JACOB’S LADDERS, CROSSBREEDING AND INFINITE SETS OF META-FUNCTIONAL EQUATIONS AS NEW SPECIES GENERATED BY THE MOTHER FORMULA

JAN MOSER

ABSTRACT. In this paper we obtain a set of meta-functional equations as new species of formulas in classical mathematical analysis. Mentioned species are generated by crossbreeding complete hybrid formula as a mother formula. Namely, they are generated by an infinite set of crossbreedings on some subsidiary infinite set of meta-functional equations with one neutral factor.

DEDICATED TO THE 160th ANNIVERSARY OF DARWIN’S ORIGIN OF SPECIES

1. Introduction

1.1. Let us remind that the following sets of values

\[ \left\{ \zeta \left( \frac{1}{2} + it \right) \right\}^2, \{f_1(t)\} = \{\sin^2 t\}, \{f_2(t)\} = \{\cos^2 t\}, \]

\[ t \in [\pi L, \pi L + U], \quad U \in (0, \pi/2), \quad L \in \mathbb{N} \]

generate the exact complete hybrid formula (see [7], (3.2), \( k_1 = k_2 = 1 \))

\[ \tilde{Z}^2(\alpha_{1,1}) \sin^2 \alpha_{0,1} + \tilde{Z}^2(\alpha_{0,1}) \cos^2 \alpha_{0,1} = \tilde{Z}^2(\beta_1^1), \]

where (comp. [9], (1.6)–(1.10))

\[ \alpha_{l,1}^r = \alpha_r(U, \pi L; f_l, |\zeta_{0,5}|^2), \quad r = 0, 1, \quad l = 1, 2, \]

\[ |\zeta_{0,5}|^2 = \left| \zeta \left( \frac{1}{2} + it \right) \right|^2, \]

\[ \beta_1^1 = \beta_1(U, \pi L; |\zeta_{0,5}|^2), \]

\[ \alpha_{0,1}^l \in (\pi L, \pi L + U), \quad \alpha_{1,1}^l, \beta_1^1 \in (\pi L, \pi L + U), \]

\[ U \in (0, \pi/2), \quad L \geq L_0 > 0, \]

\( (L_0 \text{ is sufficiently big one}) \), next, we denote by the symbol

\[ [\pi L, \pi L + U] \]

the first reverse iteration (by means of Jacob’s ladder \( \varphi_1(t) \), see [8]) of the basic segment

\[ [\pi L, \pi L + U] = [\pi L, \pi L + U], \]

Key words and phrases. Riemann zeta-function.
and, finally,
\[
Z^2(t) = \frac{d^2 \varphi_1(t)}{dt^2} = \frac{\zeta \left( \frac{1}{2} + it \right)^2}{\omega(t)},
\]
(1.4)
\[
\omega(t) = \left\{ 1 + O \left( \frac{\ln \ln t}{\ln t} \right) \right\} \ln t,
\]
(see [2], (6.1), (6.7), (7.7), (7.8), (9.1)).

**Remark 1.** The components of the main \( \zeta \)-disconnected set
\[
\Delta(\pi L, U, 1) = [\pi L, \pi L + U] \bigcup_{1}^{1} [\pi L, \pi L + U]
\]
(for our case) are separated each from other by the gigantic distance \( \rho \) (see [3], (5.12), comp. [7], (2.2)–(2.9)):
\[
\rho([\pi L, \pi L + U]; [\pi L, \pi L + U]) \sim \pi(1 - c) \frac{L}{\ln L}, \quad L \to \infty,
\]
where \( c \) stands for the Euler’s constant.

Further, let us remind that in our papers [8] – [10] we have used an asymptotic form of corresponding exact complete hybrid formula (1.2). In this paper we shall use directly the exact complete hybrid formula (1.2) for our purposes.

