QUENCHED NONEQUILIBRIUM CENTRAL LIMIT THEOREM FOR A TAGGED PARTICLE IN THE EXCLUSION PROCESS WITH BOND DISORDER

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Abstract. For a sequence of i.i.d. random variables \( \{\xi_x : x \in \mathbb{Z}\} \) bounded above and below by strictly positive finite constants, consider the nearest-neighbor one-dimensional simple exclusion process in which a particle at \( x \) (resp. \( x+1 \)) jumps to \( x+1 \) (resp. \( x \)) at rate \( \xi_x \). We examine a quenched nonequilibrium central limit theorem for the position of a tagged particle in the exclusion process with bond disorder \( \{\xi_x : x \in \mathbb{Z}\} \). We prove that the position of the tagged particle converges under diffusive scaling to a Gaussian process if the other particles are initially distributed according to a Bernoulli product measure associated to a smooth profile \( \rho_0 : \mathbb{R} \to [0,1] \).

1. Introduction

A classical problem in statistical mechanics consists in proving that the dynamics of a single particle in a mechanical system is well approximated on a large scale by a Brownian motion ([19, 9]). In a seminal paper, Kipnis and Varadhan [10] proved an invariance principle for the position of a tracer particle in the symmetric simple exclusion process. The method relies on a central limit theorem for additive functionals of Markov processes and uses time reversibility and translation invariance. This approach has been extended to interacting particle systems whose generators satisfy a sector condition or, more generally, graded sector conditions ([12] and references therein).

In [8], we proved a non-equilibrium central limit theorem for the position of a tagged particle in the one-dimensional nearest-neighbor symmetric exclusion process. We assumed that the initial state is a product measure associated to a smooth profile. Observing that the position of the tagged particle can be recovered from the density field and the total current through a bond, we deduced a central limit theorem for the tagged particle from a joint non-equilibrium central limit theorem for the density field and the current.

The evolution of random walks in random environment has attracted some attention in these last years ([20] and references therein). Recently, a quenched central limit theorem has been proved for random conductance models [18]. Here, to each bond \( \{x,y\} \) of \( \mathbb{Z}^d \) is attached i.i.d. strictly positive random variables \( \xi_{x,y} \). Under some conditions on the variables \( \xi \), the authors proved, among other results, that

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for almost all environments $\xi$, a random walk on $\mathbb{Z}^d$ which jumps from $x$ to $y$ at rate $\xi_{x,y}$ converges, when diffusively rescaled, to a Brownian motion.

In this article we consider a one-dimensional nearest-neighbor exclusion process evolving on an environment $\xi$. Each particle behaves as the random walk described above, with the additional rule that a jump is suppressed whenever a particle decides to jump over a site already occupied. Under very mild assumptions on the environment, we prove that the density field converges to the solution of a heat equation, generalizing a previous result obtained by Nagy [16].

Assuming that the environment is strictly elliptic, i.e., formed by i.i.d. random variables $\xi_{x,x+1}$ strictly bounded away from 0 and $\infty$, we prove a non-equilibrium central limit theorem for the density field, which holds for almost all realizations of the environment. Here the assumption of independence and identical distribution of the environment could be relaxed. In contrast with [6], where annealed central limit theorems are considered, we prove in this article a quenched statement.

From this result and from a non-equilibrium central limit theorem for the current, we prove the main result of the article which states a central limit theorem for the position of a tagged particle starting from a configuration in which particles are distributed according to a Bernoulli product measure associated to a smooth density profile. This central limit theorem holds for almost all environment $\xi$'s.

The approach and the main technical difficulties can be summarized in few words to the specialists. The model is in principle non-gradient due to the presence of the environment [4]. However, a functional transformation of the the empirical measure (3.5) turns it into a gradient model. The proof that the transformed empirical measure is close to the original empirical measure imposes some conditions on the environment.

The same strategy can be applied to derive a nonequilibrium central limit theorem for the density field. Here, however, to prove tightness and to show that the transformed density field is close to the original, some sharp estimates on the space time correlations are needed as well. The deduction of these estimates require a Nash type bound on the kernel of the random walk in the random conductance model, which has been proved only under a strict ellipticity condition of the environment. At this point, it remains to adapt the strategy introduced in [8] to prove the central limit theorem for the tagged particle.

While in Rome in April 2005, the second author showed to A. Faggionato the model and the method described in next section to derive the hydrodynamic behavior of this bond disorder model. At that time he thought that the approach required uniform ellipticity of the environment. A few months later and independently, Alessandra [1], generalizing Nagy’s method [16], obtained a proof of the hydrodynamic behavior requiring only the assumptions stated in Theorem 2.1 below, while the authors realized that strict ellipticity was not needed in their approach.

2. Main results

We state in this section the main results of the article. Denote by $\mathcal{X}$ the state space $\{0,1\}^\mathbb{Z}$ and by the Greek letter $\eta$ the elements of $\mathcal{X}$ so that $\eta(x) = 1$ if there is a particle at site $x$ for the configuration $\eta$ and $\eta(x) = 0$ otherwise.

Consider a sequence $\{\xi_x : x \in \mathbb{Z}\}$ of strictly positive numbers. The symmetric nearest-neighbor simple exclusion process with bond disorder $\{\xi_x : x \in \mathbb{Z}\}$ is the Markov process $\{\eta_t : t \geq 0\}$ on $\mathcal{X}$ whose generator $L_{\xi,N}$ acts on cylinder functions
$f$ as

$$(L_{\xi,N} f)(\eta) = N^2 \sum_{x \in \mathbb{Z}} \xi_x (\nabla_x f)(\eta),$$

where $(\nabla_x f)(\eta) = f(\sigma^{x,x+1}\eta) - f(\eta)$ and

$$\sigma^{x,y}\eta(z) = \begin{cases} 
\eta(y), & z = x \\
\eta(x), & z = y \\
\eta(z), & z \neq x, y.
\end{cases}$$

Notice that the process is speeded up by $N^2$.

Existence and ergodic properties of this Markov process can be proved as in the space homogeneous case ([13,16]). Moreover, the Bernoulli product measures $\nu_\alpha$ in $\{0,1\}^\mathbb{Z}$, with marginals $\nu_\alpha\{\eta(x) = 1\} = \alpha$ for $\alpha \in [0,1]$, are extremal, reversible measures.

For each profile $\rho_0 : \mathbb{R} \to [0,1]$, denote by $\nu^{\mathbb{R}_+}_{\rho_0(\cdot)}$ the product measure on $\mathcal{X}$ with marginals given by $\nu^{\mathbb{R}_+}_{\rho_0(\cdot)}\{\eta(x) = 1\} = \rho_0(x/N)$. For a measure $\mu$ on $\mathcal{X}$, let $\mathbb{P}^{\mathbb{R}_+}_{\mu}$ stand for the probability measure on the path space $D(\mathbb{R}_+, \mathcal{X})$ induced by the Markov process $\eta_t$ and the measure $\mu$.

The empirical measure associated to the process $\eta_t$ is defined by

$$\pi^N_t(du) = \frac{1}{N} \sum_{x \in \mathbb{Z}} \eta_t(x) \delta_{x/N}(du).$$

Fix $0 < \gamma < \infty$. Let $C^2_0(\mathbb{R})$ be the set of twice continuously differentiable functions $G : \mathbb{R} \to \mathbb{R}$ with compact support. Fix a profile $\rho_0 : \mathbb{R}_+ \to [0,1]$. A bounded function $\rho : \mathbb{R}_+ \times \mathbb{R} \to [0,1]$ is said to be a weak solution of the heat equation

$$\begin{aligned}
\left\{ \begin{array}{l}
\partial_t \rho = \gamma^{-1} \Delta \rho \\
\rho(0, \cdot) = \rho_0(\cdot)
\end{array} \right.
\end{aligned} \tag{2.1}$$

if

$$\langle \rho_t, G \rangle = \langle \rho_0, G \rangle + \int_0^t ds \langle \rho_s, \gamma^{-1} \Delta G \rangle \tag{2.2}$$

for all $t \geq 0$ and all $G$ in $C^2_0(\mathbb{R})$. In these equations, $\Delta$ stands for the Laplacian and $\langle \rho, H \rangle$ for the integral of $H$ with respect to the measure $\rho(u)du$. It is well known that for any bounded profile $\rho_0 : \mathbb{R} \to [0,1]$, there exists a unique weak solution of (2.1). The first main result of the article states a quenched law of large numbers for the empirical measure under weak assumptions on the environment $\{\xi_x : x \in \mathbb{Z}\}$.

**Theorem 2.1.** Assume that

$$\lim_{K \to \infty} \frac{1}{K} \sum_{x=1}^{K} \xi_x^{-1} = \gamma, \quad \lim_{K \to \infty} \frac{1}{K} \sum_{x=-K}^{-1} \xi_x^{-1} = \gamma \tag{2.3}$$

for some $0 < \gamma < \infty$. Fix a profile $\rho_0 : \mathbb{R} \to [0,1]$. Under $\mathbb{P}^{\mathbb{R}_+}_{\nu^{\mathbb{R}_+}_{\rho_0(\cdot)}}$, $\pi^N_t$ converges in probability to the weak solution of (2.1): For every continuous function with compact support $G$, every $t \geq 0$ and every $\delta > 0$,

$$\lim_{N \to \infty} \mathbb{P}^{\mathbb{R}_+}_{\nu^{\mathbb{R}_+}_{\rho_0(\cdot)}} \left[ |\langle \pi^N_t, G \rangle - \langle \rho_t, G \rangle| > \delta \right] = 0,$$

where $\rho(t,u)$ is the weak solution of (2.1).
To prove a quenched nonequilibrium central limit theorem for the empirical measure, assume that \( \{\xi_x : x \in \mathbb{Z}\} \) is a sequence of i.i.d. random variables defined on a probability space \((\Omega, P, \mathcal{F})\) such that

\[
P[\varepsilon \leq \xi_0 \leq \varepsilon^{-1}] = 1 \tag{2.4}
\]

for some \( \varepsilon > 0 \). This strong ellipticity condition is needed in Section 6 to prove sharp estimates of the decay of the space-time correlation functions. All other arguments require the weaker integrability condition: \( E[|\xi_0|^6] < \infty \).

Fix a profile \( \rho_0 : \mathbb{R} \to [0, 1] \) and an environment \( \xi \). Let \( \rho_t^{N,\xi}(x) = \mathbb{E}_{\nu_{\rho_0(\cdot)}}[\eta_t(x)] \).

A trivial computation shows that \( \rho_t^{N,\xi} : \mathbb{Z} \to [0,1] \) is the solution of the discrete linear equation

\[
\begin{align*}
\partial_t \rho_t(x) & = N\{\xi_x(\nabla_N \rho_t)(x) - \xi_{x-1}(\nabla_N \rho_t)(x-1)\}, \\
\rho_0(x) & = \rho_0(x/N),
\end{align*} \tag{2.5}
\]

where \( (\nabla_N h)(x) = N\{h(x+1) - h(x)\} \). We denote frequently \( \rho_t^{N,\xi} \) by \( \rho_t^N \).

