Two-dimensional arrays of superconducting and soft magnetic strips as dc magnetic metamaterials

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Abstract

We have theoretically investigated the magnetic response of two-dimensional (2D) arrays of superconducting and soft magnetic strips, which are regarded as models of dc magnetic metamaterials. The anisotropy of the macroscopic permeabilities depends on whether the applied magnetic field is parallel to the wide surface of the strips ($\mu_\parallel$) or perpendicular ($\mu_\perp$). For the 2D arrays of superconducting strips, $0 < \mu_\perp/\mu_0 \ll \mu_\parallel/\mu_0 \simeq 1$, whereas for the 2D arrays of soft magnetic strips, $\mu_\parallel/\mu_0 \gg \mu_\perp/\mu_0 \simeq 1$, where $\mu_0$ is the vacuum permeability. We also demonstrate that strong anisotropy of the macroscopic permeability can be obtained for hybrid arrays of superconducting and soft magnetic strips, where $\mu_\parallel/\mu_0 \gg 1 \gg \mu_\perp/\mu_0 > 0$.

(Some figures may appear in colour only in the online journal)

1. Introduction

It has been proposed that dc magnetic metamaterials can be used for magnetic field control [1–3], and their application to magnetic cloaking devices has been investigated [4–9]. Arrays of thin superconductors are candidates for dc magnetic metamaterials, because their magnetic permeability can exhibit geometrical anisotropy; the macroscopic permeability is small (i.e., $\mu_\perp/\mu_0 \ll 1$) when the applied magnetic field is perpendicular to the wide surface of the thin superconductors, whereas thin superconductors are magnetically transparent (i.e., $\mu_\parallel/\mu_0 \simeq 1$) when the applied field is parallel to the wide surface [1–3, 10]. The behavior of arrays of thin soft magnets is analogously dual to that of arrays of thin superconductors; thin soft magnets have a large permeability (i.e., $\mu_\parallel/\mu_0 \gg 1$) when the applied magnetic field is parallel to the wide surface of thin soft magnets, whereas thin soft magnets are magnetically transparent (i.e., $\mu_\perp/\mu_0 \simeq 1$) when the applied field is perpendicular to the wide surface. Because of the anisotropy in the macroscopic permeability, arrays of thin superconductors and soft magnets can behave as dc magnetic metamaterials and can be used to control dc magnetic fields.

In this paper we theoretically investigate the distribution of the magnetic field in two-dimensional (2D) arrays of superconducting strips and of soft magnetic strips, and present analytical expressions for the macroscopic permeabilities that characterize the magnetic response of the 2D arrays. We propose hybrid arrays of superconducting and soft magnetic strips that have both small perpendicular permeability, $0 < \mu_\perp/\mu_0 \ll 1$, and large parallel permeability, $\mu_\parallel/\mu_0 \gg 1$. This paper is organized as follows: the basic formalism for the two-dimensional magnetic field is laid out in section 2, the results for the 2D arrays of superconducting strips [10] are shown in section 3, the 2D arrays of soft magnetic strips are investigated in section 4, the hybrid arrays of superconducting and soft magnetic strips are examined in section 5, and a brief discussion and summary of the results are given in section 6.

2. Two-dimensional magnetic field

2.1. Local (microscopic) magnetic field

We investigate 2D arrays of superconducting and soft magnetic strips as the basic components of dc magnetic metamaterials. The thickness, $d$, of the strips is much smaller than the width, and is regarded as infinitesimal, $\epsilon = d/2 \to 0$. The length, $L_z$, of the strips along the $z$ axis is much larger...
than the width, and is regarded as infinite, \( L_c \to \infty \). The wide surface of the strips is parallel to the \( xz \) plane, and the strips are regularly arranged in the \( xy \) plane. We analyze the local (microscopic) magnetic field, \( \mathbf{H} = H_x(x, y)\mathbf{\hat{x}} + H_y(x, y)\mathbf{\hat{y}} \), in the \( xy \) plane. Outside the strips, the relationship between the local magnetic field, \( \mathbf{H} \), and the local magnetic induction, \( \mathbf{B} = B_x(x, y)\mathbf{\hat{x}} + B_y(x, y)\mathbf{\hat{y}} \), is given by \( \mathbf{B} = \mu_0 \mathbf{H} \), where \( \mu_0 \) is the vacuum permeability.

The 2D magnetic field is analyzed using the complex field [11, 12]

\[ \mathcal{H}(\zeta) = H_y(x, y) + iH_x(x, y), \quad (2.1) \]

as the analytic function of the complex variable \( \zeta = x + iy \). The complex potential is defined by \( \mathcal{G}(\zeta) = \int \mathcal{H}(\xi) \, d\xi \), and the contour lines of \( \text{Re} \mathcal{G}(x + iy) \) correspond to the magnetic field lines in the \( xy \) plane.

2.2. Macroscopic field and macroscopic permeability

In the unit cell of the 2D array, the macroscopic magnetic field \( \langle \mathbf{B} \rangle \) is calculated as the averaged line integral of \( \mathbf{H} \) at the cell edge, whereas the macroscopic magnetic permeability \( \langle \mathbf{B} \rangle = \mu_0 \langle \mathbf{H} \rangle \) is calculated as the averaged surface integral of \( \mathbf{B} \) at the cell side [10, 13, 14]. Because of the different definitions of the averaging procedures for obtaining the macroscopic fields, the macroscopic relationship, \( \langle \mathbf{B} \rangle \neq \mu_0 \langle \mathbf{H} \rangle \), generally holds, even though the microscopic relationship, \( \mathbf{B} = \mu_0 \mathbf{H} \), holds.

We consider the case where the wide surfaces of the strips are parallel to the \( xz \) plane; therefore the permeability tensor \( \mu_{\alpha\beta} \) defined by \( \langle B_{\alpha} \rangle = \mu_{\alpha\beta} \langle H_{\beta} \rangle \) has only diagonal components, \( \mu_{xx} = \mu_\parallel \) and \( \mu_{yy} = \mu_\perp \):

\[ \langle B_x \rangle = \mu_\parallel \langle H_x \rangle \quad \text{and} \quad \langle B_y \rangle = \mu_\perp \langle H_y \rangle. \quad (2.2) \]

The magnetic response to a parallel field is characterized by the parallel permeability, \( \mu_\parallel \), whereas the response to a perpendicular field is characterized by the perpendicular permeability, \( \mu_\perp \). We demonstrate later that for superconducting strip arrays, \( 0 < \mu_\perp / \mu_0 \ll \mu_\parallel / \mu_0 \simeq 1 \), whereas for soft magnetic strip arrays, \( \mu_\parallel / \mu_0 \gg \mu_\perp / \mu_0 \simeq 1 \). We also show that for the hybrid arrays of superconducting and soft magnetic strips, \( 0 < \mu_\perp / \mu_0 \ll 1 \ll \mu_\parallel / \mu_0 \).

