THE CALABI-YAU EQUATION, SYMPLECTIC FORMS AND ALMOST COMPLEX STRUCTURES

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Dedicated to Professor S.-T. Yau on the occasion of his 60th birthday.

Abstract. We discuss a conjecture of Donaldson on a version of Yau’s Theorem for symplectic forms with compatible almost complex structures and survey some recent progress on this problem. We also speculate on some future possible directions, and use a monotonicity formula for harmonic maps to obtain a new local estimate in the setting of Donaldson’s conjecture.

1 Background - Yau’s Theorem

In this section we give some background on Yau’s Theorem [Y1] in Kähler geometry, formerly known as the Calabi Conjecture. It can be stated as follows.

Theorem 1.1 Let $(M, \omega)$ be a compact Kähler manifold of complex dimension $n$. If $\sigma$ is a volume form on $M$ satisfying $\int_M \sigma = \int_M \omega^n$ then there exists a unique Kähler form $\tilde{\omega}$ in $[\omega]$ satisfying

$$\tilde{\omega}^n = \sigma.$$ (1.1)

The uniqueness part of the theorem was proved earlier by Calabi [Cal]. We will call (1.1) the Calabi-Yau equation.

Yau’s Theorem shows that the space of Kähler forms in a fixed Kähler class $\beta$ can be identified with the space of volume forms on $M$ with integral $\beta^n$ via the map $\omega \mapsto \omega^n$. Yau’s Theorem can also be stated in terms of the first Chern class of the manifold.

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Theorem 1.2 Let \((M, \omega)\) be a compact Kähler manifold of complex dimension \(n\). If \(\Psi\) is a closed real \((1,1)\)-form representing the cohomology class \(c_1(M)\) then there exists a unique Kähler metric \(\tilde{\omega} \in [\omega]\) satisfying

\[
\frac{1}{2\pi} \text{Ric}(\tilde{\omega}) = \Psi.
\] (1.2)

It is not difficult to see that Theorems 1.1 and 1.2 are equivalent. Indeed, assuming Theorem 1.1 we proceed as follows. The first Chern class \(c_1(M)\) is represented by \(\frac{1}{2\pi} \text{Ric}(\omega)\) and hence the \(\partial \bar{\partial}\)-Lemma produces a smooth function \(F\) on \(M\), which we may assume satisfies

\[
\Psi = \frac{1}{2\pi} \text{Ric}(\omega) - \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} F.
\] (1.3)

By the definition of the Ricci curvature, (1.2) is then equivalent to

\[
\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \frac{\tilde{\omega}^n}{e^F \omega^n} = 0.
\] (1.4)

Since every pluriharmonic function on \(M\) is constant, one immediately sees that solving (1.4) is equivalent to finding a Kähler form \(\tilde{\omega}\) in \([\omega]\) satisfying

\[
\tilde{\omega}^n = \sigma,
\]

namely, equation (1.1), for \(\sigma = e^F \omega^n\). Conversely, given \(\sigma = e^F \omega^n\) as in Theorem 1.1 one can define \(\Psi \in c_1(M)\) by (1.3) and see in the same way that \(\tilde{\omega}\) solving (1.2) satisfies \(\tilde{\omega}^n = \sigma\).

An immediate corollary of Theorem 1.2 is the following widely-used result, which is also sometimes referred to as Yau’s Theorem.

Corollary 1.1 If a compact Kähler manifold \((M, \omega)\) satisfies \(c_1(M) = 0\) then there exists a unique Kähler form \(\hat{\omega} \in [\omega]\) with \(\text{Ric}(\hat{\omega}) = 0\).

This result produces Ricci flat metrics on a large class of algebraic varieties, and this has had an enormous impact on algebraic geometry and string theory. For example they were used by Todorov [Td] and Siu [Si] to prove two long-standing conjectures about K3 surfaces.

We recall now the proof of Theorem 1.1. Yau used a continuity method as follows. Write \(\sigma = e^F \omega^n\) and consider the 1-parameter family of equations

\[
(*)_t \quad \tilde{\omega}_t^n = e^{tF + c_t \omega^n}, \quad t \in [0,1],
\]
for constants $c_t$ defined by $e^{-c_t} = \int_M e^{tf} \omega^n / \int_M \omega^n$. Clearly $\hat{\omega}_0 = \omega$ solves $(*)_t$ for $t = 0$. To solve $(*)_t$ for $t \in [0, 1]$, Yau proved $C^\infty$ estimates on $\hat{\omega}_t$ depending on the fixed data $M$, $\omega$, $F$. Combining these estimates with an implicit function theorem argument shows that the set

$$\{ t \in [0, 1] \mid (*)_t \text{ admits a smooth solution} \}$$

is open and closed in $[0, 1]$ and hence equal to $[0, 1]$. The Kähler form $\hat{\omega} = \hat{\omega}_1$ then solves (1.1).

The $C^\infty$ estimates of Yau can be stated as:

**Theorem 1.3** Let $(M, \omega)$ be a compact Kähler manifold of complex dimension $n$. If a Kähler form $\hat{\omega} \in [\omega]$ solves the Calabi-Yau equation

$$\hat{\omega}^n = \sigma,$$

for some volume form $\sigma$ on $M$ then there are $C^\infty$ a priori bounds on $\hat{\omega}$ depending only on $\omega$, $M$ and $\sigma$.

More precisely, we have the following. For each $k = 0, 1, 2, \ldots$, there exists a constant $A_k$ depending only on $M$, $\omega$, $\sigma$ (with smooth dependence on $\sigma$ and $\omega$) such that

$$\|\hat{\omega}\|_{C^k(g)} \leq A_k,$$

where $g$ is the Kähler metric associated to $\omega$.

Of course, we would take $\sigma = e^{tf+c} \omega^n$ in order to apply this theorem to the argument above.

Donaldson [Do] noted that the assertion of Theorem 1.1 makes sense even if the complex structure is not integrable. In that case, one can take $\omega$ to be a symplectic form compatible with an almost complex structure $J$ and seek a symplectic form $\tilde{\omega}$, cohomologous to $\omega$, satisfying the Calabi-Yau equation

$$\tilde{\omega}^n = \sigma,$$

for some given volume form. It turns out that the equation $(\omega+da)^n = \sigma$ for a 1-form $a$ satisfying $d^*a = 0$ is overdetermined for $n > 2$ and so we restrict to the case $n = 2$. We also remark that we do not expect the analogue of Theorem 1.2 to hold, due to the problem of finding a function $F$ solving (1.3), see also Conjecture 2.4 below.

Donaldson [Do] conjectured that in dimension 4, one could obtain $C^\infty$ bounds for solutions to the Calabi-Yau equation $\tilde{\omega}^2 = \sigma$. And, at least in the case when $b^+(M) = 1$, he conjectured that the analogue of Theorem
would hold. In [W2] it was shown that the estimates all reduce to a $C^0$ bound on an ‘almost-Kähler potential’ $\varphi$, and moreover, that $\Vert \varphi \Vert_{C^0}$ can be bounded (and hence the equation solved) in the case when the Nijenhuis tensor of the almost complex structure $J$ is suitably small.

In fact, Donaldson described in [Do] a more general framework which includes a conjectural almost complex version of Yau’s Theorem as a special case, with applications to symplectic forms and almost complex structures. Further analytic results in the setting where the background symplectic form is only taming the almost complex structure were given in [TWY], improving those of [W2]. We postpone the discussion of these estimates until Section 3 below.