Denote by \( \mathcal{S}(\mathbb{R}) \) the Schwartz space of rapidly decreasing functions and by \( \mathcal{S}'(\mathbb{R}) \) its dual, the space of distributions. Let \( \{Y_t^N, t \geq 0\} \) be the density fluctuation field, a \( \mathcal{S}'(\mathbb{R}) \)-valued process given by

\[
Y_t^N(G) = \frac{1}{\sqrt{N}} \sum_{x \in \mathbb{Z}} G(x/N)\{\eta_t(x) - \rho_t^{N,\xi}(x)\}
\]

for \( G \) in \( \mathcal{S}(\mathbb{R}) \). Next theorem states the almost sure convergence of the finite dimensional distributions of \( Y_t^N \) to the marginals of a centered Gaussian field.

**Theorem 2.2.** Let \( \{\xi_x : x \in \mathbb{Z}\} \) be a sequence of i.i.d. random variables satisfying assumption 2.4. Let \( \rho_0 : \mathbb{R} \to [0, 1] \) be a profile with first derivative in \( L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \). There exists a set of environments \( \Omega_0 \) with total measure such that for every \( \xi \in \Omega_0 \), every \( k \geq 1 \) and every \( 0 \leq t_1 < \cdots < t_k \), \((Y_{t_1}^N, \ldots, Y_{t_k}^N)\) converges to a centered Gaussian vector \((u_{t_1}, \ldots, u_{t_k})\) with covariance given by

\[
E[Y_s(G)Y_t(H)] = \int_{\mathbb{R}} \chi(\rho_0(u)) T_s G(u) T_t H(u) + 2\gamma^{-1} \int_0^s dr \int_{\mathbb{R}} \chi(\rho(r,u)) \nabla T_{s-r} G(u) \nabla T_{t-r} H(u)
\]

for all \( 0 \leq s \leq t, G, H \) in \( \mathcal{S}(\mathbb{R}) \). Here \( \rho \) stands for the solution of the heat equation \((2.0), \{T_r : r \geq 0\}\) for the semi-group associated to \( \gamma^{-1} \Delta \) and \( \chi(\alpha) = \alpha(1 - \alpha) \) for the compressibility in the exclusion process.

Denote by \( \nu_{\rho_0(\cdot)}^{N,N} \) the measure \( \nu_{\rho_0(\cdot)}^N \) conditioned to have a particle at the origin and by \( X_t^N \) the position at time \( t \) of the particle initially at the origin. Define \( u_t^N = u_t^{N,\xi} \) by the relation

\[
\sum_{x=0}^{u_t^N} \rho_t^{N,\xi}(x) \leq \xi_{-1} \int_0^t N^2 \{\rho_s^{N,\xi}(-1) - \rho_s^{N,\xi}(0)\} \, ds < \sum_{x=0}^{u_t^N+1} \rho_t^{N,\xi}(x), \tag{2.7}
\]

where \( \rho_t^{N,\xi} \) is the solution of \((2.0)\) with initial condition \( \rho_0^{N,\xi}(0) = 1, \rho_0^{N,\xi}(x) = \rho_0(x/N), x \neq 0 \). Let \( W_t^N = \langle X_t^N - u_t^N \rangle / \sqrt{N} \).
Theorem 2.3. Let \( \{\xi_x : x \in \mathbb{Z}\} \) be a sequence of i.i.d. random variables satisfying assumption (2.4). Let \( \rho_0 \) be an initial profile with first derivative in \( L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) and second derivative in \( L^\infty(\mathbb{R}) \). There exists a set of environments \( \Omega_0 \) with total measure and the following property. For every \( \xi \) in \( \Omega_0 \), every \( k \geq 1 \) and every \( 0 \leq t_1 < \ldots < t_k \), under \( \mathbb{P}_{\xi \rho_0} \), \((W^N_{t_1}, \ldots, W^N_{t_k})\) converges in law to a Gaussian vector \((W_{t_1}, \ldots, W_{t_k})\) with covariances given by

\[
\rho(s,u_0)\rho(t,u_1)E[W_sW_t] = \int_{-\infty}^{0} dv P[Z_s \leq v] P[Z_t \leq v] \chi(\rho_0(v)) \\
+ \int_0^\infty dv P[Z_s \geq v] P[Z_t \geq v] \chi(\rho_0(v)) \\
+ \frac{2}{\gamma} \int_s^t dr \int_{-\infty}^{\infty} dv p_{t-r}(u_t,v) p_{s-r}(u_s,v) \chi(\rho(r,v))
\]

provided \( s \leq t \). In this formula, \( Z_t = u_t + B_{t/\gamma}^0 \), where \( B_{t/\gamma}^0 \) is a standard Brownian motion starting from the origin, and \( p_{t}(v,w) \) stands for the kernel of \( B^0_{t/\gamma} \).

3. Hydrodynamic limit

We prove in this section Theorem 2.3. Fix an environment satisfying (2.3) and denote by \( \mathcal{M}_+(\mathbb{R}) \) the set of positive Radon measures in \( \mathbb{R} \). Fix \( T \geq 0 \) and a bounded profile \( \rho_0 : \mathbb{R} \to [0,1] \). Let \( \{Q_N : N \geq 1\} = \{Q_N,\xi : N \geq 1\} \) be the sequence of measures on \( D([0,T],\mathcal{M}_+(\mathbb{R})) \) induced by the Markov process \( \pi_i^N \) and the initial state \( \nu_{\rho_0}^N \).

The proof of Theorem 2.3 is divided in two steps. We first prove tightness of the sequence \( \{Q_N\}_N \), and then that all limit points of \( \{Q_N\}_N \) are supported on weak solutions of the hydrodynamic equation. It follows from these two results and the uniqueness of weak solutions of the heat equation (2.1) that \( \pi_t^N \) converges in probability to the absolutely continuous measure \( \rho(t,u)du \) whose density is the solution of (2.1) (cf. [9]).

It turns out that this program can not be accomplished for the empirical measure \( \pi_t^N \), but for a “corrected by the environment” process \( X_t^N \), which is close enough to the empirical measure \( \pi_t^N \).

3.1. Corrected empirical measure. Denote by \( C_0^2(\mathbb{R}) \) the space of twice continuously differentiable functions with compact support. For a function \( G \) in \( C_0^2(\mathbb{R}) \) and an environment \( \xi \), let \( T_\xi G : \mathbb{Z} \to \mathbb{R} \) be the sequence defined by

\[
(T_\xi G)(x) = \sum_{j < x} \xi_j^{-1} \left\{ G\left(\frac{j+1}{N}\right) - G\left(\frac{j}{N}\right) \right\}.
\]

For each each \( N \geq 1 \) and each function \( G \) in \( C_0^2(\mathbb{R}) \), the series \( \sum_x \xi_x^{-1} [G((x + 1)/N) - G(x/N)] \) is absolutely summable because \( G \) has compact support. Moreover, it follows from (2.5) that

\[
T_\xi G = T_{\xi,G}^N := \sum_{x \in \mathbb{Z}} \xi_x^{-1} \left\{ G((x + 1)/N) - G(x/N) \right\}
\]

converges to 0 as \( N \uparrow \infty \).
We introduce $T_\xi G$ for two reasons. On the one hand, we expect $(T_\xi G)(x)$ to be close to $\gamma G(x/N)$, which is the content of Lemma 3.1 below. On the other hand, $N\{ (T_\xi G)(x+1) - (T_\xi G)(x) \} \xi_x = (\nabla_N G)(x/N)$, where $\nabla_N$ stands for the discrete derivative: $(\nabla_N G)(x/N) = N\{ G(x+1/N) - G(x/N) \}$. Hence, formally, \[ \int x \sum_{x\in\mathbb{Z}}(T_\xi G)(x)\eta(x) = \frac{1}{N} \sum_{x\in\mathbb{Z}}(\nabla_N G)(x/N)\eta(x), \] where $\Delta_N$ stands for the discrete Laplacian.

Of course, $T_\xi G$ may not belong to $\ell_1(\mathbb{Z})$, the space of summable series, and the left hand side of the previous formula may not be defined. To overcome this difficulty, we modify $T_\xi G$ in order to integrate it with respect to the empirical measure. Fix an arbitrary integer $l > 0$ which will remained fixed in this section. Let $g = g_l : \mathbb{R} \to \mathbb{R}$ be defined by
\[ g(u) = \begin{cases} 0, & u < 0, \\ u/l, & 0 \leq u < l, \\ 1, & u \geq l. \end{cases} \]

For each function $G$ in $C_0^2(\mathbb{R})$, let
\[ (T_{\xi,l} G)(x) := (T_\xi G)(x) - \frac{\Delta_G}{T_\xi g}(T_\xi G)(x). \]

Notice that $T_{\xi,l}$ converges to $\gamma$ almost surely, as $N \to \infty$. In particular, by the ratio $T_{\xi,l} G / T_\xi g$ vanishes almost surely as $N \to \infty$. In the end of this section we prove the following statement.

**Lemma 3.1.** For each function $G$ in $C_0^2(\mathbb{R})$, and each environment $\xi$ satisfying (2.3), $T_{\xi,l} G$ belongs to $\ell_1(\mathbb{Z})$ and
\[ \lim_{N \to \infty} \frac{1}{N} \sum_{x\in\mathbb{Z}}|T_{\xi,l} G(x) - \gamma G(x/N)| = 0. \]

Denote by $X_t^N$ the corrected empirical measure defined by
\[ X_t^N(G) = X_{\xi,l}^N(G) = \frac{1}{N} \sum_{x\in\mathbb{Z}} T_{\xi,l} G(x) \eta_N(x) \]
for each function $G$ in $C_0^2(\mathbb{R})$.

As mentioned before, the sequence $T_{\xi,l} G(x)$ has two properties. On the one hand, in view of Lemma 3.1 it is close to $\gamma G(x/N)$ in $\ell_1(\mathbb{Z})$. In particular, the integral of $G$ with respect to the empirical measure is close to $\gamma^{-1}X_t^N(G)$ uniformly in time. On the other hand, $\gamma^{-1}X_t^N(G)$ has an integral term which can be expressed as function of the empirical measure. Indeed, for a function $G$ in $C_0^2(\mathbb{R})$, let $M_t^N(G) = M_t^{\xi,l}(G)$ be the martingale defined by
\[ M_t^N(G) = X_t^N(G) - X_0^N(G) - \int_0^t ds \langle X_s^N(G) \rangle = X_t^N(G) - X_0^N(G) - \int_0^t ds \left\{ \langle \pi_s^N(G) \rangle, \Delta G \rangle - \frac{T_{\xi,l} G}{T_\xi g} \langle \pi_s^N(G) \rangle \right\}. \]
3.2. Tightness of $\pi^N_t$. It is well known that a sequence of probability measures $\{Q_N\}_N$ on $D([0,T],\mathcal{M}_+(\mathbb{R}))$ is tight if and only if the sequence $\{Q_N(G)\}_N$ is tight for all $G \in \mathcal{C}_0^2(\mathbb{R})$, where $Q_N(G)$ is the probability measure in $D([0,T],\mathbb{R})$ corresponding to the process $(\pi^N_t, G)$.

