About Global Stable of Solutions of Logistic Equation with Delay

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Abstract. The article is devoted to the definition of all the arguments for which all positive solutions of logistic equation with delay tend to zero for $t \to \infty$. The authors have proved the acquainted Wright's conjecture on evaluation of a multitude of such arguments. An approach that enables subsequent refinement of this evaluation has been developed.

1. Introduction

Logistic equation with delay

\[ \dot{u} = r[1 - u(t - T)]u \]  

(1)

can be seen in many applied problems (e.g. [1, 2, 3]). Considerable literature is concerned with the analysis of its solution. In the context of the problem the coefficients $r$ and $T$ are positive. Initial data space for the equation (1) is the space $C_{[-T,0]}$. As the analysis of only nonnegative solutions of the equation (1) is of interest, then only nonnegative initial functions from $C_{[-T,0]}$ are revised. The solution with nonnegative initial function posed for $t = t_0$ is nonnegative for all $t > t_0$. Hereafter the term ”solution” is used only for nonnegative solutions of the equation (1).

The number of arguments in the equation (1) can be reduced by applying normalizing time substitution $t \to Tt$. As a result we get the following equation

\[ \dot{u} = \lambda[1 - u(t - 1)]u, \quad \lambda = rT. \]  

(2)

In this equation it is opportune to make a substitution $u = 1 + x$. Then we get the following equation

\[ \dot{x} = -\lambda x(t - 1)[1 + x], \quad x \geq -1. \]  

(3)

Let us enumerate acquainted and necessary for future properties of the equation (3) solutions:
(i) Every solution of the equation (3) at sufficiently large $t$ complies with the following inequation

\[ x(t) \leq \exp(\lambda) - 1. \]  

(4)
(ii) Equilibrium point stability \( x_0 \equiv 0 \) is defined by the position of characteristic quasipolynomial roots (of an equation linear on \( x_0 \) of the equation).

\[
\mu = -\lambda \exp(-\mu). \tag{5}
\]

On condition that

\[
0 < \lambda < \frac{\pi}{2} \tag{6}
\]

all roots of the equality (5) have negative real components, therefore the equilibrium point of \( x_0 \) in the equation (3) is asymptotically stable. The remarkable thing is that the equation (5) for \( \lambda = \pi/2 \) has a couple of purely imaginary roots and the rest its roots have negative real components. Nonlinear analysis of the equation (3) in small neighbourhood of \( x_0 \) cited in works of a lot of authors (e.g. [1, 5]) has shown that zero equilibrium point in the equation (3) for \( \lambda = \pi/2 \) is also asymptotically stable. Furthermore for \( \lambda > \pi/2 \) this equilibrium point is instable and stable cycle is bifurcated from it.

The following article deals with the matter of equilibrium point global stability \( u_0 \equiv 1 \) in the equation (2) or, what is the same, equilibrium point \( x_0 \equiv 1 \) in the equation (3). This means that the problem of finding such values of argument \( \lambda \) that all solutions of the equation (2) with positive initial functions (or all solutions of the equation (3) with initial functions complying with inequation \( x > -1 \)) tend to \( u_0 \) (accordingly to \( x_0 \)) for \( t \to \infty \) is set. In works [1, 2, 3, 4, 5, 6, 7, 8] it was shown that global stability region includes all values of the argument \( \lambda \), complying with inequation \( 0 < \lambda \leq 3/2 \). In the work [1] deals with a hypothesis that there is a more strong result positing that a multitude of such values of \( \lambda \) is posed by the following inequation

\[
0 < \lambda < \frac{37}{24}. \tag{7}
\]

The authors of the following work think that for argumentation of this result using the method of the work [1] you need to overcome "immeasurable" difficulties. In the following paragraphs evaluation of the inequation (7) will be proved and the procedure for continual improvement of the inequation (7) evaluation will be developed. The possibility of getting more accurate evaluations in comparison with the inequation (4) for solutions of the equation (2) is opened up as the condition of the inequation (7) is not fulfilled. Subject to the above mentioned the inequation \( \lambda \leq \pi/2 \) for values of \( \lambda \) from global stability region is trivially performed.