The outline of this survey paper is as follows. In Section 2, we discuss Donaldson’s conjecture and some applications to symplectic and almost complex geometry. In Section 3, we discuss the estimates of [TWY] and [W2]. In Section 4, we give a rough sketch of the main steps in the proof of Yau’s estimates and explain how some are generalized in [W2], [TWY]. In Section 5, we describe how a monotonicity formula for harmonic maps can be applied to give a local estimate in the setting of Donaldson’s conjecture.

## 2 Donaldson’s conjecture and applications

In this section we discuss the conjecture of Donaldson on estimates for the Calabi-Yau equation and describe some consequences. We begin by recalling some terminology. A symplectic form $\omega$ on a manifold $M$ tames an almost complex structure $J$ if $\omega(X, JX) > 0$ for all nonzero tangent vectors $X$. The symplectic form $\omega$ is compatible with $J$ if, in addition,

$$\omega(JX, JY) = \omega(X, Y), \quad \text{for all } X, Y.$$ 

In either case, the data $(\omega, J)$ determines a Riemannian metric $g_\omega$ given by

$$g_\omega(X, Y) = \frac{1}{2}(\omega(X, JY) + \omega(Y, JX)),$$

satisfying the almost-Hermitian condition $g_\omega(JX, JY) = g_\omega(X, Y)$.

In [Do], Donaldson made the following conjecture on $C^\infty$ estimates of solutions of the Calabi-Yau equation in terms of a reference taming symplectic form. His conjecture is restricted to the case of four real dimensions, for reasons that will be made clear later.

**Conjecture 2.1** Let $(M, \Omega)$ be a compact symplectic four-manifold equipped with an almost complex structure $J$ tamed by $\Omega$. Let $\sigma$ be a smooth volume
form on $M$. If $\tilde{\omega} \in [\Omega]$ is a symplectic form on $M$ which is compatible with $J$ and solves the Calabi-Yau equation

$$\tilde{\omega}^2 = \sigma,$$

then there are $C^\infty$ a priori bounds on $\tilde{\omega}$ depending only on $\Omega$, $J$ and $\sigma$.

More precisely, we have the following. For each $k = 0, 1, 2, \ldots$, there exists a constant $A_k$ depending smoothly on the data $\Omega$, $J$ and $\sigma$ such that

$$\|\tilde{\omega}\|_{C^k(g_\Omega)} \leq A_k.$$  

(2.2)

Note that the estimate (2.2) on $\tilde{\omega}$ for $k = 0$ together with the equation (2.1) immediately imply the additional estimate

$$\tilde{g} \geq c g_\Omega,$$

for some uniform constant $c = c(\Omega, J, \sigma) > 0$, where $\tilde{g}$ is the metric associated to $\tilde{\omega}$.

It is perhaps worth remarking that Conjecture 2.1 would be false in general if the cohomological condition $\tilde{\omega} \in [\Omega]$ were removed, even in the Kähler case (cf. [Do], Section 3.3). Indeed, suppose that $(M, \Omega, J)$ were a Kähler manifold admitting a sequence of Kähler classes $\beta_i$ satisfying $\beta_i^2 = \int_M \sigma$ with a non-Kähler limit $\beta_\infty = \lim_{i \to \infty} \beta_i$ in $H^{1,1}(M; \mathbb{R})$. By Yau’s theorem one could find a sequence of Kähler metrics $\tilde{\omega}_i \in \beta_i$ satisfying $\tilde{\omega}_i^2 = \sigma$. If the estimates of Conjecture 2.1 held in this case then one could take a subsequential limit of the $\tilde{\omega}_i$ to obtain a Kähler metric in $\beta_\infty$, a contradiction. (For a discussion of a related problem of the behavior of Ricci-flat metrics as the Kähler class degenerates, see [T2]).

We expect that one could replace the assumption $\tilde{\omega} \in [\Omega]$ with a weaker condition which would ensure that the cohomology class $[\tilde{\omega}]$ remains bounded and uniformly distant from the boundary of the Kähler cone in the Kähler case. By the characterizations of the Kähler cone due to Buchdahl [Bu], Lamari [La] and Demailly-Paun [DP], an element $\beta \in H^{1,1}(M; \mathbb{R})$ is Kähler if it is numerically positive on analytic cycles and if it is also a limit of Kähler classes. In light of this it seems natural to ask:

**Question 2.1** Can one replace the assumption $\tilde{\omega} \in [\Omega]$ in Conjecture 2.1 with conditions on

(a) the boundedness of $[\tilde{\omega}]$ in $H^{1,1}(M; \mathbb{R})$; and

(b) the data $[\tilde{\omega}] \cdot C$, for $J$-holomorphic curves $C$ in $M$?
Although we will see that applications of Conjecture 2.1 do require the restriction to dimension 4, we do not know any counterexample to the conjecture itself in higher dimensions. We pose as a question:

**Question 2.2** Does Conjecture 2.1 hold in any dimension?

We now describe an application of Conjecture 2.1. First, recall the well-known fact that given a general almost complex four-manifold \((M, J)\) which admits symplectic forms there may not exist a symplectic form \(\omega\) compatible with \(J\). Donaldson [Do] conjectured that the (obviously necessary) condition of the existence of a taming symplectic form for \(J\) is sufficient for the existence of a compatible \(\omega\). Combining Donaldson’s conjecture with a characterization of the existence of taming symplectic forms due to Sullivan [Su] we get:

**Conjecture 2.2** Let \((M, J)\) be a compact almost complex four-manifold with \(b^+(M) = 1\). Then the following are equivalent:

(i) There exists a symplectic form on \(M\) compatible with \(J\).

(ii) There exists a symplectic form on \(M\) taming \(J\).

(iii) There is no nonzero closed positive current on \(M\) which is of type \((1, 1)\) with respect to \(J\) and is homologous to zero.

**Proof that Conjecture 2.1 implies Conjecture 2.2** We clearly have that \((i) \Rightarrow (ii)\). The implication \((ii) \Rightarrow (iii)\) is also trivial: if \(\Omega\) tames \(J\) and \(T\) is a nonzero null-homologous closed positive \((1, 1)\) current, then

\[
0 = \langle \Omega, T \rangle = \langle \Omega^{1,1}, T \rangle > 0, \tag{2.3}
\]

because \(\Omega^{1,1}\) is positive definite. The fact that \((iii) \Rightarrow (ii)\) is a theorem of Sullivan (Theorem III.2 in [Su]).

It remains to show that \((ii) \Rightarrow (i)\). Following Donaldson’s argument (see the description in [W2]), we fix \(\Omega\) and a symplectic form taming \(J\). We then choose \(J_0\), an almost complex structure compatible with \(\Omega\), and connect it to \(J = J_1\) with a smooth path \(J_t\), \(0 \leq t \leq 1\), of almost complex structures all tamed by \(\Omega\). We then look for a symplectic form \(\omega_t\) compatible with \(J_t\) with \([\omega_t] \in [\Omega]\) and satisfying the Calabi-Yau equation

\[
\omega_t^2 = \Omega^2. \tag{2.4}
\]
Setting $\omega_0 = \Omega$ clearly solves this for $t = 0$ and the set $T$ of all $t \in [0, 1]$ such that we have a solution $\omega_t$ is open by Proposition 1 of [Do]. This openness argument crucially uses the assumption of four dimensions. Note that since $b^+(M) = 1$, the span of $[\Omega]$ in $H^2(M; \mathbb{R})$ is trivially a maximal positive subspace for the intersection form, and this ensures that we can solve (2.4) for $\omega_t$ in the same cohomology class as $\Omega$. This is the only part of the proof where we use the condition $b^+(M) = 1$.

