I. INTRODUCTION

The late fifties saw the emergence of the wormhole in the seminal papers of Misner and Wheeler [1] and Wheeler [2,3]. Electric charge was claimed to be a manifestation of the topology of a space which essentially looked like a sheet with a handle. This was given the name 'wormhole'. In spite of the elegance and simplicity of the idea, interest in the Wheeler wormhole declined over years primarily because of the rather ambitious nature of the programme which unfortunately had little connection with reality or support from experiment.

The wormhole remained dormant over the years with some isolated pieces of interesting work such as the one by Ellis [4] appearing once in a while.

The year 1988 saw a resurgence of interest in 'wormholery'—once again in terms of a bunch of new, exotic ideas which seems to have remained alive for quite some time now. Two separate directions emerged—one concerning Euclidean signature metrics and the other related to the Lorentzian ones.

Euclidean wormholes arose in the context of Euclidean quantum gravity [5,6]. The focus here was to construct a viable model for topology change where the transition amplitude could be evaluated at least in the saddle-point approximation. Giddings and Strominger [5] accomplished this task by deriving an exact wormhole solution of the Euclidean Einstein equations with matter in the form of an antisymmetric tensor field of rank three. They also showed that the transition amplitude for a topology changing process from $\mathbb{R}^3$ to $\mathbb{R}^3 \oplus S^3$ is significant only when the size of the wormhole throat is of the order of Planck length.

On the other hand, interest in Lorentzian wormholes was stimulated by the important paper of Morris and Thorne [7] where static, spherically symmetric Lorentzian wormholes were first defined and analysed in great detail with emphasis on their geometry and matter content. Soon after Morris, Thorne and Yurtsever (MTY) [8] (and almost simultaneously Novikov [9]) constructed a model time machine using static, spherically symmetric traversable wormholes. Since the publication of the MTY paper the existence/nonexistence of a time–machine (theoretically!) has been an issue of major controversy. Questions such as the classical mechanics of billiard balls [10] and the Cauchy problem for the scalar wave equation in the presence of closed timelike curves [11,12] have been addressed in detail. More interestingly, Kim and Thorne [13] and Hawking [14] have discussed at length whether vacuum polarization effects can prohibit the occurrence of closed timelike curves.

However, one major problem with the existence of traversable wormholes and time machines is that the matter required to support such a geometry essentially violates the Energy Conditions of General Relativity (GR) [8,7]. One can restrict such violation to an infinitesimally small thin shell if one adopts Visser’s approach towards wormhole construction [15]. In fact, Visser [16] has also shown that for non spherically symmetric, static wormholes one can have many null geodesics along which the Averaged Null Energy Condition is satisfied.

On the other hand, as shown recently by one of us here, if one shifts attention to nonstatic geometries one finds that WEC violation can be avoided for arbitrarily small or large intervals of time [17]. In this paper, we shall demonstrate in greater detail, through various arguments and examples that within Classical General Relativity there exist Lorentzian wormholes which are nonstatic and which do not require WEC violating matter to support them. These wormholes, as will turn out, exist for a finite (but arbitrarily small or large) time interval and
represent evolving geometries. During its evolution the shape of the wormhole changes in the embedding space—the throat radius expands or contracts and the rate of change of the embedding function increases or decreases. One can draw an analogy between these geometries and that of the usual FRW universe \((k = 1)\). The spacelike sections of the former are topologically \(R \otimes S^2\) while those of the latter are \(S^1\). In the spirit of this, one can therefore think of these spacetimes as constituting ‘wormhole universes’. Other papers which deal with evolving wormholes are due to Hochberg and Kephart and Roman. While the former discusses a possible resolution of the horizon problem using a network of dynamic (evolving) wormholes possibly present in the early universe, the latter considers an evolving geometry with an inflationary scale factor. Very recently, Wang and Letelier have constructed a class of evolving wormholes which they claim can be built out of matter satisfying both the Weak and Dominant Energy conditions. However, the approach towards constructing evolving wormholes that we discuss in this paper are somewhat different from theirs. It should also be mentioned that Visser in has discussed the stability of his wormholes by making the throat dynamic (i.e time dependent).

We state explicitly that these evolving wormholes though supported with normal matter do not necessarily violate the theorem of Topological Censorship due to Friedman, Schleich and Witt. Topological Censorship essentially states that asymptotically flat traversable wormhole spacetimes cannot exist if matter satisfies the Averaged Null Energy Condition (ANEC). More precisely, the theorem proves that there must be at least one null geodesic along which the ANEC is violated. In many of the spacetimes to be discussed here asymptotic flatness is not assumed. However, spacelike sections when embedded in \(R^3\) resemble two asymptotically flat regions connected by a bridge. We should remember that the essential features of a wormhole geometry are largely encoded in the spacelike section and in the condition for nonexistence of horizons \((g_{00} \neq 0)\). Moreover, our geometries do violate WEC in some interval of time but not always. In fact by suitably adjusting parameters one can make the timespan over which WEC is satisfied as large as one wants.

The paper is organised as follows. The next section deals with the proof of the fact that evolving wormholes can satisfy the WEC. In Section III we discuss some specific examples in \(2 + 1\) and \(3 + 1\) dimensions. Section IV illustrates a specific model of a wormhole in an FRW universe. Extra compact decaying dimensions present in the wormhole metric are dealt with in Section V where two specific models are presented. Finally, in Section VI we conclude with some remarks on future directions.

