Substitutions over infinite alphabet generating \((-\beta)\)-integers

Daniel Dombek
Department of Mathematics FNSPE
Czech Technical University in Prague
Czech Republic
dombedan@fjfi.cvut.cz

1 Introduction

This contribution is devoted to the study of positional numeration systems with negative base introduced by Ito and Sadahiro in 2009, called \((-\beta)\)-expansions. We give an admissibility criterion for more general case of \((-\beta)\)-expansions and discuss the properties of the set of \((-\beta)\)-integers, denoted by \(\mathbb{Z}_{-\beta}\). We give a description of distances within \(\mathbb{Z}_{-\beta}\) and show that this set can be coded by an infinite word over an infinite alphabet, which is a fixed point of a non-erasing non-trivial morphism.

2 Numeration with negative base

In 1957, Rényi introduced positional numeration system with positive real base \(\beta > 1\) (see [7]). The \(\beta\)-expansion of \(x\in[0,1)\) is defined as the digit string \(d_\beta(x) = 0 \cdot x_1x_2x_3\cdots\), where

\[ x_i = \lfloor \beta T_{i-1}^{-1}(x) \rfloor \quad \text{and} \quad T_\beta(x) = \beta x - \lfloor \beta x \rfloor. \]

It holds that

\[ x = \frac{x_1}{\beta} + \frac{x_2}{\beta^2} + \frac{x_3}{\beta^3} + \cdots. \]

Note that this definition can be naturally extended so that any real number has a unique \(\beta\)-expansion, which is usually denoted \(d_\beta(x) = x_0x_1x_2\cdots\), where \(\cdot\), the fractional point, separates negative and non-negative powers of \(\beta\). In analogy with standard integer base, the set \(\mathbb{Z}_\beta\) of \(\beta\)-integers is defined as the set of real numbers having the \(\beta\)-expansion of the form \(d_\beta(x) = x_0x_1x_2\cdots\).

\((-\beta)\)-expansions, a numeration system built in analogy with Rényi \(\beta\)-expansions, was introduced in 2009 by Ito and Sadahiro (see [5]). They gave a lexicographic criterion for deciding whether some digit string is the \((-\beta)\)-expansion of some \(x\) and also described several properties of \((-\beta)\)-expansions concerning symbolic dynamics and ergodic theory. Note that dynamical properties of \((-\beta)\)-expansions were also studied by Frougny and Lai (see [4]). We take the liberty of defining \((-\beta)\)-expansions in a more general way, while an analogy with positive base numeration can still be easily seen.

**Definition 1.** Let \(-\beta < -1\) be a base and consider \(x\in[l,l+1)\), where \(l\in\mathbb{R}\) is arbitrary fixed. We define the \((-\beta)\)-expansion of \(x\) as the digit string \(d(x) = x_1x_2x_3\cdots\), with digits \(x_i\) given by

\[ x_i = \lfloor -\beta T_{i-1}(x) - l \rfloor, \quad (1) \]

where \(T(x)\) stands for the generalised \((-\beta)\)-transformation

\[ T : [l,l+1) \rightarrow [l,l+1), \quad T(x) = -\beta x - \lfloor -\beta x - l \rfloor. \quad (2) \]
It holds that
\[ x = \frac{x_1}{-\beta} + \frac{x_2}{(\beta)^2} + \frac{x_3}{(\beta)^3} + \cdots \]
and the fractional point is again used in the notation, \( d(x) = 0 \cdot x_1 x_2 x_3 \cdots \).

The set of digits used in \((-\beta)\)-expansions of numbers (in the latter referred to as the alphabet of \((-\beta)\)-expansions) depends on the choice of \( l \) and can be calculated directly from (1) as
\[
\mathcal{A}_{-\beta,l} = \{ \lfloor -l(\beta + 1) \rfloor, \ldots, \lfloor -l(\beta + 1) \rfloor \}.
\]

We may demand that the numeration system possesses various properties. Let us summarise the most natural ones:

