A DIXMIER-DOUADY THEORY FOR STRONGLY SELF-ABSORRING
C*-ALGEBRAS II: THE BRAUER GROUP

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Abstract. We have previously shown that the isomorphism classes of orientable locally trivial
fields of C*-algebras over a compact metrizable space X with fiber D ⊗ K, where D is a strongly
self-absorbing C*-algebra, form an abelian group under the operation of tensor product. Moreover
this group is isomorphic to the first group \( E_1^D(X) \) of the (reduced) generalized cohomology theory
associated to the unit spectrum of topological K-theory with coefficients in D. Here we show that
all the torsion elements of the group \( E_1^D(X) \) arise from locally trivial fields with fiber \( D ⊗ M_n(C) \),
\( n \geq 1 \), for all known examples of strongly self-absorbing C*-algebras D. Moreover the Brauer group
generated by locally trivial fields with fiber \( D ⊗ M_n(C) \), \( n \geq 1 \) is isomorphic to \( \text{Tor}(E_1^D(X)) \).

Keywords: strongly self-absorbing, C*-algebras, Dixmier-Douady class, Brauer group, torsion, op-
posite algebra
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1. Introduction

Let X be a compact metrizable space. Let K denote the C*-algebra of compact operators on an
infinite dimensional separable Hilbert space. It is well-known that K ⊗ K ≅ K and \( M_n(C) ⊗ K ≅ K \).
Dixmier and Douady [7] showed that the isomorphism classes of locally trivial fields of C*-algebras
over X with fiber K form an abelian group under the operation of tensor product over C(X) and
this group is isomorphic to \( H^3(X,Z) \). The torsion subgroup of \( H^3(X,Z) \) admits the following
description. Each element of \( \text{Tor}(H^3(X,Z)) \) arises as the Dixmier-Douady class of a field A which
is isomorphic to the stabilization \( B ⊗ K \) of some locally trivial field of C*-algebras B over X with
all fibers isomorphic to \( M_n(C) \) for some integer \( n \geq 1 \), see [8], [1].

In this paper we generalize this result to locally trivial fields with fiber \( D ⊗ K \) where D is a strongly self-absorbing C*-algebra [17]. For a C*-algebra B, we denote by \( \mathcal{C}_B(X) \) the isomor-
phism classes of locally trivial continuous fields of C*-algebras over X with fibers isomorphic to B. The isomorphism classes of orientable locally trivial continuous fields is denoted by \( \mathcal{C}^0_B(X) \), see Definition 2.2. We have shown in [4] that \( \mathcal{C}_{D⊗K}(X) \) is an abelian group under the operation
of tensor product over C(X), and moreover, this group is isomorphic to the first group \( E^1_D(X) \) of a
generalized cohomology theory \( E^*_D(X) \) which we have proven to be isomorphic to the theory
associated to the unit spectrum of topological K-theory with coefficients in D, see [5]. Similarly
\( (\mathcal{C}^0_{D⊗K}(X), ⊗) ≅ E^1_D(X) \) where \( E^*_D(X) \) is the reduced theory associated to \( E^*_D(X) \). For D = C, we have, of course, \( E^1_C(X) ≅ H^3(X,Z) \).

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We consider the stabilization map $\sigma : \mathcal{C}_{D \otimes M_n(\mathbb{C})}(X) \to (\mathcal{C}_{D \otimes \mathbb{K}}(X), \otimes) \cong E_D^1(X)$ given by $[A] \mapsto [A \otimes \mathbb{K}]$ and show that its image consists entirely of torsion elements. Moreover, if $D$ is any of the known strongly self-absorbing $C^*$-algebras, we show that the stabilization map
\[
\sigma : \bigcup_{n \geq 1} \mathcal{C}_{D \otimes M_n(\mathbb{C})}(X) \to \text{Tor}(\tilde{E}_D^1(X))
\]
is surjective, see Theorem 2.10. In this situation $\mathcal{C}_{D \otimes M_n(\mathbb{C})}(X) \cong \mathcal{C}_{D \otimes M_n(\mathbb{C})}(X)$ by Lemma 2.2 and hence the image of the stabilization map is contained in the reduced group $\overline{E}_D^1(X)$. In analogy with the classical Brauer group generated by continuous fields of complex matrices $M_i$, we introduce a Brauer group $\text{Br}_D(X)$ for locally trivial fields of $C^*$-algebras with fibers $M_n(D)$ for $D$ a strongly self-absorbing $C^*$-algebra and establish an isomorphism $\text{Br}_D(X) \cong \text{Tor}(\tilde{E}_D^1(X))$, see Theorem 2.15.

Our proof is new even in the classic case $D = \mathbb{C}$ whose original proof relies on an argument of Serre, see [8, Thm.1.6], [1, Prop.2.1]. In the cases $D = \mathbb{Z}$ or $D = \mathcal{O}_\infty$ the group $\tilde{E}_D^1(X)$ is isomorphic to $H^1(X, BSU_\otimes)$, which appeared in [20], where its equivariant counterpart played a central role.

We introduced in [4] characteristic classes
\[
\delta_0 : E_D^1(X) \to H^1(X, K_0(D)_\otimes) \quad \text{and} \quad \delta_k : E_D^1(X) \to H^{2k+1}(X, \mathbb{Q}), \quad k \geq 1.
\]
If $X$ is connected, then $\tilde{E}_D^1(X) = \ker(\delta_0)$. We show that an element $a$ belongs $\text{Tor}(\tilde{E}_D^1(X))$ if and only if $\delta_0(a)$ is a torsion element and $\delta_k(a) = 0$ for all $k \geq 1$.

In the last part of the paper we show that if $A^{op}$ is the opposite $C^*$-algebra of a locally trivial continuous field $A$ with fiber $D \otimes \mathbb{K}$, then $\delta_k(A^{op}) = (-1)^k \delta_k(A)$ for all $k \geq 0$. This shows that in general $A \otimes A^{op}$ is not isomorphic to a trivial field, unlike what happens in the case $D = \mathbb{C}$. Similar arguments show that in general $[A^{op}]_{Br} \neq [-A]_{Br}$ in $\text{Br}_D(X)$ for $A \in \mathcal{C}_{D \otimes M_n(\mathbb{C})}(X)$, see Example 3.5.

We would like to thank Ilan Hirshberg for prompting us to seek a refinement of Theorem 2.10 in the form of Theorem 2.11.

2. Background and main result

The class of strongly self-absorbing $C^*$-algebras was introduced by Toms and Winter [17]. They are separable unital $C^*$-algebras $D$ singled out by the property that there exists an isomorphism $D \to D \otimes D$ which is unitarily homotopic to the map $d \mapsto d \otimes 1_D$ [6], [19].

If $n \geq 2$ is a natural number we denote by $M_n(\mathbb{C})$ the UHF-algebra $\overline{M_n(\mathbb{C})}_\otimes$. If $P$ is a nonempty set of primes, we denote by $M_{P^{\infty}}$ the UHF-algebra of infinite type $\bigotimes_{p \in P} M_{P^{\infty}}$. If $P$ is the set of all primes, then $M_{P^{\infty}}$ is the universal UHF-algebra, which we denote by $M_{\mathbb{Q}}$.

The class $D_{pi}$ of all purely infinite strongly self-absorbing $C^*$-algebras that satisfy the Universal Coefficient Theorem in KK-theory (UCT) was completely described in [17]. $D_{pi}$ consists of the Cuntz algebras $O_2$, $O_\infty$ and of all $C^*$-algebras $M_{P^{\infty}} \otimes O_\infty$ with $P$ an arbitrary set of primes. Let $D_{qd}$ denote the class of strongly self-absorbing $C^*$-algebras which satisfy the UCT and which are quasidiagonal. A complete description of $D_{qd}$ has become possible due to the recent results of Matui and Sato [13, Cor. 6.2] that build on results of Winter [18], and Lin and Niu [12]. Thus $D_{qd}$ consists of $\mathbb{C}$, the Jiang-Su algebra $\mathcal{Z}$ and all UHF-algebras $M_{P^{\infty}}$ with $P$ an arbitrary set of primes.
The class $D = D_{qd} \cup D_{pi}$ contains all known examples of strongly self-absorbing $C^*$-algebras. It is closed under tensor products. If $D$ is strongly self-absorbing, then $K_0(D)$ is a unital commutative ring. The group of positive invertible elements of $K_0(D)$ is denoted by $K_0(D)^+_\ast$.

