POLYNOMIAL SPLITTINGS OF CASSON–GORDON INVARIANTS

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Abstract. In this paper we prove that the Casson–Gordon invariants of the connected sum of two knots split when the Alexander polynomials of the knots are coprime. As one application, for any knot $K$, all but finitely many algebraically slice twisted doubles of $K$ are linearly independent in the knot concordance group.

1. Introduction

In his classification of the knot concordance groups, Levine [7] defined the algebraic concordance group, $G$, of Witt classes of Seifert matrices and a homomorphism from the knot concordance group, $C$, of knots in the 3-sphere $S^3$ to $G$. Casson and Gordon [1] proved that the kernel of Levine’s homomorphism $C \to G$, the concordance group of algebraically slice knots, is nontrivial. Gilmer [4] used the work of [1] to define a Witt type group $\Gamma^+$ and showed that there are homomorphisms $C \to \Gamma^+ \to G$. The group $\Gamma^+$ is roughly characterized by the property that a class of knots maps to zero in $\Gamma^+$ if and only if all of Levine’s invariants and the Casson–Gordon invariants of a representative of the class vanish. However, the definition of $\Gamma^+$ used here is modified from the one used by Gilmer to correct for an error in [3], as described in Section 2.

It follows from Levine’s work [8] that if the connected sum of two knots with relatively prime Alexander polynomials maps to zero in $G$, then so does each knot. We show a similar result for the Casson–Gordon invariants as follows.

Theorem 1.1. Let $K_1$ and $K_2$ be knots with relatively prime Alexander polynomials in $\mathbb{Q}[t^{\pm 1}]$. Suppose that either $K_1$ or $K_2$ has a non-singular Seifert form. Then if $K_1 \# K_2$ is zero in $\Gamma^+$ then so are both $K_1$ and $K_2$.

To demonstrate the strength of this result we study the family of $k$-twisted doubles of a given knot $K$, denoted $D_k(K)$. This family contains an infinite number of algebraically slice knots and these algebraically slice knots have been the subject of careful study. Casson and Gordon [1] found the first examples of nontrivial concordance classes in the kernel of Levine’s homomorphism using the family $D_k(U)$ where $U$ is the unknot. Since then [5, 10, 11, 12, 16] have found infinite linearly independent families of algebraically slice knots among the knots $D_k(U)$. In each case these families were very scarce: roughly one knot was chosen for each prime integer. Theorem [13] will yield that for every knot $K$ (not just the unknot $U$) the set of all algebraically slice knots in the family of knots $D_k(K)$ is (with finite exceptions) linearly independent. More precisely:

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Theorem 1.2. (a) For any knot $K$, all but finitely many algebraically slice twisted doubles of $K$ are linearly independent in $\Gamma^+$ and so in $C$.

(b) If $\sigma_r(K) \geq 0$ for all $r$, then all algebraically slice twisted doubles of $K$ except the untwisted and 2-twisted ones are linearly independent in $\Gamma^+$ and so in $C$, where $\sigma_r$ denotes the averaged Tristram–Levine signature. In addition, if $\sigma_r(K) > 0$ for some $r$, then all algebraically slice twisted doubles of $K$ except the untwisted one are linearly independent in $\Gamma^+$ and so in $C$.

Corollary 1.3. (a) All algebraically slice twisted doubles of the unknot except the two known to be slice (untwisted and 2-twisted ones) are linearly independent in the knot concordance group.

(b) There are infinitely many knots $K$ for each of which all algebraically slice twisted doubles of $K$ except the untwisted one are linearly independent in the knot concordance group.

The full set of knots $D_k(K)$ contains knots which are not algebraically slice, representing elements of infinite order, order 2, and order 4 in $G$. It has not been possible to prove that this set is linearly independent in $C$, but we do have the following theorem, based on recent work of Friedl [2]:

Theorem 1.4. (a) For any knot $K$, there is a set $K$ containing all twisted doubles of $K$ except a finite number of nonnegatively twisted ones such that no nontrivial linear combinations of elements in $K$ are ribbon.

(b) If $\sigma_r(K) \geq 0$ for all $r$, then no nontrivial linear combinations of twisted doubles of $K$ except those with 0, 1, 2 twists are ribbon. In addition, if $\sigma_r(K) > 0$ for $r = \frac{2}{5}, \frac{1}{3}, \frac{2}{3}$, then no nontrivial linear combinations of twisted doubles of $K$ except the untwisted one are ribbon.

In the past, the construction of independent knots depended on finding knots for which some branched covers had homology groups of order divisible by distinct primes. Such an approach could conceivably work with doubled knots by using high degree covers, but the argument would be far more burdensome than the one we give. A paper in preparation will address another application of Theorem 1.1 that there are examples of linearly independent algebraically slice knots having the same homology on all prime power fold branched covers, in which case no such approach could possibly work, and our approach using the splitting associated with the polynomial is definitely required.

The paper is organized as follows: In Section 2, we summarize the modified results of [4] about the Casson–Gordon invariants on slice knots and ribbon knots to correct for an error in [4]. In Section 3, we prove Theorem 1.1. In Section 4, we state that all algebraically slice twisted doubles of a knot (with finite exceptions) have infinite order in $\Gamma^+$ and similar results concerning their ribbonness. We also summarize some facts on the Casson–Gordon invariants of genus 1 knots and the Tristram–Levine signatures of satellite knots and torus knots. In Section 5, we estimate the Casson–Gordon invariants of algebraically slice $D_k(K)$ and prove Theorem 1.2 and Corollary 1.3. In Section 6, we estimate the Casson–Gordon invariants of all twisted doubles $D_k(K)$ for double branched covers and prove Theorem 1.4.

2. Gilmer’s obstructions

In this section we state the modified results of [4] about the Casson–Gordon invariants on slice knots and ribbon knots to correct for an error in [4]. Conventions
are ribbon. In fact, this is an equivalent statement of Fox’s conjecture: All slice knots are ribbon. Gilmer [3, 4] combined the slicing obstructions of [7] with those of [1] in a non-trivial way. Recently Gilmer has announced that there is an error in [3, 4] (cf. [2]).
However, the following two weaker statements are known to be valid. Note that the first statement has a weaker conclusion: the phrase “all primes $p$” is replaced by “all but finitely many primes $p$”; the second statement has a stronger hypothesis: “a slice knot $K$” is replaced by “a ribbon knot $K$.” The first statement directly follows from Gilmer’s original proof if primes $p$ are chosen so that $p$ do not divide the order of torsion of $H_1(R)$ (see [4] for the definition of $R$). The second statement is a corollary of Friedl’s recent work [2] theorem 8.3 and corollary 8.5.

**Theorem 2.1.** (a) If $F$ is a Seifert surface for a slice knot $K$ then there is a metabolizer $Z$ for the isometric structure on $H_1(F)$ such that $\tau(K, N_p^q \cap (Z \otimes \mathbb{Q}/\mathbb{Z}))$ vanishes for all prime powers $q$ and all but finitely many primes $p$.

(b) If $F$ is a Seifert surface for a ribbon knot $K$ then there is a metabolizer $Z$ for the isometric structure on $H_1(F)$ such that $\tau(K, N_p^q \cap (Z \otimes \mathbb{Q}/\mathbb{Z}))$ vanishes for all prime powers $q$ and all primes $p$.

We say that a knot $K$ has vanishing Gilmer slice (resp. ribbon) obstruction if the conclusion of Theorem 2.1(a) (resp. (b)) is satisfied for any Seifert surface $F$ of $K$. Otherwise, we say that $K$ has nonvanishing Gilmer slice (resp. ribbon) obstruction.

Gilmer [4] defined a Witt type group $\Gamma^+$ and a homomorphism from the knot concordance group $C$ to $\Gamma^+$ such that the class of a knot maps to zero if and only if it satisfies the conclusion of Theorem 2.1(b). It became unknown whether the original $\Gamma^+$ is a group and whether there is a homomorphism $C \rightarrow \Gamma^+$ since the proof of the cancellation lemma of [4, lemma 5] had a similar gap.

On the other hand, if the definition of $\Gamma^+$ is modified so that the class of a knot maps to zero if and only if it satisfies the conclusion of Theorem 2.1(a), then $\Gamma^+$ is a group and there are still homomorphisms $C \rightarrow \Gamma^+ \rightarrow \mathcal{G}$. For instance, using the notation in [4, page 12], the definition of a metabolizer for $(U, \{,\}, s, \tau_q^s)$ should be modified as done in the conclusion of Theorem 2.1(a), namely, “all primes $p$” should be replaced by “all but finitely many primes $p$.” With this new definition of metabolizer we can eliminate the gap in the proof of [4, lemma 5] and hence this new $\Gamma^+$ becomes a group. Throughout this paper, $\Gamma^+$ is this modified one.

**3. Alexander polynomial and proof of Theorem 1.1**

Let $K$ be a knot in the 3-sphere $S^3$ with Seifert surface $F$, Seifert pairing $\theta$, and Seifert matrix $A$. We define $\Delta_K(t) = \det(A - tA^t)$, called the Alexander polynomial of $K$ corresponding to the Seifert surface $F$. As is well known, the Alexander polynomial of a knot is uniquely determined up to multiplication by $\pm t^n$ in $\mathbb{Q}[t^{\pm 1}]$. Observe that the characteristic polynomial for the associated isometric structure $s$ to $K$ is $\det(xI - G) = \pm x^{2g} \det(A - (1 - x^{-1})A^t) = \pm x^{2g} \Delta_K(1 - x^{-1})$, where $g$ is the genus of $F$.

To prove Theorem 1.1 we need a generalized notion of Seifert form. Consider integral valued bilinear forms $\theta$ on finitely generated free $\mathbb{Z}$-modules $H$. Define the transpose of $\theta$, denoted $\theta^t$, by $\theta^t(x, y) = \theta(y, x)$ for all $x$ and $y$ in $H$. We say that $\theta$ is a Seifert form if the form $\theta - \theta^t$ is unimodular, i.e. the associated map $H \rightarrow \text{Hom}(H, \mathbb{Z})$, defined by $x \mapsto (\theta - \theta^t)(x, )$, is an isomorphism. A form is called non-singular if its associated map is injective.

