Fermat’s principle in black-hole spacetimes

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Abstract

Black-hole spacetimes are known to possess closed light rings. We here present a remarkably compact theorem which reveals the physically intriguing fact that these unique null circular geodesics provide the *fastest* way, as measured by asymptotic observers, to circle around spinning Kerr black holes.

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Fermat’s principle, also known as the principle of least time \([1, 2]\), asserts that among all possible null trajectories, the path taken by a ray of light between two given points \(A\) and \(B\) in a flat spacetime geometry is the path that minimizes the traveling time \(T_{A\rightarrow B}\). This remarkably elegant principle implies, in particular, that the unique null trajectory taken by a ray of light between two given points is generally distinct from the straight line trajectory which minimizes the spatial distance \(d_{AB}\) between these points.

In the present Essay we would like to highlight an intriguing and closely related physical phenomenon which characterizes curved spacetime geometries. In particular, we here raise the physically interesting question: Among all possible closed paths that circle around a black hole in a curved spacetime, which path provides the fastest way, as measured by asymptotic observers, to circle the central black hole?

We first note that, in flat spacetimes, the characteristic orbital period \(T_{\odot}\) of a test particle that moves around a spatially compact object of radius \(R\) is trivially bounded from below by the compact relation

\[
T_{\odot} \geq T_{\odot}^{\text{flat min}} = 2\pi R, \tag{1}
\]

where the equality sign in (1) is attained by massless particles that circle the compact object on the shortest possible (tangential) trajectory with \(r_{\text{fast}} = R\).

It should be emphasized, however, that the simple lower bound (1) is not valid in realistic curved spacetimes. In particular, it does not take into account the important time dilation (redshift) effect which is caused by the gravitational field of the central compact object \([4]\). In addition, the flat-space relation (1) does not take into account the well-known phenomenon of dragging of inertial frames by spinning compact objects in curved spacetimes \([4]\).

As we shall explicitly show below, due to the influences of these two interesting physical effects, the shortest possible orbital period \(T_{\odot}\) of a test particle around a central compact object, as measured by asymptotic observers, is larger than the naive flat-space estimate (1). In particular, we shall prove that, in generic curved spacetimes, the unique circular trajectory \(r = r_{\text{fast}}\) that minimizes the traveling time \(T_{\odot}\) around a central Kerr black hole is distinct from the tangential trajectory with \(r = r_{\text{short}}\) which could minimize the traveling distance around the spinning black hole.

The fastest circular orbit around a spinning Kerr black hole.— We shall analyze the physical and mathematical properties of equatorial circular trajectories around spinning Kerr black holes. In Boyer-Lindquist coordinates \((t, r, \theta, \phi)\), the asymptotically flat black-hole spacetime can be de-
scribed by the curved line element  

\[ ds^2 = -\frac{\Delta}{\rho^2}(dt - a \sin^2 \theta d\phi)^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2} [adt - (r^2 + a^2) d\phi]^2, \]

(2)

where \( M \) is the black-hole mass, \( J \equiv Ma \) is its angular momentum, \( \Delta \equiv r^2 - 2Mr + a^2 \), and \( \rho^2 \equiv r^2 + a^2 \cos^2 \theta \). The black-hole (event and inner) horizons are determined by the spatial zeros of the metric function \( \Delta \):

\[ r_\pm = M \pm (M^2 - a^2)^{1/2}. \]

(3)

We would like to identify the unique circular trajectory which minimizes the orbital period \( T_\odot \), as measured by asymptotic observers, around the central black hole. We shall therefore assume the relation \( v/c \to 1^- \) for the velocity of the orbiting test particle [5]. The corresponding radius-dependent orbital periods \( T_\odot (r) \) of the test particles can easily be obtained from the characteristic black-hole curved line element (2) with \( ds = dr = d\theta = 0 \) and \( \Delta \phi = \pm 2\pi \) [6–8]. This yields the compact functional relation

\[ T_\odot (r) = 2\pi \cdot \frac{\sqrt{r^2 - 2Mr + a^2} - 2Ma}{1 - \frac{2M}{r}} \]

(4)

for the orbital periods of co-rotating test particles around the central spinning black hole.

The physically interesting co-rotating circular orbit with \( r = r_{\text{fast}} \), which is characterized by the shortest possible orbital period \( T_{\odot \text{min}} = \min_r \{T_\odot (r)\} \) around the central spinning black hole, is determined by the functional relation \( dT(r = r_{\text{fast}})/dr = 0 \). This yields the characteristic algebraic equation

\[ r^2 - 3Mr + 2a^2 + 2a\sqrt{r^2 - 2Mr + a^2} = 0 \quad \text{for} \quad r = r_{\text{fast}}. \]

(5)

Remarkably, this equation can be solved analytically to yield the simple functional relation

\[ r_{\text{fast}} = 2M \cdot \{1 + \cos\left[\frac{2}{3} \cos^{-1}(-a/M)\right]\}. \]

(6)

for the unique orbital radius \( r_{\text{fast}}(M, a) \) which characterizes the fastest co-rotating circular trajectory (the closed circular path with the shortest possible orbital period) around the central spinning Kerr black hole.

