CHARACTERIZATION OF LINEARLY REPETITIVE CUT AND PROJECT SETS

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Abstract. We give an explicit characterization of linearly repetitive cut and project sets, answering a question of Lagarias and Pleasants. Our results allow us to calculate the Hausdorff dimension of the collection of linearly repetitive cut and project sets, for all choices of dimension and codimension of the cut and project setup, and to exhibit specific examples, for all pairs of dimension and codimension in which they exist.

1. Introduction

A Delone set $Y \subseteq \mathbb{R}^d$ is linearly repetitive if there exists a constant $C > 0$ such that, for any $r > 0$, every patch of size $r$ in $Y$ occurs in every ball of diameter $Cr$ in $\mathbb{R}^d$. This concept was introduced by Lagarias and Pleasants in [13] as a model for perfectly ordered quasicrystals. Linear repetitivity has since been explored by several authors [1 2 5 6 9 10], especially because of its connection with problems in ergodic theory. In [13, Problem 8.3], Lagarias and Pleasants asked for a characterization of all linearly repetitive cut and project sets. The main result of this paper is the following explicit characterization.

Theorem 1.1. A $d$-dimensional, aperiodic, totally irrational canonical cut and project set $Y$, defined by linear forms $\{L_i\}_{i=1}^{k-d}$, is linearly repetitive if and only if

(LR1) The sum of the ranks of the kernels of the maps $L_i : \mathbb{Z}^d \rightarrow \mathbb{R}/\mathbb{Z}$ defined by

$\mathcal{L}_i(n) = L_i(n) \mod 1$

is equal to $d(k - d - 1)$, and

(LR2) Each $L_i$ is relatively badly approximable.

Condition (LR1) is necessary and sufficient for $Y$ to have minimal patch complexity. Condition (LR2) is a Diophantine condition, which places a strong restriction on how well the subspace defining $Y$ can be

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approximated by rationals. A linear form is relatively badly approximable if it is badly approximable when restricted to rational subspaces complementary to its kernel. In the next section we will explain this in more detail and prove that it is a well defined property. Note that in the special case when $k - d = 1$, condition (LR1) is automatically satisfied, and condition (LR2) requires that the linear form defining $Y$ is badly approximable in the usual sense. Finally, the case of aperiodic $Y$ is the primary focus of most authors, and indeed that which was studied in Lagarias and Pleasants’s paper. The proof of our main theorem could easily be modified (in the presence of slightly different hypotheses) to deal with the case when $Y$ has a non-trivial group of periods but, since this is a somewhat less interesting case, we do not attempt to do so.

In [6], Besbes, Boshernitzan, and Lenz proved that an aperiodic Delone set is linearly repetitive if and only if it satisfies regularity conditions which they call (PQ) and (U). One consequence of our main theorem is that, for totally irrational, aperiodic, canonical cut and project sets, condition (PQ) by itself is actually a necessary and sufficient condition. Results of this type for analogous problems in symbolic dynamics were previously discovered by other authors, and they appear to have been anticipated to hold in greater generality (see [6, Remark 5]). These discoveries are of particular interest because of their relevance to the validity of subadditive ergodic theorems. In Section 4 we will use Theorem 1.1 together with the results of [6], to deduce the following corollary.

**Corollary 1.2.** A totally irrational aperiodic canonical cut and project set is linearly repetitive if and only if it satisfies condition (PQ) from [6]. Consequently, such a set satisfies a subadditive ergodic theorem if and only if it is linearly repetitive.

To begin to make the above statements more precise, let us now give some definitions. Let $E$ be a $d$-dimensional subspace of $\mathbb{R}^k$, and $F_\pi \subseteq \mathbb{R}^k$ a subspace complementary to $E$. Write $\pi$ for the projection onto $E$ with respect to the decomposition $\mathbb{R}^k = E + F_\pi$. Choose a set $\mathcal{W}_\pi \subseteq F_\pi$, and define $S = \mathcal{W}_\pi + E$. The set $\mathcal{W}_\pi$ is referred to as the **window**, and $S$ as the **strip**. For each $s \in \mathbb{R}^k / \mathbb{Z}^k$, we define the **cut and project set** $Y_s \subseteq E$ by

$$Y_s = \pi(S \cap (\mathbb{Z}^k + s)).$$

We adopt the conventional assumption that $\pi|_{\mathbb{Z}^k}$ is injective. We also assume in much of what follows that $E$ is a **totally irrational** subspace of $\mathbb{R}^k$, which means that the canonical projection of $E$ into $\mathbb{R}^k / \mathbb{Z}^k$
is dense. There is little loss of generality in this assumption, since any subspace of $\mathbb{R}^k$ is dense in some rational sub-torus of $\mathbb{R}^k/\mathbb{Z}^k$. Nevertheless, we will give an example in Section 6, the vertices of a Penrose tiling as a $k = 5$ and $d = 2$ set, to demonstrate how our method of proof can be directly adapted to deal with non_totally irrational subspaces.

For the problem of studying linear repetitivity, the $s$ in the definition of $Y_s$ plays only a minor role. If we restrict our attention to points $s$ for which $\mathbb{Z}^k + s$ does not intersect the boundary of $\mathcal{S}$ (these are called regular points) then, as long as $E$ is totally irrational, the sets of finite patches in $Y_s$ do not depend on the choice of $s$. In particular, the property of being linearly repetitive does not depend on the choice of $s$, as long as $s$ is taken to be a regular point. On the other hand, for points $s$ which are not regular, the cut and project set $Y_s$ may contain ‘additional’ patches coming from points on the boundary, which will make it non_repetitive, and therefore not linearly repetitive. For this reason, we will always assume that $s$ is taken to be a regular point, and we will simplify our notation by writing $Y$ instead of $Y_s$.

We focus our attention on canonical cut and project sets, which are cut and project sets in which the strip is given by $C + E$, where $C$ is the unit cube in $\mathbb{R}^k$. This is a common assumption, and it is implicit in discussions in the paper of Lagarias and Pleasants, as well as the works of many other people, that in order for $Y$ to have a well behaved structure, the window should be taken to have “nice” geometric properties.

If $E$ is totally irrational, we can write it as

$$E = \{(x, L(x)) : x \in \mathbb{R}^d\},$$

where $L : \mathbb{R}^d \to \mathbb{R}^{k-d}$ is a linear function. For each $1 \leq i \leq k - d$, we define the linear form $L_i : \mathbb{R}^d \to \mathbb{R}$ by

$$L_i(x) = L(x)_i = \sum_{j=1}^{d} \alpha_{ij} x_j,$$

and we use the points $\{\alpha_{ij}\} \in \mathbb{R}^{d(k-d)}$ to parametrize the choice of $E$.

