A COMPACTIFICATION
OF THE MODULI SCHEME OF ABELIAN VARIETIES

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Abstract. We construct a canonical compactification $\text{SQ}_g,K^{\text{toric}}$ of the moduli of abelian varieties over $\mathbb{Z}[\zeta_N,1/N]$ where $\zeta_N$ is a primitive $N$-th root of unity. It is very similar to, but slightly different from the compactification $\text{SQ}_g,K$ in [N99]. Any degenerate abelian scheme on the boundary of $\text{SQ}_g,K^{\text{toric}}$ is one of the (torically) stable quasi-abelian schemes introduced in [AN99], which is reduced and singular. In contrast with it, some of degenerate abelian schemes on the boundary of $\text{SQ}_g,K$ are nonreduced schemes.

1. Introduction.

In the article [N99] a canonical compactification $\text{SQ}_{g,K}$ of the moduli scheme $A_{g,K}$ of abelian varieties with level structure was constructed by applying the geometric invariant theory [MFK94]. It is a compactification of $A_{g,K}$ by all Kempf-stable degenerate abelian schemes. However some of the Kempf-stable degenerate abelian schemes are nonreduced by contrast with Deligne-Mumford stable curves.

The purpose of this article is to construct another canonical compactification $\text{SQ}_{g,K}^{\text{toric}}$ of $A_{g,K}$ by certain singular reduced degenerate abelian schemes only instead of Kempf-stable ones. The new compactification $\text{SQ}_{g,K}^{\text{toric}}$ is very similar to $\text{SQ}_{g,K}$. The compactifications are as functors the same if $g \leq 4$, and different if $g \geq 8$ (or maybe if $g \geq 5$). An advantage of $\text{SQ}_{g,K}^{\text{toric}}$ is that the reduced degenerate abelian schemes on the boundary $\text{SQ}_{g,K}^{\text{toric}} \setminus A_{g,K}$ are much simpler than Kempf-stable ones.

Let $R$ be a complete discrete valuation ring and $k(\eta)$ the fraction field of $R$. Given an abelian variety $(G_\eta,\mathcal{L}_\eta)$ over $k(\eta)$, we have Faltings-Chai’s degeneration data of it by a finite base change if necessary. Then there are two natural choices of $R$-flat projective degenerating families $(P,\mathcal{L})$ and $(Q,\mathcal{L})$ of abelian varieties with generic fibre isomorphic to $(G_\eta,\mathcal{L}_\eta)$ where $(Q,\mathcal{L})$ is the most naive choice, while $(P,\mathcal{L})$ is the normalization of $(Q,\mathcal{L})$ after a certain finite base change such that the closed fibre $P_0$ is reduced.

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We call the closed fibre \((P_0, L_0)\) of \((P, L)\) a torically stable quasi-abelian scheme (abbr. TSQAS) \([AN99]\), while we call \((Q_0, L_0)\) a projectively stable quasi-abelian scheme (abbr. PSQAS) \([N99]\).

Let \((K, e)\) be a finite symplectic abelian group. Since 
\[K \cong \bigoplus_{i=1}^{g} (\mathbb{Z}/e_i \mathbb{Z})^\oplus\]
for some positive integers \(e_i\) such that \(e_i|e_{i+1}\), we define \(e_{\min}(K) = e_1\) and \(e_{\max}(K) = e_g\). Let \(N = e_{\max}(K)\). The Heisenberg group \(G(K)\) is by definition a central extension of \(K\) by the group \(\mu_N\) of all \(N\)-th roots of unity.

The classical level \(K\)-structures on abelian varieties are generalized as level \(G(K)\)-structures on TSQASes.

**Theorem.** If \(e_{\min}(K) \geq 3\), the functor of \(g\)-dimensional torically stable quasi-abelian schemes with level \(G(K)\)-structure over reduced base algebraic spaces is coarsely represented by a complete reduced separated algebraic space \(SQ_{\text{toric}}^{g,K}\) over \(\mathbb{Z}[[\zeta_N, 1/N]]\).

We prove the theorem with the help of \([N99]\) and Keel-Mori \([KM97]\). Alexeev \([A99]\) treats similar problems in a different formulation.

Here is an outline of our article. In Section 2 we recall from \([N99]\) a couple of basic facts about degenerating families of abelian varieties. In Section 3 we define a rigid \(G(K)\)-structure and a level \(G(K)\)-structure on a TSQAS \((P_0, L_0)\) or their family. In Section 4 we recall from \([N99]\) the stable reduction theorem for TSQASes with rigid \(G(K)\)-structure. In Sections 5, 6 and 7 we prove the above theorem.

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2. Degenerating families of abelian varieties

The purpose of this section is to recall basic facts about degenerating families of abelian varieties. To minimize the article we try to keep the same notation as \([N99]\).

2.1. Grothendieck’s stable reduction. Let \(R\) be a complete discrete valuation ring, \(I\) the maximal ideal of \(R\) and \(S = \text{Spec} \ R\). Let \(\eta\) be the generic point of \(S\), \(k(\eta)\) the fraction field of \(R\) and \(k(0) = R/I\) the residue field.

Suppose we are given a \(g\)-dimensional polarized abelian variety \((G_\eta, L_\eta)\) over \(k(\eta)\) such that \(L_\eta\) is symmetric, ample and rigidified (that is trivial) along the unit section. Then by Grothendieck’s stable reduction theorem \([SGA7]\), \((G_\eta, L_\eta)\) can be extended to a polarized semiabelian \(S\)-scheme \((G, L)\) with \(L\) a rigidified relatively ample invertible sheaf on \(G\) as the connected Néron model of \(G_\eta\) by taking a finite extension \(K'\) of \(k(\eta)\) if necessary. The closed fibre \(G_0\) is a semiabelian scheme over \(k(0)\), namely an extension of an abelian variety \(A_0\) by a split torus \(T_0\).

From now on we restrict ourselves to the totally degenerate case, that is, the case where \(A_0\) is trivial because by \([N99]\) there is no essentially new difficulty when we consider the case \(A_0\) is nontrivial. Hence we assume that \(G_0\) is a split \(k(0)\)-torus. Let \(\lambda(L_\eta) : G_\eta \to G_\eta^t\) be the polarization
(epi)morphism. By the universal property of the (connected) Néron model $G^t$ of $G^t_n$ we have an epimorphism $\lambda : G \to G^t$ extending $\lambda(\mathcal{L}_n)$. Hence the closed fibre of $G^t$ is also a split $k(0)$-torus.

Let $S_n = \text{Spec} R/I^{n+1}$ and $G_n = G \times_S S_n$. Associated to $G$ and $\mathcal{L}$ are the formal scheme $G_{\text{for}} = \lim G_n$ and an invertible sheaf $\mathcal{L}_{\text{for}} = \lim \mathcal{L} \otimes R/I^{n+1}$. By our assumption that $G_0$ is a $k(0)$-split torus, $G_n$ turns out to be a multiplicative group scheme for every $n$ by \cite{Faltings-Chai90}, Theorem 2.3. Let $\mathcal{L}_{\text{for}}$ be a multiplicative group scheme for every $n$ by \cite{Faltings-Chai90}, p. 7]. Thus the scheme $G_{\text{for}}$ is a formal split $S$-torus. Similarly $G_{\text{for}}^t$ is a formal split $S$-torus. Let $X := \text{Hom}_Z(G_{\text{for}}, (G_{m,s})_{\text{for}})$, $Y := \text{Hom}_Z(G_{\text{for}}^t, (G_{m,s})_{\text{for}})$ and $\bar{G} := \text{Hom}_Z(X, G_{m,s})$, $\bar{G}^t = \text{Hom}(Y, G_{m,s})$. Then $G$ (resp. $G^t$) algebraizes $G_{\text{for}}$ (resp. $G_{\text{for}}^t$). The morphism $\lambda : G \to G^t$ induces an injective homomorphism $\phi : Y \to X$ and an algebraic epimorphism $\bar{\lambda} : \bar{G} \to \bar{G}^t$. For simplicity we identify the injection $\phi : Y \to X$ with the inclusion $Y \subset X$.

2.2. Fourier expansions. In the totally degenerate case $G_{\text{for}}$ (resp. $\bar{G}$) is a formal split $S$-torus (resp. a split $S$-torus). We choose and fix the coordinate $w^x (x \in X)$ of $\bar{G}$ satisfying $w^x w^y = w^{x+y}$ ($\forall x, y \in X$). Since $\mathcal{L}_{\text{for}}$ is trivial on $G_{\text{for}}$, we have

$$\Gamma(G_n, \mathcal{L}_n) = \Gamma(G, \mathcal{L}) \otimes k(\eta) \hookrightarrow \Gamma(G_{\text{for}}, \mathcal{L}_{\text{for}}) \otimes k(\eta) \hookrightarrow \prod_{x \in X} k(\eta) \cdot w^x.$$ 

Therefore, any element $\theta \in \Gamma(G_n, \mathcal{L}_n)$ can be written as a formal Fourier series $\theta = \sum_{x \in X} \sigma_x(\theta) w^x$ with $\sigma_x(\theta) \in k(\eta)$, which converges I-adically.

**Theorem 2.3.** [Faltings–Chai90] Let $k(\eta)^* = k(\eta) \setminus \{0\}$. There exists a function $a : Y \to k(\eta)^*$ and a bimultiplicative function $b : Y \times X \to k(\eta)^*$ with the following properties:

1. $b(y,x) = a(x+y) a(x)^{-1} a(y)^{-1}$, $a(0) = 1$ ($\forall x \in X, \forall y \in Y$),
2. $b(y,z) = b(z,y) = a(y+z) a(y)^{-1} a(z)^{-1}$ ($\forall y, z \in Y$),
3. $b(y,y) \in I$ ($\forall y \neq 0$), and for every $n \geq 0$, $a(y) \in I^n$ for almost all $y \in Y$,
4. $\Gamma(G_n, \mathcal{L}_n)$ is identified with the $k(\eta)$ vector subspace of formal Fourier series $\theta$ that satisfy $\sigma_{x+y}(\theta) = a(y) b(y,x) \sigma_x(\theta)$ and $\sigma_x(\theta) \in k(\eta)$ ($\forall x \in X, \forall y \in Y$).

