STABILITY AND LARGE-TIME BEHAVIOR ON 3D INCOMPRESSIBLE MHD EQUATIONS WITH PARTIAL DISSIPATION NEAR A BACKGROUND MAGNETIC FIELD

HONGXIA LIN\(^1\), JIAHONG WU\(^2\) AND YI ZHU\(^3\)

Abstract. Physical experiments and numerical simulations have observed a remarkable stabilizing phenomenon: a background magnetic field stabilizes and damps electrically conducting fluids. This paper intends to establish this phenomenon as a mathematically rigorous fact on a magnetohydrodynamic (MHD) system with anisotropic dissipation in \(\mathbb{R}^3\). The velocity equation in this system is the 3D Navier-Stokes equation with dissipation only in the \(x_1\)-direction while the magnetic field obeys the induction equation with magnetic diffusion in two horizontal directions. We establish that any perturbation near the background magnetic field \((0, 1, 0)\) is globally stable in the Sobolev setting \(H^3(\mathbb{R}^3)\). In addition, explicit decay rates in \(H^2(\mathbb{R}^3)\) are also obtained. When there is no presence of the magnetic field, the 3D anisotropic Navier-Stokes equation in \(\mathbb{R}^3\) is not well understood and the small data global well-posedness remains an intriguing open problem. This paper reveals the mechanism of how the magnetic field generates enhanced dissipation and helps stabilize the fluid.

1. Introduction

This paper deals with the stability and large-time behavior problem on a system of 3D anisotropic MHD equations near a background magnetic field. To shed some light on the potential difficulties of this problem, we briefly review several facts on the behavior of solutions to the Euler and the anisotropic Navier-Stokes equations.

It is well-known that solutions of the incompressible Euler equations

\[
\begin{cases}
\partial_t u + (u \cdot \nabla) u = -\nabla P, \\
\nabla \cdot u = 0
\end{cases}
\]

can grow rather rapidly in time. In fact, Kiselev and Sverak are able to construct a vorticity solution of the 2D Euler equations in a disk whose gradient grows double exponentially in time [42]. In the periodic setting, an example of Zlatos shows that the vorticity gradient can grow at least exponentially [88]. Choi and Jeong obtain linear in time growth for the vorticity gradient for certain smooth and compactly supported initial vorticity in \(\mathbb{R}^2\) [16]. Classical solutions to the 3D Euler equations could develop finite-time singularities ([14, 28]). Many more results in this direction can be found in a review paper by Drivas and Elgindi [25]. As a special consequence, perturbations governed by the Euler equations near the trivial solution are generally not stable. How much dissipation does one really need in order to achieve the stability? Adding the full Laplacian dissipation is certainly

2010 Mathematics Subject Classification. 35A05, 35B35, 76D03.

Key words and phrases. Background magnetic field; magnetohydrodynamic equation; partial dissipation; stability; decay rate.
sufficient. As demonstrated by Schonbek and others (see, e.g., [60–62, 70]), solutions of the Navier-Stokes equations
\[
\begin{align*}
\partial_t u + (u \cdot \nabla) u &= -\nabla P + \mu \Delta u, \\
\nabla \cdot u &= 0
\end{align*}
\]
are asymptotically stable and decay in time with explicit decay rates. When the dissipation is anisotropic and only in two directions, the Navier-Stokes equations become
\[
\begin{align*}
\partial_t u + (u \cdot \nabla) u &= -\nabla P + \mu \Delta_h u, \\
\nabla \cdot u &= 0
\end{align*}
\tag{1.1}
\]
where \( \Delta_h = \partial_1^2 + \partial_2^2 \) is the horizontal Laplacian. Due to its physical applications and special mathematical properties, (1.1) has attracted considerable interests and an array of beautiful small data global well-posedness results have been obtained (see, e.g., [12, 13, 38, 52, 54, 55, 83, 84]). New approaches have very recently been developed to tackle the large-time behavior problem and explicit decay rates have been extracted for (1.1) (see [40, 77]). If we further reduce the dissipation to be in just one direction, the resulting 3D anisotropic Navier-Stokes equations
\[
\begin{align*}
\partial_t u + (u \cdot \nabla) u &= -\nabla P + \mu \partial_1^2 u, \\
\nabla \cdot u &= 0
\end{align*}
\tag{1.2}
\]
is not well-understood. In particular, the small data global well-posedness problem remains open. In addition, very little is known on the stability properties and the large-time behavior.

This paper focuses on the following system of the 3D MHD equations with anisotropic dissipation
\[
\begin{align*}
\partial_t u + (u \cdot \nabla) u &= -\nabla P + \mu \partial_1^2 u + (B \cdot \nabla) B, \\
\nabla \cdot u &= 0 \\
\partial_t B + (u \cdot \nabla) B &= \eta \Delta_h B + (B \cdot \nabla) u, \\
\nabla \cdot B &= 0
\end{align*}
\tag{1.3}
\]
with the initial data
\[ u(x, 0) = u_0, \quad B(x, 0) = B_0. \]
Here \( u = (u_1, u_2, u_3)^T, B = (B_1, B_2, B_3)^T \) and \( P \) represent the velocity field of the fluid, the magnetic field and the scalar pressure, respectively. The constants \( \mu > 0 \) and \( \eta > 0 \) are the viscosity coefficient and the magnetic diffusivity. The MHD system (1.3) focused here is relevant in the modeling of reconnecting plasmas (see, e.g., [17, 18, 57]).

The motivation for studying (1.3) comes from two distinct sources. The first is the stabilizing phenomenon observed in physical experiments involving electrically conducting fluids. The experiments exhibit a remarkable phenomenon: a background magnetic field actually stabilizes and damps turbulent MHD fluids (see, e.g., [2–4, 19, 21, 32, 33]). We
intend to establish this phenomenon as a mathematically rigorous fact on \([1,3]\). The second is to initiate new strategies and develop innovative tools for stability and large-time behavior problems on anisotropic models.

To understand the stabilizing mechanism of a background magnetic field
\[ u^{(0)} \equiv 0, \quad B^{(0)} \equiv e_2 := (0, 1, 0), \]
which is obviously a steady-state of \([1,3]\), we study the dynamics of the perturbation \((u, b)\) with \(b = B - B^{(0)}\). Clearly \((u, b)\) satisfies the MHD equations
\[
\begin{aligned}
\partial_t u + (u \cdot \nabla) u &= -\nabla p + \mu \partial_1^2 u + (b \cdot \nabla) b + \partial_2 b, \quad x \in \mathbb{R}^3, \quad t > 0, \\
\partial_t b + (u \cdot \nabla) b &= \eta \Delta_h b + (b \cdot \nabla) u + \partial_2 u, \\
\nabla \cdot u &= \nabla \cdot b = 0, \\
u(x, 0) &= u_0(x), \quad b(x, 0) = b_0(x).
\end{aligned}
\]

(1.4)

Our main result asserts the global well-posedness and stability of \((u, b)\), and provides precise decay rates for various Sobolev norms of \((u, b)\). The precise statement of these results is presented in the following theorem. To simplify the notation, we use \(\|f\|_{L^p} \equiv \|f\|_{L^p(\mathbb{R})}\) for the norm \(\|f\|_{L^p} \equiv \|f\|_{L^p(\mathbb{R})}\) and \(\|f\|_{L^p_{1,1/2} L^2_{0,1/2}} \equiv \|f\|_{L^p_{1,1/2} L^2_{0,1/2}(\mathbb{R})}\).

**Theorem 1.1.** Assume \((u_0, b_0) \in H^3(\mathbb{R}^3)\) with \(\nabla \cdot u_0 = 0\) and \(\nabla \cdot b_0 = 0\) satisfies
\[
(u_0, b_0), \ (\partial_3 u_0, \partial_3 b_0), \ (\partial_2^2 u_0, \partial_2^2 b_0) \in L^2_{0,1/2} L^1_{0,1/2}(\mathbb{R}^3).
\]

Then there exists a sufficiently small constant \(\delta > 0\) such that, if
\[
\begin{aligned}
\|(u_0, b_0)\|_{H^3(\mathbb{R}^3)} + \|(u_0, b_0)\|_{L^1_{0,1/2} L^2_{0,1/2}(\mathbb{R}^3)} + \|(\partial_3 u_0, \partial_3 b_0)\|_{L^2_{0,1/2} L^1_{0,1/2}(\mathbb{R}^3)} \\
+ \|(\partial_2^2 u_0, \partial_2^2 b_0)\|_{L^2_{0,1/2} L^1_{0,1/2}(\mathbb{R}^3)} &\leq \delta,
\end{aligned}
\]

(1.5)

then \((1.4)\) admits a unique global solution \((u, b) \in C([0, \infty); H^3(\mathbb{R}^3))\). In addition, \((u, b)\) is stable in the sense that, for an absolute constant \(C > 0\),
\[
\begin{aligned}
\|(u, b)(t)\|_{H^3(\mathbb{R}^3)}^2 + \int_0^t \left( \|(\partial_1 u(\tau))_{H^3(\mathbb{R}^3)}^2 + \|(\partial_2 u(\tau))_{H^3(\mathbb{R}^3)}^2 + \|(\nabla b(\tau))_{H^2(\mathbb{R}^3)}^2 \right) d\tau &\leq C\delta^2
\end{aligned}
\]

for any \(t > 0\).

Furthermore, \((u, b)\) obeys the following time decay estimates, for \(0 < \varepsilon \leq \frac{1}{3C}\),
\[
\begin{aligned}
\|(u, b)\|_{L^2(\mathbb{R}^3)} &\leq C(1 + t)^{-\frac{3}{4}}, \quad \|(\nabla u, \nabla b)\|_{L^2(\mathbb{R}^3)} \leq C(1 + t)^{-1}, \\
\|(\partial_3 u, \partial_3 b)\|_{L^2(\mathbb{R}^3)} &\leq C(1 + t)^{-\frac{3}{4} + \varepsilon}, \quad \|(\partial_1 \partial_j u, \partial_1 \partial_j b)\|_{L^2(\mathbb{R}^3)} \leq C(1 + t)^{-\frac{3}{4} + \varepsilon}, \quad j = 1, 2, \\
\|(\partial_1 \partial_3 u, \partial_1 \partial_3 b)\|_{L^2(\mathbb{R}^3)} &\leq C(1 + t)^{-1 + \varepsilon}, \quad \|(\partial_2 \partial_j u, \partial_2 \partial_j b)\|_{L^2(\mathbb{R}^3)} \leq C(1 + t)^{-\frac{3}{4} + \varepsilon}, \quad j = 2, 3, \\
\|(\partial_2^2 u, \partial_2^2 b)\|_{L^2(\mathbb{R}^3)} &\leq C(1 + t)^{-\frac{5}{8}}.
\end{aligned}
\]

Theorem 1.1 rigorously confirms the smoothing and stabilizing effect of the magnetic field on the electrically conducting fluids. Without the magnetic field, the fluid motion...
is governed by the 3D anisotropic Navier-Stokes equation (1.2) alone and whether or not the velocity is stable in Sobolev spaces remains an outstanding open problem. When coupled with magnetic field, Theorem 1.1 ensures that any perturbation near a background magnetic is stable and decays to zero at explicit rates as $t \to \infty$.

We clarify the differences between Theorem 1.1 and some of the closely related results. Wu and Zhu [75] solved the stability problem for the MHD system with horizontal dissipation $\Delta_h u$ and vertical magnetic diffusion $\partial^3 b$. It appears that the situation considered here is more difficult. This is due to the handling of the velocity nonlinearity $(u \cdot \nabla) u$. When the velocity dissipation is only in one direction, the triple-product term $((u \cdot \nabla) u, u)_{H^3}$ is much more difficult to control than any triple product terms generated for the MHD system considered in [75]. In fact, this term is exactly the reason why the well-posedness problem on the 3D anisotropic Navier-Stokes (1.2) is open. One main contribution of this paper is the handling of the Navier-Stokes nonlinearity when the dissipation of the velocity is only in a single direction. The smoothing and stabilizing effect of the magnetic field on the fluids, and the elaborate construction of time-weighted energy functional are the key ingredients of this successful story. We remark that there is a very large mathematical literature on the incompressible MHD equations. In particular, there have been substantial recent developments on the well-posedness and stability problems, and significant progress has been made (see, e.g., [1, 5–11, 15, 22–24, 26, 27, 29, 31, 34–37, 39, 41, 43–51, 56, 58, 59, 63–65, 67–69, 72–74, 76, 78–82, 85–87]).

We explain the proof of Theorem 1.1. Due to the lack of velocity dissipation in two directions, we take the functional setting to be the Sobolev space $H^3$ in order to guarantee the uniqueness. The local existence follows from a standard procedure (see, e.g., [53]), so we focus on the global $a priori$ bounds of $(u, b)$. This is accomplished via the bootstrapping argument (see, e.g., [66]). A crucial step is to construct a suitable energy functional. Naturally it should include the $H^3$-norm together with the time integral pieces from the dissipative terms

$$E_0^{(1)}(t) = \sup_{0 \leq \tau \leq t} \|(u(\tau), b(\tau))\|^2_{H^3} + \int_0^t \left(\|\partial_1 u(\tau)\|^2_{H^3} + \|\nabla b(\tau)\|^2_{H^3}\right) d\tau.$$  

However, due to the lack of velocity dissipation in two directions, the triple product generated by the nonlinearity, namely $((u \cdot \nabla) u, u)_{H^3}$ can not be bounded in terms of $E_0^{(1)}(t)$. The most difficult piece is the following triple product

$$\int \partial_3^3 (u \cdot \nabla u) \cdot \partial_3^3 u \, dx.$$

Here we have used $\int$ to denote the integral in $x$ over $\mathbb{R}^3$. To distinguish the derivatives in different directions, we further write it as

$$\int \partial_3^3 (u \cdot \nabla u) \cdot \partial_3^3 u \, dx$$

$$= 3 \int \partial_3 u_h \cdot \nabla_h \partial_3^2 u \cdot \partial_3^3 u \, dx + 3 \int \partial_3^2 u_h \cdot \nabla_h \partial_3 u \cdot \partial_3^3 u \, dx + \int \partial_3^3 u_h \cdot \nabla_h u \cdot \partial_3^3 u \, dx$$

$$+ 3 \int \partial_3 u_h \cdot \partial_3^3 u \cdot \partial_3^2 u \, dx + 3 \int \partial_3^2 u_h \cdot \partial_3^2 u \cdot \partial_3^3 u \, dx + \int \partial_3^3 u_h \cdot \partial_3 u \cdot \partial_3^3 u \, dx. \quad (1.6)$$
Clearly we need to seek enhanced dissipation in the $x_2$ or the $x_3$ direction to complement the existing dissipation in the $x_1$-direction. The background magnetic field is along the $x_2$ direction and it is in this direction that the extra regularization is generated. Mathematically this is reflected in the wave structure. We explain this. To avoid unnecessary complications, we look at the linearized system of (\ref{1.4}), namely
\begin{equation}
\begin{aligned}
\partial_t u &= \mu \partial^2_x u + \partial_x b, \\
\partial_t b &= \eta \Delta b + \partial_2 u, \\
\nabla \cdot u &= \nabla \cdot b = 0.
\end{aligned}
\end{equation}

By differentiating (\ref{1.7}) in $t$ and making several substitutions, (\ref{1.7}) can be converted into the following system of wave equations
\begin{equation}
\begin{aligned}
\partial_t u - (\mu \partial^2_x + \eta \Delta) \partial_t u + \mu \eta \partial^2_x \Delta b - \partial_2^2 u &= 0, \\
\partial_t b - (\mu \partial^2_x + \eta \Delta) \partial_t b + \mu \eta \partial^2_x \Delta b - \partial_2^2 b &= 0, \\
\nabla \cdot u &= \nabla \cdot b = 0.
\end{aligned}
\end{equation}

(\ref{1.8}) is a system of anisotropic and degenerate wave equations. In comparison with (\ref{1.7}), (\ref{1.8}) exhibits much more smoothing and stabilizing properties. In particular, the two terms $\partial^2_x u$ and $\partial_2^2 b$ in (\ref{1.8}), emerged from the interaction of the velocity and the magnetic field, generates the dissipation in the $x_2$-direction. This confirms the stabilizing effect of the background magnetic field. To include this regularizing property in the energy functional, we define
\begin{equation}
E_0^{(2)}(t) = \int_0^t \| \partial_2 u(\tau) \|^2_{H^2} d\tau.
\end{equation}
We emphasize that the extra dissipative effect in the $x_2$-direction is one-derivative lower than what a standard dissipation term $\partial^2_x u$ provides. This is why this energy functional only allows the time integrability of $\| \partial_2 u \|^2_{H^2}$, not $\| \partial_2 u \|^2_{H^3}$. Combining $E_0^{(1)}$ and $E_0^{(2)}$ gives
\begin{equation}
E_0(t) = E_0^{(1)} + E_0^{(2)}
= \sup_{0 \leq \tau \leq t} \| (u(\tau), b(\tau)) \|^2_{H^1} + \int_0^t \left( \| \partial_1 u(\tau) \|^2_{H^2} + \| \partial_2 u(\tau) \|^2_{H^2} + \| \nabla b(\tau) \|^2_{H^1} \right) d\tau.
\end{equation}

However, there are still two terms in (\ref{1.6}) (the third term and the fourth term) that can not be bounded in terms of $E_0(t)$. After invoking the divergence-free condition $\partial_3 u_3 = -\partial_1 u_1 - \partial_2 u_2$, these terms are reduced to the difficult term
\begin{equation}
\int |\partial_2 u| |\partial_3^2 u| |\partial_3^2 u| dx.
\end{equation}

Due to the aforementioned weaker smoothing effect in the $x_2$-direction, (\ref{1.9}) can not be bounded by $E_0(t)$. Extra maneuvers are necessary. Our idea is to include two extra time-weighted energy functionals
\begin{equation}
\begin{aligned}
E_1(t) &= \sup_{0 \leq \tau \leq t} (1 + \tau) ||(\nabla_h u(\tau), \nabla_h b(\tau))||^2_{H^1} \\
&+ \int_0^t (1 + \tau) \left( ||\partial_1 \nabla_h u(\tau)||^2_{H^1} + ||\partial_2 \nabla_h u(\tau)||^2_{L^2} + ||\nabla^2_h b(\tau)||^2_{H^1} \right) d\tau,
\end{aligned}
\end{equation}
\begin{equation}
\begin{aligned}
E_2(t) &= \sup_{0 \leq \tau \leq t} \left( (1 + \tau) ||(u(\tau), b(\tau))||^2_{L^2} + (1 + \tau)^2 ||(\nabla_h u(\tau), \nabla_h b(\tau))||^2_{L^2} \right).
\end{aligned}
\end{equation}
(1 + \tau)^{1-2\varepsilon}\|((\partial_3 u(\tau), \partial_3 b(\tau))\|^2_{L^2} \\
+ \sum_{j=1}^{2}(1 + \tau)^{1-2\varepsilon}\|((\partial_1 \partial_j u(\tau), \partial_1 \partial_j b(\tau))\|^2_{L^2} \\
+ \sum_{j=2}^{3}(1 + \tau)^{1-2\varepsilon}\|((\partial_2 \partial_j u(\tau), \partial_2 \partial_j b(\tau))\|^2_{L^2} \\
+ (1 + \tau)^{2-2\varepsilon}\|((\partial_1 \partial_3 u(\tau), \partial_1 \partial_3 b(\tau))\|^2_{L^2} + (1 + \tau)^{1\varepsilon}\|((\partial_3^2 u(\tau), \partial_3^2 b(\tau))\|^2_{L^2})

We shall show that the inclusion of \(E_1(t)\) and \(E_2(t)\) enables us to bound the term in (1.9) suitably and thus establish a closed energy inequality. The definition of \(E_2(t)\) is certainly not simple. It takes into account of the precise time decay rate of each norm involved in \(E_2(t)\). We will resort to the integral representation of (1.4) and spectral analysis to control the terms in \(E_2(t)\). Having obtained the necessary components of the energy functional, we sum them up to form our total energy functional

\[ E(t) = E_0(t) + E_1(t) + E_2(t). \]

Our main efforts are devoted to proving the following estimate

\[ E(t) \leq C_1 F(u_0, b_0) + C_2 \left( E_1^8(t) + E_2^2(t) \right), \]  

(1.10)

where \(C_1\) and \(C_2\) are constants, and

\[ F(u_0, b_0) = \| (u_0, b_0) \|_{H^3}^2 + \| (u_0, b_0) \|_{L_{t,x}^2 L_{t,x}^{1/2}}^2 + \| (\partial_3 u_0, \partial_3 b_0) \|_{L_{t,x}^2 L_{t,x}^{1/2}}^2 + \| (\partial_3^2 u_0, \partial_3^2 b_0) \|_{L_{t,x}^2 L_{t,x}^{1/2}}^2. \]

Verifying (1.10) is a very lengthy process. For the sake of clarity, we divide the whole process into the proofs of the following inequalities

\[ E_0(t) \leq C E(0) + C E_1^8(t), \]  

(1.11)

\[ E_1(t) \leq C E(0) + C E_0(t) + C E_1^8(t), \]  

(1.12)

\[ E_2(t) \leq C \left( E_1^8(t) + E_2^2(t) \right) + C \left( \| (u_0, b_0) \|_{H^2}^2 + \| (u_0, b_0) \|_{L_{t,x}^2 L_{t,x}^{1/2}}^2 \right) \]  

\[ + \| (\partial_3 u_0, \partial_3 b_0) \|_{L_{t,x}^2 L_{t,x}^{1/2}}^2 + \| (\partial_3^2 u_0, \partial_3^2 b_0) \|_{L_{t,x}^2 L_{t,x}^{1/2}}^2 \). \]  

(1.13)

To prove (1.11), we realize that \(E_0(t)\) consists of two different types of terms \(E_0^{(1)}(t)\) and \(E_0^{(2)}(t)\), as aforementioned. The boundedness of \(E_0^{(2)}(t)\) relies on the enhanced dissipation from the wave structure. Naturally the proof of (1.11) is further split into two parts,

\[ (\| u(t) \|^2_{H^3} + \| b(t) \|^2_{H^3}) + 2 \int_0^t \left( \mu \| \partial_1 u(\tau) \|^2_{H^3} + \eta \| \nabla_b b(\tau) \|^2_{H^3} \right) d\tau \leq C E(0) + C E_1^8(t) \]

and

\[ -(\partial_2 u(t), b(t))_{H^2} + \frac{1}{2} \int_0^t \| \partial_2 u(\tau) \|^2_{H^2} - \int_0^t \left( \| \partial_2 b(\tau) \|^2_{H^2} + (\mu^2 + \eta^2) \| \Delta b(\tau) \|^2_{H^2} \right) d\tau \]

\[ \leq C E(0) + C E_1^8(t). \]
The detailed estimates are provided in Section 3. To prove (1.12), we also need to divide the terms in $E_1(t)$ into two parts,

$$\int_0^t (1 + \tau)\|\partial_2 \nabla h (\tau)\|_{L^2}^2 \, d\tau$$

and the rest of the terms. The regularization from the wave structure in (1.8) is used to gain the time integrability of the vertical derivative. More technical details are left in Section 4.

The proof of (1.13) is extremely elaborate and relies on the precise decay rates of the norms involved in $E_2(t)$. Direct energy estimates are not sufficient for this purpose. Instead we solve the system of linear equations (1.7) and recast the nonlinear system (1.4) into an integral form. This form relies on three kernel functions. They are degenerate and anisotropic in the frequency space. We first perform a detailed spectral analysis in suitably divided subdomains of the frequency space to obtain sharp and precise upper bounds for the kernel functions. The terms in $E_2(t)$ are then estimated according to the orders and directions of their derivatives. After a lengthy process, we finally obtain (1.13).

Once (1.10) is at our disposal, a direct application of the bootstrapping argument yields the desired global bounds and Theorem 1.1 then follows.

The rest of this paper is divided into four sections. Section 2 applies the bootstrapping argument to the a priori inequality (1.10) to establish Theorem 1.1. In addition, several anisotropic inequalities for products and triple products are provided here as well. They will be used in the subsequent sections. Section 3 details the proof of (1.11). Section 4 proves (1.12) while Section 5 is devoted to (1.13).

2. Proof of Theorem 1.1 and anisotropic Sobolev inequalities

This section serves two purposes. The first is to prove Theorem 1.1 by applying the bootstrapping argument to the a priori inequality in (1.10). The second is to provide anisotropic inequalities for several products and triple products, which will be used in the proofs in subsequent sections.