Our proof of Theorem 1.1 gives an explicit correspondence between the collection of $k$ to $d$ linearly repetitive cut and project sets, and the Cartesian product of the following two sets:

(S1) The set of all $(k - d)$-tuples $(L_1, \ldots, L_{k-d})$, where each $L_i$ is a badly approximable linear form in $m_i \geq 1$ variables, with the integers $m_i$ satisfying $m_1 + \cdots + m_{k-d} = d$, and

(S2) The set of all $d \times d$ integer matrices with non-zero determinant.
The fact that the set \((S_1)\) is empty when \(d < k/2\) implies that, for this range of \(k\) and \(d\) values, there are no linearly repetitive cut and project sets. On the other hand, for \(d \geq k/2\), there are uncountably many, as implied by the following corollary to our main result.

**Corollary 1.3.** For \(d < k/2\), there are no totally irrational aperiodic linearly repetitive cut and project sets. For \(d \geq k/2\), the collection of \(\{\alpha_{ij}\} \in \mathbb{R}^{d(k-d)}\) which define linearly repetitive cut and project sets is a set with Lebesgue measure 0 and Hausdorff dimension \(d\).

This corollary will be proved in Section 5. We will also show in Section 6 how to exhibit specific examples of linearly repetitive cut and project sets, for any choice of \(d \geq k/2\).

2. Definitions and preliminary results

2.1. Summary of notation. For sets \(A\) and \(B\), the notation \(A \times B\) denotes the Cartesian product. If \(A\) and \(B\) are subsets of the same Abelian group, then \(A + B\) denotes the collection of all elements of the form \(a + b\) with \(a \in A\) and \(b \in B\). If \(A\) and \(B\) are any two Abelian groups then \(A \oplus B\) denotes their direct sum.

For \(x \in \mathbb{R}\), \(\{x\}\) denotes the fractional part of \(x\) and \(||x||\) denotes the distance from \(x\) to the nearest integer. For \(x \in \mathbb{R}^m\), we set \(|x| = \max\{|x_1|, \ldots, |x_m|\}\) and \(||x|| = \max\{|||x_1||, \ldots, ||x_m|||\}. We use the symbols \(\ll, \gg, \asymp\) for the standard Vinogradov and asymptotic notation.

For cut and project sets, we use the notation \(E\) and \(F_\pi\) (referred to as the **physical space** and **internal space**, respectively), and \(\pi : \mathbb{R}^k \to E\), as above. We also set \(F_\rho = \{0\} \times \mathbb{R}^{k-d} \subseteq \mathbb{R}^k\), and we define \(\rho : \mathbb{R}^k \to E\) and \(\rho^* : \mathbb{R}^k \to F_\rho\) to be the projections onto \(E\) and \(F_\rho\) with respect to the decomposition \(\mathbb{R}^k = E + F_\rho\) (we are assuming that \(E\) is totally irrational). The notation here is suggestive of the fact that \(F_\pi\) is the subspace which gives the projection defining \(Y\) (hence the letter \(\pi\)), while \(F_\rho\) is the subspace with which we reference \(E\) (hence the letter \(\rho\)). We write \(W = \mathcal{S} \cap F_\rho\), and for convenience we also refer to this set as the **window** defining \(Y\). This slight ambiguity should not cause any confusion in the arguments below.

Finally, for \(y \in Y\) we define \(\tilde{y}\) to be the point in \(\mathcal{S} \cap \mathbb{Z}^k\) which satisfies \(\pi(\tilde{y}) = y\). Since \(\pi|_{\mathbb{Z}^k}\) is injective, this point is well defined.

2.2. Results from Diophantine approximation. Let \(L : \mathbb{R}^d \to \mathbb{R}^{k-d}\) be a linear map given by a matrix with entries \(\{\alpha_{ij}\} \in \mathbb{R}^{d(k-d)}\).
For any $N \in \mathbb{N}$, there exists an $n \in \mathbb{Z}^d$ with $|n| \leq N$ and
\[
\|L(n)\| \leq \frac{1}{Nd/(k-d)}.
\] (2.1)
This is a multidimensional analogue of Dirichlet’s Theorem, which follows from a straightforward application of the pigeonhole principle. We are interested in having an inhomogeneous version of this result, requiring the values taken by $\|L(n) - \gamma\|$ to be small, for all choices of $\gamma \in \mathbb{R}^{k-d}$. For this we will use the following ‘transference theorem,’ a proof of which can be found in \cite[Chapter V, Section 4]{8}.

**Theorem 2.1.** \cite[Chapter V, Theorem VIII]{8}. Given a linear map $L$ as above, the following statements are equivalent:

1. There exists a constant $C_1 > 0$ such that
   \[
   \|L(n)\| \geq \frac{C_1}{|n|^{d/(k-d)}},
   \]
   for all $n \in \mathbb{Z}^d \setminus \{0\}$.

2. There exists a constant $C_2 > 0$ such that, for all $\gamma \in \mathbb{R}^{k-d}$, the inequalities
   \[
   \|L(n) - \gamma\| \leq \frac{C_2}{Nd/(k-d)}, \quad |n| \leq N,
   \]
   are soluble, for all $N \geq 1$, with $n \in \mathbb{Z}^d$.

Next, with a view towards applying this theorem, let $\mathcal{B}_{d,k-d}$ denote the collection of numbers $\alpha \in \mathbb{R}^{d(k-d)}$ with the property that there exists a constant $C = C(\alpha) > 0$ such that, for all nonzero integer vectors $n \in \mathbb{Z}^d$,
\[
\|L(n)\| \geq \frac{C}{|n|^{d/(k-d)}}.
\]

The Khintchine-Groshev Theorem (see \cite{3} for a detailed statement and proof) implies that the Lebesgue measure of $\mathcal{B}_{d,k-d}$ is 0. However in terms of Hausdorff dimension these sets are large. It is a classical result of Jarnik that $\dim \mathcal{B}_{1,1} = 1$, and this was extended by Wolfgang Schmidt, who showed in \cite[Theorem 2]{17} that, for any choices of $1 \leq d < k$,
\[
\dim \mathcal{B}_{d,k-d} = d(k-d).
\]

Finally, we introduce the definition of relatively badly approximable linear forms. As mentioned in the introduction, these are linear forms that are badly approximable when restricted to rational subspaces complementary to their kernels. To be precise, suppose that $L : \mathbb{R}^d \to \mathbb{R}$
is a single linear form in \(d\) variables, and define \(L : \mathbb{Z}^d \to \mathbb{R}/\mathbb{Z}\) by \(L(n) = (n) \mod 1\). Let \(S \leq \mathbb{Z}^d\) be the kernel of \(L\), and write \(r = \text{rk}(S)\) and \(m = d - r\). We say that \(L\) is **relatively badly approximable** if \(m > 0\) and if there exists a constant \(C > 0\) and a group \(\Lambda \leq \mathbb{Z}^d\) of rank \(m\), with \(\Lambda \cap S = \{0\}\) and

\[
\|L(\lambda)\| \geq \frac{C}{|\lambda|^m} \quad \text{for all} \quad \lambda \in \Lambda \setminus \{0\}.
\]

Now suppose that \(L\) is relatively badly approximable and let \(\Lambda\) be a group satisfying the condition in the definition. Let \(F \subseteq \mathbb{Z}^d\) be a complete set of coset representatives for \(\mathbb{Z}^d/(\Lambda + S)\). We have the following lemma.

