INTTEGRABLE EXTENSIONS OF CLASSICAL ELLIPTIC INTEGRABLE SYSTEMS

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We consider two particular examples of the general construction proposed previously. We consider integrable extensions of the classical elliptic Calogero–Moser model of \( N \) particles with spin and the integrable Euler–Arnold top related to the \( SL(N, \mathbb{C}) \) group. The extended systems have extra \( N - 1 \) degrees of freedom and can be described in terms of Darboux variables.

**Keywords:** Hitchin systems, Calogero–Moser model, Euler–Arnold top

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**In memory of Mikhail Konstantinovich Polivanov**

1. Introduction and summary

It is possible to extend the phase space of an integrable system such that an integrable system exists on the extended phase space. The extension is nontrivial if the extended system is not a direct product of two noninteracting systems. We do not have a general theory of such extensions, and consider some examples based on the extensions of Hitchin-type systems [1], [2]. In our construction, the underlying system is the symplectic quotient of the extended system by the action of some Abelian group.

A particular example of a nontrivial extension, which does not fit into our scheme, however is given by Hitchin systems related to parabolic Higgs bundles (the Hitchin systems with spin variables). Symplectic quotients of these systems are the Hitchin systems defined on smooth curves (the systems without spin). An example of such a nontrivial extension is the passage from the system of \( N \) Calogero–Moser (CM) particles to the same system of particles equipped with spin [3], [4]. Here, in particular, we consider the further extension of this system to the system with extra \( N - 1 \) degrees of freedom.

In our previous paper [5], we proposed some generalizations of the parabolic Higgs bundles that lead to special extensions of integrable systems. The advantage of our construction is the existence of Darboux coordinates for the extended system. The first example of such systems is the Sutherland model with two types of spins [6], [7]. Here, we consider two more examples in detail. The first is an extension of the integrable Euler–Arnold top related to the \( SL(N, \mathbb{C}) \) group [8], [9]. The second is an extension of the spin elliptic CM system.

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Section 2 is auxiliary. We describe the symplectic structure on one of the symmetric \( SL(N, \mathbb{C}) \) spaces and on \( SL(N, \mathbb{C}) \) coorbits and establish a relation between them. In Sec. 3, we discuss the two examples.

2. Cotangent bundles to symmetric spaces and coadjoint orbits

2.1. Cotangent bundles to groups. We consider the complex groups \( GL(N, \mathbb{C}) \), \( SL(N, \mathbb{C}) \) and \( PGL(N, \mathbb{C}) \) for \( N \geq 2 \):

\[
GL(N, \mathbb{C}) = \{ f \in \text{Mat}_N \mid \text{det} f \neq 0 \},
\]

\[
SL(N, \mathbb{C}) = \{ f \in GL(N, \mathbb{C}) \mid \text{det} f = 1 \},
\]

(2.1)

\[
PGL(N, \mathbb{C}) = GL(N, \mathbb{C})/\mathbb{C}^*,
\]

where \( \mathbb{C}^* \) is the center of \( GL(N, \mathbb{C}) \). Let \( Z = Z_N \) be the center of \( SL(N, \mathbb{C}) \). Then

\[
PGL(N, \mathbb{C}) = SL(N, \mathbb{C})/Z.
\]

(2.2)

Let \( G \) be the group \( GL(N, \mathbb{C}), \) \( SL(N, \mathbb{C}) \) or \( PGL(N, \mathbb{C}) \) and \( \mathfrak{g} \) be their Lie algebras. We identify \( \mathfrak{g} \) with the Lie coalgebra \( \mathfrak{g}^* \) by means the invariant metric \( (\cdot, \cdot) \) on \( \mathfrak{g} \). Then the algebra \( \mathfrak{g} \) becomes a Poisson space. Let \( S = \sum S^a T_a \) be the decomposition with respect to a basis in \( \mathfrak{g} \). The Poisson bracket is the Lie–Poisson bracket

\[
\{ S^a, S^b \} = C^{ab}_{\ c} S^c,
\]

(2.3)

where \( C^{ab}_{\ c} \) are structure constants of the algebra \( \mathfrak{g} \). This bracket is degenerated on \( \mathfrak{g} \). Fixing \( l = \text{rank} \mathfrak{g} \) Casimir functions

\[
(S) = c_1, \quad (S^2) = c_2, \quad \ldots, \quad (S^N) = c_N
\]

(2.4)

we obtain a nondegenerate bracket on the coadjoint orbits \( \mathcal{O}_\nu \) described below. The dimension of the generic orbit is \( \dim(\mathcal{O}_\nu) = \dim G - l \). We consider only this type of orbits in what follows.

The cotangent bundle \( T^*G \) is identified with the tangent bundle \( TG \). Using its trivialization, we describe \( T^*G \) as

\[
T^*G = \mathfrak{g} \times G = \{ \zeta \in \mathfrak{g}, g \in G \}.
\]

(2.5)

In these coordinates, the canonical invariant symplectic form on \( T^*G \) takes the form

\[
\omega = D(\zeta, Dg g^{-1}) = (X, g^{-1} Dg),
\]

(2.6)

where

\[
X = g^{-1} \zeta g \in \mathfrak{g}.
\]

(2.7)

The form is invariant under the left action of the subgroup \( K \subseteq G \)

\[
\zeta \rightarrow f \zeta f^{-1}, \quad g \rightarrow fg, \quad X \rightarrow X, \quad f \in K,
\]

(2.8)

producing the left moment map

\[
\mu^L(\zeta, g) = \zeta|_{\mathfrak{k}^*} = gXg^{-1}|_{\mathfrak{k}^*},
\]

(2.9)

where \( \mathfrak{k} = \text{Lie}(K) \), and \( \mathfrak{k}^* \) is the Lie coalgebra. The right action

\[
\zeta \rightarrow \zeta, \quad g \rightarrow gf^{-1}, \quad f \in G,
\]

(2.10)

leads to the moment map

\[
\mu^R(\zeta, g) = g^{-1} \zeta g = X.
\]

