ABELIAN SCHUR GROUPS OF ODD ORDER

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Abstract. A finite group $G$ is called a Schur group if any Schur ring over $G$ is associated in a natural way with a subgroup of $\text{Sym}(G)$ that contains all right translations. It is proved that the group $C_3 \times C_3 \times C_p$ is Schur for any prime $p$. Together with earlier results, this completes a classification of the abelian Schur groups of odd order.

1. Introduction

A Schur ring or $S$-ring over a finite group $G$ can be defined as a subring of the group ring $\mathbb{Z}G$ that is a free $\mathbb{Z}$-module spanned by a partition of $G$ closed under taking inverse and containing the identity element $e$ of $G$ as a class (see Section 2 for details). An important example of such a partition is given by the orbits of the point stabilizer $K_e$ of a permutation group $K$ such that

$$G_{\text{right}} \leq K \leq \text{Sym}(G),$$

where $G_{\text{right}}$ is the group induced by the right translations of $G$. The corresponding $S$-rings are said to be schurian in honor of I. Schur who studied the $S$-rings of this type.

In fact, there are a lot of non-shurian $S$-rings. An infinite family of them can be found in paper of R. Pöschel [9], where he introduced a concept of a Schur group: the group $G$ is Schur if every $S$-ring over $G$ is schurian. A motivation for being interested in the Schur groups comes from the problem of testing isomorphism of Cayley graphs, see [6, 8].

In [4], it was proved that every finite abelian Schur group belongs to one of several explicitly given families. Recent results [7, 10] show that two of them are indeed consist of Schur groups. The main result of the present paper is given in the theorem below concerning the next family.

Theorem 1.1. For any prime $p$, all $S$-rings over a group $G = E_9 \times C_p$ are schurian. In particular, $G$ is a Schur group.

All cyclic Schur groups were classified in [3]. Therefore, as an immediate consequence of this theorem and the above mentioned results, we obtain a classification of all abelian Schur groups of odd order.

Theorem 1.2. A noncyclic abelian group of odd order is Schur if and only if it is isomorphic to $C_3 \times C_{3k}$ for an integer $k \geq 1$, or $E_9 \times C_p$ for a prime $p \geq 3$.

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In Section 6, we deduce Theorem 1.1 from Theorem 6.1 stating that any $S$-ring $A$ over the group $G = E_9 \times C_p$ is either obtained from two $S$-rings over smaller groups (and then the schurity of $A$ is under control) or is a cyclotomic $S$-ring (and then $A$ is schurian by definition). The proof of Theorem 6.1 is mainly based on Theorem 5.1 giving a sufficient condition for $A$ to be cyclotomic. This carried out in three steps. First, the $S$-rings over $E_9$ are completely described (Section 3). This enables us to prove that any class of the partition of $G$ associated with $A$ is an orbit of a suitable subgroup of $\text{Aut}(G)$ (Section 4). At the last step, we show that this subgroup can be chosen the same for all classes (Section 5). For reader’s convenience, we cite basic facts on $S$-rings in Section 2.

Notation.
As usual by $\mathbb{Z}$ we denote the ring of rational integers.
A finite field of order $q$ is denoted by $\text{GF}(q)$.
The projections of $X \subseteq A \times B$ to $A$ and $B$ are denoted by $X_A$ and $X_B$, respectively.
The set of non-identity elements of a group $G$ is denoted by $G^\#$.
The center of a group $G$ is denoted by $Z(G)$.
Let $X \subseteq G$. The subgroup of $G$ generated by $X$ is denoted by $\langle X \rangle$; we also set $\text{rad}(X) = \{ g \in G : gX = Xg = X \}$.
Let $\sigma \in \text{Aut}(G)$. The element $\sum_{x \in X} x^\sigma$ of the group ring $\mathbb{Z}G$ is denoted by $X^\sigma$, and by $X$ if $\sigma$ is the identity.
The componentwise multiplication in the ring $\mathbb{Z}G$ is denoted by $\circ$.
The group of all permutations of $G$ is denoted by $\text{Sym}(G)$.
The induced action of $G \leq \text{Sym}(\Omega)$ on an invariant set $\Delta \subseteq \Omega$ is denoted by $G\Delta$.
The cyclic group of order $n$ is denoted by $C_n$.
The elementary abelian group of order $p^k$ is denoted by $E_{p^k}$.

2. Preliminaries

2.1. Definitions. Let $G$ be a finite group. A subring $A$ of the group ring $\mathbb{Z}G$ is called a Schur ring ($S$-ring, for short) over $G$ if there exists a partition $\mathcal{S} = \mathcal{S}(A)$ of $G$ such that

(S1) $\{1_G\} \in \mathcal{S}$,
(S2) $X \in \mathcal{S} \Rightarrow X^{-1} \in \mathcal{S}$,
(S3) $A = \text{Span}\{X : X \in \mathcal{S}\}$.

In particular, condition (S3) implies that $A$ is closed with respect to the componentwise multiplication $\circ$. A group isomorphism $f : G \to G'$ is called a Cayley isomorphism from an $S$-ring $A$ over $G$ onto an $S$-ring $A'$ over $G'$ if $\mathcal{S}(A)^f = \mathcal{S}(A')$. The set of Cayley isomorphisms from $A$ to itself is denoted by $\text{Iso}_{cay}(A)$. Up to notation, the following statement is known as the Schur theorem on multipliers (see, e.g., [4, statement (1) of Theorem 2.3]).

Lemma 2.1. Let $A$ be an $S$-ring over an abelian group $G$. Then

$$Z(\text{Aut}(G)) \leq \text{Iso}_{cay}(A).$$

The elements of $\mathcal{S}$ and the number $\text{rk}(A) = |\mathcal{S}|$ are called, respectively, the basic sets and rank of the $S$-ring $A$. Any union of basic sets is called an $A$-subset of $G$ or $A$-set; the set of all of them is denoted by $\mathcal{S}(A)$.$\cup$. The latter set is closed with
Lemma 2.4. For every prime $p$ of the field spanned by the set $A$ is denoted by $G$, the set of all of them is denoted by $F$ multiplicative subgroup of a finite field $C$. 

Lemma 2.3. Let $A$ be an $S$-ring over a group $G$, $H \in \mathfrak{S}(A)$ and $X \in S(A)$. Then the cardinality of the set $X \cap Hx$ does not depend on $x \in X$.

