Periodic Solutions of Stochastic Differential Equations Driven by Lévy Noises

Xiao-Xia Guo\textsuperscript{1} · Wei Sun\textsuperscript{2}

Abstract
In this paper, we first show the well-posedness of SDEs driven by Lévy noises under mild conditions. Then, we consider the existence and uniqueness of periodic solutions of the SDEs. To establish the ergodicity and uniqueness of periodic solutions, we investigate the strong Feller property and the irreducibility of the corresponding time-inhomogeneous semigroups when both small and large jumps are allowed in the equations. Doob’s celebrated theorem on the uniqueness of invariant measures for time-homogeneous Markov processes has been generalized to obtain the uniqueness of periodic measures for time-inhomogeneous Markov processes. Some examples are presented to illustrate our results.

Keywords
Stochastic differential equation · Lévy noise · Periodic solution · Uniqueness · Strong Feller property · Irreducibility · Regular semigroup

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1 Introduction
Stochastic differential equations (SDEs) driven by Brownian motions have been studied extensively after the seminal work of Itô (1951, 1987). Since many practical dynamical systems may suffer from abrupt changes, e.g. natural disasters, epidemic
diseases, financial crises, SDEs with jumps fit better the reality. In recent years, there has been more and more interest in SDEs driven by Lévy noises. Such SDEs are finding a wide range of applications including but not restricted to nonlinear filtering, stochastic control, population genetics, option pricing and signal processing (cf. Situ 2006; Øksendal and Sulem 2004; Bao et al. 2011; Bao and Yuan 2012; Cont and Tankov 2004; Patel and Kosko 2008).

The concept of stationary or periodic solutions plays a fundamental role in studying the long-time behaviour of random dynamical systems modelled by SDEs. In particular, the uniqueness of periodic solutions is vital for ergodicity of non-autonomous random dynamical systems. Since the pioneering work of Poincaré (1881, 1882, 1885, 1886), periodic solutions have been studied for more than a century. Khasminskii (2012) systematically studied periodically varying properties of SDEs. But compared with the well-developed theory for the existence of periodic solutions, the theory of the uniqueness of periodic solutions is far from complete. The main purpose of this paper is to investigate the existence and uniqueness of periodic solutions for SDEs with drift, diffusion, small and large jumps.

In the past decade, many works have been devoted to study periodic solutions of SDEs. Here, we list some of them which are closely related to our paper. Xu et al. (2009, 2014, 2015) and Li and Xu (2013) discussed the existence and uniqueness of periodic solutions for non-autonomous SDEs with finite or infinite delay. Chen et al. (2017) obtained the existence of periodic solutions to Fokker–Planck equations through considering the $L^2$-bounded periodic solutions in distribution for the corresponding SDEs. Hu and Xu (2018) presented the existence and uniqueness theorems for periodic Markov processes on Polish spaces. Zhang et al. (2015) investigated the existence and uniqueness of periodic solutions of SDEs driven by Lévy processes. There are also many papers discussing periodic solutions of stochastic biomathematical models. For example, Hu and Li (2015) obtained the existence and uniqueness of periodic solutions of stochastic logistic equations; Zhang et al. (2017) showed that a stochastic non-autonomous Lotka–Volterra predator–prey model with impulsive effects has a unique periodic solution, which is globally attractive. It is worth pointing out that the above papers except for Zhang et al. (2015) only focused on SDEs driven by Brownian motions.

To show the uniqueness of periodic solutions, a typical method is to study the global attractiveness of solutions. The drawback of this method is that usually very strong conditions are required in order to establish the asymptotic stability. In our paper, completely different from the existing methods in the literature, we will investigate the ergodicity for non-autonomous random dynamical systems through considering the strong Feller property and the irreducibility of the time-inhomogeneous semigroups corresponding to the SDEs. It is well known that the uniqueness of invariant measures for autonomous random dynamical systems is implied by the strong Feller property and the irreducibility. We recall the reader’s attention to Dong (2008, 2018), Dong and Xie (2011), Xie (2012), Xie and Zhang (2020) for some recent works on the uniqueness of invariant measures for SDEs. However, to the best of our knowledge, no similar result has been given in the literature for the non-autonomous case. A main reason is that it is difficult to extend the Koopman–von Neumann mixing theorem (cf. Da Prato and Zabczyk 1996, Section 1.2) to the time-inhomogeneous case.
We will generalize Doob’s celebrated theorem on the uniqueness of invariant measures for time-homogeneous Markov processes (cf. Doob 1948 and Da Prato and Zabczyk 1996, Theorem 4.2.1) so as to obtain the ergodicity and uniqueness of periodic solutions for non-autonomous SDEs driven by Lévy noises, see Theorems 3.13 and 3.14. Note that Theorem 3.14 is even new for non-autonomous SDEs driven by Brownian motions and the conditions of Theorem 3.14 are novel and much weaker than those given in the literature. Regarding this point, we may compare the condition (H) of our Theorem 3.14 with conditions (i) and (ii) of Chen et al. (2017, Theorem 1.3) and conditions (16) and (17) of Zhang et al. (2015, Theorem 2). The power of Theorem 3.14 will be illustrated by the examples given in Sect. 4. In a continuation paper (Guo and Sun 2020), we have applied Theorem 3.14 to investigate periodic solutions of hybrid jump diffusion processes. Here, we would like to point out that Theorem 3.13 has independent interest and can be extended so as to study the uniqueness of periodic solutions for stochastic partial differential equations with jumps. We will consider this problem in a forthcoming paper.

The rest of this paper is organized as follows. In Sect. 2, we describe the framework of the whole paper and discuss the well-posedness of global solutions to SDEs driven by Lévy noises. Note that all the four parts of our SDEs, i.e. the drift, diffusion, small and large jumps, are modelled by general nonlinear functions. The intrinsic nonlinearity can also be explicitly seen from the examples given in Sect. 4. In Sect. 3, we consider the existence and uniqueness of periodic solutions. We will follow the method of Khasminskii (2012, Theorem 3.8) to give sufficient conditions that ensure the existence of periodic solutions. The main part of Sect. 3 is devoted to the uniqueness problem. We will use the Bismut–Elworthy–Li formula to show the strong Feller property of the time-inhomogeneous semigroups corresponding to the SDEs. For the irreducibility of semigroups, we use the method of Girsanov’s transformation, which is discussed in Da Prato and Zabczyk (1996, Theorem 7.3.1). To overcome the essential difficulty caused by the small jump part of Lévy noises, we adopt the remarkable method of Ren et al. (2010), Xie and Zhang (2020). The condition (H3) posed in Theorem 3.9 to ensure the irreducibility is novel and has its own interest. After establishing the strong Feller property and the irreducibility, we prove the key Theorem 3.13 and then use it to prove Theorem 3.14, which is the main result of this paper. Finally, in Sect. 4, we use examples to illustrate Theorem 3.14. In particular, we will show that the stochastic Lorenz equation and the stochastic equation of the lemniscate of Bernoulli have unique periodic solutions.

## 2 Well-Posedness of SDEs

In this section, we describe the framework of the whole paper. We refer the reader to Applebaum (2009) for the notation and terminology used below. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space with filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e. it is increasing, right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Suppose that $k, l, m \in \mathbb{N}$ with $k \geq m$. Denote by $\mathbb{R}_+$ the set of all non-negative real numbers. Let $\{B(t)\}_{t \geq 0}$ be a $k$-dimensional standard Brownian motion and $\mathcal{N}$ be an independent Poisson random measure on $\mathbb{R}_+ \times (\mathbb{R}^l - \{0\})$ with associated
compensator $\tilde{N}$ and intensity measure $\nu$, where we assume that $\nu$ is a Lévy measure satisfying $\int_{[0,1]} (1 \wedge |u|^2) \nu(du) < \infty$.

Throughout this paper, we fix a $\theta > 0$. We consider the following SDE:

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dB(t) + \int_{|u|<1} H(t, X(t), u)\tilde{N}(dt, du) + \int_{|u|\geq 1} G(t, X(t), u)N(dt, du)$$  \hspace{1cm} (2.1)

with $X(0) \in F_0$. We assume that the coefficient functions $b(t, x) : [0, \infty) \times \mathbb{R}^m \to \mathbb{R}^m, \sigma(t, x) : [0, \infty) \times \mathbb{R}^m \to \mathbb{R}^{m \times k}, H(t, x, u) : [0, \infty) \times \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}^m$ and $G(t, x, u) : [0, \infty) \times \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}^m$ are all Borel measurable and satisfy

$$b(t + \theta, x) = b(t, x), \quad \sigma(t + \theta, x) = \sigma(t, x), \quad H(t + \theta, x, u) = H(t, x, u), \quad G(t + \theta, x, u) = G(t, x, u)$$  \hspace{1cm} (2.2)

for any $t \geq 0, x \in \mathbb{R}^m$ and $u \in \mathbb{R}^l - \{0\}$. If the large jump term is removed from (2.1), we get the following modified SDE:

$$dZ(t) = b(t, Z(t))dt + \sigma(t, Z(t))dB(t) + \int_{|u|<1} H(t, Z(t), u)\tilde{N}(dt, du)$$  \hspace{1cm} (2.3)

with $Z(0) = X(0)$.

We put the following assumption:

(A1) $b(\cdot, 0), \sigma(\cdot, 0) \in L^2([0, \theta); \mathbb{R}^m), \int_{|u|<1} |H(\cdot, 0, u)|^2 \nu(du) \in L^1([0, \theta); \mathbb{R}^m)$. Hereafter, we use $|x|$ to denote the Euclidean norm of a vector $x$, use $\langle x, y \rangle$ to denote the Euclidean inner product of vectors $x$ and $y$, use $A^T$ to denote the transpose of a matrix $A$, and use $|A| := \sqrt{\text{trace}(A^T A)}$ to denote the trace norm of $A$.

**Lemma 2.1** Suppose that (A1) holds and there exists $L \in L^1([0, \theta); \mathbb{R}^+) \text{ such that for any } t \in [0, \theta) \text{ and } x, y \in \mathbb{R}^m,$

$$|b(t, x) - b(t, y)|^2 \leq L(t)|x - y|^2, \quad |\sigma(t, x) - \sigma(t, y)|^2 \leq L(t)|x - y|^2, \quad \int_{|u|<1} |H(t, x, u) - H(t, y, u)|^2 \nu(du) \leq L(t)|x - y|^2.$$

Then, the SDE (2.3) has a unique solution $\{Z(t), t \geq 0\}$. If in addition $\mathbb{E}[|Z(0)|^2] < \infty$, then

$$\mathbb{E}\left[|Z(t)|^2\right] < \infty, \quad \forall t \geq 0.$$

**Proof** First we assume that $\mathbb{E}[|Z(0)|^2] < \infty$. Set $Z_0(t) = Z(0)$ for $t \geq 0$. For $n = 1, 2, \ldots$, define the Picard iterations

$$Z_n(t) = Z_0(t) + \int_0^t b(s, Z_n(s))ds + \int_0^t \sigma(s, Z_n(s))dB(s) + \int_0^t \int_{|u|<1} H(s, Z_n(s), u)\tilde{N}(ds, du) + \int_0^t \int_{|u|\geq 1} G(s, Z_n(s), u)N(ds, du).$$
Similarly, we have

\[ Z_n(t) = Z_0(t) + \int_0^t b(s, Z_{n-1}(s))ds + \int_0^t \sigma(s, Z_{n-1}(s))dB(s) \]
\[ + \int_0^t \int_{|u| < 1} H(s, Z_{n-1}(s-), u) \tilde{N}(ds, du). \]

Let

\[ L(t) = L(t - k\theta) \quad \text{for} \ t \in [k\theta, (k+1)\theta), \ k \in \mathbb{N}. \]

By Doob’s martingale inequality, for \( t \geq 0 \), we have

\[
E \left[ \sup_{0 \leq s \leq t} |Z_1(s) - Z_0(s)|^2 \right] 
\leq 3E \left[ \sup_{0 \leq s \leq t} \left| \int_0^s b(v, Z_0(v))dv \right|^2 \right] + 3E \left[ \sup_{0 \leq s \leq t} \left| \int_0^s \sigma(v, Z_0(v))dB(v) \right|^2 \right] 
\leq 3E \left[ \int_0^t |b(s, Z_0(s))|^2 ds \right] + 12E \left[ \int_0^t |\sigma(s, Z_0(s))|^2 ds \right] 
\leq 3E \left[ \int_0^t 2(b(s, 0)^2 + L(s)|Z_0(s)|^2)ds \right] + 12E \left[ \int_0^t 2(\sigma(s, 0)^2 + L(s)|Z_0(s)|^2)ds \right] 
\leq C_1 (1 + E[|Z(0)|^2]),
\]

where

\[ C_1 = 6(t + 8) \int_0^t L(s)ds + 6tE \int_0^t |b(s, 0)|^2 ds + 24E \int_0^t |\sigma(s, 0)|^2 ds 
+ 24E \int_0^t \int_{|u| < 1} |H(s, 0, u)|^2 v(du)ds < \infty. \]

Similarly, we have

\[
E \left[ \sup_{0 \leq s \leq t} |Z_{n+1}(s) - Z_n(s)|^2 \right] 
\leq C_2(t)E \left[ \sup_{0 \leq s \leq t} |Z_n(s) - Z_{n-1}(s)|^2 \right] 
\leq (C_2(t))^nE \left[ \sup_{0 \leq s \leq t} |Z_1(s) - Z_0(s)|^2 \right] 
\leq (C_2(t))^nC_1 (1 + E[|Z(0)|^2]),
\]
where

\[ C_2(t) = (3t + 24) \int_0^t L(s) \, ds. \]

We claim that \( \{Z_n(t)\} \) converges in \( L^2 \) for \( t \geq 0 \). Indeed, for \( r, n \in \mathbb{N} \) with \( r < n \), we have

\[
\|Z_n(t) - Z_r(t)\|_2 \leq \sum_{i=r+1}^{n} \|Z_i(t) - Z_{i-1}(t)\|_2 \leq \{C_1(1 + \mathbb{E}[|Z(0)|^2])\}^{1/2} \sum_{i=r+1}^{n} (C_2(t))^{i/2}. \tag{2.5}
\]

Hereafter, \( \| \cdot \|_2 = \{\mathbb{E}[| \cdot |^2]\}^{1/2} \) denotes the \( L^2 \)-norm. We choose \( \varepsilon > 0 \) such that

\[
(3\varepsilon + 24) \int_t^{t+\varepsilon} L(s) \, ds < 1, \quad \forall t \geq 0.
\]

Then, for each \( 0 \leq t \leq \varepsilon \), \( \{Z_n(t)\} \) is a Cauchy sequence and hence converges to some \( Z(t) \in L^2(\Omega, \mathcal{F}, \mathbb{P}) \). Letting \( r \to \infty \), we obtain by (2.5) that

\[
\|Z(t) - Z_n(t)\|_2 \leq \{C_1(1 + \mathbb{E}[|Z(0)|^2])\}^{1/2} \sum_{i=n+1}^{\infty} (C_2(t))^{i/2},
\]

for \( n \in \mathbb{N} \cup \{0\} \). By the standard argument (cf. Applebaum 2009, Theorem 6.2.3), we can show that \( \{Z(t), 0 \leq t \leq \varepsilon\} \) is the unique solution of the SDE (2.3) on \([0, \varepsilon]\). Repeating this procedure with \( Z(0) \) replaced by \( Z(k\varepsilon) \), we obtain that \( \{Z(t), k\varepsilon \leq t \leq (k+1)\varepsilon\} \) is the unique solution of the SDE (2.3) on \([k\varepsilon, (k+1)\varepsilon]\) for any \( k \in \mathbb{N} \). Hence, \( \{Z(t), t \geq 0\} \) is the unique solution of the SDE (2.3).