3. Two-dimensional arrays of superconducting strips

In this section we briefly review the magnetic field distribution and macroscopic permeability of 2D arrays of superconducting strips reported in [10]. Each superconducting strip has a width \( 2w \), an infinitesimal thickness \( d \) (i.e., \( \epsilon = d/2 \to 0 \)), and an infinite length along the \( z \) axis. The wide surfaces of the superconducting strips are parallel to the \( xz \) plane. It is assumed that the superconducting strips are in the complete shielding state, where the magnetic field is completely shielded in the superconducting strips. The complete shielding state is achieved when the London penetration depth, \( \lambda \), is much smaller than the dimensions of the superconducting strips, \( \lambda / d \to 0 \) for thick strips or \( \lambda^2 / wd \to 0 \) for thin strips, in the Meissner state. The complete shielding state has also been observed for a weak field or large critical current density limit in the critical state model [15]. The 2D arrays of superconducting strips are exposed to an applied magnetic field \( \mathbf{H}_s = H_{s_x}\mathbf{\hat{x}} + H_{s_y}\mathbf{\hat{y}} \), which is expressed as \( H_{y x} + iH_{y y} \) in terms of the complex field.

When a 2D array of superconducting strips is exposed to a parallel magnetic field along the \( x \) axis, the magnetic field is not disturbed by thin superconducting strips for which \( \epsilon \to 0 \). Therefore, the macroscopic permeability for a parallel field is equal to the vacuum permeability, \( \mu_\parallel / \mu_0 = 1 \), for the thin-strip limit.

In contrast, when a 2D array of superconducting strips is exposed to a perpendicular magnetic field along the \( y \) axis, the magnetic field is disturbed by the superconducting strips. Because of the magnetic shielding by the superconducting strips, the macroscopic permeability for a perpendicular field is smaller than the vacuum permeability, \( 0 < \mu_\perp / \mu_0 < 1 \), depending on the geometry of the 2D array.

3.1. Rectangular array of superconducting strips

We consider a rectangular array of superconducting strips, in which the superconducting strips of width \( 2w \) are regularly arranged with a unit cell of \( 2a \times 2b \) in the \( xy \) plane, as shown in figure 1.

We employ the auxiliary complex variable, \( \eta_c \), defined as

\[ \eta_c(\zeta) \equiv \text{sn}(\zeta / \epsilon_c, k_c), \quad (3.1) \]

where \( \text{sn}(u, k) \) is the sine amplitude (i.e., the Jacobi sn function) [16]. The modulus, \( k_c \), is obtained as a function of \( b/a \) by solving

\[ b/a = K(\sqrt{1 - k_c^2})/K(k_c), \quad (3.2) \]

where \( K(k) \) is the complete elliptic integral of the first kind [16]. The \( c_\parallel \) in (3.1) is then given by

\[ c_\parallel = a/K(k_c) = b/K(\sqrt{1 - k_c^2}). \quad (3.3) \]
The complex field $\mathcal{H}(\zeta)$ and the complex potential $\mathcal{G}(\zeta) = \int_0^\zeta \mathcal{H}(\zeta') d\zeta'$ for the rectangular array of superconducting strips in the complete shielding state are [10]

$$\mathcal{H}(\zeta) = H_{0y} - \frac{\eta_0(\zeta)}{\sqrt{\eta_0(\zeta)^2 - \gamma_r^2}} + i H_{0x},$$

(3.4)

$$\mathcal{G}(\zeta) = \frac{H_{0y} c_r}{\sqrt{1 - \eta_0^2}} \times F \left( \arcsin \sqrt{\frac{k_r^2 - \gamma_r^2}{\eta_0(\zeta)^2 - \gamma_r^2}} \right) + i H_{0x} \zeta,$$

(3.5)

where $F(\varphi, k)$ is the elliptic integral of the first kind [16]. The parameters $\gamma_r$ and $\kappa_r$ in (3.4) and (3.5) are defined as

$$\gamma_r = \eta_r(w) = \text{sn}(w/c_r, k_r),$$

(3.6)

$$\kappa_r = \frac{1 - \gamma_r^2}{k_r^2 - \gamma_r^2} = \frac{k_r \text{cn}(w/c_r, k_r)}{\text{dn}(w/c_r, k_r)},$$

(3.7)

where $\text{cn}(u, k)$ and $\text{dn}(u, k)$ are the Jacobi $\text{cn}$ and $\text{dn}$ functions, respectively. We do not need to consider the details of the constants $H_{0y}$ and $H_{0x}$ in (3.4) and (3.5), because neither $H_{0x}$ nor $H_{0y}$ affects the final results of the effective permeability.

When the rectangular array of superconducting strips is exposed to a parallel magnetic field along the $x$ axis, the magnetic field is not disturbed by thin superconducting strips where $\epsilon \to 0$; that is, (3.4) shows that $\mathcal{H}(\zeta) = i H_{0x}$ for $H_{0y} = 0 \neq H_{0x}$, in this case, the macroscopic fields are $\langle B_x \rangle / \mu_0 = \langle H_x \rangle = H_{0x}$, and the macroscopic permeability for a parallel field is equal to the vacuum permeability, $\mu_{\|} / \mu_0 = 1$, for the thin-strip limit.

In contrast, when the rectangular array of superconducting strips is exposed to a perpendicular magnetic field along the $y$ axis (i.e., $H_{0y} = 0 \neq H_{0x}$), the magnetic field is disturbed by the superconducting strips. Because of the magnetic shielding by the superconducting strips, the macroscopic permeability for a perpendicular field is smaller than the vacuum permeability, $0 < \mu_{\perp} / \mu_0 < 1$, depending on the geometry of the 2D array. Figure 2 shows the magnetic field lines as the contour lines of $\text{Re} \mathcal{G}(x + iy)$ obtained from (3.5) for $H_{0x} = 0$. The magnetic field is concentrated near the gaps between the edges of the superconducting strips.

The local magnetic induction, $B_x(x, y) = \mu_0 H_x(x, y)$, and the local magnetic field, $H_x(x, y) = \text{Re} \mathcal{H}(x + iy)$, are obtained from (3.4). We examine the macroscopic perpendicular fields, $\langle B_x \rangle$ and $\langle H_x \rangle$, averaged over the unit cell of the rectangular array. The macroscopic magnetic induction $\langle B_x \rangle$ and macroscopic magnetic field $\langle H_x \rangle$ are calculated from the local fields as [10]

$$\frac{\langle B_x \rangle}{\mu_0} = \frac{1}{2\pi a} \int_{-a}^{+a} H_x(x, b) \, dx = \frac{1}{2\pi a} \int_{-a}^{+a} H_y(x, y) \, dx,$$

(3.8)

Figure 2. Magnetic field lines in the rectangular array of superconducting strips (shown as solid horizontal bars) exposed to a perpendicular magnetic field for $w/a = 0.8$ and $b/a = 0.3$.

The last expression of (3.8) is independent of $y$, because $\nabla \cdot \mathbf{B} = 0$ [10]. As shown in section A.1 in the appendix, the macroscopic fields defined by (3.8) and (3.9) are consistent with the macroscopic relationship,

$$\langle B_x \rangle / \mu_0 = \langle H_x \rangle + \langle M_y \rangle,$$

(3.10)

where $\langle M_y \rangle$ is the magnetization of the superconducting strips defined by

$$\langle M_y \rangle \equiv -\frac{1}{4\pi b} \int_{-w}^{+w} x K_x(x) \, dx$$

(3.11)

and $K_x(x) = H_x(x, -\epsilon) - H_x(x, +\epsilon)$ is the sheet current density in the superconducting strips.