**Proof of Theorem 1.1** The local (in time) well-posedness of (1.4) in $H^3$ can be shown via standard procedures (see, e.g., [53]). It suffices to establish the global bounds stated in Theorem 1.1 in order to obtain the global existence. This is accomplished by applying the bootstrapping argument to (1.10), namely

$$E(t) \leq C_1 F(u_0, b_0) + C_2 \left( E^{\frac{3}{2}}(t) + E^2(t) \right),$$

(2.1)

where

$$F(u_0, b_0) = \|(u_0, b_0)\|_{H^3}^2 + \|(u_0, b_0)\|_{L^2_x L^{\frac{3}{2}} t_{1/2}}^2 + \| (\partial_3 u_0, \partial_3 b_0) \|_{L^2_x L_{1/2}}^2 + \| (\partial_3^2 u_0, \partial_3^2 b_0) \|_{L^2_x L_{1/2}}^2.$$
A useful description of the bootstrapping argument can be found in [66, p.21]. In order to apply the bootstrapping argument, we make the ansatz that

\[ E(t) \leq M := \min \left\{ 1, \frac{1}{(4C_2)^2} \right\}. \]  

(2.2)

We then verify that \( E(t) \) actually admits a smaller bound,

\[ E(t) \leq \frac{M}{2}. \]

Inserting (2.2) in (4.1) and recalling the initial assumption (1.5), we have

\[ E(t) \leq C_1 F(u_0, b_0) + C_2 (M^\frac{1}{2} + M) E(t) \]

\[ \leq C_1 \delta^2 + 2C_2 M^\frac{1}{2} E(t) \]

\[ \leq C_1 \delta^2 + \frac{1}{2} E(t), \]

or

\[ E(t) \leq 2C_1 \delta^2. \]

If the initial data is sufficiently small, say

\[ \delta^2 \leq \frac{M}{4C_1}, \]

then we derive

\[ E(t) \leq 2C_1 \delta^2 \leq \frac{M}{2}. \]

The bootstrapping argument then implies \( T = \infty \) and asserts that for any time \( t > 0 \),

\[ E(t) \leq C \delta^2, \]

which, in particular, implies the desired global bound on the solution \((u, b)\). As a consequence, we obtain the global existence of solutions. The uniqueness is obvious due to the high regularity of the solution. The global bound on \( E_2(t) \) yields the desired decay rates stated in Theorem [1.1]. This completes the proof of Theorem [1.1]. \( \square \)

In the second part of this section, we provide several anisotropic upper bounds for products and triple products. The bounds stated in the following lemma are powerful tools in controlling the nonlinearity in terms of the anisotropic dissipation.

**Lemma 2.1.** For some constants \( C > 0 \), \( i, j, k = 1, 2, 3 \) and \( i \neq j \neq k \), we have

\[ \int |fgh| \, dx \leq C \left\| f \right\|_{L^2(\mathbb{R}^3)} \left\| \partial_i f \right\|_{L^2(\mathbb{R}^3)} \left\| g \right\|_{L^2(\mathbb{R}^3)} \left\| \partial_j g \right\|_{L^2(\mathbb{R}^3)} \left\| \partial_k h \right\|_{L^2(\mathbb{R}^3)}, \]  

(2.3)

\[ \int |fgh| \, dx \leq C \left\| f \right\|_{L^2(\mathbb{R}^3)} \left\| \partial_i f \right\|_{L^2(\mathbb{R}^3)} \left\| g \right\|_{L^2(\mathbb{R}^3)} \left\| \partial_j f \right\|_{L^2(\mathbb{R}^3)} \left\| \partial_k f \right\|_{L^2(\mathbb{R}^3)} \left\| \partial_i \partial_j f \right\|_{L^2(\mathbb{R}^3)} \left\| \partial_k h \right\|_{L^2(\mathbb{R}^3)}, \]

(2.4)

\[ \|f g\|_{L^3(\mathbb{R}^3)} \leq C \left\| f \right\|_{L^2(\mathbb{R}^3)} \left\| \partial_i f \right\|_{L^2(\mathbb{R}^3)} \left\| g \right\|_{L^2(\mathbb{R}^3)} \left\| \partial_j f \right\|_{L^2(\mathbb{R}^3)} \left\| \partial_k f \right\|_{L^2(\mathbb{R}^3)} \left\| \partial_i \partial_j f \right\|_{L^2(\mathbb{R}^3)} \left\| \partial_k g \right\|_{L^2(\mathbb{R}^3)}, \]

(2.5)

\[ \|f g\|_{L^3_{1/3} L^1_{1/3} L^2} \leq C \left\| f \right\|_{L^2(\mathbb{R}^3)} \left\| \partial_i f \right\|_{L^2(\mathbb{R}^3)} \left\| g \right\|_{L^2(\mathbb{R}^3)}. \]

(2.6)
Let us prove (2.5). By Hölder’s inequality, for \( i = 1, 2, 3 \), we have the simple fact
\[
\|f\|_{L^\infty_t L^2_x} \leq \sqrt{2} \|f\|_{L^2_t L^2_x} \|\partial_t f\|_{L^2_t L^2_x}.
\] (2.7)
By (2.7),
\[
\|fg\|_{L^2_t L^2_x} \leq \|f\|_{L^2_t L^2_x} \|g\|_{L^2_t L^2_x} \leq C \|f\|_{L^2_t L^2_x} \|\partial_2 f\|_{L^2_t L^2_x} \|\partial_3 g\|_{L^2_t L^2_x}
\leq \|f\|_{L^2_t L^2_x} \|\partial_2 f\|_{L^2_t L^2_x} \|\partial_3 g\|_{L^2_t L^2_x}.
\]
By Minkowski’s inequality, (2.7) and Hölder’s inequality,
\[
\|f\|_{L^\infty_t L^1_x L^3_y} \leq \|f\|_{L^\infty_t L^2_x L^2_y} \leq C \|f\|_{L^2_t L^2_x} \|\partial_2 f\|_{L^2_t L^2_x} \|\partial_3 f\|_{L^2_t L^2_x} \|\partial_4 f\|_{L^2_t L^2_x} \|\partial_5 g\|_{L^2_t L^2_x} \|\partial_6 g\|_{L^2_t L^2_x}
\leq C \|f\|_{L^2_t L^2_x} \|\partial_2 f\|_{L^2_t L^2_x} \|\partial_3 f\|_{L^2_t L^2_x} \|\partial_4 f\|_{L^2_t L^2_x} \|\partial_5 g\|_{L^2_t L^2_x} \|\partial_6 g\|_{L^2_t L^2_x}
\]
Therefore,
\[
\|fg\|_{L^2_t L^2_x} \leq C \|f\|_{L^2_t L^2_x} \|\partial_2 f\|_{L^2_t L^2_x} \|\partial_3 f\|_{L^2_t L^2_x} \|\partial_4 f\|_{L^2_t L^2_x} \|\partial_5 g\|_{L^2_t L^2_x} \|\partial_6 g\|_{L^2_t L^2_x}.
\]
To prove (2.6), we apply Hölder’s inequality, Minkowski’s inequality and (2.7) to obtain
\[
\|fg\|_{L^\infty_t L^1_x L^2_y} \leq C \|f\|_{L^\infty_t L^2_x L^2_y} \|g\|_{L^\infty_t L^2_x L^2_y} \leq C \|f\|_{L^\infty_t L^2_x L^2_y} \|g\|_{L^2_t L^2_x}
\leq C \|f\|_{L^2_t L^2_x} \|\partial_5 f\|_{L^2_t L^2_x} \|\partial_6 f\|_{L^2_t L^2_x} \|\partial_7 g\|_{L^2_t L^2_x}.
\]
This completes the proof of Lemma 2.1. \(\square\)

3. Estimate for \( E_0(t) \)

This section is devoted to proving the a priori estimate (1.11) for \( E_0(t) \). More precisely, we prove the following proposition. We exploit the extra smoothing reflected in the wave structure (1.3) to make up for the lack of vertical dissipation in the velocity equation. The idea is to consider a Lyapunov functional involving an inner product besides the standard \( H^2 \)-norm.

**Proposition 3.1.** Let \((u, b)\) be a solution of the system (1.4). Then, for some constant \( C > 0 \), we have
\[ E_0(t) \leq CE(0) + CE^2(t). \] (3.1)
To prove (3.1), we work with the Lyapunov functional defined by
\[ L(u, b)(t) = \| (u(t), b(t)) \|_{H^3}^2 + \lambda (\partial_2 u(t), b(t))_{H^2}, \]
where \( \lambda > 0 \) is a small parameter. Next we show the bound of \( L(u, b) \). We evaluate the time evolution of each part in this Lyapunov functional. For the sake of clarity, we divide this process into two lemmas. The first focuses on bounding \( \| (u(t), b(t)) \|_{H^3}^2 \), while the second handles the inner product \( (\partial_2 u(t), b(t))_{H^2} \).

**Lemma 3.2.** Assume \((u, b)\) is a solution to (1.4). Then we have
\[
(\|u(t)\|_{H^3}^2 + \|b(t)\|_{H^3}^2) + 2 \int_0^t \left( \mu \| \partial_1 u(\tau) \|_{H^3}^2 + \eta \| \nabla b(\tau) \|_{H^2}^2 \right) d\tau \leq CE(0) + CE^2(t).
\]

**Proof of Lemma 3.2.** First we take the \( L^2 \)-inner product of (1.4) with \((u, b)\) to obtain
\[
\frac{1}{2} \frac{d}{dt} \left( \|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 \right) + \mu \| \partial_1 u \|_{L^2}^2 + \eta \| \nabla b \|_{L^2}^2 = 0. \tag{3.2}
\]
Due to the equivalence of the norm \( \| (u(t), b(t)) \|_{H^3} \) with \( \| (u(t), b(t)) \|_{L^2} + \| (u(t), b(t)) \|_{H^3} \), it suffices to bound \( \| (u(t), b(t)) \|_{H^3} \). Applying \( \partial_i^3 (i = 1, 2, 3) \) to the equations (1.4) and taking the \( L^2 \)-inner product of the resulting equations with \( \partial_i^3 u, \partial_i^3 b \), we have
\[
\frac{1}{2} \sum_{i=1}^3 \frac{d}{dt} \left( \| \partial_i^3 u(t) \|_{L^2}^2 + \| \partial_i^3 b(t) \|_{L^2}^2 \right) + \sum_{i=1}^3 \left( \mu \| \partial_i^3 \partial_1 u \|_{L^2}^2 + \eta \| \partial_i^3 \nabla b \|_{L^2}^2 \right) = - \sum_{i=1}^3 \int \partial_i^3 (u \cdot \nabla u) \cdot \partial_i^3 u \, dx + \sum_{i=1}^3 \int \partial_i^3 (b \cdot \nabla b) \cdot \partial_i^3 u \, dx - \sum_{i=1}^3 \int \partial_i^3 (u \cdot \nabla b) \cdot \partial_i^3 b \, dx + \sum_{i=1}^3 \int \partial_i^3 (b \cdot \nabla u) \cdot \partial_i^3 b \, dx \tag{3.3}
\]
By Leibniz formula, integration by parts and \( \nabla \cdot u = 0 \), we have
\[
I_1 = - \sum_{i=1}^3 \sum_{k=1}^3 C_k^3 \int \partial_i^3 u \cdot \nabla \partial_i^{3-k} u \cdot \partial_i^3 u \, dx = \sum_{k=1}^3 C_k^3 \int \partial_i^3 u \cdot \nabla \partial_i^{3-k} u \cdot \partial_i^3 u \, dx
\]
where \( C_k^3 \) is the standard binomial coefficient. By Hölder’s inequality and Sobolev’s inequality,
\[
I_{11} = - \sum_{i=1}^3 \left( 3 \int \partial_i u \cdot \nabla \partial_i^2 u \cdot \partial_i^3 u \, dx + 3 \int \partial_i^2 u \cdot \nabla \partial_i u \cdot \partial_i^3 u \, dx + \int \partial_i^3 u \cdot \nabla u \cdot \partial_i^3 u \, dx \right) \leq C(\|u\|_{L^\infty} \| \nabla \partial_i^2 u \|_{L^2}) \leq C(\|u\|_{H^1} \| \nabla \partial_i^2 u \|_{L^2} + \| \nabla u \|_{L^4} \| \nabla \partial_i^2 u \|_{L^4} + \| \nabla u \|_{H^1} \| \nabla \partial_i^2 u \|_{L^2} \| \nabla u \|_{H^1} \| \nabla \partial_i^2 u \|_{L^2} + \| \nabla u \|_{H^1} \| \nabla \partial_i^2 u \|_{L^2}). \tag{3.4}
\]
Rewriting the terms \( I_{12} \) in components, we have

\[
I_{12} \leq 4 \int |\partial_3 u| |\partial_3^2 \nabla_h u| |\partial_3^2 u| \, dx + 6 \int |\nabla \partial_3 u| |\partial_3^2 u| \, dx \\
- 3 \int \partial_3 u \cdot \partial_3^2 u \, dx - \int \partial_3^2 u \cdot \nabla_h u \cdot \partial_3^2 u \, dx \\
\leq 4 \int |\partial_3 u| |\partial_3^2 \nabla_h u| |\partial_3^2 u| \, dx + 6 \int |\nabla \partial_3 u| |\partial_3^2 u| \, dx \\
+ 8 \int |u| |\partial_3^2 u| |\partial_1 \partial_3 u| \, dx + 4 \int |\partial_2 u| |\partial_3^2 u|^2 \, dx \\
:= I_{121} + I_{122} + I_{123} + I_{124},
\]

where we have used the divergence-free condition, \( \partial_3 u_3 = -\partial_1 u_1 - \partial_2 u_2 \). By the anisotropic inequalities (2.3) and (2.4),

\[
I_{121} + I_{122} \leq C \left( ||\partial_3 u||_{L^2} \right)^{\frac{1}{2}} ||\partial_2 \partial_3 u||_{L^2} \right)^{\frac{1}{2}} ||\partial_3^2 \nabla_h u||_{L^2} \right)^{\frac{1}{2}} ||\partial_3^2 u||_{L^2} \right)^{\frac{1}{2}} ||\partial_3 \partial_1 u||_{L^2} \right)^{\frac{1}{2}} \\
+ C \left( ||\nabla \partial_3 u||_{L^2} \right)^{\frac{1}{2}} ||\nabla \partial_3^2 u||_{L^2} \right)^{\frac{1}{2}} ||\partial_2 \partial_3^2 u||_{L^2} \right)^{\frac{1}{2}} ||\partial_3^2 u||_{L^2} \right)^{\frac{1}{2}} ||\partial_3 \partial_3 u||_{L^2} \right)^{\frac{1}{2}} ||\partial_1 \partial_3 u||_{L^2} \right)^{\frac{1}{2}} \\
\leq C \left( ||\nabla u||_{H^3} \right)^{\frac{1}{2}} \left( ||\nabla \nabla u||_{H^1} \right)^{\frac{1}{2}} + ||\partial_1 \nabla^3 u||_{L^2} \right)^{\frac{1}{2}}.
\]

Applying (2.4) again, \( I_{123} \) can be bounded by

\[
I_{123} \leq C ||u||_{H^3} \left( ||\nabla \nabla u||_{H^1} \right)^{\frac{1}{2}} + ||\partial_1 \nabla^3 u||_{L^2} \right)^{\frac{1}{2}}. \\
\]

Therefore,

\[
I_{12} \leq C ||u||_{H^3} \left( ||\nabla \nabla u||_{H^1} \right)^{\frac{1}{2}} + ||\partial_1 \nabla^3 u||_{L^2} \right)^{\frac{1}{2}} + I_{124},
\]

where \( I_{124} \) will be estimated at the end of the proof. Consequently, (3.4), together with (3.5), leads to

\[
I_1 \leq C ||u||_{H^3} \left( ||\nabla \nabla u||_{H^1} \right)^{\frac{1}{2}} + ||\partial_1 \nabla^3 u||_{L^2} \right)^{\frac{1}{2}} + I_{124}.
\]

Since \( b \) has better dissipation than \( u \), it is simpler to bound \( I_2 \). By Leibniz’s formula,

\[
I_2 = \sum_{j=1}^2 \sum_{k=1}^3 C_3^k \int \partial_3^k b \cdot \nabla \partial_3^{3-k} b \cdot \partial_3^3 u \, dx + \sum_{k=1}^3 C_3^k \int \partial_3^k b \cdot \nabla \partial_3^{3-k} b \cdot \partial_3^3 u \, dx \\
+ \int b \cdot \nabla \partial_3^3 b \cdot \partial_3^3 u \, dx \\
:= I_{21} + I_{22} + \int b \cdot \nabla \partial_3^3 b \cdot \partial_3^3 u \, dx.
\]

As in \( I_{11} \), we first have

\[
I_{21} \leq C \left( ||\nabla b||_{L^2} \right)^{\frac{1}{2}} ||\nabla \partial_3^2 b||_{L^2} \right)^{\frac{1}{2}} + ||\nabla \partial_3^2 b||_{L^2} \right)^{\frac{1}{2}} ||\nabla \nabla \nabla b||_{L^2} \right)^{\frac{1}{2}} ||\nabla \nabla u||_{L^2} \right)^{\frac{1}{2}} \\
\leq C \left( ||\nabla b||_{H^3} \right)^{\frac{1}{2}} ||\nabla \partial_3^2 b||_{L^2} \right)^{\frac{1}{2}} + ||\nabla \partial_3^2 b||_{H^3} \right)^{\frac{1}{2}} ||\nabla \nabla \nabla b||_{H^1} \right)^{\frac{1}{2}} ||\nabla \nabla u||_{L^2} \right)^{\frac{1}{2}} \\
\leq C ||\nabla b||_{H^3} \left( ||\nabla \nabla b||_{H^1} \right)^{\frac{1}{2}} + ||\nabla \nabla^3 u||_{L^2} \right)^{\frac{1}{2}}.
\]
For $I_{22}$, we further split it into two parts and then apply (2.3) to get

\[
I_{22} = \sum_{k=1}^{3} C_{3}^{k} \int \partial_{3}^{k} b \cdot \nabla_{h} \partial_{3}^{k} b \cdot \partial_{3}^{3} u \, dx + \sum_{k=1}^{3} C_{3}^{k} \int \partial_{3}^{k} b \cdot \partial_{3}^{k} \partial_{3}^{3} b \cdot \partial_{3}^{3} u \, dx
\]

\[
\leq C \sum_{k=1}^{3} \| \partial_{3}^{k} b \|_{L^{2}} \| \partial_{3}^{k} \partial_{3}^{3} b \|_{L^{2}} \| \partial_{3}^{3} \nabla_{h} \partial_{3}^{3} b \|_{L^{2}} \| \partial_{3}^{3} \nabla_{h} \partial_{3}^{3} b \|_{L^{2}} \| \partial_{3}^{3} u \|_{L^{2}} \| \partial_{3}^{3} u \|_{L^{2}}
\]

\[
+ C \sum_{k=1}^{3} \| \partial_{3}^{k} b \|_{L^{2}} \| \partial_{3}^{k} \partial_{3}^{3} b \|_{L^{2}} \| \partial_{3}^{3} \nabla_{h} \partial_{3}^{3} b \|_{L^{2}} \| \partial_{3}^{3} \nabla_{h} \partial_{3}^{3} b \|_{L^{2}} \| \partial_{3}^{3} u \|_{L^{2}} \| \partial_{3}^{3} u \|_{L^{2}}
\]

\[
\leq C (\| \nabla b \|_{H^{2}} + \| \nabla^{3} u \|_{L^{2}}) (\| \nabla b \|_{H^{2}} + \| \partial_{3}^{3} u \|_{L^{2}}).
\]  

(3.8)

Therefore, (3.7) and (3.8) yield

\[
I_{2} \leq C (\| \nabla b \|_{H^{2}} + \| \nabla^{3} u \|_{L^{2}}) (\| \nabla b \|_{H^{2}} + \| \partial_{3}^{3} u \|_{L^{2}}) + \int b \cdot \nabla \partial_{3}^{3} b \cdot \partial_{3}^{3} u \, dx.
\]  

(3.9)

We proceed to deal with $I_{3}$. $I_{3}$ is firstly divided into three parts,

\[
I_{3} = - \sum_{i=1}^{2} \sum_{k=1}^{3} C_{3}^{k} \int \partial_{i}^{k} u \cdot \nabla \partial_{i}^{3} b \cdot \partial_{3}^{3} b \, dx - \sum_{k=1}^{3} C_{3}^{k} \int \partial_{3}^{k} u_{h} \cdot \nabla_{h} \partial_{3}^{3} b \cdot \partial_{3}^{3} b \, dx
\]

\[
- \sum_{k=1}^{3} C_{3}^{k} \int \partial_{3}^{k} u_{h} \cdot \nabla_{h} \partial_{3}^{3} b \cdot \partial_{3}^{3} b \, dx
\]

\[= I_{31} + I_{32} + I_{33}.
\]

By (2.4),

\[
I_{31} + I_{32} \leq C \sum_{i=1}^{2} \sum_{k=1}^{3} \| \partial_{i}^{k} u \|_{L^{2}} \| \partial_{i} \partial_{3}^{3} b \|_{L^{2}} \| \nabla \partial_{i}^{3} b \|_{L^{2}} \| \partial_{3}^{3} \nabla_{h} \partial_{3}^{3} b \|_{L^{2}} \| \partial_{3}^{3} \nabla_{h} \partial_{3}^{3} b \|_{L^{2}}
\]

\[
+ C \sum_{k=1}^{3} \| \partial_{3}^{k} u_{h} \|_{L^{2}} \| \partial_{3}^{k} \partial_{3}^{3} b \|_{L^{2}} \| \nabla_{h} \partial_{3}^{3} b \|_{L^{2}} \| \partial_{3}^{3} \nabla_{h} \partial_{3}^{3} b \|_{L^{2}} \| \partial_{3}^{3} \partial_{3}^{3} b \|_{L^{2}} \| \partial_{3}^{3} \partial_{3}^{3} b \|_{L^{2}}
\]

\[
\leq C (\| \nabla u \|_{H^{2}} + \| \nabla b \|_{H^{2}}) (\| \partial_{3}^{3} u \|_{H^{2}} + \| \nabla b \|_{H^{2}}).
\]  

(3.10)

For $I_{33}$, we further decompose it, integrate by parts and use (2.4) to get

\[
I_{33} = - \sum_{k=2}^{3} C_{3}^{k} \int \partial_{3}^{k} u_{h} \cdot \nabla_{h} \partial_{3}^{k} b \cdot \partial_{3}^{3} b \, dx + 6 \int u_{h} \cdot \nabla_{h} b \cdot \partial_{3}^{3} b \, dx
\]

\[
\leq C \sum_{k=2}^{3} \| \partial_{3}^{k} u_{h} \|_{L^{2}} \| \partial_{3}^{k} \partial_{3}^{3} b \|_{L^{2}} \| \partial_{3}^{3} \nabla_{h} \partial_{3}^{3} b \|_{L^{2}} \| \partial_{3}^{3} \nabla_{h} \partial_{3}^{3} b \|_{L^{2}} \| \partial_{3}^{3} \partial_{3}^{3} b \|_{L^{2}} \| \partial_{3}^{3} \partial_{3}^{3} b \|_{L^{2}}
\]

\[
+ C (\| u \|_{H^{2}} \| \partial_{3} \partial_{3}^{3} b \|_{L^{2}} \| \partial_{3} \partial_{3}^{3} b \|_{L^{2}} \| \partial_{3} \partial_{3}^{3} b \|_{L^{2}} \| \partial_{3} \partial_{3}^{3} b \|_{L^{2}} \| \partial_{3} \partial_{3}^{3} b \|_{L^{2}}
\]

\[
\leq C (\| u \|_{H^{2}} + \| \nabla b \|_{H^{2}}) (\| \partial_{3}^{3} u \|_{H^{2}} + \| \nabla \nabla u \|_{H^{2}} + \| \nabla b \|_{H^{2}}).
\]  

(3.11)

where we have used $\nabla \cdot u = 0$. Combining (3.10) and (3.11) yields

\[
I_{3} \leq C (\| u \|_{H^{3}} + \| \nabla b \|_{H^{2}}) (\| \nabla \nabla u \|_{H^{2}} + \| \partial_{3}^{3} u \|_{H^{2}} + \| \nabla b \|_{H^{2}}).
\]  

