Measuring and Localizing Homology Classes

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Abstract
We develop a method for measuring and localizing homology classes. This involves two problems. First, we define relevant notions of size for both a homology class and a homology group basis, using ideas from relative homology. Second, we propose an algorithm to compute the optimal homology basis, using techniques from persistent homology and finite field algebra. Classes of the computed optimal basis are localized with cycles conveying their sizes. The algorithm runs in $O(\beta^3 n^3 \log^2 n)$ time, where $n$ is the size of the simplicial complex and $\beta$ is the Betti number of the homology group.

1 Introduction

In recent years, the problem of computing the topological features of a space has drawn much attention. There are two reasons for this. The first is a general observation: compared with geometric features, topological features are more qualitative and global, and tend to be more robust. If the goal is to characterize a space, therefore, features which incorporate topology seem to be good candidates.

The second reason is that topology plays an important role in a number of applications. Researchers in graphics need topological information to facilitate parameterization of surfaces and texture mapping [13, 4]. In the field of sensor networks, the use of homological tools is crucial for certain coverage problems [10]. Computational biologists use topology to study protein docking and folding problems [1, 8]. Finally, topological features are especially important in high dimensional data analysis, where purely geometric tools are often deficient, and full-blown space reconstruction is expensive and often ill-posed [3, 16].

Once we are able to compute topological features, a natural problem is to rank the features according to their importance. The significance of this problem
can be justified from two perspectives. First, unavoidable errors are introduced in data acquisition, in the form of traditional signal noise, and finite sampling of continuous spaces. These errors may lead to the presence of many small topological features that are not “real”, but are simply artifacts of noise or of sampling [21]. Second, many problems are naturally hierarchical. This hierarchy – which is a kind of multiscale or multi-resolution decomposition – implies that we want to capture the large scale features first. See Figure 1 for examples.

There are a variety of ways of characterizing topological spaces in the literature, including fundamental groups, homology groups, and the Euler characteristic. In this paper, we concentrate on homology groups as they are relatively straightforward to compute in general dimension, and provide a decent amount of information (more, say, than a coarse measure like the Euler characteristic).

Ranking the homology classes according to their importance involves the following three subproblems.

1. **Measuring the size of a homology class:** We need a way to quantify the size of a given homology class, and this size measure should agree with intuition. For example, in Figure 2 (center), the measure should be able to distinguish the one large class (of the 1-dimensional homology group) from the two smaller classes. Furthermore, the measure should be easy to compute, and applicable to homology groups of any dimension.

2. **Localizing a homology class:** Given the size measure for a homology class, we would like to find a representative cycle from this class which, in a precise sense, has this size. For example, in Figure 2 (center), the cycles $z_1$ and $z_2$ are well-localized representatives of their respective homology classes; whereas $z_3$ is not.

3. **Choosing a basis for a homology group:** We would like to choose a “good” set of homology classes to be the generators for the homology group (of a fixed dimension). Suppose that $\beta$ is the dimension of this group, and that we are using $\mathbb{Z}_2$ coefficients; then there are $2^\beta - 1$ nontrivial homology classes in total. For a basis, we need to choose a subset of $\beta$ of these classes, subject to the constraint that these $\beta$ generate the group. The criterion
of goodness for a basis is based on an overall size measure for the basis, which relies in turn on the size measure for its constituent classes. For instance, in Figure 3, we must choose three from the seven nontrivial 1-dimensional homology classes: \{[z_1], [z_2], [z_3], [z_1] + [z_2], [z_1] + [z_3], [z_2] + [z_3], [z_1] + [z_2] + [z_3]\}. In this case, the intuitive choice is \{[z_1], [z_2], [z_3]\}, as this choice reflects the fact that there is really only one large cycle.

Figure 3: A topological space formed from three circles. See accompanying discussion in the text.

1.1 Related Works

There is much work that has been done in the general field of computational topology [2]. Examples include fast algorithms for computing Betti numbers [11, 15], as well as techniques for relating topological spaces to their approximations [19, 5]; where the latter usually derive from sampled versions of the spaces. However, in the following we will focus only on the areas of computational topology which are most germane to the current study: persistent homology and algorithms for localizing topological features. Note that a more formal review of persistence will be given in Section 2.3.
Persistent Homology  Persistent homology [12, 7, 22, 24] is designed to track the persistences of homological features over the course of a filtration of a topological space. At first blush, it might seem that the powerful techniques of this theory are ideally suited to solving the problems we have set out. However, due to their somewhat different motivation, these techniques do not quite yield a solution. There are two reasons for this. First, the persistence of a feature depends not only on the space in which the feature lives, but also on the filtering function chosen. In the absence of a geometrically meaningful filter, it is not clear whether the persistence of a feature is a meaningful representation of its size. Second, and more importantly, the persistence only gives information for homology classes which ultimately die; for classes which are intrinsically part of the topological space, and which thus never die, the persistence is infinite. However, it is precisely these essential (or non-persistent) classes that we care about.

In more recent work, Cohen-Steiner et al. [6] have extended persistent homology in such a way that essential homology classes also have finite persistences. This extension serves to complete the theory and has some nice properties like stability, duality and symmetry for triangulated manifolds. However, the persistences thus computed still depend on the filter function, and furthermore, do not always seem to agree with an intuitive notion of size. See Figure 4.

![Figure 4: Computing the extended persistent homology of a torus using the height function as the filter function. The (birth,death time) pairs of the two 1-dimensional homology classes are $(t_1, t_2)$ and $(t_2, t_1)$, respectively. The persistences are not consistent with our intuition of their sizes.](image)

Localization of Topological Features  Zomorodian and Carlsson [23] take a different approach to solving the localization problem. Their method starts with a topological space and a cover, a set of spaces whose union contains the original space. A blowup complex is built up which contains homology classes of all the spaces in the cover. The authors then use persistent homology to identify homology classes in the blowup complex which correspond to a same homology class in the given topological space. The persistent homology algorithm produces a complete set of generators for the relevant homology group,
which forms a basis for the group. However, both the quality of the generators and the complexity of the algorithm depend strongly on the choice of cover; there is, as yet, no suggestion of a canonical cover.

Using Dijkstra’s shortest path algorithm, Erickson and Whittlesey [14] showed how to localize a one-dimensional homology class with its shortest cycle. Although not explicitly mentioned, the length of this shortest cycle can be deemed as a measure of the size of its homology class. They proved, by an application of matroid theory, that finding \( \beta \) linearly independent homology classes whose sizes have the smallest sum can be achieved by a greedy method, namely, finding the smallest homology classes one by one, subject to a linear independence constraint. Their algorithm takes \( O(n^2 \log n + n^2 \beta + n\beta^3) \) or \( O(n^2 \beta + n\beta^3) \) if \( \beta \) is nearly linear in \( n \). The authors also show how the idea carries over to finding the optimal generators of the first fundamental group, though the proof is considerably harder in this case. Note that this work is restricted to 1-dimensional homology classes in a 2-dimensional topological space. A similar measure was used by Wood et al. [21] to remove topological noise of 2-dimensional surface. This work also suffers from the dimension restriction.

1.2 Our Contributions

In this paper, we solve the three problems listed in Section 1, namely, measuring the size of homology classes, localization classes, and choosing a basis for a homology group. We define a size measure for homology classes, based on relative homology, using geodesic distance. This solves the first problem. For the second problem, we localize homology classes with cycles which are strongly related to the size measure just defined. We solve the third problem by choosing the set of linearly independent homology classes whose sizes have the minimal sum. The time complexity of our algorithm is \( O(\beta^4 n^3 \log^2 n) \), where \( n \) is the cardinality of the given simplicial complex, and \( \beta \) is the dimension of the homology group. We assume the input of our algorithm is a simplicial complex \( K \), i.e. a triangulation of the given topological space.

**Size measure and localization.** In section 3, we define the **size** of a homology class \( h \), \( S(h) \), as the radius of the smallest geodesic ball within the topological space which carries a cycle of \( h \), \( z_0 \in h \). Here a geodesic ball, \( B_r \), is the subset of the topological space consisting of points whose geodesic distance from the point \( p \) is no greater than \( r \). The intuition behind this definition will be further elaborated in Section 3.2. Any cycle of \( h \) lying within this smallest geodesic ball is a **localized cycle** of \( h \).

**Optimal homology basis.** Although there are \( 2^\beta - 1 \) nontrivial homology classes, only \( \beta \) of them are needed to construct the homology group, subject to the constraint that these classes generate the group. We choose to compute the set whose sizes have the minimal sum, which we call the **optimal homology basis**. This basis contains as few large homology classes as possible, and thus captures important features effectively.
Computing the smallest class. To compute the smallest nontrivial homology class, we find the smallest geodesic ball, $B_{\text{min}}$, which carries any nonbounding cycle of the given simplicial complex $K$. To find $B_{\text{min}}$, we visit all of the vertices of $K$ in turn. For each vertex $p$, we compute the persistent homology using the geodesic distance from $p$ as a filter. This yields the smallest geodesic ball centered on $p$ carrying any nonbounding cycle of $K$, namely, $B_p^{r(p)}$. The ball with the smallest $r(p)$ is exactly $B_{\text{min}}$. Once we find $B_{\text{min}}$, its radius, $r_{\text{min}}$, is the size of the smallest class. Any nonbounding cycle of $K$ carried by $B_{\text{min}}$ is a localized cycle of this class, and can be computed by a reduction-style algorithm.

Computing the optimal homology basis. We use matroid theory to prove that the optimal homology basis can be computed by a greedy method. We first compute the smallest homology class of the given simplicial complex $K$, as described above. We then destroy this class by sealing up one of its cycles with new simplices. Next, we compute the smallest homology class of the updated simplicial complex, $K'$, which is the second smallest class of the optimal homology basis of $K$. We then destroy this class and proceed to compute the third smallest class. The whole basis is computed in $\beta$ rounds. Theorem 4.5 establishes that this sealing technique yields the optimal homology basis. The time to compute the optimal homology basis is $O(\beta^4 n^4)$.

An improvement using finite field linear algebra. In computing the smallest geodesic ball $B_{\text{min}}$, we may avoid explicit computation of $B_p^{r(p)}$ for every $p$. Instead, Theorem 5.3 suggests we visit all of the vertices in a breadth-first fashion. For the root of the breadth-first tree, we use the explicit algorithm; for the rest of the vertices, we need only check whether a specific geodesic ball carries any nonbounding cycle of $K$. This latter task is not straightforward, as some of the nonbounding cycles in this ball may be boundaries in $K$. We use Theorem 5.5 to reduce this problem to rank computations of sparse matrices over the $\mathbb{Z}_2$ field. The time to compute the optimal homology basis with this improvement is $O(\beta^4 n^3 \log^2 n)$.

