Optimal Unbiased Estimation for Expected Cumulative Cost

Zhenyu Cui, Michael C. Fu, Yijie Peng, and Lingjiong Zhu

Abstract—We consider estimating an expected infinite-horizon cumulative cost/reward contingent on an underlying stochastic process by Monte Carlo simulation. An unbiased estimator based on truncating the cumulative cost at a random horizon is proposed. Explicit forms for the optimal distributions of the random horizon are given. Moreover, we characterize when the optimal randomized estimator is preferred over a fixed truncation estimator by considering the tradeoff between bias and variance. Numerical experiments substantiate the theoretical results.

Index Terms—unbiased simulation, optimal control, cumulative cost, efficiency of estimator.

I. INTRODUCTION

We consider estimating an expected cumulative cost \( \alpha := \mathbb{E} \left[ \int_0^\infty g(X_t, t) dt \right] \). In the special case when \( g(X_t, t) = e^{-ct} f(X_t) \) \((c > 0)\), \( \alpha \) is referred to as a cumulative discounted cost. In finance, this is related to simulating a cumulative (discounted) cash flow of a stochastic perpetuity (Fox and Glynn [1989], Blanchet and Sigman [2011]) or a mortgage-backed security (MBS) (Glasserman and Staum [2003]). In steady-state simulation, \( \alpha \) corresponds to the expected long-run behavior of the sample time-average, e.g., average waiting time in a queueing system (Whitt [2002]).

In our setting, \( \alpha \) is assumed to be unavailable in closed form, but Monte Carlo simulation can be used to estimate \( \alpha \). It is computationally infeasible to simulate the cumulative cost over an infinite horizon. Thus, a truncation technique is needed to estimate \( \alpha \), and batching is typically used to construct a confidence interval (Alexopoulos et al. [2016]). However, truncating at a fixed horizon generally leads to bias, which is difficult to quantify in statistical inference. We propose a randomized estimator that truncates at a random horizon to retrieve the unbiasedness. By doing so, an asymptotically valid confidence interval can be obtained by sampling i.i.d. sample paths of cumulative cost truncated at the random horizon, which can be justified by a classical central limit theorem. Since variability is introduced by the random horizon, the unbiasedness of the estimator may come at the cost of a larger variance. Therefore, variance reduction is particularly important for the randomization scheme, which motivates us to ask the following question: what is the optimal randomized unbiased estimator, and in what sense is it optimal?

The proposed estimator truncates the cumulative cost at a random horizon following a distribution independent of the underlying stochastic process. Our goal is to find an optimal distribution for the random horizon. We consider both a constrained optimization problem where the variance of the estimator is minimized subject to a fixed expected simulation cost/work, and an unconditional optimization problem where we seek to minimize the work-variance product (Glynn and Whitt [1992]) of the estimator. These are infinite-dimensional functional optimization problems over all possible distributions, which are difficult to solve numerically in general. However, we derive explicit forms for the optimal distributions of the random horizon by using the maximum principle for an optimal control problem (Bryson and Ho [1975]). The optimal distributions are in a shifted distribution class. For a discounted continuous cumulative cost contingent on an exponential Lévy process, the optimal randomization distributions are shifted exponential distributions. In Glynn [1983] where a special case of cumulative cost is studied, several families of randomization distributions are used to obtain unbiased estimator, and some variance reduction techniques are developed based on the asymptotic results of the randomized estimator. For a very special case, Glynn [1983] derived a randomization distribution supporting on \([0, \infty)\), which minimizes the asymptotic variance of the estimator as the computational budget goes to infinity. Our results show that this randomization distribution is not necessarily optimal in any fixed computational budget setting considered in this paper, because the support of our optimal randomization distribution is shifted away from zero under certain scenario.

Although the proposed randomized estimator eliminates bias, it inevitably increases the variance. We define a utility function as a linear combination of bias and variance. With a positive weight on the variance, we show that the optimal randomized estimator is less favorable than the fixed truncation estimator when the computational budget is sufficiently small. A threshold function of the computational budget for the weight of variance is provided, and the advantage of the optimal randomized estimator can be justified if the weight of the variance is less than this threshold.

Related work on using randomization in other settings to recover unbiasedness includes McLeish [2011] and Rhee and Glynn [2015] for stochastic differential equation (SDE) models. Rhee and Glynn [2015] use an infinite sum truncated at a random horizon independent of the underlying SDE to obtain an unbiased estimation of path functionals associated with the SDE. They derive an optimal distribution for the random horizon by minimizing the asymptotic variance of the estimator as the computational budget goes to infinity, whereas we consider a fixed computational budget setting. In addition, solving the optimal randomization distribution in Rhee and Glynn [2015] is an optimization problem with countably many arguments, which can be handled by a mathematical programming approach, whereas solving the optimal randomization distribution for our problem is a continuous-time functional optimization problem, which can be solved analytically using the maximum principle for an optimal control problem. Other work using randomization to eliminate bias includes unbiased estimation of Markov chain equilibrium expectations (Glynn and Rhee [2014]), unbiased stochastic optimization (Blanchet et al. [2015]), unbiased Bayesian inference (Lyne et al. [2015]), and unbiased maximum likelihood inference (Jacob et al. [2015]). None of the previous work provides analysis comparing randomized unbiased Monte Carlo (RUMC) method with traditional Monte Carlo (MC), e.g., a fixed truncation estimator, by taking both bias and variance into account.

Closely related to the RUMC method is the multi-level Monte Carlo (MLMC) method introduced in Giles [2008] and Giles [2015], which combines biased estimators of different step sizes to improve the convergence rate of traditional MC method. There is also recent interest in combining RUMC and MLMC for developing an unbiased
II. UNBIASED RANDOMIZED ESTIMATOR

We consider estimating the cumulative cost/reward:

\[ \alpha := \mathbb{E} \left[ \int_0^\infty g(X_s, s) ds \right], \]

where \( X := \{ X_s : s \geq 0 \} \) is the underlying stochastic process. This covers the special case \( g(X_s, s) := e^{-cS}f(X_s) (c > 0) \), which is frequently used in asset pricing. For example, \( \alpha \) can be the expected present value of a discounted cumulative cash flow contingent on the future value of an underlying asset.

Let \( N \) be a random horizon independent of the underlying stochastic process \( X \), and \( Q \) be the distribution of \( N \), which satisfies

\[ Q \in \mathcal{M}(\mathbb{R}^+) := \{ Q : Q(N \geq 0) = 1, Q(N > s) > 0, s \in [0, \infty) \}. \]

The proposed randomized unbiased estimator is

\[ I := \int_0^\infty g(X_s, s) \frac{1_{N>s}}{Q(N > s)} ds = \int_0^N g(X_s, s) \frac{1_{N>s}}{Q(N > s)} ds. \]  

Due to the independence of \( N \) and \( X \), the unbiasedness of the proposed estimator can be established straightforwardly by applying Fubini’s theorem:

\[ \mathbb{E} \left[ \int_0^\infty g(X_s, s) \frac{1_{N>s}}{Q(N > s)} ds \right] = \int_0^\infty \mathbb{E}[g(X_s, s)] ds =: \alpha. \]

In the main body of the paper, we ignore the technicality induced by possible discretization for simulating the continuous-time cost process. We consider the following three optimization problems:

1) minimize the variance of the estimator subject to a linear penalty on the computational cost/work;
2) minimize the variance of the estimator subject to a fixed pre-specified level of computational cost/work;
3) minimize the work-variance product of the estimator.

Large (small) computational cost/work corresponds to small (large) bias. The three optimizations are natural formulations of the tradeoff between bias and variance. It turns out that solving the earlier optimization problem helps solving the succeeding optimization problem(s). Throughout the paper, we assume the expectations and variances of all estimators are well-defined to avoid the problems of interest becoming meaningless.

A. Minimizing Variance with Penalty

We want to optimize over all possible distributions \( Q \in \mathcal{M}(\mathbb{R}^+) \) in order to minimize the variance of the estimator with a penalty for the computational cost:

\[ \inf_{Q \in \mathcal{M}(\mathbb{R}^+)} \left\{ \mathbb{Var} \left( \int_0^\infty g(X_s, s) \frac{1_{N>s}}{Q(N > s)} ds \right) + \lambda \mathbb{E}[Q(N)] \right\}, \quad \lambda > 0. \]

The following lemma rewrites the optimization problem (2), making it amenable for analysis.

