ON THE NORM OF THE CENTRALIZERS OF A GROUP

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Abstract. For any group $G$, let $C(G)$ denote the intersection of the normalizers of centralizers of all elements of $G$. Set $C_0 = 1$. Define $C_{i+1}(G)/C_i(G) = C(G/C_i(G))$ for $i \geq 0$. By $C_\infty(G)$ denote the terminal term of the ascending series. In this paper, we show that a finitely generated group $G$ is nilpotent if and only if $G = C_n(G)$ for some positive integer $n$.

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1. Introduction and results

For any group $G$, the norm $B_1(G)$ of $G$ is the intersection of all the normalizers of subgroups of $G$ (in fact, $B_1(G)$ is the intersection of all the normalizers of non-1-subnormal subgroups of $G$). This concept was introduced by R. Baer in 1934 and was investigated by many authors, for example, see [1] [2] [4]. It is well-known [4] that $Z(G) \leq B_1(G) \leq Z_2(G)$. More recently in [10] it has been generalized and showed that the intersection of all the normalizers of non-$n$-subnormal subgroups of $G$, say $B_n(G)$ (with the stipulation that $B_n(G) = G$ if all subgroups of $G$ are $n$-subnormal) is a nilpotent normal subgroup of $G$ of class $\mu(n)$, where $\mu(n)$ is the function of Roseblade’s Theorem.

The author in [5] showed, in view of the proof of the main theorem, that every group with finitely many $n$ of centralizers is nilpotent-by-(finite of order $(n - 1)!$). That is,

$$| G/ \bigcap_{a \in G} N_G(C_G(a)) | \leq (n - 1)!.$$  

(See Theorem 2.2 of [6] and also Theorem B of [7].) This result suggests that the behavior of centralizers has a strong influence on the structure of the group (for more information see [8] and [9]). This is the main motivation to introduce a new series of norms in groups by their normalizers of the centralizers.

Definition 1.1. For any group $G$, we define the subgroup $C(G)$ to be the intersection of the normalizers of the centralizers of $G$. That is,

$$C(G) = \bigcap_{a \in G} N_G(C_G(a)).$$  

Clearly $B_1(G) \leq C(G)$. Define the series whose terms $C_i(G)$ are characteristic subgroups as follows:

$C_{i+1}(G)/C_i(G) = C(G/C_i(G))$ for $i \geq 0$. By $C_\infty(G)$ denote the terminal term of the ascending series.
We say that a group $G$ is a $\overline{C}_n$-group ($\overline{C}_\infty$-group) if $C_n(G) = G$ for some $n \in \mathbb{N}$ ($G = C_\infty(G)$, respectively).

We give a characterization for finitely generated nilpotent groups in terms of the subgroups $C_i(G)$ of $G$, as follows:

**Theorem.** Let $G$ be a finitely generated group. Then the following statements are equivalent:

1. $G$ is nilpotent;
2. $G = C_n(G)$ for some positive integer $n$;
3. $G/C_m(G)$ is nilpotent for some positive integer $m$.

2. **Proof**

For the proof of the main theorem we need the following Lemmas.

Let $G$ be a finitely generated group. Then the following statements are equivalent:

1. $G$ is nilpotent;
2. $G = C_n(G)$ for some positive integer $n$;
3. $G/C_m(G)$ is nilpotent for some positive integer $m$.

Lemma 2.1. For any group $G$, the subgroup $C(G)$ is nilpotent of class $\leq 3$ and so it is soluble of class $\leq 2$.

Proof. Let $x \in C(G)$. Then, by definition of $C(G)$, $C_G(y^x) = C_G(y)$, for all $y \in G$. It follows that $[x, y^2] = 1$, for all $y \in C(G)$. That is, $C(G)$ is a 2-Engel group. But it is well-known that every 2-Engel group is a nilpotent group of class at most 3, completing the proof.

Remark 2.2. Since $C(G)$ is a nilpotent group of class $\leq 3$, it is easy to see that every $\overline{C}_n$-group is a soluble group of class at most $2n$.

The converse of the above Remark is not true in general. For example the symmetric group of degree 3, $S_3$ is not a $\overline{C}_1$-group.

Here we show that the class of $\overline{C}_1$-groups is closed by subgroups. In fact, we have.

Lemma 2.3. For every subgroup $H$ of $G$, we have

$$H \cap C(G) \leq C(H).$$

Proof. We have $H \cap C(G) = H \cap (\bigcap_{a \in G} N_G(C_G(a))) = \bigcap_{a \in G}(H \cap N_G(C_G(a))) = \bigcap_{a \in G} N_H(C_G(a)) \leq \bigcap_{a \in H} N_H(C_G(a)) \leq \bigcap_{a \in H} N_H(C_H(a)) = C(H)$ which is our assertion.

We denote by $Z_i(G)$ is $i$-term of the ascending central series of $G$. Here we give a very close connection between this series and the upper central series.