Applying the argument of the proof of Applebaum (2009, Theorem 6.2.3), we can show the existence and uniqueness of solutions of the SDE (2.3) for the case that \( \mathbb{E}[|Z(0)|^2] = \infty \). We omit the details here and refer the reader to Applebaum (2009, Theorem 6.2.3).

Now we put the following local Lipschitz condition.

\textbf{(A2)} For each \( n \in \mathbb{N} \), there exists \( L_n \in L^1((0, \theta); \mathbb{R}_+) \) such that for any \( t \in [0, \theta) \) and \( x, y \in \mathbb{R}^m \) with \(|x| \vee |y| \leq n\),

\[
|b(t, x) - b(t, y)|^2 \leq L_n(t)|x - y|^2, \quad |\sigma(t, x) - \sigma(t, y)|^2 \leq L_n(t)|x - y|^2,
\]

\[
\int_{|u| < 1} |H(t, x, u) - H(t, y, u)|^2 v(du) \leq L_n(t)|x - y|^2.
\]

Let \( C^{1, 2}(\mathbb{R}_+ \times \mathbb{R}^m; \mathbb{R}) \) be the space of all real-valued functions \( V(t, x) \) on \( \mathbb{R}_+ \times \mathbb{R}^m \) which are continuously differentiable with respect to \( t \) and twice continuously differentiable with respect to \( x \). Springer
differentiable with respect to $x$. For $V \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^m; \mathbb{R})$, we define

\[
\mathcal{L}V(t, x) := V_t(t, x) + \langle V_x(t, x), b(t, x) \rangle + \frac{1}{2} \text{trace}(\sigma^T(t, x)V_{xx}(t, x)\sigma(t, x))
\]

\[
+ \int_{\{|u|<1\}} [V(t, x + H(t, x, u)) - V(t, x) - \langle V_x(t, x), H(t, x, u) \rangle]v(du)
\]

\[
+ \int_{\{|u|\geq1\}} [V(t, x + G(t, x, u)) - V(t, x)]v(du).
\]

(2.6)

Hereafter, we set $V_t = \frac{\partial V}{\partial t}$, $V_x = \nabla_x V = (\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_m})$ and $V_{xx} = (\frac{\partial^2 V}{\partial x_i \partial x_j})_{m \times m}$. A Borel measurable function $f$ on $[0, \infty)$ is said to be locally integrable, denoted by $f \in L_{loc}^1([0, \infty); \mathbb{R})$, if

\[
\int_0^\tau |f(x)|dx < \infty, \quad \forall \tau > 0.
\]

Denote by $C^\infty_0(\mathbb{R}^m)$ the space of all smooth functions on $\mathbb{R}^m$ with compact support.

Further, we make the following assumption.

(H1) There exist $V_1 \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^m; \mathbb{R}_+)$ and $q \in L_{loc}^1([0, \infty); \mathbb{R})$ such that

\[
\lim_{|x| \to \infty} \left[ \inf_{t \in [0, \infty)} V_1(t, x) \right] = \infty,
\]

and for $t \geq 0$ and $x \in \mathbb{R}^m$,

\[
\mathcal{L}V_1(t, x) \leq q(t).
\]

(2.7)

(2.8)

Theorem 2.2 Suppose that (A1), (A2) and (H1) hold. Then, the SDE (2.1) has a unique solution $\{X(t), t \geq 0\}$.

Proof For $n \in \mathbb{N}$, set

\[
\hat{x}_n = xI_{\{|x| \leq n\}} + \frac{n}{|x|}I_{\{|x| > n\}}, \quad x \in \mathbb{R}^m.
\]

We define the truncated functions by

\[
b_n(t, x) = b_n(t, \hat{x}_n), \quad \sigma_n(t, x) = \sigma_n(t, \hat{x}_n),
\]

\[
H_n(t, x, u) = H_n(t, \hat{x}_n, u), \quad G_n(t, x, u) = G_n(t, \hat{x}_n, u).
\]

(2.9)

Then, $b_n$, $\sigma_n$ and $H_n$ satisfy the global Lipschitz condition and the condition (A1). Hence, by Lemma 2.1, there exists a unique solution $\{Z_n(t), t \geq 0\}$ to the SDE

\[
dZ_n(t) = b_n(t, Z_n(t))dt + \sigma_n(t, Z_n(t))dB(t) + \int_{\{|u|<1\}} H_n(t, Z_n(t), u)\tilde{N}(dt, du)
\]

(2.10)
with $Z_n(0) = X(0)$.

To allow the large jump in the equation, we will use the interlacing technique. Denote

$$B = \{ u \in \mathbb{R}^l : |u| < 1\}, \quad B^c = \{ u \in \mathbb{R}^l : |u| \geq 1\}. \quad (2.11)$$

Let $\{p(t)\}$ be the Poisson point process with values in $B^c$ associated with the Poisson random measure $N(dt, du)$, i.e.

$$N([0, t], A) = \#\{ p(s) \in A : s \in [0, t], A \in \mathcal{B}(B^c) \}. \quad (2.12)$$

Then, $\{p(t)\}$ is independent of $\{Z_n(t), t \geq 0\}, n \in \mathbb{N}$.

Define $\tau_r := \inf\{ t > 0 : N([0, t]; B^c) = r \}$, which is the $r$th jump time of $t \mapsto N([0, t]; B^c)$. Let $\{Z(t), t \geq 0\}$ be the solution of the SDE (2.3). Define

$$X_n(t) = \begin{cases} Z_n(t), & \text{for } 0 \leq t < \tau_1, \\ Z_n(\tau_1-) + G_n(\tau_1-, Z_n(\tau_1-), p(\tau_1)), & \text{for } t = \tau_1, \\ Z_n^{(1)}(t), & \text{for } \tau_1 < t < \tau_2, \\ Z_n^{(1)}(\tau_2-) + G_n(\tau_2-, Z_n^{(1)}(\tau_2-), p(\tau_2)), & \text{for } t = \tau_2, \\ \ldots, & \text{for } t \geq \tau_2. \end{cases}$$

where $\{Z_n^{(1)}(t), t \geq \tau_1\}$ is the solution of the SDE (2.3) with $Z_n^{(1)}(\tau_1) = X_n(\tau_1)$. Then, $\{X_n(t), t \geq 0\}$ is the unique solution of the following SDE:

$$dX_n(t) = b_n(t, X_n(t))dt + \sigma_n(t, X_n(t))dB(t) + \int_{\{|u|<1\}} H_n(t, X_n(t-), u)\tilde{N}(dt, du)$$

$$+ \int_{\{|u|\geq1\}} G_n(t, X_n(t-), u)N(dt, du) \quad (2.13)$$

with $X_n(0) = X(0)$.

For $n \in \mathbb{N}$, we define the stopping time

$$\beta_n = \inf\{ t \in [0, \infty) : |X_n(t)| \geq n\}.$$ 

For $t \in [0, \beta_n)$, we have

$$b_n(t, X_n(t)) = b_{n+1}(t, X_n(t)), \quad \sigma_n(t, X_n(t)) = \sigma_{n+1}(t, X_n(t)), \quad H_n(t, X_n(t), u) = H_{n+1}(t, X_n(t), u), \quad G_n(t, X_n(t), u) = G_{n+1}(t, X_n(t), u).$$

$\{\beta_n\}$ is increasing. Hence, there exists a stopping time $\beta$ such that

$$\beta = \lim_{n \to \infty} \beta_n.$$
Define
\[ X(t) = \lim_{n \to \infty} X_n(t), \quad t \in [0, \beta). \]

We now show that \( \beta = \infty \) a.s. If this is not true, then there exist \( \varepsilon > 0 \) and \( T_1 \in (0, \infty) \) such that
\[ P\{\beta \leq T_1\} > 2\varepsilon. \]

Hence, we can find a sufficiently large integer \( n_0 \) such that
\[ P\{\beta_n \leq T_1\} > \varepsilon, \quad \forall n \geq n_0. \tag{2.14} \]

By Itô’s formula and (2.8), we obtain that for \( t \geq 0 \),
\[
E[V_1(t \wedge \beta_n, X(t \wedge \beta_n))] \\
= E[V_1(0, X(0))] + \mathbb{E}\left[ \int_0^{t \wedge \beta_n} \mathcal{L}V_1(s, X(s)) \, ds \right] \\
\leq V_1(0, x) + \int_0^t q(s) \, ds.
\]

Thus,
\[
E[V_1(T_1 \wedge \beta_n, X(T_1 \wedge \beta_n))] \leq V_1(0, x) + \int_0^{T_1} q(s) \, ds,
\]
which implies that
\[
E[I_{\{\beta_n \leq T_1\}} V_1(\beta_n, X(\beta_n))] \leq V_1(0, x) + \int_0^{T_1} q(s) \, ds. \tag{2.15}
\]

Define
\[ \mu(n) = \inf \{ V_1(t, x) : (t, x) \in [0, \infty) \times \mathbb{R}^m, \ |x| \geq n \}. \]

Then, \( \lim_{n \to \infty} \mu(n) = \infty \) by the condition (2.7). From (2.14) and (2.15), it follows that
\[ \varepsilon \mu(n) < \mu(n) P\{\beta_n \leq T_1\} \leq V_1(0, x) + \int_0^{T_1} q(s) \, ds, \]
which results in a contradiction when \( n \to \infty \). Therefore,
\[ \beta = \infty \quad a.s. \tag{2.16} \]
and \( \{X(t), t \geq 0\} \) is the unique solution of the SDE (2.1) on \([0, \infty)\). \qed
3 Periodic Solutions of SDEs Driven by Lévy Noises

In this section, we will study the existence and uniqueness of periodic solutions of the SDE (2.1). Denote by $B(\mathbb{R}^m)$ the Borel $\sigma$-algebra of $\mathbb{R}^m$, and denote by $B_b(\mathbb{R}^m)$ (resp. $C_b(\mathbb{R}^m)$) the space of all real-valued bounded Borel functions (resp. continuous and bounded functions) on $\mathbb{R}^m$. For $f \in B_b(\mathbb{R}^m)$, we use $\|f\|_\infty$ to denote its supremum norm.

Recall that a stochastic process $\{X(t), t \geq 0\}$ with values in $\mathbb{R}^m$, defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, is called a Markov process if, for all $A \in B(\mathbb{R}^m)$ and $0 \leq s < t < \infty$,

$$\mathbb{P}\{X(t) \in A|\mathcal{F}_s\} = \mathbb{P}\{X(t) \in A|X(s)\}. \tag{3.1}$$

We define the transition probability function of $\{X(t), t \geq 0\}$ by

$$P(s, x, t, A) = \mathbb{P}\{X(t) \in A|X(s) = x\}, \ x \in \mathbb{R}^m, A \in B(\mathbb{R}^m).$$

$\{P(s, x, t, A)\}$ defines a semigroup of linear operators $\{P_{s,t}\}$ on $B_b(\mathbb{R}^m)$:

$$P_{s,t}f(x) := \mathbb{E}_{s,x}[f(X(t))]:= \int_{\mathbb{R}^m} f(y) P(s, x, t, dy), \ x \in \mathbb{R}^m, f \in B_b(\mathbb{R}^m).$$

$\{P_{s,t}\}$ is called the Markovian transition semigroup of $\{X(t)\}$.

**Definition 3.1** (i) A Markov process $\{X(t), t \geq 0\}$ is said to be $\theta$-periodic if for any $n \in \mathbb{N}$ and any $0 \leq t_1 < t_2 < \cdots < t_n$, the joint distribution of the random variables $X(t_1 + k\theta), X(t_2 + k\theta), \ldots, X(t_n + k\theta)$ is independent of $k$ for $k \in \mathbb{N} \cup \{0\}$. A Markovian transition semigroup $\{P_{s,t}\}$ is said to be $\theta$-periodic if $P(s, x, t, A) = P(s + \theta, x, t + \theta, A)$ for any $0 \leq s < t, x \in \mathbb{R}^m$ and $A \in B(\mathbb{R}^m)$. A family of probability measures $\{\mu_s, s \geq 0\}$ on $(\mathbb{R}^m, B(\mathbb{R}^m))$ is said to be $\theta$-periodic with respect to $\{P_{s,t}\}$ if

$$\mu_s(A) = \int_{\mathbb{R}^m} P(s, x, s + \theta, A) \mu_s(dx), \ \forall A \in B(\mathbb{R}^m), s \geq 0. \tag{3.1}$$

(ii) A stochastic process $\{X(t), t \geq 0\}$ with values in $\mathbb{R}^m$ is said to be a $\theta$-periodic solution of the SDE (2.1) if it is a solution of (2.1) and is $\theta$-periodic.

