GUMBEL DISTRIBUTION IN EXIT PROBLEMS

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Abstract. We explain the connection between the Gumbel limit for diffusion exit times and the theory of extreme values.

1. Introduction

The Gumbel distribution $\Lambda$ is one of the max-stable distributions. Its distribution function is
$$\Lambda(x) = e^{-e^{-x}}, \quad x \in \mathbb{R},$$
and its density is given by
$$\lambda(x) = e^{-x-e^{-x}}, \quad x \in \mathbb{R}.$$ (1)

The Gumbel distribution is mostly well-known as a limit law in the extreme value theory, see, e.g., [dHF06, Chapter 1]. Surprisingly, it also appears as a limiting distribution (in the limit of vanishing noise) for normalized exit times of diffusions conditioned on unlikely exit locations. The first result of this kind was obtained in [Day90] where exits through a characteristic repelling boundary were considered. In a recent paper [CGLM13] (see also references therein for related results) a similar result was obtained for the case where the prescribed exit location for the diffusion is separated from the starting point by a potential wall. Let us state the main result from [CGLM13].

Let $X_\varepsilon$ be a strong solution of the following one-dimensional stochastic differential equation driven by a drift vector field $b$ and a Wiener process $W$:
$$dX_\varepsilon(t) = b(X_\varepsilon(t))dt + \varepsilon dW(t), \quad X_\varepsilon(0) = x_0, \quad (2)$$
where $\varepsilon > 0$ is a constant diffusion coefficient and $x_0$ is a nonrandom starting point. This diffusion process is considered on a segment $[A, B]$ containing 0 and $x_0$, until the first exit time
$$\tau_\varepsilon = \inf\{t \geq 0: X_\varepsilon(t) \in \{A, B\}\}.$$ (3)

It is assumed that the drift $b$ is smooth on $[A, B]$ and satisfies $b(0) = 0$, $b'(0) > 0$, $b(x) < 0$ for $x \in [A, 0)$, and $b(x) > 0$ for $x \in (0, B]$.

Let us assume that $x_0 < 0$. Then the event $C_\varepsilon = \{X_\varepsilon(\tau_\varepsilon) = B\}$ is unlikely for small $\varepsilon$ because on $C_\varepsilon$ the process $X_\varepsilon$ has to travel against the drift. The probability of $C_\varepsilon$ decays exponentially in $\varepsilon^{-2}$ as follows from
the celebrated Freidlin–Wentzell theory of large deviations for small white noise perturbations of dynamical systems, see [FW84]. Nevertheless, one can study the behavior of $\tau_\varepsilon$ conditioned on this unlikely event.

**Theorem 1** ([CGLM13]). Under the conditions given above, there are constants $c_1, c_2, c_3$ such that as $\varepsilon \to 0$,

$$\text{Law} \left[ \tau_\varepsilon - c_1 \ln \frac{1}{\varepsilon} \bigg| C_\varepsilon \right] \Rightarrow \text{Law} \left[ c_2 Z + c_3 \right],$$

where $Z$ is a Gumbel random variable and "$\Rightarrow$" denotes weak convergence of probability measures.

Although it is clear that the behavior of both extreme values and exit times depends on the tails of the random variables involved, the precise reason why the Gumbel distribution appears as a limit in both cases has not been known. The goal of this note is to explain the connection which is provided by the theory of residual life times developed in the classical work [BdH74].

We stress that the computations in the present note are not new and go back at least to [Day90]. What is new is the realization that these computations can be viewed as a specific case of those in the heart of residual life times theory. In fact, this link between exit times and residual life times is absent in the literature known to the author (and MathSciNet).

In Section 2 we consider a simple model problem that simultaneously captures the essence of the mechanism of generating asymptotically Gumbel exit times, and allows for elementary computations. In Section 3 we explain the necessary facts on extreme values and residual life times. In Section 4, we revisit the computations of Section 2 and explain them from the point of view of the theory of residual life times.

