HOLE PROBABILITIES AND OVERCROWDING ESTIMATES
FOR PRODUCTS OF COMPLEX GAUSSIAN MATRICES

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ABSTRACT. We consider eigenvalues of a product of \( n \) non-Hermitian, independent random matrices. Each matrix in this product is of size \( N \times N \) with independent standard complex Gaussian variables. The eigenvalues of such a product form a determinantal point process on the complex plane (Akemann and Burda [1]), which can be understood as a generalization of the finite Ginibre ensemble. As \( N \to \infty \), a generalized infinite Ginibre ensemble arises. We show that the set of absolute values of the points of this determinantal process has the same distribution as \( \{R_1^{(n)}, R_2^{(n)}, \ldots\} \), where \( R_k^{(n)} \) are independent, and \( \left(R_k^{(n)}\right)^2 \) is distributed as the product of \( n \) independent Gamma variables Gamma(\( k, 1 \)). This enables us to find the asymptotics for the hole probabilities, i.e. for the probabilities of the events that there are no points of the process in a disc of radius \( r \) with its center at 0, as \( r \to \infty \). In addition, we solve the relevant overcrowding problem: we derive an asymptotic formula for the probability that there are more than \( m \) points of the process in a fixed disk of radius \( r \) with its center at 0, as \( m \to \infty \).

1. Introduction

Products of random matrices are used in different areas of research. For example, the book by Crisanti, Paladin and Vulpiani [9] describes applications of products of random matrices in statistical mechanics of disordered systems, localization, wave propagation in random media, and chaotic dynamical systems. For an application of such products to the study of compositions of random quantum operations we refer the reader to the paper by Roga, Smaczyński and Życzkowski [30]. A paper by Osborn [27] considers products of random matrices in the context of Quantum Chromodynamics. For applications to Schrödinger operators see the book by Bougerol and Lacroix [5].

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In the 1960’s and 70’s different fundamental probabilistic results on products of random matrices were obtained. In particular, the asymptotic behavior of products of independent random matrices was investigated in the work by Furstenberg and Kesten [20]. However, in the 1960’s and 70’s spectral aspects of products of random matrices did not attract any serious attention of mathematicians working in the field. Only recently a number of works appeared in which eigenvalue distributions of products of random matrices, in the limit of large matrices, were considered (see, for example, Burda, Nowak, Jarosz, Livian and Swiech [6, 7], Burda, Janik, and Waclaw [8], Götze and Tikhomirov [14], Penson and Zyczkowski [29], O’Rourke and Soshnikov [32], Forrester [19]), and where products of random matrices were studied by usual methods of Random Matrix Theory. We refer the reader to the books by Anderson, Guionnet and Zeitouni [3], Deift [10], Forrester [17], and Pastur and Shcherbina [28] for an introduction to Random Matrix Theory, and for the description of its basic methods and results.

In this article we concentrate on radial distributions of eigenvalues of products of complex non-Hermitian independent random matrices. For a properly normalized product of complex non-Hermitian independent random matrices O’Rourke and Soshnikov [32] showed (under certain assumptions on the entries of the random matrices) that the empirical spectral distribution of the eigenvalues converges to the limiting distribution, which is a power of the circular law. Forrester [19] derived a formula for the Lyapunov exponents for a product of complex Gaussian matrices. The starting point of the present research is the result obtained in Akemann and Burda [1]. They considered the product of $n$ complex non-Hermitian, independent random matrices, each of size $N \times N$ with independent identically distributed Gaussian entries (Ginibre matrices). It was shown that the eigenvalues of such a product form a complex determinantal point process which can be understood as a generalization of the classical Ginibre ensemble. It is the aim of the present paper to study in detail this complex determinantal process (which in this paper is called the generalized finite-$N$ Ginibre ensemble with parameter $n$), and its infinite analogue (which in this paper is called hereafter the generalized infinite Ginibre ensemble with parameter $n$). We show that the set of absolute values of the points of the generalized Ginibre ensemble has the same distribution as $\{R_1^{(n)}, R_2^{(n)}, \ldots\}$, where $R_k^{(n)}$ are independent, and $\left(R_k^{(n)}\right)^2$ is distributed as the product of $n$ independent Gamma variables Gamma($k, 1$), see Theorem 3.1 and Theorem 3.2. This enables us to find the asymptotics for the hole probabilities, i.e. for the probabilities of the events that there are no points of the process in a disc of radius $r$ with its center at 0, as $r \to \infty$, both for the generalized finite-$N$ and for the generalized infinite Ginibre ensembles (Theorem 3.5). In addition, we solve the relevant overcrowding problem: we derive an asymptotic formula.
for the probability of the event that there are more than \( m \) points of the generalized infinite Ginibre ensemble in a fixed disk of radius \( r \) with its center at 0, as \( m \to \infty \), see Theorem 3.6. In proving these Theorems we apply a technique similar to that developed in Kostlan [23], Krishnapur [24] and Hough, Krishnapur, Peres and Virág [21] for the case of the classical Ginibre ensemble, and for the case of random analytic functions.

This paper is organized as follows. In Section 2 we describe the result obtained in Akemann and Burda [1], and define explicitly the relevant determinantal process. In Section 3 we state main results of this paper. Theorem 3.1 and Theorem 3.2 describe the distribution of absolute values of the points of the generalized Ginibre ensembles, Theorem 3.4 gives an exact formula for the hole probabilities, Theorem 3.5 concerns the rate of the decay of the hole probabilities, and Theorem 3.6 solves the relevant overcrowding problem. The rest of the paper is devoted to proofs of these results.

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2. Products of random matrices and generalized Ginibre ensembles

In this article we consider the product \( P_n \) of \( n \) independent random matrices,

\[
P_n = X_1 X_2 \ldots X_n.
\]

Each matrix \( X_j, j = 1, \ldots, n \), is of size \( N \times N \), and with i.i.d standard complex Gaussian entries. Let \( z_1, \ldots, z_N \) be the eigenvalues of \( P_n \). We study some statistical properties of the distribution of the eigenvalues \( z_1, \ldots, z_N \) in the complex plane, both for finite and for large (\( N \to \infty \)) matrices. More explicitly, the starting point of the present work is the following result obtained recently by Akemann and Burda [1].

