Inequalities for algebraic Casorati curvatures and their applications

Dedicated to Felice Casorati

Abstract. The notion of different kind of algebraic Casorati curvatures are introduced. Some results expressing basic Casorati inequalities for algebraic Casorati curvatures are presented. Equality cases are also discussed. As a simple application, basic Casorati inequalities for different $\delta$-Casorati curvatures for Riemannian submanifolds are presented. Further applying these results, Casorati inequalities for Riemannian submanifolds of real space forms are obtained. Finally, some problems are presented for further studies.

AMS 2000 Mathematics Subject Classification: 53B20.

Keywords: Casorati curvature, algebraic Casorati curvature, Casorati inequalities.

1 Introduction

Felice Casorati was one of the great Italian mathematicians, best known for the Casorati-Weierstrass theorem in complex analysis. He was born in Pavia on December 17, 1835 and his soul departed on September 11, 1890 in Casteggio. Before his departure, in 1889, Casorati [8] defined a curvature for a regular surface in Euclidean 3-space which turns out to be the normalized sum of the squared principal curvatures. In [9], the author says that he could not check the paper [8] before printing, and advises readers to rather use a subsequent paper [10]. This curvature is now well known as the Casorati curvature. Several geometers believe that Casorati preferred this curvature over the traditional Gaussian curvature because the Casorati curvature vanishes for a surface in Euclidean 3-space if and only if both Euler normal curvatures (or principal curvatures) of the surface vanish simultaneously and thus corresponds better with the common intuition of curvature. For a hypersurface of a Riemannian manifold the Casorati curvature is defined to be the normalized sum of the squared principal normal curvatures of the hypersurface, and in general, the Casorati curvature of a submanifold of a Riemannian manifold is defined to be the normalized squared length of the second fundamental form [22]. Geometrical meaning and the importance of the Casorati curvature, discussed by several geometers, can be visualized in several research/survey papers including [19], [23], [24], [28], [30], [33], [34], [46], [56] and [57].

The paper is organized as follows. In section 2, some preliminaries regarding curvature like tensors are presented. In section 3, given an $n$-dimensional Riemannian manifold $(M, g)$, a Riemannian vector bundle $(B, g_B)$ over $M$, a $B$-valued symmetric $(1,2)$-tensor

\textsuperscript{1}Submitted to Note di Mat

1
field \( \zeta \) and a (curvature-like) tensor field \( T \) satisfying the algebraic Gauss equation, we introduce the notion of different kind of algebraic Casorati curvatures \( \hat{\delta}_{C,T,\zeta}(n-1), \delta_{C,T,\zeta}(n-1), \hat{\delta}_{C,T}(r;n-1), \delta_{C,T}(r;n-1), \hat{\delta}(n-1), \delta(n-1) \), which in special cases of Riemannian submanifolds reduce to already known \( \delta \)-Casorati curvatures. In section 4, first we prove an useful Lemma regarding a constrained extremum problem. Then we present results expressing basic Casorati inequalities for algebraic Casorati curvatures. Equality cases are also discussed. After this, application parts begin. In section 5, we obtain basic Casorati inequalities for Casorati curvatures \( \delta(r;n-1), \hat{\delta}(r;n-1), \delta(n-1), \hat{\delta}(n-1) \) for Riemannian submanifolds. In section 6, we further apply these results to obtain Casorati inequalities for Riemannian submanifolds of real space forms with very short proofs. Finally, in section 7, we present some problems for further studies.

2 Curvature like tensor

In 1967, R.S. Kulkarni introduced the notion of a curvature structure (cf. [35, §8 of Chapter 1], [36]), which is now widely known as a curvature-like tensor (field). Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold. Let \(T\) be a curvature-like tensor so that it satisfies the following properties

\[
T(X, Y, Z, W) = -T(Y, X, Z, W),
\]
\[
T(X, Y, Z, W) = T(Z, W, X, Y),
\]
\[
T(X, Y, Z, W) + T(Y, Z, X, W) + T(Z, X, Y, W) = 0
\]

for all vector fields \(X, Y, Z\) and \(W\) on \(M\). For a curvature-like tensor field \(T\), the \(T\)-sectional curvature associated with a 2-plane section \(\Pi_2\) spanned by orthonormal vectors \(X\) and \(Y\) at \(p \in M\), is given by [6]

\[
K_T(\Pi_2) = K_T(X \wedge Y) = T(X, Y, Y, X).
\]

Let \(\{e_1, e_2, \ldots, e_n\}\) be any orthonormal basis of \(T_p M\). The \(T\)-Ricci tensor \(S_T\) is defined by

\[
S_T(X, Y) = \sum_{j=1}^{n} T(e_j, X, Y, e_j), \quad X, Y \in T_p M.
\]

The \(T\)-Ricci curvature is given by

\[
\text{Ric}_T(X) = S_T(X, X), \quad X \in T_p M.
\]

The \(T\)-scalar curvature is given by [6]

\[
\tau_T(p) = \sum_{1 \leq i < j \leq n} T(e_i, e_j, e_j, e_i), \quad (2.4)
\]

Now, let \(\Pi_k\) be a \(k\)-plane section of \(T_p M\) and \(X\) a unit vector in \(\Pi_k\). If \(k = n\) then \(\Pi_n = T_p M\); and if \(k = 2\) then \(\Pi_2\) is a plane section of \(T_p M\). We choose an orthonormal basis \(\{e_1, \ldots, e_k\}\) of \(\Pi_k\). Then we define the \(T\)-\(k\)-Ricci curvature of \(\Pi_k\) at \(e_i, i \in \{1, \ldots, k\}\), denoted \((\text{Ric}_T)_{\Pi_k}(e_i)\), by

\[
(\text{Ric}_T)_{\Pi_k}(e_i) = \sum_{j=1, j \neq i}^{k} K_T(e_i \wedge e_j).
\]

(2.5)
We note that a \( T-n\text{-Ricci curvature} \) \((\text{Ric}_T)_{T_pM}(e_i)\) is the usual \( T\text{-Ricci curvature} \) of \( e_i \), denoted \( \text{Ric}_T(e_i) \). The \( T-k\text{-scalar curvature} \) \( \tau_T(\Pi_k) \) of the \( k \)-plane section \( \Pi_k \) is given by

\[
\tau_T(\Pi_k) = \sum_{1 \leq i < j \leq k} K_T(e_i \wedge e_j). \tag{2.6}
\]

We note that

\[
\tau_T(\Pi_k) = \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1, j \neq i}^{k} K_T(e_i \wedge e_j) = \frac{1}{2} \sum_{i=1}^{k} (\text{Ric}_T)_{\Pi_k}(e_i). \tag{2.7}
\]

The \( T\)-scalar curvature of \( M \) at \( p \) is identical with the \( T-n\)-scalar curvature of the tangent space \( T_pM \) of \( M \) at \( p \), that is, \( \tau_T(p) = \tau_T(T_pM) \). If \( \Pi_2 \) is a 2-plane section, \( \tau_T(\Pi_2) \) is nothing but the \( T\)-sectional curvature \( K_T(\Pi_2) \) of \( \Pi_2 \). The \( T-k\text{-normalized scalar curvature} \) of a \( k \)-plane section \( \Pi_k \) at \( p \) is defined as

\[
(\tau_T)_{\text{Nor}}(\Pi_k) = \frac{2}{k(k-1)} \tau_T(\Pi_k).
\]

The \( T\text{-normalized scalar curvature} \) at \( p \) is defined as

\[
(\tau_T)_{\text{Nor}}(p) = (\tau_T)_{\text{Nor}}(T_pM) = \frac{2}{n(n-1)} \tau_T(p).
\]

If \( T \) is replaced by the Riemann curvature tensor \( R \), then \( T\)-sectional curvature \( K_T \), \( T\)-Ricci tensor \( S_T \), \( T\)-Ricci curvature \( \text{Ric}_T \), \( T\)-scalar curvature \( \tau_T \), \( T\)-normalized scalar curvature \( (\tau_T)_{\text{Nor}} \), \( T-k\)-Ricci curvature \( (\text{Ric}_T)_{\Pi_k} \), \( T\)-k-scalar curvature \( \tau_T(\Pi_k) \), \( T\)-k-normalized scalar curvature \( (\tau_T)_{\text{Nor}}(\Pi_k) \) and \( T\)-normalized scalar curvature \( (\tau_T)_{\text{Nor}} \) become the sectional curvature \( K \), the Ricci tensor \( S \), the Ricci curvature \( \text{Ric} \), the scalar curvature \( \tau \), the normalized scalar curvature \( \tau_{\text{Nor}} \), \( k\)-Ricci curvature \( \text{Ric}_{\Pi_k} \), \( k\)-scalar curvature \( \tau(\Pi_k) \), \( k\)-normalized scalar curvature \( \tau_{\text{Nor}}(\Pi_k) \) and normalized scalar curvature \( \tau_{\text{Nor}} \), respectively.