We claim that the process $X^N_t(G)$ is tight. Recall Aldous criteria for tightness in $D([0,T],\mathbb{R})$:

**Lemma 3.2.** A sequence of probability measures $\{P_N\}_N$ in $D([0,T],\mathbb{R})$ is tight if

(i) For all $0 \leq t \leq T$ and for all $\varepsilon > 0$ there exists a finite constant $A$ such that $\sup_N P_N(|X_t| > A) < \varepsilon$,

(ii) For all $\delta > 0$,

$$\lim_{\beta \to 0} \lim_{N \to \infty} \sup_{\theta \in T} P_N(|X_t - X_{\tau}^\theta| > \delta) = 0,$$

where $T$ is the set of stopping times with respect to the canonical filtration bounded by $T$.

To prove tightness of $X^N_t(G)$ note that (i) is automatically satisfied because the number of particles per site is bounded and $T_{\xi,G}$ converges to $\gamma G$ in $\ell_1(\mathbb{Z})$.

To check condition (ii), fix a stopping time $\tau$ bounded by $T$ and $\theta \leq \beta$. Recall from formula (3.6) that we may express $X^N_{\tau + \theta}(G) - X^N_{\tau}(G)$ as the sum of a martingale difference and an integral. On the one hand, computing the quadratic variation of the martingale $M^N_t(G)$, we obtain that

$$\mathbb{E}_N[(M^N_{\tau + \theta}(G) - M^N_{\tau}(G))^2] =$$

$$\mathbb{E}_N\left[\int_\tau^{\tau + \theta} ds \frac{1}{N^2} \sum_{x \in \mathbb{Z}} \left\{\nabla_N G(x/N) - \frac{T_{\xi,G} g(x/N)}{T_{\xi,g}} \nabla_N g(x/N)\right\}(\eta^N_{s}(x + 1) - \eta^N_{s}(x))^2\right].$$

The previous expression is bounded above by $N^{-1}\{C(G) + l^{-1}(T_{\xi,G}/T_{\xi,g})^2\}$, which vanishes as $N \to \infty$ in view of (2.3).

On the other hand, since there is at most one particle per site and since $G$ belongs to $\mathcal{C}_0^2(\mathbb{R})$,

$$\left|\int_\tau^{\tau + \theta} ds \frac{1}{N} \sum_{x \in \mathbb{Z}} \left\{\Delta_N G(x/N) - \frac{T_{\xi,G} \Delta_N g(x/N)}{T_{\xi,g}} \right\} \eta^N_{s}(x)\right| \leq C_0 \beta + \frac{2\beta}{l} |T_{\xi,G}/T_{\xi,g}|$$

for some finite constant $C_0$ depending only on $\beta$. As $N \to \infty$, the second term vanishes in view of (2.3), (3.2). This proves condition (ii) of Lemma 3.2 and tightness of the process $X^N_t(G)$.

In view of Lemma 3.1

$$\sup_{0 \leq t \leq T} |\gamma(\pi^N_t,G) - X^N_t(G)| \leq \frac{1}{N} \sum_{x \in \mathbb{Z}} |\gamma G(x/N) - T_{\xi,G}(G(x)| (3.7)$$

converges to 0 as $N \to 0$. In particular, $\langle \pi^N_t, G \rangle$ is also tight, with the same limit points of $X^N_t(G)$. Since this statement holds for all $G \in \mathcal{C}_0^2(\mathbb{R})$, the sequence $Q_N$ is tight.

3.3. Uniqueness of limit points. Let $Q$ be a limit point of the sequence $\{Q_N\}_N$. Since there is at most one particle per site, $Q$ is concentrated on absolutely continuous paths $\pi(t,du) = \rho(t,u)du$, with positive density bounded by 1: $0 \leq \rho(t,u) \leq 1$. 


We have seen in the last subsection that $Q$ is also a limit point of $X_N$. Fix a function $G$ in $C_0^2(\mathbb{R})$ and recall the definition of the martingale $M_t^N(G)$ given in \cite{JaraLandim2009}. By the proof of the tightness of $X_N^t$, the expectation of the quadratic variation of $M_t^N(G)$ vanishes as $N \uparrow \infty$. In particular, in view of \cite{JaraLandim2009}, the measure $Q$ is concentrated in trajectories $\pi_t$ such that

$$
\langle \pi_t, G \rangle = \langle \pi_0, G \rangle + \int_0^t ds \langle \pi_s, \gamma^{-1} \Delta G \rangle
$$

for all $0 \leq t \leq T$, $G$ in $C_0^2(\mathbb{R})$. By the uniqueness of weak solutions of the heat equation, Theorem \ref{thm:heat} is proved. \hfill $\square$

We conclude this section with the

**Proof of Lemma \ref{lem:quenched}* $T_{\xi,t}G$ belongs to $\ell_1(\mathbb{Z})$ because it belongs to $\ell_\infty(\mathbb{Z})$ and vanishes outside a finite set. Fix a smooth function $G$ in $C_0^2(\mathbb{R})$.

Consider first the sum over $x \leq 0$. In this case $(T_{\xi,y}G)(x) = 0$ so that $(T_{\xi,t}G)(x) = (T_\xi G)(x)$. In particular,

$$
\frac{1}{N} \sum_{x \leq 0} \left| (T_{\xi,t}G)(x) - \gamma G(x/N) \right| = \frac{1}{N} \sum_{x \leq 0} \left| \frac{1}{N} \sum_{y < x} \xi^{-1}_y (\nabla_N G)(y/N) \right|
$$

where $\xi^{-1}_y = \xi^{-1} - \gamma$. Both sums in $x$ and $y$ start from $-AN$, for some $A > 0$, because $G$ has compact support. Fix $\varepsilon > 0$. Since $G'$ is uniformly continuous, there exists $\delta > 0$ such that $|G'(v) - G'(u)| \leq \varepsilon$ if $|v - u| \leq \delta$. We may therefore replace $(\nabla_N G)(y/N)$ by $G'(k\delta)$, for $k\delta \leq y/N \leq (k+1)\delta$, paying a price bounded by $C(G)\varepsilon$. After this replacement, the law of large numbers \cite{JaraLandim2009} ensures that the previous expression vanishes as $N \uparrow \infty$.

Similarly, for $x \geq lN$, $(T_{\xi,y}G)(x) = T_{\xi,y}G$ so that $(T_{\xi,t}G)(x) = (T_\xi G)(x) - T_{\xi,G}$. Therefore,

$$
(T_{\xi,t}G)(x) - \gamma G(x/N) = -\frac{1}{N} \sum_{y \geq x} \xi^{-1}_y (\nabla_N G)(y/N)
$$

and we may repeat the previous arguments to show that the sum for $x \geq lN$ vanishes as $N \uparrow \infty$.

Finally, for $0 \leq x < lN$, we estimate separately $(T_{\xi,t}G)(x) - \gamma G(x/N)$ and $(T_{\xi,G}/T_{\xi,y}G)(T_\xi G)(y/N)$. The first piece is handled as before, while the second vanishes as $N \uparrow \infty$ in view of \cite{JaraLandim2009} and because $(T_{\xi,y}G)(x)/T_{\xi,G}$ is absolutely bounded by 1. This proves Lemma \ref{lem:quenched} \hfill $\square$

4. **Fluctuations of the empirical measure**

Let $\{\xi_k : x \in \mathbb{Z}\}$ be a sequence of i.i.d. random variables defined on a probability space $(\Omega, P, F)$. We prove in this section a quenched nonequilibrium central limit theorem for the empirical measure. The proof relies on sharp estimates of the decay of the space-time correlation functions presented in Section \ref{sec:correlations} which requires the strong ellipticity condition: $P[\varepsilon \leq \xi^{-1}_0 \leq 1] = 1$ for some $\varepsilon > 0$. To stress that it is only in the estimation of the correlation functions that we need this condition, we present all other proofs under the weaker assumption that $E[\xi^{-1}_0] < \infty$. Moreover, the hypotheses of independence, identical distribution and finiteness of the sixth moment can be relaxed.

Throughout this section the index $l$ of the operator $T_{\xi,t}$ introduced in the previous section depends on $N$ as $l = l_N = N^{1/4}$. Recall that we denote by $\mathcal{S}(\mathbb{R})$ the
Schwartz space of rapidly decreasing functions. We may extend the operators $T_ξ$, $T_ξ^G$ to $S(\mathbb{R})$:

**Lemma 4.1.** Assume that $E[\xi_0^{-6}] < \infty$ and fix a function $G \in S(\mathbb{R})$. There exists a subset $Ω_G$ with total measure such that for each $ξ$ in $Ω_G$ $T_ξ G(x)$ is well defined and

$$
\lim_{N \to \infty} N^{1/4} \sup_{x \in \mathbb{Z}} |T_ξ G(x) - \gamma G(x/N)| = 0.
$$

In particular, $\lim_{N \to \infty} N^{1/4} T_ξ G = 0$. Moreover,

$$
\lim_{N \to \infty} \sup_{x \in \mathbb{Z}} |T_ξ^G(x) - \gamma G(x/N)| = 0
$$

and

$$
\lim_{N \to \infty} \frac{1}{N} \sum_{x \in \mathbb{Z}} |T_ξ^G(x) - \gamma G(x/N)| = 0.
$$

The proof of this lemma is given at the end of this section. By interpolation it follows from this result that

$$
\lim_{N \to \infty} \frac{1}{N} \sum_{x \in \mathbb{Z}} |T_ξ^G(x)|^p = \int |\gamma G(u)|^p \, du
$$

ξ-almost surely for all $1 \leq p \leq \infty$.

Recall that the definition of the density field $Y_t^N$ given just before the statement of Theorem 2.2. Denote by $Z_t^N$ the fluctuation density field corrected by the environment:

$$
Z_t^N(G) =: \frac{1}{\gamma} Y_t^N(T_ξ^G(x)) = \frac{1}{\gamma \sqrt{N}} \sum_{x \in \mathbb{Z}} (T_ξ^G(x) \{\eta_t^N(x) - \rho_t^N(x)\})
$$

for functions $G$ in $S(\mathbb{R})$.

We prove in this section a nonequilibrium central limit theorem for the density field $Z_t^N$ in random environment and deduce from this result the convergence of the finite dimensional distributions of the field $Y_t^N$ defined in Section 2. Recall that we denote by $S'(\mathbb{R})$ the Schwartz space of distributions. For a profile $ρ_0 : \mathbb{R} \to (0,1)$ and an environment $ξ = \{ξ_x : x \in \mathbb{Z}\}$ and $T > 0$, let $Q_{ρ_0}^{N,ξ}$ be the measure on $D([0,T], S'(\mathbb{R}))$ induced by the process $Z_t^N$ and the initial state $ν_{ρ_0}^{N,ξ}$.

**Proposition 4.2.** Fix a profile $ρ_0 : \mathbb{R} \to (0,1)$ with bounded and integrable first derivative. There exists a set of environments $Ω_0$ with total measure such that for each $ξ$ in $Ω_0$, $Q_{ρ_0}^{N,ξ}$ converges to a centered Gaussian field $Z_t$ with covariance given by (2.10).

The strategy of the proof of Proposition 4.2 is similar to the one adopted for the hydrodynamic limit. We prove tightness of the distributions of $Z_t^N$ in $D([0,T], S'(\mathbb{R}))$ and that all limit points of $Z_t^N$ satisfies a martingale problem which characterizes the limiting measure.