Closedness of $T$ follows from Conjecture 2.1 together with the Ascoli-Arzelà theorem. Thus we have a solution $\omega_1$ of (2.4), a symplectic form compatible with $J_1 = J$. □

**Remark 2.1** Gromov had shown in [Gr] that (ii) implies (i) holds in the special case of $M = \mathbb{P}^2$ when the symplectic form $\Omega$ is the standard one and $J$ is any almost complex structure tamed by $\Omega$.

As pointed out in [Do], Conjecture 2.2 is interesting even in the case when $J$ is integrable. Indeed, at least in the case $b^+(M) = 1$, one can use the result to give another proof of the following result of Miyaoka-Siu [M], [Si] which does not use the classification of complex surfaces (there are already such proofs by Buchdahl [Bu] and Lamari [La]).

**Theorem 2.1** If $M$ is a complex surface with $b^1(M)$ even then $M$ is Kähler.

**Proof of Theorem 2.1 assuming $b^+(M) = 1$ and Conjecture 2.2** A result of Harvey-Lawson (Theorem 26 and pag.185 in [HL]) says that if $b^1(M)$ is even then we can find a real closed 2-form $\Omega$ on $M$ such that $\Omega^{1,1}$ is positive definite. Then

$$\Omega^2 = (\Omega^{1,1})^2 + 2\Omega^{2,0} \wedge \overline{\Omega}^{2,0}$$

is a strictly positive $(2, 2)$-form, hence $\Omega$ is a symplectic form taming $J$. Then Conjecture 2.2 implies the existence of a symplectic form compatible with $J$, that is of a Kähler form. Presumably, one ought to be able to remove the assumption $b^+(M) = 1$ using appropriate generalizations of Conjecture 2.1 and Conjecture 2.2. □

On the other hand, assuming the classification of surfaces (see [BHPV], for example) and Theorem 20 of [HL], it was shown by [LZ] that:

**Theorem 2.2** Conjecture 2.2 holds in the case when $J$ is integrable, even when $b^+(M) > 1$.  

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It should be noted that, despite this result, Conjecture 2.1 is still open in the case when $J$ is integrable. This may be somewhat surprising. Indeed, if $\Omega$ tames an integrable $J$ as in the statement of Conjecture 2.1 then by Theorem 2.2 one obtains a Kähler form $\omega$ and Yau’s estimates show that $\tilde{\omega}$ can be bounded in terms of $\omega$ and $\sigma$. However, since there are no estimates available on the Kähler form $\omega$ in terms of the data $(\Omega, J)$, this falls short of what is needed for Conjecture 2.1.

Also, as one can see from the proof that Conjecture 2.1 implies Conjecture 2.2, if Conjecture 2.1 were to hold for $J$ integrable it would not (at least by the same argument) give another proof of Theorem 2.2.

We now mention another consequence of Conjecture 2.1.

**Conjecture 2.3** Let $(M, \omega)$ be a compact symplectic four-manifold with a compatible almost complex structure $J$. Assume $b^+(M) = 1$. Then for any smooth volume form $\sigma$ on $M$ with $\int_M \sigma = \int_M \omega^2$ there exists a unique symplectic form $\tilde{\omega} \in [\omega]$ on $M$ compatible with $J$, solving the Calabi-Yau equation

$$\tilde{\omega}^2 = \sigma. \quad (2.5)$$

We remark that the uniqueness part of Conjecture 2.3 is already known to hold (cf. [Dól] and also [W2]). Indeed, Donaldson proved the following stronger uniqueness result which does not require the assumption $b^+(M) = 1$. Fix a maximal positive subspace $H^+_2 \subset H^2(M; \mathbb{R})$ for the intersection form on $M$. Then if $\tilde{\omega}_1, \tilde{\omega}_2 \in [\omega] + H^+_2$ satisfy

$$\tilde{\omega}_1^2 = \tilde{\omega}_2^2, \quad (2.6)$$

it follows that $\tilde{\omega}_1 = \tilde{\omega}_2$. Of course, the case $b^+(M) = 1$ corresponds to taking $H^+_2$ to be the span of $[\omega]$. To prove this general uniqueness result we can argue as follows. Since $[\tilde{\omega}_1] - [\tilde{\omega}_2] \in H^+_2$, we have

$$\int_M (\tilde{\omega}_1 - \tilde{\omega}_2)^2 \geq 0. \quad (2.7)$$

Using (2.6), we can find a unitary frame $\theta_1, \theta_2$ with respect to $(\tilde{\omega}_1, J)$, at a fixed point $p$ in $M$, so that

$$\tilde{\omega}_1 = \sqrt{-1}\theta_1 \wedge \overline{\theta}_1 + \sqrt{-1}\theta_2 \wedge \overline{\theta}_2, \quad \tilde{\omega}_2 = \sqrt{-1}\lambda \theta_1 \wedge \overline{\theta}_1 + \frac{\sqrt{-1}}{\lambda} \theta_2 \wedge \overline{\theta}_2,$$

for some positive constant $\lambda$. Moreover,

$$(\tilde{\omega}_1 - \tilde{\omega}_2)^2 = \tilde{\omega}_1^2 \left( 2 - \left( \lambda + \frac{1}{\lambda} \right) \right) \leq 0, \quad (2.8)$$

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with equality if and only if \( \lambda = 1 \). Then from (2.7) and (2.8) we obtain \( \tilde{\omega}_1 = \tilde{\omega}_2 \) as required.

We now explain how Conjecture 2.3 follows from Conjecture 2.1.

**Proof that Conjecture 2.1 implies Conjecture 2.3** This is contained in [W2], but we outline the proof here for the reader’s convenience. Write \( \sigma = e^{F_0} \omega^2 \) for some smooth function \( F_0 \). We then consider the Calabi-Yau equations

\[
\tilde{\omega}_t^2 = e^{tF_0+c_0} \omega^2,
\]

where \( 0 \leq t \leq 1 \), each \( \tilde{\omega}_t \) is a symplectic form compatible with \( J \), with cohomology class \( [\tilde{\omega}_t] = [\omega] \) and the constants \( c_t \) are chosen so that the integrals of both sides of (2.9) match. Then we have the trivial solution \( \tilde{\omega}_0 = \omega \) at \( t = 0 \) and the set of all \( t \in [0,1] \) such that we have a solution \( \tilde{\omega}_t \in [\omega] \) is open by Proposition 1 of [Dol]. Then Conjecture 2.1 together with the Ascoli-Arzelà theorem implies closedness, and so the existence of a solution for \( t = 1 \). □

**Remark 2.2** Delanöe [Do] considered a related problem concerning the Calabi-Yau equation. He investigated solutions of \( \tilde{\omega}^n = e^{F_n} \omega^n \), on an almost-Kähler manifold \( (M, \omega, J) \) of dimension \( 2n \), of the form \( \tilde{\omega} = \omega + d(Jd\varphi) \) for a smooth real function \( \varphi \) so that \( \tilde{\omega} \) tames \( J \) (here \( J \) acts on 1-forms by duality). He showed that in real dimension 4, if there exists such a solution for every smooth function \( F \), then \( J \) must be integrable.