Let $B$ be a $C^*$-algebra. We denote by $\text{Auto}_0(B)$ the path component of the identity of $\text{Auto}(B)$ endowed with the point-norm topology. Recall that we denote by $\mathscr{C}_B(X)$ the isomorphism classes of locally trivial continuous fields over $X$ endowed with the point-norm topology. Recall that we denote by $\mathscr{C}_B(X)$ the isomorphism classes of locally trivial continuous fields over $X$ with fibers isomorphic to $B$. The structure group of $A \in \mathscr{C}_B(X)$ is $\text{Auto}(B)$, and $A$ is in fact given by a principal $\text{Auto}(B)$-bundle which is determined up to an isomorphism by an element of the homotopy classes of continuous maps from $X$ to the classifying space of the topological group $\text{Auto}(B)$, denoted by $[X, B\text{Auto}(B)]$.

**Definition 2.1.** A locally trivial continuous field $A$ of $C^*$-algebras with fiber $B$ is orientable if its structure group can be reduced to $\text{Auto}_0(B)$, in other words if $A$ is given by an element of $[X, B\text{Auto}_0(B)]$.

The corresponding isomorphism classes of orientable and locally trivial fields is denoted by $\mathscr{C}_B^0(X)$.

**Lemma 2.2.** Let $D$ be a strongly self-absorbing $C^*$-algebra satisfying the UCT. Then $\text{Aut}(M_n(D)) = \text{Aut}_0(M_n(D))$ for all $n \geq 1$ and hence $\mathscr{C}_{D \otimes M_n(C)}(X) \cong \mathscr{C}_{D \otimes M_n(C)}^0(X)$.

**Proof.** First we show that for any $\beta \in \text{Aut}(D \otimes M_n(C))$ there exist $\alpha \in \text{Aut}(D)$ and a unitary $u \in D \otimes M_n(C)$ such that $\beta = u(\alpha \otimes 1_{M_n(C)})u^*$. Let $e_{11} \in M_n(C)$ be the rank-one projection that appears in the canonical matrix units $(e_{ij})$ of $M_n(C)$ and let $1_n$ be the unit of $M_n(C)$. Then $n[1_D \otimes e_{11}] = [1_D \otimes 1_n]$ in $K_0(D)$ and hence $n[\beta(1_D \otimes e_{11})] = n[1_D \otimes e_{11}]$ in $K_0(D)$. Under the assumptions of the lemma, it is known that $K_0(D)$ is torsion free (by [17] and that $D$ has cancellation of full projections by [19] and [15]). It follows that there is a partial isometry $v \in D \otimes M_n(C)$ such that $v^*v = 1_D \otimes e_{11}$ and $vv^* = \beta(1_D \otimes e_{11})$. Then $u = \sum_{i=1}^n \beta(1_D \otimes e_{11})v(1_D \otimes e_{11}) \in D \otimes M_n(C)$ is a unitary such that the automorphism $u^*\beta u$ acts identically on $1_D \otimes M_n(C)$. It follows that $u^*\beta u = \alpha \otimes 1_{M_n(C)}$ for some $\alpha \in \text{Aut}(D)$. Since both $U(D \otimes M_n(C))$ and $\text{Auto}(D)$ are path connected by [17], [15] and respectively [6] we conclude that $\text{Aut}(D \otimes M_n(C))$ is path-connected as well.

Let us recall the following results contained in Cor. 3.7, Thm. 3.8 and Cor. 3.9 from [4]. Let $D$ be a strongly self-absorbing $C^*$-algebra.

1. The classifying spaces $B\text{Auto}(D \otimes \mathbb{K})$ and $B\text{Auto}_0(D \otimes \mathbb{K})$ are infinite loop spaces giving rise to generalized cohomology theories $E^2_D(X)$ and respectively $E^2_D^0(X)$.

2. The monoid $(\mathscr{C}_{D \otimes \mathbb{K}}(X), \otimes)$ is an abelian group isomorphic to $E^1_D(X)$. Similarly, the monoid $(\mathscr{C}_{D \otimes \mathbb{K}}^0(X), \otimes)$ is a group isomorphic to $E^1_D^0(X)$. In both cases the tensor product is understood to be over $C(X)$.

3. $E^1_{M_Q}(X) \cong H^1(X, \mathbb{Q}^\times) \oplus \bigoplus_{k \geq 1} H^{2k+1}(X, \mathbb{Q})$,

$E^1_{M_Q \otimes \mathcal{O}_\infty}(X) \cong H^1(X, \mathbb{Q}^\times) \oplus \bigoplus_{k \geq 1} H^{2k+1}(X, \mathbb{Q})$,

4. $E^1_{M_Q \otimes \mathcal{O}_\infty}(X) \cong E^1_{M_Q \otimes \mathcal{O}_\infty}^0(X) \cong \bigoplus_{k \geq 1} H^{2k+1}(X, \mathbb{Q})$.

5. If $D$ satisfies the UCT then $D \otimes M_Q \otimes \mathcal{O}_\infty \cong M_Q \otimes \mathcal{O}_\infty$, by [17]. Therefore the tensor product operation $A \mapsto A \otimes M_Q \otimes \mathcal{O}_\infty$ induces maps $\mathscr{C}_{D \otimes \mathbb{K}}(X) \rightarrow \mathscr{C}_{M_Q \otimes \mathcal{O}_\infty \otimes \mathbb{K}}(X)$, $\mathscr{C}_{D \otimes \mathbb{K}}^0(X) \rightarrow \mathscr{C}_{M_Q \otimes \mathcal{O}_\infty \otimes \mathbb{K}}^0(X)$ and hence
The first class \( \delta \) is a topological group and \( \delta(A) \in H^{2k+1}(X,\mathbb{Q}) \).

\[ \delta(A) = \left( \delta_0(A), \delta_1(A), \delta_2(A), \ldots \right), \quad \delta_k(A) \in H^{2k+1}(X,\mathbb{Q}), \]

\[ \bar{E}_D(X) \xrightarrow{\delta} \bar{E}_{M_0 \otimes \mathcal{O}_\infty}(X) \cong \bigoplus_{k \geq 1} H^{2k+1}(X,\mathbb{Q}), \]

\[ \bar{E}_D(X) \xrightarrow{\delta} \bar{E}_{M_0 \otimes \mathcal{O}_\infty}(X) \cong \bigoplus_{k \geq 1} H^{2k+1}(X,\mathbb{Q}), \]

The invariants \( \delta_k(A) \) are called the rational characteristic classes of the continuous field \( A \), see \([4]\) Def.4.6. The first class \( \delta_0 \) lifts to a map \( \delta_0 : E_D^1(X) \to H^1(X,K_0(D)_+) \) induced by the morphism of groups \( \text{Aut}(D \otimes \mathbb{K}) \to \pi_0(\text{Aut}(D \otimes \mathbb{K})) \cong K_0(D)_+ \). \( \delta_0(A) \) represents the obstruction to reducing the structure group of \( A \) to \( \text{Aut}_0(D \otimes \mathbb{K}) \).

**Proposition 2.3.** A continuous field \( A \in \mathcal{C}_{D \otimes \mathbb{K}}(X) \) is orientable if and only if \( \delta_0(A) = 0 \). If \( X \) is connected, then \( \bar{E}_D(X) \cong \ker(\delta_0) \).

**Proof.** Let us recall from \([4]\) Cor. 2.19 that there is an exact sequence of topological groups

\[ 1 \to \text{Aut}_0(D \otimes \mathbb{K}) \to \text{Aut}(D \otimes \mathbb{K}) \xrightarrow{\pi} K_0(D)_+ \to 1. \]

The map \( \pi \) takes an automorphism \( \alpha \) to \( [\alpha(1_D \otimes e)] \) where \( e \in \mathbb{K} \) is a rank-one projection. If \( G \) is a topological group and \( H \) is a normal subgroup of \( G \) such that \( H \rightarrow G \rightarrow G/H \) is a principal \( H \)-bundle, then there is a homotopy fibre sequence \( \text{G/H} \rightarrow 	ext{B} \text{H} \rightarrow 	ext{B} \text{G} \rightarrow 	ext{B(G/H)} \) and hence an exact sequence of pointed sets \( [X,G/H] \to [X,BH] \to [X,BG] \to [X,B(G/H)] \). In particular, in the case of the fibration \([1]\) we obtain

\[ [X,K_0(D)_+] \to [X,B\text{Aut}_0(D \otimes \mathbb{K})] \to [X,B\text{Aut}(D \otimes \mathbb{K})] \xrightarrow{\delta_0} H^1(X,K_0(D)_+). \]

A continuous field \( A \in \mathcal{C}_{D \otimes \mathbb{K}}(X) \) is associated to a principal \( \text{Aut}(D \otimes \mathbb{K}) \)-bundle whose classifying map gives a unique element in \( [X,B\text{Aut}(D \otimes \mathbb{K})] \) whose image in \( H^1(X,K_0(D)_+) \) is denoted by \( \delta_0(A) \). It is clear from \([2]\) that the class \( \delta_0(A) \in H^1(X,K_0(D)_+) \) represents the obstruction for reducing this bundle to a principal \( \text{Aut}_0(D \otimes \mathbb{K}) \)-bundle. If \( X \) is connected, \( [X,K_0(D)_+] = \{\ast\} \) and hence \( \bar{E}_D(X) \cong \ker(\delta_0) \).

