Second order properties and central limit theorems for geometric functionals of Boolean models

Daniel Hug*, Günter Last†, and Matthias Schulte‡

May 11, 2014

Abstract

Let $Z$ be a Boolean model based on a stationary Poisson process $\eta$ of compact, convex particles in Euclidean space $\mathbb{R}^d$. Let $W$ denote a compact, convex observation window. For a large class of functionals $\psi$, formulas for mean values of $\psi(Z \cap W)$ are available in the literature. The first aim of the present work is to study the asymptotic covariances of general geometric (additive, translation invariant, and locally bounded) functionals of $Z \cap W$ for increasing observation window $W$, including convergence rates. Our approach is based on the Fock space representation associated with $\eta$. For the important special case of intrinsic volumes, the asymptotic covariance matrix is shown to be positive definite and can be explicitly expressed in terms of suitable moments of (local) curvature measures in the isotropic case. The second aim of the paper is to prove multivariate central limit theorems including Berry-Esseen bounds. These are based on a general normal approximation result obtained by the Malliavin-Stein method.

Key words: Boolean model, intrinsic volume, additive functional, curvature measure, covariance matrix, central limit theorem, Fock space representation, Malliavin calculus, Wiener-Itô chaos expansion, Berry-Esseen bound, integral geometry

MSC (2010): Primary: 60D05; Secondary: 60F05, 60G55, 60H07, 52A22

1 Introduction

Let $\eta$ be a stationary (locally finite) Poisson process on the space $K^d$ of convex bodies in $\mathbb{R}^d$, that is, on the space of compact, convex subsets of $\mathbb{R}^d$. The Boolean model associated with $\eta$ is the stationary random closed set $Z$ defined by

$$Z := \bigcup_{K \in \eta} K,$$

(1.1)

where the Poisson process $\eta$ is identified with its support. This is a fundamental model of stochastic geometry and continuum percolation with many applications in materials science and physics [3, 7, 17, 20, 28]. The intersection of $Z$ with a compact and convex set $W \subset \mathbb{R}^d$ is a finite union of compact, convex sets, that is, an element of the convex ring $\mathcal{R}^d$. It is a common strategy in stochastic geometry to extract and explore local information about $Z$.

*Institute of Stochastics, Karlsruhe Institute of Technology, 76128 Karlsruhe, Germany, daniel.hug@kit.edu
†Institute of Stochastics, Karlsruhe Institute of Technology, 76128 Karlsruhe, Germany, guenter.last@kit.edu
‡Institute of Stochastics, Karlsruhe Institute of Technology, 76128 Karlsruhe, Germany, matthias.schulte@kit.edu
via functionals of the intersections $Z \cap W$. Perhaps the most prominent examples of such functionals on $\mathcal{R}^d$ are the intrinsic volumes $V_0, \ldots, V_d$, which contain important geometric information about the sets to which they are applied. For instance, for a set $K \subset \mathbb{R}^d$ from the convex ring, $V_d(K)$ is the volume, $V_{d-1}(K)$ is half the surface area, and $V_0(K)$ is the Euler characteristic of $K$; see [28] for more details. The intrinsic volumes have several desirable properties. In particular, they are additive, in the sense that $V_i(K \cup L) = V_i(K) + V_i(L) - V_i(K \cap L)$ for all $K, L \in \mathcal{R}^d$ and $i \in \{0, \ldots, d\}$. In addition, they are translation invariant, and continuous if restricted to the space of convex bodies.

For a stationary and isotropic Boolean model, Miles [18] and Davy [5] obtained explicit formulas expressing the mean values $\mathbb{E}V_i(Z \cap W)$ in terms of the intensity measure of $\eta$. We refer to [28] for a discussion and more recent developments related to this fundamental result.

In the following, we are especially interested in second order properties and central limit theorems of the random vector $(V_0(Z \cap W), \ldots, V_d(Z \cap W))$, for a compact and convex observation window $W$, but in fact we study more general additive functionals of $Z \cap W$, namely so called geometric functionals. A functional on the convex ring will be called geometric if it is additive, translation invariant, locally bounded and measurable (see Section 3 for details).

While previous contributions focus on second order properties and central limit theorems for volume and surface area, to the best of our knowledge we present here the first systematic mathematical investigation of second order properties and central limit theorems of all intrinsic volumes and more general geometric functionals of a stationary Boolean model $Z$. The volume functional was first studied in [2] [15], while in [8] Berry-Esseen bounds and large deviation inequalities were established. The surface area was investigated in [19], and the results were extended in [9] to more general functionals and point processes. Integrals over Boolean models are considered in [11] [24], where the volume is included as a special case and also the surface area in the latter one. Volume and surface area of a more general Boolean model based on a Poisson process of cylinders have been investigated in [10] [11]. From a geometric point of view, volume and surface area are rather special functionals of $Z$. They arise as the restriction of deterministic measures to $Z$ or the boundary of $Z$ and do not involve the curvature of the (possibly intersecting) grains. A different though mathematically non-rigorous treatment of second moments of curvature measures of an isotropic Boolean model with an interesting application to morphological thermodynamics was presented in [16].

Our first main aim in this paper is to use the Fock space representation of Poisson functionals [28] to explore the covariance structure of geometric functionals of $Z \cap W$. Combined with some new integral geometric inequalities, which are derived by methods and results from convex and integral geometry, this approach appears to be perfectly tailored to our purposes. Under the minimal assumption that the second moments of the intrinsic volumes of the typical grain are finite, we show that for two geometric functionals $\psi_1, \psi_2$, $V_d(W)^{-1}\text{Cov}(\psi_1(Z \cap W), \psi_2(Z \cap W))$ tends to some $\sigma_{\psi_1,\psi_2} \in \mathbb{R}$ as the observation window is increased in a proper way. For the case that the third moments of the intrinsic volumes of the typical grain are finite we establish a rate of this convergence in terms of the inradius of the observation window and show that it is optimal. Via the Fock space representation the asymptotic covariances can be expressed as series of second moments. In the important case of intrinsic volumes of an isotropic Boolean model they can be represented in terms of curvature based moment measures of the typical grain. In particular, the covariance structure of the two-dimensional isotropic Boolean model becomes surprisingly explicit. For a vector of geometric functionals of the Boolean model, it is shown that the asymptotic covariance matrix is positive definite under some additional conditions, which are for example satisfied for the intrinsic volumes. The second order analysis is illustrated by explicit formulas.
for intrinsic volumes of a Boolean model with deterministic spherical grains, for which our formulas reduce to three-dimensional integration of explicitly known integrands.

Our second main aim is to prove univariate and multivariate central limit theorems for geometric functionals of \( Z \cap W \). Under the same second moment assumptions as for the existence of the asymptotic covariances, we prove that convergence in distribution takes place. In the univariate case we do not need to assume that the functional on the convex ring is translation invariant. We also obtain rates of convergence under slightly stronger moment assumptions. For the multivariate central limit theorem we provide an argument that the rate is optimal. Following common belief, we guess that our convergence rate \( V_d(W)^{-1/2} \) for the univariate case is optimal as well. In the proofs we use the Malliavin-Stein method for Poisson functionals that was recently developed in [21, 23]. In a sense this method builds on the Fock space representation and the closely related Wiener-Itô chaos expansion of Poisson functionals. The main obstacle to the application of these results is the fact that, as a rule, geometric functionals of \( Z \) admit an infinite chaos expansion. We can resolve this by bounding the kernels of the chaos expansion in a monotone way.

It is conceivable that the central limit theorem can also be derived with the techniques in [9], namely by approximation with \( m \)-dependent random fields using a truncation argument. But such an approach would neither yield much information on the asymptotic covariance structure nor rates of convergence. Stabilization is another common approach to central limit theorems in stochastic geometry; see e.g. [1, 24]. It is unclear whether the intrinsic volumes (other than volume or surface area) stabilize for Boolean models with unbounded grains. Moreover, in our setting we would need to control boundary effects.

This paper is organized as follows. In the second section we briefly summarize some notation and basic facts about the Boolean model. In the third section we establish the existence of the asymptotic covariances of a vector of geometric functionals of \( Z \cap W \) and determine the rate of convergence; see Theorem 3.1. Section 4 is devoted to the positive definiteness of the asymptotic covariance matrix; see Theorem 4.1. In Section 5 we focus on intrinsic volumes and introduce a family of curvature based moment measures of the typical grain to study infinite series of second moments arising in the Fock space representation. The main result of this section (Theorem 5.2) is of some independent interest and is applied in Section 6 to derive formulas for the asymptotic covariances of the intrinsic volumes of an isotropic Boolean model in terms of the moment measures mentioned above; see Theorem 6.1. Section 7 presents some numerical results for a Boolean model with deterministic spherical grains. In Section 8 we present a general result on the normal approximation of Poisson functionals. We use this result in Section 9 to establish multivariate and univariate central limit theorems for geometric functionals of \( Z \); see Theorem 9.1 and Theorem 9.3. Appendix A contains the proof of a crucial geometric result which is applied in Section 5 (see Lemma 5.3), while Appendix B provides a description of the curvature based moment measures from Section 5 in terms of mixed measures of translative integral geometry. Appendix C contains integral formulas for the exact (non-asymptotic) covariances of intrinsic volumes, which are rather explicit in the two-dimensional case; see Theorem 12.1 and Corollary 12.2.

2 Preliminaries on the Boolean model

In this section we collect a few basic facts about the stationary Poisson process \( \eta \) and the associated Boolean model \( Z \) in Euclidean space \( \mathbb{R}^d \). All random objects occurring in this paper are defined on an abstract probability space \((\Omega, \mathcal{F}, \mathbb{P})\). We consider the Poisson process \( \eta \) as a random element in the space \( \mathcal{N} \) of all locally finite counting measures on \( \mathcal{K}^d \). Here
locally finite means that every compact subset of $\mathbb{R}^d$ is only intersected be finitely many grains. Let $\mathcal{N}$ be equipped with the smallest $\sigma$-field such that the mappings $\mu \mapsto \mu(C)$ are measurable for all $C$ in the Borel $\sigma$-field of $\mathcal{C}^d$, the space of compact subsets of $\mathbb{R}^d$. The intensity measure $\Lambda := \mathbb{E}\eta$ of $\eta$ is translation invariant and therefore (see Theorem 4.1.1 in [28]) of the form

$$\Lambda(\cdot) = \gamma \int \mathbf{1}_{\{ K + x \in \cdot \}} \, dx \, \mathbb{Q}(dK), \quad (2.1)$$

where we assume that the intensity $\gamma$ is positive and finite, $'dx'$ denotes integration over $\mathbb{R}^d$ with respect to $d$-dimensional Lebesgue measure $\lambda_d$, and $\mathbb{Q}$ is a probability measure on $\mathcal{K}_d$ satisfying

$$\int V_d(K + C) \, \mathbb{Q}(dK) < \infty, \quad C \in \mathcal{C}^d. \quad (2.2)$$

Here, as usual, $K + C := \{ x + y : x \in K, y \in C \}$ is the Minkowski sum of $K$ and $C$, and $K + x := K + \{x\}$, for $x \in \mathbb{R}^d$. Let $Z_0$ denote a typical grain, that is, a random convex set with distribution $\mathbb{Q}$. Assumption (2.2) is equivalent to

$$v_i := \mathbb{E}V_i(Z_0) < \infty, \quad i = 0, \ldots, d. \quad (2.3)$$

The intrinsic volumes are determined by the Steiner formula

$$V_d(K + B^d_r) = \sum_{i=0}^{d} \kappa_{d-i} r^{d-i} V_i(K), \quad r \geq 0, \quad K \in \mathcal{K}_d, \quad (2.4)$$

where $B^d$ is the closed unit ball centred at the origin, $B^d_r := \{ rx : x \in B^d \}$, and $\kappa_n$ denotes the volume of the $n$-dimensional unit ball.

The Boolean model is given by $Z \equiv Z(\eta)$, where

$$Z(\mu) := \bigcup_{K \in \mu} K, \quad \mu \in \mathcal{N},$$

and $K \in \mu$ means that $\mu(K) > 0$. Recall that the mapping $\mu \mapsto Z(\mu)$ from $\mathcal{N}$ to the space of all closed subsets of $\mathbb{R}^d$ (equipped with the Fell topology) is Borel measurable (see [28]). We can invoke the thinning property of Poisson processes to assume that $\mathbb{Q}(\{\emptyset\}) = 0$. Without loss of generality we assume that $\mathbb{Q}$ is concentrated on $\mathcal{K}_d^o$, the space of convex bodies with the centre of the circumscribed ball at the origin. Since the centre of the circumscribed ball of a convex body is always contained in the convex body, we have $0 \in K$ for all $K \in \mathcal{K}_d^o$.

Subsequently, we shall need integrability assumptions such as

$$\mathbb{E}V_i(Z_0)^2 < \infty, \quad i = 0, \ldots, d, \quad (2.5)$$

or

$$\mathbb{E}V_i(Z_0)^3 < \infty, \quad i = 0, \ldots, d. \quad (2.6)$$

We next introduce two basic characteristics of the Boolean model $Z$. The volume fraction $p := \mathbb{E}V_d(Z \cap [0,1]^d)$ of $Z$ can be expressed in the form

$$p = 1 - e^{-\gamma_d}. \quad (2.7)$$
The mean covariogram of the typical grain is given by
\[ C_d(x) := \mathbb{E}V_d(Z_0 \cap (Z_0 + x)), \quad x \in \mathbb{R}^d. \] (2.7)

If we assume (2.3), then \( C_d(x) \leq v_d < \infty \) and \( C_d(x) \to 0 \) as \( \|x\| \to \infty \), where \( \|x\| \) denotes the Euclidean norm of \( x \in \mathbb{R}^d \). It is well-known (see e.g. [3]) that the covariance of \( Z \) satisfies
\[ \mathbb{P}(0 \in Z, x \in Z) = p^2 + (1 - p)^2 (e^{\gamma C_d(x)} - 1). \] (2.8)

For \( W \in \mathcal{K}^d \) we define by \( C_W(x) := V_d(W \cap (W + x)), \quad x \in \mathbb{R}^d \), the set covariance function of \( W \). Combining (2.8) with Fubini’s Theorem leads to the well-known formula
\[ \text{Var} \ V_d(Z \cap W) = (1 - p)^2 \int C_W(x)(e^{\gamma C_d(x)} - 1) \, dx, \quad W \in \mathcal{K}^d. \] (2.9)

3 Covariance structure of geometric functionals

In this paper we study random variables of the form \( \psi(Z \cap W) \), where \( \psi \) is a real-valued measurable function defined on the convex ring \( \mathcal{R}^d \) whose elements are finite unions of compact, convex sets. Measurability again refers to the Borel \( \sigma \)-field generated by the Fell topology (or, equivalently, by the Hausdorff metric). We shall assume that \( \psi \) is additive, that is, \( \psi(0) = 0 \) and \( \psi(K \cup L) = \psi(K) + \psi(L) - \psi(K \cap L) \) for all \( K, L \in \mathcal{R}^d \). We shall also assume that \( \psi \) is translation invariant, that is, \( \psi(K + x) = \psi(K) \) for all \( (K, x) \in \mathcal{R}^d \times \mathbb{R}^d \), and locally bounded in the sense that its absolute value is (uniformly) bounded on compact, convex sets contained in a translate of the unit cube \( Q_1 := [-1/2, 1/2]^d \) by a constant
\[ M(\psi) := \sup\{|\psi(K)| : K \in \mathcal{K}^d, K \subset Q_1 + x, x \in \mathbb{R}^d\} < \infty. \] (3.1)

Note that this definition simplifies in the translation-invariant case since one does not need the translations of \( Q_1 \).

In the following, we call a functional \( \psi : \mathcal{R}^d \to \mathbb{R} \) geometric if it is additive, translation invariant, locally bounded, and measurable. Examples of geometric functionals are

1. mixed volumes (see Section 5.1 in [27]) of the form \( \psi(K) := V(K[k], K_1, \ldots, K_{d-k}) \), where \( k \in \{0, \ldots, d\} \) and \( K_1, \ldots, K_{d-k} \in \mathcal{K}^d \) are fixed. Up to normalization, intrinsic volumes are obtained for \( K_i = B^d, i = 1, \ldots, d - k \);

2. integrals of surface area measures (see Sections 4.1 and 4.2 in [27]) of the form
\[ \psi_k(K) := \int_{S^{d-1}} h(u) S_k(K, du), \]
where \( S^{d-1} \) is the unit sphere in \( \mathbb{R}^d \) (the boundary of \( B^d \)), \( h : S^{d-1} \to \mathbb{R} \) is measurable and bounded, and \( k \in \{0, \ldots, d - 1\} \);

3. the centred support function \( \psi(K) := h(K - s(K), u) \) in a fixed direction \( u \), where \( u \in \mathbb{R}^d \) and \( s(K) \) is the Steiner point of \( K \) (see Sections 1.7 and 5.4 in [27] and Lemma 6.1 in [28]);

4. total measures of translative integral geometry (see Section 6.4, especially p. 234, and p. 383 in [28]).
Our first result deals with the asymptotic behaviour of the covariance between two geometric functionals of \( Z \cap W \) for expanding convex observation window \( W \). In the following, we will consider sequences of convex bodies \((W_m)_{m \in \mathbb{N}}\) satisfying \( \lim_{m \to \infty} r(W_m) = \infty \), where \( r(W) \) denotes the inradius of \( W \in \mathcal{K}^d \). We describe this situation by writing \( r(W) \to \infty \) for short. With a measurable functional \( \psi : \mathcal{R}^d \to \mathbb{R} \), we associate another measurable functional \( \psi^* : \mathcal{K}^d \to \mathbb{R} \) by

\[
\psi^*(K) := \mathbb{E}\psi(Z \cap K) - \psi(K), \quad K \in \mathcal{K}^d, \tag{3.2}
\]

if \( \mathbb{E}|\psi(Z \cap W)| < \infty \). Under assumption (2.2), it follows from (3.10) that \( \psi^* \) is well defined for a geometric functional \( \psi \).

**Theorem 3.1.** Let \( \psi_1 \) and \( \psi_2 \) be geometric functionals. If (2.5) is satisfied, then the limit

\[
\sigma_{\psi_1,\psi_2} = \lim_{r(W) \to \infty} \frac{\text{Cov}(\psi_1(Z \cap W), \psi_2(Z \cap W))}{V_d(W)} \tag{3.3}
\]

exists and is given by

\[
\sigma_{\psi_1,\psi_2} = \gamma \sum_{n=1}^{\infty} \frac{1}{n!} \int \int \psi_1^*(K_1 \cap K_2 \cap \ldots \cap K_n) \psi_2^*(K_1 \cap K_2 \cap \ldots \cap K_n) \\
\Lambda^{n-1}(d(K_2, \ldots, K_n)) \mathcal{Q}(dK_1). \tag{3.4}
\]

If (2.6) holds, then there is a constant \( c_{\psi_1,\psi_2} \) such that

\[
\left| \frac{\text{Cov}(\psi_1(Z \cap W), \psi_2(Z \cap W))}{V_d(W)} - \sigma_{\psi_1,\psi_2} \right| \leq \frac{c_{\psi_1,\psi_2}}{r(W)} \tag{3.5}
\]

for \( W \in \mathcal{K}^d \) with \( r(W) \geq 1 \).

From the proof of Theorem 3.1 it also follows that the right-hand side of (3.4) is still convergent if we take the absolute values of the integrands.

We start with some preparations. Our main probabilistic tool is the following Fock space representation of Poisson functionals, derived in [13]. For any measurable \( f : \mathcal{N} \to \mathbb{R} \) and \( K \in \mathcal{K}^d \), the function \( D_K f : \mathcal{N} \to \mathbb{R} \) is defined by

\[
D_K f(\mu) := f(\mu + \delta_K) - f(\mu), \quad \mu \in \mathcal{N}, \tag{3.6}
\]

where \( \delta_K \) is the Dirac measure located at \( K \). The **difference operator** \( D_K \) and its iterations play a central role in the analysis of Poisson processes. For \( n \geq 2 \) and \( (K_1, \ldots, K_n) \in (\mathcal{K}^d)^n \) we define a function \( D_{K_1,\ldots,K_n}^n f : \mathcal{N} \to \mathbb{R} \) inductively by

\[
D_{K_1,\ldots,K_n}^n f := D_{K_1}^1 D_{K_2,\ldots,K_n}^{n-1} f,
\]

where \( D^1 := D \). Note that

\[
D_{K_1,\ldots,K_n}^n f(\mu) = \sum_{J \subset \{1,2,\ldots,n\}} (-1)^{n-|J|} f(\mu + \sum_{j \in J} \delta_{K_j}),
\]

where \( |J| \) denotes the number of elements of \( J \). This shows that the operator \( D_{K_1,\ldots,K_n}^n \) is symmetric in \( K_1, \ldots, K_n \), and that \( (\mu, K_1, \ldots, K_n) \mapsto D_{K_1,\ldots,K_n}^n f(\mu) \) is measurable. From
Lemma 3.3. Let $\psi : \mathcal{K}^d \to \mathbb{R}$ be additive. Then, for $n \in \mathbb{N}, K_1, \ldots, K_n \in \mathcal{K}^d$, and $\mu \in \mathbb{N},$

$$D^n_{K_1, \ldots, K_n} \psi(\mu) = (-1)^n (\psi(Z(\mu) \cap K_1 \cap \ldots \cap K_n \cap W) - \psi(K_1 \cap \ldots \cap K_n \cap W)).$$

Proof. This easily follows by induction. \qed

In the following, we write $c_1, c_2, \ldots$ for constants which depend only on the dimension, the intensity, and the distribution of the typical grain.

Lemma 3.3. Let $\psi$ be an additive, locally bounded, and measurable functional and assume that (2.2) is satisfied. Then, for all $n \in \mathbb{N}, K_1, \ldots, K_n \in \mathcal{K}^d$, and $W \in \mathcal{K}^d,$

$$\mathbb{E} D^n_{K_1, \ldots, K_n} f_{\psi,W} = (-1)^n \psi^*(K_1 \cap \ldots \cap K_n \cap W)$$

(3.8)

and

$$|\mathbb{E} D^n_{K_1, \ldots, K_n} f_{\psi,W}| \leq \beta(\psi) \sum_{i=0}^d V_i(K_1 \cap \ldots \cap K_n \cap W)$$

(3.9)

with a constant $\beta(\psi)$ only depending on $\psi$. Moreover, for any $A \in \mathcal{K}^d,$

$$\mathbb{E} \psi(Z \cap A)^2 < \infty.$$  

(3.10)

Proof. We start with proving that

$$\mathbb{E} |\psi(Z \cap A)| \leq c_1 M(\psi) \sum_{i=0}^d V_i(A)$$

(3.11)

for $A \in \mathcal{K}^d$. Since (3.11) is obviously true for $A = \emptyset$, we assume $A \neq \emptyset$ in the following. We define $Q(A) := \{Q_1 + z : z \in \mathbb{Z}^d, (Q_1 + z) \cap A \neq \emptyset\}$, where we recall that $Q_1 = [-1/2, 1/2]^d$.

By the inclusion-exclusion formula for additive functionals (see e.g. [27, (3.4.3)]), we have

$$|\psi(Z \cap A)| = \left| \psi\left( Z \cap A \cap \bigcup_{Q \in Q(A)} Q \right) \right| \leq \sum_{\emptyset \neq I \subset Q(A)} \left| \psi\left( Z \cap A \cap \bigcap_{Q \in I} Q \right) \right|.$$
By the inclusion-exclusion formula and taking into account that $\psi(\emptyset) = 0$, we get
\[
|\psi(Z \cap A)| \leq \sum_{\emptyset \neq I \subset Q(A)} \left| \psi \left( \bigcup_{j=1}^{N_{Q_I}} Z_j \cap A \cap \bigcap_{Q \in I} Q \right) \right|
\]
\[
\leq \sum_{\emptyset \neq I \subset Q(A)} \sum_{\emptyset \neq J \subset \{1, \ldots, N_{Q_I}\}} \left| \psi \left( \bigcap_{j \in J} Z_j \cap A \cap \bigcap_{Q \in I} Q \right) \right|
\]
\[
\leq \sum_{\emptyset \neq I \subset Q(A)} 1 \left\{ \bigcap_{Q \in I} Q \neq \emptyset \right\} 2^{N_{Q_I}} M(\psi).
\]  
(3.12)

The cubes in $Q(A)$ form a grid, hence
\[
\left| \left\{ \emptyset \neq I \subset Q(A) : \bigcap_{Q \in I} Q \neq \emptyset \right\} \right| \leq c_2 |Q(A)|.
\]

By stationarity of $\eta$, we have $\mathbb{E}2^{N_{Q_I}} = \mathbb{E}2^{N_{Q_I}}$, and thus
\[
\mathbb{E}|\psi(Z \cap A)| \leq c_3 M(\psi)|Q(A)|.
\]  
(3.13)

Here we have used that for $C \in \mathcal{K}^d$
\[
\mathbb{E}z^{NC} = \exp \left((z - 1)\gamma \mathbb{E}_d(Z_0 + C^*) \right) < \infty, \quad z \geq 0,
\]
where $C^* := \{-x : x \in C\}$ is the reflection of $C$ in the origin.

Since $|Q(A)| \leq V_d(A + \sqrt{d}B^d)$, Steiner’s formula (2.4) yields
\[
|Q(A)| \leq \sum_{i=0}^{d} \kappa_{d-i}d^{(d-i)/2}V_i(A) \leq c_4 \sum_{i=0}^{d} V_i(A).
\]  
(3.14)

Hence, (3.13) together with (3.14) yields (3.11). Combining (3.11) with Lemma 3.2 and the definition of $\psi^*$ in (3.2) shows (3.5). For $A \in \mathcal{K}^d$, we can argue as in the derivation of (3.12), and then use (3.14), to get
\[
|\psi(A)| \leq c_2 |Q(A)| M(\psi) \leq c_2c_4 M(\psi) \sum_{i=0}^{d} V_i(A).
\]  
(3.15)

Combining (3.11) and (3.15) for $A = K_1 \cap \ldots \cap K_n \cap W$ with Lemma 3.2 yields (3.9).

