Linking the parameters of diquark-quark model to the Cabibbo angle

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INTRODUCTION

One of the earliest improved or generalized versions of the Gell-Mann-Okubo (GMO) mass relation for baryons forming the \(SU(3)\) octet has been obtained by Lichtenberg, Tassie and Keleman (LTK) in a particular version of 'diquark-quark' model \[1\]. The essence of the LTK result is that the correction to GMO combination is expressed in terms of basic parameters (of dimension of mass) characterizing both the diquark and the third quark. It is important to emphasize that viewing baryons as consisting of a diquark and a separate third quark provides a number of advantages \[2\] among which one also finds the important ability to account for some nonperturbative aspects of QCD.

On the other hand, quantum groups and quantum (\(q\)-deformed) algebras \[3\] provide a very useful tools not only for application to the spectroscopy of diatomic molecules and (super)deformed nuclei (see e.g. \[3\]) but, as it was demonstrated more recently, these are very useful when applied to phenomenological description of hadron properties \[3\]. In the framework of the approach initiated in \[3\] (where the case of vector mesons was first considered) and developed in more detail in subsequent papers, see \[3\] and references given therein, the \(q\)-algebras \(U_q(su_n)\) have been adopted, in place of the Lie algebras of the groups \(SU_n\), as those describing flavor symmetries of hadrons. Such replacement allows to derive numerous results concerning hadron masses and mass sum rules, along with a number of interesting implications. Basic tool of this approach is the representation theory of the \(q\)-algebras \(U_q(su_n)\) \[3\].

In the case of baryons, it was clearly demonstrated that the approach takes into account, in a uniform and natural way, the contributions in baryon masses which reflect essentially non-polynomial \[2\] (in fact, all-order) effects of \(SU(3)\) breaking. As another important consequence, the (phenomenologically) most adequate fixed value of the deformation parameter \(q\) is linked directly to the famous Cabibbo angle \(\theta_C\) \[5\].

The goal of this note is to find a direct connection between the basic parameters involved in the aforementioned two different extensions (modifications) of the octet baryon GMO mass formula derived within the apparently differing approaches: from the evaluation of hadron masses using \(q\)-deformed counterpart of flavor symmetries \(SU(N)\), and from the diquark-quark treatment of baryons within the LTK model.

BARYON MASS RELATION FROM THE LTK DIQUARK-QUARK MODEL

We first give a short sketch of those results from \[1\] which are relevant for our further discussion. With the notation

\[
\zeta = (m_s - m_t + v_s - v_t)/2V_0,
\]

\[
\gamma_t = (\delta_t - \delta_q)/6V_0, \quad \gamma_s = (\delta_s - \delta_q)/6V_0,
\]

(here the subscript "s" or "t" refers to \(SU(3)\) sextet or \(SU(3)\) triplet diquark respectively, and "q" refers to the third quark), the octet baryon mass differences obtained in the LTK model are of the form

\[
m_\Lambda - m_N = \frac{1}{6}(\delta_t + 3\delta_s + 2\delta_q) - v_8
\]

\[
+ V_0 \left( 2\zeta(3\gamma_s - \gamma_t) + 9\gamma_s^2 - 6\gamma_s\gamma_t + 5\gamma_t^2 \right), \quad (1)
\]

\[
m_\Sigma - m_\Lambda = \frac{1}{2}(\delta_t - \delta_s) - 4V_0 \left( \zeta(\gamma_s + \gamma_t) + \gamma_s^2 - \gamma_t^2 \right), \quad (2)
\]

\[
m_\Xi - m_\Sigma = \frac{1}{2}(2\delta_s + \delta_q) - v_8 + 8V_0 \left( \zeta(3\gamma_s^2 - 3\gamma_s\gamma_t) \right). \quad (3)
\]

