A relation between entropy and transitivity of Anosov diffeomorphisms

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Abstract
It is known that transitive Anosov diffeomorphisms have a unique measure of maximal entropy (MME). Here we discuss the converse question. Transitivity of Anosov diffeomorphisms can be reached under suitable hypotheses on Lyapunov exponents on the set of periodic points and the structure of the MME. In another way, assuming together the uniqueness of MME and that every point is regular, in Oseledec’s Theorem sense, also we can get the transitivity of Anosov diffeomorphisms in this setting.

Keywords Anosov diffeomorphisms · Transitivity · Topological Entropy · Unstable Entropy

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1 Introduction
Let \((M, g)\) be a \(C^\infty\) compact, connected and boundaryless Riemannian manifold and \(f : M \to M\) be a \(C^1\)–diffeomorphism. We say that \(f\) is an Anosov diffeomorphism if there are numbers \(0 < \beta < 1 < \eta, C > 0\) and a \(Df\)–invariant continuous splitting \(T_x M = E^u_f(x) \oplus E^s_f(x)\), such that for any \(n \geq 0\),
\[
||Df^n(x) \cdot v|| \geq \frac{1}{C} \eta^n ||v||, \forall v \in E^u_f(x),
\]
\[
||Df^n(x) \cdot v|| \leq C \beta^n ||v||, \forall v \in E^s_f(x).
\]

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It is known that the bundles $E^s_f, E^u_f$ are respectively integrable to invariant stable and unstable foliations denoted $\mathcal{F}_f^s$ and $\mathcal{F}_f^u$. Given a point $x \in M$, the stable and unstable leaves that contain $x$ are respectively characterized by

$W^s_f(x) = \{y \in M | \lim_{n \to +\infty} d(f^n(x), f^n(y)) = 0\}$,

$W^u_f(x) = \{y \in M | \lim_{n \to +\infty} d(f^{-n}(x), f^{-n}(y)) = 0\}$.

If $f$ is $C^r, r \geq 1$, then $W^*_f(x), * \in \{s, u\}$ are embedded $C^r$ submanifolds of $M$.

From now on we denote by $m$ the probability Lebesgue measure on $M$ induced by $g$, and $\text{Per}(f)$ the set of periodic points of a diffeomorphism $f$.

Anosov diffeomorphisms play an important role in the theory of dynamical systems and this class of diffeomorphisms satisfies many rich dynamical properties, such as shadowing and closing lemmas and, in the case that the Anosov diffeomorphism is $C^2$ and preserves a measure $\mu$ absolutely continuous, this measure is ergodic. A step towards classifying Anosov diffeomorphisms, up to topological conjugacy, would consist in showing that every Anosov diffeomorphism is transitive.

**Definition 1.1** Let $(X, d)$ be a compact metric space and $f : X \to X$ a continuous function. We say that $f$ is transitive if for any nonempty open sets $U$ and $V$ there exists an integer $N$ such that $f^{-N}(V) \cap U \neq \emptyset$, or equivalently, there exists a point $x \in X$ with dense orbit.

There are examples of non-transitive Anosov flows, see [9], but there are no known examples of non-transitive Anosov diffeomorphisms.

For Anosov diffeomorphisms, since they are Axiom A diffeomorphisms, transitivity property is equivalent to $\Omega(f) = M$, where $\Omega(f)$ is the non-wandering set of $f$. Under transitivity assumption, Anosov diffeomorphisms have a unique measure of maximal entropy, see [4], for instance. We could ask about the converse.

**Question 1** Given $f : M \to M$ an Anosov diffeomorphism having a unique measure of maximal entropy, is $f$ transitive?

In general, this question is totally inconclusive. Note that, in case of a possible non-transitive Anosov diffeomorphism, could occur $\Omega(f) = \Omega_1 \cup \ldots \cup \Omega_s, s > 1$, a union of distinct basic sets with mutually different topological entropies, so $f$ could have a unique measure of maximal entropy but would not be transitive. We could ask under what conditions an Anosov diffeomorphism having a unique measure of maximal entropy is transitive.

