An inequality for length and volume in the complex projective plane

Mikhail G. Katz

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Abstract
We prove a new inequality relating volume to length of closed geodesics on area minimizers for generic metrics on the complex projective plane. We exploit recent regularity results for area minimizers by Moore and White, and the Kronheimer–Mrowka proof of the Thom conjecture.

Keywords Minimal surface · Regularity · Systole · Closed geodesics · Croke–Rotman inequality · Gromov’s stable systolic inequality for complex projective space · Kronheimer–Mrowka theorem

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1 Introduction
The 1-systole $\text{sys}_1$ of a Riemannian manifold $M$ is the least length of a noncontractible loop in $M$. In the 1950s, Carl Loewner proved an inequality relating the systole and the area of an arbitrary metric on the 2-dimensional torus; see [14,17,34].

Gromov obtained a variety of inequalities relating the 1-systole and the volume of $M$. We will now present those that will be used in this paper. Thus, we have the inequality

$$\text{sys}_1^2(S) \leq \frac{4}{3} \text{area}(S)$$

from Ref. [13, p. 49, Corollary 5.2.B], valid for every closed aspherical surface $S$. There are also bounds that improve as the genus $g$ grows. Thus, Gromov proved that a surface $S_g$ of genus $g$ satisfies

$$\text{sys}_1^2(S_g) \leq \frac{64}{4\sqrt{g} + 27} \text{area}(S_g)$$

1 Department of Mathematics, Bar Ilan University, Ramat Gan 5290002, Israel
An asymptotically better result is the following:

\[
\frac{\text{sys}_2^2(S_g)}{\text{area}} \leq \frac{1}{\pi} \log^2 g \left(1 + o(1)\right) \quad \text{when} \; g \to \infty
\]  

(1.3)

from Ref. [18, p. 1211, Theorem 2.2], improving the multiplicative constant in Gromov’s similar upper bound. Apart from the multiplicative constant, the asymptotic behavior \(\log^2 g\) is the correct one due to the existence of arithmetic hyperbolic surfaces satisfying lower bounds of this type; see e.g., Buser–Sarnak [3], Katz et al. [16,20,21].

The literature contains a number of results on the higher systoles, as well. Recall that the 2-systole \(\text{sys}_2\) of \(M\) can be defined as the least area of a homologically nontrivial surface in \(M\), or more generally 2-cycle with integer coefficients:

\[
\text{sys}_2(M) = \min \left\{ \text{area}(S) : [S] \in H_2(M; \mathbb{Z}) \setminus \{0\} \right\}
\]

The 2-systole of \(M\) is typically not controlled by the volume of \(M\), a phenomenon referred to as systolic freedom. For example, the complex projective plane \(\mathbb{C}P^2\) admits metrics of arbitrarily small volume such that every homologically nontrivial surface in \(\mathbb{C}P^2\) has at least unit area; see Katz–Suciu ([22], Theorem 1.1, p. 113).

Gromov has also defined a modified invariant of \(M\) called the stable 2-systole, \(\text{stsys}_2(M)\). It is a result of Federer [9] that the limits below exist and therefore can be used to define \(\text{stsys}_2\) as follows:

\[
\text{stsys}_2(M) = \min \left\{ \lim_{n \to \infty} \frac{1}{n} \text{sys}_2(n\alpha) : \alpha \in H_2(M; \mathbb{Z}) \setminus \{\text{torsion}\} \right\}
\]  

(1.4)

where \(\text{sys}_2(x)\) denotes the infimum of areas of surfaces representing the class \(x \in H_2(M; \mathbb{Z})\).

Gromov’s stable systolic inequality for \(\mathbb{C}P^n\) asserts that\n
\[
\text{stsys}_2^n(\mathbb{C}P^n) \leq n! \text{vol}(\mathbb{C}P^n),
\]

for an arbitrary metric on \(\mathbb{C}P^n\). A more general non-optimal inequality appears in [13, p. 96, item 7.4.C]. The optimal inequality for \(\mathbb{C}P^n\) is in [14, p. 262, Theorem 4.36].

The proof relies on the duality of the stable norm in homology and the comass norm in cohomology; see [14, Section 4.34, p. 261], [10, Section 4.10, p. 380], [33, Lemma 17], [1,15,19]. The 2-point homogeneous Fubini-Study metric satisfies the case of equality in this sharp inequality.

