COMBINATORIAL PATCHWORKING: BACK FROM TROPICAL GEOMETRY

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ABSTRACT. We show that, once translated to the dual setting of convex triangulations of lattice polytopes, results and methods from [ARS21], [RS18], [JRS18] and [RRS] extend to non-convex triangulations. So, while the translation of Viro’s patchworking method to the setting of tropical hypersurfaces has inspired several tremendous developments over the last two decades, we return to the original polytope setting in order to generalize and simplify some results regarding the topology of $T$-submanifolds of real toric varieties.

Bourré de complexes
Et tout a changé
Boris Vian

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Notation. Throughout the text $\mathbb{F}_2$ denotes the field $\mathbb{Z}/2\mathbb{Z}$. If $X$ is a topological space, we denote by $b_i(X)$ the $i$-th Betti number of $X$ with coefficient in $\mathbb{F}_2$. A lattice polytope $\Delta \subset \mathbb{R}^n$ is a convex polytope with vertices in $\mathbb{Z}^n$. Such a polytope defines a complex toric variety $\text{Tor}_C(\Delta)$, and $\Delta$ is said to be non-singular if it is full-dimensional and if $\text{Tor}_C(\Delta)$ is non-singular. In this case, we denote by $h^{p,q}(\Delta)$ the corresponding Hodge number of a non-singular hypersurface in $\text{Tor}_C(\Delta)$ with Newton

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polytope $\Delta$. More generally, we denote by $h^{pq}(\Delta^k)$ the $(p,q)$-Hodge number of a non-singular complete intersection of $k$ hypersurfaces in $\text{Tor}_{\mathbb{C}}(\Delta)$ with Newton polytope $\Delta$. By convention such a complete intersection is defined to be $\text{Tor}_{\mathbb{C}}(\Delta)$ when $k = 0$. The standard complex conjugation on $\mathbb{C}^*$ induces a real structure on the complex algebraic variety $\text{Tor}_{\mathbb{C}}(\Delta)$, whose real part is denoted by $\text{Tor}_{\mathbb{R}}(\Delta)$.

1. Introduction

1.1. Betti numbers of $T$-manifolds. Let us start by briefly recalling Viro’s combinatorial patchworking for projective hypersurfaces. Let $\Delta_n \subset \mathbb{R}^n$ be the standard $n$-simplex with vertices

$$0, \ (1,0,0,\ldots,0), \ (0,1,0,\ldots,0), \ \ldots, \ (0,0,\ldots,0,1).$$

Choose an integer $d \geq 1$, a triangulation $\Gamma$ of the $d$-dilation $d\Delta_n$ of $\Delta_n$, and a sign distribution on $d\Delta_n$, that is to say a function $\varepsilon : d\Delta_n \cap \mathbb{Z}^n \rightarrow \mathbb{F}_2$ (see Figure 1.1). We denote by $\tilde{d}\Delta_n$ and $\tilde{\Gamma}$ the union of all copies of $d\Delta_n$ and $\Gamma$ under successive orthogonal symmetries with respect to coordinate hyperplanes of $\mathbb{R}^n$. Extend also the sign distribution $\varepsilon$ to $\tilde{d}\Delta_n \cap \mathbb{Z}$ using the following rule (see Figure 1.1): given $(i_1, \ldots, i_n) \in d\Delta_n \cap \mathbb{Z}^n$ and $(s_1, \ldots, s_n) \in \mathbb{F}_2^n$, define

$$\varepsilon((-1)^{s_1} i_1, \ldots, (-1)^{s_n} i_n) = (-1)^{s_1 i_1 + \cdots + s_n i_n} \varepsilon(i_1, \ldots, i_n).$$

In each simplex $\sigma$ of $\tilde{\Gamma}$, separate the vertices with different signs by taking the convex hull of all middle points of edges of $\sigma$ having different signs (see Figure 1.1). Note that this convex hull is either empty (if $\varepsilon$ is constant on $\sigma$), or is a convex polyhedron of dimension $n - 1$ contained in $\sigma$. Denote by $X_{\Gamma,\varepsilon}$ the obtained $PL$-hypersurface of $\tilde{d}\Delta_n$. Finally identify, via the antipodal map, opposite pairs of points on the boundary of $\tilde{d}\Delta_n$, and denote by $X_{\Gamma,\varepsilon}$ the obtained $PL$-hypersurface of the real projective space $\mathbb{R}P^n$. It is called a $T$-hypersurface.

The subdivision $\Gamma$ is called convex, or regular, or coherent, if there exists a convex piecewise linear function $\lambda : d\Delta_n \rightarrow \mathbb{R}$ whose domains of linearity are exactly the facets of $\Gamma$. When $\Gamma$ is convex, Viro’s combinatorial patchworking Theorem [Vir84, Vir06, IV96, GKZ94] states that the $T$-hypersurface $X_{\Gamma,\varepsilon}$ is isotopic in $\mathbb{R}P^n$ to the set of real solutions of the polynomial equation

$$\sum_{(i_1, \ldots, i_n) \in \text{Vert}(\Gamma)} (-1)^{\varepsilon(i_1, \ldots, i_n) \lambda(i_1, \ldots, i_n)} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} x_{n+1}^{d-i_1 - \cdots - i_n} = 0,$$
where $\text{Vert}(\Gamma)$ denotes the set of vertices of the triangulation $\Gamma$, for $t$ a small enough positive real number. Hence, this beautiful result establishes a bridge between real algebraic geometry and combinatorics. Nevertheless, the construction of $X_{\Gamma, \varepsilon}$ makes sense even if $\Gamma$ is not convex, in which case the relations of $X_{\Gamma, \varepsilon}$ to real algebraic geometry remains unclear. We refer the interested reader to [dLV98, IS02, IS03, BB06, BB08] for some investigations in this direction.

We restricted ourselves so far to projective hypersurfaces for simplicity, nevertheless the construction of $T$-hypersurfaces actually makes sense for any lattice polytope $\Delta$ (only the gluing rule for the $2^n$ copies of $\Delta$ has to be adapted, see Section 4). One then obtains a $\mathbb{PL}$-hypersurface $X_{\Gamma, \varepsilon}$ in $\text{Tor}_R(\Delta)$ out of a triangulation $\Gamma$ of $\Delta$ and a sign distribution on $\Delta$, and Viro’s combinatorial patchworking Theorem still holds true when $\Gamma$ is convex.

In connection to Hilbert’s 16th problem, one is thus interested in exploring the range of possibilities for the topology of $X_{\Gamma, \varepsilon}$. This paper fits into this perspective. While the general situation remains poorly understood, Itenberg conjectured almost 20 years ago that for convex and unimodular triangulations, Betti numbers of $T$-hypersurfaces are bounded by diagonal sums of Hodge numbers of the corresponding projective hypersurfaces. A triangulation $\Gamma$ is said to be unimodular, or primitive, if all facets of $\Gamma$ have lattice volume 1, or equivalently are the standard $n$-simplex $\Delta_n$ up to translations and the action of $GL_n(\mathbb{Z})$. Itenberg’s conjecture has recently been proved by Renaudineau and Shaw in [RS18]. In this note we show that the approach by Renaudineau and Shaw extends to non-convex triangulations.

**Theorem 1.1.** Let $\Delta \subset \mathbb{R}^n$ be a non-singular lattice polytope, let $\Gamma$ be a unimodular triangulation of $\Delta$, and let $\varepsilon$ be a sign distribution on $\Delta$. Then one has

$$\forall p \geq 0, \quad b_p(X_{\Gamma, \varepsilon}) \leq \sum_{q \geq 0} h^{p,q}(\Delta).$$

When the subdivision $\Gamma$ is convex, this statement is exactly [RS18, Theorem 1.4]. Theorem 1.1 has first been proved independently by Haas [Haa97] and Itenberg [Ite95] for $T$-curves, and by Itenberg [Ite97] for $T$-surfaces in $\mathbb{RP}^3$. Renaudineau-Shaw’s results [RS18] sits at the crossroad of algebraic geometry and combinatorics, and Theorem 1.1 provides a combinatorial generalization. An algebro-geometric generalization of [RS18, Theorem 1.4] has recently been proposed by Ambrosi and Manzaroli in [AM22].

In this paper we actually study not only $T$-hypersurfaces, but $T$-manifolds of arbitrary codimension in arbitrary non-singular real toric manifolds. To do so, the data of a sign distribution is replaced by a real phase structure on the $k$-skeleton of $\Gamma$ in analogy to the real phase structures studied in [RRS22]. Given such real phase structure $E$, we construct a $\mathbb{PL}$-manifold $X_{\Gamma, E}$, called a $T$-manifold, of codimension $k$ in $\text{Tor}_R(\Delta)$. We refer to Section 4 for precise definitions. The next theorem is the main result of this paper, and specializes to Theorem 1.1 when $k = 1$.

**Theorem 1.2.** Let $\Delta \subset \mathbb{R}^n$ be a non-singular lattice polytope, let $\Gamma$ be a unimodular triangulation of $\Delta$, and let $E$ be a real phase structure on the $k$-skeleton of $\Gamma$. Then one has

$$\forall p \geq 0, \quad b_p(X_{\Gamma, E}) \leq \sum_{q \geq 0} h^{p,q}(\Delta^k),$$

with equality when $k = 0$ or $k = n$.

Again, when $\Gamma$ is convex, there already exists a tropical counterpart: In this case, the data of $E$ corresponds to a real phase structure, in the sense of [RRS22, Section 2], on the $k$-th stable intersection of a tropical hypersurface dual to $\Gamma$. Our $T$-manifold corresponds to the associated real patchwork, whose properties are discussed for general non-singular tropical varieties with real phase.
structures in [RRS]. In particular, Theorem 1.2 is then a straightforward combination of [RRS, Theorem 1.2, Section 2.6] with Theorem 1.4 below.

We point out that $T$-manifolds do not seem to be related to the generalization to complete intersections by Sturmfels [Stu94] of Viro’s patchworking Theorem. Sturmfels’s construction produces a $PL$-manifold of codimension $k$ out of a mixed subdivision of $k$ triangulations of $\Delta$. This $PL$-manifold depends not only on the initial triangulations but also heavily on the chosen mixed subdivision. Here our construction of $T$-manifolds does not make use of any mixed subdivision, but restricts in return to considering $k$ times the same subdivision of $\Delta$. In particular we do not know, even when $\Gamma$ is convex, whether a $T$-manifold in $\text{Tor}_R(\Delta)$ is always isotopic to the real part of a complete intersection of real algebraic hypersurfaces in $\text{Tor}_R(\Delta)$ with Newton polytope $\Delta$. So, it might be interesting to study this further and to compare the range of topological possibilities of both constructions.

Finally, note that Theorem 1.2 has been known for a long time in the cases $k = 0$ or $k = n$. In the case $k = n$, Theorem 1.2 simply states that the numbers of facets of $\Gamma$ is the lattice volume of $\Delta$, which follows from the definition of unimodularity of $\Gamma$. When $k = 0$ there exists a unique real phase structure, and in this case $X_{\Delta, E} = \text{Tor}_R(\Delta)$. Hence Theorem 1.2 is now a consequence of [BFMVH06, Section 4].

As in [RS18], we obtain as a by-product of our proof of Theorem 1.2 the following relation between the Euler characteristic of $X_{\Gamma, E}$, and the topological signature $\sigma(\Delta^k)$ of a non-singular complete intersection of $k$ hypersurfaces in $\text{Tor}_C(\Delta^k)$ with Newton polytope $\Delta$.

**Theorem 1.3.** Let $\Delta \subset \mathbb{R}^n$ be a non-singular lattice polytope, let $\Gamma$ be a unimodular triangulation of $\Delta$, and let $E$ be a real structure on the $k$-skeleton of $\Gamma$. Then one has

$$\chi(X_{\Gamma, E}) = \sigma(\Delta^k).$$

When $k = 1$, Theorem 1.3 has originally been proved by Itenberg [Ite97] when $n = 3$, and generalized by Bertrand [Ber10] for any $n$. When $\Gamma$ is convex and $k = 1$, Renaudineau and Shaw gave in [RS18] an alternative proof of Theorem 1.3, later extended to higher codimensions by Renaudineau, Rau, and Shaw [RRS]. All these proofs are combinatorial and do not use the relation of combinatorial patchworking to real algebraic geometry given by Viro’s Theorem. An algebro-geometric proof of Theorem 1.3 has been proposed by the first author in [Bru22] for hypersurfaces and convex triangulations.

As mentioned above, $T$-manifolds of higher codimensions are a priori not related to Sturmfels’ combinatorial patchworking of complete intersections. In particular when $k \geq 2$, and even if $\Gamma$ is convex, Theorem 1.3 seems to be disjoint from the results in [BB07] and [Bru22].

### 1.2. Hodge numbers, Poincaré duality, and Heredity.

As mentioned in the beginning of this introduction, this paper is built upon the observation that the aforementioned “tropical” works do not seem to use in an essential way the convexity of the triangulations dual to the tropical hypersurfaces under study. This will probably seem obvious to experts once formulated, yet it requires some efforts to rigorously prove this observation. Indeed the above works refer at some points to former results about abstract non-singular tropical manifolds (e.g. Poincaré duality [JRS18]), and explicitly use convexity arguments at some steps of their reasoning (e.g. in the computation of tropical $p$-characteristics in [ARS21, Proof of Theorem 1.8]). Altogether, our task is to translate the aforementioned results to the dual setting of unimodular triangulations and to free them from the convexity hypothesis (in particular, from any reference to tropical manifolds).

