On Sprays with Vanishing χ-Curvature

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Abstract

Every Riemannian metric or Finsler metric on a manifold induces a spray via its geodesics. In this paper, we discuss several expressions for the χ-curvature of a spray. We show that the sprays obtained by a projective deformation using the S-curvature always have vanishing χ-curvature. Then we establish the Beltrami Theorem for sprays with χ = 0.

Keywords: Sprays, Isotropic curvature, χ-curvature and S-curvature.

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1 Introduction

A spray $G$ on a manifold $M$ is a special vector field on the tangent bundle $TM$. In a standard local coordinate system $(x^i, y^j)$ in $TM$, a spray $G$ can be expressed by

$$G = y^j \frac{\partial}{\partial x^j} - 2G^i \frac{\partial}{\partial y^i},$$

where $G^i = G^i(x, y)$ are local $C^\infty$ functions on non-zero vectors with the following homogeneity: $G^i(x, \lambda y) = \lambda^2 G^i(x, y)$, $\forall \lambda > 0$. Every Finsler metric induces a spray on a manifold. Some geometric quantities of a Finsler metric are actually defined by the induced spray only. These quantities are extremely interesting to us.

For a spray $G$ on a manifold $M$, with the Berwald connection, we define two key quantities: the Riemann curvature tensor $R^i_{jkl}$ and the Berwald curvature tensor $B^i_j{}_{kl}$ (see [6]). Certain averaging process gives rise to various notions of Ricci curvature tensor. One of them is the Ricci curvature tensor: $\text{Ric}_{jl} := \frac{1}{2} \{ R^m_{jl} y^m y^l + R^m_{jm} y^l y^j \}$ ([2]). The well known Ricci curvature $\text{Ric} := \text{Ric}_{jl} y^j y^l$ has been studied for a long time by many people. Besides these quantities, we have another important quantity which is expressed in terms of the vertical derivatives of the Riemann curvature. It is the so-called χ-curvature defined by

$$\chi_k := -\frac{1}{6} \left\{ 2R^m_{k \cdot m} + R^m_{m \cdot k} \right\}. \quad \text{(1)}$$

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where $R^i_{jk} = y^j R_{j kl}^i y^l$. The $\chi$-curvature can be expressed in several forms. For an arbitrary volume form $dV$,

$$\chi_k = \frac{1}{2} \left\{ S_{k|m} y^m - S^k \right\},$$

(2)

where $S = S(G, dV)$ is the S-curvature of $(G, dV)$ ([5]). For a spray induced by a Finsler metric, the $\chi$-curvature can be expressed by

$$\chi_k = \frac{1}{2} \left\{ \sum_{p} I_{k|p} y^p y^q + \sum_{m} R^m_{k} \right\},$$

(3)

where $I_k := g^{ij} C_{ijk}$ denotes the mean Cartan torsion ([4] [1]). These are three typical expressions for the mysterious quantity $\chi$. In this paper, we shall focus on sprays with $\chi = 0$.

For a spray $G$ on a manifold $M$, in the projectively equivalent class of $G$, there is always a spray with $\chi = 0$. More precisely, for any volume form $dV$ on $M$, we may construct a spray $\hat{G}$ by a projective change:

$$\hat{G}^i := G^i - S_{n+1} y^i,$$

where $S$ is the S-curvature of $(G, dV)$. This spray $\hat{G}$ is invariant under a projective change with $dV$ fixed. This projective deformation is first introduced in [6]. We prove the following

**Theorem 1.1** Let $G$ be a spray on a manifold $M$. For any volume form $dV$, the spray $\hat{G}$ associated with $(G, dV)$ has vanishing $\chi$-curvature, $\hat{\chi} = 0$.

Note that $\hat{G}$ is projectively equivalent to $G$. Hence if $G$ is of scalar curvature, then $\hat{G}$ is of scalar curvature too. Hence it is of isotropic curvature since $\hat{\chi} = 0$. Thus $\hat{G}$ must be of isotropic curvature. We obtain the following

**Corollary 1.2** Let $G$ be a spray of scalar curvature on a manifold $M$. For any volume form $dV$, the spray $\hat{G}$ associated with $(G, dV)$ must be of isotropic curvature.

