Non-Compact SCFT and Mock Modular Forms

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Dedicated to the memories of Prof. Tohru Eguchi

Abstract

One of interesting issues in two-dimensional superconformal field theories is the existence of anomalous modular transformation properties appearing in some non-compact superconformal models, corresponding to the ‘mock modularity’ in mathematical literature. I review a series of my studies on this issue in collaboration with T. Eguchi, mainly focusing on the papers [10, 18, 22].

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1 Introduction and Summary

Two-dimensional superconformal field theories have been a central subject in the study of string theory for a long time. One of intriguing issues is the existence of anomalous modular transformation properties in some non-compact or non-rational superconformal models. This is called the ‘mock modularity’ in mathematical literature. Namely, non-trivial mixtures of discrete and continuous spectra often emerge under the modular S-transformation, which make it difficult to assure modular invariance in a simple manner. Such an anomalous modular behavior has been first observed for the massless (BPS) characters of $\mathcal{N} = 4$ superconformal algebra (SCA) [1]. Similar behavior appears in the discrete (BPS) characters in the $\mathcal{N} = 2$ Liouville theory or the $SL(2,\mathbb{R})/U(1)$-supercoset model [2, 3] (see also [4, 5]).

The mock modularity is well expressed in terms of the following meromorphic function often called the ‘Appell-Lerch sum’ [6, 7];

$$f^{(k)}(\tau, z) := \sum_{n \in \mathbb{Z}} \frac{q^{k n^2} y^{2 k n}}{1 - y q^n}, \quad (k \in \mathbb{Z}_{>0})$$

(1.1)

which roughly corresponds to the discrete representations of $(\mathcal{N} \geq 2)$ SCA. Its modular property is summarized as follows;

$$f^{(k)}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = \tau e^{2\pi i k} \frac{2}{\sqrt{2k}} \left[ f^{(k)}(\tau, z) - \frac{i}{\sqrt{2k}} \sum_{m \in \mathbb{Z}_{2k}} \int_{\mathbb{R} + i0} dp' \frac{q^{\frac{1}{2} p'^2}}{1 - e^{-2\pi (\sqrt{2k} + i m\pi)}} \Theta_{m,k}(\tau, 2z) \right],$$

(1.2)

$$f^{(k)}(\tau + 1, z) = f^{(k)}(\tau, z).$$

(1.3)

The inhomogeneous term in the R.H.S of (1.2) exhibits the mixing of the discrete and continuous spectra mentioned above. If it would be absent, the function $f^{(k)}(\tau, z)$ (2.37) would get a (weak) Jacobi form\(^1\) of weight 1 and index $k$, showing a much simpler modular behavior. It has been known [6] that one can ‘absorb’ this anomalous modularity into a suitable non-holomorphic correction term. Namely, we can define a function $\hat{f}^{(k)}(\tau, z)$ of the form

$$\hat{f}^{(k)}(\tau, z) \equiv f^{(k)}(\tau, z) + [\text{non-holomorphic correction}],$$

(1.4)

which becomes a non-holomorphic Jacobi form of weight 1 and index $k$. We shall call it the ‘modular completion’\(^2\) of $f^{(k)}(\tau, z)$ and will yield the precise definition of the correction term later.

\(^1\)A basic summary of the (weak) Jacobi form is given in Appdendix A. See [8] for more detail.

\(^2\)In the mathematical language the modular completions correspond to the ‘harmonic Maass forms’ associated to the mock modular forms.
So, what is the physical interpretation of the modular completion? When making the interpretation as characters of SCA, (1.4) corresponds to the next schematic relation
\[
\hat{\chi}_{\mathrm{dis}}(\ast; \tau, z) = \chi_{\mathrm{dis}}(\ast; \tau, z) + \sum \text{[non-hol. coefficient]} \chi_{\mathrm{cont}}(\ast; \tau, z), \quad (1.5)
\]
where \(\chi_{\mathrm{dis}}\) denotes the character of discrete (or degenerate) representation while \(\chi_{\mathrm{cont}}\) does the continuous (or non-degenerate) one. The second term of R.H.S would suggest a kind of mixing of the discrete and continuous spectra, but the precise interpretation seems to be difficult due to the non-holomorphicity of the coefficients.

A good laboratory to study this feature would be the \(SL(2, \mathbb{R})/U(1)\)-supercoset model, which is the supersymmetric extension of the two-dimensional Euclidean black-hole [9]. Among other things, it has been shown in [10] that the modular completions are naturally obtained by evaluating the torus partition function by path-integration. This is remarkable in the sense that it has revealed a physical origin of the modular completion of mock modular form. Indeed, the modular transformations are identified as nothing but the coordinate transformations on the Euclidean torus of world-sheet in any conformal field theories, and thus the good modularity has to be achieved as long as working with the path-integral. On the other hand, the ‘holomorphic factorization’ such as
\[
Z(\tau) \sim \sum_{j, \tilde{j}} N_{j, \tilde{j}} \chi_j(\tau) \overline{\chi_{\tilde{j}}(\tau)},
\]
is violated in this case, although it is naively expected in the ‘Hamiltonian formalism’ or the representation theory of SCA;
\[
Z(\tau) = \text{Tr} \left[ e^{-\beta H} \right].
\]
This incompatibility between the good modularity and holomorphicity would originate from the existence of gapless continuous spectrum of non-BPS states causing the IR-divergence in this system.

We can also derive the modular completions of discrete characters by computing the elliptic genus in a simpler manner, as we will demonstrate in the next section. Then, the elliptic genus gets non-holomorphic although the expected modularity is gained. One may summarize this feature as follows; the Hamiltonian formalism respects the holomorphicity but produces the ‘modular anomaly’, whereas the path-integration leads to the good modularity but yields the ‘holomorphic anomaly’. Closely related studies are also given in [11, 12, 13, 14, 15].

We summarize main contents of this paper as follows;

1. In section 2, we evaluate the torus partition function and elliptic genus of the \(SL(2, \mathbb{R})/U(1)\)-supercoset theory based on the path-integration. This model is a simple example of supercoset
theories with $\mathcal{N} = 2$ superconformal symmetry [16]. Nevertheless, it is fairly non-trivial to work with the partition function or the elliptic genus due to its non-compact nature or the non-rationality, and we face at the holomorphic anomaly noted above. We especially make use of the ‘spectral flow expansion’ in the computation of elliptic genus according to [17, 18], and derive simple formulas of the modular completions, which make their modular behavior manifest.

2. In section 3, as an application of our analyses on the $SL(2, \mathbb{R})/U(1)$-theory, we study the elliptic genus of the ALE-spaces\(^3\) realized as the non-compact Gepner-like orbifolds according to [20, 21];

$$\text{type II/ALE}(G) \cong SU(2)/U(1) \otimes SL(2)/U(1)|_{U(1)\text{-charge} \in \mathbb{Z}},$$

where the type of singularity $G$ for the ALE-space is naturally encoded into the data of modular invariance in the R.H.S. We especially review our work on this subject presented in [22], motivated by the recent progress in studies of new types of moonshine phenomena [23, 24, 25, 26, 27, 28].

We discuss a kind of duality between two $\mathcal{N} = 4$ superconformal systems of different central charges; one is the world-sheet CFT describing the ALE-background (or so-called the CHS-system describing NS5-branes [29]), and the other would be identified as the $\mathcal{N} = 4$ supersymmetric extension of Liouville theory. From the view points of AdS$_3$/CFT$_2$-duality in the NS5-NS1 system studied in [30], the latter should be identified as the ‘space-time CFT’ generically possessing a central charge proportional to the brane charges. Thus, this could offer a novel duality picture of moonshine phenomena; one is that of the world-sheet, and the other is of the space-time.

2 Torus Partition function and Elliptic Genus of $SL(2)/U(1)$-Supercoset Model

In this section, we review the studies on the torus partition function and elliptic genus of the $SL(2)/U(1)$-supercoset model presented mainly in [10, 17, 18], which focus on the physical

\(^3\)The ALE-space is a four dimensional non-compact hyperKähler manifold in which the simple singularity (so-called ‘ADE-type singularity’) is resolved. Needless to say, ALE($A_1$) is identified with the famous Eguchi-Hanson solution of gravitational instanton [19].
2.1 Torus Partition Function

We start with demonstrating the computation of torus partition function of the $SL(2)/U(1)$-supercoset model by performing the path-integration according to [10]. We shall incorporate the general twist angles $z, \bar{z} \in \mathbb{C}$ (in the $\bar{\Psi}$-sector) into the partition function for the reason that will get clear later. See also e.g. [31, 32, 3] for the earlier works in which closely related analyses are presented.

It is familiar that the supercoset theories are described by the SUSY gauged WZW models [33, 34, 35]. We set the level of $SL(2, \mathbb{R})$ super-WZW model to be a real positive number $k$ (the level of bosonic part is $k + 2$), which means that the central charge of this superconformal system is

$$\hat{c} \equiv \frac{c}{3} = 1 + \frac{2}{k}. \tag{2.1}$$

The world-sheet action of relevant SUSY gauged WZW model for $SL(2, \mathbb{R})/U(1)$ in our convention is written as ($\kappa \equiv k + 2$)

$$S(g, A, \psi^\pm, \bar{\psi}^\pm) := \kappa S_{\text{gWZW}}(g, A) + S_\psi(\psi^\pm, \bar{\psi}^\pm, A), \tag{2.2}$$

$$S_{\text{gWZW}}(g, A) := \kappa S_{WZW}^{SL(2, \mathbb{R})}(g) + \frac{\kappa}{\pi} \int d^2v \left\{ A_v \text{Tr} \left( \frac{\sigma_2}{2} \partial_v g g^{-1} \right) \pm \text{Tr} \left( \frac{\sigma_2}{2} g^{-1} \partial_v g \right) A_v \right\}, \tag{2.3}$$

$$S_{WZW}^{SL(2, \mathbb{R})}(g) := -\frac{1}{8\pi} \int \Sigma d^2v \left( \partial_o g^{-1} \partial_o g \right) + \frac{i}{12\pi} \int_B \text{Tr} \left( (g^{-1} dg)^3 \right), \tag{2.4}$$

$$S_\psi(\psi^\pm, \bar{\psi}^\pm, A) := \frac{1}{2\pi} \int d^2v \left\{ \psi^+(\partial_v + A_v)\psi^- + \psi^-(\partial_v - A_v)\psi^+ \right. \right.$$

$$\left. + \bar{\psi}^+(\partial_v \pm A_v)\bar{\psi}^- + \bar{\psi}^-(\partial_v \mp A_v)\bar{\psi}^+ \right\}. \tag{2.5}$$

In (2.3) and (2.5), the $+$ sign/ $-$ sign is chosen for the axial-like/vector-like gauged WZW model, which we shall denote as $S_{\text{gWZW}}^{(A)}/S_{\text{gWZW}}^{(V)}$ ($S_\psi^{(A)}/S_\psi^{(V)}$) from now on. The $U(1)$ chiral gauge transformation is defined by

$$g \mapsto \Omega_L g \Omega_R,$$

$$A_\theta \mapsto A_\theta - \Omega_L^{-1} \partial_\theta \Omega_L, \quad A_v \mapsto A_v - \Omega_R^{-1} \partial_v \Omega_R,$$

$$\psi^\pm \mapsto \Omega_L^{\pm 1} \psi^\pm, \quad \bar{\psi}^\pm \mapsto \Omega_R^{\pm 1} \bar{\psi}^\pm,$$

$$\Omega_L(v, \bar{v}), \Omega_R(v, \bar{v}) \in e^{i2\sigma_2}. \tag{2.6}$$
where we set $\epsilon \equiv +1, -1$ for the axial, vector model, respectively. The gauged WZW action $S_{gWZW}^{(A)}/S_{gWZW}^{(V)}$ is invariant under the axial/vector type gauge transformations that correspond to $\Omega_L = \Omega_R$ in (2.6). Both of the classical fermion actions $S_{\psi}^{(A)}, S_{\psi}^{(V)}$ (2.5) are invariant under general chiral gauge transformations $\Omega_L, \Omega_R$, and we assume the absence of chiral anomalies when $\Omega_L = \Omega_R$ holds (in other words, the anomalies emerge for $\Omega_L = \Omega_R^{-1}$). It is well-known that this model describes the (supersymmetric extension of) string theory on 2D Euclidean black-hole, or the ‘cigar geometry’ [9], and also known to be equivalent (mirror) to the $\mathcal{N} = 2$ Liouville theory [36].