**Remark 2.4.** If \( D = \mathbb{C} \) or \( D = \mathbb{Z} \) then \( A \) is automatically orientable since in those cases \( K_0(D)_+ \) is the trivial group.

**Remark 2.5.** Let \( Y \) be a compact metrizable space and let \( X = \Sigma Y \) be the suspension of \( Y \). Since the rational Künneth isomorphism and the Chern character on \( K^0(X) \) are compatible with the ring structure on \( K_0(C(Y) \otimes D) \), we obtain a ring homomorphism

\[ \text{ch}: K_0(C(Y) \otimes D) \to K^0(Y) \otimes K_0(D) \otimes \mathbb{Q} \to \prod_{k=0}^{\infty} H^{2k}(Y,\mathbb{Q}) =: H^{ev}(Y,\mathbb{Q}), \]

which restricts to a group homomorphism \( \text{ch}: E^0_D(Y) \to SL_1(H^{ev}(Y,\mathbb{Q})) \), where the right hand side denotes the units, which project to \( 1 \in H^0(Y,\mathbb{Q}) \). If \( A \) is an orientable locally trivial continuous field with fiber \( D \otimes \mathbb{K} \) over \( X \), then we have

\[ \delta_k(A) = \log \text{ch}(f_A) \in H^{2k}(Y,\mathbb{Q}) \cong H^{2k+1}(X,\mathbb{Q}), \]

\[ \delta_0(A) = \log \text{ch}(f_A) \in H^{2k}(Y,\mathbb{Q}) \cong H^{2k+1}(X,\mathbb{Q}), \]
where \( f_A : Y \to \Omega \text{Aut}_0(D \otimes \mathbb{K}) \simeq \text{Aut}_0(D \otimes \mathbb{K}) \) is induced by the transition map of \( A \). The homomorphism \( \log : SL_1(H^{ev}(Y, \mathbb{Q})) \to H^{ev}(Y, \mathbb{Q}) \) is the rational logarithm from [14] Section 2.5].

For the proof of [3] it suffices to treat the case \( D = M_\mathbb{Q} \otimes \mathcal{O}_\infty \), where it can be easily checked on the level of homotopy groups, but since \( E^2_\bullet(Y) \) and \( H^{ev}(Y, \mathbb{Q}) \) have rational vector spaces as coefficients this is enough.

**Lemma 2.6.** Let \( D \) be a strongly self-absorbing \( C^* \)-algebra in the class \( \mathcal{D} \). If \( p \in D \otimes \mathbb{K} \) is a projection such that \( [p] \neq 0 \) in \( K_0(D) \), then there is an integer \( n \geq 1 \) such that \( [p] \in nK_0(D)_{+}^\times \). If \( [p] \in nK_0(D)_{+}^\times \), then \( p(D \otimes \mathbb{K})p \cong M_n(D) \). Moreover, if \( n, m \geq 1 \), then \( M_n(D) \cong M_m(D) \) if and only if \( nK_0(D)_{+}^\times = mK_0(D)_{+}^\times \).

**Proof.** Recall that \( K_0(D) \) is an ordered unital ring with unit \([1_D]\) and with positive elements \( K_0(D)_{+} \) corresponding to classes of projections in \( D \otimes \mathbb{K} \). The group of invertible elements is denoted by \( K_0(D)^\times \) and \( K_0(D)_{+}^\times \) consists of classes \([p]\) of projections \( p \in D \otimes \mathbb{K} \) such that \([p] \in K_0(D)^\times \). It was shown in [3] Lemma 2.14 that if \( p \in D \otimes \mathbb{K} \) is a projection, then \([p] \in K_0(D)_{+}^\times \) if and only if \( p(D \otimes \mathbb{K})p \cong D \). The ring \( K_0(D) \) and the group \( K_0(D)_{+}^\times \) are known for all \( D \in \mathcal{D} \), [17]. In fact \( K_0(D) \) is a unital subring of \( \mathbb{Q} \), \( K_0(D)_{+}^\times = \mathbb{Q}_{+} \cap K_0(D) \) if \( D \in \mathcal{D}_{qd} \) and \( K_0(D)_{+}^\times = K_0(D) \) if \( D \in \mathcal{D}_{pq} \). Moreover:

\[
\begin{align*}
K_0(\mathbb{C}) &\cong K_0(\mathbb{Z}) \cong K_0(\mathbb{O}_{\infty}) \cong \mathbb{Z}, K_0(\mathbb{O}_2) = \{0\}, \\
K_0(M_{p\infty}) &\cong K_0(M_{p\infty} \otimes \mathbb{O}_{\infty}) \cong \mathbb{Z}[1/p] \cong \bigotimes_{p \in D} \mathbb{Z}[1/p] \cong \{np_k^1 p_k^2 \cdots p_k^r : p_k \in P, n_k \in \mathbb{Z}\}, \\
K_0(\mathbb{C})_{+}^\times &\cong K_0(\mathbb{Z})_{+}^\times = \{1\}, K_0(\mathbb{O}_{\infty})_{+}^\times = \{\pm 1\}, \\
K_0(M_{p\infty})_{+}^\times &\cong \{p_k^1 p_k^2 \cdots p_k^r : p_k \in P, n_k \in \mathbb{Z}\}, \\
K_0(M_{p\infty} \otimes \mathbb{O}_{\infty})_{+}^\times &\cong \{\pm p_k^1 p_k^2 \cdots p_k^r : p_k \in P, n_k \in \mathbb{Z}\}.
\end{align*}
\]

In particular, we see that in all cases \( K_0(D)_{+}^\times = \mathbb{N} \cdot K_0(D)_{+}^\times \), which proves the first statement. If \( p \in D \otimes \mathbb{K} \) is a projection such that \([p] \in nK_0(D)_{+}^\times \), then there is a projection \( q \in D \otimes \mathbb{K} \) such that \([q] \in K_0(D)_{+}^\times \) and \([p] = n[q] = [\text{diag}(q, q, \ldots, q)] \). Since \( D \) has cancellation of full projections, it follows then immediately that \( p(D \otimes \mathbb{K})p \cong M_n(D) \) proving the second part.

To show the last part of the lemma, suppose now that \( \alpha : D \otimes \mathcal{M}_n(\mathbb{C}) \to D \otimes \mathcal{M}_m(\mathbb{C}) \) is a \(*\)-isomorphism. Let \( e \in \mathcal{M}_n(\mathbb{C}) \) be a rank one projection. Then \( \alpha(1_D \otimes e)(D \otimes \mathcal{M}_m(\mathbb{C})) \alpha(1_D \otimes e) \cong D \).

By [3] Lemma 2.14 it follows that \( \alpha_0[1_D] = [\alpha(1_D \otimes e)] \in K_0(D)_{+}^\times \). Since \( \alpha_0 \) is unital, \( \alpha_0(n[1_D]) = m[1_D] \) and hence \( m[1_D] \in nK_0(D)_{+}^\times \). This is equivalent to \( nK_0(D)_{+}^\times = mK_0(D)_{+}^\times \).

Conversely, suppose that \( m[1_D] = nu \) for some \( u \in K_0(D)_{+}^\times \). Let \( \alpha \in \text{Aut}(D \otimes \mathbb{K}) \) be such that \([\alpha(1_D \otimes e)] = u \). Then \( \alpha(n[1_D]) = nu = m[1_D] \). This implies that \( \alpha \) maps a corner of \( D \otimes \mathbb{K} \) that is isomorphic to \( \mathcal{M}_n(D) \) to a corner that is isomorphic to \( \mathcal{M}_m(D) \). \( \square \)

**Corollary 2.7.** Let \( D \in \mathcal{D} \) and let \( \theta : D \otimes \mathcal{M}_n(\mathbb{C}) \to D \otimes \mathcal{M}_n(\mathbb{C}) \) be a unital inclusion induced by some unital embedding \( \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C}) \), where \( n \geq 2, r \geq 0 \). Let \( R \) be the set of prime factors of \( n \). Then, under the canonical isomorphism \( K_0(\mathcal{D} \otimes \mathcal{M}_n(\mathbb{C})) \cong K_0(D) \), we have

\[
\theta^{-1}_n(K_0(D \otimes \mathcal{M}_n(\mathbb{C}))_r^\times) = \bigcup_{r \in R} rK_0(D)_+^\times \subseteq K_0(D)
\]

where \( r \) runs through the set of all products of the form \( \prod_{q \in R} q^{k_q} \), \( k_q \in \mathbb{N} \cup \{0\} \).