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2. A SIMPLE EXIT PROBLEM

Here we consider the following model problem. Let us suppose that the drift is linear, i.e., $b(x) \equiv \beta x$ for an arbitrary constant $\beta > 0$ and, the constants $A$ and $B$ are equal to $-1$ and $1$, respectively. Instead of considering a fixed value of $x_0 \in (-1, 0)$, we will assume that $x_0 = -\varepsilon a$ for some $a > 0$, and then sequentially take the limits $\varepsilon \to 0$ and $a \to +\infty$. This model is different from the one considered in [CGLM13], [Day90], [Day92], [Day95], and our results are not equivalent to the results therein. However, we choose this model and the limiting procedure since this is the simplest setting that suits our goal of demonstrating a natural connection between exit problems and extreme values theory.

Let us first study the limiting behavior of $\tau_\varepsilon$ as $\varepsilon \to 0$. 
Theorem 2. Let us fix \(a > 0\) and introduce \(r = a\sqrt{2\beta}\). As \(\varepsilon \to 0\),
\[
\text{Law}\left[\tau_\varepsilon - \frac{1}{\beta} \ln \frac{1}{\varepsilon} \mid X_\varepsilon(\tau_\varepsilon) = 1\right] \Rightarrow \text{Law}\left[\frac{1}{\beta} \ln(N - r) + \frac{1}{2\beta} \ln(2\beta) \mid N > r\right],
\]
where \(N\) is a standard Gaussian random variable.

Proof: In the case of linear drift, the solution \(X_\varepsilon\) can be represented by the following Duhamel principle:
\[
X_\varepsilon(t) = e^{\beta t} \left( x_0 + \varepsilon \int_0^t e^{-\beta s} dW(s) \right)
\]
(4)
\[
= e^{\beta t} \left( -\varepsilon a + \varepsilon \int_0^t e^{-\beta s} dW(s) \right)
\]
\[
= \varepsilon e^{\beta t} \left( -a + \int_0^t e^{-\beta s} dW(s) \right),
\]
conditioned on the exit through the right end of \([-1, 1]\), i.e., on \(\{X(\tau_\varepsilon) = 1\}\).

Let us introduce
\[
I_t = \int_0^t e^{-\beta s} dW(s), \quad t \in [0, +\infty],
\]
and \(I^* = \sup_{t \geq 0} |I_t|\). Representing \(I_t\) as a time changed Wiener process, we see that \(I^*\) is a.s.-finite. Since \(|X_\varepsilon(\tau_\varepsilon)| = 1\), equation (4) implies
\[
\tau_\varepsilon \geq \frac{1}{\beta} \ln \frac{1}{\varepsilon(I^* + a)},
\]
and we can conclude that \(\tau_\varepsilon \to +\infty\) a.s. Combining this with (4) and continuity of \(I_t\) on \([0, +\infty]\), we see that
\[
\frac{X_\varepsilon(\tau_\varepsilon)}{\varepsilon e^{\beta \tau_\varepsilon}} = -a + I_{\tau_\varepsilon} \xrightarrow{a.s.} -a + I_\infty.
\]
Recalling that \(|X_\varepsilon(\tau_\varepsilon)| = 1\), we conclude that
\[
\tau_\varepsilon - \frac{1}{\beta} \ln \frac{1}{\varepsilon} \xrightarrow{a.s.} -\frac{1}{\beta} \ln(-a + I_\infty).
\]
Also,
\[
X_\varepsilon(\tau_\varepsilon) \xrightarrow{a.s.} \text{sgn}(-a + I_\infty).
\]
Now the Theorem follows since \(I_\infty\) is a centered Gaussian random variable with variance \(1/(2\beta)\), i.e., \(I_\infty = N/\sqrt{2\beta}\) for a standard Gaussian random variable \(N\).

The limiting distribution provided for the first part of the limiting procedure \((\varepsilon \to 0)\) by Theorem 2 is a linear transformation of the conditional distribution \(\text{Law}\left[-\ln(N - r) \mid N > r\right]\), where \(r = a\sqrt{2\beta}\). In fact, this conditional distribution can be computed explicitly. For example, its density is given by
\[
p_r(x) = \frac{1}{\sqrt{2\pi} e^{-x - (e^{-x} + r)^2/2}} \frac{1}{1 - G(r)},
\]
(5)
where $G$ is the standard Gaussian distribution function.

