Perturbative interpretation of relativistic symmetries in nuclei

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Perturbation theory is used systematically to investigate the symmetries of the Dirac Hamiltonian and their breaking in atomic nuclei. Using the perturbation corrections to the single-particle energies and wave functions, the link between the single-particle states in realistic nuclei and their counterparts in the symmetry limits is discussed. It is shown that the limit of $S - V = \text{const}$ and relativistic harmonic oscillator (RHO) potentials can be connected to the actual Dirac Hamiltonian by the perturbation method, while the limit of $S + V = \text{const}$ cannot, where $S$ and $V$ are the scalar and vector potentials, respectively. This indicates that the realistic system can be treated as a perturbation of spin-symmetric Hamiltonians, and the energy splitting of the pseudospin doublets can be regarded as a result of small perturbation around the Hamiltonian with RHO potentials, where the pseudospin doublets are quasidegenerate.

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It is well known that the spin symmetry (SS) breaking, i.e., the remarkable spin-orbit splitting for the spin doublets $(n, l, j = l \pm 1/2)$, is one of the most important concepts for understanding the traditional magic numbers $(2, 8, 20, 28, \ldots)$ in atomic nuclei [1, 2]. Meanwhile, a new symmetry, the so-called pseudospin symmetry (PSS) [3, 4], is introduced to explain the near degeneracy between two single-particle states with the quantum numbers $(n - 1, l + 2, j = l + 3/2)$ and $(n, l, j = l + 1/2)$ by defining the pseudospin doublets $(\tilde{n} = n - 1, \tilde{l} = l + 1, j = \tilde{l} \pm 1/2)$. The splittings of both spin and pseudospin doublets play critical roles in the shell structure evolutions. Thus, it is a fundamental task to explore the origin of SS and PSS, as well as the mechanism of their breaking.

Since the suggestion of PSS in atomic nuclei, there have been comprehensive efforts to understand its origin. Apart from the rather formal relabeling of quantum numbers, various proposals for an explicit transformation from the normal scheme to the pseudospin scheme have been discussed [5, 6]. Based on the single-particle Hamiltonian of the oscillator shell model, the origin of PSS is connected with the special ratio in the strength of the spin-orbit and orbit-orbit interactions [5]. The relation between the PSS and the relativistic mean field (RMF) theory [8] was first noted in Ref. [9], in which Bahri et al. found that the RMF theory approximately explains such a special ratio in the strength of the spin-orbit and orbit-orbit interactions.

As substantial progress, the PSS was shown to be a symmetry of the Dirac Hamiltonian, where the pseudovectorial angular momentum $\tilde{l}$ is nothing but the orbital angular momentum of the lower component of the Dirac spinor, and the equality in magnitude but difference in sign of the scalar potential $S(r)$ and vector potential $V(r)$ was suggested as the exact PSS limit by reducing the Dirac equation to a Schrödinger-like equation [11]. As a more general condition, $d(S + V)/dr = 0$ can be approximately satisfied in exotic nuclei with highly diffuse potentials [11, 12]. Meanwhile, based on this limit, the pseudospin SU(2) algebra was established [13], and the specific node structures of the pseudospin doublets were illuminated [14]. Furthermore, the Dirac Hamiltonian with spin and pseudospin SU(2) symmetries can also be derived in supersymmetric (SUSY) patterns [15]. However, since there exist no bound nuclei within $S + V = \text{const}$, the non-perturbative nature of PSS in realistic nuclei has been presented in Refs. [16–18], which is also related to the consideration of the PSS as being a dynamical symmetry [19].

On the other hand, the relativistic harmonic oscillator (RHO) potentials were used to understand the origin of PSS [20, 21]. Subsequently, the spin and pseudospin U(3) algebra was established in the Dirac Hamiltonian with RHO potentials [18, 22]. Recently, Typel pointed out that the Hamiltonian with spin U(3) symmetry is one of the simplest cases where the pseudospin symmetry-breaking potential derived in the SUSY framework vanishes [23]. Meanwhile, Marcos et al. commented that the quasi-degeneracy of the pseudospin doublets in realistic nuclei can be considered as the breaking of their degeneracy in the Dirac Hamiltonian with RHO potentials [24].

In this Rapid Communication, the perturbation theory will be used for the first time to investigate the symmetries of the Dirac Hamiltonian and their breaking in realistic nuclei. The perturbation corrections to the single-particle energies and wave functions will be accurately calculated numerically. In this way, the link between the single-particle states in realistic nuclei and their counterparts in the symmetry limits will be constructed explicitly.

