Analysis of the stress field in a wedge using the fast expansions with pointwise determined coefficients

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Abstract. The stress problem for the elastic wedge-shaped cutter of finite dimensions with mixed boundary conditions is considered. The differential problem is reduced to the system of linear algebraic equations by applying twice the fast expansions with respect to the angular and radial coordinate. In order to determine the unknown coefficients of fast expansions, the pointwise method is utilized. The problem solution derived has explicit analytical form and it’s valid for the entire domain including its boundary. The computed profiles of the displacements and stresses in a cross-section of the cutter are provided. The stress field is investigated for various values of opening angle and cusp’s radius.

1. Introduction

A lot of research is devoted to consideration of elastic infinite wedge-shaped domains. The stress field for the antiplane deformation of elastic wedge is examined in [1-3]. The plane deformation of the wedge is discussed in [4-6]. Some particular results are obtained by applying Mellin transforms [1,5,6] and using functions of a complex variable [2,3,7]. The exact solutions to the plane deformation of the elastic wedge under zero loading on its lateral edges can be derived using Wiener-Hopf method [4]. The wedge has straight cracks on its axis of symmetry. An intrusion of the wedge into the plastic half-space has been investigated in the monograph [8]. The loaded wedge with smooth edges was considered in [9]. The particular solutions for the truncated circular sector are given in [10]. Some studies are devoted to three-dimensional problems for an elastic wedge [11,12]. In [11] the explicit matrix algorithm for three-dimensional wedge problem solution is proposed. In [12] one of the wedge’s surfaces is reinforced with a coating of Winkler type. On the other surface some arbitrary boundary conditions are set. In this case, the methods of nonlinear boundary integral equations and of successive approximations are used.

In this work the new analytical method of fast expansions [13] will be applied. It allows to obtain the high accuracy solution to the problem under study in an explicit analytical form. The method of fast expansions is applicable for solution to the problems associated with partial differential [14], integro-differential [13] and ordinary differential [15] equations.

2. Materials and methods

The equilibrium equations in terms of displacements in cylindrical coordinates for the wedge-shaped
domain $\Omega$ $(0 < r_0 \leq r \leq R,$ $0 \leq \theta \leq \theta_0$) have form
\[
\begin{align*}
2 \frac{\partial}{\partial r} \left( \frac{\partial U}{\partial r} \cdot \frac{U}{r} \right) + & \frac{1 + \nu}{(1 - \nu)} \frac{\partial^2 V}{\partial r \partial \theta} - \frac{3 - \nu}{(1 - \nu)} \frac{\partial V}{\partial r} \frac{V}{r^2} + \frac{1}{(1 - \nu)} \frac{\partial^2 U}{\partial \theta^2} = 0, \\
\frac{1 + \nu}{(1 - \nu)} \frac{\partial^2 U}{\partial r \partial \theta} + & \frac{3 - \nu}{(1 - \nu)} \frac{\partial U}{\partial r} \frac{\partial (\partial V / \partial r) + V}{r} + \frac{2}{(1 - \nu)} \frac{1}{r^2} \frac{\partial^2 V}{\partial \theta^2} = 0.
\end{align*}
\]
(1)

The displacements $U$ and $V$ are defined at the rear edge $r = R$:
\[
U\big|_{r=R} = U_R(\theta), \quad V\big|_{r=R} = V_R(\theta)
\]
(2)

The rest edges are loaded by external forces
\[
\begin{align*}
\left( \frac{1}{r} \frac{\partial U}{\partial \theta} + \frac{\partial V}{r} \right) \bigg|_{\theta = \theta_0} = & \Phi_1(r), \\
\left( \frac{1}{r} \frac{\partial U}{\partial \theta} + \frac{\partial V}{r} \right) \bigg|_{\theta = \theta_0} = & \Phi_2(r), \\
\left( \frac{\partial U}{\partial r} + \nu \left( \frac{1}{r} \frac{\partial V}{\partial \theta} + \frac{U}{r} \right) \right) \bigg|_{\theta = \theta_0} = & F_1(\theta),
\end{align*}
\]
(3)

Relying on a physical meaning of the problem, the stresses and strains should be bounded throughout the wedge-shaped domain $\Omega$ (otherwise the cutter would be broken). For this reason, the additional conditions must be imposed
\[
(U,V) \in \left[ C^0(0 \leq \theta \leq \theta_0), \quad C^0(0 \leq r \leq R) \right], \quad |U,V| < \infty.
\]
(5)

The formulation of the elastic problem (1)-(5) is given in a classical form, but it is not complete. The requirement to the functions $U$ and $V$ of being smooth and bounded is essential, since it imposes specific restrictions on the boundary conditions formulation. For the case of mixed boundary conditions (2)-(4), the relations (5) lead to the necessity of the "consistency" conditions formulation. If these conditions aren't satisfied, a discontinuity will occur at the angular points $(r = R, \theta = 0)$, $(r = r_0, \theta = \theta_0)$, $(r = \theta_0, \theta = 0)$, $(r = 0, \theta = \theta_0)$, that is, the conditions (5) will be broken. Hence, the accuracy of the solution will be essentially affected.