In this work, we present partial answers to the above Question 1 connecting transitivity, volume growth of unstable leaves, Lyapunov exponents, and some regularity of the measure of maximal entropy.

Before stating our results let us introduce something concerning Lyapunov exponents.

**Definition 1.2** Let $f : M \to M$ be a $C^1$ diffeomorphism. We say that $x \in M$ is a regular point for $f$ if there are real numbers $\lambda_1(x) > \lambda_2(x) > \ldots > \lambda_l(x)$ and a splitting $T_xM = E_1(x) \oplus \ldots \oplus E_l(x)$ of the tangent space of $M$ at $x$, such that

$$\lim_{n \to \pm \infty} \frac{1}{n} \log(||Df^n(x) \cdot v||) = \lambda_i(x), \forall v \in E_i \setminus \{0\} \text{ and } 1 \leq i \leq l.$$ 

The numbers $\lambda_i(x)$ are called Lyapunov exponents of $f$ at $x$. We denote by $R(f)$ the set of regular points of $f$. 

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**Theorem 1.3** (Oseledec’s Theorem) If \( f : M \to M \) is a \( C^1 \)–diffeomorphism preserving \( \mu \), a Borel probability measure, then \( \mu(R(f)) = 1 \).

For the proof of Oseledec’s Theorem see [14] or [11].

Given a point \( x \in R(f) \), we denote by \( \Lambda^u_f(x) \) the sum of all positive Lyapunov exponent of a point \( x \). More precisely

\[
\Lambda^u_f(x) = \sum_{i=1}^t \max\{\lambda_i(x), 0\} \cdot \dim(E_i(x)).
\]

Analogously we denote by \( \Lambda^s_f(x) \) the sum of all negative Lyapunov exponent of \( x \in R(f) \).

**Theorem A** If \( f : M \to M \) is a \( C^1 \)–Anosov diffeomorphism such that \( p \mapsto \Lambda^u_f(p) \) is constant on \( \text{Per}(f) \), then \( h_{\text{top}}(f) = \Lambda^u_f \), where \( \Lambda^u_f \) here denotes the common value \( \Lambda^u_f(p), p \in \text{Per}(f) \). Additionally, suppose that \( f \) has a unique measure of maximal entropy, \( \mu \). If \( \mu \) is an absolutely continuous measure with continuous density with respect to \( m \), then \( f \) is transitive.

In Theorem A, we are not presuming that the density \( \rho := \frac{d\mu}{dm} \) is positive. If that were the case, the result would be trivial, since \( \mu \) is ergodic. The additional part of Theorem A is very easy in the \( C^2 \)–setting, by applying the SRB theory of [4], for instance. Here we deal with \( C^1 \)–context and the SRB theory doesn’t hold in general.

**Theorem B** If \( f : M \to M \) is a \( C^1 \)–Anosov diffeomorphism, such that \( R(f) = M \) and it has a unique measure of maximal entropy, then \( f \) is transitive.

**Remark 1.4** Particularly all algebraic linear Anosov automorphism on tori or infra-nilmanifolds satisfies all our hypotheses. In this way, every known Anosov diffeomorphism is conjugated to a model satisfying all hypotheses of our Theorems. So, modulo conjugacy, the class of Anosov diffeomorphisms satisfying all hypotheses of our Theorems could be bigger than the class of linear Anosov automorphisms. The great question is if the class of Anosov diffeomorphisms coincides or not with the set of linear Anosov diffeomorphisms, modulo conjugacy.

## 2 Comments and applications

We observe that when the regularity of an Anosov diffeomorphism \( f \) is only \( C^1 \), such diffeomorphism does not necessarily satisfy the absolute continuity of the unstable and stable foliations, see [15]. The absolute continuity of the unstable and stable foliations ever holds in \( C^2 \) context, it is the main ingredient in the proof of ergodicity of \( C^2 \) conservative Anosov diffeomorphisms. In \( C^2 \) setting if \( f \) is an Anosov diffeomorphism preserving an absolute continuous measure with respect to \( m \), we can conclude \( \Omega(f) = M \), see [6]. In contrast with this, for \( C^1 \) Anosov diffeomorphisms, Bowen in [5] constructed an example of Axiom A diffeomorphism with a basic set \( \Omega_s \), with empty interior and such that \( m(\Omega_s) > 0 \).

