Abstract. We present a new on-line algorithm for computing the Lempel-Ziv factorization of a string that runs in $O(N \log N)$ time and uses only $O(N \log \sigma)$ bits of working space, where $N$ is the length of the string and $\sigma$ is the size of the alphabet. This is a notable improvement compared to the performance of previous on-line algorithms using the same order of working space but running in either $O(N \log^3 N)$ time (Okanohara & Sadakane 2009) or $O(N \log^2 N)$ time (Starikovskaya 2012). The key to our new algorithm is in the utilization of an elegant but less popular index structure called Directed Acyclic Word Graphs, or DAWGs (Blumer et al. 1985). We also present an opportunistic variant of our algorithm, which, given the run length encoding of size $m$ of a string of length $N$, computes the Lempel-Ziv factorization on-line, in $O\left(m \cdot \min\left\{\frac{\log \log m \log \log N}{\log \log \log m}, \sqrt{\frac{\log m}{\log \log m}}\right\}\right)$ time and $O(m \log N)$ bits of space, which is faster and more space efficient when the string is run-length compressible.

1 Introduction

The Lempel-Ziv (LZ) factorization of a string [18], discovered over 35 years ago, captures important properties concerning repeated occurrences of substrings in the string, and has numerous applications in the field of data compression, compressed full text indices [11], and is also the key component to various efficient algorithms on strings [10,6]. Therefore, a large amount of work has been devoted to its efficient computation, especially in the off-line setting where the text is static, and the LZ factorization can be computed in as fast as $O(N)$ time assuming an integer alphabet, using $O(N \log N)$ or less bits of space (See [1] for a survey; more recent results are [12,18]). There is much less work for the on-line setting, where new characters may be appended to the end of the string. If we may use $O(N \log N)$ bits of space, the problem can be solved in $O(N \log \sigma)$ time where $\sigma$ is the size of the alphabet, by use of string indices such as suffix trees [17] and on-line algorithms to construct them [16]. However, when $\sigma$ is small and $N$ is very large (e.g. DNA), the $O(N \log N)$ bits space complexity is much larger than the $N \log \sigma$ bits of the input text, and can be prohibitive. To solve this problem, space efficient on-line algorithms for LZ factorization based on succinct data structures have been proposed. Okanohara and Sadakane [13] gave an
algorithm that runs in \( O(N \log^3 N) \) time using \( N \log \sigma + o(N \log \sigma) + O(N) \) bits of space. Later Starikovskaya \cite{starikovskaya}, achieved \( O(N \log^2 N) \) time using \( O(N \log \sigma) \) bits of space, assuming \( \log \sigma \) \( N \) characters are packed in a machine word.

In this paper, we propose a new on-line LZ factorization algorithm running in \( O(N \log N) \) time using only \( O(N \log \sigma) \) space, which is a notable improvement compared to the run-times of the previous on-line algorithms while still keeping the working space within a constant factor of the input text. Our algorithm is based on a novel application of a full text index called Directed Acyclic Word Graphs, or DAWGs \cite{asano_daung}, which, despite its elegance, has not received as much attention as suffix trees. To achieve a more efficient algorithm, we exploit an interesting feature of the DAWG structure that, unlike suffix trees, allows us to collect information concerning the left context of strings into each state in an efficient and on-line manner. We further show that the DAWG allows for an opportunistic variant of the algorithm which is more time and space efficient when the run length encoding (RLE) of the string is small. Given the RLE of size \( m \leq N \) of the string, our on-line algorithm runs in \( O(m \cdot \min \{ (\log \log m) (\log \log N), \sqrt{\log m \log \log N} \}) = o(m \log m) \) time using \( O(m \log N) \) bits of space. This improves on the off-line algorithm of \cite{barnat} which runs in \( O(m \log m) \) time using \( O(m \log N) \) bits of space.

2 Preliminaries

Let \( \Sigma = \{1, \ldots, \sigma\} \) be a finite integer alphabet. An element of \( \Sigma^* \) is called a string. The length of a string \( S \) is denoted by \(|S|\). The empty string \( \epsilon \) is the string of length 0. Let \( \Sigma^+ = \Sigma^+ - \{\epsilon\} \). For a string \( S = XYZ \), \( X \) and \( Z \) are called a prefix, substring, and suffix of \( S \), respectively. The set of prefixes and substrings of \( S \) are denoted by \( \text{Prefix}(S) \) and \( \text{Substr}(S) \), respectively. The longest common prefix (lcp) of strings \( X \) and \( Y \) is the longest string in \( \text{Prefix}(X) \cap \text{Prefix}(Y) \). The \( i \)-th character of a string \( S \) is denoted by \( S[i] \) for \( 1 \leq i \leq |S| \), and the substring of a string \( S \) that begins at position \( i \) and ends at position \( j \) is denoted by \( S[i..j] \) for \( 1 \leq i \leq j \leq |S| \). For convenience, let \( S[i..j] = \epsilon \) if \( j < i \). A position \( i \) is called an occurrence of \( X \) in \( S \) if \( S[i..i+|X|-1] = X \). For any string \( S = S[1..N] \), let \( S^{\text{rev}} = S[N] \cdots S[1] \) denote the reversed string. For any character \( a \in \Sigma \) and integer \( i \geq 0 \), let \( a^0 = \epsilon \), \( a^i = a^{i-1}a \). We call \( i \) the exponent of \( a^i \).

The default base of logarithms will be 2. Our model of computation is the unit cost word RAM with the machine word size at least \( \log N \) bits. For an input string \( S \) of length \( N \), let \( r = \log \sigma N = \frac{\log \sigma N}{\log \sigma} \). For simplicity, assume that \( \log N \) is divisible by \( \log \sigma \), and that \( N \) is divisible by \( r \). A string of length \( r \), called a meta-character, consists of \( \log N \) bits, and therefore fits in a single machine word. Thus, a meta-character can also be transparently regarded as an element in the integer alphabet \( \Sigma^r = \{1, \ldots, N\} \). We assume that given \( 1 \leq i \leq N - r + 1 \), any meta-character \( A = S[i..i+r-1] \) can be retrieved in constant time. Also, we can pre-compute an array of size \( 2^{\log \sigma N} \), occupying \( O(\sqrt{N \log N}) = o(N) \) bits in \( o(N) \) time, so \( A^{\text{rev}} = (A[r/2 + 1..r])^{\text{rev}} \) \( (A[1..r/2])^{\text{rev}} \) can be computed in constant time. We call a string on the alphabet \( \Sigma^r \) of meta-characters, a meta-string.
Any string $S$ whose length is divisible by $r$ can be viewed as a meta-string $S$ of length $n = \left\lceil \frac{|S|}{r} \right\rceil$. We write $\langle S \rangle$ when we explicitly view string $S$ as a meta-string, where $\langle S \rangle[j] = S[(j-1)r+1..jr]$ for each $j \in [1, n]$. Such range $[(j-1)r+1, jr]$ of positions will be called meta-blocks and the beginning positions $(j-1)r+1$ of meta-blocks will be called block borders. For clarity, the length $n$ of a meta-string $\langle S \rangle$ will be denoted by $||\langle S \rangle||$. Meta-strings are sometimes called packed strings. Note that $n \log N = N \log \sigma$.

### 2.1 LZ Factorization

There are several variants of LZ factorization, and as in most recent work, we consider the variant also called s-factorization [5]. The s-factorization of a string $S$ is the factorization $S = f_1 \cdots f_z$ where each s-factor $f_i \in \Sigma^+$ $(i = 1, \ldots, z)$ is defined as follows: $f_1 = S[1]$. For $i \geq 2$: if $S[f_1 \cdots f_{i-1} + 1] = c \in \Sigma$ does not occur in $f_1 \cdots f_{i-1}$, then $f_i = c$. Otherwise, $f_i$ is the longest prefix of $f_1 \cdots f_z$ that occurs at least twice in $f_1 \cdots f_i$. Notice that self-referencing is allowed, i.e., the previous occurrence of $f_i$ may overlap with itself. Each s-factor can be represented in a constant number of words, i.e., either as a single character or a pair of integers representing the position of a previous occurrence of the factor and its length. (See Fig. 4 in Appendix A. for an example.)

