Extensions and Dilations for $C^*$-dynamical Systems

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1 Introduction

Throughout this note, $A$ will denote a $C^*$-algebra with unit $1$ and $\alpha$ will denote an injective, unital endomorphism of $A$.

Definition 1.1 A (contractive) covariant representation of the pair $(A, \alpha)$ is a pair $(\pi, T)$ consisting of a $C^*$-representation $\pi$ of $A$ on a Hilbert space $H$ and a contraction operator $T$ in $B(H)$ such that

$$T\pi(\alpha(a)) = \pi(a)T,$$  \hspace{1cm} (1)
for all $a \in A$. If $T$ is an isometry, then we say that $(\pi, T)$ is isometric, while if $T$ is a coisometry, we say $(\pi, T)$ is coisometric.

Our primary objective here is to prove the following theorem and corollary.

**Theorem 1.2** If $(\pi, T)$ is a contractive covariant representation of $(A, \alpha)$ on a Hilbert space $H$, then there exists a coisometric covariant representation $(\rho, V)$ on a Hilbert space $K$ containing $H$ that extends $(\pi, T)$. That is, $\rho(a)H \subseteq H$ and $\rho(a)|H = \pi(a)$ for all $a \in A$, while $VH \subseteq H$ and $V|H = T$.

**Corollary 1.3** If $(\pi, T)$ is a contractive covariant representation of $(A, \alpha)$ on a Hilbert space $H$, then there exists a covariant representation $(\sigma, U)$ of $(A, \alpha)$ on a Hilbert space $K$ containing $H$ such that $U$ is unitary and such that $T^n = PU^n|H$ for all $n \geq 0$, where $P$ denotes the projection of $K$ onto $H$.

Theorem 1.2 is a special case of Theorem 5.10 of [5] while Corollary 1.3 is a special case of Theorem 5.22 in [5]. Our exposition is designed to give self-contained, elementary proofs of these results which avoid the technology employed in [5]. That is, we avoid the formal use of the theory of $C^*$-correspondences, which, in a sense, are the central objects of [5]. We hope that the exposition given here will aid the interested reader in his or her efforts to understand the results of [5].

In contrast to the situation for single contraction operators on Hilbert space, neither the coisometric extension $(\rho, V)$ of $(\pi, T)$ in Theorem 1.2 nor the dilation $(\sigma, U)$, with $U$ unitary, in Corollary 1.3 is unique in general. (By “unique”, here, we really mean “unique up to unitary equivalence”.) Indeed, unless $\alpha$ is an automorphism, where it is possible to obtain uniqueness by imposing minimality of $(\rho, V)$ and $(\sigma, U)$ as is done in [4], $(\rho, V)$ and $(\sigma, U)$ cannot be chosen uniquely. An additional objective of this note, then, is to explain the lack of uniqueness and to discuss a way to organize the coisometric extensions and dilations of a contractive covariant representation of $(A, \alpha)$ in a special, but important, situation.

To see why a dilation $(\sigma, U)$ of $(\pi, T)$, with $U$ unitary, might not be unique unless $\alpha$ is an automorphism, observe that the covariance equation, equation (1), implies that

$$\sigma(\alpha(a)) = U^* \sigma(a) U,$$
for all \( a \in A \). Thus, as Stacey develops in [7], the representation \( \sigma \) extends to the algebra \( A_\infty \), which is the inductive limit of the inductive system

\[
A \xrightarrow{\alpha} A \xrightarrow{\alpha} A \xrightarrow{\alpha} \cdots
\]

built from \( A \) and \( \alpha \). So, to construct \((\sigma, U)\) from \((\pi, T)\) we are faced with the problem of extending \( \pi \) to \( A_\infty \) and this extension problem does not have a unique solution. Of course, in the process of extending \( \pi \) to \( A_\infty \) to obtain \( \sigma \) we really have to “inter-leave” the construction of \( \sigma \) with the construction of \( U \). One might hope that this would cut down on the possibilities for \( \sigma \) leading to an essentially unique dilation \((\sigma, U)\). However, it doesn’t. As we will see, the same problem arises when building \((\rho, V)\). Indeed, \((\rho, V)\) lies at the center of the uniqueness problem.

