\textbf{\gamma\text{-BOUNDED REPRESENTATIONS OF AMENABLE GROUPS}}

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\textbf{Abstract.} Let $G$ be an amenable group, let $X$ be a Banach space and let $\pi: G \to B(X)$ be a bounded representation. We show that if the set \{\pi(t) : t \in G\} is \gamma\text{-bounded then} $\pi$ extends to a bounded homomorphism $w: \mathcal{C}^*(G) \to B(X)$ on the group $\mathcal{C}^*$-algebra of $G$. Moreover $w$ is necessarily \gamma\text{-bounded. This extends to the Banach space setting a theorem of Day and Dixmier saying that any bounded representation of an amenable group on Hilbert space is unitarizable. We obtain additional results and complements when $G = \mathbb{Z}$, $\mathbb{R}$ or $\mathbb{T}$, and/or when $X$ has property (\alpha).}

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1. Introduction.

The notions of $R$-boundedness and \gamma\text{-boundedness play a prominent role in various recent developments of operator valued harmonic analysis and multiplier theory, see for example [42, 39, 1, 14, 21, 22, 23, 26]. These notions are also now central in the closely related fields of functional calculi (see [27, 13, 30], abstract control theory in Banach spaces [19, 20], or vector valued stochastic integration, see [41] and the references therein. This paper is devoted to another aspect of harmonic analysis, namely Banach space valued group representations. Our results will show that \gamma\text{-boundedness is the key concept to understand certain behaviors of such representations.}

Throughout we let $G$ be a locally compact group, we let $X$ be a complex Banach space and we let $B(X)$ denote the Banach algebra of all bounded operators on $X$. By a representation of $G$ on $X$, we mean a strongly continuous mapping $\pi: G \to B(X)$ such that $\pi(tt') = \pi(t)\pi(t')$ for any $t, t'$ in $G$, and $\pi(e) = I_X$. Here $e$ and $I_X$ denote the unit of $G$ and the identity operator on $X$, respectively. We say that $\pi$ is bounded if moreover $\sup_{t \in G} \|\pi(t)\| < \infty$. Assume that $G$ is amenable and that $X = H$ is a Hilbert space. Then it follows from the Day-Dixmier unitarization Theorem (see e.g. [38, Chap. 0]) that any bounded representation of $G$ on $H$ extends to a bounded homomorphism $\mathcal{C}^*(G) \to B(H)$ from the group $\mathcal{C}^*$-algebra $\mathcal{C}^*(G)$ into $B(H)$. In general this extension property is no longer possible when $H$ is replaced by an arbitrary Banach space. To see a simple example, let $G$ be an infinite abelian group, let $1 \leq p < \infty$ and let $\lambda_p: G \to B(L^p(G))$ be the regular representation defined by letting $[\lambda_p(t)f](s) = f(s - t)$ for any $f \in L^p(G)$. Recall that $\mathcal{C}^*(G) = \mathcal{C}_0(\hat{G})$, where $\hat{G}$ denotes the dual group of $G$. Hence if $\lambda_p$ extends to a bounded homomorphism $\mathcal{C}^*(G) \to B(L^p(G))$, 

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then any function in $C_0(\widehat{G})$ is a bounded Fourier multiplier on $L^p(G)$. As is well-known, this implies that $p = 2$, see e.g. [32, Thm. 4.5.2]. (See also Corollary 6.2 for more on this.) This leads to the problem of finding conditions on a Banach space representation $\pi: G \to B(X)$ ensuring that its extension to a bounded homomorphism $C^*(G) \to B(X)$ is indeed possible.

We recall the definitions of $\gamma$-boundedness and $R$-boundedness. The latter is more classical (see [7]), but the two notions are completely similar. Let $(g_k)_{k \geq 1}$ be a sequence of complex valued, independent standard Gaussian variables on some probability space $\Sigma$. For any $x_1, \ldots, x_n$ in $X$, we let

$$\left\| \sum_k g_k \otimes x_k \right\|_{G(X)} = \left( \int_{\Sigma} \left( \sum_k \| g_k(\lambda) x_k \|_X^2 \right)^{1/2} d\lambda \right)^{1/2}.$$

Next we say that a set $F \subset B(X)$ is $\gamma$-bounded if there is a constant $C \geq 0$ such that for any finite families $T_1, \ldots, T_n$ in $F$, and $x_1, \ldots, x_n$ in $X$, we have

$$\left\| \sum_k g_k \otimes T_k x_k \right\|_{G(X)} \leq C \left\| \sum_k g_k \otimes x_k \right\|_{G(X)}.$$

In this case, we let $\gamma(F)$ denote the smallest possible $C$. This constant is called the $\gamma$-bound of $F$. Now let $(\varepsilon_k)_{k \geq 1}$ be a sequence of independent Rademacher variables on some probability space. Then replacing the sequence $(g_k)_{k \geq 1}$ by the sequence $(\varepsilon_k)_{k \geq 1}$ in the above definitions, we obtain the notion of $R$-boundedness. The corresponding $R$-bound constant of $F$ is denoted by $R(F)$. Using the symmetry of Gaussian variables, it is easy to see (and well-known) that any $R$-bounded set $F \subset B(X)$ is automatically $\gamma$-bounded, with $\gamma(F) \leq R(F)$. If further $X$ has a finite cotype, then Rademacher averages and Gaussian averages are equivalent (see e.g. [37, Chap. 3]), hence the notions of $R$-boundedness and $\gamma$-boundedness are equivalent. Clearly any $\gamma$-bounded set is bounded and if $X$ is isomorphic to a Hilbert space, then any bounded set is $\gamma$-bounded. We recall that conversely if $X$ is not isomorphic to a Hilbert space, then there exist bounded sets $F \subset B(X)$ which are not $\gamma$-bounded (see [11, Prop. 1.13]).

Our main result asserts that if $G$ is amenable and if $\pi: X \to B(X)$ is a representation such that $\{\pi(t) : t \in G\}$ is $\gamma$-bounded, then there exists a (necessarily unique) bounded homomorphism $w: C^*(G) \to B(X)$ extending $\pi$ (see Definition [2,4] for the precise meaning). Moreover $w$ is $\gamma$-bounded, i.e. it maps the unit ball of $C^*(G)$ into a $\gamma$-bounded set of $B(X)$.

If $X$ has property $(\alpha)$, we obtain the following analog of the Day-Dixmier unitarization Theorem: a representation $\pi: G \to B(X)$ extends to a bounded homomorphism $C^*(G) \to B(X)$ if and only if $\{\pi(t) : t \in G\}$ is $\gamma$-bounded. As an illustration, consider the case $G = \mathbb{Z}$ and recall that $C^*(\mathbb{Z}) = C(\mathbb{T})$. Let $T: X \to X$ be an invertible operator on a Banach space with property $(\alpha)$. We obtain that there exists a constant $C \geq 1$ such that

$$\left\| \sum_k c_k T^k \right\| \leq C \sup \left\{ \left| \sum_k c_k z^k \right| : z \in \mathbb{C}, |z| = 1 \right\}$$

for any finite sequence $(c_k)_{k \in \mathbb{Z}}$ of complex numbers, if and only if the set

$$\{T^k : k \in \mathbb{Z}\}$$

is $\gamma$-bounded.
The main result presented above is established in Section 4. Its proof makes crucial use of the transference methods available on amenable groups (see [8]) and of the Kalton-Weis ℓ-spaces introduced in the unpublished paper [28]. Sections 2 and 3 are devoted to preliminary results and background on these spaces and on group representations. In Section 5 we give a proof of the following result: if a Banach space $X$ has property ($\alpha$), then any bounded homomorphism $w: A \to B(X)$ defined on a nuclear $C^*$-algebra $A$ is automatically $R$-bounded (and even matricially $R$-bounded). This result is due to Éric Ricard (unpublished). In the case when $A$ is abelian, it goes back to De Pagter-Ricker [10] (see also [29]). Section 6 contains examples and illustrations, some of them using the above theorem. We pay a special attention to the $\gamma$-bounded representations of the classical abelian groups $\mathbb{Z}, \mathbb{R}, \mathbb{T}$.

We end this introduction with some notation and general references. First, we will use vector valued integration and Bochner $L^p$-spaces for which we refer to [15]. We let $G(X) \subset L^2(\Sigma; X)$ be the closed subspace spanned by the finite sums $\sum_k g_k \otimes x_k$, with $x_k \in X$. Next the space $\text{Rad}(X)$ is defined similarly, using the Rademacher sequence $(\varepsilon_k)_{k \geq 1}$. For any $n \geq 1$, we let $\text{Rad}_n(X) \subset \text{Rad}(X)$ be the subspace of all sums $\sum_{k=1}^n \varepsilon_k \otimes x_k$. It follows from classical duality on Bochner spaces that we have a natural isometric isomorphism

$$\text{Rad}_n(X)^{**} = \text{Rad}_n(X^{**}).$$

Second, we refer to [17] for general background on classical harmonic analysis. Given a locally compact group $G$, we let $dt$ denote a fixed left Haar measure on $G$. For any $p \geq 1$, we let $L^p(G) = L^p(G, dt)$ denote the corresponding $L^p$-space. We recall that the convolution on $G$ makes $L^1(G)$ a Banach algebra. Finally we will use basic facts on $C^*$-algebras and Hilbert space representations, for which [38] and [35] are relevant references.

For any Banach spaces $X, Y$, we let $B(Y, X)$ denote the space of all bounded operators from $Y$ into $X$, equipped with the operator norm, and we set $B(X) = B(X, X)$. Given any set $V$, we let $\chi_V$ denote the indicator function of $V$.

2. Preliminaries on $\gamma$-bounded representations.

We let $M_{n,m}$ denote the space of $n \times m$ scalar matrices equipped with its usual operator norm. We start with the following well-known tensor extension property, for which we refer e.g. to [14, Cor. 12.17].

**Lemma 2.1.** Let $a = [a_{ij}] \in M_{n,m}$ and let $x_1, \ldots, x_m \in X$. Then

$$\left\| \sum_{i,j} a_{ij} g_i \otimes x_j \right\|_{G(X)} \leq \|a\|_{M_{n,m}} \left\| \sum_j g_j \otimes x_j \right\|_{G(X)}.$$

This result does not remain true if we replace Gaussian variables by Rademacher variables and this defect is the main reason why it is sometimes easier to deal with $\gamma$-boundedness than with $R$-boundedness.

An extremely useful property proved in [7, Lem. 3.2] is that if $F \subset B(X)$ is any $R$-bounded set, then its strongly closed absolute convex hull $\overline{\text{ac}c}(F)$ is $R$-bounded as well, with
an estimate $R(\text{aco}(F)) \leq 2R(F)$. It turns out that a similar property holds for $\gamma$-bounded sets without the extra factor 2.

**Lemma 2.2.** Let $F \subset B(X)$ be any $\gamma$-bounded set. Then its closed absolute convex hull $\text{aco}(F)$ with respect to the strong operator topology is $\gamma$-bounded as well, and

$$\gamma(\text{aco}(F)) = \gamma(F).$$

**Proof.** Consider the set

$$\tilde{F} = \{zT : T \in F, \ z \in \mathbb{C}, \ |z| \leq 1\}.$$  

Applying Lemma 2.1 to diagonal matrices, we see that $\tilde{F}$ is $\gamma$-bounded and that $\gamma(\tilde{F}) = \gamma(F)$. Moreover $\text{aco}(F)$ is equal to $\text{co}(\tilde{F})$, the convex hull of $\tilde{F}$. Hence the argument in [7, Lem. 3.2] shows that $\text{aco}(F)$ is $\gamma$-bounded and that $\gamma(\text{aco}(F)) = \gamma(\tilde{F})$. The result follows at once. $\square$

Let $Z$ be an arbitrary Banach space. Following [29], we say that a bounded linear map $v: Z \to B(X)$ is $\gamma$-bounded (resp. $R$-bounded) if the set

$$\{v(z) : z \in Z, \ |z| \leq 1\}$$

is $\gamma$-bounded (resp. $R$-bounded). In this case, we let $\gamma(v)$ (resp. $R(v)$) denote the $\gamma$-bound (resp. the $R$-bound) of the latter set.

