A COMBINATORIAL PROOF OF SYMMETRY AMONG MINIMAL STAR FACTORIZATIONS

BRIDGET EILEEN TENNER

Abstract. The number of minimal transitive star factorizations of a permutation was shown by Irving and Rattan to depend only on the conjugacy class of the permutation, a surprising result given that the pivot plays a very particular role in such factorizations. Here, we explain this symmetry and provide a bijection between minimal transitive star factorizations of a permutation $\pi$ having pivot $k$ and those having pivot $k'$.

Keywords: permutation, star factorization, minimal transitive star factorization

1. Introduction

For any positive integer $n$, let $[n]$ denote the set $\{1, \ldots, n\}$. The symmetric group $S_n$ consists of all permutations of the set $[n]$. It is most convenient for our purposes to represent permutations in cycle notation. That is, write a permutation $\pi$ as the product of disjoint cycles $(x, \pi(x), \ldots, \pi^{\ell-1}(x))$ where $\ell > 0$ is minimal so that $\pi^\ell(x) = x$. If an element $x$ is fixed by $\pi$, that is, if $\pi(x) = x$, then the cycle $(x)$ may be suppressed.

Example 1.1. Let $\pi \in S_6$ be the permutation defined as follows.

```
1 2 3 4 5 6
```

Then $\pi = (421)(6)(35) = (6)(214)(53)$.

As evidenced by Example 1.1, there are multiple ways to write a given permutation in cycle notation. It is common to designate one of these to be standard.

Definition 1.2. The standard form of a permutation in cycle notation is obtained by writing each cycle so that its minimal element appears in the leftmost position, and writing the cycles from left to right in increasing order of minimal elements. Fixed points are not suppressed in the standard form.

Example 1.3. The standard form of the permutation in Example 1.1 is $(142)(35)(6)$.

There are several well studied sets of generators for the symmetric group, and effort has been made to enumerate the most efficient (that is, the shortest) ways to write a given permutation in terms of generators in one of these sets.

Definition 1.4. Given a generating set $G$ of the symmetric group $S_n$, a factorization, or decomposition of $\pi \in S_n$ as $\pi = g_1 \cdots g_\ell$ for $g_i \in G$ is minimal if $\ell$ is minimal. This $\ell$ is the $G$-length (or length if the generating set is clear from the context) of $\pi$.  

2010 Mathematics Subject Classification. 05A05, 05A15, 05A19.
When \( G = \{(i \ i + 1) : 1 \leq i < n\} \), Stanley has shown that in certain cases, including the \( G \)-longest permutation \((1n)(2(n-1))(3(n-2)) \cdots \), this number of minimal factorizations of a permutation \( \pi \) is the same as the number of standard Young tableaux of a particular shape \( \lambda(\pi) \). In a different vein, Dénès computed the number of minimal factorizations of a permutation when \( G = \{(i \ j) : 1 \leq i < j \leq n\} \).

The generating set for \( S_n \) that we consider here is the set of star transpositions.

**Definition 1.5.** Fix positive integers \( n \geq k \geq 1 \). The set \( \star_{n,k} = \{(k \ i) : i \in [n] \setminus \{k\}\} \) is the set of star transpositions with pivot \( k \).

The nomenclature refers to the fact that when the elements of \( \star_{n,k} \) are considered to define edges on the vertices \([n]\), the resulting graph is a star with center label \( k \).

**Lemma 1.6.** The set \( \star_{n,k} \) generates the symmetric group \( S_n \), for any \( k \in [n] \).

**Proof.** Since \( (i \ i + 1) = (k \ i)(k \ i + 1)(k \ i) \), we can generate all simple transpositions, and these, in turn, generate \( S_n \). \( \square \)

In [6], Pak considered minimal factorizations of a particular family of permutations into the star transpositions of Definition 1.5.

**Definition 1.7.** A factorization \( \pi = g_1 \cdots g_r \in S_n \) for \( g_i \in \star_{n,k} \) is transitive if the group generated by \( \{g_1, \ldots, g_r\} \) acts transitively on the set \([n]\). In other words, this factorization is transitive if \( \{g_1, \ldots, g_r\} = \star_{n,k} \).

In the class of permutations studied in [6], the only fixed point was the pivot itself. Thus the factorizations of these permutations are necessarily transitive. Pak’s work was generalized by Irving and Rattan, who computed the number of minimal transitive star factorizations of any permutation [4]. In that work, Irving and Rattan discovered a surprising symmetry in their enumeration, essentially saying that the choice of pivot does not affect the number of minimal transitive star factorizations of a permutation. The purpose of the current article is to provide a combinatorial proof of the symmetry that they found.