**Lemma 2.2.** Suppose that \(L\) is relatively badly approximable, with \(\Lambda\) and \(F\) as above. Then there exists a constant \(C' > 0\) such that, for any \(\lambda \in \Lambda\) and \(f \in F\), with \(L(\lambda + f) \neq 0\), we have that

\[
\|L(\lambda + f)\| \geq \frac{C'}{1 + |\lambda|^m}.
\]

**Proof.** Any element of \(F\) has finite order in \(\mathbb{Z}^d/(\Lambda + S)\). Therefore, for each \(f \in F\) there is a positive integer \(u_f\), and elements \(\lambda_f \in \Lambda\) and \(s_f \in S\), for which

\[
f = \frac{\lambda_f + s_f}{u_f}.
\]

If \(L(\lambda + f) \neq 0\) then either \(\lambda + f = s_f/u_f \neq 0\), or \(\lambda + u_f^{-1}\lambda_f \neq 0\). The first case only pertains to finitely many possibilities, and in the second case we have that

\[
\|L(\lambda + f)\| \geq u_f^{-1} \cdot \|L(u_f\lambda + \lambda_f + s_f)\| = u_f^{-1} \cdot \|L(u_f\lambda + \lambda_f)\| \geq \frac{C}{u_f|u_f\lambda + \lambda_f|^m}.
\]

Therefore, replacing \(C\) by an appropriate constant \(C' > 0\), and using the fact that \(F\) is finite, finishes the proof. \(\square\)

We can also deduce that if \(L\) is relatively badly approximable, then the group \(\Lambda\) in the definition may be replaced by any group \(\Lambda' \leq \mathbb{Z}^d\) which is complementary to \(S\). This is the content of the following lemma.
Lemma 2.3. Suppose that $L$ is relatively badly approximable. Then, for any group $\Lambda' \subseteq \mathbb{Z}^d$ of rank $m$, with $\Lambda' \cap S = \{0\}$, there exists a constant $C' > 0$ such that
\[
\| L(\lambda') \| \geq \frac{C'}{|\lambda'|^m} \quad \text{for all } \lambda' \in \Lambda' \setminus \{0\}.
\]

Proof. Let $\Lambda$ be the group in the definition of relatively badly approximable. Choose a basis $v_1, \ldots, v_m$ for $\Lambda'$, and for each $1 \leq j \leq m$ write
\[
v_j = \frac{\lambda_j + s_j}{u_j},
\]
with $\lambda_j \in \Lambda, s_j \in S$, and $u_j \in \mathbb{N}$.

Each $\lambda' \in \Lambda'$ can be written in the form
\[
\lambda' = \sum_{j=1}^{m} a_j v_j,
\]
with integers $a_1, \ldots, a_m$, and we have that
\[
\| L(\lambda') \| \geq (u_1 \cdots u_m)^{-1} \left\| L \left( \sum_{j=1}^{m} b_j \lambda_j \right) \right\|,
\]
with $b_j = a_j u_1 \cdots u_m / u_j \in \mathbb{Z}$ for each $j$. If the integers $a_j$ are not identically 0 then, since $\Lambda' \cap S = \{0\}$, it follows that
\[
\lambda := \sum_{j=1}^{m} b_j \lambda_j \neq 0.
\]

Using the relatively badly approximable hypothesis gives that
\[
\| L(\lambda') \| \geq \frac{C}{u_1 \cdots u_m \cdot |\lambda|^m}.
\]
Finally since $|\lambda| \ll |\lambda'|$, we have that
\[
\frac{C}{u_1 \cdots u_m \cdot |\lambda|^m} \geq \frac{C'}{|\lambda'|^m},
\]
for some constant $C' > 0$. \qed

2.3. Patterns in cut and project sets. Let $F_\rho, \rho, \rho^*$, and $\tilde{y}$ be defined as in Section 2.1. Assume that we are given a bounded convex set $\Omega \subseteq E$ which contains a neighborhood of 0 in $E$. Then, for each $r \geq 0$, define the patch of size $r$ at $y$, by
\[
P(y, r) := \{ y' \in Y : \rho(y' - \tilde{y}) \in r\Omega \}.
\]
In other words, \( P(y, r) \) consists of the projections (under \( \pi \)) to \( Y \) of all points of \( S \) whose first \( d \) coordinates are in a certain neighborhood of the first \( d \) coordinates of \( \tilde{y} \).

We remark that there are several different definitions of ‘patches of size \( r \)’ in the literature. For example, from the point of view of \( Y \) being contained in \( E \), it is more natural to define a patch of size \( r \) at \( y \) to be the collection of points of \( Y \) which lie within distance \( r \) of \( y \). In \[11\] we considered this definition of patch (what we called there type 1 patches), together with the definition that we have given above (type 2 patches). In fact, the two definitions of patches agree except for possibly on a constant neighborhood of their boundaries (see \[11\], Equation (4.1)). Therefore if \( Y \) is linearly repetitive for one definition of ‘patch of size \( r \)’, it will be linearly repetitive for the other, and similarly, for any other reasonable definition.

For \( y_1, y_2 \in Y \), we say that \( P(y_1, r) \) and \( P(y_2, r) \) are equivalent if

\[
P(y_1, r) = P(y_2, r) + y_1 - y_2.
\]

This defines an equivalence relation on the collection of patches of size \( r \). We denote the equivalence class of the patch of size \( r \) at \( y \) by \( P(y, r) \).

As indicated in the introduction, a cut and project set \( Y \) is linearly repetitive if there is a \( C > 0 \) such that, for every \( r > 0 \), every ball of size \( Cr \) in \( E \) contains a representative from every equivalence class of patches of size \( r \). There are two technical points which will ease our discussion below. First of all, since the points of \( Y \) are relatively dense, in the above definition we are free to restrict our attention, without loss of generality (by increasing \( C \) if necessary), to balls of size \( Cr \) centered at points of \( Y \). Secondly, the property of being linearly repetitive does not depend on the choice of \( \Omega \) used to define the patches. This follows from the fact that, if \( \Omega' \subseteq E \) is any other bounded convex set which contains a neighborhood of 0, then there are dilations of \( \Omega' \) which contain, and which are contained in, \( \Omega \).