### II. EVOLVING WORMHOLES AND ENERGY CONDITIONS

We begin our analysis with the following ansatzen for the metric and the energy-momentum tensor.

\[
\begin{align*}
\rho(r,t) &= \frac{(D - 2)}{\Omega^2} \left[ \frac{(D - 1)}{2} \left( \frac{\dot{\Omega}}{\Omega} \right)^2 \right] \\
\tau(r,t) &= \frac{(D - 2)}{\Omega^2} \left[ \frac{\ddot{\Omega}}{\Omega} - \frac{(D - 5)}{2} \left( \frac{\dot{\Omega}}{\Omega} \right)^2 \right] \\
p(r,t) &= \frac{(D - 2)}{\Omega^2} \left[ \frac{-\ddot{\Omega}}{\Omega} - \frac{(D - 5)}{2} \left( \frac{\dot{\Omega}}{\Omega} \right)^2 \right] + \left[ \frac{-\ddot{\Omega} + (D - 3)(D - 5)b + (3 - D)b'r}{2r^3\Omega^2} \right]
\end{align*}
\]
The dots denote derivatives with respect to \( t \) and the primes derivatives with respect to \( r \). The WEC \((T_{\mu\nu}u^\mu u^\nu \geq 0)\) reduces to the following inequalities for the case of a diagonal energy-momentum tensor

\[
\rho \geq 0, \quad \rho + \tau \geq 0, \quad \rho + p \geq 0 \quad \forall (r, t)
\]  

(7)

From Eqns. (4), (5), (6) one can write down three inequalities which have to be satisfied if the WEC is not to be violated. These are

\[
\frac{(D - 2)}{\Omega^2} \left( \frac{(D - 1)}{2} \left( \frac{\dot{\Omega}}{\Omega} \right)^2 + \frac{b' r + (D - 4) b}{2r^3} \right) \geq 0 \quad (8)
\]

\[
\frac{(D - 2)}{\Omega^2} \left( -\frac{\dot{\Omega}}{\Omega} + 2 \left( \frac{\dot{\Omega}}{\Omega} \right)^2 - \frac{b - b' r}{2r^3} \right) \geq 0 \quad (9)
\]

\[
\frac{(D - 2)}{\Omega^2} \left[ \frac{\ddot{\Omega}}{\Omega} + 2 \left( \frac{\dot{\Omega}}{\Omega} \right)^2 + \frac{(2D - 7)b + b' r}{2r^3\Omega^2} \right] \geq 0 \quad (10)
\]

Several important facts should be noted here in comparison with the case of a static geometry. Eq. (8) is satisfied if \( b' \leq 0 \) and \( D \geq 4 \) irrespective of the geometry being time independent/dependent. However, if it is time dependent, then one can satisfy Eq (8) even for the case when \( b' \leq 0 \). For example, for \( D = 4 \) one obtains the inequality

\[
\frac{|b'|}{r^2} \leq 3 \left( \frac{\ddot{\Omega}}{\dot{\Omega}} \right)^2 \quad (11)
\]

For every \( t = \text{constant} \) slice Eq. (11) has to hold, which means

\[
\frac{|b'|}{r^2} \leq \min \left[ 3 \left( \frac{\ddot{\Omega}}{\dot{\Omega}} \right)^2 \right] \quad (12)
\]

where \( \min \) denotes the minimum value of the function in the given time interval.

For a static geometry Eq. (9) can never be satisfied, as shown by Morris and Thorne [7]. But, for a nonstatic geometry with \( b' \geq 0 \) one can satisfy Eq.(9). We require

\[
2 \left( \frac{\ddot{\Omega}}{\dot{\Omega}} \right)^2 - \frac{\ddot{\Omega}}{\dot{\Omega}} \geq \frac{b - b' r}{2r^3} \quad (13)
\]

Interestingly, Eq. (13) is the same in all dimensions. Stated explicitly Eq (13) implies that the value of \((b - b' r)/r^3\) for all \( r \) must be less than or equal to the minimum value of the function \( 2 \left( \frac{\ddot{\Omega}}{\dot{\Omega}} \right)^2 - \frac{\ddot{\Omega}}{\dot{\Omega}} \), in the corresponding domain of \( t \). However, we need

\[
F(t) = \left[ 2 \left( \frac{\ddot{\Omega}}{\dot{\Omega}} \right)^2 - \frac{\ddot{\Omega}}{\dot{\Omega}} \right] > 0 \quad (14)
\]

Eq (14) can be written in a more precise form by introducing a function \( \chi(t) = \Omega/\ddot{\Omega} \). We have

\[
\frac{d\chi}{dt} > (-1) \quad (15)
\]

With \( b' \leq 0 \), \( D \geq 4 \) and Eq (13) holding one clearly sees that Eq.(9) is satisfied. Therefore from this very simple analysis it is clear that nonstatic spherically symmetric Lorentzian wormhole geometries can exist with the required matter not violating the WEC. However, the fact that \( \Omega(t) \) be finite everywhere and must satisfy the condition Eq. (13) implies that these wormholes exist for finite intervals of time (arbitrarily small or large). For finite and bounded \( \Omega \) which is everywhere nonzero this can be proved as follows. The finiteness and boundedness of \( \Omega(t) \) implies that it can have only a specific class of functional forms. These include (i) functions which have no extremum and asymptotically approach the constant limiting values (e.g. \( A + B \tanh \omega t \)) (ii) functions which have one extremum and asymptotically approach constant limiting values (e.g. \( A + Be^{-\omega t^2} \)) (iii) oscillatory functions which may or may not approach the limiting values (e.g. \( \sin \omega t + a \)). In all the three cases where the functions asymptotically approach limiting values \( F(t) \) tends to zero at \( t \to \pm \infty \) and the WEC is violated. For a purely oscillatory function there exists more than one extremum and at the minimum \( F(t) \) is clearly negative. Thus WEC violations can be avoided only for finite intervals of time if one chooses a finite and bounded \( \Omega(t) \).

One can also derive the fact that the WEC will be violated for some interval of time by the following argument (this was communicated to us by Matt Visser [23]).

