gl(N, N) CURRENT ALGEBRAS AND
TOPOLOGICAL FIELD THEORIES

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ABSTRACT

The conformal field theory for the gl(N, N) affine Lie superalgebra in two space-time dimensions is studied. The energy-momentum tensor of the model, with vanishing Virasoro anomaly, is constructed. This theory has a topological symmetry generated by operators of dimensions 1, 2 and 3, which are represented as normal-ordered products of gl(N, N) currents. The topological algebra they satisfy is linear and differs from the one obtained by twisting the N = 2 superconformal models. It closes with a set of gl(N) bosonic and fermionic currents. The Wess-Zumino-Witten model for the supergroup GL(N, N) provides an explicit realization of this symmetry and can be used to obtain a free-field representation of the different generators. In this free-field representation, the theory decomposes into two uncoupled components with sl(N) and U(1) symmetries. The non-abelian component is responsible for the extended character of the topological algebra, and it is shown to be equivalent to an SL(N)/SL(N) coset model. In the light of these results, the G/G coset models are interpreted as topological sigma models for the group manifold of G.
1. Introduction and summary of main results

Since their introduction by Witten [1] five years ago, topological field theories have been a subject of intensive investigation. Some of these theories have been shown to be relevant in the attempts to understand the non-perturbative structure of string theory and quantum gravity [2]. Moreover, the three-dimensional Chern-Simons gauge theory [3] has provided us with a fascinating connection between three-dimensional topology and conformal field theory in two space-time dimensions.

In Mathematics, topological field theories have become valuable tools in the study of topological invariants of low-dimensional manifolds. So, for example, the Donaldson polynomials are obtained from observables of the four-dimensional topological Yang-Mills theories [1], and two-dimensional topological gravity [4] provides a framework to study intersection theory on the moduli space of Riemann surfaces [2,5]. In three dimensions, Chern-Simons theories allow to define invariant polynomials for knots and links and, for example, when the gauge group is $SU(2)$, the Jones polynomial and its generalizations are obtained.

In general there are two fundamental classes of topological theories [6]. To the first class belong those models in which the action does not depend on the metric of the manifold on which the local fields are defined [7]. The natural observables of these models are metric-independent operators whose vacuum expectation values give rise to topological invariant quantities. Examples of this first class of theories are the $BF$ models and the already mentioned Chern-Simons theory.

The second class of topological theories have an action which does depend on the metric of the base manifold. However, these theories possess a nilpotent fermionic symmetry that allows the introduction of a BRST cohomology. The distinctive feature of this class of theories is that their energy-momentum tensor is exact within this BRST cohomology, a fact which ensures the metric-independence of any correlator involving BRST-invariant operators. This is precisely the sense in which these theories are topological. The Donaldson-Witten theory in four
dimensions [1], and the topological sigma model in two [8], are the most outstanding theories of this class. In this paper we shall consider this second type of topological field theories.

Several procedures have been proposed in order to construct models satisfying the highly non-trivial constraint of having an energy-momentum tensor which is exact with respect to a fermionic symmetry of the action. The mostly used one is that in which the topological theory is obtained from models having two or more supersymmetries. This procedure is based on a redefinition of the Lorentz group of the theory, which amounts to a modification (a “twist”) of the energy-momentum tensor $T$ of the model in such a way that it becomes the BRST variation of a fermionic operator [9]. The BRST current implementing the topological symmetry of the twisted theory is one of the supercurrents of the model we start with. Applying this twist to an $N = 2$ superconformal theory in two dimensions [10], one ends up with a topological conformal field theory [11]. The generators of this topological symmetry close an algebra (the so-called conformal topological algebra) which is a transcription of the original $N = 2$ superconformal algebra. The main drawback of this twisting procedure is the difficulty of constructing models possessing more than one supersymmetry. Furthermore, it has been shown [12] that one can relax the conditions required to the untwisted theory while keeping a sensible topological theory after the twist is performed.

Another method to generate topological conformal theories in two dimensions has been proposed by Eguchi and Yang [9]. It consists in constructing coset theories with vanishing Virasoro central charge $c$. These cosets are formulated in terms of bosonic operators. However, at the topological point $c = 0$, a fermionic symmetry making $T$ BRST-exact shows up. In a Coulomb gas approach, the generator of this topological symmetry is obtained from the screening operators of the theory. Some other procedures have been presented. In general, the requirements that a topological field theory must fulfil make their construction so restrictive that one may hope that a complete classification programme of (at least) some classes of them could be completed.
In this paper a new method to generate topological conformal field theories is presented. Our construction makes use of the properties of affine Lie superalgebras [13,14] whose bosonic and fermionic contents are matched in such a way that the net balance of commuting and anticommuting local degrees of freedom of the corresponding conformal field theory gives a vanishing result. In order to implement this balance we will be forced to consider non-semisimple Lie superalgebras and, in fact, we shall restrict ourselves to the case of $gl(N,N)$.

Our first step will be the construction of the energy-momentum tensor $T$ of the theory. This operator will be obtained by requiring the $gl(N,N)$ currents to be primary, dimension-one operators. The fact that we are dealing with a non-semisimple Lie algebra will introduce additional complications to the standard Sugawara construction of $T$. These new features were analysed by Rozasni and Saleur [15], who considered the $gl(1,1)$ case in their study of the Alexander-Conway knot polynomial in quantum field theory [16]. Their method carries over to the more general $gl(N,N)$ model. Once $T$ has been determined, we shall check that the corresponding conformal field theory has a vanishing Virasoro anomaly, and we will characterize its underlying topological symmetry. In our approach there is a manifest balance between even and odd excitations, and it is straightforward to find the symmetry relating them. Actually, this symmetry can be generated in a local way by a suitable combination of the fermionic currents of the affine Lie superalgebra. The exactness of our energy-momentum tensor with respect to this topological symmetry can be readily verified. The current algebra is used as a guiding principle for this purpose and, as a matter of fact, the odd partner of $T$ is constructed by a fermionic analogue of the Sugawara construction. In general, all the operators appearing in the topological algebra will be normal-ordered products of currents.

The elucidation of the algebra closed by $T$ and its odd companion can serve to unravel the algebraic structures underlying the topological symmetry in quantum field theory. The algebra of our model transcends the mere twisting of the $N = 2$ superconformal algebra. In general, we shall obtain an extended topological
algebra including two additional dimension-three operators (a bosonic field and its fermionic partner). An interesting aspect of this algebra is its linear character, in spite of the fact that it contains higher-spin operators. Actually, Kazama [17] found this same algebra as a consistent truncation of a larger one, satisfied by the matter system studied by Distler [18]. Therefore our model provides an explicit realization of this extended topological algebra.

It is important to point out that the model we shall construct is already “twisted”, i.e., no redefinition is needed to render the theory topological. Nevertheless, we can try to relate our algebra to the standard superconformal symmetries. It turns out that, upon “untwisting”, our algebra can be related to the $N = 1$ superconformal algebra in such a way that all generators can be arranged into $N = 1$ supermultiplets.

Another topic we shall investigate is the compatibility of the extended topological algebra with the current algebra on which it is based. By construction, $T$ closes with the currents when they are commuted, so it is quite natural to require the latter to be primary also with respect to the odd partner of $T$. It turns out, however, that only half the initial currents satisfy this condition. They close with themselves and with the generators of the topological algebra. Each of the bosonic currents in this restricted set is BRST-exact, and their fermionic partners also belong to this set. In fact, they close an algebra that is the topological analogue of the affine algebra and that we shall name topological current algebra. Our analysis will reveal that the topological symmetry can coexist with a current algebra. Actually, the non-abelian nature of the latter determines the extended character of the former. As happens with conformal field theories, one would expect that additional symmetries, such as current algebra symmetries, could serve to organize the Hilbert space of the topological theory in such a way that a well-defined representation theory could be developed. In string theory this extra symmetry could provide a dynamical principle to overcome the difficulties appearing beyond the $d = 1$ barrier [19].
An explicit realization of this topological conformal field theory can be obtained by quantizing the Wess-Zumino-Witten (WZW) model [20] for the $GL(N, N)$ supergroup. Using this realization we shall conclude that our model describes a zero-dimensional topological sigma model. Performing a Gauss-type decomposition of the of the basic $GL(N, N)$ variable of the WZW model, we shall obtain a free-field representation [21] of the extended topological algebra that will shed light on its nature and will clarify its relation with other models of two-dimensional topological matter. Within this free-field realization, our model is represented as the superposition of two uncoupled models having $sl(N)$ and $U(1)$ symmetries. Each of these two separately constitutes a topological conformal field theory. The non-abelian one realizes the extended topological algebra non-trivially, whereas for the $U(1)$ component, this algebra reduces to the one obtained by twisting the $N = 2$ superconformal models. The non-abelian theory we shall find is identical to the representation of the $G/G$ coset models [22] obtained in references [23,24,25]. This means that our extended topological algebra is realized in this type of topological theories which, from our point of view, are regarded as topological sigma models for group manifolds, i.e., as the topological analogue of the Wess-Zumino-Witten theories. These aspects of our construction will be discussed elsewhere [26].

This paper is organized as follows. In section 2, after reviewing the basic features of the $gl(N, N)$ current algebras needed in this paper, we construct the energy-momentum tensor. The topological algebra of our model is explored in section 3. The relations satisfied by this algebra have been compiled in Appendix A. The topological current algebra of our model is obtained in section 4. An example of how these topological algebras appear in the $sl(2)$ WZW model is developed in Appendix B. In section 5 the free-field representation of the extended topological algebra is worked out. Finally, in section 6 we discuss our results and indicate some possible lines of future development of our ideas.
2. $gl(N, N)$ current algebras

We begin this section giving the basic definitions and properties of the $gl(N, N)$ Lie superalgebra [13,14]. Consider a supervector space having $N$ bosonic dimensions and $N$ fermionic elements. In this supervector space we shall take an homogeneous basis having $N$ bosonic and $N$ fermionic elements. We shall label this basis in such a way that the first $N$ vectors are the bosonic ones. Accordingly, in this space, any vector is determined by a set of quantities $V^A$ with $1 \leq A \leq 2N$, the $V^a(V^{a+N})$ for $a = 1, \ldots, N$ being the bosonic (fermionic) components. In what follows a capital latin letter will denote an index running from 1 to $2N$, whereas lower-case latin indices can take any value between 1 and $N$. The grade of a given index $A$ (denoted by $g(A)$) is defined to be zero (one) if it labels a bosonic (fermionic) component. Therefore, with our conventions,

$$g(A) = \begin{cases} 0, & \text{if } 1 \leq A \leq N \\ 1, & \text{if } N + 1 \leq A \leq 2N. \end{cases}$$

(2.1)

The $gl(N, N)$ Lie superalgebra is the algebra of $2N \times 2N$ matrices acting on a supervector space with $N$ bosonic and $N$ fermionic dimensions. In an homogeneous basis we can write the general form of any element of $gl(N, N)$ as

$$X = \begin{pmatrix} B_1 & F_1 \\ F_2 & B_2 \end{pmatrix},$$

(2.2)

where $B_1$ and $B_2$ are $N \times N$ matrices whose elements are c-numbers, whereas the entries of the $N \times N$ matrices $F_1$ and $F_2$ are odd Grassmann numbers. The matrices of the form

$$B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & F_1 \\ F_2 & 0 \end{pmatrix},$$

(2.3)

are said to be homogeneous. To these homogeneous matrices we shall assign a grade. A matrix like $B$ is called even, whereas $F$ is said to be odd. Actually we
shall consider off-diagonal matrices like $F$ as odd matrices even if their entries are c-numbers. The grade of any homogeneous matrix $X$ is defined as follows:

$$g(X) = \begin{cases} 
0, & \text{if } X \text{ is even} \\
1, & \text{if } X \text{ is odd}.
\end{cases}$$  \hfill (2.4)

The grade of vector indices and matrices (eqs. (2.1) and (2.4)) are defined modulo 2 (i.e. as for a $Z_2$ grading).

The superalgebra structure for $gl(N, N)$ is introduced by defining the generalized Lie bracket for any two homogeneous matrices $X$ and $Y$:

$$[X, Y] \equiv XY - (-1)^{g(X)g(Y)} YX.$$ \hfill (2.5)

Notice that if at least one of the two matrices $X$ and $Y$ is even, their bracket $[X, Y]$ is a commutator while, on the contrary, if both $X$ and $Y$ are odd, $[X, Y]$ is an anticommutator. Using the definition of the bracket given above, it is easy to check that it satisfies a graded Jacobi identity:

$$(-1)^{g(X_1)g(X_3)} [X_1, [X_2, X_3]] + (-1)^{g(X_2)g(X_1)} [X_2, [X_3, X_1]] +$$
$$+ (-1)^{g(X_1)g(X_2)} [X_3, [X_1, X_2]] = 0.$$ \hfill (2.6)

Furthermore, the Lie bracket (2.5) acts as a graded derivation, i.e. it satisfies the identity

$$[X_1X_2, X_3] = (-1)^{g(X_2)g(X_3)} [X_1, X_3]X_2 + X_1[X_2, X_3].$$ \hfill (2.7)

Let us now define the matrices $E_{AB}$ as follows:

$$(E_{AB})_{CD} = \delta_{AC}\delta_{BD}.$$ \hfill (2.8)

Obviously the set \{ $E_{AB}$; $1 \leq A, B \leq 2N$ \} constitute a basis for the space of
2N × 2N matrices. Therefore any \( X \in gl(N, N) \) can be written as

\[
X = \sum_{A,B=1}^{2N} X^{AB} E_{AB}, \quad (2.9)
\]

where the \( X^{AB} \) are the contravariant components of \( X \) with respect to the basis \( \{ E_{AB} \} \). From our previous definitions, the grade of a matrix \( E_{AB} \) is given by

\[
g(E_{AB}) = g(A) + g(B). \quad (2.10)
\]

Notice that the \( E_{AB} \) matrices with zero grade span the c-number part of \( gl(N, N) \), whereas those with non-vanishing grade span (with odd Grassmann coefficients) the odd part of \( gl(N, N) \). It is interesting to compute the generalized Lie brackets among the \( E_{AB} \) matrices. Using their explicit form in the definition of the bracket (eq. (2.5)) it is straightforward to arrive at

\[
[E_{AB}, E_{CD}] = \sum_{P,Q=1}^{2N} F_{AB,CD}^{PQ} E_{PQ}, \quad (2.11)
\]

where we have written the result in terms of the structure constants

\[
F_{AB,CD}^{PQ} = \delta_{AP} \delta_{DQ} \delta_{BC} - (-1)^{(g(A)+g(B))(g(C)+g(D))} \delta_{CP} \delta_{BQ} \delta_{AD}. \quad (2.12)
\]

Although we have obtained the Lie brackets (2.11) from the explicit expression for the matrix elements of \( E_{AB} \) (eq. (2.8)), we can now regard eqs. (2.11) and (2.12) as the definition of the abstract Lie superalgebra \( gl(N, N) \). The \( E_{AB} \) are the generators of this algebra, while their explicit form written down in eq. (2.8) constitutes the defining (fundamental) representation of \( gl(N, N) \). From eq. (2.12) we easily obtain the (anti)symmetry properties of the structure constants

\[
F_{CD,AB}^{PQ} = - (-1)^{(g(A)+g(B))(g(C)+g(D))} F_{AB,CD}^{PQ}, \\
F_{AB,CD}^{PQ} = - (-1)^{g(Q)+g(A)} (-1)^{(g(A)+g(B))(g(P)+g(Q))} F_{QP,CD}^{BA}. \quad (2.13)
\]

These equations will be very useful in future calculations. For a matrix like (2.2)
one can define the supertrace as follows:

$$Str(X) = Tr(B_1) - Tr(B_2) = \sum_{A=1}^{2N} (-1)^{g(A)} X^{AA}, \quad (2.14)$$

where $X^{AA}$ are the diagonal contravariant components of $X$ with respect to the \{E_{AB}\} basis (see eq. (2.9)). The supertrace (2.14) can be used to define the Killing-Cartan bilinear form for $gl(N,N)$. In complete analogy with what it is done for ordinary Lie algebras, we define the following invariant bilinear form:

$$< X, Y > \equiv Str(XY), \quad (2.15)$$

where $X$ and $Y$ are two matrices in the fundamental representation of $gl(N,N)$.

