On the existence and number of invariant polynomials

Lucas Reis

Universidade de São Paulo, Instituto de Ciências Matemáticas e de Computação, São Carlos, SP 13560-970, Brazil.

Abstract

This paper explores a natural action of the group PGL\(_2(\mathbb{F}_q)\) on the set of monic irreducible polynomials of degree at least two over a finite field \(\mathbb{F}_q\). Our main results deal with the existence and number of fixed points and, in particular, we provide some improvements of previous works.

Keywords: Mobius inversion formula; group action; fixed points; enumeration formula

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1. Introduction

Let \(\mathbb{F}_q\) be the finite field with \(q\) elements, where \(q\) is a power of a prime \(p\). We have the following transformations on the polynomial ring \(\mathbb{F}_q[x]\).

Definition 1.1. For \(A \in \text{GL}_2(\mathbb{F}_q)\) with \(A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\) and \(f \in \mathbb{F}_q[x]\) a polynomial of degree \(k\), set

\[
A \circ f = (bx + d)^k f \left(\frac{ax + c}{bx + d}\right).
\]

Additionally, if \([A]\) denotes the class of \(A\) in the group PGL\(_2(\mathbb{F}_q)\), for \(f \in \mathbb{F}_q[x]\) a nonzero polynomial, set

\[
[A] \circ f = c_{f,A} \cdot (A \circ f),
\]

where \(c_{f,A} \in \mathbb{F}_q^*\) is the element of \(\mathbb{F}_q\) such that \(c_{f,A} \cdot (A \circ f)\) is monic.

Let \(\mathcal{I}_k\) be the set of monic irreducible polynomials of degree \(k\) over \(\mathbb{F}_q\). As pointed out in [5], the group PGL\(_2(\mathbb{F}_q)\) acts on the sets \(\mathcal{I}_k\) with \(k \geq 2\), via the compositions \([A] \circ f\). It is then natural to ask about the fixed points.

Definition 1.2. Let \(k \geq 2, f \in \mathcal{I}_k, [A] \in \text{PGL}_2(\mathbb{F}_q)\) and \(G\) a subgroup of PGL\(_2(\mathbb{F}_q)\).
(i) \( f \) is \([A]\)-invariant if \([A] \circ f = f\);
(ii) \( f \) is \(G\)-invariant if \([B] \circ f = f\) for any \([B] \in G\).

From the previous definition, some natural questions arise:

- Given a subgroup \(G\) of \(\text{PGL}_2(\mathbb{F}_q)\), there exists \(G\)-invariants?
- Given \(n \geq 2\) and \([A] \in \text{PGL}_2(\mathbb{F}_q)\), there exists \([A]\)-invariants of degree \(n\)? How many are they?

In this paper, we deal with the two questions above and our main results can be stated as follows.

**Theorem 1.3.** Let \(G\) be a noncyclic group of \(\text{PGL}_2(\mathbb{F}_q)\). Then any \(G\)-invariant has degree two.

**Theorem 1.4.** Let \([A] \in \text{PGL}_2(\mathbb{F}_q)\) be an element of order \(D = \text{ord}([A])\). Then, for any integer \(n > 2\), the number \(\mathcal{N}_A(n)\) of \([A]\)-invariants of degree \(n\) is zero if \(n\) is not divisible by \(D\) and, for \(n = Dm\) with \(m \in \mathbb{N}\), the following holds:

\[
\mathcal{N}_A(Dm) = \frac{\varphi(D)}{Dm} \left( c_A + \sum_{d|m, \gcd(d,D) = 1} \mu(d)(q^{m/d} + \eta_A(m/d)) \right),
\]

where \(\varphi\) is the Euler Phi function, \(\mu\) is the Mobius function, \(\eta_A : \mathbb{N} \rightarrow \mathbb{N}\) and \(c_A \in \mathbb{Z}\) are given as follows

1. \(c_A = 0\) and \(\eta_A \equiv \varepsilon\), where \(\varepsilon = -1\) or 0, according to whether \(A\) has distinct or equal eigenvalues in \(\mathbb{F}_q\), respectively;
2. \(c_A = -1\) and \(\eta_A\) is the zero function if \(A\) has symmetric eigenvalues in \(\mathbb{F}_q^2 \setminus \mathbb{F}_q^2\);
3. \(c_A = 0\) and \(\eta_A(t) = (-1)^{t+1}\) if \(A\) has non symmetric eigenvalues in \(\mathbb{F}_q^2 \setminus \mathbb{F}_q^2\).

We remark that some results in the direction of Theorems 1.3 and 1.4 were previously obtained. In [5] and [4], Theorem 1.3 is proved for the cases that \(G = \text{PGL}_2(\mathbb{F}_q)\) and \(G\) is a \(p\)-group, respectively. In addition, Theorem 5.3 of [4] entails that, if \([A] \in \text{PGL}_2(\mathbb{F}_q)\) has order \(D\) and \(n > 2\), the number \(\mathcal{N}_A(n)\) of \([A]\)-invariants of degree \(n\) equals zero if \(n\) is not divisible by \(D\) and, for \(n = Dm\) with \(m \in \mathbb{N}\),

\[
\mathcal{N}_A(n) \approx \frac{\varphi(D)}{Dm} q^m.
\]

Here, \(a_m \approx b_m\) means \(\lim_{m \to \infty} \frac{a_m}{b_m} = 1\). We observe that this asymptotic formula agrees with Theorem 1.4.

The structure of the paper is given as follows. In Section 2, we provide all the machinery that is used in the proof of our main results. Section 3 is devoted to prove Theorem 1.3 and, in Section 4, we prove Theorem 1.4.
2. Preliminaries

In this section, we provide background material that is frequently used throughout the paper.

