Engineered Swift Equilibration of a Brownian Gyrator

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In the context of stochastic thermodynamics, a minimal model for non equilibrium steady states has been recently proposed: the Brownian Gyrator (BG). It describes the stochastic overdamped motion of a particle in a two dimensional harmonic potential, as in the classic Ornstein-Uhlenbeck process, but considering the simultaneous presence of two independent thermal baths. When the two baths have different temperatures, the steady BG exhibits a rotating current, a clear signature of non equilibrium dynamics. Here, we consider a time-dependent potential, and we apply a reverse-engineering approach to derive exactly the required protocol to switch from an initial steady state to a final steady state in a finite time \( \tau \). The protocol can be built by first choosing an arbitrary quasi-static counterpart - with few constraints - and then adding a finite-time contribution which only depends upon the chosen quasi-static form and which is of order \( 1/\tau \). We also get a condition for transformations which - in finite time - conserve internal energy, useful for applications such as the design of microscopic thermal engines. Our study extends finite-time stochastic thermodynamics to transformations connecting non-equilibrium steady states.

Introduction - Fast switching through two or more modes of operation in microscopic experiments - where fluctuations dominate - is a goal for several applications: cyclical mesoscopic thermal machines such as colloids in time-dependent optical traps [1–6], thermal engines realised in bacterial baths [7, 8], realisation of bit operation under noisy environment with connection to information theory [9] and much more. Experiments and theory have recently demonstrated the existence of special protocols that in finite time realise conditions which are usually realised in infinite time: these protocols can be deduced by reverse-engineering the desired, fast, path of evolution of given observables, including the probability distribution in phase space [10, 11].

A paradigmatic example has been given in one effective dimension with a harmonic trap, realised by optical radiation confining a colloidal particle [12]. The colloidal particle has reached a steady state in the trap with a stiffness \( k^i \). Then the trap is modulated from the initial stiffness \( k^i \) to a new stiffness \( k^f \) in some finite time \( \tau \). If the change \( k^i \to k^f \) is realised in too short a time, e.g. taking ideally \( \tau = 0 \) (what is called “STEP” protocol), then the colloidal particle will take some uncontrolled additional time to reach the steady state compatible with the final stiffness. Such a “natural” time is related to the typical relaxation times of the system and can be very long, depending upon the situations. Interestingly, it is possible to design one or more “swift equilibration” (SE) protocols \( k(t) \), with \( k(0) = k^i \) and \( k(\tau) = k^f \), such that at time \( \tau \) the final steady state is reached and no additional relaxation time is needed. The shape of \( k(t) \) can be non-intuitive: when \( \tau \) is smaller than the typical relaxation times, such protocols may exhibit large excursions well outside of the range \([k^i, k^f]\). In fact, there are cases where \( k(t) \) can even become negative, posing problems to its experimental realization. Additional constraints may be introduced into the mathematical design problem, in order to limit the protocol excursion [13]. Other possibilities have been suggested, where the trap position is also modulated by additional noise [14]. The SE protocol has been demonstrated also in an atomic force microscopy experiments [15].

Here, we discuss the problem of swift equilibration in a two-dimensional harmonic trap. The generalization may seem a pure increase of dimensionality, but in fact it allows us to step outside of the realm of pure equilibrium steady states. A two-dimensional harmonic trap may be coupled to different thermostats and, in general, may exhibit rotating currents which break the time-reversal symmetry even in the steady state [16–23]. It is therefore a sound test-ground for the study of SE protocols for switching between two different non-equilibrium steady states in a finite time.

Swift equilibration protocol - Firstly, let us intro-
duce the general strategy for the SE. We adopted a different and, in a sense, more general approach with respect to [12]. Consider an experimental system that can be leveraged controlling some forcing parameters, which will be noted in a vector \( \Pi \). The instantaneous statistical state of the system can be described by a set of parameters, which will be denoted by a vector \( \gamma(t) \): for instance in a Gaussian process, as in our case below, these can be the parameters of a multivariate Gaussian. The value of the parameters \( \gamma(t) \) depends, through a dynamical equation, on the the history of the applied forcing \( \Pi \) up to time \( t \). We prepare the system in a stationary condition, given a value for the forcing \( \Pi^0 \). This means that we observe a time constant value of the system parameters \( \gamma \) that depends on the forcing \( \Pi^0 \): we note this value as \( \gamma^{st}[\Pi^0] \). Our goal is to lead the system into a new stationary state with a final set of parameters \( \gamma_{st} \). Then the evolution of parameters is so slow that the system is always in its stationary state. In this quasi-static forcing, \( \gamma(t) = \gamma^{st}[\Pi^0] \) at any time, including the final one.

On the contrary, when \( \tau \) is finite (smaller than the largest characteristic time of the system), \( \Pi(t/\tau) \) is not a SE and must be modified with appropriate finite time corrections, i.e. a finite-\( \tau \) SE protocol reads \( \Pi = \Pi^0 + \frac{1}{2} \delta \mathcal{P} \). The quantity \( \delta \mathcal{P} \), hereafter named the finite time correction to the quasi-static protocol, depends upon the choice of the quasi-static protocol \( \mathcal{P} \). This is the relevant quantity one has to know to experimentally perform the desired SE. The exact, explicit and general formula for \( \delta \mathcal{P}[\mathcal{P}(s)] \) in the case of the Brownian Gyration is the main result of this letter, and it allows a number of interesting theoretical considerations.

The Brownian Gyration - The system we consider has been introduced in [17] and then studied in [18] and [19], with an experimental realization obtained recently in [21–23]. It is widely known as Brownian Gyration (BG). Its stochastic differential equation takes the form

\[
\begin{align*}
\frac{dx}{dt} &= -(k_x x + u y)dt + \sqrt{2T_x}dW_x \\
\frac{dy}{dt} &= -(k_y y + u x)dt + \sqrt{2T_y}dW_y
\end{align*}
\]

which fairly describes an overdamped particle subject to a potential \( V(x,y) = \frac{1}{2}k_xx^2 + \frac{1}{2}k_yy^2 + u xy \) in contact with two thermal baths at temperature \( T_x \) and \( T_y \). Note that the condition of confining potential, required for the steady states, is \( k_x, k_y - u^2 > 0 \) [24]. In this article we consider the case where \( k_x, k_y, u \) may depend upon time (on the contrary we keep the temperatures constant). For compactness we denote the set of parameters by the vector \( \Pi \), i.e. \( \Pi_1 = k_x, \Pi_2 = k_y \) and \( \Pi_3 = u \). The associated FP equation reads

\[
\partial_t p = \partial_x(p\partial_x V) + \partial_y(p\partial_y V) + T_x\partial^2_x p + T_y\partial^2_y p
\]

where \( p(x,y,t) \) is the one time distribution of the stochastic process. The process is Gaussian and for Gaussian initial condition keeps the Gaussian form at all times:

\[
p(x,y,t) = \frac{\exp\left(-\frac{1}{2}\gamma_1 x^2 - \frac{1}{2}\gamma_2 y^2 - \gamma_3 xy\right)}{2\pi(\gamma_1 - \gamma_3)^{1/2}},
\]

where \( \gamma = \{\gamma_1, \gamma_2, \gamma_3\} \) depends on time. The introduction of form (3) in Eq. (2) leads to the equations governing the time evolution of \( \gamma(t) \), since:

\[
\begin{align*}
\dot{\gamma}_1 &= 2\left(k_x\gamma_1 - T_x\gamma_2 + u\gamma_3 - T_y\gamma_1^2\right) \\
\dot{\gamma}_2 &= 2\left(k_y\gamma_2 - T_y\gamma_1 + u\gamma_3 - T_x\gamma_2^2\right) \\
\dot{\gamma}_3 &= \gamma_3\left(k_x + k_y\right) + u\left(\gamma_1 + \gamma_2\right) + 2\gamma_2(3T_x\gamma_1 + T_y\gamma_2)
\end{align*}
\]

