Ground State Degeneracy of Potts Antiferromagnets: Homeomorphic Classes with Noncompact $W$ Boundaries

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We present exact calculations of the zero-temperature partition function $Z(G, q, T = 0)$ and ground-state degeneracy $W(\{G\}, q)$ for the $q$-state Potts antiferromagnet on a number of families of graphs $G$ for which (generalizing $q$ from $\mathbb{Z}^+$ to $\mathbb{C}$) the boundary $B$ of regions of analyticity of $W$ in the complex $q$ plane is noncompact, passing through $z = 1/q = 0$. For these types of graphs, since the reduced function $W_{\text{red.}} = q^{-1}W$ is nonanalytic at $z = 0$, there is no large-$q$ Taylor series expansion of $W_{\text{red.}}$. The study of these graphs thus gives insight into the conditions for the validity of the large-$q$ expansions. It is shown how such (families of) graphs can be generated from known families by homeomorphic expansion.

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I. INTRODUCTION

Nonzero ground state entropy, $S_0 \neq 0$, is an important subject in statistical mechanics. One physical example is provided by ice, for which $S_0 = 0.82 \pm 0.05 \text{ cal/(K-mole)}$, i.e., $S_0/R = 0.41 \pm 0.03$, where $R = N_{\text{Avog}} k_B$. A particularly simple model exhibiting ground state entropy without the complication of frustration is the $q$-state Potts antiferromagnet (AF) on a graph $G$ (which may or may not be a regular lattice) for $q \geq \chi(G)$, where $\chi(G)$ denotes the chromatic number of $G$, i.e., the minimum number of colors necessary to color the vertices of the lattice such that no two adjacent vertices have the same color. As is evident, there is a deep connection with graph theory here, since the zero-temperature partition function of the above-mentioned $q$-state Potts antiferromagnet on a graph $G$ satisfies

$$Z(G, q, T = 0)_{P \text{AF}} = P(G, q) \quad (1.1)$$

where $P(G, q)$ is the chromatic polynomial expressing the number of ways of coloring the vertices of the graph $G$ with $q$ colors such that no two adjacent vertices (connected by a bond of the graph) have the same color. Since the strict mathematical definition of a graph $G$ requires the number of vertices $n = v(G)$ to be finite, we shall denote

$$\lim_{n \to \infty} G = \{G\} \quad (1.2)$$

In this limit, the ground state entropy per vertex (site) is given by

$$S_0 = k_B \ln W(\{G\}, q) \quad (1.3)$$

where $W(\{G\}, q)$, the ground state degeneracy per vertex, is

$$W(\{G\}, q) = \lim_{n \to \infty} P(G, q)^{1/n} \quad (1.4)$$

As noted, in the limit $n \to \infty$, nonzero ground state entropy, $S_0(\{G\}, q) > 0$, or equivalently, ground state degeneracy $W(\{G\}, q) > 1$ generically occurs in the $q$-state Potts antiferromagnet for sufficiently large $q$ on a given lattice $G$; equivalently, the number of ways of coloring the graph subject to the constraint that no two adjacent vertices have the same color grows exponentially rapidly with the number of vertices on the graph.

Given the fact that $P(G, q)$ is a polynomial, there is a natural generalization, which we assume here, of the variable $q$ from integer to complex values. $W(\{G\}, q)$ is an analytic function in the $q$ plane except along a continuous locus of points (and at possible isolated points which will not be relevant here). Following the terminology in our earlier papers, we denote this continuous locus as $B$. In the limit as $n \to \infty$, the locus $B$ forms by means of a coalescence of a subset of the zeros of $P(G, q)$ (called chromatic zeros of $G$). Because $P(G, q)$ has real (indeed, integer) coefficients, $B$ has the basic property of remaining invariant under the replacement $q \to q^*$:

$$B(q) = B(q^*) \quad (1.5)$$

In a series of papers we have calculated and analyzed the loci $B$ for a variety of families of graphs and investigated the connections of these loci with the behavior of $W(\{G\}, q)$ for the physical values $q \in \mathbb{Z}_+$, which we have also studied by means of rigorous upper and lower bounds and large-$q$ series.
and Monte Carlo measurements [12,13,14]. In many cases the loci $B$ enclose and form the boundaries of two or more regions in the complex $q$ plane. Since this is the case for the families of graphs studied here, we shall use the abbreviated nomenclature “$W$ boundary” for $B$. We note that in other families of graphs, such as the infinitely long, finite-width homogeneous strip graphs of various lattices (with open boundary conditions) studied in Ref. [13], the loci $B$ form arcs that do not enclose any region. The study of these loci for various families of graphs also makes a very interesting connection with algebraic geometry.

With regard to the properties of the ground state entropy of the Potts antiferromagnet on a given ($n \to \infty$ limit of a graph) \{\{G\}\}, two of the most important properties of $W(\{G\}, q)$ and the associated nonanalytic locus $B$ include (i) the maximal finite value of $q$ at which $B$ crosses the real axis in the $q$ plane, which we denote $q_c$, as in our previous work; and (ii) the question of whether $B(q)$ is bounded or unbounded (noncompact). Property (i) is important because in cases where $B$ encloses regions, the behavior of the ground state degeneracy $W(\{G\}, q)$ changes qualitatively (nonanalytically) as $q$ (taken to be real) decreases through $q_c$. Concerning the importance of property (ii), we first recall that since an obvious upper bound on $P(G, q)$ describing the coloring of an $n$-vertex graph with $q$ colors is $P(G, q) \leq q^n$, and hence $W(\{G\}, q) \leq q$, it is natural to define a reduced function that is bounded as $q \to \infty$:

$$W_{\text{red}}(\{G\}, q) = q^{-1}W(\{G\}, q)$$  \hspace{1cm} (1.6)$$

(This function was denoted $W_r(\{G\}, q)$ in our Refs. [13,15].) If and only if $W_{\text{red}}(\{G\}, q)$ is analytic at $z = 1/q = 0$, there exists a Taylor series expansion of this function around this point. Such large-$q$ series expansions are a useful means for obtaining approximate values of the ground state degeneracy of the $q$-state Potts antiferromagnet for finite $q$ on regular lattices [25–27,16,17]. In turn, $W_{\text{red}}(\{G\}, q)$ is nonanalytic at $1/q = 0$ if and only if the locus $B$ is unbounded in the $q$ plane, i.e., passes through the origin of the $1/q$ plane. Thus, families of graphs with unbounded $B(q)$ do not have large-$q$ Taylor series expansions for $W_{\text{red}}(\{G\}, q)$. In Ref. [13] we drew attention to this for bipyramid graphs [23] and in Ref. [15] we presented a general method for constructing families of graphs with unbounded, noncompact loci $B$. We also presented a general (necessary and sufficient) algebraic condition such that $B$ is unbounded. Clearly it is important to understand better the differences between the families of graphs that yield $W_{\text{red}}(\{G\}, q)$ functions analytic at $1/q = 0$ and those that produce $W_{\text{red}}(\{G\}, q)$ functions that are nonanalytic at $1/q = 0$. Since $B$ forms by merging of chromatic zeros of $G$ as $v(G) \to \infty$, a necessary condition for $B$ to be noncompact in the $q$ plane, extending infinitely far from the origin, is that as $v(G) \to \infty$, the magnitudes $|q|$ of some chromatic zeros must increase without bound. That this is not a sufficient condition is illustrated by the chromatic zeros of the complete graph $K_p$. [28]: these occur at $q = 0, 1, \ldots, p - 1$ so that, as $p \to \infty$, the magnitude of the largest chromatic zero grows without bound, but the locus $B$ for this family is not only not noncompact, it is trivial, consisting of the empty set.

In the present paper, continuing our earlier studies [13,14], we shall address this problem. We shall present several additional methods for constructing families of graphs with unbounded $B$. We shall use these to obtain exact calculations of the corresponding chromatic polynomials, $W$ functions, and boundaries $B$. In particular, we shall generate an infinite number of such families via the method of homeomorphic expansion. Homeomorphic classes of graphs have been of continuing interest in graph theory [11]; for some recent theorems in a somewhat different direction from the present work, see Ref. [29]. The study of these families is valuable because it enables one gain further insight into the origin of the nonanalyticity of $W_r(\{G\}, q)$ at $1/q = 0$ and the consequent nonexistence of large-$q$ series expansions for the ground state degeneracy of the Potts antiferromagnet on such graphs. Here we shall give detailed analyses of families with loci $B$ that have
a simple structure in the vicinity of $1/q = 0$. In a companion paper, we shall present results for families of graphs with loci $\mathcal{B}$ that are more complicated in the vicinity of $1/q = 0$.

Before proceeding, we recall two subtleties in the definition of $W(\{G\}, q)$ when $q$ is not a positive integer. First, for certain ranges of real $q$, $P(G, q)$ can be negative, and, of course, when $q$ is complex, so is $P(G, q)$ in general; in these cases it is not obvious, \textit{a priori}, which of the $n$ roots

$$P(G, q)^{1/n} = \{|P(G, q)|^{1/n} e^{i \arg(P(G, q)) / n} e^{\pi i r / n}\}, \quad r = 0, 1, \ldots, n - 1 \quad (1.7)$$

to choose in eq. (1.4). Consider the function $W(\{G\}, q)$ defined via eq. (1.4) starting with $q$ on the positive real axis where $P(G, q) > 0$, and consider the maximal region in the complex $q$ plane that can be reached by analytic continuation of this function. We denote this region as $R_1$. Clearly, the phase choice in (1.7) for $q \in R_1$ is that given by $r = 0$. For families of graphs $\{G\}$ where there are regions $R_i$ of analyticity of $W(\{G\}, q)$ that are not analytically connected with $R_1$, there is no canonical choice of phase in eq. (1.7) and hence it is only possible to determine the magnitude $|W(\{G\}, q)|$ unambiguously.

A second subtlety in the definition of $W(\{G\}, q)$ concerns the fact that at certain special points $q_s$, the following limits do not commute (for any choice of $r$ in eq. (1.7)):

$$\lim_{n \to \infty} \lim_{q \to q_s} P(G, q)^{1/n} \neq \lim_{q \to q_s} \lim_{n \to \infty} P(G, q)^{1/n} \quad (1.8)$$

The set $q_s$ includes points where $P(G, q)$ has zeros whose multiplicity does not scale like the number of vertices (typically these are simple zeros). Following our earlier work [13], we define

$$W(\{G\}, q_s) \equiv W(\{G\}, q_s)|_{q \to q_s} \equiv \lim_{q \to q_s} \lim_{n \to \infty} P(G, q)^{1/n} \quad (1.9)$$

This definition has the advantage of maintaining the analyticity of $W(\{G\}, q)$ at the special points $q_s$.

A general form for the chromatic polynomial of an $n$-vertex graph $G$ is

$$P(G_n, q) = c_0(q) + \sum_{j=1}^{N_a} c_j(q) a_j(q) t^{(n)} \quad (1.10)$$

where

$$t(n) = t_1 n + t_0 \quad (1.11)$$

and $c_j(q)$ and $a_j(q)$ are certain functions of $q$. Here the $a_j(q)$ and $c_{j \neq 0}(q)$ are independent of $n$, while $c_0(q)$ may contain $n$-dependent terms, such as $(-1)^n$, but does not grow with $n$ like $(\text{const.})^n$. Obviously, the reality of $P(G, q)$ for real $q$ implies that $c_j(q)$ and $a_j(q)$ are real for real $q$. The condition that $\mathcal{B}$ does not extend to infinite distance from the origin in the $q$ plane is equivalent to the condition that for sufficiently large $|q|$, there is one leading term $a_j(q)$ in eq. (1.10). Here we recall that “leading term $a_\ell(q)$ at a point $q$” was defined in Ref. [13] as a term satisfying $|a_\ell(q)| \geq 1$ and $|a_j(q)| > |a_\ell(q)|$ for $j \neq \ell$. (If the $c_0$ term is absent and $N_a = 1$, then the sole $a_1(q)$ may be considered to be leading even if $|a_j(q)| < 1$.) In the limit as $n \to \infty$ (i.e., for fixed $p$, $r \to \infty$), the leading term $a_\ell$ in a given region determines the limiting function $W$, with $|W| = |a_\ell| t^\ell$ and the boundary $\mathcal{B}$ occurs where there is a nonanalytic change in $W$ as it switches between being determined by different leading terms $a_\ell$ in eq. (1.10).