2.1. Auxiliary lemmas

We present, without proof, some auxiliary results from [5]. We use slightly different notations and, for more details, see Sections 4 and 5 of [5].

Lemma 2.1. For any $A, B \in \text{GL}_2(\mathbb{F}_q)$ and $f \in \mathcal{I}_k$ with $k \geq 2$, the following hold.

(i) $[A] \circ f$ is in $\mathcal{I}_k$,
(ii) $[A] \circ ([B] \circ f) = [AB] \circ f$,
(iii) if $[I]$ is the identity of $\text{PGL}_2(\mathbb{F}_q)$, $[I] \circ f = f$.

Definition 2.2. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{F}_q)$ and $r$ a non-negative integer,

$F_{A,r}(x) := bx^{q^r+1} - ax^{q^r} + dx - c$.

From Theorem 4.5 of [5], we have the following result.

Lemma 2.3. Let $f$ be an irreducible polynomial of degree $Dm \geq 3$ such that $[A] \circ f = f$, where $D$ is the order of $[A]$. The following hold:

(i) there is a unique positive integer $\ell \leq D - 1$ such that $\gcd(\ell, D) = 1$ and $f$ divides $F_{A,s}(x)$, where $s = \ell \cdot \frac{Dm}{\ell} = \ell \cdot m$,
(ii) for any $r \geq 1$, the irreducible factors of $F_{A,r}$ are of degree $Dr$, of degree $Dk$ with $k < r$, $r = km$ and $\gcd(m, D) = 1$ and of degree at most 2.

Lemma 2.4 (see [5], item (a) of Lemma 5.1). Let $r \geq 1$ and let $k$ be a divisor of $r$ such that $m := r/k$ is relatively prime with $D$, the order of $[A]$. For $j$ such that $jm \equiv 1 \pmod{D}$, the irreducible factors of $F_{A,r}(x)$ of degree $Dr$ are exactly the irreducible factors of $F_{A, jk}(x)$ of degree $Dk$.

2.2. Invariants through conjugacy classes

We establish some interesting relations between the polynomials that are invariant by two conjugated elements. We start with the following result.

Lemma 2.5. Let $A, B, P \in \text{GL}_2(\mathbb{F}_q)$ such that $[B] = [P] \cdot [A] \cdot [P]^{-1}$. For any $k \geq 2$ and any $f \in \mathcal{I}_k$, we have that $[B] \circ f = f$ if and only if $[A] \circ g = g$, where $g = [P]^{-1} \circ f \in \mathcal{I}_k$.

Proof. Observe that, from Lemma 2.1, the following are equivalent:

- $[B] \circ f = f$,
- $[P] \circ ([A] \circ ([P]^{-1} \circ f)) = f$,
- $[A] \circ ([P]^{-1} \circ f) = [P]^{-1} \circ f$. 

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Definition 2.6. For $A \in \text{GL}_2(\mathbb{F}_q)$, $C_A(n)$ is the set of $[A]$-invariants of degree $n$ and $C_A = \bigcup_{n \geq 2} C_A(n)$ is the set of $[A]$-invariants.

Since the compositions $[A] \circ f$ preserve degree and permute the set of monic irreducible polynomials of a given degree, we obtain the following result.

Theorem 2.7. Let $A, B, P \in \text{GL}_2(\mathbb{F}_q)$ such that $B = PAP^{-1}$. Let $\tau : C_B \to C_A$ be the map given by $\tau(f) = [P]^{-1} \circ f$. Then $\tau$ is a degree preserving one to one correspondence. Additionally, for any $n \geq 2$, the restriction of $\tau$ to the set $C_A(n)$ is an one to one correspondence between $C_A(n)$ and $C_B(n)$. In particular, $N_A(n) = N_B(n)$.

Proof. From Lemma 2.5 $\tau$ is well defined and is a one to one correspondence. Additionally, since the compositions $[A] \circ f$ preserve degree, the restriction of $\tau$ to the set $C_A(n)$ is a one to one correspondence between $C_A(n)$ and $C_B(n)$ and, since these sets are finite, they have the same cardinality, i.e., $N_A(n) = N_B(n)$. \qed

We observe that, if $G$ is the cyclic group generated by $[A] \in \text{PGL}_2(\mathbb{F}_q)$, $f$ is $[A]$-invariant if and only if is $G$-invariant. From the previous theorem, the following corollary is straightforward.

Corollary 2.8. Let $G, H \in \text{PGL}_2(\mathbb{F}_q)$ be groups with the property that there exists $P \in \text{GL}_2(\mathbb{F}_q)$ such that $G = [P] \cdot H \cdot [P]^{-1} = \{ [P] \cdot [A] \cdot [P]^{-1} \mid [A] \in H \}$. Then there is a one to one correspondence between the $G$-invariants and the $H$-invariants that is degree preserving.

The previous results entail that in order to count polynomials that are invariant by an element (or a subgroup) of $\text{PGL}_2(\mathbb{F}_q)$, we only need to consider them up to conjugations. In this context, understanding the conjugacy classes of the elements in $\text{PGL}_2(\mathbb{F}_q)$ is crucial.

Definition 2.9. For $A \in \text{GL}_2(\mathbb{F}_q)$ such that $[A] \neq [I]$, $A$ is of type 1 (resp. 2, 3 or 4) if its eigenvalues are distinct and in $\mathbb{F}_q$ (resp. equal and in $\mathbb{F}_q$, symmetric and in $\mathbb{F}_q \setminus \mathbb{F}_q^*$ or not symmetric and in $\mathbb{F}_q^* \setminus \mathbb{F}_q$).