Following [RS18] [RRS], the proof of Theorem 1.2 is based on the computation of homology groups of two families of combinatorial cosheaves defined on the poset $\Xi$ of cell pairs of $\Gamma$. Such a cell pair is defined to be a couple $(F, \sigma)$ with $F$ a face of $\Delta$ and $\sigma$ a face of $\Gamma$ contained in $F$. A cosheaf $\mathcal{F}$ on $\Xi$ is the data of an $\mathbb{F}_2$-vector space $\mathcal{F}(F, \sigma)$ associated to each cell pair $(F, \sigma)$ of $\Gamma$, satisfying some...
compatibility relations. We refer to Section [2.1] for a precise definition, as well as for the definition of homology groups $H_q(\Gamma; F)$ of $F$. The rank of $H_q(\Gamma; F)$, as an $\mathbb{F}_2$-vector space, is denoted by $h_q(\Gamma; F)$.

In a first step of the proof of Theorem 1.2, we consider for each integers $p$ and $k$ a cosheaf $F_p^k$ on $\Xi$ that recovers the Hodge numbers $h^{p,q}(\Delta^k)$. Given a rational polyhedron $\sigma \subset \mathbb{R}^n$, we denote by $T(\sigma)$ the tangent space of $\sigma$ (that is, the linear space generated by the vectors $v - w$, $v, w \in \sigma$) and set $T_\mathbb{Z}(\sigma) = T(\sigma) \cap \mathbb{Z}^n$ and $T_\mathbb{Z}_\mathbb{F}(\sigma) = T_\mathbb{Z}(\sigma) \otimes \mathbb{F}_2$. Moreover, we use $\sigma^\perp$ as shorthand for $T_\mathbb{Z}(\sigma) \setminus (\mathbb{F}_2^\vee)$.

Now, for any $p, k \geq 0$, the cosheaf $F_p^k$ on $\Xi$ is defined by

$$\forall (F, \sigma) \in \Xi, \quad F_p^k(F, \sigma) = \sum_{\dim \tau = k} \bigwedge^p \left( \tau^\perp / F^\perp \right).$$

When $\Gamma$ is convex, these cosheaves have been introduced in the dual setting of tropical subvarieties by Itenberg, Katzarkov, Mikhalkin and Zharkov in [IKMZ19]. More precisely, $F_p^k$ corresponds to the cosheaf of framing groups $F_p$ of the $k$-th stable intersection $X^k$ of a tropical hypersurface $X$ dual to $\Gamma$. In particular, a result in [IKMZ19] shows that when $\Gamma$ is convex we have $h_q(\Gamma; F_p^k) = h_q(X^k, F_p)$.

The next theorem is the second main statement of our paper.

**Theorem 1.4.** For any $p, q$ and $k$, one has

$$h_q(\Gamma; F_p^k) = h^{p,q}(\Delta^k).$$

When $\Gamma$ is convex and $k = 1$, Theorem 1.4 has first been proved by Arnal, Renaudineau, and Shaw in [ARS21].

**Remark 1.5.** We only consider $\mathbb{F}_2$-cosheaves in this paper, since coefficients in $\mathbb{F}_2$ are sufficient for the purposes of Theorem 1.2. Nevertheless a large part of the text, in particular Theorem 1.4 and Theorem 1.6 can be extended to their obvious $\mathbb{Z}$-cosheaf version $F_p^k$, as in [ARS21, JRS18], or $F_p^k$ for an arbitrary field $K$. See also Remark 2.1 below.

At this point, it may be worthwhile to recall that all Hodge numbers $h^{p,q}(\Delta^k)$ are known since the seminal work [DK86] by Danilov and Khovanskii. Since we will follow an analogous strategy in the proof of Theorem 1.4, we briefly indicate the procedure to compute these numbers.

As a consequence of Lefschetz hyperplane section theorem, one obtains that

$$h^{p,q}(\Delta^k) = h^{p,q}(\text{Tor}_\mathbb{C}(\Delta)) \quad \text{if } p + q < n - k.$$

We refer to this property as Heredity in the following. Combining this with Danilov-Jurkiewicz theorem and Poincaré duality, one obtains all Hodge numbers $h^{p,q}(\Delta^k)$ except when $p + q = n - k$. In particular $h^{p,q}(\Delta^k) = 0$ except possibly when $p = q$ or $p + q = n - k$. Now the computation of the numbers $h^{p,n-k-p}(\Delta^k)$ follows from the computation of the $p$-characteristics

$$e_{\Delta,k,p} = \sum_{q \geq 0} (-1)^q h^{p,q}(\Delta^k).$$

Using motivic properties of $e_{\Delta,k,p}$, Danilov and Khovanskii computed explicitly all $p$-characteristics of hypersurfaces of toric varieties, and indicated an algorithm to recursively compute $p$-characteristics of complete intersections by reducing to the case of hypersurfaces. Following this algorithm, Di Rocco, Haase, and Nill obtained in [DRHN19] a closed expressions for $e_{\Delta,k,p}$ in terms of $k, p$, and $\Delta$. In conclusion all Hodge numbers $h^{p,q}(\Delta^k)$ are expressed in terms of Hodge numbers of the ambient toric variety $h^{p,q}(\text{Tor}_\mathbb{C}(\Delta))$, for which several different calculations are available (see for example [Dan78, FMSS95, Jor98]).

Hence, in order to prove Theorem 1.4 we follow the strategy from complex algebraic geometry by Danilov and Khovanskii which was also used in [ARS21]: we prove that the numbers $h_q(\Gamma; F_p^k)$ satisfy
Heredity and Poincaré duality and have the same $p$-characteristic as $h^{p,q}(\Delta^k)$. For convenience, we state this here in summary as our last main result.

**Theorem 1.6.** The numbers $h_q(\Gamma; F^k_p)$ satisfy the following properties.

1. (Heredity, Proposition 3.2) For $p + q < n - k$,
   
   $$h_q(\Gamma; F^k_p) = h_q(\Gamma; F^0_p).$$

2. (Poincaré duality, Theorem 3.3) For any $p, q$ and $k$,
   
   $$h_q(\Gamma; F^k_p) = h_{n-k-q}(\Gamma; F^k_{n-k-p}).$$

3. (p-characteristic, Proposition 3.12) For any $p$ and $k$,
   
   $$\sum_{q \geq 0} (-1)^q h_q(\Gamma; F^k_p) = \sum_{q \geq 0} (-1)^q h^{p,q}(\Delta^k).$$

When $\Gamma$ is convex and $k = 1$, the items (Heredity) and (p-characteristic) are proven in [ARS21]. When $\Gamma$ is convex, item (Poincaré duality) is proven in [JRS18].

1.3. **Real phase structures.** Given Theorem 1.4, the second step in the proof of Theorem 1.2 is the definition of real phase structures on the $k$-skeleton of $\Gamma$. This is the translation to the dual setting of the real phase structure for tropical varieties introduced in [RRS22]. Given $\sigma$ a face of $\Gamma$, we define $\sigma^\vee = (\mathbb{F}_2^n)^\vee / \sigma^+$, and denote by $\pi_{\sigma} : (\mathbb{F}_2^n)^\vee \to \sigma^\vee$ the projection map. More generally if $\sigma \subset \tau$, there is a natural projection map $\pi_{\sigma,\tau} : \tau^\vee \to \sigma^\vee$.

**Definition 1.7.** A real phase structure $\mathcal{E}$ on the $k$-skeleton of $\Gamma$ consists of a choice of a point $\mathcal{E}(\sigma) \in \sigma^\vee$ for every $\sigma \in \Gamma$ of dimension $k$ such that for any $\tau \in \Gamma$ of dimension $k + 1$ and any $s \in \tau^\vee$ the set

$$\{ \sigma \subset \tau : \text{dim}(\sigma) = k \text{ and } \pi_{\sigma,\tau}(s) = \mathcal{E}(\sigma) \}$$

has even cardinal.

Since $\sigma^\vee = \{0\}$ if $\dim \sigma = 0$, there exists a unique real phase structure on the vertices of $\Gamma$ ($k = 0$). On the other extreme ($k = n$), real phase structures on the $n$-skeleton of $\Gamma$ are also easy to describe: they consist in an arbitrary choice of an element in $(\mathbb{F}_2^n)^\vee$ for all $n$-simplex of $\Gamma$. Real phase structures on edges of $\Gamma$ ($k = 1$) provide an equivalent way to describe sign distributions on vertices of $\Delta$, see Example 4.3.

Starting from a real phase structure $\mathcal{E}$ on the $k$-skeleton of $\Gamma$, we construct the $T$-manifold $X_{\Gamma,\mathcal{E}}$ of codimension $k$ in $\text{Tor}_G(\Delta)$, and a sign cosheaf $\mathcal{S}$ on $\Xi$. We refer to Definition 4.5 for details. As an example, we depicted in Figure 1.2 a $T$-line in $\mathbb{R}P^3$ (see Example 4.4). The sign cosheaf $\mathcal{S}$ on $\Xi$ is defined by

$$\mathcal{S}(F,\sigma) = \mathbb{F}_2^{\mathcal{E}(F,\sigma)},$$

where

$$\mathcal{E}(F,\sigma) = \{ s \in F^\vee : \pi_{\sigma}(s) = \mathcal{E}(\tau) \text{ for some } \tau \subset \sigma \text{ with } \text{dim}(\tau) = k \}.$$

The sign cosheaf has first been defined for convex primitive triangulations in [RS18] for $k = 1$, and in [RRS] for $k \geq 2$. The proof of Theorem 1.2 now follows from the fact the homology groups of $\mathcal{S}$ are related to both the homology groups of $X_{\Gamma,\mathcal{E}}$ and the homology groups of $F^k_p$. This is the fundamental idea from [RS18] for hypersurfaces, later generalized in [RRS] in any codimension. Here our task is simply to check that the two important statements about $\mathcal{S}$ from [RRS] carry over to possibly non-convex triangulations. First, the groups $H_p(\Xi; \mathcal{S})$ and $H_p(X_{\Gamma,\mathcal{E}}; \mathbb{F}_2)$ are canonically isomorphic for any $p$, see Proposition 4.13. Second, there exists a filtration of cosheaves

$$0 = \mathcal{S}_{n+1} \subset \mathcal{S}_n \subset \cdots \subset \mathcal{S}_0 = \mathcal{S}.$$
such that $S_p/S_{p+1} \cong \mathcal{F}_p^k$, see Proposition 4.14. From the spectral sequence associated to this filtration it follows that
\[ h_q(\Gamma; S) \leq \sum_{p \geq 0} h_q(\Gamma; \mathcal{F}_p^k). \]
for any $p$. This completes the proof of Theorem 1.2.

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2. Preliminaries

2.1. Poset homology. A poset is a set $P$ equipped with a partial order $\leq$. In this text, all posets will be finite. A cover relation, denoted by $x < y$, is a pair $x < y$ such that there exists no $z \in P$ with $x < z < y$. A grading of $P$ is a function $\dim : P \to \mathbb{Z}$ such that $\dim(y) - \dim(x) = 1$ for every cover relation $x < y$. Given a poset $P$, we denote by $P^{\text{op}}$ the poset with inverted partial order. If $P$ is graded by the function $\dim$, we equip $P^{\text{op}}$ with the grading $-\dim$.

Given a pair $x \leq y$, we denote by $[x, y] = \{ z \in P : x \leq z \leq y \}$ the interval between $x$ and $y$. We call $\dim(y) - \dim(x)$ the codimension of $(x, y)$ or the length of $[x, y]$. A graded poset is thin or satisfies the diamond property if every interval of length 2 contains exactly 4 elements. Such an interval is called a diamond of $P$ since schematically it looks as follows.

\[ y \searrow \downarrow \swarrow z_1 \downarrow \swarrow z_2 \searrow x \]
Clearly, if $P$ is thin then $P^{\text{op}}$ is thin as well. The term thin in this context was apparently coined by Björner in [Bjö84]. We refer to this work for more background and motivation.

Let $R$ be a ring. A $R$-cosheaf $\mathcal{F}$ on a poset $P$ is a contravariant functor from $P$ (whose morphisms are the ordered pairs $x \leq y$) to the category of $R$-modules. Analogously, a sheaf on $P$ is given by a covariant functor. Since we will mostly work with cosheaves, let us spell the definition out in this case: we assign an $R$-module $\mathcal{F}(x)$ to any $x \in P$, and have $R$-linear maps $\iota_{x,y}: \mathcal{F}(y) \to \mathcal{F}(x)$ for every pair $x \leq y$ such that

\begin{equation}
\iota_{x,y} \circ \iota_{y,z} = \iota_{x,z}
\end{equation}

for all triples $x \leq y \leq z$, and $\iota_{x,x} = \text{Id}_{\mathcal{F}(x)}$ for any $x \in P$.

Let $P$ be a thin graded poset and $\mathcal{F}$ a $\mathbb{F}_2$-cosheaf on $P$. We consider the differential complex $C_\bullet(P; \mathcal{F})$ given by

\[ C_q(P; \mathcal{F}) = \bigoplus_{\dim(q) = q} \mathcal{F}(x), \quad \partial : C_q(P; \mathcal{F}) \to C_{q-1}(P; \mathcal{F}), \]

where the restriction of $\partial$ on $\mathcal{F}(x)$ is given by

\[ \partial(\alpha) = \sum_{y < x} \iota_{y,x}(\alpha). \]

Note that $\partial^2 = 0$ thanks to the diamond property of $P$ and the fact that we work over $\mathbb{F}_2$. The associated homology groups are denoted by $H_q(P; \mathcal{F})$.

**Remark 2.1.** For simplicity, in this text we restrict ourselves to cosheaves over $\mathbb{F}_2$. However, most of the cosheaves and statements that we will encounter have analogous versions over $\mathbb{Z}$ or the reader’s favourite ring. In this generality, we additionally have to equip the poset $P$ with a balanced signature (or balanced colouring). Here, a signature is a map $s: \mathcal{C}(P) \to \{+1, -1\}$ where $\mathcal{C}(P)$ denotes the set of all cover relations in $P$. A signature is called balanced if any diamond contains an odd number of $-1$’s. Then the modified map $\partial = \sum_{x < y} s(x < y) \iota_{y,x}$ still satisfies $\partial^2 = 0$. It is easy to check that all the posets that are used in the following admit a balanced signature (obtained by choosing orientations for the geometric objects in the background). For more details on balanced signatures and poset homology in this generality we refer to [BB05 Section 2.7] and [Cha19].

