Another view on the velocity at the Schwarzschild horizon

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Abstract

It is shown that a timelike radial geodesic does not become null at the event horizon.

Recently an attempt was made to demonstrate that in the Schwarzschild geometry the radial geodesics of material particles become null at the event horizon [1].

For this purpose, was derived an expression that corresponds to the velocity of a material particle following a radial trajectory as measured by an observer, also on a radial trajectory, when they intersect. The observer maintains its spacelike Kruskal coordinates unchanged and for this reason we call it a Kruskal observer. For $r > 2m$, the expression is, (eq.(20) of [1] and eq.(8) of [2]),

$$v = \frac{1 + \tanh(t/4m) \cdot \frac{dt}{dr}(1 - 2m/r)}{\tanh(t/4m) + \frac{dt}{dr}(1 - 2m/r)},$$

where $dt$ and $dr$ refer to the movement $t(r)$ of the particle. At the event horizon, where $r = 2m$ and $t = +\infty$, the value of eq.(1) is apparently indetermined and in [1] it is stated that $v = 1$, independently of the precise relationship $t(r)$.

In [2] some manipulations were made maintaining the generality of the expression, i.e. without substituting for $t(r)$. These permitted to show (eq.(13) of [2]) that the velocity is always less than 1 along the way, until it obviously turns to 0/0 at $r = 2m$. So there is no a priori reason to think it is necessarily $v = 1$.

This procedure was commented in a somewhat ungracious manner in [3] without any further explanations being made.

The best way to avoid confusion and get a definitive result seems to be to consider a specific geodesic $t(r)$, transforming (1) in a function of 1 variable.

Let us then consider a material particle in an ingoing radial geodesic parametrized by its proper time $\tau$. For this trajectory we can write, in Schwarzschild coordinates,

$$ds^2 = -d\tau^2 = - \left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2.$$
\[ E = \frac{dt}{d\tau} \left( 1 - \frac{2m}{r} \right), \]  
\[ \text{we obtain,} \]
\[ \frac{dt}{dr} = -E \left( 1 - \frac{2m}{r} \right)^{-1} \left[ E^2 - \left( 1 - \frac{2m}{r} \right) \right]^{1/2}. \]  
(4)

Each geodesic is characterized by its value of \( E \), defined by the initial conditions \( r_i \) and \( v_i \) (\( v_i \) is referred to the observer at rest at \( r_i \)).

\[ E = \left( 1 - \frac{2m}{r_i} \right) (1 - v_i^2)^{-1}. \]  
(5)

To simplify the integration of \( dt/dr \) we introduce the approximation,

\[ \left( 1 - \frac{1 - 2m/r}{E^2} \right)^{-1/2} \approx 1 + \frac{1 - 2m/r}{2E^2}, \]  
valid for small \( r - 2m \).

This way we obtain,

\[ t(r) = t_0 + \int_{r}^{r_0} \frac{s}{s-2m} ds + \frac{1}{2E^2} (r_0 - r) = t_0 + (r_0 - r) \left( 1 + \frac{1}{2E^2} \right) + 2m \ln \left| \frac{r_0 - 2m}{r - 2m} \right|. \]  
(7)

Each geodesic \( E \) begins at a point \( r_i \) at a time \( t_i \) which we may consider to be \( t_i < 0 \). While it does not reach \( r_0 \) we do not know the exact expression for \( t(r) \). \( r_0 \) is the point where the approximation (6) becomes valid and we make \( t_0 = 0 \).

Now we can insert (7) and (4) in (3) and plot \( v(r) \) for several geodesics \( E \).

![Figure 1: Velocities at the region of the horizon.](image)
In figure 1 we plot $|v|$ for two values of $E$ assuming $r_0 = 3m$. Each curve represents the velocity of one particle measured by a family of Kruskal observers, each one intersecting the particle at a different point between $3m$ and $2m$. Even though eq. (1) was defined as a velocity only for $r > 2m$, we plot that mathematical function through $r = 1.9m$ to get a clearer view that $v(r = 2m)$ is less than 1.