Since for the families of graphs studied here, the boundary $\mathcal{B}$ is noncompact in the $q$ plane, it is often more convenient to describe the boundary in the complex $z$ or $y$ planes, where
\[ z \equiv \frac{1}{q} \] (1.12)

and

\[ y \equiv \frac{1}{q - 1} \] (1.13)

The variable \( y \) is commonly used in large-\( q \) series expansions. Some useful relations between these variables are

\[ q = 1 + \frac{1}{y}, \quad z = \frac{y}{1 + y}, \quad y = \frac{z}{1 - z} \] (1.14)

We define polar coordinates as

\[ z = \zeta e^{i\theta} \] (1.15)

and

\[ y = \rho e^{i\beta} \] (1.16)

We shall use some standard notation from graph theory and combinatorics: \( v(G) \) and \( e(G) \) denote the numbers of vertices and edges in the graph \( G \), and the symbol \( q^{(n)} \) is given by

\[ q^{(p)} = p!(q - p) \prod_{s=0}^{p-1}(q - s) \] (1.17)

The terms “edge” and “bond” will be used synonymously. For any partition of \( n, n = p + u \),

\[ q^{(p)}(q - p)^{(u)} = q^{(u)}(q - u)^{(p)} = q^{(p+u)} \] (1.18)

For our results below, it will also be convenient to define the polynomial

\[ D_k(q) = \frac{P(C_k, q)}{q(q - 1)} = a^{k-2} \sum_{j=0}^{k-2} (-a)^{-j} = \sum_{s=0}^{k-2} (-1)^s \binom{k-1}{s} q^{k-2-s} \] (1.19)

where

\[ a = q - 1 \] (1.20)

and \( P(C_k, q) \) is the chromatic polynomial for the circuit graph,

\[ P(C_k, q) = a^k + (-1)^k a \] (1.21)

Some useful properties of \( D_k \) are listed in an appendix.

The organization of the paper is as follows. In Section II we review some methods and results from our previous work on families of the form \((K_p, b + G_r)\) that will be necessary for our current discussion (where \( K_p \) is the complete graph on \( p \) vertices \( \mathbb{F}_p \) and, following standard notation in the mathematical literature on graph theory, we use the symbol \( G + H \) for the “join” of these graphs; see below for definitions). Section III contains another method of constructing families of graphs with noncompact \( W \) boundaries \( \mathcal{B} \) involving the removal of bonds not just from one vertex of the \( K_p \) subgraph in \( K_p + G_r \). In Section IV we present
and discuss a number of ways to construct families of graphs with noncompact $W$ boundaries based on homeomorphic expansion of starting sets of graphs. In Section V we analyze two such methods involving homeomorphic expansion of the $K_p$ subgraph in a larger graph. Sections VI-VIII are detailed analyses of the respective chromatic polynomials, $W$ functions, and their boundaries of regions of analyticity $B$ for families of graphs obtained by specific homeomorphic expansions. In Section IX we include some remarks on general geometrical features of the families of graphs with noncompact $W$ boundaries $B$. Finally, Section X contains our conclusions.

II. FAMILIES OF THE FORM $(K_p)_b + G_r$

Since we will construct and analyze a number of new families of graphs with noncompact boundaries $B$ as homeomorphic expansions of families studied in Ref. [15] it is first necessary to review briefly the method that we formulated and used in Ref. [15] to generate such families. Consider a family of graphs $G$ and its $v(G) \to \infty$ limit, $\{G\}$. If this family already has the property that the limiting function $W(\{G\}, q)$ has a region boundary $B$ that extends to complex infinity in the $q$ plane, then we have no work to do to get such a boundary. So assume that this family is such that $W(\{G\}, q)$ has a region boundary $B$ that does not extend to complex infinity in the $q$ plane. Specifically, we start with an $r$-vertex graph $G_r$ and then form the graph $K_p + G_r$, where, as above, $K_p$ denotes the complete graph on $p$ vertices [28], and we adopt the notation $G + H$ that is commonly used in graph theory to indicate that every vertex of $G$ is connected by a bond to every vertex of $H$; this is called the “join” of $G$ and $H$ in the mathematical literature. We then remove $b$ bonds from one vertex in the $K_p$ subgraph; the resultant family is denoted $(K_p)_b + G_r$ (2.1)

(In Ref. [15] we used the notation $G \times H$ for what is called $G + H$ here, and the notation $(K_p \times G_n)_{rb}$ for the family denoted $(K_p)_b + G_r$ above, where the subscript $rb$ signified “removed bonds”.) Since each such vertex has degree $\Delta = p - 1$ (i.e., has $p - 1$ bonds connecting to it), it follows that

$$1 \leq b \leq p - 1$$

(2.2)

It was proved in Ref. [15] that in the limit $r \to \infty$, the family $(K_p)_b + G_r$ has a noncompact $W$ region boundary $B(q)$, extending infinitely far from the origin of the $q$ plane. A number of families of this type were constructed and the chromatic polynomials, resultant $W$ functions, and boundaries $B$ calculated [15]. Specifically, by means of the relation

$$P(K_p + G_r, q) = q^{(p)} P(G_r, q - p) ,$$

(2.3)

the chromatic polynomial of $(K_p)_b + G_r$ was calculated to be [15]

$$P((K_p)_b + G_r, q) = P(K_p + G_r, q) + bP(K_{p-1} + G_r, q)$$

$$= q^{(p-1)} [(q - (p - 1))P(G_r, q - p) + bP(G_r, q - (p - 1))]$$

(2.4)

Substituting the expression (1.10) yields

$$P((K_p)_b + G_r, q) = q^{(p-1)} \left\{ (q - (p - 1)) \left\{ c_0(q - p) + \sum_{j=1}^{N_r} c_j(q - p)a_j(q - p)^{(r)} \right\} \right\}$$

5
+b\left\{c_0(q-(p-1)) + \sum_{j=1}^{N_a} c_j(q-(p-1))a_j(q-(p-1))^t(r)\right\} \tag{2.5}

For a given $q$, the boundary $\mathcal{B}$ as $r \to \infty$ (with $p$ fixed) is determined by the degeneracy in magnitude of the leading terms in eq. (2.3) at this value of $q$, viz.,

$$|a_\ell(q-p)| = |a_\ell(q-(p-1))| \tag{2.6}$$

This equation, and hence also $\mathcal{B}$ are independent of $b$ for the interval (2.2), $1 \leq b \leq p-1$. In passing, we note that since the total number of vertices of $(K_p)_b + G_r$ is $v((K_p)_b + G_r) = p+r$, another way to take $n \to \infty$ is to let $p \to \infty$ with $r$ fixed, rather than letting $r \to \infty$ with $p$ fixed. However, from the viewpoint of either statistical mechanics or graph theory, this is not as interesting a limit, since for any given graph $G_r$ and for any given (finite) value of $q \in \mathbb{Z}_+$, as $p$ becomes sufficiently large, one will not be able to color the graph $(K_p)_b + G_r$ and the chromatic polynomial will vanish. We shall therefore restrict ourselves to the other way of letting the number of vertices go to infinity, viz., $r \to \infty$ with $p$ fixed.

Writing

$$a_\ell = \sum_{s=0}^{s_{\text{max}}} \alpha_{\ell,s} q^s , \tag{2.7}$$

recalling that the basic theorem that the coefficient of the highest-order term, $q^n$, in the chromatic polynomial $P(G_n,q)$ of any $n$-vertex graph $G_n$ is unity implies that

$$\alpha_{s_{\text{max}}} = 1 , \tag{2.8}$$

next dividing eq. (2.4) by $|q^{s_{\text{max}}}|$, and finally reexpressing the degeneracy equation in the more convenient variable $z$ given in eq. (1.12), we obtain

$$|(1-pz)^{s_{\text{max}}} + \sum_{s=0}^{s_{\text{max}}-1} \alpha_{\ell,s}(1-pz)^s z^{s_{\text{max}}-s}|$$

$$= |(1-(p-1)z)^{s_{\text{max}}} + \sum_{s=0}^{s_{\text{max}}-1} \alpha_{\ell,s}(1-(p-1)z)^s z^{s_{\text{max}}-s}| \tag{2.9}$$

This equation is clearly satisfied for $z = 0$, which shows that the boundary $\mathcal{B}$ for the $r \to \infty$ limit of the family of graphs $(K_p)_b + G_r$ is noncompact in the $q$ plane, passing through $z = 1/q = 0$. As noted above, it is thus more convenient to analyze $\mathcal{B}$ in the $z$ plane. We next show that $\mathcal{B}$ passes vertically through the origin of the $z$ plane for the $r \to \infty$ limit of all of the families of the type $(K_p)_b + G_r$. Writing eq. (2.9) in polar coordinates, using (1.15), we have, for $|z| = \zeta \to 0$,

$$\zeta s_{\text{max}} \cos \theta + O(\zeta^2) = 0 \tag{2.10}$$

Since $s_{\text{max}} \neq 0$, it follows that

$$\cos \theta \to \pm \frac{\pi}{2} \quad \text{as} \quad \zeta \to 0 \tag{2.11}$$

which proves the above assertion, that $\mathcal{B}$ passes vertically through the point $z = 0$.

The simplest families of graphs of the form $(K_p)_b + G_r$ are those for which $P(G_r,q)$ has, in the notation of eq. (1.10), $c_0 = 0$, $N_a = 1$, and $a_1(q)$ a linear function of $q$,
For such families, in the $q$ plane, $B$ is a vertical line with $q_\alpha$ arbitrary and

$$q_R = q_c = p - (\frac{1}{2} + \alpha_0) \quad \text{for} \quad s_{max} = 1$$

(2.13)

where $Re(q) = q_R$ and $Im(q) = q_i$. In the $z$ plane, $B$ is the circle defined by

$$|z - \frac{z_c}{2}| = \frac{z_c}{2}$$

(2.14)

with

$$z_c = \frac{1}{q_c} = \frac{1}{p - (\frac{1}{2} + \alpha_0)}$$

(2.15)

Equivalently, $B$ is the circle $|y - \frac{y_c}{2}| = \frac{y_c}{2}$ in the $y$ plane, where $y_c = 1/(p - (\frac{3}{2} + \alpha_0))$. Thus, $B$ divides the $z$ (equivalently $q$ or $y$) plane into two regions: (i) $R_1$, as defined above to include the region on the positive $z$ axis contiguous with the origin, and (ii) its complement, denoted $R_2$. We find that

$$W(\lim_{r \to \infty} [(K_p)_b + G_r], q) = q - p + 1 + \alpha_0 \quad \text{for} \quad q \in R_1$$

(2.16)

$$|W(\lim_{r \to \infty} [(K_p)_b + G_r], q)| = |q - p + \alpha_0| \quad \text{for} \quad q \in R_2$$

(2.17)

These families include the cases $G_r = (i)$ edgeless graphs $K_r$; (ii) tree graphs $T_r$; and (iii) chains of triangles with each pair of adjacent triangles intersecting on a mutual edge, i.e. $(Ch)_{3,r}$ in the notation of Ref. [15] (The general chain graph of $n_p$ $k$-sided polygons with adjacent $k$-gons intersecting on a mutual edge is denoted $(Ch)_{k,r}$; the number of vertices, $r$, is given by $r = (k - 2)n_p + 2$.) The chromatic polynomials for these families are

$$P((K_p)_b + \overline{K}_r, q) = q^p \left[(q - p)^r + b(q - (p - 1))^r\right]^{-1}$$

(2.18)

$$P((K_p)_b + T_r, q) = q^{(p+1)} \left[(q - (p + 1))^r - b(q - p)^r\right]^{-2}$$

(2.19)

$$P((K_p)_b + (Ch)_{k,r}, q) = q^{(p+1)} \left[(q - (p + 1))D_k(q - p)^{n_p} + bD_k(q - (p - 1))^{n_p}\right]$$

(2.20)

Thus

$$\alpha_0 = 0, \ -1, \ -2 \quad \text{for} \quad G_r = \overline{K}_r, \ T_r, \ (Ch)_{3,r}$$

(2.21)

so that

$$q_c = p - \frac{1}{2}, \ p + \frac{1}{2}, \ p + \frac{3}{2} \quad \text{for} \quad G_r = \overline{K}_r, \ T_r, \ (Ch)_{3,r}$$

(2.22)

The boundaries $B$ for other families have the same behavior near the origin, $z = 0$, as shown by eq. (2.10), but have different features where they cross the positive real $z$ axis. For example, consider $G_r = C_r$, which has, in the notation of eq. (1.10), $N_a = 1$ and a linear $a_1(q) = q - 1$ but also a nonzero $c_0$ term, viz.,
For \( \lim_{r \to \infty} [(K_p)_b + C_r] \), the portion of \( B \) near to \( z = 0 \) forms part of the circle (2.14), but to the right of the points \( z_{\text{int.}} = 1/q_{\text{int.}} \) and \( z^{*}_{\text{int.}} \), where

\[
q_{\text{int.}} = p + \frac{1}{2} + \frac{i\sqrt{3}}{2}
\]

the boundary bifurcates into two arcs, which cross the positive \( z \) axis at

\[
z_c = \frac{1}{q_c} = \frac{1}{p+1} \quad \text{for} \quad G_r = C_r
\]

and at \( z = 1/p \). Since we shall compare this in detail with homeomorphic expansions of this family, we show this boundary in Fig. 1. For this and similar plots given later we also show chromatic zeros calculated for a reasonably large finite value of \( r \) (in this case, \( r = 27 \)). The comparison of these chromatic zeros with the \( r \to \infty \) locus \( B \) shows quantitatively how \( B \) forms via the merging of the zeros. Of course, since for finite \( r \), the chromatic polynomial is of finite degree (equal to the number of vertices on the graph), its zeros (i.e., the chromatic zeros of the graph) are of bounded magnitude in \( q \) and hence do not track the asymptotic curve \( B \) all the way in to the origin of the \( z = 1/q \) plane. The same comment applies to the chromatic zeros that we will show for other families below.