Assume that \( |z_1| \leq \ldots \leq |z_N| \). Then the joint density of \( (z_i)_{i=1,\ldots,N} \) with respect to Lebesgue measure on \( \mathbb{C}^N \) is given by

\[
\rho_n^{(n)}(z_1, \ldots, z_N) = \left( \frac{1}{\pi^N \prod_{k=1}^N \Gamma(k)} \right)^n \prod_{k=1}^N w_n(z_k) \prod_{1 \leq i < j \leq N} |z_i - z_j|^2,
\]

where

\[
w_n(z) = \pi^{n-1} C_{0,n}^{n-0} \left( |z|^2 \left\| 0, 0, \ldots, 0 \right. \right).
\]
Here $G_{0,n}^{m,0} \left( \left| z \right|^2 \right| 0, 0, \ldots, 0 \right)$ stands for Meijer’s $G$-function with suitable choice of parameters. For Meijer’s $G$-functions we adopt the same notation and definitions as in Luke [26], Gradshtein and Ryzhik [15]. Namely, the Meijer $G$-function $G_{m,n}^{p,q} \left( x \mid a_1, a_2, \ldots, a_p \mid b_1, b_2, \ldots, b_q \right)$ is defined as

$$G_{m,n}^{p,q} \left( z \mid a_1, a_2, \ldots, a_p \mid b_1, b_2, \ldots, b_q \right) = \frac{1}{2\pi i} \int_{C} \frac{\prod_{j=1}^{m} \Gamma(b_j - s) \prod_{j=1}^{n} \Gamma(1 - a_j + s)}{\prod_{j=m+1}^{p} \Gamma(1 - b_j + s) \prod_{j=n+1}^{q} \Gamma(a_j - s)} z^s ds.$$ 

Here an empty product is interpreted as unity, $0 \leq m \leq q$, $0 \leq n \leq p$, and the parameters $\{a_k\}$ ($k = 1, \ldots, p$) and $\{b_j\}$ ($j = 1, \ldots, m$) are such that no pole of $\Gamma(b_j - s)$ coincides with any pole of $\Gamma(1 - a_k + s)$. We assume that $z \in \mathbb{C} \setminus \{0\}$. The contour of integration $C$ goes from $-i\infty$ to $+i\infty$ so that all poles $\Gamma(b_j - s)$, $j = 1, \ldots, m$, lie to the right of the path, and all poles of $\Gamma(1 - a_k + s)$, $k = 1, \ldots, n$, lie to the left of the path. If $p = 0$, then $n = 0$, and we write the corresponding Meijer $G$-function as $G_{0,q}^{m,0} \left( x \mid b_1, b_2, \ldots, b_q \right)$.

In particular, we have

$$G_{0,n}^{n,0}(t|0,0,\ldots,0) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-s} \Gamma^n(s) ds \quad (t > 0, c > 0).$$

This integral can be evaluated (see Springer and Thompson [33], Lomnicki [25]) by contour integration in the form of an infinite series:

$$G_{0,n}^{n,0}(t|0,0,\ldots,0) = \sum_{j=0}^{\infty} R(t, n, j),$$

where $R(t, n, j)$ is the residue of the integrand at the $n$th-order pole

$$s = -j \quad (j = 0, 1, \ldots),$$

i.e.

$$R(t, n, j) = \frac{1}{(n-1)!} \frac{d^{n-1}}{ds^{n-1}} \left\{ t^{-s} (s+j)^n \Gamma^n(s) \right\} \big|_{s=-j}.$$

Specifically, we find

$$G_{0,1}^{1,0}(t|0,0,\ldots,0) = \sum_{j=0}^{\infty} (-1)^j \frac{t^j}{j!} = e^{-t},$$

and

$$G_{0,2}^{2,0}(t|0,0,\ldots,0) = \sum_{j=0}^{\infty} \frac{t^j}{(j!)^2} \left( -\log t + 2\psi(j+1) \right) = 2K_0(2\sqrt{t}).$$
Here $\psi(.)$ is the Euler psi function and $K_0(.)$ is the modified Bessel function of the second kind of zero order, see, for example, Erdélyi [13].

Equation (2.1) implies that the eigenvalues of $P_n$ form a determinantal point process on the complex plane with kernel

$$K^{(n)}_N(z, \xi) = \sum_{k=0}^{N-1} \frac{(z \bar{\xi})^k}{(k!)^n}$$

with respect to the background measure $\frac{1}{\pi} w_n(z) dm(z)$. Here $dm(z)$ denotes the Lebesgue measure on the complex plane. The fact that $\frac{1}{\pi} w_n(z) dm(z)$ is indeed a probability measure for any positive integer $n$ can be checked using the formula

$$\int_0^\infty t^{\nu-1} G_{m,n}^{p,q} \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \right) dt = \frac{\prod_{j=1}^m \Gamma(b_j + \nu) \prod_{j=1}^n \Gamma(1 - a_j - \nu)}{\prod_{j=m+1}^q \Gamma(1 - b_j - \nu) \prod_{j=n+1}^p \Gamma(a_j + \nu)},$$

see Luke [26], Section 5.6.7.

We will refer to the determinantal point process formed by the eigenvalues of $P_n$ as to the generalized finite-$N$ Ginibre ensemble with parameter $n$. The reason is that once there is only one matrix in the product ($n = 1$), we have

$$w_1(z) = G_{1,0}^{1,0}(|z|^2 |0,0,\ldots,0) = e^{-|z|^2},$$

and equation (2.1) turns into

$$\rho^{(n=1)}_N(z_1, \ldots, z_N) = \frac{1}{\pi \prod_{k=1}^N \Gamma(k)} \prod_{k=1}^N e^{-|z_k|^2} \prod_{1 \leq i < j \leq N} |z_i - z_j|^2.$$  

Therefore the determinantal point process formed by eigenvalues of $P_n$ reduces to the classical Ginibre ensemble at finite $N$, i.e. to the determinant point process with the kernel

$$K^{(n=1)}_N(z, \xi) = \sum_{k=0}^{N-1} \frac{(z \bar{\xi})^k}{k!}$$

with respect to the background measure $\frac{1}{\pi} e^{-|z|^2} dm(z)$.