Let $S = U/L$ be a section of $G$. It is called an $A$-section if both $U$ and $L$ are $A$-groups. Given $X \in S(A)_U$, the module

$$A_S = \text{Span}\{\pi_S(X) : X \in S(A)_U\}$$

is an $S$-ring over the group $S$, where $\pi_S : U \to S$ is the natural epimorphism. The basic sets of $A_S$ are exactly the sets from the right-hand side of the formula.

2.2. Wreath and tensor products. Let $S = U/L$ be an $A$-section. The $S$-ring $A$ is called an $S$-wreath product if $L \leq G$ and $L \leq \text{rad}(X)$ for all basic sets $X$ outside $U$; in this case, we write

$$A = A_U \wr S A_{G/L},$$

and omit $S$ when $U = L$. When the explicit indication of the section $S$ is not important, we use the term generalized wreath product and omit $S$ in the previous notation. The $S$-wreath product is nontrivial or proper if $1 \neq L$ and $U \neq G$.

If $A_1$ and $A_2$ are $S$-rings over groups $G_1$ and $G_2$ respectively, then the subring $A = A_1 \otimes A_2$ of the ring $\mathbb{Z}G_1 \otimes \mathbb{Z}G_2 = \mathbb{Z}G$, where $G = G_1 \times G_2$, is an $S$-ring over the group $G$ with

$$S(A) = \{X_1 \times X_2 : X_1 \in S(A_1), X_2 \in S(A_2)\}.$$ 

It is called the tensor product of $A_1$ and $A_2$. The following statement was proved in [3] Lemma 2.3

Lemma 2.3. Let $A$ be an $S$-ring over an abelian group $G = C \times D$. Assume that $C$ and $D$ are $A$-groups. Then

1. $X_C$ and $X_D$ are basic sets of $A$ for all $X \in S(A)$,
2. $A \supseteq A_C \otimes A_D$, and the equality holds if $A_C$ or $A_D$ is the group ring.

2.3. Cyclotomic S-rings. An $S$-ring $A$ over a group $G$ is said to be cyclotomic if there exists $M \leq \text{Aut}(G)$ such that

$$S(A) = \text{Orb}(M,G).$$

In this case, $A$ is denoted by $\text{Cyc}(M,G)$. Obviously, the group $K = G_{\text{right}}M$ satisfies condition [1]. Thus, any cyclotomic $S$-ring is schurian. When $M$ is a multiplicative subgroup of a finite field $F$, we say that $A$ is a cyclotomic $S$-ring over $F$. For such a ring, the group $\text{Iso}_{cyc}(A)$ contains the Frobenius automorphism of the field $F$. The following statement is a special case of [1] Theorem 5.1)

Lemma 2.4. For every prime $p$, each $S$-ring over a group $C_p$ is cyclotomic.
2.4. **Duality.** Let $G$ be an abelian group. Denote by $\hat{G}$ the group dual to $G$, i.e., the group of all irreducible complex characters of $G$. It is well known that there is a uniquely determined lattice antiisomorphism between the subgroups of $G$ and $\hat{G}$ [12]. The image of the group $H$ with respect to this antiisomorphism is denoted by $H^\perp$.

For any S-ring $A$ over the group $G$, one can define the dual S-ring $\hat{A}$ over $\hat{G}$ as follows: two irreducible characters of $G$ belong to the same basic set of $\hat{A}$ if they have the same value on each basic set of $A$ (for the exact definition, we refer to [1, 2]). One can prove that $\text{rk}(\hat{A}) = \text{rk}(A)$ and the S-ring dual to $\hat{A}$ is equal to $A$. The following statement collect some facts on the dual S-rings proved in [2, Sec. 2.3].

**Lemma 2.5.** Let $A$ be an S-ring over an abelian group $G$. Then

1. the mapping $\Theta(A) \to \Theta(\hat{A})$, $H \mapsto H^\perp$ is a lattice antiisomorphism,
2. $\hat{A}_H = \hat{A}_{G/H^\perp}$ and $\hat{A}_{G/H} = \hat{A}_{H^\perp}$ for every $H \in \Theta(A)$,
3. $A = \text{Cyc}(K, G)$ for $K \leq \text{Aut}(G)$ if and only if $\hat{A} = \text{Cyc}(K, \hat{G})$,
4. $A = A_1 \otimes A_2$ if and only if $\hat{A} = \hat{A}_1 \otimes \hat{A}_2$,
5. $A$ is the $U/L$-wreath product if and only if $\hat{A}$ is the $L^\perp/U^\perp$-wreath product.

2.5. **Subdirect product.** Let $U$ and $V$ be groups. Assume that $\varphi$ and $\psi$ be homomorphisms from $U$ and from $V$ onto isomorphic groups, i.e., there exists $f \in \text{Iso}(\text{im}(\varphi), \text{im}(\psi))$.

In this situation, one can define the subdirect product of the groups $U$ and $V$ with respect to the homomorphisms $\varphi$ and $\psi$, and the isomorphism $f$ as the following subgroup of $U \times V$:

$$U \prod_f ^{\varphi, \psi} V = \{(u, v) \in U \times V : f(\varphi(u)) = \psi(v)\}.$$ 

It is easily seen that if the groups $\text{im}(\varphi)$ and $\text{im}(\psi)$ are trivial, then the subdirect product equals $U \times V$.

In what follows, we are interested in the orbits of subdirect products of permutation groups, in which the homomorphisms $\varphi$ and $\psi$ are induced by the actions of the groups on imprimitivity systems associated with normal subgroups. More exactly, let $A$ and $P$ be groups, and we are given the following data:

1. $U_0 \trianglelefteq U \leq \text{Aut}(A)$ and $V_0 \trianglelefteq V \leq \text{Aut}(P)$,
2. $X_A \in \text{Orb}(U, A)$ and $X_P \in \text{Orb}(V, P)$,
3. a bijection $f : \text{Orb}(U_0, X_A) \to \text{Orb}(V_0, X_P)$.

