A SIMPLE PROOF OF WITTEN CONJECTURE THROUGH LOCALIZATION

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Abstract. We obtain a system of relations between Hodge integrals with one \( \lambda \)-class. As an application, we show that its first non-trivial relation implies the Witten’s Conjecture/Kontsevich Theorem [13, 7].

1. Introduction

In this paper, we obtain an alternate proof of the Witten’s Conjecture [13] which claims that the tautological intersections on the moduli space of stable curves \( \overline{M}_{g,n} \) is governed by KdV hierarchy. It is first proved by M.Kontsevich [7] by constructing combinatorial model for the intersection theory of \( \overline{M}_{g,n} \) and interpreting the trivial graph summation by a Feynman diagram expansion for a new matrix integral. A.Okounkov-R.Pandharipande [12] and M.Mirzakhani [11] gave different approaches through the enumeration of branched coverings of \( \mathbb{P}^1 \) and the Weil-Petersen volume, respectively. Recently, M.Kazarian-S.Lando [5] obtained an algebro-geometric proof by using the ELSV-formula to relate the intersection indices of \( \psi \)-classes to Hurwitz numbers.

Here we take an approach using virtual functorial localization on the moduli space of relative stable morphisms \( \overline{M}_g(\mathbb{P}^1, \mu) \) [9]. \( \overline{M}_g(\mathbb{P}^1, \mu) \) consists of maps from Riemann surfaces of genus \( g \) and \( n = l(\mu) \) marked points to \( \mathbb{P}^1 \) which has prescribed ramification type \( \mu \) at \( \infty \in \mathbb{P}^1 \). As the result, we obtain a system of relations between linear Hodge integrals. It recursively expresses each linear Hodge integral by lower-dimensional ones. The first non-trivial relation of this system is 'cut-and-join relation', and is of same recursion type as that of single Hurwitz numbers [8]. Moreover, as we increase the ramification degree, we can extract a relation between absolute Gromov-Witten invariants from this relation. And we show this relation implies the following recursion relation for the correlation functions of topological gravity [1]:

\[
\langle \tilde{\sigma}_a \prod_{k \in S} \tilde{\sigma}_k \rangle_g = \sum_{k \in S} (2k + 1) \langle \tilde{\sigma}_{a+k-1} \prod_{l \neq k} \tilde{\sigma}_l \rangle_g + \frac{1}{2} \sum_{a+b=n-2} \langle \tilde{\sigma}_a \tilde{\sigma}_b \prod_{l \in S} \tilde{\sigma}_l \rangle_{g-1} \\
+ \frac{1}{2} \sum_{S=X \cup Y, a+b=n-2, g_1+g_2=g} \langle \tilde{\sigma}_a \prod_{k \in X} \tilde{\sigma}_k \rangle_{g_1} \langle \tilde{\sigma}_b \prod_{l \in Y} \tilde{\sigma}_l \rangle_{g_2} \cdots (*)
\]
which is equivalent to the Witten’s Conjecture/Kontsevich Theorem. This recursion relation (*) is also equivalent to the Virasoro constraints; i.e. (*) can be expressed as linear, homogeneous differential equations for the \( \tau \)-function \[1\]

\[
\tau(\tilde{t}) = \exp \sum_{g=0}^{\infty} \left( \exp \sum_{n} \tilde{t}_n \tilde{\sigma}_n \right)_g
\]

\[L_n \cdot \tau = 0, \quad (n \geq -1)\]

where \( L_n \) denote the differential operators

\[
L_{-1} = -\frac{1}{2} \frac{\partial}{\partial \tilde{t}_0} + \sum_{k=1}^{\infty} (k + \frac{1}{2}) \tilde{t}_k \frac{\partial}{\partial \tilde{t}_{k-1}} + \frac{1}{4} \tilde{t}_0^2
\]

\[
L_0 = -\frac{1}{2} \frac{\partial}{\partial \tilde{t}_1} + \sum_{k=0}^{\infty} (k + \frac{1}{2}) \tilde{t}_k \frac{\partial}{\partial \tilde{t}_k} + \frac{1}{16}
\]

\[
L_n = -\frac{1}{2} \frac{\partial}{\partial \tilde{t}_{n-1}} + \sum_{k=0}^{\infty} (k + \frac{1}{2}) \tilde{t}_k \frac{\partial}{\partial \tilde{t}_{k+n}} + \frac{1}{4} \sum_{i=1}^{n} \frac{\partial^2}{\partial \tilde{t}_{i-1} \partial \tilde{t}_{n-i}}
\]

As a remark, it is possible that the general recursion relation obtained from our approach implies the Virasoro conjecture for a general non-singular projective variety.