If $N = \infty$, then we call the corresponding determinantal point process the generalized infinite Ginibre ensemble with parameter $n$.

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1For a background on determinantal point processes we refer the reader to survey articles by Borodin [4], and by Hough, Krishnapur, Peres, and Virág [22].
3. Statement of results

3.1. The distribution of the moduli of eigenvalues. Our first result concerns the distribution of the moduli of the eigenvalues of \( P_n \). Recall that gamma variables \( \Gamma(k, 1) \) are those having the following density function

\[
\varrho_k^{(1)}(x) = \begin{cases} 
\frac{1}{\Gamma(k)} x^{k-1} e^{-x}, & x \geq 0, \\
0, & x < 0.
\end{cases}
\]  

**Theorem 3.1.** Let \( P_n = X_1X_2\ldots X_n \) be a product of \( n \) independent random matrices. Each matrix \( X_j, j = 1, \ldots, n \), is of size \( N \times N \), and with i.i.d. standard complex Gaussian entries. The set of absolute values of eigenvalues of \( P_n \) has the same distribution as the set \( \{ R_1^{(n)}, R_2^{(n)}, \ldots, R_N^{(n)} \} \), where the random variables \( R_1^{(n)}, R_2^{(n)}, \ldots, R_N^{(n)} \) are independent, and for each \( 1 \leq k \leq N \), the random variable \( (R_k^{(n)})^2 \) has the same distribution as the product of \( n \) independent and identically distributed gamma variables \( \Gamma(k, 1) \).

The next Theorem gives even more explicit information on the set of random variables \( \{ R_1^{(n)}, R_2^{(n)}, \ldots, R_N^{(n)} \} \).

**Theorem 3.2.** The random variable \( (R_k^{(n)})^2 \) has the density function given by the formula

\[
\varrho_k^{(n)}(x) = \begin{cases} 
\frac{1}{\Gamma(k)^n} G_{0,n}^{n,0} \left( x \left| k-1, k-1, \ldots, k-1 \right. \right), & x \geq 0, \\
0, & x < 0.
\end{cases}
\]

**Remarks.**

(a) Theorem 3.1 generalizes the result obtained by Kostlan [23] in the case of the classical Ginibre ensemble (\( n = 1 \)).

(b) The generalized finite-\( N \) Ginibre ensemble with parameter \( n \) is a determinantal process on \( \mathbb{C} \) with the kernel of the form \( K(z, \xi) = \sum_k c_k (z \xi)^k \) (where \( c_k \) are some coefficients) with respect to a radially symmetric measure. It is a known general fact (see Hough, Krishnapur, Peres and Virág [21], Section 4.7) that the set of absolute values of the points for such processes has the same distribution as a set of independent random variables. Theorem 3.1 and Theorem 3.2 describe this set of independent random variables explicitly.

(c) Theorem 3.2 is closely related to the following result on the distribution of a product of independent gamma variables (see Springer and Thompson [34], Section 3).
**Proposition 3.3.** Let $x_1, x_2, \ldots, x_n$ be $n$ independent gamma variables having density functions

$$f_k(x_k) = \begin{cases} \frac{1}{\Gamma(b_k)} x_k^{b_k-1} e^{-x_k}, & x_k \geq 0, \\ 0, & x_k < 0, \end{cases}$$

where $b_k > 0$, $1 \leq k \leq n$. Then the probability density function $g(z)$ of the product $z = x_1 x_2 \ldots x_n$ is Meijer’s $G$-function multiplied by a normalizing constant, i.e.

$$g(z) = \frac{1}{n} \prod_{i=1}^{n} \Gamma(b_i) G_{0,n}^{n,0}(z | b_1 - 1, b_2 - 1, \ldots, b_n - 1).$$

### 3.2. An exact formula for the hole probabilities.

Denote by $\mathcal{N}_{GG}^{(n)}(r; N)$ the number of points of the generalized finite-$N$ Ginibre ensemble with parameter $n$ in the disk of radius $r$ with its center at 0. Alternatively, $\mathcal{N}_{GG}^{(n)}(r; N)$ can be understood as the number of eigenvalues of the random matrix $P_n$ in the disk of radius $r$ with its center at 0. By the hole probability $\operatorname{Prob}\{\mathcal{N}_{GG}^{(n)}(r; N) = 0\}$ we mean the probability of an event that there are no points of the generalized finite-$N$ Ginibre ensemble with parameter $n$ in the disk of radius $r$ with its center at 0.

**Theorem 3.4.** The hole probability $\operatorname{Prob}\{\mathcal{N}_{GG}^{(n)}(r; N) = 0\}$ for the generalized finite-$N$ Ginibre ensemble with parameter $n$ can be written as

$$\operatorname{Prob}\{\mathcal{N}_{GG}^{(n)}(r; N) = 0\} = \prod_{k=1}^{N} G_{1,n+1,0}^{n+1,0}(r^2 | 0, 1, k, \ldots, k) \left(\frac{1}{\Gamma(k)}\right)^n.$$ 

**Remarks.**

(a) Note that there is a convenient integral representation for the Meijer $G$-function in the formula above, namely

$$G_{1,n+1,0}^{n+1,0}(r^2 | 0, 1, k, \ldots, k) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} r^{-2s} (\Gamma(k + s))^n ds, \quad c > 0.$$ 

(b) Since

$$G_{1,2,0}^{2,0}(r^2 | 0, 1_k) = \Gamma(k, r^2) = \int_{r^2}^{\infty} e^{-t} t^{k-1} dt,$$

we see that once $n = 1$ the formula in the statement of Theorem 3.4 reduces to

$$\operatorname{Prob}\{\mathcal{N}_{GG}^{(n=1)}(r; N) = 0\} = \prod_{k=1}^{N} \frac{\Gamma(k, r^2)}{\Gamma(k)}.$$
The formula just written above for \( \text{Prob}\left\{ \mathcal{N}_{GG}^{(n=1)}(r; N) = 0 \right\} \) is well known, see Grobe, Haake, and Sommers [16], Forrester [18].

