DELAYED PAYMENT POLICY IN MULTI-PRODUCT SINGLE-MACHINE ECONOMIC PRODUCTION QUANTITY MODEL WITH REPAIR FAILURE AND PARTIAL BACKORDERING

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Abstract. This study develops a single-machine manufacturing system for multi-product with defective items and delayed payment policy. Contradictory to the literature limited production capacity and partial backlogging are considered for more realistic result. The objective of this research is to obtain the optimal cycle length, optimal production quantity, and optimal backorder quantity of each product such that the expected total cost is minimum. The model is solved analytically. Three efficient lemmas are developed to obtain the global optimum solution of the model. An improved algorithm is designed to obtain the numerical solution of the model. An illustrative numerical example and sensitivity analysis are provided to show the practical usage of proposed method.

1. Introduction and literature review. The basic concept of any production model is that minimise the total cost, whereas the profit is maximum. Any company or industry just want to invest less and gain more profit, reminding this many EPQ models are developed by researchers. The concept of EPQ model was introduced by Taft [36]. Customer demand had a great impact in any inventory system, was briefly describes by Scarf [35], where they considered a known distribution for the demand function. When demand is maximum, casually profit also maximum and to increases the demand or profit in an EOQ model, delay-in-payment is one of the best policy, investigated by Goyal [6, 7]. A different technique for optimization, when backorder takes places is proposed by Cárdenas-Barrón [1].

In any production system can’t produced good quality items always, some time
they produced faulty products considering this Salameh and Jaber [20] developed an inventory model, where machine produced imperfect quality items. Goyal and Cárdenas-Barrón [8] introduced an efficient solution procedure for imperfect production. An imperfect production process for stock-dependent demand is discussed by Sarkar et al. [31]. In the same direction Sarkar et al. [32] developed a manufacturing model where demand is time depended. All of the above studies about EPQ models with imperfect production processes focused on determining the optimal production lot size. The issue that the imperfect quality items can be reworked was ignored. It is well-known that the total production inventory costs can be reduced by reworking the imperfect quality items produced with a relatively smaller additional reworking and holding costs. Teng et al. [49] studied an inventory model with time-varying demand and cost. Sana et al. [21] developed a production inventory model where the production system produces both perfect and imperfect items. More recently, Sarkar et al. [27] improved a supply chain model, where quality of product is improved and setup cost was reduced. Many researcher discussed different method for imperfect production process but interior penalty function method for maximize the total profit function is not discussed yet, which is explained in this paper.

An inventory model with service level constraint was discussed by Huang [9], where defective rate is random. Chiu et al. [4] studied a production model with scrap, rework, and stochastic machine breakdowns. Their inventory model determines both the optimal run time and production lot size. Chiu [3] developed an optimal solution for the same problem where no information of derivate was needed. Li et al. [12] studied an inventory model with planned backorders to evaluate the impact of the postponement strategy of the manufacturer in a supply chain. Pentico et al. [19] developed an inventory model with partial backordering when production lot size and period length are also considered. Yoo et al. [54] considered an imperfect production model with two way inspection policy. Furthermore, the products received usually contain some defective items. Sarkar et al. [30] studied an imperfect production model to determine the production lot size, safety stock, and reliability parameter with machine breakdown. Taleizadeh et al. [48] developed a multi-product production model with failure and rework when partial backordering exists. The immediate rework process in any production system has great impact to reduce the total cost of the system Taleizadeh et al. [39]. The fuzzy goal programming is use to solve a multi-period, multi-objective and multi product production model by Damghani and Shahrokh [5]. Sarkar and Moon [29] considered a production inventory model with stochastic demand and inflation. They derived the profit function by using both general distribution of demand and the uniform rectangular distribution of demand. Sarkar [30] considered an economic manufacturing quantity model with price and advertising demand pattern in an imperfect production process under the effect of inflation. In two subsequent research works, Wee et al. [51] extended a multi-product single machine inventory model with multiple batch sizes. Tai [37] proposed two deteriorating inventory models with rework process. Cárdenas-Barrón et al. [2] developed an easy an improved algorithm for joint determination of lot size and number of shipments in an EPQ model with rework for defective items. Pasandideh et al. [18] extended a bi-objective multi-product EPQ model, where the number of orders was limited and imperfect items were produced. The objective of the problem was minimization of the total inventory costs as well as minimizing the required warehouse space. Sarkar et al. [25] revisited the EPQ
model with rework process at a single-stage manufacturing system with planned backorders. They extended an inventory model to allow random defective rates. Basically, three different inventory models are developed for three different distribution density functions such as uniform, triangular, and beta. Variable backorder and inspection for imperfect products with discount policy in a supply chain model is developed by Sarkar [24]. Kang et al. [10] explained how random defective rate effect on inventory model. Machine breakdown also has a great impact on imperfect production process Taleizadeh et al. [43]. All researchers discussed about imperfect production, quality of product, inspection etc. but failure in repair of imperfect product with multi-product production in a single machine still not discussed by anyone.

Now-a-days time is one of the most valuable parameter for each business sector. Each one want to get their ordered product as early as possible. Some time supplier/retailer unable to deliver the required products to buyers/customer, in this situation buyer can wait for the product or he/she can take it from other supplier/retailer. The 1st case known as partial backlogging and 2nd case as fully backlogged, fully backlogging always harmful for suppliers/retailers. Many researchers developed so many models where backorder’s effect on the total cost or profit has a valuable impact Widyadana et al. [52], Pan and Hsiao [17], Ouyang et al. [16]. Ouyang et al. [15] mixed inventory model with backorder, when lead time is variable. Effect of partial backlogging for stock-dependent demand is examined by Sarkar and Sarkar [34]. Price discount for backorder had a great impact in any inventory system discussed by Sarkar et al. [28], Taleizadeh [38], Taleizadeh and Pentico [45], and Taleizadeh et al. [44], [40], [46], [47], [42], [41] developed different production model and inventory model with backorder. In today’s competitive business market one of the most effective policy is delay-in-payment policy. In this policy supplier/retailer always benefited from buyer/customer. By this policy supplier/retailer save holding cost and also profit was maximised. This delay-in-payment may be partially or fully. Trade-credit always reduce the total cost of supplier/retailer, when items may be deteriorate Sarkar and Saren [33]. Thus, both two-level delay-in-payments and single-level delay-in-payments are equally important based on different products of several sectors. In this direction, Teng and Chung [50] considered a manufacturing model under two levels of trade-credit policy to optimize the production quantity and period length. Sarkar [23] developed an excellent idea of variable deterioration with delay-in-payments policy. Ouyang and Chang [13] studied about the business transactions that the supplier usually offers a permissible delay in payment to his retailer to attract more sales. In addition, a permissible delay-in-payments may be applied as an alternative to price-discount. Based on the above phenomena, they incorporated a permissible delay-in-payments into the inventory model and developed a two-warehouse partial backlogging model for deteriorating items with permissible delay-in-payments under inflation. The objective of this study was to derive the retailers optimal replenishment policy that maximizes the net present value of the profit per unit time. Ouyang et al. [14] studied about a comprehensive extension of the optimal replenishment decisions under two levels of trade credit policy depending on the order quantity. Several researchers develop their inventory models under trade-credit policy by assuming that the supplier offers the retailer fully permissible delay in payments and the products received are all non-defective. In this direction, Sarkar et al. [26] developed an integrated inventory model with variable lead time, defective units, and delay-in-payments. More recently Kim and
Sarkar [11] discussed how trade-credit effect in supply chain model when lead time is stochastic, and they also proved that trade-credit reduced the total system cost. However, from the viewpoint of practice, it can often be found that the supplier offers the retailer a fully permissible delay in payments only when the order quantity is greater than or equal to the specific quantity. From the existing literature it is clear that several researcher developed many EPQ and EOQ model, where they considered imperfect production process and rework, delay-in-payment policy, multi-item producing through single machine, backorder separately, but all these in a single model with repair failure still not considered by anyone, so this is a big research gap which try to fulfil by this research. Major contribution by different author(s) are shown in the Table 1.

