Scattering and correlations

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Introduction

Let us consider the propagation of scalar waves with the speed $v > 0$ given by the wave equation $u_{tt} - v^2 \Delta u = 0$ outside a compact domain $D$ in the Euclidean space $\mathbb{R}^d$. Let us put $\Omega = \mathbb{R}^d \setminus D$. We can assume for example Neumann boundary conditions. We will denote by $\Delta_\Omega$ the (self-adjoint) Laplace operator with the boundary conditions. So our stationary wave equation is the Helmholtz equation

$$v^2 \Delta_\Omega f + \omega^2 f = 0$$

with the boundary conditions. We consider a bounded interval $I = [\omega^2_-, \omega^2_+] \subset [0, +\infty[$ and the Hilbert subspace $\mathcal{H}_I$ of $L^2(\Omega)$ which is the image of the spectral projector $P_I$ of our operator $-v^2 \Delta_\Omega$.

Let us compute the integral kernel $\Pi_I(x, y)$ of $P_I$ defined by:

$$P_I f(x) = \int_\Omega \Pi_I(x, y) f(y) |dy|$$

into 2 different ways:

1. From general spectral theory
2. From scattering theory.

Taking the derivatives of $\Pi_I(x, y)$ w.r. to $\omega_+$, we get a simple general and exact relation between the correlation of scattered waves and the Green’s function confirming the calculations from Sanchez-Sesma and al. [March 2006] in the case where $D$ is a disk.

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1 \( \Pi_I(x, y) \) from spectral theory

Using the resolvent kernel (Green’s function) \( G(\omega, x, y) = [(\omega^2 + v^2 \Delta_\Omega)^{-1}](x, y) \) for \( \Im \omega > 0 \) and the Stone formula, we have:

\[
\Pi_I(x, y) = -\frac{2}{\pi} \text{Im} \left( \int_{\omega_-}^{\omega_+} G(\omega + i0, x, y) \omega \, d\omega \right)
\]

Taking the derivative w.r. to \( \omega_+ \) of \( \Pi_{[\omega^2, \omega^2]}(x, y) \), we get

\[
\frac{d}{d\omega} \Pi_{[\omega^2, \omega^2]}(x, y) = -\frac{2\omega}{\pi} \text{Im}(G(\omega + i0, x, y)) \, .
\]  

(2)

2 Short review of scattering theory

They are many references for scattering theory: for example Reed-Simon, Methods of modern Math. Phys. III; Ramm, Scattering by obstacles.

Let us define for \( k \in \mathbb{R}^d \) the plane wave

\[
e_0(x, k) = e^{i< k | x >} \, .
\]

We are looking for solutions

\[
e(x, k) = e_0(x, k) + e^s(x, k)
\]

of the Helmholtz equation (1) in \( \Omega \) where \( e^s \), the scattered wave satisfies the so-called Sommerfeld radiation condition\(^1\):

\[
e^s(x, k) = \frac{e^{ik|x|}}{|x|^{(d-1)/2}} \left( e^\infty \left( \frac{x}{|x|}, k \right) + O \left( \frac{1}{|x|} \right) \right), \quad x \to \infty.
\]

The complex function \( e^\infty(\hat{x}, k) \) is usually called the scattering amplitude.

It is known that the previous problem admits an unique solution. In more physical terms, \( e(x, k) \) is the wave generated by the full scattering process from the plane wave \( e_0(x, k) \). Moreover we have a generalised Fourier transform:

\[
f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \hat{f}(k) e(x, k) \, |dk| 
\]

with

\[
\hat{f}(k) = \int_{\mathbb{R}^d} e(y, \bar{k}) f(y) \, |dy| \, .
\]

From the previous generalised Fourier transform, we can get the kernel of any function \( \Phi(-v^2 \Delta_\Omega) \) as follows:

\[
[\Phi(-v^2 \Delta_\Omega)](x, y) = (2\pi)^{-d} \int_{\mathbb{R}^d} \Phi(v^2 k^2) e(x, k) e(y, \bar{k}) \, |dk| \, .
\]  

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\(^1\)As often, we denote \( k := |k| \) and \( \bar{k} := k/k \)
3 $\Pi_I(x, y)$ from scattering theory

Using Equation (3) with $\Phi = 1_I$ the characteristic functions of some bounded interval $I = [\omega^2, \omega^2]$, we get:

$$
\Pi_I(x, y) = (2\pi)^{-d} \int_{\omega_{-} \leq k \leq \omega} e(x, k)\overline{e(y, k)}|dk| .
$$

Using polar coordinates and defining $|d\sigma|$ as the usual measure on the unit $(d-1)$-dimensional sphere, we get:

$$
\Pi_I(x, y) = (2\pi)^{-d} \int_{\omega_{-} \leq k \leq \omega} \int_{k^2 = k} e(x, k)\overline{e(y, k)}k^{d-1}dk|d\sigma| .
$$

We will denote by $\sigma_{d-1}$ the total volume of the unit sphere in $\mathbb{R}^d$: $\sigma_0 = 2$, $\sigma_1 = 2\pi$, $\sigma_2 = 4\pi, \cdots$.