We start proving tightness. For a function $G$ in $S(\mathbb{R})$, consider the martingale $M_t^N(G)$ defined by

$$
M_t^N(G) = Z_t^N(G) - Z_0^N(G) - \int_0^t \gamma_t^N(s, G) \, ds,
$$

where

$$
\gamma_t^N(s, G) = Y_s^N(\gamma^{-1} Δ_N G) - \frac{T_s^G}{T_ξ^G} Y_s^N(\gamma^{-1} Δ_N G).
$$
The quadratic variation \( \langle M^N(G) \rangle_t \) of this martingale is equal to \( \int_0^t \gamma^N_1(s,G) \, ds \), with \( \gamma^N_1(s,G) \) given by

\[
\frac{1}{\gamma^2 N} \sum_{x \in \mathbb{Z}} \xi_x^{-1} \left\{ \left( \nabla N G(x/N) - \frac{T_{\xi,G}}{T_{\xi,g}} (\nabla_N g)(x/N) \right) \right\}^2 \left( \eta^N_s(x+1) - \eta^N_s(x) \right)^2.
\]

In view of Mitoma’s criterion for the relative compactness of a sequence of measures in \( D([0, T], \mathcal{S}(\mathbb{R})) \) (cf. [15], [7], [5]), to show that the process \( Z^N_t \) is tight, it is enough to prove that

\[
\sup_{N} \sup_{0 \leq t \leq T} \mathbb{E}_{\nu_{N}(\cdot)}\left[Z^N_t(G)^2\right] < \infty, \quad \sup_{N} \sup_{0 \leq t \leq T} \mathbb{E}_{\nu_{N}(\cdot)}\left[\gamma^N_1(t,G)^2\right] < \infty \quad (4.3)
\]

for \( i = 1, 2 \) and a dense family of functions \( G \) in \( \mathcal{S}(\mathbb{R}) \). Moreover, to show that all limit points of the sequence \( Z^N_t \) are concentrated on \( C([0, T], \mathcal{S}(\mathbb{R})) \), we need to check that for each function \( G \) in \( \mathcal{S}(\mathbb{R}) \) there exists a sequence \( \delta_N = \delta(t, G, N) \), vanishing as \( N \uparrow \infty \), such that

\[
\lim_{N \to \infty} \mathbb{P}_{\nu_{N}(\cdot)}\left[\sup_{0 \leq t \leq T} |Z^N_t(G) - Z^N_t(G)| \geq \delta_N \right] = 0. \quad (4.4)
\]

To prove (4.3), consider a countable dense subset of functions \( \mathcal{S}_0(\mathbb{R}) = \{ G_k : k \geq 1 \} \) in \( \mathcal{S}(\mathbb{R}) \). Let \( \Omega_0 = \bigcap_{k \geq 1} \Omega_{G_k} \cap \Omega_{(G_k)^2} \cap \Omega_{\Delta G_k} \), where \( \Omega_{G_k} \) are the total measure sets introduced in Lemma 4.1. Fix an environment \( \xi \) in \( \Omega_0 \) and a function \( G \) in the class \( \{ G_k : k \geq 1 \} \). By Theorem 6.1

\[
\mathbb{E}_{\nu_{N}(\cdot)}\left[Z^N_t(G)^2\right] \leq \frac{1}{\gamma^2 N} \sum_{x \in \mathbb{Z}} (T_{\xi,G}(G)(x))^2 + \frac{C_1}{\gamma^2} \sqrt{T} \left( \frac{1}{N} \sum_{x \in \mathbb{Z}} \left| (T_{\xi,G}(G)(x)) \right| \right)^2
\]

for some finite constant \( C_1 \) depending only on \( \varepsilon \) and \( \rho_0 \). Since \( \xi \) belongs to \( \Omega_0 \), by (4.1), as \( N \uparrow \infty \), these expressions converge to finite expressions.

On the other hand, by definition of \( \gamma^N_i(t, G) \),

\[
\frac{\gamma^2}{2} \mathbb{E}_{\nu_{N}(\cdot)}\left[\gamma^N_1(t,G)^2\right] \leq \mathbb{E}_{\nu_{N}(\cdot)}\left[Y^N_t(\Delta_N G)^2\right] + \left\{ \frac{T_{\xi,G}}{T_{\xi,g}} \right\}^2 \mathbb{E}_{\nu_{N}(\cdot)}\left[Y^N_t(\Delta_N g)^2\right].
\]

The first term is handled in the same way as \( Z^N_t(G) \). The second term is also simple to estimate, since

\[
Y^N_t(\Delta_N g)^2 = \frac{N}{tN} \left\{ [\eta^N_t(0) - \rho^N_t(0)] - [\eta^N_t(IN) - \rho^N_t(IN)] \right\}^2,
\]

and since, by Lemma 4.1, \( (T_{\xi,G}/T_{\xi,g})^2 N^{-2} \) vanishes as \( N \uparrow \infty \) for all \( \xi \) in \( \Omega_0 \).

Finally, by definition of \( \gamma^N_2(t, G) \),

\[
\mathbb{E}_{\nu_{N}(\cdot)}\left[\gamma^N_2(t,G)^2\right] \leq \frac{2}{\gamma^2 N} \sum_{x \in \mathbb{Z}} \xi_x^{-1} \nabla_N G(x/N)^2 + \frac{2}{N^2 \gamma^2} \left( \frac{T_{\xi,G}}{T_{\xi,g}} \right)^2 \sum_{0 \leq x < IN} \xi_x^{-1}.
\]

The first term converges to a finite constant as \( N \uparrow \infty \), while the second term vanishes for \( \xi \) in \( \Omega_0 \).

Since condition (4.4) follows from the fact that no more than one particle jumps at each time, the previous estimates show that for each environment \( \xi \) in \( \Omega_0 \), the sequence \( Q^{N,\xi}_{\rho_0} \) is tight and that each limiting point is concentrated on \( C([0, T], \mathcal{S}(\mathbb{R})) \).

We consider now the question of uniqueness of limit points. Fix \( \xi \) in \( \Omega_0 \), let \( Q^\xi \) be a limit point of \( Q^{N,\xi}_{\rho_0} \) and assume without loss of generality that \( Q^{N,\xi}_{\rho_0} \) converges
to $Q^\xi$. Let $\mathfrak{A}$, $\mathfrak{B}$, $t \geq 0$, stand for the operators $\gamma^{-1}\Delta$, $\sqrt{2}\gamma^{-1}\chi(\rho(t,u))\nabla$, respectively, where $\rho$ is the solution of the heat equation (2.1) and $\chi$ is the compressibility given by $\chi(\alpha) = \alpha(1-\alpha)$.

According to Holley-Stroock [24] theory of generalized Ornstein-Uhlenbeck processes and to Stroock and Varadhan [21], there exists a unique process $Z_t$ in $C([0, +\infty), \mathcal{S}'(\mathbb{R}))$ with the following two properties: $Z_0$ is a centered Gaussian field with covariance given by

$$E[Z_0(G)Z_0(H)] = \int_\mathbb{R} G(u)H(u)\chi(\rho_0(u)) \, du$$

for all $G$, $H$ in $\mathcal{S}(\mathbb{R})$. Moreover, the processes $M_t(G)$, $m_t(G)$ defined by

$$Z_t(G) - Z_0(G) - \int_0^t Z_s(\mathfrak{A}G) \, ds \quad \text{and} \quad (M_t(G))^2 - \int_0^t \|\mathfrak{B}_sG\|^2 \, ds$$

are martingales with respect to the canonical filtration $\{\mathcal{F}_s : s \geq 0\}$ for all $G$ in $\mathcal{S}(\mathbb{R})$. Of course, it is enough to check these conditions for a dense family of functions in $\mathcal{S}(\mathbb{R})$.

Recall the definition of the class $\mathcal{S}_0(\mathbb{R})$ and fix a function $G$ in $\mathcal{S}_0(\mathbb{R})$. An elementary computation of the characteristic function $E_{\nu_{\rho_0}}(\exp\{i\theta Z_0^N(G)\})$ shows that $Z_0^N$ converges to a centered Gaussian field with covariances given by (4.5).

Recall from (1.2) the definition of the martingale $M_t^N(G)$ and fix a bounded function $U$ in $\mathcal{F}_s$. To prove that $M_t(G)$ is a martingale, it is enough to show that

$$\lim_{N \to \infty} E_{\nu_{\rho_0}}[M_t^N(G)U] = E[M_t(G)U]$$

for all $t \geq s$.

Since $Z^N_t(G)$ is bounded in $L^2(\mathcal{P}_{\nu_{\rho_0}}^N)$, (4.7) holds with $Z^N_t(G) - Z^N_0(G)$, $Z_t(G) - Z_0(G)$ in place of $M_t^N(G)$, $M_t(G)$. By Schwarz inequality and a previous estimate,

$$E_{\nu_{\rho_0}}\left[\left(\frac{T_{\xi,g}}{T_{\xi,g}} \int_0^t Y_s^N(\Delta N G) \, ds\right)^2\right] \leq \frac{Ct^2N}{T_{\xi,g}^2} \left(\frac{T_{\xi,g}}{T_{\xi,g}}\right)^2$$

vanishes as $N \uparrow \infty$ for all $\xi \in \Omega_0$. On the other hand, by Schwarz inequality, Theorem 6.1 and Lemma 4.1,

$$E_{\nu_{\rho_0}}\left[\left(\int_0^t \{Y_s^N(\Delta N G) - Z_s^N(\Delta N G)\} \, ds\right)^2\right]$$

$$\leq C t^{5/2} \left(\frac{1}{N} \sum_{x \in \mathbb{Z}} |\Delta N G(x/N) - \gamma^{-1}(T_{\xi,1}\Delta N G)(x)|^2\right)^2$$

$$+ t \frac{1}{N} \sum_{x \in \mathbb{Z}} \{\Delta N G(x/N) - \gamma^{-1}(T_{\xi,1}\Delta N G)(x)\}^2$$

vanishes as $N \uparrow \infty$ for all $\xi \in \Omega_0$. Replacing $\Delta N G$ by $\Delta G$ and recalling all previous estimates, we deduce that

$$\lim_{N \to \infty} E_{\nu_{\rho_0}}[U \int_0^t Y_s^N(\Delta G) \, ds] = \lim_{N \to \infty} E_{\nu_{\rho_0}}[U \int_0^t Y_s^N(\Delta G) \, ds]$$

$$= E[U \int_0^t Z_s(\mathfrak{A}G) \, ds]$$

because $Z^N_s(\Delta G)$ is bounded in $L^2$. This concludes the proof of (1.7).
To prove 4.1 with $m_t(G)$, $m_t^N(G) = M_t^N(G)^2 - \langle M^N(G) \rangle_t$ in place of $M_t(G)$, $M_t^N(G)$, observe first that $\mathbb{E}_{\nu_N} [M_t^N(G)^4]$ is finite in view of Theorem 6.1 so that $\mathbb{E}_{\nu_N} [UM_t^N(G)^2]$ converges to $E[U M_t(G)^2]$. To show that $\mathbb{E}_{\nu_N} [U (M_t^N(G))^2]$ converges to $E[U \int_0^t \| 2 \nu_s G \|^2 ds]$, notice that $\mathbb{E}_{\nu_N} [(M_t^N(G))^2]$ is bounded uniformly in $N$, for all $\xi$ in $\Omega_0$, and that $(M_t^N(G))^2$ can be written as

