DEFINING NEW LINEAR FUNCTIONS IN TAME EXPANSIONS
OF THE REAL ORDERED ADDITIVE GROUP

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Abstract. We explore semibounded expansions of arbitrary ordered groups; namely, expansions that do not define a field on the whole universe. For \( \mathcal{R} = \langle R, <, +, \ldots \rangle \), a semibounded o-minimal structure and \( P \subseteq R \) a set satisfying certain tameness conditions, we discuss under which conditions \( \langle R, P \rangle \) defines total linear functions that are not definable in \( \mathcal{R} \). Examples of such structures that do define new total linear functions include the cases when \( R \) is a reduct of \( \langle R, <, +, \cdot \rangle \upharpoonright (0,1)^2 \), \((x \mapsto \lambda x)_{\lambda \in I \subseteq R}\), and \( P = 2^\mathbb{Z} \), or \( P \) is an iteration sequence (for any \( I \)) or \( P = \mathbb{Z} \), for \( I = \mathbb{Q} \).

1. Introduction

This work, as the one of [9], is at the intersection of two different directions in model theory. The semibounded o-minimal structures that were first studied in the 90’s by Marker, Peterzil, Pillay [19, 23, 25] and others, they relate to Zilber’s dichotomy principle on definable groups and fields, and have continued to develop in recent years (see [5, 7, 24]). The second direction is that of tame expansions \( \mathcal{R} \) of o-minimal structures \( \mathcal{R} \). That are expansions of \( \mathcal{R} \) which are not o-minimal, yet are still considered tame for some geometric/analytic, measure theoretic, model theoretic or descriptive set theoretic properties shared by the sets definable in these structures. This area is much richer, originating to A. Robinson [28] and van den Dries [2, 3], it has largely expanded in the last two decades by many authors, and includes broad categories of structures, such as d-minimal/ noiseless/ o-minimal open core expansions of o-minimal structures, or more generally expansions of the real line that does not define the whole projective hierarchy (see for example [20], [17], [12], [10]).

The work of this paper is the direct continuation of the one of [9] where we were studying a generalisation of semiboundedness to tame although non-o-minimal settings for expansions of the real vector space \( \langle \mathbb{R}, <, +, (x \mapsto \lambda x)_{\lambda \in \mathbb{R}} \rangle \). In this paper we generalize the result of [9] to expansions of the real additive ordered group. Moreover, we give some necessary conditions for an expansion of the form \( \langle \mathcal{R}, P \rangle \), for \( \mathcal{R} \) a semibounded expansion of the real additive ordered group and \( P \subseteq \mathbb{R} \) not to define any total linear functions that are not definable in \( \mathcal{R} \). As an application, we obtain that for any reduct of semibounded expansions \( \mathcal{R} \) of an o-minimal expansion of the real additive ordered group \( \mathcal{R} \), such as \( \mathcal{R} = \langle \mathcal{R}, 2^\mathbb{Z} \rangle \) with \( \mathcal{R} = \langle \mathcal{R}, <, +, \cdot \rangle \upharpoonright [0,1]^2 \), \((x \mapsto \lambda x)_{\lambda \in I \subseteq \mathbb{R}} \) or \( \mathcal{R}, \mathbb{Z} \) with \( \mathcal{R} = \langle \mathcal{R}, <, +, \cdot \rangle \upharpoonright [0,1]^2 \), every definable smooth (that is, infinitely differentiable) function are already definable in 

Date: October 26, 2021.

2010 Mathematics Subject Classification. Primary 03C64, Secondary 22B99.

Key words and phrases. o-minimality, tame expansions, d-minimality, noiseless, semibounded.
\( \mathcal{R} \). Especially, none of these structures defines total linear functions that are not definable in \( \mathcal{R} \).

We now collect some definitions and state our results. We assume familiarity with the basics of \( o \)-minimality, as they can be found, for example, in [2]. A standard reference for semibounded \( o \)-minimal structures is [5].

For the rest of this paper, and unless stated otherwise, we fix an \( o \)-minimal expansion \( \mathcal{R} = (\mathbb{R}, <, +, \ldots) \) of the real ordered group, and an expansion \( \mathcal{R} = (\mathbb{R}, \ldots) \) of \( \mathcal{R} \). By ‘definable’ we mean definable in \( \mathcal{R} \), possibly with parameters. By \( P \) we denote a subset of \( \mathbb{R} \) of dimension 0.

We denote by \( \langle \mathcal{R}, P \rangle \# \) the expansion of \( \langle \mathcal{R}, P \rangle \) by predicates for any subsets of \( P \) for every \( k \in \mathbb{N} \). If \( \mathcal{R} = \langle \mathcal{R}, P \rangle \), we denote by \( \mathcal{R}^\# = \langle \mathcal{R}, P \rangle^\# \). If \( \mathcal{R} \) is a real closed field, we call a set definable in \( \mathcal{R} \) semialgebraic. We denote by \( \mathcal{R}_{\text{vec}} = \langle \mathcal{R}, <, +, (x \mapsto \lambda x) \rangle \), \( \mathcal{R}_{\text{res}} = \langle \mathcal{R}, <, +, (x \mapsto \lambda x) \downarrow (0, 1) \rangle \) and \( \mathcal{R}_{\text{sb}} = \langle \mathcal{R}_{\text{vec}}, \mathcal{R}_{\text{res}} \rangle \). We say that an expansion of \( \mathcal{R}_{\text{gp}} \) is purely linear if it has the form \( \langle \mathcal{R}, <, +, (x \mapsto \lambda x) \downarrow (0, 1) \rangle_{\lambda \in I}, (x \mapsto \lambda x)_{\lambda \in J} \rangle \) for some \( J \subseteq I \subseteq \mathbb{R} \). Given \( \mathcal{M} \), an expansion of the real ordered additive group, we denote by \( \Lambda(\mathcal{M}) \) the field of total linear functions definable in \( \mathcal{M} \).

Following [9], we define a generalization of semiboundedness to non \( o \)-minimal settings.

**Definition 1.1.** Let \( \mathcal{M} = \langle M, <, +, \ldots \rangle \) be an expansion of an ordered group. We call \( \mathcal{M} \) **semibounded** if there is no definable field whose domain is the whole \( \mathcal{M} \).

**Remark 1.2.** In [9] we asked the question whether or not this definition was equivalent to defining no pole (that is a bijection between a bounded interval and \( \mathbb{R} \)). In the \( o \)-minimal setting the two properties are equivalent. However in our setting they are not and we exhibit a counterexample in Section 6.

For \( \mathcal{M} \) a semibounded structure, we denote by \( \Lambda(\mathcal{M}) \) the field of total linear functions definable in \( \mathcal{M} \) (possibly with parameters). For any set \( X \subseteq \mathbb{R}^n \), we define its **dimension** as the maximum \( k \) such that some projection of \( X \) to \( k \) coordinates has non-empty interior.

In [9] we introduced certain tameness properties for expansions of the real additive ordered group ((DP) and (DIM) below).

**Definition 1.3.** Let \( Y \subseteq X \subseteq \mathbb{R}^n \) be two sets. We say that \( Y \) is a **chunk** of \( X \) if it is a cell definable in \( \mathcal{R} \), \( \dim Y = \dim X \), and for every \( y \in Y \), there is an open box \( B \subseteq \mathbb{R}^n \) containing \( y \) such that \( B \cap X \subseteq Y \). Equivalently, \( Y \) is a relatively open cell in \( X \) with \( \dim Y = \dim X \) and definable in \( \mathcal{R} \).

**Definition 1.4.** We say that \( \mathcal{R} \) has the **decomposition property** (DP) if for every definable set \( X \subseteq \mathbb{R}^n \),

(I) there is a family \( \{ Y_t \}_{t \in R^m} \) of subsets of \( R^m \) that is definable in \( \mathcal{R} \), and a definable set \( S \subseteq R^m \) with \( \dim S = 0 \), such that \( X = \bigcup_{t \in S} Y_t \),

(II) \( X \) contains a chunk.
Definition 1.5. We say that $\mathcal{R}$ has the dimension property (DIM) if for every family $\{X_t\}_{t \in A}$ definable in $\mathcal{R}$, and definable set $S \subseteq A$ so that $\dim S = 0$, we have
\[
\dim \bigcup_{t \in S} X_t = \max_{t \in S} \dim X_t.
\]

Definition 1.6. We say that $\mathcal{R}$ is noiseless if, given a definable set $X$, then it has interior or is nowhere dense. Equivalently if for $X \subseteq \mathbb{R}$, $X$ has interior or is nowhere dense (see [20]).

