Lower bounds for pseudodifferential operators with a radial symbol.

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ABSTRACT. In this paper we establish explicit lower bounds for pseudodifferential operators with a radial symbol. The proofs use classical Weyl calculus techniques and some useful, if not celebrated, properties of the Laguerre polynomials.

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1. Introduction.

If a function $F$ defined on $\mathbb{R}^{2d}$ is smooth and has bounded derivatives, the Weyl calculus associates with it a pseudodifferential operator $\operatorname{Op}_h^{Weyl}(F)$ which is bounded on $L^2(\mathbb{R}^d)$ and satisfies, for all $f$ and $g$ in $\mathcal{S}(\mathbb{R}^d)$,

\begin{equation}
< \operatorname{Op}_h^{Weyl}(F) f, g > = (2\pi h)^{-d} \int_{\mathbb{R}^{2d}} F(Z) H_h(f, g, Z) dZ,
\end{equation}

where $H_h(f, g, \cdot)$ is the Wigner function

\begin{equation}
H_h(f, g, Z) = \int_{\mathbb{R}^d} e^{-\frac{i}{h} \cdot \zeta} f \left( z + \frac{t}{2} \right) g \left( z - \frac{t}{2} \right) dt Z = (z, \zeta) \in \mathbb{R}^{2d}.
\end{equation}

For this form of the definition, see [U], [L] or [C-R], Chapter II, Proposition 14.

The different variants of Gårding’s inequality prove that, if $F \geq 0$, the operator $\operatorname{Op}_h^{Weyl}(F)$ is roughly $\geq 0$. More precisely, according to the classical Gårding’s inequality (see [HO] or [L]), the non negativity of $F$ implies the existence of a positive constant $C$, independent of $h$, such that, for all sufficiently small $h$ and for all $f$ in $\mathcal{S}(\mathbb{R}^d)$:

\begin{equation}
< \operatorname{Op}_h^{Weyl}(F) f, f > \geq -Ch\|f\|_{L^2(\mathbb{R}^d)}^2.
\end{equation}

See [L-N] for other similar results. This inequality holds for systems of operators, whereas the more precise Fefferman-Phong inequality [F-P] is valid only for scalar operators. Fefferman-Phong’s inequality states that, under the same hypotheses as Gårding’s inequality, one has, for all $h$ in $(0,1)$ and all $f$ in $\mathcal{S}(\mathbb{R}^d)$:

\begin{equation}
< \operatorname{Op}_h^{Weyl}(F) f, f > \geq -Ch^2\|f\|_{L^2(\mathbb{R}^d)}^2.
\end{equation}

See [MAR] for these semiclassical versions. Sometimes the non negativity of $F$ implies the exact non negativity of the operator, for example in the simple case when $F$ depends on $x$ or on $\xi$ only. It is possible, too, to apply Melin’s inequality. To take only one example, let $F \geq 0$ attain its minimum only once, for a nondegenerate critical point. In this case (and in other analogous situations), Melin’s inequality ensures the exact non negativity of $\operatorname{Op}_h^{Weyl}(F)$ for a sufficiently small $h$. See [B-N] or [L-L] for cases when the difference between $F(x, \xi)$ and its minimum is equivalent to a power, greater than 2, of the distance between $(x, \xi)$ and the unique point where the minimum is attained.

In this article we are interested in the case when $F$ is radial. We assume that there exists a function $\Phi$ defined on $\mathbb{R}$ such that

\begin{equation}
F(x, \xi) = \Phi(|x|^2 + |\xi|^2) \quad (x, \xi) \in \mathbb{R}^{2d}.
\end{equation}
Moreover, we suppose that $\Phi$ is nondecreasing on $[0, \infty)$ and such that $F$ is smooth, with bounded derivatives. In this case, we aim at giving an explicit lower bound on the spectrum of the operator $Op_h^W(F)$. The main result of this paper is the following theorem.

**Theorem 1.1** Let $F$ be a smooth function defined on $\mathbb{R}^{2d}$, bounded as well as all its derivatives. Assume that $F$ is of the form (1.5), where $\Phi$ is a non decreasing function defined on $[0, \infty)$.

Then for all $f$ in $S(\mathbb{R}^d)$,

\begin{equation}
< Op_h^W(F) f, f > \geq \frac{1}{h} \int_0^\infty \Phi(t)e^{-\frac{t}{h}} dt \| f \|^2_{L^2(\mathbb{R}^d)}.
\end{equation}

**Remarks**
1 - We do not need to assume that $\Phi \geq 0$ to ensure the non negativity of the operator. The non negativity of the integral suffices.
2 - In the case when $\Phi$ is not flat at the origin, let $m \geq 1$ be the smallest integer for which $\Phi^{(m)}(0) \neq 0$.

