All Moments of the Uniform Ensemble of Quantum Density Matrices

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Abstract

Given a uniform ensemble of quantum density matrices $\rho$, it is useful to calculate the mean value over this ensemble of a product of entries of $\rho$. We show how to calculate such moments in this paper. The answer involves well known results from Group Representation Theory and Random Matrix Theory. This quantum problem has a well known classical counterpart: given a uniform ensemble of probability distributions $P = (P_1, P_2, \ldots, P_N)$ where the $P_j$ are non-negative reals that sum to one, calculate the mean value over this probability simplex of products of $P$ components. The answer to the classical problem follows from an integral formula due to Dirichlet.
1 Introduction

The probability simplex of dimension \( N - 1 \) is the set of all points \( P = (P_1, P_2, \ldots, P_N) \), where the components of \( P \) are non-negative reals that sum to one. Take \( N = 3 \) for simplicity. It is useful to calculate moment integrals such as:

\[
\int d^3 P \ P_1^2 P_3, \tag{1}
\]

where \( d^3 P = dP_1 dP_2 dP_3 \), and where the point \( P \) ranges over the 2 dimensional probability simplex. As will be discussed in detail later, such integrals can be performed using an integral formula due to Dirichlet[1][2].

Now suppose we generalize this problem to the quantum regime by considering quantum density matrices instead of classical probability distributions. Suppose that \( \rho \) is a quantum density matrix (i.e., \( \rho \) is a Hermitian matrix with non-negative eigenvalues that sum to one). It is useful to calculate moment integrals such as

\[
\int D\rho \ (\rho_{1,1})^2 \rho_{1,2}. \tag{2}
\]

As will be discussed in detail later, there is a very natural way of defining the measure \( D\rho \). The goal of this paper is to show how to calculate integrals like Eq.(2). The answer involves well known results from Group Representation Theory [3][4][5][6], and Random Matrix Theory [7][8][9]. Ref.[8] by Itzykson and Zuber is especially pertinent to this paper.

2 Notation

In this section, we will introduce some notation that will be used in subsequent sections.

RHS (ditto, LHS) will mean “right hand side” (ditto, “left hand side”). For any complex number \( z \), let \( z_\Re \) (ditto, \( z_\Im \)) represent its real (ditto, imaginary) part.

As usual, \( \delta(x) \) for real \( x \) will denote the Dirac delta function \( \delta(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \). Likewise, \( \delta(x, y) \) and \( \delta_y \) will denote the Kronecker delta function. \( \delta_y \) equals 1 if \( x = y \) and it equals 0 if \( x \neq y \). \( \epsilon_{i_1,i_2,...,i_N} \) will denote the totally anti-symmetric tensor with \( N \) indices. \( \epsilon_{i_1,i_2,...,i_N} \) equals 1 if \( (i_1, i_2, \ldots, i_N) \) is an even permutation of \( (1, 2, 3, \ldots, N) \), and it equals \(-1\) for odd permutations. Let \( \theta(S) \) be the “truth function” or “indicator function”; it equals 1 if the statement \( S \) is true, and it equals 0 if \( S \) is false. For example, \( \theta(x > 0) \) is the unit-step function; it equals 1 if \( x > 0 \) and it equals 0 if \( x \leq 0 \).

The set of Hermitian \( N \times N \) matrices will be denoted by \( \text{Herm}(N) \). Any square matrix \( A \) is said to be positive semi-definite (ditto, positive definite) if the eigenvalues of \( A \) are non-negative (ditto, strictly positive). We will write \( A \geq 0 \) (ditto, \( A > 0 \)) if \( A \) is positive semi-definite (ditto, positive definite). Suppose \( A \) and \( B \) are two \( N \times N \)
matrices. We define the dot product of $A$ and $B$ by $A \cdot B = \text{tr}(A^T B) = \sum_{i,j} A_{i,j} B_{i,j}$. We will use $[\frac{\partial}{\partial A_{i,j}}]_{i,j} = \frac{\partial}{\partial A}$ to denote the matrix of the partial derivatives with respect to the entries of $A$. Note that $B \cdot \frac{\partial}{\partial A} = \text{tr}(B^T \frac{\partial}{\partial A})$

We will often denote the set $\{x_k : k \in K\}$ by $\{x_k\}_{k \in K}$, or simply by $\{x_k\}_k$. Likewise, the ordered set $(x_k : k \in K)$ will be denoted by $(x_k)_{k \in K}$ or simply by $(x_k)_k$. $\sum_{k \in K} x_k$ will often be denoted by $\sum\{x_k\}_k$ and $\prod_{k \in K} x_k$ by $\prod\{x_k\}_k$.

Suppose $\alpha$ is an $N$ component column vector, $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{N-1})^T$. One defines the Vandermonde determinant of $\alpha$ by

\[
\Delta(\alpha) = \begin{vmatrix}
1 & \alpha_0 & \alpha_0^2 & \ldots & \alpha_0^{N-1} \\
1 & \alpha_1 & \alpha_1^2 & \ldots & \alpha_1^{N-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \alpha_{N-1} & \alpha_{N-1}^2 & \ldots & \alpha_{N-1}^{N-1}
\end{vmatrix}.
\]

Often, we will express the RHS of the last equation more succinctly as follows:

\[
\Delta(\alpha) = \det(\alpha^0, \alpha^1, \ldots, \alpha^{N-1}).
\]

As is well known and easily proven:

\[
\Delta(\alpha) = \prod \{ \alpha_j - \alpha_i \}_{0 \leq i < j \leq N-1}.
\]

Vandermonde determinants are sometimes, especially in very old literature, referred to as difference products, or as basic alternants.