### 3 Algebraic Casorati curvatures

Let \((M, g)\) be an \( n \)-dimensional submanifold of an \( m \)-dimensional Riemannian manifold \((\tilde{M}, \tilde{g})\). The equation of Gauss is given by

\[
R(X, Y, Z, W) = \tilde{R}(X, Y, Z, W) + \tilde{g}(\sigma(Y, Z), \sigma(X, W)) - \tilde{g}(\sigma(X, Z), \sigma(Y, W)) \tag{3.1}
\]

for all \( X, Y, Z, W \in TM \), where \( \tilde{R} \) and \( R \) are the curvature tensors of \( \tilde{M} \) and \( M \), respectively and \( \sigma \) is the second fundamental form of the immersion of \( M \) in \( \tilde{M} \). The Ricci-Kühn equation is given by

\[
R^\perp(X, Y, N, V) = \tilde{R}(X, Y, N, V) + g([A_N, A_V]X, Y) \tag{3.2}
\]

for all \( X, Y \in TM \) and for all \( N, V \in T^\perp M \), where

\[
R^\perp(X, Y)N = \nabla^\perp_X \nabla^\perp_Y N - \nabla^\perp_Y \nabla^\perp_X N - \nabla^\perp_{[X,Y]} N,
\]

\[
[A_N, A_V] = A_N A_V - A_V A_N,
\]

with \( \nabla^\perp \) being the induced normal connection in the normal bundle \( T^\perp M \) and \( A_N \) being the shape operator in the direction \( N \).
Let $M$ be an $n$-dimensional Riemannian submanifold of an $m$-dimensional Riemannian manifold $\widetilde{M}$. A point $p \in M$ is said to be an *invariantly quasi-umbilical point* if there exist $m - n$ mutually orthogonal unit normal vectors $N_{n+1}, \ldots, N_m$ such that the shape operators with respect to all directions $N_\alpha$ have an eigenvalue of multiplicity $n - 1$ and that for each $N_\alpha$ the distinguished eigendirection is the same. The submanifold is said to be an *invariantly quasi-umbilical submanifold* if each of its points is an invariantly quasi-umbilical point. For details, we refer to [4].

Let $(M, g)$ be an $n$-dimensional Riemannian submanifold of an $m$-dimensional Riemannian manifold $(\widetilde{M}, \widetilde{g})$. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space $T_p M$ and $e_\alpha$ belongs to an orthonormal basis $\{e_{n+1}, \ldots, e_m\}$ of the normal space $T_p^\perp M$. We let

$$
\sigma_{ij}^\alpha = \widetilde{g} (\sigma (e_i, e_j), e_\alpha), \quad i, j \in \{1, \ldots, n\}, \quad \alpha \in \{n + 1, \ldots, m\}.
$$

Then, the squared mean curvature of the submanifold $M$ in $\widetilde{M}$ is defined by

$$
\|H\|^2 = \frac{1}{n^2} \sum_{\alpha = n+1}^m \left( \sum_{i=1}^n \sigma_{ii}^\alpha \right)^2,
$$

and the squared norm of second fundamental form $\sigma$ is

$$
\|\sigma\|^2 = \sum_{i,j=1}^n \widetilde{g} (\sigma (e_i, e_j), \sigma (e_i, e_j)).
$$

Let $K_{ij}$ and $\widetilde{K}_{ij}$ denote the sectional curvature of the plane section spanned by $e_i$ and $e_j$ at $p$ in the submanifold $M$ and in the ambient manifold $\widetilde{M}$, respectively. In view of (3.1), we have

$$
K_{ij} = \widetilde{K}_{ij} + \sum_{\alpha = n+1}^m (\sigma_{ii}^\alpha \sigma_{jj}^\alpha - (\sigma_{ij}^\alpha)^2). \quad (3.3)
$$

From (3.3) it follows that

$$
2\tau(p) = 2\widetilde{\tau} (T_p M) + n^2 \|H\|^2 - \|\sigma\|^2,
$$

where

$$
\widetilde{\tau} (T_p M) = \sum_{1 \leq i < j \leq n} \widetilde{K}_{ij}
$$

denotes the scalar curvature of the $n$-plane section $T_p M$ in the ambient manifold $\widetilde{M}$. From (3.4), it immediately follows that

$$
\tau_{Nor}(p) = \widetilde{\tau}_{Nor} (T_p M) + \frac{n}{n-1} \|H\|^2 - \frac{1}{n(n-1)} \|\sigma\|^2, \quad (3.5)
$$

where

$$
\tau_{Nor}(p) = \frac{2\tau(p)}{n(n-1)}, \quad \widetilde{\tau}_{Nor} (T_p M) = \frac{2\widetilde{\tau} (T_p M)}{n(n-1)}. \quad (3.6)
$$

The *Casorati curvature* $C$ [22] of the Riemannian submanifold $M$ is defined to be the normalized squared length of the second fundamental form $\sigma$, that is,

$$
C = \frac{1}{n} \|\sigma\|^2 = \frac{1}{n} \sum_{\alpha = n+1}^m \sum_{i,j=1}^n (\sigma_{ij}^\alpha)^2. \quad (3.7)
$$
For a $k$-dimensional subspace $\Pi_k$ of $T_pM$, $k \geq 2$ spanned by $\{e_1, \ldots, e_k\}$, the Casorati curvature $C(\Pi_k)$ of the subspace $\Pi_k$ is defined to be [21]

$$C(\Pi_k) = \frac{1}{k} \sum_{\alpha=n+1}^{m} \sum_{i,j=1}^{k} (\sigma^\alpha_{ij})^2.$$  

The normalized $\delta$-Casorati curvatures $\hat{\delta}_C(n-1)$, $\delta'_C(n-1)$ of a Riemannian submanifold $M$ are given by [21]

$$[\hat{\delta}_C(n-1)]_p = 2C_p - \frac{2n-1}{2n} \sup \{C(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_pM\}, \quad (3.8)$$

$$[\delta'_C(n-1)]_p = \frac{1}{2} C_p + \frac{n+1}{2n(n-1)} \inf \{C(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_pM\}. \quad (3.9)$$

In [21], the authors denoted $\delta_C(n-1)$ by $\delta_C(n-1)$. The (modified) normalized $\delta$-Casorati curvatures $\delta_C(n-1)$ of the Riemannian submanifold $M$ is given by ([39], [64])

$$[\delta_C(n-1)]_p = \frac{1}{2} C_p + \frac{n+1}{2n} \inf \{C(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_pM\}. \quad (3.10)$$

It should be noted that the normalized $\delta$-Casorati curvatures $\hat{\delta}_C(n-1)$, $\delta'_C(n-1)$ and $\delta_C(n-1)$ vanish trivially for $n=2$ [64]. In [39], the authors pointed out that the coefficient $\frac{n+1}{2n(n-1)}$ in (3.9) was inappropriate and therefore they modified the coefficient from $\frac{n+1}{2n(n-1)}$ to $\frac{n+1}{2n}$ in the definition of $\delta'_C(n-1)$ to obtain the definition of $\delta_C(n-1)$ (see also [40]). For a positive real number $r \neq n(n-1)$, letting

$$a(r) = \frac{1}{nr} (n-1) (n+r) (n^2 - n - r),$$

the normalized $\delta$-Casorati curvatures $\delta_C(r; n-1)$ and $\hat{\delta}_C(r; n-1)$ of a Riemannian submanifold $M$ are given by [22]

$$[\delta_C(r; n-1)]_p = rC_p + a(r) \inf \{C(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_pM\}, \quad (3.11)$$

if $0 < r < n(n-1)$, and

$$[\hat{\delta}_C(r; n-1)]_p = rC_p + a(r) \sup \{C(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_pM\}, \quad (3.12)$$

if $n(n-1) < r$, respectively.

In [39] the normalized $\delta$-Casorati curvatures $\hat{\delta}_C(r; n-1)$ and $\delta_C(r; n-1)$ are called as the generalized normalized $\delta$-Casorati curvatures $\delta'_C(r; n-1)$ and $\delta_C(r; n-1)$, respectively. We see that [40]

$$[\delta_C(n-1)]_p = \frac{1}{n(n-1)} \left[ \delta_C \left( \frac{n(n-1)}{2}; n-1 \right) \right]_p, \quad (3.13)$$

$$[\hat{\delta}_C(n-1)]_p = \frac{1}{n(n-1)} \left[ \hat{\delta}_C(2n(n-1); n-1) \right]_p \quad (3.14)$$

for all $p \in M$. 

5
Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold and \((B, g_B)\) a Riemannian vector bundle over \(M\). If \(\zeta\) is a \(B\)-valued symmetric \((1, 2)\)-tensor field and \(T\) a \((0, 4)\)-tensor field on \(M\) such that

\[
T(X, Y, Z, W) = g_B(\zeta(X, W), \zeta(Y, Z)) - g_B(\zeta(X, Z), \zeta(Y, W)) \tag{3.15}
\]

for all vector fields \(X, Y, Z, W\) on \(M\), then the equation (3.15) is said to be an \textit{algebraic Gauss equation} \cite{15}. Every \((0, 4)\)-tensor field \(T\) on \(M\), which satisfies (3.15), becomes a curvature-like tensor.

A typical example of an algebraic Gauss equation is given for a submanifold \(M\) of an Euclidean space, if \(B\) is the normal bundle, \(\zeta\) the second fundamental form and \(T\) the curvature tensor. Some nice situations, in which such \(T\) and \(\zeta\) satisfying an algebraic Gauss equation exist, are Lagrangian and Kaehlerian slant submanifolds of complex space forms and \(C\)-totally real submanifolds of Sasakian space forms.