We note, too, that our standing hypotheses, that \( A \) is unital and that \( \alpha \) preserves the unit and is injective, are made, fundamentally, to avoid complications with inductive limits. This may not be evident from the proofs, which proceed at a very elementary level, but closer analysis reveals that once these hypotheses are relaxed, then difficulties begin to arise. The presentation in [5] is designed to address these - and to extend to a much broader context.

2 Proofs

The proof of Theorem 1.2 is based on two lemmas. The first is

**Lemma 2.1** Given a representation \( \pi : A \to B(H) \), there is a Hilbert space \( K \), an isometry \( W : H \to K \) and a representation \( \rho \) of \( A \) in \( B(K) \) such that

\[
W^* \rho(\alpha(a))W = \pi(a),
\]

for all \( a \in A \).

Note: Equation (2) implies that \( WW^* \) commutes with \( \rho(\alpha(A)) \). This fact will play an important role throughout the computations that use this lemma.

**Proof.** If \( \pi \) has a unit cyclic vector \( \xi \), say, let \( \omega_0 \) be the state on \( \alpha(A) \) defined by the formula, \( \omega_0(\alpha(a)) = (\pi(a)\xi, \xi) \). Apply the Hahn-Banach theorem to extend \( \omega_0 \) to a state \( \omega \) on \( A \) and let \( (\rho, K) \) be the GNS representation of \( A \) determined by \( \omega \). Then \( W \) is defined by setting \( W\pi(a)\xi = \rho(\alpha(a))\xi \). In
general, we may write \( \pi \) as a direct sum of cyclic representations and apply this argument to each summand. \( \square \)

Of course the Hahn-Banach theorem introduces an arbitrariness to the extension \( \rho \) of \( \pi \) and the operator \( W \) that cannot be avoided in general; it is the source of non-uniqueness in the theory. However, it can be controlled if we have a “transfer operator” for \( \alpha \) at our disposal.

**Definition 2.2** A transfer operator for \( \alpha \) is a completely positive left inverse of \( \alpha \); i.e., a completely positive map \( \tau : A \to A \) such that \( \tau \circ \alpha(a) = a \) for all \( a \in A \).

The idea of introducing transfer operators into the study of endomorphisms of \( C^* \)-algebras is due to R. Exel [3]. On the face of it, our definition is a bit different from his. However, he works in a more general setting than ours and it is not hard to see that under our hypotheses that \( \alpha \) is injective and unital, his definition coincides with ours (see his Proposition 2.6 in particular). As Exel explains, in general a transfer operator need not exist for \( \alpha \), and when one does exist, it need not be unique. In fact, as he shows, transfer operators are just as plentiful as conditional expectations of \( A \) onto the range of \( \alpha \). To see this, note that if \( \tau \) is a transfer operator for \( \alpha \), then \( E := \alpha \circ \tau \) is a conditional expectation onto the range of \( \alpha \). Indeed, \( E \) evidently is a completely positive, unital map with range \( \alpha(A) \) and the computation,

\[
E^2 = (\alpha \circ \tau) \circ (\alpha \circ \tau) = \alpha \circ (\tau \circ \alpha) \circ \tau = \alpha \circ \tau = E,
\]

completes the proof. Conversely, if \( E \) is a conditional expectation onto the range of \( \alpha \), then \( \alpha^{-1} \circ E \) is a transfer operator for \( \alpha \). In the commutative setting, transfer operators are the subject of an active area of study because of their importance in statistical mechanics, Markov processes and the general theory of irreversible dynamical systems (see [2]).

**Remark 2.3** If \( \tau \) is a transfer operator for \( \alpha \) and if \( \pi \) is a representation of \( A \) on the Hilbert space \( H \), then \( \pi \circ \tau \) is a unital completely positive map from \( A \) to \( B(H) \). Further, if \( \rho \) is the minimal Stinespring dilation of \( \pi \circ \tau \), mapping \( A \) to \( B(K) \), and if \( W : H \to K \) is the isometry that comes in Stinespring’s theorem, then for all \( a \in A \),

\[
W^* \rho(\alpha(a))W = \pi \circ \tau(\alpha(a)) = \pi(a).
\]
By the minimality assumption the pair \((\rho, W)\) is uniquely determined up to unitary equivalence. (See the Remark after [1, Theorem 1.1.1].)

If \(\alpha\) has a transfer operator \(\tau\) and if \(\pi : A \to B(H)\) is a representation, then we shall say that the triple \((\rho, W, K)\) obtained from the minimal Stinespring dilation of the completely positive map \(\pi \circ \tau\), mapping \(A\) to \(B(H)\), is the extension of \(\pi\) adapted to \(\tau\).