Under these conditions, the sets $\Pi_A = \text{Orb}(U_0, A)$ and $\Pi_P = \text{Orb}(V_0, P)$ form imprimitivity systems of the (transitive) groups

$$U^{X_A} \leq \text{Sym}(X_A) \quad \text{and} \quad V^{X_P} \leq \text{Sym}(X_P).$$ 

Denote by $\varphi$ and $\psi$ the natural epimorphisms from $U$ onto $U^{\Pi_A}$ and from $V$ onto $V^{\Pi_P}$, respectively. Note that the permutation groups $U^{\Pi_A}$ and $V^{\Pi_P}$ are regular (this follows from the normality of the groups $U_0$ and $V_0$). Therefore, each
isomorphism from $U^\Pi_A$ onto $V^\Pi_P$ is induced by a certain bijection of the form given in condition (P3).

**Lemma 2.6.** In the above notation, let $G = A \times P$. Assuming $f \in \text{Iso}(U^\Pi_A, V^\Pi_P)$, denote by $K$ the subdirect product of the groups $U$ and $V$ with respect to the homomorphisms $\varphi$, $\psi$, and $f$. Then $K \leq \text{Aut}(A) \times \text{Aut}(P)$ and

$$
\bigcup_{Y \in \Pi_A} Y \times Y^f \in \text{Orb}(K, G).
$$

3. S-rings over $E_9$

Up to Cayley isomorphism, there are exactly ten S-rings over $E_9$. This can be checked in a straightforward way or with the help of the GAP package COCO2 [5]. In this section, we cite relevant properties of these S-ring to be used in Sections 4 and 5. The first statement can be established by inspecting the above ten S-rings one after the other.

**Theorem 3.1.** Every S-ring over a group $E_9$ is Cayley isomorphic to one of the S-rings listed below:

1. $\text{Cyc}(M, F)$, where $F = \text{GF}_9$ and $1 < M \leq F^\times$,
2. the tensor product of two S-rings over $C_3$,
3. the wreath product of two S-rings over $C_3$.

In statement (1), $M$ is a cyclic group of order 2, 4, or 8 (in the last case, the corresponding S-ring is of rank 2). In statement (2), there are three S-rings of ranks 4, 6, and 9 (the last one is $\text{Z}_{E_9}$). In statement (3), there are four S-rings of ranks 3, 4, 4, and 5. To establish some properties of this S-rings, we need an auxiliary notion.

Let $G$ be an abelian group $G$ and $X$ an orbit of a subgroup of $\text{Aut}(G)$; in particular, $X = X^{-1}$ or $X \cap X^{-1} = \emptyset$. A uniform partition $\Pi$ of $X$ is said to be regular if the condition $X = X^{-1}$ implies that

1. $\Pi^{-1} = \Pi$,
2. the permutation $Y \mapsto Y^{-1}$, $Y \in \Pi$, is either trivial or fixed point free,
3. $X \cap \sum_{Y \in \Pi} Y^{-1} = \alpha X$ for some integer $\alpha \geq 0$.

Thus, in the case $X \cap X^{-1} = \emptyset$, any uniform partition of $X$ is regular. If $X = X^{-1}$, then the partition of $X$ into one class is regular. A less trivial example is given by groups $L \triangleleft K \leq \text{Aut}(G)$: in this case, one can take any $X \in \text{Orb}(K, G)$ and $\Pi = \text{Orb}(L, X)$.

**Lemma 3.2.** Let $\mathcal{A}$ be an S-ring over a group $G = E_9$, and let $X \in S(\mathcal{A})$. Then for any regular partition $\Pi$ of $X$, there exists groups $L \leq M \leq \text{Aut}(G)$ such that

$$
\Pi = \text{Orb}(X, L) \quad \text{and} \quad X \in \text{Orb}(M, G).
$$

Moreover, the group $M$ is cyclic unless $\mathcal{A}$ is the tensor product of two S-rings of rank 2 and $|X| = 4$. In the latter case, $M = E_4$.

**Proof.** According to Theorem 3.1, we have the following cases:

1. $X$ is an orbit of a Singer subgroup $M$ of the group $\text{Aut}(G) \cong \text{GL}(2, p)$; in particular, $|X| = 2, 4, \text{or } 8$;

---

1A partition of a set is said to be uniform if all its classes have the same cardinality
(X2) $X$ is an orbit of a subgroup of $\text{Aut}(C) \times \text{Aut}(C') \leq \text{Aut}(G)$, where $C \cong C_3 \cong C'$ are such that $G = C \times C'$; in particular, $|X| = 1$, 2, or 4;

(X3) $X$ is an orbit of a subgroup $M \leq \text{Aut}(G)$ of order 6 that stabilizes a group $C \cong C_3$; in particular, $|X| = 1$, 2, 3, or 6.

Let $\Pi$ be a regular partition of $X$. Without loss of generality, we may assume that $1 < |\Pi| < |X|$. Since also $|\Pi|$ divides $|X|$, 

\[(|X|, |\Pi|) = (4, 2), (8, 2), (8, 4), (6, 2), \text{ or } (6, 3),\]

where the first pair appears in cases (X1) and (X2), the second two appear in (X1), and the last two appear in (X3). In all these cases, the set $X$ is symmetric. A simple counting argument using conditions (R2) and (R3) shows that the permutation defined in (R2) must be trivial unless $(|X|, |\Pi|) = (4, 2)$ for the case (X2), and $(|X|, |\Pi|) = (6, 2)$. Therefore, the number of possible partitions $\Pi$ of the set $X$ is 1 or 3, 3, 1, 1, and 1, respectively to cases listed in (3). Now a straightforward check in each case completes the proof. □

Lemma 3.2 shows that the group $L$ equals the kernel of the homomorphism from $M$ to $\text{Sym}(\Pi)$ induced by the action of $M$ on the $\text{Orb}(X, L)$. In what follows, we say that $(M, L)$ is a standard pair for the basic set $X$ and regular partition $\Pi$; though the standard pair is not uniquely determined, the following statement holds true.

\textbf{Lemma 3.3.} In the notation of Lemma 3.2, assume that the group $M^\Pi$ is cyclic. Then for any regular cyclic group $C \leq \text{Sym}(\Pi)$ centralizing the permutation in condition (R2), there exist $\sigma \in \text{Aut}(G)$ such that

1. $(M^\sigma, L)$ is a standard pair for $X$ and $\Pi$,
2. $(M^\sigma)^\Pi = C$.