Now, let \(\{e_1, \ldots, e_n\}\) be an orthonormal basis of the tangent space \(T_pM\) and \(e_\alpha\) belong to an orthonormal basis \(\{e_{n+1}, \ldots, e_m\}\) of the Riemannian vector bundle \((B, g_B)\) over \(M\) at \(p\). We put

\[
\zeta_{ij}^\alpha = g_B(\zeta(e_i, e_j), e_\alpha), \quad \|\zeta\|^2 = \sum_{i,j=1}^n g_B(\zeta(e_i, e_j), \zeta(e_i, e_j)),
\]

\[
\text{trace} \zeta = \sum_{i=1}^n \zeta(e_i, e_i), \quad \|\text{trace} \zeta\|^2 = g_B(\text{trace} \zeta, \text{trace} \zeta).
\]

Motivated by the definitions given in \cite{21}, \cite{22} and \cite{39} we give the following definitions.

**Definition 3.1** Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold, \((B, g_B)\) a Riemannian vector bundle over \(M\), \(\zeta\) a \(B\)-valued symmetric \((1, 2)\)-tensor field on \(M\), and \(T\) a curvature-like tensor field satisfying the algebraic Gauss equation (3.15). Then the \textit{algebraic Casorati curvature} \(C^T,\zeta\) with respect to \(T\) and the Riemannian vector bundle \((B, g_B)\) over \(M\) is defined to be

\[
C^T,\zeta = \frac{1}{n} \|\zeta\|^2 = \frac{1}{n} \sum_{\alpha=n+1}^m \sum_{i,j=1}^n \left(\zeta_{ij}^\alpha\right)^2. \tag{3.16}
\]

For a \(k\)-dimensional subspace \(\Pi_k\) of \(T_pM\), \(k \geq 2\), spanned by \(\{e_1, \ldots, e_k\}\), the \textit{algebraic Casorati curvature} \(C^T,\zeta(\Pi_k)\) of the subspace \(\Pi_k\) is defined to be

\[
C^T,\zeta(\Pi_k) = \frac{1}{k} \sum_{\alpha=n+1}^m \sum_{i,j=1}^k \left(\zeta_{ij}^\alpha\right)^2. \tag{3.17}
\]

We note that

\[
C_p^T,\zeta = C^T,\zeta(T_pM), \quad p \in M.
\]

**Definition 3.2** Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold, \((B, g_B)\) a Riemannian vector bundle over \(M\), \(\zeta\) a \(B\)-valued symmetric \((1, 2)\)-tensor field on \(M\), and \(T\) a curvature-like tensor field satisfying the algebraic Gauss equation (3.15). Then we define the following three \textit{algebraic Casorati curvatures} \(\delta_{C^T,\zeta}(n-1)\) and \(\hat{\delta}_{C^T,\zeta}(n-1)\) and by

\[
[\delta_{C^T,\zeta}(n-1)]_p = \frac{1}{2} C_p^T,\zeta + \frac{n+1}{2n} \inf \left\{C^T,\zeta(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_pM\right\}, \tag{3.18}
\]

\[
[\hat{\delta}_{C^T,\zeta}(n-1)]_p = \frac{1}{2} C_p^T,\zeta + \frac{n+1}{2n} \inf \left\{C^T,\zeta(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_pM\right\}.
\]
\[ [\hat{\delta}_{C^p}(n - 1)]_p = 2C^p_r - \frac{2n - 1}{2n} \sup \{ C^p_r : \Pi_{n-1} \text{ is a hyperplane of } T_p M \}. \quad (3.19) \]

**Definition 3.3** Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold, \((B, g_B)\) a Riemannian vector bundle over \(M\), \(\zeta\) a \(B\)-valued symmetric \((1,2)\)-tensor field on \(M\), and \(T\) a curvature-like tensor field satisfying the algebraic Gauss equation (3.15). For a positive real number \(r \neq n(n - 1)\), let

\[ a(r) = \frac{1}{nr} (n - 1) (n + r) (n^2 - n - r) \]

and define the *algebraic Casorati curvatures* \(\delta_{C^p, \zeta}(r; n - 1)\) and \(\hat{\delta}_{C^p, \zeta}(r; n - 1)\) by

\[ [\delta_{C^p, \zeta}(r; n - 1)]_p = r C^p_r + a(r) \inf \{ C^p_r : \Pi_{n-1} \text{ is a hyperplane of } T_p M \} \]

if \(0 < r < n(n - 1)\), and

\[ [\hat{\delta}_{C^p, \zeta}(r; n - 1)]_p = r C^p_r + a(r) \sup \{ C^p_r : \Pi_{n-1} \text{ is a hyperplane of } T_p M \} \]

if \(n(n - 1) < r\).

**Remark 3.4** Let \((M, g)\) be an \(n\)-dimensional Riemannian submanifold of an \(m\)-dimensional Riemannian manifold \((\hat{M}, \hat{g})\). Let the Riemannian vector bundle \((B, g_B)\) over \(M\) be replaced by the normal bundle \(T^1 M\), and the \(B\)-valued symmetric \((1,2)\)-tensor field \(\zeta\) be replaced by the second fundamental form of immersion \(\sigma\). Then the algebraic Casorati curvature \(C^p_r\) becomes the *Casorati curvature* \(C\) of the Riemannian submanifold \(M\) given by (3.7). The algebraic Casorati curvatures \(\delta_{C^p, \zeta}(n - 1)\) and \(\hat{\delta}_{C^p, \zeta}(n - 1)\) become *normalized* \(\delta\)-Casorati curvatures \(\delta_C(n - 1)\) and \(\hat{\delta}_C(n - 1)\) of the Riemannian submanifold \(M\) given by (3.10) and (3.8), respectively. Finally, algebraic Casorati curvatures \(\delta_{C^p, \zeta}(r; n - 1)\) and \(\hat{\delta}_{C^p, \zeta}(r; n - 1)\) become normalized \(\delta\)-Casorati curvatures \(\delta_C(r; n - 1)\) and \(\hat{\delta}_C(r; n - 1)\) of the Riemannian submanifold \(M\) given by (3.11) and (3.12), respectively.

Now, we present the following useful Lemma.

**Lemma 3.5** Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold, \((B, g_B)\) a Riemannian vector bundle over \(M\) and \(\zeta\) a \(B\)-valued symmetric \((1,2)\)-tensor field. Let \(T\) be a curvature-like tensor field satisfying the algebraic Gauss equation (3.15). Then

\[ nC^p_r - \| \text{trace } \zeta \|^2 = -2\tau_T. \quad (3.22) \]

**Proof.** Let \(p \in M\), the set \(\{e_1, \ldots, e_n\}\) be an orthonormal basis of the tangent space \(T_p M\) and \(e_\alpha\) belong to an orthonormal basis \(\{e_{n+1}, \ldots, e_m\}\) of the Riemannian vector bundle \((B, g_B)\) over \(M\) at \(p\). From (3.15), we get

\[ (K_T)_{ij} = T(e_i, e_j, e_j) = \sum_{\alpha = n+1}^m (\zeta^\alpha_{ij} - (\zeta^\alpha_{ij})^2), \quad (3.23) \]

which implies that

\[ 2\tau_T = \| \text{trace } \zeta \|^2 - \| \zeta \|^2 = \| \text{trace } \zeta \|^2 - nC^p_r. \quad (3.24) \]

This gives (3.22). \(\blacksquare\)

7
4 Basic Casorati inequalities

We begin with the following two Lemmas:

**Lemma 4.1** ([18, Theorem 21.4, p. 425]) Let \( \Upsilon \subset \mathbb{R}^n \) be an open convex set in \( \mathbb{R}^n \). Then a \( C^2 \) function \( f : \Upsilon \to \mathbb{R} \) is a convex function on the open convex set \( \Upsilon \) if and only if for each \( x \in \Upsilon \), the Hessian of \( f \) at \( x \), denoted \((\text{Hess} f)_x\), is a positive semidefinite matrix.

**Lemma 4.2** ([18, Corollary 21.2, p. 429]) Let \( \Upsilon \subset \mathbb{R}^n \) be an open convex set in \( \mathbb{R}^n \). Let \( f : \Upsilon \to \mathbb{R} \) be a \( C^1 \) convex function with a point \( x_0 \in \Upsilon \) such that \( \text{grad} f(x_0) = 0 \), then the point \( x_0 \) is a global minimizer of \( f \) over \( \Upsilon \).