1.2. First we shall consider the following four sets
\[
\{ \zeta(ns) \}, \quad \{ \Gamma(ns) \}, \quad \{ \text{cn}(ns, k) \}, \quad \{ J_p(ns) \},
\]
(1.6)
\[
n \in \mathbb{N}, \quad ns \in \mathbb{C}\backslash\{N, P\}, \quad k^2 \in (0, 1), \quad p \in \mathbb{Z},
\]
where
\[
\{N, P\}
\]
is the set of all zeros and poles of functions
\[
\zeta(s), \quad \Gamma(s), \quad \text{cn}(s, k), \quad J_p(s).
\]
We obtain (for example) the following result in this direction: There are the sets
\[
_{\Omega_n^m}, \quad l = 1, \ldots, 4, \quad n \in \mathbb{N}
\]
such that we have the following infinite set
\[
|\zeta(0^n_m s_l^m) ||\Gamma(0^n_m s_l^m) ||\text{cn}(0^n_m s_l^m, k) | + |J_p(0^n_m s_l^m) ||\text{cn}(0^n_m s_l^m, k) | = \]
(1.7)
\[
= |\zeta(0^n_m s_l^m) ||\Gamma(0^n_m s_l^m) ||\text{cn}(0^n_m s_l^m, k) | + |J_p(0^n_m s_l^m) ||\text{cn}(0^n_m s_l^m, k) |,
\]
\[
(m, n) \in \mathbb{N}^2, \quad s_l^m \in \Omega_l^m, \quad l = 1, 2, 3, 4
\]
of exact meta-functional equations as new species generated by the mother formula (1.2). Next, it follows from (1.7) that on the infinite set
\[
\{K(m, n)\} = \]
(1.8)
\[
= \{ |\zeta(0^n_m s_l^m) ||\Gamma(0^n_m s_l^m) ||\text{cn}(0^n_m s_l^m, k) | + |J_p(0^n_m s_l^m) ||\text{cn}(0^n_m s_l^m, k) | \},
\]
\[
(m, n) \in \mathbb{N}^2
\]
with quite complicated elements, still the commutative law
\[
K(m, n) = K(n, m)
\]
holds true.
1.3. Further, we shall consider even more complicated case of the set of four tuples:

\[
\{\zeta(s), \Gamma(s), \text{cn}(s, k), J_p(s)\}, \\
\{\Gamma(2s), \text{cn}(2s, k), J_p(2s), \zeta(2s)\}, \\
\{\text{cn}(3s, k), J_p(3s), \zeta(3s), \Gamma(3s)\}, \\
\{J_p(4s), \zeta(4s), \Gamma(4s), \text{cn}(4s, k)\}, \\
\{\zeta(5s), \Gamma(5s), \text{cn}(5s, k), J_p(5s)\}, \\
\{\Gamma(6s), \text{cn}(6s, k), J_p(6s), \zeta(6s)\}, \\
\{\text{cn}(7s, k), J_p(7s), \zeta(7s), \Gamma(7s)\}, \\
\{J_p(8s), \zeta(8s), \Gamma(8s), \text{cn}(8s, k)\}, \\
\{\zeta(9s), \Gamma(9s), \text{cn}(9s, k), J_p(9s)\}, \\
\ldots
\]

(1.10)

generated by the cyclical changes in order of symbols

\[\zeta, \Gamma, \text{cn}(k), J_p.\]

In this case there are sets (for example)

\[\Omega_l^1, \Omega_l^2, l = 1, 2, 3, 4\]

of elements

\[s_i^1, s_i^4\]

such that the following equation

\[
|\zeta(s_i^1)||\Gamma(s_i^2)||\Gamma(4s_i^3)| + |\text{cn}(s_i^3, k)||\text{cn}(4s_i^4, k)| = \\
|\zeta(4s_i^4)||J_p(4s_i^5)||\text{cn}(s_i^3, k)| + |J_p(2s_i^1)||\Gamma(4s_i^3)|
\]

(1.11)

holds true.

**Remark 2.** Of course, the exact meta-functional equation (1.11) represents only one element of an infinite sets of corresponding descendans (new species) generated by the mother formula (1.2) as we shall see.

