Spin(9)-invariant valuations on the octonionic plane

Andreas Bernig and Floriane Voide

Abstract. The dimensions of the spaces of k-homogeneous Spin(9)-

invariant valuations on the octonionic plane are computed using results

from the theory of differential forms on contact manifolds as well as oc-

tonionic geometry and representation theory. Moreover, a valuation on

Riemannian manifolds of particular interest is constructed which yields,

as a special case, an element of $\text{Val}^{5\text{spin}}(9)$.

1. Introduction

On an $n$-dimensional Euclidean vector space $V$, we consider the space

$\mathcal{K}(V)$ of compact convex subsets in $V$. A valuation is a real or complex

valued functional $\mu$ on $\mathcal{K}(V)$ with the property that

$$\mu(K \cup L) + \mu(K \cap L) = \mu(K) + \mu(L),$$

for all $K, L \in \mathcal{K}(V)$ with $K \cup L \in \mathcal{K}(V)$.

In this article we consider only continuous (with respect to the Hausdorff

topology on $\mathcal{K}(V)$) and translation invariant valuations; we denote by $\text{Val}$

the vector space of these valuations.

A valuation $\mu$ is said to be $k$-homogeneous if

$$\mu(tK) = t^k \mu(K)$$

for every $K \in \mathcal{K}(V)$ and $t > 0$. We denote by $\text{Val}_k$ the subspace of $k$-

homogeneous valuations in $\text{Val}$. By a theorem of McMullen [36] we have the

following decomposition

$$\text{Val} = \bigoplus_{k=0}^{n} \text{Val}_k. \tag{1}$$

A valuation $\mu$ is even if $\mu(-K) = \mu(K)$ and odd if $\mu(-K) = -\mu(K)$ for

all $K \in \mathcal{K}(V)$.

Let $G$ be a compact subgroup of the special linear group $\text{SO}(n)$ of $V$,

and let $\bar{G}$ be the group generated by $G$ and translations. Consider the

space $\text{Val}^G \subset \text{Val}$ of $\bar{G}$-invariant continuous valuations. It was shown by

Alesker [1] that this vector space has finite dimension if and only if $G$ acts transi-

tively on the unit sphere $S(V)$ in $V$. A classification of all compact

---

Supported by SNF-grant PP002-114715/1 and DFG-grant BE 2484/5-1.
connected Lie groups acting transitively and effectively on $S(V)$ was obtained by Montgomery-Samelson [37] and Borel [25]. The list contains the series

$$\text{SO}(n), \text{U}(n), \text{SU}(n), \text{Sp}(n), \text{Sp}(n) \cdot \text{U}(1), \text{Sp}(n) \cdot \text{Sp}(1),$$

and the three exceptional groups

$$G_2, \text{Spin}(7), \text{Spin}(9).$$

The space of continuous $\text{SO}(n)$-invariant valuations is described by a famous theorem of Hadwiger [31], which states that $\text{Val}^{\text{SO}(n)}$ has dimension $n + 1$. A basis of this space is given by the so called intrinsic volumes. Hadwiger’s theorem implies in a very straightforward way many of the most important theorems in integral geometry, compare [33] for these applications.

Hadwiger-type theorems and integral geometry of the unitary groups $\text{U}(n), \text{SU}(n)$ was extensively studied during the last few years [2, 14, 21, 22, 38, 44, 45].

For the symplectic groups $\text{Sp}(n), \text{Sp}(n) \cdot \text{U}(1), \text{Sp}(n) \cdot \text{Sp}(1)$, the dimensions of the spaces of invariant valuations were computed in [18]. A Hadwiger-type theorem is known only in the cases $n = 1$ [4, 14, 15] (note that $\text{Sp}(1) \cong \text{SU}(2)$) and $n = 2$ [24].

The cases of $G_2$ and Spin(7) have been treated by the first named author in [16].

Until now, only partial results have been obtained by Alesker in the case $G = \text{Spin}(9)$, acting on a 16-dimensional space by the spin representation. Using the theory of plurisubharmonic functions in quaternionic variables, he constructed a 2-homogeneous Spin(9)-invariant valuation in [9].

In the first section of this paper we compute the dimensions of the spaces $\text{Val}^{\text{Spin}(9)}_k$.

**Theorem 1.**

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|-----|---|---|---|---|---|---|---|---|---|---|-----|----|----|----|----|----|----|
| dim $\text{Val}^{\text{Spin}(9)}_k$ | 1 | 1 | 2 | 3 | 6 | 10 | 15 | 20 | 27 | 20 | 15 | 10 | 6 | 3 | 2 | 1 | 1 |

These dimensions are computed using the results of the first named author [18, Theorem 2.1 and Corollary 2.2], the description of the spin action due to Sudbery [41] and facts from representation theory.

Before going on, let us compare our result with the one in [16]. The two exceptional groups $G_2$ and Spin(7) act not only transitively on some sphere, but isotropically, i.e. the stabilizer acts transitively on the unit sphere of the tangent space. This allows to relate invariant valuations under these groups to invariant valuations on the tangent spaces. This approach does not work for Spin(9), since the action of this group on $S^{15}$ is not isotropic. Another difference is that (again in contrast to the cases $G_2$ and Spin(7)) there are rather many new invariant valuations. Therefore it seems a difficult
problem to write down an explicit Hadwiger-type theorem and to compute the kinematic formulas for this group.

In the second part of this paper, we present a result in the more general setting of valuations on manifold. Before stating it, we have to recall the Klain embedding.

Let \( \mu \) be an even valuation of homogeneity degree \( k \) on the Euclidean vector space \( V \). For a \( k \)-plane \( E \in \text{Gr}_k(V) \), the restriction \( \mu|_E \) is a continuous, translation invariant simple valuation. From a characterization theorem of Klain [32], it follows that \( \mu|_E = c(E) \cdot \text{vol}_E \) for some constant \( c(E) \in \mathbb{R} \).

**Definition 1.1.** The map \( \text{Kl}_\mu : \text{Gr}_k(V) \longrightarrow \mathbb{R}, \text{Kl}_\mu(E) = c(E) \) is called the **Klain function** of \( \mu \). It is a continuous function on \( \text{Gr}_k(V) \).

The induced map \( \text{Kl} : \text{Val}_k^+ \longrightarrow C(\text{Gr}_k(V)), \mu \mapsto \text{Kl}_\mu \) is injective. \( \text{Kl} \) is called the **Klain embedding**.

Let us now recall some notions related to Alesker’s theory of valuations on manifolds [5, 6, 7, 8, 12]. Roughly speaking, a smooth valuation on a manifold \( M \) of dimension \( n \) is a functional (satisfying some technical properties) on the space of smooth compact submanifolds with corners (see Section 5 for the definition). The space \( \mathcal{V}^\infty(M) \) of smooth valuations on \( M \) admits a natural filtration

\[
\mathcal{V}^\infty(M) = \mathcal{W}_0 \supset \mathcal{W}_1 \supset \cdots \supset \mathcal{W}_n.
\]

Under the action of the Euler-Verdier involution (see Section 5), each \( \mathcal{W}_k \) splits as a sum \( \mathcal{W}_k \cong \mathcal{W}_k^+ \oplus \mathcal{W}_k^- \). The quotient \( \mathcal{W}_k/\mathcal{W}_{k+1} \) is canonically isomorphic to the space of smooth sections of the vector bundle \( \text{Val}_k(TM) \) over \( M \) whose fiber at a point \( p \in M \) is given by \( \text{Val}_k(T_pM) \). If \( \mu \in \mathcal{W}_k \), we denote by \( T^\epsilon_p \mu \) the image of \( \mu \) in \( \text{Val}_k(T_pM) \). If \( \mu \in \mathcal{W}_k^\epsilon \) with \( \epsilon := (-1)^k \), then \( T^\epsilon_p \mu \in \text{Val}_k^+(T_pM) \) is even and can be described by its Klain function, which is a function on \( \text{Gr}_k(T_pM) \).

Let \( (M,g) \) be a Riemannian manifold, \( p \in M \). The sectional curvature of \( M \) at \( p \) is a function \( K_p \in C(\text{Gr}_2(T_pM)) \). Note that the Klain function of an even 2-homogeneous valuation on \( T_pM \) belongs to the same space.