- The most common requirement is that zero is an allowed digit. We see that \( 0 \in \mathcal{A}_{-\beta,l} \) is equivalent to \( 0 \in [l, l + 1) \) and consequently \( l \in (-1, 0) \). Note that this implies \( d(0) = 0 \cdot 0^a \).
- We may require that \( \mathcal{A}_{-\beta,l} = \{0, 1, \ldots, \lfloor \beta \rfloor \} \). This is equivalent to the choice \( l \in \left(-\frac{\lfloor \beta \rfloor + 1}{\beta + 1}, -\frac{\beta}{\beta + 1}\right] \).
- So far, \((-\beta)\)-expansions were defined only for numbers from \([l, l + 1)\). In Rényi numeration, the \( \beta \)-expansion of arbitrary \( x \in \mathbb{R}^+ \) (expansions of negative numbers differ only by \( \langle \rangle \) sign) is defined as \( d_\beta(x) = x_k x_{k-1} \cdots x_1 x_0 \cdot x_{-1} x_{-2} \cdots \), where \( k \in \mathbb{N} \) satisfies \( \frac{x_k}{\beta} \in [l, l + 1) \) and \( d_\beta \left( \frac{x_k}{\beta} \right) = 0 \cdot x_k x_{k-1} x_{k-2} \cdots \). The same procedure does not work for \((-\beta)\)-expansions in general. A necessary and sufficient condition for the existence of unique \( d(x) \) for all \( x \in \mathbb{R} \) is that \( -\frac{1}{\beta} [l, l + 1) \subset [l, l + 1) \).

This is equivalent to the choice \( l \in \left(-\frac{\beta}{\beta + 1}, -\frac{1}{\beta + 1}\right] \). Note that this choice is disjoint with the previous one, so one cannot have uniqueness of \((-\beta)\)-expansions and non-negative digits bounded by \( \beta \) at the same time.

Let us stress that in the following we will need 0 to be a valid digit. Therefore, we shall always assume \( l \in (-1, 0) \). Note that we may easily derive that the digits in the alphabet \( \mathcal{A}_{-\beta,l} \) are then bounded by \( \lfloor \beta \rfloor \) in modulus.

### 3 Admissibility

In Rényi numeration there is a natural correspondence between ordering on real numbers and lexicographic ordering on their \( \beta \)-expansions. In \((-\beta)\)-expansions, standard lexicographic ordering is not suitable anymore, hence a different ordering on digit strings is needed.

The so-called alternate order was used in the admissibility condition by Ito and Sadahiro and it will work also in the general case. Let us recall the definition. For the strings
\[
u, v \in (\mathcal{A}_{-\beta,l})^\mathbb{N}, \quad u = u_1 u_2 u_3 \cdots \quad \text{and} \quad v = v_1 v_2 v_3 \cdots
\]
we say that \( u \prec_{alt} v \) (\( u \) is less than \( v \) in the alternate order) if \( u_m (-1)^m < v_m (-1)^m \), where \( m = \min \{ k \in \mathbb{N} \mid u_k \neq v_k \} \). Note that standard ordering between reals in \([l, l + 1)\) corresponds to the alternate order on their respective \((-\beta)\)-expansions.

**Definition 2.** An infinite string \( x_1 x_2 x_3 \cdots \) of integers is called \((-\beta)\)-admissible (or just admissible), if there exists an \( x \in [l, l + 1) \) such that \( x_1 x_2 x_3 \cdots \) is its \((-\beta)\)-expansion, i.e. \( x_1 x_2 x_3 \cdots = d(x) \).

We give the criterion for \((-\beta)\)-admissibility (proven in [2]) in a form similar to both Parry lexicographic condition (see [6]) and Ito-Sadahiro admissibility criterion (see [5]).
Theorem 3. (2) An infinite string $x_1x_2x_3\cdots$ of integers is $(-\beta)$-admissible, if and only if
\[ l_1l_2l_3\cdots \preceq_{alt} x_ix_{i+1}x_{i+2}\cdots \prec_{alt} r_1r_2r_3\cdots, \quad \text{for all } i \geq 1, \tag{4} \]
where $l_1l_2l_3\cdots = d(l)$ and $r_1r_2r_3\cdots = d^*(l+1) = \lim_{\epsilon \to 0^+} d(l+1 - \epsilon)$.