**Definition 1.8.** Given a permutation \( \pi \in S_n \), let \( \star_k(\pi) \) denote the set of minimal transitive star factorizations of \( \pi \) having pivot \( k \). Let \( s_k(\pi) = |\star_k(\pi)| \).

As discussed in [4], if \( \pi \in S_n \) has \( m \) cycles, then each element of \( \star_k(\pi) \) has length \( n + m - 2 \). As is customary, permutations are viewed as maps, and so are multiplied from right to left.

**Example 1.9.** Consider \((142)(35)(6) \in S_6 \). Then

\[
(31)(36)(36)(32)(34)(31)(35) \in \star_3((142)(35)(6)).
\]

The main result of [4] is the following theorem, where \( x(r) \) denotes the falling factorial

\[
x(r) = x(x-1)(x-2) \cdots (x-r+1) = \frac{x!}{(x-r)!}.
\]

Note that the discussions in [4] assume that \( k = 1 \), but the proof can be extended to an arbitrary pivot \( k \).
Theorem 1.10 (4). Let \( \pi \in S_n \) have cycles of lengths \( \ell_1, \ldots, \ell_m \). Then
\[
s_k(\pi) = (n + m - 2)(m - 2)\ell_1 \cdots \ell_m
\]
for all \( k \in [n] \).

The remarkable feature of Theorem 1.10 is its symmetry: it depends only on the cycle type of \( \pi \). Given the special role played by the pivot in star transpositions, one would not expect the cycle containing the pivot to behave in the same way as the other cycles in \( \pi \). It was an open question in [4] to explain this symmetry. Goulden and Jackson looked at non-minimal factorizations into star transpositions, and uncovered the same symmetry in that setting [3], although again an explanation of this symmetry was lacking. In [2], Féray has given a proof of the symmetry in both the minimal and non-minimal situations. His argument uses the algebra of partial permutations of Ivanov and Kerov [5], but does not give a combinatorial reason for the phenomenon.

Giving such a justification, in fact a bijection between minimal transitive star factorizations with pivot \( k \) and those with pivot \( k' \), is the purpose of the current article. We do this by giving a bijection \( \phi_{\pi,k} \) between injections from \( [m-2] \) into \( [n + m - 2] \) together with elements of \( [\ell_1] \times \cdots \times [\ell_m] \), and the elements of \( \ast_k(\pi) \) (see Definition 5.9 and Theorem 5.12). This \( \phi_{\pi,k} \) takes such an injection and \( m \)-tuple, and produces a valid word for \( \ast_k(\pi) \) via the maps \( \text{tree}_{\pi,k} \) and \( \omega \), and a set of cycle enclosures via the map \( \text{cycle}_{\pi,k} \). As shown in Proposition 3.12, these determine a unique element of \( \ast_k(\pi) \).

We can compose the maps
\[
\phi_{\pi,k'} \circ (\phi_{\pi,k})^{-1}
\]
to obtain the desired combinatorial bijection between \( \ast_k(\pi) \) and \( \ast_{k'}(\pi) \). We will often describe the inverses of our procedures, such as in Proposition 5.8, to give intuition about how to apply the above composition of maps.

In Section 2, we briefly establish notation that will be used in the duration of the paper. Section 3 characterizes elements of \( \ast_k(\pi) \), relying heavily on the work of [4]. It is also in this section that we define cycle enclosures and valid words. In Section 4, we introduce the class of trees which are crucial to our bijection, and which themselves are in bijection with the set of valid words via a map \( \omega \). In Section 5, we define the maps \( \text{cycle}_{\pi,k} \) and \( \text{tree}_{\pi,k} \), and ultimately the bijection \( \phi_{\pi,k} \). Finally, the paper concludes with Section 6 in which we give intuition for understanding the symmetry of Theorem 1.10. Throughout the paper, we use running examples to illustrate each of the definitions and operations.

2. Notation and terminology

Here we establish notation and terminology that will be used throughout the present work.

Fix positive integers \( n \geq k \), and \( \pi \in S_n \). Let \( \pi \) consist of \( m \) disjoint cycles, with lengths \( \ell_1, \ldots, \ell_m \) when read from left to right in standard form. The symbol \( k \) appears in the \( p \)-th of these cycles. (Note that \( k \) will play the role of the pivot, hence the index “\( p \)”.)