Consistency with existing results. We prove in Section 6 that our result is consistent with the low dimensional optimal result of Erickson and Whittlesey [14].

2 Preliminaries

In this section, we briefly describe the background necessary for our work, including a discussion of simplicial complexes, homology groups, persistent homology, and relative homology. Please refer to [18] for further details in algebraic topology, and [12, 22, 7, 24] for persistent homology. For simplicity, we restrict
our discussion to the combinatorial framework of simplicial homology in the \( \mathbb{Z}_2 \) field.

### 2.1 Simplicial Complex

A \( d \)-dimensional simplex or \( d \)-simplex, \( \sigma \), is the convex hull of \( d + 1 \) affinely independent vertices, which means for any of these vertices, \( v_i \), the \( d \) vectors \( v_j - v_i, j \neq i \), are linearly independent. A 0-simplex, 1-simplex, 2-simplex and 3-simplex are a vertex, edge, triangle and tetrahedron, respectively. The convex hull of a nonempty subset of vertices of \( \sigma \) is its face. A simplicial complex \( K \) is a finite set of simplices that satisfies the following two conditions.

1. Any face of a simplex in \( K \) is also in \( K \).
2. The intersection of any two simplices in \( K \) is either empty or is a face for both of them.

The dimension of a simplicial complex is the highest dimension of its simplices. If a subset \( K_0 \subset K \) is a simplicial complex, it is a subcomplex of \( K \).

### 2.2 Homology Groups

Within a given simplicial complex \( K \), a \( d \)-chain is a formal sum \( d \)-simplices in \( K \), \( c = \sum_{\sigma \in K} a_{\sigma} \sigma, a_{\sigma} \in \mathbb{Z}_2 \). All the \( d \)-chains form the group of \( d \)-chains, \( C_d(K) \). The boundary of a \( d \)-chain is the sum of the \((d - 1)\)-faces of all the \( d \)-simplices in the chain. The boundary operator \( \partial_d : C_d(K) \to C_{d-1}(K) \) is a group homomorphism.

A \( d \)-cycle is a \( d \)-chain without boundary. The set of \( d \)-cycles forms a subgroup of the chain group, which is the image of the boundary operator, \( Z_d(K) = \text{ker}(\partial_d) \). A \( d \)-boundary is the boundary of a \((d+1)\)-chain. The set of \( d \)-boundaries forms a group, which is the image of the boundary operator, \( B_d(K) = \text{img}(\partial_{d+1}) \). It is not hard to see that a \( d \)-boundary is also a \( d \)-cycle. Therefore, \( B_d(K) \) is a subgroup of \( Z_d(K) \). A \( d \)-cycle which is not a \( d \)-boundary, \( z \in Z_d(K) \setminus B_d(K) \), is a nonbounding cycle.

The \( d \)-dimensional homology group is defined as the quotient group \( H_d(K) = Z_d(K)/B_d(K) \). An element in \( H_d(K) \) is a homology class, which is a coset of \( B_d(K), [z] = z + B_d(K) \) for some \( d \)-cycle \( z \in Z_d(K) \). If \( z \) is a \( d \)-boundary, \( [z] = B_d(K) \) is the identity element of \( H_d(K) \). Otherwise, when \( z \) is a nonbounding cycle, \( [z] \) is a nontrivial homology class and \( z \) is called a representative cycle of \( [z] \). Cycles in the same homology class are homologous to each other, which means their difference is a boundary.

The dimension of the homology group, which is referred to as the Betti number, \( \beta_d = \dim(H_d(K)) = \dim(Z_d(K)) - \dim(B_d(K)) \). It can be computed with a reduction algorithm based on row and column operations of the boundary matrices [18]. Various reduction algorithms have been devised for different purposes [17, 12, 22].
The following notation will prove convenient. We say that a $d$-chain $c \in C_d(K)$ is carried by a subcomplex $K_0$ when all the $d$-simplices of $c$ belong to $K_0$, formally, $c \subseteq K_0$. We denote $\text{vert}(K)$ as the set of vertices of the simplicial complex $K$, $\text{vert}(c)$ as that of the chain $c$.

In this paper, we focus on the simplicial homology over the finite field $\mathbb{Z}_2$. In this case, a chain corresponds to a $n_d$-dimensional vector, where $n_d$ is the number of $d$-simplices in $K$. Computing the boundary of a $d$-chain corresponds to multiplying the chain vector with a boundary matrix $[b_1, ..., b_{n_d}]$, whose column vectors are boundaries of $d$-simplices in $K$. By slightly abusing the notation, we call the boundary matrix $\partial_d$.

### 2.3 Persistent Homology

Given a topological space $X$ and a filter function $f : X \rightarrow \mathbb{R}$, persistent homology studies the homology classes of the sublevel sets, $X^t = f^{-1}(-\infty, t]$. A nontrivial homology class in $X^{t_1}$ may become trivial in $X^{t_2}$, $t_1 < t_2$, (formally, when induced by the inclusion homomorphism). Persistent homology tries to capture this phenomenon by measuring the times at which a homology class is born and dies. The persistence, or life time of the class is the difference between its death and birth times. Those with longer lives tell us something about the global structure of the space $X$, as described by the filter function. Note that the essential, that is, nontrivial homology classes of the given topological space $X$ will never die.

Edelsbrunner et al. [12] devised an $O(n^3)$ algorithm to compute the persistent homology. Its input are a simplicial complex $K$ and a filter function $f$, which assigns each simplex in $K$ a real value. Simplices of $K$ are sorted in ascending order according to their filter function values. This order is actually the order in which simplices enter the sublevel set $f^{-1}(-\infty, t]$ while $t$ increases. For simplicity, in this paper we call this ordering the simplex-ordering of $K$ with regard to $f$. The output of the algorithm is the birth and death times of homology classes.

The algorithm performs column operations on an overall incidence matrix, $D$, whose rows and columns correspond to simplices in $K$. An entry $D(i, j) = 1$ if and only if the simplex $\sigma_i$ belongs to the boundary of the simplex $\sigma_j$. To some extent, $D$ is a big boundary matrix which can accommodate chains of arbitrary dimension. Columns and rows of $D$ are sorted in ascending order according to the function values of simplices. The algorithm performs the column reduction from left to right, recording $\text{low}(i)$ as the lowest nonzero entry of each column $i$. If column $i$ is reduced to a zero column, $\text{low}(i)$ does not exist. To reduce column $i$, we repeatedly find column $j$ satisfying $j < i$ and $\text{low}(j) = \text{low}(i)$; we then add column $j$ to column $i$, until column $i$ becomes a zero column or we cannot find a qualified $j$ anymore.

The reduction of $D$ can be written as a matrix multiplication,

$$R = DV,$$  \hspace{1cm} (1)
where $R$ is the reduced matrix and $V$ is an upper triangular matrix. Columns of $V$ corresponding to zero columns of $R$ whose corresponding simplices are $d$-dimensional form a basis of the cycle group $Z_d(K)$.

After the reduction, each paring, low($i$) = $j$, corresponds to a homology class whose birth time is $f(\sigma_i)$ and death time is $f(\sigma_j)$. A simplex $\sigma_i$ that is not paired, namely, neither low($i$) = $j$ nor low($j$) = $i$ for any $j$, corresponds to an essential homology class, namely, a nontrivial homology class of $K$. An essential homology class only has a birth time, namely, $f(\sigma_i)$, and it never dies. Therefore, all the nontrivial homology classes of $K$ have infinite persistences.

2.4 Relative Homology

Given a simplicial complex $K$ and a subcomplex $K_0 \subseteq K$, we may wish to study the structure of $K$ by ignoring all the chains in $K_0$. We consider two $d$-chains, $c_1$ and $c_2$ to be the same if their difference is carried by $K_0$. The objects we are interested in are then defined as these equivalence classes, which form a quotient group, $C_d(K; K_0) = C_d(K)/C_d(K_0)$. We call it the group of relative chains, whose elements (cosets), are called relative chains.

The boundary operator $\partial_d : C_d(K) \to C_{d-1}(K)$ induces a relative boundary operator, $\partial^K_0 : C_d(K, K_0) \to C_{d-1}(K, K_0)$. Analogous to the way we define $Z_d(K)$, $B_d(K)$ and $H_d(K)$ in $C_d(K)$, we define the group of relative cycles, the group of relative boundaries and the relative homology group in $C_d(K, K_0)$, denoted as $Z_d(K, K_0)$, $B_d(K, K_0)$ and $H_d(K, K_0)$, respectively. An element in $Z_d(K, K_0) \setminus B_d(K, K_0)$ is a nonbounding relative cycle.

The following notation will prove convenient. We define a homomorphism $\phi_{K_0} : C_d(K) \to C_d(K, K_0)$ mapping $d$-chains to their corresponding relative chains, $\phi_{K_0}(c) = c + C_d(K_0)$. This homomorphism induces another homomorphism, $\phi_{K_0}^* : H_d(K) \to H_d(K, K_0)$, mapping homology classes of $K$ to their corresponding relative homology classes, $\phi_{K_0}^*(h) = \phi_{K_0}(z) + B_d(K, K_0)$ for any $z \in h$.

Given a $d$-chain $c \in C_d$, its corresponding relative chain $\phi_{K_0}(c)$ is a relative cycle if and only if $\partial_d(c)$ is carried by $K_0$. Furthermore, it is a relative boundary if and only if there is a $(d+1)$-chain $c' \in C_{d+1}(K)$ such that $c - \partial_{d+1}(c')$ is carried by $K_0$.

These ideas are illustrated in Figure 5. Although $z_1$ and $z_2$ are both nonbounding cycles in $K$, $\phi_{K_0}(z_1)$ is a nonbounding relative cycle whereas $\phi_{K_0}(z_2)$ is only a relative boundary. Although chains $c_1$ and $c_2$ are not cycles in $K$, $\phi_{K_0}(c_1)$ and $\phi_{K_0}(c_2)$ are relative cycles homologous to $\phi_{K_0}(z_1)$ and $\phi_{K_0}(z_2)$, respectively.

