A new generalization of Mittag-Leffler function via $q$-calculus

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Abstract
The present paper deals with a new different generalization of the Mittag-Leffler function through $q$-calculus. We then investigate its remarkable properties like convergence, recurrence relation, integral representation, $q$-derivative formula, $q$-Laplace transformation, and image formula under $q$-derivative operator. In addition to this, we consider some specific cases to give the utilization of our main results.

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Keywords: $q$-gamma functions; $q$-beta functions; Mittag-Leffler function

1 Introduction
The Swedish mathematician Gösta Mittag-Leffler discovered a special function in 1903 (see [12, 13]) defined as

$$E_\eta(u) = \sum_{m=0}^{\infty} \frac{u^m}{\Gamma(\eta m + 1)m!}, \quad (\eta, u \in \mathbb{C}; \Re(\eta) > 0),$$

(1.1)

where $\Gamma(\cdot)$ is a classical gamma function [17]. The special function defined in (1.1) is called the Mittag-Leffler function.

For the very first time, in 1905, Wiman [21] firstly proposed the generalization of the Mittag-Leffler function $E_{\eta, \kappa}(u)$ as follows:

$$E_{\eta, \kappa}(u) = \sum_{m=0}^{\infty} \frac{u^m}{\Gamma(\eta m + \kappa)m!}, \quad (\eta, \kappa \in \mathbb{C}; \Re(\eta) > 0, \Re(\kappa) > 0).$$

(1.2)

Subsequently, the generalized form of series (1.1) and (1.2) was studied by Prabhakar [16] in 1971:

$$E_{\eta, \kappa}^{\sigma}(u) = \sum_{m=0}^{\infty} \frac{u^m(\sigma)_m}{\Gamma(\eta m + \kappa m!)(\sigma)_m}, \quad (\eta, \kappa, \sigma \in \mathbb{C}; \Re(\eta) > 0, \Re(\kappa) > 0, \Re(\sigma) > 0),$$

(1.3)

where $(\sigma)_m = \frac{\Gamma(\sigma + m)}{\Gamma(\sigma)}$ denotes the Pochhammer symbol [17].
The Mittag-Leffler function plays a vital role in the solution of fractional order differential and integral equations. It has recently become a subject of rich interest in the field of fractional calculus and its applications. Nowadays some mathematicians consider the classical Mittag-Leffler function as the queen function in fractional calculus. An enormous amount of research in the theory of Mittag-Leffler functions has been published in the literature. For a detailed account of the various generalizations, properties, and applications of the Mittag-Leffler function, readers may refer to the literature (see [3, 8–10, 14, 15, 18, 20]).

The $q$-calculus is the $q$-extension of the ordinary calculus. The theory of $q$-calculus operators has been recently applied in the areas of ordinary fractional calculus, optimal control problem, in finding solutions of the $q$-difference and $q$-integral equations, and $q$-transform analysis.

In 2009, Mansoor [11] proposed a new form of $q$-analogue of the Mittag-Leffler function given as

$$e_{\eta,\kappa}(u; q) = \sum_{m=0}^{\infty} \frac{u^m}{\Gamma_q(\eta m + \kappa)}, \quad (|u| < (1 - q)^{-\eta}),$$

(1.4)

where $\eta > 0, \kappa \in \mathbb{C}$.

For other analogues of the Mittag-Leffler functions on the quantum time scale by means of the linear Caputo $q$-fractional initial value problems and of better imitation to the theory of time scales, we refer the reader to Definition 10 and Remark 11 in [1]. For the Kilbas–Saigo $q$-analogue of the Mittag-Leffler function, we refer to [2].