The notions of isometric structure, metabolizer, and Alexander polynomial extend to (algebraic) Seifert forms. Observe that if $\theta$ is a Seifert form on $H$, then the rank of $H$ must be even. Polynomials in $\mathbb{Q}[t^{\pm 1}]$ are said to be relatively prime if their greatest common divisor is a unit.
The following lemma is a refinement of [6, proposition 3]. Its geometrical origins are in the work of [3] on the knot concordance group. They did not need the splitting of a metabolizer as stated below, while we will need the splitting later.

**Lemma 3.1.** Let \( \theta_1 \) and \( \theta_2 \) be Seifert forms on \( H_1 \) and \( H_2 \). Suppose that their Alexander polynomials are relatively prime in \( \mathbb{Q}[t^{\pm 1}] \) and that either \( \theta_1 \) or \( \theta_2 \) is non-singular. Then if \( \theta_1 \oplus \theta_2 \) is null-concordant with a metabolizer \( Z \) for the associated isometric structure, then \( \theta_1 \) and \( \theta_2 \) are null-concordant with metabolizers \( Z_1 \) and \( Z_2 \) for the associated isometric structures, respectively, such that \( Z_i = Z \cap H_i, \ i = 1, 2 \), and \( Z = Z_1 \oplus Z_2 \).

**Proof.** Consider the associated isometric structures \( s_i \) to \( \theta_i, \ i = 1, 2 \). Then \( s_1 \oplus s_2 \) on \( H = H_1 \oplus H_2 \) is the associated isometric structure \( s \) to \( \theta_1 \oplus \theta_2 \). Let \( Z_i = Z \cap H_i, \ i = 1, 2 \), and let \( \varphi_i(x) = x^{2\theta_i} \Delta_{\theta_i}(1 - x^{-1}) \), where \( 2\theta_i \) is the rank of \( H_i \). Since \( \Delta_{\theta_1} \) and \( \Delta_{\theta_2} \) are relatively prime in \( \mathbb{Q}[t^{\pm 1}] \) and since either \( \theta_1 \) or \( \theta_2 \) is non-singular, \( \Delta_{\theta_1} \) and \( \Delta_{\theta_2} \) are relatively prime in \( \mathbb{Q}[t] \). Then \( \varphi_1 \) and \( \varphi_2 \) are also relatively prime in \( \mathbb{Q}[x] \). For, if \( f(x) \) is a common factor of \( \varphi_1 \) and \( \varphi_2 \), then \( (1 - t)^d f(1/(1 - t)) \), \( d = \deg f \), is a common factor of \( \Delta_{\theta_1} \) and \( \Delta_{\theta_2} \). Thus there are polynomials \( u_1 \) and \( u_2 \) in \( \mathbb{Z}[x] \) and a non-zero integer \( c \) such that \( u_1 \varphi_1 + u_2 \varphi_2 = c \).

For \( z \in Z \), there are \( z_1 \in H_1 \), and \( z_2 \in H_2 \) with \( z = z_1 + z_2 \). As stated right after the definition of the Alexander polynomial, each \( \varphi_i \) is the characteristic polynomial for \( s_i \), and hence \( \varphi_i(s_i) = 0 \). Using this and \( s(z_i) = s_i(z_i) \), we have

\[
\begin{align*}
cz_1 &= u_1(s)\varphi_1(s)z_1 + u_2(s)\varphi_2(s)z_1 \\
&= u_1(s_1)\varphi_1(s_1)z_1 + u_2(s_1)\varphi_2(s_1)z_1 \\
&= u_2(s_1)\varphi_2(s_1)z_1 \\
&= u_2(s_1)\varphi_2(s_1)z_1 + u_1(s_2)\varphi_2(s_2)z_2 \\
&= u_2(s)\varphi_2(s)z.
\end{align*}
\]

Since \( Z \) is \( s \)-invariant, \( cz_1 = u_2(s)\varphi_2(s)z \in Z \). Since \( Z \) is a direct summand of \( H \), this implies \( z_1 \in Z \), and hence \( z_1 \in Z_1 \). Similarly, \( z_2 \in Z_2 \). So \( Z = Z_1 + Z_2 \). This now implies that \( Z = Z_1 \oplus Z_2 \) since \( H = H_1 \oplus H_2 \) and \( Z_i = Z \cap H_i \).

Since \( Z \) is \( s \)-invariant, each \( Z_i \) is \( s_i \)-invariant. Since \( H/Z = H_1/Z_1 \oplus H_2/Z_2 \) is torsion free, each \( Z_i \) is a direct summand of \( H_1 \). Since the intersection pairing \( \langle , \rangle \) on \( H \) is unimodular, \( Z_i = Z_i^1 \) on \( H_1 \). Thus each \( Z_i \) is a metabolizer for \( s_i \). \( \square \)

**Proof of Theorem 2.1.** Let \( F_1 \) and \( F_2 \) be Seifert surfaces for \( K_1 \) and \( K_2 \). Then a boundary connected sum \( F_1 \sharp F_2 \) is a Seifert surface for \( K_1 \sharp K_2 \). Let \( Z \) be a metabolizer for the isometric structure on \( H_1(F_1 \sharp F_2) = H_1(F_1) \oplus H_1(F_2) \) satisfying the conclusion of Theorem 2.1 with the exceptional primes \( p_1, \ldots, p_n \), i.e. \( \tau(K_1 \sharp K_2, N^g_p \cap (Z \otimes \mathbb{Q}/\mathbb{Z})) \) vanishes for all prime powers \( q \) and all primes \( p \) except \( p_1, \ldots, p_n \).

Then by Lemma 3.1 there are metabolizers \( Z_1 \) and \( Z_2 \) for the isometric structures on \( H_1(F_1) \) and \( H_1(F_2) \), respectively, with \( Z = Z_1 \oplus Z_2 \). Let \( q \) be a power of a prime. Let \( N = \ker \varepsilon_q^g \otimes \mathbb{Q}/\mathbb{Z} \) and \( N_i = \ker \varepsilon_i^g \otimes \mathbb{Q}/\mathbb{Z} \), where \( \varepsilon^g \) and \( \varepsilon^g_i \) are the endomorphisms of \( H \) and \( H_i \), respectively, as denoted in Section 2. Then since \( \varepsilon^g = \varepsilon_q^g \otimes \varepsilon_2^g \), \( N = N_1 \oplus N_2 \), and \( Z = Z_1 \oplus Z_2 \),

\[
N \cap (Z \otimes \mathbb{Q}/\mathbb{Z}) = (N_1 \cap Z_1 \otimes \mathbb{Q}/\mathbb{Z}) \oplus (N_2 \cap Z_2 \otimes \mathbb{Q}/\mathbb{Z}).
\]

Let \( N_p \) and \( N_{i,p} \) denote the \( p \)-primary components of \( N \) and \( N_i \), respectively. Let \( \chi_1 \in N_{1,p} \cap (Z_1 \otimes \mathbb{Q}/\mathbb{Z}) \). Then \( \chi = \chi_1 \oplus 0 \) is an element in \( N_p \cap (Z \otimes \mathbb{Q}/\mathbb{Z}), \)
where $0$ stands for the trivial character in $N_2 \cap (Z_2 \otimes \mathbb{Q}/Z)$. By the additivity of Casson–Gordon invariants [10, page 335], for all primes $p$ except $p_1, \ldots, p_n$,

$$0 = \tau(K_1 \# K_2, \chi) = \tau(K_1, \chi_1) + \tau(K_2, 0).$$

Also, by [10, corollary B2] $\tau$ is determined by the algebraic concordance class of the knot if the character is trivial. This implies that $\tau(K_2, 0) = 0$ and hence $\tau(K_1, \chi_1) = 0$. Since $\chi_1$ was chosen arbitrarily, we just have found a metabolizer $Z_1$ for the isometric structure on $H_1(F_1)$ such that $\tau(K_1, N_{1,p} \cap (Z_1 \otimes \mathbb{Q}/Z))$ vanishes for all prime powers $q$ and all primes $p$ except $p_1, \ldots, p_n$, i.e. $K_1$ is zero in $\Gamma^+$. Similarly, $K_2$ is zero in $\Gamma^+$. This completes the proof.

The proof given above also works for the Gilmer ribbon obstructions, in which case there are no exceptional primes $p_1, \ldots, p_n$.

**Corollary 3.2.** Under the same conditions as in Theorem 1.1, if $K_1 \# K_2$ has vanishing Gilmer ribbon obstruction, then so do both $K_1$ and $K_2$.

4. **Twisted doubles of a knot**

In this section we state that all but finitely many algebraically slice twisted doubles of a knot have infinite order in the knot concordance group $G$, in fact, in $\Gamma^+$. We also state similar results concerning ribbonness in the line of Theorem 2.1(b). The proofs will be given in the next two sections. In preparations, we also summarize some facts on the Casson–Gordon invariants of genus 1 knots and the Tristram–Levine signatures of satellite knots and torus knots.

Let $K$ be a knot in the 3-sphere $S^3$. Let $D_k(K)$ denote the $k$-twisted double of $K$ as illustrated in Figure 1. Here, $k$ may be negative.

The following theorem is due to [8, corollary 23].

**Theorem 4.1.** The $k$-twisted double of a knot $K$ is:

(a) of infinite order in the algebraic concordance group, $G$, if $k < 0$;
(b) algebraically slice if $k \geq 0$ and $4k + 1$ is a perfect square;
(c) of order 2 in $G$ if $k > 0$, $4k + 1$ is not a perfect square, and every prime congruent to 3 mod 4 has even exponent in the prime power factorization of $4k + 1$;
(d) of order 4 in $G$ if $k > 0$ and some prime congruent to 3 mod 4 has odd exponent in the prime power factorization of $4k + 1$.