**FIG. 1.** Boundary \( B \) in the \( z = 1/q \) plane for the \( r \to \infty \) limit of the family of graphs \((K_2)_b = 1 + C_r = K_2 + C_r\). Chromatic zeros for \( r = 27 \) are shown for comparison.

Similar comments apply for other families such as \((K_p)_b + L_r\) and for the \( k \geq 4 \) members of the family \((K_p)_b + (Ch)_{k,r}[13]\), where \( L_r \) denotes a cyclic square strip of width one square, i.e. cyclic ladder graph, and \((Ch)_{k,r}\) was defined above in our discussion of the special case \( k = 3 \).
Of the families of graphs that we have discussed, with noncompact boundaries \( \mathcal{B} \) of regions of analyticity in \( W \), most have chromatic polynomials that depend on three parameters, \( p, r, \) and \( b \), and the resultant \( \mathcal{B} \) for the limit \( r \to \infty \) depends on one parameter, \( p \) (and is independent of \( b \)). The family \( (K_p)_b + (Ch)_{k,r} \) illustrates a case where the chromatic polynomial depends on four parameters – \( p, k, r, \) and \( b \) – and \( \mathcal{B} \) thus depends on two parameters: \( p \) and \( k \).

Here we present results on another such family,

\[
G_{p,b,s,u} = (K_p)_b + sK_u
\]

where \( sK_u \) denotes the disjoint union of \( s \) copies of the \( K_u \) graph (where the vertices of each of the \( s \) \( K_u \) subgraphs are not connected to each other by bonds). That is, one adjoins \( K_p \) to \( G_r = sK_u \) and then removes \( b \) bonds from one vertex of the \( K_p \) subgraph. An illustration of the special case \( p = 2, b = 1, s = 3, \) and \( u = 2 \) is shown in Fig. 2(d).

Clearly, the number of vertices in the disjoint union \( sK_u \) is \( r = v(sK_u) = su \), so that

\[
n = v((K_p)_b + sK_u) = p + su
\]

For \( s = 1 \) and \( b = 0 \), one has the simplification \( K_p + K_u = K_{p+u} \). Henceforth, we shall assume that \( b \) satisfies the restriction (2.2). For general \( s \) and \( u \), from eq. (2.4) we have

\[
P((K_p)_b + sK_u, q) = q^{(p-1)} \left\{ (q - (p - 1))[(q - p)^{(u)}]^{s} + b[(q - (p - 1))^{(u)}]^{s} \right\}
\]

This chromatic polynomial has the form

\[
P((K_p)_b + sK_u, q) = q^{(p)} \left[ (q - p)^{(u-1)} \right]^{s} P_{b,s}
\]
where \( P_{b,s} \) is a polynomial in \( q \) of degree \( s \).

Now consider the limit \( s \to \infty \) with \( p \) and \( u \) fixed. Equation (2.28) has the form of eq. (1.10), so that the degeneracy equation is

\[
|(q - p)^{(u)}| = |(q - (p - 1))^{(u)}| \tag{2.30}
\]

Taking into account that \( q = 0 \) is never a solution of eq. (2.30) and dividing by \( |q|^u \), we obtain the corresponding degeneracy equation of leading terms in the \( z \) plane:

\[
|\prod_{j=0}^{u-1} (1 - (p + j)z)| = |\prod_{j'=0}^{u-1} (1 - (p - 1 + j')z)| \tag{2.31}
\]

By the same reasoning as before, it follows that the resultant locus \( B \) which is the solution to the degeneracy equation is noncompact in the \( q \) plane, passing through \( z = 1/q = 0 \). The general solution to the degeneracy equation (2.30) in the \( q \) plane is the vertical line given by

\[
\text{Re}(q) = q_c = p - 1 + \frac{u}{2} \tag{2.32}
\]

or equivalently, in the \( z \) plane, the circle

\[
|z - \frac{z_c}{2}| = \frac{z_c}{2}, \quad z_c = \frac{1}{p - 1 + \frac{u}{2}} \tag{2.33}
\]

and similarly, in the \( y \) plane, the circle \(|y - \frac{y_c}{2}| = \frac{y_c}{2}\) where \( y_c = 1/(p - 2 + \frac{u}{2}) \). The \( q \) (or \( z \) or \( y \)) plane is divided into two regions by the boundary \( B \), namely, \( R_1 \) with \( \text{Re}(q) > p - 1 + \frac{u}{2} \), and \( R_2 \), with \( \text{Re}(q) < p - 1 + \frac{u}{2} \). The dependence of \( q_c \) on \( u \) is easy to understand: clearly, just as increasing \( p \) reduces the number of colors available to color the rest of the graph and hence increases the value of \( q_c \) (serving as the demarcation on the real axis between the region of \( q \) values with \( W > 1 \) and nonzero ground state entropy \( S_0 = k_B \ln W \), from the region where the behavior of \( W \) is different), so also increasing \( u \) for fixed \( p \) has the same effect.

We find that

\[
W(\lim_{s \to \infty} [(K_p)b + s K_u], q) = |(q - (p - 1))^{(u)}|^{1/u} \quad \text{for} \quad q \in R_1 \tag{2.34}
\]

where we recall the definition (1.17). Further,

\[
|W(\lim_{s \to \infty} [(K_p)b + s K_u], q)| = |(q - p)^{(u)}|^{1/u} \quad \text{for} \quad q \in R_2 \tag{2.35}
\]

It follows that

\[
W(\lim_{s \to \infty} [(K_p)b + s K_u], q = q_c) = 0 \quad \text{if} \quad q_c \in \mathbb{Z}_+ \quad \text{i.e. if} \quad u \quad \text{is even} \tag{2.36}
\]

Note that the zeros in \( W \) extend into region \( R_1 \), to the right of \( q_c \), if \( p + u - 2 > q_c \), i.e., if \( u \geq 3 \). For example, for the case \( u = 3 \), for which \( q_c = p + \frac{1}{2} \), the expression (2.34) has formal zeros at \( q = p - 1, q = p, \) and \( q = p + 1 \). The first two of these are not relevant since they lie to the left of \( q_c \) in the region \( R_2 \) where the expression (2.34) does not apply; however, the third zero, at \( q = p + 1 \), is in region \( R_1 \) and is a true zero of \( W \). For this illustrative value \( u = 3 \), the expression (2.35) has formal zeros at \( q = p, q = p + 1, \) and \( q = p + 2 \). The last two of these are not relevant since they lie to the right of \( q_c \) in region \( R_1 \) where the expression (2.35) does not apply, while the first one, at \( q = p \), lies in region \( R_2 \) where this expression does apply and hence is a true zero of \( W \) in this region.
It is interesting to comment on the comparative values of $q_c$ defined in the limit as $r \to \infty$ and the chromatic number $\chi$ for specific members of a family of graphs parametrized by $r$. Before doing this, one should note that there are cases where $B$ is trivial or does not cross the real axis, so that no $q_c$ is defined; an example is given by tree graphs $T_r$, for which $B = \emptyset$ and $\chi = 2$. For circuit graphs $C_r$, $\chi = 2$ if $r$ is even and $\chi = 3$ if $r$ is odd. In the limit $r \to \infty$, $B$ is the circle $|q-1| = 1$, so that $q_c = 2$; hence this family illustrates the possibilities $q_c = \chi$ and $q_c < \chi$. For the square lattice with free boundary conditions or (periodic or toroidal) boundary conditions that preserve the bipartite nature of the lattice, $\chi = 2$ while in all cases $q_c = 3$ [13], so that this illustrates the possibility $q_c > \chi$. For the infinitely long strip of the square lattice with width $L_y = 3$ vertices ($w = L_y - 1$ squares), $q_c = 2$, equal to $\chi$ for this family; for the infinitely long strip of the triangular lattice with $L_y = 3$, $q_c \approx 2.57$, less than the chromatic number $\chi = 3$ for this family, while for the infinitely long triangular strip with width $L_y = 4$, $B$ does not cross the real $q$ axis, so no $q_c$ is defined [13,18]. Clearly, there is a large variety of behavior as regards the question of whether $q_c$ for the infinite-vertex limit of a family of graphs is smaller than, equal to, or larger than the chromatic number $\chi$ for specific members of the family. As an illustration of the families discussed here, $K_2 + K_r$ has $\chi = 2$ and $q_c = 3/2 < \chi$; $\overline{K}_2 + T_r$ has $\chi = 3$ and $q_c = 5/2 < \chi$, and $\overline{K}_2 + (Ch)_3,r$ has $\chi = 4$ and $q_c = 7/2 < \chi$. We shall discuss the situation for homeomorphic expansions of graphs in these families below.

III. FAMILIES OF THE FORM $(K_p)_{(b)} + G_r$

Another way to construct families with noncompact $W$ boundaries $B$ is to remove bonds not just from one vertex of the $K_p$ subgraph but also from one or more other vertices of this subgraph that are not adjacent to the first vertex, i.e., are not connected to this first vertex by any bonds. We symbolize this family by

$$(K_p)_{(b)} + G_r$$

where the subscript $\{b\}$ refers to the removal of multiple bonds from non-adjacent vertices of the $K_p$ subgraph. For this family, the maximum number of bonds that can be removed from the $K_p$ subgraph is the number of bonds in this subgraph, namely,

$$e(K_p) = \frac{p^2}{2} = \frac{p(p - 1)}{2}$$

In general, the chromatic polynomial for such families is not quite as easily derived as that for the type of families where all bonds removed are from one vertex, where we obtained the simple formula (2.4) [15].

A particularly simple family of graphs is obtained when one removes all of the $\binom{p}{2}$ bonds in the $K_p$ subgraph, thereby yielding the subgraph $\overline{K}_p$ and the resultant family

$$\overline{K}_p + G_r$$

Clearly, for $p = 2$,

$$(K_2)_{b=1} + G_r = \overline{K}_2 + G_r$$

For the general family $\overline{K}_p + G_r$, one can use the addition-contraction theorem to reexpress $P(\overline{K}_p + G_r, q)$ as the linear combination

\[ P(K_p + G_r, q) = \sum_{j=1}^{p} S_p^{(j)} P(K_j + G_r, q) \]
\[ = \sum_{j=1}^{p} S_p^{(j)} q^{(j)} P(G_r, q - j) \]

(3.5)

where the coefficients \( S_p^{(j)} \) are the Stirling numbers of the second kind, defined by the equation (cf. eq. (1.17))

\[ q^p = \sum_{j=1}^{p} S_p^{(j)} q^{(j)} \]

(3.6)

A closed-form expression is [31]

\[ S_p^{(j)} = \frac{1}{j!} \sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k} k^p \]

(3.7)

Given that \( P(G, q) \) has the form (1.10), it follows, by the same argument as was used for the \((K_p)_b + G_r\) families with removal of \(b\) bonds connecting to a given vertex, that \( B \) is noncompact in the \(q\) plane, passing through \(z = 1/q = 0\).