**Remark 3.** The symmetry similar to (1.9) is no longer valid for (1.11).

**Remark 4.** However, there are also meta-functional equations of the type:

\[
|\zeta(1s_i^1)||\Gamma(1s_i^2)||\text{cn}(5s_i^3, k)| + |J_p(5s_i^4)||\text{cn}(1s_i^3, k)| = \\
|\zeta(5s_i^4)||J_p(5s_i^5)||\text{cn}(1s_i^3, k)| + |J_p(1s_i^1)||\text{cn}(5s_i^3)|
\]

(1.12)

and for them (1.9)-type symmetries hold true w.r.t. transposition 1 ↔ 5.

1.4. Let us notice that we present in this paper a kind of general method of generating infinite sets of meta-functional equations as the set of descendants of the mother formula (1.2).

**Remark 5.** Genericity of the presented method is based on:

(A) infinity of a set of possibilities how to make a choice of the mother formula
(B) applicability of this method to infinite set of admissible infinite sets of types

(1.6), (1.10), ...
Next, we give the list of stages leading us to new infinite sets of meta-functional

equations in present paper:

(a) We assign the set of two factorization formulas (see [4] - [7]) to the set (1.1),
(b) crossbreeding on the last set gives us the exact complete hybrid formula
(1.2) defined of the critical line $\sigma = \frac{1}{2},$
(c) next, we assign corresponding sets of level-curves to sets (1.6), (1.10).
(d) by means of mentioned level-curves we assign infinite sets of meta-functional
equations to the mother formula (1.2) in such a way that every equation
contains an identical (=neutral) factor (with respect to operation (c)),
(e) finally, the crossbreeding on the last sets gives mentioned meta-functional
equations on the complex plane as new species generated by the mother
formula (1.2).

Remark 6. Also this paper is based on new notions and methods in the theory of the
Riemann’s zeta-function we have introduced in the series of 52 papers concerning
Jacob’s ladders. These can be found in arXiv [math.CA] starting with the paper [1].

2. Set of four tuples with a simple ordering of the symbols
$\zeta, \Gamma, \text{cn}(k), J_p$

2.1. Now, we assign corresponding level-curves to each set in (1.6). Namely, we
define four sets of level-curves

\[ \{ \Omega_l \}, \ l = 1, 2, 3, 4, \ \Omega_l \in \mathbb{C}, \]

where

\[
\begin{align*}
\Omega_1 &= \Omega_1^0 (\tilde{S}_1, |\zeta(n_s)|), \\
\Omega_2 &= \Omega_2^0 (\tilde{S}_1, |\Gamma(n_s)|), \\
\Omega_3 &= \Omega_3^0 (\tilde{S}_2, |\text{cn}(n_s,k)|), \\
\Omega_4 &= \Omega_4^0 (\tilde{S}_3, |J_p(n_s)|),
\end{align*}
\]

and

\[
\begin{align*}
\tilde{S}_1 &= (U, \pi L; f_1, |\zeta_{0.5}|^2), \\
\tilde{S}_2 &= (U, \pi L; f_2, |\zeta_{0.5}|^2), \\
\tilde{S}_3 &= (U, \pi L; |k_{0.5}|^2),
\end{align*}
\]

as the loci

\[
\begin{align*}
|\zeta(n_{s1})|^0 &= \tilde{Z}^2(\alpha_{1,1}^0) = c_1, \\
|\Gamma(n_{s2})|^0 &= \sin^2 \alpha_{0,1}^0 = c_2, \\
|\text{cn}(n_{s3}, k)|^0 &= \cos^2 \alpha_{0,2}^0 = c_3, \\
|J_p(n_{s4})|^0 &= \tilde{Z}^2(\beta_1^0) = c_4,
\end{align*}
\]

and

\[ 0 < c_1, \ldots, c_4 < +\infty \]
for every admissible (see (1.4)) and fixed \( U, L, k, p \), where

\[
\theta_s \in \Omega_t
\]

**Remark 7.** Let us compare conditions (2.4) with corresponding inclusions in (1.6).

Consequently, we have the following (see (1.2), (2.3), (2.5)):

**Lemma 1.** There is the set

\[
(2.6) \quad |\zeta(s_1^0)\Gamma(s_2^0) + \tilde{Z}^2(\alpha_1^{2.1})| = |J_p(s_2^0)|, \quad n \in \mathbb{N}
\]

of exact meta-functional equations with the neutral factor

\[
\tilde{Z}^2(\alpha_1^{2.1})
\]

(comp. the point (d) in the subsection 1.4 of this text).