Next we say that a representation $\pi: G \to B(X)$ is $\gamma$-bounded (resp. $R$-bounded) if the set

$$\{\pi(t) : t \in G\}$$

is $\gamma$-bounded (resp. $R$-bounded). In this case, we let $\gamma(\pi)$ (resp. $R(\pi)$) denote the $\gamma$-bound (resp. the $R$-bound) of the latter set.

For any bounded representation $\pi: G \to B(X)$, we let $\sigma_\pi: L^1(G) \to B(X)$ denote the associated bounded homomorphism defined by

$$\sigma_\pi(k) = \int_G k(t)\pi(t) \, dt, \quad k \in L^1(G),$$

where the latter integral in defined in the strong sense. It turns out that $\sigma_\pi$ is nondegenerate, that is,

$$\text{Span}\{\sigma_\pi(k)x : k \in L^1(G), \ x \in X\}$$

is dense in $X$. Moreover, for every nondegenerate bounded homomorphism $\sigma: L^1(G) \to B(X)$, there exists a unique representation $\pi: G \to B(X)$ such that $\sigma = \sigma_\pi$, see [11, Lem. 2.4 and Rem. 2.5].

**Lemma 2.3.** Let $\pi: G \to B(X)$ be a bounded representation. Then $\pi$ is $\gamma$-bounded if and only if $\sigma_\pi$ is $\gamma$-bounded. Moreover $\gamma(\pi) = \gamma(\sigma_\pi)$ in this case.
Proof. For any \( k \in L^1(G) \) such that \( \|k\|_1 \leq 1 \), the operator \( \sigma_\pi(k) \) belongs to the strongly closed absolute convex hull of \( \{\pi(t) : t \in G\} \). Hence the ‘only if’ part follows from Lemma 2.2 and we have \( \gamma(\sigma_\pi) \leq \gamma(\pi) \).

For the converse implication, we let \( (h_i)_i \) be a contractive approximate identity of \( L^1(G) \). For any \( t \in G \), let \( \delta_t \) denote the point mass at \( t \). Then for any \( k \in L^1(G) \), and any \( x \in X \), we have

\[
\pi(t)\sigma_\pi(k)x = \sigma_\pi(\delta_t * k)x \\
= \lim_i \sigma_\pi(h_i * \delta_t * k)x \\
= \lim_i \sigma_\pi(h_i * \delta_t)\sigma_\pi(k)x.
\]

Hence if we let \( Y \subset X \) be the dense subspace defined by (2.1), we have that

\[
\lim_i \sigma_\pi(h_i * \delta_t)y = \pi(t)y, \quad y \in Y, \ t \in G.
\]

Now assume that \( \sigma_\pi \) is \( \gamma \)-bounded and let \( y_1, \ldots, y_n \in Y \) and \( t_1, \ldots, t_n \in G \). For any \( \iota \) and any \( k = 1, \ldots, n \), we have \( \|h_\iota * \delta_{t_\iota}\|_1 \leq 1 \). Hence

\[
\left\| \sum_k g_k \otimes \pi(t_\iota) y_k \right\|_{G(X)} \leq \gamma(\sigma_\pi) \left\| \sum_k g_k \otimes y_k \right\|_{G(X)}.
\]

Passing to the limit when \( \iota \to \infty \), this yields

\[
\left\| \sum_k g_k \otimes \pi(t) y_k \right\|_{G(X)} \leq \gamma(\sigma_\pi) \left\| \sum_k g_k \otimes y_k \right\|_{G(X)}.
\]

Since \( Y \) is dense in \( X \), this implies that \( \pi \) is \( \gamma \)-bounded, with \( \gamma(\pi) \leq \gamma(\sigma_\pi) \).

Let \( \lambda: G \to B(L^2(G)) \) denote the left regular representation. We recall that for any \( f \in L^2(G) \),

\[
\lambda(t)f = \delta_t * f \quad \text{and} \quad \sigma_\lambda(k) = k * f
\]

for any \( t \in G \) and any \( k \in L^1(G) \). The reduced \( C^* \)-algebra of \( G \) is defined as

\[
C^*_\lambda(G) = \sigma_\lambda(L^1(G)) \subset B(L^2(G)).
\]

We recall that \( C^*_\lambda(G) \) is equal to the group \( C^* \)-algebra \( C^*(G) \) if and only if \( G \) is amenable, see e.g. [34 (4.21)]. The notion on which we will focus on in Section 4 and beyond is the following.

**Definition 2.4.** We say that a bounded representation \( \pi: G \to B(X) \) extends to a bounded homomorphism \( w: C^*_\lambda(G) \to B(X) \) if \( w \circ \sigma_\lambda = \sigma_\pi \).

Note that there exists a bounded operator \( w: C^*_\lambda(G) \to B(X) \) such that \( w \circ \sigma_\lambda = \sigma_\pi \) if and only if there is a constant \( C \geq 0 \) such that

\[
\|\sigma_\pi(f)\| \leq C \|\sigma_\lambda(f)\|, \quad f \in L^1(G),
\]

that this extension is unique and is necessarily a homomorphism.
We refer the reader to [11] for some results concerning representations \( \pi : G \to B(X) \) extending to an \( R \)-bounded homomorphism \( w : C^*_\lambda(G) \to B(X) \) in the case when \( G \) is abelian, and their relationships with \( R \)-bounded spectral measures (see also Remark 4.5).

3. Multipliers on the Kalton-Weis \( \ell \)-spaces.

We will need abstract Hilbert space valued Banach spaces, usually called \( \ell \)-spaces, which were introduced by Kalton and Weis in the unpublished paper [28]. These \( \ell \)-spaces allow to define abstract square functions and were used in [28] to deal with relationships between \( H^\infty \) calculus and square function estimates. Similar spaces are constructed in [24] for the same purpose, in the setting of noncommutative \( L^p \)-spaces. Recently, \( \ell \)-spaces played an important role in the development of vector valued stochastic integration (see in particular [40, 41]) and for control theory in a Banach space setting [20]. In this section, we first recall some definitions and basics of \( \ell \)-spaces, and then we develop specific properties which will be useful in the next section.

Let \( X \) be a Banach space and let \( H \) be a Hilbert space. We let \( \overline{H} \) denote the conjugate space of \( H \). We will identify the algebraic tensor product \( \overline{H} \otimes X \) with the subspace of \( B(H, X) \) of all bounded finite rank operators in the usual way. Namely for any finite families \( (\xi_k)_k \) in \( H \) and \( (x_k)_k \) in \( X \), we identify the element \( \sum_k \xi_k \otimes x_k \) with the operator \( u : H \to X \) defined by letting \( u(\eta) = \sum_k \langle \eta, \xi_k \rangle x_k \) for any \( \eta \in H \).

For any \( u \in \overline{H} \otimes X \), there exists a finite orthonormal family \( (e_k)_k \) of \( H \) and a finite family \( (x_k)_k \) of \( X \) such that \( u = \sum_k e_k \otimes x_k \). Then we set

\[
\|u\|_G = \left\| \sum_k g_k \otimes x_k \right\|_{G(X)}.
\]

Using Lemma 2.1, it is easy to check that this definition does not depend on the \( e_k \)'s and \( x_k \)'s representing \( u \). Next for any \( u \in B(H, X) \), we set

\[
\|u\|_\ell = \sup \{ \|uP\|_G : P : H \to H \text{ finite rank orthogonal projection} \}.
\]

Note that the above quantity may be infinite. Then we denote by \( \ell_+(H, X) \) the space of all bounded operators \( u : H \to X \) such that \( \|u\|_\ell < \infty \). This is a Banach space for the norm \( \|u\|_\ell \). We let \( \ell(H, X) \) denote the closure of \( \overline{H} \otimes X \) in \( \ell_+(H, X) \). It is observed in [28] that \( \ell(H, X) = \ell_+(H, X) \) provided that \( X \) does not contain \( c_0 \) (we will not use this fact in this paper).

Proposition 3.1. Let \( S \in B(H) \).

1. For any finite rank operator \( u : H \to X \), we have \( \|u \circ S\|_G \leq \|u\|_G \|S\| \).

2. For any \( u \in \ell_+(H, X) \), the operator \( u \circ S \) belongs to \( \ell_+(H, X) \) and \( \|u \circ S\|_\ell \leq \|u\|_\ell \|S\| \).

Proof. Part (1) is a straightforward consequence of Lemma 2.1. Indeed suppose that \( u = \sum_i e_i \otimes x_i \) for some finite orthonormal family \( (e_i)_i \) of \( H \) and some \( x_i \in X \). Then if \( (e'_j)_j \) is
an orthonormal basis of \( \text{Span}\{S^*(e_i) : i = 1, \ldots, n\} \), we have

\[
u \circ S = \sum_{i,j} \langle e_i, S(e'_j) \rangle \overline{e'_j} \otimes x_i.
\]

Hence

\[
\|u \circ S\|_G = \left\| \sum_{i,j} \langle e_i, S(e'_j) \rangle g_j \otimes x_i \right\|_{G(X)} \leq \left\| \langle e_i, S(e'_j) \rangle \right\|_{\ell_2 \rightarrow \ell_2} \left\| \sum_i g_i \otimes x_i \right\|_{G(X)} \leq \|S\|_G \|u\|_G.
\]

To prove (2), consider an arbitrary \( u: H \rightarrow X \) and let \( P: H \rightarrow H \) be a finite rank orthogonal projection. Then \( SP \) is finite rank hence there exists a finite rank orthogonal projection \( Q: H \rightarrow H \) such that \( SP = QSP \). Applying the first part of this proof to \( uQ \), we infer that

\[
\|uSP\|_G = \|uQSP\|_G \leq \|uQ\|_G \|QSP\| \leq \|u\|_\ell \|S\|.
\]

The result follows by passing to the supremum over \( P \).

\[\square\]

**Remark 3.2.**

(1) It is clear from above that for any finite rank \( u: H \rightarrow X \), we have \( \|u\|_G = \|u\|_\ell \). More generally for any \( u: H \rightarrow X \), we have \( \|u\|_\ell = \sup\{\|uw\|_G\} \), where the supremum runs over all finite rank operators \( w: H \rightarrow H \) with \( \|w\| \leq 1 \).

(2) Let \( S \in B(H) \) and let \( \varphi_S: B(H, X) \rightarrow B(H, X) \) be defined by \( \varphi_S(u) = u \circ S \). It is easy to check (left to the reader) that the restriction of \( \varphi_S \) to \( \overline{H} \otimes X \) coincides with \( S^* \otimes I_X \).

We will now focus on the case when \( H = L^2(\Omega, \mu) \), for some arbitrary measure space \( (\Omega, \mu) \). We will identify \( H \) and \( \overline{H} \) in the usual way. We let \( L^2(\Omega; X) \) be the associated Bochner space and we recall that \( L^2(\Omega) \otimes X \) is dense in \( L^2(\Omega; X) \). There is a natural embedding of \( L^2(\Omega; X) \) into \( B(L^2(\Omega), X) \) obtained by identifying any \( F \in L^2(\Omega; X) \) with the operator

\[
u_F: f \mapsto \int_\Omega F(t) f(t) \, d\mu(t), \quad f \in L^2(\Omega).
\]

Thus we have the following diagram of embeddings, that we will use without any further reference. For example, it will make sense through these identifications to compute \( \|F\|_\ell \) for any \( F \in L^2(\Omega; X) \).

\[
\begin{align*}
L^2(\Omega; X) & \quad \mapsto \quad B(L^2(\Omega), X) \\
L^2(\Omega) \otimes X & \quad \longrightarrow \quad \ell(L^2(\Omega), X) \\
& \quad \longrightarrow \quad \ell_+(L^2(\Omega), X)
\end{align*}
\]

By a subpartition of \( \Omega \), we mean a finite set \( \theta = \{I_1, \ldots, I_m\} \) of pairwise disjoint measurable subsets of \( \Omega \) such that \( 0 < \mu(I_i) < \infty \) for any \( i = 1, \ldots, m \). We will use the natural partial order on subpartitions, obtained by saying that \( \theta \leq \theta' \) if and only if each set in \( \theta \) is a
union of some sets in $\theta'$. For any subpartition $\theta = \{I_1, \ldots, I_m\}$, we let $E_\theta : L^2(\Omega) \to L^2(\Omega)$ be the orthogonal projection defined by
\[ E_\theta(f) = \sum_{i=1}^{m} \frac{1}{\mu(I_i)} \left( \int_{I_i} f(t) \, d\mu(t) \right) \chi_{I_i}, \quad f \in L^2(\Omega). \]
It is plain that $\lim_{\theta \to \infty} \|E_\theta(f) - f\|_2 = 0$ for any $f \in L^2(\Omega)$. Now let $E_\theta^X : B(L^2(\Omega), X) \to L^2(\Omega) \otimes X$ be defined by $E_\theta^X(u) = uE_\theta$. Then the above approximation property extends as follows.