(3.12)
We now bound $I_4$. As in $I_2$, we decompose $I_4$ into three parts,

$$I_4 = \sum_{i=1}^{2} \sum_{k=1}^{3} C_i^k \int \partial_i^k b \cdot \nabla \partial_i^{3-k} u \cdot \partial_i^3 b \, dx + \sum_{k=1}^{3} C_3^k \int \partial_3^k b \cdot \nabla \partial_3^{3-k} u \cdot \partial_3^3 b \, dx + \int b \cdot \nabla \partial_1^3 u \cdot \partial_1^3 b \, dx$$

$$:= I_{41} + I_{42} + \int b \cdot \nabla \partial_1^3 u \cdot \partial_1^3 b \, dx.$$

By Hölder’s inequality and Sobolev’s inequality,

$$I_{41} \leq C \sum_{k=1}^{3} \|\nabla_h^k b\|_{L^2} \|\nabla \nabla_h^{3-k} u\|_{L^2} \|\nabla_h^3 b\|_{L^2} \leq C \|\nabla u\|_{H^2} \|\nabla_h b\|_{H^2}^2.$$  

The estimate for $I_{42}$ is more subtle. We first further split it into three terms,

$$I_{42} = 3 \int \partial_3 b \cdot \nabla \partial_3^2 u \cdot \partial_3^3 b \, dx + \sum_{k=1}^{3} C_3^k \int \partial_3^k b \cdot \nabla \partial_3^{3-k} u \cdot \partial_3^3 b \, dx$$

$$+ \sum_{k=1}^{3} C_3^k \int \partial_3^k b \partial_3^{4-k} u \cdot \partial_3^3 b \, dx$$

$$:= I_{421} + I_{422} + I_{423}.$$  

Applying (3.4) to $I_{421}$, and (2.3) to $I_{422}$ and $I_{423}$, respectively, we obtain

$$I_{421} \leq C \|\partial_3 b\|_{L^2} \|\partial_3 \partial_3^2 u\|_{L^2} \|\partial_3 \partial_3^3 b\|_{L^2} \|\nabla_h \partial_3^3 u\|_{L^2} \|\partial_3^3 b\|_{L^2} \leq C \|\nabla b\|_{H^2} (\|\nabla^2 \nabla_h u\|_{L^2}^2 + \|\nabla \nabla_h b\|_{H^2}^2),$$

and

$$I_{422} + I_{423} \leq C \sum_{k=1}^{3} \|\partial_3^k b\|_{L^2} \|\partial_3 \partial_3^3 u\|_{L^2} \|\partial_3 \partial_3^3 b\|_{L^2} \|\nabla_h \partial_3^3 u\|_{L^2} \|\partial_3 \partial_3^3 b\|_{L^2} \|\nabla_h \partial_3^3 b\|_{L^2}$$

$$+ C \sum_{k=1}^{3} \|\partial_3^k b\|_{L^2} \|\partial_3 \partial_3^3 u\|_{L^2} \|\partial_3 \partial_3^3 b\|_{L^2} \|\nabla_h \partial_3^3 u\|_{L^2} \|\partial_3 \partial_3^3 b\|_{L^2} \|\nabla_h \partial_3^3 b\|_{L^2}$$

$$\leq C (\|\nabla u\|_{H^2} + \|\nabla^2 b\|_{H^2}) (\|\partial_3 \nabla u\|_{H^2}^2 + \|\nabla_h b\|_{H^2}^2).$$

Thus,

$$I_4 \leq C (\|\nabla u\|_{H^2} + \|\nabla b\|_{H^2}) (\|\partial_3 \nabla u\|_{H^2}^2 + \|\partial_3 \nabla b\|_{H^2}^2 + \|\nabla_h b\|_{H^2}^2).$$

Inserting (3.6), (3.9), (3.12) and (3.13) in (3.3) and combining with (3.2), we conclude

$$\frac{1}{2} \frac{d}{dt} \left( \|u(t), b(t)\|_{L^2}^2 + \sum_{i=1}^{3} \|\partial_i^3 u(t), \partial_i^3 b(t)\|_{L^2}^2 \right) + \left[ \|\partial_1 u\|_{L^2}^2 + \eta \|\nabla_h u\|_{L^2}^2 \right]$$

$$+ \sum_{i=1}^{3} (\|\partial_i^3 \partial_1 u\|_{L^2}^2 + \eta \|\partial_i^3 \nabla_h b\|_{L^2}^2) \right]$$

$$\leq C (\|\partial_3 \nabla u\|_{H^2}^2 + \|\nabla b\|_{H^2}^2 + \|\partial_3 \nabla b\|_{H^2}^2 + \|\nabla_h b\|_{H^2}^2) + I_{124}.$$  

(3.14)
Integrating \((3.14)\) over \([0, t]\) yields

\[
\| (u(t), b(t)) \|_{H^3}^2 + 2 \int_0^t (\mu \| \partial_1 u(\tau) \|_{H^3}^2 + \eta \| \nabla_h b(\tau) \|_{H^3}^2) \, d\tau
\]

\[
\leq C \int_0^t \left( \| u(\tau) \|_{H^3}^2 + \| b(\tau) \|_{H^3}^2 \right) \left( \| \partial_2 u(\tau) \|_{H^3}^2 + \| \partial_1 u(\tau) \|_{H^3}^2 \right) d\tau
\]

\[
+ C \| u_0 \|_{H^3}^2 + \| b_0 \|_{H^3}^2 + C \int_0^t I_{124}(\tau) \, d\tau
\]

\[
\leq CE_0^3(t) + CE(0) + C \int_0^t I_{124}(\tau) \, d\tau.
\]

It remains to bound the integral of \(I_{124}\). By means of \((2.4)\), we have

\[
I_{124} \leq C \| \partial_2 u \|_{L^2}^\frac{1}{2} \| \partial_2^2 u \|_{L^2}^\frac{1}{2} \| \partial_3 \partial_2 u \|_{L^2}^\frac{1}{2} \| \partial_2 \partial_3 u \|_{L^2}^\frac{1}{2} \| \partial_1 \partial_3 u \|_{L^2}^\frac{1}{2}
\]

Then applying Hölder’s inequality leads to

\[
\int_0^t I_{124}(\tau) \, d\tau \leq C \sup_{0 \leq \tau \leq t} \left( 1 + \tau \right)^\frac{1}{4} \| \partial_2 u(\tau) \|_{L^2}^\frac{1}{2} \| \partial_2^2 u(\tau) \|_{L^2}^\frac{1}{2} \| \partial_3 \partial_2 u(\tau) \|_{L^2}^\frac{1}{2} \| \partial_2 \partial_3 u(\tau) \|_{L^2}^\frac{1}{2} \| \partial_1 \partial_3 u(\tau) \|_{L^2}^\frac{1}{2}
\]

\[
\times \left( \int_0^t \| \partial_2 \partial_3 u(\tau) \|_{L^2}^\frac{1}{2} \int_0^t \| \partial_1 \partial_3 u(\tau) \|_{L^2}^\frac{1}{2} \right)^\frac{1}{4}
\]

\[
\leq CE_2^\frac{1}{4}(t)E_0^\frac{1}{4}(t) \left( \int_0^t \| \partial_2 u(\tau) \|_{L^2}^\frac{1}{2} \| \partial_2^2 u(\tau) \|_{L^2}^\frac{1}{2} \| \partial_3 \partial_2 u(\tau) \|_{L^2}^\frac{1}{2} \| \partial_2 \partial_3 u(\tau) \|_{L^2}^\frac{1}{2} \| \partial_1 \partial_3 u(\tau) \|_{L^2}^\frac{1}{2}
\]

\[
\times \left( \int_0^t \| \partial_2 \partial_3 u(\tau) \|_{L^2}^\frac{1}{2} \right)^\frac{1}{4} \left( \int_0^t \| \partial_1 \partial_3 u(\tau) \|_{L^2}^\frac{1}{2} \right)^\frac{1}{4}
\]

\[
\leq CE_2^\frac{1}{4}(t)E_0^\frac{1}{4}(t) \leq CE_2^\frac{1}{4}(t).
\]

Therefore,

\[
\| (u(t), b(t)) \|_{H^3}^2 + 2 \int_0^t (\mu \| \partial_1 u(\tau) \|_{H^3}^2 + \eta \| \nabla_h b(\tau) \|_{H^3}^2) \, d\tau \leq CE_2^\frac{1}{4}(t) + CE(0).
\]

This completes the proof of Lemma 3.2 \( \square \)

Next we evaluate the inner product \((\partial_2 u(t), b(t))_{H^2}\) and prove the following lemma.

**Lemma 3.3.** Assume \((u, b)\) is a solution to \((1.4)\). Then

\[
- (\partial_2 u(t), b(t))_{H^2} + \frac{1}{2} \int_0^t \left( \| \partial_2 b(\tau) \|_{H^2}^2 - \int_0^\tau \left( \| \partial_2 b(\tau) \|_{H^2}^2 \right) \, d\tau \right)
\]

\[
\leq CE(0) + CE_2^\frac{1}{4}(t). \tag{3.15}
\]

**Proof of Lemma 3.3** Invoking the equations of \(u\) and \(b\) in \((1.4)\), we have

\[
- \frac{d}{dt} (\partial_2 u(t), b(t))_{H^2} + \| \partial_2 u \|_{H^2}^2 - \| \partial_2 b \|_{H^2}^2
\]

\[
= (\partial_2 (u \cdot \nabla u), b)_{H^2} - (\partial_2 (b \cdot \nabla b), b)_{H^2} + (\partial_2 u, u \cdot \nabla b)_{H^2} - (\partial_2 u, b \cdot \nabla u)_{H^2}
\]

\[
- \mu (\partial_2^2 u, b)_{H^2} - \eta (\partial_2 u, \Delta b)_{H^2}
\]

\[
:= I_5 + \cdots + I_{10}. \tag{3.16}
\]
By integration by parts, $I_5$ can be rewritten as

$$I_5 = - \int u \cdot \nabla u \cdot (\partial_2 b - \partial_2 \Delta b) \, dx + \int \nabla (u \cdot \nabla u) \cdot \partial_2 \nabla^2 b \, dx$$

$$= - \int u \cdot \nabla u \cdot (\partial_2 b - \partial_2 \Delta b) \, dx + \int (\nabla u \cdot \nabla u) u \cdot \partial_2 \nabla^3 b \, dx$$

$$+ \int (u \cdot \nabla) \nabla u \cdot \partial_2 \nabla^3 b \, dx.$$

Applying (2.3) and (2.4) leads to

$$I_5 \leq C\|u\|_{L^2}^\frac{1}{3} \|\partial_2 \Delta u\|_{L^2}^\frac{1}{3} \|\nabla u\|_{L^2}^\frac{1}{3} \|\partial_2 b\|_{L^2}^\frac{1}{3} \|\partial_3 \partial_2 \Delta b\|_{L^2}^\frac{1}{3} + C\|\nabla u\|_{L^2}^\frac{1}{3} \|\partial_1 \nabla u\|_{L^2}^\frac{1}{3} \|\partial_3 \partial_3 \nabla u\|_{L^2}^\frac{1}{3} \|\nabla u\|_{L^2}^\frac{1}{3} \|\partial_2 \nabla u\|_{L^2}^\frac{1}{3} + C\|\partial_2 \nabla^3 b\|_{L^2}^\frac{1}{3}$$

$$+ C\|u\|_{L^2}^\frac{1}{3} \|\partial_1 \nabla u\|_{L^2}^\frac{1}{3} \|\partial_3 \partial_3 \nabla u\|_{L^2}^\frac{1}{3} \|\nabla^2 u\|_{L^2}^\frac{1}{3} \|\partial_2 \nabla^2 u\|_{L^2}^\frac{1}{3} \|\partial_2 \nabla^3 b\|_{L^2}^\frac{1}{3}.$$

Similarly,

$$I_6 \leq C\|b\|_{H^2} \|\nabla b\|_{H^1}^2.$$

For $I_7$, we split it into two parts

$$I_7 = \int u \cdot \nabla b \cdot (\partial_2 u - \partial_2 \Delta u) \, dx + \int \Delta (u \cdot \nabla b) \cdot \partial_2 \Delta u \, dx := I_{71} + I_{72}.$$ 

By (2.4),

$$I_{71} \leq C\|u\|_{H^1} \|\partial_1 \nabla u\|_{L^2}^\frac{1}{3} \|\partial_3 \nabla u\|_{L^2}^\frac{1}{3} \|\nabla b\|_{L^2}^\frac{1}{3} \|\partial_2 \nabla b\|_{L^2}^\frac{1}{3} \|\partial_2 u + \partial_2 \Delta u\|_{L^2}^\frac{1}{3} \leq C\|u\|_{H^1} \|\nabla b\|_{L^2}^2 + \|\partial_2 \Delta u\|_{L^2}.$$

Similarly, making use of the inequality (2.4) again, we get

$$I_{72} = \int (\Delta u \cdot \nabla b + 2 \nabla u \cdot \nabla^2 b + u \cdot \nabla b) \cdot \partial_2 \Delta u \, dx$$

$$\leq C\|\Delta u\|_{L^2}^\frac{1}{3} \|\partial_1 \Delta u\|_{L^2}^\frac{1}{3} \|\nabla b\|_{L^2}^\frac{1}{3} \|\partial_2 \nabla b\|_{L^2}^\frac{1}{3} \|\nabla^2 b\|_{L^2}^\frac{1}{3} \|\partial_2 \Delta u\|_{L^2}^\frac{1}{3} + C\|\nabla b\|_{H^2} \|\nabla u\|_{H^1}^2 + \|\nabla \nabla b\|_{H^1}^2,$$

which, together with the estimate of $I_{71}$, gives

$$I_7 \leq C\|u\|_{H^2} \|\nabla b\|_{H^2}^2 (\|\nabla b\|_{H^2}^2 + \|\nabla \nabla b\|_{H^1}^2).$$

$I_8$ can be estimated with the same process as $I_7$. Firstly,

$$I_8 = - \int b \cdot \nabla u \cdot (\partial_2 u - \partial_2 \Delta u) \, dx - \int \Delta (b \cdot \nabla u) \cdot \partial_2 \Delta u \, dx = I_{81} + I_{82}.$$

Then we can derive

$$I_{81} \leq C\|\nabla u\|_{L^2}^2 + \|b\|_{H^2} \|\partial_2 \Delta u\|_{H^1}^2 + \|\partial_1 b\|_{H^1}^2.$$
and

\[ I_{82} = - \int (\Delta b \cdot \nabla u + 2 \nabla b \cdot \nabla^2 u + b \cdot \nabla \Delta u) \cdot \partial_2 \Delta u \, dx \]

\[ \leq C \| \Delta b \|_{L^2}^2 \| \partial_1 \Delta b \|_{L^2} \| \nabla u \|_{L^2}^2 \| \partial_2 \nabla u \|_{L^2}^2 \| \partial_3 \nabla u \|_{L^2}^2 \| \partial_2 \Delta u \|_{L^2} \]

\[ + C \| \nabla b \|_{L^2}^2 \| \partial_2 \nabla b \|_{L^2}^2 \| \partial_3 \nabla b \|_{L^2}^2 \| \nabla^2 u \|_{L^2}^2 \| \partial_1 \nabla^2 u \|_{L^2}^2 \| \partial_2 \Delta u \|_{L^2} \]

\[ + C \| \nabla b \|_{L^2}^2 \| \partial_2 \nabla b \|_{L^2}^2 \| \partial_3 \nabla b \|_{L^2}^2 \| \nabla \Delta u \|_{L^2}^2 \| \partial_1 \nabla \Delta u \|_{L^2}^2 \| \partial_2 \Delta u \|_{L^2} \]

\[ \leq C (\| \nabla u \|_{H^2} + \| b \|_{H^2}) (\| \partial_2 \nabla u \|_{H^2} + \| \partial_1 \nabla^2 u \|_{H^2} + \| \nabla b \|_{H^2}^2). \]

Thus,

\[ I_8 \leq C (\| \nabla u \|_{H^2} + \| b \|_{H^2}) (\| \partial_2 \nabla u \|_{H^2} + \| \partial_1 \nabla^2 u \|_{H^2} + \| \nabla b \|_{H^2}^2). \]

By Hölder’s inequality and Young’s inequality,

\[ I_9 + I_{10} = -\mu \langle \partial_2 u, \partial_1 \partial_2 b \rangle_{H^2} - \eta \langle \partial_2 u, \Delta b \rangle_{H^2} \leq \frac{1}{2} \| \partial_2 u \|_{H^2}^2 + \mu^2 \| \partial_1 \partial_2 b \|_{H^2}^2 + \eta^2 \| \Delta b \|_{H^2}^2. \]

In summary, we have obtained

\[ - \frac{d}{dt} (\| u(t) \|_{H^2}^2 + \| b(t) \|_{H^2}^2 - \lambda \| \partial_2 u(t), b(t) \|_{H^2}) + \int_0^t \left[ 2\mu \| \partial_1 u(\tau) \|_{H^2}^2 \right. \]

\[ + (2\eta - \lambda (1 + \mu^2 + \eta^2)) \| \nabla b(\tau) \|_{H^2}^2 + \frac{\lambda}{2} \| \partial_2 u(\tau) \|_{H^2}^2 \] \[ \left. d\tau \right] \]

\[ \leq CE(0) + CE^{\frac{3}{2}}(t), \]

where \( \lambda \) is a parameter. Now we select \( \lambda \) to be sufficiently small to obtain

\[ \| u(t) \|_{H^2}^2 + \| b(t) \|_{H^2}^2 + \int_0^t \left( \| \partial_1 u(\tau) \|_{H^2}^2 + \| \partial_2 u(\tau) \|_{H^2}^2 + \| \nabla b(\tau) \|_{H^2}^2 \right) d\tau \]

\[ \leq CE(0) + CE^{\frac{3}{2}}(t). \]

This completes the proof of Proposition 3.1. \( \square \)
4. Estimate for $E_1(t)$

The section proves the *a priori* inequality (1.12) for $E_1(t)$. That is, we establish the following proposition. Since the velocity equation does not have the vertical dissipation, we need to make use of the extra smoothing and stabilization revealed by the wave structure in (1.8). Our idea is to use the inner product $(1 + t)\partial_2 \nabla_h u, \nabla_h b)$ to decode this regularizing property. As a consequence, we obtain the time integrability of $(1 + t)\|\partial_2 \nabla_h u\|_{L^2}^2$. More details are given in Lemma 4.3 and its proof.

**Proposition 4.1.** For some constants $C > 0$, it holds

$$E_1(t) \leq CE(0) + CE_0(t) + CE^2(t).$$

(4.1)

We shall divide the proof of (4.1) into two main parts. The first one bounds the time-weighted energy $(1 + t)\|\nabla_h u, \nabla_h b\|_{H^1}^2$, while the second handles the inner product $(1 + t)(\partial_2 \nabla_h u, \nabla_h b)$ to generate the time-weighted dissipation $(1 + t)\|\partial_2 \nabla_h u\|_{L^2}^2$.

**Lemma 4.2.** Assume $(u, b)$ solves (1.4). Then we have

$$(1 + t)(\|\nabla_h u(t)\|_{H^1}^2 + \|\nabla_h b(t)\|_{H^1}^2) + 2 \int_0^t (1 + \tau)(\mu\|\partial_1 \nabla_h u(\tau)\|_{H^1}^2 + \eta\|\Delta_h b(\tau)\|_{H^1}^2)d\tau$$

$$\leq E_0(t) + E(0) + CE^2(t).$$

(4.2)

**Proof of Lemma 4.2.** Taking the $H^1$-inner product of (1.4) with $(\Delta_h u, \Delta_h b)$, and multiplying by $(1 + t)$, we obtain

$$\frac{1}{2} \frac{d}{dt}(1 + t)(\|\nabla_h u(t)\|_{H^1}^2 + \|\nabla_h b(t)\|_{H^1}^2) + (1 + t)(\mu\|\partial_1 \nabla_h u\|_{H^1}^2 + \eta\|\Delta_h b\|_{H^1}^2)$$

$$= \frac{1}{2}(\|\nabla_h u\|_{H^1}^2 + \|\nabla_h b\|_{H^1}^2) - (1 + t)(\nabla_h (u \cdot \nabla u), \nabla_h u)_{H^1} + (1 + t)(\nabla_h (b \cdot \nabla b), \nabla_h u)_{H^1}$$

$$- (1 + t)(\nabla_h (u \cdot \nabla b), \nabla_h b)_{H^1} + (1 + t)(\nabla_h (b \cdot \nabla u), \nabla_h b)_{H^1}$$

$$:= \frac{1}{2}(\|\nabla_h u\|_{H^1}^2 + \|\nabla_h b\|_{H^1}^2) + J_1 + J_2 + J_3 + J_4.$$ 

(4.3)

To bound $J_1$, we split $J_1$ into three parts

$$J_1 = - (1 + t)\int \nabla_h (u \cdot \nabla u) \cdot \nabla_h u \, dx + \int \nabla_h^2 (u \cdot \nabla u) \cdot \nabla_h^2 u \, dx$$

$$+ \int \nabla_h \partial_3 (u \cdot \nabla u) \cdot \nabla_h \partial_3 u \, dx$$

$$:= -(1 + t)(J_{11} + J_{12} + J_{13}).$$

By the anisotropic inequality (2.3),

$$J_{11} = \int \nabla_h u_h \cdot \nabla_h u \cdot \nabla_h u \, dx + \int \nabla_h u_3 \cdot \nabla_h u \, dx$$

$$\leq C\|\nabla_h u\|_{L^2}^\frac{1}{2} \|\partial_2 \nabla_h u\|_{L^2}^\frac{1}{2} \|\nabla_h u\|_{L^2}^\frac{1}{2} \|\partial_3 \nabla_h u\|_{L^2}^\frac{1}{2} \|\nabla_h u\|_{L^2}^\frac{1}{2} \|\partial_1 \nabla_h u\|_{L^2}^\frac{1}{2}$$

$$+ C\|\nabla_h u_3\|_{L^2}^\frac{1}{2} \|\partial_3 \nabla_h u_3\|_{L^2}^\frac{1}{2} \|\partial_3 u_3\|_{L^2}^\frac{1}{2} \|\partial_2 \partial_3 u_3\|_{L^2}^\frac{1}{2} \|\nabla_h u_3\|_{L^2}^\frac{1}{2} \|\partial_1 \nabla_h u_3\|_{L^2}^\frac{1}{2}$$

$$\leq C\|\nabla_h u\|_{L^2}^\frac{1}{2} \|\nabla_h u\|_{H^1} + C\|\nabla_h u_3\|_{L^2}^\frac{1}{2} \|\partial_3 u_3\|_{L^2}^\frac{1}{2} \|\nabla_h^2 u_3\|_{L^2} \|\nabla_h u_3\|_{H^1}.$$

(4.4)
Therefore,
\[ \int_0^1 (1 + \tau) J_{12}(\tau) d\tau \leq C \sup_{0 \leq \tau \leq 1} (1 + \tau)^{\frac{1}{2}} \| \nabla^2 u(\tau) \|_{L^2} \int_0^1 (1 + \tau)^{\frac{1}{2}} \| \nabla^2 u(\tau) \|_{H^1} d\tau \]
\[ + C \sup_{0 \leq \tau \leq 1} (1 + \tau)^{\frac{1}{2}} \| \nabla^2 u(\tau) \|_{L^2} \int_0^1 (1 + \tau)^{\frac{1}{2}} \| \nabla^2 u(\tau) \|_{H^1} d\tau \]
\[ \leq CE_1^2(t) E_1^2(t) E_0^2(t) + E_2^2(t) E_0^2(t) E_1^2(t) E_0^2(t) \]
\[ \leq CE_2^2(t). \]  

Applying (2.3) again and using Sobolev’s inequality, \( J_{12} \) can be bounded as
\[ J_{12} = \int \nabla^2 u \cdot \nabla u \cdot \nabla^2 u \, dx + 2 \int \nabla^2 u \cdot \nabla u \cdot \nabla^2 u \, dx \]
\[ \leq \| \nabla u \|_{L^2} \| \nabla^2 u \|_{L^2} + C \| \nabla^2 u \|_{L^2} \| \nabla u \|_{L^2} \| \nabla^2 u \|_{L^2} \| \nabla u \|_{L^2} \| \nabla^2 u \|_{L^2} \]
\[ \leq C \| \nabla u \|_{H^2} \| \nabla^2 u \|_{L^2} + C \| \nabla^2 u \|_{L^2} \| \nabla u \|_{L^2} \| \nabla^2 u \|_{L^2} \]
\[ \leq CE_1^2(t) E_1^2(t) \]  