**Proof.** From Lemma 2.6 we see that \( K_0(D) \cong \mathbb{Z}[1/P] \) for a (possibly empty) set of primes \( P \). The order structure is the one induced by \( (\mathbb{Q}, \mathbb{Q}_{+}) \) if \( D \) is quasidiagonal or \( K_0(D)_+^\times = \mathbb{Z}[1/P] \) if \( D \) is
purely infinite. If $R \subseteq P$, then $\theta$ induces an isomorphism on $K_0$ and the statement is true, since $\theta_*$ is order preserving and $\mathbb{Z}[1/R]^{\times} \subseteq K_0(D)^{\times}$. Thus, we may assume that $R \nsubseteq P$. Let $S = P \cup R$ and thus $K_0(D \otimes M_n) \cong \mathbb{Z}[1/S]$. The map $\theta_*$ induces the canonical inclusion $\mathbb{Z}[1/P] \hookrightarrow \mathbb{Z}[1/S]$. We can write $x \in \mathbb{Z}[1/P]$ as

$$x = m \cdot \prod_{p \in P} p^{r_p} \cdot \prod_{q \in R \setminus P} q^{k_q}$$

with $m \in \mathbb{Z}$ relatively prime to all $p \in P$ and $q \in R$, only finitely many $r_p \in \mathbb{Z}$ non-zero and $k_q \in \mathbb{N} \cup \{0\}$. From this decomposition we see that $x$ is invertible in $\mathbb{Z}[1/S]$ if and only if $m = \pm 1$. This concludes the proof since $p^{r_p} \in K_0(D)^+_n$.

Remark 2.8. Let $q \in D \otimes K$ be a projection and let $\alpha \in \text{Aut}(D \otimes K)$. As in [4] Lemma 2.14] we have that $[\alpha(q)] = [\alpha(1 \otimes e)] \cdot [q]$ with $[\alpha(1 \otimes e)] \in K_0(D)^+_n$. Thus, the condition $[q] \in nK_0(D)^+_n$ for $n \in \mathbb{N}$ is invariant under the action of $\text{Aut}(D \otimes K)$ on $K_0(D)$. Given $A \in \mathcal{E}_{D \otimes K}(X)$, a projection $p \in A$, $x_0 \in X$ and an isomorphism $\phi: A(x_0) \to D \otimes K$ the condition $[\phi(p(x_0))] \in nK_0(D)^+_n$ is independent of $\phi$. Abusing the notation we will write this as $[p(x_0)] \in nK_0(D)^+_n$.

Corollary 2.9. Let $D \in \mathcal{D}$ and let $A \in \mathcal{E}_{D \otimes K}(X)$ with $X$ a connected compact metrizable space. If $p \in A$ is a projection such that $[p(x_0)] \in nK_0(D)^+_n$ for some point $x_0$, then $(pAp)(x) \cong M_n(D)$ for all $x \in X$ and hence $pAp \in \mathcal{E}_{D \otimes M_n(\mathbb{C})}(X)$. If $p \in A$ is a projection with $[p(x_0)] \in K_0(D) \setminus \{0\}$, then $[p(x_0)] \in nK_0(D)^+_n$ for some $n \in \mathbb{N}$.

Proof. Let $V_1, \ldots, V_k$ be a finite cover of $X$ by compact sets such that there are bundle isomorphisms $\phi_i: A(V_i) \cong C(V_i) \otimes D \otimes K$. Let $p_i$ be the image of the restriction of $p$ to $V_i$ under $\phi_i$. After refining the cover $(V_i)$, if necessary, we may assume that $\|p_i(x) - p_i(y)\| < 1$ for all $x, y \in V_i$. This allows us to find a unitary $u_i$ in the multiplier algebra of $C(V_i) \otimes D \otimes K$ such that after replacing $\phi_i$ by $u_i\phi_i u_i^*$ and $p_i$ by $u_i p_i u_i^*$, we may assume that $p_i$ are constant projections. Since $X$ is connected and $[p(x_0)] \in nK_0(D)^+_n$ by assumption, it follows from $[p_i(x_0)] \in nK_0(D)^+_n$ for $x_0 \in V_i$ and the above remark that $[p_j(x)] \in nK_0(D)^+_n$ for all $1 \leq j \leq k$ and all $x \in V_j$. Then Lemma 2.6 implies $(pAp)(V_j) \cong C(V_j) \otimes M_n(D)$. By Lemma 2.6 we also have that $[p(x_0)] \neq 0$ implies $[p(x_0)] \in nK_0(D)^+_n$ for some $n \in \mathbb{N}$ proving the statement about the case $[p(x_0)] \in K_0(D) \setminus \{0\}$. 

We study the image of the stabilization map

$$\mathcal{E}_{D \otimes M_n(\mathbb{C})}(X) \to \mathcal{E}_{D \otimes K}(X)$$

induced by the map $A \mapsto A \otimes K$, or equivalently by the map

$$\text{Aut}(D \otimes M_n(\mathbb{C})) \to \text{Aut}(D \otimes M_n(\mathbb{C}) \otimes K) \cong \text{Aut}(D \otimes K).$$

Let us recall that $\mathcal{D}$ denotes the class of strongly self-absorbing C*-algebras which satisfy the UCT and which are either quasidiagonal or purely infinite.

Theorem 2.10. Let $D$ be a strongly self-absorbing C*-algebra in the class $\mathcal{D}$. Let $A$ be a locally trivial continuous field of C*-algebras over a connected compact metrizable space $X$ such that $A(x) \cong D \otimes K$ for all $x \in X$. The following assertions are equivalent:

1. $\delta_k(A) = 0$ for all $k \geq 0$.
2. The field $A \otimes M_Q$ is trivial.
(3) There is an integer \( n \geq 1 \) and a unital locally trivial continuous field \( \mathcal{B} \) over \( X \) with all fibers isomorphic to \( M_n(D) \) such that \( A \cong B \otimes \mathbb{K} \).

(4) \( A \) is orientable and \( A^{\otimes m} \cong C(X) \otimes D \otimes \mathbb{K} \) for some \( m \in \mathbb{N} \).

**Proof.** The statement is immediately verified if \( D \cong \mathcal{O}_2 \). Indeed all locally trivial fields with fiber \( \mathcal{O}_2 \otimes \mathbb{K} \) are trivial since \( \text{Aut}(\mathcal{O}_2 \otimes \mathbb{K}) \) is contractible by [1] Cor. 17 & Thm. 2.17. For the remainder of the proof we may therefore assume that \( D \not\cong \mathcal{O}_2 \).

(1) \( \iff \) (2) If \( D \in \mathcal{D}_{pt} \), then it is known that \( D \otimes M_2 \cong M_2 \). Similarly, if \( D \in \mathcal{D}_{pt} \) and \( D \not\cong \mathcal{O}_2 \) then \( D \otimes M_2 \cong \mathcal{O}_\infty \otimes M_2 \). If \( A \) is as in the statement, then \( A \otimes M_2 \) is a locally trivial field whose fibers are all isomorphic to either \( M_2 \otimes \mathbb{K} \) or to \( \mathcal{O}_\infty \otimes M_2 \otimes \mathbb{K} \). In either case, it was shown in [1] Cor. 4.1 that such a field is trivial if and only if \( \delta_k(A) = 0 \) for all \( k \geq 0 \). As reviewed earlier in this section, this follows from the explicit computation of \( E^1_{M_2}(X) \) and \( E^1_{M_2 \otimes \mathcal{O}_\infty}(X) \).