We observe that the types of $A$ and $\lambda \cdot A$ are the same for any $\lambda \in \mathbb{F}_q^*$. For this reason, we say that $[A]$ is of type $t$ if $A$ is of type $t$. The following theorem entails that any element $[A] \in \text{PGL}_2(\mathbb{F}_q)$ is conjugated to a special element of type $t$, for some $1 \leq t \leq 4$.

Theorem 2.10. Let $A \in \text{GL}_2(\mathbb{F}_q)$ such that $[A] \neq [I]$ and let $a, b$ and $c$ be elements of $\mathbb{F}_q^*$. Let $A(a) := \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, $E := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $C(b) := \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix}$ and $D(c) := \begin{pmatrix} 0 & 1 \\ c & 1 \end{pmatrix}$. Then $[A] \in \text{PGL}_2(\mathbb{F}_q)$ is conjugated to:

(i) $[A(a)]$ for some $a \in \mathbb{F}_q \setminus \{0, 1\}$ if and only if $A$ is of type 1.
(ii) $[E]$ if and only if $A$ is of type 2.
(iii) \([C(b)]\) for some non square \(b \in \mathbb{F}_q^*\) if and only if \(A\) is of type 3.
(iv) \([D(c)]\) for some \(c \in \mathbb{F}_q\) such that \(x^2 - x - c \in \mathbb{F}_q[x]\) is irreducible if and only if \(A\) is of type 4.

Proof. This results follows from the fact that two elements \([A], [B]\) in \(\text{GL}_2(\mathbb{F}_q)\) are conjugated if and only if the characteristic polynomials \(P_A(x)\) and \(P_B(x)\) of \(A\) and \(B\) are equal up to a transformation of the form \(f(x) \mapsto \lambda^{-2} f(\lambda x)\) for some \(\lambda \in \mathbb{F}_q^*\). We omit the details.

Definition 2.11. An element \(A \in \text{GL}_2(\mathbb{F}_q)\) is in reduced form if it is equal to \(A(a), E, C(b)\) or \(D(c)\) for some suitable \(a, b\) or \(c\) in \(\mathbb{F}_q\).

We finish this section giving a complete study on the order of the elements in \(\text{PGL}_2(\mathbb{F}_q)\), according to their type.

Lemma 2.12. Let \([A]\) be an element of type \(t\) and let \(D\) be its order in \(\text{PGL}_2(\mathbb{F}_q)\). The following hold:

(i) for \(t = 1\), \(D > 1\) is a divisor of \(q - 1\),
(ii) for \(t = 2\), \(D = p\),
(iii) for \(t = 3\), \(D = 2\),
(iv) for \(t = 4\), \(D\) divides \(q + 1\) and \(D > 2\).

Proof. Since any two conjugated elements in \(\text{PGL}_2(\mathbb{F}_q)\) have the same order, from Theorem 2.10 we can suppose that \(A\) is in the reduced form. From this fact, items (i), (ii) and (iii) are straightforward. For item (iv), let \(\alpha, \alpha^q\) be the eigenvalues of \(A\). We that \([A]^D = [A^D]\) and then \([A]^D = [I]\) if and only if \(A^D\) equals the identity element \(I \in \text{GL}_2(\mathbb{F}_q)\) times a constant. The latter holds if and only if \(\alpha^D\) and \(\alpha^qD\) are equal. Observe that \(\alpha^D = \alpha^qD\) if and only if \(\alpha^{(q-1)D} = 1\). In particular, since \(\alpha \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q\), we have \(D > 1\) and \(D\) divides \(q + 1\). If \(D = 2\), then \(\alpha^q = -\alpha\), a contradiction since \(A\) is not of type 3.

3. On \(G\)-invariants: the noncyclic case

Here we provide the proof of Theorem 1.3 showing the triviality of \(G\)-invariants when \(G\) is a noncyclic subgroup of \(\text{PGL}_2(\mathbb{F}_q)\). We start with the following definition.

Definition 3.1. Let \(\overline{\mathbb{F}}_q\) be the algebraic closure of \(\mathbb{F}_q\). For \(A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{F}_q)\) and \(\alpha \in \overline{\mathbb{F}}_q \setminus \mathbb{F}_q\),

\([A] \circ \alpha := \frac{d\alpha - c}{a - b\alpha}\).

According to Lemma 2.6 and Theorem 4.2 of \(\overline{\mathbb{F}}_q\), we have the following result.

Lemma 3.2. Let \(f \in \mathbb{F}_q[x]\) be a monic irreducible polynomial of degree at least two and \(\alpha \in \overline{\mathbb{F}}_q \setminus \mathbb{F}_q\). Let \(A, B \in \text{GL}_2(\mathbb{F}_q)\), the following holds:
(i) $f$ if $[A]$-invariant if and only if $f$ divides $F_{A,r}$ for some $r \geq 0$ or, equivalently, $[A] \circ \alpha = \alpha^q$;
(ii) $[A] \circ ([B] \circ \alpha) = [AB] \circ \alpha$.

From the previous lemma, we have the following result.

**Lemma 3.3.** Let $r$ be a non-negative integer and $A_1, A_2 \in \text{GL}_2(\mathbb{F}_q)$ such that $[A_1] \neq [A_2]$ in $\text{PGL}_2(\mathbb{F}_q)$. If $\alpha \in \overline{\mathbb{F}_q} \setminus \mathbb{F}_q$ is such that $[A_1] \circ \alpha = [A_2] \circ \alpha$, then $\alpha \in \mathbb{F}_q^\times$.