Consider the condition \( F(t) > 0 \). This can be written equivalently as

\[
F(t) = \Omega \frac{d^2}{dt^2} (\Omega^{-1}) > 0 \quad (16)
\]

Now assuming a well behaved \( \Omega \) we can evaluate the integral of \( F(t)\Omega^{-2} \). This gives

\[
\int_{-\infty}^{+\infty} F(t)\Omega^{-2} dt = -\int_{-\infty}^{+\infty} \left[ \frac{d}{dt}(\Omega^{-1}) \right] dt < 0 \quad (17)
\]

on integration by parts.

Since the integral is negative, the integrand must be negative somewhere. Therefore \( F(t) \) cannot be positive everywhere if \( \Omega \) is well behaved everywhere. Thus the WEC must be violated somewhere at least.

In \( 2 + 1 \) dimensions the WEC conditions turn out to be

\[
\frac{1}{\Omega^2} \left[ \frac{(\ddot{\Omega})^2}{\dot{\Omega}^2} + \frac{b + b' r}{2r^3} \right] \geq 0 \quad (18)
\]
\[
\frac{1}{\Omega^2} \left[ 2 \left( \frac{\Omega}{\dot{\Omega}} \right)^2 - \frac{\dot{\Omega}}{\Omega} + b' r - b \right] \geq 0 \quad (19)
\]

In order to satisfy both these inequalities we need the following to hold:

\[
d\left( \frac{\Omega}{\dot{\Omega}} \right) \leq \min[\rho(r,t)] \quad (20)
\]

\[
\min \left( \frac{\Omega}{\dot{\Omega}} \right)^2 \geq \frac{b - b'r}{2r^3} \quad (21)
\]

Some special exact solutions with simple equations of state will be discussed in next section.

We now move on to examine how the geodesic focussing arguments for WEC violation by static wormholes get modified once the latter are allowed to evolve. We recall that the expansion \( \theta \) of a congruence of null rays satisfies the Raychaudhuri equation (the rotation expression for \( \omega_{\mu \nu} \) comes to a focus. In the case of a static wormhole , the affine parameter, \( \tau \), is negative anywhere it has to go to a finite value of \( \tau \). Hence either the wormhole has a vanishing throat or the WEC is violated.

\[
\frac{d\theta}{d\lambda} + \frac{1}{2} \theta^2 = -R_{\mu \nu} \xi_\mu \xi_\nu - 2\sigma^2 \quad (22)
\]

Here \( \sigma_{\mu \nu} \) is the shear tensor of the geodesic bundle (which is zero for our case here). From Einstein’s equation we know that \( R_{\mu \nu} \xi^\mu \xi^\nu = T_{\mu \nu} \xi^\mu \xi^\nu \) for all null \( \xi^\nu \). So if \( T_{\mu \nu} \xi^\mu \xi^\nu \geq 0 \), \( \theta^{-1} \geq \theta^{-1} + \frac{1}{2} \alpha \) by (9). Thus if \( \theta \) is negative anywhere it has to go to \(-\infty\) at a finite value of the affine parameter, \( \lambda \), i.e. the bundle must necessarily come to a focus. In the case of a static wormhole , the expression for \( \theta \) is [23]

\[
\theta = 2\beta r' \quad (23)
\]

where \( \beta \) is a positive quantity and \( r' = \frac{dr}{d\tau} \) and \( l \) is defined by

\[
l(r) = \pm \int_{b_0}^{r} \frac{dr}{\sqrt{1 - \frac{\rho}{r}}} \quad (24)
\]

Thus if \( r' \) is negative anywhere, and it is so for \( l < 0 \), then so is \( \theta \). But \( \theta \to \infty \) only if \( r \to 0 \), since \( r' \) is always finite. Hence either the wormhole has a vanishing throat radius which is tantamount to its not being a wormhole at all, or the WEC is violated.

For the evolving case \( \theta \) is given by

\[
\theta = \frac{2\beta}{R(\tau)} \left( \frac{\dot{R}(\tau)}{R(\tau)} + r' \right) \quad (25)
\]

Here we have used real time \( \tau \) and the \( \dot{\tau} \) denotes differentiation with respect to \( \tau \). Thus as long as \( \dot{R} > \sqrt{\rho} \) i.e. the wormhole is opening out fast enough , \( \theta \) is never negative and the fact that the bundle does not focus no longer implies WEC violation. We emphasize that \( \theta > 0 \) while being a necessary condition for WEC conservation is not sufficient. To see this let us assume \( b(r) = \frac{\dot{b}}{b} \) and \( R(\tau) = \alpha \tau \), the scale factor for a Milne universe. For these choices of \( b(r) \) and \( R(\tau) \) the condition \( \rho \geq 0 \) leads to \( b_0^2 \alpha^2 \geq \frac{1}{3} \). However (18),(19) leads to \( b_0^2 \alpha^2 \geq \frac{1}{3} \) which is more stringent in the sense that if it be satisfied then the \( \rho \geq 0 \) inequality is trivially true. As for \( \theta \), \( \dot{\tau} \) passes through a minimum at \( l = \frac{1}{2 b_0} \) (note that \( r^2 (l) = b_0^2 + l^2 \)) while \( \dot{R} = \omega \). Clearly if \( \alpha > \frac{1}{\sqrt{2 b_0}} \), \( \theta > 0 \), as we would expect for WEC preservation. On the other hand, if we choose \( R(\tau) = e^{\alpha \tau} \), the inflationary scale factor, and require that \( \alpha > \frac{1}{2 b_0} \), then \( \theta > 0 \) but (19) is no longer satisfied as \( \frac{d\dot{R}}{\dot{\tau}} \neq 0 \) identically.

In other words, the focussing argument which is - ‘\( \theta < 0 \Rightarrow \text{WEC Violation for wormholes} \)’ is useful for picking out WEC violation , but not WEC conservation. For this it is necessary to examine \( T_{\mu \nu} \xi^\mu \xi^\nu \) explicitly as done above.