Similar arguments were used to study exits from neighborhoods of unstable equilibria in [Day90, Day92, Day95, BG04, Bak08, Bak10, Bak11, Bak12, BCT2, BG13].

The second part of the analysis is letting $a \to \infty$ or, equivalently, $r \to \infty$. Our next statement claims that as $r \to \infty$, under a proper normalization (shift by $\ln r$), the limit of the distributions described in Theorem 2 is the Gumbel distribution.

**Theorem 3.** Let $\lambda$ and $p_r$ be defined by (1) and (5). Then

$$
\lim_{r \to \infty} p_r(x - \ln r) = \lambda(x), \quad x \in \mathbb{R}.
$$

**Proof:** For any $x \in \mathbb{R},$

$$
p_r(x - \ln r) = \frac{\frac{1}{\sqrt{2\pi}} e^{-x - \ln r - (e^{-x - \ln r} + r)^2/2}}{1 - G(r)}
\approx \frac{\frac{1}{\sqrt{2\pi r}} e^{-r^2/2}}{1 - G(r)} e^{-x - e^{-2x}/(2r^2) - e^{-x}} \to \lambda(x), \quad r \to \infty,
$$

where we used the fact that $1 - G(r) \sim \frac{1}{\sqrt{2\pi r}} e^{-r^2/2}$.

\[\Box\]

**Corollary 1.** As $r \to \infty,$

$$
\text{Law} \left[ -\ln(N - r) - \ln r \mid N > r \right] \Rightarrow \Lambda.
$$

**Proof:** This claim is a direct consequence of Theorem 3 and Scheffé’s theorem (see [Sch47] or [Bil68, Appendix II]) stating that pointwise convergence of densities implies convergence of distributions in total variation and hence weak convergence. One can also give a direct proof:

$$
\frac{P\{-\ln(N - r) - \ln r < x\}}{P\{N > r\}} = \frac{P\{N > r + e^{-x}/r\}}{P\{N > r\}}
\sim \frac{1}{\sqrt{2\pi r + e^{-x}/r}} e^{-(r + e^{-x}/r)^2/2} \sim e^{-e^{-x}}
$$

\[\Box\]

Although this convergence statement along with Theorem 3 is a result of a straightforward computation that in some form appeared for the first time in the diffusion context in [Day90], we still need to explain the connection to the theory of extreme values and residual lifetime theory, and we proceed to recall some related basic facts.

### 3. Gumbel Distribution in Extreme Value Theory and Residual Life Time Theory

Let us start with some results from [Gne43] where basins of attraction of $\Lambda$ and other max-stable distributions were studied. If random variables
Let $X_1, X_2, \ldots$ be i.i.d. with common distribution function $F$, then
\[
P\{\max(X_1, \ldots, X_n) \leq x\} = F^n(x), \quad x \in \mathbb{R}, n \in \mathbb{N}.
\]
The following is a version of Lemma 3 in [Gne43]:

**Lemma 1.** Let $F$ be a distribution function, $\Phi$ a continuous distribution function, and let $(a_n), (b_n)$ be number sequences, $a_n > 0$ for all $n$. Then
\[
\lim_{n \to \infty} F^n(a_n x + b_n) = \Phi(x), \quad x \in \mathbb{R},
\]
if and only if
\[
\Phi(x) \neq 0 \quad \Rightarrow \quad \lim_{n \to \infty} n[1 - F(a_n x + b_n)] = -\ln \Phi(x).
\]

**Proof:** Let us recall the proof of this statement from [Gne43]. Condition (7) is equivalent to
\[
\Phi(x) \neq 0 \quad \Rightarrow \quad \lim_{n \to \infty} F(a_n x + b_n) = 1,
\]
which in turn implies
\[
\Phi(x) \neq 0 \quad \Rightarrow \quad \lim_{n \to \infty} F(a_n x + b_n) = 1,
\]
which is equivalent to (7).

If (8) holds for some sequences $(a_n)$ and $(b_n)$ with $a_n > 0$, then we will write $F \in D(\Phi)$.

Let $G$ be the distribution function of the standard Gaussian distribution. The following simple result shows that $G \in D(\Lambda)$.