We end this section by defining symbols for two expressions that arise frequently in subsequent sections. Let

\[
\mathcal{F}_N = \prod \{ j ! \}_{1 \leq j \leq N}.
\]

From Eq.(6), it follows that $\mathcal{F}_N = \Delta(0, 1, 2, \ldots, N)$. Let

\[
L_N = \frac{N(N - 1)}{2}.
\]

An $N \times N$ matrix has $L_N$ entries below (and above) its main diagonal. Also, $L_N = 1 + 2 + \cdots + (N - 2) + (N - 1)$.

### 3 Ensemble of Classical Probability Distributions

In this section, we will review the simplex moments integral and its variants.

Suppose index $b$ ranges over the integers from 1 to $N_2$. Define the following moment integral over an $N_2 - 1$ dimensional simplex:
\[ S_{N_b - 1} \equiv \prod \left\{ \int_0^\infty dx_b \right\}_b \delta(\sum_b x_b - \lambda) \prod \{x_b^{\nu_b}\}_b, \]  

(8a)

where \( \lambda \) is a positive real and the \( \nu_b \) are non-negative integers. It is well known that

\[ S_{N_b - 1} = \frac{\prod \{\nu_b!\}_b}{\nu!} \lambda^\nu, \]  

(8b)

where

\[ \nu = \sum_b \nu_b + N_b - 1. \]  

(8c)

Eqs. (8) can be generalized so that they also apply to non-integer \( \nu_b \) (just replace factorials by Gamma functions according to the prescription \( n! \to \Gamma(n + 1) \)), but we won’t bother with such generalizations in this paper. Eqs. (8) are a generalization to higher dimensions of the Beta function, given by

\[ \beta(m, n) = \int_0^1 dt t^{m-1}(1-t)^{n-1} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}. \]  

(9)

Now let \( B \) be an index that ranges over the integers from 1 to \( N_B \), where \( N_B = N_b - 1 \). The \( N_B \)-dimensional Dirichlet integral is defined by

\[ D_{N_b - 1} \equiv \prod \left\{ \int_0^\infty dx_B \right\}_B \theta(\sum_B x_B < \lambda) \prod \{x_B^{\nu_B}\}_B f(\sum_B x_B). \]  

(10a)

It is well known that

\[ D_{N_b - 1} = \frac{\prod \{\nu_B!\}_B}{\nu!} g(\lambda), \]  

(10b)

where

\[ g(\lambda) = \int_0^\lambda dt f(t) t^\nu \]  

(10c)

Note that \( g(\lambda) = \lambda^\nu \) when \( f(t) = 1 \). In Eqs. (10), \( \nu \) is defined as in Eq. (8c) but with \( \nu_{N_b} = 0 \) so that \( \sum_b \nu_b = \sum_B \nu_B \).

Proofs of Eqs. (8) for \( S_{N_b} \) and Eqs. (10) for \( D_{N_B} \) are easily found in the literature (see, for example Refs. [1][2]) so we won’t present them here. However, we do want to emphasize that these two integral formulas follow trivially from each other. One can prove as follows that the formula for \( D_{N_B} \) implies the formula for \( S_{N_b} \). Take \( \frac{d}{d\lambda} \) of Eqs. (10a) and (10b). Then use \( \frac{d}{dx} \theta(x > 0) = \delta(x) \) and the fact that \( N_B = N_b - 1 \). Conversely, one can prove as follows that the formula for \( S_{N_b} \) implies the formula for \( D_{N_B} \). Applying the operator \( \Omega = \int_0^{\lambda'} d\lambda \frac{d}{d\lambda} \) to Eq. (10a) gives

\[ \int_0^{\lambda'} d\lambda f(\lambda) S_{N_b - 1} = \frac{\prod \{\nu_b\}_b}{\nu!} \left( \int_0^{\lambda'} d\lambda f(\lambda) \lambda^{\nu - 1} \nu \right), \]  

(11)
whereas applying $\Omega$ to Eq. (8a) gives

$$
\int_0^\lambda d\lambda f(\lambda) \frac{S_{N_b-1}}{d\lambda} = (12a)
$$

$$
= \prod \left\{ \int_0^\infty d\lambda \ x_b \ x_B \right\}_b \int_0^\lambda d\lambda f(\lambda) \left( -1 \frac{\partial}{\partial x_N} \right) \delta(\sum_b x_b - \lambda) (12b)
$$

$$
= \prod \left\{ \int_0^\infty d\lambda \ x_B \right\}_B \int_0^\lambda d\lambda f(\lambda) \delta(\sum_B x_B - \lambda) (12c)
$$

$$
= D_{N_b-1} (12d)
$$

In going from line $b$ to line $c$, we have assumed that $\nu_{N_b} = 0$ and performed the $x_{N_b}$ integration.

4 Some Results from Group Representation Theory

In this section, we will review quickly some well known facts from Group Representation Theory. For more details and proofs, see, for example, Refs. [3] [4] [5] [6].

A $K$-box Young graph with at most $N$ rows is specified by an $N$-dimensional column vector of integers $\eta = (\eta_0, \eta_1, \ldots, \eta_{N-1})^T$ such that $\eta_0 \geq \eta_1 \geq \cdots \geq \eta_{N-1} \geq 0$ and $\sum_{j=0}^{N-1} \eta_j = K$. $\eta_j$ is the number of boxes in the $j$-th row, where the top row is the 0-th one. Each subsequent row has the same number or fewer boxes than the row above. If some of the last few components of $\eta$ are zero, they are often omitted. For example,

```
+---+---+---+---+
 |   |   |   |   |
```

is specified by $\eta = (6, 3, 3, 0, 0)^T$ with $K = 12$ and $N = 5$.

