On solitons in the negative refracting medium

A I Maimistov\textsuperscript{1,2}, I R Gabitov\textsuperscript{3} and E I Lyashko\textsuperscript{2}

\textsuperscript{1}Department of Solid State Physics and Nanostructures, National Research Nuclear University MEPhI, Moscow, 115409, Russia
\textsuperscript{2}Department of Physics and Technology of Nanostructures, Moscow Institute for Physics and Technology, Dolgoprudny, Moscow region, 141700 Russia
\textsuperscript{3}Department of Mathematics, University of Arizona, Tucson, Arizona 85721, USA

E-mail: aimaimistov@gmail.com

Abstract. We consider the coupled electromagnetic waves propagating in a nonlinear medium, which is featured by a positive and negative refraction indexes. The backward waves can be propagating in this case. The example of the true soliton is discussed. In general case the coupled forward and backward solitary wave can be found. They are analogues to the optical solitons.

1. Introduction

As a rule, the phase velocity ($V_{ph}$) and the Pointing vector ($S$) of the electromagnetic wave are collinear vectors. Frequently the Pointing vector is considered to be proportional to the group velocity vector. It is correct in the case of isotropic medium. Antiparallel orientation of the phase velocity and the Pointing vector (or group velocity vector) was first discussed in [1, 2]. In [3] it was indicated that antiparallel orientation of $V_{ph}$ and $S$ results in negative refraction. Subsequently, this idea was developed by Mandelstam in [4]. It has been predicted that when the real parts of the dielectric permittivity and magnetic permeability in the medium simultaneously take on negative values in some frequency range, antiparallel orientation of $V_{ph}$ and $S$ occurs [5,6] and the property of negative refraction appears [7]. The existence of the media characterized by negative refractive index (NRI) was demonstrated experimentally first in the microwave and then in the near-infrared ranges. Reviews of the properties of the NRI materials and the nonlinear phenomena in NRI materials are presented in [8-13].

2. Sine-Gordon equation. Forward and backward solitons

The Sine-Gordon equation is the famous example of the evolution equation in the nonlinear wave theory [14, 15]. This equation is derived in many fields of physics and its properties are well known. In the one dimension case the Sine-Gordon equation is completely integrable one. Solitons and multi-solitons (in the strict sense) are the solutions of this equation. Thus the Sine-Gordon equation can be considered as an etalon equation in this context.

There are two forms of the Sine-Gordon equation

$$u_{xx} - u_{tt} = a \sin u,$$

and
\[ u_{xt} = a \sin u . \] (2)

Linearization of these equations results in linear wave equations for description of the linear wave propagation

\[ u_{xx} - u_{tt} - au = 0 , \] (3)

\[ u_{xt} - au = 0 . \] (4)

From equation (3) the dispersion relation follows in the form \( \omega^2(k) = a + k^2 . \) The phase and group velocities are determined by the following expression

\[ V_{ph} = \frac{\omega}{k} = \pm \frac{(a + k^2)^{1/2}}{k} , \quad V_s = \frac{d \omega}{dk} = \pm \frac{k}{(a + k^2)^{1/2}} . \]

It is necessary to emphasize that signs of the phase and group velocities are the same. Hence, the linear waves in this case are the forward waves.

Equation (4) results in the dispersion relation in the form \( \omega(k) = ak^{-1} . \) Thus, the phase and group velocities are determined by the following expression

\[ V_{ph} = \frac{\omega}{k} = \frac{a}{k^2} , \quad V_s = \frac{d \omega}{dk} = -\frac{a}{k^2} . \]

In this case the signs of the phase and group velocities are opposite ones. Hence, the linear waves in this case are the backward waves.

If we are supposed that equation (1) is the nonlinear generalization of equation (3), we can say equation (1) describes the forward wave soliton. Oppositely, if the Sine Gordon equation (2) is the nonlinear generalization of equation (4), the solitons to equation (2) can be treated as the backward wave solitons.

Another example of the backward wave solitons has been represented in [16], where backward wave propagation was considered by using the Korteweg-de Vries equation. Propagation of the backward solitary waves (they are not solitons in the strict sense) was discussed in many papers. Review of the nonlinear phenomena in this field is presented in [12].

3. Coupled forward and backward waves

New phenomena in nonlinear optics were predicted in the case of the forward and backward wave interaction. Second and third harmonic generation in NRI metamaterials can be accompanied by quadratic and cubic backward wave soliton formation [17-20]. The forward wave and backward wave interaction is realized in a nonlinear coupler and in a nonlinear waveguide array, which consists of alternating waveguides of positive and negative refraction indexes [21-24]. Gap solitons and the bistability of continuous waves in an oppositely directed coupler represent new effects due to the positive-negative refraction phenomenon [21, 24].

