A new inequality for the von Neumann entropy

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Abstract

Strong subadditivity of von Neumann entropy, proved in 1973 by Lieb and Ruskai, is a cornerstone of quantum coding theory. All other known inequalities for entropies of quantum systems may be derived from it. Here we prove a new inequality for the von Neumann entropy which we prove is independent of strong subadditivity: it is an inequality which is true for any four party quantum state, provided that it satisfies three linear relations (constraints) on the entropies of certain reduced states.

1 Introduction

Entropy is a key concept both in classical and in quantum information theory: Shannon’s source and channel coding and half a century of work have exhibited a vast range of operational coding problems whose solution can be expressed most naturally by (Shannon-Gibbs-Boltzmann) entropies of random variables:

\[ H(X) = -\sum_x \Pr\{X = x\} \log_2 \Pr\{X = x\}. \]

Quantum information theory allows a wealth of new information processing possibilities, with the von Neumann (quantum) entropy playing a role analogous to Shannon’s (classical) entropy in classical information theory: for a density operator \( \rho \), the von Neumann entropy is

\[ S(\rho) = -\text{Tr}\rho \log_2 \rho. \]

Although the two entropy functionals exhibit similarities, they have many decidedly different properties. These properties however, because of the intimate relation of the entropy to operational properties of (classical and quantum) information, ultimately express statements about the “nature of information”. Furthermore, they are indispensable technical tools in proving the information theoretic optimality of constructions: most importantly, there are inequalities governing the relative magnitude of entropies, conditional entropies and mutual informations.

In the quantum case, there is essentially only one known inequality (all others being derivable from it): strong subadditivity. Proved by Lieb and Ruskai in 1973, it is the key result on which virtually every nontrivial quantum coding theorem relies.

We prove here a new inequality for the von Neumann entropy, which we show cannot be derived from the known ones: it is a constrained inequality in that it is not true in general but only for states satisfying three particular linear constraints on their entropies.

One starting point for our work was the desire to understand properties of quantum entropy. We were also motivated by investigations of multi-party entanglement in: there, entropy values were found which are allowed by strong subadditivity but for which the authors could not find quantum states. This led to the conjecture that it is impossible to realise those values by a quantum state, which we indeed prove here.

The structure of our paper is as follows: the next section reviews the well-established convexity framework for (linear) information inequalities. In section we state and prove our result, while in section we explain why it does not follow from the standard inequalities. In section we present a number of alternative forms of our inequality. We close in section with a discussion and a conjectured non-constrained inequality.
2 Linear inequalities

Pippenger \cite{Pippenger} initiated the programme of determining all (linear) inequalities satisfied by the classical entropy functional $H$. This question was based on the realisation of two facts (see Yeung’s work \cite{Yeung2007}): first, that in information theoretic applications, the properties one uses about the entropy to bound information quantities seem always to be

1. Nonnegativity of entropy $H(X)$.
2. Nonnegativity of conditional entropy $H(X|Y) = H(XY) - H(Y)$.
3. Nonnegativity of mutual information $I(X;Y) = H(X) + H(Y) - H(XY)$
4. Nonnegativity of conditional mutual information $I(X;Z|Y) = H(XY) + H(YZ) - H(XYZ) - H(Y)$,

for random variables $X,Y,Z$, the so-called basic inequalities.

Second, that for every number $n$ of random variables, the points in $\mathbb{R}^{2n-1}$ given by the entropies of all possible subsets of the random variables,

$$\{(H(X_S))_{\emptyset \neq S \subset \{1,\ldots,n\} : X_1,\ldots,X_n \text{ random variables}}\}$$

(where all random variables are assumed to be discrete and indeed finite range) form “almost” a convex cone (i.e., closed under nonnegative linear combinations) in the positive orthant: one only needs to go to the topological closure, denoted $\Gamma_n^*$, and called the (classical) entropy cone.

Surprisingly, the classical entropy cone can be strictly smaller than the cone cut out by the basic inequalities for all subsets of random variables (which we will call “Shannon cone” $\Gamma_n$): while the two cones coincide for $n \leq 3$, they differ for $n = 4$. Indeed, as Yeung and Zhang have shown, there are further inequalities satisfied by the entropy cone which are not dependent on the basic inequalities; i.e., they are violated by points in the Shannon cone.