The inner products of the elements of the \{E_{AB}\} basis define the metric tensor:

$$G_{AB,CD} \equiv< E_{AB}, E_{CD} >= Str(E_{AB}E_{CD}). \quad (2.16)$$

Using the explicit expressions of the matrix elements of the $E_{AB}$ matrices, one immediately gets

$$G_{AB,CD} = (-1)^{g(A)} \delta_{AD} \delta_{BC}. \quad (2.17)$$

Notice that, contrary to what happens for ordinary Lie algebras, $G_{AB,CD}$ is not symmetric. This has to be taken into account when performing explicit calculations. The inverse matrix of $G_{AB,CD}$ will be denoted by superindices. It satisfies

$$\sum_{C,D} G_{AB,CD} G^{CD,PQ} = \sum_{C,D} G^{AB,CD} G_{CD,PQ} = \delta_{AP} \delta_{BQ}. \quad (2.18)$$

These two conditions are fulfilled by
\[ G^{AB,CD} = (-1)^{g(B)} \delta_{AD} \delta_{BC}. \]  

(2.19)

Using the inverse metric tensor, we can obtain the quadratic Casimir operator of the \( gl(N, N) \) algebra:

\[ C_1 = \sum_{A,B,C,D} G^{AB,CD} E_{AB} E_{CD} = \sum_{A,B} (-1)^{g(B)} E_{AB} E_{BA}. \]  

(2.20)

From the fundamental Lie brackets (2.11) and the derivation property (2.7) one can prove, after a short calculation, that the bracket of \( C_1 \) with any element of the algebra vanishes:

\[ [C_1, E_{AB}] = 0. \]  

(2.21)

It is important to point out that (2.21) can be proved without using the explicit expressions of the \( E_{AB} \) matrices. Therefore (2.21) is valid for any representation of the abstract Lie superalgebra. For the fundamental representation (2.8) \( C_1 \) actually vanishes:

\[ (C_1)^{Fundamental}_{AB} = 0. \]  

(2.22)

The \( gl(N, N) \) superalgebra is not semisimple. It contains the element \( \sum_A E_{AA} \), which has a vanishing bracket with all the generators (in the fundamental representation this element is represented by the identity matrix). Due to this fact we have a second quadratic Casimir:

\[ C_2 = \sum_{A,B} E_{AA} E_{BB}. \]  

(2.23)

Again one easily proves using (2.7) that

\[ [C_2, E_{AB}] = 0. \]  

(2.24)
For the fundamental representation \( C_2 \) is just the unit matrix,

\[
(C_2)_{AB}^{\text{Fundamental}} = \delta_{AB}.
\] (2.25)

Let us consider now the affine Kac-Moody superalgebra based on \( gl(N, N) \). Our basic object will be an holomorphic current \( J(z) \) taking values in a \( gl(N, N) \) algebra. The contravariant components \( J^{AB}(z) \) of the currents are the coefficients of the expansion of \( J(z) \) in terms of the \( \{E_{AB}\} \) basis:

\[
J(z) = \sum_{A,B} J^{AB}(z) E_{AB}.
\] (2.26)

Notice that, since \( J(z) \) takes values in \( gl(N, N) \), the components \( J^{a,b}(z) \) and \( J^{a+N,b+N}(z) \) are bosonic (i.e. c-number valued) and, on the contrary, \( J^{a,b+N}(z) \) and \( J^{a+N,b}(z) \) are fermionic (\( 1 \leq a, b \leq N \)). The covariant components \( J_{AB}(z) \) are given by

\[
J_{AB}(z) = \text{Str}(J(z) E_{AB}) = (-1)^{g(B)} J^{BA}(z),
\] (2.27)

where in the last step we have made use of the explicit form of the metric tensor (see eq. (2.17)). In order to define the affine Kac-Moody superalgebra it is convenient to expand \( J_{AB}(z) \) in a Laurent series around \( z = 0 \):

\[
J_{AB}(z) = \sum_{n \in \mathbb{Z}} J_{AB}^n z^{-n-1}.
\] (2.28)

The \( gl(N, N) \) current algebra is obtained by requiring the modes \( J_{AB}^n \) to satisfy

\[
[J_{AB}^n, J_{CD}^m] = F_{AB,CD}^{PQ} J_{PQ}^{n+m} + k n \delta_{n+m,0} G_{AB,CD},
\] (2.29)

where \( F_{AB,CD}^{PQ} \) are the structure constants (2.12) and the central extension has been taken to be proportional to the metric tensor \( G_{AB,CD} \). In (2.29) \( k \) is a c-number constant (the level of the algebra). An alternative way of defining the
current algebra is obtained by giving the short-distance expansion of the product of two arbitrary currents:

\[ J_{AB}(z)J_{CD}(w) = \frac{k}{(z-w)^2}G_{AB,CD} + F_{APQ}^{CD} \frac{J_{PQ}(w)}{z-w}. \]  \hfill (2.30)

As is well known in the framework of the radial quantization of two-dimensional field theories, the operator product expansions (OPE) (2.30) are equivalent to the brackets (2.29). We shall indistinctly use OPE’s or brackets as best suits our convenience.

We can undertake now the construction of a two-dimensional conformal invariant theory consistent with the $gl(N,N)$ affine symmetry. This objective would be accomplished if we were able to construct an energy-momentum tensor $T$ such that the currents $J_{AB}(z)$ transform as dimension-one primary fields. The natural ansatz for $T$ is the Sugawara construction, in which $T$ is built up as a quadratic expression in the currents. In order to unambiguously define these operators, in which two or more fields evaluated at the same point are multiplied, we need to adopt a normal ordering prescription. Suppose that $A(z)$ and $B(z)$ are two local fields whose Laurent modes are $A_n$ and $B_n$,

\[ A(z) = \sum_{n \in \mathbb{Z}} A_n z^{-n-\Delta_A} \quad B(z) = \sum_{n \in \mathbb{Z}} B_n z^{-n-\Delta_B}, \]  \hfill (2.31)

where $\Delta_A$ and $\Delta_B$ are the conformal weights of $A(z)$ and $B(z)$ respectively. All the fields we shall encounter inside normal-ordered products will have integer conformal weights and, therefore, we shall assume that this condition is satisfied in the equations that follow. The normal-ordered product of two arbitrary modes $:A^n B^m:$ is defined as

\[ :A^n B^m: = \begin{cases} A^n B^m, & \text{if } m \geq 1 - \Delta_B \\ (-1)^{g(A)g(B)} B^m A^n, & \text{if } m < 1 - \Delta_B. \end{cases} \]  \hfill (2.32)

The modes $(:AB:)^n$ of the normal-ordered product of $A$ and $B$ are defined by the
equation

\[ :A(z)B(z) : = \sum_{n \in \mathbb{Z}} (: AB :)^n z^{-n-\Delta_A-\Delta_B}. \]  

(2.33)

Substituting the mode expansions of \( A(z) \) and \( B(z) \) in the left-hand side of (2.33) and using (2.32) we get

\[ (: AB :)^n = \sum_{p=1-\Delta_B}^{\infty} A^{n-p} B^p + (-1)^{g(A)g(B)} \sum_{p=\Delta_B}^{\infty} B^{-p} A^{n+p}. \]  

(2.34)

It is important to point out that the order of fields inside a normal-ordered product is relevant. Indeed, one has

\[ (: AB :)^n = (-1)^{g(A)g(B)} (: BA :)^n + \sum_{p=1-\Delta_B}^{n+\Delta_A-1} [A^{n-p}, B^p]. \]  

(2.35)

Notice that, apart from a sign, we get an extra contribution when we reverse the order of the operators \( A \) and \( B \) inside \( : AB : \). Let us examine the consequences of eq. (2.35) when \( A \) and \( B \) have a bracket of the form

\[ [A^n, B^m] = D^{n+m} + nk_D \delta_{n+m,0}, \]  

(2.36)

where \( D \) is an operator of conformal dimension \( \Delta_D \), which is easily seen to be \( \Delta_A + \Delta_B - 1 \), and \( k_D \) is a constant number. Using (2.36) in eq. (2.35), we get

\[ (: AB :)^n = (-1)^{g(A)g(B)} (: BA :)^n - (\partial D)^n - \frac{\Delta_D}{2}(\Delta_A - \Delta_B) k_D \delta_{n,0} \]  

(2.37)

In eq. (2.37) we have introduced the derivative of the operator \( D \), whose modes are given by

\[ (\partial D)^n = -(n + \Delta_D) D^n. \]  

(2.38)

Notice that when \( \Delta_A = \Delta_B \) (which will be the case in most of our calculations), the last term in the right-hand side of (2.37) disappears and only the term containing \( \partial D \) survives.
When more than two fields are multiplied, the normal order is defined inductively according to the rule

\[ A_n \cdots A_1 \equiv (\ldots (A_n A_{n-1}) \cdots A_2) A_1 \], \quad (2.39)\]
i.e., the product \( A_n \cdots A_1 \) is considered as the product of \( A_n \cdots A_2 \) with \( A_1 \) and so on. A reordering formula like (2.37) can also be obtained for normal-ordered products of more than two fields. Proceeding as we did to get eq. (2.37) and using the prescription (2.39) one obtains

\[
\begin{align*}
(ABC)^n &= (-1)^{g(A)g(B)}(ABC)^n - (\partial DC)^n - \frac{\Delta D}{2}(\Delta_A - \Delta_B)k_D C^n \\
(CAB)^n &= (-1)^{g(A)g(B)}(CBA)^n - (C\partial D)^n - \frac{\Delta D}{2}(\Delta_A - \Delta_B)C^n k_D,
\end{align*}
\] (2.40)

where we have supposed that the bracket (2.36) still holds.

The obvious candidates to become the energy-momentum tensor of the theory are the Sugawara bilinears constructed from the quadratic Casimir invariants of the theory. As we have seen, the \( gl(N, N) \) algebra has two such Casimir invariants and, therefore, we should consider the operators

\[
T_1 = \sum_{A,B} (-1)^{g(B)} J_{AB} J_{BA} : ,
\]
\[
T_2 = \sum_{A,B} J_{AA} J_{BB} : .
\] (2.41)

In order to determine the precise combination of \( T_1 \) and \( T_2 \) that makes \( J_{AB} \) a primary field with conformal weight equal to one, we must compute the brackets \([T_1^n, J_{AB}^m]\) and \([T_2^n, J_{AB}^m]\). These brackets can be obtained from a general expression that we shall now derive. Suppose we want to compute

\[
[(: A_1 A_2 :)^n, A_3^m],
\] (2.42)

where \( A_1, A_2 \) and \( A_3 \) are operators of dimensions \( \Delta_1, \Delta_2 \) and \( \Delta_3 \) whose Laurent
modes have the brackets

\[
[A_1^n, A_3^m] = A_{13}^{n+m} + nk_{13}\delta_{n+m,0} \\
[A_2^n, A_3^m] = A_{23}^{n+m} + nk_{23}\delta_{n+m,0}.
\] (2.43)

In (2.43) \(A_{13}\) and \(A_{23}\) are operators, whereas \(k_{13}\) and \(k_{23}\) are constants. Notice that the brackets among the \(gl(N, N)\) currents are of the form displayed in eq. (2.43). Using (2.35) together with the derivation property (eq. (2.7)), it is easy to arrive at the result

\[
[(: A_1 A_2 :)^n, A_3^m] = (-1)^{g(A_2)g(A_3)}([(: A_{13} A_2 :)^n + m k_{13} A_2^n + m) + \\
+ (: A_1 A_{23})^n - mA_1^{n+m}k_{23} - \sum_{j=1}^{m-\Delta_2} [A_1^{m+n-j}, A_2^j],
\] (2.44)

where \(\Delta_{23}\) is the dimension of the operator \(A_{23}\). The last term in the right-hand side of (2.44) originates when one tries to express the result in terms of normal-ordered products. Using equations (2.44) and (2.29), we obtain after an easy calculation

\[
[T_1^n, J_{AB}^m] = -2mkJ_{AB}^{n+m} + 2m(-1)^{g(B)}\delta_{AB} \sum_{C} J_{CC}^{n+m},
\] (2.45)

where we have used the fact that

\[
\sum_{A,B,C,D} (-1)^{g(B)} F_{AB,CD}^{PQ} F_{BA,RS}^{CD} = -2(-1)^{g(R)}\delta_{PQ}\delta_{RS},
\] (2.46)

which can be easily obtained from the explicit form of the structure constants (see eq. (2.12)). Eq. (2.46) is nothing but the value of the quadratic Casimir \(C_1\) for the adjoint representation. On the other hand eq. (2.45) implies that \(T_1\) does not act diagonally on the \(gl(N, N)\) currents. Therefore it seems plausible that we should
also consider the operator $T_2$. Again making use of (2.44), we obtain

$$[T_2^n, J_{AB}^m] = -2mk(-1)^{g(B)}\delta_{AB}\sum_C J_{CC}^{n+m}. \quad (2.47)$$

From a glance at the right-hand sides of eqs. (2.45) and (2.47) it is clear that we can combine $T_1$ and $T_2$ in such a way that the non-diagonal terms disappear and $J_{AB}$ becomes a dimension-one primary field. Let us define

$$T = \frac{1}{2k} T_1 + \frac{1}{2k^2} T_2. \quad (2.48)$$

From eqs. (2.45) and (2.47) we get

$$[T^n, J_{AB}^m] = -m J_{AB}^{n+m}, \quad (2.49)$$

i.e., $J_{AB}$ is indeed primary with respect to $T$.