In order to show that $\psi(Z \cap A)$ is square integrable, we first derive an upper bound for
\[
M_A(\psi) := \sup \{|\psi(L)| : L \in \mathcal{K}^d, L \subseteq A\}.
\]  
(3.16)

Let $L \in \mathcal{K}^d$ with $L \subseteq A$. Then, using the inclusion-exclusion formula for additive functionals and (3.1), we get
\[
|\psi(L)| = \left| \psi \left( L \cap \bigcup_{Q \in Q(A)} Q \right) \right| \leq 2^{|Q(A)|} M(\psi),
\]
and hence
\[
M_A(\psi) \leq 2^{|Q(A)|} M(\psi).
\]
Again by the inclusion-exclusion formula, we have
\[ |\psi(Z \cap A)| \leq (2^{NA} - 1)M_A(\psi) \leq 2^{NA}M_A(\psi), \]
and therefore
\[ \mathbb{E}\psi(Z \cap A)^2 \leq \mathbb{E}[4^{NA}] 4^{|Q(A)|} M(\psi)^2 < \infty, \]
which concludes the proof.

Lemma 3.4. There is a constant \( \beta_1 > 0 \), depending only on the dimension, such that
\[ \int V_k(W \cap (K + x)) \, dx \leq \beta_1 \sum_{i=0}^{d} V_i(W) \sum_{r=k}^{d} V_r(K), \] (3.18)
for all \( k \in \{0, \ldots, d\} \) and \( W, K \in \mathcal{K}^d \).

Proof. Using the same notation as in the proof of Lemma 3.3 and the fact that \( V_k \) is increasing and translation invariant, we obtain that
\[ \int V_k(W \cap (K + x)) \, dx \leq \sum_{\emptyset \neq I \subset Q(W)} \int V_k \left( W \cap \bigcap_{Q \in I} Q \cap (K + x) \right) \, dx \]
\[ \leq \sum_{\emptyset \neq I \subset Q(W)} 1 \left\{ \bigcap_{Q \in I} Q \neq \emptyset \right\} \int V_k(K \cap (Q_1 + x)) \, dx. \]
Let \( B' \) denote a ball of radius \( \sqrt{d}/2 \). Then
\[ \int V_k(K \cap (Q_1 + x)) \, dx \leq \int V_k(K \cap (B' + x)) \, dx \leq c_5 \sum_{r=k}^{d} V_r(K), \]
where \( c_5 \) is a constant depending only on the dimension and we have used the kinematic formula (see [28, Theorem 5.1.3]) and the rotation invariance of \( B' \). On the other hand, it was shown in the proof of Lemma 3.3 that
\[ \left| \left\{ \emptyset \neq I \subset Q(W) : \bigcap_{Q \in I} Q \neq \emptyset \right\} \right| \leq c_2 c_4 \sum_{i=0}^{d} V_i(W). \]
Combining the preceding inequalities, we obtain the assertion of the lemma.

Lemma 3.5. For \( A \in \mathcal{K}^d \) and \( n \in \mathbb{N} \),
\[ \int \sum_{k=0}^{d} V_k(A \cap K_1 \cap \ldots \cap K_n) \Lambda^n(d(K_1, \ldots, K_n)) \leq \alpha^n \sum_{k=0}^{d} V_k(A), \]
where \( \alpha = \gamma (d + 1) \beta_1 \sum_{i=0}^{d} \mathbb{E}V_i(Z_0) \) with \( \beta_1 \) as in Lemma 3.4.
Proof. In the following calculation and also later, we use the convention $\int c d\lambda^n := c$. We apply (2.14) and (3.18) to get
\[
\int \sum_{k=0}^{d} V_k(A \cap K_1 \cap \ldots \cap K_n) \lambda^n(d(K_1, \ldots, K_n))
\]
\[
= \sum_{k=0}^{d} \gamma \int \int \int V_k(A \cap K_1 \cap \ldots \cap K_{n-1} \cap (K_n + y)) dy Q(dK_n) \lambda^{n-1}(d(K_1, \ldots, K_{n-1}))
\]
\[
\leq \sum_{k=0}^{d} \gamma \int \int \beta_1 \sum_{i=0}^{d} V_i(A \cap K_1 \cap \ldots \cap K_{n-1}) \sum_{r=k}^{d} V_r(K_n) Q(dK_n) \lambda^{n-1}(d(K_1, \ldots, K_{n-1}))
\]
\[
\leq \gamma(d + 1) \beta_1 \sum_{i=0}^{d} \mathbb{E}V_i(Z_0) \int \sum_{k=0}^{d} V_k(A \cap K_1 \cap \ldots \cap K_{n-1}) \lambda^{n-1}(d(K_1, \ldots, K_{n-1})).
\]
By iterating this step $(n - 1)$ more times, we obtain the assertion. \qed

Lemma 3.6. There is a constant $\beta_2$, depending only on the dimension, such that
\[
\lambda_d(\{x \in \mathbb{R}^d : (K + x) \cap \partial W \neq \emptyset\}) \leq \beta_2 \sum_{i=0}^{d-1} V_i(W) \sum_{r=0}^{d} V_r(K)
\]
for $K, W \in \mathcal{K}^d$.

Proof. Let $Q(\partial W) := \{Q_1 + z : z \in \mathbb{Z}^d, (Q_1 + z) \cap \partial W \neq \emptyset\}$. Then we have
\[
\lambda_d(\{x \in \mathbb{R}^d : (K + x) \cap \partial W \neq \emptyset\}) \leq \sum_{Q \in Q(\partial W)} \int 1\{(K + x) \cap Q \neq \emptyset\} dx
\]
\[
= \sum_{Q \in Q(\partial W)} V_d(K + Q_1)
\]
\[
\leq |Q(\partial W)| c_0 \sum_{r=0}^{d} V_r(K).
\]
Let $\text{dist}(x, A) := \inf\{\|x - y\| : y \in A\}$ for $x \in \mathbb{R}^d$ and a closed set $A \subset \mathbb{R}^d$, and let $\partial^- W := \{x \in W : \text{dist}(x, \partial W) \leq r\}$. Then
\[
V_d(\partial^- W) \leq V_d(W + B^d_r) - V_d(W). \quad (3.19)
\]
To see this, let $p_W : \mathbb{R}^d \to W$ denote the metric projection to $W$ and consider the map $T : (W + B^d_r) \setminus W \to \mathbb{R}^d, x \mapsto 2p_W(x) - x$. Let $x \in \partial^- W$ and choose a point $y \in \partial W$ such that $\|x - y\| = \text{dist}(x, \partial W) \leq r$. Using that $y - x$ is an outer normal of $W$ at $y$ it is easy to see that $T(2y - x) = x$. Hence $\partial^- W \subset T((W + B^d_r) \setminus W)$. Since the metric projection is 1-Lipschitz it is not hard to prove that $T$ has the same property. Therefore (3.19) follows. This yields that
\[
|Q(\partial W)| \leq \lambda_d(\{x \in \mathbb{R}^d : \text{dist}(x, \partial W) \leq \sqrt{d}\}) \leq 2(V_d(W + B^d_{\sqrt{d}}) - V_d(W)) \leq c_7 \sum_{i=0}^{d-1} V_i(W),
\]
where Steiner’s formula was used. \qed
**Lemma 3.7.** Let \( W \in \mathcal{K}^d \) such that \( r(W) > 0 \) and let \( k \in \{0, \ldots, d-1\} \). Then

\[
\frac{V_k(W)}{V_d(W)} \leq \frac{2^d - 1}{\kappa_{d-k} r(W)^{d-k}}.
\]

*Proof.* Steiner’s formula and the fact that \( V_i(W) \geq 0 \), for \( i = 0, \ldots, d-1 \), imply that

\[
(2^d - 1) V_d(W) = V_d(2W) - V_d(W) \geq V_d(W + r(W)B^d) - V_d(W) = \sum_{i=0}^{d-1} \kappa_{d-i} r(W)^{d-i} V_i(W)
\]

\[
\geq \kappa_{d-k} r(W)^{d-k} V_k(W),
\]

which concludes the proof. \( \square \)

*Proof of Theorem 3.1.* Let \( W \in \mathcal{K}^d \). In order to compute the numerator in (3.3) we shall apply Lemma 3.7, with \( f = f_{\psi_1, W} \) and \( g = f_{\psi_2, W} \). From (3.10) we conclude that indeed \( \mathbb{E} f(\eta)^2 < \infty \) and \( \mathbb{E} g(\eta)^2 < \infty \). Since \( Z \) is stationary, the translation invariance of a functional \( \psi : \mathcal{R}^d \to \mathbb{R} \) implies that \( \psi^* : \mathcal{K}^d \to \mathbb{R} \) defined by (3.2) is translation invariant as well. From (3.8) we get

\[
\frac{1}{n!} \int \mathbb{E} D^n_{K_1, \ldots, K_n} f_{\psi_1, W}(\eta) \mathbb{E} D^n_{K_1, \ldots, K_n} f_{\psi_2, W}(\eta) \Lambda^n(d(K_1, \ldots, K_n))
\]

\[
= \frac{\gamma}{n!} \int \int \psi^*_1((K + x) \cap K_2 \cap \ldots \cap K_n \cap W) \psi^*_2((K + x) \cap K_2 \cap \ldots \cap K_n \cap W)
\]

\[
\Lambda^{n-1}(d(K_2, \ldots, K_n)) \mathcal{Q}(dK) dx.
\]

For \( W \in \mathcal{K}^d \) and \( n \in \mathbb{N} \), we define \( f_{W,n} : \mathcal{K}^d \to \mathbb{R} \) by

\[
f_{W,n}(K) := \frac{1}{V_d(W)} \int \psi^*_1((K + x) \cap K_2 \cap \ldots \cap K_n \cap W)
\]

\[
\psi^*_2((K + x) \cap K_2 \cap \ldots \cap K_n \cap W) \Lambda^{n-1}(d(K_2, \ldots, K_n)) dx,
\]

and \( f_n : \mathcal{K}^d \to \mathbb{R} \) by

\[
f_n(K) := \int \psi^*_1(K \cap K_2 \cap \ldots \cap K_n) \psi^*_2(K \cap K_2 \cap \ldots \cap K_n) \Lambda^{n-1}(d(K_2, \ldots, K_n)).
\]

Our aim is to prove that

\[
\sum_{n=1}^{\infty} \frac{\gamma}{n!} \int f_{W,n}(K) \mathcal{Q}(dK) \to \sum_{n=1}^{\infty} \frac{\gamma}{n!} \int f_n(K) \mathcal{Q}(dK)
\]

as \( r(W) \to \infty \). Since we want to apply the dominated convergence theorem, we provide an upper bound for \( \sum_{n=1}^{\infty} \frac{\gamma}{n!} |f_{W,n}| \), which is independent of \( W \).

It follows from (3.9) in Lemma 3.3, the translation invariance of \( V_i \) and \( \Lambda \), and the monotonicity of the intrinsic volumes that

\[
|f_{W,n}(K)| \leq \sum_{i,j=0}^{d} \frac{\beta(\psi_1) \beta(\psi_2)}{V_d(W)} \int V_i((K + x) \cap K_2 \cap \ldots \cap K_n \cap W)
\]

\[
V_j((K + x) \cap K_2 \cap \ldots \cap K_n \cap W) \Lambda^{n-1}(d(K_2, \ldots, K_n)) dx
\]

\[
\rightarrow \sum_{i,j=0}^{d} \frac{\beta(\psi_1) \beta(\psi_2)}{\Lambda^d} \int V_i((K + x) \cap K_2 \cap \ldots \cap K_n \cap W)
\]

\[
V_j((K + x) \cap K_2 \cap \ldots \cap K_n \cap W) \Lambda^{n-1}(d(K_2, \ldots, K_n)) dx
\]
\[
\begin{align*}
&\leq \sum_{i,j=0}^{d} \frac{\beta(\psi_1)\beta(\psi_2)}{V_d(W)} \int V_i(K \cap K_2 \cap \ldots \cap K_n) \Lambda^{n-1}(d(K_2, \ldots, K_n)) \\
&\quad \int V_j((K + x) \cap W) \, dx.
\end{align*}
\]

Combining this estimate with Lemma 3.4 and Lemma 3.5, we get
\[
\frac{1}{n!} |f_{W,n}(K)| \leq (d + 1) \beta(\psi_1)\beta(\psi_2) \left( \sum_{i=0}^{d} V_i(K) \right)^2 \sum_{r=0}^{d} \frac{V_r(W)}{V_d(W)} \frac{\alpha^{n-1}}{n!}.
\tag{3.20}
\]

By (2.5) the right-hand side of (3.20) is integrable. Moreover, Lemma 3.7 shows that it is uniformly bounded for \( W \in K^d \) with \( r(W) \geq 1 \), and the same holds if we sum over all \( n \in \mathbb{N} \).

Next we bound \( |f_{W,n}(K) - f_n(K)| \). By using the translation invariance of \( \psi_1^*, \psi_2^* \) and \( \Lambda \), we have
\[
\begin{align*}
&f_{W,n}(K) - f_n(K) \\
&= \frac{1}{V_d(W)} \int \left( \psi_1^*((K + x) \cap K_2 \cap \ldots \cap K_n) \cap W \right) \psi_2^*((K + x) \cap K_2 \cap \ldots \cap K_n) \cap W \\
&\quad - 1\{x \in W\} \psi_1^*((K + x) \cap K_2 \cap \ldots \cap K_n) \psi_2^*((K + x) \cap K_2 \cap \ldots \cap K_n) \\
&\quad \int dx \, \Lambda^{n-1}(d(K_2, \ldots, K_n)).
\end{align*}
\]

Note that the integrand is zero if \( x \in W \) and \( K + x \subset W \). The same holds for the case that \( x \notin W \) and \( (K + x) \cap W = \emptyset \). This means that the integrand can be only non-zero if \( (K + x) \cap \partial W \neq \emptyset \). On the other hand, the integrand is always bounded by
\[
\begin{align*}
&|\psi_1^*((K + x) \cap K_2 \cap \ldots \cap K_n) \cap W| + |\psi_2^*((K + x) \cap K_2 \cap \ldots \cap K_n) \cap W| \\
&\quad \leq 2\beta(\psi_1)\beta(\psi_2) \left( \sum_{i=0}^{d} V_i((K + x) \cap K_2 \cap \ldots \cap K_n) \right)^2,
\end{align*}
\]

where we have used Lemma 3.3 and the monotonicity of the intrinsic volumes. Hence, we obtain that
\[
\begin{align*}
&|f_{W,n}(K) - f_n(K)| \\
&\leq \frac{2\beta(\psi_1)\beta(\psi_2)}{V_d(W)} \int 1\{(K + x) \cap \partial W \neq \emptyset\} \left( \sum_{i=0}^{d} V_i((K + x) \cap K_2 \cap \ldots \cap K_n) \right)^2 \\
&\quad \int dx \, \Lambda^{n-1}(d(K_2, \ldots, K_n)) \\
&\quad \leq \frac{2\beta(\psi_1)\beta(\psi_2)}{V_d(W)} \sum_{i=0}^{d} V_i(K) \int 1\{(K + x) \cap \partial W \neq \emptyset\} \, dx \\
&\quad \int \sum_{r=0}^{d} V_r(K \cap K_2 \cap \ldots \cap K_n) \Lambda^{n-1}(d(K_2, \ldots, K_n)),
\end{align*}
\]
where we have used the fact that $V_i$ is increasing and the translation invariance of $V_i$ and $\Lambda$ in the last step. Now Lemma 3.5 and Lemma 3.6 yield that

$$|f_{W,n}(K) - f_n(K)| \leq \frac{2\beta_2\beta(\psi_1)\beta(\psi_2)\alpha^{n-1}}{V_d(W)} \left( \sum_{i=0}^{d} V_i(K) \right)^{3} \sum_{r=0}^{d-1} V_r(W).$$

Together with Lemma 3.7 this shows that for any $W \in K^d$ with $r(W) \geq 1$,

$$|f_{W,n}(K) - f_n(K)| \leq \beta(\psi_1, \psi_2)\alpha^{n-1} \left( \sum_{i=0}^{d} V_i(K) \right)^{3} \frac{1}{r(W)},$$

where $\beta(\psi_1, \psi_2)$ depends only on $d, \psi_1, \psi_2$. Therefore, for any $W \in K^d$ with $r(W) \geq 1$,

$$\left| \sum_{n=1}^{\infty} \frac{\gamma}{n!} f_{W,n}(K) - \sum_{n=1}^{\infty} \frac{\gamma}{n!} f_n(K) \right| \leq \frac{\gamma\beta(\psi_1, \psi_2)e^\alpha}{\alpha} \left( \sum_{i=0}^{d} V_i(K) \right)^{3} \frac{1}{r(W)}.$$

Thus an application of the dominated convergence theorem yields the convergence result stated in the theorem.

Under the stronger moment assumption (2.6), the rate of convergence in (3.21) follows from (3.22) by carrying out the integration with respect to $K$. \hfill \Box

If the geometric functional is the volume, the asymptotic variance has a significantly easier representation than in (3.6), namely

$$\sigma_{d,d} := \lim_{r(W) \to \infty} \frac{\text{Var} V_d(Z \cap W)}{V_d(W)} = (1-p)^2 \int (e^{\gamma C_d(x)} - 1) \, dx. \tag{3.22}$$

This follows from an application of the dominated convergence theorem to the exact variance formula (2.9). The inequalities $e^t - 1 \leq te^t, t \geq 0$, and $C_d(x) \leq v_d$ imply that

$$\int (e^{\gamma C_d(x)} - 1) \, dx \leq e^{\gamma v_d} \int \text{EV}_d(Z_0 \cap (Z_0 + x)) \, dx = e^{\gamma v_d} \text{EV}_d(Z_0)^2 < \infty.$$ 

Together with $C_W(x)/V_d(W) \leq 1$ this means that $e^{\gamma C_d(x)} - 1$ is integrable and is an upper bound for $(C_W(x)/V_d(W))(e^{\gamma C_d(x)} - 1)$. Now the observation that $C_W(x)/V_d(W) \to 1$ as $r(W) \to \infty$ for any $x \in \mathbb{R}^d$ (this follows from $V_d(W) - C_W(x) \leq V_d(\partial^e W)$, (3.19), Steiner’s formula, and Lemma 3.7) yields (3.22). In Section 6 formulas as (3.22) are derived for the other intrinsic volumes.

The following proposition shows that the rate of convergence stated in Theorem 3.1 is optimal.

**Proposition 3.8.** Assume that (2.5) is satisfied and that the typical grain is full-dimensional with positive probability. Then there is a constant $c_{d,d} > 0$ such that

$$\left| \sigma_{d,d} - \frac{\text{Var} V_d(Z \cap W)}{V_d(W)} \right| \geq \frac{c_{d,d}}{r(W)},$$

for $W \in K^d$ with $r(W) \geq 1$. 

---

13
Proof. Recall from the proof of Lemma 3.6 that dist(z, A) = inf{∥z − a∥ : a ∈ A}, for z ∈ R^d and a closed set A ⊂ R^d, and ∂^-rW = {z ∈ W : dist(z, ∂W) ≤ r} for r ≥ 0. For s ≥ 0 we define D_W(s) := {z ∈ W : dist(z, ∂W) = s}. Then

W^-s := {z ∈ W : dist(z, ∂W) ≥ s} = {z ∈ R^d : z + B^d_s ⊂ W}

is convex, the boundary of W^-s is D_W(s), and s → W^-s is strictly decreasing with respect to set inclusion, for s ∈ [0, r(W)].

It follows from (2.30) and (3.22) that

\[ \sigma_{d,d} - \frac{\Var(V_d(Z \cap W))}{V_d(W)} = (1 − p)^2 \int B^0 \frac{V_d(W) − V_d(W \cap (W + x))}{V_d(W)} (e^{γC_d(x)} − 1) \, dx. \]

Since the typical grain is full-dimensional with positive probability, there are constants τ > 0 and r_0 ∈ (0, 1/2) such that e^{γC_d(x)} − 1 ≥ τ for all x ∈ B^d_{r_0}. This means that

\[ \sigma_{d,d} - \frac{\Var(V_d(Z \cap W))}{V_d(W)} ≥ (1 − p)^2 \frac{τ}{V_d(W)} \int B^0 (V_d(W) − V_d(W \cap (W + x))) \, dx. \]

(3.23)

Denoting by B^d(x, r) the closed ball with centre x and radius r, we have

\[ \int_{B^0_{r_0}} (V_d(W) − V_d(W \cap (W + x))) \, dx = \int_{B^0_{r_0}} \int_W \left(1_{\{y ∈ W\}} − 1_{\{y ∈ W, y ∈ W + x\}}\right) dy \, dx \]

\[ = \int_W (V_d(B^d(y, r_0)) − V_d(W \cap B^d(y, r_0))) \, dy \]

\[ ≥ \int_{∂^-r_{r_0/2}W} (V_d(B^d(y, r_0)) − V_d(W \cap B^d(y, r_0))) \, dy. \]

Using the fact that V_d(B^d(y, r_0)) − V_d(W ∩ B^d(y, r_0)) ≥ c_0^d for y ∈ ∂^-r_{r_0/2}W with c > 0, we obtain

\[ \int_{B^0_{r_0}} (V_d(W) − V_d(W \cap (W + x))) \, dx ≥ c_0^d V_d(∂^-r_{r_0/2}W). \]

(3.24)

It follows from Lemma 3.2.34 in [6] that

\[ V_d(∂^-r W) = \int_0^r \mathcal{H}^{d-1}(D_W(s)) \, ds \]

for r ∈ [0, r(W)]. The discussion at the beginning of this proof implies that \( \mathcal{H}^{d-1}(D_W(\cdot)) \) is strictly decreasing on [0, r(W)]. Together with V_d(∂^-r(W)) = V_d(W) we get for r(W) ≥ r_0/2 that

\[ V_d(W) = \int_0^{r(W)} \mathcal{H}^{d-1}(D_W(s)) \, ds \]

\[ ≤ \int_0^{r(W)} \mathcal{H}^{d-1}\left(D_W\left(\frac{r_0}{2r(W)}s\right)\right) \, ds \]

\[ = \int_0^{r_0/2} \mathcal{H}^{d-1}(D_W(t)) \frac{2r(W)}{r_0} \, dt \]

\[ = \frac{2r(W)}{r_0} V_d(∂^-r_{r_0/2}W). \]

Combining this with (3.23) and (3.24) concludes the proof. □
4 Positive definiteness

Let \( \psi_0, \ldots, \psi_d \) be geometric functionals on \( \mathbb{R}^d \). We assume that \( \psi_k \), for \( k \in \{0, \ldots, d\} \), is positively homogeneous of degree \( k \) and

\[
|\psi_k(K)| \geq \hat{\beta}(\psi_k) r(K)^k, \tag{4.1}
\]

for \( K \in \mathcal{K}^d \), with a constant \( \hat{\beta}(\psi_k) \) which only depends on \( \psi_k \). By Theorem 3.1 for \( k, l \in \{0, \ldots, d\} \), the asymptotic covariances \( \sigma_{\psi_k, \psi_l} \) exist under the assumption (2.5). The following theorem shows that the asymptotic covariance matrix is positive definite. In particular, the result applies to the intrinsic volumes \( V_0, \ldots, V_d \), which also means that their asymptotic variances are strictly positive.

**Theorem 4.1.** Let the preceding assumptions and (2.5) be satisfied. Moreover assume that the typical grain \( Z_0 \) has non-empty interior with positive probability. Then the covariance matrix \( \Sigma := (\sigma_{\psi_k, \psi_l})_{k,l=0,\ldots,d} \) is positive definite.

**Proof.** For a vector \( a = (a_0, \ldots, a_d)^\top \in \mathbb{R}^{d+1} \) we have

\[
a^\top \Sigma a = \gamma \sum_{n=1}^{\infty} \frac{1}{n!} \int \left( \sum_{k=0}^{d} a_k \psi^*_k(K_1 \cap K_2 \cap \ldots \cap K_n) \right)^2 \Lambda^{n-1}(d(K_2, \ldots, K_n)) Q(dK_1).
\]

Since each summand is non-negative, the matrix \( \Sigma \) is positive definite if we can prove that one summand is greater than zero for a given \( a \in \mathbb{R}^{d+1} \setminus \{0\} \). Specifically, under the given assumptions we shall show that the summand obtained for \( n = d + 1 \) is positive. In order to show this, we shall prove that for \( K_1, \ldots, K_{d+1} \) in the support of \( Q \) and having non-empty interiors, there is a set of translation vectors \( x_2, \ldots, x_{d+1} \in \mathbb{R}^d \) of positive \( \lambda^d \) measure (recall that \( \lambda_d \) denotes \( d \)-dimensional Lebesgue measure) for which

\[
\sum_{k=0}^{d} a_k \psi^*_k(K_1 \cap (K_2 + x_2) \cap \ldots \cap (K_{d+1} + x_{d+1})) \neq 0.
\]

For the rest of the proof, we argue with a convex body \( L \in \mathcal{K}^d \). Properties which will be required of \( L \) will be provided by an application of Lemma 4.2 and \( L \) of the form \( L = K_1 \cap (K_2 + x_2) \cap \ldots \cap (K_{d+1} + x_{d+1}) \in \mathcal{K}^d \), for a set of translation vectors \( x_2, \ldots, x_{d+1} \in \mathbb{R}^d \) of positive \( \lambda^d \) measure. This will finally prove the preceding assertion and thus the theorem.