The 1-systole of a compact manifold \(M\) is always realized by a closed geodesic in \(M\). There exist inequalities for the least length \(\text{LCG}(M)\) of a nontrivial closed geodesic in \(M\). One such inequality is the Croke–Rotman inequality

\[
\text{LCG}^2(S^2) \leq 32\text{area}(S^2)
\]  

(1.6)

(see [5,35]), for an arbitrary metric on the 2-sphere. The multiplicative constant 32 is not believed to be optimal. It is conjectured that the following tight bound holds:

\[
\text{LCG}^2(S^2) \leq 2\sqrt{3}\text{area}(S^2)
\]  

(1.7)

(see e.g., Burns–Matveev [2, p. 23]), with equality attained by a triangular “pillow.”

We will exploit these geometric inequalities to prove Theorem 1.1 below.

Note that a degree \(d\) Veronese embedding \(\mathbb{C}P^1 \to \mathbb{C}P^n(d)\) (by degree \(d\) homogeneous polynomials in homogeneous coordinates) has constant Gaussian curvature. Namely, the embedding preserves the Kahler form \(\omega\). Hence it necessarily preserves the metric due to the relation \(\omega(X, Y) = g(JX, Y)\) (when the metrics are suitably normalized). Hence the
Veronese embedding is an isometry. Its image has area $\pi d$ when the sectional curvature of $\mathbb{CP}^n$ is normalized to satisfy $1 \leq K \leq 4$. The geodesics are the great circles and therefore the LCG of this degree $d$ genus 0 minimal surface grows as $\sqrt{d}$. We show that such a phenomenon cannot occur in $\mathbb{CP}^2$ even if one allows arbitrary generic metrics.

**Theorem 1.1** For a generic Riemannian metric on the complex projective plane $\mathbb{CP}^2$, every minimal surface $S$ of sufficiently high homological multiplicity admits a nontrivial closed geodesic of length controlled by the total volume of the metric: $\text{LCG}^4(S) \leq C \text{vol}(\mathbb{CP}^2)$.

Specific values for the constant $C$ are discussed below.

**Corollary 1.2** For a generic metric of unit volume on $\mathbb{CP}^2$, there exist homologically nontrivial minimal surfaces $S$ with a nontrivial closed geodesic of length at most $C$, with constant $C$ independent of the metric.

Thus, while the total volume does not control the area of a homologically nontrivial minimizing surface, it does control the length of a shortest closed geodesic on the minimizer.

More detailed versions of the theorem with explicit constants appear in Sect. 2.

Our techniques do not enable us to find closed geodesics of controlled length for an arbitrary metric on $\mathbb{CP}^2$. The existence of closed geodesics in compact Riemannian manifolds was proved by Fet [11,12], but it is unknown whether there exists a closed geodesic of length controlled by the volume; see also Rotman [36, Theorem 0.3] for related results. Sabourau [38] constructs a one-cycle sweepout of the essential sphere in the complex projective space (endowed with an arbitrary Riemannian metric) whose one-cycle length is controlled by the volume. A generalisation appears in [31].

We will use the following result of Moore [27, p. 279, Theorem 5.1.1], [28, Theorem 2] and White [40, Corollary 39], exploiting Moore [25,26].

**Theorem 1.3** (Moore; White). For a generic metric on $M$, a simple (multiplicity 1) area minimizer $S$ in a homology class in $H_2(M; \mathbb{Z})$ is a smoothly embedded surface.

Briefly, by Morgan’s result [29], in the orientable case the tangent planes to a minimizer $S$ at a self-intersection $p \in M$ must necessarily be complex lines of a common complex structure on $T_p M$, and it can be shown that such a situation cannot arise for a generic conformal factor. The nonorientable case (for homology classes with $\mathbb{Z}_2$ coefficients) was treated in White’s earlier paper [39]. Our Theorem 1.1 can be contrasted with the following open questions in the case of $\mathbb{Z}_2$ coefficients.

1. Is the area of a minimizing surface $S \in [\mathbb{CP}^1] \in H_2(\mathbb{CP}^2; \mathbb{Z}_2)$ controlled by the total volume:

$$\text{area}(S) \leq C \sqrt{\text{vol}(\mathbb{CP}^2)} \quad ?$$

2. Is the length of a shortest closed geodesic $\gamma$ on a minimizing surface $S \in [\mathbb{CP}^1] \in H_2(\mathbb{CP}^2; \mathbb{Z}_2)$ controlled by the total volume:

$$\text{length}(\gamma) \leq C \sqrt[4]{\text{vol}(\mathbb{CP}^2)} \quad ?$$

Note that a minimizing surface $S \in [\mathbb{CP}^1] \in H_2(\mathbb{CP}^2; \mathbb{Z}_2)$ can be nonorientable. In such a situation, smoothness results of White still apply, but there is no analog of the stable 2-systolic inequality (1.5).
Our proof combines the regularity results by Moore and White with the celebrated Kronheimer–Mrowka proof of the Thom conjecture (see Sect. 3), to obtain upper bounds on the closed geodesic as above. Possible generalisations are discussed in Sect. 3.