The form of the energy-momentum tensor $T$ displayed in eq. (2.48) generalizes the result of Rozanski and Saleur [15], who have obtained a similar equation for $N = 1$. As was pointed out in ref. [15], the second term in (2.48) can be regarded as a quantum correction to the Sugawara tensor $T_1$. Notice that in this case we have obtained a $\frac{1}{k^2}$ contribution, instead of the usual $k \to k + c_v$ shift that appears, for example, in the $SU(N)$ Wess-Zumino-Witten model — $c_v$ being the quadratic Casimir in the adjoint representation. In our case the fact that $k$ is not shifted in $T$ originates in the vanishing of the diagonal terms in $(C_2)^{\text{Adjoint}}$ (see eq. (2.46)), which is due to an exact cancellation between the bosonic and fermionic contributions in eq. (2.46).

We should also check that $T^n$ satisfies the Virasoro algebra. With this purpose in mind, let us compute the bracket of $T^n$ with a general quadratic bilinear built up with primary fields. Suppose that $A(z)$ and $B(z)$ are primary fields with conformal
dimensions $\Delta_A$ and $\Delta_B$ respectively. Consider the bilinear operator

$$O(z) =: A(z)B(z) :$$  \hspace{1cm} (2.50)

A calculation similar to the one that led to eq. (2.44) gives

$$[T^n, O^m] = [(\Delta_A + \Delta_B - 1)n - m]O^{n+m} + \sum_{q=1-\Delta_B}^{n-\Delta_B} (q - n\Delta_B)[A^{n+m-q}, B^q].$$  \hspace{1cm} (2.51)

Notice that $O(z)$ is a primary operator with conformal dimension $\Delta_A + \Delta_B$ only if the last term in the right-hand side of (2.51) vanishes. This is the case of $T_1$ and $T_2$,

$$[T^n, T_1^m] = (n - m)T_1^{n+m} \hspace{0.5cm} [T^n, T_2^m] = (n - m)T_2^{n+m},$$  \hspace{1cm} (2.52)

which, in particular, implies that

$$[T^n, T^m] = (n - m)T^{n+m}.$$ \hspace{1cm} (2.53)

Therefore the energy-momentum tensor operator satisfies the Virasoro algebra without central extension.

Moreover, it follows from eqs. (2.22) and (2.25) that a field transforming in the fundamental representation of the current algebra has a conformal weight given by

$$\Delta_{\text{Fundamental}} = \frac{1}{2k^2}.$$ \hspace{1cm} (2.54)

Notice that only $T_2$ contributes to (2.54). On the other hand, due to the non-simplicity of the $gl(N, N)$ superalgebra, a field transforming as an arbitrary representation of the algebra is not in general a primary field of the Virasoro algebra. This happens because, for this superalgebra, the quadratic Casimirs evaluated in general representations are non-diagonal (the adjoint representation provides an example where this phenomenon occurs, see eq. (2.46)).
It is interesting to stress the reason why the Virasoro central charge \( c \) of our theory vanishes. When one applies eq. (2.51) to obtain (2.52), it turns out that there is an exact compensation between the conformal anomaly coming from the bosonic currents and that originated from the fermionic ones. Actually, the conformal field theory we are dealing with is non-unitary, since it has an indefinite metric in its Fock space (see, for example, the signs appearing in the right-hand side of (2.41)). Moreover, in a \( gl(N, N) \) current algebra there is an exact balance between fermionic and bosonic degrees of freedom and, as a consequence, the central charge of the Virasoro algebra vanishes. In fact, from the results obtained in ref. [27] it follows that, for \( N > M \), the \( gl(N, M) \) current superalgebra shares many properties with the \( gl(N - M) \) affine (bosonic) Lie algebra. It is thus clear that, when \( N = M \), we are dealing with a limiting case which requires a separate study. From these considerations, one would expect the theory whose energy-momentum tensor is given by (2.48) to be a topological field theory. This is indeed the case as we shall show in the next section, where the nature of the theory we have constructed will be explored in detail.

3. The topological algebra

In this section we aim at establishing the topological character of the conformal field theory possessing the \( gl(N, N) \) current symmetry described in the previous section. We would like to uncover the topological symmetry of our theory and study its relation with the original \( gl(N, N) \) supersymmetry. By a topological field theory we mean a theory in which there exists a nilpotent (i.e. fermionic), BRST-type symmetry such that the energy-momentum tensor of the theory is BRST-exact. The theory we have at hand enjoys an affine \( gl(N, N) \) symmetry and, in particular, there exist fermionic nilpotent currents from which a topological BRST symmetry fulfilling our requirements can be constructed. Let us see that this is indeed the case. First of all we shall distinguish from now on between bosonic and
fermionic currents. We introduce the following notations:

\[
\Psi_{ab} \equiv J_{a+N,b} \quad \Lambda_{ab} \equiv J_{a,b+N} \\
K_{ab} \equiv J_{ab} \quad L_{ab} \equiv J_{a+N,b+N}.
\] (3.1)

Notice that \(\Psi_{ab}\) and \(\Lambda_{ab}\) are fermionic (i.e. Grassmann odd) currents, whereas \(K_{ab}\) and \(L_{ab}\) are bosonic fields. With these notations the general commutation relations (eq. (2.29)) in the [boson,boson] sector take the form

\[
[K_{ab}^n, K_{cd}^m] = \delta_{bc}K_{ad}^{n+m} - \delta_{ad}K_{cb}^{n+m} + kn\delta_{bc}\delta_{ad}\delta_{n+m,0} \\
[L_{ab}^n, L_{cd}^m] = \delta_{bc}L_{ad}^{n+m} - \delta_{ad}L_{cb}^{n+m} - kn\delta_{bc}\delta_{ad}\delta_{n+m,0} \\
[K_{ab}^n, L_{cd}^m] = 0.
\] (3.2)

For the [fermion, fermion] case we have

\[
[\Psi_{ab}^n, \Psi_{cd}^m] = [\Lambda_{ab}^n, \Lambda_{cd}^m] = 0 \\
[\Psi_{ab}^n, \Lambda_{cd}^m] = \delta_{ad}\Psi_{cb}^{n+m} + \delta_{bc}\Psi_{ad}^{m+n} - kn\delta_{bc}\delta_{ad}\delta_{n+m,0},
\] (3.3)

and finally the brackets involving both bosonic and fermionic currents are

\[
[\Psi_{ab}^n, K_{cd}^m] = \delta_{bc}\Psi_{ad}^{n+m} \\
[\Psi_{ab}^n, L_{cd}^m] = -\delta_{ad}\Psi_{cb}^{n+m} \\
[\Lambda_{ab}^n, K_{cd}^m] = -\delta_{ad}\Lambda_{cb}^{n+m} \\
[\Lambda_{ab}^n, L_{cd}^m] = \delta_{bc}\Lambda_{ad}^{n+m}.
\] (3.4)

It is also interesting to write down the energy-momentum tensor \(T\) in terms of our component currents (3.1). The contribution \(T_1\) coming from the first quadratic Casimir takes the form

\[
T_1 = \sum_{a,b} : (K_{ab}K_{ba} - L_{ab}L_{ba} + \Psi_{ab}\Lambda_{ba} - \Lambda_{ab}\Psi_{ba}) :
\] (3.5)

Defining the current of the identity \(J_E\) as

\[
J_E = \sum_a (K_{aa} + L_{aa}),
\] (3.6)

\(T_2\) is simply given by

\[
T_2 = : J_E J_E :
\] (3.7)

It is important to point out that from eq. (3.3) it follows that the two fermionic
currents $\Psi_{ab}$ and $\Lambda_{ab}$ present in our algebra are nilpotent. We would like to define a BRST current without any free index of the current algebra. The simplest way of achieving this objective is to sum over the $gl(N)$ diagonal components of one of the fermionic currents. Suppose we define:

$$Q = \sum_a \Lambda_{aa}. \quad (3.8)$$

Of course, nothing essential would change if we had chosen the $\Psi$ currents instead of the $\Lambda$’s in the definition (3.8). Obviously we have

$$[Q^n, Q^m] = 0. \quad (3.9)$$

Let us see that the current $Q$ defined above serves our purposes. Using eqs. (3.3) and (3.4) we easily obtain the brackets of $Q$ with the currents:

$$[Q^n, K^m_{ab}] = -\Lambda_{ab}^{n+m}, \quad [Q^n, L^m_{ab}] = \Lambda_{ab}^{n+m}, \quad [Q^n, \Lambda_{ab}^n] = 0 \quad (3.10)$$

$$[Q^n, \Psi_{ab}^m] = L_{ab}^{n+m} + K_{ab}^{n+m} + kn\delta_{ab}\delta_{n+m,0}.$$ 

We can use the zero-mode of $Q$ to define a BRST cohomology operator $\delta$. For any field $\Phi$ we define $\delta\Phi$ as

$$\delta\Phi = [Q^0, \Phi]. \quad (3.11)$$

Taking the brackets (3.10) into account, the BRST variations of the currents are

$$\delta K_{ab} = -\Lambda_{ab} \quad \delta L_{ab} = \Lambda_{ab} \quad \delta\Lambda_{ab} = 0 \quad (3.12)$$

$$\delta\Psi_{ab} = K_{ab} + L_{ab}.$$ 

Notice that the combination $K_{ab} + L_{ab}$ of the bosonic currents is $\delta$-exact. On the other hand by a direct calculation using (3.12) and the graded derivation property
of the bracket (eq. (2.7)) one can check that $T_1$ and $T_2$ are $\delta$-invariant,

\[
\delta T_1 = 0 \quad \delta T_2 = 0, \quad (3.13)
\]

i.e. $T_1$ and $T_2$ are closed in the BRST cohomology defined by $\delta$. Actually, as we shall prove below, $T_1$ and $T_2$ (and thus the complete energy-momentum tensor) are cohomologically exact. In order to verify this fact it is clear that we must find fermionic operators whose BRST variations give $T_1$ and $T_2$. We would like to have an expression of these odd partners of $T_1$ and $T_2$ in terms of the underlying currents of the model. The natural way to proceed is to imitate the Sugawara construction employed to obtain the energy-momentum tensor and consider operators that are bilinear in the currents. Due to the fermionic character of the operators we are looking for, it is clear that we must deal with products of a fermionic and a bosonic current evaluated at the same point. After some trial and error, we easily arrive at the desired result. In fact if we define

\[
\delta G_1 = T_1, \quad (3.15)
\]

with

\[
G_1 = \sum_{a,b} : (\Psi_{ab} K_{ba} - L_{ab} \Psi_{ba}) :, \quad (3.14)
\]

it is easy to check that

\[
\delta G_1 = T_1. \quad (3.15)
\]

In the same way, inspecting eqs. (3.12), (3.6) and (3.7), we conclude

\[
T_2 = \delta G_2, \quad (3.16)
\]

with

\[
G_2 = \sum_a : \Psi_{aa} J_E :, \quad (3.17)
\]

Therefore if we define $G$ as

\[
G = \frac{1}{2k} G_1 + \frac{1}{2k^2} G_2, \quad (3.18)
\]

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we have

\[ T = \delta G, \quad (3.19) \]

i.e. \( G \) is the odd partner of \( T \) we were looking for. Let us now verify that this new fermionic field \( G \) is primary with respect to \( T \) with conformal weight equal to two. This fact can be established by computing the bracket \([T^n, G^m]\). Using our general expression (2.51) for general bilinear operators, we immediately get

\[ [T^n, G^m] = (n - m)G_{n+m}, \quad (3.20) \]

which proves the statement.

Once we have obtained the explicit form of \( G \), we may ask ourselves what is the algebra that \( G \) closes with \( T \) and \( Q \). We shall refer to it as the topological algebra. It is interesting to point out that eq. (3.19) only determines the bracket of the zero-mode of \( Q \) with the Laurent modes of \( T \). As we want to realize our topological BRST symmetry as a local symmetry, we should worry about the general bracket \([Q^n, G^m]\). On general grounds we can write

\[ [Q^n, G^m] = T^{n+m} + nR^{m+m} + \frac{d}{2}m(m+1)\delta_{n+m,0}, \quad (3.21) \]

where \( d \) is a c-number and \( R \) is a dimension-one abelian current. Actually eq. (3.21) is the bracket obtained when the \( N = 2 \) superconformal algebra is twisted [9,11]. This twist consists in a redefinition of the energy-momentum tensor of the superconformal theory by adding the derivative of the \( U(1) \) current \( R \) appearing in the \( N = 2 \) superconformal algebra:

\[ T = T_{N=2} + \frac{1}{2} \partial R. \quad (3.22) \]

After this redefinition eq. (3.21) is satisfied — \( Q \) and \( G \) being the two supersymmetry currents of the initial theory. The Virasoro central charge for the twisted
model vanishes and thus the redefined energy-momentum tensor satisfies (2.53). Moreover, the c-number anomaly $d$ appearing in the right-hand side of (3.21) is related to the central charge of the untwisted theory as

$$d = \frac{C_{N=2}}{3}. \quad (3.23)$$

Other brackets obtained by twisting the $N = 2$ superconformal algebra are

$$[T^n, R^m] = -mR^{n+m} - \frac{d}{2}n(n+1)\delta_{n+m,0}$$

$$[T^n, Q^m] = -mQ^{n+m}$$

$$[R^n, R^m] = d\delta_{n+m,0}$$

$$[R^n, G^m] = -G^{n+m}$$

$$[R^n, Q^m] = Q^{n+m}. \quad (3.24)$$

Furthermore eq. (3.20) also holds. Actually, in the original superconformal theory, both $Q$ and $G$ behave as primary fields of conformal dimension $\frac{3}{2}$. After the redefinition (3.22), $Q$ becomes a dimension-one operator, whereas $G$ acquires conformal weight 2. This different behaviour is due to the different $R$-charges of these two operators (see eq. (3.24)).

The theory we arrive at by twisting is a model of topological matter. In case we start with a conformal invariant sigma model, $d$ is nothing but the dimension of the target space in which the bosonic sector of the theory is embedded. For this reason we shall call $d$ from now on the dimension of the topological algebra. Let us also recall [2] that, when topological matter is coupled to topological gravity, the resulting theory reproduces many features of the matrix models that describe non-critical strings.