Before we construct explicit examples it is useful to discuss briefly the embedding in \( R^3 \) of a \( \theta = \pi/2 \), \( t = t_0 \) slice, where \( t_0 \) lies in the interval in which the wormhole exists. Since our geometry is nonstatic each such slice will be different - more precisely the value of the function \( \Omega(t) \) at \( t = t_0 \) will dictate the shape and features of this slice, which will thus change as we alter \( t_0 \). The metric on such a slice takes the form

\[
\tilde{ds}^2 = \Omega^2(t_0) \left[ \frac{dr^2}{1 - \frac{\rho(r,t)}{r}} + r^2 d\phi^2 \right] \quad (26)
\]

Define

\[
\tilde{r} = \Omega(t_0) r \quad (27)
\]

Thus the metric on the slice takes the form

\[
\tilde{ds}^2 = \frac{d\tilde{r}^2}{1 - a(\tilde{r}) \Omega(t_0)} + \tilde{r}^2 d\phi^2 \quad (28)
\]

where \( a(\tilde{r}) \) is the functional form of \( b(r) \) in the \( \tilde{r} \) coordinate. The minimum value of \( \tilde{r} \) which determines the throat radius is evaluated from

\[
a(b_0) \Omega(t_0) = b_0 \quad (29)
\]

This clearly shows the dependence on \( \Omega(t) \). Using the mathematics of embedding we can write the following differential equation for the spacelike slice at \( t = t_0 \).

\[
\frac{dz(\tilde{r})}{d\tilde{r}} = \pm \left[ \frac{a(\tilde{r}) \Omega(t_0)}{\tilde{r} - a(\tilde{r}) \Omega(t_0)} \right]^{\frac{1}{2}} \quad (30)
\]

where \( z(\tilde{r}) \) is the embedding function. Integrating (28) one can obtain the \( z(\tilde{r}) \) for the slice at \( t = t_0 \).
III. EXAMPLES

A. 2+1 Dimensions

As mentioned in the previous section, the 2 + 1 dimensional case is somewhat special. Here we present two exact evolving wormhole solutions. The important point to realise is that in 2 + 1 dimensions the \( \tau = p \) constraint is automatically satisfied.

We first discuss the case \( \tau = p = 0 \). From the field equations it turns out that the conformal factor \( \Omega(t) = \exp(\pm \omega t) \) is the only solution that satisfies this constraint. This scale factor in real time is the familiar Milne universe in 2 + 1 dimensions. The only WEC inequality we have to check here is the one for \( \rho \) (i.e. \( \rho \geq 0 \)). Choosing \( b(r) = b_0 \) we end up with a requirement

\[
b_0^2 \omega^2 \geq \frac{1}{2} \tag{31}
\]

On the other hand if we choose \( \tau = p = p_0 > 0 \) (where \( p_0 \) is a constant) the solution to the differential equation governing the evolution of the conformal factor is:

\[
\Omega(t) = \frac{1}{\cosh t \sqrt{p_0}} \tag{32}
\]

A requirement for WEC conservation is

\[
b_0 \frac{2}{\pi} \leq p_0 \tanh^2 \sqrt{p_0 t} \tag{33}
\]

Thus one has to allow WEC violation at least for a small interval \((-t_0, t_0)\) in the neighborhood of \( t = 0 \). The resulting constraint on the throat radius parameter \( b_0 \) becomes

\[
b_0^2 \geq \frac{1}{2p_0 \tanh^2 t_0 \sqrt{p_0}} \tag{34}
\]

B. 3+1 Dimensions

In all the examples to be discussed here we shall choose \( b(r) = b_0 \). The WEC imposes the following restriction on \( b_0 \):

\[
b_0^2 \geq \max \left( \frac{1}{F(t)} \right) \tag{35}
\]

This condition will restrict the minimum possible value of the throat radius of the wormhole. Other choices of \( b(r) \) lead to a modified version of the above stated condition and of the subsequent analysis.

(i) \( \Omega(t) = \frac{1}{e^{-\omega t}} \)

This choice of \( \Omega \) was first discussed by Roman \[19\]. It is essentially the case of an inflationary wormhole universe. Roman’s idea was to build a model of a wormhole of large size emerging out of inflation. One can check very easily that this choice of the scale factor leads to \( F(t) = 0 \), which in turn implies that the WEC will be violated for all times. More details about this scenario can be found in the paper by Roman \[19\].

(ii) \( \Omega(t) = \exp(\pm \omega t) \)

In a sense this case is also quite unique because it leads to \( F(t) = 2\omega^2 \), which is a constant. This scale factor is reminiscent of the one used for the Milne universe. In real time it represents linearly expanding spacelike slices. The WEC is satisfied if and only if the following inequality holds:

\[
b_0^2 \omega^2 \geq \frac{1}{2} \tag{36}
\]

The time interval for which the wormhole can exist (assuming that it can grow very large) is \(-\infty < t \leq \infty\) for the \( \exp(\omega t) \) case and \(-\infty \leq t < \infty\) for the \( \exp(-\omega t) \) case. Note that one of the infinities is excluded from each of the intervals because the spacetime collapses into a singularity there.

(iii) \( \Omega(t) = \sin \omega t \)

The scale factor is the same as the one that arises in the closed FRW cosmology which starts with a ‘bang’ and ends with a ‘crunch’. Instead of the usual \( S^3 \) space-like sections we have wormhole metrics on \( R \otimes S^2 \). The expression for \( F(t) \) turns out to be

\[
F(t) = 2\omega^2(2\cot^2 \omega t + 1) \tag{37}
\]

One can easily check that \( F(t) \) has a minimum at \( \omega t = \frac{\pi}{2} \). Hence the constraint on the allowed values of \( b_0 \) turns out to be the same as in the previous case i.e. \( b_0^2 \omega^2 \geq \frac{1}{2} \). One can also say that the lifetime of this wormhole universe is \( \frac{\pi}{\omega} \). The time interval for which this universe can exist without collapsing into a singularity is \( \frac{\pi}{\omega} < t < \frac{(m+1)\pi}{\omega} \).

