Research Article

Related Fixed Point Theorems in Partially Ordered $b$-Metric Spaces and Applications to Integral Equations

Youssef Errai, El Miloudi Marhrani, and Mohamed Aamri

Laboratory of Algebra, Analysis and Applications, Hassan II University, Faculty of Sciences Ben M’Sik, Avenue Commandant Harti B.P 7955, Sidi Othmane, Casablanca, Morocco

Correspondence should be addressed to Youssef Errai; yousseferrai1@gmail.com

Received 28 November 2020; Accepted 23 February 2021; Published 16 March 2021

Academic Editor: Paul Eloe

Copyright © 2021 Youssef Errai et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this research paper, we have set some related fixed point results for generalized weakly contractive mappings defined in partially ordered complete $b$-metric spaces. Our results are an extension of previous authors who have already worked on fixed point theory in $b$-metric spaces. We state some examples and one sample of the application of the obtained results in integral equations, which support our results.

1. Introduction

The concept of $b$-metric space has been dealt with by distinctly different authors since it first appeared and became largely used. Bakhtin in [1] was the first who introduced this concept, and later on, Czerwik in [2, 3] invested it in the convergence of measurable functions. Accordingly, several interesting results about the existence of fixed points have been obtained concerning both single-valued and multivalued operators in $b$-metric spaces (see, e.g., [4, 5]). In the same way, Hussain and Shah [6] conducted a research and then came up with results from KKM mappings in cone $b$-metric spaces. Likewise, Roshan et al. [7] used the notion of almost generalized contractive mappings in ordered complete $b$-metric spaces and set some fixed and common fixed point results. Regarding partially ordered metric spaces, Ran and Reurings (see [8]) had first set up their assumptions before Nieto and Rodríguez-López used them (see [9, 10]). Afterwards, many authors presented several interesting and significant results in such spaces (see [4, 7, 11–19]).

The purpose of this research paper is to prove some fixed point theorems for generalized contractive conditions for four mappings in complete $b$-metric spaces.

Our work goes through the following steps. First, we have demonstrated a two $b$-metric space theorem with four mappings. Second, we have stated a relevant example which backs up the mentioned theorem. Third, we have given three corollaries related to the theorem. Besides, an example with only two related corollaries is provided for the second theorem. To sum up, we conclude the manuscript by an application to solve a system of integral equations.

Throughout this paper, $\mathbb{R}$ and $\mathbb{R}^+$ denote the sets of all real numbers and nonnegative real numbers, respectively.

Consistent with [3, 4], the following definitions and outcomes are going to be vital and required in the ending.

Definition 1 (see [3]). Given a nonempty set $X$. A function $d : X \times X \longrightarrow \mathbb{R}^+$ is called $b$-metric if there is a real number $s \in [1,\infty)$ such that for all $x, y, z \in X$, the following conditions hold:
\(d(x, y) = 0\) if and only if \(x = y\);
(ii) \(d(x, y) = d(y, x)\);
(iii) \(d(x, z) \leq s[d(x, y) + d(y, z)]\).

The pair \((X, d)\) is called a \(b\)-metric space.

**Definition 2** (see [20]). Let \((X, d)\) be a \(b\)-metric space. Then, a sequence \(\{x_n\}\) in \(X\) is
(a) convergent if there exists \(x \in X\) such that \(d(x_n, x) \to 0\) as \(n \to \infty\). In this case, we write \(\lim_{n \to \infty} x_n = x\).
(b) Cauchy sequence if \(d(x_n, x_m) \to 0\) as \(n, m \to \infty\).

**Definition 3.** Let \(X\) be a nonempty set. Then, \((X, d, \delta, \leq)\) is called a partially ordered two \(b\)-metric space if \(d\) and \(\delta\) are \(b\)-metrics on the partially ordered set \((X, \leq)\).

A subset \(Y\) of a partially ordered set \(X\) is said to be well ordered if every two elements of \(Y\) are comparable.

**Definition 4** (see [4]). Let \((X, \leq)\) be a partially ordered set. A mapping \(f\) on \(X\) is called dominating (resp. dominated) if \(x \leq f(x)\) (resp. \(f(x) \leq x\)) for each \(x \in X\).

**Definition 5.** Let \(\{x_n\}\) be a sequence in a \(b\)-metric space \((X, d, \delta)\), \(g, h : X \to X\), and \(x \in X\).

(i) \(x\) is said to be a coincidence point of pair \(\{g, h\}\) if \(gx = hx\)
(ii) \(\{g, h\}\) is said to be compatible if \(d(ghx_n, hgx_n) \to 0\) as \(n \to \infty\)
(iii) \(\{g, h\}\) is said to be weakly compatible if \(ghx = hgx\), where \(gx = hx\)

**Remark 6** (see [20]). In a \(b\)-metric space \((X, d)\), the following assertions hold:
(R1) A convergent sequence has a unique limit
(R2) Each convergent sequence is a Cauchy sequence
(R3) In general, a \(b\)-metric is not continuous
(R4) In general, a \(b\)-metric does not induce a topology on \(X\).

The fact in (R3) requires the following Lemma concerning \(b\)-convergent sequences to prove our findings:

**Lemma 7** (see [4]). Let \((X, d)\) be a \(b\)-metric space with \(s \in [1, \infty)\) and suppose that \(\{x_n\}\) and \(\{y_n\}\) are convergent to \(x, y\), respectively. Then, we have
\[
\frac{1}{s^2} d(x, y) \leq \liminf_{n \to \infty} d(x_n, y_n) \leq \limsup_{n \to \infty} d(x_n, y_n) \leq s^2 d(x, y).
\]

(1)
In particular, if \(x = y\), then we have \(\lim_{n \to \infty} d(x_n, y_n) = 0\). Moreover, for each \(z \in X\), we have
\[
\frac{1}{s} d(x, z) \leq \liminf_{n \to \infty} d(x_n, z) \leq \limsup_{n \to \infty} d(x_n, z) \leq s d(x, z).
\]

(2)

2. Main Results
Throughout this paper, let \(f, g, P\), and \(T\) be four self-maps on \(X\), and \(d\) and \(\delta\) are two \(b\)-metrics with constant \(s\) and \(r\), respectively. Set

\[
M_d(x, y) = \max \left\{ d(Px, Ty), d(fx, Px), d(gy, Ty), \frac{[d(Px, gy) + d(fx, Ty)]}{2s} \right\},
\]
\[
M_\delta(x, y) = \max \left\{ \delta(Px, Ty), \delta(fx, Px), \delta(gy, Ty), \frac{[\delta(Px, gy) + \delta(fx, Ty)]}{2r} \right\},
\]
\[
N_d(x, y) = \max \left\{ d(x, y), d(fx, x), d(gy, y), \frac{[d(x, gy) + d(fx, y)]}{2s} \right\},
\]
\[
N_\delta(x, y) = \max \left\{ \delta(x, y), \delta(fx, x), \delta(gy, y), \frac{[\delta(x, gy) + \delta(fx, y)]}{2r} \right\},
\]
\[
Q_d(x, y) = a_1 d(Px, Ty) + a_2 d(fx, Px) + a_3 d(gy, Ty) + a_4 d(Px, gy) + a_5 d(fx, Ty),
\]
\[
R_d(x, y) = a_1 d(x, y) + a_2 d(fx, x) + a_3 d(gy, y) + a_4 d(x, gy) + a_5 d(fx, y),
\]

(3)
with
\[ 0 < a_3, a_4, 0 \leq a_1, a_2, a_5, 0 < a_2 + a_3 \leq 1, \quad a_1 + a_2 + a_3 + 2a_4 \leq 1, \quad a_1 + a_4 + a_5 \leq 1. \] (4)

Let,
\[ \Psi = \{ \psi : [0, +\infty) \rightarrow [0, +\infty) \text{ is continuous monotone, nondecreasing and } \psi(t) = 0 \Leftrightarrow t = 0 \} , \]
\[ \Phi = \{ \varphi : [0, +\infty) \rightarrow [0, +\infty) \text{ is lower semicontinuous and } \varphi(t) = 0 \Leftrightarrow t = 0 \} . \] (5)

Finally, we consider the following assumptions:

**Assumption 8.** Let \( \{x_n\} \) and \( \{y_n\} \) be two sequences such that
(i) \( \{x_n\} \) is nonincreasing
(ii) \( \text{for all } n, y_n \leq x_n \text{ and } y_n \rightarrow u \Rightarrow u \leq x_n \text{ for all } n. \)