It is worthwhile to consider some special cases for \(G_r\). For \(G_r = K_r\), we calculate

\[ P(K_p + K_r, q) = \sum_{j=1}^{p} S_p^{(j)} q^{(j)}(q - j)^r \]

(3.8)

Hence, in the limit \(r \to \infty\), \( B \) is determined by the degeneracy equation

\[ |q - 1| = |q - p|, \quad \text{i.e.,} \quad |1 - z| = |1 - pz| \]

(3.9)

Clearly, \(z = 0\) is a solution, and \( B \) is, in the \(q\) plane, the vertical line with

\[ \text{Re}(q) = q_c = \frac{p+1}{2} \]

(3.10)

i.e., in the \(z\) plane the circle

\[ |z - \bar{z}_c| = \frac{z_c}{2}, \quad z_c = \frac{2}{p+1} \]

(3.11)

Let us compare the values of \(q_c\) for the respective \(r \to \infty\) limits of \(K_p + K_r\) and \((K_p)_b + K_r\) (where it is understood in the latter case that \(b\) bonds are removed from a vertex in the \(K_p\) subgraph). For \(p = 2\), the value of \(q_c\) in eq. (3.10) is equal to that for \(G_r = K_r\) in eq. (2.22), as follows from eq. (3.4). For \(p \geq 3\), the value of \(q_c\) in eq. (3.10) is smaller than that for \(G_r = K_r\) in eq. (2.22). This is a consequence of the fact that the graph \((K_p)_b + K_r\) can be obtained from \(K_p + K_r\) by addition of bonds and hence, by the theorem in section VII of Ref. [17], the coloring of the former graph with \(q\) colors is more constrained than the coloring of the latter graph; the same theorem implies that for \(q \in \mathbb{Z}_+\),

\[ W(\lim_{r \to \infty} [(K_p)_b + K_r], q) \leq W(\lim_{r \to \infty} [K_p + K_r], q) \]

(3.12)

In region \(R_1\) to the right of the vertical line (3.10) in the \(q\) plane,
\[ W(\lim_{r \to \infty} (\overrightarrow{K}_p + T_r), q) = q - 1 \quad \text{for} \quad q \in R_1 \]  

(3.13)

while in the complement, \( R_2 \),

\[ |W(\lim_{r \to \infty} (\overrightarrow{K}_p + T_r), q)| = |q - p| \quad \text{for} \quad q \in R_2 \]  

(3.14)

A second case is \( G_r = T_r \), i.e., the family \( \overrightarrow{K}_p + T_r \). We find

\[ P(\overrightarrow{K}_p + T_r, q) = \sum_{j=1}^{p} S_p^{(j)} q^{(j)} (q-j)(q-j-1)^{r-1} \]  

(3.15)

Hence, in the limit \( r \to \infty \), \( B \) is determined by the degeneracy equation

\[ |q - 2| = |q - p - 1|, \quad \text{i.e.,} \quad |1 - 2z| = |1 - (p + 1)z| \]  

(3.16)

In the \( q \) plane the locus of solutions to this equation is the vertical line with

\[ Re(q) = q_c = \frac{p + 3}{2} \]  

(3.17)

which again is a circle in the \( z \) plane,

\[ |z - \frac{z_c}{2}| = \frac{z_c}{2}, \quad z_c = \frac{2}{p + 3} \]  

(3.18)

For \( p = 2 \), the value of \( q_c \) in eq. (3.17) is equal to that for \( G_r = T_r \) in eq. (2.22), as follows from eq. (3.4). For \( p \geq 3 \), \( q_c \) is larger for \( \lim_{r \to \infty} [K_p + T_r] \) with \( b \) bonds removed from one vertex in \( K_p \) than for \( \lim_{r \to \infty} [\overrightarrow{K}_p + T_r] \), for the reason given above. For region \( R_1 \) to the right of the vertical line (3.17) in the \( q \) plane,

\[ W(\lim_{r \to \infty} (\overrightarrow{K}_p + T_r), q) = q - 2 \quad \text{for} \quad q \in R_1 \]  

(3.19)

while in the complement, \( R_2 \),

\[ |W(\lim_{r \to \infty} (\overrightarrow{K}_p + T_r), q)| = |q - p - 1| \quad \text{for} \quad q \in R_2 \]  

(3.20)

**IV. CONSTRUCTION OF FAMILIES OF GRAPHS WITH NONCOMPACT \( B \) VIA HOMEOMORPHIC EXPANSION**

A major result of the present paper is the construction of families of graphs with noncompact \( W \) boundaries by means of homeomorphic expansion (HE) (also called inflation) of a beginning set of families of graphs. We recall the definition that two graphs \( G \) and \( H \) are homeomorphic to each other if \( H \), say, can be obtained from \( G \) by successive insertions of degree-2 vertices on bonds of \( G \). Each such insertion subdivides an existing edge of \( G \) into two, connected by the inserted degree-2 vertex. This process is called homeomorphic expansion and its inverse is called homeomorphic reduction, i.e. the successive removal of vertices of degree 2 from a graph \( H \). Clearly, homeomorphic expansion of a graph always yields another graph. The inverse
is not necessarily true; i.e., homeomorphic reduction of a graph can produce a multigraph or pseudograph instead of a (proper) graph. Here, a multigraph is a finite set of vertices and bonds that, like a graph, has no bonds that loop around from a given vertex back to itself but, in contrast to a (proper) graph, may have more than one bond connecting two vertices. A pseudograph is a finite set of vertices and bonds that may have multiple bonds connecting two vertices and may also have looping bonds.

For example, consider homeomorphic reduction of a circuit graph $C_r$: removing one of the vertices (all of which have degree 2), one goes from $C_r$ to $C_{r-1}$, and so forth, until one gets to $C_3$. During this sequence of homeomorphic reductions, one remains within the category of graphs. However, the next homeomorphic reduction takes $C_3$ to $C_2$, which is a multigraph, not a proper graph. Removing one of the two vertices in $C_2$ produces a pseudograph consisting of a single vertex and a bond that goes out and loops back to this vertex. Thus homeomorphism is an equivalence relation on pseudographs. This complication will not be relevant here because we shall only use homeomorphic expansions, not reductions, of graphs, and the homeomorphic expansion of a proper graph always yields another proper graph. For our subsequent discussion, we shall denote the homeomorphic expansion involving the insertion of $k-2$ additional vertices (where $k ≥ 3$) on a specific bond $b$ of a graph $G$ as $HEG_{k-2,b}(G)$. Most of our studies will be of graphs in which the homeomorphic expansion of $G$ is performed systematically on each bond of $G$; in these cases, we shall denote the resultant graph as $HEG_{k-2}(G)$.

The homeomorphic expansion of a family of graphs with a compact (empty or nontrivial) locus $B$ can produce a family of graphs with either a compact or noncompact $B$; this depends on the nature of the original family and of the homeomorphic expansion. For example, if one starts with the tree graph $T_r$ and inserts $k-2$ degree-2 vertex on each of the bonds, one obtains another tree graph, $T_{r'}$, where $r' = r + (r-1)(k-2)$.

In the limit $r \to \infty$, both $T_r$ and its homeomorphic expansion, $T_{r'}$, have a trivial $B = \emptyset$. If one starts with the circuit graph $C_r$ and inserts $k-2$ degree-2 vertices on each bond of this graph, one obtains another circuit graph, $C_{r'}$, where $r' = r + r(k-2)$. In the limit $r \to \infty$, both $C_r$ and its homeomorphic expansion $C_{r'}$ have a compact $B$ given by the unit circle centered at $q = 1$, $|q-1| = 1$. We next proceed to discuss the cases of main interest here, where the homeomorphic expansion (i) leads from a family with a compact $B$ to one with a noncompact $B$ or (ii) takes a family that already has a noncompact $B$ to another that again has a noncompact $B$. We comment on the effect of a homeomorphic expansion of a graph $G$ on its girth $\gamma(G)$, defined as the length of (= number of vertices on) a minimal-length closed circuit in this graph. Clearly, a homeomorphic expansion applied to one or more bonds of a graph increases the girth of the graph if and only if these bonds lie on the minimal-length circuits of the original graph. We shall consider four main types of homeomorphic expansions, as well as combinations thereof:

1. $HEK0$

Start with the family $K_p + G_r$ (where no bonds have been removed from any vertex of the $K_p$ subgraph). If in the limit $r \to \infty$, $B$ for the $G_r$ family itself is compact (bounded) in the $q$ plane, then the same holds for the family $K_p + G_r$. Now insert degree-2 vertices on one or more bonds of the $K_p$. We shall denote this generically as an $HEK0$ homeomorphic expansion, meaning that the homeomorphic expansion acts on the $K_p$ subgraph and that originally there were $b = 0$ bonds removed from this $K_p$. Specifically, we denote the graph obtained by successive insertion of $k-2$ degree-2 vertices on a single bond $b_j$ of the $K_p$ as

$$HEK_{k-2,b_j}(K_p + G_r)$$ (4.1)

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where $k \geq 3$, and so forth for similar homeomorphic insertions on other bonds of the $K_p$. We will show below that this homeomorphic expansion leads from the family $K_p + G_r$ with compact (trivial or nontrivial) $B$ to the family (4.1) with a locus $B$ that is noncompact in the $q$ plane. The labelling convention is chosen so that as one moves along the expanded set of bonds linking a vertex of the original $K_p$ to what was originally an adjacent vertex, one traverses a total of $k$ vertices, including the original pair, i.e., $k - 2$ inserted vertices.

Next, we consider cases (nos. 2-5 below) where we begin with a family of graphs that already has a locus $B$ that is unbounded in the $q$ plane. For these the homeomorphic expansion produces another family of graphs again with an unbounded $B$:

2. **HEK**

Start with the family $(K_p)_b + G_r$ where $b$ bonds have been removed from one vertex of the $K_p$ subgraph and insert degree-2 vertices on one or more bonds of the $K_p$. We shall denote this generically as an **HEK** homeomorphic expansion, meaning that the homeomorphic expansion acts on the $K_p$ subgraph. Specifically, we denote the graph obtained by successive insertion of $k - 2$ degree-2 vertices on a single bond of the $K_p$ as

$$HEK_{k-2;b_j}[(K_p)_b + G_r]$$

where again $k \geq 3$, and so forth for similar insertions on other bonds of the $K_p$ subgraph.

3. **HEG**

Start with either $(K_p)_b + G_r$ or $(K_p)_{\{b\}} + G_r$ and add vertices to bonds in the $G_r$ subgraph. We label this type of homeomorphic expansion generically as **HEG**. Analogously to the previous category, we denote the respective graphs obtained by successive insertion of $k - 2$ degree-2 vertices on a single bond of the $G_r$ subgraph as

$$HEG_{k-2;b_j}[(K_p)_b + G_r], \quad HEG_{k-2;\{b\}}[(K_p)_{\{b\}} + G_r]$$

and the graphs obtained by successive insertions of $k - 2$ degree-2 vertices on all of the bonds of $G_r$ as

$$HEG_{k-2}[(K_p)_b + G_r], \quad HEG_{k-2}[(K_p)_{\{b\}} + G_r]$$

4. **HEC**

Start with either $(K_p)_b + G_r$ or $(K_p)_{\{b\}} + G_r$ and add vertices to the bonds connecting vertices in the $K_p$ subgraph to vertices in the $G_r$ subgraph. We label this type of homeomorphic expansion generically as **HEC**, where the “C” refers to the fact that the homeomorphic expansion is performed on the

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above-mentioned connecting bonds. We denote the respective graphs obtained by successive insertion of $k - 2$ degree-2 vertices on a single bond $b_{ij}$ connecting a vertex $v_i \in K_p$ to a vertex $v_j \in G_r$ as

\[ HEC_{k-2;b_{ij}}[(K_p)_b + G_r], \quad HEC_{k-2;b_{ij}}[(K_p)_{\{b\}} + G_r] \]  

and the graphs obtained by successive insertions of $k - 2$ degree-2 vertices on all of the bonds connecting vertices of $K_p$ to vertices of $G_r$ as

\[ HEC_{k-2}[(K_p)_b + G_r], \quad HEC_{k-2}[(K_p)_{\{b\}} + G_r] \]  

5. Combinations

Clearly, one can combine several types of homeomorphic expansion. For example, starting with $(K_p)_b + G_r$, one can add vertices both to bonds in the subgraph $K_p$, to bonds that connect $K_p$ to $G_r$, and to bonds in the $G_r$ subgraph.

In the present paper we shall concentrate on homeomorphic expansions of types (1)-(3). Our results for homeomorphic expansions of type (4) involve somewhat more complicated boundaries $B$ than those discussed here and will be dealt with in a separate paper. Composite homeomorphic expansions of type (5) can be studied by similar means. We now proceed to consider the various homeomorphic classes in more detail.

