Improved Monte-Carlo method for solving of integral Fredholm’s equations of a second kind, with confidence regions in the uniform norm

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Abstract

We offer in this article some modification of Monte-Carlo method for solving of a linear integral Fredholm’s equation of a second kind (Fredholm’s well posed problem).

We prove that the rate of convergence of offered method is optimal under natural conditions still in an uniform norm, and construct an asymptotic as well as non-asymptotic confidence region, again in the uniform norm.

Key words and phrases: Kernel, Linear integral Fredholm’s equation of a second kind, Monte-Carlo method, random variables, natural distance, Central Limit Theorem in the space of continuous functions, contraction, Kroneker’s and ordinary degree of integral operator, Gaussian random field, covariation function, tail of distribution, metric entropy, entropic integral, ordinary and subgaussian norm and space, norm of linear operator, compact operators, Neuman series, uniform norm, spectral radius, asymptotic and non-asymptotic confidence region, random variable and random vector, variance, Dependent Trial Method (DTM).

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1 Definitions. Notations. Previous results.

Statement of problem.

Let \( T = \{t\} \) be compact metrisable space equipped with probabilistic Borelian non-trivial complete measure: \( \mu(T) = 1 \). The completeness of the measure implies that each non-empty open set \( A \) has a positive measure: \( \mu(A) > 0 \).

We consider a linear Fredholm’s type equation in the space \( C(T) \) of continuous functions equipped as ordinary with the uniform norm.
of the form

$$z(t) = f(t) + \int_T K(t, s) z(s) \mu(ds) = f(t) + \int T \left( \int T K(t, s) z(s) \mu(ds) \right) , \quad t, s \in T ,$$

where the kernel $K = K(t,s)$ is a continuous function of two variables $(t,s) \in T^2$, so that $K[\cdot]$ is a linear integral (compact) operator of the form

$$K[z](t) \equiv \int T K(t, s) z(s) \mu(ds), \quad z(\cdot) \in C(T),$$

and with continuous non-zero “free” member $f = f(t), \; t \in T$.

We will write for brevity in the sequel

$$\int g(t) \mu(dt) \equiv \int T g(t) \mu(dt), \; g(\cdot) \in C(T)$$

or at last for the function $g(\cdot) \in L_1(T, \mu)$.

Put

$$\rho = \rho[K] \equiv \max_{t \in T} \int T |K(t, s)| \mu(ds), \quad 0 \leq \rho[K] \leq \overline{\rho}[K].$$

Obviously, $0 \leq \rho[K] \leq \overline{\rho}[K]$.

The value $\rho = \rho[K]$ is nothing more that the norm of operator $K$ acting in the space $C(T)$:

$$\rho[K] = \|K\|_{C(T) \to C(T)} = \sup_{0 \neq g \in C(T)} \left\{ \frac{\|K[g]\|}{\|g\|} \right\} .$$

We suppose hereafter that the operator $K[\cdot]$ satisfies in addition the contraction principle:

$$\rho[K] = \|K\|_{C(T) \to C(T)} := \sup_{0 \neq g \in C(T)} \left\{ \frac{\|K[g]\|}{\|g\|} \right\} < 1 .$$

The classical expression for solution $z(t)$ may be written by virtue of the uniformly convergent iterations of fixed point method (Neuman series): $z(t) = \lim_{d \to \infty} z_d(t)$, where $z_0(t) = f(t)$, and for $d = 0, 1, 2, \ldots$

$$z_{d+1}(t) = f(t) + \int T K(t, s) z_d(s) \mu(ds) = f(t) + K[z_d](t).$$

The iteration $z_d(t)$ has a form $z_0(t) = K^0[f](t) = f(t)$,
\[ z_1(t) = K[f](t) = \int_T K(t, s) f(s) \mu(ds), \]

and in the case when \( d \geq 2 \)

\[ z_d(t) = K^d[f](t) = \int_{T^d} K_d(t, \vec{s}) \prod_{l=1}^{d} \mu(ds_l) = \]

\[ \int_{T^d} K_d(t, s_1, s_2, \ldots, s_d) \cdot f(s_d) \cdot \prod_{l=1}^{d} \mu(ds_l) = \]

\[ \int_{T^d} \cdots \int_{T^d} K(t, s_1) \cdot \prod_{r=1}^{d-1} K(s_r, s_{r+1}) \cdot f(s_d) \cdot \prod_{l=1}^{d} \mu(ds_l) \] \hspace{1cm} (1.6)

Here \( K^d[\cdot] \) denotes the \( d \)th power of the (integral) operator \( K[\cdot] \), in contradiction to the ordinary power of the function \( K^2(t, s) \),

\[ \vec{s} = \vec{s}_d = (s_1, s_2, \ldots, s_d), \quad K_d(t, \vec{s}) = K_d(t, s_1, s_2, \ldots, s_d) := \]

\[ K(t, s_1) \cdot \prod_{r=1}^{d-1} K(s_r, s_{r+1}) \] .