If the parameters vector \( \Pi \) of Eq. (1) does not depend on time, then the time-dependence of \( \gamma(t) \) is only due to the relaxation from initial conditions. In that case, assuming that the potential is confining, a steady state is reached asymptotically, and - for ergodicity - coincides with the solution \( \partial_t p_{st} = 0 \), uniquely determined by the values \( \gamma^{st}[\Pi] \) that obey Eqs. (4) with all left-hand-sides set to zero (see Supplementary Information [25]). When \( T_x = T_y = T \) (“thermodynamic equilibrium”) the Boltzmann distribution \( p_{st} \propto e^{-V/T} \) is recovered, i.e. \( \gamma_1^* = \frac{k_x}{T}, \gamma_2^* = \frac{k_y}{T}, \gamma_3^* = \frac{u}{T} \). On the contrary, when \( T_x \neq T_y \), the steady state is not of the Boltzmann form and, most importantly, contains a current: \( J(x,y) = (-p_x\partial_x V - T_x\partial_x p_{st}, -p_y\partial_y V - T_y\partial_y p_{st}) \neq (0,0) \) which is rotational, with null divergence. The steady current breaks time-reversal invariance (detailed balance) and for this reason the BG has been
proposed as a minimal model for non-equilibrium steady states [17].

SE for the Brownian Gyrator - We look for the forcing protocol \( \Pi \) that in a finite time \( \tau \) leads the system from the stationary state \( \gamma^{eq}[\Pi] \) to the stationary state \( \gamma^{eq}[\Pi'] \). We require that the vector \( \Pi(t) \equiv \{ k_x, k_y, u \} \) has the form \( \Pi(t) = \mathcal{P}(t/\tau) + \frac{1}{\tau} \delta \mathcal{P}(t/\tau) \), where \( \mathcal{P}(s) \equiv \{ K_x, K_y, U \} \) is a given quasi-static protocol, and \( \delta \mathcal{P}(s) \equiv \{ \delta k_x, \delta k_y, \delta u \} \) is its finite time correction.

In order to accomplish our task, first we invert the relation \( \mathcal{P} = A[\gamma] \), and \( \delta \mathcal{P} = B[\gamma] \cdot \frac{d}{ds} \gamma \), see Supplementary Information [25] for full formula of both terms.

In order to close our loop, now, we need to express everything as a function of the quasi-static protocol. This is done in two steps. The first step is to invert the relation \( \mathcal{P} = A(\gamma) \). Since this relation is valid even in the \( \tau \rightarrow \infty \) limit, the result is nothing but the expression of \( \gamma^{eq}[\mathcal{P}] \) that solve Eqs. (4) in the stationary condition and parameters set to \( \mathcal{P} \). Finally, we have to express \( \frac{d}{ds} \gamma \) as a function of the quasi-stationary protocols. This is done considering that \( \frac{d}{ds} \gamma = K_x \delta k_x + K_y \delta k_y + U \delta u \) applying this operator to \( \gamma^{eq}[\mathcal{P}] \) (here and in the following \( f' \) stands for \( \frac{d}{ds} f \)). Putting back \( \gamma \) and \( \frac{d}{ds} \gamma \) in the definition of the forcing protocol, we obtain the final expression:

\[
\Pi(t) = \mathcal{P}(t/\tau) + \frac{1}{\tau} \delta \mathcal{P}(t/\tau), \\
\delta \mathcal{P}(s) = B[\mathcal{P}(s)] \cdot \mathcal{P}'(s),
\]

with a matrix \( B \) which is fully defined in the Supplementary Information [25]. We recall the operative meaning of this formula: one chooses an arbitrary [26] quasi-static protocol \( \mathcal{P}(s) = \{ K_x(s), K_y(s), U(s) \} \) and this corresponds to a particular form of \( \delta \mathcal{P}(s) = \{ \delta k_x(s), \delta k_y(s), \delta u(s) \} \) for finite time corrections. Before giving a handier expression of \( \delta \mathcal{P} \), we discuss some special cases.

Firstly, we consider the case where there is no interaction among \( x \) and \( y \), i.e. when \( u = 0 \) both at the beginning and at the end. Then it does not make sense to switch on \( u \) during the protocol, so that the choice \( U(s) = 0 \) is quite general (hence \( U' = 0 \)). The two degrees of freedom are independent, each one follows a separate equation and the finite time corrections take the characteristic log-derivative of the quasi-static protocol:

\[
\delta k_x = \frac{1}{2} \frac{K_x}{K_x}, \quad \delta k_y = \frac{1}{2} \frac{K_y}{K_y}, \quad \delta u = 0.
\]

This result coincides with that in [12].

As a second step, we consider the case of two interacting degrees of freedom \( U \neq 0 \) in contact with the same thermal bath at temperature \( T_x = T_y = T \). In this case the finite time corrections to the quasi-static forcing read:

\[
\delta k_x^\text{eq} = \frac{(\Delta + K_x \alpha) K_x + U^2 K_y - 2 K_x K_y U}{2 K_x \Delta}, \\
\delta k_y^\text{eq} = -\frac{U K_x K_y}{2 K_x \Delta}, \\
\delta u^\text{eq} = \frac{U K_x K_y}{2 K_x \Delta}.
\]

where \( K_x = K_x + K_y \) and \( \Delta = K_x K_y - U^2 \). The value for \( \delta k_{xy}^\text{eq} \) is obtained swapping the subscripts \( x \) and \( y \) in the expression for \( \delta k_y^\text{eq} \). Note that the result does not depend on the temperature \( T \). This result generalizes [12] in two dimensions.

A richer phenomenology is obtained in the realm of non-equilibrium, when \( T_x \neq T_y \) and a non zero current appears in the stationary state (a brief study of the current during the SE can be found in Supplementary Information [25]).

For instance for \( T_y = T_x + dT \), we observe interesting deviations

\[
\delta k_x = \delta k_x^\text{eq} - \frac{U (T_x + dT)}{K_x} \delta u + O(dT^2), \\
\delta k_y = \delta k_y^\text{eq} + \frac{U (T_x + dT)}{K_y} \delta u + O(dT^2), \\
\delta u = \delta u^\text{eq} - \frac{U (T_x + dT)}{K_x + K_y} \delta u + O(dT^2).
\]