We also note that a numerical study of equation (3) showed that for \( 0 < \lambda < \pi/2 \) all solutions of this equation tend to zero for \( t \to \infty \). However as can be seen analytically it is clear from the following paragraphs that only for the condition \( 0 < \lambda < 37/24 \). In the paper [9], an opinion was expressed that global stability includes \( \lambda \in [1.5, 1.5706] \). But no arguments confirming it was not given.

2. Main Structure

First we note that for every solution of \( x(t) \) (of the equation (3)) and for every whole \( n > 1 \) there is such \( t_0 \), that for \( t > t_0 \) the function \( x(t) \) is \( n \) times continuously differentiable. Moreover from the inequation (4) the possibility of finding the values of coefficients of \( a_n \) and \( b_n \) opens up for for which

\[
-b_n \leq x^{(n)}(t) \leq a_n \tag{8}
\]

For example, for \( n = 0 \) we get \( b_0 = -1, \ a_0 = \exp(\lambda) - 1 \).

Let us assume that solution \( x(t) \) of the equation (3) for some \( \alpha \in (0, 1) \) and \( M \in (0, \exp(\lambda) - 1) \) complies with the following inequations

\[
-\alpha \leq x(t) \leq M \quad (t \geq t_0). \tag{9}
\]
Then for every \( n \) from the equation (3) and the inequation (9) we can get explicit formulae for coefficients \( a_n = a_n(\alpha, M) \) and \( b_n = b_n(\alpha, M) \) that occur in the inequation (8).

Let us arbitrary settle the number \( m \geq 1 \) and label by \( C(m, \alpha, M) \) and \( S(m, \alpha, M) \) extremums of composed functions

\[
C(m, \alpha, M) = \inf \int_0^1 x(s)ds
\]  

(10)

and

\[
S(m, \alpha, M) = \sup \int_0^1 x(s)ds,
\]  

(11)

where upper and lower bounds are taken by all \( m \) times continuously differential solutions of the equation (3) complying with the inequation (9) for \( n = 1, \ldots, m \) and for which \( x(1) = 0 \).

Let us consider that the hypothesis that solution of the equation (3) is equal to zero may not be fulfilled, that is some solution can be of fixed sign (for \( t \geq t_0 \)). But then this solution, as it appears from the equation (3), steadily tends to zero for \( t \to \infty \) and so such solutions are not of interest. In this regard it is useful to highlight that these steadily tending to zero solutions can exist only on condition that \( 0 < \lambda < \exp(-1) \).

2.1. Transition to Two-dimensional Projectivity

Let us set

\[
M_1 = \exp(-\lambda C(m, \alpha, M)) - 1,
\]  

(12)

\[
\alpha_1 = 1 - \exp(-\lambda S(m, \alpha, M)).
\]  

(13)

Therefore there occurs the following two-dimensional projectivity

\[
(\alpha, M) \overset{\Phi(\alpha, M)}{\mapsto} (\alpha_1, M_1) \overset{\Phi(\alpha_1, M_1)}{\mapsto} (\alpha_2, M_2) \overset{\Phi(\alpha_2, M_2)}{\mapsto} \cdots
\]  

(14)

To make all solutions of the equation (3) complying with the inequation \( x \geq -1 \) tend to zero, it is sufficient that for all \( \alpha \in (0, 1) \) and \( M \in (0, \exp(\lambda) - 1) \) the following inequations are performed

\[
\alpha_1 < \alpha, \quad M_1 < M.
\]  

(15)

**Remark 2.1** Let us consider the iterations \((\alpha_n, M_n) \ (n = 2, 3, \ldots)\) of \( \Phi(\alpha, M) \) transformation:

\[
(\alpha_{n+1}, M_{n+1}) = \Phi(\alpha_n, M_n).
\]  

(16)

Let us assume that for some \( \lambda \) we have

\[
\alpha_0(\lambda) = \lim_{n \to \infty} \sup_{k > n} \alpha_k, \quad M_0(\lambda) = \lim_{n \to \infty} \sup_{k > n} M_k.
\]  

Then for solutions of the equation (3) the following equations are performed

\[
\lim_{t \to \infty} \inf_{\tau \geq t} x(\tau) \geq -\alpha_0(\lambda), \quad \lim_{t \to \infty} \sup_{\tau \geq t} x(\tau) \leq M_0(\lambda).
\]  

(17)
2.2. Reduction of Two-dimensional Projectivity of the Equations (12), (13) to Two Projectivities of the First Order
Here we suppose that $m \leq 2$.

