Unitarity and geometric phases are two ubiquitous properties of physical systems.

The Berry phase is the phase acquired by a system when it is subjected to a cyclic evolution, resulting only from the geometrical properties of the path traversed in the parameter space because of anholonomy [1, 2].

Unitarity represents the probability conservation in particle scattering processes described by the unitary scattering operator, $S$. The relation, $S = 1 + iT$, between the $S$-operator and the transition operator, $T$, leads to the Optical Theorem,

$$-i(T - T^\dagger) = T^\dagger T. \quad (1)$$

The matrix elements of this equation between initial and final states are expressed, in perturbation theory, in terms of Feynman diagrams. The evaluation of the right hand side requires the insertion of a complete set of intermediate states. Therefore, since $-i(T - T^\dagger) = 2 \text{Im} T$, Eq. (1) yields the computation of the imaginary part of Feynman integrals from a sum of contributions from all possible intermediate states. A Feynman diagram is thus responsible for an imaginary part of the scattering amplitudes when the intermediate, virtual particles go on-shell.

The Cutkosky-Veltman rules, implementing the unitarity conditions, allow the calculation of the discontinuity across a branch cut of an arbitrary Feynman amplitude, which corresponds to its imaginary part [3]. Accordingly, the imaginary part of a given Feynman integral can be computed by evaluating the phase-space integral obtained by cutting two internal particles, which amounts to applying the on-shell conditions and replacing their propagators by the corresponding $\delta$-function, $(p^2 - m^2 + i0)^{-1} \to (2\pi i) \delta^{(+)}(p^2 - m^2)$.

In later studies the problem of finding the discontinuity of a Feynman integral associated to a singularity was addressed in the language of homology theory and the geometric phases resulting from the curved geometry in effective momentum space experienced by the two on-shell particles going around the loop.

In a recent work [8] it has been shown that double-cuts of one-loop scattering amplitudes can be efficiently evaluated by using the well-known Generalised Cauchy Formula, also known as Cauchy-Pompeiu Formula, or Cauchy-Green Formula as well [10]. In the case of double-cuts, the 4-dimensional loop-momentum can be decomposed in terms of an ad hoc basis of four massless vectors whose coefficients depend on two complex-conjugated variables, left over as free components after imposing the two on-shell cut-constraints. Therefore, the double-cut phase-space integral is written as a two-fold integration over these two variables. The integration is finally carried out by using Generalised Cauchy Formula as an application of Stokes’ Theorem for rational function of two complex-conjugated variables. As such, the result of the phase-space integration can be naturally interpreted as the flux of a 2-form that is given by the product of the two tree-level amplitudes sewn along the cut.

### I. DOUBLE-CUT

The two-particle Lorentz invariant phase-space (LIPS) in the $K^2$-channel is defined as,

$$\int d^4\Phi = \int d^4\ell_1 \delta^{(+)}(\ell_1^2 - m_1^2) \delta^{(+)}((\ell_1 - K)^2 - m_2^2) , \quad (2)$$

where $K^\mu$ is the total momentum across the cut. We introduce a suitable parametrisation for $\ell^\mu_1$ [8,9], in terms of four massless momenta, which is a solution of the two on-shell conditions, $\ell_1^2 = m_1^2$ and $(\ell_1 - K)^2 = m_2^2$,

$$\ell_1^\mu = \frac{1}{1 + z\bar{z}}(p^\mu + z\bar{z} q^\mu + z e_+^\mu + \bar{z} e_-^\mu) + \rho K^\mu , \quad (3)$$

where $p^\mu$ and $q^\mu$ are two massless momenta with the requirements,

$$p_\mu + q_\mu = K_\mu ,$$
$$p^2 = q^2 = 0 , \quad 2 p \cdot q = 2 p \cdot K = 2 q \cdot K \equiv K^2 ; \quad (4)$$

the vectors $e_+^\mu$ and $e_-^\mu$ are orthogonal to both $p^\mu$ and $q^\mu$, with the following properties [12],

$$\epsilon_+^2 = \epsilon_-^2 = 0 = \epsilon_+ \cdot p = \epsilon_+ \cdot q ,$$
$$2 \epsilon_+ \cdot \epsilon_- = -K^2 . \quad (6)$$
The complex conjugated variables \( \rho \) and depends only on the kinematics.

\[ \rho = \frac{K^2 + m_1^2 - m_2^2 - \sqrt{\lambda(K^2, m_1^2, m_2^2)}}{2K^2} , \tag{7} \]

with the Källen function defined as,

\[ \lambda(K^2, m_1^2, m_2^2) = (K^2)^2 + (m_1^2)^2 + (m_2^2)^2 - 2K^2m_1^2 - 2K^2m_2^2 - 2m_1^2m_2^2 , \tag{8} \]

and depends only on the kinematics.

The complex conjugated variables \( z \) and \( \bar{z} \) parametrize the degrees of freedom left over by the cut-conditions.