**Assumption 9.** Let \( f, g, P, \text{ and } T \) be four self-maps on \( X \) such that either condition
(a) \( \{g, T\} \text{ is weakly compatible, } \{f, P\} \text{ is compatible and } f \text{ or } P \text{ is continuous, or} \)
(b) \( \{f, P\} \text{ is weakly compatible, } \{g, T\} \text{ is compatible and } g \text{ or } T \text{ is continuous.} \)

Our main result is the following theorem:

**Theorem 10.** We consider four self-mappings \( f, g, P, \text{ and } T \) in ordered complete two \( b \)-metric space \( (X, d, \delta, \leq) \) that fulfill the following conditions:
(i) \( \{f, g\} \) is dominated and \( \{P, T\} \) is dominating
(ii) \( fX \subseteq TX \) and \( gX \subseteq PX \)
(iii) \( \psi \in \Psi, \varphi \in \Phi \) and for every two comparable elements \( x, y \in X \), we have
\[
\begin{align*}
\psi(sd(fx, gy)) & \leq \psi(M_d(x, y)) - \varphi(M_d(x, y)), \\
\psi(r \delta(fx, gy)) & \leq \psi(M_d(x, y)) - \varphi(M_d(x, y)).
\end{align*}
\] (6)
(iv) Assumptions 8 and 9 are satisfied
Then, \( f, g, P, \text{ and } T \) have a unique common fixed point in \( X \).

**Proof.** Let \( x_0 \) be an arbitrary point in \( X \). By condition (ii), we can define inductively two sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) as follows:
\[ y_{2n+1} = fx_{2n} = Tx_{2n+1}, \quad y_{2n+2} = gx_{2n+1} = Px_{2n+2}, \quad n = 0, 1, 2, \ldots. \] (7)

We have
\[ x_{2n+1} \leq Tx_{2n+1} = fx_{2n} \leq x_{2n} \text{ and } x_{2n} \leq Px_{2n} = gx_{2n-1} \leq x_{2n-1}. \] (8)

Thus, \( x_{n+1} \leq x_n \) for all \( n \geq 0 \), and
\[
\begin{align*}
M_g(x_{2n}, x_{2n+1}) & = \max \{ \delta(y_{2n}, y_{2n+1}), \delta(y_{2n+1}, y_{2n+2}) \}, \\
M_d(x_{2n}, x_{2n+1}) & = \max \{ d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}) \}.
\end{align*}
\] (9)

**Step 1.** If \( y_{2k+1} = y_{2k} \) for some \( k \), then \( \{y_n\} \) is constant. Indeed, we have
\[
\begin{align*}
\psi(\delta(y_{2k+1}, y_{2k+2})) & \leq \psi(M_d(x_{2k}, x_{2k+1})) - \varphi(M_d(x_{2k}, x_{2k+1})), \\
\psi(d(y_{2k+1}, y_{2k+2})) & \leq \psi(M_g(x_{2k}, x_{2k+1})) - \varphi(M_g(x_{2k}, x_{2k+1})).
\end{align*}
\] (10)

which implies
\[
\begin{align*}
\delta(y_{2k+1}, y_{2k+2}) & \leq M_d(x_{2k}, x_{2k+1}) = d(y_{2k+1}, y_{2k+2}), \\
d(y_{2k+1}, y_{2k+2}) & \leq M_g(x_{2k}, x_{2k+1}) = \delta(y_{2k+1}, y_{2k+2}).
\end{align*}
\] (11)

If we insert in (10), we obtain \( \varphi(d(y_{2k+1}, y_{2k+2})) = 0 \), and then \( \{y_n\} \) is a constant sequence. Its value is a common fixed point of \( f, g, P, \text{ and } T \).

In the following, we can assume that \( d(y_{2n}, y_{2n+1}) > 0 \) for each \( n \).

**Step 2.** The sequences \( (d(y_{2n+1}, y_{2n})) \) and \( (\delta(y_{2n+1}, y_{2n})) \) are monotone nonincreasing. Indeed, since \( x_{2n} \) and \( x_{2n+1} \) are comparable, we obtain
\[
\psi(d(y_{2n+1}, y_{2n+2})) \leq \psi(s^4d(y_{2n+1}, y_{2n+2})) = \psi(s^4d(fx_{2n+1}, gx_{2n+1})) \\
\leq \psi(M_b(x_{2n}, x_{2n+1})) - \varphi(M_b(x_{2n}, x_{2n+1})) \\
\leq \psi(M_b(x_{2n}, x_{2n+1})).
\]

Consequently, if we permute \(d\) and \(\delta\), we obtain

\[
\left( \begin{array}{c}
\delta(y_{2n+1}, y_{2n+2}) \\
d(y_{2n+1}, y_{2n+2})
\end{array} \right) \leq M_b(x_{2n}, x_{2n+1}) = \delta(y_{2n+1}, y_{2n+2}).
\]

If \(\delta(y_{2n+1}, y_{2n+2}) > \delta(y_{2n}, y_{2n+1})\) for some \(n\), we obtain

\[
d(y_{2n+1}, y_{2n+2}) \leq M_b(x_{2n}, x_{2n+1}) = \delta(y_{2n+1}, y_{2n+2}).
\]

We have \(d(y_{2n+1}, y_{2n+2}) \leq d(y_{2n+1}, y_{2n})\). Otherwise, we obtain \(M_d(x_{2n}, x_{2n+1}) = d(y_{2n+2}, y_{2n+1}),\) and consequently,

\[
\psi(d(y_{2n+1}, y_{2n+2})) \leq \psi(\delta(y_{2n+1}, y_{2n+2})) \\
\leq \psi(d(y_{2n+2}, y_{2n+1})) = \psi(d(y_{2n+1}, y_{2n+2})),
\]

which gives \(\varphi(d(y_{2n+2}, y_{2n+1})) = 0\), which contradicts our hypothesis. Consequently,

\[
\left( \begin{array}{c}
\delta(y_{2n+1}, y_{2n+2}) \\
d(y_{2n+1}, y_{2n+2})
\end{array} \right) \leq M_b(x_{2n}, x_{2n+1}) = \delta(y_{2n+1}, y_{2n+1}).
\]

Taking account of (6), we obtain

\[
\psi(\delta(y_{2n+1}, y_{2n+2})) \leq \psi(d(y_{2n+1}, y_{2n})) = \psi(d(y_{2n+2}, y_{2n})),
\]

which gives

\[
\delta(y_{2n+1}, y_{2n+2}) \leq d(y_{2n+1}, y_{2n}).
\]

On the other hand, the inequality

\[
\psi(d(y_{2n+1}, y_{2n})) \leq \psi(M_b(x_{2n}, x_{2n-1})) = \psi(M_b(x_{2n}, x_{2n-1})),
\]

gives

\[
d(y_{2n+1}, y_{2n}) \leq M_b(x_{2n}, x_{2n-1}) = \delta(y_{2n+1}, y_{2n}) < \delta(y_{2n+1}, y_{2n+2}),
\]

which contradicts (18). Thus, \(\{d(y_{2n+1}, y_{2n+2})\}\) and \(\{\delta(y_{2n+1}, y_{2n+2})\}\) are monotone nonincreasing sequences. It follows that there exists a nonnegative real number \(c\) such that

\[
\lim_{n} d(y_{2n+1}, y_{2n+2}) = \lim_{n} \delta(y_{2n+1}, y_{2n+2}) = c.
\]

Using (10), we deduce that \(\psi(c) \leq \psi(c) - \varphi(c)\), and consequently, \(c = 0\).