The macroscopic perpendicular permeability $\mu_{\perp,sc,r} = \langle B_x \rangle / \langle H_x \rangle$ for the rectangular array of superconducting strips is obtained from (3.4), (3.8), and (3.9), as

$$\frac{\mu_{\perp,sc,r}}{\mu_0} = \frac{b}{a} \frac{K(\kappa_r)}{\sqrt{1 - \kappa_r^2}},$$

(3.12)

where $\kappa_r$ is given by (3.7). Simple expressions for $\mu_{\perp,sc,r}$ in the limiting cases can be obtained from (3.12). For large stack spacings, $b/a > 2$,

$$\frac{\mu_{\perp,sc,r}}{\mu_0} \approx \left[ 1 - \frac{2a}{\pi b} \ln \cos \left( \frac{\pi w}{2a} \right) \right]^{-1},$$

(3.13)

whereas for small stack spacings, $b/a \ll 1$,

$$\frac{\mu_{\perp,sc,r}}{\mu_0} \approx 1 - \frac{w}{a} + \frac{2b}{\pi a} \ln 2.$$

(3.14)

Equation (3.14) is not accurate near $w/a \approx 0$ or 1. Figure 3 shows plots of $\mu_{\perp,sc,r}/\mu_0$ versus $w/a$ obtained from (3.2), (3.3), (3.7), and (3.12). We can obtain a small perpendicular permeability, $\mu_{\perp,sc,r}/\mu_0 \ll 1$, when the gaps between the edges of the superconducting strips are small, $1 - w/a \ll 1$. 

1 For the linear magnetic materials investigated in the present paper (i.e., superconducting strips in the complete shielding state or ideal soft magnetic strips), neither $H_{0x}$ nor $H_{0y}$ affects the nonlinear response (e.g., superconducting strips in the critical state), then $H_{0x}$ and $H_{0y}$ need to be determined as functions of $B_{sc,x}$ and $B_{sc,y}$. The relationship between the complex field $H_{0x} + i H_{0y}$ at $(x, y) = (0, b)$ and the applied field $H_{ap} + i H_{ap}$ may be determined by considering the total shape (e.g., the demagnetization factor) of the magnetic metamaterials.
The relationship between \(k_h\) defined by (3.16) and \(k_i\) defined by (3.2) is expressed by
\[
\frac{c_\text{h}}{a} = 2b \left( \frac{K(1 - k_i^2)}{\sqrt{1 - k_i^2}} \right).
\]
(3.17)

The complex field \(\mathcal{H}(\zeta)\) and the complex potential \(\mathcal{G}(\zeta) = \int_{b_0}^{b} \mathcal{H}(\zeta) \, d\zeta'\) for the rectangular array of superconducting strips in the complete shielding state are [10]
\[
\mathcal{H}(\zeta) = H_{0y} \frac{\eta_h(\zeta) \sqrt{\eta_h(\zeta)^2 - k_h^{-2}}}{\sqrt{\eta_h(\zeta)^2 - \gamma_h^2}} + iH_{0x},
\]
(3.18)
\[
\mathcal{G}(\zeta) = \frac{H_{0y}c_h}{k_h \beta_h - \gamma_h^2} \times F \left( \arcsin \left( \frac{\beta_h^2 - \gamma_h^2}{\eta_h(\zeta)^2 - \gamma_h^2}, \kappa_h \right) \right) + iH_{0x}\zeta,
\]
(3.19)
where
\[
\eta_h = \eta_h(w) = \text{sn}(w/c_h, k_h),
\]
(3.20)
\[
\beta_h = \eta_h(a - w + 2ib) = \sqrt{\frac{k_h^2 - \gamma_h^2}{1 - \gamma_h^2}}.
\]
(3.21)
\[
\kappa_h = \left[ \frac{(1 - \gamma_h^2)^2}{k_h^{-2} - 1 + (1 - \gamma_h^2)^2} \right]^{1/2}
\]
\[
= \left[ 1 + \frac{k_h^{-2} - 1}{\text{cn}^4(w/c_h, k_h)} \right]^{-1/2}.
\]
(3.22)

Under a parallel magnetic field along the \(x\) axis, (3.18) shows that \(\mathcal{H}(\zeta) = iH_{0x}\) for \(H_{0y} = 0 \neq H_{0x}\), and the macroscopic permeability for a parallel field is equal to the vacuum permeability, \(\mu_0/\mu_0 = 1\), for the thin-strip limit.

In contrast, under a perpendicular magnetic field along the \(y\) axis (i.e., \(H_{0x} = 0 \neq H_{0y}\)), the magnetic field is disturbed by the superconducting strips. Figure 5 shows the magnetic field lines as the contour lines of \(\text{Re } \mathcal{G}(x + iy)\) obtained from (3.19) for \(H_{0x} = 0\). The magnetic field is concentrated near the gaps between the edges of the superconducting strips.

The definitions of the macroscopic magnetic induction \(B_y\) and the magnetization \(M_y\) for the hexagonal array are the same as those for the rectangular array, and are expressed by (3.8) and (3.11), respectively. The definition of the macroscopic magnetic field for the hexagonal array, \(H_y\), given by (3.9) is inconsistent with the macroscopic relationship given by (3.10). Therefore, we use a modified definition of \(H_y\) for the hexagonal array [10].
\[
H_y(\zeta) = \frac{1}{2b} \left[ \int_0^{2b} H_y(a, y) \, dy - \int_0^{a} H_x(x, 2b - \epsilon) \, dx \right].
\]
(3.23)
For the hexagonal array, the macroscopic quantities defined by (3.8), (3.11), and (3.23) satisfy (3.10), as shown in section A.2 in the appendix.

The macroscopic perpendicular permeability, \( \mu_{\perp,\text{sc}} = (B_{\perp})/(H_{y}) \), for the hexagonal array of superconducting strips, is obtained from (3.8), (3.18), and (3.23):

\[
\frac{\mu_{\perp,\text{sc}}}{\mu_0} = \frac{2b}{a} \frac{K(k_h)}{K(\sqrt{1-k_h^2})}.
\]

Equation (3.24) is not accurate near \( w/a \approx 0, 1/2 \) or 1. Figure 6 shows plots of \( \mu_{\perp,\text{sc}}/\mu_0 \) versus \( w/a \) obtained from (3.16), (3.17), (3.22), and (3.24). We can obtain a small perpendicular permeability, \( \mu_{\perp,\text{sc}}/\mu_0 \ll 1 \), for a wide range of \( 0.5 < w/a < 1 \), when \( b/a \ll 1 \).