(2.11)
We consider the interrelations between $T^*GL(N, \mathbb{C})$ and $T^*SL(N, \mathbb{C})$. The symplectic form $\omega$ on $T^*GL(N, \mathbb{C})$, Eq. (2.6), is invariant under the scaling

$$g \to g\lambda, \quad \zeta \to \zeta, \quad \lambda \in \mathbb{C}^*.$$  

The corresponding moment map is $\text{tr} \zeta$. We fix the gauge by the condition $\det g = 1$ (2.1). In this way, we arrive at $T^*SL(N, \mathbb{C})$ as the symplectic quotient

$$T^*SL(N, \mathbb{C}) = \mathbb{C}^* \setminus T^*GL(N, \mathbb{C}) \equiv \{ (\zeta, g) \mid \text{tr} \zeta = 0, \det g = 1 \}. \quad (2.12)$$

The condition $\det g = 1$ does not fix the gauge completely. We can further act with the center $Z$ on $g \to \gamma g$, with the condition being preserved. In this way we arrive at the cotangent bundle $T^*PGL(N, \mathbb{C})$.

The symplectic form on $T^*SL(N, \mathbb{C})$ is

$$\omega = D(\zeta, Dg g^{-1}) = (X, g^{-1}Dg), \quad \text{where} \quad (\zeta, g) \text{ satisfy (2.12)}. \quad (2.13)$$

Because $X$ is gauge invariant and $\text{tr} X = 0$, it is a section of the cotangent bundle $T^*SL(N, \mathbb{C})$.

There is a residual symmetry defined by multiplication by the central element $e_N(\gamma) \in Z$,

$$e_N(\gamma) = \exp \left( \frac{2\pi i \gamma}{N} \right), \quad \gamma = \text{diag}(1, \ldots, 1). \quad (2.14)$$

After taking the quotient, we obtain the cotangent bundle $T^*PGL(N, \mathbb{C})$.

2.2. Coadjoint orbits. We consider the cotangent bundle $T^*SL(N, \mathbb{C})$ and the left symplectic quotient $SL(N, \mathbb{C}) \setminus \nu T^*SL(N, \mathbb{C})$ with the moment $\mu^L$ in Eq. (2.9) taking values in the Cartan subalgebra $\mathfrak{h}^C \subset \mathfrak{sl}(N, \mathbb{C})$:

$$\mu^L(\zeta, g) = \zeta = \nu \in (\mathfrak{h}^C)^* \sim \mathfrak{h}^C. \quad (2.15)$$

Here, $\nu$ is a fixed regular element of $\mathfrak{h}^C$. The subgroup preserving this value is the Cartan subgroup $H^C \subset SL(N, \mathbb{C})$. Thus, the symplectic quotient is defined as

$$SL(N, \mathbb{C}) \setminus \nu T^*SL(N, \mathbb{C}) = \{ (\zeta, g) \mid \zeta = \nu \in \mathfrak{h}^C, \quad g \sim fg, \quad f \in H^C \}. \quad (2.16)$$

It is the coadjoint orbit

$$O_\nu = H^C \setminus SL(N, \mathbb{C}) = \{ S = g^{-1} \nu g \mid g \in SL(N, \mathbb{C}), \quad \nu \in \mathfrak{h}^C \}. \quad (2.17)$$

After substituting $\zeta = \nu$ in (2.6), the form $\omega$ on $T^*SL(N, \mathbb{C})$ becomes the Kirillov–Kostant form on $O_\nu$:

$$\omega^{KK} = D(\nu, Dg g^{-1}). \quad (2.18)$$

The form is invariant under the right $G^C$ action in (2.10). This transformation generates the moment

$$\mu^R = g^{-1} \nu g = S. \quad (2.19)$$

For $g = \mathfrak{sl}(N, \mathbb{C})$, Casimir functions (2.4) are

$$(S^2) = c_2, \quad \ldots, \quad (S^N) = c_N \quad \left( c_2 = \sum_{j=1}^{N} \nu_j^2, \quad \ldots, \quad c_N = \sum_{j=1}^{N} \nu_j^N \right), \quad (2.20)$$

and we assume that $c_j \neq c_k$ for $j \neq k$. The corresponding orbit is

$$O_\nu = g^{-1} \nu g, \quad g \in SL(N, \mathbb{C}), \quad \nu = \text{diag}(\nu_1, \ldots, \nu_N), \quad \text{tr} \nu = 0, \quad (2.21)$$

and $\nu_j \neq \nu_k$ for $j \neq k$. The dimension of the generic orbit is

$$\dim_C(O_\nu) = \dim_C SL(N, \mathbb{C}) - (N - 1) = N(N - 1). \quad (2.22)$$
2.3. Cotangent bundles to $T^*(SO(N,\mathbb{C})\backslash GL(N,\mathbb{C}))$. We consider the group $G = GL(N,\mathbb{C})$, its subgroup

$$K = SO(N,\mathbb{C}) = \{ g \in GL(N,\mathbb{C}) \mid g^T g = Id, \det g = 1 \}$$

(2.23)

and the quotient space $X^G = SO(N,\mathbb{C})\backslash GL(N,\mathbb{C})$. Then $X^G = \{ x \}$ is the space of complex symmetric matrices $x$ with $\det x \neq 0$. Evidently, the matrix

$$Q = g^T g, \quad g \in GL(N,\mathbb{C}),$$

(2.24)

is an element of $X^G$.

The cotangent bundle $T^*X^G = T^*(SO(N,\mathbb{C})\backslash GL(N,\mathbb{C}))$ can be identified with the symplectic quotient $SO(N,\mathbb{C})\backslash T^*G$. The form $\omega$ on $T^*GL(N,\mathbb{C})$, Eq. (2.6), is invariant under the left $SO(N,\mathbb{C})$-action:

$$\zeta \rightarrow f \zeta f^{-1}, \quad g \rightarrow fg, \quad f \in SO(N,\mathbb{C}).$$

(2.25)

The corresponding moment map assumes the form $\mu^L = \zeta|_{so(N,\mathbb{C})}$ and we take

$$\zeta|_{so(N,\mathbb{C})} = 0.$$  

(2.26)

We note that the $SO(N,\mathbb{C})$-action preserves the moment constraint $f \zeta f^{-1}|_{so(N,\mathbb{C})} = 0$ for $f \in SO(N,\mathbb{C})$.