Let us assume that for some solution of $x(t)$ for all $t \geq t_0$ the following inequation is performed

$$x(t) \geq -\alpha \quad (0 < \alpha < 1).$$

(18)

First let us note that for $m = 1$ the expression $C(1, \alpha, M)$ does not depend from $M$. Let us assume that $C(\alpha) = C(1, \alpha, M) = \cdots$.

Let

$$M_1 = \exp(-\lambda C(\alpha)) - 1.$$

Let us consider the expression $C(m, \alpha, M_1)$ and assume that

$$M_2 = \exp(-\lambda C(m, \alpha, M_1)) - 1.$$

Then let us pose $M_3, \cdots$ from the formula

$$M_{n+1} = \exp(-\lambda C(m, \alpha, M_n)) - 1 \quad (n = 2, 3, \cdots).$$

(19)

Let us note that there occur inequations

$$M_{n+1} \leq M_n.$$

We denote by $M(\alpha)$ the limit value of the sequence $M_n$:

$$M(\alpha) = \lim_{n \to \infty} M_n.$$

Herewith $M(\alpha)$ is a stable fixed-point value of the equation (19) projectivity.

On the next stage let us consider the equations

$$\alpha_1 = 1 - \exp(-\lambda S(\alpha, M(\alpha)))$$

and

$$\alpha_{n+1} = 1 - \exp(-\lambda S(\alpha_n(\alpha), M(\alpha_n(\alpha)))) \quad (n = 1, 2, \cdots).$$

(20)

Then we give the conclusive result.

**Theorem 2.2** Let us assume that for all $\alpha \in (0, 1)$ the following limit equality is performed

$$\lim_{n \to \infty} \alpha_n(\alpha) = 0.$$

Than all solutions of the equation (3) tend to zero for $t \to \infty$.

**Remark 2.3** For the validity of the Theorem 2.2 statement it is sufficient that for all $\alpha \in (0, 1)$ the inequation $\alpha_1 < \alpha$ is performed.

The next two section are concerned with the study of case when $m = 1$. As will be shown, in these case the two-dimensional projectivity of the equations (12), (13) reduces to one-dimensional one. The case when $m = 1$ the importance of getting prior evaluations for the solutions is shown. In this regard at first, the results in case of rather rough evaluations are given and then it is shown that more accurate evaluations enable to prove sufficient extension of global stability region. Finally in Section 4 for $m = 2$ based on the Theorem 2.2 of the use of more accurate evaluations evaluation of the inequation (7).
3. The Case when $m = 1$

Let $m = 1$. It is reasonable to consider only the case when $\lambda \geq 1$.

Let us show that at a accurate prior evaluation solutions it is possible to extend the range of such values of $\lambda$ at which there is zero equilibrium point global stability in the equation (3).

**Theorem 3.1** Let

\[ 0 < \lambda \leq \frac{3}{2}. \]

Then all solutions of the equation (3) tend to zero for $t \to \infty$.

**Proof.** The equation (3) for $x \neq 0$ is equivalent to the following equation

\[ (1 + x)^{-1} \dot{x} = -\lambda x(t - 1). \] (21)

Then from evaluations of the inequation (9) we get that

\[ -\lambda M \leq (1 + x)^{-1} \dot{x} \leq \lambda \alpha. \] (22)

Therefore we get the more accurate evaluations for $x(t)$

\[
\begin{align*}
-\lambda M \exp(-\lambda M(t - 1)) &\leq \dot{x} \leq \lambda \alpha \exp(\lambda \alpha(t - 1)), \\
-1 + \exp(-\lambda \alpha(1 - t)) &\leq x(t) \leq -1 + \exp(\lambda M(1 - t)).
\end{align*}
\] (23)