Thus,
\[ \int_0^1 (1 + \tau) J_{12}(\tau) d\tau \leq C \sup_{0 \leq \tau \leq 1} \| \nabla^2 u(\tau) \|_{L^2} \int_0^1 (1 + \tau)^{\frac{1}{2}} \| \nabla^2 u(\tau) \|_{H^1} d\tau \]
\[ + C \sup_{0 \leq \tau \leq 1} (1 + \tau)^{\frac{1}{2}} \| \nabla^2 u(\tau) \|_{L^2} \int_0^1 (1 + \tau)^{\frac{1}{2}} \| \nabla^2 u(\tau) \|_{H^1} d\tau \]
\[ \leq CE_1^2(t) E_1^2(t) E_0^2(t) E_1^2(t) E_0^2(t) \]
\[ \leq CE_2^2(t). \]  

The bound for \( J_{13} \) is more complicated. We first decompose it as follows,
\[ J_{13} = \int \nabla \partial_3 u \cdot \nabla u \cdot \nabla \partial_3 u \, dx + \int \nabla \partial_3 u \cdot \nabla \partial_3 u \cdot \nabla \partial_3 u \, dx + \int \partial_3 u \cdot \nabla \partial_3 u \cdot \nabla \partial_3 u \, dx \]
\[ \leq 3 \int |\nabla \partial_3 u| |\nabla \partial_3 u|^2 \, dx + 2 \int |\partial_3 u| |\nabla \partial_3 u| |\nabla \partial_3 u| \, dx + \int |\nabla \partial_3 u| |\partial_3^2 u| |\partial_3 \nabla u| \, dx \]
\[ := J_{131} + J_{132} + J_{133}. \]

By means of (2.3) and (2.4),
\[ J_{131} \leq C \| \nabla \partial_3 u \|^2_{L^2} \| \partial_3 \nabla u \|^2_{L^2} \| \partial_3 \nabla u \|^2_{L^2} \| \partial_3 \nabla \partial_3 u \|^2_{L^2} \| \partial_3 \nabla \partial_3 u \|^2_{L^2} \]
\[ \leq C \| \nabla \partial_3 u \|^2_{L^2} \| \partial_3 \nabla u \|^2_{L^2} \| \partial_3 \nabla \partial_3 u \|^2_{L^2}, \]
\[ J_{132} \leq C \| \partial_3 u \|^2_{L^2} \| \partial_3 \partial_3 u \|^2_{L^2} \| \partial_3 \partial_3 u \|^2_{L^2} \| \partial_3 \nabla u \|^2_{L^2} \| \partial_3 \nabla \partial_3 u \|^2_{L^2} \]
\[ \leq C \| \partial_3 u \|^2_{H^1} \| \partial_3 \nabla u \|^2_{L^2} \| \partial_3 \nabla \partial_3 u \|^2_{L^2}, \]
and
\[ J_{133} \leq C \| \nabla \partial_3 u \|^2_{L^2} \| \partial_3 \nabla \partial_3 u \|^2_{L^2} \| \partial_3 \nabla \partial_3 u \|^2_{L^2} \| \partial_3 \nabla \partial_3 u \|^2_{L^2} \]
\[ \leq C \| \nabla \partial_3 u \|^2_{L^2} \| \partial_3 \nabla \partial_3 u \|^2_{L^2} \| \partial_3 \nabla \partial_3 u \|^2_{L^2}. \]
Thereby, applying Hölder’s inequality gives

\[
\int_0^\tau (1 + \tau)J_{131}(\tau) \, d\tau \leq C \sup_{0 \leq \tau \leq t} ||\nabla_h u(\tau)||_{L^2}^{\frac{1}{2}} (1 + \tau)^{\frac{1}{2}} ||\partial_3 \nabla_h u(\tau)||_{L^2}^{\frac{1}{2}} \\
\times \int_0^\tau (1 + \tau)^{\frac{1}{2}} ||\partial_2 \nabla_h u(\tau)||_{L^2}^{\frac{1}{2}} ||\partial_3 \nabla_h u(\tau)||_{L^2}^{\frac{1}{2}} \, d\tau \\
\leq CE_{E_2}^\frac{1}{2}(t)E_1^\frac{1}{2}(t)E_0^\frac{1}{2}(t) \leq CE_2^\frac{1}{2}(t), \quad (4.11)
\]

\[
\int_0^\tau (1 + \tau)J_{132}(\tau) \, d\tau \leq C \sup_{0 \leq \tau \leq t} ||\partial_3 u(\tau)||_{H^1}^{\frac{1}{2}} (1 + \tau)^{\frac{1}{2}} ||\partial_3 \nabla_h u(\tau)||_{L^2}^{\frac{1}{2}} \int_0^\tau (1 + \tau)^{\frac{1}{2}} ||\nabla_h^2 u(\tau)||_{L^2}^{\frac{1}{2}} \, d\tau \\
\times (1 + \tau)^{\frac{1}{2}} ||\partial_3 \partial_1 \nabla_h u(\tau)||_{L^2}^{\frac{1}{2}} ||\partial_2 \partial_3 \nabla_h u(\tau)||_{L^2}^{\frac{1}{2}} \, d\tau \\
\leq CE_{E_0}^\frac{1}{2}(t)E_1^\frac{1}{2}(t)E_0^\frac{1}{2}(t) \leq CE_2^\frac{1}{2}(t), \quad (4.12)
\]

\[
\int_0^\tau (1 + \tau)J_{133}(\tau) \, d\tau \leq C \sup_{0 \leq \tau \leq t} ||\nabla_h u(\tau)||_{L^2}^{\frac{1}{2}} ||\nabla_h^2 u(\tau)||_{L^2}^{\frac{1}{2}} \int_0^\tau (1 + \tau)^{\frac{1}{2}} ||\nabla_h^2 u(\tau)||_{L^2}^{\frac{1}{2}} \\
\times (1 + \tau)^{\frac{1}{2}} ||\partial_3 \partial_1 \nabla_h u(\tau)||_{L^2}^{\frac{1}{2}} ||\partial_3 \nabla_h u(\tau)||_{H^1} \, d\tau \\
\leq CE_{E_2}^\frac{1}{2}(t)E_0^\frac{1}{2}(t)E_1^\frac{1}{2}(t)E_0^\frac{1}{2}(t) \leq CE_2^\frac{1}{2}(t). \quad (4.13)
\]

Adding (4.11), (4.12) and (4.13) yields

\[
\int_0^\tau (1 + \tau)J_{13}(\tau) \, d\tau \leq CE_2^\frac{1}{2}(t). \quad (4.14)
\]

Consequently, according to the estimates (4.5), (4.7) and (4.14), we derive

\[
\int_0^\tau J_1(\tau) \, d\tau \leq CE_2^\frac{1}{2}(t). \quad (4.15)
\]

In the following, we handle \( J_3 \). The terms \( J_2 \) and \( J_4 \) will be estimated together later. Firstly,

\[
J_3 = -(1 + t) \left( \int \nabla_h (u \cdot \nabla b) \cdot \nabla_h b \, dx + \int \nabla_h^2 (u \cdot \nabla b) \cdot \nabla_h^2 b \, dx \right) \\
+ \int \nabla_h \partial_3 (u \cdot \nabla b) \cdot \nabla_h \partial_3 b \, dx \\
:= -(1 + t)(J_{31} + J_{32} + J_{33}).
\]

Invoking (4.4) and (4.6), we have

\[
J_{31} = \int \nabla_h u \cdot \nabla_h b \cdot \nabla_h b \, dx + \int \nabla_h u \partial_3 b \cdot \nabla_h b \, dx \\
\leq C ||\nabla_h u||_{L^2} ||\partial_2 \nabla_h u||_{L^2} ||\partial_3 \nabla_h b||_{L^2} ||\nabla_h b||_{L^2} + C ||\nabla_h u||_{L^2} ||\partial_3 \nabla_h u||_{L^2} ||\partial_3 \nabla_h b||_{L^2} ||\nabla_h b||_{L^2} + C ||\partial_3 \nabla_h b||_{L^2} ||\partial_1 \nabla_h b||_{L^2}^{\frac{1}{2}}.
\]

\[
J_{32} = \int \nabla_h^2 u \cdot \nabla b \cdot \nabla_h b \, dx + 2 \int \nabla_h u \cdot \nabla \nabla_h b \cdot \nabla_h b \, dx \\
\leq C ||\nabla b||_{H^2} ||\nabla_h^2 u||_{L^2} ||\nabla_h b||_{L^2} + C ||\nabla_h u||_{L^2} ||\partial_2 \nabla_h u||_{L^2} ||\nabla_h b||_{L^2} ||\partial_3 \nabla_h b||_{L^2} ||\nabla_h b||_{L^2} ||\partial_3 \nabla_h b||_{L^2} ||\partial_1 \nabla_h b||_{L^2}^{\frac{1}{2}}.
\]
Then a similar argument to (4.5) and (4.7) gives

\[ \int_0^t (1 + \tau)(J_{31} + J_{32})(\tau) d\tau \leq CE^{\frac{5}{2}}(t). \]

For \( J_{33} \), we still reformulate it into several integrals

\[
J_{33} \leq 2 \int |\nabla_h u| |\partial_3 \nabla_h b|^2 dx + \int |\nabla_h b| |\partial_3 \nabla_h u_0||\partial_3 \nabla_h b| dx \\
+ \int |\partial_3 b| |\nabla_h b| |\partial_3 \nabla_h b| dx + \int |\partial_3 u| |\nabla_h b| |\partial_3 \nabla_h b| dx + \int |\nabla_h u_3| |\partial_3^2 b| |\partial_3 \nabla_h b| dx \\
:= 2J_{331} + \cdots + J_{335}.
\]

Going through a similar process as in \( J_{13} \), we are able to establish the bound for \( J_{33} \). Recalling (4.8), we have

\[
J_{331} \leq C||\nabla_h u||^\frac{1}{2}_{H^1}||\partial_2 \nabla_h u||^\frac{1}{2}_{L^2}||\partial_3 \nabla_h b||^\frac{1}{2}_{L^2}||\partial_3 \nabla u_0||^\frac{1}{2}_{L^2}||\partial_3 \nabla_h b||^\frac{1}{2}_{L^2}.
\]

Then

\[
\int_0^t (1 + \tau)J_{331}(\tau) d\tau \leq C \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{1}{2}}||\nabla_h u(\tau)||_H \int_0^t (1 + \tau)^{\frac{1}{2}}||\partial_3 \nabla_h b(\tau)||_H d\tau \leq CE^{\frac{5}{2}}(t).
\]

As in (4.9) and (4.12), \( J_{332} \) can be bounded by

\[
\int_0^t (1 + \tau)J_{332}(\tau) d\tau \leq C \sup_{0 \leq \tau \leq t} ||\partial_3 b(\tau)||_H \int_0^t (1 + \tau)^{\frac{1}{2}}||\nabla_h u(\tau)||_L^2 d\tau \leq CE^{\frac{5}{2}}(t).
\]

Also, from (4.10) and (4.13), we get

\[
\int_0^t (1 + \tau)J_{335}(\tau) d\tau \leq C \sup_{0 \leq \tau \leq t} ||\partial_3 u_3(\tau)||_H \int_0^t (1 + \tau)^{\frac{1}{2}}||\nabla_h u_3(\tau)||_L^2 d\tau \leq CE^{\frac{5}{2}}(t).
\]

The rest terms \( J_{332} \) and \( J_{334} \) can be handled as \( J_{331} \) and \( J_{333} \), respectively. Thus, we have

\[
\int_0^t (1 + \tau)(J_{332}(\tau) + J_{334}(\tau)) d\tau \leq CE^{\frac{5}{2}}(t).
\]

Consequently, we derive

\[
\int_0^t (1 + \tau)J_{33}(\tau) d\tau \leq CE^{\frac{5}{2}}(t).
\]

Combining all estimates above for \( J_{31} \) through \( J_{33} \), we conclude

\[
\int_0^t J_3(\tau) d\tau \leq CE^{\frac{5}{2}}(t). \tag{4.16}
\]
Finally we bound $J_2$ and $J_4$. $J_2$ and $J_4$ can be estimated with a nearly same argument as $J_1$ and $J_3$, respectively. We shall just sketch the proof. By integration by parts and the divergence-free condition, we split $J_2$ and $J_4$ into three parts as follows.

$$J_2 + J_4 := (J_{21} + J_{22} + J_{23})(1 + t),$$

where

$$J_{21} = \int (\nabla h b \cdot \nabla b \cdot \nabla h u + \nabla h b \cdot \nabla u \cdot \nabla h b) \, dx,$$

$$J_{22} = \int [(\nabla \nabla h b \cdot \nabla) b \cdot \nabla \nabla h u + (\nabla b \cdot \nabla) \nabla h b \cdot \nabla \nabla h u + (\nabla h b \cdot \nabla) \nabla b \cdot \nabla \nabla h u] \, dx,$$

$$J_{23} = \int [(\nabla \nabla h b \cdot \nabla) u \cdot \nabla \nabla h b + (\nabla b \cdot \nabla) \nabla h b \cdot \nabla \nabla h b + (\nabla h b \cdot \nabla) \nabla u \cdot \nabla \nabla h b] \, dx.$$  

It is easy to verify that

$$\int_0^\tau (J_2(\tau) + J_4(\tau)) \, d\tau \leq C E^{3/4}(t). \quad (4.17)$$

According to (4.4) and (4.5),

$$\int_0^\tau (1 + \tau) J_{21}(\tau) \, d\tau \leq C E^{3/4}(t).$$

For $J_{22}$, we further divide it into two parts

$$J_{22} = \int (\nabla^2_h b \cdot \nabla b \cdot \nabla^2_h u + 2 \nabla h b \cdot \nabla \nabla h b \cdot \nabla^2_h u) \, dx$$

$$+ \int (\nabla h \partial_3 b \cdot \nabla b \cdot \nabla \partial_3 h u + \partial_3 b \cdot \nabla \nabla h b \cdot \nabla \partial_3 h u + \nabla h b \cdot \nabla \partial_3 b \cdot \nabla \partial_3 h u) \, dx$$

$$:= J_{221} + J_{222}.$$  

As in (4.6) and (4.7) for $J_{12}$,

$$\int_0^\tau (1 + \tau) J_{221}(\tau) \, d\tau \leq C E^{3/4}(t).$$

For $J_{222}$, we have

$$J_{222} \leq 3 \int |\nabla h b| |\partial_3 \nabla h b| |\partial_3 \nabla h u| \, dx + 2 \int |\partial_3 b| |\nabla^2 h b| |\partial_3 \nabla h u| \, dx$$

$$+ \int |\nabla h b_3| |\partial^2_3 b| |\partial_3 \nabla h u| \, dx.$$  

Using the similarities between $J_{222}$ and $J_{131}$, $J_{132}$ and $J_{133}$, we can easily find

$$\int_0^\tau (1 + \tau) J_{222}(\tau) \, d\tau \leq C E^{3/4}(t).$$

Therefore,

$$\int_0^\tau (1 + \tau) J_{22}(\tau) \, d\tau \leq C E^{3/4}(t).$$
To bound $J_{23}$, we decompose it into
\[
J_{23} \leq \int (\nabla^2_h b \cdot \nabla u \cdot \nabla^2_h b + 2 \nabla_h b \cdot \nabla \nabla_h u \cdot \nabla^2_h b) \, dx
\]
\[+ J_{331} + 2J_{332} + J_{333} + J_{334} + \int |\nabla_h b_j| |\partial^2_3 u| |\partial_3 \nabla_h b| \, dx.
\]
The first term and the last term are similar to $J_{32}$ and $J_{335}$, respectively. Thus,
\[
\int_0^\prime (1 + \tau) J_{23}(\tau) d\tau \leq CE^2(t).
\]
Integrating (4.3) over $[0, t]$ and invoking (4.15), (4.16) and (4.17), we derive the desired estimate (4.2). This completes the proof of Lemma 4.2.

We now turn to the second lemma.

**Lemma 4.3.** Assume $(u, b)$ is a solution to (1.4). Then we have
\[
-(1 + t)(\partial_2 \nabla_h u(t), \nabla_h b(t)) + \frac{1}{2} \int_0^\prime (1 + \tau)||\partial_2 \nabla_h u(\tau)||_{L^2}^2 d\tau
\]
\[ \quad - \frac{1}{2} \int_0^\prime (1 + \tau) \left(3||\partial_2 \nabla_h b(\tau)||_{L^2}^2 + \mu^2||\nabla_h \partial_2^2 u(\tau)||_{L^2}^2 + \eta^2||\nabla_h \Delta b(\tau)||_{L^2}^2\right) d\tau
\]
\[ \leq CE(0) + \frac{1}{2} E_0(t) + CE^2(t).
\] (4.18)

**Proof of Lemma 4.3** As in (3.16), we have
\[
-\frac{d}{dt}(1 + t)(\partial_2 \nabla_h u, \nabla_h b) + (1 + t)||\partial_2 \nabla_h u||_{L^2}^2 - (1 + t)||\partial_2 \nabla_h b||_{L^2}^2
\]
\[= -(\partial_2 \nabla_h u, \nabla_h b) + (1 + t)(\partial_2 \nabla_h (u \cdot \nabla u), \nabla_h b) - (1 + t)(\partial_2 \nabla_h (b \cdot \nabla b), \nabla_h b)
\]
\[+ (1 + t)(\partial_2 \nabla_h u, \nabla_h (u \cdot \nabla u)) - (1 + t)(\partial_2 \nabla_h u, \nabla_h (b \cdot \nabla u))
\]
\[- \mu(1 + t)(\partial_2 \nabla_h \partial_2^2 u, \nabla_h b) - \eta(1 + t)(\partial_2 \nabla_h u, \nabla_h \Delta b)
\]
\[:= J_5 + \cdots + J_{11},
\]
where $(F, G)$ denotes the $L^2$-inner product of $F$ and $G$. It is clear that
\[
\int_0^\prime J_5(\tau) d\tau \leq \frac{1}{2} \int_0^\prime (||\partial_2 \nabla_h u(\tau)||_{L^2}^2 + ||\nabla_h b(\tau)||_{L^2}^2) d\tau \leq \frac{1}{2} E_0(t),
\]
\[J_{10} = \mu(1 + t)(\nabla_h \partial_2^2 u, \partial_2 \nabla_h b) \leq \frac{\mu^2}{2}(1 + t)||\partial_1^2 \nabla_h u||_{L^2}^2 + \frac{1}{2}(1 + t)||\partial_2 \nabla_h b||_{L^2}^2,
\]
\[J_{11} \leq \frac{1}{2}(1 + t)||\partial_2 \nabla h u||_{L^2}^2 + \frac{\eta^2}{2}(1 + t)||\Delta b \nabla h b||_{L^2}^2.
\]
Next we bound the nonlinear integral terms. We mainly focus on $J_6$ and $J_8$. The estimates for $J_7$ and $J_9$ can be established similarly. By integration by parts and (2.3), we have
\[
J_6 = -(1 + t) \int \nabla_h u \cdot \nabla u \cdot \partial_2 \nabla_h b \, dx - (1 + t) \int u \cdot \nabla \nabla_h u \cdot \partial_2 \nabla_h b \, dx
\]
\[\leq C(1 + t)||\nabla_h u||_{L^2}^2 ||\partial_1 \nabla h u||_{L^2}^2 ||\nabla u||_{L^2}^2 ||\partial_2 \nabla h u||_{L^2}^2 ||\partial_2 \nabla_h b||_{L^2}^2 ||\partial_3 \partial_2 \nabla_h b||_{L^2}^2.
\]
Furthermore, 

\[ \int_0^\tau J_6(\tau) d\tau \leq C \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{1}{4}} ||\nabla_h u(\tau)||_{L^2}^{\frac{3}{2}} ||\nabla u(\tau)||_{L^2}^{\frac{1}{2}} \int_0^\tau ||\nabla_h \nabla u(\tau)||_{L^2} \]

\[ \times (1 + \tau)^{\frac{1}{2}} ||\partial_2 \nabla_h b(\tau)||_{H^1} d\tau \]

\[ + C \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{1}{4}} ||\nabla_h u(\tau)||_{L^2}^{\frac{3}{2}} ||\nabla u(\tau)||_{L^2}^{\frac{1}{2}} \int_0^\tau ||\nabla_h \nabla u(\tau)||_{H^1} \]

\[ \times (1 + \tau)^{\frac{1}{2}} ||\partial_2 \nabla_h b(\tau)||_{H^1} d\tau \]

\[ \leq CE_2^\frac{3}{2}(t)E_0^1(t)E_1^\frac{1}{2}(t) \leq CE^\frac{3}{2}(t). \]

Similarly, we can bound \( J_7 \) as

\[ \int_0^\tau J_7(\tau) d\tau \leq CE^\frac{3}{2}(t). \]

For \( J_8 \), applying the anisotropic inequality (2.4) yields

\[ J_8 = (1 + t) \int \nabla_h u \cdot \nabla b \cdot \partial_2 \nabla_h u \, dx + (1 + t) \int u \cdot \nabla b \cdot \partial_2 \nabla_h u \, dx \]

\[ \leq C(1 + t)||\nabla_h u||_{L^2}^{\frac{3}{2}} ||\partial_2 \nabla_h u||_{L^2}^{\frac{1}{2}} ||\nabla b||_{L^2}^{\frac{1}{2}} ||\partial_1 \nabla b||_{L^2}^{\frac{1}{2}} ||\partial_3 \nabla b||_{L^2}^{\frac{1}{2}} ||\partial_2 \nabla_h u||_{L^2} \]

\[ + C(1 + t)||u||_{L^2}^{\frac{3}{2}} ||\partial_2 u||_{L^2}^{\frac{1}{2}} ||\partial_3 u||_{L^2}^{\frac{1}{2}} ||\nabla \nabla_h b||_{L^2}^{\frac{1}{2}} ||\partial_1 \nabla h b||_{L^2}^{\frac{1}{2}} ||\partial_2 \nabla_h u||_{L^2}. \]

Thus,

\[ \int_0^\tau J_8(\tau) d\tau \leq C \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{1}{2}} ||\nabla_h u(\tau)||_{L^2}^{\frac{3}{2}} ||\nabla b(\tau)||_{H^1}^{\frac{1}{2}} \int_0^\tau (1 + \tau)^{\frac{1}{2}} ||\partial_2 \nabla_h u(\tau)||_{L^2}^{\frac{1}{2}} ||\partial_1 \nabla b(\tau)||_{H^1}^{\frac{1}{2}} d\tau \]

\[ + C \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{1}{2}} ||\nabla \nabla_h b(\tau)||_{L^2}^{\frac{1}{2}} ||u(\tau)||_{H^1}^{\frac{1}{2}} \int_0^\tau (1 + \tau)^{\frac{1}{2}} ||\partial_1 \nabla h b(\tau)||_{L^2}^{\frac{1}{2}} ||\partial_2 \nabla_h u(\tau)||_{L^2}^{\frac{1}{2}} d\tau \]

\[ \leq CE_1(t)E_0^\frac{1}{2}(t) \leq CE^\frac{3}{2}(t). \]

Also,

\[ \int_0^\tau J_9(\tau) d\tau \leq CE^\frac{3}{2}(t). \]

Collecting all the estimates for \( J_5 \) through \( J_{11} \), and integrating over \([0, t]\), we derive the desired bound (4.18). This completes the proof of Lemma 4.3. \( \Box \)

We now putting together the two lemmas above to obtain Proposition 4.1.

**Proof of Proposition 4.1.** According to Lemma 4.2 and 4.3, the combination (4.2)+\( \lambda_1 \)(4.18) yields

\[ (1 + t)(||\nabla_h u(t)||_{H^1}^2 + ||\nabla h b(t)||_{H^1}^2 - \lambda_1(\partial_2 \nabla_h u, \nabla h b)) \]
\[ + \int_0^\tau (1 + \tau) \left( (2\mu - \frac{\mu^2}{2} \lambda_1) \| \partial_1 \nabla h u(\tau) \|_{H^1}^2 \\
+ \frac{\lambda_1}{2} (1 + \tau) \| \partial_2 \nabla h u(\tau) \|_{L^2}^2 + (2\eta - \frac{3\eta^2}{2} \lambda_1) \| \Delta h b(\tau) \|_{H^1}^2 \right) d\tau \leq CE(0) + CE_0(t) + CE_3^2(t), \]

where \( \lambda_1 \) is a parameter. If \( \lambda_1 \) is sufficiently small, then

\[ (1 + \tau) \left( \| \nabla h u(t) \|_{H^1}^2 + \| \nabla h b(t) \|_{H^1}^2 \right) \]

\[ + \int_0^\tau (1 + \tau) \left( \| \partial_1 \nabla h u(\tau) \|_{H^1}^2 + \| \partial_2 \nabla h u(\tau) \|_{L^2}^2 + \| \nabla h b(\tau) \|_{H^1}^2 \right) d\tau \leq CE(0) + CE_0(t) + CE_3^2(t). \]

This completes the proof of Proposition 4.1. □

5. Estimate for \( E_2(t) \)

This section establishes the a priori inequality (1.13) for \( E_2(t) \). That is, we prove the following proposition.