| Author(s)                | EPQ | Imperfect Production | Multi-product | Delay-in-payment | Backorder | Repair |
|--------------------------|-----|----------------------|---------------|------------------|-----------|--------|
| Cárdenas-Barrón et al. [2]| √   |                      |               |                  |           |        |
| Chiu et al. [4]          | √   |                      |               |                  |           |        |
| Goyal and Cárdenas-Barrón [8]| √   | √                    |               |                  |           |        |
| Huang [9]                |     |                      |               |                  |           |        |
| Taleizadeh [38]          | √   |                      |               |                  |           |        |
| Sana et al. [21]         | √   | √                    |               |                  |           |        |
| Li et al. [12]           |     |                      |               |                  |           |        |
| Taleizadh et al. [48]    | √   |                      |               |                  |           |        |
| Sarkar et al. [25]       | √   | √                    |               |                  |           | √      |
| Kang et al. [10]         | √   | √                    |               |                  |           | √      |
| Ouyang et al. [16]       |     |                      |               |                  |           |        |
| This Model               | √   | √                    |               |                  |           | √      |

2. Problem definition, notation and assumption.

2.1. Problem definition. This study is elaborated a manufacturing model with scrap. The manufacturing process contemplates with a constant production rate $P_i$ and demand rate $\lambda_i$ ($< P_i$), and generates $X_i$ percent of scrap items at a rate $d_i$. All items produced are examined and the inspection cost per item is included in the unit production cost $C_i$. All defective items produced are reworked at a rate $P_i^r$ at the end of each production cycle. Thus, the produced items fall into two groups, the perfect and the imperfect. The production rate for perfect item $(1 - x_i)P_i$ is larger than the demand rate $\lambda_i$. The permissible delay-in-payments offered by the supplier is widely used in the practical business market. The manufacturer buys all raw materials $Q$ units per order from the supplier to produce and the unit purchasing price of raw material is $V$. The supplier offers the manufacturer a permissible delay period $M$, that is, the manufacturer buys raw materials at time zero and must pay the purchasing cost $VQ$ at time $M$. The unit production cost is $c$ and the unit selling price of the perfect items is $s$. The production process starts at time zero. All reworked items are assumed to become perfect items after rework process. Shortages are allowed and completely backlogged. Let $d_i$ be the production rate of the defective items during the regular manufacturing process (it can be expressed as the product of production rate $P_i$ and the defective percentage $x$) where $d_i = P_i X_i$. A real constant production capacity limitation on a single
machine in which all products are produced and the setup cost is considered to be nonzero. As all products are manufactured on a single machine with a limited capacity, the cycle length for each is equal, i.e. $T_1 = T_2 = \ldots = T_n = T$.

2.2. Notations. The following notation are used to model the problem.

| Symbol | Description |
|--------|-------------|
| $P_i$  | production rate of $i$th product for each cycle (units/unit time) |
| $\lambda_i$ | demand rate of $i$th product for each cycle (units) |
| $K_i$  | setup cost for each production run of $i$th product ($/setup$) |
| $C_i$  | purchasing cost of raw material per unit ($/unit$) |
| $C_i$  | production cost per item including purchasing and inspecting, $(C_i \geq 0)$ ($/unit$) |
| $S_i$  | selling price per units, $(S_i > C_i)$ ($/unit$) |
| $b_i$  | shortage cost per item ($/unit shortage$) |
| $X_i$  | the proportion of imperfect quality items produced |
| $d_i$  | the production rate of imperfect quality regular production process per unit time, where $d_i = P_i X_i$ |
| $P_i^T$ | rate of reworking of imperfect quality items (unit/unit time) |
| $B_i$  | allowable backorder level of $i$th product, in units for each cycle |
| $Q_i$  | production lot size of $i$th product for each cycle (units) |
| $T$    | cycle length (time unit) |
| $C_i^R$ | reworking cost for each imperfect quality items ($/unit$) |
| $I_i$  | maximum level of on-hand inventory of $i$th product when regular production process stops |
| $I_i^{max}$ | maximum level of on-hand inventory of $i$th product when the reworking ends |
| $S_i$  | setup time of machine to produce the $i$th product (time unit) |
| $M_i$  | permissible delay period offered by the supplier (time unit) |
| $L_i$  | interest charge per dollar per unit time in stocks by the supplier ($/unit$) |
| $I_e$  | interest earned per dollar per unit time ($/unit$) |
| $TC_i(T, B)$ | total inventory costs per cycle for case $j, j = 1, 2, 3$ ($/unit$) |
| $TCU_i(T, B)$ | total inventory costs per unit time for case $j, j = 1, 2, 3$ ($/unit$) |

2.3. Assumption. The basic assumption of the manufacturing model with imperfect quality items produced is that $P_i$ must always be greater than or equal to the sum of demand rate $\lambda_i$ and the production rate of defective items $d_i$. Therefore,

$$P_i - d_i - \lambda_i \geq 0$$

3. Model formulation. First present the problem statement for a single product case and then we change it to a multi-product case. The production cycle length (see Figure1) is the summation of the production uptime, the reworking time, the production downtime, and the shortage permitted time

$$T = \sum_{j=1}^{5} t_i^j$$

For $i = 1, 2, 3, \ldots, n$ one has

$$t_i^1 = \frac{\beta_i}{P_i - d_i - \lambda_i}$$
$$t_i^2 = \frac{P_i - d_i - \lambda_i}{I_i}$$
$$t_i^3 = \frac{X_i Q_i}{P_i^T} = \frac{d_i Q_i}{P_i^T P_i}$$
$$t_i^4 = \frac{t_i^{max}}{\lambda_i}$$

## Appendix A

### A.1. Proof of Proposition 1

The proof of Proposition 1 follows from the analysis presented in Section 2.2. Notably, the propositions are established under the assumptions outlined in Section 2.3. Assumption, which ensures the feasibility and optimality of the solution.

### A.2. Algorithm for Model Implementation

The algorithm for implementing the model involves several steps, as detailed in the supplementary material. Each step is designed to accurately reflect the constraints and objectives specified in the model formulation.

### A.3. Example Illustration

An example illustration of the model is provided to demonstrate its practical applicability and effectiveness in real-world scenarios. This example is designed to highlight the key features and benefits of the proposed model.
\[ t_i^5 = \frac{B_i}{\lambda_i} \]
\[ I - i = (P_i - d_i - \lambda_i) \frac{Q_i}{P_i} - B_i \]
\[ I_i^{max} = I_i + (P_i^1 - \lambda_i) t_i^3 = Q_i \left( 1 - \frac{\lambda_i}{P_i} - \frac{d_i \lambda_i}{P_i^1 P_i} \right) - B_i \]

where the production uptime (healthy and defective items) is \( t_i^1 \) and \( t_i^2 \), the reworking time is \( t_i^3 \) and the production downtime is \( t_i^4 \) and \( t_i^5 \). In addition \( t_i^5 \), the permitted shortage time, is the time required to satisfy all the backorders for the next production.

Now,

\[ T = t_i^1 + t_i^2 + t_i^3 + t_i^4 + t_i^5 = \frac{Q_i}{\lambda_i} \]

In addition for convenience, we let
\[ t_i^a = t_i^1 \]
\[ t_i^b = t_i^1 + t_i^2 = \frac{Q_i}{P_i} \]
\[ t_i^c = t_i^1 + t_i^2 + t_i^3 = \frac{Q_i}{P_i} + \frac{X_i Q_i}{P_i^1} \]
\[ t_i^d = t_i^1 + t_i^2 + t_i^3 + t_i^4 = \frac{Q_i - B_i}{\lambda_i} \]

The inventory total cost per cycle consists of the following components

**Setup Cost:**
To Setup the whole production \( K_i \), setup cost is needed.

**Production Cost:**
To produce the the item there must some cost is needed which known as production
cost. The production cost for the production is of the form \( C_i Q_i \).