$$\int_0^t ds \frac{1}{\gamma^2} \sum_{x \in \mathbb{Z}} \xi_x^{-1} (\nabla_N G)(x)^2 (\eta^N_s(x + 1) - \eta^N_s(x))^2$$

plus a remainder which vanishes as $N \uparrow \infty$ for all $\xi$ in $\Omega_0$. By Theorem 6.1 and Schwarz inequality,

$$\int_0^t ds \frac{1}{\gamma^2} \sum_{x \in \mathbb{Z}} \xi_x^{-1} (\nabla_N G)(x)^2 \{ \rho^N_s(x) - \rho^N_s(x) \}$$

vanishes in $L^2$, as well as the same expression with $\tilde{\eta}^N_s(x) \tilde{\eta}^N_s(x + 1)$ in place of $\tilde{\eta}^N_s(x) = \eta^N_s(x) - \rho^N_s(x)$. The penultimate integral is thus equal to

$$\int_0^t ds \frac{1}{\gamma^2} \sum_{x \in \mathbb{Z}} \xi_x^{-1} (\nabla_N G)(x)^2 \{ \rho^N_s(x + 1) + \rho^N_s(x) - 2 \rho^N_s(x + 1) \rho^N_s(x) \}$$

plus a remainder which vanishes in $L^2(\mathbb{P}_{\nu_N}^\gamma)$. This shows that $\mathbb{E}_{\nu_N} [U (M_t^N(G))^2]$ converges to $E[U \int_0^t \| 2 \nu_s G \|^2 ds]$ and concludes the proof of uniqueness.

**Proof of Theorem 2.2** By Lemma 4.2 and Theorem 6.1 for each $t \geq 0$ and $G$ in $\mathcal{S}(\mathbb{R})$, $Z_t^N(G) - Y^N_t(G)$ vanishes in $L^2$ on $\xi$-almost surely as $N \uparrow \infty$. In particular, we may deduce from the central limit theorem for $Z^N_t$ the convergence of the finite dimensional distributions of $Y^N_t$.

We conclude this section with the

**Proof of Lemma 4.1** Fix $G$ in $\mathcal{S}(\mathbb{R})$ and recall that $\tilde{\xi}^{-1}_x = \xi^{-1}_x - \gamma$. For $N$ fixed, $T_{\xi} G(x)$ is well defined because $\sum_{-k \leq x \leq k} \tilde{\xi}^{-1}_x (\nabla_N G)(x/N)$ is a Cauchy sequence in $L^2(\mathbb{P})$.

By Doob’s inequality, for each $x < y$, $A > 0$,

$$P \left[ \max_{x < z \leq y} \left| \frac{1}{N} \sum_{w = x+1}^{z} \tilde{\xi}^{-1}_w (\nabla_N G)(w/N) \right| > A \right] \leq \frac{C_0(G) E[\tilde{\xi}^{-6}_x]}{A^6 N^3}$$

for some finite constant $C_0$ depending on $G$. Take $A = N^{-{(1/4) + \epsilon}}$ for some $0 < \epsilon < 1/12$, estimate the right hand side by $C_0(G) E[\tilde{\xi}^{-6}_x] N^{6\epsilon - 3/2}$ and let $y \uparrow \infty$ to conclude by Borel-Cantelli that

$$N^{1/4} \max_{x < z} \left| \frac{1}{N} \sum_{w = x+1}^{z} \tilde{\xi}^{-1}_w (\nabla_N G)(w/N) \right|$$

vanishes, as $N \uparrow \infty$, almost surely. This proves the first statement of the lemma.

Since

$$T_{\xi} G = \frac{1}{N} \sum_{x \in \mathbb{Z}} \tilde{\xi}^{-1}_x (\nabla_N G)(x/N) = \frac{1}{N} \sum_{x \in \mathbb{Z}} \tilde{\xi}^{-1}_x (\nabla_N G)(x/N),$$

$T_{\xi} G$ is absolutely bounded by $\sup_{x \in \mathbb{Z}} |T_{\xi} G(x) - \gamma G(x/N)|$ and the second claim of the lemma follows from the first one.
To prove the last two statements, notice that \((T_{\xi,t}G)(x) - \gamma G(x/N)\) is absolutely bounded by \(R_{\xi,G}^N(x) + |T_{\xi,t}G| \mathbf{1}\{0 < x < l_N \}\), where \(R_{\xi,G}^N(x)\) is equal to
\[
\frac{1}{N} \sum_{y<x} \hat{\xi}_y^{-1} \nabla_N G(y/N) \quad \text{for } x \leq 0, \quad \frac{1}{N} \sum_{y \geq x} \hat{\xi}_x^{-1} \nabla_N G(x/N) \quad \text{for } x > 0.
\]
In particular, by the first part of the proof and since \(l_N = N^{1/4}\), \(\sup_{x \in \mathbb{Z}} |T_{\xi,t}G(x) - \gamma G(x/N)|\), \(l_N T_{\xi,G}\) vanishes, as \(N \uparrow \infty\), \(\xi\)-almost surely. On the other hand, by Tchebychev and Hölder inequality,
\[
P\left( \frac{1}{N} \sum_{x \in \mathbb{Z}} R_{\xi,G}^N(x) > A \right) \leq \frac{C_0}{A^4 N^4} \sum_{x \in \mathbb{Z}} (1 + |x|^{3(1+\epsilon)}) E\left[ R_{\xi,G}^N(x)^4 \right]
\]
for some \(\epsilon > 0\) and some finite constant \(C_0 = C_0(\epsilon)\). Since \(G\) belongs to \(S(\mathbb{R})\), the previous expectation is less than or equal to \(C_0 E[\xi_0^{-4}]N^{-2}F_G(x/N)\) for some rapidly decreasing positive function \(F_G\). The left hand side is thus bounded above by \(C_0(\epsilon)E[\xi_0^{-4}]N^{3(1-\epsilon)}A^{-4}\). Choosing \(0 < \epsilon < 1/7\), \(A = N^{-\epsilon}\) we conclude the proof of the last statement of the lemma with a Borel-Cantelli argument. \(\Box\)

5. Central limit theorem for a tagged particle.

We prove in this section Theorem 2.6. Unless otherwise stated, we assume throughout this section that \(\rho_0\) is an initial condition with first derivative in \(L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})\) and second derivative in \(L^\infty(\mathbb{R})\), and that the environment satisfies the assumptions of the previous section. The proof follows closely the approach presented in [5]. We omit therefore some details.

We first consider the current through a bond. For each \(x \in \mathbb{Z}\), denote by \(J_{x,x+1}^N(t)\) the current over the bond \(\{x, x+1\}\) in the time interval \([0, t]\). This is the total number of particles which jumped from \(x\) to \(x+1\) minus the total number of particles which jumped from \(x+1\) to \(x\) in the time interval \([0, t]\).

The current \(J_{x,x+1}^N(t)\) can be related to the occupation variables \(\eta_t(x)\) through the formula
\[
J_{x-1,x}^N(t) - J_{x,x+1}^N(t) = \eta_t(x) - \eta_t(x).
\]
(5.1)
The first result states a law of large numbers for the current through a bond assuming that the environment satisfies condition (2.3).

Proposition 5.1. Consider a sequence \(\{\xi_x : x \in \mathbb{Z}\}\) satisfying (2.3) and a profile \(\rho_0 : \mathbb{R}_+ \to [0,1]\) satisfying the assumptions stated at the beginning of this section. For every \(\delta > 0\),
\[
\lim_{N \to \infty} \mathbb{P}_N^{\xi_0,\rho_0} \left[ \frac{J_{0,1}^N(t)}{N} + \int_0^t \gamma^{-1}(\partial_u \rho)(s,0) \, ds \right] > \delta \right] = 0 ,
\]
where \(\rho(t,u)\) is the solution of (2.1).

Proof. Fix \(a > 0\). Identity (5.1) and a summation by parts give that
\[
\frac{1}{aN^2} \sum_{x=1}^{aN} \xi_x^{-1} \left\{ J_{x,x+1}^N(t) - J_{0,1}^N(t) \right\} = \frac{1}{aN^2} \sum_{x=1}^{aN} \left\{ \eta_t^N(x) - \eta_t^N(0) \right\} \sum_{k=x}^{aN} \xi_k^{-1} .
\]
(5.2)
Since the right hand side is of order \(a\), the law of large numbers for \(J_{0,1}^N(t)/N\) follows from a law of large numbers for \(a^{-1}N^{-2} \sum_{x=1}^{aN} \xi_x^{-1} J_{x,x+1}^N(t)\). We may rewrite this...
latter expression as
\[
\frac{1}{aN^2} \sum_{x=1}^{aN} \xi_x^{-1} M^N_{x,x+1}(t) + \frac{1}{a} \int_0^t \{ \eta_t^N(1) - \eta_t^N(aN+1) \} \, ds ,
\] (5.3)
where
\[
M^N_{x,x+1}(t) = J^N_{x,x+1}(t) - N^2 \int_0^t \xi_x^N(x) - \xi_x^N(x+1) \, ds ,
\]
x in \( \mathbb{Z} \), are orthogonal martingales with quadratic variation \( \langle M^N_{x,x+1} \rangle_t \) given by
\[
N^2 \int_0^t \xi_x^N(x) - \xi_x^N(x+1) \, ds .
\]

In view of (2.3) and of the explicit expression for the quadratic variation of the orthogonal martingales \( M^N_{x,x+1} \), the first term in (5.3) vanishes in \( L^2(\mathbb{P}^N_{\rho_0(\cdot)}) \) as \( N \uparrow \infty \). On the other hand, by Theorem 6.1, the variance of the second term in (5.3) vanishes as \( N \uparrow \infty \). Its expectation is equal to
\[
\frac{1}{a} \int_0^t \{ \rho_{x-}(1) - \rho_{x+}(aN+1) \} \, ds .
\]

By Lemma 6.6, this integral converges to \( a^{-1} \int_0^t \{ \rho_{x}(0) - \rho_{x}(a) \} \, ds \), where \( \rho \) is the solution of (2.1). It remains to let \( a \downarrow 0 \) to conclude the proof. \( \square \)

We prove now a quenched nonequilibrium central limit theorem for the current. Let \( \bar{J}^N_{x,x+1}(t) = J^N_{x,x+1}(t) - E^N_{\rho_0(\cdot)}[J^N_{x,x+1}(t)] \).