**Proof.** From hypothesis, $[B] \circ \alpha = \alpha$, where $[B] = [A_1][A_2]^{-1}$. However, if $[B]$ is not the identity, the equality $[B] \circ \alpha = \alpha$ yields a polynomial equation of degree at most 2 in $\alpha$ with coefficients in $\mathbb{F}_q$. Therefore, $\alpha \in \mathbb{F}_q^\times$. \hfill $\Box$

### 3.1. Proof of Theorem 1.3

We observe that it suffices to prove the theorem in the case that $G$ is a noncyclic group of $\text{PGL}_2(\mathbb{F}_q)$, generated by two elements $\{A_1, A_2\} \subseteq \text{PGL}_2(\mathbb{F}_q)$. Let $D_1$ and $D_2$ be the orders of $[A_1]$ and $[A_2]$, respectively. We recall that, for any element $[A] \in \text{PGL}_2(\mathbb{F}_q)$ of order $d$, the $[A]$-invariants have degree two or degree divisible by $d$. In particular, if there exists a monic irreducible polynomial $f \in \mathbb{F}_q[x]$ of degree $n \geq 3$ that is $G$-invariant, then $n$ is divisible by $D_1$ and $D_2$.

Therefore, $n = \frac{D_1D_2}{D} \cdot n_0$ for some positive integer $n_0$, where $D = \gcd(D_1, D_2)$. In addition, from Lemmas 2.3 and 2.4 we conclude that there exist positive integers $j_1 \leq D_1$ and $j_2 \leq D_2$ such that $\gcd(j_1, D_1) = \gcd(j_2, D_2) = 1$ and $f$ divides both $F_{A_1^{j_1}}(x)$ and $F_{A_2^{j_2}}(x)$. In other words, if we set $[B_i] = [A_i]^{j_i}$ for $i = 1, 2$, we have that

$$[B_1] \circ \alpha = \alpha^{q^{D_2n_0/D}} \quad \text{and} \quad [B_2] \circ \alpha = \alpha^{q^{D_1n_0/D}},$$

(2)

for any root $\alpha \in \overline{\mathbb{F}_q} \setminus \mathbb{F}_q^\times$ of $f$. In particular, we have the following equalities

$$[B_1B_2] = ([B_1] \circ \alpha)^{q^{D_1n_0/D}} = \alpha^{q^{(D_1+D_2)n_0/D}} = ([B_2] \circ \alpha)^{q^{D_2n_0/D}} = [B_2B_1] \circ \alpha.$$  

In addition, from Eq. (2), we have that

$$[B_1]^{D_1/D} \circ \alpha = \alpha^{q^{D_1D_2n_0/D}} = [B_2]^{D_2/D} \circ \alpha.$$  

Since $\gcd(j_1, D_1) = \gcd(j_2, D_2) = 1$, the elements $[B_1]$ and $[B_2]$ have orders $D_1$ and $D_2$, respectively, and they also generate $G$. In addition, since $\alpha$ is not in $\overline{\mathbb{F}_q}$, the previous equalities and Lemma 2.4 entail that $[B_1] \cdot [B_2] = [B_2] \cdot [B_1]$ and $[B_1]^{D_1/D} = [B_2]^{D_2/D}$. Now, the proof of Theorem 1.3 follows from the following result.

**Proposition 3.4.** Let $G$ be an abelian subgroup of $\text{PGL}_2(\mathbb{F}_q)$, generated by two elements $g_1, g_2$ of orders $d_1$ and $d_2$. If $d = \gcd(d_1, d_2)$ and $g_1^{d_1/d} = g_2^{d_2/d}$, then $G$ is cyclic.
Proof. We observe that the structure of $G$ is not changed up to conjugation by an element of $\text{PGL}_2(\mathbb{F}_q)$. In particular, we can suppose that $g_1$ is in reduced form. We have four cases to consider:

(i) If $g_1$ of type 1, $g_1 = [A(a)]$ for some $a \in \mathbb{F}_q \setminus \{0, 1\}$. It is direct to verify that the centralizer of $g_1$ equals the group $\{[A(b)]; b \in \mathbb{F}_q^*\}$. In particular, $g_2 = [A(b)]$ for some $b \in \mathbb{F}_q^*$. Since $\mathbb{F}_q^*$ is cyclic, the subgroup of $\mathbb{F}_q^*$ generated by $a$ and $b$ is also cyclic. If $\theta$ is any generator for such a group, it follows that $G$ is generated by $g_3 = [A(\theta)]$.

(ii) If $g_1$ is of type 2, $g_1 = [E]$ has order $p = d_1$. In particular, from Lemma [2.12] we have that either $d_2 = p$ or $d_2$ is not divisible by $p$. In particular, either $d_2 = p$ or $d_1 = 1$. If $d_2 = p$, we have that $d = p$ and so $g_1 = g_2$. Hence, $G$ is generated by $g_1$. If $d_1 = 1$, we have that the orders of $g_1$ and $g_2$ are relatively prime and so $G$ is generated by $g_3 = g_1g_2$.

(iii) If $g_1$ is of type 3, $g_1$ has order $d_1 = 2$. If $d_2$ is even, we have that $d = 2$ and so $g_1 = g_2^{d_2/2}$, hence $G$ is generated by $g_1$. If $d_2$ is odd, the orders of $g_1$ and $g_2$ are relatively prime and so $G$ is generated by $g_3 = g_1g_2$.