**Theorem 2.** On every Riemannian manifold \((M,g)\), there exists a valuation \( \mu \in \mathcal{W}_2^+ \) such that for every \( p \in M \), the Klain function of the induced valuation \( T^2_p \mu \) on \( T_pM \) is the sectional curvature of \( M \) at \( p \).

The example \( M = \mathbb{O}P^2 \) (octonionic projective plane) yields an even, homogeneous of degree 2, \( \text{Spin}(9) \)-invariant valuation on \( \mathbb{O}^2 = \mathbb{R}^{16} \), which is not a multiple of the second intrinsic volume.

**Acknowledgements.** We are very grateful to Franz Schuster and Gil Solanes for their suggestions and comments on the first drafts of this paper.

### 2. Forms and valuations

We begin by recalling the ideas and notations from [18], leading to Theorem 2.3 and Corollary 2.4 below.
Let $V$ be an $n$-dimensional Euclidean vector space. The unit sphere in $V$ will be denoted by $S(V)$. The sphere bundle $SV := V \times S(V)$ is a contact manifold with contact form given by

$$\alpha_{(x,v)}(w) = (v, d\pi(w)), \quad w \in T_{(x,v)}SV,$$

where $\pi : SV \to V$ is the canonical projection. We denote by $Q := \ker \alpha$ the contact distribution.

The space of complex valued differential forms on $SV$ is denoted by $\Omega^*(SV)$. It has a bigrading

$$\Omega^*(SV) = \bigoplus_{k,l} \Omega^{k,l}(SV),$$

where $\Omega^{k,l}(SV)$ denotes the space of differential forms of bidegree $(k,l)$ on $SV$.

**Definition 2.1.** Given $K \in \mathcal{K}(V)$, the oriented $(n-1)$-dimensional Lipschitz submanifold of $SV$

$$N(K) := \{(x,v) \in SV | v \text{ is an outer unit normal vector of } K \text{ at } x\}$$

is called the normal cycle of $K$.

For basic properties of $N(K)$ we refer to [28].

**Definition 2.2.** A translation invariant valuation $\mu \in \text{Val}$ is called smooth if there exist $\omega \in \Omega^{n-1}(SV)^{tr}$ and $\varphi \in \Lambda^n V^* \otimes \mathbb{C} = \Omega^n(V)^{tr}$ such that

$$\mu(K) = \int_{N(K)} \omega + \int_K \varphi,$$

for every $K \in \mathcal{K}(V)$, where the superscript $\text{tr}$ means translation invariance. We denote by $\text{Val}^\infty$ the space of translation invariant smooth valuations.

Smooth valuations are dense in the space of continuous translation invariant valuations. In recent years, several algebraic operations like product, convolution and Alesker-Fourier transform were introduced on $\text{Val}^\infty$ [3, 10, 17, 20, 29]. Moreover, these algebraic structures are closely related to kinematic formulas, compare [21, 22, 23, 44].

The space $\text{Val}^\infty$ fits into an exact sequence, as was shown in [39] and [18] and will be explained now. We consider the following subspaces of $\Omega^{k,l}(SV)^{tr}$.

$$\mathcal{I}^{k,l}(SV)^{tr} := \{ \omega \in \Omega^{k,l}(SV)^{tr} | \omega = \alpha \wedge \xi + d\alpha \wedge \psi, \xi \in \Omega^{k-1,l-1}(SV)^{tr}, \psi \in \Omega^{k-1,l}(SV)^{tr} \},$$

$$\Omega_v^{k,l}(SV)^{tr} := \{ \omega \in \Omega^{k,l}(SV)^{tr} | \alpha \wedge \omega = 0 \},$$

$$\Omega_h^{k,l}(SV)^{tr} := \Omega^{k,l}(SV)^{tr} / \Omega_v^{k,l}(SV)^{tr},$$

$$\Omega_p^{k,l}(SV)^{tr} := \Omega^{k,l}(SV)^{tr} / \mathcal{I}^{k,l}(SV)^{tr}.$$
The letters v,h,p stand for \textit{vertical}, \textit{horizontal}, \textit{primitive} respectively.

Let $L$ be multiplication by the 2-form $d\alpha$. Then $L : \Omega_h^{k-1,l-1}(SV)^{tr} \to \Omega_h^{k,l}(SV)^{tr}$ is an injection for $k + l \leq n$ and

$$\Omega_p^{k,l}(SV)^{tr} \cong \Omega_h^{k,l}(SV)^{tr} / L\Omega_h^{k-1,l-1}(SV)^{tr}. \quad (4)$$

Let $G$ be a compact subgroup of $SO(V)$ acting transitively on $S(V)$. By $\bar{G}$ we denote the group generated by $G$ and translations. By a slight abuse of notation, the superscript $G$ stands for translation and $G$-invariance.

We consider the operators

$$d_Q : \Omega_p^{k,l}(SV)^G \to \Omega_p^{k,l+1}(SV)^G$$

induced by the exterior derivative $d : \Omega^{k,l}(SV)^G \to \Omega^{k,l+1}(SV)^G$, and

$$\text{nc} : \Omega_p^{k,n-k-1}(SV)^G \to \text{Val}_k^G \quad \omega \mapsto \int_{N(\cdot)} \omega.$$ 

This map is well defined, since $N(K)$ is a legendrian cycle, hence vanishes on $\mathcal{I}^{k,n-k-1}(SV)^G$.

\textbf{Theorem 2.3 \cite{18}.} For $0 < k < n$, the sequence

$$0 \rightarrow (\Lambda^k V^*)^G \otimes \mathbb{C} \rightarrow \Omega_p^{k,0}(SV)^G \xrightarrow{d_Q} \ldots$$

$$\ldots \xrightarrow{d_Q} \Omega_p^{k,n-k-1}(SV)^G \xrightarrow{\text{nc}} \text{Val}_k^G \rightarrow 0$$

is exact.

Note that exactness on the left part of the sequence was shown by Rumin \cite{39}, while exactness of the right hand side is a consequence of results in \cite{19}.

The theorem and \cite{18} yield the following corollary.

\textbf{Corollary 2.4 \cite{18}.} For $0 \leq k, l \leq n$, set

$$b_k := \dim(\Lambda^k V)^G,$$

$$b_{k,l} := \dim \Omega_h^{k,l}(SV)^G,$$

and $b_k := 0, b_{k,l} := 0$ for other values of $k$ and $l$. Then for $0 \leq k \leq n$:

$$\dim \text{Val}_k^G = \sum_{l=0}^{n-k-1} (-1)^{n-k-l-1}(b_{k,l} - b_{k-1,l-1}) + (-1)^{n-k}b_k. \quad (5)$$

The first named author \cite{18} used this corollary to determine the dimensions of the spaces of invariant valuations on quaternionic vector spaces. In the present paper, we will study the case of the octonionic plane $\mathbb{O}^2$. 
3. The group Spin(9)

Let us first recall the definition of the spin groups Spin(n) for general n.

On a vector space V with dimension n we consider the special orthogonal group SO(V) \cong SO(n) of V.

It is well known that the fundamental group of SO(n) is given by \( \pi_1(\text{SO}(n)) = \mathbb{Z}/2\mathbb{Z} \) for \( n \geq 3 \). This implies that, for \( n \geq 3 \), SO(n) has a connected double covering, called the spin group Spin(n).

Explicit constructions and descriptions of these groups can be found in [26, 30]. Note that the Lie algebra of Spin(n) equals the Lie algebra of SO(n), that is the space \( \mathfrak{so}(n) \) of anti-symmetric matrices.

The group Spin(9) admits a representation on a 16-dimensional space, called the spin representation. We will use the following description of the spin representation on the Lie algebra level due to Sudbery ([41]). Recall that \( \mathbb{O} \) denotes the 8-dimensional normed division algebra of the octonions. \( \mathbb{O} \) is neither commutative nor associative. However, it is alternative, i.e. for all \( a, b, c \in \mathbb{O} \),

\[ [a, b, c] = -[b, a, c], \]

where \( [a, b, c] := a(bc) - (ab)c \) is the associator. The center of \( \mathbb{O} \) is \( \mathbb{R} \) and we denote by \( \mathbb{O}' \) its orthogonal complement in \( \mathbb{O} \), i.e. the 7-dimensional space of pure octonions.