It will be convenient to introduce alternative notations of gauged WZW actions;

\[
S_{gWZW}^{(A)}(g, h_L, h_R) := S_{WZW}^{SL(2,\mathbb{R})}(h_L gh_R) - S_{WZW}^{SL(2,\mathbb{R})}(h_L h_R^{-1}),
\]

\[
S_{gWZW}^{(V)}(g, h_L, h_R) := S_{WZW}^{SL(2,\mathbb{R})}(h_L gh_R) - S_{WZW}^{SL(2,\mathbb{R})}(h_L h_R).\]

They are equivalent with (2.3) under the identification of gauge field;

\[
A_v \frac{\sigma_2}{2} = \partial_v h_L h_R^{-1}, \quad A_v \frac{\sigma_2}{2} = \epsilon \partial_v h_R h_R^{-1},
\]

where we set $\epsilon = +1, (-1)$ for the axial (vector) model as before.

Now, transforming the world-sheet coordinates as $v = e^{iw}, \bar{v} = e^{-i\bar{w}}$, we define the world-sheet torus $\Sigma$ by the identifications $(w, \bar{w}) \sim (w + 2\pi, \bar{w} + 2\pi) \sim (w + 2\pi \tau, \bar{w} + 2\pi \bar{\tau})$ $(\tau \equiv \tau_1 + i\tau_2, \tau_2 > 0)$. Including the ‘angle parameter’ $z$ coupled with the $U(1)_R$-charge in $\mathcal{N} = 2$ SCA, the torus partition function is written as

\[
Z(\tau, z) = \int_{\Sigma} \frac{d^2u}{\tau_2} \int \mathcal{D}[g, A[u], \psi^\pm, \bar{\psi}^\pm] \times \exp \left[ -\kappa S_{gWZW}^{(A)}(g, A[u + \frac{2}{k}z]) - S_{\psi}^{(A)}(\psi^\pm, \bar{\psi}^\pm, A[u + \frac{k + 2}{k}z]) \right],
\]

where $\frac{d^2u}{\tau_2}$ is the modular invariant measure of modulus parameter $u$, and we work in the $\tilde{R}$-sector for world-sheet fermions. We here adopted the notation ‘$A[u]$’ for the gauge field $A \equiv (A_u dw + A_{\bar{w}} d\bar{w}) \frac{\sigma_2}{2}$ to emphasize the modulus dependence, explicitly written as

\[
A[u]_w = \partial_w X + i\partial_w Y - \frac{u}{2\tau_2}, \quad A[u]_{\bar{w}} = \partial_{\bar{w}} X - i\partial_{\bar{w}} Y - \frac{\bar{u}}{2\tau_2}
\]

where $X, Y$ are real scalar fields parameterizing the chiral gauge transformations;

\[
\Omega_L = e^{(X+iY)\frac{\sigma_2}{2}}, \quad \Omega_R \left(\equiv \Omega^\dagger_L\right) = e^{(X-iY)\frac{\sigma_2}{2}}.
\]

Note that the modulus parameter $u$ is normalized so that it correctly couples with the zero-modes of $U(1)$-currents $J^3 \equiv j^3 + \psi^+ \bar{\psi}^-, \bar{J}^3 \equiv \bar{j}^3 + \bar{\psi}^+ \bar{\psi}^-$ which should be gauged (where $j^3$, ...
\( j^3 \) are the bosonic parts). On the other hand, the complex parameter \( z \) precisely couples with the zero-modes of \( \mathcal{N} = 2 \) \( U(1)_R \)-currents \( J, \tilde{J} \) in the Kazama-Suzuki model [16];

\[
J = \psi^+ \psi^- + \frac{2}{k} \tilde{J}^3 \equiv \frac{k+2}{k} \psi^+ \psi^- + \frac{2}{k} j^3, \tag{2.13}
\]

(\( \tilde{J} \) is defined in the same way.)

We can evaluate this path-integration by separating the degrees of freedom of chiral gauge transformations \( X, Y \) according to the standard quantization procedures of gauged WZW models [35, 33, 34]. We have to path-integrate the compact boson \( Y \) explicitly, while the non-compact boson \( X \) is decoupled as a gauge volume. By using the definitions of gauged WZW actions (2.7), (2.8) and a suitable change of integration variables, we obtain

\[
Z(\tau, z) = \int_{\Sigma} \frac{d^2 u}{\tau_2} \int \mathcal{D}[g, Y, \bar{Y}, \psi^+, \psi^-, b, \bar{b}, c, \bar{c}] 
\times \exp \left[ -\kappa S^{(V)}_{gWZW} \left( \frac{e^{iY_2}, h^{u+\frac{2}{k} z}}{2}, h^{u+\frac{2}{k} z} \right) + \kappa S^{(A)}_{gWZW} \left( \frac{e^{iY_2}, h^{u+\frac{2}{k} z}}{2}, h^{u+\frac{2}{k} z} \right) \right] 
\times \exp \left[ -2 \kappa S^{(A)}_{gWZW} \left( \frac{e^{iY_2}, h^{u+\frac{2}{k} z}}{2}, h^{u+\frac{2}{k} z} \right) - S^{(A)}_{\psi} \left( \psi^+ \psi^-, a[u + \frac{k+2}{k} z] \right) \right] 
\times \exp \left[ -S_{\psi}(b, \bar{b}, c, \bar{c}) \right], \tag{2.14}
\]

where the ghost variables have been introduced to rewrite the Jacobian factor. It is most non-trivial to evaluate the path-integration of the compact boson \( Y \). Its world-sheet action is evaluated as

\[
S^{(A)}_{Y}(Y, u, z) \equiv -\kappa S^{(A)}_{gWZW} \left( \frac{e^{iY_2}, h^{u+\frac{2}{k} z}}{2}, h^{u+\frac{2}{k} z} \right) + 2 \kappa S^{(A)}_{gWZW} \left( e^{iY_2}, h^{u+\frac{2}{k} z}, h^{u+\frac{2}{k} z} \right) 
\equiv \frac{k}{\pi} \int_{\Sigma} d^2 u \partial_w \partial_u Y^w \partial_u \bar{Y}^u - \frac{2\pi}{\tau_2} \bar{c} |z|^2, \tag{2.15}
\]

where we introduced the twisted scalar filed \( Y^w \equiv Y + \frac{1}{\tau_2} \text{Im}(w \bar{u}) \) that satisfies the following boundary condition \((u \equiv s_1 \tau + s_2, 0 \leq s_1, s_2 \leq 1)\);

\[
Y^w(w + 2\pi, \bar{w} + 2\pi) = Y^w(w, \bar{w}) - 2\pi (m_1 + s_1), \\
Y^w(w + 2\pi \tau, \bar{w} + 2\pi \bar{\tau}) = Y^w(w, \bar{w}) + 2\pi (m_2 + s_2), \quad (m_1, m_2 \in \mathbb{Z}). \tag{2.16}
\]

Note that the linear couplings between the \( U(1) \)-currents \( i\partial_w Y, i\partial_\bar{w} Y \) and the \( \mathcal{N} = 2 \) moduli \( z, \bar{z} \) are precisely canceled out. This should be the case, since the \( \mathcal{N} = 2 U(1) \)-current \( J \) possesses no contribution from the compact boson \( Y \), which should be gauged away.

Because of the boundary condition (2.16), the zero-mode integral yields the summation over winding sectors weighted by the factor \( e^{-\frac{2\pi}{\tau_2} |m_1 \tau + m_2 + u|^2} \equiv e^{-\frac{2\pi}{\tau_2} (m_1 + s_1) \tau + (m_2 + s_2)^2} \) determined by
the ‘instanton action’. After all, we achieve the next formula of partition function \[10\];

\[
Z(\tau, z) = N e^{\frac{2\pi}{\tau_2} (c|z|^2 - \frac{k+4}{k} \pi^2 z^2)} \sum_{m_1, m_2 \in \mathbb{Z}} \int_{\Sigma} \frac{d^2 u}{\tau_2} \left| \frac{\theta_1 (\tau, u + \frac{k+2}{k} z)}{\theta_1 (\tau, u + \frac{2}{k} z)} \right|^2 e^{-4\pi \frac{w_{\tau \theta}}{\tau_2}} e^{-\frac{2\pi}{\tau^2}|m_1 \tau + m_2 + u|^2} \\

\equiv N e^{\frac{2\pi}{\tau_2} (c|z|^2 - \frac{k+4}{k} \pi^2 z^2)} \int_{\mathbb{C}} \frac{d^2 u}{\tau_2} \left| \frac{\theta_1 (\tau, u + \frac{k+2}{k} z)}{\theta_1 (\tau, u + \frac{2}{k} z)} \right|^2 e^{-4\pi \frac{w_{\tau \theta}}{\tau_2}} e^{-\frac{\pi}{\tau_2}|u|^2},
\]

(2.17)

where \(N\) is a normalization constant. This is identified as the Euclidean cigar model whose asymptotic circle has the radius \(\sqrt{\alpha k}\).

The \(u\)-integration leads to the obvious IR-divergence that originates from the non-compactness of cigar geometry. Therefore, we need to introduce a regularization scheme. We adopt the following regularized partition function \(^4\) \((u \equiv u_1 + iu_2 \equiv s_1 \tau + s_2, \; z \equiv z_1 + iz_2, \; u_1, u_2, s_1, s_2, z_1, z_2 \in \mathbb{R})\)

\[
Z_{\text{reg}}(\tau, z, \bar{z}; \epsilon) = N e^{\frac{2\pi}{\tau_2} (c|z|^2 - \frac{k+4}{k} \pi^2 z^2)} \int_{\mathbb{C}} \frac{d^2 u}{\tau_2} \sigma_\epsilon (\tau, u, z, \bar{z}) \left| \frac{\theta_1 (\tau, u + \frac{k+2}{k} z)}{\theta_1 (\tau, u + \frac{2}{k} z)} \right|^2 e^{-4\pi \frac{w_{\tau \theta}}{\tau_2}} e^{-\frac{\pi}{\tau_2}|u|^2},
\]

(2.18)

where we introduced a regularization factor \(\sigma_\epsilon (\tau, u, z, \bar{z}) \; (\epsilon > 0)\) defined as

\[
\sigma_\epsilon (\tau, u, z, \bar{z}) := \prod_{m_1, m_2 \in \mathbb{Z}} \left[ 1 - e^{-\frac{1}{\tau_2} \left\{ (s_1 + m_1) \tau + (s_2 + m_2) \right\}} \right].
\]

(2.19)

All the singularities of the integrand located at \(u + \frac{2}{k} z \in \mathbb{Z}\tau + \mathbb{Z}\) that originate from the \(\theta_1\)-factor are removed by inserting (2.19) and the \(u\)-integral converges as long as \(\epsilon > 0\). We will later regard \(\sigma_\epsilon (\tau, u, z, \bar{z})\) as a holomorphic function with respect to complex variables \(s_1, s_2\).

Now, the \(u\)-integral in (2.18) is finite as long as \(\epsilon > 0\), and because of the modular invariance of \(\sigma_\epsilon (\tau, u, z, \bar{z})\), \(Z_{\text{reg}}(\tau, z, \bar{z}; \epsilon)\) is defined so as to preserve the expected modular invariance;

\[
Z_{\text{reg}}(\tau + 1, z, \bar{z}; \epsilon) = Z_{\text{reg}}(\tau, z, \bar{z}; \epsilon), \quad Z_{\text{reg}} \left( -\frac{1}{\tau}, \frac{z}{\tau}, \frac{\bar{z}}{\tau}; \epsilon \right) = Z_{\text{reg}}(\tau, z, \bar{z}; \epsilon).
\]

(2.20)

The partition function \(Z_{\text{reg}}(\tau, z, \bar{z}; \epsilon)\) logarithmically diverges in the limit \(\epsilon \to +0\), and the divergent part is identified with the contributions from the strings propagating in the asymptotic cylindrical region of the cigar geometry. It implies that the characteristic behavior around

\(^4\)We shall denote the regularized partition function as ‘\(Z_{\text{reg}}(\tau, z, \bar{z}; \epsilon)\)’ rather than ‘\(Z_{\text{reg}}(\tau, z; \epsilon)\)’ although \(\bar{z}\) is just the complex conjugate of \(z\) in (2.18). This is because we will later treat \(z\) and \(\bar{z}\) as two independent complex variables to derive the elliptic genus, while \(\bar{\tau}\) is always the complex conjugate of \(\tau\) in this section.
\[ \epsilon \to +0 \] is given by
\[
Z_{\text{reg}}(\tau, z, \bar{z}; \epsilon) = C \ln \epsilon \left| e^{2\pi \frac{i}{\tau_2} z^2} \right| \frac{\theta_4(\tau, z)^2}{\eta(\tau)^3} \times \sum_{n, w \in \mathbb{Z}} \int_0^\infty dp q^{\frac{p^2}{4} + \frac{i}{4} \left( \frac{1}{\sqrt{2} \pi} + \sqrt{k} \right)^2} q^{\frac{p^2}{4} + \frac{i}{4} \left( \frac{1}{\sqrt{2} \pi} - \sqrt{k} \right)^2} y^{\frac{a+b}{2}} y^{-\frac{a+b}{2}} + Z_{\text{finite}}(\tau, z, \bar{z}) + O(\epsilon, \epsilon \ln \epsilon), \]
(2.21)

where \( C \) is some positive constant independent of \( \epsilon \). The leading part proportional to the volume factor \( |\ln \epsilon| \) (the ‘asymptotic part’) is expressible in terms of the extended continuous characters [2, 3] as in the free theory when the level \( k \) is rational. We here denote the term of order \( O(\epsilon^0) \) as \( Z_{\text{finite}}(\tau, z, \bar{z}) \), which is still finite after taking the \( \epsilon \to +0 \) limit and is modular invariant. This part can be directly extracted from \( Z_{\text{reg}}(\tau, z, \bar{z}; \epsilon) \) as given in [37];
\[
Z_{\text{finite}}(\tau, z, \bar{z}) = \lim_{\epsilon \to +0} \left[ 1 - \epsilon \ln \epsilon \frac{\partial}{\partial \epsilon} \right] Z_{\text{reg}}(\tau, z, \bar{z}; \epsilon). \]
(2.22)

We emphasize that \( Z_{\text{finite}}(\tau, z, \bar{z}) \) is uniquely determined irrespective of the adopted regularization scheme, even though the overall constant \( C \) in the asymptotic part as well as the correction terms of \( O(\epsilon, \epsilon \ln \epsilon) \) will depend on the method of regularization.