In our $gl(N,N)$ theory eqs. (3.21) and (3.24) are satisfied for $d = 0$, i.e.

$$d_{gl(N,N)} = 0. \quad (3.25)$$

In order to check (3.25) one has to use the explicit expressions of $Q$, $G$ and $T$, together with the basic brackets (eqs. (3.2) – (3.4)) in our general equation for
the brackets of an arbitrary bilinear operator (eq. (2.44)). In so doing, one also obtains the explicit form of the $R$ current, which turns out to be

$$R = \frac{1}{2} \sum_a (K_{aa} - L_{aa}).$$  \hfill (3.26)$$

It is important to emphasize that, in our approach, we do not have to perform a modification of the supersymmetric theory we started with in order to obtain a topological field theory, i.e. no twist is needed. Actually, as we shall now show, the topological algebra of our $gl(N,N)$ theory differs from the one obtained from the $N=2$ superconformal models. In fact the topological algebra of the former can be regarded as an extension of the latter. The difference between these two algebras shows up when computing the bracket $[G^n, G^m]$, which vanishes in the twisted $N=2$ algebra. The calculation of this bracket in our case is rather involved, so let us first give some of the intermediate steps, which will turn out to be very useful in what follows; in fact these results are interesting by themselves. First of all we compute the brackets among the odd companion $G$ of the energy-momentum tensor and the currents of the model, which can be obtained as particular cases of our general equation (2.44). Since $G$ is the sum of two contributions, we have to consider brackets involving these two terms separately. First of all we write down the brackets of $G_1$ and the bosonic currents,

$$[G^n_1, K^{m}_{ab}] = G^{n+m}_{1,ab} + m\delta_{ab} \sum_c \Psi^{n+m}_{cc} - mk\Psi^{n+m}_{ab}$$

$$[G^n_1, L^{m}_{ab}] = - G^{n+m}_{1,ab} - m\delta_{ab} \sum_c \Psi^{n+m}_{cc} - mk\Psi^{n+m}_{ab},$$ \hfill (3.27)

where $G_{1,ab}$ is a fermionic bilinear operator given by

$$G_{1,ab} =: \sum_c (\Psi_{ac}K_{cb} - L_{ac}\Psi_{cb}) : .$$ \hfill (3.28)

In the same way, the brackets of $G_1$ and the fermionic currents $\Psi$ and $\Lambda$ are given
by

\[ [G^n_1, \Psi^m_{ab}] = 0 \]

\[ [G^n_1, \Lambda^m_{ab}] = T^{n+m}_{1,ab} + k m (K^{n+m}_{ab} - L^{n+m}_{ab}) - m \delta_{ab} J^{n+m}_E, \tag{3.29} \]

with

\[ T^{n+m}_{1,ab} = \sum_c : (K_{ac} K_{cb} - L_{ac} L_{cb} + \Psi_{ac} \Lambda_{cb} - \Lambda_{ac} \Psi_{cb}) :. \tag{3.30} \]

The brackets involving \( G^2 \) are calculated in the same way, with the result

\[ [G^n_2, K^m_{ab}] = G^{n+m}_{2,ab} - m k \delta_{ab} \sum_c \Psi^{n+m}_{cc} \]

\[ [G^n_2, L^m_{ab}] = - G^{n+m}_{2,ab} + m k \delta_{ab} \sum_c \Psi^{n+m}_{cc} \tag{3.31} \]

\[ [G^n_2, \Psi^m_{ab}] = 0 \]

\[ [G^n_2, \Lambda^m_{ab}] = T^{n+m}_{2,ab} + m k \delta_{ab} J^{n+m}_E, \]

where now \( G_{2,ab} \) and \( T_{2,ab} \) are given by

\[ G_{2,ab} = : \Psi_{ab} J_E : \]

\[ T_{2,ab} = : (K_{ab} + L_{ab}) J_E :. \tag{3.32} \]

Notice that \( G_i = \sum_{a,b} G_{i,ab} \) and \( T_i = \sum_{a,b} T_{i,ab} \), for \( i = 1, 2 \).

Using the basic results (3.27), (3.29) and (3.31) we can obtain the brackets between the different terms of \( G \). After some calculations we get

\[ [G^n_1, G^m_1] = 2 \sum_{a,b} : G_{1,ab} \Psi_{ba} : + : \Psi_{aa} \partial \Psi_{bb} :)^{n+m} \]

\[ [G^n_1, G^m_2] = [G^m_2, G^m_1] = 2 k \sum_{a,b} : \partial \Psi_{aa} \Psi_{bb} :)^{n+m} \tag{3.33} \]

\[ [G^n_2, G^m_2] = 0. \]

In order to get (3.33) one has to pursue the same steps we followed to obtain eq. (2.44). Notice that this last equation is not applicable here, since the brackets
(3.27), (3.29) and (3.31) are not of the form displayed in eq. (2.43). Taking eq. (3.28) into account, we arrive at

\[ [G^n, G^m] = W^{n+m}, \quad (3.34) \]

where \( W \) is a bosonic operator, trilinear in the currents, whose explicit expression is

\[ W = \frac{1}{2k^2} : Tr(G_1 \Psi) : + \frac{1}{2k^2} : Tr(\partial \Psi Tr \Psi) : . \quad (3.35) \]

In eq. (3.35) we have used a trace notation to represent the double \( gl(N, N) \) summation of eq. (3.33); this notation simplifies our equations greatly and will be frequently used from now on.

As was announced above, \( W \) does not vanish in general. Notice that \( W \) is a spin-three operator which, as we shall show below, behaves as a Virasoro primary field. The presence of this \( W \) operator in the right-hand side of (3.34) will force us to extend the topological algebra. In this extended algebra one should include the brackets of \( W \) with all other generators (i.e. with \( Q, R, T \) and \( G \)). In principle this process will introduce new fields and there is no guarantee that the algebra will close with a finite number of generators. A priori, this situation is similar to the one presented when one analyses \( W \)-algebras which, in general, only close with a finite number of fields if they are non-linear. However, in our case, the situation is quite different. We shall check below that, in order to close the algebra, we will need to introduce just one additional operator (a BRST partner of \( W \)) and nevertheless the algebra will remain linear. Before launching into the calculations supporting this conclusion, let us study in what cases \( W \) is not identically zero. Having this purpose in mind, we shall reorder \( W \) using the rules developed in the previous section. The basic idea consists in getting the fermionic currents appearing in the trilinear expression of \( W \) to contiguous places; in so doing, we shall be able to use in some cases the nilpotency of these currents to conclude the vanishing of the corresponding contribution to \( W \). Using (2.40) we can rewrite the first term of eq.
\[ T r(G_1 \Psi) := \sum_{a,b,c} : (\Psi_{ac} K_{cb} \Psi_{ba} - L_{ac} \Psi_{cb} \Psi_{ba}) : . \] (3.36)

Let us now reorder the first term in the right-hand side of (3.36) using the general equation (2.40). Taking the bracket of \( K \) and \( \Psi \) displayed in eq. (3.4) into account, we can readily write:

\[ \sum_{a,b,c} : \Psi_{ac} K_{cb} \Psi_{ba} := - \sum_{a,b,c} : K_{ab} \Psi_{bc} \Psi_{ca} : - N \sum_{a,b} : \partial \Psi_{ab} \Psi_{ba} : . \] (3.37)

This result can be plugged back in eq. (3.36), after which we get

\[ W = - \frac{1}{2k^2} : T r[(K + L) \Psi \Psi] : + \frac{1}{2k^2} : [T r\partial \Psi T r\Psi - N T r(\partial \Psi \Psi)] : . \] (3.38)

This reordered expression shows that \( W \) vanishes in the \( gl(1,1) \) theory. Indeed, in this case we only have a single \( \Psi \) current and therefore we can eliminate the trace operation appearing in (3.38). For \( N = 1 \), due to the nilpotency of \( \Psi \), the first term in the right-hand side of (3.38) vanishes, while the other two terms cancel with each other. Thus, only for \( gl(N,N) \) with \( N > 1 \) do we really have a topological algebra extended by a dimension-three operator. Let us now come back to the general case. An important piece of information is obtained by computing the BRST variation of \( W \). Concentrating on the first term in (3.35), which is proportional to \( : T r(G_1 \Psi) : \) and using the BRST variation of the currents (eq. (3.12)), we get

\[ \delta T r(\Psi : G_1 : ) = : T r[T_1 \Psi - G_1(K + L)] : . \] (3.39)

Freely reordering the fields in the right-hand side of eq. (3.39) we get a complete cancellation of all the terms trilinear in the currents. However, we know that with this reordering we generate (as in eq. (2.40)) bilinear terms containing the
derivatives of the fields. In our case we are left with
\[
\delta Tr(: G_1 \Psi :) = - : Tr(\partial K + \partial L) Tr \Psi : + : Tr \partial \Psi Tr (K + L) : =
\]
\[
= - \delta (Tr : \partial \Psi \Psi :),
\]
where in the last step we have used eq. (3.12) again to express the result as a total \(\delta\)-variation. Comparing eq. (3.40) with the expression of \(W\) (eq. (3.38)), we conclude that \(W\) is \(\delta\)-closed, namely,
\[
\delta W = 0.
\] (3.41)

Remarkably \(W\) is also \(\delta\)-exact. It is easy to guess the expression whose BRST variation gives \(W\). For this purpose eq. (3.38) is very illustrative. Consider the first term in this expression. Given that \(\delta \Psi = K + L\), one is led to suppose that \(\delta \Psi^3\) is proportional to \(W\) and that the last two terms in (3.38) containing derivatives of \(\Psi\) will originate as the result of the change of order of the fields inside the normal-ordered product. That this is indeed the case can be checked by a direct calculation that we now describe. Using (3.12) we get
\[
\delta Tr(: \Psi^3 :) = : Tr[(K + L) \Psi \Psi - \Psi(K + L) \Psi + \Psi \Psi(K + L)] :
\] (3.42)
The three terms in eq. (3.42) have the same structure. If we try to convert the second and third term into the first one (which has the order appearing in the expression of \(W\)) we generate derivative terms terms as follows:
\[
\frac{1}{3} \delta Tr(: \Psi^3 :) = : Tr[(K + L) \Psi \Psi] : - : Tr \partial \Psi Tr \Psi : + N : Tr(\partial \Psi \Psi) :.
\] (3.43)
Comparing eqs. (3.43) and (3.38), it is evident that upon defining the fermionic operator \(V\) as
\[
V \equiv - \frac{1}{6k^2} Tr(: \Psi^3 :),
\] (3.44)
one has
\[
W = \delta V,
\] (3.45)
which is the desired result. Notice that for \(gl(1,1)\) current algebras \(V\) vanishes
identically, which is consistent with the fact that $W$ is also zero in this case.

One might wonder if there is a redefinition of $T$, $G$, and $R$ such that the algebra they close is just the twisted $N = 2$ superconformal algebra, *i.e.*, such that $G$ is nilpotent and $W$ can be set to zero. In other words, we are asking ourselves to what extent is it unavoidable to deal with the extended algebra instead of the usual (non-extended) one. We could try, for example, to exploit the fact that the choice for the BRST partner of $T$ is not unique. Indeed, we could add to $G$ a BRST variation of a dimension-two bosonic operator. It is easy to convince oneself that, if we want to keep the expressions for our generators local in the currents, then there is only one possibility for $G$ (*i.e.* the one we have chosen in eq. (3.18)). As our $Q$-symmetry carries $R$-charge $+1$ and $G$ must have $R$-charge $−1$ in the topological algebra (see eq. (3.24)), we need a dimension-two bosonic operator with $R$-charge equal to $−2$ in order to preserve the $R$-symmetry. Moreover, we can only add $sl(N)$ singlet operators to $G$. Taken the fact that the bosonic currents $K$ and $L$ have vanishing $R$-charge, and that the $\Lambda$'s possess $R$-charge equal to $+1$, we are forced to consider operators constructed with two $\Psi$ currents. So the only possible terms satisfying all the above conditions and whose $\delta$-variations can be added to $G$ are of the form $Tr(\Psi^2)$ and $Tr(\Psi)Tr(\Psi)$, which vanish identically due to eq. (3.3).

By inspecting eqs. (3.19) and (3.45) one immediately notices a clear parallelism. Inspired by this analogy one could be led to think that the implementation of eq. (3.45) by means of the local current $Q(z)$ would oblige us to consider new operators, which would show up when $V$ is anticommutated with the non-zero modes of $Q$. Fortunately this is not the case, as we are now going to establish, and our extended topological algebra closes with the only addition of $W$ and its BRST partner $V$ to the generators $T$, $G$, $R$, and $Q$ of the twisted $N = 2$ superconformal algebra.

As we have an explicit representation of the operators in terms of the $gl(N, N)$ currents, we can compute the brackets involving $V$ and $W$ directly. However in
some cases this direct calculation is rather long and tedious. Fortunately the same result can be obtained in a much simpler way by exploiting consistency conditions of the brackets already obtained explicitly. The fact that in the bracket of $G$ with itself only one operator ($i.e.$ $W$) appears is crucial in order to solve these consistency conditions. The generalized Jacobi identity (2.6) will become the basic tool in this approach. For example, if we calculate the bracket of $Q^r$ with both sides of eq.(3.34) and apply (2.6), we obtain

$$[Q^r, W^{n+m}] = -[G^n, [G^m, Q^r]] - [G^m, [Q^r, G^n]].$$

(3.46)

After making use of eqs. (3.21), (3.20) and (3.24) in the right-hand side of (3.46) we arrive at

$$[Q^r, W^n] = 0,$$

(3.47)

which generalizes eq. (3.41).

In the same way we can prove that $W$ is a Virasoro primary field with conformal dimension three. We start by computing the bracket of $T^r$ with both sides of eq. (3.34). Using again the Jacobi identity and the Virasoro primary character of $G$ (eq. (3.20)) we get

$$[T^r, W^{n+m}] = -[G^n, [G^m, T^r]] + [G^m, [T^r, G^n]] = (r - m)[G^n, G^{m+r}] + (r - n)[G^m, G^{n+r}],$$

(3.48)

which, after using (3.34) again, yields the desired result:

$$[T^r, W^n] = (2r - n)W^{r+n}.$$  

(3.49)

Let us now study the behaviour of $W$ under the $R$-symmetry. Computing the bracket of $R$ with eq. (3.34) and proceeding as above we obtain:

$$[R^r, W^{n+m}] = -[G^n, G^{m+r}] - [G^m, G^{n+r}].$$

(3.50)

If we now take the basic eq. (3.34) into account, we see that $W$ has charge $-2$
with respect to the abelian current $R$:

$$[R^r, W^m] = -2W^{r+m}, \quad (3.51)$$

On the other hand from (2.6) we have

$$[Q^r, [G^n, W^m]] = [G^n, [W^m, Q^r]] - [W^m, [Q^r, G^n]], \quad (3.52)$$

and using eqs. (3.47), (3.21), (3.49) and (3.51) we get:

$$[Q^r, [G^n, W^m]] = (2n - m)W^{n+m+r}. \quad (3.53)$$

Putting $r = 0$ in (3.53) we obtain the BRST variation of $[G^n, W^m]$

$$\delta([G^n, W^m]) = (2n - m)W^{n+m}. \quad (3.54)$$

Comparing this last equation with (3.45) one is tempted to write

$$[G^n, W^m] = (2n - m)V^{n+m}. \quad (3.55)$$

This equation actually holds as we shall prove below using an independent argument. Let us assume for a moment that eq. (3.55) is satisfied and see what the consequences are. Substituting back this result into (3.53) we get:

$$[Q^r, V^n] = W^{r+n}. \quad (3.56)$$

It is instructive to compare this last equation with eq. (3.21) relating the energy-momentum tensor to its BRST partner. As was stated previously, contrary to what happens with $T$ and $G$, the relation between $W$ and its fermionic counterpart $V$ does not involve any new operator.
There are still several other brackets which we have to compute in order to completely check the closure of the topological algebra. First of all, some of them can be shown to vanish by a simple inspection of their explicit form. For example, taking into account that $\Psi$ anticommutes with itself and that

$$\left[ \mathcal{G}_{1,ab}^n, \Psi_{cd}^m \right] = 0, \quad (3.57)$$

we can write

$$\left[ V^r, V^n \right] = 0$$
$$\left[ V^r, W^n \right] = 0 \quad (3.58)$$
$$\left[ G^r, V^n \right] = 0,$$

where in order to prove the last equality we must use eqs. (3.29) and (3.31). Moreover it is a straightforward consequence of our general equation (2.51) that

$$\left[ T^r, V^n \right] = (2r - n)V^{r+n}, \quad (3.59)$$

i.e., $V$ is a dimension-three primary field.