\textbf{Proof.} The statement is trivial if $|\Pi| \leq 3$, because $\text{Sym}(\Pi)$ contains a unique cyclic subgroup of order $|\Pi|$. Since $|X| \leq 8$ and $|\Pi|$ divides $|X|$, we may assume that

$(|X|, |\Pi|) = (4, 4), (6, 6), (8, 8), \text{ or } (8, 4).$

The condition on $C$ implies that $C \leq \text{Aut}(G)$ in the first three cases. This proves the required statement in these cases, because any two cyclic subgroups of $\text{Aut}(G) \cong \text{GL}(2, 3)$ of the same order at least 4 are conjugate. In case $(8, 2)$, the condition on $C$ implies that permutation in condition (R2) belongs to $C$. This leaves exactly three possibilities for $C$, and for each of them $C = M^\Pi$, where $M$ is one of the three Singer subgroups of $\text{Aut}(G)$. □

4. Dense S-rings over $E_g \times C_p$: basic sets

Throughout this section, we assume that $p > 3$ and $G = A \times P$, where $A = E_g$ and $P = C_p$. In what follows, $A$ is a dense S-ring over $G$, which means that $A$ and $P$ are $A$-subgroups of $G$. By Lemma 2.1, we have

\[(\tau) \times \text{Aut}(P) = Z(\text{Aut}(G)) \leq \text{Iso}_{\text{cay}}(A),\]

where $\tau \in \text{Aut}(A)$ is the involution taking $a$ to $a^{-1}$.

Let $X$ be a basic set of the S-ring $A$. In view of Lemma 2.3, the projections $X_A \subset A$ and $X_P \subset P$ are basic sets of the S-rings $A_A$ and $A_P$. Therefore

$X_A \times X_P \in S(A)^{\cup}$. 
By Lemmas 3.2 and 2.4 there exist groups $U(X) \leq \text{Aut}(A)$ and $V(X) \leq \text{Aut}(P)$ such that

$$X_A \in \text{Orb}(U, A) \quad \text{and} \quad X_P \in \text{Orb}(V, P),$$

where $U = U(X)$ and $V = V(X)$. For any element $a \in X_a$, each basic set inside $X_A \times X_P$ intersects $\{a\} \times X_P$. Since the group $V \leq \text{Aut}(G)$ acts transitively on the latter set, formula (4) implies that $V$ acts transitively on $S(A)_{X_A \times X_P}$.

**Lemma 4.1.** Let $\Pi_P(X) = \{X(a)\}_{a \in X_A}$, where

$$X(a) = \{x \in X_P : (a, x) \in X\}.$$

Then there exists a group $V_0 \leq V$ such that

$$\Pi_P = \text{Orb}(V_0, X_P).$$

In particular, $\Pi_P$ is an imprimitivity system for the group $V^{X_P}$.

**Proof.** Denote by $V_0$ the subgroup of $V$ leaving $X$ fixed (as a set). From formula (4), it follows that each set $X_a$ is contained in some $Y \in \text{Orb}(V_0, X_P)$. On the other hand, $X(a)$ cannot be smaller than $Y$ by the definition of $V_0$. Thus, $X_a = Y$. \qed

Let us define an equivalence relation $\sim$ on the set $X_A$ by setting $a \sim b$ if and only if $X(a) = X(b)$. In particular, all the elements of $X_A$ are $\sim$-equivalent if and only if $X = X_A \times X_P$. Denote by $\Pi_A = \Pi_A(X)$ the partition of $X_A$ into the classes of the equivalence relation $\sim$. From our definitions and Lemma 4.1, it follows that the mapping

$$f : \Pi_A \to \Pi_P, \quad [a] \mapsto X(a),$$

is a well-defined bijection, where $[a]$ denotes the class of the equivalence relation $\sim$ that contains $a \in X_A$. Moreover,

$$X = \bigcup_{Y \in \Pi_A} Y \times f(Y)$$

**Lemma 4.2.** $\Pi_A$ is a regular partition of $X$ in the sense of Section 3.

**Proof.** It is easily seen that $[a]y = X \cap Ay$ for all $a \in X_A$ and $y \in X(a)$. By Lemma 3.2, this implies that the partition $\Pi_A$ is uniform. Without loss of generality, we may assume that $X_A$ is symmetric. By formula (11) and since $\text{Aut}(P)$ acts transitively on $A_{X_A \times X_P}$, there exists $\sigma \in V$ such that $(X^\tau)^\sigma = X$. In view of equality (7) and Lemma 4.1, this implies that

$$([a] \times X(a))^{\sigma^\tau} = [a]^{-1} \times X(a)^\sigma = [b] \times X(b)$$

for all $a \in X_A$, where the element $b \in X_P$ is defined by the condition $X(a)^\sigma = X(b)$. Thus, $[a]^{-1} = [b]$ and the partition $\Pi_A$ satisfies condition (R1). Furthermore, if $[a] = [a]^{-1}$ for some $a \in X_A$, then $X^\tau = X$ by formula (7), and hence the permutation $[a] \to [a]^{-1}$ is trivial. This shows that $\Pi_A$ satisfies condition (R2). Finally, again by formula (7) we have

$$\sum_{Y \in \Pi_A} \frac{Y \times f(Y)}{\Pi_A} = A \circ (X, X^\tau) = \alpha X_A + \xi$$

for some integer $\alpha \geq 0$ and $\xi \in A$ such that $X_A \circ \xi = 0$. This shows that $\Pi_A$ satisfies condition (R3). \qed
From Lemmas 4.2 and 3.2, it follows that given \( X \in S(A) \) there exists a standard pair \((U, U_0)\) for the set \( X_A \in S(A) \) and regular partition \( \Pi_A \). The following statement is the main result of this section.

**Theorem 4.3.** In the above notation, the set \( X \) is an orbit of the subdirect product \( K = K(X) \) of the groups \( U \) and \( V \) with respect to the homomorphisms

\[
\varphi : U \rightarrow U^{\Pi_A} \quad \text{and} \quad \psi : V \rightarrow V^{\Pi_P},
\]

and the isomorphism \( f : U^{\Pi_A} \rightarrow V^{\Pi_P} \) induced by bijection \( \Phi \). Furthermore, \( K \subseteq \text{Aut}(G) \)

**Proof.** According to our notation, we are in the situation described by the conditions (P1), (P2), and (P3). Moreover, the group \( V^{\Pi_P} \) is cyclic. First, assume that the group \( U^{\Pi_A} \) is also cyclic. Then by Lemma 3.3 for \((M, L) = (U, U_0)\), \( \Pi = \Pi_A \), and \( C = f^{-1}V^{\Pi_P}f \), the standard pair can be chosen so that

\[
U^{\Pi_A} = fU^{\Pi_A}f^{-1},
\]
i.e., \( f \in \text{Iso}(U^{\Pi_A}, V^{\Pi_P}) \). Thus, from Lemma 2.6 and relation (7) it immediately follows that

\[
K \leq \text{Aut}(A) \times \text{Aut}(P) = \text{Aut}(G) \quad \text{and} \quad X \in \text{Orb}(K, G),
\]
as required.