Recently, Sharma and Jain [19] introduced the following $q$-analogue of the generalized Mittag-Leffler function:

$$E_{\eta,\kappa}^\sigma(u; q) = \sum_{m=0}^{\infty} \frac{(q^\sigma; q)_m u^m}{(q; q)_m \Gamma_q(\eta m + \kappa)},$$

(1.5)

$$\left( \eta, \kappa, \sigma \in \mathbb{C}; \Re(\eta) > 0, \Re(\kappa) > 0, \Re(\sigma) > 0, |q| < 1 \right).$$

2 Prelude

In the theory of $q$-series (see [6]), for complex $\lambda$ and $0 < q < 1$, the $q$-shifted factorial is defined as follows:

$$\lambda; q)_m = \begin{cases} 1; & m = 0, \\
(1 - \lambda)(1 - \lambda q) \cdots (1 - \lambda q^{m-1}); & m \in \mathbb{N}, \end{cases}$$

(2.1)

which is equivalent to

$$\lambda; q)_m = \frac{(\lambda; q)_{\infty}}{(\lambda q^m; q)_{\infty}}$$

(2.2)

and its extension naturally is

$$\lambda; q)_{\eta} = \frac{(\lambda; q)_{\infty}}{(\lambda q^{\eta}; q)_{\infty}}, \quad \eta \in \mathbb{C},$$

(2.3)

where the principal value of $q^\eta$ is taken.
For \( s, t \in \mathbb{R} \), the \( q \)-analogue of the exponent \((s - t)^m\) is

\[
(s - t)^{(m)} = \begin{cases} 
1; & m = 0, \\
\prod_{i=0}^{m-1} (s - tq^i); & m \neq 0 
\end{cases}
\]

and connected by the following relationship:

\[
(s - t)^{(m)} = s^m(t/s;q)_m \quad (s \neq 0).
\]

Obviously, its expansion for \( t \in \mathbb{R} \) is as follows:

\[
(s - t)^{(m)} = \frac{s^m(t/s;q)_{\infty}}{(q^m t/s;q)_{\infty}}, \quad (s;q)_r = \frac{(s;q)_{\infty}}{(sq^r;q)_{\infty}}.
\]

Note that

\[
(s - t)^{(r)} = s^r(t/s;q)_r.
\]

The \( q \)-analogue of binomial coefficient is defined for \( s, t > 0 \) as

\[
\binom{s}{t}_q = \frac{(s)_q!}{(t)_q!(s-t)_q!} = \frac{(s;q)_r}{(t;q)_{r-t}(q;q)_{r-t}} = \binom{s}{t}_q.
\]

The definition can be generalized in the following way. For arbitrary complex \( r \), we have

\[
\binom{r}{m}_q = \frac{(q^{-r};q)_m(-1)^m(q^{-r})_{m-r}}{(q;q)_m} = \frac{\Gamma_q(r + 1)}{\Gamma_q(m + 1)\Gamma_q(r - m + 1)}.
\]

where \( \Gamma_q(u) \) is the \( q \)-gamma function.

The \( q \)-gamma and \( q \)-beta functions [6] are defined by

\[
\Gamma_q(u) = \frac{(q;u)_\infty}{(q^{1-u};q)_\infty},
\]

for \( u \in \mathbb{R} \setminus \{0, -1, -2, -3, \ldots\}; |q| < 1 \).

Clearly,

\[
\Gamma_q(u + 1) = [u]_q \Gamma_q(u)
\]

and

\[
B_q(\eta, \kappa) = \frac{\Gamma_q(\eta)\Gamma_q(\kappa)}{\Gamma_q(\eta + \kappa)} = \int_0^1 u^{\eta-1} (qu; q)_\infty \frac{d_u u}{(q^\kappa u;q)_\infty} = \int_0^1 u^{\eta-1} (qu; q)_{\kappa-1} \frac{d_q u}{(q^\kappa u;q)_\infty}.
\]

Also, the \( q \)-difference operator and \( q \)-integration of a function \( f(u) \) defined on a subset of \( \mathbb{C} \) are given by [6] respectively:

\[
D_q f(u) = \frac{f(u) - f(uq)}{u(1 - q)} \quad (u \neq 0, q \neq 1), (D_q f)(0) = \lim_{u \to 0} (D_q f)(u)
\]