The parameters \(\delta_t, \delta_s\) and \(\delta_q\) (of dimension of mass) in these expressions are a measure of the violation of \(SU(3)\) and, added properly to the respective \(SU(3)\) invariant masses \(m_t, m_s\) and \(m_q\), provide necessary mass splittings in the diquark triplet, diquark sextet, and in the third quark \(SU(3)\) triplet. For further details concerning definition and physical meaning of all the involved parameters (including \(v_s, v_t, v_8\) and \(V_0\)) see ref. \[1\].
The modified (improved) version of GMO relation obtained in the LTK diquark-quark model is
\[ \frac{3}{2} m_A + \frac{1}{2} m_{\Sigma} - m_N - m_\Xi = C_{\text{LTK}} \equiv \mu_s (3 \xi_{ts}^2 + 18 \xi_{ts} - 13) \]
(4)
where for convenience we have set
\[ \mu_s = V_0 \gamma_s^2, \quad \xi_{ts} = \gamma_s / \gamma_t. \]
(5)
The quantity \( \mu_s \) must be positive since it can also be inferred by using the decuplet mass combination \[1\]: \( 8 \mu_s = 2 m_{\Xi_{2}} - m_{\Sigma_{2}} - m_0 = +9.8 \text{ MeV}. \) From the viewpoint of agreement with data, it is clear that there exists a continuum of values for the pair \( (\mu_s, \xi_{ts}) \), determined in the \( \mu_s - \xi_{ts} \) plane by the curve \( \mu_s (3 \xi_{ts}^2 + 18 \xi_{ts} - 13) = \text{const.} \), which provide the agreement of eq.(4) with data. To reduce maximally such sort of non-uniqueness, one needs some additional criteria. To this end, LTK exploited the diquark-quark model to just maximize).

Negative sign of the solution \( \xi_{ts} = -3 \) implies that the mass difference \( \delta_s \) must be greater (less) than \( \delta_q \) when the mass difference \( \delta_s \) is less (greater) than \( \delta_q \). However, this value \( \xi_{ts} = -3 \) is in conflict with empirics: it supplies negative value to the r.h.s. of (4) thus providing the correction to GMO in wrong direction.

QUANTUM-GROUP BASED BARYON MASS RELATIONS

Now let us turn to another modification of the GMO mass sum rule, namely the q-deformed mass relation obtained, using the quantum algebras \( U_q(su_n) \) taken for flavor symmetries, in the form \[2, 3, 10\]
\[ [2] m_N + \frac{[2] m_\Xi}{[2] - 1} = [3] m_A + \left( \frac{[2]^2}{[2] - 1} - [3] \right) m_\Sigma \]
(6)
with \([3]_q = ([2]_q)^2 - 1, \) and with \( A_q, B_q \) being certain polynomials in \([2]_q \equiv [2]_q = q + q^{-1} \) whose sets of zeros are completely different. This q-analog yields, as particular cases, the familiar Gell-Mann - Okubo relation \[11\] \( m_N + m_\Xi = \frac{3}{2} m_A + \frac{1}{2} m_\Sigma \) (known to hold with the 0.58% accuracy) at the 'classical' value \( q = 1 \), and the whole infinite set of new mass sum rules \[7, 4\]
\[ m_N + \frac{1}{[2]_q - 1} m_\Xi = \frac{[3]_q}{[2]_q - 1} m_A + \left( \frac{[2]_q}{[2]_q - 1} - \frac{[3]_q}{[2]_q} \right) m_\Sigma, \]
(7)
where \( q_n = \exp(i \pi / n) \). It should be mentioned that for each such value \( q_n \), the respective sum rule shows better agreement with data than GMO one.

The phenomenologically most adequate mass relation among those contained in the series \[7, \] namely
\[ m_N + m_\Xi / ([2]_q - 1) = m_A / ([2]_q - 1) + m_\Sigma, \]
(8)
shows the remarkable 0.07% accuracy. This, most accurate, mass sum rule corresponds to the value \( q_7 = e^{i \pi / 7} \), for which a clear physical meaning was suggested in ref. \[8, 12\] where the value \( q_7 \) was directly linked to the Cabibbo angle, i.e. \( \frac{1}{2} \ln q_7 = \frac{\pi}{7} = 2 \theta_C \).

Now let us present the optimal mass relation \( 9 \) in the form of GMO combination with a correction to it:
\[ \frac{3}{2} m_A + \frac{1}{2} m_{\Sigma} - m_N - m_\Xi = C_q, \]
(9)
\( C_q \equiv ([2]_q - 1)^{-1} (m_\Xi - m_\Lambda) - \frac{1}{2} (m_\Sigma - m_\Lambda). \)

Formulæ \( 9 - 13 \) encode highly nonlinear dependence of mass on SU(3)-breaking. This makes them radically different from the classical GMO result accounting only first order effects in SU(3)-breaking.