It follows that

\[
\begin{align*}
x_0(t) &= \begin{cases} 
-\alpha, & t \in [0,t_0], \\
-1 + \exp(-\lambda \alpha(1 - t)), & t \in (t_0, 1],
\end{cases} \\
x^0(t) &= \begin{cases} 
M, & t \in [0, t^0], \\
-1 + \exp(\lambda M(1 - t)), & t \in (t^0, 1].
\end{cases}
\end{align*}
\]

For $C(1, \alpha, M)$ and $S(1, \alpha, M)$ the following equations are true

\[
C(1, \alpha, M) = \int_0^1 x_0(s)ds = -\alpha + \frac{1}{\lambda} + (1 - \alpha)(\lambda \alpha)^{-1}\ln(1 - \alpha),
\]

\[
S(1, \alpha, M) = \int_0^1 x^0(s)ds = M + \frac{1}{\lambda} - (1 + M)(\lambda M)^{-1}\ln(1 + M).
\]

And in this context the two-dimensional projectivity of the equations (12), (13) reduce to one-dimensional projectivity of the equality

\[
f(\lambda, \beta) = 1 - \exp[-1 - \lambda m(\beta) + m^{-1}(\beta)(1 + m(\beta))]\ln(1 + m(\beta))
\]

and

\[
m(\beta) = -1 + \exp[-1 + \lambda \beta - \beta^{-1}(1 - \beta)\ln(1 - \beta)].
\]

Let us note that $f'(0) = \left(\lambda - \frac{1}{2}\right)^2$, so $f'(0) < 1$ for $\lambda < \frac{3}{2}$.

Thus, based on more accurate evaluations we succeeded to show that for $0 < \lambda < \frac{3}{2}$ all solutions of the equation (3) tend to zero for $t \to \infty$. Let us note that for $\lambda \leq \frac{\pi}{2}$, for every solution at larger $t$ the following evaluation is performed

\[ x(t) \geq -\beta_0 = -0.5254171515. \]

The theorem is proved.
4. The Case when \( m = 2 \)

Now we formulate our main result.

**Theorem 4.1** Let \( 0 < \lambda < \frac{37}{24} \). Then all solutions of the equation (3), where certain functions corresponding the valuation \( x(s) > -1 \), tend to zero for \( t \to \infty \).

Proof. Let us arbitrary settle solution \( x(t) \) of the equation (3). Let us assume that for all \( t \geq t_0 \), the following evaluations are performed

\[
x(t) \geq -\alpha \quad (1 > \alpha > 0).
\]

Let us label by \( M \) the supremum of this solution, i.e.

\[
\sup_{t>t_0} x(t) = M.
\]

Then for the value \( M \) from the previous formulations comes the evaluation

\[
M \leq \exp \left[ -\lambda \alpha \left( 1 - \frac{1}{2\lambda} \right) \right] - 1 = M_1(\alpha, \lambda).
\]

In this context we can take \( a_2 = a_2(M_1) \) and with regard to (26)

\[
a_2 = \lambda^2 M_1(1 + 2M_1).
\]

Let us consider the inequations (23) to improve the achieved evaluation. Then for \( x_0(t) \) we get the following function

\[
x_0(t) = \begin{cases} 
-\alpha, & -1 \leq t \leq t_1, \\
-\alpha + \frac{1}{2} a_2 (t - t_1)^2, & t_1 \leq t \leq t_2, \\
-1 + \exp(\lambda \alpha t), & t_2 < t \leq 0,
\end{cases}
\]

Here

\[
t_2 = \frac{\ln \left( \frac{a_2 \Delta_0^2}{2} - \alpha + 1 \right)}{\alpha \lambda}, \quad t_1 = t_2 - \Delta_0, \quad \Delta_0 = \frac{a_2 - \sqrt{a_2^2 + a_0}}{\alpha \lambda a_2},
\]
\[ a_0 = 2\alpha^2\lambda^2a_2(\alpha - 1). \]