Note that for a symmetric protocol \( K_x = K_y = K \) the finite time correction to \( u \) becomes of the second order in \( dT \). For such a symmetric protocol one has that \( \delta k_{xy}^\text{eq} = \delta k_{yx}^\text{eq} \) and both are proportional to a logarithmic derivative, as happens to the non interacting case: \( \delta k_{xy}^\text{eq} = \delta k_{yx}^\text{eq} = \frac{1}{2} \frac{d}{ds} \log(K^2 - U^2) \) (which is minus the “free energy” of the system divided by \( 2T \), see Supplementary Information [25]). In the same symmetric case, one can consider a quasi-static protocol involving only weak interactions \( U \ll 1 \). In this case the general non-equilibrium case reads:

\[
\delta k_x = \frac{1}{2} \frac{K_x}{K_x} - \frac{U (T_x + T_y)}{K_x + K_y} U + O(U^2), \\
\delta k_y = \frac{1}{2} \frac{K_y}{K_y} - \frac{U (T_x + T_y)}{K_x + K_y} U + O(U^2), \\
\delta u = \frac{U}{2} \frac{K_y}{K_y} - \frac{U}{2} \frac{K_x}{K_x} + O(U^2).
\]
Note that the corrections to $\delta k_x$ and $\delta k_y$ at first order in $U$ are different: the finite time correction to a symmetric quasi-static protocol should not be the same, since the symmetry is broken by the non equilibrium condition $T_y \neq T_x$. Nevertheless we note that they differ only by a factor $T_x/T_y$. It turns out that this is a general mathematical feature of the solution for the general (non symmetric) quasi-static protocol. In fact, in the general case, the finite time corrections $\delta k_{xy}$ and $\delta u$ have a striking mathematical structure:

$$
\begin{align*}
\delta k_x &= \delta k_x^{eq} + \frac{1}{T_y} \frac{d}{ds} F, \\
\delta k_x &= \delta k_x^{eq} - \frac{1}{T_y} \frac{d}{ds} F, \\
\delta u &= \delta u^{eq} + (T_y - T_x) J + \frac{d}{ds} G,
\end{align*}
\tag{6}
$$

where $F$ and $G$ are functions of $K_x$ and $U$. The function $J$ remains finite in the equilibrium limit $T_y \to T_x$, while $F$ and $G$ vanish. The explicit expressions for $F, G$ and $J$ are quite simple and are given in the Supplementary Information [25]. Equations (6) are the main result of this work.

Several comments about this equation are in order. Firstly we note that during a quasi-static protocol ($\tau \to \infty$) the internal energy is constant, as one could expect: this constant value does not depend on the forcing parameters, neither on $k_{x,y}$ nor on $u$. More interestingly, the finite time correction to the internal energy, can be written with a single differential form $\frac{d}{ds} \Theta$, where $\Theta = \frac{K_x T_y + K_y T_x}{4(K_x K_y - U^2)}$. An immediate reward of this result is that it allows to identify a specific class of quasi-static protocols, that we call the finite time isothermal protocols, defined as the SE protocols that keep $\Theta$ constant. Using such a quasi-static protocol one can perform a SE procedure with constant internal energy in any finite time $\tau$, provided to force the system with the appropriate finite time corrections (6). In Fig. 1 we show simple numerical simulations giving a demonstration of our results.

**Conclusions** - Here we have proposed a general framework for studying finite time transformations in stochastic processes under the important request of connecting two steady states without the need of further relaxation time (“swift equilibration”). Our general framework is based upon the idea of fixing an arbitrary quasi-static protocol and then computing the finite-time corrections to it. We have applied our idea to a model (“Brownian gyration”) with a harmonic potential in contact with two different thermal baths, a minimal non-equilibrium generalization of the celebrated Ornstein-Uhlenbeck process. In this sense, the model can be considered as the harmonic oscillator or the “perfect gas” for non-equilibrium steady states. Despite the linearity of the model, the problem of SE discloses a rich and promising terrain for theoretical explorations. We give the exact explicit expression for the general SE, and also a
simple condition to obtain finite-time transformations that conserve internal energy. The existence of experimental realizations of the steady Brownian gyrator [16, 21–23] let us foresee interesting experimental investigations of our procedures in the next future. An important theoretical perspective concerns the research of optimal protocols with respect to work or other thermodynamic relevant quantities, for instance a suitable definition of finite-time adiabatic transformations [27] to the case where two thermal baths are present.

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[24] The concavity condition is necessary to reach a steady state but it can be relaxed in transient states. However it can become necessary in experimental realizations as discussed for instance in [13, 14, 28].
[25] See Supplemental Material at [URL will be inserted by publisher], which includes Refs. [13], [29], and [30], for details on analytical and numerical results.
[26] In the following we always consider continuous functions, which attain the values $\Pi'$ and $\Pi''$ at the border with zero derivatives. However, some of these requirements can be relaxed, if one admits jumps in the finite time forcing $\Pi(t)$, as explained in [13].
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Supplementary material for: Engineered Swift Equilibration of a Brownian Gyrator

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SE FORMULATION THROUGH QUASI-STATIC PROTOCOLS

Here we detail the procedure to obtain the expression for the forcing in the SE protocol. We start from the Fokker-Planck equation of the stochastic process in study:

$$\partial_t p = \partial_x [p \partial_x V + T_x \partial_x p] + \partial_y [p \partial_y V + T_y \partial_y p],$$

which can also be written as a continuity equation $\partial_t p = -\nabla \cdot \mathbf{J}(p)$ defining a current probability $\mathbf{J}$, and where the potential $V$ depends on time through a set of forcing parameters $\Pi(t) = \{k_x(t), k_y(t), u(t)\}$ as

$$V(x, y; \Pi(t)) = \frac{1}{2} k_x(t) x^2 + \frac{1}{2} k_y(t) y^2 + u(t) x y.$$  

The FP equation is satisfied by the process propagator $p(x, y; t|x_0, y_0; t_0)$, which fully defines our continuous Markov process. However the FP equation is satisfied also by the single time distribution:

$$p(x, y; t) = \int p(x, y; t|x_0, y_0; t_0)p(x_0, y_0; t_0)dx_0dy_0$$

provided the correct initial conditions are imposed

$$p(x, y; t)|_{t=t_0} = p(x, y; t_0).$$

In our case, we want that at $t_0 = 0$ the system is stationary $p(x, y; 0) = p_0(x, t)$, where $p_0$ satisfies $\nabla \cdot \mathbf{J}(p_0) = 0$, with the forcing parameters in $\mathbf{J}$ given by the initial values $\Pi^0 = \{K_x(0), K_y(0), U(0)\}$.

For the quadratic potential $V$, the stationary solution is Gaussian:

$$p_0(x, y; \Gamma) = \frac{\sqrt{\gamma_1^2 x^2 - \gamma_2^2 y^2 - \gamma_3^2 x y}}{2\pi} \exp \left( -\frac{1}{2} \gamma_1^2 x^2 - \frac{1}{2} \gamma_2^2 y^2 - \gamma_3^2 x y \right)$$

where the vector $\gamma = \{\gamma_1^s, \gamma_2^s, \gamma_3^s\}$ depends on the initial value of the forcing $\Pi^0$ as:

$$\gamma_1^s = \frac{[K_x T_y (K_x + K_y) - U^2 (T_y - T_x)] (K_x + K_y)}{T_x T_y (K_x + K_y)^2 + (T_y - T_x)^2 U^2}$$

$$\gamma_2^s = \frac{[K_y T_x (K_x + K_y) + U^2 (T_y - T_x)] (K_x + K_y)}{T_x T_y (K_x + K_y)^2 + (T_y - T_x)^2 U^2}$$

$$\gamma_3^s = \frac{(K_x T_y + K_y T_x) (K_x + K_y) U}{T_x T_y (K_x + K_y)^2 + (T_y - T_x)^2 U^2}.$$  

Note that for $T_x = T_y = T$ the stationary distribution recovers the expected Gibbs distribution $P_0 = Z^{-1} \exp(-V/T)$, since $\gamma_1^s = K_x/T$, $\gamma_2^s = K_y/T$, and $\gamma_3^s = U/T$ (and $Z$ is the normalization constant, see later).

