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On a viscous two-fluid channel flow including evaporation

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Abstract: In this contribution a particular plane steady-state channel flow including evaporation effects is investigated from analytical point of view. The channel is assumed to be horizontal. The motion of two heavy viscous immiscible fluids is governed by a free boundary value problem for a coupled system of Navier-Stokes and Stephan equations. The flow domain is unbounded in two directions and the free interface separating partially both liquids is semi-infinite, i.e. infinite in one direction. The free interface begins in some point $Q$ where the half-line $\Sigma$ separating the two parts of the channel in front of $Q$ ends. Existence and uniqueness of a suitable solution in weighted HÖLDER spaces can be proved for small data (i.e. small fluxes) of the problem.

Keywords: Navier-Stokes equations, Stephan equations, Free boundary value problem, Semi-infinite inner channel wall

MSC: 35R35, 35Q30, 76D03, 76D05

1 Introduction

In this paper we are concerned with the investigation of a particular free boundary value problem (BVP) for a two-fluid non-isothermal channel flow. The infinite channel is assumed to be horizontal and it contains a partial inner wall (cf. the thin red line $\Sigma_1$ in Figure 1) which is semi-infinite. The flow problem is assumed to be stationary and 2D. In Figure 1 the blue line denotes the lower channel wall which moves with constant speed $R$ in $x_1$-direction. The red line $\Sigma_2$ denotes the upper channel wall that is at rest. Finally, by the cyan curve $\Gamma$ we understand the a priori unknown free interface between the two fluid layers. It has the representation $x_2 = \varphi(x_1)$ where the function $\varphi$ has to be found as well as the flow fields for velocity $v(x)$, for the pressure $p(x)$ and for the temperature $\theta(x)$.

Models of the described kind are quite important in many technological and scientific applications. Corresponding examples may be found in the field of materials science, particularly in coating and solidification processes with evaporation or in crystal-growth processes (cf. [1-12]). The investigations of such problems are performed from technical point of view as well as from analytical and/or numerical point of view. It was our main objective to obtain statements about the existence and/or uniqueness of free BVP for evaporation problems.

The flow describes a coupled heat-and mass transfer (Stephan equations). The (positive) fluxes $F_m$ are prescribed in each fluid layer $\Omega_m (m = 1, 2)$ (cf. Fig. 1). The lower liquid layer is characterized by red color whereas the upper one is marked by green color. Both liquids are heavy, viscous, heat-conducting,
incompressible and immiscible. Therefore, the mathematical model can incorporate evaporation effects. The surface tension $\sigma(\theta)$ is temperature-depending in a known manner.

\[ \sigma(\theta) = a\theta - b, \quad (a, b > 0) \]

which is frequently used in the literature. This leads to an effect which is called Benard-Marangoni-effect or thermo-capillary convection.

### 2 Mathematical model

The governing equations (Navier - Stokes & Stephan) of the problem which yield in $\Omega := \Omega_1 \cup \Omega_2$ read as follows

\[
\begin{cases}
(\nu \cdot \nabla) \nu - \nu \nabla^2 \nu + \frac{1}{\eta} \nabla p = g e_g, \\
\nabla \cdot \nu = 0, \\
(\nu \cdot \nabla) \theta - \lambda \nabla^2 \theta = 0.
\end{cases}
\]

They are supplied by the boundary conditions at the lower moving wall $\Sigma_0$:

$$
\nu|_{\Sigma_0} = R, \quad \theta|_{\Sigma_0} = \theta_0 = 0.
$$

Let us emphasize that the value 0 in Eq. (2) does not represent the absolute temperature but some dimensionless value which is in fact the difference to some reference temperature related to a characteristic temperature difference.