(2) \( \Rightarrow \) (3) Assume now that \( A \otimes M_2 \) is trivial, i.e. \( A \otimes M_2 \cong C(X) \otimes D \otimes M_2 \otimes \mathbb{K} \). Let \( p \in \mathcal{A} \otimes M_2 \) be the projection that corresponds under this isomorphism to the projection \( 1 \otimes e \in C(X) \otimes D \otimes M_2 \otimes \mathbb{K} \) where \( 1 \) is the unit of the \( C^\ast \)-algebra \( C(X) \otimes D \otimes M_2 \) and \( e \in \mathbb{K} \) is a rank-one projection. Then \( \|p(x)\| \neq 0 \) in \( K_0(A(x) \otimes M_2) \) for all \( x \in X \) (recall that \( D \not\cong \mathcal{O}_2 \)). Let us write \( M_2 \) as the direct limit of an increasing sequence of its subalgebras \( M_{k(i)}(\mathbb{C}) \). Then \( A \otimes M_2 \) is the direct limit of the sequence \( A_i = A \otimes M_{k(i)}(\mathbb{C}) \). It follows that there exist \( i \geq 1 \) and a projection \( p_i \in A_i \) such that \( \|p - p_i\| < 1 \). Then \( \|p(x) - p_i(x)\| < 1 \) and so \( [p_i(x)] \neq 0 \) in \( K_0(A(x)) \) for each \( x \in X \), since its image in \( K_0(A(x) \otimes M_2) \) is equal to \( [p(x)] \neq 0 \). Let us consider the locally trivial unital field \( \mathcal{B} := p_i(A \otimes M_{k(i)}(\mathbb{C})) \). Since the fibers of \( A \otimes M_{k(i)}(\mathbb{C}) \) are isomorphic to \( D \otimes \mathbb{K} \otimes M_{k(i)}(\mathbb{C}) \cong D \otimes \mathbb{K} \), it follows by Corollary [2.9] that there is \( n \geq 1 \) such that all fibers of \( \mathcal{B} \) are isomorphic to \( M_n(D) \). Since \( \mathcal{B} \) is isomorphic to a full corner of \( A \otimes \mathbb{K} \), it follows by [3] that \( A \otimes \mathbb{K} \cong \mathcal{B} \otimes \mathbb{K} \). We conclude by noting that since \( A \) is locally trivial and each fiber is stable, then \( A \cong A \otimes \mathbb{K} \) by [9] and so \( A \cong B \otimes \mathbb{K} \).

(3) \( \Rightarrow \) (2) This implication holds for any strongly self-absorbing \( C^\ast \)-algebra \( D \). Let \( A \) and \( B \) be as in (3). Let us note that \( B \otimes M_2 \) is a unital locally trivial field with all fibers isomorphic to the strongly self-absorbing \( C^\ast \)-algebra \( D \otimes M_2 \). Since \( \text{Aut}(D \otimes M_2) \) is contractible by [1] Thm. 2.3, it follows that \( B \otimes M_2 \) is trivial. We conclude that \( A \otimes M_2 \cong (B \otimes M_2) \otimes \mathbb{K} \cong C(X) \otimes D \otimes M_2 \otimes \mathbb{K} \).

(2) \( \iff \) (4) This equivalence holds for any strongly self-absorbing \( C^\ast \)-algebra \( D \) if \( A \) is orientable. In particular we do not need to assume that \( D \) satisfies the UCT. In the UCT case we note that since the map \( K_0(D) \to K_0(D \otimes M_2) \) is injective, it follows that \( A \) is orientable if and only if \( A \otimes M_2 \) is orientable, i.e. \( \delta_0(A) = 0 \) if and only if \( \delta_0(A) = 0 \). Since \( \delta_0(A) = 0 \), \( A \) is determined up to isomorphism by its class \( [A] \in \hat{E}^1_B(X) \). To complete the proof it suffices to show that the kernel of the map \( \tau : \hat{E}^1_B(X) \to \hat{E}^1_{D \otimes M_2}(X) \), \( \tau[A] = [A \otimes M_2] \), consists entirely of torsion elements. Consider the natural transformation of cohomology theories:

\[
\tau \otimes \text{id}_{\mathbb{Q}} : \hat{E}^1_B(X) \otimes \mathbb{Q} \to \hat{E}^1_{D \otimes M_2}(X) \otimes \mathbb{Q} \cong \hat{E}^1_{D \otimes M_2}(X).
\]

If \( D \neq \mathbb{C} \), it induces an isomorphism on coefficients since \( \hat{E}^1_{D \otimes \mathbb{C}}(pt) = \pi_i(\text{Aut}_0(D \otimes \mathbb{K})) \cong K_i(D) \) by [4] Thm.2.18 and since the map \( K_i(D) \otimes \mathbb{Q} \to K_i(D \otimes \mathbb{K}) \) is bijective. We conclude that the kernel of \( \tau \) is a torsion group. The same property holds for \( D = \mathbb{C} \) since \( \hat{E}^1_C(X) \) is a direct summand of \( \hat{E}^1_C(X) \) by [4] Cor.3.8.

\( \square \)
Theorem 2.11. Let $D$, $X$ and $A$ be as in Theorem 2.10 and let $n \geq 2$ be an integer. The following assertions are equivalent:

1. The field $A \otimes M_{n^\infty}$ is trivial.
2. There is a $k \in \mathbb{N}$ and a unital locally trivial continuous field $B$ over $X$ with all fibers isomorphic to $M_{n^k}(D)$ such that $A \cong B \otimes \mathbb{K}$.
3. $A$ is orientable and $A^\otimes M_{n^k} \cong C(X) \otimes D \otimes \mathbb{K}$ for some $k \in \mathbb{N}$.

Proof. By reasoning as in the proof of Theorem 2.10 we may assume that $D \not= \mathbb{O}_2$.

$(1) \Rightarrow (2)$: By assumption the continuous field $A \otimes M_{n^\infty}$ is trivializable and hence it satisfies the global Fell condition of $[4]$. This means that there is a full projection $p_\infty \in A \otimes M_{n^\infty}$ with the property that $p_\infty(x) \in K_0(A(x) \otimes M_{n^\infty})^+$ for all $x \in X$. Let $\nu_i: M_{n^i}(\mathbb{C}) \to M_{n^\infty}$ be a unital inclusion map. Since $A \otimes M_{n^\infty}$ is the inductive limit of the sequence

$$A \to A \otimes M_{n^1}(\mathbb{C}) \to \cdots \to A \otimes M_{n^i}(\mathbb{C}) \to A \otimes M_{n^{i+1}}(\mathbb{C}) \to \cdots$$

there is an $i \in \mathbb{N}$ and a full projection $p \in A \otimes M_{n^i}(\mathbb{C})$ with $\|(id_A \otimes \nu_i)(p) - p_\infty\| < 1$. Fix a point $x_0 \in X$. Let $\theta: A(x_0) \otimes M_{n^i}(\mathbb{C}) \to A(x_0) \otimes M_{n^\infty}$ be the unital inclusion induced by $\nu_i$. Note that $\theta_*([p(x_0)]) = ([id_A(x_0) \otimes \nu_i]_*([p(x_0)])) = [p_\infty(x_0)] \in K_0(A(x_0) \otimes M_{n^\infty})^+$. By Corollary 2.11 this implies that $[p(x_0)] \in rK_0(A(x_0))^+$ for some $r \in \mathbb{N}$ that divides $n^k$ for some $k \in \mathbb{N} \cup \{0\}$. Then $B_0 := p(A \otimes M_{n^i}(\mathbb{C}))p \in \mathcal{C}_{D\otimes\mathbb{K}}(X)$ by Corollary 2.9. Write $n^k = mr$ with $m \in \mathbb{N}$. It follows that $B := B_0 \otimes M_{n^i}(\mathbb{C}) \in \mathcal{C}_{D\otimes\mathbb{K}}(X)$. The fact that $B \otimes \mathbb{K} \cong A$ follows just as in step $(2) \Rightarrow (3)$ in the proof of Theorem 2.10.

$(2) \Rightarrow (1)$: This is just the same argument as step (3) $\Rightarrow (2)$ in the proof of Theorem 2.10.