Pippenger \cite{Pippenger} observed that a similar situation occurs in the quantum case: with an underlying state multipartite state $\rho$, denote the entropy of its restriction to subsystems $A,\ldots$, or groups of subsystems $AB,\ldots$, by $S(A)$, $S(AB)$, etc. Then, there is a “von Neumann” cone $\Sigma_n$, defined by the basic inequalities

1. Nonnegativity of entropy $S(A)$.
2. Nonnegativity of the quantity $S(A|B) + S(A) = S(AB) - S(B) + S(A)$ (this is known as the Araki-Lieb inequality \cite{Araki1984})
3. Nonnegativity of $S(C|A) + S(C|B) = S(CA) + S(CB) - S(A) - S(B)$ (this replaces nonnegativity of the conditional entropy, and is called “weak monotonicity”).
4. Nonnegativity of quantum mutual information $I(A;B) = S(A) + S(B) - S(AB)$
5. Nonnegativity of quantum conditional mutual information $I(A;C|B) = S(AB) + S(BC) - S(ABC) - S(B)$.

(Note that the names of the quantities are given based on straightforward analogy, with no operational significance implied at this point.)

The latter two are simply subadditivity and strong subadditivity \cite{Lieb1973} of the quantum entropy. The properties 2) and 3) above, can actually be derived from 4) and 5) (and vice versa) by viewing the state as the restriction of a pure state on the given parties plus one, and the fact that for a pure state, the entropy of a subset of the parties equals the entropy of the complementary set. This is a consequence of linear algebra, namely the Schmidt decomposition of bipartite pure states, whose coefficients are the eigenvalues of both reduced states (for a more detailed discussion, see \cite{Yuan2002}). Note also that choosing trivial $B$ (i.e., with Hilbert space $\mathbb{C}$) reduces weak monotonicity to the Araki-Lieb inequality, and conditional mutual information to mutual information (compare the classical case). Thus all non-trivial inequalities may be derived from strong subadditivity.
And there is the cone of the closure of all points realized by entropies of the $2^n - 1$ nontrivial marginals, the (quantum) entropy cone $\Sigma_n^*$, of states on tensor products of $n$ finite quantum systems (that it is indeed a cone is proved in the same way as for the classical case [5]).

The faces of the cone $\Sigma_n$ are given by certain entropies being zero (which means that the corresponding subsystem is in a pure state), certain mutual informations being zero (which means that certain pairs of subsystems are in a product state), certain conditional mutual informations being zero, etc. The latter is fully analogous to the classical case of a Markov chain, where $A$ and $C$ are independent conditional on $B$ (which we call the “pivot” of the chain), as explained in [4].

All this raises the following natural question: are there any further linear inequalities for the quantum entropy than those above? To be precise: is $\Sigma_n^* \subsetneq \Sigma_n$, and if so, can we find a hyperplane intersecting the interior of $\Sigma_n$ but having $\Sigma_n^*$ entirely in one halfspace?

## 3 The new inequality

Our main result is the following theorem, which gives an answer to the question in its first form, and provides evidence for a positive answer to the second.

**Theorem 1** Let $\rho^{ABCD}$ be a state of a quadrupartite quantum system, such that strong subadditivity is saturated for the three triples $ABC$, $CAB$ and $ADB$ (pivot always in the middle). Then,

$$I(C; D) \geq I(C; AB).$$

The angles in the figure represent the strong subadditivity constraints which are saturated in the conditions of theorem [6] (i.e. strong subadditivity is saturated for the triples $ABC$, $CAB$, $ADB$; pivot always in the middle). Theorem [6] then states that under these conditions, the correlation between $C$ and $D$ is not smaller than that between $C$ and $AB$.

**Proof.** The proof relies heavily on the recent characterisation of states which saturate strong subadditivity [4], which is stated below as proposition [2].

First of all, since we have strong subadditivity saturated for $CAB$,

$$\rho^{CAB} = \bigoplus_i p_i \rho_i^{Ca} \otimes \rho_i^{Cb},$$

by proposition [2]. To this we apply proposition [2] once more, for the triple $ABC$: there exists the recovery map $R_{B \rightarrow C'}$ (we duplicate $C$, and attach a prime, to distinguish the two incarnations of $C$) mapping $\rho^{AB}$ to $\rho^{ABC'}$. It maps the above $\rho^{CAB}$ to

$$\rho^{CABC'} = \bigoplus_{ij} p_{ij} p_j^{C_i} \rho_i^{Ca} \otimes \rho_i^{Cja} \otimes \rho_j^{C'c}.$$  

(Notice that the states on the far right, by the structure of $R_{B \rightarrow C'}$, can only depend on $j$, the Hilbert space sector measured by the map, as described in proposition [2] below.)