**Proposition 5.2.** There exists a total measure set \( \Omega_0 \subset \Omega \) with the following property. For each \( \xi \) in \( \Omega_0 \), each \( k \geq 1 \) and each \( 0 \leq t_1 < \cdots < t_k \), the random vector \( N^{-1/2}(\bar{J}^N_{1,0}(t_1), \ldots, \bar{J}^N_{k,0}(t_k)) \) converges in law to a Gaussian vector \( (J_{t_1}, \ldots, J_{t_k}) \) with covariances given by
\[
E[J_sJ_t] = \int_{-\infty}^0 dv \, P[B_s \leq v] P[B_t \leq v] \chi(\rho_0(v)) + \int_0^\infty dv \, P[B_s \geq v] P[B_t \geq v] \chi(\rho_0(v)) + 2\gamma^{-1} \int_0^s dv \int_{-\infty}^v dr \, P(t_{s-r}(0,v)p_{s-r}(0,v) \chi(\rho(r,v))
\]
provided \( s \leq t \). In this formula, \( B_t = B_t^{0/\gamma} \), where \( B_t^0 \) is a standard Brownian motion starting from the origin, and \( p_t(v,w) \) is the kernel of \( B_t \).

**Proof.** The proof of this proposition is similar to the one of Theorem 2.3 in [S]. Some details are therefore omitted.

Let \( H_0(u) = 1\{ u \geq 0 \} \) and define the sequence \( \{ G_n : n \geq 1 \} \) of approximations of \( H_0 \) by
\[
G_n(u) = \{ 1 - (u/n) \}^+1\{ u \geq 0 \} .
\]
We claim that for every \( t \geq 0 \),
\[
\lim_{n \to \infty} E^N_{\rho_0(\cdot)} \left[ N^{-1/2} \bar{J}_{-1,0}(t) - Y_t^N(G_n) + Y_0^N(G_n) \right]^2 = 0 \quad (5.4)
\]
uniformly in $N$. The proof of \eqref{5.3} relies on the estimates of the two point space-time correlation functions, presented in Lemma \ref{2.2} and follows closely the proof of Proposition 3.1 in \cite{8}. We leave the details to the reader.

Fix $t \geq 0$ and $n \geq 1$. By approximating $G_n$ in $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ by a sequence \{\(H_{n,k} : k \geq 1\)\} of smooth functions with compact support, recalling Theorem \ref{2.6} we show that $Y^N_t(G_n)$ converges in law to a Gaussian variable denoted by $Y_t(G_n)$.

By \eqref{5.4}, \{\(Y^N_t(G_n) - Y^N_0(G_n) : n \geq 1\)\} is a Cauchy sequence uniformly in $N$. In particular, $Y^N_t(G_n) - Y^N_0(G_n)$ is a Cauchy sequence and converges to a Gaussian limit denoted by $Y_t(H_0) - Y_0(H_0)$. Therefore, by \eqref{5.3}, $N^{-1/2} \mathcal{J}_{-1,0}(t)$ converges in law to $Y_t(H_0) - Y_0(H_0)$.

The same argument show that any vector $N^{-1/2}(\mathcal{J}_{-1,0}(t_1), \ldots, \mathcal{J}_{-1,0}(t_k))$ converges in law to $(Y_{t_1}(H_0) - Y_0(H_0), \ldots, Y_{t_k}(H_0) - Y_0(H_0))$. The covariances can be computed since by \eqref{2.6}

$$E \left[ \{Y_t(H_0) - Y_0(H_0)\} \{Y_s(H_0) - Y_0(H_0)\} \right]$$

$$= \lim_{n \to \infty} E \left[ \{Y_t(G_n) - Y_0(G_n)\} \{Y_s(G_n) - Y_0(G_n)\} \right]$$

$$= \lim_{n \to \infty} \left\{ \int_{\mathbb{R}} \left\{ (T_tG_n)(T_sG_n) + G_n^2 - (T_tG_n)G_n - (T_sG_n)G_n \right\} \chi(\rho(u)) + 2\gamma^{-1} \int_{0}^{s} dr \left( \nabla T_{-r}G_n \nabla T_{s-r}G_n \chi(\rho(r,u)) \right) \right\}.$$

A long but elementary computation permits to recover the expression presented in the statement of the proposition. \hfill \Box

We turn now to the behavior of a tagged particle. Let $\nu^{N,*}_{\rho_0(\cdot)}$ be the product measure $\nu^N_{\rho_0(\cdot)}$ conditioned to have a particle at the origin. All our previous results for the process starting from $\nu^N_{\rho_0(\cdot)}$ remain in force for the process starting from $\nu^{N,*}_{\rho_0(\cdot)}$, since we can couple both processes in such a way that they differ at most at one site at any given time.

Denote by $X^N_t$ the position at time $t \geq 0$ of the particle initially at the origin. Since the relative ordering of particles is conserved by the dynamics, a law of large numbers for $X^N_t$ is a consequence of the hydrodynamic limit and the law of large numbers for the current \cite{17, 13, 8}. In fact, the distribution of $X^N_t$ can be obtained from the joint distribution of the current and the empirical measure via the relation

$$\{X^N_t \geq n\} = \{J_{-1,0}(t) \geq \sum_{x=0}^{n-1} \eta_t(x)\} \quad (5.5)$$

for all $n \geq 0$ and a similar relation for $n \leq 0$.

**Theorem 5.3.** Consider a sequence $\{\xi_x : x \in \mathbb{Z}\}$ satisfying \ref{2.6} and a profile $\rho_0 : \mathbb{R}^+ \to [0,1]$ satisfying the assumptions presented at the beginning of this section. For every $t \geq 0$, $X_t/N$ converges in $\mathbb{P}_{\nu^N_{\rho_0(\cdot)}}$-probability to $u_t$, the solution of

$$\int_{0}^{u_t} \rho(t,u)du = -\frac{1}{\gamma} \int_{0}^{t} (\partial_0 \rho)(s,0)ds. \quad (5.6)$$
Notice that \( u_t \) satisfies the differential equation
\[
\dot{u}_t = \frac{1}{\gamma} \frac{(\partial_u \rho)(t, u_t)}{\rho(t, u_t)}.
\]

The proof of this result is similar to the one of Theorem 2.5 in [8] and left to the reader.

It remains to prove a central limit theorem for the position of the tagged particle. **Proof of Theorem 2.3.** Recall the definition of \( u_t^N \) presented just before the statement of the theorem, assume that \( u_t^N > 0 \) and fix \( a \in \mathbb{R} \). By equation (5.5), the set \( \{X_t \geq u_t^N + a\sqrt{N} \} \) is equal to the set in which
\[
\mathcal{J}_{-1,0}(t) \geq \sum_{x=0}^{u_t^N} \eta_t(x) + \sum_{x=1}^{a\sqrt{N}-1} \eta_t(x + u_t^N) - \left\{ \mathbb{E}_{\nu_0^{N,*,1}}[\mathcal{J}_{-1,0}(t)] - \sum_{x=0}^{u_t^N} \rho_t^{N,*,1}(x) \right\}.
\]

We claim that second term on the right hand side of this equation divided by \( \sqrt{N} \) converges to its mean in \( L^2 \). Indeed, by Theorem 6.1 its variance is bounded by \( C_0(\varepsilon, \rho_0)\alpha N^{-1/2} \) for some finite constant \( C_0 \). Notice that we are taking expectations with respect to a measure, \( \nu_0^{N,*,1} \), whose associated profile does not have a bounded first derivative. However, coupling this measure with \( \nu_0^{N,*,1} \), in such a way that they differ at most by one particle at every time, we can still show that the variance is bounded by \( C_0(\varepsilon, \rho_0)\alpha N^{-1/2} \) as claimed. The same ideas, the linearity of equation (2.4) and Nash estimate, stated in Proposition 6.2 below, show that \( \rho_t^{N,*,1} \) converges uniformly on compact sets to the the solution of the heat equation (2.1) because \( \rho_t^N \) converges in view of Lemma 6.6.

To compute the expectation of the second term on the right hand side of (5.7), observe that the middle term in (5.7) is equal to \( \mathbb{E}_{\nu_0^{N,*,1}}[\mathcal{J}_{-1,0}(t)] \). By the proof of the law of large numbers for the current, this middle expression divided by \( N \) converges to \( -\gamma^{-1} \int_0^t (\partial_u \rho)(s, 0) ds \). In particular, by the law of large numbers for the empirical measure and by relation (5.6), \( N^{-1} u_t^N \) converges to \( u_t \). Hence, by the uniform convergence of \( \rho_t^{N,*,1} \),
\[
\frac{1}{\sqrt{N}} \sum_{x=1}^{a\sqrt{N}-1} \rho_t^{N,*,1}(x + u_t^N)
\]
converges to \( a\rho(t, u_t) \) and so does in probability the second term on the right hand side of (5.7).

By definition of \( u_t^N \), the third term on the right hand side is absolutely bounded by 1.

Finally, by (5.4), for fixed \( t, N^{-1/2} \{\mathcal{J}_{-1,0}(t) - \sum_{x=0}^{u_t^N} \eta_t(x) \} \) behaves as \( Y_t^N(G_n) - Y_0^N(G_n) - Y_t^N(\{0, u_t^N/N\}) \), as \( N \uparrow \infty, n \uparrow \infty \). Repeating the arguments presented in the proof of Proposition 5.2 since \( u_t^N/N \) converges to \( u_t \), we show that this latter variable converges in law to a centered Gaussian variable, denoted by \( W_t \), and which is formally equal to \( Y_t(H_{u_t}) - Y_0(H_0) \), where \( H_a(u) = 1\{u \geq a\} \).

Up to this point we proved that
\[
\lim_{N \to \infty} \mathbb{P}_{\nu_0^{N,*,1}} \left[ \frac{X_t^N - u_t^N}{\sqrt{N}} \geq a \right] = P[W_t \geq a \rho(t, u_t)]
\]
provided $u_t > 0$. Analogous arguments permit to prove the same statement in the case $u_t = 0$, $a > 0$. By symmetry around the origin, we can recover the other cases: $u_t < 0$ and $a$ and $a < 0$.

Putting all these facts together, we conclude that for each fixed $t$, $(X_t - u_t^N)/\sqrt{N}$ converges in distribution to the Gaussian $W_t/\rho(t, u_t) = [Y_t(H_{u_t}) - Y_0(H_0)]/\rho(t, u_t)$. The same arguments show that any vector $(N^{-1/2}[X_{t_1} - u_{t_1}^N], \ldots, N^{-1/2}[X_{t_k} - u_{t_k}^N])$ converges to the corresponding centered Gaussian vector. It remains to compute the covariances, which can be derived as in the proof of Proposition 5.2. Details are left to the reader. □

6. Correlation estimates

We assume throughout this section that $\{\xi_e : e \in Z\}$ is a sequence of numbers bounded below and above: $0 < \varepsilon < \xi_e < \varepsilon^{-1}$ for all $e$, and that the profile $\rho_0 : \mathbb{R} \to [0, 1]$ has bounded first derivative. Recall that $\rho_1^N(x) = \mathbb{E}_{\rho^N} [\eta(x)]$ satisfies equation (2.4).