Remark 4. Ito and Sadahiro have described the admissibility condition for their numeration system considered with $l = -\frac{\beta}{\beta + 1}$. This choice imply for any $\beta$ the alphabet of the form $\mathcal{A}_{-\beta,1} = \{0,1,\ldots, [\beta]\}$. They have shown that in this case the reference strings used in the condition in Theorem 3 (i.e. $d(l) = l_1l_2l_3\cdots$ and $d^*(l+1) = r_1r_2r_3\cdots$) are related in the following way:
\[ r_1r_2r_3\cdots = 0l_1l_2l_3\cdots \]
if $d(l)$ is not purely periodic with odd period length, and,
\[ r_1r_2r_3\cdots = (0l_1l_2\cdots l_q-1(l_q-1))^{\omega}, \]
if $d(l) = (l_1l_2\cdots l_q)^{\omega}$, where $q$ is odd.

Remark 5. Besides Ito-Sadahiro case and the general one, we may consider another interesting example, the choice $l = -\frac{1}{2}$, $\beta \notin 2\mathbb{Z} + 1$. This leads to a numeration defined on “almost symmetric” interval $[-\frac{1}{2}, \frac{1}{2})$ with symmetric alphabet
\[ \mathcal{A}_{-\beta,-\frac{1}{2}} = \left\{ \left[-\frac{\beta + 1}{2}, \frac{\beta + 1}{2}\right], 0, 1, \ldots, \left[-\frac{\beta + 1}{2}\right] \right\}. \]

Note that we use the notation $(-a) = \overline{a}$ for shorter writing of negative digits. If we denote the reference strings as usual, i.e. $d\left(-\frac{1}{2}\right) = l_1l_2l_3\cdots$ and $d^*(\frac{1}{2}) = r_1r_2r_3\cdots$, the following relation can be shown:
\[ r_1r_2r_3\cdots = \overline{l_1l_2l_3}\cdots \]
if $d(l)$ is not purely periodic with odd period length, and,
\[ r_1r_2r_3\cdots = (\overline{l_1l_2\cdots l_q-1(l_q-1)}l_1l_2\cdots l_q-1(l_q-1))^{\omega}, \]
if $d(l) = (l_1l_2\cdots l_q)^{\omega}$, where $q$ is odd.

4 \hspace{1cm} \hspace{0.5cm} (-\beta)-\text{integers}

We have already discussed basic properties of $(-\beta)$-expansions and the question of admissibility of digit strings. In the following, $(-\beta)$-admissibility will be used to define the set of $(-\beta)$-integers.

Let us define a “value function” $\gamma$. Consider a finite digit string $x_1x_2\cdots x_q$, then $\gamma(x_1,\cdots, x_q) = \sum_{i=1}^{k-1} x_i(-\beta)^i$.

Definition 6. We call $x \in \mathbb{R}$ a $(-\beta)$-integer, if there exists a $(-\beta)$-admissible digit string $x_kx_{k-1}\cdots x_00^{\omega}$ such that $d(x) = x_kx_{k-1}\cdots x_1x_0 \bullet 0^{\omega}$. The set of $(-\beta)$-integers is then defined as
\[ \mathbb{Z}_{-\beta} = \{ x \in \mathbb{R} \mid x = \gamma(a_{k-1}a_{k-2}\cdots a_1a_0), a_{k-1}a_{k-2}\cdots a_1a_00^{\omega} \text{ is } (-\beta)-\text{admissible} \}, \]
or equivalently
\[ \mathbb{Z}_{-\beta} = \bigcup_{i \geq 0} (-\beta)^i T^{-i}(0). \]
Note that \((-\beta)\)-expansions of real numbers are not necessarily unique. As was said before, uniqueness holds if and only if \(l \in (-\frac{1}{\beta+1}, -\frac{1}{\beta+1}]\). Let us demonstrate this ambiguity on the following example.

**Example 7.** Let \(\beta\) be the greater root of the polynomial \(x^2 - 2x - 1\), i.e. \(\beta = 1 + \sqrt{2}\), and let \([l, l+1) = \left(-\frac{1}{\beta+1}, \frac{1}{\beta+1}\right)\). Note that \([l, l+1)\) is not invariant under division by \((-\beta)\).