Immediate corollaries are that $D_k(K)$ is algebraically slice if and only if $k = l(l + 1)$ for an integer $l \geq 0$ and that $D_k(K)$ has infinite order in $\Gamma^+$ if $k < 0$.

We will further prove the following two theorems. For a nonnegative integer $n$ let $nD_k(K)$ denote the connected sum of $n$ copies of $D_k(K)$. The first of the following
two theorems concerns the order of algebraically slice $D_k(K)$ in $\Gamma^+$. The second concerns the Gilmer ribbon obstructions of $nD_k(K)$ for not only algebraically slice but all twisted doubles $D_k(K)$.

**Theorem 4.2.** (a) For any knot $K$, the algebraically slice $k$-twisted double $D_k(K)$ has infinite order in $\Gamma^+$ for all but finitely many $k$.

(b) If $\sigma_r(K) \geq 0$ for all $r$, the algebraically slice $D_k(K)$ has infinite order in $\Gamma^+$ for any $k \neq 0, 2$, where $\sigma_r(K)$ denotes the averaged Tristram–Levine signature of $K$ (details will be given later). In addition, if $\sigma_r(K) > 0$ for some $r$, then $D_2(K)$ has infinite order in $\Gamma^+$ as well.

**Theorem 4.3.** (a) For any knot $K$, there is a set $I$ of all integers except a finite number of nonnegative integers such that, for any $k \in I$ and any integer $n \neq 0$, $nD_k(K)$ has nonvanishing Gilmer ribbon obstruction.

(b) If $\sigma_r(K) \geq 0$ for all $r$, for any integer $n \neq 0$ and any integer $k \neq 0, 1, 2$, $nD_k(K)$ has nonvanishing Gilmer ribbon obstruction. In addition, if $\sigma_r(K) > 0$ for $r = 3, 6$ for any integer $n \neq 0$, $nD_1(K)$ and $nD_2(K)$ have nonvanishing Gilmer ribbon obstruction as well.

We devote the remaining two sections to proving these theorems. Before that, we summarize some useful facts in the rest of this section.

### 4.1. Casson–Gordon invariants of a genus 1 knot.

We state the work of [13] theorem 7 that gives a formula for $\tau$ for genus 1 knots in terms of the classical signatures. We remark that [3] first found the formula for the 2-fold branched cover case and algebraically slice case.

For a knot $K$ and a character $\chi$, $\tau(K, \chi)$ is defined to be an element of the Witt group $W(\mathbb{C}(t), J) \otimes_{\mathbb{Q}} \mathbb{Q}$, where $J$ denotes the involution on $\mathbb{C}(t)$ given by complex conjugation and by the map $t \mapsto t^{-1}$ and $W(\mathbb{C}(t), J)$ is the Witt group of finite dimensional hermitian inner product spaces. For details, see [1]. Let $W(\mathbb{R})$ denote the Witt group of finite dimensional inner product spaces over $\mathbb{R}$. The signature function $\sigma: W(\mathbb{R}) \to \mathbb{Z}$ is an isomorphism. Also there is a natural map $W(\mathbb{R}) \to W(\mathbb{C}(t), J)$ given by tensoring with $\mathbb{C}(t)$ over $\mathbb{R}$. Composing this map with $\sigma$ and with $Q$ gives a homomorphism $\rho: Q \to W(\mathbb{C}(t), J) \otimes_{\mathbb{Q}} Q$. Note that for each complex number $\xi$ with $|\xi| = 1$, there is a homomorphism $\sigma_\xi: W(\mathbb{C}(t), J) \otimes_{\mathbb{Q}} \mathbb{Q} \to \mathbb{Q}$ (see [11]). It is easy to see that $\sigma_1 \circ \rho$ is the identity.

For any real number $r$, define $A_r(K) = (1 - e^{2\pi ir}) A + (1 - e^{-2\pi ir}) A^t$, where $A$ is a Seifert matrix of $K$, and define $\sigma_r(K)$ to be $\sigma(A_r)$ if $A_r$ is non-singular and elsewhere to be the average of the one-sided limits of $\sigma(A_r)$. This $\sigma_r(K)$ is a concordance invariant and is equal to the Tristram–Levine signature $\tau$ [17] of $K$ except perhaps at finitely many $r$. It is an immediate consequence of the definition that $\sigma_r(K) = \sigma_{1-r}(K)$. Thus we only need to consider $\sigma_r(K)$ for $0 \leq r \leq \frac{1}{2}$.

The following is due to [13]. The case for $q = 2$ and the case for algebraically slice knots are due to [3] [11].

**Theorem 4.4.** Let $F$ be a genus one Seifert surface of a knot $K$ and let $A = \begin{pmatrix} a & -b \\ -b & a \end{pmatrix}$ be the Seifert matrix of $K$ with respect to a basis $\{x, y\}$ of $H_1(F)$. Let $q$ and $d$ be powers of primes, $s$ an integer relatively prime to $d$, and $N^q = \ker((G^q - (G - I))^s) \otimes_{\mathbb{Z}/d} \mathbb{Z} \subset H_1(F; \mathbb{C}/\mathbb{Z})$, where $G = (A - A^t)^{-1} A$. Suppose that $x \otimes s/d \in N^q$ and $d \mid a$. Then the multiplicative inverse, $m^*$, of $m$ mod $d$ exists
and \( x \otimes s/d \) defines a character \( \chi \) for which
\[
\tau(K, \chi) = \rho \sum_{i=0}^{q-1} \left( \sigma_{\frac{s}{d}}(J_x) + \frac{2(d-s_i)s_i a}{d^2} - s_i \sigma(\frac{s}{d}) \right),
\]
where \( J_x \) is a simple closed curve on \( F \) representing \( x \) and, for \( i = 0, \ldots, q-1 \), \( s_i \) is an integer such that \( 0 < s_i < d \) and \( s_i \equiv (1 + m^*)s \mod d \).

In particular, (a) if \( q = 2 \), then
\[
\tau(K, \chi) = \rho \left( 2\sigma(\frac{s}{d})(J_x) + \frac{4(d-s_0)s_0 a}{d^2} - s_0 \sigma(\frac{s}{d}) \right).
\]

(b) If \( a = 0 \), then \( d \mid (m+1)^q - m^q \) implies \( x \otimes s/d \in N^q \) and
\[
\tau(K, \chi) = \rho \sum_{i=0}^{q-1} \sigma_{\frac{s}{d}}(J_x).
\]

4.2. Satellite knots and torus knots. Let \( K \) be a knot in \( S^3 \). By an axis for \( K \) of winding number \( w \) we mean an unknotted simple closed curve \( \gamma \) in \( S^3 - K \) having linking number \( w \) with \( K \). Let \( V \) be a solid torus complementary to a tubular neighborhood of \( \gamma \), with \( K \) contained in the interior of \( V \). There is a preferred generator \( v \) for \( H_1(V) \), specified by the condition \( lk(v, \gamma) = +1 \). For any knot \( C \) in \( S^3 \) there is an untwisted orientation-preserving embedding \( h: V \to S^3 \) taking \( V \) onto a tubular neighborhood of \( C \) such that \( C \) represents \( h_*(v) \) in \( H_1(hV) \). We say that the knot \( h(K) \), denoted \( C(K) \), is a satellite of \( C \) with orbit \( K \), axis \( \gamma \), and winding number \( w \).

The following is \([9 \text{ theorem } 2]\).

**Theorem 4.5.** Let \( C(K) \) be a satellite of \( C \) with orbit \( K \) and winding number \( w \). Then
\[
\sigma_r(C(K)) = \sigma_{wr}(C) + \sigma_r(K).
\]

Let \( T_{m,n} \) denote the \((m,n)\) torus link. To fix orientation conventions \( T_{2,2} \) is the positive Hopf link. We will use the work of \([9]\) on the signatures of \( T_{m,n} \) to prove the following proposition. For a real number \( z \), let \( \lfloor z \rfloor \) denote the greatest integer that is less than or equal to \( z \).

**Proposition 4.6.** Let \( k > 0 \) and \( l \geq 2 \) be integers and suppose \( 0 \leq r \leq \frac{1}{2} \).

(a) If \( r \neq (2d+1)/2(2k+1) \) for any integer \( d \),
\[
\sigma_r(T_{2,2k+1}) = -2 \left[ r(2k+1) + \frac{1}{2} \right].
\]

(b) For any integer \( t \) with \( 1 \leq t \leq l/2 \), \( \sigma_r(T_{l,-1}) \) increases from \(-2(t-1)^2 + 2l(t-1) \) to \(-2t^2 + 2(l+1)t - 2 \) over the interval \( (t-1)/l \leq r \leq t/(l+1) \) and decreases from \(-2t^2 + 2(l+1)t - 2 \) to \(-2t^2 + 2lt \) over the interval \( t/(l+1) \leq r \leq l/t \).