Finally we consider the analogue of Theorem 1.2. Suppose, as in Conjecture 2.1, that \( M \) admits a symplectic form \( \Omega \) taming an almost complex structure \( J \). Let \( \nabla \) be an affine connection \( M \). We say that \( \nabla \) is an **almost-Hermitian connection** if

\[
\nabla J = 0 = \nabla g_0.
\]

It is well-known (see e.g. [KN]) that almost-Hermitian connections always exist, and we will assume that \( \nabla \) is one of them. Choose a local unitary frame \( \{e_1, \ldots, e_n\} \) for \( g_0 \), and let \( \{\theta^1, \ldots, \theta^n\} \) be a dual coframe. Then locally there exists a matrix of complex valued 1-forms \( \{\theta^j_i\} \), called the connection 1-forms, such that

\[
\nabla e_i = \theta^j_i e_j.
\]

Applying \( \nabla \) to \( g_0(e_i, e_j) \) we see that \( \{\theta^j_i\} \) satisfies the skew-Hermitian property

\[
\theta^j_i + \overline{\theta^i_j} = 0.
\]
Now define the torsion $\Theta = (\Theta^1, \ldots, \Theta^n)$ of $\nabla$ by

$$d\theta^i = -\theta^j_i \wedge \theta^j + \Theta^i, \quad \text{for } i = 1, \ldots, n. \quad (2.10)$$

Notice that the $\Theta^i$ are 2-forms. Equation (2.10) is known as the first structure equation. Define the curvature $\Psi = \{\Psi^i_j\}$ of $\nabla$ by

$$d\theta^i_j = -\theta^k_i \wedge \theta^k_j + \Psi^i_j. \quad (2.11)$$

Note that $\{\Psi^i_j\}$ is a skew-Hermitian matrix of 2-forms. Equation (2.11) is known as the second structure equation. Differentiating (2.11) we see that the real 2-form $\sqrt{-1} \Psi^i_i$ is closed (here we are summing over $i$), and by Chern-Weil theory it represents the first Chern class $c_1(M, \Omega)$. Associated to an almost-Hermitian manifold $(M, g_\Omega, J)$ is a unique canonical connection $\nabla$ satisfying the conditions:

(i) $\nabla J = 0 = \nabla g_\Omega$.

(ii) The torsion $(\Theta^i)$, viewed as a $T'M$-valued 2-form, has vanishing $(1,1)$-part.

We will denote by $\text{Ric}(g_\Omega, J)$ the 2-form $\sqrt{-1} \Psi^i_i$ computed with the canonical connection. In general it is not of type $(1,1)$, but if $(M, g_\Omega, J)$ is Kähler then $\text{Ric}(g_\Omega, J)$ is just the standard Ricci form. We then have the following conjecture, which should be compared with Theorem 1.2.

**Conjecture 2.4** Let $(M, \Omega)$ be a compact symplectic four-manifold with $b^+(M) = 1$ equipped with an almost complex structure $J$ tamed by $\Omega$. Then for every smooth function $F$ on $M$ there exists a unique symplectic form $\tilde{\omega} \in [\Omega]$ compatible with $J$ satisfying

$$\text{Ric}(\tilde{g}, J) = \text{Ric}(g_\Omega, J) + \frac{1}{2}d(\mathcal{J}dF). \quad (2.12)$$

Notice that if $J$ is integrable then $\frac{1}{2}d(\mathcal{J}dF) = -\sqrt{-1} \partial \bar{\partial} f$, and if $(M, g_\Omega, J)$ is Kähler then the $\partial \bar{\partial}$-lemma implies that by varying $F$, the right hand side of (2.12) can be made equal to any representative of $2\pi c_1(M)$. We do not expect this to hold in general.

*Proof that Conjecture 2.1 implies Conjecture 2.4* We are free to add a constant to $F$ so that it satisfies

$$\int_M e^F g_\Omega = \int_M \frac{\Omega^2}{2}. \quad (2.13)$$
Since Conjecture 2.1 implies Conjecture 2.3, we can find a unique $\tilde{\omega} \in [\Omega]$ compatible with $J$ satisfying

$$\frac{\tilde{\omega}^2}{2} = e^F dV_{g_\Omega}. \tag{2.14}$$

This can be written locally in terms of the metrics $\tilde{g}$ and $g_\Omega$ as

$$\det \tilde{g} = e^F \det g_\Omega, \tag{2.15}$$

and the computation to derive (3.16) in [TWY] gives

$$\frac{1}{2} d(JdF) = \frac{1}{2} d \left( Jd\log \frac{\det \tilde{g}}{\det g_\Omega} \right) = \text{Ric}(\tilde{g}, J) - \text{Ric}(g_\Omega, J), \tag{2.16}$$

as required. The uniqueness statement follows easily once one notices that conversely (2.16) and (2.13) imply (2.15) and so also (2.14). \qed

### 3 Estimates for the Calabi-Yau equation

In this section we describe a number of estimates for the Calabi-Yau equation which make some progress towards Conjecture 2.1.

In [TWY], it was shown that Conjecture 2.1 holds, in any dimension, assuming a positive curvature condition on the fixed metric $g_\Omega$. The key to this result is to work with a good choice of local frame, an important technique for these kinds of problems (cf. [To1]).

As in the previous section we let $(M, \Omega)$ be a compact symplectic four-manifold, $J$ be an almost complex structure tamed by $\Omega$ and $g_\Omega$ be the associated almost-Hermitian metric. Let $\nabla$ be the canonical connection of $(M, g_\Omega, J)$ and we define a modified curvature tensor $\mathcal{R}_{ijkl}$ as follows:

$$\mathcal{R}_{ijkl} = R^j_{ikl} + 4N^r_{ij}N^l_{kj},$$

where $R^j_{ikl}$ is the $(1,1)$-part of the curvature of $\nabla$ and $N^r_{ij}$ is the Nijenhuis tensor, which can also be viewed as the $(0,2)$-part of the torsion of $\nabla$. In the case when the data $(g_\Omega, J)$ is Kähler, the tensor $\mathcal{R}_{ijkl}$ coincides with the usual curvature tensor. We write $\mathcal{R} \geq 0$ if the modified curvature tensor is nonnegative in the Griffiths sense, that is, if

$$\mathcal{R}_{ijkl}X^k\bar{X}^jY^k\bar{Y}^l \geq 0, \quad \text{for all (1,0) vectors } X, Y.$$

Then in [TWY] it is shown that:
Theorem 3.1 If $\mathcal{R}(g_\Omega, J) \geq 0$ then Conjecture [2.1] holds. Moreover, the analogous conjecture holds for manifolds of any even dimension.

We note that this gives the first examples of non-Kähler manifolds for which Conjecture [2.1] holds. In the case when $M = \mathbb{P}^n$ and $(g_{FS}, J)$ is the Fubini-Study metric, we have

$$\mathcal{R}_{\tilde{g} \bar{\tau}}(g_{FS}, J) = (g_{FS}) \tilde{\tau}(g_{FS})_{k\bar{\tau}} + (g_{FS}) \tau(g_{FS})_{k\tau},$$

and hence the condition $\mathcal{R} \geq 0$ holds whenever the data $(g_\Omega, \Omega)$ is not too far from the Fubini-Study metric. We note that such results cannot be obtained using Yau’s theorem and an implicit function type argument, since we require the estimates to hold for all volume forms $\sigma$.

We also remark that the proof of Theorem 3.1 does not make use of the condition $\tilde{\omega} \in [\Omega]$. However, this does not contradict the discussion in Section 2 on the necessity of a cohomological assumption, since the nonnegativity of $\mathcal{R}$ must impose restraints on the topology of $M$.