### 3.3. The decay of the hole probabilities.

A basic quantity of interest is the decay of the hole probability as \( r \to \infty \). We investigate the decay of the hole probabilities both for the generalized finite-\( N \) Ginibre ensemble with parameter \( n \) (the case of products of \( n \) random matrices, each of which is of size \( N \)), and for the infinite generalized Ginibre ensemble (the case of product of \( n \) infinite random matrices).

#### Theorem 3.5. (A) (The case of products of random matrices of finite size \( N \)).

As \( r \to \infty \),

\[
\text{Prob}\left\{ \mathcal{N}_{GG}^{(n)}(r; N) = 0 \right\} = \frac{(2\pi)^{\frac{n(n+1)}{2}}}{n^\frac{N}{2} \prod_{k=1}^{N} (\Gamma(k))^n} \exp\left\{ -nNr^\frac{2}{n} + N\left(n - \frac{1}{n}\right) \log(r) \right\} \left(1 + O\left(r^{-\frac{2}{n}}\right)\right).
\]

This implies that for the product of \( n \) matrices of finite size \( N \) we have

\[
\lim_{r \to \infty} \left( \frac{1}{r^\frac{2}{n}} \log \left[ \text{Prob}\left\{ \mathcal{N}_{GG}^{(n)}(r; N) = 0 \right\} \right] \right) = -nN.
\]

#### (B) (The case of products of infinite random matrices).

The following limiting relation holds true

\[
\lim_{r \to \infty} \left( \frac{1}{r^\frac{2}{n}} \log \left[ \text{Prob}\left\{ \mathcal{N}_{GG}^{(n)}(r; N = \infty) = 0 \right\} \right] \right) = -\frac{n}{4}.
\]

#### Remarks.

(a) For the classical (infinite) Ginibre ensemble we have \( n = 1 \), and Theorem 3.5 (B) says that

\[
\frac{1}{r^4} \text{Prob}\left\{ \log \mathcal{N}_{GG}^{(n=1)}(r; N = \infty) = 0 \right\} \to -\frac{1}{4},
\]

as \( r \to \infty \). This asymptotic result for the classical Ginibre ensemble is well known, see Grobe, Haake, and Sommers [16], Forrester [18], Hough, Krishnapur, Peres, and Virág [21], Akemann, Phillips, and Shifrin [2] for different proofs and related results.

(b) The result of Theorem 3.5 (B) can be compared with the decay of hole probabilities for the zeros of the Gaussian analytic function,

\[
f(z) = \sum_{n=0}^{\infty} \frac{a_n z^n}{\sqrt{n!}},
\]
EIGENVALUES OF PRODUCTS OF COMPLEX GAUSSIAN MATRICES

where \( a_n \) are i.i.d. standard complex Gaussian random variables. Namely, it was proved by Sodin and Tsirelson [31] that the hole probability for the zeros decays like \( \exp\{-cr^4\} \). Theorem 3.5 says that for the generalized infinite Ginibre ensemble with parameter \( n \) the hole probability decays like \( \exp\{-Cr^4n\} \).

Thus we have the same behavior only for \( n = 1 \).

3.4. Overcrowding. Consider a disk with a fixed radius \( r > 0 \), and with its center at 0. Recall that \( \mathcal{N}_{GG}^{(n)}(r; N = \infty) \) denotes the number of points of the generalized infinite Ginibre ensemble with parameter \( n \) in this disk. The problem is to estimate the probability of the event that in this disc there are more than \( m \) points of the ensemble, i.e. to estimate \( \text{Prob}\{\mathcal{N}_{GG}^{(n)}(r; N = \infty) \geq m\} \).

We are especially interested in the decay of this probability, \( \text{Prob}\{\mathcal{N}_{GG}^{(n)}(r; N = \infty) \geq m\} \), as \( m \to \infty \).

In this article we prove the following Theorem.

**Theorem 3.6.** Let \( \mathcal{N}_{GG}^{(n)}(r; N = \infty) \) be the number of points of the infinite generalized Ginibre ensemble in the disk of radius \( r \) around 0. Then for a fixed \( r > 0 \)

\[
\text{Prob}\{\mathcal{N}_{GG}^{(n)}(r; N = \infty) \geq m\} = \exp\left\{-\frac{1}{2}nm^2\log(m)(1 + o(1))\right\},
\]

as \( m \to \infty \).

**Remarks.**

(a) Theorem 3.6 is a generalization of the result obtained by Krishnapur (see Krishnapur [24], Section 2.1) for the classical infinite Ginibre ensemble (the case corresponding to \( n = 1 \)).

(b) In the context of zeros of Gaussian analytic function the overcrowding problem was formulated by Yuval Peres, and was studied in detail by Krishnapur in [24]. It was shown that the probability of the event that in the disk with a fixed radius \( r \) around 0 there are more than \( m \) zeros of the Gaussian analytic function decays in the same way as \( \text{Prob}\{\mathcal{N}_{GG}^{(n=1)}(r; N = \infty) \geq m\} \).

4. Proofs of Theorem 3.1 and 3.2

Let \( r_1, \ldots, r_N \) be the moduli of the eigenvalues \( z_1, \ldots, z_N \) of \( P_n \), i.e. \( r_1 = |z_1|, r_2 = |z_2|, \ldots, r_N = |z_N| \). Thus \( r_1 \leq \ldots \leq r_N \). We want to find the joint density of \( (r_i)_{i=1,\ldots,N} \).