Remark 1.7. Note that Fornasiero calls noiseless structures i-minimal. It is also the denomination we have used in [9].

As we saw in [9, Lemma 4.4, remark 4.5], (DIM) is equivalent to noislessness. Moreover, in our setting noiselessness implies (DPI).

Fact 1.8. (see [29] main theorem) Let $\mathcal{R} = \langle \mathcal{R}, P \rangle$ being noiseless. Then $\mathcal{R}^\#$ has (DPI). Moreover, given a definable set $X$, we can find a family of cells $\{X_t : t \in A\}$ definable in $\mathcal{R}$ and $S \subseteq A$, a set of dimension 0 so that $X = \bigcup_{t \in S} X_t$ and for every projection $\pi$, for every $t, t' \in S$ either $\pi(X_t) \cap \pi(X_{t'}) = \emptyset$ or they are equal.

Moreover, in our setting, we may further simplify the assumptions.

Definition 1.9. We say that $\mathcal{R}$ has the interior or isolated point property if given any definable set $X \subseteq \mathbb{R}$, then $X$ has interior or has an isolated point.

Proposition 1.10. Let $\mathcal{R} = \langle \mathcal{R}, P \rangle$ has the interior or isolated point property. Then $\mathcal{R}^\#$ is noiseless and has (DP).

Proof. Noiselessness is straightforward since a dense set without interior has no isolated point. (DPI) follows from Fact 1.8.

For (DPII), let $X \subseteq \mathbb{R}^n$ be a set of dimension $m$. We do an induction on $n$. For $n = 0$, the result is trivial. We assume that $n > 1$ and that for any $m' \leq n' < n$, (DPII) holds. The case $m = n$ is straightforward since $X$ would have interior and that an open box is a chunk of $X$. We assume $m < n$. By a simple induction, we may further assume that $m = n - 1$. Let $\pi$ be the projection on the first $n - 1$-coordinates and without loss of generality, we may assume that $\dim(\pi(X)) = n$. By Fact 1.8, there is a family of cells $\{X_t : t \in A\}$ and $S \subseteq A$ a small set that satisfy the conditions of Fact 1.8. Let $t \in S$ so that $\dim(\pi(X_t)) = n$ and let $Z = \pi(X_t)$. By assumption on $\{X_t : t \in S\}$, $\pi^{-1}(Z) \cap X$ is composed of disjoint graphs of functions with domain $Z$. I.e $\pi^{-1}(Z) \cap X = \bigcup_{t \in S'} X_t$ for some $S' \subseteq S$, for every $t \in S', X_t$ is the graph of a continuous function with domain $Z$ and for every $t, t' \in S'$, $X_t \cap X_{t'} = \emptyset$. Since for every $x \in Z$ there is at least one isolated point in $\pi^{-1}(x) \cap X$, the set $Y = \{x \in \bigcup_{t \in S'} X_t : x$ is isolated in $\pi^{-1}(\pi(x)) \cap X\}$ has dimension $n$. By Fact 1.8, we may find a compact set $W \subseteq Y$, definable in $\mathcal{R}$ and so that $\dim(W) = m$.

We just have to show that $W \cap \pi^{-1}(\text{int}(\pi(W)))$ is a chunk of $X$. Since $W$ is compact, $\{(x, \pi^{-1}(\pi(x)) \cap X) : x \in W\}$ has an infimum that is not 0. This gives us the result.

Remark 1.11. Note that (DPII) implies the interior or isolated point property. Therefore, in the case where $\mathcal{R}$ expands $\mathbb{R}_{gp}$, the conditions of [8], [9] are equivalent to (DPII) alone.
Therefore we may rephrase the conditions of [9] into \(\tilde{\mathcal{R}}\) has the interior or isolated point property.

In Section 3 we generalize [8, Theoreme 1.5] to our setting.

**Theorem 1.12.** Let \(\tilde{\mathcal{R}} \subseteq \langle \mathcal{R}, P \rangle^\#\) having the interior or isolated point property. Then every definable \(\mathcal{C}^\infty\)-function with a domain definable in \(\mathcal{R}\) is definable in \(\langle \mathcal{R}, \Lambda(\tilde{\mathcal{R}}) \rangle\).

Moreover, if \(\mathcal{R}\) is purely linear, more can be said:

**Theorem 1.13.** Let \(\mathcal{R} = \langle \mathbb{R}, <, +, (x \mapsto \lambda x)_{\lambda \in (0,1)} \rangle\) be purely linear. Let \(\tilde{\mathcal{R}} \subseteq \langle \mathcal{R}, P \rangle^\#\) having the interior or isolated point property. Then every definable \(\mathcal{C}^1\)-function with a domain definable in \(\mathcal{R}\) is definable in \(\langle \mathcal{R}, \Lambda(\tilde{\mathcal{R}}) \rangle\). Moreover, \(\Lambda(\tilde{\mathcal{R}}) \subseteq I\).

With these results, we may further study some examples.

**Definition 1.14** ([21]). Let \(f : \mathbb{R} \to \mathbb{R}\) be a bijection definable in \(\mathcal{R}\), and \(f^n\) the \(n\)-th compositional iterate of \(f\). We say that \(\mathcal{R}\) is \(f\)-bounded if for every function definable in \(\mathcal{R}\) \(g : \mathbb{R} \to \mathbb{R}\), there is \(n \in \mathbb{N}\) such that ultimately \(f^n > g\).

Let \(c \in \mathbb{R}\) and \(f\) a function definable in \(\mathcal{R}\) such that \(\mathcal{R}\) is \(f\)-bounded, and such that \((f^n(c))_n\) is growing and unbounded. We call \((f^n(c))_n\) an iteration sequence.

Before studying examples, we need to define two important class of structures.

**Definition 1.15.** We say that \(\mathcal{R}\) is \(d\)-minimal if given any definable family \(\{X_t : t \in A\}\) of subsets of \(\mathbb{R}\) of dimension 0, there is \(N\) so that for every \(t \in A\), \(X_t\) is the union of at most \(N\) discrete sets. Note that if \(\mathcal{R}\) is \(d\)-minimal then it has the interior or isolated point property. Note also that, as shown in [12], if \(\mathcal{R}\) is \(d\)-minimal then \(\tilde{\mathcal{R}}^\#\) is \(d\)-minimal too.

We say that \(\mathcal{R}\) is locally \(o\)-minimal if any bounded definable set of dimension 0 is finite. Note that local \(o\)-minimality implies \(d\)-minimality since a set of dimension 0 needs to be discrete.

**Example 1.16.** We consider the following structures:

1. \(\langle \mathbb{R}, \mathbb{Z} \rangle^\#\) with \(\Lambda(\mathcal{R}) = \mathbb{Q}\),
2. \(\langle \mathcal{R}, P \rangle^\#\), where \(P\) is an iteration sequence for some \(o\)-minimal expansion \(\mathcal{R}'\) of \(\mathcal{R}\).
3. \(\langle \mathcal{R}, \alpha^x \rangle^\#\), for \(\alpha > 0\)
4. \(\langle \mathbb{R}_{vec}, P \rangle^\#\), where \(P > 0\) is the range of a decreasing sequence with limit 0.