Theorem A allows us to study the topological entropy for a class of \( C^1 \) perturbations of Anosov diffeomorphisms that act as identity on an invariant direction. These perturbations have special importance in the problems related to increasing or decreasing center Lyapunov exponents in the partially hyperbolic context.
Consider \( A : \mathbb{T}^3 \to \mathbb{T}^3 \) a linear Anosov automorphism such that \( T^3 \to \mathbb{T}^3 = E_s^A \oplus E^s_A \oplus E^u_A \) and the corresponding eigenvalues \( \lambda_s^A, \lambda^s_A, \lambda^u_A \) satisfy \( \lambda^s_A < \lambda^u_A < 0 < \lambda_s^A \). The stable bundle of \( A \) is the sum \( E_s^A \oplus E^s_A \).

In the remarkable work of Baraviera and Bonatti [3], to perturb center Lyapunov exponents, they construct smooth perturbations \( \psi : \mathbb{T}^3 \to \mathbb{T}^3 \), which are sufficient \( C^1 \) — close to the identity such that

- \( \psi \) are volume preserving,
- \( \psi \) leave invariant the direction \( E^u_A \),
- \( \psi \) acts on the direction \( E^u_A \) as the identity.

Moreover, by the construction in [3], the map \( f = A \circ \psi : \mathbb{T}^3 \to \mathbb{T}^3 \) is a smooth and volume preserving Anosov diffeomorphism such that

\[
\lambda^s_f = \int_{\mathbb{T}^3} \log(||Df(x)||E^s_f||)dm > \lambda^s_A.
\]

Since \( \psi \) are smooth perturbations sufficiently \( C^1 \) — close to the identity, \( A \) and \( f \) are conjugated by a conjugacy \( h \) and consequently have the same topological entropy. Moreover, \( \psi \) act on the direction \( E^u_A \) as the identity, then

\[
\lambda^u_f = \int_{\mathbb{T}^3} \log(||Df(x)||E^u_f||)dm = \lambda^u_A.
\]

By applying the Pesin formula

\[
\lambda^u_A = h_m(f),
\]

the volume \( m \) is also the measure of maximal entropy for \( f \).

Let us we consider a \( C^1 \) perturbation \( \psi \), non necessarily \( C^1 \) close to the identity, requiring only that \( \psi \) acts as the identity on the direction \( E^u_A \). If \( f = A \circ \psi \) is still Anosov, such that \( \dim E^u_f = 1 \), our Theorem A allows to conclude that \( h_{\text{top}}(f) = h_{\text{top}}(A) \).

Our Theorem B is a \( C^1 \) version of Corollary 1.6 of [12]. In that work, since the setting is \( C^2 \) we don’t need to assume the uniqueness of the measure of maximal entropy.

To finalize our comments we mention that M. Brin in [7] obtained sufficient conditions for transitivity by a pinching condition on the uniform expanding and uniform contraction constants, to get a global product structure and so \( \Omega(f) = M \). Due to the scarcity of results guaranteeing the transitivity of Anosov diffeomorphisms, we believe that the techniques applied to prove our Theorems may be of special interest in the study of the transitivity of Anosov diffeomorphisms.

### 3 Useful technical preliminaries

To prove Theorems A and B we will need some tools related with uniform convergence of the limit which defines Lyapunov exponents. This uniformity is related with volume growth of the unstable foliation and consequently with the entropy along the unstable leaves.

Let us recall results given in [1] and [8].

**Lemma 3.1** Let \( \mathcal{M} \) be the space of \( f \) — invariant measures, \( \phi \) be a continuous function on \( M \). If \( \alpha < \int \phi d\mu < \lambda, \forall \mu \in \mathcal{M}, \) then there exists \( N \) such that for every \( n \geq N \), holds

\[
\alpha < \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i(x)) < \lambda.
\]
A relation between entropy and transitivity of Anosov diffeomorphisms

for all \( x \in M \).