**Step 3.** \(\{y_k\}\) is a Cauchy sequence.

Assume that there exist \(\epsilon, \tau > 0\) for which we can find subsequences \(\{y_{2m_k}\}\) and \(\{y_{2n_k}\}\) of \(\{y_{2n}\}\) such that \(2n_k > 2m_k + k > k\) for all \(k\), we have

\[
d(y_{2m_k}, y_{2n_k}) \geq \epsilon, d(y_{2m_k}, y_{2n_k} - 2) < \epsilon,
\]

\[
\delta(y_{2m_k}, y_{2n_k}) \geq \tau, \delta(y_{2m_k}, y_{2n_k} - 2) < \tau.
\]

Using the triangle inequality in \(b\)-metric spaces, we obtain

\[
\epsilon \leq d(y_{2m_k}, y_{2n_k}) \leq sd(y_{2m_k}, y_{2n_k}) + s^2d(y_{2n_k-2}, y_{2n_k}) + s^2d(y_{2n_k}, y_{2n_k-1})
\]

\[
< \epsilon s + s^2d(y_{2n_k-2}, y_{2n_k-1}) + s^2d(y_{2n_k}, y_{2n_k-1}),
\]

which leads to

\[
\epsilon \leq \lim_{k \to \infty} \sup d(y_{2n_k}, y_{2m_k}) \leq \epsilon s,
\]

and since

\[
\epsilon \leq d(y_{2n_k}, y_{2m_k}) \leq sd(y_{2n_k}, y_{2n_k-1}) + s^2d(y_{2n_k-1}, y_{2n_k})
\]

\[
\leq sd(y_{2n_k}, y_{2n_k-1}) + s^2d(y_{2n_k-1}, y_{2n_k}) + s^2d(y_{2n_k}, y_{2m_k}),
\]

we obtain

\[
\frac{\epsilon}{s} \leq \lim_{k \to \infty} \sup d(y_{2n_k-1}, y_{2m_k}) \leq s^2 \epsilon.
\]

By the same arguments, we obtain

\[
\frac{\epsilon}{s^2} \leq \lim_{k \to \infty} \sup d(y_{2n_k+1}, y_{2n_k-1}) \leq s \epsilon s,
\]

and consequently

\[
\frac{\epsilon}{s} \leq \lim_{k \to \infty} \sup d(y_{2n_k+1}, y_{2n_k}).
\]

Similarly, we obtain

\[
\epsilon \leq \lim \inf_{k \to \infty} d(y_{2n_k}, y_{2m_k}) \leq \epsilon s.
\]
\[
\varepsilon \leq \lim \inf_{k \to \infty} d(\mathbf{x}_{2n+1}, \mathbf{x}_{2n-1}) \leq \varepsilon s^2.
\]

Using (24) and (26), we obtain

\[
\varepsilon \leq \lim \sup_{k \to \infty} M_d(\mathbf{x}_{2m}, \mathbf{x}_{2n-1}) \leq \varepsilon s^3.
\]

Moreover, we have

\[
\frac{\varepsilon}{s^2} \leq \lim \inf_{k \to \infty} d(\mathbf{y}_{2m}, \mathbf{y}_{2n}) \leq \varepsilon s^3.
\]

By the same arguments, we obtain

\[
\frac{\tau}{r} \leq \lim \sup_{k \to \infty} M_\delta(\mathbf{x}_{2m}, \mathbf{x}_{2n-1}) \leq \tau r^3.
\]

Using (31), we get

\[
M_d(x_{2m}, x_{2n-1}) = \max \left\{ d(\mathbf{y}_{2m}, \mathbf{y}_{2n-1}), d(\mathbf{y}_{2m}, \mathbf{y}_{2n}), d(\mathbf{y}_{2m}, \mathbf{y}_{2n-1}), \frac{d(\mathbf{y}_{2m}, \mathbf{y}_{2n}) + d(\mathbf{y}_{2m}, \mathbf{y}_{2n-1})}{2s} \right\}.
\]

Then,

\[
\frac{\varepsilon}{2s} + \frac{\varepsilon}{2s^3} = \min \left\{ \frac{\varepsilon}{s}, \frac{\varepsilon + \varepsilon s^2}{2s} \right\}
\]

\[
\leq \max \left\{ \lim \inf_{k \to \infty} d(\mathbf{y}_{2m}, \mathbf{y}_{2n-1}), \lim \inf_{k \to \infty} d(\mathbf{y}_{2m}, \mathbf{y}_{2n}), \frac{\lim \inf_{k \to \infty} d(\mathbf{y}_{2m}, \mathbf{y}_{2n}) + \lim \inf_{k \to \infty} d(\mathbf{y}_{2m}, \mathbf{y}_{2n-1})}{2s} \right\}
\]

\[
\leq \max \left\{ \varepsilon s^2, \frac{s \varepsilon s + \varepsilon s^3}{2s} = \varepsilon s^2. \right\}
\]

Consequently,

\[
\psi(\varepsilon s^3) \leq \psi\left(\frac{\varepsilon}{s^3} \lim \sup_{k \to \infty} d(\mathbf{y}_{2m}, \mathbf{y}_{2n}) \right)
\]

\[
\leq \psi\left(\lim \sup_{k \to \infty} M_\delta(\mathbf{x}_{2m}, \mathbf{x}_{2n-1}) \right)
\]

\[
- \lim \inf_{k \to \infty} \psi(M_\delta(\mathbf{x}_{2m}, \mathbf{x}_{2n-1}))
\]

\[
\leq \psi(\varepsilon s^3) - \psi\left(\lim \inf_{k \to \infty} M_\delta(\mathbf{x}_{2m}, \mathbf{x}_{2n-1}) \right),
\]

which implies that

\[
\psi(\varepsilon s^3) \leq \psi(\varepsilon s^3) \leq \psi(\varepsilon s^3) - \psi\left(\lim \inf_{k \to \infty} M_\delta(\mathbf{x}_{2m}, \mathbf{x}_{2n-1}) \right),
\]

\[
\psi(\varepsilon s^3) \leq \psi(\varepsilon s^3) \leq \psi(\varepsilon s^3) - \psi\left(\lim \inf_{k \to \infty} M_\delta(\mathbf{x}_{2m}, \mathbf{x}_{2n-1}) \right).
\]

It follows that

\[
\psi\left(\lim \inf_{k \to \infty} M_\delta(\mathbf{x}_{2m}, \mathbf{x}_{2n-1}) \right) = 0.
\]
We have also

$$\varphi \left( \lim_{k \to \infty} \inf \left( M_d(x_{2m_k}, x_{2n_k-1}) \right) \right) = 0. \quad (41)$$

Then,

$$\lim_{k \to \infty} \inf \left( M_d(x_{2m_k}, x_{2n_k-1}) \right) = \lim_{k \to \infty} \inf \left( M_d(x_{2m_k-1}, x_{2n_k-1}) \right) = 0, \quad (42)$$

which contradicts with (36) and (37). It follows that \{y_{2n}\} is a Cauchy sequence with \(b\)-metric \(d\); it is also a Cauchy sequence in \((X, \delta)\). And then there exists \(y \in X\) such that

$$\lim_{n \to \infty} \sup_{x \in X} M_d(Px_{2n+2}, y) = \lim_{n \to \infty} \sup_{x \in X} M_d(Px_{2n+1}, y) = 0.$$

Step 4. \(y\) is a common fixed point of \(f, g, P,\) and \(T\).

Since \(P\) is continuous, we have

$$\lim_{n \to \infty} P^2x_{2n+2} = \lim_{n \to \infty} Pf_{x_{2n}} = Py. \quad (44)$$

Moreover, the pair \(\{f, P\}\) is compatible, then \(\lim_{n \to \infty} d(Pf_{x_{2n}}, Pf_{x_{2n}}) = 0.\) The triangle inequality implies

$$\lim_{n \to \infty} Pf_{x_{2n}} = Py. \quad (45)$$

Since \(Px_{2n+1} = g_{x_{2n+1}} \leq x_{2n+1},\) we obtain by (6),

$$\lim_{n \to \infty} f_{x_{2n}} = \lim_{n \to \infty} Tx_{2n+1} = \lim_{n \to \infty} g_{x_{2n+1}} = \lim_{n \to \infty} Px_{2n+2} = y. \quad (43)$$

Using Lemma (7), we obtain

$$\left\{ \begin{array}{l}
\psi \left( \frac{1}{s^2} d(Pf_{x_{2n+2}}, g_{x_{2n+1}}) \right) \leq \psi \left( M_d(Pf_{x_{2n+2}}, x_{2n+1}) \right) - \varphi \left( M_d(Pf_{x_{2n+2}}, x_{2n+1}) \right), \\
\psi \left( \frac{1}{r^2} d(Pf_{x_{2n+2}}, g_{x_{2n+1}}) \right) \leq \psi \left( M_d(Pf_{x_{2n+2}}, x_{2n+1}) \right) - \varphi \left( M_d(Pf_{x_{2n+2}}, x_{2n+1}) \right).
\end{array} \right. \quad (46)$$