**Remark 1.17.** An iteration sequence for \(\mathcal{R}\) has the form \((f^n(c))_n\) for \(f : \mathbb{R} \to \mathbb{R}\) a growing function definable in \(\mathcal{R}\) such that \(\mathcal{R}\) is \(f\)-bounded. Since \(\mathcal{R}\) is semibounded, it is ultimately affine and thus we may assume that \(f\) is of the form \(x \mapsto ax + b\) (for \(a > 1 \in \Lambda(\mathcal{R}), b \in \mathbb{R}\)). Therefore, for \(c \in \mathbb{R}\), \(f^n(c) = a^n c + \sum_{0 \leq i < n} a^i b\). Then

\[
f^n(c) = a^n c + \frac{a^n - 1}{a - 1} b = \frac{1}{a - 1} (a^n ((a - 1)c + b) - b).
\]

Thus

\[
\langle \mathcal{R}, (x \mapsto \beta x), (f^n(c))_n \rangle = \langle \mathcal{R}, (x \mapsto \beta x), (a^n)_n \rangle
\]

for \(\beta = (a - 1)c + b\).
Therefore, in Example 1.16 the special case of (2) where \( P \) is an iteration sequence for \( \mathcal{R} \) is actually a reduct of (3). The other cases of (2) includes \( P = \alpha^N \) where \( \alpha \notin \Lambda(\mathcal{R}) \) and any iteration sequences for \( \mathcal{R} \) or other non semibounded o-minimal structures.

In section 4 we prove the following proposition:

**Proposition 1.18.** The structures of Example 1.16 are d-minimal. moreover, (1)-(2) are locally o-minimal.

In Section 5 we prove the following Proposition:

**Proposition 1.19.** Let \( \tilde{\mathcal{R}} \) be (1)-(3) of Example 1.16. Then a definable smooth function over a semialgebraic domain is definable in \( \mathcal{R} \).

Moreover, we discuss some conditions that would allow \( \langle \mathcal{R}, P \rangle^\# \) to define linear functions that are not definable in \( \mathcal{R} \).

We restrict further the shape of \( \mathcal{R} \) and establish the following proposition:

**Proposition 1.20.** Let \( \mathcal{R} \) be purely linear. Let \( \tilde{\mathcal{R}} \) be any of the structures of Example 1.16. Then every definable \( C^1 \) functions with a domain definable in \( \mathcal{R} \) is definable in \( \mathcal{R} \).

In Section 6 we discuss some questions and give some examples of semibounded d-minimal structure that are not locally o-minimal and so that for every algebraic set \( X \subseteq \mathbb{R}^n \) not definable in \( \mathcal{R} \), \( \langle \tilde{\mathcal{R}}, X \rangle \) defines any bounded set in the projective hierarchy. We also discuss the difference between being semibounded in the sens of Definition 1.1 and defining no poles. We finish with some natural questions that arise from this work.

**Structure of the paper.** In Section 2 we fix some notations, and establish basic properties for semibounded structures. In Section 3 we prove Theorems 1.12 and 1.13. In Section 4 we prove Proposition 1.18. In Section 5 we prove Propositions 1.19, 1.20. In Section 6 we discuss some questions.

**Acknowledgment:** We would like to thank Philipp Hieronymi and Eric Walsberg for raising the questions that led to this work. We would also like to specially thank Pantelis Eleftheriou for asking questions, for his comments and his corrections.

## 2. Preliminaries

In this section, we fix some notation and prove basic facts about semibounded structures.

If \( A, B \subseteq \mathbb{R} \), we denote \( \frac{A}{B} = \{a/b : a \in A, b \in B\} \). If \( t \in \mathbb{R} \), we write \( \frac{A}{t} \) for \( \frac{A}{\mathbb{Z}} \).

By a \( k \)-cell, we mean a cell of dimension \( k \). If \( S \subseteq \mathbb{R}^n \) is a set, its closure is denoted by \( \overline{S} \), with sole exception \( \mathbb{R} \), which denotes the real field. By an open box \( B \subseteq \mathbb{R}^n \), we mean a set of the form

\[
B = (a_1, b_1) \times \ldots \times (a_n, b_n),
\]

for some \( a_i < b_i \in \mathbb{R} \cup \{\pm \infty\} \). By an open set we always mean a non-empty open set.
When we write equality between two structures, we mean the two structures have the same collection of definable sets.

Let $\mathcal{M}$ be an expansion of $\mathbb{R}_{gp}$. By $P_{ind}(\mathcal{M})$, we mean the structure induced on $P$ in $\langle \mathcal{M}, P \rangle$ that is

$$\langle P, \{S \subseteq P^k : S \text{ is definable in } \langle \mathcal{M}, P \rangle \} \rangle.$$

**Definition 2.1.** We say that $X \subseteq \mathbb{R}^n$ is a cone if there are a bounded definable set $B \subseteq \mathbb{R}^n, v_1, \ldots, v_l \in \Lambda(\mathcal{R})^n$ some linearly independent vectors over $\mathcal{R}$ such that

$$X = B + \sum_{a \in (\mathbb{R}^{>0})^l} \sum_{i} a_i v_i := \{ x \in \mathbb{R}^n : \exists b \in B, \forall \lambda \in (\mathbb{R}^{>0})^l, x = b + \sum_{i} a_i v_i \}.$$  

Moreover, for every $x \in X$, there are unique $b \in B$ and $a \in (\mathbb{R}^{>0})^l$ with 

$$x = b + \sum_{i} a_i v_i.$$  

**Fact 2.2** (Structure Theorem: decomposition into cones [4 Fact 1.6 (5)]). Let $X \subseteq \mathbb{R}^n$ be a set definable in $\mathcal{R}$. Then $X$ can be decomposed into finitely many cones.

**Fact 2.3.** Let $f : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be a function definable in $\mathcal{R}$. Then there is an interval $B \subseteq \mathbb{R}$ and an affine function $\lambda : \mathbb{R}^n \to \mathbb{R}, x \mapsto \sum \lambda_i x_i + b$ where $\lambda_i \in \Lambda(\mathcal{R})$ such that for every $x \in X$, $f(x) \in \lambda(x) + B$.

**Proof.** Easy to see, using [5 Fact 1.6]. \qed

**Proposition 2.4.** Let $\mathcal{R}$ be a reduct of $\mathbb{R}_{sb}$. Then $\mathcal{R}$ has the form $\langle \mathcal{R}', (x \mapsto \lambda x)_{\lambda \in I} \rangle$, for some semibounded o-minimal expansion of $\mathbb{R}_{gp}$ so that $\Lambda(\mathcal{R}') = \mathbb{Q}$.

**Proof.** First, as a reduct of $\mathbb{R}_{sb}$, $\mathcal{R}$ is o-minimal and semibounded. Let $X$ be a set definable in $\mathcal{R}$. By a simple induction on the structure of the cell, we may assume that $X$ is the graph of a function $f$. By Fact 2.2 we may assume that there is $C$ a bounded set definable in $\mathcal{R}$ and $v_1, \ldots, v_n \in \mathbb{R}^{n+m}$ some linearly independent vectors so that 

$$\Gamma(f) = C + \sum_{a \in (\mathbb{R}^{>0})^n} \sum_{i} a_i v_i.$$  

It is clear that $f$ is definable in $\langle \mathcal{R}, (x \mapsto v_{i,j} x)_{i \leq n, j \leq n+m} \rangle$. What remains to show is that $\mathcal{R}$ defines $(x \mapsto v_{i,j} x)$ for every $i, j$. By permuting coordinates, we just have to show that $x \mapsto v_{1,1} x$ is definable in $\mathcal{R}$. Moreover, without loss of generality, we may assume that 

$$\Gamma(f) = \sum_{a \in (\mathbb{R}^{>0})^n} \sum_{i} a_i v_i.$$  

We proceed by induction on $n$. For $n = 1$, we may assume $f : \mathbb{R}^{>0} \to \mathbb{R}^m$, $x \mapsto (x v_{1,j})_j$ and 

$$v_{1,1} x = \pi_1(f(x)).$$  

We assume the result to holds for $n$ and prove it for $n+1$. Let $Y = \sum_{a \in (\mathbb{R}^{>0})^n} \sum a_i v_i$. Observe that $Y \subseteq fr(\Gamma(f)), Y$ is definable in $\mathcal{R}$ and dim$(Y) = n$. We apply the induction hypothesis to $Y$ to get the result. \qed

We finish this section by exposing a central result by Friedmann and Miller:
Definition 2.5 ([12]). We say that a set \( Q \subseteq \mathbb{R} \) is **sparse** if for every function definable in \( \mathcal{R} \): \( f : \mathbb{R}^k \to \mathbb{R} \), \( \dim f(Q^k) = 0 \).