The boundary conditions at the walls at rest $\Sigma_1(k = 1, 2)$ look like:

$$
\nu|_{\Sigma_1} = 0, \quad \theta|_{\Sigma_1} = \theta_k.
$$
Let us explain that the boundary conditions (3) for $k = 1$ mean both sides $\Sigma_1$ of the partial inner wall $\Sigma_1$. Finally, the conditions at the free interface distance from the singularity points. For $k = 1$ we have

$$\begin{align*}
[\theta]|_{r} &= 0, \\
v \cdot n|_{r} &= \frac{\partial \tilde{\theta}}{\partial n}|_{r}, \\
\frac{d}{dr} \sqrt{1 + \phi(x, r)} &= \frac{1}{\sigma(\theta)} [-p + n \cdot S(v) n]|_{r}, \\
\int_{S_{f}(\theta)} v \cdot n \, dx &= F_k. \quad (k = 1, 2)
\end{align*}$$

As a consequence one gets the relation: $\lim_{x_1 \to + \infty} \varphi(x_1) - \text{const.} = h_\infty$.

In order to prove the unique solvability of the BVP in appropriate functional spaces the following two-cycle iteration scheme was applied.

$$[\Gamma^{(0)} \to \Omega^{(0)}] \to (v^{(0)}, p^{(0)}, \theta^{(0)}) \to [\Gamma^{(1)} \to \Omega^{(1)}]$$

$$\to (v^{(1)}, p^{(1)}, \theta^{(1)}) \to \ldots \to [\Gamma^{(k)} \to \Omega^{(k)}] \to (v^{(k)}, p^{(k)}, \theta^{(k)}) \to \ldots$$

This scheme was introduced by V.V. Pukhnachev and V.A. Solonnikov about 45 years ago (cf. e.g. [13, 14] or [15]). The two-cycle iteration scheme was also applied in the papers [14, 15] and by the author in [9, 16]. In the references [11, 17, 18] other methods are used to handle different free BVP.

The scheme (5) is very senseful in cases where the free boundary is semi-infinite. In a first cycle the three flow fields $v, p, \theta$ are computed in a flow domain with fixed boundaries neglecting one of the boundary conditions - mostly the normal stress condition (4)3, i.e. the 5th equation in (4). This first cycle is then divided into several steps: The linear problem with fixed boundary containing the corresponding estimates for the solution, a model problem at the separation point $Q$ for the determination of the weight functions, the regularity of the solutions at infinity and then the nonlinear problem with fixed boundary.

In a second stage the neglected boundary condition is used in order to compute a new shape of the free boundary (and simultaneously a new shape of the entire flow domain). This equation is usually

$$K(x_1) := \frac{1}{\sigma(\theta)} [-p + n \cdot S(v)n]|_{r},$$

where $K(x_1)$ denotes the curvature of $\Gamma$ in $x_1$, and it is equal to the left-hand side of Eq. (4)3. In both cycles a related linear problem is solved and the continuous dependence of the solutions on the boundary data is also proved. Then BANACH’s fixed point argument related to some contraction operator $B$ shows the remaining parts for small data.

## 3 Function spaces

First of all we define some weighted Hölder spaces. Let $B$ be an arbitrary domain in $\mathbb{R}^2$ and $N \subset B$ a manifold of dimension $n < 2$. Define further $\rho_N(x) := \text{dist}(x, N).$ By $\beta = (\beta_1, \beta_2)$ we understand a multiindex, and $|\beta|$ is the integer part of $r$. Then by $C^r(B)(r > 0$, non-integer) we mean the well-known Hölder space with a finite norm $|u|_{C^r(B)}$. Now we obtain the subsequent weighted Hölder space $C^r_{\varphi}(B, N)$ of functions with the finite norm

$$|u|_{C^r_{\varphi}(B, N)} = \sum_{|\beta| < r} \sup_{x \in B\setminus N} \rho_N^{\beta} |D^\beta u(x)| + \sum_{|\beta| = r} \sup_{x \in B\setminus N} \rho_N^{\beta} \sup \frac{|D^\beta u(x) - D^\beta u(y)|}{|x - y|^r}. \quad (6)$$

Let us remark that the weight functions in (6) represent some kind of power functions with respect to the distance from the singularity points. For $r > s > 0, r, s$ non-integer) we get the space $C^r_{\varphi}(B, N)$ having the norm

$$|u|_{C^r_{\varphi}(B, N)} := |u|_{C^s_{\varphi}(B, N)} + \sum_{s < |\beta| < r} \sup_{x \in B\setminus N} \rho_N^{\beta} |D^\beta u(x)| + \sum_{|\beta| = s} \sup_{x \in B\setminus N} \rho_N^{\beta} \sup \frac{|D^\beta u(x) - D^\beta u(y)|}{|x - y|^r}. \quad (7)$$
The spaces \( C^r(B_1 \cup B_2, N) \) represent the natural generalization of the last ones to the case of two separate subdomains \( B_k \) as we have.