For \( L \in \mathcal{K}^d \), let \( N_1(L) \) be the number of grains of \( \eta \) that intersect \( L \), but do not contain it, and let \( N_2(L) \) be the number of grains of \( \eta \) that contain \( L \). Then \( N_1(L) \) and \( N_2(L) \) are independent, Poisson distributed random variables with parameters

\[
s_1(L) = \Lambda(\{K \in \mathcal{K}^d : K \cap L \neq \emptyset \text{ and } L \not\subset K\}),
\]

\[
s_2(L) = \Lambda(\{K \in \mathcal{K}^d : L \subset K\}).
\]

If \( N_2(L) \neq 0 \), then \( L \subset Z \), and therefore

\[
\psi_k(Z \cap L) - \psi_k(L) = \psi_k(L) - \psi_k(L) = 0.
\]

If \( N_1(L) = N_2(L) = 0 \), then \( Z \cap L = \emptyset \), and hence

\[
\psi_k(Z \cap L) - \psi_k(L) = 0 - \psi_k(L) = -\psi_k(L).
\]
This leads to
\[
\psi_k^*(L) = \mathbb{E}\left[\psi_k(Z \cap L) - \psi_k(L)\right] = -\exp(-s_1(L) - s_2(L))\psi_k(L) + R_k(L), \tag{4.2}
\]
where
\[
R_k(L) = \mathbb{E}\{N_1(L) \geq 1, N_2(L) = 0\}\left(\psi_k(Z \cap L) - \psi_k(L)\right).
\]
Next we bound \(R_k(L)\) from above. So assume that \(N_1(L) \neq 0\). Let \(K_1, \ldots, K_{N_1(L)}\) denote the grains of \(\eta\) which hit \(L\), but do not contain \(L\). With the definition of \(M_L(\psi_k)\) from (3.16), we obtain from (3.17) that
\[
|\psi_k(Z \cap L) - \psi_k(L)| \leq |\psi_k(Z \cap L)| + |\psi_k(L)| \leq 2^{N_1(L)} M_L(\psi_k).
\]
In the following, let \(R(K)\) stand for the radius of the circumscribed ball of \(K \in \mathcal{K}^d\). For \(A \in \mathcal{K}^d\) with \(A \subset L\), let \(a \in \mathbb{R}^d\) be the centre of the circumscribed ball of \(A\), hence \(A - a \subset 2R(A)Q_1\). Since \(\psi_k\) is locally bounded and homogeneous of degree \(k\), we get
\[
|\psi_k(A)| = (2R(A))^k|\psi_k((2R(A))^{-1}(A - a))| \leq (2R(A))^k M(\psi_k),
\]
and hence
\[
M_L(\psi_k) \leq (2R(L))^k M(\psi_k). \tag{4.3}
\]
Thus, in the present case, we have
\[
|\psi_k(Z \cap L) - \psi_k(L)| \leq 2^{N_1(L)}(2R(L))^k M(\psi_k).
\]
Hence the remainder term can be bounded from above by
\[
|R_k(L)| \leq \mathbb{E}\left[\{N_1(L) \geq 1, N_2(L) = 0\}2^{N_1(L)}(2R(L))^k M(\psi_k)\right]
= \exp(-s_2(L))(2R(L))^k M(\psi_k)\exp(s_1(L))(1 - \exp(-2s_1(L)))
\leq \exp(-s_2(L))(2R(L))^k M(\psi_k)\exp(s_1(L)2s_1(L)).
\]
Next we derive an upper bound for \(s_1(L)\). By definition and the reflection invariance of Lebesgue measure, we have
\[
s_1(L) = \gamma \int \int 1\{1(L + x) \cap K \neq \emptyset, L + x \not\subset K\} \, dx \, dQ(dK).
\]
To bound the inner integral from above, we can assume that \(L \in \mathcal{K}^d_o\), by the translation invariance of Lebesgue measure. If the integrand is non-zero, then \(x \in (K + R(L)B^d) \setminus K\) or \(x \in \partial K^-_{R(L)}\). Then the inequality (3.19) implies that the inner integral is bounded from above by \(2V_{4d}((K + R(L)B^d) \setminus K\). Hence, if \(R(L) \leq 1\), Steiner’s formula and our moment assumption yield that
\[
s_1(L) \leq c_8 R(L),
\]
where as before \(c_8\) denotes a constant depending on \(\gamma, Q\). Hence, if \(R(L)\) is sufficiently small, then \(s_1(L) \leq 1\), and thus
\[
|R_k(L)| \leq 6 \cdot (2R(L))^k M(\psi_k) s_1(L) \exp(-s_2(L))
\leq 6 \cdot 2^k \cdot c_8 M(\psi_k) R(L)^{k+1} \exp(-s_2(L)). \tag{4.4}
\]
We also have from (4.3) that
\[ |\exp(-s_1(L) - s_2(L))\psi_k(L)| \leq M_L(\psi_k) \exp(-s_2(L)) \leq (2R(L))^k M(\psi_k) \exp(-s_2(L)). \quad (4.5) \]
Hence, if \( R(L) \) is sufficiently small, we deduce from (4.2), (4.4), and (4.5) that
\[ |\psi_k^*(L)| \leq \tilde{\beta}(\psi_k) R(L)^k \exp(-s_2(L)), \quad (4.6) \]
where \( \tilde{\beta}(\psi_k) \) is a constant depending on \( \gamma, \mathbb{Q}, \psi_k \). In addition,
\[ |\exp(-s_1(L) - s_2(L))\psi_k(L)| \geq \exp(-s_2(L))(\tilde{\beta}(\psi_k)/3) r(L)^k, \quad (4.7) \]
if \( s_1(L) \leq 1 \), with \( \tilde{\beta}(\psi_k) \) as in (4.1).

Let \( k_0 \) be the smallest \( k \in \{0, \ldots, d\} \) such that \( a_k \neq 0 \). Then, if \( R(L) \) is sufficiently small, we get
\[
\left| \sum_{k=0}^{d} a_k \psi_k^*(L) \right| = \left| \sum_{k=k_0}^{d} a_k \psi_k^*(L) \right|
\]
\[ = \left| -a_{k_0} \exp(-s_1(L) - s_2(L))\psi_{k_0}(L) + a_{k_0} R_{k_0}(L) + \sum_{k=k_0+1}^{d} a_k \psi_k^*(L) \right|
\]
\[ \geq |a_{k_0}| |\exp(-s_1(L) - s_2(L))\psi_{k_0}(L)| - |a_{k_0} R_{k_0}(L)| - \sum_{k=k_0+1}^{d} |a_k| |\psi_k^*(L)|
\]
\[ \geq \exp(-s_2(L)) \left( |a_{k_0}| \left( \tilde{\beta}(\psi_{k_0})/3 \right) r(L)^{k_0} - \beta^* R(L)^{k_0+1} \right),
\]
where we used (4.4) and (4.7), for \( k = k_0 \), and (4.6) for \( k \geq k_0 + 1 \). Here we denote by \( \beta^* \) a constant which depends on \( a_{k_0}, \ldots, a_d, \psi_{k_0}, \ldots, \psi_d, \gamma, \mathbb{Q} \). The lower bound thus obtained is positive if \( R(L) \) is sufficiently small and \( R(L)/r(L) \leq c_0 \), for some constant \( c_0 \). The proof is completed by an application of Lemma 4.2 below.

The following lemma on the ratio of circumradius and inradius of translates of convex bodies is a key argument in the proof of Theorem 4.1.

Lemma 4.2. For all \( K_1, \ldots, K_d+1 \in \mathcal{K}^d \) with non-empty interior there is a constant \( c_0 > 0 \) such that
\[ \chi_d^d \left( \left\{ (x_2, \ldots, x_{d+1}) \in (\mathbb{R}^d)^d : R(L) \leq c_0 r(L) \text{ and } R(L) \leq r \text{ for } L = K_1 \cap \bigcap_{i=2}^{d+1} (K_i + x_i) \right\} \right) > 0 \]
for all \( r > 0 \).

Proof. Let \( u_1, \ldots, u_{d+1} \in \mathbb{R}^d \) be unit vectors whose endpoints are the vertices of a regular simplex. For \( i = 1, \ldots, d+1 \) let \( x_i \) be a point in the boundary of \( K_i \) which has \( u_i \) as an exterior normal vector. The support cone \( S(K_i, x_i) \) of \( K_i \) at \( x_i \) (cf. [27, p. 70]) then satisfies
\[ K_i - x_i \subset S(K_i, x_i) := \text{cl} \left( \bigcup_{t>0} t(K_i - x_i) \right) \subset H^-(K_i, u_i) - x_i, \]
where $H^-(K_i, u_i)$ is the supporting half-space of $K_i$ with exterior unit normal $u_i$ and $\text{cl}$ denotes the closure. By [23, Theorem 12.2.2] it follows that $t(K_i - x_i) \to S(K_i, x_i)$ in the topology of closed convergence as $t \to \infty$. Moreover, since $K_1, \ldots, K_{d+1}$ have non-empty interiors, there are vectors $z_1, \ldots, z_{d+1} \in \mathbb{R}^d$ such that the origin is an interior point of

$$S_0 := \bigcap_{i=1}^{d+1} (S(K_i, x_i) + z_i) \subset \bigcap_{i=1}^{d+1} (H^-(K_i, u_i) - x_i + z_i)$$

and the circumradius of the intersection on the right-hand side is less than 1 (say). Then [27, Theorem 1.8.8] and [28, Theorem 12.3.3] imply that

$$S_0 = \lim_{t \to \infty} \left( t \bigcap_{i=1}^{d+1} (K_i + x_i(t)) \right),$$

where $x_i(t) := -x_i + t^{-1}z_i$ and the convergence is with respect to the Hausdorff distance. Since the inradius and the circumradius of the intersection of translates of convex bodies are continuous with respect to the translations as long as the intersection has non-empty interior, there is some $t_0 > 1$ such that the ratio between inradius and circumradius of

$$t \bigcap_{i=1}^{d+1} (K_i + x_i(t))$$

is close to the corresponding ratio of $S_0$, for $t \geq t_0$, and therefore also

$$1 \leq \frac{R \left( \bigcap_{i=1}^{d+1} (K_i + x_i(t)) \right)}{r \left( \bigcap_{i=1}^{d+1} (K_i + x_i(t)) \right)} < \tilde{c}_0,$$

with a constant $\tilde{c}_0 > 1$ which depends only on $K_1, \ldots, K_{d+1}$. Moreover, for $t \geq t_0 > 1$ we have

$$R \left( \bigcap_{i=1}^{d+1} (K_i + x_i(t)) \right) \leq R \left( \bigcap_{i=1}^{d+1} (H^-(K_i, u_i) - x_i + z_i) \right) < 1$$

and thus

$$R \left( \bigcap_{i=1}^{d+1} (K_i + x_i(t)) \right) < \frac{1}{t}.$$

Therefore, if $r < 1/(2t_0)$ the proof of the lemma is completed by remarking that the intersections are continuous with respect to translations as long as the intersection has non-empty interior and by using the translation invariance of Lebesgue measure. Clearly, this proves the lemma for all $r > 0$. \hfill \Box

**Remark 4.3.** So far we established the positive definiteness of the asymptotic covariance matrix of $d + 1$ functionals $\psi_k, k \in \{0, \ldots, d\}$, where $\psi_k$ was assumed to be homogeneous of degree $k$. The choice of $d + 1$ functionals of the given degrees of homogeneity is motivated by the example of the intrinsic volumes. However, it should be noted that the preceding argument immediately extends to any finite number of functionals $\psi_1, \ldots, \psi_m$ having different non-negative degrees of homogeneity $k_1, \ldots, k_m$ and satisfying a lower bound corresponding to (4.1), that is, $|\psi_i(K)| \geq \beta(\psi_i) r(K)^{k_i}$ for all $K \in \mathcal{K}^d$.  

18
5 Some integral formulas for intrinsic volumes

We shall see in the next section that in the particularly important case of intrinsic volumes and under the assumption of isotropy the asymptotic covariances of Theorem 3.1 can be expressed in terms of the numbers

\[ \rho_{i,j} := \gamma \sum_{n=1}^{\infty} \frac{1}{n!} \int V_i(K_1 \cap \ldots \cap K_n)V_j(K_1 \cap \ldots \cap K_n) \Lambda^{n-1}(d(K_2, \ldots, K_n)) \, \Omega(dK_1). \tag{5.1} \]

In this section we study these numbers without isotropy assumption on $Z$. The results are of independent interest.

For $W \in \mathcal{K}^d$ and $i, j \in \{0, \ldots, d\}$, we define

\[ \rho_{i,j}(W) := \sum_{n=1}^{\infty} \frac{1}{n!} \int V_i(K_1 \cap \ldots \cap K_n \cap W)V_j(K_1 \cap \ldots \cap K_n \cap W) \Lambda^n(d(K_1, \ldots, K_n)), \tag{5.2} \]

which is a finite window version of $\rho_{i,j}$. The numbers $\rho_{i,j}(W)$ are further studied in Appendix C. The relationship between (5.1) and (5.2) is given in the next corollary.

**Corollary 5.1.** Let $i, j \in \{0, \ldots, d\}$. If (2.5) is satisfied, then $\rho_{i,j} < \infty$ and

\[ \lim_{r(W) \to \infty} \frac{\rho_{i,j}(W)}{V_d(W)} = \rho_{i,j}. \tag{5.3} \]

If (2.6) is satisfied, then there is a constant $c_{i,j}$ such that

\[ \left| \rho_{i,j} - \frac{\rho_{i,j}(W)}{V_d(W)} \right| \leq \frac{c_{i,j}}{r(W)} \]

for $W \in \mathcal{K}^d$ with $r(W) \geq 1$.

**Proof.** This can be proved in a similar way as Theorem 3.1. \qed

The previous corollary describes $\rho_{i,j}$ as the limit of $V_d(W)^{-1}\rho_{i,j}(W)$ for observation windows with $r(W) \to \infty$. It is, however, more convenient to work with the series representation (5.1). We shall see that this series can be expressed in terms of a finite family of (curvature) measures $H_{i,j}$ to be introduced below.

For $j \in \{0, \ldots, d\}$ and $K \in \mathcal{K}^d$, we let $\Phi_j(K; \cdot)$ denote the $j$-th curvature measure of $K$ (see [23]). In particular, $\Phi_d(K; \cdot)$ is the restriction of Lebesgue measure to $K$ while $\Phi_{d-1}(K; \cdot)$ is half the $(d-1)$-dimensional Hausdorff measure restricted to the boundary of $K$ (if the affine hull of $K$ has full dimension). Furthermore, $\Phi_j(K; \mathbb{R}^d) = V_j(K)$ for all $j \in \{0, \ldots, d\}$. For $j \in \{0, \ldots, d-1\}$, $n \in \mathbb{N}$, and $K_1, \ldots, K_n \in \mathcal{K}^d$ we define

\[ \Phi_j(K_1, \ldots, K_n; \cdot) := \Phi_j(K_1 \cap \ldots \cap K_n; \partial K_1 \cap \ldots \cap \partial K_n \cap \cdot). \tag{5.4} \]

Since $\Phi_j(K_1; \cdot)$, $j \in \{0, \ldots, d-1\}$, is concentrated on the boundary $\partial K_1$ of $K_1$, this definition is consistent with the case $n = 1$. For $i \in \{1, \ldots, d-1\}$ and $k \in \{1, \ldots, d-i\}$, we define a measure $H_{i,d}^k$ on $\mathbb{R}^d$ by

\[ H_{i,d}^k := \gamma \int \int \int 1\{y - z \in \cdot\} 1\{z \in K_1 \cap \ldots \cap K_k\} \Phi_i(K_1, \ldots, K_k; dy) \, dz \]

\[ \Lambda^{k-1}(d(K_1, \ldots, K_{k-1})) \, \Omega(dK_k), \tag{5.5} \]

\[ H_{i,d}^0 := \gamma \int \int \int 1\{y - z \in \cdot\} 1\{z \in K_1 \cap \ldots \cap K_k\} \Phi_i(K_1, \ldots, K_k; dy) \, dz \]

\[ \Lambda^{d-k}(d(K_1, \ldots, K_{k-1})) \, \Omega(dK_k), \tag{5.6} \]

where $\mathbf{1}$ denotes the characteristic function.
with the appropriate interpretation of the case $k = 1$.

For $i, j \in \{1, \ldots, d-1\}$, $k \in \{1, \ldots, d - i\}$, $l \in \{1, \ldots, d - j\}$, and $m \in \{0, \ldots, k \wedge l\}$ we define a measure $H_{i,j}^{k,l,m}$ on $\mathbb{R}^d$ by

$$H_{i,j}^{k,l,m} := \gamma \int \int \int 1 \{y - z \in \cdot\} 1 \{y \in K_{k+1}^o \cap \ldots \cap K_{k+l-m}^o, \, z \in K_{1}^o \cap \ldots \cap K_{k-m}^o\} \Phi_i(K_1, \ldots, K_k; dy) \Phi_j(K_{k+1-m}, \ldots, K_{k+l-m}; dz) \Lambda^{k+l-m-1}(d(K_1, \ldots, K_{k+l-1})) Q(dK_{k+l-m}), \quad (5.6)$$

where $K^o$ denotes the interior of $K \in \mathcal{K}^d$ and with the appropriate interpretation of the cases $m = k$ or $m = l$. Let

$$H_{i,d} := \sum_{k=1}^{d-i} \frac{1}{k!} H_{i,d}^k, \quad i \in \{1, \ldots, d-1\},$$

$$H_{i,j} := \sum_{k=1}^{d-i} \sum_{l=1}^{d-j} \sum_{m=0}^{k \wedge l} \frac{1}{m!(k-m)!(l-m)!} H_{i,j}^{k,l,m}, \quad i, j \in \{1, \ldots, d-1\},$$

and, for $j \in \{0, \ldots, d-1\},$

$$h_{0,j} := \sum_{l=1}^{d-j} \gamma \int \int \Phi_j(K_1, \ldots, K_l; \mathbb{R}^d) \Lambda^{l-1}(d(K_1, \ldots, K_{l-1})) Q(dK_l). \quad (5.7)$$

Moreover, we define $H_{d,d}(dx) := (1 - e^{-\gamma C_d(x)}) dx$, $H_{0,j} := H_{j,0} := h_{0,j} \delta_0$ for $j \in \{0, \ldots, d-1\}$, and $H_{0,d} := H_{d,0} := (1 - e^{-\gamma V_d}) \delta_0$, where $\delta_0$ is the Dirac measure concentrated at the origin and $C_d(x)$ is the mean covariogram of the typical grain as defined in (2.7).

Subsequently, we assume that

$$Q(\{K \in \mathcal{K}^d : V_d(K) > 0\}) = 1, \quad (5.8)$$

that is, the typical grain almost surely has non-empty interior.

**Theorem 5.2.** Assume that (2.3) and (5.8) are satisfied. Then the measures $H_{i,j}$ are all finite. Moreover, the limits (5.3) are given by

$$\rho_{i,j} = \int e^{\gamma C_d(x)} H_{i,j}(dx), \quad i, j \in \{0, \ldots, d\}. \quad (5.9)$$

For $j = d$ the result remains true without assumption (5.8).

In particular, we thus have

$$\rho_{d,d} = \int \left(e^{\gamma C_d(x)} - 1\right) dx, \quad (5.10)$$

$\rho_{0,d} = e^{\gamma V_d} - 1$, and $\rho_{0,j} = e^{\gamma V_d} h_{0,j}$ for $j \in \{0, \ldots, d-1\}$.

The proof of Theorem 5.2 is based on the following geometric auxiliary result, whose proof will be given in Appendix A. Here we use the abbreviation $[n] = \{1, \ldots, n\}$. 

20
Lemma 5.3. Let \( i \in \{0, \ldots, d-1\} \) and \( n \geq 1 \). Assume that (5.8) is satisfied and \( K_1 \in K^d \) has non-empty interior. Then

\[
\Phi_i(K_1 \cap \ldots \cap K_n; \cdot) = \sum_{k=1}^{d-i} \sum_{I \in [n]} \Phi_i \left( \bigcap_{r \in I} K_r; \cdot \cap \bigcap_{r \in I} \partial K_r \cap \bigcap_{s \notin I} K_s^0 \right)
\]

for \( \Lambda^{n-1} \)-a.e. \( (K_2, \ldots, K_n) \in (K^d)^{n-1} \).

Proof of Theorem 5.2. We start with showing that the measures \( H_{i,j} \) are finite. Let \( i, j \in \{1, \ldots, d-1\} \), \( k \in \{1, \ldots, d-i\} \), \( l \in \{1, \ldots, d-j\} \), and \( m \in \{0, \ldots, k \land l\} \). Then

\[
H_{i,j}^{k,l,m}(\mathbb{R}^d) \leq \gamma \int \int \int 1\{K_1 \cap \ldots \cap K_{k+l-m} \neq \emptyset\} \Phi_i(K_1, \ldots, K_k; dy)
\]

\[
\Phi_j(K_{k+1-m}, \ldots, K_{k+l-m}; dz) \Lambda^{k+l-m-1}(d(K_1, \ldots, K_{k+l-m-1})) Q(dK_{k+l-m})
\]

\[
\leq \gamma \int \int V_0(K_1 \cap \ldots \cap K_{k+l-m}) V_i(K_1) V_j(K_{k+l-m})
\]

\[
\Lambda^{k+l-m-1}(d(K_1, \ldots, K_{k+l-m-1})) Q(dK_{k+l-m}).
\]

For \( k + l - m = 1 \) the right-hand side is finite because of assumption (2.5). Otherwise we obtain by Lemma 3.4 and Lemma 3.3 that

\[
H_{i,j}^{k,l,m}(\mathbb{R}^d)
\]

\[
\leq \gamma^2 \alpha^{k+l-m-2} \int \int \int \sum_{r=0}^{d} V_r((K_1 + x) \cap K_{k+l-m}) V_i(K_1) V_j(K_{k+l-m}) dx Q^2(d(K_1, K_{k+l-m}))
\]

\[
\leq (d + 1) \gamma^2 \alpha^{k+l-m-2} \beta_1 \int \int \sum_{r=0}^{d} V_r(K_1) \sum_{r=0}^{d} V_r(K_{k+l-m})
\]

\[
V_i(K_1) V_j(K_{k+l-m}) Q^2(d(K_1, K_{k+l-m})).
\]

Now it follows from (2.5) that the right-hand side is finite. Similar (but easier) arguments show that the other measures are also finite.

To prove that the series (5.11) is given by (5.9), we distinguish different cases and start with \( i = j = d \). Then we have

\[
\rho_{d,d} = \gamma \sum_{n=1}^{\infty} \frac{1}{n!} \int \int V_d(K_1 \cap \ldots \cap K_n)^2 \Lambda^{n-1}(d(K_2, \ldots, K_n)) Q(dK_1)
\]

\[
= \sum_{n=1}^{\infty} \frac{\gamma^n}{n!} \int \ldots \int 1\{y \in K_1 \cap (K_2 + x_2) \cap \ldots \cap (K_n + x_n)\}
\]

\[
1\{z \in K_1 \cap (K_2 + x_2) \cap \ldots \cap (K_n + x_n)\} dy dz dx_2 \ldots dx_n Q^n(d(K_1, \ldots, K_n))
\]

\[
= \sum_{n=1}^{\infty} \frac{\gamma^n}{n!} \int \int V_d((K_2 - y) \cap (K_2 - z)) \cdots V_d((K_n - y) \cap (K_n - z))
\]

\[
1\{y \in K_1\} 1\{z \in K_1\} dy dz Q^n(d(K_1, \ldots, K_n))
\]
\[
= \sum_{n=1}^{\infty} \frac{\gamma^n}{n!} \int \int (\mathbb{E}V_d(Z_0 \cap (Z_0 + y - z)))^{n-1} 1\{y, z \in K_1\} \, dy \, dz \, Q(dK_1)
\]
\[
= \sum_{n=1}^{\infty} \frac{\gamma^n}{n!} \int \int (\mathbb{E}V_d(Z_0 \cap (Z_0 + y)))^{n-1} 1\{y + z \in K_1\} 1\{z \in K_1\} \, dy \, dz \, Q(dK_1)
\]
\[
= \sum_{n=1}^{\infty} \frac{\gamma^n}{n!} \int C_d(y)^n \, dy = \int (e^{C_d(y)} - 1) \, dy.
\]
For \(i = 0\) and \(j = d\) we get
\[
\rho_{0,d} = \gamma \sum_{n=1}^{\infty} \frac{1}{n!} \int V_d(K_1 \cap \ldots \cap K_n) \Lambda^{n-1}(d(K_2, \ldots, K_n)) \, Q(dK_1)
\]
\[
= \sum_{n=1}^{\infty} \frac{\gamma^n}{n!} \int \ldots \int 1\{z \in K_1 \cap (K_2 + x_2) \cap \ldots \cap (K_n + x_n)\} \, dz \, dx_2 \ldots \, dx_n \, Q^n(d(K_1, \ldots, K_n))
\]
\[
= \sum_{n=1}^{\infty} \frac{\gamma^n}{n!} \int V_d(K_1)V_d(K_2) \cdots V_d(K_n) \, Q^n(d(K_1, \ldots, K_n)),
\]
and hence
\[
\rho_{0,d} = \sum_{n=1}^{\infty} \frac{1}{n!} (\gamma v_d)^n = e^{\gamma v_d} - 1.
\]
This and the preceding calculation remain true without assumption (5.8), the reason being that the integration in (5.1) can effectively be restricted to full-dimensional grains.