(iv) If $g_1$ is of type 4, $g_1 = [D(c)]$ for some $c \in \mathbb{F}_q$ such that $x^2 - x - c$ is irreducible over $\mathbb{F}_q$. It is direct to verify that the centralizer of $g_1$ equals the group $G_0 = \{[A_t]; t \in \mathbb{F}_q\} \cup \{[I]\}$, where $A_t = t \cdot I + D(c)$. In particular, $g_1, g_2 \in G_0$ and so $G$ is a subgroup of $G_0$. We observe that $D(c)^2 = D(c) + cI$ and then, for any $s, t \in \mathbb{F}_q$, the following holds:

$$[A_s] \cdot [A_t] = \begin{cases} [I] & \text{if } s + t + 1 = 0, \\ [A_{h(s,t)}], h(s,t) = \frac{st+c}{st+1} & \text{otherwise}. \end{cases}$$

However, since $x^2 - x - c \in \mathbb{F}_q[x]$ is irreducible, $K = \mathbb{F}_q[x]/(x^2 - x - c)$ is the finite field with $q^2$ elements. Let $k\alpha + w$ be a primitive element of $K = \mathbb{F}_{q^2}$, where $\alpha$ is a root of $x^2 - x - c$ and $k, w \in \mathbb{F}_q$ with $k \neq 0$. We claim that, in this case, $G_0$ is the cyclic group generated by $[A_{w/k}]$. In fact, we have $[A_{w/k}] \in G_0$ and, for any $s \in \mathbb{F}_q$, $\alpha + s = (k\alpha + w)^r$ for some $r \in \mathbb{N}$. Hence

$$(kx + w)^r \equiv x + s \pmod{x^2 - x - c},$$

and then $(kA + wI)^r = A + sI = A_s$. Taking equivalence classes in $\text{PGL}_2(\mathbb{F}_q)$ we obtain $[A_{w/k}]^r = [A_s]$, hence $G_0$ is cyclic. Since $G$ is a subgroup of $G_0$, it follows that $G$ is also cyclic.

3.2. A remark on quadratic invariants

We observe that, if $f \in \mathbb{F}_q[x]$ is a quadratic irreducible polynomial and $H$ is a subgroup of $\text{PGL}_2(\mathbb{F}_q)$ generated by elements $[B_1], \ldots, [B_s]$, $f$ is $H$-invariant if and only if $B_i \circ f = \lambda_i \cdot f$ for some $\lambda_i \in \mathbb{F}_q$, if we write $f(x) = x^2 + ax + b$, the equalities $B_i \circ f = \lambda_i \cdot f$ yield a system of $s$ equations, where $a, b$ and the $\lambda_i$'s
are variables. The coefficients of these equations arise from the entries of each $B_i$. Given the elements $B_i$, we can easily discuss the solutions of this system. We remark that we can actually have monic irreducible quadratic polynomials that are invariant by noncyclic groups. For instance, if $q$ is odd and $b \in \mathbb{F}_q^*$ is not a square, the polynomial $f = x^2 - b$ is invariant by the (noncyclic) group generated by the elements

$$H_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad H_2 = \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix}.$$ 

For $q$ even, if $c \in \mathbb{F}_q$ is such that $x^2 + x + c$ is irreducible over $\mathbb{F}_q$, this polynomial is invariant by the (noncyclic) group generated by the elements

$$H_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad H_2 = \begin{pmatrix} 0 & 1 \\ c & 1 \end{pmatrix}.$$

4. On the number of $[A]$-invariants

In this section we provide complete enumeration formulas for the number of $[A]$-invariants of degree $n > 2$, proving Theorem 1.4. We naturally study the number of $[A]$-invariants according to the type of $A$. From Theorem 2.7, it suffices to consider elements of $\text{PGL}_2(\mathbb{F}_q)$ of type $1 \leq t \leq 4$ in reduced form. We start with elements of type 1 and 2. If $A$ has type $t = 1$ (resp. $t = 2$) and is in reduced form, we see that $A \circ f$ corresponds to $f(ax)$ (resp. $f(x + 1)$).

From definition, $[A] \circ f = f$ if and only if $A \circ f = \lambda \cdot f$ for some $\lambda \in \mathbb{F}_q^*$. A comparison on the leading coefficient (resp. the constant term) of the equality $f(x + 1) = \lambda \cdot f(x)$ (resp. $f(ax) = \lambda \cdot f(x)$), entails the following fact:

"If $A$ is of type 1 or 2 in reduced form and $f \in \mathbb{F}_q[x]$ is a monic irreducible polynomial of degree $k \geq 2$, then $[A] \circ f = f$ if and only if $A \circ f = f$.”

Therefore, we are looking for the monic irreducible polynomials $f \in \mathbb{F}_q[x]$ of degree $n$ that satisfies $f(x) = f(ax)$ or $f(x) = f(x + 1)$. The number of monic irreducible polynomials satisfying such identities was provided in Theorems 2 and 4 of [2]. Combining these theorems with Theorem 2.7 we easily obtain the following result.

**Lemma 4.1.** Let $[A] \in \text{PGL}_2(\mathbb{F}_q)$ be an element of type $t \leq 2$ and order $D$. Then, for any integer $n > 2$, the number $N_A(n)$ of $[A]$-invariants of degree $n$ is zero if $n$ is not divisible by $D$ and, for $n = Dm$ with $m \in \mathbb{N}$, the following holds:

$$N_A(Dm) = \frac{\varphi(D)}{Dm} \sum_{d|m, \gcd(d,D)=1} \mu(d)(q^{m/d} - \varepsilon),$$

where $\varepsilon = 1$ if $t = 1$ and $\varepsilon = 0$, $D = p$ if $t = 2$. 