Consequently,

$$\left\{ \begin{array}{l}
\psi \left( \frac{1}{s^2} d(Py, y) \right) \leq \psi \left( \frac{1}{r^2} \delta(Py, y) \right) - \varphi \left( \frac{1}{s^2} d(Py, y) \right), \\
\psi \left( \frac{1}{r^2} \delta(Py, y) \right) \leq \psi \left( \frac{1}{s^2} d(Py, y) \right) - \varphi \left( \frac{1}{r^2} d(Py, y) \right),
\end{array} \right. \quad (47)$$

and then

$$\psi \left( \frac{1}{s^2} d(Py, y) \right) \leq \psi \left( \frac{1}{s^2} d(Py, y) \right) - \varphi \left( \frac{1}{s^2} d(Py, y) \right), \quad (48)$$

$$\psi \left( \frac{1}{r^2} \delta(Py, y) \right) \leq \psi \left( \frac{1}{r^2} \delta(Py, y) \right) - \varphi \left( \frac{1}{r^2} \delta(Py, y) \right), \quad (49)$$

which gives \(\varphi((1/s^2)d(Py, y)) = 0\) or equivalently \(Py = y.\)
Moreover, \( \nu \leq T \nu = f \gamma \leq y \) gives

\[
\psi(d(T \nu, g \nu)) = \psi(d(f \nu, g \nu)) \leq \psi(M_{d}(y, v)) - \psi(M_{d}(y, v)),
\]
\[
\psi(\delta(T \nu, g \nu)) = \psi(\delta(f \nu, g \nu)) \leq \psi(M_{d}(y, v)) - \psi(M_{d}(y, v)),
\]

with \( M_{d}(y, v) = d(g \nu, T \nu) \) and \( M_{d}(y, v) = \delta(g \nu, T \nu) \). Consequently,

\[
\begin{align*}
\psi(d(T \nu, g \nu)) & \leq \psi(\delta(g \nu, T \nu)) - \psi(\delta(g \nu, T \nu)), \\
\psi(\delta(T \nu, g \nu)) & \leq \psi(d(g \nu, T \nu)) - \psi(d(g \nu, T \nu)),
\end{align*}
\]

which implies \( g \nu = T \nu \). And since \( \{ g, T \} \) is weakly compatible, we obtain

\[
g \nu = g f \nu = g T \nu = T g \nu = T f \nu = T y.
\]

If \( n \to +\infty \), and using Lemma (7), we obtain

\[
\begin{align*}
\frac{1}{s} d(y, g \nu) & \leq \liminf_{n} M_{d}(x_{2n}, y) \leq \limsup_{n} M_{d}(x_{2n}, y) \leq s d(y, g \nu), \\
\frac{1}{s} \delta(y, g \nu) & \leq \liminf_{n} M_{d}(x_{2n}, y) \leq \limsup_{n} M_{d}(x_{2n}, y) \leq s \delta(y, g \nu).
\end{align*}
\]

If \( n \to +\infty \) in (56), and using Lemma (7), we obtain

\[
\begin{align*}
\psi(s d(y, g \nu)) & \leq \psi(\frac{1}{s} s d(y, g \nu)) \leq \psi(\frac{1}{s} \delta(y, g \nu)) - \psi(\frac{1}{s} \delta(y, g \nu)), \\
\psi(\delta(y, g \nu)) & \leq \psi(\frac{1}{s} \delta(y, g \nu)) \leq \psi(s d(y, g \nu)) - \psi(\frac{1}{s} d(y, g \nu)),
\end{align*}
\]

which implies \( d(y, g \nu) = \delta(y, g \nu) = 0 \) and then \( y = g \nu \).

We conclude that \( f \nu = g \nu = P \nu = T \nu = y \).

If \( f \) is continuous, the proof is the same. The same goes for condition (b) of Assumption 9.

Step 5. Suppose that \( u \) and \( v \) are two common fixed points of \( f, g, P, \) and \( T \) but \( d(u, v) > 0 \) with \( u \) and \( v \) are comparable. We have

\[
\begin{align*}
\psi(d(u, v)) = \psi(d(fu, gv)) & \leq \psi(s d(fu, gv)) \leq \psi(M_{d}(u, v)) - \psi(M_{d}(u, v)), \\
\psi(\delta(u, v)) = \psi(\delta(fu, gv)) & \leq \psi(r \delta(fu, gv)) \leq \psi(M_{d}(u, v)) - \psi(M_{d}(u, v)),
\end{align*}
\]

with \( M_{d}(u, v) = \delta(u, v) \) and \( M_{d}(u, v) = d(u, v) \).

So from (59), we have

\[
\begin{align*}
\psi(d(u, v)) & \leq \psi(\delta(u, v)) - \phi(\delta(u, v)), \\
\psi(\delta(u, v)) & \leq \psi(d(u, v)) - \phi(d(u, v)).
\end{align*}
\]

Hence, \( \psi(d(u, v)) \leq \psi(d(u, v)) - \phi(d(u, v)) \). Therefore, \( u = v \).

Example 11. Let \( X = [0, 1] \); we define on \( X \) the \( b \)-metrics \( d(x, y) = (x - y)^{2} \) with \( s = 2 \) and \( \delta(x, y) = |x - y|^{3} \) with \( r = 4 \). We endow \( X \) with the partial order \( \leq \) given by

\[ y \leq x \iff x \leq y \quad \text{for all} \quad x, y \in X. \]

And we define \( f, g, P, \) and \( T \) on \( X \) by

\[
\begin{align*}
f_{x} & = \begin{cases} 0, & \text{if} \ x \leq \frac{1}{8}, \\
\frac{1}{64}, & \text{if} \ x \in \left(\frac{1}{8}, 1\right].
\end{cases} \\
g_{x} & = 0 \quad \text{for all} \ x \in X, \\
T_{x} & = \begin{cases} x, & \text{if} \ x \in \left(0, \frac{1}{8}\right], \\
1, & \text{if} \ x \in \left[\frac{1}{8}, 1\right].
\end{cases} \\
P_{x} & = \begin{cases} 0, & \text{if} \ x = 0, \\
\frac{1}{8}, & \text{if} \ x \in \left(0, \frac{1}{8}\right], \\
1, & \text{if} \ x \in \left[\frac{1}{8}, 1\right].
\end{cases}
\end{align*}
\]

Obviously, conditions (i) and (ii) of Theorem 10 are satisfied. Moreover, \( \{ f, P \} \) is weakly compatible, \( \{ g, T \} \) is compatible and \( g \) is continuous.

The control functions \( \psi, \phi : [0, \infty) \to [0, \infty) \) are defined as \( \psi(t) = (9/8)t \) and \( \phi(t) = (1/8)t \) for all \( t \in [0, \infty) \).

To prove that \( f, g, P, \) and \( T \) satisfy (6), for this, we consider the following cases:

(i) \( f \) if \( x \in [0, 1/8] \) and \( y \in [0, 1] \), then \( d(fx, gy) = 0 = \delta(fx, gy) \) and (6) are satisfied.

(ii) \( f \) if \( x \in (1/8, 1] \) and \( y = 0 \), then

\[
\begin{align*}
\psi(s d(fx, gy)) & = \psi(16d(fx, gy)) = \psi\left(\frac{1}{256}\right) = \frac{9}{2048} < 1 = \delta(Px, Tg) \leq M_{d}(x, y), \\
& = \psi(M_{d}(x, y)) - \phi(M_{d}(x, y)),
\end{align*}
\]
\[ \psi(r^{4}\delta(fx,gy)) = \psi(256\delta(fx,gy)) = \psi \left( \frac{1}{1024} \right) = \frac{9}{8192} \]

\[ < 1 = d(Px,Ty) \leq M_{d}(x,y) \]

\[ = \psi(M_{d}(x,y)) - \varphi(M_{d}(x,y)). \]

(iii) If \( x \in (1/8,1] \) and \( y \in (0,1/8) \), then

\[ \psi(s^{4}d(fx,gy)) = \psi(16d(fx,gy)) = \psi \left( \frac{1}{256} \right) = \frac{9}{2048} \]

\[ < \frac{343}{512} \leq \delta(Px,Ty) \leq M_{d}(x,y) \]

\[ = \psi(M_{d}(x,y)) - \varphi(M_{d}(x,y)). \]

(iv) If \( x \in (1/8,1] \) and \( y \in (1/8,1) \),

\[ \psi(s^{4}d(fx,gy)) = \psi(16d(fx,gy)) = \psi \left( \frac{1}{256} \right) = \frac{9}{2048} \]

\[ < \frac{49}{64} \leq d(Px,Ty) \leq M_{d}(x,y) \]

\[ = \psi(M_{d}(x,y)) - \varphi(M_{d}(x,y)). \]

Thus, all conditions of Theorem (10) are satisfied. Moreover, 0 is the unique common fixed point of \( f, g, P, \) and \( T \).