For application purposes, we prove the following

**Lemma 4.3** Let
\[
\Upsilon = \left\{ (x^1, \ldots, x^n) \in \mathbb{R}^n : x^1 + \cdots + x^n = k \right\}
\]
be a hyperplane of \( \mathbb{R}^n \), and \( f : \mathbb{R}^n \to \mathbb{R} \) a quadratic form given by
\[
f(x^1, \ldots, x^n) = a \sum_{i=1}^{n-1} (x^i)^2 + b (x^n)^2 - 2 \sum_{1 \leq i < j \leq n} x^i x^j, \quad a > 0, \ b > 0. \tag{4.1}
\]

Then the constrained extremum problem
\[
\min_{(x^1, \ldots, x^n) \in \Upsilon} f \tag{4.2}
\]
has a global solution given by
\[
\begin{cases}
x^1 = x^2 = \cdots = x^{n-1} = \frac{k}{a+1}, \\
x^n = \frac{k}{b+1} = \frac{n-1}{b} \left( \frac{k}{a+1} \right) = (a-n+2) \frac{k}{a+1},
\end{cases} \tag{4.3}
\]
provided that
\[
b = \frac{n-1}{a-n+2}. \tag{4.4}
\]

**Proof.** First we note that the set \( \Upsilon \) is an open convex set in \( \mathbb{R}^n \) and the function \( f \) is a \( C^\infty \) function (and hence a \( C^2 \) function). Now we compute the matrix for the Hessian \( \text{Hess} f \) of the function \( f \). The partial derivatives of the function \( f \) are
\[
\begin{cases}
\frac{\partial f}{\partial x^i} = 2 (a+1) x^i - 2 \sum_{\ell=1}^{n} x^\ell, \quad i \in \{1, \ldots, n-1\}, \\
\frac{\partial f}{\partial x^n} = 2 (b+1) x^n - 2 \sum_{\ell=1}^{n} x^\ell.
\end{cases} \tag{4.5}
\]
From (4.5), we have
\[
\begin{align*}
\frac{\partial^2 f}{\partial (x^i)^2} &= 2a, \quad i \in \{1, \ldots, n-1\}, \\
\frac{\partial^2 f}{\partial x^i \partial x^j} &= -2, \quad i, j \in \{1, \ldots, n-1\}, \\
\frac{\partial^2 f}{\partial x^i \partial x^n} &= -2, \quad i \in \{1, \ldots, n-1\}, \\
\frac{\partial^2 f}{\partial (x^n)^2} &= 2b.
\end{align*}
\]
(4.6)

Thus, in the standard frame of \(\mathbb{R}^n\), the Hess \(f\) has the matrix given by
\[
\begin{pmatrix}
2 & a & -1 & \cdots & -1 & -1 \\
-1 & a & \cdots & -1 & -1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-1 & -1 & \cdots & a & -1 \\
-1 & -1 & \cdots & -1 & b
\end{pmatrix}.
\]

We note that for any \(X = (X^1, \ldots, X^n) \in T_x \Upsilon, x \in \Upsilon\), it follows that \(\sum_{\ell=1}^{n} X^\ell = 0\). Consequently, for any \(X = (X^1, \ldots, X^n) \in T_x \Upsilon, x \in \Upsilon\) we have
\[
\text{Hess} f (X, X) \geq 0.
\]

Thus, for each \(x \in \Upsilon\), the Hessian \((\text{Hess} f)_x\) of \(f\) at \(x\) is positive semidefinite. In view of Lemma 4.1, this implies that the \(C^2\) function \(f\) is a convex function on the open convex set \(\Upsilon\).

For an optimal solution \((x^1, \ldots, x^n)\) of the problem (4.2), the vector \(\text{grad} f\) is normal to \(\Upsilon\), equivalently, it is collinear with the vector \((1, 1, \ldots, 1)\). From (4.5), for a critical point \(x = (x^1, \ldots, x^n)\) of the function \(f\) we have
\[
\begin{align*}
(a + 1) x^i - \sum_{\ell=1}^{n} x^\ell &= 0, \quad i \in \{1, \ldots, n-1\}, \\
(b + 1) x^n - \sum_{\ell=1}^{n} x^\ell &= 0.
\end{align*}
\]
(4.7)

From (4.7), it follows that a critical point \((x^1, \ldots, x^{n-1}, x^n)\) of the function \(f\) has the form
\[
x^1 = \cdots = x^{n-1} = t, \quad x^n = \frac{n-1}{b} t.
\]
(4.8)

Since
\[
x^1 + x^2 + \cdots + x^n = k,
\]
in view of (4.8), a critical point \((x^1, \ldots, x^n)\) of the considered problem is given by (4.3). Solving one of the following three relations appearing in (4.3)
\[
\frac{k}{b+1} = \frac{n-1}{b} \left( \frac{k}{a+1} \right) = (a - n + 2) \frac{k}{a+1},
\]
we get the equivalent relation given by (4.4). Consequently, in view of Lemma 4.2, the point \((x^1, \ldots, x^n)\) given by (4.3) is a global minimum point. Inserting (4.3) into (4.1) we have \(f (x^1, \ldots, x^n) = 0\). ■

Now, we present the following Theorem, involving the Casorati inequalities for algebraic Casorati curvatures \(\delta_{\mathcal{C}r, \zeta} (r; n-1)\) and \(\hat{\delta}_{\mathcal{C}r, \zeta} (r; n-1)\).
Theorem 4.4 Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold, \((B, g_B)\) a Riemannian vector bundle over \(M\) and \(\zeta\) a \(B\)-valued symmetric \((1,2)\)-tensor field. Let \(T\) be a curvature-like tensor field satisfying the algebraic Gauss equation (3.15). Then

\[
(\tau_T)_{Nor}(p) \leq \frac{1}{n(n-1)} [\delta_{\nabla r, \zeta}(r; n-1)]_p, \quad 0 < r < n(n-1),
\]

(4.9)

\[
(\tau_T)_{Nor}(p) \leq \frac{1}{n(n-1)} [\delta_{\nabla r, \zeta}(r; n-1)]_p, \quad n(n-1) < r.
\]

(4.10)

If

\[
\inf \{C^T,\zeta(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_p M\}
\]

(resp. \(\sup \{C^T,\zeta(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_p M\}\)

is attained by a hyperplane \(\Pi_{n-1}\) of \(T_p M\), \(p \in M\), then the equality sign holds in (4.9) (resp. (4.10)) if and only if with respect to a suitable orthonormal tangent frame \(\{e_1, \ldots, e_n\}\) and a suitable orthonormal frame \(\{e_{n+1}, \ldots, e_m\}\) of the Riemann vector bundle \((B, g_B)\), the components of \(\zeta\) satisfy

\[
\zeta^\alpha_{ij} = \begin{cases} 0 & i, j \in \{1, \ldots, n\}, \ i \neq j \ \alpha \in \{n+1, \ldots, m\}, \\ \zeta_{11}^\alpha = \zeta_{22}^\alpha = \cdots = \zeta_{n-1,n-1}^\alpha = \frac{r}{n(n-1)} \zeta_{nn}^\alpha & \alpha \in \{n+1, \ldots, m\}. \end{cases}
\]

(4.11)

(4.12)

Proof. Let \(p \in M\) and the set \(\{e_1, \ldots, e_n\}\) be an orthonormal basis of the tangent space \(T_p M\) and \(e_\alpha\) belong to an orthonormal basis \(\{e_{n+1}, \ldots, e_m\}\) of the Riemannian vector bundle \((B, g_B)\) over \(M\) at \(p\). We consider the following function

\[
\mathcal{P} = rC^T,\zeta + a(r)C^T,\zeta(\Pi_{n-1}) - 2\tau_T(p).
\]

(4.13)

where \(\Pi_{n-1}\) is a hyperplane of \(T_p M\). In view of (3.22), the relation (4.13) becomes

\[
\mathcal{P} = (n + r)C^T,\zeta + a(r)C^T,\zeta(\Pi_{n-1}) - \|\text{trace } \zeta\|^2.
\]

(4.14)

Without loss of generality, assume that the hyperplane \(\Pi_{n-1}\) is spanned by \(e_1, \ldots, e_{n-1}\). Then from (4.14) it follows that

\[
\mathcal{P} = \frac{n + r}{n} \sum_{\alpha=n+1}^m \sum_{i,j=1}^n (\zeta_{ij}^\alpha)^2 + \frac{2a(r)}{n-1} \sum_{\alpha=n+1}^m \sum_{i,j=1}^{n-1} (\zeta_{ij}^\alpha)^2 - \frac{2}{n-1} \sum_{\alpha=n+1}^m \left( \sum_{i=1}^n \zeta_{ii}^\alpha \right)^2.
\]

(4.15)

The function \(\mathcal{P}\) is a quadratic polynomial in the components of the tensor \(\zeta\) and can be written as

\[
\mathcal{P} = \sum_{\alpha=n+1}^m \left\{ 2 \left( \frac{r}{n} + \frac{a(r)}{n-1} + 1 \right) \sum_{1 \leq i < j \leq n-1} (\zeta_{ij}^\alpha)^2 + 2 \left( \frac{r}{n} + 1 \right) \sum_{i=1}^{n-1} (\zeta_{ii}^\alpha)^2 \right\}
\]

\[
+ \left( \frac{r}{n} + \frac{a(r)}{n-1} \right) \sum_{i=1}^{n-1} (\zeta_{ii}^\alpha)^2 + \frac{r}{n} (\zeta_{nn}^\alpha)^2 - 2 \sum_{1 \leq i < j \leq n} \zeta_{ii}^\alpha \zeta_{jj}^\alpha \right\}
\]

\[
\geq \sum_{\alpha=n+1}^m \left\{ \left( \frac{r}{n} + \frac{a(r)}{n-1} \right) \sum_{i=1}^{n-1} (\zeta_{ii}^\alpha)^2 + \frac{r}{n} (\zeta_{nn}^\alpha)^2 - 2 \sum_{1 \leq i < j \leq n} \zeta_{ii}^\alpha \zeta_{jj}^\alpha \right\}.
\]