See [1] or [8] for the proofs of the above Lemma. The version \( \int \phi d\mu > \alpha \) is not written in the aforementioned papers, but it analogously takes place.

An important tool for us is the notion of topological entropy along an expanding foliation. In [10] the authors deal with a notion of topological entropy \( h_{\text{top}}(f, \mathcal{W}) \) of an invariant expanding foliation \( \mathcal{W} \) of a diffeomorphism \( f \). Among other results, they establish the variational principle in this sense and also a relation between \( h_{\text{top}}(f, \mathcal{W}) \) and volume growth of \( \mathcal{W} \).

Here \( W(x) \) denotes the leaf of \( \mathcal{W} \) by \( x \). Given \( \delta > 0 \), we denote by \( W(x, \delta) \) the \( \delta \)-ball centered in \( x \) on \( W(x) \), with the induced Riemannian distance, which is denoted by \( d_W \).

Given \( x \in M, \varepsilon > 0, \delta > 0 \) and \( n \geq 1 \) an integer number, let \( N_W(f, \varepsilon, n, x, \delta) \) be the maximal cardinality of a set \( S \subset W(x, \delta) \) such that \( \max_{j=0, \ldots, n-1} d_W(f^j(z), f^j(y)) \geq \varepsilon \), for any distinct points \( y, z \in S \).

**Definition 3.2** The unstable entropy of \( f \) on \( M \), with respect to the expanding foliation \( \mathcal{W} \) is given by

\[
\begin{align*}
    h_{\text{top}}(f, \mathcal{W}) &= \lim_{\delta \to 0} \sup_{x \in M} h_{\text{top}}^{\mathcal{W}}(f, W(x, \delta)), \\
    h_{\text{top}}^{\mathcal{W}}(f, W(x, \delta)) &= \lim_{\varepsilon \to 0} \limsup_{n \to +\infty} \frac{1}{n} \log(N_W(f, \varepsilon, n, x, \delta)).
\end{align*}
\]

Define \( \mathcal{W} \)-volume growth by

\[
\chi_{\mathcal{W}}(f) = \sup_{x \in M} \chi_{\mathcal{W}}(x, \delta),
\]

where

\[
\chi_{\mathcal{W}}(x, \delta) = \limsup_{n \to +\infty} \frac{1}{n} \log(Vol(f^n(W(x, \delta)))).
\]

Note that, since we are supposing \( \mathcal{W} \) an expanding foliation, the above definition is independent of \( \delta \) and the Riemannian metric.

**Theorem 3.3** (Theorem C and Corollary C.1 of [10]) With the above notations

\[
h_{\text{top}}(f, \mathcal{W}) = \chi_{\mathcal{W}}(f).
\]

Moreover, \( h_{\text{top}}(f) \geq h_{\text{top}}(f, \mathcal{W}) \).

### 3.1 Attractors and repellers

Consider \( \Omega(f) = \Omega_1 \cup \ldots \cup \Omega_n \) the spectral decomposition corresponding to an axiom A diffeomorphism \( f : M \to M \). We say that a basic set \( \Omega_i \) is an attractor if and only if \( \bigcup_{x \in \Omega_i} W^s_f(x) \) is an open neighborhood of \( \Omega_i \). A basic set \( \Omega_j \) is an repeller if and only if \( \bigcup_{x \in \Omega_j} W^u_f(x) \) is an open neighborhood of \( \Omega_j \).

A general property of attractors is that they are \( u \)-saturated, it means that if \( \Omega_i \) is an attractor and \( x \in \Omega_i \), then \( W^u_f(x) \subset \Omega_i \). Analogously, repellers are \( s \)-saturated. A well-known direct consequence of these facts is that if \( \Omega_i \) is simultaneously attractor and repeller, then \( \Omega_i = M \), by the connectedness of \( M \).
Theorem 3.4 If $f : M \rightarrow M$ is an expansive homeomorphism having shadowing property, then $f$ has attractors and repellers. Particularly, Anosov diffeomorphisms have attractors and repellers basic sets.

For more about attractors and repellers, we refer [2], Chapter 3.