Given the initial condition (2), with $p(x, y; t_0) = p_0(x, y)$, the solution for a general forcing protocol $\Pi(t)$ keeps a Gaussian form, but the parameters $\gamma(t)$ do not satisfy the stationary relation (3-5). In fact, inserting a time dependent Gaussian distribution $P(x, y; \gamma(t))$ in (1), it can be easily shown that the FP equation is satisfied as long as:

$$\frac{1}{2} \frac{d}{dt} \gamma_1 = \gamma_1 k_x - T_x \gamma_1^2 + u \gamma_3 - T_y \gamma_3^2$$

$$\frac{1}{2} \frac{d}{dt} \gamma_2 = \gamma_2 k_y - T_y \gamma_2^2 + u \gamma_3 - T_x \gamma_3^2$$

$$\frac{d}{dt} \gamma_3 = \gamma_3 (k_x + k_y) + u (\gamma_1 + \gamma_2) - 2 \gamma_3 (\gamma_1 T_x + \gamma_2 T_y).$$

Such equations provide the values of the distribution parameters $\gamma(t) = \{\gamma_1(t), \gamma_2(t), \gamma_3(t)\}$ as a function of the forcings $\Pi(t)$. Obviously the expressions for $\gamma^s$, which do not depend on time, solve these three equations for $k_x = K_x$, $k_y = K_y$, and $u = U$. 
Our SE program is to compute the forcing protocols $\Pi(t)$, with the initial values $\Pi_i = \Pi(0)$ and the final values $\Pi_f = \Pi(\tau)$ such that $\gamma(\tau) = \gamma_{st}[\Pi']$ (starting from $\gamma(0) = \gamma_{st}[\Pi]$).

We look for an expression of the forcing protocol written as the sum of an arbitrary quasi-static protocol plus some finite corrections: $\Pi(t) = \mathcal{P}(t/\tau) + \frac{1}{\tau} \delta \mathcal{P}(t/\tau)$. The meaning of the quasi-static protocol $\mathcal{P}$ is that in our scheme if we perform the SE in a very large $\tau$, then $\Pi \equiv \mathcal{P}$, or more precisely $\operatorname{lim}_{\tau \rightarrow \infty} \Pi(\sigma \tau) = \mathcal{P}(\sigma)$.

For the Brownian Gyraotor we use the following notation:

$$
\begin{align*}
    k_x(t) &= K_x(t/\tau) + \frac{1}{\tau} \delta k_x(t/\tau) \\
    k_y(t) &= K_y(t/\tau) + \frac{1}{\tau} \delta k_y(t/\tau) \\
    u(t) &= U(t/\tau) + \frac{1}{\tau} \delta u(t/\tau),
\end{align*}
$$

that is $\Pi(t) = \{k_x(t), k_y(t), u(t)\}$, $\mathcal{P}(s) = \{K_x(s), K_y(s), U(s)\}$ and $\delta \mathcal{P}(s) = \{\delta k_x(s), \delta k_y(s), \delta u(s)\}$.

In order to carry on our program, we need to invert the equations (6-8), so to obtain:

$$
\begin{align*}
    k_x &= \frac{1}{2(\gamma_1 + \gamma_2)(\gamma_1^2 - \gamma_3^2)} \left[ \frac{d\gamma_1}{dt} \gamma_2^2 + \gamma_3 \left( \frac{d\gamma_2}{dt} \gamma_1^2 - \frac{d\gamma_1}{dt} \gamma_2^2 \right) + \frac{d\gamma_1}{dt} \gamma_1 \gamma_2 - 2\gamma_3 \gamma_2 \frac{d\gamma_3}{dt} \right] + \\
    &\quad + \gamma_1 T_x + \frac{\gamma_3^2}{\gamma_1 + \gamma_2} (T_y - T_x) \\
    k_y &= \frac{1}{2(\gamma_1 + \gamma_2)(\gamma_1^2 - \gamma_3^2)} \left[ \left( \frac{d\gamma_1}{dt} \gamma_1^2 - \frac{d\gamma_2}{dt} \gamma_1 \gamma_2 \right) \gamma_3 + \frac{d\gamma_2}{dt} \gamma_1 \gamma_2 - \frac{d\gamma_1}{dt} \gamma_2 \gamma_3 \right] + \\
    &\quad + \gamma_2 T_y + \frac{\gamma_3^2}{\gamma_1 + \gamma_2} (T_x - T_y) \\
    u &= \frac{1}{2(\gamma_1 + \gamma_2)(\gamma_1^2 - \gamma_3^2)} \left[ 2\frac{d\gamma_2}{dt} \gamma_1 \gamma_2 - \gamma_3 \frac{d(\gamma_1 \gamma_2)}{dt} \right] + \frac{\gamma_3}{\gamma_1 + \gamma_2} (T_x \gamma_1 + T_y \gamma_2).
\end{align*}
$$

Now we impose that the distribution parameters depend on time through the rescaled time $s = t/\tau$. As a consequence $\frac{d\tau}{dt} = \frac{1}{2} \gamma'$ (where we indicated with prime the derivative with respect to $s$). Hence, in the large $\tau$ limit $\frac{d\tau}{dt} \rightarrow 0$, and the Eqs. (9-11) become the expressions of the quasi-static protocol as a function of the distribution parameters. They read:

$$
\begin{align*}
    \mathcal{K}_x &= \gamma_1 T_x + \frac{\gamma_3^2}{\gamma_1 + \gamma_2} (T_y - T_x) \\
    \mathcal{K}_y &= \gamma_1 T_y + \frac{\gamma_3^2}{\gamma_1 + \gamma_2} (T_y - T_x) \\
    \mathcal{U} &= \frac{\gamma_3}{\gamma_1 + \gamma_2} (\gamma_1 T_x + \gamma_2 T_y).
\end{align*}
$$

These are nothing but the inversion of the expressions of the stationary parameters $\gamma_{st}$ given in (3-5), where $\mathcal{K}_x$, $\mathcal{K}_y$ and $\mathcal{U}$ now depend on (the rescaled) time and we substitute $\gamma_{st}$ with $\gamma(t)$.

If we invert again Eqs. (12-14), we obtain expressions formally identical to Eqs. (3-5):

$$
\begin{align*}
    \gamma_1 &= \frac{\mathcal{K}_x T_y (\mathcal{K}_x + \mathcal{K}_y) - \mathcal{U}^2 (T_y - T_x)}{T_x T_y (\mathcal{K}_x + \mathcal{K}_y)^2 + (T_y - T_x)^2 \mathcal{U}^2} (\mathcal{K}_x + \mathcal{K}_y) \\
    \gamma_2 &= \frac{\mathcal{K}_x T_y (\mathcal{K}_x + \mathcal{K}_y) + \mathcal{U}^2 (T_y - T_x)}{T_x T_y (\mathcal{K}_x + \mathcal{K}_y)^2 + (T_y - T_x)^2 \mathcal{U}^2} (\mathcal{K}_x + \mathcal{K}_y) \\
    \gamma_3 &= \frac{(\mathcal{K}_x T_y + \mathcal{K}_y T_x)(\mathcal{K}_x + \mathcal{K}_y) \mathcal{U}}{T_x T_y (\mathcal{K}_x + \mathcal{K}_y)^2 + (T_y - T_x)^2 \mathcal{U}^2},
\end{align*}
$$

where $\gamma_1$, $\gamma_2$ and $\gamma_3$ are the stationary parameters and $\mathcal{K}_x$, $\mathcal{K}_y$ and $\mathcal{U}$ are the quasi-static protocols.
but with a completely different meaning: they represent the time dependent distribution parameters \( \gamma(t) \) during the SE protocol as a function of the quasi-static protocol \( P(s = t/\tau) \). We stress that the system in general is not stationary for \( 0 < t < \tau \), since for a finite time \( \tau \), \( \Pi(t) \neq P(s = t/\tau) \).