The rest of this paper is organized as follows: In section 2, we recall the recursion formula obtained in [6] and derive cut-and-join relation as its special case. In section 3, we prove asymptotic formulas for the coefficients in the cut-and-join relation. Then we derive first two relations of the system of relations between linear Hodge integrals, and show that the cut-and-join relation implies (*).

* Please refer to [6] for miscellaneous notations.

2. Recursion Formula

The following recursion formula was derived in [6].

**Theorem 2.1.** For any partition \( \mu \) and \( e \) with \( |e| < |\mu| + l(\mu) - \chi \), we have

\[
[\lambda^{l(\mu)-\chi}] \sum_{|\nu|=|\mu|} \Phi_{\mu,\nu}^{e}(-\lambda) z_{e} D_{\nu,e}^{e} (\lambda) = 0
\]

where the sum is taken over all partitions \( \nu \) of the same size as \( \mu \).

Here \( [\lambda^a] \) means taking the coefficient of \( \lambda^a \), and \( D_{\nu,e}^{e} \) consists of linear Hodge integrals as follows;

\[
D_{g,\nu,e} = \frac{1}{l(e)! \ | \operatorname{Aut} \nu |} \left[ \prod_{i=1}^{l(\nu)} \nu_i^{\psi_i} \right] \int_{\mathcal{M}_{g,l(\nu)+l(e)}} \Lambda_{g}^{e}(1) \prod_{j=1}^{l(e)} (1 - \psi_j)^{e_j} \prod_{i=1}^{l(\nu)} (1 - \nu_i \psi_i)
\]

where \( \Lambda_{g}^{e}(t) \) is the dual Hodge bundle;

\[
\Lambda_{g}^{e}(t) = t^g - \lambda_1 t^{g-1} + \cdots + (-1)^g \lambda_g
\]
Introduce formal variable $p_i$, $q_j$ such that $p_\nu = p_{\nu_1} \times \cdots \times p_{\nu_{\ell(\nu)}}$, $q_\epsilon = q_{\epsilon_1} \times \cdots \times q_{\epsilon_{\ell(\epsilon)}}$, and form a generating series to define $\mathcal{D}_{\nu,e}^*$ as follows:

$$\mathcal{D}(\lambda, p, q) = \sum_{|\nu| \geq 1} \sum_{g \geq 0} \lambda^{2g-2+|\nu|} P_\nu q_\epsilon D_{g,\nu}$$

$$\mathcal{D}^*(\lambda, p, q) = \exp(\mathcal{D}(\lambda, p, q)) = \sum_{\nu, \mu \geq 1} \lambda^{-|\nu|+(\nu, \mu)} P_\nu q_\epsilon \mathcal{D}^*_{\nu,\mu} = \sum_{\nu, \epsilon \geq 1} P_\nu q_\epsilon \mathcal{D}_{\nu,\epsilon}^*(\lambda)$$

The convoluted term $\Phi_{\nu,\mu}^*(-\lambda)$ consists of double Hurwitz numbers as follows:

$$\Phi_{\nu,\mu}^*(\lambda) = \lambda^{-|\nu|+(\nu, \mu)} \sum_{\chi} H_\chi^*(\nu, \mu) \frac{\lambda^{-|\nu|+(\nu, \mu)} - 1}{|\nu|+(\nu, \mu)}! \Phi^*(\lambda; p^0, p^{\infty}) = 1 + \sum_{\nu, \mu} \Phi_{\nu,\mu}^* (\lambda) p_\nu p_\mu^\infty$$