**Definition 3.2** Let $0 \leq s_0 < t_0 < \infty$. A Markovian transition semigroup $\{P_{s,t}\}$ is said to be regular at $(s_0, t_0)$ if all transition probability measures $P(s_0, x, t_0, \cdot)$, $x \in \mathbb{R}^m$, are mutually equivalent. $\{P_{s,t}\}$ is said to be Feller (resp. strongly Feller) at $(s_0, t_0)$ if $P_{s_0,t_0} f \in C_b(\mathbb{R}^m)$ for any $f \in C_b(\mathbb{R}^m)$ (resp. $B_b(\mathbb{R}^m)$). $\{P_{s,t}\}$ is said to be irreducible at $(s_0, t_0)$ if $P(s_0, x, t_0) > 0$ for any $x \in \mathbb{R}^m$ and any non-empty open subset $A$ of $\mathbb{R}^m$. $\{P_{s,t}\}$ is said to be regular, Feller, strongly Feller, irreducible if it is regular, Feller, strongly Feller, irreducible at any $(s_0, t_0)$, respectively.
3.1 Feller and Strong Feller Properties of Time-Inhomogeneous Semigroups

Let \( \{Z_n(t), t \geq 0\} \) be the solution of the SDE (2.10). By the standard argument (cf. Applebaum 2009, Theorem 6.4.5), we can show that \( \{Z_n(t), t \geq 0\} \) is a Markov process. Further, we obtain by the interlacing structure that the solution \( \{X_n(t), t \geq 0\} \) of the SDE (2.13) is also a Markov process. By the proof of Theorem 2.2 and approximation, we find that the solution \( \{X(t), t \geq 0\} \) of \( \{X_n(t), t \geq 0\} \) is Markov on \( \mathbb{R}^m \). In this subsection, we will show that the transition semigroup \( \{P_{s,t}\} \) of \( \{X(t), t \geq 0\} \) is Feller and strongly Feller under suitable conditions.

Let \( \{X^x(t)\} \) be the unique solution to the SDE (2.1) with \( X^x(0) = x \in \mathbb{R}^m \) and let \( \{X_n^x(t)\} \) be the unique solution to the SDE (2.13) with \( X_n^x(0) = x \in \mathbb{R}^m \). Denote by \( \{P_{s,t}\} \) the transition semigroup of \( \{X_n(t), t \geq 0\} \).

We make the following assumption for the operator \( L \), which is defined in (2.6).

\((H^2_w)\) There exists \( V_2 \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^m; \mathbb{R}_+) \) such that

\[
\lim_{|x| \to \infty} \inf_{t \in [0, \infty)} V_2(t, x) = \infty,
\]

and

\[
\sup_{x \in \mathbb{R}^m, t \in [0, \infty)} \mathcal{L}V_2(t, x) < \infty.
\]

Obviously, \((H^2_w)\) implies \((H^1)\).

**Lemma 3.3** Suppose that \((A1), (A2)\) and \((H^2_w)\) hold. If \( \{P^n_{s,t}\} \) is Feller for every \( n \in \mathbb{N} \), then \( \{P_{s,t}\} \) is Feller. If \( \{P^n_{s,t}\} \) is strongly Feller for every \( n \in \mathbb{N} \), then \( \{P_{s,t}\} \) is strongly Feller.

**Proof** To simplify notation, we only give the proof for the case that \( s = 0 \). The proof for the case that \( s > 0 \) is completely similar. For \( n \in \mathbb{N} \), define

\[
\tau_n(x) = \inf\{t \in [0, \infty) : |X_n^x(t)| \geq n\}.
\]

By Itô’s formula, we have

\[
\mathbb{E}[V_2(t \land \tau_n, X_n^x(t \land \tau_n))] = \mathbb{E}[V_2(0, X_n^x(0))] + \mathbb{E}\left[ \int_0^t \mathcal{L}V_2(v, X_n^x(v \land \tau_n))dv \right].
\]

By (3.2)–(3.4), we get

\[
P\{\tau_n(x) < t\} \leq \frac{V_2(0, x) + t \sup_{y \in \mathbb{R}^m, t \in [0, \infty)} \mathcal{L}V_2(t, y)}{\inf_{|y| > n, t \in [0, \infty)} V_2(t, y)} \to 0 \quad \text{as} \quad n \to \infty.
\]
Note that

\[ X^x(t) = X^x_n(t), \quad t < \tau_n(x). \]

Then, for \( f \in B_b(\mathbb{R}^m) \) and \( x, y \in \mathbb{R}^m \), we have

\[
|\mathbb{E}[f(X^x(t))] - \mathbb{E}[f(X^y(t))]| \\
= |\mathbb{E}[f(X^x(t))I_{t<\tau_n(x)}] + \mathbb{E}[f(X^y(t))I_{t\geq \tau_n(x)}]| \\
- |\mathbb{E}[f(X^x(t))I_{t\geq \tau_n(y)}] - \mathbb{E}[f(X^y(t))I_{t\geq \tau_n(y)}]| \\
= |\mathbb{E}[f(X^x_n(t))] - \mathbb{E}[f(X^y_n(t))]| + |\mathbb{E}[f(X^x(t))I_{t\geq \tau_n(x)}]| \\
+ |\mathbb{E}[f(X^y(t))I_{t\geq \tau_n(y)}]| \\
\leq |\mathbb{E}[f(X^x_n(t))] - \mathbb{E}[f(X^y_n(t))]| + |\mathbb{E}[f(X^x(t))I_{t\geq \tau_n(x)}]| \\
+ |\mathbb{E}[f(X^y(t))I_{t\geq \tau_n(y)}]| \\
\leq |\mathbb{E}[f(X^x_n(t))] - \mathbb{E}[f(X^y_n(t))]| + 2\|f\|_{\infty}P(\tau_n(x) \leq t) + 2\|f\|_{\infty}P(\tau_n(y) \leq t).
\]

(3.6)

Letting \( n \to \infty \) and then \( y \to x \) in (3.6), we obtain by (3.5) that \( \{P_{0,t}\} \) is Feller if \( \{P^n_{0,t}\} \) is Feller for every \( n \in \mathbb{N} \), and \( \{P_{0,t}\} \) is strongly Feller if \( \{P^n_{0,t}\} \) is strongly Feller for every \( n \in \mathbb{N} \).

Now we consider the Feller property of \( \{P_{s,t}\} \). We need the following additional assumption.

\( \mathbf{(B)} \)

(i) \( G(t, x, u) \) is continuous in \( x \) for each \( t \in [0, \theta) \) and \( |u| \geq 1 \).

(ii) For each \( n \in \mathbb{N} \), there exist \( \gamma_n > m \) and \( M_n \in L^1([0, \theta); \mathbb{R}_+) \) such that for any \( t \in [0, \theta) \) and \( x, y \in \mathbb{R}^m \) with \( |x| \vee |y| \leq n \),

\[
\int_{\{|u|<1\}} |H(t, x, u) - H(t, y, u)|^{\gamma_n} v(du) \leq M_n(t)|x - y|^{\gamma_n},
\]

\[
\int_{\{|u|<1\}} |H(t, x, u)|^{\gamma_n} v(du) \leq M_n(t)(1 + |x|)^{\gamma_n}.
\]

**Theorem 3.4** Suppose that \( \mathbf{(A1)}, \mathbf{(A2)}, \mathbf{(B)} \) and \( \mathbf{(H^2)} \) hold. Then, \( \{P_{s,t}\} \) is Feller.

**Proof** Under the condition \( \mathbf{(B)} \), similar to the first part of the proof of Applebaum (2009, Theorem 6.7.2 and Note 1 (p. 402)), we can show that \( \{P^n_{s,t}\} \) is Feller for every \( n \in \mathbb{N} \). Therefore, the proof is complete by Lemma 3.3. \( \Box \)

Next we consider the strong Feller property of \( \{P_{s,t}\} \). We need the following assumptions.

\( \mathbf{(A3)} \)

(i) \( b(\cdot, 0) \in L^2([0, \theta); \mathbb{R}^m), \sigma(\cdot, 0) \in L^\infty([0, \theta); \mathbb{R}^m), \int_{\{|u|<1\}} |H(\cdot, 0, u)|^{2} v(du) \in L^1([0, \theta); \mathbb{R}^m) \).

(ii) For each \( n \in \mathbb{N} \), there exists \( L_n \in L^\infty([0, \theta); \mathbb{R}_+) \) such that for any \( t \in [0, \theta) \) and \( x, y \in \mathbb{R}^m \) with \( |x| \vee |y| \leq n \),

\[
|b(t, x) - b(t, y)|^2 \leq L_n(t)|x - y|^2, \quad |\sigma(t, x) - \sigma(t, y)|^2 \leq L_n(t)|x - y|^2.
\]
\[
\int_{\{|u|<1\}} |H(t, x, u) - H(t, y, u)|^2 v(du) \leq L_n(t)|x - y|^2.
\]

(A4) For any \( t \in [0, \theta) \) and \( x \in \mathbb{R}^m \), \( Q(t, x) := \sigma(t, x)\sigma^T(t, x) \) is invertible and
\[
\sup_{|x| \leq n, t \in [0, \theta)} |Q^{-1}(t, x)| < \infty, \ \forall n \in \mathbb{N}.
\] (3.7)

Obviously, (A3) implies (A1) and (A2).

Let \( J \) be a non-negative function in \( C_0^\infty(\mathbb{R}^m) \) satisfying
\[
J(x) = 0 \text{ for } |x| \geq 1 \text{ and } \int_{\mathbb{R}^m} J(x)dx = 1.
\]

For \( \varepsilon > 0 \), define
\[
J_\varepsilon(x) = \varepsilon^{-m}J(\varepsilon^{-1}x).
\]

Let \( u \) be a locally integrable function on \( \mathbb{R}^m \). We define
\[
u^\varepsilon(x) := J_\varepsilon * u(x) := \int_{\mathbb{R}^m} J_\varepsilon(x - y)u(y)dy.
\]

We have the following standard lemma.

Lemma 3.5 Let \( \varepsilon > 0 \). (i) If \( u \) is a bounded function on \( \mathbb{R}^m \), then\[
\|u^\varepsilon\|_\infty \leq \|u\|_\infty.
\]

(ii) If \( u \) is a continuous function on \( \mathbb{R}^m \), then \( \lim_{\varepsilon \to 0} u^\varepsilon(x) = u(x) \) uniformly on any compact subset of \( \mathbb{R}^m \).

(iii) If \( u \) is Lipschitz continuous on \( \mathbb{R}^m \) with Lipschitz constant \( L \), then
\[
\|u^\varepsilon - u\|_\infty \leq \varepsilon \quad \text{and} \quad \|\nabla u^\varepsilon\|_\infty \leq L.
\]

(iv) Let \( h(t, x, u) : [0, \infty) \times \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}^m \) be Borel measurable. If there exists \( \{L(t), t \geq 0\} \) such that
\[
\int_{\{|u|<1\}} |h(t, x, u) - h(t, y, u)|^2 v(du) \leq L(t)|x - y|^2, \quad t \geq 0, \ x, y \in \mathbb{R}^m.
\]

Then,
\[
\int_{\{|u|<1\}} |h^\varepsilon(t, x, u) - h^\varepsilon(t, y, u)|^2 v(du) \leq L(t)|x - y|^2, \quad t \geq 0, \ x, y \in \mathbb{R}^m,
\]
and
\[
\int_{|\nu|<1} |\nabla_x h^\varepsilon(t, x, u)|^2 v(\text{d}u) \leq L(t), \quad t \geq 0, \ x \in \mathbb{R}^m.
\]

**Lemma 3.6** Let \( \eta(t, x) : [0, \infty) \times \mathbb{R}^m \to \mathbb{R}^{m \times k} \) be Borel measurable. Suppose that (i) \( \eta(\cdot, 0) \in L^\infty([0, \theta); \mathbb{R}^m) \) and there exists \( L \in L^\infty([0, \theta); \mathbb{R}_+) \) such that for any \( x, y \in \mathbb{R}^m, \)
\[
|\eta(t, x) - \eta(t, y)|^2 \leq L(t)|x - y|^2. \tag{3.8}
\]
(ii) For any \( t \in [0, \theta) \) and \( x \in \mathbb{R}^m, \eta(t, x)\eta^T(t, x) \) is invertible. Moreover, for any \( n \in \mathbb{N}, \) there exists \( \kappa_n > 0 \) such that
\[
\langle y, \eta(t, x)\eta^T(t, x)y \rangle \geq \kappa_n|y|^2, \quad \forall y \in \mathbb{R}^m, |x| \leq n, \ t \in [0, \theta). \tag{3.9}
\]
Then, for any \( n \in \mathbb{N}, \) if \( \varepsilon \) is sufficiently small, we have
\[
\langle y, \eta^\varepsilon(t, x)(\eta^\varepsilon(t, x))^T y \rangle \geq \frac{\kappa_n}{2} |y|^2, \quad \forall y \in \mathbb{R}^m, |x| \leq n, \ t \in [0, \theta). \]