**Lemma 3.3.**

1. For any $u \in \ell(L^2(\Omega), X)$, $\lim_{\theta \to \infty} \|E_\theta^X(u) - u\|_\ell = 0$.
2. For any $u \in L^2(\Omega; X)$, $\lim_{\theta \to \infty} \|E_\theta^X(u) - u\|_{L^2(\Omega; X)} = 0$.

**Proof.** By Remark 3.2 (2), the restriction of $E_\theta^X$ to $L^2(\Omega) \otimes X$ coincides with $E_\theta \otimes I_X$, hence (1) holds true if $u \in L^2(\Omega) \otimes X$. According to Proposition 3.1, we have $\|E_\theta^X : \ell(L^2(\Omega), X) \to \ell(L^2(\Omega), X)\| \leq 1$. Since $L^2(\Omega) \otimes X$ is dense in $\ell(L^2(\Omega), X)$, part (1) follows by equicontinuity. The proof of (2) is identical. \qed

**Lemma 3.4.** For any $u \in B(L^2(\Omega), X)$ and any subpartition $\theta_0$ of $\Omega$,
\[ \|u\|_\ell = \sup \{ \|uE_\theta\|_G : \theta \text{ subpartition of } \Omega, \theta \geq \theta_0 \}. \]

**Proof.** Let $P : L^2(\Omega) \to L^2(\Omega)$ be a finite rank orthogonal projection, and let $(h_1, \ldots, h_n)$ be an orthonormal basis of its range. Then
\[ uP = \sum_k h_k \otimes u(h_k) \quad \text{and} \quad uE_\theta P = \sum_k h_k \otimes uE_\theta(h_k) \]
for any subpartition $\theta$. Since $E_\theta(h_k) \to h_k$ for any $k = 1, \ldots, n$, we deduce that
\[ \|uP\|_G = \lim_{\theta \to \infty} \|uE_\theta P\|_G. \]
By Proposition 3.3 this implies that $\|uP\|_G \leq \sup_{\theta \geq \theta_0} \|uE_\theta\|_G$ and the result follows at once. \qed

Let $\phi : \Omega \to B(X)$ be a bounded strongly measurable function. We may define a multiplication operator $T_\phi : L^2(\Omega; X) \to L^2(\Omega; X)$ by letting
\[ [T_\phi(F)](t) = \phi(t)F(t), \quad F \in L^2(\Omega; X). \]
Consider the associated bounded set
\[ F_\phi = \left\{ \frac{1}{\mu(I)} \int_I \phi(t) \, d\mu(t) : I \subset \Omega, 0 < \mu(I) < \infty \right\}. \]

The following is an analog of [24, Prop. 4.4] and extends [28, Prop. 4.11].
Proposition 3.5. If the set $F_\phi$ is $\gamma$-bounded, there exists a (necessarily unique) bounded operator

$$M_\phi: \ell(L^2(\Omega), X) \rightarrow \ell_+(L^2(\Omega), X),$$

such that $M_\phi$ and $T_\phi$ coincide on the intersection $\ell(L^2(\Omega), X) \cap L^2(\Omega; X)$. Moreover we have

$$\|M_\phi\| \leq \gamma(F_\phi).$$

Proof. Let $E \subset L^2(\Omega)$ be the dense subspace of all simple functions and let $u \in E \otimes X$. There exists a subpartition $\theta_0 = (A_1, \ldots, A_N)$ and some $x_1, \ldots, x_N$ in $X$ such that

$$u = \sum_j \chi_{A_j} \otimes x_j.$$

Let $\theta = (I_1, \ldots, I_m)$ be another subpartition and assume that $\theta_0 \leq \theta$. Thus there exist $\alpha_{ij} \in \{0, 1\}$ for $i = 1, \ldots, m$ and $j = 1, \ldots, N$ such that $\chi_{A_j} = \sum_i \alpha_{ij} \chi_{I_i}$ for any $j$. Consequently, we have

$$u = \sum_{i,j} \alpha_{ij} \chi_{I_i} \otimes x_j$$

and

$$[T_\phi(u)](t) = \sum_{i,j} \alpha_{ij} \chi_{I_i}(t) \phi(t)x_j.$$

For any $i = 1, \ldots, m$, let

$$T_i = \frac{1}{\mu(I_i)} \int_{I_i} \phi(t) d\mu(t).$$

Then a thorough look at the definition of $E_\phi^X$ shows that

$$E_\phi^X(T_\phi(u)) = \sum_{i,j} \alpha_{ij} \chi_{I_i} \otimes T_i(x_j).$$

Since $(\mu(I_i)^{-\frac{1}{2}} \chi_{I_i})_i$ is an orthonormal family of $L^2(\Omega)$, this implies that

$$\|E_\phi^X(T_\phi(u))\|_G = \left\| \sum_{i,j} \alpha_{ij} \mu(I_i)^{\frac{1}{2}} g_i \otimes T_i(x_j) \right\|_{G(X)}.$$

Likewise,

$$\|u\|_G = \left\| \sum_{i,j} \alpha_{ij} \mu(I_i)^{\frac{1}{2}} g_i \otimes x_j \right\|_{G(X)}.$$

Since each $T_i$ belongs to the set $F_\phi$, this implies that $\|E_\phi^X(T_\phi(u))\|_G \leq \gamma(F_\phi)\|u\|_G$. Taking the supremum over $\theta$ and applying Lemma 3.4, we obtain that $T_\phi(u) \in \ell_+(L^2(\Omega), X)$, with

$$\|T_\phi(u)\|_\ell \leq \gamma(F_\phi)\|u\|_G.$$

This induces a bounded operator $M_\phi: \ell(L^2(\Omega), X) \rightarrow \ell_+(L^2(\Omega), X)$ coinciding with $T_\phi$ on $E \otimes X$ and verifying $\|M_\phi\| \leq \gamma(F_\phi)$.

To show that $M_\phi$ and $T_\phi$ coincide on $\ell(L^2(\Omega), X) \cap L^2(\Omega; X)$, let $u$ belong to this intersection and note that by construction, $M_\phi(E_\phi^X(u)) = T_\phi(E_\phi^X(u))$ for any subpartition $\theta$. Then the equality $M_\phi(u) = T_\phi(u)$ follows from Lemma 3.3. \qed
In the rest of this section, we consider natural tensor extensions of the spaces and multipliers considered so far. Let $N \geq 1$ be a fixed integer and let $(e_1, \ldots, e_N)$ denote the canonical basis of $\ell_N^2$. We let $\ell_N^2 \otimes L^2(\Omega)$ be the Hilbert space tensor product of $\ell_N^2$ and $L^2(\Omega)$. For any bounded operator $u: \ell_N^2 \otimes L^2(\Omega) \to X$ and any $k = 1, \ldots, N$, let $u_k: L^2(\Omega) \to X$ be defined by $u_k(f) = u(e_k \otimes f)$. Then the mapping $u \mapsto \sum_k e_k \otimes u_k$ induces an algebraic isomorphism

\[
B(\ell_N^2 \otimes L^2(\Omega), X) \simeq \ell_N^2 \otimes B(L^2(\Omega), X).
\]

Let us now see the effects of this isomorphism on the special spaces considered so far. Let

\[
\ell_N^2 \otimes L^2(\Omega) = L^2(\Omega_N).
\]

Then it is clear that under the identification (3.2), an operator $u: L^2(\Omega_N) \to X$ belongs to $L^2(\Omega_N; X)$ if and only if $u_k$ belongs to $L^2(\Omega; X)$ for any $k = 1, \ldots, N$. Moreover this induces an isometric isomorphism identification

\[
L^2(\Omega_N; X) = \ell_N^2(L^2(\Omega; X)).
\]

Likewise it is easy to check (left to the reader) that $u: L^2(\Omega_N) \to X$ belongs to $\ell_+(L^2(\Omega_N), X)$ if and only if $u_k$ belongs to $\ell_+(L^2(\Omega), X)$ (resp. $\ell(L^2(\Omega), X)$) for any $k = 1, \ldots, N$, which leads to algebraic isomorphisms

\[
\ell_+(L^2(\Omega_N), X) \simeq \ell_N^2 \otimes \ell_+(L^2(\Omega), X) \quad \text{and} \quad \ell(L^2(\Omega_N), X) \simeq \ell_N^2 \otimes \ell(L^2(\Omega), X).
\]

Now let $\phi: \Omega \to B(X)$ be a bounded strongly measurable function as before and let $\phi_N: \Omega_N \to B(X)$ be defined by

\[
\phi_N(t, k) = \phi(t), \quad t \in \Omega, \ k = 1, \ldots, N.
\]

As in (3.1), we may associate a set $F_{\phi_N} \subset B(X)$ to $\phi_N$. A moment’s thought shows that $F_{\phi} \subset F_{\phi_N} \subset \text{co}(F_{\phi})$. Hence $F_{\phi_N}$ is $\gamma$-bounded if and only if $F_{\phi}$ is $\gamma$-bounded and we have

\[
\gamma(F_{\phi_N}) = \gamma(F_{\phi})
\]

in this case. It is clear that under the identifications (3.3), the associated multiplier operator $M_{\phi_N}: \ell(L^2(\Omega_N), X) \to \ell_+(L^2(\Omega_N), X)$ satisfies

\[
M_{\phi_N} = I_{\ell_N^2} \otimes M_{\phi}.
\]

4. Characterization of $\gamma$-bounded representations of amenable groups

Throughout we let $G$ be a locally compact group equipped with a left Haar measure and for any measurable $I \subset G$, we simply let $|I|$ denote the measure of $I$. If $\pi: G \to B(X)$ is any bounded representation and $\|\pi\| = \sup_{t \in G} \|\pi(t)\|$, it is plain that for any $I \subset G$ and any $z \in X$, we have

\[
\|\pi\|^{-1}|I|^{\frac{1}{2}}\|z\| \leq \left( \int_I \|\pi(t)z\|^2 \, dt \right)^{\frac{1}{2}} \leq \|\pi\||I|^{\frac{1}{2}}\|z\|.
\]
The first part of the following lemma is an analog of this double estimate when the space \( L^2(G;X) \) is replaced by \( \ell_+(L^2(G),X) \). In the second part, we apply the principles explained at the end of the previous section.

**Lemma 4.1.** Let \( \pi: G \to B(X) \) be a \( \gamma \)-bounded representation and let \( I \subset G \) be any measurable subset of \( G \) with finite measure.

1. For any \( z \in X \), the function \( t \mapsto \chi_I(t)\pi(t)z \) belongs to \( \ell_+(L^2(G),X) \) and we have
   \[
   \gamma(\pi)^{-1}|I|^\frac{1}{2}\|z\| \leq \|t \mapsto \chi_I(t)\pi(t)z\|_\ell \leq \gamma(\pi)|I|^\frac{1}{2}\|z\|.
   \]

2. Let \( N \geq 1 \) be an integer. Let \( z_1, \ldots, z_N \in X \) and let \( F_k(t) = \chi_I(t)\pi(t)z_k \) for any \( k = 1, \ldots, N \). Then
   \[
   \gamma(\pi)^{-1}|I|^\frac{1}{2}\left\| \sum_k e_k \otimes z_k \right\|_G \leq \left\| \sum_k e_k \otimes F_k \right\|_\ell \leq \gamma(\pi)|I|^\frac{1}{2}\left\| \sum_k e_k \otimes z_k \right\|_G.
   \]

**Proof.** Part (1) is a special case of part (2) so we only need to prove the second statement. The upper estimate is a simple consequence of Proposition 3.5 applied with \( \pi = \phi \), and the discussion at the end of Section 3. Indeed, let \( F_\pi \) be the set associated with \( \pi: G \to B(X) \) as in (3.1). For any \( I \subset G \) with \( 0 < |I| < \infty \), the operator \( |I|^{-1} \int_I \pi(t) \, dt \) belongs to the strong closure of the absolute convex hull of \( \{\pi(t) : t \in G\} \). Hence \( \gamma(F_\pi) \leq \gamma(\pi) \) by Lemma 2.2.