If we take \( P \) and \( T \) as the identity maps on \( X \) in Theorem (10), we conclude the following corollary.

**Corollary 12.** We consider two self-mappings \( f \) and \( g \) in ordered complete two \( b \)-metric space \( (X,d,\delta,\ll) \) that fulfill the following conditions:

(i) \( \{ f, g \} \) is dominated

(ii) \( \psi \in \Psi, \varphi \in \Phi \) and for every two comparable elements \( x, y \in X \), we have

\[ \psi(s^{4}d(fx,gy)) \leq \psi(N_{\delta}(x,y)) - \varphi(N_{\delta}(x,y)), \]

\[ \psi(r^{4}\delta(fx,gy)) \leq \psi(N_{\delta}(x,y)) - \varphi(N_{\delta}(x,y)). \]

(iii) Assumption 8 is satisfied

Then \( f \) and \( g \) have a unique common fixed point in \( X \).

If we take \( \psi(t) = t \) for \( t \in [0,\infty) \) in Corollary (12), we have the following corollary.

**Corollary 13.** We consider two self-mappings \( f \) and \( g \) in ordered complete two \( b \)-metric space \( (X,d,\delta,\ll) \) that fulfill the following conditions:

(i) \( \{ f, g \} \) is dominated

(ii) \( \varphi \in \Phi \) and for every two comparable elements \( x, y \in X \), we have

\[ s^{4}d(fx,gy) \leq N_{\delta}(x,y) - \varphi(N_{\delta}(x,y)), \]

\[ r^{4}\delta(fx,gy) \leq N_{\delta}(x,y) - \varphi(N_{\delta}(x,y)). \]

(iii) Assumption 8 is satisfied

Then, \( f \) and \( g \) have a unique common fixed point in \( X \).

**Corollary 14.** We consider two self-mappings \( f \) and \( g \) in ordered complete two \( b \)-metric space \( (X,d,\delta,\ll) \) that fulfill the following conditions:

(i) \( \{ f, g \} \) is dominated

(ii) \( k \in (0,1) \) and for every two comparable elements \( x, y \in X \), we have

\[ s^{4}d(fx,gy) \leq kN_{\delta}(x,y), \]

\[ r^{4}\delta(fx,gy) \leq kN_{\delta}(x,y). \]

(iii) Assumption 8 is satisfied

Then, \( f \) and \( g \) have a unique common fixed point in \( X \).

**Theorem 15.** We consider two self-mappings \( f, g, P, \) and \( T \) in ordered complete two \( b \)-metric space \( (X,d,\delta,\ll) \) that fulfill the following conditions:

(i) \( \{ f, g \} \) is dominated and \( \{ P, T \} \) is dominating

(ii) \( fx \leq TX \) and \( gX \leq PX \)
(iii) \( \psi \in \Psi, \varphi \in \Phi \) and for every two comparable elements \( x, y \in X \), we have

\[
\psi(s^i d(f x, g y)) \leq \psi(Q_d(x, y)) - \varphi(Q_d(x, y)). \tag{69}
\]

(iv) Assumptions 8 and 9 are satisfied

Then, \( f, g, P, \) and \( T \) have a unique common fixed point in \( X \).

**Proof.** If we follow similar arguments to those given in Theorem (10), we have the following steps.

**Step 1.** If \( y_{2k+1} = y_{2k} \) for some \( k \), then \( (y_n)_n \) is constant.

So, from (69), we have

\[
\psi(d(y_{2k+1}, y_{2k+2})) \leq \psi(s^i d(y_{2k+1}, y_{2k+2})) = \psi(s^i d(f x_{2k}, g x_{2k+1})) \leq \psi(Q_d(x_{2k}, x_{2k+1})) - \varphi(Q_d(x_{2k}, x_{2k+1}))), \tag{70}
\]

where

\[
Q_d(x_{2k}, x_{2k+1}) = a_1 d(P x_{2k}, T x_{2k+1}) + a_2 d(f x_{2k}, P x_{2k}) + a_3 d(g x_{2k+1}, T x_{2k+1}) + a_4 d(P x_{2k}, g x_{2k+1}) + a_5 d(f x_{2k}, T x_{2k+1}) = a_1 d(y_{2k}, y_{2k+1}) + a_2 d(y_{2k+1}, y_{2k+2}) + a_3 d(y_{2k+2}, y_{2k+1}) + a_4 d(y_{2k+1}, y_{2k+2}) + a_5 d(y_{2k+2}, y_{2k+1}) = a_3 d(y_{2k}, y_{2k+1}) + a_4 d(y_{2k+1}, y_{2k+2}) \leq (a_3 + a_4)s d(y_{2k+1}, y_{2k+2}). \tag{71}
\]

So, from (70) and (71), we obtain

\[
\psi(d(y_{2k+1}, y_{2k+2})) \leq \psi((a_3 + a_4)s d(y_{2k+1}, y_{2k+2})) - \varphi(Q_d(x_{2k}, x_{2k+1}))) \leq \psi(d(y_{2k+1}, y_{2k+2})) - \varphi(Q_d(x_{2k}, x_{2k+1})))), \tag{72}
\]

which gives \( \psi(Q_d(x_{2k}, x_{2k+1}))) = 0 \), and so, \( Q_d(x_{2k}, x_{2k+1}))) = 0 \), since \( a_3, a_4 > 0 \) which further implies that \( y_{2k} = y_{2k+1} = y_{2k+2} \). Hence, \( (y_n) \) is a constant sequence and \( y_{2k} \) is a common fixed point of \( f, g, P, \) and \( T \).

In the following, we can assume that \( d(y_{2n}, y_{2n+1}) \) for each \( n \).

**Step 2.** The sequence \( (d(y_{2n}, y_{2n+1})) \) are monotone nonincreasing.

Since \( x_2n \) and \( x_{2n+1} \) are comparable, from (69), we have

\[
\psi(d(y_{2n+1}, y_{2n+2})) \leq \psi(s^i d(y_{2n+1}, y_{2n+2})) = \psi(s^i d(f x_{2n}, g x_{2n+1})) \leq \psi(Q_d(x_{2n}, x_{2n+1})) - \varphi(Q_d(x_{2n}, x_{2n+1}))) \leq \psi(Q_d(x_{2n}, x_{2n+1}))). \tag{73}
\]

Hence,

\[
d(y_{2n+1}, y_{2n+2}) \leq Q_d(x_{2n}, x_{2n+1}), \tag{74}
\]

where

\[
Q_d(x_{2n}, x_{2n+1}) = a_1 d(P x_{2n}, T x_{2n+1}) + a_2 d(f x_{2n}, P x_{2n}) + a_3 d(g x_{2n+1}, T x_{2n+1}) + a_4 d(P x_{2n}, g x_{2n+1}) + a_5 d(f x_{2n}, T x_{2n+1}) = a_1 d(y_{2n}, y_{2n+1}) + a_2 d(y_{2n+1}, y_{2n+2}) + a_3 d(y_{2n+2}, y_{2n+1}) + a_4 d(y_{2n+1}, y_{2n+2}) + a_5 d(y_{2n+2}, y_{2n+1}) = (a_1 + a_2) d(y_{2n}, y_{2n+1}) + a_3 d(y_{2n+2}, y_{2n+1}) + a_4 d(y_{2n+1}, y_{2n+2}) \leq (a_1 + a_2 + a_3 + a_4) s \max \{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}) \} \leq max \{d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}) \}. \tag{75}
\]

If for some \( n, d(y_{2n+2}, y_{2n+1}) \geq d(y_{2n+1}, y_{2n+2}) > 0 \), then (74) gives that \( Q_d(x_{2n}, x_{2n+1}) = d(y_{2n+2}, y_{2n+1}) \), and from (69), we have

\[
\psi(d(y_{2n+1}, y_{2n+2})) \leq \psi(s^i d(y_{2n+2}, y_{2n+1})) = \psi(Q_d(x_{2n}, x_{2n+1})) - \varphi(Q_d(x_{2n}, x_{2n+1}))) = \psi(d(y_{2n+2}, y_{2n+1})) - \varphi(d(y_{2n+2}, y_{2n+1}))). \tag{76}
\]

And \( \varphi(d(y_{2n+2}, y_{2n+1})) = 0 \) or equivalently \( d(y_{2n+2}, y_{2n+1}) = 0 \), a contradiction.