If $S_K$ is the symmetric group (permutation group) on $K$ letters, then the classes of $S_K$ are specified by a $K$-tuple of non-negative integers $i = (i_1, i_2, \ldots, i_K)$ such that $1i_1 + 2i_2 + \cdots + Ki_k = K$. $C(i) = (1^{i_1}, 2^{i_2}, \ldots, K^{i_K})$ represents the class of elements of $S_K$ with $i_1$ cycles of length 1, $i_2$ cycles of length 2, $\ldots$, and $i_K$ cycles of length $K$. The order (i.e., number of elements) of $C(i)$ is given by

$$
| C(i) | = \frac{K!}{(1^{i_1}i_1)!(2^{i_2}i_2)!\cdots(K^{i_K}i_K)!}
$$

(13)

There is a one-to-one onto correspondence between: (1) the irreps of $S_K$, (2) the classes of $S_K$, (3) the Young graphs with $K$ boxes. Tables 1 to 4 give the characters for each (irrep, class) pair of $S_K$, where $K$ ranges from 1 to 4.

The irreps of the unitary group $U(N)$ (and also the irreps of $GL(N)$) are in one-to-one onto correspondence with the Young graphs with at most $N$ rows. We
will specify the irreps of $U(N)$ by $\eta^N$, where $\eta$ is an $N$-dimensional column vector of integers that specifies a Young graph. (The superscript $N$ serves to distinguish this from an irrep of a symmetric group). In terms of tensors, the number $K$ of boxes of the Young graph corresponds to the number of tensor indices, \{1, 2, \ldots, N\} corresponds to the range of the tensor indices, and the Young graph gives the symmetry properties of the tensor.

Weyl showed that the dimension of the irrep $\eta^N$ of $U(N)$ is given by:

$$\dim(\eta^N) = \frac{\det((\eta + \delta)^{N-1}, (\eta + \delta)^{N-2}, \ldots, (\eta + \delta)^0)}{\mathcal{F}_{N-1}}$$

where $\delta = (N-1, N-2, \ldots, 1, 0)^T$.

Let $\chi^{\eta^N}(A)$ represent the character of $A \in U(N)$ in the irrep $\eta^N$ of $U(N)$. Inspired by an identity due to Frobenius, Weyl derived the following two expressions for $\chi^{\eta^N}(A)$. First,

$$\chi^{\eta^N}(A) = \frac{\det(\alpha^{\eta_0 + N-1}, \alpha^{\eta_0 + N-2}, \ldots, \alpha^{\eta_{N-1}})}{\det(\alpha^{N-1}, \alpha^{N-2}, \ldots, \alpha^0)},$$

where $\alpha$ is the vector of eigenvalues of the $N \times N$ matrix $A$. Second,

$$\chi^{\eta^N}(A) = \sum_{\mathcal{C}(i)} \frac{|\mathcal{C}(i)|}{K!} \chi^{\eta}(\mathcal{C}(i)) t_1^{i_1} t_2^{i_2} \cdots t_K^{i_K},$$

where $K$ is the number of boxes in the Young graph $\eta$, and where the sum is over all classes $\mathcal{C}(i)$ of the permutation group $S_K$, $|\mathcal{C}(i)|$ is the order of class $\mathcal{C}(i)$, $\chi^{\eta}(\mathcal{C}(i))$ is the character of $\mathcal{C}(i)$ in the irrep $\eta$ of $S_K$, and where $t_1, t_2, \cdots$ are defined by
Table 3: Characters for each (irrep, class) pair of the permutation group $S_3$.

| K=3 | (1$^3$) | (1, 2) | (3) | ← class | ← order |
|-----|---------|--------|-----|---------|---------|
| Characters ↓
| 1 | 3 | 2 |
| 1 | 1 | 1 |
| 2 | 0 | -1 |
| 1 | -1 | 1 |

↑ irrep

Table 4: Characters for each (irrep, class) pair of the permutation group $S_4$.

| K=4 | (1$^4$) | (1$^2$, 2) | (1, 3$^2$) | (2$^2$) | (4) | ← class | ← order |
|-----|---------|------------|-----------|-------|-----|---------|---------|
| Characters ↓
| 1 | 6 | 8 | 3 | 6 |
| 1 | 1 | 1 | 1 | 1 |
| 3 | 1 | 0 | -1 | -1 |
| 2 | 0 | -1 | 2 | 0 |
| 3 | -1 | 0 | -1 | 1 |
| 1 | -1 | 1 | 1 | 1 |

↑ irrep

\[ t_r = \text{tr}(A^r) = \sum_{j=0}^{N-1} (\alpha_j)^r, \quad (17) \]

where \( \{\alpha_j\}_j \) are the eigenvalues of \( A \).

Eq.(14) for \( \text{dim}(\eta^N) \) follows from Eq.(15) and the fact that \( \text{dim}(\eta^N) = \chi^N(I) \), where \( I \) is the identity matrix. Let RHS1 denote the RHS of Eq.(15) and RHS2 denote the RHS of Eq.(16). Frobenius was the first to prove that RHS1=RHS2, but his proof, which is discussed in Hamermesh[4], made no mention of \( U(N) \). Weyl gave a new proof[6] in which \( U(N) \) was crucial.