The actual forcings \( \Pi(t) \) are given by the Eqs. (9-11) and we want to express \( \Pi(t) \) as a function of the quasi-static protocol \( P \). In order to do this, we have to substitute the expression of \( \gamma(t) \) as a function of \( P \) back in Eqs. (9-11) and we also need the expressions of \( \frac{d\gamma}{dt} \) as a function of \( P \) (and its derivative). Using the expression of \( \gamma \) as a function of \( P \) in Eqs. (15-17), and

\[
\frac{d\gamma(t)}{dt} = \frac{1}{\tau} \left[ K'_{xy} \partial K_x + K'_{yx} \partial K_y + U' \partial t \right] \gamma
\]

we can obtain the expressions for \( \frac{d\gamma}{dt} \) as a function of \( P \) and \( \partial \). (As before and in the following the prime denotes the derivatives with respect to \( s \), for instance \( P' \equiv \frac{dP}{ds} \mid_{s=t/\tau} \). Substituting the expressions found for \( \gamma(t) \) (Eqs. (15-17)) and \( \frac{d\gamma}{dt} \) (not explicitly reported here) as a function of \( P \) and \( \partial \), in Eqs. (9-11) we can get the final result in the form \( \Pi(t) = P(s = t/\tau) + \frac{1}{\tau} \delta P(s = t/\tau) \), where \( \delta P(s) = B[\partial P(s)] \partial P(s) \), and \( B \) is a matrix. More explicitly, the expression will be:

\[
k_x(t) = \left\{ K_x(s) + \frac{1}{\tau} \left[ B_{11}K_x'(s) + B_{12}K_y'(s) + B_{13}U'(s) \right] \right\}_{s=t/\tau} \quad (18)
\]

\[
k_y(t) = \left\{ K_y(s) + \frac{1}{\tau} \left[ B_{21}K_x'(s) + B_{22}K_y'(s) + B_{23}U'(s) \right] \right\}_{s=t/\tau} \quad (19)
\]

\[
u(t) = \left\{ U(s) + \frac{1}{\tau} \left[ B_{31}K_x'(s) + B_{32}K_y'(s) + B_{33}U'(s) \right] \right\}_{s=t/\tau} \quad (20)
\]

The finite time corrections to the quasi static protocols \( \delta P \), that is \( \delta k_x, \delta k_y \) and \( \delta u \), are the terms in square brackets involving the elements of the \( B \) matrix. Note that the choice of the quasi-static protocol is completely arbitrary. The only conditions on the arbitrary quasi-static protocol is that it has to satisfy \( P(0) = \Pi^1, P(1) = \Pi^f \). (Apparently, one should also impose \( P'(0) = P'(1) = 0 \), but these constraints can be relaxed if one admits jumps in the finite time protocol \( \Phi(t) \) at the boundaries, as explained in [1]).

The full expressions for the matrix elements are cumbersome and not really instructive (we give them below, for the sake of completeness). However we noted that the expressions for \( \delta k_x, \delta k_y \) and \( \delta u \) can be written in a much more compact and practical form, as we now explain.

Firstly, we consider the equilibrium case \( T_x = T_y = T \). In this case, the expressions for \( \delta P = \delta P^{eq} \) are:

\[
\delta k_x^{eq} = \frac{(U^2 - K_xK_y - K_y^2)K_x - U^2K_x' + 2K_yU'U'}{2(K_x + K_y)(U^2 - K_xK_y)} \quad (22)
\]

\[
\delta k_y^{eq} = \frac{(U^2 - K_xK_y - K_y^2)K_y - U^2K_y' + 2K_xU'U'}{2(K_x + K_y)(U^2 - K_xK_y)} \quad (23)
\]

\[
\delta u^{eq} = \frac{K_yU'K_x' + K_xU'K_y' - 2K_xK_yU'}{2(K_x + K_y)(U^2 - K_xK_y)} \quad (24)
\]

The interesting observation is that these expressions do not depend explicitly on \( T \). So, even in the general case \( T_x \neq T_y \), we can extract the terms composing \( \delta P^{eq} \) and work with the rest \( \delta P - \delta P^{eq} \) which is the only part explicitly depending on \( T_x \) and \( T_y \). This corresponds to a (not so simpler) matrix \( B^{uneq} = B - B^{eq} \). However, a careful inspection reveals some striking regularities. For instance it is easy to verify that

\[
\delta k_x B_{12}^{eq} = \partial k_x B_{12}^{eq} \quad (26)
\]

\[
\delta k_x B_{13}^{eq} = \partial k_y B_{13}^{eq} \quad (27)
\]

\[
\partial k_y B_{13}^{eq} = \delta U B_{12}^{eq} \quad (28)
\]
This suggests that a single generating function $F(K_x, K_y, U)$ can be used to express $B_{ij}^{\text{neq}}$:

$$B_{11}^{\text{neq}} = \frac{1}{T_x} \partial_{K_x} F$$  \hspace{1cm} (29)$$

$$B_{12}^{\text{neq}} = \frac{1}{T_x} \partial_{K_y} F$$  \hspace{1cm} (30)$$

$$B_{13}^{\text{neq}} = \frac{1}{T_x} \partial_{\alpha} F,$$  \hspace{1cm} (31)

so that Eqs. (26-28) hold for the Schwarz equalities $\partial_\alpha \partial_\beta F = \partial_\beta \partial_\alpha F$ (where $\alpha$ and $\beta$ are two arbitrary arguments of the functions $F$). The reason for the factor $\frac{1}{T_x}$ is that one can show that the symmetry between $\delta_k$ and $\delta_k\gamma$ (to the exchange of the subscripts $x \leftrightarrow y$) implies that:

$$B_{21}^{\text{neq}} = -\frac{1}{T_y} \partial_{K_x} F$$  \hspace{1cm} (32)$$

$$B_{22}^{\text{neq}} = -\frac{1}{T_y} \partial_{K_y} F$$  \hspace{1cm} (33)$$

$$B_{23}^{\text{neq}} = -\frac{1}{T_y} \partial_{\beta} F.$$  \hspace{1cm} (34)

The function $F$ is defined as:

$$F = \frac{1}{2} \frac{T_x T_y}{T_y - T_x} \log \left[ 1 + \frac{U^2}{(K_x + K_y)^2} \right] + c.$$

where $c(T_x, T_y)$ is an arbitrary constant (with respect to $K_x, K_y$ and $U$).  As a partial final result we have just obtained that:

$$k_x(t) = \left. \left\{ K_x(s) + \frac{1}{\tau} \left[ \delta k_x^\text{eq}(s) + \frac{1}{T_x} \frac{d}{ds} F(K_x, K_y, U) \right] \right\} \right|_{s=t/\tau},$$  \hspace{1cm} (36)$$

$$k_y(t) = \left. \left\{ K_y(s) + \frac{1}{\tau} \left[ \delta k_y^\text{eq}(s) - \frac{1}{T_y} \frac{d}{ds} F(K_x, K_y, U) \right] \right\} \right|_{s=t/\tau}.$$  \hspace{1cm} (37)