Assuming the spherical symmetry, the radial Dirac
In the case of the spin and pseudospin SU(2) symmetry limits shown in Ref. [12], the Dirac Hamiltonian with exact symmetries reads

\[ H_{0}^{SS} = \left( \begin{array}{cc} \Sigma + M & -\frac{d}{dr} + \frac{\kappa}{r} - \Delta_{0} - M \\ \frac{d}{dr} + \frac{\kappa}{r} - \Delta_{0} - M & \Sigma + M \end{array} \right), \]

\[ H_{0}^{PSS} = \left( \begin{array}{cc} \Sigma_{0} + M & -\frac{d}{dr} + \frac{\kappa}{r} - M \\ \frac{d}{dr} + \frac{\kappa}{r} - M & \Sigma_{0} + M \end{array} \right), \]

whose eigenenergies are denoted as \( E_{0} \) in general, and the corresponding symmetry-breaking potentials are

\[ W^{SS} = \left( \begin{array}{cc} 0 & 0 \\ 0 & \Delta_{0} - \Delta \end{array} \right), \quad W^{PSS} = \left( \begin{array}{cc} \Sigma - \Sigma_{0} & 0 \\ 0 & 0 \end{array} \right). \]

In contrast to adopting the Schrödinger-like equations in the previous studies [10, 16, 19, 27], it is clearly shown that the operators \( H, \Sigma_{0} \) and \( W \) used in the present calculations are all Hermitian, and they do not contain any singularity. This allows us to perform the order-by-order perturbation calculations. In addition, it should also be noticed that only \( W \) corresponds to the symmetry-breaking potential within the present decomposition, thus the ambiguity caused by the strong cancellations among the different terms in the Schrödinger-like equations can also be avoided. Therefore, the present method can provide a clear and quantitative way for investigating the perturbative nature of SS and PSS. This method can be universally applied to the cases that the nature of the symmetry is either perturbative or non-perturbative. For the case where the nature of the symmetry is perturbative, the link between the single-particle states in realistic nuclei and their counterparts in the symmetry limits can be constructed. For the case where the nature of the symmetry is non-perturbative, the divergence of the perturbation series can be found explicitly.

In the present calculations, as illustrated with dashed and dash-dotted lines in Fig. 1, the constant potentials in Eqs. (4) are chosen as \(-\Delta_{0} - M = -350 \text{ MeV} \) and \( \Sigma_{0} + M = 900 \text{ MeV} \). We have checked that the convergence of the perturbation calculations is not sensitive to these values.

In Fig. 2 taking the spin doublets \( k = 1f \) and the pseudospin doublets \( k = 1d \) as examples, the values of \( |W_{mk}/(E_{m} - E_{k})| \) are plotted as functions of the energy differences \( E_{m} - E_{k} \). For the SS case, the unperturbed eigenstates are chosen as those of \( H_{0}^{SS} \) in panel (a), while the unperturbed eigenstates are chosen as those of \( H \) in panel (b), namely, the former perturbation calculations are performed from a spin-symmetric Hamiltonian \( H_{0}^{SS} \) to the realistic Hamiltonian \( H \), whereas the latter ones are performed from \( H \) to \( H_{0}^{SS} \). For the PSS case, since there are no bound states in the pseudospin-symmetric Hamiltonian \( H_{0}^{PSS} \), the perturbation calculations are performed only from \( H \) to \( H_{0}^{PSS} \) as shown in panel (c).

For the completeness of the single-particle basis, the single-particle states \( m \) must include not only the states in the

\[ W_{mk} = \langle \Psi_{m} \rangle \langle W \Psi_{k} \rangle, \]

determines whether \( W \) can be treated as a small perturbation and governs the convergence of the perturbation series [20].

\[ \frac{W_{mk}}{E_{k} - E_{m}} \ll 1 \quad \text{for} \quad m \neq k, \]
it is about 0.6 for the PSS case because different compo-

cases of the single-particle states in the Dirac

FIG. 2: (Color online) Values of $|W_{mk}/(E_m - E_k)|$ vs the

energy differences $E_m - E_k$ for the spin doublets $k = 1f$

(panel (a) and (b)) and the pseudospin doublets $k = 1d$

(panel (c)). The unperturbed eigenstates are chosen as those of $H_0^{\text{SS}}$ (panel (a)) and $H$ (panels (b) and (c)).

The single-particle states $m$ include the states in the Dirac

sea and Fermi sea.