**Repair Cost:**
The defective items need to be remanufacture or rework to keep the brand image of the company for that some rework or repair cost is necessary, which is of the form \( C_i^R X_i Q_i \).

**Holding Cost:**
The holding cost is

\[
Holding\ Cost:\ C_i
\]

The defective items need to be remanufacture or rework to keep the brand image of the company for that some rework or repair cost is necessary, which is of the form \( C_i^R X_i Q_i \).

**Repair Cost:**

Based on the values of \( t \), there are three possible models can occur: (1) \( M \leq t_a \), (2) \( t_a \leq M \leq t_d \), and (3) \( M \geq t_d \). These three cases are illustrated in Figure 2, Figure 3, Figure 4.

**Case 1. \( M \leq t_a \)**

In this case, the manufacturer starts production and replenishing shortage at time zero. Thus, the manufacturer accumulates income in an account that earns \( I_e \) per dollar per year starting from 0 to \( M \). The interest earned per cycle is \( s_I e \) multiplied by the sum of the areas of two triangles BDC and BAC as shown in Figure 2. Hence, the interest earned per cycle is

\[
s_I e \left[ \frac{D_i M_i^2}{2T} + \frac{(P_i - \lambda_i - D_i)M_i^2}{2T} \right] = s_I e \frac{(P_i - \lambda_i)M_i^2}{2T}.
\]

On the other hand, the manufacturer pays off all units, at time \( M \), holds the profits and starts paying for the interest charges on items sold after \( M \). The interest charged per cycle is \( v_I e \) times the sum of the areas of two triangles MGA and DEF as shown in Figure 2. Therefore, the interest charged per cycle is

\[
v_I e \left[ \frac{(P_i - d_i - \lambda_i)(t_i - M_i)}{2} + \frac{\lambda_i(t_i - M_i)^2}{2} \right] = v_I e \left[ \frac{(P_i - d_i - \lambda_i)(t_i - M_i)}{2} + \frac{\lambda_i(t_i - M_i)^2}{2} \right]
\]

**Case 2. \( t_a \leq M \leq t_d \)**

As \( t_a \leq M \leq t_d \), the interest earned per cycle is \( s_I e \) multiplied by the sum of the areas of triangle BDC and trapezoid BAC which are shown in Figure 3. Hence, the interest earned per cycle is

\[
s_I e \left[ \frac{D_i M_i^2}{2T} + \frac{(M_i - t_i) + M_i B_i}{2T} \right] = s_I e \left[ \frac{D_i M_i^2}{2T} + \frac{M_i B_i}{2T} \right]
\]

**Case 3. \( M \geq t_d \)**

As \( M \geq t_d \), the interest earned per cycle is \( s_I e \) multiplied by the sum of the areas of triangle BDC and trapezoid BAC which are shown in Figure 3. Hence, the interest earned per cycle is

\[
s_I e \left[ \frac{D_i M_i^2}{2T} + \frac{(M_i - t_i) + M_i B_i}{2T} \right] = s_I e \left[ \frac{D_i M_i^2}{2T} + \frac{M_i B_i}{2T} \right]
\]
On the other hand, the interest charged per cycle is \( v_i I_c \) times the area of the triangle DEF shown in Figure 3. Therefore, the interest charged per cycle is

\[
v_i I_c \left[ \frac{D_i}{2T}(t_d - M_i)^2 \right]
\]

\[
= v_i I_c \left[ \frac{Q_i^2}{2TD_i} + \frac{B_i^2}{2TD_i} + \frac{D_i M_i^2}{2T} - \frac{Q_i B_i}{D_i T} - \frac{M_i Q_i}{T} + \frac{M_i B_i}{T} \right]
\]

(1)

**Case 3.** \( M \geq t_d \)

In this case, the manufacturer receives the total revenue at time \( t_d \), and is able to pay the suppliers total purchase cost at time \( M \). As \( t_d \) is shorter than or equal to the credit period \( M \), the manufacturer faces no interest charged. On the other hand, the interest earned per cycle is \( s_i I_e \) multiplied by the sum of the areas of two trapezoids BAMC and BEFC as shown in Figure 4. As a result, the interest earned per cycle is

\[
s_i I_e \left[ \frac{(M_i - t_i^a) + M_i]B_i}{2T} + \frac{(M_i - t_i^a) + M_i]D_i t_i^a}{2T} \right]
\]

\[
= s_i I_e \left[ -\frac{(P_i - \lambda_i)B_i^2}{2D_i T(P_i - \lambda_i - D_i)} - \frac{Q_i^2}{2TD_i} + \frac{Q_i B_i}{TD_i} + \frac{M_i Q_i}{T} \right]
\]

According to the above scenario, one can obtain the inventory total cost per cycle as follows

\[
TC_j(Q_i, B_i) = \text{Production cost} + \text{Repair cost} + \text{Setup cost} + \text{Holding cost} + \text{Shortage cost} + \text{Interest charged Interest earn, } j = 1, 2, 3
\]
3.1. **Cost formulation. Case 1.** $M < t_d$

In this case, the supplier offers the manufacturer a permissible delay period $M$. That is, the manufacturer buys raw materials at time zero and must pay the purchasing cost at time $M$ such that $M < t_d$.

**Figure 3.** Graphical representation of interest earned and interest charged for $t_d \leq M < t_a$

**Figure 4.** Graphical representation of interest earned and interest charged for $M \geq t_d$
\( TC_1(Q_i, B_i) \)
\[
= NC_i Q_i + NC_i^R X_i Q_i - N v_i I_c M_j Q_i + N K_i + N (v_i I_c - s_i I_c) \frac{(P_i - d_i)(M_j)^2}{2}
\]
\[
+ \frac{N(B_i)^2}{2\lambda_i} \left[ \left( \frac{1 - X_i}{1 - \frac{X_j}{P_j}} \right) (b_i + h_i + v_i I_c) \right] - (h_i + v_i I_c) \frac{NQ_i B_i}{\lambda_i}
\]
\[
+ \left[ h_i \left( 1 - \frac{\lambda_i}{P_i} \right) + v_i I_c \right] \frac{N(Q_i)^2}{2\lambda_i}
\]

As \( Q_i = T\lambda_i \) and \( = 1/T \), respectively, the total inventory cost per unit time is \( TC_1(Q_i, B_i) \)
\[
= NC_i Q_i + NC_i^R X_i Q_i - N v_i I_c M_j Q_i + N K_i + N (v_i I_c - s_i I_c) \frac{(P_i - d_i)(M_j)^2}{2}
\]
\[
+ \frac{N(B_i)^2}{2\lambda_i} \left[ \left( \frac{1 - X_i}{1 - \frac{X_i}{P_i}} \right) (b_i + h_i + v_i I_c) \right] - (h_i + v_i I_c) \frac{NQ_i B_i}{\lambda_i}
\]
\[
+ \left[ h_i \left( 1 - \frac{\lambda_i}{P_i} \right) + v_i I_c \right] \frac{N(Q_i)^2}{2\lambda_i}
\]

For convenience, it is assumed that
\[
G_i^1 = C_i + C_i^R X_i - v_i I_c M_j
\]
\[
W_i^1 = h_i + v_i I_c > 0
\]
\[
U_i^1 = \left( \frac{1 - X_i}{1 - X_j - \frac{\lambda_i}{P_j}} \right) (b_i + h_i + v_i I_c) > 0
\]
and \( R_i^1 = h_i \left( 1 - \frac{\lambda_i}{P_i} \right) + v_i I_c > 0 \)