Lemma 2.4. For any group $G$, we have

$$Z_{i+1}(G) \leq C_i(G) \subseteq R_{2i}(G).$$

Proof. We let $C_i = C_i(G)$, and proceed by induction on $i$. First we show that $Z_{i+1}(G) \leq C_i(G)$. It is clear, if $i = 1$. Assume that $x \in Z_{i+1}(G)$. So $[x, y] \in Z_i(G)$ for all $y \in G$ and so, by the induction hypothesis, $[x, y] \in C_{i-1}$. It follows that $y^x = yt$ for some $t \in C_{i-1}$ and therefore

$$C_G/C_{i-1}(y^xC_{i-1}) = C_G/C_{i-1}(yC_{i-1}).$$
Lemma 2.7. Let $\square$ the proof.

G group. Then according to Lemma 2.3, Proof. We argue by induction on $\forall x \in G$ and so $x \in R_{2i+2}(G)$, and this completes the proof. □

We note that it is not true in general that $Z_{i+1}(G) = C_i(G)$. For instance, if $G$ is the dihedral group of size 32, then $Z_3(G) = C_2(G) = Z_4(G) = G$. In fact, we have the following Lemma.

Lemma 2.5. Let $G$ be a dihedral group of degree $n$, $D_n$. Then

$$C_i(G) = Z_{2i}(G),$$

for any $i \geq 0$.

Proof. Suppose that $n = 2^m m$, where $(2, m) = 1, \alpha \geq 0$. It is easy to see that

$$|C_1(D)| = \begin{cases} 1 & \alpha \leq 1; \\ 2 & \alpha = 2; \\ 4 & \alpha \geq 3. \end{cases}$$

It follows, by Lemma 2.4, that $Z_{i+1}(G) \leq C_i(G)$ and so $Z_{i+1}(G) = C_i(G)$. Hence $C_i(G) = Z_{2i}(G)$ (note that $Z_j(G)/Z_{i}(G) = Z_{i+j}(G)/Z_{i}(G)$ for any $i, j \geq 0$.) □

The class of nilpotent groups is not closed under forming extensions. However, we have the following well-known result, due to P. Hall (this result is often very useful for proving that a group is nilpotent).

Theorem (P. Hall). Let $N$ be a normal subgroup of a group $G$. If $G/N'$ and $N$ are nilpotent, then $G$ is nilpotent.

Here we show that the following statement (note that the subgroup $C(G)$ is nilpotent).

Lemma 2.6. For any finitely generated group $G$, we have

$$G/C(G)$$

is nilpotent $\iff$ $G$ is nilpotent.

Proof. Let $G/C(G)$ is a finitely generated nilpotent group and $x \in G$. By definition of $C(G)$, $N_C(C_G(x))/C(G)$ is a subgroup of $G/C(G)$ and so, as $G/C(G)$ is nilpotent, it is subnormal subgroup of $G/C(G)$. It follows that $N_{C(G)}(C_G(x))$ is subnormal subgroup of $G$, written $N_{C(G)}(C_G(x)) \leq G$. Hence

$$\langle x \rangle \leq C_G(x) \leq N_{C(G)}(C_G(x)) \leq G.$$ 

Therefore $\langle x \rangle \leq G$. That is, every cyclic subgroup of $G$ is a subnormal subgroup of $G$. So $G$ is a finitely generated Baer group, where a group $G$ is said to be Baer if for every $x \in G$ the cyclic subgroup $\langle x \rangle$ is subnormal in $G$. But it is well-known that Baer groups are locally nilpotent. Hence $G$ is a nilpotent group and this completes the proof. □

Lemma 2.7. Let $H$ be a subgroup of finitely generated group $G$. Then we have

$$H/C_i(G)$$

is nilpotent $\iff$ $H$ is nilpotent.

Proof. We argue by induction on $i$. Let $H/C(G)$ is a finitely generated nilpotent group. Then according to Lemma 2.3 $C(G) = C(G) \cap H \leq C(H)$ and so

$$H/C(H) \cong (H/C(G))/(C(H)/C(G)).$$
From which it follows that $H/C(H)$ is nilpotent and so, by Lemma 2.6 $H$ is nilpotent. Now assume that $i > 1$ and $H/C_{i+1}(G)$ is a nilpotent group. In this case we have 

$$H/C_{i+1}(G) \cong (H/C_i(G))/(C_{i+1}(G)/C_i(G)).$$

Applying the induction hypothesis (note that $C_{i+1}(G)/C_i(G) = C(G/C_i(G))$ and $H/C_i(G) \leq G/C_i(G)$), we thus conclude that $H/C_i(G)$ is nilpotent and so, again by the induction hypothesis, $H$ is nilpotent, completing the proof. \hfill \Box

We can now deduce the main Theorem.

**Proof of the Theorem.** According to Lemma 2.4 every nilpotent group of class $n+1$ is a $C_n$-group. Now assume that $G$ is a finitely generated $C_n$-group. According to Lemma 2.4 and Remark 2.2 we conclude that $G$ is finitely generated $2n$-Engel soluble group, so it is well-known (see [3]) that $G$ is nilpotent. Finally Lemma 2.7 completes the proof.

**Remark 2.8.** In view of Lemma 2.4 one can see that every nilpotent group (note necessarily finitely generated) of class $n+1$ is a $C_n$-group.

Finally, we state the following Question.

**Question 2.9.** Is the nilpotency class of every nilpotent $C_n$-group bounded by $n$?

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