### 2.2 Tools

Let $B$ be a bit array of length $N$. For any position $x$ of $B$, let $\text{rank}(B, x)$ denote the number of 1’s in $B[1..x]$. For any integer $j$, let $\text{select}(B, j)$ denote the position of the $j$th 1 in $B$. For any pair of position $x, y$ $(x \leq y)$ of $B$, the number of 1’s in $B[x..y]$ can be expressed as $\text{pc}(B, x, y) = \text{rank}(B, y) - \text{rank}(B, x - 1)$. Dynamic bit arrays can be maintained to support rank/select queries and flip operations in $O(\log N)$ time, using $N + o(N)$ bits of space (e.g. Raman et al. [14]).

Directed Acyclic Word Graphs (DAWG) are a variant of suffix indices, similar to suffix trees or suffix arrays. The DAWG of a string $S$ is the smallest partial deterministic finite automaton that accepts all suffixes of $S$. Thus, an arbitrary string is a substring of $S$ iff it can be traversed from the source of the DAWG. While each edge of the suffix tree corresponds to a substring of $S$, an edge of a DAWG corresponds to a single character.

Theorem 1 (Blumer et al. [4]). The numbers of states, edges and suffix links of the DAWG are $O(|S|)$, independent of the alphabet size $\sigma$. The DAWG augmented with the suffix links can be constructed in an on-line manner in $O(|S| \log \sigma)$ time using $O(|S| \log |S|)$ bits of space.

We give a more formal presentation of DAWGs below. Let $\text{EndPos}_S(u) = \{ j \mid u = S[i..j], 1 \leq i \leq j \leq N \}$. Define an equivalence relation on $\text{Substr}(S)$ such that for any $u, w \in \text{Substr}(S)$, $u \equiv_S w \iff \text{EndPos}_S(u) = \text{EndPos}_S(w)$, and denote the equivalence class of $u \in \text{Substr}(S)$ as $[u]_S$. When clear from the
context, we abbreviate the above notations as \( \text{EndPos}_u \), \( \equiv \), and \([u]\), respectively. Note that for any two elements in \([u]\), one is a suffix of the other (or vice versa).

We denote by \( \overline{u} \) the longest member of \([u]\). The states \( V \) and edges \( E \) of a DAWG can be characterized as \( V = \{ [u] \mid u \in \text{Substr}(S) \} \) and \( E = \{ ([u], a, [ua]) \mid u, ua \in \text{Substr}(S), u \not= ua \} \). We also define the set \( G \) of labeled reversed edges, called suffix links, by \( G = \{ ([au], a, [ua]) \mid u, au \in \text{Substr}(S), u = \overline{au} \} \). An edge \( ([u], a, [ua]) \in E \) is called a primary edge if \(|\overline{u}| + 1 = |\overline{ua}|\), and a secondary edge otherwise. We call \([ua]\) a primary (resp. secondary) child of \([u]\) if the edge is primary (resp. secondary). (See Fig. 2 in Appendix for examples.) By storing \(|\overline{u}|\) at each state \([u]\), we can determine whether an edge \( ([u], a, [ua]) \) is primary or secondary in \( O(1) \) time using \( O(|S| \log |S|) \) bits of total space.

Whenever a state \( u \) is created during the on-line construction of the DAWG, it is possible to assign the position \( \text{pos}_{[u]} = \min \text{EndPos}_S(u) \) to that state. If state \( u \) is reached by traversing the DAWG from the source with string \( p \), this means that \( p = S[\text{pos}_{[u]} - |p| + 1..\text{pos}_{[u]}] \), and thus the first occurrence \( \text{pos}_{[u]} - |p| + 1 \) of \( p \) can be retrieved, using \( O(|S| \log |S|) \) bits of total space.

For any set \( P \) of points on a 2-D plane, consider query \( \text{find\_any}(P, I_h, I_t) \) which returns an arbitrary element in \( P \) that is contained in a given orthogonal range \( I_h \times I_t \) if such exists, and returns \text{nil} \) otherwise. A simple corollary of the following result by Blelloch [3]:

**Theorem 2** (Blelloch [3]). The 2D dynamic orthogonal range reporting problem on \( n \) elements can be solved using \( O(n \log n) \) bits of space so that insertions and deletions take \( O(\log n) \) amortized time and range reporting queries take \( O(\log n + k \log n / \log \log n) \) time, where \( k \) is the number of output elements.

is that the query \( \text{find\_any}(P, I_h, I_t) \) can be answered in \( O(\log n) \) time on a dynamic set \( P \) of points. It is also possible to extend the \( \text{find\_any} \) query to return, in \( O(\log n) \) time, a constant number of elements contained in the range.

### 3 On-line LZ Factorization with Packed Strings

The problem setting and high-level structure of our algorithm follows that of Starikovskaya [15], but we employ somewhat different tools. The goal of this section is to prove the following theorem.

**Theorem 3.** The s-factorization of any string \( S \in \Sigma^* \) of length \( N \) can be computed in an on-line manner in \( O(N \log N) \) time and \( O(N \log \sigma) \) bits of space.

By on-line, we assume that the input string \( S \) is given \( r \) characters at a time, and we are to compute the s-factorization of the string \( S[1..jr] \) for all \( j = 1, \ldots, n \). Since only the last factor can change for each \( j \), the whole s-factorization need not be re-calculated so we will focus on describing how to compute each s-factor \( f_i \) by extending \( f_i \) while a previous occurrence exists. We show how to maintain dynamic data structures using \( O(N \log \sigma) \) bits in \( O(N \log N) \) total time that allow us to (1) determine whether \(|f_i| < r \) in \( O(1) \) time, and if so, compute
in \( O(|f_i| \log N) \) time (Lemma 1), (2) compute \( f_i \) in \( O(|f_i| \log N) \) time when \( |f_i| \geq r \) (Lemma 3), and (3) retrieve a previous occurrence of \( f_i \) in \( O(|f_i| \log N) \) time (Lemma 3). Since \( \sum_{i=1}^N |f_i| = N \), these three lemmas prove Theorem 3.

The difference between our algorithm and that of Starikovskaya can be summarized as follows: For (1), we show that a dynamic succinct bit-array that supports rank/select queries and flip operations can be used, as opposed to a suffix trie employed in [15]. This allows our algorithm to use a larger meta-character size of \( r = \log_\sigma N \) instead of \( \log_\sigma N / 4 \) in [15], where the 1/4 factor was required to keep the size of the suffix trie within \( O(N \log \sigma) \) bits. Hence, our algorithm can pack characters more efficiently into a word. For (2), we show that by using a DAWG on the meta-string of length \( n = N/r \) that occupies only \( O(N \log \sigma) \) bits, we can reduce the problem of finding valid extensions of a factor to dynamic orthogonal range reporting queries, for which a space efficient dynamic data structure with \( O(\log n) \) time query and update exists [3].

In contrast, Starikovskaya’s algorithm uses a suffix tree on the meta-string and dynamic wavelet trees requiring \( O(\log^2 n) \) time for queries and updates, which is the bottleneck of her algorithm. For (3), we develop an interesting technique for the case \( |f_i| < r \) which may be of independent interest.

In what follows, let \( l_i = \sum_{k=1}^{i-1} |f_k| \). Although our presentation assumes that \( N \) is known, this can be relaxed at the cost of a constant factor by simply restarting the entire algorithm when the length of the input string doubles.