Fermi sea, but also those in the Dirac sea. Since the

spherical symmetry is adopted, only the states $m$ and $k$

with the same quantum numbers $l$ and $j$ lead to non-

vanishing matrix elements $W_{mk}$. It is seen that the values

of $|W_{mk}/(E_m - E_k)|$ decrease as a general tendency

when the energy differences $|E_m - E_k|$ increase. From

the mathematical point of view, this property provides

natural cut-offs of the single-particle states in the per-

turbation calculations.

Although the potentials satisfy $|\Delta_0 - \Delta| \gg |\Sigma - \Sigma_0|$, the largest value of $|W_{mk}/(E_m - E_k)|$ is roughly 0.10 ($H_0^{\text{SS}}$ to $H$) or 0.06 ($H$ to $H_0^{\text{SS}}$) for the SS case, whereas it is about 0.6 for the PSS case because different components of the Dirac spinors are involved:

$$W_{mk}^{\text{SS}} = (F_m |(\Delta_0 - \Delta)|G_k),$$

(8a)

$$W_{mk}^{\text{PSS}} = (G_m |(\Sigma - \Sigma_0)|G_k),$$

(8b)

where for the Fermi states the upper component $G(r) \sim O(1)$, and the lower component $F(r) \sim O(1/10)$. This indicates that the criterion in Eq. (5) can be well fulfilled for the SS case, but questionable for the PSS case.

Let us then examine the perturbation corrections to the single-particle energies of the spin doublets $1f$ and pseudospin doublets $1d$. In panel (a) of Fig. 3 by choosing the unperturbed eigenstates as those of $H_0^{\text{SS}}$, the single-particle energies obtained at the exact spin symmetry limit, and their counterparts obtained by the first-, second-, and third-order perturbation calculations, as well as those obtained by the self-consistent RMF theory, are shown from left to right. Meanwhile, the corresponding results obtained by choosing the unperturbed eigenstates as those of $H$ are shown in panels (b) and (c) of Fig. 3. It can be seen clearly that the spin-orbit splitting is well reproduced by the second-order perturbation calculations as shown in panel (a), and in the reversed way the energy degeneracy of the spin doublets can be well restored as shown in panel (b). This verifies that for studying the relationship between the eigenstates of $H_0$ and $H$ by perturbation theory, it is equivalent to use the definitions $H = H_0 + W$ and $H_0 = H - W$. In contrast, as shown in panel (c), the energy degeneracy

FIG. 3: (Color online) Single-particle energies of spin doublets

$1f$ (panels (a) and (b)) and pseudospin doublets $1d$ (panel

(c)) obtained at the exact symmetry limits, and by the first-

second-, and third-order perturbation calculations, as well

as those by the RMF theory. The unperturbed eigenstates are

chosen as those of $H_0^{\text{SS}}$ (panel (a)) and $H$ (panels (b) and

(c)), respectively.
Ref. [22], the Dirac Hamiltonian this statement in a perturbative way. 

generacy appearing in the spin-symmetric Hamiltonian be alternatively considered as the breaking of their de-

as the symmetry limit. explicit way that the nature of PSS is non-perturbative, 

realistic system can be treated as a perturbation of the 

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tivity of the wave functions for the pseudospin doublets 

the second-order perturbation corrections, but the iden-

nian in realistic nuclei with the symmetry limit of 

bridge can be constructed to connect the Dirac Hamilto-

identity of the wave functions for the spin dou-

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FIG. 4: (Color online) Upper panels: Same as Fig. 2, but for the case of the RHO potentials. Lower panels: Same as Fig. 3, but for all single-particle states in the pf major shell. The unperturbed eigenstates are chosen as those of \( H_{0}^{\text{RHO}} \) (panels (a) and (c)) and \( H \) (panels (b) and (d)), respectively.

of the pseudospin doublets cannot be restored up to the third-order perturbation calculations, and there exist no bound eigenstates of \( H_{0}^{\text{PSS}} \). Thus, the link between the pseudospin doublets in realistic nuclei and their counter-

parts in the \( S + V = \text{const} \) limit is unclear. In addition, choosing the eigenstates of \( H \) as the unperturbed eigenstates, the perturbation corrections to the single-

particle wave functions are also evaluated. It is found that the identity of the wave functions for the spin dub-

bles, \( G_{0}(1f_{5/2}) = G_{0}(1f_{7/2}) \), can be well reproduced by the second-order perturbation corrections, but the identity of the wave functions for the pseudospin doublets \( F_{0}(1f_{5/2}) = F_{0}(2p_{3/2}) \), cannot be fulfilled.