## 2 Closed geodesics on area minimizers

One possible tool in investigating geometric inequalities is the existence of minimizing surfaces in homology classes. Such existence is guaranteed by Geometric Measure Theory; see Chang [4], De Lellis et al. [6–8]. It is in the nature of the techniques used that the geometry of the minimizer is rather inexplicit. Geometric inequalities (such as (2.1) below) can be viewed as providing some control over the geometry of the minimizer.

While the volume of a metric on \( \mathbb{C}P^2 \) does not control the 2-systole (as discussed in Sect. 1), it turns out that the volume does control suitable 1-dimensional invariants (such as the LCG) of area minimizing surfaces \( S \subseteq \mathbb{C}P^2 \).

### Theorem 2.1
Consider a generic metric on \( \mathbb{C}P^2 \). Then for \( n \) sufficiently large, an area-minimizing surface \( S \in n[\mathbb{C}P^1] \) satisfies

\[
\text{LCG}(S) \leq 8\sqrt{2} \cdot \sqrt{2! \text{vol}(\mathbb{C}P^2)}. \tag{2.1}
\]

Note that we left the factor \( 2! = 2 \) under the radical sign in inequality (2.1) as an allusion to Gromov’s stable systolic inequality (1.5), used in the proof of (2.1).

### Corollary 2.2
For a generic metric of unit volume on \( \mathbb{C}P^2 \), there exist homologically nontrivial minimal surfaces with a closed geodesic of length at most \( 8 \cdot 2^{3/4} \).

### Proof of Theorem 2.1
We first consider the two values \( n = 1, 2 \). Suppose we have a bound

\[
\text{sys}_2(2[\mathbb{C}P^1]) \leq 4\text{stsys}_2(\mathbb{C}P^2). \tag{2.2}
\]

Then the area of a minimizing surface \( S \in n[\mathbb{C}P^1], n = 1, 2 \), is controlled by the volume of \( \mathbb{C}P^2 \) via Gromov’s stable systolic inequality (1.5). If \( S \) is the sphere, then \( \text{LCG}(S) \) is controlled by the area of \( S \) via the Croke–Rotman inequality (1.6). If \( S \) is aspherical, then even better bounds exist such as Gromov’s inequality (1.1). By (2.2) and (1.6), in either case we have

\[
\text{LCG}^2(S) \leq 2^7\text{stsys}_2(\mathbb{C}P^2).
\]

It follows that

\[
\text{LCG}^4(S) \leq 2^{14}\text{stsys}_2^2(\mathbb{C}P^2) \leq 2^{14}2!\text{vol}(\mathbb{C}P^2),
\]

proving inequality (2.1) in these cases.

It remains to consider the case when (2.2) is violated; in other words, \( \text{sys}_2(2[\mathbb{C}P^1]) > 4\text{stsys}_2(\mathbb{C}P^2) \). The rest of the proof is as in the proof of the sharper bound of Theorem 3.1 below. \( \square \)
3 Inequality in the assumption of the tight Croke–Rotman inequality

**Theorem 3.1** Suppose the tight Croke–Rotman inequality (1.7) holds. Then given a generic metric on $\mathbb{CP}^2$, for $n$ sufficiently large, an area-minimizing surface $S \in n[\mathbb{CP}^1]$ satisfies

$$\text{LCG}(S) \leq 2 \sqrt[4]{\frac{2}{\pi}} \sqrt{2! \text{vol}(\mathbb{CP}^2)}.$$  

(3.1)

In other words, we have

$$\text{LCG}(S) \leq 4 \sqrt{\frac{\pi}{\sqrt{\pi^2/2}}} \sqrt{2! \text{vol}(\mathbb{CP}^2)}.$$  

(3.2)

**Corollary 3.2** For a generic metric of unit volume on $\mathbb{CP}^2$, in the assumption of (1.7), there exist homologically nontrivial minimal surfaces with a closed geodesic of length at most 4.

Note that for the Fubini–Study metric on $\mathbb{CP}^2$, one has

$$\text{LCG} \left( \frac{\pi}{\sqrt{\pi^2/2}} \right) = \sqrt{\pi \sqrt{2}},$$

not far from the value of the constant in (3.2).