Here $H_\chi^*(\nu, \mu)$ is the double Hurwitz number with ramification type $\nu, \mu$ with Euler characteristic $\chi$. The recursion formula (1) was derived by integrating point-classes over the relative moduli space $\mathcal{M}_g(\mathbb{P}^1, \mu)$, and the 'cut-and-join relation' is only the first term in this much more general formula. This can also be seen as follows: Denote by $J_{ij}(\mu)$, $C_i(\mu)$ for the cut-and-join partitions of $\mu$ [11] and consider the following identity obtained by localization method:

$$0 = \int_{\mathcal{M}_g(\mathbb{P}^1, \mu)} \prod_{k=0}^{r-2} (H - k) = \text{Contribution from the graph that is mapped to } p_r$$

$$+ \text{Contribution from the graphs that are mapped to } p_{r-1}$$

It is straightforward to show that preimages of $p_r$ and $p_{r-1}$ under the branching morphism $\text{Br} : \mathcal{M}_g(\mathbb{P}^1, \mu) \rightarrow \mathbb{P}^r$ are the unique graph $\Gamma_r$ and the 'cut-and-join graphs' of $\Gamma_r$, respectively. Hence we recover the 'cut-and-join relation' as the restriction of (1) to the first two fixed points $\{p_r, p_{r-1}\}$.

$$r \Gamma_r = \sum_{i=1}^{n} \left[ \sum_{j \neq i}^{\mu_i + \mu_j} \Gamma_{ij}^{g_1} + \sum_{p=1}^{\mu_i - 1} \sum_{\eta}^{1+\lambda_{\eta}} \Gamma_{ij}^{g_1} \right]$$

where $\Gamma$'s are the contributions from 'cut-and-join' graphs defined as follows:

- Original graph that is mapped to the branching point $p_r$:
  $$\Gamma_r = \frac{1}{|\text{Aut} \mu|} \prod_{i=1}^{\mu_1} \mu_i^{\mu_i} \int_{\mathcal{M}_{g,n}} \frac{A^\nu y}{(1 - \mu_i \psi_i)}$$

- Join graph that is obtained by joining $i$-th and $j$-th marked points:
  $$\Gamma_{ij}^{g_1} = \frac{1}{|\text{Aut} \eta|} \prod_{k=1}^{n-1} \eta_k^{\nu_k} \int_{\mathcal{M}_{g,n-1}} \frac{A^\nu y}{(1 - \eta_k \psi_k)}$$

- Cut graph that is obtained by pinching around the $i$-th marked point:
  $$\Gamma_{C1}^{g_1} = \frac{1}{|\text{Aut} \nu|} \prod_{k=1}^{n+1} \nu_k^{\nu_k} \int_{\mathcal{M}_{g-1,n+1}} \frac{A^\nu y}{(1 - \nu_k \psi_k)}$$
• Cut graph that is obtained by splitting around the \( i \)-th marked point:

\[
\Gamma^i C_2 = \prod_{k=1}^{n+1} v_k^{\nu_k} \prod_{s=1,2} \frac{1}{|\text{Aut } \nu_s|} \int_{\mathcal{M}_{g_s,n_s}} \frac{\Lambda_{g_s}^\vee(1)}{\prod (1-\nu_{s,k}\psi_k)}, \quad \nu \in C_i(\mu)
\]

As was mentioned in [10], this 'cut-and-join relation' recovers the ELSV formula since this relation is of the same type as the recursion formula for single Hurwitz numbers [8], hence giving the identification of the graph contributions with single Hurwitz numbers:

\[
H_{g,\mu} = \frac{r!}{|\text{Aut } \mu|} \left[ \prod_{i=1}^{l(\mu)} \mu_i^{\mu_i} \right] \int_{\mathcal{M}_{g,l(\mu)}} \frac{\Lambda_g(1)}{\prod (1-\mu_i\psi_i)}
\]

which is the ELSV formula. When there’s no confusion, we will denote by \( \eta = \eta^{ij} \) for the join-partition and \( \nu = \nu^{ij} \) for the cut-partition of splitting \( \mu_i = p + (\mu_i - p) \) for some \( 1 \leq p < \mu_i \). Also denote by \( \nu_1 \) and \( \nu_2 \) for the splitting of cut-partition \( \nu \) such that \( \nu_1 \cup \nu_2 = \nu \) with \( p \in \nu_1, \mu_i - p \in \nu_2 \). Note that in the \( \Gamma_{C_2} \)-type contribution, unstable vertices (i.e. \( g = 0 \) and \( n=1,2 \)) are included. We can also use any set \( \{ p_{k_0}, \ldots, p_{k_n} \} \), \( n > 0 \) of fixed points and obtain relations between linear Hodge integrals. And these can be applied to derive deeper relations.