But the two states obtained by tracing out $C$ and $C'$, respectively (and identifying $C$ with $C'$ again), must coincide:

$$\rho^{CAB} = \bigoplus_{ij} p_{ij} p_j^{C_i} \rho_i^{Ca} \otimes \rho_i^{Cja} \otimes \rho_j^{C},$$

$$\| \rho^{ABC} = \bigoplus_{ij} p_{ij} p_j^{C_i} \rho_i^{Ca} \otimes \rho_i^{Cja} \otimes \rho_j^{C}.$$
Comparing these two, for a given sector labelled \( ij \), with \( p_{ij} > 0 \), we obtain that both \( \rho_{ij}^{\text{Ca}} \) and \( \rho_{ij}^{\text{Bo}} \) are actually product states:

\[
\begin{align*}
\rho_{ij}^{\text{Ca}} &= \rho_{ij}^{a} \otimes \rho_{ij}^{\text{aL}}, \\
\rho_{ij}^{\text{Bo}} &= \rho_{ij}^{b} \otimes \rho_{ij}^{\text{bL}}.
\end{align*}
\]

That the right hand sides contain both \( i \) and \( j \) (whereas the left hand sides mention only \( i \) and only \( j \), respectively) is no error, but in fact the main point: it means that for actually occurring \( ij \), i.e., \( p_{ij} > 0 \), the state of \( C \) belonging to this sector depends only on \( i \) and only on \( j \) — in other words, it must be a common function of \( i \) and \( j \); \( \rho_{ij}^{C} \), with \( k = f(i) = g(j) \), with certain (deterministic) functions \( f \) and \( g \).

We note that if all the \( p_{ij} \) are strictly positive, then the only way for \( \rho_{ij}^{C} \) to be consistent with eq. \( \Pi \) is for it to be constant, i.e. independent of \( i \) and \( j \). Situations in which \( \rho_{ij}^{C} \) can vary with \( i \) and \( j \) are only possible when some of the \( p_{ij} \) are zero. As an illustration, consider a state \( \rho^{\text{ABC}} \) which has \( i, j = 1, 2, 3 \), and \( p_{11} > 0, p_{22} > 0, p_{32} > 0, p_{33} > 0 \), but \( p_{ij} = 0 \) otherwise; then the non-constant possibility \( \rho_{ij}^{C} = \rho_{ij}^{C} \) for \( i = j = 1 \) and \( \rho_{ij}^{C} = \rho_{2} \neq \rho_{1} \) for \( i, j = (2, 2), (2, 3), (3, 2), (3, 3) \) (i.e. \( f(1) = g(1) = 1, f(2) = f(3) = g(2) = g(3) = 2 \)) is consistent with eq. \( \Pi \).

Returning to the general situation, with \( k \) as described in the previous paragraph but one, we can rewrite \( \rho^{\text{ABC}} \) again,

\[
\rho^{\text{ABC}} = \bigoplus_{ij} p_{ij} \rho_{ij}^{a} \otimes \rho_{ij}^{b} \otimes \rho_{ij}^{C}.
\]

In fact, let us introduce quantum registers \( K_{A} \) and \( K_{B} \) holding \( k \) explicitly (of course, by our observation, they will be perfectly correlated) — and note that their content can be extracted locally at \( A \) and \( B \), respectively, without disturbing the state \( \rho^{\text{ABC}} \), by a measurement of the orthogonal subspace sector \( i \) and \( j \), respectively:

\[
\rho^{K_{A}K_{B}K_{C}} = \bigoplus_{ij} p_{ij} |k\rangle \langle k| K_{A} \otimes \rho_{ij}^{a} \otimes \rho_{ij}^{b} \otimes \rho_{ij}^{C}.
\]

We shall use the following convention: some of the registers are classical (such as \( K_{A} \) and \( K_{B} \)) inasmuch they come with a distinguished basis, and the global state is written as a mixture of states which have the classical registers in one of their distinguished basis states. These classical registers we identify with random variables, by the same name, for example \( K_{A} \) with distribution

\[
\Pr\{K_{A} = k\} = p_{k} = \sum_{i: f(i) = k} p_{i}.
\]

This will allow us to speak about the state in quantum theoretical language, and interchangeably about its classical properties in random variable language. For example, as random variables, \( K_{A} = K_{B} \) with probability 1, by our earlier observation.