**Proof** We follow the argument of the proof of Dong (2018, Lemma 2.2). For \( x, y \in \mathbb{R}^m, \)
we have
\[
\langle y, \eta^\varepsilon(t, x)(\eta^\varepsilon(t, x))^T y \rangle
= \sum_{i,j=1}^m \sum_{r=1}^m y_i y_j \eta^\varepsilon_{ir}(t, x)\eta^\varepsilon_{jr}(t, x)
= \sum_{i,j=1}^m \sum_{r=1}^m y_i y_j \int_{\mathbb{R}^m} J_\varepsilon(x - z)\eta_{ir}(t, z)dz \int_{\mathbb{R}^m} J_\varepsilon(x - z')\eta_{jr}(t, z')dz'
= \sum_{i,j=1}^m \sum_{r=1}^m y_i y_j \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} J_\varepsilon(z) J_\varepsilon(z')\eta_{ir}(t, x - z)\eta_{jr}(t, x - z')dz dz'
= \sum_{i,j=1}^m \sum_{r=1}^m y_i y_j \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} J_\varepsilon(z) J_\varepsilon(z')\eta_{ir}(t, x - z)[\eta_{ir}(t, x - z') - \eta_{jr}(t, x - z)]dz dz'
\quad + \sum_{i,j=1}^m \sum_{r=1}^m y_i y_j \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} J_\varepsilon(z) J_\varepsilon(z') \eta_{ir}(t, x - z)\eta_{jr}(t, x - z)dz dz'. \tag{3.10}
\]
By (3.9), we get
\[
\sum_{i,j=1}^m \sum_{r=1}^m y_i y_j \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} J_\varepsilon(z) J_\varepsilon(z') \eta_{ir}(t, x - z)\eta_{jr}(t, x - z)dz dz'
= \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} J_\varepsilon(z) J_\varepsilon(z') \langle y, \eta(t, x)\eta^T(t, x)y \rangle dz dz'.
\]
By (3.8), we get

\[ \langle \nabla, \varepsilon \rangle \frac{\beta_1}{2} \leq k_n |y|^2, \quad \forall y \in \mathbb{R}^n, \quad |x| \leq n, \quad t \in [0, \theta). \]  

(3.11)

By (3.10), we get

\[ \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} J_\varepsilon(z) J_\varepsilon(z') \eta_{ir} (t, x - z) (\eta_{ir} (t, x - z') - \eta_{jr} (t, x - z)) |dz| dl \]

\[ \leq \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} J_\varepsilon(z) J_\varepsilon(z') |\eta_{ir} (t, 0)| |\eta_{ir} (t, x - z) - \eta_{jr} (t, x - z)| |dz| dl \]

\[ + \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} J_\varepsilon(z) J_\varepsilon(z') |\eta_{ir} (t, x - z) - \eta_{jr} (t, 0)| |\eta_{ir} (t, x - z') - \eta_{jr} (t, x - z)| |dz| dl \]

\[ \leq \left[ |\eta_{ir} (t, 0)| \sqrt{L(t)} + (|x| + \varepsilon) L(t) \right] \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} J_\varepsilon(z) J_\varepsilon(z') |z - z'| |dz| dl \]

\[ \leq 2 \varepsilon \left[ |\eta_{ir} (t, 0)| \sqrt{L(t)} + (|x| + \varepsilon) L(t) \right]. \]  

(3.12)

Therefore, the proof is complete by (3.10)–(3.12).

For \( n \in \mathbb{N} \), let \( b_n, \sigma_n, H_n, G_n \) be defined as in (2.9). Denote

\[ Q_n (t, x) := \sigma_n (t, x) \sigma_n^T (t, x). \]

Suppose that conditions (A3) and (A4) hold. If we replace \( b, \sigma, H, Q \) with \( b_n, \sigma_n, H_n, Q_n \), then the corresponding conditions still hold. By Lemma 2.1, the following SDE

\[ dZ_{n, \varepsilon} (t) = b_n (t, Z_{n, \varepsilon} (t)) dt + \sigma_n (t, Z_{n, \varepsilon} (t)) dB(t) + \int_{|u| < 1} H_n (t, Z_{n, \varepsilon} (t - u)) N (du, d\bar{N}(dr)) \]  

(3.13)

has a unique solution \( \{Z_{n, \varepsilon} (t), t \geq 0\} \) with \( Z_{n, \varepsilon} (0) = x \).

**Lemma 3.7** Suppose that (A3) and (A4) hold. Let \( T > 0 \). Then, there exists a constant \( M_T > 0 \) such that for all \( \varphi \in B_b (\mathbb{R}^m) \) and \( 0 \leq s \leq t \leq T \),

\[ |E_{s, \varepsilon} \{ \varphi (X_n (t)) \} - E_{s, \varepsilon} \{ \varphi (X_n (t)) \} | \leq \frac{M_T}{\sqrt{T - s}} \| \varphi \|_{\infty} |x - y|, \quad \forall x, \ y \in \mathbb{R}^m. \]

**Proof** We only give the proof for the case that \( s = 0 \). The proof for the case that \( s > 0 \) is completely similar. To simplify notation, we omit the subscript “\( n \)” in the following proof.

**Step 1.** Let \( \{Z_\varepsilon (t), t \geq 0\} \) be the solution of the SDE (3.13). Denote by \( C_b^2 (\mathbb{R}^m) \) the space of all continuously differentiable functions on \( \mathbb{R}^m \) with bounded second-order partial derivatives. First, we prove the following Bismut–Elworthy–Li formula: for any \( \varphi \in C_b^2 (\mathbb{R}^m) \), \( t > 0 \) and \( \varepsilon \in \mathbb{R}^m \), we have

\[ \langle \nabla, E_{0, \varepsilon} \{ \varphi (Z_\varepsilon (t)) \} \rangle, h \rangle = \frac{1}{t} E_{0, \varepsilon} \left\{ \varphi (Z_\varepsilon (t)) \int_0^t \left( (\sigma_\varepsilon (t - v, Z_\varepsilon (v)))^T \varphi (Z_\varepsilon (v)) \right) \right\}. \]

(3.14)
In fact, for $\varphi \in C_b^2(\mathbb{R}^m)$, $\psi(t, x) := \mathbb{E}_{0,x}[\varphi(Z_\varepsilon(t))]$ is the unique solution of the following equation:

$$\begin{align*}
\begin{cases}
\frac{d\psi(t, x)}{dt} = \widetilde{L}\psi(t, x), \\
\psi(0, x) = \varphi(x), \quad t \geq 0, \ x \in \mathbb{R}^m,
\end{cases}
\end{align*}$$

where

$$\widetilde{L}\psi(t, x) := \langle \psi_x(t, x), b^\varepsilon(t, x) \rangle + \frac{1}{2} \text{trace}((\sigma^\varepsilon(t, x))^T \psi_x(t, x) \sigma^\varepsilon(t, x))$$

$$+ \int_{|u| < 1} \left[ \psi(t, x + H^\varepsilon(t, x, u)) - \psi(t, x) - (\psi_x(t, x), H^\varepsilon(t, x, u)) \right] \nu(du).$$

Applying Itô’s formula to the process $\{\psi(t - v, Z_\varepsilon(v)), v \in [0, t]\}$, we get

$$\varphi(Z_\varepsilon(t)) = \psi(t, x) + \int_0^t \left[ \frac{\partial}{\partial v} \psi((t - v, Z_\varepsilon(v)) + \widetilde{L}\psi((t - v, Z_\varepsilon(v))) \right] \, dv$$

$$+ \int_0^t \langle \psi_x((t - v, Z_\varepsilon(v)), \sigma^\varepsilon(t - v, Z_\varepsilon(v)) \rangle \, dB(v)$$

$$+ \int_0^t \int_{|u| < 1} [\psi(t - v, Z_\varepsilon(v -)) + H^\varepsilon(t - v, Z_\varepsilon(v -), u)) - \psi(t - v, Z_\varepsilon(v -))] \tilde{N}(dv, du)$$

$$= \psi(t, x) + \int_0^t \langle \psi_x((t - v, Z_\varepsilon(v)), \sigma^\varepsilon(t - v, Z_\varepsilon(v)) \rangle \, dB(v)$$

$$+ \int_0^t \int_{|u| < 1} [\psi(t - v, Z_\varepsilon(v -)) + H^\varepsilon(t - v, Z_\varepsilon(v -), u)) - \psi(t - v, Z_\varepsilon(v -))] \tilde{N}(dv, du).$$

By Lemma 3.5, following the argument of Xie (2012, Theorem 3.1), we can show that there exists a positive constant $C_T$, which is independent of $\varepsilon$, such that

$$\mathbb{E}_{0,x} \{ |\nabla Z_\varepsilon(v)|^2 \} \leq C_T |h|^2, \quad \forall v \in [0, T].$$

Note that the transition semigroup $P^\varepsilon_{0,t}$ of $Z_\varepsilon(t)$ is given by $P^\varepsilon_{0,t} \varphi(x) = \mathbb{E}_{0,x}[\varphi(Z_\varepsilon(t))]$. Multiplying both sides of (3.15) by $\int_0^t ((\sigma^\varepsilon(t - v, Z_\varepsilon(v)))^T (\sigma^\varepsilon(t - v, Z_\varepsilon(v)) (\sigma^\varepsilon(t - v, Z_\varepsilon(v)))^T)^{-1} \nabla Z_\varepsilon(v) h, \ dB(v)$, and taking expectation we get

$$\mathbb{E}_{0,x} \left\{ \varphi(Z_\varepsilon(t)) \int_0^t ((\sigma^\varepsilon(t - v, Z_\varepsilon(v)))^T (\sigma^\varepsilon(t - v, Z_\varepsilon(v)) (\sigma^\varepsilon(t - v, Z_\varepsilon(v)))^T)^{-1} \nabla Z_\varepsilon(v) h, \ dB(v) \right\}$$

$$= \int_0^t \mathbb{E}_{0,x} \{ ((\sigma^\varepsilon(t - v, Z_\varepsilon(v)))^T \psi_x((t - v, Z_\varepsilon(v), (\sigma^\varepsilon(t - v, Z_\varepsilon(v)), (\sigma^\varepsilon(t - v, Z_\varepsilon(v)))h, \ dB(v)) \}.$$
\[
- v, Z_\varepsilon(v)) \right]^T [\sigma^\varepsilon(t - v, Z_\varepsilon(v))(\sigma^\varepsilon(t - v, Z_\varepsilon(v)) \right]^T - 1 \nabla_x Z_\varepsilon(v)h]\right] \text{d}v
\]

\[
= \int_0^t \mathbb{E}_{0,v} \{\langle \psi_x(t - v, Z_\varepsilon(v)), \nabla_x Z_\varepsilon(v)h \rangle \} \text{d}v
\]

\[
= \int_0^t \nabla_x \mathbb{E}_{0,v} \{\langle P_0,v \varphi(Z_\varepsilon(v)), h \rangle \} \text{d}v
\]

\[
= t \langle \nabla_x P_0,v \varphi(x), h \rangle.
\]

Then, (3.14) holds.

By (3.7) and Lemma 3.6, there exists \( K > 0 \) such that if \( \varepsilon \) is sufficient small then

\[
\sup_{x \in \mathbb{R}^m, t \in [0, \theta)} |[\sigma^\varepsilon(t, x)(\sigma^\varepsilon(t, x)) \right]^T - 1| \leq K.
\]

Thus, we obtain by (3.14) that

\[
|\langle \nabla_x \mathbb{E}_{0,v} \{\varphi(Z_\varepsilon(t))\}, h \rangle |^2
\]

\[
\leq \frac{1}{t^2} \| \varphi \|^2_{\infty} \mathbb{E}_{0,v} \left\{ \int_0^t |(\sigma^\varepsilon(t - v, Z_\varepsilon(v)) \right]^T [\sigma^\varepsilon(t - v, Z_\varepsilon(v)) \right)^T - 1 \nabla_x Z_\varepsilon(v)h|^2 \text{d}v \right\}
\]

\[
\leq \frac{K}{t^2} \| \varphi \|^2_{\infty} \mathbb{E}_{0,v} |\nabla_x Z_\varepsilon(v)h|^2 \text{d}v. \tag{3.17}
\]

Define

\[
M_T := \sqrt{K C_T}.
\]

By (3.16) and (3.17), we get

\[
|\mathbb{E}_{0,v} \{\varphi(Z_\varepsilon(t))\} - \mathbb{E}_{0,v} \{\varphi(Z_\varepsilon(t))\} | \leq \frac{M_T}{\sqrt{t}} \| \varphi \|_{\infty} |x - y|, \quad \forall x, y \in \mathbb{R}^m. \tag{3.18}
\]

Denote by \text{Var}(\cdot) the total variation norm of a signed measure. Let \( C > 0 \) and \( t > 0 \). We claim that the following conditions are equivalent:

(i) For all \( \varphi \in C^2_b(\mathbb{R}^m) \) and \( x, y \in \mathbb{R}^m \), \( |P_{0,t}^\varepsilon \varphi(x) - P_{0,t}^\varepsilon \varphi(y)| \leq C \| \varphi \|_{\infty} |x - y| \);

(ii) For all \( \varphi \in B_b(\mathbb{R}^m) \) and \( x, y \in \mathbb{R}^m \), \( |P_{0,t}^\varepsilon \varphi(x) - P_{0,t}^\varepsilon \varphi(y)| \leq C \| \varphi \|_{\infty} |x - y| \);

(iii) For all \( x, y \in \mathbb{R}^m \), \( \text{Var}(P_{0,t}^\varepsilon(x, \cdot) - P_{0,t}^\varepsilon(y, \cdot)) \leq C |x - y| \).

In fact, since each function in \( B_b(\mathbb{R}^m) \) can be approximated pointwise by functions in \( C^2_b(\mathbb{R}^m) \), we have that for all \( x, y \in \mathbb{R}^m \),

\[
\sup_{\varphi \in \mathcal{K}_1} |P_{0,t}^\varepsilon \varphi(x) - P_{0,t}^\varepsilon \varphi(y)| = \sup_{\varphi \in \mathcal{K}_2} |P_{0,t}^\varepsilon \varphi(x) - P_{0,t}^\varepsilon \varphi(y)|,
\]
where
\[ \mathcal{K}_1 = \{ \varphi \in C_b(\mathbb{R}^m) : \| \varphi \|_\infty < 1 \}, \quad \mathcal{K}_2 = \{ \varphi \in C^2_b(\mathbb{R}^m) : \| \varphi \|_\infty < 1 \}. \]

Hence,
\[ \sup_{\varphi \in \mathcal{K}_1} |P_{0,t}^\varepsilon \varphi(x) - P_{0,t}^\varepsilon \varphi(y)| = \text{Var}(P_{0,t}^\varepsilon \varphi(0, x, \cdot) - P_{0,t}^\varepsilon \varphi(0, y, \cdot)). \]

Then, (i) implies (iii). On the other hand, if (iii) holds then for all \( \varphi \in B_b(\mathbb{R}^m) \),
\[ |P_{0,t}^\varepsilon \varphi(x) - P_{0,t}^\varepsilon \varphi(y)| = \left| \int_{\mathbb{R}^m} \varphi(z) [P^\varepsilon(0, x, t, dz) - P^\varepsilon(0, y, t, dz)] \right| \leq \| \varphi \|_\infty \text{Var}(P_{0,t}^\varepsilon \varphi(0, x, \cdot) - P_{0,t}^\varepsilon \varphi(0, y, \cdot)) \leq C \| \varphi \|_\infty |x - y|, \]

which implies that (ii) holds. By the equivalence of (i)–(iii), we conclude that (3.18) holds for any \( \varphi \in B_b(\mathbb{R}^m) \).