Recall that an element of type 3 in reduced form equals $C(b)$ for some non-square $b \in \mathbb{F}_q^*$, and $[C(b)]$ has order two. We observe that, from definition, $[C(b)] \circ f = f$ if and only if $x^{2m} f \left(\frac{b}{x}\right) = \lambda \cdot f(x)$, for some $\lambda \in \mathbb{F}_q^*$. Since $b$ is not a square in $\mathbb{F}_q^*$, the polynomial $x^2 - b$ is irreducible over $\mathbb{F}_q$. In particular, if $\theta \in \mathbb{F}_q^2$ is a root of $x^2 - b$, evaluating both sides of the previous equality at $x = \theta$, we obtain $b^{m} f(\theta) = \lambda f(\theta)$. If $f$ is monic irreducible and has degree $2m \geq 3$, $f$ is not divisible by $x^2 - b$ and so $f(\theta) \neq 0$. Therefore, $\lambda = b^m$, i.e., $f$ is $[C(b)]$-invariant if and only if $x^{2m} f \left(\frac{b}{x}\right) = b^m \cdot f(x)$. The number of monic irreducible polynomials satisfying the previous identity was obtained in Corollary 7 of [3]. Combining this corollary with Theorem 2.7, we easily obtain the following result.

**Lemma 4.2.** Let $[A] \in \text{PGL}_2(\mathbb{F}_q)$ be an element of type 3. In particular, its order is $D = 2$. Then, for any integer $n > 2$, the number $N_A(n)$ of $[A]$-invariants of degree $n$ is zero if $n$ is not divisible by $D$ (i.e., $n$ is odd) and, for $n = 2m$ with $m \in \mathbb{N}$, the following holds:

$$N_A(2m) = \frac{1}{2m} \left(-1 + \sum_{d|m, \gcd(d,2)=1} \mu(d)q^{m/d}\right).$$

In particular, cases 1 and 2 of Theorem 1.4 are now proved.

### 4.1. Elements of type 4

Here we establish the last case of Theorem 1.3 that corresponds to elements of type 4. Again, we only consider elements of type 4 in reduced form. We emphasize that the previous enumeration formulas for elements of type $t \leq 3$ are based in the Mobius Inversion Formula and its generalizations. This inversion formula is often employed when considering the enumeration of irreducible polynomials with specified properties. We recall a nice generalization of this result.

**Theorem 4.3.** Let $\chi : \mathbb{N} \to \mathbb{C}$ be a completely multiplicative function (which is, in other words, an homomorphism between the monoids $(\mathbb{N}, \cdot)$ and $(\mathbb{C}, \cdot)$). Also let $L, K : \mathbb{N} \to \mathbb{C}$ be two functions such that

$$L(n) = \sum_{d|n} \chi(d) \cdot K \left(\frac{n}{d}\right), n \in \mathbb{N}.$$

Then,

$$K(n) = \sum_{d|n} \chi(d) \cdot \mu(d) \cdot L \left(\frac{n}{d}\right), n \in \mathbb{N}.$$

An interesting class of completely multiplicative functions is the class of Dirichlet Characters and, for instance, the principal Dirichlet character modulo $d$ is the function $\chi_d : \mathbb{N} \to \mathbb{N}$ such that $\chi_d(n) = 1$ if $\gcd(d,n) = 1$ and $\chi_d(n) = 0$, otherwise. We first present a direct consequence of the results contained in Subsection 2.1.
Lemma 4.4. Let $A$ be an element of $\text{GL}_2(\mathbb{F}_q)$ and let $D$ be the order of $[A]$ in $\text{PGL}_2(\mathbb{F}_q)$. Then, for any $m \in \mathbb{N}$, the $[A]$-invariants of degree $Dm > 2$ are exactly the irreducible factors of degree $Dm$ of $F_{A,m}$, where $j$ runs over the positive integers $\leq D - 1$ such that $\gcd(j, D) = 1$.

Proof. According to Lemma 2.3, the $[A]$-invariants of degree $Dm > 2$ are exactly the irreducible factors of degree $Dm$ of $F_{A,t,m}$, where $\ell$ runs over the positive integers $\leq D - 1$ such that $\gcd(\ell, D) = 1$. Additionally, according to Lemma 2.4, for each $\ell$, the following holds: if we set $j(\ell)$ as the least positive solution of $j\ell \equiv 1 \pmod{D}$, the irreducible factors of degree $Dm$ of $F_{A,t,m}$ are exactly the irreducible factors of degree $Dm$ of $F_{A(\ell),m}$. Clearly $j(\ell)$ runs over the positive integers $j \leq D - 1$ such that $\gcd(j, D) = 1$ (that is, $j(\ell)$ is a permutation of the numbers $\ell$).

Now, it suffices to count the irreducible polynomials of degree $Dm$ that divide the polynomials $F_{A,m}$ for $j \leq D - 1$ and $\gcd(D, j) = 1$. In this case, it is crucial to study the coefficients of $A^j$. When $A$ is an element of type 4 in reduced form, we can obtain a complete description on the powers of $A$.

Proposition 4.5. If $A = D(c)$ is an element of type 4, then

$$A^j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} = \delta \begin{pmatrix} \alpha^{qj+1} - \alpha^{qj} & \alpha^{q(j+1)+1} - \alpha^{qj+1} \\ \alpha^j - \alpha^{qj} & \alpha^{j+1} - \alpha^{q(j+1)} \end{pmatrix}, j \in \mathbb{Z}, \quad (3)$$

where $\alpha$ is an eigenvalue of $A$ and $\delta = (\alpha - \alpha^q)^{-1}$. In particular, if $D$ is the order of $[A]$, $c_j \neq 0$ for $1 \leq j \leq D - 1$.