Next we turn to \(i = 0\) and \(j \in \{0, \ldots, d - 1\}\). Then, using \(V_j(L) = \Phi_j(L; \mathbb{R}^d)\), for \(L \in \mathcal{K}^d\), and Lemma 5.3 we get
\[
\rho_{0,j} = \gamma \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{l=1}^{d-j} \sum_{J \subseteq [n]} \sum_{|J| = l} \int \Phi_j \left( \bigcap_{r \in J} K_r; \bigcap_{r \in J} \partial K_r \cap \bigcap_{s \in J} K_s^o \right) \Lambda^{n-1}(d(K_2, \ldots, K_n)) \, Q(dK_1)
\]
\[
= \gamma \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{l=1}^{d-j} \sum_{J \subseteq [n]} \sum_{|J| = l} \int \int 1\{z \in \bigcap_{s \notin J} K_s^o\} \Phi_j(K,J;d) \, \Lambda^{n-1}(d(K_2, \ldots, K_n)) \, Q(dK_1),
\]
where \(\Phi_j(K,J;\cdot) = \Phi_j(K_{j_1}, \ldots, K_{j_l}; \cdot)\) for \(J = \{j_1, \ldots, j_l\}\) (see (5.4)). At this stage and also later we use the covariance property
\[
\int h(y) \Phi_i(K_1, \ldots, K_l; dy) = \int h(y + x) \Phi_i(K_1 - x, \ldots, K_l - x; dy), \quad x \in \mathbb{R}^d, \quad (5.11)
\]
which holds for all measurable \(h : \mathbb{R}^d \to [0, \infty]\). This follows from the definition (5.4) and [28 Theorem 14.2.2]. Using (5.11) and then the invariance of \(\Lambda\) under translations, it is easy to check that, for instance,
\[
\int \int \int 1\{z \in K_1^o \cap \ldots \cap K_n^o\} \Phi_j(K_{1,\ldots,i};dz) \Lambda^{n-1}(d(K_1, \ldots, K_{i-1}) \cap K_{i+1}^o, \ldots, K_n^o) \, Q(dK_1)
\]
\[
= \int \int \int 1\{z \in K_1^o \cap \ldots \cap K_n^o\} \Phi_j(K_{1,\ldots,i};dz) \Lambda^{n-1}(d(K_2, \ldots, K_n) \cap K_{i+1}^o, \ldots, K_n^o) \, Q(dK_1).
\]
From such symmetry relations we deduce that

\[
\rho_{0,j} = \gamma \sum_{l=1}^{d-j} \sum_{n=1}^\infty \frac{1}{n!} \left( \begin{array}{c} n \\ l \end{array} \right) \iiint 1\{z \in K_{l+1}^\cap \ldots \cap K_n^\cap\} \\
\Phi_j(K_1, \ldots, K_l; dz) \Lambda^{n-1}(d(K_2, \ldots, K_n)) \mathcal{Q}(dK_1) \\
= \gamma \sum_{l=1}^{d-j} \sum_{n=1}^\infty \frac{1}{n!} \left( \begin{array}{c} n \\ l \end{array} \right) \gamma^{n-l} \iiint V_d(K_{l+1}) \ldots V_d(K_n) \Phi_j(K_1, \ldots, K_l; \mathbb{R}^d) \\
\mathcal{Q}^{n-l}(d(K_{l+1}, \ldots, K_n)) \Lambda^{l-1}(d(K_2, \ldots, K_l)) \mathcal{Q}(dK_1) \\
= e^{\gamma \nu_d} h_{0,j}.
\]

Next we address the case \( i \in \{1, \ldots, d - 1\} \) and \( j = d \). Using again Lemma 5.3 and a symmetry argument (as above), we obtain

\[
\rho_{i,d} = \gamma \sum_{n=1}^\infty \frac{1}{n!} \sum_{k=1}^{d-i} \sum_{l \in [n] \mid l = k} \iiint \mathbb{1}\{y \in \cap_{r \notin I} K_r^\cap\} V_d(K_1 \cap \ldots \cap K_n) \\
\Phi_i(K_I; dy) \Lambda^{n-1}(d(K_2, \ldots, K_n)) \mathcal{Q}(dK_1) \\
= \gamma \sum_{n=1}^\infty \frac{1}{n!} \sum_{k=1}^{d-i} \left( \begin{array}{c} n \\ k \end{array} \right) \iiint \mathbb{1}\{y \in K_{k+1}^\cap \ldots \cap K_n^\cap\} \mathbb{1}\{z \in K_1 \cap \ldots \cap K_n\} \\
dz \Phi_i(K_1, \ldots, K_k; dy) \Lambda^{n-1}(d(K_2, \ldots, K_n)) \mathcal{Q}(dK_1).
\]

Then we interchange the order of summation to get

\[
\rho_{i,d} = \gamma \sum_{k=1}^{d-i} \sum_{n=k}^\infty \frac{\gamma^{n-k}}{k!(n-k)!} \int \ldots \int \mathbb{1}\{x_{k+1} \in (K_{k+1}^\cap - y) \cap (K_{k+1}^\cap - z)\} \\
\ldots \mathbb{1}\{x_n \in (K_n^\cap - y) \cap (K_n - z)\} \mathcal{Q}(dK_{k+1}) \ldots \mathcal{Q}(dK_n) \, dx_{k+1} \ldots dx_n \\
\mathbb{1}\{z \in K_1 \cap \ldots \cap K_k\} \Phi_i(K_1, \ldots, K_k; dy) \, dz \Lambda^{k-1}(d(K_2, \ldots, K_k)) \mathcal{Q}(dK_1) \\
= \gamma \sum_{k=1}^{d-i} \sum_{n=k}^\infty \frac{\gamma^{n-k}}{k!(n-k)!} \iiint \mathbb{E} V_d(Z_0 \cap (Z_0 + y - z))^n \mathbb{1}\{z \in K_1 \cap \ldots \cap K_k\} \\
\Phi_i(K_1, \ldots, K_k; dy) \, dz \Lambda^{k-1}(d(K_2, \ldots, K_k)) \mathcal{Q}(dK_1) \\
= \gamma \sum_{k=1}^{d-i} \frac{1}{k!} \iiint e^{\gamma \nu_d(y-z)} \mathbb{1}\{z \in K_1 \cap \ldots \cap K_k\} \\
\Phi_i(K_1, \ldots, K_k; dy) \, dz \Lambda^{k-1}(d(K_2, \ldots, K_k)) \mathcal{Q}(dK_1),
\]
which yields that
\[ \rho_{i,j} = \frac{\gamma}{n!} \sum_{k=1}^{d-i} \frac{1}{n!} \int e^{\gamma C_d(z)} H_{i,d}^{k}(x) = \int e^{\gamma C_d(z)} H_{i,d}(dx). \]

Finally, we consider the case where \( i, j \in \{1, \ldots, d-1\} \). Again by Lemma 6.3 we get
\[
\rho_{i,j} = \gamma \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k=1}^{d-i} \sum_{l=1}^{d-j} \sum_{k \cap l} \sum_{m=0}^{k \cap l} \left[ \Phi_i(K_1; dy) \Phi_j(K_J; dz) \right] \Lambda^{n-1}(d(K_2, \ldots, K_n)) Q(dK_1) \\
= \gamma \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k=1}^{d-i} \sum_{l=1}^{d-j} \sum_{m=0}^{k \cap l} \sum_{I \cap J \neq \emptyset} \left[ \Phi_i(K_1; dy) \Phi_j(K_J; dz) \right] \Lambda^{n-1}(d(K_2, \ldots, K_n)) Q(dK_1).
\]

A symmetry argument shows (as before) that for each choice of \( I, J \) such that \( |I| = k, |J| = l \) and \( |I \cap J| = m \), the preceding integral has the same value. There are \( \binom{k}{m} \binom{n-k}{l-m} \) possible choices of \( I, J \) with these properties. Thus we obtain
\[
\rho_{i,j} = \gamma \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k=1}^{d-i} \sum_{l=1}^{d-j} \sum_{m=0}^{k \cap l} \left[ \Phi_i(K_1; dy) \Phi_j(K_J; dz) \right] \Lambda^{k+l-m-1}(d(K_1, \ldots, K_{k+l-m-1})) Q(dK_{k+l-m}) \\
= \gamma \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k=1}^{d-i} \sum_{l=1}^{d-j} \sum_{m=0}^{k \cap l} \frac{\{(n \geq k + l - m)\gamma^{n-(k+l-m)}\}}{m!(k-m)!(l-m)!(n-(k+l-m))!} \\
\int \cdots \int x_r \in (K_r^o - y) \cap (K_r^o - z) \\
\cdots dx_{k+l-m+1} \cdots dx_{n} Q(dK_{k+l-m+1}) \cdots Q(dK_n) \\
\{(y \in K_{k+1}^o \cap \ldots \cap K_{k+l-m}^o) \{z \in K_{k+1}^o \cap \ldots \cap K_{k-m}^o\} \}
\Phi_i(K_1, \ldots, K_k; dy) \Phi_j(K_{k+1-m}, \ldots, K_{k+1-m}; dz) \\
\Lambda^{k+l-m-1}(d(K_1, \ldots, K_{k+l-m-1})) Q(dK_{k+l-m})
\]

24
and hence
\[ \rho_{i,j} = \gamma \sum_{k=1}^{d-i} \sum_{l=1}^{d-j} \sum_{m=0}^{k\wedge l} \frac{1}{m!(k-m)!(l-m)!} \int \int \int \sum_{n=k+l-m}^{\infty} \frac{(\gamma C_d(y-z))^n}{(n-(k+l-m))!} \]
\[ \times 1\{y \in K_{k+1}^0 \cap \ldots \cap K_{k+l-m}^0\} 1\{z \in K_k^0 \cap \ldots \cap K_{k-l}^0\} \]
\[ \Phi_i(K_1, \ldots, K_k; dy) \Phi_j(K_{k-l}, \ldots, K_{k+l-m}; dz) \]
\[ \Lambda^{k+l-m-1}(d(K_1, \ldots, K_{k+l-m-1})) \mathbb{Q}(dK_{k+l-m}) \]
\[ = \sum_{k=1}^{d-i} \sum_{l=1}^{d-j} \sum_{m=0}^{k\wedge l} \frac{1}{m!(k-m)!(l-m)!} \int e^{\gamma C_d(x)} H_{i,j}^{k,l,m}(dx). \]

This concludes the proof of the theorem. \( \square \)

The measures \( H_{i,j}^{1,1,m}, m \in \{0, 1\} \), can be expressed in terms of the mixed moment measures
\[ M_{i,j} := \mathbb{E} \int \int 1\{(y,z) \in \cdot\} \Phi_i(Z_0; dy) \Phi_j(Z_0; dz), \quad i,j \in \{0, \ldots, d\}, \]
and the functions \( C_j : \mathbb{R}^d \to [0, \infty) \), defined by
\[ C_j(x) := \mathbb{E} \Phi_j(Z_0; Z_0^0 + x), \quad x \in \mathbb{R}^d, \ j \in \{0, \ldots, d\}. \]

**Lemma 5.4.** Assume that (2.5) is satisfied. Then, for any \( i,j \in \{0, \ldots, d-1\} \),
\[ H_{i,d}^{1} = \gamma \int 1\{y - z \in \cdot\} M_{i,d}(d(y, z)), \quad (5.12) \]
\[ H_{i,j}^{1,1,0} = \gamma^2 \int 1\{y - z \in \cdot\} C_i(y - z) M_{j,d}(d(z, y)), \quad (5.13) \]
\[ H_{i,j}^{1,1,1} = \gamma \int 1\{y - z \in \cdot\} M_{i,j}(d(y, z)). \quad (5.14) \]

**Proof.** Equations (5.12) and (5.14) follow directly from the definitions, while (5.13) follows from an easy calculation using the covariance property (5.11). \( \square \)

Lemma 5.4 implies in particular that
\[ H_{d-1,d} = \gamma \int 1\{y - z \in \cdot\} M_{d-1,d}(d(z, y)), \quad (5.15) \]
\[ H_{d-1,d-1} = \gamma^2 \int 1\{y - z \in \cdot\} C_{d-1}(y - z) M_{d-1,d}(d(z, y)) \]
\[ + \gamma \int 1\{y - z \in \cdot\} M_{d-1,d-1}(d(y, z)). \quad (5.16) \]
These formulas will be used in the next section.
6 Covariance structure in the isotropic case

In this section we assume that the typical grain is isotropic, that is, its distribution $Q$ is invariant under rotations and that the moment assumption (2.5) is satisfied. Our aim is to derive more explicit formulas for the asymptotic covariances

$$\sigma_{i,j} := \lim_{r(W) \to \infty} \frac{\text{Cov}(V_i(Z \cap W), V_j(Z \cap W))}{V_d(W)}, \quad i, j \in \{0, \ldots, d\}; \quad (6.1)$$

confer the statement of Theorem 3.1.

Using the iterated version of the local kinematic formula [28, Theorem 5.3.2], which is obtained by combining [28, Theorem 6.4.1, (b) and (6.15)] and [28, Theorem 6.4.2, (6.20)], we obtain for $j \in \{0, \ldots, d - 1\}$ and $l \in \{1, \ldots, d - j\}$ that

$$\gamma \int \Phi_j(K_1, \ldots, K_l, \mathbb{R}^d) \Lambda^{l-1}(d(K_1, \ldots, K_l-1)) Q(dK_l) = \sum_{m_1, \ldots, m_l = j}^{d-1} \sum_{m_1 + \ldots + m_l = (l-1)d+j} c_d \prod_{i=1}^l c_d^{m_i} \gamma v_{m_i},$$

where, as in [28, (5.4)],

$$c_d^{m_j} := \frac{m_j! k_m}{j^{k_j}}, \quad m, j \in \{0, \ldots, d\}.$$

Combining this with (5.7) and Theorem 5.2 and under the assumption (5.8), we get

$$\rho_{0,j} = e^{\gamma v_d} P_j(\gamma v_j, \ldots, \gamma v_{d-1}), \quad j \in \{0, \ldots, d - 1\}, \quad (6.2)$$

where $P_j$ (a multivariate polynomial on $\mathbb{R}^{d-j}$ of degree $d$) is defined by

$$P_j(t_j, \ldots, t_{d-1}) := c_d^j \sum_{l=1}^{d-j} \frac{1}{l!} \sum_{m_1, \ldots, m_l = j}^{d-1} \prod_{i=1}^l c_d^{m_i} t_{m_i}. \quad (6.3)$$

The following main result of this section shows that the asymptotic covariances (6.1) are linear combinations of the numbers $\rho_{i,j}$ given by (5.1). To describe the coefficients, we define for any $j \in \{0, \ldots, d - 1\}$ and $l \in \{j, \ldots, d\}$ a polynomial $P_{j,l}$ on $\mathbb{R}^{d-j}$ of degree $l - j$ by

$$P_{j,l}(t_j, \ldots, t_{d-1}) := 1\{l = j\} + c_d^l \sum_{s=1}^{l-j} \frac{(-1)^s}{s!} \sum_{m_1, \ldots, m_s = j}^{d-1} \prod_{i=1}^s c_d^{m_i} t_{m_i} \quad (6.3)$$

and complement this definition by $P_{d,d} := 1$.

**Theorem 6.1.** Assume that the typical grain is isotropic and suppose that (2.5) holds. Then

$$\sigma_{i,j} = (1 - p)^2 \sum_{k=i}^d \sum_{l=j}^d P_{i,k} (\gamma v_i, \ldots, \gamma v_{d-1}) P_{j,l} (\gamma v_j, \ldots, \gamma v_{d-1}) \rho_{k,l},$$

for all $i, j \in \{0, \ldots, d\}$. 

26
Proof. The formula preceding Theorem 9.1.4 in [28] is the finite volume version of the fundamental result of [18] and [5] on the densities of intrinsic volumes. Using this result, we obtain for all \( i \in \{0, \ldots, d-1\} \) and \( A \in \mathcal{K}^d \) that

\[
\mathbb{E}V_i(Z \cap A) - V_i(A) = -(1 - p) \sum_{k=i}^{d} V_k(A) P_{i,k}(\gamma v_i, \ldots, \gamma v_{d-1}).
\]  

(6.4)

For \( i = d \), equation (6.4) is a direct consequence of stationarity and the definition \( P_{d,d} = 1 \). Using this formula in (5.1), we obtain the assertion from (5.1).

**Corollary 6.2.** Assume that (2.5) is satisfied. Then, for \( i, j \in \{d-1, d\} \), the assertions of Theorem 6.1 remain true in the general stationary case (without isotropy assumption). Moreover,

\[
\sigma_{d,d} = (1 - p)^2 \int (e^{\gamma C_d(x)} - 1) \, dx,
\]

\[
\sigma_{d-1,d} = -(1 - p)^2 \gamma v_{d-1} \int (e^{\gamma C_d(x)} - 1) \, dx + (1 - p)^2 \gamma \int e^{\gamma C_d(x-y)} M_{d-1,d}(d(x,y)).
\]

If, in addition, (5.8) holds, then

\[
\sigma_{d-1,d-1} = (1 - p)^2 \gamma^2 v_{d-1}^2 \int (e^{\gamma C_d(x)} - 1) \, dx
\]

\[
+ (1 - p)^2 \gamma^2 \int e^{\gamma C_d(x-y)} (C_{d-1}(x-y) - 2v_{d-1}) M_{d-1,d}(d(y,x))
\]

\[
+ (1 - p)^2 \gamma \int e^{\gamma C_d(x-y)} M_{d-1,d-1}(d(x,y)).
\]

Proof. The formula preceding Theorem 9.1.4 in [28] does not require isotropy for \( j = d - 1 \). Therefore, for \( i, j \in \{d-1, d\} \), the proof of Theorem 6.1 applies without this assumption.

By definition (6.3), \( P_{d-1,d-1} = P_{d,d} = 1 \) and \( P_{d-1,d}(\gamma v_{d-1}) = -\gamma v_{d-1} \). Therefore we obtain from Theorem 6.1 that \( \sigma_{d,d} = (1 - p)^2 \rho_{d,d} \),

\[
\sigma_{d-1,d} = (1 - p)^2 (\rho_{d-1,d} - \gamma v_{d-1} \rho_{d,d}),
\]

\[
\sigma_{d-1,d-1} = (1 - p)^2 (\rho_{d-1,d-1} - \gamma v_{d-1} \rho_{d-1,d} - \gamma v_{d-1} \rho_{d,d-1} + \gamma^2 v_{d-1}^2 \rho_{d,d}).
\]

Inserting first (5.10), (5.9), and then (5.15) and (5.10), we obtain the result.

Together with Corollary 6.2, the next corollary provides rather explicit formulas for the asymptotic covariance in the two-dimensional isotropic case.

**Corollary 6.3.** Let \( d = 2 \), assume that the typical grain is isotropic, and suppose that (2.5) and (5.8) are satisfied. Then

\[
\sigma_{0,0} = (1 - 2p)(1 - p)\gamma + (1 - p)(2p - 3)\frac{\gamma^2 v_1^2}{\pi}
\]

\[
+ (1 - p)^2 \left( \gamma - \frac{\gamma^2 v_1^2}{\pi} \right)^2 \int (e^{\gamma C_2(x)} - 1) \, dx
\]

\[
+ (1 - p)^2 \int \chi(x-y) M_{1,2}(d(x,y)) + \frac{4}{\pi^2} (1 - p)^2 \gamma^3 v_1^2 \int e^{\gamma C_2(x-y)} M_{1,1}(d(x,y)),
\]

\[
27
\]
\[ \sigma_{0,1} = (1 - p)^2 \gamma v_1 + (1 - p)^2 \left( \frac{\gamma^2 v_1^3}{\pi} \right) \int (e^{\gamma C_2(x)} - 1) \, dx \]
\[ + (1 - p)^2 \int \tilde{\chi}(x - y) \, M_{1,2}(d(y, x)) - (1 - p)^2 \frac{2 \gamma^2 v_1}{\pi} \int e^{\gamma C_2(x - y)} \, M_{1,1}(d(x, y)), \]
\[ \sigma_{0,2} = p(1 - p) - (1 - p)^2 \left( \gamma - \frac{\gamma^2 v_1^2}{\pi} \right) \int (e^{\gamma C_2(x)} - 1) \, dx \]
\[ - (1 - p)^2 \frac{2 \gamma^2 v_1}{\pi} \int e^{\gamma C_2(x - y)} \, M_{1,2}(d(x, y)), \]

where
\[ \chi(z) := e^{\gamma C_2(z)} \left( \frac{4 \gamma^4 v_1^2}{\pi^2} (C_1(z) - v_1) + \frac{4 \gamma^3 v_1}{\pi} \right), \]
\[ \tilde{\chi}(z) := e^{\gamma C_2(z)} \left( \frac{3 \gamma^3 v_1^2}{\pi^2} - \frac{2 \gamma^3 v_1}{\pi} C_1(z) - \gamma^2 \right). \]

The proof of \( \sigma_{0,2} \) remains true without assumption \((5.8)\).

**Proof.** We have \( P_{0,0}(t_0, t_1) = 1, P_{0,1}(t_0, t_1) = -\frac{2}{\pi} t_1, P_{0,2}(t_0, t_1) = -t_0 + \frac{1}{\pi} t_1^2, P_{1,1}(t_1) = 1, P_{1,2}(t_1) = -t_1, \) and \( P_{2,2}(t_1) = 1. \) Moreover, we have \( P_0(t_0, t_1) = t_0 + \frac{1}{\pi} t_1^2 \) and \( P_1(t_1) = t_1. \) Using \((6.2)\), Theorem \(5.2\) and Lemma \(5.4\) we obtain
\[ \rho_{0,0} = e^{\gamma v_2} \left( \gamma + \frac{\gamma^2 v_1^2}{\pi} \right), \quad \rho_{0,1} = e^{\gamma v_2} \gamma v_1, \quad \rho_{0,2} = e^{\gamma v_2} - 1, \]
\[ \rho_{1,1} = \frac{2}{\pi} \int e^{\gamma C_2(y - z)} C_1(y - z) \, M_{1,2}(d(z, y)) + \frac{\gamma}{\pi} \int e^{\gamma C_2(y - z)} \, M_{1,1}(d(y, z)), \]
\[ \rho_{1,2} = \frac{1}{\pi} \int e^{\gamma C_2(y - z)} \, M_{1,2}(d(y, z)), \quad \rho_{2,2} = \int \left( e^{\gamma C_2(x)} - 1 \right) \, dx. \]

The result follows by substituting these expressions into Theorem \(6.1\). \( \square \)

The proof of Theorem \(6.1\) also yields the following non-asymptotic result for which definition \((5.2)\) should be recalled.

**Theorem 6.4.** Assume that the typical grain is isotropic and that \((2.5)\) holds. Let \( W \in \mathcal{K}_d \) and \( i, j \in \{0, \ldots, d\} \). Then
\[ \text{Cov}(V_i(Z \cap W), V_j(Z \cap W)) \]
\[ = (1 - p)^2 \sum_{k=i}^d \sum_{l=j}^d P_{i,k}(\gamma v_i, \ldots, \gamma v_{d-1}) P_{j,l}(\gamma v_j, \ldots, \gamma v_{d-1}) \rho_{k,l}(W). \]

Further discussion of the case \( d = 2 \) will be provided in Appendix C.
7 The spherical Boolean model

In this section we show how some of the formulas of Section 6 can be used to determine explicitly the covariances of a stationary and isotropic Boolean model whose typical grain is the unit ball $B^d$. In this particular case, we get from Corollary 6.2 that

$$\sigma_{d-1,d} = (1-p)^2 \bar{\gamma} \left[ -v_{d-1} \int e^{\gamma C_d(x)}(x) dx + \frac{1}{2} \int \int_{B^d} e^{\gamma C_d(x-y)} dy \mathcal{H}^{d-1}(dx) \right],$$

where $C_d(x) = V_d(B^d \cap (B^d + x))$ and $\mathcal{H}^j$ denotes the $j$-dimensional Hausdorff measure. Clearly, $\bar{C}_d(t) := V_d(B^d \cap (B^d + tv))$, for $t \geq 0$ and $v \in S^{d-1}$, is independent of the choice of the unit vector $v$ and

$$\bar{C}_d(t) = 2 \kappa_{d-1} \int_0^1 \sqrt{1 - u^{d-1}} du = 2 \pi \frac{d}{(d+1)} \int_0^1 \sqrt{1 - u^{d-1}} du, \quad t \in [0, 2].$$

Introducing polar coordinates, we get

$$F_d(\gamma) := v_{d-1} \int e^{\gamma C_d(x)}(x) - 1 dx = v_{d-1} d \kappa_d \int_0^2 e^{\gamma C_d(t)}(t) - 1 t^{d-1} dt =: v_{d-1} f_d(\gamma),$$

where $v_{d-1} = d \kappa_d/2$. On the other hand, for an arbitrary unit vector $v \in S^{d-1}$, by the rotation invariance of $B^d$ we get

$$G_d(\gamma) := \frac{1}{2} \int_{S^{d-1}} \int_{B^d} e^{\gamma C_d(x-y)} dy \mathcal{H}^{d-1}(dx) = v_{d-1} \int_{B^d} e^{\gamma C_d(v-y)} dy.$$

We parametrize $y$ in the form

$$y = tv + \sqrt{1 - t^2} sw, \quad t \in [-1, 1], s \in [0, 1], w \in v^\perp \cap S^{d-1},$$

and hence we obtain

$$G_d(\gamma) = v_{d-1} (d-1) \kappa_{d-1} \int_0^1 \int_0^1 s^{d-2} \sqrt{1 - t^2}^{d-1} \exp \left( \gamma \bar{C}_d \left( \sqrt{(1-t)^2 + (1-t^2)s^2} \right) \right) ds dt$$

$$= v_{d-1} (d-1) \kappa_{d-1} \int_0^1 \int_0^1 \exp \left( \gamma \bar{C}_d \left( \sqrt{(2-t)^2 + (2-t)s^2} \right) \right) s^{d-2} \sqrt{2-t}^{d-1} ds dt$$

$$= : v_{d-1} g_d(\gamma).$$

Therefore we have

$$\sigma_{d-1,d} = (1-p)^2 \bar{\gamma} v_{d-1} ( - f_d(\gamma) + g_d(\gamma) ),$$

which shows that the sign of the covariance $\sigma_{d-1,d}$ is completely determined by the sign of the function $g_d - f_d$.

It is preferable to plot the covariances as functions of the intensity. Here we have

$$\sigma_{d-1,d}(\gamma) = \gamma e^{-2\kappa_d} v_{d-1} (g_d(\gamma) - f_d(\gamma)).$$

Figure 1 shows the result for various dimensions.