**Lemma 2.4** Given \((\pi, T)\) acting on \(H\), choose \(\rho, W\) and \(K\) as in Lemma 2.1. Define the following objects:

\[
\Delta := (I - TT^*)^{1/2} \quad \text{note that } \Delta \text{ commutes with } \pi(A);
\]

\[
D_* := \rho(A)W \Delta, H \subseteq K; \quad D_* := \Delta, W^*|D_*; \quad \text{and the representation } \hat{\pi} : A \to B(D_*), \text{ where } \hat{\pi}(a) := \rho(a)|D_*.
\]

Then \(\left[ \begin{array}{cc} \pi & T \\ \hat{\pi} & D_* \end{array} \right] \), acting on \(H \oplus D_*\), gives a contractive covariant representation of \(A\) such that \(\left[ \begin{array}{cc} T & D_* \\ 0 & 0 \end{array} \right] \) is a partial isometry and which, when restricted to \(H\), gives \((\pi, T)\).

**Proof.** This is a simple matrix computation:

\[
\left[ \begin{array}{cc} T & D_* \\ 0 & 0 \end{array} \right] \left[ \begin{array}{c} \pi(a) \\ \hat{\pi}(a) \end{array} \right] = \left[ \begin{array}{cc} T & D_* \\ 0 & 0 \end{array} \right] \left[ \begin{array}{c} \pi(a) \\ \hat{\pi}(a) \end{array} \right] = \left[ \begin{array}{cc} T \pi(a) & D_* \hat{\pi}(a) \\ 0 & 0 \end{array} \right] = \left[ \begin{array}{cc} T \pi(a) & \Delta, W^* \rho(\alpha(a))WW^*|D_* \\ 0 & 0 \end{array} \right] = \left[ \begin{array}{cc} \pi(a)T & \Delta, W^* \rho(\alpha(a))WW^*|D_* \\ 0 & 0 \end{array} \right] = \left[ \begin{array}{cc} \pi(a)T & \Delta, \pi(a)W^*|D_* \\ 0 & 0 \end{array} \right] = \left[ \begin{array}{cc} \pi(a)T & \pi(a)D_* \\ 0 & 0 \end{array} \right] = \left[ \begin{array}{cc} \pi(a) & \hat{\pi}(a) \\ 0 & 0 \end{array} \right] \left[ \begin{array}{cc} T & D_* \\ 0 & 0 \end{array} \right]
\]

\(\square\)
As is customary in the dilation theory of single operators, $\Delta_*$ is called the defect operator of $T^*$ and $D_*$ is called the associated defect space.

**Proof of Theorem 1.2.** Lemma 2.4 is the “zeroth” step in an inductive construction. We apply Lemma 2.1 again to $\hat{\pi}$, which is a representation of $A$ on $D_*$, to obtain a Hilbert space $K_1$, an isometric embedding $W_1 : D_* \rightarrow K_1$ and a rep. $\pi_1 : A \rightarrow B(K_1)$ such that $W_1^*\pi_1(\alpha(a))W_1 = \hat{\pi}(a)$, for all $a \in A$. Observe that $W_1W_1^*$ commutes with $\pi_1(\alpha(A))$. Set

1. $D_1* := \overline{\pi_1(A)W_1D_*} \subseteq K_1$,
2. $D_1* := W_1^*$, and
3. $\hat{\pi}_1 : A \rightarrow B(D_1*)$, $\hat{\pi}_1(a) := \pi_1(a)|D_1*$

Note that $W_1D_* \subseteq D_1*$ and that while $\hat{\pi}_1(\alpha(A))W_1D_* \subseteq W_1D_*$, $\hat{\pi}_1(A)D_*$ need not be contained in $D_*$. Inductively, we obtain sequences $\{D_{k*}\}_{k \geq 1}$, $\{W_k\}_{k \geq 1}$, $\{\pi_k\}_{k \geq 1}$, $\{\hat{\pi}_k\}_{k \geq 1}$, and $\{D_k\}_{k \geq 1}$, where for $k > 1$, $D_{k*}$ and $K_k$ are Hilbert spaces, $W_k : D_{k-1*} \rightarrow K_k$ is an isometry, $\pi_k : A \rightarrow B(K_k)$ is a $C^*$-representation, and $\hat{\pi}_k : A \rightarrow B(D_{k*})$ is a $C^*$-representation such that the equations

$$W_k^*\pi_k(\alpha(a))W_k = \hat{\pi}_{k-1}(a),$$

$$D_{k*} := \overline{\pi_k(A)W_kD_{k-1*}} \subseteq K_{k-1},$$

$$D_{k*} := W_k^*,$$

and

$$\hat{\pi}_k : A \rightarrow B(D_{k*}), \hat{\pi}_k(a) := \pi_k(a)|D_{k*},$$

are satisfied.

