Mathematical Modelling of Drying Kinetics in the Natural Solar Drying of Mango

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Abstract. There are several drying kinetic models. Some of them are theoretical, semi-theoretical or empirical. Among them, several are based on the Fick’s second law. Those models generally use some constant coefficients for the whole drying process. The present model uses the water speed of transfer. It is correlated to the drying rate, to the drying time and to the modelling parameters. Finally, the drying rate and the moisture content are expressed as functions of those parameters and the drying time. The modelling parameters, which are time dependent, are calculated from the experimental drying data. In a previous work, our mathematical model was applied in the modelling of natural solar drying of plantain banana. In the present work, the model is applied in the modelling of natural solar drying of mango. The drying rate is estimated with a high accuracy, as the relative errors between experiment and theory are in the range 0.03 % - 7.7 %. By way of comparison, the modelling performed using the characteristic drying curve yields relative errors varying between 11.37 % and 84.47 %. In the same way, the moisture content is estimated by our model with a high accuracy, as the relative errors lie in the range 0.003 % - 1.1 %. The modelling of the moisture content carried out using the Henderson and Pabis model yields an estimate with relative errors varying between 1.8 % and 5.26 %.

1. Introduction
It is well known that in most developing countries, the agricultural losses are high after harvesting. In Côte d’Ivoire, the percentage of waste for mango is higher than 30%. Solar drying is a traditional technique used by farmers to preserve their productions, in order to reduce the wastes. Therefore it is worth to have a better knowledge about this technique, in order to have a good understanding of the products behaviour during the drying process.

Drying is a physical phenomenon. It involves liquid, vapor and heat transfers. The study of the mass transfer leads to the introduction of the moisture content X. The variation of X with time defines the drying rate which is expressed by the following relationship:

\[- \frac{dX}{dt} = - \frac{1}{m_s} \frac{dm}{dt}\] (1)

In equation (1), \(m\) is the mass of the product. \(m_0 = m - m_e\) is the dry mass; \(m_e\) is la mass of water in the product. The drying rate, which is a function of time, is expressed as:
\[ \frac{dX}{dt} = f(t) \]  

This curve generally presents, during the drying process, three distinct phases, in relation with the product moisture content [1].

The first phase is the transient phase which is generally very short. This phase corresponds to the warming up of the product. It is not always observed.

The second phase is the constant drying rate phase. The free water of the product is evaporated during this period. It is assuming that the constancy of the drying rate means that there is an equilibrium characterised by the fact that the energy consumption to remove water from the surface of the product by evaporation is equal to the heat transfer to the surface by the ambient air. As a result, the following relationship may be written [2]

\[ \frac{dX}{dt} = \frac{hc_s p}{m_e L v} (Ta - Tp) = -k \]  \hspace{1cm} (3)

In equation (3), k is a constant. hc is the mass transfer coefficient. The constant drying rate phase goes on until the critical moisture content \( X_{cr} \) is reached.

In the third phase, there is a continuous decrease of the drying rate. It is characterised by a slowing down of the water migration towards the surface of the product.

Most of the models are elaborated to study the drying kinetics in the third phase. Those classical diffusion models use the moisture ratio \( X_r \) which is expressed as:

\[ X_r = \frac{X - X_e}{X_{cr} - X_e} \]  \hspace{1cm} (4)

\( X_e \) is the equilibrium moisture content. The Fick’s diffusion law is used in several models. It is expressed as [3]:

\[ \frac{dX_r}{dt} = D \frac{d^2X_r}{dr^2} \]  \hspace{1cm} (5)

In equation (5), r (m) is the diffusion path, t is the time. The diffusion coefficient \( D \) (m²/s) (or effective diffusivity) is constant. The analytical solution of equation (5), given by Cranck, is expressed as [4]:

\[ X_r = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} \exp\left[-(2n+1)^2 \frac{\pi^2 D t}{4 L^2}\right] \]  \hspace{1cm} (6)

L is the half thickness of the drying product. For a drying time large enough, the other terms of the series are considered to be negligible as compared with the first term for which \( n = 0 \) [5]. Equation (6) is therefore expressed as:

\[ X_r = \frac{8}{\pi^2} \exp\left(-\frac{\pi^2 D t}{4 L^2}\right) \]  \hspace{1cm} (7)

The diffusion coefficient \( D \) is considered as constant during all the drying process. The drying kinetics of several foodstuffs have been successfully modeled by using the Fick’s diffusion model.

Several semi-empirical models have also been used to interpret the experimental results, such as the Newton model, the Page model, the Henderson and Pabis model, the logarithmic model, the Peleg model and the Weibull model [6–7].