This result is quite useful, even for the numerical computation of the forcing. Once decided the desired (arbitrary) quasi-static protocol, one can compute iteratively the finite time correction with a simple Euler discretization of the derivative $\frac{d}{ds} F$, without bothering of the cumbersome matrix $B$. Unfortunately, a similar trick does not work exactly the same for the three matrix elements $B_{ij}^{\text{neq}}$ of the matrix $B^{\text{neq}}$. In fact one can show that the "Schwarz" conditions (26-28) do not hold for such elements. However we discovered that if we subtract the coefficient of the first order contribution in $T_y - T_x$, which is

$$\mathcal{J} = \frac{(K_y K_y' - K_x K_x') U}{(T_y K_x + T_x K_y)(K_x + K_y)^2}$$

we can again express the rest of $B^{\text{neq}}$ via a generating function $G$:

$$G = \sqrt{\frac{T_x T_y}{T_y - T_x} \left\{ \arctan \left[ \frac{T_y - T_x}{\sqrt{T_x T_y}} \frac{U}{K_y + K_x} \right] - \frac{U}{K_y + K_x} + d, \right\}}$$

where $d$ is a constant, as $c$. More precisely, we get that the final expression for the forcing $u$ is:

$$u(t) = \left. \left\{ U(s) + \frac{1}{\tau} \left[ \delta u^\text{eq}(s) + (T_y - T_x) \mathcal{J}(s) + \frac{d}{ds} G(K_x, K_y, U) \right] \right\} \right|_{s=t/\tau}. \hspace{1cm} (38)$$

The equations Eqs. (36-38) are the main result of this work.
THE MATRIX B

For the sake of completeness we provide the explicit expression of the matrix B, which in its more compact form read:

\[
\begin{align*}
B_{11} &= \frac{\mathcal{U}^2 (T_y - T_x)^2}{T_x (T_y - T_x) (K_y + K_x)} \left( 1 + \frac{\mathcal{U}^2 (T_y - T_x)^2}{T_x T_y (K_y + K_x)^2} \right)^{-1} - \frac{K_y K_y + K_y^2 - \mathcal{U}^2}{2 (K_y + K_x) (-K_x K_y + \mathcal{U}^2)} \\
B_{12} &= \frac{\mathcal{U}^2 (T_y - T_x)^2}{T_x (T_y - T_x) (K_y + K_x)} \left( 1 + \frac{\mathcal{U}^2 (T_y - T_x)^2}{T_x T_y (K_y + K_x)^2} \right)^{-1} - \frac{2 (K_y + K_x) (-K_x K_y + \mathcal{U}^2)}{2 (K_y + K_x) (-K_x K_y + \mathcal{U}^2)} \\
B_{13} &= \frac{\mathcal{U} (T_y - T_x)^2}{T_x (-T_y + T_x) (K_y + K_x)} \left( 1 + \frac{\mathcal{U}^2 (T_y - T_x)^2}{T_x T_y (K_y + K_x)^2} \right)^{-1} + \frac{K_y \mathcal{U}}{(K_y + K_x) (-K_x K_y + \mathcal{U}^2)} \\
B_{21} &= \frac{\mathcal{U}^2 (T_y - T_x)^2}{T_y (T_x - T_y) (K_y + K_x)} \left( 1 + \frac{\mathcal{U}^2 (T_y - T_x)^2}{T_x T_y (K_y + K_x)^2} \right)^{-1} - \frac{2 (K_y + K_x) (-K_x K_y + \mathcal{U}^2)}{2 (K_y + K_x) (-K_x K_y + \mathcal{U}^2)} \\
B_{22} &= \frac{\mathcal{U}^2 (T_y - T_x)^2}{T_y (T_x - T_y) (K_y + K_x)} \left( 1 + \frac{\mathcal{U}^2 (T_y - T_x)^2}{T_x T_y (K_y + K_x)^2} \right)^{-1} - \frac{K_y K_y + K_y^2 - \mathcal{U}^2}{2 (K_y + K_x) (-K_x K_y + \mathcal{U}^2)} \\
B_{23} &= \frac{\mathcal{U} (T_y - T_x)^2}{T_y (T_x - T_y) (K_y + K_x)} \left( 1 + \frac{\mathcal{U}^2 (T_y - T_x)^2}{T_x T_y (K_y + K_x)^2} \right)^{-1} + \frac{K_y \mathcal{U}}{(K_y + K_x) (-K_x K_y + \mathcal{U}^2)} \\
B_{31} &= \frac{(T_y - T_x) K_y \mathcal{U}}{(K_x T_y + K_y T_x) (K_y + K_x)^2} - \frac{\mathcal{U}}{(K_y + K_x)^2} \left( 1 + \frac{\mathcal{U}^2 (T_y - T_x)^2}{T_x T_y (K_y + K_x)^2} \right)^{-1} - 1 + \frac{K_x K_y}{(K_y + K_x) (-K_x K_y + \mathcal{U}^2)} \left( \frac{(T_y - T_x) K_y \mathcal{U}}{(K_x T_y + K_y T_x) (K_y + K_x)^2} - \frac{\mathcal{U}}{(K_y + K_x)^2} \left( 1 + \frac{\mathcal{U}^2 (T_y - T_x)^2}{T_x T_y (K_y + K_x)^2} \right)^{-1} - 1 \right) \\
B_{32} &= -\frac{(T_y - T_x) K_x \mathcal{U}}{(K_x T_y + K_y T_x) (K_y + K_x)^2} - \frac{\mathcal{U}}{(K_y + K_x)^2} \left( 1 + \frac{\mathcal{U}^2 (T_y - T_x)^2}{T_x T_y (K_y + K_x)^2} \right)^{-1} - 1 + \frac{K_x K_y}{(K_y + K_x) (-K_x K_y + \mathcal{U}^2)} \left( \frac{(T_y - T_x) K_x \mathcal{U}}{(K_x T_y + K_y T_x) (K_y + K_x)^2} - \frac{\mathcal{U}}{(K_y + K_x)^2} \left( 1 + \frac{\mathcal{U}^2 (T_y - T_x)^2}{T_x T_y (K_y + K_x)^2} \right)^{-1} - 1 \right) \\
B_{33} &= (K_y + K_x)^{-1} \left( 1 + \frac{\mathcal{U}^2 (T_y - T_x)^2}{T_x T_y (K_y + K_x)^2} \right)^{-1} - 1 - \frac{K_x K_y}{(K_y + K_x) (-K_x K_y + \mathcal{U}^2)} \left( \frac{(T_y - T_x) K_y \mathcal{U}}{(K_x T_y + K_y T_x) (K_y + K_x)^2} - \frac{\mathcal{U}}{(K_y + K_x)^2} \left( 1 + \frac{\mathcal{U}^2 (T_y - T_x)^2}{T_x T_y (K_y + K_x)^2} \right)^{-1} - 1 \right)
\end{align*}
\]

\section*{INTERNAL ENERGY AND ISOTHERMAL PROTOCOLS}

We consider the internal energy of the system, defined as:

\[
E = \int V(x, y) p(x, y, t) dx dy = \frac{1}{2} k_x \Sigma_{11} + \frac{1}{2} k_y \Sigma_{22} + u \Sigma_{12},
\]

where \(\Sigma\) is the covariances matrix of the Gaussian propagator \(\mathcal{P}\), i.e. \(\Sigma_{11} = \langle x^2 \rangle = \gamma_2 / \det \Gamma\), \(\Sigma_{22} = \langle y^2 \rangle = \gamma_3 / \det \Gamma\), and \(\Sigma_{12} = \Sigma_{21} = \langle xy \rangle = -\gamma_1 / \det \Gamma\), and \(\det \Gamma = \gamma_1 \gamma_2 - \gamma_3^2\).