(4.16)
For \( \alpha = n + 1, \ldots, m \), we consider a quadratic form
\[
f_\alpha : \mathbb{R}^n \to \mathbb{R}
\]
given by
\[
f_\alpha (\zeta_{11}^\alpha, \ldots, \zeta_{nn}^\alpha) = \left( \frac{r}{n} + \frac{a(r)}{n - 1} \right) \sum_{i=1}^{n-1} (\zeta_{ii}^\alpha)^2 + \frac{r}{n} (\zeta_{nn}^\alpha)^2 - 2 \sum_{1 \leq i < j \leq n} \zeta_{ii}^\alpha \zeta_{jj}^\alpha \quad (4.17)
\]
and the constrained extremum problem
\[
\min f_\alpha,
\]
subject to the condition
\[
\zeta_{11}^\alpha + \cdots + \zeta_{nn}^\alpha = k_\alpha,
\]
where \( k_\alpha \) is a real constant. Comparing (4.17) with (4.1), we see that
\[
a = \left( \frac{r}{n} + \frac{a(r)}{n - 1} \right), \quad b = \frac{r}{n},
\]
which verifies the relation
\[
b = \frac{n - 1}{a - n + 2}
\]
of (4.4). Thus applying Lemma 4.3, we see that the critical point
\[
\zeta^c = (\zeta_{11}^\alpha, \zeta_{22}^\alpha, \ldots, \zeta_{n-1n-1}^\alpha, \zeta_{nn}^\alpha)
\]
given by
\[
\zeta_{11}^\alpha = \zeta_{22}^\alpha = \cdots = \zeta_{n-1n-1}^\alpha = \frac{r}{(n-1)(n+r)} k_\alpha, \quad \zeta_{nn}^\alpha = \frac{n}{n+r} k_\alpha \quad (4.18)
\]
is a global minimum point. Inserting (4.18) into (4.17) we have \( f_\alpha (\zeta^c) = 0 \). Hence we have
\[
\mathcal{P} \geq 0, \quad (4.19)
\]
which in view of (4.13) gives
\[
\frac{2 \tau_T(p)}{n(n-1)} \leq \frac{r}{n(n-1)} C_p^{T,\zeta} + \frac{a(r)}{n(n-1)} C^{T,\zeta}(\Pi_{n-1}) \quad (4.20)
\]
for every tangent hyperplane \( \Pi_{n-1} \) of \( T_pM \).

If \( 0 < r < n(n-1) \), then \( a(r) > 0 \) and taking the infimum over all the tangent hyperplanes \( \Pi_{n-1} \) of \( T_pM \), the relation (4.20) gives the inequality (4.9). If \( n(n-1) < r \), then \( a(r) < 0 \), and taking the supremum over all the tangent hyperplanes \( \Pi_{n-1} \) of \( T_pM \), the relation (4.20) gives the inequality (4.10).

Suppose that
\[
\inf \{ C^{T,\zeta}(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_pM \}
\]
is attained by a hyperplane \( \Pi_{n-1} \) spanned by \( e_1, \ldots, e_{n-1} \). Then the equality sign holds in (4.9) (resp. (4.10)) if and only if we have the equality in all the previous inequalities. Thus the equality sign is true in the inequality (4.9) (resp. (4.10)) if and only if the relations (4.11) and (4.12) are true. \( \square \)

Now, we have the following two results.
Theorem 4.5 Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold, \((B, g_B)\) a Riemannian vector bundle over \(M\) and \(\zeta\) a \(B\)-valued symmetric \((1,2)\)-tensor field. Let \(T\) be a curvature-like tensor field satisfying the algebraic Gauss equation (3.15). Then the \(T\)-normalized scalar curvature \((\tau_T)_{\text{Nor}}\) is bounded above by the algebraic Casorati curvature \(\tilde{\delta}_{CR,\zeta}(n-1)\) given by (3.18), that is,

\[
(\tau_T)_{\text{Nor}}(p) \leq [\delta_{CR,\zeta}(n-1)]_p. \tag{4.21}
\]

If

\[
\inf \{C^{T,\zeta}(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_pM\}
\]

is attained by a hyperplane \(\Pi_{n-1}\) of \(T_pM\), then the equality sign holds in (4.21) if and only if with respect to suitable orthonormal tangent frame \(\{e_1, \ldots, e_n\}\) and orthonormal frame \(\{e_{n+1}, \ldots, e_m\}\), the components of \(\zeta\) satisfy

\[
\zeta^\alpha_{ij} = 0 \quad i, j \in \{1, \ldots, n\}, \quad i \neq j \quad \alpha \in \{n+1, \ldots, m\}, \tag{4.22}
\]

\[
\zeta^\alpha_{11} = \zeta^\alpha_{22} = \cdots = \zeta^\alpha_{n-1n-1} = \frac{1}{2} \zeta^\alpha_{nn}. \tag{4.23}
\]

Proof. Using

\[
[\delta_{CR,\zeta}(n-1)]_p = \frac{1}{n(n-1)} \left[ \delta_{CR,\zeta} \left( \frac{n(n-1)}{2} ; n-1 \right) \right]_p
\]

in (4.9), we get (4.21). Taking \(2r = n(n-1)\) in (4.12) we get (4.23). □

Theorem 4.6 Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold, \((B, g_B)\) a Riemannian vector bundle over \(M\) and \(\zeta\) a \(B\)-valued symmetric \((1,2)\)-tensor field. Let \(T\) be a curvature-like tensor field satisfying the algebraic Gauss equation (3.15). Then the \(T\)-normalized scalar curvature \((\tau_T)_{\text{Nor}}\) is bounded above by the algebraic Casorati curvature \(\tilde{\delta}_{CR,\zeta}(n-1)\), that is,

\[
(\tau_T)_{\text{Nor}}(p) \leq [\tilde{\delta}_{CR,\zeta}(n-1)]_p. \tag{4.25}
\]

If

\[
\sup \{C^{T,\zeta}(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_pM\}
\]

is attained by a hyperplane \(\Pi_{n-1}\) of \(T_pM\), then the equality sign in (4.25) is true if and only if with respect to a suitable orthonormal tangent frame \(\{e_1, \ldots, e_n\}\) and a suitable orthonormal frame \(\{e_{n+1}, \ldots, e_m\}\) of the Riemann vector bundle \((B, g_B)\), the components of \(\zeta\) satisfy

\[
\zeta^\alpha_{ij} = 0, \quad i, j \in \{1, \ldots, n\}, \quad i \neq j \quad \alpha \in \{n+1, \ldots, m\}, \tag{4.26}
\]

\[
\zeta^\alpha_{11} = \zeta^\alpha_{22} = \cdots = \zeta^\alpha_{n-1n-1} = 2 \zeta^\alpha_{nn}, \quad \alpha \in \{n+1, \ldots, m\}. \tag{4.27}
\]

Proof. Using

\[
[\tilde{\delta}_{CR,\zeta}(n-1)]_p = \frac{1}{n(n-1)} \left[ \tilde{\delta}_{CR,\zeta} \left( 2n(n-1) ; n-1 \right) \right]_p
\]

in (4.10), we get (4.25). Taking \(r = 2n(n-1)\) in (4.12) we get (4.27). □
5 Casorati inequalities for Riemannian submanifolds

Theorem 5.1 Let \((M, g)\) be an \(n\)-dimensional Riemannian submanifold of \(m\)-dimensional Riemannian manifold \((\widetilde{M}, \widetilde{g})\). Then the generalized normalized \(\delta\)-Casorati curvatures \(\delta_{C}(r; n-1)\) and \(\widehat{\delta}_{C}(r; n-1)\) satisfy

\[
\tau_{\text{Nor}}(p) \leq \frac{1}{n(n-1)}[\delta_{C}(r; n-1)]_{p} + \overline{\tau}_{\text{Nor}}(T_{p}M), \quad 0 < r < n(n-1), \quad (5.1)
\]

\[
\tau_{\text{Nor}}(p) \leq \frac{1}{n(n-1)}[\widehat{\delta}_{C}(r; n-1)]_{p} + \overline{\tau}_{\text{Nor}}(T_{p}M), \quad n(n-1) < r. \quad (5.2)
\]

If

\[
\inf\{C(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_{p}M\}
\]

(resp. \(\sup\{C(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_{p}M\}\))

is attained by a hyperplane \(\Pi_{n-1}\) of \(T_{p}M\), \(p \in M\), then the equality sign holds in (5.1) (resp. (5.2)) for all \(p \in M\) if and only if \((M, g)\) is an invariantly quasi-umbilical submanifold with trivial normal connection in \((\widetilde{M}, \widetilde{g})\), such that with respect to suitable tangent orthonormal frame \(\{e_{1}, \ldots, e_{n}\}\) and normal orthonormal frame \(\{e_{n+1}, \ldots, e_{m}\}\), the shape operators \(A_{\alpha} \equiv A_{e_{\alpha}}, \alpha \in \{n+1, \ldots, m\}\), take the following forms:

\[
A_{n+1} = \begin{pmatrix}
a & 0 & 0 & \cdots & 0 & 0 \\
0 & a & 0 & \cdots & 0 & 0 \\
0 & 0 & a & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & a & 0 \\
0 & 0 & 0 & \cdots & 0 & \frac{n(n-1)}{r}
\end{pmatrix}, \quad A_{n+2} = \cdots = A_{m} = 0. \quad (5.3)
\]

Proof. Let \((M, g)\) be an \(n\)-dimensional Riemannian submanifold of an \(m\)-dimensional Riemannian manifold \((\widetilde{M}, \widetilde{g})\). Let the Riemannian vector bundle \((B, g_{B})\) over \(M\) be replaced by the normal bundle \(T^{\perp}_{M}\), and the \(B\)-valued symmetric \((1,2)\)-tensor field \(\zeta\) be replaced by the second fundamental form of immersion \(\sigma\). In (3.15), we set

\[
T(X, Y, Z, W) = R(X, Y, Z, W) - \tilde{R}(X, Y, Z, W)
\]

with \(R\) the Riemann curvature tensor on \(M\) and \(\zeta = \sigma\). Then we see that

\[
(\tau_{T})_{\text{Nor}}(p) = \tau_{\text{Nor}}(p) - \overline{\tau}_{\text{Nor}}(T_{p}M),
\]

\[
\delta_{C,\zeta}(r; n-1) = \delta_{C}(r; n-1),
\]

\[
\widehat{\delta}_{C,\zeta}(r; n-1) = \widehat{\delta}_{C}(r; n-1).
\]

Using these facts in (4.9) and (4.10), we get (5.1) and (5.2), respectively.