Then, \( d(y_{2n}, y_{2n+1}) \leq Q_d(x_{2n}, x_{2n+1}) = d(y_{2n+1}, y_{2n+2}) \). By the same arguments, we have

\[
d(y_{2n+1}, y_{2n+2}) \leq Q_d(x_{2n+1}, x_{2n+2}) = d(y_{2n+2}, y_{2n+1}) \tag{77}
\]

Therefore, \( \{d(y_n, y_{n+1})\} \) is a nonincreasing sequence, and so, there exists \( c \geq 0 \) so that

\[
\lim_{n \to +\infty} d(y_n, y_{n+1}) = \lim_{n \to +\infty} Q_d(x_n, x_{n+1}) = c. \tag{78}
\]
Now, we demonstrate that \( c = 0 \). Suppose that \( c > 0 \). Since we have

\[
\psi(d(y_{2n+1}, y_{2n+2})) \leq \psi(s^2 d(y_{2n+2}, y_{2n+1})) \leq \psi(Q_d(x_{2n}, x_{2n+1}))) \quad (87)
\]

which implies

\[
\frac{\varepsilon}{s^2} (a_1 s + a_4 s^2 + a_5) \leq \limsup_{k \to \infty} Q_d(x_{2n}, x_{2m+1}) \leq (a_1 + a_4 + a_5) \varepsilon s^3,
\]

which implies

\[
\frac{\varepsilon}{s^2} (a_1 + a_4 + a_5) \leq \limsup_{k \to \infty} Q_d(x_{2n}, x_{2m+1}) \leq \varepsilon s^3.
\]

Similarly, we have

\[
\frac{\varepsilon}{s^2} (a_1 + a_4 + a_5) \leq \liminf_{k \to \infty} Q_d(x_{2n}, x_{2m+1}) \leq \varepsilon s^3.
\]

As

\[
\psi(s^2 d(y_{2n+1}, y_{2n})) = \psi(s^2 d(fx_{2n}, gx_{2m+1})) \leq \psi(Q_d(x_{2n}, x_{2m+1})) - \phi(Q_d(x_{2n}, x_{2m+1})).
\]

Taking the upper limit as \( k \to \infty \) and using \( (85) \) and \( (89) \), we obtain

\[
\psi(\varepsilon s^3) \leq \psi(s^2 \limsup_{k \to \infty} d(x_{2n}, x_{2m+1})) \leq \psi(s \limsup_{k \to \infty} d(x_{2n}, x_{2m+1})) - \liminf_{k \to \infty} \phi(Q_d(x_{2n}, x_{2m+1})),
\]

which implies that

\[
\phi \left( \liminf_{k \to \infty} Q_d(x_{2n}, x_{2m+1}) \right) = 0.
\]

So, \( \liminf_{k \to \infty} Q_d(x_{2n}, x_{2m+1}) = 0 \), which contradicts \( (90) \). It follows that \( \{ y_{2n} \} \) is a Cauchy sequence in \( X \). Since \( X \) is complete, there exists \( y \in X \) so that

\[
\lim_{n \to \infty} f x_{2n} = \lim_{n \to \infty} g x_{2n+1} = \lim_{n \to \infty} P x_{2n+2} = y.
\]

Step 4. \( y \) is a common fixed point of \( f, g, P, \) and \( T \).

Since \( P \) is continuous, we have

\[
\lim_{n \to \infty} P^2 x_{2n+2} = Py, \quad \lim_{n \to \infty} P f x_{2n} = Py.
\]

Using the triangle inequality in \( b \)-metric space, we get

\[
d(f P x_{2n}, Py) \leq s [d(f P x_{2n}, Pf x_{2n}) + d(Pf x_{2n}, Py)].
\]
Since the pair \( \{ f, P \} \) is compatible, \( \lim_{n \to \infty} d(fP_{x_{2n}}, P_{x_{2n}}) = 0 \). So, passing the upper limit when \( n \to \infty \) from the above inequality and Lemma (7), we have

\[
\limsup_{n \to \infty} d(fP_{x_{2n}}, Py) \\
\leq s \left[ \limsup_{n \to \infty} d(fP_{x_{2n}}, Pf_{x_{2n}}) + \limsup_{n \to \infty} d(Pf_{x_{2n}}, Py) \right] = 0.
\]

(97)

Hence,

\[
\lim_{n \to \infty} fP_{x_{2n}} = Py.
\]

(98)

As \( P_{x_{2n+2}} = g_{x_{2n+1}} \leq x_{2n+1} \) so, from (69), we have

\[
\psi(s^2d(fP_{x_{2n+2}}, g_{x_{2n+1}})) \leq \psi(Q_d(P_{x_{2n+2}}, x_{2n+1})) - \varphi(Q_d(P_{x_{2n+2}}, x_{2n+1})),
\]

where

\[
Q_d(P_{x_{2n+2}}, x_{2n+1}) = a_1d(P^2_{x_{2n+2}}, Tx_{2n+1}) + a_2d(fP_{x_{2n+2}}, P^2_{x_{2n+2}}) + a_3d(g_{x_{2n+1}}, Tx_{2n+1}) + a_4d(P^2_{x_{2n+2}}, g_{x_{2n+1}}) + a_5d(fP_{x_{2n+2}}, Tx_{2n+1}).
\]

(100)

Now, by using Lemma (7), we get

\[
\limsup_{n \to \infty} Q_d(P_{x_{2n+2}}, x_{2n+1}) \leq a_1s^2d(Py, y) + a_2s^2d(Py, y) + a_3s^2d(Py, y) \leq (a_1 + a_2 + a_5)s^2d(Py, y).
\]

(101)

Similarly,

\[
\frac{1}{s^2}(a_1 + a_4 + a_3)d(Py, y) \leq \liminf_{n \to \infty} Q_d(P_{x_{2n+2}}, x_{2n+1}).
\]

(102)

Hence, by taking the upper limit in (99) and using Lemma (7), we obtain

\[
\psi(s^2d(Py, y)) \leq \psi(s^2d(Py, y)) - \varphi\left(\frac{1}{s^2}(a_1 + a_4 + a_5)d(Py, y)\right),
\]

(103)

which gives \( \varphi(1/s^2(a_1 + a_4 + a_5)d(Py, y)) = 0 \) or equivalently \( Py = y \). Now, since \( g_{x_{2n+1}} \leq x_{2n+1} \) and \( g_{x_{2n+1}} \to y \) as \( n \to \infty \), then \( y \leq x_{2n+1} \), and from (69), we have

\[
\psi(s^2d(fy, g_{x_{2n+1}})) \leq \psi(Q_d(y, x_{2n+1})) - \varphi(Q_d(y, x_{2n+1})),
\]

(104)

where

\[
Q_d(y, x_{2n+1}) = a_1d(Py, Tx_{2n+1}) + a_2d(fy, Py) + a_3d(gy_{x_{2n+1}}, Ty_{2n+1}) + a_4d(Py, gy_{x_{2n+1}}) + a_5d(fy, Ty_{2n+1}).
\]

(105)

By using Lemma (7), we get

\[
\limsup_{n \to \infty} Q_d(y, x_{2n+1}) \leq a_1s(a_2d(Py, y) + a_3s^2d(y, y) + a_4s(a_5sdf y, y) + a_5sd(fy, y) + a_5sdf y, y) \leq (a_4 + a_5sdf y, y) \leq df y, y) \leq (a_4 + a_5sdf y, y).
\]

(106)

Similarly,

\[
\left(a_3 + \frac{a_5}{s}\right)d(df y, y) \leq \liminf_{n \to \infty} Q_d(y, x_{2n+1}).
\]

(107)

Taking the upper limit as \( n \to \infty \) in (104) and using Lemma (7), we have

\[
\psi(s^2d(fy, y)) = \psi\left(\frac{1}{s}d(df y, y)\right) \leq \psi(sdf y, y) - \liminf_{n \to \infty} (\varphi(Q_d(y, x_{2n+1}))) \leq \psi(s^2d(fy, y)) - \psi\left(\frac{1}{s}d(df y, y)\right) \leq \psi\left(\left(a_3 + \frac{a_5}{s}\right)d(df y, y)\right).
\]

(108)

which implies \( \varphi((a_2 + a_5s)(df y, y)) = 0 \). So, \( fy = y \).

On the other hand, we have \( fX \subseteq TX \), so there exists a point \( v \in X \) such that \( fy = Tv \).

