)^2(s - 2)^3(s - 1)^4s^4} \right),
\end{align*}
\]

respectively\(^1\); these determinants must vanish, as they are derived from overdetermined systems. Since equations (10)–(11) have no integer roots \( s \), we have reached a contradiction. \( \square \)

3.3. Extremal Type II Codes of Lengths At Most 32. The approach used to prove Theorem 3.2 may also be applied to show that extremal Type II codes of lengths \( n = 8, 24 \), and 32 are generated by their minimal-weight codewords. In these cases the determinants

\[
\begin{align*}
2^{7}3^{3}7^{1} & \left( \frac{3s - 10}{(s - 1)s} \right), \\
2^{15}3^{2}5^{1}7^{1}11^{2}23^{1} & \left( \frac{7s^2 - 98s + 344}{(s - 2)(s - 1)^2s^2} \right), \\
2^{17}3^{1}5^{2}7^{1}29^{1}31^{1} & \left( \frac{7s^2 - 126s + 584}{(s - 2)(s - 1)^2s^2} \right)
\end{align*}
\]

are obtained; none have integral roots \( s \). We therefore recover the following result.

**Theorem 3.3.** If \( C \) is an extremal Type II code of length \( n = 8, 24, \) or 32, then \( C = C_{w_0}(C) \).

Technically, Theorem 3.3 has been known (if only implicitly), as the extremal Type II codes of lengths \( n = 8, 24, \) and 32 have been fully classified [Ple72, PS75, CP80, CP92]. Our methods, however, let us prove that the extremal Type II codes of these lengths are generated by their minimal codewords without appeal to the classification results or to the explicit forms of these codes.

There is no analog of Theorems 3.2 and 3.3 for extremal Type II codes of length \( n = 16 \). Indeed, the extremal Type II code with tetrad subcode \( d_{16} \) has codewords of weight 8 that cannot be obtained as linear combinations of codewords of weight 4. As expected, following the method used to prove Theorem 3.2 in the case \( n = 16 \) yields the determinant

\[-93184 \left( \frac{s - 8}{(s - 1)s} \right),\]

which vanishes for \( s = 8 \).

3.4. Extremal Type II Codes of Lengths 56 and 96. Now, we prove an analog of Theorem 3.3 for extremal Type II codes of lengths \( n = 56, 96 \).

**Lemma 3.4.** If \( C \) is an extremal Type II code of length \( n \), and \( w > 0 \) is such that \( C_w \neq \emptyset \), then for each \( j \) (1 \( \leq j \leq n \)) there exists \( c \in C_w \) such that \( c_j = 1 \).

**Proof.** By Theorem 2.7, \( C_w \) is a 1-design. We then have from Corollary 2.4 that

\[
\sum_{c \in C_w} c_j = \frac{|C_w|}{|\Omega_w|} \sum_{v \in \Omega_w} v_j > 0.
\]

The result follows immediately. \( \square \)

\(^1\) These determinants were computed using the formula of Corollary 2.5. We omit the equations obtained from the zonal spherical harmonic polynomials of the highest degrees when there are more than \( \frac{n^2}{4} + 2 \) equations obtained by this method.
We now state and prove the configuration result for extremal Type II codes of lengths 56 and 96.

**Theorem 3.5.** If $C$ is an extremal Type II code of length $n = 56, 96$, then

$$C = C_{w_0}(C).$$

**Proof.** Seeking a contradiction, we suppose that $C_{w_0}(C) \neq C$, and consider $C_{w_0}(C) ^ \perp$. We must have $C_{w_0}(C) ^ \perp \neq C_{w_0}(C)$, since otherwise we would have $C_{w_0}(C) = C$ by Lemma 3.4. Thus, there is some equivalence class $[\hat{c}] \in (C_{w_0}(C) ^ \perp)/(C_{w_0}(C))$ with minimal-weight representative $\hat{c}$ of weight $\text{wt}(\hat{c}) = s > 0$.

Corollary 2.5 yields $t(n) + 1$ equations in the variables $N_{2j'}(C; \hat{c})$ ($0 \leq j' \leq w_0/4$). Combining these equations with (11), we obtain a system of $t(n) + 2$ equations in the variables $N_{2j'}(C; \hat{c})$ ($0 \leq j' \leq w_0/4$). For $n = 56, 96$, these inhomogeneous systems have (extended) matrices with determinants

$$-2^{27}3^35^411^213^217^353^3 \left( \frac{(s-16)(3s^3-112s^2+1368s-5120)}{(s-4)(s-3)(s-2)^2(s-1)^3s^3} \right),$$

$$-2^{53}3^57^211^213^217^119^223^329^331^243^147^289^2 \cdot \left[ \frac{(s-24)(68s^5-6936s^4+289901s^3-6153306s^2+65640728s-277774080)}{(s-6)(s-5)(s-4)^2(s-3)^3(s-2)^4(s-1)^5s^5} \right].$$

where $S_{96}(s)$ is the rational function

$$S_{96}(s) = \left( \frac{(s-24)(68s^5-6936s^4+289901s^3-6153306s^2+65640728s-277774080)}{(s-6)(s-5)(s-4)^2(s-3)^3(s-2)^4(s-1)^5s^5} \right).$$

These determinants must vanish, but the only integral roots of (12) and (13) are multiples of 4. Therefore, $C_{w_0}(C) ^ \perp$ is doubly even, and it follows that $C_{w_0}(C) ^ \perp$ is self-orthogonal. Then, $\dim(C_{w_0}(C) ^ \perp) \leq n/2$ and so $\dim(C_{w_0}(C)) \geq n/2$. We must therefore have $C_{w_0}(C) = C$. \qed

**Remarks.** After the our results were first circulated in [Kom09b] and [EK13], Harada [Har15] showed the following result that generalizes our configuration results for extremal Type II codes of lengths 24, 48, 72, and 96.

**Theorem (Har15).** If $C$ is an extremal Type II code of length $24m$ for a positive integer $m \leq 6$, and $w = 4k$ for some integer $k$ with $m < k < 5m$ and $(m, k) \neq (6, 18)$, then $C = C_w(C)$.

Taking $k = m + 1$ for $m = 1, 2, 3, 4$ recovers our results for $n = 24m$. Harada’s approach is different from ours, as it uses Mendelsohn’s relations [Men71].

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2 Note that the variables $N_j(C; \hat{c})$ vanish for $j$ not of the form $2j'$ with $0 \leq j' \leq w_0/4$, as the conclusion of Lemma 3.1 holds for $[\hat{c}] \in (C_{w_0}(C) ^ \perp)/(C_{w_0}(C))$. 
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