**Lemma 1.** The optimization problem (2) is equivalent to

\[ \inf_{Q \in \mathcal{M}(\mathbb{R}^+)} \left\{ \int_0^\infty g(X_s, s) \frac{1_{N>s}}{Q(N > s)} ds + \mu \int_0^\infty Q(N > t) Q(N > s) ds \right\}, \quad \lambda > 0, \]

where

\[ \Gamma(s) := \int_0^\infty \mathbb{E}[g(X_t, t) g(X_s, s)] dt. \]

**Proof:** Notice that

\[ \mathbb{E} \left[ \left( \int_0^\infty g(X_s, s) \frac{1_{N>s}}{Q(N > s)} ds \right)^2 \right] = \int_0^\infty \mathbb{E}[g(X_s, s)] ds^2. \]

By Fubini’s theorem and the independence between \( X \) and \( N \), we have

\[ \int_0^\infty \mathbb{E}[g(X_s, s)] ds = \int_0^\infty \mathbb{E}[g(X_s, s)] ds = \int_0^\infty \mathbb{E}[g(X_s, s)] ds =: \alpha. \]

Since \( \int_0^\infty \mathbb{E}[g(X_s, s)] ds \) is independent of \( Q \), we can drop it in the optimization, which leads to the conclusion.

Following a similar procedure in the proof of Lemma 1, we have

\[ \mathbb{E} \left[ \left( \int_0^\infty g(X_s, s) ds \right)^2 \right] = \int_0^\infty \Gamma(s) ds \int_0^\infty Q(N > s) ds. \]

The random truncation increases the variance by noticing \( \int_0^\infty \Gamma(s) ds \leq \int_0^\infty \frac{\Gamma(s)}{Q(N > s)} ds \), but the increased variance is compensated by a decreased computational cost. The objective of the optimization problem (3) is a functional of \( \Gamma(s) \), \( \lambda \), and \( Q \). Thus, we expect the optimal randomization distribution \( Q^* \) for the optimization problem (3) should be determined by \( \Gamma(s) \) and \( \lambda \). Basically, \( \Gamma(s) \) captures how fast the cost process \( g(X_t, t) \) decays after time \( s \), while \( \lambda \) is the unit cost for computing the cumulative cost.

**Assumption 1.** \( \Gamma(s) \) is a non-negative and non-increasing smooth function.

The non-negativity and monotonicity in Assumption 1 can be justified if the cost process \( g(X_s, s) \) is non-negative (or non-positive) and \( \mathbb{E}[g(X_t, t) g(X_s, s)] \) is non-increasing in \( s \). In the case where the cost process has both positive and negative parts, we can decompose it into the difference of two non-negative processes and estimate the cumulative cost of both processes separately. Under Assumption 1, we have an explicit form for the optimal distribution given in the following theorem.
Theorem 1. Under Assumption 1, for the optimization problem (3), we have
\[
Q^*(N > s) = \begin{cases} 
1 & \text{for } s \leq s^*, \\
\sqrt{\frac{2(N-s)}{N}} & \text{for } s > s^*, 
\end{cases}
\] (4)
where \(s^* = \inf\{s \in [0, \infty) : \Gamma(s) \leq \lambda/2\}\). The minimum variance is given by
\[
\inf_{Q \in \mathcal{M}(R^+)} \left\{ 2 \int_0^\infty \frac{\Gamma(s)}{Q(N > s)} ds + \lambda \int_0^\infty Q(N > s) ds \right\} = 2 \int_0^{s^*} \Gamma(s) ds + 2 \sqrt{2\lambda} \int_{s^*}^\infty \sqrt{\Gamma(s) ds + \lambda s^*}.
\]

Proof: For the optimization problem (3), we have
\[
Q^*(N > s) = \arg \min \{x : L(x; s, \lambda)\},
\]
where
\[
L(x; s, \lambda) := \frac{2}{x} \Gamma(s) + \lambda x.
\]
Notice that the function \(L(x; s, \lambda)\) decreases for \(x < \sqrt{\frac{2(N-s)}{N}}\) and increases for \(x > \sqrt{\frac{2(N-s)}{N}}\). In addition, for any \(Q \in \mathcal{M}(R^+)\), \(Q(N > s)\) is required to decrease from 1 to 0 as \(s\) goes from 0 to \(\infty\). Then, we can calculate
\[
\inf_{Q \in \mathcal{M}(R^+)} \left\{ 2 \int_0^\infty \frac{\Gamma(s)}{Q(N > s)} ds + \lambda \int_0^\infty Q(N > s) ds \right\} = 2 \int_0^{s^*} \Gamma(s) ds + 2 \sqrt{2\lambda} \int_{s^*}^\infty \sqrt{\Gamma(s) ds + \lambda s^*}.
\]
Combining the arguments above leads to the conclusion. \(\square\)

Remark 1. It is straightforward to know that if \(\Gamma(0) \leq \lambda/2\), then \(s^* = 0\) and if \(\Gamma(0) > \lambda/2\), then \(s^* \in (0, \infty)\) and \(\Gamma(s^*) = \lambda/2\). We can see the following insight from the explicit form of the optimal distribution \(Q^*\): large \(\lambda\) and small \(\Gamma(s)\) correspond to small \(Q^*(N > s)\), which means the distribution of the random truncation concentrates on the domain where \(N\) is small. This insight intuitively makes sense. Large unit cost \(\lambda\) for computing the cumulative cost favors a small truncation size. Small \(\Gamma(s)\) roughly indicates that the cost process decays fast after time \(s\), which thus encourages us to put more computational effort before time \(s\). If \(\Gamma(s)\) is non-monotone, the optimal \(Q^*\) can be solved by an optimal control problem, which can be found in the appendix.

In the following, we consider the exponential Lévy process, which includes geometric Brownian motion (GBM) as a special case; see e.g. Fu et al. [2017]. Let \(Y_t\) be a Lévy process with the characteristic triplet \((\mu, \sigma^2, \nu)\), and its characteristic exponent is given by \(\phi(\cdot)\), which is uniquely characterized by the Lévy-Khintchine formula: \(\mathbb{E}[e^{i\theta Y_t}] = e^{\mathbb{E}e^{i\theta Y_t}}\). Then, \(X_t = e^{\beta Y_t}\) is an exponential Lévy process. Let \(f(x) = x^\beta\) for a fixed \(\beta\), so \(g(X_t, t) = e^{-\beta X_t}\).

Define
\[
\phi_1(\beta) := \phi(\beta) - c, \quad \phi_2(2\beta) := \phi(2\beta) - 2c,
\] and we also assume that \(\phi_1(\beta) < 0\) and \(\phi_2(2\beta) < 0\) in order for related integrals to be well-defined. Then, we have
\[
\mathbb{E}[g(X_t, t)g(X_s, s)] = \mathbb{E}[e^{-(t+s)\phi_1(\beta)Y_t + Y_s}] = e^{-(t+s)\phi_1(\beta) + \phi_2(2\beta)}.
\]

\[
\Gamma(s) = \int_s^\infty \mathbb{E}[g(X_t, t)g(X_s, s)] dt = \int_s^\infty e^{-(t-s)\phi_1(\beta) + \phi_2(2\beta)} dt = \frac{1}{\phi_1(\beta)} e^{-s\phi_2(2\beta)}.
\]

Corollary 1. If \(\{X_t\}\) is an exponential Lévy process with characteristic exponent \(\phi\) and \(f(x) = x^\beta\), then under the optimal randomization distribution \(Q^*\). \(N\) is a shifted exponential random variable with the probability density function given by
\[
q^*(s) := |\phi_2(2\beta)| \left[ \frac{1}{2\lambda \phi_1(\beta)} \right]^{-\frac{1}{2}} |\phi_1(\beta)|^{-\frac{1}{2}} \Gamma\left(\frac{1}{2}\phi_1(\beta)\right) |\phi_2(2\beta)| s \mathbb{I}_{\{s > s^*\}},
\]
where the optimal shift \(s^*\) is given by:
\[
s^* := \begin{cases} 
0 & \text{if } \lambda \geq 2\Gamma(0) = \frac{2}{|\phi_1(\beta)|}, \ s^* = 0, \text{Otherwise, } \\
\frac{|\phi_2(2\beta)|}{|\phi_1(\beta)|} \log \left(\frac{1}{2\lambda \phi_1(\beta)}\right) & \text{if } \lambda < \frac{2}{|\phi_1(\beta)|}, 
\end{cases}
\]
where \(\phi_1\) and \(\phi_2\) are given by (5).

Proof: Note that for \(\lambda \geq 2\Gamma(0) = \frac{2}{|\phi_1(\beta)|}\), \(s^* = 0\). Otherwise, \(\Gamma(s^*) = \frac{\lambda}{2}\) so that
\[
s^* = -\frac{1}{|\phi_2(2\beta)|} \log \left(\frac{1}{2\lambda \phi_1(\beta)}\right)
\]
We conclude that the optimal \(Q^*\) is given by
\[
Q^*(N > s) := \begin{cases} 
1 & \text{for } 0 \leq s \leq s^*, \\
\sqrt{\frac{2}{\lambda |\phi_1(\beta)|}} e^{-\frac{1}{2}|\phi_2(2\beta)| s} & \text{for } s > s^*, 
\end{cases}
\]
which completes the proof by differentiation. \(\square\)

B. Constrained Optimization

In this section, we consider the second optimization problem, in which we aim to minimize the variance of the randomized estimator given that the computational budget is fixed at a level \(m\):
\[
\inf_{Q \in \mathcal{M}(R^+):\mathbb{E}[Q(N)]=m} \left\{ \mathbb{E} \left[ \int_0^\infty g(X_s, s)^2 \frac{1_{\{N > s\}} ds}{Q(N > s)} \right] \right\}.
\] (6)
An explicit characterization for the optimal distribution is obtained in the following theorem by using the maximum principle of an optimal control problem.