**Proposition 3.1** (Triality principle, [40]). For \( T \in \mathfrak{so}(\mathbb{O}) \), there exist unique elements \( T^\flat, T^\sharp \in \mathfrak{so}(\mathbb{O}) \) satisfying the following generalization of the derivation equation:

\[ T(xy) = (T^\sharp x)y + x(T^\flat y). \]

**Theorem 3.2** ([41]).

1. The Lie algebra \( \mathfrak{so}(9) \) of Spin(9) can be represented as

\[ \mathfrak{so}(9) = A_2'(\mathbb{O}) \oplus \mathfrak{so}(\mathbb{O}'), \]

where \( A_2'(\mathbb{O}) \) is the set of traceless antihermitian 2 \times 2 matrices with entries in \( \mathbb{O} \).

2. The spin representation \( \rho \) of \( \mathfrak{so}(9) \) on

\[ S := \{2 \times 1 \text{ column vectors with entries in } \mathbb{O}\} \cong \mathbb{O}^2 \cong \mathbb{R}^{16} \]

is given by

\[ \rho(A)(x) := A \cdot x, \quad A \in A_2'(\mathbb{O}) \quad \text{(matrix multiplication)}, \]

\[ \rho(T)(x) := T^\sharp x, \quad T \in \mathfrak{so}(\mathbb{O}') \quad \text{(componentwise action)}, \]

where \( T^\sharp \in \mathfrak{so}(\mathbb{O}') \subset \mathfrak{so}(\mathbb{O}) \) is uniquely defined by the triality principle.

**Corollary 3.3.** The stabilizer of the Spin(9)-action on \( S^{15} \) is Spin(7). Its action on the tangent space \( T_{(1,0)}S^{15} = \mathbb{O}' \oplus \mathbb{O} \) is the sum of the standard representation on \( \mathbb{O}' = \mathbb{R}^7 \) and the spin representation on \( \mathbb{O} = \mathbb{R}^8 \).
For the details of these computations, we refer to [43].

Let us also recall some facts from the representation theory for odd dimensional orthogonal Lie algebras, again referring to [26, 30] for details.

An irreducible representation of $\mathfrak{so}(2m + 1)$ can be represented by an element of the lattice $\Lambda \subset \mathbb{R}^m$ generated by $L_1, \ldots, L_m$ and $(L_1 + \cdots + L_m)/2$, i.e. by an element of the form

$$\sum_{i=1}^{m} \lambda_i L_i$$

with $\lambda_1 \geq \cdots \geq \lambda_m \geq 0$ and the $\lambda_i$’s are either all integers or all half integers. We encode this information in the vector $[\lambda_1, \ldots, \lambda_m]$, called **highest weight**, and the associated irreducible representation is denoted by $\Gamma_{[\lambda_1, \ldots, \lambda_m]}$.

**Proposition 3.4.** [30] For $k = 1, \ldots, m - 1$, the exterior power $\Lambda^k(V_{st})$ of the standard representation $V_{st}$ of $\mathfrak{so}(2m+1)$ is the irreducible representation with highest weight

$$[1, \ldots, 1, 0, \ldots, 0]_k \text{ terms}.$$  

The spin representation $S$ is the irreducible representation with highest weight

$$[1/2, \ldots, 1/2].$$

These $m$ representations are called fundamental representations. In fact, the representation ring $R$ is a polynomial ring on the isomorphism classes of the fundamental representations

$$R = \mathbb{Z}[[V_{st}], [\Lambda^2 V_{st}], \ldots, [\Lambda^{m-1} V_{st}], [S]].$$

It is useful to work with characters of representations, since the character carries the essential information about the representation. Let us denote by $\mathbb{Z}[\Lambda]$ the integral group ring on the abelian group $\Lambda$.

The character of the irreducible representation $\Gamma_{[\lambda_1, \ldots, \lambda_m]} = \Gamma_\lambda$ can be computed by Weyl’s character formula ([30])

$$\text{Char}(\Gamma_\lambda) = \frac{x_j^{\lambda_i+m-i+1/2} - x_j^{-(\lambda_i+m-i+1/2)}}{x_j^{m-i+1/2} - x_j^{-(m-i+1/2)}} \in \mathbb{Z}[\Lambda], \quad (6)$$

where $x_j^{\pm 1}$ and $x_j^{\pm 1/2}$ are the elements of $\mathbb{Z}[\Lambda]$ corresponding to the weights $\pm L_j$ and $\pm \frac{1}{2}L_j$ respectively.

We denote by $B_i$ the character of $\Lambda^i V_{st}$ and by $B$ the character of the spin representation $S$. Any $\mathfrak{so}(2m + 1)$-representation $V \in R$ can be expressed as a polynomial in the fundamental representations $V_{st}, \Lambda^2 V_{st}, \ldots, \Lambda^{m-1} V_{st}, S$, and its character can be computed as the same polynomial in $B_1, \ldots, B_{m-1}, B$ ([30]).
In practice, there is no easy way to find the decomposition of an arbitrary representation as an element of $R = \mathbb{Z}[[V_{st}], [\Lambda^2 V_{st}], \ldots, [\Lambda^m V_{st}], [S]]$. But we will see that the only representations of interest in our case are exterior powers of irreducible representations or sums of irreducible representations. To compute the character of a representation given in this form, we can use the following recurrence formula.

**Theorem 3.5** (Adams formula [30]). Define the Adams operator $\psi^k : \mathbb{Z}[\Lambda] \to \mathbb{Z}[\Lambda]$ by $\psi^k(x_j) = x_j^k$. Then for any $\mathfrak{so}(2m + 1)$-representation $V$

$$\text{Char}(\Lambda^d V) = \frac{1}{d} \sum_{k=1}^{d}(-1)^{k-1} \psi^k(\text{Char} V) \text{Char}(\Lambda^{d-k} V).$$

This formula allows us to compute inductively the character of $\Lambda^k V$. The next step is to write the obtained polynomial as a linear combination of characters of irreducible representations. Since a character uniquely determines the associated representation, we conclude that, if

$$\text{Char}(V) = \bigoplus n_\lambda \text{Char}(\Gamma_\lambda),$$

then

$$V = \bigoplus n_\lambda \Gamma_\lambda.$$

To decompose the character of a representation in characters of irreducible representations, we need two observations

- The leading monomial of the character of the irreducible representation $\Gamma_{[\lambda_1, \ldots, \lambda_m]}$ is $x_1^{\lambda_1} \cdots x_m^{\lambda_m}$. Recall that the leading monomial of a polynomial is the monomial of highest degree (with respect to the lexicographic order).
- If the leading monomial of the character of a representation $V$ is $n_\lambda x_1^{\lambda_1} \cdots x_m^{\lambda_m}$, then the leading monomial of the character of $V - n_\lambda \Gamma_{[\lambda_1, \ldots, \lambda_m]}$ is of strictly lower degree.

Therefore we apply the following algorithm to decompose the character of a representation $V$ in irreducible characters :

a) Find the leading monomial of $\text{Char}(V) : n_\lambda x_1^{\lambda_1} \cdots x_m^{\lambda_m}$.

b) Compute $\text{Char}(\Gamma_{[\lambda_1, \ldots, \lambda_m]})$ with the help of Weyl’s character formula.

c) Compute $\text{Char}(V - n_\lambda \Gamma_{[\lambda_1, \ldots, \lambda_m]})$.

d) Find the leading monomial of the new polynomial. If it is not a constant, start over with b), otherwise we have the decomposition of $V$.

After finitely many steps, we obtain the decomposition of $\text{Char}(V)$.

4. Dimension of $\text{Val}^{\text{Spin}(9)}$

In Section 3 we sketched an algorithm to determine the characters of exterior powers of some given representation of $\mathfrak{so}(2m + 1)$. In this section, we will apply this algorithm in order to compute the constants $b_k, b_{k,l}$. We
denote by $V$ the 16-dimensional space $\mathbb{O}^2$, with the spin representation of $\text{Spin}(9)$.