Each tree discussed in this work is ordered: it has a designated root node, and an ordering is specified for the children of each vertex.
3. Characterization of minimal transitive star factorizations

In this section we give a description of \( \ast_k(\pi) \). The interested reader is referred to \([4]\) for more information. While the discussions in \([4]\) assume that \( k = 1 \), the proofs of the results cited below can be extended to an arbitrary pivot value \( k \).

**Example 3.1.** Consider the permutation \( \pi = (142)(35)(6) \in \mathcal{S}_6 \). By Theorem 1.10

\[
\sigma_1(\pi) = \frac{(6 + 3 - 2)!}{6!} \cdot 3 \cdot 2 \cdot 1 = 42.
\]

The elements of \( \ast_3(\pi) \), that is, the 42 minimal transitive star factorizations with pivot 3 of \( \pi \), are given below.

\[
\begin{align*}
(35)(36)(31)(32)(34)(31) & \quad (35)(36)(36)(32)(34)(31)(32) & \quad (35)(36)(36)(34)(31)(32)(34) \\
(36)(35)(31)(32)(34)(31) & \quad (36)(36)(35)(32)(34)(31)(32) & \quad (36)(36)(35)(34)(31)(32)(34) \\
(36)(36)(31)(32)(34)(31) & \quad (36)(36)(36)(32)(34)(31)(32) & \quad (36)(36)(36)(34)(31)(32)(34) \quad \ldots
\end{align*}
\]

Note that, in some of these, the identity product \( (36)(36) = (1) \) is included. This is done so that the entire product factorization is transitive.

The following statements are easy to prove, and are discussed in \([4]\).

**Lemma 3.2** \((4)\).  
(a) The cycle

\[
(k \quad a_2 \quad a_3 \quad \cdots \quad a_\ell)
\]

admits exactly one minimal \( k \)-star factorization:

\[
(k \quad a_\ell)(k \quad a_{\ell-1}) \cdots (k \quad a_3)(k \quad a_2).
\]

(b) The cycle

\[
(b_1 \quad b_2 \quad \cdots \quad b_\ell),
\]

where \( b_i \neq k \) for all \( i \), admits \( \ell \) different minimal \( k \)-star factorizations:

\[
(k \quad b_\ell)(k \quad b_{i+\ell-1})(k \quad b_{i+\ell-2}) \cdots (k \quad b_{i+1})(k \quad b_i),
\]

where the subscripts are taken modulo \( \ell \).

It is helpful to introduce terminology to identify the different possibilities described in Lemma 3.2(b).
Definition 3.3. Suppose that the standard form of a permutation $\pi$ contains the cycle $C = (b_1 \ b_2 \cdots \ b_t)$, and suppose that $\delta \in \ast_k(\pi)$, with $k \neq b_j$ for all $j$. Let $i$ be such that $\delta$ contains the subword

$$(k \ b_i)(k \ b_{i+\ell-1})\cdots(k \ b_{i+1})(k \ b_i),$$

with subscripts taken modulo $\ell$. (Note that the transpositions in this subword do not necessarily appear consecutively in the factorization $\delta$, as shown in Example 3.5.) Then the cycle $C$ is enclosed by $b_i$.

Definition 3.4. Given a permutation $\pi$ and $\delta \in \ast_k(\pi)$, the cycle enclosures of $\delta$ are the set of letters that enclose all cycles in $\pi$ except the cycle containing $k$.

Example 3.5. Continuing our running example, consider $\delta = (35)(34)(31)(32)(36)(35)(34) \in \ast_3((142)(35)(6))$. Then (142) is enclosed by 4, and the cycle enclosures of $\delta$ are $\{4, 6\}$.

Before stating Lemma 3.8 which is crucial to the description of $\ast_k(\pi)$, we must make the following definition.

Definition 3.6. Given $\delta = (k \ \delta_1)(k \ \delta_2)\cdots(k \ \delta_{n+m-2}) \in \ast_k(\pi)$, define a word $\omega(\delta) \in [m]^{n+m-2}$ so that if $\delta_i$ appears in the $j$th cycle in the standard form of $\pi$, then the $i$th letter of $\omega(\delta)$ is $j$.

