ALGEBRAIC ASPECTS OF THE FRACTIONAL QUANTUM HALL EFFECT

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Abstract

Some algebraic issues of the FQHE are presented. First, it is shown that on the space of Laughlin wavefunctions describing the $\nu = 1/m$ FQHE, there is an underlying $W_\infty$ algebra, which plays the role of a spectrum generating algebra and expresses the symmetry of the ground state. Its generators are expressed in a second quantized language in terms of fermion and vortex operators. Second, we present the naturally emerging algebraic structure once a general two-body interaction is introduced and discuss some of its properties.

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1. Introduction

The Quantum Hall Effect \(^1,2\) (QHE) appears in two-dimensional systems of electrons in the presence of a strong perpendicular uniform magnetic field \(B\). It is characterized by quantized values of the Hall conductivity, which turns out to be proportional to the filling factor \(\nu\), the ratio between the number of electrons and the degeneracy of the Landau levels.

The main idea behind the QHE is the existence of a gap, which gives rise to an incompressible ground state at densities proportional to \(\nu\), \(\rho = \nu B/2\pi\). This phenomenon is easily understood in the case of integer \(\nu\) (IQHE), where the first \(\nu\) Landau levels are completely filled, because of the large energy gap between adjacent Landau levels. As a result, the IQHE can be understood solely in terms of noninteracting fermions. The existence of a gap is less obvious in the case of noninteger \(\nu\) (FQHE), where Landau levels are only partially filled. In this case, it is believed that the repulsive Coulomb interactions among electrons are important in generating strong correlations which eventually produce a new ground state with a gap.

In an attempt to explain the FQHE for \(\nu = 1/m\), where \(m\) is an odd integer, Laughlin proposed\(^3\) a set of wavefunctions which turn out to be quite close to the true solutions, at least numerically, for a large class of repulsive potentials\(^4\). Using the connection to the two-component, one-dimensional plasma, Laughlin showed that the ground state corresponds to an incompressible configuration of uniform density \(\rho = \nu B/2\pi\). Based on Laughlin wavefunctions, a hierarchy scheme\(^4-5\) has been developed to explain the other rational values of \(\nu\). Jain has also proposed\(^6\) a scheme for constructing generalized wavefunctions which successfully account for the experimentally observed filling fractions.

In the limit of very strong magnetic fields we consider all the electrons confined in the lowest Landau level (LLL), since the energy separation of adjacent Landau levels is proportional to the magnetic field. In ref.[7] we have presented a field theoretic formulation for electrons in the LLL and emphasized the existence of a \(W_\infty\) algebra, which emerged
as the algebra of unitary transformations preserving the LLL condition and the particle number. In the thermodynamic limit, in the case of the completely filled LLL ($\nu = 1$) and in the absence of interactions the system is invariant under $W_\infty$ transformations. In the presence of a confining potential the symmetry is reduced. Some of the $W_\infty$ generators annihilate the ground state while the others create excitations, which include the low lying gapless edge excitations$^{[8-9]}$. Because the confining potential is itself a member of the algebra, $W_\infty$ plays the role of a spectrum generating algebra and expresses the symmetry of the ground state. This point has also been discussed extensively in ref.[10].

In this paper we first generalize these ideas to the $\nu = 1/m$ FQHE as expressed in terms of the Laughlin wavefunctions. Because of the specific form of these wavefunctions we find the existence of an isomorphic algebra which captures the symmetry of the Laughlin ground state.* The generators of this algebra are expressed in a second quantized language and provide a one-parameter family of $W_\infty$ representations, the parameter being related to the filling fraction $\nu$.

Going beyond the Laughlin wavefunctions and the hierarchy scheme associated with them, one would like to understand the underlying dynamics responsible for the FQHE and the nature of the true ground state. It is essential then to include in the analysis of the problem the Coulomb interactions among the electrons. Following the spirit of our earlier algebraic treatment of the IQHE we propose to study the new algebraic structure emerging from the introduction of the two-body interactions projected onto the lowest Landau level. We find that $W_\infty$ is naturally extended to a new infinite algebra, which we refer to as the $X_\infty$ algebra, which contains $W_\infty$ as a subalgebra.

This paper is organized as follows. In section 2 we give a brief discussion of the field theory of fermions in the lowest Landau level and introduce the $W_\infty$ algebra as the algebra of unitary transformations which preserve the LLL condition and the particle number. In

* General arguments hinting at the existence of some infinite dimensional algebra underlying the $\nu = 1/m$ Laughlin wavefunctions were mentioned in ref.[10].
section 3 we discuss the role of $W_\infty$ algebra in the presence of a confining potential. In section 4 we project the Coulomb potential onto the lowest Landau level and identify the Haldane potential whose zero energy eigenstates are the Laughlin states. We also present a second quantized expression for the vortex operator and following Read’s idea \cite{11} we construct an operator that generates the Laughlin ground state from the Fock vacuum. In section 5 we present the infinite dimensional algebra which expresses the symmetry of the Laughlin ground state. This is isomorphic to $W_\infty$. Its generators are given in terms of fermionic and vortex operators. In section 6 we analyze the enhanced infinite algebraic structure once two-body interactions are introduced and we conclude with discussions in section 7.

2. Field Theory of Fermions in the Lowest Landau Level

The many-body Hamiltonian of a system of massive (mass $M$) fermions in a uniform magnetic field in two space dimensions is given by

$$H_0 = \frac{1}{2M} \sum_{a=1}^{N} (\Pi_a)^2$$

$$= \frac{1}{2M} \sum_{a=1}^{N} (\Pi_a^x + i \Pi_a^y)(\Pi_a^x - i \Pi_a^y) + \frac{B}{2M} N,$$  \hspace{1cm} (2.1)

where

$$\Pi^i = p_i - A^i(x) = -i \frac{\partial}{\partial x^i} - A^i(x), \hspace{1cm} i = x, y$$  \hspace{1cm} (2.2)