To complete the proof, we show that the group \( U^{\Pi_A} \) must be cyclic. Assume on the contrary that this is not true. Then \( |\Pi_A| = 4 \). Moreover, by statement (1) of Lemma 3.2, the S-ring \( A_A \) is the tensor product of two trivial S-rings and \( |X| = 4 \). In particular, there are two distinct \( A_A \)-groups \( C \) and \( D \), each of order 3 and

\[
X = C^\# \times D^\#.
\]

Note that both \( C \) and \( D \) are also \( A \)-groups, and \( G = C \times (DP) = D \times (CP) \). By statement (1) of Lemma 2.8, this implies that \( Y = X_{DP} \) and \( Z = X_{CP} \) are basic sets of \( A \). It is easily seen that

\[
Y_P = Z_P = X_P \quad \text{and} \quad |\Pi_P(Y)| = |\Pi_P(Z)| = 2 \quad \text{and} \quad \Pi_P(Y) \neq \Pi_P(Z).
\]

By Lemma 4.1 this implies that the transitive cyclic group \( V(X)^{X_P} \) has two distinct imprimitivity systems, each with exactly two blocks, a contradiction. \( \square \)

For distinct basic sets \( X \) and \( Y \) of the S-ring \( A \), the groups \( K(X) \) and \( K(Y) \) are not necessarily equal: even if the standard pairs for \( X_A \) and \( Y_A \) are equal, the subdirect products \( K(X) \) and \( K(Y) \) may correspond to different isomorphisms \( f \). The following statement provide a sufficient condition for \( Y \) to be an orbit of \( K(X) \).

In what follows, we set

\[
\mathcal{G}(A)' = \{ H \in \mathcal{G}(A) : \ G = H \times H' \ \text{for some} \ H' \in \mathcal{G}(A) \}.
\]

**Lemma 4.4.** Let \( A \) be a dense S-ring over \( G \), \( X, Y \in S(A) \), and \( K = K(X) \). Then \( Y \in \text{Orb}(K, G) \) if at least one of the following conditions is satisfied:

1. \( Y = X_H \) for some \( H \in \mathcal{G}(A)' \),
2. \( Y = X_\sigma \) for some \( \sigma \in Z(\text{Aut}(G)) \),
3. \( X_A = Y_A \).
Proof. Under condition (1), the $A$-groups $H$ and $H'$ are $K$-invariant. Therefore, it is easily seen that

$$X \in \text{Orb}(K, G) \Rightarrow X_H, X_{H'} \in \text{Orb}(K, G),$$

and we are done. Now assume that condition (2) is satisfied. Since the automorphism $\sigma$ centralizes $K \leq \text{Aut}(G)$, we conclude that

$$Y = X^\sigma \in \text{Orb}(K^\sigma, G) = \text{Orb}(K, G),$$

as required. To complete the proof, it suffices to note that condition (3) is a consequence of conditions (1) and (2). \hfill \Box

5. Dense S-rings over $E_9 \times C_p$ are cyclotomic

The main result of the present section is given in the following theorem. Along the proof, we freely use the notation introduced in Section 4.

Theorem 5.1. Every dense S-ring over $E_9 \times C_p$ is cyclotomic.

Proof. Let $A$ be a dense S-ring over the group $G = A \times P$ with $A = E_9$ and $P = C_p$ for a prime $p > 3$ (for $p = 2, 3$, the required statement can be verified by enumeration of the S-rings over small groups [13]). We divide the proof into three separate cases depending on which statement of Theorem 4.3 holds for the S-ring $A$.

Case 1: $A_A = \text{Cyc}(M, \mathbb{F})$, where $\mathbb{F} = \mathbb{GF}_9$ and $1 < M \leq \mathbb{F}^\times$. In this case, $M$ is a cyclic group of order $m \in \{2, 4, 8\}$. Fix a basic set

$$X \in \mathcal{S}(A)_{G \setminus (A \cup P)}; |X_A| = m.$$

Denote by $K$ the group $K(X)$ defined in Theorem 4.3. First assume that $|M| \neq 4$.

Then $rk(A_A) = 2$ or $\mathcal{G}(A_A)$ contains all subgroups of $A$. It easily follows that

$$S(A) = \{(X_H)\sigma : H \in \mathcal{G}(A), \sigma \in \text{Z}(\text{Aut}(G))\}.$$  

By Lemma 4.4, this implies that $A = \text{Cyc}(K, G)$.

Now let $|M| = 4$. In this case, for any $Y \in S(A)$, the set $Y_A$ is either trivial, or is equal to $X_A$ or to $A^\# \setminus X_A$. By Lemma 4.4, it suffices to verify that in the last case, $X$ and $Y$ are orbits of a certain group $K' \leq \text{Aut}(G)$ (except for one case, $K'$ will be equal to $K$). To this end, set

$$k_X = |\Pi_A(X)| \quad \text{and} \quad k_Y = |\Pi_A(Y)|.$$  

Each of the numbers $k_X$ and $k_Y$ divides $|X| = |Y| = 4$, and hence is equal to 1, 2, or 4. Let us analyze all these possibilities. It is convenient to denote the eight nontrivial elements of the group $A$ by $a_i^{\pm 1}, i = 1, 2, 3, 4$, so that the orbits $X_A$ and $Y_A$ of the group $M$ are of the form:

$$X_A = \{a_1, a_3, a_1^{-1}, a_3^{-1}\} \quad \text{and} \quad Y_A = \{a_2, a_4, a_2^{-1}, a_4^{-1}\}.$$  