Example 3.7. Rewrite each $\delta \in \ast_3((142)(35)(6))$ from Example 3.1 as the word $\omega(\delta)$.

| 2331111 | 2331111 | 2331111 | 2113311 | 2113311 | 2113311 |
| 3321111 | 3321111 | 3321111 | 1133112 | 1133112 | 1133112 |
| 3311112 | 3311112 | 3311112 | 2111331 | 2111331 | 2111331 |
| 2311113 | 2311113 | 2311113 | 1113312 | 1113312 | 1113312 |
| 3111132 | 3111132 | 3111132 | 2111333 | 2111333 | 2111333 |
| 2133111 | 2133111 | 2133111 | 1111233 | 1111233 | 1111233 |
| 1331112 | 1331112 | 1331112 | 1111332 | 1111332 | 1111332 |

The repetition in this list is due to Lemma 3.2(b).

The following lemma completely characterizes the possible words $\omega(\delta)$ that may exist for $\delta$ a minimal transitive star factorization of a permutation $\pi$.

Lemma 3.8 (I). Let $\omega \in [m]^{n+m-2}$ be a word on $[m]$. There exists $\delta \in \ast_k(\pi)$ such that $\omega = \omega(\delta)$ if and only if the following statements hold for $\omega$:

- the symbol $p$ appears $\ell_p - 1$ times,
- the symbol $j$ appears $\ell_j + 1$ times for all $j \in [m]\{p\}$,
- the word $\omega(\delta)$ contains no subword $ijij$ for $i \neq j$, and
- the word $\omega(\delta)$ contains no subword $j pij$ for $j \neq p$.

Definition 3.9. If $\omega \in [m]^{n+m-2}$ satisfies the requirements of Lemma 3.8 that $\omega$ is a valid word for $\ast_k(\pi)$. Let $\mathcal{W}_k(\pi)$ be the set of valid words for $\ast_k(\pi)$.

In fact, the information contained in a minimal transitive star factorization with pivot $k$ is equivalent to the information contained in its cycle enclosures and its image under $\omega$. We now explain this precisely. Encoding a minimal transitive star factorization in this way will be key to the description of the bijection $\phi_{\pi,k}$. 
Definition 3.10. Let \((\ell'_1, \ldots, \ell'_{m-1}) = (\ell_1, \ldots, \ell_p, \ldots, \ell_m)\). Define a map
\[
\rho_{\pi,k} : W_k(\pi) \times [\ell'_1] \times \cdots \times [\ell'_{m-1}] \rightarrow *_k(\pi)
\]
as follows. Consider \((\omega, c_1, \ldots, c_{m-1}) \in W_k(\pi) \times [\ell'_1] \times \cdots \times [\ell'_{m-1}]\). Write the \(p\)th cycle in the standard form of \(\pi\) as \((k \ a_2 \ \cdots \ a_{\ell_p})\). Replace the \(\ell_p - 1\) copies of \(p\) in \(\omega\) by the star transpositions
\[
(k \ a_{\ell_p})(k \ a_{\ell_p-1}) \cdots (k \ a_3)(k \ a_2),
\]
in order.

For \(i < p\), suppose that the \(i\)th cycle of \(\pi\) when written in standard form is \((b_1 \ b_2 \ \cdots \ b_{\ell_i})\). Replace the \(\ell'_i + 1 = \ell_i + 1\) copies of \(i\) in \(\omega\) by the star transpositions
\[
(k \ b_{c_i})(k \ b_{c_i+\ell_i-1}) \cdots (k \ b_{c_i+1})(k \ b_{c_i}),
\]
in order, where the subscripts are taken modulo \(\ell_i\).

For \(i > p\), suppose that the \(i\)th cycle of \(\pi\) when written in standard form is \((b_1 \ b_2 \ \cdots \ b_{\ell_i})\). Replace the \(\ell'_{i-1} + 1 = \ell_i + 1\) copies of \(i\) in \(\omega\) by the star transpositions
\[
(k \ b_{c_{i-1}})(k \ b_{c_{i-1}+\ell_i-1}) \cdots (k \ b_{c_{i-1}+1})(k \ b_{c_i}),
\]
in order, where the subscripts are taken modulo \(\ell_i\).

This uniquely determines a minimal transitive star factorization with pivot \(k\), which we denote \(\rho_{\pi,k}(\omega, c_1, \ldots, c_{m-1})\).

Example 3.11. Take \(2111331 \in W_3((142)(35)(6)), 2 \in [3], \) and \(1 \in [1]\). Then
\[
\rho_{(142)(35)(6),3}(2111331, 2, 1) = (35)(34)(31)(32)(36)(36)(34).
\]

Proposition 3.12. The map \(\rho_{\pi,k}\) is a bijection.

Proof. The operation of \(\rho_{\pi,k}\) is easily reversible: take \(\delta \in *_k(\pi)\), let \(\omega = \omega(\delta)\), and define the \(c_i\) from the indices of the set of cycle enclosures of \(\delta\).