**Proposition 4.1.** The numbers $b_k = \dim(\Lambda^k(\mathbb{O}^2))^{\text{Spin}(9)}$ are given by

$$b_k = \begin{cases} 
1 & \text{for } k = 0, 8, 16, \\
0 & \text{for all other values of } k.
\end{cases}$$

**Proof.** We compute the exterior powers of the spin representation of $\mathfrak{so}(9)$ as follows.

*Case $k = 0$:* $\Lambda^0(V) = \mathbb{C}$ is the trivial representation, so $b_0 = 1$.

*Case $k = 1$:* $\Lambda^1(V) = V = \Gamma_{[1/2,1/2,1/2]}$ is irreducible, so $b_1 = 0$.

*Case $2 \leq k \leq 8$:* With Adams formula and the algorithm described above, we can decompose $\Lambda^k V$ in irreducible representations as follows:

$$
\begin{align*}
\Lambda^2 V &= \Gamma_{[1,1,1,0]} \oplus \Gamma_{[1,1,0,0]} \\
\Lambda^3 V &= \Gamma_{[3/2,3/2,1/2,1/2]} \oplus \Gamma_{[3/2,1/2,1/2,1/2]} \\
\Lambda^4 V &= \Gamma_{[2,2,0,0]} \oplus \Gamma_{[2,1,1,1]} \oplus \Gamma_{[2,1,0,0]} \oplus \Gamma_{[1,1,1,1]} \\
\Lambda^5 V &= \Gamma_{[5,2,3,2,1,1/2]} \oplus \Gamma_{[5,2,1/2,1/2,1/2]} \oplus \Gamma_{[3/2,3,2,3/2,2]} \oplus \Gamma_{[3,2,3,1,1/2,1/2]} \oplus \Gamma_{[3,2,1/2,1/2,1/2]} \\
\Lambda^6 V &= \Gamma_{[3,1,1,1,0]} \oplus \Gamma_{[3,1,0,0,0]} \oplus \Gamma_{[2,2,1,1]} \oplus \Gamma_{[2,1,1,1]} \oplus \Gamma_{[2,1,0,0,0]} \oplus \Gamma_{[2,1,1,0]} \oplus \Gamma_{[1,1,1,1,0]} \\
\Lambda^7 V &= \Gamma_{[7/2,1/2,1/2,1/2]} \oplus \Gamma_{[5,2,3,2,3/2,1/2]} \oplus \Gamma_{[5,2,1/2,1,1/2,1/2]} \oplus \Gamma_{[5,2,1/2,1,1/2,1/2]} \oplus \Gamma_{[2,1,1,1,0]} \oplus \Gamma_{[2,1,0,0,0]} \oplus \Gamma_{[2,1,1,1,0]} \oplus \Gamma_{[1,1,1,1,0]} \oplus \Gamma_{[1,1,1,0,0]} \oplus \Gamma_{[0,0,0,0,0]}.
\end{align*}
$$

Hence $b_k = 1$ if and only if $k = 8$.

*Case $9 \leq k \leq 16$:* Since there is an isomorphism of $\text{Spin}(9)$-modules

$$\Lambda^k V \cong \Lambda^{16-k} V,$$

the only non-zero $b_k$ is $b_{16}$. \qed

**Proposition 4.2.** The numbers $b_{k,l} = \dim \Omega^k_{\mathbb{O}^2}(\text{Spin}(9))$ are given by the following table, together with the symmetry relations $b_{k,l} = b_{15-k,15-l} = b_{k,15-l} = b_{15-k,l}$:
\begin{tabular}{|c|cccccccc|}
\hline
\(k\) & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
\(l\) & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
0 & 0 & 1 & 0 & 0 & 1 & 2 & 1 & 0 & 4 \\
1 & 0 & 2 & 2 & 3 & 5 & 7 & 10 & 9 & 16 \\
2 & 0 & 2 & 7 & 7 & 10 & 22 & 28 & 24 & 32 \\
3 & 1 & 3 & 7 & 18 & 30 & 39 & 50 & 63 & 72 \\
4 & 2 & 5 & 10 & 30 & 56 & 68 & 88 & 116 & 128 \\
5 & 1 & 7 & 22 & 39 & 68 & 116 & 150 & 162 & 176 \\
6 & 0 & 10 & 28 & 50 & 88 & 150 & 204 & 210 & 224 \\
7 & 4 & 9 & 24 & 63 & 116 & 162 & 210 & 266 & 280 \\
\hline
\end{tabular}

**Proof.** We first describe the spaces \(\Omega_{h}^{k,l}(SV)^{\text{Spin}(9)}\).

Let \(v = (1,0)^T \in S^{15} \subset O^2\). Since Spin(9) acts transitively on \(SV\), there is an isomorphism

\[ \Psi : \Omega^{k,l}(SV)^{\text{Spin}(9)} \rightarrow \Lambda^{k,l}(T_{(0,v)}SV)^{\text{Stab}((0,v))} = \Lambda^{k,l}(T_{(0,v)}SV)^{\text{Spin}(7)} \]

\[ \omega \mapsto \omega(0,v). \]

The tangent space \(T_{(0,v)}(SV)\) decomposes as

\[ T_{(0,v)}(SV) = \mathbb{R}v \oplus T \oplus T, \]

where \(T := T_vS^{15} = O' \oplus O\).

The isomorphism thus becomes

\[ \Psi : \Omega^{k,l}(SV)^{\text{Spin}(9)} \rightarrow \Lambda^{k,l}(\mathbb{R} \oplus O' \oplus O \oplus O' \oplus O)^{\text{Spin}(7)}. \]

Recall that

\[ \Omega^{k,l}(SV)^{G} := \Omega^{k,l}(SV)^{G}/\Omega^{k,l}(SV)^{G}, \]

where

\[ \Omega^{k,l}(SV)^{G} := \{ \omega \in \Omega^{k,l}(SV)^{G} | \alpha \wedge \omega = 0 \}. \]

Since

\[ \Psi(\alpha)(\lambda v + w) = \alpha(0,v)(\lambda v + w) = \langle v, \lambda v \rangle = \lambda, \]

for all \(w \in T \oplus T\), \(\Psi\) induces an isomorphism on the space of horizontal forms by

\[ \Omega^{k,l}(SV)^{\text{Spin}(9)} \rightarrow \Lambda^{k,l}(O' \oplus O \oplus O' \oplus O)^{\text{Spin}(7)} \]

\[ \omega \mapsto \omega(0,v). \]

So the coefficients \(b_{k,l}\) can be computed as

\[ b_{k,l} = \dim \Lambda^{k,l}(O' \oplus O \oplus O' \oplus O)^{\text{Spin}(7)} \]

\[ = \dim(\Lambda^{k}(O' \oplus O) \otimes \Lambda^{l}(O' \oplus O))^{\text{Spin}(7)} \]

\[ = \dim \text{Hom}_{\text{Spin}(7)}(\Lambda^{k}(O' \oplus O), \Lambda^{l}(O' \oplus O)), \]

where the last equation follows from the self-duality of \(\Lambda^{k}(O' \oplus O)\).
Therefore, if
\[ \Lambda^i(O' \oplus O) = \sum_{\lambda} n^{(i)}_{\lambda} \Gamma_{\lambda}, \quad i = 0, \ldots, 7 \]
is the decomposition into irreducible parts, Schur’s lemma implies that
\[ b_{k,l} = \sum_{\lambda} n^{(k)}_{\lambda} n^{(l)}_{\lambda}. \tag{7} \]

We first compute with Weyl’s character formula (6) the character
\[ \text{Char}(\Gamma_{[1,0,0]}) + \text{Char}(\Gamma_{[1/2,1/2,1/2]}) \]
of \( O \oplus O' \), then apply Adam’s formula and the same algorithm as before. The result is the following table, whose \( i \)-th column contains in the line indexed by \([\lambda_1, \lambda_2, \lambda_3]\) the coefficient of \( \Gamma_{[\lambda_1, \lambda_2, \lambda_3]} \) in the decomposition of \( \Lambda^i(O \oplus O') \).