2.2. Next, we can write (see (2.6))

\[
(2.7) \quad |\zeta(s_1^0)\Gamma(s_2^0) + \tilde{Z}^2(\alpha_1^{2.1})| = |J_p(s_2^0)|, \quad (m, n) \in \mathbb{N}^2, \quad m \neq n.
\]

Now we use the operation of crossbreeding to obtain the following Theorem (comp. [6], [7] on the set (2.7), in this context elimination of \( \tilde{Z}^2(\alpha_1^{1.2}) \))

**Theorem 1.** There is infinite set

\[
(2.8) \quad |\zeta(s_1^0)\Gamma(s_2^0) + \tilde{Z}^2(\alpha_1^{2.1})| = |J_p(s_2^0)|, \quad (m, n) \in \mathbb{N}^2, \quad m \neq n, \quad k^2 \in (0, 1), p \in \mathbb{Z},
\]

(for every admissible and fixed \( k \) and \( p \)) of exact meta-functional equations as new species generated by the mother formula (1.2).

2.3. We see immediately that for elements of the set

\[
(2.9) \quad \{K(m, n) = \{|\zeta(s_1^0)\Gamma(s_2^0) + \tilde{Z}^2(\alpha_1^{2.1})|, \quad (m, n) \in \mathbb{N}^2
\}
\]

the following is true:

**Corollary 1.**

\[
(2.10) \quad K(m, n) = K(n, m), \quad \forall (m, n) \in \mathbb{N}^2,
\]

(the case \( m = n \) is the trivial one).

3. **Set of four tuples with cyclically ordered symbols \( \zeta, \Gamma, cn(k), J_p \)**

3.1. Further, we assign corresponding level curves to elements of the first four sets in (1.10). Namely

\[
\Omega_l^m, \quad m, l = 1, 2, 3, 4
\]
as the loci

\[ \Omega_1^1 : \]
\[ |\zeta(s_1^1)| = \tilde{Z}^2(\alpha_{1}^{1,1}) = c_1, \]
\[ |\Gamma(s_2^1)| = |\sin^2(\alpha_{0}^{1,1})| = c_2, \]
\[ |\text{cn}(s_3^1, k)| = \cos^2(\alpha_{0}^{2,1}) = c_3, \]
\[ |J_p(s_4^1)| = \tilde{Z}^2(\beta_{1}^1) = c_4, \]
\[ s_1^1 \in \Omega_1^1 \]
\[ \Omega_1^3 : \]
\[ |\text{cn}(3s_3^3, k)| = c_1, \]
\[ |J_p(3s_3^3, k)| = c_2, \]
\[ |\zeta(3s_3^3)| = c_3, \]
\[ s^3_3 \in \Omega_1^3 \]
\[ 0 < c_1, c_2, c_3, c_4 < +\infty, \]

where (comp. [2,3])

\[ \Omega_1^1 = \Omega_1^1(\tilde{S}_1, |\zeta(1s)|), \ldots, \Omega_4^3 = \Omega_4^3(\tilde{S}_3, |\text{cn}(4s, k)|) \]

for every admissible (see (1.3), (2.8)) and fixed \(U, L, k, p\).

Thus, we have (see (1.2), (3.1)) the following Lemma 2.

\[ |\zeta(s_1)| |\Gamma(s_2^1)| + \tilde{Z}^2(\alpha_{1}^{1,1}) |\text{cn}(s_3, k)| = |J_p(s_4)|, \]  \hspace{1cm} (3.3)
\[ |\Gamma(2s_2^1)| |\text{cn}(2s_2, k)| + \tilde{Z}^2(\alpha_{1}^{2,1}) |J_p(2s_2^1, k)| = |\zeta(2s_4)|, \]  \hspace{1cm} (3.4)
\[ |\text{cn}(3s_3^3, k)| |J_p(3s_3^3, k)| + \tilde{Z}^2(\alpha_{1}^{2,1}) |\zeta(3s_3^3)| = |\Gamma(3s_4)|, \]  \hspace{1cm} (3.5)
\[ |J_p(4s_4^3)| |\zeta(4s_4)| + \tilde{Z}^2(\alpha_{1}^{1,1}) |\Gamma(4s_4^3)| = |\text{cn}(4s_4^3, k)| \]  \hspace{1cm} (3.6)

of transmutations of the mother formula (1.2) with the neutral factor \( \tilde{Z}^2(\alpha_{1}^{1,1}) \).