The proof will now be completed by showing two things: first, that \( I(C; AB) \) equals the Holevo quantity of the ensemble of the \( \rho_{k}^{C} \), which is \( I(K_{A}; C) \); second, that \( k \) is also “known at \( D \)” by which we mean that there is a measurement on \( D \) extracting a random variable \( K_{D} \) perfectly correlated with \( K_{A} = K_{B} \).

First, the first claim: the system \( AB \), by our above characterisation, falls into orthogonal sectors \( (AB)_{k} \), labelled by \( k \), and the state in this sector is some \( \sigma_{k}^{(AB)C} \otimes \rho_{k}^{C} \), because it is a convex combination of states \( \rho_{ij}^{a} \otimes \rho_{ij}^{b} \otimes \rho_{ij}^{C} \), with \( ij \) consistent with \( k \), so they all have the same state \( \rho_{k}^{C} \) on \( C \). Hence, there is a quantum operation extracting \( K_{A} \) from \( AB \) (a coarse-graining of the disturbance-free measurement of \( i \) and \( j \)), as well as a reverse, creating \( \sigma_{k}^{(AB)C} \) in \( AB \) from \( K_{A} \). By monotonicity of the quantum mutual information, \( I(C; AB) = I(K_{A}; C) \).

The second claim is seen as follows: using the third constraint, \( I(A; B|D) = 0 \), with the monotonicity of the quantum conditional mutual information under the local maps extracting \( K_{A} \) (from \( A \)) and \( K_{B} \) (from \( B \)), gives \( I(K_{A}; K_{B}|D) = 0 \). Proposition \( \Box \) guarantees the existence of a measurement (whose outcome we think of being stored in a classical register \( K_{D} \)) such that conditional on each
measure outcome, $K_A$ and $K_B$ are in a product state. It is straightforward to check that then $I(K_A; K_B|K_D) = 0$, and because $K_A$ and $K_B$ are perfectly correlated, they must also be perfectly correlated with $K_D$: $K_A = K_B = K_D$ with probability 1, as random variables.

These two facts, by monotonicity of the quantum mutual information under quantum operations, finally yield $I(C; AB) = I(K_A; C) = I(K_D; C) \leq I(C; D)$. \hfill \Box

**Proposition 2 (Hayden, Jozsa, Petz and Winter [4])** A state $\rho^{ABC}$ saturating strong subadditivity at pivot $B$, i.e., satisfying $S(AB) + S(BC) = S(ABC) + S(B)$, must have the following form. There exists an orthogonal decomposition of $B$’s Hilbert space $\mathcal{H}_B$ into subspaces $\mathcal{H}_{B_j}$, each of which has a natural presentation as tensor product of two Hilbert spaces:

$$\mathcal{H}_B = \bigoplus_j \mathcal{H}_{b^L_j} \otimes \mathcal{H}_{b^R_j},$$

such that (with states $\rho^{Ab^L_j}_{j}$ on $\mathcal{H}_A \otimes \mathcal{H}_{b^L_j}$ and $\rho^{b^R_j}_{j}$ on $\mathcal{H}_{b^R_j} \otimes \mathcal{H}_C$)

$$\rho^{ABC} = \bigoplus_j p_j \rho^{Ab^L_j}_{j} \otimes \rho^{b^R_j}_{j}.$$ 

This can be operationally rephrased as follows: there is a quantum operation $R_{B\rightarrow C}$ from $B$ to $BC$ such that $\rho^{ABC} = (\text{id}_A \otimes R_{B\rightarrow C})\rho^{AB}$, which has the following form:

1. Perform a projective measurement associated with an orthogonal decomposition of $B$ into sectors $B_j$.
2. Each sector has a tensor product structure $B_j = b^L_j b^R_j$; having measured $j$ in step 1, the map discards the state on $b^R_j$ and replaces it by $\rho^{b^R_j}_{j}$ on the composite system $b^R_j C$.