The Lie algebra $g = gl(N,\mathbb{C})$ can be decomposed into the sum of antisymmetric and symmetric matrices

$$gl(N,\mathbb{C}) = so(N,\mathbb{C}) + p^G.$$  

(2.27)

The space of symmetric matrices $p^g$ can be regarded as the tangent space to $X^G = \{ SO(N,\mathbb{C})g \}$ at the point $g = Id$. These subspaces (2.27) are orthogonal with respect to the invariant metric on $gl(N,\mathbb{C})$. Its restriction to $p^G$ is nondegenerate.

Moment condition (2.26) means that $\zeta \in p^G$. In this way, the symplectic quotient $SO(N,\mathbb{C})\backslash T^*GL(N,\mathbb{C})$ is defined by the set of pairs $(g, \zeta)$ that satisfy the equivalence relation

$$\{(g, \zeta) \sim (fg, f \zeta f^{-1}), \quad f \in SO(N,\mathbb{C}), \quad g \in GL(N,\mathbb{C}), \quad \zeta \in p^G\}.$$  

(2.28)

Because $g \in GL(N,\mathbb{C})$ is defined up to the left multiplication by $SO(N,\mathbb{C})$, we obtain the cotangent bundle $T^*X^G$, $X^G = SO(N,\mathbb{C})\backslash GL(N,\mathbb{C})$

$$T^*X^G = SO(N,\mathbb{C})\backslash T^*GL(N,\mathbb{C}).$$  

(2.29)

It follows from this definition of $T^*X^G$ that

$$\dim T^*X^G = 2(\dim GL(N,\mathbb{C}) - \dim SO(N,\mathbb{C})) = N(N + 1).$$  

(2.30)

The symplectic form on $T^*X^G$ coincides with (2.6):

$$\omega = D(\zeta, Dgg^{-1}),$$  

(2.31)

where the pair $(\zeta, g)$ satisfies (2.28).

Let $g \in GL(N,\mathbb{C})$ and $\zeta \in p^G$. We consider the element

$$X^G = g^{-1} \zeta g \in gl(N,\mathbb{C}).$$  

(2.32)
It is a section of the bundle $T^*X^G$. In these terms, $\omega$ in (2.31) becomes

$$\omega = D(X^G, g^{-1}Dg).$$

(2.33)

Because $X^G \in gl(N, \mathbb{C})$, it can be expanded with respect to a basis $\{T_a\}$ of the algebra $gl(N, \mathbb{C})$ as $X^G = \sum X^a T_a$. The coefficients $X^a$ form a Lie–Poisson algebra on $gl(N, \mathbb{C})$. As in (2.3),

$$\{X^a, X^b\} = C_{c}^{ab} X^c \quad (a, b = 1, \ldots, N^2).$$

(2.34)

As above, these brackets have $N$ Casimir functions $c_j = (X^j), (j = 1, \ldots, N)$. But they are not the Casimir functions of the Poisson algebra on $T^*X^G$.

It follows from (2.25) that $T^*X^G$ can be described by the gauge-invariant variables

$$P = g^{-1}\zeta(g^T)^{-1}, \quad Q = g^Tg.$$  

(2.35)

where $(\zeta, g)$ satisfies (2.28). We note that $P^T = P$ and $Q^T = Q$ ($Q \in X^G$). In these variables,

$$X^G = PQ.$$  

(2.36)

Let $e^j$ be a basis in $\mathfrak{p}^G (j = 1, \ldots, \text{dim}\mathfrak{p}^G)$ with the pairing $(e^j, e^k) = \delta_{jk}$. We expand $Q$ and $P$ in this basis: $Q = \sum_j Q^j e^j, \quad P = \sum_j P^j e^j$. In these variables, the Poisson algebra on $T^*X$ is canonical:

$$\{P^j, Q^k\} = \delta^{jk} \quad (j, k = 1, \ldots, \text{dim}\mathfrak{t}).$$

(2.37)

To prove this, we define another invariant symplectic form on $T^*X^G$,

$$\omega^{X^G} = \omega(\zeta, g) - \omega(\zeta, (g^T)^{-1}),$$  

(2.38)

where $\omega$ is given in (2.31). By direct calculation, we find that

$$\omega^{X^G} = (DP, DQ).$$

(2.39)

In this way, we obtain Darboux brackets (2.37).

The symplectic form $\omega$ on $T^*X^G$, Eq. (2.33), and $\omega^X$ in (2.39) are invariant under the right action of the group $G$ in (2.10). Similarly to (2.9), the moment corresponding to the action is

$$\mu^G_{\lambda}(\zeta, g) = g^{-1}\zeta g \in \mathfrak{g}^{\mathfrak{Lie}\ast(G)},$$

(2.40)

(see (2.26)), or

$$\mu^G_{\lambda} (2.32) X^G (2.36) = PQ.$$  

(2.41)

2.4. Cotangent bundle $T^*(SO(N, \mathbb{C}) \setminus SL(N, \mathbb{C}))$ and coadjoint orbits. In what follows, we need the quotient space $X^S = SO(N, \mathbb{C}) \setminus SL(N, \mathbb{C})$. It is the space of complex symmetric matrices with $\det g = 1$,

$$X^S = \{g \in SL(N, \mathbb{C}) \mid g^T = g\}.$$  

(2.42)

In fact, it is a symmetric pseudo-Riemannian space (see definitions in [10], [11]).
Similarly to (2.28), the cotangent bundle $T^*X^S$ is a result of the symplectic reduction of the cotangent bundle $T^*SL(N, C)$ under the left action of the subgroup $SO(N, C)$

$$T^*X^S = T^*(SO(N, C) \backslash SL(N, C)) = SO(N, C) \backslash T^*SL(N, C).$$