Let us now give the promised proof of eq. (3.55). First of all, using eqs. (3.19) and (3.45) we have

$$\delta(\left[ G^r, V^n \right]) = \left[ T^r, V^n \right] - \left[ G^r, W^n \right]. \quad (3.60)$$

By virtue of the last equation in (3.58), the left-hand side of (3.60) vanishes. The primary character of $V$ with respect to $T$ (eq. (3.59)) is easily seen to imply (3.55). Now, using eqs. (3.41) and (3.45) it follows that

$$\delta(\left[ V^r, W^n \right]) = \left[ W^r, W^n \right]. \quad (3.61)$$

And, after using the second equation in (3.58), we conclude that

$$\left[ W^r, W^n \right] = 0. \quad (3.62)$$

In order to complete the algebra it remains to compute the bracket of $R$ with
The Jacobi identity implies:

\[ [R^r, [G^m, W^m]] = -[G^m, [W^m, R^r]] - [W^m, [R^r, G^m]]. \] (3.63)

As \( G \) and \( W \) have \( R \)-charges \(-1\) and \(-2\) (see eqs. (3.24) and (3.51)) it follows that

\[ [R^r, [G^m, W^m]] = -2[G^m, W^{m+r}] + [W^m, G^{m+r}]. \] (3.64)

Using (3.55) in both sides of (3.64) one readily concludes that

\[ [R^r, V^n] = -3V^{r+n}, \] (3.65)

i.e., \( V \) has \( R \)-charge equal to \(-3\). This bracket completes the set of relations defining the extended topological algebra. We have gathered all the brackets we have obtained in Appendix A, where we have written the form of the algebra for arbitrary dimension \( d \). Up to now, we have obtained a representation for \( d = 0 \) only. In the next section we shall see that there exists a modification of the generators giving rise to a \( d \neq 0 \) extended algebra.

As was mentioned in section 1, the extended topological algebra we have found has been previously studied by Kazama [17], who characterized algebraically some non-trivial extensions of the twisted \( N = 2 \) superconformal algebra. The algebra compiled in Appendix A was obtained form a truncation of a larger algebra in which there are two extra operators, one fermionic and the other bosonic, of dimensions zero and one respectively. This larger extended algebra is satisfied by the \( bc \) system studied by Distler [18], which is relevant in two-dimensional quantum gravity. Nevertheless, as far as we know, an explicit representation of the algebra displayed in Appendix A was not known.

To summarize, in this section we have been able to obtain the algebra encoding the topological symmetry of our \( gl(N, N) \) theory. The generators of this algebra have dimensions 1, 2, and 3 and have been represented as normal-ordered products.
of currents. Only for the $gl(1,1)$ case does our model possess a twisted $N = 2$ superconformal algebra. In general we have to deal with an extended topological algebra, which has the important property of closing with a set of $gl(N)$ bosonic and fermionic currents. This last question will be further explored in the next section.

4. The topological current algebra and $d \neq 0$ extensions

Let us now discuss the relation of our topological algebra with the underlying current algebra from which the former has been obtained. The brackets of $T$ and the different currents are fixed by the fact that the $J_{AB}$ are primary dimension-one operators (see eq. (2.49)). It would therefore seem natural to expect the (anti)commutator of $G$ with the currents to be given by an odd analogue of eq. (2.49). However this is not the case, as can be verified by combining eqs. (3.27), (3.29) and (3.31). One has

$$[G^n, K^m_{ab}] = G^{n+m}_{ab} - \frac{m}{2} \Psi^{n+m}_{ab},$$

$$[G^n, L^m_{ab}] = -G^{n+m}_{ab} - \frac{m}{2} \Psi^{n+m}_{ab},$$

$$[G^n, \Psi^m_{ab}] = 0,$$

$$[G^n, \Lambda^m_{ab}] = T^{n+m}_{ab} + \frac{m}{2} (K^{n+m}_{ab} - L^{n+m}_{ab}),$$

where

$$G_{ab} = \frac{1}{2k} G_{1,ab} + \frac{1}{2k^2} G_{2,ab},$$

$$T_{ab} = \frac{1}{2k} T_{1,ab} + \frac{1}{2k^2} T_{2,ab}. \quad (4.2)$$

Therefore we conclude that, contrary to what happens with $T$, the algebra of its topological partner $G$ and the currents does not close, since new dimension-two operators ($G_{ab}$ and $T_{ab}$) appear in the right-hand side of (4.1). These new fields have to be introduced in the algebra and when this is done we get a proliferation of additional operators. Performing a similar calculation with $V$ and $W$ we would
reach an equivalent conclusion. For example the bracket \([W, L_{ab}]\) gives rise to new dimension-three operators. Quite remarkably, however, there is a subset of currents that closes with the extended topological algebra without introducing new fields. Consider first of all the bosonic currents. By inspecting (4.1) one readily realizes that the diagonal combination

\[ J_{ab} \equiv K_{ab} + L_{ab} \]  

(4.3)
satisfies

\[ [G^n, J_{ab}^m] = -m \Psi_{ab}^{n+m}, \]  

(4.4)

which is indeed the odd analogue of

\[ [T^n, J_{ab}^m] = -m J_{ab}^{n+m}. \]  

(4.5)

Thus \(G\) closes with half the bosonic currents. The situation for the fermionic ones is much simpler. It is evident that \(G\) closes with the \(\Psi\)'s only. Recall (see eq. (3.12)) that the \(\Psi\)'s are the fermionic partners of the \(J\)'s. In fact using (3.10) we can write

\[ [Q^n, \Psi_{ab}^m] = J_{ab}^{n+m} + kn \delta_{ab} \delta_{n+m,0} \]  

\[ [Q^n, J_{ab}^m] = 0. \]  

(4.6)

Eq. (4.6) relating \(\Psi\) to its BRST partner \(J\) has a close resemblance to eq. (3.21) relating \(Q\) and \(T\). This similarity is reinforced when one looks at the algebra closed by \(J\) and \(\Psi\):

\[ [J^n_{ab}, J^m_{cd}] = \delta_{bc} J^{n+m}_{ad} - \delta_{ad} J^{n+m}_{cb} \]  

\[ [\Psi^n_{ab}, J^m_{cd}] = \delta_{bc} \Psi^{n+m}_{ad} - \delta_{ad} \Psi^{n+m}_{cb} \]  

(4.7)

\[ [\Psi^n_{ab}, \Psi^m_{cd}] = 0. \]

Notice that no central extension appears in (4.7). In fact the first bracket in (4.7) is the basic commutator of an affine, zero level, \(gl(N)\) current algebra. We can
summarize the situation as follows: \((G, T)\) and \((\Psi, J)\) are topological doublets of operators with dimensions two and one respectively. In both cases the two members of this topological multiplet close an algebra without any central element. However a \(c\)-number anomaly is obtained when the current of the topological symmetry is anticommutated with the fermionic member of the doublet (of course, in the case of the dimension-two doublet \((G, T)\), the abelian current \(R\) is also obtained when \(Q\) acts on \(G\)). Based on this similarity, if a system is endowed with bosonic and fermionic currents satisfying relations like eqs. (4.6) and (4.7), we will say that it possesses a topological current algebra, in the same sense that the algebra in Appendix A is the topological version of the Virasoro algebra. This type of algebras are present in other topological theories. This fact is illustrated in Appendix B, where an \(sl(2)\) WZW model at zero level is studied as a topological field theory [28].

For completeness we should compute the bracket of all the generators appearing in the topological algebra with the currents. An easy calculation gives how \(J\) and \(\Psi\) behave under the \(R\) symmetry:

\[
\left[ R^m, J_{ab}^m \right] = \delta_{ab} \delta_{n+m,0} \\
\left[ R^m, \Psi_{ab}^m \right] = -\Psi_{ab}^{n+m}.
\]  

(4.8)

Finally, by simple inspection we get:

\[
\left[ W^n, \Psi_{ab}^m \right] = \left[ V^n, \Psi_{ab}^m \right] = 0.
\]  

(4.9)

Performing a \(\delta\)-variation of this equation and using eqs. (3.41), (3.45) and (3.12) one obtains

\[
\left[ W^n, J_{ab}^m \right] = \left[ V^n, J_{ab}^m \right] = 0.
\]  

(4.10)

Altogether eqs. (4.4)–(4.10) imply that our conformal topological algebra is compatible with the \(gl(N)\) topological current algebra generated by \(J_{ab}\) and \(\Psi_{ab}\).
As was proved above, our \( gl(N, N) \) theory satisfies the extended topological algebra with the parameter \( d \) equal to zero (see eq. (3.25)). Let us ask ourselves if it is possible to modify the theory in such a way that it continues to be topological but the parameter \( d \) does not vanish. In other words, we would like to find out whether or not there are marginal directions (in the moduli space of topological theories to which our model belongs) such that one can move away from the \( d = 0 \) point. With this objective in mind, notice that we can deform our energy-momentum tensor by adding a two-dimensional term that preserves its BRST-exactness. Indeed, due to the last equality in eq. (3.12), it follows that for any c-number constant matrix \( \alpha_{ab} \), the transformation

\[
T \rightarrow T + \sum_{a,b} \alpha_{ab} \partial J_{ab} \tag{4.11}
\]

keeps the operator \( T \) \( \delta \)-exact. Moreover the transformed operator \( T \) satisfies the extended topological algebra if \( G \) and \( R \) are modified as

\[
G \rightarrow G + \sum_{a,b} \alpha_{ab} \partial \Psi_{ab} \\
R \rightarrow R + \sum_{a,b} \alpha_{ab} J_{ab} \tag{4.12}
\]

and the spin-three operators \( V \) and \( W \) remain unchanged. It is important to stress the fact that, under the transformations (4.11) and (4.12), the algebra retains its form, the only modification introduced being the change of the parameter \( d \) (see below). In terms of modes, the twist performed in eqs. (4.11) and (4.12) is equivalent to

\[
T^n \rightarrow T^n - (n + 1) \sum_{a,b} \alpha_{ab} J_{ab}^n \\
G^n \rightarrow G^n - (n + 1) \sum_{a,b} \alpha_{ab} \Psi_{ab}^n \\
R^n \rightarrow R^n + \sum_{a,b} \alpha_{ab} J_{ab}^n \tag{4.13}
\]
The dimension of the modified algebra now becomes

$$d = 2kTr(\alpha).$$

(4.14)

Therefore, in principle we have (at least) $N^2$ possible modular parameters in the space of topological theories related to our original $gl(N,N)$ model. This number of parameters is drastically reduced if we require to our twisted theory to possess a current algebra symmetry. Consider first the bosonic currents. An easy calculation shows that the commutators of the modified operator $T$ with $K$ and $L$ are given by

$$[T^n, K_{ab}^m] = -mK_{ab}^{n+m} + (n+1) \left( \sum_c (\alpha_{bc}K_{ac}^{n+m} - \alpha_{ca}K_{cb}^{n+m}) - kn\alpha_{ba}\delta_{n+m,0} \right)$$

$$[T^n, L_{ab}^m] = -mL_{ab}^{n+m} + (n+1) \left( \sum_c (\alpha_{bc}L_{ac}^{n+m} - \alpha_{ca}L_{cb}^{n+m}) + kn\alpha_{ba}\delta_{n+m,0} \right).$$

(4.15)

In view of this result it is clear that the only chance to have a $gl(N)$ current algebra is by requiring the second term in the right-hand side of (4.15) to vanish. In general this only takes place when the matrix $\alpha$ is proportional to the unit matrix. Accordingly let us write

$$\alpha_{ab} = \alpha \delta_{ab}.$$  

(4.16)

With this restriction we immediately obtain from (4.15) that the currents $J$ appearing in the topological current algebra are primary dimension-one operators for the twisted energy-momentum operator. However the combinations $K_{ab} - L_{ab}$ of the bosonic currents pick up an anomaly which is proportional to the dimension $d$ of the algebra. According to eq. (4.14) this parameter $d$ is now equal to $2k\alpha$. A straightforward calculation shows that when eq. (4.16) holds, the $\Psi_{ab}$ are also primary and eq. (4.4)–(4.10) are satisfied. Therefore the only remaining variable $\alpha$ parametrizes a line in the moduli space of conformal topological theories, such
that all models contained in this line enjoy the $gl(N)$ topological current algebra described above. From this viewpoint $\alpha = 0$ corresponds to a model in which this symmetry is enhanced and we have a full set of $gl(N, N)$ conserved currents. Notice that, when $\alpha_{ab}$ has the form displayed in eq. (4.16), $T$, $R$ and $G$ are twisted along the direction of the unit of the $gl(N)$ algebra (see eqs. (4.11) and (4.12)). In order to fix this deformation parameter $\alpha$, one needs to impose further requirements. For example, for the $gl(1, 1)$ model, the standard minimal topological matter [29] with dimension $n^2 + 2$, $n$ being an arbitrary integer, is obtained by putting $\alpha = \frac{n}{2k(n+2)}$.

In the next section we shall check, using a free field representation, that in this case our theory is equivalent to a twisted $N = 2$ minimal superconformal model. In the general $gl(N, N)$ case a criterion to fix $\alpha$ is lacking.