(iv) \( \Omega(t) = (\omega t)^\nu \quad \nu \text{ integral or fractional} \)

This case is important because the scale factors that arise in the dust-filled or radiation dominated FRW cosmologies with flat spacelike sections are obtained by considering special values of the \( \nu \) used above. We shall deal with these special cases later in a separate section.

For general \( \nu \) the expression for \( F(t) \) is given as

\[
F(t) = \frac{2\nu(\nu + 1)}{t^2} \tag{38}
\]

Therefore as \( t \to \pm \infty \) \( F(t) \to 0 \). The constraint on \( b_0 \) turns out to be dependent on \( t \).

\[
b_0^2 \geq \max \left( \frac{t^2}{2\nu(\nu + 1)} \right) \tag{39}
\]

Thus the evolving wormhole with this type of scale factor can exist only for a finite interval of time (similar to
the previously discussed cases). The lower bound on \( b_0 \) is decided by the maximum time \( t \) up to which we wish the wormhole to exist with the matter threading the geometry satisfying the WEC. Beyond this time the matter will violate the WEC. As we shall see later this condition sets the most stringent restriction on the size of the wormhole.

We now move on to a special class of scale factors which exhibit ‘flashes’ of WEC violation. This essentially means that the matter threading the wormhole violates the Energy Conditions only for small intervals of time and is normal at all other times. Whether such matter is physically possible is as yet unknown. However, the intervals during which the WEC is violated can be chosen to be very small (this obviously leads to a constraint on the throat radius). We shall deal with two representative examples. Many more can constructed without much of a problem.

(a) \( \Omega(t) = \sin \omega t + a \quad a > 1 \)

This scale factor reminds us of the ‘bounce’ type solutions in cosmology which were constructed in order to avoid the big–bang singularity. For a nonstatic wormhole with the scale factor chosen as above the function \( F(t) \) turns out to be:

\[
F(t) = 2\omega^2 \left[ \frac{2 - \sin^2 \omega t + a \sin \omega t}{(\sin \omega t + a)^2} \right]
\]  

This function is plotted in Fig 1.

Note that \( F(t) \) goes to zero in the vicinity of \( \omega t = 3\pi/2 \) stays negative for a while and then subsequently becomes positive again. This happens once in every period, i.e at \( \omega t \) values given by \((2n + 1) \pi/2\). Thus the WEC will be violated during those intervals of time. The time domain over which the WEC is violated is, as mentioned before, related to the throat radius parameter of the wormhole. One chooses this interval in the following way. Assume a minimum, positive and finite value of \( F(t) \) say \( F_0 \). In the neighborhood of the point where \( F(t) \) is zero, this value occurs at points \( \omega t_0 - \delta \) and \( \omega t_0 + \delta \). If the shape function is assumed to be constant \( (b(r) = b_0) \) then the relation between the throat radius parameter and the minimum value of \( F(t) \) turns out to be:

\[
b_0^2 \geq \frac{1}{F_0}
\]

Hence the interval during which the WEC is violated is \( 2\delta \). The smaller this time interval the larger the minimum allowed value of \( b_0 \).

(b) \( \Omega(t) = \frac{t^2 + a^2}{t^2 + b^2} \quad b > a > 0 \)

In this case also the scale factor is such that the geometry is never singular. Asymptotically it becomes a static wormhole. The function \( F(t) \) turns out to be:

\[
F(t) = \frac{4(b^2 - a^2)(3t^2 - a^2)}{(t^2 + b^2)(t^2 + a^2)^2}
\]

This function is shown plotted in Fig 2. At \( t = \pm a/\sqrt{3} \) \( F(t) = 0 \). For \(-a/\sqrt{3} < t < a/\sqrt{3} \) \( F(t) \) is negative. One can carry out an analysis similar to the one for (a)–the only difference being that WEC violation occurs here only in the neighborhood of the interval mentioned above and at \( \pm \infty \).

Further examples can be constructed by choosing other forms of \( \Omega(t) \). Two worth mentioning follow from constraints on the matter stress energy for an evolving wormhole geometry. For the perfect fluid with \( \rho = \frac{\tilde{F}}{2} \) we end up with the scale factor of a closed FRW universe while for traceless matter in general i.e matter obeying only \(-\rho + \tau + 2p = 0 \) we get a linear \( \Omega(t) \) i.e \( \Omega(t) = at + b \).

**IV. A WORMHOLE IN A FLAT FRW UNIVERSE**

A probable realisation of an evolving wormhole could be obtained by thinking of it as part of an asymptotically FRW universe i.e. by imagining the asymptotically flat parts of an evolving wormhole geometry as constituting a flat \( (k = 0) \) FRW spacetime. Mathematically one therefore chooses a metric which represents an evolving wormhole with the scale factor identical to that of either the matter or the radiation dominated FRW model. Thus

\[
ds^2 = -d\tau^2 + \tau^n \left( \frac{dr^2}{1 - \frac{b(r)}{\tau}} + r^2 d\Omega^2 \right)
\]

where we have changed our time coordinate from the conformal time used in the earlier discussions to real time. The exponent \( n \) takes on the values \( \frac{2}{3} \) and \( \frac{2}{5} \) for the radiation and matter dominated cases of the flat FRW universes respectively. The Einstein equations lead to the following expressions for the matter density and the pressures.

\[
\rho(r, \tau) = \frac{b'}{(r\tau)^{2n}} + \frac{3n^2}{(\tau r)^2}
\]

\[
p_1(r, \tau) = -\frac{b}{( \tau)^{2n} - \frac{n(3n - 2)}{(\tau r)^2}}
\]

\[
p_2(r, \tau) = \frac{b - b' r}{2r^3(\tau)^{2n}} - \frac{n(3n - 2)}{(\tau r)^2}
\]

The WEC inequality \( \rho + p_1 \geq 0 \) reduces to the following:

\[
\frac{b' - b}{2r^3(\tau)^{2n}} + \frac{2n}{(\tau r)^2} \geq 0
\]

As expected all the essential properties of the matter stress energy of the flat FRW model follow from the expressions for \( \rho, p_1 \) and \( p_2 \). As \( r \to \infty \), only the \( \tau \)-dependent terms survive, and we get \( \rho = \rho_{FRW} = \frac{1}{3(\tau r)^2} \),
\[ p_1 = p_2 = 0 \] for dust and \( q = p_1 = p_2 = \frac{1}{\gamma(\tau)} \) for pure radiation.