It is clear from each of Lemmas 3.2 and 3.8 that the cycle containing the pivot in a permutation \(\pi\) behaves differently with regard to elements of \(*_k(\pi)\). This emphasizes the unexpected nature of the symmetry in Theorem 1.10.

4. A CLASS OF TREES

In [4], a correspondence was given between \(*_1(\pi)\) and a particular class of trees. We will similarly utilize a graphical approach to explain the symmetry of Theorem 1.10. However, this is the extent of the similarity in approach between [4] and the current work: the details of our correspondence, and the trees themselves, differ from those in [4].

Definition 4.1. If a node in an ordered tree has any children, then it is a parent. If a nonempty ordered tree contains at most one parent (the root), then it is a sapling.

Example 4.2. Below are three examples of saplings.

We will work with a set \(T_k(\pi)\) of trees, defined here and later, equivalently, in Definition 5.3 (see Proposition 5.8).
Definition 4.3. Let $T_p$ be the sapling with $\ell_p$ nodes, where every leaf is labeled $p$. For $i \in [m]\{p\}$, let $T_i$ be the sapling with $\ell_i + 1$ nodes, where every leaf in the tree is labeled $i$.

We now describe the set $\mathcal{T}_k(\pi)$ of ordered trees specific to our work here.

Definition 4.4. Consider the following iterative procedure.

1. $T(0) = T_p$.
2. $T(j + 1)$ is obtained from $T(j)$ by taking some $T_i$ that has not already been added, and inserting it into $T(j)$ by making the root of $T_i$ a new child of some parent node in $T(j)$, and giving this root the label $i$.

Let $\mathcal{T}_k(\pi)$ consist of all possible $T(m - 1)$ so obtained.

Example 4.5. Consider $\pi = (142)(35)(6)$. The following two trees are elements of $\mathcal{T}_1(\pi)$, where here $p = 1$.

```
    1  3  1  2
   /  /  /  /  \
  3  2  2  1
```

The following two trees are elements of $\mathcal{T}_3(\pi)$, where now $p = 2$.

```
    3  2  1
   /  /  /  \
  3  1  1  1
```
```
    3  1
   /  /  \
  3  1  1
```

Each tree in the set $\mathcal{T}_k(\pi)$ corresponds to a word in $[m]^{n+m-2}$.

Definition 4.6. Given $T \in \mathcal{T}_k(\pi)$, we obtain a word $\omega(T)$ by reading the labels of the non-root nodes in the order seen via a depth-first search.

Example 4.7. Continuing Example 4.5, the first pair of trees, elements of $\mathcal{T}_1(\pi)$, map to the words 1331222 and 1223321 respectively, while the second pair of trees, elements of $\mathcal{T}_3(\pi)$, map to the words 3321111 and 3111132 respectively.

The use of the letter $\omega$ to denote the maps in both Definitions 3.6 and 4.6 is not coincidental.

Proposition 4.8. Given a word $\omega \in [m]^{n+m-2}$, we have $\omega = \omega(T)$ for some $T \in \mathcal{T}_k(\pi)$ if and only if $\omega = \omega(\delta)$ for some $\delta \in \mathcal{W}_k(\pi)$.

Proof. This follows from Lemma 3.8 and Definition 4.4.

In other words, Proposition 4.8 describes a bijection between $\mathcal{T}_k(\pi)$ and $\mathcal{W}_k(\pi)$. Although phrased in different terms, the set of valid words for $\mathcal{W}_k(\pi)$ is enumerated in [4]:

$$|\mathcal{T}_k(\pi)| = |\mathcal{W}_k(\pi)| = (n + m - 2)_{(m-2)}\ell_p.$$
5. Bijective construction of minimal transitive star factorizations

In this section, we use the characterization of $\star_k(\pi)$ of Proposition 3.12 to give a map
\[ \phi_{\pi,k} : \{[m-2] \leftrightarrow [n+m-2]\} \times [\ell_1] \times \cdots \times [\ell_m] \rightarrow \star_k(\pi), \]
where \( \{X \leftrightarrow Y\} \) denotes the set of injections from \( X \) into \( Y \). More precisely, we will obtain an element of \( \mathcal{W}_k(\pi) \) and a set of cycle enclosures, which, by Proposition 3.12, define a unique element of \( \star_k(\pi) \). This element will be the image of \( \phi_{\pi,k}(x) \).