**Proposition 4.1.** The joint density of \( (r_i)_{i=1,\ldots,N} \) is given by

\[
\frac{(2\pi)^N}{\pi^N \prod_{k=1}^{N} \Gamma(k)^n} \text{per}[r_i^{2j-1}]_{i,j=1}^{N} \prod_{j=1}^{N} w_n(r_j).
\]
Proof. Let $z_1, \ldots, z_N$ be the eigenvalues of $P_n$. The joint density of $(z_i)_{i=1,\ldots,N}$ is given by formula (2.1). Set $z_i = r_i e^{i \theta_i}, \ i = 1, \ldots, N$.

We have

$$\prod_{1 \leq i < j \leq N} |z_i - z_j|^2 = \left| \sum_{\sigma \in S(N)} (-1)^{\text{sgn}(\sigma)} \prod_{j=1}^{N} z_j^{\sigma(j)-1} \right|^2$$

$$= \sum_{\sigma, \sigma' \in S(N)} (-1)^{\text{sgn}(\sigma) + \text{sgn}(\sigma')} \prod_{j=1}^{N} r_j^{\sigma(j)-1} e^{i(\sigma(j)-1)\theta_j} \prod_{k=1}^{N} r_k^{\sigma'(k)-1} e^{-i(\sigma'(k)-1)\theta_k}.$$

This gives

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{1 \leq i < j \leq N} |z_i - z_j|^2 d\theta_1 \cdots d\theta_N = (2\pi)^N \sum_{\sigma \in S(N)} \prod_{j=1}^{N} r_j^{2\sigma(j)-2}.$$

Therefore the joint density of $(r_i)_{i=1,\ldots,N}$ is

$$(2\pi)^N \left( \frac{1}{\pi^N \prod_{k=1}^{N} \Gamma(k)} \right)^n \prod_{k=1}^{N} w_n(r_k) \sum_{\sigma \in S(N)} \prod_{j=1}^{N} r_j^{2\sigma(j)-2} \prod_{j=1}^{N} r_j$$

$$= \frac{(2\pi)^N}{(\pi^N \prod_{k=1}^{N} \Gamma(k))^n \text{per}[r_i^{2j-1}]_{i,j=1}^{N}} \prod_{j=1}^{N} w_n(r_j).$$

\[\square\]

Proposition 4.1 implies that the random variables $y_1 = r_1^2$, $y_2 = r_2^2$, \ldots, $y_N = r_N^2$ have the joint density given by

$$\text{per} \left[ \frac{y_i^{j-1}}{\pi^{n-1} \Gamma(j)} w_n(\sqrt{y_i}) \right]_{i,j=1}^{N}.$$

Observe that formula (2.2) implies

$$\int_0^{\infty} y^{j-1} w_n(\sqrt{y}) dy = \pi^{n-1} \Gamma^n(j).$$

Therefore the vector of squares of absolute values of eigenvalues of $P_n$ (in uniform order) has the density

$$\frac{1}{N!} \text{per} \left[ q_j^{(n)}(y_i) \right]_{i,j=1}^{N},$$
where the functions \( \varrho_j^{(n)}(y) \), \( 1 \leq j \leq N \), are probability density functions defined by

\[
\varrho_j^{(n)}(y) = \begin{cases} \frac{y^{j-1}w_n(\sqrt{y})}{\pi^{n-1}(\Gamma(j))^n}, & y \geq 0, \\ 0, & y < 0. \end{cases}
\]

Since

\[ y^{j-1}w_n(\sqrt{y}) = y^{j-1}G_{0,n}^{n,0}(y|0,0,\ldots,0) = G_{0,n}^{n,0}(y|j-1,j-1,\ldots,j-1), \]

we conclude that \( \varrho_j^{(n)}(y) \) can be rewritten as in equation (3.2).

To complete the proof of Theorem 3.2 we use the following well known fact (see, for example, Kostlan [23], Lemma 1.5). Assume we are given an \( N \)-tuplet of independent random variables \( (A_i) \), \( 1 \leq i \leq N \), with the corresponding densities \( \varrho_i \), \( 1 \leq i \leq N \). Define a new \( N \)-tuplet of random variables, \( (B_i) \), \( 1 \leq i \leq N \), as a random permutation of the vector \( (A_i) \), \( 1 \leq i \leq N \) (these random permutations are equal to each other in probability). Then the joint density of the random vector \( (B_i) \), \( 1 \leq i \leq N \), is

\[
\frac{1}{N!} \text{per} \left[ \varrho_j(B_i) \right]_{i,j=1}^N.
\]

Considering squares of moduli of unordered eigenvalues of \( P_n \) as random variables \( (B_i) \), \( 1 \leq i \leq N \), we obtain the statement of Theorem 3.2.

Theorem 3.1 follows from Theorem 3.2 and from the result by Springer and Thompson [34] on the distribution of a product of independent gamma variables, see Proposition 3.3. Namely, Proposition 3.3 and Theorem 3.2 imply that each random variable \( \left( R_i^{(n)} \right)^2 \), \( 1 \leq i \leq N \), has the same distribution as the product of \( n \) identically distributed and independent gamma variables having density function (3.1). Theorem 3.1 is proved.

\[ \square \]

5. Proof of Theorem 3.4

From Theorem 3.1 we conclude that

\[
\text{Prob} \left\{ N_{GC}^{(n)}(r;N) = 0 \right\} = \prod_{k=1}^{N} \text{Prob} \left\{ \left( R_k^{(n)} \right)^2 > r^2 \right\},
\]

where the random variables \( R_1^{(n)}, R_2^{(n)}, \ldots, R_N^{(n)} \) are those introduced in the statement of Theorem 3.1. By Theorem 3.2

\[
\text{Prob} \left\{ \left( R_k^{(n)} \right)^2 > r^2 \right\} = \frac{1}{(\Gamma(k))^n} \int_{r^2}^{\infty} G_{0,n}^{n,0}(x|k-1,\ldots,k-1)dx.
\]
The last integral can be computed explicitly using the formula
\[
\int_1^\infty x^{-\rho}(x-1)^{\sigma-1} G_{p,q}^{m,n}(\alpha \mid a_1, \ldots, a_p \mid b_1, \ldots, b_q) \, dx = \Gamma(\sigma) G_{p+1,q+1}^{m+1,n}(\alpha \mid a_1, \ldots, a_p, \rho) - \rho - \sigma, \quad b_1, \ldots, b_q),
\]
see Gradshtein and Ryzhik [15], 7.811.3. This gives

\[
\text{Prob}\left\{ (R_k^{(n)})^2 > r^2 \right\} = \frac{r^2 G_{1,n+1}^{n+1,0}(r^2 \mid -1, k-1, \ldots, k-1)}{(\Gamma(k))^n}.
\]