Furthermore, \( C_0^r(\Omega, I) \) is the set of functions \( \nu \) vanishing for \(|x| \gg 1\) and satisfying the boundary conditions

\[
\nu|_{\Sigma} = 0 \quad (j = 0, 1, 2), \quad \nu \cdot n|_{\Gamma} = 0, \quad |\nu|_{I} = 0.
\]

Also, \( J_{\nu}^o(\Omega, I) := \{ \nu \in C_0^o(\Omega, I), \ \text{div} \nu = 0 \} \) and we need the solenoidal spaces with DIRICHLET - norm like \( D(\Omega) := C_0^o(\Omega, I) \) and \( H(\Omega) := F_{\Omega}^o(\Omega, I) \). The DIRICHLET norm is then defined by

\[
\| u_x \|^2_{\Omega, I} := \| u_x \|^2_{L^2(\Omega)} = \int_{\Omega} \sum_{i=1}^2 \left( \frac{\partial u}{\partial x_i} \right)^2 \ dx.
\]

By \( C_{\nu,0}^o(\Omega, I) \) we understand the set of scalar fields \( \theta(x) \) vanishing for \(|x| \gg 1\) and \( \theta|_{\Sigma} = 0 \ (j = 0, 1, 2) \), \( |\theta|_{I} = 0 \). The symbol \( D_\nu(\Omega) \) is an abbreviation for the set \( C_{\nu,0}^o(\Omega, I) \). The spaces to which the solution belongs are of the subsequent type. First, define some subdomains of the infinite channel (with \( m = 1, 2 \), i.e. \( \Omega^0 = \{ x_1 \in \mathbb{R}, 0 < x_1 < 2 \}, \quad I = \{ x_1 \in \mathbb{R}, x_1 > 1 \}, \quad \Omega^m = \{ x \in \Omega_m, |x_1| < 2 \}, \quad \Omega^\infty = \{ x \in \Omega_m, |x_1| > 1 \} \).

Finally, the weighted HÖLDER spaces containing the generalized solutions, are

\[
C^\infty_{\theta, m}(\Omega_m) = \{ u(x), u|_{\Omega_m} \in C^1(\Omega_m, Q), \exp(zx_1)u(x)|_{\Omega_m} \in C^0(\Omega_m), \exp(-zx_1)u(x)|_{\Omega_m} \in C^0(\Omega_m) \}.
\]

They are essentially used throughout this paper and their norms are given by

\[
\| u \|^r_{m, \theta} := |u|_{C^0(\Omega_m, Q)} + |\exp(zx_1)u|_{C^0(\Omega_m)} + |\exp(-zx_1)u|_{C^0(\Omega_m)}.
\]

The weight functions here in formula (8) are exponential functions and they decay at the infinities. As above, we obtain for our double channel \( C^\infty_{\theta, m}(\Omega) := C^\infty_{\theta, m}(\Omega_1 \cup \Omega_2, Q) \).

At the end, for functions of one real variable we deal with the space \( C^\infty_{\theta, m}(\mathbb{R}^1) \) supplied with the norm

\[
\| f \|^r_{\mathbb{R}^1, \theta} := |f|_{C^0(\mathbb{R}, Q)} + |f(x_1) \exp(zx_1)|_{C^0}.
\]

### 4 On the Basic Flow for Large \( x_1 \)

In this section we are interested in getting an appropriate starting (or initial) solution for the iteration scheme (5). For this purpose, and under the assumptions

\[
v_2 = 0, \quad \frac{\partial v_1}{\partial x_1} = 0, \quad \frac{\partial \theta}{\partial x_1} = 0,
\]

we calculate for given values \( F_1, F_2, R, \theta_0 = 0, \theta_1, \theta_2 \) and associated rheological parameters the flow fields and values \( v(x), p(x), \theta(x), p_0, h_\infty, \theta_\infty \). The value \( \theta_\infty \) which has not been defined before describes the (asymptotic) value of the temperature \( \theta \) at the free interface when \( x_1 \) goes to \( +\infty \). Let us emphasize that the assumptions guarantee solution fields that are uniform and unidirectional (not depending on main-stream direction \( x_1 \)).