Given a poset $P$, a subset $U \subset P$ is open if it is closed under taking larger elements, that is,

\[ x \in U, x < y \quad \implies \quad y \in U. \]

If $P$ is thin and $U \subset P$ is open, then $U$ (with the restricted partial order) is thin as well. Given a cosheaf $\mathcal{F}$ on $P$, we denote the restriction to $U$ by $\mathcal{F}|_U$. For simplicity, we write $C_q(U; \mathcal{F})$ and $H_q(U; \mathcal{F})$ instead of $C_q(U; \mathcal{F}|_U)$ and $H_q(U; \mathcal{F}|_U)$, respectively, for the restricted differential complexes and homology groups.

Given two posets $P$ and $Q$, the product $P \times Q$ is a poset with partial order

\[ (x, x') \leq (y, y') \quad \iff \quad x \leq y \quad \text{and} \quad x' \leq y'. \]

If $P$ and $Q$ are graded, $P \times Q$ can be graded by setting $\dim(x, y) = \dim(x) + \dim(y)$. If $P$ and $Q$ are thin, then $P \times Q$ is thin as well. Indeed, an interval of length 2 in the product is either constant in one factor or has the form

\[
\begin{align*}
(x, y') &\leq (y, y') \\
(y, x') &\leq (x, x')
\end{align*}
\]
with \( x \prec y \) and \( x' \prec y' \). Given a cosheaf \( \mathcal{F} \) on an open subset \( U \subset P \times Q \), the associated differential complex \( C_\bullet(U; \mathcal{F}) \) can be refined into the following bicomplex

\[
\cdots \xrightarrow{\partial_1} E_{r+1,s+1}^0 \xrightarrow{\partial_1} E_{r,s+1}^0 \xrightarrow{\partial_1} E_{r-1,s+1}^0 \xrightarrow{\partial_1} \cdots \\
\cdots \xrightarrow{\partial_2} E_{r+1,s}^0 \xrightarrow{\partial_2} E_{r,s}^0 \xrightarrow{\partial_2} E_{r-1,s}^0 \xrightarrow{\partial_2} \cdots \\
\cdots \xrightarrow{\partial_2} E_{r+1,s-1}^0 \xrightarrow{\partial_2} E_{r,s-1}^0 \xrightarrow{\partial_2} E_{r-1,s-1}^0 \xrightarrow{\partial_2} \cdots \\
\cdots \xrightarrow{\partial_2} \\
\cdots \\
\cdots
\]

where

\[
E_{r,s}^0 = \bigoplus_{\substack{(x,x') \in U \\ \dim(x) = r \\ \dim(x') = s}} \mathcal{F}(x,x'), \quad C_q(U; \mathcal{F}) = \bigoplus_{r+s=q} E_{r,s}^0,
\]

and the restrictions of the differentials on \( \mathcal{F}(x,x') \) are given by

\[
\partial_1(\alpha) = \sum_{x < y} \iota_{(x,x'),(y,x')}(\alpha) \quad \text{and} \quad \partial_2(\alpha) = \sum_{x' < y'} \iota_{(x,x'),(x,y')}(\alpha).
\]

The two canonical filtrations of the bicomplex given by

\[
F_i = \bigoplus_{r,s,i} E_{r,s}^0 \quad \text{and} \quad F'_i = \bigoplus_{r,s,i} E_{r,s}^0
\]

give rise to two spectral sequences, both converging to \( H_\bullet(U; \mathcal{F}) \). We refer to [BT82, Section 14] for more details. To fix our index notation in the spectral sequences, we describe in Figures 2.1 and 2.2 the first pages of these two spectral sequences. In particular, a basic application of spectral sequences gives the following.

**Lemma 2.2.** Suppose that \( H_r(E_{\bullet,s}^0) = 0 \) for all \( r \neq r_0 \). Then

\[
\forall q \in \mathbb{Z}, \quad H_q(U; \mathcal{F}) = \frac{\text{Ker} \left( \partial_2 : H_{r_0}(E_{\bullet,q-r_0}) \to H_{r_0}(E_{\bullet,q-r_0-1}) \right)}{\text{Im} \left( \partial_2 : H_{r_0}(E_{\bullet,q-r_0+1}) \to H_{r_0}(E_{\bullet,q-r_0}) \right)}.
\]

Analogously if \( H_s(E_{r,\bullet}^0) = 0 \) for all \( s \neq s_0 \), then

\[
\forall q \in \mathbb{Z}, \quad H_q(U; \mathcal{F}) = \frac{\text{Ker} \left( \partial_1 : H_{s_0}(E_{q-s_0,\bullet}) \to H_{s_0}(E_{q-s_0-1,\bullet}) \right)}{\text{Im} \left( \partial_1 : H_{s_0}(E_{q-s_0+1,\bullet}) \to H_{s_0}(E_{q-s_0,\bullet}) \right)}.
\]
2.2. The cosheaves $F^k_p$. We now turn to the special cases of poset homology that are of interest to us. Recall that $\Delta$ is a non-singular lattice polytope in $\mathbb{R}^n$ and $\Gamma$ is a primitive triangulation of $\Delta$. We denote by $\text{Fac}(\Delta)$ the face lattice of $\Delta$, graded by the dimension. The triangulation $\Gamma$ is a poset with respect to inclusion and also graded by the dimension. We define the following open set of the poset $\text{Fac}(\Delta) \times \Gamma^{\text{op}}$:

$$\Xi = \{(F, \sigma) : \sigma \subset F\} \subset \text{Fac}(\Delta) \times \Gamma^{\text{op}}.$$  

We emphasize that we use the inverted order in the second factor $\Gamma^{\text{op}}$, that is,

$$(G, \tau) \leq (F, \sigma) \iff \sigma \subseteq \tau \subseteq G \subseteq F.$$  

**Remark 2.3** (For the attention of tropical experts). When convex, the triangulation $\Gamma$ is dual to a (generalized) polyhedral subdivision $S$ of the tropical toric variety $\text{Tor}_T(\Delta)$. As usual $\text{Tor}_T(\Delta)$ is a disjoint union of (tropical) toric orbits $B_F$ which are in one-to-one correspondence with faces $F$ of $\Delta$. A cell $\sigma \in \Gamma$ has a dual cell $\sigma^\vee_F \in S$ contained in $B_F$ for each face $F$ of $\Delta$ containing $\sigma$. In particular the correspondence $(F, \sigma) \in \Xi \mapsto \sigma^\vee_F \in S$ establishes an isomorphism of posets between $\Xi$ and the face poset of $S$. 

**Figure 2.1.** The spectral sequence for the filtration $F_i = \bigoplus_{r,s : s \leq i} E^0_{r,s}$.
The poset $\Xi$ is graded by $\dim(F, \sigma) = \dim(F) - \dim(\sigma)$, which takes values in $\mathbb{Z}_{\geq 0}$. It is well-known that $\text{Fac}(\Delta)$ and $\Gamma$ are thin. Hence $\Xi$ is thin as well.

Recall that we define cosheaves $\mathcal{F}_p^k$ on $\Xi$ for all $p, k = 0, \ldots, n$ given by

$$\mathcal{F}_p^k(F, \sigma) = \bigwedge_{\tau, \sigma}^p \left( \tau^\perp / F^\perp \right).$$

In particular $\mathcal{F}_p^k(F, \sigma) = 0$ if $p > \dim F - k$ or $\dim \sigma < k$. Note that $\mathcal{F}_p^0(F, \sigma) = \bigwedge_{\tau, \sigma}^p (\mathbb{F}_2^\perp / F^\perp)$ only depends on $F$, and that for all $p$

$$\mathcal{F}_p^0(F, \sigma) \subset \mathcal{F}_p^{p-1}(F, \sigma) \subset \cdots \subset \mathcal{F}_p^1(F, \sigma) \subset \mathcal{F}_p^0(F, \sigma).$$

The next lemma gives more detailed information about this filtration.

**Lemma 2.4.** Given $1 \leq k \leq n$, and $0 \leq p \leq n - k$, one has

$$\dim \mathcal{F}_p^k(F, \sigma) = \binom{\dim F}{p} - \sum_{l=0}^{k-1} \binom{\dim \sigma}{l} \binom{\dim F - \dim \sigma}{p - \dim \sigma + l}$$
In particular if \( p \leq \dim \sigma - k \), then
\[
\mathcal{F}_p^k(F, \sigma) = \mathcal{F}_p^0(F, \sigma).
\]

**Proof.** Let \( m \) be the dimension of the face \( F \). Since \( \sigma \) is a primitive simplex, there exists a basis \((e_1, \ldots, e_n)\) of \( \mathbb{Z}^n \) such that \( e_1, \ldots, e_{\dim \sigma} \) are the edges of \( \sigma \) adjacent to one of its vertices \( v_0 \), and \((e_1, \ldots, e_m)\) is a basis of \( T_{\mathbb{Z}}(F) \). We denote by \((e_1^\vee, \ldots, e_n^\vee)\) the basis of \((\mathbb{Z}^n)^\vee\) dual to \((e_1, \ldots, e_n)\). We identify the quotient space \((\mathbb{Z}^n)^\vee/F^\perp\) with the linear space generated by \((e_1^\vee, \ldots, e_m^\vee)\). Given two positive integers \(a, b\), we denote by \(\mathcal{P}(a, b)\) the set of subsets of \(\{1, \ldots, a\}\) of cardinal \(b\).

There is a bijection between faces of \(\sigma\) of dimension \(k\) containing \(v_0\), and elements of \(\mathcal{P}(\dim \sigma, k)\). The face of \(\sigma\) corresponding to such subset \(I\) is denoted by \(\tau_I\). A basis of \(\tau_I^\perp/F^\perp\) is then \((e_i^\vee)_{i \in \{1, \ldots, m\} \setminus I}\), and a basis of \(\bigwedge^p \tau_I^\perp/F^\perp\) is given by
\[
(e_i^\vee \wedge \cdots \wedge e_p^\vee)_{\{i_1, \ldots, i_p\} \subset \{1, \ldots, m\} \setminus I}.
\]

Defining
\[
\mathcal{J} = \{ J \in \mathcal{P}(m, p) | \exists I \in \mathcal{P}(\dim \sigma, k), J \cap I = \emptyset \},
\]
we obtain that the vector space
\[
\sum_{I \in \mathcal{P}(\dim \sigma, k)} \bigwedge^p \left( \tau_I^\perp/F^\perp \right)
\]
amits
\[
(e_i^\vee \wedge \cdots \wedge e_p^\vee)_{\{i_1, \ldots, i_p\} \in \mathcal{J}}
\]
as a basis. Hence it has dimension
\[
|\mathcal{J}| = \binom{m}{p} - |\{ J \in \mathcal{P}(m, p) | \forall I \in \mathcal{P}(\dim \sigma, k), J \cap I \neq \emptyset \}|
\]
\[
= \binom{m}{p} - \sum_{l=0}^{p-1-k} |\mathcal{P}(\dim \sigma, \dim \sigma - l) \times |\mathcal{P}(\dim F - \dim \sigma, p - \dim \sigma + l)|
\]
\[
= \binom{\dim F}{p} - \sum_{l=0}^{p-1-k} \binom{\dim \sigma}{l} \binom{\dim F - \dim \sigma}{p - \dim \sigma + l}.
\]

To end the proof of the lemma, it remains to observe that
\[
\mathcal{F}_p^k(F, \sigma) = \sum_{I \in \mathcal{P}(\dim \sigma, k)} \bigwedge^p \left( \tau_I^\perp/F^\perp \right).
\]
Indeed, let \(\tau\) be a face of \(\sigma\) of dimension \(k\) which does not contain the vertex \(v_0\). Without loss of generality, we may assume that \((e_1 + e_2, \ldots, e_1 + e_{k+1})\) is a basis of \(T_{\mathbb{R}^2}(\tau)\), and so that
\[
(e_1^\vee + e_2^\vee + \cdots + e_{k+1}^\vee, e_{k+1}^\vee, \cdots e_m^\vee)
\]
is a basis of \(\tau^\perp/F^\perp\). In particular we see that
\[
\bigwedge^p \left( \tau^\perp/F^\perp \right) \subset \sum_{I \in \mathcal{P}(\dim \sigma, k)} \bigwedge^p \left( \tau_I^\perp/F^\perp \right),
\]
which ends the proof. \(\Box\)

The definition of the cosheaves \(\mathcal{F}_p^k\) is motivated by their relation to the homology of linear subspaces of complex tori. Even if we will not strictly speaking use the following proposition in the text, it is important to keep it in mind.
Lemma 2.7. Furthermore, the group \( \sum \) \( H_p(M_{\dim \sigma, k} \times (\mathbb{C}^*)^{\dim F - \dim \sigma}; \mathbb{F}_2) \), where \( M_{n, k} \) is a generic linear space of codimension \( k \) in \((\mathbb{C}^*)^n\).

Remark 2.6. Proposition 2.5 combined with the Lefschetz hyperplane section theorem provides an alternative geometric proof of Lemma 2.4.

2.3. Homology of non-singular toric varieties. Here we relate numbers \( h_q(\Gamma; \mathcal{F}_p^0) \) to Hodge numbers of the toric variety \( \text{Tor}_\mathbb{C}(\Delta) \) The next paragraph recasts in convenient (for us) notations known results about computations of homology and Chow groups of toric varieties.