Note that in Eqs.(15) and (16), \( A \) is a unitary matrix. Hence, its eigenvalues are of the form \( \alpha_j = e^{i\beta_j} \) for some real \( \beta_j \). However, the equation RHS1=RHS2 can be analytically continued to complex \( \alpha_j \) with \( |\alpha_j| \neq 1 \). This is because both RHS1 and RHS2 are linear combinations of monomials of the form \( \alpha_0^{p_0} \alpha_1^{p_1} \ldots \alpha_{N-1}^{p_{N-1}} \), where \( p_0, p_1, \ldots, p_{N-1} \) are non-negative integers. Since the \( \beta_j \)'s are arbitrary reals, the coefficient in RHS1 of any fixed monomial \( \alpha_0^{p_0} \alpha_1^{p_1} \ldots \alpha_{N-1}^{p_{N-1}} \) must equal the coefficient in RHS2 of that same monomial.
Table 5 was calculated using Eq.(16). A very similar table can be found in Ref.[8]. Table 6 was derived from the information in Table 5.

| K | irrep | $\chi^{\eta^N}(A)$ | dim($\eta^N$) = $\chi^{\eta^N}(I)$ |
|---|------|--------------------|-----------------------------------|
| 1 | t₁ | $1$                | $N$                              |
| 2 | $\frac{1}{2}t_1^2 + \frac{1}{2}t_2$ | $\frac{1}{2}N(N + 1)$           |
|   | $\frac{1}{2}t_1^2 - \frac{1}{2}t_2$ | $\frac{1}{2}N(N - 1)$           |
| 3 | $\frac{1}{6}t_1^3 + \frac{1}{3}t_1t_2 + \frac{1}{3}t_3$ | $\frac{1}{6}N(N + 1)(N + 2)$    |
|   | $\frac{1}{3}t_1^3 + 0t_1t_2 - \frac{1}{3}t_3$ | $\frac{1}{3}N(N + 1)(N - 1)$    |
|   | $\frac{1}{6}t_1^3 - \frac{1}{2}t_1t_2 + \frac{1}{3}t_3$ | $\frac{1}{6}N(N - 1)(N - 2)$    |
| 4 | $\frac{1}{24}t_1^4 + \frac{1}{8}t_1^2t_2 + \frac{1}{3}t_1^2t_2 + \frac{1}{6}t_1t_2 + \frac{1}{2}t_1t_2 + \frac{1}{4}t_1t_3 + \frac{1}{3}t_1t_3 + \frac{1}{4}t_2t_3 + \frac{1}{3}t_2t_3 + \frac{1}{4}t_3t_4$ | $\frac{1}{24}N(N + 1)(N + 2)(N + 3)$ |
|   | $\frac{1}{8}t_1^2 + \frac{1}{2}t_1^2t_2 - \frac{1}{6}t_1^2t_2 + \frac{1}{3}t_1t_2 - \frac{1}{2}t_1t_2 + \frac{1}{3}t_1t_3 + \frac{1}{3}t_1t_3 - \frac{1}{4}t_1t_3 + \frac{1}{4}t_2t_3 + \frac{1}{4}t_2t_3 + \frac{1}{3}t_2t_3 + \frac{1}{3}t_2t_3 + \frac{1}{4}t_3t_4$ | $\frac{1}{8}N(N + 1)(N + 2)(N - 1)$ |
|   | $\frac{1}{24}t_4^4 + \frac{1}{8}t_4^2t_2 + \frac{1}{3}t_4^2t_2 + \frac{1}{6}t_4t_2 + \frac{1}{4}t_4t_2 + \frac{1}{3}t_4t_3 + \frac{1}{3}t_4t_3 - \frac{1}{4}t_4t_3 + \frac{1}{4}t_2t_3 + \frac{1}{4}t_2t_3 + \frac{1}{3}t_2t_3 + \frac{1}{3}t_2t_3 + \frac{1}{4}t_3t_4$ | $\frac{1}{24}N(N - 1)(N - 2)(N - 3)$ |

Table 5: $K$ is the number of boxes in the Young graph. The irreps of $U(N)$ and $GL(N)$ are in 1-1 correspondence with the Young graphs with at most $N$ rows. $\chi^{\eta^N}(A)$ is the character of $A \in U(N)$ in the irrep $\eta^N$ of $U(N)$. \(\dim(\eta^N)\) is the dimension of irrep $\eta^N$.

## 5 Ensemble of Quantum Density Matrices

In this section, we will generalize the simplex moments integral Eqs.(8) to the quantum realm. To go from classical to quantum physics, we will replace probability distributions by quantum density matrices. In the quantum case, we will need to do integrals over a manifold of matrices. Such integrals are used in several fields of mathematical physics. They are crucial to the field of Random Matrix Theory [7].

Let

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}.$$  

(18)

It is easy to check that $U$ is unitary. For any complex number $z$, let $z_R$ (ditto, $z_I$) represent its real (ditto, imaginary) part. Note that

8
\[
\begin{array}{|c|c|}
\hline
K & \sum_{\text{all irreps } \eta^N \text{ with } K \text{ boxes}} \dim(\eta^N) \chi^\eta^N(A) \\
\hline
0 & 1 \text{ (convenient definition)} \\
1 & Nt_1 \\
2 & \frac{N}{2}(Nt_1^2 + t_2) \\
3 & \frac{N}{6}(N^2t_1^3 + 3Nt_1t_2 + 2t_3) \\
4 & \frac{N}{24}(N^3t_1^4 + 6N^2t_2t_1^2 + 8Nt_3t_1 + 3Nt_2^2 + 6t_4) \\
\hline
\end{array}
\]

Table 6: This table was derived using the information in Table [3].