Note that $[z_1]$ and $[z_2]$ are both nontrivial homology classes in $K$. But their correspondences in the relative homology group may not necessarily be nontrivial. We can see that $\phi_{K_0}^*([z_1])$ is a nontrivial relative homology class, whereas $\phi_{K_0}^*([z_2])$ is trivial. We say that the class $[z_2]$ is carried by $K_0$. This concept plays an important role in our definition of the size measure. Further details will be given in Section 3.2.
Figure 5: A disk with two holes, whose triangulation is $K$. Simplices of $K$ lying completely in the dotted rectangle form a subcomplex $K_0$. The 1-dimensional relative homology group $H_1(K, K_0)$ has dimension 1, although $H_1(K)$ has dimension 2. The nontrivial class $[z_2]$ is carried by $K_0$.

2.5 Rank Computations of Sparse Matrices over Finite Fields

Wiedemann [20] presented a randomized algorithm to capture the rank of a sparse matrix over finite field. His method performs a binary search for the rank. For an $m \times n$ sparse matrix $A$, the algorithm starts with $s = \min(m, n)/2$. It tests if $s > \text{rank}(A)$ or not, and then decides whether $s = s/2$ or $s = 3s/2$. For each $s$, $s \times m$ and $s \times n$ matrices $P$ and $Q$ are randomly generated for several times. If $PAQ$ is singular all the times, $s > \text{rank}(A)$ with high probability. The expected time of the algorithm is $O(n(\omega + n \log n) \log n)$, where $n$ is the maximal dimension of the matrix and $\omega$ is the total number of nonzero entries in $A$.

3 Defining the Problem

In this section, we provide a technique for ranking homology classes according to their importance. Specifically, we solve the three problems mentioned in Section 1 by providing

- a meaningful size measure for homology classes that is computable in arbitrary dimension;
- localized cycles which are consistent with the size measure of their homology classes;
- and an optimal homology basis which distinguishes large classes from small ones effectively.
3.1 The Discrete Geodesic Distance

In order to measure the size of homology classes, we need a notion of distance. As we will deal with a simplicial complex $K$, it is most natural to introduce a discrete metric, and corresponding distance functions. We define the discrete geodesic distance from a vertex $p \in \text{vert}(K)$, $f_p : \text{vert}(K) \rightarrow \mathbb{Z}$, as follows.

For any vertex $q \in \text{vert}(K)$, $f_p(q) = \text{dist}(p,q)$ is the length of the shortest path connecting $p$ and $q$, in the 1-skeleton of $K$; it is assumed that each edge length is one, though this can easily be changed. We may then extend this distance function from vertices to higher dimensional simplices naturally. For any simplex $\sigma \in K$, $f_p(\sigma)$ is the maximal function value of the vertices of $\sigma$, $f_p(\sigma) = \max_{q \in \text{vert}(\sigma)} f_p(q)$. Finally, we define a geodesic ball $B_{r_p} p$, $p \in \text{vert}(K)$, $r \geq 0$, as the subset of $K$, $B_{r_p} p = \{ \sigma \in K | f_p(\sigma) \leq r \}$. It is straightforward to show that these subsets are in fact subcomplexes.

3.2 Measuring the Size of a Homology Class

Using notions from relative homology, we proceed to define the size of a homology class as follows. Given a simplicial complex $K$, assume we are given a collection of subcomplexes $L = \{ L \subseteq K \}$. Furthermore, each of these subcomplexes is endowed with a size. In this case, we define the size of a homology class $h$ as the size of the smallest $L$ carrying $h$. Here we say a subcomplex $L$ carries $h$ if $h$ has a trivial image in the relative homology group $H_d(K,L)$, namely, $\phi^*_L(h) = B_d(K,L)$. In Figure 5, the class $[z_2]$ is carried by $K_0$, whereas $[z_1]$ is not.

Definition 3.1. The size of a class $h$, $S(h)$, is the size of the smallest measurable subcomplex carrying $h$, formally,

$$S(h) = \min_{L \in L} \text{size}(L) \quad \text{s.t.} \quad \phi^*_L(h) = B_d(K,L).$$

To facilitate computation, we prove the following theorem.

Theorem 3.2. The size of a homology class $h$, is the size of the smallest measurable subcomplex carrying one of its cycles, $z \in h$, formally,

$$S(h) = \min_{L \in L} \text{size}(L) \quad \text{s.t.} \quad \exists z \in h : z \subseteq L,$$

Proof. As we know, for any cycle $z \in h$, the relative chain $\phi_L(z)$ is a relative boundary if and only if there is a $(d+1)$-chain $c' \in C_{d+1}(K)$ such that $z - \partial_{d+1}(c')$ is carried by $L$. This means that $h$ is carried by $L$ if and only if there exists some cycle $z \in h$ carried by $L$. 

In this paper, we take $L$ to be the set of discrete geodesic balls, $L = \{ B_{p}^r | p \in \text{vert}(K), r \geq 0 \}$. The size of a geodesic ball is naturally its radius $r$. Combining the size definition and the theorem we have just proven, we define the size measure of homology classes as follows.

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Definition 3.3. The size of a homology class is the radius of the smallest geodesic ball carrying one of its cycles, formally,

$$S(h) = \min r \text{ s.t. } \exists p \in \text{vert}(K) \text{ and } z \in h : z \subseteq B^r_p.$$ 

This smallest geodesic ball is denoted as $B_{\text{min}}(h)$ for convenience, whose radius is $S(h)$.

In Figure 2 (right), the three geodesic balls centered at $p_1$, $p_2$ and $p_3$ are the smallest geodesic balls carrying nontrivial homology classes $[z_1]$, $[z_2]$ and $[z_3]$, respectively. Their radii are the size of the three classes. In Figure 6, the smallest geodesic ball carrying a nontrivial homology class is the pink one centered at $p_2$ \(^1\), not the one centered at $p_1$. Note that these geodesic ball may not look like Euclidean balls in the embedding space.

![Figure 6: On a tube, the smallest geodesic ball is centered at $p_2$, not $p_1$.](image)

3.3 A Localized Cycle

We would like to localize a homology class with a cycle which conveys its size. Define the radius of a cycle $z$ as,

$$\text{rad}(z) = \min_{p \in \text{vert}(K)} \max_{q \in \text{vert}(z)} \text{dist}(p, q),$$

which is a natural extension of the canonical definition of radius, e.g. of a Euclidean ball. We define the localized cycles of a homology class $h$ as the one with the minimal radius, namely, $z_0 = \text{argmin}_{z \in h} \text{rad}(z)$.

Based on Theorem 3.2, it is not hard to see that the size of a class $h$ is equal to the minimal radius of its cycles, namely, $S(h) = \min_{z \in h} \text{rad}(z)$, which is exactly the radius of its localized cycles. Thus, this definition of localized cycles agrees with our size measure for homology classes.

Given a homology class $h$, any of its cycles carried by $B_{\text{min}}(h)$ has the radius $S(h)$, and thus is localized. In Figure 2, $z_1$ and $z_2$ are localized cycles of $[z_1]$ and $[z_2]$ because they are carried by $B_{\text{min}}([z_1])$ and $B_{\text{min}}([z_1])$, respectively.

Remark 3.4. Another quantity which can describe the size of a cycle is the diameter

$$\text{diam}(z) = \max_{p, q \in \text{vert}(z)} \text{dist}(p, q).$$

\(^1\)This geodesic ball actually carries the shortest cycle of the class using the definition of Erickson and Whittlesey [14]. We will discuss this in Section 6.
We deliberately avoid this quantity because we conjecture computing the cycle with the minimal diameter, \(\text{argmin}_{z \in h} \text{diam}(z)\), is NP-complete. On the other hand, our definition of a localized cycle gives a 2-approximation of the minimal diameter, formally,

\[
\text{diam} \left( \text{argmin}_{z \in h} \text{rad}(z) \right) \leq 2 \text{min}_{z \in h} \text{diam}(z),
\]

which can be shown to be a tight bound.

### 3.4 The Optimal Homology Basis

There are \(2^{\beta_d} - 1\) nontrivial homology classes. However, we only need \(\beta_d\) of them to form a basis. The basis should be chosen wisely so that we can easily distinguish important homology classes from noise. See Figure 3 for an example. There are \(2^3 - 1 = 7\) nontrivial homology classes; we need three of them to form a basis. We would prefer to choose \([[z_1], [z_2], [z_3]]\) as a basis, rather than \([[z_1] + [z_2] + [z_3], [z_2] + [z_3], [z_3]]\). The former indicates that there is one big cycle in the topological space, whereas the latter gives the impression of three large classes.

In keeping with this intuition, the \textit{optimal homology basis} is defined as follows.

**Definition 3.5.** The optimal homology basis is the basis for the homology group whose elements’ size have the minimal sum, formally,

\[
\mathcal{H}_d = \text{argmin}_{\{h_1, \ldots, h_{\beta_d}\}} \sum_{i=1}^{\beta_d} S(h_i), \text{s.t. dim}\{h_1, \ldots, h_{\beta_d}\} = \beta_d.
\]

This definition guarantees that large homology classes appear as few times as possible in the optimal homology basis. In Figure 3, the optimal basis will be \([[z_1], [z_2], [z_3]]\), which has only one large class.

### 4 The Algorithm

In this section, we introduce an algorithm to measure and localize the optimal homology basis as defined in Definition 3.5. We first introduce an algorithm to measure and localize the smallest homology class, namely, \textbf{Measure-Smallest}(K), which uses the persistent homology algorithm. Based on this procedure, we provide the algorithm \textbf{Measure-All}(K), which measures and localizes the optimal homology basis. The algorithm takes \(O(\beta_d^4 n^4)\) time, where \(\beta_d\) is the Betti number and \(n\) is the cardinality of the input \(K\).
4.1 Measuring and Localizing the Smallest Homology Class

The procedure \texttt{Measure-Smallest}(K) measures and localizes the smallest nontrivial homology class, namely, the one with the smallest size,

\[ h_{\text{min}} = \arg\min_{h \in \mathcal{H}_d(K) : h \neq \mathcal{B}_d(K)} S(h). \]

The output of this procedure will be a pair \((S_{\text{min}}, z_{\text{min}})\), where \(S_{\text{min}} = S(h_{\text{min}})\) and \(z_{\text{min}}\) is a localized cycle of \(h_{\text{min}}\). According to the definitions, this pair is determined by the smallest geodesic ball carrying \(h_{\text{min}}\), namely, \(B_{\text{min}}(h_{\text{min}})\). Once this ball is computed, its radius is \(S_{\text{min}}\), and a cycle of \(h_{\text{min}}\) carried by this ball is \(z_{\text{min}}\).