**Proposition 5.1.** Let \((u, b)\) be a solution to the system (1.4). Then it holds

\[ E_2(t) \leq C \left( E_3^2(t) + E_2^2(t) \right) + C \left( \| (u_0, b_0) \|_{H^1}^2 + \| (u_0, b_0) \|_{L^1_t L^2_x}^2 \\
+ \| (\partial_3 u_0, \partial_3 b_0) \|_{L^1_t L^2_x}^2 + \| (\partial_3^2 u_0, \partial_3^2 b_0) \|_{L^1_t L^2_x}^2 \right). \]  

(5.1)

We remark that energy estimates are no longer sufficient for the proof of (5.1). We resort to the integral representation of (1.4). To convert (1.4) into an integral representation, we take the Fourier transform of (1.4), solve the linearized system and represent the nonlinear system into an integral form via Duhamel’s principle. The integral representation involves three key kernel functions, which are degenerate and anisotropic. Due to the anisotropy, we divide the frequency space into subdomains to obtain sharp upper bounds on the kernel functions. This is done in Proposition 5.4. Once these bounds are at our disposal, we then estimate the \( L^2 \)-norms of \((u, b)\) and its derivatives via the integral representation. For the sake of clarity, we divide the rest of this section into two subsections.

5.1. **Integral representation and bounds for the kernels.** The subsection derives the integral representation of (1.4) and establishes optimal upper bounds for the kernel functions. First we recall two basic tools. The first one specifies the decay rate of a general heat operator associated with a fractional Laplacian operator. Here the fractional Laplacian operator can be defined through the Fourier transform

\[ \widehat{\Delta^\alpha} f(\xi) = |\xi|^\alpha \widehat{f}(\xi). \]

The decay rate is stated in the following lemma, whose proof can be found in many references (see, e.g., [71]).
Lemma 5.2. Assume \( \alpha \geq 0 \) and \( \beta > 0 \) are real numbers. Let \( 1 \leq p \leq q \leq \infty \). Then there exists a constant \( C > 0 \) such that, for any \( t > 0 \),
\[
\| \Lambda^\alpha e^{-\Lambda^\beta t} f \|_{L^q(\mathbb{R}^d)} \leq C t^{-\frac{\alpha}{p} - \frac{\beta}{q} + \frac{1}{2}} \| f \|_{L^p(\mathbb{R}^d)}.
\]

The second tool is an elementary inequality providing upper bounds for a convolution type integral. Its proof is straightforward.

Lemma 5.3. Assume \( 0 < s_1 \leq s_2 \). Then, for some constant \( C > 0 \),
\[
\int_0^t (1 + t - \tau)^{-s_1} (1 + \tau)^{-s_2} d\tau \leq \begin{cases} 
C(1 + t)^{-s_1}, & \text{if } s_2 > 1, \\
C(1 + t)^{-s_1} \ln(1 + t), & \text{if } s_2 = 1, \\
C(1 + t)^{1-s_1-s_2}, & \text{if } s_2 < 1.
\end{cases}
\] (5.2)

Now we derive an integral representation of (1.4). Applying the Leray-Hopf projection operator \( P = I - \nabla \Delta^{-1} \nabla \cdot \) to the velocity equation in (1.4) and taking the Fourier transform of the resulting equations, we have
\[
\partial_t \left( \begin{array}{c} \hat{u} \\ \hat{b} \end{array} \right) = A \left( \begin{array}{c} \hat{u} \\ \hat{b} \end{array} \right) + \left( \begin{array}{c} \hat{N}_1 \\ \hat{N}_2 \end{array} \right),
\] (5.3)
where
\[
A = \begin{pmatrix} -\mu \xi_1^2 & i\xi_2 \\ i\xi_2 & -\eta |\xi_h|^2 \end{pmatrix}, \quad N_1 = \mathbb{P}(b \cdot \nabla b - u \cdot \nabla u), \quad N_2 = b \cdot \nabla u - u \cdot \nabla b
\]
with \( |\xi_h|^2 = \xi_1^2 + \xi_2^2 \). To diagonalize \( A \), we compute the eigenvalues of \( A \),
\[
\lambda_1 = \frac{-(\mu \xi_1^2 + \eta |\xi_h|^2) - \sqrt{\Gamma}}{2}, \quad \lambda_2 = \frac{-(\mu \xi_1^2 + \eta |\xi_h|^2) + \sqrt{\Gamma}}{2},
\]
where
\[
\Gamma = (\mu \xi_1^2 + \eta |\xi_h|^2)^2 - 4(\mu \eta \xi_1^2 |\xi_h|^2 + \xi_2^2).
\]
The corresponding eigenvectors are
\[
\rho_1 = \begin{pmatrix} \lambda_1 + \eta |\xi_h|^2 \\ i\xi_2 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} \lambda_2 + \eta |\xi_h|^2 \\ i\xi_2 \end{pmatrix}.
\]
Therefore, the matrix \( A \) can be diagonalized as
\[
A = (\rho_1, \rho_2) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} (\rho_1, \rho_2)^{-1}.
\] (5.4)
We can now represent (5.3) as

\[
\left( \begin{array}{c} \hat{u} \\ \hat{b} \end{array} \right) = e^{At} \left( \begin{array}{c} \hat{u}_0 \\ \hat{b}_0 \end{array} \right) + \int_0^t e^{A(t-\tau)} \left( \begin{array}{c} \hat{N}_1(\tau) \\ \hat{N}_2(\tau) \end{array} \right) d\tau.
\]

By making \( e^{At} \) more explicit via (5.4), we obtain the integral representation

\[
\hat{u}(\xi, t) = \hat{Q}_1(t)\hat{u}_0 + \hat{Q}_2(t)\hat{b}_0 + \int_0^t (\hat{Q}_1(t-\tau)\hat{N}_1(\tau) + \hat{Q}_2(t-\tau)\hat{N}_2(\tau)) d\tau, \quad (5.5)
\]

\[
\hat{b}(\xi, t) = \hat{Q}_2(t)\hat{u}_0 + \hat{Q}_3(t)\hat{b}_0 + \int_0^t (\hat{Q}_2(t-\tau)\hat{N}_1(\tau) + \hat{Q}_3(t-\tau)\hat{N}_2(\tau)) d\tau, \quad (5.6)
\]

where

\[
\hat{Q}_1(t) = \eta|\xi_h|^2G_1(t) + G_2(t), \quad \hat{Q}_2(t) = i\xi_2G_1(t), \quad \hat{Q}_3(t) = -\eta|\xi_h|^2G_1(t) + G_3(t), \quad (5.7)
\]

with

\[
G_1(t) = \frac{e^{i\gamma t} - e^{A_1 t}}{\lambda_2 - \lambda_1}, \quad G_2(t) = \frac{\lambda_2 e^{i\gamma t} - \lambda_1 e^{A_1 t}}{\lambda_2 - \lambda_1} = e^{i\gamma t} + \lambda_1 G_1(t),
\]

\[
G_3(t) = \frac{\lambda_2 e^{i\gamma t} - \lambda_1 e^{A_1 t}}{\lambda_2 - \lambda_1} = e^{A_1 t} - \lambda_1 G_1(t).
\]

We remark that when \( \lambda_1 = \lambda_2 \), the representation in (5.5) and (5.6) remains valid if we replace \( G_1 \) by its limiting form

\[
G_1(t) = \lim_{\lambda_2 \to \lambda_1} \frac{e^{i\gamma t} - e^{A_1 t}}{\lambda_2 - \lambda_1} = t e^{A_1 t}.
\]

Next we investigate the behaviors of the kernels \( \hat{Q}_i(\xi, t) \) \((i = 1, 2, 3)\), which play a crucial role in the estimate of \( E_2(t) \). There kernels are anisotropic and degenerate. To obtain precise and sharp upper bounds, we divide the frequency space into subdomains and classify the behavior of the kernel functions in each subdomain.

**Proposition 5.4.** The domain \( \mathbb{R}^3 \) is split into two subdomains, \( \mathbb{R}^3 = A_1 \cup A_2 \) with

\[
A_1 := \left\{ \xi \in \mathbb{R}^3 : \sqrt{\xi} \leq \frac{\mu \xi_1^2 + \eta|\xi_h|^2}{2} \text{ or } 3(\mu \xi_1^2 + \eta|\xi_h|^2)^2 \leq 16(\mu \xi_1^2|\xi_h|^2 + \xi_2^2) \right\},
\]

\[
A_2 := \left\{ \xi \in \mathbb{R}^3 : \sqrt{\xi} > \frac{\mu \xi_1^2 + \eta|\xi_h|^2}{2} \text{ or } 3(\mu \xi_1^2 + \eta|\xi_h|^2)^2 > 16(\mu \xi_1^2|\xi_h|^2 + \xi_2^2) \right\}.
\]

Then we have

1. There exist two constants \( C > 0 \) and \( c_0 > 0 \) such that, for any \( \xi \in A_1 \),

\[
\text{Re} \lambda_1 \leq -\frac{\mu \xi_1^2 + \eta|\xi_h|^2}{2}, \quad \text{Re} \lambda_2 \leq -\frac{\mu \xi_1^2 + \eta|\xi_h|^2}{4},
\]

\[
|G_1(t)| \leq t e^{-\frac{\mu \xi_1^2 + \eta|\xi_h|^2}{4}}, \quad |\hat{Q}_i(\xi, t)| \leq Ce^{-c_0|\xi_h|^2 t}, \quad i = 1, 2, 3.
\]

2. There is a constant \( C > 0 \) such that, for any \( \xi \in A_2 \),

\[
\lambda_1 < -\frac{3(\mu \xi_1^2 + \eta|\xi_h|^2)}{4}, \quad \lambda_2 \leq -\frac{\mu \xi_1^2|\xi_h|^2 + \xi_2^2}{\mu \xi_1^2 + \eta|\xi_h|^2},
\]
Now we bound \( \Gamma \) namely \( \Gamma \), where we have used the simple fact that \( x e^{\lambda t} \) is an imaginary number. If \( |\lambda| \leq |\sqrt{\Gamma}| \) then, for some constants \( C > 0 \), \( c_1 > 0 \), \( c_2 > 0 \), \( c_3 > 0 \) and \( i = 1, 2, 3 \),

\[
|Q_i(t)| \leq C e^{-c_1 |\xi|^2 t}, \quad \text{if} \quad \xi \in A_{21},
\]

\[
|Q_i(t)| \leq C e^{-c_1 |\xi|^2 t}, \quad \text{if} \quad \xi \in A_{22},
\]

\[
|Q_i(t)| \leq C (e^{-c_1 |\xi|^2 t} + e^{-c_2 |\xi|^2 t}), \quad \text{if} \quad \xi \in A_{23}.
\]

**Proof of Proposition 5.4** (1) For \( \xi \in A_1 \), \( \sqrt{\Gamma} \leq \frac{\mu \xi_1^2 + \eta |\xi_h|^2}{2} \). Through the direct estimates and the mean-value theorem, we have

\[
-\frac{3(\mu \xi_1^2 + \eta |\xi_h|^2)}{4} \leq \Re \lambda_1 \leq -\frac{\mu \xi_1^2 + \eta |\xi_h|^2}{2},
\]

\[
\Re \lambda_2 \leq -\frac{\mu \xi_1^2 + \eta |\xi_h|^2}{4}, \quad |G_1(t)| \leq te^{-\frac{\mu \xi_1^2 + \eta |\xi_h|^2}{4}}.
\]

(5.8)

To bound the kernel functions \( \widehat{Q}_1(t) \) and \( \widehat{Q}_3(t) \), we consider two cases: \( \lambda_1 \) is a real number and \( \lambda_1 \) is an imaginary number. If \( \lambda_1 \) is a real number, for some pure constant \( c_0 \) dependent of \( \mu \) and \( \eta \), we have

\[
|\widehat{Q}_1(t)| = \left| \eta \xi_h^2 G_1(t) + \lambda_1 G_1(t) + e^{\lambda_1 t} \right|
\]

\[
\leq C(\mu \xi_1^2 + \eta |\xi_h|^2)te^{-\frac{\mu \xi_1^2 + \eta |\xi_h|^2}{4}} + e^{-\frac{\mu \xi_1^2 + \eta |\xi_h|^2}{4}},
\]

\[
\leq Ce^{-c_0 |\xi_h|^2 t},
\]

where we have used the simple fact that \( x e^{-x} \leq C \) for \( x \geq 0 \). If \( \lambda_1 \) is an imaginary number, namely \( \Gamma < 0 \), then

\[
|\lambda_1|^2 = \mu \xi_1^2 |\xi_h|^2 + \xi_2^2, \quad \Gamma = 4|\lambda_1|^2 - (\mu \xi_1^2 + \eta |\xi_h|^2)^2.
\]

Clearly, (5.8) implies

\[
\left| \eta \xi_h^2 G_1(t) + e^{\lambda_1 t} \right| \leq Ce^{-c_0 |\xi_h|^2 t}.
\]

Now we bound \( |\lambda_1 G_1(t)| \). We further divide the consideration into two subcases: \( |\lambda_1| \leq |\sqrt{\Gamma}| \) and \( |\lambda_1| \geq |\sqrt{\Gamma}| \). In the case when \( |\lambda_1| \leq |\sqrt{\Gamma}| \), by the definition of \( G_1 \), we obtain

\[
|\lambda_1 G_1(t)| = \frac{|\lambda_1|}{|\sqrt{\Gamma}|} |e^{\lambda_1 t} - e^{\lambda_2 t}| \leq Ce^{-\frac{\mu \xi_1^2 + \eta |\xi_h|^2}{4}}.
\]

In the case when \( |\lambda_1| \geq |\sqrt{\Gamma}| = \sqrt{-\Gamma} \), we have

\[
|\lambda_1|^2 \geq 4|\lambda_1|^2 - (\mu \xi_1^2 + \eta |\xi_h|^2)^2,
\]
\[ \sqrt{3}|\lambda_1| \leq \mu \xi_1^2 + \eta |\xi_h|^2. \]

Thus,
\[ |\lambda_1 G_1(t)| \leq \frac{1}{\sqrt{3}}(\mu \xi_1^2 + \eta |\xi_h|^2)|G_1| \leq C(\mu \xi_1^2 + \eta |\xi_h|^2)te^{-\frac{\mu \xi_1^2 + \eta |\xi_h|^2}{2}} \leq Ce^{-c_0|\xi_h|^2t}. \]

Consequently, if \( \lambda_1 \) is an imaginary number, we derive
\[ |\hat{Q}_1(t)| \leq Ce^{-c_0|\xi_h|^2t}. \]

In summary, for \( \xi \in A_1 \),
\[ |\hat{Q}_1(t)| \leq Ce^{-c_0|\xi_h|^2t}. \]

Similarly, we have
\[ |\hat{Q}_3(t)| = \left| -\eta |\xi_h|^2 G_1(t) - \lambda_1 G_1(t) + e^{4t} \right| \leq Ce^{-c_0|\xi_h|^2t}. \]

Now we bound \( \hat{Q}_2(t) \). As in the estimate of \( \hat{Q}_1(t) \), we consider the following two cases: \( |\xi_2| \leq |\sqrt{\Gamma}| \) and \( |\xi_2| \geq |\sqrt{\Gamma}| \). In the first case \( |\xi_2| \leq |\sqrt{\Gamma}| \), by the definition of \( Q_2(t) \) in (5.7),
\[ |\hat{Q}_2(t)| = \left| \frac{\xi_2}{\sqrt{\Gamma}} \right| e^{4t} \leq Ce^{-c_0|\xi_h|^2t}. \]

In the second case, \( |\xi_2| \geq |\sqrt{\Gamma}| \),
\[ |(\mu \xi_1^2 + \eta |\xi_h|^2)^2 - 4(\mu \eta \xi_1^2 |\xi_h|^2 + \xi_2^2)| \leq \xi_2^2, \]

or
\[ -\xi_2^2 \leq (\mu \xi_1^2 + \eta |\xi_h|^2)^2 - 4(\mu \eta \xi_1^2 |\xi_h|^2 + \xi_2^2) \leq \xi_2^2, \]

which implies
\[ 3\xi_2^2 + 4\mu \eta \xi_1^2 |\xi_h|^2 \leq (\mu \xi_1^2 + \eta |\xi_h|^2)^2. \]

In particular,
\[ \sqrt{3}|\xi_2| \leq \mu \xi_1^2 + \eta |\xi_h|^2. \]

Therefore,
\[ |\hat{Q}_2(t)| \leq \frac{1}{\sqrt{3}}(\mu \xi_1^2 + \eta |\xi_h|^2) |G_1(t)| \leq Ce^{-c_0|\xi_h|^2t}. \]

(2) For \( \xi \in A_2 \), we have \( \frac{\mu \xi_1^2 + \eta |\xi_h|^2}{2} \leq \sqrt{\Gamma} \leq \mu \xi_1^2 + \eta |\xi_h|^2 \). It then follows that
\[ - (\mu \xi_1^2 + \eta |\xi_h|^2) \leq \lambda_1 < -\frac{3}{4}(\mu \xi_1^2 + \eta |\xi_h|^2), \]
\[ \lambda_2 = \frac{\Gamma - (\mu \xi_1^2 + \eta |\xi_h|^2)^2}{2(\mu \xi_1^2 + \eta |\xi_h|^2 + \sqrt{\Gamma})} \leq \frac{-\mu \eta \xi_1^2 |\xi_h|^2 + \xi_2^2}{\mu \xi_1^2 + \eta |\xi_h|^2}. \]

Therefore,
\[ |G_1(t)| \leq \frac{1}{\lambda_2 - \lambda_1} (e^{4t} + e^{4t}) < \frac{2}{\mu \xi_1^2 + \eta |\xi_h|^2} \left( e^{-\frac{1}{4}(\mu \xi_1^2 + \eta |\xi_h|^2)t} + e^{-\frac{\mu \eta \xi_1^2 |\xi_h|^2}{\mu \xi_1^2 + \eta |\xi_h|^2}} \right). \]
As a consequence,
\[
|\widehat{Q}_1(t)| = \left| \eta|\xi_h|^2 G_1(t) + \lambda_1 G_1(t) + e^{4t} \right|
\leq 2(\mu|\xi|^2 + \eta|\xi_h|^2)|G_1(t)| + e^{4t}
\leq C(e^{-\frac{3}{2}(\mu|\xi|^2 + \eta|\xi_h|^2)t} + e^{\frac{\mu|\xi|^2 + \eta|\xi_h|^2}{\mu|\xi|^2 + \eta|\xi_h|^2}t}).
\]

Similarly,
\[
|\widehat{Q}_3(t)| = \left| -\eta|\xi_h|^2 G_1(t) - \lambda_1 G_1(t) + e^{4t} \right| < C(e^{-\frac{3}{2}(\mu|\xi|^2 + \eta|\xi_h|^2)t} + e^{\frac{\mu|\xi|^2 + \eta|\xi_h|^2}{\mu|\xi|^2 + \eta|\xi_h|^2}t}).
\]

Due to \(\sqrt{r} > \frac{\mu|\xi|^2 + \eta|\xi_h|^2}{2} \), we find
\[
\frac{3}{4}(\mu|\xi|^2 + \eta|\xi_h|^2)^2 > 4(\mu|\xi|^2|\xi_h|^2 + \xi^2) \geq \xi^2.
\]

Therefore,
\[
|\widehat{Q}_2(t)| < C(\mu|\xi|^2 + \eta|\xi_h|^2) |G_1(t)| \leq C(e^{-\frac{3}{2}(\mu|\xi|^2 + \eta|\xi_h|^2)t} + e^{\frac{\mu|\xi|^2 + \eta|\xi_h|^2}{\mu|\xi|^2 + \eta|\xi_h|^2}t}).
\]

Finally, according to the upper bound for \(|\widehat{Q}_i(t)| \ (i = 1, 2, 3)\), by further division of \(A_2\) into \(A_{21}, A_{22}\) and \(A_{23}\), we can establish more definite upper bound. For \(\xi \in A_{21}, \xi_1 \sim \xi_2\), we have
\[
\frac{\mu|\xi|^2|\xi_h|^2 + \xi^2}{\mu|\xi|^2 + \eta|\xi_h|^2} \sim |\xi_h|^2 + 1.
\]

For \(\xi \in A_{22}, \xi_1 \gg \xi_2\), there exists a \(c_1 > 0\) small sufficiently such that
\[
\frac{\mu|\xi|^2|\xi_h|^2 + \xi^2}{\mu|\xi|^2 + \eta|\xi_h|^2} \geq c_1|\xi_h|^2.
\]

The behavior \(\xi \in A_{23}\) can be similarly identified. This completes the proof of Proposition 5.1. \(\square\)

5.2. Proof of Proposition 5.1 With these preparations at our disposal, we are now ready to prove Proposition 5.1. Since the process is complicated and long, the proof is divided into three lemmas. To do so, we make the following decomposition for \(E_2(t)\),
\[
E_2(t) = E_{21}(t) + E_{22}(t) + E_{23}(t),
\]
where
\[
E_{21}(t) = \sup_{0 \leq s \leq t} (1 + \tau)(|(u, b)(\tau)|)\|_{L^2},
\]
\[
E_{22}(t) = \sup_{0 \leq s \leq t} (1 + \tau)^2(\|\nabla_{n} u(\tau), \nabla_{n} b(\tau))\|_{L^2} + \sup_{0 \leq s \leq t} (1 + \tau)^{1-2\varepsilon}(\|\partial_{3} u(\tau), \partial_{3} b(\tau))\|_{L^2},
\]
\[
E_{23}(t) = \sup_{0 \leq s \leq t} \sum_{k=1}^{2} (1 + \tau)^{2-2\varepsilon}(\|\partial_{1} \partial_{k} u(\tau), \partial_{1} \partial_{k} b(\tau))\|_{L^2} + \sup_{0 \leq s \leq t} (1 + \tau)^{2-2\varepsilon}(\|\partial_{1} \partial_{3} u(\tau), \partial_{1} \partial_{3} b(\tau))\|_{L^2}.
\]
Next we present the estimates for \( E_2(t) \), \( E_{21}(t) \) and \( E_{23}(t) \), which will be shown in three lemmas. Proposition 5.1 then follows as an immediate consequence.