If we want to find the \((-\beta)\)-expansion of number \(x \notin [l, l+1)\), we have to find such \(k \in \mathbb{N}\) that \(\frac{1}{\beta^k} x \in [l, l+1)\), compute \(d\left(\frac{1}{\beta^k} x\right)\) by definition and then shift the fractional point by \(k\) positions to the right. The problem is that, in general, different choices of the exponent \(k\) may give different \((-\beta)\)-admissible digit strings which all represent the same number \(x\).

Let us find possible \((-\beta)\)-expansions of 1. It can be shown that \(\frac{1}{\beta^k} x \in [l, l+1)\) if and only if \(k \in \mathbb{N} \setminus \{0, 2, 4, 6, 8\}\) and there are 5 \((-\beta)\)-admissible digit strings representing 1, computed from \((-\beta)\)-expansions of \(\frac{1}{\beta^k} x\) for \(k = 1, 3, 5, 7, 9\) respectively:

\[
1 \cdot 0^\omega = 120 \cdot 0^\omega = 13210 \cdot 0^\omega = 132210 \cdot 0^\omega = 13222210 \cdot 0^\omega.
\]

Let us mention some straightforward observations on the properties of \(\mathbb{Z}_{-\beta}\):

- \(\mathbb{Z}_{-\beta}\) is nonempty if and only if 0 \(\in \mathcal{A}_{-\beta,l}\), i.e. if and only if \(l \in (-1, 0]\).
- The definition implies \(-\beta \mathbb{Z}_{-\beta} \subset \mathbb{Z}_{-\beta}\).
- A phenomenon unseen in Rényi numeration arises, there are cases when the set of \((-\beta)\)-integers is trivial, i.e. when \(\mathbb{Z}_{-\beta} = \{0\}\). This happens if and only if both numbers \(\frac{1}{\beta}\) and \(-\frac{1}{\beta}\) are outside of the interval \([l, l+1)\). This can be reformulated as

\[
\mathbb{Z}_{-\beta} = \{0\} \iff \beta < -\frac{1}{l} \quad \text{and} \quad \beta \leq \frac{1}{l+1},
\]

and it can be seen that the strictest limitation for \(\beta\) arises when \(l = -\frac{1}{2}\). This implies for any choice of \(l \in \mathbb{R}\):

\[
\mathbb{Z}_{-\beta} \neq \emptyset \quad \text{and} \quad \beta \geq 2 \Rightarrow \mathbb{Z}_{-\beta} \supseteq \{0\}.
\]

- It holds that \(\mathbb{Z}_{-\beta} = \mathbb{Z}\) if and only if \(\beta \in \mathbb{N}\).

**Remark 8.** As was shown in Example 7 in a completely general case of \((-\beta)\)-expansions, there is a problem with ambiguity. Because of this, in the following we shall limit ourselves to the choice \(l \in \left(-\frac{1}{\beta+1}, -\frac{1}{\beta+1}\right]\). Note that we allow Ito-Sadahiro case \(l = -\frac{1}{\beta+1}\), which also contains ambiguities, but only in countably many cases, which can be avoided by introducing a notion of strong \((-\beta)\)-admissibility.

**Definition 9.** Let \(x_1x_2x_3 \cdots \in \mathcal{A}_{-\beta,l}\). We say that

\[
x_1x_2x_3 \cdots \text{ is strongly } (-\beta)\text{-admissible} \quad \text{if} \quad 0x_1x_2x_3 \cdots \text{ is } (-\beta)\text{-admissible}.
\]

**Remark 10.** Note that if \(l \in \left(-\frac{1}{\beta+1}, -\frac{1}{\beta+1}\right]\), the notions of strong admissibility and admissibility coincide. In the case \(l = -\frac{1}{\beta+1}\), the only numbers with non-unique expansions are those of the form \((-\beta)^k l\), which have exactly two possible expansions using digit strings \(l_1l_2l_3 \cdots\) and \(1l_1l_2l_3 \cdots\). While both are \((-\beta)\)-admissible, only the latter is also strongly \((-\beta)\)-admissible.
In order to describe distances between adjacent \((-\beta)-\)integers, we will study ordering of finite digit strings in the alternate order. Denote by \(\mathcal{S}(k)\) the set of infinite \((-\beta)-\)admissible digit strings such that erasing a prefix of length \(k\) yields \(0^\omega\), i.e. for \(k \geq 0\), we have