**Remark 2.6.** If \( \langle \mathcal{R}, P \rangle \) is i-minimal, then \( P \) is sparse.

**Fact 2.7 ([12] Last claim in the proof of Theorem A]).** Assume \( P \subseteq \mathbb{R} \) is sparse. Let \( A \subseteq \mathbb{R}^{n+1} \) be definable in \( \tilde{\mathcal{R}}^\# \) such that for every \( x \in \mathbb{R}^n \), \( A_x \) has no interior. Then there is a function definable in \( \mathcal{R} \): \( f : \mathbb{R}^{m+n} \to \mathbb{R} \) such that for every \( x \in \mathbb{R}^n \),

\[
A_x \subseteq f(P^m \times \{x\}).
\]

3. **Proof of Theorem 1.12, 1.13**

The goal of this section is only to establish Theorems 1.12 and 1.13.

**Theorem 3.1.** We assume that \( \tilde{\mathcal{R}} \subseteq \langle \mathcal{R}, P \rangle^\# \) has the interior or isolated point property. Let \( f : X \subseteq \mathbb{R}^n \to \mathbb{R}^m \) be a \( \mathcal{C}^\infty \)-function definable in \( \mathcal{R} \) with a domain definable in \( \mathcal{R} \). Then \( f \) is definable in \( \langle \mathcal{R}, \Lambda(\tilde{\mathcal{R}}) \rangle \).

**Proof.** First, by [9][Theorem 1.7], \( f \) is definable in \( \mathcal{R}_{ab} \) (by the exact same proof, actually, since it is not necessary that \( \langle \mathcal{R}_{ab}, P \rangle \) has the interior or isolated point property. Take for example (1) of Example 1.16). By Proposition 2.4 \( \langle \mathcal{R}, f \rangle = \langle \mathcal{R}, (x \mapsto \lambda x)_{\lambda \in I} \rangle \) for some \( I \) and by definition of \( \Lambda(\mathcal{R}) \) and since \( f \) is definable in \( \mathcal{R} \), \( f \) is definable in \( \langle \mathcal{R}, \Lambda(\tilde{\mathcal{R}}) \rangle \). \( \square \)

**Proposition 3.2.** Let \( \mathcal{R} \) be a linear reduct of \( \mathcal{R}_{ab} \) of the form \( \langle \mathcal{R}, <, +, ((x \mapsto \lambda x)_{\lambda \in I}, (x \mapsto \lambda x)_{\lambda \in J}) \rangle \). We assume that \( \langle \mathcal{R}, P \rangle \) has the interior or isolated point property and \( \tilde{\mathcal{R}} \subseteq \langle \mathcal{R}, P \rangle^\# \). Let \( f \) be a \( \mathcal{C}^1 \) function that has a domain that is definable in \( \mathcal{R} \) and that is definable in \( \tilde{\mathcal{R}} \). Then \( f \) is definable in \( \langle \mathcal{R}, \Lambda(\tilde{\mathcal{R}}) \rangle \). Moreover, \( \Lambda(\tilde{\mathcal{R}}) \subseteq I \).

Before proceeding to the proof, we first need a basic fact. We don’t prove it.

**Fact 3.3.** Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a function. Then \( f \) is linear if and only if for every \( x \in \mathbb{R}^{n-1}, \sigma \) a coordinate permutation, \( g : y \mapsto f(\sigma(x, y)) \) is linear.

**Proof.** (of Proposition 3.2) By Fact 3.3, we may assume that \( n = 1 \) and that \( \text{dom}(f) = \mathbb{R} \). Moreover, since \( \{X_t : t \in S\} \) is a family of linear functions, there are \( \{a_t, b_t \in \mathbb{R} : t \in S\} \) so that for every \( t \in S \) \( X_t \) is the restriction of \( x \mapsto a_t x + b_t \) to its domain. Let \( S' = \{t \in S : \dim(X_t) = 1\} \). By (DIM),

\[
\dim(\Gamma(f) \setminus \bigcup_{t \in S'} X_t) = 0
\]

and since \( f \) is continuous,

\[
\Gamma(f) = \bigcup_{t \in S'} X_t.
\]

Since \( f \) is \( \mathcal{C}^1 \),

\[
g : \bigcup_{t \in S'} \text{dom}(X_t) \to \mathbb{R}, \quad x \in \text{dom}(X_t) \mapsto a_t
\]

can be extended by continuity to \( \mathbb{R} \). Therefore, since \( \dim(\text{Im}(f')) = 0, \text{Im}(f') = \{\ast\} \) and for every \( t, t' \in S' \) \( a_{t'} = a_t \). Moreover by continuity of \( f, b_t = b_{t'} \) and \( f \) is then linear with coefficients in \( I \) and is definable in \( \langle \mathcal{R}, <, +, (x \mapsto \lambda x)_{\lambda \in I} \rangle \). \( \square \)
Remark 3.4. Note that, assuming definable completeness (see [22]. Equivalently, asking for the intermediate value property for definable continuous functions, or that any interval is definably connected), Theorems 1.12, 1.13 holds for non archimedean expansions of a semibounded expansion of an additive abelian ordered group by a set of dimension 0 so that the interior or isolated point property holds.

4. D-MINIMALITY

In this section, we establish d-minimality for the structures of Example 1.16.

4.1. local o-minimality of (1)-(2). In this subsection, we establish local o-minimality for \( \langle \mathbb{R}, \mathbb{Z} \rangle \) (if \( \Lambda(\mathbb{R}) = \mathbb{Q} \)) and \( \langle \mathbb{R}, P \rangle \) with \( P \) an iteration sequence for some expansion of \( \mathbb{R} \).

Fact 4.1. The structure \( \langle \mathbb{R}, <, +, \mathbb{Z} \rangle \) is locally o-minimal.

Proof. This is a consequence of the local o-minimality of \( \langle \mathbb{R}, <, +, (x \mapsto \sin(2\pi x)) \rangle \), proved in [30, Theorem 2.7], since \( \langle \mathbb{R}, <, +, \mathbb{Z} \rangle \) is a reduct of it. □

Lemma 4.2. Let \( P \subseteq \mathbb{R} \) with \( \dim P = 0 \) and assume that \( \langle \mathbb{R}, <, +, \Lambda(\mathbb{R}), P \rangle \) is locally o-minimal. Then \( \langle \mathbb{R}, P \rangle \) is locally o-minimal.

Proof. Let \( X \subseteq \mathbb{R} \) be a definable set of dimension 0. We show that \( X \) is discrete. Since \( P \) is countable, it is sparse and by Fact 2.7 there is a function definable in \( \mathbb{R} \) \( f : \mathbb{R}^k \rightarrow \mathbb{R} \) such that \( X \subseteq f(P^k) \).

Let \( B \) be a bounded interval. We just have to show that \( B \cap f(P^k) \) is finite. By Fact 2.3 there is \( g : \mathbb{R}^k \rightarrow \mathbb{R} \), a linear function (definable in \( \mathbb{R} \)) and a bounded interval \( C = (-a, a) \) so that for every \( x \in P^k, f(x) \in g(x) + B \). Since \( \langle \mathbb{R}, <, +, \Lambda(\mathbb{R}), P \rangle \) is locally o-minimal, \( g(P^k) \cap (B + C) \) is finite, \( f(P^k) \cap B \) is finite too and we have the result. □

Remark 4.3. Note that assuming that \( \langle \mathbb{R}, <, +, \Lambda(\mathbb{R}), P \rangle \) is locally o-minimal and not only \( \langle \mathbb{R}, <, +, P \rangle \) is necessary; for example, take \( \langle \mathbb{R}, <, +, (x \mapsto \sqrt{2} x) \rangle \), that is a semibounded o-minimal expansion of \( \langle \mathbb{R}, <, + \rangle \) but \( \langle \mathbb{R}, <, +, (x \mapsto \sqrt{2} x), \mathbb{Z} \rangle \) is not d-minimal as shown in [14].