The truncated sum

\[ z^{(M)}(t) \overset{\text{def}}{=} f(t) + \sum_{d=1}^{M} z_d(t), \quad M = 2, 3, \ldots \]

gives a following error estimate

\[ \|z(\cdot) - z^{(M)}(\cdot)\| \leq \|f\| \cdot \frac{\rho^{M+1}}{1 - \rho}. \]

In order to calculate each integral \( z_d(t) \), we offer the classical Monte-Carlo method, which is named as ”Dependent Trial Method”, see [9], [10], [19], [20]. Indeed, let us introduce a \( d \)-dimensional random vector \( \xi^{(d)} \):

\[ \xi^{(d)} = \{\xi_1^d, \xi_2^d, \ldots, \xi_d^d\}, \]

where all the random variables \( \xi_k^d \) are (common) independent and have the distributions \( \mu \):

\[ \mathbf{P} \left( \xi_k^d \in A \right) = \mu(A). \]

Let also \( \xi_i^{(d)} \) be independent copies of the r.v. \( \xi^{(d)} \). The consistent as \( n(d) \to \infty \), here and in the sequel \( n(d) \), \( d = 1, 2, \ldots \) are ”Great” integer numbers, unbiased Monte-Carlo approximation \( x_{d,n(d)}(t) \) for the integral \( x_d(t) \) may has a form
The so-called Depending Trial Method (DTM), see [9] [10], [20], and correspondingly consistent as $\min_d n(d) \to \infty$ an unbiased Monte-Carlo estimation $x^{(M)}_\vec{n}(t)$ for $x^{(M)}(t)$ may be written as

$$z^{(M)}_{\vec{n}}(t) := f(t) + \sum_{d=1}^{M} x_{d,n(d)}(t) = f(t) + \sum_{d=1}^{M} \frac{1}{n(d)} \sum_{j=1}^{n(d)} K_d(t, \vec{\xi}_i^{(d)}) \cdot f(\xi^{(d)}_i),$$  \hspace{1cm} (1.8)

with probability one in the uniform norm:

$$\mathbf{P}\left( ||z^{(M)}_{\vec{n}}(t) - z^{(M)}(t)|| \to 0 \right) = 1,$$

the Law of Large Numbers (LLN) in the Banach space $C(T)$.

This method was investigated in [20], see also [10], [4], [5], [8], [11]-[12], [13], [17], [21] etc. It was proved in particular in [20] that

$$\text{Var}_M(\vec{n}) \overset{\text{def}}{=} \max_t \text{Var}\left\{ z^{(M)}_{\vec{n}}(t) \right\} \leq ||f||^2 \cdot \sum_{d=1}^{M} [n(d)]^{-1}(\rho_2)^d,$$  \hspace{1cm} (1.9)

including (formally) the case $M = \infty$.

We introduced above the value

$$\rho_2 = \rho_2[K] \overset{\text{def}}{=} \max_{t \in T} \left( \int_T (K(t,s))^2 \mu(ds) \right),$$  \hspace{1cm} (1.10)

so that $\rho_2[K]$ represents the norm: $\rho_2[K] = ||K(2)||_{(C(T) \to C(T))}$ of a Kroneker’s square $K(2)[\cdot]$ of the integral operator $K$, i.e. the compact linear integral operator acting inside the space $C(T)$ with the continuous kernel $K^2(t, s)$:

$$K^{(2)}[g](t) \overset{\text{def}}{=} \int_T K^2(t, s) g(s) \mu(ds).$$  \hspace{1cm} (1.11)

Of course, $\rho \leq ||\rho_2||^2$.

More detail information about integral operators may be found in the classical books of N.Dunford and J.Schwartz [6], [7].

One can calculate the general amount $N$ of elapsed independent random variables with distribution $\mu$ as follows
\[ N = \sum_{d=1}^{M} d \cdot n(d). \]  

Solving the following conditional extremal problem

\[ \text{Var}_M(\bar{n}) \to \min \]

subject to the restriction

\[ \sum_{d=1}^{M} d \cdot n(d) \leq N, \] \hspace{1cm} (1.13)

the authors of the article [20] obtained the following estimate

\[ \min \text{Var}_M(\bar{n})/\left[ \sum_{d=1}^{M} d \cdot n(d) \leq N \right] \approx \frac{1}{N}. \] \hspace{1cm} (1.14)

On the other words, the speed of convergence \( z_n^{(M)}(t) \to z^{(M)}(t) \) is equal to \( 1/\sqrt{N} \), alike as in the classical Monte-Carlo method. At the same is true also in the uniform norm, see [20].

