REDUCIBILITY OF 1-D QUANTUM HARMONIC OSCILLATOR EQUATION
WITH UNBOUNDED OSCILLATION PERTURBATIONS

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ABSTRACT. We build a new estimate relative with Hermite functions based upon oscillatory integrals and Langer’s turning point theory. From it we show that the equation
\[ i\partial_t u = -\partial_x^2 u + x^2 u + \epsilon(x)^\mu W(\nu x, \omega t)u, \quad u = u(t, x), \quad x \in \mathbb{R}, \quad 0 \leq \mu < \frac{1}{3}, \]
can be reduced in \( H^1(\mathbb{R}) \) to an autonomous system for most values of the frequency vector \( \omega \) and \( \nu \), where \( W(\varphi, \theta) \) is a smooth map from \( \mathbb{T}^d \times \mathbb{T}^n \) to \( \mathbb{R} \) and odd in \( \varphi \).

1. INTRODUCTION OF THE MAIN RESULTS

1.1. Statement of the Results. In this paper we consider 1-d quantum harmonic oscillator equation
\[ i\partial_t u = H(\omega t)u, \quad x \in \mathbb{R}, \]
\[ H(\omega t) = -\partial_x^2 + x^2 + \epsilon(x)^\mu W(\nu x, \omega t), \]
where \( 0 \leq \mu < \frac{1}{3} \). \( W(\varphi, \theta) \) is defined on \( \mathbb{T}^d \times \mathbb{T}^n \) and satisfies
\[ W(\varphi, \theta) = -W(\varphi, \theta), \quad \forall (\varphi, \theta) \in \mathbb{T}^d \times \mathbb{T}^n. \]
In order to state the results we need to introduce some notations and spaces, \( C^\beta(\mathbb{R}^n, X) \). Assume that \( X \) is a complex Banach space with the norm \( \| \cdot \|_X \). Let \( C^\beta(\mathbb{R}^n, X) \), \( 0 < b < 1 \), be the space of H"older continuous functions \( f : \mathbb{R}^n \to X \) with the norm
\[ \| f \|_{C^\beta(\mathbb{R}^n, X)} := \sup_{0 < |z_1 - z_2| < 2\pi} \frac{\| f(z_1) - f(z_2) \|_X}{|z_1 - z_2|^b} + \sup_{z \in \mathbb{R}^n} \| f(z) \|_X. \]
If \( b = 0 \), then \( \| f \|_{C^\beta(\mathbb{R}^n, X)} \) denotes the sup-norm. For \( \beta = [\beta] + b \) with \( 0 \leq b < 1 \), we denote by \( C^\beta(\mathbb{R}^n, X) \) the space of functions \( f : \mathbb{R}^n \to X \) with H"older continuous partial derivatives and \( \partial^\mu f \in C^\beta(\mathbb{R}^n, X) \) for all multi-indices \( \nu = (\nu_1, \cdots, \nu_n) \in \mathbb{N}^n \), where \( |\nu| := |\nu_1| + \cdots + |\nu_n| \leq \beta \) and \( X_\nu = \mathcal{L}(\prod_{i=1}^n Y_i, X) \) with the standard norm and \( Y_i := \mathbb{R}^n, i = 1, \cdots, |\nu| \). We define the norm
\[ \| f \|_{C^\beta(\mathbb{R}^n, X)} := \sum_{|\nu| \leq \beta} \| \partial^\mu f \|_{C^\beta(\mathbb{R}^n, X)}. \]
\( C^\beta(\mathbb{T}^n, X) \). Denote by \( C^\beta(\mathbb{T}^n, X) \) the space of all functions \( f \in C^\beta(\mathbb{R}^n, X) \) that are of period \( 2\pi \) in all variables. We define \( \| f \|_{C^\beta(\mathbb{T}^n, X)} := \| f \|_{C^\beta(\mathbb{R}^n, X)} \).

Linear Space. Let \( s \in \mathbb{R} \), we define the complex weighted-\( \ell^2 \)-space
\[ \ell^2_s := \{ \xi = (\xi_j \in \mathbb{C}, \ j \in \mathbb{Z}_+) \mid \| \xi \|_s < \infty \}, \quad \text{where} \quad \| \xi \|_s^2 = \sum_{j \in \mathbb{Z}_+} |j|^s |\xi_j|^2. \]

Hermite functions. The harmonic oscillator operator \( T = -\frac{d^2}{dx^2} + x^2 \) has eigenfunctions \((h_j)_{j \geq 1}\), so called the Hermite functions, namely,
\[ Th_j = (2j - 1)h_j, \quad \| h_j \|_{L^2(\mathbb{R})} = 1, \quad j \geq 1. \]
\( \mathcal{H}^p \). Let \( p \geq 0 \) be an integer we define
\[ \mathcal{H}^p := \{ u \in \mathcal{H}^p(\mathbb{R}, \mathbb{C}) \mid x \mapsto x^\alpha \partial_x^\beta u \in L^2(\mathbb{R}) \text{ for any } \alpha, \beta \in \mathbb{N}, \ 0 \leq \alpha + \beta \leq p \}. \]
To a function \( u \in \mathcal{H}^p \) we associate the sequence \((u_j)_{j \geq 1} \) of its Hermite coefficients by the formula \( u(x) = \sum_{j \geq 1} u_j h_j(x) \). For \( p \geq 0, u \in \mathcal{H}^p \) \( \iff \) \((u_j)_{j \geq 1} \in \ell^p \) and we define its norm by
\[ \|u\|_p = \|(u_j)_{j \geq 1}\|_p = \left( \sum_{j \geq 1} j^p |u_j|^2 \right)^{\frac{1}{2}}. \]
For simplicity we define \( \alpha = \frac{1}{12} - \frac{1}{4} \) and \( \beta(n, \mu) = 18(n + 3)(2 + \alpha^{-1})(2 + 5\alpha^{-1}) \). Our intent is to prove the following

**Theorem 1.1.** Assume that \( W(\varphi, \theta) \) satisfies (1.2) and \( W(\varphi, \theta) \) is \( C^s(\mathbb{T}^d \times \mathbb{T}^n) \) with \( s \geq d + [1 \vee \tau] + n + 3 + \beta \) and \( \beta > \beta(n, \mu) \) for all \( 0 \leq \epsilon < \epsilon_* \), there exists a closed set \( \Omega_* \times D_* \subset [A, B]^d \times [1, 2]^n \) and for all \( (\nu, \omega) \in \Omega_* \times D_* \), the linear Schrödinger equation (1.1) reduces to a linear autonomous equation in the space \( \mathcal{H}^1 \).

More precisely, there exists \( \epsilon_* > 0 \) such that for all \( 0 \leq \epsilon < \epsilon_* \) there exists a closed set \( \Omega_{\epsilon} \times D_{\epsilon} \), and for \( (\nu, \omega) \in \Omega_{\epsilon} \times D_{\epsilon} \), there exist a linear isomorphism \( \Psi_{\omega, \epsilon} : L(\mathcal{H}^1) \to \mathcal{H}^1 \) with \( 0 \leq \nu' \leq 1 \), unitary on \( L^2(\mathbb{R}) \), \( \mathcal{V}_{\omega, \epsilon}(\theta) \) is \( C^s(\mathbb{T}^n, L(\mathcal{H}^1)) \cap C^s(\mathbb{T}^n, L(\mathcal{H}^1, \mathcal{H}^1)) \) with \( i \not\in \mathbb{Z} \) and \( \nu' \leq \frac{2}{\theta} \beta \) and a bounded Hermitian operator \( Q_1 \in L(\mathcal{H}^1) \) such that \( t \mapsto u(t, \cdot) \in \mathcal{H}^1 \) satisfies (1.1) if and only if \( t \mapsto v(t, \cdot) = \Psi_{\omega, \epsilon}^{-1}(\omega t) u(t, \cdot) \) satisfies the autonomous equation
\[ i\partial_t v = -v_{xx} + x^2 v + \epsilon Q_1(v), \]
and
\[ \|\Psi_{\omega, \epsilon}^{-1}(\omega t) - id\|_{C^s(\mathbb{T}^n, L(\mathcal{H}^1))} \leq C\epsilon \frac{\nu' \beta}{\theta^2} \] for all \( 0 \leq \epsilon < \epsilon_* \), and
\[ \|\Psi_{\omega, \epsilon}^{-1}(\omega t) - id\|_{C^s(\mathbb{T}^n, L(\mathcal{H}^1, \mathcal{H}^1))} \leq C\epsilon^{-1} \frac{\nu' \beta}{\theta^2} \] for all \( 0 \leq \epsilon < \epsilon_* \), and
\[ \|Q_1\|_{L(\mathcal{H}^1, \mathcal{H}^1)} \leq K_1 \]
for all \( 0 \leq \epsilon < \epsilon_* \). Our intent is to prove the following

**Remark 1.2.** The sets \( \Omega_{\epsilon} \) satisfies \( \text{Meas}([A, B]^d \setminus \Omega_{\epsilon}) = \mathcal{O}(\gamma) \) when \( \gamma \to 0 \), while \( D_{\epsilon} \) satisfies \( \text{Meas}([-1, 1]^n \setminus D_{\epsilon}) \leq c(\beta, \nu, \mu) \epsilon^{\frac{3}{2} + \alpha^{-1}(2 + 5\alpha^{-1})} \).

**Remark 1.3.** For any \( \nu \in \Omega_{\epsilon} \subset [A, B]^d \), we have \( (k, \nu) \geq \frac{\nu' \beta}{\theta^2} \) where \( k \neq 0 \) and \( \tau > d - 1 \).

**Remark 1.4.** The derivative in (1.4) is in the sense of Whitney.

A consequence of the above theorems and corollary is that in the considered range of parameters all the Sobolev norms, i.e. the \( \mathcal{H}^s \) norms of the solutions are bounded forever and the spectrum of the Floquet operator is pure point.

Consider 1-d quantum harmonic oscillator equation
\[ i\partial_t u = \mathcal{H}_0(u(t))u, \ x \in \mathbb{R}, \]
\[ \mathcal{H}_0(u(t)) := -\partial_{xx} + x^2 + \epsilon X(x, \omega t), \]
where
\[ X(x, \theta) = \langle x \rangle^\mu \sum_{k \in \Lambda} (a_k(\theta) \sin kx + b_k(\theta) \cos kx) \] with \( k \in \Lambda \subset \mathbb{R} \setminus \{0\} \) with \( |\Lambda| < \infty \) and \( 0 \leq \mu < \frac{1}{2} \).
Corollary 1.5. Assume that \( a_k(\theta) \) and \( b_k(\theta) \) are bounded and \( \beta \) as in Theorem 1.1. There exists \( \epsilon_0 > 0 \) such that for all \( 0 < \epsilon < \epsilon_0 \) there exists a closed set \( D_\epsilon \subset [1, 2]^n \) such that for all \( \omega \in D_\epsilon \) the linear Schrödinger equation (1.5) reduces to a linear autonomous equation in the space \( H^1 \).

More precisely, there exists \( \epsilon_0 > 0 \) such that for all \( 0 < \epsilon < \epsilon_0 \) there exists a closed set \( D_\epsilon \) and for \( \omega \in D_\epsilon \), there exist a linear isomorphism \( \Psi_{\omega}^{\infty, 2}(\theta) \in \mathcal{L}(H^s) \) with \( 0 \leq s' \leq 1 \), unitary on \( L^2(\mathbb{R}) \), where \( \Psi_{\omega}^{\infty, 2}(\theta) - id \in C^\infty([0, \omega(0, H_n^n)) \cap C([0, \omega(0, H_n^n)))) \) with \( \nu \notin \mathbb{Z} \) and \( \nu \leq 2 \beta \) and a bounded Hermitian operator \( Q_2 \in \mathcal{L}(H^1) \) such that \( t \mapsto u(t, \cdot) \in H^1 \) satisfies (1.1) if and only if \( t \mapsto v(t, \cdot) = \Psi_{\omega}^{\infty, 2}(\omega t)u(t, \cdot) \) satisfies the autonomous equation

\[
i\partial_t v = -v_{xx} + x^2 v + \varepsilon Q_2(v).
\]

Furthermore,

\[
\|\Psi_{\omega}^{\infty, 2}(\theta) - id\|_{\mathcal{L}(C_t(\mathbb{T}^n, \mathcal{L}(H^\alpha, H^{\alpha+1})))} \leq C \varepsilon^{\frac{1}{2}} \beta^{-1}, \quad (\theta, \omega) \in \mathbb{T}^n \times D_\epsilon,
\]

and

\[
\|\Psi_{\omega}^{\infty, 2}(\theta) - id\|_{\mathcal{L}(C_t(\mathbb{T}^n, \mathcal{L}(H^{\alpha}, H^{\alpha+1})))} \leq C \varepsilon^{\frac{1}{2}} \beta^{-1}, \quad (\theta, \omega) \in \mathbb{T}^n \times D_\epsilon,
\]

for \( 0 \leq s' \leq 1 \), and for any \( p \in \mathbb{N} \) and \( \omega \in D_\epsilon \), there exists a constant \( K_p \) depending on \( n, \beta \),

\[
\|Q_2\|_{\mathcal{L}(H^p, H^{p+\alpha})} + \|\partial_\theta Q_2\|_{\mathcal{L}(H^p, H^{p+\alpha+1})} \leq K_2.
\]

Remark 1.6. The set \( D_\epsilon \) satisfies \( \text{Meas}([1, 2]^n \setminus D_\epsilon) \leq c(\beta, n, \mu) \varepsilon^{\frac{3}{(2\alpha+1)(2\alpha+3\mu+1)}} \).

1.2. Related results and a critical lemma. In the following we recall some relevant results.

For 1d harmonic oscillator see \([12]\) and \([16]\) for periodic in time bounded perturbations. Refer to \([24], [47]\) and \([48]\) for 1d harmonic oscillators with quasi-periodic in time bounded perturbations.

In \([1]\) Bambusi and Graffi first proved the reducibility of 1d Schrödinger equation with an unbounded time quasi-periodic perturbation. In \([1]\) they assumed that the potential grows at infinity like \( |x|^{2l} \) with a real \( l > 1 \) and the perturbation is bounded by \( 1 + |x|^\beta \) with \( \beta < l - 1 \); reducibility in the limiting case \( \beta = l - 1 \) was obtained by Liu and Yuan in \([34]\). Recently, the results in \([1]\) and \([34]\) have been improved in \([2, 3]\), in which Bambusi firstly obtained the reducibility results for 1d harmonic oscillators with unbounded perturbations. In \([3]\) Bambusi proved the reducibility when the symbol of the perturbation grows at most like \( (\xi^2 + x^2)^{\beta/2} \) with \( \beta < 2 \). In \([2]\) he generalized the class of the symbol to which the perturbation belongs (see \([4]\)). However, in \([2, 7]\), Bambusi wrote “we also remark that the assumption that the functions \( a_i \) are symbols rules out cases like \( a_i(x, \omega t) = \cos(x - \omega t) \).” The terms “\( a_i \)” are exactly the oscillatory ones considered in this paper.

More applications of pseudodifferential calculus can be found in several papers (see e.g. \([7, 8, 11, 17, 22, 36, 37, 40]\)). We mention that the above results are limited in the one dimensional case, while some higher dimensional results on this problem have been recently obtained \([6, 18, 23, 32, 38]\).

The related techniques have been used for a control on the growth of Sobolev norms in \([5, 39]\).

The proof of Theorem 1.1 is based upon the KAM in \([32]\) and the following estimate of Hermite functions.

Lemma 1.7. Suppose \( h_m(x) \) satisfies (1.3). For any \( k \neq 0 \) and for any \( m, n \geq 1 \),

\[
\left| \int_\mathbb{R} (x)^\mu e^{ikx} h_m(x) \overline{h_n(x)} dx \right| \leq C(|k| \vee |k|^{-1}) m^{-\frac{\mu}{2} + \frac{n}{4}} n^{-\frac{\mu}{2} + \frac{n}{4}},
\]

where \( C \) is an absolute constant and \( 0 \leq \mu < \frac{1}{4} \).

By Theorem 2.1 in sect. 2 and Lemma 1.7 we prove Thm. 1.1 in section 3. For the readers’ convenience we give a fast introduction of Langer’s turning point theory from \([45]\) at the beginning of section 4. The lengthy proof for Lemma 1.7 is then given after it. Section 5 is divided into two parts. In the first part we present Theorem 5.1 without proof. In the second we give some lemmas.
Notations. For \( k \in \mathbb{Z}^n \), \( |k| = \sum_{j=1}^{n} |k_j| \). We use \( \langle x \rangle = \sqrt{1 + x^2} \). \( \langle \cdot , \cdot \rangle \) is the standard scalar product in \( \mathbb{R}^n \) or \( \ell^2 \). \( \| \cdot \| \) is an operator-norm or \( \ell^2 \)-norm. We use the notations \( 1 \lor \tau := \max\{1, \tau\} \), \( \mathbb{Z}_+ = \{1, 2, \cdots \} \) and \( \mathbb{N} = \{0, 1, 2, \cdots \} \) and \( \mathbb{T} = \mathbb{R}/2\pi \mathbb{Z} \). For a positive number \( a \), \( |a| \) means the largest integer not larger than \( a \). We use the notation \( f(x) = O(g(x)) \) for \( x \to \infty \) if \( |f(x)| \leq C|g(x)| \) when \( x \) is large enough. We denote \( T^\alpha_\rho = \{(a + ib) \in \mathbb{C}^n/2\pi \mathbb{Z}^n| \max_j |b_j| < \rho \} \). The notation \( \text{Meas} \) stands for the Lebesgue measure in \( \mathbb{R}^n \).

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2. A KAM Theorem

We introduce the KAM Theorem from [32] especially for 1d case. We remark that KAM theory is almost well-developed for nonlinear Hamiltonian PDEs in 1-d context. See [10, 20, 25, 26, 27, 28, 29, 30, 31, 33, 35, 42, 46, 50] for 1-d KAM results. Comparing with 1-d case, the KAM results for multidimensional PDEs are relatively few. Refer to [13, 15, 19, 21, 23, 41] for \( n \)-d results. See [9] for an almost complete picture of recent KAM theory.

2.1. Setting. Linear space. For \( p \geq 0 \) we define \( Y_p := \ell^2_\rho \times \ell^2_\rho = \{ \zeta = (\zeta_a = (\xi_a, \eta_a) \in \mathbb{C}^2)_{a \in \mathbb{Z}_+^n} \| \| \zeta \|_{L} < \infty \} \) with \( \| \zeta \|_{L}^2 := \sum_{a \in \mathbb{Z}_+^n} (|\xi_a|^2 + |\eta_a|^2) a^p \). We provide the space \( Y_p, p \geq 0 \), with the symplectic structure \( i \sum_{a \in \mathbb{Z}_+^n} d\xi_a \wedge d\eta_a \).

Infinite matrices. We denote by \( \mathcal{M}_{\alpha} \) the set of infinite matrices \( A : \mathcal{E} \times \mathcal{E} \to \mathbb{C} \) with the norm \( |A|_\alpha := \sup_{a, b \in \mathbb{Z}_+^n} (ab)^n |A_{ab}| < +\infty \). We also denote \( \mathcal{M}_{\alpha}^+ \) be the subspace of \( \mathcal{M}_{\alpha} \) satisfying that an infinite matrix \( A \in \mathcal{M}_{\alpha}^+ \) if \( |A|_{\alpha+} := \sup_{a, b \in \mathbb{Z}_+^n} (ab)^n (1 + |a - b|) |A_{ab}| < +\infty \).