Then one can see that

\[ \frac{1}{h} \int_0^\infty \Phi(t)e^{-\frac{t}{h}} dt = \Phi(0) + \Phi^{(m)}(0)h^m + O(h^{m+1}). \]

3 - The result can be applied to symbols $F$ depending on the distance from another point $(x_0, \xi_0)$ for, if $\tau F(x, \xi) = F(x + x_0, \xi + \xi_0)$ and $Tf(u) = e^{i(\xi_0/h)(u-x_0)}f(u-x_0)$, then

\[ < Op_h^W(\tau F) f, g > = < Op_h^W(F)Tf, Tg > . \]

We are greatly indebted to N. Lerner for the reference [A-G].

2. **Proof of Theorem 1.1.**

We denote by $(H_n)_{n \geq 0}$ the sequence of the Hermite functions. It is a Hermitian basis of $L^2(\mathbb{R})$, satisfying

\begin{equation}
(D^2 + x^2)H_n = (2n + 1)H_n.
\end{equation}

For each multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$, we set:

\begin{equation}
u_\alpha(x) = \prod_{j=1}^d H_{\alpha_j}(x_j).
\end{equation}

These functions form a Hermitian basis of $L^2(\mathbb{R}^d)$.

We shall need the Laguerre polynomials as well, which are defined by

\begin{equation}L_n(x) = e^x \frac{d^n}{dx^n} (x^n e^{-x}).\end{equation}

One has:

\begin{align}
L_0(x) &= 1 & L_1(x) &= 1 - x & L_2(x) &= \frac{x^2}{2} - 2x + 1.
\end{align}

Theorem 1.1 is a consequence of the following proposition, in which the parameter $h$ is equal to 1 and the Weyl operator $Op_1^W(F)$ is denoted by $Op^W(F)$.
Proposition 2.1 Under the hypotheses of Theorem 1.1 one has, for all multi-indices $\alpha$ and $\beta$ such that $\alpha \neq \beta$:

\begin{equation}
\langle Op^{Weyl}(F)u_{\alpha}, u_{\beta} \rangle = 0.
\end{equation}

For each multi-index $\alpha$:

\begin{equation}
\langle Op^{Weyl}(F)u_{\alpha}, u_{\alpha} \rangle = 2^{-d} \left[ \Phi(0)V_{\alpha}(0) + \frac{1}{2} \int_0^{\infty} \Phi'(t/2)V_{\alpha}(t)dt \right],
\end{equation}

with

\begin{equation}
V_{\alpha}(X) = 4e^{-\frac{\pi}{2} \sum_{k=0}^{d-1} C_{d-1}^{k}T_{|\alpha|+k}(X)},
\end{equation}

where we set, for all integer $n$,

\begin{equation}
T_n(X) = \left[ \sum_{k=0}^{n-1} (-1)^k L_k(X) \right] + \frac{(-1)^n}{2} L_n(X).
\end{equation}

Proof of (2.5). Let $\alpha$ and $\beta$ be two different multi-indices and let $j \leq d$ be such that $\alpha_j \neq \beta_j$. Set $P_j = D_j^2 + x_j^2$. According to (2.1) we have:

\[2(\alpha_j - \beta_j) < \langle Op^{Weyl}(F)u_{\alpha}, u_{\beta} \rangle = \langle Op^{Weyl}(F)P_ju_{\alpha}, u_{\beta} \rangle - \langle Op^{Weyl}(F)u_{\alpha}, P_ju_{\beta} \rangle.\]

The fact that $F$ is radial implies that $x_j \frac{\partial F}{\partial x_j} - \xi_j \frac{\partial F}{\partial x_j} = 0$ which, in turn, implies that $Op^{Weyl}(F)$ and $P_j$ commute, thanks to properties of the Weyl calculus. Consequently, the right term of the above inequality is equal to 0, which proves (2.5).

Proof of (2.6). For each multi-index $\alpha$, the Wigner function $H(u_{\alpha}, u_{\alpha})$ (where the parameter $h$, equal to 1, is omitted), satisfies:

\begin{equation}
H(u_{\alpha}, u_{\alpha})(x, \xi) = 2^d(-1)^{|\alpha|}e^{-((|x|^2+|\xi|^2)/2)} \prod_{j=1}^{d} L_{\alpha_j}(2(x_j^2 + \xi_j^2)).
\end{equation}

See, for example, [FO] or [J-L-V]. Hence, if $F$ is as in Theorem 1.1,

\[\langle Op^{Weyl}(F)u_{\alpha}, u_{\alpha} \rangle = (2\pi)^{-d}2^{-d}(-1)^{|\alpha|} \int_{\mathbb{R}^{2d}} \Phi(|x|^2 + |\xi|^2)e^{-((|x|^2+|\xi|^2)/2)} \prod_{j=1}^{d} L_{\alpha_j}(2(x_j^2 + \xi_j^2))dx d\xi.
\]