Theorem 2. Under Assumption 1, for the optimization problem (6),
(i) If \(m > \int_0^\infty \sqrt{\Gamma(s)} ds/\sqrt{\Gamma(0)}\), then we have
\[
Q^*(N > s) := \begin{cases} 
1 & \text{for } s \leq s^*, \\
\sqrt{\frac{\Gamma(s)}{\Gamma(0)}} & \text{for } s > s^*, 
\end{cases}
\]
where \(s^*\) is the unique positive solution to the following equation:
\[
s^* + \int_{s^*}^\infty \sqrt{\Gamma(s)} ds = m,
\]
and the minimum variance is given by
\[
\inf_{Q \in \mathcal{M}(R^+):\mathbb{E}[Q(N)]=m} \left\{ \mathbb{E} \left[ \int_0^\infty g(X_s, s)^2 \frac{1_{\{N > s\}} ds}{Q(N > s)} \right] \right\} = 2 \int_0^{s^*} \Gamma(s) ds + 2 \Gamma(s^*) (m - s^*) - \alpha^2.
\]
(ii) If \(m \leq \int_0^\infty \sqrt{\Gamma(s)} ds/\sqrt{\Gamma(0)}\), then
\[
Q^*(N > s) := \frac{m \sqrt{\Gamma(s)}}{\int_0^\infty \sqrt{\Gamma(s)} ds},
\] (7)
and the minimum variance is given by
\[
\inf_{Q \in \mathcal{M}(\mathbb{R}^+): E^Q[N]=m} \left\{ \begin{array}{c}
\text{Var} \left( \int_0^\infty g(X_s, s) \frac{1_{\{N>s\}}}{Q(N>s)} ds \right) \\
= \frac{2}{m^2} \left( \int_0^\infty \sqrt{\Gamma(s)} ds \right)^2 - \alpha^2.
\end{array} \right.
\]

Proof: Consider the following infinite horizon optimal control problem:
\[
\inf_{\{u(s) \in [0,1] : s \in [0,\infty)\}} \left\{ \int_0^\infty 2\Gamma(s) \frac{u(s)}{u(s)} ds \right\},
\]
\[
\text{s.t. } z(s) = u(s), \quad s \in [0,\infty),
\]
\[
z(0) = 0, \quad \lim_{t \to \infty} z(t) = m.
\]

We introduce the Hamiltonian:
\[
H(z(s), u(s), p(s), s) := \frac{2\Gamma(s)}{u(s)} + p(s)u(s),
\]
where \(p(s)\) is an adjoint variable. For \(s \in [0,\infty)\), the optimal control \(u^*(t)\) satisfies the following maximum principle (Halkin [1974]):
\[
\begin{align*}
\text{Optimal condition : } u^*(s) &= \inf u \in [0,1] H(z(s), u, p(s), s), \\
\text{Adjoint equation : } p(s) &= -H_z(z(s), u(s), p(s), s) = 0.
\end{align*}
\]

From the adjoint equation, we know there exists \(\gamma \in \mathbb{R}\) such that \(p(s) \equiv \gamma\) for \(s \in [0,\infty)\). For \(\gamma \leq 0\), we control \(u^*(s) \equiv 0\) on \([0,\infty)\) satisfies the optimal condition, but it cannot satisfy the state constraint in (8). Thus, we have \(\gamma \in \mathbb{R}^+\). As in the proof of Theorem 1, the optimal condition implies
\[
\begin{align*}
\begin{cases}
\frac{1}{2\sqrt{\gamma}} s^* \\
\frac{1}{\gamma} m - s^*
\end{cases}
\end{align*}
\]
for \(s \leq s^*\), for \(s > s^*\),

where \(s^* = \inf \{ s \in [0,\infty) : \Gamma(s) \leq \gamma/2 \}\). By the state constraint in (8),
\[
\lim_{t \to \infty} z(t) = \int_0^\infty u^*(s) ds = s^* + \int_0^\infty \frac{2\Gamma(s)}{\gamma} ds = m,
\]
we have \(s^* < m\) and
\[
\gamma = \frac{2}{(m - s^*)^2} \left( \int_0^\infty \sqrt{\Gamma(s)} ds \right)^2.
\]

From Remark 1, we know that if \(\Gamma(0) \leq \gamma/2\), then \(s^* = 0\), which implies
\[
\Gamma(0) \leq \frac{1}{m^2} \left( \int_0^\infty \sqrt{\Gamma(s)} ds \right)^2;
\]
if \(\Gamma(0) > \gamma/2\), then \(s^* \in (0,\infty)\) and \(\Gamma(s^*) = \gamma/2\), which implies
\[
\Gamma(s^*) = \frac{1}{(m - s^*)^2} \left( \int_{s^*}^\infty \sqrt{\Gamma(s)} ds \right)^2,
\]
or equivalently,
\[
s^* + \int_{s^*}^\infty \sqrt{\Gamma(s)} ds = m.
\]

Define
\[
G(s) := s + \int_{s^*}^\infty \sqrt{\Gamma(s)} ds - m.
\]

We have \(G(m) > 0\), and
\[
G'(s) = \frac{-\Gamma(s) \int_{s^*}^\infty \sqrt{\Gamma(s)} ds}{2\Gamma(s) \sqrt{\Gamma(s)}} > 0.
\]

Thus, equation \(G(s) = 0\) has a unique solution on \((0,\infty)\) and only if \(\Gamma(0) < 0\), or equivalently,
\[
\Gamma(0) > \frac{1}{m^2} \left( \int_0^\infty \sqrt{\Gamma(s)} ds \right)^2.
\]

Summarizing the above arguments, the maximum principle and state constraint in (6) offer a unique \(u^*(s)\) on \([0,\infty)\), which is the optimal control.

With Assumption 1, the optimal control \(u^*(s)\) is non-increasing on \([0,\infty)\) and \(\lim_{s \to \infty} u^*(t) = 0\). By noticing that the optimization (6) is equivalent to
\[
\inf_{Q \in \mathcal{M}(\mathbb{R}^+): E^Q[N]=m} \left\{ \int_0^\infty 2\Gamma(s) \frac{u(s)}{u(s)} ds \right\},
\]
we know that \(Q^*(N > s) = u^*(s)\) on \([0,\infty)\) is the optimal distribution for the optimization problem (6). The rest of the proof is a straightforward calculation. ■

Remark 2. Optimization problem (2) can also be viewed as an optimal control problem but without a state constraint in (8), which is imposed by the computational budget constraint. When the computational budget is smaller than a threshold, i.e., \(m \leq \int_0^\infty \sqrt{\Gamma(s)} ds / \sqrt{\Gamma(0)}\), we have \(s^* = 0\), so that the distribution \(Q^*\) given by (7) is supported on \(\mathbb{R}^+\). Increasing the computational budget \(m\) on the range \(0, \int_0^\infty \sqrt{\Gamma(s)} ds / \sqrt{\Gamma(0)}\) would make the tail of the distribution \(Q^*\) heavier, which indicates that the distribution of the optimal randomization shifts more weight toward the domain when \(N\) is large as the computational budget \(m\) increases. When the computational budget is larger than a threshold, i.e., \(m > \int_0^\infty \sqrt{\Gamma(s)} ds / \sqrt{\Gamma(0)}\), we have \(s^* > 0\), which indicates that the truncation size \(N\) would be almost surely larger than a certain threshold under the optimal randomization if the computational budget is larger than a certain threshold.

As an illustration, we then show the optimal distribution for the optimization (6) when \(X_t\) is an exponential Lévy process.

Corollary 2. If \(\{X_t\}\) is an exponential Lévy process with characteristic exponent \(\phi\) and \(f(x) = x^\beta\), then when \(m|\phi_2(2\beta)| \leq 2\), the optimal distribution \(Q^*\) is given by
\[
Q^*(N > s) = \frac{m}{2} \phi_2(2\beta)|e^{- \frac{s}{2\phi_2(2\beta)}}|
\]
for any \(0 < s < \infty\). On the other hand, when \(m|\phi_2(2\beta)| > 2\), the optimal \(Q^*\) is given by
\[
Q^*(N > s) := \frac{1}{2} \phi_2(2\beta)(-1 - e^{- \phi_2(2\beta) - \phi_2(2\beta)}),
\]
where \(\phi_2\) is given by (5).