and $B$ is a uniform external magnetic field defined by $\vec{\nabla} \times \vec{A} = -B$. In what follows we shall omit the last constant term in (2.1). The corresponding Schrödinger wave function $\Psi(x_1, x_2, \cdots x_N)$ is a totally antisymmetric function of $x$’s. As is well known, the spectrum consists of infinitely degenerate levels of energy $E_n = \omega n$, where $\omega = B/M$ is the cyclotron frequency (in units $\hbar = c = e = 1$). These are the Landau levels. For large $B$ or small $M$ the energy gap between Landau levels is big and to a good approximation we can consider the fermions restricted to the lowest Landau level. In this case the fermionic wavefunction
obeys
\[
(\Pi_a^x - i\Pi_a^y)\Psi(x_1, x_2, \cdots x_N) = 0 \tag{2.3}
\]
In the symmetric gauge, where \(A^x = By/2, A^y = -Bx/2\), the LLL condition is written as
\[
(\partial_{\bar{z}_a} + \frac{1}{2}z_a)\Psi(x_1, x_2, \cdots x_N) = 0 \tag{2.4}
\]
where \(z_a = \sqrt{B/2}(x_a + iy_a), \bar{z}_a = \sqrt{B/2}(x_a - iy_a)\). The solution to (2.4) is given by
\[
\Psi(x_1, x_2, \cdots x_N) = f(\bar{z}) \exp\left(-\frac{1}{2} \sum_{a=1}^{N} |z_a|^2\right) \tag{2.5}
\]
where \(f(\bar{z})\) is a polynomial in the variables \(\bar{z}_a\).

The above many-body quantum mechanical system given in (2.3)-(2.5) can be further expressed in a second quantized language. The fermion operator satisfies the constraint
\[
(\partial_{\bar{z}} + \frac{1}{2}z) \Psi(x, y, t) = 0 \tag{2.6}
\]
which again produces the LLL condition (2.4). A general solution of (2.6) has the form
\[
\Psi(x, y, t) = \sqrt{\frac{B}{2\pi}} e^{-\frac{1}{2} |z|^2} \sum_{n=0}^{\infty} \frac{\bar{z}^n}{\sqrt{n!}} C_n(t)
\equiv \sqrt{\frac{B}{2\pi}} e^{-\frac{1}{2} |\bar{z}|^2} \psi(\bar{z}, t) \tag{2.7}
\]
where the modes \(C_n\) satisfy the usual anticommutation relations \(\{C_n, C_m^\dagger\} = \delta_{nm}\). The constrained fermion operators no longer satisfy the usual anticommutation relations. One can show that
\[
\{\psi(\bar{z}, t), \psi^\dagger(z', t)\} = e^{z'\bar{z}} \tag{2.8}
\]
For later purposes let us define \(|n>\) and \(|z>\) to be the number basis and coherent basis representations for a harmonic oscillator
\[
a^\dagger a|n> = n|n> \quad a|z> = z|z> \quad [a, a^\dagger] = 1 \tag{2.9}
\]
The right hand side of eq.(2.8) is essentially a δ-function in a coherent state representation.

In the absence of an external potential and interactions the maximal symmetry the system can possess is the $W_\infty$ symmetry$^{[7,10]}$. The $W_\infty$ transformation is defined as a unitary transformation in the space of $C_n$’s:

$$C_n(t) = u_{nm}C_m(t) = \langle n|u|m\rangle C_m(t)$$  (2.10)

An infinitesimal unitary transformation is generated by a hermitian operator which we write as $\hat{\xi}(\hat{a},\hat{a}^\dagger)\hat{\xi}$ with the antinormal ordering symbol, where $\xi$ is a real function when $\hat{a}$ and $\hat{a}^\dagger$ are replaced by $z$ and $\bar{z}$ respectively. Then using (2.7) we obtain the following infinitesimal transformation for $\Psi$:

$$\delta\Psi(x,y,t) = i\sqrt{\frac{B}{2\pi}} e^{-\frac{1}{2}|z|^2} \xi(\partial_\bar{z},\bar{z})\frac{1}{2}\sum_n \langle z|n\rangle C_n(t) = i\xi(\partial_\bar{z} + \frac{1}{2}z,\bar{z})\hat{\xi}\Psi(x,y,t)$$  (2.11)

where $\hat{\xi}$ indicates that the derivatives are placed on the left of $z$ and $\bar{z}$. The fermion density $\rho = \Psi^\dagger\Psi$ transforms as

$$\delta\rho(x,y,t) = i(\hat{\xi}(\partial_\bar{z} + z,\bar{z})\hat{\xi} - \hat{\xi}(z,\partial_\bar{z} + \bar{z})\hat{\xi})\rho(x,y,t)$$  (2.12)

while the fermion number remains invariant

$$\int dxdy \delta\rho(x,y,t) = 0$$  (2.13)

The generator of this $W_\infty$ transformation is given by

$$\rho[\xi] \equiv \int dxdy \xi(z,\bar{z})\rho(x,y,t) = \int d^2z e^{-|z|^2} \psi^\dagger(z)\xi(\partial_\bar{z},\bar{z})\hat{\xi}\psi(\bar{z})$$  (2.14)

where $d^2z \equiv \frac{B}{2\pi} dxdy$. The operators $\rho[\xi]$ satisfy an infinite dimensional algebra given by

$$[\rho[\xi_1],\rho[\xi_2]] = \frac{i}{B}\rho[\{\xi_1,\xi_2\}]$$  (2.15)

where

$$\{\xi_1,\xi_2\} = iB\sum_{n=1}^\infty \frac{(-)^n}{n!} (\partial_\bar{z}_n^\dagger \xi_1 \partial_\bar{z}_n^\dagger \xi_2 - \partial_\bar{z}_n^\dagger \xi_2 \partial_\bar{z}_n^\dagger \xi_1)$$  (2.16)
is the so called Moyal bracket. By choosing $\xi(z, \bar{z}) = z^l \bar{z}^m$ we obtain the commutation relation

$$\left[ \rho_{rs}, \rho_{lm} \right] = \sum_{n=1}^{\min(s,l)} \frac{(-)^n}{n!} \frac{l!s!}{(l-n)!(s-n)!} \rho_{r+l-n,s+m-n} - (s \leftrightarrow m, r \leftrightarrow l)$$

(2.17)

where $\rho_{lm} = \int d\mathbf{x} z^l \bar{z}^m \rho(\mathbf{x}) \equiv \rho[z^l \bar{z}^m]$. The Lie algebra (2.15) and its representation (2.17) in the specific basis are manifestations of the $W_\infty$ algebra\cite{12}, which in this case is the algebra of $U(\infty)$. It corresponds to unitary transformations (linear in the space of $C$’s) which preserve the lowest Landau level condition and the particle number.