|     | \( i = 0 \) | \( i = 1 \) | \( i = 2 \) | \( i = 3 \) | \( i = 4 \) | \( i = 5 \) | \( i = 7 \) |
|-----|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| \([0,0,0]\) | 1           |             |             |             |             |             |             |
| \([1/2,1/2,1/2]\) | 1           | 1           | 2           | 3           | 4           |             | 5           | 6           |
| \([1,0,0]\) | 1           | 1           | 1           | 2           | 3           | 5           | 3           |
| \([1,1,0]\) | 2           | 1           | 1           | 5           |             | 6           |             | 4           |
| \([1,1,1]\) |             | 2           | 4           | 3           | 4           | 7           |             |
| \([1/2,1/2,1/2]\) | 1           | 2           | 3           |             | 5           | 6           |             | 7           |
| \([1,1,1]\) |             |             |             |             |             |             |             |             |
| \([3/2,3/2,3/2]\) | 1           |             |             |             |             |             |             |             |
| \([2,0,0]\) | 1           | 2           | 1           |             | 1           |             | 3           |
| \([2,1,0]\) |             | 1           | 1           | 2           |             | 3           |             | 4           |
| \([2,2,0]\) |             |             | 1           |             |             |             |             | 2           |
| \([2,2,1]\) |             |             |             | 1           | 2           |             |             | 1           |
| \([2,2,2]\) |             |             |             |             |             | 1           |             |             |
| \([3/2,3/2,3/2]\) | 1           | 1           | 2           |             |             |             |             | 2           |
| \([2,2,1]\) |             |             |             |             |             |             |             |             |
| \([2,2,2]\) |             |             |             |             |             |             |             |             |
| \([3,0,0]\) |             |             |             |             |             |             |             | 1           |
| \([3,1,0]\) |             |             |             |             |             |             |             | 1           |
| \([3,1,1]\) |             |             |             |             |             |             |             |             |

The values for \( b_{k,l} \) stated in the proposition follow from (7) and this table. \( \square \)

Proof of Theorem \( \Box \) Theorem follows from Proposition \( \Box \) and (5). \( \square \)
5. A canonical valuation on Riemannian manifolds

We first recall the background material for valuations on manifolds. We refer to the articles [5], [6] and the lectures notes [11] as a general reference for the material presented in the following.

Let $M$ be a smooth oriented $n$-dimensional manifold.

**Definition 5.1.** A closed subset $P \subset M$ is a submanifold with corners if it is locally diffeomorphic to $\mathbb{R}^i_{\geq 0} \times \mathbb{R}^j$, with integers $i, j$.

Denote by $P_M(M)$ the space of compact submanifolds with corners in $M$.

Consider the oriented projectivization $P_M$ of the cotangent bundle $T^*M$

$$P_M := (T^*M \setminus \{\Sigma\})/\mathbb{R}_{>0} \cong S^*M,$$

where $\{\Sigma\}$ is the zero section of $T^*M$ and $S^*M$ denotes the cosphere bundle of $M$.

An element of $P_M$ can be thought of as a pair $(p, H)$ with $p \in M$ and $H \subset T_pM$ an oriented hyperplane.

**Definition 5.2.** For $P \in P(M)$, we define the following sets: The tangent cone to $P$ at $x$ is given by

$$T_xP := \{v \in T_xM \mid \exists \text{ smooth curve } c : \mathbb{R} \to P \text{ with } c(0) = x, c'(0) = v\}.$$

The dual tangent cone is

$$T^*_xP := \{\xi \in T^*_xM \mid \langle \xi, v \rangle \leq 0 \forall v \in T_xP\}.$$

The normal cycle of $P$ is given by

$$N(P) := \bigcup_{x \in P} (T^*_xP \setminus \{0\})/\mathbb{R}_{>0}.$$

It is well known that $N(P)$ is a $(n-1)$-dimensional Lipschitz submanifold of $P_M$, which can be oriented in a canonical way.

If $M = V$ is a vector space, this definition coincides with Definition [24].

**Definition 5.3.** A valuation on a manifold $M$ is a finitely additive functional $\mu : P(M) \to \mathbb{R}$, i.e. for any $P_1, P_2 \in P(M)$ such that $P_1 \cup P_2, P_1 \cap P_2 \in P(M)$,

$$\mu(P_1 \cup P_2) = \mu(P_1) + \mu(P_2) - \mu(P_1 \cap P_2).$$

A valuation is said to be smooth if there exist $\gamma \in \Omega^{n-1}(S^*M)$ and $\varphi \in \Omega^n(M)$ such that

$$\mu(P) = \int_{N(P)} \gamma + \int_P \varphi.$$

We denote by $\mathcal{V}^\infty(M)$ the space of smooth valuations on $M$. Note that any pair $(\varphi, \gamma) \in \Omega^n(M) \times \Omega^{n-1}(SM)$ defines a valuation. However, different pairs may define the same valuation, compare [19] for the description of the kernel of this map.

If $\mu \in \mathcal{V}^\infty(M)$ is represented by a pair $(\varphi, \gamma)$, then the pair $((-1)^n \varphi, (-1)^n s^* \gamma)$ defines a valuation $\sigma \mu \in \mathcal{V}^\infty(M)$. Here $s : S^*M \to S^*M$ is the involution.
$(p, [\xi]) \mapsto (p, [-\xi]), p \in M, \xi \in T^*_p M \setminus \{0\}$. The operation $\sigma$ is well-defined and is called Euler-Verdier involution [6].

From now on, we let $(M, g)$ be a Riemannian manifold. There is a canonical identification $S^*M \cong SM$, so that we may consider $N(P)$ as a submanifold of SM.

It will be more convenient to work with a subset in $TM$ instead of $SM$.

**Definition 5.4.** For a compact submanifold with corners $P \subset M$, its disc bundle $N_1(P) \subset TM$ is the sum of $P \times \{0\}$ and the image of $[0, 1] \times N(P)$ under the homothety in the second factor

$$N_1(P) = \iota_* (P) + F_* ([0, 1] \times N(P)),$$

where $\iota : M \hookrightarrow TM$, $p \mapsto (p, 0)$ is the natural inclusion and $F : \mathbb{R} \times SM \to TM$, $(t, (p, v)) \mapsto (p, tv)$ is the homothety map.

Clearly $N_1(P)$ is an $n$-dimensional Lipschitz submanifold of $TM$ with boundary, and we have

$$\partial N_1(P) = N(P).$$

If a smooth form $\gamma \in \Omega^{n-1}(SM)$ extends to an $(n-1)$-form $\tilde{\gamma} \in \Omega^{n-1}(TM)$, then Stoke’s theorem implies

$$\int_{N(P)} \gamma = \int_{N_1(P)} d\tilde{\gamma}.$$

Conversely, we have the following.

**Lemma 5.5.** Any smooth $n$-form $\omega$ on $TM$ defines a smooth valuation by

$$\mu(P) = \int_{N_1(P)} \omega,$$

for $P \in \mathcal{P}(M)$.

**Proof.** Let us write

$$\mu(P) = \int_{N_1(P)} \omega = \int_P \iota^* \omega + \int_{[0, 1] \times N(P)} F^* \omega = \int_P \iota^* \omega + \int_{N(P)} \int_0^1 F^* |_{(t, \cdot)} \left( \frac{\partial}{\partial t}, \cdot \right) dt = \varphi + \int_{N(P)} \gamma,$$

with $\varphi \in \Omega^n(M)$ and $\gamma \in \Omega^{n-1}(SM)$. \qed
Let us denote by $\text{Val}^\infty (TM)$ the bundle whose fiber over a point $p$ is the space $\text{Val}^\infty (T_p M)$. Then (11) implies a grading
\[
\text{Val}^\infty (TM) = \bigoplus_{k=0}^n \text{Val}^\infty_k (TM).
\]

**Theorem 5.6 (6, 11).** There exists a canonical filtration of $\mathcal{V}^\infty (M)$ by closed subspaces
\[
\mathcal{V}^\infty (M) = \mathcal{W}_0 \supset \mathcal{W}_1 \supset \cdots \supset \mathcal{W}_n,
\]
such that the associated graded space $\text{gr}_W \mathcal{V}^\infty (M) := \bigoplus_{k=0}^n \mathcal{W}_k/\mathcal{W}_{k+1}$ is canonically isomorphic to the space $C^\infty (M, \text{Val}^\infty (TM))$ of smooth sections of the infinite-dimensional vector bundle $\text{Val}^\infty (TM) \to M$.