Let \( \mathcal{W} = S \cap F_\rho \). There is a natural action of \( \mathbb{Z}^k \) on \( F_\rho \), given by

\[
n.w = \rho^*(n) + w = w + (0, n_2 - L(n_1)),
\]

for \( n = (n_1, n_2) \in \mathbb{Z}^k = \mathbb{Z}^d \times \mathbb{Z}^{k-d} \) and \( w \in F_\rho \). For each \( r \geq 0 \) we define the \textbf{\( r \)-singular points} of \( \mathcal{W} \) by

\[
\text{sing}(r) := \mathcal{W} \cap \left( (-\rho^{-1}(r\Omega) \cap \mathbb{Z}^k) \cdot \partial \mathcal{W} \right),
\]

and the \textbf{\( r \)-regular points} by

\[
\text{reg}(r) := \mathcal{W} \setminus \text{sing}(r).
\]
The following result follows from the proof of [11, Lemma 3.2] (see also [12]).

**Lemma 2.4.** Suppose that $W$ is a parallelotope generated by integer vectors. For every equivalence class $\mathcal{P} = \mathcal{P}(y, r)$, there is a unique connected component $U$ of $\text{reg}(r)$ with the property that, for any $y' \in Y$,

$\mathcal{P}(y', r) = \mathcal{P}(y, r)$ if and only if $\rho^*(y') \in U$.

For each equivalence class $\mathcal{P} = \mathcal{P}(y, r)$ we define $\xi_\mathcal{P}$, the frequency of $\mathcal{P}$, by

$$\xi_\mathcal{P} := \lim_{R \to \infty} \frac{\#\{y' \in Y : |y'| \leq R, \mathcal{P}(y', r) = \mathcal{P}(y, r)\}}{\#\{y' \in Y : |y'| \leq R\}}.$$

It is not difficult to show that, in our setup, the limit defining $\xi_\mathcal{P}$ always exists. Lemma 2.4 combined with the Birkhoff Ergodic Theorem, proves that, for totally irrational $E$, the frequencies of equivalence classes of patches are given by the volumes of connected components of $\text{reg}(r)$. This is the full content of [11, Lemma 3.2] (this idea is also implicit in [4]), and we record it here.

**Lemma 2.5.** If $E$ is totally irrational then for any $r > 0$ and any equivalence class $\mathcal{P} = \mathcal{P}(y, r)$, the frequency $\xi_\mathcal{P}$ is equal to the volume of the connected component $U$ in the statement of Lemma 2.4.

Finally, in our proof of Theorem 1.1 it will be convenient to replace the canonical window by a square window. For this we use the following lemma, the proof of which, in the language of tiling theory, shows that the cut and project sets obtained from the two windows are ‘mutually locally derivable’ (we will not use this terminology, but the interested reader can see [16, Section 1.3] for details).

**Lemma 2.6.** Let $Y_1$ be a totally irrational $k$ to $d$ cut and project set, constructed with the window

$$\mathcal{W}_1 = \left\{ \sum_{i=d+1}^{k} t_i e_i : 0 \leq t_i < 1 \right\} \subseteq F_p,$$

and let $Y_2$ be a cut and project set formed from the same data as $Y_1$, but with the canonical window. Then $Y_1$ is linearly repetitive if and only if $Y_2$ is.

**Proof.** We will show that there is a constant $c > 0$ with the property that, for all sufficiently large $r$, the collection of all points in a ball of size $r$ in $Y_1$ uniquely determines the points in a ball of size $r - c$ in $Y_2$ and, in the other direction, that every collection of points in a ball
of size $r$ in $Y_2$ uniquely determines the points in a ball of size $r - c$ in $Y_1$. Using the observations at the beginning of this section (and, in particular, [11, Equation 4.1]) this easily implies that one of the sets is linearly repetitive if and only if both are.

Write $\mathcal{W}_2$ for the canonical window (in $F_\rho$), and let $\mathcal{W}' \subseteq F_\rho$ be the image under $\rho^*$ of the parallelotope generated by the vectors $e_1, \ldots, e_d$.

Then it is clear that

$$\mathcal{W}_2 = \mathcal{W}_1 + \mathcal{W'},$$

and the points in $Y_2 \setminus Y_1$ correspond precisely to integer points which are mapped by $\rho^*$ into $\mathcal{W}_2 \setminus \mathcal{W}_1$.

For each $1 \leq i \leq d$, let $v_i = \pi(e_i)$, and for each subset $I \subseteq \{1, \ldots, d\}$, let

$$v_I = \sum_{i \in I} v_i,$$

with $v_\emptyset$ taken to be 0. For each $y \in Y_2$, let $I^{(1)}_y \subseteq \{1, \ldots, d\}$ denote the collection of indices $i$ for which

$$y + v_i \notin Y_1,$$

and, similarly, let $I^{(2)}_y$ denote the collection of indices $i$ for which

$$y + v_i \notin Y_2.$$

Then, by what we said in the previous paragraph,

$$Y_2 = \{y + v_I : y \in Y_1, I \subseteq I^{(1)}_y\}.$$

It follows that we can find a constant $c > 0$ such that, for every $x \in E$ and $r > c$,

$$Y_2 \cap B(x, r - c/2) = \{y + v_I : y \in Y_1 \cap B(x, r), I \subseteq I^{(1)}_y\} \cap B(x, r - c/2).$$

In the other direction, we have that

$$Y_1 = Y_2 \setminus \{y \in Y_2 : I^{(2)}_y \neq \emptyset\},$$

which means that

$$Y_1 \cap B(x, r) = \{y : y \in Y_2 \cap B(x, r), I^{(2)}_y = \emptyset\}.$$

We can assume that $c$ has been chosen so that $|v_i| \leq c/2$ for all $i$. Therefore we have verified the assertion at the beginning of the proof, that balls of size $r$ in either one of sets, $Y_1$ or $Y_2$, uniquely determine balls of size $r - c$ in the other. \qed
2.4. Subadditive ergodic theorems and (PQ). In order to facilitate the proof of Corollary 1.2 below, we briefly gather together some definitions and results from [6].

Let $\mathcal{P}$ be an equivalence class of patches of size $r$, for some $r > 0$, and let $B$ be a bounded subset of $E$. Write $\#_\mathcal{P} B$ for the maximum number of disjoint patches of size $r$ in $B$ which are in the equivalence class $\mathcal{P}$, and define

$$\nu'(\mathcal{P}) = r^d \cdot \liminf_{|C| \to \infty} \frac{\#_\mathcal{P} C}{|C|},$$

where $C$ runs over all cubes in $E$. We say that $Y$ satisfies condition (PQ) if

$$\inf_{\mathcal{P}} \nu'(\mathcal{P}) > 0,$$

where the infimum is taken over all equivalence classes of patches, for all $r > 0$.