$(1) \Leftrightarrow (3)$: The orientability of $A$ follows from Theorem 2.10. Observe that the elements $[A] \in \mathcal{C}_{D\otimes\mathbb{K}}(X) = \tilde{E}_D^1(X)$ such that $n^k[A] = 0$ or equivalently $A^\otimes M_{n^k}$ is trivializable for some $k \in \mathbb{N} \cup \{0\}$ coincide precisely with the elements in the kernel of the group homomorphism $\tilde{E}_D^1(X) \to \tilde{E}_D^1(X) \otimes \mathbb{Z}[\frac{1}{n}]$. Since $\mathbb{Z}[\frac{1}{n}]$ is flat, it follows that $X \to \tilde{E}_D^1(X) \otimes \mathbb{Z}[\frac{1}{n}]$ still satisfies all axioms of a generalized cohomology theory. In particular, we have the following commutative diagram of natural transformations of cohomology theories:

$$
\begin{array}{ccc}
\tilde{E}_D^1(X) & \longrightarrow & \tilde{E}_{D\otimes M_{n^\infty}}(X) \\
\downarrow & & \downarrow \cong \\
\tilde{E}_D^1(X) \otimes \mathbb{Z}[\frac{1}{n}] & \longrightarrow & \tilde{E}_{D\otimes M_{n^\infty}}(X) \otimes \mathbb{Z}[\frac{1}{n}]
\end{array}
$$

where the isomorphism on the right hand side can be checked on the coefficients. A similar argument shows that for $D \not= \mathbb{C}$ the bottom homomorphism is an isomorphism. Thus the kernel of the left vertical map agrees with the one of the upper horizontal map in this case. For $D = \mathbb{C}$ we can use that $\tilde{E}_C^1(X)$ embeds as a direct summand into $\tilde{E}_Z^2(X)$ via the natural $*$-homomorphism $\mathbb{C} \to \mathbb{Z}$ [4 Cor. 4.8]. In particular, $\tilde{E}_C^1(X) \otimes \mathbb{Z}[\frac{1}{n}] \to \tilde{E}_Z^2(X) \otimes \mathbb{Z}[\frac{1}{n}]$ is injective. 

\begin{cor}
Corollary 2.12. Let $D$ and $X$ be as in Theorem 2.10. Then any element $x \in \tilde{E}_D^1(X)$ with $nx = 0$ is represented by the stabilization of a unital locally trivial field over $X$ with all fibers isomorphic to $M_{n^k}(D)$ for some $k \geq 1$. Moreover if $A \in \mathcal{C}_{D\otimes\mathbb{K}}(X)$, then $A \otimes M_Q$ is trivial $\iff$ $A \otimes M_{n^\infty}$ is trivial for some $n \in \mathbb{N}$ $\iff$ $A$ is orientable and $n^k[A] = 0$ in $\tilde{E}_D^1(X)$ for some $k \in \mathbb{N}$ and some $n \in \mathbb{N}$.
\end{cor}
(An example from \[1\] for \(D = \mathbb{C}\) shows that in general one cannot always arrange that \(k = 1\).)

**Proof.** The first part follows from Theorem \[2.11\] Indeed, condition (3) of that theorem is equivalent to requiring that \(A\) is orientable and \(n^k[A] = 0\) in \(\bar{E}^1_D(X)\). The second part follows from Theorems \[2.10\] and \[2.11\]. \(\square\)

**Definition 2.13.** Let \(D\) be a strongly self-absorbing \(C^*\)-algebra. If \(X\) is connected compact metrizable space we define the Brauer group \(Br_D(X)\) as equivalence classes of continuous fields \(A \in \bigcup_{n \geq 1} \mathcal{C}_{M_n(D)}(X)\). Two continuous fields \(A_i \in \mathcal{C}_{M_n(D)}(X), i = 1, 2\) are equivalent, if

\[
A_1 \otimes p_1 C(X, M_{N_i}(D))p_1 \cong A_2 \otimes p_2 C(X, M_{N_i}(D))p_2,
\]

for some full projections \(p_i \in C(X, M_{N_i}(D))\). We denote by \([A]_{Br}\) the class of \(A\) in \(Br_D(X)\). The multiplication on \(Br_D(X)\) is induced by the tensor product operation, after fixing an isomorphism \(D \otimes D \cong D\). We will show in a moment that the monoid \(Br_D(X)\) is a group.

**Remark 2.14.** It is worth noting the following two alternative descriptions of the Brauer group. (a) If \(D \in D\) is quasidiagonal, then two continuous fields \(A_i \in \mathcal{C}_{M_n(D)}(X), i = 1, 2\) have equal classes in \(Br_D(X)\), if and only if \(A_1 \otimes p_1 C(X, M_{N_1}({\mathbb{C}}))p_1 \cong A_2 \otimes p_2 C(X, M_{N_2}({\mathbb{C}}))p_2\), for some full projections \(p_i \in C(X, M_{N_i}({\mathbb{C}}))\). (b) If \(D \in D\) is purely infinite, then two continuous fields \(A_i \in \mathcal{C}_{M_n(D)}(X), i = 1, 2\) have equal classes in \(Br_D(X)\), if and only if \(A_1 \otimes p_1 C(X, M_{N_1}(\mathcal{O}_\infty))p_1 \cong A_2 \otimes p_2 C(X, M_{N_2}(\mathcal{O}_\infty))p_2\), for some full projections \(p_i \in C(X, M_{N_i}(\mathcal{O}_\infty))\). In order to justify (a) we observe that if \(D\) is quasidiagonal, then every projection \(p \in C(X, M_N(D))\) has a multiple \(p(m) := p \otimes 1_{M_m}(\mathbb{C})\) such that \(p(m)\) is Murray-Von Neumann equivalent to a projection in \(C(X, M_{N_m}(\mathbb{C})) \otimes 1_D \subset C(X, M_{N_m}(\mathbb{C})) \otimes D\) and that \(A_i \otimes D \cong A_i\) by \[9\]. For (b) we note that if \(D\) is purely infinite, then every projection \(p \in C(X, M_N(D))\) has a multiple \(p \otimes 1_{M_m}(\mathbb{C})\) that is Murray-Von Neumann equivalent to a projection in \(C(X, M_{N_m}(\mathcal{O}_\infty)) \otimes 1_D\).

One has the following generalization of a result of Serre, \[8\] Thm.1.6).

**Theorem 2.15.** Let \(D\) be a strongly self-absorbing \(C^*\)-algebra in \(D\).

(i) \(Tor(\bar{E}^1_D(X)) = ker\left(\bar{E}^1_D(X) \xrightarrow{\bar{\delta}} \bigoplus_{k \geq 1} H^{2k+1}(X, \mathbb{Q})\right)\)

(ii) The map \(\theta : Br_D(X) \to Tor(\bar{E}^1_D(X)), [A]_{Br} \mapsto [A \otimes \mathbb{K}]\) is an isomorphism of groups.

**Proof.** (i) was established in the last part of the proof of Theorem \[2.10\].

(ii) We denote by \(L_p\) the continuous field \(p C(X, M_N(D))p\). Since \(L_p \otimes \mathbb{K} \cong C(X, D \otimes \mathbb{K})\) it follows that the map \(\theta\) is a well-defined morphism of monoids.

We use the following observation. Let \(\theta : S \to G\) be a unital surjective morphism of commutative monoids with units denoted by 1. Suppose that \(G\) is a group and that \(\{s \in S : \theta(s) = 1\} = \{1\}\). Then \(S\) is a group and \(\theta\) is an isomorphism. Indeed if \(s \in S\), there is \(t \in S\) such that \(\theta(t) = \theta(s)^{-1}\) by surjectivity of \(\theta\). Then \(\theta(st) = \theta(s)\theta(t) = 1\) and so \(st = 1\). It follows that \(S\) is a group and that \(\theta\) is injective.

We are going to apply this observation to the map \(\theta : Br_D(X) \to Tor(\bar{E}^1_D(X))\). By condition (3) of Theorem \[2.10\] we see that \(\theta\) is surjective. Let us determine the set \(\theta^{-1}(\{0\})\). We are going to show that if \(B \in \mathcal{C}_{D\otimes M_n(C)}(X)\), then \([B \otimes \mathbb{K}] = 0\) in \(\bar{E}^1_D(X)\) if and only if

\[
B \cong p(C(X) \otimes D \otimes M_N(\mathbb{C}))p \cong L_{C(X,D)}(p C(X,D)^N)
\]
for some selfadjoint projection \( p \in C(X) \otimes D \otimes M_N(\mathbb{C}) \cong M_N(C(X, D)) \). Let \( B \in \mathcal{E}_D \otimes M_n(\mathbb{C})(X) \) be such that \([B \otimes 1] = 0\) in \( E_D^1(X)\). Then there is an isomorphism of continuous fields \( \phi : B \otimes K \xrightarrow{\cong} C(X) \otimes D \otimes K \). After conjugating \( \phi \) by a unitary we may assume that \( p := \phi(1_B \otimes \varepsilon_{11}) \in C(X) \otimes D \otimes M_N(\mathbb{C}) \) for some integer \( N \geq 1 \). It follows immediately that the projection \( p \) has the desired properties. Conversely, if \( B \cong p(C(X) \otimes D \otimes M_N(\mathbb{C})) \) \( p \) then there is an isomorphism of continuous fields \( B \otimes K \cong C(X) \otimes D \otimes K \) by [3]. We have thus shown that that \( \theta([B]_{Br}) = 0 \) iff and only if \([B]_{Br} = 0\).