In particular, if \( (t-1)/l \leq r \leq t/l \), then
\[
-2(t-1)^2 + 2l(t-1) \leq \sigma_r(T_{l,-1}) \leq -2t^2 + 2(l+1)t - 2.
\]

**Proof.** Define \( f_{m,n}(r) = \frac{1}{2} \) (jump in \( \sigma_r(T_{m,n}) \) at \( r \)). Then \([9]\) showed that if \( m, n > 0 \) and \( 0 \leq r \leq \frac{1}{2} \),
\[
f_{m,n}(r) = \begin{cases} (-1)^{\lfloor a/n \rfloor + \lfloor b/m \rfloor} & \text{if } a, b, mn, mr, nr \in \mathbb{Z}, \, mr, nr \notin \mathbb{Z} \text{ with } mn = am + bn, \\ 0 & \text{otherwise}. \end{cases}
\]

Note that \( f_{m,n}(r) \) is nonzero only if \( r = s/mn \) for an integer \( s \) with \( m \nmid s \) and \( n \nmid s \).
To prove (a), let \( m = 2 \) and \( n = 2k + 1 \). Then \( f_{2,2k+1}(r) \) is nonzero only when \( r = s/(2k+1) \) for odd \( s \) with \( 1 \leq s \leq 2k-1 \). Note that \( mn r = s = (-ks)(2) + (s)(2k+1) \). So we have

\[
f_{2,2k+1}\left(\frac{s}{2(2k+1)}\right) = (-1)^{\lceil -ks/(2k+1) \rceil + \lfloor s/2 \rfloor}
\]

Write \( s = 2d - 1 \) for \( 1 \leq d \leq k \). Then \( \lceil -ks/(2k+1) \rceil = -d \) and \( \lfloor s/2 \rfloor = d - 1 \). Thus

\[
f_{2,2k+1}(r) = \begin{cases} 
-1 & \text{if } r = (2d - 1)/2(2k+1) \text{ with } 1 \leq d \leq k, \\
0 & \text{otherwise.}
\end{cases}
\]

Since \((2d - 1)/2(2k+1) \leq r\) if and only if \( d \leq r(2k+1) + \frac{1}{2} \), we see

\[
\sigma_r(T_{2,2k+1}) = \sum_{d=1}^{\lceil r(2k+1)+1/2 \rceil} 2f_{2,2k+1} \left( \frac{2d - 1}{2(2k+1)} \right) = -2 \left[ r(2k+1) + \frac{1}{2} \right].
\]

To prove (b), note first that \( f_{l,-l-1} = -f_{l,l+1} \) since \( T_{l,-l-1} \) is the mirror image of \( T_{l,l+1} \). Let \( m = l \) and \( n = l + 1 \). For any integer \( s \) with \( 0 < s < l(l+1)/2, l \nmid s \), and \( l + 1 \nmid s \), we see \( s = -(s)(l) + (s)(l+1) \) and hence

\[
f_{l,-l-1} \left( \frac{s}{l(l+1)} \right) = -f_{l,l+1} \left( \frac{s}{l(l+1)} \right) = -(-1)^{\lceil -s/(l+1) \rceil + \lfloor s/l \rfloor}.
\]

Let \( t \) be an integer with \( 1 \leq t \leq l/2 \). Note \((l+1)(t-1) < lt < (l+1)t \). We will consider two cases: If \((l+1)(t-1) < s < lt\), then \( \lceil -s/(l+1) \rceil = -t \) and \( \lfloor s/l \rfloor = t-1 \) and hence \( f_{l,-l-1} \left( s/(l(l+1)) \right) = 1 \). If \( lt < s < (l+1)t \), then \( \lceil -s/(l+1) \rceil = -t \) and \( \lfloor s/l \rfloor = t \) and hence \( f_{l,-l-1} \left( s/(l(l+1)) \right) = -1 \).

Since the sets \( \{ s \in \mathbb{Z} \mid (t-1)/l < s/l(l+1) < t/(l+1) \} \) and \( \{ s \in \mathbb{Z} \mid t/(l+1) < s/l(l+1) < t/l \} \) have \( l - t \) and \( t - 1 \) elements, respectively, \( \sigma_r(T_{l,-l-1}) \) changes by \( 2(t-1) - 2(t-1) = 2l - 4t + 2 \) over the interval \((t-1)/l < r < t/l \). Incorporating these, we have

\[
\sigma_r(T_{l,-l-1}) = \begin{cases} 
\sum_{t=1}^{l-1} (2l - 4t' + 2) & \text{if } r = (t-1)/l \\
\sum_{t=1}^{l-1} (2l - 4t' + 2) + 2(l-t) & \text{if } r = t/l + 1
\end{cases}
\]

\[
= \begin{cases} 
-2(t-1)^2 + 2l(t-1) & \text{if } r = (t-1)/l \\
-2t^2 + 2(l+1)t - 2 & \text{if } r = t/(l+1)
\end{cases}
\]

Now (b) follows. \( \square \)

5. Algebraically slice twisted doubles

In this section we will estimate the Casson–Gordon invariants of algebraically slice \( D_k(K) \) and prove Theorem \[2\] Theorem \[3\] and Corollary \[4\].

As mentioned in Section \[1\] \( D_k(K) \) is algebraically slice if and only if \( k = l(l+1) \) for an integer \( l \geq 0 \). A Seifert matrix for \( D_l(l+1)(K) \) corresponding to the Seifert surface \( F \) in Figure \[5\] is \( \begin{pmatrix} -1 & 1 \\ 0 & l(l+1) \end{pmatrix} \). The matrix \( G = (A - A^t)^{-1}A \) associated with
the isometric structure $s$ is \( \begin{pmatrix} 0 & -l(l+1) \\ -1 & 1 \end{pmatrix} \) and has eigenvectors

\[
v^+ = \begin{pmatrix} l+1 \\ 1 \end{pmatrix} \quad \text{corresponding to eigenvalue } -l
\]

\[
v^- = \begin{pmatrix} -l \\ 1 \end{pmatrix} \quad \text{corresponding to eigenvalue } l+1.
\]

With rational coefficients, $G$ is diagonalizable with respect to the basis \( \{v^+, v^-\} \).

For a positive integer $n$, let $nD_{l(l+1)}(K)$ be the connected sum of $n$ copies of $D_{l(l+1)}(K)$. We put a subscript $n$ on objects corresponding to $nD_{l(l+1)}(K)$. For example, $F_n$ denotes the Seifert surface of $nD_{l(l+1)}(K)$ obtained by boundary connected summing $n$ copies of the Seifert surface $F$ of $D_{l(l+1)}(K)$.

### 5.1. Metabolizer of $nD_{l(l+1)}(K)$

Let $Z_n$ be a metabolizer of the associated isometric structure of $nD_{l(l+1)}(K)$. (It should be remarked that unlike $F_n$, $s_n$, and $\theta_n$, $Z_n$ needs not be a direct sum of metabolizers of $D_{l(l+1)}(K)$.) Since $G$ is diagonalizable with respect to the basis \( \{v^+, v^-\} \) with rational coefficients, $G_n$ associated with $s_n$ is diagonalizable with respect to the basis \( \{v_j^+, v_j^-\} \) for which the $j$-th coordinate is the only nonzero $v_j^\pm$ under the identification of $H_1(F_n)$ with the direct sum of $n$ copies of $H_1(F)$. Note that \( \{v_1^+, \ldots, v_n^+\} \) is a basis for the eigenspace of $G_n$ corresponding to the eigenvalue $-l$ and \( \{v_1^-, \ldots, v_n^-\} \) is a basis for the eigenspace of $G_n$ corresponding to $l+1$.

The following lemma shows that $Z_n$ contains an eigenvector of $G_n$ for which the Casson–Gordon invariant can be easily estimated as will be shown later.

**Lemma 5.1.** There are integers $e \geq \frac{n}{2}$, $a > 0$, and $a_{e+1}, \ldots, a_n \in \mathbb{Z}$ such that $Z_n$ contains either \( a(\sum_{j=1}^{e} v_j^+) + \sum_{j=e+1}^{n} a_j v_j^+ \) or \( a(\sum_{j=1}^{e} v_j^-) + \sum_{j=e+1}^{n} a_j v_j^- \).

**Proof.** From a basic result of linear algebra, $Z_n \otimes \mathbb{Q}$ has a basis consisting of eigenvectors of $G_n$ since $Z_n$ is invariant under $G_n$ and $G_n$ is diagonalizable over $\mathbb{Q}$. In particular, $Z_n \otimes \mathbb{Q} = E^+ \oplus E^-$, where $E^\pm$ are the eigenspaces of $G_n$ restricted to $Z_n \otimes \mathbb{Q}$. Since the rank of $Z_n$ is $n$, one of $E^\pm$ has rank greater than or equal to $n/2$. Suppose that $E^+$ has rank $e \geq n/2$. Using the Gauss-Jordan algorithm and rearranging basis elements, we may assume that a basis for $E^+$ consists of vectors of the form $v_j^+ + u_j$, $1 \leq j \leq e$, where $u_j$ are linear combinations of $v_{e+1}, \ldots, v_n^+$. Adding these basis elements together we see that $Z_n \otimes \mathbb{Q}/\mathbb{Z}$ contains a vector $\sum_{j=1}^{e} v_j^+ + \sum_{j=e+1}^{n} b_j v_j^+$ for some $b_{e+1}, \ldots, b_n \in \mathbb{Q}$. Multiplying the vector by a nonzero integer gives a desired vector. The same argument works if rank $E^- \geq n/2$. \qed

### 5.2. Estimation of the Casson–Gordon invariants

We use Theorem 4.3(b) to estimate $\tau$ of $D_{l(l+1)}(K)$ for the character corresponding to $v^\pm$ in this subsection.

To change basis of $H_1(F)$ to either \( \{x^+ = v^+, y = (-1,0)\} \) or \( \{x^- = v^-, y = (-1,0)\} \), let

\[
P^+ = \begin{pmatrix} l+1 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad P^- = \begin{pmatrix} -l & -1 \\ 1 & 0 \end{pmatrix}.
\]
The Seifert matrices with respect to these bases \( \{x^\pm, y\} \) are
\[
A^+ = (P^+)^t AP^+ = \begin{pmatrix} 0 & l + 1 \\ l & -1 \end{pmatrix} \quad \text{and} \quad A^- = (P^-)^t AP^- = \begin{pmatrix} 0 & -l \\ -l - 1 & -1 \end{pmatrix},
\]
respectively. Let \( m^+ = -l - 1 \) and \( m^- = l \). We can apply Theorem 4.4(b) to \( A^\pm \) with \( x = x^\pm \) and \( m = m^\pm \), respectively, since \( a = 0 \). Observe that we can choose
\[
J_{x^+} = J_{x^-} = T_{l, -l - 1} \# K,
\]
as shown in Figure 2 when \( l = 3 \). Here, the property \( T_{m,n} = T_{-m,-n} \) has been used.