Parameter. In the paper \( \omega \) will play the role of a parameter belonging to \( D_0 = [1, 2]^n \). All the constructed functions will depend on \( \omega \) with \( C^1 \) regularity. When a function is only defined on a Cantor subset of \( D_0 \) the regularity is understood in Whitney sense.

A class of quadratic Hamiltonians. Let \( D \subset D_0 \), \( \alpha > 0 \) and \( \sigma > 0 \). We denote by \( \mathcal{M}_{\alpha}(D, \sigma) \) the set of mappings as \( T^\alpha_\rho \times D \ni (\theta, \omega) \mapsto Q(\theta, \omega) \in \mathcal{M}_{\alpha} \) which is real analytic on \( \theta \in T^\alpha_\rho \) and \( C^1 \) continuous on \( \omega \in D \). This space is equipped with the norm \( |Q|_{D, \sigma} := \sup_{|\omega| = D_0, |\theta| < \sigma} |\partial^{k}_{\omega} Q(\theta, \omega)|_{\alpha} \).

The subspace of \( \mathcal{M}_{\alpha}(D, \sigma) \) formed by \( F(\theta, \omega) \) such that \( \partial^{k}_{\omega} F(\theta, \omega) \in \mathcal{M}_{\alpha+}^+, |k| = 0, 1 \), is denoted by \( \mathcal{M}_{\alpha}^+(D, \sigma) \) and equipped with the norm \( |F|_{D, \sigma} := \sup_{|\omega| = D_0, |\theta| < \sigma} |\partial^{k}_{\omega} F(\theta, \omega)|_{\alpha+} \). The subspace of \( \mathcal{M}_{\alpha}(D, \sigma) \) that are independent of \( \theta \) will be denoted by \( \mathcal{M}_{\alpha}(D) \) and for \( N \in \mathcal{M}_{\alpha}(D) \),

\[
|N|_{\alpha} := \sup_{|\omega| = D_0, |k| = 0, 1} |\partial^{k}_{\omega} N(\omega)|_{\alpha}.
\]

\( C^1 \) norm of operator in \( \omega \). Given \( (\theta, \omega) \in T^\alpha_\rho \times D, \Phi(\theta, \omega) \in \mathcal{L}(Y_r, Y_{r'}) \) being \( C^1 \) operator with respect to \( \omega \) in Whitney sense, we define the \( C^1 \) norm of \( \Phi(\theta, \omega) \) with respect to \( \omega \) by

\[
\|\Phi\|_{\mathcal{L}(Y_r, Y_{r'})} := \sup_{(\theta, \omega) \in T^\alpha_\rho \times D, \ |k| = 0, 1, \ |\zeta|_{r} \neq 0} \frac{\|\partial^{k}_{\omega} \Phi(\theta, \omega) \zeta\|_{r'}}{\|\zeta\|_{r}},
\]

where \( r, r' \in \mathbb{R} \).
2.2. The reducibility theorem. In this subsection we state an abstract reducibility theorem for quadratic $t$-quasiperiodic Hamiltonian of the form

$$H(t, \xi, \eta) = \langle \xi, N\eta \rangle + \varepsilon \langle \xi, P(\omega t)\eta \rangle, \quad (\xi, \eta) \in Y_1 \subset Y_0,$$

and the associated Hamiltonian system is

$$\begin{cases} \dot{\xi} = -iN\xi - i\varepsilon P'(\omega t)\xi, \\
\dot{\eta} = iN\eta + i\varepsilon P(\omega t)\eta,
\end{cases}$$

where $N = \text{diag}\{\lambda_a, \; a \in \mathbb{Z}_+\}$ satisfying the following assumptions:

**Hypothesis A1 - Asymptotics.** There exist positive constants $c_0, \; c_1, \; c_2$ such that

$$c_1 a \geq \lambda_a \geq c_2 a \text{ and } |\lambda_a - \lambda_b| \geq c_0 |a - b|, \; a, b \in \mathbb{Z}_+.$$

**Hypothesis A2 - Second Melnikov condition in measure estimates.** There exist positive constants $\alpha_1, \alpha_2$ and $c_3$ such that the following holds: for each $0 < \kappa < 1/4$ and $K > 0$ there exists a closed subset $D' := D'(\kappa, K) \subset D_0$ with $\text{Meas}(D_0 \setminus D') \leq c_3 K^{\alpha_1} \kappa^{\alpha_2}$ such that for all $\omega \in D'$, $k \in \mathbb{Z}^n$ with $0 < |k| \leq K$ and $a, b \in \mathbb{Z}_+$ we have $|\langle k, \omega \rangle + \lambda_a - \lambda_b| \geq \kappa(1 + |a - b|)$. Then we have the following reducibility results.

**Theorem 2.1.** Given a non autonomous Hamiltonian (2.1), we assume that $(\lambda_a)_{a \in \mathbb{Z}_+}$ satisfies Hypothesis A1-A2 and $P(\theta) \in C^0(\mathbb{T}^n, \mathcal{M}_a)$ with $\alpha > 0$ and $\beta > \max\{9(2 + \frac{1}{9})^{-\frac{1}{2}}, \; 9n, \; 24\}$ where $\gamma_1 = \max\{\alpha_1, n + 3\}, \; \gamma_2 = \frac{9\alpha_2}{\gamma_1 + 9(2 + \frac{1}{9})^{-\frac{1}{2}}}, \; \delta \in (0, \frac{\gamma_1}{\gamma_2})$.

Then there exists $\varepsilon_*(n, \beta, \delta) > 0$ such that if $0 \leq \varepsilon < \varepsilon_*(n, \beta, \delta)$, there exist

(i) a Cantor set $D_\varepsilon \subset D_0$ with $\text{Meas}(D_0 \setminus D_\varepsilon) \leq c(n, \beta, \delta)\varepsilon^{\frac{2}{9}n - \tau};$

(ii) a $C^1$ family in $\omega \in D_\varepsilon$ (in Whitney sense), linear, unitary and symplectic coordinate transformation $\Phi_\infty^\omega(\theta) : Y_0 \to Y_0, \; \theta \in \mathbb{T}^n, \; \omega \in D_\varepsilon$, of the form

$$(\xi_+, \eta_+) \mapsto (\xi, \eta) = \Phi^\omega_\infty(\theta)(\xi_+, \eta_+) = (M_\omega(\theta)\xi_+, M_\omega(\theta)\eta_+),$$

where $\Phi^\omega_\infty(\theta) = \text{id} \in C^1(\mathbb{T}^n, \mathcal{L}(Y_0, Y_{2a})) \cap C^1(\mathbb{T}^n, \mathcal{L}(Y_{s'}, Y_{s''}))$ with $0 \leq s' \leq s'' < 1, \; \varepsilon \leq \frac{2}{9} \beta, \; \iota \notin \mathbb{Z}$ and satisfies

$$\|\Phi^\omega_\infty - \text{id}\|_{C^1(\mathbb{T}^n, \mathcal{L}(Y_0, Y_{2a}))} \leq C(n, \beta, \iota)\varepsilon^{\frac{1}{9}(\frac{2}{9} - \iota)}, \;$$

and

$$\|\Phi^\omega_\infty - \text{id}\|_{C^1(\mathbb{T}^n, \mathcal{L}(Y_{s'}, Y_{s''}))} \leq C(n, \beta, \iota)\varepsilon^{\frac{1}{9}(\frac{2}{9} - \iota)}.$$

(iii) a $C^1$ family of autonomous quadratic Hamiltonians in normal forms

$$H_\infty(\xi_+, \eta_+) = \langle \xi_+, N_\infty(\omega)\eta_+ \rangle = \sum_{j \geq 1} \lambda_j^\infty(\xi_j^+ + \eta_j^+), \; \omega \in D_\varepsilon,$$

where $N_\infty(\omega) = \text{diag}\{\lambda_j^\infty\}$ is diagonal and is close to $N$, i.e.

$$[N_\infty(\omega) - N]^{D_\varepsilon} \leq c(n, \beta)\varepsilon,$$

such that

$$H(t, \Phi^\omega_\infty(\omega t)(\xi_+, \eta_+)) = H_\infty(\xi_+, \eta_+), \; t \in \mathbb{R}, \; (\xi_+, \eta_+) \in Y_1, \; \omega \in D_\varepsilon.$$

Furthermore $\Phi^\omega_\infty(\theta)$ and $\Phi^\omega_\infty(\theta)^{-1}$ are bounded operators from $Y_{s'}$ into itself for $0 \leq s' \leq 1$ and they satisfy:

$$\|M_\omega(\theta) - \text{id}\|_{\mathcal{L}(E_j^+, E_j^+)}, \|M_\omega(\theta)^{-1} - \text{id}\|_{\mathcal{L}(E_j^-, E_j^-)} \leq c\varepsilon^{1/2}.$$

In this section we will apply Theorem 2.1 to the equation (1.1) to prove Theorem 1.1. For readers’ convenience, we rewrite the equation

$$i\partial_t u = -\partial_x^2 u + x^2 u + \varepsilon(x)^\mu W(\nu x, \omega t)u, \quad u = u(t, x), \quad x \in \mathbb{R},$$  \hspace{1cm} (3.1)$$

where $0 \leq \mu < \frac{1}{4}$ and the potential $W(\varphi, \theta): \mathbb{T}^d \times \mathbb{T}^n \to \mathbb{R}$ satisfies all the conditions in Theorem 1.1. Following [14], we expand $u$ and $\overline{\nu}$ on the Hermite basis $\{h_j\}_{j \geq 1}$, namely, $u = \sum_{j \geq 1} \xi_j h_j$ and $\overline{\nu} = \sum_{j \geq 1} \eta_j h_j$. And thus (3.1) can be written as a nonautonomous Hamiltonian system

$$\begin{cases}
\dot{\xi}_j = -\frac{\partial H}{\partial \eta_j} = -i(2j-1)\xi_j - i\varepsilon \frac{\partial}{\partial \eta_j} \overline{p(t, \xi, \eta)}, \quad j \geq 1, \\
\dot{\eta}_j = \frac{1}{2} \frac{\partial H}{\partial \xi_j} = i(2j-1)\eta_j + i\varepsilon \frac{\partial}{\partial \xi_j} \overline{p(t, \xi, \eta)}, \quad j \geq 1,
\end{cases}$$  \hspace{1cm} (3.2)$$

where

$$H(t, \xi, \eta) = n(\omega) + p(t, \xi, \eta) = \langle \xi, N\eta \rangle + \varepsilon \langle \xi, P(\omega t)\eta \rangle, \quad (\xi, \eta) \in Y_1 \subset Y_0,$$  \hspace{1cm} (3.3)$$

and $n(\omega) := \sum_{j \geq 1} (2j-1)\xi_j \eta_j$ and $P_1(\omega t) = \int_{\mathbb{R}} \langle x \rangle^\mu W(\nu x, \theta) h_i(x) \overline{h_j(x)} dx$. Here the external parameters are the frequencies $\omega = (\omega_j)_{j \leq j \leq n} \in D_0 := [1, 2]^n$. The proofs for the following two lemmas are standard.

**Lemma 3.1.** When $\lambda_\alpha = 2a - 1$, $a \in \mathbb{Z}_+$, Hypothesis A1 holds true with $c_0 = c_2 = 1$ and $c_1 = 2$.

**Lemma 3.2.** When $\lambda_\alpha = 2a - 1$, $a \in \mathbb{Z}_+$, Hypothesis A2 holds true with $D_0 = [0, 2\pi]^n$, $\alpha_1 = n + 1$, $\alpha_2 = 1$, $c_3 = c(\alpha)$ and

$$D' := \{ \omega \in [0, 2\pi]^n \mid \langle k, \omega \rangle + j \geq \kappa(1 + |j|), \quad \text{for all } j \in \mathbb{Z} \text{ and } k \in \mathbb{Z}^n \setminus \{0\} \}.$$  

For the following we define the set $R^*_{\gamma, k} = \{ \nu \in \mathbb{R}^d : \langle \langle k, \nu \rangle \rangle \leq \frac{\gamma}{|k|} \}$ for $k \neq 0$ and $R^*_{\gamma} = \bigcup_{0 \neq k \in \mathbb{Z}^d} R^*_{\gamma, k}$. It is well known that $Meas(R^*_{\gamma, k} \cap [A, B]^d) = O(\gamma/|k|^{r+1})$, and thus $Meas(R^*_{\gamma} \cap [A, B]^d) \leq O(\gamma)$. We define the set $\Omega_{\gamma} := [A, B]^d \setminus R^*_{\gamma}$ and then $Meas([A, B]^d \setminus \Omega_{\gamma}) = O(\gamma)$ as $\gamma \to 0$ and $\tau > d - 1$.

**Lemma 3.3.** If $W(\varphi, \theta) \in C^s(\mathbb{T}^d \times \mathbb{T}^n)$ with $s \geq d + [1 \vee \tau] + n + \beta_1 + 3$ and $\nu \in \Omega_{\gamma}$ and $\beta_1 \in \mathbb{N}$, then there exists $\alpha > 0$ such that the matrix function $P(\theta)$ defined by

$$(P(\theta))^i_j = \int_{\mathbb{R}} \langle x \rangle^\mu W(\nu x, \theta) h_i(x) \overline{h_j(x)} dx, \quad i, j \geq 1,$$

belongs to $C^{\beta_1}(\mathbb{T}^n, \mathcal{M}_\alpha)$ with $\alpha = \frac{1}{12} - \frac{\mu}{4}$.

**Proof.** We divide the proof into several steps.

(a) We show that $P(\theta) \in \mathcal{M}_\alpha$. Since $W(\nu x, \theta) = \sum_{k \in \mathbb{Z}^d, l \in \mathbb{Z}^n} \hat{W}(k, l) e^{ik \cdot \nu x} e^{il \theta}$, then

$$\begin{align*}
(P(\theta))^i_j &= \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}^d, l \in \mathbb{Z}^n} \hat{W}(k, l) e^{ik \cdot \nu x} e^{il \theta} \langle x \rangle^\mu h_i(x) \overline{h_j(x)} dx \\
&= \sum_{l \in \mathbb{Z}^n} e^{il \theta} \int_{\mathbb{R}} \sum_{k \neq 0} \hat{W}(k, l) e^{ik \cdot \nu x} h_i(x) \overline{h_j(x)} dx,
\end{align*}$$

where...
where we use (1.2). Note $\nu \in \Omega_\gamma$, we have $|k \cdot \nu| \geq \frac{2}{|k|^{\gamma}}$ for $\tau > d - 1$. Thus by Lemma 1.7

$$|(P(\theta))_1^2| \leq \frac{C(\gamma)}{i^\alpha j^{\alpha}} \sum_{l \in \mathbb{Z}^d} \sum_{k \neq 0} |\hat{W}(k, l)| \cdot |k|^{1/\tau}. \quad (3.4)$$

Denote $s_1 = [1 \vee \tau] + d + 1$ and $s_2 = n + 1$. When $W(\phi, \theta) \in C^{s_1 + s_2}(\mathbb{T}^d \times \mathbb{T}^n)$, we have

$$|\hat{W}(k, l)| \leq \frac{d^{s_1} n^{s_2}}{\sum_{|\alpha| = s_1} s_1^{s_1} |l|^{s_2} \sup_{|\nu| = s_2} |\partial_{\nu}^s \partial_{\phi}^s W(\phi, \theta)|, \quad \forall k \neq 0, l \neq 0,$$

and

$$|\hat{W}(k, 0)| \leq \frac{d^{s_1}}{\sum_{|\alpha| = s_1} s_1^{s_1} |\nu|^{s_2} \sup_{|\nu| = s_2} |\partial_{\nu}^s \partial_{\phi}^s W(\phi, \theta)|. \quad (3.5)$$

From (3.4) and (3.5), $|P(\theta)|_\alpha \leq C(\gamma, d, \tau, n) \sup_{|\alpha| \leq |1/\tau| + d + 1} |\partial_{\nu}^s \partial_{\phi}^s W(\phi, \theta)|$ which follows $P(\theta) \in \mathcal{M}_\alpha$.

(b) We show that $P(\theta) \in C^0(\mathbb{T}^n, \mathcal{M}_\alpha)$. For $\theta_1, \theta_2 \in \mathbb{R}^n$, $i, j \geq 1,$

$$|P(\theta_1) - P(\theta_2)|_\alpha = \left| \int_\mathbb{R} (x)^\mu (W(\nu x, \theta_1) - W(\nu x, \theta_2))h_i(x)\hat{h}_j(x)dx \right|$$

$$= \left| \sum_{l \in \mathbb{Z}^n} \left( e^{i\theta_1} - e^{i\theta_2} \right) \left( \sum_{k \neq 0} \hat{W}(k, l) \int_\mathbb{R} e^{ik \cdot \nu x} (x)^\mu h_i(x)\hat{h}_j(x)dx \right) \right|$$

$$\leq C(\gamma, d, \tau, n) \frac{\|\theta_1 - \theta_2\|}{i^\alpha j^{\alpha}} \sup_{|\alpha| \leq s_1} |\partial_{\nu}^s \partial_{\phi}^s W(\phi, \theta)|.$$
which means that $P(\theta)$ is Fréchet differentiable on $\theta_0 \in \mathbb{T}^n$ and $P'(\theta_0) = A$.

(d) By a straightforward computation we can show that $P(\theta) \in C^1(\mathbb{T}^n, \mathcal{M}_\alpha)$ since
\[ \| P'(\theta_1) - P'(\theta_2) \|_{\mathcal{L}(\mathbb{R}^n, \mathcal{M}_\alpha)} \leq C(\gamma, d, \tau, n) \cdot \| \theta_1 - \theta_2 \| \sup_{|\alpha| \leq |1 + \tau| + d + 1, |\nu| \leq n + 3} |\partial^\alpha_{\nu} \partial_\theta^{\tau+1} W| \]

(e) Inductively, we assume that $P(\theta) \in C^m(\mathbb{T}^n, \mathcal{M}_\alpha), m \leq \beta_1 - 1$, with
\[ \left( P^{(m)}(\theta)(\xi_1, \ldots, \xi_m) \right)_i = \int_{\mathbb{R}} (x)^\mu W^{(m)}_\theta(\nu x, \theta)(\xi_1, \ldots, \xi_m) h_i(x) h_j(x) dx \]

satisfying
\[ \| P^{(m)}(\theta) \|_{\mathcal{L}_m(\mathbb{R}^n, \mathcal{M}_\alpha)} \leq C(\gamma, d, \tau, n) \sup_{|\alpha| \leq |1 + \tau| + d + 1, |\nu| \leq n + m + 1} |\partial^\alpha_{\nu} \partial_\theta^{\tau} W| \]

where $\mathcal{L}_m(\mathbb{R}^n, \mathcal{M}_\alpha)$ denotes the multi-linear operator space $\mathcal{L}(\mathbb{R}^n \times \ldots \times \mathbb{R}^n, \mathcal{M}_\alpha)$. Then we show that $P(\theta) \in C^{m+1}(\mathbb{T}^n, \mathcal{M}_\alpha)$ with
\[ \left( P^{(m+1)}(\theta)(\xi_1, \ldots, \xi_{m+1}) \right)_i = \int_{\mathbb{R}} (x)^\mu W^{(m+1)}_\theta(\nu x, \theta)(\xi_1, \ldots, \xi_{m+1}) h_i(x) h_j(x) dx, \]

and
\[ \| P^{(m+1)}(\theta) \|_{\mathcal{L}_{m+1}(\mathbb{R}^n, \mathcal{M}_\alpha)} \leq C(\gamma, d, \tau, n) \sup_{|\alpha| \leq |1 + \tau| + d + 1, |\nu| \leq n + m + 2} |\partial^\alpha_{\nu} \partial_\theta^{\tau+1} W| \]

We follow the method in steps (c) and (d), and divide the proof into two parts (e1) and (e2) respectively.