**Definition 2.4.** By taking a finite base change of $S$ if necessary the functions $b$ and $a$ can be extended respectively to $X \times X$ and $X$ so that the previous relations between $b$ and $a$ are still true on $X \times X$. Let $R^* = R \setminus \{0\}$ and $k(0)^* = k(0) \setminus \{0\}$. Then we define integer-valued functions $A : X \to \mathbb{Z}$, $B : X \times X \to \mathbb{Z}$ and $b(y,x) \in R^*$, $\bar{a}(y) \in R^*$ by

$$B(y,x) = \text{val}_s(b(y,x)),$$  
$$A(x) = \text{val}_s(a(x)) = B(x,x)/2 + r(x)/2,$$  
$$b(y,x) = \bar{b}(y,x) s^{B(y,x)},$$  
$$a(x) = \bar{a}(x)s^{(B(x,x)+r(x))/2}.$$
for some \( r \in \text{Hom}_\mathbb{Z}(X, \mathbb{Z}) \), where \( B \) is positive definite by Theorem 2.3. We set \( a_0 = \bar{a} \mod I \) and \( b_0 = \bar{b} \mod I \). Hence \( a_0(x), b_0(y, x) \in k(0)^* \).

**Definition 2.5.** Let \( X \) be a lattice of rank \( g \), \( X_\mathbb{R} = X \otimes \mathbb{R} \), and let \( B : X \times X \to \mathbb{Z} \) be a positive definite symmetric integral bilinear form, which determines a distance \( \| \cdot \|_B \) on \( X_\mathbb{R} \) by \( \| x \|_B := \sqrt{B(x, x)} \,(x \in X_\mathbb{R}) \). For any \( \alpha \in X_\mathbb{R} \) we say that \( a \in X \) is \( \alpha \)-nearest if

\[
\| a - \alpha \|_B = \min \{ \| b - \alpha \|_B ; b \in X \}.
\]

We define a Delaunay cell \( \sigma \) to be the closed convex hull of all lattice elements which are \( \alpha \)-nearest for some \( \alpha \in X_\mathbb{R} \). All the Delaunay cells constitute a locally finite decomposition of \( X_\mathbb{R} \), which we call the Delaunay decomposition \( \text{Del}_B \). Let \( \text{Del} := \text{Del}_B \), and \( \text{Del}(c) \) the set of all the Delaunay cells containing \( c \in X \). For \( \sigma \in \text{Del}(c) \), we define \( C(c, \sigma) \) to be the cone spanned by edges of \( \sigma \) at \( c \). See [N99, p. 662].

**Definition 2.6.** We define

\[
\tilde{R} := R[a(x)w^x \vartheta; x \in X] \simeq R[\xi_x \vartheta; x \in X],
\]

\[
\xi_x := s^{B(x, x)/2+r(x)/2}w^x,
\]

\[
\zeta_{x,c} := s^{B(a(c,x)+r(x)/2)w^x} \,(x + c \in C(c, \sigma))
\]

where \( \tilde{R} \) is the graded algebra with \( \deg(a(x)w^x \vartheta) = 1 \) and \( \deg a = 0 \) for \( a \in R \), while \( \sigma \in \text{Del}(c) \) is a Delaunay \( g \)-cell with \( x + c \in C(c, \sigma) \). Let \( \tilde{Q} := \text{Proj}(\tilde{R}) \). We define an action \( S_y \) on \( \tilde{Q} \) by

\[
S_y^*(a(x)w^x \vartheta) = a(x + y)w^{x+y} \vartheta \quad \text{for} \quad y \in Y.
\]

which induces a natural action on \( \tilde{P} \), the normalization of \( \tilde{Q} \), denoted by the same \( S_y \). By \( \tilde{\mathcal{L}} \) we denote \( O_{\text{Proj}(1)} \) on \( \tilde{Q} \) as well as its pullback to \( \tilde{P} \).

**Theorem 2.7.** 1. The quotients \( (\tilde{P}_\text{for}, \tilde{\mathcal{L}}_\text{for})/Y \) and \( (\tilde{Q}_\text{for}, \tilde{\mathcal{L}}_\text{for})/Y \) are flat projective formal \( S \)-schemes.

2. There exist flat projective \( S \)-schemes \( (P, \mathcal{L}) \) and \( (Q, \mathcal{L}) \) such that their formal completions \( (P_{\text{for}}, \mathcal{L}_{\text{for}}) \) and \( (Q_{\text{for}}, \mathcal{L}_{\text{for}}) \) along the closed fibres are respectively isomorphic to \( (\tilde{P}_\text{for}, \tilde{\mathcal{L}}_\text{for})/Y \) and \( (\tilde{Q}_\text{for}, \tilde{\mathcal{L}}_\text{for})/Y \).

3. \( P \) is the normalization of \( Q \).

**Proof.** This follows from [EGA, III, 5.4.5] and [M72]. See also [N99].

**Definition 2.8.** Let \( \text{Del}(P_0) \) be the Delaunay decomposition corresponding to \( P_0 \). By taking a finite base change of \( S \) if necessary, we may assume that \( dA(\alpha(\sigma)) \in \text{Hom}(X, \mathbb{Z}) \) for any Delaunay cell \( \sigma \in \text{Del}(P_0) \). By [AN99] this implies that \( P_0 \) is reduced. We call the closed fibre \( (P_0, \mathcal{L}_0) \) of \( (P, \mathcal{L}) \) a torically stable quasi-abelian scheme (abbr. a TSQAS) over \( k(0) := R/I \).

In what follows we always assume that \( dA(\alpha(\sigma)) \in \text{Hom}(X, \mathbb{Z}) \) for any \( \sigma \in \text{Del}(P_0) \). Hence \( P_0 \) is reduced.
We quote a few theorems from [AN99] and [N93].

**Theorem 2.9.** Let \((\bar{P}_0, \bar{L}_0)\) be the closed fibre of \((\bar{P}, \bar{L})\) and \(\tilde{\zeta}_{x,c} := \zeta_{x,c} \otimes k\) the restriction to \(P_0\). Then

1. \(\bar{P}_0\) is covered with infinitely many affine schemes of finite type
   \[ U_0(c) := \text{Spec } k[\tilde{\zeta}_{x,c}; x \in X] \quad (c \in X). \]

2. \(R_0(c) := k[\tilde{\zeta}_{x,c}; x \in X]\) is a \(k\)-algebra of finite type. Let \(x_i \in X\). If \(x_i\)'s belong to one and the same Delaunay cell (resp. otherwise), then
   \[ \tilde{\zeta}_{x_1,c} \cdots \tilde{\zeta}_{x_m,c} = \tilde{\zeta}_{x_1 + \cdots + x_m,c} \quad (\text{resp. } 0). \]

**Theorem 2.10.** Let \((P_0, \mathcal{L}_0)\) be the closed fibre of \((P, \mathcal{L})\). Let \(\sigma\) and \(\tau\) be Delaunay cells in \(\text{Del}(P_0)\).

1. For each \(\sigma \in \text{Del}(P_0)\) there exists a \(G\)-invariant subscheme \(O(\sigma)\) of \(P_0\) which is a torus of dimension \(\dim \sigma\) over \(k(0)\),
2. \(\sigma \subset \tau\) iff \(O(\sigma)\) is contained in \(\overline{O(\tau)}\), the closure of \(O(\tau)\) in \(P_0\),
3. \(P_0 = \bigcup_{\sigma \in \text{Del}(P_0)} \text{mod } Y O(\sigma)\).

**Theorem 2.11.** Let \(n > 0\). Then

1. \(h^0(P_0, \mathcal{L}_0^n) = [X : Y^n]\), \(h^i(P_0, \mathcal{L}_0^n) = 0\) \((i > 0)\), and
   \[
   \Gamma(P_0, \mathcal{L}_0) = \left\{ \sum_{x \in X} c(x)\xi_x; \begin{array}{ll}c(x + y) = b_0(y, x) a_0(y) c(x) \in k, \forall x \in X, \forall y \in Y, \end{array} \right. \}
   \]
2. \(\mathcal{L}_0^n\) is very ample for \(n \geq 2g + 1\).

2.12. **The group schemes \(G\) and \(G^\sharp\).** We review [N99, 3.12] to recall the notation. By choosing a suitable base change of \(S\) we assume \(dA(\alpha(\sigma)) \in \text{Hom}(X, \mathbb{Z})\) for any \(\sigma \in \text{Del}_B\). Then \(P_0\) is reduced. Then \(G\) is realized as an open subscheme of \(P\). In fact, for any Delaunay \(g\)-cell \(\sigma \in \text{Del}(0)\), there is an open smooth subscheme \(G(\sigma) \subset P\) such that

1. \(G(\sigma) \simeq G, G(\sigma)_\eta = P_\eta, G(\sigma)_0 = O(\sigma)\),
2. \(G(\sigma)_{\text{for}}\) is a formal \(S\)-torus.

We define \(G^\sharp = G^\sharp(\sigma) := \bigcup_{x \in (X/Y)} S_x(G(\sigma)) \subset P\). Then \(G^\sharp\) is a group scheme over \(S\) such that \(G^\sharp_\eta = P_\eta\). It is uniquely determined by \(P\), independent of the choice of \(\sigma\). See [N99, 4.13] for the detail.

2.13. **The Heisenberg group.** Let \(K(\mathcal{L}_\eta)\) be the kernel of \(\lambda(\mathcal{L}_\eta) : G_\eta \to G^\sharp_\eta\). It is a subgroup scheme of \(G_\eta\) representing the functor defined by

\[
K(\mathcal{L}_\eta)(U) = \{ x \in G_\eta(U); T^*_x(\mathcal{L}_\eta,U) \simeq \mathcal{L}_\eta,U \}
\]

for an \(k(\eta)\)-scheme \(U\) where \(\mathcal{L}_\eta,U\) is the pullback of \(\mathcal{L}_\eta\) to \((G_\eta) \times_{k(\eta)} U\). Let \(\mathcal{L}_\eta^{\text{tsr}}\) be the \(G_\eta\)-torsor on \(G_\eta\) associated with the invertible sheaf \(\mathcal{L}_\eta\). Let
Lemma 2.14. The flat closure $K_S^\sharp(\mathcal{L}_\eta)$ of $K(\mathcal{L}_\eta)$ in $G^2$ is finite over $S$.