Proof: Let us recall that \(\Gamma(s) = \frac{1}{|\phi_1(s)|} e^{-s \phi_2(2\beta)}\), and we have
\[
\int_0^\infty \sqrt{\Gamma(s)} ds = \frac{2}{\sqrt{|\phi_1(\beta)| \cdot |\phi_2(2\beta)|}}.
\]

Therefore, when
\[
\frac{1}{|\phi_1(\beta)|} \Gamma(0) \leq \frac{1}{m^2} \left( \int_0^\infty \sqrt{\Gamma(s)} ds \right)^2 = \frac{4}{m^2 |\phi_1(\beta)| \cdot |\phi_2(2\beta)|^2},
\]
the optimal \(Q^*\) is given by
\[
Q^*(N > s) = \frac{m}{2} \phi_2(2\beta)|e^{- \frac{s}{2\phi_2(2\beta)}}|, \quad s \in (0,\infty).
\]
When \( m|\phi_2(2\beta)| > 2 \), the optimal \( Q^* \) is given by
\[
Q^*(N > s) := \begin{cases} 
\frac{1}{\sqrt{2}} \frac{1}{\sqrt{|\phi_1(\beta)|}} e^{-\frac{1}{2} |\phi_2(2\beta)|} & \text{for } s \leq s^*, \\
\frac{1}{(m-s)^2} \left( \int_{s^*}^{\infty} \sqrt{\gamma} |\phi_1(\beta)| \right)^2 & \text{for } s > s^*, 
\end{cases}
\]
where \( s^* = \Gamma^{-1}(\gamma/2) = -\frac{1}{|\phi_2(2\beta)|} \log(\frac{1}{2} |\phi_1(\beta)|) \) and
\[
\gamma = \frac{2}{(m-s)^2} \left( \int_{s^*}^{\infty} \sqrt{\gamma} |\phi_1(\beta)| \right)^2.
\]
This completes the proof.

C. Minimization of the Work Variance Product

In this section, we consider the third optimization problem, which is to minimize the product of the variance and the expected value of \( N \), i.e.,
\[
\inf_{Q \in \mathcal{M}(\mathbb{R}^+)} \left\{ \text{Var} \left( \int_0^{\infty} g(X_s,s) \frac{1}{Q(N > s)} ds \right) \cdot E Q[N] \right\},
\]
where \( g(x,s) \) is the optimal control law. A key observation is that this optimization problem is equivalent to first minimizing over the variance conditional on \( E Q[N] = m \) and then minimizing over all possible values of fixed levels \( m \geq 0 \), i.e.,
\[
\inf_{m \geq 0} \left\{ m \cdot \left[ \int_0^{\infty} \frac{\Gamma(s)}{Q^*(N > s)} ds - \left( \int_0^{\infty} E[g(X_s,s)] ds \right)^2 \right] \right\}.
\]

Theorem 3. Under Assumption 1, for the optimization problem (9), we have
\[
Q^*(N > s) := \begin{cases} 
\frac{1}{\sqrt{\Gamma(s)}} & \text{for } s \leq s^*, \\
\frac{1}{(m-s)^2} \left( \int_{s^*}^{\infty} \sqrt{\gamma} |\phi_1(\beta)| \right)^2 & \text{for } s > s^*, 
\end{cases}
\]
where \( s^* \) is the unique positive solution to the following equation:
\[
\frac{\alpha^2}{2} + s^* \Gamma(s^*) - \int_0^{s^*} \Gamma(s) ds = 0.
\]
The minimum value of the work-variance product is given by
\[
\inf_{Q \in \mathcal{M}(\mathbb{R}^+)} \left\{ \text{Var} \left( \int_0^{\infty} g(X_s,s) \frac{1}{Q(N > s)} ds \right) \cdot E Q[N] \right\} = 2(m^*)^2 \Gamma(s^*),
\]
where
\[
m^* = s^* + \frac{\int_0^{\infty} \sqrt{\Gamma(s)} ds}{\sqrt{\Gamma(s^*)}}.
\]

Proof. We have
\[
\inf_{m \geq 0} \left\{ m \cdot \left[ \int_0^{\infty} \frac{\Gamma(s)}{Q^*(N > s)} ds - \left( \int_0^{\infty} E[g(X_s,s)] ds \right)^2 \right] \right\} = \min \left\{ \inf_{0 \leq m \leq \frac{2m}{\sqrt{\Gamma(s^*)}}} \left\{ 2m \int_0^{\infty} \frac{\Gamma(s)}{Q^*(N > s)} ds \right\} - m \left( \int_0^{\infty} E[g(X_s,s)] ds \right)^2 \right\},
\]

or equivalently,
\[
\inf_{m \geq 0} \left\{ m \cdot \left[ \int_0^{\infty} \frac{\Gamma(s)}{Q^*(N > s)} ds - \left( \int_0^{\infty} E[g(X_s,s)] ds \right)^2 \right] \right\}.
\]

We first establish the following lemma before proving the main result of this section.

Lemma 2. Under Assumption 1, we have
\[
\int_0^{\infty} \Gamma(s) ds > \frac{\alpha^2}{2}.
\]

Proof. By definition, we have
\[
2 \int_0^{\infty} \Gamma(s) ds = 2 \int_0^{\infty} E[g(X_s,s)g(X_t,t)] dt ds = 2 \int_0^{\infty} \int_s^{\infty} E[g(X_s,s)g(X_t,t)] ds dt = 2 \int_0^{\infty} \int_s^{\infty} E[g(X_s,s)] ds \right)^2 \right) \]
\[
= \left( \mathbb{E} \left( \int_0^{\infty} g(X_s,s) ds \right) \right)^2 + \text{Var} \left( \int_0^{\infty} g(X_s,s) ds \right) > \alpha^2,
\]
noticing that \( \alpha = E \left[ \int_0^{\infty} g(X_s,s) ds \right] \). This completes the proof. ■

Our goal is to minimize \( K(s^*) \), and we have that the optimal solution \( s^{**} \) must satisfy the first-order condition:
\[
0 = K'(s^*) = \frac{\Gamma(s^*)}{\Gamma(s^{**})} \int_{s^*}^{s^{**}} \sqrt{\Gamma(s)} ds \left( \frac{\alpha^2}{2} + s^* \Gamma(s^*) - \int_{s^*}^{s^{**}} \Gamma(s) ds \right).
\]
Recall that \( \Gamma'(s^*) < 0 \), and thus the first-order condition \( K'(s^*) = 0 \) is equivalent to
\[
\frac{\alpha^2}{2} + s^* \Gamma(s^*) - \int_{s^*}^{s^{**}} \Gamma(s) ds = 0.
\]
Denote
\[H(s^*) = \frac{\alpha^2}{2} + s^* \Gamma(s^*) - \int_0^{s^*} \Gamma(s) ds, \quad 0 < s^* < \infty,\]
and we have \(H(0) = \alpha^2/2 > 0\). Noticing that \(\int_0^{\infty} \Gamma(s) ds < \infty\), we have 
\[\lim_{s^* \to \infty} s^* \Gamma(s^*) = 0.\]
Therefore, we can solve \(H(s^*) = 0\) such that \(s^* \Gamma(s^*) = \int_0^{s^*} \Gamma(s) ds\), which
completes the proof.

**Remark 3.** From the proof of Theorem 2, we know \(m^\ast\) is increasing with respect to \(s^\ast\). Therefore, there exists a unique \(m^\ast\) such that
\[m^\ast = s^\ast + \frac{\int_0^{\infty} \sqrt{\Gamma(s^\ast)} ds}{\sqrt{\Gamma(0)}} \geq \frac{\int_0^{\infty} \sqrt{\Gamma(s)} ds}{\sqrt{\Gamma(0)}},\]
which is the optimal expected computational budget for minimizing the work-variance product. Notice that the support of the optimal distribution \(Q^\ast\) is always shifted away from zero for minimizing the work-variance product.

**Corollary 3.** If \(\{X_t\}\) is an exponential Lévy process with characteristic exponent \(\phi\) and \(f(x) = x^\alpha\), then
\[Q^\ast(N > s) := \left\{ \begin{array}{ll} 1 & \text{for } 0 \leq s \leq s^\ast, \\ e^{-s^\ast \phi(2\beta)} & \text{for } s > s^\ast, \end{array} \right.\]
where \(m^\ast = \sqrt{s^\ast \phi(2\beta)} + \frac{\alpha}{\phi(2\beta)}\) and \(\phi_2\) is defined by (5), and \(s^\ast\) is the unique positive solution to the following transcendental algebraic equation:
\[\frac{1}{\phi(1)} + 2s^\ast e^{-s^\ast |\phi_2(2\beta)|} - \frac{2}{\phi(2\beta)} \left(1 - e^{-s^\ast |\phi_2(2\beta)|}\right) = 0,\]
which can be solved in a closed form:
\[s^\ast = W\left(\frac{e}{2} \left(\frac{2}{\phi_2(2\beta)} - \frac{1}{\phi(1)}\right)\right) + \frac{1}{\phi_2(2\beta)},\]
where \(W(\cdot)\) is the Lambert-W function and \(\phi_1\) and \(\phi_2\) are defined by (5). Furthermore,
\[m^\ast = W\left(\frac{e}{2} \left(\frac{2}{\phi_2(2\beta)} - \frac{1}{\phi(1)}\right)\right) + \frac{1}{\phi_2(2\beta)}.\]