The change of variables $t_j = 2(x_j^2 + \xi_j^2)$ allows to write:

\[\langle Op^{Weyl}(F)u_{\alpha}, u_{\alpha} \rangle = (2\pi)^{-d}2^{-d}(-1)^{|\pi/2|} \int_{(0,\pi)\times \mathbb{R}^{d}} \Phi((t_1 + \cdots + t_d)/2)e^{-\frac{1}{2}(t_1+\cdots+t_d)} \prod_{j=1}^{d} L_{\alpha_j}(t_j)dt_1...dt_\alpha.
\]

This equality can be written as

\[\langle Op^{Weyl}(F)u_{\alpha}, u_{\alpha} \rangle = (2\pi)^{-d}2^d(\pi/2)^d \int_{0}^{\infty} \Phi(X/2)U_{\alpha}(X)dX,
\]
with:

\[ U_{\alpha}(X) = (-1)^{|\alpha|} e^{-\frac{X}{2}} \int_{\Omega_d(X)} L_{\alpha_d}(X - t_1 - \ldots - t_{d-1}) \prod_{j=1}^{d-1} L_{\alpha_j}(t_j) dt_1 \ldots dt_{d-1}, \]

where

\[ \Omega_d(X) = \{(t_1, \ldots, t_{d-1}), \quad t_j > 0, \quad t_1 + \ldots + t_{d-1} < X \}. \]

The equality (2.6) will be a consequence of an integration by parts using the following lemma.

**Lemma 2.2** We have:

\[
(2.10) \quad U_{\alpha}(X) = -V'_{\alpha}(X)
\]

where \( V_{\alpha} \) is defined by (2.7) and (2.8).

**Proof of Lemma 2.2.** One knows (cf [M-O-S], section 5.5.2) that

\[
(2.11) \quad \int_0^X L_{\alpha_1}(t)L_{\alpha_2}(X - t)dt = L_{\alpha_1 + \alpha_2}(X) - L_{\alpha_1 + \alpha_2 + 1}(X).
\]

It follows, by induction on \( d \), that

\[
\int_{\Omega_d(X)} L_{\alpha_d}(X - t_1 - \ldots - t_{d-1}) \prod_{j=1}^{d-1} L_{\alpha_j}(t_j) dt_1 \ldots dt_{d-1} = \sum_{k=0}^{d-1} C_{d-1}^k (-1)^k L_{|\alpha|+k}(X).
\]

Hence

\[
U_{\alpha}(X) = e^{-\frac{X}{2}} \sum_{k=0}^{d-1} C_{d-1}^k (-1)^k L_{|\alpha|+k}(X).
\]

Using the recurrence relation \( L'_{k+1}(t) = L'_k(t) - L_k(t) \), we prove (for example by induction) that for all integer \( n \):

\[
\frac{d}{dt} e^{-\frac{t}{2}} T_n(t) = \frac{(-1)^{n+1}}{4} L_n(t) e^{-\frac{t}{2}}.
\]

The equality (2.10) of the Lemma follows from (2.7) and from the above identities.

**End of the proof of Theorem 1.1.** We shall begin by proving (1.6) for \( h = 1 \). Set

\[
(2.12) \quad S_n(X) = \sum_{k=0}^n (-1)^k L_k(X).
\]

Using the recurrence relation \( L'_{k+1}(t) = L'_k(t) - L_k(t) \), one verifies, by induction, that for all \( n \):

\[
T'_n(X) = \frac{1}{2} S_{n-1}(X).
\]

Since \( L_n(0) = 1 \) for all \( n \), we see that \( T_n(0) = 1/2 \) and that

\[
T_n(X) = \frac{1}{2} + \frac{1}{2} \int_0^X S_{n-1}(t) \, dt.
\]

According to [A-G], Theorem 12 (see [F] as well), \( S_n(X) \geq 0 \) for all \( n \geq 0 \) and for all \( X \geq 0 \). Therefore \( T_n(X) \geq 1/2 \) for all \( n \) and \( X \), and, using (2.7):

\[
(2.13) \quad V_{\alpha}(X) \geq 2^d e^{-\frac{X}{2}}.
\]
Since $T_n(0) = 1/2$, $V_\alpha(0) = 2^d$. Hence, if $\Phi' \geq 0$, one gets:

\begin{equation}
\Phi(0)V_\alpha(0) + \frac{1}{2} \int_0^\infty \Phi'(t/2)V_\alpha(t)dt \geq 2^d \int_0^\infty \Phi(t)e^{-t}dt.
\end{equation}

The inequality (1.6), for $h = 1$, follows from (2.5), (2.6) and (2.14). For an arbitrary $h > 0$, it suffices to apply the above result to the function $F_h(x, \xi) = F(h^{1/2}x, h^{1/2}\xi)$, that is to say, to the function $\Phi_h(t) = \Phi(ht)$.

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