A Note Concerning the
Turbulent Boundary Layer Drag at Large Reynolds Numbers

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Abstract. A correlation is obtained for the drag coefficient $c_f'$ of the turbulent boundary layer as a function of the effective boundary layer Reynolds number $Re$ that we previously introduced. A comparison is performed also with another correlation for the drag coefficient as a function of the traditional Reynolds number $Re_{\theta}$, based on the momentum thickness of the boundary layer proposed recently by R. D. Watson, R. M. Hall and J. B. Anders (NASA Langley Research Center) on the basis of different set of experimental data. We show that the correlation obtained by us agrees with experimental data from the Illinois Institute of Technology, but is incompatible with the data obtained in the Royal Institute of Technology at Stockholm. On the other hand, both sets of data are in disagreement with the Langley correlation.
1 Introduction

The drag coefficient for the turbulent boundary layer is determined as follows:

\[ c_f' = \frac{2u_*^2}{U^2}, \]  

where \( u_* \) is the friction velocity, \( u_* = \sqrt{\tau/\rho} \), \( \tau \) the wall shear stress, \( \rho \) the fluid density, \( U \) is the free stream velocity. It is apparently the most practically important characteristic of the turbulent boundary layer. In the present note a correlation will be derived between the drag coefficient \( c_f' \) and the effective Reynolds number \( Re \), introduced previously (1,2) and determined as follows. It was shown (2) that when the free stream turbulence is low, the average velocity \( u \) distribution in the intermediate region of the boundary layer between the viscous sublayer and the free stream consists of two different self-similar structures described by the scaling (power) laws \( \phi = A\eta^\alpha \) and \( \phi = B\eta^\beta \). Here \( \phi = u/u_* \) and \( \eta = u_* y/\nu \) are dimensionless variables (recently the notations \( U^+ \) and \( y^+ \) instead of classical notations \( \phi \) and \( \eta \) have become more popular), \( \nu \) the fluid kinematic viscosity, \( y \) the distance from the boundary. The interface between the two regions described by different scaling laws is sharp at \( y = \lambda \), so that the viscous sublayer and the first region adjacent to it form the sharply bounded wall layer of thickness \( \lambda \). Furthermore, we showed (2) that at sufficiently large Reynolds numbers the mean velocity distribution in the first region can be represented in the form obtained earlier for pipes (see (3))

\[ \phi = \frac{u}{u_*} = \left( \frac{1}{\sqrt{3}} \ln Re + \frac{5}{2} \right) \left( \frac{u_* y}{\nu} \right)^{\frac{1}{\ln Re}} \]  

where \( Re = U\Lambda/\nu \) is a Reynolds number, defined by a characteristic length scale \( \Lambda \) proportional to the thickness \( \lambda \) of the wall layer (4). The procedure for determining the effective Reynolds number and the characteristic length \( \Lambda \), described in detail in (1,2), is not more complicated, perhaps even simpler than the procedure for determining the momentum thickness \( \theta \) (1,4).

If the effective Reynolds number \( Re \) is in fact a governing characteristic of the turbulent boundary layer, like the Reynolds number based on the average velocity and pipe diameter for flow in pipes, then the drag coefficient \( c_f' \) should be a function of this parameter. Therefore it was natural to correlate the experimental data for drag coefficient and the effective
Reynolds number determined for the same experiments. Such correlation will be presented and discussed below.

On the other hand, Watson, Hall and Anders (5) at the NASA Langley Research Center recently obtained a different correlation of the drag coefficient with the parameter traditional in turbulent boundary layer studies — the Reynolds number $Re_\theta$ based on the momentum thickness. A comparison of the two correlations is performed in the present Note.

2 Correlation of the drag coefficient and effective Reynolds number for the turbulent boundary layer

According to the model presented by us previously (see (3)) the drag coefficient for the turbulent pipe flows $c$ as a function of the Reynolds number $Re = \bar{u}d/\nu$ is represented in the form

$$c = \frac{8u^2}{\bar{u}^2} = \frac{8}{\Psi^{2/(1+\alpha)}} , \quad \alpha = \frac{3}{2\ln Re} .$$

Here $\bar{u}$ is the mean velocity (bulk discharge per unit time divided by the cross-section area), and

$$\Psi(\alpha) = \frac{e^{3/2}(\sqrt{3} + 5\alpha)}{2^\alpha \alpha(1 + \alpha)(2 + \alpha)} .$$

As $Re \to \infty$, $\Psi(\alpha) \simeq (2e^{3/2}/\sqrt{3}) \ln Re$, so that asymptotically the relation is valid

$$c \simeq \frac{6}{e^3 \ln^2 Re} \simeq \frac{0.3}{\ln^2 Re} . \quad [3]$$

The relation [3] is asymptotically covariant. This means that a replacement of $Re$ by certain $Re'$, equal to Constant times $Re$, leaves the asymptotics [3] invariant. Contrary to that, a redefinition of $c$, e.g. an introduction of a different factor before $u^2/\bar{u}^2$ or the use of the maximum velocity instead of mean velocity $\bar{u}$, will change the Reynolds-number dependence of the drag coefficient.