Without loss of generality, we may assume that $a_2 = a_1a_3$ and $a_4 = a_1a_3^{-1}$, and also

$$f_X([a_1]) = f_Y([a_2]) \quad \text{and hence} \quad f_X([a_1]) = f_Y([a_2]),$$

where $f_X$ and $f_Y$ are the bijections defined by formula (8) for $X$ and $Y$, respectively.
Claim 1: \( \{k_X, k_Y\} \neq \{4, 2\} \) and \( \{k_X, k_Y\} \neq \{4, 1\} \). Assume, for instance, that \( k_X = 4 \). Then a straightforward calculation shows that
\[
(X_A X) \cap (Y_A \times X_P) = a_2 \ X(a_1, a_3) \cup a_4 \ X(a_1, a_3^{-1}) \cup a_2^{-1} \ X(a_1^{-1}, a_3^{-1}) \cup a_4^{-1} \ X(a_1^{-1}, a_3),
\]
where \( X(a_i, a_j) = X(a_i) \cup X(a_j) \) for all \( i, j \). Note that the left-hand side of (11) is an \( \mathcal{A} \)-set, because \( X_A, X, Y_A, \) and \( X_P \) are basic sets of \( \mathcal{A} \). Furthermore, assumption (10) implies that it contains \( a_2 X(a_1) \) and hence intersects \( Y \) nontrivially. Thus,
\[
Y \subseteq (X_A X) \cap (Y_A \times X_P).
\]
On the other hand, from the form of the right-hand side of (11), it follows that \( k_Y \neq 1 \), and if \( k_Y = 2 \), then the cyclic group \( V(Y) = V(X) \) has two different imprimitivity systems, each consisting two blocks (Lemma 4.1). Since this is impossible, the claim is proved.

Claim 2: if \( \{k_X, k_Y\} = \{1, 2\} \), then \( X, Y \in \text{Orb}(K', G) \) for some group \( K' \) such that \( K < K' \leq \text{Aut}(G) \). Without loss of generality, we may assume that \( k_X = 1 \) and \( k_Y = 2 \). Note that the Frobenius automorphism \( k' \) of the field \( F \) is an automorphism of the S-ring \( \mathcal{A}_A \). It follows that \( X_A, Y_A \in \text{Orb}(U', A) \), where \( U' = (U, k') \). Set
\[
K' = U' \prod_{f} V,
\]
where \( \varphi' : U' \to U'/U'_0 \) and \( U'_0 = \langle M_0, k' \rangle \) with \( M_0 \) being the subgroup of \( M \) of order 2. Then by Lemma 2.6 applied to the set \( X \) and trivial bijection \( f : \{X_A\} \to \{X_P\} \), and to the set \( Y \) with the bijection \( f_Y \), we conclude that \( X \) and \( Y \) are orbits of the group \( K' \leq \text{Aut}(G) \).

By Claims 1 and 2, to complete the proof of the Case 1, we may assume that \( k_X = k_Y := k \). If now \( k = 1 \) or \( 2 \), then the bijection (10) is unique and hence \( Y \in \text{Orb}(K, G) \). Assume that \( k = 4 \). In this case, the groups \( K(X) \) and \( K(Y) \) may correspond to subdirect products with different bijections \( f_X \) and \( f_Y \). Namely, there are two possibilities:
\[
\text{either } f_X([a_3]) = f_Y([a_4]) \text{ or } f_X([a_3]) = f_Y([a_3^{-1}]).
\]
However, the first case is impossible, because the \( \mathcal{A} \)-set
\[
(XY^{-1}) \cap A^\# = \{a_2^\pm 1, a_3^\pm 1\}
\]
intersects each of the two different basic sets \( X_A \) and \( Y_A \) nontrivially. Since in the last case, \( Y \in \text{Orb}(K, G) \), we are done.

Case 2: \( \mathcal{A}_A = \mathcal{A}_C \otimes \mathcal{A}_D \), where \( C \) and \( D \) are subgroups of \( A \) such that \( A = C \times D \) and \( |C| = |D| = 3 \). First assume that one of the S-rings \( \mathcal{A}_C \) or \( \mathcal{A}_D \) is the group ring, say the first one. Then by statement (2) of Lemma 2.3 for \( G_1 = C \) and \( G_2 = D \mathcal{P} \), we have
\[
\mathcal{A} = ZC \otimes \mathcal{A}_D \mathcal{P}.
\]
Since $DP$ is a cyclic group and $D, P$ are $A_{DP}$-groups, the classification of S-rings over a cyclic group $Cpq$ with primes $p$ and $q$ implies that the S-ring $A_{DP}$ is cyclotomic (see [6]). Since also the S-ring $\mathbb{Z}C$ is cyclotomic, formula (13) shows that so is the S-ring $A$.

To complete this case assume that both $A_C$ and $A_D$ are S-rings of rank 2. Take any $X \in S(A) \setminus A$ such that $X_A = C^# \times D^#$. Then one can easily verify that formula (8) holds. By Lemma 4.4, this implies that $A = \text{Cyc}(K,G)$ with $K = K(X)$.

Case 3: $A_A = A_C \wr A_{A/C}$, where $C \leq A$ is a group of order 3. Depending on whether $A_{A/C}$ is of rank 2 or 3, the set $S(A) \setminus A$ consists of one set of cardinality 6 or two sets of cardinalities 3, respectively. Fix a basic set

$$X \in S(A) \setminus A \times P^#.$$  

First, assume that $A_C$ is of rank 3. Since the number $|\Pi_A(X)|$ divides 6, there exists a standard pair $(U, U_0)$ for $X_A$ and $\Pi_A(X)$ such that the set $\text{Orb}(U_0, C)$ consists of singletons. Assume that the group $K = K(X)$ is associated with this pair. Then obviously

$$Y \in \text{Orb}(K,G) \quad \text{for all} \quad Y \in S(A) \times P^#.$$  

Next, we observe that $X_A \times X_P$ is the union of $X$ and $X^{-1}$, where depending on whether $A_{A/C}$ is of rank 2 or 3, we have

$$X = X^{-1} \quad \text{and} \quad |X_A| = 6 \quad \text{or} \quad X \cap X^{-1} = \emptyset \quad \text{and} \quad |X_A| = |X_A^{-1}| = 3,$$

respectively. This easily implies that any $Y \in S(A)$ contained in $(A \setminus C) \times P$ is of the form $X^\sigma$ for some $\sigma \in \text{Z}(\text{Aut}(G))$. Thus, again $Y \in \text{Orb}(K,G)$ by Lemma 4.4 and hence $A = \text{Cyc}(K,G)$.