Step 2. In the following, we will prove that the solutions of the SDEs (3.13) converge to the solution of the SDE (2.10) in mean square, i.e.
\[ \lim_{\varepsilon \to 0} \mathbb{E}_{0,x} \left[ \sup_{t \in [0,T]} |Z_{\varepsilon}(t) - Z(t)|^2 \right] = 0. \] (3.19)

In fact, by Lemma 3.5, we obtain that for \( w \in [0, T] \),
\[
\mathbb{E}_{0,x} \left[ \sup_{t \in [0,w]} |Z(t) - Z_{\varepsilon}(t)|^2 \right] \\
= \mathbb{E}_{0,x} \left[ \sup_{t \in [0,w]} \int_0^t \left[ (b(v, Z(v)) - b^\varepsilon(v, Z(v))) + (b^\varepsilon(v, Z(v)) - b^\varepsilon(v, Z_\varepsilon(v))) \right] dv \\
+ \int_0^t \left[ (\sigma(v, Z(v)) - \sigma^\varepsilon(v, Z(v))) + (\sigma^\varepsilon(v, Z(v)) - \sigma^\varepsilon(v, Z_\varepsilon(v))) \right] dB(v) \\
+ \int_0^t \left[ (H(v, Z(v-), u) - H^\varepsilon(v, Z(v-), u)) + (H^\varepsilon(v, Z(v-), u) - H^\varepsilon(v, Z_\varepsilon(v-), u)) \right] dv \\
\leq 6w \int_0^w L(v) \varepsilon^2 dv + 48 \int_0^w L(v) \varepsilon^2 dv \\
+ (6w + 48) \int_0^w L(v) \mathbb{E}_{0,x} \left[ \sup_{t \in [0,v]} |Z(t) - Z_{\varepsilon}(t)|^2 \right] dv \\
\leq (6T + 48) \varepsilon^2 \int_0^T L(v) dv + (6T + 48) \int_0^w L(v) \mathbb{E}_{0,x} \left[ \sup_{t \in [0,v]} |Z(t) - Z_{\varepsilon}(t)|^2 \right] dv.
\]
Define
\[ \vartheta = (6T + 48) \int_0^T L(v) dv. \]

Then, Gronwall’s inequality implies that
\[ \mathbb{E}_{0,x} \left[ \sup_{t \in [0,T]} \left| Z(t) - Z_b(t) \right|^2 \right] \leq \vartheta \varepsilon^2 e^{\vartheta}. \]

Letting \( \varepsilon \to 0 \), we obtain (3.19).

Finally, combining (3.18) and (3.19), we obtain that for any \( \varphi \in B_b(\mathbb{R}^m) \), \( t \in (0, T] \) and \( x, y \in \mathbb{R}^m \),
\[ \left| \mathbb{E}_{0,x}[\varphi(Z(t))] - \mathbb{E}_{0,y}[\varphi(Z(t))] \right| \leq \frac{M_T}{\sqrt{t}} \| \varphi \|_{\infty} |x - y|. \quad (3.20) \]

**Step 3.** Let \( B \), \( B^c \) and the Poisson point process \( \{p(t)\} \) be defined as in (2.11) and (2.12), respectively. Define \( \tau_1 := \inf\{t > 0 : N([0, t]; B^c) = 1\} \), which is the first jump time of \( t \mapsto N([0, t]; B^c) \). Let \( \{Z(t), t \geq 0\} \) be the solution of the SDE (2.10). Then, \( \{p(t)\} \) is independent of \( \{Z(t), t \geq 0\} \).

Denote by \( P_{0,t}^Z \) and \( P_{0,t}^X \), the transition semigroups of \( Z(t) \) and \( X(t) \), respectively. Then, for \( \varphi \in B_b(\mathbb{R}^m), t \geq 0 \) and \( x \in \mathbb{R}^m \), we have
\[ P_{0,t}^Z \varphi(x) = \mathbb{E}_{0,x}[\varphi(Z(t))], \quad P_{0,t}^X \varphi(x) = \mathbb{E}_{0,x}[\varphi(X(t))]. \]

Note that
\[ P_{0,t}^X \varphi(x) = \mathbb{E}_{0,x}[\varphi(X(t)) ; t < \tau_1] + \mathbb{E}_{0,x}[\varphi(X(t)) ; t \geq \tau_1] \]
\[ = \mathbb{E}_{0,x}[\varphi(Z(t)) ; t < \tau_1] + \mathbb{E}_{0,x}\{I_{[\tau_1 \leq t]} \mathbb{E}_{\tau_1, X(\tau_1)}[\varphi(X(t))]\} \]
\[ = e^{-v(B^c) \tau_1} P_{0,t}^Z \varphi(x) \]
\[ + \int_0^t \int_{|u| \geq 1} \int_{\mathbb{R}^m} e^{-v(B^c) v} P_{v,t}^X \varphi(w + G(v, w, u)) P_{0,v}^Z(x, dv) v(du) dv. \]
\[ (3.21) \]

By (3.18) and (3.21), we obtain that for \( t \in (0, T] \),
\[ \left| \mathbb{E}_{0,x}[\varphi(X(t))] - \mathbb{E}_{0,y}[\varphi(X(t))] \right| \]
\[ \leq \left| \mathbb{E}_{0,x}[\varphi(Z(t))] - \mathbb{E}_{0,y}[\varphi(Z(t))] \right| \]
\[ + \int_0^t \int_{|u| \geq 1} \int_{\mathbb{R}^m} e^{-v(B^c) v} P_{v,t}^X \varphi(w + G(v, w, u))[P_{0,v}^Z(x, dv) - P_{0,v}^Z(y, dv)] v(du) dv \]
\[ \leq \frac{M_T}{\sqrt{t}} \| \varphi \|_{\infty} |x - y| + \int_0^t v(B^c) e^{-v(B^c) v} \frac{M_T}{\sqrt{v}} \| \varphi \|_{\infty} |x - y| dv \]
\[ M_T \| \varphi \|_{\infty} |x - y| \left( \frac{1}{\sqrt{t}} + 2 \nu(B^c) \sqrt{t} \right) \]
\[ \leq \frac{M_T + 2 \nu(B^c) T}{\sqrt{t}} \| \varphi \|_{\infty} |x - y|. \]

Therefore, the proof is complete. \(\square\)

**Theorem 3.8** Suppose that (A3), (A4) and (H^2) hold. Then, \( \{P_{s,t}\} \) is strongly Feller.

**Proof** By Lemma 3.7, we know that \( \{P^n_{s,t}\} \) is strongly Feller for every \( n \in \mathbb{N} \). Hence, Lemma 3.3 implies that \( \{P_{s,t}\} \) is strongly Feller. \(\square\)

### 3.2 Irreducibility of Time-Inhomogeneous Semigroups

Let \( \{X(t), \ t \geq 0\} \) be the solution of the SDE (2.1). In this subsection, we will show that the transition semigroup \( \{P_{s,t}\} \) of \( \{X(t), \ t \geq 0\} \) is irreducible. Denote by \( B_{b,loc}(\mathbb{R}^\alpha) \) and \( B_{b,loc}([0, \infty) \times \mathbb{R}^m; \mathbb{R}^m) \) the sets of all locally bounded Borel measurable functions on \( \mathbb{R}^\alpha \) and maps from \([0, \infty) \times \mathbb{R}^m \) to \( \mathbb{R}^m \), respectively. Let \( f \) be a function on \([0, \infty) \times \mathbb{R}^m \). For \( \rho > 0 \), we define \( f^\rho(t, x) = f(t, \rho x), \ t \geq 0, x \in \mathbb{R}^m \).

We make the following assumption.

**H^3** There exists \( V_3 \in C^{1,2}([0, \infty) \times \mathbb{R}^m; \mathbb{R}^\alpha) \) satisfying the following conditions:

(i) \[
\lim_{|x| \to \infty} \left[ \inf_{t \in [0, \infty)} V_3(t, x) \right] = \infty. \quad (3.22)
\]

(ii) For any \( \rho \geq 1 \), there exist \( q_\rho \in B_{b,loc}(\mathbb{R}^\alpha) \) and \( W_\rho(t, x) \in B_{b,loc}([0, \infty) \times \mathbb{R}^m; \mathbb{R}^m) \) satisfying for each \( n \in \mathbb{N} \) there exists \( R_n \in L^1_{loc}([0, \infty); \mathbb{R}^\alpha) \) such that for any \( t \in [0, \infty) \) and \( x, y \in \mathbb{R}^m \) with \( |x| \vee |y| \leq n \),
\[
|W_\rho(t, x) - W_\rho(t, y)|^2 \leq R_n(t)|x - y|^2,
\]

and for \( t \geq 0 \) and \( x \in \mathbb{R}^m \),
\[
\mathcal{L} V_3^\rho(t, x) \leq q_\rho(t), \quad (3.23)
\]
\[
V_3^\rho(t, x) \leq \langle W_\rho, \nabla_x V_3^\rho(t, x) \rangle. \quad (3.24)
\]

**Theorem 3.9** Suppose that (A1), (A2), (A4) and (H^3) hold. Then, \( \{P_{s,t}\} \) is irreducible.

**Proof** To simplify notation, we only give the proof for the case that \( s = 0 \). The proof for the case that \( s > 0 \) is completely similar. Let \( x, y \in \mathbb{R}^m \) with \( x \neq y \) and \( T > 0 \). For \( r \in \mathbb{N} \), we consider the following SDE:
Thus, we obtain by (3.26) that there exists $\rho \geq 2.2$, we can show that the SDE (3.25) has a unique solution $X(t)$.

By conditions (A1), (A2) and (H3), following the argument of the proof of Theorem 2.2, we can show that the SDE (3.25) has a unique solution $\{X(t), t \geq 0\}$ with $X(0) = x$.

By Itô’s formula and (3.24), we get

$$
\mathbb{E}[e^{rt}V_3^{\rho}(t, X(t) - y)] 
= \mathbb{E}[V_3^{\rho}(0, x - y)] + \mathbb{E}\left[\int_0^t e^{rv}L^fV_3^{\rho}(v, X(v) - y)dv\right]
+ \mathbb{E}\left[\int_0^t e^{rv}[rV_3^{\rho}(v, X(v) - y) + q_\rho(v)]dv\right]
\leq \mathbb{E}[V_3^{\rho}(0, x - y)] + \mathbb{E}\left[\int_0^t e^{rv}[-rV_3^{\rho}(v, X(v) - y) + q_\rho(v)]dv\right]
+ \mathbb{E}\left[\int_0^t e^{rv}V_3^{\rho}(v, X(v) - y)dv\right]
\leq \mathbb{E}[V_3^{\rho}(0, x - y)] + \int_0^t q_\rho(v)e^{rv}dv.
$$

Hence,

$$
\mathbb{E}[V_3^{\rho}(t, X(t) - y)] \leq \mathbb{E}[V_3^{\rho}(0, x - y)] + \frac{(1 - e^{-rt})\sup_{0 \leq v \leq T}|q_\rho(v)|}{r}.
$$

By (3.22), for any $0 < a < |x - y|$, there exists $\rho_a \geq 1$ such that

$$
\{z \in \mathbb{R}^m : |z| \geq a\} \subset \{z \in \mathbb{R}^m : V^{\rho_a}(T, z) \geq 1\}.
$$

Thus, we obtain by (3.26) that there exists $r_a \in \mathbb{N}$ such that

$$
\mathbb{P}(|X^{r_a}(T) - y| \geq a) 
\leq \mathbb{P}\{V_3^{\rho_a}(T, X^{r_a}(T) - y) \geq 1\}
\leq \mathbb{E}[V_3^{\rho_a}(T, X^{r_a}(T) - y)]
\leq \mathbb{E}[V_3^{\rho_a}(0, x - y)] + \frac{(1 - e^{-r_aT})\sup_{0 \leq v \leq T}|q_\rho(v)|}{r_a}.
$$
For $K \in \mathbb{N}$, define
\[
\tau_K := \inf \{ t : |X^{ra}(t)| \geq K \}.
\]
Similar to (2.16), we can show that there exists $K \in \mathbb{N}$ such that
\[
P(\tau_K \leq T) < \frac{1}{2},
\]
which together with (3.27) implies that
\[
P(\tau_K \leq T) + P(|X^{ra}(T) - y| \geq a) < 1.
\]
(3.28)

Define
\[
\alpha(t) := -r_a(\sigma(t, X^{ra}(t)))^T[\sigma(t, X^{ra}(t))(\sigma(t, X^{ra}(t)))^T]^{-1}W_{\rho_a}(t, X^{ra}(t) - y),
\]
\[
\tilde{B}(t) := B(t) + \int_0^{t \wedge \tau_K} \alpha(v)dv,
\]
and
\[
\mathcal{M}(t) = \exp\left( \int_0^{t \wedge \tau_K} \alpha(v)dB(v) - \frac{1}{2} \int_0^{t \wedge \tau_K} |\alpha(v)|^2dv \right).
\]

By (A1), (A2), (A4) and (H3), we get
\[
E\left\{ \exp\left( \frac{1}{2} \int_0^{t \wedge \tau_K} |\alpha(v)|^2dv \right) \right\} < \infty,
\]
i.e. Novikov’s condition is satisfied. Then, $\{\mathcal{M}(t)\}$ is a martingale. We define a new probability measure by
\[
\mathbb{Q}(F) = \int_F \mathcal{M}(T)d\mathbb{P}, \quad F \in \mathcal{F}_T.
\]
Thus, we obtain by Girsanov’s theorem that, under $\mathbb{Q}$, $\tilde{B}(t)$ is still a Brownian motion and $N(dt, du)$ is a Poisson random measure with the same compensator $\nu(du)dt$.