Let $\mathcal{B}$ denote the collection of all bounded subsets of $\mathbb{R}^d$, and suppose that $F$ is a function from $\mathcal{B}$ to $\mathbb{R}$. We say that $F$ is subadditive if, for any disjoint sets $B_1, B_2 \in \mathcal{B}$, we have the inequality

$$F(B_1 \cup B_2) \leq F(B_1) \cup F(B_2).$$

We say that $F$ is $Y$-invariant if

$$F(B) = F(x + B),$$

whenever $x + (B \cap Y) = (x + B) \cap Y$. Finally, we say that $Y$ satisfies a subadditive ergodic theorem if, for all subadditive $Y$-invariant functions $F$, the limit

$$\lim_{|C| \to \infty} \frac{F(C)}{|C|}$$

exists. As above, the limit is taken over all cubes $C \subseteq E$.

It follows from [6, Theorem 1] that $Y$ satisfies a subadditive ergodic theorem if and only if it satisfies condition (PQ).

3. Proof of Theorem 1.1

Using Lemma 2.6, we assume without loss of generality that the window used to construct $Y$ is given by (2.2). We also identify $\mathcal{W}$ with a subset of $\mathbb{R}^{k-d}$, in the obvious way. Recall that if $Y$ is linearly repetitive with respect to one convex patch shape $\Omega$, then it is linearly repetitive with respect to all convex patch shapes. The precise shape $\Omega$ which we will use will be specified later in the proof, but until then everything we say will apply to any fixed choice of such a shape.
For \( r > 0 \) let \( c(r) \) denote the number of equivalence classes of patches of size \( r \). If \( Y \) is linearly repetitive then there exists a constant \( C > 0 \) such that \( c(r) \) is bounded above by \( Cr^d \), for all \( r > 0 \). For the first part of the proof of Theorem 1.1 we will show that condition (LR1) is necessary and sufficient for a bound of this type to hold.

For each \( 1 \leq i \leq k - d \), let \( S_i \subseteq \mathbb{Z}^d \) denote the kernel of the map \( L_i \), and let \( r_i \) be the rank of \( S_i \). Furthermore, for each subset \( I \subseteq \{1, \ldots, k - d\} \) let

\[
S_I = \bigcap_{i \in I} S_i,
\]

and let \( r_I \) be the rank of \( S_I \). For convenience, set \( S_\emptyset = \mathbb{Z}^d \) and \( r_\emptyset = d \). For any pair \( I, J \subseteq \{1, \ldots, k - d\} \), the sum set \( S_I + S_J \) is a subgroup of \( \mathbb{Z}^d \), and it therefore has rank at most \( d \). On the other hand we have that

\[
\text{rk}(S_I + S_J) = \text{rk}(S_I) + \text{rk}(S_J) - \text{rk}(S_I \cap S_J),
\]

which gives the inequality

\[
r_I + r_J \leq d + r_{I \cup J}. \tag{3.1}
\]

As one application of this inequality we see immediately that

\[
r_1 + r_2 + \cdots + r_{k-d} \leq d + r_{12} + r_3 + \cdots + r_{k-d} \\
\leq 2d + r_{123} + r_4 + \cdots + r_{k-d} \\
\vdots \\
\leq d(k - d - 1) + r_{12\ldots(k-d)} \\
= d(k - d - 1). \tag{3.2}
\]

The last equality here uses the assumption that \( Y \) is aperiodic.

From Lemma 2.4, we know that \( c(r) \) is equal to the number of connected components of \( \text{reg}(r) \). Let the map \( \mathcal{C} : \mathbb{Z}^{d(k-d)} \to \mathcal{W} \) be defined by

\[
\mathcal{C}(n^{(1)}, \ldots, n^{(k-d)}) = (\{L_1(n^{(1)})\}, \ldots, \{L_{k-d}(n^{(k-d)})\}),
\]

for \( n^{(1)}, \ldots, n^{(k-d)} \in \mathbb{Z}^d \). Identify \( \mathbb{Z}^d \) with the set \( \mathcal{Z} = \mathbb{Z}^k \cap \langle e_1, \ldots, e_d \rangle_{\mathbb{R}}, \) and for each \( r > 0 \) let \( \mathcal{Z}_r \subseteq \mathbb{Z}^d \) be defined by

\[
\mathcal{Z}_r = -\rho^{-1}(r\Omega) \cap \mathcal{Z}.
\]

Since our window \( \mathcal{W} \) is a fundamental domain for the integer lattice in \( F_\rho \), there is a one to one correspondence between points of \( Y \) and elements of \( \mathcal{Z} \). This correspondence is given explicitly by mapping a
point \( y \in Y \) to the vector in \( Z \) given by the first \( d \) coordinates of \( \bar{y} \). Also, notice that if \( n \in \mathbb{Z}^k \) and \( -n.0 \in \mathcal{W} \), then it follows that

\[-n.0 = \{ \{ L_1(n_1, \ldots, n_d) \}, \ldots, \{ L_{k-d}(n_1, \ldots, n_d) \} \}.

These observations together imply that the collection of all vertices of connected components of \( \text{reg}(r) \) is precisely the set \( C(\mathbb{Z}^{k-d}_r) \), which in turn implies that

\[ c(r) \asymp |C(\mathbb{Z}^{k-d}_r)|. \]

The values of the function \( C \) define a natural \( \mathbb{Z}^{d(k-d)} \) action on \( \mathcal{W} \). Therefore we may regard the set \( C(\mathbb{Z}^{d(k-d)}) \) as a group, isomorphic to \( \mathbb{Z}^{d(k-d)}/\ker(C) \cong \mathbb{Z}^d/S_1 \oplus \cdots \oplus \mathbb{Z}^d/S_{k-d} \).

If (LR1) holds then we have that

\[ \text{rk}(C(\mathbb{Z}^{d(k-d)})) = d(k - d) - \sum_{i=1}^{k-d} r_i = d, \]

and from this it follows that

\[ |C(\mathbb{Z}^{k-d}_r)| \asymp r^d. \]

On the other hand, if (LR1) does not hold then by \((3.2)\) we have that

\[ \text{rk}(C(\mathbb{Z}^{d(k-d)})) > d, \]

which implies that

\[ |C(\mathbb{Z}^{k-d}_r)| \gtrsim r^{d+1}. \]

We conclude that \( c(r) \ll r^d \) if and only if condition (LR1) holds, so (LR1) is a necessary condition for linear repetitivity.