3.2. Now, we use the operation of crossbreeding (see [2,7]) on every two different elements of the set

(3.7) \( \{(3.3), (3.4), (3.5), (3.6)\} \).

Remark 8. Let the symbol

(3.3) \times (3.6) \Rightarrow

stand for we obtain by crossbreeding of the elements (3.3) and (3.6).

Consequently, we obtain the following statement.
Theorem 2. There is the following set of exact meta-functional equations as other transmutations of the mother formula (3.2):

\((3.3) \times (3.4) \Rightarrow (3.8)\)

\[\zeta(s_1^4) \mid \Gamma(s_1^2) \mid J_p(2s_2^2) + \zeta(2s_2^2) \mid cn(s_3^4) \mid k = \]
\[= \mid \Gamma(2s_2^2) \mid cn(3s_2^4) \mid cn(s_4^t) \mid k + \mid J_p(s_1^4) \mid J_p(2s_2^2)\]

\((3.3) \times (3.5) \Rightarrow (3.9)\)

\[\zeta(s_1^4) \mid \zeta(3s_2^4) \mid \Gamma(s_1^2) \mid \Gamma(3s_2^4) \mid cn(s_3^4) \mid k = \]
\[= \mid cn(s_3^4) \mid cn(3s_3^4) \mid J_p(3s_2^4) \mid + \mid J_p(s_1^4) \mid J_p(3s_2^4)\]

\((3.3) \times (3.6) \Rightarrow (3.10)\)

\[\zeta(s_1^4) \mid \Gamma(s_1^2) \mid \Gamma(4s_4^4) \mid cn(s_4^t) \mid k = \]
\[= \mid \zeta(4s_4^4) \mid J_p(4s_4^4) \mid cn(s_3^4) \mid k \]

\((3.4) \times (3.5) \Rightarrow (3.11)\)

\[\zeta(3s_3^4) \mid \Gamma(2s_2^2) \mid \Gamma(3s_2^4) \mid J_p(2s_2^2) = \]
\[= \mid cn(3s_3^4) \mid \mid J_p(2s_2^2) \mid J_p(3s_2^4) \mid + \mid J_p(s_1^4) \mid J_p(3s_2^4)\]

\((3.4) \times (3.6) \Rightarrow (3.12)\)

\[\mid \Gamma(2s_2^2) \mid \Gamma(4s_4^4) \mid cn(2s_2^2) \mid k + \mid cn(4s_4^t) \mid J_p(2s_2^2) = \]
\[= \mid J_p(2s_2^2) \mid J_p(4s_4^4) \mid cn(4s_4^t) \mid \zeta(4s_4^4) \mid \zeta(4s_4^4)\]

\((3.5) \times (3.6) \Rightarrow (3.13)\)

\[\mid cn(3s_3^4) \mid \mid J_p(3s_3^4) \mid \Gamma(4s_4^4) + \mid cn(4s_4^t) \mid \zeta(3s_3^4) \mid = \]
\[= \mid J_p(4s_4^4) \mid cn(3s_3^4) \mid \zeta(3s_3^4) \mid + \mid J_p(3s_3^4) \mid \Gamma(4s_4^4)\]

3.3. Cyclical ordering of the symbols \(\zeta, \Gamma, cn(k), J_p\) in the set (1.10) gives the following generalization of our Lemma 2.