### 3.1 Algorithm for \( |f_i| < r \)

Consider a bit array \( M_k[1..N] \). For any meta-character \( A \in \Sigma^r \), let \( M_k[A] = 1 \) iff \( S[l + 1..l + r] = A \) for some \( 0 \leq l \leq k - r \), i.e., \( M_k[A] \) indicates whether \( A \) occurs as a substring in \( S[1..k] \). For any short string \( t \) (\( |t| < r \)), let \( D_t \) be the lexicographically smallest and largest meta-characters having \( t \) as a prefix, namely, the bit-representation\(^1\) of \( D_t \) is the concatenation of the bit-representation of \( t \) and \( 0^{(r-|t|)} \log \sigma \), and the bit-representation of \( U_t \) is the concatenation of the bit-representation of \( t \) and \( 1^{(r-|t|)} \log \sigma \). These representations can be obtained from \( t \) in constant time using standard bit operations. Then, the set of meta-characters that have \( t \) as a prefix can be represented by the interval \( tr(t) = [D_t, U_t] \). It holds that \( t \) occurs in \( S[1..k-r+r] \) iff some element in \( M_k[D_t..U_t] \) is 1, i.e. \( pc(M_k, D_t, U_t) > 0 \). Therefore, we can check whether or not a string of length up to \( r \) occurs at some position \( p \leq l_i \) by using \( M_{l_i+r-1} \).

For any \( 0 \leq m \leq r \), let \( t_m = S[l_i + 1..l_i + m] \). We have that \( |f_i| < r \) iff \( M_{l_i+r-1}[t_r] = 0 \), which can be determined in \( O(1) \) time. Assume \( |f_i| < r \), and let \( m_i = \max\{m \mid 0 \leq m < r, pc(M_{l_i+r-1}, D_{t_m}, U_{t_m}) > 0\} \), where \( m_i = 0 \) indicates that \( S[l_i + 1] \) does not occur in \( S[1..l_i] \). From the definition of \( s \)-factorization, we have that \( |f_i| = \max(1, m_i) \). Notice that \( m_i \) can be computed by \( O(|f_i|) \) rank queries on \( M_{l_i+r-1} \), due to the monotonicity of \( pc(M_{l_i+r-1}, D_{t_m}, U_{t_m}) \) for increasing values of \( m \). To maintain \( M_k \) we can use rank/select dictionaries for a dynamic bit array of length \( N \) (e.g. [15]) mentioned in Section 2. Thus we have:

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\(^1\)Assume that \( 0^{|\log N|} \) and \( 1^{|\log N|} \) correspond to meta-characters 1 and \( N \), respectively.
Lemma 1. We can maintain in $O(N \log N)$ total time, a dynamic data structure occupying $N + o(N)$ bits of space that allows whether or not $|f_i| < r$ to be determined in $O(1)$ time, and if so, $f_i$ to be computed in $O(|f_i| \log N)$ time.

3.2 Algorithm for $|f_i| \geq r$.

To compute $f_i$ when $|f_i| \geq r$, we use the DAWG for the meta-string $\langle S \rangle$ which we call the packed DAWG. While the DAWG for $S$ requires $O(N \log N)$ bits, the packed DAWG only requires $O(N \log \sigma)$ bits. However, the complication is that only substrings with occurrences that start at block borders can be traversed from the source of the packed DAWG. In order to overcome this problem, we will augment the packed DAWG and maintain the set $\text{Points}_{[u]} = \{(A^{rev}, X) \mid ([u], X, [u]X) \in E, A^{rev}X \in \text{Substr}(\langle S \rangle)\}$ for all states $[u]$ of the packed DAWG. A pair $(A^{rev}, X) \in \text{Points}_{[u]}$ represents that there exists an occurrence of $A^{rev}X$ in $\langle S \rangle$, in other words, the longest element $\overleftarrow{u}$ corresponding to the state can be extended by $X$ and still have an occurrence in $\langle S \rangle$ immediately preceded by $A$.

Lemma 2. For meta-string $S$ and its packed DAWG $(V, E, G)$, the the total number of elements in $\text{Points}_{[u]}$ for all states $[u] \in V$ is $O(||\langle S \rangle||)$.

Proof. Consider edge $([u], X, [u]X) \in E$. If $\overleftarrow{u}X \neq \overrightarrow{u}X$, i.e., the edge is secondary, it follows that there exists a unique meta-character $A = \langle S \rangle|_{\text{pos}(u)X} - ||\overleftarrow{u}X||$ such that $A^{rev}X \equiv (S)\overleftarrow{u}X$, namely, any occurrence of $\overleftarrow{u}X$ is always preceded by $A$ in $\langle S \rangle$. If $\overleftarrow{u}X = \overrightarrow{u}X$, i.e., the edge is primary, then, for each distinct meta-character $A$ preceding an occurrence of $\overleftarrow{u}X = \overrightarrow{u}X$ in $\langle S \rangle$, there exists a suffix link $([A\overleftarrow{u}X], X, [u]X) \in G$. Therefore, each point $(A^{rev}, X) \in \text{Points}_{[u]}$ can be associated to either a secondary edge from $[u]$ or one of the incoming suffix links to its primary child $[u]X$. (See also Fig. [4] in Appendix A.) Since each state has a unique longest member, each state has exactly one incoming primary edge. Therefore, the total number of elements in $\text{Points}_{[u]}$ for all states $[u]$ is equal to the total number of secondary edges and suffix links, which is $O(||\langle S \rangle||)$.

Lemma 3. For string $S \in \Sigma^*$ of length $N$, we can, in $O(N \log \sigma)$ total time and bits of space and in an on-line manner, construct the packed DAWG $(V, E, G)$ of $S$ as well as maintain $\text{Points}_{[u]}$ for all states $[u] \in V$ so that $\text{find\_any}(\text{Points}_{[u]}; I_h, I_v)$ for an orthogonal range $I_v \times I_h$ can be answered in $O(\log n)$ time.

Proof. It follows from Theorem [1] that the packed DAWG can be computed in an on-line manner, in $O(N \log \sigma)$ time and bits of space, since the size of the alphabet for meta-strings is $O(N)$ and the length of the meta-string is $n = \overleftarrow{N}$. To maintain and support $\text{find\_any}$ queries on $\text{Points}$ efficiently, we use the dynamic data structure by Blelloch [3] mentioned in Theorem [2]. Thus from Lemma [2] the total space requirement is $O(N \log \sigma)$ bits. Since each insert operation can be performed in amortized $O(\log n)$ time (no elements are deleted in our algorithm), what remains is to show that the total number of insert operations to
Points is $O(n)$. This is shown below by a careful analysis of the on-line DAWG construction algorithm \footnote{4}. (See Algorithm \footnote{4} in Appendix B. for pseudo-code.)

Assume we have the packed DAWG for a prefix $u = \langle S \rangle[1..\|u\|]$ of meta-string $\langle S \rangle$. Let $B = \langle S \rangle[\|u\|+1]$ be the meta-character that follows $u$ in $\langle S \rangle$. We group the updates performed on the packed DAWG when adding $B$, into the following two operations: (a) the new sink state $[uB]$ is created, and (b) a state is split.

First, consider case (a). Let $u_0 = u$, and consider the sequence $[u_1], \ldots, [u_q]$ of states such that the suffix link of $[u_j]$ points to $[u_{j+1}]$ for $0 \leq j < q$, and $[u_q]$ is the first state in the sequence which has an out-going edge labeled by $B$. Note that any element of $[u_{j+1}]$ is a suffix of any element of $[u_j]$. The following operations are performed. (See also Fig. 4 in Appendix A.) (a-1) The primary edge from the old sink $[u]$ to the new sink $[uB]$ is created. No insertion is required for this edge since $[uB]$ has no incoming suffix links. (a-2) For each $1 \leq j < q$ a secondary edge $([u_j], B, [uB])$ is created, and the pair $\langle C_j, B \rangle$ is inserted to Points$_{[u_j]}$, where $C_j$ is the unique meta-character that immediately precedes $R_jB$ in $uB$, i.e., $C_j = (uB)[pos_{uB} - |R_jB|]$. (a-3) Let $([u_q], B, w)$ be the edge with label $B$ from state $[u_q]$. The suffix link of the new sink state $[uB]$ is created and points to $w$. Let $e = ([v], B, w)$ be the primary incoming edge to $w$, and $A$ be the meta-character that labels the suffix link (note that $[v]$ is not necessarily equal to $[u_q]$). We then insert a new pair $(A_{rev}, B)$ into Points$_{[v]}$.