Let $p^S$ be the subspace in the Lie algebra $sl(N, C)$ orthogonal to the Lie subalgebra $so(N, C)$,

$$sl(N, C) = so(N, C) + p^S, \quad p^S = \{ \zeta \in sl(N, C) \mid \zeta^T = \zeta, \text{tr} \zeta = 0 \}.$$

In these terms, the cotangent bundle $T^*X^S$ can be identified with the set of pairs $(g, \zeta)$, where $g$ is given in (2.42):

$$T^*X^S = \{(g, \zeta), \quad g \in X^S, \quad \zeta \in p^S\}.$$

Hence, the symplectic form $\omega$ on $T^*X^S$ is

$$\omega^S = D(\zeta, Dgg^{-1}) = D(X^S, g^{-1}Dg), \quad g \in X^S, \quad X = g^{-1}\zeta g.$$

It follows from (2.30) that

$$\dim T^*X^S = N(N + 1) - 2 = \dim O_\nu + 2(N - 1).$$

In terms of Darboux variables, $X^S = \mathcal{P} \mathcal{Q}$, where $\mathcal{P}$ and $\mathcal{Q}$ are complex symmetric matrices with $\det g = 1$ and $\text{tr} X^S = 0$.

We consider the symplectic action $(g, \zeta) \rightarrow (\lambda g, \zeta)$ of $\mathbb{C}^*$ on $T^*X^G$. Because this action commutes with the $SO(N, C)$-action, similarly to (2.12), we have another realization of $T^*X^S$:

$$T^*X^S = \mathbb{C}^* \backslash T^*X^G.$$

The Darboux variables are those in (2.35),

$$\mathcal{P} = g^{-1}\zeta(g^T)^{-1}, \quad \mathcal{Q} = g^T, \quad \det g = 1, \quad \text{tr} \zeta = 0,$$

and the form can be written as

$$\omega^X = \omega^S(\zeta, g) - \omega^S(\zeta, (g^T)^{-1}),$$

where $\omega^S$ is (2.44) and

$$\omega^X = (D\mathcal{P}, D\mathcal{Q}).$$

The additional constraints then become

$$\text{tr} X^S = \text{tr} \mathcal{P} \mathcal{Q} = 0, \quad \det \mathcal{Q} = 1.$$

Because the Cartan subgroup $H^C \subset SL(N, C)$ lies in $X^S$ ($H^C \not\subset SO(N, C)$), we can consider the additional left action of $H^C$ on $T^*X^S = SO(N, C) \backslash \nu SL(N, C)$:

$$g \rightarrow fg, \quad \zeta \rightarrow f\zeta f^{-1}, \quad f \in H^C.$$

Let $h^C \subset sl(N, C)$ be the Cartan subalgebra $h^C = \text{Lie}(H^C)$ and $\nu \in h^C$ be a regular element. The symplectic reduction of $T^*X^S$ with respect to this action is defined by the moment constraint equation

$$\mu^L = \text{Pr} \zeta |_{h^C} = \nu.$$
Table 1

| SympLECTic action | Constraints | Definition |
|-------------------|-------------|------------|
| 1. $\mathbb{C}^*$ | $\det g = 1, \ \text{tr} \ \zeta = 0$ | (2.12) |
| 2. $\text{SO}(N, \mathbb{C})$ | $g^T = g, \ \zeta^T = \zeta$ | (2.29) |
| 3. $\text{SO}(N, \mathbb{C})$ | $g^T = g, \ \zeta^T = \zeta$ | (2.43) |
| 4. $\mathbb{C}^*$ | $\det g = 1, \ \text{tr} \ \zeta = 0$ | (2.46) |
| 5. $\text{H}^C \setminus \nu$ | $g \in \text{SL}(N, \mathbb{C}), \ \zeta = \nu \in \mathfrak{h}^C$ | (2.52) |
| 6. $\text{SL}(N, \mathbb{C}) \setminus \nu$ | $g \in \mathcal{X}^S, \ \zeta = \nu \in \mathfrak{h}^C$ | (2.17) |

and the gauge invariant variable $\mathcal{X}^S$ in (2.32). We take $\nu$ to be the same as in (2.16). Therefore, $\mathcal{X}^S = g^{-1} \nu g \in \mathfrak{g}^C$ is an element of the coadjoint orbit $\mathcal{O}_\nu$, Eq. (2.17). This means that

$$
\mathcal{O}_\nu = \{ \mathcal{X} = g^{-1} \nu g \mid g \in G^C \} = \mathcal{H}^C \setminus \nu \ T^* \mathcal{X}^S. 
$$

To summarize, we have obtained the description of symplectic manifolds incorporated in the following commutative diagram, where the arrows denote the symplectic reductions (see also Table 1):

\[
\begin{array}{ccc}
T^*GL(N, \mathbb{C}) & \xrightarrow{1} & T^*SL(N, \mathbb{C}) \\
\downarrow & & \downarrow \\
T^*\mathcal{X}^G & \xrightarrow{2} & T^*\mathcal{X}^S \\
\downarrow & & \downarrow \\
\mathcal{O}_\nu & & \end{array}
\]

3. Extensions of elliptic integrable systems

Let $\Sigma_\tau = \mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z})$ be an elliptic curve. The Lax operator $L(z)$ is a meromorphic $(1, 0)$ form on $\Sigma_\tau$ taking values in the Lie algebra $\mathfrak{sl}(N, \mathbb{C})$. It satisfies some quasiperiodicity conditions with respect to the shifts by the lattice vectors and has a simple pole at $z = 0$. The residue and the quasiperiodicities fix $L(z)$. Let

$$
\text{Res} \ L(z)_{|z=0} = \mathbf{W}. \quad (3.1)
$$

We consider two types of residues:

1) $\mathbf{W} = \mathbf{S} \in \mathcal{O}_\nu$ and 2) $\mathbf{W} = \mathbf{X} \in T^*\mathcal{X}^G$ or $\mathbf{W} = \mathbf{X} \in T^*\mathcal{X}^S$. \quad (3.2)

The first case corresponds to standard integrable systems. Formula (2.45) suggests that the Lax operator in the second case defines an extension of the standard integrable systems. We prove that this extension is also integrable.

