The “Wrong Minimal Surface Equation” does not have the Bernstein property

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A celebrated result of S. Bernstein [3] states that every solution of the minimal surface equation over the entire plane $\mathbb{R}^2$ has to be an affine linear function. Since the paper of Bernstein appeared in 1927, many different proofs and generalizations of this beautiful theorem were given, namely to higher dimensions and to more general equations, for a careful account we refer to the paper by Simon [6] and to the monograph by Dierkes-Hildebrandt-Tromba [4] chap. 3.

In his paper [5] Simon posed the question whether the equation

$$ (1 + u_x^2)u_{xx} + 2u_xu_yu_{xy} + (1 + u_y^2)u_{yy} = 0 $$

(1)

has the Bernstein property i.e. whether every $C^2$-solution defined on all of $\mathbb{R}^2$ necessarily has to be affine.

We here show by a very simple argument that this is not the case.

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To start with we consider $u \in C^2(\mathbb{R}^2)$ to be a solution of the elliptic equation (1) with $u_{xy} \equiv 0$ in the whole of $\mathbb{R}^2$. Then $u$ has the form $u(x, y) = h(x) + g(y)$, with $g, h \in C^2(\mathbb{R})$ and the equation (1) becomes: $(1 + (h'(x))^2)h''(x) + (1 + (g'(y))^2)g''(y) = 0$. We put $(1 + (h'(x))^2)h''(x) = c$ and hence $(1 + (g'(y))^2)g''(y) = -c$, $c \in \mathbb{R}$, and choose $c = 1$.

(To get the linear solutions take $c = 0$.)

By separation of variables we solve the equation $(1 + f^2)f' = 1$ with $f = h'(x)$ and obtain $(1 + f^2)df = dx$, or $f + \frac{f^3}{3} = x$. By Cardano’s formulae:

$$ h'(x) = f(x) = \frac{1}{\sqrt{2}} \left( \frac{3}{\sqrt[3]{9x^2 + 4}} + \frac{3}{\sqrt[3]{9x^2 + 4 - 3x}} \right). $$

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An integration yields:

\[ h(x) = \frac{-1}{\sqrt[3]{1024}} \left\{ 9x \left( \frac{\sqrt[3]{9x^2 + 4} - 3x}{\sqrt[3]{9x^2 + 4} + 3x} \right) + \sqrt[3]{9x^2 + 4} \left( \frac{\sqrt[3]{9x^2 + 4} - 3x}{\sqrt[3]{9x^2 + 4} + 3x} \right) \right\}. \]

Similarly, we get

\[ g(y) = \frac{1}{\sqrt[3]{1024}} \left\{ 9y \left( \frac{\sqrt[3]{9y^2 + 4} - 3y}{\sqrt[3]{9y^2 + 4} + 3y} \right) + \sqrt[3]{9y^2 + 4} \left( \frac{\sqrt[3]{9y^2 + 4} - 3y}{\sqrt[3]{9y^2 + 4} + 3y} \right) \right\} = -h(y). \]

Thus, the non-linear \(C^2\)-function

\[ u(x, y) = h(x) - h(y) \]

solves (1) in the whole plane \(\mathbb{R}^2\).

\[ u(x, y) \]

**Remark.** The above \(u\) solves also the elliptic equation

\[ (1 + u_x^2)u_{xx} - 2u_xu_yu_{xy} + (1 + u_y^2)u_{yy} = 0. \]

More generally, we have the following

**Theorem.** Let \(F_i \in C^1(\mathbb{R}, \mathbb{R})\) be bijective with positive derivative \(F'_i = f_i > 0\) for \(i = 1, 2\). Then the equation

\[ f_1(u_x) \cdot u_{xx} + 2B \cdot u_{xy} + f_2(u_y) \cdot u_{yy} = 0, \]  

with an arbitrary \(B\) (depending on \(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}\)) has non-linear entire \(C^2\)-solutions in \(\mathbb{R}^2\), i.e. (2) does not have the Bernstein property.

(For the ellipticity of (2) assume \(|B| < \sqrt{f_1(u_x)f_2(u_y)}\).)

**Proof.** We proceed analogously as above: To the end we construct a \(C^2\)-solution \(u\) with \(u_{xy} \equiv 0\), i.e. \(u(x, y) = h(x) + g(y)\), with \(g, h \in C^2(\mathbb{R})\). Thus, our equation (2) becomes:

\[ f_1(h'(x))h''(x) + f_2(g'(y))g''(y) = 0. \]

Put \(f_1(h'(x))h''(x) = c\) and \(f_2(g'(y))g''(y) = -c\), with an arbitrary constant \(c \in \mathbb{R}\). For the linear solutions take \(c = 0\). Since we are interested in non-linear ones, let us choose \(c = 1\):

By separation of variables we get:

\[ F_1(h'(x)) = x \quad \text{and} \quad F_2(g'(y)) = -y. \]
Since $F_i (i = 1, 2)$ is bijective in the whole of $\mathbb{R}$, a non-linear entire $C^2$—solution is given by

$$u(x, y) = \int F_1^{-1}(x) dx + \int F_2^{-1}(-y) dy,$$

wherein $F_i^{-1}$ is the bijective continuous inverse of $F_i (i = 1, 2)$.