\[
\begin{pmatrix} z \\ z^* \end{pmatrix} = U \begin{pmatrix} z_R \sqrt{2} \\ z_\Im \sqrt{2} \end{pmatrix}.
\] (19)

This transformation rule motivates us to generalize the volume element \(dx\) in real space to a volume element \(d^2z\) in complex space according to:

\[
dx \rightarrow d^2z = 2dz_Rdz_\Im.
\] (20)

We also generalize the Dirac delta function \(\delta(x)\) in real space to a Dirac delta function \(\delta^2(z)\) in complex space:

\[
\delta(x) \rightarrow \delta^2(z) = \frac{1}{2}\delta(z_R)\delta(z_\Im).
\] (21)

Thus,

\[
\int_{-\infty}^{\infty} dx \delta(x) = 1 \rightarrow \int d^2z \delta^2(z) = 1.
\] (22)

Next we want to give a convenient definition of a volume element \(\mathcal{D}\rho\) for the manifold of all Hermitian \(N \times N\) matrices \(\rho\). We define

\[
\mathcal{D}\rho = \prod_x \left\{d\rho_{xx} \right\} x \prod_{x', x < x'} \left\{d^2\rho_{xx'} \right\}.
\] (23)

This definition of \(\mathcal{D}\rho\) prompts us to define:

\[
\delta(\rho) = \prod_x \left\{\delta(\rho_{xx}) \right\} x \prod_{x', x < x'} \left\{\delta^2(\rho_{xx'}) \right\},
\] (24)

so that

\[
\int \mathcal{D}\rho \delta(\rho) = 1.
\] (25)

The manifold of real points \(x\) with volume element \(dx\) corresponds to the manifold of Hermitian \(N \times N\) matrices \(\rho\) with volume element \(\mathcal{D}\rho\). But what is the matrix counterpart of the manifold of complex points \(z\) with volume element \(d^2z\)?
A natural candidate for this is the manifold of all complex $N \times N$ matrices $A$. We define its volume element by:

$$D^2 A = \prod \left\{ d^2 A_{xx'} \right\}_{x,x'}. \quad (26)$$

Note that $D^2 A$ is a product of twice as many $dx$-like real-space volume elements as $D^2 \rho$. Finally, we define

$$\delta^2 (A) = \prod \left\{ \delta^2 (A_{xx'}) \right\}_{x,x'}. \quad (27)$$

so that

$$\int D^2 A \delta^2 (A) = 1. \quad (28)$$

Given two Hermitian $N \times N$ matrices $\rho$ and $\omega$, one has

$$\omega \cdot \rho = \sum_x \omega_{xx} \rho_{xx} + 2 \sum_{x<y} \left[ (\omega_{xy})_R (\rho_{xy})_R - (\omega_{xy})_I (\rho_{xy})_I \right]. \quad (29)$$

Combining the last equation and the identity $\delta(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx}$ for real $x$, yields

$$\int D\omega \exp(i\omega \cdot \rho) = (2\pi)^N \delta^2 (\rho). \quad (30)$$

Let $X$ be a Hermitian $N \times N$ matrix with eigenvalues $\{\chi_j\}_j$. We need to consider a real valued function $f(X)$ that is invariant under unitary transformations of its argument $X$; that is, $f(UXU^\dagger) = f(X)$ for any unitary $N \times N$ matrix $U$. $f(X)$ depends only on the eigenvalues of $X$, so $f(X) = F(\chi)$ for some function $F : \mathbf{R}^N \rightarrow \mathbf{R}$. Note that not all functions from $\mathbf{R}^N$ to $\mathbf{R}$ qualify for the job of $F$. $F(\chi)$ must also depend on $\chi$ in a symmetrical way: since all $N \times N$ permutation matrices belong to $U(N)$, $F(\chi)$ must be invariant under permutations of its arguments $\{\chi_j\}_j$. Henceforth, we will indulge in a convenient abuse of notation by replacing the symbol $F$ by $f$ so that $f(X) = f(\chi)$.

Let $X$ and $A$ be Hermitian $N \times N$ matrices with eigenvalues $\{\chi_j\}_j$ and $\{\alpha_j\}_j$, respectively. Let $f(X)$ be a real valued function of $X$ that is invariant under unitary transformations of its argument $X$. Finally, let $\xi$ be an arbitrary complex number. The following two integral formulas are well known: First (see Ref. [7]),

$$\int DX f(X) = \frac{(2\pi)^L}{F_N} \prod \left\{ \int_{-\infty}^{\infty} d\chi_j \right\}_j \Delta^2 (\chi) f(\chi), \quad (31)$$

and, second (see Refs. [8] [9]),

$$\int DX e^{\xi \mathrm{tr}(AX)} f(X) = \left( \frac{2\pi}{\xi} \right)^L \prod \left\{ \int_{-\infty}^{\infty} d\chi_j \right\}_j \frac{\Delta(\chi)}{\Delta(\alpha)} e^{\xi (\chi \cdot \alpha)} f(\chi). \quad (32)$$
In the spirit of non-rigorous, applied mathematics, I will not specify precise sufficiency conditions on \( A \), \( f(\cdot) \) and \( \xi \) under which Eqs. (31) and (32) are valid. I leave it to more competent pure mathematicians to figure this out.

Proofs of Eqs. (31) and (32) are easily found in the literature so we won’t present them here. Proving Eq. (32) involves solving a partial differential equation (an initial value problem for a diffusion equation).