We consider two types of quasiperiodicities. The first one defines the integrable Euler–Arnold $\text{SL}(N, \mathbb{C})$ top and the second, the elliptic CM system with spin.
3.1. Extension of integrable Euler–Arnold $SL(N, \mathbb{C})$ top.

**Euler–Arnold $SL(N, \mathbb{C})$ top.** To define the Euler–Arnold top, we introduce a special basis in the $GL(N, \mathbb{C})$ and $SL(N, \mathbb{C})$ groups and the corresponding Lie algebras. We consider two matrices

$$Q = \text{diag}(e_N(1), \ldots, e_N(m), 1, \ldots, 1), \quad e_N(z) = \exp \frac{2\pi i}{N} z,$$

$$\Lambda = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}. \quad (3.3)$$

We note that $Q^N = \Lambda^N = \text{Id}$ and

$$Q^m \Lambda^n = e_N(-mn) \Lambda^n Q^m. \quad (3.4)$$

We consider two-dimensional lattices in $\mathbb{C}$:

$$\Gamma_N = Z_N^{(2)} = (\mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z}), \quad \bar{\Gamma}_N = \bar{Z}_N^{(2)} = Z_N^{(2)} \setminus (0, 0). \quad (3.5)$$

The matrices $Q^a \Lambda^a$, $a = (a_1, a_2) \in Z_N^{(2)}$, generate a basis in the $GL(N, \mathbb{C})$ group and in the algebra $gl(N, \mathbb{C})$, while $Q^a \Lambda^a$, $\alpha = (\alpha_1, \alpha_2) \in \bar{Z}_N^{(2)}$, generate a basis in the Lie algebra $sl(N, \mathbb{C})$. More precisely, we define the generators

$$T_a = \frac{N}{2\pi i} e_N\left(\frac{a_1 a_2}{2}\right) Q^a \Lambda^a. \quad (3.6)$$

They are almost invariant on the $\mathbb{Z} \oplus \mathbb{Z}$ lattice: $T_{a_1+N,a_2+N} = \pm T_{a_1,a_2}$. Therefore, $T_{-a_1,-a_2} = \pm T_{N-a_1,N-a_2}$.

From (3.4), we find the multiplication in $GL(N, \mathbb{C})$:

$$T_a T_b = \kappa_{a,b} T_{a+b}, \quad \kappa_{a,b} = \frac{N}{2\pi i} e_N\left(-\frac{a \times b}{2}\right), \quad (a \times b = a_1 b_2 - a_2 b_1). \quad (3.7)$$

Similarly, the commutation relations for the algebra $sl(N, \mathbb{C})$ in the basis $T_a$, $\alpha \in \bar{\Gamma}_N$, take the form

$$[T_\alpha, T_\beta] = C(\alpha, \beta) T_{\alpha + \beta}, \quad C(\alpha, \beta) = N \sin \frac{\pi}{N}(\alpha \times \beta). \quad (3.8)$$

We define an invariant form on $gl(N, \mathbb{C})$:

$$(T_a, T_b) = \frac{4\pi^2}{N^2} \text{tr}(T_a \cdot T_b) = \delta_{a+b,0}. \quad (3.9)$$

We consider the decomposition of an element $S \in sl(N, \mathbb{C})$:

$$S = \sum_{\alpha \in \bar{\Gamma}_N} S_\alpha T_\alpha. \quad$$

From (3.8), the corresponding Poisson bracket is

$$\{S_\alpha, S_\beta\} = C(\alpha, \beta) S_{\alpha + \beta}. \quad (3.10)$$

We fix Casimir functions (2.20). This means that $S$ belongs to the corresponding coadjoint orbit $O_\nu$, Eq. (2.17).
Let $\wp(x)$ be the Weierstrass function and $\wp_\alpha = \wp(\frac{\alpha_1 + \alpha_2 x}{N})$, $\alpha \in \tilde{\Gamma}_N$. The integrable Euler–Arnold $SL(N, \mathbb{C})$ top is defined by the Hamiltonian

$$H = \frac{1}{2} \sum_{\alpha \in \tilde{\Gamma}_N} S_\alpha \wp_\alpha S_{-\alpha}$$

(3.11)

and Lie–Poisson bracket (3.10). The phase space $\mathcal{M}^{ET}$ of the elliptic top is the coadjoint orbit

$$\mathcal{M}^{ET} \sim \mathcal{O}_\nu.$$  

(3.12)

To find the commuting integrals of motion, we use the Lax operator.

**Lax operator.** In this case, the Lax operator $L(z)$ has the quasiperiodicities

$$L^{ET}(z + 1) = QL^{ET}(z)Q^{-1}, \quad L^{ET}(z + \tau) = \Lambda L^{ET}(z)\Lambda^{-1}.$$  

(3.13)

It can be defined in terms of the Kronecker function. The Kronecker function $\phi(u, z)$ is related to the elliptic curve $\Sigma_\tau$ and takes the form

$$\phi(u, z) = \frac{\vartheta(u + z)\vartheta'(0)}{\vartheta(u)\vartheta(z)},$$

(3.14)

where $\vartheta(z)$ is the theta function

$$\vartheta(z|\tau) = q^{1/8} \sum_{n \in \mathbb{Z}} (-1)^n e^{\pi i (n(n+1)\tau + 2nz)}.$$  

(3.15)

In addition, we need the Eisenstein function $E_1(z) = \partial_z \ln \vartheta(z)$. The $\eta_1$ constant is extracted from the asymptotics of $E_1(z)$,

$$E_1(z) \sim \frac{1}{z} + \eta_1 z + \cdots.$$  

The Weierstrass function $\wp$ is related to $E_1(z)$ as

$$\wp(u) = -\partial_z E_1(u) - 2\eta_1,$$

(3.16)

is double-periodic, and has a second-order pole at $z = 0$. The basis of the double-periodic functions on $\Sigma_\tau$ are

$$\{1, (-\partial_z)^{k-2} \wp(z), \ k = 2, 3, \ldots \}.$$  

(3.17)

They have poles of orders $0, 2, 3, \ldots$.