By (3.28), we have
\[
\mathbb{Q}(\{\tau_K \leq T \} \cup \{|X^{ra}(T) - y| \geq a\}) < 1.
\]
(3.29)

Note that $X^{ra}(t)$ also solves the following SDE:
\[
X(t \wedge \tau_K) = x + \int_0^{t \wedge \tau_K} b(v, X(v))dv + \int_0^{t \wedge \tau_K} \sigma(v, X(v))d\tilde{B}(v)
\]


\[
+ \int_{0}^{t \wedge \tau} \int_{|u| < 1} H(v, X(v -), u) \tilde{N}(dv, du)
\]

\[
+ \int_{0}^{t \wedge \tau} \int_{|u| \geq 1} G(v, X(v -), u)N(dv, du).
\]

Set \( \eta_K := \inf \{ t : |X(t)| \geq K \} \). By the weak uniqueness of the solutions of the SDE (2.1), we know that the law of \( \{ (X(t) \mathbb{1}_{|t| < \eta_K}) \} \) under \( \mathbb{P} \) is the same as that of \( \{ (X^{ra}(t) \mathbb{1}_{|t| < \tau_K}) \} \) under \( \mathbb{Q} \). Hence, we obtain by (3.29) that

\[
\mathbb{P}(|X(T) - y| \geq a) \leq \mathbb{P}((\eta_K \leq T) \cup \{ \eta_K > T, |X(T) - y| \geq a \})
\]

\[
= \mathbb{Q}((\tau_K \leq T) \cup \{ \tau_K > T, |X^{ra}(T) - y| \geq a \})
\]

\[
< 1,
\]

which implies that

\[
\mathbb{P}(|X(T) - y| < a) > 0.
\]

Since \( a, x, y, T \) are arbitrary, we conclude that \( \{ P_{s,t} \} \) is irreducible.

### 3.3 Existence of Periodic Solutions

In this subsection, we investigate the existence of periodic solutions of the SDE (2.1). We make the following assumption for the operator \( \mathcal{L} \), which is defined in (2.6).

\( (H^2) \) There exists \( V_2 \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^m; \mathbb{R}_+) \) such that

\[
\lim_{|x| \to \infty} \sup_{t \in [0, \infty)} \mathcal{L}V_2(t, x) = -\infty.
\]

(3.32)

Obviously, \( (H^2) \) implies \( (H^2_w) \).

**Lemma 3.10** Let \( \{ X(t), t \geq 0 \} \) be the unique solution of the SDE (2.1). Then, its transition semigroup \( \{ P_{s,t} \} \) is \( \theta \)-periodic.

**Proof** Define

\[
\overline{B}(t) = B(t + \theta) - B(\theta).
\]
Then, we obtain by (2.1) and (2.2) that

\[ X(t + \theta) \]
\[ = X(0) + \int_0^t \int_{|u| < 1} H(s, X(s), u) \tilde{N}(ds, du) + \int_0^t \int_{|u| \geq 1} G(s, X(s), u)N(ds, du) \]
\[ + \int_0^t \int_{|u| < 1} b(s, X(s))ds + \int_0^t \sigma(s, X(s))dB(s) \]
\[ + \int_0^t + \int_{|u| < 1} H(s, X(s), u) \tilde{N}(ds, du) + \int_0^t \int_{|u| \geq 1} G(s, X(s), u)N(ds, du) \]
\[ = X(\theta) + \int_0^t \int_{|u| < 1} b(\theta + r, X(\theta + r))dr + \int_0^t \sigma(\theta + r, X(\theta + r))d\mathbb{B}(r) \]
\[ + \int_0^t \int_{|u| < 1} H(\theta + r, X(\theta + r), u) \tilde{N}(dr, du) \]
\[ + \int_0^t \int_{|u| \geq 1} G(\theta + r, X(\theta + r), u)N(dr, du) \]
\[ = X(\theta) + \int_0^t \int_{|u| < 1} b(r, X(\theta + r))dr + \int_0^t \sigma(r, X(\theta + r))d\mathbb{B}(r) \]
\[ + \int_0^t \int_{|u| < 1} H(r, X(\theta + r), u) \tilde{N}(dr, du) \]
\[ + \int_0^t \int_{|u| \geq 1} G(r, X(\theta + r), u)N(dr, du). \]

Hence,

\[ dX(t + \theta) = b(t, X(t + \theta))dt + \sigma(t, X(t + \theta))d\mathbb{B}(t) \]
\[ + \int_{|u| < 1} H(r, X(r + \theta), u) \tilde{N}(dr, du) \]
\[ + \int_{|u| \geq 1} G(r, X(r + \theta), u)N(dr, du). \]  \hspace{1cm} (3.33)

By (3.33), we find that \( \{X(t + \theta), t \geq 0\} \) is a weak solution of the SDE (2.1). From the weak uniqueness of solutions, we know that \( \{X(t), t \geq 0\} \) and \( \{X(t + \theta), t \geq 0\} \) have the same distribution. Therefore,

\[ P(s, x, t, A) = P(s + \theta, x, t + \theta, A), \quad \forall 0 \leq s < t, x \in \mathbb{R}^m, A \in \mathcal{B}(\mathbb{R}^m). \]

\[ \square \]

**Theorem 3.11** Suppose that (A1), (A2), (B) and (H2) hold. Then, the SDE (2.1) has a \( \theta \)-periodic solution.
Proof Let \( \{X(t), t \geq 0\} \) be the unique solution of the SDE (2.1). For \( n \in \mathbb{N} \), define
\[
\beta_n = \inf \{ t \in [0, \infty) : |X(t)| \geq n \}.
\]
For \( t \geq 0 \), by Itô’s formula, we get
\[
\mathbb{E}[V_2(t \wedge \beta_n, X(t \wedge \beta_n))] = \mathbb{E}[V_2(0, X(0))] + \mathbb{E} \left[ \int_0^{t \wedge \beta_n} \mathcal{L}V_2(s, X(s))ds \right].
\] (3.34)

Define
\[
A_n := -\sup_{|x| > n, t \in [0, \infty)} \mathcal{L}V_2(t, x).
\]
By (3.32), we get
\[
\lim_{n \to \infty} A_n = \infty.
\] (3.35)

We have that
\[
\mathcal{L}V_2(t, X(s)) \leq -I_{\{|X(s)| \geq n\}} A_n + \sup_{|x| < n, t \in [0, \infty)} \mathcal{L}V_2(t, x).
\]
By (3.30), (3.31) and (3.34), we know that there exist positive constants \( c_1 \) and \( c_2 \) such that for large \( n \),
\[
\mathbb{E} \left[ \int_0^{t \wedge \beta_n} I_{\{|X(s)| \geq n\}}ds \right] \leq \frac{c_1 t + c_2}{A_n}.
\] (3.36)

Denote \( B_n = \{ x \in \mathbb{R}^m : |x| < n \} \) and \( B^c_n = \{ u \in \mathbb{R}^m : |x| \geq n \} \). Letting \( n \to \infty \) in (3.36), we obtain by (3.35) that
\[
\lim_{n \to \infty} \lim_{T \to \infty} \frac{1}{T} \int_0^T P(0, x, u, B^c_n)du = 0.
\] (3.37)

By (3.31), there exists \( \lambda > 0 \) such that
\[
\mathcal{L}V_2(t, x) \leq \lambda, \ \forall t \geq 0, x \in \mathbb{R}^m.
\] (3.38)

By (3.34) and (3.38), we get
\[
\mathbb{E}[V_2(t, X(t))] \leq \lambda t + V_2(0, x).
\]
Together with Chebyshev’s inequality, this implies that
\[
P(0, x, t, B^c_n) \leq \frac{\lambda t + V_2(0, x)}{\inf_{|x| > n, t \in [0, \infty)} V_2(t, x)}.
\] (3.39)
By (3.30) and (3.39), we find that there exists a sequence of positive integers \( \gamma_n \uparrow \infty \) such that

\[
\lim_{n \to \infty} \left\{ \sup_{x \in B_{\gamma_n}, t \in (0, \theta)} P(0, x, t, B_c^n) \right\} = 0. \tag{3.40}
\]

By (3.37), (3.40), Lemma 3.10 and Khasminskii (2012, Theorem 3.2 and Remark 3.1), we conclude that the SDE (2.1) has a \( \theta \)-periodic solution. Here, we would like to call the reader’s attention to a missing condition in Khasminskii (2012, Theorem 3.2), which was pointed out by Hu and Xu recently. According to Hu and Xu (2018, Theorem 2.1 and Remark A.1), Khasminskii (2012, Theorem 3.2 and Remark 3.1) holds under the additional assumption that \( \{ P_{s,t} \} \) is a Feller semigroup. By Theorem 3.4, \( \{ P_{s,t} \} \) is a Feller semigroup and hence we can apply (Khasminskii 2012, Theorem 3.2 and Remark 3.1) to show that the SDE (2.1) has a \( \theta \)-periodic solution. \( \square \)

### 3.4 Uniqueness of Periodic Solutions

In this subsection, we investigate the uniqueness of periodic solutions of the SDE (2.1). We put the following assumption, which implies conditions \((H^2)\) and \((H^3)\).

\( (H) \) There exists \( V \in C^{1,2}([0, \infty) \times \mathbb{R}^m; \mathbb{R}_+) \) satisfying the following conditions.

(i)

\[
\lim_{|x| \to \infty} \left[ \inf_{t \in [0, \infty)} V(t, x) \right] = \infty,
\]

and

\[
\lim_{n \to \infty} \sup_{|x| > n, t \in [0, \infty)} \mathcal{L}V(t, x) = -\infty.
\]

(ii) For any \( \rho \geq 1 \), there exists \( W_\rho(t, x) \in B_{b,loc}([0, \infty) \times \mathbb{R}^m; \mathbb{R}_+) \) satisfying for each \( n \in \mathbb{N} \) there exists \( R_n \in L^1_{loc}((0, \infty); \mathbb{R}_+) \) such that for any \( t \in [0, \infty) \) and \( x, y \in \mathbb{R}^m \) with \( |x| \vee |y| \leq n \),

\[
|W_\rho(t, x) - W_\rho(t, y)|^2 \leq R_n(t)|x - y|^2,
\]

and

\[
\sup_{x \in \mathbb{R}^m, t \in [0, \infty)} \mathcal{L}V^\bullet_\rho(t, x) < \infty,
\]

\[
V^\bullet_\rho(t, x) \leq \langle W_\rho, \nabla_x V^\bullet_\rho(t, x) \rangle, \quad \forall t \geq 0, x \in \mathbb{R}^m.
\]

**Lemma 3.12** Let \( 0 \leq s < t < t_1 \). If a Markovian semigroup \( \{ P_{s,t} \} \) is strongly Feller at \( (t, t_1) \) and irreducible at \( (s, t) \), then it is regular at \( (s, t_1) \).
Proof Suppose that \( \{P_{s,t}\} \) is strongly Feller at \((t, t_1)\) and irreducible at \((s, t)\). Assume that for some \(x_0 \in \mathbb{R}^m\) and \(A \in \mathcal{B}(\mathbb{R}^m)\), \(P_{s,t_1}(x_0, A) > 0\). Since

\[
P_{s,t_1}(x_0, A) = \int_{\mathbb{R}^m} P(s, x_0, t, dy) P(t, y, t_1, A),
\]

there exists \(y_0 \in \mathbb{R}^m\) such that \(P(t, y_0, t_1, A) > 0\). Since \(P_{s,t_1}\) is strongly Feller at \((t, t_1)\), \(P_{s,t_1}I_A \in C_b(\mathbb{R}^m)\). Hence, there exists \(r_0 > 0\) such that \(P_{s,t_1}(y, A) > 0\) for all \(y \in B(y_0, r_0)\), where \(B(y_0, r_0) := \{y \in \mathbb{R}^m : |y - y_0| < r_0\}\). Consequently, for arbitrary \(x \in \mathbb{R}^m\), we have

\[
P_{s,t_1}(x, A) = \int_{\mathbb{R}^m} P(s, x, t, dy) P(t, y, t_1, A)
\geq \int_{B(y_0, r_0)} P(s, x, t, dy) P(t, y, t_1, A),
\]

where we have used the fact that \(P_{s,t}(x, B(y_0, r_0)) > 0\) and \(P_{s,t_1}(y, A) > 0\) for all \(y \in B(y_0, r_0)\). Thus, if \(P_{s,t_1}(x_0, A) > 0\) for some \(x_0 \in \mathbb{R}^m\) then \(P_{s,t_1}(x, A) > 0\) for all \(x \in \mathbb{R}^m\). Therefore, the regularity of \(\{P_{s,t}\}\) at \((s, t_1)\) follows. 

\[\square\]

Theorem 3.13 Let \(\{P_{s,t}\}\) be a stochastically continuous \(\theta\)-periodic Markovian semigroup and \(\{\mu_s\}\) be a family of \(\theta\)-periodic (probability) measures with respect to \(\{P_{s,t}\}\). If \(\{P_{s,t}\}\) is regular at \((s, s + \theta)\) for any \(s \in [0, \theta)\), then \(\{\mu_s\}\) is the unique \(\theta\)-periodic measures with respect to \(\{P_{s,t}\}\).

Proof Let \(\{\mu_s\}\) be a family of \(\theta\)-periodic measures with respect to \(\{P_{s,t}\}\). Suppose that \(\{P_{s,t}\}\) is regular at \((s, s + \theta)\) for any \(s \in [0, \theta)\).

Step 1. We first show that \(\mu_s\) is ergodic for any \(s \geq 0\). That is, if \(A \in \mathcal{B}(\mathbb{R}^m)\) and

\[
P(s, x, s + \theta, A) = I_A(x), \quad \mu_s - a.s., \tag{3.41}
\]

then \(\mu_s(A) = 0\) or \(\mu_s(A) = 1\).