We will show that the map \( \phi_{\pi,k} \) is a bijection for every \( k \), thus obtaining a bijection from the set \( \star_k(\pi) \) to the set \( \star_{k'}(\pi) \): the composition of maps
\[ \phi_{\pi,k'} \circ (\phi_{\pi,k})^{-1}. \]

It is easiest to define the map \( \phi_{\pi,k} \) via two preliminary operations.

**Definition 5.1.** Let \((\ell'_1, \ldots, \ell'_{m-1}) = (\ell_1, \ldots, \ell_p, \ldots, \ell_m)\). Fix \((c_1, \ldots, c_{m-1}) \in [\ell'_1] \times \cdots \times [\ell'_{m-1}]\), and \( i \in [m] \setminus \{p\} \). If the \( i \)th cycle of \( \pi \) in standard form is \((b_{i,1}, b_{i,2}, \ldots, b_{i,\ell_i})\), then set
\[ \text{cycle}_{\pi,k}(c_1, \ldots, c_{m-1}) = \{b_{i,c_i} : i < p\} \cup \{b_{i,c_{i-1}} : i > p\}. \]

We now compute \( \text{cycle} \) for our running example, as well as for a more complicated example that we will similarly examine throughout this section.

**Example 5.2.** We have \( \text{cycle}_{(142)(35)(6),3}(2,1) = \{4, 6\} \) because the second symbol in \((142)\) is 4 and the first symbol in \((6)\) is 6.

**Example 5.3.** We have \( \text{cycle}_{(18)(297)(3)(46),5}(1,1,2,1) = \{1, 3, 5, 6\} \) because the first symbol in \((18)\) is 1, the first symbol in \((3)\) is 3, the second symbol in \((46)\) is 6, and the first symbol in \((5)\) is 5.

The second operation, \( \text{tree}_{\pi,k} \), is rather more complex. The idea is to take an injection \( f : [m-2] \leftrightarrow [n+m-2] \) and a value \( c \in [\ell_p] \), and to reinterpret them as a particular tree. In this tree, all nodes except the root will have been labeled by values in \([m]\).

**Definition 5.4.** Fix \((f, c) \in \{[m-2] \leftrightarrow [n+m-2]\} \times [\ell_p]\). We now outline the procedure for producing the tree \( \text{tree}_{\pi,k} \).

1. Label \([n+m-2] \backslash f([m-2])\), in increasing order, by “1,” \ldots, “2,” \ldots, “m,” \ldots, where each \( i \) appears \( \ell_i \) times.
2. Label the elements in the set \( f([m-2]) \) by the labels “\( f(i) \)” as appropriate.
3. Change the \( c \)th occurrence of “\( p \)” to “\( f(0) \)”.
4. Create \( m \) factors in \([n+m-2]\) by inserting \( m-1 \) bars: just after the rightmost “\( i \)” for each \( i < p \) and just before the leftmost “\( i \)” for each \( i > p \).
5. Create a sapling \( T_i \) with leaves labeled as in the \( i \)th factor. Any leaves labeled by \( f \) are hooks, and the \( T_i \) are ornaments.
6. Let \( T(0) = T_p \).
7. For \( i \in [m-2] \), if the ornament \( T_j \) containing “\( f(i) \)” has not yet been attached in \( T(i-1) \), then attach it to \( T(i-1) \) by identifying its root with the hook “\( f(i-1) \)” (called using the hook), and label the identified node “\( j \)” otherwise take no action; the resulting tree is \( T(i) \).
There remains at least 1 unattached ornament and at least 1 unused hook. The attaching process and the fact that there were equally many ornaments and hooks at the start of the process means that there are the same number of unattached ornaments as unused hooks. Let these be $T_1, \ldots, T_r$ and $f(h_1), \ldots, f(h_r)$, in increasing order of subscripts. For each $j$, attach the root of $T_{ij}$ to the hook “$f(h_j)$” as before. The result is $\text{tree}_{\pi,k}(f, c)$.

We demonstrate Definition 5.4 with two examples.

**Example 5.5.** Continuing Example 5.2, take $(142)(35)(6) \in \mathcal{S}_6$, where $n = 6$ and $m = 3$. Let $k = 3$, so $p = 2$. Let $f \in \{[1] \leftrightarrow [7]\}$ be defined by $f(1) = 3$, and let $c = 2$. Definition 5.4 produces the following work, where we represent the initial interval $[7]$ as a sequence of dots.