Under the assumptions (9) the governing Eqs. (1) take the subsequent reduced form:

\[
-v \nabla^2 v_1 + \frac{\partial p}{\eta \partial x_1} = 0, \quad \frac{\partial p}{\partial x_2} = -\eta g, \quad -\lambda \nabla^2 \theta = 0,
\]

where the second equation replaces the continuity Eq. (1). Now it is possible to divide the original problem into three independent problems for the flow fields. Let us start with the problem for velocities \( v \):

\[
\begin{cases}

v_1 \eta_1 \frac{d^2 v_1}{dx_2^2} = \frac{dp}{dx_1}, \\
v_2 \eta_2 \frac{d^2 v_2}{dx_2^2} = \frac{dp}{dx_1}, \\
v_1^{(1)}(1) = 0, \\
v_1^{(2)}(1) = R, \\
\int_{-h}^{h} v_1(x_2) \ dx_2 = F_1, \\
\int_{-h}^{h} v_1(x_2) \ dx_2 = F_2.
\end{cases}
\]

(10)
For the pressure \( p \) one obtains the following equations
\[
\frac{\partial p^{(1)}}{\partial x_2} = -\eta_1 g,
\quad p^{(1)}_{x_2=0} = p^{(2)}_{x_2=0},
\quad \frac{\partial p^{(2)}}{\partial x_2} = -\eta_2 g,
\] (11)

Finally, the problem for temperature \( \theta \) reads
\[
\begin{align*}
\frac{d^2 \theta^{(1)}}{dx_2^2} &= 0, \\
\theta^{(1)}_{x_2=0} = \theta_0 = 0, \\
\theta^{(2)}_{x_2=1} = \theta_2,
\end{align*}
\] (12)

In Eqs. (10), (11), (12) the superscripts \((k), (k = 1, 2)\) or \((+)\) denote the corresponding fluid layer and the subregion \( x_1 \geq 1 \). The solutions of these three (independent) problems are of NUSSELT type (cf. also [19]) and allow the representation
\[
\begin{align*}
v^{(1)}_1(x_2) &= \begin{cases} 0.5a_1x_2^2 + b_1x_2 + R, & 0 \leq x_2 \leq h_\infty \\
0.5a_2(x_2^2 - 1) + b_2(x_2 - 1), & h_\infty \leq x_2 \leq 1 ' 
\end{cases} \\
v^{(2)}_1(x_2) &\equiv 0, \quad p_0 = a_1v_1\eta_1 = a_2v_2\eta_2,
\end{align*}
\] (13)

The coefficients in (13) are given by
\[
\begin{align*}
a_1 &= \left[-\frac{3F_1 - Rh_\infty}{h_\infty^2} - 3\frac{F_2}{r(1 - h_\infty^2)}\right], \quad a_2 = ra_1, \\
b_1 &= \left[2(1 + h_\infty)\frac{F_1 - Rh_\infty}{h_\infty^2} + h_\infty\frac{F_2}{r(1 - h_\infty^2)}\right], \quad b_2 = rb_1.
\end{align*}
\]

Note, that the values \( h_\infty \) and \( p_0 \) are already known for these expressions (see Eq. (16) below). That is why it follows \( \theta_\infty = (\lambda_1(1 - h_\infty) + \lambda_2h_\infty) \) and for the complete temperature and pressure fields one obtains
\[
\theta^{(+)}_1(x_2) = \begin{cases} \frac{\partial x_2}{\eta_1h_\infty}, & 0 \leq x_2 \leq h_\infty \\
\theta_\infty + \frac{\theta_1 - \theta_2}{h_\infty}, & h_\infty \leq x_2 \leq 1,
\end{cases}
\] (14)

\[
p^{(+)}_1(x) = \begin{cases} p_0x_2 - \eta_1gx_2 + k, & 0 \leq x_2 \leq h_\infty \\
p_0x_2 - \eta_2gx_2 - h_\infty - \eta_1gh_\infty + k, & h_\infty \leq x_2 \leq 1,
\end{cases}
\] (15)