The well-known Beltrami Theorem in Riemannian geometry says that for two projectively equivalent Riemannian metrics $g_1, g_2$, the metric $g_1$ is of constant curvature if and only if $g_2$ is of constant curvature. In particular, if a Riemannian metric $g$ is locally projectively flat, then it is of constant curvature since $g$ is locally projectively equivalent to the standard Euclidean metric. This theorem can be extended to sprays with $\chi = 0$.

**Theorem 1.3** For two projectively equivalent sprays $G_1, G_2$ with $\chi = 0$, $G_1$ is of isotropic curvature if and only if $G_2$ is of isotropic curvature. In particular, if a spray $G$ is locally projectively flat with $\chi = 0$, then it is of isotropic curvature.

Sprays or Finsler metrics with $\chi = 0$ deserve further study. Spherically symmetric metrics with $\chi = 0$ have been studied in [9].

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2 Preliminaries

A spray $G$ on a manifold $M$ is a vector field on the tangent bundle $TM$ which is locally expressed in the following form

$$G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i},$$

where $G^i = G^i(x, y)$ are local $C^\infty$ functions on $TU = U \times \mathbb{R}^n$,

$$G^i(x, \lambda y) = \lambda^2 G^i(x, y), \quad \lambda > 0.$$

Put

$$N^i_j := \frac{\partial G^i}{\partial y^j}, \quad \Gamma^i_{jk} = \frac{\partial^2 G^i}{\partial y^j \partial y^k}.$$

Let $\omega^i := dx^i$ and $\omega^{n+i} := dy^i + N^i_j dx^j$ and $\omega^i := \Gamma^i_{jk} dx^k$. We have

$$d\omega^i = \omega^j \wedge \omega^i_{jk}.$$

Put

$$\Omega^i_j := d\omega^i - \omega^k \wedge \omega^i_{jk}.$$

We obtain two quantities $R$ and $B$:

$$\Omega^i_j = 1/2 R^i_{jk} \omega^k \wedge \omega^l - B^i_{jk} \omega^k \wedge \omega^{n+l},$$

where $R^i_{jk} + R^i_{kj} = 0$.

$$R^i_{jk} = \frac{\delta \Gamma^i_{jl}}{\delta x^k} - \frac{\delta \Gamma^i_{jk}}{\delta x^l} + \Gamma^i_{ks} \Gamma^s_{jl} - \Gamma^s_{jk} \Gamma^i_{ls},$$

$$B^i_{jk} = \frac{\partial \Gamma^i_{kl}}{\partial y^j}.$$  \hspace{1cm} (4)

We have the first set of Bianchi identities

$$R^i_{jkl} + R^i_{kjl} + R^i_{ljk} = 0$$  \hspace{1cm} (5)

$$B^i_{jk} = B^i_{kj}.$$  \hspace{1cm} (6)

In fact $B^i_{jk}$ is symmetric in $j, k, l$ and $y^l B^i_{jk} = 0$. Put

$$R^i_{kl} := y^l R^i_{jkl}, \quad R^i_{jkl} := R^i_{jkl}, \quad R^i_{k} := y^i R^i_{jkl}, \quad R^i_{k} := y^i R^i_{jkl}.$$

The two-index Riemann curvature tensor $R^i_{kl}$ and the four-index Riemann curvature tensor $R^i_{jkl}$ determine each other by the following identity:

$$R^i_{jkl} = 1/3 \{ R^i_{kjl} - R^i_{ljk} \},$$  \hspace{1cm} (7)
We also have

\[ R^i_{jk} = \frac{1}{3} \left\{ 2R^i_{kj} + R^i_{j,k} \right\}, \quad (8) \]
\[ R^i_{kl} = \frac{1}{3} \left\{ R^i_{k,l} - R^i_{l,k} \right\}, \quad (9) \]

where \( T^*_{*,k} \) is the vertical covariant derivative, namely, \( T^*_{*,k} = \frac{\partial}{\partial y^k} (T^*_*) \).

Further covariant derivatives yield the second set of Bianchi identities:

\[ R^i_{jk\mid m} + R^i_{jm\mid k} + R^i_{mk\mid j} + B^i_{j\mid m} R^p_{k\mid l} R^p_{m\mid k} + B^i_{j\mid l} R^p_{m\mid k} R^p_{l\mid m} = 0 \quad (10) \]
\[ R^i_{jkl\mid m} = B^i_{j\mid m} - B^i_{j\mid km\mid l} \quad (11) \]
\[ B^i_{j\mid kl\cdot m} = B^i_{j\mid km\cdot l} \quad (12) \]

Contracting (10) with \( y_j \) yields

\[ R^i_{kl\mid m} + R^i_{lm\mid k} + R^i_{mk\mid l} = 0 \quad (13) \]

Contracting (13) with \( y_l \) yields

\[ R^i_{k\mid m} - R^i_{m\mid k} + R^i_{mk\mid j} y^j = 0 \quad (14) \]

### 3 The \( \chi \)-curvature

The \( \chi \)-curvature defined by the Riemann curvature tensor in (1) can be expressed in several ways.