See [N99, Lemma 4.14].

Definition 2.15. Let $\mathcal{L}^{\text{tsr}}$ be the $G_m$-torsor on $G^2$ associated with the invertible sheaf $\mathcal{L}_{[G^2]}$. Let $\mathcal{G}^\sharp_{S}(\mathcal{L}_\eta) := \mathcal{L}^{\text{tsr}}_{[K_S^\sharp(\mathcal{L}_\eta)]}$ and $e^\sharp_S$ an extension of $e^{\mathcal{L}_\eta}_S$ to $K_S^\sharp(\mathcal{L}_\eta)$. By [MB85, IV, 7.1 (ii)] $K_S^\sharp(\mathcal{L}_\eta)$ is a group scheme over $S$ extending $\mathcal{G}(\mathcal{L}_\eta)$, which is a central extension of $K_S^\sharp(\mathcal{L}_\eta)$ by $G_{m,S}$ with $e^\sharp_S$ the commutator form. The bimultiplicative form $e^\sharp_S$ on $K_S^\sharp(\mathcal{L}_\eta)$ is nondegenerate alternating by [MB85, IV, 2.4] and by Lemma 2.14.

Definition 2.16. We define

$$
K(P, \mathcal{L}) := K_S^\sharp(\mathcal{L}_\eta), \quad G(P, \mathcal{L}) := \mathcal{G}^\sharp_{S}(\mathcal{L}_\eta),
$$

$$
G(G_{\eta}, \mathcal{L}_\eta) := \mathcal{G}(\mathcal{L}_\eta) = G(P, \mathcal{L}) \otimes k(\eta),
$$

$$
K(P_0, \mathcal{L}_0) := K(P, \mathcal{L}) \otimes k(0), \quad G(P_0, \mathcal{L}_0) := G(P, \mathcal{L}) \otimes k(0).
$$

The natural projection from $\mathcal{L}^{\text{tsr}}$ to $G^2$ makes $G(P, \mathcal{L})$ a central extension of $K(P, \mathcal{L})$ by $G_{m,S}$ with commutator form $e^\sharp_S$.

1. $G(P, \mathcal{L}) (\text{resp. } G(P_0, \mathcal{L}_0))$ the Heisenberg group scheme of $(P, \mathcal{L})$ (resp. $(P_0, \mathcal{L}_0)$). Later in Definition 2.14 we define a finite version $G(P, \mathcal{L})$ of $G(P, \mathcal{L})$.

Lemma 2.17. Let $G^\sharp \subset P$ be the group $S$-scheme in 2.12. Then

1. $\Gamma(Q, \mathcal{L}) = \Gamma(P, \mathcal{L}) = \Gamma(G^\sharp, \mathcal{L})$, and it is an irreducible $\mathcal{G}(P, \mathcal{L})$-module of weight one, that is by definition, any $G(P, \mathcal{L})$-submodule of $\Gamma(P, \mathcal{L})$ of weight one is of the form $J\Gamma(P, \mathcal{L})$ for some ideal $J$ of $R$,

2. $\Gamma(P_0, \mathcal{L}_0)$ is an irreducible $G(P_0, \mathcal{L}_0)$-module of weight one.

See [MB85, V, 2.4.2; VI, 1.4.7] and [N99, Lemma 5.12].
Lemma 2.18. We define a morphism \( \lambda(\mathcal{L}_0) : G_0^* \to \text{Pic}^0(P_0) \) by
\[
\lambda(\mathcal{L}_0)(a) = T_a^*(\mathcal{L}_0) \otimes \mathcal{L}_0^{-1}
\]
for any \( U \)-valued point \( a \) of \( G_0^* \). Then the proof of the first assertion is the same as [N99, Lemma 5.14].

Proof. First we note that \( G_0^* \cong G_0 \times (X/Y) \) in general, and that in the totally degenerate case \( G_0 \cong \text{Hom}_\mathbb{Z}(X, G_m) \), while in the general case \( G_0 \) is a \( \text{Hom}_\mathbb{Z}(X, G_m) \)-torsor over an abelian variety \( A_0 \) whose extension class is determined uniquely by \( (P_0, \mathcal{L}_0) \).

Next we recall \( \text{Pic}^0(P_0) \) is a \( k(0) \)-scheme by [FGA, 232, Corollaire 6.6]. Then the proof of the first assertion is the same as [N99, Lemma 5.14].

Next we prove the second assertion. We see as in the case of abelian varieties that \( K(P_0, \mathcal{L}_0) \) is the maximal subscheme of \( G_0^* \) such that the sheaf \( m_\#(\mathcal{L}) \otimes p_2^*(\mathcal{L})^{-1} \) is trivial on \( K(P_0, \mathcal{L}_0) \times P_0 \) where \( m : G_0^* \times P_0 \to P_0 \) is the action of \( G_0^* \) and \( p_2 : G_0^* \times P_0 \to P_0 \) is the second projection. We also see as in the case of abelian varieties that \( \mathcal{G}(P_0, \mathcal{L}_0) \) is the scheme representing the functor similar to \( \text{Pic} \):
\[
\mathcal{G}(P_0, \mathcal{L}_0)(U) = \{(a, \phi) ; a \in K(P_0, \mathcal{L}_0)(U) \text{ and } \phi \text{ is an isomorphism of } T_a^*(\mathcal{L}_0) \text{ with } \mathcal{L}_0 \}
\]
for any \( k(0) \)-scheme \( U \). By the first assertion and 2.13 \( K(P_0, \mathcal{L}_0) \) and \( \mathcal{G}(P_0, \mathcal{L}_0) \) are independent of the choice of a Delaunay \( g \)-cell \( \sigma \). \( \square \)

Definition 2.19. Let \( k \) be an algebraically closed field and \( (P_0, \mathcal{L}_0) \) be a TSQAS over \( k = k(0) \). Then we define
\[
\begin{align*}
\epsilon_{\text{min}}(K(P_0, \mathcal{L}_0)) &= \max\{n > 0; \ker(n \cdot \text{id}_{G_0^*}) \subset K(P_0, \mathcal{L}_0)\}, \\
\epsilon_{\text{max}}(K(P_0, \mathcal{L}_0)) &= \min\{n > 0; \ker(n \cdot \text{id}_{G_0^*}) \supset K(P_0, \mathcal{L}_0)\}.
\end{align*}
\]
where \( G_0^\# \) is the closed fibre of \( G^* \) in 2.12. If the order of \( K(P_0, \mathcal{L}_0) \) and the characteristic of \( k(0) \) are coprime, then \( K(P_0, \mathcal{L}_0) \cong \oplus_{i=1}^g (\mathbb{Z}/e_i\mathbb{Z})^{\oplus 2} \) for some positive integers \( e_i \) with \( e_i | e_{i+1} \). From this it follows \( \epsilon_{\text{min}}(K(P_0, \mathcal{L}_0)) = e_1 \) and \( \epsilon_{\text{max}}(K(P_0, \mathcal{L}_0)) = e_g \).

Theorem 2.20. Let \( (Q_0, \mathcal{L}_0) \) be a projectively stable quasi-abelian scheme over \( k(0) \). If \( \epsilon_{\text{min}}(K(P_0, \mathcal{L}_0)) \geq 3 \), \( \Gamma(Q, \mathcal{L}) \otimes k(0) \) is very ample on \( Q_0 \).

Proof. See [N99, Theorem 6.3]. \( \square \)

Corollary 2.21. Suppose \( \epsilon_{\text{min}}(K(P_0, \mathcal{L}_0)) \geq 3 \). Then \( \Gamma(P, \mathcal{L}) \) is base-point free and defines a finite morphism \( \Phi_\mathcal{L} \) of \( P \) into the projective space. The image of \( P \) by \( \Phi_\mathcal{L} \) is isomorphic to \( Q \).

Proof. Let \( \nu : P \to Q \) be the normalization morphism. By definition \( \mathcal{L}_P = \nu^*(\mathcal{L}_Q) \). By Lemma 2.17 we have \( \Gamma(P, \mathcal{L}_P) = \nu^*\Gamma(Q, \mathcal{L}_Q) \). Hence \( \Gamma(P, \mathcal{L}_P) \) is base-point free by Theorem 2.20 so that it defines a finite \( S \)-morphism.
\( \Phi_{P,\ell} : P \to P(\Gamma(P, L_P)) \), which factors through \( Q \). Since by Theorem 2.24 \( \Gamma(Q, L) \) is very ample on \( Q \), the image \( \Phi_{P,\ell}(P) \) is \( Q \).

3. Level \( G(K) \)-structure

Let \( \zeta_N \) be a primitive \( N \)-th root of unity and \( \mathcal{O}_N := \mathbb{Z}[\zeta_N, 1/N] \). For simplicity we write \( \mathcal{O} = \mathcal{O}_N \).

**Definition 3.1.** Let \( H \) be a finite abelian group such that \( e_{\text{max}}(H) \), the maximal order of elements in \( H \), is equal to \( N \). Now we regard \( H \) as a constant finite abelian group \( \mathcal{O} \)-scheme. Let \( H^\vee := \text{Hom}_{\mathcal{O}}(H, G_{m,\mathcal{O}}) \) be the Cartier dual of \( H \). We set \( K := K(H) = H \oplus H^\vee \) and define a bimultiplicative (or simply a \textit{bilinear}) form \( e_K : K \times K \to G_{m,\mathcal{O}} \) by

\[
e_K(z \oplus \alpha, w \oplus \beta) = \beta(z)\alpha(w)^{-1}
\]

where \( z, w \in H, \alpha, \beta \in H^\vee \). We note that \( H \) is a maximally isotropic subgroup of \( K \), unique up to isomorphism.