### 2.2 Elliptic Genus

We next consider the elliptic genuses [38] of the \( SL(2, \mathbb{R})/U(1) \)-theory. This is obtained by formally setting \( \bar{z} = 0 \) while fixing \( z \) at a general complex value in \( Z_{\text{finite}}(\tau, z, \bar{z}) \) (2.22). We also need to divide the partition function by the factor
\[
e^{2\pi \frac{i}{\tau_2} (|z|^2 - \bar{z}^2)} \equiv e^{2\pi \frac{i}{\tau_2} (\frac{i}{\tau} + \frac{1}{\tau})^2} \sim e^{\frac{\pi}{\tau_2} \frac{i}{\tau} z^2}, \]
(2.23)
in order to include the anomaly factor correctly. In fact, we can uniquely determine this factor by requiring the following conditions due to the analysis given in [10];

- The elliptic genus should have the correct modular property;
\[
\mathcal{Z}(\tau + 1, z) = \mathcal{Z}(\tau, z), \quad \mathcal{Z} \left( -\frac{1}{\tau}, \frac{z}{\tau} \right) = e^{i\pi \hat{\delta} z^2} \mathcal{Z}(\tau, z), \quad \mathcal{Z}(\tau, z) = \sum \text{ch}_{\text{dia}}(*, 0; \tau, z) + \text{[subleading terms]}, \quad \mathcal{Z}(\tau, z) = \sum \text{ch}_{\text{dia}}(*, 0; \tau, z) + \text{[subleading terms]}, \]
(2.24)

- The elliptic genus should be expanded by the discrete characters (B.2) around \( \tau \sim i\infty \) as
with no extra overall factor, where the ‘subleading terms’ include spectral flow sectors as well as non-holomorphic corrections.

In this way, the elliptic genus is written as

$$Z(\tau, z) = \lim_{\epsilon \to +0} e^{-\frac{\pi i}{\tau_2} \bar{z}^2} Z_{\text{finite}}(\tau, z, \bar{z} = 0)$$

$$\equiv \lim_{\epsilon \to +0} e^{-\frac{\pi i}{\bar{\tau}_2} \bar{z}^2} Z_{\text{reg}}(\tau, z, \bar{z} = 0; \epsilon).$$ \hspace{1cm} (2.26)

The equality of second line is owing to the simple fact that the asymptotic term in (2.21) drops off when setting $\bar{z} = 0$. We thus obtain the path-integral expression of elliptic genus as

$$Z(\tau, z) = \lim_{\epsilon \to +0} ke^{\frac{\pi i}{\tau_2} z^2} \int_{\Sigma} \frac{d^2u}{\tau_2} \sigma_\epsilon(\tau, u, z, 0) \frac{\theta_1(\tau, u + \frac{k+2}{k} z)}{\theta_1(\tau, u + \frac{2}{k} z)} e^{2\pi i z u} e^{-\frac{\pi i}{\tau_2} |\mu|^2}. \hspace{1cm} (2.27)$$

The $u$-integral is not easy to perform since the Gaussian factor $e^{-\frac{\pi i}{\tau_2} |u|^2}$ breaks the periodicity of the integrand. If this factor was absent, the relevant integral would reduce to a simple period integral over a torus $\Sigma \equiv \mathbb{C}/\Lambda$, ($\Lambda \equiv \mathbb{Z} \tau + \mathbb{Z}$). One way to avoid this complication is given by using the following identity $[17, 18]$, which we shall call the ‘spectral flow expansion’;

$$Z(\tau, z) = \sum_{\lambda \equiv n_1 \tau + n_2 \in \Lambda} (-1)^{n_1 + n_2 + n_1 n_2} s_\lambda(\frac{\epsilon}{2}) \cdot Z^{(\infty)}(\tau, z). \hspace{1cm} (2.28)$$

where $s_\lambda^{(\epsilon)}$ denotes the spectral flow operator defined by

$$s_\lambda^{(\epsilon)} \cdot f(\tau, z) := q^{\alpha \omega} y^{2 \alpha \omega} e^{2\pi i \omega \beta} f(\tau, z + \lambda)$$

$$\equiv e^{2\pi i \frac{\omega}{\tau_2} \lambda_2 (\lambda + 2z)} f(\tau, z + \lambda),$$

$$\lambda \equiv \alpha \tau + \beta \equiv \lambda_1 + i \lambda_2, \hspace{0.5cm} \alpha, \beta, \lambda_1, \lambda_2 \in \mathbb{R}. \hspace{1cm} (2.29)$$

Recall that the elliptic genus of a complex $D$-dimensional manifold is a Jacobi form with index $\frac{D}{2}$ $[8]$. Here $\hat{e}$ (2.1) is the effective dimension of a target manifold described by a superconformal field theory with a central charge $c$. Thus the suffix $\frac{\epsilon}{2}$ of the flow operator $s_\lambda^{(\frac{\epsilon}{2})}$ denotes the index of the elliptic genus $Z(\tau, z)$ describing the cigar geometry.

On the other hand, $Z^{(\infty)}(\tau, z)$ is defined as the elliptic genus of ‘$\mathbb{Z}_\infty$-orbifold’ of the cigar model, or equivalently the universal cover of trumpet in the $T$-dual picture. Namely,

$$Z^{(\infty)}(\tau, z) := \lim_{\epsilon \to +0} ke^{\frac{\pi i}{\tau_2} z^2} \int_{\Sigma} \frac{d^2\omega}{\tau_2} \int_{\Sigma} \frac{d^2u}{\tau_2} \sigma_\epsilon(\tau, u, z, 0) \frac{\theta_1(\tau, \mu + \frac{k+2}{k} z)}{\theta_1(\tau, \mu + \frac{2}{k} z)} e^{2\pi i z u} e^{-\frac{\pi i}{\tau_2} |u + \omega|^2}$$

$$= \lim_{\epsilon \to +0} \frac{e^{\frac{\pi i}{\tau_2} z^2}}{ke^{\frac{\pi i}{\tau_2} z^2}} \int_{\Sigma} \frac{d^2u}{\tau_2} \sigma_\epsilon(\tau, u, z, 0) \frac{\theta_1(\tau, u + \frac{k+2}{k} z)}{\theta_1(\tau, u + \frac{2}{k} z)} e^{2\pi i z u}. \hspace{1cm} (2.30)$$

9
In the second line, due to the periodicity of the integrand under \( u \rightarrow u + \nu, \ \nu \in \Lambda \) except for the factor \( e^{-\frac{\pi}{kT_2}|u+\omega|^2} \), we made use of an obvious relation \( \int_{\Sigma} \frac{d^2\omega}{\tau_2} \int_{\mathbb{C}} \frac{d^2u}{\tau_2} [\cdots] = \int_{\mathbb{C}} \frac{d^2u}{\tau_2} \int_{\Sigma} \frac{d^2\omega}{\tau_2} [\cdots] \), and carried out the Gaussian integral over \( \omega \).

Computation is now easy. Since the integrand of (2.30) is holomorphic and periodic with respect to the integration variables \( s_1, s_2 \) with \( u \equiv s_1\tau + s_2 \), one may regard it as a double period integral. Thus, by deforming the integration contour as

\[
s_1 \in [0,1] + i \frac{z}{kT_2}, \quad s_2 \in [0,1] - i \frac{z\tau}{kT_2},
\]

we can directly evaluate (2.30) with the help of the identity (A.5) as

\[
Z^{(\infty)}(\tau, z) = \lim_{\epsilon \to +0} e^{-\frac{\pi z^2}{kT_2}} \int_{\Sigma_{\tau_2}} \frac{d^2u}{\tau_2} \sigma_\epsilon(\tau, u, 0, 0) \frac{\theta_1(\tau, u + z)}{\theta_1(\tau, u)} e^{2\pi iz\frac{\omega}{\tau_2}} = e^{-\frac{\pi z^2}{kT_2}} \int_{\Sigma_{\tau_2}} \frac{d^2u}{\tau_2} \frac{\theta_1(\tau, z)}{\theta_1(\tau, u)} \sum_{n \in \mathbb{Z}} \int_{\Sigma_{\tau_2}} \frac{d^2u}{\tau_2} e^{2\pi inu} e^{2\pi iz\frac{u}{\tau_2}} = e^{-\frac{\pi z^2}{kT_2}} \frac{\theta_1(\tau, z)}{2\pi z\eta(\tau)^3}. \quad (2.32)
\]

In the second line, we have used

\[
\sigma_\epsilon(\tau, u, 0, 0) = \prod_{m_1, m_2 \in \mathbb{Z}} \left[ 1 - e^{-\frac{\pi}{kT_2}|u + m_1\tau + m_2|^2} \right],
\]
as well as the fact that the \( u \)-integral converges even in the absence of \( \sigma \)-factor, contrary to the previous evaluation of torus partition function.

Substituting (2.32) back into the formula of spectral flow expansion (2.28), we finally obtain the following very simple expression of elliptic genus of cigar model;

\[
Z(\tau, z) = \sum_{\lambda \equiv n_1\tau + n_2 \in \Lambda} (-1)^{n_1+n_2+n_1n_2} s^{(\frac{\hat{z}}{\lambda})}_\lambda \left[ e^{-\frac{\pi z^2}{kT_2}} \frac{\theta_1(\tau, z)}{2\pi z\eta(\tau)^3} \right] = \frac{\theta_1(\tau, z)}{2\pi \eta(\tau)^3} \sum_{\lambda \in \Lambda} s^{(\frac{\hat{z}}{\lambda})}_\lambda \left[ e^{-\frac{\pi z^2}{kT_2}} \frac{z^2 + |\lambda|^2 + 2\lambda z}{z} \right] = \frac{\theta_1(\tau, z)}{2\pi \eta(\tau)^3} \sum_{\lambda \in \Lambda} \rho^{(1/k)}(\lambda, z) \equiv \frac{\theta_1(\tau, z)}{2\pi \eta(\tau)^3} \sum_{\lambda \in \Lambda} \rho^{(1/k)}(\lambda, z) \equiv \frac{\theta_1(\tau, z)}{2\pi \eta(\tau)^3} \sum_{\lambda \in \Lambda} \rho^{(1/k)}(\lambda, z). \quad (2.33)
\]

In the last line, we introduced the conventional symbol

\[
\rho^{(\kappa)}(\lambda, z) := s^{(\kappa)}_{\lambda} \cdot e^{-\frac{\pi z^2}{kT_2}} \equiv e^{-\frac{\pi z^2}{kT_2}} |\lambda|^2 + 2\lambda z + z^2]. \quad (2.34)
\]
More explicitly, (2.33) is rewritten as
\[
Z(\tau, z) = \frac{\theta_1(\tau, z)}{2\pi \eta(\tau)^3} \sum_{m, n \in \mathbb{Z}} q^{\frac{1}{2} m^2} y^{\frac{1}{2} m} e^{2\pi i m \tau} e^{-\frac{\pi i}{\eta(\tau)}(z + m\tau + n)^2} z + m\tau + n. \quad (2.35)
\]

Here the double power series of \( \lambda \in \Lambda \) absolutely converges due to the Gaussian factor, and thus (2.33) exhibits the good modular behavior. One can easily confirm that the elliptic genus (2.33) possesses the modular property as a weak Jacobi form with weight 0 and index \( \hat{c}/2 \) given in (2.24), and also
\[
\frac{1}{2} \cdot Z(\tau, z) = (-1)^{n_1 + n_2 + n} Z(\tau, z), \quad (\forall n_1, n_2 \in \mathbb{N} \mathbb{Z}), \quad (2.36)
\]
for the case of \( k = N/K, N, K \in \mathbb{Z}_{>0} \).