Proof. Since $A$ is of type 4, $A$ is a diagonalizable matrix over $\mathbb{F}_q$ but not over $\mathbb{F}_q$ and we can write

$$A = M \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^q \end{pmatrix} M^{-1}, \quad \text{where} \quad M = \begin{pmatrix} \alpha^q & \alpha \\ -1 & -1 \end{pmatrix}$$

is an invertible matrix and $\alpha$ is an eigenvalue of $A$. From now, Eq. (3) follows by direct calculations. We see that $c_j = 0$ if and only if $\alpha^j = \alpha^{qj}$. The latter is equivalent to $[A]^j = [I]$ and so $j$ must be divisible by $D$. In particular, for $1 \leq j \leq D - 1$, $c_j \neq 0$.

From Lemma 2.3 in general, the irreducible factors of $F_{A,m}$ have degree divisible by $D$. The problem relies on counting the irreducible polynomials of degree one and two. From the previous proposition, we describe the linear and quadratic irreducible factors of $F_{A,m}$ as follows.

Lemma 4.6. Suppose that $A = D(c)$ is an element of type 4 and order $D$. For any positive integers $j$ and $m$ such that $j \leq D - 1$ and $\gcd(j, D) = 1$, the polynomial $F_{A,m} \in \mathbb{F}_q[x]$ has degree $q^m + 1$, is free of linear factors and has at most one irreducible factor of degree 2. In addition, $F_{A,m}$ has an irreducible factor of degree 2 if and only if $m$ is even and, in this case, this irreducible factor is $x^2 + c^{-1}x - c^{-1}$.
Proof. From definition, \( F_{A_i,m} = b_j x^{q^m} + a_j x^{q^m} + d_j x - c_j \). From Proposition\( \ref{prop:irreducible-polynomials} \), \( c_j \neq 0 \) if \( 1 \leq j \leq D - 1 \). Therefore, hence \( F_{A_i,m} \) has degree \( q^m + 1 \) if \( j \leq D - 1 \). We split the proof into cases, considering the linear and quadratic irreducible polynomials.

(i) If \( F_{A_i,m} \) has a linear factor, there exists \( \gamma \in \mathbb{F}_q \) such that \( F_{A_i,m}(\gamma) = 0 \).

This case, \( \gamma^q = \gamma \) and a direct calculation yields \( F_{A_i,m}(\gamma) = b_j \gamma^2 + (d_j - a_j) \gamma - c_j \) and so \( \gamma \) is a root of \( p_j(x) = b_j x^2 + (d_j - a_j) x - c_j \). Let \( \alpha, \alpha^q \) be the eigenvalues of \( A = D(c) \), hence \( \alpha^q + \alpha = 1 \) and \( \alpha^{q+1} = -c \). From Eq. \( \ref{eq:proof} \) and the previous equalities, we can easily deduce that \( d_j - a_j = c_j \) and \( b_j = c c_j \). Therefore, \( p_j(x) \) equals \( c x^2 + x - 1 \) (up to a constant). This shows that \( \gamma \) is a root of \( x^2 + c^{-1}x - c^{-1} \). However, since \( A = D(c) \) is of type 4, its characteristic polynomial \( p(x) = x^2 - x - c \) is irreducible over \( \mathbb{F}_q \) and so \( x^2 p(\frac{1}{x}) = x^2 + c^{-1}x - c^{-1} \). In particular, \( \gamma \) is not an element of \( \mathbb{F}_q \).

(ii) If \( F_{A_i,m} \) has an irreducible factor of degree 2, there exists \( \gamma \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q \) such that \( F_{A_i,m}(\gamma) = 0 \). We observe that, in this case, \( \gamma^q = \gamma \). For \( m \) even, \( F_{A_i,m}(\gamma) = b_j \gamma^2 + (d_j - a_j) \gamma - c_j \) and in the same way as before we conclude that \( x^2 + c^{-1}x - c^{-1} \) is the only quadratic irreducible factor of \( F_{A_i,m} \). If \( m \) is odd, \( \gamma^q = \gamma \) and equality \( F_{A_i,m}(\gamma) = 0 \) yields \( b_j \gamma^{q+1} - a_j \gamma^q + d_j \gamma - c_j = 0 \). Raising the \( q \)-th power in the previous equality and observing that \( \gamma^q = \gamma \), we obtain

\[
b_j \gamma^{q+1} - a_j \gamma + d_j \gamma^q - c_j = 0,
\]

and so \( (\gamma^q - \gamma)(a_j + d_j) = 0 \). Since \( \gamma \) is not in \( \mathbb{F}_q \), the last equality implies that \( a_j = -d_j \). However, from Eq. \( \ref{eq:proof} \), we obtain \( \alpha^{q+1} - \alpha^{q+j} = \alpha^{q(q+1)} - \alpha^{q+j} \). Therefore, \( (\alpha^q - \alpha)(\alpha^q + \alpha) = 0 \). Recall that, since \( A = D(c) \) is of type 4, \( \alpha \) is not in \( \mathbb{F}_q \), i.e., \( \alpha^q \neq \alpha \). Therefore \( \alpha^q = -\alpha \) and then \( \alpha^{2q} = \alpha^2 \). Again, from Eq. \( \ref{eq:proof} \), this implies that \( |A|^{2q} = 1 \), hence \( 2j \) is divisible by \( D \). However, since \( j \) and \( D \) are relatively prime, it follows that \( D \) divides 2. This is a contradiction, since any element of type 4 has order \( D > 2 \) (see Lemma\( \ref{lem:order-of-2} \)).