We first present an algorithm to compute the smallest geodesic ball carrying \(h_{\text{min}}\), i.e. \(B_{\text{min}}(h_{\text{min}})\). Second, we introduce the technique for finding \(z_{\text{min}}\) from the computed ball. The two corresponding procedures are \texttt{Bmin} and \texttt{Localized-Cycle}. See Algorithm 1 for pseudocode of the procedure \texttt{Measure-Smallest}(K).

\begin{algorithm}[H]
\caption{Measure-Smallest(K)}
\begin{algorithmic}[1]
\State \textbf{Goal:} measuring and localizing \(h_{\text{min}}\).
\State \textbf{Input:} \(K\): the given simplicial complex.
\State \textbf{Output:} \(S_{\text{min}}, z_{\text{min}}\): the size and a localized cycle of \(h_{\text{min}}\).
\State \(r_{\text{min}}, p_{\text{min}} = \text{Bmin}(K)\)
\State \(S_{\text{min}} = r_{\text{min}}\)
\State \(z_{\text{min}} = \text{Localized-Cycle}(p_{\text{min}}, r_{\text{min}}, K)\)
\end{algorithmic}
\end{algorithm}

4.1.1 Computing \(B_{\text{min}}(h_{\text{min}})\)

It is straightforward to see that \(B_{\text{min}}(h_{\text{min}})\) is also the smallest geodesic ball carrying any nontrivial homology class of \(K\). It can be computed by computing and comparing the smallest geodesic balls centered at all vertices carrying nontrivial classes. See Algorithm 2 for the procedure.

\textbf{Theorem 4.1.} Procedure \texttt{Bmin}(K) computes \(B_{\text{min}}(h_{\text{min}})\).

\textbf{Proof.} For each vertex \(p\), we compute the smallest geodesic ball centered at \(p\) carrying any nontrivial homology class, namely, \(B_p^{r(p)}\). We apply the persistent homology algorithm to \(K\) with the filter function \(f_p\). Notice that a geodesic ball \(B_p^r\) is the sublevel set \(f_p^{-1}(-\infty, r] \subseteq K\). Nontrivial homology classes of \(K\) are essential homology classes in the persistent homology algorithm. (For clarity, in the rest of this paper, we may use “essential homology classes” and “nontrivial homology classes of \(K\)” interchangeably.) Therefore, the birth time of the first essential homology class is \(r(p)\), and the subcomplex \(f_p^{-1}(-\infty, r(p)]\) is \(B_p^{r(p)}\).

When all the \(B_p^{r(p)}\)'s are computed, we compare their radii and pick the smallest one as \(B_{\text{min}}(h_{\text{min}})\). \qed
Algorithm 2 $B_{\text{min}}(K)$

| Goal: | computing $B_{\text{min}}(h_{\text{min}})$. |
|-------|------------------------------------------|
| Input: | $K$: the given simplicial complex. |
| Output: | $p_{\text{min}}, r_{\text{min}}$, the center and radius of $B_{\text{min}}(h_{\text{min}})$. |

1: $r_{\text{min}} = +\infty$
2: for $p \in \text{vert}(K)$ do
3: apply the persistent homology algorithm to $K$ with filter function $f_p$
4: $r(p)$ = birth time of the first essential homology class
5: if $r(p) < r_{\text{min}}$ then
6: $p_{\text{min}} = p$
7: $r_{\text{min}} = r(p)$
8: end if
9: end for

Once $B_{\text{min}}(h_{\text{min}})$ is computed, its radius is the size of $h_{\text{min}}$. Any cycle of $h_{\text{min}}$ carried by $B_{\text{min}}(h_{\text{min}})$ is a localized cycle of $h_{\text{min}}$. Next, we explain how to compute one such localized cycle.

4.1.2 Computing a Localized Cycle of $h_{\text{min}}$

The procedure $\text{Localized-Cycle}(p_{\text{min}}, r_{\text{min}}, K)$ computes a localized cycle of $h_{\text{min}}$. We assume that $B_{\text{min}}(h_{\text{min}})$, the smallest geodesic ball carrying the smallest homology class, carries exactly one nontrivial homology class, (i.e. $h_{\text{min}}$ itself). Any cycle carried by this ball which is nonbounding in $K$ is a cycle of $h_{\text{min}}$, and thus is a localized cycle of $h_{\text{min}}$. Therefore, we first compute a basis for all the cycles carried by $B_{\text{min}}(h_{\text{min}})$. Second, we check elements in this basis one by one until we find one which is nonbounding in $K$. See Algorithm 3 for the procedure. Note that we use the algorithm of Wiedemann [20] for rank computation, because the related matrices are sparse.

Theorem 4.2. The procedure $\text{Localized-Cycle}(p_{\text{min}}, r_{\text{min}}, K)$ computes a localized cycle of $h_{\text{min}}$.

Proof. The cycles carried by $B_{\text{min}}(h_{\text{min}})$ form a vector space

$$Z_d(K) \cap C_d(B_{\text{min}}(h_{\text{min}})).$$

We compute its basis by column reducing the boundary matrix restricted to $B_{\text{min}}(h_{\text{min}})$. After the reduction, each zero column corresponds to an element of the basis. More specifically, we compute the basis as follows. We first construct a matrix $\partial_d'$ with columns of the boundary matrix $\partial_d$ whose corresponding simplices belong to $B_{\text{min}}(h_{\text{min}})$. Next we perform a column reduction on this

---

2 This assumption may not necessarily be true. It is possible that $B_{\text{min}}(h_{\text{min}})$ carries two or more nontrivial classes. Suppose $p_{\text{min}}$ is the center of $B_{\text{min}}(h_{\text{min}})$. Then the proof can be easily modified to deal with this case, by fixing an order on simplices with the same function value $f_{p_{\text{min}}}$, and simulating this order on $f_{p_{\text{min}}}$, i.e. treating $f_{p_{\text{min}}} (\sigma_1) < f_{p_{\text{min}}} (\sigma_2)$ if $\sigma_1$ comes before $\sigma_2$ (even though $f_{p_{\text{min}}}(\sigma_1) = f_{p_{\text{min}}}(\sigma_2)$).
Algorithm 3 Localized-Cycle($p_{\text{min}}, r_{\text{min}}, K$)

Goal: compute a localized cycle of $h_{\text{min}}$.

Input: $p_{\text{min}}, r_{\text{min}}$: the center and radius of $B_{\text{min}}(h_{\text{min}})$.

$K$: the given simplicial complex.

Output: $z_{\text{min}}$: a localized cycle of $h_{\text{min}}$.

1: $\text{rank}_0 = \text{rank}(\partial_{d+1})$
2: construct $\partial_d'$ by picking columns of $\partial_d$ whose corresponding simplices belong to $B_{\text{min}}(h_{\text{min}})$
3: reduce $\partial_d'$ and get $R$ and $V$
4: for $z = \text{columns in } V \text{ corresponding to zero columns in } R$ do
5: $\text{rank}_1 = \text{rank}([z, \partial_{d+1}])$
6: if $\text{rank}_1 \neq \text{rank}_0$ then
7: $z_{\text{min}} = z$
8: break
9: end if
10: end for

matrix from left to right, like in the persistent homology algorithm. The reduction corresponds to a matrix multiplication

\[ R = \partial_d' V, \]

where $R$ is the reduced matrix and $V$ is an upper triangular matrix. The columns in $V$ corresponding to zero columns in $R$ form the basis of cycles carried by $B_{\text{min}}(h_{\text{min}})$.

Next, we check elements in this basis one by one to find one which is nonbounding in $K$. An element of this basis, $z$, is nonbounding in $K$ if and only if it cannot be expressed as a linear combination of boundaries of $K$. Since columns of the boundary matrix $\partial_{d+1}$ generate $B_d(K)$, we just need to compute the rank of the matrix $[z, \partial_{d+1}]$ and compare it with the rank of $\partial_{d+1}$. The cycle $z$ is nonbounding in $K$ if and only if these two ranks are different.

4.2 The Optimal Homology Basis

In this section, we present the algorithm for computing the optimal homology basis defined in Definition 3.5, namely, $\mathcal{H}_d$. We first show that the optimal homology basis can be computed in a greedy manner. Second, we introduce an efficient greedy algorithm.

4.2.1 Computing $\mathcal{H}_d$ in a Greedy Manner

Recall that the optimal homology basis is

\[ \mathcal{H}_d = \arg\min_{\{h_1, \ldots, h_{\beta_d}\}} \sum_{i=1}^{\beta_d} S(h_i) \quad \text{s.t. dim}\{h_1, \ldots, h_{\beta_d}\} = \beta_d. \]
We use matroid theory [9] to show that we can compute the optimal homology basis with a greedy method. Let $H$ be the set of nontrivial $d$-dimensional homology classes (i.e. the homology group minus the trivial class). Let $L$ be the family of sets of linearly independent nontrivial homology classes. Then we have the following theorem. The same result has been mentioned in [14].

**Theorem 4.3.** The pair $(H, L)$ is a matroid when $\beta_d > 0$.

**Proof.** We show $(H, L)$ is a matroid by proving the following properties.

1. The set $H$ is finite and nonempty as $\text{card}(H) = 2^{\beta_d} - 1$.

2. For any set of linearly independent nontrivial homology classes, its subsets are also linearly independent. Therefore, elements in $L$ are independent subsets of $H$, and $L$ is hereditary.

3. For any two sets of linearly independent classes $l_1, l_2 \in L$ such that $\text{card}(l_1) < \text{card}(l_2)$, we can always find a homology class $h \in l_2 \setminus l_1$ such that $l_1 \cup \{h\}$ is still linearly independent. Otherwise, any element in $l_2$ is dependent on $l_1$. This means $\dim(l_2) \leq \dim(l_1) = \text{card}(l_1) < \text{card}(l_2)$, which contradicts the linear independence of $l_2$. Therefore, $(H, L)$ satisfies the exchange property.

We construct a weighted matroid by assigning each nontrivial homology class its size as the weight. This weight function is strictly positive because a nontrivial homology class can not be carried by a geodesic ball with radius zero. According to matroid theory, we can compute the optimal homology basis $H_d = \arg\min_{l \in L} \sum_{h \in l} S(h)$.

with a naive greedy method as follows.