Since

\[
z^{\sigma} G_{p,q}^{m,n}(z \mid a_1, \ldots, a_p) = G_{p,q}^{m,n}(z \mid a_1 + \sigma, \ldots, a_p + \sigma),
\]
(see, for example, Luke [26], Section 5.4) we can rewrite the expression for

\[
\text{Prob}\left\{ (R_k^{(n)})^2 > r^2 \right\}
\]

as

\[
\text{Prob}\left\{ (R_k^{(n)})^2 > r^2 \right\} = \frac{G_{1,n+1}^{n+1,0}(r^2 \mid 1, k, \ldots, k)}{(\Gamma(k))^n},
\]

and the result of Theorem 3.4 follows. □

6. PROOF OF THEOREM 3.5

6.1. Proof of the asymptotic formula for the hole probability for the generalized finite-\(N\) Ginibre ensemble. We use Theorem 3.4, which expresses the hole probability \(\text{Prob}\left\{ N_{GG}^{(n)}(r; N) = 0 \right\}\) in terms of the Meijer \(G\)-functions. The following asymptotic formula holds true (see Luke [26], Section 5.7)

\[
G_{p,q}^{q,0}(z \mid a_1, a_2, \ldots, a_p \mid b_1, b_2, \ldots, b_q) \sim \frac{(2\pi)^{(\sigma-1)/2}}{\sigma^{1/2}} \exp\left\{ -\sigma z^{1/\sigma} \right\} z^\theta \sum_{k=0}^\infty M_k z^{-k/\sigma},
\]

where

\[
|z| \to \infty, \quad |\arg z| \leq (\sigma + \epsilon)\pi - \delta, \quad \delta > 0.
\]

In the formula above the parameters \(\sigma\) and \(\epsilon\) are defined by

\[
\sigma = q - p,
\]

and

\[
\epsilon = \frac{1}{2} \quad \text{if} \quad \sigma = 1, \quad \epsilon = 1 \quad \text{if} \quad \sigma > 1.
\]
The parameter $\theta$ is defined by the formula

$$\sigma \theta = \left\{ \frac{1}{2}(1 - \sigma) + \sum_{k=1}^{q} b_k - \sum_{k=1}^{p} a_k \right\}.$$

Finally, the coefficients $M_k$'s are independent of $z$ and can be found explicitly. In particular, $M_0 = 1$.

In our case $p = 1$, $q = n + 1$, $\sigma = n$, and $z = r^2$. It is not hard to find the parameter $\theta$ as well. The result is

$$\theta = k - \frac{1}{2} - \frac{1}{2n}.$$

This gives

$$G_{1,n+1}^{n+1,0}(r^2, 0, \frac{1}{k}, \ldots, k) = \frac{(2\pi)^{\frac{2n+1}{n^2}}}{\Gamma(k)} \exp \left\{ -nr^2 \right\} \left[ 1 + O \left( r^{-\frac{2}{n}} \right) \right],$$

as $r \to \infty$. We insert this asymptotic expression into the formula in the statement of Theorem 3.4. The statement of Theorem 3.5 (A) follows immediately. \hfill $\Box$

6.2. Proof of the asymptotic formula for the hole probability for the generalized infinite Ginibre ensemble.

6.2.1. An upper bound for the hole probability. To estimate the hole probabilities we use the following standard fact (called the Markov inequality).

**Proposition 6.1.** Suppose $\varphi : \mathbb{R} \to \mathbb{R}$ is a positive valued function, and let $A$ be a Borel subset of $\mathbb{R}$. Then

$$\inf \{ \varphi(y) : y \in A \} \cdot \text{Prob} \{ X \in A \} \leq \mathbb{E} \varphi(X).$$

**Proof.** See, for example, Durrett [12], Section 1.6, Theorem 1.6.4. \hfill $\Box$

**Proposition 6.2.** We have

$$\text{Prob} \left\{ \mathcal{N}_{GG}^{(n)}(r; N = \infty) = 0 \right\} \leq \exp \left\{ -\frac{n}{4} \frac{r^4}{n} + O \left( \log(r) \right) \right\},$$

as $r \to \infty$.

**Proof.** Let $\alpha \geq 0$, $A = (r^2, \infty)$, and $\varphi(x) = x^\alpha$. Then by the Markov inequality (Proposition 6.1) we have

$$\text{Prob} \left\{ \left( R_k^{(n)} \right)^2 > r^2 \right\} \leq \int_0^\infty t^{\alpha} C_{0,n}^{n,0}(t|k-1, k-1, \ldots, k-1) \frac{dt}{(\Gamma(k))^n(r^2)^\alpha}.$$
(we have used the fact that the random variable \((R_{n_k}^{(n)})^2\) has the density function given by formula (3.2), see Theorem 3.2). By formula (2.2)
\[
\int_0^\infty t^\alpha G_{0,n}^{n,0}(t|k-1,k-1,\ldots,k-1)dt = (\Gamma(k+\alpha))^n.
\]
Therefore,
\[
\text{Prob}\left\{ (R_{n_k}^{(n)})^2 > r^2 \right\} \leq \Gamma(k+\alpha) \frac{n}{(\Gamma(k))^n (r^2)^\alpha}.
\]
Next we use the following well known inequality (see, for example, Digital Library of Mathematical Functions [11], § 5.6)