### 3. Degree Analysis

In this section, we study asymptotic behaviour of the 'cut-and-join relation' and obtain a system of relations between linear Hodge integrals. The Hodge integral terms in the graph contributions can be expanded as follows:

\[
\int_{\mathcal{M}_{g,n}} \frac{\Lambda_g(1)}{\prod (1-\mu_i\psi_i)} = \sum_k \prod \mu_i^{k_i} \int_{\mathcal{M}_{g,n}} \prod \psi_i^{k_i} + \text{lower degree terms}
\]

where \( \tilde{k} = (k_1, \ldots, k_n) \) are multi-indices running over condition \( \sum k_i = 3g - 3 + n \). Hence the top-degree terms consist of Hodge-integral of \( \psi \)-classes and lower degree terms involve \( \lambda \)-classes. This will give a system of relations between Hodge integrals involving one \( \lambda \)-class. More precisely, integrals will be determined recursively by either lower-dimensional or lower-degree \( \lambda \)-class integrals. The following asymptotic formula is crucial in degree analysis.

**Proposition 3.1.** As \( n \to \infty \), we have for \( k, l \geq 0 \)

\[
e^{-n} \sum_{p+q=n} \frac{p^{p+k+1}q^{q+l+1}}{p!q!} \longrightarrow \frac{1}{2} \left[ \frac{(2k+1)!!(2l+1)!!}{2^{k+l+2}(k+l+2)!} \right] \eta^{k+l+2} + o(n^{k+l+2})
\]

\[
e^{-n} \sum_{p+q=n} \frac{p^{p+k+1}q^{-1}}{p!} \longrightarrow \frac{n^{k+\frac{1}{2}}}{\sqrt{2\pi}} - \left[ \frac{(2k+1)!!}{2^{k+1}k!} \right] n^k + o(n^k)
\]
Proof. Let \( m \) be an integer such that \( 1 < m < n \) and consider three ranges of \( p, q \) as follows:

\[
R_l = \{(p, q) \mid p > n - m \text{ and } q < m\} \\
R_c = \{(p, q) \mid m \leq p, q \leq n - m\} \\
R_r = \{(p, q) \mid p < m \text{ and } q > n - m\}
\]

Recall the Stirling’s formula:

\[
n! = \sqrt{2\pi n^{n+1/2}} \left(1 + \frac{1}{12n} + \cdots\right)
\]

For the summation over \( R_c \), let \( m = n\epsilon \) and \( p = nx \) for some \( \epsilon, x \in \mathbb{R}_{>0} \) so that \( m, p \in \mathbb{N} \), then we have

\[
e^{-n} \sum_{p=m}^{m-n} \frac{p^{p+k+1} q^{q+1}}{p! q!} = \sum_{p=m}^{m-n} \frac{1}{2\pi} p^{k+\frac{3}{2}} q^{l+\frac{3}{2}} [1 + o(1)]
\]

\[
= \frac{n^{k+l+2}}{2\pi} \sum_{p=m}^{m-n} x^{k+\frac{3}{2}} (1-x)^{l+\frac{3}{2}} \frac{1}{n} + o(n^{k+l+2})
\]

\[
\rightarrow \frac{n^{k+l+2}}{2\pi} \int_{\epsilon}^{1-\epsilon} x^{k+\frac{3}{2}} (1-x)^{l+\frac{3}{2}} dx + o(n^{k+l+2}) \quad \text{as } n \text{ goes to } \infty
\]

\[
= \frac{n^{k+l+2}}{2\pi} \frac{(2k+1)!!(2l+1)!!}{(2(k+l)+3)!!} \int_{\epsilon}^{1-\epsilon} \frac{(1-x)^{k+l+\frac{3}{2}}}{\sqrt{x}} dx + o(n^{k+l+2}) + O(\sqrt{\epsilon})
\]

\[
= \frac{1}{2} \left( \frac{(2k+1)!!(2l+1)!!}{2k+l+2} \right) n^{k+l+2} + o(n^{k+l+2}) + O(\sqrt{\epsilon})
\]