### 2.3 Relations to Mock Modular Forms

It is amusing that the elliptic genus of \( SL(2)/U(1) \)-theory is written in terms of the modular completion of the mock modular form, which was first shown in [11, 10]. The relevant mock modular form\(^5\) is called in many literature as the ‘Appell-Lerch sum’ [6, 7], defined by
\[
f_u^{(k)}(\tau, z) := \sum_{n \in \mathbb{Z}} \frac{q^{kn^2} y^{2kn}}{1 - ywq^n}, \quad (q \equiv e^{2\pi i \tau}, \quad y \equiv e^{2\pi iz}, \quad w \equiv e^{2\pi i u}), \quad (2.37)
\]
and its ‘modular completion’ is explicitly given as [6];
\[
\tilde{f}_u^{(k)}(\tau, z) := f_u^{(k)}(\tau, z) - \frac{1}{2} \sum_{m \in \mathbb{Z}_{2k}} R_{m,k}(\tau, u) \Theta_{m,k}(\tau, 2z), \quad (k \in \mathbb{Z}_{>0}), \quad (2.38)
\]
with\(^6\)
\[
R_{m,k}(\tau, u) := \sum_{\nu \in m + 2k \mathbb{Z}} \left[ \text{sgn}(\nu + 0) - \text{Erf} \left\{ \sqrt{\frac{\pi \tau_2}{k}} \left( \nu + 2k \frac{u_2}{\tau_2} \right) \right\} \right] w^{-\nu} q^{-\frac{\nu^2}{4k}}, \quad (\tau_2 \equiv \text{Im } \tau, \quad u_2 \equiv \text{Im } u). \quad (2.39)
\]

Note that \( R_{m,k} \) is non-holomorphic because of the inclusions of \( \tau_2 \) and \( u_2 \). The holomorphic function \( f_u^{(k)}(\tau, z) \) shows a complicated modular transformation law, which is often called the ‘mock modularity’. In the context in physics this function with \( u = 0 \) is closely related to the character formulas of massless representations of \( \mathcal{N} = 4 \) superconformal algebra (SCA) [1]

---

\(^5\)See e.g. [39] for the precise definitions and mathematically rigid terminologies for mock modular forms.

\(^6\)See Appendix A for the convention of error function Erf(\(x\)), theta function \( \Theta_{m,k}(\tau, z) \) and Jacobi forms.
as well as the ‘extended characters’ (spectral flow sums) of $\mathcal{N} = 2$ SCA [4, 2]. The modular property of $f^{(k)}(\tau, z) \equiv f^{(k)}_{u=0}(\tau, z)$ is summarized in (1.2) and (1.3).

It is crucial that the anomalous modular transformation law of the ‘holomorphic part’ $f^{(k)}_{u}(\tau, z)$ is compensated by the second term in the R.H.S. of (2.38), and the ‘completed’ one $\hat{f}^{(k)}(\tau, z)$ behaves as a Jacobi form of weight 1 and index $k$. This means that the elliptic genus of $SL(2)/U(1)$-supercoset is described by a non-holomorphic generalization of a Jacobi form.

Let us clarify the precise relation between our computation of elliptic genus $Z(\tau, z)$ (2.33) (or (2.35)) and the modular completion $\hat{f}^{(k)}(\tau, z)$ of $f^{(k)}_{u}(\tau, z)$. In the case of $k = N/K$, $(N, K \in \mathbb{Z}_{>0})$, the formula (2.33) is rewritten as

$$Z(\tau, z) = \frac{\theta_{1}(\tau, z)}{2\pi \eta(\tau)^{3}} \sum_{a, b \in \mathbb{Z}_{N}} s^{(N/K)}_{a+b} \cdot \hat{f}^{(N/K)}(\tau, \frac{z}{N}).$$

This expression has been originally derived in [10]. Equating this formula with (2.33) in the special case of $N = 1, K = 1/k \in \mathbb{Z}_{>0}$, one can rewrite $\hat{f}^{(k)}_{u}(\tau, z)$ in a compact form;

$$\hat{f}^{(K)}(\tau, z) = \frac{i}{2\pi} \sum_{\nu \in \Lambda} \frac{\rho^{(K)}(\nu, z)}{z + \nu} = \frac{i}{2\pi} \sum_{m, n \in \mathbb{Z}} q^{Km^2} e^{-\frac{\pi K(z + m\tau + n)^2}{2}} y^{2Km} e^{-\tau\frac{z + m\tau + n}{2}}.$$

This is a fairly non-trivial identity and a direct proof of it has been presented in Appendix C of [18] with the ‘$u$-parameter’ included.

We also point out a nice relation [10, 17];

$$Z(\tau, z) = \sum_{v, a \in \mathbb{Z}_{N}} \sum_{v + Ka \in \mathbb{Z}_{N}} \hat{\chi}_{\text{dis}}^{(N,K)}(v, a; \tau, z) = \sum_{v, a \in \mathbb{Z}_{N}} \sum_{m \in \mathbb{Z}} \hat{\text{ch}}_{\text{dis}}^{(N,K)}(v, m; \tau, z),$$

where $\hat{\chi}_{\text{dis}}^{(N,K)}$, $\hat{\text{ch}}_{\text{dis}}^{(N,K)}$ denote the modular completions of the discrete extended and irreducible characters (B.4), (B.1) of $\mathcal{N} = 2$ SCA with $\hat{c} = 1 + \frac{2}{k} \equiv 1 + \frac{2K}{N}$.

In fact, this would be expected from the viewpoints of representation theory of $\mathcal{N} = 2$ SCA in the manner similar to the $SU(2)/U(1)$ Kazama-Suzuki model and the $\mathcal{N} = 2$ minimal characters [40]. However, it should be emphasized that $Z(\tau, z)$ is expanded by the modular completions, not by the characters themselves. We then likewise obtain

$$\hat{\chi}_{\text{dis}}^{(v, a; \tau, z)} = \frac{\theta_{1}(\tau, z)}{2\pi \eta(\tau)^{3}} \sum_{b \in \mathbb{Z}_{N}} \sum_{\lambda \in \mathbb{Z}_{\tau + b + N\Lambda}} e^{-2\pi i \frac{z}{N}(v + Ka)} \cdot \frac{\rho^{(1/2)}(\lambda, z)}{z + \lambda},$$

$$\hat{\text{ch}}_{\text{dis}}^{(\nu, m; \tau, z)} = \frac{\theta_{1}(\tau, z)}{2\pi \eta(\tau)^{3}} \sum_{n \in \mathbb{Z}} e^{-2\pi i \frac{z}{N}(v + m)} \cdot \frac{\rho^{(1/2)}(m\tau + n, z)}{z + m\tau + n}.$$
In the case of \( k = N \in \mathbb{Z}_{>0} \ (\hat{c} = 1 + \frac{2}{N} \), in other words), we also introduce the function \( \hat{F}^{(N)}(v, a; \tau, z) \) defined by

\[
\hat{\chi}_{\text{dis}}(v, a; \tau, z) \equiv \frac{\theta_1(\tau, z)}{i\eta(\tau)^3} \hat{F}^{(N)}(v, a; \tau, z),
\]

(2.45)

for the convenience of arguments in the next section. This function is explicitly written as

\[
\hat{F}^{(N)}(v, a; \tau, z) = \sum_{n \in a + \mathbb{Z}} \frac{(yq^n)^{\frac{N}{2}}}{1 - yq^n} y^{2n} q^{\frac{n^2}{N}} - \frac{1}{2} \sum_{j \in \mathbb{Z}_2} R_{v+Nj,N}(\tau, 0) \Theta_{v+Nj+2aN} \left( \frac{2z}{N} \right) \equiv \frac{i}{2\pi} \sum_{m \in a+NZ} \sum_{n \in \mathbb{Z}} e^{-2\pi i \frac{v}{N}(v+a)} \rho(m\tau + n, z). (2.46)
\]

The second line is readily derived from (2.43), and again manifestly exhibits the good modular property of it.

Some comments are in order:

1. In the above analysis we assumed the axial-like gauging, in other words, the cigar model. So, what happens if instead taking the vector-like gauging? As was examined in [17], in the case of \( k = N/K \) (\( N \) and \( K \) are coprime positive integers), the vector-like \( SL(2)/U(1) \)-theory is identified with the \( \mathbb{Z}_N \)-orbifold of cigar model. The elliptic genus of the vector-like model is computed as

\[
Z_{\text{vector}}(\tau, z) = \frac{\theta_1(\tau, z)}{i\eta(\tau)^3} \hat{f}^{(NK)} \left( \frac{2z}{N} \right) = \sum_{v \in \mathbb{Z}_N} \hat{\chi}_{\text{dis}}(v, 0; \tau, z).
\]

(2.47)

The expression in the first line has been originally derived in [11] for the case of \( K = 1 \).

2. In addition to the elliptic genera, the ‘finite part’ of torus partition function \( Z_{\text{finite}}(\tau, z, \bar{z}) \) (2.22) is also expressible in terms of the modular completions of extended characters \( \hat{\chi}_{\text{dis}} \). For instance, in the cigar model we obtain [10]

\[
Z_{\text{finite}}(\tau, z, \bar{z}) = e^{-2\pi \frac{\hat{c}}{2} z^2} \sum_{v=0}^{N-1} \sum_{aL,aR \in \mathbb{Z}_N \atop v + K(aL + aR) \in N\mathbb{Z}} \hat{\chi}_{\text{dis}}(v, a_L; \tau, z) \overline{\hat{\chi}_{\text{dis}}(v, a_R; \tau, \bar{z})}. (2.48)
\]

Again the modular completions \( \hat{\chi}_{\text{dis}}(v, a; \tau, z) \) play the similar role to those for the \( \mathcal{N} = 2 \) minimal characters in the \( SU(2)/U(1) \) Kazama-Suzuki model.
3. What is the role of the ‘$u$-parameter’ of the function (2.38) in the $SL(2)/U(1)$-gauged WZW model? It is actually identified with the continuous twist parameter of the general spin structure of the world-sheet fermions. More precisely, setting $u \equiv \alpha \tau + \beta$, ($\forall \alpha, \beta \in \mathbb{R}$), we can extend the analyses given above to those with the world-sheet fermions of $\psi^\pm(w)$, $\tilde{\psi}^\pm(\bar{w})$ satisfy the twisted boundary condition (with respect to the cylinder coordinate $w$, $\bar{w}$);

$$
\begin{align*}
\psi^\pm(w + 2\pi) &= e^{\mp 2\pi i \alpha} \psi^\pm(w), & \psi^\pm(w + 2\pi \tau) &= e^{\mp 2\pi i \beta} \psi(w), \\
\tilde{\psi}^\pm(\bar{w} + 2\pi) &= e^{\pm 2\pi i \alpha} \tilde{\psi}^\pm(\bar{w}), & \tilde{\psi}^\pm(\bar{w} + 2\pi \bar{\tau}) &= e^{\pm 2\pi i \beta} \tilde{\psi}^\pm(\bar{w}).
\end{align*}
$$

(2.49)

The calculations are almost parallel though including some technical complications, and the twist parameter $u$ eventually turn out to be identified with the ‘$u$-parameter’ of the function $\hat{u}^\ast(\tau, z)$ (2.38). See [18] for the detailed arguments. Closely related studies including such a twisting based on a different approach have been given in [12, 15].

3 Applications to Gepner-like Orbifolds and Moonshine Phenomena

In this section we review the main studies given in [22].