Let \( \mu_N := \text{Spec} \mathcal{O}[x]/(x^N - 1) \) be the group scheme of all \( N \)-th roots of unity. We define group \( \mathcal{O} \)-schemes \( \mathcal{G}(K) \) and \( G(K) \) by

\[
\mathcal{G}(K) := \{(a, z, \alpha) ; a \in G_{m,\mathcal{O}}, z \in H, \alpha \in H^\vee \} ,
\]

\[
G(K) := \{(a, z, \alpha) ; a \in \mu_N, z \in H, \alpha \in H^\vee \}
\]

endowed with group scheme structure

\[
(a, z, \alpha) \cdot (b, w, \beta) = (ab\beta(z), z + w, \alpha + \beta).
\]

Let \( V(K) \) be the group algebra \( \mathcal{O}[H^\vee] \) of \( H^\vee \) over \( \mathcal{O} \), and an \( \mathcal{O} \)-basis \( v(\chi) (\chi \in H^\vee) \) of \( V(K) \). The group scheme \( G(K) \) and \( \mathcal{G}(K) \) act on \( V(K) \) by

\[
U(K)(a, z, \alpha)(v(\chi)) = a\chi(z)v(\chi + \alpha)
\]

where \( a \in \mu_N \) or \( a \in G_{m,\mathcal{O}}, z \in H \) and \( \alpha \in H^\vee \). Let \( \mathcal{G}(K) = U(K)G(K) \).

**Definition 3.2.** Let \( k \) be a field over \( \mathcal{O} \). Any \( G(K) \otimes k \)-module \( V \) is of weight one if every \( a \in \mu_N \subset G(K) \otimes k \) acts on \( V \) as scalar multiplication \( a \cdot \text{id}_V \). Then we say that the action of \( G(K) \) on \( V \) is of weight one.

**Remark 3.3.** By [M60] \( V(K) \otimes k \) is an irreducible \( G(K) \otimes k \)-module of weight one, unique up to equivalence. Any finite dimensional \( G(K) \)-module of weight one over \( k \) is a direct sum of copies of \( V(K) \otimes k \).

Let \( R \) be a discrete valuation ring, \( k(0) = R/I \) and \( S = \text{Spec} R \). If the order of \( K(P, L) \) and the characteristic of \( k(0) \) are coprime, then \( K(P, L) \) is a reduced flat finite group \( S \)-scheme, étale over \( S \). Hence by taking a finite base change if necessary, we may assume by [N99, Section 7] that \( (K(P, L), e^S_L) \simeq (K_S, e_{K,S}) \) and \( G(P, L) \simeq \mathcal{G}(K)_S \) for a suitable \( K \).

**Definition 3.4.** The (finite) Heisenberg group scheme \( G(P, L) \) of \((P, L)\) is defined to be the unique subgroup scheme of \( G(P, L) \) mapped isomorphically onto \( G(K)_S \) when \( G(P, L) \simeq \mathcal{G}(K)_S \). See [N99, Section 7]. Let \( G(P_0, L_0) = G(P, L) \otimes k(0) \).
Definition 3.5. A pair \((P_0, \mathcal{L}_0)\) is called a \(g\)-dimensional \(K\)-symplectic torically stable quasi-abelian scheme or a \(K\)-symplectic TSQAS over \(k\) if

(i) \((P_0, \mathcal{L}_0)\) is a \(g\)-dimensional torically stable quasi-abelian scheme over \(k\); that is, a closed fibre of some \((P, \mathcal{L})\) in Theorem 2.5.

(ii) \((K(P_0, \mathcal{L}_0), e_{S_0}^* \otimes \tilde{k}) \simeq (K, e_K) \otimes \tilde{k}\).

Suppose \(e_{\min}(K) \geq 3\) in what follows.

Definition 3.6. Let \((Z, L)\) a \(K\)-symplectic TSQAS over \(k\). A level \(G(K)\)-structure \((\phi, \rho)\) on \((Z, L)\) is defined to be a pair of a finite \(k\)-morphism \(\phi : Z \to \mathbf{P}(V(K) \otimes k)\) and a central extension isomorphism \(\rho : G(K) \otimes \mathcal{O} k \to G(Z, L)\) which induce a weight one \(\rho\)-equivariant isomorphism of line bundles (namely the centers acting as scalar multiplication of weight one)

(i) \(\phi^*(O_{\mathbf{P}(V(K) \otimes k)}(1)) \simeq L\).

The isomorphism (i) induces a weight one \(\rho\)-equivariant isomorphism of \(k\)-vector spaces by Lemma 2.17

\[
H^0(\phi^*) : H^0(O_{\mathbf{P}(V(K))}(1)) \otimes k = V(K) \otimes k \simeq H^0(Z, L)
\]

where \(G(K)\) acts on \(V(K) \otimes \mathcal{O} k\) as a conjugate to \(U(K)\).

The level \(G(K)\)-structure \((\phi, \rho)\) is called a rigid \(G(K)\)-structure if

(ii) \(\rho = G(H^0(\phi^*)) \circ (U(K) \otimes \mathcal{O} k)\)

where \(G(H^0(\phi^*))(\bar{g}) := H^0(\phi^*) \circ \bar{g} \circ (H^0(\phi^*))^{-1}\) for any \(\bar{g} \in \bar{G}(k)\), that is, the action of \(G(K)\) on \(V(K) \otimes \mathcal{O} k\) is just \(U(K)\).

If (i) is satisfied, we denote \((Z, L, G(Z, L), \phi, \rho)\) by \((Z, \phi, \rho)_{\text{LEV}}\) because \(L\) and \(G(Z, L)\) are uniquely determined by \(\phi\) and \(\rho\). If (i) and (ii) are true, we denote it by \((Z, \phi, \rho)_{\text{RIG}}\).

Definition 3.7. Let \((Z_i, L_i, G(Z_i, L_i), \phi_i, \rho_i)\) be \(k\)-TSQASes with level \(G(K)\)-structure \((i = 1, 2)\). They are defined to be isomorphic if there is a \(k\)-isomorphism \(f : Z_1 \simeq Z_2\) such that

(i) \(L_1 \simeq f^* L_2\),

(ii) \(\rho_1 = G(f^*) \circ \rho_2\)

where \(G(f^*)(g) = f^* g(f^*)^{-1}\) for any \(g \in G(Z_2, L_2)\).

In this case we write \((Z_1, \phi_1, \rho_1)_{\text{LEV}} \simeq (Z_2, \phi_2, \rho_2)_{\text{LEV}}\). By (ii) we have \(G(Z_1, L_1) = G(f^*)G(Z_2, L_2)\). We define \((Z_i, \phi_i, \rho_i)_{\text{RIG}}\) to be isomorphic if \((Z_i, \phi_i, \rho_i)_{\text{LEV}}\) are isomorphic.

Lemma 3.8. Let \((Z_i, \phi_i, \rho_i)_{\text{RIG}}\) be \(k\)-TSQASes with rigid \(G(K)\)-structure \((i = 1, 2)\). Then the following are equivalent:

1. \((Z_1, \phi_1, \rho_1)_{\text{RIG}} \simeq (Z_2, \phi_2, \rho_2)_{\text{RIG}}\).
2. there is a \(k\)-isomorphism \(f : Z_1 \simeq Z_2\) with \(\phi_1 = \phi_2 \circ f\).

Proof. Though the definition of rigid \(G(K)\)-structure is slightly different from [No92, Section 9], the proof of this lemma proceeds in the same manner as [No92, Lemma 9.7]. \(\square\)
Lemma 3.9. Let \((Z, L)\) be a\(K\)-symplectic TSQAS over \(k\). For any level \(G(K)\)-structure \((\phi, \rho)\) on \((Z, L)\) there exists a rigid \(G(K)\)-structure \((\phi(\rho), \rho)\) such that \((Z, \phi(\rho), \rho)_{\text{LEV}} \simeq (Z, \phi, \rho)_{\text{LEV}}\). If \(L\) is very ample, then the rigid \(G(K)\)-structure \((\phi(\rho), \rho)\) is unique.

Proof. We choose and fix a symplectic isomorphism \(\sigma : (K, e_K) \otimes k(0) \simeq (K(P_0, \mathcal{L}_0), e_{S,0})\) and a central extension isomorphism \(\rho : G(K) \otimes k(0) \simeq G(P_0, \mathcal{L}_0)\) in a compatible way. We may assume \(k = k(0)\). By the uniqueness of weight one \(G(K)\otimes k\)-module there is a \(k\)-isomorphism \(H^0(\phi(\rho)^*): V(K) \otimes k \to \Gamma(P_0, \mathcal{L}_0)\) such that
\[
\rho(g)(H^0(\phi(\rho)^*)(w)) = H^0(\phi(\rho)^*)U(K)(g)(w) \quad (\forall g \in G(K), w \in V(K))
\]
Hence \(\rho = G(H^0(\phi(\rho)^*))U(K)\). By Corollary \ref{cor:2.21}, \(\phi(\rho)\) is a finite \(k\)-morphism of \(Z\) into \(P(V(K) \otimes k)\). Uniqueness of \(\phi(\rho)\) follows from irreducibility of \(U(K)\) and Schur’s lemma when \(L\) is very ample. \(\square\)

We note that if \(L\) is not very ample, then there might be an automorphism \(\psi\) of \((Z, L)\) which keeps \(\phi\) and \(\rho\) invariant.

Definition 3.10. Let \(T\) be a noetherian \(O\)-scheme. Then a quintuplet \((P, \mathcal{L}, G(P, \mathcal{L}), \phi, \rho)\) is called a torically stable quasi-abelian \(T\)-scheme (abbr. \(a\) \(T\)-TSQAS) of relative dimension \(g\) with level \(G(K)\)-structure if

(i) \(P\) is a proper flat \(T\)-scheme with a relatively ample invertible sheaf \(\mathcal{L}\),
(ii) \(\phi\) is a finite \(T\)-morphism of \(P\) into \(P(V(K) \otimes_{O_T} O_T)\),
(iii) \(G(P, \mathcal{L})\) is a finite flat reduced group \(T\)-scheme acting on \((P, \mathcal{L})\),
(iv) \(\rho : G(K) \to G(P, \mathcal{L})\) is an isomorphism of group \(T\)-schemes,
(v) \(\phi^*(O_{P(V(K))}^{(1)} \otimes_{O_T} M) \simeq \mathcal{L}\) is a \(\rho\)-equivariant isomorphism (of line bundles on \(P\)) for some weight one action of \(G(K)_T\) on \(O_{P(V(K))}^{(1)}\),
(vi) for any geometric point \(s\) of \(T\), \((P_s, \phi_s, \rho_s)\) is a TSQAS of dimension \(g\) over \(k(s)\) with level \(G(K)\)-structure

where \(M\) is some invertible \(O_T\)-module with trivial \(G(K)_T\)-action and \(\pi : \pi : P \to T\) is the structure morphism.