The Kronecker function has the quasiperiodicities

$$\phi(u, z + 1) = \phi(u, z), \quad \phi(u, z + \tau) = e^{-2\pi i u} \phi(u, z).$$

(3.18)

and has a first-order pole at $z = 0$:

$$\phi(u, z) \sim \frac{1}{z} + E_1(u) + \frac{1}{2} z (E_1^2(u) - \wp(u)) + \cdots.$$  

(3.19)

It is related to the Weierstrass function $\wp$ as

$$\phi(u, z)\phi(-u, z) = \wp(z) - \wp(u).$$  

(3.20)
Let
\[ \varphi_m(z) = e_N(-m_2 z) \phi\left( -\frac{m_1 + m_2 \tau}{N}, z \right), \quad m = (m_1, m_2). \]

Then
\[ \varphi_m(z + 1) = e_N(-m_2) \varphi_m(z), \quad \varphi_m(z + \tau) = e_N(m_1) \varphi_m(z). \]

The Lax operator assumes the form
\[ L^{\text{ET}}(z) = \sum_{\alpha \in \tilde{\Gamma}_N} S_{\alpha} \varphi_{\alpha}(z) T_{\alpha}. \quad (3.21) \]

It follows from (3.18) and (3.7) that \( L(z) \) has the needed quasiperiodicities (3.13). We note that there is no constant term because it does not satisfy Eq. (3.13). From (3.19),
\[ \text{Res} L^{\text{ET}}(z)|_{z=0} = \sum_{\alpha \in \tilde{\Gamma}_N} S_{\alpha} T_{\alpha}. \quad (3.22) \]

These properties imply that \( \{L(z)^k\} \) are double-periodic functions with poles of the order up to \( k \). Therefore, they can be expanded in a basis made of the Weierstrass functions and their derivatives, Eq. (3.17):
\[ \langle (L^{\text{ET}})^k(z) \rangle = I_{0,k} + I_{2,k} \varphi(z) + \cdots + I_{k,k} (\partial_z)^{k-2} \varphi(z). \quad (3.23) \]

The term \( I_{1,k} \) is absent because by the residue theorem there are no periodic functions with one simple pole on \( \Sigma_\tau \). In particular, it follows from (3.20) that
\[ \langle (L^{\text{ET}})^2(z) \rangle = I_{0,2} + I_{2,2} \varphi(z), \quad I_{0,2} = -2H^{\text{ET}} \quad (3.24) \]
(see (3.11)).

Thus, there are \( \frac{N(N+1)}{2} - 1 \) independent quantities \( I_{s,k} \) \( (k = 2, 3, \ldots, N, s = 0, 2, \ldots, k) \):
\[ I_{0,n} \quad I_{2,N} \quad \ldots \quad I_{N,N}, \]
\[ \ldots \ldots \ldots, \]
\[ I_{0,3} \quad I_{2,3} \quad I_{3,3}, \]
\[ I_{0,2} \quad I_{2,2}. \quad (3.25) \]

It follows from the existence of the classical \( r \)-matrix [9] that they Poisson commute pairwise. Therefore, they play the role of conservation laws of the elliptic top hierarchy on \( SL(N, \mathbb{C}) \). We have a tower of \( \frac{1}{2} N(N+1) - 1 \) independent integrals of motion. We note that \( I_{k,k} = (\mathcal{S}^k) \) with \( k = 2, 3, \ldots, N \) are Casimir functions (2.20). By fixing their values, we obtain coadjoint orbit (2.17). In this way, we eventually have \( N(N - 1)/2 \) integrals of motion. The number of integrals is equal to \( \frac{1}{2} \dim O_\nu \) (see (2.22)). Therefore, the elliptic top is a completely integrable system.

**Extension of the top.** We consider case 2 in (3.2). In the decomposition of Lax operator (3.21), we replace the coefficients \( S_{\alpha} \) with \( X_{\alpha} \):
\[ L^{\text{ETT}}(z) = \sum_{\alpha \in \tilde{\Gamma}_N} X_{\alpha} \varphi_{\alpha}(z) T_{\alpha}. \quad (3.26) \]
where
\[ X = \sum_{\alpha \in \Gamma_N} X_{\alpha} T_{\alpha}. \]  

(3.27)

We set \( \text{tr } X = X_0 = 0 \) here (see the first constraint in (2.49)). This condition corresponds to an \( T^*SL(N, \mathbb{C}) \)-bundle, defining which requires imposing the second constraint in (2.49), the one corresponding to gauge fixing. In this case, it is convenient to work with a system that is not fully gauged. In other words, we use \( X^S \) of form (3.27) in our construction.

Using expansion (3.23), we construct a tower of commuting integrals (3.25). In terms of the \( X_\alpha \) variables, we have \( N(N-1)/2 \) integrals of motion, as previously. But this number is less than \( \frac{1}{2} \dim T^*\mathcal{X}^G \) (see Eq. (2.30)). We recall that \( X^G \) is a special section of the bundle \( T^*\mathcal{X}^G \). The coordinates on \( T^*\mathcal{X}^G \) are \( \mathcal{P} \) and \( \mathcal{Q} \) defined in (2.35). In these terms, the \( I_{k,\ell} = ((X^G)^k) \) are no longer Casimir functions with respect to Darboux bracket (2.37). With respect to this bracket, we have \( \frac{1}{2} \dim T^*\mathcal{X}^S \) (see Eq. (2.45)).