Secondly, the pressures become increasingly disparate as we approach the throat \( (p_2 - p_1 = \frac{3\rho_c}{2(\tau)}b_s \geq 0) \). From the geometrical viewpoint this is because the curvature in the \( \theta \phi \) directions is different from that in the \( r \) direction. Thus, inside the wormhole, matter is subject to anisotropic stresses and to remain in equilibrium it must respond by generating anisotropic strains.

Lastly let us look at the WEC condition. For concreteness and simplicity we choose \( b(r) = b_0 \). From the WEC inequality \( p + p_1 \geq 0 \) we arrive at the condition \( b_0 R(\tau) \geq \frac{1}{\sqrt{2n}} \). This quite simply leads to the statement that the throat radius \( b_0 R(\tau) \) at time \( \tau \) must exceed the horizon size \( c \tau \) (upto a constant factor given by \( \sqrt{1/2n} \) which is of \( O(1) \), being exactly equal to 1 and \( \sqrt{3} \) for pure radiation and dust respectively). Now for an FRW evolution the horizon always moves faster than the scale \( R(\tau) \) and hence eventually overtakes \( b_0 R(\tau) \), causing WEC violation to begin at the throat and slowly spread outward. Furthermore, since wormholes presumably arise through quantum gravitational processes, they can reasonably be assumed to form in the Planckian era, with a radius which is typically of Planck size at Planck time. If their subsequent evolution is FRW-like, WEC violation occurs within a few Planck times. To avoid this, the wormhole must inflate to a size much larger than the horizon. This is precisely what happens in the early universe (inflationary epoch). However, as we have shown earlier an inflationary scale factor leads to WEC violation at all times.

V. THE CASE OF COMPACT EXTRA DIMENSIONS

A. Exponential Inflation

It is well known from the work of Roman \(^{13} \) and from the analysis presented in the previous sections that with an inflationary scale factor one cannot avoid the violation of the WEC even for a finite interval of time. This is somewhat depressing, because if one believes in the existence of wormholes, then one possible way in which they can appear on macroscopic scales is by growing very large during the inflationary epoch. We now demonstrate through an example that with compact extra dimensions \( (S^2) \) an inflationary wormhole can exist for a finite interval of time with matter satisfying the WEC.

The assumption for the metric is:

\[ ds^2 = -dt^2 + a_1^2(t) \left( \frac{dr^2}{1 - b/r} + r^2 d\Omega_2^2 \right) + a_2^2(t) \left( d\chi^2 + \sin^2 \chi d\xi^2 \right) \]

where \( a_1(t) \) and \( a_2(t) \) are the scale factors for the wormhole and the compact extra dimensions respectively.

We assume

\[ a_1(t) = e^{\omega t}, \quad a_2(t) = (t_0 - t)^{1/2}, \quad b(r) = b_0 \]

Instead of writing down the Einstein equation in all its glory, we straightaway move on to the WEC inequalities. \( \rho, \tau, p_1, p_2, p_3 \) and \( p_4 \) are the six non-zero diagonal components of the energy momentum tensor. We have essentially four inequalities as \( p_1 = p_2 \) and \( p_3 = p_4 \) by symmetry.

\[ \rho \geq 0 \Rightarrow 3\omega^2 + \frac{1 - 3\omega}{t_0 - t} + \frac{1}{4(t_0 - t)^2} \geq 0 \]

\[ \rho + \tau \geq 0 \Rightarrow -\frac{b_0 e^{-2\omega t}}{r^3} + \frac{2}{(t_0 - t)} \left[ -\omega + \frac{1}{2(t_0 - t)} \right] \geq 0 \]

\[ \rho + p_1 \geq 0 \Rightarrow \frac{b_0 e^{-2\omega t}}{2r^3} + \frac{2}{(t_0 - t)} \left[ -\omega + \frac{1}{2(t_0 - t)} \right] \geq 0 \]

\[ \rho + p_3 \geq 0 \Rightarrow -3\omega^2 + \frac{2 - 3\omega}{t_0 - t} + \frac{1}{2(t_0 - t)^2} \geq 0 \]

The quantity reminiscent of the \( F(t) \) is the second term in the second and third inequalities. If this term is positive (which is possible if \( t > t_0 - \frac{1}{2\omega} \) ) then the third inequality is satisfied and the second one yields a bound on the allowed domain of \( b_0 \). The first inequality is satisfied for all \( \omega \leq \frac{1}{3} \) while the fourth one gives another lower bound on \( t \) :

\[ t \geq t_0 - \frac{\sqrt{(2 - 3\omega)^2 + 24\omega^2} + (2 - 3\omega)}{12\omega^2} \]

Of the two lower bounds on \( t \) one has to choose the more stringent one. For instance if \( \omega = \frac{1}{3} \) then the first bound gives \( t \geq t_0 - 1.5 \) and the second one implies \( t \geq t_0 - 2.2 \). Thus, if we assume \( t \geq t_0 - 1.5 \) the other inequality is automatically satisfied. The wormhole can exist for the interval \( t_0 - 1.5 < t < t_0 \) with matter satisfying the WEC. The bound on the throat radius is:

\[ b_0^2 \geq \max \left[ \frac{e^{-2\omega t}(t_0 - t)^2}{\omega(t - t_0 + 1/2\omega)} \right] \]