**Lemma 5.5.** Assume that \((u, b)\) is a solution to \((1.4)\). Then we have

\[
E_2(t) \leq CE^2(t) + C\left(\|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2\right). \tag{5.10}
\]

**Proof of Lemma 5.5.** Recalling (5.5) and (5.6), and applying Plancherel’ theorem, we have

\[
\|u(t)\|_{L^2(\mathbb{R}^3)} = \|\hat{u}(t)\|_{L^2(\mathbb{R}^3)} \leq \|\hat{Q}_1(t)\hat{u}_0\|_{L^2(\mathbb{R}^3)} + \|\hat{Q}_2(t)\hat{b}_0\|_{L^2(\mathbb{R}^3)}
\]

\[
+ \int_0^t \|\hat{Q}_1(t - \tau)\hat{N}_1(\tau)\|_{L^2(\mathbb{R}^3)} d\tau + \int_0^t \|\hat{Q}_2(t - \tau)\hat{N}_2(\tau)\|_{L^2(\mathbb{R}^3)} d\tau, \tag{5.11}
\]

\[
\|b(t)\|_{L^2(\mathbb{R}^3)} = \|\hat{b}(t)\|_{L^2(\mathbb{R}^3)} \leq \|\hat{Q}_2(t)\hat{u}_0\|_{L^2(\mathbb{R}^3)} + \|\hat{Q}_3(t)\hat{b}_0\|_{L^2(\mathbb{R}^3)}
\]

\[
+ \int_0^t \|\hat{Q}_2(t - \tau)\hat{N}_1(\tau)\|_{L^2(\mathbb{R}^3)} d\tau + \int_0^t \|\hat{Q}_3(t - \tau)\hat{N}_2(\tau)\|_{L^2(\mathbb{R}^3)} d\tau. \tag{5.12}
\]

We shall only provide the estimates for \(\|u\|_{L^2(\mathbb{R}^3)}\). \(\|b\|_{L^2(\mathbb{R}^3)}\) can be estimated in a similar way and admits the same bound as \(u\) due to the similarity of (5.12) with (5.11). We focus on the first term and the third term on the right side in (5.11). The estimates for the rest can be established similarly. By Proposition 5.4 and Lemma 5.2,

\[
\|\hat{Q}_1(t)\hat{u}_0\|_{L^2(\mathbb{R}^3)} \leq C\|e^{-\zeta_0|\xi|^2} \hat{u}_0\|_{L^2(\mathbb{R}^3)} + C\|e^{-c_3\tau} \hat{u}_0\|_{L^2(\mathbb{R}^3)}
\]

\[
= C\|\|e^{-\zeta_0(\Lambda_1^2 + \Lambda_2^2)\tau} \hat{u}_0\|_{L^2_{x_1,x_2}}\|_{L^2_{x_3}} + C\|e^{-c_3\tau} \hat{u}_0\|_{L^2}
\]

\[
\leq (1 + t)^{-\frac{4}{3}}(\|u_0\|_{L^2_{x_1,x_2}}\|_{L^2_{x_3}} + \|u_0\|_{L^2}), \tag{5.13}
\]

where we have used the fact \(e^{-c_3\tau}(1 + t)^m \leq C(c_3, m)\) for any \(m \geq 0\). For the third term, according to the upper bound for \(\hat{Q}_1(t)\),

\[
\int_0^t \|\hat{Q}_1(t - \tau)\hat{N}_1(\tau)\|_{L^2(\mathbb{R}^3)} d\tau \leq \int_0^t \|\hat{Q}_1(t - \tau)\hat{M}_1(\tau)\|_{L^2(\mathbb{R}^3)} d\tau
\]

\[
\leq C \int_0^t \|e^{-\zeta_0|\xi|^2(t - \tau)} \hat{M}_1(\tau)\|_{L^2(\mathbb{R}^3)} d\tau + C \int_0^t e^{-c_3(t - \tau)} \|\hat{M}_1(\tau)\|_{L^2(\mathbb{R}^3)} d\tau
\]

\[
= C \int_0^{t-1} \|e^{-\zeta_0|\xi|^2(t - \tau)} \hat{M}_1(\tau)\|_{L^2(\mathbb{R}^3)} d\tau + C \int_{t-1}^t \|e^{-\zeta_0|\xi|^2(t - \tau)} \hat{M}_1(\tau)\|_{L^2(\mathbb{R}^3)} d\tau
\]

\[
+ C \int_0^t e^{-c_3(t - \tau)} \|\hat{M}_1(\tau)\|_{L^2(\mathbb{R}^3)} d\tau, \tag{5.14}
\]
where \( M_1 = b \cdot \nabla b - u \cdot \nabla u \) and we have used the fact that the projection operator \( P \) is bounded in \( L^2 \). Observing the simple facts, for any positive number \( m \),

\[
(1 + t - \tau)^{-m} \geq 2^{-m} \quad \text{for} \quad \tau \in [t - 1, t] \quad \text{and} \quad e^{-c^2} (1 + t)^m \leq C(c, m) \quad \text{for} \quad t > 0,
\]

we have

\[
\int_{t-1}^{t} \|e^{-\tilde{c}(t-\tau)} \tilde{M}_1(\tau)\|_{L^2(\mathbb{R}^3)} d\tau \leq 2^m \int_{t-1}^{t} (1 + t - \tau)^{-m} \|\tilde{M}_1(\tau)\|_{L^2(\mathbb{R}^3)} d\tau.
\]

Then (5.14) can be further bounded as

\[
\int_{0}^{t} \|\tilde{Q}_1(t-\tau) \tilde{N}_1(\tau)\|_{L^2(\mathbb{R}^3)} d\tau 
\]

\[
\leq C \int_{0}^{t-1} \|e^{-\tilde{c}(t-\tau)} \tilde{M}_1(\tau)\|_{L^2(\mathbb{R}^3)} d\tau + C \int_{0}^{t} (1 + t - \tau)^{-m} \|\tilde{M}_1(\tau)\|_{L^2(\mathbb{R}^3)} d\tau. \tag{5.15}
\]

Next we bound the terms on the right side in (5.15). It suffices to estimate the integral involving \( u \cdot \nabla u \). The integral of \( b \cdot \nabla b \) admits the same bound. As in (5.13), we have

\[
\int_{0}^{t-1} \|e^{-\tilde{c}(t-\tau)} u \cdot \nabla u(\tau)\|_{L^2(\mathbb{R}^3)} d\tau \leq C \int_{0}^{t} (1 + t - \tau)^{-\frac{3}{2}} \|u \cdot \nabla u(\tau)\|_{L^2(\mathbb{R}^3)} d\tau.
\]

By (2.6),

\[
\|u \cdot \nabla u\|_{L^2_{x_1 x_2 x_3}} \leq C\|u_h\|_{L^2_{x_1 x_2 x_3}}^{\frac{1}{2}} \|\partial_3 u_h\|_{L^2_{x_1 x_2 x_3}}^{\frac{1}{2}} \|\nabla_h u\|_{L^2_{x_1 x_2 x_3}} + C\|u_3\|_{L^2_{x_1 x_2 x_3}}^{\frac{1}{2}} \|\partial_3 u_3\|_{L^2_{x_1 x_2 x_3}}^{\frac{1}{2}} \|\partial_3 u\|_{L^2_{x_1 x_2 x_3}}. \tag{16.16}
\]

By Lemma 5.3,

\[
\int_{0}^{t-1} \|e^{-\tilde{c}(t-\tau)} u \cdot \nabla u(\tau)\|_{L^2(\mathbb{R}^3)} d\tau 
\]

\[
\leq C \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{1}{2}} \|u_h(\tau)\|_{L^2_{x_1 x_2 x_3}}^{\frac{1}{2}} (1 + \tau)^{\frac{1}{2} - \frac{1}{2}\varepsilon} \|\partial_3 u_h(\tau)\|_{L^2_{x_1 x_2 x_3}}^{\frac{1}{2}} (1 + \tau) \|\nabla_h u(\tau)\|_{L^2_{x_1 x_2 x_3}} 
\]

\[
\times \int_{0}^{t} (1 + t - \tau)^{-\frac{3}{2}} (1 + \tau)^{-\frac{3}{2} + \frac{1}{2}\varepsilon} d\tau 
\]

\[
+ C \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{1}{2}} \|u_3(\tau)\|_{L^2_{x_1 x_2 x_3}}^{\frac{1}{2}} (1 + \tau)^{\frac{1}{2}} \|\partial_3 u_3(\tau)\|_{L^2_{x_1 x_2 x_3}}^{\frac{1}{2}} (1 + \tau)^{-\frac{1}{2} - \frac{1}{2}\varepsilon} \|\partial_3 u(\tau)\|_{L^2_{x_1 x_2 x_3}} 
\]

\[
\times \int_{0}^{t} (1 + t - \tau)^{-\frac{3}{2}} (1 + \tau)^{-\frac{3}{2} + \frac{1}{2}\varepsilon} d\tau 
\]

\[
\leq CE_2(t)(1 + t)^{-\frac{1}{2}}. \tag{5.17}
\]

Applying Hölder’s inequality and Sobolev’s inequality, the second integral involving \( u \cdot \nabla u \) in (5.15) can be bounded as

\[
\int_{0}^{t} (1 + t - \tau)^{-m} \|u \cdot \nabla u(\tau)\|_{L^2(\mathbb{R}^3)} d\tau \leq C \int_{0}^{t} (1 + t - \tau)^{-m} \|u(\tau)\|_{L^4_{x_1 x_2 x_3}} \|\nabla u(\tau)\|_{L^2_{x_1 x_2 x_3}} d\tau 
\]

\[
\leq C \int_{0}^{t} (1 + t - \tau)^{-m} \|u(\tau)\|_{L^2_{x_1 x_2 x_3}}^{\frac{1}{2}} \|\nabla u(\tau)\|_{L^2_{x_1 x_2 x_3}}^{\frac{1}{2}} \|\nabla^2 u(\tau)\|_{L^2_{x_1 x_2 x_3}}^{\frac{1}{2}} d\tau 
\]

\[
\leq CE_2(t) \int_{0}^{t} (1 + t - \tau)^{-m}(1 + \tau)^{-\frac{1}{2} + \frac{1}{2}\varepsilon} d\tau \leq CE_2(t)(1 + t)^{-\frac{1}{2}},
\]
where \( m > 1 \). As a consequence, we have

\[
\int_0^\infty \| \tilde{Q}_1(t - \tau) \tilde{u} \cdot \nabla u(\tau) \|_{L^2(\mathbb{R}^3)} \, d\tau \leq C(1 + t)^{-\frac{1}{4}} E_2(t).
\]

Thereby, we infer

\[
\int_0^\infty \| \tilde{Q}_1(t - \tau) \tilde{N}_1(\tau) \|_{L^2(\mathbb{R}^3)} \, d\tau \leq C(1 + t)^{-\frac{1}{4}} E_2(t).
\] (5.18)

The second term and the fourth term admit the similar bound as (5.13) and (5.18), respectively. Therefore, we can conclude

\[
(1 + t)^{\frac{1}{4}} \| u(t) \|_{L^2} \leq C \left( E(t) + \| (u_0, b_0) \|_{L^2_{1/2} L^1_{1/2}}^2 + \| (u_0, b_0) \|_{L^2}^2 \right),
\]

which means

\[
(1 + t) \| u(t) \|_{L^2}^2 \leq C \left( E^2(t) + \| (u_0, b_0) \|_{L^2_{1/2} L^1_{1/2}}^2 + \| (u_0, b_0) \|_{L^2}^2 \right).
\]

Also, \( \| b \|_{L^2} \) obeys the same bound. This complete the proof of Lemma 5.5.

\[\square\]

**Lemma 5.6.** Let \( (u, b) \) be a solution to (1.4). Then we have

\[
E_{22}(t) \leq C E^2(t) + C (\| (u_0, b_0) \|_{L^2_{1/2} L^1_{1/2}}^2 + \| \partial_3 u_0, \partial_3 b_0 \|_{L^2_{1/2} L^1_{1/2}}^2 + \| \nabla u_0, \nabla b_0 \|_{L^2}^2). \tag{5.19}
\]

**Proof of Lemma 5.6** By differentiating (5.5) and (5.6), we have, for \( i = 1, 2, 3 \),

\[
\partial_i u(\xi, t) = \tilde{Q}_1(t) \partial_i u_0 + \tilde{Q}_2(t) \partial_i b_0 + \int_0^t (\tilde{Q}_1(t - \tau) \partial_i \tilde{N}_1(\tau) + \tilde{Q}_2(t - \tau) \partial_i \tilde{N}_2(\tau)) \, d\tau,
\]

\[
\partial_i b(\xi, t) = \tilde{Q}_2(t) \partial_i u_0 + \tilde{Q}_3(t) \partial_i b_0 + \int_0^t (\tilde{Q}_2(t - \tau) \partial_i \tilde{N}_1(\tau) + \tilde{Q}_3(t - \tau) \partial_i \tilde{N}_2(\tau)) \, d\tau.
\]

As in the proof of Lemma 5.5, we focus on the \( \| \partial_i u(t) \|_{L^2} \). Clearly,

\[
\| \partial_i u(t) \|_{L^2(\mathbb{R}^3)} = \| \tilde{Q}_1(t) \partial_i u_0 \|_{L^2(\mathbb{R}^3)} + \| \tilde{Q}_2(t) \partial_i b_0 \|_{L^2(\mathbb{R}^3)} + \| \tilde{Q}_1(t - \tau) \partial_i \tilde{N}_1(\tau) \|_{L^2(\mathbb{R}^3)} \, d\tau + \| \tilde{Q}_2(t - \tau) \partial_i \tilde{N}_2(\tau) \|_{L^2(\mathbb{R}^3)} \, d\tau.
\]

It suffices to bound \( H_1, H_2, H_3, H_4 \) and \( H_1, H_2, H_3, H_4 \) share similar estimates as \( H_1, H_2, H_3, H_4 \).

(1) \( i = 1 \) or \( i = 2 \).

We focus on the case \( i = 2 \). The case \( i = 1 \) is similar. By Proposition 5.4, Lemma 5.2 and Minkowski’s inequality,

\[
H_{21} \leq C \| e^{-\tilde{\xi}_0 \tilde{\alpha}_1^2 t} \tilde{Q}_2 u_0 \|_{L^2(\mathbb{R}^3)} + C \| e^{-\tilde{\xi}_3 t} \tilde{Q}_2 u_0 \|_{L^2(\mathbb{R}^3)}
\]

\[
= C \left( \| e^{-\tilde{\xi}_0 \tilde{\alpha}_1^2 t} \tilde{Q}_2 u_0 \|_{L^2_{1/2} L^1_{1/2}}^2 + C e^{-\tilde{\xi}_3 t} \| \tilde{Q}_2 u_0 \|_{L^2} \right).
\]
For that from the estimates (5.16) and (5.17), we obtain
\[
0 \leq (1 + t)^{-\frac{1}{2}} \left\| e^{-\zeta_0 t^2} u_0 \right\|_{L^2} + C (1 + t)^{-1} \left\| \partial_t u_0 \right\|_{L^2}
\]
\[
\leq C (1 + t)^{-\frac{1}{2}} \left\| e^{-\zeta_0 t^2} u_0 \right\|_{L^2} + C (1 + t)^{-1} \left\| \partial_t u_0 \right\|_{L^2}
\]
\[
\leq C (1 + t)^{-1} (\left\| u_0 \right\|_{L^2} + \left\| \partial_t u_0 \right\|_{L^2}).
\]
(5.21)

Similarly,
\[
H_{22} \leq C (1 + t)^{-1} (\left\| b_0 \right\|_{L^2} + \left\| \partial_t b_0 \right\|_{L^2}).
\]
(5.22)

For \(H_{23}\), similarly to (5.15), we first bound it by
\[
H_{23} \leq C \int_0^t \left\| e^{-\zeta_0 \xi (1 - \tau)} \partial_2 M_1(\tau) \right\|_{L^2(\mathbb{R}^2)} d\tau
+ C \int_0^t (1 + t - \tau)^{-m} \left\| \partial_2 M_1(\tau) \right\|_{L^2(\mathbb{R}^2)} d\tau,
\]
where \(M_1 = b \cdot \nabla b - u \cdot \nabla u\). We consider the first term involving \(u \cdot \nabla u\) in (5.23). Firstly, from the estimates (5.16) and (5.17), we obtain
\[
\int_0^t \left\| e^{-\zeta_0 \xi (1 - \tau)} \partial_2 (u \cdot \nabla u)(\tau) \right\|_{L^2(\mathbb{R}^2)} d\tau \leq \int_0^t (1 + t - \tau)^{-1} \left\| u \cdot \nabla u(\tau) \right\|_{L^2} d\tau
\]
\[
\leq CE_2(t) \int_0^t (1 + t - \tau)^{-1} \left\{ (1 + \tau)^{-\frac{3}{2} + \frac{1}{2} \varepsilon} + (1 + \tau)^{-\frac{3}{4} + \varepsilon} \right\} d\tau
\]
\[
\leq CE_2(t)(1 + t)^{-1}.
\]

For the second term in (5.23), it follows from Hölder’s inequality and Sobolev’s inequality that
\[
\left\| \partial_2 (u \cdot \nabla u) \right\|_{L^2} \leq \left\| \partial_2 u \right\|_{L^2} \left\| \nabla u \right\|_{L^\infty} + \left\| u \right\|_{L^\infty} \left\| \partial_2 \nabla u \right\|_{L^2} + \left\| u \right\|_{L^4} \left\| \partial_2 \partial_3 u \right\|_{L^4}
\]
\[
\leq C \left( \left\| \partial_2 u \right\|_{L^2} \left\| \nabla u \right\|_{L^4} + \left\| u \right\|_{L^4} \left\| \nabla^2 u \right\|_{L^2} \right)
\]
\[
+ C \left\| u \right\|_{L^2} \left\| \partial_2 \partial_3 u \right\|_{L^2} \left\| \nabla \partial_2 \partial_3 u \right\|_{L^2}.
\]
(5.24)

Therefore, for \(m > 2\), we derive
\[
\int_0^t (1 + t - \tau)^{-m} \left\| \partial_2 (u \cdot \nabla u)(\tau) \right\|_{L^2(\mathbb{R}^2)} d\tau
\]
\[
\leq C \sup_{0 \leq \tau \leq t} (1 + \tau) \left\| \partial_2 u(\tau) \right\|_{L^2} \left\| \nabla^2 u(\tau) \right\|_{L^2} \left\| \nabla \partial_2 u(\tau) \right\|_{L^2}
\]
\[
+ C \sup_{0 \leq \tau \leq t} (1 + \tau)^{-\frac{3}{2} + \frac{1}{2} \varepsilon} \left\| \nabla^2 u(\tau) \right\|_{L^2} \left\| \nabla \partial_2 u(\tau) \right\|_{L^2} \left\| \nabla \partial_2 \partial_3 u(\tau) \right\|_{L^2}
\]
\[
+ C \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{1}{2}} \left\| u_3(\tau) \right\|_{L^2} \left\| \nabla u_3(\tau) \right\|_{L^2} \left\| \nabla \partial_2 \partial_3 u(\tau) \right\|_{L^2}
\]
\[
\times \int_0^t (1 + t - \tau)^{-m} (1 + \tau)^{-\frac{3}{2} + \frac{1}{2} \varepsilon} d\tau
\]
\[
+ C \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{1}{2}} \left\| u_3(\tau) \right\|_{L^2} \left\| \nabla u_3(\tau) \right\|_{L^2} \left\| \nabla \partial_2 \partial_3 u(\tau) \right\|_{L^2}
\]
\[
\times \int_0^t (1 + t - \tau)^{-m} (1 + \tau)^{-\frac{3}{2} + \frac{1}{2} \varepsilon} d\tau
\]
\[
\leq CE_2^2(t)E_0^4(t)(1 + t)^{-\frac{2}{3}} + CE_2(t)(1 + t)^{-\frac{5}{3} + \frac{4}{3}E} + CE_2^3(t)E_0^4(t)(1 + t)^{-\frac{2}{3} + \frac{1}{2}E}
\leq CE(t)(1 + t)^{-1}.
\]

(5.25)

Consequently,
\[
\int_0^t \| \hat{Q}_1(t - \tau)\partial_3(u \cdot \nabla u)(\tau) \|_{L^2(\mathbb{R}^3)} d\tau \leq CE(t)(1 + t)^{-1}.
\]

Similarly,
\[
\int_0^t \| \hat{Q}_1(t - \tau)\partial_3(b \cdot \nabla b)(\tau) \|_{L^2(\mathbb{R}^3)} d\tau \leq CE(t)(1 + t)^{-1}.
\]

Hence,
\[
H_{23} \leq CE(t)(1 + t)^{-1},
\]

(5.26)

Similarly,
\[
H_{24} \leq CE(t)(1 + t)^{-1}.
\]

(5.27)

(5.21), (5.22), (5.26) and (5.27) yield
\[
(1 + t)\| \partial_3 u(t) \|_{L^2} \leq CE(t) + C(\| (u_0, b_0) \|_{L^3_xL^1_t} + \| (\partial_3 u_0, \partial_3 b_0) \|_{L^2}).
\]

Similarly,
\[
(1 + t)\| \partial_3 b(t) \|_{L^2} \leq CE(t) + C(\| (u_0, b_0) \|_{L^3_xL^1_t} + \| (\partial_3 u_0, \partial_3 b_0) \|_{L^2}).
\]

For \(i = 1\), \(\| (\partial_1 u, \partial_1 b) \|_{L^2}\) obeys a similar bound to \(\| (\partial_3 u, \partial_3 b) \|_{L^2}\) with only a minor modification of (5.24) and (5.25),
\[
(1 + t)\| (\partial_1 u(t), \partial_1 b(t)) \|_{L^2} \leq CE(t) + C(\| (u_0, b_0) \|_{L^3_xL^1_t} + \| (\partial_1 u_0, \partial_1 b_0) \|_{L^2}).
\]

(2) \(i = 3\)

Invoking the estimate (5.13), we have
\[
H_{31} \leq C\| e^{-\tilde{c}_0\tilde{\xi}_1^2(t - \tau)} \partial_3 u_0 \|_{L^2(\mathbb{R}^3)} + C\| e^{-c_3f \tau} \partial_3 u_0 \|_{L^2(\mathbb{R}^3)} \leq C(1 + t)^{-\frac{5}{3}}(\| \partial_3 u_0 \|_{L^3_xL^1_t} + \| \partial_3 u_0 \|_{L^2}).
\]

(5.28)

\(H_{33}\) can be similarly estimated as \(H_{23}\),
\[
H_{33} \leq C \int_0^\tau \| e^{-\tilde{c}_0\tilde{\xi}_1^2(t - \tau)} \partial_3 M_1(\tau) \|_{L^2(\mathbb{R}^3)} d\tau + C \int_0^\tau (1 + t - \tau)^{-m} \| \partial_3 M_1(\tau) \|_{L^2(\mathbb{R}^3)} d\tau.
\]

(5.29)

Firstly, we have
\[
\int_0^\tau \| e^{-\tilde{c}_0\tilde{\xi}_1^2(t - \tau)} \partial_3(u \cdot \nabla u)(\tau) \|_{L^2(\mathbb{R}^3)} d\tau \leq C \int_0^\tau (1 + t - \tau)^{-\frac{1}{2}} \| \partial_3(u \cdot \nabla u)(\tau) \|_{L^2_xL^1_t} d\tau
\]

Applying the estimate (2.6) yields
\[
\| \partial_3(u \cdot \nabla u) \|_{L^2_xL^1_t} \leq C(\| \partial_3 u_0 \|_{L^2_xL^1_t} + \| \partial_3 u_0 \|_{L^2_xL^1_t} + \| \partial_3 u_0 \|_{L^2_xL^1_t} + \| \partial_3 u_0 \|_{L^2_xL^1_t} + \| \partial_3 u_0 \|_{L^2_xL^1_t}).
\]
As a consequence, we arrive at

\[ \int_0^{\tau-1} \left| e^{-\tau_0 \| \partial_3 u \|_{L^2}} \partial_3 (u \cdot \nabla u)(\tau) \right|_{L^2} \, d\tau \leq CE_2(t) \int_0^{\tau-1} (1 + t - \tau)^{-\frac{11}{4} + \frac{1}{4}e} + (1 + \tau)^{-\frac{7}{4} + \frac{1}{4}e} + (1 + \tau)^{1-} \right] \, d\tau \leq CE(t)(1 + t)^{-\frac{1}{2} + e}. \]  

(5.31)

To bound the second term in (5.29), we apply Hölder’s and Sobolev’s inequalities to obtain

\[ \int_0^{\tau} (1 + t - \tau)^{-m} \| \partial_3 u \|_{L^2} \| \nabla u(\tau) \|_{L^\infty} + \| u(\tau) \|_{L^\infty} \| \nabla \partial_3 u(\tau) \|_{L^2} \, d\tau \leq C \int_0^{\tau} (1 + t - \tau)^{-m} (1 + \tau)^{-\frac{1}{2} + e} \, d\tau \leq CE(t)(1 + t)^{-\frac{1}{2}}. \]  

(5.32)

The estimates (5.31) and (5.32) then lead to

\[ \int_0^{\tau} \left| Q_1(t - \tau) \partial_3 (u \cdot \nabla u)(\tau) \right|_{L^2} \, d\tau \leq CE(t)(1 + t)^{-\frac{1}{2} + e}. \]

Therefore,

\[ H_{33} \leq CE(t)(1 + t)^{-\frac{1}{2} + e}. \]  

(5.33)

Similarly,

\[ H_{32} + H_{34} \leq C(\| \partial_3 b_0 \|_{L^2_{t} L^1_{y_{1}y_{2}}} + \| \partial_3 b_0 \|_{L^2} + E(t)(1 + t)^{-\frac{1}{2} + e}). \]  

(5.34)

Finally, by the estimates (5.28), (5.33) and (5.34), we conclude

\[ (1 + t)^{-\frac{1}{2} - e} \| \partial_3 u(t) \|_{L^2} \leq CE(t) + C(\| \partial_3 u_0, \partial_3 b_0 \|_{L^2_{t} L^1_{y_{1}y_{2}}} + \| \partial_3 u_0, \partial_3 b_0 \|_{L^2}). \]

This completes the proof of Lemma 5.6. \( \square \)

Next we bound \( E_{23}(t) \), which involves the second-order derivatives of \( (u, b) \).