We discuss now the general case in dimension $2n$, with no curvature assumptions. Suppose we are in the setting of Conjecture [2.1] so that $\Omega$ is a symplectic form taming $J$ while $\tilde{\omega} \in [\Omega]$ is a symplectic form compatible with $J$ and satisfying

$$\tilde{\omega}^n = \sigma.$$

Inspired by the Kähler case we define a function $\varphi$ by

$$\tilde{\Delta} \varphi = 2n - \text{tr}_{\tilde{g}\Omega}, \quad (3.1)$$

together with the normalization $\sup_M \varphi = 0$. This definition is well-posed since it easy to see (cf. (3.2) in [TWY]) that

$$\text{tr}_{\tilde{g}\Omega} = 2n \frac{\tilde{\omega}^{n-1} \wedge \Omega}{\tilde{\omega}^n}, \quad (3.2)$$

and thus $\text{tr}_{\tilde{g}\Omega}$ has average $2n$ with respect to $\tilde{\omega}^n$. Note that if $J$ were integrable, and $\Omega$, $\tilde{\omega}$ Kähler with respect to $J$ then $\varphi$ would correspond to the usual Kähler potential defined by

$$\tilde{\omega} = \Omega + \sqrt{-1} \partial \bar{\partial} \varphi, \quad \sup_M \varphi = 0.$$

We have the following result [TWY].
Theorem 3.2 Fix an arbitrary constant $\alpha > 0$. Then, with the notation given above, there are $C^\infty$ a priori bounds on $\tilde{\omega}$ depending only on $\Omega$, $J$, $\sigma$, $\alpha$ and

$$I_\alpha(\varphi) := \int_M e^{-\alpha \varphi} \Omega^n.$$ 

This reduces Conjecture 2.1 to establishing a uniform bound on the quantity $I_\alpha(\varphi)$ for some sufficiently small $\alpha > 0$. In the setting where $J$ is integrable and $\Omega$, $\tilde{\omega}$ are Kähler forms, the quantity $I_\alpha(\varphi)$ is always uniformly bounded when $\alpha$ is small, by a very general result which is independent of the Calabi-Yau equation [H], [T]. This gives then in particular an alternative proof of Yau’s theorem (also, cf. [W1]).

Finally, we note that, as a consequence of the proof of Theorem 3.2 we also have:

Theorem 3.3 With the notation given above, there are $C^\infty$ a priori bounds on $\tilde{\omega}$ depending only on $\Omega$, $J$, $\sigma$ and $\|\tilde{\omega}\|_{C^0(g_\Omega)}$.

That is, the $C^\infty$ bounds on $\tilde{\omega}$ for Conjecture 2.1 follow from a $C^0$ bound on $\tilde{\omega}$. In fact, the result of Theorem 3.3 is already contained in [W2] in the special case when $\Omega$ is compatible with $J$. We also mention that Donaldson [Do] proved a related result that a $C^0$ bound on $\tilde{\omega}$ together with a BMO type estimate on $\tilde{\omega}$ is enough to give Conjecture 2.1.

4 Methods

In this section we will briefly outline the key estimates of Yau (Theorem 1.3) and describe, informally, those arguments and estimates that still hold in the setting of Donaldson’s conjecture.

Let $\tilde{\omega}$ solve the Calabi-Yau equation

$$\tilde{\omega}^n = e^F \omega^n,$$

on a compact Kähler manifold $M$, for some smooth function $F$. Let $\varphi$ be the Kähler potential, defined by

$$\tilde{\omega} = \omega + \sqrt{-1} \partial \bar{\partial} \varphi, \quad \int_M \varphi \omega^n = 0,$$

The key steps in proving $C^\infty$ a priori bounds on $\tilde{\omega}$ are as follows.
Step 1. The inequality
\[ \text{tr} \tilde{g} \leq Ce^{A(\varphi - \inf_M \varphi)}, \]  
holds for uniform constants \( A, C \).

Step 2. The Kähler potential \( \varphi \) satisfies \( \| \varphi \|_{C^0} \leq C \), for a uniform \( C \).

Step 3. If \( \| \tilde{\omega} \|_{C^0} \) is uniformly bounded, we have \( \| \tilde{\omega} \|_{C^1(g)} \leq C \), for a uniform \( C \).

Step 4. Given a Hölder bound \( \| \tilde{\omega} \|_{C^\beta(g)} \leq C \) for some \( \beta > 0 \), we have, for each \( k = 2, 3, \ldots \), the estimates \( \| \tilde{\omega} \|_{C^k(g)} \leq A_k \), for uniform \( A_k \).

The proof of Step 1 uses the maximum principle and the key inequality:
\[ \tilde{\Delta} \log \text{tr} \tilde{g} \geq -C_1 \text{tr} \tilde{g} - C_2, \]  
for uniform constants \( C_1 \) and \( C_2 \). Observe that applying the Laplace operator of \( \tilde{g} \) to the quantity \( \text{tr} \tilde{g} \) gives rise to three terms. Ignoring first order derivatives for the moment, one sees that two derivatives landing on \( \tilde{g} \) give a term involving the Ricci curvature of \( \tilde{g} \), which can be controlled using the Calabi-Yau equation. When two derivatives land on \( g \) this gives the full curvature tensor of \( g \), and the resulting term can be bounded by \( (\text{tr} \tilde{g})(\text{tr} \tilde{g}) \).

Finally, the first order derivatives give rise to a positive quantity
\[ g^{i\ell} \tilde{g}^{p\ell} \tilde{g}^{k\ell} \nabla_i \tilde{g}_{kq} \nabla_j \tilde{g}_{p\ell}, \]  
which can be used to control the negative term \( -|d\text{tr} \tilde{g}|^2 / (\text{tr} \tilde{g})^2 \) produced from differentiating the logarithm function.

Once (4.2) is established, the estimate (4.1) follows immediately from the maximum principle applied to the quantity \( (\log \text{tr} \tilde{g} - A\varphi) \) for a constant \( A \) chosen sufficiently large. The point is that the Kähler potential \( \varphi \) satisfies the equation
\[ \tilde{\Delta} \varphi = 2n - \text{tr} \tilde{g} g, \]  
and so by choosing \( A \) larger than \( C_1 \), the bad term \( -C_1 \text{tr} \tilde{g} \) in (4.2) can be replaced by a good positive term. Then using the Calabi-Yau equation one sees that the quantities \( \text{tr} \tilde{g} \) and \( \text{tr} \tilde{g} \) are basically equivalent.

Note that the same inequality (4.1) holds for other equations in Kähler geometry such as the equation for Kähler-Einstein metrics with negative Ricci curvature [Y1], [Au].
Step 2 was achieved using the celebrated Moser iteration method of Yau. We illustrate the basic idea by describing how to obtain a uniform $L^2$ estimate of $\varphi$. By the Calabi-Yau equation,

$$\int_M \varphi(\omega^n - \tilde{\omega}^n) \leq C \int_M |\varphi| \omega^n. \quad (4.4)$$

On the other hand, since $\tilde{\omega} = \omega + \sqrt{-1} \partial \bar{\partial} \varphi$,

$$\int_M \varphi(\omega^n - \tilde{\omega}^n) = -\int_M \varphi \sqrt{-1} \partial \bar{\partial} \varphi \wedge \sum_{i=0}^{n-1} (\omega^i \wedge \tilde{\omega}^{n-1-i})$$

$$\geq \int_M \sqrt{-1} \partial \varphi \wedge \bar{\partial} \varphi \wedge \omega^{n-1}, \quad (4.5)$$

after integrating by parts. Combining (4.4) and (4.5) with the Poincaré inequality

$$\int_M |\varphi|^2 \omega^n \leq C \int_M |\nabla \varphi|^2 \omega^n,$$

gives $||\varphi||_{L^2(\omega)} \leq C$. This idea can then be extended by an iteration process to give $L^p$ estimates $||\varphi||_{L^p} \leq C(p)$ by using the quantity $\varphi |\varphi|^a$ for $a > 0$ in the above calculation instead of $\varphi$ and applying the Sobolev inequality instead of the Poincaré inequality. The $C^0$ bound of $\varphi$ then follows after checking that the constants $C(p)$ remain bounded as $p \to \infty$.