Let now $A_C$ and $A_{A/C}$ be of rank 2 and 3, respectively. Then

$$A^\perp, P^\perp \in \mathcal{S}(\tilde{A})$$

by statement (1) of Lemma 2.5. It follows that $\tilde{A}$ is a dense S-ring over the group $\tilde{G}$. The statements (2) and (3) of that lemma imply that the restriction of $\tilde{A}$ to the group $A^\perp$ is the wreath product of the S-rings of rank 3 and rank 2. By the previous paragraph, we conclude that $\tilde{A}$ is a cyclotomic S-ring over $\tilde{G}$. By statement (3) of Lemma 2.5, this proves that $A$ is a cyclotomic S-ring over $G$.

In the remaining case, both $A_C$ and $A_{A/C}$ are of rank 2. It follows that $A_A$ is of rank 3 and

$$S^\#(A) = \{C^#, A \setminus C\}.$$  

Fix arbitrary basic sets $X, Y \in S(A)$ such that

$$X_A = A \setminus C, \quad Y_A = C^#, \quad X_P = Y_P \neq 1_P.$$  

By Lemma 4.4 it suffices to verify that $X$ and $Y$ are orbits of a certain group $K' \leq \text{Aut}(G)$. To this end, we define the numbers $k_X$ and $k_Y$ by formula (12). Then from Lemmas 1.1 and 1.2 it follows that

$$k_X \in \{1, 2, 3, 6\} \quad \text{and} \quad k_Y \in \{1, 2\}.$$  

As in Case 1, not each combination for the pair $(k_X, k_Y)$ is possible.
Claim 3: \((k_X, k_Y) \neq (6, 1)\) and \((k_X, k_Y) \neq (3, 2)\). Let us consider the first case. Denote by \(r\) the cardinality of \(X_P\). Then
\[
|X| = r \quad \text{and} \quad |Y| = 2r.
\]
The set \(G'(A \cup P)\) is partitioned into basic sets \(X^\sigma\) and \(Y^\sigma\), where \(\sigma \in Z(\Aut(G))\). Since \(|X^\sigma| = |X|, |Y^\sigma| = |Y|,\) and \(|X_P| = |Y_P| = (p-1)/r\), we obtain
\[
|S(A)| = |S(A_A)| + |S(A_P)| - 1 + \frac{7(p-1)}{r}.
\]
Next, let \(\pi : G \to G/C\) be the natural epimorphism. Since \(k_X = 6\), each of the sets \(\pi(X^\sigma)\) is of cardinality \(r\). It follows that
\[
|S(A_{G/C})| = |S(A_A/C)| + |S(A_{C/P}/C)| - 1 + \frac{2(p-1)}{r}.
\]
For the S-ring \(\hat{A}\) dual to \(A\), relation (13) holds. Since the S-rings \(A_A\) and \(A_{G/A}\) as well as \(A_P\) and \(A_{G/P}\) are isomorphic, statement (2) of Lemma 2.5 and equality (16) yield
\[
|S(\hat{A}_{G'A\setminus(A\cup P)}| = \frac{7(p-1)}{r}.
\]
Furthermore, \(C^\perp \cong C_{3p}\) is an \(\hat{A}\)-group and the restriction of \(\hat{A}\) to this group is isomorphic to the S-ring \(A_{G/C}\). Therefore from equality (17), it follows that
\[
|S(\hat{A}_{C^\perp\setminus(A\cup P)}| = \frac{2(p-1)}{r}.
\]
Now using equalities (18) and (19), we conclude that there are exactly \(5(p-1)/r\) basic sets of \(\hat{A}\) outside \(A^\perp, P^\perp,\) and \(C^\perp\). All of these basic sets are obtained from any one of them by applying an automorphism from \(Z(\Aut(\hat{G}))\). Therefore they have the same size, say \(k\). This implies that
\[
|\hat{G} \setminus (A^\perp \cup P^\perp \cup C^\perp)| = 6(p-1).
\]
On the other hand, the S-rings \(A_P = \Cyc(V, P)\) and \(\hat{A}_{P^\perp}\) are isomorphic, where \(V = V(X) = V(Y)\) is a subgroup of \(\Aut(P)\) of order \(r\). Therefore the nonidentity basic sets of the latter S-ring are of cardinality \(r\). It follows that \(r\) divides \(k\), which contradicts equality (20).

The proof of the Claim 3 for the case \((k_X, k_Y) = (3, 2)\) differs from the previous argument only in the values of the parameters. Namely, here \(|X| = 2r\) and \(|Y| = r\). Therefore, the last summands on the right-hand sides in formulas (16) and (18) are equal to \(5(p-1)/r\), and those in formulas (17) and (19) are \((p-1)/r\). Thus, equality (20) leads to the equality \(4k/r = 6\), which is also impossible, because \(r\) divides \(k\). The claim is proved. \(\square\)

Let us return to the remaining part of Case 3. By Claim 3 and formula (15), there are six possibilities for the pair \((k_X, k_Y)\). For each of them, we define a group \(K' \leq \Aut(G)\) by formula (12), where the standard pair \((U', U'_0)\) is given in the second and third columns of Table 1 below; in the fourth column contains the sizes of the \(U'_0\)-orbits.

A straightforward check shows that in each case, \(Y \in \Orb(K', G)\), as required. \(\square\)
We deduce Theorem 1.1 in the end of the section from the theorem below giving a complete description of all S-rings over the group $E_9 \times C_p$.

**Theorem 6.1.** Let $A$ be an S-ring over a group $G = E_9 \times C_p$, where $p > 3$ is a prime. Then one of the following statements holds:

1. $A$ is trivial or cyclotomic,
2. $A$ is the tensor product of a trivial S-ring and an S-ring over $C_3$,
3. $A$ is a proper S-wreath product with $|S| \leq 3$.

**Proof.** If the S-ring $A$ is dense, then we are done with statement (1) by Theorem 5.1. Assume that $A$ is not dense. Then $A$ or $P$ is not an $A$-subgroup of $G$. By the duality (see Lemma 2.5), we may assume that $A \not\in G(A)$. Denote by $C$ the maximal $A$-subgroup of the group $A$. Clearly, this group is trivial or of order 3. The lemma below is a special case of [4, Lemma 6.2].

**Lemma 6.2.** In the above notation, one of the following statements holds:

1. $A = A_C \wr A_{G/C}$ and also $rk(A_{G/C}) = 2$,
2. $A$ is the $U/L$-wreath product, where $P \leq L < G$ and $U = CL$.