Next we determine the correlation coefficient $C_{d-1,d}(\gamma)$, as a function of the intensity $\gamma$. For this we also have to determine explicitly $\sigma_{d,d}$ and $\sigma_{d-1,d-1}$, which requires some further calculations. First, we have

$$\sigma_{d,d} = (1-p)^2 \int (e^{\gamma C_d(x)} - 1) dx = (1-p)^2 f_d(\gamma),$$

29
hence
\[
\sqrt{\sigma_{d,d}} = (1 - p)\sqrt{f_d}(\gamma);
\]
second,
\[
\sigma_{d-1,d-1} = (1 - p)^2 \gamma^2 \left[ (v_{d-1})^2 f_d(\gamma) + \int e^{\gamma C_d(x-y)} C_{d-1}(x-y) M_{d-1,d}(d(y,x)) \right.
\]
\[
- 2v_{d-1} \int e^{\gamma C_d(x-y)} M_{d-1,d}(d(y,x)) + \frac{1}{\gamma} \int e^{\gamma C_d(x-y)} M_{d-1,d-1}(d(x,y)) \right].
\]

In order to determine \( C_{d-1}(x) \) explicitly, we introduce the surface area of a spherical cap of height \( 1 - h \), denoted by \( \text{Cap}_d(h) \), for \( h \in (0, 1] \), that is,
\[
\text{Cap}_d(h) = (d - 1)\kappa_{d-1} \int_0^1 \sqrt{1 - s^2} ds.
\]

Therefore
\[
C_{d-1}(x) = \frac{1}{2} \mathcal{H}^{d-1}(S_{d-1} \cap (B^d + x)) = \frac{1}{2} \text{Cap}_d(||x||/2)
\]
\[
= \frac{1}{2} (d - 1)\kappa_{d-1} \int_0^1 \sqrt{1 - s^2}^{d-3} ds =: C_{d-1}(||x||),
\]
for \( 0 < ||x|| \leq 2 \). Let \( v \in S^{d-1} \) be fixed. Then, arguing as in the derivation of (7.1), we obtain
\[
\int e^{\gamma C_d(x-y)} C_{d-1}(x-y) M_{d-1,d}(d(y,x)) = v_{d-1} \int_{B^d} e^{\gamma C_d(x-v)} C_{d-1}(x-v) dx
\]
Finally, since
\[
C_d(\sqrt{2-t})^{d-1} \left( \sqrt{(2-t)^2 + t(2-t)\sigma^2} \right) \]
we have
\[
\tilde{C}_d \left( \sqrt{(2-t)^2 + t(2-t)s^2} \right) \]
Furthermore, we have
\[
\gamma C_d \left( \sqrt{(2-t)^2 + t(2-t)s^2} \right)
\]
shown in Figure 2.

This finally implies that
\[
\frac{\sigma_{d-1,d-1}}{(1-p)^2\gamma^2} = \left( \frac{d\kappa_d}{2} \right)^2 f_d(\gamma) - \frac{(d\kappa_d)^2}{2} g_d(\gamma) + \frac{d\kappa_d(d-1)\kappa_{d-1}}{2} h_d(\gamma) + \frac{d\kappa_d(d-1)\kappa_{d-1}}{4\gamma} k_d(\gamma).
\]
Finally, since
\[
M_{d-1,d-1} = \frac{1}{4} \int_{S^{d-1}} \int_{S^{d-1}} 1\{(y,z) \in \cdot\} \mathcal{H}^{d-1}(dy) \mathcal{H}^{d-1}(dz),
\]
we get (with an arbitrary unit vector \(v_0\))
\[
\int e^{\gamma C_d(x-y)} M_{d-1,d-1}(d(x,y))
\]
\[
= \frac{d\kappa_d}{4} \int_{S^{d-2}} \int_0^\pi \exp \left( \gamma C_d (v_0 - [\cos \theta v_0 + \sin \theta v]) \right) \sin^{d-2} \theta \, d\theta \, \mathcal{H}^{d-2}(dv)
\]
\[
= \frac{d\kappa_d(d-1)\kappa_{d-1}}{4} \int_0^\pi \sin^{d-2} \theta \exp \left( \gamma \tilde{C}_d \left( \sqrt{2(1-\cos \theta)} \right) \right) \, d\theta
\]
\[
= \frac{d\kappa_d(d-1)\kappa_{d-1}}{4} \int_0^2 \sqrt{s(2-s)}^{d-3} \exp \left( \gamma \tilde{C}_d \left( \sqrt{2(2-s)} \right) \right) \, ds
\]
\[
=: \frac{d\kappa_d(d-1)\kappa_{d-1}}{4} k_d(\gamma).
\]
Hence, we have
\[
\frac{\sigma_{d-1,d-1}}{(1-p)^2\gamma^2} = \left( \frac{d\kappa_d}{2} \right)^2 f_d(\gamma) - \frac{(d\kappa_d)^2}{2} g_d(\gamma) + \frac{d\kappa_d(d-1)\kappa_{d-1}}{2} h_d(\gamma) + \frac{d\kappa_d(d-1)\kappa_{d-1}}{4\gamma} k_d(\gamma).
\]
This finally implies that
\[
\text{Cor}_{d-1,d}(\gamma) = \frac{\frac{d\kappa_d}{2} (g_d(\gamma) - f_d(\gamma))}{\sqrt{f(\gamma)} \sqrt{\left( \frac{d\kappa_d}{2} \right)^2 f_d(\gamma) - \frac{(d\kappa_d)^2}{2} g_d(\gamma) + \frac{d\kappa_d(d-1)\kappa_{d-1}}{2} h_d(\gamma) + \frac{d\kappa_d(d-1)\kappa_{d-1}}{4\gamma} k_d(\gamma)}}.
\]
From these considerations we also deduce the plausible fact that
\[
\lim_{\gamma \downarrow 0} \text{Cor}_{d-1,d}(\gamma) = \lim_{\gamma \downarrow 0} \frac{1}{\sqrt{\gamma} \int C_d(x) \, dx} \sqrt{\frac{1}{\gamma} \left( \frac{d\kappa_d}{2} \right)^2} = 1,
\]
which is confirmed by our numerical calculations. Plots of \(\text{Cor}_{d-1,d}(\gamma)\) for \(d = 2, \ldots, 6\) are shown in Figure 2.
In a similar way, the formulas from Corollary 6.3 can be specified in the case of a planar Boolean model with the unit circle as deterministic typical grain. Then we have
\[ \chi(r, \gamma) = 4\gamma^3 e^{\gamma C_2(r)} (\gamma C_1(r) - \pi \gamma + 1) \]
\[ \tilde{\chi}(r, \gamma) = \gamma^2 e^{\gamma C_2(r)} (3\pi \gamma - 2C_1(r)\gamma - 1), \]
and, for instance,
\[ \sigma_{0,0}(\gamma) = (1 - 2p)(1 - p)\gamma + (1 - p)(2p - 3)\gamma^2 \pi + (1 - p)^2 \gamma^2 (1 - \pi \gamma)^2 f_2(\gamma) \]
\[ + (1 - p)^2 2\pi \int_0^2 \int_0^1 \chi \left( \sqrt{(2 - t)^2 + t(2 - t)s^2}, \gamma \right) \sqrt{t(2 - t)} \, ds \, dt \]
\[ + (1 - p)^2 \gamma^3 4\pi \int_0^\pi \exp \left( \gamma \tilde{C}_2 \left( \sqrt{2(1 - \cos(t))} \right) \right) \, dt, \]
where \( p = p(\gamma) = 1 - e^{-\pi \gamma} \). Moreover,
\[ \sigma_{0,1}(\gamma) = (1 - p)^2 \gamma^3 \pi + (1 - p)^2 \gamma^2 \pi (1 - \pi \gamma) f_2(\gamma) \]
\[ + (1 - p)^2 2\pi \int_0^2 \int_0^1 \tilde{\chi} \left( \sqrt{(2 - t)^2 + t(2 - t)s^2}, \gamma \right) \sqrt{t(2 - t)} \, ds \, dt \]
\[ - (1 - p)^2 \gamma^3 2\pi \int_0^\pi \exp \left( \gamma \tilde{C}_2 \left( \sqrt{2(1 - \cos(t))} \right) \right) \, dt, \]
\[ \sigma_{0,2}(\gamma) = p(1 - p) - (1 - p)^2 \gamma (1 - \pi \gamma) f_2(\gamma) - (1 - p)^2 2\gamma^2 \pi g_2(\gamma), \]
\[ \sigma_{2,2}(\gamma) = (1 - p)^2 f_2(\gamma). \]
The variances and covariances as well as the correlation functions for the planar case are plotted in Figure 3 and Figure 4.

The unique zero $\gamma_0$ of $\sigma_{0,1}$ satisfies $\gamma_0 \in [0.90785, 0.90786]$, whereas $\sigma_{0,2}$ has two zeros $\gamma_1, \gamma_2$ with $\gamma_1 \in [0.13336, 0.13337]$ and $\gamma_2 \in [1.097998, 1.097999]$. The unique zero $\gamma_3$ of $\sigma_{1,2}$ satisfies $\gamma_3 \in [0.369200, 0.369201]$; see also Figure 1.

A local minimum of $\sigma_{0,1}$ is in $[0, 0.2239]$ with value $\approx 0.0010234$, $\sigma_{0,1}$ has a local minimum in $[0, 0.2237]$ with value $\approx 0.0104$. A local maximum of $\sigma_{0,1}$ is in $[0, 0.535, 0.537]$ with value $\approx 0.06515$, a local maximum of $\sigma_{0,2}$ is in $[0, 0.511, 0.513]$ with value $\approx 0.52577$. A global maximum of $\sigma_{0,1}$ is in $[0, 0.369200, 0.369201]$ with value $\approx 0.067755$, and a global minimum is in $[1.294, 1.296]$ with value $\approx -0.03179$.

Close to the global maximum of $\sigma_{0,1}$ is the global maximum of $\sigma_{0,2}$, which is in $[0.0524, 0.0526]$ with value $\approx 0.070517$. Moreover, $\sigma_{0,2}$ has a local minimum in $[0.3674, 0.3676]$ with value $\approx -0.1673672$, and a local maximum in $[1.36, 1.37]$ with value $\approx 0.0053$. Furthermore, $\sigma_{0,2}$ has a global minimum in $[0.43, 0.45]$ with value $\approx -0.77$, and a local maximum in $[3.1, 3.3]$ with value $\approx 0.4644$. For large intensities, $\sigma_{0,1} \approx -0.76$ and $\sigma_{0,2} \approx -0.759$ and $\sigma_{0,2} \approx 0.45$.

8 Normal approximation via the Malliavin-Stein method

In this section we prepare the central limit theorems for geometric functionals of a Boolean model by proving a general result on the normal approximation of Poisson functionals. Our approach is based on recent results from [21, 23].

Throughout this section, let $\eta$ be a Poisson process on a measurable space $(X, \mathcal{X})$ with a $\sigma$-finite intensity measure $\lambda$, see [12, Chapter 12]. Consider a $[-\infty, \infty]$-valued random variable $F$ such that $\mathbb{P}(|F| < \infty) = 1$ and $F = f(\eta)$ $\mathbb{P}$-a.s. for some measurable $f : \mathbb{N} \to \mathbb{R}$. Any such $f$ is called representative of the Poisson functional $F$. If $f$ is a (fixed) representative of $F$ we define

$$D^n_{x_1, \ldots, x_n} F := D^n_{x_1, \ldots, x_n} f(\eta), \quad n \in \mathbb{N}, \ x_1, \ldots, x_n \in X,$$
where $D^n$ is the $n$-th iterated difference operator used in Section 3. If $\tilde{f}$ is another representative of $F$, then the multivariate Mecke equation (see e.g. [13, (2.10)]) implies that $D^n x_1,\ldots,x_n f(\eta) = D^n x_1,\ldots,x_n \tilde{f}(\eta)$ $\mathbb{P}$-a.s. and for $\lambda^n$-a.e. $(x_1,\ldots,x_n) \in X^n$. Let $L^2_\eta$ denote the space of all Poisson functionals $F$ such that $\mathbb{E} F^2 < \infty$. For $F \in L^2_\eta$ we define $f_n : X^n \to \mathbb{R}$ by

$$f_n(x_1,\ldots,x_n) = \frac{1}{n!} \mathbb{E} D^n x_1,\ldots,x_n F,$$

It was shown in [13] that $f_n$ belongs to the space $L^2_s(\lambda^n)$ of symmetric functions on $X^n$ that are square-integrable with respect to $\lambda^n$. Now the Fock space representation (see [13]) tells us that

$$\text{Var} F = \sum_{n=1}^{\infty} n! \|f_n\|^2_n,$$

(8.1)

where $\| \cdot \|_n$ denotes the norm in $L^2(\lambda^n)$. Moreover, it is known from [13] that $F$ has the representation

$$F = \mathbb{E} F + \sum_{n=1}^{\infty} I_n(f_n),$$

(8.2)

where $I_n(\cdot)$ stands for the $n$-th multiple Wiener-Itô integral, and the right-hand side converges in $L^2(\mathbb{P})$. The identity (8.2) is called Wiener-Itô chaos expansion of $F$. The multiple Wiener-Itô integrals are defined for square integrable symmetric functions and are orthogonal in the sense that

$$\mathbb{E} I_n(f) I_m(g) = \begin{cases} n!(f,g)_n, & n = m, \\ 0, & n \neq m, \end{cases}$$

for $f \in L^2_s(\lambda^n)$, $g \in L^2_s(\lambda^m)$, and $n, m \geq 1$, where $(\cdot,\cdot)_n$ denotes the scalar product in $L^2(\lambda^n)$. If the condition

$$\sum_{n=1}^{\infty} n n! \|f_n\|^2_n < \infty$$

(8.3)
is satisfied, the difference operator (3.6) has the representation
\[ D_x F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(x, \cdot)) \] (8.4)
\( \mathbb{P} \)-a.s. for \( \lambda \)-a.e. \( x \in X \) (see e.g. [13, Theorem 3.3]). From now on, we write \( F \in \text{dom} \, D \) if \( F \in L^2_\eta \) satisfies (8.3). For a Poisson functional \( F \in L^2_\eta \) such that \( \sum_{n=1}^{\infty} n^2 n! \| f_n \|_{\infty}^2 < \infty \) the Ornstein-Uhlenbeck generator is given by
\[ LF = -\sum_{n=1}^{\infty} n I_n(f_n) \]
and its pseudo-inverse is
\[ L^{-1} F = -\sum_{n=1}^{\infty} \frac{1}{n} I_n(f_n) \] (8.5)
for \( F \in L^2_\eta \). These operators together with the difference operator and the Skorohod integral, which is not used in this paper, are called Malliavin operators. Combining (8.4) and (8.5), we see that
\[ D_x L^{-1} F = -\sum_{n=1}^{\infty} I_{n-1}(f_n(x, \cdot)) \] (8.6)
\( \mathbb{P} \)-a.s. for \( \lambda \)-a.e. \( x \in X \). More details on the Wiener-Itô chaos expansion and the Malliavin operators can be found in [13] and the references therein. In [21, 23], the Malliavin operators and Stein’s method are combined to derive bounds for the normal approximation of Poisson functionals. In the following, we evaluate bounds obtained by this technique, which is called Malliavin-Stein method.

To measure the distance between two real-valued random variables \( A, B \), we use the Wasserstein distance that is given by
\[ d_W(A, B) = \sup_{h \in \text{Lip}(1)} |E h(A) - E h(B)|. \]
Here, \( \text{Lip}(1) \) stands for the set of all functions \( h : \mathbb{R} \rightarrow \mathbb{R} \) with a Lipschitz constant less than or equal to one. For two \( m \)-dimensional random vectors \( A, B \) we define
\[ d_3(A, B) = \sup_{h \in \mathcal{H}} |E h(A) - E h(B)|, \]
where \( \mathcal{H} \) is the set of all three times continuously differentiable functions \( h : \mathbb{R}^m \rightarrow \mathbb{R} \) such that
\[ \max_{i,j=1,\ldots,m} \sup_{x \in \mathbb{R}^m} \left| \frac{\partial^2 h}{\partial x_i \partial x_j}(x) \right| \leq 1 \quad \text{and} \quad \max_{i,j,k=1,\ldots,m} \sup_{x \in \mathbb{R}^m} \left| \frac{\partial^3 h}{\partial x_i \partial x_j \partial x_k}(x) \right| \leq 1. \]
Convergence in the Wasserstein distance or in the \( d_3 \)-distance implies convergence in distribution.

In the following, we establish an upper bound for the \( d_3 \)-distance between a Gaussian random vector and a random vector \( F = (F^{(1)}, \ldots, F^{(m)}) \) of Poisson functionals \( F^{(1)}, \ldots, F^{(m)} \in L^2_\eta \). Each of these components has a Wiener-Itô chaos expansion
\[ F^{(k)} = \mathbb{E} F^{(k)} + \sum_{n=1}^{\infty} I_n(f_n^{(k)}) \]}
with \( f_n^{(k)} \in L^2_\alpha(\mathbb{R}^n) \), \( n \in \mathbb{N} \). We also state a bound for the Wasserstein distance between the normalization of a Poisson functional \( F \) and a standard Gaussian random variable.

We need to introduce some notation. Consider functions \( g_1 : X^{n_1} \to \mathbb{R} \) and \( g_2 : X^{n_2} \to \mathbb{R} \), where \( n_1, n_2 \in \mathbb{N} \). The tensor product \( g_1 \otimes g_2 \) is the function on \( X^{n_1+n_2} \) which maps each \( (x_1, \ldots, x_{n_1+n_2}) \) to \( g_1(x_1, \ldots, x_{n_1}) g_2(x_{n_1+1}, \ldots, x_{n_1+n_2}) \). This definition can be iterated in the obvious way. Fix two integers \( i, j \geq 1 \) and consider functions \( f : X^i \to \mathbb{R} \) and \( g : X^j \to \mathbb{R} \).

Let \( \sigma \) be a partition of \( I_{ij} := \{1, \ldots, 2i+2j\} \) and let \( |\sigma| \) be the number of blocks (i.e., subsets) of \( \sigma \). The function \( (f \otimes f \otimes g \otimes g)_\sigma : X^{\sigma} \to \mathbb{R} \) is defined by replacing all variables whose indices belong to the same block of \( \sigma \) by a new common variable. Let \( \pi = \{J_1, \ldots, J_4\} \) be the partition of \( I_{ij} \) into the sets \( J_1 := \{1, \ldots, i\} \), \( J_2 := \{i+1, \ldots, 2i\} \), \( J_3 := \{2i+1, \ldots, 2i+j\} \), and \( J_4 := \{2i+j+1, \ldots, 2i+2j\} \). Let \( \Pi_{ij} \) be the set of all partitions \( \sigma \) of \( I_{ij} \) such that \( |J \cap J'| \leq 1 \) for all \( J \in \pi \) and all \( J' \in \sigma \). By \( \Pi_{ij} \) we denote the set of all partitions \( \sigma \in \Pi_{ij} \) such that

(i) \( \{1,2i+1\}, \{i+1,2i+j+1\} \in \sigma \) or \( \{1, i+1, 2i+1, 2i+j+1\} \in \sigma \);
(ii) each block of \( \sigma \) has at least two elements;
(iii) for every partition of \( \{1, 2, 3, 4\} \) in two disjoint non-empty sets \( M_1, M_2 \) there are \( u \in M_1, v \in M_2 \) such that \( J_u \) and \( J_v \) are both intersected by one block of \( \sigma \).

Let \( \Pi_{ij}^{(1)} \) (resp. \( \Pi_{ij}^{(2)} \)) be the set of all partitions \( \sigma \in \Pi_{ij} \) such that \( \{1,2i+1\}, \{i+1,2i+j+1\} \in \sigma \) (resp. \( \{1, i+1, 2i+1, 2i+j+1\} \in \sigma \)).

Now we are able to state the main result of this section:

**Theorem 8.1.** Assume that \( F^{(k)} \in \text{dom} \ D \) and

\[
\int |(f^{(k)}_i \otimes f_i^{(k)} \otimes f_j^{(l)} \otimes f_j^{(l)})_\sigma| \, d\lambda|\sigma| < \infty \tag{8.7}
\]

for all \( \sigma \in \Pi_{ij}, \ i, j \in \mathbb{N}, \) and \( k, l \in \{1, \ldots, m\} \). Assume that that there are \( a > 0 \) and \( b \geq 1 \) such that

\[
\int |(f^{(k)}_i \otimes f_i^{(k)} \otimes f_j^{(l)} \otimes f_j^{(l)})_\sigma| \, d\lambda|\sigma| \leq \frac{a b^{i+j}}{(i!)^2 (j!)^2} \tag{8.8}
\]

for all \( \sigma \in \Pi_{ij}, \ i, j \in \mathbb{N}, \) and \( k, l \in \{1, \ldots, m\} \). Let \( F := (F^{(1)}, \ldots, F^{(m)}) \) and let \( N \) be a centred Gaussian random vector with a given positive semidefinite covariance matrix \( (\sigma_{k,l})_{k,l=1,\ldots,m} \). Then

\[
d_3(F - E F, N) \leq \frac{m}{2} \sum_{k,l=1}^{m} |\sigma_{k,l} - \text{Cov}(F^{(k)}, F^{(l)})| \\
+ \left( \frac{m}{2} + \frac{m}{4} \sum_{n=1}^{m} \sqrt{\text{Var} F^{(n)}} \right) 2^{13/2} m^2 \sum_{i=1}^{\infty} i^{17/2} \frac{b^i}{[i/14]^!} \sqrt{a}.
\]

In the univariate case, we have the following result for the Wasserstein distance:

**Corollary 8.2.** Let \( F \in \text{dom} \ D \) such that \( \text{Var} F > 0 \) and the assumptions (S.7) and (S.8) are satisfied and let \( N \) be a standard Gaussian random variable. Then

\[
d_w \left( \frac{F - E F}{\sqrt{\text{Var} F}}, N \right) \leq 2^{13/2} \sum_{i=1}^{\infty} i^{17/2} \frac{b^i}{[i/14]^!} \sqrt{a} \sqrt{\text{Var} F}.
\]
We prepare the proof of Theorem 8.1 by two lemmas and a proposition.

**Lemma 8.3.** Let $i, j \geq 1$, $f \in L^2_\mathbb{R}(\lambda^i)$, $g \in L^2_\mathbb{R}(\lambda^j)$, and assume that
\[
\int |(f \otimes f \otimes g \otimes g)_\sigma| \, d\lambda^{\sigma} < \infty, \quad \sigma \in \Pi_{ij}.
\]
Then
\[
\text{Var} \left( \int I_{i-1}(f(z, \cdot)) I_{j-1}(g(z, \cdot)) \lambda(dz) \right) = \sum_{\sigma \in \Pi_{ij}^{(1)}} \int (f \otimes f \otimes g \otimes g)_\sigma \, d\lambda^{\sigma}, \quad (8.9)
\]
\[
\mathbb{E} \int I_{i-1}(f(z, \cdot))^2 I_{j-1}(g(z, \cdot))^2 \lambda(dz) = \sum_{\sigma \in \Pi_{ij}^{(2)}} \int (f \otimes f \otimes g \otimes g)_\sigma \, d\lambda^{\sigma}. \quad (8.10)
\]

**Proof.** Combining the formulas
\[
\mathbb{E} \left( \int I_{i-1}(f(z, \cdot)) I_{j-1}(g(z, \cdot)) \lambda(dz) \right)^2
\]
\[
= \int \int \mathbb{E} I_{i-1}(f(y, \cdot)) I_{i-1}(f(z, \cdot)) I_{j-1}(g(y, \cdot)) I_{j-1}(g(z, \cdot)) \lambda(dy) \lambda(dz),
\]
and
\[
\mathbb{E} \int I_{i-1}(f(z, \cdot)) I_{j-1}(g(z, \cdot)) \lambda(dz) = \begin{cases} (i - 1)! \langle f, g \rangle_i, & i = j, \\ 0, & i \neq j, \end{cases} \quad (8.11)
\]
with Theorem 3.1 in [14] (see also [30, 22]) proves the first equation. The second identity is a consequence of
\[
\mathbb{E} \int I_{i-1}(f(z, \cdot))^2 I_{j-1}(g(z, \cdot))^2 \lambda(dz) = \int \mathbb{E} I_{i-1}(f(z, \cdot))^2 I_{j-1}(g(z, \cdot))^2 \lambda(dz)
\]
and, again, Theorem 3.1 in [14]. \qed

**Proposition 8.4.** Let $F^{(1)}, \ldots, F^{(m)} \in \text{dom} D$ be such that (8.7) holds. Let $F := (F^{(1)}, \ldots, F^{(m)})$ and let $N$ be a centred Gaussian random vector with a given positive semidefinite covariance matrix $(\sigma_{k,l})_{k,l=1,\ldots,m}$. Then
\[
d_3(F - \mathbb{E} F, N) \leq \frac{m}{2} \sum_{k,l=1}^m |\sigma_{k,l} - \text{Cov}(F^{(k)}, F^{(l)})|
\]
\[
+ \left( \frac{m}{2} + \frac{m}{4} \sum_{n=1}^m \sqrt{\text{Var } F^{(n)}} \right) \sum_{k,l=1}^m \sum_{i,j=1}^\infty ij \left( \sum_{\sigma \in \Pi_{ij}} \int |(f_i^{(k)} \otimes f_i^{(l)} \otimes f_j^{(l)} \otimes f_j^{(l)})_\sigma| \, d\lambda^{\sigma} \right).
\]

**Proof.** From [23, Theorem 4.2] it is known that
\[
d_3(F - \mathbb{E} F, N) \leq \frac{m}{2} \left( \sum_{k,l=1}^m \mathbb{E} \left( |\sigma_{k,l} - \int D_z F^{(k)}(-D_z L^{-1} F^{(l)})| \lambda(dz) \right)^2 \right)
\]
\[
+ \frac{1}{4} \mathbb{E} \left( \sum_{k=1}^m |D_z F^{(k)}| \right) \sum_{l=1}^m \left| D_z L^{-1} F^{(l)} \right| \lambda(dz). \quad (8.12)
\]
We bound the two summands on the above right-hand side separately. For the first one we have

\[ \sqrt{\sum_{k,l=1}^{m} \mathbb{E} \left( \sigma_{k,l} - \int D_z F^{(k)} (-D_z L^{-1} F^{(l)}) \lambda(dz) \right)^2} \]

\[ \leq \sum_{k,l=1}^{m} \sqrt{\mathbb{E} \left( \sigma_{k,l} - \text{Cov}(F^{(k)}, F^{(l)}) + \text{Cov}(F^{(k)}, F^{(l)}) - \int D_z F^{(k)} (-D_z L^{-1} F^{(l)}) \lambda(dz) \right)^2} \]

\[ \leq \sum_{k,l=1}^{m} |\sigma_{k,l} - \text{Cov}(F^{(k)}, F^{(l)})| + \sqrt{\mathbb{E} \left( \text{Cov}(F^{(k)}, F^{(l)}) - \int D_z F^{(k)} (-D_z L^{-1} F^{(l)}) \lambda(dz) \right)^2}. \]

From \(8.4\) and \(8.6\) and the covariance version of \(8.1\) (see \(8.7\)) we obtain that

\[ \mathbb{E} \left( \int D_z F^{(k)} (-D_z L^{-1} F^{(l)}) \lambda(dz) - \text{Cov}(F^{(k)}, F^{(l)}) \right)^2 \]

\[ = \mathbb{E} \left( \int \sum_{i=1}^{\infty} i g_i^{(k)}(z) \sum_{j=1}^{\infty} g_j^{(l)}(z) \lambda(dz) - \sum_{n=1}^{\infty} n! (f_n^{(k)}, f_n^{(l)})_n \right)^2, \]

where

\[ g_n^{(l)}(z) := I_{n-1}(f_n^{(l)}(z, \cdot)). \]

Using the triangle inequality and \(8.11\), we obtain

\[ \sqrt{\mathbb{E} \left( \int D_z F^{(k)} (-D_z L^{-1} F^{(l)}) \lambda(dz) - \text{Cov}(F^{(k)}, F^{(l)}) \right)^2} \]

\[ \leq \sum_{i,j=1}^{\infty} \sqrt{\mathbb{E} \left( \int g_i^{(k)}(z) g_j^{(l)}(z) \lambda(dz) - \mathbb{E} \int g_i^{(k)}(z) g_j^{(l)}(z) \lambda(dz) \right)^2} \]

\[ = \sum_{i,j=1}^{\infty} \sqrt{\text{Var} \left( \int g_i^{(k)}(z) g_j^{(l)}(z) \lambda(dz) \right)}^{1/2} \]

\[ \leq \sum_{i,j=1}^{\infty} \left[ \sum_{\sigma \in \Pi_{ij}} \int_{\mathcal{P}} |(f_i^{(k)} \otimes f_j^{(k)} \otimes f_i^{(l)} \otimes f_j^{(l)})_{\sigma}| d\lambda[\sigma] \right], \]

where we have applied \(8.9\) in Lemma \(8.3\) to get the final inequality.