1. Sort elements in $H$ into an order which is monotonically increasing according to size, namely, $\text{seq}(H) = (h_1, h_2, ..., h_{2^{\beta_d} - 1}), h_i \in H$, such that $S(h_i) \leq S(h_j) \ \forall \ i < j$.

2. Repeatedly pick the smallest class from $\text{seq}(H)$ that is linearly independent of those we have already picked, until no more elements are qualified.

3. The selected $\beta_d$ classes $\{h_{i_1}, h_{i_2}, ..., h_{i_{\beta_d}}\}$ form the optimal homology basis $H_d$. (Note that the $h$’s are ordered by size, i.e. $S(h_{i_k}) \leq S(h_{i_{k+1}})$.)

However, we cannot compute the exponentially long sequence $\text{seq}(H)$ (exponential in $\beta_d$) directly. Next, we present our greedy algorithm which is polynomial.
4.2.2 Computing $\mathcal{H}_d$ with a Sealing Technique

In this section, we introduce the algorithm for computing $\mathcal{H}_d$. Instead of computing the exponentially long sequence $\text{seq}(H)$ directly, our algorithm uses a sealing technique and takes time polynomial in $\beta_d$.

We start by measuring and localizing the smallest homology class of the given simplicial complex $K$, which is also the first class we choose for $\mathcal{H}_d$. We destroy this class by sealing up one of its cycles – i.e. the localized cycle we computed – with new simplices. Next, we measure and localize the smallest homology class of the augmented simplicial complex $K'$. This class is the second smallest homology class in $\mathcal{H}_d$. We destroy this class again and proceed for the third smallest class in $\mathcal{H}_d$. This process is repeated for $\beta_d$ rounds, yielding $\mathcal{H}_d$.

We destroy a homology class by sealing up the class’s localized cycle, which we have computed. To seal up this cycle $z$, we add (a) a new vertex $v$; (b) a $(d+1)$-simplex for each $d$-simplex of $z$, with vertex set equal to the vertex set of the $d$-simplex together with $v$; (c) all of the faces of these new simplices. In Figure 7, a 1-cycle with four edges, $z_1$, is sealed up with one new vertex, four new triangles and four new edges.

We assign the new vertices $+\infty$ geodesic distance from any vertices with which they share an edge in the original complex $K$. Whenever we run the persistent homology algorithm, all of the new simplices have $+\infty$ filter function values. Furthermore, in the procedure $\text{Measure-Smallest}(K')$, we will not consider any geodesic ball centered at these new vertices. In other words, the geodesic distance from these new vertices will never be used as a filter function. Algorithm 4 contains the pseudocode.

Algorithm 4 Measure-All($K$)

Goal: compute the optimal homology basis, $\mathcal{H}_d$.

Input: $K$: the given simplicial complex.

Output: $\mathcal{H}_d$: the optimal homology basis.

1: $K' = K$
2: $\mathcal{H}_d = \emptyset$
3: for $i = 1$ to $\beta_d$ do
4: $h = (S,z) = \text{Measure-Smallest}(K')$
5: $\mathcal{H}_d = \mathcal{H}_d \cup \{h\}$
6: seal $z$ with new simplices, augment $K'$ accordingly
7: $\forall \sigma \in K' \setminus K, p \in K, f_p(\sigma) = +\infty$
8: end for

Next, we prove that this algorithm does compute the optimal homology basis $\mathcal{H}_d$. We will prove in Theorem 4.5 that $\text{Measure-All}(K)$ produces the same result as the naive greedy method presented in the previous section. We begin by proving a lemma, based on the assumption in Footnote 2 that $h_{\text{min}}$ is the only not trivial homology class carried by $B_{\text{min}}(h_{\text{min}})$.

Lemma 4.4. Given a simplicial complex $K$, if we seal up its smallest homology class $h_{\text{min}}(K)$, any other nontrivial homology class of $K$, $h$, is still nontrivial.
in the augmented simplicial complex $K'$. In other words, any cycle of $h$ is still nonbounding in $K'$.

Proof. As we deal with two complexes $K$ and $K'$ with $K \subseteq K'$, we let $I : C_d(K) \to C_d(K')$ and $I^* : H_d(K) \to H_d(K')$ be the maps induced by inclusion. Also, for a chain $c$, let $|c|$ be the simplicial complex composed of simplices from $c$ and their faces.

We proceed by contradiction. Let $z_{\text{min}} \in h_{\text{min}}(K)$ be the localized cycle of $h_{\text{min}}(K)$ that we seal up. For any nontrivial class $h \in H_d(K)$, $h \neq h_{\text{min}}(K)$, suppose $I^*(h)$ is trivial. We will show that there exists a cycle in $h$ which is carried by $B_{\text{min}}(h_{\text{min}})$, which contradicts the fact that $h_{\text{min}}$ is the only nontrivial class carried by $B_{\text{min}}(h_{\text{min}})$.

Suppose $I^*(h)$ is trivial. For any cycle $z \in h$, its corresponding $I(z)$ is the boundary of a $(d+1)$-chain in $K'$. As $z$ is nonbounding in $K$, it must be the case that at least one of the simplices of this $(d+1)$-chain must be new. That is

$$I(z) = \partial_{d+1} \left( \sum_{\sigma \in K' \setminus K} a_{\sigma} \sigma + \sum_{\tau \in K} a_{\tau} \tau \right),$$

where at least one $a_{\sigma} \neq 0$. But there exists a cycle $z'$ which is homologous to $z$ in $K$, with $z' = z - \partial_{d+1}(\sum_{\tau \in K} a_{\tau} \tau)$, which yields, finally, that $I(z') = \partial_{d+1}(\sum_{\sigma \in K' \setminus K} a_{\sigma} \sigma)$. In other words, $I(z')$ is the boundary of a $(d+1)$-chain all of whose simplices are new. Any simplex of $|I(z')|$ is a face of the new simplices and belongs to the original complex $K$, and thus belongs to $|I(z_{\text{min}})|$. It follows that $I(z')$ is carried by the simplicial complex corresponding to $I(z_{\text{min}})$, $|I(z_{\text{min}})|$; and hence, $z'$ is carried by $|z_{\text{min}}|$. Consequently, $z'$ and $h$ are carried by $B_{\text{min}}(h_{\text{min}})$, which leads to the desired contradiction. □

Theorem 4.5. The procedure Measure-All($K$) computes $H_d$.

Proof. We prove the theorem by showing that the sealing up technique produces the same result as the naive greedy algorithm, namely, $H_d = \{h_{i_1}, h_{i_2}, ..., h_{i_{\beta_d}}\}$.

We show that for any $l \leq \beta_d$, after computing and sealing up the first $l - 1$ classes of $H_d$, i.e. $\{h_{i_1}, ..., h_{i_{l-1}}\}$, the next class we choose is exactly $h_{i_l}$. In other words, the localized cycle and size of the smallest class of the augmented simplicial complex $K^{l-1}$ are equal to that of $h_{i_l}$.

First, any class between $h_{i_{l-1}}$ and $h_{i_l}$ in seq($H$) will not be chosen. Any such class $h_j$ is linearly dependent on classes that have already been chosen, namely, $\{h_{i_1}, ..., h_{i_{l-1}}\}$. Since these classes have been sealed up, a cycle of $h_j$ is a boundary in $K^{l-1}$. Thus, $h_j$ cannot be chosen.

Second, Lemma 4.4 leads to the fact that for any class in seq($H$) that is not linearly dependent on $\{h_{i_1}, ..., h_{i_{l-1}}\}$, it is nontrivial in $K^{l-1}$.

Third, the smallest class of $K^{l-1}$, $h_{\text{min}}(K^{l-1})$, corresponds to $h_{i_l}$: any new simplex belonging to $K^{l-1} \setminus K$ will not change the computation of the geodesic balls $B_p$ with finite radius $r$, and thus will change neither the size measurement.
nor the localization. Thus, the $h_{\text{min}}(K^{l-1})$ computed by the sealing technique is identical to $h_i$, computed by the naive greedy method, in terms of the size and the localized cycle.

The algorithm is illustrated in Figure 7. The rectangle, $z_1$, and the octagon, $z_2$, are the localized cycles of the smallest and the second smallest homology classes ($S([z_1]) = 2, S([z_2]) = 4$). The nonbounding cycle $z_3 = z_1 + z_2$ corresponds to the largest nontrivial homology class $[z_3] = [z_1] + [z_2]$ ($S([z_3]) = 5$). After the first round, we choose $[z_1]$ as the smallest class in $\mathcal{H}_1$. Next, we destroy $[z_1]$ by sealing up $z_1$, which yields the augmented complex $K'$. This time, we choose $[z_2]$, giving $\mathcal{H}_1 = \{[z_1], [z_2]\}$.

Figure 7: Left: the original complex $K$. Right: the augmented complex $K'$ after sealing up the smallest class, $[z_1]$.

4.3 Complexity

We analyze the complexity of the non-refined algorithm. Denote $n$ and $m$ as the upper bounds of the total numbers of simplices of the original complex $K$ and the intermediate complex $K'$, respectively. The algorithm runs the procedure Measure-Smallest $\beta_d$ times with the input $K'$, and thus runs the procedures $B\text{min}$ and $\text{Localized-Cycle}$ $\beta_d$ times with the input $K'$.

The procedure $B\text{min}$ runs the persistent homology algorithm on the intermediate complex, $K'$, using filter function $f_p$ for each vertex of the original complex, $K$. Therefore, each time $B\text{min}$ is called, it takes $O(nm^3)$ time.

The procedure $\text{Localized-Cycle}$ runs the persistent homology algorithm once, and Wiedemann’s rank computation algorithm $O(m)$ times. The matrices used for rank computations are $[z, \partial_{d+1}]$ which have $O(m)$ nonzero entries. Therefore, each time $\text{Localized-Cycle}$ is called, it takes $O(m^3 \log^2 m)$ time.

In total the whole algorithm takes $O(\beta_d(nm^3 + m^3 \log^2 m)) = O(\beta_d nm^3)$ time. Next, we bound $m$, the size of the intermediate simplicial complex $K'$. During the algorithm, we seal up $\beta_d$ nonbounding cycles. For each sealing, the number of newly added simplices is bounded by the number of simplices of the sealed cycle. As we have shown, each cycle we seal up only contains simplices in the original complex $K$. Therefore, the number of new simplices used to seal
up each cycle is $O(n)$. The size of the intermediate simplicial complex, $K'$, is $O(\beta d n)$ throughout the whole algorithm.