Next, consider case (b). After performing (a), node $w$ is split if the edge $([v_1], B, w)$ is secondary. Let $[v_1] = [v]$, and let $[v_1], \ldots, [v_h]$ be the parents of the state $w$ of the packed DAWG for $u$, sorted in decreasing order of their longest member. Then, it holds that there is a suffix link from $[v_h]$ to $[v_{h+1}]$ and any element of $[v_{h+1}]$ is a suffix of any element of $[v_h]$ for any $1 \leq h < k$. Assume $R_jB$ is the longest suffix of $uB$ that has another (previous) occurrence in $uB$. (Namely, $[v_i]$ is equal to the state $[u_q]$ of (a-2) above.) If $i > 1$, then the state $w$ is split into two states $[v_{iB}]$ and $[v_{iB}]$ such that $[v_{iB}] \cup [v_{iB}] = w$ and any element of $[v_{iB}]$ is a proper suffix of any element of $[v_{iB}]$. The following operations are performed. (See also Fig. 5 in Appendix A.) (b-1) The secondary edge from $[v_i]$ to $w$ becomes the primary edge to $[v_iB]$, and for all $i < j \leq k$ the secondary edge from $[v_j]$ to $w$ becomes a secondary edge to $[v_jB]$. The primary and secondary edges from $[v_i]$ to $w$ for all $1 \leq h < i$ become the primary and secondary ones from $[v_h]$ to $[v_iB]$, respectively. Clearly the sets Points$_{[v_i]}$ for all $1 \leq h < i$ are unchanged. Also, since any edge $([v_j], B, [v_{iB}])$ are all secondary, the sets Points$_{[v_j]}$ for all $i < j \leq k$ are unchanged. Moreover, the element of Points$_{[v_i]}$ that was associated to the secondary edge to $w$, is now associated to the suffix link from $[v_iB]$ to $[v_{iB}]$. Hence, Points$_{[v_i]}$ is also unchanged. Consequently, there are no updates due to edge redirection. (b-2) All outgoing edges of $[v_iB]$ are copied as outgoing edges of $[v_iB]$. Since any element of $[v_iB]$ is a suffix of any element of $[v_iB]$, the copied edges are all secondary. Hence, we insert a pair to Points$_{[v_iB]}$ for each secondary edge, accordingly.

Thus, the total number of insert operations to Points for all states is linear in the number of update operations during the on-line construction of the packed DAWG, which is $O(n)$ due to \footnote{4}. This completes the proof. $\square$
For any string \( f \) and integer \( 0 \leq m \leq \min(|f|, r - 1) \), let strings \( \alpha_m(f), \beta_m(f), \gamma_m(f) \) satisfy \( f = \alpha_m(f)\beta_m(f)\gamma_m(f), |\alpha_m(f)| = m, \) and \( |\beta_m(f)| = j' r \) where \( j' = \max\{j \geq 0 \mid m + jr \leq |f|\} \). We say that an occurrence of \( f \) in \( S \) has offset \( m \) \( (0 \leq m \leq r - 1) \), if, in the occurrence, \( \alpha_m(f) \) corresponds to a suffix of a meta-block, \( \beta_m(f) \) corresponds to a sequence of meta-blocks (i.e. \( \beta_m(f) \in \text{Substr}(\langle S \rangle) \)), and \( \gamma_m(f) \) corresponds to a prefix of a meta-block.

Let \( f^m \) denote the longest prefix of \( S[l_i + 1..N] \) which has a previous occurrence in \( S \) with offset \( m \). Thus, \( |f_i| = \max_{0 \leq m < r} |f^m_i| \). In order to compute \( f^m \), the idea is to find the longest prefix \( u \) of meta-string \( \langle \beta_m(S[l_i + 1..N]) \rangle \) that can be traversed from the source of the packed DAWG while assuring that at least one occurrence of \( u \) in \( S \) is immediately preceded by a meta-block that has \( \alpha_m(S[l_i + 1..N]) \) as a suffix. It follows that \( u = \beta_m(f^m) \).

**Lemma 4.** Given the augmented packed DAWG \((V, E, G)\) of Lemma 3 of meta-string \( \langle S \rangle \), the longest prefix \( f \) of any string \( P \) that has an occurrence with offset \( m \) in \( S \) can be computed in \( O(\frac{|f^m|}{m} \log n + r \log m) \) time.

**Proof.** We first traverse the packed DAWG for \( \langle S \rangle \) to find \( \beta_m(f) \). This traversal is trivial for \( m = 0 \), so we assume \( m > 0 \). For any string \( t \) \( (|t| < r) \), let \( L_t \) and \( R_t \) be, respectively, the lexicographically smallest and largest meta-character which has \( t \) as a suffix, namely, the bit-representation of \( L_t \) is the concatenation of \( 0^{(r-|t|)} \log \sigma \), and the bit-representation of \( t \), and the bit-representation of \( R_t \) is the concatenation of \( 1^{(r-|t|)} \log \sigma \), and the bit-representation of \( t \). Then, the set of meta-characters that have \( t^{rev} \) as a prefix, (or, \( t \) as a suffix when reversed), can be represented by the interval \( hr(t) = [L_t^{rev}, R_t^{rev}] \). Suppose we have successfully traversed the packed DAWG with the prefix \( u = \langle \beta_m(P) \rangle \| u \| + 1 \) and want to traverse with the next meta-character \( X = \langle \beta_m(P) \rangle \| u \| + 1 \). If \( u = \overline{\alpha_m(P)} \), i.e. only primary edges were traversed, then there exists an occurrence of \( \alpha_m(P)uX \) with offset \( m \) in string \( S \) iff \( \text{find}_\text{any}(\text{Points}_{|u|}, hr(\alpha_m(P))), [X, X] \) \( \neq \text{nil} \). Otherwise, if \( u \neq \overline{\alpha_m(P)} \), all occurrences of \( u \) (and thus all extensions of \( u \) that can be traversed) in \( S \) is already guaranteed to be immediately preceded by the unique meta-character \( A = \langle S \rangle \| u \| + 1 \) such that \( A^{rev} \in hr(\alpha_m(P)) \). Thus, there exists an occurrence of \( \alpha_m(P)uX \) with offset \( m \) in string \( S \) iff \( [(u), X, [uX]) \in E \). We extend \( u \) until \( \text{find}_\text{any} \) returns \( \text{nil} \) or no edge is found, at which point we have \( \alpha_m(P)u = \alpha_m(f)\beta_m(f) \).