(\( \Re(\eta), \Re(\kappa) > 0 \)).
and 

\[ \int_0^u f(t) d(t; q) = u(1 - q) \sum_{m=0}^{\infty} q^m f(uq^m). \] (2.12)

3 Generalized q-Mittag-Leffler function and its properties

In this section, we generalize definition (1.5) by introducing the following relation for \((qc, q)m\):

\[ \frac{(qc; q)_m}{(q; q)_m} = \frac{B_q(\sigma, c - \sigma)}{B_q(\sigma, c - \sigma)}. \] (3.1)

Now, we define the generalization of Mittag-Leffler function (1.5) using the above relation as follows:

\[ E^{(\sigma; \eta, \kappa)}_{(c; q)}(u; q) = \sum_{m=0}^{\infty} \frac{B_q(\sigma + m, c - \sigma)}{B_q(\sigma, c - \sigma)} \frac{u^m}{(q; q)_m \Gamma_q(\eta m + \kappa)}, \] (3.2)

where \(B_q(\cdot)\) is the \(q\)-analogue of beta function.

We enumerate the relations as particular cases of \(q\)-analogue of the generalized Mittag-Leffler function with other special functions as given below.

(i) On setting \(c = 1\) in (3.2), we obtain

\[ E^{(\sigma; 1)}_{(1; q)}(u; q) = \sum_{m=0}^{\infty} \frac{(q^c; q)_m}{(q; q)_m} \frac{u^m}{\Gamma_q(\eta m + \kappa)} = E^{(\sigma; \eta, \kappa)}_{(1; q)}(u; q), \] (3.3)

which is given by equation (1.5).

(ii) Again, on setting \(\sigma = 1\) in (3.2), we obtain

\[ E^{(1; \eta, \kappa)}_{(c; q)}(u; q) = \sum_{m=0}^{\infty} \frac{u^m}{\Gamma_q(\eta m + \kappa)} = e_{\eta, \kappa}(u; q), \] (3.4)

the function \(e_{\eta, \kappa}(u; q)\) can be termed as \(q\)-analogue of the Mittag-Leffler function defined in (1.4).

(iii) On setting \(\eta = \kappa = \sigma = 1\), in (3.2), we obtain

\[ E^{(1; 1)}_{(1; q)}(u; q) = \sum_{m=0}^{\infty} \frac{(q^c; q)_m}{(q, q)_m} \frac{u^m}{(q; q)_m} = \varphi_0(q^c; -; q, u), \] (3.5)

where the function \(\varphi_0(q^c; -; q, u) = (1 - q)^{-c}\) can be termed as \(q\)-binomial function.

(iv) On setting \(c = c + \sigma\), in (3.2), we obtain \(q\)-analogue of the Mittag-Leffler function \(E^{(\sigma; \eta, \kappa)}_{(c; q)}(u; q)\) defined in (1.5).

4 Convergence of \(E^{(\sigma; \eta, \kappa)}_{(c; q)}(u; q)\)

**Theorem 4.1** The \(q\)-analogue of the generalized Mittag-Leffler function defined by the summation formula (3.2) converges absolutely for \(|u| < (1 - q)^{-\eta}\) provided that \(0 < q < 1\), \(\eta > 0\), \(\Re(\sigma) > \Re(\eta)\), \(c, \sigma \in \mathbb{C}\).
Proof Writing the summation formula (3.2) as \( E^{(\sigma, c)}_{\eta, \kappa}(u; q) = \sum_{m=0}^{\infty} \sum_{n=0}^{m} \) and by applying the ratios formula, we find

\[
\lim_{m \to \infty} \frac{s_{m+1}}{s_m} = \lim_{m \to \infty} \frac{B_q(\sigma + m + 1, c - \sigma)}{B_q(\sigma + m, c - \sigma)} \frac{(q', q)_{m+1}}{(q', q)_m} \frac{(q, q)_m}{\Gamma_q(\eta m + \kappa)} = \frac{1}{\Gamma_q(\kappa)} + \sum_{m=1}^{\infty} \frac{B_q(\sigma + m, c - \sigma)}{B_q(\sigma, c - \sigma)} \frac{(1 - q') (q' + m)}{(q, q)_m} \frac{u^m}{\Gamma_q(\eta m + \kappa)}.
\]