During the SE, as for the \(\gamma(t)\), the expressions for \(\Sigma\) can be written in terms of the quasi-static protocol, and read:

\[
\begin{align*}
\Sigma_{11} &= \frac{1}{K_x K_y - \mathcal{U}^2} \left[ K_y T_y + \frac{\mathcal{U}^2}{K_x + K_y} (T_y - T_x) \right] = \frac{1}{\Delta} \left[ K_y T_y + \frac{\mathcal{U}^2}{K_x} (T_y - T_x) \right] \\
\Sigma_{22} &= \frac{1}{K_x K_y - \mathcal{U}^2} \left[ K_x T_x - \frac{\mathcal{U}^2}{K_x + K_y} (T_x - T_y) \right] = \frac{1}{\Delta} \left[ K_x T_x - \frac{\mathcal{U}^2}{K_y} (T_x - T_y) \right] \\
\Sigma_{12} &= \Sigma_{21} = -\frac{1}{K_x K_y - \mathcal{U}^2} \left[ K_x T_x + K_y T_y \right] = -\frac{\mathcal{U}}{K_x} \frac{\chi}{\Delta},
\end{align*}
\]

where \(\chi = K_x T_y + K_y T_x\), \(\Delta = K_x K_y - \mathcal{U}^2\), and \(K_s = K_x + K_y\).
Using these expressions together with (36-38) for the forcing protocols \( \Pi = P + \frac{1}{\tau} \delta \mathcal{P} \), one can compute the explicit expression of the internal energy \( E \).

Strikingly, it turns out, after a careful algebra, that this expression is very simple:

\[
E = \frac{T_x + T_y}{2} + \frac{1}{\tau} \frac{d}{ds} \left[ \frac{K_x T_x + K_y T_y}{4(K_x K_y - U'^2)} \right]_{s=0}/\tau.
\]

Interestingly, the finite time correction to the internal energy, which has to be in the form \( E_x K_x' + E_y K_y' + E_u U' \), with three (cumbersome) coefficients depending on the quasi-static protocol, can actually be written with a single differential form, since \( E_x = \partial_x \Theta, E_y = \partial_y \Theta, E_u = \partial_u \Theta \), where \( \Theta \equiv \frac{1}{2} \Theta' \). Note that \( 2\Theta'/\tau \) looks like an extra temperature of the system during the forcing.

Choosing a quasi-static protocol satisfying

\[
\frac{K_x T_x + K_y T_y}{K_x K_y - U'^2} = \text{constant},
\]

one can perform a SE procedure with constant internal energy in a finite time \( \tau \), provided to correct the quasi-static protocol with its appropriate finite time corrections (36-38).

Figure 1. In the upper panels (a) and (b) we show the forcing for the SE used in simulations. In (a) the forcings obtained with the quasi-static protocols defined in Eqs. (54) (black), and in Eqs. (55) (red) applied during a long time \( \tau = 10 \). The SE is so slow that the forcings \( k_x \) (continuous line), \( k_y \) (dotted line), and \( u \) (dashed line) are very close to the quasi-static protocols. In (b) the same quasi-static protocols are used, but for a much shorter SE, with \( \tau = 0.1 \). Note the large finite time corrections for \( k_x \) and \( u \), as well as the perturbation of \( k_y \), which was almost constant in the slow SE. An other interesting observation, in this case, is that, despite the quasi-static isothermal protocol (53) is very close to the cubic quasi-static protocol (54), their corrections are very different, especially for \( k_x \). In the lower panes, the internal energy measured during the different SE are shown. In (c) the internal energy measured during the slow SE (\( \tau = 10 \)): the internal energy is almost constant, exactly as for a quasi-static SE, for both the protocols. In (d) the internal energy during the fast SE protocols are shown. The internal energy during the fast cubic SE shows large variations, in contrast with the isothermal protocol, where the internal energy is constant.

**NUMERICAL SIMULATIONS**

We performed some numerical simulations in order to further check the (exact) relation (53). We considered two different quasi-static protocols. In the first one the parameters, which will be indicated as “cubic” in the following, are defined by:

\[
\begin{align*}
K_x(s) &= 1 + 3s^2 - 2s^3 \\
K_y(s) &= 2 \\
U(s) &= 3s^2 - 2s^3
\end{align*}
\]

(54)
Figure 2. In these figures we show the marginal distribution $p(x; t) = \int p(x, y; t) dy$ (blue points) and $p(y; t) = \int p(x, y; t) dx$ (red points) at the beginning and the end of the SE protocols considered in this work. In (a) we show the initial marginal distributions for all the SE protocols considered: note that since we choose $u(0) = 0$, the marginal distributions are of the Gibbs-Boltzmann form $p(x; 0) \propto \exp(-k_x x^2/T_x)$ and $p(y; 0) \propto \exp(-k_y y^2/T_y)$ (green lines). Note that, despite we consider different temperatures $T_x = 1$ and $T_y = 2$, $p(x; t)$ is equal to $p(y; t)$ despite the two degrees of freedom are different: this is a special artifact due to the choice we made for $k_x(0) = 1$ and $k_y(0) = 2$, so we have $k_x(0)/T_x = k_y(0)/T_y$. In (b) and (c) we show the marginal distributions at the end of the SE protocols: in this case since $u(\tau) = 1$, the marginal distributions have no more the Gibbs-Boltzmann form. The marginal distributions $p(x; \tau)$ and $p(y; \tau)$ are different, but are independent on $\tau$. In both cases, $\tau = 10$ and $\tau = 0.1$, the marginal distribution coincide with the theoretical prediction (green line) that does not depend on $\tau$.

and the initial values of the forcing are $\Pi^i = \{1, 2, 0\}$, while the final values are $\Pi^f = \{2, 2, 1\}$. The second protocol has the same expression for $K_x$ and $U$, while $K_x$ is chosen according to Eq. 53. The constant is chosen in order to have the same final value of $K_x(1) = 2$. The final expressions for this isothermal protocol become:

$$\begin{align*}
K_x(s) &= \frac{2(3s^2 - 2s^2)2^2(2T_x + T_y) + 3T_x}{4s^2 T_y} \\
K_y(s) &= 2 \\
U(s) &= 3s^2 - 2s^3
\end{align*}$$

(55)

The initial values for the isothermal protocols are $\Pi^i = \{6T_x/(4T_x + T_y), 2, 0\}$, while the final values are, as before, $\Pi^f = \{2, 2, 1\}$. In order to have exactly the same initial and final values for the two protocols, we choose $T_x = 1$ and $T_y = 2$. In Figure 1, in the two upper plots, we show the forcing corresponding to the cubic and isothermal protocol for two different values of the total time $\tau$. On the left we show a slow protocol $\tau = 10$, where the finite time corrections are very small and the protocol has the same shape of the quasi-static choice. On the right we show a protocol two order of magnitude faster: $\tau = 0.1$. In all the plots, the cubic protocol is in black, while the isothermal protocol is in red. Firstly, we also performed a check on the initial and final distributions, as shown in Fig. 2. The marginal distributions $p(x; t)$ and $p(y, t)$ are computed for the cubic protocol. We collected the statistics on $10^7$ trajectories for each SE protocol considered. In the figure we show that the final (marginal) distributions are the same irrespectively of the protocol duration $\tau$ and, since the very slow $\tau = 10$ case is quasi-static in practice, the final distributions are identical to the stationary distributions. Similar results are obtained for the isothermal protocol.