Next we assume that (LR1) holds and we prove that, under this assumption, condition (LR2) is necessary and sufficient in order for \( Y \) to be linearly repetitive. First of all, suppose that \( I \) and \( J \) were disjoint, nonempty subsets of \( \{1, \ldots, k-d\} \) for which

\[ r_I + r_J < d + r_{I \cup J}. \]

Then, by the same argument used in \((3.2)\), we would have that

\[ \sum_{i=1}^{k-d} r_i \leq d(k - d - 3) + r_{(I \cup J)^c} + r_I + r_J < d(k - d - 1). \]

This clearly contradicts (LR1). Therefore if (LR1) holds then, by \((3.1)\), we have that

\[ r_I + r_J = d + r_{I \cup J}, \]

whenever \( I \) and \( J \) are disjoint and nonempty.
For each $1 \leq i \leq k - d$, define $J_i = \{1, \ldots, k - d\} \setminus \{i\}$, and let $\Lambda_i = S_{J_i}$. Write $m_i = r_{J_i}$ for the rank of $\Lambda_i$. Then, by what was established in the previous paragraph, we have that

$$m_i + r_i = d.$$ 

If $n$ is any nonzero vector in $\Lambda_i$, then $n$ is in $S_j$ for all $j \neq i$. Since $Y$ is aperiodic, this means that $n \notin S_i$, which gives that

$$\text{rk}(\Lambda_i + S_i) = m_i + r_i - \text{rk}(\Lambda_i \cap S_i) = d.$$ 

Furthermore, for any $j \neq i$, the fact that $\Lambda_j \subseteq S_i$ implies that $\Lambda_j \cap \Lambda_i = \{0\}$, so

$$\text{rk}(\Lambda_1 + \cdots + \Lambda_{k-d}) = \sum_{i=1}^{k-d} \text{rk}(\Lambda_i) = \sum_{i=1}^{k-d} (d - r_i) = d.$$ 

For each $i$, let $F_i \subseteq \mathbb{Z}^d$ be a complete set of coset representatives for $\mathbb{Z}^d/(\Lambda_i + S_i)$. Also, write $\Lambda = \Lambda_1 + \cdots + \Lambda_{k-d}$, and let $F \subseteq \mathbb{Z}^d$ be a complete set of representatives for $\mathbb{Z}^d/\Lambda$. What we have shown so far implies that all of the sets $F_1, \ldots, F_{k-d}$, and $F$ are finite.

Again thinking of $\mathbb{Z}^d$ as being identified with the set $\mathcal{Z}$, let

$$\mathcal{Z}_{r,\Lambda} = \mathcal{Z}_r \cap \Lambda, \quad \mathcal{Z}_{r,\Lambda_i} = \mathcal{Z}_r \cap \Lambda_i, \quad \text{and} \quad \mathcal{Z}_{r, S_i} = \mathcal{Z}_r \cap S_i.$$ 

For each $i$, choose a basis $\{v_{ij}^{(i)}\}_{j=1}^{m_i}$ for $\Lambda_i$, and define

$$\Omega_i' = \left\{ \sum_{j=1}^{m_i} t_j v_{ij}^{(i)} : -1/2 \leq t_i < 1/2 \right\},$$

and

$$\Omega' = \Omega_1' + \cdots + \Omega_{k-d}',$$

so that $\Omega'$ is a fundamental domain for $\mathbb{R}^d/\Lambda$. We now specify $\Omega$ to be the subset of points in $E$ whose first $d$ coordinates lie in $\Omega'$. In other words,

$$\Omega = E \cap \rho^{-1}(\Omega').$$

Notice that every $n \in \Lambda$ has a unique representation of the form

$$n = \sum_{i=1}^{k-d} \sum_{j=1}^{m_i} a_{ij} v_{ij}^{(i)}, \quad a_{ij} \in \mathbb{Z}.$$ 

Using this representation, we have that

$$\mathcal{L}(n) = \mathcal{C}((n_{ij}^{(i)})_{i=1}^{m_i}),$$
where, for each \( i \), the vector \( n^{(i)} \in \mathbb{Z}^d \) is given by
\[
n^{(i)} = \sum_{j=1}^{m_i} a_{ij} v_j^{(i)}.
\]
This gives a one to one correspondence between elements of \( \mathcal{L}(\Lambda) \) and elements of the set
\[
\mathcal{C}(\Lambda_1 \times \cdots \times \Lambda_{k-d}) = \mathcal{L}_1(\Lambda_1) \times \cdots \times \mathcal{L}_{k-d}(\Lambda_{k-d}).
\]
We will combine this observation with the facts that
\[
\mathcal{L}(\mathbb{Z}^d) = \mathcal{L}(\Lambda + F)
\]
and
\[
\mathcal{C}(\mathbb{Z}^{d(k-d)}) = \mathcal{C}((\Lambda_1 + F_1) \times \cdots \times (\Lambda_{k-d} + F_{k-d}))
\]
in order to study the spacings between points of the sets \( \mathcal{L}(\mathbb{Z}_r) \) and \( \mathcal{C}(\mathbb{Z}_r^{k-d}) \).

First of all, it is clear that
\[
\mathcal{L}(\mathbb{Z}_r) \supseteq \mathcal{L}_1(\mathbb{Z}_{r,\Lambda_1}) \times \cdots \times \mathcal{L}_{k-d}(\mathbb{Z}_{r,\Lambda_{k-d}}), \tag{3.3}
\]
and that
\[
\mathcal{C}(\mathbb{Z}_r^{k-d}) \supseteq \mathcal{L}_1(\mathbb{Z}_{r,\Lambda_1}) \times \cdots \times \mathcal{L}_{k-d}(\mathbb{Z}_{r,\Lambda_{k-d}}). \tag{3.4}
\]

Since all of the sets \( F_1, \ldots, F_{k-d} \), and \( F \) are finite, there is a constant \( \kappa > 0 \) with the property that, for all sufficiently large \( r \),
\[
\mathbb{Z}_r \subseteq \mathbb{Z}_{r+k,\Lambda} + F, \quad \text{and} \quad \mathbb{Z}_r \subseteq \mathbb{Z}_{r+k,\Lambda_1} + S_i + F_i,
\]
for each \( 1 \leq i \leq k - d \). For the second inclusion here we are using the definition of \( \Omega \) and the fact that \( \Lambda_j \subseteq S_i \) for all \( j \neq i \). These inclusions imply that
\[
\mathcal{L}(\mathbb{Z}_r) \subseteq \mathcal{L}(\mathbb{Z}_{r+k,\Lambda}) + \mathcal{L}(F)
\]
\[
\subseteq \mathcal{L}_1(\mathbb{Z}_{r+k,\Lambda_1} + F) \times \cdots \times \mathcal{L}_{k-d}(\mathbb{Z}_{r+k,\Lambda_{k-d}} + F), \tag{3.5}
\]
and that
\[
\mathcal{C}(\mathbb{Z}_r^{k-d}) \subseteq \mathcal{C}((\mathbb{Z}_{r+k,\Lambda_1} + F_1) \times \cdots \times (\mathbb{Z}_{r+k,\Lambda_{k-d}} + F_{k-d}))
\]
\[
= \mathcal{L}_1(\mathbb{Z}_{r+k,\Lambda_1} + F_1) \times \cdots \times \mathcal{L}_{k-d}(\mathbb{Z}_{r+k,\Lambda_{k-d}} + F_{k-d}). \tag{3.6}
\]

Now we are positioned to make our final arguments.