For a prime power \( q \), a prime \( p \) dividing \( |(m + 1)^q - m^q| = (l + 1)^q - l^q \), and any integer \( s \) relatively prime to \( p \), Theorem 4.4(b) implies that \( x^\pm \otimes s/p \) defines a character. Abusing notation, \( x^\pm \otimes s/p \) will also denote its corresponding character. We have
\[
\sigma_1^{\tau} \left(D_{l(l+1)}(K), x^\pm \otimes \frac{s}{p}\right) = \sum_{i=0}^{q-1} \sigma_{\frac{q}{p}} (T_{l, l+1} \# K),
\]
where \( s_i \) are integers such that \( 0 < s_i < p \) and \( s_i \equiv (1 + (m^\pm)^i)s \mod p \).

Since a connected sum \( K_1 \# K_2 \) can be considered as a satellite of \( K_1 \) with orbit \( K_2 \) and winding number 1, by Theorem 4.5 we have

**Lemma 5.2.** For any prime power \( q \), any prime \( p \) dividing \( (l + 1)^q - l^q \), and any integer \( s \) with \( p \nmid s \),
\[
\sigma_1^{\tau} \left(D_{l(l+1)}(K), x^\pm \otimes \frac{s}{p}\right) = \sum_{i=0}^{q-1} \left( \sigma_{\frac{q}{p}} (T_{l, l+1}) + \sigma_{\frac{q}{p}} (K) \right),
\]
where \( s_i \) are integers such that \( 0 < s_i < p \) and \( s_i \equiv (1 + (m^\pm)^i)s \mod p \).

Now we need to estimate \( \sigma_r(T_{l, l+1}) \).
Lemma 5.3. (a) 
\[ \sigma_r(T_{l_0-l-1}) \geq \begin{cases} 0 & \text{if } l \geq 1, \\ 2 & \text{if } l \geq 2 \text{ and } 1/l(l+1) < r \leq \frac{1}{2}. \end{cases} \]

(b) For any constant \( C_0 \), there is an integer \( l_0 \geq 2 \) such that, for any \( l \geq l_0 \), any prime power \( q \), any prime \( p \) dividing \((l + 1)^q - l^q\), and any \( m = m^\pm \), there is an integer \( s \) such that
\[
\sum_{i=0}^{q-1} \sigma_{s_p}(T_{l_0-l-1}) > qC_0,
\]
where \( s_i \) are integers such that \( 0 < s_i < p \) and \( s_i \equiv (1 + m^\pm)j \mod p \).

Proof. If \( l = 1 \) then \( T_{l_0-2} \) is the unknot and \( \sigma_1(T_{l_0-2}) = 0 \) for any \( r \). Now suppose that \( l \geq 2 \). From Proposition 4.6(b) \( \sigma_r(T_{l_0-l-1}) \) has local minima \(-2t^2 + 2lt\) at the integers \( t \) with \( 0 \leq t \leq l/2 \). Observe that the function \(-2t^2 + 2lt\) is increasing over \( 0 \leq t \leq l/2 \) and has \( 0 \) at \( t = 0 \). Also, a close look at the proof of Proposition 4.6(b) reveals \( f_{l_0-l-1}(1/l(l+1)) = 1 \) and hence \( \sigma_r(T_{l_0-l-1}) \geq 2 \) if \( 1/l(l+1) < r \leq \frac{1}{2} \). Now (a) follows.

To show (b), let \( l_0 \) be an integer such that \( l_0 \geq 2 \) and \( \frac{4l_0^2}{4} - l_0 - 2 \geq 2C_0 \). Let \( l \geq l_0 \), \( q \) a prime power, and \( p \) a prime dividing \((l + 1)^q - l^q\). It is easy to see that \( 1 + (m^\pm)^* \equiv 0 \mod p \). For simplicity, let \( a = 1 + (m^\pm)^* \) and \( e \) the multiplicative order of \( a \mod p \). Then \( e \) divides \( p - 1 \) and let \( f = (p - 1)/e \). For any integer \( z \) not divisible by \( p \), let \( E(z) = \{a^iz \in \mathbb{Z}/p\} \) be integers such that \( 0 \) are integers \( z_1, \ldots, z_j \) such that \( \cup_j E(z_j) = \mathbb{Z}/p^* \) and \( E(z_j) \) are all disjoint.

Let \( P = \{x \in \mathbb{Z}/p^* \mid p/4 \leq x \leq 3p/4\} \). We will show that there is a \( s \) for which at least half of \( s_0, \ldots, s_{q-1} \) belong to \( P \). Note \( |P| \geq (p-1)/2 = |\mathbb{Z}/p^*|/2 \). Since \( \{E(z_j)\}_{j=1}^{j=\ldots,j} \) is a partition of \( \mathbb{Z}/p^* \), there is \( j \) such that \( |E(z_j) \cap P| \geq |E(z_j)|/2 = e/2 \). For simplicity, assume \( j = 1 \).

Let \( G = \{(c, d) \in \mathbb{Z} \times \mathbb{Z} \mid 0 \leq c \leq q-1 \text{ and } 0 \leq d \leq e-1 \} \). Define a function \( \phi: G \rightarrow E(z_1) \) by \( \phi(c, d) = a^c d z_1 \mod p \). Since, for each fixed \( c \), \( \phi(c, d) \) is \( a \) a \( q \) to 1 map. Since \( |E(z_1) \cap P| \geq |E(z_1)|/2 \), \( |\phi^{-1}(E(z_1) \cap P)| \geq q/2 \).

For each fixed \( d \) with \( 0 \leq d \leq e-1 \), let \( G_d = \{(c, d) \in G \mid 0 \leq c \leq q-1 \} \). Then \( \{G_d\}_{d=0}^{d=e-1} \) is a partition of \( G \) and hence there is an integer \( d_0 \) such that \( |G_{d_0} \cap \phi^{-1}(E(z_1) \cap P)| \geq |G_{d_0}|/2 = q/2 \). Let \( s = a^{d_0} z_1 \) and, for \( i = 0, \ldots, q-1 \), let \( s_i \) be integers such that \( 0 < s_i < p \) and \( s_i \equiv a^i \mod p \). Then \( \phi(G_{d_0}) = \{s_0, s_1, \ldots, s_{q-1}\} \) is a partition of \( \mathbb{Z}/p \) and at least half of \( s_i \)’s belong to \( P \).

Note that if \( s_i \in P \), then \( \frac{1}{4} \leq s_i/p \leq \frac{1}{4} \). Let \( t = [l/4 + 1] \). Then \( l/4 < t \leq l/4 + 1 \) and \( (t-1)/l \leq \frac{1}{4} \leq s_i/p \). By Proposition 4.6(b), if \( s_i \in P \),
\[
\sigma_{s_p}(T_{l_0-l-1}) \geq -2(t-1)^2 + 2l(t-1)
\]
\[
> -2 \left(\frac{l}{4} - 1\right)^2 + 2 \left(\frac{l}{4} - 1\right)
\]
\[
= \frac{3}{8} l^2 - l - 2
\]
\[
\geq 2C_0
\]
by the definition of \( l_0 \).
Summing these, we have

\[ \sum_{i=0}^{q-1} \sigma_{\frac{q}{p}}(T_{i,-t-1}) = \left( \sum_{s_i \in P} + \sum_{s_i \notin P} \right) \sigma_{\frac{q}{p}}(T_{i,-t-1}) \geq \sum_{s_i \in P} \sigma_{\frac{q}{p}}(T_{i,-t-1}) \quad \text{by (a)} \]

\[ > \frac{q}{2} \cdot 2C_0 = qC_0. \]

This completes the proof. \(\square\)

5.3. Homology of prime power fold cyclic branched covers. We need some algebraic background. The resultant of two non-constant integral polynomials \(f(t)\) and \(g(t)\) is defined as follows: We may factor completely the polynomials \(f\) and \(g\) in some extension ring of \(\mathbb{Z}\) as: \(f(t) = a \prod_{i=1}^{n}(t - \alpha_i)\) and \(g(t) = b \prod_{j=1}^{m}(t - \beta_j)\).

Then the resultant of \(f\) and \(g\), denoted \(R(f,g)\), is \(a^m b^n \prod_{i,j} (\alpha_i - \beta_j)\). It is easy to see that \(R(f,g) = 0\) if and only if \(f\) and \(g\) have a common root in a field over \(\mathbb{Z}\). We remark that this is also valid when working modulo a prime \(p\).

It is known by Fox (see [13] for a proof) that the order of the homology of the \(n\)-fold cyclic cover of \(S^3\) branched over a knot \(J\) is the absolute value of the resultant, \(|R(t^n - 1, \Delta_J(t))|\), of \(t^n - 1\) and the Alexander polynomial \(\Delta_J(t)\) of \(J\).

**Proposition 5.4.** For any knot \(K\) and any integer \(k > 0\), there are infinitely many distinct primes each of which divides \(|H_1(M^q)|\) for some prime \(q\), where \(M^q\) is the \(q\)-fold cyclic cover of \(S^3\) branched over \(D_k(K)\).

**Proof.** The Alexander polynomial of \(D_k(K)\) is \(\Delta_k = -kt^2 + (2k + 1)t - k\). Let \(R_n\) denote \(|R(t^n - 1, \Delta_k)|\).

First, we will show that \(R_{q_1}\) and \(R_{q_2}\) are relatively prime for distinct primes \(q_1\) and \(q_2\). Suppose to the contrary there is a prime \(p\) dividing both \(R_{q_1}\) and \(R_{q_2}\). Then, working modulo \(p\), for \(j = 1, 2\), \(\Delta_k\) and \(t^{q_j} - 1\) have a common root \(r_j\) in an extension field \(F_p\) of \(p\) elements.