For $n \geq 1$, denote by $\mathcal{E}_n$ the subsets of $Z$ with $n$ points. For each $x_n = \{x_1, \ldots, x_n\}$ in $\mathcal{E}_n$, let

$$\varphi_t(x_n) = \mathbb{E}_{\rho^N} \left[ \prod_{i=1}^{n} \{\eta(x_i) - \rho_1^N(x_i)\} \right].$$

**Theorem 6.1.** Fix a finite time interval $[0, T]$ and an initial profile $\rho_0$ with bounded first derivative. There are constants $C_n$, depending only on $\varepsilon$, $\rho_0$, $n$ and $T$, such that

$$\sup_{\mathcal{E}_n} \sup_{t \in [0, T]} |\varphi_t(x_{2n})| \leq \frac{C_{2n}}{n} \quad \text{and} \quad \sup_{\mathcal{E}_n} \sup_{t \in [0, T]} |\varphi_t(x_{2n+1})| \leq \frac{C_{2n+1} \log N}{N^{n+1}}.$$

The proof of this theorem follows closely the proof of [3] for the simple exclusion process without environment. We start with a Nash estimate for the transition probability of a random walk in elliptic environment [11], [2], [3]. Denote by $\mathcal{L}_1$ the generator of a random walk in the bond environment $\xi$:

$$(\mathcal{L}_1 f)(x) = \xi_x \left( f(x+1) - f(x) \right) + \xi_{x'} \left( f(x+1) - f(x) \right).$$

Let $p_t^\xi(x, y)$ be the transition probability associated to the generator $\mathcal{L}_1$.

**Proposition 6.2.** There exists a finite constant $C_0(\varepsilon)$, depending only on $\varepsilon$, such that $p_t^\xi(x, y) \leq C_0(\varepsilon)N^{-1/2} + C_0(\varepsilon)N^{-1/2}$ for all $x, y$ in $\mathbb{Z}$, $t \geq 0$.

The proof of Theorem 6.2 relies also on a comparison between the semigroup associated to the evolution of $n$ exclusion particles with the semigroup associated to $n$ independent particles. For $n \geq 1$, denote by $\mathcal{L}_n$ the generator corresponding to the evolution of $n$ exclusion particles in the environment $\xi$:

$$(\mathcal{L}_n h)(x_n) = \sum_{i=1}^{n} \mathbf{1}_{\{x + e_i \in \mathcal{E}_n\}} \xi_{x_i} [h(x_n + e_i) - h(x_n)]$$

$$+ \sum_{i=1}^{n} \mathbf{1}_{\{x - e_i \in \mathcal{E}_n\}} \xi_{x_{i-1}} [h(x_n - e_i) - h(x_n)]$$

for every function $h : \mathcal{E}_n \to \mathbb{R}$. In this formula, for $1 \leq i \leq n$, $e_i$ stands for the $i$-th canonical vector in $\mathbb{R}^n$ and $x_n$ is understood as the vector $(x_1, \ldots, x_n)$. Denote by
$S_n(t)$ the semigroup associated to $\mathcal{L}_n$ and by $S^0_n(t)$ the semigroup associated to $n$ independent particles evolving in the environment $\xi$.

A bounded symmetric function $f : \mathbb{Z}^2 \to \mathbb{R}$ is said to be definite positive provided

$$\sum_{x,y} f(x,y) \beta_x \beta_y \geq 0$$

for every sequence $\{\beta_x : x \in \mathbb{Z}\}$ in $\ell_1(\mathbb{Z})$ with $\sum_x \beta_x = 0$. A bounded symmetric function $f : \mathbb{Z}^n \to \mathbb{R}$ is said to be definite positive if it is so for each pair of coordinates. From [14] we have that

**Proposition 6.3.** Let $f : \mathbb{Z}^n \to \mathbb{R}$ be a bounded, symmetric, definite positive function. Then,

$$S_n(t)f \leq S^0_n(t)f$$

for all $t \geq 0$.

Theorem 6.1 is based on an induction argument. Observe first that

$$\frac{d}{dt} \varphi_t(x_n) = (\mathcal{L}_n \varphi_t)(x_n) + \Gamma_t(x_n), \quad (6.1)$$

where

$$\Gamma_t(x_n) = 2N^2 \sum_{x \in \mathbb{Z}, x \neq x_n} \xi_x [\varphi_t(x_n^{x+1}) - \varphi_t(x_n^x)] [\rho^N_t(x+1) - \rho^N_t(x)]$$

$$- N^2 \sum_{x \in \mathbb{Z}} \xi_x \varphi_t(x_n^{x,x+1}) [\rho^N_t(x+1) - \rho^N_t(x)]^2.$$

Here and below $x_n^y$, $x_n^{y,z}$ stand for the configuration $x_n \setminus \{y\}$, $x_n \setminus \{y, z\}$, respectively.

In view of the differential equation (6.1), we can represent $\varphi_t(x_n)$ as an expectation with respect to a random walk in an environment $\xi$ with sources at the boundary $\partial \mathcal{E}_n = \{x_n \in \mathcal{E}_n; \min_{i \neq j} |x_i - x_j| = 1\}$: Denote by $\mathbb{E}_{x_n}$ (resp. $\mathbb{E}^0_{x_n}$) the expectation with respect to $n$ exclusion (resp. independent) particles starting at $x_n$. Since $\varphi_0(x_n) = 0$, we have that

$$\varphi_t(x_n) = \int_0^t ds \mathbb{E}_{x_n}[\Gamma_s(x_n(t - s))]. \quad (6.2)$$

Since $\varphi_t(x) = 0$ for all $x$ in $\mathbb{Z}$, $t \geq 0$, to start the induction argument, set $n = 2$ and remark that the first term in the definition of $\Gamma_t$ vanishes. On the other hand, by [6,3] below, the derivative $\nabla_N \rho^N_t$ is uniformly bounded. Since the environment is elliptic and $\varphi_t(\phi) = 1$, $\Gamma_t(x)$ is absolutely bounded by $C_0(\varepsilon, \rho_0)1\{x_2 \in \partial \mathcal{E}_2\}$ for some finite constant $C_0$. The function $f : \mathbb{Z}^2 \to \mathbb{R}$ defined by $f(x,y) = 2 \mathbf{1}\{x = y\} + \mathbf{1}\{|x-y| = 1\}$ is bounded, symmetric and definite positive. Therefore, by Proposition 6.3 by the integral representation (6.2) of $\varphi_t$ and by the previous estimate of $\Gamma_t$,

$$\varphi_t(x,y) \leq C(\varepsilon, \rho_0) \int_0^t \mathbb{E}^0_{(x,y)}[f(x_2(s))] ds.$$

It remains to apply Proposition 6.2 and to integrate in time, keeping in mind that time is speeded up by $N^2$, to obtain that

$$\sup_{x \neq y \in \mathbb{Z}} |\varphi_t(x,y)| \leq \frac{C(\varepsilon, \rho_0) \sqrt{t}}{N^2} \leq \frac{C(\varepsilon, \rho_0, T)}{N}.\quad (6.3)$$
for all $0 \leq t \leq T$.

To extend this estimate to $n \geq 3$, we need to exploit the non-trivial cancellations in the first term of the definition of $\Gamma_t$. For $n \geq 1$, let

$$A_t^n = \sup_{x_n \in \mathcal{E}_n} |\varphi_t(x_n)|,$$
$$B_t^n = \sup_{x \in \mathbb{Z}} \sup_{x_{n-1} \in \mathcal{E}_{n-1}} |\varphi_t(x_{n-1} \cup \{x\}) - \varphi_t(x_{n-1} \cup \{x+1\})|.$$

We claim that there exists a finite sequence of constants $C(\varepsilon, \rho_0, n)$, $n \geq 2$, such that

$$A_t^n \leq C(\varepsilon, \rho_0, n) \int_0^t \left\{ NB_s^{n-1} + A_s^{n-2} \right\} \frac{ds}{N\sqrt{t-s}}; \quad (6.3)$$
$$B_t^n \leq C(\varepsilon, \rho_0, n) \int_0^t \left\{ NB_s^{n-1} + A_s^{n-2} \right\} \frac{ds}{1 + N^2(t-s)}.$$

Theorem 6.1 follows from these bounds and elementary computations.

It remains to prove the estimates (6.3). The first one is simpler and follows the same steps presented for $n = 2$. Fix $n \geq 3$ and a configuration $x_n$ in $\mathcal{E}_n$. Assume that the particles are evolving according to a stirring process. By (6.2) and by definition of $A_t^n$, $B_t^n$,

$$|\varphi_t(x_n)| \leq \int_0^t ds \mathbb{E}_{x_n} \left[ |\Gamma_s(x_n(t-s))| \right]$$
$$\leq C(\varepsilon, \rho_0, n) \int_0^t ds \mathbb{P}_{x_n} \left[ x_n(t-s) \in \partial \mathcal{E}_n \right] \left\{ NB_s^{n-1} + A_s^{n-2} \right\}$$

for some finite constant $C(\varepsilon, \rho_0, n)$. Since $x_n(t-s)$ belongs to the boundary of $\mathcal{E}_n$, there are at least two particles at distance one. By definition of the stirring process, any pair of particles evolves according to a symmetric exclusion process in the environment $\xi$. In particular, comparing the original process with independent particles and applying Nash estimate, we can bound the probability appearing in the last displayed formula by $C\{N^2(t-s)\}^{-1/2}$. This proves the first estimate in (6.3).

We now turn to $B_t^n$. Since $B^n_1 = 0$, fix $n \geq 2$, $x$ in $\mathbb{Z}$ and $x_{n-1}$ in $\mathcal{E}_{n-1}$ such that $x$, $x+1 \notin x_{n-1}$. Consider $n+1$ particles evolving on $\mathbb{Z}$ according to the following rules. They start from $x_{n-1}$, $x$, $x+1$ and evolve according to a stirring process. However, when the particles starting at $x$ and $x+1$ are at distance 1, each one jumps, independently from the other, to the site occupied by the other at the rate determined by the environment. Once these particles occupy the same site, they remain together for ever. Notice that the two distinguished particles behave until they meet exactly as two independent particles.

Denote by $\mathbb{P}_{x_{n-1}, x, x+1}$, $\mathbb{E}_{x_{n-1}, x, x+1}$ the probability and the expectation corresponding to the evolution just described. Let $\tau$ be the coalescence time of the distinguished particles and let $x_n(t, x)$, $x_n(t, x+1)$ be the configuration at time $t$ of the system starting from $x_{n-1} \cup \{x\}$, $x_{n-1} \cup \{x+1\}$, respectively. By construction,
\( x_n(t, x) = x_n(t, x + 1) \) for \( t \geq \tau \). In particular,
\[
\varphi_t(x_n \cup \{x\}) - \varphi_t(x_n \cup \{x + 1\}) = \int_0^t ds \mathbb{E}_{x, x+1} [\Gamma_s(x_n(t, x)) - \Gamma_s(x_n(t, x + 1))] = \int_0^t ds \mathbb{E}_{x, x+1} [\mathbf{1}\{\tau > t-s\} [\Gamma_s(x(t-s, x)) - \Gamma_s(x(t-s, x + 1))]].
\]

By definition of \( A^k \) and \( B^k \), this expression is less than or equal to
\[
C_0 \sum_{y=x+1} \int_0^t ds \{NB_s^{n-1} + A_s^{n-2}\} \mathbb{P}_{x, x+1} [\tau > t-s, x_n(t-s, y) \in \partial E_n]
\]
for some finite constant \( C_0 = C_0(\varepsilon, \rho_0, n) \). In view of Nash estimate, replacing the indicator function \( \mathbf{1}\{\tau > t-s\} \) by \( \mathbf{1}\{\tau > (t-s)/2\} \) and applying the Markov property at time \((t-s)/2\), we bound the previous expression by
\[
C_0 \int_0^t ds \{NB_s^{n-1} + A_s^{n-2}\} \frac{1}{\sqrt{1 + N^2(t-s)}} \mathbb{P}_{x, x+1} [\tau > (t-s)/2].
\]

By Lemma 6.4 below, the probability appearing in the previous formula is bounded above by \( C(\varepsilon) \{1 + N^2(t-s)\}^{-1/2} \). This concludes the proof of estimate (6.3) and the one of Theorem 6.1.