Suppose first of all that (LR2) holds. Let \( U \) be any connected component of \( \text{reg}(r) \). Then \( U \) is a \((k-d)\)-dimensional box, with faces parallel
to the coordinate hyperplanes, and with vertices in the set \( C(\mathbb{Z}_r^{k-d}) \). Therefore we can write \( U \) in the form

\[
U = \{ x \in W : \ell_i < x_i < r_i \},
\]

where for each \( i \), the values of \( \ell_i \) and \( r_i \) are elements of the set \( L_i(\mathbb{Z}_r) \). By equation (3.6), together with Lemma 2.2, there is a constant \( c_1 > 0 \) such that, for every \( i \),

\[
r_i - \ell_i \geq c_1 r_m.
\]

Next we will show that there is a constant \( c_2 > 0 \) such that, for all sufficiently large \( r \), the orbit of every point in \( F_\rho/\mathbb{Z}^{k-d} \) under the action of \( \mathbb{Z}_{c_2r} \) intersects every connected component of \( \text{reg}(r) \). Then Lemma 2.4 will imply that \( Y \) is linearly repetitive. To show that there is such a constant \( c_2 \), we use (3.3) and Theorem 2.1. Each one of the linear forms \( L_i \) is a badly approximable linear form in \( m_i \) variables, when restricted to \( \Lambda_i \). Therefore, by (T2) of Theorem 2.1 there is a constant \( \eta > 0 \) with the property that, for all sufficiently large \( r \) and for each \( i \), the collection of points \( L_i(\mathbb{Z}_{c_2r}, \Lambda_i) \) is \( \eta/(c_2 r^{m_i}) \)-dense in \( \mathbb{R}/\mathbb{Z} \). Choosing \( c_2 > 3c_1/\eta \) completes the proof of this part of the theorem, verifying that (LR1) and (LR2) together imply linear repetitivity.

For the final part, suppose that (LR1) holds and (LR2) does not. Then one of the linear forms \( L_i \) is not relatively badly approximable, and we assume without loss of generality that it is \( L_1 \). Let \( c_2 \) be any positive constant, and consider the collection of points \( L(\mathbb{Z}_{c_2r}) \). By (3.5), the first coordinates of these points are a subset of

\[
L_1(\mathbb{Z}_{c_2r}, \Lambda_1 + F).
\]

There are at most \( c_2 \delta r^{m_1} - 1 \) points in the latter set, for some constant \( \delta \) depending on \( \Lambda_1 \). Therefore, thinking of the points as being arranged in increasing order in \([0, 1)\), there must be two consecutive points which are at least \( 1/(c_2 \delta r^{m_1}) \) apart. On the other hand, by (3.4) and our hypothesis on \( L_1 \), we can choose \( r \) large enough so that there is a connected component \( U \) of \( \text{reg}(r) \), given as in (3.7), with

\[
r_1 - \ell_1 < \frac{1}{c_2 \delta r^{m_1}}.
\]

From these two observations it is clear that there is some point in \( F_\rho/\mathbb{Z}^{k-d} \) whose orbit under \( \mathbb{Z}_{c_2r} \) does not intersect \( U \). Since \( c_2 > 0 \) was arbitrary, this means that \( Y \) is not linearly repetitive. Therefore, (LR1) and (LR2) are necessary conditions for linear repetitivity, and the proof of Theorem 1.1 is complete.
4. Proof of Corollary 1.2

It follows from [6, Theorem 2] that $Y$ is linearly repetitive if and only if it satisfies conditions (PQ) and (U) (we will not define condition (U), since it is unnecessary for our purposes). Therefore, if $Y$ is linearly repetitive then it satisfies condition (PQ) and, by [6, Theorem 1], it satisfies a subadditive ergodic theorem.

In the other direction, suppose that $Y$ satisfies condition (PQ) (equivalently, that $Y$ satisfies a subadditive ergodic theorem). Then the fact that $\nu(P) > 0$ for all equivalence classes $P$ implies that there is a constant $C > 0$ with the property that, for any equivalence class $P = P(y, r)$, we have that

$$\xi_P \geq \frac{C}{r^d}. \quad (4.1)$$

It follows, first of all, that the number of patches of size $r$ is $\ll r^d$. By the first part of our proof of Theorem 1.1, this implies that condition (LR1) is satisfied.

Now suppose that condition (LR2) is not satisfied and assume, without loss of generality, that the linear form $L_1$ defining $E$ is not relatively badly approximable. Then, for any $\epsilon > 0$, we can choose $r > 0$ so that there is a connected component $U$ of $\text{reg}(r)$, given as in (3.7), with

$$r_1 - \ell_1 < \frac{\epsilon}{r m_1}.$$

We can also assume that $U$ has been chosen so that

$$r_i - \ell_i \ll \frac{1}{r m_i}, \quad \text{for each } 2 \leq i \leq k - d.$$

Then $U$ has volume $\ll \epsilon/r^d$, where the implied constant depends only on $E$. Since $\epsilon$ can be taken arbitrary small, this together with (4.1) and Lemma 2.5 implies that (PQ) is not satisfied. Therefore, (PQ) implies (LR1) and (LR2), which, by Theorem 1.1, implies linear repetitivity.