Thus we have indicated through the above example that an inflationary wormhole with compact extra dimensions can exist for a finite interval of time with matter satisfying the WEC.
B. Kaluza Klein Inflation

We now move on to a more realistic case of a Kaluza–Klein cosmology. Here, at early times, we have a Kaluza–Klein universe which is an exact solution of the field equations in higher dimensional GR. The wormhole, as we shall see in this case, can comfortably inflate without doing any violence to the WEC. To see how this works out, we assume the topology of the universe to be $R \otimes W \otimes S^D$, where $R \otimes W$ refers to the wormhole spacetime and the $S^D$ corresponds to an extra dimensional 2–sphere. The metric describing such an universe has the form:

$$ds^2 = -dt^2 + a_1^2(t) \left[ \frac{dr^2}{1 - \frac{2b}{r}} + r^2 d\Omega_D^2 \right] + a_2^2(t) d\Omega_D^2$$

where $a_1(t)$ and $a_2(t)$ are the scale factors associated with the wormhole and the compact D–sphere respectively.

From the Einstein Field Equations we arrive at the following four energy condition inequalities:

$$3 \left( \frac{\dot{a}_1}{a_1} \right)^2 + 3D \frac{\dot{a}_1 \ddot{a}_2}{a_1 a_2} + D(D - 1) \left[ \frac{\dot{a}_2}{a_2} + 1 \right] + \frac{b'}{r^2 a_1^2} \geq 0$$

$$-2 \frac{\ddot{a}_1}{a_1} + 2 \left( \frac{\dot{a}_1}{a_1} \right)^2 + D \frac{\dot{a}_1 \ddot{a}_2}{a_1 a_2} - D \frac{\ddot{a}_2}{a_2} + \frac{b' r - b}{r^3 a_1^2} \geq 0$$

$$-2 \frac{\ddot{a}_1}{a_1} + 2 \left( \frac{\dot{a}_1}{a_1} \right)^2 + D \frac{\dot{a}_1 \ddot{a}_2}{a_1 a_2} - D \frac{\ddot{a}_2}{a_2} + \frac{b' r + b}{2r^3 a_1^2} \geq 0$$

$$-(D - 1) \frac{\ddot{a}_2}{a_2} - 3 \frac{\ddot{a}_1}{a_1} + 3 \frac{\dot{a}_1 \ddot{a}_2}{a_1 a_2} - (D - 1) \left[ \frac{\dot{a}_2}{a_2} + 1 \right] \geq 0$$

We can explicitly check that these are satisfied, e.g., for $b(r) = b_0$ and for perfect fluid matter obeying $p = (\gamma - 1) \rho$ asymptotically.

The exact solution for $\gamma = 2$ is:

$$dt = \frac{a_1(\tau) d\tau}{\sqrt{D - 1}}, \quad a_1^D(\tau) = \kappa_1 \tan^{a_1^D}\tau, \quad a_2^{D-1}(\tau) = \kappa_2 \tan^{a_2^{D-1}}\tau$$

with $0 \leq a^2 \leq \frac{d \tau}{\pi + D - 1}$ and $0 \leq \tau \leq \pi$ (d and D being the number of normal and extra dimensions respectively and the $\kappa_i$ are arbitrary constants) is generic in its behaviour as far as higher dimensional cosmologies are concerned.

It is easily verified that each of the $b(r)$ independent terms in the WEC inequalities stated above reduce to $\frac{2}{2^D - 2} a^2 / \sin^2 \tau$ for $d = 4$ and $D = 2$. The first, third and the fourth inequalities are then automatically satisfied whereas the second one constrains the allowed values of $b_0$. It follows then that wormholes can inflate and subsequently evolve through the FRW era all the way to the present, without violating the WEC. Such wormholes must however be as large as the visible universe today. This conclusion, though observationally disappointing is, by itself not fatal because we are dealing with an asymptotically flat FRW universe, which being infinite can accommodate an infinite number of visible universes.

It is nevertheless, of interest in this context to enquire if an oscillatory $R(t)$ (which can, in principle keep the throat from growing indefinitely) is consistent with WEC preservation. The answer is clearly no, if we require $R(t)$ to go through a minimum smoothly. Indeed, at such a minimum $R = 0$ and $\ddot{R} > 0$ which implies that $\frac{-d}{dt} \left( \frac{\dot{a}_1}{a_1} \right) < 0$ and that, consequently, the L.H.S of the second WEC inequality cannot be positive for all $r$ and $t$. On the other hand, if we let $R(t)$ have a cusp at the minimum $-R(t) = A + \sin \omega t$ is a concrete example—0 the WEC can be satisfied for all values of $t$. The discontinuities in $\dot{R}(t)$ inherent in such choices of $R(t)$ are of course unphysical but their existence indicates that WEC violation can be restricted to arbitrarily small intervals of time (and, equally important perhaps, to length scales where quantum gravity is important). This is, in fact the temporal analogue of the spatial confinement of exotic matter to an infinite slimy thin shell at the joint between two asymptotically flat spaces in wormholes obtained by suturing two such spacetimes together. The important difference is that in this case the energy–time uncertainty relation can be invoked to allow arbitrarily energy violations for the delta function time intervals in question. The possibilities multiply if we introduce extra dimensions. The scale factor, $a_1(t)$, can now in fact pass smoothly through a minimum since $\dot{a}_1(t) = 0$ reduces the time–dependent factor in the second WEC inequality to $-D \frac{\ddot{a}_2}{a_2} - 2 \frac{\ddot{a}_1}{a_1}$, and the $-D \frac{\ddot{a}_2}{a_2}$ term can, for appropriate choice, of $a_2$, overcome the negative contribution from $-2 \frac{\ddot{a}_1}{a_1}$. However, if a wormhole with an oscillating throat is simultaneously part of an asymptotically flat FRW universe, then its scale factor, $R(t)$, must necessarily be $r$ as well as $t$ dependent. This $r$ dependence is also necessary to keep the slowing down in the expansion of the asymptotic region from causing the WEC violation which it inevitably would, if the entire space were governed by a single scale factor. For a scenario, involving extra dimensions the oscillations in $a_1(r, t)$ at the throat could plausibly be coupled to those in $a_2(r, t)$; indeed they could well be the result of such a coupling. A complete theory of astrophysical wormholes based on $r$ and $t$ dependent scale factors, which exploits these possibilities will be reported elsewhere.
VI. CONCLUDING REMARKS