The boundary conditions (2) and (4) at the arcs $r = r_0, r = R$ are to be conformed with the conditions (3) at the rays $\theta = 0, \theta = \theta_0$. For this purpose the predetermined functions $U_R(\theta), V_R(\theta)$ at the arc $r = R$ are replaced by some new functions $U'_R(\theta), V'_R(\theta)$ and the boundary conditions (2) are rewritten as:
\[
\forall \theta \in [0,\theta_0], \quad U\big|_{r=r_0} = U'_R(\theta), \quad V\big|_{r=R} = V'_R(\theta)
\]
(6)

The functions $U'_R(\theta), V'_R(\theta)$ are supposed to coincide with the original functions $U_R(\theta), V_R(\theta)$ outside the $\varepsilon$–neighborhoods of the angular points. And inside the $\varepsilon$–neighborhoods these functions are such that they take on some unknown values at points $\theta = 0$ and $\theta = \theta_0$:
\[ \forall \theta \in [\varepsilon, \theta_0 - \varepsilon], \ U_1^*(\theta) = U_R(\theta), \ V_1^*(\theta) = V_R(\theta), \]

\[ U_{r1}^*(0) = U_{r1}^*, \ V_{r1}^*(0) = V_{r1}^*, \ U_R^*(\theta_0) = U_{r2}^*, \ V_R^*(\theta_0) = V_{r2}^*, \]

(7)

where the constants \( U_{r1}^*, V_{r1}^*, U_{r2}^*, V_{r2}^* \) are assumed to be unknown in advance and they are to be determined throughout the solution process.

Further, the boundary functions of sixth order [13] will be used to determine the displacements \( U \) and \( V \), so the functions \( U_R^*(\theta), V_R^*(\theta) \) will be smoothly conjugated with \( U_R(\theta), V_R(\theta) \) from (2) at the points \( (R, \varepsilon) \) and \( (R, \theta_0 - \varepsilon) \) up to sixth order derivatives as it is required by the condition (5).

Assuming fixed condition at the cutter’s rear edge \( r = R \), the relations (7) reduce

\[ \forall \theta \in [\varepsilon, \theta_0 - \varepsilon], \ U_1^*(\theta) = 0, \ V_1^*(\theta) = 0, \ U_{r1}^*(0) = U_{r1}^*, \ V_{r1}^*(0) = V_{r1}^*, \ U_R^*(\theta_0) = U_{r2}^*, \ V_R^*(\theta_0) = V_{r2}^*. \]

(8)

Thus, under the "fixed boundary" we will imply the equality of the displacement components to zero only for \( \varepsilon \leq \theta \leq \theta_0 - \varepsilon \), and inside the \( \varepsilon \) – neighborhoods of the angular points the displacements are determined by some additional relations \( U_R^*(\theta), V_R^*(\theta) \) satisfying the conditions (5) and (8).

Assuming the smallness of \( \varepsilon \) – neighborhoods the concrete relations for \( U_R^*(\theta), V_R^*(\theta) \) are not given. This is sufficient to derive the solution to the problem in a finite form.

At the cusp of the cutter at points \( (r_0, 0), (r_0, \theta_0) \) some "consistency" conditions are also to be satisfied. In order to derive these conditions we will make use of the following considerations. For the boundary conditions defined in (2)-(4) the unique solution to the boundary problem exists. However, discontinuities of the stresses might occur at the angular points if this circumstance isn’t envisaged in advance when the boundary conditions are being specified. That is, the stresses will depend on the direction of approach to the angular points for arbitrarily chosen functions \( F_1(\theta) \) and \( F_2(\theta) \). This drawback can be avoided by specifying two "consistency" conditions at every point \( (r_0, 0), (r_0, \theta_0) \).

The first two conditions can be obtained using the symmetry property of the stress tensor \( \sigma_{ii} = \sigma_{\theta\theta} \).