Since the associated linear problem is completely decomposed, we got the same polynomial equation for the determination of the value \( h_\infty \) as in the former paper [17].
\[
0 = r(r - 1)Rh_\infty + \left[-4r(r - 1)F_1 - r(r - 1)F_2\right]h_\infty^2
+ \left[6r(2r - 5)F_1 + 2r(2r - 3)F_2\right]h_\infty^3 + \left[2r(2r - 1)R + 3R - 2rF_1 - 2rF_2\right]h_\infty^4
+ \left[r^2R + 4r(r - 1)F_1\right]h_\infty - r^2F_1.
\] (16)

In [17] the subsequent two lemmas were proved.

**Lemma 4.1.** If \( F_2 > 0, \) then Eq. (16) has at least one root \( h_\infty \) within the open interval \([0, 1]\).

**Lemma 4.2.** If \( F_1, F_2 \geq 0, \) then Eq. (16) has at most three different roots \( h_\infty \in ]0, 1[.\)

Note that in the subregion \( \Omega^{(-)} := \Omega^{(-)} \cup \Omega^{(+)} \), i.e. for \( x_1 \leq -1 \), the corresponding problems are even simpler due to the fact that there is no free boundary. In order not to repeat simple things we restrict the presentation to the basic solution \( \nu^{(-)}, p^{(-)}, \theta^{(-)} \) in the double - channel which can also be determined very simple by straightforward calculations in the left part \( \Omega^{(-)} \) of the (double) - channel. The corresponding velocities and temperatures do not depend on \( x_1 \). In \( \Omega^{(+)} \) one obtains
\[
\begin{align*}
\nu^{(1)}_1(x_2) &= \left[\frac{3R - 6F_1}{\eta_2}\right]x_2^2 + \left[\frac{6R + 6F_1}{\eta_2}\right]x_2 + R, \\
\nu^{(1)}_2(x) &\equiv 0, \\
p^{(1)}_1(x) &= 2v_1\eta_1\left[\frac{3R - 6F_1}{\eta_2}\right]x_1 - \eta_1gx_2 + \eta_1k.
\end{align*}
\] (17)
In \( \Omega_2 \) one gets, respectively,
\[
\begin{align*}
\psi^{(i)}(x_2) &= -\frac{6F_2}{(1-h_2)^2} x_2^3 + \frac{6(1-h_2)F_2}{(1-h_2)^2} x_2 - \frac{6h_2 F_2}{1-h_2}, \\
\psi^{(i)}(x) &\equiv 0, \\
\theta^{(i)}(x_2) &= \theta_1 + \frac{h_2}{1-h_2}(\theta_2 - \theta_1), \\
\phi^{(i)}(x) &= -\frac{12v_2 \eta_2 F_2}{(1-h_2)^2} x_2 - \eta_2 gx_2 + k_2.
\end{align*}
\]

It is well-known that the pressure \( p \) can be determined only up to an additive constant in channel flows (cf. \( k_1, k_2 \) in formulae (17), (18)).

5 The free interface equation

Eq. (16) coincides with equation (A.13) from [17] for horizontal channels. Recall that the final thickness \( h_\infty \) is a function of \( F_1, F_2, R \) and of the rheological parameters of the fluids. It can have up to three different values in the open interval \([0, 1]\) for the same parameter set (cf. [17]). Furthermore, by \( \psi^{(i)}(x_1) \) we denote the infinitely differentiable solution of the following free BVP.

\[
\begin{align*}
\frac{d}{dx_1} \frac{\psi'(x_1)}{\sqrt{1+\psi'(x_1)^2}} - \frac{2\pi \eta_2}{a} \psi \psi' &= -\frac{2\pi \eta_2}{a} \psi h_\infty, \\
\phi(0) &= 0, \\
\lim_{x_1 \to \infty} \psi(x_1) &= h_\infty.
\end{align*}
\]

which can be obtained from the 5-th condition (4) of (4) by setting \( v = 0, p = \text{const.}, \theta = 0 \) as the initial solution for \( F_1 = F_2 = R = \theta_0 = \theta_2 = 0 \). Let \( \xi = \xi(x_1) \) be a smooth cut-off function vanishing for \( |x_1| \leq 1 \) and being equal to 1 for \( |x_1| \geq 2 \). Finally, assume that \( \eta_1 > \eta_2 \) is satisfied. This makes physically sense.