We want to improve in this report the described before algorithm, namely, we intend slightly reduce the number of expended random variables with at the same exactness.

We will ground also the convergence of our approximation in the uniform norm still with the classical rate and describe the building of confidence region in this norm, asymptotical or not.

Some modern results about the Monte-Carlo solutions of inequations be found in an articles [8], [11]-[12], [13], [20]; see also a monograph of Prem K.Kythe, Pratap Puri [17].

Throughout this paper, the letters \( C, C_j(\cdot) \) etc. will denote a various positive finite constants which may differ from one formula to the next even within a single string of estimates and which does not depend on the essentially variables \( t, x, y, u \) etc.

We make no attempt sometimes to obtain the best values for these constants.

One of the new applications, namely, in the reliability theory, of these equations may be found for instance in [10]; see also the reference therein.

Another denotation: we define for arbitrary random variable \( \xi \) it centering \( \xi^{(0)} \) as an ordinary linear operator

\[ \xi^{(0)} \overset{\text{def}}{=} \xi - \mathbf{E}\xi, \] \hspace{1cm} (1.15)

the “pure random part”, which is defined for arbitrary r.v. from the space \( L_1(\Omega, \mathbf{P}) \), so that \( \mathbf{E}\xi^{(0)} = 0 \) and \( \text{Var} \left[ \xi^{(0)} \right] = \text{Var} \xi. \)
2 Structure of offered solution. Rough estimate of convergence.

Let us introduce the following (functional) recursion: \( x_{0,0}(t) := f(t), \ t \in T \) and for the values \( m = 0, 1, 2, \ldots, M - 1, \) where \( M = \text{const} \geq 2 \)

\[
x_{m+1,n(m+1)}(t) := f(t) + \frac{1}{n(m+1)} \sum_{l=1}^{n(m+1)} K \left( t, \xi_l^{(m)} \right) x_{m,n(m)} \left( \xi_l^{(m)} \right), \tag{2.0}
\]

which is in turn some modification of the Depending Trial Method.

We describe here a rough investigations of this approach; the rigorous reasoning will be represented in the next sections.

Note first of all that the amount of all elapsed random variables with distribution \( \mu \) in (2.0), which we will denote again by \( N, \) is

\[
N = \sum_{d=1}^{M} n(d), \tag{2.1}
\]

As before, \( \{\xi_l^{(m)}\} \) are common independent random variables with distribution \( \mu \) and \( \{n(m)\} \) are certain sequence of natural positive numbers.

It will be presumed furthermore that

\[
\forall m = 1, 2, \ldots, M \implies n(m) \to \infty. \tag{2.2}
\]

All the functions \( x_{m,n(m)}(t), \ m = 1, 2, \ldots, M \) are random processes (r.p.), or more generally random fields (r.f). For instance,

\[
x_{1,n(1)}(t) = f(t) + \frac{1}{n(1)} \sum_{l=1}^{n(1)} K \left( t, \xi_l^{(1)} \right) f \left( \xi_l^{(1)} \right). \tag{2.3}
\]

The last expression may be rewritten under some conditions, which be clarified below, as follows.

\[
x_{1,n(1)}(t) = x_{1}(t) + \frac{1}{\sqrt{n(1)}} \tau_{1,n(1)}(t), \tag{2.4}
\]

where the r.f. \( \tau_{1,n(1)}(t) \) is uniformly relative the index \( n(1) \) subgaussian:

\[
\sup_{n(1) \geq 1} P \left( \sup_{t \in T} \left| \tau_{1,n(1)}(t) \right| > u \right) \leq \exp \left( -C_1 \ u^2 \right), \tag{2.5}
\]

where \( C_1 = C_1[f, K] = \text{const} > 0. \) As \( n(1) \to \infty \) the sequence of r.f. \( \{\tau_{1,n(1)}(t)\} \) converges weakly in the space \( C(T) \) by virtue of CLT in this space to the centered continuous a.e. Gaussian random field \( \tau_1(t) = \tau_{1,\infty}(t) \) with covariation function
\[ R_1(t_1, t_2) = \text{Cov} (\tau_1(t_1), \tau_1(t_2)) = \mathbf{E} r_1^0(t_1) r_1^0(t_2) = \]
\[
\int_T K(t_1, s) K(t_2, s) f^2(s) \mu(ds) - \int_T K(t_1, s) f(s) \mu(ds) \cdot \int_T K(t_2, s) f(s) \mu(ds),
\]
and wherein
\[
\sup_{t \in T} \text{Var} \left[ x_{1,n(1)}(t) \right] \leq \frac{D(1)}{n(1)}, \quad D(1) = \text{const} = \max_{t \in T} R_1(t, t) < \infty. \quad (2.7)
\]