Let us remark that there exist product structures on $\mathcal{V}^\infty (M)$ and on $\text{Val}^\infty (T_p M)$ and the isomorphism of the theorem is an isomorphism of graded algebras. However, we will not need the product structure in this paper.

Let us describe the isomorphism more explicitly. Let $\mu \in \mathcal{W}_k$ and $p \in M$. Let $\tau : U \to V \subset \mathbb{R}^n$ be a coordinate chart around $p$. The differential $d\tau_p : T_p M \to T_{\tau(p)} V \cong \mathbb{R}^n$ is a linear isomorphism. The valuation $T^k_p \mu \in \text{Val}^\infty (T_p M)$ is defined by
\[
T^k_p \mu (P) := \frac{1}{k!} \frac{d^k}{dt^k} \bigg|_{t=0} (\tau^{-1} \ast \mu (\tau(p)) + t(d\tau_p (P) - \tau(p))), \quad P \in \mathcal{P}(T_p M).
\]

It is independent of the choice of $\tau$. Strictly speaking, by this definition we obtain an element of $\mathcal{V}^\infty (T_p M)^tr$ (translation invariant smooth valuations on $T_p M$), but this latter space can be canonically identified with $\text{Val}^\infty (T_p M)$ [6].

Under the action of the Euler-Verdier involution, each $\mathcal{W}_k$ splits as $\mathcal{W}^+_k \oplus \mathcal{W}^-_k$, where $\mathcal{W}^\pm_k$ denotes the ($\pm 1$)-eigenspace of $\sigma$ acting on $\mathcal{W}_k$. If $\mu \in \mathcal{W}^\pm_k, \epsilon \in \{\pm 1\}$, then $T^k_p \mu \in \text{Val}_k (T_p M)$ has parity $(-1)^k \epsilon$.

Let us introduce a filtration on the space of $n$-forms on $T^* M$, following [5].

For every $(p, \xi) \in T^* M$ we define
\[
\mathcal{W}_k (\Omega^n (T^* M))|_{(p, \xi)} := \left\{ \omega \in \Lambda^n T^*_p \omega (T^* M) \mid \omega |_F \equiv 0 \text{ for all } F \subset T_{(p, \xi)} (T^* M) \right\},
\]
where $\pi : T^* M \to M$ is the projection. Then we have the filtration
\[
\Omega^n (T^* M) = \mathcal{W}_0 (\Omega^n (T^* M)) \supset \mathcal{W}_1 (\Omega^n (T^* M)) \supset \cdots \supset \mathcal{W}_n (\Omega^n (T^* M)) \supset \mathcal{W}_{n+1} (\Omega^n (T^* M)) = 0.
\]

For $\epsilon \in \{\pm 1\}$ set
\[
\mathcal{W}^\epsilon_k (\Omega^n (T^* M)) := \{ \omega \in \mathcal{W}^\pm_k (\Omega^n (T^* M)) : \bar{s}^\epsilon \omega = (-1)^{n+\epsilon} \omega \},
\]
where $\bar{s} : T^* M \to T^* M, (x, \xi) \mapsto (x, -\xi)$. 

Theorem 5.7 ([5]). The map $\Xi : \Omega^n(TM) \to \mathcal{V}_{\infty}(M)$ given by

$$(\Xi(\omega))(P) := \int_{N_1(P)} \omega$$

is surjective. More precisely, for $k \in \{0, 1, \ldots, n\}$ and $\epsilon \in \{\pm 1\}$, the map $\Xi$ maps $W^\epsilon_k(\Omega^n(TM))$ onto $W^\epsilon_k$ surjectively.

Let now $(M, g)$ be a Riemannian manifold. Let $\Gamma(M)$ denote the space of vector fields on $M$, let $R$ be the Riemann curvature tensor of $M$ and $K$ its sectional curvature.

For $p \in M$, the tensor $R_p$ is an element of $\text{Sym}^2 \Lambda^2 T_p M \subset \Lambda^2 T_p M \otimes \Lambda^2 T_p M$. Let $R^*_p$ be the image of $R_p$ under the maps $\Lambda^2 T_p M \otimes \Lambda^2 T_p M \cong \Lambda^2 (T_p M \oplus T_p M) \cong \Lambda^n (T_{(p,v)} TM)$, where the first isomorphism is induced by the Hodge-$*$ operator.

We define an $n$-form $\omega \in \Omega^n(TM)$ by

$$(\omega_{(p,v)})(p,v) := 1/\kappa_{n-2} R^*_p \in \Lambda^n (T_{(p,v)} TM),$$

where the normalization constant $\kappa_{n-2}$ is the volume of the $(n-2)$-dimensional unit ball, that is

$$\kappa_{n-2} = \frac{\pi^{(n-2)/2}}{\Gamma \left( \frac{n-2}{2} + 1 \right)}.$$

By Lemma 5.5, we may associate a smooth valuation to $\omega$.

Definition 5.8. The sectional curvature valuation $\mu^\text{sec} \in \mathcal{V}_{\infty}(M)$ is defined by

$$\mu^\text{sec}(P) := \int_{N_1(P)} \omega, \quad P \in \mathcal{P}(V).$$

Theorem 5.9. The valuation $\mu^\text{sec}$ defined above has filtration index 2 and belongs to the $(+1)$-eigenspace of the Euler-Verdier involution, i.e. $\mu^\text{sec} \in W^+_2$. For each $p \in M$, the Klain function of the induced valuation $T^\text{2,sec}_p \in \text{Val}_{\infty,+}^+(T_p M)$ equals the sectional curvature of $M$ at $p$.

Proof. In order to show that $\mu^\text{sec} \in W_2$, we use Theorem 5.7. It thus suffices to show that, for $(p,v) \in TM$

$$R^*_p |_F \equiv 0$$

for all $F \subset T_{(p,v)} (TM)$ with $\dim(F \cap T_{(p,v)} (\pi^{-1} p)) = \dim(F \cap V_{(p,v)}) > n-2$, where $V_{(p,v)} \cong T_p M$ is the vertical subspace of $T_{(p,v)} (TM)$. However, this is immediate from the fact that $R^*_p \in \Lambda^2 (T_p M) \otimes \Lambda^{n-2} (T_p M)$.

Since $\tilde{s}^* \omega = (-1)^n \omega$, $\mu^\text{sec} \in W^+_2$ and hence $T^\text{2,sec}_p \in \text{Val}_{\infty,+}^+(T_p M)$ for all $p \in M$.

Finally, fix $p \in M$ and let us compute the Klain function of $T^\text{2,sec}_p \in \text{Val}_{\infty,+}^+(T_p M)$. 

Let \( E \in \text{Gr}_2(T_pM) \), and let \( D^2 \) be the 2-dimensional unit ball in \( E \). Consider the exponential map \( \text{exp} : T_pM \to M \), and set \( \tau := \text{exp}^{-1} \). Then we have \( \tau(p) = 0 \) and \( d\tau_p = \text{id}_{|T_pM} \). Using (8) we obtain

\[
T_p^2 \mu^\sec(D^2) = \frac{1}{2} \left. \frac{d^2}{dt^2} \right|_{t=0} \left( (\tau^{-1})^* \mu^\sec(\tau(p) + t(d\tau_p(D^2) - \tau(p))) \right)
\]

\[
= \frac{1}{2} \left. \frac{d^2}{dt^2} \right|_{t=0} \mu^\sec(\tau^{-1}(tD^2))
\]

\[
= \frac{1}{2} \left. \frac{d^2}{dt^2} \right|_{t=0} \int_{N_1(\tau^{-1}(tD^2))} \omega.
\]

Define \( S_t := \tau^{-1}(tD^2) \in \mathcal{P}(M) \). Then

\[
\mu^\sec(S_t) = \int_{N_1(S_t)} \omega
\]

\[
= \int_{N_1(S_t)} \omega(q,v)(v^h_1, v^h_2, v^v_3, \ldots, v^v_n) dq dv,
\]

where \( v^h_1, v^h_2 \) are horizontal lifts (i.e. lifts in the horizontal subspace \( H(q,v) \)) of an orthonormal basis \( \{v_1, v_2\} \) of \( T_qS_t \) and \( v^v_3, \ldots, v^v_n \) are vertical lifts (i.e. lifts in the vertical subspace \( V(q,v) \)) of an orthonormal basis of the orthogonal complement of \( T_qS_t \) in \( T_qM \).