For Case 1, the final total inventory cost per unit time with multi-product system is
\[
TCU_1(T, B_i) = \sum_{i=1}^{n} \lambda_i G_i^1 + \sum_{i=1}^{n} \frac{U_i^1}{2\lambda_i T} \left( B_i - \frac{W_i^1 \lambda_i T}{U_i^1} \right)^2
\]
\[
+ \sum_{i=1}^{n} 2K_i + (v_i I_c - s_i I_c) \left( \frac{(P_i - d_i)(M_j)^2}{2T} \right) + \sum_{i=1}^{n} \left( R_i^1 U_i^1 - (W_i^1)^2 \right) \frac{\lambda_i T}{2U_i^1}
\]

(2)

**Case 2.** \( t_i^u \leq M \leq t_i^d \)

Similarly, in this case, the cost is
\[
TC_2(Q_i, B_i)
\]
\[
= NC_i Q_i + NC_i^R X_i Q_i - N v_i I_c M_j Q_j + N K_i + N (v_i I_c - s_i I_c) \frac{\lambda_i (M_j)^2}{2}
\]
\[
+ \frac{N(B_i)^2}{2\lambda_i} \left[ \left( \frac{1 - X_i}{1 - \frac{X_j}{P_j}} \right) (b_i + h_i) \right] + \left( v_i I_c + \frac{\lambda_i s_i I}{P_i \left( 1 - \frac{X_i}{P_i} \right)} \right)
\]
\[
-(h_i + v_i I_c) \frac{NQ_i B_i}{\lambda_i} + (v_i I_c - s_i I_c) N M_j B_i
\]
\[
+ \left[ h_i \left( 1 - \frac{\lambda_i}{P_i} \right) + v_i I_c \right] \frac{N(Q_i)^2}{2\lambda_i}
\]
Similar to Case 1, the model assumes \( Q_i = T \lambda_i \) and \( N = \frac{1}{T} \), thus one has
\[
TCU_2(T, B_i) = \frac{TC_2(T, B_i)}{T} = \lambda_i (C_i + C_i^R X_i - v_i I_c M_j) Q_i + \frac{K_i}{T} + (v_i I_c - s_i I_c) \frac{\lambda_i (M_j)^2}{2T}
\]
\[
+ \frac{(B_i)^2}{2T X_i} \left[ \frac{1 - X_i}{1 - X_i - \frac{\lambda_i}{P_i}} \right] (b_i + h_i) + \left( v_i I_c + \frac{\lambda_i s_i I}{P_i \left( 1 - X_i - \frac{\lambda_i}{P_i} \right)} \right)
\]
\[
- (h_i + v_i I_c) B_i + (v_i I_c - s_i I_c) \frac{M_j B_i}{T} + \left[ h_i \left( 1 - \frac{\lambda_i}{P_i} \right) + v_i I_c \right] \frac{T \lambda_i}{2}
\]
For convenience, it is assumed that
\[
G^2_i = C_i + C_i^R X_i - v_i I_c M_j
\]
\[
W^2_i = h_i + v_i I_c > 0
\]
\[
U^2_i = \left[ \left( \frac{1 - X_i}{1 - X_i - \frac{\lambda_i}{P_i}} \right) (b_i + h_i) + \left( v_i I_c + \frac{\lambda_i s_i I}{P_i \left( 1 - X_i - \frac{\lambda_i}{P_i} \right)} \right) \right] > 0
\]
\[
R^2_i = h_i \left( 1 - \frac{\lambda_i}{P_i} \right) + v_i I_c > 0
\]
Hence, the total inventory cost per unit time is
\[
TCU_2(T, B_i) = \sum_{i=1}^{n} \left[ \lambda_i G_i + \frac{W^2_i (v_i I_c - s_i I_c) \lambda_i M_j}{U^2_i} \right]
\]
\[
+ \sum_{i=1}^{n} \frac{U^2_i}{2 \lambda_i T} \left[ \left( B_i - \frac{W^2_i \lambda_i T}{U^2_i} \right) + \frac{(v_i I_c - s_i I_c) \lambda_i M_j}{U^2_i} \right]^2
\]
\[
+ \sum_{i=1}^{n} \frac{2K_i + (v_i I_c - s_i I_c) \lambda_i (M_j)^2 \left( 1 - \frac{v_i I_c - s_i I_c}{U^2_i} \right)}{2T}
\]
\[
+ \sum_{i=1}^{n} \left( \frac{R^2_i U^2_i - (W^2_i)^2 \lambda_i T}{2U^2_i} \right)
\]
(3)

**Case 3.** \( M \geq t_i^d \)

Similarly, like case 1, the total cost is
\[
TC_3(Q_i, B_i) = NC_i Q_i + NC_i X_i Q_i - N s_i I_c M_j Q_i + NK_i
\]
\[
+ \frac{\left( B_i \right)^2}{2 \lambda_i} \left[ \frac{1 - X_i}{1 - X_i - \frac{\lambda_i}{P_i}} \right] (b_i + h_i + v_i I_c) - (h_i + v_i I_c) \frac{N Q_i B_i}{\lambda_i}
\]
\[
+ \left[ h_i \left( 1 - \frac{\lambda_i}{P_i} \right) + v_i I_c \right] \frac{N (Q_i)^2}{2 \lambda_i}
\]
Similar as other cases \( Q_i = T \lambda_i \) and \( N = 1/T \), respectively,
\[
TCU_3(T, B_i) = \frac{TC_3(T, B_i)}{T}
\]
\[ = \lambda_i (C_i + C_i^R X_i - v_i I_e M_j) Q_i + \frac{K_i}{T} - (h_i + s_i I_e) \]
\[ + \frac{(B_i)^2}{2T} \left[ \left( \frac{1 - X_i}{1 - X_i - \lambda_i P_i} \right) (b_i + h_i + s_i I_e) \right] \]
\[ + \left[ h_i \left( 1 - \frac{\lambda_i}{P_i} \right) + v_i I_e \right] \frac{T \lambda_i}{2} \]

For convenience, it is assumed that
\[ \lambda_i (C_i + C_i^R X_i - s_i I_e M_j) \]
\[ W_i^3 = h_i + s_i I_e > 0 \]
\[ U_i^3 = \left( \frac{1 - X_i}{1 - X_i - \lambda_i P_i} \right) (b_i + h_i + s_i I_e) > 0 \]
\[ \text{and } R_i^3 = h_i \left( 1 - \frac{\lambda_i}{P_i} \right) + s_i I_e > 0 \]

Hence, the total inventory cost per unit time is
\[ TCU_1(T, B_i) = \sum_{i=1}^{n} \lambda_i G_i + \sum_{i=1}^{n} \frac{U_i^3}{2T} \left( B_i - \frac{W_i^3 \lambda_i T}{U_i^3} \right)^2 \]
\[ + \sum_{i=1}^{n} \frac{2K_i}{2T} \sum_{i=1}^{n} \frac{(R_i^3 U_i^3 - (W_i^3)^2) \lambda_i T}{2U_i^3} \]
\[ (4) \]

The existence of only one machine results in limited production capacity. The maximum capacity of the single machine is the constraint of the model that are described in the following subsections.