We could ask if the same result holds for d-minimality instead of local o-minimality. It is not the case, see Section 6 for a counterexample.

In order to prove local o-minimality for Example 1.16(2), we first need some more general proposition about the structure induced on \( P \) by \( \bar{R} \).

Proposition 4.4. The structure induced on \( P \) by \( \langle \bar{R}, P \rangle \) is \( \langle P, < \rangle \).

Proof. First, note that if \( P \) is an iteration sequence for \( \bar{R}' \) some expansion of \( \bar{R} \), and that the structure induced on \( P \) by \( \bar{R}' \) is \( \langle P, < \rangle \) then the structure induced on \( P \) by \( \bar{R} \) is also \( \langle P, < \rangle \). Therefore we may assume that \( P \) is an iteration sequence for \( \bar{R} \).

In [11] Proof of Lemma 1.1], using the quantifier elimination result from [21], Fornasiero is actually showing that \( P_{ind} \) is o-minimal. By [20], \( P_{ind} \) is strongly o-minimal and by [27, Theorem 3.1], \( P_{ind}(\bar{R}) = \langle P, < \rangle \). □

In the following, \( s : P \rightarrow P \) denote the successor function.
Definition 4.5. Let $X \subseteq P$ be definable in $\langle P, \langle \rangle \rangle$. We say that $X$ is an $s$-cell if it is a point or an interval of the form $(a, +\infty)$ for some $a \in \mathfrak{g}$. Let $X \subseteq P^n$ and let $\pi$ be the projection on the first $n-1$ coordinates. We say that $X$ is an $s$-cell if, $\pi(X)$ is one and $X$ either has the form

$$\{(x, y) : x \in \pi(X), y = f(x)\}$$

or

$$\{(x, y) : x \in \pi(X), g(x) > y > f(x)\}$$

for some function $f, g : P^{n-1} \to P \cup \{\infty\}$ of the form $x \mapsto a$ for some $a \in P \cup \{\infty\}$ or $x \mapsto s^k(\pi_i(x))$ for some $i \in \{1, \ldots, n-1\}$ and $k \in \mathbb{Z}$ and where $\{\#\{P \cap (f(x), g(x))\} : x \in \pi(X)\}$ is unbounded.

Fact 4.6. Let $X$ be a set definable in $\langle P, \langle \rangle \rangle$. By a simple quantifier elimination result we get that there is decomposition of $X$ into finitely many s-cells.

Lemma 4.7. Let $X \subseteq P^n$ be an infinite s-cell. Then is $Y \subseteq X, K \subseteq \{1, \ldots, n\}, a_i \in P$ and $j_i \in \mathbb{Z}$ for $i \in \{1, \ldots, n\}, a \in P$ so that

$Y = \{(x_1, \ldots, x_n) : \text{there is } x > a \in P \text{ for } i \in K \text{ } x_i = s^{j_i}x, \text{ and for } i \notin K, x_i = a_i\}$.

Proof. We do an induction on $n$. For $n = 1$, since $X$ is infinite the result follows from the definition of an s-cell. We assume the result holds for $n$ and that $X \subseteq P^{n+1}$. For $\pi$ the projection on the first $n$-coordinates, if $\pi(X)$ is finite then it is a singleton $\{a\}$ and since $X$ is infinite, $X = \{(a, y) : y > b\}$ for some $b \in P$ and we have the result.

If $\pi(X)$ is infinite then we apply the induction hypothesis to $\pi(X)$ to get $X_1$ of the wanted form. It is easy to see that there are some functions $f, g$ definable in $\langle P, \langle \rangle \rangle$ of the form $x \mapsto s^k(\pi_i(x))$ or constant so that $\pi^{-1}(X_1) \cap X$ either has the form

$$\{(x, y) : \text{there is } z > a \in P \text{ for } i \in K \text{ } x_i = s^{j_i}(z), \text{ and for } i \notin K, x_i = a_i \text{ and } y = f(z)\}$$

that is of the right form, or has the form

$$\{(x, y) : \text{there is } z > a \in P \text{ for } i \in K \text{ } x_i = s^{j_i}(z), \text{ and for } i \notin K, x_i = a_i \text{ and } g(z) > y > f(z)\}.$$

Then

$$\{(x, y) : \text{there is } z > a \in P \text{ for } i \in K \text{ } x_i = s^{j_i}(z), \text{ and for } i \notin K, x_i = a_i \text{ and } y = s(f(z))\} \subseteq X$$

and has the right form.

Corollary 4.8. The structures (1)-(2) of Example 1.16 are locally o-minimal.

Proof. By Lemma 4.2 we just have to show that $\langle \mathcal{R}, P \rangle$ are locally o-minimal for $\mathcal{R}$ a pure linear structure. For (1), it is by Fact 2.6. For (2), towards a contradiction, we assume that there is a definable set $X \subseteq \mathbb{R}$ of dimension 0 so that 0 is an accumulation point of $X$. By Fact 2.7 there is a linear function $f : \mathbb{R}^k \to \mathbb{R}$ so that $X \subseteq \overline{f(P^k)}$.

Claim 4.9. There is a decreasing sequence $(x_n)_n \in X^\mathbb{N}$ so that $\lim_n x_n = 0$ and $\{x_n : n \in \mathbb{N}\}$ is definable.

Proof. By [21], $\langle \mathcal{R}', P \rangle$ is d-minimal and, as a reduct, $\langle \mathcal{R}, P \rangle$ is d-minimal too. Therefore, we can do an induction on the number $n$ of discrete sets needed to decompose $Y = \{x \in \overline{X} : x > 0\}$. Without loss of generality, we may assume that:

(*) the number of discrete sets needed to decompose $B \cap Y$ for any boxes $B \ni 0$ is $n$. 

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If \( n = 1 \), since \( \overline{X} \) is closed, then \( Y \) is the range of a sequence that goes to 0.

We assume that the result holds for \( n \) and prove it for \( n + 1 \). Let \( Y' = \{ x \in Y : x \) is not isolated in \( Y \} \). By a simple induction, one can show that \( Y' \) admits a decomposition into \( n \) discrete sets. By definition \( Y' \) is closed in its convex hull and by (*) \( 0 \in \overline{Y'} \). We apply the induction hypothesis to \( Y' \) to get the result. \( \square \)

Let \( Y = \{ x_n : n \in \mathbb{N} \} \) be the set produced by Claim 4.9. Note that \( d \)-minimal structures have definable choice. For \( n \in \mathbb{N} \) let

\[
y_n \text{ be a choice among } f(P^k) \cap [x_{n+1}, x_n) \text{ if it exists and } y_n = y_{n-1} \text{ otherwise}.\]

By replacing \( Y \) by \( Y' = \{ y_n : n \in \mathbb{N} \} \), we may assume that \( Y' \subseteq f(P^k) \). Let \( \rho : Y \to P^k \) be a choice function so that \( \rho(y) \in f^{-1}(y) \). Let \( Z = \rho(Y) \). Since \( Y \) is infinite, \( Z \subseteq P^k \) is too and by Fact 4.6 it contains an infinite s-cell. By Lemma 4.4 there is \( Z' \subseteq Z \) so that \( Z' \) has the form

\[
\{(x_1, \ldots, x_n) : x > a \in P \text{ for } i \in I \} x_i = s^j(x), \text{ and for } i \notin I, x_i = a_i \}.
\]

Thus \( f(Z') = \mu(P > a) \) for some linear function \( \mu \) and \( f(Z') \) is closed and discrete. Moreover, since \( Z' \) is infinite, that \( f(Z') \subseteq f(Z) = Y \) and that \( f|_{Z'} \) is an embedding, \( f(Z') \) is the range of a decreasing sequence with limit 0. That is a contradiction. \( \square \)

4.2. \( d \)-minimality of any expansion of a linear structure by any sequence with limit 0.

**Theorem 4.10.** We assume that \( P \subseteq \mathbb{R}^>0 \) is discrete and closed in its convex hull (thus, it has at most two accumulation points, and we assume one to be 0 and \( P > 0 \)). Moreover, we assume that \( \langle \mathcal{R}, P^{>1} \rangle \) is locally o-minimal. Then \( \mathcal{R}^\# \) is \( d \)-minimal.