Since \((1 - q') = (1 - q'^{m+1}) - q' (1 - q')\), the above equation reduces to

\[
E^{(\sigma, c)}_{\eta, \kappa}(u; q) = \frac{1}{\Gamma_q(\kappa)} + \sum_{m=1}^{\infty} \frac{B_q(\sigma + m, c - \sigma)}{B_q(\sigma, c - \sigma)} \frac{(1 - q'^{m+1}) (q'^{m+1})_m}{(q, q)_m} \frac{u^m}{\Gamma_q(\eta m + \kappa)} + \frac{q'}{\Gamma_q(\kappa)} \sum_{m=1}^{\infty} \frac{B_q(\sigma + m, c - \sigma)}{B_q(\sigma, c - \sigma)} \frac{(1 - q^{m+1}) (q^{m+1})_m}{(q, q)_m} \frac{u^m}{\Gamma_q(\eta m + \kappa)}.
\]

On replacing \(m\) with \(m + 1\) in the second summation, it becomes

\[
E^{(\sigma, c)}_{\eta, \kappa}(u; q) = \frac{1}{\Gamma_q(\kappa)} + \sum_{m=1}^{\infty} \frac{B_q(\sigma + m + 1, c - \sigma)}{B_q(\sigma, c - \sigma)} \frac{(q'^{m+1})_m}{(q, q)_m} \frac{u^{m+1}}{\Gamma_q(\eta m + \kappa)} + \frac{q'}{\Gamma_q(\kappa)} \sum_{m=1}^{\infty} \frac{B_q(\sigma + m, c - \sigma)}{B_q(\sigma, c - \sigma)} \frac{(q^{m+1})_m}{(q, q)_m} \frac{u^m}{\Gamma_q(\eta m + \eta + \kappa)},
\]

which leads to the required result (5.1).

5 Recurrence relations
Theorem 5.1 If \(\eta, \kappa, \sigma \in \mathbb{C}, \Re(\eta) > 0, \Re(\kappa) > 0, \Re(\sigma) > 0, \) and \(\sigma \neq c,\) then

\[
E^{(\sigma, c + 1)}_{\eta, \kappa}(u; q) = E^{(\sigma, c + 1)}_{\eta, \kappa}(u; q) - u q^{\sigma} E^{(\sigma, c + 1)}_{\eta, \kappa}(u; q).
\]

Proof Using definition (3.2), we obtain

\[
E^{(\sigma, c)}_{\eta, \kappa}(u; q) = \sum_{m=0}^{\infty} \frac{B_q(\sigma + m, c - \sigma)}{B_q(\sigma, c - \sigma)} \frac{(q', q)_m}{(q, q)_m} \frac{u^m}{\Gamma_q(\eta m + \kappa)}.
\]

6 Some elementary properties of the generalized \(q\)-Mittag-Leffler function
We begin with the following theorem, which shows the integral representation of the generalized \(q\)-Mittag-Leffler function.
Theorem 6.1 (Integral representation) For the generalized $q$-Mittag-Leffler function, we have

$$E_{\eta,\kappa}^{(\sigma,\nu)}(u; q) = \frac{1}{B_q(\sigma, c - \sigma)} \int_0^1 t^{\sigma-1} \frac{(tq; q)_\infty}{(tq^{-\sigma}; q)_\infty} E_{\eta,\kappa}^{(c)}(tu; q) \, dq \, t,$$

(6.1)

provided that $\eta, \kappa, \sigma \in \mathbb{C}$, $\Re(\eta) > 0$, $\Re(\kappa) > 0$, $\Re(\sigma) > 0$, and $\sigma \neq c$.