**Lemma 5.7.** Let \( (u, b) \) be a solution to (1.4). Then it holds

\[ E_{23}(t) \leq CE^2(t) + C(\| (u_0, b_0) \|_{L^2_{t} L^1_{y_{1}y_{2}}} + \| (\partial_3 u_0, \partial_3 b_0) \|_{L^2_{t} L^1_{y_{1}y_{2}}}^2 + \| (\partial_3^2 u_0, \partial_3^2 b_0) \|_{L^2_{t} L^1_{y_{1}y_{2}}}^2 + \| (\Delta u_0, \Delta b_0) \|_{L^2}^2). \]  

(5.35)
Proof of Theorem 5.7. First of all, we have, for \( i, j = 1, 2, 3 \),
\[
\partial_i \partial_j \mu(\xi, t) = \tilde{Q}_1(t) \partial_i \partial_j \mu_0 + \tilde{Q}_2(t) \partial_i \partial_j b_0 \\
+ \int_0^t (\tilde{Q}_1(t - \tau) \partial_i \partial_j N_1(\tau) + \tilde{Q}_2(t - \tau) \partial_i \partial_j N_2(\tau)) d\tau,
\]
(5.36)
\[
\partial_i \partial_j b(\xi, t) = \tilde{Q}_2(t) \partial_i \partial_j \mu_0 + \tilde{Q}_3(t) \partial_i \partial_j b_0 \\
+ \int_0^t (\tilde{Q}_2(t - \tau) \partial_i \partial_j N_1(\tau) + \tilde{Q}_3(t - \tau) \partial_i \partial_j N_2(\tau)) d\tau.
\]
Throughout the proof, we only show the bound of \( \| \partial_i \partial_j \mu(t) \|_{L^2} \). The estimates for \( \| \partial_i \partial_j b(t) \|_{L^2} \) can be obtained similarly. Taking the \( L^2 \) norm on both side of (5.36), we have
\[
\| \partial_i \partial_j \mu(t) \|_{L^2(\mathbb{R}^3)} = \| \tilde{Q}_1(t) \partial_i \partial_j \mu_0 \|_{L^2(\mathbb{R}^3)} + \| \tilde{Q}_2(t) \partial_i \partial_j b_0 \|_{L^2(\mathbb{R}^3)} \\
+ \int_0^t \| \tilde{Q}_1(t - \tau) \partial_i \partial_j N_1(\tau) \|_{L^2(\mathbb{R}^3)} d\tau + \int_0^t \| \tilde{Q}_2(t - \tau) \partial_i \partial_j N_2(\tau) \|_{L^2(\mathbb{R}^3)} d\tau.
\]
\[
= K_{ij1} + K_{ij2} + K_{ij3} + K_{ij4}.
\]
We focus on \( K_{ij1} \) and \( K_{ij3} \). The bound for the other terms can be established in a similar way. The proof will be split into four cases: \( i = 1, j = 1, 2; i = 1, j = 3; i = 2, j = 2, 3; i = j = 3 \).

(1) \( i = 1, j = 1, 2 \).

It suffices to investigate the case \( i = 1, j = 2 \). The case \( i = 1, j = 1 \) can be dealt with similarly. By Lemma 5.2,
\[
K_{121} \leq C \| e^{-\gamma_0 \xi_0^2 (t - \tau)} \partial_1 \partial_2 \mu_0 \|_{L^2(\mathbb{R}^3)} + C \| e^{-c_3 \xi_1^2 \partial_1 \partial_2 u_0} \|_{L^2(\mathbb{R}^3)} \\
\leq C(1 + t)^{-\frac{2}{3}} \left( \| u_0 \|_{L^2_{\mathbb{R}^3}, L^4_{\mathbb{R}^3}^1} + \| \partial_1 \partial_2 u_0 \|_{L^2} \right).
\]
(5.37)
Similarly,
\[
K_{122} \leq C(1 + t)^{-\frac{2}{3}} \left( \| b_0 \|_{L^2_{\mathbb{R}^3}, L^4_{\mathbb{R}^3}^1} + \| \partial_1 \partial_2 b_0 \|_{L^2} \right).
\]
(5.38)
For \( K_{123} \), we first give a different bound from the ones in Lemma 5.5 and Lemma 5.6
\[
K_{123} \leq C \int_0^{t-1} \| e^{-\gamma_0 \xi_0^2 (t - \tau)} \partial_1 \partial_2 M_1(\tau) \|_{L^2(\mathbb{R}^3)} d\tau + C \int_{t-1}^t \| e^{-\gamma_0 \xi_0^2 (t - \tau)} \partial_1 \partial_2 M_1(\tau) \|_{L^2(\mathbb{R}^3)} d\tau \\
+ C \int_0^t e^{-c_3 (t - \tau)} \| e^{-c_3 \xi_1^2 \partial_1 \partial_2 M_1(\tau)} \|_{L^2(\mathbb{R}^3)} d\tau.
\]
For \( \tau \in [t-1, t] \), we have \( e^{-c_3 (t - \tau)} \geq e^{-c_3} \) and thus
\[
\int_{t-1}^t \| e^{-\gamma_0 \xi_0^2 (t - \tau)} \partial_1 \partial_2 M_1(\tau) \|_{L^2(\mathbb{R}^3)} d\tau \leq e^{c_3} \int_{t-1}^t \| e^{-\gamma_0 \xi_0^2 (t - \tau)} \partial_1 \partial_2 M_1(\tau) \|_{L^2(\mathbb{R}^3)} d\tau.
\]
As a consequence, for a constant \( c_4 > 0 \),
\[
K_{123} \leq C \int_0^{t-1} \| e^{-\gamma_0 \xi_0^2 (t - \tau)} \partial_1 \partial_2 M_1(\tau) \|_{L^2(\mathbb{R}^3)} d\tau \\
+ C \int_0^t e^{-c_3 (t - \tau)} \| e^{-c_4 \xi_1^2 \partial_1 \partial_2 M_1(\tau)} \|_{L^2(\mathbb{R}^3)} d\tau
\]
Hence, 
\[ \int_0^t \|e^{-\frac{c_0}{2}(t-\tau)} \partial_3 \partial_2(u \cdot \nabla u)(\tau)\|_{L^2(\mathbb{R}^3)} d\tau \]
\[ \leq C \int_0^t (1 + t - \tau)^{-\frac{3}{2}} \|u \cdot \nabla u(\tau)\|_{L^2_t L^4_x} d\tau \]
\[ \leq CE(t) \int_0^t (1 + t - \tau)^{-\frac{3}{2}} \left( (1 + \tau)^{-\frac{3}{2}} + (1 + \tau)^{-\frac{3}{2} + \frac{\delta}{4}} \right) d\tau \]
\[ \leq CE(t)(1 + t)^{-\frac{3}{2} + \frac{\delta}{4}}. \]

Hence,
\[ K_{1231} \leq CE(t)(1 + t)^{-\frac{3}{2} + \frac{\delta}{4}}. \quad (5.39) \]

For \( K_{1232} \), according to Lemma 5.2, we have
\[ K_{1232} \leq C \int_0^t e^{-c_3(t-\tau)}(t - \tau)^{-\frac{1}{2}} \|\partial_2(u \cdot \nabla u - b \cdot \nabla b)(\tau)\|_{L^2(\mathbb{R}^3)} d\tau. \]

By the anisotropic inequality (2.5),
\[ \|\partial_2(u \cdot \nabla u)\|_{L^2(\mathbb{R}^3)} \leq C \|\partial_2 u\|_{L^4} \|\partial_2^2 u\|_{L^2} \|\partial_2 \partial_3 u\|_{L^2} \|\partial_2^2 \partial_3 u\|_{L^2} \|\nabla u\|_{L^2} \|\nabla \partial_1 u\|_{L^2} \]
\[ + C \|u\|_{L^4} \|\partial_1 u\|_{L^4} \|\partial_2 u\|_{L^2} \|\partial_1 \partial_2 u\|_{L^2} \|\nabla \partial_2 u\|_{L^2} \|\partial_2 \partial_3 u\|_{L^2}. \]

Hence,
\[ \int_0^t e^{-c_3(t-\tau)}(t - \tau)^{-\frac{1}{2}} \|\partial_2(u \cdot \nabla u)(\tau)\|_{L^2(\mathbb{R}^3)} d\tau \]
\[ \leq CE(t) \int_0^t e^{-c_3(t-\tau)}(t - \tau)^{-\frac{1}{2}} \|\nabla u\|_{L^2} \|\nabla \partial_1 u\|_{L^2} \|\nabla \partial_2 u\|_{L^2} \|\partial_2 \partial_3 u\|_{L^2} d\tau \]
\[ \leq CE(t)(1 + t - \tau)^{-\frac{3}{2}} \left( (1 + \tau)^{-\frac{3}{2} + \frac{\delta}{4}} \right) d\tau, \]

where we have used the simple fact: \( e^{-ct}(1+t)^m \leq C(m) \) for any \( t \geq 0, m \geq 0 \). Furthermore, selecting \( m > 2 \), and then applying Hölder inequality with \( 1 < p < 2 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \), we infer
\[ \int_0^t e^{-c_3(t-\tau)}(t - \tau)^{-\frac{1}{2}} \|\partial_2(u \cdot \nabla u)(\tau)\|_{L^2(\mathbb{R}^3)} d\tau \]
\[ \leq CE(t) \left( \int_0^t e^{-c_3(t-\tau)}(t - \tau)^{-\frac{1}{2}} d\tau \right)^{\frac{p}{2}} \left( \int_0^t (1 + t - \tau)^{-m} (1 + \tau)^{-\frac{3}{2} + \frac{\delta}{4}} d\tau \right)^{\frac{q}{2}} \]
\[ \leq CE(t)(1 + t)^{-\frac{3}{2} + \frac{\delta}{4}}. \quad (5.40) \]
where we have used fact that the integration \( \int_0^\infty x^{s-1}e^{-x}dx (s > 0) \) converges to \( \Gamma(s) \). Consequently,

\[
K_{1232} \leq CE(t)(1 + t)^{-\frac{s}{2} + e}. \tag{5.41}
\]

(5.39) and (5.41) lead to

\[
K_{123} \leq CE(t)(1 + t)^{-\frac{s}{2} + e}. \tag{5.42}
\]

With a similar argument, we obtain

\[
K_{124} \leq CE(t)(1 + t)^{-\frac{s}{2} + e}. \tag{5.43}
\]

Combining the estimates (5.37), (5.38), (5.43) and (5.42), we derive

\[
(1 + t)^{\frac{s}{2} - e}||\partial_1 \partial_3 u(t)||_{L^2} \leq CE(t) + C(||(u_0, b_0)||_{L^3_t L^1_y} + ||(\partial_1 \partial_2 u_0, \partial_1 \partial_2 b_0)||_{L^2}).
\]

Similarly, we can also obtain

\[
(1 + t)^{\frac{s}{2} - e}||\partial_1^2 u(t)||_{L^2} \leq CE(t) + C(||(u_0, b_0)||_{L^3_t L^1_y} + ||(\partial_1^2 u_0, \partial_1^2 b_0)||_{L^2}).
\]

(2) \( i = 1, j = 3 \).

Firstly, from (5.21), we have

\[
K_{131} \leq C(1 + t)^{-1}(||\partial_3 u_0||_{L^3_t L^1_y} + ||\partial_1 \partial_3 u_0||_{L^2}). \tag{5.44}
\]

For \( K_{133} \), similarly to \( K_{123} \), we first bound it as

\[
\begin{align*}
K_{133} & \leq C \int_0^\infty \left| e^{-c_0 \delta_3^2 (t-\tau)} \partial_1 \partial_3 M_1(\tau) \right|_{L^2(\mathbb{R}^3)} d\tau \\
& \quad + C \int_0^\infty \left| e^{-c_1 (t-\tau)} \partial_1 \partial_3 M_1(\tau) \right|_{L^2(\mathbb{R}^3)} d\tau \\
& \leq C \int_0^\infty (1 + t - \tau)^{-1} ||\partial_3 M_1(\tau)||_{L^3_t L^1_y} d\tau \\
& \quad + C \int_0^\infty \left( 1 + \frac{3}{2} \right) ||\partial_3 M_1(\tau)||_{L^2} d\tau \\
& \quad + C ||\partial_3 M_1||_{L^2} + C ||\partial_3^2 u_0||_{L^2}.
\end{align*}
\]

Invoking (5.30) and (5.31), we get

\[
K_{1331} \leq CE_2(t) \int_0^\tau \left[ (1 + \tau)^{-\frac{s}{2} + e} + (1 + \tau)^{-\frac{s}{2} + e} + (1 + \tau)^{-1} \right] d\tau \\
\leq CE(t)(1 + t)^{-1 + e}. \tag{5.45}
\]

For \( K_{1332} \), by Hölder’s inequality and Sobolev’s inequality, we first have

\[
||\partial_3 (u \cdot \nabla u)||_{L^2} \leq ||\partial_3 u||_{L^4} ||\nabla u||_{L^2} + ||u||_{L^\infty} ||\partial_3 u||_{L^2}
\]

\[
\leq C ||\partial_3 u||_{L^2}^\frac{1}{3} ||\partial_3 \nabla u||_{L^2} ||\nabla \nabla u||_{L^2}^\frac{2}{3} + C ||\partial_3 u||_{L^2}^\frac{1}{3} ||\partial_3 \nabla u||_{L^2} ||\nabla \nabla u||_{L^2}^\frac{2}{3}
\]

\[
+ C ||\nabla u||_{L^2} ||\nabla \nabla u||_{L^2} + C ||\nabla u||_{L^2} ||\nabla \nabla u||_{L^2} ||\partial_3^2 u||_{L^2}.
\]
Then, for \( m > 1 \),
\[
\int_0^t e^{-c_3(t-\tau)} (t - \tau)^{-\frac{4}{3}} \| \partial_3 (u \cdot \nabla u) (\tau) \|_{L^2} \, d\tau 
\leq CE_2(t) \int_0^t e^{-c_3(t-\tau)} (t - \tau)^{-\frac{4}{3}} [ (1 + \tau)^{\frac{2}{3} + \varepsilon} + (1 + \tau)^{-\frac{1}{3} + \varepsilon} + (1 + \tau)^{-1 + \varepsilon}] \, d\tau 
\leq CE(t) \int_0^t e^{-c_3(t-\tau)} (t - \tau)^{-\frac{4}{3}} (1 + \tau)^{-1 + \varepsilon} \, d\tau 
\leq CE(t) (1 + t)^{-1 + \varepsilon},
\]
where we have used a similar derivation with (5.40) for the last inequality. Thus, we get
\[
K_{1332} \leq CE(t) (1 + t)^{-1 + \varepsilon}.
\]
which, together with (5.45), gives
\[
K_{133} \leq CE(t) (1 + t)^{-1 + \varepsilon}. \tag{5.46}
\]
Therefore, by (5.44) and (5.46), we conclude
\[
(1 + t)^{-1 + \varepsilon} \| \partial_1 \partial_3 u(t) \|_{L^2} \leq CE(t) + C(\| (\partial_3 u_0, \partial_3 b_0) \|_{L^2_{x_1} L^2_{x_2}} + \| (\partial_1 \partial_3 u_0, \partial_1 \partial_3 b_0) \|_{L^2}). \tag{5.47}
\]

(3) \( i = 2, j = 2, 3 \).

It suffices to bound \( \| \partial_2 \partial_3 u \|_{L^2} \). Firstly, a similar argument with (5.21) yields
\[
K_{231} \leq C \| e^{-c_3(t-\tau)} \|_{L^1(\mathbb{R}^3)} \| \partial_2 \partial_3 u_0 \|_{L^2(\mathbb{R}^3)} + C \| e^{-c_3(t-\tau)} \|_{L^1(\mathbb{R}^3)} \| \partial_2 \partial_3 u(t) \|_{L^2(\mathbb{R}^3)} 
\leq C (1 + t)^{-1} (\| \partial_3 u_0 \|_{L^2_{x_1} L^2_{x_2}} + \| \partial_2 \partial_3 u_0 \|_{L^2}). \tag{5.48}
\]

As in \( H_{23} \), \( K_{233} \) is firstly bounded by
\[
K_{233} \leq C \int_0^t \| e^{-c_3(t-\tau)} \partial_2 \partial_3 M_1(\tau) \|_{L^2(\mathbb{R}^3)} \, d\tau 
+ \frac{C}{t} \int_0^t (1 + t - \tau)^{-m} \| \partial_3 M_1(\tau) \|_{L^2(\mathbb{R}^3)} \, d\tau 
\leq C \int_0^t (1 + t - \tau)^{-1} \| \partial_3 M_1(\tau) \|_{L^2_{x_1} L^2_{x_2}} \, d\tau 
+ \frac{C}{t} \int_0^t (1 + t - \tau)^{-m} \| \partial_2 \partial_3 M_1(\tau) \|_{L^2(\mathbb{R}^3)} \, d\tau 
\leq K_{2331} + K_{2332}.
\]
Now we estimate \( K_{2331} \). Recalling the bound (5.45) gives
\[
K_{2331} \leq CE_2(t) (1 + t)^{-1 + \varepsilon}. \tag{5.49}
\]

By Hölder’s inequality and Sobolev’s inequality,
\[
\| \partial_2 \partial_3 (u \cdot \nabla u) \|_{L^2} \leq \| \partial_2 \partial_3 u \cdot \nabla u \|_{L^2} + \| \partial_2 u \cdot \nabla \partial_3 u \|_{L^2} + \| \partial_3 u \cdot \nabla \partial_2 u \|_{L^2} + \| u \cdot \nabla \partial_2 \partial_3 u \|_{L^2} 
\leq \| \nabla u \|_{L^\infty} \| \nabla \partial_2 u \|_{L^2} + \| \partial_3 u \|_{L^2} \| \nabla \partial_2 u \|_{L^2} + \| u \cdot \nabla \partial_2 \partial_3 u \|_{L^2}
\]
where we have used the anisotropic inequality \((2.5)\) for \(\|u \cdot \nabla \partial_3 u\|_{L^2}\). Thus,

\[
\int_0^t (1 + t - \tau)^{-m} \|\partial_2 \partial_3 (u \cdot \nabla u)(\tau)\|_{L^2(\mathbb{R}^3)} d\tau \\
\leq C \int_0^t (1 + t - \tau)^{-m} \left( \|\nabla^2 u\|_{L^2} \|\nabla^3 u\|_{L^2} \|\nabla \partial_2 u\|_{L^2} + \|\partial_2 u\|_{L^2} \|\nabla \partial_2 u\|_{L^2} \|\nabla \partial_3 u\|_{L^2} \|\nabla^2 \partial_3 u\|_{L^2} \right) d\tau \\
+ C \int_0^t (1 + t - \tau)^{-m} \left( \|\nabla^2 u\|_{L^2} \|\nabla^3 u\|_{L^2} \|\nabla \partial_2 u\|_{L^2} \|\nabla \partial_3 u\|_{L^2} \|\nabla \partial_2 \partial_3 u\|_{L^2} \right) d\tau \\
:= L_1 + L_2.
\]

By means of \((5.2)\), for \(m > 1\), we infer

\[
L_1 \leq C \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{3}{2} - \varepsilon} \|\nabla \partial_2 u(\tau)\|_{L^2} \|\nabla^2 u(\tau)\|_{H^1} \int_0^t (1 + t - \tau)^{-m} (1 + \tau)^{-\frac{3}{4} + \varepsilon} d\tau \\
+ \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{3}{4}} \|\nabla \partial_3 u(\tau)\|_{L^2} \|\nabla^2 u(\tau)\|_{H^1} \|\nabla \partial_2 u(\tau)\|_{L^2} \|\nabla^2 u(\tau)\|_{H^1} \\
\times \int_0^t (1 + t - \tau)^{-m} (1 + \tau)^{-\frac{3}{4} + \varepsilon} d\tau \\
\leq E_2^\frac{1}{4}(t) E_0^\frac{1}{4}(t) (1 + t)^{-\frac{3}{4} + \varepsilon} \leq C E(t)(1 + t)^{-\frac{3}{4} + \varepsilon}.
\]

For \(L_2\), applying Hölder’s inequality yields, for \(m > 1\),

\[
L_2 \leq C \sup_{0 \leq \tau \leq t} (1 + \tau)^{\frac{3}{4}} \|u(\tau)\|_{L^2} \|\nabla \partial_2 u(\tau)\|_{L^2} \|\nabla \partial_3 u(\tau)\|_{L^2} \|\nabla \partial_3 u(\tau)\|_{L^2} \|\nabla \partial_3 u(\tau)\|_{L^2} \\
\times \|\nabla \partial_2 \partial_3 u(\tau)\|_{L^2} \|\nabla \partial_2 \partial_3 u(\tau)\|_{L^2} \|\nabla \partial_2 \partial_3 u(\tau)\|_{L^2} \|\nabla \partial_2 \partial_3 u(\tau)\|_{L^2} \\
\times \int_0^t (1 + t - \tau)^{-m} (1 + \tau)^{-\frac{3}{4} + \varepsilon} \|\nabla \partial_2 \partial_3 u(\tau)\|_{L^2} \|\nabla \partial_2 \partial_3 u(\tau)\|_{L^2} d\tau \\
\leq CE_2^\frac{1}{4}(t) E_0^\frac{1}{4}(t) (1 + t)^{-\frac{3}{4} + \varepsilon} \left( \int_0^t (1 + t - \tau)^{-m} (1 + \tau)^{-\frac{3}{4} + \varepsilon} d\tau \right)^\frac{1}{2} \left( \int_0^t \|\nabla \partial_2 \partial_3 u(\tau)\|_{L^2}^2 d\tau \right)^\frac{1}{2} \\
\leq CE_2^\frac{1}{4}(t) E_0^\frac{1}{4}(t) (1 + t)^{-\frac{3}{4} + \varepsilon} \leq C E(t)(1 + t)^{-\frac{3}{4} + \varepsilon}.
\]

Therefore,

\[
\int_0^t (1 + t - \tau)^{-m} \|\partial_2 \partial_3 (u \cdot \nabla u)(\tau)\|_{L^2(\mathbb{R}^3)} d\tau \leq C E(t)(1 + t)^{-\frac{3}{4} + \varepsilon}.
\]

Thus,

\[
K_{2332} \leq C E(t)(1 + t)^{-\frac{3}{4} + \varepsilon},
\]

which, together with \((5.48)\), gives

\[
K_{233} \leq C E(t)(1 + t)^{-\frac{3}{4} + \varepsilon}.
\]
\(K_{232}\) and \(K_{234}\) can be bounded with similar arguments as those for \(K_{231}\) and \(K_{233}\), respectively. Therefore, by (5.47) and (5.49), we conclude

\[
(1 + t)^{\frac{3}{2} - \varepsilon} \| \partial_2 \partial_3 u(t) \|_{L^2} \leq CE(t) + C(\| (\partial_3 u_0, \partial_3 b_0) \|_{L^2_{3, \tau_{1, 1, 2}}^1} + \| (\partial_2 \partial_3 u_0, \partial_2 \partial_3 b_0) \|_{L^2}).
\]

Similarly,

\[
(1 + t)^{\frac{3}{2} - \varepsilon} \| \partial_3 u(t) \|_{L^2} \leq CE(t) + C(\| (u_0, b_0) \|_{L^2_{3, \tau_{1, 1, 2}}^1} + \| (\partial_3^2 u_0, \partial_3^2 b_0) \|_{L^2}).
\]

(4) \(i = j = 3\).