4 Proof of theorem A

Consider $X$ a compact metric space and $f : X \rightarrow X$ a continuous map. The first fact that we observe is that if $f$ has a unique measure of maximal entropy, $\mu$, then it is ergodic. This fact is a consequence of Jacobs Theorem.

Lemma 4.1 (Jacobs Theorem) Suppose that $X$ is a completely separable metric space and $f : X \rightarrow X$ a continuous function. Given any Borel invariant probability measure $\mu$ let $\{ \mu_P : P \in \mathcal{P} \}$ be its ergodic decomposition. Then,

$$h_\mu(f) = \int_M h_{\mu_P}(f) d\hat{\mu}(P),$$

if one side is infinite, so is the other side.

For a proof, we refer to [13].

When the topological entropy is finite, by Variational Principle, we observe that in the case where $\mu$ is the unique measure of maximal entropy, then, necessarily, it is an ergodic component.

The proof of Theorem A is a consequence of two lemmas. Denote $J^u_x = |\det Df(x) : E^u_x \rightarrow E^u_{f(x)}|$. We can use Anosov Closing Lemma to get

$$\Lambda^u_f(x) := \lim_{n \rightarrow +\infty} \frac{1}{n} \log(J^u f^n(x)) = \Lambda^u_f,$$

for any $x \in R$. In fact, consider $\varepsilon > 0$ arbitrary. Since $f$ is $C^1$ take $\delta > 0$ such that $1 - \varepsilon < J^u f(x) < 1 + \varepsilon$, if $d(x, y) < \delta$. Using that $x \in R$, there is a sequence of integers $n_k, k = 1, 2, \ldots$ such that $d(x, f^{n_k}(x)) < \delta'$, for some $\delta' > 0$ such that every $\delta'$-pseudo orbit is indeed $\delta$ shadowed by a periodic orbit of a point $p_k$ such that $f^{n_k}(p_k) = p_k$. So

$$\frac{J^u f^{n_k}(x)}{J^u f^{n_k}(p_k)} = \frac{\prod_{j=0}^{n_k-1} J^u f(f^j(x))}{\prod_{j=0}^{n_k-1} J^u f(f^j(p_k))} \Rightarrow (1 - \varepsilon)^{n_k} < \frac{J^u f^{n_k}(x)}{\Lambda^u_f^{n_k}} < (1 + \varepsilon)^{n_k}.$$

We get $\Lambda^u_f + \log(1 - \varepsilon) \leq \Lambda^u_f(x) = \lim_{k \rightarrow +\infty} \frac{1}{n_k} \log(J^u f^{n_k}(x)) \leq \Lambda^u_f + \log(1 + \varepsilon)$, so taking $\varepsilon$ going to zero, we obtain $\Lambda^u_f(x) = \Lambda^u_f$, for any $x \in R$.

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By Ruelle’s inequality we obtain $h_{\mu}(f) \leq \Lambda^u_f$, and using the Variational Principle, we get

$$h_{\text{top}}(f) \leq \Lambda^u_f.$$  \hfill (4.2)

Since the identity Eq. 4.1 holds on a full probability set, using Lemma 3.1 and the Birkhoff Ergodic Theorem with $\phi(x) = \log(J^u f(x))$, $x \in M$, we conclude that the limit given in the expression Eq. 4.1 is uniform. So for any $\varepsilon > 0$, there is $N > 0$ an integer number such that for any $n \geq N$ and $x \in M$, holds

$$J^u f^n(x) > e^{n(\Lambda^u_f - \varepsilon)}.$$  \hfill (4.3)

So given $B^u(x, \delta)$ a $u$–ball centered in $x$ with radius $\delta > 0$, we have

$$Vol_u(f^n(B^u(x, \delta))) = \int_{B^u(x, \delta)} J^u f^n(x) dVol_u(x) > e^{n(\Lambda^u_f - \varepsilon)} Vol_u(B^u(x, \delta)),$$  \hfill (4.4)

where $Vol_u$ denotes the $u$–dimension volume along unstable leaves induced by the Riemannian metric of $M$.