As pointed out by Kazama [17], a hint to understand the nature of our extended topological algebra is provided by the observation that it can be obtained by twisting an $N = 1$ superconformal model in which extra supermultiplets are present. In order to check this point, let us define the twisted energy-momentum tensor $\tilde{T}$ as follows:

$$\tilde{T} = T - \frac{1}{2} \partial R. \quad (4.17)$$

Notice that this twist is performed along a $U(1)$ direction orthogonal to the identity current $J_E$, which corresponds to the twist (4.11) when the eq. (4.16) is imposed. Another difference between (4.11) and (4.17) is that in this latter case there is no free parameter. It is an easy exercise to prove that the modes of the operator $\tilde{T}$ satisfy the commutation relations

$$[\tilde{T}^n, \tilde{T}^m] = (n - m)\tilde{T}^{n+m} + \frac{d}{4} n(n^2 - 1) \delta_{m+n,0}, \quad (4.18)$$

which correspond to a central charge

$$c = 3d. \quad (4.19)$$

The coefficient in the second term of (4.17) has been chosen in such a way that,
with respect to the modified tensor $\tilde{T}$, the $U(1)$ current $R$ becomes a primary dimension-one operator. Let us define the following operators:

\[
\begin{align*}
\tilde{G}^n & = G^{n-\frac{1}{2}} \\
\tilde{Q}^n & = Q^{n+\frac{1}{2}} \\
\tilde{V}^n & = V^{n-\frac{3}{2}} \\
\tilde{W}^n & = W^{n-1}.
\end{align*}
\] (4.20)

It is easy to check that $\tilde{G}^n$, $\tilde{Q}^n$ and $\tilde{V}^n$ are the Laurent modes of dimension-$\frac{3}{2}$ primary fields, whereas $\tilde{W}^n$ corresponds to a dimension-2 Virasoro primary. Thus the deformation (4.17) changes the dimensions of the operators appearing in our algebra. Moreover we can find a linear combination of the dimension-$\frac{3}{2}$ fields mentioned above such that behaves as supercurrent. Indeed if we define

\[
\tilde{G}^n = \tilde{Q}^n + \tilde{G}^n - \frac{1}{2}\tilde{V}^n,
\] (4.21)

we can immediately check using the brackets of Appendix A that

\[
[\tilde{G}^n, \tilde{G}^m] = 2\tilde{T}^{n+m} + d(n^2 - \frac{1}{4})\delta_{m+n,0},
\] (4.22)

which corresponds to the anticommutator among the modes of the supercurrent of an $N = 1$ superconformal algebra. It is easy to verify that the other fields appearing in the topological algebra can be accommodated into $N = 1$ supermultiplets. Recall that such multiplets are composed by two fields $(A, B)$ in such a way that they are connected by the supercurrent. In fact if $\Delta_A$ and $\Delta_B$ are the conformal dimensions of the fields $A$ and $B$ ($\Delta_B = \Delta_A + \frac{1}{2}$) one has

\[
\begin{align*}
[\tilde{G}^n, A^m] & = B^{n+m} \\
[\tilde{G}^n, B^m] & = ((2\Delta_A - 1)n - m)A^{n+m}.
\end{align*}
\] (4.23)

Two of such supermultiplets can be formed. First of all, if we define

\[
S = -\tilde{Q} + \tilde{G} - \frac{3}{2}\tilde{V},
\] (4.24)

it can be checked after a short calculation that $S$ is the supersymmetric companion of the $U(1)$ current $R$, i.e., $(R, S)$ behave as in eq. (4.23) with $\Delta_A = 1$. The two
remaining fields \( \tilde{V} \) and \( \tilde{W} \) are the components of the multiplet \((\tilde{V}, \tilde{W})\) which satisfy eq.(4.24) with conformal weight \( \frac{3}{2} \).

5. The \( GL(N, N) \) Wess-Zumino-Witten model and free fields

In this section we shall study a two-dimensional field theory possessing a \( gl(N, N) \) current algebra. This model is nothing but a Wess-Zumino-Witten (WZW) model for the \( GL(N, N) \) supergroup. Our basic variable will now be a function \( g \) taking values in \( GL(N, N) \). As is well known, in a vicinity of the unit, \( g \) can be represented locally as the exponential of an element of the \( gl(N, N) \) algebra. Let us introduce complex coordinates for our two-dimensional base manifold according to the conventions:

\[
\begin{align*}
z &= x + iy \\
\bar{z} &= x - iy
\end{align*}
\]

\[
\partial = \frac{1}{2}(\partial_x - i\partial_y) \\
\bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y),
\]

where \( x \) and \( y \) are arbitrary real coordinates. The action for the WZW model is given by [20]

\[
S[g] = k\Gamma[g] = \frac{k}{2\pi} \int d^2x \text{Str}(g^{-1}\partial g g^{-1}\bar{\partial} g) + \\
+ \frac{ik}{12\pi} \int d^3x \epsilon^{\mu\nu\rho} \text{Str}(g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\rho g),
\]

where \( k \) is a c-number constant (the level of the current algebra) and, as is well known, the three-dimensional integral appearing in the right-hand side of (5.2) is taken over a manifold whose boundary is our two-dimensional base manifold. The functional \( \Gamma \) defined in (5.2) satisfies the remarkable Polyakov-Wiegmann (PW) cocycle property [30]:

\[
\Gamma[gh] = \Gamma[g] + \Gamma[h] + \frac{1}{\pi} \int d^2x \text{Str}(g^{-1}\bar{\partial} g \partial hh^{-1}).
\]

This equation can be demonstrated by a direct calculation. The only difference between the proof of the PW property for groups and supergroups is that, in the
latter case, we have to deal with supertraces instead of ordinary traces. As the supertrace of elements of \( GL(N, N) \) satisfies the cyclic property, eq. (5.3) can be obtained following the same steps as in the case of ordinary groups.

One of the main advantages of working with an explicit representation like the one based on the action (5.2) is that one can convert the model into a theory of free fields. The standard procedure to achieve this objective has been described in ref [21]. The first step consists in using a gaussian representation of the (super)group variable. Any element of \( GL(N, N) \) can be decomposed as

\[
g = g_L g_D g_U = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & \chi \\ 0 & 1 \end{pmatrix}, \tag{5.4}
\]

where we have used an \( N \times N \) block notation in which \( \alpha \) and \( \beta \) (\( \chi \) and \( \lambda \)) are \( N \times N \) bosonic (respectively fermionic) matrices. Using the PW property (eq. (5.3)) we can readily write the action \( S \) in terms of the variables introduced in (5.4):

\[
S[g] = k\gamma[\alpha] - k\gamma[\beta] - \frac{k}{\pi} \int d^2x Tr(\alpha \partial \chi \beta^{-1} \bar{\partial} \lambda), \tag{5.5}
\]

where \( \gamma \) is the general WZW functional for the \( gl(N) \) group, whose explicit expression can be obtained by replacing supertrace by trace in eq. (5.2). The currents implementing the \( gl(N, N) \) Kac-Moody symmetry are given by Witten’s bosonization formula. Concentrating only on the holomorphic part of the algebra we have:

\[
J = -k\partial gg^{-1}. \tag{5.6}
\]

The different contravariant components of the current can be read from the matrix elements of \( J \) as follows (see eq. (2.26)):

\[
J = \begin{pmatrix} J^{ab} & J^{a,b+N} \\ J^{a+N,b} & J^{a+N,b+N} \end{pmatrix}. \tag{5.7}
\]
Substituting the gaussian decomposition (5.4) in eq. (5.6) we obtain:

\[
\begin{align*}
J_{ab} &= k(\alpha \partial \chi^\beta - 1) - \partial \alpha_{\alpha\beta}^{ab} \\
J_{a,b+N} &= -k(\alpha \partial \chi^\beta - 1)^{ab} \\
J_{a+N,b} &= k(\lambda \partial \chi^\beta - 1) - \lambda \partial \alpha_{\alpha\beta}^{ab} \\
J_{a+N,b+N} &= -k(\lambda \partial \chi^\beta - 1 + \partial \beta_{\beta\gamma}^{ab}) 
\end{align*}
\]

The action (5.5) and the current components (5.8) are greatly simplified in terms of the fermionic variables

\[
\begin{align*}
\xi_{ab} &= \lambda_{\gamma\beta}^{ba} \\
\eta_{ab} &= k(\alpha \partial \chi^\beta - 1)^{ba}.
\end{align*}
\]

In fact the fermionic part of the action \( S \) takes a free-field form

\[
S[g] = k\gamma[\alpha] - k\gamma[\beta] - \frac{1}{\pi} \int d^2x Tr(\eta \partial \xi).
\]

Notice that the operator product expansion (OPE) between \( \eta \) and \( \xi \) that follows from (5.10) is simply

\[
\eta_{ab}(z)\xi_{cd}(w) = -\frac{\delta_{ad}\delta_{bc}}{z - w}. \tag{5.11}
\]

The bosonic \( gl(N) \) currents corresponding to the \( \alpha \) and \( \beta \) variables appearing in (5.10) are

\[
\begin{align*}
J_{(\alpha)}^{ab} &= -k(\partial \alpha_{\alpha\beta}^{ab}) \\
J_{(\beta)}^{ab} &= k(\partial \beta_{\beta\gamma}^{ab}).
\end{align*}
\]

In terms of the free fermionic fields \( \eta, \xi \) and the currents (5.12), the different components of \( J \) simplify greatly. First of all, using eqs. (2.27) and (3.1), we can write

\[
\begin{align*}
K_{ab} &= J_{ab} = J^{ba} \\
L_{ab} &= J_{a+N,b+N} = -J^{b+N, a+N} \\
\Psi_{ab} &= J_{a+N,b} = J^{b, a+N} \\
\Lambda_{ab} &= J_{a,b+N} = -J^{b+N, a}.
\end{align*}
\]
and, taking eqs. (5.8), (5.9) and (5.12) into account, one gets:

\[
\begin{align*}
K_{ab} &= - (\xi \eta)_{ab} + j^{(\alpha)}_{ab} \\
L_{ab} &= - (\eta \xi)_{ab} + j^{(\beta)}_{ab} \\
\Psi_{ab} &= - \eta_{ab} \\
\Lambda_{ab} &= (\xi \eta \xi)_{ab} - (j^{(\alpha)} \xi)_{ab} - (\xi j^{(\beta)})_{ab} + k \partial \xi_{ab}.
\end{align*}
\] (5.14)

Up to now, we have treated our fields as classical objects. In a quantum theory, however, a change of variables as the one written down in eq. (5.9) should be accompanied by the corresponding Jacobian, which is a functional determinant that takes the change in the measure of the corresponding path integral into account. These quantum corrections can introduce additional terms in the naive effective action (5.10), which can change the propagators and couplings of the theory and, therefore, the corresponding OPE’s between the different operators. On the other hand [21] the free-field expression of the currents can also be affected by these quantum corrections, since extra terms can originate from the regularization of singular operator products in eq. (5.8). We will not attempt to compute this functional determinant. Instead we are going to check whether or not the currents (5.14) satisfy the OPE’s of the \( gl(N, N) \) affine algebra at level \( k \). We shall see that this is not the case if the OPE’s dictated from the classical action (5.10) are used. Therefore, we must allow for a slight modification of these basic OPE’s in order to have a representation of the \( gl(N, N) \) Kac-Moody algebra. For example, the bosonic \( gl(N) \) currents \( j^{(\alpha)} \) and \( j^{(\beta)} \) must satisfy the OPE’s

\[
\begin{align*}
\hat{j}^{(\alpha)}_{ab}(z) \hat{j}^{(\alpha)}_{cd}(w) &= k_{\alpha} \delta_{ad} \delta_{bc} \frac{1}{(z-w)^2} + \frac{1}{z-w} [\delta_{bc} \hat{j}^{(\alpha)}_{ad}(w) - \delta_{ad} \hat{j}^{(\alpha)}_{cb}(w)] \\
\hat{j}^{(\beta)}_{ab}(z) \hat{j}^{(\beta)}_{cd}(w) &= k_{\beta} \delta_{ad} \delta_{bc} \frac{1}{(z-w)^2} + \frac{1}{z-w} [\delta_{bc} \hat{j}^{(\beta)}_{ad}(w) - \delta_{ad} \hat{j}^{(\beta)}_{cb}(w)],
\end{align*}
\] (5.15)

where we have allowed for a finite renormalization of the levels. These constants \( k_{\alpha} \) and \( k_{\beta} \) can be determined from the products of two bosonic \( gl(N, N) \) currents.
Performing a direct calculation using the free-field representation (5.14) and the operator products (5.11) and (5.15) we get:

\[
K_{ab}(z)K_{cd}(w) = \left(\frac{k_{\alpha} + N}{2} \right) \frac{\delta_{ad} \delta_{bc}}{(z - w)^2} + \frac{1}{z - w} \left[ \delta_{bc} K_{ad}(w) - \delta_{ad} K_{cb}(w) \right]
\]

\[
L_{ab}(z)L_{cd}(w) = \left(\frac{k_{\beta} + N}{2} \right) \frac{\delta_{ad} \delta_{bc}}{(z - w)^2} + \frac{1}{z - w} \left[ \delta_{bc} L_{ad}(w) - \delta_{ad} L_{cb}(w) \right].
\] (5.16)

Comparing the right-hand side of (5.16) with eq. (3.2) we can determine the unknown constants \(k_{\alpha}\) and \(k_{\beta}\):

\[
k_{\alpha} = k + N, \quad k_{\beta} = -k - N.
\] (5.17)

Notice that the \(gl(N)\) levels of \(j^{(\alpha)}\) and \(j^{(\beta)}\) are renormalized by the same quantity \(N\) with respect to the values \((k\) and \(-k\) respectively) dictated by the classical action (5.10). At the classical level \(j^{(\alpha)}\) and \(j^{(\beta)}\) are uncoupled. However it is natural to suppose that the jacobian of the change of variables (5.9) could induce terms that couple these two currents in the quantum effective action. Actually this coupling is needed in order to reproduce the \(gl(N, N)\) current algebra with our free-field representation. If we compute the OPE of \(K\) and \(L\) we get

\[
K_{ab}(z)L_{cd}(w) = -\frac{\delta_{ab} \delta_{cd}}{(z - w)^2} + \frac{\delta_{ab} \delta_{cd}}{(z - w)^2} \left[ j_{ab}^{(\alpha)}(z) j_{cd}^{(\beta)}(w) \right],
\] (5.18)

and therefore in order to get a vanishing result (as in (3.2)) we need to impose

\[
\frac{\delta_{ab} \delta_{cd}}{(z - w)^2} = 0
\] (5.19)

Notice the different index structure of the double pole in eqs. (5.19) and (5.15). In this last equation, the coefficient of the \(1/(z - w)^2\) term is proportional to the \(gl(N)\) Cartan-Killing form, whereas in (5.19) the coupling only affects the abelian
subalgebra of $gl(N)$ (see below). We can now compute the singular terms in the product expansion of fermionic and bosonic currents with the result

$$
\Psi_{ab}(z)K_{cd}(w) = \frac{\delta_{bc}}{z-w}\Psi_{ad}(w)
$$

$$
\Psi_{ab}(z)L_{cd}(w) = -\frac{\delta_{ad}}{z-w}\Psi_{cb}(w)
$$

$$
\Lambda_{ab}(z)K_{cd}(w) = -\frac{\delta_{ad}}{z-w}\Lambda_{cb}(w)
$$

$$
\Lambda_{ab}(z)L_{cd}(w) = \frac{\delta_{bc}}{z-w}\Lambda_{ad}(w).
$$

Eq. (5.20) is in agreement with the brackets listed in eq. (3.4). The products it remains to check are those in which two fermionic currents are multiplied. After some calculation we get

$$
\Lambda_{ab}(z)\Lambda_{cd}(w) = 0
$$

$$
\Psi_{ab}(z)\Psi_{cd}(w) = 0
$$

$$
\Psi_{ab}(z)L_{cd}(w) = \frac{k\delta_{bc}\delta_{ad}}{(z-w)^2} - \frac{1}{z-w}[\delta_{ad}K_{cb}(w) + \delta_{bc}L_{ad}(w)].
$$