Now, \( \gamma_m(f) \) is a prefix of meta-character \( B = \langle \beta_m(P) \rangle \| u \| + 1 \). When \( u = \overline{\alpha_m(P)} \), we can compute \( \gamma_m(f) \) by asking \( \text{find}_\text{any}(\text{Points}_{|u|}, hr(\alpha_m(P))), tr(B[1..j]) \) for \( 0 \leq j < r \). The maximum \( j \) such that \( \text{find}_\text{any} \) does not return \( \text{nil} \) gives \( |\gamma_m(f)| \). If \( u \neq \overline{\alpha_m(P)} \), \( \gamma_m(f) \) is the longest lcp between \( B \) and any outgoing edge from \( [u] \). This can be computed in \( O(\log n + |\gamma_m(f)|) \) time by maintaining outgoing edges from \( [u] \) in balanced binary search trees, and finding the lexicographic predecessor/successor \( B^-, B^+ \) of \( B \) in these edges, and computing the lcp between them. (See Fig. 4 in Appendix.) The lemma follows since each \( \text{find}_\text{any} \) query takes \( O(\log n) \) time. \( \square \)

From the proof of Lemma 4, \( \beta_m(f^m) \) can be computed in \( O(\frac{|f^m|}{m} \log n) \) time, and for all \( 0 \leq m < r \), this becomes \( O(|f_i| \log n) \) time. However, for comput-
ing $\gamma_m(f_i^m)$, if we simply apply the algorithm and use $O(r \log n)$ time for each $f_i^m$, the total time for all $0 \leq m < r$ would be $O(r^2 \log n)$ which is too large for our goal. Below, we show that all $\gamma_m(f_i^m)$ are not required for computing $\max_{0 \leq m < r} |f_i^m|$, and this time complexity can be reduced.

Consider computing $F_m = \max_{0 \leq x \leq m} |f_i^x|$ for $m = 0, \ldots, r - 1$. We first compute $\hat{f}_i^m = \alpha_m(f_i^m)\beta_m(f_i^m)$ using the first part of the proof of Lemma 4. We shall compute $\gamma_m(f_i^m)$ only when $F_m$ can be larger than $F_{m-1}$ i.e., $|f_i^m| + |\gamma_m(f_i^m)| > F_{m-1}$. Since $|\gamma_m(f_i^m)| < r$, this will never be the case if $|f_i^m| \leq F_{m-1} - r + 1$, and will always be the case if $|\hat{f}_i^m| > F_{m-1}$. For the remaining case, i.e., $0 \leq F_{m-1} - |f_i^m| < r - 1$, $F_m > F_{m-1}$ iff $|\gamma_m(f_i^m)| > F_{m-1} - |\hat{f}_i^m|$. If $u = \frac{\sqrt{r}}{2}$, this can be determined by a single $\text{find\_any}$ query with $j = F_{m-1} - |\hat{f}_i^m| + 1$ in the last part of the proof of Lemma 4 and if so, the rest of $\gamma_m(f_i^m)$ is computed using the $\text{find\_any}$ query for increasing $j$. When $u = \frac{\sqrt{r}}{2}$, whether or not the lcp between $B$ and $B^+$ or $B^-$ is greater than $F_{m-1} - |\hat{f}_i^m|$ can be checked in constant time using bit operations.

From the above discussion, each $\text{find\_any}$ or predecessor/successor query for computing $\gamma_m(f_i^m)$ updates $F_m$, or returns nil. Therefore, the total time for computing $F_r = |f_i|$ is $O((r + |f_i|) \log n) = O(|f_i| \log n)$.

A technicality we have not mentioned yet, is when and to what extent the packed DAWG is updated when computing $f_i$. Let $F$ be the length of the current longest prefix of $S[l_i+1..N]$ with an occurrence less than $l_i + 1$, found so far while computing $f_i$. A self-referencing occurrence of $S[l_i+1..l_i+F]$ can reach up to position $l_i + F - 1$. When computing $f_i$ using the packed DAWG, $F$ is increased by at most $r$ characters at a time. Thus, for our algorithm to successfully detect such self-referencing occurrences, the packed DAWG should be built up to the meta-block that includes position $l_i + F - 1 + r$ and updated when $F$ increases. This causes a slight problem when computing $f_i^m$ for some $m$; we may detect a substring which only has an occurrence larger than $l_i$ during the traversal of the DAWG. However, from the following lemma, the number of such future occurrences that update $F$ can be limited to a constant number, namely two, and hence by reporting up to three elements in each $\text{find\_any}$ query that may update $F$, we can obtain an occurrence less than $l_i + 1$, if one exists. These occurrences can be retrieved in $O(\log N)$ time in this case, as described in Section 5.3.

**Lemma 5.** During the computation of $f_i^m$, there can be at most two future occurrences of $f_i^m$ that will update $F$.

*Proof.* As mentioned above, the packed DAWG is built up to the meta string $\langle S[1..s] \rangle$ where $s = \lfloor \frac{\hat{l}_i + F + r - 1}{r} \rfloor$. An occurrence of $f_i^m$ possibly greater than $\hat{l}_i$ can be written as $p_{m,k} = \lfloor \frac{\hat{l}_i + F + r - 1}{r} \rfloor - m + 1 + kr$, where $k = 0, 1, \ldots$. For the occurrence to be able to update $F$ and also be detected in the packed DAWG, it must hold that $s > p_{m,k} + F$. Since $l_i + F + 2r - 2 \geq s > p_{m,k} + F \geq l_i - m + 1 + kr + F$, $k$ should satisfy $(2 - k)r \geq 1 - m$, and thus can only be 0 or 1. \[\square\]

**Lemma 6.** We can maintain in a total of $O(N \log N)$ time, a dynamic data structure occupying $O(N \log \sigma)$ bits of space that allows $f_i$ to be computed in $O(|f_i| \log N)$ time, when $|f_i| \geq r$. 

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3.3 Retrieving a Previous Occurrence of \( f_i \)

If \(|f_i| \geq r\), let \( f_i = f_i^m, A^{rev} \in hr(\alpha_m(f_i)), u = \beta_m(f_i), \) and \( X = tr(\gamma_m(f_i)) \) where \( A \) and \( X \) were found during the traversal of the packed DAWG. We can obtain the occurrence of \( f_i \) by simple arithmetic on the ending positions stored at each state, i.e., from \( \text{pos}_{[uX]} \) if \( uX \neq uX \) or \( m = 0 \), from \( \text{pos}_{[\text{Au}X]} \) otherwise. State \([\text{Au}X]\) can be reached in \( O(\log N) \) time from state \([uX]\), by traversing the suffix link in the reverse direction.

For \(|f_i| < r\), \( f_i \) is a substring of a meta-character. Let \( A_i \) be one of the previously occurring meta-characters with prefix \( f_i \) for which \( M_{i,r-1}[A_i] = 1 \), thus giving a previous occurrence of \( f_i \). \( A_i \) can be any meta-character in the range \( tr(f_i) = [D_t,m, U_t,m] \) with a set bit, so \( A_i \) can be retrieved in \( O(\log N) \) time by \( A_i = \text{select}(M_{i,r-1}, \text{rank}(M_{i,r-1}, U_t,m)). \) Unfortunately, we cannot afford to explicitly maintain previous occurrences for all \( N \) meta-characters, since this would cost \( O(N \log N) \) bits of space. We solve this problem in two steps.

First, consider the case that a previous occurrence of \( f_i \) crosses a block border, i.e. has an occurrence with some offset \( 1 \leq m \leq |f_i| - 1 \), and \( f_i = \alpha_m(f_i) \gamma_m(f_i). \) For each \( m = 1, \ldots , |f_i| - 1 \), we ask \( \text{find}_\text{any}(\text{Points}(\gamma_m(f_i)), tr(\gamma_m(f_i))). \) If a pair \((A^{rev}, X)\) is returned, this means that \( AX \) occurs in \( \langle S \rangle \) and \( A[r - m + 1..r] = \alpha_m(f_i) \) and \( X[1..\gamma_m(f_i)] = \gamma_m(f_i). \) Thus, a previous occurrence of \( f_i \) can be computed from \( \text{pos}_{[AX]} \). The total time required is \( O(|f_i| \log n) \). If all the \( \text{find}_\text{any} \) queries returned \text{nil}, this implies that no occurrence of \( f_i \) crosses a block border and \( f_i \) occurs only inside meta-blocks. We develop an interesting technique to deal with this case.