Let
\[
M_\pi: \ell(L^2(G),X) \to \ell_+(L^2(G),X)
\]
be the multiplier operator associated with \( \pi \). Then for any \( I \subset G \) and any \( z \in X \), the function \( t \mapsto \chi_I(t)\pi(t)z \) is equal to \( M_\pi(\chi_I \otimes z) \). Thus according to (3.4), we have
\[
\sum_k e_k \otimes F_k = M_\pi \left( \sum_k e_k \otimes \chi_I \otimes z_k \right).
\]

Moreover \( (|I|^{-\frac{1}{2}}e_k \otimes \chi_I)_k \) is an orthonormal family of \( \ell^2_N \otimes L^2(G) \), hence
\[
\left\| \sum_k e_k \otimes \chi_I \otimes z_k \right\|_\ell = |I|^{\frac{1}{2}} \left\| \sum_k e_k \otimes z_k \right\|_G.
\]

Consequently we have
\[
\left\| \sum_k e_k \otimes F_k \right\|_\ell \leq \gamma(F_{\pi_N}) \left\| \sum_k e_k \otimes \chi_I \otimes z_k \right\|_\ell \leq \gamma(\pi)|I|^\frac{1}{2}\left\| \sum_k e_k \otimes z_k \right\|_G.
\]

We now turn to the lower estimate, for which we will use duality. For any \( \varphi_1, \ldots, \varphi_N \) in \( X^* \), we set
\[
\left\| (\varphi_1, \ldots, \varphi_N) \right\|_{\ell^\infty} = \sup \left\{ \left| \sum_{k=1}^N \langle \varphi_k, x_k \rangle \right| : x_1, \ldots, x_N \in X, \left\| \sum_{k=1}^N g_k \otimes x_k \right\|_{G(X)} \leq 1 \right\}.
\]

We fix some \( I \subset G \) with \( 0 < |I| < \infty \). Then we consider \( z_1, \ldots, z_N \) in \( X \) and the functions \( F_1, \ldots, F_N \) in \( L^2(G;X) \) given by \( F_k(t) = \chi_I(t)\pi(t)z_k \). By Hahn-Banach there exist \( \varphi_1, \ldots, \varphi_N \)
in $X^*$ such that
\[ \|(\varphi_1, \ldots, \varphi_N)\|_{\ell^*} = 1 \quad \text{and} \quad \left\| \sum_k e_k \otimes z_k \right\|_G = \sum_k \langle \varphi_k, z_k \rangle. \]

Using the latter equality and Lemma \ref{lemma:3.3} (2), we thus have
\[ |I| \left\| \sum_k e_k \otimes z_k \right\|_G = \sum_k \int_I \langle \varphi_k, z_k \rangle \, dt \]
\[ = \sum_k \int_I \langle \pi(t^{-1})^* \varphi_k, \pi(t)z_k \rangle \, dt \]
\[ = \sum_k \int_G \langle \chi_I(t) \pi(t^{-1})^* \varphi_k, F_k(t) \rangle \, dt \]
\[ = \lim_{\theta \to \infty} \sum_k \int_G \langle \chi_I(t) \pi(t^{-1})^* \varphi, [E^X_\theta(F_k)](t) \rangle \, dt. \]

Let $\theta = (I_1, \ldots, I_m)$ be a subpartition of $G$ such that $I = I_1 \cup \cdots \cup I_n$ for some $n \leq m$ and let
\[ J_\theta = \sum_k \int_G \langle \chi_I(t) \pi(t^{-1})^* \varphi_k, [E^X_\theta(F_k)](t) \rangle \, dt \]
be the above sum of integrals. For any $i = 1, \ldots, n$, let
\[ T_i = \frac{1}{|I_i|} \int_{I_i} \pi(t) \, dt \quad \text{and} \quad S_i = \frac{1}{|I_i|} \int_{I_i} \pi(t^{-1}) \, dt. \]

For any $k$ we have
\[ E^X_\theta(F_k) = \sum_{i=1}^n \chi_{I_i} \otimes T_i(z_k). \]

We deduce that
\[ J_\theta = \sum_{k=1}^N \sum_{i=1}^n \int_{I_i} \langle \pi(t^{-1})^* \varphi_k, T_i(z_k) \rangle \, dt \]
\[ = \sum_{k=1}^N \sum_{i=1}^n \int_{I_i} \langle \varphi_k, \pi(t^{-1})T_i(z_k) \rangle \, dt \]
\[ = \sum_{k=1}^N \sum_{i=1}^n |I_i| \langle \varphi_k, S_i T_i(z_k) \rangle. \]

According to the definition of the $\ell^*$-norm, this identity implies that
\[ |J_\theta| \leq \left\| \sum_k g_k \otimes \left( \sum_i |I_i| S_i T_i(z_k) \right) \right\|_{G(X)}. \]
Let \( a : \ell^2_{nN} \to \ell^2_{N} \) be defined by
\[
a \left( (c_{ik})_{1 \leq i \leq n, 1 \leq k \leq N} \right) = \left( \sum_{i} c_{ik} |I_i|^{\frac{1}{2}} \right)_k, \quad c_{ik} \in \mathbb{C}.
\]

Let \( c = (c_{ik}) \) in \( \ell^2_{nN} \). Using Cauchy-Schwarz and the fact that \( |I| = \sum_i |I_i| \), we have
\[
\|a(c)\|_2^2 = \sum_k \left( \sum_i c_{ik} |I_i|^{\frac{1}{2}} \right)^2 \leq \sum_k \left( \sum_i |c_{ik}|^2 \right) \left( \sum_i |I_i| \right) = |I| \|c\|_2.
\]

Hence \( \|a\| \leq |I|^{\frac{1}{2}} \). Let \( (g_{ik})_{i,k \geq 1} \) be a doubly indexed family of independent standard Gaussian variables. According to Lemma 2.1, the latter estimate implies that
\[
\left\| \sum_{i,k} g_{ik} \otimes |I_i|^{\frac{1}{2}} y_{ik} \right\|_{G(X)} \leq |I|^{\frac{1}{2}} \left\| \sum_{i,k} g_{ik} \otimes y_{ik} \right\|_{G(X)}
\]
for any \( y_{ik} \) in \( X \). We deduce that
\[
|J_\theta| \leq |I|^{\frac{1}{2}} \left\| \sum_{i,k} g_{ik} \otimes |I_i|^{\frac{1}{2}} S_i T_i(z_k) \right\|_{G(X)}.
\]

Next observe that by convexity again, we have \( \gamma(\{S_1, \ldots, S_n\}) \leq \gamma(\pi) \). The latter estimate therefore implies that
\[
|J_\theta| \leq \gamma(\pi) |I|^{\frac{1}{2}} \left\| \sum_{i,k} g_{ik} \otimes |I_i|^{\frac{1}{2}} T_i(z_k) \right\|_{G(X)}.
\]

Since \( (|I_i|^{-\frac{1}{2}} \chi_{I_i})_i \) is an orthonormal family of \( L^2(G) \), we have, using (4.1),
\[
\left\| \sum_{i,k} g_{ik} \otimes |I_i|^{\frac{1}{2}} T_i(z_k) \right\|_{G(X)} = \left\| \sum_{i,k} e_k \otimes \chi_{I_i} \otimes T_i(z_k) \right\|_G
\]
\[
= \left\| (\ell^2_N \otimes E_\theta^X) \left( \sum_k e_k \otimes F_k \right) \right\|_G \leq \left\| \sum_k e_k \otimes F_k \right\|_\ell.
\]

Hence
\[
|J_\theta| \leq \gamma(\pi) |I|^{\frac{1}{2}} \left\| \sum_k e_k \otimes F_k \right\|_\ell,
\]
and passing to the limit when \( \theta \to \infty \), this yields the lower estimate. \( \square \)

**Remark 4.2.** The above lemma remains true if \( \pi(t) \) is replaced by \( \pi(t^{-1}) \). This follows either from the proof itself, or by considering the representation \( \pi^{op} : G^{op} \to B(X) \) defined by \( \pi^{op}(t) = \pi(t^{-1}) \). Here \( G^{op} \) denotes the opposite group of \( G \), i.e. \( G \) equipped with the reverse product.
The following notion was introduced in [29]. For any $C^*$-algebra $A$, the space $M_N(A)$ of $N \times N$ matrices with entries in $A$ is equipped with its unique $C^*$-norm.

**Definition 4.3.** Let $A$ be a $C^*$-algebra and let $w: A \to B(X)$ be a bounded linear map.

1. We say that $w$ is matricially $\gamma$-bounded if there is a constant $C \geq 0$ such that

$$
\left\| \sum_{i,j=1}^{N} g_i \otimes w(a_{ij})x_j \right\|_{G(X)} \leq C \left\| [a_{ij}] \right\|_{M_N(A)} \left\| \sum_{j=1}^{N} g_j \otimes x_j \right\|_{G(X)}
$$

for any $N \geq 1$, for any $[a_{ij}] \in M_N(A)$ and for any $x_1, \ldots, x_N \in X$. In this case we let $\|w\|_{\text{Mat} - \gamma}$ denote the smallest possible constant $C$.

2. We say that $w$ is matricially $R$-bounded if (4.2) holds when the Gaussian sequence $(g_k)_k$ is replaced by a Rademacher sequence $(\varepsilon_k)_k$, and we let $\|w\|_{\text{Mat} - R}$ denote the smallest possible constant in this case.

Two simple comments are in order (see [29, Remark 4.2] for details). First, restricting (4.2) to the case when $[a_{ij}]$ is a diagonal matrix, we obtain that any matricially $\gamma$-bounded map $w: A \to B(X)$ is $\gamma$-bounded, with

$$
\gamma(w) \leq \|w\|_{\text{Mat} - \gamma}.
$$

Second, if $X = H$ is a Hilbert space, then $\gamma$-matricial boundedness coincides with complete boundedness and we have $\|w\|_{\text{Mat} - \gamma} = \|w\|_{cb}$ (the completely bounded norm of $w$). Similar comments apply to $R$-boundedness.

The proof of our main result below uses transference techniques from [8] in the framework of $\ell$-spaces.

**Theorem 4.4.** Let $G$ be an amenable locally compact group and let $\pi: G \to B(X)$ be a bounded representation. The following assertions are equivalent.

1. $\pi$ is $\gamma$-bounded.
2. $\pi$ extends to a bounded homomorphism $w: C^*_\lambda(G) \to B(X)$ (in the sense of Definition 2.4) and $w$ is $\gamma$-bounded.

In this case, $w$ is matricially $\gamma$-bounded and

$$
\gamma(\pi) \leq \gamma(w) \leq \|w\|_{\text{Mat} - \gamma} \leq \gamma(\pi)^2.
$$

**Proof.** Assume (ii) and let $\sigma_\pi: L^1(G) \to B(X)$ be induced by $\pi$. Then $\sigma_\pi = w \circ \sigma_\lambda$ and $\sigma_\lambda$ is a contraction. Hence $\sigma_\pi$ is $\gamma$-bounded, with $\gamma(\sigma_\pi) \leq \gamma(\sigma_\lambda)$. Then (i) follows from Lemma 2.3 and we have $\gamma(\pi) \leq \gamma(\sigma_\pi)$.