**Proof.** By [12], it is sufficient to show that \( \mathcal{R} \) is \( d \)-minimal. Since \( P \) is countable, it is sparse and by Fact 2.27 we just need to show that, for \( f : \mathbb{R}^{n+m} \to \mathbb{R} \) definable in \( \mathcal{R} \), there are \( g_i : \mathbb{R}^{n+i} \to \mathbb{R} \) some functions definable in \( \mathcal{R} \) so that for every \( a \in \mathbb{R}^n \), \( f(a, P^m) \) is the union of finitely many discrete sets. In the case where \( P \cap [0,1] \) is finite then \( \mathcal{R} \) is locally o-minimal and the result follows. Thus, we may assume that \( P \cap [0,1] \) is the range of a decreasing sequence \( (x_n) \) with limit 0.

Since \( f \) is definable in \( \mathcal{R} \), it has the form \( x \mapsto \sum \lambda_i x_i + b \). Since for \( a, a' \in \mathbb{R}^n \), \( f(a, P^k) \) and \( f(a', P^k) \) are translate of each others we may assume that \( n = b = 0 \).

We proceed by induction on \( m \). For \( m = 0 \), the result is trivial. We assume that the result holds for \( m < k \) and prove it for \( k \). We first consider the image under \( f \) of \( Z = (P^m) \cap [0,1]^m \). Let \( x \) be a non-isolated point of \( \overline{f(Z)} \) and \( ((x_i,s)_{i\leq m})_s \) be a sequence so that

\[
\lim_s f((x_i,s)_i) = x
\]

and that there is no subsequence that is ultimately constant in each coordinates. For \( A \subseteq \{1, \ldots, k\} \), let \( f_A : \mathbb{R}^{|A|} \to \mathbb{R} \), \( x \mapsto \sum_i a_{j_i} x_i \), where \( j_i \) is the \( i \)-th element of \( A \). By induction,

\[
\bigcup_{A \subseteq \{1, \ldots, k\}} f_A(P^{|A|}) \text{ is the union of finitely many discrete sets.}
\]

We show that \( x \in \bigcup_{A \subseteq \{1, \ldots, k\}} \overline{f_A(P^{|A|})} \).
By taking subsequences, we may assume that for every $i$, $(x_{i,s})$, is monotone (constant or decreasing). By assumption they can not all be ultimately constant. Thus, there is $i$ so that $(x_{i,s})$ is strictly decreasing: $(\lambda_i x_{i,s})$ goes to zero and $x \in \bigcup_{A \subseteq \{1, \ldots, k\}} f_A(P[A])$. Therefore,

$$f(Z) \setminus \bigcup_{A \subseteq \{1, \ldots, k\}} f_A(P[A])$$

and we have d-minimality by induction.

Now, we consider the case $f((P^{>1})^l \times (P^{<1})^n)$ for some $l \neq 0$. Let $K = \{1, \ldots, l\}$, $K' = \{l + 1, \ldots, k\}$. Since $f$ is linear, there is a box $B$ of finite radius so that

$$f_{K'}((P^{<1})^n) \subseteq B.$$ 

Moreover, by local o-minimality of $\langle R, P^{>1}\rangle$, for every $x \in f((P^{>1})^l \times (P^{<1})^n)$, there are only finitely many $y \in (P^{>1})^l$ so that $x \in f_K(y) + B$ and thus

$$x \in f_{K'}((P^{<1})^n) + f_K(y)$$ 

for some $y \in (P^{>1})^l$. Therefore

$$f((P^{>1})^l \times (P^{<1})^n)) = f_{K'}((P^{<1})^n) + f_K((P^{>1})^l)$$

and we have the result by applying the induction hypothesis to $f_{K'}((P^{<1})^l)$. \hfill \square

5. Proof of Proposition 1.19

5.1. Defining new linear functions. In this subsection, we expose some preliminary results and questions related to the question:

**Question 5.1.** Does $\overline{R}$ defines a total linear function that is not definable in $R$?

For the rest of this section, we assume that there is $\lambda \in R \setminus A(R)$.

Let us start a discussion about some restrictions that $\overline{R}$ should satisfy in order to define $x \mapsto \lambda x$. First, a general lemma.

**Lemma 5.2.** Let $\{g_t : t \in A\}$ be a family of functions definable in $R$ such that for every $t \in A$,

$$g_t = (x \mapsto \lambda x) \upharpoonright (0, x_t).$$

Then there is a uniform bound over the $x_t$’s.

**Proof.** If there was not, $x \mapsto \lambda x$ would be definable in $R$. \hfill \square

**Proposition 5.3.** We assume that $f : x \mapsto \lambda x$ is definable in $\overline{R}$. Let $\{f_t : (a_t, b_t) \rightarrow R : t \in S\} \subseteq \{f_t : t \in A\}$ be the families of functions given by $DP(I)$ applied to $f$ (the second one being definable in $R$). Then there is a uniform bound over $|\text{dom}(f_t)|$ for $t \in S$.

**Proof.** First of all, we may assume that for every $t \in A$, $f_t$ is linear and that $f_t$ is some restriction of $x \mapsto \lambda x$. We define $g_t : (0, b_t - a_t)$ by

$$g_t : x \mapsto f_t(x + a_t) - f_t(a_t).$$

We have the result by applying Lemma 5.2. \hfill \square

The following proposition shows that more can be said in the d-minimal setting. But first, we need a lemma.
Lemma 5.4. Let \( P \subseteq \mathbb{R} \) be a set of dimension 0 so that \( \langle \mathbb{R}, P \rangle \) is \( d \)-minimal and let \( A \subseteq \mathbb{R} \) be a definable unbounded set of dimension 0. Then the set of the size of complementary intervals of \( A \) is unbounded.

Proof. Without loss of generality, we may assume that \( A \) is closed. We do an induction on \( k \), the number of discrete sets needed to decompose \( A \). If \( k = 1 \), then \( A \) is closed and discrete and by [15], \( \hat{R} \) defines the whole projective hierarchy. That is a contradiction with \( d \)-minimality.

We assume that the result holds for \( m < k \). For \( Y \) a set of dimension 0, we denote by \( X_Y \) the set of complementary intervals of \( Y \) and we denote by \( |X_Y| \) the set of size of the complementary intervals of \( Y \). Let \( B = \{ x \in A : x \) is isolated in \( A \} \) and \( C = A \setminus B \). Note that \( C \) is closed and \( B \) is discrete. Note also that, using a simple induction, one can easily show that \( C \) admits a decomposition into \( k - 1 \) discrete sets and that \( C \subseteq \mathbb{B} \). Let
\[
D = C \cup (B \setminus \bigcup_{x \in C} [x - 1, x + 1])
\]
It is easy to see that, by construction,
\[
|X_D| + 1 \leq |X_A|
\]
and thus, if \( |X_D| \) is unbounded then \( |X_A| \) is unbounded too. Moreover, \( C \) does admit a decomposition into \( k - 1 \) discrete sets \( C_1, \ldots, C_l \). By definition, for any \( i \)
\[
C_i \subseteq \mathbb{B}
\]
and thus, \( C_i \cup (B \setminus \bigcup_{x \in C} [x - 1, x + 1]) \) is discrete. Thus \( D \) does admit a decomposition into \( k - 1 \) discrete sets and, by applying the induction hypothesis to \( D \), we have that \( |X_D| \) is unbounded. As said previously, it implies that \( |X_A| \) is unbounded and we have the result. \( \square \)

Proposition 5.5. Let \( P' \subseteq \mathbb{R} \) be a set of dimension 0. We assume that \( \langle \mathbb{R}, P' \rangle \) is \( d \)-minimal. Then \( x \mapsto \lambda x \) is not definable in \( \langle \mathbb{R}, P' \rangle^\# \).