\[ \mathcal{S}(k) = \{a_{k-1}a_{k-2} \cdots a_00^\omega \mid a_{k-1}a_{k-2} \cdots a_00^\omega \text{ is } (-\beta)-\text{admissible}\}, \]

in particular \(\mathcal{S}(0) = \{0^\omega\}\). For a fixed \(k\), the set \(\mathcal{S}(k)\) is finite. Denote by \(\text{Max}(k)\) the string \(a_{k-1}a_{k-2} \cdots a_00^\omega\) which is maximal in \(\mathcal{S}(k)\) with respect to the alternate order and by \(\text{max}(k)\) its prefix of length \(k\), i.e. \(\text{Max}(k) = \text{max}(k)0^\omega\). Similarly, we define \(\text{Min}(k)\) and \(\text{min}(k)\). Thus,

\[ \text{Min}(k) \preceq_{\text{alt}} r \preceq_{\text{alt}} \text{Max}(k), \quad \text{for all digit strings } r \in \mathcal{S}(k). \]

With this notation we can give a theorem describing distances in \(\mathbb{Z}_{-\beta}\) valid for cases \(l \in \left[-\frac{\beta}{\beta+1}, -\frac{1}{\beta+1}\right]\).

Note that for case \(l = \frac{\beta}{\beta+1}\) it was proven in \([I]\).

**Theorem 11.** Let \(x < y\) be two consecutive \((-\beta)-\)integers. Then there exist a finite string \(w\) over the alphabet \(\mathcal{A}_{-\beta,1}\), a non-negative integer \(k \in \{0, 1, 2, \ldots\}\) and a positive digit \(d \in \mathcal{A}_{-\beta,1} \setminus \{0\}\) such that \(w(d-1)\text{Max}(k)\) and \(wd\text{Min}(k)\) are strongly \((-\beta)-\)admissible strings and

\[
\begin{align*}
    x &= \gamma(w(d-1)\text{max}(k)) &< & y = \gamma(wd\text{min}(k)) &\text{ for } k \text{ even,} \\
    x &= \gamma(wd\text{min}(k)) &< & y = \gamma(w(d-1)\text{max}(k)) &\text{ for } k \text{ odd.}
\end{align*}
\]

In particular, the distance \(y - x\) between these \((-\beta)-\)integers depends only on \(k\) and equals to

\[
\Delta_k := \left|(-\beta)^k + \gamma(\text{min}(k)) - \gamma(\text{max}(k))\right|. \tag{5}
\]

## 5 Coding \(\mathbb{Z}_{-\beta}\) by an infinite word

Note that in order to get an explicit formula for distances from Theorem\([\text{X}]\) knowledge of reference strings \(\text{min}(k)\) and \(\text{max}(k)\) is necessary. These depend on both reference strings \(d(l)\) and \(d^*(l+1)\). Concerning the form of \(\text{min}(k)\) and \(\text{max}(k)\) we provide the following proposition.

**Proposition 12.** Let \(\beta > 1\). Denote \(d(l) = l_1l_2l_3 \cdots, d^*(l+1) = r_1r_2r_3 \cdots\).

- \(\text{min}(0) = \text{max}(0) = \varepsilon\).
- For \(k \geq 1\) either \(\text{min}(k) = l_1l_2 \cdots l_k\) or there exists \(m(k) \in \{0, \ldots, k-1\}\) such that
  \[
  \text{min}(k) = \begin{cases} 
  l_1l_2 \cdots (l_{k-m(k)}+1)\text{min}(m(k)) & \text{if } k-m(k) \text{ even} \\
  l_1l_2 \cdots (l_{k-m(k)}-1)\text{max}(m(k)) & \text{if } k-m(k) \text{ odd}
  \end{cases}
  \]
- For \(k \geq 1\) either \(\text{max}(k) = r_1r_2 \cdots r_k\) or there exists \(m'(k) \in \{0, \ldots, k-1\}\) such that
  \[
  \text{max}(k) = \begin{cases} 
  r_1r_2 \cdots (r_{k-m'(k)}-1)\text{max}(m'(k)) & \text{if } k-m'(k) \text{ even} \\
  r_1r_2 \cdots (r_{k-m'(k)}+1)\text{min}(m'(k)) & \text{if } k-m'(k) \text{ odd}
  \end{cases}
  \]
Computing \( \min(k) \) and \( \max(k) \) for a general choice of \( l \) may lead to difficult discussion, however, in special cases an important relation between \( d(l) \) and \( d^*(l+1) \) arises and eases the computation. Examples were given in Remarks \([\text{4,5]}\).