We are now able to conclude that \( BrD(X) \) is a group and that \( \theta \) is injective by the general observation made earlier.

**Definition 2.16.** Let \( D \) be a strongly self-absorbing C*-algebra. Let \( A \) be a locally trivial continuous field of C*-algebras with fiber \( D \otimes K \). We say that \( A \) is a torsion continuous field if \( A^{\otimes k} \) is isomorphic to a trivial field for some integer \( k \geq 1 \).

**Corollary 2.17.** Let \( A \) be as in Theorem 2.10. Then \( A \) is a torsion continuous field if and only if \( \delta_0(A) \in H^1(X, K_0(D) \mathbb{Z}) \) is a torsion element and \( \delta_k(A) = 0 \in H^{2k+1}(X, \mathbb{Q}) \) for all \( k \geq 1 \).

**Proof.** Let \( m \geq 1 \) be an integer such that \( m\delta_0(A) = 0 \). Then \( \delta_0(A^{\otimes m}) = 0 \). We conclude by applying Theorem 2.10 to the orientable continuous field \( A^{\otimes m} \).

3. **Characteristic classes of the opposite continuous field**

Given a C*-algebra \( B \) denote by \( B^{op} \) the opposite C*-algebra with the same underlying Banach space and norm, but with multiplication given by \( b^{op}a^{op} = (a-b)^{op} \). The conjugate C*-algebra \( \overline{B} \) has the conjugate Banach space as its underlying vector space, but the same multiplicative structure. The map \( a \mapsto a^\star \) provides an isomorphism \( B^{op} \to \overline{B} \). Any automorphism \( \alpha \in \text{Aut}(B) \) yields in a canonical way automorphisms \( \overline{\alpha} : \overline{B} \to \overline{B} \) and \( \alpha^{op} : B^{op} \to B^{op} \) compatible with \( * : B^{op} \to B \). Therefore we have group isomorphisms \( \theta : \text{Aut}(B) \to \text{Aut}(\overline{B}) \) and \( \text{Aut}(B) \to \text{Aut}(B^{op}) \). Note that \( \alpha \in \text{Aut}(B) \) is equal to \( \theta(\alpha) \) when regarded as set-theoretic maps \( B \to B \). Given a locally trivial continuous field \( A \) with fiber \( B \), we can apply these operations fiberwise to obtain the locally trivial fields \( A^{op} \) and \( \overline{A} \), which we will call the opposite and the conjugate field. They are isomorphic to each other and isomorphic to the conjugate and the opposite C*-algebras of \( A \).

A real form of a complex C*-algebra \( A \) is a real C*-algebra \( A^R \) such that \( A \cong A^R \otimes \mathbb{C} \). A real form is not necessarily unique [2] and not all C*-algebras admit real forms [16]. If two C*-algebras \( A \) and \( B \) admit real forms \( A^R \) and \( B^R \), then \( A^R \otimes_R B^R \) is a real form of \( A \otimes B \).

**Example 3.1.** All known strongly self-absorbing C*-algebras \( D \in D \) admit a real form.

Indeed, the real Cuntz algebras \( O_2^R \) and \( O_\infty^R \) are defined by the same generators and relations as their complex versions. Alternatively \( O_\infty^R \) can be realized as follows. Let \( H^R_R \) be a separable infinite dimensional real Hilbert space and let \( F^R(H_R) = \bigoplus_{n=0}^\infty H^R_{n} \) be the real Fock space associated to it. Every \( \xi \in H^R_R \) defines a shift operator \( s_\xi(\eta) = \xi \otimes \eta \) and we denote the algebra spanned by the \( s_\xi \) and their adjoints \( s_\xi^\star \) by \( O_\infty^R \). If \( F(H_R \otimes \mathbb{C}) \) denotes the Fock space associated to the complex Hilbert space \( H = H_R \otimes \mathbb{C} \), then we have \( F^R \otimes \mathbb{C} \cong F(H) \). If we represent \( O_\infty \) on \( F(H) \) using the above construction, then the map \( s_\xi + is_\eta \mapsto s_{\xi+i\eta} \) induces an isomorphism \( O_\infty^R \otimes \mathbb{C} \to O_\infty \). Likewise define \( M^R_2 \) to be the infinite tensor product \( M_2(\mathbb{R}) \otimes M_3(\mathbb{R}) \otimes M_4(\mathbb{R}) \otimes \ldots \)
Since $M_n(C) \cong M_n(R) \otimes C$, we obtain an isomorphism $M^R \otimes C \cong M_0$ on the inductive limit. Let $K^R$ be the compact operators on $H_R$ and $K$ those on $H$, then we have $K^R \otimes C \cong K$. Thus, $C^* \otimes O_\infty \otimes K$ is the complexification of the real $C^*$-algebra $M^R \otimes O_\infty \otimes K^R$.

The Jiang-Su algebra $Z$ admits a real form $Z^R$ which can be constructed in the same way as $Z$. Indeed, one constructs $Z^R$ as the inductive limit of a system

$$\cdots \to C([0,1], M_{p_nq_n}(R)) \xrightarrow{\phi_n} C([0,1], M_{p_{n+1}q_{n+1}}(R)) \to \cdots$$

where the connecting maps $\phi_n$ are defined just as in the proof of [11] Prop. 2.5 with only one modification. Specifically, one can choose the matrices $u_0$ and $u_1$ to be in the special orthogonal group $SO(p_nq_n)$ and this will ensure the existence of a continuous path $u_t$ in $O(p_nq_n)$ from $u_0$ to $u_1$ as required.

If $B$ is the complexification of a real $C^*$-algebra $B^R$, then a choice of isomorphism $B \cong B^R \otimes C$ provides an isomorphism $\pi: B \to \overline{B}$ via complex conjugation on $C$. On automorphisms we have $\text{Ad}_{c^{-1}}: \text{Aut}(B) \to \text{Aut}(B)$. Let $\eta = \text{Ad}_{c^{-1}} \circ \theta: \text{Aut}(B) \to \text{Aut}(B)$. Now we specialize to the case $B = D \otimes K$ with $D \in D$ and study the effect of $\eta$ on homotopy groups, i.e. $\eta_*: \pi_{2k}(\text{Aut}(B)) \to \pi_{2k}(\text{Aut}(B))$. By [11] Theorem 2.18 the groups $\pi_{2k+1}(\text{Aut}(B))$ vanish.

Let $R$ be a commutative ring and denote by $[K^0(S^{2k}) \otimes R]^\times$ the group of units of the ring $K^0(S^{2k}) \otimes R$. Let $[K^0(S^{2k}) \otimes R]^\times_1$ be the kernel of the morphism of multiplicative groups $[K^0(S^{2k}) \otimes R]^\times \to R^\times$. This is the group of virtual rank 1 vector bundles with coefficients in $R$ over $S^{2k}$. Let $c_S: K^0(S^{2k}) \to K^0(S^{2k})$ and $c_R: K^0(D) \to K^0(D)$ be the ring automorphisms induced by complex conjugation.

**Lemma 3.2.** Let $D$ be a strongly self-absorbing $C^*$-algebra in the class $D$, let $R = K^0(D)$ and let $k > 0$. There is an isomorphism $\pi_{2k}(\text{Aut}(D \otimes K)) \to [K^0(S^{2k}) \otimes R]^\times_1$ such that the following diagram commutes

$$\pi_{2k}(\text{Aut}(D \otimes K)) \xrightarrow{\eta_*} \pi_{2k}(\text{Aut}(D \otimes K))$$

$$[K^0(S^{2k}) \otimes R]^\times_1 \xrightarrow{c_S \otimes c_R} [K^0(S^{2k}) \otimes R]^\times_1$$

**Proof.** Observe that $\pi_{2k}(\text{Aut}(D \otimes K)) = \pi_{2k}(\text{Aut}_0(D \otimes K))$ (for $k > 0$) and $\text{Aut}_0(D \otimes K)$ is a path connected group, therefore $\pi_{2k}(\text{Aut}(D \otimes K)) = [S^{2k}, \text{Aut}_0(D \otimes K)]$. Let $e \in K$ be a rank 1 projection such that $c(1_D \otimes e) = 1_D \otimes c$. It follows from the proof of [11] Theorem 2.22 that the map $\alpha \mapsto \alpha(1 \otimes e)$ induces an isomorphism $[S^{2k}, \text{Aut}_0(D \otimes K)] \to K_0(C(S^{2k}) \otimes D)^\times_1 = 1 + K_0(C_0(S^{2k} \setminus x_0) \otimes D)$. We have $\eta(\alpha)(1 \otimes e) = c^{-1}(\alpha(c(1 \otimes e))) = c^{-1}(\alpha(1 \otimes e))$, i.e. the isomorphism intertwines $\eta$ and $c^{-1}$.