The field operator $\Psi(x, y, t)$ is expanded, in equation (2.7), in terms of one-body angular momentum eigenstates. Since we work in the infinite plane, no boundaries are present, all different angular momentum eigenstates are degenerate. In order to confine the electrons in a finite area one can introduce an external potential $V_c(\mathbf{x})$ which splits the degeneracy and associates higher energy to higher angular momentum states. In particular, we choose a harmonic oscillator potential whose projection onto the lowest Landau level takes the form

$$V_c = \lambda \int d^2 z e^{-|z|^2} \psi^\dagger(z) \partial \bar{z} \bar{z} \psi(\bar{z})$$

(2.18)

where $\lambda$ is a positive constant.

Since the confining potential is a member of the algebra, $V_c = \lambda \rho_{11}$, eq.(2.15) plays the role of a spectrum generating algebra.

$W_\infty$ algebra and $\nu = 1$ QHE

For the case of the completely filled LLL ($\nu = 1$), the presence of the confining potential selects a unique ground state which is the minimum angular momentum state

$$\Psi_1^0(x_1, x_2, \ldots, x_N) = \prod_{i<j} (\bar{z}_i - \bar{z}_j) \exp(-\frac{1}{2} \sum_{i=1}^{N} |z_i|^2)$$

(3.1)

This corresponds to a circular droplet configuration which is incompressible. Compression of the droplet corresponds to lowering its angular momentum, which would require an
electron to jump to the next Landau level. However this is not energetically allowed due
to the big energy gap. On the other hand, deformations that would result in transitions
to states with higher angular momentum are allowed and cost some energy due to the
confining potential. These higher angular momentum states are states of the form
\[ \Psi_1(x_1, x_2, ..., x_N) = \prod_{i<j}(\bar{z}_i - \bar{z}_j)P(\bar{z}_1, ..., \bar{z}_N)\exp(-1/2 \sum_{i=1}^{N}|z_i|^2) \] (3.2)
where \( P(\bar{z}_1, ..., \bar{z}_N) \) is a symmetric polynomial. If the polynomial \( P \) is homogeneous, the
above states are energy eigenstates (given (2.18)) with the energy depending on the degree
of the polynomial. In particular when the degree of the polynomial \( P \) is much smaller than
\( N \), of order \( O(1) \), the states (3.2) correspond to edge excitations\[8-9\].

It is clear from our discussion in the previous section that a deformation of the ground
state which produces the excited states (3.2) has to be generated by a \( W_\infty \) transformation.
Using eq.(2.8) we find that
\[ \rho[l]|\Psi_0^1| = \int d^2z_1...d^2z_N e^{-\sum_i|z_i|^2} \sum_k \xi(\partial_{\bar{z}_k}, \bar{z}_k) \prod_{i<j}(\bar{z}_i - \bar{z}_j)|z_1...z_N > \] (3.3)
where \( |\Psi_0^1> \) is the ground state of the completely filled lowest Landau level
\[ |\Psi_0^1> = \int d^2z_1...d^2z_N e^{-\sum_i|z_i|^2} \prod_{i<j}(\bar{z}_i - \bar{z}_j)|z_1...z_N > \] (3.4)
and \( |z_1...z_N> = 1/\sqrt{N!}\psi^\dagger(z_1)\psi^\dagger(z_N)|0> \). Since in general
\[ \sum_k \xi(\partial_{\bar{z}_k}, \bar{z}_k) \prod_{i<j}(\bar{z}_i - \bar{z}_j) = \prod_{i<j}(\bar{z}_i - \bar{z}_j)P(\bar{z}_1, ..., \bar{z}_N) \] (3.5)
where \( \xi(z, \bar{z}) \) is a polynomial in \( z, \bar{z} \) and \( P \) is a symmetric polynomial, we find that
\[ \rho[l]|\Psi_0^1> = 0 \quad \text{if} \quad l > k \]
\[ \rho[l]|\Psi_0^1> = |\Psi_1> \quad \text{if} \quad l \leq k \] (3.6)
The first line in eq.(3.6) expresses the symmetry of the ground state and the second line
the generation of excitations\[10\]. In particular the excitations generated by \( \rho[l,l+i] \), with
\( i << N \) correspond to edge excitations\[8-9\].
Since the Hamiltonian is a member of $W_{\infty}$, the excitation spectrum is also dictated by $W_{\infty}$

$$[V_c, \rho_{l,t+k}] = \lambda \, k \, \rho_{l,t+k} \quad (3.7)$$

Therefore the $W_{\infty}$ algebra serves as the spectrum generating algebra for the excitations (3.2) and captures the infinite symmetry of the ground state (as $N \to \infty$).

4. Laughlin States and Vortices

While the IQHE can be understood in terms of noninteracting fermions, repulsive Coulomb interactions and the resulting correlations among the electrons are believed to play a crucial role in the emergence of FQHE.

Since our analysis has been restricted to LLL electrons, which is believed to capture the essential physics, we have to project any relevant potential onto the lowest Landau level[^13]. The projected Coulomb interaction is given by

$$V_{\text{int}} = \int d\vec{x}d\vec{x'} : \rho(\vec{x})\rho(\vec{x'}) : v(|\vec{x}-\vec{x'}|) = \int d^2z d^2z' e^{-|z|^2} e^{-|z'|^2} \psi^\dagger(z)\psi^\dagger(z')\psi(z')\psi(z) v(\sqrt{\frac{2}{B}} |z-z'|) \quad (4.1)$$

where

$$v(|\vec{x}-\vec{x'}|) = \frac{1}{|\vec{x}-\vec{x'}|} \quad (4.2)$$

Next let us define the operator

$$O_{mn} = \int d^2z d^2z' e^{-|z|^2} e^{-|z'|^2} (\frac{z-z'}{\sqrt{2}})^m (\frac{z+z'}{\sqrt{2}})^n \frac{1}{\sqrt{n!m!}} \psi(z)\psi(z') \quad (4.3)$$

This is an operator which annihilates a pair of LLL electrons with relative angular momentum $m$ (necessarily an odd integer) and center of mass angular momentum (relative to the origin) $n$. $O_{mn}$ is defined as the hermitian conjugate of $O_{mn}$. Equation (4.1) can be written in terms of $O, O'$ as

$$V_{\text{int}} = \sum_{m=\text{odd}} v_m \sum_{n=0}^\infty O_{mn} O_{mn} \quad (4.4)$$
where
\[ v_m = \int d^2 z \ e^{-|z|^2} \frac{|z|^{2m}}{m!} v(\frac{2}{\sqrt{B}}|z|) \] (4.5)

The above expressions are general and hold for any two-body interaction. In the case of the Coulomb interaction the parameters \( v \) are monotonically decreasing, \( v_1 > v_3 > \ldots > 0 \).

