Pontryagin forms on \((4r - 2)\)-manifolds and symplectic structures on the spaces of Riemannian metrics

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Abstract

The Pontryagin forms on the 1-jet bundle of Riemannian metrics, are shown to provide in a natural way diffeomorphism-invariant pre-symplectic structures on the space of Riemannian metrics for the dimensions \(n \equiv 2 \pmod{4}\). The equivariant Pontryagin forms provide canonical moment maps for these structures. In dimension two, the symplectic reduction corresponding to the pre-symplectic form and its moment map attached to the first Pontryagin form, is proved to coincide with the Teichmüller space endowed with the Weil-Petersson symplectic form.

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1 Introduction

The aim of this paper is to show how the classical Chern-Weil construction of Pontryagin classes, when applied to the universal Levi-Civita connection on the principal bundle of linear frames over the 1-jet bundle of Riemannian metrics, naturally provides diffeomorphism-invariant pre-symplectic structures on the space of Riemannian metrics in dimensions $n = 4r - 2$. Similarly, the Berline-Vergne construction of equivariant characteristic classes, when applied to the previous bundle, provides a canonical moment map for these pre-symplectic structures. In dimension two, we apply the Marsden-Weinstein symplectic reduction to the pre-symplectic form and its moment map corresponding to the first Pontryagin polynomial, and we prove that the reduction coincides with the Teichmüller space endowed with the Weil-Petersson symplectic form multiplied by the scalar factor $\frac{1}{2\pi^2}$.

The general idea is to consider the constructions of Riemannian geometry as depending on an arbitrary Riemannian metric in order to obtain diffeomorphisms invariant objects. This is equivalent to work in the product space $M \times \mathcal{M}$ instead of $M$. Moreover, as usually the objects constructed depend only on the metric and its derivatives up to a certain order $r$, this infinite-dimensional manifold can be replaced by the finite-dimensional bundle of metrics $\mathcal{M}$, or by its $r$-jet bundle $J^r \mathcal{M}$. Working on finite-dimensional manifolds, allows us to use local differential methods and we have the jet bundle geometry (e.g. see [17, 16]) at our disposal. The relationship between both spaces is given by the evaluation map $\text{ev}_r : M \times \mathcal{M} \rightarrow J^r \mathcal{M}$, $\text{ev}_r(x, g) = j^r x g$.

We work on the first jet bundle $J^1 \mathcal{M}$ due to the fact that the Levi-Civita connection of a metric depends on the first derivatives of that metric. Thus, if we pull the linear frame bundle $F M$ back to $J^1 \mathcal{M}$, then we obtain a principal $GL(n, \mathbb{R})$-bundle $\bar{\pi} : q^* F M \rightarrow J^1 \mathcal{M}$ admitting a canonical connection $\omega$ (called the universal Levi-Civita connection), which is invariant under the action of diffeomorphisms of $M$ (for details see [10]). According to the Chern-Weil theory of characteristic classes, by evaluating the $k$-th Pontryagin polynomial at the curvature $\Omega$ of the universal Levi-Civita connection, we obtain a closed $4k$-form on $J^1 \mathcal{M}$, the so-called universal $k$-th Pontryagin form $p_k(\Omega)$, which is invariant under the natural action of the diffeomorphism group of $M$. If $4k \leq \dim M$, then pulling $p_k(\Omega)$ back via $j^1 g$, we obtain the $k$-th Pontryagin form $p_k(\Omega^g)$ corresponding to the metric $g$. As the space of Riemannian metrics on $M$ is contractible, the maps $j^1 g$ for different $g$’s are homotopic. Hence, we recover the well-known result according to which the cohomology class of $p_k(\Omega^g)$ is independent of the metric $g$ chosen. Hence, for $4k \leq \dim M$, the universal Pontryagin forms determine the Pontryagin classes of $M$. Moreover, as $\dim J^1 \mathcal{M} > \dim M$, non-zero universal Pontryagin forms of degree greater than $\dim M$, exist. These are precisely the forms under consideration below.

According to [9], a differential $r$-form $\alpha \in \Omega^r (J^1 \mathcal{M})$ on $J^1 \mathcal{M}$ with $r > \dim M = n$ determines a $(r - n)$-form on the space of Riemannian metrics $\mathcal{M}$ given by $\mathcal{M}[\alpha] = \int_M ev^*_M \alpha \in \Omega^{r-n}(\mathcal{M})$. In particular, if $4r - n = 2$, i.e., $\dim M = 4r - 2$, then $\mathcal{M}[p_k(\Omega)]$ is a closed differential 2-form on $\mathcal{M}$,
i.e., a pre-symplectic structure $\sigma$ on $\mathfrak{Met}_M$, which is invariant under the action of the orientation-preserving diffeomorphism group $\text{Diff}^+ M$ on $\mathfrak{Met}_M$.

In addition, the pre-symplectic structure $\sigma$ admits a canonical moment map $\mu$, which is obtained as follows. Since the universal Levi-Civita connection is invariant under the action of the diffeomorphism group of $M$, the Berline-Vergne construction of equivariant characteristic classes, provides a canonical equivariant extension for the universal $k$-th Pontryagin form, called the equivariant $k$-th Pontryagin form. This equivariant extension provides an equivariant extension of $\sigma$, which is known (e.g., see [1]) to be equivalent to provide a moment map $\mu$ for $\sigma$.

Therefore, for $\dim M = 4r - 2$, the space of Riemannian metrics is endowed with the pre-symplectic structure $\sigma$ and the moment map $\mu$ corresponding to the universal and equivariant $r$-th Pontryagin forms.

In [9] the equivariant forms obtained in the present paper are shown to be related to the expressions of local gravitational anomalies given by the Atiyah-Singer index theorem for families. Moreover, it is also shown that for the study of the problem of locality in quantum field theory, it is essential that these forms are obtained form forms on the jet bundle.

As we have a pre-symplectic structure and a moment map, we can study the corresponding symplectic reduction. Below, we analyze in detail the two-dimensional case, $\dim M = 2$. Some partial results of the analogous situation in higher dimensions can be found in [11]. For a surface, we obtain concrete expressions of $\sigma$ and $\mu$ (see Proposition 5.2), and the Marsden-Weinstein quotient is proved to coincide, up to the scalar factor $\frac{1}{2\pi^2}$, with the Teichmüller space of $M$ endowed with the Weil-Petersson symplectic form.