**Lemma 7.** For string \( S[1..k] \) and increasing values of \( 1 \leq k \leq N \), we can maintain a data structure in \( O(N \log N) \) total time and \( O(N \log \sigma) \) bits of space that, given any meta-character \( A \), allows us to retrieve a meta-character \( A' \) that corresponds to a meta block of \( S \), and some integer \( d \) such that \( A'[1 + d..r] = A[1..r - d] \) and \( 0 \leq d \leq d_{A,k} \), in \( O(\log N) \) time, where \( d_{A,k} = \min\{(l - 1) \mod r | 1 \leq l \leq k - r + 1, A = S[l..l + r - 1]\}. \) (Also see Fig. 7 in Appendix A.)

**Proof.** Consider a tree \( T_k \) where nodes are the set of meta-characters occurring in \( S[1..k] \). The root is \( \langle S \rangle[1] \). For any meta-character \( A \neq \langle S \rangle[1] \), the parent \( B \) of \( A \) must satisfy \( B[2..r] = A[1..r - 1] \) and \( A \neq B \). Given \( A \), its parent \( B \) can be encoded by a single character \( B[1] \in \Sigma \) that occupies \( \log \sigma \) bits and can be recovered from \( B[1] \) in \( A \) in constant time by simple bit operations. Thus, together with \( M_0 \) used in Section 3.1 which indicates which meta-characters are nodes of \( T_k \), the tree can be encoded in \( O(N \log \sigma) \) bits. We also maintain another bit vector \( X_k \) of length \( N \) so that we can determine in constant time, whether a node in \( T_k \) corresponds to a meta-block. The lemma can be shown if we can maintain the tree for increasing \( k \) so that for any node \( A \) in the tree, either \( A \) corresponds to a meta-block \( (d_{A,k} = 0) \), or, \( A \) has at least one ancestor at most \( d_{A,k} \) nodes above it that corresponds to a meta-block. Assume that we have \( T_{k-1} \), and want to update it to \( T_k \). Let \( A = S[k - r + 1..k] \). If \( A \) previously corresponded to or the new occurrence corresponds to a meta-block, then, \( d_{A,k} = 0 \) and we simply set \( X_k[A] = 1 \) and we are done. Otherwise, let \( B = S[k - r..k - 1] \) and
denote by $C$ the parent of $A$ in $T_{k-1}$, if there was a previous occurrence of
$A$. Based on the assumption on $T_{k-1}$, let $x_B \leq d_{B,k-1} = d_{B,k}$ and $x_C$ be the
distance to the closest ancestor of $B$ and $C$, respectively, that correspond to a
meta-block. We also have that $d_{A,k-1} \geq x_C + 1$. If $(k-r) \mod r \geq x_C + 1$,
then $d_{A,k} = \min\{(k-r) \mod r, d_{A,k-1}\} \geq x_C + 1$, i.e., the constraint is already
satisfied and nothing needs to be done. If $(k-r) \mod r < x_C + 1$ or there was
no previous occurrence of $A$, we have that $d_{A,k} = (k-r) \mod r$. Notice that
in such cases, we cannot have $A = B$ since that would imply $d_{A,k} = d_{A,k-1} \neq
(k-r) \mod r$, and thus by setting the parent of $A$ to $B$, we have that there
exists an ancestor corresponding to a meta-block at distance $x_B + 1 \leq d_{B,k} + 1 \leq
(k-r-1) \mod r + 1 = d_{A,k}$.

Thus, what remains to be shown is how to compute $x_C$ in order to determine
whether $(k-r) \mod r < x_C + 1$. Explicitly maintaining the distances to the
closest ancestor corresponding to a meta-block for all $N$ meta-characters will
take too much space ($O(N \log \log N)$ bits). Instead, since the parent of a given
meta-character can be obtained in constant time, we calculate $x_C$ by simply
going up the tree from $C$, which takes $O(x_C) = O(\log N)$ time. Thus, the update
for each $k$ can be done in $O(\log N)$ time, proving the lemma.

Using Lemma 7 we can retrieve a meta-character $A'$ that corresponds to a
meta-block and an integer $0 \leq d \leq d_{A,k}$ such that $A'[1+d..r] = A_i[1..r-d]$, in
$O(\log N)$ time. Although $A'$ may not actually occur $d$ positions prior to an
occurrence of $A_i$ in $S[1..k]$, $f_i$ is guaranteed to be completely contained in $A'$ since
it overlaps with $A_i$, at least as much as any meta-block actually occurring prior
to $A_i$ in $S[1..k]$. Thus, $f_i = A_i[1..|f_i|] = A'[1+d..d+|f_i|]$, and $(\text{pos}_{f_i} - 1)r + 1 + d
is a previous occurrence of $f_i$. The following lemma summarizes this section.

**Lemma 8.** We can maintain in $O(N \log N)$ total time, a dynamic data structure
occupying $O(N \log \sigma)$ bits of space that allows a previous occurrence of $f_i$
to be computed in $O(|f_i| \log N)$ time.

### 4 On-line LZ factorization based on RLE

For any string $S$ of length $N$, let $RLE(S) = a_1^{p_1} a_2^{p_2} \cdots a_m^{p_m}$ denote the run length encoding of $S$. Each $a_k^{p_k}$ is called an RL factor of $S$, where $a_k \neq a_{k+1}$ for any
$1 \leq k < m$, $p_k \geq 1$ for any $1 \leq h \leq m$, and therefore $m \leq N$. Each RL factor
can be represented as a pair $(a_k, p_k) \in \Sigma \times [1..N]$, using $O(\log N)$ bits of space.
As in the case with packed strings, we consider the on-line LZ factorization
problem, where the string is given as a sequence of RL factors and we are to
compute the $s$-factorization of $S_j = a_1^j \cdots a_{m}^j$ for all $j = 1, \ldots, m$. Similar to
the case of packed strings, we construct the DAWG of $RLE(S)$ of length $m$,
which we will call the RLE-DAWG, in an on-line manner. The RLE-DAWG
has $O(m)$ states and edges and each edge label is an RL factor $a_k^{p_k}$, occupying
a total of $O(m \log N)$ bits of space. We can show that the first RL-factor of
$f_i$ (corresponding to the offset in the case of packed string), can be determined
very easily, and therefore greatly simplifies the algorithm. Moreover, we can show
that the problem of finding valid extensions of the $s$-factor can be reduced to the simpler dynamic predecessor/successor problem, and by using the linear-space dynamic predecessor/successor data structure of [2], we obtain the following result. (See Appendix for full proof.)

**Theorem 4.** Given an RLE($S$) = $a_1^{p_1}a_2^{p_2} \cdots a_m^{p_m}$ of size $m$ of a string $S$ of length $N$, we can compute in an on-line manner the $s$-factorization of $S$ in $O \left( m \cdot \min \left\{ \frac{(\log \log m)(\log \log N)}{\log \log \log N}, \sqrt{\frac{\log m}{\log \log m}} \right\} \right)$ time using $O(m \log N)$ bits of space.

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Appendix A: Figures

Fig. 1. The s-factorization of the string $S = abaababaaaaababab$ is $a$, $b$, $a$, $aba$, $baba$, $aaaa$, $b$, $babab$. Each factor can be represented as a single character or a pair of integers representing the position of a previous occurrence of the factor and its length, i.e., $a$, $b$, $(1, 1)$, $(1, 3)$, $(5, 4)$, $(10, 4)$, $(2, 1)$, $(5, 5)$. Notice that the previous occurrence of an s-factor may overlap with itself. For example, in the case of $f_5 = baba$, the two occurrences of $f_5$ are at positions 5 and 7.