The conditions of equality cases (4.11) and (4.12) become

\[
\sigma_{ij}^{\alpha} = 0 \quad i, j \in \{1, \ldots, n\}, \quad i \neq j \quad \alpha \in \{n+1, \ldots, m\} \quad (5.4)
\]

and

\[
\sigma_{11}^{\alpha} = \sigma_{22}^{\alpha} = \cdots = \sigma_{n-1n-1}^{\alpha} = \frac{r}{n(n-1)} \sigma_{nn}^{\alpha}, \quad \alpha \in \{n+1, \ldots, m\}, \quad (5.5)
\]
respectively. Thus the equality sign holds in both the inequalities (5.1) and (5.2) if and only if (5.4) and (5.5) are true.

The interpretation of the relations (5.4) is that the shape operators with respect to all normal directions $e_\alpha$ commute, or equivalently, that the normal connection $\nabla^\bot$ is flat, or still, that the normal curvature tensor $R^\bot$, that is, the curvature tensor of the normal connection, is trivial. Furthermore, the interpretation of the relations (5.5) is that there exist $m-n$ mutually orthogonal unit normal vectors $\{e_{n+1}, \ldots, e_m\}$ such that the shape operators with respect to all directions $e_\alpha$ ($\alpha \in \{e_{n+1}, \ldots, e_m\}$) have an eigenvalue of multiplicity $n-1$ and that for each $e_\alpha$ the distinguished eigendirection is the same (namely $e_n$), that is, the submanifold is invariantly quasi-umbilical [4].

Thus from the relations (5.4) and (5.5), we conclude that the equality holds in (5.1) and/or (5.2) for all $p \in M$ if and only if the Riemannian submanifold $M$ is invariantly quasi-umbilical with trivial normal connection $\nabla^\bot$ in $\tilde{M}$, such that with respect to suitable orthonormal tangent and normal orthonormal frames, the shape operators take the form given by (5.3).

\textbf{Theorem 5.2} Let $(M, g)$ be an $n$-dimensional Riemannian submanifold of $m$-dimensional Riemannian manifold $(\tilde{M}, \tilde{g})$. Then the normalized $\delta$-Casorati curvature $\delta_C(n-1)$ satisfies

$$\tau_{\text{Nor}}(p) \leq [\delta_C(n-1)]_p + \tau_{\text{Nor}}(T_p M).$$

If

$$\inf\{C(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_p M\}$$

is attained by a hyperplane $\Pi_{n-1}$ of $T_p M$, $p \in M$, then the equality sign holds if and only if $M$ is an invariantly quasi-umbilical submanifold with trivial normal connection in $\tilde{M}$, such that with respect to suitable orthonormal tangent frame $\{e_1, \ldots, e_n\}$ and normal orthonormal frame $\{e_{n+1}, \ldots, e_m\}$, the shape operators $A_\alpha \equiv A_{e_\alpha}$, $\alpha \in \{n+1, \ldots, m\}$, take the following forms

$$A_{n+1} = \begin{pmatrix}
a & 0 & 0 & \ldots & 0 & 0 \\
0 & a & 0 & \ldots & 0 & 0 \\
0 & 0 & a & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & a & 0 \\
0 & 0 & 0 & \ldots & 0 & 2a
\end{pmatrix}, \quad A_{n+2} = \cdots = A_m = 0.$$ (5.7)

\textbf{Proof.} Using (3.13) in (5.1), we get (5.6). Putting $2r = n(n-1)$ in (5.3) we get (5.7).

\textbf{Theorem 5.3} Let $(M, g)$ be an $n$-dimensional Riemannian submanifold of $m$-dimensional Riemannian manifold $(\tilde{M}, \tilde{g})$. Then the normalized $\delta$-Casorati curvature $\tilde{\delta}_C(n-1)$ satisfies

$$\tau_{\text{Nor}}(p) \leq [\tilde{\delta}_C(n-1)]_p + \tau_{\text{Nor}}(T_p M).$$

If

$$\sup\{C(\Pi_{n-1}) : \Pi_{n-1} \text{ is a hyperplane of } T_p M\}$$

is attained by a hyperplane $\Pi_{n-1}$ of $T_p M$, $p \in M$, then the equality sign holds if and only if $(M, g)$ is an invariantly quasi-umbilical submanifold with trivial normal connection in $(\tilde{M}, \tilde{g})$, such that with respect to suitable orthonormal tangent frame $\{e_1, \ldots, e_n\}$ and normal
orthonormal frame \( \{ e_{n+1}, \ldots, e_m \} \), the shape operators \( A_\alpha \equiv A_{e_\alpha}, \alpha \in \{ n+1, \ldots, m \} \), take the following forms:

\[
A_{n+1} = \begin{pmatrix}
  a & 0 & 0 & \cdots & 0 & 0 \\
  0 & a & 0 & \cdots & 0 & 0 \\
  0 & 0 & a & \cdots & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & 0 & \cdots & a & 0 \\
  0 & 0 & 0 & \cdots & 0 & \frac{1}{2} a
\end{pmatrix}, \quad A_{n+2} = \cdots = A_m = 0. \tag{5.9}
\]

**Proof.** Using (3.14) in (5.2), we get (5.8). Putting \( r = 2n(n-1) \) in (5.3) we get (5.9). \( \blacksquare \)

### 6 Casorati inequalities for submanifolds of real space forms

An \( m \)-dimensional Riemannian manifold \((\tilde{M}, \tilde{g})\) with constant sectional curvature \( c \), denoted \( \tilde{M}(c) \), is called a **real space form**, and its Riemann curvature tensor \( \tilde{R} \) is then given by

\[
\tilde{R}(X, Y, Z, W) = c \{ \tilde{g}(Y, Z) \tilde{g}(X, W) - \tilde{g}(X, Z) \tilde{g}(Y, W) \} \tag{6.1}
\]

for all vector fields \( X, Y, Z, W \) on \( \tilde{M} \). The model spaces for real space forms are the Euclidean spaces \((c = 0)\), the spheres \((c > 0)\), and the hyperbolic spaces \((c < 0)\). For an \( n \)-dimensional Riemannian submanifold \((M, g)\) of a real space form \( \tilde{M}(c) \) it is easy to see that

\[
\tilde{\tau}_{Nor}(T_p M) = c. \tag{6.2}
\]

**Theorem 6.1** [22, Theorem 2.1 and Corollary 3.1] Let \((M, g)\) be an \( n \)-dimensional Riemannian submanifold of \( m \)-dimensional real space form \( \tilde{M}(c) \). Then

\[
\tau_{Nor}(p) \leq \frac{1}{n(n-1)} [\delta_c(r; n-1)]_p + c, \quad 0 < r < n(n-1), \tag{6.3}
\]

\[
\tau_{Nor}(p) \leq \frac{1}{n(n-1)} [\tilde{\delta}_c(r; n-1)]_p + c, \quad n(n-1) < r. \tag{6.4}
\]

The equality sign holds in (6.3) (resp. (6.4)) for all \( p \in M \) if and only if \((M, g)\) is an invariantly quasi-umbilical submanifold with trivial normal connection in \( \tilde{M}(c) \), such that with respect to suitable tangent orthonormal frame \( \{ e_1, \ldots, e_n \} \) and normal orthonormal frame \( \{ e_{n+1}, \ldots, e_m \} \), the shape operators \( A_\alpha \equiv A_{e_\alpha}, \alpha \in \{ n+1, \ldots, m \} \), take the forms given by (5.3).

**Proof.** Using (6.2) in (5.1) and (5.2) we get (6.3) and (6.4), respectively. \( \blacksquare \)

**Theorem 6.2** (Theorem 4.1, [64]) Let \((M, g)\) be an \( n \)-dimensional Riemannian submanifold of \( m \)-dimensional real space form \( \tilde{M}(c) \). Then the normalized \( \delta \)-Casorati curvature \( \delta_c(n-1) \) satisfies

\[
\tau_{Nor}(p) \leq [\delta_c(n-1)]_p + c. \tag{6.5}
\]

Moreover, the equality sign holds for all \( p \in M \) if and only if \((M, g)\) is an invariantly quasi-umbilical submanifold with trivial normal connection in \( \tilde{M}(\tilde{g}) \), such that with respect to suitable orthonormal tangent frame \( \{ e_1, \ldots, e_n \} \) and normal orthonormal frame \( \{ e_{n+1}, \ldots, e_m \} \), the shape operators \( A_\alpha \equiv A_{e_\alpha}, \alpha \in \{ n+1, \ldots, m \} \), take the forms given by (5.7).
Proof. Using (3.13) in (6.3), we get (6.5).