This leads to the following relations:

\[ \Phi_1(r_0) = F_2(0), \ \Phi_2(r_0) = F_2(\theta_0) \]

(9)

These conditions impose the restrictions on a choice of the function \( F_1(\theta) \).

The remaining two conditions are derived by formally substituting the radial loading \( F_1(\theta) \) at the cusp from relations (4) by some new function \( F_1^*(\theta) \). This function is supposed to coincide with the original function \( F_1(\theta) \) inside the interval \( \delta \leq \theta \leq \theta_0 - \delta \), and it takes on some unknown values \( F_1^*(0) = F_{11}^* \) and \( F_1^*(\theta_0) = F_{12}^* \) at the angular points

\[ \forall \theta \in [\delta, \theta_0 - \delta] \}, \ F_1^*(\theta) = F_1(\theta), \ F_1^*(0) = F_{11}^*, \ F_1^*(\theta_0) = F_{12}^*, \]

(10)

where the constants \( F_{11}^*, F_{12}^* \) are assumed to be determined throughout the solution process.

Following the condition (5), the function \( F_1^*(\theta) \) should be smoothly conjugated with \( F_1(\theta) \) at the points \( (r_0, \delta) \) and \( (r_0, \theta_0 - \delta) \) up to sixth order derivatives. Assuming the smallness of \( \delta \) – neighborhoods the concrete relation for \( F_1^*(\theta) \) is not given to simplify the construction of the solution. It is only sufficient to suppose that the function \( F_1^*(\theta) \) satisfies the conditions (5) and (10).

Further, we need to define \( \Phi_1(r) \) in the boundary conditions (3) to solve the problem. As an example, the simplest rapidly decreasing functions were chosen. The main loading throughout the
sample processing is applied to the upper edge \((\theta = \theta_0)\) of the cutter

\[ \Phi_2(r) = \Phi_{\alpha_0} \left( 1 - \frac{r}{R} \right), \quad \Phi_4(r) = \Phi_{\alpha_4} \left( 1 - \frac{r}{R} \right) \]

(11)

The lower edge \((\theta = 0)\) is assumed to be zero loaded, that is \(\Phi_1(r) = \Phi_3(r) = 0\).

Under the conditions (9)-(11), the loadings \(F_1(\theta), F_2(\theta)\) on the cutter's cusp are expressed in the simplest linear form

\[
\forall \theta \in [\delta, \theta_0 - \delta] \Rightarrow F_i^*(\theta) = F_i(\theta) = F_{i0} \frac{\theta}{\theta_0}, \quad F_i^*(0) = F_{i1}, \quad F_i^*(\theta_0) = F_{i2}, \quad F_2(\theta) = \Phi_{\alpha_2} \left( 1 - \frac{r}{R} \right) \frac{\theta}{\theta_0}.
\]

(12)

The constant coefficients \(F_{i0}, \Phi_{\alpha_2}, \Phi_{\alpha_4}\) in relations (11), (12) depend on pressing force applied to the sample being processed and they are determined experimentally.

From the previous considerations, it follows that instead of relations (4), the consistent boundary conditions should be used

\[
\left( \frac{\partial U}{\partial r} + V \left( \frac{1}{r} \frac{\partial V}{\partial \theta} + \frac{U}{r} \right) \right)_{r=0} = F_i^*(\theta), \quad \left( \frac{1}{r} \frac{\partial U}{\partial \theta} + \frac{\partial V}{\partial r} - \frac{V}{r} \right)_{r=0} = F_2(\theta),
\]

(13)

where the function \(F_i^*(\theta)\) must satisfy the conditions (12). The form of function \(F_i(\theta)\) is not changed as it satisfies the consistency conditions (9) due to relations (11).

Thus, the original boundary value problem (1)-(4) reduces to its altered formulation involving consistent boundary conditions (3), (8), (13) with 6 additional unknown constants

\[ U_{r1}^*, U_{r2}^*, V_{r1}^*, V_{r2}^*, F_{i1}^*, F_{i2}^* \]

(14)

to be determined throughout the solution process.