The definition of \( \omega \) implies that

\[
\mu^\sec(S_t) = \frac{1}{\kappa_{n-2}} \text{vol}_{n-2}(D^{n-2}) \int_{S_t} R^*_q(v_1, v_2, v_3, \ldots, v_n) dq,
\]

where \( D^{n-2} \) denotes the \((n-2)\)-dimensional unit ball, since, by definition, \( R^*_q \) is constant on each fiber.

Using the isomorphism given by the Hodge-\( * \) operator, we obtain

\[
\frac{1}{\kappa_2 t^2} \mu^\sec(S_t) = \frac{1}{\kappa_2 t^2} \int_{S_t} R_q(v_1, v_2, v_1, v_2) dq
\]

\[
= \frac{1}{\kappa_2 t^2} \int_{S_t} K(T_qS_t) dq
\]

\[
\to K(E), \quad \text{as } t \to 0.
\]

By definition of \( T_p^2 \mu^\sec \), we obtain

\[
T_p^2 \mu^\sec(D^2) = \left. \frac{d^2}{dt^2} \right|_{t=0} \mu^\sec(S_t)
\]

\[
= \kappa_2 K(E)
\]

\[
= \text{vol}_2(D^2) K(E),
\]

and therefore

\[
\text{Kl}_{T^2 \mu^\sec}(E) = K(E),
\]
i.e. the Klain function of $T_p^2 \mu^{\sec}$ coincides with the sectional curvature $K$ of $M$ at $p$.  

**Examples.** Let $G$ be a Lie group acting isotropically on $M$, i.e. $G$ acts transitively on the unit sphere bundle $SM$. Let us fix a point $p \in M$ and let $H \subset G$ be the stabilizer of $p$. Then the isomorphism from Theorem 5.6 restricts to an isomorphism

$$gr_W V^\infty(M)^G \cong C^\infty(M, Val^\infty(TM))^G \cong Val^H(T_p M).$$

In particular, we have the following isomorphism

$$(W_k/W_{k+1})^G \cong Val^H_k(T_p M), \rho \mapsto T_p^k \rho. \quad (9)$$

Since the form $\omega$ is $G$-invariant, the same holds true for the valuation $\mu^{\sec}$. Hence $T_p^2 \mu^{\sec} \in Val^H_2(T_p M)$.

(1) **The complex projective space.**

Let $M := \mathbb{C}P^n$ with its Fubini-Study metric, $T_p M = \mathbb{C}^n$, $G := U(n+1)$, $H := \text{Stab}_p = U(1) \times U(n)$. The action of $H$ on $\mathbb{C}^n$ is not faithful, factoring out the kernel leaves us with the canonical action of $U(n)$ on $\mathbb{C}^n$.

It is well-known that the $U(n)$-orbits of $\text{Gr}_2(\mathbb{C}^n)$ are characterized by their Kähler angles (compare e.g. Tasaki [42] for a more general statement). More precisely, let a plane $E$ be generated by two orthogonal vectors $v, w$. Then the Kähler angle $\varphi(E) \in [0, \frac{\pi}{2}]$ is defined by the equation

$$\cos^2 \varphi(E) = \langle v, iw \rangle^2.$$

Thus $\varphi$ measures the angle between the complex planes spanned by $\{v, iv\}$ and $\{w, iw\}$ (cf. [34]).

The sectional curvature of a plane $E$ is given by $K(E) = 1 + 3 \cos^2 \varphi(E)$, hence the Klain function of $T_p^2 \mu^{\sec}$ is given by

$$\text{Kl}_{T_p^2 \mu^{\sec}}(E) = 1 + 3 \cos^2 \varphi(E).$$

Let us relate the sectional curvature valuation

$$T_p^2 \mu^{\sec} \in \text{Val}_2(T_p \mathbb{C}P^n)^{U(n)} = \text{Val}_2(\mathbb{C}^n)^{U(n)}$$

with known valuations from hermitian integral geometry. Several bases of $\text{Val}^{U(n)}$ have been constructed in [21], for instance hermitian intrinsic volumes and Tasaki valuations. In particular, a basis for $\text{Val}^{U(n)}$ is given by the Tasaki valuations $\tau_{2,i}$, $i = 0, 1$. Their Klain functions are

$$\text{Kl}_{\tau_{2,0}} = 1, \quad \text{Kl}_{\tau_{2,1}} = \cos^2 \varphi,$$

where $\varphi$ is the Kähler angle. Hence the sectional curvature valuation $T_p^2 \mu^{\sec}$ can be written in terms of the Tasaki valuations as

$$T_p^2 \mu^{\sec} = \tau_{2,0} + 3 \tau_{2,1}.$$
(2) Let $M := \mathbb{HP}^n$, the quaternionic projective space with its standard metric, $G := \text{Sp}(n+1)$, $T_p M \cong \mathbb{H}^n$, $H := \text{Stab}_p = \text{Sp}(n) \times \text{Sp}(1)$. Again the action of $H$ on $T_p M$ is not faithful. Factoring out the kernel we obtain an action of $\text{Sp}(n) \cdot \text{Sp}(1)$ on $\mathbb{H}^n$.

The $\text{Sp}(n) \cdot \text{Sp}(1)$-orbits on $\text{Gr}_2(\mathbb{H}^n)$ were described in [24]. Given a 2-plane $E \subset \mathbb{H}^n$, choose an orthonormal basis $u_1, u_2$ of $E$. Then the standard quaternionic hermitian scalar product of $u_1, u_2$ is a pure quaternion. The orbit of $E$ is uniquely characterized by the norm $\lambda$ of this quaternion. Moreover, there exist an $\text{Sp}(n) \cdot \text{Sp}(1)$ and translation invariant continuous valuation $\tau$ whose Klain function equals $\lambda^2$. These statements are shown in the case $n = 2$ in [24, Theorems 1 and 2], but the proofs work for higher $n$ as well.

On the other hand, the sectional curvature of $\mathbb{HP}^n$ at the 2-plane $E$ equals $1 + 3\lambda^2$, compare e.g. [35]. Since $\dim \text{Val}_{\text{Spin}(9)^2} = 2$ [15], we obtain by comparing the Klain functions $T^2_p \mu^\text{sec} = \mu_2 + 3\tau$, where $\mu_2$ is the second intrinsic volume (whose Klain function is 1).

(3) The octonionic projective plane. Due to the non-associativity of $\mathbb{O}$, the concept of octonionic projective space only makes sense in dimension 2 ([13]). For $M = \mathbb{O}^2$, we have $T_p M = \mathbb{O}^2$, $G := F_4$ and $H := \text{Stab}_p = \text{Spin}(9)$.

For $(a, b), (c, d) \in \mathbb{O}^2$ with $\|(a, b)\| = \|(c, d)\| = 1$ and $\langle (a, b), (c, d) \rangle = 0$, the sectional curvature of the plane generated by $(a, b), (c, d)$ is given by (cf. [27])

$$K(E((a, b), (c, d))) = 4 \left[ \|a \wedge c\|^2 + \|b \wedge d\|^2 + \frac{1}{4} ||a||^2 ||d||^2 + \frac{1}{4} ||b||^2 ||c||^2 + \frac{1}{2} \langle ab, cd \rangle - \langle ad, bc \rangle \right],$$

where $\|a \wedge b\|^2$ is

$$\|a \wedge b\|^2 = \det \begin{pmatrix} \langle a, a \rangle & \langle a, b \rangle \\ \langle b, a \rangle & \langle b, b \rangle \end{pmatrix} = ||a||^2 ||b||^2 - \langle a, b \rangle^2.$$

In particular,

$$K(E((1, 0), (1, 0))) = 4, \quad K(E((1, 0), (0, 1))) = 1.$$

Alesker [9] constructed a valuation $\tau_{\text{oct}}$ on $\mathbb{O}^2$ which is $\text{Spin}(9)$-invariant and of degree of homogeneity 2, called the octonionic pseudo-volume. Its Klain function satisfies

$$\mathbf{Kl}_{\tau_{\text{oct}}} (E((1, 0), (1, 0))) = 0, \quad \mathbf{Kl}_{\tau_{\text{oct}}} (E((1, 0), (0, 1))) = 1.$$