**Proof:** In the case of exponential Lévy process, we have \(\Gamma(s) = \frac{1}{\phi(0)} e^{-s|\phi_2(2\beta)|}\), then the first-order condition \(K'(s) = 0\) is equivalent to
\[\frac{1}{\phi(1)} + 2s^\ast e^{-s^\ast |\phi_2(2\beta)|} - \frac{2}{\phi_2(2\beta)} \left(1 - e^{-s^\ast |\phi_2(2\beta)|}\right) = 0.\] (10)
In order to solve (10), we denote \(y = s^\ast + \frac{1}{\phi(2\beta)}\), then we can rewrite the algebraic equation into the following equivalent form:
\[y \cdot e^y = \frac{2}{\phi(2\beta)} - \frac{1}{\phi(1)}\], (11)
and note that the right-hand side is positive due to the Lévy-Khintchine theorem and Jensen’s inequality, i.e., \(2\phi(1) > |\phi_2(2\beta)|\) always holds. By the definition of the Lambert-W function, we can recognize that the solution to (11) is given explicitly by
\[y = \frac{2}{\phi(2\beta)} - \frac{1}{\phi(1)}\].
Then we have the desired solutions of \(s^\ast\) and \(m^\ast\). Note that \(b > 0\), and the Lambert-W function is uniquely defined, then this establishes the uniqueness of \(s^\ast\). Applying the results of Theorem 3 completes the proof.

**III. RANDOMIZATION VS. FIXED TRUNCATION**

As discussed in the last section, randomization inevitably increases the variance, although it eliminates the bias. Obviously, small bias and variance are desirable in practice. Basically, whether the optimal randomization is favorable or not depends on the tradeoff between bias and variance. Thus, we consider a utility function as follows:
\[U_w(I_m) := -(E[I_m] - \alpha)^2 - w\text{Var}(I_m), \quad w \geq 0,\]
where \(I_m\) is an estimator of \(\alpha\) subject to computational budget \(m\).

A large \(w\) indicates more weight on the variance and less weight on the bias in the tradeoff of these two factors. We denote the optimal randomized estimator with expected computational budget \(m\) as \(I_{m}^\ast\) and the fixed truncation estimator with computational budget \(m\) as \(I_{m}^f\), defined by
\[I_{m}^f := \int_0^m g(X_s, s) ds,\]
which completes the proof.

**Proposition 1.** For any \(w > 0\), when \(m\) is sufficiently small, \(U_w(I_{m}^f) < U_w(I_{m}^\ast)\).

**Proof:** Note that
\[U_w(I_{m}^f) = -(E[I_{m}^f] - \alpha)^2 - w\text{Var}(I_{m}^f), \quad U_w(I_{m}^\ast) = \inf\{E[I_{m}^\ast] - \alpha)^2 - w\text{Var}(I_{m}^\ast)\}.\]

From Theorem 2, we have that for \(m \leq \int_0^\infty \sqrt{\Gamma(s)} ds / \sqrt{\Gamma(0)}\),
\[Q^\ast(N > s) = \frac{m^{\sqrt{\Gamma(0)}}}{\int_0^\infty \sqrt{\Gamma(s)} ds / \sqrt{\Gamma(0)}},\]
which completes the proof.

**Remark 4.** The proposition indicates that as long as the variance is of a concern to a practitioner, the optimal randomized estimator would not be favored if the computational budget is small enough. When \(m\) is small, the distribution of \(N\) must be very skewed in order to make \(E[N] = m\) and the estimator unbiased at the same time. Specifically, \(Q^\ast(N > s)\) is small for \(s > m\), which leads to a very
large variance, because $Q(N > s)$ appears in the denominator of the expression $2 \int_0^\infty \frac{\Gamma(s)}{Q(N > s)} ds$.

In general, we define the following threshold level:

$$w(m) := \inf_{Q \in M(\mathbb{R}^+) : \mathbb{E}[Q] = m} \frac{\mathbb{E} \int_0^m g(X_s, s) ds}{\text{Var} \left( \int_0^m g(X_s, s) ds \right)}$$

such that $U_w(I_m^m) > U_w(I_m^m)$ for all $0 < w \leq w(m)$. This threshold $w(m)$ represents the maximum weight that a practitioner can put onto the variance such that the optimal randomized estimator is more favorable than the fixed truncation estimator. Similar to the proof in Proposition 1, it is straightforward to show that $\lim_{m \to 0} w(m) = 0$.

As discussed previously,

$$\inf_{Q \in M(\mathbb{R}^+) : \mathbb{E}[Q] = m} \frac{\mathbb{E} \int_0^m g(X_s, s) ds}{\text{Var} \left( \int_0^m g(X_s, s) ds \right)} > \text{Var} \left( \int_0^\infty g(X_s, s) ds \right)$$

In the case where $\int_0^m g(X_s, s) ds$ and $\int_0^\infty g(X_s, s) ds$ are positively correlated, $\text{Var} \left( \int_0^m g(X_s, s) ds \right) > \text{Var} \left( \int_0^\infty g(X_s, s) ds \right)$, which implies that $w(m)$ is well defined and non-negative.

**Proposition 2.** If $\{X_t\}$ is an exponential Lévy process with characteristic exponent $\phi$ and $f(x) = x^\beta$, then

$$w(m) = \begin{cases} \Delta_1, & 0 < m \leq \frac{2}{|\phi(2\beta)|}, \\ \Delta_2 = \Delta_1, & m > \frac{2}{|\phi(2\beta)|}, \end{cases}$$

where

$$\Delta_1 := \frac{e^{-2|\phi(2\beta)|m}}{|\phi(2\beta)|^2},$$

$$\Delta_2 := \frac{1}{|\phi(2\beta)|^2},$$

and $\phi_1$ and $\phi_2$ are given by (5).

**Proof:** Recall that

$$\mathbb{E}[g(X_t, s)g(X_s, s)] = e^{(t-s)|\phi_1(2\beta)| + 2|\phi_2(2\beta)|},$$

and

$$\Gamma(s) = \int_s^\infty e^{(t-s)|\phi_1(2\beta)| + 2|\phi_2(2\beta)|} dt = \frac{1}{|\phi_1(2\beta)|} e^{-s|\phi_2(2\beta)|}.$$

We have

$$\left( \mathbb{E} \int_0^m g(X_s, s) ds \right)^2 = \left( \int_0^m e^{\phi_1(2\beta)s} ds \right)^2 = \Delta_1,$$

and

$$\left( \mathbb{E} \int_0^\infty g(X_s, s) ds \right)^2 = \frac{1}{|\phi_1(2\beta)|^2}.$$

In addition,

$$\text{Var} \left( \int_0^m g(X_s, s) ds \right) = \mathbb{E} \left[ \left( \int_0^m g(X_s, s) ds \right)^2 \right] - \left( \mathbb{E} \int_0^m g(X_s, s) ds \right)^2$$

$$= 2\mathbb{E} \left[ \int_0^m \int_0^m g(X_s, s) g(X_t, t) dt ds \right] - \left( \mathbb{E} \int_0^m g(X_s, s) ds \right)^2$$

$$= 2\int_0^m \int_s^m e^{(t-s)|\phi_1(2\beta)| + \phi_2(2\beta)} dt ds - \left( \int_0^m e^{\phi_1(2\beta)} ds \right)^2 = \Delta_4.$$
Proposition 3. For the CIR process, we have
\[
w(m) = \begin{cases} 
\frac{\pi}{2} \int_0^\infty \Gamma(s) ds \sqrt{\Gamma(0)}, & \text{if } 0 < m \leq \frac{\int_0^\infty \sqrt{\Gamma(s)} ds}{\sqrt{\Gamma(0)}}; \\
\frac{\pi}{2} \int_0^\infty \Gamma(s) ds \sqrt{\Delta_1 - \Delta_2}, & \text{if } m > \frac{\int_0^\infty \sqrt{\Gamma(s)} ds}{\sqrt{\Gamma(0)}},
\end{cases}
\]
where
\[
\Delta_1 := 2 \left( A \frac{1 - e^{-2cs^*}}{2c} + B \frac{1 - e^{-(k+2cs^*)}}{k+c} + C \frac{1 - e^{-2(k+cs^*)}}{2(k+c)} \right) + 2m(\tau + \theta) \left( A e^{-2cs^*} + B e^{-(k+2cs^*)} + C e^{-2(k+cs^*)} \right) - \alpha^2;
\]
\[
\Delta_2 := \frac{\theta^2}{c} (1 - e^{-cm})^2 + (X_0 - \theta) \left( \frac{\sigma^2}{k + c} \right) \left( 1 - e^{-(2k+cs^*)m} \right) \left( \frac{c}{k + c} \right) + \frac{2\theta(X_0 - \theta)}{c} \left( 1 - e^{-(k+2cs^*)m} \right) - \frac{(X_0 - \theta)}{c} \left( 1 - e^{-(k+cs^*)m} \right) \left( \frac{c}{k + c} \right) + \theta \sigma^2 \frac{(X_0 - \theta)}{k(1+c)} \left( 1 - e^{-(2k+cs^*)m} \right) - \frac{(X_0 - \theta)}{c} \left( 1 - e^{-(k+cs^*)m} \right) \left( \frac{c}{k + c} \right)
\]
and \( \Gamma(0) = A + B + C \). Here \( s^* \) is the unique positive solution to the following equation:
\[
s^* + \int_s^\infty \sqrt{\Gamma(s)} ds \sqrt{\Gamma(s^*)} = m.
\]
The proof of Proposition 3 can be found in the Appendix.