We claim that the three polynomial \(\Delta_k\), \(t^{q_1} - 1\), and \(t^{q_2} - 1\) have a common root, \(r\), mod \(p\). If \(r_1 = r_2\), this is obvious. Suppose \(r_1 \neq r_2\). Then \(\Delta_k\) has two distinct roots \(r_1\) and \(r_2\) mod \(p\) and hence \(\Delta_k\) must be quadratic over \(F_p\). In particular, \(k \neq 0\) in \(F_p\).

Thus, \(r_1 r_2 = (-k)/(k-1) = 1\) or \(r_2 = 1/r_1\). So, \(r_3^k = 1/r_1^k = 1\) mod \(p\) and \(r_2\) is a common root, \(r\), mod \(p\) of the three polynomials.

Since \(q_1\) and \(q_2\) are distinct primes, there are \(a\) and \(b\) such that \(aq_1 + bq_2 = 1\). So \(r = r^{aq_1+bq_2} = (r^{q_1})^a(r^{q_2})^b \equiv 1\) mod \(p\). This implies that 1 is a root of \(\Delta_k\) mod \(p\). However, this is a contradiction since \(\Delta_k(1) = 1 \neq 0\) mod \(p\). Thus there are no primes \(p\) dividing both \(R_{q_1}\) and \(R_{q_2}\), implying they are relatively prime.

It now suffices to show that \(R_q > 1\) for any large primes \(q\). We will show that \(R_q \to \infty\) as prime \(q \to \infty\). If \(G\) is a matrix associated with the isometric structure of \(D_k(K)\) and \(N^q\) is the kernel of \((G^q - (G - I)^q) \otimes 1_{q/2}\) as before, then \(R_q = |\det(G^q - (G - I)^q)|\) since \(R_q = |H_1(M^q)| = |N^q|\). A Seifert matrix for \(D_k(K)\) corresponding to the Seifert surface in Figure 11 is \(A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\). So \(G = (A - A^t)^{-1} A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\). Let \(u = (1 + \sqrt{4k + 1})/2, w = (1 - \sqrt{4k + 1})/2\), and \(P = \begin{pmatrix} u & w \\ -w & u \end{pmatrix}\). Then \(P^{-1}(G^q - (G - I)^q)P = \begin{pmatrix} 0 & 0 \\ 0 & w^q \end{pmatrix}\) and \(P^{-1}(G - I)P = \begin{pmatrix} -w & 0 \\ 0 & -u \end{pmatrix}\). We now see that, for any odd integer \(q\),

\[ P^{-1}(G^q - (G - I)^q)P = (u^q + w^q)I \]
and hence $R_q = (u^q + w^q)^2$. Since $|u| > 1$, $|w/u| < 1$, and
\[
\sqrt{R_q} = |u^q + w^q| \geq |u|^q - |w|^q = |u|^q \left(1 - \frac{|w|^q}{u^q}\right),
\]
$R_q \to \infty$ as prime $q \to \infty$. In particular, $R_q > 1$ for any large primes $q$. This completes the proof. \qed

5.4. Proofs of Theorem 4.2, Theorem 1.2, and Corollary 1.3.

Proof of Theorem 4.2. Let $M = \min_{0 < r < 1} \sigma_r(K)$. For $C_0 = 2|M|$, there is $l_0$ satisfying the conclusion of Lemma 5.3(b). Let $l \geq l_0$.

Suppose to the contrary that $n > 0$ and $nD_{l(l+1)}(K)$ is zero in $\Gamma^+$. Then $nD_{l(l+1)}(K)$ has vanishing Gilmer slice obstruction. Use the same notation $F$, $M^q$, $N$, etc. as before for $D_{l(l+1)}(K)$ and put a subscript $n$ on the objects corresponding to $nD_{l(l+1)}(K)$. Let $Z_n$ be a metabolizer satisfying the conclusion of Theorem 2.1(a) for the surface $F_n$. By Lemma 5.4, $Z_n$ contains an integral vector $v$ that is either $a(\sum_{j=1}^{e} v_j^+ + \sum_{j=e+1}^{n} a_j v_j^+$ or $a(\sum_{j=1}^{e} v_j^-) + \sum_{j=e+1}^{n} a_j v_j^-$. Since $a > 0$ and $e \geq n/2$. By Proposition 5.4 we can find a prime $p$ and an odd prime $q$ such that $p$ divides $|H_1(M^q)|$, $p$ does not divide $a$, and $\tau(nD_{l(l+1)}(K), (N_0^q)_p \cap (Z_n \otimes \mathbb{Q}/\mathbb{Z})$ vanishes, where $(N_0^q)_p$ denotes the $p$-primary component subgroup of $N_0^q$.

Recall that $N^q \cong \ker((G^q - (G-I)^q) \otimes \text{id}_{\mathbb{Q}/\mathbb{Z}})$. From the proof of Proposition 5.3, $G^q - (G-I)^q$ is the identity matrix multiplied by an integer $h = |u^q + w^q| = (l+1)^q - l^q$ when $q$ is odd and $k = l(l+1)$. Thus $N^q = \{z \otimes 1/h \mid z \in H_1(F)\}$. Since $N_n^q$ is the direct sum of $n$ copies of $N^q$, we have $N_n^q = \{z_n \otimes 1/h \mid z_n \in H_1(F_n)\}$. Since the prime $p$ divides $|H_1(M^q)|$, $p$ divides $h$ and hence $z_n \otimes 1/p \in (N_n^q)_p$ for any $z_n \in H_1(F_n)$. In particular, for the vector $v \in Z_n$ chosen above, $v \otimes 1/p \in (N_n^q)_p$. Moreover, since $v \in Z_n$, $v \otimes 1/p \in (N_n^q)_p \cap (Z_n \otimes \mathbb{Q}/\mathbb{Z})$ and hence $\sigma_1 \tau(nD_{l(l+1)}(K), v \otimes 1/p) = 0$.

On the other hand, let $s$ be the constant from Lemma 5.3(b) determined by $l$, $p$, and $q$ chosen above together with $m = m^+$ or $m^-$ depending on whether $v = a(\sum_{j=1}^{e} v_j^+ + \sum_{j=e+1}^{n} a_j v_j^+$ or $a(\sum_{j=1}^{e} v_j^-) + \sum_{j=e+1}^{n} a_j v_j^-$. Since $p$ was chosen not to divide $a$, by multiplying an integer to $v$, we may further assume that $a \equiv s \mod p$. For $i = 0, \ldots, q-1$, $j = e+1, \ldots, n$, let $s_i$, $s_{ij}$ be integers such that $0 \leq s_i, s_{ij} < p$, $s_i \equiv (1 + m^s)a_j \mod p$, and $s_{ij} \equiv (1 + m^s)a_j \mod p$. Since $v \otimes 1/p = \sum_{j=1}^{e} v_j^+ \otimes a_j/p + \sum_{j=e+1}^{n} v_j^+ \otimes a_j/p$, we have
\[
\sigma_1 \tau \left(nD_{l(l+1)}(K), v \otimes \frac{1}{p}\right) = \sigma_1 \tau \left(nD_{l(l+1)}(K), \sum_{j=1}^{e} v_j^+ \otimes \frac{a_j}{p} + \sum_{j=e+1}^{n} v_j^+ \otimes \frac{a_j}{p}\right).
\]

By the additivity of $\sigma_1 \tau$, it is equal to
\[
\sum_{j=1}^{e} \sigma_1 \tau \left(D_{l(l+1)}(K), x^+ \otimes \frac{a_j}{p}\right) + \sum_{j=e+1}^{n} \sigma_1 \tau \left(D_{l(l+1)}(K), x^+ \otimes \frac{a_j}{p}\right),
\]
which is, by Lemma 5.2, equal to
\[
\sum_{j=1}^{e} \left(\sum_{i=0}^{q-1} \sigma_{\frac{s_i}{p}(T_{l_i-1})} + \sum_{i=0}^{q-1} \sigma_{\frac{s_i}{p}(K)}\right) + \sum_{j=e+1}^{n} \left(\sum_{i=0}^{q-1} \sigma_{\frac{s_{ij}}{p}(T_{l_i-1})} + \sum_{i=0}^{q-1} \sigma_{\frac{s_{ij}}{p}(K)}\right).
\]
By Lemma 5.3, we now see that
\[ \sigma_1 \tau \left( nD_l(t+1)(K), v \otimes \frac{1}{p} \right) > eqC_0 + nqM \geq q(2e|\mathcal{M}| + n\mathcal{M}) \geq 0. \]

Thus \( \sigma_1 \tau \left( nD_l(t+1)(K), v \otimes \frac{1}{p} \right) \neq 0 \), which is a contradiction. This proves (a).

Next, suppose that \( l \geq 2 \) and \( \sigma_r(K) \geq 0 \) for all \( r \). Under the same contradiction hypothesis and notation as above except: let \( s = \lfloor p/2 \rfloor \), instead of choosing it from Lemma 5.3(b). Observe that \( 1/l(l + 1) < 1/4 \leq s/p \leq 1/2 \) and hence \( \sigma_{s/p}(T_{l,-l-1}) \geq 2 \) by Lemma 5.3(a). We may assume \( a \equiv s \mod p \) as before. Then we have
\[ \sigma_1 \tau \left( nD_l(t+1)(K), v \otimes \frac{1}{p} \right) = \sigma_{2q}(T_{l,-l-1}) + \text{other terms} \geq \sigma_q(T_{l,-l-1}) \geq 2. \]

This is a contradiction, completing the proof of the first part of (b).