Let \( \Sigma \subset \mathbb{R}^n \) be a unimodular fan defining a non-singular toric variety \( \text{Tor}_\mathbb{C}(\Sigma) \). We can construct a differential complex

\[
E_{n,p} \quad \| \quad E_{n-1,p} \quad \| \quad E_{p,p}
\]

where the differential maps a vector \( v \in \bigwedge^p (\mathbb{F}_2^\Sigma/\tau_2(\sigma)) \) to the sum of the natural projections of \( v \) to \( \bigwedge^p (\mathbb{F}_2^\Sigma/\tau_2(\sigma)) \) for all cones \( \sigma \) of \( \Sigma \) that contain \( \tau \) as a facet, see e.g. [Jor98, Chapter 2]. We denote by \( H_{r,p}(\Sigma) \) the \( r \)-th homology group of this complex.

Lemma 2.7. The Borel-Moore homology group \( H_i^{BM}(\text{Tor}_\mathbb{C}(\Sigma); \mathbb{F}_2) \) splits as a (non-canonical) direct sum

\[
H_i^{BM}(\text{Tor}_\mathbb{C}(\Sigma); \mathbb{F}_2) = \bigoplus_{r+p=i} H_{r,p}(\Sigma).
\]

Furthermore, the group \( H_{p,p}(\Sigma) \) is isomorphic to \( A_p(\text{Tor}_\mathbb{C}(\Sigma)) \otimes \mathbb{F}_2 \), where \( A_p(\text{Tor}_\mathbb{C}(\Sigma)) \) is the \( p \)-th Chow group of \( \text{Tor}_\mathbb{C}(\Sigma) \).

Proof. The first part of the statement is contained in [Jor98, Theorem 2.4.1, Proposition 2.4.5]. Indeed, the complex \( E_{r, \bullet} \) is the first page of the spectral sequence for \( H_i^{BM}(\text{Tor}_\mathbb{C}(\Sigma); \mathbb{F}_2) \) induced by the stratification of \( \text{Tor}_\mathbb{C}(\Sigma) \) into torus orbits, and this spectral sequence degenerates at the second page. An isomorphism between \( H_{p,p}(\Sigma) \) and \( A_p(\text{Tor}_\mathbb{C}(\Sigma)) \otimes \mathbb{F}_2 \) is given by [FMSS95, Theorem 1], [FS97, Proposition 1.1] and the universal coefficient theorem.

Corollary 2.8. If \( \Sigma \) is the cone generated by a basis of \( \mathbb{Z}^n \), then

\[ H_{n,n}(\Sigma) = \mathbb{F}_2 \quad \text{and} \quad H_{r,p}(\Sigma) = 0 \quad \text{if} \ (r, p) \neq (n, n). \]

Proof. In this case \( \text{Tor}_\mathbb{C}(\Sigma) = \mathbb{C}^n \), and one computes easily that

\[ H_{2n}^{BM}(\mathbb{C}^n) = \mathbb{F}_2 \quad \text{and} \quad H_i^{BM}(\mathbb{C}^n) = 0 \quad \text{if} \ i \neq 2n. \]

Since \( E_{r,p} = 0 \) if \( r \) or \( p \) is not in \( \{0, \ldots, n\} \), the result follows from Lemma 2.7.

Corollary 2.9. For all integers \( p, q \) one has

\[ h_q(\Gamma; \mathcal{F}_p^0) = h^{p,q}(\text{Tor}_\mathbb{C}(\Delta)). \]

Proof. We compute the groups \( H_q(\Gamma; \mathcal{F}_p^0) \) using the filtration \( F_i^q = \bigoplus_{r,s : r \leq i} E_{r,s}^0 \) (see Figure 2.2). Each space \( E_{r,s}^0 \) splits into the direct sum

\[ E_{r,s}^0 = \bigoplus_{\dim F = r} E_{F,s}^0 \quad \text{with} \quad E_{F,s}^0 = \bigoplus_{\dim \sigma = -s} F_p^0(F, \sigma), \]
and the differential complex (from Figure 2.2 rotated by 90 degrees)

\[ \frac{\partial_2}{\partial_2} E_{r,s+1}^0 \rightarrow \frac{\partial_2}{\partial_2} E_{r,s}^0 \rightarrow \frac{\partial_2}{\partial_2} E_{r,s-1}^0 \rightarrow \cdots \]

splits into the direct sums of the differential complexes

\[ \frac{\partial_2}{\partial_2} E_{F,r,s+1}^0 \rightarrow \frac{\partial_2}{\partial_2} E_{F,r,s}^0 \rightarrow \frac{\partial_2}{\partial_2} E_{F,r,s-1}^0 \rightarrow \cdots \]

Since \( F_p^0(F, \sigma) = \bigwedge^p \left( (\mathbb{F}_2^n)^{\vee} / F^{\perp} \right) \) only depends on \( F \), the latter differential complex is nothing but the standard simplicial differential complex of \( F \) with the subdivision induced by \( \Gamma \), with coefficients in \( \bigwedge^p \left( (\mathbb{F}_2^n)^{\vee} / F^{\perp} \right) \). Since \( F \) is a contractible topological space, we obtain

\[ H_0(E_{F,r,*}^0) = \bigoplus_{\dim F = r} \bigwedge^p \left( (\mathbb{F}_2^n)^{\vee} / F^{\perp} \right) \quad \text{and} \quad H_s(E_{F,r,*}^0) = 0 \quad \forall s > 0, \]

from which we deduce that

\[ H_0(E_{r,*}^0) = \bigoplus_{\dim F = r} \bigwedge^p \left( (\mathbb{F}_2^n)^{\vee} / F^{\perp} \right) \quad \text{and} \quad H_s(E_{r,*}^0) = 0 \quad \forall s > 0. \]

Hence by Lemma 2.2 we obtain that the homology \( H_*(\Gamma; F_p^0) \) is the homology of the complex

\[ 0 \rightarrow \bigwedge^p (\mathbb{F}_2^n)^{\vee} \rightarrow \bigoplus_{\dim F = n-1} \bigwedge^p \left( (\mathbb{F}_2^n)^{\vee} / F^{\perp} \right) \rightarrow \cdots \rightarrow \bigoplus_{\dim F = p} \bigwedge^p \left( (\mathbb{F}_2^n)^{\vee} / F^{\perp} \right) \rightarrow 0, \]

where the differential maps a vector \( v \in \bigwedge^p \left( (\mathbb{F}_2^n)^{\vee} / F^{\perp} \right) \) to the sum of the natural projections of \( v \) to \( \bigwedge^p \left( (\mathbb{F}_2^n)^{\vee} / G^{\perp} \right) \) for all facets \( G \) of \( F \). By Lemma 2.7 we obtain

\[ H_i(\text{Tor}_C(\Delta); \mathbb{F}_2) = \bigoplus_{r+p = i} H_r(\Gamma; F_p^0). \]

On the other hand, the Jurkiewicz–Danilov Theorem [Dan78, Theorem 10.8] says that integer homology and Chow groups of \( \text{Tor}_C(\Delta) \) are torsion-free and

\[ b_{2p}(\text{Tor}_C(\Delta)) = \text{rk}(A_p(\text{Tor}_C(\Delta))) = h^{p,p}(\text{Tor}_C(\Delta)) \quad \text{and} \quad b_{2p+1}(\text{Tor}_C(\Delta)) = 0. \]

Hence \( h_p(\Gamma; F_p^0) = \text{rk}(A_p(\text{Tor}_C(\Delta))) = h^{p,p}(\text{Tor}_C(\Delta)) \) by Lemma 2.7 and we deduce that \( h_q(\Gamma; F_p^0) = 0 = h^{p,q}(\text{Tor}_C(\Delta)) \) if \( p \neq q \).

\[ \square \]

3. Homology of \( F_p^k \)

Here we prove Theorem 1.4 following the strategy indicated in the introduction: we first prove a heredity statement (Proposition 3.2) in Section 3.1, then Poincaré duality (Theorem 3.3) in Section 3.2, and end the proof by equating tropical and complex \( p \)-characteristics in Section 3.3.

3.1. Heredity. Given \( (F, \sigma) \in \Xi \), we denote by \( F_\sigma \) the minimal face of \( \Delta \) containing \( \sigma \).

**Lemma 3.1.** Given integers \( p, k \geq 0 \), consider the spectral sequence from Figure 2.2 with respect to \( \Xi \) and \( F = F_p^k \). Then the first page satisfies

\[ E_{r,s}^1 = \bigoplus_{\dim \sigma = -s} F_{p+\dim F_\sigma - n}(F_\sigma, \sigma) \quad \text{and} \quad E_{r,s}^1 = 0 \quad \text{if} \ r \neq n. \]
Proposition 3.2

Proof. Note that the complex $E^0_{*,s}$ splits into the direct sum

$$E^0_{*,s} = \bigoplus_{\dim \sigma = s} E^0_{*,\sigma} \quad \text{with} \quad E^0_{*,\sigma} = \bigoplus_{F \supseteq \sigma} F^k_p(F, \sigma).$$

We denote by $H_*(\sigma; F^k_p)$ the homology (with respect to $\partial_1$) of these subcomplexes. It is hence sufficient to prove

$$H_n(\sigma; F^k_p) = F^k_{n+\dim F-n}(F, \sigma) \quad \text{and} \quad H_r(\sigma; F^k_p) = 0 \quad \text{if} \ r \neq n.$$

for all $\sigma \in \Gamma$. To do so, let us fix suitable coordinates. Since $\Delta$ is a non-singular polytope, there exists a basis $(e_1, \cdots, e_n)$ of $\mathbb{Z}^n$ such that

- $(e_1, \cdots, e_{r_0})$ is a basis of $T_{\mathbb{Z}}(F_{\sigma})$;
- for any $F \supseteq F_{\sigma}$, a basis for $T_{\mathbb{Z}}(F)$ can be obtained by completing $(e_1, \cdots, e_{r_0})$ with vectors from $(e_{r_0+1}, \cdots e_n)$.

The choice of the basis $(e_1, \cdots, e_n)$ induces a direct sum decomposition

$$\tau^\perp/F_{\tau}^\perp = \tau^\perp/F_{\tau}^\perp \perp F_{\tau}^\perp/F_{\tau}^\perp$$

for all $k$-dimensional faces $\tau$ of $\sigma$ and all faces $F$ of $\Delta$ containing $F_{\sigma}$ which is compatible with the differential $\partial_1$. Hence we have the direct sum decomposition

$$F^k_p(F, \sigma) = \sum_{\tau \subset \sigma, \dim \tau = k} \bigwedge^p \left( \tau^\perp/F_{\tau}^\perp \right)$$

$$= \sum_{\tau \subset \sigma, \dim \tau = k} \left( \bigoplus_{a+b=p} \bigwedge^a \left( \tau^\perp/F_{\tau}^\perp \right) \oplus \bigwedge^b \left( F_{\tau}^\perp/F_{\tau}^\perp \right) \right)$$

$$= \bigoplus_{a+b=p} \left( F^k_{a}(F_{\sigma}, \sigma) \oplus \bigwedge^b \left( F_{\sigma}^\perp/F_{\sigma}^\perp \right) \right),$$

and the differential $\partial_1$ respects the $(a,b)$-direct sum decomposition. As a consequence we obtain

$$H_r(\sigma; F^k_p) = \bigoplus_{a+b=p} F^k_{a}(F_{\sigma}, \sigma) \otimes H_r(D_b),$$

where $D_b$ is the differential complex

$$0 \rightarrow \bigwedge^b F_{\sigma}^\perp \rightarrow \bigoplus_{\dim F = n-1, F \supseteq F_{\sigma}} \bigwedge^b \left( F_{\sigma}^\perp/F_{\sigma}^\perp \right) \rightarrow \cdots \rightarrow \bigoplus_{\dim F = \dim F_{\sigma} + b} \bigwedge^b \left( F_{\sigma}^\perp/F_{\sigma}^\perp \right) \rightarrow 0$$

with differential mapping a vector $v \in \bigwedge^b \left( F_{\sigma}^\perp/F_{\sigma}^\perp \right)$ to the sum of the natural projections of $v$ to $\bigwedge^b \left( F_{\sigma}^\perp/G^\perp \right)$ for all facets $G$ of $F$. By Corollary 2.8 we have

$$H_n(D_{n-\dim F_{\sigma}}) = \mathbb{F}_2 \quad \text{and} \quad H_r(D_b) = 0 \quad \text{if} \ (r, b) \neq (n, n - \dim F_{\sigma}),$$

which proves the claim. \hfill \Box

As a consequence we obtain that homology of $F^k_p$ is partially inherited from homology of $F^0_p$.

Proposition 3.2 (Heredity). If $p, q$ and $k$ are such that $p + q < n - k$, then

$$H_q(\Gamma; F^k_p) = H_q(\Gamma; F^0_p).$$
Proof. In view of Lemma 2.2 and Lemma 3.1, it is sufficient to prove that
\[ \bigoplus_{\dim \sigma = -s} \mathcal{F}^k_{p+\dim F_{\sigma-n}}(F_{\sigma}, \sigma) = \bigoplus_{\dim \sigma = -s} \mathcal{F}^0_{p+\dim F_{\sigma-n}}(F_{\sigma}, \sigma) \]
for all \( s \) such that \( q = n + s \leq n - k - p \). But for such \( s \) and \( \sigma \) with \( \dim(\sigma) = -s \), we get
\[ p + \dim F_{\sigma} - n \leq p \leq -s - k = \dim \sigma - k \]
and hence by Lemma 2.4 the equality holds. \( \square \)

3.2. Poincaré duality. In this section, we prove the following theorem.

Theorem 3.3 (Poincaré duality). For any \( k, p \) and \( q \), we have canonical isomorphisms
\[ \text{Hom}(H_q(\Gamma; \mathcal{F}^k_p), F_2) \cong H_{n-k-q}(\Gamma; \mathcal{F}^k_{n-k-p}) \]

We prove Theorem 3.3 in several steps. We start to show in Section 3.2.1 that the poset \( \Xi \) is the face poset of a regular \( CW \)-complex, from which we deduce in Section 3.2.2 that the considered homology groups are invariants under barycentric subdivision. We then follow the strategy from [JRS18], that is, we consider the sheaf dual to \( \mathcal{F}^k_p \) and its cohomology in Section 3.2.3, define the cap product in Section 3.2.4, and we mix all these ingredients in Section 3.2.5 to eventually prove Theorem 3.3.