For instance, the Henderson and Pabis model, is expressed as:

\[ X_r = a \exp(-Kt) \]  \hspace{1cm} (8)

The parameters of the model which are \( a \) and \( K \) are considered to be constant. This constancy of the modelling parameters is stated in most of the semi-empirical and empirical models.
In some cases, the modeling of drying kinetics is performed by using a dimensionless function \( F(X_r) \) defined by the following relationship [8]:

\[
\frac{dx}{dt} = \left( \frac{dx}{dt} \right)_0 F(X_r)
\]

The variation of \( F(X_r) \) defines what is called the characteristic drying curve. Following the work of Langrish et al cited by Haoua [8], \( F(X_r) \) is a polynomial function expressed as

\[
F(X_r) = \sum A_n X_r^n
\]

All those models use constant coefficients for the whole drying process. As a matter of fact, the modifications undergone by a product during the drying process may have an influence on the modeling parameters. On that basis, a new theoretical model was elaborated, which uses parameters that vary during the drying process.

2. Some few recalls about the theory of the model
In a previous work, the model was applied in the modeling of drying kinetics of natural solar drying of plantain banana [9]. In that previous work, the whole theoretical basis of the model was established. A summary of the model is given below.

The models is elaborated to study the decreasing drying rate phase. At the beginning of this phase, the product contains a solid phase and a liquid phase. The water flow in the solid matrix is considered. The mass of the solid matrix is \( m_s \). Over a time interval \( \Delta t \), the water flow speed \( V_e \) is assumed to be constant. The mass flux \( dm/dt \) over this time interval is written as:

\[
\dot{m} = \frac{dm}{dt} = \frac{\Delta m}{\Delta t} = \rho \cdot V_e \cdot Sp
\]

\( Sp \) is the cross-sectional area of the product. \( \rho \) is the density of liquid water. A one-dimensional flow is considered in a \( y \) direction. The speed of transfer, \( V_e \), which is also called \( St \), is expressed as:

\[
St = V_e = \Delta y/\Delta t
\]

From equations (11) and (12) one gets:

\[
\Delta y = \frac{\Delta m}{\rho \cdot Sp}
\]

\[
St = \frac{1}{\rho \cdot Sp} \cdot \frac{\Delta m}{\Delta t}
\]

The variation of the moisture content is \( dX = dm/m_s \). Then from equations (12), (13) and (14) one gets the following relationship for the speed of transfer:

\[
St = C \frac{dx}{dt}
\]

\( C \) is expressed as:

\[
C = \frac{m_s}{\rho \cdot Sp}
\]

\( C \) is constant if the influence of shrinkage is neglected and if the water density is constant.

A differential analysis of the speed of transfer was also performed in order to find a second relationship which expresses \( St \). The differential of \( St \) is expressed as:

\[
dSt = \frac{\partial St}{\partial t} dt + \frac{\partial St}{\partial y} dy
\]
Since $X = X(t)$, one can write:

$$\frac{\partial S_t}{\partial t} = \frac{\partial S_t}{\partial X} \cdot \frac{\partial X}{\partial t}$$

(18)

Then one gets the following differential equation:

$$\frac{dS_t}{dt} - \lambda S_t - \beta \frac{dX}{dt} = 0$$

(19)

In equation (19), $\lambda$ and $\beta$ are defined as:

$$\beta = \frac{\partial S_t}{\partial X}; \quad \lambda = \frac{\partial S_t}{\partial y}$$

(20)

$S_t$ is calculated for a time interval $\Delta t$. For that interval, it is assumed that $dX/dt$ is constant. But $dX/dt$ varies from one interval to the other. The solution of the differential equation (19) is:

$$S_t = K \left[ \frac{dX}{dt} - \left( \frac{dX}{dt} \right)_{i-1} \exp(\lambda \Delta t) \right]$$

(21)

The initial condition is: $S_t = 0$ for $t = 0$

In relationship (21), the parameter $K$ is defined as: $K = -\beta/\lambda$.

An interval $\Delta t$ is the time after which a $\Delta m$ measurement of the product sample mass is carried out during the drying experiment. That corresponds to a step. Therefore from step i-1 to step i, $S_t$ is written as:

$$S_t = K \left[ \frac{dX}{dt} \right]_i - \left( \frac{dX}{dt} \right)_{i-1} \exp(\lambda \Delta t)$$

(22)

The modelling parameters $K$ and $\lambda$ are linked to the particular interval $\Delta t$ considered.