Let us introduce the following covariation functions, more exactly, the sequence of ones
\[ R_{m+1}(t_1, t_2) \overset{\text{def}}{=} \int_T K(t_1, s) K(t_2, s) x_m^2(s) \mu(ds) - \int_T K(t_1, s) x_m(s) \mu(ds) \cdot \int_T K(t_2, s) x_m(s) \mu(ds), \quad m = 0, 1, 2, \ldots. \quad (2.8)\]

Define also for arbitrary (continuous) covariation function \( R = R(t_1, t_2) \) its linear transform (operator)
\[ V_K[R](t_1, t_2) \overset{\text{def}}{=} \int_T K(t_1, s) K(t_2, s) R(s, s) \mu(ds). \quad (2.9)\]

Evidently, the function \( (t_1, t_2) \rightarrow V_K[R](t_1, t_2), \ t_1, t_2 \in T \) is also a continuous covariation function.

One can substitute the expression (2.4) for the (random) function \( x_{1,n(1)}(t) \) into the recursion (2.0) in order to obtain the value for \( x_{2,n(2)} : x_{2,n(2)}(t) = \)
\[
 f(t) + \frac{1}{n(2)} \cdot \left\{ \sum_{j=1}^{n(2)} K \left( t, \xi_j^{(2)} \right) \left[ x_1(\xi_j^{(2)}) + n(1)^{-1/2} \xi_{1,n(1)}(\xi_j^{(2)}) \right] \right\} = \\
 f(t) + \frac{1}{n(2)} \cdot \left\{ \sum_{j=1}^{n(2)} K \left( t, \xi_j^{(2)} \right) x_1(\xi_j^{(2)}) \right\} + \\
n(2)^{-1} n(1)^{-1/2} \left\{ \sum_{j=1}^{n(2)} K \left( t, \xi_j^{(2)} \right) \xi_{1,n(1)}(\xi_j^{(2)}) \right\} = \\
x_2(t) + \frac{1}{\sqrt{n(2)}} \tau_2(t) + \frac{1}{\sqrt{n(2)n(1)}} \tau_{2,1}(t),
\]
where the centered non-wise correlated r.f. \( \tau_2(t), \tau_{2,1}(t) \) have correspondingly covariation functions.
\[
\text{Cov}(\tau_2(t_1), \tau_2(t_2)) = R_2(t_1, t_2)
\]

and
\[
R_{2,1}(t_1, t_2) \overset{\text{def}}{=} \text{Cov}(\tau_{2,1}(t_1), \tau_{2,1}(t_2)) = V_K[R_1](t_1, t_2).
\] (2.10)

Let us denote for simplicity
\[
\sigma(m + 1) := \frac{1}{\sqrt{n(m + 1)}}, \quad \sigma(m + 1, m) := \frac{1}{\sqrt{n(m + 1) n(m)}},
\]
\[
\sigma(m + 1, m, m - 1) := \frac{1}{\sqrt{n(m + 1) n(m) n(m - 1)}}, \ldots,
\]
\[
\sigma(m + 1, m, m - 1, \ldots, m - k) = \frac{1}{\sqrt{n(m + 1) n(m) n(m - 1) n(m - 2) \ldots n(m - k)}}, \quad k = 0, 1, \ldots, m - 1;
\]

so that
\[
\sigma(m + 1, m, m - 1, \ldots, 1) = \frac{1}{\sqrt{\prod_{l=-1}^{m-1} n(m - l)}}.
\]

We deduce passing to the more general case the following decomposition:
\[
x_{m+1,n(m+1)}(t) = x_{m+1}(t) + \sigma(m + 1) \tau_{m+1}(t) + \sigma(m + 1, m) \tau_{m+1,m}(t) + \sigma(m + 1, m, m - 1) \tau_{m+1,m,m-1}(t) + \ldots + \sigma(m + 1, m, m - 1, \ldots, 1) \tau_{m+1,m,m-1,\ldots,2,1}(t),
\] (2.11)

where the centered and non-correlated r.f. \(\{\tau_k(t)\}\) have the covariation function correspondingly
\[
\text{Cov}(\tau_{m+1}(t_1), \tau_{m+1}(t_2)) = R_{m+1}(t_1, t_2);
\]
\[
R_{m+1,m}(t_1, t_2) := \text{Cov}(\tau_{m+1,m}(t_1), \tau_{m+1,m}(t_2)) = V_K[R_{m,m-1}](t_1, t_2);
\]
\[
R_{m+1,m,m-1}(t_1, t_2) := \text{Cov}(\tau_{m+1,m,m-1}(t_1), \tau_{m+1,m,m-1}(t_2)) = V_K[R_{m,m-1,m-2}](t_1, t_2);
\]

\[
R_{m+1,m,m-1}(t_1, t_2) := \text{Cov}(\tau_{m+1,m,m-1}(t_1), \tau_{m+1,m,m-1}(t_2)) = V_K[R_{m,m-1,m-2}](t_1, t_2);
\]
\[ R_{m+1, m-1, \ldots, 1}(t_1, t_2) := \text{Cov} (\tau_{m+1, m, m-1, \ldots, 1}(t_1), \tau_{m+1, m, m-1, \ldots, 1}(t_2)) = \]

\[ V_K[R_{m, m-2, \ldots, 1}(t_1, t_2)], \]  

a recursion: \( m = 2, 3, \ldots, M - 1. \)