Of course, Eq. (31) follows from Eq. (32) when \( \xi \to 0 \). To check this quickly, assume (without loss of generality) that the eigenvalues \( \alpha_j \) are very small and widely separated:

\[
0 < \alpha_0 \ll \alpha_1 \ll \cdots \ll \alpha_{N-1} \ll 1 .
\]  

(33)

Then, keep the largest term in the expansion of \( \Delta(\alpha) \):

\[
\Delta(\alpha) \approx \alpha_0 \alpha_1 \cdots \alpha_{N-1} .
\]

(34)

Furthermore, keep only the largest term in the Taylor expansion of \( \exp(\xi \chi \cdot \alpha) \) and in the multinomial expansion of \( (\chi \cdot \alpha)^{L_N} \):

\[
\exp(\xi \chi \cdot \alpha) \approx \frac{(\xi \chi \cdot \alpha)^{L_N}}{L_N!} \approx \frac{\xi^{L_N}}{L_N!} \frac{L_N!}{F_{N-1}} (\chi_0 \chi_1 \cdots \chi_{N-1}) (\alpha_0 \alpha_1 \cdots \alpha_{N-1}) .
\]

(35)

Use these approximations on the integrand on the RHS of Eq. (32). Also replace the \( \chi \)-dependent part of the integrand, that is, \( \Delta(\chi)\chi_0 \chi_1 \cdots \chi_{N-1} \), by its totally symmetric part \( \Delta^2(\chi)/N! \). This converts the RHS of Eq. (32) into the RHS of Eq. (31) in the limit \( \xi \to 0 \).

We are almost ready to present our generalization of the simplex moments integral. But first we need to prove two lemmas.

**Lemma 5.1** Suppose \( \beta_0, \beta_1, \ldots, \beta_{N-1} \) are non-negative integers. Then

\[
\epsilon_{k_0 k_1 \cdots k_{N-1}} \prod \{(k_j + \beta_j)!\}_j = \prod \{\beta_j!\}_j \Delta(\beta)
\]

(36)

**proof:**

Note that the LHS of Eq. (36) can be expressed as \( \det(M) \), where the matrix \( M \) has entries \( M_{i,j} = (i + \beta_j)! \). Eq. (36) can be proven easily using mathematical induction and simple properties of determinants. QED

**Lemma 5.2** Suppose \( \beta_0, \beta_1, \ldots, \beta_{N-1} \) are non-negative integers. Then

\[
\prod \left\{ \int_0^\infty dx_j \right\}_j \delta(\sum_j x_j - 1) \Delta(x)x_0^{\beta_0}x_1^{\beta_1} \cdots x_{N-1}^{\beta_{N-1}} = \frac{\prod \{\beta_j!\}_j \Delta(\beta)}{(\sum_j \beta_j + L_N + N - 1)!}
\]

(37)
proof: Replace $\Delta(x)$ by
\[
\Delta(x) = \epsilon_{k_0 k_1 \cdots k_{N-1}} x_0^{k_0} x_1^{k_1} \cdots x_{N-1}^{k_{N-1}} .
\] (38)
in the LHS of Eq.(37). Then use the simplex moments integral Eqs.(8) to get
\[
LHS = \frac{\epsilon_{k_0 k_1 \cdots k_{N-1}} \prod \{(k_j + \beta_j)!\}}{\nu} ,
\] (39)
where
\[
\nu = \sum_j (\beta_j + k_j) + N - 1 = \sum_j \beta_j + L_N + N - 1 .
\] (40)
Finally, apply Eq.(36). QED

The volume of a simplex ($\nu_b = 0 \ \forall b$ in Eq.(8) ) is the simplest case of the simplex moments integral. To warm up, we first present the quantum version of this simplest case:

**Claim 5.1** Suppose $X \in \text{Herm}(N)$. Then
\[
\int \mathcal{D}X \ \theta(X \geq 0) \delta(\text{tr}X - 1) = \frac{(2\pi)^{L_N} F_{N-1}^{N-1}}{(N^2 - 1)!} .
\] (41)
proof: Let LHS (ditto, RHS) stand for the left (ditto, right) hand side of Eq.(41). Then
\[
LHS = \frac{(2\pi)^{L_N} \prod \left\{ \int_0^\infty d\chi_j \right\}}{F_N} \Delta^2(\chi) \delta(\sum_j \chi_j - 1)
\] (42a)
\[
= \frac{(2\pi)^{L_N} \prod \left\{ \int_0^\infty d\chi_j \right\}}{F_N} \Delta(0, 1, 2, \ldots, N - 1) \delta(\sum_j \chi_j - 1)
\] (42b)
\[
= \frac{(2\pi)^{L_N} \Delta(0, 1, 2, \ldots, N - 1)}{(2L_N + N - 1)!}
\] (42c)
\[
= RHS .
\] (42d)
In going from line $a$ to line $b$, we replaced one of the $\Delta(\chi)$ by $N! \chi_0^0 \chi_1^1 \cdots \chi_{N-1}^{N-1}$; this was valid because the rest of the integrand was totally anti-symmetric under permutations of the $\{\chi_j\}$'s. To go from line $b$ to line $c$, we applied Eq.(37). To go from line $c$ to line $d$, we used $\Delta(0, 1, 2, \ldots, N - 1) = F_{N-1}$. QED

Finally, we are ready to present the main result of this paper, a generalization of the simplex moments integral to quantum mechanics. Actually, we will give a moment generating function and calculate moments from that.
Claim 5.2 Suppose $X, A \in \text{Herm}(N)$, and $\xi$ is a complex number. If both sides of the following formula exist, then

$$\int D X e^{\xi \text{tr}(AX)} \theta(X \geq 0) \delta(\text{tr}X - 1) = \sum_{K=0}^{\infty} \xi^K \frac{(2\pi)^L N \mathcal{F}_{N-1}}{(K + N^2 - 1)!} \sum_{\eta : \text{boxes}} \dim(\eta^N) \chi^{\eta^N}(A),$$

(43)

where we define $\dim(\eta^N) \chi^{\eta^N}(A) = 1$ for $K = 0$.

proof:

We will prove this claim when $\xi = 1$. The more general case can be obtained from this by scaling $A$ (i.e., replacing $A$ by $\xi A$, where $\xi$ is real, and then analytically continuing $\xi$ to complex values.) Let LHS (ditto, RHS) stand for the left (ditto, right) hand side of Eq.(43).