(e1) We show that $P^{(m)}(\theta)$ is Fréchet differentiable and for $\forall \theta_0 \in \mathbb{R}^n, i, j \in \mathbb{Z},$
\[ \left( P^{(m+1)}(\theta_0)(\xi_1, \ldots, \xi_{m+1}) \right)_i = \int_{\mathbb{R}} (x)^\mu W^{(m+1)}_\theta(\nu x, \theta_0)(\xi_1, \ldots, \xi_{m+1}) h_i(x) h_j(x) dx, \]

with
\[ \| P^{(m+1)}(\theta) \|_{\mathcal{L}_{m+1}(\mathbb{R}^n, \mathcal{M}_\alpha)} \leq C(\gamma, d, \tau, n) \sup_{|\alpha| \leq |1 + \tau| + d + 1, |\nu| \leq n + m + 2} |\partial^\alpha_{\nu} \partial_\theta^{\tau} W| \]

In fact, given $\theta_0 \in \mathbb{R}^n$, we define for $\xi_1, \ldots, \xi_{m+1} \in \mathbb{R}^n, i, j \in \mathbb{Z},$
\[ (\mathcal{B}(\xi_1, \ldots, \xi_{m+1}))_i := \int_{\mathbb{R}} (x)^\mu W^{(m+1)}_\theta(\nu x, \theta_0)(\xi_1, \ldots, \xi_{m+1}) h_i(x) h_j(x) dx. \]

Since
\[ |(\mathcal{B}(\xi_1, \ldots, \xi_{m+1}))_i| \leq C(n) \|\xi_1\| \cdots \|\xi_{m+1}\| \sum_{k \neq 0, k \in \mathbb{Z}^d} \|W(k, l)\| \|l\|^{m+1} \int_{\mathbb{R}} (x)^\mu e^{ik\nu x} h_i(x) h_j(x) dx \]

\[ \leq \frac{C(\gamma, d, \tau, n) \|\xi_1\| \cdots \|\xi_{m+1}\|}{i^{\alpha} j^{\nu}} \sup_{|\alpha| \leq |1 + \tau| + d + 1, |\nu| \leq n + m + 2} |\partial^\alpha_{\nu} \partial_\theta^{\tau} W(\varphi, \theta)|, \]

it follows that $\|\mathcal{B}\|_{\mathcal{L}_{m+1}(\mathbb{R}^n, \mathcal{M}_\alpha)} \leq C(\gamma, d, \tau, n) \sup_{|\alpha| \leq |1 + \tau| + d + 1, |\nu| \leq n + m + 2} |\partial^\alpha_{\nu} \partial_\theta^{\tau} W(\varphi, \theta)|$. By a similar computation,
\[ \| P^{(m)}(\theta) - P^{(m)}(\theta_0) - \mathcal{B}(\theta - \theta_0) \|_{\mathcal{L}_m(\mathbb{R}^n, \mathcal{M}_\alpha)} \]
\[ \leq C(\gamma, n) \cdot \sum_{k \in \mathbb{Z}^n, k \neq 0, l \in \mathbb{Z}^n} |k|^{1 + \tau} \|W(k, l)\| \|l\|^{m+2} \cdot \|\theta - \theta_0\|^2 \]
Thus, \( P^{(m)}(\theta) \) is Fréchet differentiable and \( P^{(m+1)}(\theta_0) = \mathcal{B} \).

(c2) For simplicity, we denote \((ii)^{(m+1)} := (i) \times \cdots \times (i) \in \mathcal{L}_{m+1}(\mathbb{R}^n)\). Note

\[
\int_\mathbb{R} (x)^m W_{\theta}^{(m+1)}(\nu x, \theta) (x, \cdots, x_{m+1}, h_i(x) h_j(x)) dx
\]

therefore,

\[
\left\langle P^{(m+1)}(\theta_1) - P^{(m+1)}(\theta_2) \right\rangle (\xi_1 \cdots \xi_{m+1})^j
\leq \sup_{|\alpha| \leq |1+1|, \ |\nu| \leq n+m+3} \left\| |\alpha| \cdot \theta \cdot |\nu| \cdot \sup_{|\alpha| \leq |1+1|, \ |\nu| \leq n+m+3} |\partial_\nu^\alpha \partial_\theta^\nu W|.
\]

It follows that

\[
\| P^{(m+1)}(\theta_1) - P^{(m+1)}(\theta_2) \|_{\mathcal{L}_{m+1}(\mathbb{R}^n, \mathcal{M}_a)} \leq C(\gamma, d, \tau, n) \cdot \left\| \theta_1 - \theta_2 \right\| \sup_{|\alpha| \leq |1+1|, \ |\nu| \leq n+m+3} \left\| |\alpha| \cdot \theta \cdot |\nu| \cdot \sup_{|\alpha| \leq |1+1|, \ |\nu| \leq n+m+3} |\partial_\nu^\alpha \partial_\theta^\nu W|.
\]

which means that \( \| P^{(m+1)}(\theta_1) - P^{(m+1)}(\theta_2) \|_{\mathcal{L}_{m+1}(\mathbb{R}^n, \mathcal{M}_a)} \to 0 \) as \( \| \theta_1 - \theta_2 \| \to 0 \). Thus we finish the induction. \( \square \)

Proof of Theorem 1.1: It is clear that the Schrödinger equation (3.1) is equivalent to Hamiltonian system (3.2) with \( \lambda_a = 2a - 1 \). By lemmas given above, we can apply Theorem 2.1 to (3.2) with

\( \gamma_1 = n + 3, \quad \gamma_2 = \frac{\gamma_1}{2}, \quad \text{and} \quad \delta = \frac{\gamma_2}{2} \).

This leads to Theorem 1.1.

More precisely, in the new coordinates given in Theorem 2.1, \((\xi, \eta) = (\mathcal{M}_\omega \xi_+, M_\eta \eta_+), \) system (3.2) becomes autonomous and the systems are changed into the following:

\[
\left\{ \begin{array}{l}
\dot{\xi}_{+,a} = -i\lambda_a^\infty(\omega) \xi_{+,a}, \\
\dot{\eta}_{+,a} = i\lambda_a^\infty(\omega) \eta_{+,a},
\end{array} \right. \quad a \in \mathbb{Z}_+.
\]

Hence the solution starts from \((\xi_+(0), \eta_+(0))\) is given by

\[
(\xi_+(t), \eta_+(t)) = (e^{-itN_\omega \xi_+(0)}, e^{itM_\eta \eta_+(0)}), \quad t \in \mathbb{R},
\]

where \( N_\omega = \text{diag}\{\lambda_a^\infty\}_{a \in \mathbb{Z}_+} \). Then the solution \( u(t, x) \) of (1.1) corresponding to the initial data \( u_0(x) = \sum_{a \geq 1} \xi_a(0) h_a(x) \in \mathcal{H}^1 \) is formulated by \( u(t, x) = \sum_{a \geq 1} \xi_a(t) h_a(x) \) with \( \xi(t) = \mathcal{M}_\omega(\omega t)e^{-itN_\omega} M_\eta^1(0) \xi(0) \), where we use the fact \((\mathcal{M}_\omega)^{-1} = M_\eta^1\).

Let us define the transformation \( \psi_{\omega}^{\infty,1}(\theta) \) by

\[
\psi_{\omega}^{\infty,1}(\theta)(\sum_{a \geq 1} \xi_a h_a(x)) := \sum_{a \geq 1} (M_\eta^1(\theta) \xi_a) h_a(x) = \sum_{a \geq 1} \xi_{+,a} h_a(x).
\]

u(t, x) satisfies (1.1) if and only if \( \nu(t, x) = \psi_{\omega}^{\infty,1}(\omega t) u(t, x) \) satisfies the autonomous equation

\[
i\partial_t v = (-\partial_{xx} + |x|^2)v + \varepsilon Q_1 v,
\]

where

\[
\varepsilon Q_1(\sum_{a \in \mathbb{Z}_+} \xi_a h_a(x)) = \sum_{a \in \mathbb{Z}_+} ((\lambda_a^\infty - \lambda_a) \xi_a h_a(x) = \sum_{a \in \mathbb{Z}_+} (\lambda_a^\infty - \lambda_a) \xi_a h_a(x).
\]
For the rest estimates see lemma 3.4 below and (2.2).

**Lemma 3.4.**
\[ \| \Psi_{\omega}^{-1}(\cdot) - i d \|_{C^{1}(\Omega, \mathbb{C})} \leq C e^{C_{\beta} (1 - \delta^{-1})} , \]
and
\[ \| \Psi_{\omega}^{-1}(\cdot) - i d \|_{C^{1}(\Omega, \mathbb{C})} \leq C e^{C_{\beta} (1 - \delta^{-1})} , \]

where \( \omega \) is defined in Theorem 2.1 and \( 0 \leq s' \leq 1 \).

We delay the above proof in section 5.

4. Estimates on eigenfunctions

4.1. Langer’s turning point. We now introduce Langer’s turning point method based on the contents in Chapter 22.27 of [45]. For other application of Langer’s turning point theory, see [48, 49].

Consider the function
\[ \psi''(x) + (\lambda - q(x))\psi(x) = 0, \quad x > 0, \]
where \( q(x) \) increases steadily to \( +\infty \), \( q(x) \) is three times differentiable, and for \( x > x_0 \) for some positive constant \( x_0 \), \( q'(x) \) is nondecreasing, and as \( x \to \infty \)
\[ \frac{q'(x)}{q(x)} = O\left( \frac{1}{x} \right), \quad \frac{q''(x)}{q(x)} = O\left( \frac{1}{x} \right), \quad \frac{q'''(x)}{q(x)} = O\left( \frac{1}{x} \right). \]

We also suppose that there exists a unique \( X > 0 \) such that \( \lambda = q(X) \).

Then for constant \( a \geq 1 \), \( \ln \frac{q(ax)}{q(x)} = \int_x^{ax} \frac{q'(t)}{q(t)} dt = O(\int_x^{ax} \frac{1}{t} dt) = O(\ln a) \), which means \( q(ax) = O(q(x)) \), and similarly for \( q'(x) \) and \( q''(x) \). Since \( q(x) = \int_x^{a x} q'(t) dt + q(0) \leq xq'(x) + q(0) \), it follows \( q'(x) \geq C q(x)^2 \), when \( x \geq x_1 \) for some \( x_1 > 0 \). Now set \( \eta(x) = (\lambda - q(x))\frac{4}{3} \psi(x) \) and \( \zeta(x) = \int_x^{X} \lambda - q(t) \frac{1}{t^2} dt \), where
\[ \arg \zeta(x) = \begin{cases} \frac{\pi}{2}, & (x > X), \\ -\frac{\pi}{2}, & (x < X). \end{cases} \]

Then the equation (4.1) is transformed into
\[ \frac{d^2 \eta}{dz^2} + \eta + \left( \frac{q''(x)}{q(x) q(x - q(x))} + \frac{5q'(x)^2}{10q(x) q(x - q(x))} \right) \eta = 0 \]
and this may be expressed as
\[ \frac{d^2 \eta}{dz^2} + \left( 1 + \frac{5}{36q^2} \right) \eta = f(x) \eta, \]
where \( f(x) = \frac{q'(x)}{q(x)} - \frac{q''(x)}{4q(x) q(x - q(x))} - \frac{5q'(x)^2}{10q(x) q(x - q(x))} \). As we know, Bessel equation \( \frac{d^2 G}{dz^2} + \left( 1 + \frac{5}{36q^2} \right) G = 0 \) has two linearly independent solutions \( J_\frac{1}{2}(\zeta) \) and \( H_\frac{1}{2}^{(1)}(\zeta) \), where \( J_\nu(x) \) and \( H_\nu^{(1)}(x) \) are the first kind Bessel function and one of the third kind Bessel function, respectively. By the property of Bessel function that \( x J_\nu(x) H_\nu^{(1)}(x) - J_\nu'(x) H_\nu^{(1)}(x) = \frac{2}{\pi} \), then (4.2) is formally equivalent to the integral equation
\[ \eta = \left( \frac{\pi^2}{2} \right)^{\frac{1}{2}} H_\frac{1}{2}^{(1)}(\zeta) + \frac{\pi}{2} \int_x^\infty \left( H_\frac{1}{2}^{(1)}(\zeta) J_\frac{1}{2}(\theta) - J_\frac{1}{2}^{(1)}(\zeta) H_\frac{1}{2}(\theta) \right) \zeta^{\frac{1}{2}} \theta^{\frac{1}{2}} f(t) (\lambda - q(t))^{\frac{1}{2}} \eta(t) dt, \]
where we write \( \zeta = \zeta(x) \) and \( \theta = \zeta(t) \) for convenience. Set
\[ \alpha(x) = e^{-i\zeta} \left( \frac{\pi^2}{2} \right)^{\frac{1}{2}} H_\frac{1}{2}^{(1)}(\zeta), \quad \beta(x) = e^{i\zeta} \left( \frac{\pi^2}{2} \right)^{\frac{1}{2}} J_\frac{1}{2}(\zeta), \quad \chi(x) = e^{-i\zeta} \eta(x), \]
then
\[ \chi(x) = \alpha(x) + i \int_{x}^{\infty} \left( \alpha(x)\beta(t) - e^{2i(\theta - \zeta)}\beta(x)\alpha(t) \right) f(t)(\lambda - q(t))^{\frac{1}{2}}d\chi(t)dt. \]

Clearly, \( \alpha(x), \beta(x) \) are bounded, and \( \Im(\theta - \zeta) = \Im(f_{x}'(\lambda - q(u))^{\frac{1}{2}}du) \geq 0 \). To give the estimate of solution of (4.2) or (4.1), we first present two preparation lemmas and delay the proofs in the Appendix.

**Lemma 4.1.** ([45]) For fixed \( \lambda \), if \( x > 2X \), then
\[ \int_{x}^{\infty} |f(t)| |\lambda - q(t)|^{\frac{1}{2}} dt \leq \frac{C}{x(q(x))^{\frac{1}{2}}} \]
where \( C \) is a constant independent of \( x \) and \( \lambda \).

**Lemma 4.2.** ([45]) \( \int_{0}^{\infty} |f(x)||\lambda - q(x)|^{\frac{1}{2}} dx = O\left(\frac{1}{x^{\lambda x^{\frac{1}{2}}}}\right), \quad \lambda \rightarrow \infty.\)

From these two lemmas, we can prove that the iteration converges. In fact, if we denote
\[ \int_{0}^{\infty} |f(t)||\lambda - q(t)|^{\frac{1}{2}} dt = M_{0} = O\left(\frac{1}{x^{\lambda x^{\frac{1}{2}}}}\right), \]
and \( |\alpha(x)| \beta(t) - e^{2i(\theta - \zeta)}\beta(x)\alpha(t) | \leq M \) uniformly, then
\[ |\chi_{0}(x)| = |\alpha(x)| \leq C, \quad |\chi_{1}(x) - \chi_{0}(x)| \leq CM_{0} \]
and generally, if \( |\chi_{n}(x) - \chi_{n-1}(x)| \leq CM_{n}M_{0}^{n} \), then
\[ |\chi_{n+1}(x) - \chi_{n}(x)| = \left| \int_{x}^{\infty} \left( \alpha(x)\beta(t) - e^{2i(\theta - \zeta)}\beta(x)\alpha(t) \right) f(t)(\lambda - q(t))^{\frac{1}{2}} (\chi_{n}(t) - \chi_{n-1}(t)) dt \right| \]
\[ \leq CM_{n+1}M_{0}^{n} \int_{x}^{\infty} |f(t)| |\lambda - q(t)|^{\frac{1}{2}} dt \leq CM_{n+1}M_{0}^{n+1}. \]
Thus,
\[ |\chi_{n}(x)| \leq |\chi_{0}(x)| + |\chi_{1}(x) - \chi_{0}(x)| + \cdots + |\chi_{n}(x) - \chi_{n-1}(x)| \]
\[ \leq C(1 + MM_{0} + \cdots + M^{n}M_{0}^{n}) \leq C \frac{1}{1 - MM_{0}}. \]

If \( \lambda \) is sufficiently large, then \( MM_{0} < 1 \), and by the theorem of dominated convergence, when \( n \rightarrow \infty \), \( \chi_{n}(x) \rightarrow \chi(x) = \alpha(x) + O\left(\frac{1}{x^{\lambda x^{\frac{1}{2}}}}\right) \) uniformly w.r.t \( x \), which means that \( \chi(x) \) is bounded. Next we show that
\[ \chi(x) = \alpha(x) \left( 1 + O\left(\frac{1}{x^{\lambda x^{\frac{1}{2}}}}\right) \right). \]

In fact, similar as Lemma 5.4, if \( \zeta(x) < -c_{0} \) or \( i\zeta(x) < -c_{0} \), where \( c_{0} \) are arbitrary two positive constants, we can prove that \( |\alpha(x)| > C \) and (4.3) holds. While for \( 0 < |\zeta(x)| \leq c_{0} \) we have \( |\beta(x)| \leq C|\alpha(x)| \). Thus,
\[ |\chi(x) - \alpha(x)| = \left| \int_{x}^{\infty} \left( \alpha(x)\beta(t) - e^{2i(\theta - \zeta)}\beta(x)\alpha(t) \right) f(t)(\lambda - q(t))^{\frac{1}{2}} \chi(t) dt \right| \leq \frac{C|\alpha(x)|}{x^{\lambda x^{\frac{1}{2}}}}. \]

Hence we have

**Lemma 4.3.** ([45]) When \( \lambda > c_{1} > 0 \) large enough such that
\[ M \int_{0}^{\infty} |f(t)||\lambda - q(t)|^{\frac{1}{2}} dt < 1, \]
the solution of (4.1) can be written as
\[ \psi(x) = (\lambda - q(x))^{-\frac{1}{4}(\frac{\zeta(x)}{2})} H_{\frac{1}{2}}^{(1)}(\zeta)(1 + O\left(\frac{1}{x^{\lambda x^{\frac{1}{2}}}}\right))). \]

Since \( M \int_{x}^{\infty} |f(t)||\lambda - q(t)|^{\frac{1}{2}} dt \leq \frac{MC}{x(q(x))^{\frac{1}{2}}} \), where \( M, C \) are independent of \( x \) and \( \lambda \), it follows
\[ M \int_{c_{2}}^{\infty} |f(t)||\lambda - q(t)|^{\frac{1}{2}} dt < \frac{1}{2} \]
for some positive constant \( c_{2} \). Hence we have
Lemma 4.4. [45] For any fixed $\lambda$, when $x > \max \{2X, c_2\}$, the solution of (4.1) can be written as $\psi(x) = \psi_1(x) + \psi_2(x)$, where $\psi_1(x) = (\lambda - q(x))^{-\frac{1}{2}}(\frac{\pi}{2})^{\frac{1}{4}} H_{\frac{1}{2}}^{(1)}(\zeta)$, and $|\psi_2(x)| \leq \frac{C}{x^{(q(x))\frac{1}{4}}}|\psi_1(x)|$.