Therefore, from the perturbative point of view, the bridge can be constructed to connect the Dirac Hamilton-

nian in realistic nuclei with the symmetry limit of \( S + V = \text{const} \), but not \( S + V = \text{const} \). This indicates that the realistic system can be treated as a perturbation of the spin-symmetric Hamiltonian. This also confirms in an explicit way that the nature of PSS is non-perturbative, as the Dirac Hamiltonian with \( S + V = \text{const} \) is regarded as the symmetry limit.

However, it has been pointed out that the energy splitting of the pseudospin doublets in realistic nuclei could be alternatively considered as the breaking of their degeneracy appearing in the spin-symmetric Hamiltonian with RHO potentials [23, 24]. In the following, we assess this statement in a perturbative way.

In the case of the spin \( U(3) \) symmetry limit shown in Ref. [22], the Dirac Hamiltonian \( H \) in Eq. (2) is split as

\[
H = H_{0}^{\text{RHO}} + W^{\text{RHO}},
\]

with

\[
H_{0}^{\text{RHO}} = \left( \frac{\Sigma_{\text{HO}} + M}{\epsilon_{r}} - \frac{\Sigma}{\epsilon_{r}} + \frac{\Delta}{\epsilon_{r}} - \Delta_{0} - M \right),
\]

and

\[
W^{\text{RHO}} = \left( \begin{array}{cc} \Sigma - \Sigma_{\text{HO}} & 0 \\ 0 & \Delta_{0} - \Delta \end{array} \right),
\]

where \( \Sigma_{\text{HO}}(r) = c_{0} + c_{2}r^{2} \) has the form of a harmonic oscillator. Here, \( H_{0}^{\text{RHO}} \) leads to the energy degeneracy of the whole major shell, and \( W^{\text{RHO}} \) is identified as the corresponding symmetry-breaking potential. In the present investigation, we choose \( -\Delta_{0} - M = -350 \text{ MeV} \) as in \( H_{0}^{\text{SS}} \) and \( c_{0} + M = 865 \text{ MeV} \). As discussed before, the perturbative properties are not sensitive to these two constants. Meanwhile, the coefficient \( c_{2} \) is chosen as 1.00 MeV/fm² to minimize the perturbations to the pf states.

In the upper panels of Fig. 4 the values of \( |W_{m_{k}}(E_{m} - E_{k})| \) for the pseudospin doublets \( k = 1\bar{d} \) are shown as functions of the energy differences \( E_{m} - E_{k} \). It is found that the general patterns shown in panels (a) and (b) are the same as those in panels (a) and (b) of Fig. 2 respectively, and the largest perturbation correction is roughly 0.16 (\( H_{0}^{\text{RHO}} \) to \( H \)) or 0.10 (\( H \) to \( H_{1}^{\text{RHO}} \)). This indicates that the criterion in Eq. (11) is fulfilled, even though not as well as in the SS case. In the lower panels of Fig. 4 the single-particle energies of the states in the pf major shell obtained at the exact symmetry limit and by the self-consistent RMF theory, as well as their counterparts obtained in the first-, second-, and third-order perturbation calculations, are shown. As shown in panel (c), not only the spin-orbit splitting, but also the pseudospin-orbit splitting, is well reproduced by
the third-order perturbation calculations, and in the reversed way the energy degeneracy of all the states in the pf major shell can be well restored as shown in panel (d). Thus, the link between the pf states in realistic nuclei and their counterparts in the symmetry limit with RHO potential can be explicitly established. Furthermore, it is found that the single-particle wave functions of $H (H_{0}^{\text{RHO}})$ can also be reproduced by the second-order perturbation calculations starting from $H_{0}^{\text{RHO}} (H)$.

Therefore, the bridge connecting the Dirac Hamiltonian in realistic nuclei and that with RHO potentials can be clearly constructed using perturbation theory. This indicates that the energy splitting of the pseudospin doublets can be regarded as a result of small perturbation around the Dirac Hamiltonian with RHO potentials, where the degeneracy of the pseudospin doublets appears.

Of course, one must always keep in mind that the actual picture in nuclear spectra is generally more complex than that of a simple potential model. Still, there are well-identified cases where the core polarization effects are small enough (correlation and polarization diagrams may compensate each other and give relatively large spectroscopic factors) to allow for the notion of single-particle state to hold.

In summary, the symmetries of the Dirac Hamiltonian and their breaking in realistic nuclei are investigated in the framework of perturbation theory. The present framework can provide a clear and quantitative way for investigating the perturbative nature of SS and PSS. By examining the perturbation corrections to the single-particle energies and wave functions, the link between the single-particle states in realistic nuclei and their counterparts in the symmetry limits has been established. It is found that the symmetry limits of $S - V = \text{const}$ and RHO potentials can be connected to the Dirac Hamiltonian in realistic nuclei by perturbation theory, but not $S + V = \text{const}$. In other words, it is suggested that the realistic system can be treated as a perturbation of spin-symmetric Hamiltonians, and the energy splitting of the pseudospin doublets can be regarded as a result of small perturbation around the Hamiltonian with RHO potentials, where the pseudospin doublets are quasidegenerate.