4. Two-dimensional arrays of soft magnetic strips

We investigate the magnetic field distribution and macroscopic permeability of 2D arrays of soft magnetic strips. The dimensions of the soft magnetic strips are the same as those of the superconducting strips shown in section 3: each soft magnetic strip has a width of \( 2w \), an infinitesimal thickness of \( d \) (i.e., \( \epsilon = d/2 \rightarrow 0 \)), and an infinite length along the \( z \) axis. The wide surfaces of the soft magnetic strips are parallel to the \( xz \) plane. The soft magnetic strips are treated as ideal soft magnets, with an infinite permeability, zero hysteresis, and an infinite saturation field [17]. In the ideal soft magnet, the relationship between \( B \) and \( H \) is given by \( B = \mu_n H \), where

\[
\mu_n \rightarrow \infty. \text{ Outside the ideal soft magnet, } H = B/\mu_0 \text{ has only a perpendicular component at the surface [18].}
\]

Figure 6. Effective permeability of the hexagonal array of superconducting strips in a perpendicular magnetic field, \( \mu_{\perp,\text{sc}} \), as a function of \( w/a \) for \( b/a = 5, 2, 1, 0.5, 0.2, \text{ and } 0.1 \). The dashed line corresponds to \( \mu_{\perp,\text{sc}}/\mu_0 = 1 \) for \( b/a \rightarrow \infty \), and the chained line of \( \mu_{\perp,\text{sc}}/\mu_0 = 1 - w/a \) is shown for comparison with figure 3. The effective permeability of the hexagonal array of soft magnetic strips in a parallel magnetic field, \( \mu_{\parallel,\text{sc}} \), corresponds to the inverse of \( \mu_{\perp,\text{sc}} \), that is, \( \mu_{\perp,\text{sc}}/\mu_0 = \mu_0/\mu_{\parallel,\text{sc}} \).

When the 2D array of soft magnetic strips is exposed to a perpendicular magnetic field along the \( y \) axis, the magnetic field is not disturbed by thin soft magnetic strips of \( \epsilon \rightarrow 0 \). Therefore, the macroscopic permeability for a perpendicular field is equal to the vacuum permeability, \( \mu_{\perp}/\mu_0 = 1 \), for the thin-strip limit.

When the 2D array of soft magnetic strips is exposed to a parallel magnetic field along the \( x \) axis, on the other hand, the magnetic field is disturbed by soft magnetic strips. The macroscopic permeability of a perpendicular field is larger than the vacuum permeability, \( \mu_{\parallel}/\mu_0 > 1 \), depending on the geometry of the 2D array.

4.1. Rectangular array of soft magnetic strips

We consider a rectangular array of soft magnetic strips, in which soft magnetic strips of width \( 2w \) are regularly arranged with a unit cell of \( 2a \times 2b \) in the \( xy \) plane, as shown in figure 7. The geometry of the rectangular array of soft magnetic strips is exactly the same as that of the rectangular array of superconducting strips shown in figure 1.

The complex field, \( \mathcal{H}(\zeta) \), and the complex potential, \( \mathcal{G}(\zeta) = \int_0^{\zeta} \mathcal{H}(\zeta') \, d\zeta' \), for the rectangular array of soft magnetic strips based on the ideal soft magnet model are given by

\[
\mathcal{H}(\zeta) = H_y + i H_x \frac{\eta_{\gamma}(\zeta)}{\sqrt{\eta_{\gamma}(\zeta)^2 - Y_t^2}}, \tag{4.1}
\]

where \( \eta_{\gamma}(\zeta) = \frac{2b}{a} \frac{K(k_h)}{K(\sqrt{1-k_h^2})} \cdot \frac{2w}{a} \ln 2 \) for \( 0 < w/a < 1/2 \), and \( \frac{2b}{a} \left( \frac{2w}{a} - 1 + \frac{8b}{\pi a} \ln 2 \right) \) for \( 1/2 < w/a < 1 \).

Figure 6. Effective permeability of the hexagonal array of superconducting strips in a perpendicular magnetic field, \( \mu_{\perp,\text{sc}} \), as a function of \( w/a \) for \( b/a = 5, 2, 1, 0.5, 0.2, \text{ and } 0.1 \). The dashed line corresponds to \( \mu_{\perp,\text{sc}}/\mu_0 = 1 \) for \( b/a \rightarrow \infty \), and the chained line of \( \mu_{\perp,\text{sc}}/\mu_0 = 1 - w/a \) is shown for comparison with figure 3. The effective permeability of the hexagonal array of soft magnetic strips in a parallel magnetic field, \( \mu_{\parallel,\text{sc}} \), corresponds to the inverse of \( \mu_{\perp,\text{sc}} \), that is, \( \mu_{\perp,\text{sc}}/\mu_0 = \mu_0/\mu_{\parallel,\text{sc}} \).

When the 2D array of soft magnetic strips is exposed to a perpendicular magnetic field along the \( y \) axis, the magnetic field is not disturbed by thin soft magnetic strips of \( \epsilon \rightarrow 0 \). Therefore, the macroscopic permeability for a perpendicular field is equal to the vacuum permeability, \( \mu_{\perp}/\mu_0 = 1 \), for the thin-strip limit.

When the 2D array of soft magnetic strips is exposed to a parallel magnetic field along the \( x \) axis, on the other hand, the magnetic field is disturbed by soft magnetic strips. The macroscopic permeability of a perpendicular field is larger than the vacuum permeability, \( \mu_{\parallel}/\mu_0 > 1 \), depending on the geometry of the 2D array.

4.1. Rectangular array of soft magnetic strips

We consider a rectangular array of soft magnetic strips, in which soft magnetic strips of width \( 2w \) are regularly arranged with a unit cell of \( 2a \times 2b \) in the \( xy \) plane, as shown in figure 7. The geometry of the rectangular array of soft magnetic strips is exactly the same as that of the rectangular array of superconducting strips shown in figure 1.

The complex field, \( \mathcal{H}(\zeta) \), and the complex potential, \( \mathcal{G}(\zeta) = \int_0^{\zeta} \mathcal{H}(\zeta') \, d\zeta' \), for the rectangular array of soft magnetic strips based on the ideal soft magnet model are given by

\[
\mathcal{H}(\zeta) = H_y + i H_x \frac{\eta_{\gamma}(\zeta)}{\sqrt{\eta_{\gamma}(\zeta)^2 - Y_t^2}}, \tag{4.1}
\]
Figure 7. Rectangular array of soft magnetic (SM) strips. The solid horizontal bars show the cross section of the soft magnetic strips in the xy plane. In the nth layer at y = 2nb, the nth strip is situated at |x - 2na| < w, where m = 0, ±1, ±2, ..., ±∞ and n = 0, ±1, ±2, ..., ±∞.

\[
\mathcal{G}(\zeta) = H_{0y} \zeta + \frac{i H_{0x} c_\tau}{\sqrt{1 - k^2}} \times F \left( \text{arcsin} \frac{k^2 - \gamma^2}{\eta(\zeta) - \gamma^2}, \kappa \right),
\tag{4.2}
\]  

where \( \eta, \kappa, c_\tau, \gamma, \) and \( \kappa \) are defined by (3.1)-(3.3), (3.6), and (3.7), respectively. The behavior of the soft magnetic strips is analogously dual to that of the superconducting strips; (4.1) and (4.2) are obtained simply by exchanging \( H_{0y} \leftrightarrow i H_{0x} \) in (3.4) and (3.5), respectively [19].