Given that the number of vertices in the original graph is a linear function of the two (positive integer) parameters $p$ and $r$, the number $n$ of vertices of the homeomorphic expansion is a linear function of $p$, $r$, and $k$. There are therefore three basic ways of producing the limit $n \to \infty$, namely ($L$ denotes limit)

\[ L_p : p \to \infty \text{ with } r \text{ and } k \text{ fixed} \]  

\[ L_r : r \to \infty \text{ with } p \text{ and } k \text{ fixed} \]  

\[ L_k : k \to \infty \text{ with } p \text{ and } r \text{ fixed} \]

We have explained above (after eq. (2.6)) why the limit $L_p$ is not very interesting. From the viewpoint of the present work on boundaries $B$ that are noncompact in the $q$ plane, the limit $L_k$ is also not of primary interest, since it generically yields a compact boundary $B$, as we shall illustrate below. Hence we shall concentrate on the limit $L_r$ here.

V. H\(E\)K0 AND H\(E\)K HOMEOMORPHIC CLASSES

Let us consider first the homeomorphic expansions of type (1), namely, $HEK_{k-2,b_{ij}}(K_p + G_r)$. By use of the deletion-contraction theorem and eq. (2.3), we find that, for arbitrary $G_r$,

\[
P(H\(E\)K_{k-2,b_{ij}}(K_p + G_r), q) = D_kP(K_p + G_r, q) + [D_k + (-1)^{k-1}]P(K_{p-1} + G_r, q) \\
= D_kq(p)P(G_r, q - p) + [D_k + (-1)^{k-1}]q^{(p-1)}P(G_r, q - (p - 1)) \]  

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It follows by the same argument as that given with eq. (2.4) that as \( r \to \infty \), this family \( HEK_{k-2; b} (K_p + G_r) \) has a \( W \) boundary \( B \) that is noncompact in the \( q \) plane. This is true independent of whether the family \( G_r \) has this property. Indeed, as is evident from a comparison of eq. (5.1) with eq. (2.4), for a given \( G_r \),

\[
B \left( \lim_{r \to \infty} HEK_{k-2; b} (K_p + G_r) \right) = B \left( \lim_{r \to \infty} [(K_p)_b + G_r] \right)
\]  

(5.2)

That is, if we start with the family \( K_p + G_r \), remove \( b \) bonds from a given vertex in the \( K_p \) subgraph, and take \( r \to \infty \), the resultant boundary \( B \) is the same as if we had instead homeomorphically added some number \( k-2 \geq 1 \) degree-2 vertices to a bond in the \( K_p \) subgraph. One can, of course, continue this process with homeomorphic expansions of other bonds of the \( K_p \). Note that if one uses the limit \( L_k \) in eq. (4.3) to get \( n \to \infty \) for this class of homeomorphic expansions, then, for arbitrary (finite) \( G_r \), the resultant boundary is the compact locus comprised by the unit circle \(|a| = 1\) (where \( a \) was defined in eq. (1.20)), i.e.,

\[
B_{L_k} = q \quad \text{such that} \quad |q - 1| = 1
\]  

(5.3)

We next consider the homeomorphic expansions of type (2), namely, \( HEK_{k-2; b} [(K_p)_b + G_r] \). Here, the boundary \( B \) for the \( r \to \infty \) of the beginning family \( (K_p)_b + G_r \) is already noncompact in the \( q \) plane, and the families generated by the homeomorphic expansion maintain this property. As an illustration, consider the family \( (K_3)_1 + G_r \) and homeomorphically expand one of the two remaining bonds, denoted \( b_j \), in the \( (K_3)_1 = T_3 \) subgraph. We find for the resultant chromatic polynomial the result

\[
P(HEK_{k-2; b} [(K_3)_1 + G_r], q) = D_k q^{(3)} P(G_r, q - 3)
\]

\[
+ [2D_k + (-1)^{k-1}] q^{(2)} P(G_r, q - 2)
\]  

(5.4)

Again, using the same reasoning that we employed before with eq. (2.4), we deduce that as \( r \to \infty \), this family \( HEK_{k-2; b} [(K_3)_1 + G_r] \) has a noncompact locus \( B \) in the \( q \) plane. As was the case for the \( HEK0 \) class, for the limit \( L_k \), the resultant boundary is the unit circle \(|q - 1| = 1\) given in eq. (5.3).

VI. THE FAMILY \( T_{k,r} = HEG_{k-2}(K_2 + T_r) \)

We next proceed to analyze in detail several homeomorphic classes of graph families of the form \( HEG \). One interesting family of graphs with noncompact \( B(q) \) is obtained by homeomorphic expansion starting with the family \((K_p)_b + T_r\) for \( r \geq 2 \) and adding vertices on each of the bonds in the \( T_r \) subgraph. We let \( k \) be the number of vertices on \( T_r \) between each pair of vertices that connect with the \( K_p \), including this pair (which were originally adjacent on \( T_r \) before the homeomorphic expansion). We denote this family as

\[
T_{p,b,k,r} = HEG_{k-2}[(K_p)_b + T_r]
\]  

(6.1)

where as above, \( r \geq 2 \) and \( 1 \leq b \leq p - 1 \). The number of vertices is given by

\[
v(T_{p,b,k,r}) = (r - 1)(k - 1) + p + 1
\]  

(6.2)

For the lowest value \( k = 2 \),

\[
T_{p,b,k=2,r} = (K_p)_b + T_r
\]  

(6.3)
which we studied previously [13]. We shall thus concentrate on the cases \( k \geq 3 \) here. It suffices for our present purposes to consider the simplest nontrivial case \( p = 2 \) and hence \( b = 1 \), for which

\[
(K_2)_{b=1} + T_r = K_2 + T_r
\]  

(6.4)

An illustration of a graph of this type is given in Fig. 2(a). For brevity, we define

\[
T_{k,r} = T_{p=2,h=1,k,r} = HEG_{k-2}(K_2 + T_r)
\]  

(6.5)

We observe that the chromatic number is

\[
\chi(T_{k,r}) = \begin{cases} 2 & \text{for } k \text{ odd} \\ 3 & \text{for } k \text{ even} \end{cases}
\]  

(6.6)

The girth, i.e., the length of a minimal-length circuit on this graph, is

\[
\gamma(T_{k,r}) = k + 1
\]  

(6.7)

By the deletion-contraction theorem, we find the recursion relation

\[
P(T_{k,r}, q) = [D_3D_k + (-1)^{k-1}]P(T_{k,r-1}, q) + q(q-1)D_k(D_{k+1})^{r-2}
\]  

(6.8)

Solving this, we obtain the chromatic polynomial

\[
P(T_{k,r}, q) = A_k[D_3D_k + (-1)^{k-1}]^{r-2} + q(q-1)(D_{k+1})^{r-1}
\]  

(6.9)

where

\[
A_k = (q-3)P(C_{k+1}, q) + (q-1)P(C_{k-1}, q)
\]  

(6.10)

\( P(T_{k,r}, q) \) has the general factors \( q(q-1) \) for \( k \) odd and \( q(q-1)(q-2) \) for \( k \) even. As is evident from eq. (6.9), \( P(T_{k,r}, q) \) has the form of eq. (1.10) with

\[
a_1 = D_{k+1}
\]  

(6.11)

and

\[
a_2 = D_3D_k + (-1)^{k-1}
\]  

(6.12)

(recall that \( D_3 = q-2 \)). In the limit \( r \to \infty \) (with \( k \) fixed [32]) the locus \( B \) is determined by the degeneracy of magnitudes

\[
|a_1| = |a_2|
\]  

(6.13)

This equation can be simply expressed in terms of the variable \( a = q-1 \) which was defined above in eq. (1.20) (and should not be confused with the variables \( a_1(q) \) and \( a_2(q) \)). To do this, we multiply both sides of eq. (6.13) by \( |q(q-1)| = |(a+1)a| \) (the spurious solutions at \( q = 0 \) and \( q = 1 \) thereby introduced are discarded) and use eq. (1.22) to get

\[
|a(a^k - a^{k-1} + 2(-1)^{k+1})| = |a(a^k + (-1)^{k+1})|
\]  

(6.14)
i.e., after dividing by \(|a|^{k+1}\),

\[
|1 - y + 2(-1)^{k+1}y^k| = |1 + (-1)^{k+1}y^k|
\]

(6.15)

Since \(y = 0\) is a solution of this equation, \(B\) is noncompact in the \(q\) plane, passing through \(y = z = 0\). In polar coordinates, with \(y = pe^{i\beta}\), eq. (6.13) yields

\[
\rho\left[3\rho^{2k-1} + \rho - 2 \cos \beta + 2(-1)^{k-1}\rho^{k-1}\left\{\cos(k\beta) - 2\rho \cos((k-1)\beta)\right\}\right] = 0
\]

(6.16)

As \(\rho \to 0\), it follows that \(\cos \beta = 0\), i.e. \(\beta = \pm \pi/2\), so that \(B\) approaches \(y = z = 0\) vertically.

To calculate the point at which the boundary \(B\) crosses the real \(q\), or equivalently, \(y\) or \(z\) axes, we set \(\beta = 0\) in eq. (6.16); for \(\rho \neq 0\) this gives

\[
3\rho^{2k-1} + \rho - 2 + 2(-1)^{k-1}\rho^{k-1}(1 - 2\rho) = 0
\]

(6.17)

(Note that \(Re(q) > 0 \iff Re(z) > 0\) and \(Re(q) > 1 \iff Re(y) > 0\) so that setting \(\beta = 0\), i.e. making \(y\) real and positive, implies that \(q\) is real and \(q > 1\); we comment later on the interval \(0 < q < 1\).) Since \(\rho = |y|\) is real and positive, we are only interested in roots of eq. (6.17) that have this property. For both even and odd \(k\), there is only one such root, which is thus \(y_c\) (equivalently, \(z_c\)), the minimum nonzero value of \(y\) (\(z\)) respectively, at which the boundary \(B\) crosses the positive real axis in the \(y\) (\(z\)) plane, corresponding to the maximal finite real point \(q_c\) at which the pre-image of this boundary \(B\) crosses the positive real axis in the \(q\) plane. For \(k\) odd, this single root is \(\rho = y = 1\) (the expression on the left-hand side contains a factor \((\rho - 1)\)) so that \(z_c = 1/2\), and

\[
q_c = 2 = \chi \quad \text{for} \quad k \quad \text{odd}
\]

(6.18)

For \(k\) even, the single such root of eq. (6.17) has a value that increases monotonically from \(\rho = y_c = 2/3\) (equiv. \(z_c = 2/5\)) for \(k = 2\) toward the limit \(\rho = y_c = 1\) (equiv. \(z_c = 1/2\)) as \(k \to \infty\) through even integers. Correspondingly, for \(k\) even, \(q_c\) decreases monotonically from \(5/2\) for \(k = 2\) toward \(2\) in the same limit; some values are \(q_c = 2.2564, 2.1736, 2.1315\) for \(k = 4, 6, 8\). Hence,

\[
2 < q_c < \frac{5}{2} < \chi = 3 \quad \text{for} \quad k \quad \text{even}
\]

(6.19)

To show that \(B\) does not cross the positive real \(q\) axis in the interval \(0 < q < 1\), we take \(\beta = \pi\), i.e., \(y\) real and negative, in eq. (6.16), which, for \(\rho \neq 0\), yields the equation

\[
3\rho^{2k-1} + \rho + 2 - 2\rho^{k-1} - 4\rho^k = 0
\]

(6.20)

This range \(-\infty < y < 0\) includes both the interval \(0 < q < 1\) and the interval \(-\infty < q < 0\). We find that in this range of \(y\), eq. (6.20) has no real positive roots for \(\rho\) except for \(\rho = 1\), i.e., \(y = -1\) or equivalently, \(q = 0\), but this is just a spurious solution introduced when we multiplied both sides of eq. (6.13) by \(|q(q - 1)|\) to express it as eq. (6.14). Hence, \(B\) crosses the positive \(z\) or equivalently \(q\) axis only once for this family of graphs.

We proceed to discuss the boundary \(B\) for specific values of \(k\) further. For \(k = 2\) \([\underline{15}]\) \(B\) is the vertical line with \(Re(q) = 5/2\) in the \(q\) plane, or equivalently, the circles \(|z - z_c|/2 = z_c/2\), \(|y - y_c|/2 = y_c/2\) in the \(z\) and \(y\) planes, where \(z_c = 2/5\), \(y_c = 2/3\) as given above.
FIG. 3. Boundary $B$ in the $z = 1/q$ plane for the $r \to \infty$ limit of the family of graphs $HEG_{k-2}(\overline{K}_2 + T_r)$ with $k =$ (a) 3 (b) 4. Chromatic zeros for $r = 16$ are shown for comparison.
FIG. 4. As in Fig. 3 for (a) 5 (b) 6. Chromatic zeros for \( r = (a) 10 \) (b) 8 are shown for comparison.