Note that this method differs somewhat from Yau’s original proof, which was rather more involved and made use of Step 1. Alternative proofs of Step 2 have been given by Kolodziej [Ko], Blocki [Bl] and also by the second author, where it was shown in [W1] that (4.1) implies a uniform estimate of the potential $\varphi$.

Observe that once Steps 1 and 2 have been established, the Calabi-Yau equation implies immediately that the metrics $g$ and $\tilde{g}$ are uniformly equivalent. Step 3 then follows from a maximum principle argument applied to the third order derivatives of $\varphi$. This computation was inspired by an estimate of Calabi [Ca2]. The idea is to compute the Laplace operator of $\tilde{g}$ applied to a quantity $S$, the norm-squared of the tensor $\nabla_i \nabla_j \nabla_k \varphi$, with the norm taken with respect to $\tilde{g}$. A lengthy calculation gives:

$$\tilde{\Delta} S \geq -C_1 S - C_2.$$

One can then apply the maximum principle to $(S + A \text{tr}_g \tilde{g})$ for a sufficiently large constant $A$, making use of the fact that $\tilde{\Delta} \text{tr}_g \tilde{g}$ contains the positive term $A_3$ which is equivalent to $S$. This then gives the bound for $S$ as
required in Step 3. Step 4 follows from standard elliptic estimates after differentiating the Calabi-Yau equation. This completes the outline of Yau’s estimates.

We now discuss how these estimates can be extended in the non-integrable case. Assume that we are in the setting of Conjecture 2.1 so that $M$ is a compact 4-manifold equipped with an almost complex structure $J$ and $\Omega$ is a symplectic form taming $J$. We have a symplectic form $\tilde{\omega} \in [\Omega]$ compatible with $J$ and satisfying the Calabi-Yau equation $\tilde{\omega}^2 = \sigma$ for some volume form $\sigma$ (in fact, much of what we say here carries over easily to any dimension).

It turns out that Step 1 holds: we have the estimate
\[
\tilde{\Delta} \log g_{0\tilde{g}} \geq -C_1 \log g_{0\tilde{g}} - C_2.
\] (4.6)
This was first proved in [W2] in the case when $\Omega$ is compatible with $J$, using normal coordinates and careful estimates of the terms involving the Nijenhuis tensor. In [TWY] it was shown that (4.6) holds even if $\Omega$ only tames $J$. The method of [TWY], simplifying the arguments in [W2], was to use the method of moving frames and the canonical connection, as described in Section 2. From (4.6), the analogue of (3.1) then follows immediately with the potential $\varphi$ defined by (3.1).

Moreover, it was shown in [TWY] that under the assumption $R(g_0, J) \geq 0$ discussed in Section 3 we have the stronger inequality
\[
\tilde{\Delta} \log g_{0\tilde{g}} \geq -C_2.
\] (4.7)
Then (4.7) together with the Calabi-Yau equation and an iteration argument, starting with the $L^1$ estimate on $\log g_{0\tilde{g}}$, gives a uniform upper bound on the quantity $\log g_{0\tilde{g}}$.

Step 2 cannot be carried out in the same way as in Yau’s theorem due to the lack of a $\partial \bar{\partial}$-Lemma. This seems to be the missing ingredient in a direct proof of Conjecture 2.1 along these lines.

For Step 3, it was shown in [TWY] that the analogue of the third-order estimate does indeed hold in this setting, although the computation is significantly more involved. An alternative approach to Step 3 was carried out in [W2], in the case when $\Omega$ is compatible with $J$, using the method of Evans and Krylov [Ev], [Kr] (see also [Tr]). This argument exploits the concavity of the log det function and gives a Hölder bound on $\tilde{\omega}$. While this is weaker than the estimate $\|\tilde{\omega}\|_{C^1(g_0)}$ obtained by the maximum principle, it is sufficient for the purpose of obtaining higher order estimates.

Step 4 follows from standard elliptic theory as in the Kähler case, and is discussed in [Do], [W2], [TWY].
Returning to Step 2: a Moser type iteration argument making use of Step 1 gives instead an estimate

$$-\inf_M \varphi \leq C_\alpha + \log \left( \int_M e^{-\alpha \varphi} dV_{g_0} \right)^{1/\alpha}$$

for any strictly positive $\alpha > 0$. Combining this result with Steps 1, 2 and 4 gives the proof of Theorem 3.2.

5 A monotonicity formula

In this section we will describe how a monotonicity formula for harmonic maps can be used to give a local estimate for solutions to the Calabi-Yau equation.

In general if $(M, J)$ is an almost complex manifold and $g$ is a Riemannian metric which satisfies $g(X, Y) = g(JX, JY)$ for all $X, Y$, then we can define a real 2-form $\omega$, not necessarily closed, by setting

$$\omega(X, Y) = g(JX, Y).$$

In this case we call the data $(M, g, \omega, J)$ an almost-Hermitian manifold. Let us recall briefly the notion of a harmonic map. If $f : (M, g) \to (M', g')$ is a mapping between Riemannian manifolds, its differential $df$ can be viewed as a section of $T^*M \otimes f^*T M'$. This bundle has a natural connection induced from the Levi-Civita connections of $g$ and $g'$, and we define the Laplacian of the map $f$ to be $\Delta f = \text{tr}_g(\nabla df)$, which is a section of $f^*T M'$. If we pick local coordinates $\{x^\alpha\}$ on $M$ and $\{y^i\}$ on $M'$ then, writing $f$ in components $\{f^i\}$, we have

$$(\Delta f)^i = g^{\alpha\beta} \frac{\partial^2 f^i}{\partial x^\alpha \partial x^\beta} - g^{\alpha\beta} \Gamma^\gamma_{\alpha\beta} \frac{\partial f^i}{\partial x^\gamma} + g^{\alpha\beta} \Gamma^\gamma_{jk} \frac{\partial f^j}{\partial x^\alpha} \frac{\partial f^k}{\partial x^\beta},$$

where $\Gamma^\gamma_{\alpha\beta}$ and $\Gamma^\gamma_{jk}$ are the Christoffel symbols of $g$ and $g'$ respectively. A map $f$ is called harmonic if $\Delta f = 0$. We have the following result of Lichnerowicz [Li].

**Theorem 5.1** Let $(M, g, \omega, J)$ and $(M', g', \omega', J')$ be two almost-Hermitian manifolds of real dimension $2n$ and $2n'$ respectively, and $f : M \to M'$ be a $(J, J')$-holomorphic map, that is a map that satisfies

$$df \circ J = J' \circ df.$$
If
\[ d(\omega^{n-1}) = (d\omega')^{2,1} = 0, \tag{5.3} \]
then \( f \) is harmonic.

Notice that (5.3) is satisfied if \( \omega \) and \( \omega' \) are closed (the converse is also true if \( n = 2 \)), but the theorem fails for general almost-Hermitian manifolds that do not satisfy the assumption (5.3) (see (9.11) in \([EL]\)). As an aside, we note here that harmonic maps between Riemannian manifolds satisfy a Schwarz lemma \([GH]\) and so do holomorphic maps between Kähler manifolds \([Y2]\). For a general Schwarz lemma on holomorphic maps between almost-Hermitian manifolds, see \([To1]\).