3.2.1. Geometric realization of \( \Xi \). Recall that any regular \( CW \)-complex has an underlying face poset, where the partial order is given by inclusion among faces.

Proposition 3.4. The poset \( \Xi \) is the face poset of a regular \( CW \)-structure for \( \Delta \).

Given a poset \( P \), we denote by \( O(P) \) the order complex of \( P \). This is the simplicial complex whose vertices are the elements of \( P \) and whose faces are the flags in \( P \). (We use the term flag instead of chain in order to avoid conflicts with chains in the sense of homology). Given a flag \( X = (x_0 < \cdots < x_l) \), we denote its length by \( l(X) = l \), its first element by \( \text{first}(X) = x_0 \) and its last element by \( \text{last}(X) = x_l \). Recall that \( \Gamma \) is a triangulation of \( \Delta \), and its barycentric subdivision is the simplicial complex \( O(\Gamma) \). Given a flag \( \sigma_* \in O(\Gamma) \), we denote by \( S(\sigma_*) \) the corresponding simplex in the barycentric subdivision of \( \Gamma \).

Definition 3.5. For any \( (F, \sigma) \in \Xi \), we define
\[ O(F, \sigma) = \{ \sigma_* \in O(\Gamma) : \sigma \subset \text{first}(\sigma_*) \text{ and } \text{last}(\sigma_*) \subset F \}, \]
\[ C(F, \sigma) = \bigcup_{\sigma_* \in O(F, \sigma)} S(\sigma_*) \subset \Delta. \]

Note that \( \dim C(F, \sigma) = \dim(F, \sigma) \). When \( \Gamma \) is convex, the subdivision
\[ \Delta = \bigcup_{(F, \sigma) \in \Xi} C(F, \sigma) \]
is combinatorially isomorphic to the dual subdivision \( S \) of the tropical toric variety \( \text{Tor}_T(\Delta) \) mentioned in Remark 2.3.

Example 3.6. We illustrate in Figure 3.1 some examples of cells \( C(F, \sigma) \) in the case when \( \Delta \) is the standard unimodular triangle \( \Delta_2 \) in \( \mathbb{Z}^2 \) equipped with the trivial subdivision.

Example 3.7. We depicted in Figure 3.2 the \( CW \)-structure of \( 6\Delta_2 \) induced by the subdivision depicted in Figure 1.1b.

We claim that the cells \( C(F, \sigma) \) are the cells of the regular \( CW \)-structure of \( \Delta \) promised in Proposition 3.4.
Proof of Proposition 3.4. We first note that

\[(G, \tau) \leq (F, \sigma) \iff O(G, \tau) \subset O(F, \sigma) \iff C(G, \tau) \subset C(F, \sigma),\]

so the order on \(\Xi\) coincides with the order by inclusion of the sets \(C(F, \sigma)\). Hence we need to check that these sets form a regular CW-complex. That is to say, we need to prove that each cell \(C(F, \sigma)\) is a ball and that its boundary \(\partial C(F, \sigma)\) is a union of cells.

We denote by \(\Omega(F, \sigma)\) the star fan of \(F \cap \Gamma\) with respect to the simplex \(\sigma\). By definition, \(\Omega(F, \sigma)\) is a polyhedral fan in \(T(F)/T(\sigma)\) whose cones are in one-to-one correspondence with \(\sigma \subset \tau \subset F\). The support \(C\) of \(\Omega(F, \sigma)\) is a full-dimensional polyhedral cone in \(T(F)/T(\sigma)\), more precisely it is the inner cone of \(F\) at \(\sigma\). We denote the barycentric subdivision of \(\Omega(F, \sigma)\) by \(\Omega^b(F, \sigma)\). We can notice that the poset of cones of \(\Omega^b(F, \sigma)\) is exactly \(O(F, \sigma)\). Note that, when \((G, \tau) \leq (F, \sigma)\), the simplicial complex \(C(G, \tau)\) is simplicially homeomorphic to its image in \(T(F)/T(\sigma)\) under the canonical projection map. Moreover, up to translation, for each \(\sigma^* \in O(F, \sigma)\) the image of \(S(\sigma^*)\) in \(T(F)/T(\sigma)\) is a simplex obtained by cutting the corresponding cone in \(\Omega^b(F, \sigma)\) with a transversal affine hyperplane. In summary \(C(F, \sigma)\) is simplicially homeomorphic to the simplicial complex obtained by intersecting \(\Omega^b(F, \sigma)\) with the unit ball \(B^{\dim(F, \sigma)} \subset T(F)/T(\sigma)\). But \(C \cap B^{\dim(F, \sigma)}\) is a ball whose boundary
is $\partial C \cap B^{\dim(F,\sigma)} \cup (C \cap \partial B^{\dim(F,\sigma)})$, which under the simplicial homeomorphism corresponds to chains $\sigma_*$ such that either $\sigma \subseteq \text{first}(\sigma_*)$ or $\text{last}(\sigma_*) \subseteq F$. Such flags belong to $O(G,\tau)$ for some $(G,\tau) < (F,\sigma)$, which proves the claim. \hfill \square

3.2.2. The barycentric subdivision. The partial order on a poset $P$ induces a poset structure on the order complex $O(P)$: given two flags $X$ and $Y$ in $O(P)$, we put $X \leq Y$ if $X$ is obtained from $Y$ by removing some pieces in the flag. In particular, we have last$(X) \leq \text{last}(Y)$ if $X \leq Y$, and any cosheaf $\mathcal{F}$ on $P$ induces a cosheaf on $O(P)$, still denoted by $\mathcal{F}$ and defined by

$$\mathcal{F}(X) = \mathcal{F}(\text{last}(X)) \quad \forall X \in O(P).$$

Given an open subset $U \subset P$, we set

$$K(U) = \{ X \in O(P) : \text{last}(X) \in U \} = O(P) \setminus O(P \setminus U).$$

Note that $O(U) \subset K(U)$ and that $K(U)$ is an open subset of $O(P) = K(P)$. In particular, both $O(U)$ and $K(U)$ are thin.

There is a chain map $\text{Bar}: C_*(U; \mathcal{F}) \to C_*(K(U); \mathcal{F})$ defined as follows. We write a chain $\alpha \in C_q(U; \mathcal{F})$ as

$$\alpha = \sum_{x \in U} \alpha(x)[x]$$

meaning that $\alpha(x)$ denotes the coefficient of $\alpha$ in the direct summand $\mathcal{F}(x)$. Now, we define $\text{Bar}(\alpha) \in C_q(K(U); \mathcal{F})$ by setting for any flag $X$ of length $q$

$$\text{Bar}(\alpha)(X) = \alpha(\text{last}(X)).$$

Here, it is understood that $\alpha(x) = 0$ if $\dim(x) \neq q$. It is straightforward to check that $\text{Bar}$ is a chain map.

**Proposition 3.8.** Let $\mathcal{F}$ be a cosheaf on $\Xi$ and let $U \subset \Xi$ be an open subset. Then the chain map $\text{Bar}$ from above induces isomorphisms of homology groups

$$\text{Bar}: H_q(U; \mathcal{F}) \cong H_q(K(U); \mathcal{F}).$$

**Proof.** We denote by $Q = \Xi \setminus U$ the closed complement of $U$. We first note that by definition the differential complexes $C_*(U; \mathcal{F})$ and $C_*(K(U); \mathcal{F})$ coincide with the relative complexes $C_*(\Xi, Q; \mathcal{F})$ and $C_*(O(\Xi), O(Q); \mathcal{F})$, respectively. By Proposition 3.4, $\Xi$ represents a regular CW complex with support $\Delta$, and $U$ and $Q$ represent an open and closed subsets of $\Delta$, respectively. Moreover, $C_*(\Xi, Q; \mathcal{F})$ and $C_*(O(\Xi), O(Q); \mathcal{F})$ are just the cellular and simplicial complexes for computing the relative homology of $Q \subset \Delta$. It is well-known that $\text{Bar}$ provides an isomorphism of homology groups, cf. [Hat02, Theorems 2.27 and 2.35]. \hfill \square

3.2.3. Cohomology and Mayer-Vietoris. Given a cosheaf $\mathcal{F}$ on $P$, its dual sheaf is defined via

$$\mathcal{G}(x) = \text{Hom}(\mathcal{F}(x), \mathbb{F}_2)$$

with induced restriction maps $i^*: \mathcal{G}(x) \to \mathcal{G}(y)$ for $x \leq y$. As for cosheaves, there is an induced sheaf on the order complex $O(P)$ given by $\mathcal{G}(X) = \mathcal{G}(\text{last}(X))$. Given an open set $U \subset P$, we denote by $C^*(O(U); \mathcal{G})$ the simplicial cochain complex with coboundary map $\partial^*$. The associated cohomology groups are denoted by $H^*(O(U); \mathcal{G})$.

Given an inclusion of open sets $U' \subset U \subset P$, we have chain maps $C_*(K(U); \mathcal{F}) \to C_*(K(U'); \mathcal{F})$ and $C^*(O(U); \mathcal{G}) \to C^*(O(U'); \mathcal{G})$ given by restriction. We use the notation $\alpha|_U$ for a restricted chain as usual. These restriction maps induce the Mayer-Vietoris exact sequences

$$0 \to C_*(K(U \cup V); \mathcal{F}) \to C_*(K(U); \mathcal{F}) \oplus C_*(K(V); \mathcal{F}) \to C_*(K(U \cap V); \mathcal{F}) \to 0$$
and
\[ 0 \to C^\ast(O(U \cup V); \mathcal{G}) \to C^\ast(O(U); \mathcal{G}) \oplus C^\ast(O(V); \mathcal{G}) \to C^\ast(O(U \cap V); \mathcal{G}) \to 0 \]
for two open subsets \( U, V \subset P \). The exactness is a straightforward consequence of \( K(U \cup V) = K(U) \cup K(V), K(U \cap V) = K(U) \cap K(V) \) and \( O(U \cup V) = O(U) \cup O(V), O(U \cap V) = O(U) \cap O(V) \) (the second last equality uses that \( U \) and \( V \) are open).

### 3.2.4. The cap product.
We denote the dual sheaves of the cosheaves \( \mathcal{F}_p^k \) on \( \Xi \) by \( \mathcal{G}_p^k = \text{Hom}(\mathcal{F}_p^k, \mathbb{F}_2) \). Note that for \( x \in \Xi \) we have the contraction maps
\[
\langle \cdot, \cdot \rangle_x : \mathcal{G}_p^k(x) \times \mathcal{F}_p^k(x) \to \mathcal{F}_{p'-p}^k(x)
\]
which are induced by the usual contraction maps
\[
\bigwedge^p V \times \bigwedge^{p'} V \to \bigwedge^{p'-p} V
\]
for a vector space \( V \). Note that these maps are compatible with the (co-)sheaf structures, that is, given \( x \leq y, \varphi \in \mathcal{G}_p^k(x) \) and \( \alpha \in \mathcal{F}_p^k(y) \), we have
\[
\langle \varphi, \iota_\alpha \rangle_x = \iota(\varphi, \alpha)_y.
\]

Given a flag \( X = (x_0 < \cdots < x_l) \), we set
\[
X_{\leq k} = (x_0 < \cdots < x_k), \\
X_{> k} = (x_{l-k} < \cdots < x_l), \\
X^i = (x_0 < \cdots < x_i < \cdots < x_l).
\]

**Definition 3.9.** The cap product on an open set \( U \subset \Xi \) is the map defined by
\[
\cap : C^q(O(U); \mathcal{G}_p^k) \times C^{q'}(K(U); \mathcal{F}_p^k) \to C^{q'-q}(K(U); \mathcal{F}_{p'-p}^k), \quad (\varphi, \alpha) \mapsto \sum_{X \in K(U)} \iota \langle \varphi(X_{> q}), \alpha(X) \rangle_{\text{last}(X)} [X_{\leq q'-q}].
\]

If \( X = (x_0 < \cdots < x_{q'}) \) is such that \( x_{q'-q} \notin U \), then \( \varphi \) is not defined on \( X_{> q} \); this is exactly the case when \( X_{\leq q'-q} \notin K(U) \). We set \([X_{\leq q'-q}] = 0\) in such cases meaning that these terms are removed from the sum in the above definition. Given open subsets \( U \subset V \subset \Xi \), it is obvious that \( \cap \) commutes with the restriction maps from above, that is, \((\varphi \cap \alpha)|_U = \varphi|_U \cap \alpha|_U\). Moreover, with a calculation identical to the classical case, see [Hat02 Page 240], one verifies that
\[
\partial(\varphi \cap \alpha) = \beta \cap \partial \alpha - (-1)^{q'-q} \partial^* \beta \cap \alpha.
\]
Hence we obtain induced maps on homology
\[
\cap : H^q(O(U); \mathcal{G}_p^k) \times H^{q'}(K(U); \mathcal{F}_p^k) \to H^{q'-q}(K(U); \mathcal{F}_{p'-p}^k).
\]

### 3.2.5. Poincaré duality.
For fixed value \( 0 \leq k \leq n \) and \( m = n - k \), we define the \( k \)-th fundamental class as the chain \( \Omega^k \in C_m(\Xi; \mathcal{F}_m^k) \) as follows. Given \( x = (F, \sigma) \in \Xi \) with \( \dim(x) = m \), the vector space \( \sigma^\perp/F^\perp \) has dimension \( m \). Hence
\[
\mathcal{F}_m^k(x) = \bigwedge^m \left( \sigma^\perp/F^\perp \right) \cong \mathbb{F}_2,
\]
and we denote by \( \text{Vol}(x) \) the generator in \( \mathcal{F}_m(x) \). We now define
\[
\Omega^k = \sum_{\sigma \in \Gamma, \dim(\sigma) = k} \text{Vol}((\Delta, \sigma)) [(\Delta, \sigma)] \in C_m(\Xi; \mathcal{F}_m^k).
\]

**Lemma 3.10.** We have \( \partial \Omega^k = 0 \).
Proof. Elements \((F, \tau) \in C_{m-1}(\Xi; \mathcal{F}_m^k)\) that may contribute to \(\partial \Omega^k\) are of two kinds.