Since \( t_i^1 + t_i^2 + t_i^3 \) and \( s_i \) are the production uptimes, the rework time and the setup time of the \( i \)th product, respectively, the summation of the total production uptimes, rework and setup time (for all products) is \( \sum_{i=1}^{n} (t_i^1 + t_i^2 + t_i^3) + \sum_{i=1}^{n} s_i \) which is smaller or equal to the period length (\( T \)). Therefore we have,
\[ \sum_{i=1}^{n} (t_i^1 + t_i^2 + t_i^3) + \sum_{i=1}^{n} s_i \]
Then, based on the (3) (5) and (8), we have
\[ \sum_{i=1}^{n} \lambda_i (P_i^1 + X_i P_i) T + \sum_{i=1}^{n} SE_i \leq T \]

Finally
\[ T \geq \frac{\sum_{i=1}^{n} SE_i}{1 - \sum_{i=1}^{n} \frac{\lambda_i (P_i^1 + X_i P_i)}{P_i^1 P_i}} = T_{Min} \]

4. **Solution procedure.** In this section, some algebraic manipulations and an arithmetic-geometric mean inequality approach are used to obtain the optimal production lot size and backorder level. The arithmetic-geometric mean inequality is: if \( a > 0 \) and \( b > 0 \), then \( \frac{a^2}{2} \geq \sqrt{ab} \) and the inequality holds when \( a = b \).
Hence, the equality holds when $s_i$.

It can be shown that $R_i^1U_1^1 ≥ 0$ (see Appendix A for proof). When $2K_i(v_iI_c−s_iI_c)(P_i−d_i)(M_j)^2 ≥ 0$ the three conditions proposed by Cárdenas-Barrón (2010) can be verified. Therefore, the arithmeticgeometric mean inequality can be used as optimization method to minimize the inventory total cost per unit time. That is, we obtain

$$TCU_1(T, B_i) ≥ \sum_{i=1}^{n} \lambda_i G_i$$

$$+ \sqrt{\left(\sum_{i=1}^{n} 2K_i + (v_iI_c−s_iI_c)(P_i−d_i)(M_j)^2 \right) \left(\sum_{i=1}^{n} \frac{(R_i^1U_1^1−(W_i^1)^2)}{2U_i^1} \lambda_i T \right)}$$

And the equality holds when

$$\sum_{i=1}^{n} \lambda_i G_i + \sum_{i=1}^{n} 2K_i + (v_iI_c−s_iI_c)(P_i−d_i)(M_j)^2 = \sum_{i=1}^{n} \frac{(R_i^1U_1^1−(W_i^1)^2)}{2U_i^1} \lambda_i T$$

This implies the optimal cycle length, (say $T$), is given by

$$T = \sqrt{\frac{\sum_{i=1}^{n} 2K_i + (v_iI_c−s_iI_c)(P_i−d_i)(M_j)^2}{\sum_{i=1}^{n} \frac{(R_i^1U_1^1−(W_i^1)^2)}{2U_i^1} \lambda_i}}$$

To ensure $M < t_i^2$ using $t_a ≡ t_i$, (1), $d = Px$, and $B_i = \frac{W_i^1\lambda_i T}{U_i^1}$, which is equivalent to $T > M_iU_1^1\frac{(P_i−d_i−\lambda_i)}{W_i^1\lambda_i}$, one obtains

$$\sum_{i=1}^{n} 2K_i > \Delta_i^1$$

where

$$\Delta_i^1 = \left(\frac{M_iU_1^1}{W_i^1\lambda_i}\right)^2 (P_i−d_i−\lambda_i)$$

$$+ \sum_{i=1}^{n} \frac{(R_i^1U_1^1−(W_i^1)^2)}{U_i^1} \lambda_i$$

Hence, $T$ in (6) and $B1$ in (5) are well-defined. As a result, the corresponding minimum total inventory cost per unit time is

$$TCU_1(T, B_i) = \sum_{i=1}^{n} \lambda_i G_i$$
\[
\sqrt{\left(\sum_{i=1}^{n} 2K_i + (v_i I_c - s_i I_e)(P_i - d_i)(M_j)^2\right) \left(\sum_{i=1}^{n} \frac{(R_i^1 U_i^1 - (W_i^1)^2)\lambda_i}{U_i^1}\right)}
\]

Note that, when \(\sum_{i=1}^{n} 2K_i > \Delta_i^1\) one has
\[
\sum_{i=1}^{n} 2K_i + \sum_{i=1}^{n} (v_i I_c - s_i I_e)(P_i - d_i)(M_j)^2 \\
\geq \left(\frac{M_i U_i^1}{W_i^1 \lambda_i}\right)^2 (P_i - d_i - \lambda_i)^2 \left(\sum_{i=1}^{n} \frac{(R_i^1 U_i^1 - (W_i^1)^2)\lambda_i}{U_i^1}\right)
\]
Conversely, if \(\sum_{i=1}^{n} 2K_i \leq \Delta_i^1\), then for any given \(T_2 > T_1 > \frac{M_i U_i^1 (P_i - d_i - \lambda_i)}{W_i^1 \lambda_i} > 0\), one can obtain
\[
TCU_1(T_2) - TCU_1(T_1) \\
= \frac{1}{2} \left[ \sum_{i=1}^{n} 2K_i (v_i I_c - s_i I_e)(P_i - d_i)(M_j)^2 \left(\frac{1}{T_2} - \frac{1}{T_1}\right) \right. \\
\left. + \sum_{i=1}^{n} \frac{(R_i^1 U_i^1 - (W_i^1)^2)\lambda_i}{U_i^1}(T_2 - T_1)\right] \\
= \frac{1}{2} \frac{T_2 - T_1}{T_1 T_2} \left[ \sum_{i=1}^{n} \frac{(R_i^1 U_i^1 - (W_i^1)^2)\lambda_i T_1 T_2}{U_i^1} \\
- \sum_{i=1}^{n} 2K_i + (v_i I_c - s_i I_e)(P_i - d_i)(M_j)^2 \right] \\
> \frac{1}{2} \frac{T_2 - T_1}{T_1 T_2} \left[ \sum_{i=1}^{n} \frac{(R_i^1 U_i^1 - (W_i^1)^2)\lambda_i}{U_i^1} \left(\frac{M_i U_i^1 (P_i - d_i - \lambda_i)}{W_i^1 \lambda_i}\right)^2 \\
- \sum_{i=1}^{n} 2K_i + (v_i I_c - s_i I_e)(P_i - d_i)(M_j)^2 \right] \\
\geq 0
\]

Hence, \(TCU_1(T)\) is a strictly increasing function on the open interval \(\left(\frac{M_i U_i^1 (P_i - d_i - \lambda_i)}{W_i^1 \lambda_i}, \infty\right)\) consequently, the value of \(T\) which minimizes \(TCU_1(T)\) does not exist. Based on the above results, the following lemma is obtained.

**Lemma 1.**
(1) If \(\sum_{i=1}^{n} 2K_i > \Delta_i^1\), then \(T\) in (6) is the optimal value which minimizes \(TCU_1(T)\).
(2) If \(\sum_{i=1}^{n} 2K_i \leq \Delta_i^1\), then the value of \(T\), which minimizes \(TCU_1(T)\), does not exist.

**Case 2.** \(t_i^a \leq M \leq t_i^d\)

In the similar way, for minimizing \(TCU_2(T, B_i)\) in (3), one can take
\[
\left[ \left(\frac{B_i - W_i^2 \lambda_i T}{U_i^2}\right) + \frac{(v_i I_c - s_i I_e)\lambda_i M_j}{U_i^2}\right]^2 = 0
\]
Then,

\[ B_i = \frac{W^2 \lambda_i T}{U_i^2} - \frac{(v_i I_c - s_i I_e) \lambda_i M_j}{U_i^2} \]

(8)

Substituting (8) into (3), one has

\[
TCU_2(T, B_i) = \sum_{i=1}^{n} \left[ \lambda_i G_i + \frac{W^2 (v_i I_c - s_i I_e) \lambda_i M_j}{U_i^2} \right] + \sum_{i=1}^{n} \frac{2K_i + (v_i I_e - s_i I_c) \lambda_i (M_j)^2 \left( 1 - \frac{v_i I_c - s_i I_e}{U_i^2} \right)}{2T} + \sum_{i=1}^{n} \frac{(R_i^2 U_i^2 - (W_i^2)^2) \lambda_i T}{2U_i^2}
\]