Note that this scalar factor $\frac{1}{2\pi^2}$ is precisely what is needed for the cohomology class of this form to coincide with Mumford’s tautological class $\kappa_1$ (e.g., see [19]).

2 Preliminaries on the geometry of the bundle of metrics

In this section we recall some results appeared in [10]. Let $q: \mathcal{M}_M \to M$ be the bundle of Riemannian metrics of an $n$-dimensional smooth manifold $M$, i.e., $\mathcal{M}_M = \{g_x \in S^2(T^*_x M) : g_x$ is positive definite on $T_x M\}$, which is a convex open subset in $S^2(T^* M)$. The global sections of this bundle are the Riemannian metrics on $M$.

We denote by $\text{Diff}^+ M \subset \text{Diff} M$ the subgroup of orientation preserving diffeomorphisms, and we set $\mathcal{G} = \text{Diff} M \times \mathbb{R}$, $\mathcal{G}^+ = \text{Diff}^+ M \times \mathbb{R}$. Every coordinate system $(U, x^i)$ on $M$ induces coordinates $(q^{-1}U, x^i, y_{ij})$ on $\mathcal{M}_M$ by setting $g_x = y_{ij}(g_x)(dx^i)_x \otimes (dx^j)_x$, $y_{ij} = y_{ji}$, for every Riemannian metric $g$ on $U$. We set $(y^{ij}) = (y_{ij})^{-1}$.

Let $\pi: FM \to M$ be the bundle of linear frames of $M$. The lift of $\phi \in \text{Diff} M$ (resp. $X \in \mathfrak{X}(M)$) to $FM$ is denoted by $\tilde{\phi}: FM \to FM$ (resp. $\tilde{X} \in \mathfrak{X}(FM)$), see [13], VI.1–2. Let $q_1^* FM = J^1 \mathcal{M}_M \times_M FM$ be the pull-back of $FM$ to $J^1 \mathcal{M}_M$.
via \( q_1: J^1\mathcal{M}_M \to M \). There are two canonical projections

\[
\begin{array}{ccc}
q_1^*FM & \xrightarrow{\bar{q}_1} & FM \\
\pi \downarrow & & \downarrow \pi \\
J^1\mathcal{M}_M & \xrightarrow{q_1} & M
\end{array}
\]

The first projection \( \bar{\pi}: q_1^*FM \to J^1\mathcal{M}_M \) is a principal \( \text{Gl}(n, \mathbb{R}) \)-bundle and the second projection \( \bar{q}_1: q_1^*FM \to FM \) is \( \text{Gl}(n, \mathbb{R}) \)-equivariant.

The group \( \text{Diff}M \) acts naturally on \( \mathcal{M}_M \) and on \( FM \); hence it also acts on \( q_1^*FM \). If \( \phi \in \text{Diff}M \) (resp. \( X \in \mathfrak{x}(M) \)), its lift to \( q_1^*FM \) is \( \hat{\phi} = (\hat{\phi}^{(1)}, \hat{\phi}) \) (resp. \( \hat{X} = (\hat{X}^{(1)}, \hat{X}) \)), where \( \hat{\phi}: \mathcal{M}_M \to \mathcal{M}_M, \hat{\phi}^{(1)}: J^1\mathcal{M}_M \to J^1\mathcal{M}_M \) (resp. \( \hat{X} \in \mathfrak{x}(\mathcal{M}_M), \hat{X}^{(1)} \in \mathfrak{x}(J^1\mathcal{M}_M) \)) are the natural lifts of \( \phi \) (resp. \( X \)). As \( \bar{q}_1 \) is a \( \text{Diff}M \)-equivariant map, we have \((\bar{q}_1)_*(\hat{X}) = \hat{X} \). If \( X = X^i \partial/\partial x^i \), then we have

\[
\hat{X} = X^i \frac{\partial}{\partial x^i} - \sum_{i<j} \left( \frac{\partial X^r}{\partial x^i} y_{kj} + \frac{\partial X^k}{\partial x^j} y_{ki} \right) \frac{\partial}{\partial y_{ij}}.
\]

For every \( t \in \mathbb{R} \), we define \( \varphi_t \in \text{Aut}FM \) (resp. \( \hat{\varphi}_t \in \text{Diff}(J^1\mathcal{M}_M) \), resp. \( \hat{\varphi}_t \in \text{Aut}(q_1^*FM) \)) as follows:

\[
\begin{align*}
\varphi_t(u) &= \exp(-\frac{1}{2}t) \cdot u, \\
\hat{\varphi}_t(j^1_\mathfrak{g}^*g) &= j^1_\mathfrak{g}(\exp(t) \cdot g), \\
\hat{\varphi}_t(j^1_\mathfrak{g}^*g, u) &= (j^1_\mathfrak{g}(\exp(t) \cdot g), \exp(-\frac{1}{2}t) \cdot u).
\end{align*}
\]

We denote by \( \xi \in \mathfrak{x}(FM) \) (resp. \( \hat{\xi} \in \mathfrak{x}(J^1\mathcal{M}_M) \), resp. \( \hat{\xi} \in \mathfrak{x}(q_1^*FM) \)) the infinitesimal generator of the 1-parameter group \((\varphi_t)\) (resp. \((\hat{\varphi}_t)\), resp. \((\hat{\varphi}_t)\)) defined above. We have \( q_1^*\pi_*(\hat{\xi}) = 0, q_1^*\pi_*(\hat{\xi}) = \xi \), and \( \hat{\pi}_*(\hat{\xi}) = \xi \) and

\[
\hat{\xi} = y_{ij} \frac{\partial}{\partial y_{ij}} + y_{ij,k} \frac{\partial}{\partial y_{ij,k}},
\]

where \((x^i, y_{ij}, y_{ij,k})\) is the coordinate system induced on \( J^1\mathcal{M}_M \).

The group \( \mathcal{G} = \text{Diff}M \times \mathbb{R} \) acts by automorphisms of the principal \( \text{Gl}(n, \mathbb{R}) \)-bundle \( \bar{\pi}: q_1^*FM \to J^1\mathcal{M}_M \), inducing a \( \mathcal{G} \)-action on the associated bundles to \( q_1^*FM \) (such as \( q_1^*TM, q_1^*T^*M \), etc.), as well as in the space of sections and differential forms with values on these bundles.