It follows from the condition (vi) that the isomorphism (v) induces a \(\rho\)-equivariant isomorphism of \(O_T\)-Modules
\[
R^0\pi_*(\phi^*) : V(K) \otimes_{O_T} M \simeq \pi_*(\mathcal{L}).
\]
We denote \((P, \mathcal{L}, G(P, \mathcal{L}), \phi, \rho)\) by \((P, \phi, \rho)_{\text{LEV}}\). We call \((P, \phi, \rho)_{\text{LEV}}\) a \(T\)-TSQAS with rigid \(G(K)\)-structure and denote it by \((P, \phi, \rho)_{\text{RIG}}\) if
\[
\text{(vii) } R^0\pi_*(\phi^*) = G(R^0\pi_*(\phi^*)) \circ U(K)_T.
\]

Definition 3.11. Let \((P_i, \phi_i, \rho_i)_{\text{LEV}} := (P_i, \mathcal{L}_i, G(P_i, \mathcal{L}_i), \phi_i, \rho_i)\) \((i = 1, 2)\) be \(T\)-TSQASes with level \(G(K)\)-structure. We define them to be isomorphic if there exist a \(T\)-isomorphism \(f : P_1 \to P_2\) and an invertible \(O_T\)-module \(M\) with trivial \(G(K)_T\)-action such that

(i) \(\mathcal{L}_1 \simeq f^*\mathcal{L}_2 \otimes_{O_T} M\),
(ii) \(\rho_1 = G(f^*) \circ \rho_2\).
If \((P_i, \phi_i, \rho_i)\) are rigid \(G(K)\)-structures, then (i) and (ii) is by Lemma 3.5 equivalent to

(iii) \(\phi_1 = \phi_2 \circ f\).

An algebraic space \(T\) is by definition the isomorphism class of an étale representative \(U \to T\) with étale equivalence relation \(R \subset U \times U\). See [K71]. Let \(p_i : R \to U\) be the composite of the immersion \(R \subset U \times U\) with \(i\)-th projection \((i = 1, 2)\). A \(T\)-TSQAS \((Z, \psi, \rho)_{\text{lev}}\) with level \(G(K)\)-structure is a \(U\)-TSQAS \((Z_U, \psi_U, \rho_U)_{\text{lev}}\) whose pullbacks by \(p_i\) are isomorphic as \(R\)-TSQASes with level \(G(K)\)-structure.

**Definition 3.12.** We define the functor \(\mathcal{SQ}_{g,K}^\text{toric}\) as follows. For any noetherian \(\mathcal{O}_N\)-scheme \(T\), we set

\[
\mathcal{SQ}_{g,K}^\text{toric}(T) = \text{the set of torically stable quasi-abelian } \ T\text{-schemes } (P, \phi, \rho)_{\text{lev}} \text{ of relative dimension } g \\
\text{with level } G(K)\text{-structure modulo } T\text{-iso}.
\]

**3.13. General rigid \(G(K)\)-structures.** Even if \((Z, L)\) is neither a TSQAS nor a PSQAS, we can also speak of a rigid (or level) \(G(K)\)-structure. If \(G(K)\) acts on a polarized \(k\)-scheme \((Z, L)\) with \(L\) very ample (or equivalently, \(L\) is \(G(K)\)-linearized [MF94, p. 30]), then \(H^0(Z, L)\) becomes a \(G(K)\)-module in a natural manner. Let \(\rho\) be the action of \(G(K)\) on \(H^0(Z, L)\) and \(P := P(V(K) \otimes \mathcal{O}_k)\). If \(\phi : (Z, L) \to (P, O_P(1))\) is a \(G(K)\)-equivariant closed immersion such that \(H^0(\phi^*) : V(K) \otimes \mathcal{O}_k \to H^0(Z, L)\) is an isomorphism with \(\rho = \phi(H^0(\phi^*)) \circ (U(K) \otimes k)\), then we call the triplet \((Z, \phi, \rho)\) a rigid \(G(K)\)-structure on \((Z, L)\), which we denote \((Z, \phi, \rho)_{\text{rig}}\). In particular, if \(Z\) is a \(G(K)\)-invariant closed subscheme of \(P\), then \((Z, i, U(K))\) is a rigid \(G(K)\)-structure on \((Z, O_P(1) \otimes O_Z)\) where \(i\) is the inclusion of \(Z\) in \(P\). If \(\rho\) is of weight one, then Lemmas 3.8, 3.9 are true as well.

**4. The stable reduction theorem**

**4.1. The rigid \(G(K)\)-structure we start from.** Let \(R\) be a complete discrete valuation ring, \(k(\eta)\) (resp. \(k(0)\)) the fraction field (resp. the residue field) of \(R\), and \(S = \text{Spec } R\). Let \((G_\eta, \mathcal{L}_\eta)\) be a polarized abelian variety over \(k(\eta)\) ample and \(K(\mathcal{L}_\eta) := \ker \lambda(\mathcal{L}_\eta)\). Let \(e_{\mathcal{L}_\eta}\) be the Weil pairing of \(K(\mathcal{L}_\eta)\).

Suppose that the order of \(K(\mathcal{L}_\eta)\) and the characteristic of \(k(0)\) are coprime. Then there exists a finite symplectic constant abelian group \(Z\)-scheme \((K, e_K)\) such that \((K, e_K) \otimes_Z k(\eta) \simeq (K(\mathcal{L}_\eta), e_{\mathcal{L}_\eta})\). Let \(N = e_{\text{max}}(K)\). We also may assume that \(R\) contains a primitive \(N\)-th root \(\zeta_N\) of unity. If \(e_{\text{min}}(K) \geq 3\), then by Lemma 3.9 \((G_\eta, \mathcal{L}_\eta)\) has a unique rigid \(G(K)\)-structure because \(\mathcal{L}_\eta\) is very ample.

**Theorem 4.2.** Let \(R\) be a complete discrete valuation ring and \(S = \text{Spec } R\). Let \((G_\eta, \mathcal{L}_\eta)\) be a polarized abelian variety over \(k(\eta)\). Let \(K(\mathcal{L}_\eta) := \ker \lambda(\mathcal{L}_\eta)\) and \(N = e_{\text{max}}(K(G_\eta, \mathcal{L}_\eta))\). Assume that
Let $\rho$ be such that $N$ is enough such that

Then after a suitable finite base change if necessary, there exist flat projective schemes $(P, \mathcal{L})$ and $(Q, \mathcal{L})$, semiabelian group schemes $G$ and $G^2$, the flat closure $K(P, \mathcal{L})$ of $K(\mathcal{L}_\eta)$ in $G^2$, a symplectic form $e_S^\mathcal{L}$ on $K(P, \mathcal{L})$ extending $e^\mathcal{L}$ and the Heisenberg group schemes $G(P, \mathcal{L})$ and $G(P, \mathcal{L})$ of $(P, \mathcal{L})$, all of these being defined over $S$, such that

1. $P$ is reduced over $S$,
2. $(G, \mathcal{L})$ and $(G^2, \mathcal{L})$ are open subschemes of $(P, \mathcal{L})$,
3. $G^2 = K(P, \mathcal{L}) \cdot G$,
4. $(G_\eta, \mathcal{L}_\eta) \simeq (G_\eta^2, \mathcal{L}_\eta) \simeq (P_\eta, \mathcal{L}_\eta) \simeq (Q_\eta, \mathcal{L}_\eta)$,
5. there exists a constant finite symplectic abelian group $\mathbf{Z}$-scheme $(K, e_K)$ such that $(K(P, \mathcal{L}), e^\mathcal{L}_S) \simeq (K, e_K)_S$ and $G(P, \mathcal{L}) \simeq G(K)_S$,
6. $\Gamma(G^2, \mathcal{L}) \simeq \Gamma(P, \mathcal{L}) \simeq \Gamma(Q, \mathcal{L}) \simeq V(K)^{\otimes_{O_N}} R$ and they are irreducible $G(P, \mathcal{L})$-modules of weight one, unique up to equivalence.

See [N99] for the proof of it and for the details of $(Q, \mathcal{L})$.

5. The scheme parametrizing TSQASes

Let $K$ be a symplectic finite abelian group with $e_{\min}(K) \geq 3$. Let $N = e_{\max}(K)$ and $\mathcal{O} = \mathcal{O}_N$. In what follows we fix $K$ and $\mathcal{O}$.

5.1. The scheme $H_1 \times H_2$. Choose and fix a coprime pair of natural integers $d_1$ and $d_2$ such that $d_1 > d_2 \geq 2g + 1$ and $d_k \equiv 1 \mod N$. This pair does exist because it is enough to choose prime numbers $d_1$ and $d_2$ large enough such that $d_k \equiv 1 \mod N$ and $d_1 > d_2$. We choose integers $q_k$ such that $q_1 d_1 + q_2 d_2 = 1$.

Now consider a $G(K)$-module $W_k(K) := W_k \otimes V(K) \simeq V(K)^{\otimes N_k}$ where $N_k = d_k^2$ and $W_k$ is a free $\mathcal{O}$-module of rank $N_k$ with trivial $G(K)$-action. Let $\rho_k$ be the natural action of $G(K)$ on $W_k(K)$.

Let $H_k$ be the Hilbert scheme parametrizing all pure $g$-dimensional polarized subschemes $(Z_k, M_k)$ of the projective space $\mathbf{P}(W_k(K))$ such that

$$\chi(Z_k, n M_k) = n^g d_k^n \sqrt{|K|} \quad \text{for } k = 1, 2.$$

Let $X_k$ be the universal subscheme of $\mathbf{P}(W_k(K))$ over $H_k$, $X$ the product of $X_1$ and $X_2$ over $\mathcal{O}$. Hence $X$ is a subscheme of $\mathbf{P}(W_1(K)) \times_{\mathcal{O}} \mathbf{P}(W_2(K))$ over $H_1 \times_{\mathcal{O}} H_2$.

5.2. The scheme $U_1$. Let $(X, L)$ be a polarized $\mathcal{O}$-scheme with $L$ very ample and $P(n)$ an arbitrary polynomial. Let $\text{Hilb}^P(X)$ be the Hilbert scheme parametrizing all subschemes $Z$ of $X$ with $\chi(Z, n L_Z) = P(n)$. As is well known $\text{Hilb}^P(X)$ is a projective $\mathcal{O}$-scheme.