We have shown in this paper that wormholes with normal matter are a realistic possibility even in the domain of classical GR. The existence of these geometries is not dependent on the Averaged Energy Conditions. The reason to believe in this is related to the fact that the Averaged Energy Conditions are not always satisfied. The role of extra compact decaying dimensions have been dealt with in the context of two simple models—one involving an exponential inflation and the other a Kaluza–Klein type inflation.

It is quite possible that the matter required for spacetimes which exhibit 'flashes' of WEC violation might actually satisfy the Averaged Null or Averaged Weak Energy Conditions. The reason to believe in this is related to the fact that the Averaged Energy Conditions are global (one integrates over the quantity $T_{\mu\nu}^{\xi\xi}$, along a certain timelike/null geodesic). However we have not carried out this calculation although we hope to do it in due course.

We have also not dealt with the question of human traversability in the context of these evolving geometries. Such an analysis, is however, not too difficult. One has to replace the static observer’s frame in the case of a static wormhole with the comoving frame for the evolving geometry. Then one carries out a simple Lorentz transformation to go into the frame of the traveller. The Riemann tensor components are obtained in this traveller’s frame and they lead to the tidal force constraints. A similar analysis also holds for the acceleration constraint. The major difference is that all constraints depend on time and one has to find by extremization the time at which the inequalities yield the most stringent condition. After this extremization with respect to time one has to extremize w.r.t $r$ to obtain the final condition which when satisfied would make the wormhole traversable.

Since evolving wormholes are nonstatic, the study of quantum field theory in these backgrounds may result in particle creation. The relevant calculation would be fruitful to pursue. It requires, however, the exact solutions of the scalar wave/Maxwell or Dirac equations which at first sight may be rather difficult to obtain. Numerical analyses can nevertheless be done to provide hints into the number density and distributions of the created particles.

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[1] C.W.Misner and J.A.Wheeler, Annals of Physics 2, 525 (1957)
[2] J.A.Wheeler, Annals of Physics 2, 604 (1957)
[3] J.A.Wheeler, Geometrodynamics (Academic, New York, 1962)
[4] H.Ellis, Journal of Mathematical Physics 14, 104 (1973)
[5] S.W.Hawking, Physical Review D 37, 904 (1988)
[6] S.B.Giddings and A.Strominger, Nuclear Physics B 306, 890 (1988)
[7] M.S.Morris, American Journal of Physics 56, 39 (1988)
[8] M.S.Morris, K.S.Thorne and U.Yurtsever, Physical Review Letters 61, 1466 (1988)
[9] I.D.Novikov, Soviet Physics JETP 68, 439 (1989)
[10] F.Echeverria,G.Klinkhammer and K.S.Thorne, Physical Review D 44, 1077 (1991)
[11] J.L.Friedman, M.S.Morris, I.D.Novikov, F.Echeverria, G.Klinkhammer, K.S.Thorne and U.Yurtsever, Physical Review 42, 1915 (1990)
[12] J.L.Friedman and M.S.Morris, Physical Review Letters 66, 401 (1991)
[13] S.Y.Kim and K.S.Thorne, Physical Review D 43, 3929 (1991)
[14] S.W.Hawking, Physical Review D 46, 603 (1992)
[15] M.Visser, Nuclear Physics B 328, 203 (1989)
[16] M.Visser, Physical Review D 39, 3182 (1989)
[17] Sayan Kar, Physical Review D, 862 (1994)
[18] D.Hochberg and T.W.Keplah, Physical Review Letters 70, (1993)
[19] T.A.Roman, Physical Review D 47, 1370 (1993); Computers in Physics 8, 480 (1994)
[20] A. Wang and P. Letelier, gr-qc/9506004 and 9506004
[21] J. L. Friedman, K. Schleich and D. M. Witt, Physical Review Letters 71, 1486 (1993)
[22] C.W.Misner, K.S.Thorne and J.A.Wheeler, Gravitation (W.H.Freeman, San Francisco, 1973)
[23] Matt Visser, private communication
[24] A.K.Raychaudhuri, Physical Review 98, 1123 (1955)
[25] R.M.Wald, General Relativity (University of Chicago Press, Chicago, 1985)
[26] K. Ghoruku and T.Soma, Physical Review D 46, 1507 (1992)
[27] D. Sahdev, Physical Review D 30, 2495 (1984); Physics Letters, B 137, 155 (1984)
FIGURE CAPTIONS

Fig. 1: $F(t)$ vs $t$ for $\Omega(t) = \sin t + a$ \hspace{1cm} (a)$\omega = 1, a = 1.5(b)\omega = 1, a = 2(c)\omega = 1, a = 6$

Fig. 2: $F(t)$ vs $t$ for $\Omega(t) = \left(\frac{t^2 + a^2}{t^2 + b^2}\right)$ \hspace{1cm} $b > a > 0(a)a = 1, b = 1.5(b)a = 1, b = 2(c)a = 1, b = 3$