Fig. 2. The DAWG for string $ababbbb$. The solid and dashed arcs represent edges and suffix links, respectively. For simplicity, only a subset of suffix links is shown. The numbers in each state $[u]$ is $\text{pos}_{[u]}$. For example, consider the non-empty suffixes of substring $ababb$. They are represented by 3 different states $[ababb] = \{ababb, babb, abbb\}$, $[bb] = \{bb\}$, and $[b] = \{b\}$, because $\text{EndPos}_{[ababb]} = \{5\}$, $\text{EndPos}_{[bb]} = \{5, 6, 7\}$, and $\text{EndPos}_{[b]} = \{2, 4, 5, 6, 7\}$. The suffix link of $[ababb]$ is labeled by character $a$ and points to $[bb]$. This is because $bb$ is the longest member of $[bb]$ and $abb$ is a member of $[ababb]$. The edge $([abab], b, [ababb])$ is primary, while the edge $([ab], b, [ababb])$ is secondary.
Fig. 3. A snapshot of the on-line construction of the packed DAWG of meta string \( \langle S \rangle = ABCBA \) for string \( S = aaaaaababaaaa \), where the block size \( r = 3 \), \( A = aaaa \), \( B = aab \), and \( C = aba \). The solid and dashed arcs represent edges and suffix links, respectively. For simplicity, only the new suffix links are shown at each step. The number in each state \( [u] \) is \( pos_{[u]} \) in \( \langle S \rangle \).
Fig. 4. Using the augmented packed DAWG to compute the longest prefix $f$ of $P$ that occurs with offset $m$ in $S$ (Lemma 3). The left figure is a part of the packed DAWG for meta-string $\langle S \rangle$, where bold solid arcs represent primary edges, regular solid arcs represent secondary edges, and the dashed arcs represent suffix links. The right figure is the set $\text{Points}_{[CA]}$ for state $[CA]$. The pairs $(A^{rev}, C)$, $(D^{rev}, C)$, and $(E^{rev}, C)$ correspond to the incoming suffix links of state $[CAC]$, which is a primary child of $[CA]$ with the incoming primary edge labeled by $C$. On the other hand, the pair $(B^{rev}, D)$ corresponds to the outgoing secondary edge of $[CA]$ labeled by $D$. Assume that for some $0 \leq m < r$ and string $P = \alpha_m(P)\text{CAB}$, where $hr(\alpha_m(P)) = [A^{rev}, E^{rev}]$. Assume also that we have traversed the packed DAWG with $\beta_m(P)[1..2] = CA$, and want to traverse with the next meta-character $B$. Since $CA = \overleftarrow{CA}$ and there is no point in range $hr(\alpha_m(P)) \times [B, B]$, we see that $\alpha_m(P)\text{CAB}$ does not occur with offset $m$ in $S$, so $\beta_m(f) = CA$. We then compute $\gamma_m(f)$ which is a prefix of $B$, by querying a point in range $hr(\alpha_m(P)) \times tr[1..j]]$ for all $0 \leq j < |\gamma_m(f)|$. Next, consider what happens for another string $P' = \alpha_m(P')\text{AB}$, where $C^{rev} \in hr(\alpha_m(P'))$. If we have traversed with $\beta_m(P')[1..1] = A$ from the source, we are at state $[A] = [CA]$, and $A \neq \overleftarrow{A} = CA$. At this point, it is guaranteed that all occurrences of $A$ (and all extensions to $A$ that can be traversed on the packed DAWG) will be immediately preceded by $C$. Thus we only need to check outgoing edges. Since there is no outgoing edge labeled with the next meta-character $B$, $\gamma_m(f')$ is the longest lcp between labels of outgoing edges from $[CA]$, which is the lcp between the successor $B^+ = C$ and $B$ (the predecessor $B^- of B$ is nil).
Fig. 5. Illustration for case (a) of Lemma 3. Primary edge \([u_0], B, [u_0]B\) and secondary edges \([u_j], B, [u_0]B\) are created for all \(1 \leq j < q\). The suffix link of the new sink \([u_0]B\) points to state \(w\).

Fig. 6. Illustration for case (b) of Lemma 3. State \(w\) is split into two states \([v_1]B\) and \([v_k]B\) due to the update of the packed DAWG. The bold arcs represent primary edges. The outgoing edges of \(w\) may or may not be primary, however, the copied outgoing edges from \([v_i]B\) are all secondary.

Fig. 7. Finding a meta-block \(A'\) that has \(A_i[1..r - d]\) as a suffix (Lemma 4). We can find \(A'\) and \(d \leq d_{A,k}\) from \(A_i\) in \(O(\log N)\) time. Notice that \(A_i\) does not necessarily occur at the position depicted, but if we assume that there exists an occurrence of \(A_i\) such that its prefix \(f_i\) is inside a meta-block, \(f_i\) is guaranteed to be completely contained in \(A'\) since \(A'\) overlaps with \(A_i\) at least as much as any meta-block actually occurring prior to \(A_i\). Thus, a previous occurrence of \(f_i\) can be retrieved from \(pos_{A'}\), \(d\), and \(|f_i|\).
Appendix B: Pseudo Codes

Algorithm 1: On-line DAWG construction algorithm by Blumer et al. [4]

1 Procedure builddawg(w);
2 Create state source; cursink = source;
3 for each letter a of w do cursink = update(cursink, a);
4 return source;

5 Procedure update(cursink, a);
6 Create state newsink and primary edge (cursink, a, newsink);
7 curstate = cursink; suffixstate = undefined;
8 while curstate ≠ source and suffixstate = undefined do
9   curstate = suflink(curstate);
10  if ∃cstate s.t. (curstate, a, cstate) ∈ E then
11     create secondary edge (curstate, a, newsink);
12  else if (curstate, a, cstate) ∈ E is primary then
13     suffixstate = cstate;
14  else /* (curstate, a, cstate) ∈ E is secondary */
15     suffixstate = split(curstate, a, cstate);

16 Procedure split(pstate, a, cstate);
17 Create state newcstate;
18 Change secondary edge (pstate, a, cstate) to primary edge (pstate, a, newcstate);
19 for every edge (cstate, c, dest) ∈ E do
20   create secondary edge (newcstate, c, dest);
21 suflink(newcstate) = suflink(cstate);
22 suflink(cstate) = newcstate;
23 curstate = pstate;
24 while curstate ≠ source do
25   curstate = suflink(curstate);
26  if ∃cstate s.t. (curstate, a, cstate) ∈ E then
27     Change secondary edge (curstate, a, cstate) to secondary edge (curstate, a, newcstate);
28  else
29     break out of the while loop;
30 return newcstate;
Appendix C: Proof of Theorem 4

Here we provide a proof for Theorem 4 and show how to compute the s-factorization of a string $S$ from $RLE(S)$, efficiently and on-line. We begin with the following lemma.

**Lemma 9.** Each RL factor $a^p_k$ of $RLE(S)$ is covered by at most 2 s-factors of string $S$.

**Proof.** Consider an s-factor $f_i$ that starts at the $j$th position in the RL factor $a^p_k$, where $1 < j \leq p_k$. Since $a^{p_k-j+1}_k$ is both a suffix and a prefix of $a^p_k$, we have that the s-factor extends at least to the end of $a^p_k$. This implies that each RL factor $a^p_k$ is always covered by at most 2 s-factors. □

Let $z$ be the number of s-factors of string $S$. It immediately follows from Lemma 9 that $z \leq 2m$. This allows us to describe the complexity of our algorithm without using $z$. Lemma 9 also implies that if an s-factor $f_i$ intersects with an RL factor $a^p_k$, then the first RL factor of $f_i$ is always a suffix $a^k_r$ of $a^p_k$ with $p \leq p_k$. This simplicity allows us to perform on-line s-factorization from RLE efficiently. A proof for Theorem 4 follows:

**Proof.** Let $RLE(S) = a^{p_1}_1a^{p_2}_2 \cdots a^{p_m}_m$. For any $1 \leq k \leq h \leq m$, let $RLE(S)[k..h] = a^{p_k}_k a^{p_{k+1}}_{k+1} \cdots a^{p_h}_h$. Let $\text{Substr}(RLE(S)) = \{RLE[S][k..h] | 1 \leq k \leq h \leq m\}$.