By Jensen’s inequality and the definitions of the Malliavin operators, we obtain for the second summand in \(8.12\) that

\[ \int \mathbb{E} \left( \sum_{k=1}^{m} |D_z F^{(k)}| \right)^2 \sum_{l=1}^{m} |D_z L^{-1} F^{(l)}| \lambda(dz) \]

\[ \leq m \sum_{k,l=1}^{m} \int \mathbb{E}(D_z F^{(k)})^2 |D_z L^{-1} F^{(l)}| \lambda(dz) \]

38
\[
= m \sum_{k,l=1}^{\infty} i j \int \mathbb{E} g_i^{(k)}(z) g_j^{(k)}(z) |D_z L^{-1} F(l)| \lambda(dz)
\leq m \sum_{k,l=1}^{\infty} i j \left( \int \mathbb{E} g_i^{(k)}(z)^2 g_j^{(k)}(z)^2 \lambda(dz) \right)^{1/2} \left( \int \mathbb{E} (D_z L^{-1} F(l))^2 \lambda(dz) \right)^{1/2}.
\]

Combining (8.6) and (8.11) with (8.1), we get
\[
\int \mathbb{E} (D_z L^{-1} F(l))^2 \lambda(dz) = \sum_{n=1}^{\infty} (n-1)!|f_n^{(l)}|^2_n \leq \text{Var} F(l).
\]

Now (8.10) in Lemma 8.3 concludes the proof. \(\square\)

**Lemma 8.5.** For any integers \(i, j \geq 1\),
\[
\bar{\Pi}_{i,j} \leq \frac{(i!)^2(j!)^2}{\max\{i,j\}/7!}.
\]

**Proof.** For a fixed partition \(\sigma \in \bar{\Pi}_{ij}\) let \(k_{uv}\) with \(u, v \in \{1, 2, 3, 4\}\) and \(u < v\) be the number of blocks \(A \in \sigma\) such that \(|A \cap J_u| = |A \cap J_v| = 1\) and \(A \cap (J_u \cup J_v) = A\). We define \(k_{uvw}\) for \(u, v, w \in \{1, 2, 3, 4\}\) and \(u < v < w\) and \(k_{1234}\) in the same way. For a possible combination of fixed numbers \(k_{12}, \ldots, k_{1234}\) the number of partitions \(\sigma \in \bar{\Pi}_{ij}\) having this form is less than
\[
\frac{(i!)^2(j!)^2}{k_{12}k_{13}k_{14}k_{23}k_{24}k_{34}k_{123}k_{124}k_{134}k_{234}k_{1234}!} \leq \frac{(i!)^2(j!)^2}{\max\{i,j\}/7!}.
\]

To get this inequality, we have used the fact that
\[
k_{12} + k_{13} + k_{14} + k_{23} + k_{124} + k_{134} + k_{1234} = i,
\]
whence one of the factors in the denominator is at least \([i/7]\). For a similar reason there must be a factor in the denominator that is at least \([j/7]\).

Moreover, there are less than \(\max\{i+j+1\}^{11}\) possible choices for \(k_{12}, \ldots, k_{1234}\), which concludes the proof. \(\square\)

Note that we have not used the first and the third condition of the definition of \(\bar{\Pi}_{ij}\) in the proof of Lemma 8.5, whence the inequality even holds for a larger class of partitions. Now we are prepared for the proofs of Theorem 8.1 and Corollary 8.2.

**Proof of Theorem 8.1.** We aim at applying Proposition 8.4. Combining Lemma 8.5 and assumption (8.8), we get
\[
\sum_{i,j=1}^{\infty} i j \sqrt{\sum_{\sigma \in \bar{\Pi}_{ij}} \left| \left( f_i^{(k)} \otimes f_i^{(k)} \otimes f_j^{(l)} \otimes f_j^{(j)} \right)_{\sigma} \right| d\lambda[\sigma]} \leq \sum_{i,j=1}^{\infty} i j \sqrt{\frac{\max\{i,j+1\}^{11} b^{i+j} a}{\max\{i,j\}/7!}}.
\]

A straightforward computation and the inequality \(\sqrt{m} \geq \lceil m/2 \rceil\) for \(m \in \mathbb{N}\) show that
\[
\sum_{i,j=1}^{\infty} i j \sqrt{\frac{\max\{i,j+1\}^{11} b^{i+j} a}{\max\{i,j\}/7!}} \leq 2 \frac{12}{7} \sum_{1 \leq j \leq i} i^2 \sqrt{\frac{\max\{i,j\}^{11} b^{i+j} a}{\max\{i,j\}/7!}} \leq 2 \frac{12}{7} \sum_{i=1}^{\infty} i^{7/2} \frac{b^i}{[i/14]!},
\]
where the right-hand side converges. Thus, Theorem 8.1 is a consequence of Proposition 8.4. \(\square\)
Proof of Corollary 8.2. From [21, Theorem 3.1] we know that

\[
\frac{dW}{\sqrt{\text{Var}
F}(F - \mathbb{E}F, N)} \leq \frac{1}{\text{Var} F} \sqrt{\mathbb{E}
\left(\text{Var} F - \int D_z F(-D_z L^{-1} F) \lambda(dz)\right)^2} + \frac{1}{(\text{Var} F)^{3/2}} \int \mathbb{E}(D_z F)^2 |D_z L^{-1} F| \lambda(dz),
\]

where we can bound the right-hand side in the same way as in the proofs of Proposition 8.4 and Theorem 8.1.

9 Central limit theorems for geometric functionals

In the following we use the general normal approximation results of the previous section to derive central limit theorems for geometric functionals of the Boolean model (1.1). We establish central limit theorems under the minimal moment assumption (2.5), but we need a stronger moment assumption in order to derive rates of convergence. For the Berry-Esseen bounds we assume that the typical grain \(Z_0\) of the Boolean model satisfies the moment assumption

\[
\mathbb{E}V_i(Z_0)^{3+\varepsilon} < \infty, \quad i \in \{0, \ldots, d\},
\]

for a fixed \(\varepsilon \in (0, 1]\). This allows us to state central limit theorems with rates of convergence depending on \(\varepsilon\).

Theorem 9.1. Let \(\psi_1, \ldots, \psi_m\) be geometric functionals on \(\mathcal{R}^d\) and let \(\Psi := (\psi_1, \ldots, \psi_m)\). Assume that (2.5) is satisfied and let \(N\) be an \(m\)-dimensional centred Gaussian random vector with covariance matrix \((\sigma_{\psi_k, \psi_l})_{k,l=1,\ldots,m}\) given by (3.3). Then

\[
\frac{1}{\sqrt{V_d(W)}} (\Psi(Z \cap W) - \mathbb{E}\Psi(Z \cap W)) \overset{d}{\to} N \quad \text{as} \quad r(W) \to \infty.
\]

If (9.1) is satisfied, there is a constant \(c_{\psi_1,\ldots,\psi_m}\) depending on \(\psi_1, \ldots, \psi_m, \Lambda\) and \(\varepsilon\) such that

\[
\frac{d}{\sqrt{V_d(W)}} (\Psi(Z \cap W) - \mathbb{E}\Psi(Z \cap W)), N \leq \frac{c_{\psi_1,\ldots,\psi_m}}{r(W)^\min\{\varepsilon d/2, 1\}}
\]

for \(W \in \mathcal{K}^d\) with \(r(W) \geq 1\).

Remark 9.2. We will see in the proof of Theorem 9.1 that the translation invariance of \(\psi_1, \ldots, \psi_m\) is only used to ensure the existence of an asymptotic covariance matrix. Hence, such a multivariate central limit theorem still holds for functionals \(\psi_1, \ldots, \psi_m\) which are not translation invariant, if we can establish the existence of an asymptotic covariance matrix. In this case the rate of convergence depends on the rate of convergence for the covariances.

In the univariate case we can rescale by the square root of the variance, whence the existence of the asymptotic variance is not necessary. Thus, translation invariance of the functional is not required. We only need to assume that the variance does not degenerate as \(r(W) \to \infty\), which, for instance, holds under the conditions of Section 4.
Theorem 9.3. Assume that (2.5) is satisfied and let \( \psi \) be an additive, locally bounded, and measurable functional on \( \mathcal{R}^d \) with constants \( r_0 \geq 1 \) and \( \sigma_0 > 0 \) such that

\[
\frac{\text{Var} \psi(Z \cap W)}{V_d(W)} \geq \sigma_0
\]

(9.3)

for \( W \in \mathcal{K}^d \) with \( r(W) \geq r_0 \). Denote by \( N \) a standard Gaussian random variable. Then

\[
\frac{\psi(Z \cap W) - E\psi(Z \cap W)}{\sqrt{\text{Var} \psi(Z \cap W)}} \xrightarrow{d} N \quad \text{as} \quad r(W) \to \infty.
\]

If (9.1) is satisfied, there is a constant \( c_\psi \) depending on \( \psi, \Lambda, \sigma_0, r_0, \) and \( \varepsilon \) such that

\[
d_W \left( \frac{\psi(Z \cap W) - E\psi(Z \cap W)}{\sqrt{\text{Var} \psi(Z \cap W)}}, N \right) \leq \frac{c_\psi}{V_d(W)^{\min\{\varepsilon/2,1/2\}}},
\]

(9.4)

for \( W \in \mathcal{K}^d \) with \( r(W) \geq r_0 \).

Remark 9.4. By replacing in (9.1) the volume of \( W \) by the volume of its inball, we obtain a rate of order \( r(W)^{-\min\{\varepsilon/d,2,d/2\}} \). Comparing (9.2) and (9.4), we see that for \( \varepsilon = 1 \) and \( d \geq 3 \) the rate of convergence in the multivariate case is weaker than in the univariate case. This is caused by the slow rate of convergence in Theorem 3.1. Since we need to bound

\[
\sum_{k,l=1}^m \left| \sigma_{\psi_k,\psi_l} - \text{Cov}(\psi_k(Z \cap W), \psi_l(Z \cap W)) \right|
\]

in order to apply Theorem 8.1. In the univariate analogue, which is Corollary 8.3, we normalize with the exact variance and do not have such a term. If we replace the Gaussian random vector \( N \) by a centred Gaussian random vector \( N(W) \), having the covariance matrix of \( V_d(W)^{-1/2}(\Psi(Z \cap W)) \), the sum above vanishes and we obtain

\[
d_3 \left( \frac{1}{\sqrt{V_d(W)}}(\Psi(Z \cap W) - E\Psi(Z \cap W)), N(W) \right) \leq \frac{c_{\psi_1,\ldots,\psi_m}}{V_d(W)^{\min\{\varepsilon/2,1/2\}}},
\]

which is the same rate as in the univariate case.

For \( k, l \in \{1, \ldots, m\} \) we obtain by choosing \( g(x) = x_k x_l / 2 \) as test function in the definition of the \( d_3 \)-distance that

\[
d_3 \left( \frac{1}{\sqrt{V_d(W)}}(\Psi(Z \cap W) - E\Psi(Z \cap W)), N \right) \geq \frac{1}{2} \left| \sigma_{\psi_k,\psi_l} - \text{Cov}(\psi_k(Z \cap W), \psi_l(Z \cap W)) \right|.
\]

Hence Proposition 3.8 shows that the rate in (9.4) is optimal for \( \varepsilon = 1 \) and \( d \geq 2 \).

We organize the proofs of Theorem 9.1 and Theorem 9.3 such that we first impose the moment assumption (9.1) and establish (9.2) and (9.4). In a second step we prove that convergence in distribution is still obtained (without convergence rates) under the weaker moment assumption (2.3).

Proof of (9.2) in Theorem 9.1 under assumption (9.1). From now on, we write

\[
f_i^{(k)}(K_1, \ldots, K_i) := \frac{(-1)^i}{i!} \psi_k^{(i)}(K_1 \cap \ldots \cap K_i \cap W)
\]

41
for \( K_1, \ldots, K_i \in \mathcal{K}^d, 1 \leq k \leq m, \) and \( i \geq 1. \) It is a direct consequence of (3.8) that \( f_{ij}^{(k)} \) is the \( i \)-th kernel of the Wiener-Itô chaos expansion of the Poisson functional \( \psi_k(Z \cap W). \)

The assumption \( \psi_k(Z \cap W) \in \text{dom} D \) and the integrability condition (8.8) are satisfied since the kernels are bounded by (3.9) for every \( W \in \mathcal{K}^d \) and the measure of the grains hitting \( W \) is also finite.

In the sequel we check assumption (8.8) for the cases \( \sigma = \tilde{\Pi}_{ij}^{(1)} \) and \( \sigma = \tilde{\Pi}_{ij}^{(2)} \) separately. We start with the first case. Let \( k, l \in \{1, \ldots, m\} \) and \( \sigma \in \tilde{\Pi}_{ij}^{(1)}. \) From (3.9) in Lemma 3.3 it follows that

\[
\int |(f_{ij}^{(k)} \otimes f_{ij}^{(k)} \otimes f_{ij}^{(l)} \otimes f_{ij}^{(l)})_\sigma| d\Lambda^{[\sigma]}
\leq \frac{(\beta(\psi_k)\beta(\psi_l))^2}{(i!)^2(j!)^2} \int \prod_{p=1}^{d} \sum_{r=0}^{d} V_r \left( \bigwedge_{n \in N_p(\sigma)} K_n \cap W \right) \Lambda^{[\sigma]}(d(K_1, \ldots, K_{|\sigma|}))
\]

with non-empty sets \( N_p(\sigma) \subset \{1, \ldots, |\sigma|\}, p = 1, \ldots, 4, \) depending on \( \sigma. \) Every \( n \in \{1, \ldots, |\sigma|\} \) is contained in at least two of these sets. By removing the index \( n \) from the sets until it occurs only in one set, we increase the integral and can use Lemma 3.3 to integrate over \( K_n. \) Due to the special structure of \( \sigma \in \tilde{\Pi}_{ij}^{(1)}, \) we obtain by iterating this step and using the abbreviation

\[
h_W(A) = \sum_{r=0}^{d} V_r(A \cap W)
\]

that

\[
\int |(f_{ij}^{(k)} \otimes f_{ij}^{(k)} \otimes f_{ij}^{(l)} \otimes f_{ij}^{(l)})_\sigma| d\Lambda^{[\sigma]}
\leq \frac{(\beta(\psi_k)\beta(\psi_l))^2}{(i!)^2(j!)^2} \int h_W(K_1) h_W(K_1 \cap K_2) h_W(K_2 \cap K_3) h_W(K_3) \Lambda^3(d(K_1, K_2, K_3))
= \frac{(\beta(\psi_k)\beta(\psi_l))^2}{(i!)^2(j!)^2} \alpha^{[\sigma]-3} \int \left( \int h_W(K_1) h_W(K_1 \cap K_2) \Lambda(dK_1) \right)^2 \Lambda(dK_2).
\]

For a fixed \( K_2 \in \mathcal{K}^d, \) Lemma 3.4 implies the second inequality in

\[
\int h_W(K_1) h_W(K_1 \cap K_2) \Lambda(dK_1) \leq \gamma \mathbb{E} \left[ \sum_{r=0}^{d} V_r(Z_0) \int \sum_{s=0}^{d} V_s((Z_0 + x) \cap K_2 \cap W) \, dx \right]
\leq (d + 1)\gamma \beta_1 \mathbb{E} \left[ \left( \sum_{r=0}^{d} V_r(Z_0) \right)^2 \right] \sum_{s=0}^{d} \mathbb{E}(V_s(K_2 \cap W)).
\]

Putting \( c_0 = (d + 1)\gamma \beta_1 \mathbb{E} \left( \sum_{r=0}^{d} V_r(Z_0) \right)^2 \) and applying Lemma 3.4 again, we get

\[
\int \left( \int h_W(K_1) h_W(K_1 \cap K_2) \Lambda(dK_1) \right)^2 \Lambda(dK_2)
\leq c_0^2 \int \left( \sum_{r=0}^{d} V_r(K_2 \cap W) \right)^2 \Lambda(dK_2)
\]

42
A further application of Lemma 3.4 yields the second inequality in

\[ \leq \gamma c_9^2 \mathbb{E} \left[ \sum_{r=0}^{d} V_r(Z_0) \int \sum_{s=0}^{d} V_s((Z_0 + x) \cap W) \, dx \right] \]

\[ \leq (d + 1)\gamma \beta_1 c_9^2 \mathbb{E} \left[ \left( \sum_{r=0}^{d} V_r(Z_0) \right)^2 \right] \sum_{s=0}^{d} V_s(W) = c_{10} \sum_{r=0}^{d} V_r(W) \]

with a constant \( c_{10} \) depending on \( \Lambda \). Finally, since \( |\sigma| \leq i + j \) for \( \sigma \in \tilde{\Pi}_{ij} \) we have

\[ \int |(f_i^{(k)} \otimes f_i^{(k)} \otimes f_j^{(l)} \otimes f_j^{(l)})_\sigma| \, d\Lambda^{[\sigma]} \leq \frac{c_{10} (\beta(\psi_k) \beta(\psi_l))^2}{(i!)^2(j!)^2} \alpha^{[\sigma]-3} \sum_{r=0}^{d} V_r(W) \leq \frac{a_1 b_1^{i+j}}{(i!)^2(j!)^2} \]

(9.5)

with \( a_1 = \max_{1 \leq k, l \leq m} \alpha^{-3} c_{10} (\beta(\psi_k) \beta(\psi_l))^2 \sum_{r=0}^{d} V_r(W) \) and \( b_1 = \max\{\alpha, 1\} \). Now it follows from Lemma 3.7 that there is a constant \( c_{11} \) depending on \( \psi_1, \ldots, \psi_m \) and \( \Lambda \) such that

\[ \frac{a_1}{V_d(W)^2} \leq \frac{c_{11}}{V_d(W)} \]

(9.6)

for \( W \in \mathcal{K}_d \) with \( r(W) \geq 1 \).

For \( \sigma \in \Pi_{ij}^{(2)} \) we obtain from (3.9) in Lemma 3.3 and Lemma 3.5 as above that

\[ \int |(f_i^{(k)} \otimes f_i^{(k)} \otimes f_j^{(l)} \otimes f_j^{(l)})_\sigma| \, d\Lambda^{[\sigma]} \leq \frac{(\beta(\psi_k) \beta(\psi_l))^2}{(i!)^2(j!)^2} \alpha^{[\sigma]-1} \int \left( \sum_{r=0}^{d} V_r(K_1 \cap W) \right)^4 \Lambda(dK_1). \]

A further application of Lemma 3.4 yields the second inequality in

\[ \int \left( \sum_{r=0}^{d} V_r(K_1 \cap W) \right)^4 \Lambda(dK_1) \]

\[ \leq \gamma \mathbb{E} \left[ \left( \sum_{r=0}^{d} \min\{V_r(Z_0), V_r(W)\} \right)^3 \int \sum_{s=0}^{d} V_s((Z_0 + x) \cap W) \, dx \right] \]

\[ \leq (d + 1)\gamma \beta_1 \mathbb{E} \left[ \left( \sum_{r=0}^{d} \min\{V_r(Z_0), V_r(W)\} \right)^3 \sum_{s=0}^{d} V_s(Z_0) \right] \sum_{u=0}^{d} V_u(W). \]

Consequently, we have

\[ \int |(f_i^{(k)} \otimes f_i^{(k)} \otimes f_j^{(l)} \otimes f_j^{(l)})_\sigma| \, d\Lambda^{[\sigma]} \leq \frac{a_2 b_2^{i+j}}{(i!)^2(j!)^2} \]

(9.7)

with

\[ a_2 = (d + 1)\gamma \beta_1 \max_{1 \leq k, l \leq m} \frac{(\beta(\psi_k) \beta(\psi_l))^2}{\alpha} \mathbb{E} \left[ \left( \sum_{r=0}^{d} \min\{V_r(Z_0), V_r(W)\} \right)^3 \sum_{s=0}^{d} V_s(Z_0) \right] \sum_{u=0}^{d} V_u(W) \]

43
and $b_2 = \max\{\alpha, 1\}$. Since
\[
\frac{1}{V_d(W)^2} \mathbb{E} \left[ \left( \sum_{r=0}^{d} \min\{V_r(Z_0), V_r(W)\} \right)^3 \sum_{s=0}^{d} V_s(Z_0) \right] \sum_{u=0}^{d} V_u(W)
\leq \mathbb{E} \left[ \left( \sum_{r=0}^{d} V_r(Z_0) \right)^3 \sum_{s=0}^{d} \frac{\min\{V_s(Z_0), V_s(W)\}}{V_d(W)} \right] \sum_{u=0}^{d} V_u(W)
\leq \mathbb{E} \left[ \left( \sum_{r=0}^{d} V_r(Z_0) \right)^3 \sum_{s=0}^{d} \frac{V_s(Z_0)^2 V_s(W)^{1-\varepsilon}}{V_d(W)} \right] \sum_{u=0}^{d} V_u(W),
\]
Lemma 3.7 and the moment assumption (9.1) imply that there is a constant $c_{12}$ depending on $\psi_1, \ldots, \psi_m$, $\Lambda$, and $\varepsilon$ such that
\[
\frac{a_2}{V_d(W)^2} \leq \frac{c_{12}}{V_d(W)^\varepsilon}
\tag{9.8}
\]
for $W \in K^d$ with $r(W) \geq 1$.

If we rescale the Poisson functionals by $V_d(W)^{-\frac{1}{2}}$, (9.6) and (9.8) together with (9.3) and (9.7) imply that (8.8) holds with $a = \max\{a_1, a_2\} V_d(W)^{-2}$ and $b = \max\{b_1, b_2\}$. Then $a$ is of the order $V_d(W)^{-\min\{1, \varepsilon\}}$. Now (9.2) is a consequence of Theorem 3.1 and Theorem 8.1. \hfill \Box

**Proof of (9.4) in Theorem 9.3 under assumption (9.1).** By the same arguments as in the previous proof and analogous choices for $a_1, a_2$, the conditions of Corollary 8.2 are satisfied with $a = \max\{a_1, a_2\}$. It follows from assumption (9.3) that
\[
\frac{a}{(\text{Var } \psi(Z \cap W))^2} = \frac{\max\{a_1, a_2\}}{(\text{Var } \psi(Z \cap W))^2} \leq \frac{1}{\sigma_0^2} \frac{\max\{a_1, a_2\}}{V_d(W)^2}
\]
for $W \in K^d$ with $r(W) \geq r_0$. Combining this with (9.6) and (9.8) concludes the proof. \hfill \Box

For $W \in K^d$ we define the set
\[
\mathcal{M}_W = \{K \in K^d : V_j(K) \leq \sqrt{V_d(W)}, \ j \in \{0, \ldots, d\}\}
\]
and the stationary Boolean model
\[
Z^W = \bigcup_{K \in \eta \setminus \mathcal{M}_W} K.
\]
The idea of the proofs of Theorem 9.1 and Theorem 9.3 under the weaker moment assumption (2.5) is to approximate the Boolean model $Z$ by the Boolean model $Z^W$. A similar truncation has been used in [11] to prove the central limit theorem for the volume of a more general Boolean model based on a Poisson process of cylinders. We prepare the proofs by the following lemma.

**Lemma 9.5.** Let $\psi$ be an additive, locally bounded, and measurable functional on $\mathcal{R}^d$ and assume that (2.5) is satisfied. Then
\[
\lim_{r(W) \to \infty} \frac{\mathbb{E} \left[ \left( \psi(Z \cap W) - \mathbb{E} \psi(Z \cap W) \right) - \left( \psi(Z^W \cap W) - \mathbb{E} \psi(Z^W \cap W) \right) \right]^2}{V_d(W)} = 0
\]
and
\[
\limsup_{r(W) \to \infty} \frac{\mathbb{E} \left( \psi(Z \cap W) - \mathbb{E} \psi(Z \cap W) \right)^2 + \mathbb{E} \left( \psi(Z^W \cap W) - \mathbb{E} \psi(Z^W \cap W) \right)^2}{V_d(W)} < \infty.
\]

44
Proof. Define

\[ g_{n,W}(K_1,\ldots,K_n) := \frac{(-1)^n}{n! \sqrt{V_d(W)}} (E\psi(Z \cap K_1 \cap \ldots \cap K_n \cap W) - \psi(K_1 \cap \ldots \cap K_n \cap W)). \]

Further we define

\[ h_{n,W}(K_1,\ldots,K_n) := \frac{(-1)^n}{n! \sqrt{V_d(W)}} (E\psi(Z_W \cap K_1 \cap \ldots \cap K_n \cap W) - \psi(K_1 \cap \ldots \cap K_n \cap W)) \]

for \( K_1,\ldots,K_n \in M_W \) and \( h_{n,W}(K_1,\ldots,K_n) := 0 \) if there is a \( j \in \{1,\ldots,n\} \) such that \( K_j \notin M_W \).