5. Proof of Corollary 1.3

Our proof of Theorem 1.1 demonstrates how, to each aperiodic totally irrational canonical linearly repetitive cut and project set, we may associate a subgroup $\Lambda \leq \mathbb{Z}^d$ of finite index, with decomposition

$$\Lambda = \Lambda_1 + \cdots + \Lambda_{k-d},$$

so that each $L_i$ is badly approximable, when viewed as a linear form in $m_i$ variables, restricted to $\Lambda_i$. The first part of Corollary 1.3 clearly follows from the fact that the integers $m_i \geq 1$ have sum equal to $d$. 
In the other direction, suppose that \( d \geq k - d \). If we start with \( k - d \) positive integers \( m_i \), with sum equal to \( d \), and a collection of badly approximable linear forms \( L_i : \mathbb{R}^{m_i} \to \mathbb{R} \) then, thinking of
\[
\mathbb{R}^d = \mathbb{R}^{m_1} + \cdots + \mathbb{R}^{m_{k-d}},
\]
any canonical cut and project set arising from the subspace
\[
E = \{(x, L_1(x), \ldots, L_{k-d}(x)) : x \in \mathbb{R}^d\}
\]
is linearly repetitive, by the proof of Theorem 1.1. It follows that the collection of \( \{\alpha_{ij}\} \in \mathbb{R}^{d(k-d)} \) which define linearly repetitive cut and project sets is a countable union (over all allowable choices of \( \Lambda_i \) above) of sets of Lebesgue measure 0 and Hausdorff dimension at most
\[
\dim \mathcal{B}_{m_1,1} + \cdots + \dim \mathcal{B}_{m_{k-d},1} = m_1 + \cdots + m_{k-d} = d.
\]

Since the canonical cut and project sets corresponding to \( \Lambda_i = \mathbb{Z}^{m_i} \) are all linearly repetitive, the Hausdorff dimension of this set is equal to \( d \).

6. Examples

6.1. The Penrose tiling as a non-totally irrational cut and project set. Let \( \zeta = \exp(2\pi i/5) \) and let \( Y \) be a canonical cut and project set defined using the two dimensional subspace \( E \) of \( \mathbb{R}^5 \) generated by the vectors
\[
(1, \text{Re}(\zeta), \text{Re}(\zeta^2), \text{Re}(\zeta^3), \text{Re}(\zeta^4))
\]
and
\[
(0, \text{Im}(\zeta), \text{Im}(\zeta^2), \text{Im}(\zeta^3), \text{Im}(\zeta^4)).
\]
Well known results of de Bruijn [7] and Robinson [15] show that the set \( Y \) is the image under a linear transformation of the collection of vertices of a Penrose tiling, and in fact that all Penrose tilings can be obtained in a similar way from cut and project sets. The fact that \( Y \) is linearly repetitive can be deduced directly from the definition of the Penrose tiling as a primitive substitution. However, we will show how the machinery developed in this paper can be used to prove linear repetitivity from the definition of \( Y \) as a cut and project set.

Note first that \( E \) is not a totally irrational subspace of \( \mathbb{R}^5 \), since it is contained in the rational subspace orthogonal to \((1, 1, 1, 1, 1)\). In this case Theorem 1.1 does not apply directly, but the proof in Section 3 is still robust enough to allow us to draw the desired conclusions. Set
\[
\alpha_1 = \cos(2\pi/5), \alpha_2 = \cos(4\pi/5), \beta_1 = \sin(2\pi/5), \text{ and } \beta_2 = \sin(4\pi/5),
\]
so that
\[
E = \{(x, x\alpha_1 + y\beta_1, x\alpha_2 + y\beta_2, x\alpha_2 - y\beta_2, x\alpha_1 - y\beta_1) : x, y \in \mathbb{R}\}.
\]
After making the change of variables $x_1 = x$ and $x_2 = x\alpha_1 + y\beta_1$, we can write $E$ as

$$E = \{(x, L_1(x), L_2(x), L_3(x)) : x = (x_1, x_2) \in \mathbb{R}^2\}.$$ 

The functions $L_i$ are linear forms which (using the fact that $4\alpha_1^2 + 2\alpha_1 - 1 = 0$) are given by

$$L_1(x) = -x_1 + 2\alpha_1 x_2,$$
$$L_2(x) = -2\alpha_1 x_1 - 2\alpha_1 x_2,$$
$$L_3(x) = 2\alpha_1 x_1 - x_2.$$ 

Write $L_i : \mathbb{Z}^2 \to \mathbb{R}/\mathbb{Z}$ for the restriction of $L_i$ to $\mathbb{Z}^2$, modulo 1, and notice that $L_1 + L_2 + L_3 = 0$. This means that the orbit of 0 under the natural $\mathbb{Z}^2$-action of $E$ on $F/\mathbb{Z}^3$ is contained in the two dimensional rational subtorus with equation $x + y + z = 0$. The kernels of the forms $L_i$ are all rank 1 subgroups of $\mathbb{Z}^2$, and it follows that the number of connected components of $\text{reg}(r)$ which intersect the rational subtorus is $\asymp r^2$.

Since the forms are linearly dependent, we can understand the orbit of a point in $F/\mathbb{Z}^3$ under the $\mathbb{Z}^2$-action by considering only the values of $L_1$ and $L_3$. The number $\alpha_1$ is a quadratic irrational, hence it is badly approximable, and the rest of our proof of Theorem 1.1 applies with little modification, to show that there is a constant $c > 0$ such that the orbit of every point in $W$, under the action of $c\Omega \cap \mathbb{Z}^2$, intersects every connected component of $\text{reg}(r)$. Therefore $Y$ is linearly repetitive.

6.2. Explicit examples for all $d \geq k - d$. For $d \geq k/2$ it is easy to give examples of subspaces $E$ satisfying the hypotheses of Theorem 1.1. Write $d = m_1 + \cdots + m_{k-d}$, with positive integers $m_i$, and for each $i$ let $K_i$ be an algebraic number field, of degree $m_i + 1$ over $\mathbb{Q}$. Suppose that the numbers $1, \alpha_{i1}, \ldots, \alpha_{im_i}$ form a $\mathbb{Q}$-basis for $K_i$, and define $L_i : \mathbb{R}^{m_i} \to \mathbb{R}$ to be the linear form with coefficients $\alpha_{i1}, \ldots, \alpha_{im_i}$. Then, using the decomposition $\mathbb{R}^d = \mathbb{R}^{m_1} + \cdots + \mathbb{R}^{m_{k-d}}$, let

$$E = \{(x, L_1(x), \ldots, L_{k-d}(x)) : x \in \mathbb{R}^d\}.$$ 

The collection of points

$$\{(L_1(n), \ldots, L_{k-d}(n)) : n \in \mathbb{Z}^d\}$$

is dense in $\mathbb{R}^{k-d}/\mathbb{Z}^{k-d}$, and it follows from this that the subspace $E$ is totally irrational. The intersection of the kernels of the corresponding maps $L_i$ is $\{0\}$, so any canonical cut and project set formed from $E$ will be aperiodic. Condition (LR1) of Theorem 1.1 is clearly satisfied. Furthermore, by a result of Perron [14], each of the linear forms $L_i$ is badly
approximable. Therefore (LR2) is also satisfied, and any canonical cut
and project set formed from $E$ is linearly repetitive.

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