The projectivity \( M_{n+1}(\alpha, M_n, \lambda) \) is expressed in the following way
\[ M_{n+1} = \exp[-\lambda C(2, \alpha, M_n)] - 1. \]  \hfill (27)

Consequent iterations of the projectivity
\[ M_1 \rightarrow M_2 \rightarrow \cdots. \]

Lead to the conclusion that the desired upper evaluation of \( x(t) \) solution doesn’t rank over the fixed-point value of (27) projectivity.

\[ \text{Figure 3. Graph of the function } f(\lambda, \beta) \text{ where } \lambda = \frac{37}{24} \]

\[ \text{Figure 4. Graph of the function } f(\lambda, \beta) \text{ where } \lambda = \frac{\pi}{2}, \alpha_0 = 0.4809292426 \]

After definition of the fixed point \( M(\alpha) \), the function \( x^0(t) \) on which the desired value \( S(2, \alpha, M(\alpha)) \) is realized, is written as
\[
x^0(t) = \begin{cases} 
M(\alpha), & 0 \leq t \leq \tau_1, \\
M(\alpha) + \frac{1}{2} b_2(t - \tau_1)^2, & \tau_1 < t \leq \tau_2, \\
-1 + \exp(-\lambda M(\alpha)(t - 1)), & \tau_2 < t \leq 1.
\end{cases}
\]

Here
\[ \tau_2 = 1 - \frac{\ln \left( \frac{b_0 \Delta_2^2 + M(\alpha) + 1}{M(\alpha)\lambda} \right)}{M(\alpha)\lambda}, \quad \tau_1 = \tau_2 - \Delta_1, \quad \Delta_1 = -\frac{b_2 + \sqrt{b_2^2 - b_0}}{M(\alpha)\lambda b_2}, \]
\[ b_0 = 2M(\alpha)^2\lambda^2b_2(M(\alpha) + 1), \quad b_2 = -\lambda^2\alpha(1 + M(\alpha))^2. \]

Then for equations \( C(2, \alpha, M(\alpha)) \) and \( S(2, \alpha, M(\alpha)) \) we have the formulas
\[
C(2, \alpha, M(\alpha)) = \int_{-1}^{0} x_0(s) ds = [6a_2^2\alpha^2\lambda^2(1 - \alpha) \ln \left( \frac{1}{2}(a_2 - \sqrt{a_2^2 + a_0}) \right) +
+3\sqrt{a_2^2 + a_0a_2^2 - 2a_2^2 + (-6a_4\lambda^2 + 6a_3\lambda^2)a_2^2 - (a_2^2 + a_0)^2} \cdot (6a_2^2\alpha^3\lambda^3)^{-1},
\]

\[ 7 \]
\[ S(2, \alpha, M(\alpha)) = \int_0^1 x^0(s)ds = [6M^2(\alpha)\lambda^2(M^2(\alpha) + M(\alpha)\lambda - M(\alpha) - 2)b_2^2 - \\
-6b_2^2M^2(\alpha)\lambda^2(M(\alpha) + 1)\ln \left( \frac{\lambda(-b_2 - \sqrt{b_2^2 - b_0^2})}{M^2(\alpha)\lambda^2b_2} + M(\alpha) + 1 \right) - \\
-9\sqrt{b_2^2 - b_0b_2} - 10b_2^2 + 6b_2b_0 - (b_2^2 - b_0^2)\cdot (6b_2^2M^3(\alpha)\lambda^3)^{-1}. \]

Hence, that if \( 0 < \lambda < \frac{37}{24} \) then all orbits of the mapping (27) tend to zero for \( n \to \infty \). The theorem is proved.

Intimate, that \( f'_n(\lambda, 0) = (\lambda - \frac{13}{24})^2 \), thus \( f'_n(\lambda, 0) < 1 \) for \( \lambda < \frac{37}{24} \).

5. Conclusions

Wright’s conjecture [1] that under condition of the inequation (7) zero equilibrium point of the equation (3) is globally stable, has been proved. An approach of subsequent refinement of corresponding evaluations has been developed.

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