For a given projective scheme $T$ and a given flat projective $T$-scheme $(X, L)$ and an arbitrary polynomial $P(n)$, let $\text{Hilb}_{\text{conn}}^P(X/T)$ be the scheme
parametrizing all connected subschemes $Z$ of $X$ with $\chi(Z, n_LZ) = P(n)$ projected to one point of $T$. Then $\text{Hilb}^P_{\text{conn}}(X/T)$ is a projective $T$-subscheme of $\text{Hilb}^P(X)$.

Let $p_k : X_1 \times_\mathcal{O} X_2 \to X_k$ be the $k$-th projection. Let $X = X_1 \times_\mathcal{O} X_2$ and $H = H_1 \times_\mathcal{O} H_2$. The aim of the subsequent sections is to construct a new compactification of the moduli scheme of abelian varieties as the quotient of a certain subscheme of $\text{Hilb}^P_{\text{conn}}(X/H)$ by $\text{GL}(W_1) \times \text{GL}(W_2)$.

Let $B$ be the pullback to $X$ of a very ample invertible sheaf on $H$. Let $M_i = p_i^*(\mathcal{O}_{\text{P}(W_i(K))}(1))$ and $M = d_2M_1 + d_1M_2 + B$. Then $M$ is a very ample invertible sheaf on $X$. Now we define $U_1$ to be the subset of $\text{Hilb}^P_{\text{conn}}(X/H)$ consisting of all subschemes $Z$ such that

(i) $p_{k|Z}$ is an isomorphism ($i = 1, 2$),
(ii) $d_2L_1 = d_1L_2$,
(iii) $Z$ is $G(K)$-stable

where $P(n) = (2nd_1d_2)^q \sqrt{|K|}$ and $L_i = M_i \otimes O_Z$.

We prove that $U_1$ is a nonempty closed $\mathcal{O}$-subscheme of $\text{Hilb}^P_{\text{conn}}(X/H)$. The condition $d_2L_1 = d_1L_2$ is closed, while the condition that $p_{i|Z}$ is an isomorphism is open and closed. The $G(K)$-stability of $Z$ is equivalent to the condition that $Z \in \text{Hilb}^P_{\text{conn}}(X/H)$ is fixed by the natural $G(K)$-action on $X$ and $H$. Hence it is a closed condition. Hence $U_1$ is a closed, hence a projective $\mathcal{O}$-subscheme of $\text{Hilb}^P_{\text{conn}}(X/H)$.

It remains to show $U_1 \neq \emptyset$. Let $k$ be an algebraically closed field over $\mathcal{O}$, and $(A, L)$ a polarized abelian variety over $k$ with $G(A, L) \simeq G(K)$. Since $\epsilon_{\text{min}}(K) \geq 3$, $L$ is very ample and $(A, d_iL) \in H_i$. Moreover $\Gamma(A, L) \simeq V(K) \otimes k$. Hence there is a unique rigid $G(K)$-structure of $(A, L)$ by Lemma 3.9 in other words, there is a unique $G(A, L)$-$G(K)$ equivariant closed immersion $\phi : A \to \text{P}(V(K) \otimes k)$ of $(A, L)$. In particular $\phi(A) \simeq A$ is a $G(K)$-stable subscheme of $\text{P}(V(K) \otimes k)$. Since $L$ has a $G(A, L)$-linearization of weight one, $d_iL$ has a $G(A, L)$-linearization of weight $d_i$ too. Since $d_i \equiv 1 \mod N$, and since $\alpha^N = 1$ for any $\alpha \in \mu_N$, $d_iL$ has a $G(A, L)$-linearization of weight one. Hence $\Gamma(A, d_iL)$ is a direct sum of $V(K) \otimes k$. Since $\Gamma(A, d_iL)$ is very ample, we can choose a $G(A, L)$-$G(K)$-equivariant closed immersion $\phi_i : A \to \text{P}(W_i(K))$. Then $\phi_i(A)$ is a $G(K)$-stable subscheme. Therefore $(\phi_1(A), \phi_2(A)) \in H_1 \times H_2$. Let $Z \subset \phi_1(A) \times \phi_2(A) \simeq A \times A$ be the inverse image of the diagonal. Since $Z \simeq A$, we see that

\[
\chi(Z, n(d_2L_1 + d_1L_2 + B)) = \chi(A, 2nd_1d_2L) = (2nd_1d_2)^q \sqrt{|K|} = P(n),
\]

which proves $Z \in \text{Hilb}^P_{\text{conn}}(X/H)$. Thus $Z \in U_1(k)$. Hence $U_1 \neq \emptyset$.

**Lemma 5.3.** Let $k$ be an algebraically closed field over $\mathcal{O}$. Let $Z \in U_1(k)$ and $L = q_1L_1 + q_2L_2$. Then $L_i = d_iL$.

**Proof.** One sees readily $d_iL = d_i(q_1L_1 + q_2L_2) = (d_1q_1 + d_2q_2)L_i = L_i$. \qed
5.4. The scheme $U_2$. Let $q_i$ be the integers chosen above. Let $U_2$ be the open subscheme of $U_1$ consisting of all subschemes $Z$ of $X$ such that (i)-(iii) are true and

(iv) $L$ is ample where $L = q_1 L_1 + q_2 L_2$,
(v) $\chi(Z, nL) = n^g \sqrt{|K|}$,
(vi) $H^q(Z, nL) = 0$ for $q > 0$ and $n > 0$,
(vii) $\Gamma(Z, L)$ is base point free,
(viii) the pullback $H^0(p_i^*): W_i(K) \otimes k \to \Gamma(Z, L_i)$ is surjective for $i = 1, 2$,
(ix) $Z$ is reduced.

It is clear that (iv)-(ix) are open conditions. Note that the surjectivity in (viii) implies an $G(K)$-equivariant isomorphism in view of (iii).

We note $U_2 \neq \emptyset$. In fact, letting $k$ be an algebraically closed field over $\mathcal{O}$ we choose a polarized abelian variety $(A, L)$ over $k$ with $G(A, L) \cong G(K)$. Then $L$ is very ample and $(A, d_i L) \in H_i$ (identified with $\phi_i(A)$), and the inverse image $Z$ of the diagonal ($\simeq A$) belongs to $U_1(k)$ as we saw in 5.2. Since $L_1 = d_1 L$ by Lemma 5.3, all the conditions (iv)-(viii) are true for $Z$ as is well known. Hence $Z \in U_2(k)$. Hence $U_2 \neq \emptyset$.

5.5. The scheme $U_3$. Next we recall that the locus $U_{g, K}$ of abelian varieties is an open subscheme of $U_2$. The condition on $Z \in U_{g, K}$ that the natural action of $G(K)$ on $Z$ is contained in $\text{Aut}^0(Z)$ is open. Since $L$ is very ample, this condition implies that if $(Z, L)$ is a polarized abelian variety, then the restriction of the $G(K)$-action to $Z$ reduces to $K(Z, L)$.

Now we define $U_3$ to be a reduced subscheme in $U_2$ whose underlying set is the union of all the irreducible components of $U_2$ over which at least one of the geometric fibres of $X$ is a polarized abelian variety $(Z, L)$ with $L_i = d_i L$ and $L$ very ample such that

(x) the restriction to $Z$ of the $G(K)$-action on $X$ is contained in $\text{Aut}^0(Z)$.

By definition $U_3$ is the closure of $U_{g, K}$ in $U_2$ with reduced structure. Note that it is an $\mathcal{O}$-subscheme of $\text{Hilb}_{\text{conn}}^P(X/H)$.

6. The fibres over $U_3$

6.1. $(S_k)$ and $(R_k)$. Here we recall the conditions $(S_k)$ and $(R_k)$:

$(S_k)$ $\text{depth}(A_p) \geq \inf(k, \text{ht}(p))$ for all $p \in \text{Spec}(A)$,
$(R_k)$ $A_p$ is regular for all $p \in \text{Spec}(A)$ with $\text{ht}(p) \leq k$.

Lemma 6.2. Let $A$ be a noetherian local ring. Then
1. (Serre) $A$ is normal if and only if $(R_1)$ and $(S_2)$ are true for $A$,
2. $A$ is reduced if and only if $(R_0)$ and $(S_1)$ are true for $A$.

See [M70, Theorem 39] and [EGA] IV, 5.8.5 and 5.8.6).

Lemma 6.3. Let $R$ be a discrete valuation ring, $S := \text{Spec } R$, $\eta$ the generic point of $S$ and $k(\eta)$ the fraction field of $R$. Let $(Z_k, \phi_k, \rho_k)_{R_{\text{SG}}}$ be flat proper schemes over $S$ with rigid $G(K)$-structure in the sense of [3.13] such that $\phi_k$
is a closed immersion of $Z_k$ into $\mathbf{P}(V(K) \otimes \mathcal{O})$ and $H^0(\phi_k^*): V(K) \otimes \mathcal{O}$ $R \to H^0(Z_k, L_k)$ is a $G(K)$-isomorphism. If $(Z_k, \phi_k, \rho_k)_{\text{rig}}$ are isomorphic abelian varieties over $k(\eta)$, then they are isomorphic over $S$.