All in all, we finally add the enumeration formula for the number of \( [A] \)-invariants when \( [A] \) is of type 4, completing the proof of Theorem\( \ref{thm:enumeration-formula} \).

**Theorem 4.7.** Suppose that \( A \) is an element of type 4 and set \( D = \text{ord}([A]) \). Then \( \mathcal{N}_A(n) = 0 \) if \( n \) is not divisible by \( D \) and, for \( n = Dm \),

\[
\mathcal{N}_A(Dm) = \frac{\varphi(D)}{Dm} \sum_{d|m} (q^{m/d} + \epsilon(m/d)) \mu(d),
\]

where \( \epsilon(s) = (-1)^{s+1} \). 

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Proof. From Theorem 2.7, we can suppose that $A$ is in the reduced form, i.e., $A = D(c)$ for some $c$ such that $x^2 - x - c$ is irreducible over $\mathbb{F}_q$. For each positive integer $j$ such that $j \leq D - 1$ and $\gcd(j, D) = 1$, let $n(j)$ be the number of irreducible factors of degree $Dm$ of $F_{A^j,m}$. From Lemma 4.4, it follows that

$$N_A(Dm) = \sum_{j \leq D - 1, \gcd(j,D)=1} n(j).$$

Fix $j$ such that $j \leq D - 1$ and $\gcd(j, D) = 1$. According to Lemma 2.3, the irreducible factors of $F_{A^j,m}$ are of degree $Dm$, of degree $Dk$, where $k$ divides $m$ and $\gcd(mk, D) = 1$ and of degree at most 2. For each divisor $k$ of $m$ such that $\gcd(mk, D) = 1$, let $P_{k,m}$ be the product of all irreducible factors of degree $Dk$ of $F_{A^j,m}$ and let $L(k)$ be the number of such irreducible factors. Also, set

$$\varepsilon_m(x) = \gcd(F_{A^j,m}(x), x^2 + c^{-1}x - c^{-1}).$$

Therefore, from Lemma 4.6, we obtain the following identity

$$F_{A^j,m}/\varepsilon_m(x) = \prod_{k|m, \gcd(mk, D)=1} P_{k,m}.$$

From Lemma 4.6, $F_{A^j,m}$ has degree $q^m + 1$ and the degree of $\varepsilon_m(x)$ is either 0 or 2, according to whether $m$ is odd or even. In particular, if we set $\epsilon(m) = (-1)^{m+1}$, taking degrees on the last equality we obtain:

$$q^m + 1 - \deg(\varepsilon_m(x)) = q^m + \epsilon(m) = \sum_{k|m, \gcd(mk, D)=1} \mathcal{L}(k) \cdot (kD) = \sum_{k|m} \mathcal{L}(k) \cdot (kD) \cdot \chi_D \left( \frac{m}{k} \right),$$

where $\chi_D$ is the principal Dirichlet character modulo $D$. From Theorem 4.3 we obtain

$$\mathcal{L}(k) \cdot kD = \sum_{d|k} (q^{k/d} + \epsilon(k/d)) \cdot \mu(d) \cdot \chi_D(d)$$

for any $k \in \mathbb{N}$. Therefore,

$$\mathcal{L}(m) = \frac{1}{Dm} \sum_{d|m} (q^{m/d} + \epsilon(m/d)) \cdot \mu(d) \cdot \chi_D(d) = \frac{1}{Dm} \sum_{d|m, \gcd(d,D)=1} (q^{m/d} + \epsilon(m/d)) \mu(d).$$

From definition, $n(j) = \mathcal{L}(m)$ and so

$$N_A(Dm) = \sum_{j \leq D - 1, \gcd(j,D)=1} n(j) = \varphi(D) \cdot \mathcal{L}(m) = \frac{\varphi(D)}{Dm} \sum_{d|m, \gcd(d,D)=1} (q^{m/d} + \epsilon(m/d)) \mu(d).$$

\qed
4.2. A remark on previous results

In [1], the author explores the degree distribution of the polynomials
\[(ax + b)x^{qm} - (cx + d).\]

In the context of \([A]\)-invariants, Theorem 5 of [1] can be read as follows:

"If \([A] \in \text{PGL}_2(\mathbb{F}_q)\) has order \(D\) and \(m \geq 3\), then the number of \([A]\)-invariants of degree \(Dm\) equals
\[
\frac{\varphi(D)}{Dm} \sum_{d|m, \gcd(d, D) = 1} \mu(d)q^{m/d}. \tag{4}
\]

We observe that Eq. (4) disagrees with Theorem 1.4 and turns out to be incorrect in many cases. For instance, if \(q = 2\) and \(A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\), then \(A\) is diagonalizable over \(\mathbb{F}_4 \setminus \mathbb{F}_2\) and \([A]\) has order 3. If \(r = 3^k\), Theorem 1.4 entails that
\[
N_{[A]}(3^k) = N_{[A]}(3^{k+1}) = \frac{2}{3^{k+1}}(2^{3^k} + 1) = N. \tag{5}
\]

However, Eq. (4) provides \(N_{[A]}(3^{k+1}) = \frac{2}{3^{k+1}} \cdot 2^{3^k}\), that is not even an integer.

The lack of accuracy in Eq. (4) is, perhaps, due to the miscalculation of the linear and quadratic factors of the polynomials \((ax + b)x^{qm} - (cx + d)\) in [1].

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