Of course, all the centered continuous random fields \( \tau_{m+1, m, m-1, \ldots, m-k}(t), 1 \leq k \leq m - 1, m = 1, 2, \ldots, M \) dependent on the vector \( \vec{n}_m = \vec{n} = \{n(m), n(m-1), \ldots, n(1)\} : \)

\[ \tau_{m+1, m, m-1, \ldots, m-k}(t) = \tau^{(\vec{n})}_{m+1, m, m-1, \ldots, m-k}(t). \]

They are uniformly subgaussian:

\[
\sup_{\vec{n}} P \left( \sup_{t \in T} |\tau_{m+1, m, m-1, \ldots, m-k}(t) | > u \right) \leq \exp \left( -C(K(\cdot, \cdot), f(\cdot); m + 1, m, m - 1, \ldots, m - k) u^2 \right), \tag{2.13}
\]

\( C(K(\cdot, \cdot), f(\cdot); m + 1, m, m - 1, \ldots, m - k) > 0, \) and converges weakly in distribution in the space \( C(T) \) as \( \min_m n(m) \to \infty \) to the continuous centered independent Gaussian random fields with covariations described in (2.12).

Note that the variance \( \text{Var} \left[ x_{M, n(M)}(t) \right] \) allows the following estimate

\[
C^{-1}(K, f) \max_{t \in T} \text{Var} \left[ x_{M, n(M)}(t) \right] \leq \frac{1}{n(M)} + \frac{1}{n(M) n(M-1)} + \frac{1}{n(M) n(M-1) n(M-2)} + \ldots + \frac{1}{n(M) n(M-1) n(M-2) \ldots n(2) n(1)}, \tag{2.14}
\]

\( C(K, f) \in (0, \infty), \) and besides if for instance both the functions \( f(\cdot), K(\cdot, \cdot) \) are positive, the inverse inequality to the one in (2.14) holds true relative the other constant.

One can choose for instance \( n(M) := N/2, \) and further

\[
n(M-1) := N/4, n(M-2) := N/8, \ldots, n(M-k) := N/2^{k+1}, \tag{2.15}
\]

to make sure that the offered algorithm in comparison to the previous described before obeys at the same speed of convergence but requires significantly less random variables, cf. the relations (2.1) and (1.12).

It will be presumed in the sequel of course that

\[
N \gg 2^{M+1}. \tag{2.16}
\]
3 Central Limit Theorem for described solution.
Asymptotical confidence interval in the uniform norm.

We recall now some notions from the theory of the Central Limit Theorem (CLT) in the separable Banach space of continuous functions $C(T)$. More detail information about this theory may be found in [2], [5], [15], [18], [22] etc.

Let $\nu = \nu(t)$, $t \in T$ be centered: $E\nu(t) = 0$, $t \in T$ numerical valued random field having finite second moment:

$$\forall t, s \in T \exists R(t, s) = \text{Cov}(\nu(t), \nu(s)).$$  \hspace{1cm} (3.0)

Let also $\nu_i(t)$ be independent copies of $\nu(t)$. Denote

$$S_n(t) := n^{-1/2} \sum_{i=1}^{n} \nu_i(t).$$  \hspace{1cm} (3.1)

The r.f. $\nu(\cdot)$ satisfies by definition CLT in the space $C(T)$, if r.f. $\nu(t)$ is continuous a.e. and the sequence of r.f. $S_n(\cdot)$ converges as $n \to \infty$ weakly in distribution to the Gaussian r.f. $S_\infty(t) = S(t)$, also continuous everywhere, having at the same first two moments as the source r.f. $\nu(t)$.

This implies that for arbitrary continuous bounded functional $F : C(T) \to R$

$$\lim_{n \to \infty} E F(S_n) = E F(S).$$  \hspace{1cm} (3.2)

As a consequence from (3.2)

$$\lim_{n \to \infty} P(\max_{t \in T} |S_n(t)| > u) = P(\max_{t \in T} |S(t)| > u), \quad u > 0.$$  \hspace{1cm} (3.3)

Note that the asymptotical as $u \to \infty$ behavior as well as non-asymptotical at $u \geq 1$ estimates are well known, see e.g. [21], [18]. This circumstance allows to construct a confidence region in the Monte-Carlo parametric computations, see [9], [10], [18].