3.1 Gepner-like Orbifolds describing ALE-spaces

As an interesting application of our previous analyses on $SL(2)/U(1)$, let us consider the non-compact extensions of Gepner models [41] describing the Calabi-Yau compactifications, initiated by [20, 21]. Especially, the type II string theory defined on ALE space of the type $G (\equiv A_m, D_m, E_m)$ is described by the superconformal system expressed schematically as [20]

$$
type \text{II}/\text{ALE}(G) \cong SU(2)/U(1) \otimes SL(2)/U(1)|_{U(1)-\text{charge} \in \mathbb{Z}},
$$

where $SU(2)/U(1)$, $SL(2)/U(1)$ respectively denotes the Kazama-Suzuki supercoset theories and ‘$U(1)$-charge $\in \mathbb{Z}$’ means the orbifolding with respect to the total $U(1)_R$-charge measured
by the total $U(1)_R$-current $J^{\text{tot}}(z) \equiv J^{SU(2)/U(1)}(z) + J^{SL(2)/U(1)}(z)$. Note that the superconformal
symmetry gets enhanced to $\mathcal{N} = 4$, since the total central charge is equal $\hat{c} = 2$ [42]. It is quite
interesting that the type of blown-up singularity of ALE-space, denoted by $G$, is naturally
encoded into the type of modular invariance of affine $SU(2)$ [43, 44].

We shall now focus on the simplest case of $A_{N-1}$, in which the relevant superconformal
system is expressed as

$$\text{type II/ALE}(A_{N-1}) \cong \left. \frac{SU(2)}{U(1)} \right|_{\mathbb{Z}_N\text{-orbifold}} \otimes \left. \frac{SL(2)}{U(1)} \right|_{\mathbb{Z}_N\text{-orbifold}},$$

with the total central charge

$$c = \frac{3(N-2)}{N} + \frac{3(N+2)}{N} = 6,$$

($\hat{c} = 2$, in other words). Here the orbifolding for the total $U(1)$-charge reduces to the $\mathbb{Z}_N$-
orbitolding and effectively represented in terms of the integral spectral flows, similarly to [42] for the compact Gepner models.

The elliptic genus of $SU(2)/U(1)$-sector, or equivalently, the $\mathcal{N} = 2$-minimal model is given
by the well-known formula [40];

$$\mathcal{Z}^{(\text{min})}(\tau, z) = \sum_{\ell=0}^{N-2} \text{ch}^{(\tilde{R})}_{\ell, \ell+1}(\tau, z) \equiv \frac{\theta_1(\tau, \frac{N-1}{N} z)}{\theta_1(z)},$$

where $\text{ch}^{(\tilde{R})}_{\ell, m}(\tau, z)$ denotes the character of $\mathcal{N} = 2$ minimal model with level $N-2$ ($\hat{c} = 1 - \frac{2}{N}$)
in the $\tilde{R}$-sector, of which expression is given in Appendix A. To construct the Gepner-like $\mathbb{Z}_N$-
orbitolding we also need the ‘spectrally flowed’ elliptic genus which is again expanded in terms of the minimal characters;

$$\mathcal{Z}^{(\text{min})}_{[a, b]}(\tau, z) := (-1)^{a+b+ab} s^{(\frac{N-2}{2N})}_{a \tau + b} \cdot \mathcal{Z}^{(\text{min})}(\tau, z)$$

$$\equiv \sum_{\ell=0}^{N-2} e^{2\pi i \frac{b}{N}(\ell+1-a)} \text{ch}^{(\tilde{R})}_{\ell, \ell+1-2a}(\tau, z) \quad (^7 a, b \in \mathbb{Z}_N).$$

For the $SL(2)/U(1)$-sector, on the other hand, we here adopt the vector-like model$^7$, and rewrite the elliptic genus (2.47) as $\mathcal{Z}^{SL(2)/U(1)}(\tau, z)$, that is,

$$\mathcal{Z}^{SL(2)/U(1)}(\tau, z) = \frac{\theta_1(\tau, z)}{i \eta(\tau)^3} \tilde{f}(N) \left( \tau, \frac{z}{N} \right)$$

$$= \sum_{v \in \mathbb{Z}_N} \hat{\chi}^{(\tilde{R})}_{\text{dis}}(v, 0; \tau, z).$$

$^7$When instead taking the axial-like model (2.40), the arguments given below are almost parallel.
The spectrally flowed one is defined in the manner similar to (3.3);

\[ Z_{SL(2)/U(1)}^{\text{flowed}}(\tau, z) := (-1)^{a+b+ab} \frac{N^2}{a^2+b^2} \cdot Z_{SL(2)/U(1)}(\tau, z) \]

\[ \equiv \sum_{v=0}^{N-1} e^{2\pi i \frac{b}{N} (v+a)} \chi^{(R)}(N-a, v, a, \tau, z) \quad (\forall a, b \in \mathbb{Z}_N) . \]  \hspace{1cm} (3.5)

The elliptic genus of the ALE space of \( A_{N-1} \)-type is now written as

\[ Z_{\text{ALE}(A_{N-1})}(\tau, z) = \frac{1}{N} \sum_{a, b \in \mathbb{Z}_N} Z_{\text{min}}^{(a, b)}(\tau, z) Z_{SL(2)/U(1)}^{(a, b)}(\tau, z) \]

\[ \equiv \sum_{r=1}^{N-1} \sum_{a \in \mathbb{Z}_N} \text{ch}_{r-1, r-2a}^{(R)}(\tau, z) \hat{\chi}_{\text{dis}}^{(R)}(N, a, \tau, z) \]

\[ \equiv -\frac{\theta_1(\tau, z)}{\eta(\tau)^3} \sum_{r=1}^{N-1} \sum_{a \in \mathbb{Z}_N} \text{ch}_{r-1, r-2a}^{(R)}(\tau, z) \hat{F}^{(N)}(r, a, \tau, -z) . \]  \hspace{1cm} (3.6)

In the third line we introduced the function \( \hat{F}^{(N)}(v, a; \tau, z) \) defined in (2.45), and made use of

\[ \hat{F}^{(N)}(v, a; \tau, z) = -\hat{F}^{(N)}(-v, -a; \tau, -z), \quad (\forall v, a \in \mathbb{Z}_N) . \]  \hspace{1cm} (3.7)

It is well-known that the elliptic genus of K3-surface is given as [42, 45]

\[ Z_{K3}(\tau, z) = 8 \left[ \left( \frac{\theta_3(\tau, z)}{\theta_3(\tau, 0)} \right)^2 + \left( \frac{\theta_4(\tau, z)}{\theta_4(\tau, 0)} \right)^2 + \left( \frac{\theta_2(\tau, z)}{\theta_2(\tau, 0)} \right)^2 \right] \equiv 2\phi_{0,1}(\tau, z) . \]  \hspace{1cm} (3.8)

Here, \( \phi_{0,1}(\tau, z) \) denotes the standard notation of (holomorphic) weak Jacobi form of weight 0, index 1, which is known to be unique up to normalization. Since the ALE-space would be regarded as the ‘non-compact K3-surface’, it is natural to express (3.6) in the form as

\[ Z_{\text{ALE}(A_{N-1})}(\tau, z) = \alpha \phi_{0,1}(\tau, z) + \frac{\theta_1(\tau, z)^2}{\eta(\tau)^6} \hat{H}^{(N)}(\tau), \]  \hspace{1cm} (3.9)

where \( \hat{H}^{(N)}(\tau) \) is a non-holomorphic modular form of weight 2. In fact, one can confirm by straightforward calculations that the R.H.S of (3.6) behaves as a non-holomorphic weak Jacobi form of weight 0 and index 1. Because of the uniqueness of the holomorphic Jacobi form mentioned above, one can readily determine the coefficient \( \alpha \) by evaluating the Witten index of (3.6);

\[ Z_{\text{ALE}(A_{N-1})}(\tau, 0) = N - 1, \]  \hspace{1cm} (3.10)

leading to \( \alpha = \frac{N - 1}{12} \) (see e.g. [3]).
On the other hand, it is much more difficult to determine the ‘non-holomorphic part’ \( \hat{H}^{(N)}(\tau) \). By using the identity
\[
\sum_{m \in \mathbb{Z}_{2N}} \text{ch}_{\ell,m}(\tau,z) \Theta_{m,N} \left( \tau, -\frac{2z}{N} \right) = -\frac{\theta_1(\tau,z)}{i\pi} \oint_{w=0} \frac{dw}{w} \frac{\Theta^{[-1]}_{\ell+1,N}(\tau,2w)}{\theta_1(\tau,2w)},
\]
which is essentially the branching relation of \( SU(2)/U(1) \) Kazama-Suzuki supercoset (A.16), one can derive a simple formula
\[
\hat{H}^{(N)}(\tau) = \frac{\eta(\tau)^3}{i\pi} \oint_{w=0} \frac{dw}{w} \frac{\hat{f}^{(N)}(\tau,w)}{i\theta_1(\tau,2w)} e^{(N-2)G_2(\tau)w^2}.
\]
where \( G_2(\tau) \) is the (unnormalized) 2nd Eisenstein series (A.7). See [22] for the detailed computation. To summarize, we have obtained
\[
Z_{\text{ALE}(A_{N-1})}(\tau,z) = \frac{N-1}{12} \phi_{0,1}(\tau,z) + \frac{\theta_1(\tau,z)^2}{\eta(\tau)^6} \hat{H}^{(N)}(\tau)
\equiv \frac{N-1}{12} \phi_{0,1}(\tau,z) + \frac{\theta_1(\tau,z)^2}{\eta(\tau)^3} \frac{1}{i\pi} \oint_{w=0} \frac{dw}{w} \frac{\hat{f}^{(N)}(\tau,w)}{i\theta_1(\tau,2w)} e^{(N-2)G_2(\tau)w^2}.
\]

Now, what can we say about the non-holomorphic function \( \hat{H}^{(N)}(\tau) \)? Physically, it may express the spectrum of \( \mathcal{N} = 4 \) massive representations that measures the ‘deviation’ from the compact K3-background. Mathematically, it is indeed a (completed) mock modular form of weight 2. Substituting the formula (2.41) into (3.13), we obtain the next expression [22];
\[
\hat{H}^{(N)}(\tau) = \frac{1}{4\pi^2} \left[ N\hat{G}_2(\tau) + \frac{\partial}{\partial w} \sum_{\lambda \in \Lambda'} e^{-\frac{\pi}{T_2} N\left\{ |\lambda|^2 + 2\lambda w + w^2 \right\}} \right]_{w=0}
= \frac{1}{4\pi^2} \left[ N\hat{G}_2(\tau) - \sum_{\lambda \in \Lambda'} \frac{e^{-\frac{\pi}{T_2} N|\lambda|^2}}{\lambda^2} \left\{ 1 + \frac{2\pi N}{T_2} |\lambda|^2 \right\} \right],
\]
where we denoted \( \Lambda' \equiv \Lambda - \{0\} \) and set \( \hat{G}_2(\tau) \equiv G_2(\tau) - \frac{\pi}{T_2} \) (the ‘completion of \( G_2(\tau) \)’ (A.9)). Its holomorphic part \( H^{(N)}(\tau) \) would be obtained by formally taking the limit \( \tau \to -i\infty \) while keeping \( \tau \) finite in the expression of (3.14). Thus, we guess
\[
H^{(N)}(\tau) \sim \frac{N}{4\pi^2} \hat{G}_2(\tau) + \frac{1}{4\pi^2} \frac{\partial}{\partial w} \sum_{\lambda=m \tau + n \in \Lambda'} q^{Nm^2} e^{2\pi i (2N) mw} \frac{e^{2\pi i (2N) mw}}{\lambda + w} \right|_{w=0}.
\]
However, the double series appearing in (3.15) does not converge, and thus we have to be more careful. To this end, we introduce the symbol of the ‘principal value’;
\[
\sum_{n \neq 0}^P a_n := \lim_{N \to \infty} \sum_{n=1}^N (a_n + a_{-n}), \quad \sum_{n \in \mathbb{Z}}^P a_n := a_0 + \sum_{n \neq 0}^P a_n.
\]
and the correct expression of $H^{(N)}(\tau)$ should be

$$H^{(N)}(\tau) = \frac{N}{4\pi^2} G_2(\tau) + \frac{1}{4\pi^2} \frac{\partial}{\partial w} \left[ \sum_{m\neq 0} \sum_{n \in \mathbb{Z}} P \frac{q^{Nm^2} e^{2\pi i (2N) mw}}{\lambda + w} + \sum_{n \neq 0} \frac{1}{w + n} \right] \bigg|_{w=0}$$

$$\equiv \frac{N-1}{12} - 2N \sum_{n=1}^{\infty} \frac{q^n}{1-q^n} + 2 \sum_{m=1}^{\infty} q^{Nm^2} \left[ \frac{q^m}{(1-q^m)^2} + Nm \frac{1+q^m}{1-q^m} \right]. \tag{3.17}$$