Lemma 3.

\[\mid \zeta([(4m + 1)s_1^{4m+1}] \mid \Gamma([(4m + 1)s_2^{4m+1}] + \tilde{Z}^2(\alpha_1^{2,1}) \mid cn([4m + 1)s_3^{4m+1}, k]) = \]
\[= \mid J_p([(4m + 1), s_4^{4m+1}]\),
\[\mid \Gamma([(4m + 2)s_2^{4m+2}] \mid cn([4m + 2)s_2^{4m+2}, k]) + \tilde{Z}^2(\alpha_1^{2,1}) \mid J_p([(4m + 2)s_3^{4m+2}]\]
\[= \mid \zeta([(4m + 2), s_4^{4m+2}]\).

\[(3.14) \mid cn([(4m + 3)s_1^{4m+3}, k]) \mid J_p([(4m + 3)s_2^{4m+3}] + \tilde{Z}^2(\alpha_1^{2,1}) \mid [((4m + 3)s_3^{4m+3}] = \]
\[= \mid \Gamma([(4m + 3), s_4^{4m+3}]\],
\[\mid J_p([(4m + 4)s_2^{4m+4}, k]) \mid [((4m + 4)s_2^{4m+4}] + \tilde{Z}^2(\alpha_1^{2,1}) \mid \Gamma([(4m + 4)s_3^{4m+4}]\]
\[= \mid cn([(4m + 4), s_4^{4m+4}, k]\),
\[m \in \mathbb{N}_0,\]

where the symbols

\[s_1^{4m+1} \in \Omega_1^{4m+1}, \ldots, s_4^{4m+1} \in \Omega_2^{4m+1}, \ldots\]

one obtains as a natural continuation of the scheme (3.1), see (1.10).
Next, we use the symbol

\[(4m + q) \times (4m + r) \Rightarrow m, n \in \mathbb{N}_0, q, r = 1, 2, 3, 4\]

in the sense of our Remark 8, where \(4m + q\) and \(4m + r\) are sequential numbers of equations in (3.14). Consequently, we obtain the following generalization of our Theorem 2.

**Theorem 3.** There are six infinite classes of exact meta-functional equations as new species generated by the mother formula (1.2):

\[(4m + 1) \times (4n + 2) \Rightarrow \]

\[
|\zeta[(4m + 1)s_1^{4m+1}]| |\Gamma[(4m + 1)s_2^{4m+1}]| |J_p[(4n + 2)s_3^{4n+2}]| +
+ |\zeta[(4n + 2)s_4^{4n+2}]| \cn[(4m + 1)s_3^{4m+1}, k]| =
= |\Gamma[(4n + 2)s_4^{4n+2}]| \cn[(4n + 2)s_2^{4m+2}, k]| |\cn[(4m + 1)s_3^{4m+1}, k]| +
+ |J_p[(4m + 1)s_4^{4m+1}]| |J_p[(4n + 2)s_3^{4n+2}]|,
\]

(3.15)

\[(4m + 3) \times (4n + 4) \Rightarrow \]

\[
|\cn[(4m + 3)s_1^{4m+3}, k]| |\Gamma_p[(4m + 3)s_2^{4m+3}]| |\Gamma[(4n + 4)s_4^{4n+4}]| +
+ |\cn[(4n + 4)s_4^{4n+4}, k]| |\zeta[(4m + 3)s_3^{4m+3}]| =
= |J_p[(4n + 4)s_1^{4n+4}]| |\zeta[(4m + 3)s_3^{4m+3}]| |\zeta[(4n + 4)s_2^{4n+4}]| +
+ |\Gamma[(4m + 3)s_4^{4m+3}]| |\Gamma[(4n + 4)s_3^{4n+4}]|,
\]

\(m, n \in \mathbb{N}_0\).

### 3.4. Further, as a subset of such four consecutive sets in (1.10) that corresponds to the rows

\[4m + 1, 4m + 2, 4m + 3, 4m + 4\]

we call the \(m\)-th cell of the set (1.10).

**Remark 9.** We may assume that results of some interactions between the \(m\)-th and \(n\)-th cells are expressed by the equations (3.15). Namely:

(a) If \(m = n\), then we have the case of internal interactions between all mutually different elements of the \(m\)-th cell.

(b) If \(m \neq n\), then we have the case of external interactions between every element (=corresponding four tuple) of the \(m\)-th cell with every element of the \(n\)-th cell.