When the rectangular array of soft magnetic strips is exposed to a perpendicular magnetic field along the y axis, the magnetic field is not disturbed by thin soft magnetic strips where \( \epsilon \rightarrow 0 \); that is, (4.1) shows that \( \mathcal{H}(\zeta) = H_{0y} \) for \( H_{0x} = 0 \neq H_{0y} \). In this case, the macroscopic fields are \( \langle B_x \rangle / \mu_0 = \langle H_y \rangle = H_{0y} \), and the macroscopic permeability for a perpendicular field is equal to the vacuum permeability, \( \mu_{\perp} / \mu_0 = 1 \), for the thin-strip limit.

In contrast, when the rectangular array of soft magnetic strips is exposed to a parallel magnetic field along the x axis (i.e., \( H_{0y} = 0 \neq H_{0x} \)), the magnetic field is disturbed by the soft magnetic strips. The macroscopic permeability for a parallel field is larger than the vacuum permeability, \( \mu_{\parallel} / \mu_0 > 1 \), depending on the geometry of the 2D array. Figure 8 shows the magnetic field lines as the contour lines of \( \text{Re} \mathcal{G}(x + iy) \) obtained from (4.2) for \( H_{0y} = 0 \).

The macroscopic parallel fields, \( \langle B_x \rangle \) and \( \langle H_x \rangle \), averaged over the unit cell of the rectangular array are defined by

\[
\langle B_x \rangle / \mu_0 = \frac{1}{2b} \int_0^{2b} H_x(a, y) \, dy,
\tag{4.3}
\]

\[
\langle H_x \rangle = \frac{1}{2a} \int_{-a}^{a} H_x(x, b) \, dx = \frac{1}{2a} \int_{-a}^{a} H_x(x, y) \, dx.
\tag{4.4}
\]

Figure 8. Magnetic field lines in the rectangular array of soft magnetic strips (shown as solid horizontal bars) exposed to a parallel magnetic field for \( w/a = 0.8 \) and \( b/a = 0.3 \).

The last expression of (4.4) is independent of \( y \), because \( \nabla \times \mathbf{H} = 0 \) [10]. As shown in section A.3 in the appendix, (4.3) and (4.4) are consistent with

\[
\frac{\langle B_x \rangle / \mu_0}{\mu_0} = \langle H_y \rangle + \langle M_y \rangle,
\tag{4.5}
\]

where \( \langle M_y \rangle \) is the magnetization arising from the soft magnetic strips, defined as [10]

\[
\langle M_y \rangle \equiv \frac{1}{2ab} \int_{-w}^{+w} x \sigma_m(x) \, dx.
\tag{4.6}
\]

The expression \( \sigma_m(x) = H_y(x, +\epsilon) - H_y(x, -\epsilon) \) corresponds to the effective sheet magnetic charge in the soft magnetic strips [17, 18].

The macroscopic parallel permeability, \( \mu_{\parallel m, r} = \langle B_x \rangle / \langle H_x \rangle \), for the rectangular array of soft magnetic strips is obtained from (4.1), (4.3), and (4.4), as

\[
\frac{\mu_{\parallel m, r}}{\mu_0} = \frac{a}{b} \frac{K_b(1 - \kappa^2)}{K_b(\kappa)},
\tag{4.7}
\]

where \( \kappa \) is given by (3.7). Note that \( \mu_{\perp m, r} \) given by (3.12) and \( \mu_{\parallel m, r} \) given by (4.7) hold the simple relationship \( \mu_{\parallel m, r} = \mu_{\perp m, r}^0 \mu_{\perp m, r} \). Figure 3 shows plots of \( \mu_0 / \mu_{\parallel m, r} \) versus \( w/a \) obtained from (3.2), (3.3), (3.7), and (4.7). We can obtain a large parallel permeability, \( \mu_{\parallel m, r} / \mu_0 \gg 1 \), when the gaps between the edges of the soft magnetic strips are small, \( 1 - w/a \ll 1 \).

4.2 Hexagonal array of soft magnetic strips

We next consider a hexagonal array of soft magnetic strips, in which soft magnetic strips of width \( 2w \) are regularly arranged with a unit cell of \( 2a \times 2b \) in the xy plane, as shown in figure 9. The geometry of the hexagonal array of soft magnetic strips is exactly the same as that of the hexagonal array of superconducting strips shown in figure 4.

The complex field, \( \mathcal{H}(\zeta) \), and the complex potential \( \mathcal{G}(\zeta) = \int_0^\zeta \mathcal{H}(\zeta') \, d\zeta' \) for the hexagonal array of soft magnetic strips based on the ideal soft magnet model are given by

\[
\mathcal{H}(\zeta) = H_{0y} + i H_{0x} \frac{\eta_0(\zeta) \sqrt{\eta(\zeta)^2 - k^2}}{\sqrt{\eta(\zeta)^2 - \gamma^2 \sqrt{\eta(\zeta)^2 - \beta^2}}},
\tag{4.8}
\]
Figure 9. Hexagonal array of soft magnetic strips. Solid horizontal bars show the cross sections of the soft magnetic strips in the xy plane. In the even layer at y = 4nb, the nth strip is situated at [x − 2ma] < w, whereas in the odd layer at y = (4n + 2)b, the nth strip is situated at [x − (2m + 1)a] < w, where m = 0, ±1, ±2, . . . , ±∞ and n = 0, ±1, ±2, . . . , ±∞.

\[ \mathcal{G}(\zeta) = H_{0y} \zeta + \frac{iH_{0y}c_h}{k_h \sqrt{\beta_h^2 - \gamma_h^2}} \times F \left( \arcsin \sqrt{\frac{\beta_h^2 - \gamma_h^2}{\eta_h(\zeta)^2 - \gamma_h^2}}, k_h \right) \]  

(4.9)

where \( \eta_h, k_h, c_h, \beta_h, \gamma_h, \) and \( k_h \) are defined by (3.15)–(3.17), (3.20), (3.21), and (3.22), respectively. Equations (4.8) and (4.9) are obtained simply by exchanging \( H_{0y} \leftrightarrow iH_{0y} \) in (3.18) and (3.19), respectively.

When the hexagonal array of soft magnetic strips is exposed to a perpendicular magnetic field along the y axis, (4.8) shows that \( \mathcal{H}(\zeta) = H_{0y} \) for \( H_{0x} = 0 \neq H_{0y} \). The macroscopic permeability for a perpendicular field is equal to the vacuum permeability, \( \mu_{\perp}/\mu_0 = 1 \), for the thin-strip limit.

In contrast, when the hexagonal array of soft magnetic strips is exposed to a parallel magnetic field along the x axis (\( H_{0x} \neq 0 \) and \( H_{0y} = 0 \)), the magnetic field is disturbed by the soft magnetic strips. Figure 10 shows the magnetic field lines as the contour lines of \( \text{Re} \mathcal{G}(x + iy) \) obtained from (4.9).

The definitions of the macroscopic magnetic field, \( \langle H_x \rangle \), and the magnetization, \( \langle M_i \rangle \), for the hexagonal array are the same as those used for the rectangular array, and are given by (4.4) and (4.6), respectively. However, the definition of the macroscopic magnetic induction, \( \langle B_i \rangle \), for the hexagonal array given by (4.3) is inconsistent with the macroscopic relationship given by (4.5). Therefore, we use the modified definition of \( \langle B_i \rangle \) for the hexagonal array,

\[ \frac{\langle B_x \rangle}{\mu_0} = \frac{1}{2b} \left[ \int_0^{2b} H_x(a, y) \, dy + \int_a^0 H_x(x, 2b - \epsilon) \, dx \right] \]  

(4.10)

For the hexagonal array, the macroscopic quantities defined by (4.4), (4.6), and (4.10) satisfy (4.5), as shown in section A.4 in the appendix.