**Lemma 2.** Let $b_n$ satisfy $1 - G(b_n) = n^{-1}$ and let $a_n = b_n^{-1}$. Then
\[
\lim_{n \to \infty} G^n(a_n x + b_n) = \Lambda(x), \quad x \in \mathbb{R}.
\]

**Proof:** Noticing that as $n \to \infty$, one has $b_n \to \infty$, $a_n \to 0$, $a_n x + b_n \to \infty$, we can write
\[
n(1 - G(a_n x + b_n)) \sim n \frac{1}{2\pi} \frac{1}{(a_n x + b_n)} e^{-(a_n x + b_n)^2/2} - \frac{b_n^2}{2} e^{b_n a_n x} e^{-a_n^2 x^2/2} - n(1 - G(b_n)) e^{-x} \sim e^{-x},
\]
i.e.,
\[
\lim_{n \to \infty} n(1 - G(a_n x + b_n)) = -\ln \Lambda(x), \quad x \in \mathbb{R},
\]
and the result follows from Lemma 3. □
Let us now turn to residual life times. For a random variable $X$ with distribution function $F(x) = \mathbb{P}\{X \leq x\}$ and tail
\begin{equation}
R(x) = \mathbb{P}\{X > x\} = 1 - F(x),
\end{equation}
the distribution tail of residual life time after time $t$ is
\begin{equation}
R_r(x) = \mathbb{P}\{X - r > x | X > r\} = \frac{R(r + x)}{R(t)}.
\end{equation}
In [BdH74], a theory of scaling limits for residual life times was developed and connections with the theory of scaling limits of extreme values were established. The following is a version of Theorem 3 from [BdH74].

\textbf{Theorem 4 ([BdH74]).} Let $F$ be a distribution function, let $R$ and $R_r$ be defined by (12) and (13), and let $(a_n), (b_n)$ be number sequences satisfying $a_n > 0$ for all $n$. Then the following two conditions are equivalent:

1. For all $x \in \mathbb{R}$, $F(x) < 1$ and $\lim_{n \to \infty} F^n(a_n x + b_n) = \Lambda(x)$.
2. There are functions $a(r) > 0$ and $b(r)$ such that
\[
\lim_{n \to \infty} \frac{R(a(r)x + b(r))}{R(r)} = -\ln \Lambda(x) = e^{-x}, \quad x \in \mathbb{R}.
\]

\textbf{Sketch of proof:} Suppose condition 1 is satisfied. Lemma 1 implies that
\begin{equation}
\lim_{n \to \infty} n R(a_n x + b_n) = -\ln \Lambda(x) = e^{-x}, \quad x \in \mathbb{R},
\end{equation}
and condition 2 follows with $a$ and $b$ satisfying $a(r) = a_n$ and $b(r) = b_n$ for all $r$ such that $(n + 1)^{-1} \leq R(r) < n^{-1}$. We omit the proof of the converse implication and only mention that it is also based on (14). \hfill \Box

\textbf{Remark 1.} In fact, the function $b$ in condition 2 of the theorem can be chosen to be $b(r) = r$ (see [BdH74], Corollary 2), so that condition 2 can be rewritten as
\begin{equation}
\lim_{n \to \infty} \frac{R(r + a(r)x)}{R(r)} = -\ln \Lambda(x) = e^{-x}, \quad x \in \mathbb{R}.
\end{equation}

4. Logarithmic transformation of residual life times

Let $X$ be a random variable with distribution function $F$ and tail function $R = 1 - F$. Motivated by the results of Section 2, we would like to study the limiting behavior of $\text{Law}[\ln(X - r) | X > r]$, so we are interested in
\[
H_r(x) = \mathbb{P}\{-\ln(X - r) \leq x | X > r\}.
\]

\textbf{Theorem 5.} Let $F \in D(\Lambda)$. Let $R$ be defined by (12). If (15) holds for some function $a(\cdot)$, then
\[
\lim_{n \to \infty} H_r(x - \ln a(r)) = \Lambda(x).
\]
Proof: We can rewrite \( H_r(x) \) as
\[
H_r(x) = P\{X - r > e^{-x}|X > r\} = \frac{R(r + e^{-x})}{R(r)}.
\]
Since
\[
H_t(x - \ln a(r)) = \frac{R(r + a(r)e^{-x})}{R(r)},
\]
Theorem 4, Remark 1, and the basic identity
\[
(16) \quad -\ln \Lambda(e^{-x}) = \Lambda(x),
\]
imply our claim. \( \square \)

Theorem 5 can be applied to the Gaussian distribution as the following lemma shows.