Finally, substitute $\beta d n$ for $m$. We conclude that the algorithm takes $O(\beta d n m^3) = O(\beta d n (\beta d n)^3) = O(\beta^4 d n^4)$ time.

5 An Improvement Using Finite Field Linear Algebra

In this section, we present an improvement on the algorithm presented in the previous section, more specifically, an improvement on the procedure $B_{\text{min}}(K)$. The idea is based on the finite field linear algebra behind the homology.

We first observe that for neighboring vertices, $p_1$ and $p_2$, the persistence diagrams using $f_{p_1}$ and $f_{p_2}$ as filter functions are close. In Theorem 5.3, we prove that the birth times of the first essential homology classes using $f_{p_1}$ and $f_{p_2}$ differ by no more than 1. This observation suggests that for each $p$, instead of computing $B_r(p)$, we may just test whether a certain geodesic ball carries any essential homology class. Second, with some algebraic insight, we reduce the problem of testing whether a geodesic ball carries any essential homology class to the problem of comparing dimensions of two vector spaces. Furthermore, we use Theorem 5.5 to reduce the problem to rank computations of sparse matrices on the $\mathbb{Z}_2$ field, for which we have ready tools (of Wiedemann [20]).

In doing so, we improve the complexity of computing the optimal homology basis to $O(\beta^4 d n^3 \log^2)$.

Remark 5.1. This complexity is close to that of the persistent homology algorithm, whose complexity is $O(n^3)$. Given the nature of the problem, it seems likely that the persistence complexity is a lower bound. If this is the case, the current algorithm is nearly optimal.

Remark 5.2. Cohen-Steiner et al. [8] provided a linear algorithm to maintain the persistent diagram while changing the filter function. However, this algorithm is not directly applicable in our context. The reason is that it takes $O(n)$ time to update the persistent diagram for a transposition in the simplex-ordering. In our case, even for filter functions of two neighboring vertices, it may take $O(n^2)$ transpositions to transform one simplex-ordering into the other. Therefore, updating the persistent diagram while changing the filter function takes $O(n^2) \times O(n) = O(n^3)$ time. This is the same amount of time it would take to compute the persistent diagram from scratch.

In this section, we assume that $K$ has a single component; multiple components can be accommodated with a simple modification. For convenience, we use “carrying nonbounding cycles” and “carrying essential homology classes” interchangeably, because a geodesic ball carries essential homology classes of $K$ if and only if it carries nonbounding cycles of $K$.
5.1 The Stability of Persistence Leads to An Improvement

Cohen-Steiner et al. [7] proved that the change, suitably defined, of the persistence of homology classes is bounded by the changes of the filter functions. Since the filter functions of two neighboring vertices, $f_{p_1}$ and $f_{p_2}$, are close to each other, the birth times of the first nonbounding cycles in both filters are close as well. This leads to Theorem 5.3.

**Theorem 5.3.** If two vertices $p_1$ and $p_2$ are neighbors, the birth times of the first nonbounding cycles for filter functions $f_{p_1}$ and $f_{p_2}$ differ by no more than 1.

*Proof.* We first prove that the filter functions are close for two neighboring vertices $p_1$ and $p_2$, formally,

$$|f_{p_1} - f_{p_2}|_{\infty} \leq 1. \tag{2}$$

For any vertex $q$, we can connect $q$ and $p_2$ by concatenating the edge $(p_1, p_2)$ to the shortest path connecting $q$ and $p_1$. Therefore the geodesic distance between $q$ and $p_2$ is no greater than one plus the geodesic distance between $q$ and $p_1$, formally,

$$f_{p_2}(q) \leq 1 + f_{p_1}(q).$$

It is trivial to see that we can switch $p_1$ and $p_2$ in this equation. Therefore, we have

$$|f_{p_1}(q) - f_{p_2}(q)| \leq 1.$$

It is not hard to extend this equation from any vertex $q \in \text{vert}(K)$ to any simplex $\sigma \in K$. Therefore, Equation (2) is proven.

Next, we show that the birth times of the first nonbounding cycles in the two filter functions are close, formally,

$$|f_{p_1}(z') - f_{p_2}(z'')| \leq 1, \tag{3}$$

where $z'$ and $z''$ are the first nonbounding cycles in the filters $f_{p_1}$ and $f_{p_2}$, respectively. Here by slightly abusing the notation, we denote $f(z)$ as the birth time of the cycle $z$ in the filter $f$.

It is not hard to see that the birth time of any cycle $z$ is the maximum of the function values of its simplices, and thus, is the maximum of the function values of its vertices, formally,

$$f(z) = \max_{q \in \text{vert}(z)} f(q).$$

We prove Equation (3) by contradiction. Suppose

$$f_{p_1}(z') - f_{p_2}(z'') \geq 2.$$
We know that for any vertex \( q \in \text{vert}(z'') \),
\[
f_{p2}(q) \leq f_{p2}(z'') \leq f_{p1}(z') - 2.
\]
From Equation (2), we have
\[
f_{p1}(q) \leq f_{p2}(q) + 1 \leq f_{p1}(z') - 1, \forall q \in \text{vert}(z''),
\]
\[
\Rightarrow f_{p1}(z'') = \max_{q \in \text{vert}(z'')} f_{p1}(q) \leq f_{p1}(z') - 1.
\]
This contradicts the fact that \( z' \) is the first nonbounding cycle in the filter \( f_{p1} \).
Therefore, the assumption is wrong, and
\[
f_{p1}(z'') - f_{p2}(z'') \leq 1.
\]
Similarly, we can prove that
\[
f_{p2}(z'') - f_{p1}(z') \leq 1.
\]
In summary, we have proven Equation (3), and consequently, proven the theorem.

This theorem suggests a way to avoid computing \( B^r(p) \) for all \( p \in K \). Recall that \( r(p) \) is the radius of the smallest geodesic ball centered at \( p \) that carries any nonbounding cycle. Based on this theorem, we know that for any vertex \( p_i \), \( r(p_i) \geq r(p_j) - 1 \) for any neighbor \( p_j \). Since our objective is to find the minimum of the \( r(p) \)'s, we can do a breadth-first search through all the vertices with global variables \( r_{\text{min}} \) recording the smallest \( r(p) \) we have found, and \( p_{\text{min}} \) recording the corresponding center \( p \).

We start by applying the persistent homology algorithm on \( K \) with filter function \( f_{p0} \). Initialize \( r_{\text{min}} \) as the birth time of the first nonbounding cycle of \( K \), \( r(p_0) \), and \( p_{\text{min}} \) as \( p_0 \). Next, we do a breadth-first search through the rest vertices. For each vertex \( p_i, i \neq 0 \), we know there exists a neighbor \( p_j \) such that \( r(p_j) \geq r_{\text{min}} \). Therefore,
\[
r(p_i) \geq r(p_j) - 1 \geq r_{\text{min}} - 1.
\]
We only need to test whether the geodesic ball \( B^{r_{\text{min}}-1}(p) \) carries any nonbounding cycle of \( K \). If so, \( r_{\text{min}} \) is decremented by one, and \( p_{\text{min}} \) is updated to \( p \).

However, testing whether the subcomplex \( B^{r_{\text{min}}-1}(p) \) carries any nonbounding cycle of \( K \) is not as easy as computing nonbounding cycles of the subcomplex. A nonbounding cycle of \( B^{r_{\text{min}}-1}(p) \) may not be nonbounding in \( K \) as we require. For example, in Figure 8, we want to compute the smallest geodesic ball centered at \( p \) carrying any nonbounding cycle of \( K \), \( B^{r(p)}(p) \). The gray geodesic ball in the first figure does not carry any nonbounding cycle of \( K \), although it carries its own nonbounding cycles. The geodesic ball in the second figure carries nonbounding cycles of \( K \) and is the ball we want, namely, \( B^{r(p)}(p) \). Therefore, we need algebraic tools to distinguish nonbounding cycles of \( K \) from those of the subcomplex \( B^{r_{\text{min}}-1}(p) \).
5.2 Testing Whether a Subcomplex Carries Nonbounding Cycles of $K$

In this subsection, we present the procedure for testing whether a subcomplex $K_0$ carries any nonbounding cycle of $K$. A chain in $K_0$ is a cycle if and only if it is a cycle of $K$. However, solely from $K_0$, we are not able to tell whether a cycle carried by $K_0$ bounds or not in $K$. Instead, we write the set of cycles carried by $K_0$, $Z^K_0(K)$, and the set of boundaries of $K$ carried by $K_0$, $B^K_0(K)$, as sets of linear combinations with certain constraints. Consequently, we are able to test whether any cycle carried by $K_0$ is nonbounding in $K$ by comparing the dimensions of $Z^K_0(K)$ and $B^K_0(K)$. Theorem 5.5 shows that these dimensions can be computed by rank computations of sparse matrices.

5.2.1 Expressing $Z^K_0(K)$ and $B^K_0(K)$ as Sets of Linear Combinations with Certain Constrains

The set of cycles and the set of boundaries of $K$ carried by $K_0$ are

$$Z^K_0(K) = Z_d(K) \cap C_d(K_0) \quad \text{and} \quad B^K_0(K) = B_d(K) \cap C_d(K_0),$$

respectively. Since $Z_d(K)$, $B_d(K)$ and $C_d(K_0)$ are all vector spaces, $Z^K_0(K)$ and $B^K_0(K)$ are both vector spaces. Furthermore, since $B_d(K)$ is a subspace of $Z_d(K)$, $B^K_0(K)$ is a subspace of $Z^K_0(K)$. It is not hard to show that the subcomplex $K_0$ carries nonbounding cycles of $K$ if and only if the dimensions of these two vector spaces are different.