IV. NUMERICAL EXPERIMENT

We consider the example of discounted cost \( g(X_s, s) = e^{-cs} X_s^\beta \) (\( c > 0 \)) with \( X_s \) being the geometric Brownian motion (GBM) model, which is a special case of the exponential Lévy process. The GBM model is governed by the following SDE:
\[
dX_t = \mu X_t dt + \sigma X_t dW_t, \quad X_0 = x_0.
\]
Then, we have \( \phi(\beta) = \left( \mu - \frac{\sigma^2}{2} \right) \beta + \frac{\sigma^2}{2} \beta^2 \). In addition, we can compute
\[
\Gamma(s) = x_0^{2\beta} e^{s \phi(2\beta)} / \phi(\beta),
\]
and
\[
\alpha = \int_0^\infty e^{-cs} x_0^\beta e^{s \phi(\beta)} ds = \frac{x_0^\beta}{c - \phi(\beta)} = \frac{x_0^\beta}{\phi(\beta)}.
\]
From Corollary 3, we know that the optimal randomization distribution is a shifted exponential distribution. The survival function of the shifted exponential distribution family is given by
\[
Q(N > s) = \begin{cases} 
1 - e^{-\eta(s-\delta)} & \text{for } s \leq \delta, \\
\eta & \text{for } s > \delta.
\end{cases}
\]
Let \( \eta^* := |\phi(2\beta)|/2 \), which is the rate in the optimal randomization distribution. The variance-work product of the shifted exponential distribution family can be expressed as a function of \( \delta \) and \( \eta^* \):
\[
p(\delta, \eta^*) := \var \left( \int_0^\infty g(X_s, s) \frac{1}{Q(N > s)} ds \right) - \mathbb{E}^Q[N]
\]
\[
= \left( 2 \int_0^\infty \frac{\Gamma(s)}{Q(N > s)} ds - \alpha^2 \right) \left( \int_0^\infty Q(N > s) ds \right) - \left( \int_0^\infty \frac{1 - e^{-\delta \phi(2\beta)}}{\phi(\beta)} ds \right) \left( \int_0^\infty e^{-\delta \phi(2\beta)} ds \right) - \alpha^2 \left( \delta + \frac{1}{\eta} \right).
\]
We first set the parameters by \( x_0 = 1, \mu = 0.1, \sigma = 0.35, c = 0.6, \beta = 0.5 \). In this case, \( \alpha = 1.7689, \eta^* = 0.55, s^* = 4.7971 \), and the minimum work-variance product value is \( p(s^*, \eta^*) = 0.68358 \).

In Figure 1, the red line is the minimum work-variance product, and the blue line is the work-variance product function of a non-shifted exponential distribution, i.e., \( p(0, \eta) \) with \( \eta \in [0.01, |\phi(2\beta)|] \). We can see the variance-work product of the non-shifted exponential distribution family is strictly larger than the minimum work-variance product, which is consistent with the Theorem result that the support of the optimal randomized distribution is always shifted away from zero. The work-variance product increases tremendously when \( \eta \) grows larger than 0.6. A large \( \eta \) would lead to a light tail of the survival function, which causes a large variance.

In Figure 2, the blue line in the top graph plots the work-variance product function \( p(s^*, \eta) \) with \( \eta \in [0.01, |\phi(2\beta)|] \), while the blue line in the bottom graph is the work-variance product function \( p(\delta, \eta^*) \) with \( \delta \in [2/|\phi(2\beta)|, 20] \). We can see the two work-variance product functions deviate from the optimal value \( p(s^*, \eta^*) \) except at the optimal point. The right panel in Figure 2 also substantiates the uniqueness of \( s^* \) in Theorem 3.

Then, we plot the threshold level \( w(m) \) given by Proposition 2 for this example. From Figure 3, we can see that even the optimal weight for all computational budget \( m \), the level of threshold \( w(m) \) is less than 0.13. This indicates that the advantage of the optimal randomized estimator over the fixed truncation estimator can only be justified under the scenario where the bias is the paramount concern.

For \( m = 1 \) in \( U_w (I_m) \), this utility corresponds to the mean squared error (MSE), which is a widely used metric for the efficiency of an estimator. Figure 3 implies that the MSE of the optimal randomized estimator is always larger than the MSE of the fixed truncation estimator. Moreover, we prove in the Appendix that this conclusion holds for all exponential Lévy processes. The numerical results for
control theory offers a new perspective to deal with the RUMC in the utility function for more general RUMC problems. The optimal estimating the optimal randomized distribution and the threshold level of the optimal randomized estimator via a utility function taking shifted exponential distributions. Moreover, we justify the advantage of an unbiased randomized estimator for simulating an expected cumulative cost contingent on the cumulative cost. The optimal distributions are in a shifted distribution class. For a discounted continuous cumulative cost contingent on the CIR process can be found in the appendix, which are similar to those for the exponential Lévy process.

V. CONCLUSION

In this paper, we derive an explicit form for the optimal distribution of an unbiased randomized estimator for simulating an expected cumulative cost. The optimal distributions are in a shifted distribution class. For a discounted continuous cumulative cost contingent on the exponential Lévy process, the optimal randomization distributions are shifted exponential distributions. Moreover, we justify the advantage of the optimal randomized estimator via a utility function taking both bias and variance into consideration. Future research lies in deriving the optimal randomized distribution and the threshold level in the utility function for more general RUMC problems. The optimal control theory offers a new perspective to deal with the RUMC problems.

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Zheng, Z. and Glynn, P. W. (2017). A CLT for infinitely stratified estimators, with applications to debiased MLMC. *ESAIM: Proceedings and Surveys*, 59:104–114.
Zhenyu Cui received B.S. in Actuarial Science from the University of Hong Kong, China, and Ph.D. in Statistics from the University of Waterloo, Canada respectively in 2008 and 2013. After working for two years at the City University of New York, he joined Stevens Institute of Technology in 2015.

Michael C. Fu (S’89- M’89-SM’06-F’08) received degrees in mathematics and electrical engineering and computer science from MIT, Cambridge, MA, in 1985 and the Ph.D. degree in applied math from Harvard University in 1989. Since 1989, he has been with the University of Maryland, College Park. He has also served as the Operations Research Program Director at the National Science Foundation and is a Fellow of the Institute for Operations Research and the Management Sciences.

Yijie Peng received the B.E. degree in mathematics from Wuhan University, China, and the Ph.D. degree in management science from Fudan University, China, in 2007 and 2014, respectively. After working as a research fellow at Fudan University and George Mason University, he joined the Department of Industrial Engineering and Management at Peking University in July, 2017.

Lingjiong Zhu received BA in mathematics from University of Cambridge in 2008, and PhD in mathematics from New York University in 2013. After working at Morgan Stanley and University of Minnesota, he joined Florida State University as an Assistant Professor in 2015.

APPENDIX

Optimal Randomization for Non-Monotone $\Gamma(s)$

Let $z(s) := Q(N > s)$. Then, finding the optimal distribution can be viewed as finding the state corresponding to the following optimal control problem:

$$\sup_{u(s) \in (-\infty,0]} \int_0^\infty -\left(\frac{2\Gamma(s)}{z(s)} + \lambda z(s)\right) ds$$

s.t. $z(s) = u(s), \quad z(0) = 1, \quad \lim_{t \to \infty} z(t) = 0.$

Notice that the constraint $u(s) \in (-\infty,0]$ makes sure the state $z(s)$ is non-increasing. The maximum principle gives a necessary condition of the optimal control:

$$\dot{p}(s) = \frac{2\Gamma(s)}{z^2(s)} + \lambda, \quad u^*(s) = \arg \max_{-\infty < u \leq 0} H(z(s), u(s), p(t), s),$$

where

$$H(z(s), u(s), p(t), s) := -\left(\frac{2\Gamma(s)}{z(s)} + \lambda z(s)\right) + p(s)u.$$  

The necessary and sufficient condition for the optimal control can be given by the following Hamilton-Jacobi-Bellman (HJB) partial differential equation:

$$\hat{V}(z, s) + \min_u \left\{ \nabla_z \hat{V}(z, s) u + \left(\frac{2\Gamma(s)}{z} + \lambda z\right) \right\} = 0,$$

subject to the terminal condition $\lim_{s \to \infty} V(x, t) = 0.$

MSE Comparison for Exponential Lévy Process

Define

$$MSE_1 := \text{Var}(I_m) + (\mathbb{E}[I_m] - \alpha)^2,$$

$$MSE_2 := \text{Var}(I_N).$$

Notice that the randomized estimator is unbiased. Thus, the MSE of the randomized estimator is its variance.