**Example 1.** Taking $F_i(t) = t$ we find that $u(x, y) = x^2 - y^2$ solves the elliptic equation

$$u_{xx} + u_{xy} + u_{yy} = 0.$$

**Example 2.** With $f_i(t) = 1 + t^2$ and $F_i(t) = t + \frac{t^3}{3}$ we obtain the equation \([1]\).

**Example 3.** Take $f_i(t) = \frac{1}{\sqrt{1 + t^2}}$ and $F_i(t) = \text{arsinh } t$ respectively,

then $u(x, y) = \cosh(x) - \cosh(y)$ solves

$$\frac{u_{xx}}{\sqrt{1 + u_x^2}} + \frac{u_{yy}}{\sqrt{1 + u_y^2}} = 0$$

or

$$\sqrt{1 + u_y^2} \cdot u_{xx} - u_{xy} + \sqrt{1 + u_x^2} \cdot u_{yy} = 0$$

also

$$\sqrt{1 + u_y^2} \cdot u_{xx} + 2\tilde{B}u_{xy} + \sqrt{1 + u_x^2} \cdot u_{yy} = 0,$$

with an arbitrary $\tilde{B}$ such that $|\tilde{B}| < \frac{1}{4}(1 + u_y^2)(1 + u_x^2)$.

**Corollary.** With the notation of the theorem we obtain:

The function

$$u = u(x, y) = \int F_1^{-1}(x) dx + \int F_2^{-1}(-y) dy$$

also solves the equation

$$\frac{u_{xx}}{f_2(u_y)} + 2\tilde{B}u_{xy} + \frac{u_{yy}}{f_1(u_x)} = 0,$$

with an arbitrary $\tilde{B}$, i.e. this equation does not have the Bernstein property in $\mathbb{R}^2$.

(For the ellipticity of our last equation assume $|\tilde{B}| < \frac{1}{\sqrt{f_2(u_y)f_1(u_x)}}$.)

**Remark.** Both the bijectivity and the strict positivity of $F_i'$ respectively are essential for the conclusion of the theorem.

In fact we have the following counterexamples:

**Example** Take $f_i(t) = \frac{1}{1 + t^2}$ and $F_i(t) = \text{arctan}(t)$ respectively (so $F_i : \mathbb{R} \to \mathbb{R}$ is not bijective), then we get:

$$u(x, y) = \ln(\cos y) - \ln(\cos x) \in C^2 \left(\left(-\frac{\pi}{2}; \frac{\pi}{2}\right) \times \left(-\frac{\pi}{2}; \frac{\pi}{2}\right)\right)$$

which solves

the minimal surface equation $(1 + u_y^2)u_{xx} - 2u_xu_yu_{xy} + (1 + u_x^2)u_{yy} = 0$

and the equation $\frac{u_{xx}}{1 + u_x^2} + \frac{u_{yy}}{1 + u_y^2} = 0$ respectively, clearly not on all of $\mathbb{R}^2$. 

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The condition $F'_i > 0$ cannot be replaced by the strong monotonicity of $F_i$ (for $i = 1$ or $i = 2$). Otherwise the solution $u$ can develop singularities:

**Example** Take $f_i'(t) = t^2$ and $F_i(t) = \frac{1}{3} t^3$ respectively, then we obtain:

$$u(x, y) = \frac{9}{4} \left( |x|^{4/3} - |y|^{4/3} \right)$$

which solves the equation

$$u_x^2 u_{xx} + 2 u_x u_y u_{xy} + u_y^2 u_{yy} = 0. \quad (3)$$

Aronsson presented this $C^{1,1/3}(\mathbb{R}^2)$ — “singular solution” of (3) in [2]:

$u$ is $C^1$ in $\mathbb{R}^2$, $C^\infty$ in each open quadrant and the coordinate axes are lines of singularity for $u$.

Interestingly, the equation (3) has the Bernstein-property, see [1].

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**Added in Proof.** Quite recently, I have found the presentation of P. A. Bezborodov at the International Conference on Analysis and Geometry (1999, Novosibirsk, Russia) where a similar result was stated, however no proofs were given, cf. Bezborodov, P. A., Kontrprimier k gipoteze Sa imona, Tezisy Trudov Meжdunarodno i konferencii po analizu i geometrii, Novosibirsk, 30 avg.-3 sent. 1999. – Novosibirsk: Izd-во ИМ СО РАН, 1999. – С. 10–11. (BEZBORODOV P. A., A Counterexample to Simon’s Conjecture, Novosibirsk, 1999, in Russian.)