To derive the second line, we substituted (A.7) as well as familiar identities

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \equiv \zeta(2) = \frac{\pi^2}{6}, \quad \frac{i}{2\pi} \sum_{n \in \mathbb{Z}} P \frac{1}{z+n} = \frac{1}{2} + \frac{y}{1-y} = -\frac{1}{2} + \frac{1}{1-y}, \quad (y \equiv e^{2\pi i z}).$$

Curiously, this function $H^{(N)}(\tau)$ plays a crucial role in the ‘umbral moonshine’ phenomena [24, 25, 26, 27, 28] as was first suggested in [26]. More precise statement will be presented in the next subsection. It should be also equated to the ‘second helicity supertrace’ (supersymmetric index) counting the space-time BPS states computed in [27, 28] based on a different string-theoretical construction. Indeed, one finds the ‘number theoretical’ formula for this index;

$$\chi^{(k,d)}_2(\tau) = \left( \frac{k}{d} - d \right) E_2(\tau) - 24 F^{(k,d)}_2(\tau), \tag{3.18}$$

$$F^{(k,d)}_2(\tau) = \left( d \sum_{kr^s,ks^r>0} \frac{-k}{d} \sum_{d^2r^s,ks>0} s q^{r^s} \right), \tag{3.19}$$

on page 12 of [28] (derived in ref.[57] of [28], more precisely), where $E_2(\tau)$ denotes the normalized 2nd Eisenstein series;

$$E_2(\tau) \left( \equiv \frac{3}{\pi^2} G_2(\tau) \right) = 1 - 24 \sum_{n=1}^{\infty} \frac{q^n}{1-q^n}.$$ 

The precise relation between $H^{(N)}(\tau)$ and the index $\chi^{(k,c)}_2(\tau)$ is written as

$$H^{(N)}(\tau) = \frac{1}{12} \chi^{(N,1)}_2(\tau). \tag{3.20}$$

This identity has been proved in [22] (Appendix D or ‘Addendum’) by directly evaluating the both sides of (3.20).
3.2 ‘Duality’ in $\mathcal{N} = 4$ Liouville Theory and Umbral Moonshine

As is familiar, the type II string theory on an ALE space discussed in the previous subsection is also interpreted as the NS5-brane system by T-duality [20]. Let us focus on the ALE($A_{N-1}$)-case, or the stack of $N$-NS5 branes equivalently, for simplicity. In this picture the world-sheet CFT is identified as the ‘CHS-system’ [29], which is described by a non-compact boson $\phi$ with the linear dilaton charge $Q_\phi = \sqrt{\frac{2}{N}}$, $SU(2)$-WZW model with level $N - 2$, and four Majorana fermions at least in a free field realization. This is an $\mathcal{N} = 4$ superconformal system of level 1 ($c = 3\hat{c} = 6$) as noticed in the previous subsection.

It is interesting that we have another $\mathcal{N} = 4$ theory described by the same field contents, but with different linear dilaton charge $Q_\phi \equiv -(N - 1)\sqrt{\frac{2}{N}}$. This second $\mathcal{N} = 4$ system has the total central charge

$$c (\equiv 3\hat{c}) = (1 + 3Q_\phi^2) + \frac{3(N - 2)}{N} + 4 \cdot \frac{1}{2} = 6(N - 1).$$

In other words, the relevant $\mathcal{N} = 4$ SCA has the level $N - 1$ ($N - 2$ comes from $SU(2)$-WZW, while the 4 fermions add 1 to the level).

Actually, the same field content without the linear dilaton lead to a simple free field realization of the large $\mathcal{N} = 4$ superconformal algebra, and the modification by including the linear dilaton term turns out to induce the small $\mathcal{N} = 4$ SCA in the cases of only two different values of linear dilaton; one has $\hat{c} = 2$, and another has $\hat{c} = 2(N - 1)$, as mentioned above. See [22] for the detailed computation. These types of reductions from the large $\mathcal{N} = 4$ to the small $\mathcal{N} = 4$ theories have been already found in [46, 47], and also potentially utilized in [48] in order to construct the Feigin-Fuchs representation of $\mathcal{N} = 4$ SCFT. Thus, one may naturally define the ‘$\mathcal{N} = 4$ Liouville theory’ with $\hat{c} = 2(N - 1)$ by including a suitable Liouville potential, which should be constructed in the manner similar to the screening charges presented in [48].

In the context of string theory on the NS5-NS1 background, the first one ($\hat{c} = 2$) is identified with the world-sheet CFT for the ‘short string’ sector (or that describing the ‘Coulomb branch tube’), while the second one ($\hat{c} = 2(N - 1)$) corresponds to the ‘long string’ sector (or the ‘Higgs branch tube’) [30]. They are expected to be dual to each other from the viewpoints of AdS$_3$/CFT$_2$-duality.

Although the possible Liouville potentials for the $\mathcal{N} = 4$ Liouville theory have very complicated forms, we can evaluate its elliptic genus due to the invariance under marginal deforma-
tions. In [22], we have shown that the elliptic genus is given as

\[
\mathcal{Z}_{N=4 \text{ Liouville}}(\tau, z) = (-1)^{N-1} \widehat{\text{ch}}_0^{(R)}(N - 1, 0; \tau, z)
\equiv \frac{2\theta_1(\tau, z)^2}{i\eta(\tau)^2}\widehat{f}^{(N)}(\tau, z),
\] (3.21)

where \(\widehat{\text{ch}}_0^{(R)}(N - 1, 0; \tau, z)\) denotes the modular completion of \(N = 4\) massless character of level \(N - 1\), isospin 0 in the \(\bar{R}\)-sector\(^8\).

We can also rewrite (3.21) as

\[
\mathcal{Z}_{N=4 \text{ Liouville}}(\tau, z) = \frac{\theta_1(\tau, z)}{i\eta(\tau)^2} \sum_{r=1}^{N-1} \sum_{a \in \mathbb{Z}_N} \text{ch}^{(R)}_{r-1,r+2a}(\tau, z) \widehat{F}^{(N)}(r, a; \tau, (N - 1)z),
\] (3.22)

by using the identity

\[
2\frac{\theta_1(\tau, z)}{\theta_1(\tau, 2z)} \widehat{F}^{(N)}(\tau, z) = \sum_{\ell=0}^{N-2} \sum_{a \in \mathbb{Z}_N} \text{ch}^{(R)}_{\ell+1 \ell+2a}(\tau, z) \widehat{F}^{(N)}(\ell + 1, a; \tau, (N - 1)z),
\] (3.23)

which was proved in [22]. By comparing (3.22) with (3.6), we can observe a very simple ‘duality correspondence’;

\[
\widehat{F}^{(N)}(v, a; \tau, -z) \text{ for ALE}(A_{N-1}) \quad \longleftrightarrow \quad \widehat{F}^{(N)}(v, a; \tau, (N - 1)z) \text{ for } N = 4 \text{ Liouville of level } N - 1.
\] (3.24)

Another useful realization of the duality is given as

\[
\mathcal{Z}_{\text{ALE}(A_{N-1})}(\tau, z) = \frac{N - 1}{12} \phi_{0,1}(\tau, z) + \frac{\theta_1(\tau, z)^2}{2\pi i} \int_{w=0}^1 \frac{dw}{w} \frac{e^{(N-2)G_2(\tau)w^2}}{\theta_1(\tau, w)^2} \mathcal{Z}_{N=4 \text{ Liouville}}(\tau, w)
\equiv \frac{N - 1}{12} \phi_{0,1}(\tau, z) + \frac{\theta_1(\tau, z)^2}{\eta(\tau)^6} \frac{1}{8\pi^3 i} \int_{w=0}^1 \frac{dw}{w} \frac{e^{(N-1)G_2(\tau)w^2}}{\sigma(\tau, w)^2} \mathcal{Z}_{N=4 \text{ Liouville}}(\tau, w),
\] (3.25)

where we introduced the Weierstrass \(\sigma\)-function (A.6) in the second line. This is just derived from the identity (3.13).

Now, let us make some comments on the relation with the analyses on the ‘umbral moonshine’ [24, 25, 26, 27, 28]. In [26], the authors studied (the holomorphic part of) the extension\(^8\)

---

\(^8\)This is actually the unique modular completion of \(\mathcal{N} = 4\) massless characters since they are independent of the value of isospin \(\ell\), as was discussed in [18].
of (3.6) with general modular coefficients determined by the simply-laced root system \( X \) corresponding to each Niemeier lattice. We have rank \( X = 24 \) by definition, and let \( N \) be the Coxeter number of \( X \). A Niemeier lattice is explicitly expressed as

\[
X = \prod_i X_i, \quad \sum_i \text{rank } X_i = 24, \tag{3.26}
\]

where each \( X_i \) is the irreducible component of root system possessing the common Coxeter number \( N \).

We then define

\[
Z^{[c=2]}_X(\tau, z) := \sum_i Z_{\text{ALE}(X_i)}(\tau, z)
\equiv -\frac{\theta_1(\tau, z)}{\eta(\tau)^3} \sum_{r,s=1}^{N-1} \sum_{a \in \mathbb{Z}_N} N_{r,s}^{X_i} \text{ch} \left( \tilde{\mathcal{R}} \right)_{r,s} \tilde{F}^{(N)}(s, a; \tau, -z) \tag{3.27}
\]

where we set

\[
N_{r,s}^{X} \equiv \sum_i N_{r,s}^{X_i}, \tag{3.28}
\]

and \( N_{r,s}^{X_i} \) denotes the modular invariant coefficients of \( SU(2)_{N-2} \) associated to the simply-laced root system \( X_i \) [43, 44]. One may identify \( Z_{\text{ALE}(X_i)}(\tau, z) \) as the elliptic genus of the ALE space associated to the simple singularity of the type \( X_i \). In [26] it was suggested that the root system \( X = \prod_i X_i \) should be identified as the geometrical data of various K3-singularities.

Since we assume rank \( X \equiv \sum_i \text{rank } X_i = 24 \), we can rewrite (3.27) as

\[
Z^{[c=2]}_X(\tau, z) = 2 \phi_{0,1}(\tau, z) - \frac{\theta_1(\tau, z)^2}{\eta(\tau)^3} \tilde{h}^X(\tau). \tag{3.29}
\]

Here the non-holomorphic function \( \tilde{h}^X(\tau) \) is the completion of mock modular form of weight 1/2 which can be evaluated similarly to \( \tilde{H}^{(N)}(\tau) \) given in (3.13). Then, the umbral moonshine (the version of [26]) claims that the ‘umbral group’\(^9\) \( G_X \) should act on the holomorphic part \( h_X(\tau) \) of \( \tilde{h}^X(\tau) \).

For example, let us take \( X = A_1^{24} \). Then, by comparing the holomorphic parts of both sides, we obtain from (3.29)

\[
Z_{K3}(\tau, z) = 24 \text{ch}_0^{\tilde{R}} (k = 1, 0; \tau, z) + \frac{\theta_1(\tau, z)^2}{\eta(\tau)^3} h^X(\tau), \tag{3.30}
\]

\(^9\)The umbral group is defined as the symmetry group of the Niemeier lattice labeled by \( X \) modulo the Weyl group associated to the root system \( X \) [24, 25].
where \( \text{ch}_0^R(k = 1, 0; \tau, z) \) denotes the \( \mathcal{N} = 4 \) massless character of level \( k = 1 \) and isospin \( \ell \), \( \ell = 0 \). Notice that \( \mathcal{Z}_{k3}(\tau, z) = 2\phi_{0,1}(\tau, z) \), \( \mathcal{Z}^{[\ell=2]}_X(\tau, z) = 24\mathcal{Z}_{\text{ALE}(A_4)}(\tau, z) \) hold, and we have the identity

\[
[\text{hol. part of } \mathcal{Z}_{\text{ALE}(A_4)}(\tau, z)] = \text{ch}_0^R(k = 1, 0; \tau, z),
\]

as shown in [3]. Then, the umbral group \( G_X \) is no other than the Mathieu group \( M_{24} \), and we obtain

\[
h^X(\tau) \equiv -24 \frac{H^{(2)}(\tau)}{\eta(\tau)^3}
= 2q^{-\frac{1}{2}} \left[ -1 + 45q + 231q^2 + 770q^3 + 2277q^4 + 5796q^5 + \cdots \right].
\]

(3.32)

All the numerical coefficients of \( q^{n-\frac{1}{2}} \) \( (n \geq 1) \) in (3.32) are known to be strictly equal the dimensions of some (reducible, in general) representations of \( M_{24} \). This remarkable fact is no other than the ‘Mathieu moonshine’ first discovered by [23].