The bundle \( q_1^*TM \to J^1\mathcal{M}_M \) is endowed with a universal metric given by \( g \left( (j^1_\mathfrak{g}^*g, X), (j^1_\mathfrak{g}^*g, Y) \right) = g_x(X, Y), \forall X, Y \in T_xM \), which is invariant under the action of the group \( \mathcal{G} = \text{Diff}M \times \mathbb{R} \) defined above (cf. \([10]\)).

We denote by \( \omega^g \) the Levi-Civita connection form of \( g \) and by \( \nabla^g \) the covariant derivation law on the associated vector bundles. The \( \mathfrak{gl}(n, \mathbb{R}) \)-valued 1-form on \( q_1^*FM \) defined by \( \omega_{\text{hor}}(X) = \omega^g((\bar{q}_1)_*X), \forall X \in T_{(\mathfrak{g}(\mathfrak{g}, u))}(q_1^*FM) \), is a \( \mathcal{G} \)-invariant connection form on the principal \( \text{Gl}(n, \mathbb{R}) \)-bundle \( \bar{\pi}: q_1^*FM \to J^1\mathcal{M}_M \) (see \([10]\)), but unfortunately, it is not \( \mathfrak{g} \)-Riemannian. In fact, the connection \( \omega_{\text{hor}} \) induces a derivation law \( \nabla^{\text{hor}} \) on the associated bundles to \( q_1^*FM \), and we
have $\nabla^\omega_{\text{hor}} g = \theta$, where $\theta = (dy_{ij} - y_{ij,k} dx^k) \otimes dx^i \otimes dx^j$ is the $V(q)$-valued 1-form determining the contact structure on $J^1 \mathcal{M}_M$ and we have used the natural identification $V(q) \cong q^* S^2 T^* M = \mathcal{M}_M \times_M S^2 T^* M$. 

Fortunately, if $\vartheta \in \Omega^1(J^1 \mathcal{M}_M, \text{End} TM)$ is the form given by

$$(2.3) \quad \vartheta = g^{-1} \theta = y^{\alpha j} (dy_{\alpha a} - y_{\alpha a,k} dx^k) \otimes dx^i \otimes \frac{\partial}{\partial x^j},$$

then we can define a connection form on $q_1^* FM$—called the ‘universal Levi-Civita connection’—as follows: $\omega = \omega_{\text{hor}} + \frac{1}{2} \vartheta$, which is $G$-invariant and $g$-Riemannian; i.e., $\nabla^\omega g = 0$ (see [10]). Then, the connection form $\omega$ is reducible to a connection on the principal $O(n)$-sub-bundle

$$OM = \{ (j^1 g, u_x) \in q_1^* FM : u_x \text{ is } g_x\text{-orthonormal} \} \subset q_1^* FM.$$ 

In fact, it is the only $G$-invariant connection on $OM$ (see [10]). We consider the usual identification (e.g., see [13 II, Example 5.2]) between differential forms on the base manifold of a principal bundle taking values in the adjoint bundle and differential forms of the adjoint type on that principal bundle. With this identification we have $\Omega, \Omega_{\text{hor}} \in \Omega^2(J^1 \mathcal{M}_M, \text{End} TM)$, where $\Omega, \Omega_{\text{hor}}$ are the curvature forms of $\omega, \omega_{\text{hor}}$, respectively.

As $V(q) \cong q^* S^2 T^* M$, an element $h \in \Omega^0(M, S^2 T^* M)$ determines a vertical vector field $H$ on $\mathcal{M}_M$, defined by $H(g_x) = (g_x, h_x)$, $g_x \in \mathcal{M}_M$. Locally, $H = h_{ij} \partial/\partial y_{ij}$ if $h = h_{ij} dx^i \otimes dx^j$.

Some important formulas used below, are the following (see [10]):

$$(2.4) \quad \Omega_{\text{hor}} = (\partial \Gamma^i_{jk} \wedge dx^k + \Gamma^i_{as} \Gamma^s_{j,} dx^s \wedge dx^j) dx^j \otimes \frac{\partial}{\partial x^i},$$

$$(2.5) \quad \Gamma^i_{jk} = \frac{1}{2} y^{\alpha i} (y_{\alpha j,k} + y_{ak,j} - y_{jk,a}),$$

$$(2.6) \quad \Omega = (\Omega_{\text{hor}})_A - \frac{1}{2} \vartheta \wedge \vartheta.$$ 

The universal $k$-th Pontryagin form of $M$, $p_k(\Omega) \in \Omega^{4k}(J^1 \mathcal{M}_M)$, is defined as the form obtained by means of the Chern-Weil theory of characteristic classes by applying the $k$-th Pontryagin polynomial to the curvature $\Omega$ of the universal Levi-Civita connection $\omega$. These forms are closed, $G$-invariant and satisfy the following universal property (see [10]): for every Riemannian metric $g$ we have $(j^1 g)^* (p_k(\Omega)) = p_k(\Omega^g)$, where $\Omega^g \in \Omega^2(M, \text{End} TM)$ is the curvature form of the Levi-Civita connection of the metric $g$. Hence the Pontryagin forms of degree equal to or less than $n$ determine the Pontryagin classes of $M$. However, the key point is that there are non-zero Pontryagin forms of degree greater than $n$ (as $\dim(J^1 \mathcal{M}_M) > n$). For example, as $p_1(X) = -\frac{1}{12} \text{tr}(X^2)$, $X \subset \mathfrak{so}(n)$, we have

$$p_1(\Omega) = -\frac{1}{12} \text{tr}(\Omega \wedge \Omega) \in \Omega^4(J^1 \mathcal{M}_M).$$

For $n = 2$ this form does not vanish, as we see below.

We denote the covariant differential of $X \in \mathfrak{X}(M)$ with respect to the Levi-Civita connection of a metric $g$ on $M$ by $\nabla^g X \in \Omega^1(M, TM) \cong \Omega^2(M, \text{End} TM)$. 