Figure 10. Magnetic field lines in the hexagonal array of soft magnetic strips (shown as solid horizontal bars) exposed to a parallel magnetic field for \( w/a = 0.8 \) and \( b/a = 0.3 \).

The macroscopic parallel permeability, \( \mu_{\parallel, sm, h} = \langle B_x \rangle/\langle H_x \rangle \), for the hexagonal array of soft magnetic strips is obtained from (4.4), (4.8), and (4.10), as

\[ \frac{\mu_{\parallel, sm, h}}{\mu_0} = \frac{a}{2b} \frac{K(\sqrt{1 - \kappa_h^2})}{K(k_h)} \]  

(4.11)

where \( k_h \) is given by (3.22). Note that \( \mu_{\perp, sm, h} \) given by (3.24) and \( \mu_{\parallel, sm, h} \) given by (4.11) hold the simple relationship \( \mu_{\parallel, sm, h} = \mu_{\perp, sm, h}^2/\mu_{\perp, sm, h} \). Figure 6 shows plots of \( \mu_{\parallel}/\mu_{\perp, sm, h} \) versus \( w/a \) obtained from (3.16), (3.17), (3.22), and (3.24). We can obtain a large parallel permeability, \( \mu_{\parallel, sm, h}/\mu_0 \gg 1 \), for a wide range of \( 0.5 < w/a < 1 \), when \( b/a \ll 1 \).

5. Hybrid arrays of superconducting and soft magnetic strips

We investigate the magnetic field distribution and macroscopic permeability of 2D arrays composed of both superconducting strips and soft magnetic strips. Here we consider the case when both superconducting strips and soft magnetic strips are parallel to the \( xz \) plane.

5.1. Rectangular array of superconducting and soft magnetic strips

We consider the hybrid array shown in figure 11, which is composed of the rectangular array of superconducting strips shown in figure 1 and the rectangular array of soft magnetic strips shown in figure 7.

The complex field \( \mathcal{T}(\zeta) \) for the hybrid rectangular array of superconducting and soft magnetic strips is given by

\[ \mathcal{T}(\zeta) = H_{0y} \frac{\eta_y(\zeta)}{\sqrt{\eta_y(\zeta)^2 - \gamma_{rs}^2}} + iH_{0x} \frac{\eta_y(\zeta - a)}{\sqrt{\eta_y(\zeta - a)^2 - \gamma_{rm}^2}} \]  

(5.1)

We do not investigate the case when soft magnetic strips are vertical to superconducting strips, because the anisotropy in the macroscopic permeability is weak for such vertical hybrid arrays.
Equation (5.4) is not accurate near \( w_s + w_m \leq a \). The solid horizontal bars show the cross sections of the superconducting (SC) strips and soft magnetic (SM) strips in the \( xy \) plane. The hybrid array is a combination of the rectangular array of superconducting strips and soft magnetic strips in figure 1 and the rectangular array of soft magnetic strips shown in figure 7. The origin of the soft magnetic strip array is shifted from \((x, y) = (0, 0)\) to \((a, 0)\).

where \( \gamma_{rs} = \eta_l(w_s) = \text{sn}(w_s/c_1, k_1), \gamma_m = \eta_l(w_m - a) = \text{cn}(w_m/c_1, k_1)/\text{dn}(w_m/c_1, k_1) \), and \( \eta_l(z) \) is given by (3.1). Equation (5.1) corresponds to the combination of (3.4) and (4.1). In a perpendicular magnetic field, \( H_{0x} = 0 \neq H_{0y} \), the field distribution is determined by the arrangement of the superconducting strips, and is not affected by the thin soft magnetic strips. In a parallel magnetic field, \( H_{0y} = 0 \neq H_{0x} \), on the other hand, the field distribution is determined by the arrangement of the soft magnetic strips, and is not affected by the thin superconducting strips.

The resulting macroscopic permeability for a perpendicular field \( \mu_{hyb, r} \parallel \) and that for a parallel field \( \mu_{hyb, r} \| \) are respectively given by

\[
\mu_{hyb, r} = \frac{b}{a} \frac{K(\kappa_{rs})}{K(\sqrt{1 - \kappa_{rs}^2})}, \quad (5.2)
\]

\[
\mu_{hyb, r} = \frac{a}{b} \frac{K(\sqrt{1 - \kappa_{m}^2})}{K(\kappa_{m})}, \quad (5.3)
\]

where \( \kappa_{rs} = k_c \text{cn}(w_s/c_1, k_1)/\text{dn}(w_s/c_1, k_1) \) and \( \kappa_{m} = k_c \text{cn}(w_m/c_1, k_1)/\text{dn}(w_m/c_1, k_1) \). For small stack periodicity, \( b/a \ll 1 \), (5.2) and (5.3) reduce to

\[
\frac{\mu_{hyb, r}}{\mu_0} \approx 1 - \frac{w_s}{a} + \frac{2b}{\pi a} \ln 2, \quad (5.4)
\]

\[
\frac{\mu_{hyb, r}}{\mu_0} \approx \left( 1 - \frac{w_m}{a} + \frac{2b}{\pi a} \ln 2 \right)^{-1}. \quad (5.5)
\]

Equation (5.4) is not accurate near \( w_s/a \approx 0 \) or 1, and (5.5) is not accurate near \( w_m/a \approx 0 \) or 1. If \( w_s = w_m = a/2 \) and \( b/a \ll 1 \), then \( \mu_{hyb, r}/\mu_0 \approx \mu_0/\mu_{hyb, r} \approx 1/2 \). Figure 12 shows plots of \( \mu_{hyb, r}/\mu_0 \) and \( \mu_{hyb, r}/\mu_{0} \) versus \( w_s/a = 1 - w_m/a \) for the case where \( w_s + w_m = a \), calculated from (5.2) and (5.3).

5.2. Hexagonal array of superconducting and soft magnetic strips

We next consider the hybrid array shown in figure 13, which is composed of the hexagonal array of superconducting strips shown in figure 4 and the hexagonal array of soft magnetic strips shown in figure 9.