**Theorem 6.3** (Theorem 1 and Corollary 3, [21]) Let \((M, g)\) be an \(n\)-dimensional Riemannian submanifold of \(m\)-dimensional real space form \(\tilde{M}(c)\). Then the normalized \(\delta\)-Casorati curvature \(\hat{\delta}_C(n-1)\) satisfies

\[
\tau_{Nor}(p) \leq [\hat{\delta}_C(n-1)]_p + c.
\] (6.6)

Moreover, the equality sign holds for all \(p \in M\) if and only if \((M, g)\) is an invariantly quasi-umbilical submanifold with trivial normal connection in \((\tilde{M}, \tilde{g})\), such that with respect to suitable orthonormal tangent frame \(\{e_1, \ldots, e_n\}\) and normal orthonormal frame \(\{e_{n+1}, \ldots, e_m\}\), the shape operators \(A_\alpha \equiv A_{e_\alpha}\), \(\alpha \in \{n+1, \ldots, m\}\), take the forms given by (5.9).

**Proof.** Using (3.14) in (6.4), we get (6.6).

### 7 Further studies

In this section, we present some problems. Similar problems can be formulated in those situations, where Riemann curvature tensor of the ambient manifold has some nice well known form.

**Problem 7.1** Like in [14], to obtain Casorati inequalities for conformally flat submanifolds of a real space form.

**Problem 7.2** Riemannian manifolds of quasi-constant curvature (cf. [5], [16], [26], [42], [58]) represent a good generalization of real space forms. To obtain Casorati inequalities for submanifolds of quasi-constant curvature manifolds. To study Casorati ideal submanifolds of quasi-constant curvature manifolds.

**Problem 7.3** To obtain Casorati inequalities for submanifolds of generalized complex space forms (cf. [32], [45], [55], [51]).

**Problem 7.4** Like the improved Chen-Ricci inequalities [53], to improve Casorati inequalities for Lagrangian [13] and Kaehlerian slant submanifolds [12] of a complex space form, if possible.

**Problem 7.5** To obtain Casorati inequalities for different kind of submanifolds of locally conformal Kaehler space forms (cf. [25], [54]).

**Problem 7.6** Like the improved Chen-Ricci inequalities [53], to improve Casorati inequalities for Lagrangian submanifolds of a locally conformal Kaehler space form (under some conditions), if possible.

**Problem 7.7** To obtain Casorati inequalities for submanifolds of Kaehler manifolds of quasi constant holomorphic sectional curvatures (cf. [27], [2]).

**Problem 7.8** To obtain Casorati inequalities for different kind of submanifolds of Bochner-Kaehler manifolds [17].
Problem 7.9 Like the improved Chen-Ricci inequalities [53], to improve Casorati inequalities for Lagrangian submanifolds of Bochner-Kaehler manifolds, if possible.

Problem 7.10 Like the improved Chen-Ricci inequalities [53], to improve Casorati inequalities for Lagrangian submanifolds of a quaternionic space form [31], if possible.

Problem 7.11 To obtain Casorati inequalities for different kind of submanifolds [52] of generalized \((\kappa, \mu)\) space forms [7] and in particular generalized Sasakian space forms [1] and Sasakian space forms.

Problem 7.12 Like the improved Chen-Ricci inequalities [53], to improve Casorati inequalities for Legendrian submanifolds of a Sasakian space form (cf. [50], [3]).

Problem 7.13 To obtain Casorati inequalities for different kind of submanifolds of different kind of manifolds equipped with a semi-symmetric metric connection (cf. [47], [59], [43]).

Problem 7.14 To obtain Casorati inequalities for centroaffine hypersurfaces [44].

Acknowledgements. The author is thankful to Professor Ugo Gianazza (gianazza@imati.cnr.it), Claudio Gnoli (claudio.gnoli@unipv.it), Claudia Olivati and Anna Bendiscioli from University of Pavia, Italy for their help in tracing the original papers of Felice Casorati ([8], [9]).

References

[1] P. Alegre, D.E. Blair, A. Carriazo, Generalized Sasakian space forms, Israel J. Math. 141 (2004), 157-183. MR2063031 (2005f:53057)

[2] C.L. Bejan, M. Benyounes, Kähler manifolds of quasi-constant holomorphic sectional curvature, J. Geom. 88 (2008), 1-14.

[3] D.E. Blair, Riemannian geometry of contact and symplectic manifolds, Second edition. Progress in Mathematics, 203. Birkhäuser Boston, Inc., Boston, MA, 2010. MR2682326 (2012d:53245)

[4] D.E. Blair, A.J. Ledger, Quasi-umbilical, minimal submanifolds of Euclidean space, Simon Stevin 51 (1977/78), no. 1, 3-22. MR0461304 (57 #1289)

[5] V. Boju, M. Popescu, Espaces à courbure quasi-constante, J. Diff. Geom. 13 (1978) 373-383. MR0551566 (81c:53041)

[6] J. Bolton, F. Dillen, J. Fastenakels, L. Vrancken, A best possible inequality for curvature-like tensor fields, Math. Inequal. Appl. 12 (2009), no. 3, 663-681. MR2540985 (2010g:53055)

[7] A. Carriazo, V. Martin-Molina, M.M. Tripathi, Generalized \((\kappa, \mu)\)-space forms, Mediterranean J. Math. 10 (2013), no. 1, 475-496. MR3019118.

[8] F. Casorati, Nuova definizione della curvatura delle superficie e suo confronto con quella di Gauss (New definition of the curvature of the surface and its comparison with that of Gauss), Rend. Inst. Matem. Accad. Lomb. Series II 22 (1889), no. 8, 335-346.
[9] F. Casorati, *Ristampa della Nota: Nuova misura della curvatura della superficie inserta nel fasc. VIII del presente volume deli Rendiconti* (Reprint Note: New measure of the curvature of the surface in fasc. VIII of this Volume), Rend. Inst. Matem. Accad. Lomb. Series II 22 (1889), no. 10, 842.

[10] F. Casorati, *Mesure de la courbure des surfaces suivant l’idée commune. Ses rapports avec les mesures de courbure gaussienne et moyenne* (Measuring the curvature of the surfaces along the common idea. Its relations with Gaussian and mean curvature measurements) Acta Math. 14 (1890), no. 1, 95-110.

[11] B.-Y. Chen, *Geometry of submanifolds*, Pure and Applied Mathematics, No. 22. Marcel Dekker, Inc., New York, 1973. MR0353212 (50 #5697)

[12] B.-Y. Chen, *Geometry of slant submanifolds*, Katholieke Universiteit Leuven, Louvain, 1990. 123 pp. MR1099374 (92d:53047)

[13] B.-Y. Chen, *Riemannian geometry of Lagrangian submanifolds*, Taiwanese J. Math. 5 (2001), no. 4, 681-723. MR1870041 (2002k:53154)

[14] B.-Y. Chen, *A general inequality for conformally flat submanifolds and its applications*, Acta Math. Hungar. 106 (2005), no. 3, 239–252. MR2129528 (2006a:53060)

[15] B.-Y. Chen, F. Dillen, L. Verstraelen, *δ-invariants and their applications to centroaffine geometry*, Differential Geom. Appl. 22 (2005), 341-354. MR2166127 (2006i:53010)

[16] B.-Y. Chen, K. Yano, *Hypersurfaces of a conformally flat space*, Tensor (N.S.) 26 (1972), 318-322. MR0331283 (48 #9617)

[17] B.-Y. Chen, K. Yano, *Manifolds with vanishing Weyl or Bochner curvature tensor*, J. Math. Soc. Japan 27 (1975), 106-112. MR0355895 (50 #8369)

[18] E.K.P. Chong, S. H. Žak, *An introduction to Optimization*, Second Edition, John Wiley & Sons, Inc., 2001.

[19] S. Decu, *Extrinsic and intrinsic principal directions of ideal submanifolds*, Bull. Transilv. Univ. Braşov Ser. III 1(50) (2008), 93-97. MR2478009. MR2478009 (2010b:53094)

[20] S. Decu, *Optimal inequalities involving Casorati curvature of slant submanifolds in quaternion space forms*, Riemannian Geometry and Applications - Proceedings RIGA 2014, 87-96, Editura Univ. Bucur., Bucharest, 2014. MR3330257.