Most of our theoretical understanding of the \( \nu = 1/m \) FQHE is based on the Laughlin wavefunctions
\[ \Psi_m(\vec{x}_1, \ldots, \vec{x}_N) = \exp\left( -\frac{1}{2} \sum_i |z_i|^2 \right) \prod_{i<j} (\bar{z}_i - \bar{z}_j)^m P(\bar{z}_1, \ldots, \bar{z}_N) \] (4.6)
where \( m \) is an odd integer.

Since the Laughlin states involve electron pairs whose relative angular momentum is larger than or equal to \( m \), we obtain
\[ O_{m'n} |\Psi_m\rangle = 0 \quad \text{for } m' < m \text{ and all } n \] (4.7)
So the space of Laughlin states can be thought of as the space of zero energy eigenstates of a truncated Coulomb potential
\[ V_{\text{int}}^m = \sum_{n=\text{odd}}^{m-2} v_n \sum_l \bar{O}_{nl} O_{nl} \] (4.8)
This is the Haldane potential\(^{[14]}\).

Here and in the next section we are going to assume that the relevant two-body interaction is (4.8) and the resulting physical space is the space of the Laughlin wavefunctions (4.6). We shall show that in this restricted space there is an underlying \( W_\infty \) algebra, which plays the same role which the usual \( W_\infty \) algebra (2.15) had in the \( \nu = 1 \) case, namely part of it expresses the infinite symmetry of the Laughlin ground state and part of it generates excitations.

Before we derive the specific form of the generator of this \( W_\infty \) algebra, it is useful to mention some of the properties of the Laughlin states and the quasihole (vortex) operator.
In particular we shall derive a second quantized expression for the quasihole operator which will emerge later in the expression of the $W_\infty$ generator in the $\nu = 1/m$ case.

Let us define the operator

$$\alpha(\bar{z}) = \int d^2 z' e^{-|z'|^2} \ln(\bar{z} - z') \psi(\bar{z}') \psi(z')$$

(4.9)

One can show by a straightforward calculation that

$$\psi(z') e^{n\alpha(\bar{z})} = (\bar{z} - z')^n e^{n\alpha(\bar{z})} \psi(z')$$

$$e^{n\alpha(\bar{z})} \psi^\dagger(z') = (\bar{z} - \partial_z')^n \psi^\dagger(z') e^{n\alpha(\bar{z})}$$

(4.10)

where $n$ is an integer.

We first remark that although the operator $\alpha(\bar{z})$ is ill defined because of a logarithmic singularity, the exponentiated form $e^{\alpha(\bar{z})}$ is well defined. The proof of this goes as follows. We first observe, using (4.10), that

$$e^{\alpha(\bar{z})}|z_1...z_N> = \prod_{i=1}^{N}(\bar{z} - \partial_{z_i})|z_1...z_N>$$

(4.11)

We can then show that

$$e^{\alpha(\bar{z})} = |0><0| + \sum_{N=1}^{\infty} \int \prod_{i=1}^{N} d^2 z_i e^{-|z_i|^2 (\bar{z} - \bar{z}_i)} |z_1...z_N><z_1...z_N|$$

(4.12)

This is true since $|z_1...z_N>$ form a complete basis and the rhs of the above equation satisfies (4.11). Further using the fact that the projector on the ground state can be expressed in terms of a normal product as

$$|0><0| = : \exp(- \int d^2 z e^{-|z|^2} \psi^\dagger(z) \psi(z)) :$$

(4.13)

it is straightforward to derive that

$$e^{\alpha(\bar{z})} = : \exp(\int d^2 z' e^{-|z'|^2} \psi^\dagger(z') \psi(z')[(\bar{z} - \bar{z}') - 1]) :$$

(4.14)
Expressions (4.12) and (4.14) are free of singularities.

We now claim that $e^{\alpha(\bar{z})}$ is the creation operator of a vortex (quasihole),* since using (4.11) or (4.12) we have

$$e^{\alpha(\bar{z})}|\Psi_0^m\rangle = \int d^2z_1...d^2z_N e^{-\sum_i |z_i|^2} \prod_{i<j} (\bar{z}_i - \bar{z}_j)^m \prod_i (\bar{z} - \bar{z}_i) |z_1 z_2 \cdots z_n\rangle$$  \hspace{1cm} (4.15)

where

$$|\Psi_0^m\rangle = \int d^2z_1...d^2z_N e^{-\sum_i |z_i|^2} \prod_{i<j} (\bar{z}_i - \bar{z}_j)^m |z_1 z_2 \cdots z_n\rangle$$  \hspace{1cm} (4.16)

and $e^{-\frac{1}{2} \sum_i |z_i|^2} \prod_{i<j} (\bar{z}_i - \bar{z}_j)^m \prod_i (\bar{z} - \bar{z}_i)$ is the quasihole wavefunction[3]. In fact using (4.15), (4.16) we can also show that

$$\psi(\bar{z})|\Psi_0^0\rangle_N = \sqrt{N} e^{m\alpha(\bar{z})} |\Psi_0^0\rangle_{N-1}$$  \hspace{1cm} (4.17)

where the subscripts indicate the fermion number. This verifies that $m$ vortices are equivalent to one hole.

Further, following Read’s idea[11] we can construct the operator

$$q_m^\dagger = \int d^2z e^{-|z|^2} \psi^\dagger(z) e^{m\alpha(\bar{z})}$$  \hspace{1cm} (4.18)

such that the $N$-body Laughlin ground state is created out of the vacuum as

$$|\Psi_0^0\rangle = \frac{1}{\sqrt{N!}} (q_m^\dagger)^N |0\rangle$$  \hspace{1cm} (4.19)

By using (4.10) and the commutativity of $\alpha$’s, it is not difficult to prove that

$$q_m^\dagger q_m^\dagger = -(-)^m q_m^\dagger q_m^\dagger$$  \hspace{1cm} (4.20)

Thus for $m$ odd, $q_m^\dagger$ ($q_m$) is a bosonic operator.