**Lemma 3.1**

\[ \chi_k = -\frac{1}{2} R^m_{k\mid m} = -\frac{1}{2} R^m_{m\mid k} y^j. \quad (15) \]

**Proof:** It follows from (8). Q.E.D.

Lemma 3.1 tells us that if \( R^m_{m\mid k} = 0 \), then \( \chi = 0 \).

Put

\[ T^i_k := R^i_k - \left\{ \frac{1}{2} R^i_{j\mid k} - \frac{1}{2} R^i_{k\mid j} y^j \right\}, \quad (16) \]

where \( R := \frac{1}{n-1} R^m_m \). By definition, \( G \) is of isotropic curvature if \( T^i_k = 0 \). Note that

\[ \text{trace}(T) := T^m_m = 0. \]

By a direct computation, we can obtain another expression for \( \chi_k \).

**Lemma 3.2**

\[ \chi_k = -\frac{1}{2} T^m_{m\mid k}. \quad (17) \]
Lemma 3.2 tells us that if \( G \) is of isotropic curvature, then \( \chi = 0 \).

Recall the definition of the Weyl curvature

\[
W^i_k := A^i_k - \frac{1}{n+1} A^m_{k,m} y^i,
\]

where \( A^i_k := R^i_k - R^i_k \). Clearly,

\[ W^m_{k,m} = 0. \]

We obtain a nice formula for the Weyl curvature.

**Lemma 3.3** The Weyl curvature is given by

\[
W^i_k = R^i_k - \left\{ R^i_k - \frac{1}{2} R_{k,y^i} \right\} + \frac{3}{n+1} \chi y^i. \tag{19}
\]

**Proof:** One can easily rewrite \( W^i_k \) as

\[
W^i_k = R^i_k - \left\{ R^i_k - \frac{1}{2} R_{k,y^i} \right\} - \frac{1}{2(n+1)} \left\{ 2 R^m_{k,m} + (n-1) R_{k} \right\} y^i.
\]

By (1), we prove the lemma. Q.E.D.

Given a volume \( dV = \sigma(x)dx^1 \cdots dx^n \), the S-curvature of \((G, dV)\) is defined by

\[
S := \Pi y^m \frac{\partial}{\partial x^m} \left( \ln \sigma \right).
\]

We have the following expression for \( \chi \).

**Lemma 3.4** ([2])

\[
\chi_k = \frac{1}{2} \left\{ S_{k|m} y^m - S_k \right\}. \tag{20}
\]

In local coordinates, by (20), one can easily get

\[
\chi_k = \frac{1}{2} \left\{ \Pi_{x^m y^k y^m} - \Pi_{x^k} - 2 \Pi_{y^k y^m} G^m \right\}, \tag{21}
\]

where \( \Pi := \frac{\partial G^m}{\partial y^m} \).

Clearly, \( \chi \) is independent of \( dV \).

## 4 Sprays with \( \chi = 0 \)

A spray is said to be \( S \)-closed if in local coordinates, \( \Pi = \frac{\partial G^m}{\partial y^m} \) is a closed local 1-form. The spray induced by a Riemannian metric \( g = g_{ij}(x)y^i y^j \) is \( S \)-closed.

In fact

\[
\Pi = y^k \frac{\partial}{\partial x^k} \left[ \ln \sqrt{\det(g_{ij}(x))} \right], \tag{22}
\]

where \( G^m = g_{ij}(x) y^i y^j \).
By (22), for any volume form $dV = \sigma(x)dx^1 \wedge \cdots \wedge dx^n$, the S-curvature of $(G, dV)$ is a closed 1-form,

$$S = y^k \frac{\partial}{\partial x^k} \ln \varphi(x),$$

where $\varphi(x) = \sqrt{\det(g_{ij}(x))}/\sigma(x)$.