In addition, suppose that \( \sigma_{r_0}(K) > 0 \) for some \( r_0 \). We only need to check the case \( l = 1 \). By the definition of the averaged signature \( \sigma_r \), there are \( r_1 \) and \( r_2 \) such that \( 0 < r_1 < r_2 \leq \frac{p}{2} \) and \( \sigma_r(K) > 0 \) for any \( r \) with \( r_1 < r < r_2 \). We can take \( p \) arbitrary large as above so that \( r_1 < s/p < r_2 \) for some integer \( s \). Apply a similar argument as above. The difference from the previous case is that the role of \( \sigma_{s/p}(T_{l,-l-1}) \) is switched with that of \( \sigma_{s/p}(K) \).

\( \square \)

Proof of Theorem 1.2. The Alexander polynomial of \( D_t(K) \) is \( \Delta_k = -kt^2 + (2k + 1)t - k \). We will see that all \( \Delta_k \) are coprime in \( \mathbb{Q}[t^{\pm 1}] \). Let \( k \neq l \) and let \( g \) be the greatest common divisor of \( \Delta_k \) and \( \Delta_l \) in \( \mathbb{Q}[t^{\pm 1}] \). Then \( g \) divides \( l\Delta_k - k\Delta_l = (l-k)t \) that is a unit in \( \mathbb{Q}[t^{\pm 1}] \). So, \( \Delta_k \) and \( \Delta_l \) are relatively prime in \( \mathbb{Q}[t^{\pm 1}] \) for any distinct pair of integers \( k \) and \( l \).

Let \( \mathcal{L} \) be the set of integers \( l \geq 0 \) for which \( D_l(t+1)(K) \) has finite order in \( \Gamma^+ \). Theorem 1.2 implies \( \mathcal{L} \) contains all but finitely many nonnegative integers. Note that under the hypotheses of (b) \( \mathcal{L} \) contains all but 0, 1 or only 0. Suppose that there are distinct \( l_i \in \mathcal{L} \) such that \( e_1D_{l_1}(t+1)(K) \# e_2D_{l_2}(t+1)(K) \# \cdots \# e_nD_{l_n}(t+1)(K) \) is zero in \( \Gamma^+ \) for some integers \( e_1, \ldots, e_n \). Since \( \Delta_{e_1D_{l_1}(t+1)(K)}(t) = (\Delta_{l_1(t+1)})^{e_1} \), all \( \Delta_{e_iD_{l_i}(t+1)(K)}(t) \) are pairwise coprime. Also, note that all \( e_iD_{l_i}(t+1)(K) \) have nonsingular Seifert forms. Applying Theorem 1.1 inductively, each \( e_iD_{l_i}(K) \) must be zero in \( \Gamma^+ \) and hence each \( e_i \) must be 0. This completes the proof.

\( \square \)

Proof of Corollary 1.3. Part (a) is an immediate consequence of Theorem 1.2(b) since \( \sigma_r(K) = 0 \) for any \( r \) if \( K \) is the unknot. For part (b), the torus knots \( T_{l,-l-1} \) for \( l > 1 \) are such examples with \( \sigma_r \geq 0 \) for all \( r \) and \( \sigma_r > 0 \) for some \( r \).

\( \square \)

6. Non-ribbonness of linear combinations of twisted doubles

In this section we will estimate the Casson–Gordon invariants of all twisted doubles \( D_k(K) \) for double branched covers and prove Theorem 1.3 and Theorem 1.4.

6.1. Estimation of the Casson–Gordon invariants. We estimate the Casson–Gordon invariants of \( D_k(K) \) for \( k \geq 0 \) in this subsection. Recall from Theorem 1.2 that if \( k \geq 0 \), then \( D_k(K) \) has finite order in the algebraic concordance group \( \mathcal{G} \).

A Seifert matrix for \( D_k(K) \) corresponding to the Seifert surface in Figure 11 is \[
\begin{pmatrix}
1 & 0 \\
2 & 1
\end{pmatrix}^t
\begin{pmatrix}
-1 & 1 \\
0 & k
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
2 & 1
\end{pmatrix} =
\begin{pmatrix}
4k + 1 & 2k + 1 \\
k & k
\end{pmatrix}.
\]
Lemma 6.1. We will show since \( D \) runs over all prime power characters.

In this case, \( a = 4k + 1, m = -(2k + 1) \), and \( b = k \) following the notation of Theorem 4.4.

We consider only the case \( q = 2 \). The map \( \varepsilon^2 \) is represented by the matrix

\[
G^2 - (G - I)^2 = \left[ (A - A^t)^{-1} A \right]^2 - \left[ (A - A^t)^{-1} A^t \right]^2 = \begin{pmatrix} -(4k + 1) & -2k \\ 2(4k + 1) & 4k + 1 \end{pmatrix}.
\]

Let \( p \) be a prime dividing \( 4k + 1 \) and let \( s \) be an integer with \( 0 < s < p \). Note that \( p \) is odd and \( \varepsilon^2 \otimes id_{\mathbb{Q}/\mathbb{Z}}(x \otimes s/p) = 0 \) in \( H_1(F) \otimes \mathbb{Q}/\mathbb{Z} \), i.e. \( x \otimes s/p \) is in the kernel of \( \varepsilon^2 \otimes id_{\mathbb{Q}/\mathbb{Z}} \). Note that \( J_x \), a simple closed curve on \( F \) representing \( x = (1, 2) \), can be represented by \( K(T_{2,2k+1}) \), a satellite knot of \( K \) with orbit \( T_{2,2k+1} \) as shown in Figure 3. Note \( \sigma_{s/p}(K(T_{2,2k+1})) = \sigma_{2s/p}(K) + \sigma_{s/p}(T_{2,2k+1}) \) by Theorem 4.3.

By Theorem 4.4 \( x \otimes s/p \) defines a character \( \chi_{s/p} \) and

\[
\sigma_1 \tau(D_k(K), \chi) = 2\sigma_{2s/p}(K) + 2\sigma_{s/p}(T_{2,2k+1}) + \frac{4(p - s)s(4k + 1)}{p^2} - \sigma_{s/p}(D_k(K)).
\]

Since \( D_k(K) \) has finite algebraic order and since \( \sigma_{s/p} \) is additive under connected sums, \( \sigma_{s/p}(D_k(K)) = 0 \). Thus, we have

\[
\sigma_1 \tau(D_k(K), \chi_{s/p}) = 2\sigma_{2s/p}(K) + 2\sigma_{s/p}(T_{2,2k+1}) + 4\left(\frac{s}{p}\right)\left(1 - \frac{s}{p}\right)(4k + 1).
\]

We will show

**Lemma 6.1.** Let \( \mathcal{M} = 2\min_{0 < r < 1} \sigma_r(K) \).

(a) For any \( k \geq 3 \),

\[
\min_{\chi} \sigma_1 \tau(D_k(K), \chi) \geq \mathcal{M} - \frac{4}{4k + 1},
\]

where \( \chi \) runs over all prime power characters.

(b) Let \( s \) be an integer such that \( p = 4s \pm 1 \). Then, for any constant \( C_0 \), there is \( k_0 \geq 1 \) such that, for any \( k \geq k_0 \),

\[
\sigma_1 \tau(D_k(K), \chi_{s/p}) > C_0.
\]

(c) Suppose that \( \mathcal{M} \geq 0 \). If \( k \geq 3 \) then \( \sigma_1 \tau(D_k(K), \chi_r) \) may have only one non-positive value \(-4/(4k + 1)\) at \( r = 1/(4k + 1) \), and if we let \( c = (4k + 1)/p \)

\[
\sigma_1 \tau(D_k(K), \chi_{s/p}) > \begin{cases} cs - 2 & \text{if } p = 4s + 1, \\ cs - c/2 - 2 & \text{if } p = 4s - 1. \end{cases}
\]
Proof. Let \( r = s/p \), where \( 0 < s < p \). Since \( \sigma_1 \tau(D_k(K), \chi_r) = \sigma_1 \tau(D_k(K), \chi_{1-r}) \), it suffices to compute \( \sigma_1 \tau \) for \( \chi_{s/p} \) when \( 1 \leq s \leq (p-1)/2 \). Then \( 1/(4k+1) \leq r \leq 2k/(4k+1) \). For \( 1/(4k+1) \leq r \leq 2k/(4k+1) \), let

\[
f(r) = \frac{1}{2}(\sigma_1 \tau(D_k(K), \chi_r) - 2\sigma_2r(K)) = \sigma_r(T_{2,2k+1}) + 2r(1-r)(4k + 1).
\]

From Proposition 4.6(a), we have \(-2(2k+1)r - 1 \leq \sigma_r(T_{2,2k+1})\). Let

\[
g(r) = -2(2k+1)r - 1 + 2r(1-r)(4k + 1) = -2(4k + 1)r^2 + 4kr - 1.
\]

Then \( g(r) \leq f(r) \). Observe that \( g \) is a quadratic polynomial in \( r \) with maximum at \( r = k/(4k+1) \) and that

\[
g\left(\frac{s}{4k+1}\right) = \begin{cases} 
-1 & \text{if } s = 2k, \\
-3/(4k+1) & \text{if } s = 1 \text{ and } 2k - 1, \\
(4k-9)/(4k+1) & \text{if } s = 2 \text{ and } 2k - 2.
\end{cases}
\]

To prove (a) and the first part of (c), assume that \( k \geq 3 \). Then \( g((2k-2)/(4k+1)) = (4k-9)/(4k+1) > 0 \) and hence \( f(r) \geq g(r) > 0 \) if \( 2/(4k+1) \leq r \leq (2k-2)/(4k+1) \). Now, we will compute \( f(r) \) when \( r = 1/(4k+1), (2k-1)/(4k+1), \) and \( 2k/(4k+1) \). By Proposition 4.6(a),

\[
\sigma_r(T_{2,2k+1}) = \begin{cases} 
-2 & \text{if } r = 1/(4k+1), \\
-2k & \text{if } r = (2k-1)/(4k+1), 2k/(4k+1).
\end{cases}
\]

So,

\[
f(r) = \begin{cases} 
-2/(4k+1) & \text{if } r = 1/(4k+1), \\
2(k-2)/(4k+1) & \text{if } r = (2k-1)/(4k+1), \\
2k/(4k+1) & \text{if } r = 2k/(4k+1).
\end{cases}
\]

Since \( k \geq 3 \), \( f(r) \) can be negative only when \( r = 1/(4k+1) \), and \( f(1/(4k+1)) \) is the minimum. Thus, \( \min_\chi \sigma_1 \tau(D_k(K), \chi) = \min_r (2\sigma_2r(K) + 2f(r)) \geq \mathcal{M} - 4/(4k+1) \). This proves (a). Assuming \( \mathcal{M} \geq 0 \), we have the first statement of (c).