Let \(A \in \mathcal{B}(\mathbb{R}^m)\) satisfying \(\mu_s(A) > 0\). We will show that \(\mu_s(A) = 1\). By (3.41), we get

\[
\mu_s(\{x \in A : P(s, x, s + \theta, A) = 1\}) = \mu_s(A).
\]

Then, there exists \(x_0 \in A\) such that \(P(s, x_0, s + \theta, A) = 1\). Since all probabilities \(P(s, x, s + \theta, A), x \in \mathbb{R}^m\), are mutually equivalent, we get \(P(s, x, s + \theta, A) = P(s, x_0, s + \theta, A) = 1, \forall x \in \mathbb{R}^m\). It follows that

\[
\mu_s(A) = \int_{\mathbb{R}^m} P(s, y, s + \theta, A)\mu_s(dy) = 1.
\]
Step 2. Next we show that for any $s \geq 0$ and $\varphi \in L^2(\mathbb{R}^m; \mu_s)$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} P_{s,s+i\theta}\varphi = \int_{\mathbb{R}^m} \varphi d\mu_s \text{ in } L^2(\mathbb{R}^m; \mu_s). \quad (3.42)$$

Note that

$$\left\| \frac{1}{n} \sum_{i=1}^{n} P_{s,s+i\theta}\varphi \right\|_{L^2(\mathbb{R}^m; \mu_s)} \leq \frac{1}{n} \sum_{i=1}^{n} \left( \int_{\mathbb{R}^m} P_{s,s+i\theta}(\varphi^2) d\mu_s \right)^{1/2} = \left( \int_{\mathbb{R}^m} \varphi^2 d\mu_s \right)^{1/2}. \quad (3.43)$$

Hence, in order to prove (3.42), it is sufficient to show that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} P_{s,s+i\theta}\varphi = \int_{\mathbb{R}^m} \varphi d\mu_s \text{ weakly in } L^2(\mathbb{R}^m; \mu_s). \quad (3.43)$$

Define $\Xi := \mathbb{R}^Z$. Let $\mathcal{G}$ be the $\sigma$-algebra generated by the set of all cylindrical sets on $\Xi$. By the Kolmogorov extension theorem, there exists a unique probability measure $\mathbb{P}_{\mu_s}$ on $(\Xi, \mathcal{G})$ such that

$$\mathbb{P}_{\mu_s}((\omega \in \Xi : \omega_{n_1} \in A_1, \omega_{n_2} \in A_2, \ldots, \omega_{n_k} \in A_k, n_1 < n_2 < \cdots < n_k))$$

$$= \int_{A_1} \mu_s(dx_1) \int_{A_2} P(s, x_1, s + \theta, dx_2) \cdots \int_{A_k} P(s, x_{k-1}, s + (k-1)\theta, dx_k),$$

$$\forall A_i \in \mathcal{B}(\mathbb{R}^m), \ 1 \leq i \leq k, \ k \in \mathbb{N}.$$ 

Define $\Theta : \Xi \to \Xi$ by $(\Theta \omega)_l = \omega_{l+1}$, $l \in \mathbb{Z}$. Then, $\Theta$ is a measure preserving transformation. Let $\varphi \in L^2(\mathbb{R}^m; \mu_s)$ and define

$$\xi = \varphi(\omega_0). \quad (3.44)$$

We have $\xi \in L^2(\Omega, \mathcal{F}, \mathbb{P}_{\mu_s})$. Then, we obtain by Birkhoff’s ergodic theorem (cf. Petersen 1983, Theorem 2.3) that there exits $\xi^* \in L^2(\Xi, \mathcal{G}, \mathbb{P}_{\mu_s})$ such that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \xi(\Theta^i) = \xi^*, \ P_{\mu_s}$-a.s. and in $L^2(\Xi, \mathcal{G}, \mathbb{P}_{\mu_s}). \quad (3.45)$$

Define

$$U_i \eta(\omega) = \eta(\Theta^i \omega), \ \eta \in L^2(\Xi, \mathcal{G}, \mathbb{P}_{\mu_s}), \ \omega \in \Xi, \ i \in \mathbb{N}. \quad (3.46)$$
Then, we obtain by (3.45) that
\[ U_i \xi^* = \xi^*, \quad i \in \mathbb{N}. \] (3.47)

Define \( G_j = \sigma \{ \omega_u : u \leq j \}, j \in \mathbb{Z} \), and \( G_{[r,j]} = \sigma \{ \omega_u : r \leq u \leq j \} \) for \( r \leq j \in \mathbb{Z} \). Following the argument of the proof of Da Prato and Zabczyk (1996, Lemma 3.2.2), we can show that for any \( \eta \) which is \( G_{[-j,j]} \)-measurable, \( j \in \mathbb{N} \),
\[ \mathbb{E}_{\mu_s}[|E_{\mu_s}(U_j \eta|G_{[0,0]}) - \xi^*|^2] \leq 10 \mathbb{E}_{\mu_s}[|\eta - \xi^*|^2]. \]

Following the argument of the proof of Da Prato and Zabczyk (1996, Proposition 2.2.1), we can show that for arbitrary \( F \in G \) and \( \varepsilon > 0 \) there exists a cylindrical set \( C \) such that
\[ \mathbb{P}_{\mu_s}(F \setminus C) + \mathbb{P}_{\mu_s}(C \setminus F) < \varepsilon. \]
Then, there exists a sequence \( \{\eta_j\} \) of \( G_{[-j,j]} \)-measurable elements of \( L^2(\Xi, G, P_{\mu_s}) \) such that
\[ \lim_{j \to \infty} \mathbb{E}_{\mu_s}[U_j \eta_j|G_{[0,0]}] = \xi^* \text{ in } L^2(\Xi, G, P_{\mu_s}). \]

Moreover, there exists \( \{\varphi_j\} \subset L^2(\Xi, G, \mathbb{P}_{\mu_s}) \) such that
\[ \mathbb{E}_{\mu_s}[U_j \eta_j|G_{[0,0]}] = \varphi_j(\omega_0), \quad \mathbb{P}_{\mu_s} \text{-a.s.} \]
Without loss of generality, we can assume that
\[ \lim_{j \to \infty} \varphi_j(\omega_0) = \xi^*, \quad \mathbb{P}_{\mu_s} \text{-a.s. and in } L^2(\Xi, G, P_{\mu_s}). \]

Define
\[ \phi(x) = \begin{cases} \lim_{j \to \infty} \varphi_j(x), & \text{if the limit exists,} \\ 0, & \text{otherwise.} \end{cases} \]
Then, we have \( \xi^*(\omega) = \phi(\omega_0), \mathbb{P}_{\mu_s} \text{-a.s.} \). By (3.46) and (3.47), we get
\[ \phi(\omega_0) = \xi^*(\omega) = \xi^*(\Theta^j \omega) = \phi((\Theta^j \omega)_0) = \phi(\omega_i), \quad i \in \mathbb{N}. \] (3.48)
We claim that \( \phi \) is a constant. In fact, define \( \Lambda = \phi^{-1}(\alpha, \infty) \) for \( \alpha \in \mathbb{R} \). Then, we obtain by (3.48) that
\[ P(s, x, s + \theta, \Lambda) = \mathbb{E}_{s,x} [\chi_{\Lambda}(\omega_1)] = \mathbb{E}_{s,x} [\chi_{\Lambda}(\omega_0)] = \chi_{\Lambda}(x), \quad \mu_s \text{-a.s.} \]
Since $\mu_s$ is ergodic by Step 1, we have that $\mu(\Lambda) = 0$ or 1. Since $\alpha \in \mathbb{R}$ is arbitrary, we conclude that $\phi$ is a constant. Thus, $\xi^s$ is a constant. Therefore, we obtain by (3.44) and (3.45) that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \varphi(\omega_i) = \int_{\mathbb{R}^m} \varphi \, d\mu_s \text{ in } L^2(\Sigma, \mathcal{G}, P_{\mu_s}).$$

(3.49)

Let $\zeta \in L^2(\mathbb{R}^m; \mu_s)$. Then, we obtain by (3.49) that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \langle P_{s, s+i\theta} \varphi, \zeta \rangle_{L^2(\mathbb{R}^m; \mu_s)} = \int_{\mathbb{R}^m} \varphi \, d\mu_s \int_{\mathbb{R}^m} \zeta \, d\mu_s.$$

Since $\zeta \in L^2(\mathbb{R}^m; \mu_s)$ is arbitrary, the proof of (3.43) is complete.

Step 3. Finally, we show that $\{\mu_s\}$ is the unique $\theta$-periodic measures. Suppose that $\{\mu'_s\}$ is a different family of $\theta$-periodic measures with respect to $\{P_{s,t}\}$. Then, there exist $s \geq 0$ and $\Gamma \in B(\mathbb{R}^m)$ such that

$$\mu_s(\Gamma) \neq \mu'_s(\Gamma).$$

By Step 2, there exists a sequence $\{T_n \uparrow \infty\}$ such that

$$\lim_{n \to \infty} \frac{1}{T_n} \sum_{i=1}^{T_n} p(s, x, s + i\theta, \Gamma) = \mu_s(\Gamma), \mu_s-a.s.,$$

and

$$\lim_{n \to \infty} \frac{1}{T_n} \sum_{i=1}^{T_n} p(s, x, s + i\theta, \Gamma) = \mu'_s(\Gamma), \mu'_s-a.s.$$

Define

$$\left\{ x \in \mathbb{R}^m : \lim_{n \to \infty} \frac{1}{T_n} \sum_{i=1}^{T_n} p(s, x, s + \theta, \Gamma) = \mu_s(\Gamma) \right\} = A,$$

and

$$\left\{ x \in \mathbb{R}^m : \lim_{n \to \infty} \frac{1}{T_n} \sum_{i=1}^{T_n} p(s, x, s + \theta, \Gamma) = \mu'_s(\Gamma) \right\} = B.$$

It is clear that $A \cap B = \emptyset$ and $\mu_s(A) = \mu'_s(B) = 1$. Thus, $\mu_s$ and $\mu'_s$ are singular. However, since $\{P_{s,t}\}$ is regular at $(s, s+\theta)$ and $\{\mu_s\}$ and $\{\mu'_s\}$ are $\theta$-periodic measures (cf. (3.1)), we must have that $\mu_s$ and $\mu'_s$ are equivalent to $P(s, x, s + \theta, \cdot), x \in \mathbb{R}^m$. We have arrived at a contradiction. $\Box$
Now, we can state and prove the main result of this paper.

**Theorem 3.14** Suppose that (A3), (A4) and (H) hold. Then,

(i) The SDE (2.1) has a unique $\theta$-periodic solution \{X(t), t \geq 0\};

(ii) The Markovian transition semigroup \{Ps,t\} of \{X(t), t \geq 0\} is strongly Feller and irreducible;

(iii) Let $\mu_s(A) = P(X(s) \in A)$ for $A \in \mathcal{B}(\mathbb{R}^m)$ and $s \geq 0$. Then, for any $s \geq 0$ and $\varphi \in L^2(\mathbb{R}^m; \mu_s)$, we have

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} Ps,s+i\theta \varphi = \int_{\mathbb{R}^m} \varphi d\mu_s \text{ in } L^2(\mathbb{R}^m; \mu_s).$$

**Proof** By (A3), (H) and Theorem 2.2, we know that the SDE (2.1) has a unique solution. Further, by (A3), (A4), (H) and Theorem 3.8, we know that \{Ps,t\} is strongly Feller. Hence, we conclude that the SDE (2.1) has a $\theta$-periodic solution by following the argument of the proof of Theorem 3.11. The uniqueness of the $\theta$-periodic solution is a direct consequence of Theorems 3.8 and 3.9 and Lemmas 3.12 and 3.13. Finally, the last assertion of the theorem follows from the proof of Lemma 3.13 (see (3.42)). ☐

4 Examples

In this section, we use three examples to illustrate Theorem 3.14.

**Example 4.1** Suppose that the coefficient functions $b, \sigma, H, G$ of the SDE (2.1) are all Borel measurable and satisfy (2.2), (A3) and (A4). In addition, we assume that there exist $r, c_1, c_2 > 0$ such that

$$\langle b(t, x), x \rangle + |\sigma(t, x)|^2 \leq -c_1|x|^r + c_2, \quad \forall t \in [0, \theta), x \in \mathbb{R}^m, \quad (4.1)$$

and for any $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$\int_{|u|<1} |H(t, x, u)|^2 v(du) + |x| \int_{|u|\geq1} |G(t, x, u)| v(du)$$

$$+ \int_{|u|\geq1} |G(t, x, u)|^2 v(du) \leq \varepsilon |x|^r + c_\varepsilon, \quad \forall t \in [0, \theta), x \in \mathbb{R}^m.$$ 

Note that in (4.1) we only assume that $r > 0$. This condition is weaker than the condition $r > 1$ that adopted in Xie and Zhang (2020, (2.11)).