* Our explicit second quantized form of the vortex operators satisfies the consistent operator equations discussed in ref.[2] pg. 294.
5. \( \mathcal{W}_\infty \) algebra and Laughlin states for \( \nu = 1/m \)

As we mentioned earlier, in the presence of the Haldane potential all Laughlin states of the form (3.2) are degenerate. The introduction of a confining potential splits the degeneracy of these states just as in the \( \nu = 1 \) case. The ground state corresponds to the state with \( P = 1 \), while states with \( P \neq 1 \) correspond to excited states of higher energy, including edge excitations when the degree of the polynomial \( P \) is much smaller than \( N \). The states (3.2) form a Hilbert space \( \mathcal{H}_m \).

Repeating the steps of section 3, we shall try to construct operators which generate the states (4.6) by acting on the Laughlin ground state. More generally we are interested in finding the transformations which preserve the particle number and leave \( \mathcal{H}_m \) invariant. In doing so we utilize the one-to-one mapping between \( \nu = 1 \) states (3.2) and \( \nu = \frac{1}{2p+1} \) Laughlin states (4.6).

Let \( |\Psi^0_m\rangle \) be the \( N \)-body Laughlin ground state \( (m = 2p+1) \). Obviously the wanted operator cannot be \( \rho[\xi] \) since

\[
\rho[\xi]|\Psi^0_m\rangle = \int d^2z_1...d^2z_N e^{-\sum|z_i|^2}\sum_k \frac{\xi(\partial_{\bar{z}_k}, \bar{z}_k)^{\frac{1}{m}}}{} \prod_{i<j}(\bar{z}_i - \bar{z}_j)^m|z_1...z_N\rangle
\]

and in general

\[
\sum_k \frac{\xi(\partial_{\bar{z}_k}, \bar{z}_k)^{\frac{1}{m}}}{} \prod_{i<j}(\bar{z}_i - \bar{z}_j)^m \neq \prod_{i<j}(\bar{z}_i - \bar{z}_j)^mP(\bar{z}_1,..., \bar{z}_N)
\]

where \( \xi(z, \bar{z}) \) is a polynomial in \( z, \bar{z} \) and \( P \) is some symmetric polynomial (\( P \) could be also zero). The operator \( \rho[\xi] \) does not keep \( \mathcal{H}_m \) invariant since its action does not preserve the fundamental property of the Laughlin wavefunctions to behave like \( (\bar{z}_i - \bar{z}_j)^m \).

Let us define the following operators

\[
U_{2p} = \sum_{N=2}^{\infty} \int d^2z_1...d^2z_N e^{-\sum|z_i|^2} \prod_{i<j}(\bar{z}_i - \bar{z}_j)^{2p}|z_1...z_N\rangle <z_1...z_N| \]

\[
\dot{U}_{2p} = \sum_{N=2}^{\infty} \int d^2z_1...d^2z_N e^{-\sum|z_i|^2} \prod_{i<j}(\bar{z}_i - \bar{z}_j)^{-2p}|z_1...z_N\rangle <z_1...z_N| \]

(5.3)
The operator $\tilde{U}_{2p}$ is in general singular but its action on $\mathcal{H}_m$ is well defined. In particular

$$\tilde{U}_{2p} : \mathcal{H}_m \to \mathcal{H}_1$$
$$U_{2p} : \mathcal{H}_1 \to \mathcal{H}_m$$  \hspace{1cm} (5.4)$$

Using eqs.(3.3) and (3.5) we find that

$$U_{2p} \rho[\xi] \tilde{U}_{2p} |\Psi_0^m\rangle = \int d^2z_1...d^2z_N e^{-\sum |z_i|^2} \prod_{i<j}(\bar{z}_i - \bar{z}_j)^{2p} \sum_i \xi(\partial_{\bar{z}_i}, \bar{z}_i)\sum_{k<l}(\bar{z}_k - \bar{z}_l) |z_1...z_N\rangle$$

$$= \int d^2z_1...d^2z_N e^{-\sum |z_i|^2} \prod_{i<j}(\bar{z}_i - \bar{z}_j)^m P(\bar{z}_1, ..., \bar{z}_N) |z_1...z_N\rangle$$  \hspace{1cm} (5.5)$$

It is clear now that

$$W_{2p}[\xi] \equiv U_{2p} \rho[\xi] \tilde{U}_{2p} \hspace{1cm} W_{2p} : \mathcal{H}_m \to \mathcal{H}_m$$  \hspace{1cm} (5.6)$$

is the generator of transformations which preserve the particle number and keep $\mathcal{H}_m$ invariant and also generate excitations in $\mathcal{H}_m$ the same way that $\rho[\xi]$ generated excitations in $\mathcal{H}_1$. In particular

$$(W_{2p})_{lk} |\Psi_0^m\rangle = 0 \hspace{1cm} \text{if} \hspace{1cm} l > k$$
$$(W_{2p})_{lk} |\Psi_0^m\rangle = |\Psi_m\rangle \hspace{1cm} \text{if} \hspace{1cm} l \leq k$$  \hspace{1cm} (5.7)$$

Therefore in the thermodynamic limit the Laughlin ground state is annihilated by the infinitely many generators $W_{2p}$. From the above construction it is obvious that on the space of states $|\Psi_m\rangle$, the operators $W_{2p}[\xi]$ satisfy the same algebra as the $\rho[\xi]$ operators, namely the $W_\infty$ algebra

$$[W_{2p}[\xi_1], W_{2p}[\xi_2]] |\Psi_m\rangle = W_{2p}[\{\xi_1, \xi_2\}] |\Psi_m\rangle$$  \hspace{1cm} (5.8)$$

where $\{\}$ is the Moyal bracket defined in (2.16). Each $W_{2p}$, $p = 0, 1, ...$ provides a representation for $W_\infty$ algebra.
We would further like to obtain a second quantized expression for these operators in terms of fermionic creation and annihilation operators. Let us explicitly write down the transformation of a state in $\mathcal{H}_m$ under the action of $W_{2p}[\xi]$.

$$\delta_{\xi}^{2p}|\Psi_m> \equiv W_{2p}[\xi]|\Psi_m> = U_{2p} \rho[\xi] \tilde{U}_{2p}|\Psi_m>$$

$$= \int d^2 z_1 ... d^2 z_N e^{-\sum |z_i|^2} \prod_{i<j}(z_i - \bar{z}_j)^{2p} \sum_i \xi(\partial_{z_i}, \bar{z}_i) \frac{i}{4} \prod_{k<l}(\bar{z}_k - \bar{z}_l)^{-2p} F(z_1...z_N)|z_1...z_N>$$

where $F(z_1...z_N) = \prod_{i<j}(z_i - \bar{z}_j)^{2p+1} P(z_1...z_N)$.