Let us now describe how we can code the set of \((\beta)-\)integers by an infinite word over the infinite alphabet \( \mathbb{N} \).

Let \( (z_n)_{n \in \mathbb{Z}} \) be a strictly increasing sequence satisfying
\[
  z_0 = 0 \quad \text{and} \quad \mathbb{Z}_{-\beta} = \{z_n \mid n \in \mathbb{Z}\}.
\]
We define a bidirectional infinite word over an infinite alphabet \( v_{-\beta} \in \mathbb{N}^\mathbb{Z} \), which codes the set of \((\beta)-\)integers. According to Theorem \([\text{11]}\) for any \( n \in \mathbb{Z} \) there exist a unique \( k \in \mathbb{N} \), a word \( w \) with prefix 0 and a letter \( d \) such that
\[
  z_{n+1} - z_n = |\gamma(w(d-1)\max(k)) - \gamma(wd\min(k))|.
\]
We define the word \( v_{-\beta} = (v_i)_{i \in \mathbb{Z}} \) by \( v_n = k \).

**Theorem 13.** Let \( v_{-\beta} \) be the word associated with \((\beta)-\)integers. There exists an antimorphism \( \Phi : \mathbb{N}^* \to \mathbb{N}^* \) such that \( \Psi = \Phi^2 \) is a non-erasing non-identical morphism and \( \Psi(v_{-\beta}) = v_{-\beta} \). \( \Phi \) is always of the form
\[
  \Phi(2l) = S_2l(2l+1)\bar{R}_{2l} \quad \text{and} \quad \Phi(2l+1) = R_{2l+1}(2l+2)\bar{S}_{2l+1},
\]
where \( \bar{u} \) denotes the reversal of the word \( u \) and words \( R_j, S_j \) depend only on \( j \) and \( \min(k), \max(k) \) with \( k \in \{j, j+1\} \).

The proof is based on the self-similarity of \( \mathbb{Z}_{-\beta} \), i.e. \( \beta \mathbb{Z}_{-\beta} \subset \mathbb{Z}_{-\beta} \), and on the following idea. Let \( x = \gamma(w(d-1)\max(k)) < y = \gamma(wd\min(k)) \) be two neighbours in \( \mathbb{Z}_{-\beta} \) with gap \( \Delta_k \) and suppose only \( k \) even. If we multiply both \( x \) and \( y \) by \((\beta)\), we get a longer gap with possibly more \((\beta)-\)integers in between. It can be shown that between \( -\beta y \) and \( -\beta x \) there is always a gap \( \Delta_{k+1} \). Hence the description is of the form \( \Phi(k) = S_k(k+1)\bar{R}_k \), where the word \( S_k \) codes the distances between \((\beta)-\)integers in \( \gamma(\min(k)\beta) ; \gamma(wd\min(k+1)) \) and, similarly, \( R_k \) encodes distances within the interval \( \gamma(w(d-1)\max(k)\beta) ; \gamma(w(d-1)\max(k+1)) \).

As it turns out, in some cases (mostly when reference strings \( l_1l_2l_3 \cdots \) and \( r_1r_2r_3 \cdots \) are eventually periodic of a particular form) we can find a letter-to-letter projection to a finite alphabet \( \Pi : \mathbb{N} \to \mathcal{B} \) with \( \mathcal{B} \subset \mathbb{N} \), such that \( u_{-\beta} = \Pi v_{-\beta} \) also encodes \( \mathbb{Z}_{-\beta} \) and it is a fixed point of a non-erasing antimorphism \( \varphi = \Pi \circ \Phi \) over the finite alphabet \( \mathcal{B} \). Clearly, the square of \( \varphi \) is then a non-erasing morphism over \( \mathcal{B} \) which fixes \( u_{-\beta} \).