The characteristic form of the evolution equations describing the forward and backward wave interaction is

\[ i(e_{1,x} + v_1^{-1}e_{1,t}) = P_1[e_1, e_2] , \quad i(-e_{2,x} + v_2^{-1}e_{2,t}) = P_2[e_1, e_2] , \] (5)

where \( e_1 \) (\( e_2 \)) is a slowly varying envelope of forward (backward) wave, \( v_1 \) (\( v_2 \)) is the group velocity of forward (backward) wave, and \( P_1 \) (\( P_2 \)) is a functional describing the dispersion and coupling effects. For example, in the second harmonic generation case these functional can be written as

\[ P_1[e_1, e_2] = -s_1 e_{1,t} + e_2 e_1^* , \quad P_2[e_1, e_2] = -s_2 e_{2,t} + e_1^2 . \]
At this point the forward and backward wave propagating in cubic nonlinear medium will be considered. Let us assume that the phase velocities are collinear but the group velocities are oppositely directed. The normalized system of evolution equations can be written in following form

\[ i(\epsilon_{1,x} + v_1^2 \epsilon_{1,t}) = s_1 \epsilon_{1,t} + (|\epsilon_1|^2 + |\epsilon_2|^2) \epsilon_1, \]  
\[ i(-\epsilon_{2,x} + v_2^2 \epsilon_{2,t}) = s_2 \epsilon_{2,t} + (|\epsilon_1|^2 + |\epsilon_2|^2) \epsilon_2, \]

where \( s_1 \) (\( s_2 \)) is the coefficient of group velocity dispersion for the forward (backward) wave.

If the following ansatz

\[ \epsilon_1(y) = a_1(y) e^{i\omega_1 t + i\kappa_1 x}, \quad \epsilon_2(y) = a_2(y) e^{i\omega_2 t + i\kappa_2 x}, \quad y = \tau_p^{-1}(t - x/V + t_0), \]

is used, then the family of steady state solutions of the system of equations (6)-(7) can be found. Here \( \omega_{1,2}, \kappa_{1,2}, \tau_p \) and \( t_0 \) are parameters of normalization. Parameter \( V \) is the solitary wave velocity.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{soliton_envelope.png}
\caption{Real envelope of the forward soliton (full line) and backward soliton (dashed line) as function of characteristic variable \( y \).}
\end{figure}

The kind of steady state solutions of system of equations (6)-(7) is defined by the boundary conditions at infinity (\( y \to \pm \infty \)). Thus, there are bright-bright solitary waves (i.e., a coupled bright forward wave and a bright backward wave) and dark-bright solitary waves (for example, a coupled bright forward wave and a dark backward wave). The first kind of the solitary wave is shown in figure 1. At some condition on the parameters \( \omega_{1,2}, \kappa_{1,2} \) and \( v_{1,2} \) the amplitudes \( a_1 \) and \( a_2 \) can be the same.

4. Acknowledgments
The research was supported by Russian Science Foundation (project 14-22-00098). I.R. Gabitov was partially supported by NSF grant DMS-0509589, ARO-MURI award 50342-PH-MUR.

References
[1] Lamb H 1904 Proc. London Math. Soc. 1 473
[2] Pocklington H C 1905 Nature 71 607
[3] Schuster A 1904 An Introduction to the Theory of Optics (London: Edward Arnold)
[4] Mandel'shtam L I 1945 Zh. Eksp. Teor. Fiz. 15 475
[5] Sivukhin D V 1957 Opt. Spektrosk. 3 308
[6] Pafomov V E 1959 Sov.Phys. JETP 9 1321
[7] Veselago V G 1968 Sov. Phys. Usp. 10 509
[8] Veselago V, Braginsky L, Shklover V and Hafner Ch 2006 J. Comput. and Theor. Nanoscience. 3 189
[9] Agranovich V M and Gartstein Yu N 2006 Phys. Usp. 49 1029
[10] Litchinitser N M, Gabitov I R, Maimistov A I and Shalaev V M 2008 Negative refractive index metamaterials in optics Progress in Optics vol. 51 ed. E Wolf. (Amsterdam: Elsevier) chapter 1 pp. 1-68
[11] Maimistov A I and Gabitov I R 2007 Eur. Phys. J. Special Topics 147 265
[12] Lapine M, Shadrivov I V and Kivshar Yu S 2014 Rev. Mod. Phys. 86 1093
[13] Maimistov A I and Gabitov I R 2015 Nonlinear Optical Effects in Positive-Negative Refractive Index Materials. Nonlinear, Tunable and Active Metamaterials ed. I V Shadrivov, M Lapine and Yu S Kivshar (Springer-Verlag, Berlin) pp 133–158
[14] Barone A, Esposito F, Magee C J and Scott A C 1971 Riv.Nuovo Cimento A 1 227
[15] Scott A. 2003 Nonlinear Science: Emergence and Dynamics of Coherent Structures, 2nd ed. (Oxford: OUP).
[16] Weina Cui, Yongyuan Zhu, Hongxia Li and Sumei Liu 2010 Phys.Lett. A 374 380
[17] Maimistov A I, Gabitov I R and Kazantseva E V Optics and Spectroscopy 102 90
[18] Roppo V, Centini M, Sibilia C, Bertolotti M, de Ceglia D, Scalora M, Akozbek N, Bloemer M J, Haus J. W, Kosareva O G and Kandidov V P 2007 Phys. Rev. A 76 033829 (12 pages)
[19] Roppo V, Ciraci C, Cojocaru C and Scalora M 2010 J.Opt.Soc.Amer. B 27 1671
[20] Elyutin S O, Maimistov A I and Gabitov I R 2010 JETP 111 157
[21] Litchinitser N M, Gabitov I R and Maimistov A I 2007 Phys. Rev. Lett. 99 113902
[22] Kazantseva E V, Maimistov A I and Ozhenko S S 2009 Phys. Rev. A 80 43833 (7 pages)
[23] Kazantseva E V and Maimistov A I 2013 Quantum Electron. 43 807
[24] Ryzhov M S and Maimistov A I 2012 Quantum Electron. 42 1034