- \(F\) is a facet of \(\Delta\) and \(\dim(\tau) = k\). In this case for dimensional reasons
  \[
  \mathcal{F}_m(F, \tau) = \bigwedge^{m} \left( \tau^\perp / F^\perp \right) = 0,
  \]
  so this cell does not contribute to \(\partial \Omega^k\).

- \(F = \Delta\) and \(\dim(\tau) = k + 1\). This case reduces to the case when \(k = 1\) and \(\Delta = \tau\) is the standard simplex in \(\mathbb{R}^n\). Denoting by \((e_1^\vee, \cdots, e_n^\vee)\) the basis of \((\mathbb{Z}^n)^\vee\) dual to the standard basis of \(\mathbb{Z}^n\), the elements \(\text{Vol}((\Delta, \sigma))\) where \(\sigma\) ranges over facets of \(\Delta\) are
  \[
  e_1^\vee, \cdots, e_n^\vee, e_1^\vee + e_2^\vee + \cdots + e_n^\vee.
  \]
  Hence
  \[
  \sum_{\text{dim}(\tau) = k} \text{Vol}((\Delta, \sigma)) = 0,
  \]
  and this cell does not contribute to \(\partial \Omega^k\) neither.

Altogether we obtain that \(\partial \Omega^k = 0\).

We are now ready to formulate the main theorem. By abuse of notation, we write \(\Omega^k\) for \(\text{Bar}(\Omega^k) \in C_{m}(O(\Xi); \mathcal{F}_m^k)\). Given an open set \(U \subset \Xi\), we write \(\Omega^k|_U\) for the restriction to \(C_{m}(K(U); \mathcal{F}_m^k)\).

**Theorem 3.11.** Let \(U \subset \Xi\) be an open set. Then the cap product with the fundamental class
\[
\cap \Omega|_U: H^q(O(U); \mathcal{G}_k^p) \to H_{m-q}(K(U); \mathcal{F}_m^k),
\]
\[
\varphi \mapsto \varphi \cap \Omega^k|_U
\]
is an isomorphism for all \(0 \leq p, q \leq m\).

**Proof.** We start by proving the statement on open stars \(U = U(x) = \{y \in \Xi : y > x\}\). Note that \(O(U)\) is contractible in the following sense: Consider the map \(\rho: C^q(O(U); \mathcal{G}_k^p) \to C^{q-1}(O(U); \mathcal{G}_k^p)\)
given by
\[
\rho(\varphi)(x) = \begin{cases} 
0 & \text{if first}(X) = x, \\
\varphi(x < X) & \text{otherwise}.
\end{cases}
\]
It is easy to check that \(\partial^* \rho + \rho \partial^* = \text{Id}\) for \(q > 0\). For \(q = 0\) the sum is equal to \(\text{Id} - \Phi \circ \text{pr}_x\), where \(\text{pr}_x: C^0(O(U); \mathcal{G}_k^p) \to C^0_\kappa(x)\) is the projection and \(\Phi: C^0_\kappa(x) \to C^0_\kappa(O(U); \mathcal{G}_k^p)\) is the map given by \(\Phi(\alpha)(y) = (\kappa_x \alpha)(y)\). It follows that \(H^q(O(U); \mathcal{G}_k^p) = 0\) for \(q > 0\) and \(H^0(O(U); \mathcal{G}_k^p) = \mathcal{G}_k^p(x)\). On the homology side, we have \(H_{m-q}(K(U); \mathcal{F}_m^k) = H_{m-q}(U; \mathcal{F}_m^k)\) by Proposition 3.8.

Let \(x = (F, \sigma)\) and set \(s = n - \dim(F)\) and \(l = \dim(x)\) and \(d = \dim(\sigma)\). The proof that Poincaré duality holds is now a rewriting, in the case of coefficients in \(\mathbb{F}_2\), of the proof of Poincaré duality over \(\mathbb{Z}\) from [JRS18 Section 5]. For the sake of brevity, we use notation from [JRS18]. First note that \(\mathcal{F}_p^k(x)\) is equal to the corresponding \(\mathcal{F}_p\) group over \(\mathbb{F}_2\) for the matroidal fan
\[
V = H^k \times \mathbb{R}^d \times \mathbb{T}^s \subset \mathbb{R}^{d+l} \times \mathbb{T}^s
\]
at \((0,0,-\infty)\), where \(H^k\) denotes the \(k\)-th stable self-intersection of the standard tropical hyperplane in \(\mathbb{R}^d\). Hence, \(H_{m-q}(U; \mathcal{F}_m^k) = H_{m-q}(V; \mathcal{F}_m^k) \cap \mathcal{F}_m^k(x) = \text{Hom}(\mathcal{F}_p, \mathbb{F}_2)\). Our cap product is equal to the one from [JRS18 Definition 4.11]: For \(q = 0\), given \(\varphi \in \mathcal{G}_k^p(x)\), the class \(\varphi \cap \Omega^k|_U\) is represented by the chain obtained from contracting all coefficients of the fundamental chain with \(\varphi\). Hence by [JRS18 Corollary 5.9] Poincaré duality holds for stars \(U = U(x)\).

Note that \(U(x) \cap U(y) = U(x \cap y)\) if the join \(x \cap y\) exists and \(U(x) \cap U(y) = \emptyset\) otherwise. Hence for the general case, we can copy the proof of [JRS18 Theorem 5.3], that is, we proceed by induction.
over the number of stars that are needed to cover an open set $U$. The induction step is provided by the Mayer-Vietoris commutative diagram (we omit the (co-)sheaves for simplicity):

$$0 \longrightarrow C^*(O(U \cup V)) \longrightarrow C^*(O(U)) \oplus C^*(O(V)) \longrightarrow C^*(O(U \cap V)) \longrightarrow 0$$

Since the triangulation $\Gamma$ is finite, the result is proved.

Proof of Theorem 3.3. We apply Theorem 3.1 to $U = \Xi$ which proves

$$H^0(O(\Xi); \mathcal{G}^p) \cong H_{m-q}(K(\Xi); \mathcal{F}_{m-p}) = H_{m-q}(O(\Xi); \mathcal{F}_{m-p}).$$

By the universal coefficient theorem, we have

$$H^0(O(\Xi); \mathcal{G}^p) \cong \text{Hom}(H_q(O(\Xi); \mathcal{F}_p), \mathbb{F}_2).$$

Finally, by Proposition 3.8 we have

$$H_{m-q}(O(\Xi); \mathcal{F}_{m-p}) \cong H_{m-q}(\Xi; \mathcal{F}_{m-p}), \quad \text{Hom}(H_q(O(\Xi); \mathcal{F}_p), \mathbb{F}_2) = \text{Hom}(H_q(\Xi; \mathcal{F}_p), \mathbb{F}_2),$$

which proves the claim. □

### 3.3. Euler characteristics

So far, we proved that

$$h_q(\Gamma; \mathcal{F}_p^k) = h^{p,q}(\Delta^k) \text{ if } p + q \neq n - k.$$ 

Hence the proof of Theorem 1.4 can be now reduced to an Euler characteristic computation, which is the content of the following proposition.

**Proposition 3.12.** For any $p \geq 0$, one has

$$\chi(\mathcal{F}_p^k) = \sum_{q \geq 0} (-1)^q h^{p,q}(\Delta^k).$$

We prove Proposition 3.12 in this section, up to a combinatorial lemma which is proved in its turn in Appendix A. To avoid any ambiguity, note that our definition of the binomial coefficient $\binom{a}{b}$ for any integers $a$ and $b$ is

$$\binom{a}{b} = \begin{cases} 
\frac{a(a-1)(a-2)\cdots(a-b+1)}{b!} & \text{if } b \geq 0 \\
0 & \text{if } b < 0 
\end{cases}$$

For example

$$\binom{-1}{b} = (-1)^b \quad \forall b \geq 0, \quad \text{and} \quad \binom{0}{b} = \begin{cases} 
1 & \text{if } b = 0 \\
0 & \text{if } b \neq 0 
\end{cases} \quad \forall b \in \mathbb{Z}.$$ 

Given $k, p \geq 0$, we define

$$e_{\Delta,k,p} = \sum_{q \geq 0} (-1)^q h^{p,q}(\Delta^k).$$

The integer $e_{\Delta,k,p}$ is the $p$-characteristic of a non-singular complete intersection $X$ of $k$ hypersurfaces in $\text{Tor}_C(\Delta)$ with Newton polytope $\Delta$. That is to say, if $\chi_y(\Delta^k)$ denotes the Hirzebruch genus of such complete intersection, one has

$$\chi_y(\Delta^k) = \sum_{p \geq 0} e_{\Delta,k,p} y^p.$$ 

Given a face $F$ of $\Delta$, we denote by $e^o_{F,k,p}$ the $p$-characteristic of the intersection of $X$ with the torus orbit of $X$ corresponding to the face $F$. By additivity of the $p$-characteristic, we have

$$e_{\Delta,k,p} = \sum_{F \text{ face of } \Delta} e^o_{F,k,p}.$$
Hence thanks to the next proposition, the \( p \)-characteristic \( e_{\Delta,k,p} \) can be expressed in terms of the combinatoric of \( \Delta \). Given a face \( F \) of \( \Delta \), we denote by \( \nu_F(i) \) the number of simplices of dimension \( i \) of \( \Gamma \) contained in \( F \). Note that by [DLRS10, Theorem 9.3.25], the integer \( \nu_F(i) \) only depends on \( F \) and not on a particular choice of unimodular subdivision \( \Gamma \).

**Proposition 3.13.** For any face \( F \) of dimension \( m \) of \( \Delta \), one has

\[
e^o_{F,k,p} = (-1)^m \binom{m}{p} + \sum_{i \geq 0} \alpha_{m,k,p,i} \times \nu_F(i),
\]

where

\[
\alpha_{m,k,p,i} = \sum_{l,u \geq 1} (-1)^u \binom{k}{l} \binom{u-1}{i} \binom{m+l}{m-p+u} \binom{u-1}{i}.
\]

**Proof.** Given \( l \geq 0 \), we denote by

\[
\lambda_F(l) = \text{Card}(lF \cap \mathbb{Z}^n)
\]

the number of lattice points contained in the \( l \)-th dilate of \( F \). By [DRHN19, Theorem 1.6], one has

\[
e^o_{F,k,p} = (-1)^m \binom{m}{p} + (-1)^m \sum_{l \geq 1} (-1)^l \binom{k}{l} \left( \sum_{b_1 \geq b_2 \geq 0} (-1)^{B_i} \right) \left( \sum_{b_i \geq 0} \binom{m+l}{m-p+u} \binom{u-1}{i} \lambda_F(l+\sum_{i} b_i) \right)
\]

\[
= (-1)^m \binom{m}{p} + (-1)^m \sum_{l \geq 1, u \geq l} (-1)^{l+u} \binom{k}{l} \binom{m+l}{m-p+u} \lambda_F(l+u),
\]

where \( \rho(\kappa,l) \) is the number of ordered partitions of a non-negative integer \( \kappa \) into \( l \) non-negative integers. Since

\[
\rho(\kappa,l) = \binom{\kappa+l-1}{l-1},
\]

we obtain after the change of variable \( u = \kappa + l \)

\[
e^o_{F,k,p} = (-1)^m \binom{m}{p} + (-1)^m \sum_{l \geq 1, u \geq l} (-1)^u \binom{k}{l} \binom{u-1}{i} \binom{m+l}{m-p+u} \lambda_F(u).
\]

By [DLRS10] Theorem 9.3.25], we have

\[
\lambda_F(u) = \sum_{i \geq 0} \binom{u-1}{i} \nu_F(i).
\]

Hence we obtain

\[
e^o_{F,k,p} = (-1)^m \binom{m}{p} + (-1)^m \sum_{l \geq 1, u \geq l} (-1)^u \binom{k}{l} \binom{u-1}{i} \binom{m+l}{m-p+u} \binom{u-1}{i} \nu_F(i).
\]

Since the \( \binom{u-1}{i} \) is 0 if \( 1 \leq u < l \), we may sum over \( u \geq 1 \) in the above sum. That is, the proposition is proved.