Proof By the definition of $q$-analogue of beta function, we can rewrite equation (3.2) as follows:

$$E_{\eta,\kappa}^{(\sigma,\nu)}(u; q) = \sum_{m=0}^{\infty} \left\{ \int_0^1 t^{\sigma+1+m} \frac{(tq; q)_\infty}{(tq^{-\sigma}; q)_\infty} \, dq \, t \right\} \frac{1}{B_q(\sigma, c - \sigma)} \times \frac{(q^\nu; q)_m}{\Gamma_q(\eta m + \kappa)} \frac{u^m}{(q; q)_m}$$

which leads to the required result (6.1). □

Theorem 6.2 For $\eta, \kappa, \sigma \in \mathbb{C}$, $\Re(\eta) > 0$, $\Re(\kappa) > 0$, $\Re(\sigma) > 0$, $c \neq \sigma$, then for any $m \in \mathbb{N}$, we have

$$D_q^m[u^{\sigma-1} E_{\eta,\kappa}^{(\sigma,\nu)}(\lambda u^n; q)] = u^{\sigma-m-1} E_{\eta,\kappa}^{(\sigma,\nu)}(\lambda u^n; q).$$

(6.2)

Proof By considering the function

$$f(u) = u^{\sigma-1} E_{\eta,\kappa}^{(\sigma,\nu)}(\lambda u^n; q).$$

In view of (2.11) and using definition (3.2), we obtain

$$D_q[f(u)] = \sum_{m=0}^{\infty} \frac{B_q(\sigma + m + 1, c - \sigma)}{B_q(\sigma, c - \sigma)} \frac{(q^\nu; q)_m}{(q; q)_m} \times \frac{\lambda^m}{1-q} \frac{u^{\sigma+m+1}}{\Gamma_q(\eta m + \kappa + 1)}.$$

Since, according to the functional equation (2.9), the right-hand side of the above expression can be written as

$$\sum_{m=0}^{\infty} \frac{B_q(\sigma + m + 1, c - \sigma)}{B_q(\sigma, c - \sigma)} \frac{(q^\nu; q)_m}{(q; q)_m} \frac{\lambda^m}{1-q} \frac{u^{\sigma+m+1}}{\Gamma_q(\eta m + \kappa + 1)} = u^{\sigma-2} E_{\eta,\kappa}^{(\sigma,\nu)}(\lambda u^n; q).$$

Conclusively, we obtain

$$D_q[f(u)] = u^{\sigma-2} E_{\eta,\kappa}^{(\sigma,\nu)}(\lambda u^n; q).$$

Iterating the above result $m - 1$ times, we obtain the required result (6.2). □
Theorem 6.3 Let $\xi, \zeta, \sigma, \kappa \in \mathbb{C}; \Re(\xi), \Re(\kappa), \Re(\sigma) > 0; \xi \neq 0, -1, -2, \ldots$, then

$$
\int_{0}^{1} u^{\xi-1} (1 - qu)^{(\zeta-1)} E_{\eta, \kappa}^{(\sigma; q)} (xu^\rho; q) \, dq \, du
$$

\begin{equation}
= \sum_{m=0}^{\infty} B_{q}(\sigma + m, c - \sigma)(q^\zeta; q)_m \frac{x^m \Gamma_q(\xi + \rho m) \Gamma_q(\xi)}{B_q(\sigma, c - \sigma)(q; q)_m} \frac{\Gamma_q(\eta m + \kappa) \Gamma_q(\xi)}{\Gamma_q(\eta m + \kappa + \rho m)}.
\end{equation}

(6.3)