According with the method of fast expansions [13] the displacements \(U(r, \theta)\) and \(V(r, \theta)\) can be expressed as the superposition of boundary function and truncated Fourier sine series

\[
U = M_0^U(r, \theta) + \sum_{m=1}^{N_1} u_m(r) \sin m\pi \frac{\theta}{\theta_0}, \quad V = M_0^V(r, \theta) + \sum_{m=1}^{N_1} v_m(r) \sin m\pi \frac{\theta}{\theta_0},
\]

(15)

where \(N_1\) is the number of Fourier series terms retained, \(M_0^U(r, \theta)\) and \(M_0^V(r, \theta)\) are the boundary functions of sixth order

\[
M_0^U(r, \theta) = A_6(r) \left( 1 - \frac{\theta}{\theta_0} \right) + A_5(r) \frac{\theta}{\theta_0} + A_4(r) \left( \frac{\theta^2}{2} - \frac{\theta^3}{6} \frac{\theta_0^3}{3} \right) + A_3(r) \left( \frac{\theta^3}{6\theta_0} - \frac{\theta_0^3}{6} \right) + A_2(r) \left( \frac{\theta^4}{24} - \frac{\theta_0^4}{120\theta_0} \right) + A_1(r) \left( \frac{\theta^5}{120\theta_0} - \frac{\theta_0^5}{5040\theta_0} \right) + A_0(r) \left( \frac{\theta^6}{720\theta_0} - \frac{\theta_0^6}{5040\theta_0} \right)
\]

(16)

\[
\begin{align*}
&+ A_6(r) \left( \frac{\theta^6}{120\theta_0} \right) + A_5(r) \left( \frac{\theta^7}{24} - \frac{\theta_0^7}{120\theta_0} \right) + A_4(r) \left( \frac{\theta^8}{24} - \frac{\theta_0^8}{120\theta_0} \right) \right) + A_3(r) \left( \frac{\theta^9}{720\theta_0} - \frac{\theta_0^9}{5040\theta_0} \right) + A_2(r) \left( \frac{\theta^{10}}{5040\theta_0} - \frac{\theta_0^{10}}{2160\theta_0} \right) + A_1(r) \left( \frac{\theta^{11}}{5040\theta_0} - \frac{\theta_0^{11}}{2160\theta_0} \right) + A_0(r) \left( \frac{\theta^{12}}{5040\theta_0} - \frac{\theta_0^{12}}{2160\theta_0} \right)
\end{align*}
\]
This representation of the displacement components involves $16 + 2N_t$ unknown functions

$$A_j(r) = A_0(r) + B_1(r) + B_2(r) + B_3(r), \quad m = 1, 2, \ldots, N_t,$$

where $A_0(r) = \sum_{n=1}^{N_t} a_n(r) \sin n\pi z$ and $B_j(r) = \sum_{n=1}^{N_t} b_n(r) \sin n\pi z$.

For $j = 1, 2, 3$, the solutions for $A_j(r)$ and $B_j(r)$ are given by

$$A_j(r) = A_{0j}(r) + \sum_{n=1}^{N_t} a_{nj}(r) \sin n\pi z,$$

and

$$B_j(r) = B_{0j}(r) + \sum_{n=1}^{N_t} b_{nj}(r) \sin n\pi z.$$

The functions $a_{nj}(r)$ and $b_{nj}(r)$ are determined by the boundary conditions and the geometry of the problem. The full expressions for $A_j(r)$ and $B_j(r)$ are quite complex and involve series expansions in terms of the radial coordinate $r$.
\[ v_{m}(r) = M_{6}^{(m)}(r) + \sum_{n=1}^{N_{1}}v_{n}^{(m)}\sin n\pi z, \quad m = 1 \div N_{1}, \quad M_{6}^{(m)}(r) = \psi^{(m)}(1-z) + \psi_{2}^{(m)}z + \psi_{3}^{(m)} \left( \frac{z^{2}}{2} - \frac{z^{3}}{6} - \frac{z}{3} \right) \]

\[ + \psi_{4}^{(m)} \left( \frac{z^{6}}{6} - \frac{z^{3}}{6} \right) + \psi_{5}^{(m)} \left( \frac{z^{4}}{24} - \frac{z^{5}}{120} - \frac{z^{3}}{18} + \frac{z}{45} \right) + \psi_{6}^{(m)} \left( \frac{z^{5}}{120} - \frac{z^{3}}{36} - \frac{7z}{360} \right) \]

\[ + \psi_{7}^{(m)} \left( \frac{z^{6}}{720} - \frac{z^{7}}{5040} + \frac{z^{5}}{360} - \frac{z^{3}}{270} + \frac{z}{945} \right) + \psi_{8}^{(m)} \left( \frac{z^{7}}{5040} - \frac{z^{5}}{720} + \frac{7z^{3}}{2160} - \frac{31z}{15120} \right) \]  

Substituting the relations (16), (17) and (19)-(22) into the differential equations (1) and into the boundary conditions (3), (8), (13) and differentiating with respect to \( r \) and \( \theta \) gives the representation of the original boundary value problem containing \( 2(N_{1}+8)(N_{2}+8) \) unknown constant coefficients

\[ a_{n}^{(j)}, \quad b_{n}^{(j)}, \quad \psi_{n}^{(m)}, \quad j=1 \div 8, \quad n=1 \div N_{2}+8, \quad m=1 \div N_{1}. \]  

As the resultant expressions are quite cumbersome, they are not given here and further referred to as double fast expansion form of the original problem.