□

**Proof of Proposition 3.12.** We have to prove that

\[
\chi(F^k_p) = e_{\Delta,k,p}.
\]
By Lemma 2.4 we have
\[
\chi(F^p) = \sum_{(F, \sigma) \in \Xi} (-1)^{\dim F + \dim \sigma} \left( \binom{\dim F}{p} - \sum_{l=0}^{k-1} \binom{\dim \sigma}{l} \binom{\dim F - \dim \sigma}{p - \dim \sigma + l} \right)
\]
\[
= \sum_{F \text{ face of } \Delta} (-1)^{\dim F} \sum_{i \geq 0} (-1)^i \left( \binom{\dim F}{p} - \sum_{l=0}^{k-1} \binom{i}{l} \binom{\dim F - i}{p - i + l} \right) \nu_F(i).
\]
Hence it is enough to prove that for any face $F$ of $\Delta$ of dimension $m$, one has
\[
e_F^{(p)} = (-1)^m \sum_{i \geq 0} (-1)^i \left( \binom{m}{p} - \sum_{l=0}^{k-1} \binom{i}{l} \binom{m - i}{p - i + l} \right) \nu_F(i).
\]
Since $\sum_{i \geq 0} (-1)^i \nu_F(i) = 1$, we have
\[
(-1)^m \sum_{i \geq 0} (-1)^i \left( \binom{m}{p} - \sum_{l=0}^{k-1} \binom{i}{l} \binom{m - i}{p - i + l} \right) \nu_F(i)
\]
\[
= (-1)^m \left( \binom{m}{p} - \sum_{i \geq 0} (-1)^i \sum_{l=0}^{k-1} \binom{i}{l} \binom{m - i}{p - i + l} \right) \nu_F(i).
\]
Hence by Proposition 3.13 we are reduced to prove that
\[
\forall m, k, p, i \geq 0, \quad (-1)^{i+1} \sum_{l=0}^{k-1} \binom{i}{l} \binom{m - i}{p - i + l} = \sum_{l,u \geq 1} (-1)^u \binom{k}{l} \binom{u - 1}{l - 1} \binom{m + l}{m - p + u} \binom{u - 1}{i},
\]
which is done in Lemma A.1 below.

\[\square\]

4. Real phase structures and $T$-manifolds

4.1. Real phase structures. Recall that, for a face $\sigma$ of $\Gamma$, we use the notation
\[
\sigma^\vee = (\mathbb{F}_2^n)^\vee / \sigma^\perp = T_{\mathbb{F}_2}(\sigma)^\vee,
\]
and that $\pi_{\sigma}: (\mathbb{F}_2^n)^\vee \to \sigma^\vee$ and $\pi_{\sigma, \tau}: \tau^\vee \to \sigma^\vee$ denote the canonical projection map, the latter being defined only when $\sigma \subset \tau$. The translation of the real phase structures studied in [RRS22] to our polytope setting is as follows.

**Definition 4.1.** A real phase structure $\mathcal{E}$ on the $k$-skeleton of $\Gamma$ consists of a choice of a point $\mathcal{E}(\sigma) \in \sigma^\vee$ for every $\sigma \in \Gamma$ of dimension $k$ such that, for every $\tau \in \Gamma$ of dimension $k+1$ and $s \in \tau^\vee$, the cardinal $n_{\mathcal{E}}(\tau, s)$ of the set
\[
\{ \sigma \preceq \tau : \pi_{\sigma, \tau}(s) = \mathcal{E}(\sigma) \}
\]
is even.

**Remark 4.2.** The parity condition can be restated as follows: for any $\sigma \preceq \tau$, the preimage $\pi_{\sigma, \tau}^{-1}(\mathcal{E}(\sigma))$ is an $\mathbb{F}_2$-affine line in $\tau^\vee$. The condition states that the union of these lines covers any $s \in \tau^\vee$ an even number of times (possibly 0 times). It is shown in [RRS22, Lemma 2.8] that $n_{\mathcal{E}}(\tau, s)$ takes only the values 0 and 2, and that the union of lines form a so-called necklace of lines.

**Example 4.3.** Let us have a look at the special cases $k = 0, 1$, and $n$. 

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(1) When \( k = 0 \), the space \( \sigma^\vee = (\mathbb{P}_2^3)^\vee/\sigma^\perp = \{0\} \) is trivial for any vertex \( \sigma \) of \( \Gamma \). Then the unique choice \( \mathcal{E}_0(\sigma) = 0 \) defines a real phase structure on the 0-skeleton of \( \Gamma \) since \( n_{\mathcal{E}_0}(\tau, s) = 2 \) for any edge \( \tau \) of \( \Gamma \) and any \( s \in \tau^\vee \).

(2) In the other extreme case \( k = n \), the condition from Definition 4.1 is empty. In particular any choice of a point \( \mathcal{E}(\sigma) \in (\mathbb{P}_2^3)^\vee \) for every full-dimensional \( \sigma \) defines a real phase structure on the \( n \)-skeleton of \( \Gamma \).

(3) When \( k = 1 \), a real phase structure can be alternatively described by a sign distribution \( \varepsilon : \Delta \cap \mathbb{Z}^n \to \mathbb{F}_2 \) up to flipping all signs, that is, replacing \( \varepsilon \) with \( \varepsilon + 1 \). The relation is as follows: given an edge \( \sigma \in \Gamma \) with end points \( v \) and \( w \), we identify \( \sigma^\vee \) with \( \mathbb{F}_2 \) and require that

\[
\varepsilon(v) + \varepsilon(w) + \mathcal{E}(\sigma) = 1.
\]

In laymen terms: if \( \mathcal{E}(\sigma) = 0 \), we require \( v \) and \( w \) to have different signs, and conversely if \( \mathcal{E}(\sigma) \neq 0 \), then \( v \) and \( w \) are asked to have the same sign. The condition that \( n_{\mathcal{E}}(\tau, 0) \) is even for any triangle \( \tau \) ensures that the above equation is consistent when going around the three edges in \( \tau \), and hence such \( \varepsilon \) is uniquely defined after fixing some initial sign. Vice versa, given a sign distribution the above equation defines a real phase structure on edges of \( \Gamma \).

Note that up to coordinate change, the union of the three lines \( \pi_{\sigma, \tau}^\vee(\mathcal{E}(\sigma)) \) for \( \sigma \subset \tau \) is one of the two arrangements depicted in Figure 4.1. The case on the right hand side occurs exactly when all vertices of \( \tau \) have equal sign, the left hand side picture occurring when one vertex of \( \tau \) has different sign than the other two.

**Figure 4.1.** The two types of necklaces for real phase structures on edges.

**Example 4.4.** As an example of a real phase structure for values \( 1 < k < n \), we consider \( \Delta = \Gamma = \Delta_3 \) the standard simplex in \( \mathbb{R}^3 \). Denote by \( \sigma_x, \sigma_y, \sigma_z \), and \( \sigma_u \) the 2-dimensional face of \( \Delta_3 \) contained in the hyperplane with equation \( x = 0, y = 0, z = 0 \), and \( x + y + z = 1 \), respectively. Identifying \( \mathbb{F}_2^3 \) and \( (\mathbb{P}_2^3)^\vee \) thanks to the canonical basis of \( \mathbb{R}^3 \), we have

\[
\sigma_x^\vee = \mathbb{F}_2^3/\mathbb{F}_2(1, 0, 0), \quad \sigma_y^\vee = \mathbb{F}_2^3/\mathbb{F}_2(0, 1, 0), \quad \sigma_z^\vee = \mathbb{F}_2^3/\mathbb{F}_2(0, 0, 1), \quad \sigma_u^\vee = \mathbb{F}_2^3/\mathbb{F}_2(1, 1, 1).
\]

Hence an example of a real phase structure on the 2-skeleton of \( \Delta_3 \) is given by

\[
\mathcal{E}(\sigma_x) = \pi_x(0, 0, 0) = \pi_x(1, 0, 0), \quad \mathcal{E}(\sigma_y) = \pi_y(1, 0, 0) = \pi_y(1, 1, 0),
\]

\[
\mathcal{E}(\sigma_z) = \pi_z(0, 0, 0) = \pi_z(0, 0, 1), \quad \mathcal{E}(\sigma_u) = \pi_u(1, 1, 0) = \pi_u(0, 0, 1).
\]

where \( \pi_a \) is a shorthand for the projection \( \pi_{\Delta_3, \sigma_a} \).

4.2. **T-manifolds.** Here we define the \( T \)-manifold \( X_{\Gamma, \mathcal{E}} \) associated to a real phase structure \( \mathcal{E} \) on the \( k \)-skeleton of the barycentric subdivision \( \Gamma \). We start by describing the case \( k = 0 \), for which there is a unique real phase structure \( \mathcal{E}_0 \) by Example 4.3(1), and which contains all \( T \)-manifolds with a given non-singular polytope \( \Delta \).

To construct the \( T \)-manifold \( X_{\Gamma, \mathcal{E}_0} \), we glue \( 2^n \) disjoint copies of \( \Delta \), labelled by \( s \in (\mathbb{P}_2^3)^\vee \) and denoted \( \Delta(s) \), by identifying faces \( F \subset \Delta(s) \) and \( F \subset \Delta(t) \) if and only if \( s + t \in F^\perp \). This is a classical construction in toric geometry, and \( X_{\Gamma, \mathcal{E}_0} \) is homeomorphic to the real part \( \text{Tor}_\mathbb{R}(\Delta) \) of...
Tor\(_C(\Delta)\) by [GKZ94, Theorem 5.4]. The CW-structure on \(\Delta\) given by the cells \(C(F, \sigma)\) from Section 3.2 induces a CW-structure on \(X_{\Gamma, \mathcal{E}_0}\). Given \(s \in (\mathbb{P}^2)^\vee\), we denote by \(C(F, \sigma, s)\) the copy of \(C(F, \sigma)\) in \(X_{\Gamma, \mathcal{E}_0}\) which is the image of the copy of \(C(F, \sigma)\) in \(\Delta(s)\). Note that \(C(G, \tau, t) \subset C(F, \sigma, s)\) if and only if \((G, \tau) \preceq (F, \sigma)\) and \(s + t \in G^\perp\). In particular \(C(F, \sigma, s) = C(F, \sigma, t)\) if and only if \(s + t \in F^\perp\), that is to say \(C(F, \sigma, s)\) only depends on the class of \(s\) in \((\mathbb{P}^2)^\vee / F^\perp\).

**Definition 4.5.** Let \(E\) be a real phase structure on the \(k\)-skeleton of \(\Gamma\). For \((F, \sigma) \in \Xi\), we set
\[
\mathcal{E}(F, \sigma) = \{ s \in F^\vee : \pi_\sigma(s) = \mathcal{E}(\tau) \text{ for some } \tau \subset \sigma \text{ with } \dim(\tau) = k \},
\]
and
\[
\Xi(\mathcal{E}) = \{(F, \sigma, s) : (F, \sigma) \in \Xi, \ s \in \mathcal{E}(F, \sigma)\}.
\]
The \(T\)-manifold \(X_{\Gamma, \mathcal{E}}\) is the space
\[
X_{\Gamma, \mathcal{E}} = \bigcup_{(F, \sigma, s) \in \Xi(\mathcal{E})} C(F, \sigma, s) \subset X_{\Gamma, \mathcal{E}_0}.
\]

**Remark 4.6.** Given that the CW-structure on \(\Delta\) is constructed as a coarsening of the barycentric subdivision of \(\Gamma\), the latter induces in its turn a simplical structure on \(X_{\Gamma, \mathcal{E}}\) refining the given CW-structure. The simplices of this subdivision are labelled by elements
\[
(\sigma_1 \subset \cdots \subset \sigma_l, s)
\]
where \(s \in F^\vee_{\sigma_l}\) is such that there exists \(\sigma \subset \sigma_1\) with \(\dim(\sigma) = k\) and \(\pi_\sigma(s) = \mathcal{E}(\sigma)\). (Recall that \(F_\sigma\) denotes the minimal face of \(\Delta\) containing \(\sigma\).)

Let us describe the \(T\)-manifolds associated to the preceding examples of real phase structures.

**Example 4.7.** When \(k = 0\) the construction from Definition 4.5 recovers the space \(X_{\Gamma, \mathcal{E}_0}\), hence the notation is consistent.

**Example 4.8.** When \(k = n\), each full-dimensional simplex \(\sigma\) of \(\Gamma\) contributes a single point
\[
C(\Delta, \sigma, \mathcal{E}(\sigma)) \in \Delta(\mathcal{E}(\sigma)) \subset X_{\Gamma, \mathcal{E}}.
\]
In particular \(X_{\Gamma, \mathcal{E}}\) is a union of \(\text{Vol}(\Delta)\) points, where \(\text{Vol}(\Delta)\) denotes the lattice volume of \(\Delta\).

**Example 4.9.** The \(T\)-curve in \(\mathbb{RP}^2\) corresponding to the patchworking from Figure 1.1 is depicted in Figure 4.2. More generally when \(k = 1\) it is well known, see for example [TV00, Section 2], that the

![Figure 4.2. The T-sextic from the patchworking of Figure 1.1](image-url)
$T$-hypersurface $X_{\Gamma,E}$ constructed in Section 4.1 is ambient isotopic in $X_{\Gamma,\Delta}$ to the $T$-hypersurface $X_{\Gamma,E}$ from Definition 4.9 via an isotopy which fixes the simplices of $\Gamma$. Given an edge $\sigma$ with endpoints $v$ and $w$, and $s \in (\mathbb{R}^2)^\vee$, it follows from Example 4.3 that $\pi_{\sigma}(s) = \mathcal{E}(\sigma)$ if and only if $v$ and $w$ have different signs in the extended sign distribution with respect to the orthant $s$. It is hence sufficient to compare the two constructions for a single orthant, say for $s = 0$. Using the simplicial descriptions from Remark 4.6, we note that $X_{\Gamma,E} \cap \Delta(0) \subset \Delta$ is the simplicial complex consisting of the simplices in the barycentric subdivision of $\Gamma$ labelled by flags

$$\sigma_1 \subseteq \cdots \subseteq \sigma_l$$

subject to the conditions that $\dim(\sigma_1) \geq 1$ and $0 \in \mathcal{E}(\Delta, \sigma_1)$. Let us restrict our attention to the cells contained in a fixed simplex $\sigma \in \Gamma$, that is we require $\sigma_k \subset \sigma$. It is now straightforward to check that the union of such cells is ambient isotopic in $\sigma$ to the convex hull of the midpoints of edges in $\sigma$ with different signs.

Example 4.10. The $T$-line in $\mathbb{R}P^3$ corresponding to the real phase structure from Example 4.4 is depicted in Figure 4.2.