Let \( x_t \) be a random walk in the environment \( \{\xi_x : x \in \mathbb{Z}\} \) starting from \( x_0 = 0 \). Denote by \( P \) the probability measure on the path space \( D(\mathbb{R}_+, \mathbb{Z}) \) induced by \( x_t \). For each \( a \neq 0 \), let \( \tau_a \) be the first time the random walk \( x_t \) reaches \( a \):
\[
\tau_a =: \inf\{t \geq 0; x_t = a\}.
\]

**Lemma 6.4.** There exists a finite constant \( C_0 = C_0(\varepsilon) \), depending only \( \varepsilon \), such that
\[
P(\tau_a > t) \leq \frac{C_0 a}{\sqrt{1 + t}}
\]
for all \( t > 0 \).

**Proof.** Define the function \( u : \mathbb{Z} \to \mathbb{R} \) by \( u(0) = 0 \), \( u(x+1) - u(x) = \xi_x^{-1} \). Since the environment is elliptic, \( \varepsilon \leq u(x)/x \leq \varepsilon^{-1} \) for all \( x \neq 0 \). Moreover, an elementary computation shows that \( u(x_t) \) is a martingale of quadratic variation \( \langle u(x) \rangle_t \) given by
\[
\int_0^t (\xi_{x_t}^{-1} + \xi_{x_t}^{-1}) ds.
\]
Fix \( b < 0 < a \) and set \( \tau = \min\{\tau_a, \tau_b\} \). By Doob’s optional sampling theorem, \( E[u(x_\tau)] = 0 \) and \( E[u(x_\tau)^2 - (u(x))_\tau] = 0 \). Therefore,
\[
P(\tau_a < \tau_b) = \frac{-u(b)}{u(a) - u(b)} - u(a)u(b) = E \int_0^\tau (\xi_{x_t}^{-1} + \xi_{x_t}^{-1}) ds,
\]
so that \( E[\tau] \leq -u(a)u(b)(2\varepsilon)^{-1} \). In particular,
\[
P(\tau_a > t) \leq P(\tau > t) + P(\tau_a > \tau_b) \leq \frac{-u(a)u(b)}{2\varepsilon t} + \frac{u(a)}{u(a) - u(b)}.
\]
Minimizing over \( b > 0 \) we conclude the proof of the lemma.
The same ideas provide a bound on the coalescence time of two independent particles in the environment $\xi$. Fix $x$ in $\mathbb{Z}$ and consider two independent random walks $X_t$, $Y_t$, on the environment $\xi$ such that $X_0 = x$, $Y_0 = x + 1$. For $b > 0$, let $\tau^*$, $\tau_0$ be the first time such that $Y_t = X_t$, $Y_t = Y_t + b$, respectively.

Recall the definition of the function $u$ defined in the proof of Lemma 6.4. Since $X_t$, $Y_t$ are independent, $M_t = u(Y_t) - u(X_t) - 1$ is a martingale. Repeating the arguments presented in the proof of Lemma 6.4, we obtain that

$$P(\tau^* > t) \leq \frac{C_0}{\sqrt{1 + t}}$$

for all $t > 0$ and some finite constant $C_0$ depending only on $\varepsilon$. Of course, when the time is speeded up by $N^2$, $t$ is replaced by $tN^2$.

A bound on the space-time correlations can be deduced from Theorem 6.1. For $x$, $y$ in $\mathbb{Z}$ and $s \leq t$, let

$$\psi_{s,t}(y;x) = \mathbb{E}_{\nu_N} \left[ \{ \eta_s(y) - R^s (y) \} \{ \eta_t(x) - R^t (x) \} \right].$$

**Lemma 6.5.** There exists a finite constant $C_0$, depending only on $\varepsilon$, $\rho_0$ such that

$$\sup_{x,y \in \mathbb{Z}} |\psi_{s,t}(y;x)| \leq \frac{C_0}{N} \left\{ \sqrt{s} + \frac{1}{\sqrt{t-s}} \right\}$$

for all $0 \leq s \leq t$.

**Proof.** Fix $s \geq 0$ and $y$ in $\mathbb{Z}$. For $t \geq s$, $x$ in $\mathbb{Z}$, let $\psi_t(x) = \psi_{s,t}(y;x)$. Notice that $\psi_t$ satisfies the Cauchy problem

$$\begin{cases} \frac{d}{dt} \psi_t(x) = L_1 \psi_t(x) \\ \psi_s(x) = 1 \{x \neq y\} \varphi_s(x,y) + 1 \{x = y\} \varphi_t(y)(1 - \varphi_s(y)) \end{cases},$$

where $L_1$ is the generator defined at the beginning of this section. It remains to recall Nash estimate for the semigroup and the proof of Theorem 6.1 in which we showed that $\varphi_s(x,y)$ is bounded by $C \sqrt{s}/N$.

We conclude this section with a result on the solution of the discrete linear equation (6.3).

**Lemma 6.6.** Let $\rho_0 : \mathbb{R} \to [0,1]$ be a profile whose first derivative $\rho_0'$ belongs to $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and whose second derivative $\rho_0''$ belongs to $L^\infty(\mathbb{R})$. The solution $\rho^N_t$ of equation (6.3) converges uniformly on compact sets of $\mathbb{R} \times \mathbb{R}$ to the solution of (2.1). In particular, for all $t \geq 0$ and all function $G$ in $C^1_0(\mathbb{R})$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{x \in \mathbb{Z}} G(x/N) \rho^N_t(x) = \int G(u) p(t,u) du,$$

$$\lim_{N \to \infty} \frac{1}{N} \sum_{x \in \mathbb{Z}} G(x/N) (\nabla_N \rho^N_t)(x) = \int G(u) (\partial_u p)(t,u) du,$$

where $p_t(u)$ is the solution of linear heat equation (2.1).

**Proof.** Consider the initial condition $\rho^\xi_0 = \rho_0^{N,\xi} : \mathbb{Z} \to \mathbb{R}$ defined by $\rho^\xi_0(0) = \rho^{N,\xi}(0)$,

$$\nabla_N \rho^\xi_0)(x) = \xi_x^{-1}(\nabla N \rho^{N,\xi})(x).$$

By the estimates presented at the beginning of Section 4, $\rho^\xi_0(x) - \gamma \rho_0(x)$ vanishes, as $N \uparrow \infty$, uniformly in $x$. 

Denote by $\rho^\xi_t = \rho^{N,\xi}_t$ the solution of equation (2.4) with initial condition $\rho^\xi_0$. We claim that the sequence $\{\rho^{N,\xi}_t : N \geq 1\}$ is equicontinuous on each compact set of $\mathbb{R}_+ \times \mathbb{R}$. The proof relies on uniform bounds of $\rho^\xi_t$, $\nabla_N \rho^\xi_t$, $(d/dt)\rho^\xi_t$.

First of all, by the maximum principle,
\[
\inf_{x \in \mathbb{Z}} \rho^\xi_0(x) \leq \inf_{x \in \mathbb{Z}} \rho^\xi_t(x) \leq \sup_{x \in \mathbb{Z}} \rho^\xi_t(x) \leq \sup_{x \in \mathbb{Z}} \rho^\xi_0(x).
\]

Denote by $\nabla_\xi$ the discrete derivative defined by $(\nabla_\xi h)(x) = N\xi \{h(x+1) - h(x)\}$. $(\nabla_\xi \rho^\xi_t)$ satisfies the equation
\[
\frac{d}{dt}(\nabla_\xi \rho^\xi_t)(x) = N^2 \xi \{ (\nabla_\xi \rho^\xi_t)(x) + (\nabla_\xi \rho^\xi_t)(x+1) - 2(\nabla_\xi \rho^\xi_t)(x) \}.
\]

In particular, $\nabla_\xi \rho^\xi_t$ satisfies the maximum principle and is uniformly bounded because we assumed the initial condition to have a bounded derivative.

Let $\dot{\rho}^\xi_t = (d/dt)\rho^\xi_t$. By definition,
\[
\dot{\rho}^\xi_t(x) = N \{ (\nabla_\xi \rho^\xi_t)(x) - (\nabla_\xi \rho^\xi_t)(x-1) \} = (\nabla_N \nabla_\xi \rho^\xi_t)(x-1).
\]

Since
\[
\frac{d}{dt} \rho^\xi_t(x) = N^2 \xi \left[ (\nabla_N \nabla_\xi \rho^\xi_t)(x) - (\nabla_N \nabla_\xi \rho^\xi_t)(x-1) \right] - \xi_{x-1} \left[ (\nabla_N \nabla_\xi \rho^\xi_t)(x-1) - (\nabla_N \nabla_\xi \rho^\xi_t)(x-2) \right],
\]

$\dot{\rho}^\xi_t(x) = (\nabla_N \nabla_\xi \rho^\xi_t)(x-1)$ satisfies a maximum principle. By definition of $\rho^\xi_0$, $\nabla_\xi$,
\[
(\nabla_N \nabla_\xi \rho^\xi_0)(x-1) = N \{ (\nabla_\xi \rho^\xi_0)(x) - (\nabla_\xi \rho^\xi_0)(x-1) \} = (\Delta_N \rho_0)(x)
\]

In particular, $(d/dt)\rho^\xi_t(x)$ is uniformly bounded because we assumed the initial condition to have bounded second derivative.

Notice that the previous bound does not hold for the initial condition $\rho_0$ since $\nabla_N \nabla_\xi \rho_0$ is of order $N$. This explains the introduction of $\rho^\xi_0$.

The estimates just obtained prove the equicontinuity of the sequence $\{\rho^{N,\xi}_t : N \geq 1\}$ on each compact set of $\mathbb{R}_+ \times \mathbb{R}$. Since every limit point is a weak solution of the heat equation, by uniqueness of weak solutions, $\rho^\xi_t$ converges uniformly on compact sets to the solution of (2.1) with initial condition $\gamma \rho_0$.

Since $\rho^\xi_0 - \gamma \rho_0$ converges uniformly to 0, by the maximum principle, $\rho^\xi_N$ converges uniformly on compact sets to the solution of (2.1). This concludes the proof of the lemma.

Let $h_t(x) = \xi_x (\nabla_N \rho^N_t)(x)$. A simple computation shows that $(d/dt)h_t(x) = \xi_x (\Delta_N h_t)(x)$. Hence, $h_t$ satisfies a maximum principle and
\[
\sup_{t \geq 0} \sup_{x \in \mathbb{Z}} |(\nabla_N \rho_t)(x)| \leq \sup_{x \in \mathbb{Z}} |(\nabla_N \rho_0)(x)| \quad (6.5)
\]

because $\nabla_N \rho^N_t$ is absolutely bounded above and below by $C(\epsilon)|h_t|$.

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