Remark 4.5. In the application $q(x) = x^2$, $\lambda_n = 2n - 1$ with $n \in \mathbb{Z}_+$. Then for $\lambda_n > c_1$, i.e. $n > \frac{c_1 + 1}{2}$, $h_n(x) = \psi_1^{(n)}(x) + \psi_2^{(n)}(x)$, where $x > 0$ and $\psi_1^{(n)}(x) = (\lambda_n - x^2)^{-\frac{1}{2}}(\frac{\pi}{2})^{\frac{1}{4}} H_{\frac{1}{2}}^{(1)}(\zeta_n)$ and $\psi_2^{(n)}(x) = \psi_1^{(n)}(x)O(\frac{n}{\lambda_n})$. While for $\lambda_n \leq \frac{c_2^2}{4} := c_3$ and $x > c_2$, we have $h_n(x) = \psi_1^{(n)}(x) + \psi_2^{(n)}(x)$, where $|\psi_2^{(n)}(x)| \leq \frac{C}{x^2}|\psi_1^{(n)}(x)|$.

Remark 4.6. For the following we denote $m_0 = \max \{\frac{c_1 + 1}{2}, \frac{c_2 + 1}{2}\}$.

4.2. Proof of Lemma 1.7. A well-known fact is that $h_n(x) = (n!2^n\pi^\frac{n}{2})^{-\frac{1}{2}}e^{-\frac{x^2}{2}}H_n(x)$ where $H_n(x)$ is the Hermite polynomial of degree $n$ and $h_n(x)$ is an even or odd function of $x$ according to whether $n$ is odd or even([44]). From the symmetry of $h_n(x)$, we only need to consider

\[
\int_0^{+\infty} \langle x \rangle^\mu e^{ikx} h_n(x) \overline{h_n(x)} dx \quad \text{for } 1 \leq m \leq n.
\]

Rewrite

\[
\int_0^{+\infty} \langle x \rangle^\mu e^{ikx} h_n(x) \overline{h_n(x)} dx = \int_0^{X_n} + \int_{X_n}^{+\infty}.
\]

From Remark 4.5 and Remark 4.6 and $m > m_0$,

\[
h_m(x) = \langle \lambda_m - x^2 \rangle^{-\frac{1}{2}}(\frac{\pi}{2})^{\frac{1}{4}} H_{\frac{1}{2}}^{(1)}(\zeta_m) + \langle \lambda_m - x^2 \rangle^{-\frac{1}{2}}(\frac{\pi}{2})^{\frac{1}{4}} H_{\frac{1}{2}}^{(1)}(\zeta_m) O(\frac{1}{\lambda_m})
\]

\[
:= \psi_1^{(m)}(x) + \psi_2^{(m)}(x),
\]

where $\zeta_m(x) = \int_x^{X_m} \frac{1}{\lambda_m - t^2} dt$ with $X_m = \lambda_m(X_m > 0)$. While for $m \leq m_0$, by Lemma 4.4 and $x > 2X_m$, $h_m(x) = \psi_1^{(m)}(x) + \psi_2^{(m)}(x)$, where $\psi_1^{(m)}(x) = (\lambda_m - x^2)^{-\frac{1}{2}}(\frac{\pi}{2})^{\frac{1}{4}} H_{\frac{1}{2}}^{(1)}(\zeta_m)$, and $|\psi_2^{(m)}(x)| \leq \frac{C}{x^2}|\psi_1^{(m)}(x)|$. Now we estimate (4.4) in the following three cases:

1) $m, n < C_* := 2^m m_0^3$; 2) $m \leq m_0$ and $n \geq C_* ; 3)m, n > m_0$.

Lemma 4.7. When $n, m < C_*$,

\[
\int_0^{+\infty} \langle x \rangle^\mu e^{ikx} h_m(x) \overline{h_n(x)} dx \leq \frac{C}{n^{\frac{1}{4}}m^{\frac{1}{4}}}.\]

Proof. Since $m < C_*$, then for $x > X_0$, $h_m(x) = \psi_1^{(m)}(x) + \psi_2^{(m)}(x)$, where $\psi_1^{(m)}(x) = (\lambda_m - x^2)^{-\frac{1}{2}}(\frac{\pi}{2})^{\frac{1}{4}} H_{\frac{1}{2}}^{(1)}(\zeta_m)$, and $|\psi_2^{(m)}(x)| \leq C|x^2|\psi_1^{(m)}(x)|$ and $X_0$ is a positive constant depending on $C_*$ only. $h_n(x)$ has a similar decomposition. When $x \leq X_0$, by Hölder inequality and $n, m < C_*$,

\[
\left| \int_0^{X_0} \langle x \rangle^\mu e^{ikx} h_m(x) \overline{h_n(x)} dx \right| \leq X_0^\mu \leq \frac{C}{n^{\frac{1}{4}}m^{\frac{1}{4}}}.
\]

When $x > X_0$, $|X_m^2 - x^2|^{-\frac{1}{2}} < 1$, and by Lemma 5.4, $\left| \int_{X_0}^{X_0} (\frac{\pi}{2})^{\frac{1}{4}} H_{\frac{1}{2}}^{(1)}(\zeta_m) \right| \leq e^{-|\zeta_m|}$. By Lemma 5.5 and $x > X_0$, $|\zeta_m| \geq \frac{2\sqrt{2}X_m}{X_0^2}$, $X_m \geq x - X_0$. Thus,

\[
\left| \int_{X_0}^{+\infty} \langle x \rangle^\mu e^{ikx} h_m(x) \overline{h_n(x)} dx \right| \leq \int_{X_0}^{+\infty} \langle x \rangle^\mu e^{-2(x - X_0)} dx \leq C e^{2X_0} \leq \frac{C}{n^{\frac{1}{4}}m^{\frac{1}{4}}}.
\]
Lemma 4.8. For $m \leq m_0$ and $n \geq C_3$,
\[
\left| \int_0^{+\infty} (x)^{\mu} e^{ikx} h_m(x) \overline{h_n(x)} dx \right| \leq \frac{C}{n^{1/2} \cdot m^{1/2} - \frac{3}{4}}.
\]

Proof. We split the integral into two parts
\[
\int_0^{+\infty} (x)^{\mu} e^{ikx} h_m(x) \overline{h_n(x)} dx = \int_0^{X_n^+} + \int_{X_n^+}^{+\infty}.
\]

When $x > 2X_m$, by Lemma 4.4 we have $|h_m(x)| \leq 2(x^2 - X_m^2)^{-\frac{3}{4}} |\lambda_x| \lambda_x H_1^{(1)}(\lambda_x) \leq 2e^{-\lambda_x|n|}$. On the other hand by Lemma 5.4, $|h_n(x)| \leq C(X_n^2 - x^2)^{-\frac{3}{4}}$. Thus,
\[
\left| \int_0^{X_n^+} (x)^{\mu} e^{ikx} h_m(x) \overline{h_n(x)} dx \right| \leq C \int_0^{X_n^+} (x)^{\mu} (X_n^2 - x^2)^{-\frac{3}{4}} dx \leq C X_n^{-\frac{3}{4} + \frac{3}{4}} \leq \frac{C}{n^{1/2} \cdot m^{1/2} - \frac{3}{4}}.
\]

When $x \geq X_n^+, n \geq m_0$, by Lemma 5.5, $e^{-\lambda_x|n|} \leq e^{-C(x - X_n)}$. Thus, by Hölder inequality,
\[
\left| \int_{X_n^+}^{+\infty} (x)^{\mu} e^{ikx} h_m(x) \overline{h_n(x)} dx \right| \leq C \left( \int_{X_n^+}^{+\infty} (x)^{2\mu} e^{-C_3 x} dx \right)^{\frac{1}{2}} \leq e^{-C_3 X_n^+}.
\]

Now we turn to the third case that is $m, n > m_0$. Rewrite $\int_0^{+\infty} (x)^{\mu} e^{ikx} h_m(x) \overline{h_n(x)} dx = \int_0^{X_n} + \int_{X_n}^{+\infty}$. We first turn to the integral $\int_{X_n}^{+\infty}$. In the following part of this section we will denote $\mathcal{F}(x) = (x)^{\mu} e^{ikx} \psi_1^{(m)}(x) \overline{\psi_1^{(n)}(x)}$ for simplicity.

4.3. the integral on $[X_n, +\infty)$. In fact in this part we have

Lemma 4.9. When $m_0 < m \leq n$,
\[
\left| \int_{X_n}^{+\infty} (x)^{\mu} e^{ikx} h_m(x) \overline{h_n(x)} dx \right| \leq \frac{C}{m^{1/2} \cdot n^{1/2} - \frac{3}{4}}.
\]

Lemma 4.9 results from the following two lemmas.

Lemma 4.10. For $m_0 < m \leq n$,
\[
\left| \int_{2X_n}^{+\infty} (x)^{\mu} e^{ikx} h_m(x) \overline{h_n(x)} dx \right| \leq e^{-Cn}.
\]

Proof. Since $h_n(x) := \psi_1^{(n)}(x) + \psi_2^{(n)}(x)$, we only need to prove $\left| \int_{2X_n}^{+\infty} \mathcal{F}(x) dx \right| \leq e^{-Cn}$, since the other three integrals have better estimates, where $\psi_2^{(n)}(x) = O(\frac{1}{n}) \psi_1^{(n)}(x)$. Recall that $\psi_1^{(n)}(x) = (\lambda_n - x^2)^{-\frac{3}{4}} \sqrt{\frac{\alpha_n}{2}} H_1^{(1)}(\lambda_n)$, then by Lemma 5.4, $\sqrt{\frac{\alpha_n}{2}} H_1^{(1)}(\lambda_n) \leq e^{-\lambda_n|n|}$ when $x \geq 2X_n$. By Lemma 5.5, when $x \geq 2X_n$, $|\lambda_n| \geq \frac{4\pi}{3} X_n(x - X_n) \geq \frac{4\pi}{3} (x - X_n) + \frac{4\pi}{3} X_n^2$. Therefore,
\[
\left| \int_{2X_n}^{+\infty} \mathcal{F}(x) dx \right| \leq \int_{2X_n}^{+\infty} (x)^{\mu} (x^2 - \lambda_n)^{-\frac{3}{4}} (x^2 - \lambda_n)^{-\frac{3}{4}} e^{-|\lambda_n|} e^{-|\lambda_n|} dx \leq \int_{2X_n}^{+\infty} (x)^{\mu} (x^2 - \lambda_n)^{-\frac{3}{4}} (x^2 - \lambda_n)^{-\frac{3}{4}} e^{-|\lambda_n|} dx.
\]
\[ \leq C e^{-\frac{n}{2}X_n^2} \int_{2X_n}^{+\infty} \langle x \rangle^{\mu} e^{-\frac{n}{2}x} \, dx \leq e^{-Cn}. \]

By Lemma 4.11. For \( m_0 < m \leq n \),
\[ \left| \int_{X_n}^{2X_n} \langle x \rangle^{\mu} e^{ikx} h_m(x) \, dx \right| \leq \frac{C}{m^{1/2} - \frac{1}{2} n^{1/2} - \frac{1}{4}}. \]

**Proof.** As above we only need to estimate \( |I| = \left| \int_{X_n}^{2X_n} \mathcal{F}(x) \, dx \right| \leq \frac{C}{m^{1/2} - \frac{1}{2} n^{1/2} - \frac{1}{4}}. \) We divide \( I \) into two parts as
\[
|I| = \left| \left( \int_{X_n}^{2X_n} + \int_{X_n}^{X_n + X_n^{1/2}} \right) \mathcal{F}(x) \, dx \right|
\leq CX_n^\mu \left( \int_{X_n}^{2X_n} + \int_{X_n}^{X_n + X_n^{1/2}} \right) \left| \psi_1^{(m)}(x) \psi_1^{(m)}(x) \right| \, dx.
\]

By Lemma 5.5, when \( x \geq X_n + X_n^{1/2}, \, |\zeta_n| \geq \frac{2\pi}{\sqrt{n}} (x - X_n)^{3/2} \geq \frac{2\pi}{\sqrt{n}} X_n. \) Thus,
\[
\int_{X_n + X_n^{1/2}}^{2X_n} \left| \psi_1^{(m)}(x) \psi_1^{(m)}(x) \right| \, dx \leq C \int_{X_n + X_n^{1/2}}^{2X_n} (x^2 - \lambda_n)^{-\frac{1}{4}} (x^2 - \lambda_n)^{-\frac{1}{4}} e^{-|\zeta_n|} \, dx
\leq C e^{-\frac{n}{2}X_n^2} \int_{X_n + X_n^{1/2}}^{2X_n} (x^2 - \lambda_n)^{-\frac{1}{4}} \, dx \leq C e^{-\frac{n}{2}X_n^2}.
\]

On the other hand,
\[
\int_{X_n}^{X_n + X_n^{1/2}} \left| \psi_1^{(m)}(x) \psi_1^{(m)}(x) \right| \, dx \leq C \int_{X_n}^{X_n + X_n^{1/2}} (x^2 - \lambda_n)^{-\frac{1}{4}} (x^2 - \lambda_n)^{-\frac{1}{4}} \, dx
\leq C \int_{X_n}^{X_n + X_n^{1/2}} (x^2 - \lambda_n)^{-\frac{1}{4}} \, dx \leq CX_n^{-\frac{1}{4}} \int_{X_n}^{X_n + X_n^{1/2}} (x - X_n)^{-\frac{1}{4}} \, dx \leq CX_n^{-\frac{1}{4}}.
\]

Therefore, \( |I| \leq CX_n^{\mu - \frac{1}{4}} \leq \frac{C}{m^{1/2} - \frac{1}{2} n^{1/2} - \frac{1}{4}}. \)

In the following we will estimate the integral on \([0, X_n]\), for which we have to discuss two different cases, namely, \( X_n \geq 2X_m \) or \( X_m \leq X_n \leq 2X_m \) with \( n \geq m > m_0 \).

### 4.4. the integral on \([0, X_n]\) for the case \( X_n \geq 2X_m \)
To simplify the following proof we will use the following notation in the remained parts. We define \( f_m(x) = \int_0^\infty e^{-t} t^{-\frac{1}{2}} \left( 1 + \frac{i t}{2 \zeta_m} \right)^{-\frac{1}{4}} \, dt \) and \( f_n(x) = \int_0^\infty e^{-t} t^{-\frac{1}{2}} \left( 1 + \frac{i t}{2 \zeta_n} \right)^{-\frac{1}{4}} \, dt \). When \( x \in [0, X_m] \), from a straightforward computation
\[
\psi_1^{(m)}(x) = (X_m - x^2)^{-\frac{1}{4}} \sqrt{\frac{\pi \zeta_m}{2}} H^{(1)}_{1/4}(\zeta_m)
\leq (X_m - x^2)^{-\frac{1}{4}} \frac{e^{i(\zeta_m - \frac{\pi}{4} - \frac{1}{4})}}{\Gamma(\frac{1}{4})} \int_0^\infty e^{-t} t^{-\frac{1}{2}} \left( 1 + \frac{i t}{2 \zeta_m} \right)^{-\frac{1}{4}} \, dt
= C (X_m - x^2)^{-\frac{1}{4}} e^{i\zeta_m(x)} f_m(x).
\]
Similarly, when $x \in [0, X_m]$, \( \psi^{(n)}(x) = C(X_n^2 - x^2)^{-\frac{1}{2}} e^{-x \zeta_m(x) f_n(x)} \). We also define \( \Psi(x) = (X_m^2 - x^2)^{-\frac{1}{4}} f_m(x) f_n(x) \) and \( g(x) = (\zeta_m(x) - \zeta_m(x) - k) f_n(x) = \sqrt{X_n^2 - x^2} - \sqrt{X_m^2 - x^2} - k \) with \( x \in [0, X_m] \). We will use the derivative of \( \Psi \) for many times, i.e.

\[
\Psi'(x) = \frac{1}{2} x (X_m^2 - x^2)^{-\frac{1}{4}} f_m(x) f_n(x) \\
+ \frac{1}{2} x (X_m^2 - x^2)^{-\frac{1}{4}} (X_n^2 - x^2)^{-\frac{3}{4}} f_m(x) f_n(x) \\
+ (X_m^2 - x^2)^{-\frac{1}{2}} (X_n^2 - x^2)^{-\frac{1}{2}} (f_m(x) f_n(x) + f_m(x) f_n(x)).
\]

From \( x \in [0, X_m] \) we obtain \( |f_m(x)| \leq \Gamma(\frac{k}{2}) \) and \( |f_n(x)| \leq \Gamma(\frac{k}{2}) \). By a straightforward computation we have

**Corollary 4.12.** For \( x \in [0, X_m] \) and \( m \leq n \),

\[
|\Psi'(x)| \leq C \left( x (X_m^2 - x^2)^{-\frac{1}{2}} (X_n^2 - x^2)^{-\frac{3}{4}} + x (X_m^2 - x^2)^{-\frac{1}{2}} (X_n^2 - x^2)^{-\frac{1}{2}} \right)
\]

\[
\frac{(X_m^2 - x^2)^{\frac{1}{2}} (X_n^2 - x^2)^{-\frac{3}{4}} X_m(X_m - x)^3}{X_n(X_n - x)^3} + \frac{(X_m^2 - x^2)^{-\frac{1}{2}} (X_n^2 - x^2)^{\frac{1}{2}}}{X_n(X_n - x)^3}
\]

\[
= C(J_1 + J_2 + J_3 + J_4) \leq C(J_1 + J_3).
\]

Our main intent in this subsection is to set up

**Lemma 4.13.** For \( k \neq 0 \), if \( X_n \geq 2X_m \), then

\[
\left| \int_0^{X_n} \langle x \rangle^0 e^{ikx} h_m(x) h_n(x) dx \right| \leq \frac{C(|k| \vee 1)^{\frac{1}{2}}}{m^{\frac{1}{4}} n^{\frac{1}{2}} - \frac{n}{4}},
\]

where \( m_0 < m \leq n \).

We first have

**Lemma 4.14.** For \( k \neq 0 \), if \( X_n \geq 2X_m \), then

\[
\left| \int_0^{X_n - X_m^{\frac{1}{2}}} \langle x \rangle^0 e^{ikx} h_m(x) h_n(x) dx \right| \leq \frac{C(|k| \vee 1)^{\frac{1}{2}}}{m^{\frac{1}{4}} n^{\frac{1}{2}} - \frac{n}{4}},
\]

where \( m_0 < m \leq n \).