In terms of the Darboux coordinates, the Hamiltonian takes the form
\[ H_{\text{EET}} = \frac{1}{2} \sum_{\alpha \in \Gamma_N} X_{\alpha} \mathcal{P}_{\alpha} X_{-\alpha} - \frac{1}{2} \sum_{\alpha \in \Gamma_N} (\mathcal{P}\mathcal{Q})_{\alpha} \mathcal{P}_{\alpha} (\mathcal{P}\mathcal{Q})_{-\alpha}. \]  

(3.28)

It is related to the Lax operator as in (3.24),
\[ ((L_{\text{EET}},2))(z) = I_{0,2} + I_{2,2} \mathcal{P}(z), \quad I_{0,2} = -2H_{\text{EET}}. \]

We now write the equations of motion. To explicitly define \( \mathcal{P}\mathcal{Q} \) in (3.26), we introduce a basis \( Y_a \) in the space of symmetric matrices \( p^G \), Eq. (2.27):
\[ Y_a = \frac{1}{2}(T_a + T_a^T) = \frac{1}{2}(T_{a1,a2} + e_N(a_1 a_2) T_{a1,-a2}) = \]
\[ = \frac{N}{4\pi i} e_N \left( \frac{a_1 a_2}{2} \right) Q^{a_1} (\Lambda^{a_2} + \Lambda^{-a_2}), \quad a \in \Gamma_N. \]

Then
\[ Y_a Y_b = \sum_{\gamma \in \Gamma_N} f_{a,b}^\gamma T_\gamma, \quad \text{where} \quad f_{a,b}^\gamma = f_{a,b}, \quad \gamma = a + b \]
and
\[ Y_a Y_b = \sum_{\gamma \in \Gamma_N} f_{a,b}^\gamma T_\gamma, \quad \text{where} \quad f_{a,b}^\gamma = f_{a,b}, \quad \gamma = a + b, \]
\[ f_{a,b} = \frac{N}{2\pi i} \left( e_N(a_1 b_2 - a_1 b_2) + e_N(-a_1 a_2 - b_1 b_2 + a_2 b_1 - a_1 b_2) + e_N(-a_1 a_2 - a_2 b_1 - a_1 b_2) + e_N(b_1 b_2 + a_1 b_2 + a_2 b_1) \right). \]

We have the decompositions
\[ \mathcal{P} = \sum_{a \in \Gamma_N} p_a Y_a, \quad \mathcal{Q} = \sum_{b \in \Gamma_N} q_b Y_b, \quad a_1, a_2 = 0, 1, \ldots, N - 1. \]  

(3.29)

Because \( X = \sum_{\gamma \in \Gamma_N} X_\gamma T_\gamma, \ X_\gamma = (\mathcal{P}\mathcal{Q})_\gamma \), we have
\[ X_\gamma = \sum_{(m,n) \in \Gamma_N} \mathcal{P}_m \mathcal{Q}_n f_{m,n}. \]
In view of $\text{tr} X = 0$, we have the quadratic constraints
$$
\sum_{(m,-m) \in \Gamma_N} p_m q_{-m} f_{m,-m} = 0.
$$
The corresponding equations of motion are
$$
\partial_t q_a = \sum_b q_b f_{a,b} \mathcal{P}^b q_a + f_{a,-b} p_{-b} q_{-b} - m q_m - n p_m,
\partial_t p_a = -\sum_b p_b f_{a,b} \mathcal{P}^b p_a + f_{a,-b} q_{-b} p_{-b} - m q_m - n p_m.
$$

### 3.2. Extension of Calogero–Moser system.

**Elliptic Calogero–Moser system with spin.** We consider the Chevalley basis \( \{ e_j = E_{jj}, E_{jK} \} \) in the Lie algebra \( \mathfrak{sl}(N, \mathbb{C}) \). The matrices \( S = \sum S_{jk} E_{jk} \) define the spin variables and the diagonal matrices
$$
\mathbf{u} = \sum_j u_j e_j, \quad \mathbf{v} = \sum_j v_j e_j, \quad \sum_j u_j = \sum_j v_j = 0,
$$
define the coordinates and momenta of \( N \) particles. They have the canonical brackets \( \{ v_j, u_k \} = \delta_{jk} \). The coefficients \( (S_{jj}, S_{jk}) \) satisfy the Poisson–Lie algebra \( \mathfrak{sl}(N, \mathbb{C}) \). The polynomials \( c_2 = \text{tr} S^2, \ldots, c_N = \text{tr} S^N \) (cf. (2.20)) are Casimir functions with respect to these brackets. Fixing them, we can invert the brackets and obtain a nondegenerate form \( \omega^{\mathbf{KK}} = (S, g^{-1} Dg) \) on the orbit \( S = g^{-1} \nu g \) in (2.18). The variables introduced above are coordinates on the phase space:
$$
\tilde{\mathcal{M}}^{\text{CM}} = \{ \mathbf{v}, \mathbf{u}, S \}.
$$

There is a residual gauge symmetry acting only on the spin variables \( H^C \in SL(N, \mathbb{C}) \):
$$
g \rightarrow gh^{-1}, \quad S \rightarrow hS h^{-1}.
$$
The corresponding moment constraints are \( S_{jj} = 0 \). To obtain nondegenerate brackets, we should in addition fix the gauge, for example, as \( S_{j,j+1} = 1 \). The brackets for the \( S_{jk} \) then become the Dirac brackets corresponding to these constraints. We then obtain the phase space for the elliptic CM system with spin:
$$
\mathcal{M}^{\text{CM}} = \tilde{\mathcal{M}}^{\text{CM}} // H^C = \{ (\mathbf{v}, \mathbf{u}), S_{jk} \text{ with } j \neq k, S_{j,j+1} = 1 \}.
$$
It has the dimension
$$
\dim_{\mathbb{C}} \mathcal{M}^{\text{CM}} = N(N - 1).
$$
We note that it coincides with \( \dim \mathcal{O}_\nu \) in (2.22).