By Eq. 4.4 we get $\chi^{F^u}(f) \geq \Lambda^u_f - \varepsilon$, for any $\varepsilon > 0$. From Theorem 3.3 we obtain $h_{\text{top}}(f, F^u) \geq \Lambda^u_f$, and

$$h_{\text{top}}(f) \geq \Lambda^u_f.$$  \hfill (4.5)

Now the Eqs. 4.2 and 4.5, we get $h_{\text{top}}(f) = \Lambda^u_f$. \hfill $\square$

**Lemma 4.3** Let $f : M \to M$ be a $C^1$ – Anosov diffeomorphism such that $p \mapsto \Lambda^u_f(p)$ is constant on $\text{Per}(f)$. If $f$ has a unique measure of maximal entropy $\mu$ and it is absolutely continuous with respect to $m$, such that the density $\rho = \frac{d\mu}{dm}$ is continuous, then $\mu(U) > 0$, for any non empty open set $U \subset M$, consequently $f$ is transitive.

**Proof** As in the previous Lemma, consider $\Lambda^u_f$ the common value $\Lambda^u_f(p), p \in \text{Per}(f)$. Let $\rho$ be the continuous density of $\mu$ and consider the compact set $H_0 = \rho^{-1}([0])$, in particular $\mu(H_0) = 0$. Suppose that there exists a non empty open set $U \subset M$, such that $\mu(U) = 0$, so $U \subset H_0$. In fact, if there was some $x \in U$, such that $\rho(x) > 0$, then by continuity of $\rho$, we could choose an open set $V \subset U$, such that $x \in V$ and $\rho(t) > \delta > 0$, for some $\delta > 0$ and every $t \in V$. With this $0 < \mu(V) < \mu(U)$, that contradicts the fact $\mu(U) = 0$.

By invariance of $\mu$, given $n \in \mathbb{Z}$, the set $f^n(U)$ is open set and $\mu(f^n(U)) = 0$, so as before $f^n(U) \subset H_0$. We conclude that $U \subset f^{-n}(H_0)$, for any $n \in \mathbb{Z}$.

Define $\Lambda = \bigcap_{n \in \mathbb{Z}} f^{-n}(H_0)$, we note that $U \subset \Lambda$, and $\Lambda$ is a non empty and compact hyperbolic set invariant by $f$.

Since $U \subset \Lambda$, we can take a local unstable arc $W$ of $W^u_f(x)$ fully contained in $U \subset \Lambda$. Since $\Lambda$ contains an open unstable arc $W$, proceeding as in Lemma 4.2 we get

$$h_{\text{top}}(f|\Lambda, W^u_{f|\Lambda}) \geq \Lambda^u_f,$$

and by Theorem 3.3 we get

$$h_{\text{top}}(f|\Lambda) \geq h_{\text{top}}(f|\Lambda, W^u_{f|\Lambda}) \geq \Lambda^u_f = h_{\text{top}}(f) \geq h_{\text{top}}(f|\Lambda).$$

So $h_{\text{top}}(f|\Lambda) = h_{\text{top}}(f)$, since $f|\Lambda$ is expansive, then $f|\Lambda$ admits a measure of maximal entropy $\nu$, such that $\nu(\Lambda) = 1$. Define $\overline{\nu}$ a Borel measure such that $\overline{\nu}(B) = \nu(B \cap \Lambda)$. Note that $\mu(\Lambda) = 0 = \overline{\nu}(M \setminus \Lambda)$. So $\mu$ and $\overline{\nu}$ are mutually singular measures of maximal entropy of $f$. It is a contradiction with the assumption of $f$ having a unique measure of maximal entropy.
Since $\mu$ is ergodic, for any non empty open sets $U, V \subset M$, we have $\mu(U), \mu(V) > 0$. By ergodicity, for $\mu$ a.e $x \in U$,

$$
\lim_{n \to +\infty} \frac{1}{n} \sum_{j=1}^{n-1} I_V(f^j(x)) = \mu(V) > 0,
$$

so $f^j(x) \in V$, for infinitely many $j > 0$, thus $f$ is transitive.