The second quantized expression for $W_{2p}[\xi]$ would satisfy

$$\delta_{\xi}^{2p}|\Psi_m> = \int d^2 z_1 ... d^2 z_N e^{-\sum |z_i|^2} F(z_1...z_N) \frac{1}{\sqrt{N!}} \sum_{i=1}^N \psi^\dagger(z_1) \psi^\dagger(z_2)...[W_{2p}, \psi^\dagger(z_i)]...\psi^\dagger(z_N)|0>$$

$$= (\delta_{\xi}^{2p})|\Psi_m> = \int d^2 z e^{-|z|^2} \psi^\dagger(z)e^{2p\alpha(z)} \frac{i}{4} \xi(\partial_{\bar{z}}, \bar{z}) \frac{i}{4} e^{-2p\alpha(\bar{z})} \psi(z)$$

where $\alpha(\bar{z})$ is the operator defined in (4.10) and $e^{\alpha(\bar{z})}$ is the vortex operator. One can directly verify that $W_{2p}[\xi]$ in (5.11) satisfies the $W_\infty$ algebra on $\mathcal{H}_m$ and therefore provides a representation for $W_\infty$. Equation (5.11) can be further written as

$$W_{2p}[\xi] = \int d^2 z e^{-|z|^2} \psi^\dagger(z) \frac{i}{4} \xi(\partial_{\bar{z}}, \bar{z}) \frac{i}{4} e^{-2p\alpha(\bar{z})} \psi(z)$$

The term $2p \int d^2 z' \frac{\rho(z', \bar{z}')}{(\bar{z} - \bar{z}')}$ plays the role of a gauge potential and it is similar to the one induced by the Chern-Simons interaction.

Each of the two different forms (5.11) and (5.12) for $W_{2p}$ stresses a different physical interpretation. In (5.12) $W_{2p}[\xi]$ is expressed in terms of the usual fermions with a complicated interaction, similar to a Chern-Simons interaction, while in (5.11) one can think of $W_{2p}[\xi]$ as a bilinear (like $\rho[\xi]$ in $\nu = 1$ case) of free composite fermions, made out of the usual fermions and $2p$ vortices.
Having identified the operators $W_{2p}$ which create the excited states, it is interesting to find their spectrum. The Haldane potential $V_{int}^m$ in (4.8) obviously commutes with $W_{2p}$ on the space of the wavefunctions (3.2). On the other hand the confining potential $V_c$ is not of the form $W_{2p}[\xi]$. However if $V_c$ is a harmonic oscillator potential as in (2.18), one can show that

$$V_c = \lambda [(W_{2p})_{11} + pN(N - 1)] \quad (5.13)$$

where $N$ is the fermion number operator, $N = \int d^2 z e^{-|z|^2} \psi(\bar{z})\psi(z)$, which commutes with any $W_{2p}$. As a result in the space $\mathcal{H}_m$

$$[V_c, (W_{2p})_{l,l+k}] = \lambda k (W_{2p})_{l,l+k} \quad (5.14)$$

Comparing this with (3.7) we conclude that the slope of the excitation spectrum of the Laughlin type states (3.2) is identical with the one in the $\nu = 1$ case.

6. Two-body Interactions and $X_\infty$ algebra

As we showed in the previous section the existence of a $W_\infty$ algebra in the case of the $\nu = 1/m$ FQHE described by Laughlin wavefunctions has to do with the specific form of these wavefunctions and the one-to-one mapping with the $\nu = 1$ states. Although the Laughlin states are quite successful in describing the $\nu = 1/m$ FQHE, one would like to understand better the underlying dynamics, especially the role of the Coulomb repulsive interactions, and the spectrum that emerges from it.

As we discussed in section 2, in the absence of two-body interactions, the naturally emerging algebraic structure is the $W_\infty$ algebra in (2.15). It is essentially the algebra of bilinears $C^{\dagger}C$ which provide the most general unitary transformations (linear in the space of $C$’s) which preserve the particle number and the lowest Landau level condition. The introduction of general two-body interactions projected onto the LLL via the operators $O_{nm}, \bar{O}_{nm}$ as in eqs. (4.3)-(4.5), suggests the extension of $W_\infty$ algebra by including bilinears in $C$’s and $C^{\dagger}$’s, which change the particle number by two. We call this extended
algebra, the $X_\infty$ algebra.

In order to see more clearly the algebraic structure, it is instructive to recall the situation in the case of a finite number $N$ of fermions. It is well known that one can construct a Clifford algebra from $C_n$ and $C_n^\dagger$ $(n = 1, 2, \cdots, N)$.

\begin{equation}
\gamma_n = C_n + C_n^\dagger \quad \gamma_{N+n} = -i(C_n - C_n^\dagger)
\end{equation}

\begin{equation}
\{\gamma_i, \gamma_j\} = 2\delta_{ij} \quad i, j = 1, 2, \cdots, 2N
\end{equation}

$\Sigma_{ij} = \frac{1}{2}[\gamma_i, \gamma_j]$ form an $O(2N)$ Lie algebra and all the fermion states form the basis of one $2^N$ dimensional spinor representation of $O(2N)$.\cite{[15]} From (6.1) one sees that the generators $\Sigma_{ij}$ of $O(2N)$ are the bilinears of all $C_n$ and $C_n^\dagger$ $(n = 1, 2, \cdots, N)$. On the other hand the generators of $U(N)$ are $C_n^\dagger C_m$ $(n, m = 1, 2, \cdots, N)$, only the particle-number-preserving bilinears. As $N \to \infty$, $U(N) \to W_\infty$ and $O(2N) \to X_\infty$. Any LLL state is a component of a spinor representation of $X_\infty$.