We want to express these two vector spaces as linear combinations such that we can compute their dimensions using algebraic tools. We first express the vector spaces, $B_d(K)$ and $Z_d(K)$ as sets of linear combinations. Since $B_d(K)$ is the column space of $\partial_{d+1}$, a boundary of $K$ can be written as the linear

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Figure 8: Computing $B^r_p(p)$ in a torus with tail. The ball in the second figure is what we want, although the one in the first figure has nontrivial topology.
combination of column vectors of $\partial_{d+1}$. The boundary group can be written as the set of linear combinations

$$B_d(K) = \{\partial_{d+1} \gamma \mid \gamma \in \mathbb{Z}_2^{n_{d+1}}\}.$$

The cycle group $Z_d(K)$ is the union of $B_d(K)$ and all the nonbounding cycles of $K$. Suppose we are given a basis for $H_d(K)$, $\{h_1, ..., h_{\beta_d}\}$, together with a cycle for each $h_i$, namely, $z_i \in h_i$. Elements in $h_i$ can be written as $z_i + \partial_{d+1} \gamma$. Furthermore, elements in $Z_d(K)$ can be written as linear combinations of $\{b_1, ..., b_{n_{d+1}}, z_1, ..., z_{\beta_d}\}$, where the $b_j$’s are the column vectors of $\partial_{d+1}$. We have

$$Z_d(K) = \{\hat{Z}_d \gamma \mid \gamma \in \mathbb{Z}_2^{(n_{d+1} + \beta_d)}\},$$

where $\hat{Z}_d = [\partial_{d+1}, \hat{H}_d]$ and $\hat{H}_d = [z_1, ..., z_{\beta_d}]$.

**Remark 5.4.** In our algorithm, the boundary matrix $\partial_{d+1}$ is given. We can also precompute the matrix $\hat{H}_d$ by computing an arbitrary basis of $H_d(K)$ and representative cycles of classes in this basis. More details will be provided in Section 5.3.

Since $C_d(K_0)$ is the set of chain vectors whose $i$-th entry is zero for any simplex $\sigma_i \notin K_0$, we can write $Z_d^{K_0}(K)$ and $B_d^{K_0}(K)$ as elements of $Z_d(K)$ and $B_d(K)$ whose $i$-th entries are zero. Consequently, we can write them as linear combinations with certain constraints,

$$B_d^{K_0}(K) = \{\partial_{d+1} \gamma \mid \gamma \in \mathbb{Z}_2^{n_{d+1}}, \partial_{d+1} \gamma = 0 \forall \sigma_i \notin K_0\}$$

$$Z_d^{K_0}(K) = \{\hat{Z}_d \gamma \mid \gamma \in \mathbb{Z}_2^{n_{d+1} + \beta_d}, \hat{Z}_d \gamma = 0 \forall \sigma_i \notin K_0\}$$

where $\partial_{d+1}$ and $\hat{Z}_d$ are the $i$-th rows of the matrices $\partial_{d+1}$ and $\hat{Z}_d$, respectively.

### 5.2.2 Computing Dimensions by Computing Ranks of Sparse Matrices

With the following theorem, we can compute the dimensions of these two vector spaces $Z_d^{K_0}(K)$ and $B_d^{K_0}(K)$ by matrix rank computations.

**Theorem 5.5.** For any matrix $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$, $\dim(\{A \gamma \mid A_2 \gamma = 0\}) = \text{rank}(A) - \text{rank}(A_2)$

**Proof.** For simplicity, denote $\alpha$ as $(\text{rank}(A) - \text{rank}(A_2))$. There are rank($A$) linearly independent rows in $A$, rank($A_2$) linearly independent rows in $A_2$. Therefore, there are $\alpha$ rows in $A_1$ that are linearly independent, and not linearly dependent on rows of $A_2$. Choose one such set of rows from $A_1$, $A_1' = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_\alpha \end{bmatrix}$. Since all the rows of $A$ are dependent on rows in $A_1'$ and $A_2$, for any $\gamma \in \text{nullspace}(A_2)$, $A_1' \gamma$ is determined by $A_1' \gamma$.

Proving the theorem is equivalent to showing that $A_1' \gamma$ can be an arbitrary vector in the vector space $\mathbb{Z}_2^2$. It is sufficient to show that for any row of $A_1'$, $a_i$, the following two statements are both true:

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1. There exist $\gamma_0, \gamma_1 \in \text{nullspace}(A_2)$, such that $a_i \gamma_0 = 0$ and $a_i \gamma_1 = 1$.

2. For any $\gamma \in \text{nullspace}(A_2)$, $a_i \gamma$ does not linearly depend on the products $a_j \gamma$ for the rest of the rows $a_j$ in $A_1'$.

For the first statement, choose $\gamma_0 = 0 \in \text{nullspace}(A_2)$, which satisfies $a_i \gamma_0 = 0$. Now we show that $\gamma_1$ exists by contradiction. Suppose $a_i \gamma = 0$ for all $\gamma \in \text{nullspace}(A_2)$. This implies that

$$\text{nullspace}(A_2) \subseteq \text{nullspace}\left(\begin{bmatrix} a_i \\ A_2 \end{bmatrix}\right) \Rightarrow \text{rank}\left(\begin{bmatrix} a_i \\ A_2 \end{bmatrix}\right) \leq \text{rank}(A_2).$$

This contradicts the linear independence of $a_i$ with regard to $A_2$. Therefore, $a_i \gamma$ can be either 0 or 1 for $\gamma \in \text{nullspace}(A_2)$. In fact, this statement is generally true for any row vector $a$ which is linearly independent of the rows in $A_2$.

For the second statement, again we prove by contradiction. Suppose $a_i \gamma = \sum (a_j \gamma)$ for some rows of $A_1'$, the $a_j$'s. Define a row vector $a_0 = a_i - \sum (a_j)$. We have

$$a_0 \gamma = (a_i - \sum (a_j)) \gamma = 0.$$ 

Since $a_0$ is linearly independent of $A_2$, this contradicts the first statement we have just proved. By contradiction, the second statement is true.

In conclusion, for all $\gamma \in \text{nullspace}(A_2)$, $A \gamma$ depends on $A_1' \gamma$, whose range space has dimension $\alpha$.

It is trivial to see that the order of the rows in these matrices does not interfere with the correctness of the theorem. Consequently, the matrix $A_2$ can be a certain subset of the rows of $A$, not necessarily the last few rows. Therefore, we can compute the dimensions of $B_d^{K_0}(K)$ and $Z_d^{K_0}(K)$ as

$$\dim(B_d^{K_0}(K)) = \text{rank}(\partial_{d+1}) - \text{rank}(\partial_{d+1}^{K_0}),$$

$$\dim(Z_d^{K_0}(K)) = \text{rank}(\hat{Z}_d) - \text{rank}(\hat{Z}_d^{K_0}),$$

where $\partial_{d+1}^{K_0}$ and $\hat{Z}_d^{K_0}$ are the matrices formed by rows of $\partial_{d+1}$ and $\hat{Z}_d$ whose corresponding simplices do not belong to $K_0$.

We test whether $K_0$ carries any nonbounding cycle of $K$ by testing whether these two dimensions are different. As we know, columns in $\hat{H}_d$ correspond to $\beta_d$ nonbounding cycles whose classes form a homology basis. Therefore, the ranks of $\hat{Z}_d$ and $\partial_{d+1}$ differ by $\beta_d$. $K_0$ carries nonbounding cycles of $K$ if and only if

$$\text{rank}(\hat{Z}_d^{K_0}) - \text{rank}(\partial_{d+1}^{K_0}) \neq \beta_d.$$

5.2.3 Procedure \textbf{Contain-Nonbounding-Cycle}(K, K_0, \hat{H}_d)

With all the facts in hand, we are now ready to state the algorithm for testing whether a subcomplex carries any nonbounding cycle of $K$. We use the algorithm of Wiedemann [20] for the rank computation. See Algorithm 5 for the pseudocode.
Algorithm 5 Contain-Nonbounding-Cycle($K, K_0, \hat{H}_d$)

Goal: test whether $K_0$ carries nonbounding cycles of $K$.

Input: $K$: the given simplicial complex.

$K_0$: the subcomplex.

$\hat{H}_d$: $\beta_d$ linearly independent nonbounding cycles of $K$.

Output: Boolean.

1: $\hat{Z}_d = [\partial_{d+1}, \hat{H}_d]
2: \text{compute } \partial_{d+1}^{K \setminus K_0} \text{ and } \hat{Z}_d^{K \setminus K_0} \text{ by picking rows of } \partial_{d+1} \text{ and } \hat{Z}_d \text{ whose corresponding simplices do not belong to } K_0
3: \text{if } \text{rank}(\hat{Z}_d^{K \setminus K_0}) - \text{rank}(\partial_{d+1}^{K \setminus K_0}) \neq \beta_d \text{ then}
4: \quad \text{return true}
5: \quad \text{else}
6: \quad \text{return false}
7: \text{end if}

5.3 The Improved Algorithm

Next we present the improved version of the procedure $B_{\min}(K)$. Theorem 5.3 suggests performing a breadth-first search with a global variable $r_{\min}$ and testing whether $B_{p_{\min}}^{r_{\min} - 1}$ contains nonbounding cycles of $K$ for each $p$. We use the procedure $\text{Contain-Nonbounding-Cycle}(K, K_0, \hat{H}_d)$ presented in the previous subsection for the testing. See Algorithm 6.

Algorithm 6 $B_{\min}(K)$

Goal: computing $B_{\min}(h_{\min})$, improved version.

Input: $K$: the given simplicial complex.

Output: $p_{\min}, r_{\min}$: the center and radius of $B_{\min}(h_{\min})$.

1: precompute $\hat{H}_d$
2: compute a breadth-first ordering of vert($K$), ($p_1$, ..., $p_{n_0}$).
3: apply the persistent homology algorithm on $K$ with filter function $f_{p_t}$
4: $r_{\min}$ = the birth time of the first essential homology class
5: $p_{\min} = p_1$
6: for $i = 2$ to $n_0$ do
7: \quad if $\text{Contain-Nonbounding-Cycle}(K, B^{r_{\min} - 1}_{p_i}, \hat{H}_d)$ then
8: \quad \quad $r_{\min} = r_{\min} - 1$
9: \quad \quad $p_{\min} = p_i$
10: \quad end if
11: end for

Precomputing $\hat{H}_d$. The improved algorithm requires the computation of the matrix $\hat{H}_d$, which consists of $\beta_d$ nonbounding cycles representing elements of a basis of $H_d(K)$. For this purpose, any basis is acceptable. We can precompute $\hat{H}_d$ in a similar way to the procedure $\text{Localized-Cycle}(p_{\min}, r_{\min}, K)$ (Algorithm
3). More specifically, we perform a column reduction on the boundary matrix \( \partial_d \) to compute a basis for the cycle group \( \mathbb{Z}_d(K) \). We check elements in this basis one by one until we collect \( \beta_d \) of them forming \( \hat{H}_d \). For each cycle \( z \) in this cycle basis, we check whether \( z \) is linearly independent of the \( d \)-boundaries and the nonbounding cycles we have already chosen, i.e. whether

\[
\text{rank}(\lfloor z, \partial_{d+1}, \hat{H}'_d \rfloor) \neq \text{rank}(\lfloor \partial_{d+1}, \hat{H}'_d \rfloor),
\]

where \( \hat{H}'_d \) consists of cycles we have already chosen for \( \hat{H}_d \). More details are omitted due to the space limitation.