**Proof.** Let $H = \text{Hilb}^{P(n)}(\mathbf{P}(V(K)))$ be the Hilbert scheme parametrizing all subschemes of $\mathbf{P}(V(K))$ with Hilbert polynomial $P(n) = n^g \sqrt{|K|}$ and $X_{\text{univ}}$ the universal subscheme of $\mathbf{P}(V(K))$ over $H$. Then $\phi_k$ induces a unique morphism $\text{Hilb}(\phi_k): S \to H$ such that $Z_k$ is the pullback by $\text{Hilb}(\phi_k)$ of $X_{\text{univ}}$. By the assumption there is a $k(\eta)$-isomorphism $f_\eta: Z_{1,\eta} \to Z_{2,\eta}$ by Lemma 3.8 such that $\phi_{1,\eta} = \phi_{2,\eta} \circ f_\eta$. It follows that $\text{Hilb}(\phi_{1,\eta}) = \text{Hilb}(\phi_{2,\eta})$. Since $H$ is separated, $\text{Hilb}(\phi_1) = \text{Hilb}(\phi_2)$, hence $\phi_1(Z_1) = \phi_2(Z_2)$. This implies that there is an $S$-isomorphism $f: Z_1 \to Z_2$ extending $f_\eta$ such that $\phi_1 = \phi_2 \circ f$. This proves $(Z_1, \phi_1, \rho_1)_{\text{rig}} \simeq (Z_2, \phi_2, \rho_2)_{\text{rig}}$ by Lemma 3.8. □

**Lemma 6.4.** Let $R$ be a discrete valuation ring, $S := \text{Spec } R$, $\eta$ the generic point of $S$ and $k(\eta)$ the fraction field of $R$. Let $h$ be a morphism from $S$ into $U_3$. Let $(Z, \mathcal{L})$ be the pullback by $h$ of the universal subscheme $Z_{\text{univ}}$, universal for $\text{Hilb}^{\text{conn}}_{\text{univ}}(X/H)$, such that $(Z_\eta, \mathcal{L}_\eta)$ is a polarized abelian variety. Then $(Z, \mathcal{L})$ is isomorphic to a (modified) Mumford’s family $(P, \mathcal{N})$ in Theorem 2.4 after a finite base change if necessary.

**Proof.** By the assumption on $h$, $(Z_\eta, \mathcal{L}_\eta)$ is a polarized abelian variety over $k(\eta)$ such that the action of $G(K)$ is contained in $\text{Aut}^0(Z_\eta) \cap \text{Aut}(Z_\eta, \mathcal{L}_\eta)$. After a suitable finite base change we may assume by Theorem 4.2 that there is a flat projective family $(P, \mathcal{N})$ associated with the degeneration data of $(Z_\eta, \mathcal{L}_\eta)$ such that

1. $(P, \mathcal{N})$ is a $K$-symplectic $S$-TSQAS, in particular, $P_0$ is connected reduced and $P$ is normal,
2. $(P_\eta, \mathcal{N}_\eta) \simeq (Z_\eta, \mathcal{L}_\eta)$,
3. $G(P, \mathcal{N}) \simeq G(K)_S$,
4. there is a $G(P, \mathcal{N})$-equivariant polarized finite morphism

$$\psi: (P, \mathcal{N}) \to (\mathbf{P}(V(K))_S, \mathcal{O}_{\mathbf{P}(V(K))_S}(1))$$

such that $\psi_\eta$ is a closed immersion.

Let $(Q, \mathcal{N}_Q)$ be an $S$-PSQAS extending $(P_\eta, \mathcal{N}_\eta)$ to $S$. The scheme $Q$ was defined in Section 3. By Lemma 2.21 $Q$ is the image of $P$ by the morphism $\psi: P \to \mathbf{P}(V(K) \otimes \mathcal{O})$ defined by $\Gamma(P, \mathcal{N})$, while $\mathcal{N}_Q$ is the restriction of $\mathcal{O}_{\mathbf{P}(V(K))}(1)_S$ to $Q$. Then $\psi: P \to Q$ is the normalization of $Q$ in view of Theorem 2.7.

Let $\pi: Z \to S$ be the flat family given at the start. Then $\Gamma(Z, \mathcal{L})$ is a free $R$-module of rank $\sqrt{|K|}$ by (ii). It is a $G(K)$-module of weight one, hence $G(K)$-isomorphic to $V(K) \otimes R$ after a finite base change. See Remark 3.3. By (vii) $\Gamma(Z, \mathcal{L})$ is base point free, which defines a $G(K)$-equivariant finite morphism $q: Z \to \mathbf{P}(V(K) \otimes R)$ such that $q_\eta$ is a closed immersion of $Z_\eta$ because $e_{\text{min}}(K) \geq 3$. Let $W$ be the flat closure of $q_\eta(Z_\eta)$ in $\mathbf{P}(V(K) \otimes R)$, and $\mathcal{L}_W$ the restriction of $\mathcal{O}_{\mathbf{P}(V(K))}(1)_S$. Since $Z_\eta$ is reduced, $W_{\text{red}}$ is the
flat closure of \( q_\eta(Z_\eta) \). Hence \( W \) is reduced. Since \( Z_\eta \) is irreducible, so is \( W \). Since \( Z_0 \) is reduced, so is \( Z \), hence \( q \) factors through \( W \). It follows that \( q : Z \to W \) is a finite surjective birational morphism.

By \( \textbf{3.13} \) and Lemma \( \textbf{3.9} \) the \( S \)-scheme \( (W, \mathcal{L}_W) \) has a unique rigid \( G(K) \)-structure \( (W, i_W, U(K))_{\text{RIG}} \), while \( (Q, \mathcal{N}_Q) \) has a unique rigid \( G(K) \)-structure \( (Q, i_Q, U(K))_{\text{RIG}} \) by Theorem \( \textbf{4.2} \) where \( i_W \) and \( i_Q \) are natural inclusions of \( W \) and \( Q \) into \( \mathbf{P}(V(K))_S \). Since we have

\[
(W, i_W, U(K))_{\text{RIG}} \simeq (Q, i_Q, U(K))_{\text{RIG}} \simeq (Z, i_{Z_\eta}, \rho_{Z_\eta})_{\text{RIG}},
\]

\((W, i_W, U(K))_{\text{RIG}} \) and \((Q, i_Q, U(K))_{\text{RIG}} \) are \( S \)-isomorphic by Lemma \( \textbf{6.3} \). It follows the action of \( G(K) \) on \((W, \mathcal{L}_W)\) is the same as that of \( G(W, \mathcal{L}_W) \).

Hence to complete our proof of the theorem it suffices to prove that \( Z \) is also the normalization of \( W \). Since \( q \) is finite and birational, it suffices to prove that \( Z \) is normal. By Lemma \( \textbf{6.3} \) it suffices to check that \( (R_1) \) and \( (S_2) \) are true for \( O_Z \). Since \( Z_0 \) is reduced, it is smooth at a generic point of any irreducible component of it. Hence \( Z \) is smooth at any codimension one point of \( Z \) supported by \( Z_0 \). Since \( Z_\eta \) is smooth, \( Z \) is codimension one nonsingular everywhere. This is \((R_1)\).

Next we prove \((S_2)\). Since \( \pi : Z \to S \) is flat, any regular parameter \( t \) of \( R \) is not a zero divisor of \( O_Z \). Let \( p \) be a prime ideal of \( O_Z \). If \( p \cap R \neq 0 \), then \( t \in p \) and \( q := p/IO_Z \) is a prime ideal of \( O_Z \) with \( \text{ht}(q) = \text{ht}(p) - 1 \). Since \( Z_0 \) is reduced, hence \((S_1)\) for \( Z_0 \) is true by Lemma \( \textbf{6.3} \). It follows that \( \text{depth}(O_{Z_0})_p = \text{depth}(O_{Z_0})_q + 1 \geq \inf(1, \text{ht}(q)) + 1 = \inf(2, \text{ht}(p)) \). If \( p \cap R = 0 \), then \( k(\eta) \subset (O_Z)_p \) and \( (O_Z)_p = (O_{Z_0})_pO_{Z_\eta} \). Hence \( \text{depth}(O_Z)_p = \dim(O_{Z_\eta})_p \geq \inf(2, \text{ht}(p)) \) because \( Z_\eta \) is nonsingular. This proves \((S_2)\).

Therefore \( Z \) is normal by Lemma \( \textbf{6.2} \). It follows that \( Z \) is the normalization of \( W \) and \((Z, \mathcal{L}) \simeq (P, \mathcal{N}) \).

**Corollary 6.5.** Let \((Z_0, \mathcal{L}_0)\) be the closed fibre of \((Z, \mathcal{L})\) in Lemma \( \textbf{6.4} \). Then \((Z_0, \mathcal{L}_0)\) is a \( K \)-symplectic TSQAS such that the action of \( G(K) \) on \((Z_0, \mathcal{L}_0)\) is \( G(Z_0, \mathcal{L}_0) \).

**Proof.** By the proof of Lemma \( \textbf{6.4} \), we see that \( Z \) is the normalization of \( W \) and \((W, i_W, \rho_W)_{\text{RIG}} \simeq (Q, i_Q, \rho_Q)_{\text{RIG}} \). The normalization morphism of \( Z \) onto \( W \) is \( G(K) \)-equivariant and the action of \( G(K) \) on \((W, \mathcal{L}_W)\) is \( G(W, \mathcal{L}_W) \) by the proof of Lemma \( \textbf{6.4} \). Hence the action of \( G(K) \) on \((Z, \mathcal{L})\) is \( G(Z, \mathcal{L}) \). This proves the corollary.

**Corollary 6.6.** Let \( \text{Spec} \ k \) be a geometric point over \( O \) and \( Z \subset U_3(k) \).

Let \( L = M \otimes O_Z \) under the notation of \( \textbf{5.2} \). Then \((Z, L)\) is a \( K \)-symplectic TSQAS such that the \( G(K) \)-action on \((Z, L)\) induced from that on \( W_i(K) \) is exactly \( G(Z, L) \).

**Proof.** It follows from Corollary \( \textbf{6.5} \) that \((Z, L)\) is a \( K \)-symplectic TSQAS. The remaining assertion follows from Definition \( \textbf{2.13} \) and Theorem \( \textbf{4.2} \).

### 7. The Geometric Quotient

\( \text{Lt } N = e_{\text{max}}(K) \) and \( O = O_N = Z[\zeta_N, 1/N] \).
Lemma 7.1. Let $k$ be an algebraically closed field over $O$.

1. $U_3$ is $GL(W_1) \times GL(W_2)$-invariant,
2. Let $(Z, L) \in U_3(k)$ and $(Z', L') \in U_3(k)$ where $L = M \otimes O_Z$ and $L' = M \otimes O_{Z'}$. If $(Z, L) \simeq (Z', L')$ as polarized varieties with $G(K)$-linearization, then $(Z', L')$ belongs to the $GL(W_1) \times GL(W_2)$-orbit of $(Z, L)$.