\[
(2\pi)^{\frac{1}{2}} \exp \left\{ -z + (z - \frac{1}{2}) \log z \right\} \leq \Gamma(z) \leq (2\pi)^{\frac{1}{2}} \exp \left\{ -z + (z - \frac{1}{2}) \log(z) + \frac{1}{12z} \right\},
\]
as \(z \geq 1\), to obtain
\[
\text{Prob}\left\{ (R_{n_k}^{(n)})^2 > r^2 \right\} \leq \exp \left\{ -nz + (z - \frac{1}{2}) \log(z) + \frac{1}{12z} \right\}.
\]
Set \(\alpha = r^\frac{2}{n} - k\). Then we can rewrite the inequality above as
\[
\text{Prob}\left\{ (R_{n_k}^{(n)})^2 > r^2 \right\} \leq \exp \left\{ -nr^\frac{2}{n} + \frac{n}{2} \log \frac{k}{r^\frac{2}{n}} - nk \log \frac{k}{r^\frac{2}{n}} + \frac{n}{12r^\frac{2}{n}} \right\}.
\]
This gives
\[
\prod_{k=1}^\infty \text{Prob}\left\{ (R_{n_k}^{(n)})^2 > r^2 \right\} \leq \prod_{k=1}^{r^\frac{2}{n}} \text{Prob}\left\{ (R_{n_k}^{(n)})^2 > r^2 \right\}
\]
\[
\leq \exp \left\{ -nr^\frac{2}{n} + n \frac{r^\frac{2}{n}}{2} (r^\frac{2}{n} + 1) + n \frac{r^\frac{2}{n}}{2} \sum_{k=1}^{r^\frac{2}{n}} \log \left( \frac{k}{r^\frac{2}{n}} \right) - n \sum_{k=1}^{r^\frac{2}{n}} k \log \left( \frac{k}{r^\frac{2}{n}} \right) + \frac{n}{12} \right\},
\]
where \(r^\frac{2}{n}\) is considered as an integer (this assumption should not affect our estimate). The sums in the exponent can be estimated using the Euler-MacLaurin sum formula. We write it in the form

\[
\sum_{k=1}^L f(k) = \int_1^L f(t)dt + \frac{1}{2} (f(L) + f(1)) + O(f'(L)).
\]
This formula gives
\[
\sum_{k=1}^{r \frac{1}{n}} \log \left( k \frac{1}{r \frac{1}{n}} \right) = -r \frac{1}{n} + 1 + \frac{1}{n} \log(r) + O \left( r^{-\frac{2}{n}} \right),
\]
and
\[
\sum_{k=1}^{r \frac{1}{n}} k \log \left( k \frac{1}{r \frac{1}{n}} \right) = -\frac{r}{4} + O(1),
\]
as \( r \to \infty \). Using these estimates we find
\[
\text{Prob} \left\{ \mathcal{N}_{GG}^{(n)}(r; N = \infty) = 0 \right\} = \prod_{k=1}^{\infty} \text{Prob} \left\{ \left( R_k^{(n)} \right)^2 > r^2 \right\}
\leq \exp \left\{ -\frac{n}{4} r \frac{1}{n} + O \left( \log(r) \right) \right\},
\]
as \( r \to \infty \).

6.2.2. A lower bound for the hole probability.

**Proposition 6.3.** We have
\[
\text{Prob} \left\{ \mathcal{N}_{GG}^{(n)}(r; N = \infty) = 0 \right\} \geq \exp \left\{ -\frac{n}{4} r \frac{1}{n} + O \left( r \frac{2}{n} \log(r) \right) \right\},
\]
as \( r \to \infty \).

**Proof.** It is known that
\[
\prod_{k=1}^{\infty} \text{Prob} \left\{ \left( R_k^{(n=1)} \right)^2 > r^2 \right\} \geq \exp \left\{ -\frac{1}{4} r^4 + O(r^2 \log(r)) \right\},
\]
see, for example, Hough, Krishnapur, Peres and Virág [21], Section 7.2. Consider the set of the random variables \( \{R_1^{(n)}, R_2^{(n)}, \ldots\} \). We know that the random variables \( R_k^{(n)} \) are independent, and \( \left( R_k^{(n)} \right)^2 \) has the same distribution as the product of \( n \) independent and identically distributed gamma variables Gamma(\( k, 1 \)). In particular, the random variable \( \left( R_k^{(n=1)} \right)^2 \) is itself the gamma variable Gamma(\( k, 1 \)). We conclude that the random variable \( \left( R_k^{(n)} \right)^2 \) has the same distribution as the random variable \( \left[ \left( R_k^{(n=1)} \right)^2 \right]^n \). This immediately implies
\[
\text{Prob} \left\{ \left( R_k^{(n)} \right)^2 > r^2 \right\} \geq \left[ \text{Prob} \left\{ \left( R_k^{(n=1)} \right)^2 > r \frac{1}{n} \right\} \right]^n.
\]
Using inequalities (6.3) and (6.4) we find
\[
\text{Prob}\left\{ N_{GG}(n) ; N = \infty \right\} = 0 = \prod_{k=1}^{\infty} \text{Prob}\left\{ \left( R_k(n) \right)^2 > r^2 \right\}
\geq \exp\left\{ -\frac{n}{4} r^4 + O(r^2 \log(r)) \right\},
\]
as \( r \to \infty. \)
□