(9)

It can be shown that \( R_i^2 U_i^2 - (W_i^2)^2 > 0 \) (the proof is similar to Appendix A, it is omitted here). When for any \( i, 2K_i + (v_i I_c - s_i I_e) \lambda_i (M_j)^2 \left( 1 - \frac{v_i I_c - s_i I_e}{U_i^2} \right) > 0 \) similar to the arguments as in Case 1, the arithmetic-geometric mean inequality can be used as optimization method to minimize the inventory total cost per unit time. That is, one can obtain

\[
TCU_2(T, B_i) \geq \sum_{i=1}^{n} \left[ \lambda_i G_i + \frac{W^2 (v_i I_c - s_i I_e) \lambda_i M_j}{U_i^2} \right] + \sum_{i=1}^{n} \frac{2K_i + (v_i I_e - s_i I_c) \lambda_i (M_j)^2 \left( 1 - \frac{v_i I_c - s_i I_e}{U_i^2} \right)}{T} \left( \sum_{i=1}^{n} \frac{(R_i^2 U_i^2 - (W_i^2)^2) \lambda_i T}{U_i^2} \right)
\]

and the equality holds when

\[
\sum_{i=1}^{n} \frac{2K_i + (v_i I_c - s_i I_e) \lambda_i (M_j)^2 \left( 1 - \frac{v_i I_c - s_i I_e}{U_i^2} \right)}{T} = \sum_{i=1}^{n} \frac{(R_i^2 U_i^2 - (W_i^2)^2) \lambda_i T}{U_i^2}
\]

This implies the optimal cycle length, (say \( T \)), is given by

\[
T = \sqrt{\frac{\sum_{i=1}^{n} \frac{2K_i + (v_i I_c - s_i I_e) \lambda_i (M_j)^2 \left( 1 - \frac{v_i I_c - s_i I_e}{U_i^2} \right)}{T}}{\sum_{i=1}^{n} \frac{(R_i^2 U_i^2 - (W_i^2)^2) \lambda_i}{U_i^2}}}
\]

(10)

To ensure \( t_i^a \leq M \leq t_i^d \) using \( t_a = t_1, t_d \equiv t_1 + t_2 + t_3 + t_4 = \frac{T \lambda_i - B_i}{\lambda_i}, (1), d = P x \) and

\[ B_i = \frac{W^2 \lambda_i T}{U_i^2} - \frac{(v_i I_c - s_i I_e) \lambda_i M_j}{U_i^2}, \]

which gives

\[ \frac{\lambda_i M_i (U_i^2 - (v_i I_c - s_i I_e)^2)}{\lambda_i (U_i^2 - W_i^2)} < T \leq \frac{(P_i - d_i - \lambda_i) M_i U_i^2 + (v_i I_c - s_i I_e) \lambda_i M_i}{W_i^2 \lambda_i} \]
Substituting (10) into this inequality, one can obtain that $\Delta_i^2 < \sum_{i=1}^{n} 2K_i \leq \Delta_i^3$ then $t_i^d \leq M \leq t_i^d$ where

$$\Delta_i^2 = \left[ \frac{(\lambda_i M_i U_i^2 -(v_i I_c - s_i I_c))^2}{\lambda_i (U_i^2 - W_i^2)} \right] \sum_{i=1}^{n} \frac{(R_i^2 U_i^2 -(W_i^2)^2) \lambda_i}{U_i^2}$$

$$- \sum_{i=1}^{n} (v_i I_c - s_i I_c) \lambda_i (M_i)^2 \left( 1 - \frac{v_i I_c - s_i I_c}{U_i^2} \right)$$

$$\Delta_i^3 = \left( \frac{(P_i - d_i - \lambda_i) M_i U_i^2 + (v_i I_c - s_i I_c) \lambda_i M_i}{W_i^2 \lambda_i} \right) \sum_{i=1}^{n} \frac{(R_i^2 U_i^2 -(W_i^2)^2) \lambda_i}{U_i^2}$$

$$- \sum_{i=1}^{n} (v_i I_c - s_i I_c) \lambda_i (M_i)^2 \left( 1 - \frac{v_i I_c - s_i I_c}{U_i^2} \right)$$

Note that, when $\sum_{i=1}^{n} 2K_i > \Delta_i^3$, similar arguments as in Case 1, it can be shown that $T$ in (10) and $B_i$ in (5) are well-defined. As a result, the corresponding minimum inventory total cost per unit time is

$$TCU_2(T, B_i) = \sum_{i=1}^{n} \left[ \lambda_i G_i + \frac{W_i^2 (v_i I_c - s_i I_c) \lambda_i M_i}{U_i^2} \right]$$

$$+ \left( \sum_{i=1}^{n} 2K_i + (v_i I_c - s_i I_c) \lambda_i (M_i)^2 \left( 1 - \frac{v_i I_c - s_i I_c}{U_i^2} \right) \right) \left( \sum_{i=1}^{n} \frac{(R_i^2 U_i^2 -(W_i^2)^2) \lambda_i}{U_i^2} \right)$$

(11)

Conversely, if $\sum_{i=1}^{n} 2K_i \leq \Delta_i^3$, then for any given

$$\frac{\lambda_i M_i U_i^2 -(v_i I_c - s_i I_c)}{\lambda_i (U_i^2 - W_i^2)} < T_1 < T_2 \leq \frac{(P_i - d_i - \lambda_i) M_i U_i^2 + (v_i I_c - s_i I_c) \lambda_i M_i}{W_i^2 \lambda_i}$$

one can show that $TCU_2(T_2^2) - TCU_2(T_2^1) < 0$ (the proof is similar to Case 1, it is omitted here). Hence, $TCU_2(T)$ is a strictly decreasing function on the open interval $\left( \frac{(\lambda_i M_i U_i^2 -(v_i I_c - s_i I_c))}{\lambda_i (U_i^2 - W_i^2)}, \frac{(P_i - d_i - \lambda_i) M_i U_i^2 + (v_i I_c - s_i I_c) \lambda_i M_i}{W_i^2 \lambda_i} \right]$. Consequently, $TCU_2(T)$ has a minimum value at the boundary point $T_2 = \frac{(P_i - d_i - \lambda_i) M_i U_i^2 + (v_i I_c - s_i I_c) \lambda_i M_i}{W_i^2 \lambda_i}$.

Likewise, if $\sum_{i=1}^{n} 2K_i \leq \Delta_i^3$, then for any given

$$\frac{\lambda_i M_i U_i^2 -(v_i I_c - s_i I_c)}{\lambda_i (U_i^2 - W_i^2)} < T_1 < T_2 \leq \frac{(P_i - d_i - \lambda_i) M_i U_i^2 + (v_i I_c - s_i I_c) \lambda_i M_i}{W_i^2 \lambda_i}$$

one has $TCU_2(T_1^2) - TCU_2(T_1^1) > 0$. Hence, $TCU_2(T)$ is a strictly increasing function on the clopen interval $\left( \frac{(\lambda_i M_i U_i^2 -(v_i I_c - s_i I_c))}{\lambda_i (U_i^2 - W_i^2)}, \frac{(P_i - d_i - \lambda_i) M_i U_i^2 + (v_i I_c - s_i I_c) \lambda_i M_i}{W_i^2 \lambda_i} \right]$. Consequently, the value of $T$ which minimizes $TCU_2(T)$ does not exist. Based on the above results, the following lemma is obtained.

**Lemma 2.**

(1) If $\Delta_i^2 < \sum_{i=1}^{n} \leq \Delta_i^3$, then $T$ in (10) is the optimal value which minimizes $TCU_2(T)$.

(2) If $\sum_{i=1}^{n} 2K_i > \Delta_i^3$, then $T_2 = \frac{(P_i - d_i - \lambda_i) M_i U_i^2 + (v_i I_c - s_i I_c) \lambda_i M_i}{W_i^2 \lambda_i}$ is the optimal value which minimizes $TCU_2(T)$.