The complex field \( \mathcal{H}(\zeta) \) for the hybrid hexagonal array of superconducting and soft magnetic strips is given by

\[
\mathcal{H}(\zeta) = H_{0y} - \eta_h(\zeta) \sqrt{\eta_h(\zeta)^2 - k_h^2} - \frac{\eta_h(\zeta - a) \sqrt{\eta_h(\zeta - a)^2 - k_h^2}}{\sqrt{\eta_h(\zeta - a)^2 - \gamma_{hs}^2 \eta_h(\zeta - a)^2 - \beta_{hs}^2}}, \quad (5.6)
\]

where \( \gamma_{hs} = \eta_l(w_s) = \text{sn}(w_s/c_h, k_h) \), \( \gamma_{hm} = \eta_l(w_m - a) = \text{cn}(w_m/c_h, k_h)/\text{dn}(w_m/c_h, k_h) \), and \( \eta_l(z) \) is given by (3.15). Equation (5.6) corresponds to the combination of (3.18) and (4.8). In a perpendicular magnetic field, \( H_{0x} = 0 \neq H_{0y} \), the field distribution is determined by the arrangement of the superconducting strips, and is not affected by the thin soft magnetic strips. In a parallel magnetic field, \( H_{0y} = 0 \neq H_{0x} \), on the other hand, the field distribution is determined by the arrangement of the soft magnetic strips, and is not affected by the thin superconducting strips.
The resulting macroscopic permeability for a perpendicular field, \( \mu_{\perp, \text{hyb}} \), and that for a parallel field, \( \mu_{||, \text{hyb}} \), are given by

\[
\frac{\mu_{\perp, \text{hyb}}}{\mu_0} = \frac{2b}{a} \frac{K(k_{\text{hs}})}{K(\sqrt{1 - k_{\text{hs}}^2})},
\]

for a perpendicular field, and

\[
\frac{\mu_{||, \text{hyb}}}{\mu_0} = \frac{a}{2b} \frac{K(\sqrt{1 - k_{\text{hm}}^2})}{K(k_{\text{hm}})},
\]

for a parallel field, respectively, where

\[
k_{\text{hs}} = \left[ 1 + \frac{k_{m}^{-2} - 1}{\text{cn}^4(w_s/c_n, k_{\text{hs}})} \right]^{-1/2},
\]

\[
k_{\text{hm}} = \left[ 1 + \frac{k_{m}^{-2} - 1}{\text{cn}^4(w_m/c_n, k_{\text{hm}})} \right]^{-1/2}.
\]

For a small stack periodicity, \( b/a \ll 1 \), (5.7) and (5.8) reduce to

\[
\frac{\mu_{\perp, \text{hyb}}}{\mu_0} \approx \begin{cases} 
1 - \frac{2w_s}{a} + \frac{8b}{\pi a} \ln 2 & \text{for } 0 < w_s/a < 1/2, \\
\left( \frac{2b}{a} \right)^2 \left( \frac{2w_s}{a} - 1 + \frac{8b}{\pi a} \ln 2 \right)^{-1} & \text{for } 1/2 < w_s/a < 1,
\end{cases}
\]

\[
\frac{\mu_{||, \text{hyb}}}{\mu_0} \approx \begin{cases} 
1 - \frac{2w_m}{a} + \frac{8b}{\pi a} \ln 2 & \text{for } 0 < w_m/a < 1/2, \\
\left( \frac{a}{2b} \right)^2 \left( \frac{2w_m}{a} - 1 + \frac{8b}{\pi a} \ln 2 \right)^{-1} & \text{for } 1/2 < w_m/a < 1.
\end{cases}
\]

Equation (5.11) is not accurate near \( w_s/a \simeq 0, 1/2, \) or 1, and (5.12) is not accurate near \( w_m/a \simeq 0, 1/2, \) or 1. If \( w_s = w_m = a/2 \) and \( b/a \ll 1 \), then \( \mu_{\perp, \text{hyb}} / \mu_0 \simeq \mu_{||, \text{hyb}} / \mu_0 \simeq 2b/a \ll 1 \). Figure 14 shows plots of \( \mu_{\perp, \text{hyb}} / \mu_0 \) and \( \mu_{||, \text{hyb}} / \mu_0 \) versus \( w_s/a = 1 - w_m/a \) for the case where \( w_s + w_m = a \), calculated from (5.7) to (5.8).

6. Discussion and summary

One of the most interesting applications of dc magnetic metamaterials is magnetic cloaking. We explore the possibility of dc magnetic cloaking with a cylindrical tube of the magnetic metamaterial occupying the region \( R_1 < \rho < R_2 \), where \( R_1 \) and \( R_2 \) are the inner and outer radii, respectively, and \( (\rho, \theta, z) \) denotes the cylindrical coordinates. When the metamaterial tube is exposed to a transverse magnetic field which is perpendicular to the \( z \) axis, the magnetic field inside the metamaterial tube \( (0 < \rho < R_1) \) should be zero, whereas the magnetic field outside the tube \( (\rho > R_2) \) should be undisturbed. This cylindrical cloaking can be achieved when the radial and azimuthal permeabilities are respectively given by [4–7]

\[
\mu_{\rho}/\mu_0 = 1 - R_1/\rho, \quad \mu_0/\mu_0 = (1 - R_1/\rho)^{-1}. \tag{6.1}
\]

Therefore, anisotropic permeabilities where \( 0 < \mu_{\rho}/\mu_0 < 1 < \mu_0/\mu_0 < \infty \) and \( \mu_{\rho}/\mu_0 = \mu_0/\mu_0 \) are required; the hybrid hexagonal array of superconducting and soft magnetic strips investigated in section 5.2 may achieve this. If
superconducting strips and soft magnetic strips are arranged such that their wide surfaces are perpendicular to the radial direction of the cylindrical metamaterial tube, the permeabilities should follow $\mu_\perp = \mu_\rho$ and $\mu_\parallel = \mu_0$. Equations (5.7) and (5.8) show that $0 < \mu_\perp/\mu_0 < 1 < \mu_\parallel/\mu_0 < \infty$ and $\mu_\perp/\mu_0 = \mu_\rho/\mu_1$, for superconducting and soft magnetic strips of identical widths, $w_s = w_m$. By adjusting the width, $w_s = w_m$, and the array periodicity, $a$, as a function of $\rho$, (6.1) can be satisfied approximately. However, the magnetic cloaking would be incomplete, because of the $\mu_\perp \to 0$ and $\mu_\parallel \to \infty$ singularities at $\rho \to R_1$ in (6.1). Other types of magnetic cloaking device, composed of superconductor–magnet bilayers which avoid these singular permeabilities, have also been proposed and experimentally verified [7–9].

We have theoretically investigated the field distribution in infinite 2D arrays of thin superconducting and soft magnetic strips, which are essential structures for dc magnetic metamaterials. The geometry of the thin strips produced anisotropy in the macroscopic permeability, $\mu_\perp$, when the applied magnetic field was perpendicular to the wide surface of the strips, and in $\mu_\parallel$, when it was parallel. The macroscopic permeability of the 2D arrays of superconducting strips showed that $0 < \mu_\perp/\mu_0 < \mu_\parallel/\mu_0 \simeq 1$. The behavior of the soft magnetic strips was analogously dual to that of the superconducting strips, and the macroscopic permeability of the 2D arrays of the soft magnetic strips showed that $1 \simeq \mu_\perp/\mu_0 \ll \mu_\parallel/\mu_0$. Hybrid arrays of the superconducting and soft magnetic strips exhibited strongly anisotropic macroscopic permeability, $0 < \mu_\perp/\mu_0 \ll \mu_\parallel/\mu_0$. We have also investigated two array configurations, showing that hexagonal arrays were better for producing strongly anisotropic permeability than rectangular arrays.