Define

$$V(t, x) = |x|^2.$$

We will show that $V$ satisfies the condition (H). Without loss of generality, we assume that $\rho = 1$. Let

$$W_1(t, x) = \frac{x}{2}.$$
Then,

\[ V(t, x) = \langle W_1, V_x(t, x) \rangle. \]

We have

\[
\int_{\{|u|<1\}} [V(t, x + H(t, x, u)) - V(t, x) - \langle V_x(t, x), H(t, x, u) \rangle] v(du) \\
= \int_{\{|u|<1\}} \frac{1}{2} H^T(t, x, u) V_{xx}(x + \theta_1 H(t, x, u)) H(t, x, u) v(du) \\
\leq \int_{\{|u|<1\}} |H(t, x, u)|^2 v(du),
\]

and

\[
\int_{\{|u|\geq1\}} [V(t, x + G(t, x, u)) - V(t, x)] v(du) \\
= \int_{\{|u|\geq1\}} \left[ 2\langle x, G(t, x, u) \rangle + |G(t, x, u)|^2 \right] v(du) \\
\leq 2|x| \int_{\{|u|\geq1\}} |G(t, x, u)| v(du) + \int_{\{|u|\geq1\}} |G(t, x, u)|^2 v(du).
\]

Hence, for any \( \varepsilon > 0 \), we have

\[
\mathcal{L}V(t, x) = \langle V_x(t, x), b(t, x) \rangle + \frac{1}{2} \text{trace}(\sigma^T(t, x)V_{xx}(t, x)\sigma(t, x)) \\
+ \int_{\{|u|<1\}} [V(t, x + H(t, x, u)) - V(t, x) - \langle V_x(t, x), H(t, x, u) \rangle] v(du) \\
+ \int_{\{|u|\geq1\}} [V(t, x + G(t, x, u)) - V(t, x)] v(du) \\
\leq -c_1|x|^r + c_2 + 2(\varepsilon|x|^r + c_\varepsilon).
\]

Then,

\[
\lim_{n \to \infty} \sup_{|x|>n, t\in[0,\infty)} \mathcal{L}V(t, x) = -\infty,
\]

and

\[
\sup_{x\in\mathbb{R}^m, t\in[0,\infty)} \mathcal{L}V(t, x) < \infty.
\]

Thus, the condition (H) is satisfied. Therefore, all assertions of Theorem 3.14 hold.
**Example 4.2** (Stochastic Lorenz equation) The Lorenz equation is a remarkable mathematical model for atmospheric convection, which was introduced by Lorenz (1963). In recent years, many papers have been devoted to the Lorenz equation with noises (cf. Agarwal and Wettlaufer 2016 and the references therein). We consider the following Lorenz equation with multiplicative Lévy noise:

\[
\begin{align*}
\frac{dX_1(t)}{dt} &= (-\alpha(t) X_1(t) + \alpha(t) X_2(t)) + \sum_{j=1}^{3} \sigma_{1j}(t, X(t))dB_j(t) \\
&\quad + \int_{\{|u|<1\}} H_1(t, X(t), u, \tilde{N}(dt, du) + \int_{\{|u|\geq 1\}} G_1(t, X(t), u, N(dt, du), \\
\frac{dX_2(t)}{dt} &= (\mu(t) X_1(t) - X_2(t) - X_1(t) X_3(t)) + \sum_{j=1}^{3} \sigma_{2j}(t, X(t))dB_j(t) \\
&\quad + \int_{\{|u|<1\}} H_2(t, X(t), u, \tilde{N}(dt, du) + \int_{\{|u|\geq 1\}} G_2(t, X(t), u, N(dt, du), \\
\frac{dX_3(t)}{dt} &= (-\beta(t) X_3(t) + X_1(t) X_2(t)) + \sum_{j=1}^{3} \sigma_{3j}(t, X(t))dB_j(t) \\
&\quad + \int_{\{|u|<1\}} H_3(t, X(t), u, \tilde{N}(dt, du) + \int_{\{|u|\geq 1\}} G_3(t, X(t), u, N(dt, du). \\
\end{align*}
\]

We assume that \(\alpha(t), \beta(t), \mu(t) : [0, \infty) \to \mathbb{R}_+\) are continuously differentiable with period \(\theta\) and satisfy

\[
\min\{\alpha(t) : t \in [0, \theta]\} > 0, \quad \min\{\beta(t) : t \in [0, \theta]\} > 0.
\]

The functions \(\sigma(t, x) : [0, \infty) \times \mathbb{R}^3 \to \mathbb{R}^{3 \times 3}, H(t, x, u) : [0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^l \to \mathbb{R}^3\) and \(G(t, x, u) : [0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^l \to \mathbb{R}^3\) are all Borel measurable and satisfy (2.2), (A3) and (A4). Moreover, for any \(\varepsilon > 0\) there exists \(c_\varepsilon > 0\) such that

\[
|\sigma(t, x)|^2 + \int_{\{|u|<1\}} |H(t, x, u)|^2 \nu(du) \\
+ \int_{\{|u|\geq 1\}} |G(t, x, u)|^2 \nu(du) \leq \varepsilon |x|^2 + c_\varepsilon, \quad \forall t \in [0, \theta], x \in \mathbb{R}^m.
\]

Define

\[
V(t, x) = \dot{x}_1^2 + \dot{x}_2^2 + (x_3 - \alpha(t) - \mu(t))^2.
\]

We will show that \(V\) satisfies the condition (H). Without loss of generality, we assume that \(\rho = 1\). The verification for the case that \(\rho > 1\) is completely similar. We have
that
\[ \frac{\partial V}{\partial x_1} = 2x_1, \quad \frac{\partial V}{\partial x_2} = 2x_2, \quad \frac{\partial V}{\partial x_3} = 2(x_3 - \alpha(t) - \mu(t)). \]

Let
\[ W_1(t, x) = \frac{1}{2}(x_1, x_2, x_3 - \alpha(t) - \mu(t)). \]

Then,
\[ V(t, x) = \langle W_1, V_x(t, x) \rangle. \]

For any \( \varepsilon > 0 \), we have
\[
\mathcal{L}V(t, x) = 2(\alpha(t) + \mu(t) - x_3)(\alpha'(t) + \mu'(t)) \\
- 2\alpha(t)x_1^2 - 2x_2^2 - 2\beta(t)(x_3^2 - (\alpha(t) + \mu(t))x_3) + |\sigma(t, x)|^2 \\
+ \sum_{i=1}^{3} \int_{|u|<1} |H_i(t, x, u)|^2 v(du) + \sum_{i=1}^{2} \int_{|u|\geq 1} [(x_i + G_i(t, x, u))^2 - x_i^2] v(du) \\
+ \int_{|u|\geq 1} [(x_3 + G_3(t, x, u) - \alpha(t) - \mu(t))^2 - (x_3 - \alpha(t) - \mu(t))^2] v(du) \\
\leq 2(\alpha(t) + \mu(t) + |x_3|)(|\alpha'(t)| + |\mu'(t)|) \\
- 2\left[ \alpha(t)x_1^2 + x_2^2 + \beta(t)x_3^2 - \beta(t)(\alpha(t) + \mu(t))|x_3| \right] + |x|^2 + c_\varepsilon \\
+ 2(|x_1| + |x_2| + |x_3| + \alpha(t) + \mu(t)) |v(|u| \geq 1)|^{1/2} |\varepsilon|^{1/2} (|x|^2 + c_\varepsilon)^{1/2}. 
\]

Then,
\[
\lim_{n \to \infty} \sup_{|x|>n, t \in [0, \infty)} \mathcal{L}V(t, x) = -\infty, 
\]
and
\[
\sup_{x \in \mathbb{R}^m, t \in [0, \infty)} \mathcal{L}V(t, x) < \infty. 
\]

Thus, the condition (H) is satisfied. Therefore, the stochastic Lorenz equation (4.2) has a unique \( \theta \)-periodic solution \( \{X(t), t \geq 0\} \) and assertions (ii) and (iii) of Theorem 3.14 hold.

**Example 4.3** (Equation of the lemniscate of Bernoulli with Lévy noise) In this example, we consider the stochastic equation of the lemniscate of Bernoulli, which generalizes (Chen et al. 2019, Example 3.20) to the non-autonomous case with Lévy noise.

For \( x = (x_1, x_2) \in \mathbb{R}^2 \), define
\[ I(x) = (x_1^2 + x_2^2)^2 - 4(x_1^2 - x_2^2). \]
Let
\[ \mathcal{V}(I) = \frac{I^2}{2(1 + I^2)^{3/4}}, \quad \mathcal{H}(I) = \frac{I}{(1 + I^2)^{3/8}}. \]

Consider the vector field
\[ b(x) = -\left[ \mathcal{V}_x(I) + \left( \frac{\partial \mathcal{H}(I)}{\partial x_2}, -\frac{\partial \mathcal{H}(I)}{\partial x_1} \right)^T \right]. \]

We have
\[
\begin{align*}
\frac{d\mathcal{V}(I)}{dI} &= \frac{I(I^2 + 4)}{4(1 + I^2)^{7/4}}, \\
\frac{d\mathcal{H}(I)}{dI} &= \frac{I^2 + 4}{4(1 + I^2)^{11/4}}, \\
\frac{\partial I}{\partial x_1} &= 4x_1(x_1^2 + x_2^2) - 8x_1, \\
\frac{\partial I}{\partial x_2} &= 4x_2(x_1^2 + x_2^2) + 8x_2.
\end{align*}
\]

Define
\[ f(I) = \frac{d\mathcal{V}(I)}{dI}, \quad g(I) = \frac{d\mathcal{H}(I)}{dI}. \]

Then,
\[ \mathcal{V}_x(I) = \frac{d\mathcal{V}(I)}{dI} \left( \frac{\partial I}{\partial x_1}, \frac{\partial I}{\partial x_2} \right)^T, \]

and
\[
\begin{align*}
b_1(x) &= -f(I)(4x_1(x_1^2 + x_2^2) - 8x_1) - g(I)(4x_2(x_1^2 + x_2^2) + 8x_2), \\
b_2(x) &= -f(I)(4x_2(x_1^2 + x_2^2) + 8x_2) - g(I)(-4x_1(x_1^2 + x_2^2) + 8x_1).
\end{align*}
\]

We consider the following SDE:
\[
\begin{align*}
dX_1(t) &= b_1(X(t))dt + \sigma_{11}(t, X(t))dB_1(t) + \sigma_{12}(t, X(t))dB_2(t) \\
&\quad + \int_{\{|u| < 1\}} H_1(t, X(t) - u)\tilde{N}(dt, du) + \int_{\{|u| \geq 1\}} G_1(t, x(t) - u)N(dt, du), \\
\end{align*}
\]
\[
\begin{align*}
dX_2(t) &= b_2(X(t))dt + \sigma_{21}(t, X(t))dB_1(t) + \sigma_{22}(t, X(t))dB_2(t) \\
&\quad + \int_{\{|u| < 1\}} H_2(t, X(t) - u)\tilde{N}(dt, du) + \int_{\{|u| \geq 1\}} G_2(t, x(t) - u)N(dt, du),
\end{align*}
\]

where \( \sigma(t, x) : [0, \infty) \times \mathbb{R}^2 \to \mathbb{R}^{2 \times 2}, H(t, x, u) : [0, \infty) \times \mathbb{R}^2 \times \mathbb{R}^l \to \mathbb{R}^2 \) and \( G(t, x, u) : [0, \infty) \times \mathbb{R}^2 \times \mathbb{R}^l \to \mathbb{R}^2 \) are all Borel measurable and satisfy (2.2), (A3).
and (A4). Moreover, for any \( \varepsilon > 0 \) there exists \( c_\varepsilon > 0 \) such that
\[
|\sigma(t, x)|^2 + \int_{|u| < 1} |H(t, x, u)|^2 v(du) + |x| \int_{|u| \geq 1} |G(t, x, u)| v(du) + \int_{|u| < 1} |G(t, x, u)|^2 v(du) \leq \varepsilon |x|^2 + c_\varepsilon, \quad \forall t \in [0, \theta), x \in \mathbb{R}^m.
\]

Define
\[
V(t, x) = \mathcal{V}(I(x)).
\]

We will show that \( V \) satisfies the condition (H). Without loss of generality, we assume that \( \rho = 1 \). The verification for the case that \( \rho > 1 \) is completely similar. Let
\[
W_1(t, x) = x.
\]

Then,
\[
V(t, x) \leq \frac{2I^2 + \frac{I^4}{4}}{(1 + I^2)^{7/4}} \leq \frac{(I + \frac{I^3}{4})(4x_1^2 + x_2^2)^2 - 8(x_1^2 - x_2^2))}{(1 + I^2)^{7/4}} = (W_1, V_x(t, x)), \quad \forall t \geq 0, x \in \mathbb{R}^2.
\]

By (4.4), (4.5) and (4.7), we get
\[
\lim_{|x| \to \infty} \frac{\langle V_x(t, x), b(x) \rangle}{|x|^2} < 0. \tag{4.9}
\]

By direct calculation, we find that there exists a constant \( \varrho_1 > 0 \) such that
\[
\left| \frac{\partial^2 V(x)}{\partial x_1^2} \right|, \left| \frac{\partial^2 V(x)}{\partial x_2^2} \right|, \left| \frac{\partial^2 V(x)}{\partial x_1 \partial x_2} \right| \leq \varrho_1, \quad \forall x \in \mathbb{R}^2. \tag{4.10}
\]

Then,
\[
\int_{|u| < 1} [V(t, x + H(t, x, u)) - V(t, x) - \langle V_x(t, x), H(t, x, u) \rangle] v(du)
\]
\[
= \int_{|u| < 1} \frac{1}{2} H^T(t, x, u)V_{xx}(x + \kappa_1 H(t, x, u))H(t, x, u) v(du)
\]
\[
\leq \varrho_1 \int_{|u| < 1} |H(t, x, u)|^2 v(du), \tag{4.11}
\]
where $\kappa_1 \in (0, 1)$. By (4.3), (4.4) and (4.6), we find that there exists a constant $\varrho_2 > 0$, which is independent of $t, x$, such that

\[
\int_{\{|u| \geq 1\}} [V(t, x + G(t, x, u)) - V(t, x)] v(du) \\
= \int_{\{|u| \geq 1\}} V_x(x + \kappa_2 G(t, x, u)) G(t, x, u) v(du) \\
\leq \varrho_2 \int_{\{|u| \geq 1\}} (1 + |x| + |G(t, x, u)|) |G(t, x, u)| v(du),
\]

(4.12)

where $\kappa_2 \in (0, 1)$.

Note that

\[
\mathcal{L} V(t, x) = \langle V_x(t, x), b(t, x) \rangle + \frac{1}{2} \text{trace}(\sigma^T(t, x) V_{xx}(t, x) \sigma(t, x)) \\
+ \int_{\{|u| < 1\}} [V(t, x + H(t, x, u)) - V(t, x) - \langle V_x(t, x), H(t, x, u) \rangle] v(du) \\
+ \int_{\{|u| \geq 1\}} [V(t, x + G(t, x, u)) - V(t, x)] v(du).
\]

Then, we obtain by (4.9)–(4.12) that

\[
\lim_{n \to \infty} \sup_{|x| > n, t \in [0, \infty)} \mathcal{L} V(t, x) = -\infty,
\]

and

\[
\sup_{x \in \mathbb{R}^m, t \in [0, \infty)} \mathcal{L} V(t, x) < \infty.
\]

Thus, the condition (H) is satisfied. Therefore, the stochastic equation of the lemniscate of Bernoulli (4.8) has a unique $\theta$-periodic solution $\{X(t), t \geq 0\}$ and assertions (ii) and (iii) of Theorem 3.14 hold.

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