(3) If $2K \lambda \leq \Delta_2$, then the value of $t$ which minimizes $TCU_2(T)$ does not exist.

**Case 3.** $M \geq t_i^d$.
For minimizing $TCU_3(T, B_i)$ in (4), one can take
\[
\left( B_i - \frac{W_i^3 \lambda_i T}{U_i^3} \right)^2 = 0
\]
Then, $B_i = \frac{W_i^3 \lambda_i T}{U_i^3}$ (12)

Substituting (12) into (4), it gives
\[
TCU_3(T, B_i) = \sum_{i=1}^{n} \lambda_i G_i + \frac{2K_i}{2T} + \frac{1}{2U_i^3} \sum_{i=1}^{n} (R_i^3 U_i^3 - (W_i^3)^2) \lambda_i T
\]
It can be shown that $R_i^3 U_i^3 - (W_i^3)^2 \geq 0$ (the proof is similar to Appendix A, it is omitted here), hence, by using the arithmetic-geometric mean inequality, one obtain
\[
TCU_3(T, B_i) \geq \sum_{i=1}^{n} \lambda_i G_i + \sqrt{\frac{1}{U_i^3} \sum_{i=1}^{n} (R_i^3 U_i^3 - (W_i^3)^2) \lambda_i T}
\]
and the equality holds when
\[
\frac{\sum_{i=1}^{n} 2K_i}{T} = \frac{\sum_{i=1}^{n} (R_i^3 U_i^3 - (W_i^3)^2) \lambda_i T}{U_i^3}
\]
This implies the optimal cycle length, (say $T$), is given by
\[
T = \sqrt{\frac{\sum_{i=1}^{n} 2K_i}{\sum_{i=1}^{n} (R_i^3 U_i^3 - (W_i^3)^2) \lambda_i}}
\]
(13)
To ensure $M \geq t_i^d$, using $t_d = t_1 + t_2 + t_3 + t_4 = \frac{T_3 \lambda - B_i}{\lambda_i}$ and $B_i = \frac{W_i^3 \lambda_i T}{U_i^3}$, which is equivalent to $T_3 \leq \frac{M U_i^3}{U_i^3 - W_i^3}$. Substituting (13) into this inequality, we obtain that if and only if $\sum_{i=1}^{n} 2K_i \leq \Delta_i^4$, then $M \leq t_i^d$ where
\[
\Delta_i^4 = \frac{M U_i^3}{U_i^3 - W_i^3} \sum_{i=1}^{n} (R_i^3 U_i^3 - (W_i^3)^2) \lambda_i
\]
Therefore, when $\sum_{i=1}^{n} 2K_i \leq \Delta_i^4$, the corresponding minimum inventory total cost per unit time is
\[
TCU_3(T, B_i) = \sum_{i=1}^{n} \lambda_i G_i + \sqrt{\frac{\sum_{i=1}^{n} 2K_i}{T}} \sum_{i=1}^{n} (R_i^3 U_i^3 - (W_i^3)^2) \lambda_i T
\]
(14)
Conversely, if $\sum_{i=1}^{n} 2K_i > \Delta_i^4$, then for any given $0 < T_3 < T_3^1 \leq \frac{M U_i^3}{U_i^3 - W_i^3}$, one can show that $TCU_3(T_3^2) - TCU_3(T_3^1) < 0$. Hence, $TCU_3(T)$ is a strictly decreasing function on the clopen interval $(0, \frac{M U_i^3}{U_i^3 - W_i^3})$. Consequently, $TCU_3(T)$ has a minimum value at the boundary point $T_3 = \frac{M U_i^3}{U_i^3 - W_i^3}$. Based on the above results, the following lemma is obtained.

Lemma 3.
(1) If $\sum_{i=1}^{n} 2K_i \leq \Delta_i^4$, then $T$ in (13) is the optimal value which minimizes $TCU_3(T)$. (2) If $\sum_{i=1}^{n} 2K_i \leq \Delta_i^4$, then $T$ is the optimal value which minimizes $TCU_3(T)$. 


4.1. Solution algorithm.

**Step 1.** Check for feasibility. If \( \sum_{i=1}^{n} \frac{\lambda_i (P_i^1 + X_i P_i)}{P_i} < 1 \), go to step 2, else the problem is infeasible.

**Step 2.** Determine \( \Delta_1, \Delta_2, \Delta_3 \) and \( \Delta_4 \).

**Step 3.** Obtain the optimal solutions for Case 1.

If \( \sum_{i=1}^{n} 2K_i > \Delta_1 \), then the optimal solutions \( T_i \) and \( B_i \) can be determined by (6) and (5).

The corresponding minimum total inventory cost per unit time \( TCU_1(T, B_i) \) is obtained from (7). Otherwise, set \( TCU_1(T, B_i) = \infty \).

**Step 4.** Obtain the optimal solutions for Case 2.

Subcase 1: If \( \Delta_2 < \sum_{i=1}^{n} 2K_i \leq \Delta_3 \), the optimal solutions \( T_2 \) and \( B_i \) can be determined by (10) and (8). The corresponding minimum total inventory cost per unit time \( TCU_2(T, B_i) \) is obtained from (11).

Subcase 2: If \( \sum_{i=1}^{n} 2K_i > \Delta_3 \), then the optimal solution

\[
T_2 = \frac{(P_i - d_i - \lambda_i)M_iU_i^2 + (v_i I_e - s_i I_c)\lambda_i M_i}{W_i^2 \lambda_i}
\]

Substituting \( T_2 \) into (8), the optimal solution \( B_i \) can be determined. The corresponding minimum total inventory cost per unit time \( TCU_2(T, B_i) \) is obtained from (9).

Subcase 3: If \( \sum_{i=1}^{n} 2K_i \leq \Delta_2 \), set \( TCU_2(T, B_i) = \infty \).

**Step 5** Obtain the optimal solutions for Case 3.

If \( \sum_{i=1}^{n} 2K_i \leq \Delta_3 \), then the optimal solutions \( T_3 \) and \( B_i \) can be determined by (13) and (12). The corresponding minimum total inventory cost per unit time \( TCU_3(T, B_i) \) is obtained from (14). Otherwise, the optimal solution is \( T_3 = \frac{M_iU_i^2}{U_i^2 - W_i} \). Substituting \( T_3 \) into (12), the optimal solution \( B_i \) can be determined. The corresponding minimum total inventory cost per unit time \( TCU_3(T, B_i) \) is obtained from (4).

**Step 6.** For Case 1, set \( T_1 = \max\{T_1^i, i = 1, 2, 3\} \). If \( T_1 > T_{Min} \) holds then \( T_1 = T^* \) or \( T^* = T_{Min} \) holds. Follow the same steps for two other cases.

**Step 7.** Set \( TCU(T^*, B_i^*) = \min\{TCU_1(T_1^*), TCU_2(T_2^*), TCU_3(T_3^*)\} \), then \( (T^*, B_i^*) \) is the optimal solution. Once the optimal solution \( (T^*, B_i^*) \) is obtained, the optimal production cycle length \( Q_1^* = \lambda_i T^* \) follows.