![Tree Diagram](tree_diagram.png)

**Example 5.6.** Continuing Example 5.3, take $(18)(297)(3)(46)(5) \in \mathcal{S}_9$, where $n = 9$ and $m = 5$. Let $k = 9$, so $p = 2$. Let $g \in \{[3] \leftrightarrow [12]\}$ be defined by $g(1) = 3$, $g(2) = 1$, and $g(3) = 12$, and let $c = 1$. Definition 5.4 produces the following work, where we represent the initial interval $[12]$ as a sequence of dots.

![Tree Diagram](tree_diagram.png)
Lemma 5.7. For \( f \in \{[m-2] \rightarrow [n+m-2]\} \) and \( c \in [\ell_p]\), we have \( \text{tree}_{\pi,k}(f,c) \in \mathcal{T}_k(\pi) \).

Proof. We first must show that \( \text{tree}_{\pi,k}(f,c) \) is a tree. What needs to be shown is that the last step, where unused hooks and unattached ornaments are identified, produces a tree. At that stage, an ornament has not yet been attached to the tree if and only if none of its children are hooks. Thus any unused hooks must appear in \( T(m-2) \), so attaching hooks and ornaments as described does not create any cycles.

That this tree is an element of \( \mathcal{T}_k(\pi) \) follows from Definition 5.4. \( \square \)

Proposition 5.8. The map \( \text{tree}_{\pi,k} \) is a bijection.

Proof. We will show that the map is reversible.

Fix \( T \in \mathcal{T}_k(\pi) \). This tree can be decomposed into \( m \) saplings based on the labels of the nodes. Thus each sapling is associated with a value in \( [m] \). Call this the rank of the sapling. Let \( S_i \) be the sapling of rank \( i \). (Note that the \( m \) saplings can equivalently be identified by taking each parent together with all of its children that are leaves.)

If, in \( T \), the root of the sapling \( S \) is the parent of the root of the sapling \( S' \), then \( S \) shelters \( S' \) and write \( S > S' \). If a sapling shelters nothing, then it is free. Every sapling is sheltered by exactly one other sapling, except for \( S_p \), which is sheltered by nothing.

Make maximal sequences \( S_{j,1} > S_{j,2} > \ldots > S_{j,r_j} \); that is, \( S_{j,1} = S_p \) for all \( j \), and all \( S_{j,r_j} \) are free. Index the free saplings by \( j \), so that their ranks are increasing with respect to \( j \).
Let the roots of
\[ S_{1,2}, S_{1,3}, \ldots, S_{1,r_1}, S_{2,2}, S_{2,3}, \ldots, S_{2,r_2}, S_{3,2}, S_{3,3}, \ldots, S_{3,r_3}, \ldots \]
be relabeled \( f(0), f(1), \ldots, f(m-2) \) respectively, with the provision that once a root has
been labeled, the associated sapling is skipped when allocating the subsequent labels. Let \( T' \) be the result ing ordered tree after changing these \( m-1 \) labels in \( T \).

Now look in \( T' \) at the \( m \) saplings, and recall the ranks of each sapling as defined at the outset. Working in increasing order of rank, write down the labels (as designated by \( T' \)) of all of the children of the root of each sapling from left to right (note that this includes non-leaf children). Upon completion, we have written down \( \phi \) of the children of the root of each sapling from left to right (note that this includes non-leaf children). Therefore we have written down \( \omega \) of the children of the root of each sapling from left to right (note that this includes non-leaf children). Upon completion, we have written down \( \phi \) of the children of the root of each sapling from left to right (note that this includes non-leaf children). Therefore we have written down \( \omega \) of the children of the root of each sapling from left to right (note that this includes non-leaf children).

Definition 5.9. Consider \( (f, c_1, \ldots, c_m) \in \{[m-2] \leftrightarrow [n + m - 2] \} \times [\ell_1] \times \cdots \times [\ell_m] \). Let \( \phi_{\pi,k}(f, c_1, \ldots, c_m) \) be the element of \( \pi,k \) obtained, using Proposition 3.12 from the valid word \( \omega(\text{tree}_{\pi,k}(f, c_p)) \) and the set of cycle enclosures \( \text{cycle}_{\pi,k}(c_1, \ldots, c_p, \ldots, c_m) \).

Example 5.10. With the values determined in Examples 5.2 and 5.5 we have
\[ \phi_{(142)(35)(6),3}(f, 2, 2, 1) = (35)(34)(31)(32)(36)(36)(34) \in \pi,(142)(35)(6)) \]  

Example 5.11. With the values determined in Examples 5.3 and 5.6 we have
\[ \phi_{(18)(297)(3)(46)(5),9}(g, 1, 1, 1, 2, 1) = (91)(95)(95)(96)(94)(96)(98)(93)(93)(91)(92)(97) \]  
\[ \in \pi,((18)(297)(3)(46)(5)) \]  

Theorem 5.12. For any \( k \in [n] \), the map \( \phi_{\pi,k} \) is a bijection.