In order to determine the unknown constant coefficients (23), the pointwise method, which was developed and tested in [15] will be employed. Following this technique it is necessary to discretize the wedge-shaped domain \( \Omega \) in a uniform grid with \( (N_{1}+6)(N_{2}+6) \) nodes \( (r_{i}, \theta_{j}) \) so that the interval \([0, \theta_{i}]\) is divided by \( N_{1}+6 \) inner points \( \theta_{i} = k\theta_{j}/(N_{1}+5) \), \( k = 0, 1, \ldots, N_{1}+5 \) and the interval \([r_{i}, R]\) by \( N_{2}+6 \) inner points \( r_{i} = r_{0} + s(R-r_{0})/(N_{2}+5) \), \( s = 0, 1, \ldots, N_{2}+5 \). Substituting the values of each node \( (r_{i}, \theta_{j}) \) into the double fast expansion of governing differential equations (1) gives \( 2(N_{1}+8)(N_{2}+6) \) linear algebraic equations in terms of unknown coefficients (23).

Further, it is required to discretize the boundary of the domain \( \Omega \) in the same way. Dividing the interval \([0, \theta_{0}]\) by \( N_{1}+7 \) inner computational points \( \theta_{k} = k\theta_{0}/(N_{1}+6) \), \( k = 0, 1, \ldots, N_{1}+6 \) and substituting these values into the relations (8) and (13) gives \( 4(N_{1}+7) \) linear equations. Similarly, dividing the interval \([r_{0}, R]\) by \( N_{2}+7 \) inner computational points \( r_{s} = r_{0} + s(R-r_{0})/(N_{2}+6) \), \( s = 0, 1, \ldots, N_{2}+6 \) and substituting these values into the relations (3) gives \( 4(N_{2}+7) \) more linear equations. Thus, from the boundary conditions (3), (8) and (13) we obtain \( 4(N_{1}+7) + 4(N_{2}+7) \) linear equations in terms of unknown coefficients (23).

All in all, we derive the consistent system of \( 2(N_{1}+8)(N_{2}+8) \) linear algebraic equations in the unknowns (23) and 6 additional equations for the constants \( U_{1}, U_{2}, V_{1}, V_{2}, F_{11}, F_{12} \) following from the consistency conditions. This system then has been solved numerically in Maple software.

3. Results and discussion

As an example the numerical results for solution to the boundary value problem (1), (3), (8), (13) for the cutter made of instrumental quick-cutting steel of grade R18 are provided. This steel has the following values of elastic moduli [16, 17]

\[ \sigma_{0.2} = 5.1 \cdot 10^{8} \text{ Pa}, \quad \nu = 0.33, \quad E = 2.28 \cdot 10^{11} \text{ Pa}, \quad \lambda = 1.66 \cdot 10^{11}, \quad \mu = 8.57 \cdot 10^{10}, \]

where \( \sigma_{0.2} \) is the 0.2% offset yield stress.

Since the solution determined by the expressions (15)-(17), (19)-(22) approximately satisfies the differential equations (1) and the boundary conditions (3), (8), (13), then we will use a relative residual \( \delta = \delta(r, \theta)/\max |f_{i}(r, \theta)| \) to estimate an accuracy of the solution derived, where \( f_{i}(r, \theta) \) denotes...
the \(i\)-th term of the equations (1) or boundary conditions (3), (8), (13).

The distributions of the relative residual \(\delta_D\) of the approximate solution (15) to the equations (1) retaining only three terms \((N_1 = 3)\) in first expansion, and thirty terms \((N_2 = 30)\) in the second one are shown in figure 1. These results were computed for the following values of loadings and wedge dimensions

\[
\Phi_{\omega} = 10^{-8}, \quad \Phi_{w_1} = 10^{-8}, \quad F_{01} = 10^{-8}, \quad r_0 = 10^{-5} \text{ m}, \quad R = 10^{-3} \text{ m}, \quad \theta_0 = 5\pi/180
\]

(24)

It can be seen (refer to figure 1) that the maximal values of relative residual \(\delta_D\) are concentrated nearby wedge's cusp and they don't exceed \(1.6 \times 10^{-6}\) and \(2.5 \times 10^{-8}\) for the first and second differential equations, respectively.