We must recall also some used facts about the so - called subgaussian random variables. The r.v. $\xi$ defined on some probability space is said to be subgaussian, write: $\xi \in \text{Sub}$, if there exists a non-negative constant $\tau = \tau(\xi) = \tau(\text{Law}(\xi))$, for which

$$\forall \lambda \in R \Rightarrow \max_{\pm} E \exp(\pm \lambda \xi) \leq \exp(0.5 \lambda^2 \tau^2).$$  \hspace{1cm} (3.4)

This notion was introduced at first by Kahane J.P. in [14]. See also [2], [3], [5], [10], [18], [19], [22] etc.

Evidently, if $\xi \in \text{Sub}$, $\xi \neq 0$, then $E\xi = 0$ and

$$\max \{P(\xi \geq u), P(\xi \leq -u)\} \leq \exp \left( -0.5 \frac{u^2}{\tau^2} \right), \quad u \geq 0;$$  \hspace{1cm} (3.5)
and inverse conclusion up to multiplicative constant is also true.

The minimal non-negative value $\tau = \tau(\xi)$ satisfying (3.4) for all the values $\lambda \in \mathbb{R}$, is named as subgaussian norm of the r.v. $\xi$, write $||\xi||_{\text{Sub}}$:

$$||\xi||_{\text{Sub}} \overset{\text{def}}{=} \max_{\lambda \neq 0} \sup \left\{ \ln \mathbb{E} \exp(\pm \lambda \xi) \right\}^{1/2} / |\lambda|.$$

(3.6)

Buldygin V.V. and Kozachenko Yu.V. in [2], see also [3] proved that the functional $\xi \rightarrow ||\xi||_{\text{Sub}}$ is really the norm and the space $\text{Sub}$ forms the complete rearrangement invariant (r.i.) in the classical sense [1], chapters 1, 2 Banach space. For instance, if $\xi_1, \xi_2 \in \text{Sub}$, $c_1, c_2 = \text{const} \in \mathbb{R}$, then

$$||c_1\xi_1 + c_2\eta||_{\text{Sub}} \leq |c_1|||\xi_1||_{\text{Sub}} + |c_2| ||\eta||_{\text{Sub}}, \ c_1, c_2 = \text{const} \in \mathbb{R}.$$  

If in addition both the r.v. $\xi, \eta$ are independent, then

$$||c_1\xi_1 + c_2\eta||_{\text{Sub}} \leq \left[ c_1^2 ||\xi||^2_{\text{Sub}} + c_2^2 ||\eta||^2_{\text{Sub}} \right]^{1/2},$$

(3.7)

and analogously for the linear combination of several independent random variables. In particular, if $\eta_i, i = 1, 2, \ldots, n; \eta = \eta_1$ are identical distributed centered subgaussian random variables, then

$$\sup_n ||n^{-1/2} \sum_{i=1}^n \eta_i||_{\text{Sub}} = ||\eta||_{\text{Sub}}.$$

(3.7a)

The centered Gaussian distributed r.v. $\xi$ is also subgaussian and wherein $||\xi||_{\text{Sub}} = [\text{Var}(\xi)]^{1/2}$. A more interest fact: let the mean zero r.v. $\eta$ be bounded: $||\eta||_{\infty} < \infty$; then it is also subgaussian and herewith

$$||\eta||_{\text{Sub}} \leq ||\eta||_{\infty},$$

(3.8)

and the last estimate is in general case non-improvable.

For instance, the Rademacher distributed r.v. $\eta: \mathbb{P}(\eta = 1) = \mathbb{P}(\eta = -1) = 1/2$ is subgaussian and $||\eta||_{\text{Sub}} = 1$.

Let us return to the source r.f. $\nu = \nu(t), \ t \in T$. Suppose now that it is uniformly subgaussian relative the parameter $t$:

$$D = D_\nu := \sup_{t \in T} ||\nu(t)||_{\text{Sub}} < \infty.$$  

We introduce so-called natural finite semi-distance function $\Delta(t, s) = \Delta_\nu(t, s)$ as follows

$$\Delta(t, s) = \Delta_\nu(t, s) \overset{\text{def}}{=} ||\nu(t) - \nu(s)||_{\text{Sub}}.$$  

(3.9)

Evidently,

$$\text{diam}_\Delta T \overset{\text{def}}{=} \sup_{t, s \in T} \Delta(t, s) \leq 2D;$$  

(3.10)
and the correspondent metric entropy function \( H(T, \Delta, \epsilon) \) for the set \( T \) relative the distance \( \Delta[\nu](t_1, t_2) \) at the point \( \epsilon, \epsilon \in (0, \text{diam}_T) \).