We adopted simple models for superconductors and soft magnets; the magnetic field was completely shielded in the superconductors, and the soft magnets had an infinite permeability, zero hysteresis, and an infinite saturation field. More realistic models of superconductors and soft magnets could be investigated by numerical simulations [3, 10, 20]. We focused on two-dimensional arrays of strips that have infinite length $L_z \to \infty$ along the $z$ axis. Three-dimensional rectangular arrays of superconducting square plates (i.e., $L_z = 2w$ in our notation) were numerically investigated by Navau et al [3], who showed that the lower limit of the macroscopic permeability is $\mu_\perp/\mu_0 = 1 - (w/a)^2$, in contrast to the lower limit $\mu_\perp/\mu_0 = 1 - w/a$ for the two-dimensional rectangular array, shown as the chained line in figure 3. Numerical simulation for such realistic three-dimensional arrays of superconducting and soft magnetic plates should also be investigated as future works. Furthermore, the details of magnetic metamaterial design should be investigated for magnetic cloaking and other possible applications.

Appendix. Macroscopic relationship between $(B)$, $(H)$, and $(M)$

In this appendix we examine the definition of the macroscopic magnetic induction $(B)$ and that of the macroscopic magnetic field $(H)$ to be consistent with the macroscopic relationship between $(B)$, $(H)$, and the magnetization $(M)$.

A.1. Rectangular array of superconducting strips

Because the current density in superconducting strips is given by $j_z = \partial H_y/\partial x - \partial H_x/\partial y$, the magnetization of superconducting strips $(M_s)$ defined by (3.11) is calculated as

$$4ab(M_s) = - \int_{-a}^{+a} \, dx \int_{-b}^{+b} \, dy \left( \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right)$$

$$= - \int_{0}^{2b} \, dy \left( \int_{-a}^{+a} \, x \frac{\partial H_y}{\partial x} \, dx \right)$$

$$+ \int_{-a}^{+a} \, dx \left( \int_{-b}^{+b} \, \frac{\partial H_x}{\partial y} \, dy \right)$$

$$= - \int_{0}^{2b} \, dy \left[ 2aH_y(a, y) - \int_{-a}^{+a} \, H_y(x, y) \, dx \right]$$

$$+ \int_{-a}^{+a} \, dx \left[ H_x(x, 2b - \epsilon) - H_x(x, -\epsilon) \right],$$

(A.1)

where we used $H_y(-a, y) = H_y(a, y)$. For the rectangular array of superconducting strips, substitution of $H_x(x, 2b - \epsilon) = H_x(x, -\epsilon)$ into (A.1) yields

$$\langle M_s \rangle = - \frac{1}{2b} \int_{0}^{2b} \, H_y(a, y) \, dy$$

$$+ \frac{1}{4ab} \int_{0}^{2b} \, dy \left( \int_{-a}^{+a} \, H_y(x, y) \, dx \right).$$

(A.2)

Using (3.8) and (3.9), we verify that (A.2) corresponds to (3.10). In other words, the definitions of (3.8) and (3.9) are consistent with (3.10).

A.2. Hexagonal array of superconducting strips

For the hexagonal array of superconducting strips, the boundary condition of $H_y(x, -\epsilon) = -H_y(a - x, 2b - \epsilon)$ leads to

$$\int_{-a}^{+a} \, x[H_y(x, 2b - \epsilon) - H_y(x, -\epsilon)] \, dx$$

$$= 2 \int_{a}^{0} \, x[H_y(x, 2b - \epsilon) + H_y(a - x, 2b - \epsilon)] \, dx$$

$$= 2a \int_{0}^{a} \, H_y(x, 2b - \epsilon) \, dx.$$  

(A.3)

Equation (A.1) is also valid for the hexagonal array of superconducting strips, and substitution of (A.3) into (A.1) yields

$$\langle M_s \rangle = - \frac{1}{2b} \left[ \int_{0}^{2b} \, H_y(a, y) \, dy - \int_{0}^{a} \, H_x(x, 2b - \epsilon) \, dx \right]$$

$$+ \frac{1}{4ab} \int_{0}^{2b} \, dy \left( \int_{-a}^{+a} \, H_y(x, y) \, dx \right).$$

(A.4)

Using (3.8) and (3.23), we verify that (A.4) corresponds to (3.10). In other words, the definitions of (3.8) and (3.23) are consistent with (3.10).
A.3. Rectangular array of soft magnetic strips

Because the effective magnetic charge density in soft magnetic strips is given by \( \rho_m = \mu_0 (\partial H_x / \partial x + \partial H_y / \partial y) \), the magnetization of soft magnetic strips \( \langle M_s \rangle \) defined by (4.6) is calculated as

\[
4ab \langle M_s \rangle = \int_{-a}^{+a} dx \int_{-\epsilon}^{+\epsilon} dy \left( \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} \right)
= \int_{-a}^{+a} dy \int_{-\epsilon}^{+\epsilon} x dx \left( \frac{\partial H_x}{\partial x} \right)
+ \int_{-\epsilon}^{+\epsilon} x dx \int_{-a}^{+a} \frac{\partial H_y}{\partial y} dy
= \int_{-a}^{+a} dy \left[ 2a H_x(a, y) - \int_{-a}^{+a} H_x(x, y) dx \right]
+ \int_{-\epsilon}^{+\epsilon} x dx \left[ H_y(x, 2b - \epsilon) - H_y(x, -\epsilon) \right],
\]

(A.5)

where we used \( H_x(-a, y) = H_x(a, y) \). For the rectangular array of soft magnetic strips, substitution of \( H_x(x, 2b - \epsilon) = H_y(x, -\epsilon) \) into (A.5) yields

\[
\langle M_s \rangle = \frac{1}{2b} \int_{-\epsilon}^{+\epsilon} dy \int_{-a}^{+a} H_x(a, y) dx
= -\frac{1}{4ab} \int_{-\epsilon}^{+\epsilon} dy \int_{-a}^{+a} H_x(x, y) dx.
\]

(A.6)

Using (4.3) and (4.4), we verify that (A.6) corresponds to (4.5). In other words, the definitions of (4.3) and (4.4) are consistent with (4.5).

A.4. Hexagonal array of soft magnetic strips

For the hexagonal array of soft magnetic strips, the boundary condition of \( H_x(x, -\epsilon) = -H_x(a - x, 2b - \epsilon) \) leads to

\[
\int_{-a}^{+a} x \left[ H_x(x, 2b - \epsilon) - H_x(x, -\epsilon) \right] dx
= 2 \int_{-a}^{a} x \left[ H_x(x, 2b - \epsilon) + H_x(a - x, 2b - \epsilon) \right] dx
= 2a \int_{-a}^{a} H_x(x, 2b - \epsilon) dx.
\]

(A.7)

Equation (A.5) is also valid for the hexagonal array of soft magnetic strips, and substitution of (A.7) into (A.5) yields

\[
\langle M_s \rangle = \frac{1}{2b} \left[ \int_{-\epsilon}^{+\epsilon} dy \int_{-a}^{+a} H_x(a, y) dx + \int_{-a}^{+a} dy \int_{-\epsilon}^{+\epsilon} H_x(x, 2b - \epsilon) dx \right]
- \frac{1}{4ab} \int_{-\epsilon}^{+\epsilon} dy \int_{-a}^{+a} H_x(x, y) dx.
\]

(A.8)

Using (4.4) and (10), we verify that (A.8) corresponds to (4.5). In other words, the definitions of (4.4) and (10) are consistent with (4.5).

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