**Proof.** First we estimate the main part

\[
\int_0^{X_m - X_m^{\frac{1}{2}}} \mathcal{F}(x) dx = C \int_0^{X_m - X_m^{\frac{1}{2}}} \langle x \rangle^0 e^{i(\zeta_m(x) - \zeta_m(x) + k) \Psi(x)} dx,
\]

by oscillatory integrals, where \( \Psi(x) = (X_m^2 - x^2)^{-\frac{1}{4}} (X_n^2 - x^2)^{-\frac{3}{4}} \cdot f_m(x) f_n(x) \). We discuss two different cases.

Case 1: \( k \leq \frac{X_n}{4} \). In this case, we have

\[
g(x) = \sqrt{X_n^2 - x^2} - \sqrt{X_n^2 - x^2} - k = \frac{X_n}{2} - k \geq \frac{X_n}{4}.
\]

Thus, by Lemma 5.6,

\[
\left| \int_0^{X_m - X_m^{\frac{1}{2}}} e^{i \zeta_m(x) - \zeta_m(x) + k} X_n \langle x \rangle^0 \Psi(x) dx \right|
\]
Clearly,
\[
\left| \langle x \rangle^\mu \Psi (X_m - X_m^{\frac{1}{2}}) \right| \leq C X_n^{-1} \left( \| \langle x \rangle^\mu \Psi \|_1 \right) \int_0^{X_m - X_m^{\frac{1}{2}}} |\langle x \rangle^\mu \Psi |(x) \, dx
\]

Similarly, when \( m \leq n \),
\[
\left| \langle x \rangle^\mu \Psi (X_m - X_m^{\frac{1}{2}}) \right| \leq C X_n^{-1} \left( \| \langle x \rangle^\mu \Psi \|_1 \right) \int_0^{X_m - X_m^{\frac{1}{2}}} (2 \langle x \rangle^\mu (J_1 + J_3) + \mu \langle x \rangle^\mu - 1 \langle x \rangle^\mu |\Psi(x)|) \, dx
\]

Similarly, when \( m \leq n \),
\[
\left| \langle x \rangle^\mu \Psi (X_m - X_m^{\frac{1}{2}}) \right| \leq C X_n^{-1} \left( \| \langle x \rangle^\mu \Psi \|_1 \right) \int_0^{X_m - X_m^{\frac{1}{2}}} (J_1 + J_3) \, dx + C \mu X_n^{-1} \int_0^{X_m - X_m^{\frac{1}{2}}} \langle x \rangle^\mu - 1 |\Psi(x)| \, dx
\]

Clearly,
\[
X_n^\mu \left| \Psi (X_m - X_m^{\frac{1}{2}}) \right| \leq C X_n^\mu \left( X_n^2 - (X_m - X_m^{\frac{1}{2}})^2 \right)^{-\frac{1}{2}} \left( X_n^2 - (X_m - X_m^{\frac{1}{2}})^2 \right)^{-\frac{1}{2}} \leq C X_m^{-\frac{1}{2} + \mu},
\]

and
\[
\int_0^{X_m - X_m^{\frac{1}{2}}} \mu \langle x \rangle^\mu - 1 |\Psi(x)| \, dx \leq C \left( X_n^2 - (X_m - X_m^{\frac{1}{2}})^2 \right)^{-\frac{1}{2}} \left( X_n^2 - (X_m - X_m^{\frac{1}{2}})^2 \right)^{-\frac{1}{2}} \int_0^{X_m} \mu x^{\mu - 1} \, dx
\]

\[
\leq C \left( X_n^2 - (X_m - X_m^{\frac{1}{2}})^2 \right)^{-\frac{1}{2}} \left( X_n^2 - (X_m - X_m^{\frac{1}{2}})^2 \right)^{-\frac{1}{2}} X_m \leq C X_m^{-\frac{1}{2} + \mu},
\]

together with
\[
\int_0^{X_m - X_m^{\frac{1}{2}}} J_1 \, dx \leq C \int_0^{X_m - X_m^{\frac{1}{2}}} x(X_n^2 - x^2)^{-\frac{1}{2}}(X_n^2 - x^2)^{-\frac{1}{2}} \, dx \leq C X_m^{-\frac{1}{2}},
\]

and
\[
\int_0^{X_m - X_m^{\frac{1}{2}}} J_3 \, dx \leq C X_n^{-1} \int_0^{X_m - X_m^{\frac{1}{2}}} (X_m - x)^{-3} \, dx \leq C X_m^{-\frac{1}{2}},
\]

we obtain
\[
\left| \int_0^{X_m - X_m^{\frac{1}{2}}} \mathcal{F}(x) \, dx \right| \leq C X_m^{-\frac{1}{2} + \mu} X_n^{-1}. \]

Now we turn to the remained three terms. Since \( m_0 < m \leq n \),
\[
\left| \int_0^{X_m - X_m^{\frac{1}{2}}} \langle x \rangle^\mu e^{i k x} \psi_2^{(m)}(x) \overline{\psi_1^{(n)}(x)} \, dx \right| \leq C \int_0^{X_m - X_m^{\frac{1}{2}}} X_m^{-2 + \mu} (X_n^2 - x^2)^{-\frac{1}{2}}(X_n^2 - x^2)^{-\frac{1}{2}} \, dx
\]
\[
\leq C X_m^{-\frac{1}{2} + \mu} X_n^{-\frac{1}{2}} \leq C n^{-\frac{1}{2} + \frac{3}{4}}.
\]

Similarly, when \( m_0 < m \leq n \), we have
\[
\left| \int_0^{X_m - X_m^{\frac{1}{2}}} \langle x \rangle^\mu e^{i k x} \psi_2^{(m)}(x) \overline{\psi_2^{(n)}(x)} \, dx \right| \leq C n^{-1 + \frac{3}{4}}. \]

Thus,
\[
\left| \int_0^{X_m - X_m^{\frac{1}{2}}} \langle x \rangle^\mu e^{i k x} h_m(x) \overline{h_n(x)} \, dx \right| \leq \frac{C}{n^{\frac{1}{2} - \frac{3}{4}}} \leq \frac{C}{m^{\frac{1}{2} - \frac{3}{4} n^{\frac{1}{2} - \frac{1}{4}}}}, \quad m_0 < m \leq n.
\]

Case 2: \( k > \frac{\lambda_0}{4} > 0 \).

Since \( m \leq n \), we have \( m \leq 2k^2 + 1 \) and \( n \leq 8k^2 + 1 \), it follows
\[
\left| \int_0^{X_m - X_m^{\frac{1}{2}}} \langle x \rangle^\mu e^{i k x} h_m(x) \overline{h_n(x)} \, dx \right| \leq C X_m^\mu \leq C X_m^\mu \frac{m^{\frac{1}{2}} n^{\frac{1}{2}}}{m^{\frac{1}{2} - \frac{3}{4} n^{\frac{1}{2} - \frac{1}{4}}}} \leq \frac{C k^{\frac{1}{2}}}{m^{\frac{1}{2} - \frac{3}{4} n^{\frac{1}{2} - \frac{1}{4}}}}.
\]
Combining with these two cases we finish the proof.

**Lemma 4.15.** If $X_n \geq 2X_m$, 
\[
\left| \int_{X_m}^{X_n} \langle x \rangle^\mu e^{ikx} h_m(x) \overline{h_n(x)} dx \right| \leq \frac{C}{m^{\frac{\mu}{2} + \frac{n}{2} - \frac{\mu}{4}}},
\]
where $m_0 < m \leq n$.

**Proof.** Firstly,
\[
\left| \int_{X_m}^{X_n} \mathcal{F}(x) dx \right| \leq C \int_{X_m}^{X_n} \langle x \rangle^\mu (X_m^2 - x^2)^{-\frac{\mu}{2}} (X_n^2 - x^2)^{-\frac{\mu}{4}} dx 
\]
\[
\leq CX_m^{-\frac{\mu}{4} + \mu} (X_m^2 - X_n^2)^{-\frac{\mu}{4}} \int_{X_m - X_n}^{X_m} (X_m - x)^{-\frac{\mu}{4}} dx 
\]
\[
\leq CX_m^{-\frac{\mu}{4} + \mu} (X_n^2 - \frac{X_m^2}{4})^{-\frac{\mu}{4}} X_m^{-\frac{\mu}{4} + \frac{\mu}{4}} \leq CX_m^{-\frac{\mu}{4} + \frac{\mu}{4}} X_n^{-\frac{\mu}{4} + \frac{\mu}{4}}.
\]
Similarly,
\[
\left| \int_{X_m}^{X_n} \langle x \rangle^\mu e^{ikx} \phi_j^{(m)}(x) \overline{\phi_j^{(n)}(x)} dx \right| \leq CX_m^{-\frac{\mu}{4} + \frac{\mu}{4}} X_n^{-\frac{\mu}{4} + \frac{\mu}{4}}, \quad j_1, j_2 \in \{1, 2\}.
\]
Thus, we finish the proof.

**Lemma 4.16.** If $X_n \geq 2X_m$,
\[
\left| \int_{X_m}^{X_n} \langle x \rangle^\mu e^{ikx} h_m(x) \overline{h_n(x)} dx \right| \leq \frac{C}{m^{\frac{\mu}{2} + \frac{n}{2} - \frac{\mu}{4}}},
\]
where $m_0 < m \leq n$.

**Proof.** When $X_n > 2X_{m_0}$ large enough, $X_m + X_m^{-\frac{1}{4}} \leq \frac{X_n}{2} + 1 \leq \frac{3}{2} X_n$. Thus,
\[
\left| \int_{X_m}^{X_m + X_m^{-\frac{1}{4}}} \mathcal{F}(x) dx \right| \leq CX_m^{-\frac{\mu}{4} + \mu} \int_{X_m - X_m^{-\frac{1}{4}}}^{X_m} (x^2 - X_m^2)^{-\frac{\mu}{4}} (X_n^2 - x^2)^{-\frac{\mu}{4}} dx 
\]
\[
\leq CX_m^{-\frac{\mu}{4} + \mu} (X_n^2 - (X_m + X_m^{-\frac{1}{4}})^2)^{-\frac{\mu}{4}} \int_{X_m}^{X_m + X_m^{-\frac{1}{4}}} (x - X_m)^{-\frac{\mu}{4}} dx 
\]
\[
\leq CX_m^{-\frac{\mu}{4} + \frac{\mu}{4}} X_n^{-\frac{\mu}{4} + \frac{\mu}{4}}.
\]
By $X_n \geq 2X_m$, it follows $X_n - X_n^{-\frac{1}{4}} \geq \frac{3}{2} X_m$. By Lemma 5.5,
\[
\left| \int_{X_m + X_m^{-\frac{1}{4}}}^{2X_m} \mathcal{F}(x) dx \right| \leq CX_m^{-\frac{\mu}{4} + \mu} \int_{X_m + X_m^{-\frac{1}{4}}}^{2X_m} (x^2 - X_m^2)^{-\frac{\mu}{4}} (X_n^2 - x^2)^{-\frac{\mu}{4}} e^{ikm} dx 
\]
\[
\leq CX_m^{-\frac{\mu}{4} + \mu} (X_n^2 - (X_n - X_n^{-\frac{1}{4}})^2)^{-\frac{\mu}{4}} \int_{X_m + X_m^{-\frac{1}{4}}}^{2X_m} (x - X_m)^{-\frac{\mu}{4}} e^{-(x - X_m)} dx 
\]
\[
\leq CX_m^{-\frac{\mu}{4} + \mu} X_n^{-\frac{\mu}{4}} \int_{0}^{\infty} t^{-\frac{\mu}{4}} e^{-t} dt \leq CX_m^{-\frac{\mu}{4} + \frac{\mu}{4}} X_n^{-\frac{\mu}{4} + \frac{\mu}{4}}.
\]
If \( x \geq \frac{3}{2}X_m \), then \( x - X_m \geq \frac{1}{2}x \), and thus
\[
\left| \int_{\frac{x}{2}X_m}^{X_n} F(x) \, dx \right| \leq C \int_{\frac{x}{2}X_m}^{X_n} \frac{\langle x \rangle^\mu (x^2 - X_m^2)}{x^2} \, dx \leq C X_m^{-\frac{1}{2}} \left( X_n^2 - (X_n - X_m) \right)^{\frac{1}{2}} \int_{\frac{x}{2}X_m}^{X_n} (x - X_m)^{-\frac{1}{2}+\mu} e^{-t(x-X_m)} \, dt.
\]
Finally,
\[
\left| \int_{X_n-X_n^{\frac{1}{2}}}^{X_n} F(x) \, dx \right| \leq C X_m^{\mu} \int_{X_n-X_n^{\frac{1}{2}}}^{X_n} (x^2 - X_m^2)^{-\frac{1}{2}} (X_n - X_m)^{-\frac{1}{2}} \, dx \leq C X_m^{\frac{1}{2}} (X_n - X_m)^{-\frac{1}{2}} \int_{X_n-X_n^{\frac{1}{2}}}^{X_n} (X_n - x)^{-\frac{1}{2}} \, dx \leq C X_m^{-\frac{1}{2} + \mu} \leq C X_m^{-\frac{1}{2}} X_n^{-\frac{1}{2}}.
\]
Combining with all the above estimates we have
\[
\left| \int_{X_m}^{X_n} \langle x \rangle^\mu e^{ikx} h_m(x) \, dx \right| \leq \frac{C}{m^{\frac{\mu}{2} - n^{\frac{1}{2} - \frac{1}{2}}}},
\]
which leads to
\[
\left| \int_{X_m}^{X_n} \langle x \rangle^\mu e^{ikx} h_m(x) \, dx \right| \leq \frac{C}{m^{\frac{\mu}{2} - n^{\frac{1}{2} - \frac{1}{2}}}}.
\]

Lemma 4.13 follows from Lemma 4.14, 4.15 and Lemma 4.16.

4.5. the integral on \([0, X_n]\) for the case \(X_m \leq X_n \leq 2X_m\). For the case \(X_m \leq X_n \leq 2X_m\), we split the integral into
\[
\int_{0}^{X_n} \langle x \rangle^\mu e^{ikx} h_m(x) \, dx = \int_{0}^{X_m^{\frac{1}{2}}} \langle x \rangle^\mu e^{ikx} h_m(x) \, dx + \int_{X_m^{\frac{1}{2}}}^{X_m-X_m^{\frac{1}{2}}} \langle x \rangle^\mu e^{ikx} h_m(x) \, dx + \int_{X_m-X_m^{\frac{1}{2}}}^{X_n} \langle x \rangle^\mu e^{ikx} h_m(x) \, dx.
\]

Lemma 4.17. For \(X_m \leq X_n \leq 2X_m\),
\[
\left| \int_{0}^{X_m^{\frac{1}{2}}} \langle x \rangle^\mu e^{ikx} h_m(x) \, dx \right| \leq \frac{C}{m^{\frac{\mu}{2} - n^{\frac{1}{2} - \frac{1}{2}}} \mu} \text{ with } C > 0,
\]
where \(m_0 < m \leq n\).

The proof is simple.

Next we estimate the integral on \([X_m^{\frac{1}{2}}, X_m-X_m^{\frac{1}{2}}]\). We first discuss two cases for \(k > 0\): 1. \(k > X_m^{\frac{1}{2}}\). 2. \(0 < k \leq X_m^{\frac{1}{2}}\). The first case is simple, but the second one is much complex. We will discuss five subcases for the second one. For \(k < 0\), the proof is easy to handle.

Lemma 4.18. If \(k > X_m^{\frac{1}{2}}\) and \(X_m \leq X_n \leq 2X_m\),
\[
\left| \int_{X_m^{\frac{1}{2}}}^{X_m-X_m^{\frac{1}{2}}} \langle x \rangle^\mu e^{ikx} h_m(x) \, dx \right| \leq \frac{Ck}{m^{\frac{\mu}{2} - n^{\frac{1}{2} - \frac{1}{2}}}}.
\]
Proof. If \( k > X_m^\frac{1}{2} \), it follows \( m < k^0 \) and \( n < 4k^0 \). By Hölder inequality,

\[
\left| \int_{X_m^\frac{1}{2}}^{X_m^\frac{1}{4}} (x)^\mu e^{ikx} h_m(x) \overline{h_n(x)} dx \right| \leq CX_m^{\frac{1}{2}} \frac{m^{\frac{1}{2}} n^{\frac{1}{2}}}{m^{\frac{1}{2}} n^{\frac{1}{2}}} \leq \frac{Ck}{m^{\frac{1}{2}} n^{\frac{1}{2}}}. 
\]

\( \square \)

**Lemma 4.19.** For \( 0 < k \leq X_m^\frac{1}{2}, X_m \leq X_n \leq 2X_m \), if \( 0 \leq X_m^2 - X_m^\frac{3}{2} \leq kX_m^\frac{2}{3} \), then

\[
\left| \int_{X_m^\frac{1}{2}}^{X_m^\frac{1}{4}} (x)^\mu e^{ikx} h_m(x) \overline{h_n(x)} dx \right| \leq \frac{C(k^{-1} \vee 1)}{m^{\frac{1}{2}} n^{\frac{1}{2}}},
\]

where \( m_0 < m \leq n \).