**Proof of Theorem 3.1** We first consider the two values $n = 1, 2$. Let $\epsilon > 0$ (to be specified later). Suppose we have the bound

$$\text{sys}_2(2[\mathbb{CP}^1]) \leq 2(1 + \epsilon) \text{stsys}_2(\mathbb{CP}^2).$$  

(3.3)

Then the area of a minimizing surface $S \in n[\mathbb{CP}^1]$, $n = 1, 2$, is controlled by the volume of $\mathbb{CP}^2$ via Gromov’s inequality (1.5). If $S$ is the sphere, then LCG($S$) is controlled by the area of $S$ via the tight Croke–Rotman inequality (1.7) by the hypothesis of our theorem. If $S$ is aspherical, then even better bounds exist such as Gromov’s inequality (1.1). In either case, by (1.7) and (3.3) we have

$$\text{LCG}^2(S) \leq 4 \sqrt{3}(1 + \epsilon) \text{stsys}_2(\mathbb{CP}^2).$$

Gromov’s stable systolic inequality (1.5) then gives

$$\text{LCG}^4(S) \leq \left( 4 \sqrt{3}(1 + \epsilon) \right)^2 \text{stsys}_2^2(\mathbb{CP}^2) \leq \left( 4 \sqrt{3}(1 + \epsilon) \right)^2 2! \text{vol}(\mathbb{CP}^2).$$

Thus,

$$\text{LCG}(S) \leq 2 \sqrt[4]{\frac{2}{\pi}} \sqrt{1 + \epsilon^2} \sqrt{2! \text{vol}(\mathbb{CP}^2)}.$$  

(3.4)

For $\epsilon$ sufficiently small, (3.4) implies inequality (3.1) in these cases. Thus, we can assume that inequality (3.3) is violated; in other words,

$$\text{sys}_2(2[\mathbb{CP}^1]) > 2(1 + \epsilon) \text{stsys}_2(\mathbb{CP}^2).$$

In this case we will exploit higher multiples of the class $[\mathbb{CP}^1]$ in order to prove the bound (3.1). Let $n \geq 3$ be the least value such that

$$\text{sys}_2(n[\mathbb{CP}^1]) \leq (1 + \epsilon) \text{stsys}_2(\mathbb{CP}^2).$$  

(3.5)
Such an $n$ exists due to the existence of the limit in (1.4). Then an area minimizer $S \in n[\mathbb{C}P^1]$ is simple (of multiplicity 1) and connected. By White’s Theorem 1.3 for generic metrics, $S$ is smoothly embedded. For such surfaces, the genus $g$ satisfies

$$g \geq \frac{1}{2}(n - 1)(n - 2) \quad (3.6)$$

by Kronheimer–Mrowka [24]. Since $n \geq 3$, the bound (3.6) implies that the surface $S$ is aspherical, and that $g \geq \frac{(n - 2)^2}{2}$ and therefore

$$\sqrt{g} \geq \frac{n - 2}{\sqrt{2}} \quad (3.7)$$

Now bounds (1.2) and (3.7) imply that

$$\text{sys}_1(S)^2 \leq \frac{64}{\sqrt{2}(n - 2) + 27} \cdot \text{area}(S)$$

$$= \frac{64}{\sqrt{2}n - 4\sqrt{2} + 27} \cdot \text{area}(S)$$

$$< \frac{8\sqrt{2}}{n} \cdot \text{area}(S). \quad (3.8)$$

Therefore by (3.5) and (3.8), we have

$$\text{sys}_1^2(S) \leq \frac{8\sqrt{2}\text{area}(S)}{n} \leq 8\sqrt{2}(1 + \epsilon)\text{stsys}_2(\mathbb{C}P^2).$$

Thus

$$\text{sys}_1^4(S) \leq (8\sqrt{2}(1 + \epsilon))^2\text{stsys}_2^2(\mathbb{C}P^2)$$

$$\leq (8\sqrt{2}(1 + \epsilon))^22!\text{vol}(\mathbb{C}P^2)$$

by Gromov’s stable systolic inequality. Thus

$$\text{sys}_1(S) \leq 2^{\frac{7}{4}}(1 + \epsilon)^{\frac{3}{2}}(2!\text{vol}(\mathbb{C}P^2))^{\frac{1}{4}}$$

for generic metrics on $\mathbb{C}P^2$ in this case. As $\epsilon \to 0$ we obtain the bound of Theorem 3.1 with a constant arbitrarily close to the required one. In principle $n$ (and therefore also a connected simple minimizer $S \in n[\mathbb{C}P^1]$) could depend on $\epsilon$. However, for sufficiently large $n$ we can exploit the bound (1.3) with better asymptotic behavior than Gromov’s bound (1.2), proving the inequality.

\[\square\]

**Question 3.3** Does the inequality admit a generalisation for 2-dimensional classes with positive self-intersection for arbitrary Riemannian metrics on Kahler manifolds exploiting [30], or even on symplectic manifolds using [32]?

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