Now we consider the internal energy, as shown in the lower plots of Fig. 1. In the lower left plot, the slow protocols are shown. As expected, since the finite time corrections to the protocols are negligible, the energy is constant for both the cubic and the isothermal protocol. On the other hand, for the fast protocols, while the internal energy shows a large variation during the SE, the isothermal protocol keeps the energy constant, as predicted by our computations. The theoretical predictions for the internal energy can not be distinguished, in the plots, by the simulation results.
WORK AND "FREE ENERGY" FOR STATIONARY EQUILIBRIUM TRANSFORMATIONS

In a transformation one can always define the average thermodynamic work \[ W(t) = \int_0^t dt' \langle \partial_t V(x, y, t') \rangle = \int_0^t \left[ \frac{1}{2} \dot{k}_x \langle x^2 \rangle + \frac{1}{2} \dot{k}_y \langle y^2 \rangle + \dot{u} \langle xy \rangle \right] dt'. \] (56)

Since in our settings we have \( \mathbf{\Pi}(t) = \mathbf{P}(t/\tau) + \frac{1}{\tau} \delta \mathbf{P}(t/\tau) \), hence \( \dot{\mathbf{\Pi}} = \frac{1}{\tau} \dot{\mathbf{P}} + \frac{1}{\tau^2} \delta \mathbf{P} \), and \( \Sigma(t) = \Sigma[\mathbf{P}(t/\tau)] \), we have

\[ W(t) = \int_0^{t/\tau} w_0(s) ds + \int_0^{t/\tau} w_1(s) ds \]

where

\[ w_0 = \frac{1}{2} k' \Sigma_{11} + \frac{1}{2} k'_y \Sigma_{22} + \mathcal{U} \Sigma_{12} \]

(57)

\[ w_1 = \frac{1}{\tau} \left[ \frac{1}{2} \delta k' \Sigma_{11} + \frac{1}{2} \delta k'_y \Sigma_{22} + \delta u' \Sigma_{12} \right] \]

(58)

Note that the first integral is the work needed/produced during a stationary quasi-static protocol, while the second the correction for a finite time SE protocol.

Using (49-51) one can write the stationary contribution to the work as:

\[ w_0 = \frac{T_x (K_x + K_y) K_{xy} + (T_y - T_x) \mathcal{U}^2}{2(k_y + k_x)(k_x k_y - \mathcal{U}^2)} K_x + \frac{T_y (K_x + K_y) K_{xy} - (T_y - T_x) \mathcal{U}^2}{2(k_y + k_x)(k_x k_y - \mathcal{U}^2)} K_y - \frac{\mathcal{U}(T_y K_y + T_x K_x)}{(k_y + k_x)(k_x k_y - \mathcal{U}^2)} \mathcal{U} \]

The stationary contribution to the work, for an equilibrium \( T_x = T_y = T \) system, can be written as:

\[ w_0^* = \frac{d}{ds} T \log \sqrt{k_x k_y - \mathcal{U}^2} \]

(59)

This equation agrees with equilibrium statistical mechanics. In fact, at equilibrium, we can compute the partition function

\[ Z = \int dx dy e^{-\frac{V(x, y)}{T}} \]

which because of the gaussian form of the potential \( V(x, y) = \frac{1}{4} k_x x^2 + \frac{1}{4} k_y y^2 + u xy \) reads

\[ Z = \frac{2 \pi T}{\sqrt{k_x k_y - u^2}} \]

During a stationary SE protocol, one has \( k_x = K_x, k_y = K_y, \) and \( u = \mathcal{U} \), this leads finally to the free energy during the quasi-stationary equilibrium transformation:

\[ F = -T \log Z = T \log \sqrt{K_x K_y - \mathcal{U}^2} - T \log 2 \pi T \]

This allows to rewrite the equilibrium stationary contribution to the work as:

\[ w_0(s) = \frac{d}{ds} F(K_x, K_y, \mathcal{U}) \]

If we consider the case of a non interacting protocol:

\[ w_0 = \frac{1}{2} T_x \frac{K'_x}{K_x} + \frac{1}{2} T_y \frac{K'_y}{K_y} \]

we note that we can rewrite it in terms of a time derivative:

\[ w_0 = T_x \frac{d}{ds} \log \sqrt{K_x} + T_y \frac{d}{ds} \log \sqrt{K_y} \]

This can be easily understood, reasoning as in the previous paragraph, in terms of independent equilibrium of the two degree of freedoms, each one with its own free energy:

\[ F_{x,y} = T_{x,y} \log \sqrt{K_{x,y}} - T_{x,y} \log \sqrt{2 \pi T_{x,y}} \]
PROBABILITY CURRENT

A relevant quantity for the Brownian Gyrator is the probability current. In particular, for \( T_x \neq T_y \) there can be a stationary non zero current, since the system is out-of-equilibrium.

It is convenient to consider the mean velocity [3]:

\[
\nu(x, y, t) = \frac{J(x, y, t)}{p(x, y, t)}
\]

During a SE this quantity depends on the quasi-static protocols and its derivative. In the stationary state (or during an quasi-static SE) it reads:

\[
\nu_{x}^{st}(x, y) = \frac{(T_y - T_x) \mathcal{U} \left\{ \left[ \mathcal{U}^2(T_y - T_x) + K_y T_x (K_x + K_y) \right] y + \left| \mathcal{U}(K_x T_y + K_y T_x) \right| x \right\}}{\mathcal{U}^2(T_y - T_x)^2 + T_y T_x (K_x + K_y)^2}
\]

(60)

\[
\nu_{y}^{st}(x, y) = \frac{(T_y - T_x) \mathcal{U} \left\{ \left[ \mathcal{U}^2(T_y - T_x) - K_x T_y (K_x + K_y) \right] x - \left| \mathcal{U}(K_x T_y + K_y T_x) \right| y \right\}}{\mathcal{U}^2(T_y - T_x)^2 + T_y T_x (K_x + K_y)^2}
\]

(61)

which is zero if \( T_y = T_x \) or \( \mathcal{U} = 0 \).

During the SE this quantity has finite time corrections proportional to \( K'_x, K'_y, \mathcal{U}' \) (not shown here). However, it turns out that these corrections do not contribute to the curl of the mean velocity, which reads during a finite time SE:

\[
\nabla \times \nu(x, y, t) = -\frac{(T_y - T_x)(K_x + K_y)(K_x T_y + K_y T_x) \mathcal{U}}{\mathcal{U}^2(T_y - T_x)^2 + T_y T_x (K_x + K_y)^2}
\]

Note that this quantity is homogeneous in space and depends on the quasi-stationary protocol only (hence it depends on time through \( t/\tau \)). Again this quantity is zero for \( T_y = T_x \) or \( \mathcal{U} = 0 \).

The divergence of the mean velocity is zero in the stationary state, while has finite and homogeneous in space finite time corrections (not shown).

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[2] C. Jarzynski, Comptes Rendus Physique 8, 495 (2007).
[3] U. Seifert, Rep. Prog. Phys. 75, 126001 (2012).