**5 Proof of theorem B**

**Lemma 5.1** Let $f : M \to M$ be an Anosov diffeomorphism. If $R(f) = M$, then $x \mapsto \Lambda^*_f(x), * \in \{s, u\}$ are constant functions.

**Proof** The argument here is similar to the Hopf argument, using local product structure.

Let $x_0$ be an arbitrary point on $M$ and consider $\alpha = \Lambda^*_u f(x_0)$. Since $f$ have local product structure, there is an open neighborhood $V$ of $x_0$, such that, given $z \in V$, there is a point $z' \in V \cap W^u_f(z) \cap W^s_f(x_0)$, so since every point is regular

$$
\alpha = \lim_{n \to +\infty} \frac{1}{n} \log(J^u f^n(x_0)) = \lim_{n \to +\infty} \frac{1}{n} \log(J^u f^n(z')) = \lim_{n \to -\infty} \frac{1}{n} \log(J^u f^n(z)) = \Lambda^u_f(z).
$$

The map $x \mapsto \Lambda^u_f(x)$ is locally constant, since $M$ is connected it implies $\Lambda^u_f(x) = \Lambda^u_f$, for any $x \in M$. Particularly the functions $x \mapsto \Lambda^*_f(x), * \in s, u$ are constant on $Per(f)$.

Let us give the proof of Theorem B.

**Proof of Theorem B** Since $R(f) = M$, by Lemma 5.1 there are constants $\Lambda^u_f$ and $\Lambda^s_f$, such that for any $p \in Per(f)$ holds $\Lambda^u_f(p) = \Lambda^u_f$ and $\Lambda^s_f(p) = \Lambda^s_f$. By the first part of Theorem A

$$
htop(f) = \Lambda^u_f = -\Lambda^s_f.
$$

Let $\mu_1$ be the measure of maximal entropy of $f$ and $\Omega(f) = \Omega_1 \cup \ldots \cup \Omega_n$ the spectral decomposition corresponding $f$. By ergodicity of $\mu_1$ we can assume that $\mu_1(\Omega_1) = 1$. In this way

$$
htop(f) = h_{\mu_1}(f) = h_{\mu_1}(f|_{\Omega_1}). \tag{5.1}
$$

By Theorem 3.4, consider $\Omega_i$ and $\Omega_j$ an attractor and a repeller basic sets of $f$, respectively. We claim that $i = j = 1$. Suppose without lost generality $i \neq 1$. Since $\Omega_i$ is $u$—saturated, by Theorem 3.3 and Lemma 4.2 we get

$$
htop(f) = h_{\mu_1}(f|_{\Omega_1}) = \Lambda^u_f.
$$

Since $f|_{\Omega_i}$ is expansive, let $\nu_i$ be a measure of maximal entropy of $f|_{\Omega_i}$ and define $\mu_i$ such that $\mu_i(A) = \nu_i(A \cap \Omega_i)$, for every Borel set $A$, so

$$
htop(f) = h_{\mu_1}(f|_{\Omega_1}) = h_{\mu_1}(f). \tag{5.2}
$$

Since we are supposing $i \neq 1$, we conclude that $\mu_1$ and $\mu_i$ are mutually singular measures and (5.1) and (5.2) leads us to a contradiction with the assumption of $f$ having a unique measure of maximal entropy. We conclude $i = 1$ and taking $f^{-1}$, we get $j = 1$.

Finally $\Omega_1$ is simultaneously attractor and repeller, so $\Omega_1 = M$ and $f$ is transitive. □
6 Further questions

**Question 2** Let \( f : M \to M \) be an Anosov diffeomorphism. If \( p \mapsto \Lambda_f^u(p), p \in Per(f) \) is constant, is \( f \) transitive?

Denote by \( Jf(x) \) the jacobian of \( f \) at \( x \), meaning that \( Jf(x) = |\det(Df(x) : T_xM \to T_{f(x)}M)| \).

**Question 3** Let \( f : M \to M \) be an Anosov diffeomorphism and suppose that for any point \( p \) such that \( f^n(p) = p \) holds \( Jf^n(p) = 1 \). Is \( f \) transitive?

The above question was positively answered in \( C^2 \) setting, see [12].

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