Taking the same derivative as before, we get:

$$
\frac{d}{d\omega} \Pi_{[\omega^2, \omega^2]}(x, y) = (2\pi)^{-d} \frac{d\omega^{d-1}}{vd} \int_{|k| = \omega} e(x, k)\overline{e(y, k)}|d\sigma| .
$$

Let us look at $e(x, k)$ as a random wave with $k = \omega/v$ fixed. The point-point correlation of such a random wave $C_{\omega}^{\text{scatt}}(x, y)$ is given by:

$$
C_{\omega}^{\text{scatt}}(x, y) = \frac{1}{\sigma_{d-1}} \int_{vk = \omega} e(x, k)\overline{e(y, k)}|d\sigma| .
$$

Then we have:

$$
\frac{d}{d\omega} \Pi_{[\omega^2, \omega^2]}(x, y) = (2\pi)^{-d} \frac{d\omega^{d-1}\sigma_{d-1}}{vd} C_{\omega}^{\text{scatt}}(x, y) .
$$

4 Correlation of scattered plane waves and Green’s function: the scalar case

From Equations (2) and (4), we get:

$$(2\pi)^{-d} \frac{d\omega^{d-1}\sigma_{d-1}}{vd} C_{\omega}^{\text{scatt}}(x, y) = -\frac{2\omega}{\pi} \text{Im}(G(\omega + i0, x, y)) .$$

Hence

$$C_{\omega}^{\text{scatt}}(x, y) = -\frac{2^{d+1}\pi^{d-1}\omega^d}{\sigma_{d-1}\omega^{d-2}} \text{Im}(G(\omega + i0, x, y)) .$$

For later use, we put

$$\gamma_d = \frac{2^{d+1}\pi^{d-1}}{\sigma_{d-1}} .$$

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5 The case of elastic waves

We will consider the elastic wave equation in the domain $\Omega$:

$$\hat{H}u - \omega^2 u = 0,$$

with self-adjoint boundary conditions. We will assume that, at large distances, we have

$$\hat{H}u = -a \Delta u - b \text{grad} \text{div} u,$$

where $a$ and $b$ are constants:

$$a = \frac{\mu}{\rho}, \quad b = \frac{\lambda + \mu}{\rho}$$

with $\lambda, \mu$ the Lamé’s coefficients and $\rho$ the density of the medium. We will denote $v_P := \sqrt{a + b}$ (resp. $v_S := \sqrt{a}$) the speeds of the $P -$ (resp. $S -$) waves near infinity.

5.1 The case $\Omega = \mathbb{R}^d$

We want to derive the spectral decomposition of $\hat{H}$ from the Fourier inversion formula. Let us choose, for $k \neq 0$, by $\hat{k}, \hat{k}_1, \ldots, \hat{k}_{d-1}$ an orthonormal basis of $\mathbb{R}^d$ with $\hat{k} = \frac{k}{k}$ such that these vectors depends in a measurable way of $k$. Let us introduce $P^k_P = \hat{k}\hat{k}^*$ the orthogonal projector onto $\hat{k}$ and $P^k_S = \sum_{j=1}^{d-1} \hat{k}_j\hat{k}_j^*$ so that $P_P + P_S = \text{Id}$. Those projectors correspond respectively to the polarisations of $P -$ and $S -$ waves.

We have

$$\Pi_I(x, y) = (2\pi)^{-d} \int_{\omega \in I} \omega^{d-1} d\omega \left( v_P^{-d} \int_{v_P k = \omega} e^{i\hat{k}(x-y)} P^k_P d\sigma + v_S^{-d} \int_{v_S k = \omega} e^{i\hat{k}(x-y)} P^k_S d\sigma \right).$$

using the plane waves

$$e^O_P(x, k) = e^{i\hat{k}x} \hat{k}$$

and

$$e^O_{S,j}(x, k) = e^{i\hat{k}_j x} \hat{k}_j$$

we get the formula$^2$:

$$\Pi_I(x, y) = (2\pi)^{-d} \int_{\omega \in I} \omega^{d-1} d\omega \left( v_P^{-d} \int_{v_P k = \omega} |e^O_P(x, k)\rangle\langle e^O_P(y, k)| d\sigma + v_S^{-d} \sum_{j=1}^{d-1} \int_{v_S k = \omega} |e^O_{S,j}(x, k)\rangle\langle e^O_{S,j}(y, k)| d\sigma \right).$$

$^2$We use the “bra-ket” notation of quantum mechanics: $|e\rangle\langle f|$ is the operator $x \rightarrow \langle f|x|e$ where the brackets are linear w.r. to the second entry and anti-linear w.r. to the first one.
5.2 Scattered plane waves

There exists scattered plane waves

\[ e_P(x, k) = e_P^O(x, k) + e_P^s(x, k) \]

\[ e_{S,j}(x, k) = e_{S,j}^O(x, k) + e_{S,j}^s(x, k) \]

satisfying the Sommerfeld condition and from which we can deduce the spectral decomposition of \( \hat{H} \).

5.3 Correlations of scattered plane waves and Green’s function

Following the same path as for scalar waves, we get an identity which holds now for the full Green’s tensor \( \text{Im}G(\omega + iO, x, y) \):

\[
\begin{align*}
\text{Im}G(\omega + iO, x, y) &= -\gamma_d^{-1}\omega^{d-2}\left(\frac{1}{\sigma_{d-1}v_P^d} \int_{v_Pk=\omega} |e_P(x, k)\rangle\langle e_P(y, k)| d\sigma + \right. \\
&\left. \frac{1}{\sigma_{d-1}v_S^d} \sum_{j=1}^{d-1} \int_{v_Sk=\omega} |e_{S,j}(x, k)\rangle\langle e_{S,j}(y, k)| d\sigma \right),
\end{align*}
\]

with \( \gamma_d \) defined by Equation (5).

This formula expresses the fact that the correlation of scattered plane waves randomised with the appropriate weights (\( v_P^{-d} \) versus \( v_S^{-d} \)) is proportional to the Green’s tensor. Let us insist on the fact that this true everywhere in \( \Omega \) even in the domain where \( a \) and \( b \) are not constants.