6.2.3. Proof of Theorem 3.5 (B). The statement of Theorem 3.5 (B) follows from Proposition 6.2 and Proposition 6.3. □

7. Proof of Theorem 3.6

Recall that the set of absolute values of points of the generalized Ginibre ensemble has the same distribution as the set \( \{ R_1(n), R_2(n), \ldots \} \), where \( R_k(n) \) are independent, and \( \left( R_k(n) \right)^2 \) has the same distribution as the product of \( n \) independent and identically distributed gamma variables having density function Gamma\((k,1)\), see Theorem 3.1. This implies
\[
\left( R_k(n) \right)^2 \overset{d}{=} \left( \xi^{(1)}_1 + \ldots + \xi^{(1)}_k \right) \cdot \ldots \cdot \left( \xi^{(n)}_1 + \ldots + \xi^{(n)}_k \right),
\]
where \( \xi^{(j)}_i, 1 \leq i \leq k, 1 \leq j \leq n \) are i.i.d. exponential random variables with mean 1. Therefore we can write
\[
\text{Prob}\left\{ \left( R_k(n) \right)^2 < r^2 \right\} \geq \text{Prob}\left\{ \xi^{(1)}_1 + \ldots + \xi^{(1)}_k < \frac{r^2}{n} \right\} \cdot \ldots \cdot \text{Prob}\left\{ \xi^{(n)}_1 + \ldots + \xi^{(n)}_k < \frac{r^2}{n} \right\}
\geq \prod_{j=1}^{k} \text{Prob}\left\{ \xi^{(j)}_1 < \frac{r^2}{nk} \right\} \cdot \ldots \cdot \prod_{j=1}^{k} \text{Prob}\left\{ \xi^{(j)}_1 < \frac{r^2}{nk} \right\}.
\]
For an exponential random variable \( \xi \) with mean 1 we have
\[
\text{Prob}\{ \xi < x \} \geq \frac{x}{2}, \quad 0 < x < 1.
\]
This gives
\[
\text{Prob}\left\{ \left( R_k(n) \right)^2 < r^2 \right\} \geq \left( \frac{r^2}{2k} \right)^{nk},
\]
and we obtain
\[
\text{Prob}\left\{ N_{GG}(n) ; N = \infty \right\} \geq m \geq \prod_{k=1}^{m} \text{Prob}\left\{ \left( R_k(n) \right)^2 < r^2 \right\}
\geq \prod_{k=1}^{m} \frac{r^{2k}}{2^{nk} k^{nk}} = \frac{r^{m(m+1)}}{2^{m(m+1)}} \exp\left\{ -n \sum_{k=1}^{m} k \log(k) \right\}.
\]
The Euler-MacLaurin formula (equation 6.2) gives
\[
\sum_{k=1}^{m} k \log(k) = \frac{m(m+1)}{2} \log(m) - \frac{m^2}{4} + O(\log(m)),
\]
as \(m \to \infty\). We use this estimate to get a lower bound for \(\Pr\{N_{GG}^{(n)}(r; N = \infty) \geq m\}\), namely
\[
\Pr\{N_{GG}^{(n)}(r; N = \infty) \geq m\} \geq \exp\left\{-\frac{1}{2} nm^2 \log(m)(1 + o(1))\right\},
\]
as \(m \to \infty\). To obtain an upper bound for \(\Pr\{N_{GG}^{(n)}(r; N = \infty) \geq m\}\) we use the Markov inequality (Proposition 6.1) with \(A = \{0, r^2\}\), \(\phi(x) = x - \alpha\), \(\alpha \geq 0\). This gives
\[
\Pr\left\{\left(R_k^{(n)}\right)^2 < r^2\right\} \leq \left(\int_0^\infty t^{-\alpha} G_{0,n}^{n,0}(t|k-1, k-1, \ldots, k-1)dt\right) \frac{(r^2)^{\alpha} (\Gamma(k-\alpha))^n}{(\Gamma(k))^n},
\]
where we have used formula (2.2). By inequality (6.1)
\[
\Pr\left\{\left(R_k^{(n)}\right)^2 < r^2\right\} \leq \exp\left\{n\alpha + \alpha \log(r^2) + (n + n \alpha - n) \log(k) + \frac{n}{12k}\right\}.
\]
Choosing \(\alpha = k - \frac{1}{2}\), we obtain
\[
\Pr\left\{\left(R_k^{(n)}\right)^2 < r^2\right\} \leq \exp\left\{(k - \frac{1}{2}) (n + \log(r^2) - n \log(k)) + \frac{n}{6}\right\}.
\]
In addition, by simple probabilistic arguments
\[
\Pr\{N_{GG}^{(n)}(r; N = \infty) \geq m\} \leq \Pr\left\{\sum_{k=1}^{m^2} \left(\left(R_k^{(n)}\right)^2 < r^2\right) \geq m\right\}
\]
\[
+ \sum_{k=m^2+1}^\infty \Pr\left\{\left(R_k^{(n)}\right)^2 < r^2\right\}.
\]
(Here \(\mathbb{I}(\cdot)\) stands for the characteristic function of a set). The second term on the right hand side of the inequality above can be estimated as follows. By
inequality (7.4)

$$\text{Prob}\left\{ \left( R_k^{(n)} \right)^2 < r^2 \right\} \leq \exp \left\{ -nk \log(k)(1 + o(1)) \right\},$$
as \( k \to \infty \). Therefore,

$$\sum_{k=m^2+1}^{\infty} \text{Prob}\left\{ \left( R_k^{(n)} \right)^2 < r^2 \right\} \leq \exp \left\{ -nm^2 \log(m^2)(1 + o(1)) \right\},$$
as \( m \to \infty \). Now let us estimate the first term on the right hand side of inequality (7.5). We have

$$\text{Prob}\left\{ \sum_{k=1}^{m^2} \mathbb{1} \left( \left( R_k^{(n)} \right)^2 < r^2 \right) \geq m \right\} \leq \left( \frac{m^2}{m} \right)^m \prod_{k=1}^{m} \text{Prob}\left\{ \left( R_k^{(n)} \right)^2 < r^2 \right\}.$$

Using \( \left( \frac{m^2}{m} \right) < m^{2m} \), inequality (7.4), and equation (7.1) we obtain

$$\text{Prob}\left\{ \sum_{k=1}^{m^2} \mathbb{1} \left( \left( R_k^{(n)} \right)^2 < r^2 \right) \geq m \right\} \leq \exp \left\{ 2m \log(m) + \frac{nm}{6} + \sum_{k=1}^{m} (k - \frac{1}{2}) (n + \log(r^2) - n \log(k)) \right\}$$

$$= \exp \left\{ -\frac{nm^2 \log(m)}{2} (1 + o(1)) \right\},$$
as \( m \to \infty \). The statement of Theorem 3.6 follows from inequalities (7.2) and (7.6). \( \square \)

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