[21] S. Decu, S. Haesen, L. Verstraelen, *Optimal inequalities involving Casorati curvature*, Bull. of the Transilvania Univ. of Braşov Ser. B 14(49), supplement (2007), 85-93. MR2446793 (2009e:53075)

[22] S. Decu, S. Haesen, L. Verstraelen, *Optimal inequalities characterising quasi-umbilical submanifolds*, J. Inequal. Pure Appl. Math. 9 (2008), no. 3, Article ID 79, 07 pp. MR2443744 (2009k:53118)

[23] S. Decu, A. Pantic, M. Petrovic-Torgasev, L. Verstraelen, *Ricci and Casorati principal directions of δ (2) Chen ideal submanifolds*, Kragujevac J. Math. 37 (2013), no. 1, 25-31. MR3073695.
[24] S. Decu, M. Petrović-Torgašev, A. Šebeković, L. Verstraelen, *Ricci and Casorati principal directions of Wintgen ideal submanifolds*, Filomat 28 (2014), no. 4, 657-661. MR3360059.

[25] S. Dragomir, L. Ornea, *Locally conformal Kähler geometry*, Progress in Mathematics, 155. Birkhäuser Boston, Inc., Boston, MA, 1998. MR1481969 (99a:53081)

[26] G. Ganchev, V. Mihova, *Riemannian manifolds of quasi-constant sectional curvatures*, J. Reine Angew. Math. 522 (2000), 119-141. MR1758579 (2001c:53044)

[27] G. Ganchev, V. Mihova, *Kähler manifolds of quasi-constant holomorphic sectional curvatures*, Cent. Eur. J. Math. 6 (2008), no. 1, 43-75.

[28] V. Ghișoiu, *Casorati curvatures of ideal submanifolds*, Bull. Transilvania Univ. Brașov Ser. III 1(50) (2008), 149-159. MR2478015 (2010a:53092)

[29] V. Ghișoiu, *Inequalities for the Casorati curvatures of slant submanifolds in complex space forms*, In: Riemannian Geometry and Applications. Proceedings RIGA 2011, pp. 145-150. Ed. Univ. Bucuresti, Bucharest (2011). MR2918364.

[30] S. Haesen, D. Kowalczyk, L. Verstraelen, *On the extrinsic principal directions of Riemannian submanifolds*, Note Math. 29 (2009), no. 2, 41-53. MR2789830 (2012b:53102)

[31] S. Ishihara, *Quaternion Kählerian manifolds*, J. Differential Geometry 9 (1974), 483-500. MR0348687 (50 #1184)

[32] U.K. Kim, *On 4-dimensional generalized complex space forms*, J. Australian Math. Soc. (Series A) 66(1999), 379-387. MR1694210 (2000c:53093)

[33] J.J. Koenderink, *Surface shape, the science and the looks* in “Handbook of experimental phenomenology: Visual perception of shape, space and appearance” edited by L. Albertazzi, John Wiley and Sons Ltd., Chichester, 2013, 165-180.

[34] D. Kowalczyk, *Casorati curvatures*, Bull. Transilvania Univ. Brașov Ser. III 1(50) (2008), 2009-2013. MR2478021 (2009k:53149)

[35] R.S. Kulkarni, *Curvature and metric*, Ph.D. Thesis, Harvard University, Cambridge, Mass., 1967.

[36] R.S. Kulkarni, *Curvature structures and conformal transformations*, Bull. Amer. Math. Soc. 75 1969 91–94. MR0233306 (38 #1628)

[37] C.W. Lee, J.W. Lee, G.E. Vilcu, *A new proof for some optimal inequalities involving generalized normalized δ-Casorati curvatures*, J. Inequal. Appl. 2015, 2015:310, 9 pp. MR2404717.

[38] C.W. Lee, J.W. Lee, G.E. Vilcu, D.W. Yoon, *Optimal inequalities for the Casorati curvatures of submanifolds of generalized space forms endowed with semi-symmetric metric connections*, Bull. Korean Math. Soc. 52 (2015), no. 5, 1631-1647. MR3406025

[39] C.W. Lee, D.W. Yoon, J.W. Lee, *Optimal inequalities for the Casorati curvatures of submanifolds of real space forms endowed with semi-symmetric metric connections*, J. Inequal. Appl. 2014, 2014:327, 9 pp. MR3344114.
[40] J.W. Lee, G.E. Vilcu, *Inequalities for generalized normalized $\delta$-Casorati Curvatures of slant submanifolds in quaternion space forms*, Taiwanese J. Math. 19 (2015), no. 3, 691-702. MR3353248.

[41] S. Man, L. Zhang, P. Zhang, *Some inequalities for submanifolds in a Riemannian manifold of nearly quasi-constant curvature*, Filomat (to appear)

[42] A.L. Mocanu, *Les variétés a curvure quasi-constant de type Vranceanu* (French) [Manifolds with quasi-constant curvature of Vranceanu type], Proceedings of the National Conference on Geometry and Topology (Romanian) (Tîrgovişte, 1986), 163-168, Univ. Bucureşti, Bucharest, 1988. MR0980015 (90a:53030)

[43] Z. Nakao, *Submanifolds of a Riemannian manifold with semi-symmetric metric connections*, Proc. Am. Math. Soc. 54 (1976), no. 1, 261-266. MR0445416 (56 #3758)

[44] K. Nomizu, T. Sasaki, *Affine Differential Geometry. Geometry of Affine Immersions*, Cambridge Tracts in Mathematics, vol. 111, Cambridge University Press, Cambridge, 1994. MR1311248 (96e:53014)

[45] Z. Olszak, *On the existence of generalized complex space forms*, Israel J. Math. 65(1989), 214-218. MR0998671 (90c:53091)

[46] B. Ons, L. Verstraelen, *Some geometrical comments on vision and neurobiology: seeing Gauss and Gabor walking by, when looking through the window of the Parma at Leuven in the company of Casorati*, Kragujevac J. Math. 35 (2011), no. 2, 317-325. MR2881154.

[47] E. Pak, *On the pseudo-Riemannian spaces*, J. Korean Math. Soc., 6 (1969), 23-31. MR0334025 (48 #12344)

[48] X.L. Pan, P. Zhang, L. Zhang, *Inequalities for Casorati curvatures of submanifolds in a Riemannian manifold with quasi-constant curvature*, (Chinese) J. Shandong Univ. Nat. Sci. 50 (2015), no. 9, 84-87, 94. MR3444205.

[49] V. Slesar, B. Şahin, G. E. Vilcu, *Inequalities for the Casorati curvatures of slant submanifolds in quaternionic space forms*, J. Inequal. Appl. 2014:123, 10 pp. MR3346822.

[50] S. Tanno, *Sasakian manifolds with constant $\varphi$-holomorphic sectional curvature*, Tôhoku Math. J. (2) 21 (1969), 501-507. MR0251667 (40 #4894)

[51] F. Tricerri, L. Vanhecke, *Curvature tensors on almost Hermitian manifolds*, Trans. Am. Math. Soc. 267(1981), 365-398. MR0626479 (82j:53071)

[52] M. M. Tripathi, *Almost semi-invariant submanifolds of trans-Sasakian manifolds*, J. Indian Math. Soc. (N.S.) 62(1996), no. 1-4, 225-245. MR1458496 (99a:53029)

[53] M.M. Tripathi, *Improved Chen-Ricci inequality for curvature-like tensors and its applications*, Differential Geom. Appl. 29 (2011), no. 5, 685-698. MR2831825 (2012j:53069)

[54] I. Vaisman, *On locally conformal almost Kähler manifolds*, Israel J. Math. 24 (1976), 339-351. MR0418003 (54 #6048)

[55] L. Vanhecke, *Almost Hermitian manifolds with $J$-invariant Riemann curvature tensor*, Rend. Sem. Mat. Univ. e Politec. Torino 34(1975/76), 487-498. MR0436034 (55 #8985)
[56] L. Verstraelen, *The geometry of eye and brain*, Soochow J. Math. 30 (2004), no. 3, 367-376. MR2093862.

[57] L. Verstraelen, *Geometry of submanifolds I, The first Casorati curvature indicatrices*, Kragujev. J. Math. 37 (2013), no. 1, 5-23. MR3073694.

[58] Gh. Vrânceanu, Lectii de geometrie diferentială Vol. IV (Romanian) [Lectures on differential geometry], Editura Academiei Republicii Socialiste România, Bucharest 1968. MR0244868 (39 #6181)

[59] K. Yano, *On semi-symmetric metric connections*, Rev. Roumaine Math. Pures Appl. 15 (1970), 1579-1586. MR0275321 (43 #1078)

[60] L. Zhang, X. Pan, P. Zhang, *Inequalities for Casorati curvature of Lagrangian submanifolds in complex space forms*, Adv. Math. China, 2015.

[61] P. Zhang, *Inequalities for Casorati curvatures of submanifolds in real space forms*, http://vixra.org/pdf/1408.0135v1.pdf.

[62] P. Zhang, L. Zhang, *Remarks on inequalities for the Casorati curvatures of slant submanifolds in quaternion space forms*, J. Inequal. Appl. 2014, 2014: 452, 6 pp. MR3346887.

[63] P. Zhang, L. Zhang, *Casorati inequalities for submanifolds in a Riemannian manifold of quasi-constant curvature with a semi-symmetric metric connection*, Symmetry, 2016, 8(4): 19.

[64] P. Zhang, L. Zhang, *Inequalities for Casorati curvatures of submanifolds in real space forms*, http://arxiv.org/abs/1408.4996v7. to appear in Advances in Geometry, DOI: 10.1515/advgeom-2016-0009

Department of Mathematics
Institute of Science
Banaras Hindu University
Varanasi 221005, India
Email: mmtripathi66@yahoo.com