**Proposition 4.** For the exponential Lévy process, we have that for all $0 < m < \infty,$

$$MSE_1 < MSE_2.$$

**Proof:** If $m > 2/|\phi_2(\beta)|$, we have

$$MSE_1 - MSE_2 = \frac{2(1 - e^{-m|\phi_2(\beta)|}) + 2e^{-m|\phi_2(\beta)|} - 2e^{-m|\phi_2(\beta)|}}{|\phi_1(\beta)|^2} + e^{-2|\phi_2(\beta)|}m + 2 + 2e^{-m|\phi_2(\beta)|} + 2e^{-m|\phi_2(\beta)|} + 2e^{-m|\phi_2(\beta)|}.$$  

Note that we can expand the term

$$\frac{(1 - e^{-m|\phi_2(\beta)|})^2}{|\phi_1(\beta)|^2} - \frac{2e^{-m|\phi_2(\beta)|}}{|\phi_1(\beta)|^2} + \frac{2e^{-2m|\phi_2(\beta)|}}{|\phi_1(\beta)|^2}.$$  

Plugging in this expression into the above, we can further simplify the above expression to

$$MSE_1 - MSE_2 = \frac{2(1 - e^{-m|\phi_2(\beta)|}) + 2e^{-m|\phi_2(\beta)|} - 2e^{-m|\phi_2(\beta)|}}{|\phi_1(\beta)|^2} + e^{-2|\phi_2(\beta)|}m + 2 + 2e^{-m|\phi_2(\beta)|} + 2e^{-m|\phi_2(\beta)|} + 2e^{-m|\phi_2(\beta)|} + 2e^{-m|\phi_2(\beta)|}.$$  

Then we group the above terms by those related to $e^{-m|\phi_1(\beta)|}$ and those related to $e^{-m|\phi_2(\beta)|}$, and have we

$$MSE_1 - MSE_2 = \frac{2}{|\phi_1(\beta)|} \left( e^{-m|\phi_2(\beta)|} \left( 1 - e^2 + \frac{1}{|\phi_2(\beta)|} - |\phi_1(\beta)| \right) \right).$$

From the Lévy-Khintchine formula and the Jensen inequality, it is easy to establish that $|\phi_2(\beta)| < 2|\phi_1(\beta)|$ always holds. Thus we have

$$\frac{1 + e^2}{|\phi_2(\beta)|} > \frac{1}{|\phi_1(\beta)|},$$

or equivalently we have

$$-1 - e^2 \left( \frac{1}{|\phi_2(\beta)|} \right) < -1 \left( \frac{1}{|\phi_1(\beta)|} \right).$$
Using this fact, we have

\[
MSE_1 - MSE_2 = \frac{2}{|\phi_1(\beta)|} \left( e^{-m|\phi_2(2\beta)|} \left( \frac{1}{|\phi_2(2\beta)|} - \frac{|\phi_2(2\beta)| - |\phi_1(\beta)|}{2} \right) \right)
\]

Then we group the above terms by those related to \(e^{-m|\phi_2(2\beta)|}\) and those related to \(e^{-m|\phi_1(\beta)|}\):

\[
MSE_1 - MSE_2 < \frac{2}{|\phi_1(\beta)|} \left( e^{-m|\phi_2(2\beta)|} \left( \frac{2}{|\phi_2(2\beta)|} - \frac{|\phi_2(2\beta)| - |\phi_1(\beta)|}{2} \right) \right) + e^{-m|\phi_1(\beta)|} \left( \frac{1}{|\phi_1(\beta)|} - \frac{|\phi_2(2\beta)| - |\phi_1(\beta)|}{2} \right)
\]

From the Lévy-Khintchine formula and the Jensen inequality, it is easy to establish that \(|\phi_2(2\beta)| < 2|\phi_1(\beta)|\) always holds. Thus we have

\[
e^{-m|\phi_1(\beta)|} < -1
\]

Thus we have

\[
MSE_1 - MSE_2 < \frac{2}{|\phi_1(\beta)|} \left( e^{-m|\phi_2(2\beta)|} \left( \frac{2}{|\phi_2(2\beta)|} - \frac{|\phi_2(2\beta)| - |\phi_1(\beta)|}{2} \right) \right) + e^{-m|\phi_1(\beta)|} \left( \frac{1}{|\phi_1(\beta)|} - \frac{|\phi_2(2\beta)| - |\phi_1(\beta)|}{2} \right)
\]

Note that the right hand side of (14) is exactly the same as the right hand side of (13), thus following similar arguments, we can establish that \(MSE_1 - MSE_3 < 0\). This completes the proof.

**Derivations for the CIR process**

We can compute

\[
\alpha = \int_0^\infty e^{-cx}E[X_s]ds = \int_0^\infty e^{-cx} \left( X_0 e^{-\kappa s} + \theta(1 - e^{-\kappa s}) \right) ds
\]

\[
= \frac{\theta X_0 - \theta}{\kappa + c}.
\]

We have the following expression for its cross moment for \(s < t\)

\[
E[X_sX_t] = \theta^2 e^{-\kappa(t-s)}(X_0 - \theta) \left( \theta + \frac{\sigma^2}{\kappa} \right) + e^{-\kappa s}(X_0 - \theta)
\]

\[
= \frac{\theta^2 e^{-\kappa s} + \theta e^{-\kappa(t-s)}(X_0 - \theta)}{\kappa} + \frac{\sigma^2}{\kappa} \left( \theta - X_0 \right)^2 + \frac{\theta^2 e^{-\kappa t}}{2\kappa}.
\]

Define the process \(Y_t := e^{-ct}X_t\), then we have

\[
E[Y_sY_t] = e^{-c(t-s)}E[X_sX_t]
\]

\[
= \theta^2 e^{-c(t-s)} + \frac{\sigma^2}{\kappa} \left( \theta - X_0 \right)^2 + \frac{\theta^2 e^{-c(t-s)}}{2\kappa}.
\]
For \( f(x) = x \), we have
\[
\Gamma(s) = \int_{s}^{\infty} \mathbb{E}[e^{-cs} f(X_{t}) e^{-ct} f(X_{t})] dt = \int_{s}^{\infty} \mathbb{E}[Y_{t} Y_{t}] dt
\]
\[
= \frac{\theta^{2}}{c} e^{-2cs} + \frac{\sigma^{2}}{c^{2}} (X_{0} - \theta)^{2} + \frac{\sigma^{2}}{2c} \left( \theta - 2X_{0} \right)
\]
\[
+ \frac{e^{-2(\kappa + c)s}}{\kappa + c} \left( \theta - 2X_{0} \right)
\]
\[
= \frac{\theta^{2}}{c} e^{-2cs} + \frac{\sigma^{2}}{c^{2}} (X_{0} - \theta)^{2} + \frac{\sigma^{2}}{2c} \left( \theta - 2X_{0} \right)
\]
\[
\quad + \frac{e^{-2(\kappa + c)s}}{\kappa + c} \left( (\theta - 2X_{0})^{2} + \frac{\sigma^{2}}{2c} (\theta - 2X_{0}) \right)
\]
\[
\quad + \frac{\theta^{2}}{c} e^{-2cs} + \frac{\sigma^{2}}{2c} (\theta - 2X_{0}) \tag{15}
\]

Here we have to determine the sufficient conditions to be imposed onto the parameters in order to have the Assumption 1 to be satisfied. A sufficient condition is given by
\[
\theta(X_{0} - \theta) > 0, \quad (\theta - X_{0})^{2} + \frac{\sigma^{2}}{2c} \left( \theta - 2X_{0} \right) > 0,
\]
which is equivalent to requiring
\[
X_{0} > \theta + \frac{\sigma^{2}}{2\kappa} + \sqrt{\frac{\sigma^{2}}{2\kappa} \left( \theta + \frac{\sigma^{2}}{2\kappa} \right)},
\]
Then we calculate the ingredients for the determination of the optimal randomization distribution. For example, in the case of solving the optimization problem (2), from the result in Theorem 1, we have
\[
Q^{*}(N > s) = \left\{ \frac{1}{\sqrt{2}\left( A e^{-2cs} + B e^{-(\kappa + 2c)s} + C e^{-2(\kappa + c)s} \right)} \right\} \text{ s} \geq s^{*},
\]
\[
\text{ s} > s^{*},
\]
where \( s^{*} = \inf \{s \in [0, \infty) : A e^{-2cs} + B e^{-(\kappa + 2c)s} + C e^{-2(\kappa + c)s} \leq \lambda/2 \} \).