5
Similarly, the covariant differential of a tensor field $h \in \Omega^0(M, \bigodot^r T^*M)$ is denoted by $\nabla^g h \in \Omega^1(M, \bigodot^r T^*M) \cong \Omega^0(M, \bigodot^{r+1} T^*M)$.

Let $\eta^g : \bigodot^r T^*M \to T^*M \otimes \text{End}TM$ be the vector-bundle homomorphism $\eta^g(a \otimes b \otimes c) = b \otimes a \otimes c^2$. We set $\nabla^g h = \eta^g(\nabla^g h) \in \Omega^1(M, \text{End}TM)$.

We denote by $(\nabla^g X)_S$ and $(\nabla^g X)_A$ the $g$-symmetric and $g$-skew-symmetric parts of $\nabla^g X$ respectively, and a similar notation is also used for $\nabla^g h$. We obtain $\delta^g h = \text{tr}(\nabla^g h) \in \Omega^1(M)$, where $\delta^g h$ is the $g$-divergence of $h$. If $X = X^i \partial / \partial x^i$, $h = h_{ij} \partial^i \otimes \partial^j$, then

$$\nabla^g X = \left( \frac{\partial x^i}{\partial x^j} + \Gamma^i_{jk} X^k \right) dx^j \otimes \frac{\partial}{\partial x^i},$$

$$\nabla^g h = g^{ib} \left( \frac{\partial h_{ib}}{\partial x^k} - h_{ab} \Gamma^a_{ik} - h_{ia} \Gamma^i_{bk} \right) dx^i \otimes \left( dx^k \otimes \frac{\partial}{\partial x^j} \right),$$

$$\delta^g h = g^{ib} \left( \frac{\partial h_{ib}}{\partial x^j} - h_{ab} \Gamma^a_{ij} - h_{ia} \Gamma^i_{bj} \right) dx^i.$$

For every $X \in \mathfrak{X}(M)$ we denote by $q_1^* X \in \Omega^0(J^1 \mathcal{M}_M, TM)$ the section given by $(q_1^* X)(j_{1 g}) = (j_{1 g}, X_x)$.

The covariant differential $\nabla^{\omega_{hor}}(q_1^* X) \in \Omega^1(J^1 \mathcal{M}_M, TM)$ is $q_1$-horizontal; hence, it can be viewed as a section $\nabla X$ of the bundle $q_1 \text{End}TM$. Similarly, if $h \in \Omega^0(M, \bigodot^2 \mathcal{M}_M)$, then $\nabla^{\omega_{hor}}(q_1^* h) \in \Omega^1(J^1 \mathcal{M}_M, \bigodot^2 \mathcal{M}_M)$ is $q_1$-horizontal, and $\nabla h = \eta^g(\nabla^{\omega_{hor}}(q_1^* h))$ can be considered to be an element of $\Omega^0(J^1 \mathcal{M}_M, T^*M \otimes \text{End}TM) \subset \Omega^1(J^1 \mathcal{M}_M, \text{End}TM)$.

The following Lemma will be used to obtain the explicit expression of the pre-symplectic forms.

**Lemma 2.1.** If $H$ is the vertical vector field on $\mathcal{M}_M$ determined by the section $h$ of $S^2 T^*M$, then $\iota_H \Omega = (\nabla h)_\Lambda - (g^{-1}(q_1^* h) \circ \partial)_\Lambda$.

**Proof.** Given $j_{1 g} \in J^1 \mathcal{M}_M$, we consider a normal system of coordinates for $g$ at $x$. By virtue of (2.26) and (2.31), we have

$$\Omega_{j_{1 g}} = \frac{1}{2} \left( d\Gamma_{jk}^{i} \wedge dx^k - d\Gamma_{ik}^{j} \wedge dx^k - (dy_{ia} \wedge dy_{aj}) \right)_{j_{1 g}} \otimes \left( dx^i \otimes \frac{\partial}{\partial x^j} \right)_{j_{1 g}}$$

$$= \frac{1}{2} \left( dy_{ki,j} \wedge dx^k - dy_{kj,i} \wedge dx^k - (dy_{ia} \wedge dy_{aj}) \right)_{j_{1 g}} \otimes \left( dx^i \otimes \frac{\partial}{\partial x^j} \right)_{j_{1 g}}.$$

By contracting this form with $H^{(1)} = h_{ij} \partial / \partial y_{ij} + (\partial h_{ij} / \partial x^k) \partial / \partial y_{ij,k}$, we obtain

$$\iota_{H^{(1)}} \Omega_{j_{1 g}} = \frac{1}{2} \left( \frac{\partial h_{ki}}{\partial x^j} - \frac{\partial h_{kj}}{\partial x^i} \right) dx^k - h_{ia} dy_{aj} + h_{aj} dy_{ia} \otimes \left( dx^i \otimes \frac{\partial}{\partial x^j} \right)_{j_{1 g}}$$

$$= (\nabla h)_\Lambda - (g^{-1}(q_1^* h) \circ \partial)_\Lambda.$$

\[\square\]
3 Pre-symplectic structures on $\mathcal{M} M$

Let $p: E \to M$ be a locally trivial fibre bundle over a compact connected and oriented $n$-manifold without boundary. In [8], a map $\Xi: \Omega^{n+k}(J^r E) \to \Omega^k(\Gamma(E))$ has been defined, which provides a geometrical interpretation of the forms on the jet bundle with degree greater than the dimension of the base manifold. If $ev_r: M \times \Gamma(E) \to J^r E$ is the evaluation map $ev_r(x, s) = j^r_x s$, then $\Xi[\alpha] = \int_M ev_r^*\alpha \in \Omega^k(\Gamma(E))$. The map $\Xi$ commutes with the exterior differential and the action of the automorphims of the bundle, and hence maps closed (resp. invariant) forms to closed (resp. invariant) forms.