Proof. By Fact [2.7] there is an \( \mathbb{R} \)-definable function \( f : \mathbb{R}^{n+1} \to \mathbb{R} \) so that for every \( x \in \mathbb{R} \)
\[
\lambda x \in f(x, P^m).
\]
By Fact [2.3] there is an \( \mathbb{R} \)-definable affine function \( g : \mathbb{R}^{n+1} \to \mathbb{R} \) and a box \( B \) (of radius \( \delta \)) so that \( f(x) \in g(x) + B \). We may assume that \( g \) is affine of the form \( y \mapsto \sum_i \mu_i y_i + c \) for some \( \mu_i \in I \) and \( c \in \mathbb{R} \). We first observe that for every \( x \in \mathbb{R} \)
\[
\lambda x \in g(x, P^m) + B.
\]
Let \( h : \mathbb{R}^n \to \mathbb{R} \), \( y \mapsto \sum_{i>1} \mu_i y_i + c \). By Lemma [5.4] since \( h(P^m) \) is definable in \( \langle \mathbb{R}, P' \rangle \), there is a complementary interval \( E \) of \( h(P^m) \) so that
\[
|E| > 2\delta.
\]
Let \( y \) be the middle point of \( E \). Let \( x = y/\lambda \). Since \( \lambda \notin I, \lambda \neq \mu_1 \) and let \( z = \frac{\lambda x + \delta}{\lambda - \mu_1} \). We have that
\[
\lambda z \in f(z, P^m) + B = \mu_1 z + h(P^m) + B
\]
and
\[
(\lambda - \mu_1)z = y + \delta \in h(P^m) + B.
\]
This is a contradiction with the choice of \( y \). \( \square \)

We can then give some sufficient conditions for \( \hat{R}^\# \) to define new linear functions.
Proposition 5.6. Let $P' \subseteq \mathbb{R}$ be an unbounded set of dimension 0 so that \langle R, (x \mapsto \lambda x), P' \rangle has the interior or isolated point property and such that \{d(x, P' \setminus \{x\}) : x \in P'\} is bounded. Then $x \mapsto \lambda x$ is definable in \langle R, (x \mapsto \lambda x)_{(0,1)}, P' \cup \lambda P' \rangle^\#.

Proof. First of all, we may assume that $P'$ is closed. Let \{(a_i, b_i)\} be the (bounded) family of intervals of $(0, \infty) \setminus P'$. We first observe that the function $g : x \mapsto \lambda x$ on $P'$ is definable in \langle R, P' \cup \lambda P' \rangle^\#. We define $x \mapsto \lambda x$ on $(0, \infty)$ by

$$x \in (a_i, b_i) \mapsto g(a_i) + \lambda(x - a_i).$$

\[\square\]

We can then relate Question 5.1 to:

Question 5.7. Is there a set $P \subseteq \mathbb{R}$ such that $P$ is closed unbounded, of dimension 0, \langle R, (x \mapsto \lambda x), P \rangle has the interior or isolated point property and \{d(x, P \setminus \{x\}) : x \in P\} is bounded?

5.2. Proof of Proposition 1.19.

Proof. By Theorem 1.13, a smooth function definable with an open domain definable in $R$ needs to be definable in \langle R, \Lambda(\mathcal{R}), P \rangle. For (1), since for $\lambda \in \mathbb{R} \setminus \mathbb{Q}$, $\langle R, \mathbb{Z}, \lambda \mathbb{Z} \rangle$ defines a dense-codense set (take for example $\mathbb{Z} - \lambda \mathbb{Z}$), $\Lambda(\mathcal{R}) = \mathbb{Q}$.

For (2), if $P$ is fast for some expansion of $\mathcal{R}$ then it is fast for $\mathcal{R}$ and we have the result. If $P$ is an iteration sequence for a semibounded expansion of $\mathcal{R}$, by Remark 1.17 we may assume it to be of the form $\alpha^n$ and \langle $\mathcal{R}, \alpha^M \rangle$ as well as \langle $\mathcal{R}, \alpha^2 \rangle$ are d-minimal by [2]. If $P$ is an iteration sequence for $\mathcal{R}'$ some non semibounded expansion of $\mathcal{R}$, it is sufficient to show the following claim.

Claim 5.8. $P$ is fast for $\mathcal{R}$.

Proof. First, we show that $\mathcal{R}'$ is not linearly bounded. Let $f$ be a non linear function definable in $\mathcal{R}'$. We assume that $f$ is strictly growing, positive and linearly bounded. Let $a$ be the infimum of $\{b \in \mathbb{R} : bx > f(x) \text{ eventually}\}$. If $b = 0$ then $f^{-1}$ is not linearly bounded. If not then let $g = f - ax$ and $g^{-1}$ is not linearly bounded.

Let $f$ be a non linearly bounded function definable in $\mathcal{R}'$. Since $P$ is an iteration sequence for $\mathcal{R}'$, $P$ is of the form $(g^n(a))$ for some $a \in \mathbb{R}$. Moreover, since there is $n \in \mathbb{N}$ so that $g^n > f$ eventually, $g$ is neither linearly bounded and for every $b \in \mathbb{R}$

$$bx/g(x) \rightarrow 0$$

and $P$ is fast for $\mathcal{R}$. \[\square\]

Thus \langle $\mathcal{R}, P$ \rangle is d-minimal by [13]. We apply Proposition 5.5 to get that $\Lambda(\mathcal{R}) = \Lambda(\mathcal{R})$ and we get the result. \[\square\]

5.3. Proof of Proposition 1.20.

Proposition 5.9. Let $\mathcal{R}$ be a linear reduct of $\mathcal{R}_{ab}$ and let $\overline{\mathcal{R}}$ be one of (1)-(4) of Example 1.16. Then every definable $C^1$-function with a domain definable in $\mathcal{R}$ is linear and definable in $\mathcal{R}$.

Proof. First all these structures are d-minimal by Proposition 1.18 and thus satisfy the interior or isolated point property. By Theorem 1.13 a $C^1$ function with a domain definable in $\mathcal{R}$ is linear. For (1)-(3), since these structures are reduct of \langle $\overline{\mathcal{R}}, (0,1)^2 \rangle$, we have the result by applying Proposition 1.19. For (4), since $P$ is
bounded, for every function $f : \mathbb{R}^k \to \mathbb{R}$ definable in $\mathcal{R}$, we have that $f(P^k)$ is bounded. Let us assume that $g : x \mapsto \lambda x$ is definable in $\mathcal{R}$. Let $\{X_t : t \in S\}$ be the small family of cells given by Fact 1.8 so that $\Gamma(g) = \bigcup_{t \in S} X_t$. Each $X_t$ is the graph of a linear function and let $Y$ be the set of left and right endpoints of $\pi(X_t)$ for every $t \in S$ (and for $\pi$ the projection on the first coordinates). Since $Y$ has dimension 0, it is bounded and thus one the $X_t$ is cofinal in $\mathbb{R}$. Therefore $g$ is definable in $\mathcal{R}$ and that is a contradiction. □

6. Discussion and Questions

In this section we first exhibit a counterexample that shows that Proposition 4.2 does not need to hold with the assumption of d-minimality in place of local o-minimality. It also provides some new examples of d-minimal structures that are not reducts of a d-minimal expansion of the real field. Second, we answer a question raised in [9] [Question 2.6] about some equivalence between usual notion of semiboundedness in our setting. We exhibit a counterexample that shows that Definition 1.1 is not equivalent to not defining a pole. We finish by raising some questions.

**Proposition 6.1.** The structure $\langle \mathbb{R}, <, +, (x \mapsto \lambda x)_{\lambda \in \mathbb{R}}, \{1/n : n \in \mathbb{N}\} \rangle$ is d-minimal and $\langle \mathbb{R}, <, +, \cdot_{|[0,1]^2}, 1/\mathbb{N} \rangle$ is not d-minimal. The latter even defines the trace on bounded sets of any set in the projective hierarchy.