Using (5) we note that the farthest point $r_i$ where a geodesic with $E < 1$ can begin is $r_i = 2m/(1 - E^2)$. For $E = 1/2$ we get $r_i = (8/3)m$ and that is why that curve in the figure does not reach $r = 3m$.

We can also get valuable information from the plot of $v$ at $r = 2m$ as a function of $E$.

![Figure 2: Velocities at the horizon.](image)

In figure 2 we can see that $|v|$ tends to 1 only at the limits of the domain of $E$. That is when $E = +\infty$ which is a null geodesic and when $E = 0$. From eq.(5) we see the latter corresponds to a geodesic with $r_i = 2m$ which is at rest relatively to the horizon. This corresponds to the well known result that a particle at rest on the horizon must be a photon and its velocity is 1 relatively to a radial observer.

For all the other cases the modulus of the velocity plotted in figure 2 is less than 1.

For greater values of $E$, $v$ is negative which means the particle follows the observer and reaches it from one side. For smaller values of $E$, $v$ is positive which means the observer follows the particle and see it approaching from the other side of the $r$ axis.

This result is in agreement with the one presented in [4]. There a new set of coordinates is introduced. The transformations are, essentially,
\[
\begin{align*}
  x_0 &= (w - r)/\sqrt{2} \\
  x_1 &= (w + r)/\sqrt{2} \\
  x_2 &= 2m\theta \\
  x_3 &= 2m\varphi
\end{align*}
\]

(8)

where \( w \) is the ingoing Eddington-Finkelstein coordinate,

\[
w(t, r) = t + r + 2m \ln \left| \frac{r - 2m}{2m} \right|.
\]

(9)

In these coordinates the metric takes the form,

\[
ds^2 = \left[ -\frac{1}{2} \left( 1 - \frac{2\sqrt{2}m}{x_1 - x_0} \right) - 1 \right] dx_0^2 - \left( 1 - \frac{2\sqrt{2}m}{x_1 - x_0} \right) dx_0 dx_1 + \left[ -\frac{1}{2} \left( 1 - \frac{2\sqrt{2}m}{x_1 - x_0} \right) + 1 \right] dx_1^2 + \\
+ \left( \frac{x_1 - x_0}{2\sqrt{2}m} \right)^2 \left[ dx_2^2 + \sin^2\left( \frac{x_2}{2m} \right) dx_3^2 \right],
\]

(10)

which at the horizon \((x_1 - x_0 = 2\sqrt{2})\) (and with \(\theta = \pi/2\)) reduces to the Minkowski form,

\[
ds^2 = -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2.
\]

(11)

Let us now define the Janis observer as the one who maintains the spacelike coordinates \(x_1, x_2, x_3\) constant. Like the Kruskal observer it follows a radial geodesic. The velocity of a material particle that moves along an ingoing geodesic \(dt/dr\) as measured by this observer is,

\[
v_1 = \frac{dx_1}{dx_0} = \left( 1 + \frac{dr}{dt} \frac{x}{r - 2m} + \frac{dr}{dt} \right) \left( 1 + \frac{dr}{dt} \frac{x}{r - 2m} - \frac{dr}{dt} \right).
\]

(12)

This expression is written as a function of only 1 variable (\(r\)). Inserting \(dr/dt\) from (4) and approximating the squared factor in that expression analogously to what was made in eq.(6), we obtain,

\[
v_1^2 = \left( \frac{1}{2E} - E + \frac{1 - \frac{2m}{2E}}{2E} \right)^2.
\]

(13)

For \(r = 2m\) we get,

\[
v_1^2(r = 2m) = \left( \frac{1 - 2E^2}{1 + 2E^2} \right)^2,
\]

(14)

which is eq.(10) of [4].
From here we see that $v_1 < 1$ unless $E = 0$ or $E = +\infty$. In fact the graph of this function $v_1(E)$ is identical to the one in figure 2.

This is the general behaviour for the relative velocity of two moving material particles at the Schwarzschild horizon.

References

[1] Mitra,A. (1999), astro-ph/9904162
[2] Tereno,I. (1999), astro-ph/9905144
[3] Mitra,A. (1999), astro-ph/9905175
[4] Janis,A. (1973) Phys.Rev.D 8, 2360