**Remark 3** One can easily construct states where our inequality is strict, and others where it is tight: a state of the form

$$\rho^{ABCD} = \bigoplus_j q_j \rho^{A}_{j} \otimes \rho^{B}_{j} \otimes \phi^{CD}_{j},$$

with $\phi^{CD}_{j}$ being arbitrary and having the marginal states $\tau_j$ and $\zeta_j$ on $C$ and $D$, respectively, will generically have $I(C; D) > I(C; AB)$. If however $\phi^{CD}_{j} = \tau^{C}_{j} \otimes \zeta^{D}_{j}$, with mutually orthogonal $\tau^{C}_{j}$, we have equality.

Also, it is easy to see that no proper subset of our three constraints can imply $I(C; D) \geq I(C; AB)$:

1. Saturation of strong subadditivity for $ABC$ and $CAB$: consider

$$\rho^{ABCD} = \frac{1}{2}(|000\rangle\langle 000| + |111\rangle\langle 111|)^{ABC} \otimes |00\rangle\langle 00|^{AD}.$$

It satisfies these two constraints (and may more), but has $I(C; D) = 0$ and $I(C; AB) = 1$.

2. Saturation of strong subadditivity for $ABC$ and $ADB$: consider

$$\rho^{ABCD} = \frac{1}{2}(|000\rangle\langle 000| + |111\rangle\langle 111|)^{BC} \otimes |00\rangle\langle 00|^{AD}.$$

It satisfies these two constraints (and many more), but has $I(C; D) = 0$ and $I(C; AB) = 1$.

**Remark 4** It is worth pointing out that not every application of proposition 2 along the lines of our proof of theorem 1 yields a nontrivial result, even though it may seem so at first sight: for example, consider a tripartite state $\rho^{ABC}$ which saturates strong subadditivity for $ABC$. Then, the characterisation of such states implies that $\rho^{AC}$ is separable, which is well-known to imply $S(AC) \geq S(C)$. Since this inequality is false for general states, have we found a new constrained inequality? Actually no: it can be checked immediately that in generality,

$$2S(A|C) + I(A; C|B) = [S(A|B) + S(A|C)] + [I(A; B|C)] \geq 0,$$

by the basic inequalities (weak monotonicity and strong subadditivity): hence, if $I(A; C|B) = 0$, then necessarily $S(A|C) \geq 0$. 

5
4 Why the constrained inequality is new

In the introduction we have explained already why for three parties there cannot be an information inequality independent of the basic ones, as \( \Sigma_3 = \Sigma_3 \) \[8, 6\]. Indeed, as one can see from these papers there cannot even be a constrained inequality, since on each of the 8 extremal rays of \( \Sigma_3 \) there are (nonzero) entropy vectors realised by certain states.

The four party case is studied in \[8, 6\] with particular interest in the insights to be gained about multi-party entanglement. There it is shown that \( \Sigma_4 \) has 76 extremal rays, which fall naturally into 8 classes by symmetries (permutation of the parties). For 6 of them \[8, 6\] gives states realising entropy vectors on the rays. The two remaining classes are represented by the rays spanned by the following vectors (the first row gives the combinations of subsystems in lexicographic order; below are their “entropies”):

|   | A | B | C | D | AB | AC | AD | BC | BD | CD | ABC | ABD | ACD | BCD | ABCD |
|---|---|---|---|---|----|----|----|----|----|----|-----|-----|-----|-----|------|
| I | 1 | 3 | 3 | 2 | 2  | 4  | 3  | 3  | 3  | 3  | 4   | 4   | 4   | 3   | 3  |
| II| 2 | 3 | 3 | 3 | 4  | 4  | 4  | 4  | 4  | 4  | 6   | 5   | 5   | 5   | 2   |

Clearly, if one could find states realising these vectors (or nonzero multiples), this would prove \( \Sigma_4^* = \Sigma_4 \).

It is readily verified that both these rays satisfy the condition of theorem \[8, 6\] but not the conclusion: both vectors given above have \( I(C; D) = S(C) + S(D) - S(CD) = 0 \) but \( I(C; AB) = S(C) + S(AB) - S(ABC) = 2 > 0 \).

**Corollary 5** There are no quantum states of finite systems realising entropy vectors on the rays I and II above. In fact, in the face of the cone \( \Sigma_4 \) described by the three constraint equations of theorem \[8, 6\] the new inequality \( I(C; D) \geq I(C; AB) \) cuts off a slice, which contains the rays I and II.

In other words, the two entropy vectors satisfy all the basic inequalities, but by theorem \[8, 6\] there can be no non-trivial quantum state with entropy vector in these rays.