Assume we have already computed $f_1, \ldots, f_{i-1}$ and we are computing a new s-factor $f_i$ from the $(\ell_i + 1)$th position of $S$. Let $a^d$ be the RL factor which contains the $(\ell_i + 1)$th position, and let $j$ be the position in the RL factor where $f_i$ begins.

Firstly, consider the case where $2 \leq j \leq d$. Let $p = d - j + 1$, i.e., the remaining suffix of $a^d$ is $a^p$. It follows from Lemma 9 that $a^p$ is a prefix of $f_i$. In the sequel, we show how to compute the rest of $f_i$. For any out-going edge $e = ([u], [ub^q], [ub^q])$ of a state $[u]$ of the RLE-DAWG for $RLE(S)[1..j]$ and each character $a \in \Sigma$, define $\text{mpe}_{[u]}(a, b^q) = \max\{p | a^p \text{Substr}(RLE(S)[1..j]) \cup \{0\}$. That is, $\text{mpe}_{[u]}(a, b^q)$ represents the maximum exponent of the RL factor with character $a$, that immediately precedes $\overline{u} b^q$ in $RLE(S)[1..j]$. For each pair $(a, b)$ of characters for which there is an out-going edge $([u], [ub^q], [ub^q])$ from state $[u]$ and $\text{mpe}_{[u]}(a, b^q) > 0$, we insert a point $\text{mpe}_{[u]}(a, b^q, q)$ into $\text{Pts}_{[u], a, b}$. By similar arguments to the case of packed DAWGs, each point in $\text{Pts}_{[u], a, b}$ corresponds to a secondary edge, or a suffix link (labeled with $a^p$ for some $p$) of a primary child, so the total number of such points is bounded by $O(m)$.

Suppose we have successfully traversed the RLE-DAWG by $u$ with an occurrence that is immediately preceded by $a^p$ (i.e., $a^p u$ is a prefix of s-factor $f_i$), and we want to traverse with the next RL factor $b^q$ from state $[u]$.

If $u = \overline{u}$, i.e., only primary edges were traversed, then we query $\text{Pts}_{[u], a, b}$ for a point with maximum $x$-coordinate in the range $[0, N] \times [q, N]$. Let $(x, y)$
Fig. 8.  Black and white points are dominant and non-dominant points, respectively. Left: If there is a point \((x, q)\) with \(x \geq p\), then we can traverse from state \([u]\) to \([ub^q]\) and continue. Right: If there is no such point, then the last RL factor of \(f_i\) is \(b^q\). In both cases, the \(y\)-coordinate successor of \(q\) among the dominant points is a point with maximum \(x\)-coordinate in range \([0, N] \times [q, N]\).

be such a point. If \(x \geq p\), then since \(y \geq q\), there must be a previous occurrence of \(a^p ub^q\), and hence \(a^p ub^q\) is a prefix of \(f_i\). If there is an outgoing edge of \([u]\) labeled by \(b^q\), then we traverse from \([u]\) to \([ub^q]\) and update the RLE-DAGW with the next RL factor. Otherwise, it turns out that \(f_i = a^p ub^q\). If \(x < p\) or no such point existed, then we query for a point with maximum \(y\)-coordinate in the range \([p, N] \times [0, q]\). If \((x', y')\) is a such a point, then \(f_i = a^p ub^{y'}\).

Otherwise (if \(u \neq \varepsilon\)), then all occurrences of \(u\) in \(S[1..\ell]\) is immediately preceded by the unique RL factor \(a^g\). Thus, there exists an occurrence of \(a^g ub^q\) iff \(([u], b^q, [ub^q]) \in E\). If there is no such edge, then the last RL factor of \(f_i\) is \(b^q\), where \(y = \min(\max\{k | ([u], b^k, [ub^k]) \in E\} \cup \{q\})\).

Secondly, let us consider the case where \(j = 1\). Let \([([e], a^g, [a^g])\) be the edge which has maximum exponent \(g\) for the character \(a\) from the source state \([e]\). If \(g < d\), then \(f_i = a^g\). Otherwise, \(a^d\) is a prefix of \(f_i\), and we traverse the RLE-DAGW in a similar way as above, while checking an immediately preceding occurrence of \(a^d\).

If we use priority search trees by McCreight (SIAM J. Comput. 14(2), 257–276, 1985) and balanced binary search trees, the above queries and updates are supported in \(O(\log m)\) time using a total of \(O(m \log N)\) bits of space. We can do better based on the following observation. For a set \(T\) of points in a 2D plane, a point \((p, q) \in T\) is said to be dominant if there is no point \((p', q') \in T\) satisfying both \(p' \geq p\) and \(q' \geq q\). Let \(Dom_{[u], a, b}\) denote the set of dominant points of \(Pts_{[u], a, b}\). Now, a query for a point with maximum \(x\)-coordinate in range \([0, N] \times [q, N]\) reduces to a successor query on the \(y\)-coordinates of points in \(Dom_{[u], a, b}\) (see also Fig. 8). On the other hand, a query for a point with maximum \(y\)-coordinate in range \([p, N] \times [0, q]\) reduces to a successor query on the \(x\)-coordinate of points in \(Dom_{[u], a, b}\) (see also Fig. 9). Hence, it suffices to maintain only the dominant points.

When a new dominant point is inserted into \(Dom_{[u], a, b}\) due to an update of the RLE-DAGW, then all the points that have become non-dominant are
Fig. 9. Black and white points are dominant and non-dominant points, respectively. Assume the $x$-coordinate of the point with maximum $x$-coordinate in range $[0, N] \times [q, N]$ is less than $p$. Then the $x$-coordinate successor of $p$ among the dominant points is a point with maximum $y$-coordinate in range $[p, N] \times [0, q]$.

deleted from $\text{Dom}_{[u],a,b}$. We can find each non-dominant point by a single predecessor/successor query. Once a point is deleted from $\text{Dom}_{[u],a,b}$, it will never be re-inserted to $\text{Dom}_{[u],a,b}$. Hence, the total number of insert/delete operations is linear in the size of $\text{Dom}_{[u],a,b}$, which is $O(m)$ for all the states of the RLE-DAWG. Using the data structure of [2], predecessor/successor queries and insert/delete operations are supported in $O\left(\min\left\{\frac{(\log \log m)(\log \log N)}{\log \log \log N}, \sqrt{\frac{\log m}{\log \log m}}\right\}\right)$ time, using a total of $O(m \log N)$ bits of space.

Each state of the RLE-DAWG has at most $m$ children and the exponents of the edge labels are in range $[1, N]$. Hence, assuming an integer alphabet $\Sigma = \{1, 2, \ldots, N\}$ and using the data structure of [2], we can search branches at each state in $O\left(\min\left\{\frac{(\log \log m)(\log \log N)}{\log \log \log N}, \sqrt{\frac{\log m}{\log \log m}}\right\}\right)$ time, using a total of $O(m \log N)$ bits of space. A final technicality is how to access the set $\text{Dom}_{[u],a,b}$ which is associated with a pair $(a, b)$ of characters. To access $\text{Dom}_{[u],a,b}$ at each state $[u]$, we maintain two level search structures, one for the first characters and the other for the second characters of the pairs. At each state $[u]$ we can access $\text{Dom}_{[u],a,b}$ in $O\left(\min\left\{\frac{(\log \log m)(\log \log N)}{\log \log \log N}, \sqrt{\frac{\log m}{\log \log m}}\right\}\right)$ time with a total of $O(m \log N)$ bits of space, again using the data structure of [2]. This completes the proof for Theorem 4. \qed