**Proof.** We first estimate

\[
\left| \int_{X_m^\frac{1}{2}}^{X_m^\frac{1}{4}} \mathcal{F}(x) dx \right| = \left| C \int_{X_m^\frac{1}{2}}^{X_m^\frac{1}{4}} (x)^\mu e^{i(\zeta_m - \zeta_n + kx)} \Psi(x) dx \right|.
\]

Since

\[
g(X_m - X_m^{\frac{1}{2}}) = \frac{X_n^2 - X_m^2}{\sqrt{X_n^2 - (X_m - X_m^{\frac{1}{2}})^2 + X_m^2 - (X_m - X_m^{\frac{1}{2}})^2}} - k \leq \frac{kX_m^{\frac{2}{3}}}{\sqrt{X_n^2 - X_m^2 + X_m - X_m^{\frac{1}{2}}} - k = \frac{k}{2}},
\]

together with \( g'(x) \geq 0 \), one obtains \( |g(x)| \geq \frac{k}{2} \) for \( x \in [X_m^{\frac{1}{2}}, X_m - X_m^{\frac{1}{2}}] \). Then by Lemma 5.6,

\[
\left| \int_{X_m^{\frac{1}{2}}}^{X_m} \frac{(x)^\mu e^{i(\zeta_m - \zeta_n + kx)}}{(X_m^2 - x^2)^{-\frac{1}{4}} (X_m^2 - x^2)^{-\frac{1}{4}} : f_m(x) \overline{f_n(x)} dx} \right| \leq Ck^{-1} \left[ \int_{X_m^{\frac{1}{2}}}^{X_m} |(x)^\mu \Psi'(x)| dx + \int_{X_m^{\frac{1}{2}}}^{X_m} |(x)^\mu \Psi(x)| dx \int_{X_m^{\frac{1}{2}}}^{X_m} |(x)^\mu \Psi(x)| dx \right].
\]

From

\[
|\Psi(X_m - X_m^{\frac{1}{2}})| \leq C \left( X_n^2 - (X_m - X_m^{\frac{1}{2}})^2 \right)^{-\frac{1}{4}} \left( X_n^2 - (X_m - X_m^{\frac{1}{2}})^2 \right)^{-\frac{1}{2}} \leq CX_m^{-\frac{3}{4}},
\]

and \( \int_{X_m^{\frac{1}{2}}}^{X_m} (x)^{\mu-1} |\Psi(x)| dx \leq CX_m^{-\frac{3}{4}} \). By Corollary 4.12, \( |\Psi'(x)| \leq C(J_1 + J_3) \). From

\[
\int_{X_m^{\frac{1}{2}}}^{X_m} J_1 dx \leq C \int_{X_m^{\frac{1}{2}}}^{X_m} x(X_m^2 - x^2)^{-\frac{1}{4}} (X_m^2 - x^2)^{-\frac{1}{4}} dx \leq CX_m^{-\frac{3}{4}},
\]

together with \( \int_{X_m^{\frac{1}{2}}}^{X_m} J_3 dx \leq C \int_{X_m^{\frac{1}{2}}}^{X_m} \leq CX_m^{-\frac{3}{4}} \), it follows by Corollary 4.12

\[
\left| \int_{X_m^{\frac{1}{2}}}^{X_m} \mathcal{F}(x) dx \right| \leq C k^{-1} X_m^{-\frac{3}{4}} + \mu.
\]
The estimates for the rest three terms are much simpler. In fact, when \( m > m_0 \),
\[
\left| \int_{X_m^\alpha - \frac{1}{32}X_m^\frac{1}{2}}^{X_m^\alpha} \langle x \rangle^\mu e^{ikx \psi_2^{(m)}(x)} \psi_1^{(n)}(x) dx \right| \leq CX_m^{-2+\mu} \int_{X_m^\alpha - \frac{1}{32}X_m^\frac{1}{2}}^{X_m^\alpha} (X_m^\alpha - x)^{-\frac{\mu}{2}} dx
\]
\[
\leq CX_m^{\frac{\mu}{4} + \mu} \int_0^{X_m^\alpha - \frac{1}{32}X_m^\frac{1}{2}} (X_m^\alpha - x)^{-\frac{\mu}{4}} dx \leq CX_m^{\frac{\mu}{4} + \mu}.
\]

The other two terms have same estimates. Therefore,
\[
\left| \int_{X_m^\alpha - \frac{1}{32}X_m^\frac{1}{2}}^{X_m^\alpha} \langle x \rangle^\mu e^{ikx} h_n(x) dx \right| \leq C\left(\frac{k^{-1} \lor 1}{m \frac{1}{12} \frac{1}{4} n \frac{1}{12} \frac{1}{4}}\right).
\]

\[\square\]

**Lemma 4.20.** For \( 0 < k \leq X_m^\frac{1}{2} \), \( X_m \leq X_n \leq 2X_m \), if \( kX_m^\frac{5}{4} \leq X_n^2 - X_m^2 \leq kX_m^\frac{5}{4} \), then
\[
\left| \int_{X_m^\alpha - \frac{1}{32}X_m^\alpha}^{X_m^\alpha} \langle x \rangle^\mu e^{ikx} h_n(x) dx \right| \leq \frac{C(k^{-1} \lor 1)}{m \frac{1}{12} \frac{1}{4} n \frac{1}{12} \frac{1}{4}},
\]
where \( m_0 < m \leq n \).

**Proof.** When \( kX_m^\frac{5}{4} \leq X_n^2 - X_m^2 \leq kX_m^\frac{5}{4} \),
\[
\left| \int_{X_m^\alpha - \frac{1}{32}X_m^\alpha}^{X_m^\alpha} \langle x \rangle^\mu e^{ikx} h_n(x) dx \right| \leq C \int_{X_m^\alpha - \frac{1}{32}X_m^\alpha}^{X_m^\alpha} \langle x \rangle^\mu (X_m^\alpha - x^2)^{-\frac{\mu}{2}} dx \leq Cn^{-\frac{\mu}{8} + \frac{5}{4} m - \frac{1}{8} + \frac{5}{4}}.
\]

In the following we estimate the integral on \([X_m^\alpha, X_m - \frac{1}{32}X_m^\alpha]\). Since \( m > m_0 \) large enough,
\[
g(X_m - \frac{1}{32}X_m^\alpha) \geq kX_m^\frac{5}{4} \left\{ \sqrt{kX_m + \frac{1}{10}X_m^\frac{1}{2}} + \sqrt{1 + \frac{1}{10}X_m^\frac{1}{2}} \right\} - k \geq \frac{kX_m^\frac{5}{4}}{2} - k = \frac{k}{3}.
\]

On the other hand,
\[
g(X_m - X_m^\alpha) \leq \frac{kX_m^\frac{5}{4}}{2X_m^\alpha - (X_m - X_m^\alpha)^2} - k \leq \frac{kX_m^\frac{5}{4}}{2X_m^\alpha} - k = -\frac{k}{2}.
\]

We denote \( g(a) = -kX_m^{-\frac{5}{4}} \), then by the monotonicity of \( g(x) \), \( X_m - X_m^\alpha < a < X_m - \frac{1}{32}X_m^\alpha \). In the following we estimate the integral on \([a, X_m - \frac{1}{32}X_m^\alpha]\) firstly. A straightforward computation shows us \( g'(x) > 0 \), therefore,
\[
|g'(x)| \geq \left| \frac{a}{\sqrt{X_m^2 - a^2}} - \frac{a}{\sqrt{X_m^\alpha - a^2}} \right| = \frac{a(g(a) + k)}{\sqrt{X_m^2 - a^2} \sqrt{X_m^\alpha - a^2}} \geq CkX_m^{-\frac{5}{4}},
\]

By Lemma 5.6,
\[
\left| \int_{X_m - \frac{1}{32}X_m^\alpha}^{X_m^\alpha} \langle x \rangle^\mu e^{ikx \psi_2^{(m)}(x)} \psi_1^{(n)}(x) dx \right| \leq Ck\frac{1}{4} \left\{ (\langle x \rangle^\mu \psi)(X_m - \frac{1}{32}X_m^\alpha) \right\} + \int_{X_m - \frac{1}{32}X_m^\alpha}^{X_m^\alpha} |(\langle x \rangle^\mu \psi)'(x)| dx
\]
\[
\leq Ck\frac{1}{4} \left\{ (\langle x \rangle^\mu \psi)(X_m - \frac{1}{32}X_m^\alpha) \right\} + X_m^\mu \int_{X_m - \frac{1}{32}X_m^\alpha}^{X_m^\alpha} |\psi'(x)| dx
\]
We compute the right terms in (4.6) one by one. Clearly,

\[
\left| \Psi(X_m - \frac{1}{32} X_m^{\frac{1}{2}}) \right| \leq C \left( X_m^2 - (X_m - \frac{1}{32} X_m^{\frac{1}{2}})^2 \right)^{-\delta} \left( X_m^2 - (X_m - \frac{1}{32} X_m^{\frac{1}{2}})^2 \right)^{-\delta} \leq CX_m^{-\frac{3}{2}}.
\]

From \(|\Psi'(x)| \leq C(J_1 + J_3)\) and

\[
\int_a^{X_m - \frac{1}{32} X_m^{\frac{1}{2}}} J_1 dx \leq C \int_a^{X_m - \frac{1}{32} X_m^{\frac{1}{2}}} x(X_m^2 - x^2)^{-\frac{1}{4}}(X_m^2 - x^2)^{-\frac{1}{4}} dx \leq CX_m^{-\frac{3}{2}} \int_a^{X_m - \frac{1}{32} X_m^{\frac{1}{2}}} (X_m - x)^{-\frac{1}{4}} dx \leq CX_m^{-\frac{3}{2}},
\]

and \(\int_a^{X_m - \frac{1}{32} X_m^{\frac{1}{2}}} J_3 dx \leq CX_m^{-\frac{3}{2}}\), and \(\int_a^{X_m - \frac{1}{32} X_m^{\frac{1}{2}}} |\psi|^u_1 dx \leq CX_m^{-\frac{3}{2} + \mu}\), we obtain

\[
\left| \int_a^{X_m - \frac{1}{32} X_m^{\frac{1}{2}}} (x)^u e^{ikx} \psi_1^{(m)}(x) \psi_1^{(m)}(x) dx \right| \leq C k^{-\frac{1}{2}} X_m^{-\frac{3}{4} + \mu} \leq \frac{C k^{-\frac{1}{2}}}{m^{\frac{1}{12}} n^{\frac{1}{12}} - \frac{\mu}{4}}.
\]

Next we estimate the integral on \([X_m^{\frac{1}{2}}, a]\). From \(|g(x)| \geq kX_m^{-\frac{1}{4}}\) and Lemma 5.6, one obtains

\[
\left| \int_{X_m^{\frac{1}{2}}}^a (x)^u e^{ikx} \psi_1^{(m)} \right| \leq C k^{-1} X_m^{-\frac{3}{4}} \left[ \left| (x)^u \psi \right| + \int_{X_m^{\frac{1}{2}}}^a \left| (x)^u \psi' \right| dx \right] \leq C k^{-1} X_m^{-\frac{3}{4}} \left[ \left| a \right|^u \psi(a) + \left| a \right|^u \psi'(x) dx + \psi(X_m^{\frac{1}{2}}) - \psi(x) \right].
\]

Clearly,

\[
\left| \Psi(a) \right| \leq C \left( X_m^2 - (X_m - \frac{1}{32} X_m^{\frac{1}{2}})^2 \right)^{-\delta} \left( X_m^2 - (X_m - \frac{1}{32} X_m^{\frac{1}{2}})^2 \right)^{-\delta} \leq CX_m^{-\frac{3}{2}}.
\]

and \(\int_{X_m^{\frac{1}{2}}}^a (x)^u dx \leq CX_m^{-\frac{3}{4} + \mu}\). Thus,

\[
\left| \int_{X_m^{\frac{1}{2}}}^a \mathcal{F}(x) dx \right| \leq C k^{-1} X_m^{-\frac{3}{4} + \mu} \leq \frac{C k^{-1}}{m^{\frac{1}{12}} n^{\frac{1}{12}} - \frac{\mu}{4}}.
\]

Combining with all the estimates in this part, one obtains

\[
\left| \int_{X_m^{\frac{1}{2}}}^{X_m - X_m^{\frac{1}{2}}} F(x) dx \right| \leq \frac{C(k^{-1} + 1)}{m^{\frac{1}{12}} n^{\frac{1}{12}} - \frac{\mu}{4}}.
\]

The estimates for the remained three terms are much better. Therefore,

\[
\left| \int_{X_m^{\frac{1}{2}}}^{X_m - X_m^{\frac{1}{2}}} (x)^u e^{ikx} \psi_1^{(m)} h_1(x) h_m(x) dx \right| \leq \frac{C(k^{-1} + 1)}{m^{\frac{1}{12}} n^{\frac{1}{12}} - \frac{\mu}{4}}.
\]

\[\square\]

We delay the proofs of Lemma 4.21, 4.22, 4.23 into section 5.
Lemma 4.21. For $0 < k \leq X_m^\frac{1}{4}$, $X_m \leq X_n \leq 2X_m$, if $kX_m^\frac{3}{4} \leq X_n^2 - X_m^2 \leq kX_m$, then

$$\left| \int_{X_m^\frac{1}{4}}^{X_m} (x) e^{ikx} h_m(x) \overline{h_n(x)} dx \right| \leq \frac{C(k^{-1} \lor 1)}{m^{\frac{1}{8} - \frac{2}{n} - \frac{1}{4}}},$$

where $m_0 < m \leq n$.

Lemma 4.22. For $0 < k \leq X_m^\frac{1}{4}$, $X_m \leq X_n \leq 2X_m$, if $kX_m \leq X_n^2 - X_m^2 \leq 4kX_m$, then

$$\left| \int_{X_m^\frac{1}{4}}^{X_m} (x) e^{ikx} h_m(x) \overline{h_n(x)} dx \right| \leq \frac{C(k^{-1} \lor 1)}{m^{\frac{1}{8} - \frac{2}{n} - \frac{1}{4}}},$$

where $m_0 < m \leq n$.

Lemma 4.23. For $0 < k \leq X_m^\frac{1}{4}$, $X_m \leq X_n \leq 2X_m$, if $X_n^2 - X_m^2 \geq 4kX_m$, then

$$\left| \int_{X_m^\frac{1}{4}}^{X_m} (x) e^{ikx} h_m(x) \overline{h_n(x)} dx \right| \leq \frac{C(k^{-1} \lor 1)}{m^{\frac{1}{8} - \frac{2}{n} - \frac{1}{4}}},$$

where $m_0 < m \leq n$.

From Lemma 4.18 to Lemma 4.23, we have

Lemma 4.24. For $\forall k > 0, X_m \leq X_n \leq 2X_m$, then

$$\left| \int_{X_m^\frac{1}{4}}^{X_m} (x) e^{ikx} h_m(x) \overline{h_n(x)} dx \right| \leq \frac{C(k \lor k^{-1})}{m^{\frac{1}{8} - \frac{2}{n} - \frac{1}{4}}},$$

where $m_0 < m \leq n$.

Lemma 4.25. For $\forall k < 0, X_m \leq X_n \leq 2X_m$, then

$$\left| \int_{X_m^\frac{1}{4}}^{X_m} (x) e^{ikx} h_m(x) \overline{h_n(x)} dx \right| \leq \frac{C(|k|^{-1} \lor 1)}{m^{\frac{1}{8} - \frac{2}{n} - \frac{1}{4}}},$$

where $m_0 < m \leq n$.

For the proof see in section 5.

Combining with Lemma 4.24 and Lemma 4.25, we have

Lemma 4.26. For $\forall k \neq 0, X_m \leq X_n \leq 2X_m$, then

$$\left| \int_{X_m^\frac{1}{4}}^{X_m} (x) e^{ikx} h_m(x) \overline{h_n(x)} dx \right| \leq \frac{C(|k| \lor |k|^{-1})}{m^{\frac{1}{8} - \frac{2}{n} - \frac{1}{4}}},$$

where $m_0 < m \leq n$.

Now we turn to the last part of integral. In fact for this part we have

Lemma 4.27. $\forall k \neq 0, X_m \leq X_n \leq 2X_m$, 

$$\left| \int_{X_n}^{X_m} (x) e^{ikx} h_m(x) \overline{h_n(x)} dx \right| \leq \frac{C}{m^{\frac{1}{8} - \frac{2}{n} - \frac{1}{4}}},$$

where $m_0 < m \leq n$. 

Proof. Firstly,
\[
\left| \int_{X_n - X_n^{-\frac{1}{2}}}^{X_n} F(x) \, dx \right| \leq C \int_{X_n - X_n^{-\frac{1}{2}}}^{X_n} \langle x \rangle^\mu (X_n^2 - x^2)^{-\frac{1}{2}} (x^2 - X_n^2)^{-\frac{1}{2}} \, dx
\]
\[
\leq C X_n^\mu \int_{X_n - X_n^{-\frac{1}{2}}}^{X_n} (x^2 - X_n^2)^{-\frac{1}{2}} \, dx
\]
\[
\leq C X_n^{\frac{1}{2} + \mu} \int_{X_n - X_n^{-\frac{1}{2}}}^{X_n} (x - X_n)^{-\frac{1}{2}} \, dx
\]
\[
\leq C X_n^{\frac{1}{2} + \mu} X_n^{\frac{1}{2}} \leq \frac{C}{m^{\frac{1}{2}} n^{\frac{1}{2} + \frac{1}{2}}}.
\]
It follows \[
\left| \int_{X_n - X_n^{-\frac{1}{2}}}^{X_n} \langle x \rangle^\mu e^{ikx} h_m(x) h_n(x) \, dx \right| \leq \frac{C}{m^{\frac{1}{2}} n^{\frac{1}{2} + \frac{1}{2}}}.
\] For the remained integral on \([X_m, X_n]\) we estimate the integral in two different cases.

Case 1. \(X_n - X_n^{-\frac{1}{2}} \geq X_m + X_m^{-\frac{1}{2}}\). We split the integral into three parts. The first part satisfies
\[
\left| \int_{X_m + X_m^{-\frac{1}{2}}}^{X_m + X_m^{-\frac{1}{2}}} F(x) \, dx \right| \leq \int_{X_m + X_m^{-\frac{1}{2}}}^{X_m + X_m^{-\frac{1}{2}}} \langle x \rangle^\mu (x^2 - X_m^2)^{-\frac{1}{2}} (x^2 - X_m^2)^{-\frac{1}{2}} \, dx
\]
\[
\leq C X_m^\mu X_m^{-\frac{1}{2}} X_m^{-\frac{1}{2}} (X_m - X_m - X_m^{-\frac{1}{2}})^{-\frac{1}{2}} \int_{X_m + X_m^{-\frac{1}{2}}}^{X_m + X_m^{-\frac{1}{2}}} (x - X_m)^{-\frac{1}{2}} \, dx
\]
\[
\leq C X_m^\mu X_m^{-\frac{1}{2}} X_m^{-\frac{1}{2}} X_m^{-\frac{1}{2}} \leq C n^{\frac{1}{2} + \frac{1}{2}} n^{-\frac{1}{2}}.
\]
By Lemma 5.5, when \(x \geq X_m + X_m^{-\frac{1}{2}}, i \xi_m \leq -(x - X_m)\), then the second part satisfies
\[
\left| \int_{X_m + X_m^{-\frac{1}{2}}}^{X_m + X_m^{-\frac{1}{2}}} F(x) \, dx \right| \leq C \int_{X_m + X_m^{-\frac{1}{2}}}^{X_m + X_m^{-\frac{1}{2}}} \langle x \rangle^\mu (x^2 - X_m^2)^{-\frac{1}{2}} (x^2 - X_m^2)^{-\frac{1}{2}} e^{i \xi_m} \, dx
\]
\[
\leq C (2X_m)^\mu X_m^{-\frac{1}{2}} (X_m^2 - (X_m - X_m^{-\frac{1}{2}})^2)^{-\frac{1}{2}} \int_{X_m + X_m^{-\frac{1}{2}}}^{X_m + X_m^{-\frac{1}{2}}} (x - X_m)^{-\frac{1}{2}} e^{i \xi_m} \, dx
\]
\[
\leq C (2X_m)^\mu X_m^{-\frac{1}{2}} X_m^{-\frac{1}{2}} \int_0^{\infty} t^{-\frac{1}{2}} e^{-t} \, dt \leq C n^{\frac{1}{2} + \frac{1}{2}} m^{-\frac{1}{2}}.
\]
The last part satisfies
\[
\left| \int_{X_n - X_n^{-\frac{1}{2}}}^{X_n} F(x) \, dx \right| \leq \int_{X_n - X_n^{-\frac{1}{2}}}^{X_n} \langle x \rangle^\mu (x^2 - X_n^2)^{-\frac{1}{2}} (x^2 - X_n^2)^{-\frac{1}{2}} \, dx
\]
\[
\leq C (2X_n)^\mu (X_n - X_n^{-\frac{1}{2}})^{\frac{1}{2}} X_n^{-\frac{1}{2}} \int_{X_n - X_n^{-\frac{1}{2}}}^{X_n} (x - X_n)^{-\frac{1}{2}} \, dx
\]
\[
\leq C n^{\frac{1}{2} + \frac{1}{2}} n^{-\frac{1}{2}}.
\]
Thus, in Case 1, \[
\left| \int_{X_n}^{X_n} \langle x \rangle^\mu e^{ikx} h_m(x) h_n(x) \, dx \right| \leq \frac{C}{m^{\frac{1}{2}} n^{\frac{1}{2} + \frac{1}{2}}}.
\]
Case 2. \(X_n - X_n^{-\frac{1}{2}} < X_m + X_m^{-\frac{1}{2}}\). In fact,
\[
\left| \int_{X_m}^{X_n} F(x) \, dx \right| \leq \int_{X_m}^{X_n} \langle x \rangle^\mu (x^2 - X_m^2)^{-\frac{1}{2}} (x^2 - X_m^2)^{-\frac{1}{2}} \, dx
\]
\[
\leq C (2X_m^\mu) X_n^{-\frac{1}{2}} X_n^{-\frac{1}{2}} \int_{X_m}^{X_n} (x - X_m)^{-\frac{1}{2}} (x - X_m)^{-\frac{1}{2}} \, dx
\]
\begin{equation}
\leq C(2X_m^\nu X_m^{-\frac{1}{2}} X_n^{-\frac{1}{2}} \int_{X_m}^{X_n} (x - X_m)^{-\frac{1}{2}} dx
\leq C X_m^\nu X_m^{-\frac{1}{2}} X_n^{-\frac{1}{2}} (X_n - X_m)^{\frac{1}{2}} \leq C n^{-\frac{1}{2}} m^{-\frac{1}{2}} n^{-\frac{1}{2}}
\end{equation}
where we use the symmetric property of $(x - X_m)^{-\frac{1}{2}} (X_n - x)^{-\frac{1}{2}}$ on the interval $[X_m, X_n]$. Thus,
\begin{equation}
\left| \int_{X_m}^{X_n} (x)^\mu e^{ikx} h_m(x) \overline{h_n(x)} dx \right| \leq \frac{C}{m^{\frac{1}{2}} - n^{\frac{1}{2}} n^{-\frac{1}{2}}}.\end{equation}
Combining with all the above estimates we complete the proof. \hfill \Box

From all lemmas in this subsection we have

**Lemma 4.28.** For $\forall k \neq 0$, \( \int_0^{+\infty} \langle x \rangle^\mu e^{ikx} h_m(x) \overline{h_n(x)} dx \leq \frac{C(|k|^{-1} \vee |k|)}{m^{\frac{1}{2}} - n^{\frac{1}{2}} n^{-\frac{1}{2}}} \). From Lemma 4.28 and the symmetry of $h_m(x)$, we complete the proof of Lemma 1.7.