Now, the difference function \( \omega(x_1) := (\phi(x_1) - \psi^{(i)}(x_1)) \) is equivalent to \( \exp(-\sqrt{g(\eta_1 - \eta_2)/\hat{a}} x_1) \) as \( x_1 \to +\infty \). For the unknown function \( \omega(x_1) \) we get a two-point BVP like BVP (8.8) from [20] subtracting Eq. (19) from Eq. (4). A difference to BVP (8.8) consists in the following. We have to replace \( \beta_1 \) by \( g(\eta_1 - \eta_2)/\hat{a} \) everywhere and, furthermore, we have to introduce the operator \( \Xi^{(3)} \) by

\[
\Xi^{(3)} \omega := \frac{\hat{b}}{\tilde{a}_\theta} \frac{\sigma(\theta)}{\sigma(\theta)} \omega = \frac{\sigma(0) - \sigma(\theta)}{\sigma(\theta)} \omega.
\]

The remaining part of the proof of the main theorem is a slightly modified repetition of the proof of Theorem 8.1 in [20]. First of all, one has to study the dependence of the solution to the nonlinear auxiliary problem with fixed boundary on small variations of the boundary. After getting the corresponding estimates one applies BANACH’s fixed point principle to the subsequent operator equation. Instead of the operator Eq. (8.10) from [20] we have to study the following one:

\[
\omega = \mathcal{L}(\Xi^{(1)} \omega + \Xi^{(2)} \omega + \Xi^{(3)} \omega) =: \mathcal{B} \omega
\]

with \( \Xi^{(3)} \) given in (20) and the other parts taken from [20]. Since \( \Xi^{(3)} \) is a contraction operator for small \( \theta \), we can conclude as in [20] that \( \mathcal{B} \) is a contraction operator in the ball \( \| \omega \|_{\lambda_{2,1}, \psi} < \epsilon \). Consequently, we have proved the main result of this paper.

6 Results

Let us formulate the main result of this contribution. A sketch of the proof has been given before. A very detailed application of this method can be found in the thesis [16] as well as in the article [20].

**Theorem 6.1.** There exist positive real numbers \( \hat{s}, \hat{z} \leq \min(z_0, \sqrt{1/\hat{a}}) \) such that for arbitrary \( s \in ]0, \hat{s}[ \), \( z \in ]0, \hat{z}[ \) and for sufficiently small values (\( |F_1|, |F_2|, |R|, |\theta_1|, |\theta_2| \)) and for values \( h_\infty \) fulfilling the condition

\[
|h_\infty - h_1| < \frac{2\hat{a}}{g(\eta_1 - \eta_2)}
\]
the complete mathematical model has a unique solution \( (v, p, \theta, \varphi) \) which can be represented in the form

\[
v = \xi(-x_1)u^+(x_1) + \xi(x_1)u^-(x_1) + w, \quad \varphi(x_1) = \varphi^+(x_1) + \omega(x_1),
\]

\[
p = \xi(-x_1)p^+(x_1) + \xi(x_1)p^-(x_1) + q; \quad \theta = \xi(-x_1)\theta^+(x_1) + \xi(x_1)\theta^-(x_1) + \theta_0,
\]

where \( \xi \) is the cut-off function described above, \((w^-,p^-,\theta^-)\) is the basic exact solution given by (17), (18) in both channels on left-hand side. The function \( \varphi^+ \) is the solution to the free boundary problem (19). Moreover, \( \theta_0, w \in C_{s+2}^\rho(\Omega), q \in C_{s+1}^\rho(\Omega^0 \cup \Omega^+), \nabla q \in C_{s-2}^\rho(\Omega) \) and \( \omega \in C_{s+1}^{\rho+}(\mathbb{R}^+ \cup \mathbb{R}) \) hold.

**Remark 6.2.** If Eq. (16) has more than one real root \( h \) between 0 and 1 then the statements of Theorem 6.1 remain true in the neighbourhood of each value.

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