Let us mention that \((\beta)-\)integers in the Ito-Sadahiro case \( l_2 = \frac{1\beta}{l+1} \) are also subject of \([\text{8]}\). For \( \beta \) with eventually periodic \( d(l) \), Steiner finds a coding of \( \mathbb{Z}_{-\beta} \) by a finite alphabet and shows, using only the properties of the \((\beta)-\)transformation, that the word is a fixed point of a non-trivial morphism. Our approach is of a combinatorial nature, follows a similar idea as in \([\text{11]}\) and shows existence of an antimorphism for any base \( \beta \).

To illustrate the results, let us conclude this contribution by an example.

**Example 14.** Let \( \beta \) be the real root of \( x^3 - 3x^2 - 4x - 2 \) (\( \beta \) Pisot, \( \approx 4.3 \)) and \( l = -\frac{1}{2} \). The admissibility condition gives us for any admissible digit string \( (x_i)_{i \geq 0} \):
\[
  201^a_\text{alt} x_ix_{i+1}x_{i+2} \cdots \leq_\text{alt} 20_{\Pi_0} \quad \text{for all} \ x \geq 0.
\]

We obtain
\[
  \min(0) = \varepsilon, \quad \min(1) = 2, \quad \min(2) = 20
\]
and
\[
\min(2k+1) = 20(11)^{k-1}0, \quad \min(2k+2) = 20(11)^k \quad \text{for} \ k \geq 1.
\]
Clearly it holds that \(\max(i) = \min(i)\) for all \(i \in \mathbb{N}\).

Theorem [7] gives us the following distances within \(\mathbb{Z}_{-\beta}\):
\[
\Delta_0 = 1, \quad \Delta_1 = -1 + \frac{4}{\beta} + \frac{2}{\beta^2}, \quad \text{and} \quad \Delta_{2k} = 1 - \frac{2}{\beta} - \frac{2}{\beta^2}, \quad \Delta_{2k+1} = 1 + \frac{2}{\beta} + \frac{2}{\beta^2} \quad \text{for} \ k \geq 1.
\]

Finally, the antimorphism \(\Phi : \mathbb{N}^* \to \mathbb{N}^*\) is given by
\[
0 \to 0^210^2, \\
1 \to 2, \\
2 \to 3,
\]
and for \(k \geq 1\)
\[
2k+1 \to 0^210(2k+2)010^2, \\
2k+2 \to 2k+3.
\]

It can be easily seen that a projection from \(\mathbb{N}\) to a finite alphabet exists and a final antimorphism \(\varphi : \{0,1,2,3\}^* \to \{0,1,2,3\}^*\) is of the form
\[
0 \to 0^210^2, \\
1 \to 2, \\
2 \to 3, \\
3 \to 0^2102010^2.
\]

Bibliography

[1] P. Ambrož, D. Dombek, Z. Masáková, E. Pelantová, Numbers with integer expansion in the numeration system with negative base, preprint (2011), 25pp. [arXiv:0912.4597v3 [math.NT]]
[2] D. Dombek, Z. Masáková, E. Pelantová, Number representation using generalized \((-\beta)\)-transformation, preprint (2011), 22pp. [arXiv:1102.3079v1 [cs.DM]]
[3] S. Fabre, Substitutions et \(\beta\)-systèmes de numération, Theoret. Comput. Sci. 137, 219–236 (1995). doi:10.1016/0304-3975(95)91132-A
[4] Ch. Frougny and A. C. Lai, On negative bases, Proceedings of DLT 09, Lectures Notes in Computer Science 5583 (2009). doi:10.1007/978-3-642-02737-6_20
[5] S. Ito and T. Sadahiro, Beta-expansions with negative bases, INTEGERS 9, 239–259 (2009). doi:10.1515/INTEG.2009.023
[6] W. Parry, On the \(\beta\)-expansions of real numbers, Acta Math. Acad. Sci. Hung. 11, 401–416 (1960).
[7] A. Rényi, Representations for real numbers and their ergodic properties, Acta Math. Acad. Sci. Hung. 8, 477–493 (1957).
[8] W. Steiner, On the structure of \((-\beta)\)-integers, preprint (2010), 15pp. arXiv:1011.1755v1 [math.NT]
[9] W. P. Thurston, Groups, tilings, and finite state automata, AMS Colloquium Lecture Notes, American Mathematical Society, Boulder (1989).