We now consider the setting of Conjecture 2.1 (in any dimension \( 2n \)) and derive an equation for the Laplacian of the identity map from \( M \) to itself with respect to two different metrics on \( M \). Note that the symplectic form \( \Omega \) is only taming \( J \) and so the 2-form associated to \( g_\Omega \) and \( J \) by (5.1) is not \( \Omega \) but rather its \((1,1)\)-part \( \hat{\Omega} \). For convenience, we will from now on denote \( g_\Omega \) by \( g \).

We would like to apply Theorem 5.1 to the identity map \( I : (M, g, \hat{\Omega}, J) \to (M, \tilde{g}, \tilde{\omega}, J) \), but because \( \hat{\Omega}^{n-1} \) is not closed in general we cannot do this directly. However, \( \hat{\omega} \) is compatible with \( J \) and Lichnerowicz’s proof of Theorem 5.1 \([Li]\) shows that the last term on the right hand side of (5.2) is independent of \( \tilde{g} \). Taking \( f = I \), we see that \( \Delta I \) can be uniformly bounded by quantities depending only on the metric \( g_\Omega \).

We will use this to derive a monotonicity formula, analogous to that of Price \([P]\). Let \( \xi \) be any smooth vector field on \( M \) and \( u \) be any smooth map from \( M \) to itself. We recall that the energy density of \( u \) is the quantity
\[ e(u) = \text{tr}_g(u^* \tilde{g}) = g^{ij} \partial u^k \partial u^\ell \frac{\partial u^i}{\partial x^j} \frac{\partial u^p}{\partial x^k} \tilde{g}_{kl}. \]

Then integration by parts gives
\[ \frac{1}{2} \int_M \text{div} \xi \cdot e(u) dV_g - \int_M g^{ij} \xi_k \partial u^k \partial u^\ell \frac{\partial u^i}{\partial x^j} \frac{\partial u^p}{\partial x^k} \tilde{g}_{kl} dV_g = \int_M (\Delta u)^i \xi_j \frac{\partial u^k}{\partial x^i} \frac{\partial u^p}{\partial x^j} \tilde{g}_{ik} dV_g. \tag{5.4} \]

Whenever \( u \) is harmonic, the right hand side of (5.4) vanishes and the equation says that \( u \) is a critical point of the Dirichlet integral when we reparametrize the domain \( M \) by diffeomorphisms. Such maps satisfy a monotonicity formula \([P]\). If we now consider the case when \( u \) is the identity map, we see that it is not necessarily harmonic, but the right hand side of
\[ \int_M (\Delta I)^i \xi^k \tilde{g}_{ik} dV_g, \]

is given by

\[(5.5)\]

with \((\Delta I)^i\) uniformly bounded in terms of \(g\). The energy density of \(I\) is \(\text{tr}_g \tilde{g}\).

The monotonicity formula of Price then does not apply directly, but we can trace through its proof (we will follow the proof of Theorem 1 in [GB]) to obtain the following result.

**Theorem 5.2** Let \((M, \Omega)\) be a compact \(2n\)-dimensional symplectic manifold, \(J\) an almost complex structure tamed by \(\Omega\) and \(\tilde{\omega}\) another symplectic form compatible with \(J\). We also let \(g\) and \(\tilde{g}\) be the associated Riemannian metrics. Then there exist constants \(r_0, A > 0\) that depend only on \((M, g)\) such that given any \(p \in M\) and any \(0 < r < \rho < r_0\) we have

\[ \frac{e^{Ar}}{r^{2n-2}} \int_{B_g(p, r)} \text{tr}_g \tilde{g} dV_g \leq \frac{e^{A\rho}}{\rho^{2n-2}} \int_{B_g(p, \rho)} \text{tr}_g \tilde{g} dV_g, \]

\[(5.6)\]

where \(B_g(p, r)\) denotes the geodesic ball in the metric \(g\) centered at \(p\) of radius \(r\).

The reason why this holds is the following. Using partitions of unity we can assume that the domain is a ball in \(\mathbb{R}^{2n}\) and the metric \(g\) is close to being Euclidean. With the notation of [GB], we take \(\xi\) to be the radial vector field multiplied by a cutoff function \(\eta\). We substitute this into \((5.4)\) and make use of \((5.5)\). Comparing with the proof of Theorem 1 in [GB] the only new term that appears is the quantity \((5.5)\) which can be bounded by

\[ C \int_{B_{(1+s)r}} \eta r \text{tr}_g \tilde{g} dV_g, \]

and this can be absorbed into another term of the same kind in [GB]. This proves \((5.6)\).

We now restrict to the 4-dimensional case \(n = 2\) and we assume that \(\tilde{\omega}\) is cohomologous to \(\Omega\) and satisfies the Calabi-Yau equation \((2.1)\). Then using \((2.1)\) we see that

\[ C^{-1} \text{tr}_g \tilde{g} \leq \text{tr}_g \tilde{g} \leq C \text{tr}_g \tilde{g}, \]

\[(5.7)\]

for a uniform constant \(C\). Moreover \((3.2)\) and Stokes’ Theorem imply that the \(L^1\) norm of \(\text{tr}_g \tilde{g}\) is uniformly bounded

\[ \int_M \text{tr}_g \tilde{g} dV_g \leq C \int_M \text{tr}_g \tilde{g} \Omega^2 \leq C \int_M \tilde{\omega} \wedge \Omega = C \int_M \Omega^2 \leq C, \]

\[(5.8)\]

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and so (5.7), (5.8) together with the monotonicity formula (5.6) give the following corollary.

**Corollary 5.1** Using the notation as above, with $\tilde{g}$ solving the Calabi-Yau equation, there exists a uniform constant $C$ such that

$$\int_{B_g(p,r)} \text{tr}_g \tilde{g} \, dV_g \leq Cr^2,$$

(5.9)

for all $p \in M$ and $r > 0$ small.

By analogy with the theory of harmonic maps we expect the following $\varepsilon$-regularity result:

**Conjecture 5.1** Let $(M, \Omega)$ be a compact symplectic four-manifold equipped with an almost complex structure $J$ tamed by $\Omega$. Let $\tilde{\omega}$ be another symplectic form cohomologous to $\Omega$ and compatible with $J$. Given a smooth volume form $\sigma$, we assume that $\tilde{\omega}$ satisfies the Calabi-Yau equation

$$\tilde{\omega}^2 = \sigma.$$

Then there exist constants $\varepsilon, C, r_0 > 0$ that depend only on $\Omega, J$ and $\sigma$ such that if

$$\frac{1}{r^2} \int_{B_g(p,r)} \text{tr}_g \tilde{g} \, dV_g \leq \varepsilon,$$

for some $p \in M$ and some $0 < r < r_0$, then

$$\sup_{B_g(p,r/2)} \text{tr}_g \tilde{g} \leq C \frac{1}{r^4} \int_{B_g(p,r)} \text{tr}_g \tilde{g} \, dV_g.$$

Such a result holds for harmonic maps [Sc] so one may wonder why it cannot just be applied directly in this case. The point is that a crucial step in the proof of the $\varepsilon$-regularity in [Sc] is the differential inequality

$$\Delta \text{tr}_g \tilde{g} \geq -C_0 \text{tr}_g \tilde{g} - C_1 (\text{tr}_g \tilde{g})^2,$$

where $\Delta$ is the Laplacian of $g$, the constant $C_0$ depends on the Ricci curvature of $g$ while $C_1$ depends on the whole Riemann curvature tensor of $\tilde{g}$. In the setting of the Calabi-Yau equation this is not controlled, and we are forced to use the Laplacian of $\tilde{g}$ instead. The computation

$$\tilde{\Delta} \text{tr}_g \tilde{g} \geq -C_2 - C_3 (\text{tr}_g \tilde{g})^2,$$
appears in [W2], or (3.19) of [TWY], where now $C_2$ and $C_3$ only depend on the fixed data. But since the Sobolev constant of $\tilde{g}$ is not bounded a priori, the strategy of proof in [Sc] breaks down.