Thus, theorem \[8, 6\] cannot be derived from the constraints in its statement using only the basic inequalities, and so the new inequality is indeed independent of all previously known inequalities.

5 Alternative forms of the inequality

We have presented the new inequality in theorem \[8, 6\] in a form which reflects our way of proving it. Writing out the mutual informations in terms of entropies, one notices that some terms cancel, and we arrive at the following reformulation of our result:

**Theorem 1’** Let \( \rho^{ABCD} \) be a state of a quadripartite quantum system such that

\[
\begin{align*}
S(AB) + S(BC) - S(B) - S(ABC) &= 0, \\
S(CA) + S(AB) - S(A) - S(CAB) &= 0, \\
S(AD) + S(DB) - S(D) - S(ADB) &= 0.
\end{align*}
\]

Then, \( S(ABC) + S(D) \geq S(AB) + S(CD) \), i.e., \( I(ABC; D) \geq I(AB; CD) \).

We present this reformulation mainly because it may help understanding and applying the result.

There is another one, however, which is less trivial: we can apply the purification trick that is used to relate strong subadditivity and weak monotonicity (see section 2).

In detail, we construct a purification \( \Psi^{ABCDE} \) of the given state \( \rho^{ABCD} \) and can apply theorem \[8, 6\] or 1’ to three situations: strong subadditivity saturated for the triples \( ABC \), \( CAB \) and \( AEB \); second, for \( ACE \), \( CAE \) and \( ADE \); third, for \( ABE \), \( EAB \) and \( ADB \). If we then systematically eliminate all entropies involving \( E \) by substituting the complementary group, we get the following statements:

**Theorem 1’’** Let \( \rho^{ABCD} \) be a state of a quadripartite quantum system. Consider the following three properties this state could have:

\[
\begin{align*}
I(A;C|B) &= I(B;C|A) = I(A;B|CD) = 0 \quad (1) \\
S(C|A) + S(C|BD) &= I(A;C|BD) = S(A|D) + S(A|BC) = 0 \quad (2) \\
S(B|A) + S(B|CD) &= S(A|B) + S(A|CD) = I(A;B|D) = 0. \quad (3)
\end{align*}
\]
Then,

\[(1) \implies S(C|AB) + S(C|ABD) \geq 0, \]
\[(2) \implies S(C|D) + S(C|BD) \leq 0, \]
\[(3) \implies S(D) + S(CD) \geq S(AB) + S(ABC). \]

\[\square\]

6 Discussion

Although we believe that the discovery of a new constrained information inequality is interesting in itself, our theorem is not enough to conclude \(\Sigma^*_4 \subsetneq \Sigma_4\) because it may be that there are states realising entropy vectors arbitrarily close to the points I and II in the previous section. Such a possibility could be ruled out by finding a \textit{unconstrained} inequality satisfied by \(\Sigma^*_4\) but violated by points on the rays I and II. Note that indeed for \(n = 3\), in both the quantum and classical version of the question, the set of entropic vectors is not closed, so is not identical to the entropy cone \(\Sigma^*_3, \Gamma^*_3\). On the other hand, it is still the case that the extremal rays are indeed populated by distributions/states.

We may remark that in the classical variant of the question, Yeung and Zhang also at first only found a constrained inequality \cite{1}, and only somewhat later their unconstrained inequality in \cite{2}, whose proof indeed uses ideas from constrained inequalities.

We think, however, that our result provides some evidence towards the existence of such an inequality for the quantum entropy cone; in fact, we believe that it way well be possible to prove an inequality ruling out the approximability of I and II, based on the following: in \cite{3}, it is conjectured that there is a robust version of that paper’s main theorem — characterising the states that come close to saturating strong subadditivity. It seems likely that with such a theorem one could perform an approximation version of the proof of theorem \cite{4} and conclude a new “constrained” inequality if the three constraint equations of theorem \cite{4} are only almost satisfied. In other words, there would be a trade-off between the degree by which \(I(A;C|B), I(C;B|A), I(A;B|D)\) are nonzero, and the negativity of \(I(C;D) - I(C;AB)\).

This rationalises the following conjecture, with which we close the paper:

\textbf{Conjecture 6} There exist positive constants \(\kappa_1, \kappa_2\) and \(\kappa_3\), such that for all qudripartite states,

\[\kappa_1 I(A;C|B) + \kappa_2 I(C;B|A) + \kappa_3 I(A;B|D) + [I(C;D) - I(C;AB)] \geq 0.\]

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