In view of Lemma 3.2 and the Fock space representation (3.7) the assertions of this lemma are equivalent to

\[ \lim_{r(W) \to \infty} \sum_{n=1}^{\infty} n! \|g_{n,W} - h_{n,W}\|_n^2 = 0 \] (9.9)

and

\[ \limsup_{r(W) \to \infty} \sum_{n=1}^{\infty} n! (\|g_{n,W}\|^2 + \|h_{n,W}\|^2) < \infty, \] (9.10)

which we shall prove in the following. For \( n \in \mathbb{N} \) we have

\[
\|g_{n,W} - h_{n,W}\|_n^2 = \gamma \iint (g_{n,W}(K_1 + x, K_2, \ldots, K_n) - h_{n,W}(K_1 + x, K_2, \ldots, K_n))^2 \, dx \Lambda^{n-1}(d(K_2, \ldots, K_n)) \frac{Q(\delta K_1)}{n}. \]

Our aim is to apply the dominated convergence theorem to the outer integral. For any \( K_1 \in \mathcal{K}_d \) it follows from Lemma 3.3, Lemma 3.4, and Lemma 3.5 similarly as in (3.20) that

\[
\iint (g_{n,W}(K_1 + x, K_2, \ldots, K_n) - h_{n,W}(K_1 + x, K_2, \ldots, K_n))^2 \, dx \Lambda^{n-1}(d(K_2, \ldots, K_n)) \leq 2 \iint (g_{n,W}(K_1 + x, K_2, \ldots, K_n)^2 + h_{n,W}(K_1 + x, K_2, \ldots, K_n)^2) \, dx \Lambda^{n-1}(d(K_2, \ldots, K_n)) \leq 2(d+1)β_1 β(ψ)^2 \left( \sum_{i=0}^{d} V_i(K_1) \right)^2 \sum_{i=0}^{d} \frac{V_i(W)}{V_d(W)} \frac{\alpha^{n-1}}{n!^2}, \]

where we have used that for the Boolean model \( Z_W \) we can take the same constants as for \( Z \).

The right-hand side of the previous inequality is uniformly bounded for \( r(W) \geq 1 \) because of Lemma 3.7 and it is integrable due to (2.5). Thus limit and summation in (9.9) can be interchanged. By the same arguments, the second inequality above yields (9.10).

Next we show that

\[ \lim_{r(W) \to \infty} \iint (g_{n,W}(K_1 + x, K_2, \ldots, K_n) - h_{n,W}(K_1 + x, K_2, \ldots, K_n))^2 \, dx \Lambda^{n-1}(d(K_2, \ldots, K_n)) = 0 \] (9.11)

for any \( K_1 \in \mathcal{K}_o^d \).
For $K_1, \ldots, K_n \in M_W$ we have
\[
g_{n,W}(K_1, \ldots, K_n) - h_{n,W}(K_1, \ldots, K_n)
= \frac{1}{n!} \frac{(-1)^n}{\sqrt{V_d(W)}} \mathbb{E}\left[\psi(Z \cap K_1 \cap \ldots \cap K_n \cap W) - \psi(Z_W \cap K_1 \cap \ldots \cap K_n \cap W)\right].
\]

Let us denote by $Z_1, \ldots, Z_{N_{K_1 \cap W}}$ the grains of $\eta$ that intersect $K_1 \cap W$ and are not in $M_W$. Then $N_{K_1 \cap W}$ is a Poisson distributed random variable with mean
\[
\Lambda(\{K \notin M_W : K \cap K_1 \cap W \neq \emptyset\}).
\]
Since $Z \cap K_1 \cap W = (Z_W \cup Z_1 \cup \ldots \cup Z_{N_{K_1 \cap W}}) \cap K_1 \cap W$, it follows from the inclusion-exclusion formula that
\[
|\psi(Z \cap K_1 \cap \ldots \cap K_n \cap W) - \psi(Z_W \cap K_1 \cap \ldots \cap K_n \cap W)|
\leq \sum_{\emptyset \neq J \subseteq \{1, \ldots, N_{K_1 \cap W}\}} \left|\psi\left(Z_W \cap \bigcap_{j \in J} Z_j \cap K_1 \cap \ldots \cap K_n \cap W\right)\right|
\quad + \sum_{\emptyset \neq J \subseteq \{1, \ldots, N_{K_1 \cap W}\}} \left|\psi\left(\bigcap_{j \in J} Z_j \cap K_1 \cap \ldots \cap K_n \cap W\right)\right|.
\]
Denoting by $P_W$ the distribution of the restriction of $\eta$ to $M_W$, we obtain by (3.11) and the monotonicity of the intrinsic volumes that
\[
\int \left|\psi\left(Z(\mu) \cap \bigcap_{j \in J} Z_j \cap K_1 \cap \ldots \cap K_n \cap W\right)\right| P_W(d\mu)
\leq c_1 M(\psi) \sum_{i=0}^{d} V_i \left(\bigcap_{j \in J} Z_j \cap K_1 \cap \ldots \cap K_n \cap W\right) \leq c_1 M(\psi) \sum_{i=0}^{d} V_i(K_1 \cap \ldots \cap K_n \cap W).
\]
Applying (3.15) and the monotonicity of the intrinsic volumes yields
\[
\left|\psi\left(\bigcap_{j \in J} Z_j \cap K_1 \cap \ldots \cap K_n \cap W\right)\right| \leq c_2 c_4 M(\psi) \sum_{i=0}^{d} V_i \left(\bigcap_{j \in J} Z_j \cap K_1 \cap \ldots \cap K_n \cap W\right)
\leq c_2 c_4 M(\psi) \sum_{i=0}^{d} V_i(K_1 \cap \ldots \cap K_n \cap W).
\]
Since the restrictions of $\eta$ to $M_W$ and to its complement are stochastically independent, altogether we have that
\[
|g_{n,W}(K_1, \ldots, K_n) - h_{n,W}(K_1, \ldots, K_n)|
\leq \frac{(c_1 + c_2 c_4) M(\psi)}{n! \sqrt{V_d(W)}} \mathbb{E}\left[2^{N_{K_1 \cap W}} - 1\right] \sum_{i=0}^{d} V_i(K_1 \cap \ldots \cap K_n \cap W)
\leq \frac{\hat{c}(\psi)}{n! \sqrt{V_d(W)}} \left(\exp(p_{W}(K_1)) - 1\right) \sum_{i=0}^{d} V_i(K_1 \cap \ldots \cap K_n \cap W)
\]
46
with \( p_W(K_1) = \Lambda(\{ K \notin M_W : K \cap K_1 \neq \emptyset \}) \) and \( \hat{\beta}(\psi) = (c_1 + c_2a)M(\psi) \).

If there is a \( j \in \{ 1, \ldots, n \} \) such that \( K_j \notin M_W \), we have \( g_{n,W} - h_{n,W} = g_{n,W} \), and it follows from Lemma 5.3 that

\[ |g_{n,W}(K_1, \ldots, K_n)| \leq \frac{\beta(\psi)}{n! \sqrt{V_d(W)}} \sum_{k=0}^{d} V_k(K_1 \cap \ldots \cap K_n \cap W). \]

For a fixed \( K_1 \in K_0^d \) and \( r(W) \) sufficiently large such that \( K_1 \in M_W \), we have that

\[
\int \int (g_{n,W}(K_1 + x, K_2, \ldots, K_n) - h_{n,W}(K_1 + x, K_2, \ldots, K_n))^2 \, dx \, \Lambda^{n-1}(d(K_2, \ldots, K_n))
\]

\[ \leq \int \int \left( \frac{\hat{\beta}(\psi)^2}{(n!)^2 V_d(W)} \left( \sum_{k=0}^{d} V_k((K_1 + x) \cap K_2 \cap \ldots \cap K_n \cap W) \right)^2 \right) \left( \exp(p_W(K_1)) - 1 \right)^2 \]

\[ + \frac{\beta(\psi)^2}{(n!)^2 V_d(W)} \left( \sum_{k=0}^{d} V_k((K_1 + x) \cap K_2 \cap \ldots \cap K_n \cap W) \right)^2 \sum_{i=2}^{n} 1 \{ K_i \notin M_W \} \]

\[ dx \, \Lambda^{n-1}(d(K_2, \ldots, K_n)). \]

Lemma 3.4 and Lemma 3.5 imply that

\[
\int \int (g_{n,W}(K_1 + x, K_2, \ldots, K_n) - h_{n,W}(K_1 + x, K_2, \ldots, K_n))^2 \, dx \, \Lambda^{n-1}(d(K_2, \ldots, K_n))
\]

\[ \leq \frac{\hat{\beta}(\psi)^2(d + 1)\beta_1 \alpha^{n-1}}{(n!)^2} \frac{1}{V_d(W)} \sum_{i=0}^{d} V_i(W) \left( \sum_{r=0}^{d} V_r(K_1) \right)^2 \left( \exp(p_W(K_1)) - 1 \right)^2 \]

\[ + \frac{\beta(\psi)^2(d + 1)\beta_1 \alpha^{n-2}}{(n!)^2} \frac{1}{V_d(W)} \sum_{i=0}^{d} V_i(W) \left( \sum_{r=0}^{d} V_r(K_1) \right)^2 (n - 1)p_W(K_1). \]

Then Lemma 3.7 and \( p_W(K_1) \to 0 \) as \( r(W) \to \infty \) show that the right-hand side vanishes for \( r(W) \to \infty \). This proves (9.11) so that the dominated convergence theorem yields (9.9), which concludes the proof.

**Proof of Theorem 9.1 under assumption (2.5).** The triangle inequality for the \( d_3 \)-distance yields

\[
d_3 \left( \frac{1}{\sqrt{V_d(W)}} (\Psi(Z \cap W) - \mathbb{E}\Psi(Z \cap W)), N \right)
\]

\[
\leq d_3 \left( \frac{1}{\sqrt{V_d(W)}} (\Psi(Z \cap W) - \mathbb{E}\Psi(Z \cap W)), \frac{1}{\sqrt{V_d(W)}} (\Psi(Z \cap W) - \mathbb{E}\Psi(Z \cap W)) \right)
\]

\[ + d_3 \left( \frac{1}{\sqrt{V_d(W)}} (\Psi(Z \cap W) - \mathbb{E}\Psi(Z \cap W)), N \right). \]

In the sequel we show that both terms on the right-hand side of (9.12) vanish as \( r(W) \to \infty \). By Lemma 5.5 the first expression is bounded by

\[
m \sqrt{\mathbb{E}\|\Psi(Z \cap W) - \mathbb{E}\Psi(Z \cap W)\|^2/V_d(W)} + \|\Psi(Z \cap W) - \mathbb{E}\Psi(Z \cap W)\|^2/V_d(W)
\]

\[
\times \sqrt{\mathbb{E}\|\Psi(Z \cap W) - \Psi(Z \cap W) - (\Psi(Z \cap W) + \mathbb{E}\Psi(Z \cap W))\|^2/V_d(W)},
\]

47
where $\| \cdot \|$ stands for the Euclidean norm in $\mathbb{R}^m$. Since, by Lemma 9.3, the first factor is bounded and the second factor vanishes as $r(W) \to \infty$, the first expression on the right-hand side of (9.12) vanishes as $r(W) \to \infty$.

The restriction of $\eta$ to $M_W$ is a Poisson process in $\mathcal{K}^d$ of intensity $\gamma$ and with the typical grain $Z_{0,W}$ given by

$$Z_{0,W} = \begin{cases} Z_0, & Z_0 \in M_W, \\ \emptyset, & Z_0 \notin M_W. \end{cases}$$

Obviously, $Z_{0,W}$ satisfies (2.5).

By applying Theorem 8.1 to the vector $\Psi(Z_W \cap W)$ of Poisson functionals depending on the restriction of $\eta$ to $M_W$, we shall prove that

$$\lim_{r(W) \to \infty} d_3 \left( \frac{1}{\sqrt{V_d(W)}} (\Psi(Z_W \cap W) - E(\Psi(Z_W \cap W)), N) \right) = 0. \quad (9.13)$$

Theorem 8.1 yields this without a rate of convergence if

$$\lim_{r(W) \to \infty} \frac{\text{Cov}(\psi_k(Z_W \cap W), \psi_l(Z_W \cap W))}{V_d(W)} = \sigma_{\psi_k,\psi_l} \quad (9.14)$$

for $k, l \in \{1, \ldots, m\}$ and if (8.8) holds with a fixed $b \geq 1$ and $a \geq 0$ depending on $W$ such that $a$ tends to zero as $r(W) \to \infty$.

The condition (9.14) is satisfied because of Lemma 9.5 and Theorem 8.1. The inequalities (9.5) and (9.7) also hold for the Boolean model $Z_W$ with the same $a_1, b_1, b_2$ as in the proof of (9.2) under assumption (9.1) and

$$a_2 = c_{13} \max_{1 \leq k,l \leq m} \frac{(\beta(\psi_k)\beta(\psi_l))^2}{\alpha} E \left[ \left( \sum_{r=0}^{d} \min\{V_r(Z_{0,W}), V_r(W)\} \right)^3 \sum_{s=0}^{d} V_s(Z_{0,W}) \right] \sum_{u=0}^{d} V_u(W)$$

with $c_{13} = (d+1)\gamma \beta_1$. This is the case since the derivations of (9.5) and (9.7) require only finite second moments and for the Boolean model $Z_W$ all inequalities proven in Section 3 hold with the same constants as for the original Boolean model $Z$. Consequently, (8.8) is satisfied with $a = \max\{a_1, a_2\}/V_d(W)^2$ and $b = \max\{b_1, b_2\}$. Since (8.8) only requires that the second moments are finite, we obtain that $a_1/V_d(W)^2$ tends to zero as $r(W) \to \infty$. On the other hand, $\lim_{r(W) \to \infty} d_2/V_d(W)^2 = 0$ is equivalent to

$$\lim_{r(W) \to \infty} E \left[ \frac{1}{V_d(W)} \left( \sum_{r=0}^{d} \min\{V_r(Z_{0,W}), V_r(W)\} \right)^3 \sum_{s=0}^{d} V_s(Z_{0,W}) \right] = 0. \quad (9.15)$$

This expectation can be rewritten as

$$\int \frac{1}{V_d(W)} \left( \sum_{r=0}^{d} \min\{V_r(K), V_r(W)\} \right)^3 \sum_{s=0}^{d} V_s(K) 1\{K \in M_W\} Q(dK).$$

For $K \in \mathcal{K}_0^d \cap M_W$ we have $V_r(K) \leq \sqrt{V_d(W)}$, for $r \in \{0, \ldots, d\}$, and therefore

$$\frac{1}{V_d(W)} \left( \sum_{r=0}^{d} \min\{V_r(K), V_r(W)\} \right)^3 \sum_{s=0}^{d} V_s(K) 1\{K \in M_W\} \leq (d+1)^2 \left( \sum_{r=0}^{d} V_r(K) \right)^2,$$

48
which is independent of \( W \) and integrable with respect to \( Q \). For any fixed \( K \in \mathcal{K}_d \) the left-hand side vanishes as \( r(W) \to \infty \) so that the dominated convergence theorem implies (9.15) and hence \( a \) tends to zero as \( r(W) \to \infty \). Finally, Theorem 8.1 yields (9.13), which concludes the proof of Theorem 9.1.

**Remark 9.6.** As discussed in Remark 9.4, (9.2) still holds if we replace the centred Gaussian random vector \( N \) with the asymptotic covariance matrix by a centred Gaussian random vector \( N(W) \) with the exact covariance matrix. This can be even done if the functionals are not translation invariant since in this case we do not need Theorem 3.1. The second part of the proof of Theorem 9.1 still holds because (9.14) does not apply in this situation. This means that under condition (2.5) for additive, locally bounded, and measurable functionals \( \psi_1, \ldots, \psi_m \),

\[
\lim_{r(W) \to \infty} d_3 \left( \frac{1}{\sqrt{V_d(W)}} (\Psi(Z \cap W) - \mathbb{E}\Psi(Z \cap W)), N(W) \right) = 0.
\]

**Proof of Theorem 9.3 under assumption (2.5).** For \( m = 1 \) and a centred Gaussian random variable \( N(W) \) with variance \( \text{Var} \psi(Z \cap W)/V_d(W) \) the previous remark implies that

\[
\lim_{r(W) \to \infty} d_3 \left( \frac{\psi(Z \cap W) - \mathbb{E}\psi(Z \cap W)}{\sqrt{\text{Var} \psi(Z \cap W)}}, N(W) \right) = 0. \tag{9.16}
\]

It follows from the definition of the \( d_3 \)-distance that for random vectors \( A, B \) and any \( c > 0 \),

\[
d_3(cA, cB) \leq \max\{1, c\}^3 d_3(A, B).
\]

With \( c_W := \sqrt{V_d(W)}/\sqrt{\text{Var} \psi(Z \cap W)} \) and a standard Gaussian random variable \( N \), this yields

\[
d_3 \left( \frac{\psi(Z \cap W) - \mathbb{E}\psi(Z \cap W)}{\sqrt{\text{Var} \psi(Z \cap W)}}, N \right) \leq \max\{1, c_W\}^3 d_3 \left( \frac{\psi(Z \cap W) - \mathbb{E}\psi(Z \cap W)}{\sqrt{V_d(W)}}, N(W) \right).
\]

Since \( c_W \) is bounded by assumption (9.3), (9.16) concludes the proof.

**Remark 9.7.** In Theorem 9.3 it is possible to weaken the assumption that the Poisson process is stationary. In the proof we only need to find upper bounds for the kernels and some integrals. This is, for instance, still possible if the intensity measure is of the form

\[
\Lambda(\cdot) = \int \int 1\{K + x \in \cdot\} f(x) \, dx \, Q(dK)
\]

with a non-negative bounded function \( f : \mathbb{R}^d \to \mathbb{R} \). Now we always get upper bounds if we replace this intensity measure by the measure in (2.1) with \( \gamma = \sup_{x \in \mathbb{R}^d} |f(x)| < \infty \).

For the multivariate central limit this argument does not work in general since its proof makes use of Theorem 3.1 which depends on the translation invariance of the intensity measure. But if one can prove by other methods the existence of an asymptotic covariance matrix, it is still possible to weaken the stationarity assumption as described above.
10 Appendix A

Lemma 5.5 is implied by Lemma 10.3 below. First, we provide two auxiliary results.

**Lemma 10.1.** Let $K_1, \ldots, K_m \in \mathbb{R}^d$, $m \geq 2$, be convex bodies with non-empty interiors. Then, for $\mathcal{H}^{d(m-1)}$-almost all $(t_2, \ldots, t_m) \in \mathbb{R}^{d(m-1)}$, if $K_1 \cap (K_2 + t_2) \cap \cdots \cap (K_m + t_m) \neq \emptyset$, then $(K_1)^o \cap (K_2 + t_2)^o \cap \cdots \cap (K_m + t_m)^o \neq \emptyset$.

**Proof.** The assertion is proved by induction over $m \geq 2$. For $m = 2$, the assertion holds, since any $t_2 \in \mathbb{R}^d$ such that $K_1 \cap (K_2 + t_2) \neq \emptyset$ and $K_2^o \cap (K_2 + t_2) = \emptyset$ is contained in the boundary of $K_1 + (-K_2)$, which has $d$-dimensional Hausdorff measure zero.

We assume that $A(m)$ has been proved for some $m \geq 2$ and $K_1, \ldots, K_{m+1}$ are given. Let us denote by $C \subset (\mathbb{R}^d)^{m-1}$ the set of all $(t_2, \ldots, t_m) \in (\mathbb{R}^d)^{m-1}$ such that $A(m)$ is satisfied for $K_1, \ldots, K_m$. Further, let $M$ be the set of all $(t_2, \ldots, t_{m+1}) \in (\mathbb{R}^d)^{m+1}$ such that $A(m+1)$ is not satisfied for $K_1, \ldots, K_{m+1}$. Then

$$
\mathcal{H}^{dm}(M) = \int_C \int_{\mathbb{R}^d} 1_M(t_2, \ldots, t_{m+1}) \mathcal{H}^d(t_{m+1}) \mathcal{H}^{d(m-1)}(d(t_2, \ldots, t_m)).
$$

Let $(t_2, \ldots, t_{m+1}) \in M$, $(t_2, \ldots, t_m) \in C$ and set $L := K_1 \cap (K_2 + t_2) \cap \cdots \cap (K_m + t_m)$. Then $L \cap (K_{m+1} + t_{m+1}) \neq \emptyset$ if and only if $(K_1)^o \cap (K_2 + t_2)^o \cap \cdots \cap (K_m + t_m)^o \neq K_1 \cap (K_2 + t_2) \cap \cdots \cap (K_m + t_m) \cap (K_{m+1} + t_{m+1}) \neq \emptyset$. In particular, $L \neq \emptyset$ and $L^o = (K_1)^o \cap (K_2 + t_2)^o \cap \cdots \cap (K_m + t_m)^o$. Since $L \neq \emptyset$, we obtain $L^o \cap (K_{m+1} + t_{m+1})^o \neq \emptyset$, which implies, for fixed $t_2, \ldots, t_m$, that $t_{m+1}$ lies in a subset of $\mathbb{R}^d$ of $\mathcal{H}^d$-measure zero. Therefore the inner integral in (10.1) vanishes, which proves $\mathcal{H}^{dm}(M) = 0$.

For the following lemma, we use basic notions from geometric measure theory.

**Lemma 10.2.** Let $m \geq 1$, let $K_1, \ldots, K_m$ be convex bodies in $\mathbb{R}^d$. If $m \leq d$, then for $\mathcal{H}^{d(m-1)}$-almost all translates $(t_2, \ldots, t_m) \in \mathbb{R}^{d(m-1)}$, the intersection $\partial K_1 \cap (\partial K_2 + t_2) \cap \cdots \cap (\partial K_m + t_m)$ has finite $(d-m)$-dimensional Hausdorff measure. For $m > d$, the intersection is the empty set for almost all translation vectors.

**Proof.** Since for $m = 1$ there is nothing to show, we assume that $m \in \{2, \ldots, d\}$. Let $W := \partial K_1 \times \cdots \times \partial K_m \subset \mathbb{R}^{dm}$, let $Z \subset \mathbb{R}^{d(m-1)}$ be the compact image set of the Lipschitz map $T: W \to Z \subset \mathbb{R}^{d(m-1)}$, $K_1, \ldots, K_m \mapsto (x_1 - x_2, \ldots, x_1 - x_m)$. Then the assumptions of the coarea theorem [10, Theorem 3.2.22 (2)] are satisfied. Thus, for $\mathcal{H}^{d(m-1)}$-almost all $(t_2, \ldots, t_m) \in Z$, the set $T^{-1}\{t_2, \ldots, t_m\}$ has finite $\mathcal{H}^{d-m}$ measure. Identify $\mathbb{R}^{dm}$ with $(\mathbb{R}^d)^m$ and denote by $\pi_1: (\mathbb{R}^d)^m \to \mathbb{R}^d$ the projection to the first component, which is a Lipschitz map. Then

$$
\partial K_1 \cap (\partial K_2 + t_2) \cap \cdots \cap (\partial K_m + t_m) = \pi_1(T^{-1}\{t_2, \ldots, t_m\})
$$

has finite $(d-m)$-dimensional Hausdorff measure for $\mathcal{H}^{d(m-1)}$-almost all $(t_2, \ldots, t_m) \in Z$. (If $(t_2, \ldots, t_m) \notin Z$, the intersection is the empty set.)

The assertion for $m > d$ easily follows from the one for $m = d$.

Recall that $[n] = \{1, \ldots, n\}$ for $n \in \mathbb{N}$.

**Lemma 10.3.** Let $K_1, K'_2, \ldots, K'_n \subset \mathbb{R}^d$ be convex bodies with non-empty interiors, $n \geq 1$, and let $i \in \{0, \ldots, d-1\}$. Then

$$
\Phi_i(K_1 \cap \cdots \cap K_n; \cdot) = \sum_{\emptyset \neq I \subset [n]} \Phi_i \left( \bigcap_{r \in I} K_r \cap \bigcap_{r \in I} \partial K_r \cap \bigcap_{s \notin I} K'_s \right),
$$

50
for almost all translates $K_i$ of $K'_i$ for $i = 2, \ldots, n$.

**Proof.** There is nothing to show for $n = 1$, thence we assume that $n \geq 2$. By Lemma 10.1 and Lemma 10.2, we can assume that $K_1, \ldots, K_n$ have a common interior point and for $\emptyset \neq I \subset [n]$ each intersection $\cap_{r \in I} \partial K_r$ has finite $(d - |I|)$-dimensional Hausdorff measure if $|I| \leq d$, and otherwise is the empty set.

Since $\Phi_i(K_1 \cap \ldots \cap K_n; \cdot) = \sum_{\emptyset \neq I \subset [n]} \Phi_i \left( K_1 \cap \ldots \cap K_n; \cdot \cap \bigcap_{r \in I} \partial K_r \cap \bigcap_{s \notin I} K^\circ_s \right)$, the measure property yields that

$$\Phi_i(K_1 \cap \ldots \cap K_n; \cdot) = \sum_{\emptyset \neq I \subset [n]} \Phi_i \left( \bigcap_{r \in I} K_r; \cdot \cap \bigcap_{r \in I} \partial K_r \cap \bigcap_{s \notin I} K^\circ_s \right).$$

The intersection $U := \bigcap_{s / I} K^\circ_s$ is open, $U' := \bigcap_{r \in I} \partial K_r \cap \bigcap_{s / I} K^\circ_s \subset U$, and $K_1 \cap \ldots \cap K_n \cap U = \bigcap_{r \in I} K_r \cap U$. Hence, since $\Phi_i$ is locally defined (see [27, p. 206]), it follows that

$$\Phi_i(K_1 \cap \ldots \cap K_n; \cdot) = \sum_{\emptyset \neq I \subset [n]} \Phi_i \left( \bigcap_{r \in I} K_r; \cdot \cap \bigcap_{r \in I} \partial K_r \cap \bigcap_{s / I} K^\circ_s \right).$$

Since $\bigcap_{r \in I} \partial K_r$ has finite $(d - |I|)$-dimensional Hausdorff measure for $|I| \in \{1, \ldots, d\}$, and is the empty set for $|I| > d$, we conclude that if $d \geq |I| > d - i$, then $\bigcap_{r \in I} \partial K_r$ has $i$-dimensional Hausdorff measure zero. A special case of [4] Theorem 5.5 then yields that

$$\Phi_i \left( \bigcap_{r \in I} K_r; \cdot \cap \bigcap_{r \in I} \partial K_r \cap \bigcap_{s / I} K^\circ_s \right) = 0,$$

which completes the proof. \qed

## Appendix B

In Section 5, we introduced certain measures in (5.3), (5.6), and (5.7) whose definitions involve an integration with respect to the intensity measure of the underlying stationary Poisson process. We now provide an alternate description (without detailed proofs) of these measures in terms of mixed measures of translative integral geometry. Throughout this section, we assume that condition (5.8) is satisfied.