Assume (i). Our proof of (ii) will be divided into two parts. We first show that for any $k \in L^1(G)$, we have

$$
\|\sigma_\pi(k)\| \leq \gamma(\pi)^2 \|\sigma_\lambda(k)\|.
$$

This implies the existence of $w: C^*_\lambda(G) \to B(X)$ extending $\pi$. Then we will show (4.6), which implies that $w$ is actually $\gamma$-bounded. Although (4.3) is a special case of (4.6), establishing that estimate first makes the proof easier to read.
Let $k \in L^1(G)$ and assume that $k$ has a compact support $\Gamma \subset G$. Let $V \subset G$ be an arbitrary open neighborhood of the unit $e$, with $0 < |V| < \infty$. We let $T: L^2(G) \to L^2(G)$ be the multiplication operator defined by letting $T(f) = \chi_V f$ for any $f \in L^2(G)$. Then we let $S: L^2(G) \to L^2(G)$ be defined by

$$(Sg)(s) = \int_G k(t)g(ts) \, dt, \quad g \in L^2(G), \ s \in G.$$ 

Under the natural duality between $L^2(G)$ and itself, $S$ is the transposed map of $\sigma_\lambda(k)$, hence

$$\|S\| = \|\sigma_\lambda(k)\|.$$ 

Let $x \in X$. The set $\Gamma^{-1}V \subset G$ has a positive and finite measure, hence applying Lemma 4.1 (and Remark 4.2), we see that the function $\tilde{u} = u \circ S \circ T \in B(L^2(G), X)$. Consider an arbitrary $f \in L^2(G)$. For any $h \in L^2(G)$,

$$u(h) = \int_G h(s)\chi_{\Gamma^{-1}V}(s)\pi(s^{-1})x \, ds,$$

hence according to the definitions of $T$ and $S$, we have

$$\tilde{u}(f) = \int_G \left( \int_G k(t)\chi_V(ts)f(ts) \, dt \right) \chi_{\Gamma^{-1}V}(s)\pi(s^{-1})x \, ds.$$ 

Using Fubini (which is applicable because $\chi_V f$ is integrable) and the left invariance of $ds$, this implies

$$\tilde{u}(f) = \int_G k(t) \left( \int_G \chi_V(ts)f(ts)\chi_{\Gamma^{-1}V}(s)\pi(s^{-1})x \, ds \right) \, dt$$

$$= \int_G k(t) \left( \int_G \chi_V(s)f(s)\chi_{\Gamma^{-1}V}(t^{-1}s)\pi(s^{-1}t)x \, ds \right) \, dt$$

$$= \int_G \chi_V(s)f(s) \left( \int_G k(t)\chi_{\Gamma^{-1}V}(t^{-1}s)\pi(s^{-1}t)x \, dt \right) \, ds.$$ 

Since $k$ is supported in $\Gamma$ we deduce that

$$(4.5) \quad \tilde{u}(f) = \int_G \chi_V(s)f(s) \left( \int_G k(t)\pi(s^{-1}t)x \, dt \right) \, ds.$$ 

Let $y = \sigma_\pi(k)x$. For any $s \in G$, we have

$$\int_G k(t)\pi(s^{-1}t)x \, dt = \int_G k(t)\pi(s^{-1})\pi(t)x \, dt = \pi(s^{-1})y.$$ 

Thus (4.5) shows that $\tilde{u}$ is the bounded operator associated to the function

$$\tilde{F}: s \mapsto \chi_V(s)\pi(s^{-1})y.$$
By Proposition 3.1 we have \( \| \tilde{F} \|_\ell \leq \|ST\|\|F\|_\ell \). Applying (4.4) and the fact that \( T \) is a contraction, we therefore obtain that
\[
\| s \mapsto \chi_V(s)\pi(s^{-1})y \|_\ell \leq \| \sigma_\lambda(k) \|_\ell \| s \mapsto \chi_{\Gamma^{-1}V}(s)\pi(s^{-1})x \|_\ell.
\]
Applying Lemma 4.1 (and Remark 4.2) twice we deduce that
\[
|V|^\frac{1}{2} \| y \| \leq \gamma(\pi)^2 \| \sigma_\lambda(k) \| \| \Gamma^{-1}V \|^{\frac{1}{2}} \| x \|,
\]
and hence
\[
\| \sigma_\pi(k)x \| \leq \gamma(\pi)^2 \left( \frac{\| \Gamma^{-1}V \|}{|V|} \right)^{\frac{1}{2}} \| \sigma_\lambda(k) \| \| x \|.
\]
We now apply the assumption that \( G \) is amenable. According to Folner’s condition (see e.g. [8, Chap. 2]), we can choose \( V \) such that \( \frac{\| \Gamma^{-1}V \|}{|V|} \) is arbitrarily close to 1. This yields (4.3) when \( k \) is compactly supported. Since \( \sigma_\lambda \) and \( \sigma_\pi \) are continuous, this actually implies (4.3) for any \( k \in L^1(G) \).

We now aim at showing that \( w: C^*_\lambda(G) \to B(X) \) is matricially \( \gamma \)-bounded and that \( \| w \|_{\text{Mat}-\gamma} \leq \gamma(\pi)^2 \). In fact the argument is essentially a repetition of the above one, modulo standard matrix manipulations. We fix some integer \( N \geq 1 \) and consider \( x_1, \ldots, x_N \) in \( X \). According to Definition 4.3 it suffices to show that for any \([k_{ij}] \in M_N \otimes L^1(G)\), we have
\[
\| \sum_{i,j} g_{ij} \otimes \sigma_\pi(k_{ij})x_j \|_{G(X)} \leq \gamma(\pi)^2 \| [\sigma_\lambda(k_{ij})] \|_{M_N(C^*_\lambda(G))} \| \sum_j g_j \otimes x_j \|_{G(X)}.
\]
In the sequel we let
\[
x = \sum_{j=1}^N e_j \otimes x_j \in \ell_N^2 \otimes X.
\]
Let us identify \( M_N \otimes L^1(G) \) with \( L^1(G; M_N) \) in the natural way and let \( k \in L^1(G; M_N) \) be the \( M_N \)-valued function corresponding to \([k_{ij}]\). Then
\[
(I_{M_N} \otimes \sigma_\pi)([k_{ij}]) = \int_G (k(t) \otimes \pi(t)) \ dt \quad \text{in} \quad M_N \otimes B(X).
\]
Using the isometric identification
\[
\ell_N^2 \otimes L^2(G) = L^2(G; \ell_N^2),
\]
we can regard \( M_N(C^*_\lambda(G)) \) as a \( C^* \)-subalgebra of \( B(L^2(G; \ell_N^2)) \). In this situation, it is easy to check that the matrix \([\sigma_\lambda(k_{ij})]\) corresponds to the operator valued convolution \( g \mapsto k \ast g \) defined by
\[
(k \ast g)(s) = \int_G k(t)[g(t^{-1}s)] \ dt, \quad g \in L^2(G; \ell_N^2), \ s \in G.
\]
Thus showing (4.6) amounts to show that
\[
\| \int_G (k(t) \otimes \pi(t))x \ dt \|_G \leq \gamma(\pi)^2 \| k \ast \cdot : L^2(G; \ell_N^2) \to L^2(G; \ell_N^2) \| \| x \|_G.
\]
As in the first part of the proof, we may and do assume that \( k \) has a compact support, which we denote by \( \Gamma \), and we fix an arbitrary open neighborhood \( V \subset G \) of \( e \), with \( 0 < |V| < \infty \). We let \( \hat{T} = I_{\ell^2_N} \otimes T : L^2(G; \ell^2_N) \to L^2(G; \ell^2_N) \) be the multiplication operator by \( \chi_V \) and we let \( \hat{S} : L^2(G; \ell^2_N) \to L^2(G; \ell^2_N) \) be the transposed map of \( g \mapsto k \ast g \). Let \( y_1, \ldots, y_N \) in \( X \) such that
\[
\int_G (k(t) \otimes \pi(t)) x \, dt = \sum_{k=1}^N e_k \otimes y_k.
\]
Next for any \( k = 1, \ldots, N \), let
\[
F_k(s) = \chi_{\Gamma^{-1}V}(s) \pi(s^{-1}) x_k \quad \text{and} \quad \hat{F}_k(s) = \chi_V(s) \pi(s^{-1}) y_k.
\]
Then the argument in the first part of this proof and the identification (4.7) show that
\[
\left\| \sum_k e_k \otimes \hat{F}_k \right\|_\ell \leq \left\| \hat{S} T \right\| \left\| \sum_k e_k \otimes F_k \right\|_\ell,
\]
and hence
\[
\left\| \sum_k e_k \otimes \hat{F}_k \right\|_\ell \leq \left\| k \ast : L^2(G; \ell^2_N) \to L^2(G; \ell^2_N) \right\| \left\| \sum_k e_k \otimes F_k \right\|_\ell.
\]
Now using Lemma 4.1 (2) and arguing as in the first part of the proof, we deduce (4.8). \( \square \)

**Remark 4.5.** If \( G \) is an abelian group and \( \hat{G} \) denotes its dual group, then the Fourier transform yields a natural identification \( C_0^\times(G) = C_0(\hat{G}) \). Since abelian groups are amenable, Theorem 4.4 provides a 1-1 correspondence between \( \gamma \)-bounded representations \( G \to B(X) \) and \( \gamma \)-bounded nondegenerate homomorphisms \( C_0(\hat{G}) \to B(X) \).

It is shown in [11, Prop. 2.2] (see also [10]) that any \( \gamma \)-bounded nondegenerate homomorphism \( w : C_0(\hat{G}) \to B(X) \) is of the form
\[
w(h) = \int_{\hat{G}} h \, dP, \quad h \in C_0(\hat{G}),
\]
where \( P \) is a regular strong operator \( \sigma \)-additive spectral measure from the \( \sigma \)-algebra \( \mathcal{B}(\hat{G}) \) of Borel subsets of \( \hat{G} \) into \( B(X) \). Moreover the range of this spectral measure is \( \gamma \)-bounded. Conversely, for any such spectral measure, (4.9) defines a \( \gamma \)-bounded nondegenerate homomorphism \( w : C_0(\hat{G}) \to B(X) \). (In [10, 11], the authors consider \( R \)-boundedness only but their results hold as well for \( \gamma \)-boundedness.)

Hence we obtain a 1-1 correspondence between \( \gamma \)-bounded representations \( G \to B(X) \) and regular, \( \gamma \)-bounded, strong operator \( \sigma \)-additive spectral measures \( \mathcal{B}(\hat{G}) \to B(X) \).

**Remark 4.6.**

(1) The above theorem should be regarded as a Banach space version of the Day-Dixmier unitarization Theorem which asserts that any bounded representation of an amenable group \( G \) on some Hilbert space \( H \) is unitarizable (see [38, Chap. 0]). Indeed when \( X = H \), the main implication ‘(i) \( \Rightarrow \) (ii)’ of Theorem 4.4 says that any bounded representation \( \pi : G \to B(H) \) extends to a completely bounded homomorphism \( w : C_0^\times(G) \to B(H) \), with \( \|w\|_{cb} \leq \)
According to Haagerup’s similarity Theorem \[18\], this implies the existence of an isomorphism $S: H \to H$ such that $\|S^{-1}\|\|S\| \leq \|\pi\|^2$ and $S^{-1}w(\cdot)S: C^*_\lambda(G) \to B(H)$ is a $*$-representation. Equivalently, $S^{-1}\pi(\cdot)S$ is a unitary representation.

(2) We cannot expect an extension of Theorem 4.4 for general (= non amenable) groups. See \[38, Chap. 2\] for an account on non unitarizable representations of groups on Hilbert space, and relevant open problems.