Proof. The sets \( \{[m-2] \leftrightarrow [n + m - 2] \} \times [\ell_1] \times \cdots \times [\ell_m] \) and \( \pi,k \) have the same cardinality, by Theorem 1.10. The map \( \text{cycle}_{\pi,k} \) is certainly injective, and \( \text{tree}_{\pi,k} \) is a bijection, by Proposition 5.8. The correspondence between pairs of valid words and cycle enclosures and elements of \( \pi,k \) is a bijection, by Proposition 3.12.

Therefore \( \phi_{\pi,k} \) is a bijection. \( \square \)

6. Conclusion

Theorem 5.12 yields the desired combinatorial bijection between
\[ \{[m-2] \leftrightarrow [n + m - 2] \} \times [\ell_1] \times \cdots \times [\ell_m] \]  
and \( \pi,k \). It also explains the symmetry of Theorem 1.10. That is, note that by Lemma 3.8 no occurrence of \( p \) in a valid word can sit between two occurrences of \( j \neq p \). Thus, if there are any other symbols appearing in a valid word (that is, if \( m > 1 \)), then the “first” of these must appear in one of the \( \ell_p \) spots between or outside of the \( \ell_p - 1 \) occurrences of \( p \). Choose \( c_i \in [\ell_i] \) for each \( i \). Then \( c_p \) determines where this “first” symbol appears relative to the \( ps \) in the valid word. This information, together with the element of \( \{[m-2] \leftrightarrow [n + m - 2] \} \) yields the valid word via \( \text{tree}_{\pi,k} \). For \( i \neq p \), the \( c_i \) determines the cycle enclosure of the \( i \)th
cycle. This explains that the role of the pivot \( k \) affects only in what way the value \( c_p \in [\ell_p] \) is interpreted by the bijection.

Thus, we have obtained a bijection

\[
\phi_{\pi,k'} \circ (\phi_{\pi,k})^{-1} : \ast_k(\pi) \rightarrow \ast_{k'}(\pi).
\]

Finally, we demonstrate this bijection using the ongoing example of this article.

**Example 6.1.** Let \( k = 3, \pi = (142)(35)(6) \), and \( k' = 1 \). Let us find the element of \( \ast_{k'}(\pi) \) corresponding to \( (35)(34)(31)(32)(36)(36)(34) \in \ast_k(\pi) \). From Example 5.10 we know that

\[
(\phi_{\pi,k})^{-1} ((35)(34)(31)(32)(36)(36)(34)) = (f : 1 \mapsto 3, 2, 2, 1).
\]

Now we apply \( \phi_{\pi,k'} \) to \( (f : 1 \mapsto 3, 2, 2, 1) \), obtaining

\[
\phi_{\pi,k'} \circ (\phi_{\pi,k})^{-1} ((35)(34)(31)(32)(36)(36)(34)) = (12)(15)(13)(15)(16)(16)(14) \in \ast_{k'}(\pi).
\]

**References**

[1] J. Dénès, The representation of a permutation as the product of a minimal number of transpositions and its connection with the theory of graphs, *Publ. Math. Institute Hung. Acad. Sci.* 4 (1959), 63–70.

[2] V. Féray, Partial Jucys-Murphy elements and star factorizations, to appear in *European J. Combin.*

[3] I. P. Goulden and D. M. Jackson, Transitive powers of Young-Jucys-Murphy elements are central, *J. Algebra* 321 (2009), 1826–1835.

[4] J. Irving and A. Rattan, Minimal factorizations of permutations into star transpositions, *Discrete Math.* 309 (2009), 1435–1442.

[5] V. Ivanov and S. Kerov, The algebra of conjugacy classes in symmetric groups, and partial permutations, *J. Math. Sci. (New York)* 107 (2001), 4212–4230.

[6] I. Pak, Reduced decompositions of permutations in terms of star transpositions, generalized Catalan numbers and \( k \)-ary trees, *Discrete Math.* 204 (1999), 329–335.

[7] R. P. Stanley, On the number of reduced decompositions of elements of Coxeter groups, *European J. Combin.* 5 (1984), 359–372.

Department of Mathematical Sciences, DePaul University, Chicago, IL 60614

E-mail address: bridget@math.depaul.edu