If Conjecture 5.1 were proved, then together with (5.9) it would strongly suggest that the blow-up set of a family of Calabi-Yau equations has real codimension at least 2. It is tempting to speculate that this set should actually be represented by a $J$-holomorphic curve, see [Do], and that this might ultimately lead to a proof of Conjecture 2.1. Results roughly along these lines have been proved by Taubes [Ta] for solutions of the Seiberg-Witten equations, which exhibit less nonlinearity than the Calabi-Yau equation.

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References

[Au] Aubin, T. Équations du type Monge-Ampère sur les variétés kählériennes compactes, Bull. Sci. Math. (2) 102 (1978), no. 1, 63–95.

[BHPV] Barth, W.P., Hulek, K., Peters, C.A.M., Van de Ven, A. Compact complex surfaces, Springer, Berlin, 2004.

[Bl] Blocki, Z. On uniform estimate in Calabi-Yau theorem, Sci. China Ser. A 48 (2005), suppl., 244–247.

[Bu] Buchdahl, N. On compact Kähler surfaces, Ann. Inst. Fourier (Grenoble) 49 (1999), no.1, 287–302.

[Ca1] Calabi, E. On Kähler manifolds with vanishing canonical class, in Algebraic geometry and topology. A symposium in honor of S. Lefschetz, pp. 78–89. Princeton University Press, Princeton, N. J., 1957.

[Ca2] Calabi, E. Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jörgens, Michigan Math. J. 5 (1958) 105–126.

[De] Delanöe, P. Sur l’analogue presque-complexe de l’équation de Calabi-Yau, Osaka J. Math. 33 (1996), no. 4, 829–846.
[DP] Demailly, J.-P., Paun, M. Numerical characterization of the Kähler cone of a compact Kähler manifold, Ann. of Math. (2) 159 (2004), no. 3, 1247–1274.

[Do] Donaldson, S.K. Two-forms on four-manifolds and elliptic equations, in Inspired by S.S. Chern, World Scientific, 2006.

[EL] Eells, J., Lemaire, L. A report on harmonic maps, Bull. London Math. Soc. 10 (1978), no. 1, 1–68.

[Ev] Evans, L.C. Classical solutions of fully nonlinear, convex, second order elliptic equations, Comm. Pure Appl. Math 25 (1982), 333–363.

[GH] Goldberg, S.I., Har’El, Z. A general Schwarz lemma for Riemannian manifolds, Bull. Soc. Math. Grèce (N.S.) 18 (1977), no. 1, 141–148.

[Gr] Gromov, M. Pseudoholomorphic curves in symplectic manifolds, Invent. Math. 82 (1985), no. 2, 307–347.

[GB] Große-Brauckmann, K. Interior and boundary monotonicity formulas for stationary harmonic maps, Manuscripta Math. 77 (1992), no. 1, 89–95.

[HL] Harvey, R., Lawson, H.B. An intrinsic characterization of Kähler manifolds, Invent. Math. 74 (1983), no. 2, 169–198.

[H] Hörmander, L. An introduction to complex analysis in several variables, Van Nostrand, Princeton 1973.

[KN] Kobayashi, S. and Nomizu, K. Foundations of differential geometry. Vol I. Interscience Publishers, John Wiley & Sons, New York-London, 1963.

[Ko] Kolodziej, S. The complex Monge-Ampère equation, Acta Math. 180 (1998), no. 1, 69–117.

[Kr] Krylov, N.V. Boundedly nonhomogeneous elliptic and parabolic equations, Izvestia Akad. Nauk. SSSR 46 (1982), 487–523. English translation in Math. USSR Izv. 20 (1983), no. 3, 459–492.

[La] Lamari, A. Courants kählériens et surfaces compactes, Ann. Inst. Fourier (Grenoble) 49 (1999), no.1, 263–285.

[LZ] Li, T.-J., Zhang, W. Tamed symplectic forms on complex manifolds, preprint, arXiv: 0708.2520 [math.SG].
[Li] Lichnerowicz, A. Applications harmoniques et variétés käihleriennes, in 1968/1969 Symposia Mathematica, Vol. III (INDAM, Rome, 1968/69) pp. 341–402 Academic Press, London.

[M] Miyaoka, Y. Kähler metrics on elliptic surfaces, Proc. Japan Acad. 50 (1974), 533–536.

[P] Price, P. A monotonicity formula for Yang-Mills fields, Manuscripta Math. 43 (1983), no. 2-3, 131–166.

[Sc] Schoen, R.M. Analytic aspects of the harmonic map problem in Seminar on nonlinear partial differential equations (Berkeley, Calif., 1983), 321–358, Math. Sci. Res. Inst. Publ., 2, Springer, New York, 1984.

[Si] Siu, Y.T. Every $K^3$ surface is Kähler, Invent. Math. 73 (1983), no. 1, 139–150.

[Su] Sullivan, D. Cycles for the dynamical study of foliated manifolds and complex manifolds, Invent. Math. 36 (1976), 225–255.

[Ta] Taubes, C.H. SW $\Rightarrow$ Gr: from the Seiberg-Witten equations to pseudo-holomorphic curves, J. Amer. Math. Soc. 9 (1996), no. 3, 845–918.

[Ti] Tian, G. On Kähler-Einstein metrics on certain Kähler manifolds with $c_1(M) > 0$, Invent. Math. 89 (1987), 225–246.

[Td] Todorov, A.N. Applications of the Kähler-Einstein-Calabi-Yau metric to moduli of K3 surfaces, Invent. Math. 61 (1980), no. 3, 251–265.

[To1] Tosatti, V. A general Schwarz Lemma for almost-Hermitian manifolds, Comm. Anal. Geom. 15 (2007), no.5, 1063-1086.

[To2] Tosatti, V. Limits of Calabi-Yau metrics when the Kähler class degenerates, preprint, arXiv:0710.4579, to appear in J. Eur. Math. Soc. 2008.

[TWY] Tosatti, V., Weinkove, B. and Yau, S.-T. Taming symplectic forms and the Calabi-Yau equation, Proc. London Math. Soc. 97 (2008), no.2, 401-424.

[Tr] Trudinger, N.S. Fully nonlinear, uniformly elliptic equations under natural structure conditions, Trans. Amer. Math. Soc. 278 (1983), no. 2, 751–769.
[W1] Weinkove, B. *On the J-flow in higher dimensions and the lower boundedness of the Mabuchi energy*, J. Differential Geom. **73** (2006), no. 2, 351–358.

[W2] Weinkove, B. *The Calabi-Yau equation on almost-Kähler four-manifolds*, J. Differential Geom. **76** (2007), no. 2, 317–349.

[Y1] Yau, S.-T. *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I*, Comm. Pure Appl. Math. **31** (1978), no. 3, 339–411.

[Y2] Yau, S.-T. *A general Schwarz lemma for Kähler manifolds*, Amer. J. Math. **100** (1978), no. 1, 197–203.