For $j \in \{0, \ldots, d - 1\}$, $K_1, \ldots, K_k \in {\cal K}^d$, and $r_1, \ldots, r_k \in \{j, \ldots, d\}$ with $r_1 + \ldots + r_k = (k - 1)d + j$ we denote by $\Phi^{(j)}_{r_1, \ldots, r_k}(K_1, \ldots, K_k; \cdot)$ the mixed measures of translative integral geometry. Subsequently, we shall repeatedly use the translative integral formula

$$\int \cdots \int f(x_1, x_1 - x_2, \ldots, x_1 - x_n) \Phi_j(K_1 \cap (K_2 + x_2) \cap \ldots \cap (K_n + x_n); dx_1)\,dx_2 \ldots \,dx_n$$

$$= \sum_{r_1, \ldots, r_n = j}^{d} \sum_{r_1 + \ldots + r_n = (n - 1)d + j} \int f(x_1, \ldots, x_n) \Phi^{(j)}_{r_1, \ldots, r_n}(K_1, \ldots, K_n; d(x_1, \ldots, x_n)), \quad (11.1)$$

which holds for an arbitrary non-negative, measurable function $f$ (see [28 (6.16)]). In addition to the properties of mixed measures stated in [28 Theorem 6.4.1 (b)], we need the fact that the mixed measures are translation covariant (independently in each argument), which follows
In the case from the special case \( \text{(6.14)} \) together with an approximation argument. Moreover, we use the special form \( \text{(2.11)} \) of the intensity measure without further mentioning.

Applying \( \text{(11.11)} \) and the decomposability property of the mixed measures (of translative integral geometry), stated in \( \text{[28, Theorem 6.4.1]} \), the range of summation is effectively restricted to \( r_i \leq d - 1 \) and we obtain

\[
h_{0,j} = \sum_{l=1}^{d-j} \frac{\gamma^l}{l!} \sum_{r_1, \ldots, r_l = j}^{d-1} \sum_{r_1 + \ldots + r_l = (l-1)d+j} \mathbb{E} \Phi^{(j)}_{r_1, \ldots, r_l}(Z_1, \ldots, Z_l; \mathbb{R}^d).
\]

Here and in the following, we denote by \( Z_1, Z_2, \ldots \) independent copies of the typical grain.

Next we turn to the measure \( H^k_{i,d} \) in the form

\[
H^k_{i,d} = \gamma^k \sum_{r_1, \ldots, r_k = i}^{d-1-\gamma^l} \sum_{r_1 + \ldots + r_k = (k-1)d+i}^{d-1} \mathbb{E} \int 1\{z \in \cdot\} \Phi^{(i)}_{r_1, \ldots, r_k}(Z_1, \ldots, Z_k; x^k_{n=1}(Z_n + z)) dz.
\]

In particular, for \( k = 1 \) we thus recover \( \text{(5.12)} \) as a very special case.

Next we turn to the measure \( H^{k,l,m}_{i,j} \) introduced in \( \text{(5.6)} \), for \( i, j \in \{1, \ldots, d-1\}, k \in \{1, \ldots, d-i\}, l \in \{1, \ldots, d-j\}, \) and \( m \in \{0, \ldots, k \land l\} \). If \( m = 0 \), we obtain

\[
H^{k,l,0}_{i,j} := \gamma^{k+l} \sum_{s_1, \ldots, s_k = i}^{d-1} \sum_{s_1 + \ldots + s_k = (k-1)d+i}^{d-1} \mathbb{E} \int \int 1\{x \in \cdot\} 1\{x \in Z_{k+l}\}
\]

\[
\Phi^{(i)}_{s_1, \ldots, s_k}(Z_1, \ldots, Z_k; x^k_{n=1}(Z_n^o + x)) \Phi^{(j)}_{s_1, \ldots, s_k}(Z_{k+1}, \ldots, Z_{k+l}; x^k_{n=1+k}(Z_n^o - x)) dx.
\]

In the case \( m > 0 \), we get

\[
H^{k,l,m}_{i,j} = \gamma^{k+l-m} \sum_{s_1, \ldots, s_{k-m+1} = i}^{d-1} \sum_{s_1 + \ldots + s_{k-m+1} = (k-m)d+i}^{d-1} \mathbb{E} \int \int 1\{y \in \cdot\}
\]

\[
\left\{ y \in \bigcap_{n=1}^{k-m} (x_n - Z_n^o) \cap \bigcap_{n=k-m+1}^{k} (Z_n - x_n) \cap \bigcap_{n=k+1}^{k+l-m} (Z_n^o - x_n) \right\}
\]

\[
\Phi^{(i)}_{s_1, \ldots, s_k-m+1}(Z_1, \ldots, Z_{k-m}, \bigcap_{n=k-m+1}^{k} (Z_n - x_n); d(x_1, \ldots, x_{k-m}, y))
\]

\[
\Phi^{(j)}_{s_k-m+1, \ldots, s_{k+l-m}}(Z_{k+1-m}, \ldots, Z_{k+l-m}; d(x_{k+1-m}, \ldots, x_{k+l-m})).
\]

For special choices of indices, these formulas simplify. For instance, \( \text{(6.13)} \) is recovered from the special case \( k = l = 1 \) and \( m = 0 \). Now we use that \( \Phi^{(i)}_i = \Phi_i \) (see \( \text{[28, p 229]} \)). If \( m = k = \{1, \ldots, l\} \), we obtain

\[
H^{k,k}_{i,j} = \gamma^l \sum_{r_1, \ldots, r_l = j}^{d-1} \sum_{r_1 + \ldots + r_l = (l-1)d+j} \mathbb{E} \int \int 1\{y \in \cdot\} 1\{y \in \bigcap_{n=1}^{k} (Z_n - x_n) \cap \bigcap_{n=k+1}^{l} (Z_n^o - x_n) \}
\]

\[
\Phi_i \left( \bigcap_{n=1}^{k} (Z_n - x_n); dy \right) \Phi^{(j)}_{r_1, \ldots, r_l}(Z_1, \ldots, Z_l; d(x_1, \ldots, x_l)).
\]
In the particular case \( m = k = 1 \leq l \), this yields

\[
H_{i,j}^{1,1,1} = \gamma^l \sum_{r_1, \ldots, r_l=j, r_1+\ldots+r_l=(l-1)d+j} \mathbb{E} \int \mathbf{1}_{\{y-x_1 \in \cdot\}} \mathbf{1}_{\{y-x_1 \in \bigcap_{n=2}^{l}(Z_n^\circ - x_n)\}}
\]

\[
\Phi_i(Z_1; dy) \Phi_{r_1,\ldots,r_l}(Z_1, \ldots, Z_l; d(x_1, \ldots, x_l)).
\]

If also \( l = 1 \), we recover (5.14).

The special case \( m = l \in \{1, \ldots, k\} \) can be discussed similarly or can be recovered by symmetry considerations.

It appears to be desirable to express the measures \( H_{i,j} \) in terms of geometric mean values associated with a (single) typical grain. We have seen that this is possible for \( H_{d,d} \), for \( H_{d-1,d} \) (see (5.15)), for \( H_{d-1,d-1} \) (see (5.16)), and, in the isotropic case, for \( H_{0,j} = h_{0,j}(0) \), \( j \in \{0, \ldots, d-1\} \); see (5.7), (5.9), and (6.2). We do not know whether this can be achieved also for some of the other measures.

### 12 Appendix C

In Section 5 it was shown that \( \rho_{i,j}(W)/V_d(W) \) is close to \( \rho_{i,j} \) if \( r(W) \) is large, where \( \rho_{i,j} \) and \( \rho_{i,j}(W) \) are given in (5.1) and (5.2) by their series expansions. Throughout this section, we assume that (5.8) is satisfied. Theorem 5.2 provides a succinct description of \( \rho_{i,j} \) in terms of the finite measures \( H_{i,j} \). For a stationary and isotropic Boolean model, the local quantities \( \rho_{i,j}(W) \) are a crucial ingredient in the local covariances provided in Theorem 6.4. Therefore, we are interested in describing the deviations

\[
r_{i,j}(W) := \rho_{i,j}(W) - \int C_W(x)e^{\gamma C_d(x) H_{i,j}(dx)}, \quad i, j \in \{0, \ldots, d\}, \tag{12.1}
\]

in more detail. Recall that \( C_W(y) = V_d(W \cap (W + y)), \ y \in \mathbb{R}^d \), is the set covariance (function) of \( W \in \mathcal{K}^d \). Clearly, we have \( C_W(y) = C_W(-y) \) and \( C_W(y)/V_d(W) \to 1 \) as \( r(W) \to \infty \), for \( y \in \mathbb{R}^d \). Since the measures \( H_{i,j} \) are bounded and \( C_d(x) \leq \mathbb{E} V_d(Z_0) \), for all \( x \in \mathbb{R}^d \), it follows from Theorem 5.2 that \( r_{i,j}(W)/V_d(W) \to 0 \) as \( r(W) \to \infty \). We also point out that the subsequent arguments show that \( r_{i,j}(W) \geq 0 \).

Now we start to analyze the remainder terms \( r_{i,j}(W) \) in (12.1) more carefully. By definition (5.2) we have

\[
\rho_{i,j}(W) = \sum_{n=1}^{\infty} \frac{\rho_{i,j}(W,n)}{n!},
\]

where

\[
\rho_{i,j}(W,n) := \int V_i(K_1 \cap \ldots \cap K_n \cap W) V_j(K_1 \cap \ldots \cap K_n \cap W) \Lambda^n(d(K_1, \ldots, K_n)).
\]

In the simple case where \( i = 0 \) and \( j = d \), we can use

\[
\int \cdots \int V_d((K_1 + x_1) \cap \ldots \cap (K_n + x_n) \cap W) \, dx_1 \ldots dx_n = V_d(K_1) \cdots V_d(K_n) V_d(W),
\]

to obtain from (2.1) that \( \rho_{0,d}(W) = V_d(W)(e^{\gamma v_d} - 1) \) and therefore \( r_{0,d}(W) = 0 \).
In the case $i = j = d$ we obtain by a similar argument as in the proof of Theorem 5.2 that
\[ \rho_{d,d}(W,n) = \gamma^n \int C_W(y) C_d(y)^n \, dy, \]
and hence we get $r_{d,d}(W) = 0$.

Next we consider the case where $i = 0$ and $j \in \{0, \ldots, d - 1\}$. Define $K_0 := W$ and $[n] := \{0, 1, \ldots, n\}$. Lemma 5.3 implies that
\[ \rho_{0,j}(W,n) = d - j \sum_{l=1}^{n} \sum_{J \subseteq [n], |J| = l} \int 1\{z \in \bigcap_{r \in J} K_r^\circ\} \Phi_j(K_J; dz) \Lambda^n(d(K_1, \ldots, K_n)), \]
where $\Phi_j(K_J; \cdot) := \Phi_j(K_{j_1}, \ldots, K_{j_l}; \cdot)$ for $J = \{j_1, \ldots, j_l\}$. We split the summation into two parts,
\[ \rho_{0,j}(W,n) = \rho_{0,j}(W,n,1) + r_{0,j}(W,n), \]
where the first summand takes into account all subsets $J$ with $0 \notin J$ and the second summand those subsets with $0 \in J$. By symmetry, the first summand is given by
\[ \rho_{0,j}(W,n,1) = \sum_{l=1}^{d-j} \binom{n}{l} \int \int 1\{z \in W^\circ, z \in K_{i+1}^\circ \cap \ldots \cap K_n^\circ\} \Phi_j(K_1, \ldots, K_l; dz) \Lambda^n(d(K_1, \ldots, K_n)). \]
Carrying out the integration with respect to $K_{i+1}, \ldots, K_n$ and using the form (2.1) of $\Lambda$, we obtain that
\[ \rho_{0,j}(W,n,1) = \sum_{l=1}^{d-j} \gamma^{n-l} v_d n^{n-l} \int \int 1\{z \in W^\circ\} \Phi_j(K_1, \ldots, K_l; dz) \Lambda^{l-1}(d(K_1, \ldots, K_l)). \]
Using (5.11) and then the invariance of $\Lambda$ under translations, we get
\[ \rho_{0,j}(W,n,1) = \sum_{l=1}^{d-j} \gamma^{n-l} v_d n^{n-l} \int \int \int \int 1\{z \in W^\circ\} \Phi_j(K_1, \ldots, K_{l-1}, K_l + x; dz) \Lambda^{l-1}(d(K_1, \ldots, K_l)) \mathcal{Q}(dK_l) \, dx \]
\[ = \sum_{l=1}^{d-j} \gamma^{n-l} v_d n^{n-l} \int \int \int \int \int 1\{z + x \in W^\circ\} \Phi_j(K_1, \ldots, K_l; dy) \Lambda^{l-1}(d(K_1, \ldots, K_{l-1})) \mathcal{Q}(dK_l) \, dx. \]
Therefore
\[ \sum_{n=1}^{\infty} \frac{1}{n!} \rho_{0,j}(W,n,1) = V_d(W) e^{\gamma v_d h_{0,j}}. \]
In the same way we obtain that

\[ r_{0,j}(W,n) = V_j(W) \gamma^n v_d^n + \sum_{l=1}^{d-j-1} \gamma^{n-l} v_d^{n-l} \binom{n}{l} \iint \Phi_j(W - x, K_1, \ldots, K_l; \mathbb{R}^d) \]

\[ \Lambda^{l-1}(d(K_1, \ldots, K_{l-1})) \mathbb{Q}(dK_l) \, dx. \]

This finally shows that, for \( j \in \{0, \ldots, d-1\} \),

\[ r_{0,j}(W) = V_j(W) (e^{\gamma d} - 1) + \gamma e^{\gamma d} \sum_{l=1}^{d-j-1} \frac{1}{l!} \iint \Phi_j(W - x, K_1, \ldots, K_l; \mathbb{R}^d) \]

\[ \Lambda^{l-1}(d(K_1, \ldots, K_{l-1})) \mathbb{Q}(dK_l) \, dx. \quad (12.2) \]

Next we note that, for \( i \in \{1, \ldots, d-1\} \),

\[ r_{i,d}(W) = \iint 1\{z \in W\} (e^{\gamma_d(y-z)} - 1) \Phi_i(W; dy) \, dz \]

\[ + \sum_{k=1}^{d-i-1} \frac{\gamma}{k!} \int \cdots \int e^{\gamma_d(y-z)} 1\{z + x \in W\} 1\{z \in K_1 \cap \ldots \cap K_k\} \]

\[ \Phi_i(W - x, K_1, \ldots, K_k; dy) \, dz \, dx \, \Lambda^{k-1}(d(K_1, \ldots, K_{k-1})) \mathbb{Q}(dK_k). \quad (12.3) \]

We do not prove this fact, but proceed directly to summarize the similar but more involved case where \( i, j \in \{1, \ldots, d-1\} \). For the moment we fix \( n \in \mathbb{N} \). Lemma implies that

\[ \rho_{i,j}(W,n) = \sum_{k=1}^{d-i} \sum_{l=1}^{d-j} \sum_{I \subseteq [n]} \sum_{J \subseteq [n]} \iint 1\{y \in \bigcap_{r \in I} K_r^o, z \in \bigcap_{r \in J} K_r^o\} \]

\[ \Phi_i(K_I; dy) \Phi_j(K_J; dz) \Lambda^n(d(K_1, \ldots, K_n)). \]

Distinguishing the cases \( 0 \notin I, 0 \in I, 0 \notin J \) and \( 0 \in J \), we split the summation into four parts,

\[ \rho_{i,j}(W,n) = \rho_{i,j}(W,n,1) + r_{i,j}(W,n,2) + r_{j,i}(W,n,2) + r_{i,j}(W,n,3), \]

where the first and the fourth summand are for the cases \( 0 \notin I, 0 \notin J \) and \( 0 \in I, 0 \in J \), and the second and the third summand refer to the symmetric cases \( 0 \in I, 0 \notin J \) and \( 0 \notin I, 0 \in J \) so that the third summand does not require further consideration. We define

It can be shown that

\[ \sum_{n=1}^{\infty} \frac{\rho_{i,j}(W,n,1)}{n!} = \int C_W(x) e^{\gamma_d(x)} H_{i,j}(dx). \]

The remainder terms

\[ r_{i,j}(W,l) := \sum_{n=1}^{\infty} \frac{r_{i,j}(W,n,l)}{n!}, \quad l \in \{2, 3\}. \]
are given by

\[
R_{i,j}(W,2) = \sum_{k=0}^{d-i-1} \sum_{l=0}^{d-j} \sum_{m=0}^{k\land l} \frac{\gamma}{m!(k-m)!(l-m)!} \int \cdots \int 1\{z+x \in W\} e^{\gamma C_d(y-z)} \quad (12.4)
\]

\[
1\{y \in K_{k+1}^o \cap \cdots \cap K_{k+l-m}^o, z \in K_1^o \cap \cdots \cap K_{k-m}^o\} \Phi_i(W - x, K_1, \ldots, K_k; dy)
\]

\[
\Phi_j(K_{k+1-m}, \ldots, K_{k+l-m}; dz) \Lambda^{k+l-m-1}(d(K_1, \ldots, K_{k+l-m-1})) Q(dK_{k+l-m}) dx,
\]

and

\[
R_{i,j}(W,3) = \sum_{k=0}^{d-i-1} \sum_{l=0}^{d-j-1} \sum_{m=0}^{k\land l} \frac{1\{k+l \geq 1\} \gamma}{m!(k-m)!(l-m)!} \int \cdots \int e^{\gamma C_d(y-z)} \quad (12.5)
\]

\[
1\{y \in K_{k+1}^o \cap \cdots \cap K_{k+l-m}^o, z \in K_1^o \cap \cdots \cap K_{k-m}^o\} \Phi_i(W - x, K_1, \ldots, K_k; dy)
\]

\[
\Phi_j(W - x, K_{k+1-m}, \ldots, K_{k+l-m}; dz) \Lambda^{k+l-m-1}(d(K_1, \ldots, K_{k+l-m-1}))
\]

\[
Q(dK_{k+l-m}) dx + \int (e^{\gamma C_d(y-z)} - 1) \Phi_i(W; dy) \Phi_j(W; dz).
\]

We can now summarize as follows.

**Theorem 12.1.** Let \( Z \) be a stationary Boolean model such that (2.5) and (5.8) are satisfied. Let \( W \in K^d \). Then the remainder terms (12.1) are given by \( r_{i,j}(W) = 0 \) if \( i \in \{0, d\} \) and \( j = d \), by (12.2) for \( i = 0 \) and \( j \in \{0, \ldots, d-1\} \), by (12.3) for \( i \in \{1, \ldots, d-1\} \) and \( j = d \), and by

\[
r_{i,j}(W) = r_{i,j}(W,2) + r_{j,i}(W,2) + r_{i,j}(W,3), \quad i, j \in \{1, \ldots, d-1\}.
\]

Here \( r_{i,j}(W,2) \) and \( r_{i,j}(W,3) \) are given by (12.4) and (12.5), respectively.

Together with Theorem 6.4, the preceding theorem yields explicit integral representations for \( \text{Cov}(V_i(Z \cap W), V_j(Z \cap W)) \) for an isotropic Boolean model. We restrict ourselves to giving the formulas for the remainder terms (12.1) in the case \( d = 2 \). In higher dimensions the formulas get more complicated.

**Corollary 12.2.** Let \( Z \) be a stationary Boolean model such that (2.5) and (5.8) are satisfied. Assume that \( d = 2 \), and let \( W \in K^2 \). Then \( r_{0,2}(W) = r_{2,2}(W) = 0 \). Moreover,

\[
r_{1,2}(W) = \int \Phi_1(W; W + x)(e^{\gamma C_2(x)} - 1) dx,
\]

\[
r_{1,1}(W) = 2\gamma \int \Phi_1(W; W + z - y) e^{\gamma C_2(y-z)} M_{1,2}(d(y, z))
\]

\[
+ \int (e^{\gamma C_2(y-z)} - 1) \Phi_1(W; dy) \Phi_1(W; dz),
\]

\[
r_{0,1}(W) = V_1(W')(e^{\gamma v_2} - 1).
\]

If the typical grain is also isotropic, then

\[
r_{0,0}(W) = \frac{2\gamma v_1}{\pi} e^{\gamma v_2} V_1(W) + (e^{\gamma v_2} - 1).
\]
Acknowledgement: We acknowledge the support of the German Research Foundation (DFG) through the research unit “Geometry and Physics of Spatial Random Systems” under the grant HU 1874/3-1.

References

[1] Baryshnikov, Y. AND Yukich, J.E. (2005). Gaussian limits for random measures in geometric probability. *Ann. Appl. Probab.* 15, 213–253.

[2] Baddeley, A. (1980). A limit theorem for statistics of spatial data. *Adv. in Appl. Probab.* 12, 447–461.

[3] Chiu, S.N., Stoyan, D., Kendall, W.S. AND Mecke, J. (2013). *Stochastic Geometry and its Applications.* Third Edition, Wiley, Chichester.

[4] Colesanti, A. AND Hug, D. (2000). Hessian measures of semi-convex functions and applications to support measures of convex bodies. *Manuscripta Math.* 101, 209–238.

[5] Davy, P. (1976). Projected thick sections through multi-dimensional particle aggregates. *J. Appl. Probab.* 13, 714–722, Correction: *J. Appl. Probab.* 15 (1978), 456.

[6] Federer, H. (1969). *Geometric Measure Theory.* Springer, Berlin.

[7] Hall, P. (1988). *Introduction to the Theory of Coverage Processes.* Wiley, New York.

[8] Heinrich, L. (2005). Large deviations of the empirical volume fraction for stationary Poisson grain models. *Ann. Appl Probab.* 15, 392–420.

[9] Heinrich, L. AND Molchanov, I.S. (1999). Central limit theorems for a class of random measures associated with germ-grain models. *Adv. in Appl. Probab.* 31, 283–314.

[10] Heinrich, L. AND Spiess, M. (2009). Berry-Esseen bounds and Cramr-type large deviations for the volume distribution of Poisson cylinder processes. *Lith. Math. J.* 49, 381–398.

[11] Heinrich, L. AND Spiess, M. (2013). Central limit theorems for volume and surface content of stationary Poisson cylinder processes in expanding domains. *Adv. in Appl. Probab.* 45, 312–331.

[12] Kallenberg, O. (2002). *Foundations of Modern Probability.* Second Edition, Springer, New York.

[13] Last, G. AND Penrose, M.D. (2011). Fock space representation, chaos expansion and covariance inequalities for general Poisson processes. *Probab. Theory Related Fields* 150, 663–690.

[14] Last, G., Penrose, M.D., Schulte, M. AND Thäle, C. (2013). Moments and central limit theorems for some multivariate Poisson functionals. *Adv. in Appl. Probab.* (to appear), arXiv:1205.3033

[15] Mase, S. (1982). Asymptotic properties of stereological estimators for stationary random sets. *J. Appl. Probab.* 19, 111–126.

[16] Mecke, K.R. (2001). Exact moments of curvature measures in the Boolean model. *Journal of Statistical Physics* 102, 1343–1381.

[17] Meester, R. AND Roy, R. (1996). *Continuum Percolation.* Cambridge University Press, Cambridge.

[18] Miles, R.E. (1976). Estimating aggregate and overall characteristics from thick sections by transmission microscopy. *J. Microsc.* 107, 227–233.
[19] Molchanov, I.S. (1995). Statistics of the Boolean model: from the estimation of means to the estimation of distributions. *Adv. in Appl. Probab.* 21, 63–86.

[20] Molchanov, I.S. (1996). *Statistics of the Boolean Models for Practitioners and Mathematicians.* Wiley, Chichester.

[21] Peccati, G., Solé, J.L., Taqqu, M.S. and Utzet, F. (2010). Stein’s method and normal approximation of Poisson functionals. *Ann. Probab.* 38, 443–478.

[22] Peccati, G. and Taqqu, M.S. (2010). *Wiener Chaos: Moments, Cumulants and Diagram Formulae: A survey with computer implementation.* Springer, Berlin.

[23] Peccati, G. and Zheng, C. (2010). Multi-dimensional Gaussian fluctuations on the Poisson space. *Electron. J. Probab.* 15, 1487–1527.

[24] Penrose, M.D. (2007). Gaussian limits for random geometric measures. *Electron. J. Probab.* 12, 989–1035.

[25] Rataj, J., Zähle, M. (2005). General normal cycles and lipschitz manifolds of bounded curvature. *Ann. Global Anal. Geom.* 27, 135–156.

[26] Reitzner, M. and Schulte, M. (2011). Central limit theorems for U-statistics of Poisson point processes. *Ann. Probab.* (to appear), [arXiv:1104.1039](https://arxiv.org/abs/1104.1039).

[27] Schneider, R. (1993). *Convex Bodies: the Brunn-Minkowski Theory.* Cambridge University Press, Cambridge.

[28] Schneider, R. and Weil, W. (2008). *Stochastic and Integral Geometry.* Springer, Berlin.

[29] Schulte, M. (2012). A central limit theorem for the Poisson-Voronoi approximation. *Adv. in Appl. Math.* 49, 285–306.

[30] Surgailis, D. (1984). On multiple Poisson stochastic integrals and associated Markov semi-groups. *Probab. Math. Statist.* 38, 217–239.