Applying Eq.(32) yields

$$LHS = (2\pi)^L N \prod \left\{ \int_{-\infty}^{\infty} d\chi_j \right\} \delta(\sum_j \chi_j - 1).$$

(44)

Recall the multinomial expansion:

$$(x_1 + x_2 + \cdots + x_N)^K = \sum_{\vec{k}} \delta(\sum_r k_r, K) \frac{K!}{\prod k_r!} x_1^{k_1} x_2^{k_2} \cdots x_N^{k_N}. \quad \text{(45)}$$

Using the Taylor expansion of $\exp(\cdot)$ and the multinomial expansion, we get

$$\exp(\chi \cdot \alpha) = \sum_{K' = 0}^{\infty} \frac{(\chi \cdot \alpha)^K}{K'!}.$$  

(46a)

$$= \sum_{K' = 0}^{\infty} \sum_{\vec{k}} \delta(\sum_r k_r, K') \prod \left\{ \frac{(\chi r \alpha r)^{k_r}}{k_r!} \right\}.$$  

(46b)

Now apply to Eq.(44) this expansion of $\exp(\chi \cdot \alpha)$ and Eq.(37):

$$LHS = (2\pi)^L N \sum_{K' = 0}^{\infty} \sum_{\vec{k}} \delta(\sum_r k_r, K') \frac{1}{\Delta(\alpha) \prod \left\{ \alpha_r^{k_r} \right\}} \frac{\Delta(\vec{k})}{(\sum_r k_r + L_N + N - 1)!}.$$  

(47)

The function being summed over $\vec{k}$ is a product of a totally anti-symmetric function of $\vec{k}$ times $\prod \left\{ \alpha_r^{k_r} \right\}_r$. Hence we may replace $\prod \left\{ \alpha_r^{k_r} \right\}_r$ by its totally anti-symmetric part:
\[ \prod \{\alpha_r^{k_r}\}_r \to \frac{1}{N!} \det(\alpha^{k_0}, \alpha^{k_1}, \ldots, \alpha^{k_{N-1}}). \] 

(48)

After doing this we will have a sum over \( \mathbf{k} \) of a totally symmetric function of \( \mathbf{k} \) so we can replace:

\[ \sum_{\mathbf{k}} \to N! \sum_{\mathbf{k}} \theta(0 \leq k_0 < k_1 < \cdots < k_{N-1}). \] 

(49)

(Terms with \( k_i = k_{i+1} \) vanish.) We can also change the lower limit of the \( K' \) sum from \( K' = 0 \) to \( K' = L_N \), because terms with \( K' < L_N \) do not contribute. Let \( K = K' - L_N \). Let us change variables from \( K' \) to \( K \). All these changes yield:

\[
\text{LHS} = (2\pi)^{L_N} \sum_{K=0}^{\infty} \sum_{\mathbf{k}} \theta(0 \leq k_0 < k_1 < \cdots < k_{N-1}) \delta(\sum_r k_r, K + L_N) \frac{\Delta(\mathbf{k})}{(K + N^2 - 1)!} \frac{\det(\alpha^{k_0}, \alpha^{k_1}, \ldots, \alpha^{k_{N-1}})}{\det(\alpha^0, \alpha^1, \ldots, \alpha^{N-1})}.
\]

(50)

Next consider the following change of variables:

\[
\begin{align*}
\eta_{N-1} &= k_0 \\
\eta_{N-2} &= k_1 - 1 \\
& \quad \vdots \\
\eta_1 &= k_{N-2} - (N - 2) \\
\eta_0 &= k_{N-1} - (N - 1)
\end{align*}
\]

(51)

Note that \( \eta_0 - \eta_1 = k_{N-1} - k_{N-2} - 1 \geq 0, \eta_1 - \eta_2 = k_{N-2} - k_{N-3} - 1 \geq 0, \) etc.

Also, \( \sum_j \eta_j = K' - L_N = K \). Hence, \( \eta \) specifies a Young graph with \( K \) boxes and at most \( N \) rows.

Define a column vector \( \delta = (N - 1, N - 2, \ldots, 1, 0)^T \). Let \( R \) be the \( N \times N \) matrix that has ones on the non-principal diagonal and zeros everywhere else. For example, for \( N = 2 \), \( R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). For any \( N \) dimensional column vector \( \eta \), define its \( R \) transform \( \eta^R \) by \( \eta^R = R\eta \). \( R \) just reverses the entries of \( \eta \). Eqs.(51) can now be stated succinctly as \( \eta = k^R - \delta \). Note that \( R^2 = 1 \). Thus, \( \det(RAR) = \det(A) \), for any square matrix \( A \) of the same dimension as \( R \). The right and left \( R \)'s reverse the order of the columns and of the rows of \( A \), but this does not change the value of \( \det(A) \).