On $H \oplus D_* \oplus D_1* \oplus D_2* \oplus \cdots$ set

$$\rho := \begin{bmatrix}
\pi \\
\hat{\pi} \\
\hat{\pi}_1 \\
\hat{\pi}_2 \\
\vdots \\
\vdots 
\end{bmatrix}$$
and

\[ V := \begin{bmatrix}
  T & D_* & 0 & D_1* & 0 & D_2* & \cdots \\
  0 & 0 & D_1* & 0 & D_2* & \cdots \ & \vdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
\end{bmatrix} \]

Then a straightforward calculation reveals that \((\rho, V)\) is a covariant representation that extends \((\pi, T)\), where \(V\) is a coisometry. □

**Remark 2.5** Of course, the multiple uses of Lemma 2.1 contribute to the nonuniqueness of \((\rho, V)\). However, if there is a transfer operator \(\tau\) for \(\alpha\) that is fixed in advance and if all the sequences, \(\{D_k*\}_{k\geq 1}\), \(\{W_k\}_{k\geq 1}\), \(\{\pi_k\}_{k\geq 1}\), \(\{\hat{\pi}_k\}_{k\geq 1}\), and \(\{D_k\}_{k\geq 1}\), are adapted to \(\tau\) as in Remark 2.3, then it is easy to see that \((\rho, V)\) is unique up to unitary equivalence. If these sequences are adapted to \(\tau\), then we shall say that \((\rho, V)\) is adapted to \(\tau\).

To prove Corollary 1.3 we could dilate \((\rho, V)\) using [5, Theorem 3.3]. As is proved there, in contrast to coisometric extensions, the isometric dilation of a contractive covariant representation is uniquely determined (provided, of course, it is minimal in a well-known sense that we recapitulate in Theorem 2.6). Further, as is shown in [6, Theorem 2.18], the dilation for \((\rho, V)\) will be coisometric and isometric. For completeness, we give a proof of these results in our special situation which avoids the overhead of the theory of \(C^*-\)correspondences.

**Theorem 2.6** Let \((\pi, T)\) be a contractive covariant representation of \((A, \alpha)\) on a Hilbert space \(H\). Then there is a Hilbert space \(K\) containing \(H\) and an isometric covariant representation \((\eta, W)\) of \((A, \alpha)\) on \(K\) such that \(\eta(a)H \subseteq H\) and \(\eta(a)|H = \pi(a)\), for all \(a \in A\), and such that for all \(n \geq 0\), \(T^n = PW^n|H\), where \(P\) denotes the projection of \(K\) onto \(H\). Further:

1. \((\eta, W)\) is uniquely determined by \((\pi, T)\) up to unitary equivalence if it is assumed (as may be arranged) that the smallest subspace \(K\) containing \(H\) that is invariant under \(W\) is \(K\), i.e., if it is assumed that \((\eta, W)\) is minimal; and
2. if $T$ is a coisometry, then so is $W$.

**Proof.** The proof is to build the lower right-hand corner of the Schaeffer matrix for the minimal unitary dilation of $T$ and to check that the representation $\pi$ can be extended to the Hilbert space of this dilation. For this purpose, let $\Delta$ be the square root of $I - T^*T$, i.e., the defect operator of $T$. Thanks to the covariance equation, equation (1), $\Delta$ commutes with $\pi \circ \alpha$ and so the closure of the range of $\Delta$, $\mathcal{D}$ - the defect space of $T$, reduces $\pi \circ \alpha$. If we let $K := H \oplus \mathcal{D} \oplus \mathcal{D} \oplus \cdots$ and on $K$ define $\eta$ and $W$ by the infinite matrices

$$\eta(a) := \begin{bmatrix} \pi & \pi \circ \alpha |\mathcal{D} & \pi \circ \alpha^2 |\mathcal{D} & \pi \circ \alpha^3 |\mathcal{D} & \cdots \end{bmatrix},$$