Consider the following diagram of rings:

$$K^0(S^{2k}) \otimes R \xrightarrow{c_S \otimes c_R} K^0(S^{2k}) \otimes R$$

$$K_0(C(S^{2k}) \otimes D) \xrightarrow{p \mapsto c^{-1}(p)} K_0(C(S^{2k}) \otimes D)$$
The vertical maps arise from the Künneth theorem. Since $K_1(D) = 0$, these are isomorphisms. Since $c_S$ corresponds to the operation induced on $K_0(C(S^{2k}))$ by complex conjugation on $K$, the above diagram commutes. □

**Remark 3.3.** (i) If $D \in \mathcal{D}$ then $R = K_0(D) \subset \mathbb{Q}$ with $[1_D] = [1_{D^R}] = 1$. Thus $c^{-1}(1_D) = 1_D$ and this shows that the above automorphism $c_R$ is trivial. The $K^0$-ring of the sphere is given by $K^0(S^{2k}) \cong \mathbb{Z}[X_k]/(X_k^2)$. The element $X_k$ is the $k$-fold reduced exterior tensor power of $H - 1$, where $H$ is the tautological line bundle over $S^2 \cong \mathbb{C}P^1$. Since $c_S$ maps $H - 1$ to $1 - H$, it follows that $X_k$ is mapped to $-X_k$ if $k$ is odd and to $X_k$ if $k$ is even. We have $[K^0(S^2) \otimes R]_1^X = \{1 + t X_k \mid t \in R\} \subset R[X_k]/(X_k^2)$. Thus, $c_S$ maps $1 + t X_k$ to its inverse $1 - t X_k$ if $k$ is odd and acts trivially if $k$ is even.

(ii) By [4] Theorem 2.18 there is an isomorphism $\pi_0(\text{Aut}(D \otimes \mathbb{K})) \cong K_0(D)^{\times}$ given by $[\alpha] \mapsto [\alpha(1 \otimes e)]$. Arguing as in Lemma 3.2 we see that the action of $\eta$ on this groups is given by $c_R = \text{id}$.

**Theorem 3.4.** Let $X$ be a compact metrizable space and let $A$ be a locally trivial continuous field with fiber $D \otimes \mathbb{K}$ for a strongly self-absorbing C*-algebra $D \in \mathcal{D}$. Then we have for $k \geq 0$:

$$\delta_k(A^op) = \delta_k(A) = (-1)^k \delta_k(A) \in H^{2k+1}(X, \mathbb{Q}).$$

**Proof.** Let $D^R$ be a real form of $D$. The group isomorphism $\eta: \text{Aut}(D \otimes \mathbb{K}) \to \text{Aut}(D \otimes \mathbb{K})$ induces an infinite loop map $B\eta: B\text{Aut}(D \otimes \mathbb{K}) \to B\text{Aut}(D \otimes \mathbb{K})$, where the infinite loop space structure is the one described in [4] Section 3. If $f: X \to B\text{Aut}(D \otimes \mathbb{K})$ is the classifying map of a locally trivial field $A$, then $B\eta \circ f$ classifies $A$. Thus the induced map $\eta_*: E^1_D(X) \to E^1_D(X)$ has the property that $\eta_*[A] = [A]$.

The unital inclusion $D^R \to B^R := D^R \otimes \mathcal{O}_\infty \otimes M^R_Q$ induces a commutative diagram

$$\begin{align*}
\text{Aut}(D \otimes \mathbb{K}) \xrightarrow{\eta} \text{Aut}(D \otimes \mathbb{K}) \\
\downarrow \quad \downarrow \\
\text{Aut}(B \otimes \mathbb{K}) \xrightarrow{\eta} \text{Aut}(B \otimes \mathbb{K})
\end{align*}$$

with $B := B^R \otimes \mathbb{C}$. From this we obtain a commutative diagram

$$\begin{align*}
E^1_D(X) \xrightarrow{\eta_*} E^1_D(X) \\
\delta \downarrow \quad \delta \downarrow \\
E^1_B(X) \xrightarrow{\eta_*} E^1_B(X)
\end{align*}$$

As explained earlier, $B \cong M^R_Q \otimes \mathcal{O}_\infty$. Recall that $E^1_{M^R_Q \otimes \mathcal{O}_\infty}(X) \cong H^1(X, \mathbb{Q}^\times) \oplus \bigoplus_{k \geq 1} H^{2k+1}(X, \mathbb{Q})$. By Lemma 3.2 and Remark 3.3 (i) the effect of $\eta$ on $H^{2k+1}(X, \pi_{2k}(\text{Aut}(B))) \cong H^{2k+1}(X, \mathbb{Q})$ is given by multiplication with $(-1)^k$ for $k > 0$. By Remark 3.3 (ii) $\eta$ acts trivially on $H^1(X, \pi_{0}(\text{Aut}(B))) = H^1(X, \mathbb{Q}^\times)$. □

**Example 3.5.** Let $Z$ be the Jiang-Su algebra. We will show that in general the inverse of an element in the Brauer group $Br_Z(X)$ is not represented by the class of the opposite algebra. Let $Y$ be the space obtained by attaching a disk to a circle by a degree three map and let $X_n = S^n \wedge Y$ be $n^{th}$ reduced suspension of $Y$. Then $E^1_Z(X_3) \cong K^0(X_2)^\times \cong 1 + K^0(X_2)$ by [4] Thm.2.22.
Since this is a torsion group, $Br_{\mathcal{Z}}(X_3) \cong E^1_{\mathcal{Z}}(X_3)$ by Theorem 2.15. Using the K"unneth formula, $Br_{\mathcal{Z}}(X_3) \cong 1 + \bar{K}^0(S^2) \otimes K^0(Y) \cong 1 + \mathbb{Z}/3$. Reasoning as in Lemma 3.2 with $X_3$ in place of $S^{2k}$, we identify the map $\eta : E^2_{\mathcal{Z}}(X_3) \to E^2_{\mathcal{Z}}(X_3)$ with the map $K^0(X_2)^{\times} \to K^0(X_2)^{\times}$ that sends the class $x = [V_1] - [V_2]$ to $\bar{x} = [\overline{V}_1] - [\overline{V}_2]$, where $\overline{V}_i$ is the complex conjugate bundle of $V_i$. If $V$ is a complex vector bundle, and $c_1$ is the first Chern class, $c_1(V) = -c_1(\overline{V})$ by [10] p.206. Since conjugation is compatible with the K"unneth formula, we deduce that $x = \bar{x}$ for $x \in \overline{K}^0(X_2)^{\times}$. Indeed, if $\beta \in \overline{K}^0(S^2)$, $y \in \overline{K}^0(Y)$ and $x = 1 + \beta y$, then $\bar{x} = 1 + (-\beta)(-y) = x$. Let $A$ be a continuous field over $X_3$ with fibers $M_N(\mathbb{Z})$ such that $[A]_{Br} = 1 + \beta y$ in $Br_{\mathcal{Z}}(X_3) \cong 1 + \overline{K}^0(S^2) \otimes \overline{K}^0(Y) \cong 1 + \mathbb{Z}/3$, where $\beta$ a generator of $\overline{K}^0(S^2)$ and $y$ is a generator of $\overline{K}^0(Y)$. Then $[A]_{Br} = 1 + (-\beta)(-y) = [A]_{Br}$ and hence 

$$[\overline{A} \otimes_{C(X_3)} A]_{Br} = (1 + \beta y)^2 = 1 + 2\beta y \neq 1.$$ 

**Corollary 3.6.** Let $X$ be a compact metrizable space and let $A$ be a locally trivial continuous field with fiber $D \otimes \mathbb{K}$ with $D$ in the class $D$. If $H^{4k+1}(X, \mathbb{Q}) = 0$ for all $k \geq 0$, then there is an $N \in \mathbb{N}$ such that 

$$(A \otimes_{C(X)} A^{op}) \otimes N \cong C(X, D \otimes \mathbb{K}).$$

**Proof.** If $H^{4k+1}(X, \mathbb{Q}) = 0$, then $\delta_{2k}(A \otimes_{C(X)} A^{op}) = 0$ for all $k \geq 0$. Moreover, $\delta_{2k+1}(A \otimes_{C(X)} A^{op}) = \delta_{2k+1}(A) - \delta_{2k+1}(A) = 0$. The statement follows from Corollary 2.17. 

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