For the first part we just apply Theorem 4.10. The second part follows directly from:

**Theorem 6.2.** Let $P \subseteq [0,1]$ be the range of a decreasing sequence with limit 0. Then for $\langle \mathbb{R}_{sb}, P \rangle$ to satisfy one of the following properties

(1) is d-minimal
(2) is noiseless
(3) has the interior or isolated point property
(4) satisfies that every definable set $X \subseteq \mathbb{R}$ has interior or is null (in the sense of Lebesque)
(5) defines any bounded set in the projective hierarchy

is equivalent for $\langle \mathbb{R}, P \rangle$ to satisfy the corresponding property among (1)-(5).

**Proof.** Since $\langle \mathbb{R}_{sb}, P \rangle$ is a reduct of $\langle \mathbb{R}, P \rangle$, we only have to prove the direct direction and we assume that $\langle \mathbb{R}_{sb}, P \rangle$ has one of (1)-(5).

First of all, we observe that $P$ is definable in $\mathcal{R}$-sparse since it is countable.

For (2)-(4), let $X \subseteq \mathbb{R}$ be definable in $\langle \mathbb{R}, P \rangle$ so that $X$ has dimension 0 and is (2) dense-codense somewhere, (3) has no isolated point, (4) is not null. Since these properties are local, we may assume that $X$ is bounded. Moreover, by sparseness, there is $f : \mathbb{R}^k \to \mathbb{R}$ so that $X \subseteq f(P^k)$. Since $P^k$ and $X$ are bounded, we may assume that $f$ is semibounded and this is a contradiction with assumptions (2)-(4).

For (1), using exactly the same argument we obtain that for every definable family $\{X_t : t \in A\}$ of sets of dimension 0 all contained in a bounded set, there is a uniform bound on the number of discrete sets needed to decompose each $X_t$. In the general case, let $\{X_t : t \in A\}$ be a definable family of sets of dimension 0. We just have to observe that for any bounded interval $0 \in B$, $B \cap (X_t - X_t)$ needs at least the same cardinal of discrete sets (possibly infinitely many) than $X_t$ to be decomposed.
This gives us the result by applying the first part to \( \{ (X_t - X_t) \cap B : t \in A \} \) for some bounded interval \( 0 \in B \).

For (5), by [10][Theorem 7], it is sufficient to prove that if \( (\mathbb{R}, P) \) defines a dense \( \omega \)-orderable set then so does \( (\mathbb{R}_{sb}, P) \). Let \( X \) be dense \( \omega \)-orderable definable in \( (\mathbb{R}, P) \). As previously, we may assume that \( X \) is bounded and there is a semi-bounded function \( f : [0, 1]^k \to \mathbb{R} \) so that \( X \subseteq f(P^k) \). Since \( X \) is dense \( \omega \)-orderable, so does \( f(P^k) \) and we get the result.  \( \square \)

In [9], we asked whether or not Definition 1.1 was equivalent to defining a pole. The answer is no and here is an example:

Proposition 6.3. The structure \( (\mathbb{R}, <, +, \{ f_t : x \in \mathbb{R} \mapsto tx : t \in 2^{\mathbb{Z}} \}) \) defines a pole but does not define a field on \( \mathbb{R} \).

Proof. First, observe that the function \( x \in 2^\mathbb{Z} \mapsto x^{-1} \) is definable by

\[ x \mapsto y \text{ so that } f_x(y) = 1. \]

Therefore the family of functions \( \{ g_t : x \in [t, 2t) \mapsto -(x - t)/2t + 1/2t : t \in 2^{-\mathbb{N}} \} \) is also definable and it is not hard to see that

\[ g : (0, 1) \to \mathbb{R}, x \in [t, 2t) \mapsto g_t(x) \text{ is a pole.} \]

Moreover, as shown by Delon in [11], \( (\mathbb{R}, <, +, \{ f_t \}, 2^{\mathbb{Z}}, (x \mapsto \lambda x)_{\lambda \in \mathbb{Q}}) \) is model complete and by a simple analysis of the formulas, we get that there is no definable non almost everywhere locally linear functions. Since \( (\mathbb{R}, <, +, \{ f_t \}) \) is a reduct of \( (\mathbb{R}, 2^{\mathbb{Z}}) \), that is d-minimal, and if a global field is definable then the graph of its multiplication should contain some chunk \( Y \) that is semialgebraic and not semilinear. That is a contradiction and we get the result.  \( \square \)

We finish with some natural questions that arise from the current work. Around Proposition 6.1, we may ask different questions:

Question 6.4. Is there a non-linear bounded function \( f : (0, 1) \to \mathbb{R} \) so that \( (\mathbb{R}, <, +, f) \) is \( \omega \)-minimal (and thus, semibounded) and that \( (\mathbb{R}, <, +, f, 1/\mathbb{N}) \) is d-minimal?

Actually, Walsberg answered a similar question in [31][Part 3] and we can derive an example for Question 6.4.

Example 6.5. We know that \( \widetilde{\mathbb{R}} = (\mathbb{R}_{sb}, 2^{-\mathbb{N}}) \) is d-minimal. Let \( (\mathbb{R}, 1/\mathbb{N}) \) be the pushforward of \( \widetilde{\mathbb{R}} \) by \( \log_2 \). This structure answers Question 6.4.

Question 6.6. We may ask exactly the same question as 6.4 but with \( \mathbb{R}_{\text{vec}} \) in place of \( (\mathbb{R}, <, +) \).

Question 6.7. Is it true that for every decreasing sequence \( (x_n) \to 0 \) there is a restricted analytic functions \( f \), or just an \( \omega \)-minimal function \( f \) so that \( (\mathbb{R}_{\text{vec}}, f, (x_n)) \) is d-minimal?

Remark 6.8. Of course, we could replace d-minimality by any of the properties described in [20]. That are noiseless, the interior or isolated point property, the interior or lebesgue-null property, defining the projective hierarchy . . .

Actually these questions are related to the more general question:
Question 6.9. Let $\mathcal{R}$ be any real closed field expanding $\langle \mathbb{R}, < \rangle$. Let $P \subseteq \mathbb{R}$ be a sequence set. At which conditions $\langle \mathcal{R}, P \rangle$ is tame?

We finish by asking a question around a possible generalization of Section 5.1 to non archimedian settings.

Question 6.10. Let $\mathcal{R} = \langle \mathbb{R}, <, +, \cdot \rangle$ be a non archimedian semibounded structure and $P \subseteq \mathbb{R}$ a set of dimension 0 so that $\mathcal{R}$ has (DP), is definably complete and $\mathcal{R}$ defines $(x \mapsto \lambda x)_{(0, \alpha)}$ but does not define $(x \mapsto \lambda x)_{(0, \beta)}$ for every $\beta \gg \alpha$. Is it possible to define $(x \mapsto \lambda x)_{(0, \beta)}$ for some $\beta \gg \alpha$ but not to define $(x \mapsto \lambda x)$?

Remark 6.11. Assuming that there is a $d$-minimal expansion $\mathcal{R} = \langle \mathcal{R}, P \rangle^\#$ of $\mathbb{R}_{sp}$ that defines $x \mapsto \lambda x$ (and such that $\lambda \notin \Lambda(\mathcal{R})$), we may easily define a structure that answers Question 6.10. By (DP) in $\mathcal{R}$, there is $\{X_t : t \in S\}$ a definable family of restrictions of $x \mapsto \lambda x$ to some domain. Let $Y$ be the set of endpoints of $\{\pi(X_t) : t \in S\}$ and $Z = \Gamma(x \mapsto \lambda x)_{\cdot, Y}$. Let $Z_\omega = \{(0, \alpha) \times (0, \lambda) : \alpha \in \mathbb{R}\}$. Let $\mathcal{M}$ be a non standard model of $\mathcal{R}^\#$ and let $\mathcal{R}' = \langle \mathcal{R}, <, +, \cdot \rangle$ be the non standard model of $\mathcal{R}$ contained in it. Let $Z_\omega^*$ be the interpretation of $Z_\omega$ in $\mathcal{M}$ and let $\alpha \gg 1$ be a short element of $R$. It is then easy to see that $\langle \mathcal{R}', Z_\omega^* \rangle$ has all the desired properties (as a reduct of $\mathcal{M}$), defines $(x \mapsto \lambda x)_{(0, \alpha)}$ but does not define the total $x \mapsto \lambda x$.

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