Let us next consider the type \( X \) generalization of (3.22), which is related to (3.27) via the duality correspondence like (3.24);

\[
\mathcal{Z}^{[\ell=2(N-1)]}_X(\tau, z) := \frac{\theta_1(\tau, z)}{i\eta(\tau)^3} \sum_{r,s=1}^{N-1} \mathcal{N}_{r,s}^X \text{ch}_0^R(\tau, z) \hat{F}^{(N)}(s, a; \tau, (N - 1)z).
\]

(3.33)

Similarly to (3.29), the R.H.S of (3.33) can be decomposed as

\[
\mathcal{Z}^{[\ell=2(N-1)]}_X(\tau, z) = \Phi^{X}_{0,N-1}(\tau, z) - \frac{\theta_1(\tau, z)^2}{\eta(\tau)^3} \sum_{r=1}^{N-1} \hat{h}_r^X(\tau) \chi^{(N-2)}_{r-1}(\tau, 2z)
\equiv \Phi^{X}_{0,N-1}(\tau, z) - \frac{2\theta_1(\tau, z)^2}{i\eta(\tau)^3\theta_1(\tau, 2z)} \sum_{r=1}^{N-1} \hat{h}_r^X(\tau) \Theta^{[-]}_{r,N}(\tau, 2z).
\]

(3.34)

In the above expression \( \Phi^{X}_{0,N-1}(\tau, z) \) is a weak Jacobi form of weight 0, index \( N - 1 \), which is holomorphic with respect to \( \tau \), but generically meromorphic with respect to \( z \). \( \chi^{(k)}_r(\tau, z) \) is the affine \( SU(2) \) character of level \( k \), isospin \( \ell/2 \), and \( \hat{h}_r^X(\tau) \) are the completions of vector valued mock modular forms of weight 1/2. The function \( \hat{h}_r^X(\tau) \) are uniquely determined by imposing the ‘optimal growth condition’ given in [25];

\[
\lim_{\tau \to \infty} q^{\frac{r}{2}N} \left| \hat{h}_r^X(\tau) \right| < \infty, \quad (\forall r = 1, \ldots, N - 1),
\]

(3.35)

and the umbral group \( G_X \) acts on its holomorphic part \( h^X_\tau(\tau) \) [25].

We here note the duality relation which is the natural extension of (3.25);

\[
\mathcal{Z}^{[\ell=2]}_X(\tau, z) = 2\phi_{0,1}(\tau, z) + \frac{\theta_1(\tau, z)^2}{\eta(\tau)^6} \int_{w=0}^{1} \frac{dw}{w} \frac{e^{(N-1)G_2(\tau)w^2}}{\sigma(\tau, w)^2} \mathcal{Z}^{[\ell=2(N-1)]}_X(\tau, w).
\]

(3.36)
Now, substituting the decompositions (3.29), (3.34) into the formula (3.36), we find

\[ \hat{h}^X(\tau) = \sum_{r=1}^{N-1} \hat{h}_r^X(\tau) \chi_{r-1}^{(N-2)}(\tau,0) \equiv \frac{1}{\eta(\tau)^3} \sum_{r=1}^{N-1} \hat{h}_r^X(\tau) S_{r,N}(\tau), \] (3.37)

where we introduced the ‘unary theta function’ [6]

\[ S_{r,N}(\tau) := \frac{1}{2\pi i} \partial_z \Theta_{r,N}(\tau,2z) \bigg|_{z=0} \equiv \sum_{n \in r+2NZ} nq^{\frac{n^2}{4}}, \] (3.38)

In fact, the contour integral \( \oint dw \frac{e^{(N-1)G_2(\tau)w^2}}{w^{\sigma(\tau,w)^2}} \Phi_{0,N-1}^X(\tau,w) \) has to be a holomorphic modular form of weight 2, and thus vanishes. This is the duality relation between the expansion coefficients of massive representations of \( Z_{\hat{c}=2}^X(\tau,z) \) and \( Z_{\hat{c}=2(N-1)}^X(\tau,z) \). In the case of Mathieu moonshine \( (N=2, X = A_2^{24}) \) one has the self-dual situation \( \hat{h}^{A_2^{24}}(\tau) = \hat{h}^{A_2^{24}}_{r=1}(\tau) \). In general holomorphic parts of \( \hat{h}_r^X(\tau) \) should reproduce mock modular form of umbral moonshine on which the umbral group \( G_X \) should act [24].

In this section, we have discussed how the umbral moonshine can be reproduced from the two \( \mathcal{N} = 4 \) superconformal systems that are related by the duality correspondence. This could provide a novel duality picture of moonshine phenomena based on superstring theory, or the AdS_3/CFT_2-correspondence; one is that of the world-sheet and the other is of the space-time.

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Appendix A: Notations and Useful Formulas

In Appendix A we summarize the notations adopted in this paper and related useful formulas. We assume $\tau \equiv \tau_1 + i\tau_2$, $\tau_2 > 0$ and set $q := e^{2\pi i\tau}$, $y := e^{2\pi i z}$.

**Theta functions:**

\[\theta_1(\tau, z) = i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n-1/2)^2/2} y^{n-1/2} \equiv 2 \sin(\pi y) q^{1/8} \prod_{m=1}^{\infty} (1 - q^m)(1 - yq^m)(1 - y^{-1} q^m),\]
\[\theta_2(\tau, z) = \sum_{n=-\infty}^{\infty} q^{(n-1/2)^2/2} y^{n-1/2} \equiv 2 \cos(\pi y) q^{1/8} \prod_{m=1}^{\infty} (1 - q^m)(1 + yq^m)(1 + y^{-1} q^m),\]
\[\theta_3(\tau, z) = \sum_{n=-\infty}^{\infty} q^{n^2/2} y^n \equiv \prod_{m=1}^{\infty} (1 - q^m)(1 + yq^{m-1/2})(1 + y^{-1} q^{m-1/2}),\]
\[\theta_4(\tau, z) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2/2} y^n \equiv \prod_{m=1}^{\infty} (1 - q^m)(1 - yq^{m-1/2})(1 - y^{-1} q^{m-1/2}).\]  

(A.1)

\[\Theta_{m,k}(\tau, z) = \sum_{n=-\infty}^{\infty} q^{k(n+m/2)^2} y^{k(n+m/2)}.\]  

(A.2)

We use abbreviations; $\theta_1(\tau) \equiv \theta_1(\tau, 0)$ ($\theta_1(\tau) \equiv 0$), $\Theta_{m,k}(\tau) \equiv \Theta_{m,k}(\tau, 0)$. We also set

\[\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).\]  

(A.3)

The spectral flow properties of theta functions are summarized as follows ($m, n \in \mathbb{Z}$):

\[\theta_1(\tau, z + m\tau + n) = (-1)^{m+n} q^{-m^2/2} y^{-m} \theta_1(\tau, z),\]
\[\theta_2(\tau, z + m\tau + n) = (-1)^n q^{-m^2/2} y^{-m} \theta_2(\tau, z),\]
\[\theta_3(\tau, z + m\tau + n) = q^{-m^2/2} y^{-m} \theta_3(\tau, z),\]
\[\theta_4(\tau, z + m\tau + n) = (-1)^m q^{-m^2/2} y^{-m} \theta_4(\tau, z),\]
\[\Theta_{a,k}(\tau, 2(z + m\tau + n)) = e^{2\pi ina} q^{-km^2} y^{-2km} \Theta_{a+2km,k}(\tau, 2z).\]  

(A.4)

The next identity is useful for our calculations;

\[
\frac{\theta_1(\tau, u + z)}{\theta_1(\tau, u)} = \frac{\theta_1(\tau, z)}{i\eta(\tau)^3} \sum_{n \in \mathbb{Z}} \frac{w^n}{1 - yq^n},
\]
\[
\left( y \equiv e^{2\pi i z}, w \equiv e^{2\pi i u}, \tau_2 > 0, 0 < \frac{u_2}{\tau_2} < 1 \right) .
\]  

(A.5)
We introduce the Weierstrass $\sigma$-function;

$$
\sigma(\tau, z) := e^{\frac{1}{2}G_2(\tau)z^2} \frac{\theta_1(\tau, z)}{2\pi \eta(\tau)^3}
\equiv z \prod_{\omega \in \Lambda'} \left( 1 - \frac{z}{\omega} \right) e^{\frac{\omega z}{2} + \frac{1}{2}(\omega^2 z^2)}.
\quad (\Lambda' \equiv \Lambda - \{0\}) \quad \text{(A.6)}
$$

where $G_2(\tau)$ is the (unnormalized) second Eisenstein series;

$$
G_2(\tau) := \sum_{n \in \mathbb{Z} - \{0\}} \frac{1}{n^2} + \sum_{m \in \mathbb{Z} - \{0\}} \sum_{n \in \mathbb{Z}} \frac{1}{(m\tau + n)^2}
\equiv \frac{\pi^2}{3} \left[ 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n} \right],
\quad (A.7)
$$

It is useful to note the anomalous $S$-transformation formula of $G_2(\tau)$;

$$
G_2 \left( -\frac{1}{\tau} \right) = \tau^2 G_2(\tau) - 2\pi i \tau.
\quad \text{(A.8)}
$$

We also set

$$
\widehat{G}_2(\tau) := G_2(\tau) - \frac{\pi}{\tau_2},
\quad \text{(A.9)}
$$

which is a non-holomorphic modular form of weight 2.

### Spectral flow operator

(see also [8])

$$
s_{\lambda}^{(k)} \cdot f(\tau, z) := e^{2\pi i \frac{1}{\tau_2} \lambda z(\lambda + 2z)} f(\tau, z + \lambda)
\equiv q^{k^2} y^{2k} e^{2\pi i \alpha \beta} f(\tau, z + \alpha \tau + \beta),
\quad (\lambda \equiv \alpha \tau + \beta, \ \forall \alpha, \beta \in \mathbb{R}).
\quad \text{(A.10)}
$$

An important property of the spectral flow operator $s_{\lambda}^{(k)}$ is the modular covariance, which precisely means the following:

Assume that $f(\tau, z)$ is an arbitrary function with the modular property;

$$
f(\tau + 1, z) = f(\tau, z), \quad f \left( -\frac{1}{\tau}, \frac{z}{\tau} \right) = e^{2\pi i \frac{1}{\tau^2} z^2} \tau^\alpha f(\tau, z),
$$

then, we obtain for $\forall \lambda \in \mathbb{C}$

$$
s_{\lambda}^{(k)} \cdot f(\tau + 1, z) = s_{\lambda}^{(k)} \cdot f(\tau, z), \quad s_{\lambda}^{(k)} \cdot f \left( -\frac{1}{\tau}, \frac{z}{\tau} \right) = e^{2\pi i \frac{1}{\tau^2} z^2} \tau^\alpha s_{\lambda}^{(k)} \cdot f(\tau, z).
$$
The next ‘product formula’ is also useful;

\[ s^{(\kappa)\lambda} \cdot s^{(\kappa)\lambda'} = e^{-2\pi i \frac{\kappa}{\tau_2} \text{Im}(\lambda \lambda')} s^{(\kappa)\lambda} = e^{-4\pi i \frac{\kappa}{\tau_2} \text{Im}(\lambda \lambda')} s^{(\kappa)\lambda} \cdot s^{(\kappa)\lambda}, \quad (A.11) \]

in other words,

\[ s^{(\kappa)\alpha' + \beta} \cdot s^{(\kappa)\alpha' + \beta'} = e^{-2\pi i \kappa (\alpha' - \alpha') (\beta + \beta')} s^{(\kappa)\alpha' + \beta} = e^{-4\pi i \kappa (\alpha' - \alpha') \text{Im}(\lambda \lambda')} s^{(\kappa)\alpha' + \beta}. \]

We should note that the spectral flow operators do not commute with each other in general.

**Error function** :

\[ \text{Erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt, \quad (x \in \mathbb{R}) \quad (A.12) \]

The next identity is elementary but useful;

\[ \text{sgn}(\nu + 0) - \text{Erf}(\nu) = \frac{1}{i\pi} \int_{\mathbb{R} - i\nu} dp \frac{e^{-2\nu^2}}{p - i\nu}, \quad (\nu \in \mathbb{R}), \quad (A.13) \]

**weak Jacobi forms** :

The weak Jacobi form [8] for the full modular group \( \Gamma(1) = SL(2, \mathbb{Z}) \) with weight \( k(\in \mathbb{Z} \geq 0) \) and index \( r(\in \frac{1}{2}\mathbb{Z} \geq 0) \) is defined by the conditions

(i) modularity :

\[ \Phi \left( \frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) = e^{2\pi i \nu (c^{-2}a + d)} (c\tau + d)^k \Phi(\tau, z), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1). \quad (A.14) \]

(ii) double quasi-periodicity :

\[ \Phi(\tau, z + m\tau + n) = (-1)^{2r(m+n)} q^{-rm^2} y^{-2rn} \Phi(\tau, z), \quad (\forall m, n \in \mathbb{Z}). \quad (A.15) \]

In this paper, we shall use this terminology in a broader sense. We allow a half integral index \( r \), and more crucially, allow non-holomorphic dependence on \( \tau \), while we keep the
holomorphicity with respect to $z$ $^{10}$.