5. Representations of nuclear $C^*$-algebras on spaces with property (α)

We say that a Banach space $X$ has property $(\alpha)$ if there is a constant $\alpha \geq 1$ such that

\[(5.1) \quad \left\| \sum_{i,j} \varepsilon_i \otimes \varepsilon_j \otimes t_{ij} x_{ij} \right\|_{\text{Rad}(\text{Rad}(X))} \leq \alpha \sup_{i,j} |t_{ij}| \left\| \sum_{i,j} \varepsilon_i \otimes \varepsilon_j \otimes x_{ij} \right\|_{\text{Rad}(\text{Rad}(X))}
\]

for any finite families $(x_{ij})_{i,j}$ in $X$ and $(t_{ij})_{i,j}$ in $\mathbb{C}$. This class was introduced in \[36\] and has played an important role in several recent issues concerning functional calculi and unconditionality (see \[7, 10, 12, 27, 29\]). We note that Banach spaces with property $(\alpha)$ have a finite cotype (because they cannot contain the $\ell_\infty$’s uniformly). Thus Rademacher averages and Gaussian averages are equivalent on them. Hence $R$-boundedness and $\gamma$-boundedness (as well as matricial $R$-boundedness and matricial $\gamma$-boundedness) are equivalent notions on these spaces. The class of spaces with property $(\alpha)$ is stable under taking subspaces and comprises Banach lattices with a finite cotype. On the opposite, non trivial noncommutative $L^p$-spaces do not belong to this class. For a space $X$ with property $(\alpha)$ we let $\alpha(X)$ denote the smallest constant $\alpha$ satisfying (5.1).

Let $A$ be a $C^*$-algebra and let $w: A \to B(X)$ be a bounded homomorphism. Assume that $X$ has property $(\alpha)$. It was shown in \[11, Cor. 2.19\] that if $A$ is abelian, then $w$ is automatically $R$-bounded. By \[29\], $w$ is actually matricially $R$-bounded. When $G$ is an amenable group, the $C^*$-algebra $C^*_\lambda(G)$ is nuclear (see e.g. \[34, (1.31)\]). Thus in view of Theorem 4.4 the question whether any bounded homomorphism $w: A \to B(X)$ is automatically $R$-bounded (or matricially $R$-bounded) when $A$ is nuclear became quite relevant. A positive answer to this question was shown to me by Éric Ricard. I thank him for letting me include this result in the present paper.

**Theorem 5.1.** Let $X$ be a Banach space with property $(\alpha)$ and let $A$ be a nuclear $C^*$-algebra. Any bounded homomorphism $w: A \to B(X)$ is matricially $R$-bounded. If further $w$ is nondegenerate, then

$$\|w\|_{\text{Mat}_R} \leq K_X \|w\|^2,$$

where $K_X \geq 1$ is a constant only depending on $\alpha(X)$.

We need two lemmas. In the sequel we let $(\varepsilon_j)_{j \geq 1}$, $(\theta_i)_{i \geq 1}$ and $(\eta_k)_{k \geq 1}$ denote Rademacher sequences. For simplicity we will often use the same notations $\varepsilon_j, \theta_i, \eta_k$ to denote values...
of these variables. We start with a double estimate which will lead to the result stated in Theorem 5.1 in the case when \( A \) is finite-dimensional. When

\[
A = \bigoplus_{k=1}^{N} M_{n_k},
\]

we let \((E_{ij}^k)_{1 \leq i, j \leq n_k}\) denote the canonical basis of \( M_{n_k} \), for any \( k = 1, \ldots, N \).

**Lemma 5.2.** Let \( X \) be a Banach space with property \((\alpha)\), let \( n_1, \ldots, n_N \) be positive integers, and let

\[
w: \bigoplus_{k=1}^{N} M_{n_k} \to B(X)
\]

be any unital homomorphism. Then for any \( x \in X \), we have

\[
C_X^{-1} \|w\|^{-2} \|x\| \leq \left\| \sum_{k=1}^{N} n_k^{\frac{1}{2}} \varepsilon_j \otimes \eta_k \otimes w(E_{1j}^k)x \right\|_{\text{Rad}(\text{Rad}(X))} \leq C_X \|w\| \|x\|,
\]

where \( C_X \geq 1 \) is a constant only depending on \( \alpha(X) \).

**Proof.** Let \( \varepsilon_j = \pm 1, \theta_i = \pm 1 \) and \( \eta_k = \pm 1 \) for \( j, i, k \geq 1 \). We let

\[
\Delta_r = \sum_{k=1}^{N} n_k^{\frac{1}{2}} \varepsilon_j \theta_i E_{1j}^k \quad \text{and} \quad \Delta_c = \sum_{k=1}^{N} n_k^{\frac{1}{2}} \varepsilon_j \theta_i E_{1j}^k.
\]

It is plain that

\[
\|\Delta_r\| = \|\Delta_c\| = 1 \quad \text{and} \quad \Delta_r \Delta_c = \sum_{k=1}^{N} E_{11}^k.
\]

Since \( w \) is a homomorphism, we have

\[
w(\Delta_c) \left( \sum_{k,j} \varepsilon_j \eta_k w(E_{1j}^k)x \right) = \left( \sum_{k,i} n_k^{\frac{1}{2}} \varepsilon_j \theta_i w(E_{1i}^k) \right) \left( \sum_{j,k} \varepsilon_j \eta_k w(E_{1j}^k)x \right)
\]

\[
= \sum_{k,j,i} n_k^{\frac{1}{2}} \varepsilon_j \eta_k \theta_i w(E_{ij}^k)x.
\]

We deduce that

\[
(5.3) \quad \left\| \sum_{k,j,i} n_k^{\frac{1}{2}} \varepsilon_j \eta_k \theta_i w(E_{ij}^k)x \right\| \leq \|w\| \left\| \sum_{k,j} \varepsilon_j \eta_k w(E_{1j}^k)x \right\|.
\]

Continuing the above calculation, we obtain further that

\[
w(\Delta_r) \left( \sum_{k,j,i} n_k^{\frac{1}{2}} \varepsilon_j \eta_k \theta_i w(E_{ij}^k)x \right) = w(\Delta_r \Delta_c) \left( \sum_{k,j} \varepsilon_j \eta_k w(E_{1j}^k)x \right)
\]

\[
= \left( \sum_k w(E_{11}^k) \right) \left( \sum_{k,j} \varepsilon_j \eta_k w(E_{1j}^k)x \right)
\]

\[
= \sum_{k,j} \varepsilon_j \eta_k w(E_{1j}^k)x.
\]
Consequently, 
\[(5.4) \quad \left\| \sum_{k,j} \varepsilon_j \eta_k w(E_{ij}^k)x \right\| \leq \|w\| \left\| \sum_{k,j,i} n_k^{-\frac{1}{2}} \varepsilon_j \eta_k \theta_i w(E_{ij}^k)x \right\|.
\]

Now let \( U = [u_{ij}^1] \oplus \cdots \oplus [u_{ij}^N] \) be a fixed unitary of \( \bigoplus_{k=1}^N M_{n_k} \). Then consider the diagonal (unitary) elements
\[
V = \sum_{k=1}^N \sum_{i=1}^{n_k} \eta_k \theta_i E_{ii}^k \quad \text{and} \quad W = \sum_{k=1}^N \sum_{j=1}^{n_k} \varepsilon_j E_{jj}^k.
\]

Then \( VUW \) is a unitary and
\[
w(VUW)x = \sum_{k,j,i} \varepsilon_j \eta_k \theta_i u_{ij}^k w(E_{ij}^k)x.
\]

Since \( w \) is unital, we deduce that
\[(5.5) \quad \|w\|^{-1}\|x\| \leq \left\| \sum_{k,j,i} \varepsilon_j \eta_k \theta_i u_{ij}^k w(E_{ij}^k)x \right\| \leq \|w\|\|x\|.
\]

Let us apply the above with the special unitary \( U \) defined by
\[
u_{ij}^k = n_k^{-\frac{1}{2}} \exp\left\{ \frac{2\pi \sqrt{-1}}{n_k^{\frac{1}{2}}}(ij) \right\}, \quad k = 1, \ldots, N, \ i, j = 1, \ldots, n_k.
\]

Its main feature is that \( |u_{ij}^k| = n_k^{-\frac{1}{2}} \) for any \( i, j, k \). Since \( X \) has property (\( \alpha \)), this implies that for some constant \( C_X \geq 1 \) only depending on \( \alpha(X) \), we have
\[
\left\| \sum_{k,j,i} n_k^{-\frac{1}{2}} \varepsilon_j \otimes \eta_k \otimes \theta_i \otimes w(E_{ij}^k)x \right\|_{\text{Rad}(\text{Rad}(\text{Rad}(X)))} \leq C_X \left\| \sum_{k,j,i} \varepsilon_j \otimes \eta_k \otimes \theta_i \otimes u_{ij}^k w(E_{ij}^k)x \right\|_{\text{Rad}(\text{Rad}(\text{Rad}(X)))}
\]
and
\[
\left\| \sum_{k,j,i} \varepsilon_j \otimes \eta_k \otimes \theta_i \otimes u_{ij}^k w(E_{ij}^k)x \right\|_{\text{Rad}(\text{Rad}(\text{Rad}(X)))} \leq C_X \left\| \sum_{k,j,i} n_k^{-\frac{1}{2}} \varepsilon_j \otimes \eta_k \otimes \theta_i \otimes w(E_{ij}^k)x \right\|_{\text{Rad}(\text{Rad}(\text{Rad}(X)))}.
\]

Combining with \(5.3\), \(5.4\) and \(5.5\), we get the result. \(\square\)

For any integer \( m \geq 1 \), we let
\[
\sigma_{m,X} : M_m \rightarrow B(\text{Rad}_m(X))
\]
be the canonical homomorphism defined by letting \( \sigma_{m,X}(a) = a \otimes I_X \) for any \( a \in M_m \). According to [29, Lem. 4.3], the mappings \( \sigma_{m,X} \) are uniformly \( R \)-bounded. The same proof shows they are actually uniformly matricially \( R \)-bounded. We record this fact for further use.

**Lemma 5.3.** Let \( X \) be a Banach space with property (\( \alpha \)). Then
\[
D_X := \sup_{m \geq 1} \|\sigma_{m,X}\|_{\text{Mat}-R} < \infty.
\]
Proof of Theorem 5.1. Throughout we let $w: A \to B(X)$ be a bounded homomorphism. By standard arguments, it will suffice to consider the case when $w$ is nondegenerate. The proof will be divided into three steps.

First step: we assume that $A$ is finite-dimensional, $w$ is unital and $\|w\| = 1$. Thus (5.2) holds for some positive integers $n_1, \ldots, n_N$. Let $m = n_1 + \cdots + n_N$, so that $A \subset M_m$ in a canonical way. Let $(\varepsilon_{jk})_{j,k \geq 1}$ be a doubly indexed family of independent Rademacher variables, and let $S: X \to \text{Rad}_m(X)$ be defined by

$$S(x) = \sum_{k=1}^{N} \sum_{j=1}^{n_k} \varepsilon_{jk} \otimes w(E^k_{1j})x, \quad x \in X.$$ 

Let $Y \subset \text{Rad}_m(X)$ be the range of $S$. According to Lemma 5.2 and the assumption that $X$ has property $(\alpha)$, $S$ is an isomorphism onto $Y$ and there exist a constant $B_X \geq 1$ only depending on $\alpha(X)$ such that

$$(5.6) \quad \|S\| \leq B_X \quad \text{and} \quad \|S^{-1}: Y \to X\| \leq B_X.$$ 

Let $a = [a^1_{ij}] \oplus \cdots \oplus [a^N_{ij}] \in A$. For any $x \in X$, we have

$$[\sigma_{m,X}(a)](S(x)) = \sum_{k,j,i} \varepsilon_{ik} \otimes a^k_{ij} w(E^k_{1j})x.$$ 

On the other hand we have for any $k, i$ that $E^k_{1i}a = \sum_j a^k_{ij} E^k_{1j}$. Hence

$$w(E^k_{1i})w(a)x = \sum_j a^k_{ij} w(E^k_{1j})x,$$

and then

$$\sum_{k,j,i} \varepsilon_{ik} \otimes a^k_{ij} w(E^k_{1j})x = \sum_{k,j,i} \varepsilon_{ik} \otimes w(E^k_{1j})w(a)x = S(w(a)x).$$