Let $\mathcal{M} M = \Gamma(M,\mathcal{M}_M)$ denote the space of Riemannian metrics on $M$. As $\mathcal{M} M$ is an open subset of $\mathcal{S}^2(M) = \Gamma(M,\mathcal{S}^2 T^* M)$, we have the canonical identification $T_g \mathcal{M} M \cong S^2(M)$ for any $g \in \mathcal{M} M$. The group $G = Diff M \times \mathbb{R}$ acts in a natural way on $\mathcal{M} M$ by setting,

$$G \times \mathcal{M} M \to \mathcal{M} M,$$

$$((\phi, t), g) \mapsto \exp(t) \cdot (\phi^{-1})^* g.$$

In the rest of this section, we assume $\dim M = n = 4r - 2$, $r \in \mathbb{N}$, and $f \in \mathcal{L}^{2r(n)}$ denotes a Weil polynomial of degree $2r$. Hence $f(\Omega) \in \Omega^{n+2}(J^1 \mathcal{M}_M)$ and $\sigma = \Xi[f(\Omega)] \in \Omega^2(\mathcal{M} M)$ is a $G^*$-invariant pre-symplectic form on the space $\mathcal{M} M$ of Riemannian metrics. The explicit expression for $\sigma$ is as follows:

**Theorem 3.1.** For every $g \in \mathcal{M} M$, $h, k \in T_g \mathcal{M} M \cong S^2(M)$ we have

$$\sigma_g(h, k) = -2r(2r - 1) \int_M f \left( (\nabla^g h)_\Lambda, (\nabla^g k)_\Lambda, \Omega^g, (2r-2, \Omega^g) \right) - 2r \int_M f \left( (g^{-1} h \circ g^{-1} k)_\Lambda, \Omega^g, (2r-1, \Omega^g) \right).$$

**Proof.** By virtue of [8] Proposition 11] we have

$$\sigma_g(h, k) = \int_M (j^1 g)^* (t_{K^{(1)}} t_{H^{(1)}} f(\Omega, \ldots, \Omega)).$$

Moreover, using Lemma [21] we obtain

$$t_{K^{(1)}} t_{H^{(1)}} f(\Omega, \ldots, \Omega) = 2r t_{K^{(1)}} f (t_{H^{(1)}} \Omega, \Omega, \ldots, \Omega)$$

$$= 2r t_{K^{(1)}} f \left( (\nabla h)_\Lambda - (g^{-1} (q^r_1 h) \circ \vartheta)_\Lambda, \Omega, \ldots, \Omega \right)$$

$$= 2r f \left( - (g^{-1} (q^r_1 h) \circ g^{-1} (q^r_1 k))_\Lambda, \Omega, \ldots, \Omega \right)$$

$$- 2r(2r - 1)f \left( (\nabla k)_\Lambda - (g^{-1} (q^r_1 h) \circ \vartheta)_\Lambda, \Omega, \ldots, \Omega \right).$$

$$\nabla$$
Using \((j^1 g)^* \vartheta = 0\), \((j^1 g)^* \Omega = \Omega^g\), and \((j^1 g)^* (\hat{\nabla} h)_A = (\hat{\nabla}^g h)_A\), the result follows.

If \(M\) is an oriented compact connected surface and \(f = p_1\) is the first Pontryagin polynomial, then

\[
\sigma_g(h, k) = \frac{1}{4\pi^2} \int_M \text{tr} \left( (g^{-1} h \circ g^{-1} k)_A \wedge \Omega^g \right) + \frac{1}{4\pi^2} \int_M \text{tr} \left( (\hat{\nabla}^g h)_A \wedge (\hat{\nabla}^g k)_A \right)
\]

for every \(g \in \text{Met}_M\), \(h, k \in T_g \text{Met}_M \sim S^2(M)\). (A simpler expression for this form is obtained in Proposition 5.2 below.)

If \(\dim M = 6\), i.e., \(r = 2\), the basic Weil polynomials of degree 4, are \(p_2\) and \((p_1)^2\). If we set \(t_k(X) = \text{tr}(X^k)\), \(X \in \mathfrak{so}(6; \mathbb{R})\), then

\[
p_1 = -\frac{1}{32\pi^2} t_2,
p_2 = \frac{1}{128\pi^2} \left[ (t_2)^2 - 2t_4 \right].
\]

For \(f = t_4\), the formula in Theorem 3.1 yields,

\[
\sigma_g(h, k) = -12 \int_M \text{tr} \left( (\hat{\nabla}^g h)_A \wedge (\hat{\nabla}^g k)_A \wedge \Omega^g \wedge \Omega^g \right) - 4 \int_M \text{tr} \left( (g^{-1} h \cdot g^{-1} k)_A \wedge \Omega^g \wedge \Omega^g \wedge \Omega^g \right),
\]

and for \(f = (t_2)^2\), similarly we obtain

\[
\sigma_g(h, k) = -4 \int_M \text{tr} \left( (\hat{\nabla}^g h)_A \wedge (\hat{\nabla}^g k)_A \right) \wedge \text{tr} \left( \Omega^g \wedge \Omega^g \right) - 8 \int_M \text{tr} \left( (\hat{\nabla}^g h)_A \wedge \Omega^g \right) \wedge \text{tr} \left( (\hat{\nabla}^g k)_A \wedge \Omega^g \right) - 4 \int_M \text{tr} \left( (g^{-1} h \cdot g^{-1} k)_A \wedge \Omega^g \right) \wedge \text{tr} \left( \Omega^g \wedge \Omega^g \right).
\]

4 Equivariant Pontryagin forms & moment maps

First, we recall the definition of equivariant cohomology in the Cartan model (e.g. see [21]). Let a connected Lie group \(G\) act on a manifold \(N\) and let \(\mathfrak{g} \to \mathfrak{X}(N)\), \(X \mapsto X_N\) be the induced Lie algebra homomorphism, \(X_N\) being the infinitesimal generator of the flow \(L_\exp(-tX)\) and \(L_g : N \to N\) given by \(L_g(x) = g \cdot x\), \(\forall g \in G\), \(\forall x \in N\). Let \(\Omega_G(N) = \mathcal{P}^*(\mathfrak{g}, \Omega^*(N))^G\) be the space of \(G\)-invariant polynomials on \(\mathfrak{g}\) with values in \(\Omega^*(N)\). We assign degree \(2k + r\) to the polynomials in \(\mathcal{P}^k(\mathfrak{g}, \Omega^*(N))\). The space of \(G\)-equivariant differential \(q\)-forms is

\[
\Omega^q_G(N) = \bigoplus_{2k+r=q} (\mathcal{P}^k(\mathfrak{g}, \Omega^*(N)))^G.
\]
Lemma 4.1. Let \( f, \sigma \) be the characteristic form associated to \( f \) and \( \sigma \) is \( G \)-equivariant. The universal Levi-Civita connection is \( \text{Civita connection} \) really provides canonical equivariant extensions: As the universal construction of equivariant characteristic classes of Berline and Vergne (see [20]) but for the universal Pontryagin forms, the classical construction of equivariant characteristic classes of Berline and Vergne (see [3][4][5]) really provides canonical equivariant extensions: As the universal Levi-Civita connection is \( G \)-invariant, for every \( f \in \mathcal{I}_k^{(n)} \) the \( G \)-equivariant characteristic form associated to \( f \) and \( \omega \), is a \( G \)-equivariant extension of \( f(\Omega, \xi, \Omega) \), given by \( f(\Omega)(X, t) = f(\Omega - \omega(X + t\xi), (k, \Omega - \omega(X + t\xi)) \).