The present investigation is based on simple RMF theory with only scalar and vector potentials. It would be interesting to study the corresponding symmetry limits in systems with non-local potentials or tensor interactions such as those encountered in relativistic Hartree-Fock approaches [28–30]. The analysis done in this work is easy to be generalized to such investigations.

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[1] O. Haxel, J. H. D. Jensen, and H. E. Suess, Phys. Rev. 75, 1766 (1949).
[2] M. Goeppert-Mayer, Phys. Rev. 75, 1969 (1949).
[3] A. Arima, M. Harvey, and K. Shimizu, Phys. Lett. B 30, 517 (1969).
[4] K. Hecht and A. Adler, Nucl. Phys. A 137, 129 (1969).
[5] A. Bohr, I. Hamamoto, and B. R. Mottelson, Phys. Scr. 26, 267 (1982).
[6] O. Castaños, M. Moshinsky, and C. Quesne, Phys. Lett. B 277, 238 (1992).
[7] A. L. Blokhin, C. Bahri, and J. P. Draayer, Phys. Rev. Lett. 74, 4149 (1995).
[8] B. D. Serot and J. D. Walecka, Advances in Nuclear Physics Vol. 16: The Relativistic Nuclear Many Body Problem (Plenum Press, New York, 1986).
[9] C. Bahri, J. P. Draayer, and S. A. Moszkowski, Phys. Rev. Lett. 68, 2133 (1992).
[10] J. N. Ginocchio, Phys. Rev. Lett. 78, 436 (1997).
[11] J. Meng, K. Sugawara-Tanabe, S. Yamaji, P. Ring, and A. Arima, Phys. Rev. C 58, R628 (1998).
[12] J. Meng, K. Sugawara-Tanabe, S. Yamaji, and A. Arima, Phys. Rev. C 59, 154 (1999).
[13] J. N. Ginocchio and A. Leviatan, Phys. Lett. B 425, 1 (1998).
[14] A. Leviatan and J. N. Ginocchio, Phys. Lett. B 518, 214 (2001).
[15] A. Leviatan, Phys. Rev. Lett. 92, 202501 (2004).
[16] P. Alberto, M. Fiolhais, M. Malheiro, A. Delfino, and M. Chiapparini, Phys. Rev. C 65, 034307 (2002).
[17] R. Lisboa, M. Malheiro, P. Alberto, M. Fiolhais, and A. S. de Castro, Phys. Rev. C 81, 064324 (2010).
[18] J. N. Ginocchio, J. Phys. Conf. Ser. 267, 012037 (2011).
[19] P. Alberto, M. Fiolhais, M. Malheiro, A. Delfino, and M. Chiapparini, Phys. Rev. Lett. 86, 5015 (2001).
[20] T.-S. Chen, H.-F. Lü, J. Meng, S.-Q. Zhang, and S.-G. Zhou, Chin. Phys. Lett. 20, 358 (2003).
[21] R. Lisboa, M. Malheiro, A. S. de Castro, P. Alberto, and M. Fiolhais, Phys. Rev. C 69, 024319 (2004).
[22] J. N. Ginocchio, Phys. Rev. Lett. 95, 252501 (2005).
[23] S. Typel, Nucl. Phys. A 806, 156 (2008).
[24] S. Marcos, M. López-Quelle, R. Niembro, and L. Savushkin, Eur. Phys. J. A 37, 251 (2008).
[25] W. H. Long, J. Meng, N. Van Giai, and S.-G. Zhou, Phys. Rev. C 69, 034319 (2004).
[26] W. Greiner, Quantum Mechanics: An Introduction (Springer-Verlag Berlin Heidelberg, 1994).
[27] S. Marcos, M. López-Quelle, R. Niembro, L. N. Savushkin, and P. Bernardos, Phys. Lett. B 513, 30 (2001).
[28] W. H. Long, H. Sagawa, J. Meng, and N. Van Giai, Phys. Lett. B 639, 242 (2006).
[29] W. H. Long, H. Sagawa, N. Van Giai, and J. Meng, Phys. Rev. C 76, 034314 (2007).
[30] H. Liang, W. H. Long, J. Meng, and N. Van Giai, Eur. Phys. J. A 44, 119 (2010).