**Lemma 3.0; see e.g. [5], [15], [18] etc.** If for the subgaussian random field
\[
\nu = \nu(t), \ t \in T \Rightarrow D_\nu < \infty
\]  
and the following so-called *entropic* integral converges:
\[
I = I(\nu) := \int_0^{\text{diam}_T} H^{1/2}(T, \Delta, \epsilon) \, d\epsilon < \infty,
\]
then the r.f. \( \nu(t) \) satisfies CLT in the space \( C(T) \) and moreover
\[
\sup_{n} \mathbb{P} \left( \sup_{t \in T} |S_n(t)| > u \right) \leq \exp \left( - C_2(I) \left( u/D \right)^2 \right), \quad C_2 > 0.
\]

**Example 3.1.** Note that the condition (3.11b) is fulfilled if for example \( T \) is bounded subset of the whole Euclidean space \( \mathbb{R}^d \) equipped with ordinary distance \( ||t - s|| \) and if
\[
\Delta_\nu(t, s) \leq C \ ||t - s||^\alpha, \ \exists \alpha \in (0, 1], \ \exists C \in (0, \infty), \quad (3.12a),
\]
Hölder’s condition.

In order to formulate and prove the CLT for our solution, we must introduce some new notations.
\[
\beta = \beta(K) := \max_{t_1, t_2 \in T} \int_T |K(t_1, s) K(t_2, s)| \, \mu(ds) < \infty, \quad (3.13)
\]

\[
d = d(t_1, t_2) = d(K)(t_1, t_2) := \left( \int_T \left[ K(t_1, s) - K(t_2, s) \right]^2 \, \mu(ds) \right)^{1/2}, \quad (3.14)
\]
\[
d_{m+1}(t_1, t_2) = d_{m+1}(K)(t_1, t_2) := \\
\left\{ \int_T \left[ K(t_1, s) - K(t_2, s) \right]^2 \ x_m^2(s) \, \mu(ds) \right\}^{1/2}. \quad (3.14a)
\]

All the introduced functions \( d(t_1, t_2), d_{l}(t_1, t_2) \) are finite semi-distance functions defined on the set \( T^2 \).

**Lemma 3.1.** Suppose that
\[
\int_0^1 H^{1/2}(T, d, \epsilon) \, d\epsilon < \infty. \quad (3.15)
\]
Then each centered random fields mentioned in (2.11): \( \tau_{m+1}(t), \tau_{m+1,m}(t), \tau_{m+1,m,m-1}(t), \ldots, \tau_{m+1,m,m-1,...,2,1}(t) \) satisfies the
CLT as $\min_m n(m) \to \infty$ in the space $C(T)$ with covariation functions described correspondingly in (2.12).

**Proof.** We deduce

$$|R_1(t, s)| \leq \beta ||f||^2, \ R_2(t, s)| \leq \beta ||x_1||^2$$

and in general

$$|R_{m+1}(t, s)| \leq \beta ||x_m||^2, \ m = 1, 2, \ldots.$$ 

As long as

$$||x_m|| \leq \frac{||f||}{1 - \rho}, \ m \geq 1,$$

we obtain

$$|R_{m+1}(t, s)| \leq \beta \frac{||f||^2}{(1 - \rho)^2}. \quad (3.16)$$

Further,

$$|R_{m+1,m}(t, s)| \leq \beta \max_{t,s} |R_{m,m-1}(t, s)|,$$

therefore $R_{(M)} \overset{def}{=} \max_{t,s \in T} \{|R_{M+1}(t, s)| + |R_{M+1,M}(t, s)| + \ldots + |R_{M+1,M,M-1, \ldots,1}(t, s)| \} \leq$

$$||f||^2 \frac{\beta + \beta^2 + \ldots + \beta^M}{(1 - \rho)^2} = ||f||^2 \frac{|\beta - \beta^{M+1}|}{|1 - \beta| (1 - \rho)^2}. \quad (3.17)$$

Evidently, if $\beta < 1$, then

$$R_{(M)} \leq ||f||^2 \frac{\beta}{(1 - \beta) (1 - \rho)^2}. \quad (3.17a)$$

We continue to estimate:

$$d_{m+1}(t_1, t_2) \leq \frac{||f|| d(t_1, t_2)}{1 - \rho},$$

and analogously $\tilde{d}_{m,k}(t_1, t_2) :=$

$$d_{R(m+1,m,\ldots,m-k)}(t_1, t_2) \overset{def}{=} ||\tau_{m+1,m,m-1,\ldots,m-k}(t_1) - \tau_{m+1,m,m-1,\ldots,m-k}(t_2)||_{\text{Sub}} \leq$$

$$\frac{||f|| d(t_1, t_2)}{(1 - \rho)^k}. \quad (3.18)$$
On the other words, both the distance functions $d_{R(m+1,m,...,m-k)}(t_1,t_2)$ and $d(t_1,t_2)$ are linear equivalent. Following, the entropic integral for the metric $d_{R(m+1,m,...,m-k)}(t_1,t_2)$ is finite as well as one for the metric $d$, see (3.15).