As \( n \to \infty \), we can send \( \epsilon \to 0 \). For the summation over \( R_l \) and \( R_r \), the top-degree terms belong to \( O(n^{k+1/2}) \) and \( O(n^{l+1/2}) \), respectively. Since we assume \( k,l \geq 0 \), both cases belong to \( o(n^{k+l+2}) \), and this proves the first formula. For the second formula, \( R_l \) has highest order of \( n^{k+1/2} \) and one can show that the leading term in the asymptotic behaviour is \( n^{k+1/2}/\sqrt{2}\pi \). After integration by parts, \( R_c \) gives the second highest term in the asymptotic behaviour

\[
e^{-n} \sum_{p=m}^{m-n} \frac{p^{p+k+1} q^{q-1}}{p! q!} = \sum_{p=m}^{m-n} \frac{1}{2\pi} p^{k+\frac{3}{2}} q^{l-\frac{3}{2}} [1 + o(1)] = \frac{n^{k}}{2\pi} \sum_{p=m}^{m-n} x^{k+\frac{1}{2}} (1-x)^{-3/2} \frac{1}{n} + o(n^{k})
\]

\[
\rightarrow \frac{n^{k}}{2\pi} \int_{\epsilon}^{1} x^{k+\frac{1}{2}} (1-x)^{-3/2} dx + o(n^{k}) \quad \text{as } n \text{ goes to } \infty
\]

\[
= \frac{n^{k+1/2}}{\sqrt{2}\pi} - \frac{n^{k}}{2\pi} (2k+1) \int_{\epsilon}^{\delta} \frac{x^{k+\frac{1}{2}}}{\sqrt{1-x}} dx + o(n^{k})
\]

\[
= \frac{n^{k+1/2}}{\sqrt{2}\pi} - \left[ \frac{(2k+1)!!}{2^{k+1} k!} \right] n^{k} + o(n^{k}) + O(\sqrt{\epsilon})
\]

This proves the second formula. \( \Box \)
Let $\mu_i = Nx_i$ for some $x_i \in \mathbb{R}$ and $N \in \mathbb{N}$. By taking general values of $x_i$, we can assume, without loss of generality, that $|\text{Aut} \mu| = 1$. As the ramification degree tends to infinity, i.e. as $N \to \infty$, the Hodge integral expansion (3) tends to

$$
\prod_{i=1}^{n} \int_{\mathcal{M}_{g,n}} x_i^{k_i-1/2} \psi_i^{k_i} + O(e^{N N^m}) \quad \to \quad e^{\mu} \int_{\mathcal{M}_{g,n}} x_i^{k_i-1/2} \psi_i^{k_i} + O(e^{N N^m})
$$

where $m = 3g - 3 + n - (n/2)$ is the highest degree of $N$ in (3). Same expansion applies to each term in (2). By taking out the common factor $e^{\mu}$ and applying the asymptotic formula (3.1), we find that

$$
\Gamma_{C_1} = \frac{N^{m+1/2}}{2} \sum_{k+l=k_i-2} \frac{(2k+1)!(2l+1)!}{2^{k+l+2}(k+l+2)!} \int_{\mathcal{M}_{g-1,n+1}} x_i^{k_i-1/2} \psi_i^{k_i} \psi_j^{k_j} + O(N^m)
$$

$$
\Gamma_{C_2} = N^{m+1/2} \prod_{j \neq i} x_j^{k_j-1/2} \int_{\mathcal{M}_{g,n-1}} x_i^{k_i} \psi_i^{k_i} - \frac{(2k_i+1)!}{2^{k_i+1}k_i} \int_{\mathcal{M}_{g,n}} x_i^{k_i} \psi_i^{k_i} + O(N^m)
$$

$$
\Gamma_{ij} = N^{m+1/2} \prod_{l \neq i,j} x_l^{k_l-1/2} \int_{\mathcal{M}_{g,n-1}} x_i^{k_i} \psi_i^{k_i} + O(N^m)
$$

Putting them together in the ‘cut-and-join relation’ (2) yields a system of relations between Hodge integrals with one $\lambda$-class as follows: First, we have a system of relations given by the spectrum of $N$-degree. Secondly, each relation given by some fixed $N$-degree stratum can be viewed as a polynomial in $x_i$’s;

$$
R_{\hat{m}}(x_1, \ldots, x_n) = \sum_{(s_1, \ldots, s_n)} C(s_1, \ldots, s_n) x_1^{s_1} \cdots x_n^{s_n}
$$

where $\hat{m}$ is a half integer less than or equal to $m + 1$ and the coefficient $C(s_i)$ of the homogeneous polynomial $x_1^{s_1} \cdots x_n^{s_n}$ involves linear Hodge integrals. Since $x_i$’s are independent variables, we obtain vanishing relations for each of $C(s_i)$’s. In particular, the first few vanishing relations are given as follows:

- For $N^{m+1}$-stratum, we have a trivial identity:

$$
(x_1 + \cdots + x_n) \prod_{i=1}^{n} \int_{\mathcal{M}_{g,n}} x_i^{k_i-1/2} \psi_i^{k_i} = 0
$$
• From $N^{m+1/2}$-stratum, we obtain a relation between cut-and-join graphs:

\[
\sum_{i=1}^{n} \left[ \frac{(2k_i + 1)!!}{2^{k_i+1}k_i!} x_i^{k_i} \prod_{j \neq i} x_j^{k_j - 1/2} \int_{\mathcal{M}_{g,n}} \prod_{j \neq i} \psi_j^{k_j} \right] - \sum_{j \neq i} \frac{(x_i + x_j)^{k_i+k_j-1/2}}{2\pi} \prod_{l \neq i,j} x_l^{k_l - 1/2} \int_{\mathcal{M}_{g,n-1}} \psi_{k_i+k_j-1} \prod_{l \neq i,j} \psi_l^{k_l} \\
- \frac{1}{2} \sum_{k+l = k_i - 2} (2k+1)!!(2l+1)!! \sum_{i,j} x_i^{k_i} \prod_{j \neq i} x_j^{k_j - 1/2} \int_{\mathcal{M}_{g,n-1}} \psi_{k_i+k_j-1} \prod_{l \neq i,j} \psi_l^{k_l} \int_{\mathcal{M}_{g,n-2}} \psi_{k_i} \prod_{j \neq i} \psi_j^{k_j} \bigg] = 0 \quad \cdots (**) 
\]

• Lower degree strata will give relations for Hodge integrals involving non-trivial $\lambda$-class in terms of lower-dimensional ones. For example, the relation given by the $N^{m}$-stratum recovers the $\lambda_1$-expression.

And the first non-trivial relation (**) implies the Witten’s Conjecture (*):

**Theorem 1.** The relation (***) implies (*).

**Proof.** Introduce formal variables $s_i \in \mathbb{R}_{>0}$ and recall the Laplace Transformation:

\[
\int_{0}^{\infty} \frac{x^{k-1/2}}{\sqrt{2\pi}} e^{-x/2s} dx = (2k - 1)!! s^{k+1/2}, \quad \int_{0}^{\infty} x^k e^{-x/2s} dx = k! (2s)^{k+1} 
\]

Applying Laplace Transformation to the $N^{m+1/2}$-stratum gives the following relation:

\[
\sum_{i=1}^{n} \left[ s_i^{k_i + 1}(2k_i + 1)!! \prod_{j \neq i} s_j^{k_j + 1/2}(2k_j - 1)!! \int_{\mathcal{M}_{g,n}} \prod_{j \neq i} \psi_j^{k_j} \right] - \sum_{a+b=k_i - 2} s_i^{k_i + 1}(2a + 1)!!(2b + 1)!! \prod_{j \neq i} s_j^{k_j + 1/2}(2k_j - 1)!! \\
\times \left( \int_{\mathcal{M}_{g-1,n+1}} \psi_{1}^{a} \psi_{2}^{b} \prod_{i,j} \psi_{1}^{k_i} + \sum_{g_1 + g_2 = g} \int_{\mathcal{M}_{g_1,n+1}} \psi_{1}^{g_1} \prod_{i} \psi_{1}^{k_i} \int_{\mathcal{M}_{g_2,n+1}} \psi_{1}^{g_2} \prod_{i} \psi_{1}^{k_i} \right) \\
- \sum_{j \neq i} \frac{(2w + 1)!!}{\sqrt{s_i} + \sqrt{s_j}} \left( s_i s_j^{w+2} + s_i^{3/2} s_j^{w+3/2} + \cdots + s_i^{w+2} s_j \right) \\
\times \prod_{l \neq i,j} s_l^{k_l + 1/2}(2k_l - 1)!! \int_{\mathcal{M}_{g,n-1}} \psi_{1}^{w} \prod_{i} \psi_{1}^{k_i} \bigg] = 0 
\]
where \( w = k_i + k_j - 1 \). The last term is derived from direct integration;