The boundary \( B \) is shown in the \( z \) plane for \( 3 \leq k \leq 6 \) in Figs. 3 and 4. In region \( R_1 \) containing the portion of the positive \( z \) axis analytically connected to the point \( z = 0 \),

\[
W(\lim_{r \to \infty} T_{k,r}, q) = (a_1)^{1/(k-1)} = (D_{k+1})^{1/(k-1)} \quad \text{for} \quad q \in R_1
\]
In region $R_2$ occupying the rest of the $z$ plane for $2 \leq k \leq 4$, and the rest of the $z$ plane except for the interiors of the other closed loops for $k \geq 5$, 

$$|W(\lim_{r \to \infty} T_{k,r}, q)| = |a_2|^{1/(k-1)} = |D_3 D_k + (-1)^{k-1}|^{1/(k-1)} \quad \text{for} \quad q \in R_2$$  \hspace{1cm} (6.22)

For odd $k$, as $q$ decreases toward 2 one has, using eqs. (6.21) and (11.4),

$$\lim_{q \searrow 2} W(\lim_{r \to \infty} T_{k,r}, q) = 1$$  \hspace{1cm} (6.23)

and, as $q$ increases toward 2, one has, using eq. (6.22),

$$\lim_{q \nearrow 2} |W(\lim_{r \to \infty} T_{k,r}, q)| = 1$$  \hspace{1cm} (6.24)

The equality of the right-hand sides of eqs. (6.23) and (6.24) follows from the fact that $q = 2$ is on the boundary $B$ for odd $k$. In each of the additional regions enclosed by closed curves,

$$|W(\lim_{r \to \infty} T_{k,r}, q)| = |a_1|^{1/(k-1)} = |D_{k+1}|^{1/(k-1)}$$  \hspace{1cm} (6.25)

For comparison, we show chromatic zeros for reasonably large finite values $r$ in each of these figures.

We find several general features of the boundary $B$ for the $r \to \infty$ limit of this family of graphs. The first feature is that $B$ has support only for non-negative $Re(z)$ or equivalently, non-negative $Re(q)$:

$$z \in B \Rightarrow Re(z) \geq 0 \quad \text{i.e.,} \quad Re(q) \geq 0$$  \hspace{1cm} (6.26)

Indeed, the only place on the boundary $B$ where $Re(z)$ vanishes is at the origin $z = 0$ itself:

$$(z \in B \quad \text{and} \quad Re(z) = 0) \Rightarrow z = 0$$  \hspace{1cm} (6.27)

Thus, in the vicinity of the point $z = 0$, the curve comprising $B$ bends to the right as one increases $Im(z)$ above zero or decreases it below zero; we have shown above that this curve crosses the point $z = 0$ vertically.

The second feature is that $B$ has no multiple points. The third feature concerns the number of different connected components $N_{\text{comp.}}$ comprising $B$. In the present case, this is simply related to the number of regions $N_{\text{reg.}}$ by the equation $N_{\text{reg.}} = N_{\text{comp.}} + 1$. One might hope that there would be a general mathematical theorem that would state the number of different connected components $N_{\text{comp.}}$ of the solution set of a given polynomial equation without requiring an explicit solution. However this question, which is related to the sixteenth Hilbert Problem, still remains unanswered. One can make use of an upper bound on $N_{\text{comp.}}$ contained in the Harnack theorem, which is $N_{\text{comp.}} \leq g + 1$, where $g$ denotes the genus of the algebraic curve. For plane algebraic curves of the type relevant here,

$$\sum_{m,n} c_{m,n} y^m R^n = 0$$  \hspace{1cm} (6.28)

with maximal degree $d$, where $y_R = Re(y)$, $y_I = Im(y)$, the genus is $g = (d-1)(d-2)/2$. Thus, for a given case, one writes the degeneracy equation (6.13) or its equivalent in the $z$ plane, out into cartesian components. The case $k = 2$ has already been discussed and leads to the equation $(y_R - 1/3)^2 + y_I^2 = (1/3)^2$, or equivalently, $(z_R - 1/5)^2 + z_I^2 = (1/5)^2$ where $z_R = Re(z)$ and $z_I = Im(z)$. These equations have homogeneous degree $d = 2$, hence genus $g = 0$, so Harnack’s theorem yields the bound $N_{\text{comp.}} \leq 1$ which, together with the fact that $N_{\text{comp.}} \geq 1$ implies that $N_{\text{comp.}} = 1$, in agreement with the elementary
solution of the explicit equation above. However, for higher values of $k$, the Harnack upper bound is not very restrictive. For $k \geq 3$, eq. (6.15), when written out in cartesian components, is

$$
[y^2 + y_1^2] F(y_2, y_1) = 0
$$

(6.29)

where $F(y_2, y_1) = 3(y_2^{2k-2} + y_1^{2k-2}) + $ lower order terms. Because the first factor is positive definite, eq. (6.15) thus reduces to $F(y_2, y_1) = 0$, of homogeneous degree $d = 2k - 2$ and hence genus $g = (2k - 3)(k - 2)$. The Harnack theorem then yields the upper bound $N_{\text{comp.}} \leq 2k^2 - 7k + 7$. As an example, for $k = 3, 4, 5, 6$, this bound has the respective values 4, 11, 22, 37, while the actual values are $N_{\text{comp.}} = 1, 1, 3, 3$, and so forth for higher $k$.

For both of the cases $k = 3, 4$, as was true for $k = 2$, $B$ divides the $z$ plane into two regions: (i) the region $R_1$ including the vicinity of the positive real axis contiguous with the origin, $z = 0$; and (ii) the region $R_2$ occupying the rest of the $z$ plane outside of $R_1$. In the $z$ plane, the chromatic zeros tend to cluster on the regions of the curve $B$ in the “northeast” and “southeast” directions. Indeed, as $k$ increases from 3 to 4, $B(z)$ develops protuberances in these “northeast” and “southeast” directions. Since for a given $r$ and $k$ the chromatic zeros are bounded, they avoid the origin of the $z$ plane, as can be seen in the figures. Between $k = 4$ and $k = 5$ there occurs a qualitative change in $B$, namely, that, whereas it consisted of a single component for $2 \leq k \leq 4$, it consists of three disconnected components for $5 \leq k \leq 7$, comprised of a self-conjugate closed curve together with a complex-conjugate pair of closed curves. Hence, while $B$ divides the $z$ plane into four regions for $2 \leq k \leq 4$, it divides this plane into four regions for $5 \leq k \leq 7$. In a figurative language, one can think of the northeast and southeast bulges that are evident in the $k = 4$ case as breaking off to form the two separate closed curves that first appear for $k = 5$. We also observe that the additional closed curves and associated disconnected regions appear to line up approximately along the vertical line with $\text{Re}(z) = 1/2$. It is interesting to note that this line maps to unit circles in the $y$ and $q$ planes:

$$
z : \text{Re}(z) = \frac{1}{2}, \text{Im}(z) \text{ arbitrary} \iff |y| = 1 \iff |q - 1| = 1
$$

(6.30)

Thus, the disconnected phases that appear in the $z$ plane are clustered around the unit circle $|y| = 1$ in the $y$ plane. In the $q$ plane, portions of the boundary $B$ are also clustered around the circle $|q - 1| = 1$ while one portion extends infinitely far from the origin.

**VII. THE FAMILY** $C_{k,r} = HEG_{k-2}(K_2 + C_r)$

We obtain a further infinite family of graphs with noncompact $B(q)$ by the same steps as in the previous section, but replacing the subgraph $T_r$ by the circuit subgraph $C_r$. That is, we homeomorphically expand the family $(K_p)_b + C_r$ for $r \geq 2$ by adding vertices on each of the bonds in the $C_r$ subgraph. As before, we let $k$ be the number of vertices on $C_r$ between each pair of vertices that connect with the $K_p$ subgraph, including this pair. We denote this family as

$$
C_{p,b,k,r} = HEG_{k-2}[(K_p)_b + C_r]
$$

(7.1)

where as above, $r \geq 2$ and $1 \leq b \leq p - 1$. For a given set of parameters $p$, $k$, and $r$, the number of vertices in this family is

$$
v(C_{p,b,k,r}) = r(k - 1) + p
$$

(7.2)
For the lowest value, \( k = 2 \),

\[
C_{p,b,k=2,r} = (K_p)_b + C_r
\quad (7.3)
\]

which we studied previously [13]; accordingly, we concentrate here on the cases \( k \geq 3 \). It suffices for our present purposes to consider the simplest nontrivial case \( p = 2 \) and hence \( b = 1 \), for which

\[
(K_2)_{b=1} + C_r = \bar{K}_2 + C_r
\quad (7.4)
\]

As before, a short notation is useful:

\[
C_{k, r} = C_{p=2,b=1,k,r} = HEG_{k-2}(\bar{K}_2 + C_r)
\quad (7.5)
\]

An illustration of a graph of this type is given in Fig. [3(b)]. The chromatic number and girth are the same as for \( T_{k,r} \), given by the right-hand sides of eqs. (5.4) and (5.7). By the deletion-contraction theorem, we find the recursion relation

\[
P(C_{k,r}, q) = D_k P(T_{k,r}, q) + (-1)^{k-1} P(C_{k,r-1}, q)
\quad (7.6)
\]

For the lowest value of \( r \), namely, \( r = 2 \), we have

\[
P(C_{k,2}, q) = \left[D_k + (-1)^{k-1}\right] P(T_{k,2}, q) + (-1)^k (D_3)^2 P(C_{k,2}, q)
\quad (7.7)
\]

Solving the recursion relation (7.6) with (7.7), we calculate the chromatic polynomial. As before, we express this in the form of eq. (1.10). (In order to render the polynomial property manifest, one divides through by the factors in the denominator, thereby generating a series, using the identity \((x^m - 1)/(x - 1) = \sum_{j=0}^{m-1} x^j\).)

We get

\[
P(C_{k,r}, q) = D_k\left[A_k a_2 \left(a_2 + (-1)^{k-1}\right) \left[a_2^{-2} - (-1)^{(k-1)r}\right] a_2^{-1} + q(q-1)a_2^2 \left(a_1 + (-1)^{k-1}\right) \left[a_1^{-2} - (-1)^{(k-1)r}\right] a_1^{-1} + (-1)^{(k-1)r} P(C_{k,2}, q)\right]
\quad (7.8)
\]

where \( A_k \) was defined in eq. (6.10). \( P(C_{k,r}, q) \) has the general factors \( q(q-1) \) for \( k \) odd and \( q(q-1)(q-2) \) for \( k \) even. \( P(C_{k,r}, q) \) has the form of eq. (1.10) with \( a_1 \) and \( a_2 \) as given in eqs. (6.11) and (6.12), together with

\[
a_3 = 1
\quad (7.9)
\]

For large \(|q|\) (small \(|z|\)), the terms \( a_1 \) and \( a_2 \) have larger magnitudes than \( a_3 \), so that in the limit \( r \to \infty \) (with \( k \) fixed [24]) the boundary \( B \) is determined by the degeneracy equation \(|a_1| = |a_2|\), and the same analysis goes through as before for the family \( HEG_{k-2}(\bar{K}_2 + T_r) \), with the same conclusions that \( B \) is noncompact in the \( q \) plane, passing through the point \( z = y = 0 \) vertically. For the lowest case \( k = 2 \), i.e., \( \bar{K}_2 + C_r \), \( B \) consists of three regions, as shown in Fig. [3]: \( R_1 \) including the real axis in the interval \( 0 \leq z \leq z_c \) with \( z_c = 1/3 \) as given by eq. (2.24); \( R_3 \) centered around \( z = 2/5 \) and occupying the interval \( z_c \leq z \leq 1/2 \) on the real axis; and \( R_2 \) occupying the complement of the \( z \) plane extending outward to \(|z| \to \infty \). This region
diagram may be contrasted with the simpler one for $\overline{K}_2 + T_r$, which was just a circle, separating the $z$ plane into two regions.

In Figs. 3 and 4 we show the boundary $\mathcal{B}$ in the $z$ plane for $3 \leq k \leq 6$. Again, for comparison, we show chromatic zeros for reasonably large finite values $r$ in each of these figures. The property that $\mathcal{B}$ passes through $z = 0$ vertically is evident in these figures, as are the properties (6.26) and (6.27). For each value of $k$, the region $R_1$ is the one occupying the non-negative interval of the $z$ axis adjacent to the origin. The region $R_2$ occupies the rest of the $z$ plane for $2 \leq k \leq 4$, and the rest of the $z$ plane except for the interiors of the other closed loops for $k \geq 5$. For odd $k$, there are two regions on the real $z$ axis: $R_1$ for $0 \leq z \leq 1/2$ and $R_2$ for $z < 0$ and $z > 1/2$, while for even $k \geq 4$ there are these phases and, in addition, a third “pocket” phase $R_3$ whose right-hand boundary with $R_2$ occurs at $z = 1/2$ and whose left-hand boundary with $R_1$ occurs at the point $z = z_c$ slightly less than $1/2$ (see further below).