For general $k$, it is clear from the definition that $X_{\Gamma,E}$ is a pure-dimensional simplicial complex of dimension $n - k$.

Proposition 4.11. The $T$-manifold $X_{\Gamma,E}$ is PL-smooth.

Proof. We show that the links of all vertices of $X_{\Gamma,E}$ with respect to the simplicial structure from Remark 4.6 are PL-spheres, see [RS82, Section 2.21]. Let $\sigma$ be a simplex of $\Gamma$ and $s \in F_\sigma^\vee$ be such that there exists $\tau \subset \sigma$ with $\dim(\tau) = k$ and $\pi_\tau(s) = \mathcal{E}(\tau)$. Then the link of $X_{\Gamma,E}$ at $(\sigma, s)$ is the simplicial complex given by completions

$$(\sigma_1 \subseteq \cdots \subseteq \sigma_\lambda = \sigma \subseteq \cdots \subseteq \sigma_l, s')$$

with $s = \pi_\sigma(s')$. By standard arguments, this complex is the join of the complexes corresponding to completions "before" and "after" $\sigma$, respectively, hence it is sufficient to prove that those latter are PL-spheres.

The complex of completions "before" $\sigma$ is labelled by flags $\sigma_1 \subseteq \cdots \subseteq \sigma_\lambda = \sigma$ such that there exists $\tau \subset \sigma_1$ with $\dim(\tau) = k$ and $\pi_\tau(s) = \mathcal{E}(\tau)$. Then the statement is proven in [RRS22, Proposition 2.21] in its dual version, that is, for (subfans of) the normal fan of $\sigma$ in the real vector space dual to the linear span of $F_\sigma$.

The complex of completions "after" $\sigma$ is labelled by elements $(\sigma \subset \sigma_1 \subset \cdots \subset \sigma_l, s')$ such that $s = \pi_\sigma(s')$. After a coordinate change we can assume that the inner cone of $\Delta$ at $\sigma$, which is also the support of the star fan of $\Gamma$ at $\sigma$, is equal to $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}_{\geq 0}$, with $n_1 = \dim F_\sigma - \dim \sigma$ and $n_2 = n - \dim F_\sigma$. The $2^{n_2}$ choices for $s'$ correspond to the $2^{n_2}$ orthants of $\mathbb{R}^{n_2}$ and gluing these copies produces the fan in $\mathbb{R}^{n_1+n_2}$ obtained by reflecting the fan in $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}_{\geq 0}$ to the other orthants. Hence the complex in question is the link of the barycentric subdivision of this complete fan, and is hence a PL-sphere.

4.3. The sign cosheaf. We now define the analogue in our setting of the sign cosheaf for tropical varieties with real phase structures defined in [RS18] and [RRS]. Note that given $(G, \tau) \leq (F, \sigma)$, the image of $\mathcal{E}(F, \sigma)$ under the canonical projection $\pi_{G,F} : F^\vee \to G^\vee$ is contained in $\mathcal{E}(G, \tau)$.

Definition 4.12. Let $\mathcal{E}$ be a real phase structure on the $k$-skeleton of $\Gamma$. The sign cosheaf on $\Xi$ is given by

$$\mathcal{S}(F, \sigma) = \mathbb{F}_2^{\mathcal{E}(F, \sigma)},$$

with the cosheaf map for $(G, \tau) \leq (F, \sigma)$ that sends the basis element $e_s$, with $s \in \mathcal{E}(F, \sigma)$, to the basis element $e_{\pi_{G,F}(s)}$. 

Following [RS18, RRS], we explain in the next two propositions how homology groups of \( S \) are related to both homology groups of \( X_{\Gamma,E} \) and homology groups of \( \mathcal{F}_p^k \).

**Proposition 4.13.** For every integer \( q \), the groups \( H_q(\Xi;S) \) and \( H_q(X_{\Gamma,E};\mathbb{F}_2) \) are isomorphic.

**Proof.** We have a canonical identification between the chain complex of \( \mathbb{F}_2 \) on the poset \( \Xi(\mathcal{E}) \) and the chain complex of \( S \) on \( \Xi \):

\[
C_\bullet(\Xi(\mathcal{E});\mathbb{F}_2) = \bigoplus_{(F,\sigma,s) \subseteq \Xi(\mathcal{E})} \mathbb{F}_2 = \bigoplus_{(F,\sigma) \in \Xi} \mathbb{F}_2^{E(F,\sigma)} = C_\bullet(\Xi;S).
\]

To check that the two differentials agree, one simply has to observe that given \((G, \tau) \leq (F, \sigma)\), the only element of the form \((G, \tau, t)\) in the boundary of the cell \((F, \sigma, s)\) is \((G, \tau, \pi_G(F(s)))\). Hence the two differential complexes are canonically isomorphic, and we have the group isomorphisms

\[
H_q(X_{\Gamma,E};\mathbb{F}_2) \cong H_q(\Xi(\mathcal{E});\mathbb{F}_2) \cong H_q(\Xi;S)
\]

as announced. \( \square \)

**Proposition 4.14** ([RS18, RRS]). Let \( \mathcal{E} \) be a real phase structure on the \( k \)-skeleton of \( \Gamma \). There exists a filtration of cosheaves on \( \Xi \)

\[
0 = S_{n+1} \subset S_n \subset \cdots \subset S_0 = S
\]

such that

\[
S_p/S_{p+1} \cong \mathcal{F}_p^k \quad \forall 0 \leq p \leq n.
\]

**Proof.** We can use the exact same filtration and arguments as in [RS18] Definition 4.5 and Lemma 4.8 (for \( k = 1 \)) and [RRS] Definition 2.27 and Proposition 2.29 (for arbitrary \( k \)). \( \square \)

**Proof of Theorem 1.2** To finish the proof, we use the same spectral sequence argument as in [RS18] Theorem 1.5. The spectral sequence associated to the filtration from Proposition 4.14 has 0-th page \( E_{p,q}^0 = C_q(\Xi;S_p/S_{p+1}) \). Hence by Proposition 4.14, we have \( E_{p,q}^1 = H_q(\Xi;\mathcal{F}_p^k) \). Therefore \( \dim(E_{p,q}^1) = h_q(\Gamma;\mathcal{F}_p^k) = h^{p,q}(\Delta^k) \) by Theorem 1.4. On the other hand, by Proposition 4.13 we have \( b_q(X_{\Gamma,E};\mathbb{F}_2) = \dim(H_q(\Xi;S)) = \sum_p \dim(E_{p,q}^1) \). Theorem 1.2 then follows from the inequality \( \dim(E_{p,q}^\infty) \leq \dim(E_{p,q}^1) \). \( \square \)

**Proof of Theorem 1.3** Using the same arguments as in the proof of Theorem 1.2, we get

\[
\chi(X_{\Delta,E}) = \sum_{p,q} (-1)^p \dim(E_{p,q}^\infty) = \sum_{p,q} (-1)^q \dim(E_{p,q}^1) = \sum_{p,q} (-1)^q h^{p,q}(\Delta^k) = \sigma(\Delta^k),
\]

where the last equality follows from Hodge index theorem, see for example [Voi02 Theorem 6.33]. \( \square \)

**Appendix A. A combinatorial lemma**

Given integers \( m, k, p, i \), we define the following numbers

\[
\alpha_{m,k,p,i} = \sum_{l,u \geq 1} (-1)^u \binom{k}{l} \binom{u-1}{l} \binom{m+l}{m-p+u} \binom{u-1}{i},
\]

and

\[
\beta_{m,k,p,i} = (-1)^{i+1} \sum_{l=0}^{k-1} \binom{i}{l} \binom{m-i}{p-i+l}
\]

**Lemma A.1.** For any integers \( p \in \mathbb{Z} \) and \( m, k, i \geq 0 \), one has

\[
\alpha_{m,k,p,i} = \beta_{m,k,p,i}.
\]
Proof. First note that by Pascal’s relation, we have
\[ \alpha_{m,k,p,i} = \alpha_{m-1,k,p,i} + \alpha_{m-1,k,p-1,i} \]
and
\[ \beta_{m,k,p,i} = \beta_{m-1,k,p,i} + \beta_{m-1,k,p-1,i} \].
Hence it is enough to prove the \( m = 0 \) case:
\[ \alpha_{0,k,p,i} = \beta_{0,k,p,i} \].
To do so, we prove that both sequences of numbers satisfy the same recursion, and have the same initial values.

**Step 1: rewriting of \( \alpha_{0,k,p,i} \).** We have
\[ \alpha_{0,k,p,i} = \sum_{l,u \geq 1} (-1)^u \binom{k}{l} \binom{u-1}{i} \binom{l-1}{u-p} \binom{u-1}{i} \binom{l}{u-p} \binom{l}{u-p} \]
Since
\[ \binom{k}{l} \binom{l}{u-p} = \binom{k}{u-p} \binom{l+p-u}{u} \],
we obtain
\[ \alpha_{0,k,p,i} = \sum_{u \geq 1} (-1)^u \binom{k}{u-p} \binom{u-1}{i} \left( \sum_{l \geq 1} \binom{u-1}{l-1} \binom{k+p-u}{l+p-u} \right) \binom{u-1}{i} \binom{u-1}{k+p-u} \binom{u-1}{i} \binom{u-1}{l+p-u} \].
We deduce from [GKP94, Identity 5.23] that for any \( u \geq 1 \),
\[ \sum_{l \geq 1} \binom{u-1}{l-1} \binom{k+p-u}{l+p-u} = \binom{k+p-1}{p} \],
which gives
\[ \alpha_{0,k,p,i} = \sum_{u \geq 1} (-1)^u \binom{k}{u-p} \binom{u-1}{i} \binom{k+p-1}{p} \binom{u-1}{i} \binom{u-1}{k+p-u} \binom{u-1}{i} \binom{u-1}{l+p-u} \].
Since \( k \geq 0 \), we deduce from [GKP94, Identity 5.24] that
\[ \sum_{u \geq 1} (-1)^u \binom{k}{u-p} \binom{u-1}{i} = \begin{cases} (-1)^{k+p-1} \binom{-1}{i-k} & \text{if } p > 1 \\ (-1)^k (-1)^{k-1} (1) = (-1)^k (1) & \text{if } p = 0 \end{cases} \].
We finally obtain
\[ \alpha_{0,k,p,i} = \begin{cases} (-1)^{k+p-1} \binom{k+p-1}{p} & \text{if } p \neq 0 \\ 0 & \text{if } p = 0 \text{ and } i \geq k \\ (-1)^{i+1} & \text{if } p = 0 \text{ and } i \leq k - 1 \end{cases} \].
Step 2: recursion for $\alpha_{0,k,p,i}$. Applying twice Pascal’s relation, we have for $p \geq 2$ and $k, i \geq 1$

$$
\alpha_{0,k,p,i} = (-1)^{k+p} \binom{p-1}{i-k} \binom{k+p-2}{p} + (-1)^{k+p} \binom{p-1}{i-k} \binom{k+p-2}{p-1}
$$

$$
= -\alpha_{0,k-1,p,i-1} + (-1)^{k+p} \binom{p-2}{i-k} \binom{k+p-2}{p-1} + (-1)^{k+p} \binom{p-2}{i-k-1} \binom{k+p-2}{p-1}
$$

$$
= -\alpha_{0,k-1,p,i-1} - \alpha_{0,k,p-1,i} - \alpha_{0,k,p-1,i-1}.
$$

One sees easily from Step 1 that this recursion actually holds for all $p \in \mathbb{Z}$. Hence all numbers $\alpha_{0,k,p,i}$ can be computed from the following initial values:

$$
\alpha_{0,k,p,i} = 0 \text{ for } p < 0, \quad \alpha_{0,0,p,i} = 0, \quad \alpha_{0,k,p,0} = \begin{cases} 
-1 & \text{if } p = 0 \text{ and } k \geq 1 \\
0 & \text{otherwise}
\end{cases}
$$

Step 3: recursion for $\beta_{0,k,p,i}$. We have

$$
\beta_{0,k,p,i} = (-1)^{i+1} \sum_{l=0}^{k-1} \binom{i}{l} \binom{p-i}{l} + (-1)^{p+1} \sum_{l=0}^{k-1} \binom{i}{l} \binom{p-i}{l}.
$$

In particular we see that

$$
\beta_{0,k,p,i} = 0 \quad \text{if } p < 0.
$$

Applying again twice Pascal’s rule, we obtain

$$
\beta_{0,k,p,i} = (-1)^{p+1} \sum_{l=0}^{k-1} (-1)^l \binom{i}{l} \binom{p-2+l}{p-1-i+l} + (-1)^{p+1} \sum_{l=0}^{k-1} (-1)^l \binom{i}{l} \binom{p-2+l}{p-i+l}
$$

$$
= -\beta_{0,k,p-1,i} + (-1)^{p+1} \sum_{l=0}^{k-1} (-1)^l \binom{i-1}{l-1} \binom{p-2+l}{p-i+l} + (-1)^{p+1} \sum_{l=0}^{k-1} (-1)^l \binom{i-1}{l} \binom{p-2+l}{p-i+l}
$$

$$
= -\beta_{0,k,p-1,i} - \beta_{0,k-1,p,i-1} - \beta_{0,k,p-1,i-1}.
$$

In particular we see that both sequences $\alpha_{0,k,p,i}$ and $\beta_{0,k,p,i}$ satisfy the same recursion. Furthermore we have the following initial values

$$
\beta_{0,k,p,i} = 0 \quad \text{for } p < 0, \quad \beta_{0,0,p,i} = 0, \quad \beta_{0,k,p,0} = \begin{cases} 
-1 & \text{if } p = 0 \text{ and } k \geq 1 \\
0 & \text{otherwise}
\end{cases}
$$

Since both sequences $\alpha_{0,k,p,i}$ and $\beta_{0,k,p,i}$ have the same initial values, they coincide.

□

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