As a simple computation shows, we have

**Lemma 4.1.** Let \( \omega \in \Omega^1(FM, \mathfrak{g}(n, \mathbb{R})) \) be the connection form of a linear connection \( \nabla \). For every vector field \( X \in \mathfrak{X}(M) \) we have

1. The 0-form \( \omega(X) \in \Omega^0(FM, \mathfrak{g}(n, \mathbb{R})) \) is of adjoint type.
2. If \( \nabla \) is symmetric, the 0-form on \( M \) with values on \( \text{End} TM \) corresponding to \( \omega(X) \) coincides with \( \nabla X \).
3. The 0-form \( \omega(\xi) \in \Omega^0(FM, \mathfrak{g}(n, \mathbb{R})) \) is of adjoint type and it corresponds to \( -\frac{1}{2} \text{id}_{TM} \in \Omega^0(M, \text{End} TM) \).

**Proposition 4.2.** The explicit expression for the \( G \)-equivariant characteristic form associated to the Weil polynomial \( f \) is as follows:

\[
f(\Omega)(X, t) = f\left(\Omega - (\nabla X)_\Lambda, (k, \Omega - (\nabla X)_\Lambda\right) = \sum_{i=1}^k (-1)^{k-i} \binom{k}{i} f(\Omega, \cdots, \Omega, (\nabla X)_\Lambda, \cdots, (\nabla X)_\Lambda).
\]

**Proof.** Let \( \omega \) be the universal Levi-Civita connection form on \( q_1^*FM \) and let \( \xi \) be the vector field introduced in the section 2. We have \( \omega(\xi) = 0 \), and for every \( X \in \mathfrak{X}(M) \) the form \( \omega(X) \) is a 0-form of adjoint type on \( q_1^*FM \), and the corresponding \( \text{End} TM \)-valued 0-form on \( J^1M \) is \( (\nabla X)_\Lambda \). In fact, from (2.1), (2.2), (2.3), and (2.4) we have

\[
\text{grad} \left( X^{(1)} \right) = -y^{bj} \left( y_{aj} \Gamma^a_{ki} + y_{ai} \Gamma^a_{kj} \right) X^k + \frac{\partial X^k}{\partial x^i} y_{kj} + \frac{\partial X^k}{\partial x^j} y_{ki} \ dx^i \otimes \frac{\partial}{\partial x^j}
\]

\[
= -2 (\nabla X)_\Lambda,
\]

\[
\text{grad} \left( \xi \right) = y^{ia} y_{aj} dx^j \otimes \frac{\partial}{\partial x^i} = \text{id}_{TM}.
\]

The 0-forms \( \omega_{\text{hor}}(X), \omega_{\text{hor}}(\xi) \) are of adjoint type due to the fact that \( \omega_{\text{hor}} \) is invariant under the action of \( \text{Diff} M \times \mathbb{R} \). If \( \alpha \in \Omega^0(J^1M, \text{End} TM) \) is
the 0-form taking values in $\text{End}TM$ corresponding to $\omega_{\text{hor}}(\hat{X})$, then from the formula $\omega_{\text{hor}}(\hat{X})(u,j^1_x g) = \omega^g_0((q_1)_* \hat{X}) = \omega_0^g(\hat{X})$ and Lemma 4.1 we obtain
\[
\alpha(j^1_x g) = (j^1_x g, (\nabla^g X)(x)) = (\nabla X)(j^1_x g).
\]
Hence $\alpha = \nabla X$. Accordingly, the form corresponding to $\omega(\hat{X}) = \omega_{\text{hor}}(\hat{X}) + \frac{1}{2} \theta(1)$ is $(\nabla X)_A = \nabla X - (\nabla X)_G$.

If $\beta \in \Omega^0(J^1 \mathcal{M}_M, \text{End}TM)$ is the form corresponding to $\omega_{\text{hor}}(\hat{\xi})$, then we have $\omega_{\text{hor}}(\xi)(u,j^1_x g) = \omega^g_0((q_1)_* \xi) = \omega^g_0(\xi)$. Hence
\[
\beta(j^1_x g) = (j^1_x g, -\frac{1}{2} \text{id}_{T_x M}) = -\frac{1}{2} q_1^* \text{id}(j^1_x g),
\]
that is, $\beta = -\frac{1}{2} q_1^* \text{id}_{T_x M}$. Therefore, the form corresponding to $\omega(\hat{\xi})$ vanishes, since $\beta + \frac{1}{2} \theta(\xi) = 0$.

For example, the first equivariant Pontryagin form is given by
\[
p_1(\Omega_g)(X, t) = -\frac{1}{8\pi^2} (\text{tr}(\Omega \wedge \Omega) - 2\text{tr}((\nabla X)_A \circ \Omega) + \text{tr}((\nabla X)_A \circ (\nabla X)_A)).
\]

Finally, we recall the relationship between equivariant extensions of a pre-symplectic form and moment maps (e.g., see [11]). If $\omega$ is a pre-symplectic form on $N$, then an equivariant extension of $\omega$ is given by $\omega^g = \omega + \mu$, where $\mu : g \to \Omega^0(N)$ is a $G$-invariant linear map satisfying $i_{\xi_N} \omega = d(\mu(X))$, i.e., $\mu$ is a (co-)moment map for $\omega$. Hence, to give an equivariant extension for a pre-symplectic form is equivalent to giving a moment map for it.