Suppose that \( g'v \not\in Tv \). Since \( v \not\in Tv = fvy \), from (69), we have

\[
\psi(d(Tv, g'v)) = \psi(d(fy, g'v)) \leq \psi(Q_d(y, v)) - \varphi(Q_d(y, v)),
\]

(109)

where

\[
Q_d(y, v) = a_1d(Py, Tv) + a_4d(fy, Py) + a_3d(gy_{x_{2n+1}}, Ty_{2n+1}) + a_4d(Py, gy_{x_{2n+1}}) + a_5d(fy, Ty_{2n+1})
\]

\[
= (a_1 + a_4)d(gy_{x_{2n+1}}, Ty_{2n+1}) \leq d(gy_{x_{2n+1}}, Ty_{2n+1}).
\]

(110)

So, from (109), we have

\[
\psi(d(Tv, g'v)) \leq \psi(d(gy_{x_{2n+1}}, Ty_{2n+1})) - \varphi(Q_d(y, v)).
\]

(111)
This implies that \( \varphi(Q_d(y, v)) = 0 \), so \( d(gv, T v) = 0 \) which shows a contradiction; therefore, \( gv = T v \). Since the pair \( \{ g \), \( T \} \) is weakly compatible, \( gy = gf y = T v = T y \), \( g y = T y \), and \( y \) is the coincidence point of \( g \) and \( T \).

Since \( f x_n \to x_2n \) and \( f x_n \to y \) as \( n \to \infty \), it implies that \( y \leq x_2n \), and from (69), we obtain

\[
\psi(s^4 d(f x_{2n}, g y)) \leq \psi(Q_d(x_{2n}, y)) - \varphi(Q_d(x_{2n}, y)),
\]

where

\[
Q_d(x_{2n}, y) = a_1 d(P x_{2n}, T y) + a_2 d(f x_{2n}, P x_{2n}) + a_3 d(g y, T y)
\]

\[
+ a_4 d(P x_{2n}, g y) + a_5 d(f x_{2n}, T y).
\]

By using Lemma (7), we have

\[
\frac{1}{s} (a_1 + a_4 + a_5) d(y, g y) \leq \liminf_{n \to \infty} Q_d(x_{2n}, y)
\]

\[
\leq \limsup_{n \to \infty} Q_d(x_{2n}, y)
\]

\[
\leq s(a_1 + a_4 + a_5) d(y, g y)
\]

\[
\leq s d(y, g y).
\]

Passing the upper limit as \( n \to \infty \), and using (114), we have

\[
\psi(s^4 d(y, g y)) = \psi(s^4 \frac{1}{s} d(y, g y))
\]

\[
\leq \psi(s d(y, g y)) - \varphi(\frac{1}{s} (a_1 + a_4 + a_5) d(y, g y))
\]

\[
\leq \psi(s^4 d(y, g y)) - \varphi(\frac{1}{s} (a_1 + a_4 + a_5) d(y, g y)),
\]

which implies that \( y = g y \); therefore, \( fy = g y = P y = T y = y \).

If \( f \) is continuous, the proof is the same. The same goes for condition (b) of Assumption 9.

Step 5. Suppose that \( u \) and \( v \) are two common fixed points of \( f \), \( g \), \( P \), and \( T \), but \( d(u, v) > 0 \) with \( u \) and \( v \) are comparable. By assumption, we can apply (69) to obtain

\[
\psi(d(u, v)) = \psi(d(f u, g v)) \leq \psi(s^4 d(f u, g v))
\]

\[
\leq \psi(Q_d(u, v)) - \varphi(Q_d(u, v)),
\]

where

\[
Q_d(u, v) = a_1 d(P u, T v) + a_2 d(f u, P u) + a_3 d(g v, T v)
\]

\[
+ a_4 d(P u, g v) + a_5 d(f u, T v) = a_1 d(u, v)
\]

\[
+ a_4 d(u, v) + a_5 d(u, v) = (a_1 + a_4 + a_5) d(u, v)
\]

\[
\leq d(u, v).
\]

Hence,

\[
\psi(d(u, v)) \leq \psi(d(u, v)) - \varphi(Q_d(u, v)),
\]

implies \( \varphi(Q_d(u, v)) = 0 \), which is a contradiction. Therefore, \( u = v \). The converse is clear.

**Example 16.** We consider the data of the previous example and take \( a_1 = 3/32 \), \( a_2 = a_3 = 3/64 \), \( a_4 = 5/16 \), and \( a_5 = 7/64 \).

This permits only to verify the condition (69) of Theorem (15). We shall distinguish two cases:

(i) If \( x \in [0, 1/8] \) and \( y \in [0, 1] \), then \( d(f x, g y) = 0 \) and (69) is satisfied

(ii) If \( x \in (1/8, 1) \) and \( y \in [0, 1] \), then

\[
\psi(s^4 d(f x, g y)) = \psi(16 d(f x, g y)) = \psi(1 - \frac{1}{2^4}) = \frac{9}{2^3}
\]

\[
< \frac{9}{2^{11}} \frac{1323}{128} \frac{3}{2^{21}} = a_4 d(f x, P x)
\]

\[
\leq Q_d(x, y) = \psi(Q_d(x, y)) - \varphi(Q_d(x, y)).
\]

Thus, (69) is satisfied for all \( x, y \in X \). Therefore, all conditions of Theorem (15) are satisfied. Moreover, 0 is the unique common fixed point of \( f \), \( g \), \( P \), and \( T \).

If we take \( P \) and \( T \) as the identity maps on \( X \) in Theorem (15), we conclude the following corollary.

**Corollary 17.** We consider two self-mappings \( f \) and \( g \) in ordered complete two \( b \)-metric space \((X, d, \delta, \ll)\) that fulfill the following conditions:

(i) \( \{ f, g \} \) is dominated

(ii) \( \psi \in \Psi, \varphi \in \Phi \) and for every two comparable elements \( x, y \in X \), we have

\[
\psi(s^4 d(f x, g y)) \leq \psi(R_d(x, y)) - \varphi(R_d(x, y)).
\]

(iii) Assumption 8 is satisfied

Then, \( f \) and \( g \) have a unique common fixed point in \( X \).

If we take \( \psi(t) = t \) for \( t \in [0, \infty) \) in Corollary (17), we have the following corollary.

**Corollary 18.** We consider two self-mappings \( f \) and \( g \) in ordered complete two \( b \)-metric space \((X, d, \delta, \ll)\) that fulfill the following conditions:

(i) \( \{ f, g \} \) is dominated
(ii) $\varphi \in \Phi$ and for every two comparable elements $x, y \in X$, we have
\[
s^\alpha d(fx, gy) \leq R_\alpha(x, y) - \varphi(R_\alpha(x, y)).
\] (121)

(iii) Assumption 8 is satisfied

Then, $f$ and $g$ have a unique common fixed point in $X$.

3. Applications

Consider the following system of integral equations:
\[
\begin{align*}
x(t) &= \int_0^T K(t, u, y(u))du + L_1(t), \\
y(t) &= \int_0^T K(t, u, x(u))du + L_2(t),
\end{align*}
\] (122)

where $T > 0$, $K : [0, T] \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ and $L_1, L_2 : [0, T] \rightarrow \mathbb{R}$ are continuous functions.

Let $X = C([0, T]) \times C([0, T])$, where $C([0, T])$ is the set of real continuous on $[0, T]$.

We endow $X$ with the two $b$-metric:
\[
\begin{align*}
d((x, y), (x_1, y_1)) &= \max \left\{ \max_{t \in [0, T]} |x(t) - x_1(t)|^p, \max_{t \in [0, T]} |y(t) - y_1(t)|^q \right\}, \\
\delta((x, y), (x_1, y_1)) &= \max \left\{ \max_{t \in [0, T]} |x(t) - x_1(t)|^p, \max_{t \in [0, T]} |y(t) - y_1(t)|^q \right\},
\end{align*}
\] (123)

for all $(x, y), (x_1, y_1) \in X$. Clearly, $(X, d, \delta)$ is a complete two $b$-metric space with $s = 2^{p-1}, r = 2^{q-1}$, and $p, q \in (1, \infty)$ with $1/p + 1/q = 1$. We endow $X$ with the partial order $\preceq$ given by
\[
(x, y) \preceq (x_1, y_1) \Leftrightarrow x(t) \leq x_1(t), y(t) \leq y_1(t) \quad \text{for all } t \in [0, T].
\] (124)

Clearly, the space $(X, d, \delta, \preceq)$ is an ordered complete two $b$-metric space.