The most convenient (mathematically) basis to describe the $X_\infty$ algebra is in terms of bilinears $\psi(\bar{z})$ and $\psi^\dagger(z)$

\begin{equation}
U(z, z') = \psi^\dagger(z)\psi(z'), \quad O(\bar{z}, \bar{z'}) = \psi(\bar{z})\psi(z') \quad \bar{O}(z, z') = \psi^\dagger(z')\psi^\dagger(z)
\end{equation}

The commutation relations are

\begin{align*}
[U(z_1, \bar{z}_2), U(z'_1, \bar{z}'_2)] &= e^\bar{z}_2 z'_1 U(z_1, \bar{z}'_2) - e^{\bar{z}'_2} z_1 U(z'_1, \bar{z}_2) \\
[U(z_1, \bar{z}_2), O(\bar{z}'_1, z'_2)] &= e^{\bar{z}'_1} z_1 O(\bar{z}'_1, z_2) - e^{\bar{z}_1} z'_1 O(\bar{z}_1, z_2) \\
[U(z_1, \bar{z}_2), \bar{O}(z'_1, z'_2)] &= e^\bar{z}_2 z'_1 \overline{O}(z'_1, z_1) - e^{\bar{z}'_2} z_1 \overline{O}(z'_1, z_1) \\
[O(\bar{z}_1, \bar{z}_2), \bar{O}(z'_1, z'_2)] &= e^\bar{z}_1 z'_1 + \bar{z}_1 z'_2 - e^{\bar{z}_1} z'_1 + \bar{z}_1 z_2 \\
&\quad + e^\bar{z}_2 z'_1 U(z'_1, \bar{z}_2) - e^{\bar{z}_2} z'_1 U(z'_1, \bar{z}_2) + e^{\bar{z}_2} z_1 U(z_2', \bar{z}_1) - e^{\bar{z}_2} z_1 U(z_2', \bar{z}_1)
\end{align*}

\begin{equation}
[O(\bar{z}_1, \bar{z}_2), O(\bar{z}'_1, z'_2)] = [\bar{O}(\bar{z}_1, \bar{z}_2), \bar{O}(z'_1, z'_2)] = 0
\end{equation}

Since the Hamiltonian in the presence of an external potential and a two-body interaction involves the operators $\rho_{kl}, O_{kl}, \bar{O}_{kl}$, it is useful to write down the commutation
relations (CR) of these objects, which will provide a different basis for the $X_\infty$ algebra. The CR of $\rho$'s is given in (2.15) and is the usual $W_\infty$ algebra. It is straightforward to derive the CR between $\rho$ and $\bar{O}$. We find

$$[\rho_{kl}, \bar{O}_{mn}] = (-1)^{\gamma+\delta} (\delta+n)!(l+m)\delta! (l-\delta)\gamma!(k-\gamma)! \left( -\frac{1}{\sqrt{2}} \right)^{k+l-2} \frac{k!}{\sqrt{n!m!}} \sum_{\delta=0}^{l} \sum_{\gamma=\max(0,k-l-m+\delta)} \frac{(-1)^{\gamma+\delta}(\delta+n)!(l+m-\delta)!}{\delta!(l-\delta)\gamma!(k-\gamma)!} \sum_{\delta=0}^{\min(k,n+\delta)} \left\{ \begin{array}{ll} \hat{O}_{l+m-k+\gamma-\delta,n+\delta-\gamma} & \text{if } l+m+n \geq k \\ 0 & \text{if } l+m+n \leq k \end{array} \right. \sum_{\gamma=\max(0,k-l-m+\delta)} \left( -\frac{1}{\sqrt{2}} \right)^{k+l-2} \frac{k!}{\sqrt{n!m!}} \sum_{\delta=0}^{\min(k,n+\delta)} \left\{ \begin{array}{ll} \hat{O}_{l+m-k+\gamma-\delta,n+\delta-\gamma} & \text{if } l+m+n \geq k \\ 0 & \text{if } l+m+n \leq k \end{array} \right.$$

By taking the complex conjugate of the above expression we find

$$[\rho_{kl}, \bar{O}_{mn}]^\dagger = -[\rho_{lk}, O_{mn}] \quad (6.6)$$

The CR between $O$ and $\bar{O}$ is more complicated. First we have to express $U(z_1, \bar{z}_2)$ in terms of the density operator. We find that

$$U(z_1, \bar{z}_2) \equiv \psi^\dagger(z_1)\psi(\bar{z}_2) = \frac{1}{4\pi} e^{\bar{z}_2z_1} \int d\alpha d\beta \rho(\alpha, \beta) \exp[-iz_1(\frac{\alpha-i\beta}{2})] \exp[-i\bar{z}_2(\frac{\alpha+i\beta}{2})] \quad (6.7)$$

where $\rho(\alpha, \beta)$ is the Fourier transform of the fermion density

$$\rho(\alpha, \beta) = \int d^2 z e^{-|z|^2} e^{i(\alpha Rez + \beta Imz)} \psi^\dagger(z)\psi(\bar{z}) \quad (6.8)$$

Expanding now (6.7) in powers of $z_1, \bar{z}_2$ we find

$$U(z_1, \bar{z}_2) = \frac{1}{4\pi} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \hat{\rho}_{kl} \frac{z_1^k\bar{z}_2^l}{k!l!} \quad (6.9)$$

where

$$\hat{\rho}_{kl} = (-i)^{k+l} \int d\alpha d\beta \rho(\alpha, \beta) \left( \frac{\alpha-i\beta}{2} \right)^k \left( \frac{\alpha+i\beta}{2} \right)^l \quad (6.10)$$

Given (6.9) and (6.10) we can easily find that

$$[\bar{O}_{n0}, O_{l0}] = -2\delta_{ln} \frac{1}{\pi} \left( \frac{n!!}{(n+l)!} \right)^2 \sum_{k=0}^{\mu_{nl}} \frac{2^{n-k}}{(\mu_{nl} - k)!((\Delta_{nl} + k)!)k!} \times \left\{ \begin{array}{ll} \hat{\rho}_{k\Delta_{nl}+k} & \text{if } l \geq n \\ \hat{\rho}_{\Delta_{nl}+k,k} & \text{if } n \geq l \end{array} \right. \quad (6.11)$$
where \( \mu_{nl} = \min(n,l) \), \( M_{nl} = \max(n,l) \) and \( \Delta_{nl} = \max(n,l) - \min(n,l) \).