Note that at large Reynolds number the expression for the drag coefficient should be identical for all shear flows provided the Reynolds number is properly determined. This identity was demonstrated for velocity profiles (1,2). Therefore it is logical to suggest that the expression for the drag coefficient in the boundary layer be of the form

$$c'_f = \frac{2u^2}{U^2} = \frac{\text{Const}}{\ln^2 Re} . \quad [4]$$
Here $Re$ is the effective Reynolds-number for the boundary layer, introduced in (1,2).

Processing of the experimental data by Naguib (6) and Nagib and Hites (see (7)) confirms (see Figure 1) the correlation [4], and gives the value of the Constant in [4] equal to 0.26, so that the correlation [4] takes the form

$$c'_f = \frac{0.26}{(\ln Re)^2}.$$ [5]

At the same time the data of Österlund (8) reveal a systematic deviation from [5] (Figure 2). We suggest one of the reasons for this disagreement is the following fact: The friction velocity $u_*$ was measured in (6,7) whereas it was calculated in (8) using the universal logarithmic law with inappropriately low values of the constants.

3 The correlation between the drag coefficient $c'_f$ and the Reynolds number $Re_\theta$ based on the momentum thickness

Watson, Hall and Anders (5) proposed recently the following correlation between the drag coefficient $c'_f$ and the traditional Reynolds number $Re_\theta$:

$$c'_f = 0.0097 Re_\theta^{-0.144}.$$ [6]

The correlation [6] was based on different experimental data covering the range of $Re_\theta$ from $3 \cdot 10^4$ to $6 \cdot 10^5$.

Our comparison showed (see Figures 3 and 4) that both the data of the experiments of (6),(7) (Illinois Institute of Technology) and (8) (Royal Institute of Technology, Stockholm) disagree systematically with the Langley correlation [6]. The comparison of our correlation [5] with the results of Langley experiments was for us impossible due to the incompleteness of the data available to us.

We notice that the power type asymptotics with small values of the exponent similar to [6] can be approximated by formulas similar to [5] and vice versa. Indeed, let us approximate $1/\ln^2 x$ by a power function

$$\frac{1}{\ln^2 x} = Gx^{-\gamma}.$$
The constants $G$ and $\gamma$ can be determined for instance from the condition that at a certain point $x_0$ the functions $Gx^{-\gamma}$ and $1/\ln^2 x$ be equal as well as their derivatives. This condition gives $\gamma = 2/\ln x_0$, and $G = x_0^2 (\ln x_0)^{-2}$. So, if the correlation [5] has a physical meaning, the empirically obtained correlations of the power type similar to [6] can be in fact their approximations.

4 Conclusion

A correlation between the drag coefficient and the effective Reynolds number for the turbulent boundary layers at large Reynolds numbers is proposed. It is in satisfactory agreement with the data of Naguib (6), and Nagib and Hites (7).

The data of Österlund (8) for the drag coefficient as a function of the effective Reynolds number are in systematic disagreement with the correlation [5]. Therefore there is a systematic disagreement between the data (6), (7) on one hand, and (8) on the other.

All the data of (6), (7) and (8) are in substantial systematic disagreement with the correlation [6] proposed in (5). A detailed analysis of the experimental data of (5) is needed. If such a correlation can be firmly established, it will provide the relation between the characteristic length scale $\Lambda$ and the momentum thickness $\theta$.

Acknowledgments. This work was supported in part by the Applied Mathematics subprogram of the U.S. Department of Energy under contract DE–AC03–76–SF00098, and in part by the National Science Foundation under grants DMS 94–16431 and DMS 97–32710.

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Figure Captions

Figure 1: The correlation [5] is in general agreement with the experimental data of Naguib (6) (diamonds) and Nagib and Hites (7) (squares).

Figure 2: There is a systematic substantial disagreement of the correlation [5] and the data of Österlund (8) (circles).

Figure 3: There is a systematic disagreement between the correlation [6] and the experimental data (6,7).

Figure 4: There is a systematic disagreement between the correlation [6] and the experimental data (8).
\[ \frac{0.26}{\ln^2(Re)} \]
\[ 0.26 \ln^2(Re) \]

\[ 2 (u^* U)^2 \]
$c_r \propto 2(u^* / U)^2$

$0.0097 \text{Re}^{-0.144}$
$c_f = \frac{2(\theta_u/U)^2}{0.0097 \text{Re}^{-0.144}}$