The distribution of relative residual \(\delta_G\) along the region's boundary as well as residual's magnitudes for all boundary conditions (3), (8), (13) are essentially the same. The typical behavior of relative residual \(\delta_G\) at the edges \(\theta = \theta_0\) and \(r = r_0\) of the domain \(\Omega\) is shown in figure 2.

![Figure 1](image1.png)

**Figure 1.** The relative residual of the differential equations: (a) (1); (b) (1)\(_2\).

![Figure 2](image2.png)

**Figure 2.** The relative residual of the boundary conditions: (a) \(\delta_G|_{\theta=\theta_0}\); (b) \(\delta_G|_{r=r_0}\).
It is apparent that the accuracy achieved (refer to figures 1, 2) is acceptable for the most technical purposes. It should be noted that the similar high accuracy approximate solutions were accomplished in [13-15] where the method of fast expansions has also been involved.

The computed profiles of the displacements $U(r, \theta)$, $V(r, \theta)$ and stress components $\sigma_r$, $\sigma_\theta$, $\sigma_{r\theta}$ in a cross-section $\Omega$ of the cutter are shown in figure 3 and figure 4, respectively.

Figure 3. The displacement components: (a) $U(r, \theta)$; (b) $V(r, \theta)$.

Figure 4. The stress components: (a) $\sigma_r$; (b) $\sigma_\theta$; (c) $\sigma_{r\theta}$. 
The numerical results corresponding to the parameters' values (24) satisfy the yield condition [18]

\[ \sigma_{0,2} = 5.1 \cdot 10^4 \text{ Pa} \geq \bar{\sigma}, \quad \bar{\sigma} = \sqrt{\left(\sigma_r^*\right)^2 + \left(\sigma_\theta^*\right)^2 + \left(\sigma_z^*\right)^2 + 2(\sigma_{r\theta})^2} = 3.5 \cdot 10^4 \text{ Pa}, \]

where \( \sigma_r^* = \sigma_r - \frac{1}{3}(\sigma_r + \sigma_\theta), \quad \sigma_\theta^* = \sigma_\theta - \frac{1}{3}(\sigma_r + \sigma_\theta), \quad \sigma_z^* = \sigma_z - \frac{1}{3}(\sigma_r + \sigma_\theta), \quad \sigma_z = 0. \)

The distribution of \( \bar{\sigma} \) in the cutter's cross-section is shown in figure 5. It is obvious that the maximal stress \( \bar{\sigma} \) in a cutter occurs at its edge \( \theta = \theta_0 \) in the vicinity of point \( r = 0.0035 \text{ m}. \)

![Figure 5. The distribution of \( \bar{\sigma} \).](image)

The computed values of \( \bar{\sigma} \) for various opening angles \( \theta_0 \) and radii \( r_0 \) are presented in table 1. Examining these data it can be concluded that the displacements and stresses decline with an increase of opening angle \( \theta_0 \) and they raise with a decrease of the cusp's radius \( r_0 \). These results are consistent with the general elasticity and strength of materials theories.

| \( \theta_0 \) | \( r_0 \), m | \( 1 \cdot 10^{-4} \) | \( 1 \cdot 10^{-5} \) | \( 1 \cdot 10^{-6} \) |
|----------------|--------------|----------------|----------------|----------------|
| \( \pi/180 \)   | 1.959 \cdot 10^6 | 2.043 \cdot 10^8 | 2.527 \cdot 10^8 |                  |
| \( 2\pi/180 \)  | 5.207 \cdot 10^7 | 5.432 \cdot 10^7 | 5.918 \cdot 10^7 |                  |
| \( 3\pi/180 \)  | 2.634 \cdot 10^7 | 2.831 \cdot 10^7 | 2.984 \cdot 10^7 |                  |
| \( 4\pi/180 \)  | 2.052 \cdot 10^7 | 2.143 \cdot 10^7 | 2.167 \cdot 10^7 |                  |

4. Conclusion

Thus, the application of the method of fast expansions allowed to obtain the solution to the problem in an explicit analytical form which is valid for the entire domain \( \Omega \) including its boundary. The displacement and stress profiles derived can be utilized in the analysis of the stress state arising in wedge-shaped cutters.

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