Without loss of generality, we may assume that $A$ is as in statement (2) of Lemma 6.2 otherwise this lemma implies that either $A$ is trivial (if $C = 1$) and statement (1) of Theorem 6.1 holds, or statement (3) of this theorem holds with $S = C/C$. Furthermore, if $U < G$ then $|U/L| \leq 3$ and statement (3) of Theorem 6.1 holds. Thus, we may also assume that $U = G$. Then $C \not\leq L$, for otherwise $G = L$, a contradiction. Since $|C| = 3$, it follows that $C \cap L = e$. Thus,

$$G = C \times L \quad \text{and} \quad |L| = 3p.$$ 

Note that the S-ring $A_L$ is circulant. Moreover, since $A$ is not an $A$-group, the subgroup of $L$ of order 3 is not an $A_L$-group. According to [4], this implies that

$$rk(A_L) = 2 \quad \text{or} \quad A_L = A_P \wr A_{L/P}.$$ 

Assume that $A_C = ZC$. Then $A = A_C \otimes A_L$ by statement 2 of Lemma 6.2. In particular, statement (2) of Theorem 6.1 holds, whenever $rk(A_L) = 2$. On the other hand, if $A_L$ is not trivial, then $A$ is obviously the $CP/P$-wreath product and statement (3) of Theorem 6.1 holds. Thus, we may assume that

$$rk(A_C) = 2.$$ 

Denote by $L_0$ the trivial subgroup of $L$ if $rk(A_L) = 2$, and the group $P$ otherwise. In view of (21), we have $L_0 \in \mathcal{G}(A)$. In particular, $L \setminus L_0$ is an $A$-set.

**Table 1. Standard groups for Case 3**

| $(k_X, k_Y)$ | $U'$ | $U'_0$ | $Orb(U'_0, A)$ |
|--------------|------|--------|----------------|
| (1, 1)       | $C_6$ | $C_6$  | [1, 2, 6]       |
| (1, 2)       | $D_{12}$ | Sym(3) | [1, 1, 1, 6]   |
| (2, 1)       | $D_{12}$ | Sym(3) | [1, 2, 3, 3]   |
| (2, 2)       | $C_6$ | $C_3$  | [1, 3, 3, 1, 1] |
| (3, 1)       | $C_6$ | $C_2$  | [1, 2, 2, 2]   |
| (6, 2)       | $C_6$ | 1      | [1, ..., 1]    |
Lemma 6.3. If $X \in S(A)$ is contained in $C^\# \times (L \setminus L_0)$, then $X = X_C \times X_L$.

Proof. For all $X \in S(A)$ contained in $C^\# \times L^\#$, we have

$$|X_L| \leq |X| \leq |C^\#| |X_L| = 2|X_L|.$$ 

It follows that $X_C \times X_L$ is the union of two basic sets $X$ and $X'$ of the same cardinality. Now if $X = X'$, then $X = X_C \times X_L$ and we are done. In the remaining case,

$$|X| = \frac{|X_C| \cdot |X_L|}{2} = \frac{2|X_L|}{2} = |X_L|. \quad (22)$$

Assume that $X_L \subseteq L \setminus L_0$. From the definition of the group $L_0$, it follows that $|L \setminus L_0| \leq 3p - 1$. Therefore, equality (22) yields

$$|X| = |X_L| \leq |L \setminus L_0| \leq 3p - 1. \quad (23)$$

On the other hand, if $L_0 = 1$, then $\text{rk}(A_L) = 2$ and hence $X_L = L \setminus L_0$. Furthermore, if $L_0 = P$, then $A_L = A_P \mid A_L/P$ and hence $X_L = P$ or $X = L \setminus P$. However, the first case is impossible, because by Lemma 6.3 for $H = P$ the number $|X|$ must be even. Thus, in any case, $X_L = L \setminus L_0$ and hence

$$X \cup X' = C^\# \times (L \setminus L_0).$$

The the right-hand side includes the set $C_0 := A \setminus C$ of cardinality 6. Therefore, at least one of $X$ or $X'$, say $X$, contains three elements from $C_0$. According to [4, Lemma 6.1], this implies that

$$|X| \geq |(X \cap C_0)P| \geq 3p,$$

which contradicts inequality (23). \hfill \Box

From Lemma 6.3 it follows that if $\text{rk}(A_L) = 2$, then $A = A_C \otimes A_L$ and we are done with statement (2) of Theorem 6.1. To complete the proof, in view of (22) we may assume that $X_L = L \setminus L_0$ and hence

$$X \cup X' = C^\# \times (L \setminus L_0).$$

The right-hand side includes the set $C_0 := A \setminus C$ of cardinality 6. Therefore, at least one of $X$ or $X'$, say $X$, contains three elements from $C_0$. According to [4, Lemma 6.1], this implies that

$$|X| \geq |(X \cap C_0)P| \geq 3p,$$

which contradicts inequality (23). \hfill \Box

From Lemma 6.3 it follows that if $\text{rk}(A_L) = 2$, then $A = A_C \otimes A_L$ and we are done with statement (2) of Theorem 6.1. To complete the proof, in view of (22) we may assume that $X_L = L \setminus L_0$ and hence

$$X \cup X' = C^\# \times (L \setminus L_0).$$

The right-hand side includes the set $C_0 := A \setminus C$ of cardinality 6. Therefore, at least one of $X$ or $X'$, say $X$, contains three elements from $C_0$. According to [4, Lemma 6.1], this implies that

$$|X| \geq |(X \cap C_0)P| \geq 3p,$$

which contradicts inequality (23). \hfill \Box

Proof of the Theorem 1.1. For $p \leq 3$, the statement follows from the computational results obtained in [13, p. 498]. Let $p > 3$ and $A$ an $S$-ring over $G$. Then by Theorem 6.1 this ring is obviously schurian if statement (1) of this theorem holds. In case of statement (2), the $S$-ring $A$ being the tensor product of two schurian $S$-rings is also schurian. To complete the proof, we may assume that $A$ is a proper $S$-wreath product with $|S| \leq 3$. Then the $S$-ring $A_S$ is either trivial or a group ring. Thus, the group $\text{Aut}(A_S)$ is permutation isomorphic to $S_{\text{right}}$, $\text{Sym}(3)$, or $\text{Alt}(3)$. In any case, according to a criterion of schurity of a generalized wreath product [7, Corollary 10.3], the $S$-ring $A$ is schurian. \hfill \Box
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