We have the following

**Proposition 4.1** If a spray is S-closed, then $\chi = 0$. In particular, if for some volume form $dV = \sigma dx^1 \cdots dx^n$, the S-curvature of $(G, dV)$ is a closed 1-form, then $\chi = 0$.

**Proof:** By assumption,

$$S = \Pi - y^m \frac{\partial}{\partial x^m} (\ln \sigma) = \eta_k y^k,$$

with $(\eta_k)_x^i = (\eta_i)_x^k$. Then by (21), $\chi_k = 0$. Q.E.D.

Let $\tilde{F}$ be a Finsler metric and $G$ be a spray on a manifold $M$. The spray coefficients $\tilde{G}^i$ of $\tilde{F}$ can be expressed as follows

$$\tilde{G}^i = G^i + Py^i + \frac{1}{2} F_{\cdot k}^{ik} \left\{ \tilde{F}_{k|mn} y^m - \tilde{F}_{|k} \right\}. \quad (23)$$

where $P = \tilde{F}_{|mn} y^m/(2\tilde{F})$. Thus $\tilde{F}$ is projectively equivalent to $G$ if and only if

$$\tilde{F}_{k|mn} y^m - \tilde{F}_{|k} = 0. \quad (24)$$

This is the generalized version of the famous Rapcsák Theorem. By (20), we obtain the following

**Theorem 4.2** Let $G$ be a spray with $\chi = 0$ and $dV$ be a volume form. If for the S-curvature $S$ of $(G, dV)$, $\tilde{F} = |S|$ is a Finsler metric, then it is projectively equivalent to $G$.

### 5 Sprays of isotropic curvature

A spray $G$ is said to be of scalar curvature if

$$R^i_k = R\delta^i_k - \tau_k y^i, \quad (25)$$

where $\tau_k$ is a positively homogeneous function of degree one with $\tau_k y^k = R$. This is equivalent to that $W^i_k = 0$. By (19), we see that (25) is equivalent to the following

$$R^i_k = R\delta^i_k - \frac{1}{2} R_k y^i - \frac{3}{n+1} \chi_k y^i. \quad (26)$$

The $\chi$-curvature characterizes sprays of isotropic curvature among sprays of scalar curvature. By (26), we obtain the following
Theorem 5.1 ([3]) Let $G$ be a spray of scalar curvature. $G$ is of isotropic curvature if and only if $\chi = 0$.

**Proof of Theorem 1.3:** If $G_1$ is of isotropic curvature, then $G_2$ is of scalar curvature by the projective equivalence. Since $\chi = 0$, we see that $G_2$ is of isotropic curvature by Proposition 5.1. Q.E.D.

If $G$ is of isotropic curvature, then

$$R_{jkl}^i = \frac{1}{2} \left\{ R_{l,j}^i \delta_k^i - R_{k,j}^i \delta_l^i \right\},$$

$$R_{ikl}^i = \frac{1}{2} \left\{ R_{i,k}^i \delta_l^i - R_{k,l}^i \delta_i^i \right\}.$$

Assume that $G$ is of isotropic curvature. By (10), we obtain

$$(R_{i,j|m} - R_{m,j|i})\delta_k^i + (R_{m,j|k} - R_{k,j|m})\delta_l^i + (R_{k,j|l} - R_{l,j|k})\delta_m^i = 0. \quad (27)$$

This yields

$$(R_{i|m} - R_{m|i})\delta_k^i + (R_{m|k} - R_{k|m})\delta_l^i + (R_{k|l} - R_{l|k})\delta_m^i = 0. \quad (28)$$

Contracting (28) with $y^m$ yields

$$(R_{i|m} y^m - 2R_{i|i})\delta_k^i + (2R_{k|l} y^m)\delta_l^i + (R_{k|i} - R_{l|k})y^j = 0. \quad (29)$$

Taking trace $i = k$ in (29), we obtain

$$(n - 2)(R_{i|m} y^m - 2R_{i|i}) = 0. \quad (30)$$

**Theorem 5.2** If $G$ is an $n$-dimensional spray of isotropic curvature $R$ ($n \geq 3$), then $R$ satisfies

$$\frac{1}{2} R_{i|m} y^m - R_{i|i} = 0. \quad (31)$$

**Proof:** By assumption $n \geq 3$, we obtain from (30),

$$R_{i|i} - \frac{1}{2} R_{i|m} y^m = 0.$$

Q.E.D.