Proof. First we prove (2). Let $f : (Z, L) \to (Z', L')$ be an isomorphism with $G(K)$-linearization. Hence $(Z, d_i L)$ and $(Z', d_i L')$ are isomorphic as polarized schemes with $G(K)$-linearization. By the assumptions on $(Z, L)$ and $(Z', L')$, we see first $d_i L$ and $d_i L'$ are very ample. Hence $(Z, d_i L)$ and $(Z', d_i L') \in H_i(k)$ ($i = 1, 2$). Thus we see

1. there are commutative diagrams of $G(K)$-equivariant isomorphisms

\[
\begin{array}{c}
(Z, d_i L) \\
\downarrow \psi_i
\end{array}
\quad \xrightarrow{f_i} \quad
\begin{array}{c}
(Z', d_i L') \\
\downarrow \psi_i'
\end{array}
\]

\[
\begin{array}{c}
(P(W_i(K)), O_{P(W_i(K))}(1)) \\
\downarrow \psi_i
\end{array}
\quad \xrightarrow{F_i} \quad
\begin{array}{c}
P(W_i(K)), O_{P(W_i(K))}(1)) \\
\downarrow \psi_i'
\end{array}
\]

where $\psi_i$ and $\psi_i'$ are closed immersion of $(Z, d_i L)$ into $P(W_i(K))$.

2. there are commutative diagrams of $G(K)$-equivariant isomorphisms

\[
\begin{array}{c}
H^0(Z, d_i L) \\
\uparrow H^0(\psi_i^*)
\end{array}
\quad \xleftarrow{H^0(f_i^*)} \quad
\begin{array}{c}
H^0(Z', d_i L') \\
\uparrow H^0(\psi_i'^*)
\end{array}
\]

\[
H^0(O_{P(W_i(K))}(1)) \otimes k \xleftarrow{H^0(F_i^*)} H^0(O_{P(W_i(K))}(1)) \otimes k
\]

where $H^0(O_{P(W_i(K))}(1)) = W_i(K)$.

Let $\rho_i$ and $\rho_i'$ be the $G(K)$-actions on $W_i(K) = W_i \otimes O V(K)$ defined in 7.1.

In particular we have

\[
\rho_i(g) \circ H^0(F_i^*) = H^0(F_i^*) \circ \rho_i(g)
\]

\[
\rho_i(g) = \rho_i'(g) = (id_{W_i} \otimes U(K))(g).
\]

Note that Schur’s lemma for $U(K)$ is true over any $O$-algebra. See [N99, Remark 7.15]. Hence it follows from irreducibility of $U(K)$ that $H^0(F_i^*) = h_i^* \otimes id_{V(K)}$ for some $h_i^* \in GL(W_i)$. Let $\sigma(h_i^*)$ be the transformation of $P(W_i(K))$ induced from $h_i^* \otimes id_{V(K)}$. Then $\psi_i \circ f_i = \sigma(h_i^*) \circ \psi_i$. This proves (2). (1) is clear from the proof of (2).

7.2. The geometric and categorical quotient. Let $G$ be a group scheme, $X$ a scheme and $\sigma : G \times X \to X$ the action. We say that the action $\sigma$ on $X$ is proper if the morphism $\Psi := (\sigma, p_2) : G \times X \to X \times X$ is proper. We consider the following conditions:

1. $\phi \circ \sigma = \phi \circ p_2$.
2. $\phi$ is surjective and the image of $\Psi$ is $X \times_Y X$.
3. $\phi$ is submersive, that is, $U$ is open in $Y$ if and only if $\phi^{-1}(U)$ is open in $X$.
4. a given space $Z$ and a morphism $\psi : X \to Z$ such that $\psi \circ \sigma = \psi \circ p_2$, then there is a unique morphism $\chi : Y \to Z$ such that $\psi = \chi \circ \phi$.

A pair $(Y, \phi)$ consisting of an algebraic space $Y$ and a morphism $\phi : X \to Y$ is called a geometric quotient (resp. a categorical quotient) of $X$ if the conditions 1, 2, 3, and 4 are satisfied.

The pair $(Y, \phi)$ is called a uniform geometric quotient (resp. a uniform categorical quotient) if for any $Y$-flat $Y'$ $(Y', \phi')$ is a geometric quotient (resp. a categorical quotient) of $X \times_S Y'$ by $G$ where $\phi' := \phi \times_Y Y'$.

**Theorem 7.3.** Let $G = GL(W_1) \times GL(W_2)$. Then

1. The action of $G$ on $U_{g,K}$ is proper and free.
2. The action of $G$ on $U_3$ is proper and has finite stabilizer.
3. The uniform geometric and uniform categorical quotient of $U_3$ by $GL(W_1) \times GL(W_2)$ exists as a separated algebraic space.

**Proof.** Let $k$ be a closed field. Let $Z \in U_3(k)$ and $h \in G$. Suppose $h \cdot Z = Z$. Then $h = (h_1, h_2)$ for some $h_k \in GL(W_k)$ and each $h_k$ keeps $L_k$ invariant, hence $h$ keeps $L$ invariant. This implies that $h$ is an automorphism of $(Z, L)$ with $G(K)$-linearization. In particular, $h$ induces a linear transformation $H^0(h, L)$ of $\Gamma(Z, L)$, which commutes with $U(K)(g)$ for any $g \in G(K)$. Thus $H^0(h, L)$ on $\Gamma(Z, L)$ is a scalar matrix.

We assume that $H^0(h, L)$ is the identity on $\Gamma(Z, L)$. We shall prove that it is an automorphism of $Z$ of finite order.

First we consider the totally degenerate case, that is, $Z$ is a union of normal torus embeddings. We may assume $(Z, L) = (P_0, L_0)$ by Corollary 6.6. Since $H^0(h, L)$ is the identity on $\Gamma(Z, L)$, $h$ induces the identity of $(Q_0)_{red}$ with the notation in Section 3. By Lemma 2.21 the linear system $\Gamma(Z, L)$ defines a finite morphism into $P(\Gamma(Z, L))$, whose image is $(Q_0)_{red}$. Since $\Gamma(Z, L)$ is by Theorem 2.11 the $k$-vector space consisting of finite sums of monomials $\xi_a$ with $a \in Del^0(Z) := \text{the set of Delaunay vectors of Del}(Z)$, each monomial $\xi_a$ in the sums is invariant under $h$.

For a given Delaunay $g$-cell $\sigma \in Del(Z)(0)$, we define $X(\sigma)$ to be the sublattice of the lattice $X \simeq Z^g$ generated by Delaunay vectors of $\sigma$ starting from the origin. Let $N(\sigma)$ be the index $[X : X(\sigma)]$ and $N(Z)$ the least common multiple of $N(\sigma)$ for all Delaunay $g$-cells of Del$(Z)(0)$. Then $\zeta_{b,0}$ is multiplied by an $N(Z)$-th root of unity (depending on $b$) because $N(Z)\cdot b \in X(\sigma)$ if $b \in C(0, \sigma)$. Thus $h$ is of order at most $N(Z)$. Hence the action of $G$ has finite stabilizer.

If $(Z, L)$ is a polarized abelian variety and if $H^0(h, L)$ is the identity, then $h$ is the identity because $L$ is very ample. This proves that the $G$-action on $U_{g,K}$ is free. The general case where $A_0$ in 2.3 is nontrivial follows from the totally degenerate case and the fact that the automorphism group of any polarized abelian variety $(A, L)$ is finite. This proves that the $G$-action on $U_3$ has finite stabilizer.
It remains to prove that the action of $G$ is proper. This is reduced to proving the following claim:

Let $R$ be a discrete valuation ring $R$ with fraction field $k(\eta)$, $S = \text{Spec } R$. Let $\sigma : G \times U_3 \to U_3$ be the action and $\Psi = (\sigma, p_2) : G \times U_3 \to U_3 \times U_3$. Then for any pair $(\phi, \psi_\eta)$ consisting of a morphism $\phi : S \to U_3 \times U_3$ and a morphism $\psi_\eta : \text{Spec } k(\eta) \to G \times U_3$ such that $\psi_\eta \circ \Psi = \phi \otimes_R \eta(\eta)$, there is a morphism $\psi : S \to G \times U_3$ such that $\psi \circ \Psi = \phi$ and $\psi \otimes_R \eta(\eta) = \psi_\eta$.

This is again reduced to proving the following claim:

Suppose we are given an $S$-TSQAS $(Z, \phi_Z, \rho_Z)_{\text{RIG}}$ with rigid $G(K)$-structure and a modified Mumford family $(P, \phi_P, \rho_P)_{\text{RIG}}$ with rigid $G(K)$-structure over $S$ such that

$$(Z, \phi_Z, \rho_Z)_{\text{RIG}} \simeq (P, \phi_P, \rho_P)_{\text{RIG}} \text{ over } \text{Spec } k(\eta)$$

Then they are isomorphic over $S$.

In fact, the second claim follows from the proof of Lemma 3.4. This proves properness of the action $\Psi$. The third assertion follows from [KM97]. \qed
that \((\pi_A)_*(d_1L_A) = W_i \otimes \mathcal{O} V(K)\) as \(G(K)\)-modules. Choosing a local trivialization of \(W_i\) we have a local morphism \(\eta_i : A_{g,K} \to U_{g,K}\) so that the composite of \(\eta_i\) and the natural morphism of \(U_{g,K}\) to \(A_{g,K}^{\text{toric}}\) defines a morphism from \(A_{g,K}\) to \(A_{g,K}^{\text{toric}}\), which is the inverse of \(\bar{\eta}\). This proves that \(\bar{\eta}\) is an isomorphism.

**Theorem 7.6.** Let \(K\) be a finite symplectic abelian group with \(e_{\min}(K) \geq 3\) and \(N = e_{\max}(K)\). The functor \(SQ^{\text{toric}}_{g,K}\) is reductively and coarsely represented over \(\mathcal{O}_N\) by a complete reduced separated algebraic space \(SQ^{\text{toric}}_{g,K}\).

**Proof.** We choose and fix any pair \(d_1\) and \(d_2\) as before. Let \(SQ^{\text{toric}}_{g,K}\) be the uniform geometric and uniform categorical quotient of \(U_3\) by \(\text{GL}(W_1) \times \text{GL}(W_2)\). It is a complete separated algebraic space by Theorems 4.2 and 7.3. Since \(U_3\) is reduced, so is \(SQ^{\text{toric}}_{g,K}\). The rest is immediate.

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