In the case of solving the constrained optimization when we are given the expected computational work \( m > 0 \), from the characterization in Theorem 2, we have
\[
Q^{*}(N > s) = \left\{ \frac{1}{\sqrt{2}\left( A e^{-2cs} + B e^{-(\kappa + 2c)s} + C e^{-2(\kappa + c)s} \right)} \right\} \text{ s} \geq s^{*},
\]
\[
\text{ s} > s^{*},
\]
where
\[
\frac{\int_{s}^{\infty} \sqrt{A e^{-2cs} + B e^{-(\kappa + 2c)s} + C e^{-2(\kappa + c)s}} ds}{\sqrt{A e^{-2cs} + B e^{-(\kappa + 2c)s} + C e^{-2(\kappa + c)s}}} = m.
\]

For the minimization of the variance-work product, from Theorem 3, we have
\[
Q^{*}(N > s) = \left\{ \frac{1}{\sqrt{2}\left( A e^{-2cs} + B e^{-(\kappa + 2c)s} + C e^{-2(\kappa + c)s} \right)} \right\} \text{ s} \geq s^{*},
\]
\[
\text{ s} > s^{*},
\]
where \( s^{**} \) is the unique positive solution to the following equation:
\[
\frac{\alpha^{2}}{2} + s^{**}(A e^{-2cs^{**}} + B e^{-(\kappa + 2c)s^{**}} + C e^{-2(\kappa + c)s^{**}})
\]
\[
- \int_{0}^{s^{**}} \left( A e^{-2cs} + B e^{-(\kappa + 2c)s} + C e^{-2(\kappa + c)s} \right) ds = 0. \tag{16}
\]

We can simplify the equation (16) as
\[
\frac{\alpha^{2}}{2} + s^{**}(A e^{-2cs^{**}} + B e^{-(\kappa + 2c)s^{**}} + C e^{-2(\kappa + c)s^{**}})
\]
\[
- \left( A \frac{1 - e^{-2cs^{**}}}{2c} + B \frac{1 - e^{-(\kappa + 2c)s^{**}}}{\kappa + 2c} + C \frac{1 - e^{-2(\kappa + c)s^{**}}}{2(\kappa + c)} \right) = 0.
\]

To calculate the optimal work-variance product, recall that the optimal level of \( m \) is given by
\[
m^{**} = s^{**} + \frac{\int_{s}^{m^{**}} \sqrt{A e^{-2cs} + B e^{-(\kappa + 2c)s} + C e^{-2(\kappa + c)s}} ds}{\sqrt{A e^{-2cs} + B e^{-(\kappa + 2c)s} + C e^{-2(\kappa + c)s}} + \sigma^{2}} \tag{18}
\]

Then the optimal work-variance product in the case of the CIR process is given by
\[
m^{**} 2 \int_{0}^{s^{**}} \Gamma(s) ds + 2 \sqrt{\Gamma(s^{**})} \int_{s^{**}}^{\infty} \sqrt{\Gamma(s) ds - \alpha^{2}}
\]
\[
= m^{**} \left( \frac{\alpha^{2}}{2} + 2s^{**} \Gamma(s^{**}) + 2 \Gamma(s^{*}) \right) \left( m^{**} - s^{**} \right) - \alpha^{2}
\]
\[
= 2 \left( m^{**} \right)^{2} \Gamma(s^{**})
\]
\[
= 2(m^{**})^{2}(A e^{-2cs^{**}} + B e^{-(\kappa + 2c)s^{**}} + C e^{-2(\kappa + c)s^{**}}).
\]

Note that in the second equality above we have utilized the defining equation characterizing \( s^{**} \).

**Proposition 3**

*Proof:* We have the following calculations:
\[
E \left[ \int_{m}^{\infty} g(X_{s}, s) ds \right] = \int_{m}^{\infty} e^{-cs} E[X_{s}] ds
\]
\[
= \int_{m}^{\infty} e^{-cs} (X_{0} e^{-\kappa s} + \theta(1 - e^{-\kappa s})) ds
\]
\[
= \frac{\theta}{c} e^{-cm} + X_{0} - \theta e^{-(c + \kappa)m},
\]
and
\[
\text{Var} \left[ \int_{m}^{s} g(X_{s}, s) ds \right]
\]
\[
= E \left[ \left( \int_{m}^{s} g(X_{s}, s) ds \right)^{2} \right] - \left( E \int_{m}^{m} g(X_{s}, s) ds \right)^{2}
\]
\[
= 2 \int_{m}^{s} E[Y_{s} Y_{t}] dtds
\]
\[
= \left( \int_{0}^{m} e^{-cs} (X_{0} e^{-\kappa s} + \theta(1 - e^{-\kappa s})) ds \right)^{2}. \tag{17}
\]

For the first term of the right hand side of (17), we have
\[
2 \int_{m}^{m} \int_{0}^{m} E[Y_{s} Y_{t}] dtds = \frac{\theta^{2}}{c} (1 - e^{-cm})^{2} + (X_{0} - \theta) \left( \theta + \frac{\sigma^{2}}{\kappa + 2c} \right) \frac{2}{\kappa + 2c}
\]
\[
\times \left( 1 - e^{-(\kappa + 2c)m} \frac{m}{\kappa + 2c} - e^{-(\kappa + 2c)m} \frac{m}{\kappa + 2c} \right)
\]
\[
+ \frac{\theta^{2}}{2c \kappa} \left( X_{0} - \theta \right) \left( 1 - e^{-(\kappa + 2c)m} \frac{m}{\kappa + 2c} - e^{-(\kappa + 2c)m} \frac{m}{\kappa + 2c} \right)
\]
\[
+ \frac{\theta^{2}}{2c \kappa} \left( X_{0} - \theta \right) \left( 1 - e^{-2cm} \frac{m}{\kappa + 2c} - e^{-2cm} \frac{m}{\kappa + 2c} \right).
\]

The second term on the right hand side of (17) can be calculated as
\[
\left( \int_{0}^{m} e^{-cs} (X_{0} e^{-\kappa s} + \theta(1 - e^{-\kappa s})) ds \right)^{2}
\]
\[
= \left( \frac{\theta}{c} e^{-cm} + \theta \frac{1 - e^{-cm}}{c} \right)^{2}.
\]
For the optimal randomized distribution, we have
\[
\inf_{Q \in \mathcal{M}(\mathbb{R}^+): Q[N] = m} \text{Var} \left( \int_0^N g(X_s, s) \frac{ds}{Q(N > s)} \right)
\]

\[
= \text{Var} \left( \int_0^N g(X_s, s) \frac{ds}{Q^*(N > s)} \right)
\]

\[
= 2 \int_0^\infty \frac{\Gamma(s)}{Q^*(N > s)} ds + 2 \int_s^\infty \frac{\Gamma(s)}{Q^*(N > s)} ds - \alpha^2
\]

\[
= 2 \int_0^s \frac{\Gamma(s)}{Q^*(N > s)} ds + 2 \int_s^\infty \sqrt{\Gamma(s)} ds - \alpha^2
\]

\[
= 2 \int_0^s \frac{\Gamma(s)}{Q^*(N > s)} ds \pm 2 \Gamma(s^*) (m - s^*) \alpha^2
\]

\[
= 2 \left( \frac{1 - e^{-2c}}{2c} + B \frac{1 - e^{-(\kappa + 2c)s^*}}{\kappa + 2c} + C \frac{1 - e^{-2(\kappa + c)s^*}}{2(\kappa + c)} \right)
\]

\[
+ 2(m - s^*)(Ae^{-2cs^*} + Be^{-(\kappa + 2c)s^*} + Ce^{-2(\kappa + c)s^*}) - \alpha^2,
\]

where we have utilized the characterization equation of \( s^* \) in the part (i) of the (modified) Theorem 2 for the constrained optimization.

(ii) If \( m \leq \int_0^\infty \sqrt{\Gamma(s)} ds / \sqrt{\Gamma(0)} \), then we have
\[
\inf_{Q \in \mathcal{M}(\mathbb{R}^+): Q[N] = m} \text{Var} \left( \int_0^N g(X_s, s) \frac{ds}{Q(N > s)} \right)
\]

\[
= \text{Var} \left( \int_0^N g(X_s, s) \frac{ds}{Q^*(N > s)} \right) = 2 \int_0^\infty \Gamma(s) ds - \alpha^2
\]

\[
= \frac{2}{m} \left( \int_0^\infty \sqrt{\Gamma(s)} ds \right)^2 - \alpha^2.
\]

Then it is straightforward to prove the conclusion.


\textbf{Numerical Results for CIR Process}

We consider a CIR process, and use the following parameter sets: \( \kappa = 3, \theta = 0.2, \sigma = 0.3, c = 0.6, x_0 = 0.5 \). In Figure 4, we can see that the non-optimal shifted distributions lead to larger variance-work products than that of the optimal shifted distribution. In Figure 5, the MSE of the optimal randomized estimator is larger than the MSE of the fixed truncation estimator. In Figure 6, the threshold function \( w(m) \) is plotted, and we can see that the threshold function first increases and then decreases with the optimal value less than 0.14.