**Character Formulas for $\mathcal{N} = 2$ Minimal Model:**

The character formulas of the level $k \mathcal{N} = 2$ minimal model ($\hat{c} = k/(k + 2)$) $^{49, 50}$ are described as the branching functions of the Kazama-Suzuki coset $^{16}$ $SU(2)_k \times U(1)_2/U(1)_{k+2}$ defined by

$$
\chi^{(k)}_{\ell}(\tau, w) \Theta_{s,2}(\tau, w - z) = \sum_{m \in \mathbb{Z}_{2(k+2)}} \chi^\ell_{m}(\tau, z) \Theta_{m, k+2}(\tau, w - 2z/(k + 2)) ,
$$

$$
\chi^\ell_{m}(\tau, z) \equiv 0, \quad \text{for } \ell + m + s \in 2\mathbb{Z} + 1 , \quad (A.16)
$$

where $\chi^{(k)}_{\ell}(\tau, z)$ is the spin $\ell/2$ character of $SU(2)_k$;

$$
\chi^{(k)}_{\ell}(\tau, z) = \frac{\Theta_{\ell+1,k+2}(\tau, z) - \Theta_{-\ell-1,k+2}(\tau, z)}{\Theta_{1,2}(\tau, z) - \Theta_{-1,2}(\tau, z)} \equiv \sum_{m \in \mathbb{Z}_k} c^{(k)}_{\ell,m}(\tau) \Theta_{m,k}(\tau, z) . \quad (A.17)
$$

The branching function $\chi^\ell_{m}(\tau, z)$ is explicitly calculated as follows;

$$
\chi^\ell_{m}(\tau, z) = \sum_{r \in \mathbb{Z}_k} c^{(k)}_{\ell,m-s+4r}(\tau) \Theta_{2m+(k+2)\ell-4r,2k(k+2)}(\tau, z/(k + 2)) . \quad (A.18)
$$

Then, the character formulas of unitary representations are written as

$$
\begin{align*}
\text{ch}^{(\text{NS})}_{\ell,m}(\tau, z) &= \chi^\ell_{m}(\tau, z) + \chi^\ell_{m}(\tau, z), \\
\text{ch}^{(\text{NS})}_{\ell,m}(\tau, z) &= \chi^\ell_{m}(\tau, z) - \chi^\ell_{m}(\tau, z), \\
\text{ch}^{(R)}_{\ell,m}(\tau, z) &= \chi^\ell_{m}(\tau, z) + \chi^\ell_{m}(\tau, z), \\
\text{ch}^{(R)}_{\ell,m}(\tau, z) &= \chi^\ell_{m}(\tau, z) - \chi^\ell_{m}(\tau, z). \quad (A.19)
\end{align*}
$$

$^{10}$According to the original terminology of $^{8}$, the ‘weak Jacobi form’ of weight $k$ and index $r$ ($k, r \in \mathbb{Z}_{\geq 0}$) means that $\Phi(\tau, z)$ should be Fourier expanded as

$$
\Phi(\tau, z) = \sum_{n \in \mathbb{Z}_{\geq 0}} \sum_{\ell \in \mathbb{Z}} c(n, \ell) q^n y^\ell,
$$

in addition to the conditions (A.14) and (A.15). It is called the ‘Jacobi form’ if it further satisfies the condition: $c(n, \ell) = 0$ for $n, \ell$ s.t. $4nr - \ell^2 < 0$. 

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Appendix B: Summary of Modular Completions with General Spin Structures

In Appendix B, we summarize the definitions of modular completions of the irreducible and extended characters of $\mathcal{N} = 2$ SCA given in [10, 17]. We again assume $\tau \equiv \tau_1 + i\tau_2$, $\tau_2 > 0$ and set $q \equiv e^{2\pi i\tau}$, $y \equiv e^{2\pi iz}$, $w \equiv e^{2\pi i\alpha \tau + \beta}$.

**Modular Completions of Irreducible Characters:**

\[
\hat{\text{ch}}_{\text{dis}}^{(\mathcal{R})}(\lambda, n; \tau, z) := \frac{\theta_1(\tau, z)}{2\pi \eta(\tau)^3} \sum_{\nu \in \lambda + k\mathbb{Z}} \left\{ \int_{\mathbb{R} + i(k-\nu)} dp - \int_{\mathbb{R} - i\nu} dp \right\} \frac{e^{-\pi \tau_2 \nu^2} y^{\frac{\nu}{k}} q^{\frac{\nu^2}{k}}}{1 - y^{q^n}}
\]

\[
\equiv \hat{\text{ch}}_{\text{dis}}^{(\mathcal{R})}(\lambda, n; \tau, z) + \frac{\theta_1(\tau, z)}{2\pi \eta(\tau)^3} \sum_{\nu \in \lambda + k\mathbb{Z}} \int_{\mathbb{R} - i\nu} dp \frac{e^{-\pi \tau_2 \nu^2} y^{\frac{\nu}{k}} q^{\frac{\nu^2}{k}}}{p - i\nu} (y^{q^n})^{\frac{\nu}{k}} y^{\frac{2\nu}{k}} q^{\frac{\nu^2}{k}}
\]

\[
\equiv (-1)^n s_{n\tau}^{(\hat{\lambda})} \cdot \hat{\text{ch}}_{\text{dis}}^{(\mathcal{R})}(\lambda, 0; \tau, z), (\lambda \in \mathbb{R}, n \in \mathbb{Z}), \quad (B.1)
\]

and the irreducible character is defined by [49]

\[
\text{ch}_{\text{dis}}^{(\mathcal{R})}(\lambda, n; \tau, z) := \frac{\theta_1(\tau, z)}{i\eta(\tau)^3} \frac{(y^{q^n})^{\frac{\lambda}{k}}}{1 - y^{q^n}} y^{\frac{2\lambda}{k}} q^{\frac{\lambda^2}{k}}
\]

\[
\equiv (-1)^n s_{n\tau}^{(\hat{\lambda})} \cdot \text{ch}_{\text{dis}}^{(\mathcal{R})}(\lambda, 0; \tau, z), \quad (0 \leq \lambda \leq k, n \in \mathbb{Z}). \quad (B.2)
\]

Here $\text{ch}_{\text{dis}}^{(\mathcal{R})}(\lambda, n; \tau, z)$ denotes the (modular completion) of the character associated to the $n$-th spectral flow of discrete irrep. generated by the Ramond vacua;

\[
h = \frac{\hat{c}}{8}, \quad Q = \frac{\lambda}{k} - \frac{1}{2} \quad (0 \leq \lambda \leq k). \quad (B.3)
\]

Note that the modular completion $\hat{\text{ch}}_{\text{dis}}^{(\mathcal{R})}(\lambda, n)$ has the periodicity under $\lambda \rightarrow \lambda + k$, which is obvious from the definition (B.2), while $\text{ch}_{\text{dis}}^{(\mathcal{R})}(\lambda, n)$ does not.
Modular Completions of Extended Characters:

We assume \( k = N/K \), \((N, K \in \mathbb{Z}_{>0})\), or equivalently, \( \hat{c} = 1 + \frac{2K}{N} \).

\[
\hat{\chi}^{(R)}_{\text{dis}}(v, a; \tau, z) := \sum_{n \in a + NZ} \hat{\text{ch}}_{\text{dis}} \left( \frac{v}{K}, n; \tau, z \right)
\equiv \frac{\theta_1(\tau, z)}{2\pi \eta(\tau)^3} \sum_{r \in v + NZ} \sum_{n \in a + NZ} \left\{ \int_{\mathbb{R} + i(N-0)} dp - \int_{\mathbb{R} - i0} dp \ yq^n \right\} \frac{e^{-\pi r^2/NK}}{p - ir} \frac{(yq^n)^\frac{a}{2}}{1 - yq^n} y^{\frac{2n}{K}} q^{\frac{n^2}{K}},
\]

\[
\chi^{(R)}_{\text{dis}}(v, a; \tau, z) := \sum_{n \in a + NZ} \text{ch}_{\text{dis}} \left( \frac{v}{K}, n; \tau, z \right)
\equiv \frac{\theta_1(\tau, z)}{2\pi \eta(\tau)^3} \sum_{n \in a + NZ} \left( \int_{\mathbb{R} + i(N-0)} dp - \int_{\mathbb{R} - i0} dp \ yq^n \right) \frac{e^{-\pi r^2/NK}}{p - ir} \frac{(yq^n)^\frac{a}{2}}{1 - yq^n} y^{\frac{2n}{K}} q^{\frac{n^2}{K}}, \quad (a \in \mathbb{Z}_N, \ 0 \leq v \leq N - 1). \tag{B.5}
\]

We note that \( \hat{\chi}^{(R)}_{\text{dis}}(v, a; \tau, z) \) is the extended discrete character introduced in [2, 3]. Again the modular completion \( \chi^{(R)}_{\text{dis}}(v, a) \) is periodic under \( v \to v + N \), while \( \chi^{(R)}_{\text{dis}}(v, a) \) is not.

The modular and spectral flow properties of \( \hat{\text{ch}}_{\text{dis}}, \hat{\chi}_{\text{dis}} \) are given as follows [10, 17]:

\[
\hat{\text{ch}}_{\text{dis}}(\lambda, n; \tau + 1, z) = e^{2\pi i \frac{\lambda}{k}(\lambda + n)} \hat{\text{ch}}_{\text{dis}}(\lambda, n; \tau, z), \tag{B.6}
\]

\[
\hat{\text{ch}}_{\text{dis}} \left( \lambda, n; \frac{-1}{\tau}, \frac{z}{\tau} \right) = e^{i\pi \frac{z^2}{k} \frac{1}{k}} \int_{0}^{k} d\lambda' \sum_{n' \in \mathbb{Z}} e^{2\pi i \frac{\lambda' - (\lambda + 2n)(\lambda' + 2n')}{2K}} \hat{\text{ch}}_{\text{dis}}(\lambda', n'; \tau, z), \tag{B.7}
\]

\[
\hat{\text{ch}}_{\text{dis}}(\lambda, n; \tau + r; z + s) = (-1)^{r+s} e^{2\pi i \frac{\lambda - 2Kn}{k}} q^{\frac{-2r}{K}} y^{-\hat{c}r} \hat{\text{ch}}_{\text{dis}}(\lambda, n + r; \tau, z), \quad (r, s \in \mathbb{Z}), \tag{B.8}
\]

\[
\hat{\chi}_{\text{dis}}(v, a; \tau + 1, z) = e^{2\pi i \frac{\lambda}{K}(v + Ka)} \hat{\chi}_{\text{dis}}(v, a; \tau, z), \tag{B.9}
\]

\[
\hat{\chi}_{\text{dis}} \left( v, a; \frac{-1}{\tau}, \frac{z}{\tau} \right) = e^{i\pi \frac{z^2}{k} \frac{1}{N}} \sum_{v' = 0}^{N-1} \sum_{a' \in \mathbb{Z}_N} e^{2\pi i \frac{\lambda' - (v + 2Kn)(v' + 2Kn')}{2NK}} \hat{\chi}_{\text{dis}}(v', a'; \tau, z), \tag{B.10}
\]

\[
\hat{\chi}_{\text{dis}}(v, a; \tau, z + r; z + s) = (-1)^{r+s} e^{2\pi i \frac{\lambda - 2Kn}{k}} q^{\frac{-2r}{K}} y^{-\hat{c}r} \hat{\chi}_{\text{dis}}(v, a + r; \tau, z), \quad (r, s \in \mathbb{Z}). \tag{B.11}
\]
We also note the formula for Witten indices;

\[
\lim_{z \to 0} \text{ch}_{\text{dis}}^{(R)}(\lambda, n; \tau, z) = \lim_{z \to 0} \hat{\text{ch}}_{\text{dis}}^{(R)}(\lambda, n; \tau, z) = \delta_{n,0},
\]

\[
\lim_{z \to 0} \chi_{\text{dis}}^{(R)}(v, a; \tau, z) = \lim_{z \to 0} \hat{\chi}_{\text{dis}}^{(R)}(v, a; \tau, z) = \delta_{a,0}^{(N)} \equiv \begin{cases} 
1 & a \equiv 0 \pmod{N} \\
0 & \text{otherwise.}
\end{cases}
\]  

(B.12)
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