For a spray $G$, we introduce a new quantity $\eta = \eta_k dx^k$,

$$\eta_k := \frac{1}{2} R_{k|m} y^m - R_{k|k}, \quad (32)$$

where $R := \frac{1}{n-1} \text{Ric}$.

For a spray of isotropic curvature $R$ on $n$-dimensional manifold $M$ ($n \geq 3$), By Theorem 5.2, $\eta = 0$.  

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Let $L := \tilde{F}^2$ be a Finsler metric and $G$ be a spray on a manifold. By (23), the spray coefficients of $\tilde{F}$ can be expressed as

$$\tilde{G}^i = G^i + \frac{1}{4} \hat{g}^{ik} L_{m|y^m} y^i - \frac{1}{8} \hat{g}^{ik} L_{m|y^m} L_{-k} + \frac{1}{4} \hat{g}^{ik} \left\{ L_{-k|y^m} - L_{-|k} \right\} \quad (33)$$

We obtain

$$\tilde{G}^i = G^i + \frac{1}{4} \hat{g}^{ik} L_{|y^m} + \frac{1}{2} \hat{g}^{ik} \left\{ \frac{1}{2} L_{-k|y^m} - L_{-|k} \right\}. \quad (33)$$

In [7], we introduced a notion of dually flat Finsler metrics. This concept can be generalized as follows. $\tilde{F}$ is said to be dually equivalent to $G$ if $L := \tilde{F}^2$ satisfies

$$\tilde{G}^i = G^i + \frac{1}{4} \hat{g}^{ik} L_{|k}.$$

By (33), this is equivalent to

$$\frac{1}{2} L_{-k|y^m} - L_{-|k} = 0. \quad (34)$$

For a spray $G$ on an $n$-dimensional manifold $M$ with isotropic scalar curvature $R$. Assume that $R$ is a Finsler metric, by Theorem 5.2, $R$ satisfies (31). Thus one can see that $R$ is dually equivalent to $G$.

6 Projective change by the S-curvature

Let $G$ be a spray and $dV$ be a volume form on an $n$-dimensional manifold $M$. We deform $G$ to another spray $\hat{G}$ by

$$\hat{G}^i := G^i - \frac{S}{n + 1} y^i,$$

where $S$ denotes the S-curvature of $(G, dV)$. From the definition, we see that $\hat{G}$ is projectively equivalent to $G$.

Lemma 6.1 Let $G$ be a spray and $dV$ a volume form on a manifold $M$. Let $\hat{G}$ be the spray associated with $(G, dV)$. Then the S-curvature of $(G, dV)$ vanishes. Hence, $\hat{\chi} = 0$.

Proof: Recall

$$\hat{\chi}_k = \frac{1}{2} \left\{ \hat{S}_{m,k} y^m - \hat{S}_{-k} \right\}.$$ 

On the other hand, $\hat{G}^i = G^i + P y^i$ with $P = -\frac{S}{n + 1}$. Thus

$$\hat{S} = S + (n + 1) P = 0.$$ 

This yields that $\hat{\chi} = 0$. Q.E.D.
Lemma 6.2 If $G_1$ and $G_2$ are two projectively equivalent sprays on a manifold $M$, then for any volume form $dV$, the spray $\hat{G}_1$ associated with $(G_1, dV)$ and $G_2$ associated with $(G_2, dV)$ are equal, i.e., $G_1 = G_2$.

Proof: It is easy to see that if $G_1 = G_2 + Py^i$, then

$$S_1 = S_2 + (n + 1)P.$$  

Then

$$\hat{G}_1 = G_1 - \frac{S_1}{n + 1}y^i = [G_2 + Py^i] - \frac{S_2 + (n + 1)P}{n + 1}y^i = G_2 - \frac{S_2}{n + 1}y^i = \hat{G}_2.$$  

Q.E.D.

Proof of Corollary 1.2: First by definition, $\hat{G}$ is projectively equivalent to $G$. Thus $\hat{G}$ is of scalar curvature. Since $\hat{\chi} = 0$, by Lemma 5.1, we see that $\hat{G}$ is of isotropic curvature. Q.E.D.

By the above lemma, any geometric quantity of $\hat{G}$ is a projective invariant of $G$ with respect to a fixed volume form $dV$. Further, if the geometric quantity of $\hat{G}$ is independent of the volume form $dV$, then the quantity is a projective quantity of $G$.