The Hamiltonian of the elliptic CM system with spin has the form
$$
H^{\text{CM}} = \frac{1}{2} (\mathbf{v}, \mathbf{v}) + \sum_{j \neq k} S_{jk} S_{kj} \varphi(u_j - u_k).
$$
The Lax operator corresponding the CM has the quasiperiodicities
$$
L^{\text{CM}}(z + 1) = L^{\text{CM}}(z), \quad L^{\text{CM}}(z + \tau) = R L^{\text{CM}}(z) R^{-1},
$$
where \( R = \text{diag}(e(u_1), \ldots, e(u_N)) \). It follows from (3.18) and (3.19) that
$$
L^{\text{CM}}(z) = \sum_{j=1}^N v_j e_j + \sum_{j \neq k} S_{jk} \varphi(u_j, z) E_{jk}, \quad u_{jk} = u_j - u_k.
$$
In terms of the Lax operator, the Hamiltonian of system (3.34) is defined as in (3.24). We obtain \( \frac{N(N-1)}{2} - 1 \) integrals of motion similarly.
Extension of the system. We now assume that the residue of the second Lax operator in (3.2) is the section \( X \) of the cotangent bundle \( T^*X^S \), Eq. (2.7):

\[
X = \sum_{j=1}^{N} X_{jj} e_j + \sum_{j \neq k} X_{jk} E_{jk}.
\]

Then, as in (3.36), the corresponding Lax operator \( L_{ECM} \) of the extended CM model has the form

\[
L_{ECM}(z) = \sum_{j=1}^{N} v_j e_j + \sum_{j \neq k} X_{jk} \phi(u_{jk}, z) E_{jk}.
\] (3.37)

We here set \( X_{jj} = 0 \). These constraints occur, similarly to the foregoing, after the symplectic reduction to the subgroup of diagonal matrices \( H^C \). The symplectic quotient is the phase space of the extended CM system (cf. (3.30)):

\[
\widetilde{M}_{ECM} = \{v, u, X\}.
\]

Then

\[
M_{ECM} = \widetilde{M}_{ECM} / H^C = \{(v, u), X_{jk} \text{ with } j \neq k, X_{j,j+1} = 1\},
\] (3.38)

It coincides with \( \dim C T^*X^G \) in (2.45). The Hamiltonian of the system takes the form (see(3.34))

\[
H_{ECM} = \frac{1}{2}(v, v) + \sum_{j \neq k} X_{jk} X_{kj} \phi(u_j - u_k).
\]

As in the case of the elliptic top, we use expansion (3.24) to define a tower of commuting integrals (3.25). As previously, we have \( \frac{N(N+1)}{2} - 1 \) integrals of motion. The quantities \( I_{k,k} = ((X)^k) \) are no longer Casimir functions with respect to the Darboux bracket in (2.37). With respect to this bracket, we have \( \frac{N(N+1)}{2} - 1 \) integrals. This number is less than \( \frac{1}{2} \dim T^*X^G \) in (2.30), but coincides with \( \frac{1}{2} \dim T^*X^S \) in (2.45).

We note that \( H^C \subset SL(N, \mathbb{C}) \) lies in \( X^S \) (\( H^C \not\subset SO(N, \mathbb{C}) \)). We can consider the additional left action of the subgroup \( H^C \) on the moduli space \( M_{ECM} \). The subgroup does not act on the variables \( v \) and \( u \) and acts on \( g \to hg \) as in (2.50). We found previously that \( O_\nu = H \setminus \nu T^*X^S \) (see (2.52)). Because the left action does not affect the variables \( (v, u) \), it follows that

\[
H^C \setminus \nu M_{ECM} = M_{ET} \sim O_\nu
\] (3.39)

(see (3.12)).

We describe the phase space of the extended CM system in terms of the Darboux variables. We consider the basis

\[
Y_{jk} = \frac{1}{2}(E_{jk} + E_{kj}), \quad e_j = E_{jj}
\]
in the space of symmetric matrices. As in (3.29),

\[
P = \sum_{j} p_j e_j + \sum_{j > k} p_{jk} Y_{jk}, \quad Q = \sum_{j} q_j e_j + \sum_{j > k} q_{jk} Y_{jk}.
\] (3.40)

Then

\[
X = P Q = \sum_{a} X_a e_a + \sum_{a \neq b} X_{ab} E_{ab},
\]

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Because

\[ e_l Y_{jk} = f_{ijk}^a E_{ab}, \quad f_{ijk}^a = \frac{1}{2} (\delta_{ij} \delta_{la} \delta_{kb} + \delta_{ik} \delta_{ja} \delta_{lb}), \]

\[ Y_{jk} e_l = f_{jk}^a E_{ab}, \quad f_{jk}^a = \frac{1}{2} (\delta_{jk} \cdot 23 \delta_{ja} \delta_{kb} + \delta_{lk} \delta_{ja} \delta_{lb}), \]

\[ Y_{jk} Y_{mn} = f_{jkmn}^a E_{ab}, \quad f_{jkmn}^a = \frac{1}{4} (\delta_{aj} \delta_{bn} \delta_{km} + \delta_{aj} \delta_{bm} \delta_{kn} + \delta_{aj} \delta_{bn} \delta_{jm} + \delta_{ak} \delta_{bm} \delta_{jn}), \]

\[ f_{jkmn}^a = \frac{1}{4} (\delta_{aj} \delta_{jn} \delta_{km} + \delta_{ak} \delta_{km} \delta_{jn} + \delta_{ak} \delta_{km} \delta_{jm} + \delta_{ka} \delta_{km} \delta_{jn}), \]

we obtain

\[ X_a = p_a q_a + \sum_{jkmn} f_{jkmn}^a p_{jk} q_{mn}, \]

\[ X_{ab} = \sum_{ljk} p_{lj} f_{ljk}^a + \sum_{jkmn} f_{jkmn}^a p_{jk} q_{mn}. \]  

(3.41)

The Hamiltonian \( H^{ECM} \) (3.39) is a quartic polynomial in the Darboux variables. In addition, there are the quadratic constraints \( X_a = 0 \).

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