### 3.5. Next, we present some property connected with the external interactions mentioned above. Namely, we have (similarly to (3.15))

\[(4m + 1) \times (4n + 1) \Rightarrow \]

\[
|\zeta[(4n + 1)s_1^{4n+1}]| |\Gamma[(4m + 1)s_2^{4m+1}]| |\cn[(4n + 1)s_3^{4n+1}, k]| +
+ |J_p[(4n + 1)s_4^{4n+1}]| |\cn[(4m + 1)s_3^{4m+1}, k]| =
= |\zeta[(4n + 1)s_3^{4n+1}]| |\Gamma[(4n + 1)s_2^{4n+1}]| |\cn[(4m + 1)s_3^{4m+1}, k]| +
+ |J_p[(4n + 1)s_4^{4n+1}]| |\cn[(4m + 1)s_2^{4n+1}, k]|.
\]

(3.16)
Now, if we put
\[
G(4m + 1, 4n + 1) = \\
|\zeta[(4m + 1)s_1^{4m+1}]||\Gamma[(4m + 1)s_2^{4m+1}]||\text{cn}[(4n + 1)s_3^{4n+1}, k]| + \\
|J_p[(4n + 1)s_4^{4n+1}]||\text{cn}[(4m + 1)s_3^{4m+1}, k]|
\]
then we obtain for the set
\[
\{G(4m + 1, 4n + 1), \ m, n \in \mathbb{N}_0, \ m \neq n\}
\]
the following

**Corollary 2.**

\[(3.17) \quad G(4m + 1, 4n + 1) = G(4n + 1, 4m + 1), \]

and similarly

\[(3.18) \quad G(4m + q, 4n + q) = G(4n + q, 4m + q), \ q = 2, 3, 4. \]

I would like to thank Michal Demetrian for his moral support of my study of Jacob’s ladders.

**References**

[1] J. Moser, ‘Jacob’s ladders and almost exact asymptotic representation of the Hardy-Littlewood integral’, Math. Notes 88, (2010), 414-422, arXiv: 0901.3937.

[2] J. Moser, ‘Jacob’s ladders, structure of the Hardy-Littlewood integral and some new class of nonlinear integral equations’, Proc. Steklov Inst. 276 (2011), 208-221, arXiv: 1103.0359.

[3] J. Moser, ‘Jacob’s ladders, reverse iterations and new infinite set of \(L_2\)-orthogonal systems generated by the Riemann zeta-function, arXiv: 1402.2098.

[4] J. Moser, ‘Jacob’s ladders, factorization and metamorphoses as an appendix to the Riemann functional equation for \(\zeta(s)\) on the critical line’, Proc. Steklov Inst. 296 (2017), pp. 92-102, arXiv: 1506.00442v1.

[5] J. Moser, ‘Jacob’s ladders, interactions between \(\zeta\)-oscillating systems and \(\zeta\)-analogue of an elementary trigonometric identity’, arXiv: 1609.09293v1, Proc. Steklov Inst. 299, 189-204, 2017.

[6] J. Moser, ‘Jacob ladders, crossbreeding, secondary crossbreeding and synergetic phenomena generated by the Riemann’s zeta-function and some elementary functions on disconnected sets of the critical line’, arXiv: 1806.07095v1.

[7] J. Moser, ‘Jacob’s ladders and grafting of the complete hybrid formulas into \(\zeta\)-synergetic meta-functional equation for the Riemann’s zeta-function’, arXiv: 1809.05327v1.

[8] J. Moser, ‘Jacob ladders and infinite set of transmutations of asymptotic complete hybrid formula on level curves in Gauss’ plane’, arXiv: 1905.00708v1.

[9] J. Moser, ‘Jacob’s ladders and exact meta-functional equations on level curves as global quantitative characteristics of synergetic phenomenons excited by the functions \(|\zeta\left(\frac{1}{2} + it\right)|^2\)’, arXiv: 1906.02440v1.

[10] J. Moser, ‘Jacob’s ladders and completely new exact synergetic formula for Jacobi’s elliptic functions together with Bessel’s functions excited by the function \(|\zeta\left(\frac{1}{2} + it\right)|^2\)’, arXiv: 1907.1281v1.