When we compare the $\Psi\Lambda$ product in (5.21) with the corresponding anticommutator (eq. (3.3)), we notice that the right-hand side of the result obtained in (5.21) differs in sign from the result we should have found according to eq. (3.3). This problem can be arranged by reversing the sign of $\Psi$ in our free-field representation. Notice that none of eqs. (5.20) is altered by this change. Therefore we replace the third equation in (5.14) by

$$
\Psi_{ab} = \eta_{ab}.
$$

Once we have represented the currents by free fields, we can obtain the expression of the different operators appearing in the topological algebra. Let us begin by considering the energy-momentum tensor. First of all we denote the Sugawara
bilinears for the \( j^{(\alpha)} \) and \( j^{(\beta)} \) currents by \( T_1^{(\alpha)} \) and \( T_1^{(\beta)} \) respectively:

\[
T_1^{(\alpha)} = \sum_{a,b} : j_{ab}^{(\alpha)} j_{ba}^{(\alpha)} :
\]
\[
T_1^{(\beta)} = \sum_{a,b} : j_{ab}^{(\beta)} j_{ba}^{(\beta)} :.
\]  

(5.23)

The operators \( T_1 \) and \( T_2 \) obtained with the first and second Casimirs of \( gl(N,N) \) are represented as

\[
T_1 = T_1^{(\alpha)} - T_1^{(\beta)} - N \sum_a (\partial j_{aa}^{(\alpha)} + \partial j_{aa}^{(\beta)}) + 2k \sum_a : \eta_{ab} \partial \xi_{ba} :
\]
\[
T_2 = \sum_{a,b} : (j_{aa}^{(\alpha)} + j_{aa}^{(\beta)})(j_{bb}^{(\alpha)} + j_{bb}^{(\beta)}) :.
\]  

(5.24)

Adopting a trace notation, we can now write down the expression of \( T \) as

\[
T = : Tr(\eta \partial \xi) : + \frac{1}{2k} [T_1^{(\alpha)} - T_1^{(\beta)} - N Tr(\partial j^{(\alpha)} + \partial j^{(\beta)})]
\]
\[
+ \frac{1}{2k^2} [ : Tr(j^{(\alpha)} + j^{(\beta)}) Tr(j^{(\alpha)} + j^{(\beta)}) :].
\]  

(5.25)

As a check one may verify that the currents \( J_{AB} \) represented as in eqs. (5.14) and (5.22) are primary with respect to the operator (5.25). In this calculation one must use the singular product expansion of \( T_1^{(\alpha)} \) and \( T_1^{(\beta)} \) with \( j^{(\alpha)} \) and \( j^{(\beta)} \). These can be obtained by using eqs. (5.15) and (5.19). One has

\[
T_1^{(\alpha)}(z) j_{ab}^{(\alpha)}(w) = \frac{2(k_\alpha + N)}{(z - w)^2} j_{ab}^{(\alpha)}(w) + \frac{2(k_\alpha + N)}{z - w} \partial j_{ab}^{(\alpha)}(w) -
\]
\[
- \frac{2\delta_{ab}}{(z - w)^2} \sum_c j_{cc}^{(\alpha)}(w) - \frac{2\delta_{ab}}{z - w} \sum_c \partial j_{cc}^{(\alpha)}(w).
\]  

(5.26)

This expression can be obtained in a way similar to the one employed to prove eq. (2.45). Moreover, due to the coupling (5.19), the product \( T_1^{(\alpha)} j^{(\beta)} \) contains
singular terms. It can be easily proved that

\[ T_1^{(\alpha)}(z)j_{ab}^{(\beta)}(w) = \frac{2\delta_{ab}}{(z-w)^2} \sum_c j_c^{(\alpha)}(w) + \frac{2\delta_{ab}}{z-w} \sum_c \partial j_c^{(\alpha)}(w). \quad (5.27) \]

Of course there are equations similar to (5.26) and (5.27) in which the labels \( \alpha \) and \( \beta \) are interchanged. It is now straightforward to check that the \( J_{AB} \) currents satisfy

\[ T(z)J_{AB}(w) = \frac{1}{(z-w)^2} J_{AB}(w) + \frac{1}{(z-w)} \partial J_{AB}(w), \quad (5.28) \]

which is the OPE equivalent to the bracket (2.49). Other generators appearing in the topological algebra can be equally computed:

\[
\begin{align*}
G &= \frac{1}{2k} : Tr[\eta(j^{(\alpha)} - j^{(\beta)})] : - \frac{N}{2k} Tr(\partial \eta) + \frac{1}{2k^2} : Tr(j^{(\alpha)} + j^{(\beta)}) Tr(\eta) : \\
Q &= - \frac{1}{2k} Tr[\xi(j^{(\alpha)} + j^{(\beta)}) - \xi \eta \xi - k \partial \xi] : \\
R &= \frac{1}{2} Tr(j^{(\alpha)} - j^{(\beta)}) + : Tr(\eta \xi) : \\
W &= - \frac{1}{2k^2} : Tr[(j^{(\alpha)} + j^{(\beta)}) \eta \eta] : + \frac{1}{2k^2} : [Tr \partial \eta Tr \eta - NT \partial(\eta \eta)] : \\
V &= - \frac{1}{6k^2} : Tr(\eta^3) : .
\end{align*}
\]

Eqs. (5.25) and (5.29) provide an explicit representation of our topological algebra in terms of two bosonic currents \( j^{(\alpha)} \) and \( j^{(\beta)} \) and two fermionic fields \( \eta \) and \( \xi \). However, as we pointed out above, the abelian components of \( j^{(\alpha)} \) and \( j^{(\beta)} \) are coupled (see eq. (5.19)). A free-field representation of an algebra should be given in terms of a set of uncoupled fields. As the coupling of the traceless components of the currents vanishes, it will be convenient to separate the \( sl(N) \) parts of \( j^{(\alpha)} \) and \( j^{(\beta)} \) from the remaining \( U(1) \) contributions. This decomposition will serve us to clarify the nature of the extended topological algebra. Actually we show below that our algebra is intimately linked to the non-abelian topological current algebra underlying our theory.
We split our currents as:

\[ j^{(\alpha)}_{ab} = I^{(\alpha)}_{ab} + \delta_{ab} \sqrt{\frac{N - k}{N}} j_{1} \]

\[ j^{(\beta)}_{ab} = I^{(\beta)}_{ab} - \delta_{ab} \sqrt{\frac{N + k}{N}} j_{2}, \tag{5.30} \]

where the \( sl(N) \) currents \( I^{(\alpha)} \) and \( I^{(\beta)} \) are traceless:

\[ \sum_{a} I^{(\alpha)}_{aa} = \sum_{a} I^{(\beta)}_{aa} = 0. \tag{5.31} \]

The coefficients multiplying the abelian currents \( j_{1} \) and \( j_{2} \) have been chosen for later convenience. Using eqs. (5.15) and (5.19) we can write the OPE’s satisfied by our \( sl(N) \) currents

\[ I^{(\alpha)}_{ab}(z) I^{(\alpha)}_{cd}(w) = (k - N) \frac{\delta_{ad} \delta_{bc} - \frac{1}{N} \delta_{ab} \delta_{cd}}{(z - w)^{2}} + \frac{\delta_{bc} I^{(\alpha)}_{ad}(w) - \delta_{ad} I^{(\alpha)}_{cb}(w)}{z - w} \]

\[ I^{(\beta)}_{ab}(z) I^{(\beta)}_{cd}(w) = - (k + N) \frac{\delta_{ad} \delta_{bc} - \frac{1}{N} \delta_{ab} \delta_{cd}}{(z - w)^{2}} + \frac{\delta_{bc} I^{(\beta)}_{ad}(w) - \delta_{ad} I^{(\beta)}_{cb}(w)}{z - w}. \tag{5.32} \]

The \( U(1) \) currents have the following singular product expansions:

\[ j_{1}(z) j_{1}(w) = j_{2}(z) j_{2}(w) = - \frac{1}{(z - w)^{2}} \]

\[ j_{1}(z) j_{2}(w) = - \frac{N}{(N^{2} - K^{2})^{1/2}} \frac{1}{(z - w)^{2}}. \tag{5.33} \]

Other products of currents appearing in the decomposition (5.30) are regular. The OPE’s of eq. (5.33) can be reproduced by representing \( j_{1} \) and \( j_{2} \) in terms of two bosonic fields \( \varphi \) and \( \phi \). Let us assume that these two fields obey the OPE

\[ \varphi(z) \phi(w) = - \log(z - w). \tag{5.34} \]
Then eq. (5.33) is satisfied by the following combinations of $\partial \varphi$ and $\partial \phi$:

\[ j_1 = \frac{1}{(N-k)^{\frac{3}{2}}} \partial \varphi + \frac{(N-k)^{\frac{3}{2}}}{2} \partial \phi \]
\[ j_2 = \frac{1}{(N+k)^{\frac{3}{2}}} \partial \varphi + \frac{(N+k)^{\frac{3}{2}}}{2} \partial \phi. \]  
(5.35)

Let us now perform a decomposition of the fermionic fields $\eta$ and $\xi$ similar to the one adopted in the bosonic sector of the theory:

\[ \eta_{ab} = \rho_{ab} + k \frac{\delta_{ab}}{\sqrt{N}} \mu \]
\[ \xi_{ab} = \gamma_{ab} + \frac{\delta_{ab}}{k \sqrt{N}} \zeta, \]
(5.36)

where the $\gamma$ and $\rho$ fields are traceless, i.e.:

\[ \sum_a \rho_{aa} = \sum_a \gamma_{aa} = 0. \]  
(5.37)

The only non-vanishing OPE’s are:

\[ \rho_{ab}(z) \gamma_{cd}(w) = -\frac{\delta_{ad} \delta_{bc} - \frac{1}{N} \delta_{ab} \delta_{cd}}{z-w} \]
\[ \mu(z) \zeta(w) = -\frac{1}{z-w}. \]  
(5.38)

Substituting eqs. (5.30), (5.35) and (5.36) into the expression of the energy-momentum tensor of the theory (eq. (5.25)) we obtain $T$ as the sum of an $sl(N)$ contribution and a $U(1)$ part:

\[ T = T_{sl(N)} + T_{U(1)}. \]  
(5.39)

where

\[ T_{sl(N)} = \frac{1}{2k} [ : Tr(I^{(\alpha)} I^{(\alpha)}) : - : Tr(I^{(\beta)} I^{(\beta)}) : ] + : Tr(\rho \partial \gamma) : \]
\[ T_{U(1)} = - : \partial \phi \partial \varphi : + \frac{N^2}{2} \partial^2 \phi + : \mu \partial \zeta : . \]  
(5.40)

This additivity of the $sl(N)$ and $U(1)$ components is satisfied by all the generators of the topological algebra.
It is interesting to notice that both operators $T_{sl(N)}$ and $T_{U(1)}$ have a vanishing Virasoro anomaly. This means that the $sl(N)$ and $U(1)$ parts of our theory constitute separate topological conformal field theories. In fact they are the irreducible components of our topological theory. There is however a fundamental difference between these two components: while the non-abelian part non-trivially satisfies the extended topological algebra, the algebra of the $U(1)$ component closes without introducing spin-three operators. In this sense we can say that the non-abelian part is responsible for the extension of the algebra. Another important point is that the dimension $d$ of these two algebras is not zero, although their sum vanishes. Actually the dimension of the non-abelian part exactly equals the dimension of the $SL(N)$ group manifold. This fact suggests that our theory can be regarded as a topological sigma model for the group manifold $SL(N)$ (i.e. as a topological version of the $SL(N)$ WZW model) together with a compensating abelian component.

Let us first study the $U(1)$ component of our theory. The generators of the topological algebra have the following form:

$$G = : \mu \partial \varphi : - \frac{N^2}{2} \partial \mu$$

$$Q = : \zeta \partial \phi : + \sqrt{N} \partial \zeta$$

$$R = \sqrt{N} (\partial \varphi + \frac{N}{2} \partial \phi) + : \mu \zeta : .$$

The $U(1)$ contribution to $V$ and $W$ vanishes and the parameter $d$ appearing in the algebra closed by the operators (5.41) is $1 - N^2$. In order to see how the BRST symmetry relates our fields, let us compute the transformation generated on them by $Q$. After a simple calculation we get

$$\delta \varphi = - \zeta \quad \delta \phi = 0$$

$$\delta \mu = - \partial \phi \quad \delta \zeta = 0,$$

which implies that $\zeta$ is the BRST copy of $\varphi$ and $\mu$ is the partner of the abelian current $\partial \phi$. In the $gl(1,1)$ case, this $U(1)$ contribution is the only one. Its topological dimension in this case vanishes. As explained in the previous section, by
twisting the theory as in eq. (4.11) and (4.12), we can get a non-zero $d$ value. It is easy to verify that, in so doing, we get the standard representation [29] of the twisted $N = 2$ superconformal algebra.