Next, to prove (b) and the second statement of (c), we compute \( f(s/p) \) when \( p = 4s \pm 1 \). Let \( c = (4k+1)/p \). By Proposition 4.6(a) we have

\[
\sigma^\dagger(T_{2,2k+1}) = -2 \left[ \frac{s(2k+1)}{p} + \frac{1}{2} \right] = -2 \left[ \frac{cs + 1}{2} + \frac{s}{2p} \right]
\]

since \( 2k+1 = (cp+1)/2 \). Observe that \( 0 < s/(2p) = s/(8s \pm 2) \leq \frac{1}{8} \) since \( s \geq 1 \) and so \( 8s \pm 2 \geq 6s \). Since \( (cs + 1)/2 \) is either an integer or an integer plus \( \frac{1}{2} \), \( \sigma_{s/p}(T_{2,2k+1}) = -2 \lfloor (cs + 1)/2 \rfloor \). Then

\[
f\left(\frac{s}{p}\right) = -2 \left[ \frac{cs + 1}{2} + \frac{2s}{p} \left( 1 - \frac{s}{p} \right) cp \right] \geq -\left( cs + 1 + \frac{2cs(p-s)}{p} \right) = \frac{cs(p-2s)}{p} - 1.
\]
If \( p = 4s + 1 \), then \( (p-2s)/p = (2s+1)/(4s+1) > \frac{1}{2} \). If \( p = 4s - 1 \), then \( s/p = s/(4s-1) > \frac{1}{4} \). Thus we have

\[
f\left(\frac{s}{p}\right) = \begin{cases} \frac{cs}{2} - 1 & \text{if } p = 4s + 1, \\ \frac{cs}{2} - \frac{c}{4} - 1 & \text{if } p = 4s - 1. \end{cases}
\]

\[ \geq \frac{cs}{4} - 1. \]

Note \( cs/(4k+1) = s/p = s/(4s \pm 1) \geq \frac{1}{5} \) or \( cs \geq (4k+1)/5 \). Thus, if \( k \) is sufficiently large then so is \( cs/4 - 1 \). This completes the proof. \( \Box \)

6.2. Proofs of Theorem 4.3 and Theorem 1.4

Proof of Theorem 4.3. Let \( M = 2\min_{0 < r < 1} \sigma_1(K) \). By Lemma 6.1(b) there is \( k_0 > 0 \) such that \( \sigma_1\tau(D_k(K), \chi_{s/p}) > |M| + 1 \) for any \( k \geq k_0 \). Let \( I = \{ k \in \mathbb{Z} \mid k \leq 0 \} \). We will show that \( I \) is a set satisfying the conclusion of (a).

If \( k < 0 \) then \( D_k(K) \) has infinite order in \( G \) by Theorem 4.1. Thus, for any \( n > 0 \), \( nD_k(K) \) has no metabolizer for the isometric structure and hence has nonvanishing Gilmer ribbon obstruction. From now on, assume \( k \geq k_0 \).

Suppose to the contrary that \( nD_k(K) \) vanishing Gilmer ribbon obstruction for a positive integer \( n \). Then it satisfies the conclusion of Theorem 2.1(b) for \( q = 2 \). Let \( F \) be the Seifert surface for \( D_k(K) \) as depicted in Figure 4. We take \( F_n \) as the boundary connected sum of \( n \) copies of \( F \) so that \( F_n \) is a Seifert surface of \( nD_k(K) \). Then there is a metabolizer \( Z_n \) for the isometric structure on \( H_1(F_n) \) such that \( \tau(nD_k(K), (N_n^2)_p \cap (Z_n \otimes \mathbb{Q}/\mathbb{Z})) \) vanishes for all primes \( p \), where \( (N_n^2)_p \) is the \( p \)-primary component of the kernel, \( N_n^2 \), of \( \varepsilon_n^2 \otimes \text{id}_{\mathbb{Q}/\mathbb{Z}} \).

Since \( \varepsilon_n^2 = (4k+1) \) is the direct sum of \( n \) copies of the map \( \varepsilon^2 : H_1(F) \rightarrow H_1(F) \) corresponding to \( D_k(K) \), \( N_n^2 \) is the direct sum of \( n \) copies of \( N^2 = \ker(\varepsilon^2 \otimes \text{id}_{\mathbb{Q}/\mathbb{Z}}) \). Note that \( \ker(\varepsilon^2 \otimes \text{id}_{\mathbb{Q}/\mathbb{Z}}) \cong (G^2 - (G-I)^2)^{-1}(\mathbb{Z} \oplus \mathbb{Z}) = (-1)^{1-k/(4k+1)}(\mathbb{Z} \oplus \mathbb{Z}) \), where \( G \) is the matrix of the isometric structure on \( H_1(F) \) with respect to the basis \( \{x, y\} \) as defined in subsection 6.1. So \( N^2 \) is generated by an element \( (1/(4k+1), 0) = x \otimes 1/(4k+1) \). Thus every character in \( (N_n^2)_p \) is a direct sum of characters of the form \( x \otimes s/p^e \) and \( (N_n^2)_p \) is isomorphic to \((\mathbb{Z}/p^e)^n\), where \( p^e \) is the maximal power of \( p \) dividing \( 4k+1 \). Gilmer [11, lemma 2] showed that \( |N_n^2| = |N_n^2 \cap (Z_n \otimes \mathbb{Q}/\mathbb{Z})|^2 \) and hence \(|(N_n^2)_p| = |(N_n^2)_p \cap (Z_n \otimes \mathbb{Q}/\mathbb{Z})|^2 \).

Using this and the Gauss-Jordan algorithm, Livingston and Gilmer [12] proved Theorem 1.2 showed that \( (N_n^2)_p \cap (Z_n \otimes \mathbb{Q}/\mathbb{Z}) \) has an element in \((\mathbb{Z}/p^e)^n \cong (N_n^2)_p \) having the first \( n - n_0 \) entries equal to \( p^{e-1} \) and all the remaining \( n_0 \) entries divisible by \( p^{e-1} \) for some \( n_0 \leq n/2 \). Let \( s \) be an integer for which \( p = 4s \pm 1 \). Multiplying by \( s \), we see that \( (N_n^2)_p \cap (Z_n \otimes \mathbb{Q}/\mathbb{Z}) \) has an element \( \chi \) of the form \( (x \otimes s/p, \ldots, x \otimes s/p, x \otimes s_1/p, \ldots, x \otimes s_{n_0}/p) \), where \( s_i \) can be any integers. Thus, \( \sigma_1\tau(nD_k(K), \chi) = 0 \) by the contradiction hypothesis.

On the other hand, by the additivity of \( \sigma_1\tau \), we have

\[
\sigma_1\tau(nD_k(K), \chi) = (n - n_0)\sigma_1\tau(D_k(K), \chi_{s/p}) + \sum_{i=1}^{n_0} \sigma_1\tau(D_k(K), \chi_{s_i/p}).
\]
Observe that $\sigma_1 \tau(D_k(K), \chi_{s/p}) > |M| + 1 > |M| + 4/(4k + 1)$ for any $k \geq k_0$. Now by Lemma 6.1(a),
\[ \sigma_1 \tau(nD_k(K), \chi) > (n - n_0) \left( \frac{|M|}{4k + 1} + \frac{4}{4k + 1} \right) + n_0 \left( \frac{4}{4k + 1} \right) \geq 0 \]
since $n - n_0 \geq n_0$. So $\sigma_1 \tau(nD_k(K), \chi) > 0$. This is a contradiction, proving (a).

Next, assume $M \geq 0$ and $k \geq 3$. Let $\chi$ denote the character as given above again. If $4k + 1$ is a composite number, i.e. $(4k + 1)/p > 1$, then $\sigma_1 \tau(D_k(K), \chi) > 0$ by Lemma 6.1(c) since none of $s/p$ and $s_i/p$ is $1/(4k + 1)$. Now assume $4k + 1 = p$. Then $c = 1$ and $s = k$, where $c$ and $s$ are those in Lemma 6.1(c), and $\sigma_1 \tau(D_k(K), \chi_{s/p}) > k - 2 > 4/(4k + 1)$. We now have
\[ \sigma_1 \tau(nD_k(K), \chi) > (n - n_0) \frac{4}{4k + 1} + n_0 \frac{-4}{4k + 1} \geq 0. \]
This proves the first part of (b).

For $k = 1, 2$, an elementary computation shows:

| $k$ | $r$ | 1 | 2 |
|-----|-----|---|---|
| $\frac{1}{2} \sigma_1 \tau(D_k(K), \chi_r) - \sigma_2 \tau(K)$ | $\frac{1}{3}$ | $\frac{2}{3}$ | $\frac{1}{3}$ | $\frac{2}{3}$ |

So if $\sigma_2 \tau(K) > 0$ for $r = \frac{1}{5}, \frac{1}{9}, \frac{3}{9}$, then $\sigma_1 \tau(D_k(K), \chi_{s/(4k+1)})$ are all positive for $k = 1, 2$. Note that $\sigma_4(K) = \sigma_4(K) = \sigma_4(K)$. This completes the proof. \hfill $\square$

**Proof of Theorem 1.4.** The exact same proof of Theorem 1.2 works here by applying all the counter-parts for the ribbon case: For instance, Corollary 6.2 instead of Theorem 6.1. \hfill $\square$

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