Such nonpolynomiality in SU(3)-breaking effectively accounted by the quantum-group based model, for the case of octet baryon masses was demonstrated in \[7. \] For this goal, the explicit dependence on hypercharge \( Y \) and isospin \( I \) of matrix elements for isoplet masses was analyzed. Typical contribution to octet baryon mass contains such expressions as, e.g., the terms \(( \Sigma [Y/2] [Y/2 + 1] q - [I] [I + 1] q) \) or \(( \Sigma [Y/2 - 1] [Y/2 - 2] q - [I] [I + 1] q) \), with multipliers depending on the labels \( m_{15}, m_{55} \) of a chosen dynamical representation. This shows nontrivial explicit dependence on hypercharge and the factor \([I] [I + 1] q \) (q-deformed SU(2) Casimir). Since the q-bracket is \([n]_q = \frac{\sin(n q)}{\sin(q)}\) if \( q = \exp(i h) \), baryon masses depend on \( Y \) and \( I \) (that is, on SU(3)-breaking effects) in highly nonlinear - nonpolynomial - fashion. The ability to account highly nonlinear SU(3)-breaking effects, due to the use of the quantum counterpart \( U_q(su_n) \) of usual flavor symmetries, is similar to the result \[13\] that by exploiting appropriate free q-deformed structure one is able to efficiently describe the properties of (undeformed) quantum-mechanical system with complicated interaction.

DIQUARK-QUARK PARAMETERS AND q-PARAMETER

Now let us relate the results \[11\] and \[9\] of two different approaches. To this end we form, taking the mass differences \( m_\Xi - m_\Lambda \) (multiplied with some \( w \)) and \( m_{\Sigma} - m_\Lambda \) from (1)-(3), the particular combination involved in the
r.h.s. $C_q$, of (9) and equate it to the r.h.s. of (4). Then, imposing
\[ 4w \mu_s \left( \xi_{ts}^2 - 6\xi_{ts} + 5 \right) = \mu_s \left( 5\xi_{ts}^2 + 18\xi_{ts} - 15 \right), \tag{10} \]
\[ w \left[ \frac{1}{3} (\delta_t + \delta_s + \delta_q) - v_s + 2(\gamma_s - \gamma_t)(m_s - m_t + v_s - v_t) \right] + \frac{1}{2} (\delta_s - \delta_t) + (\gamma_s + \gamma_t)(m_s - m_t + v_s - v_t) = 0, \tag{11} \]
with some $w$, guarantees validity of eq. (11) and its correspondence with eq. (4). The solution of (10), namely
\[ w = -1 + \frac{9\xi_{ts}^2 - 6\xi_{ts} + 5}{4(\xi_{ts}^2 - 6\xi_{ts} + 5)}, \tag{12} \]
when put in (11) gives a particular constraint on the parameters of the LTK quark-diquark model.

As result, we are led to the explicit relation between the ($\text{value } q_7 = \exp(i\pi/7)$ of) $q$-parameter in our $q$-GMO and the ratio $\xi_{ts} = \gamma_t/\gamma_s$ of LTK quark-diquark model:
\[ [2]_7 - 1 = 4(\xi_{ts}^2 - 6\xi_{ts} + 5) \tag{13} \]
\[ 9\xi_{ts}^2 - 6\xi_{ts} + 5. \]

Note that the tilda over $\delta_t$, $\delta_t$, and $\delta_s$ in the relation (12) is put in order to indicate that these values are now the optimized ones corresponding to all-order account of $SU(3)$ symmetry breaking in octet baryon masses, as encoded in the r.h.s. of (13).

Since $[2]_7 = 2 \cos \frac{\pi}{7} \simeq 1.80194$, the obtained relation yields as its solutions the two values namely $\xi_{ts}^{(+)} = 0.741$ and $\xi_{ts}^{(-)} = -6.705$. By the very derivation, both these values guarantee the validity of sum rule (11) to within 0.07%. The both values differ substantially from that adopted by LTK and provide positive correction to GMO, see the comment in sec.2 about the negative value of $\xi_{ts}$. Moreover, our positive value $\xi_{ts}^{(+)}$ reflects (even qualitatively) different, from the case of $\xi_{ts}^{(-)}$, physical situation. This value implies that the mass difference $\delta_t$ is greater (lesser) than $\delta_q$ at the same time as the mass difference $\delta_s$ is greater (lesser) than $\delta_q$, since
\[ \delta_t - \delta_q = 0.741(\delta_s - \delta_q), \]
that is
\[ \delta_t - \delta_s = 0.259(\delta_q - \delta_s). \]
Thus, it should necessarily be either
\[ \delta_q > \delta_t > \delta_s \quad \text{or} \quad \delta_s > \delta_t > \delta_q. \tag{14} \]

The deduced inequalities, we hope, give more realistic picture of the hierarchy among the parameters involved in the LTK diquark-quark model.