The map $\mathbb{Z}$ introduced above, naturally extends to a map on the spaces of equivariant differential forms $\mathbb{Z} : \Omega^{n+k}_G(J^1 \mathcal{M}_M) \to \Omega^k_G(\mathfrak{met}M)$ that commutes with the Cartan differential (see [13]). By applying this map to the equivariant Pontryagin forms, we obtain equivariant extensions of the pre-symplectic structures on $\mathfrak{met}M$, or equivalently, canonical moment maps for them, given by
\[
\mu : \mathfrak{X} = \mathbb{R} \to \Omega^0(\mathfrak{met}M),
\]
\[
\mu(X, t)_g = -2t \int_M f \left( (\nabla^g X)_A \wedge \Omega^g, \Omega^g, \ldots, (\Omega^g)^{2r-1} \wedge \Omega^g \right).
\]

In the two-dimensional case, for $f = p_1$, the formula (4.8) yields the following expression:
\[
\mu : \mathfrak{X} = \mathbb{R} \to \Omega^0(\mathfrak{met}M),
\]
\[
\mu(X, t)_g = \frac{1}{16\pi} \int_M \text{tr}((\nabla^g X)_A \circ \Omega^g).
\]
Similarly, if $\dim M = 6$, then for $f = t_4$ we obtain the following moment map:
\[
\mu(X, t)_g = -4 \int_M \text{tr}((\nabla^g X)_A \circ \Omega^g \wedge \Omega^g \wedge \Omega^g),
\]
and for $f = (t_2)^2$, we have
\[
\mu(X, t)_g = -4 \int_M \text{tr}((\nabla^g X)_A \circ \Omega^g) \wedge \text{tr}(\Omega^g \wedge \Omega^g).
\]
5 Symplectic reduction in dimension 2

From now on, we assume that \( M \) is a compact, orientable surface, so that \( n = 2 \).

By applying the preceding considerations to the first Pontryagin polynomial \( p_1 \) we obtain a canonical \( G \)-invariant pre-symplectic structure \( \sigma \) on the space of Riemannian metrics \( \mathfrak{M} \text{et} M \) and a moment map \( \mu \) for it, given by (3.7) and (4.9). In this section we apply the Marsden-Weinstein procedure of symplectic reduction to the pre-symplectic manifold \((\mathfrak{M} \text{et} M, \sigma)\) with respect the moment map \( \mu \). First we obtain simpler expressions of \( \sigma \) and \( \mu \) in the 2-dimensional case.

The following results easily follows by a direct calculation in normal coordinates.

**Lemma 5.1.** We have

a) \( \text{Pfaff}((\hat{\nabla}^g h)_A) = -\frac{1}{2} *_g (\delta^g h - d(\text{tr}_g h)) \) for every \( h \in S^2(M) \).

b) \( (\hat{\nabla}^g X)_A = \frac{1}{2} g^{-1}(dX^A) \) for every \( X \in \mathfrak{X}(M) \).

**Proposition 5.2.** The expressions of \( \sigma \) and \( \mu \) are as follows:

\[
\sigma_g(h, k) = \frac{1}{4\pi} \int_M S^g \text{tr} \left( g^{-1} h \circ g^{-1} k \circ g^{-1} \text{vol}_g \right) \text{vol}_g - \frac{1}{8\pi} \int_M \left( \delta^g h - d(\text{tr}_g h) \right) \wedge \left( \delta^g k - d(\text{tr}_g k) \right)
\]

\[
\mu_g(X, t) = \frac{1}{4\pi} \int_M dS^g \wedge X^A,
\]

for every \( g \in \mathfrak{M} \text{et} M, h, k \in S^2(M) \cong T_g \mathfrak{M} \text{et} M, X \in \mathfrak{X}(M) \) and \( t \in \mathbb{R} \).

**Remark 5.3.** In the previous proposition \( S^g \) denotes the scalar curvature of the metric \( g \), i.e., twice the Gauss curvature.

**Proof.** The curvature tensor of a Riemannian metric \( g \) on a surface is given by (e.g., see [13]),

\[
\Omega^g = S^g \left( g^{-1} \text{vol}_g \right) \otimes \text{vol}_g \in \Omega^2(M, \text{EndTM}),
\]

and we have

\[
\text{tr} \left( (g^{-1} h \circ g^{-1} k) \circ \Omega^g \right) = S^g \text{tr} \left( (g^{-1} h \circ g^{-1} k) \circ g^{-1} \text{vol}_g \right) \text{vol}_g = S^g \text{tr} \left( g^{-1} h \circ g^{-1} k \circ g^{-1} \text{vol}_g \right) \text{vol}_g,
\]

where the last equality is due to the fact that the trace of the product of a symmetric and a skew-symmetric endomorphism vanishes. This gives the first term in the expression of \( \sigma \). For the second term we have

\[
\text{tr} \left( (\hat{\nabla}^g h)_A \wedge (\hat{\nabla}^g k)_A \right) = -2\text{Pfaff}((\hat{\nabla}^g h)_A) \wedge \text{Pfaff}((\hat{\nabla}^g k)_A) = -\frac{1}{2} *_g (\delta^g h - d(\text{tr}_g h)) \wedge *_g (\delta^g k - d(\text{tr}_g k)) = -\frac{1}{2} (\delta^g h - d(\text{tr}_g h)) \wedge (\delta^g k - d(\text{tr}_g k)),
\]

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where the last equality is due to the fact that, in dimension 2, the following formula holds: $\star g \alpha \wedge \star g \beta = \alpha \wedge \beta$ for $\alpha, \beta \in \Omega^1(M)$.