The finiteness of the diameter of the set $T$ relative both the considered distances follows straightforwardly from the estimate (3.17).

It remains to apply the proposition of Lemma 3.0.

Let us formulate and prove one of the main results of this report.

**Theorem 3.1. CLT in the space of continuous functions for our solutions.**

Assume that all the formulated before conditions, including the restrictions (2.15) and (2.16) are fulfilled. We propose that the sequence of random fields

$$\gamma_n(t) = \gamma(t) := \sqrt{n(M)} \left( x_{M,n(M)}(t) - x_M(t) \right)$$  \hspace{1cm} (3.19)

converges weakly in distribution as $n(M) \to \infty$ in the space $C(T)$ to the continuous a.e centered Gaussian random field $\gamma(t)$ with covariation function

$$\text{Cov}(\gamma(t), \gamma(s)) = R_M(t,s).$$  \hspace{1cm} (3.20)

**Proof.** We exploit the decomposition (2.11):

$$x_{M,n(M)}(t) = x_M(t) +$$

$$\sigma(M) \tau_M(t) + \sigma(M, M-1) \tau_{M,M-1}(t) + \sigma(M, M-1, M-2) \tau_{M,M-1,M-2}(t) + \ldots =$$

$$x_M(t) + \sigma(M) \tau_M(t) + \Sigma_2 = \Sigma_1 + \Sigma_2.$$  \hspace{1cm} (3.21)

With regard to the first member $\Sigma_1$: it is centered satisfied the CLT in the space $C(T)$ with parameters showed in (3.20); we need to ground that the second member $\Sigma_2$ in (3.21) tends to zero in the uniform norm with probability one.

It is sufficient to consider only the next member in (2.11); the remains ones may be investigated analogously.

We observe using (2.15), (2.16) and Lemma 3.1

$$P_N(u) := \mathbf{P}(||x_{M,M-1}(\cdot)|| > u) = \mathbf{P} \left( \left( n(M) n(M-1) \right)^{-1/2} ||\tau_{M,M-1}(\cdot)|| > u \right) \leq$$

$$\exp \left( - C_4 N u^2 \right), \ u > 0.$$  \hspace{1cm} (3.22)

Observe that

$$\forall u > 0 \Rightarrow \sum_{N>1} P_N(u) < \infty,$$

therefore
\( P ( ||x_{M,M-1}(\cdot)|| \to 0) = 1, \)

by virtue of Lemma Borel-Cantelli.

Note that if the numbers \( n(K) \) are chosen in accordance with relation (2.16), then the sequence of r.f.

\[
\gamma_n(t) = \gamma_{\tilde{n}}(t) := \sqrt{N/2} \cdot \left( x_{M,n(M)}(t) - x_M(t) \right)
\]

converges weakly in distribution as \( N \to \infty \) in the space \( C(T) \) to the continuous a.e centered Gaussian random field \( \gamma(t) \) with the same covariation function

\[
\text{Cov}(\gamma(t), \gamma(s)) = R_M(t,s). \tag{3.24}
\]

4 Non-asymptotic confidence region in the uniform norm.

We construct in this section the non-asymptotic confidence interval for the \( x_M(t) \).

**Theorem 4.1.** The following non-asymptotical estimate is valid under the same assumptions as in the foregoing theorem 3.1:

\[
\sup_{N \geq 4} P \left( \sup_{t \in T} \left| \sqrt{N/2} \cdot \sup_{\tilde{n} \in n} \left( x_{M,n(M)}(t) - x_M(t) \right) \right| > u \right) \leq \exp(-C_3(K,f;M) u^2), \quad u > 0. \tag{4.1}
\]

**Proof.** Let us consider the following r.f., more precisely, the sequence of random fields

\[
v_N(t) = v_{N,M,n}(t) := \sqrt{n(M)} \left( x_{M,n(M)}(t) - x_M(t) \right), \quad t \in T.
\]

It follows immediately from the estimates (3.17), (3.18) that

\[
\sup_{N \geq 4} \sup_{t, \tilde{t} \in T} ||v_N(t)||_{\text{Sub}} = C_4 < \infty, \tag{4.2}
\]

\[
\sup_{N \geq 4} ||v_N(t) - v_N(s)||_{\text{Sub}} = C_5, \quad d(t,s) < \infty. \tag{4.3}
\]

Our proposition (4.1) follows from one in the Lemma 3.0.
5 Concluding remarks.

A. The offered in this preprint method may be easily generalized on the systems of integral equations, and perhaps on the some non-linear ones.

B. Perhaps, the offered here method may be used for the solving by the Monte-Carlo method for integral equations containing kernels discontinuously depending on some parameter (parameters), in the spirit of the article [10].

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