As was the case for $\lim_{r \to \infty} [HEG_{k-2}(\overline{K}_2 + T_r)]$, we find

$$ W(\lim_{r \to \infty} C_{k,r}, q) = (a_1)^{1/(k-1)} = (D_{k+1})^{1/(k-1)} \quad \text{for } q \in R_1 $$

and

$$ |W(\lim_{r \to \infty} C_{k,r}, q)| = |a_2|^{1/(k-1)} = |D_k|^{1/(k-1)} \quad \text{for } q \in R_2 $$

Hence for $k = 3$ the equalities (6.23) and (6.24) hold, with $T_{k,r}$ replaced by $C_{k,r}$. For $k \geq 3$, additional regions appear as “outgrowths” on the right-hand boundary of $R_1$ in the $z$ plane, bounded on the left by $R_1$ and on the right by $R_2$. We label these regions generically as $R_{1-2,j}$, where $j \geq 3$ indexes the region. For example, there are two such regions for $k = 3$, which are complex-conjugates of each other, and there are three such regions for $k = 4$, consisting of a complex-conjugate pair and the region $R_3$ that includes an interval of the real $z$ axis between $z = 0.42495$ and $z = 1/2$. For arbitrary $k \geq 3$, in these outgrowth regions lying between $R_1$ and $R_2$, the leading term in the limit $r \to \infty$ is the last one in eq. (7.8), viz., $(-1)^{(k-1)}P(C_{k,2}, q)$, so that

$$ |W(\lim_{r \to \infty} C_{k,r}, q)| = 1 \quad \text{for } q \in R_{1-2,j} $$

For $k \geq 5$, additional sets of regions occur that are not connected with $R_1$; these consist of complex-conjugate pairs, lying near to the vertical line in the $z$ plane with $Re(z) = 1/2$. For the cases that we have studied, each of these disconnected sets of regions actually consists of two: for the ones with $Im(z) > 0$, one part occupying a “northeast” position and the other a “southwest” position, as is evident in the figures. We label these regions disconnected from $R_1$ as $R_{disc,NE;j}$, $R_{disc,SW;j}$, and their complex-conjugates as $R_{disc,NE;j}^*$, $R_{disc,SW;j}^*$. We find that

$$ |W(\lim_{r \to \infty} C_{k,r}, q)| = |a_1|^{1/(k-1)} = |D_{k+1}|^{1/(k-1)} \quad \text{for } q \in R_{disc,SW;j}, R_{disc,SW;j}^* $$

$$ |W(\lim_{r \to \infty} C_{k,r}, q)| = 1 \quad \text{for } q \in R_{disc,NE;j}, R_{disc,NE;j}^* $$

As was noted before in the case of the $r \to \infty$ limit of the family $T_{k,r}$, this clustering of the disconnected phases along the vertical line $Re(z) = 1/2$ is equivalent to their clustering around the unit circle in the $y$ plane.

For odd $k$, $\mathcal{B}$ crosses the real $z$ axis away from $z = 0$ (equivalently the real $q$ axis) once, at $z = 1/2$ (i.e., $q = 2$). For even $k$, $\mathcal{B}$ crosses the real $z$ axis at two points away from the origin $z = 0$: (i) at $z = 1/2$, and
(ii) at a value of \( z \) that increases monotonically from \( z = 1/3 \) for \( k = 2 \), approaching \( z = 1/2 \) as \( k \to \infty \) through even integers. Hence, taking into account that

\[
\chi(C_{k,r}) = \begin{cases} 
2 & \text{for } k \text{ odd} \\
3 & \text{for } k \text{ even}
\end{cases}
\]  

we have

\[
q_c = 2 = \chi \quad \text{for odd } \ k \tag{7.16}
\]

while

\[
2 < q_c \leq 3 = \chi \quad \text{for even } \ k \tag{7.17}
\]

Equations (7.16) and (7.17) may be compared with the analogous eqs. (6.18) and (6.19) for the family \( T_{k,r} = HEG_{k-2}(R_2 + T_r) \). To derive the above results, we observe first that for the case of odd \( k \), this follows from the same analysis as applied to the \( r \to \infty \) limit of the family \( HEG_{k-2}(R_2 + T_r) \) in the previous section, since the crossing is determined by the degeneracy condition \( |a_1| = |a_2| \). For even \( k \), the crossing point at the smaller value of \( z \) is determined by the degeneracy condition of leading terms

\[
\rho^{2k-2} - \rho^{2k-2} + \rho^2 + 1 - 2\rho(\rho^{2k-2} + 1) \cos \beta = 0 \tag{7.18}
\]

For \( \beta = 0 \), this equation always has a root at \( \rho = y_c \), that increases monotonically from 1/2 (i.e., \( z_c = 1/3 \)) for \( k = 2 \) through 0.738984 (\( z = 0.424951 \)) for \( k = 4 \), toward \( \rho = y_c = 1 \) (\( z_c = 1/2 \)) as \( k \) goes to infinity through even values; equivalently, \( q_c \) decreases from 3 at \( k = 2 \) toward 2 as \( k \to \infty \) through even values. Some explicit values are \( q_c = 2.35321, 2.21486, 2.15442 \) for \( k = 4, 6, 8 \). The crossing point at the larger value of \( z \) (for even \( k \)), viz., \( z = 1/2 \), is determined by the degeneracy condition of leading terms \( |a_2| = |a_3| \). Multiplying by \( |q(q - 1)| \) and reexpressing this in terms of \( y \) as before, we obtain

\[
\frac{1 - y + 2(-1)^{k-1}y^k}{1 + y} \cos((k - 1)\beta) = 0 \tag{7.19}
\]

The number of multiple points on \( \mathcal{B} \) is

\[
N_{m,p} = 2(k-1), \quad \text{for even } \ k \tag{7.20}
\]
comprised of $k - 1$ complex-conjugate pairs, and

$$N_{m.p.} = 2k - 3, \quad \text{for odd } k \quad (7.22)$$

consisting of a real one at $z = 1/2$, i.e., $q = 2$, together with $k - 2$ complex-conjugate pairs.

It is interesting to compare the boundaries $\mathcal{B}$ and associated region diagrams for the $r \to \infty$ limits of the two families of graphs $HEG_{k-2}(\overline{K}_2 + T_r)$ and $HEG_{k-2}(\overline{K}_2 + C_r)$ for a given value of $k$. We have done this above for $k = 2$. For $k \geq 3$, one sees that the diagrams for $HEG_{k-2}(\overline{K}_2 + C_r)$ look somewhat similar to those for $HEG_{k-2}(\overline{K}_2 + T_r)$ with the differences that (i) there are additional “outgrowth” regions contiguous to the right-hand side of $R_1$; (ii) the additional disconnected regions that appear for $k \geq 5$ are themselves composed of additional regions; (iii) $\mathcal{B}$ contains multiple points; and (iv) for even $k$ there are three, rather than just two regions along the real $z$ axis. Owing to the presence of the multiple points on $\mathcal{B}$ (i.e. singular points, in the sense of algebraic geometry), the analysis of bounds on the number of disconnected components, $N_{\text{comp}}$, is more difficult than for algebraic curves without singular points, such as the curves $\mathcal{B}$ for $\lim_{r \to \infty} T_{k,r}$. 

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FIG. 5. Boundary $B$ in the $z = 1/q$ plane for the $r \to \infty$ limit of the family of graphs $HEG_{k-2}(K^2 + C_r)$ with $k$ = (a) 3 (b) 4. Chromatic zeros for $r = 12$ are shown for comparison.
FIG. 6. As in Fig. 5 for \( k = (a) 5 \text{ (b) 6} \). Chromatic zeros for \( r = (a) 10 \text{ (b) 8} \) are shown for comparison.

VIII. THE FAMILY \( S_{k,r} = HEG_{k-2}(\overline{K}_3 + T_r) \)

To illustrate one infinite family of graphs with noncompact \( \mathcal{B} \) of the type \( HEG_{k-2}(\overline{K}_p + G_r) \) with \( p > 2 \), we discuss an infinite family constructed by homeomorphic expansion of the bonds of the subgraph \( G_r = T_r \).
in the graph \((\overline{K}_3 + T_r)\). The beginning family, \(\overline{K}_3 + G_r\), is of type (3.3) (and can be obtained by complete removal of all three of the bonds in the \(K_3\) subgraph in \(K_3 + G_r\)). We denote this family as

\[ S_{k,r} = HEG_{k-2}(\overline{K}_3 + T_r) \]  

(8.1)

The number of vertices is given by

\[ v(S_{k,r}) = (r - 1)(k - 1) + 4 \]  

(8.2)

An example of a graph of this type is shown in Fig. 2(c). The chromatic number and girth are the same as those of the graph \(T_{k,r}\), given in eqs. (6.6) and (6.7). By the deletion-contraction theorem, we find the recursion relation

\[ P(S_{k,r}, q) = \left( (q - 3)D_k + (-1)^{k-1} \right) P(S_{k,r-1}, q) + 3D_k P(T_{k,r-1}, q) - q(q - 1)D_k(D_{k+1})^{r-2} \]  

(8.3)

Solving this, we obtain the chromatic polynomial

\[ P(S_{k,r}, q) = 3 \left[ (q - 1)D_{k+1} - P(T_{k,2}, q) \right] \left[ a_3^{r-2} - a_2^{r-2} \right] + q(q - 1)a_1 \left[ a_1^{r-2} - a_3^{r-2} \right] \]

\[ + a_3^{r-2} P(S_{k,2}, q) \]  

(8.4)

where \(a_1\) and \(a_2\) were given in eqs. (5.11) and (5.12),

\[ a_3 = (q - 3)D_k + (-1)^{k-1} \]  

(8.5)

and

\[ P(S_{k,2}, q) = (q - 2)P(T_{k,2}, q) + q(q - 1)^3D_{k-1} \]  

(8.6)

In the limit \(r \to \infty\) with \(k\) fixed [32], the locus \(B\) is determined by the degeneracy of magnitudes of leading terms

\[ |a_1| = |a_3| \]  

(8.7)

(\(a_2\) is never a leading term in this case). The degeneracy equation in the \(y\) variable takes the form

\[ |1 - 2y + 3(-1)^{k+1}y^{k}| = |1 + (-1)^{k+1}y^{k}| \]  

(8.8)

Since \(y = 0\) is a solution of this equation, \(B\) is noncompact in the \(q\) plane, passing through \(y = z = 0\). In polar coordinates, with \(y = \rho e^{i\beta}\), eq. (8.8) yields

\[ \rho \left[ 2\rho^{2k-1} + \rho - \cos \beta + (-1)^{k-1}\rho^{k-1} \left\{ \cos(k\beta) - 3\rho \cos((k - 1)\beta) \right\} \right] = 0 \]  

(8.9)

As \(\rho \to 0\), it follows that \(\cos \beta = 0\), i.e. \(\beta = \pm \pi/2\), so that \(B\) approaches \(y = z = 0\) vertically.

To calculate the point at which the boundary \(B\) crosses the real \(q\), or equivalently, \(y\) or \(z\) axes, we set \(\beta = 0\) in eq. (8.9): for \(\rho \neq 0\) this gives

\[ 2\rho^{2k-1} + \rho - 1 + (-1)^{k-1}\rho^{k-1}(1 - 3\rho) = 0 \]  

(8.10)
For both even and odd $k$, eq. (8.10) has only one acceptable (real positive) root for $\rho$, which is thus $y_c$. For odd $k$ this root is $y_c = 1$ (equivalently, $z_c = 1/2$, $q_c = 2$). For even $k$, the real positive crossing point increases monotonically from $y_c = 0.647799$ ($z_c = 0.39313$) for $k = 4$ to $y_c = 1$ ($z_c = 1/2$) as $k$ goes to infinity through even values; i.e., $q_c$ decreases monotonically from 2.54369 to 2 over the same range.