**Lemma 3.** If \( F \) is the standard Gaussian distribution function \( G \), then one can choose \( a(t) = t^{-1} \) in (15).

Proof: The following computation for \( R = 1 - G \) is essentially the same as (6) in the proof of Corollary 1:
\[
\frac{R(t + x/t)}{R(t)} \sim \frac{1}{\sqrt{2\pi (t+x/t)}} e^{-((t+x/t)^2/2)} \sim e^{-x}, \quad t \to \infty.
\]
\( \square \)

Now Corollary 1 responsible for the Gumbel limit for exit times can be seen as a direct consequence of Theorem 5 and Lemma 3. This puts the Gumbel limit of exit times in the context of the residual life time theory. Interestingly, this connection of exit times to the residual times theory depends on a trivial but seemingly random property (16) of the Gumbel distribution.

**References**

[Bak08] Yuri Bakhtin. Exit asymptotics for small diffusion about an unstable equilibrium. *Stochastic Process. Appl.*, 118(5):839–851, 2008.

[Bak10] Yuri Bakhtin. Small noise limit for diffusions near heteroclinic networks. *Dyn. Syst.*, 25(3):413–431, 2010.

[Bak11] Yuri Bakhtin. Noisy heteroclinic networks. *Probab. Theory Related Fields*, 150(1-2):1–42, 2011.

[Bak12] Yuri Bakhtin. Decision making times in mean-field dynamic Ising model. *Ann. Henri Poincaré*, 13(5):1291–1303, 2012.

[BC12] Yuri Bakhtin and Joshua Correll. A neural computation model for decision-making times. *J. Math. Psych.*, 56(5):333–340, 2012.

[BdH74] A. A. Balkema and L. de Haan. Residual life time at great age. *Ann. Probability*, 2:792–804, 1974.

[BG04] Nils Berglund and Barbara Gentz. On the noise-induced passage through an unstable periodic orbit. I. Two-level model. *J. Statist. Phys.*, 114(5-6):1577–1618, 2004.

[BG13] Nils Berglund and Barbara Gentz. On the noise-induced passage through an unstable periodic orbit II: The general case. Preprint arXiv:1208.2557, to appear in SIAM J. Math. Anal., 2013.
[Bil68] Patrick Billingsley. *Convergence of probability measures*. John Wiley & Sons Inc., New York, 1968.

[CGLM13] Frédéric Cérou, Arnaud Guyader, Tony Lelièvre, and Florent Malrieu. On the length of one-dimensional reactive paths. *ALEA, Lat. Am. J. Probab. Math. Stat.*, 10(1):359–389, 2013.

[Day90] Martin V. Day. Some phenomena of the characteristic boundary exit problem. In *Diffusion processes and related problems in analysis, Vol. I (Evanston, IL, 1989)*, volume 22 of *Progr. Probab.*, pages 55–71. Birkhäuser Boston, Boston, MA, 1990.

[Day92] Martin V. Day. Conditional exits for small noise diffusions with characteristic boundary. *Ann. Probab.*, 20(3):1385–1419, 1992.

[Day95] Martin V. Day. On the exit law from saddle points. *Stochastic Process. Appl.*, 60(2):287–311, 1995.

[dHF06] Laurens de Haan and Ana Ferreira. *Extreme value theory*. Springer Series in Operations Research and Financial Engineering. Springer, New York, 2006. An introduction.

[FW84] M. I. Freidlin and A. D. Wentzell. *Random perturbations of dynamical systems*, volume 260 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1984. Translated from the Russian by Joseph Szücs.

[Gne43] B. Gnedenko. Sur la distribution limite du terme maximum d’une série aléatoire. *Ann. of Math. (2)*, 44:423–453, 1943.

[Sch47] Henry Scheffé. A useful convergence theorem for probability distributions. *Ann. Math. Statistics*, 18:434–438, 1947.

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