What we find most intriguing is the fact that the spin-dependent radii \( r_{\text{fast}}(M, a) \) of the fastest circular trajectories, as given by the functional expression (6), exactly coincide with the corresponding radii \( r_\gamma (M, a) \) of the null circular geodesics [9] which characterize the spinning Kerr black-hole spacetimes. One therefore concludes that co-rotating null circular geodesics (closed light rings)
provide the fastest way, as measured by asymptotic observers, to circle around generic Kerr black holes.

It is physically interesting to define the dimensionless ratio [see Eqs. (1) and (3)]

$$\Theta(\bar{a}) \equiv \frac{T_{\odot \text{min}}}{2\pi r_+}; \quad \bar{a} \equiv a/M,$$

(7)

which characterizes the unique closed circular trajectories [with \(r = r_{\text{fast}}(\bar{a})\)] that minimize the orbital periods around the central spinning black holes. As emphasized above, a naive flat-space calculation predicts the relation \(\Theta_{\text{flat}}^\text{min} \equiv T_{\odot \text{min}}^\text{flat}/2\pi R = 1\) [see Eq. (1)]. However, substituting Eqs. (3) and (4) into (7), one finds the characteristic inequality [10]

$$\Theta_{\text{Kerr}}(\bar{a}) > 1$$

(8)

for all Kerr black-hole spacetimes in the physically allowed regime \(\bar{a} \in [0, 1]\). In particular, the dimensionless function \(\Theta_{\text{Kerr}}(\bar{a})\) exhibits a non-trivial (non-monotonic) functional dependence on the dimensionless black-hole rotation parameter \(\bar{a}\) with the property [11]

$$\min_{\bar{a}} \{\Theta_{\text{Kerr}}(\bar{a})\} \approx 2 - \frac{3(13 - 7\sqrt{3})}{88} \quad \text{at} \quad \bar{a}_{\text{Kerr}}^\text{min} \approx 1 - \frac{126 - 45\sqrt{3}}{1936}.$$

(9)

**Summary.** — Fermat’s principle asserts that, in a flat spacetime geometry, the path taken by a ray of light is unique in the sense that it represents the spatial trajectory with the shortest possible traveling time between two given points [1, 2]. This intriguing principle implies, in particular, that the paths taken by light rays are generally distinct from the straight line trajectories which could minimize the traveling distances between two given points.

In the present short Essay we have highlighted an intriguing and closely related phenomenon in curved black-hole spacetimes. In particular, we have raised the physically interesting question: Among all possible trajectories that circle around a spinning Kerr black hole, which closed trajectory provides the *fastest* way, as measured by asymptotic observers, to circle the central black hole?

Our compact theorem has revealed the physically intriguing fact that the equatorial null circular geodesics (closed light rings), which characterize the curved black-hole spacetimes, provide the fastest way to circle around spinning Kerr black holes. In particular, we have explicitly proved that, in analogy with the Fermat principle in flat spacetime geometries, the unique curved trajectories \(r = r_{\text{fast}}(M, a)\) [see Eq. (3)] which minimize the traveling times \(T_\odot\) of test particles around central black holes are distinct from the tangential trajectories \(r = r_+(M, a)\) [see Eq. (3)] which could minimize the traveling distances around the black holes.
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[1] A. Lipson, S. G. Lipson, H. Lipson, *Optical Physics* 4th Edition, Cambridge University Press (2010).
[2] See also https://en.wikipedia.org/wiki/Fermat%27s_principle.
[3] We use natural units in which $G = c = 1$.
[4] S. Chandrasekhar, *The Mathematical Theory of Black Holes*, (Oxford University Press, New York, 1983).
[5] In the present essay we consider geodesic as well as non-geodesic trajectories of test particles around the central Kerr black holes. It is well known that the null circular geodesic of the black-hole spacetime, on which massless particles that move exactly at the speed of light can circle the central black hole, is characterized by a well defined radius $r_\gamma = r_\gamma(M, a)$. It is important to stress the fact that, by using man-made rockets which are based on non-gravitational forces, massive particles can also circle the central black hole on non-geodesic trajectories with orbital velocities that, in principle, may approach arbitrarily close to the speed of light and with orbital radii that may differ from the unique radius $r_\gamma(M, a)$ of the black-hole null circular geodesic.
[6] Here the upper/lower signs correspond respectively to co-rotating/counter-rotating trajectories of the test particles around the central spinning black holes.
[7] In the present essay we are interested in circular trajectories that minimize the orbital periods $T_\circ$ around the central spinning black holes as measured by asymptotic observers. We shall therefore focus on co-rotating circular orbits.
[8] S. Hod, Phys. Rev. D 84, 104024 (2011).
[9] See equation (2.18) of J. M. Bardeen, W. H. Press and S. A. Teukolsky, Astrophys. Jour. 178, 347 (1972).
[10] In particular, one finds $\Theta^{Kerr}(\bar{a} \to 0) \to 3\sqrt{3}/2$ in the limit of non-rotating Schwarzschild black holes and $\Theta^{Kerr}(\bar{a} \to 1) \to 2$ in the limit of maximally spinning (extremal) Kerr black holes.
[11] Here we have used the series expansion $\Theta^{Kerr}(\bar{a}) = 2 + \sqrt{2}(\sqrt{3} - 2) \cdot \sqrt{\epsilon} + (14/3 - 2\sqrt{3}) \cdot \epsilon + O(\epsilon^{3/2})$ in the near-extremal $0 \leq \epsilon = 1 - \bar{a} \ll 1$ regime.