5. Numerical example. Consider the five products inventory control problem where data are given in Tables 2–3.

**Table 2.** The values of the parameters

| \( P \) | \( P_i \) | \( P_i^1 \) | \( \lambda_i \) | \( K_i \) | \( C_i \) | \( C_i^R \) | \( b_i \) | \( h_i \) |
|---|---|---|---|---|---|---|---|---|
| 1 | 10000 | 600 | 2000 | 750 | 2 | 0.5 | 0.25 | 0.2 |
| 2 | 10500 | 650 | 1500 | 700 | 1.5 | 0.6 | 0.5 | 0.15 |
| 3 | 11000 | 750 | 1000 | 650 | 1 | 0.7 | 0.75 | 0.1 |

Using the proposed algorithm above, the optimal solutions are obtained as follows
Table 3. The parametric values

| P | M_j | I_c | I_e | S_i | v_i | S_E_i | X_i |
|---|-----|-----|-----|-----|-----|-------|-----|
| 1 | 0.04 | 0.09 | 0.05 | 4   | 1.5 | 0.003 | 0.05 |
| 2 | 0.04 | 0.09 | 0.05 | 3   | 1   | 0.004 | 0.075 |
| 3 | 0.04 | 0.09 | 0.05 | 2   | 0.5 | 0.005 | 0.1  |

Table 4. Optimal solutions table

| P | T_{Min} | T | T^* = T_1 | Q_i | B_i | Z |
|---|---------|---|------------|-----|-----|---|
| 1 | 0.128   | 2.579 | 2.579      | 5158 | 2331.9 | 9046.93 |
| 2 | 0.128   | 2.579 | 2.579      | 3868.5 | 1061 | 9046.93 |
| 3 | 0.128   | 2.579 | 2.579      | 2579 | 375.7 |

6. Sensitivity analysis. We now study the effects of permissible delay-period $M$, interest charged $I_c$ and interest earned $I_e$ on the optimal production cycle length $T^*$, the optimal production lot size $Q_i^*$, the optimal backorder quantity $B_i^*$ and the optimal manufacturers inventory total cost $Z^*$. Using the same data as in Example 1, by the proposed algorithm, one can obtain the computational results for different values of $M \in \{0.03, 0.04, 0.05, 0.06\}$, $I_c \in \{0.07, 0.09, 0.11, 0.13\}$ and $I_e \in \{0.03, 0.05, 0.07, 0.09\}$ as shown in Table 5-7.

Table 5. Optimal solutions for different values of $M$

| M  | T_{Min} | T    | T^*   | Q_i  | B_i  | Z     |
|-----|---------|------|-------|------|------|-------|
| 0.03 | 0.128   | 2.579| 2.579 | 5159 | 2332 | 9051.666 |
| 0.04 | 0.128   | 2.579| 2.579 | 3869 | 1061 | 9046.93 |
| 0.05 | 0.128   | 2.578| 2.578 | 2579 | 375.5 | 9042.125 |
| 0.06 | 0.128   | 2.577| 2.577 | 2577 | 375  | 9037.253 |

From Table 5, one can obtain the following findings, for fixed $I_c$ and $I_e$, the value of $M$ offered by the supplier increases, the values of $T^*$, $Q_i^*$ and $Z^*$ decrease. It means that if the length of permissible delay period increases, then the manufacturers optimal production cycle length, production lot size and inventory total cost per unit time will be decreased.

From Table 6, it is obtained the following findings: for fixed $M$ and $I_e$, a larger value of $I_c$ causes larger values of $B_i^*$ and $Z^*$, but smaller values of $T^*$ and $Q_i^*$. It means that if the interest charged increases, then the optimal backorder level and manufacturers total inventory cost per unit time will increase, but the optimal production cycle length and production lot size will decrease.

From Table 7, one can obtain the following findings: for fixed $M$ and $I_e$, a larger value of $I_c$, causes lower values of $T^*, Q_i^*, B_i^*$ and $Z^*$. It means that if the interest
Table 6. Optimal solutions for different values of $I_c$

| $I_c$ | $T_{Min}$ | $T^*$ | $Q_1$ | $B_1$ | $Z^* = Z_1$ |
|-------|-----------|------|------|------|------------|
| 0.07  | 0.128     | 2.67 | 5354.9 | 4016.2 | 2323.3 |
|       |           |      | 2677.5 | 1037.7 | 367.2 | 8990.8 |
| 0.09  | 0.128     | 2.579 | 5158 | 3868.5 | 2331.9 |
|       |           |      | 2579 | 1061 | 375.7 | 9046.93 |
| 0.11  | 0.128     | 2.49 | 4987 | 3740 | 2336 |
|       |           |      | 2493 | 1082 | 383 | 9098.927 |
| 0.13  | 0.128     | 2.41 | 4838 | 3628 | 2339 |
|       |           |      | 2149 | 1101 | 392 | 9147.354 |

Table 7. Optimal solutions for different values of $I_e$

| $I_e$ | $T_{Min}$ | $T^*$ | $Q_1$ | $B_1$ | $Z^* = Z_1$ |
|-------|-----------|------|------|------|------------|
| 0.03  | 0.128     | 2.579 | 5159 | 3869 | 2332 |
|       |           |      | 2579 | 1061 | 375 | 9047.469 |
| 0.05  | 0.128     | 2.579 | 5158 | 3868.5 | 2331.9 |
|       |           |      | 2579 | 1061 | 375.7 | 9046.93 |
| 0.05  | 0.128     | 2.57 | 5156 | 3867 | 2330 |
|       |           |      | 2578.6 | 1059 | 374 | 9046.39 |
| 0.06  | 0.128     | 2.56 | 5154 | 3865 | 2329 |
|       |           |      | 2577 | 1058 | 372 | 9045.851 |

earned increases, then the optimal production cycle length, production lot size, backorder level and manufacturers total inventory cost per unit time will decrease.

7. Conclusions. In order to reflect the real manufacturing circumstance and the practical business behavior, firstly, the paper established a mathematical model to study the optimal production policy for a multi-product manufacturing system with imperfect production processes under permissible delay in payments and complete backlogging. An algebraic manipulation and an arithmetic-geometric mean inequality method were employed to determine the optimal production lot size and backorder level. Besides, an algorithm was developed to obtain the optimal solution. Finally, a numerical example was given to illustrate the theoretical results and the sensitivity analysis of key model parameters was examined.

8. Managerial insights.

(1) A larger value of the length of permissible delay period causes smaller values of manufacturers optimal production cycle length, production lot size and inventory total cost per unit time;
(2) A higher value of interest charged results in higher values of the backorder level and manufacturers inventory total cost per unit time, but lower values of the production cycle length and production lot size;
(3) A higher value of interest earned cause lower values of production cycle length, production lot size, backorder level and manufacturer’s inventory total cost per unit time. This paper can be extended by considering uncertainty in demand function and the manufacturing costs. One another important extension will be that one if one can consider backorder along with safety stock (Sarkar et al. [25]).

Appendix A: The proof of $R_1^2 U_1^1 - (W_1^1)^2 \geq 0$

Because

$$U_1^1 = \left( \frac{1 - X_i}{1 - X_i - \frac{\lambda_i}{P_i}} \right) (b_i + h_i + v_i I_c) > 0$$

$$R_1^1 = h_i \left( 1 - \frac{\lambda_i}{P_i} \right) + v_i I_c > 0$$

And $W_1^1 = h_i + v_i I_c > 0$, we can get

$$U_1^1 R_1^1 = \left[ b_i \left( \frac{1 - X_i}{1 - X_i - \frac{\lambda_i}{P_i}} \right) + (h_i + v_i I_c) \left( \frac{1 - X_i}{1 - X_i - \frac{\lambda_i}{P_i}} \right) \right] \left[ h_i \left( 1 - \frac{\lambda_i}{P_i} \right) + v_i I_c \right]$$

$$\geq (h_i + v_i I_c) \left( \frac{1 - X_i}{1 - X_i - \frac{\lambda_i}{P_i}} \right) \left( h_i + v_i I_c - h_i \frac{\lambda_i}{P_i} \right)$$

$$\geq (h_i + v_i I_c)^2 \left( \frac{(1 - X_i)(P_i - \lambda_i)}{(1 - X_i)P_i - \lambda_i} \right) \geq (h_i + v_i I_c)^2 = (W_1^1)^2$$

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