Firstly, we have

\[
K_{331} \leq C \| e^{-\zeta_0 |\xi|} \partial_2^2 u_0 \|_{L^2(\mathbb{R}^3)} + C \| e^{-\zeta_1 t} \partial_3^2 u_0 \|_{L^2(\mathbb{R}^3)} \leq C (1 + t)^{-\frac{3}{2}} (\| \partial_3^2 u_0 \|_{L^2_{3, \tau_{1, 1, 2}}^1} + \| \partial_3^2 u_0 \|_{L^2}).
\]

(5.50)

\(K_{233}\) can be bounded as

\[
K_{333} \leq C \int_0^\tau e^{-\zeta_0 |\xi| (t - \tau)} \partial_2^2 M_1(\tau) \|_{L^2(\mathbb{R}^3)} d\tau + C \int_0^\tau (1 + t - \tau)^{-m} \| \partial_3^2 M_1(\tau) \|_{L^2(\mathbb{R}^3)} d\tau \\
\leq C \int_0^\tau (1 + t - \tau)^{-\frac{3}{2}} \| \partial_3^2 M_1(\tau) \|_{L^2_{3, \tau_{1, 1, 2}}^1} d\tau + C \int_0^\tau (1 + t - \tau)^{-m} \| \partial_3^2 M_1(\tau) \|_{L^2(\mathbb{R}^3)} d\tau \\
: = K_{3331} + K_{3332}.
\]

We consider the integral

\[
\int_0^\tau (1 + t - \tau)^{-\frac{1}{2}} \| \partial_3^2 (u \cdot \nabla u)(\tau) \|_{L^2_{3, \tau_{1, 1, 2}}^1} d\tau.
\]

(5.51)

It follows from (2.6) that

\[
\| \partial_3^2 (u \cdot \nabla u) \|_{L^2_{3, \tau_{1, 1, 2}}^1} = \| \partial_3^2 u_j \partial_3 u + 2 \partial_3 u_j \partial_3 \partial_5 u + u_j \partial_5 \partial_3^2 u \|_{L^2_{3, \tau_{1, 1, 2}}^1} \\
\leq C (\| \partial_3 u \|_{L^2}^2 \| \partial_3^2 u \|_{L^2}^2 \| \partial_3^2 u \|_{L^2}^2 + \| \nabla u \|_{L^2} \| \partial_3 \nabla u \|_{L^2} \| \partial_3^2 u \|_{L^2} \\
+ \| \partial_3 u \|_{L^2} \| \partial_3^2 u \|_{L^2} \| \partial_3^2 u \|_{L^2} + \| \partial_3 u \|_{L^2} \| \partial_3^2 u \|_{L^2} \| \partial_3 \nabla u \|_{L^2} \\
+ \| u \|_{L^2} \| \partial_3 u \|_{L^2} \| \partial_3^2 u \|_{L^2} + \| u \|_{L^2} \| \partial_3^2 u \|_{L^2} \| \partial_3 \nabla u \|_{L^2} \\
+ \| u \|_{L^2} \| \partial_3^2 u \|_{L^2} \| \partial_3^2 u \|_{L^2} + \| u \|_{L^2} \| \partial_3^2 u \|_{L^2} \| \partial_3 \nabla u \|_{L^2} \\
\| \partial_3^2 \nabla u \|_{L^2} ).
\]

(5.52)

Inserting (5.52) in (5.51), and using Lemma 5.3, the first three terms can be bounded by

\[
\int_0^\tau (1 + t - \tau)^{-\frac{1}{2}} (\| \nabla u(\tau) \|_{L^2} \| \partial_3 \nabla u(\tau) \|_{L^2} \| \partial_3^2 u(\tau) \|_{L^2}) \\
+ \| \partial_3 u(\tau) \|_{L^2} \| \partial_3^2 u(\tau) \|_{L^2} \| \nabla u(\tau) \|_{L^2} \\
+ \| u(\tau) \|_{L^2} \| \partial_3^2 u(\tau) \|_{L^2} \| \nabla u(\tau) \|_{L^2} ) d\tau \\
\leq CE_2(t) \int_0^\tau (1 + t - \tau)^{-\frac{1}{2}} ((1 + t + \frac{3}{2}, \frac{\tau}{2}, \frac{\tau}{2}, \frac{\tau}{2}) + (1 + t - \tau)^{-\frac{3}{2}} + \frac{\tau}{2}) d\tau.
\]
As a consequence of (5.50) and (5.54),

\[ + CE^\frac{1}{2}(t)E_0^\frac{1}{2}(t) \int_0^t (1 + t - \tau)^{-\frac{1}{2}}(1 + \tau)^{-\frac{1}{2}} d\tau \]

\[ \leq CE(t)\left((1 + t)^{-\frac{1}{2}} + (1 + t)^{-\frac{1}{4}}\right). \]

The last term needs more subtle estimates. We resort to Hölder’s inequality and the integrability of \( \|\partial_3^2 \nabla u\|_{L^2} \).

\[
\int_0^t (1 + t - \tau)^{-\frac{1}{2}}\|u_h(\tau)\|_2^\frac{1}{2}\|\partial_3^2 u(\tau)\|_2^\frac{1}{2}\|\partial_3 \nabla u(\tau)\|_{L^2} d\tau \\
\leq CE^\frac{1}{2}(t) \int_0^t (1 + t - \tau)^{-\frac{1}{2}}(1 + \tau)^{-\frac{1}{2} + \frac{1}{4}}\|\partial_3^2 \nabla u(\tau)\|_{L^2} d\tau \\
\leq CE^\frac{1}{2}(t)\left(\int_0^t (1 + t - \tau)^{-1}(1 + \tau)^{-1 + \frac{s}{4}} d\tau\right)^{\frac{1}{2}}\left(\int_0^t \|\partial_3^2 \nabla u(\tau)\|_{L^2}^2 d\tau\right)^{\frac{1}{2}} \\
\leq CE^\frac{1}{2}(t)E_0^\frac{1}{2}(t)(1 + t)^{-\frac{1}{4}}.
\]

Combining all the estimates above, we get

\[
\int_0^t (1 + t - \tau)^{-\frac{1}{2}}\|\partial_3^2 (u \cdot \nabla u)(\tau)\|_{L^2_{x_1}L^2_{x_2}} d\tau \leq CE(t)(1 + t)^{-\frac{1}{4}}.
\]

Thus,

\[ K_{3331} \leq CE(t)(1 + t)^{-\frac{1}{4}}. \]

(5.53)

Finally, applying Hölder’s inequality and Sobolev’s inequality, for \( m > 1 \), we infer

\[
\int_0^t (1 + t - \tau)^{-m}\|\partial_3^2 (u \cdot \nabla u)(\tau)\|_{L^2(\mathbb{R}^3)} d\tau \\
\leq C \int_0^t (1 + t - \tau)^{-m}(\|\nabla^2 u(\tau)\|_{L^2}\|\nabla u(\tau)\|_{L^2} + \|u(\tau)\|_{L^2}\|\nabla \partial_3^2 u(\tau)\|_{L^2}) d\tau \\
\leq C \int_0^t (1 + t - \tau)^{-m}(\|\nabla^2 u(\tau)\|_{L^2}\|\nabla^2 u(\tau)\|_{L^2} + \|u(\tau)\|_{L^2}\|\nabla \partial_3^2 u(\tau)\|_{L^2} + \|\nabla u(\tau)\|_{L^2}\|\nabla \partial_3^2 u(\tau)\|_{L^2}) d\tau \\
\leq CE^\frac{1}{2}(t)E_0^\frac{1}{2}(t)\left(\int_0^t (1 + t - \tau)^{-m}(1 + \tau)^{-\frac{1}{4}} d\tau + \int_0^t (1 + t - \tau)^{-m}(1 + \tau)^{-\frac{1}{4} + \frac{s}{4}} d\tau\right) \\
\leq CE(t)(1 + t)^{-\frac{1}{4}}.
\]

Thus,

\[ K_{3332} \leq CE(t)(1 + t)^{-\frac{1}{4}}. \]

which, together with (5.53), yields

\[ K_{333} \leq CE(t)(1 + t)^{-\frac{1}{4}}. \]

(5.54)

As a consequence of (5.50) and (5.54),

\[ (1 + t)^{\frac{3}{4}}\|\partial_3^2 u(\tau)\|_{L^2} \leq CE(t) + C(\|\partial_3^2 u_0\|_{L^2_{x_1}L^2_{x_2}} + \|\partial_3^2 b_0\|_{L^2}). \]
Combining all the estimates for the four cases above, we derive the desired estimate (5.35). This completes the proof of Lemma 5.7.

Proposition 5.1 then follows from the estimates (5.9), (5.10), (5.19) and (5.35). This completes the proof of Proposition 5.1.

Acknowledgments

Lin was partially supported by the National Natural Science Foundation of China (NNSFC) under Grant 11701049 and the China Postdoctoral Science Foundation under Grant 2017M622989. Wu was partially supported by the National Science Foundation of the United States under grant DMS 2104682 and the AT&T Foundation at Oklahoma State University. Zhu is partially supported by Shanghai Sailing Program under Grant 18YF1405500 and NNSFC under Grant 11801175.

References

[1] H. Abidi and M. Paicu, Global existence for the magnetohydrodynamic system in critical spaces. Proc. Roy. Soc. Edinburgh Sect. A 138 (2008), 447-476.
[2] A. Alemany, R. Moreau, P. Sulem and U. Frisch, Influence of an external magnetic field on homogeneous MHD turbulence, J. Méc. 18 (1979), 277-313.
[3] A. Alexakis, Two-dimensional behavior of three-dimensional magnetohydrodynamic flow with a strong guiding field, Phys. Rev. E 84 (2011), 056330.
[4] H. Alfvén, Existence of electromagnetic-hydrodynamic waves, Nature 150 (1942), 405–406.
[5] C. Bardos, C. Sulem and P.L. Sulem, Longtime dynamics of a conductive fluid in the presence of a strong magnetic field, Trans. Am. Math. Soc. 305 (1988), 175-191.
[6] R. Beekie, S. Friedlander and V. Vicol, On Moffatt’s magnetic relaxation equations, Comm. Math. Phys. 390 (2022), 1311-1339.
[7] N. Boardman, H. Lin and J. Wu, Stabilization of a background magnetic field on a 2D magnetohydrodynamic flow, SIAM J. Math. Anal. 52 (2020), 5001–5035.
[8] Y. Cai and Z. Lei, Global well-posedness of the incompressible magnetohydrodynamics, Arch. Ration. Mech. Anal. 228 (2018), 969-993.
[9] C. Cao, D. Regmi and J. Wu, the 2D MHD equations with horizontal dissipation and horizontal magnetic diffusion, J. Differential Equations 254 (2013), 2661-2681.
[10] C. Cao and J. Wu, Global regularity for the 2D MHD equations with mixed partial dissipation and magnetic diffusion, Adv. Math. 226 (2011), 1803-1822.
[11] C. Cao, J. Wu and B. Yuan, The 2D incompressible magnetohydrodynamics equations with only magnetic diffusion, SIAM J. Math. Anal. 46 (2014), 588-602.
[12] J. Chemin, B. Desjardins, I. Gallagher and E. Grenier, Mathematical Geophysics. An introduction to rotating fluids and the Navier-Stokes equations, Oxford Lecture Series in Mathematics and its Applications, Vol. 32, The Clarendon Press/Oxford University Press, Oxford, 2006.
[13] J. Chemin and P. Zhang, On the global well-posedness to the 3-D incompressible anisotropic Navier-Stokes equations, Comm. Math. Phys. 272 (2007), 529-566.
[14] J. Chen and T. Hou, Finite time blowup of 2D Boussinesq and 3D Euler equations with $C^{1,\alpha}$ velocity and boundary, Comm. Math. Phys. 383 (2021), 1559–1667.
[15] W. Chen, Z. Zhang and J. Zhou, Global well-posedness for the 3-D MHD equations with partial diffusion in the periodic domain, Sci. China Math. 65 (2022), 309–318.
[16] K. Choi and I. Jeong, Infinite growth in vorticity gradient of compactly supported planar vorticity near Lamb dipole. [arXiv:2108.01811]
[17] I. Craig and Y. Litvinenko, Wave energy dissipation by anisotropic viscosity in magnetic X-points, *Astrophysical J.* 667 (2007), 1235-1242.

[18] I. Craig and Y. Litvinenko, Anisotropic viscous dissipation in three-dimensional magnetic merging solutions, *Astronomy & Astrophysics* 501 (2009), 755-760.

[19] P.A. Davidson, Magnetic damping of jets and vortices, *J. Fluid Mech.* 299 (1995), 153–186.

[20] P.A. Davidson, The role of angular momentum in the magnetic damping of turbulence, *J. Fluid Mech.* 336 (1997), 123-150.

[21] P.A. Davidson, *An Introduction to Magnetohydrodynamics*, Cambridge University Press, Cambridge, England, 2001.

[22] W. Deng and P. Zhang, Large time behavior of solutions to 3-D MHD system with initial data near equilibrium, *Arch. Rational Mech. Anal.* 230 (2018), 1017-1102.

[23] B. Dong, Y. Jia, J. Li and J. Wu, Global regularity and time decay for the 2D magnetohydrodynamic equations with fractional dissipation and partial magnetic diffusion, *J. Math. Fluid Mech.* 20 (2018), 1541-1565.

[24] B. Dong, J. Li and J. Wu, Global regularity for the 2D MHD equations with partial hyperresistivity, *Intern. Math Research Notices*, 14 (2019), 4261-4280.

[25] T. Drivas and T. Elgindi, Singularity formation in the incompressible Euler equation in finite and infinite time, arXiv:2203.17221.

[26] L. Du and D. Zhou, Global well-posedness of two-dimensional magnetohydrodynamic flows with partial dissipation and magnetic diffusion, *SIAM J. Math. Anal.* 47 (2015), 1562-1589.

[27] G. Duvaut and J. Lions, Inequations en thermoelasticite et magnetohydrodynamique, *Arch. Rational Mech. Anal.* 46 (1972), 241-279.

[28] T. Elgindi, Finite-time singularity formation for $C^{1,\alpha}$-solutions to the incompressible Euler equations on $\mathbb{R}^1$, *Ann. of Math.* 194 (2021), 647-727.

[29] C. Fefferman, D. McCormick, J. Robinson and J.L. Rodrigo, Higher order commutator estimates and local existence for the non-resistive MHD equations and related models, *J. Funct. Anal.* 267 (2014), 1035–1056.

[30] C. Fefferman, D. McCormick, J. Robinson and J. Rodrigo, Local existence for the non-resistive MHD equations in nearly optimal Sobolev spaces, *Arch. Ration. Mech. Anal.* 223 (2017), 677–691.

[31] W. Feng, F. Hafeez and J. Wu, Influence of a background magnetic field on a 2D magnetohydrodynamic flow, *Nonlinearity* 34 (2021), 2527-2562.

[32] B. Gallet, M. Berhanu and N. Mordant, Influence of an external magnetic field on forced turbulence in a swirling flow of liquid metal, *Phys. Fluids* 21 (2009), 085107.

[33] B. Gallet and C.R. Doering, Exact two-dimensionalization of low-magnetic-Reynolds-number flows subject to a strong magnetic field, *J. Fluid Mech.* 773 (2015), 154–177.

[34] L. He, L. Xu and P. Yu, On global dynamics of three dimensional magnetohydrodynamics: nonlinear stability of Alfvén waves, *Ann. PDE* 4 (2018), Art.5, 105 pp.

[35] X. Hu, Global existence for two dimensional compressible magnetohydrodynamic flows with zero magnetic diffusivity, arXiv: 1405.0274.

[36] X. Hu and F. Lin, Global existence for two dimensional incompressible magnetohydrodynamic flows with zero magnetic diffusivity, arXiv: 1405.0082.

[37] X. Hu and D. Wang, Global existence and large-time behavior of solutions to the three-dimensional equations of compressible magnetohydrodynamic flows, *Arch. Ration. Mech. Anal.*, 197 (2010), 203–238.

[38] D. Ifrimie, A uniqueness result for the Navier-Stokes equations with vanishing vertical viscosity, *SIAM J. Math. Anal.* 33 (2002), 1483–1493.

[39] R. Ji and J. Wu, The resistive magnetohydrodynamic equation near an equilibrium, *J. Differential Equations* 268 (2020), 1854-1871.

[40] R. Ji, J. Wu and W. Yang, Stability and optimal decay for the 3D Navier-Stokes equations with horizontal dissipation, *J. Differential Equations* 290 (2021), 57-77.

[41] Q. Jiu, D. Niu, J. Wu, X. Xu and H. Yu, The 2D magnetohydrodynamic equations with magnetic diffusion, *Nonlinearity* 28 (2015), 3935-3956.
A. Kiselev and V. Sverak, Small scale creation for solutions of the incompressible two-dimensional Euler equation, Ann. Math. 180 (2014), 1205-1220.

S. Lai, J. Wu and J. Zhang, Stabilizing phenomenon for 2D anisotropic magnetohydrodynamic System near a background magnetic field, SIAM J. Math. Anal. 53 (2021), 6073-6093.

S. Lai, J. Wu and J. Zhang, Stabilizing effect of magnetic field on the 2D ideal magnetohydrodynamic flow with mixed partial damping, Calc. Var. Partial Differential Equations 61 (2022), Paper No. 126.

J. Li, W. Tan and Z. Yin, Local existence and uniqueness for the non-resistive MHD equations in homogeneous Besov spaces, Adv. Math. 317 (2017), 786-798.

H. Lin and L. Du, Regularity criteria for incompressible magnetohydrodynamics equations in three dimensions, Nonlinearity 26 (2013), 219-239.

H. Lin, R. Ji, J. Wu, L. Yan, Stability of perturbations near a background magnetic field of the 2D incompressible MHD equations with mixed partial dissipation, J. Funct. Anal. 279 (2020), 108519.

F. Lin, L. Xu, and P. Zhang, Global small solutions to 2-D incompressible MHD system, J. Differential Equations 259 (2015), 5440-5485.

F. Lin and P. Zhang, Global small solutions to an MHD-type system: the three-dimensional case. Comm. Pure Appl. Math. 67 (2014), 531-580.

F. Lin and T. Zhang, Global small solutions to a complex fluid model in three dimensional, Arch. Ration. Mech. Anal. 216 (2015), 905-920.

C. Liu, D. Wang, F. Xie and T. Yang, Magnetic effects on the solvability of 2D MHD boundary layer equations without resistivity in Sobolev spaces, J. Funct. Anal. 279 (2020), 108637, 45 pp.

Y. Liu and P. Zhang, Global solutions of 3-D Navier-Stokes system with small unidirectional derivative. Arch. Ration. Mech. Anal. 235 (2020), 1405-1444.

A. Majda and A. Bertozzi, Vorticity and Incompressible Flow, Cambridge University Press, 2002.

M. Paicu, Équation anisotrope de Navier-Stokes dans des espaces critiques, Rev. Mat. Iberoamericana 21 (2005), 179–235.

M. Paicu, Équation periodique de Navier-Stokes sans viscosité dans une direction, Comm. Partial Differential Equations 30 (2005), 1107–1140.

R. Pan, Y. Zhou and Y. Zhu, Global classical solutions of three dimensional viscous MHD system without magnetic diffusion on periodic boxes, Arch. Rational Mech. Anal. 227 (2018), 637-662.

E. Priest and T. Forbes, Magnetic reconnection, MHD theory and Applications, Cambridge University Press, Cambridge, 2000.

X. Ren, J. Wu, Z. Xiang and Z. Zhang, Global existence and decay of smooth solution for the 2-D MHD equations without magnetic diffusion, J. Funct. Anal. 267 (2014), 503-541.

X. Ren, Z. Xiang and Z. Zhang, Global well-posedness for the 2D MHD equations without magnetic diffusion in a strip domain, Nonlinearity, 29 (2016), 1257-1291.

M. Schonbek, $L^2$ decay for weak solutions of the Navier-Stokes equations, Arch. Ration. Mech. Anal. 88 (1985), 209–222.

M. Schonbek, Lower bounds of rates of decay for solutions to the Navier-Stokes equations, J. Amer. Math. Soc. 4 (1991), 423-449.

M. Schonbek and T. Schonbek, On the boundedness and decay of moments of solutions of the Navier Stokes equations, Adv. Differential Equations 5 (2000), 861-898.

M. Sermange and R. Temam, Some mathematical questions related to the MHD equations, Comm. Pure Appl. Math. 36 (1983), 635-664.

H. Shang and Y. Zhai, Stability and large time decay for the three-dimensional anisotropic magnetohydrodynamic equations, Z. Angew. Math. Phys. 73 (2022), Paper No. 71, 22 pp.

Z. Tan and Y. Wang, Global well-posedness of an initial-boundary value problem for viscous non-resistive MHD systems, SIAM J. Math. Anal. 50 (2018), 1432-1470.

T. Tao, Nonlinear Dispersive Equations: Local and Global Analysis, CBMS Regional Conference Series in Mathematics, Providence, RI: American Mathematical Society, 2006.

R. Wan, On the uniqueness for the 2D MHD equations without magnetic diffusion, Nonlin. Anal. Real World Appl. 30 (2016), 32-40.

D. Wei and Z. Zhang, Global well-posedness of the MHD equations in a homogeneous magnetic field, Anal. PDE 10 (2017), 1361–1406.
[69] D. Wei and Z. Zhang, Global well-posedness for the 2-D MHD equations with magnetic diffusion, Commun. Math. Res. 36 (2020), 377–389.

[70] M. Wiegner, Decay results for weak solutions to the Navier-Stokes equations on $R^d$, J. London Math. Soc. 35 (1987), 303–313.

[71] J. Wu, Dissipative quasi-geostrophic equations with $L^p$ data, Electron J. Differential Equations 2001 (2001), 1-13.

[72] J. Wu, The 2D magnetohydrodynamic equations with partial or fractional dissipation, Lectures on the analysis of nonlinear partial differential equations, Morningside Lectures on Mathematics, Part 5, MLM5, pp. 283-332, International Press, Somerville, MA, 2018.

[73] J. Wu and Y. Wu, Global small solutions to the compressible 2D magnetohydrodynamic system without magnetic diffusion, Adv. Math. 310 (2017), 759–888.

[74] J. Wu, Y. Wu and X. Xu, Global small solution to the 2D MHD system with a velocity damping term, SIAM J. Math. Anal. 47 (2015), 2630-2656.

[75] J. Wu and Y. Zhu, Global solutions of 3D incompressible MHD system with mixed partial dissipation and magnetic diffusion near an equilibrium, Adv. Math. 377 (2021), 107466.

[76] J. Wu and Y. Zhu, Enhanced dissipation for the third component of 3D anisotropic Navier-Stokes equations, preprint.

[77] L. Xu and P. Zhang, Global small solutions to three-dimensional incompressible magnetohydrodynamical system with zero dissipation, Nonlinear Anal. Real World Appl. 41 (2018), 53-65.

[78] L. Xu, P. Zhang and Z. Zhang, Global solvability of a free boundary three-dimensional incompressible viscoelastic fluid system with surface tension, Arch. Ration. Mech. Anal. 208 (2013), 753-803.

[79] K. Yamazaki, Remarks on the global regularity of the two-dimensional magnetohydrodynamics system with zero dissipation, Nonlinear Anal. 94 (2014), 194-205.

[80] K. Yamazaki, Global regularity of logarithmically supercritical MHD system with zero diffusivity, Appl. Math. Lett. 29 (2014), 46-51.

[81] L. Xu, P. Zhang and Z. Zhang, Global solvability of a free boundary three-dimensional incompressible viscoelastic fluid system with surface tension, Arch. Ration. Mech. Anal. 208 (2013), 753-803.

[82] A. Zlatos, Exponential growth of the vorticity gradient for the Euler equation on the torus, Adv. Math. 260 (2016), 5450-5480.

[83] B. Yuan and J. Zhao, Global regularity of 2D almost resistive MHD equations, Nonlin. Anal. Real World Appl. 41 (2018), 53-65.

[84] T. Zhang, Global wellposed problem for the 3-D incompressible anisotropic Navier-Stokes equations in an anisotropic space, Comm. Math. Phys. 287 (2009), 211-224.

[85] T. Zhang, Erratum to: Global wellposed problem for the 3-D incompressible anisotropic Navier-Stokes equations in an anisotropic space, Comm. Math. Phys. 295 (2010), 877-884.

[86] T. Zhang, An elementary proof of the global existence and uniqueness theorem to 2-D incompressible non-resistive MHD system, (2014), [arXiv:1404.5681]

[87] T. Zhang, Global solutions to the 2D viscous, non-resistive MHD system with large background magnetic field, J. Differential Equations 260 (2016), 5450-5480.

[88] Y. Zhou and Y. Zhu, Global classical solutions of 2D MHD system with only magnetic diffusion on periodic domain, J. Math. Phys. 59 (2018), 081505.

[89] A. Zlatos, Exponential growth of the vorticity gradient for the Euler equation on the torus, Adv. Math. 268 (2015), 396-403.

1 COLLEGE OF MATHEMATICS AND PHYSICS, AND GEOMATHEMATICS KEY LABORATORY OF SICHUAN PROVINCE, CHENGDU UNIVERSITY OF TECHNOLOGY, CHENGDU 610059, P.R. CHINA

Email address: linhongxia18@126.com

2 DEPARTMENT OF MATHEMATICS, OKLAHOMA STATE UNIVERSITY, STILLWATER, OK 74078, UNITED STATES

Email address: jiahong.wu@okstate.edu

3 DEPARTMENT OF MATHEMATICS, EAST CHINA UNIVERSITY OF SCIENCE AND TECHNOLOGY, SHANGHAI, 200237, P. R. CHINA

Email address: Zhuyim@ecust.edu.cn