\[
\frac{N^{k+\frac{1}{2}}}{2\sqrt{2\pi}} \int_0^\infty \int_0^\infty (x_i + x_j)^{k+\frac{1}{2}} e^{-x_i y_i} e^{-x_j y_j} dx_i dx_j = \frac{N^{k+\frac{1}{2}}}{2\sqrt{2\pi}} \int_0^\infty \frac{r^{k+\frac{1}{2}} e^{-r\frac{1}{2} y_i} e^{-r\frac{1}{2} y_j} dr}{\sqrt{y_i} + \sqrt{y_j} (2y_i y_j)^k + \frac{3}{2}}
\]

under change of variable \( r = x_i + x_j \) and \( s = x_i - x_j \). Considering this as a polynomial in \( s_i \)'s, we can isolate out coefficients to obtain

\[
(\#) \cdots (2k_i + 1)!! \prod_{j \neq i} (2k_j - 1)!! \int_{\mathcal{M}_{g,n}} \prod \psi_i^{k_i} = \sum_{j \neq i} (2w + 1)!! \prod_{l \neq i,j} (2k_l - 1)!! \int_{\mathcal{M}_{g,n-1}} \psi_w \prod \psi_i^{k_i} + 
\]

\[
\sum_{a + b = k_i - 2} (2a + 1)!!(2b + 1)!! \int_{\mathcal{M}_{g,n+1}} \psi_a \psi_b \prod \psi_i^{k_i} + \sum \int_{\mathcal{M}_{g_1,n_1}} \psi_a \int_{\mathcal{M}_{g_2,n_2}} \psi_b \prod \psi_i^{k_i} \]

The reason for getting 1 as coefficient in the Join-case is due to the following expansion:

\[
\frac{1}{\sqrt{s_i} + \sqrt{s_j}} (s_i s_j^{w+2} + s_i^{3/2} s_j^{w+3/2} + \cdots + s_i^{w+2} s_j)
\]

\[
= \frac{1}{\sqrt{s_j}} (1 - \sqrt{s_i} + s_i s_j - (s_i s_j)^{3/2} + \cdots (s_i s_j^{w+2} + s_i^{3/2} s_j^{w+3/2} + \cdots + s_i^{w+2} s_j)
\]

\[
= \cdots + 1 \cdot s_i^{k_i+1} s_j^{k_j+1/2} + \cdots
\]

In the notations of (*), we have \( \tilde{\sigma}_n = (2n + 1)!!e_n = (2n + 1)!!g^n \) and

\[
\langle \tilde{\sigma}_{k_1} \cdots \tilde{\sigma}_{k_n} \rangle_g = \prod_{i=1}^n (2k_i + 1)!! \int_{\mathcal{M}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n}
\]

After multiplying a common factor \( \prod_{l \neq i} (2k_l + 1) \) on both sides of (\#), we obtain

\[
\langle \tilde{\sigma}_n \prod_{k \in S} \tilde{\sigma}_k \rangle_g = \sum_{k \in S} (2k + 1) \langle \tilde{\sigma}_n \tilde{\sigma}_{k-1} \prod_{l \neq k} \tilde{\sigma}_l \rangle_g + \frac{1}{2} \sum_{a + b = n - 2} \langle \tilde{\sigma}_a \tilde{\sigma}_b \prod_{l \neq k} \tilde{\sigma}_l \rangle_{g-1}
\]

\[
+ \frac{1}{2} \sum_{S \cup Y, a+b=n-2, g_1+g_2=g} \langle \tilde{\sigma}_a \prod_{k \in S} \tilde{\sigma}_k \rangle_{g_1} \langle \tilde{\sigma}_b \prod_{l \in S} \tilde{\sigma}_l \rangle_{g_2}
\]

which is the desired recursion relation (*). The factor \( 2k + 1 \) comes from missing \( j \)-th marked point in the Join-graph contribution, and the extra \( 1/2 \)-factor on Cut-graph contributions is due to graph counting conventions. Hence we derived Witten’s Conjecture / Kontsevich Theorem through localization on the relative moduli space.
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