In order to find the more general CR for arbitrary modes \( \bar{O} \) and \( O \) it is useful to introduce the translation operators

\[
A = \int d^2 z e^{-|z|^2} \bar{\psi}^\dagger(z) \psi(\bar{z}) \\
A^\dagger = \int d^2 z e^{-|z|^2} \bar{\psi}^\dagger(z) \bar{\psi}(\bar{z})
\]

with the following commutation relations

\[
[A, A^\dagger] = \int d^2 z e^{-|z|^2} \bar{\psi}^\dagger(z) \psi(\bar{z}) \\
[A, O_{nm}] = - \sqrt{2(m+1)} O_{n,m+1} \\
[A, \bar{O}_{nm}] = \sqrt{2m} \bar{O}_{n,m-1} \\
[A, \tilde{\rho}_{nm}] = - \tilde{\rho}_{n,m+1} \\
[A^\dagger, \bar{O}_{nm}] = \sqrt{2(m+1)} \bar{O}_{n,m+1} \\
[A^\dagger, \tilde{\rho}_{nm}] = \tilde{\rho}_{n+1,m}
\]

Using (6.13) we can express \( O_{nm} \) and \( \bar{O}_{nm} \) in terms of \( O_{n0} \) and \( \bar{O}_{n0} \) respectively

\[
O_{nm} = (-\frac{1}{\sqrt{2}})^m \frac{1}{\sqrt{m!}} \partial_p [e^{pA} O_{n0} e^{-pA}]_{p=0} \\
\bar{O}_{nm} = (\frac{1}{\sqrt{2}})^m \frac{1}{\sqrt{m!}} \partial_q [e^{q\bar{A}^\dagger} \bar{O}_{n0} e^{-q\bar{A}}]_{q=0}
\]

and further the general commutator of arbitrary \( \bar{O} \) and \( O \) modes in terms of (6.11)

\[
[\bar{O}_{nm}, O_{ls}] = (-\frac{1}{\sqrt{2}})^s (\frac{1}{\sqrt{2}})^m \frac{1}{\sqrt{s!m!}} \partial^p \partial^q [e^{pA} e^{q\bar{A}^\dagger} \bar{O}_{n0} e^{-q\bar{A}} e^{-pA} e^{-2pq}]_{p=q=0}
\]

Inserting (6.11) in the above expression we find

\[
[\bar{O}_{nm}, O_{ls}] = -2\delta_{nl} \delta_{sm} + \frac{1}{\pi} \sqrt{\frac{m!s!n!!}{2}} \sum_{r=0}^{\mu_{ms}} \sum_{k=0}^{\mu_{nl}} \frac{1}{2^{r+k} r! k!} \left\{ \begin{array}{ll}
\tilde{\rho}_{\Delta_{nl}+\Delta_{ms}+k+r, k+r} & \text{if } n \geq l, m \geq s \\
\tilde{\rho}_{\Delta_{nl}+k+r, \Delta_{ms}+k+r} & \text{if } n \geq l, s \geq m \\
\tilde{\rho}_{\Delta_{ms}+k+r, \Delta_{nl}+k+r} & \text{if } l \geq n, m \geq s \\
\tilde{\rho}_{k+r, \Delta_{nl}+\Delta_{ms}+k+r} & \text{if } l \geq n, s \geq m
\end{array} \right.
\]

Equations (2.15), (6.5), (6.6) and (6.16) express the \( X_\infty \) algebra.
There are many interesting issues regarding this algebra. First it is worth noticing the structure of the CR (6.16). It suggests that in the limit where the second term on the rhs of (6.16) is zero, $\hat{O}_{mn}$ and $O_{mn}$ play the role of creation and annihilation operators respectively and the Hamiltonian can be easily diagonalized in the resulting Fock space.

We mentioned before that the Laughlin wavefunctions are eigenstates of the Haldane potential. We expect this to be the case as we tune in a weak confining potential. In section 5 we showed that in the space of Laughlin wavefunctions the spectrum generating algebra is isomorphic to $W_\infty$. On the other hand, following the analysis of section 6, there is a corresponding $X_\infty$ algebra. It is interesting to understand the relation between these algebraic structures.

7. Discussions

In this paper we have presented an algebraic treatment of the QHE.

In the case of the IQHE $\nu = 1$, where the Coulomb interactions can be neglected, such an approach has revealed the emergence of the $W_\infty$ algebra, which plays the role of a spectrum generating algebra and expresses the symmetry of the ground state\[7,10].

In the case of the FQHE $\nu = 1/m$ as expressed by the Laughlin wavefunctions, an isomorphic algebra emerges. This has to do with the specific form of the Laughlin wavefunctions and their one-to-one mapping to the $\nu = 1$ states. The generators of this $W_\infty$ algebra are expressed in a second quantized language and they are written in terms of fermion and vortex operators. A byproduct of the isomorphism of the algebraic structures associated to $\nu = 1$ and $\nu = 1/m$ cases is the fact that in the presence of a harmonic oscillator potential the dispersion relation of the corresponding edge states is identical.

It is interesting to see if this infinite algebraic structure survives and if so its specific representation once we consider generalized wavefunctions describing $\nu \neq 1/m$ FQHE.

Although the phenomenology of the FQHE based on variational wavefunctions successfully accounts for the observed plateaus of the Hall conductivity at specific filling fractions,
the precise nature of the true ground state and the creation of a gap in the presence of Coulomb interactions is not quite understood analytically. An algebraic approach to this problem leads to a new extended infinite algebraic structure, the $X_{\infty}$ algebra. In section 6 we mentioned some interesting issues regarding this algebra. Its significance in analyzing the spectrum of FQHE is under investigation.

Acknowledgements

I thank Satoshi Iso and Bunji Sakita for their collaboration at an early stage of this work. Especially, I am grateful to Bunji Sakita for many valuable discussions and suggestions.

This work was supported in part by the U.S. Department of Energy under the contract number DE-FG02-85ER40231. I thank Bunji Sakita for his hospitality in City College CUNY where part of this work was done and for partial support from his NSF grant, PHY 90-20495.

While this work was being typed, two papers appeared, ref.[16] and ref.[17], with some partially overlapping results and ideas.

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