This shows that $\sigma_{m,X}(a)S = Sw(a)$. Thus $Y$ is invariant under the action of $\sigma_{m,X|A}$ and if we let $\sigma: A \to B(Y)$ be the homomorphism induced by $\sigma_{m,X}$, we have shown that

$$w(a) = S^{-1}\sigma(a)S, \quad a \in A.$$ 

Appealing to (5.6), this implies that

$$\|w\|_{\text{Mat}_R} \leq \|S^{-1}\|\|S\|\|\sigma\|_{\text{Mat}_R} \leq \|S^{-1}\|\|S\|\|\sigma_{m,X}\|_{\text{Mat}_R} \leq B^2_X D_X.$$ 

Second step: we merely assume that $A$ is finite-dimensional and $w$ is unital. Let $U$ be the unitary group of $A$ and let $d\tau$ denote the Haar measure on $U$. We define a new norm on $X$ by letting

$$|||x||| = \left( \int_U \|w(U)x\|^2 \, d\tau(U) \right)^{\frac{1}{2}}, \quad x \in X.$$ 

Since $w$ is unital, this is an equivalent norm on $X$ and

$$(5.7) \quad \|w\|^{-1}|||x||| \leq |||x||| \leq \|w\|||x||, \quad x \in X.$$
Let $\tilde{X}$ be the Banach space $(X, ||| \cdot |||)$ and let $\tilde{w}: A \to B(\tilde{X})$ be induced by $w$. It readily follows from (5.7) that
$$\|w\|_{\text{Mat}-R} \leq \|w\|^2 \|\tilde{w}\|_{\text{Mat}-R}.$$ 
Using Fubini’s Theorem it is easy to see that we further have
$$\alpha(\tilde{X}) \leq \alpha(X).$$

The first step shows that we have $\|\tilde{w}\|_{\text{Mat}-R} \leq K$ for some constant $K$ only depending on $\alpha(\tilde{X})$. The above observation shows that $K$ does actually depend only on $\alpha(X)$, and we therefore obtain an estimate $\|w\|_{\text{Mat}-R} \leq K_X\|w\|^2$.

**Third step: A is infinite dimensional and w is nondegenerate.** We will use second duals in a rather standard way. However the fact that $X$ may not be reflexive leads to some technicalities. Observe that using Connes’s Theorem [9] and arguing e.g. as in [38, p. 135] (see also [33]), we may assume that there exists a directed net $(A_\lambda)_\lambda$ of finite dimensional von Neumann subalgebras of $A^{**}$ such that

$$A^{**} = \bigcup_\lambda A^{**}_\lambda.$$ 

Let $u: A \to B(X^{**})$ be the homomorphism defined by letting $u(a) = w(a)^{**}$ for any $a \in A$. According to [29, Lem. 2.3], there exists a (necessarily unique) $w^{**}$-continuous homomorphism $\hat{u}: A^{**} \to B(X^{**})$ extending $u$. We claim that

$$\hat{u}(1)x = x, \quad x \in X.$$ 

Indeed let $(a_t)_t$ be a contractive approximate identity of $A$ and note that since $w$ is nondegenerate, $w(a_t)$ converges strongly to $I_X$. This implies that $u(a_t)x = w(a_t)x \to x$. Since $a_t \to 1$ in the $w^*$-topology of $A^{**}$, we also have that $u(a_t)x \to \hat{u}(1)x$ weakly, which yields the above equality.

Let $Z \subset X^{**}$ be the range of the projection $\hat{u}(1): X^{**} \to X^{**}$. The above property means that $X \subset Z$. For any $\lambda$, we let $\tilde{u}_\lambda: A_\lambda \to B(Z)$ denote the unital homomorphism induced by the restriction of $\tilde{u}$ to $A_\lambda$. Since $X$ has property $(\alpha)$, its second dual $X^{**}$ has property $(\alpha)$ as well and $\alpha(X^{**}) = \alpha(X)$, by (1.1). Moreover $\|\tilde{u}_\lambda\| \leq \|\tilde{u}\| = \|u\| = \|w\|$. Hence by the second step of this proof, we have a uniform estimate

$$\|\tilde{u}_\lambda\|_{\text{Mat}-R} \leq K_X\|w\|^2.$$ 

Consider $[a_{ij}] \in M_n(A)$ and assume that $|||a_{ij}||| \leq 1$. Let us regard $[a_{ij}]$ as an element of $M_n(A^{**})$. Then by Kaplansky’s density Theorem (see e.g. [25, Thm. 5.3.5]), there exist a net $(\lambda_s)_s$ and, for any $s$, a matrix $[a_{ij}^s]$ belonging to the unit ball of $M_n(A_{\lambda_s})$, such that for any $i, j = 1, \ldots, n$, $a_{ij}^s \to a_{ij}$ in the $w^*$-topology of $A^{**}$. Then for any $x_1, \ldots, x_n$ in $X$ and $\varphi_1, \ldots, \varphi_n$ in $X^*$, we have

$$\lim_s \sum_{i,j} \langle \varphi_i, \tilde{u}_{\lambda_s}(a_{ij}^s)x_j \rangle = \sum_{i,j} \langle \varphi_i, w(a_{ij})x_j \rangle.$$
Applying (5.8) we deduce that
\[ \left\| \sum_{i,j} \varepsilon_i \otimes w(a_{ij}) x_j \right\|_{\text{Rad}(X)} \leq K_X \|w\|^2 \left\| \sum_j \varepsilon_j \otimes x_j \right\|_{\text{Rad}(X)}. \]

\[ \square \]

**Remark 5.4.** When \( X = H \) is a Hilbert space, the above proof yields \( K_H = 1 \), and we recover the classical result that any bounded homomorphism \( u: A \to B(H) \) on a nuclear \( C^* \)-algebra is completely bounded, with \( \|u\|_{cb} \leq \|u\|^2 \) (see [5] [6] [38]).

**Remark 5.5.** Let \( \| \| \gamma \) be a cross-norm on \( \ell^2 \otimes \ell^2 \) (in the sense that \( \| z_1 \otimes z_2 \|_\gamma = \| z_1 \| \| z_2 \| \) for all \( z_1, z_2 \) in \( \ell^2 \)) and let \( \ell^2 \otimes \gamma \ell^2 \) denote the completion of the normed space \( (\ell^2 \otimes \ell^2, \| \|_\gamma) \). Assume moreover that any bounded operator \( a: \ell^2 \to \ell^2 \) has a bounded tensor extension \( a \otimes I_\ell^2: \ell^2 \otimes \gamma \ell^2 \to \ell^2 \otimes \gamma \ell^2 \). It follows from the above results that if the Banach space \( \ell^2 \otimes \gamma \ell^2 \) has property \((\alpha)\), then \( \| \|_\gamma \) is equivalent to the Hilbert tensor norm \( \| \|_2 \), and hence \( \ell^2 \otimes \gamma \ell^2 \approx S^2 \),

the space of Hilbert-Schmidt operators on \( \ell^2 \). Indeed by the closed graph theorem, there is a constant \( K \geq 1 \) such that \( \| a \otimes I_\ell^2 \| \leq K \|a\| \) for any \( a \in B(\ell^2) \). Let \( w: B(\ell^2) \to B(\ell^2 \otimes \gamma \ell^2) \) be the bounded homomorphism defined by \( w(a) = a \otimes I_\ell^2 \). According to Lemma 5.2, there is a constant \( C \geq 1 \) such that for any \( n \geq 1 \),

\[ C^{-1} \|x\| \leq \left\| \sum_{k=1}^{n} \varepsilon_k \otimes w(E_{1k}) x \right\|_{\text{Rad}(\ell^2 \otimes \ell^2)} \leq C \|x\| \]

whenever \( x \) is a linear combination of the \( e_i \otimes e_j \), with \( 1 \leq i, j \leq n \). For any scalars \((s_{ij})_{1 \leq i, j \leq n}\) and any \( \varepsilon_k = \pm 1 \), we have

\[ \sum_{k=1}^{n} \varepsilon_k w(E_{1k}) \left( \sum_{i,j=1}^{n} s_{ij} e_i \otimes e_j \right) = e_1 \otimes \left( \sum_{i,j=1}^{n} \varepsilon_i s_{ij} e_j \right). \]

Hence for \( x = \sum_{i,j=1}^{n} s_{ij} e_i \otimes e_j \), we have

\[ \left\| \sum_{k=1}^{n} \varepsilon_k \otimes w(E_{1k}) x \right\|_{\text{Rad}(\ell^2 \otimes \gamma \ell^2)} = \left\| \sum_{i=1}^{n} \varepsilon_i \otimes \left( \sum_{j=1}^{n} s_{ij} e_j \right) \right\|_{\text{Rad}(\ell^2)} = \left( \sum_{i=1}^{n} \left\| \sum_{j=1}^{n} s_{ij} e_j \right\|^2 \right)^{1/2} = \left( \sum_{i,j=1}^{n} |s_{ij}|^2 \right)^{1/2} = \|x\|_2. \]

This shows that \( \|x\| \approx \|x\|_2 \) and the result follows by density.

That result is a variant of [31] Thm 2.2, a classical unconditional characterization of \( S^2 \).
6. Examples and Applications

In the case when $X$ has property $(\alpha)$, Theorem 5.1 leads to a simplified version of Theorem 4.4 as follows.

**Corollary 6.1.** Let $G$ be an amenable group and assume that $X$ has property $(\alpha)$. Let $\pi: G \to B(X)$ be a bounded representation. Then $\pi$ is $R$-bounded if and only if it extends to a bounded homomorphism $w: C^*_\alpha(G) \to B(X)$.

*Proof.* Since $G$ is amenable, the $C^*$-algebra $C^*_\alpha(G)$ is nuclear. Hence any bounded homomorphism $w: C^*_\alpha(G) \to B(X)$ is $R$-bounded, by Theorem 5.1. The equivalence therefore follows from Theorem 4.4.

The following is a noncommutative generalization of the fact that if $G$ is an infinite abelian group $G$ and $p \neq 2$, there exist bounded functions $\hat{G} \to \mathbb{C}$ which are not bounded Fourier multipliers on $L^p(G)$.

**Corollary 6.2.** Let $G$ be an infinite amenable group and let $1 \leq p < \infty$. Let $\lambda_p: G \to B(L^p(G))$ be the ‘left regular representation’ defined by letting $[\lambda_p(t)f](s) = f(t^{-1}s)$ for any $f \in L^p(G)$. Then $\lambda_p$ extends to a bounded homomorphism $C^*_\alpha(G) \to B(L^p(G))$ (if and) only if $p = 2$.

*Proof.* Assume that $\lambda_p$ has an extension to $C^*_\alpha(G)$. Since $L^p(G)$ has property $(\alpha)$, Corollary 6.1 ensures that $\{\lambda_p(t) : t \in G\}$ is $R$-bounded. According to [11, Prop. 2.11], this implies that $p = 2$. (The latter paper considers abelian groups only but the proof works as well in the non abelian case.)

We will now focus on the three classical groups $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{T}$. We wish to mention the remarkable work of Berkson, Gillespie and Muhly [2, 3] on bounded representations of these groups on UMD Banach spaces. Roughly speaking, their results say that when $G = \mathbb{Z}$, $\mathbb{R}$ or $\mathbb{T}$, and $X$ is UMD, any bounded representation $\pi: G \to B(X)$ gives rise to a spectral family $E_\pi$ of projections allowing a natural spectral decomposition of $\pi$ (see [2, 3] for a precise statement). According to Remark 4.5, our results imply that if $\pi: G \to B(X)$ is actually $\gamma$-bounded, then $E_\pi$ is induced by a spectral measure.

Representations $\pi: \mathbb{Z} \to B(X)$ are of the form $\pi(k) = T^k$, where $T: X \to X$ is a bounded invertible operator. Furthermore $C^*_\alpha(\mathbb{Z})$ coincides with $C(\mathbb{T})$. In the next statement, we let $\kappa \in C(\mathbb{T})$ be the function defined by $\kappa(z) = z$, and we let $\sigma(T)$ denote the spectrum of $T$. We refer to [10] for some background on spectral decompositions and scalar type operators.