The speed of transfer $S_t$ is expressed by relationships (15) and (22). By equaling them, one gets the following relationship:

$$\left( \frac{dX}{dt} \right)_i = \frac{K}{K-\beta} \left( \frac{dX}{dt} \right)_{i-1} \exp(\lambda \Delta t)$$

(23)

By introducing the dimensionless parameter $T = K/C$, equation (23) becomes:

$$\left( \frac{dX}{dt} \right)_i = \frac{T}{T-1} \left( \frac{dX}{dt} \right)_{i-1} \exp(\lambda \Delta t)$$

(24)

equations (23) and (24) for which $K > C$ and $T > 1$ allow the calculation of the drying rate from step i-1 to step i, when the parameters $K$, $C$ and $\lambda$ are known.

$dX/dt$ is assimilated to $\Delta X/\Delta t$. Moreover, it is stated that $\Delta X = X_{i-1} - X_i$. Then equation (24) allows the calculation of the moisture content at step i, $X_i$, when the moisture content at step i-1 is known and for a variation $\Delta t$ of the drying time. One gets:

$$X_i(\Delta t) = \left[ -\frac{T}{T-1} \left( \frac{dX}{dt} \right)_{i-1} \Delta t \right] \exp(\lambda \Delta t) + X_{i-1}$$

(25)

Hence from equations (24) and (25), the evolution with time of the drying rate and the moisture content can be evaluated, when the modelling parameters $K$, $C$ and $\lambda$ are known.

3. Materials and methods

The natural solar drying of the mango samples was carried out by using an experimental hurdle-dryer. It has a wooden frame. The 1 m² hurdle is a wire netting which is laid on four 70 cm supports, as shown in figure 1 (a)
Figure 1. Some elements of the experimental device

The temperatures were measured by using Norme DI 7360 - type platinum resistance thermometers (100 Ω at 0°C, precision ± 0.1°C). The mass measurements were made by using a Sartorius balance (precision ± 0.1 g). The global solar radiations were recorded by using an Eppley PsP 47527-F3 – type pyranometer (precision ±10 W/m²), which is shown in figure 1 (b).

The product sample of mango used for the experimentations is a die whose initial mass is 4.8 g.

4. Results and discussions

The model is applied here to the study of natural solar drying of the mango sample. The experimental study of the solar drying of this product was already made [10]. The sample product is characterised by:

\[ m_i = 4.8 \text{ g; } S_p = 9.10^{-4} \text{ m}^2; \ m_s = 0.766 \text{ g; } C = 0.85539.10^{-3} \text{ m; } X_i = 5 \text{ g water/g dry basis; } X_f = 0.375 \text{ g water/g dry basis.} \]

4.1. Determination of the model parameters

Equations (14) allows the calculation of \( St \) for each step, since the mass variation \( \Delta m \) has been determined for the different time intervals \( \Delta t \).

4.1.1 Assessment of the water transfer speed \( St \). Only one example of \( St \) calculation is given here. Table 1 displays the remaining results. All the \( S_i \) values are calculated from the experimental data.
The calculations are made for each step. A step is characterised by a time interval \( \Delta t \) after which a new measurement of the product sample mass is carried out. A mass variation \( \Delta m \) is observed during each step.

For step 1, we have: \( \Delta t_1 = 0.666 \text{ hr} \); \( \Delta m_1 = 0.50 \text{ g} \).

From equation (13), one gets: \( \Delta y_1 = 0.5583473.10^{-3} \text{ m} \). Then for step 1, one gets: \( S_1^t = \Delta y_1/ \Delta t_1 = 0.8384.10^{-3} \text{ m.hr}^{-1} \).

For step 2, we have: \( \Delta m_2 = 0.40 \text{ g} \); \( \Delta t_2 = 0.666 \text{ hr} \); \( \Delta y_2 = 0.4466778.10^{-3} \text{ m} \); \( S_2^t = 0.670687. 10^{-3} \text{ m.hr}^{-1} \).

It should be noted that in table 1 steps 2 and 3 have the same value of \( S_2^t \) which is 0.670687.10^{-3} m.hr ^{-1}. They are characterised by the same drying rate \( \frac{dX}{dt} = 0.75075075 \text{ gg}^{-1}\text{hr}^{-1} \). Those two steps 2 and 3 lie in the constant drying rate phase.