5. **Appendix**

5.1. **a new reducibility theorem in** \( L^2(\mathbb{R}) \). If $\mu = 0$ and the perturbation terms $\varepsilon W(\nu x, \theta)$ are analytic on $\theta$ in the equation (1.1), we can prove the reducibility in $L^2(\mathbb{R})$ instead of $\mathcal{H}^1(\mathbb{R})$. More clearly, consider 1-d quantum harmonic oscillator equation
\begin{equation}
i \partial_t \psi = H_\varepsilon(\omega t) \psi, \quad x \in \mathbb{R},
H_\varepsilon(\omega t) := -\partial_{xx} + x^2 + \varepsilon X(x, \omega t),
\end{equation}
where $W(\varphi, \theta)$ is defined on $\mathbb{T}^d \times \mathbb{T}^n$ and satisfies (1.2) and for any $\varphi \in \mathbb{T}^d$ and all $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\partial_\alpha W(\varphi, \theta)$ is analytic on $\mathbb{T}^n$ and continuous on $\mathbb{T}^d \times \mathbb{T}^n$, where $0 \leq |\alpha| = \alpha_1 + \cdots + \alpha_d \leq d(1 \vee \tau) + d + 2$ and $\tau > d - 1$.

**Theorem 5.1.** Assume that $W(\varphi, \theta)$ satisfies all the above assumptions. There exists $\varepsilon_* > 0$, such that for all $0 \leq \varepsilon < \varepsilon_*$ there exists a closed set $\Omega_\varepsilon \times \Omega_1(\varepsilon) \subset [A, B]^d \times [1, 2]^n$ and for any $(\nu, \omega) \in \Omega_\varepsilon \times \Omega_1(\varepsilon)$ the linear Schrödinger equation (5.1) reduces to a linear autonomous equation in $L^2(\mathbb{R})$.

**Remark 5.2.** The set $\Omega_\gamma$ is defined as Theorem 1.1, while $\Omega_1(\varepsilon)$ satisfies $\text{Meas}(\Omega_1(\varepsilon)) \rightarrow 1$ when $\varepsilon \rightarrow 0$.

Similarly, we consider 1-d quantum harmonic oscillator equation
\begin{equation}
i \partial_t \psi = \mathcal{H}_\varepsilon(\omega t) \psi, \quad x \in \mathbb{R},
\mathcal{H}_\varepsilon(\omega t) := -\partial_{xx} + x^2 + \varepsilon X(x, \omega t),
\end{equation}
where \begin{equation}
X(x, \theta) = \sum_{k \in \Lambda} (a_k(\theta) \sin kx + b_k(\theta) \cos kx)
\end{equation}
with $k \in \Lambda \subset \mathbb{R} \setminus \{0\}$ with $|\Lambda| < \infty$, $a_k(\theta)$ and $b_k(\theta)$ are real analytic on $\mathbb{T}_\mu^n$ and continuous on $\mathbb{T}_\mu^n$.

**Corollary 5.3.** Assume that $X(x, \theta)$ satisfies all the above assumptions. There exists $\varepsilon_* > 0$, such that for all $0 \leq \varepsilon < \varepsilon_*$ there exists a closed set $\Omega_2(\varepsilon) \subset [1, 2]^n$ and for any $\omega \in \Omega_2(\varepsilon)$ the linear Schrödinger equation (5.2) reduces to a linear autonomous equation in $L^2(\mathbb{R})$.

The above proofs are based on the KAM theorem in [24] and Lemma 1.7. We omit the details.
5.2. some lemmas. Proof of Lemma 3.4: Recall that \( \Psi_\infty^{-1}(\theta)(\sum_{a \geq 1} \xi_a h_a(x)) = \sum_{a \geq 1} (M^T_\omega(\theta) \xi_a) h_a(x) \).

(a) By the estimate in Theorem 2.1, for \( \theta \in \mathbb{R}^n \),

\[
\| \Psi_\infty^{-1}(\theta) - id \|_{\mathcal{L}(\mathcal{H}', \mathcal{H}')} = \sup_{\|u\|_{\mathcal{H}'} = 1} \| (\Psi_\infty^{-1}(\theta) - id) u \|_{\mathcal{H}'}
\]

\[
= \sup_{\|\xi\|_{\mathcal{H}'} = 1} \| (M^T_\omega - Id) \xi \|_{\mathcal{H}'} \leq C(n, \beta, \iota) e^{\frac{3}{2} (\frac{1}{2} \beta - 1)}.
\]

(b) For \( b = t - [t] \in (0, 1) \), \( \theta_1, \theta_2 \in \mathbb{R}^n \),

\[
\| \Psi_\infty^{-1}(\theta_1) - \Psi_\infty^{-1}(\theta_2) \|_{\mathcal{L}(\mathcal{H}', \mathcal{H}')} = \frac{1}{\|\theta_1 - \theta_2\|_b} \| M^T_\omega(\theta_1) - M^T_\omega(\theta_2) \|_{\mathcal{L}(\mathcal{H}')}.
\]

Thus,

\[
\| \Psi_\infty^{-1} - id \|_{C^1(\mathcal{H}', \mathcal{L}(\mathcal{H}', \mathcal{H}'))} = \| M^T_\omega - Id \|_{C^1(\mathcal{H}', \mathcal{L}(\mathcal{H}', \mathcal{H}'))} \leq C(n, \beta, \iota) e^{\frac{3}{2} (\frac{1}{2} \beta - 1)}.
\]

(c) Denote \( \langle A(\theta), \eta \rangle u := \sum_{a \geq 1} \langle (A(\theta), \eta) \xi_a \rangle h_a(x) \) for \( \eta \in \mathbb{R}^n \) where \( A := (M^T_\omega - Id)^\iota \). Note that \( M^T_\omega - Id \in C^1(\mathcal{H}', \mathcal{L}(\mathcal{H}', \mathcal{H}')) \), then for \( \theta \in \mathbb{R}^n \),

\[
\| A(\theta) \|_{\mathcal{L}(\mathcal{H}', \mathcal{L}(\mathcal{H}', \mathcal{H}'))} = \sup_{\|u\|_{\mathcal{H}'} = 1, \|\eta\| = 1} \| \langle A(\theta), \eta \rangle u \|_{\mathcal{H}'}
\]

\[
= \sup_{\|\xi\|_{\mathcal{H}'} = 1, \|\eta\| = 1} \| \langle A(\theta), \eta \rangle \xi \|_{\mathcal{H}'}
\]

\[
= \| A(\theta) \|_{\mathcal{L}(\mathcal{H}', \mathcal{L}(\mathcal{H}', \mathcal{H}'))} \leq \| M^T_\omega - Id \|_{C^1(\mathcal{H}', \mathcal{L}(\mathcal{H}', \mathcal{H}'))}.
\]

Thus \( A(\theta) \in \mathcal{L}(\mathcal{H}', \mathcal{L}(\mathcal{H}', \mathcal{H}')) \) and for given \( \theta_0 \in \mathbb{R}^n \),

\[
\| \Psi_\infty^{-1}(\theta) - \Psi_\infty^{-1}(\theta_0) - \langle A(\theta_0), \theta - \theta_0 \rangle \|_{\mathcal{L}(\mathcal{H}', \mathcal{H}')} = \sup_{\|u\|_{\mathcal{H}'} = 1} \| (\Psi_\infty^{-1}(\theta) - \Psi_\infty^{-1}(\theta_0) - \langle A(\theta_0), \theta - \theta_0 \rangle) u \|_{\mathcal{H}'}
\]

\[
= \| M^T_\omega(\theta) - M^T_\omega(\theta_0) - \langle A(\theta_0), \theta - \theta_0 \rangle \|_{\mathcal{L}(\mathcal{H}', \mathcal{H}')}.
\]

Since \( M^T_\omega - Id \in C^1(\mathcal{H}', \mathcal{L}(\mathcal{H}', \mathcal{H}')) \), it follows \( \Psi_\infty^{-1} - id \in C^1(\mathcal{H}', \mathcal{L}(\mathcal{H}', \mathcal{H}')) \) and \( A = (\Psi_\infty^{-1} - id)^\iota \). Inductively, we can show that

\[
\| \Psi_\infty^{-1} - id \|_{C^1(\mathcal{H}', \mathcal{L}(\mathcal{H}', \mathcal{H}'))} \leq C(n, \beta, \iota) e^{\frac{3}{2} (\frac{1}{2} \beta - 1)}.
\]

The rest is similar. \( \square \)

Proof of Lemma 4.21. Since \( m > m_0 \) large enough, we have

\[
g(X_m - \frac{1}{32} X_m^\iota) \geq \frac{kX_m^\iota}{\sqrt{kX_m + \frac{k^2}{16}}} - k \geq \frac{kX_m^\iota}{\sqrt{kX_m + \frac{k^2}{16}}} - k = \frac{k}{3}.
\]

and \( g(X_m^\frac{1}{2}) \leq \frac{kX_m}{2 \sqrt{X_m}} - k = \frac{kX_m}{\sqrt{X_m}} - k < -\frac{k}{3} \). Thus, if we denote \( g(a) = -\frac{k}{3} \), then \( \frac{1}{3} X_m < a < X_m - \frac{1}{32} X_m^\iota \). We estimate the integral on \( [a, X_m - \frac{1}{32} X_m^\iota] \) firstly.

Since \( g'(a) = \frac{a (g(a) + k)}{\sqrt{X_m^2 - a^2}} \geq CkX_m^{-1} \), and by Lemma 5.6, we have

\[
\left| \int_a^{X_m - \frac{1}{32} X_m^\iota} \langle x \rangle^\mu e^{\frac{\lambda x^2 - \frac{k}{16} X_m^\iota}{X_m^2}} (X_m^2 - x^2)^{-\frac{1}{2}} (X_m^2 - x^2)^{-\frac{1}{2}} f_m(x) f_n(x) dx \right|
\]

\[
\leq Ck^{-\frac{1}{2}} X_m^{\iota} \left[ \left\| \langle (x) \mu \rangle \Psi(X_m - \frac{1}{32} X_m^\iota) \right\| + \int_a^{X_m - \frac{1}{32} X_m^\iota} \| \langle (x) \mu \rangle Y(x) \| dx \right]
\]
\[
\begin{align*}
&\leq Ck^{-\frac{1}{2}}X_m^{\frac{5}{8}} \left| X_m^{\mu} \right| \Psi(X_m - \frac{1}{32} X_m^\frac{5}{8}) + X_m^{\mu} \int_a^{X_m - \frac{1}{32} X_m^\frac{5}{8}} |\Psi'(x)| \, dx \\
&\quad + \mu \int_a^{X_m - \frac{1}{32} X_m^\frac{5}{8}} (x)^{\mu - 1} |\Psi(x)| \, dx .
\end{align*}
\]

Clearly,
\[
\left| \Psi(X_m - \frac{1}{32} X_m^\frac{5}{8}) \right| \leq C \left( X_m^2 - (X_m - \frac{1}{32} X_m^\frac{5}{8})^2 \right)^{-\frac{1}{4}} \left( X_m^2 - (X_m - \frac{1}{32} X_m^\frac{5}{8})^2 \right)^{-\frac{1}{4}} \leq CX_m^{-\frac{5}{8}},
\]
and \( \int_a^{X_m - \frac{1}{32} X_m^\frac{5}{8}} (x)^{\mu - 1} |\Psi(x)| \, dx \leq CX_m^{-\frac{5}{8} + \mu} \). From \( |\Psi'(x)| \leq C(J_1 + J_3) \) and
\[
\int_a^{X_m - \frac{1}{32} X_m^\frac{5}{8}} J_1 \, dx \leq C \int_a^{X_m - \frac{1}{32} X_m^\frac{5}{8}} x(X_m^2 - x^2)^{-\frac{3}{4}}(X_m^2 - x^2)^{-\frac{1}{4}} \, dx \\
\leq CX_m X_m^{-\frac{3}{8}} \int_a^{X_m - \frac{1}{32} X_m^\frac{5}{8}} (X_m - x)^{-\frac{3}{4}} \, dx \leq CX_m^{-\frac{3}{8}},
\]
and \( \int_a^{X_m - \frac{1}{32} X_m^\frac{5}{8}} J_3 \, dx \leq C \int_a^{X_m - \frac{1}{32} X_m^\frac{5}{8}} \frac{(X_m^2 - x^2)^{\frac{3}{4}}(X_m^2 - x^2)^{-\frac{1}{4}}}{X_m(X_m - x)^3} \, dx \leq CX_m^{-\frac{3}{8}} \), we obtain
\[
\left| \int_a^{X_m - \frac{1}{32} X_m^\frac{5}{8}} \mathcal{F}(x) \, dx \right| \leq Ck^{-\frac{1}{2}} X_m^{-\frac{5}{8} + \mu} \leq \frac{Ck^{-\frac{1}{2}}}{m^{\frac{11}{12}} - n^{\frac{1}{12}} - \frac{4}{9}}.
\]

Next we estimate the integral on \( |X_m - \frac{1}{32} X_m^\frac{5}{8}, X_m - X_m^\frac{5}{8}| \). From \( |g(x)| \geq \frac{b}{3} \) and Lemma 5.6, one obtains
\[
\left| \left( \int_a^{X_m - \frac{1}{32} X_m^\frac{5}{8}} (x)^{\mu} e^{i k x \Phi_{\mu}(m)} \right) \right| \leq Ck^{-1} \left( \left| (x)^{\mu} \Psi(X_m - x_m^\frac{5}{8}) \right| + \int_a^{X_m - \frac{1}{32} X_m^\frac{5}{8}} |(x)^{\mu} \Psi'(x)| \, dx \right) \\
\leq Ck^{-1} \left[ X_m^{\mu} |\Psi(X_m - \frac{1}{32} X_m^\frac{5}{8})| + X_m^{\mu} \int_a^{X_m - \frac{1}{32} X_m^\frac{5}{8}} |\Psi'(x)| \, dx \\
+ \mu \int_a^{X_m - \frac{1}{32} X_m^\frac{5}{8}} (x)^{\mu - 1} |\Psi(x)| \, dx \right].
\]

Clearly, \( \left| \Psi(X_m - X_m^\frac{5}{8}) \right| \leq C \left( X_m^2 - (X_m - X_m^\frac{5}{8})^2 \right)^{-\frac{1}{4}} \left( X_m^2 - (X_m - X_m^\frac{5}{8})^2 \right)^{-\frac{1}{4}} \leq CX_m^{-\frac{5}{8}}, \) and \( \int_a^{X_m - \frac{1}{32} X_m^\frac{5}{8}} (x)^{\mu - 1} |\Psi(x)| \, dx \leq CX_m^{-\frac{5}{8} + \mu} \). Therefore,
\[
\left| \left( \int_a^{X_m - \frac{1}{32} X_m^\frac{5}{8}} (x)^{\mu} e^{i k x \Phi_{\mu}(m)} \right) \right| \leq Ck^{-1} X_m^{-\frac{5}{8} + \mu} \leq \frac{Ck^{-1}}{m^{\frac{11}{12}} - n^{\frac{1}{12}} - \frac{4}{9}}.
\]

A straightforward computation shows that the integral on \( |X_m^\frac{5}{8}, a| \) has a better estimate. Combining with all the estimates in this part, one obtains \( \left| \int_a^{X_m^\frac{5}{8}} \mathcal{F}(x) \, dx \right| \leq \frac{C(k^{-1} + 1)}{m^{\frac{11}{12}} - n^{\frac{1}{12}} - \frac{4}{9}} \). It is easy to check that the estimates for the remained three terms are better. Thus,
\[
\left| \int_a^{X_m^\frac{5}{8}} (x)^{\mu} e^{i k x h_m(x)} \, dx \right| \leq \frac{C(k^{-1} + 1)}{m^{\frac{11}{12}} - n^{\frac{1}{12}} - \frac{4}{9}}.
\]
Proof of Lemma 4.22. Since $m > m_0$ large enough,

$$g(\frac{2\sqrt{2}}{3} X_m) \geq \frac{kX_m}{\sqrt{4kX_m + \frac{X_m^2}{9} + \frac{X_m^2}{9}}} - k \geq \frac{kX_m}{\frac{2}{5}X_m} - k = \frac{k}{5}.\]$$

Thus, by Lemma 5.6,

$$\left| \int_{X_m^{\frac{1}{4}}}^{X_m^{-\frac{3}{4}}} \langle x \rangle^\mu e^{i\frac{x^2}{X_m^{\frac{3}{4}} + kX_m^{-\frac{1}{4}}}} \frac{X_m^2}{X_m^2 - x^2} f_m(x) dx \right| \leq \frac{Ck^{-1}}{m \frac{5}{4} + n \frac{1}{4}} \left[ \left| \langle x \rangle^\mu \Psi(X_m - X_m^{\frac{1}{4}}) \right| + \int_{X_m^{\frac{1}{4}}}^{X_m^{-\frac{3}{4}}} \left| \langle x \rangle^\mu \Psi(x) \right| dx \right].$$