Since we have shown that $\text{Val}_{\text{Spin}(9)^2}$ is of dimension 2, the valuation $T^2_p \mu^\text{sec}$ can be expressed as a linear combination of the second
intrinsic volume and the octonionic pseudo-volume. Comparing the values of the Klain functions, we obtain

\[ T^2_p \mu^{\text{sec}} = 4\mu_2 - 3\tau_{\text{oct}}. \]

REFERENCES

[1] Semyon Alesker. On P. McMullen’s conjecture on translation invariant valuations. Adv. Math., 155(2):239–263, 2000.
[2] Semyon Alesker. Description of translation invariant valuations on convex sets with solution of P. McMullen’s conjecture. Geom. Funct. Anal., 11(2):244–272, 2001.
[3] Semyon Alesker. The multiplicative structure on continuous polynomial valuations. Geom. Funct. Anal., 14(1):1–26, 2004.
[4] Semyon Alesker. SU(2)-invariant valuations. Milman, V. D. (ed.) et al., Geometric aspects of functional analysis. Papers from the Israel seminar (GAFA) 2002–2003. Berlin: Springer. Lecture Notes in Mathematics 1850, 21–29 (2004), 2004.
[5] Semyon Alesker. Theory of valuations on manifolds. I: Linear spaces. Isr. J. Math., 156:311–339, 2006.
[6] Semyon Alesker. Theory of valuations on manifolds. II. Adv. Math., 207(1):420–454, 2006.
[7] Semyon Alesker. Theory of valuations on manifolds: a survey. Geom. Funct. Anal., 17(4):1321–1341, 2007.
[8] Semyon Alesker. Theory of valuations on manifolds. IV. New properties of the multiplicative structure. In Geometric aspects of functional analysis, volume 1910 of Lecture Notes in Math., pages 1–44. Springer, Berlin, 2007.
[9] Semyon Alesker. Plurisubharmonic functions on the octonionic plane and Spin(9)-invariant valuations on convex sets. J. Geom. Anal., 18(3):651–686, 2008.
[10] Semyon Alesker. A Fourier type transform on translation invariant valuations on convex sets. Israel J. Math., 181:189–294, 2011.
[11] Semyon Alesker. New structures on valuations and applications. In Eduardo Gallego and Gil Solanes, editors, Integral Geometry and Valuations, Advanced Courses in Mathematics - CRM Barcelona, pages 1–45. Springer Basel, 2014.
[12] Semyon Alesker and Joseph H. G. Fu. Theory of valuations on manifolds. III. Multiplicative structure in the general case. Trans. Amer. Math. Soc., 360(4):1951–1981, 2008.
[13] John C. Baez. The octonions. Bull. Amer. Math. Soc. (N.S.), 39(2):145–205, 2002.
[14] Andreas Bernig. A Hadwiger type theorem for the special unitary group. Geom. Funct. Anal., 19:356–372, 2009.
[15] Andreas Bernig. A product formula for valuations on manifolds with applications to the integral geometry of the quaternionic line. Comment. Math. Helv., 84(1):1–19, 2009.
[16] Andreas Bernig. Integral geometry under G2 and Spin(7). Israel J. Math., 184:301–316, 2011.
[17] Andreas Bernig. Algebraic integral geometry. In Global Differential Geometry, volume 17 of Springer Proceedings in Mathematics, pages 107–145. Springer, Berlin Heidelberg, 2012.
[18] Andreas Bernig. Invariant valuations on quaternionic vector spaces. J. Inst. Math. Jussieu, 11:467–499, 2012.
[19] Andreas Bernig and Ludwig Bröcker. Valuations on manifolds and Rumin cohomology. J. Differ. Geom., 75(3):433–457, 2007.
[20] Andreas Bernig and Joseph H. G. Fu. Convolution of convex valuations. Geom. Dedicata, 123:153–169, 2006.
[21] Andreas Bernig and Joseph H. G. Fu. Hermitian integral geometry. *Ann. of Math.*, 173:907–945, 2011.
[22] Andreas Bernig, Joseph H. G. Fu, and Gil Solanes. Integral geometry of complex space forms. *Geom. Funct. Anal.*, 24(2):403–492, 2014.
[23] Andreas Bernig and Daniel Hug. Kinematic formulas for tensor valuations. Preprint [arXiv:1402.2750](http://arxiv.org/abs/1402.2750).
[24] Andreas Bernig and Gil Solanes. Classification of invariant valuations on the quaternionic plane. *J. Funct. Anal.*, 267:2933–2961, 2014.
[25] Armand Borel. Some remarks about Lie groups transitive on spheres and tori. *Bull. Amer. Math. Soc.*, 55:580–587, 1949.
[26] Theodor Bröcker and Tammo tom Dieck. *Representations of compact Lie groups*, volume 98 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. Translated from the German manuscript, Corrected reprint of the 1985 translation.
[27] Robert B. Brown and Alfred Gray. Riemannian manifolds with holonomy group Spin(9). In *Differential geometry (in honor of Kentaro Yano)*, pages 41–59. Kinokuniya, Tokyo, 1972.
[28] Joseph H. G. Fu. Curvature measures of subanalytic sets. *Amer. J. Math.*, 116(4):819–880, 1994.
[29] Joseph H.G. Fu. Algebraic integral geometry. In Eduardo Gallego and Gil Solanes, editors, *Integral Geometry and Valuations*, Advanced Courses in Mathematics - CRM Barcelona, pages 47–112. Springer Basel, 2014.
[30] William Fulton and Joe Harris. *Representation theory. A first course*. Graduate Texts in Mathematics. Springer-Verlag, New York, 1991.
[31] Hugo Hadwiger. *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1957.
[32] Daniel A. Klain. A short proof of Hadwiger’s characterization theorem. *Mathematika*, 42(2):329–339, 1995.
[33] Daniel A. Klain and Gian-Carlo Rota. *Introduction to geometric probability*. Lincei Lincei. Cambridge University Press, Cambridge, 1997.
[34] Wilhelm Klingenberg. *Riemannian geometry*, volume 1 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin-New York, 1982.
[35] Vivian Yoh Kraines. Topology of quaternionic manifolds. *Trans. Amer. Math. Soc.*, 122:357–367, 1966.
[36] Peter McMullen. Valuations and Euler-type relations on certain classes of convex polytopes. *Proc. London Math. Soc. (3)*, 35(1):113–135, 1977.
[37] Deane Montgomery and Hans Samelson. Transformation groups of spheres. *Ann. of Math. (2)*, 44:454–470, 1943.
[38] Heungui Park. Kinematic formulas for the real subspaces of complex space forms of dimension 2 and 3. PhD-thesis University of Georgia 2002.
[39] Michel Rumin. Differential formulas on contact manifolds. (Formes différentielles sur les variétés de contact.). *J. Differ. Geom.*, 39(2):281–330, 1994.
[40] Anthony Sudbery. Octonionic description of exceptional Lie superalgebras. *J. Math. Phys.*, 24(8):1986–1988, 1983.
[41] Anthony Sudbery. Division algebras, (pseudo)orthogonal groups and spinors. *J. Phys. A*, 17(5):939–955, 1984.
[42] Hirooyuki Tasaki. Generalization of Kähler angle and integral geometry in complex projective spaces. In *Steps in differential geometry (Debrecen, 2000)*, pages 349–361. Inst. Math. Inform., Debrecen, 2001.
[43] Floriane Voide. Spin(9)-invariant valuations. PhD-thesis Goethe University Frankfurt 2013.
[44] Thomas Wannerer. Integral geometry of unitary area measures. *Adv. Math.*, 263:1–44, 2014.
[45] Thomas Wannerer. The module of unitarily invariant area measures. *J. Differential Geom.*, 96(1):141–182, 2014.

Institut für Mathematik, Johann Wolfgang Goethe-Universität Frankfurt, Robert-Mayer-Str. 10, 60054 Frankfurt, Germany

E-mail address: bernig@math.uni-frankfurt.de
E-mail address: voide@math.uni-frankfurt.de