*Table 1. Some drying characteristics of mango*

| Step | \( \Delta t \) (hr) | \( \Delta y \) (m) | \( S_t \) (m.hr\(^{-1}\)) | \( \frac{dX}{dt} \) (gg/hr\(^{-1}\)) | \( \Delta m \) (g) |
|------|------------------|------------------|------------------|------------------|------------------|
| 1    | 0.666            | 0.55835.10^{-3}  | 0.8384.10^{-3}   | 0.9384           | 0.50             |
| 2    | 0.666            | 0.44668.10^{-3}  | 0.67069.10^{-3}  | 0.7507           | 0.40             |
| 3    | 0.666            | 0.44668.10^{-3}  | 0.67069.10^{-3}  | 0.7507           | 0.40             |
| 4    | 0.666            | 0.33501.10^{-3}  | 0.50301.10^{-3}  | 0.5631           | 0.30             |
| 5    | 0.666            | 0.37967.10^{-3}  | 0.57008.10^{-3}  | 0.6381           | 0.34             |
| 6    | 1.666            | 0.29034.10^{-3}  | 0.17427.10^{-3}  | 0.1951           | 0.26             |
| 7    | 0.333            | 0.11167.10^{-3}  | 0.33534.10^{-3}  | 0.3754           | 0.10             |

### 4.1.2. Assessment of the parameter \( \lambda \)

According to the theoretical definition, \( \lambda \) is given by the following relationship: \( \lambda = \frac{\Delta S_t}{\Delta y} \)

Its value is calculated from one step to the following.

Let steps 1 and 2 be considered. Their characteristics are the followings, as shown in table 1:

- for step 1: \( S_1^t = 0.8384.10^{-3} \text{ m.hr}^{-1} \); \( \Delta y_1 = 0.5583473.10^{-3} \text{ m} \)
- for step 2: \( S_2^t = 0.670687. 10^{-3} \text{ m.hr}^{-1} \); \( \Delta y_2 = 0.4466778.10^{-3} \text{ m} \)

\( S_t^i \) denotes the value of \( S_t \) for the step i. For a step variation i-j from step i to step j, \( S_t \) goes through a variation \( \Delta S_t \), while \( \Delta y \) undergoes a variation \( \Delta y_j \).

For instance, let steps 1 and 2 be considered. \( S_t \) varies from \( S_t^1 \) to \( S_t^2 \), for a variation \( \Delta y = \Delta y_2 \). As a consequence, for a step gap 1-2, the parameter \( \lambda \) is given by

\[
\lambda = \frac{\Delta S_t}{\Delta y} = \frac{S_t^2 - S_t^1}{\Delta y_2} = -0.37548 \text{ hr}^{-1}
\]

Table 2 displays the values of \( \lambda \), calculated for the different step gaps.

It should be noted that for the step gaps 2-3, step 2 and step 3 lie in the constant drying rate phase, so that \( \Delta S_t = 0 \) and \( \lambda = 0 \).

### 4.1.3. Assessment of the parameter \( K \)

This parameter is calculated from equation (22). If \( S_t^i \) is the value of \( S_t \) at step i, \( K \) is calculated relatively to the step gaps i-1 - i when \( S_t^i \) and \( \lambda \) have been calculated for this step gap. \( K \) is expressed as:

\[
K = \frac{S_t^i}{\left(\frac{dX}{dt}\right)_i - \left(\frac{dX}{dt}\right)_{i-1} \exp(\lambda t)}
\]
The values of $K$ are displayed in table 2, together with the corresponding values of $T$ and $\lambda$. $K$ is not calculated for the step gap 2-3 because step 2 and step 3 lie in the constant drying rate phase.

All the parameters are time dependent.

**Table 2. Some values of the modelling parameters**

| Step gap | $t$ (hr) | $\lambda$ (hr$^{-1}$) | $K$ (m)     | $T$      |
|----------|----------|------------------------|-------------|----------|
| 1-2      | 1.333    | -0.37548               | 33.577.10$^{-3}$ | 39.2534  |
| 2-3      | 2.000    | 0.00000                | -           | -        |
| 3-4      | 2.666    | -0.5005                | 20.019.10$^{-3}$ | 23.4038  |
| 4-5      | 3.333    | 0.176647               | 119.306.10$^{-3}$ | 139.476  |
| 5-6      | 5.000    | -1.36326               | 1.34856.10$^{-3}$ | 1.57655  |
| 6-7      | 5.333    | 1.442379               | 5.58754.10$^{-3}$ | 6.53215  |

4.2. Assessment of the drying rates

This assessment is made in the decreasing drying rate phase which begins with the step 4, by using equation (24). The drying rate at instant $t = 0$ is therefore the one of step 3. The results are given in table 3, which displays the experimental and theoretical values of the drying rates, as well as the relative errors.