$a \in A$, and

$$W := \begin{bmatrix} T & & & \\
\Delta & I_{\mathcal{D}} & & \\
& I_{\mathcal{D}} & & \\
& & & \ddots \end{bmatrix},$$

then a straightforward calculation shows that $(\eta, W)$ is an isometric covariant representation that dilates $(\pi, T)$. Further, it is evident that $(\eta, W)$ is minimal in the sense that $K$ is the smallest subspace containing $H$ that reduces $W$. It is also evident that if $T$ is a coisometry, then so is $W$, i.e., in this event, $W$ is unitary. The uniqueness assertion, 1., is immediate from the uniqueness of the minimal isometric dilation of a contraction. See, for example, the proof on page 37 of [8]. □

**Definition 2.7** The isometric dilation $(\eta, W)$ of a contractive covariant representation $(\pi, T)$ constructed in Theorem 2.6 is called the minimal isometric dilation of $(\pi, T)$.
Proof of Corollary 1.3: As we indicated above, we simply apply Theorem 2.6 to a coisometric extension \((\rho, V)\) of \((\pi, T)\). The resulting covariant representation \((\sigma, U)\) will be isometric and coisometric by the assertion 2. of the theorem. □

It may be helpful to have a matricial picture for \((\sigma, U)\). We follow the notation developed above in the proof of Theorem 1.2. Let \(q_0\) be the projection of \(D_*\) onto \(D_* \oplus W\Delta_*H\) and for \(k \geq 1\), let \(q_k\) be the projection of \(D_{k*}\) onto \(D_{k*} \oplus W_kD_{(k-1)*}\). Also, let \(\Delta := (I - T^*T)^{1/2}\) and define

\[
\mathcal{D} := \Delta H \oplus q_0(D_*) \oplus q_1(D_{1*}) \oplus \cdots \subseteq H \oplus D_* \oplus D_{1*} \oplus D_{2*} \oplus \cdots
\]

- the Hilbert space for \((\rho, V)\). Here \(\mathcal{D}\) is the defect space for \(V\). It contains the defect space of \(T, \Delta H\), as a summand. On \(\mathcal{D}\) set

\[
\rho_1(a) := \text{diag}(\pi \circ \alpha(a), \hat{\pi} \circ \alpha(a), \hat{\pi}_1 \circ \alpha(a), \cdots).
\]

Since \(\hat{\pi}_k(A)W_kD_{(k-1)*} \subseteq W_kD_{(k-1)*}\), \(\rho_1\) is well defined. Observe that \(\rho_1\) really is the restriction of \(\rho \circ \alpha\) to \(\mathcal{D}\). Let

\[
X := (-T^*W^*|W\Delta_*H|) \oplus q_0 : D_* \to \mathcal{D}.
\]

On \(\cdots D_{2*} \oplus D_{1*} \oplus D_* \oplus H \oplus D \oplus D \oplus \cdots\), then, \(U\) is represented matricially as

\[
U = \begin{bmatrix}
\vdots & & & & & & \\
& 0 & & & & & \\
& D_{2*} & 0 & & & & \\
& & D_{1*} & 0 & & & \\
& & & D_* & (T) & & \\
& & & & q_2 & q_1 & X & \Delta & 0 \\
& & & & & & I_D & 0 & \\
& & & & & & & I_D & 0 \\
& & & & & & & & \cdots & \cdots \\
\end{bmatrix},
\]
and $\sigma$ is represented as

$$
\sigma = \begin{bmatrix}
\ddots & \hat{\pi}_2 \\
\hat{\pi}_1 & \hat{\pi} \\
\pi_1 & \rho_1 \\
\rho_1 \circ \alpha_1 & \rho_1 \circ \alpha_2 \\
& \ddots
\end{bmatrix}.
$$

**Remark 2.8** Finally, we note that if $\tau$ is a transfer operator for $\alpha$, then a contractive covariant representation $(\pi, T)$ has a unique dilation (up to unitary equivalence) $(\sigma, U)$ where $U$ is unitary and is adapted to $\tau$ in the sense that $(\sigma, U)$ is the minimal isometric dilation of the essentially unique adapted coisometric extension $(\rho, V)$ of $(\pi, T)$.

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