**LINKING THE LTK MODEL PARAMETERS TO THE CABIBBO ANGLE**

Now let us discuss the already mentioned connection between the $q$-parameter and the Cabibbo angle. As it was shown in [14], the weak mixing is properly modelled by the $q$-deformation. On the other hand, there is the important relation $\theta_W = 2(\theta_{12} + \theta_{23} + \theta_{13})$, found in [15], which connects $\theta_W$ with the Cabibbo angle $\theta_{12} \equiv \theta_C$ (and the $\theta_{13}, \theta_{23}$ mixings with 3rd family, which we neglect). This relates the mixing in bosonic (interaction) sector with that in fermionic (matter) sector of the electroweak model. Combined with (8) this implies: the Cabibbo angle can be linked to $q$-parameter of a quantum-group (or $q$-algebra) based structure applied in the fermion sector. One can thus infer
\[ \theta_8 = \frac{\pi}{7} = 2 \theta_C. \tag{15} \]

The latter formula suggests for Cabibbo angle the exact value $\frac{\pi}{7}$. (Remind that for the $q$-deformed analog of decuplet mass formula [12] [16], we have $\theta_{10} = \theta_C$.) As a final result of this note we deduce the following corollary concerning direct link of the Cabibbo angle to the (optimized) parameters of the diquark-model.

Indeed, using $\theta_7 = \frac{1}{4} \ln q_7 = 2 \theta_C$ as given in (15), we obtain from eq. (13) the formula in question, that is
\[ \cos(2\theta_C) = \frac{13\xi_{ts}^2 - 30\xi_{ts} + 25}{2(9\xi_{ts}^2 - 6\xi_{ts} + 5)}, \quad \xi_{ts} = \frac{\delta_t - \delta_q}{\delta_s - \delta_q}. \tag{16} \]

This remarkable formula connects the Cabibbo angle with the parameters of the LTK model, characterizing the diquark and the third quark, whose optimized values are such that the inequalities (14) should hold.

**APPENDIX: LTK MODEL PARAMETERS AND $q$-PARAMETER, GENERAL CASE**

In the appendix we demonstrate that, in principle, the connection with the result (11) of LTK model can be founded *in the general case* of an arbitrary member from the infinite discrete set presented in (17).

Indeed, eq. (7) can be rewritten as
\[ 3m_A + \frac{3}{2} m_N - m_\Sigma - m_\Xi = \left( \frac{3}{2} |q_n| - 1 \right) (m_\Sigma - m_A) + \frac{1}{|q_n| - 1} (m_\Xi - m_\Sigma - m_\Xi) + m_A. \tag{7'} \]

Now take mass differences from (1)-(3). Multiply $m_\Xi - m_\Sigma$ with some $x^{-1}$, $m_\Sigma - m_\Lambda$ with the appropriate
\[ y = \frac{x(x + 2)}{x + 1} - \frac{1}{2}. \]
then form the particular combination that corresponds to the r.h.s. of \((7')\) and equate it to the r.h.s. of \((4)\). The outlined procedure gives us: the relation
\[
4 \left( \frac{x(x + 2)}{x + 1} - \frac{1}{2} \right) (\xi_{ts}^2 - 1) - 24x^{-1}(\xi_{ts} - 1)
\]
\[= 7\xi_{ts}^2 - 6\xi_{ts} + 7 \tag{17}\]
and the additional constraint
\[
\left( \frac{x(x + 2)}{x + 1} - \frac{1}{2} \right) \left\{ \frac{1}{3} (\delta_t - \delta_s) - 4\mu_s \zeta(1 + \xi_{ts}) \right\}
\]
\[+ x^{-1} \left\{ \frac{1}{3} (2\delta_s - \delta_q) - v_8 + 8\mu_s \zeta \right\}
\]
\[\quad - \frac{1}{3} (\delta_t + \delta_s + \delta_q) + v_8 + 4\mu_s \zeta(\xi_{ts} - 1) = 0 \tag{18}\]
with \(\xi_{ts}\) given in \((5)\), when hold simultaneously, provide validity of eq.(4). Moreover, the latter is directly correspondent with the eq.(7‘) if the identification
\[x \leftrightarrow [2]_{q_n} - 1\]
is made. Not going into further details, let us only mention that just at \(q = \exp(i\pi/7)\) (i.e., in the case most adequate phenomenologically) we have the equality
\[[3]_{q/I}/[2]_{q/I} = ([2]_{q/I} - 1)^{-1},\]
or, put in another way,
\[\frac{x(x + 2)}{x + 1} = x^{-1}.\]
With this simplification, we recover the relations \((12)-(13)\) of the distinguished particular case considered above. It is this remarkable phenomenological validity of \((8)\) or \((9)\) that makes it possible to link the LTK diquark-quark model parameters to the Cabibbo angle, as expressed in eq. \((16)\).