Consider the mapping $F : X \rightarrow X$ defined as follows: $F(x, y) = (F_1(x, y), F_2(x, y))$ with:
\[
\begin{align*}
F_1(x, y)(t) &= \int_0^T K(t, u, y(u))du + L_1(t), \\
F_2(x, y)(t) &= \int_0^T K(t, u, x(u))du + L_2(t),
\end{align*}
\] (125)

Suppose that the following hypotheses hold:

(i) there exists a continuous function $\varepsilon : [0, T] \times [0, T] \rightarrow \mathbb{R}$, for all $t, u \in [0, T]$, $i = p, q$ and for all comparable $x, y \in C([0, T])$, we have
\[
|K(t, u, x(u)) - K(t, u, y(u))|^i \leq \varepsilon(t, u)|x(u) - y(u)|^{i(p-1)}. 
\] (126)

(ii) $\max_{t \in [0, T]} (16T)^{r-1} \max_{t \in [0, T]} \varepsilon(t, u)du < 1$.

**Theorem 19.** Under the assumptions (i) and (ii), the system integral equations (122) have a unique solution in the set $C([0, T])$.

**Proof.** By the condition (i), we have for all $t \in [0, T]$ and for all comparable $x, y \in X$,
\[
|F_1(x, y)(t) - F_1(x_1, y_1)(t)|^p 
\leq \left( \int_0^T |K(t, u, y(u)) - K(t, u, y_1(u))| |du| \right)^p 
\leq \left( \int_0^T \varepsilon(t, u)|y(u) - y_1(u)|^{i(p-1)} |du| \right)^p 
\leq T^{q-1}\left( \int_0^T \varepsilon(t, u)|y(u) - y_1(u)|^{i(p-1)} |du| \right)^p 
\leq T^{q-1}\left( \int_0^T \varepsilon(t, u) |y(u) - y_1(u)|^{i(p-1)} |du| \right)^p 
\leq T^{q-1}\left( \int_0^T \varepsilon(t, u) |y(u) - y_1(u)|^{i(p-1)} |du| \right)^p 
\leq \left( \int_0^T \varepsilon(t, u) |y(u) - y_1(u)|^{i(p-1)} |du| \right)^p.
\] (127)

By a similar calculus, we obtain
\[
|F_2(x, y)(t) - F_2(x_1, y_1)(t)|^p 
\leq \left( \int_0^T \varepsilon(t, u) |y(u) - y_1(u)|^{i(p-1)} |du| \right)^p.
\] (128)

It follows that
\[
d(F(x, y), F(x_1, y_1)) 
\leq \left( \int_0^T \varepsilon(t, u) |y(u) - y_1(u)|^{i(p-1)} |du| \right)^p.
\] (129)

Similarly,
\[
\delta(F(x, y), F(x_1, y_1)) 
\leq \left( \int_0^T \varepsilon(t, u) |y(u) - y_1(u)|^{i(p-1)} |du| \right)^p.
\] (130)
which implies that

\[
\begin{align*}
&k^* d(f(x, y), g(x, y), (x_1, y_1)) \leq kN_d(x, y, (x_1, y_1)), \\
&k^* d(f(x, y), g(x, y), (x_1, y_1)) \leq kN_d(x, y, (x_1, y_1)),
\end{align*}
\]  

(131)

with

\[
k = \max_{i \in \{p, q\}} \max_{t \in I} \int_0^t \varepsilon(t, u)du < 1.
\]  

(132)

Thus, all the hypotheses of Corollary (14) (with \( F = f = g \)) are satisfied, and hence, the mapping \( F \) has a unique fixed point in \( X \), which is a solution of the system of nonlinear integral equations (122).

4. Conclusion

In this manuscript, we have obtained interesting results on the coupled fixed point theorems that give a generalization of well-known results in the field. We have shown that these results are very useful to solve the system of integral equations.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] I. A. Bakhtin, “The contraction mapping principle in quasi-metric spaces,” *Functional Analysis*, vol. 30, pp. 26–37, 1989.

[2] S. Czerwik, “Contraction mappings in \( b \)-metric spaces,” *Acta Mathematica et Informatica Universitatis Ostraviensis*, vol. 1, no. 1, pp. 5–11, 1993.

[3] S. Czerwik, “Nonlinear set-valued contraction mappings in \( b \)-metric spaces,” *Atti del Seminario Matematico e Fisico dell’Università di Modena*, vol. 46, pp. 263–276, 1998.

[4] A. Aghajani, M. Abbas, and J. Roshan, “Common fixed point of generalized weak contractive mappings in partially ordered \( b \)-metric spaces,” *Mathematica Slovaca*, vol. 64, no. 4, pp. 941–960, 2014.

[5] M. Boriceanu, “Fixed point theory for multivalued generalized contraction on a set with two \( b \)-metrics,” *Studia Universitatis Babes-Bolyai, Mathematica*, vol. 3, 2009.

[6] N. Hussain and M. H. Shah, “KKM mappings in cone \( b \)-metric spaces,” *Computers & Mathematics with Applications*, vol. 62, no. 4, pp. 1677–1684, 2011.

[7] J. R. Roshan, V. Parvaneh, S. Sedghi, N. Shobkolaei, and W. Shatanawi, “Common fixed points of almost generalized \( (\psi, \phi) \) \( s \)-contractive mappings in ordered \( b \)-metric spaces,” *Fixed Point Theory and Applications*, vol. 2013, no. 1, 2013.

[8] A. C. M. Ran and M. C. B. Reurings, “A fixed point theorem in partially ordered sets and some applications to matrix equations,” *Proceedings of the American Mathematical Society*, vol. 1, pp. 1435–1443, 2004.

[9] J. J. Nieto and R. Rodríguez-López, “Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations,” *Order*, vol. 22, no. 3, pp. 223–239, 2005.

[10] J. J. Nieto and R. Rodríguez-López, “Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations,” *Acta Mathematica Sinica, English Series*, vol. 23, no. 12, pp. 2205–2212, 2007.

[11] P. N. Dutta and B. S. Choudhury, “A generalisation of contraction principle in metric spaces,” *Fixed Point Theory and Applications*, vol. 2008, no. 1, Article ID 406368, 2008.

[12] N. Hussain, V. Parvaneh, J. R. Roshan, and Z. Kadelburg, “Fixed points of cyclic weakly \( (\psi, \phi, l, a, b) \)-contractive mappings in ordered \( b \)-metric spaces with applications,” *Fixed Point Theory and Applications*, vol. 2013, no. 1, 2013.

[13] J. R. Roshan, V. Parvaneh, and I. Altun, “Some coincidence point results in ordered \( b \)-metric spaces and applications in a system of integral equations,” *Applied Mathematics and Computation*, vol. 226, pp. 725–737, 2014.

[14] J. R. Roshan, V. Parvaneh, and Z. Kadelburg, “Common fixed point theorems for weakly isotone increasing mappings in ordered \( b \)-metric spaces,” *Journal of Nonlinear Sciences and Applications*, vol. 7, no. 4, pp. 229–245, 2014.

[15] V. Parvaneh, J. R. Roshan, and S. Radenović, “Existence of tripled coincidence points in ordered \( b \)-metric spaces and an application to a system of integral equations,” *Fixed Point Theory and Applications*, vol. 2013, no. 1, 2013.

[16] A. Mujahid, T. Nazir, and S. Radenović, “Common fixed points of four maps in partially ordered metric spaces,” *Applied Mathematics Letters*, vol. 24, no. 9, pp. 1520–1526, 2011.

[17] A. Aghajani and R. Arab, “Fixed points of \( (\psi, \phi, \theta) \)-contractive mappings in partially ordered \( b \)-metric spaces and application to quadratic integral equations,” *Fixed Point Theory and Applications*, vol. 2013, no. 1, 2013.

[18] R. Allahyari, R. Arab, and A. S. Haghighi, “A generalization on weak contractions in partially ordered \( b \)-metric spaces and its application to quadratic integral equations,” *Journal of Inequalities and Applications*, vol. 2014, no. 1, 2014.

[19] Y. Errai, E. M. Marhrani, and M. Aamri, “Some related fixed points theorems of weak contraction with two partially ordered metric spaces,” *Advances in Fixed Point Theory*, vol. 7, no. 3, pp. 372–390, 2017.

[20] M. Boriceanu, M. Bota, and A. Petruşel, “Multivalued fractals in \( b \)-metric spaces,” *Central European Journal of Mathematics*, vol. 8, no. 2, pp. 367–377, 2010.