**Table 3. Experimental and theoretical values of the drying rate**

| $t$ (hrs) | $(dX/dt)_{\text{exp}}$ (gg$^{-1}$hr$^{-1}$) | $(dX/dt)_{\text{th}}$ (gg$^{-1}$hr$^{-1}$) | relative error (\%) |
|-----------|------------------------------------------|------------------------------------------|---------------------|
| 0.000     | 0.750750                                 | -                                        | -                   |
| 0.666     | 0.563063                                 | 0.56194                                 | 0.2                 |
| 1.333     | 0.638138                                 | 0.63793                                 | 0.03                |
| 3.000     | 0.195078                                 | 0.18006                                 | 7.7                 |
| 3.333     | 0.375375                                 | 0.372363                                | 0.8                 |

By way of comparison, the modelling of the drying rates was performed using the characteristic drying curve. The moisture ratio $X_r$ was expressed as:

$$X_r = \frac{x-x_f}{x_i-x_f}$$ (27)

In equation (27), $X_i$ is the final moisture content of the sample product.

The experimental values calculations of $F(X_r)$ were carried out by using equation (9). $(dX/dt)_0$ is the initial drying rate, i.e. the drying rate of the step 1. The following characteristic drying curve was obtained, in the decreasing drying rate phase:

$$F(X_r) = -3.0734X_r^3 + 4.4079X_r^2 - 0.3344X_r$$ (28)

This polynomial function was obtained with a determination coefficient $r^2 = 0.891$. This characteristic drying curve is shown in figure 2.
Figure 2. Plotting of $F(X_r)$ against the moisture ratio $X_r$.

Then the characteristic drying curve was used to estimate the drying rates, by using the relation (9). Figure 3 shows the variation with time of the experimental and theoretical drying rates by using the characteristic drying curve.

![Figure 3](image)

Figure 3. Variation with time of the experimental and theoretical drying rates by using the characteristic drying curve.

The deviations are important, with relative errors varying from 11.37% to 84.4%.

The current theoretical model yields obviously better results than those obtained by using the characteristic drying curve.
4.3. Assessment of the moisture content

The prediction of the moisture contents was performed, in the decreasing drying rate phase, by using equation (25). Table 4 displays the results of the prediction by the current model, as well as the experimental values of the moisture content. The relative errors between theory and experiment vary from 0.005% to 1%. Hence the current theoretical model yields an excellent prediction of the moisture content.

Table 4. Experimental and theoretical moisture contents in the decreasing drying rate phase

| t (hrs) | Xexp (g water/g) | Xth (g water/g) | relative Error (%) |
|---------|-----------------|-----------------|--------------------|
| 0.000   | 3.375           | -               | -                  |
| 0.666   | 3.000           | 3.00074         | 0.025              |
| 1.333   | 2.575           | 2.57514         | 0.005              |
| 3.000   | 2.250           | 2.27502         | 1.1                |
| 3.333   | 2.125           | 2.12600         | 0.047              |

By way of comparison, the Henderson and Pabis model was applied in the natural solar drying of mango. The following relationship was obtained ($r^2 = 0.958$):

$$X_r = 0.6005\exp(-0.137t)$$

(29)

The moisture ratio $X_r$ is expressed by equation (27). By applying this Henderson and Pabis model, the relative errors between experiment and theory range from 1.79% to 5.26%. Hence the Henderson and Pabis model yields good prediction of the moisture ratio. But the prediction by the current theoretical model yields better results.

5. Conclusion

In this work, the modelling of the water transfer speed $St$ was first performed. Then the theoretical relations of the mathematical model were established for the assessment of the drying rate and the moisture content. The results proved that the best fit of the drying rate and the moisture content are obtained by applying the current theoretical model, if compared with other existing models. The study showed also that the modelling parameters are time dependant.

Appendix

Nomenclature

| Symbol | Description |
|--------|-------------|
| Aw     | Water activity |
| D      | Diffusion coefficient (m$^2$s$^{-1}$) |
| hc     | Masse transfer coefficient (JK$^{-1}$m$^2$s$^{-1}$) |
| Lv     | Latent heat of vaporisation (JKg$^{-1}$) |
| m      | Mass (Kg) |
| $\dot{m}$ | Mass flux (Kgs$^{-1}$) |
| S      | Surfaces (m$^2$) |
| St     | Water transfer speed (ms$^{-1}$) |
| t      | Drying time (hr) |
| Ve     | Liquid phase speed (ms$^{-1}$) |
| X      | Moisture content (g water/g dry basis) |
Greek letters

\( \lambda \)  Characteristic diffusion frequency, s\(^{-1}\)
\( \rho \)  Densité (Kgm\(^{-3}\))

Subscripts

a       air
cr      critical
e       equilibrium
f        final
i        initial
r        ratio
s       solid matrix

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