**Proposition 6.3.** Let $T: X \to X$ be a bounded invertible operator.

1. The set $\{T^k : k \in \mathbb{Z}\}$ is $\gamma$-bounded if and only if there exists a $\gamma$-bounded unital homomorphism $w: C(\mathbb{T}) \to B(X)$ such that $w(\kappa) = T$.

2. Assume that $X$ has property $(\alpha)$. Then the following are equivalent.
   (i) The set $\{T^k : k \in \mathbb{Z}\}$ is $R$-bounded.
(ii) There is a bounded unital homomorphism \( w : C(\mathbb{T}) \to B(X) \) such that \( w(\kappa) = T \).

(iii) \( T \) is a scalar type spectral operator and \( \sigma(T) \subset \mathbb{T} \).

Proof. Part (1) corresponds to Theorem 4.4 when \( G = \mathbb{Z} \) and in part (2), the equivalence between (i) and (ii) is given by Corollary 6.1. The implication ‘(iii) \( \Rightarrow \) (ii)’ follows from [16, Thm. 6.24]. Conversely, assume (ii). Then by [29, Lem. 3.8], \( \sigma(T) \subset \mathbb{T} \) and there is a bounded unital homomorphism \( v : C(\sigma(T)) \to B(X) \) (obtained by factorizing \( w \) through its kernel) such that \( v(\kappa) = T, \sigma(v(f)) = f(\sigma(T)) \) for any \( f \in C(\sigma(T)) \), and \( v \) is an isomorphism onto its range. Since \( X \) has property (\( \alpha \)), it cannot contain \( c_0 \). Hence by [15, VI, Thm. 15], any bounded map \( C(\sigma(T)) \to X \) is weakly compact. Applying [16, Thm. 6.24], we deduce the assertion (iii). \( \square \)

Turning to representations of the real line, let \( (T_t)_{t \in \mathbb{R}} \) be a bounded \( c_0 \)-group on \( X \), and let \( A \) denote its infinitesimal generator. It spectrum \( \sigma(A) \) is included in the imaginary axis \( i\mathbb{R} \). Let \( \text{Rat} \subset C_0(\mathbb{R}) \) denote the subalgebra of all rational functions \( g \) with poles lying outside the real line and such that \( \deg(g) \leq -1 \). Rational functional calculus yields a natural definition of \( g(iA) \) for any such \( g \). The following is the analog of Proposition 6.3 for the real line and has an identical proof. Note that a special case of that result is announced in [43, Cor. 7.6], as a consequence of some unpublished work of Kalton and Weis.

**Proposition 6.4.** Let \( (T_t)_{t \in \mathbb{R}} \) be a bounded \( c_0 \)-group with generator \( A \).

1. The set \( \{T_t : t \in \mathbb{R}\} \) is \( \gamma \)-bounded if and only if there exists a \( \gamma \)-bounded nondegenerate homomorphism \( w : C_0(\mathbb{R}) \to B(X) \) such that \( w(g) = g(iA) \) for any \( g \in \text{Rat} \).

2. Assume that \( X \) has property (\( \alpha \)). Then the following are equivalent.

   (i) The set \( \{T_t : t \in \mathbb{R}\} \) is \( R \)-bounded.

   (ii) There is a bounded nondegenerate homomorphism \( w : C_0(\mathbb{R}) \to B(X) \) such that \( w(g) = g(iA) \) for any \( g \in \text{Rat} \).

   (iii) \( A \) is a scalar type spectral operator.

Let \( (X_n)_{n \in \mathbb{Z}} \) be an unconditional decomposition of a Banach space \( X \). For any bounded sequence \( \theta = (\theta_n)_{n \in \mathbb{Z}} \) of complex numbers, let \( T_\theta : X \to X \) be the associated multiplier operator defined by

\[
T_\theta \left( \sum_n x_n \right) = \sum_n \theta_n x_n, \quad x_n \in X_n.
\]

We say that the decomposition \( (X_n)_{n \in \mathbb{Z}} \) is \( \gamma \)-unconditional (resp. \( R \)-unconditional) if the set

\[
\{T_\theta : \theta \in \ell^\infty_\mathbb{Z}, \|\theta\|_\infty \leq 1\} \subset B(X)
\]

is \( \gamma \)-bounded (resp. \( R \)-bounded).

For any bounded representation \( \pi : \mathbb{T} \to B(X) \), and any \( n \in \mathbb{Z} \), we let \( \hat{\pi}(n) \) denote the \( n \)th Fourier coefficient of \( \pi \), defined by

\[
\hat{\pi}(n) = \frac{1}{2\pi} \int_0^{2\pi} \pi(t)e^{-int} \, dt.
\]
Equivalently, \( \hat{\pi}(n) = \sigma_{\pi}(t \mapsto e^{-int}) \). Each \( \hat{\pi}(n) : X \to X \) is a bounded projection, the ranges \( \hat{\pi}(n)X \) form a direct sum and \( \bigoplus_n \pi(n)X \) is dense in \( X \). However \( (\hat{\pi}(n)X)_{n \in \mathbb{Z}} \) is not a Schauder decomposition in general. (Indeed, take \( X = L^1(\mathbb{T}) \) and let \( \pi \) be the regular representation of \( \mathbb{T} \) on \( L^1(\mathbb{T}) \). Then \( \hat{\pi}(n)f = \hat{f}(-n)e^{-int} \) for any \( f \), and the Fourier decomposition on \( L^1(\mathbb{T}) \) is not a Schauder decomposition.)

**Proposition 6.5.** Let \( \pi : \mathbb{T} \to B(X) \) be a bounded representation.

1. \( \pi \) is \( \gamma \)-bounded if and only if \( (\hat{\pi}(n)X)_{n \in \mathbb{Z}} \) is a \( \gamma \)-unconditional decomposition of \( X \).
2. Assume that \( X \) has property \( (\alpha) \). Then \( \pi \) is \( R \)-bounded if and only if \( (\hat{\pi}(n)X)_{n \in \mathbb{Z}} \) is an unconditional decomposition of \( X \).

**Proof.** Assume that \( \pi \) extends to a bounded homomorphism \( w : c_{0,\mathbb{Z}} \to B(X) \). Then for any finitely supported scalar sequence \( (\theta_n)_{n \in \mathbb{Z}} \), we have

\[
w((\theta_n)_n) = \sum_n \theta_{-n}\pi(n).
\]

Since \( w \) is nondegenerate and bounded, this implies that \( (\hat{\pi}(n)X)_{n \in \mathbb{Z}} \) is an unconditional decomposition of \( X \). It is clear that \( (\hat{\pi}(n)X)_{n \in \mathbb{Z}} \) is actually \( \gamma \)-unconditional if and only if \( w \) is \( \gamma \)-bounded. The result therefore follows from Theorem 4.4 and Corollary 6.1. □

In the last part of this section, we are going to discuss the failure of the equivalence (i) \( \iff \) (ii) in Proposition 6.3 (2), when \( X \) is not supposed to have property \( (\alpha) \). We use ideas from [12] and [29]. Let \( (P_n)_{n \geq 1} \) be a sequence of bounded projections on some Banach space \( X \). We say that this sequence is unconditional if \( (P_nX)_{n \geq 1} \) is an unconditional decomposition of \( X \), and we say that \( (P_n)_{n \geq 1} \) has property \( (\alpha) \) if further there is a constant \( \alpha \geq 1 \) such that

\[
\left\| \sum_{i,j} \varepsilon_i \otimes t_{ij}P_j(x_i) \right\|_{\text{Rad}(X)} \leq \alpha \sup_{i,j} |t_{ij}| \left\| \sum_{i,j} \varepsilon_i \otimes P_j(x_i) \right\|_{\text{Rad}(X)}
\]

(6.1)

for any finite families \( (x_j)_j \) in \( X \) and \( (t_{ij})_{i,j} \) in \( \mathbb{C} \). If \( (P_n)_{n \geq 1} \) is unconditional, then we have a uniform equivalence

\[
\left\| \sum_{i,j} \varepsilon_i \otimes P_j(x_i) \right\|_{\text{Rad}(X)} \approx \left\| \sum_{i,j} \varepsilon_i \otimes \varepsilon_j \otimes P_j(x_i) \right\|_{\text{Rad(\text{Rad}(X))}}.
\]

Hence if \( X \) has property \( (\alpha) \), any unconditional sequence \( (P_n)_{n \geq 1} \) on \( X \) has property \( (\alpha) \). Conversely, let \( P_n : \text{Rad}(X) \to \text{Rad}(X) \) be the canonical projection defined by letting

\[
P_n\left(\sum_{j \geq 1} \varepsilon_j \otimes x_j\right) = \varepsilon_n \otimes x_n.
\]

Then \( (P_n)_{n \geq 1} \) is unconditional on \( \text{Rad}(X) \) for any \( X \), and this sequence has property \( (\alpha) \) on \( \text{Rad}(X) \) if and only if \( X \) has property \( (\alpha) \).

Here is another typical example. For any \( 1 \leq p < \infty \), let \( S^p \) denote the Schatten \( p \)-class on \( \ell^2 \) and regard any element of \( S^p \) as a bi-infinite matrix \( a = [a_{ij}]_{i,j \geq 1} \) in the usual way. We
let $E_{ij}$ denote the matrix units of $B(ℓ^2)$ and write $a = \sum_{i,j} a_{ij}E_{ij}$ for simplicity. For any $n \geq 1$, let $P_n : S^p \to S^p$ be the ‘$n$th column projection’ defined by

$$P_n\left(\sum_{i,j} a_{ij}E_{ij}\right) = \sum_i a_{in}E_{in}.$$ 

It is clear that the sequence $(P_n)_{n \geq 1}$ is unconditional on $S^p$. However if $p \neq 2$, $(P_n)_{n \geq 1}$ does not have property $(\alpha)$. This follows from the lack of unconditionality of the matrix decomposition on $S^p$. Indeed, let $a = \sum_{i,j} a_{ij}E_{ij}$, let $(t_{ij})_{i,j}$ be a finite family of complex numbers and set $x_i = \sum_j a_{ij}E_{ij}$ for any $i \geq 1$. Then

$$\left\| \sum_{i,j} \varepsilon_i \otimes P_j(x_i) \right\|_{\text{Rad}(X)} = \left\| [a_{ij}] \right\|_{S^p} \quad \text{and} \quad \left\| \sum_{i,j} \varepsilon_i \otimes t_{ij}P_j(x_i) \right\|_{\text{Rad}(X)} = \left\| [t_{ij}a_{ij}] \right\|_{S^p}.$$ 

Hence (6.1) cannot hold true.

**Proposition 6.6.** Assume that $X$ has a finite cotype and admits a sequence $(P_n)_{n \geq 1}$ of projections which is unconditional but does not have property $(\alpha)$. Then there exists an invertible operator $T : X \to X$ such that the set $\{T^k : k \in \mathbb{Z}\}$ is not $R$-bounded, but there exists a bounded unital homomorphism $w : C(\mathbb{T}) \to B(X)$ such that $w(κ) = T$.

**Proof.** Let $(ζ_j)_{j \geq 1}$ be a sequence of distinct points of $\mathbb{T}$. Since $(P_n)_{n \geq 1}$ is unconditional, one defines a bounded unital homomorphism $w : C(\mathbb{T}) \to B(X)$ by letting

$$w(f) = \sum_{j=1}^{∞} f(ζ_j)P_j, \quad f \in C(\mathbb{T}).$$ 

Arguing as in [29, Remark 4.6], we obtain that $w$ is not $R$-bounded. Let $T = w(κ)$, this is an invertible operator. If $\{T^k : k \in \mathbb{Z}\}$ were $R$-bounded, then $w$ would be $R$-bounded as well, by Theorem 4.4 and the cotype assumption.

According to the above discussion, Proposition 6.6 applies on $S^p$ for any $1 \leq p \neq 2 < ∞$, as well as on any space of the form $\text{Rad}(X)$ when $X$ does not have property $(\alpha)$ but has a finite cotype. This leads to the following general question:

**When $X$ does not have property $(\alpha)$, find a characterization of bounded invertible operators $T : X \to X$ such that $π : k \in \mathbb{Z} \mapsto T^k$ extends to a bounded homomorphism $C(\mathbb{T}) \to B(X)$.**

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