For the expression of the moment map, we have

$$\text{tr} \left( \left( \nabla^g X \right) \circ \Omega^g \right) = \frac{1}{2} S^g \text{tr} \left( g^{-1} dX^g \circ g^{-1} \text{vol}_g \right) \text{vol}_g$$

$$= - S^g \operatorname{Pfaff}(g^{-1} dX^g) \operatorname{Pfaff}(g^{-1} \text{vol}_g) \text{vol}_g$$

$$= - S^g dX^g,$$

and hence

$$\mu_g(X, t) = \frac{1}{4\pi^2} \int_M \text{tr} \left( \left( \nabla^g X \right) \circ \Omega^g \right) = - \frac{1}{4\pi^2} \int_M S^g dX^g = \frac{1}{4\pi^2} \int_M dS^g \wedge X^g.$$ 

\[\square\]

Remark 5.4. The form $\sigma$ is not a symplectic form, as it is degenerate. In fact, we have $\sigma_g(g, h) = 0$ for every $g, h \in S^2(M)$, as $\delta g = 0$, $\text{tr} g = 2$, and hence

$$\sigma_g(g, h) = \frac{1}{4\pi^2} \int_M S^g \text{tr} \left( g^{-1} g^{-1} h \circ g^{-1} \text{vol}_g \right) \text{vol}_g$$

$$= \frac{1}{4\pi^2} \int_M S^g \text{tr} \left( g^{-1} h \circ g^{-1} \text{vol}_g \right) \text{vol}_g = 0.$$

**Corollary 5.5.** Let $g \in \mathfrak{M}(M)$ be a Riemannian metric on an oriented compact connected surface. Then, $\mu_g(X, t) = 0$, $\forall X \in \mathfrak{X}(M)$, $\forall t \in \mathbb{R}$, if and only if the scalar curvature $S^g$ of $g$, is constant. Hence $\mu^{-1}(0) = \mathfrak{M}_{\text{const}} M$ is the space of metrics of constant curvature.

From now on, we assume the genus of $M$ is $\gamma > 1$. By the Gauss-Bonnet theorem, the space of metrics of constant scalar curvature $-1$ can be identified to $\mathfrak{M}_{-1} M \cong \mathfrak{M}_{\text{const}} M / \mathbb{R}$. Hence the Marsden-Weinstein quotient

$$\mu^{-1}(0) / (\text{Diff}^+ M \times \mathbb{R}) \cong \mathfrak{M}_{-1} M / \text{Diff}^+ M \cong \mathcal{M}_\gamma$$

is the moduli space of complex surfaces of genus $\gamma$.

As the moduli space presents singularities due to the fact that the action of $\text{Diff}^+ M$ on $\mathfrak{M}_{-1} M$ is not free, it is customary to replace $\text{Diff}^+ M$ by the connected component of the identity $\text{Diff}^0 M \subset \text{Diff} M$. The action of $\text{Diff}^0 M$ on $\mathfrak{M}_{-1} M$ is free, and the quotient space $\mathfrak{M}_{-1} M / \text{Diff}^0 M = \mathcal{T}(M)$ is the Teichmüller space of $M$.

The restriction of $\sigma$ to $\mathfrak{M}_{-1} M$ projects onto a canonical pre-symplectic form $\varpi$ on $\mathcal{T}(M)$. Below we show that $\varpi$ basically coincides with the Weil-Petersson symplectic form.

First let us recall some results about the Teichmüller space and the Weil-Petersson metric. We follow the exposition in [18].

The group $\text{Diff}^0 M$ acts properly and freely on the manifold $\mathfrak{M}_{-1} M$, and the quotient space $\mathcal{T}(M) = \mathfrak{M}_{-1} M / \text{Diff}^0 M$ is a differentiable manifold of
dimension \(6\gamma - 6\) called the Teichmüller space of \(M\). For every \(g \in \text{Met}_{-1}M\), we have the identification
\[
T_{[g]}T(M) \cong S^2(g)^{TT} = \{h \in S^2(M) : \operatorname{tr}_g h = 0, \delta^g h = 0\}.
\]

On the space \(\text{Met}M\) there exists a canonical Riemannian metric \(G\) (see [13, 18]) given by
\[
G_{g}(h, k) = \frac{1}{2\pi} \int_M \operatorname{tr} (g^{-1} h \circ g^{-1} k) \operatorname{vol}_g,
\]
for every \(g \in \text{Met}M\), and every \(h, k \in S^2(M) \cong T_g \text{Met}M\), which is invariant under the action of the diffeomorphisms group on \(\text{Met}M\). The metric \(G\) induces a Riemannian metric on \(T(M)\), which coincides with the Weil-Petersson metric.

Moreover, the manifold \(T(M)\) is endowed with a complex structure \(J\), given by
\[
J_{[g]}(h) = -\operatorname{vol}_g g^{-1} h.
\]
The metric \(G\) is compatible with the complex structure \(J\), and \(T(M)\) is a Kähler manifold. Hence \(T(M)\) is endowed with a canonical symplectic structure \(\sigma_{WP}\), called the Weil-Petersson symplectic form, given by
\[
(\sigma_{WP})_{[g]}(h, k) = G_g(Jh, k), \quad \forall g \in \text{Met}_{-1}, \forall h, k \in (S^2(M))^{TT}_{g}.
\]

**Theorem 5.6.** We have \(\sigma = \frac{1}{2\pi \sigma_{WP}}\). Hence, the symplectic reduction of \((\text{Met}M, \sigma)\) is \((T(M), \frac{1}{2\pi \sigma_{WP}})\).

**Proof.** By virtue of Theorem 5.2 for every \(g \in \text{Met}_{-1}M\), \(h, k \in S^2(g)^{TT}\) we have
\[
\sigma_g(h, k) = -\frac{1}{4\pi} \int_M \operatorname{tr} (g^{-1} h \circ g^{-1} k \circ g^{-1} \operatorname{vol}_g) \operatorname{vol}_g
\neq -\frac{1}{4\pi} \int_M \operatorname{tr} (g^{-1} \operatorname{vol}_g \circ g^{-1} h \circ g^{-1} k) \operatorname{vol}_g
\neq \frac{1}{2\pi} G_g(Jh, k)
\neq \frac{1}{2\pi} (\sigma_{WP})_{g}(h, k),
\]
and the result follows.

**Remark 5.7.** The preceding result provides an alternative proof of the fact that the Weil-Petersson metric on \(T(M)\) is Kähler, as we know that \(\sigma\) is closed by its very definition, and accordingly \(\sigma_{WP}\) is also closed.

**Remark 5.8.** In [6, 7, 12] the Teichmüller space with the Weil-Petersson symplectic form, is obtained by a symplectic reduction from the space of complex structures. A comparison of our constructions with this Donaldson-Fujiki's approach can be found in [11].

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