Similarly, $\left| \int_{X_m^{\frac{1}{4}}}^{X_m^{-\frac{3}{4}}} F(x) dx \right| \leq \frac{Ck^{-1}}{m \frac{5}{4} + n \frac{1}{4}}$. Next we estimate the integral on $[X_m^{\frac{2}{3}}, \frac{2\sqrt{2}}{3} X_m]$. Note that

$$g(x) = \frac{x(X_m^2 - X_m^2)}{\sqrt{X_m^2 - x^2}} \geq CkX_m^{-\frac{3}{4}},$$

by Lemma 5.6 we have

$$\left| \int_{X_m^{\frac{1}{4}}}^{X_m^{-\frac{3}{4}}} \langle x \rangle^\mu e^{i\frac{x^2}{X_m^{\frac{3}{4}} + kX_m^{-\frac{1}{4}}}} \frac{X_m^2}{X_m^2 - x^2} f_m(x) dx \right| \leq \frac{Ck^{-\frac{1}{2}} X_m^{-\frac{1}{2}}}{m \frac{5}{4} + n \frac{1}{4}} \left[ \left| \langle x \rangle^\mu \Psi(X_m - X_m^{\frac{1}{4}}) \right| + \int_{X_m^{\frac{1}{4}}}^{X_m^{-\frac{3}{4}}} \left| \langle x \rangle^\mu \Psi(x) \right| dx \right].$$

Clearly,

$$\left| \Psi(\frac{2\sqrt{2}}{3} X_m) \right| \leq C \left( X_m^{\frac{1}{2}} - (\frac{2\sqrt{2}}{3} X_m^2)^{-\frac{1}{4}} \left( X_m^{\frac{1}{2}} - (\frac{2\sqrt{2}}{3} X_m^2)^2 \right)^{-\frac{1}{4}} \right) \leq C X_m^{-1},$$

and $\int_{X_m^{\frac{1}{4}}}^{X_m^{-\frac{3}{4}}} \langle x \rangle^\mu \Psi(x) dx \leq C X_m^{-1+\mu}$. A straightforward computation shows that the remained term has the same estimate. Thus, $\left| \int_{X_m^{\frac{1}{4}}}^{X_m^{-\frac{3}{4}}} F(x) dx \right| \leq \frac{Ck^{-\frac{1}{2}} X_m^{-\frac{1}{2}+\mu}}{m \frac{5}{4} + n \frac{1}{4}} \leq \frac{Ck^{-\frac{1}{2}}}{m \frac{5}{4} + n \frac{1}{4}}. \quad \Box$
Proof of Lemma 4.23. We first estimate
\[ \left| \int_{X_m}^{X_m} F(x) \, dx \right| = \left| C \int_{X_m}^{X_m} (x) e^{i(\zeta_n + kx)} \Psi(x) \, dx \right|. \]
From \( g(x) = \frac{X_m^2 - x^2}{\sqrt{X_m^2 - x^2} + \sqrt{X_m^2 - x^2}} - k \geq \frac{4kX_m}{X_n + X_m} - k \geq \frac{k}{3} \) and Lemma 5.6, we have
\[ \left| \int_{X_m}^{X_m} (x) e^{i(\zeta_n + kx)} \frac{1}{k + \sqrt{k^2 + 1}} (X_m^2 - x^2)^{-\frac{1}{2}} (X_n^2 - x^2)^{-\frac{1}{2}} f_m(x) f_n(x) \, dx \right| \]
\[ \leq Ck^{-1} \left[ \left| (x) \right| \Psi(X_m - X_m^\frac{1}{2}) \right| + \int_{X_m}^{X_m} \left| (x) \Psi'(x) \right| \, dx \]
\[ \leq Ck^{-1} \left[ X_m^\mu \left| \Psi(X_m - X_m^\frac{1}{2}) \right| + X_m^\mu \int_{X_m}^{X_m} \left| \Psi'(x) \right| \, dx \right. \]
\[ \left. + \mu \int_{X_m}^{X_m} (x) \left| \Psi(x) \right| \, dx \right]. \]
The remained proof is similar as Lemma 4.19. \( \square \)

Proof of Lemma 4.25. We estimate
\[ \left| \int_{X_m}^{X_m} F(x) \, dx \right| = \left| C \int_{X_m}^{X_m} (x) e^{i(\zeta_n + kx)} \Psi(x) \, dx \right| \]
firstly. Since \( g(x) = \sqrt{X_m^2 - x^2} - \sqrt{X_m^2 - x^2} - k \geq -k = |k| \) and \( g'(x) > 0 \), by Lemma 5.6 we have
\[ \left| \int_{X_m}^{X_m} (x) e^{i(\zeta_n + kx)} \frac{1}{k + \sqrt{k^2 + 1}} (X_m^2 - x^2)^{-\frac{1}{2}} (X_n^2 - x^2)^{-\frac{1}{2}} f_m(x) f_n(x) \, dx \right| \]
\[ \leq C \left[ \left| (x) \right| \Psi(X_m - X_m^\frac{1}{2}) \right| + \int_{X_m}^{X_m} \left| (x) \Psi'(x) \right| \, dx \]
\[ \leq C|k|^{-1} \left[ X_m^\mu \left| \Psi(X_m - X_m^\frac{1}{2}) \right| + X_m^\mu \int_{X_m}^{X_m} \left| \Psi'(x) \right| \, dx + \mu \int_{X_m}^{X_m} \left| (x) \Psi(x) \right| \, dx \right]. \]
From
\[ \left| \Psi(X_m - X_m^\frac{1}{2}) \right| \leq C \left( X_m^2 - (X_m - X_m^\frac{1}{2})^2 \right)^{-\frac{1}{2}} \leq CX_m^{-\frac{1}{2}} \]
and \( \int_{X_m}^{X_m} (x) \left| \Psi(x) \right| \, dx \leq CX_m^{-\frac{1}{2} + \mu} \), together with
\[ \int_{X_m}^{X_m} J_1 \, dx \leq C \int_{X_m}^{X_m} x(X_m^2 - x^2)^{-\frac{1}{2}} (X_n^2 - x^2)^{-\frac{1}{2}} \, dx \leq CX_m^{-\frac{1}{2}}, \]
and \( \int_{X_m}^{X_m} J_3 \, dx \leq CX_m^{-\frac{1}{2}} \), we have
\[ \left| \int_{X_m}^{X_m} F(x) \, dx \right| \leq \frac{C|k|^{-1}}{m^{\frac{1}{2}} M^{\frac{1}{2} + \frac{\mu}{2}}} - \frac{1}{3}. \]
Since the other three terms we have better estimates, it follows
\[ \left| \int_{X_m}^{X_m} (x) e^{ikx} \psi(x) \, dx \right| \leq \frac{C|k|^{-1} \psi(1)}{m^{\frac{1}{2}} M^{\frac{1}{2} + \frac{\mu}{2}} - \frac{1}{3}}. \]
For the following lemma we denote \( d_1 = \min\{ \Gamma\left(\frac{\nu}{2}\right), \Gamma\left(\frac{\nu+1}{2}\right)\} \).

**Lemma 5.4.** Bessel function of third kind \( H^{(1)}_{\nu}(z) \) satisfies the following:

\[
\left| \sqrt{\frac{\pi z}{2}} H^{(1)}_{\frac{\nu+1}{2}}(z) \right| \leq 1, \quad z \in (-\infty, -c_0), \tag{5.4}
\]

\[
\left| \sqrt{\frac{\pi z}{2}} H^{(1)}_{\frac{\nu+3}{2}}(z) \right| \leq \frac{20}{d_1} |z|^\frac{\nu+1}{2}, \quad z \in [-c_1, 0), \tag{5.5}
\]

\[
\left| \sqrt{\frac{\pi z}{2}} H^{(1)}_{\frac{\nu+1}{2}}(z) \right| \leq \frac{C e^{c_2}}{d_1} \max\{|z|^\frac{\nu}{2}, |z|^\frac{\nu+1}{2}\}, \quad z \in (0, c_2]i, \tag{5.6}
\]

\[
\left| \sqrt{\frac{\pi z}{2}} H^{(1)}_{\frac{\nu+1}{2}}(z) \right| \leq e^{-|z|}, \quad z \in (c_3, \infty)i, \tag{5.7}
\]

where \( c_0 > 0, c_1 \in (0, 1), c_2, c_3 \) can be arbitrary positive numbers and \( C \) is a positive constant.

**Proof.** As in [51], we have the following equalities

\[
\forall \ z \in \mathbb{C}\backslash\{0\}, \Re\left(\nu + \frac{1}{2}\right) > 0,
\]

\[
J_{\nu}(z) = \frac{1}{\sqrt{\pi} \Gamma\left(\nu + \frac{1}{2}\right)} \left(\frac{2}{z}\right)^{\nu} \int_{-1}^{1} (1-t^2)^{-\frac{\nu}{2}} e^{itz} \, dt. \tag{5.8}
\]

\[
H_{\nu}^{(1)}(z) = \frac{i}{\sin(\nu \pi)} \left( J_{\nu}(z) e^{-i\nu \pi} - J_{-\nu}(z) \right). \tag{5.9}
\]

\[
H_{\nu}^{(1)}(z) = \sqrt{\frac{2}{\pi z}} \frac{e^{iz\frac{\nu}{2}}}{\Gamma\left(\nu + \frac{1}{2}\right)} \int_{0}^{\infty} e^{-t^\nu - \frac{i\nu}{2}} \left( 1 + \frac{it}{2z} \right)^{-\frac{\nu}{2}} \, dt. \tag{5.10}
\]

(1) When \( z \in (-\infty, -c_0), \ z = -|z|. \) From (5.10) we have

\[
\left| \sqrt{\frac{\pi z}{2}} H^{(1)}_{\frac{\nu+1}{2}}(z) \right| = \frac{1}{\Gamma\left(\frac{\nu+1}{2}\right)} \left| \int_{0}^{\infty} e^{-t^\nu - \frac{i\nu}{2}} \left( 1 - \frac{it}{2|z|} \right)^{-\frac{\nu}{2}} \, dt \right| \leq \frac{1}{\Gamma\left(\frac{\nu+1}{2}\right)} \int_{0}^{\infty} e^{-t^\nu - \frac{i\nu}{2}} \, dt = 1.
\]

Then we obtain (5.4).

(2) When \( z \in [-c_1, 0), \ z = -|z|. \) From (5.8) we have

\[
\left| \sqrt{\frac{\pi z}{2}} J_{\frac{\nu+1}{2}}(z) \right| = \frac{|z|^\frac{\nu}{2}}{2 \pi \Gamma\left(\frac{\nu+1}{2}\right)} \left| \int_{-1}^{1} (1-t^2)^{-\frac{\nu}{2}} e^{-|t||z|} \, dt \right|.
\]

Since \( \int_{-1}^{1} (1-t^2)^{-\frac{\nu}{2}} e^{-|t||z|} \, dt \leq 2 \int_{0}^{1} (1-t)^{-\frac{\nu}{2}} e^{-|t||z|} \, dt = \frac{2d}{\nu}, \) it follows

\[
\left| \sqrt{\frac{\pi z}{2}} J_{\frac{\nu}{2}}(z) \right| \leq \frac{3d}{\Gamma\left(\frac{\nu}{2}\right)}, \quad z \in [-c_1, 0). \tag{5.11}
\]

Similarly, we have

\[
\left| \sqrt{\frac{\pi z}{2}} J_{-\frac{\nu}{2}}(z) \right| \leq \frac{12d}{\Gamma\left(\frac{\nu}{2}\right)}, \quad z \in [-c_1, 0). \tag{5.12}
\]

From (5.9), (5.11) and (5.12) we obtain (5.5). Similarly, we obtain (5.6).

(3) When \( z \in (c_3, +\infty)i, \ z = |z|. \) From (5.10) we have

\[
\left| \sqrt{\frac{\pi z}{2}} H^{(1)}_{\frac{\nu+1}{2}}(z) \right| = \frac{e^{-|z|}}{\Gamma\left(\frac{\nu+1}{2}\right)} \left| \int_{0}^{\infty} e^{-t^\nu - \frac{i\nu}{2}} \left( 1 + \frac{t}{2|z|} \right)^{-\frac{\nu}{2}} \, dt \right| \leq \frac{e^{-|z|}}{\Gamma\left(\frac{\nu+1}{2}\right)} \int_{0}^{\infty} e^{-t^\nu - \frac{i\nu}{2}} \, dt = e^{-|z|}.
\]

\[\Box\]
By a straightforward computation we have

**Lemma 5.5.** For $x \geq X_n$, $|\zeta_n(x)| \geq \frac{\alpha}{X_n^3} (x - X_n)^{\frac{1}{2}}$. Furthermore, if we suppose $x \geq X_n + X_n^{-\frac{1}{3}}$ and $X_n > 2$, $|\zeta_n(x)| \geq \frac{\alpha}{X_n^3} (x - X_n) > x - X_n$. If $0 \leq x \leq X_n$, $|\zeta_n(x)| \geq \frac{\alpha}{X_n^3} (X_n - x)^{\frac{1}{2}}$, while $|\zeta_n(x)| \geq \frac{\alpha}{X_n^3}$ if $0 \leq x \leq X_n - X_n^{-\frac{1}{3}}$.

The next lemma is from [43].

**Lemma 5.6.** Suppose $\phi$ is real-valued and smooth in $(a, b)$, $\psi$ is complex-valued, and that $|\phi^{(b)}(x)| \geq 1$ for all $x \in (a, b)$. Then

$$\left| \int_a^b e^{i\lambda \phi(x)} \psi(x) dx \right| \leq c_k \lambda^{-1/k} \left[ |\psi(b)| + \int_a^b |\psi'(x)| dx \right]$$

holds when $k \geq 2$ or $k = 1$ and $\phi'(x)$ is monotonic. The bound $c_k$ is independent of $\phi$, $\psi$ and $\lambda$.

The following lemma is standard in Fourier analysis.

**Lemma 5.7.** For any $\varphi \in \mathbb{T}^d$ and all $\alpha = (\alpha_1, \ldots, \alpha_d)$, if $\partial^\alpha W(\varphi, \theta)$ is analytic on $\mathbb{T}_p^n$ and continuous on $\mathbb{T}^d \times \mathbb{T}_p^n$, where $0 \leq |\alpha| = \alpha_1 + \cdots + \alpha_d \leq d(1 \lor r) + d + 2$, then

$$|\widehat{W}(k, l)| \leq \sup_{\varphi \in \mathbb{T}^d} \sup_{|\theta| < p} |\partial^\alpha W(\varphi, \theta)| \leq Ce^{-|\rho|}$$

where

$$\hat{\alpha}_j = \begin{cases} \alpha_j, & \text{if } k_j \neq 0, \\ 0, & \text{if } k_j = 0, \end{cases}$$

and $C$ is a constant which is independent of $k$ and $\widehat{W}(k, l) = \frac{1}{(2\pi)^{dr+n}} \int_{\mathbb{T}^{dr+n}} W(\varphi, \theta)e^{-i\varphi - i\theta} d\varphi d\theta$.

Proof of Lemma 4.1.

$$\int_x^\infty |f(t)(\lambda - q(t))^{\frac{1}{2}}| dt \leq C \left( \int_x^\infty \frac{(q(t) - \lambda)^{\frac{1}{2}}}{|\zeta(t)|^2} dt + \int_x^\infty \frac{q''(t)}{(q(t) - \lambda)^{\frac{5}{2}}} dt + \int_x^\infty \frac{q^2(t)}{(q(t) - \lambda) \frac{3}{2}} dt \right).$$

We estimate the right terms as the following. Since $\int_x^\infty \frac{(q(t) - \lambda)^{\frac{1}{2}}}{|\zeta(t)|^2} dt = \frac{1}{|\zeta(x)|}$ and $x > 2X$, we have

$$|\zeta(x)| = \int_x^\infty (q(t) - \lambda)^{\frac{1}{2}} dt > C \int_x^\infty (q(t))^{\frac{1}{2}} dt > C x(q(x))^{\frac{1}{2}}.$$ 

Thus, $\int_x^\infty \frac{q''(t)}{(q(t) - \lambda)^{\frac{5}{2}}} dt \leq C \frac{C}{x(q(x))^{\frac{1}{2}}}$. From $\frac{q''(x)}{q(x)} = O\left(\frac{1}{x}\right)$ and $\frac{q''(x)}{q'(x)} = O\left(\frac{1}{x}\right)$,

$$\int_x^\infty \frac{q''(t)}{(q(t) - \lambda)^{\frac{5}{2}}} dt + \int_x^\infty \frac{q^2(t)}{(q(t) - \lambda) \frac{3}{2}} dt \leq C \left( \int_x^\infty \frac{q''(t)}{(q(t))^{\frac{1}{2}}} dt + \int_x^\infty \frac{q(t)q'(t)}{t(q(t))^{\frac{1}{2}}} dt \right)$$

$$\leq C \int_x^\infty \frac{q'(t)}{t(q(t))^{\frac{1}{2}}} dt \leq \frac{C}{x(q(x))^{\frac{1}{2}}}.$$ 

This proves the lemma.

Proof of Lemma 4.2. We only give a sketch of the proof. Since the integral has a singularity $X$, we split the integral into

$$\int_0^X + \int_X^{X'} + \int_X^{X''} + \int_X^\infty = I_1 + I_2 + I_3 + I_4,$$
where \(X' \geq \frac{1}{2}X\) and \(X'' \leq 2X\).

For \(I_3\), by directly integrating by parts twice, we obtain

\[
|\zeta(x)| = \int_X^x \frac{q'(t)(q(t) - \lambda)\frac{\pi}{2}}{q'(t)} dt
= \frac{2(q(x) - \lambda)\frac{\pi}{2}}{3q^2(x)} \left(1 + \frac{2(q(x) - \lambda)q''(x)}{5q^2(x)} - S\right),
\]

where \(\frac{(q(x) - \lambda)q''(x)}{q^2(x)} = O(\frac{1}{x^2})\), and \(S = \frac{2q(x)}{5(q(x) - \lambda)\frac{\pi}{2}} \int_X^x (q(t) - \lambda)\frac{\pi}{2} d\left(\frac{q''(t)}{q^2(t)}\right) = O\left(\frac{(X - x)^2}{X^2}\right)\), for \(X \leq x \leq X''\). So we can choose a suitable \(X''\) so that \(\frac{2(q(x) - \lambda)q''(x)}{5q^2(x)} \leq \frac{1}{4}\), and \(|S|\) is much smaller. Thus

\[
-\frac{1}{\zeta^2(x)} = \frac{9q^2(x)}{4(q(x) - \lambda)^3} \left[1 - \frac{4(q(x) - \lambda)q''(x)}{5q^2(x)} + O\left(\frac{1}{X^2}\right)\right]
= \frac{9q^2(x)}{4(q(x) - \lambda)^3} - \frac{9q''(x)}{5(q(x) - \lambda)^2} + O\left(\frac{1}{X^2}\right).
\]

Hence \(f(x) = O(\frac{1}{X' - x})\), and \(I_3 = O\left(\frac{1}{X' - x}\right)\int_X^x \frac{1}{(q(x) - \lambda)^2} dx = O\left(\frac{1}{X' - x}\right)\).

Similar argument can be applied to \(I_2\). The estimates for \(I_1, I_4\) are easy. We omit it. This proves the lemma. \(\square\)

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