For example, suppose \( \det(A) \) is the Vandermonde determinant \( \Delta(v) \) of a vector \( v \). Assume \( v \) is 3 dimensional for concreteness. Then \( \det(v^0, v^1, v^2) = \det((v^R)^2, (v^R)^1, (v^R)^0) \).

Changing the summation variable from \( \mathbf{k} \) to \( \eta \) in Eq.(50) yields:
\[
LHS = \frac{(2\pi)^{LN} \mathcal{F}_{N-1}}{(N^2 - 1)!} + (2\pi)^{LN} \sum_{K=1}^{\infty} \sum_{\eta^N} \Delta(\eta^R + \delta^R) \gamma, \tag{52}
\]

where \( \gamma \) is defined by

\[
\gamma = \frac{\det(\alpha^{\eta_{N-1}}, \alpha^{\eta_{N-2}+1}, \ldots, \alpha^{\eta_{0}+N-1})}{\det(\alpha^0, \alpha^1, \ldots, \alpha^{N-1})}. \tag{53}
\]

Reversing the order of the columns of both the denominator and numerator determinants of \( \gamma \) yields:

\[
\gamma = \frac{\det(\alpha^{\eta_{0}+N-1}, \alpha^{\eta_{N-2}}, \ldots, \alpha^{\eta_{N-1}})}{\det(\alpha^{N-1}, \alpha^{N-2}, \ldots, \alpha^0)} = \chi^{\eta^N}(A). \tag{54}
\]

Reversing columns and rows in \( \Delta(\eta^R + \delta^R) \) leads to Weyl’s formula for the dimension of \( \eta^N \).

\[
\Delta(\eta^R + \delta^R) = \det[(\eta + \delta)^{N-1}, (\eta + \delta)^{N-2}, \ldots, (\eta + \delta)^0] = \dim(\eta^N)\mathcal{F}_{N-1}. \tag{55}
\]

Applying Eqs. (54) and (55) to Eq. (52) finally yields LHS = RHS. QED

Eq. (43) gives a generating function that can be used to calculate moments over a uniform ensemble of density matrices. For example, we can calculate the mean value of \( X_{i1,j1} \cdot X_{i2,j2} \). Such moments will have indices attached because \( X \) is a matrix. To avoid having free indices, we will bind them to constant matrices. So instead of calculating the mean value of \( X_{i1,j1} \cdot X_{i2,j2} \), we will calculate the mean value of \( (C_1 \cdot X)(C_2 \cdot X) \), where \( C_1 \) and \( C_2 \) are constant matrices.

For any \( N \times N \) matrices \( A \) and \( C \), consider how the operator \( C \cdot \frac{\partial}{\partial A} \) acts on a power \( A^n \) for some integer \( n \). \( C \cdot \frac{\partial}{\partial A} \) replaces one \( A \) at a time by a \( C \):

\[
C \cdot \frac{\partial}{\partial A} A^n = \sum_{i=0}^{n-1} A^i C A^{n-1-i}. \tag{56}
\]

For example, if we define the operator \( \Omega \) by

\[
\Omega = \prod \left\{ C_j \cdot \frac{\partial}{\partial A} \right\}_{1 \leq j \leq 4}, \tag{57}
\]

then:

\[
\Omega t_1^4 = \sum_P \text{tr}(C_1)\text{tr}(C_2)\text{tr}(C_3)\text{tr}(C_4), \tag{58a}
\]

\[
\Omega t_2 t_1 = \sum_P \text{tr}(C_1 C_2)\text{tr}(C_3)\text{tr}(C_4), \tag{58b}
\]

\[
\Omega t_2^2 = \sum_P \text{tr}(C_1 C_2)\text{tr}(C_3 C_4), \tag{58c}
\]
\[ \Omega t_3 t_1 = \sum_P \text{tr}(C_1 C_2 C_3) \text{tr}(C_4) , \quad (58d) \]

\[ \Omega t_4 = \sum_P \text{tr}(C_1 C_2 C_3 C_4) . \quad (58e) \]

In Eqs.(58), \( t_r = \text{tr}(A^r) \) as before, and the sums run over all permutations \( P \) on 4 letters. \( P \) acts on the subscripts \( \{1, 2, 3, 4\} \).

Define

\[ Z(A) = \int \mathcal{D}X \ e^{\text{tr}(AX)} \theta(X \geq 0) \delta(\text{tr}X - 1) , \quad (59) \]

\[ I(C_1) = \int \mathcal{D}X \ (C_1 \cdot X) \theta(X \geq 0) \delta(\text{tr}X - 1) , \quad (60) \]

\[ I(C_1, C_2) = \int \mathcal{D}X \ (C_1 \cdot X)(C_2 \cdot X) \theta(X \geq 0) \delta(\text{tr}X - 1) . \quad (61) \]

Then, by virtue of Table 6 and Eq.(43), one has

\[ I(C_1) = \lim_{A \to 0} \left( C_1 \cdot \frac{\partial}{\partial A} \right) Z(A) \]

\[ = \frac{(2\pi)^{LN} F_{N-1}}{(N^2)!} N \text{tr}(C_1) , \quad (62b) \]

and

\[ I(C_1, C_2) = \lim_{A \to 0} \left( C_1 \cdot \frac{\partial}{\partial A} \right) \left( C_2 \cdot \frac{\partial}{\partial A} \right) Z(A) \]

\[ = \frac{(2\pi)^{LN} F_{N-1}}{(N^2 + 1)!} N \left[ N \text{tr}(C_1) \text{tr}(C_2) + \text{tr}(C_1 C_2) \right] . \quad (63b) \]

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