In particular,

$$
\int_{0}^{1} u^{\xi-1} (1 - qu)^{(\zeta-1)} E_{\eta, \kappa}^{(\sigma; q)} (xu^\rho; q) \, dq \, du = \Gamma_q(\xi) E_{\eta, \kappa+\xi}(x; q).
$$

(6.4)

Proof By using definition (3.2), the left-hand side of equation (6.3) can be written as

$$
\int_{0}^{1} u^{\xi-1} (1 - qu)^{(\zeta-1)} \sum_{m=0}^{\infty} B_{q}(\sigma + m, c - \sigma)(q^\zeta; q)_m \frac{u^m x^m}{B_q(\sigma, c - \sigma)(q; q)_m} \frac{\Gamma_q(\eta m + \kappa) \Gamma_q(\xi)}{\Gamma_q(\eta m + \kappa + \rho m)} \, dq \, du.
$$

Interchanging the order of summation and integration and in view of equation (2.10), we obtain the required result (6.3).

In equation (6.3) replacing $\eta = \rho$, $\xi = \kappa$, then in view of equation (3.2), we can clearly obtain (6.4). \(\square\)

Theorem 6.4 (q-Laplace transform) The q-analogue of the generalized Laplace transform is defined as follows:

$$
qL_s \left[ E_{\eta, \kappa}^{(\sigma; q)} (xu^\rho; q) \right] = \frac{1}{s} \sum_{m=0}^{\infty} B_{q}(\sigma + m, c - \sigma)(q^\zeta; q)_m \frac{\Gamma_q(1 + \rho m)}{B_q(\sigma, c - \sigma)(q; q)_m} \frac{u^m x^m}{\Gamma_q(\eta m + \kappa)} \left(\frac{1 - q}{s^\rho}\right)^m
$$

provided that $\kappa, \sigma, s \in \mathbb{C}; \Re(\beta), \Re(\kappa), \Re(s) > 0$.

Proof The $q$-Laplace transform of a suitable function is given by means of the following $q$-integral [7]:

$$
qL_s \{f(u)\} = \frac{1}{(1 - q)} \int_{0}^{s^{-1}} \frac{E_{q}^{mu}(u)}{E_{q}^{mu}(u)} \, dq \, du
$$

(6.6)

The $q$-extension of the exponential function [6] is given by

$$
E_{q}^{u} = \phi_0(-,-; q, -u) = \sum_{m=0}^{\infty} q^{(m)} u^m (q; q)_m = (-u; q)_\infty
$$

(6.7)

and

$$
e_{q}^{u} = \phi_0(0,-; q, -u) = \sum_{m=0}^{\infty} \frac{u^m (q; q)_m}{(q; q)_m} = \frac{1}{(u; q)_\infty}, \quad |u| < 1.
$$

(6.8)
By using the above $q$-exponential series and the $q$-integral equation (2.12), we can write equation (6.6) as

$$qL_s\{f(u)\} = \frac{(q; q)_\infty}{s} \sum_{j=0}^{\infty} \frac{q^{j+1}}{(q; q)_j} f(s^{-1}q^j). \quad (6.9)$$

Using definition (3.2) and the definition of $q$-Laplace transform, we obtain

$$qL_s\left[ E_{\eta, \kappa}(\sigma; c)(u; q) \right] = \frac{(q; q)_\infty}{s} \sum_{j=0}^{\infty} \frac{q^j}{(q; q)_j} \times \sum_{m=0}^{\infty} \frac{B_q(\sigma + m, c - \sigma) (q^m; q)_m [u(s^{-1}q^j)^\sigma]^m}{B_q(\sigma, c - \sigma) (q; q)_m} \frac{u^{\eta m + \kappa}}{\Gamma_1(q^{\eta m + \kappa})}. \quad (7.1)$$

On interchanging the order of summation and writing the $j$ series as $\phi_0$, which can be summed up as $\frac{1}{(q; q)_\infty}$, and after some simplifications, we obtain the required result (6.5).