In Fig. 7 we show the respective boundaries $B$ in the $z$ plane for $k = 3, 4$. Note that for $k = 3, B$ is close to being, but is not, a circle. Again, for comparison, we show chromatic zeros for reasonably large finite values of $r$ in each of these figures. As is evident from Fig. 7 and from the figures for higher values of $k$, $B$ for $\lim_{r \to \infty} S_{k,r}$ displays some similarities with that for $\lim_{r \to \infty} T_{k,r}$. Hence we do not show these figures for higher values of $k$ here. We remark, however, that there are some differences; for example, the number of connected components, $N_{\text{comp.}}$, is equal to the respective values 1,1,3,3,5,5 for $3 \leq k \leq 8$ for $\lim_{r \to \infty} T_{k,r}$ but is equal to 1,1,3,3,5,5 for the present case, $\lim_{r \to \infty} S_{k,r}$ with $3 \leq k \leq 8$. As was true with $N_{\text{comp.}}$ for $\lim_{r \to \infty} T_{k,r}$, for $k \geq 4$, the values of $N_{\text{comp.}}$ are only weakly bounded by the Harnack theorem.

For each value of $k$, the region $R_1$ is the one occupying the non-negative interval of the $z$ axis adjacent to the origin. The region $R_2$ occupies the rest of the $z$ plane for $2 \leq k \leq 4$, and the rest of the $z$ plane except for the interiors of the other closed loops for $k \geq 5$. We find

$$W(\lim_{r \to \infty} S_{k,r}, q) = (a_1)^{1/(k-1)} = (D_{k+1})^{1/(k-1)} \quad \text{for } q \in R_1$$  \hspace{1cm} (8.11)

and

$$|W(\lim_{r \to \infty} S_{k,r}, q)| = |a_3|^{1/(k-1)} = |(q - 3)D_k + (-1)^{k-1}|^{1/(k-1)} \quad \text{for } q \in R_2$$  \hspace{1cm} (8.12)

For $k = 5, 6$ ($k = 7, 8$) there is one pair (are two pairs) of complex-conjugate closed loops disconnected from $R_1$, which we denote $R_{\text{disc},j}$ and $R^*_{\text{disc},j}$. In these regions we find

$$|W(\lim_{r \to \infty} S_{k,r}, q)| = (a_1)^{1/(k-1)} = (D_{k+1})^{1/(k-1)} \quad \text{for } q \in R_{\text{disc},j}, R^*_{\text{disc},j}$$  \hspace{1cm} (8.13)

The feature observed in the $r \to \infty$ limits of the families $T_{k,r}$ and $C_{k,r}$ that the disconnected regions $R_{\text{disc},j}$ and $R^*_{\text{disc},j}$ tend to lie along the line $z = 1/2$ is also present in this case.
FIG. 7. Boundary $\mathcal{B}$ in the $z = 1/q$ plane for the $r \to \infty$ limit of the family of graphs $HEG_{k-2}(\overline{K}_3 + T_r)$ with $k =$ (a) 3 (b) 4. Chromatic zeros for $r = 14$ are shown for comparison.

IX. GENERAL CONDITION FOR NONCOMPACT $\mathcal{B}$

The key ingredient to construct families of graphs with noncompact $W$ boundaries $\mathcal{B}$ in the $q$ plane and resultant reduced functions $W_{\text{red}}(\{G\}, q)$ that are nonanalytic at $z = 1/q = 0$ is to produce a chromatic
polynomial with the feature that there are two leading terms with a degeneracy condition\(^{(2.6)}\) that has a solution at \(z = 0\) (cf. eq. (2.9)). A general statement of this condition was given as the theorem of Section IV of Ref. [15]. From our studies in the present paper, we can add some geometrical comments to this algebraic condition.

An important property of our families of graphs with noncompact \(W\) boundaries is that none of these families is a regular lattice graph. This is in accord with the derivation of the large–\(q\) expansion for regular lattices [25,26]. (Of course, the property that \(\{G\}\) is a regular lattice is not a necessary condition for the associated boundary \(\mathcal{B}\) to be compact in the \(q\) plane and hence \(W_{\text{red}}(\{G\}, q)\) to be analytic at \(z = 0\).) A basic feature of a regular lattice graph is that (except for boundary vertices, which yield a negligible effect in the thermodynamic limit and are absent if one uses periodic boundary conditions) all vertices have the same coordination number (= degree \(\Delta\) in usual graph theory terminology). A generalization of this is encountered in duals of Archimedean lattices, such as the \([4 \cdot 8^2]\) (union-jack) lattice, where the vertices fall into a finite number of sets with different coordination numbers (e.g. \(\Delta = 4\) and \(\Delta = 8\) for the \([4 \cdot 8^2]\) lattice).

In both of these classes of lattices, there are no vertices with infinite degree.

In contrast, a property of the families of graphs that we have studied with noncompact loci \(\mathcal{B}(q)\) in the limit \(r \to \infty\) is that in this limit they all contain an infinite number of different, non-overlapping (and non-self-intersecting) circuits, each of which passes through at least two fixed, nonadjacent vertices. This immediately implies that these aforementioned nonadjacent vertices have degrees \(\Delta\) that go to infinity in this limit. In the families that we constructed earlier [13] and some of the additional families discussed in sections II and III of the present work, this property was produced by starting with a family of the form \(K_p + G_r\) and removing one or more bonds from the \(K_p\) subgraph, thereby rendering two or more vertices that were originally adjacent no longer adjacent. As we showed in Section IV, it can also be produced by homeomorphically adding one or more vertices to a bond in the \(K_p\) subgraph, thereby again rendering two vertices that were originally adjacent no longer so. It should be mentioned that the condition of having two or more vertices with \(\lim_{r \to \infty} \Delta(r) = \infty\) is not, by itself, sufficient to produce a noncompact \(\mathcal{B}(q)\). For example, consider the family of “\(p\)-wheels”

\[
(Wh)_{n}^{(p)} = K_p + C_r
\]

(9.1)

(where \(n = p + r\) is the number of vertices) that we constructed in Ref. [13]. The degree of each of the \(p\) vertices in the \(K_p\) subgraph of the graph \(K_p + C_r\) is \(\Delta = p - 1 + r\), so that in the limit \(r \to \infty\) (with \(p\) fixed), this degree \(\Delta \to \infty\). However, the corresponding boundary \(\mathcal{B}\) is compact; specifically, we showed that it is the unit circle \(|q - (p + 1)| = 1\) [14]. Furthermore, we observe that the presence of non-adjacent vertices with degrees \(\Delta\) such that \(\lim_{r \to \infty} \Delta = \infty\) is also not, by itself sufficient to guarantee that \(\mathcal{B}(q)\) is noncompact. This is easily seen by considering, for example, two \(p = 1\) \(p\)-wheel graphs, \((Wh)_{n}^{(1)}\) whose central vertices are connected by a tree graph containing at least two bonds. As \(n \to \infty\), the degrees \(\Delta\) of the central vertices go to infinity, but again the resultant \(W\) boundary \(\mathcal{B}\) is compact; indeed, it is the same as for a single \(p = 1\) \(p\)-wheel, viz., \(|q - 2| = 1\). Thus, a necessary feature is observed to be the presence, in the limit as the number of vertices of the graph goes to infinity, of an infinite number of different, non-overlapping (non-self-intersecting) circuits, each of which pass through the two or more nonadjacent vertices.
X. CONCLUSIONS

In this paper we have further explored a fundamental problem in statistical mechanics – nonzero ground state entropy – using as a theoretical laboratory the \(q\)-state Potts antiferromagnet. We have presented a number of exact calculations of the zero-temperature partition function \(Z(G, q, T = 0)\) or equivalently the chromatic polynomial \(P(G, q)\), and the corresponding limiting function representing the ground-state degeneracy, \(W(\{G\}, q)\), for this model on various families of graphs \(G\) for which the boundary \(B\) of regions of analyticity of \(W\) in the complex \(q\) plane is noncompact, passing through \(z = 1/q = 0\). The study of these graphs thus gives insight into the conditions for the validity of the large-\(q\) Taylor series expansions of the reduced function \(W_{\text{red}}(\{G\}, q)\). In addition to families obtained by removal of bonds from nonadjacent vertices in the \(K_p\) subgraph of \(K_p + G_r\), we have constructed and investigated a number of families of graphs by the powerful method of homeomorphic expansion from respective starting families. We have shown how the families thus obtained have, in the limit of an infinite number of vertices, noncompact boundaries in the \(q\) plane which pass vertically through the origin of the \(z = 1/q\) plane and have support for \(\text{Re}(q) \geq 0\).

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XI. APPENDIX 1

In this Appendix we gather together some convenient formulas concerning the function \(D_k\), defined in eq. (1.19) in the text. One has

\[
D_k(q = 0) = (-1)^k (k - 1)
\]

(11.1)

and

\[
D_k(q = 1) = (-1)^k
\]

(11.2)

Since for \(k\) even, the circuit graph \(C_k\) is bipartite, which is equivalent to the fact that the chromatic number \(\chi(C_{k \text{ even}}) = 2\), and thus \(P(C_{k \text{ even}}, q = 2) = 2\), it follows that

\[
D_{k \text{ even}}(q = 2) = 1
\]

(11.3)

Since for \(k\) odd, \(\chi(C_{k \text{ odd}}) = 3\) and \(P(C_{k \text{ odd}}, q = 2) = 0\), we have

\[
D_{k \text{ odd}}(q = 2) = 0
\]

(11.4)

This zero results from a linear factor, i.e.,

\[
D_{k \text{ odd}}(q = 2) = (q - 2)\text{Pol}(q)
\]

(11.5)

where \(\text{Pol}(q)\) is a polynomial of degree \(k - 3\) in \(q\) with \(\text{Pol}(q = 2) \neq 0\). Two identities that we have derived and used for our calculations are listed below:
\[ D_k - aD_{k-1} = (-1)^k \]  
(11.6)

\[ D_{k+1} - D_k = D_3D_k + (-1)^{k+1} \]  
(11.7)

The proofs follow immediately from the definition of \( D_k \).
We recall the strict mathematical definition of a graph as a finite collection of vertices and bonds connecting (all or a subset of the) vertices, such that there are (i) no loops, i.e., bonds connecting a vertex to itself and (ii) no multiple bonds connecting a given pair of vertices. We denote the limit as the number of vertices of a family of graphs $G$ goes to infinity as $\{G\}$.

Relevant earlier papers on $\mathcal{B}$ are Refs. [20]–[23]. (The locus $\mathcal{B}$ for ladder graphs given in Refs. [21] and [22] were not quite correct; the correct locus was given in Ref. [13]; see also the comment in Ref. [18]).

It sometimes happens that a portion of $\mathcal{B}$ is comprised by a line segment $q_1 \leq q \leq q_2$ lying on the positive real $q$ axis and, furthermore, that either (i) no part of $\mathcal{B}$ crosses the positive real $q$ axis, so no crossing point $q_c$ is defined, or (ii) a part of $\mathcal{B}$ does cross the positive real $q$ axis at a value $q_c$, but $q_c < q_2$, so that, as one moves down the positive real $q$ axis toward smaller $q$, the first nonanalyticity that $W\{\{G\}, q\}$ has occurs at $q_2$ rather than at $q_c$. From our studies of many families of graphs, we have found a few examples where this type of behavior occurs, such as for the infinitely long strip of the square lattice of width $w = 4$ vertices ($w = 3$ squares) $\mathcal{B}$, an infinitely long strip of the $3^4 \cdot 4^2$ Archimedean lattice of width $L_y = 3$ vertices, and a homeomorphic expansion of a width $L_y = 2$ strip of the triangular lattice.

The complete graph $K_n$ on $n$ vertices is defined as the graph where each of these vertices is connected to all of the other $n - 1$ vertices by bonds. The complement $\overline{G}$ of a graph $G$ contains the same set of vertices such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in $G$. The complement $\overline{K_n}$ of $K_n$ is thus the graph with $n$ vertices and no bonds.

R. C. Read and E. G. Whitehead, Discrete Math, in press. We thank Professor Read for kindly giving us a copy.
We recall the statement of the addition-contraction theorem: let $G$ be a graph, and let $v$ and $v'$ be two non-adjacent vertices in $G$. Form (i) the graph $G_{\text{add.}}$ in which one adds a bond connecting $v$ and $v'$, and (ii) the graph $G_{\text{contr.}}$ in which one identifies $v$ and $v'$. Then the chromatic polynomial for coloring $G$ with $q$ colors, $P(G,q)$, satisfies $P(G,q) = P(G_{\text{add.}},q) + P(G_{\text{contr.}},q)$.

In the technical terminology of algebraic geometry, a multiple point on an algebraic curve is a point where several branches of the curve intersect. If all $n_i$ of the branches have different tangents, the multiple point is said to have index $n_i$. Additional theorems and results can be found in the references provided. For recent discussions, see, e.g., I. Itenberg, C. R. Acad. Sci. Paris 317 I (1993) 277; G. Mikhalkin, in Contemp. Math. 182 (1995) 73; and V. Kharlamov et al., eds., AMS Translations v. 173 (American Mathematical Society, Providence, 1996).