5.4 Complexity

We analyze the complexity of the improved algorithm. Denote \( n \) and \( m \) as the cardinalities of \( K \) and \( K' \), respectively. As we know, \( m = O(\beta_d n) \). Similar to the analysis of the non-refined algorithm, the improved algorithm \texttt{Measure-All}(\( K \)) runs the procedures \texttt{Bmin} and \texttt{Localized-Cycle} \( \beta_d \) times, with \( K' \) as the input. The procedure \texttt{Localized-Cycle} takes \( O(m^3 \log^2 m) \) time.

The improved procedure \texttt{Bmin} precomputes \( \hat{H}_d \) once, applies the persistent homology algorithm on \( K' \) once, and runs the procedure \texttt{Contain-Nonbounding-Cycle} \( O(n) \) times. Precomputing \( \hat{H}_d \) runs the rank computation \( O(m) \) times on matrices with \( O(m + \beta_d) = O(m) \) columns and \( O(\beta_dm) \) nonzero entries, and thus takes \( O(m^3 \log m(\beta_d + \log m)) \) time. The persistent homology algorithm takes \( O(m^3) \) time. The procedure \texttt{Contain-Nonbounding-Cycle} performs rank computations on matrices with \( O(m + \beta_d) = O(m) \) columns and \( O(\beta_dm) \) nonzero entries, and thus takes \( O(m^2 \log m(\beta_d + \log m)) \) time. Therefore, the procedure \texttt{Bmin} takes \( O(m^3 \log m(\beta_d + \log m) + m^3 + nm^2 \log m(\beta_d + \log m)) = O(m^3 \log m(\beta_d + \log m)) \) time.

Therefore, the whole improved algorithm takes \( O(\beta_d m^3 \log m(\beta_d + \log m)) = O(\beta_d^4 n^3 \log^2 n) \) time.

6 Consistency with Existing Works in Low Dimension

Erickson and Whittlesey [14] measured a 1-dimensional homology class using the length of its shortest cycle. They computed the optimal homology basis by finding the set of nonbounding and linearly independent cycles whose lengths have the minimal sum. Their algorithm works for 1-dimensional homology classes in 2-manifolds.

We prove in Theorem 6.2 that our measure, \( S(h) \), is quite close to their measure for 1-dimensional homology classes. For ease of exposition, we first prove in Lemma 6.1 that by slightly modifying our algorithm of computing the localized cycle, we can localize the smallest 1-dimensional homology class, \( h_{\text{min}} \), with a representative cycle whose length is no more than \( 2S(h) + 1 \). We start with the modification.
Recall that in the procedure $\text{Localized-Cycle}(p_{\min}, r_{\min}, K)$, a localized cycle of $h_{\min}$ is computed, given the smallest geodesic ball carrying $h_{\min}$, $B_{\min}(h_{\min})$, whose center and radius are $p_{\min}$ and $r_{\min}$, respectively. More specifically, we compute a basis of the cycles carried by $B_{\min}(h_{\min})$ by performing a column reduction on $\partial_d'$, a submatrix of the boundary matrix, $\partial_d$. The submatrix is constructed by picking columns of $\partial_d$ whose corresponding simplices belong to $B_{\min}(h_{\min})$.

**A Modification** When the relevant dimension $d = 1$, we modify our algorithm as follows. Before performing a column reduction on the submatrix $\partial_1'$, we sort its rows and columns in ascending order according to the function value $f_{p_{\min}}$ of their corresponding 1-simplices, that is, edges. For edges with the same function value, we sort them in ascending order according to the minimal function value of their vertices. After the sorting, we perform a column reduction on $\partial_1'$ to compute a basis for the cycles carried by $B_{\min}(h_{\min})$. The rest is the same as the original algorithm.

Next, we prove that this modification will produce a localized cycle of $h_{\min}$ whose length is no greater than $2S(h_{\min}) + 1$.

**Lemma 6.1.** The modified algorithm localizes the smallest 1-dimensional homology class, $h_{\min}$, with a 1-cycle with no more than $2S(h_{\min}) + 1$ edges.

**Proof.** For simplicity, we prove the case when $K$ has only one connected component. The general case follows simply.

Because of the properties of the geodesic distance, we observe the following two facts.

1. For any edge, the function values of its vertices differ in no more than 1.

2. For each vertex, $q \neq p_{\min}$, there exists at least one edge with vertices $q$ and $q'$, such that

$$f_{p_{\min}}(q') = f_{p_{\min}}(q) - 1.$$ 

By **lower edges**, we denote edges whose two vertices have different function values.

These facts imply that in the modified algorithm, a column is reduced to a nonzero column only if its corresponding edge is a lower edge. To see this, notice that in the simplex-ordering corresponding to the sorted $\partial_d'$, for any vertex $q \neq p_{\min}$, among all the edges adjacent to it, lower edges must appear first. During the reduction, $q$ must be paired with one of its lower edges. Since $p_{\min}$ corresponds to the 0-dimensional essential homology class, it is not paired by any edge. Therefore, any edge paired with a vertex is a lower edge. Any column which is reduced to a nonzero column corresponds to a lower edge.

The localized cycle we compute, $z_{\min}$, is one of the columns of $V$, corresponding to zero columns in $R$, where $R = \partial_1' V$. Let it be the $i$-th column,
corresponding to $\sigma_i$. It is straightforward to see that only columns corresponding to lower edges are used to reduce column $i$ of $\partial_1'$. Consequently, in the computed localized cycle, any edge beside $\sigma_i$ is a lower edge, and thus has two vertices whose function values differ in one. Since edge $i$ has the function value $S(h_{\text{min}})$, $z_{\text{min}}$ has no more than $2S(h_{\text{min}}) + 1$ edges.

For example, in Figure 9, $B_{\text{min}}(h_{\text{min}})$ is centered at $p_1$ with radius two. Using the modified algorithm, edge $p_3p_4$ corresponds to the nonbounding cycle. Its column is reduced using edges $p_1p_2$, $p_2p_3$, $p_4p_5$ and $p_1p_5$, which are all lower edges. The computed localized cycle has length $5 = 2S(h_{\text{min}}) + 1$.

![Figure 9: Edge $p_3p_4$ corresponds to the localized cycle whose length is $2S(h_{\text{min}}) + 1$.](image)

Based on this Lemma, we prove that our result is close to the result of [14], in which size of a 1-dimensional homology class is the length or its shortest representative cycle, namely,

$$S_E(h) = \min_{z \in h} \text{length}(z), h \in H_1(K).$$

**Theorem 6.2.** For a 1-dimensional homology class $h$,

$$2S(h) \leq S_E(h) \leq 2S(h) + 1.$$

**Proof.** Lemma 6.1 shows that there exists a representative cycle of $h$ with no more than $2S(h) + 1$ edges. Therefore, the shortest representative cycle of $h$ has no more than $2S(h) + 1$ edges. We have

$$S_E(h) \leq 2S(h) + 1.$$

Next, we show that

$$2S(h) \leq S_E(h). \quad (4)$$

Pick the shortest representative cycle $z_0$ with length $S_E(h)$. Choose any vertex $p \in z_0$ as the center to build a smallest geodesic ball carrying $z_0$. The radius of this ball is $S_E(h)/2$ when $S_E(h)$ is even, and $(S_E(h) - 1)/2$ when $S_E(h)$ is odd. Since $S(h)$ is no greater than this radius, Equation (4) is proved. \qed
This theorem shows that our measure tightly bounds the one by Erickson and Whittlesey. Furthermore, we know the localized cycles computed are almost the shortest ones.

**Corollary 6.3.** The localized cycle of $h$ computed by the modified algorithm has at most one more edge than the shortest representative cycle of $h$.

**Remark 6.4.** In fact, the algorithm can be further modified to generate exactly the same result as the one by Erickson and Whittlesey. We omit this because it involves more technical details and does not provide any new insights.

**Remark 6.5.** Our modified algorithm can compute the shortest representative cycle for 1-dimensional homology classes no matter what dimension $K$ is, whereas most of the existing works in low dimension require $K$ to be dimension two.

## 7 Conclusion

In this paper, we have defined a size measure of homology classes, found cycles localizing these classes, as well as computed an optimal homology basis for the homology group. An $O(\beta^4 n^4)$ brute force algorithm has been presented, which measures and localizes the optimal homology basis by applying the persistent homology algorithm on the simplicial complex $\beta n$ times. Aided by Theorem 5.3 and 5.5, we have improved the algorithm to $O(\beta^4 n^3 \log^2 n)$. Finally, we have shown that our result is similar to the existing optimal result in low dimensions.

**Future directions.** We intend to extend our work in two directions.

1. In this paper, a localized cycle $z_0 \in h$ satisfies the condition

$$\text{rad}(z_0) = \min_{p \in K} \max_{q \in \text{vert}(z_0)} \text{dist}(p, q) = \min_{z \in h} \text{rad}(z).$$

Can we localize $h$ with a representative cycle using other size measures? Examples of such measures are:

- $\text{card}(z_0) = \min_{z \in h} \text{card}(z)$,
- $\text{diam}(z_0) = \max_{p, q \in \text{vert}(z_0)} \text{dist}(p, q) = \min_{z \in h} \text{diam}(z)$, and
- $\text{radZ}(z_0) = \min_{p \in \text{vert}(z_0)} \max_{q \in \text{vert}(z_0)} \text{dist}_{z_0}(p, q) = \min_{z \in h} \text{radZ}(z)$,

where $\text{card}(z)$ is number of simplices in the cycle $z$ and $\text{dist}_{z_0}(q, p)$ is the geodesic distance between $p$ and $q$ within the representative cycle $z_0$. We conjecture computing $z_0$ satisfying the first two constraints are NP-complete.

2. Can we extend the results if we replace the discrete geodesic distance with continuous metric defined on the underlying space of the simplicial complex?
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