In order to analyse the non-abelian component of our theory, let us adopt a more standard notation for the $sl(N)$ Lie algebra. Let $T^i$ ($i = 1, \ldots, N^2 - 1$) be a set of generators of $sl(N)$ chosen in such a way that $Tr(T^iT^j) = \delta^{ij}$. They satisfy commutation relations $[T^i, T^j] = f^{ijk}T^k$, where $f^{ijk}$ are the (totally anti-symmetric) structure constants. If $A$ is an arbitrary element of the Lie algebra, its components $A_{ab}$ with respect to the $E_{ab}$ matrices and $A^i$ with respect to the generators $T^i$ are related by the expressions

$$A^i = \sum_{ab} A_{ab}(T^i)_{ab}, \quad A_{ab} = \sum_i A^i(T^i)_{ba}. \quad (5.43)$$

Using these equations, the $sl(N)$ contribution to the operators of eq. (5.29) can be written as

$$G = \frac{1}{2k} : \rho^i(I^{(\alpha)}^i - I^{(\beta)}^i) :$$
$$Q = - : \gamma^i(I^{(\alpha)}^i + I^{(\beta)}^i) + \frac{1}{2} f^{ijk} \gamma^j \rho^k :$$
$$R = : \rho^i \gamma^i :$$
$$W = \frac{1}{4k^2} : f^{ijk}(I^{(\alpha)}^i + I^{(\beta)}^i) \rho^j \rho^k : - \frac{N}{2k^2} : \partial \rho^i \rho^i :$$
$$V = \frac{1}{12k^2} : f^{ijk} \rho^j \rho^k \rho^k :,$$

where we sum over repeated indices. As we mentioned above, the dimension $d$ appearing in the algebra satisfied by the operators of eq. (5.44) is

$$d_{sl(N)} = N^2 - 1.$$

Let us denote by $I$ the $sl(N)$ contribution to the current $J$. Its components along
the $T^i$ generators are given by

$$T^i = I^{(\alpha)^i} + I^{(\beta)^i} + f^{ijk} \rho^j \rho^k,$$

(5.45)

It is instructive to write the BRST transformation of the non-abelian fields:

$$\delta I^{(\alpha)^i} = f^{ijk} \gamma^j I^{(\alpha)^k} + (N - k) \partial \gamma^i$$

$$\delta I^{(\beta)^i} = f^{ijk} \gamma^j I^{(\beta)^k} + (N + k) \partial \gamma^i$$

$$\delta \gamma^i = \frac{1}{2} f^{ijk} \gamma^j \gamma^k$$

$$\delta \rho^i = I^{(\alpha)^i} + I^{(\beta)^i} + f^{ijk} \gamma^j \rho^k,$$

(5.46)

The BRST transformation of our $gl(N)$ fields has, in fact, the standard form of a non-abelian BRST symmetry. This transformation law implies that the field $\rho$ is the partner of $I$. They close an $sl(N)$ algebra without central extension:

$$T^i(z)T^j(w) = f^{ijk} \frac{T^k(w)}{z - w}$$

$$\rho^i(z)T^j(w) = f^{ijk} \frac{\rho^k(w)}{z - w}.$$

(5.47)

Notice that, although the level $k$ does not appear in the current algebra, it shows up in the BRST variations of the currents. In fact the last two terms in the first two equations (5.46) are proportional to $k_\alpha$ and $k_\beta$, which are the central extensions appearing in (5.32).

The form of the energy-momentum tensor $T$, its BRST partner $G$ and the two dimension-one currents $Q$ and $R$, is identical to the one found in refs. [23,24,25] for the $sl(N)/sl(N)$ topological coset models. This means that the extended topological symmetry we have obtained is realized in the $G/G$ theories. Moreover, the deformations (4.11) and (4.12) found in the previous section, when restricted to
the $sl(N)$ component, take the form:

\[
T \rightarrow T + \sum_i \alpha^i \partial T^i \\
G \rightarrow G + \sum_i \alpha^i \partial \rho^i \\
R \rightarrow R + \sum_i \alpha^i \mathcal{T}^i.
\]  

(5.48)

The parameter $d_{sl(N)}$ is not changed from its value $N^2 - 1$ under this deformation. This can be easily verified by explicit calculation but, in fact, it is clear from (4.14) that only the $U(1)$ sector of our theory changes its parameter $d$ under this transformation. Remarkably, when the $\alpha^i$ are taken equal to one for the generators of the $sl(N)$ Cartan subalgebra and zero otherwise, it has been noticed in refs. [24,25] that the deformed $sl(N)/sl(N)$ model is equivalent to a system of $(p, q)$ $W_N$ minimal matter coupled to $W_N$ gravity (plus some extra topological sectors). With our notations, this equivalence is valid when $k$ is equal to $\frac{p^2}{2q}$, with $p, q$ integers. Actually this is not surprising in view of the interpretation of the $G/G$ models as the topological analogue of the WZW model. A similar result was obtained in refs. [31,32] using the quantum Hamiltonian reduction of the WZW model. It is interesting to point out that, since the dimension-three generators $V$ and $W$ are not altered under the transformation (5.48), the algebra of the modified theory retains its extended character after the deformation has been performed. The fact that our extended algebra is present in the minimal topological matter systems leads us to conjecture that, in fact, we are dealing with the basic topological algebra.
6. Discussion and outlook

The theories with a current algebra symmetry are the basic building blocks from which all known rational conformal field theories can be constructed. It might be that a similar statement could apply to the two-dimensional topological conformal field theories. In this paper we have constructed a model possessing both topological and current algebra symmetries. The currents that close with the topological algebra are of a special type: they form a topological multiplet, composed by a bosonic current and its BRST partner, and they obey an algebra without central extension. A c-number anomaly appears, however, when the BRST symmetry acts on the currents. Contrary to what happens with the Virasoro algebra, in order to incorporate a current algebra with a non-abelian topological symmetry, one has to extend the topological algebra by including spin-three generators. Although we do not have a general proof of this statement, this conclusion is likely to hold for any BRST symmetry acting on the currents in a non-abelian way (i.e. as an odd rotation with central extension).

An important point we have not considered here is the determination of the spectrum of physical states of our theory. Closely related to this question is the analysis of the topological invariant observables of the model. In view of the connection found in section 5 with the $G/G$ coset models, it is clear that one could invoke the results of refs. [24,25], in which the cohomology of the Fock space of the topological $G/G$ theory has been studied.

One of our motivations to study a theory with an affine $gl(N,N)$ symmetry was the relation found in ref. [15] between the $gl(1,1)$ theory and the Alexander-Conway knot polynomial. It would be interesting to analyse the implications of the topological symmetry found here in the study of knot invariants. In particular, for $N > 1$, our analysis could lead to formulate a non-abelian generalization of the Alexander-Conway knot polynomial. It is interesting to notice in this respect that, as was stressed in ref. [23], the $G/G$ coset models are the two-dimensional analogues of the three-dimensional Chern-Simons theory for the gauge group G.
This fact is a clue that may help to understand the precise relation between the Alexander-Conway and the Jones polynomials, a problem that remains open in knot theory.

Finally, let us point out that, although we have restricted ourselves to two-dimensional space-times, we could try to use a $gl(N, N)$ symmetry as a tool to generate topological field theories in any number of dimensions. In the particular case of four dimensions, it would be interesting to investigate if one can generate models of topological matter and, in that case, study their coupling to gravity. We expect to report on this and related issues in a near future.

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APPENDIX A

Below we collect the brackets of the extended topological algebra for an arbitrary dimension $d$:

\[
[T^n, T^m] = (n - m) T^{n+m}
\]
\[
[Q^n, Q^m] = 0
\]
\[
[Q^n, G^m] = T^{n+m} + n R^{n+m} + \frac{d}{2} m(m + 1) \delta_{n+m, 0}
\]
\[
[T^n, R^m] = -m R^{n+m} - \frac{d}{2} n(n + 1) \delta_{n+m, 0}
\]
\[
[T^n, Q^m] = -m Q^{n+m}
\]
\[
[T^n, G^m] = (n - m) G_{n+m}
\]
\[
[R^n, R^m] = dn \delta_{n+m, 0}
\]
\[
[R^n, Q^m] = Q^{n+m}
\]
\[
[R^n, G^m] = -G^{n+m}
\]
\[
[G^n, G^m] = W^{n+m}
\]
\[
[Q^n, W^m] = 0
\]
\[
[R^n, W^m] = -2 W^{n+m}
\]
\[
[T^n, W^m] = (2n - m) W^{n+m}
\]
\[
[G^n, W^m] = (2n - m) V^{n+m}
\]
\[
[Q^n, V^m] = W^{n+m}
\]
\[
[R^n, V^m] = -3 V^{n+m}
\]
\[
[T^n, V^m] = (2n - m) V^{n+m}
\]
\[
[G^n, V^m] = 0
\]
\[
[W^n, W^m] = 0
\]
\[
[V^n, W^m] = 0
\]
\[
[V^n, V^m] = 0.
\]
In this appendix we shall show how a topological current algebra of the type discussed in section 4 appears in another context [28]. The model we shall analyse is the zero-level $sl(2)$ WZW model. As the central charge for the $sl(2)_k$ current algebra is $c = \frac{3k}{k+2}$, it is clear that we shall be dealing with a conformal field theory with vanishing Virasoro anomaly, which we would expect to be topological. At the topological point $k = 0$, the $sl(2)$ currents close a bosonic algebra without central extension, and there exists a BRST symmetry making the energy-momentum tensor and the currents cohomologically exact.

Let us start our analysis by recalling the free-field representation of the $sl(2)$ affine algebra. It can be obtained [21] by means of a gaussian decomposition of the $SL(2)$ group similar to the one performed in section 5. Let be $\phi$ a real scalar field and $\beta$ and $\gamma$ a bosonic system of $1$– and $0$–differentials obeying the OPE’s

$$\phi(z)\phi(w) = -\log(z-w)$$
$$\beta(z)\gamma(w) = -\frac{1}{z-w}. \tag{B.1}$$

The $sl(2)_k$ currents are represented as

$$J_+ = \beta$$
$$J_- = -\beta\gamma^2 - i\sqrt{2(k+2)}\gamma\partial\phi - k\partial\gamma$$
$$J_3 = \beta\gamma + i\sqrt{\frac{k+2}{2}}\partial\phi. \tag{B.2}$$

Using eq. (B.2) it is straightforward to check that they satisfy the OPE’s

$$J_3(z)J_\pm(w) = \pm\frac{1}{z-w}J_\pm(w)$$
$$J_+(z)J_-(w) = \frac{k}{(z-w)^2} + \frac{2}{z-w}J_3(w). \tag{B.3}$$

The energy-momentum tensor of the theory can be written in terms of these free
fields as
\[ T = \frac{1}{2(k + 2)} : [J_+ J_- + J_- J_+ + 2(J_3)^2] : = \]
\[ = - \beta \partial \gamma - \frac{1}{2} (\partial \phi)^2 - \frac{i}{\sqrt{2(k + 2)}} \partial^2 \phi. \tag{B.4} \]

In order to see how the BRST symmetry appears at the topological point, it is convenient to replace the scalar field \( \phi \) by a Grassmann \((b, c)\) system. These new fields are related to \( \phi \) in the following way:
\[ b = : e^{i\phi} : \quad c = : e^{-i\phi} :. \tag{B.5} \]

From the expansion (B.1), one gets
\[ b(z)c(w) = \frac{1}{z - w}. \tag{B.6} \]

On the other hand \( T \) can be written in terms on \( b \) and \( c \) as
\[ T = - \beta \partial \gamma - jb dc + (1 - j)\partial bc \tag{B.7} \]
where \( j = \frac{1}{2} + \frac{1}{\sqrt{2(k+2)}} \) is the spin of the field \( b \), whereas \( c \) has conformal weight \( 1 - j \). Taking into account that \( : bc : = i \partial \phi \), one can immediately express the currents in terms of \( b \) and \( c \). Let us now put \( k = 0 \) everywhere. The \( c \) field has dimension zero at this point and, therefore, the fermionic nilpotent operator
\[ Q = : \beta c :. \tag{B.8} \]
has dimension one. Let us see that this is the BRST current we are looking for. First of all \( Q \) induces the following transformation on the fields:
\[ \delta b = \beta \quad \delta \beta = 0 \]
\[ \delta \gamma = - c \quad \delta c = 0. \tag{B.9} \]

Now, putting \( k = 0 \) in eq. (B.2), one realizes that the \( J^a \) operators are BRST-exact, \( i.e. \) that there exist fermionic fields \( \Psi^a \) such that \( J^a = \delta \Psi^a \). The explicit
expression of these new fields are

\[ \Psi_+ = b \quad \Psi_- = -b\gamma^2 \quad \Psi_3 = b\gamma. \]  \hspace{1cm} (B.10)

The \( \Psi \)'s are primary dimension-one operators with respect to \( T \). They close the following algebra with the bosonic currents

\[ \Psi_a(z)J_b(w) = \frac{f^{abc}}{z - w}\Psi_c(w) \quad \Psi_a(z)\Psi_b(w) = 0, \]  \hspace{1cm} (B.11)

which, together with eq. (B.2) (for \( k = 0 \)), constitute what we have called a topological current algebra. In (B.11) \( f^{abc} \) are the \( sl(2) \) structure constants (see eq. (B.3)). The BRST partner of \( T \), denoted as usual by \( G \), can be obtained as a Sugawara bilinear:

\[ G = \frac{1}{4} :[\Psi_+J_- + \Psi_-J_+ + 2\Psi_3J_3]: = -b\partial\gamma :. \]  \hspace{1cm} (B.12)

Notice that the global coefficient \( \frac{1}{4} \) is the same as in eq. (B.4) for \( k = 0 \). The operators \( Q, T \) and \( G \) close a (non-extended) topological algebra with abelian current \( R \) and dimension \( d \) equal to

\[ R = bc : \quad d = 1. \]  \hspace{1cm} (B.13)

It is interesting to obtain the OPE’s of the BRST current \( Q \) and the \( sl(2) \) currents. The singular terms in the expansion of \( Q(z)\Psi^a(w) \) are given by

\[ Q(z)\Psi_+(w) = \frac{J_+(w)}{z - w} \]
\[ Q(z)\Psi_-(w) = \frac{2}{(z - w)^2}\gamma(w) + \frac{1}{z - w}J_-(w) \]  \hspace{1cm} (B.14)
\[ Q(z)\Psi_3(w) = \frac{-1}{(z - w)^2} + \frac{1}{z - w}J_3(w). \]

It is worth pointing out that the central extensions appearing in eq. (B.14) are not \( c \)-numbers. In fact the same type of phenomenon occurs when the products
\(Q(z)J^a(w)\) are computed:

\[
\begin{align*}
Q(z)J_+(w) &= Q(z)J_3(w) = 0 \\
Q(z)J_-(w) &= 2 \frac{c(w)}{(z-w)^2}.
\end{align*}
\] (B.15)

Notice that the fields appearing in the double pole singularity in eqs. (B.14) and (B.15) are the zero-dimensional fields \(\gamma\) and \(c\). From (B.9) it follows that they belong to the same topological multiplet. Other OPE’s of the generators of the topological algebra with the currents can also be computed. For example, if we multiply \(G\) by \(J\) and \(\Psi\) we get

\[
\begin{align*}
G(z)J_a(w) &= \frac{\Psi_a(w)}{(z-w)^2} + \frac{\partial \Psi_a(w)}{z-w} \\
G(z)\Psi_a(w) &= 0.
\end{align*}
\] (B.16)

which mean that \(J\) and \(\Psi\) are also primary with respect to \(G\) in the sense discussed in section 4. The behaviour of the currents under the \(R\)-symmetry is determined by the expansions

\[
\begin{align*}
R(z)J_+(w) &= 0 \\
R(z)J_-(w) &= -\frac{2}{(z-w)^2} \gamma(w) \\
R(z)J_3(w) &= \frac{1}{(z-w)^2} \\
R(z)\Psi_a(w) &= \frac{\Psi_a(w)}{z-w}.
\end{align*}
\] (B.17)

There is a fundamental difference between this system and the one studied at the end of section 5. Although the currents \(J\) and \(\Psi\) close a non-abelian algebra, the BRST operator \(Q\) does not act as a non-abelian symmetry on the currents. This is consistent with the fact that the topological dimension of this system is not equal to the dimension of \(SL(2)\).
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