Analogously to the massless case \[8\], corresponding to the field-theoretic double-cut \( \Delta \) in \[11\] is the flux of a 2-form. It corresponds to an integral over the complex tangent bundle of the Riemann sphere, where the curvature 2-form, \( \Omega \), is defined as \[14\].

\[ \Omega = \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2} \cdot \tag{13} \]

The parameter \( \rho \) is the pseudo-threshold,

\[ \rho = \frac{K^2 + m_1^2 - m_2^2 - \sqrt{\lambda(K^2, m_1^2, m_2^2)}}{2K^2} \]

where the integration contour has to be chosen as enclosing all the complex \( z \)-poles.

In this letter we rather want to focus on what links Eq.\[11\] and Eq.\[12\], namely Stokes' Theorem \[8\], and on the geometrical interpretation of its consequence: the double-cut \( \Delta \) in \[11\] is the flux of a 2-form. It corresponds to an integral over the complex tangent bundle of the Riemann sphere, where the curvature 2-form, \( \Omega \), is defined as \[14\].

\[ \Omega = \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2} . \tag{13} \]

The product \( A_{L}^{\text{tree}} A_{R}^{\text{tree}} \) is a rational function of \( z \) and \( \bar{z} \), hence it can be written as ratio of two polynomials, \( P \) and \( Q \),

\[ A_{L}^{\text{tree}}(z, \bar{z}) A_{R}^{\text{tree}}(z, \bar{z}) = \frac{P(z, \bar{z})}{Q(z, \bar{z})} , \tag{14} \]

with the following relations among their degrees,

\[ \text{deg}_z P = \text{deg}_z Q , \quad \text{deg}_z Q = \text{deg}_z P . \tag{15} \]

\section{II. OPTICAL THEOREM}

In the double-cut integral \[11\], we did not make any assumptions on the tree-level amplitudes sewn along the cut, thus providing a general framework to the integration method developed in \[8\]. If we now choose \( A_{L}^{\text{tree}} = A_{m-2}^{*\text{tree}} \), that is the conjugate scattering amplitude of a process \( m \to 2 \), and \( A_{R}^{\text{tree}} = A_{n-2}^{\text{tree}} \), that is the amplitude of a process \( n \to 2 \), then \( \Delta \) reads,

\[ \Delta = \int d^4 \Phi A_{m-2}^{*\text{tree}} A_{n-2}^{\text{tree}} = -i \left\{ A_{m-2}^{\text{one-loop}} - A_{n-2}^{*\text{one-loop}} \right\} = 2 \operatorname{Im} \left\{ A_{m-2}^{\text{one-loop}} \right\} , \tag{16} \]

which is the definition of the two-particle discontinuity of the one-loop amplitude \( A_{n-m}^{\text{one-loop}} \) across the branch cut in the \( K^2 \)-channel, corresponding to the field-theoretic version of the Optical Theorem \[11\] for one-loop Feynman amplitudes.

On the other side, because of Stokes' Theorem in \[11\], one has,

\[ \Delta = (1 - 2 \rho) \int dz \wedge d\bar{z} A_{m-2}^{*\text{tree}} A_{n-2}^{\text{tree}} \frac{1}{(1 + |z|^2)^2} = \]

\[ (1 - 2 \rho) \int dz \int d\bar{z} A_{m-2}^{*\text{tree}} A_{n-2}^{\text{tree}} \frac{1}{(1 + |z|^2)^2} , \tag{17} \]

which provides a geometrical interpretation of the imaginary part of one-loop scattering amplitudes, as a flux of a complex 2-form through a surface bounded by the contour of the \( z \)-integral (the contour should enclose all the
poles in $z$ exposed in the integrand after the integration in $\bar{z}$ [8].

Given the equivalence of [10] and [17], a correspondence between the imaginary part of scattering amplitudes and the anholonomy of Berry’s phase does emerge, since the latter is indeed defined as the flux of a 2-form in presence of curved space [1, 2]. In this context, one could establish a parallel description between the Aharonov-Bohm (AB) effect and the double-cut of one-loop Feynman integrals.

In the AB-effect [11], an electron-beam splits with half passing by either side of a long solenoid, before being re-combined. Although the beams are kept away from the solenoid, so they encounter no magnetic field ($B = 0$), they arrive at the recombination with a phase-difference that is proportional to the magnetic flux through a surface encircled by their paths. The non-trivial anholonomy in this case is a consequence of Stokes’ Theorem, where the 2-form Berry curvature is written as the differential of the 1-form vector potential ($\nabla \times A$).

In the case of the double-cut of one-loop Feynman integrals, we could describe the evolution of the system depicted in Fig.1 from the left to the right. The two particles produced in the $A_L$-scattering, going around the loop and initiating the $A_R$-process, at the $A_R$-interaction point would experience a phase-shift due to the non-trivial geometry in effective momentum space induced by the on-shell conditions. As in the AB-effect, the anholonomy phase-shift is a consequence of Stokes’ Theorem, and here it corresponds to the imaginary part of the one-loop Feynman amplitude.

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