Lemma 6.3 Let $G$ be a spray and $dV$ a volume form on a manifold $M$. For the spray $\hat{G}$ associated with $(G, dV)$, the Riemann curvature of $\hat{G}$ is given by

$$\hat{R}^i_k = R^i_k + \tau\delta^i_k - \frac{1}{2}\tau_{kj}y^j + \frac{3\chi_k}{n + 1}y^j,$$  

where

$$\tau := \left(\frac{S}{n + 1}\right)^2 + \frac{1}{n + 1}S_{\mid\mid m}y^m.$$  

Proof: By a direct argument. Q.E.D.

By (35), we get the projective Ricci curvature tensor $\hat{\text{Ric}}_{jl} := \frac{1}{2}\{\hat{R}^m_{jl} + \hat{R}^m_{lj}\}$ and the projective Ricci curvature $\hat{\text{Ric}} := \hat{\text{Ric}}_{jl}y^jy^l$.

$$\hat{\text{Ric}}_{jl} = \text{Ric}_{jl} + \frac{n - 1}{2}\tau_{jl} + H_{jl},$$  

$$\hat{\text{Ric}} = \text{Ric} + (n - 1)\tau,$$  

where

$$\tau := \left(\frac{S}{n + 1}\right)^2 + \frac{1}{n + 1}S_{\mid\mid m}y^m.$$
where $\hat{\text{Ric}} = \hat{\text{Ric}}_{jil}y^l y^i$ is the Ricci curvature of $\hat{\mathcal{G}}$ and

$$H_{ij} := \frac{1}{2}\left\{\chi_{i,j} + \chi_{j,i}\right\}.$$ 

It is natural to consider other quantities of $\hat{\mathcal{G}}$, such as the Berwald curvature defined in (4) and the T-curvature defined in (16)

$$\hat{B}_{j}^{i} = \frac{\partial^{3} \hat{G}^{i}}{\partial y^{j} \partial y^{k} \partial y^{l}},$$

$$\hat{T}_{i}^{j} = \hat{R}^{i}_{j} - \left\{\hat{R}^{i}_{k} - \frac{1}{2}\hat{R}_{k}y^{j}\right\}.$$ 

Clearly, $\hat{B}$ and $\hat{T}$ are projective invariants with a fixed volume form $dV$. We have the following

**Proposition 6.4** Let $G$ be a spray on a manifold and $\hat{\mathcal{G}}$ a spray associated with $(G,dV)$ for some volume form $dV$. Then the Berwald curvature $\hat{B}$ and $\hat{T}$ are independent of $dV$, hence they are projective invariants of $G$. In fact $\hat{B} = D$ is the Douglas curvature and $\hat{T} = W$ is the Weyl curvature of $G$.

Here we provide another description of the Douglas curvature and the Weyl curvature of a spray.

Let $G$ be a spray and $\hat{\mathcal{G}}$ be the spray associated with $(G,dV)$ for some volume form $dV$. Let $\hat{\eta}$ be the quantity of $\hat{\mathcal{G}}$ defined in (32). Then $\hat{\eta}$ is a projective invariant of $G$ for a fixed volume form $dV$. In fact, $\hat{\eta} = W^0$ the so-called Berwald-Weyl curvature ([6]). If $G$ is of scalar curvature, then $\hat{\mathcal{G}}$ is of isotropic curvature. Thus $\hat{\eta} = 0$ when $n = \text{dim} M \geq 3$ by Theorem 5.2.

**Proposition 6.5** Let $G$ be a spray on a manifold and $\hat{\mathcal{G}}$ a spray associated with $(G,dV)$ for some volume form $dV$. Assume that $G$ is of scalar curvature. Then the projective invariant $\hat{\eta} = 0$ in dimension $n \geq 3$.

## 7 Examples

In this section, we shall give some sprays of isotropic curvature.

**Example 7.1** Let $F = \alpha + \beta$ be a Randers metric on an $n$-dimensional manifold $M$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on $M$. Let $\nabla \beta = b_lij y^dx^j$ denote the covariant derivative of $\beta$ with respect to $\alpha$. Let

$$r_{ij} := \frac{1}{2}(b_{ij} + b_{j|i}),$$

$$s_{ij} := \frac{1}{2}(b_{ij} - b_{j|i}),$$

$$s_j := b^is_{ij},$$

$$q_{ij} := r_{im}s_{mj}^i,$$

$$t_{ij} := s_{im}s_{mj}^i,$$

$$t_j := b^it_{ij}.$$
Let
\[ \hat{G}^i := G_i^\alpha + \alpha s_i^\alpha. \]  
(39)