7 Kober-type fractional $q$-calculus operators

Agarwal [4] established Kober-type fractional $q$-integral operator in the following manner:

$$(I_q^{\nu, \mu} f)(u) = \frac{u^{\nu-\mu}}{\Gamma_q(u)} \int_0^u (u - t q)^{\nu-1} t^{\nu-\mu} f(t) \, dq, \quad (7.1)$$

where $\Re(\mu) > 0$. Also, Garg et al. [5] introduced Kober fractional $q$-derivative operator given by

$$(D_q^{\nu, \mu} f)(u) = \prod_{i=0}^{m-1} \left( [v + i]_q + u q^{\nu+i} D_q I_q^{\nu, \mu} f(t) \right)(u), \quad (7.2)$$

where $m = [\Re(\mu)] + 1, m \in \mathbb{N}$.

The image formulas of the power function $u^m$ under the above operators [5] are given as follows:

$$I_q^{\nu, \mu} \left\{ u^m \right\} = \frac{\Gamma_q(v + m + 1)}{\Gamma_q(v + \mu + m + 1)} u^m, \quad (7.3)$$

$$D_q^{\nu, \mu} \left\{ u^m \right\} = \frac{\Gamma_q(v + \mu + m + 1)}{\Gamma_q(v + m + 1)} u^m. \quad (7.4)$$

Theorem 7.1 The following assumption holds true:

$$I_q^{\nu, \mu} \left[ E_{\eta, \kappa}^{(\sigma, \rho)}(u; q) \right] = \sum_{m=0}^{\infty} \frac{B_q(\sigma + m, c - \sigma) (q^m; q)_m}{B_q(\sigma, c - \sigma) (q; q)_m} \times \frac{\Gamma_q(v + m + 1)}{\Gamma_q(v + \mu + m + 1)} \frac{u^{\eta \rho + m + \kappa}}{\Gamma_1(q^{\eta \rho + m + \kappa})}, \quad (7.5)$$

particularly,

$$I_q^{\nu, \mu} E_{\eta, \kappa}^{(\nu+1, 1)}(u; q) = \frac{\Gamma_q(v + 1)}{\Gamma_q(v + \mu + 1)} E_{\eta, \kappa}^{(\nu+1, 1)}(u; q), \quad (7.6)$$

provided that if $\eta, c > 0, \kappa, \sigma, u \in \mathbb{C}; \Re(\kappa), \Re(\sigma) > 0$. 


Proof The proof of (7.5) can easily be obtained by making use of definition (3.2) and result (7.3).

Now, on setting $\sigma = v + \mu$ in definition (3.2), we obtain result (7.6).

Theorem 7.2 The following assumption holds true:

$$D^\nu_{q}\{E_{\nu,k}(u,q)\} = \sum_{m=0}^{\infty} \frac{B_q(\sigma + m, c - \sigma)}{B_q(\sigma, c - \sigma)} \frac{(q^m u)^m}{(q^m u)^m} \frac{\Gamma_q(v + \mu + m + 1)}{\Gamma_q(v + \mu + m + 1)} \frac{\Gamma_q(\eta m + \kappa)}{\Gamma_q(\eta m + \kappa)},$$

(7.7)

particularly,

$$D^\nu_{q}\{E_{\nu,k}^{(v+1;1)}(u,q)\} = \frac{\Gamma_q(v + \mu + 1)}{\Gamma_q(v + 1)} E_{\nu,k}^{(v+\mu;1)}(u,q)$$

(7.8)

provided that if $\eta, c > 0, \kappa, \sigma, u \in \mathbb{C}; \Re(k), \Re(\sigma) > 0$.

Proof The proof of (7.7) can easily be obtained by making use of definition (3.2) and result (7.4). Similarly, on setting $\sigma = v + 1$ in definition (3.2), we obtain result (7.8).

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