In fact \( \hat{G} \) is the spray associated with \((G, dV_\alpha)\). It is proved that \( \hat{G} \) is of scalar curvature if and only if the Riemann curvature \( \bar{R}_k^i \) of \( \alpha \) and the covariant derivatives of \( \beta \) satisfy the following equations ([8])
\[ \bar{R}_k^i = \kappa \left\{ \alpha^2 \delta_k^i - y_k y_i \right\} \]
\[ + \alpha^2 t_k^i + t_{00} \delta_k^i - t_{k0} y_i - t_o^i y_k - 3 s_i^\alpha \delta_k^0, \]  
(40)
\[ s_{ij|k} = \frac{1}{n-1} \left\{ a_{ik} s_m^j|_m - a_{jk} s_i^m\right\}. \]  
(41)

where \( \kappa = \kappa(x) \) is a scalar function on \( M \). In this case, \( \hat{G} \) is actually of isotropic curvature. \( \bar{R}_k^i = \hat{R}_k^i - \frac{1}{2} \bar{R}_{k|i} y^i \). By a simple computation, we obtain a formula for \( \hat{R} := \frac{1}{n-1} \text{Ric} \):
\[ \hat{R} = \kappa \alpha^2 + t_{00} + \frac{2}{n-1} \alpha s_m^m. \]

Example 7.2 Consider a spray on an open subset \( U \subset R^2 \),
\[ G = y^1 \frac{\partial}{\partial x^1} + y^2 \frac{\partial}{\partial x^2} - 2 G^1 \frac{\partial}{\partial y^1} - 2 G^2 \frac{\partial}{\partial y^2}, \]
where
\[ G^1 = B(y^1)^2 + 2 C y^1 y^2 + D(y^2)^2 + \frac{1}{3} (f_{x^1}(y^1)^2 + f_{x^2}(y^1 y^2)) \]
\[ G^2 = -A(y^1)^2 - 2 B y^1 y^2 - C(y^2)^2 + \frac{1}{3} (f_{x^1}(y^1)^2 + f_{x^2}(y^1 y^2)^2). \]

where
\[ A = A(x^1, x^2), \ B = B(x^1, x^2), \ C = C(x^1, x^2), \ D = D(x^1, x^2), \ f = f(x^1, x^2) \]
are \( C^\infty \) functions on \( U \). The geodesics are the graphs of \( x^2 = \phi(x^1) \)
\[ \phi'' = 2A(x^1, \phi) + 6B(x^1, \phi) \phi' + 6C(x^1, \phi) (\phi')^2 + 2D(x^1, \phi) (\phi')^3. \]

We have
\[ \Pi = \frac{\partial G^m}{\partial y^m} = f_{x^1} y^1 + f_{x^2} y^2. \]

Thus \( \chi_k = 0 \). Further computation shows that \( G \) is of isotropic curvature.
References

[1] X. Cheng and Z. Shen, *Finsler Geometry — An approach via Randers spaces*, Springer-Verlag, (2012)

[2] B. Li and Z. Shen, *Ricci curvature tensor and non-Riemannian quantities*, Canadian Mathematical Bulletin, 58(2015), 530-537.

[3] B. Li and Z. Shen, *On sprays of isotropic curvature*, International Journal of Mathematics, 29 (2018), https://doi.org/10.1142/S0129167X18500039

[4] Z. Shen, *Finsler manifolds with nonpositive flag curvature and constant S-curvature*, Mathematische Zeitschrift, 249(2005), 625-639.

[5] Z. Shen, *On some non-Riemannian quantities in Finsler geometry*, Canad. Math. Bull. 56(2013), 184-193.

[6] Z. Shen, *Differential Geometry of Spray and Finsler Spaces*, Kluwer Academic Publishers, 2001.

[7] Z. Shen, *Riemann-Finsler geometry with applications to information geometry*, Chinese Ann. of Math. Ser. B 27 (2006), 73-94.

[8] Z. Shen and G. C. Yildirim, *A characterization of Randers metrics of scalar flag curvature*, Recent Developments in Geometry and Analysis, Advanced Lectures in Mathematics 23 (2013), 345-358.

[9] H. Zhu, *On a class of Finsler metrics with special curvature properties*, Balkan Journal of Geometry and Its Applications, 23 (2018), 97-108.

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