On a Conjecture of Givental

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Abstract

These brief notes record our puzzles and findings surrounding Givental’s recent conjecture which expresses higher genus Gromov-Witten invariants in terms of the genus-0 data. We limit our considerations to the case of a complex projective line, whose Gromov-Witten invariants are well-known and easy to compute. We make some simple checks supporting his conjecture.

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1 Brief Summary

These notes are brief sketches of our troubles and findings surrounding a work of Givental [5]. Let $F_g$ be the generating function in the small phase space for genus-$g$ Gromov-Witten (GW) invariants of a manifold $X$ with a semi-simple Frobenius structure on $H^*(X, \mathbb{Q})$. Then, Givental’s conjecture, whose equivariant counter-part he has proved [5], is

$$e^{\sum_{g \geq 2} \lambda^{g-1} F_g(t)} = \left[ \sum_{k,l \geq 0} \sum_{i,j} V_{k,l}^{ij} \Delta_i \Delta_j \prod_j \tau(\lambda \Delta_j; \{q^j_n\}) \right]_{q^k_n = T^k_n}, \quad (1)$$

where $i,j = 1, \ldots, \dim H^*(X, \mathbb{Q})$; $\tau$ is the KdV tau-function governing the intersection theory on the Deligne-Mumford space $\overline{M}_{g,n}$; and $V_{kl}^{ij}, \Delta_j,$ and $T^j_n$ are functions of the small phase space coordinates $t \in H^*(X, \mathbb{Q})$ and are defined by solutions to the flat-section equations associated with the genus-0 Frobenius structure of $H^*(X, \mathbb{Q})$ [5]. This remarkable conjecture organizes the higher genus GW-invariants in terms of the genus-0 data and the $\tau$-function for a point. The motivation for our work lies in verifying the conjecture for $X = \mathbb{P}^1$, which is the simplest example with a semi-simple Frobenius structure on its cohomology ring and whose GW-invariants can be easily computed.

We have obtained two particular solutions to the flat-section equations (1), an analytic one encoding the two-point descendant GW-invariants of $\mathbb{P}^1$ and a recursive one corresponding to Givental’s fundamental solution. According to Givental, both of these two solutions are supposed to yield the same data $V_{kl}^{ij}, \Delta_j,$ and $T^j_n$. Unfortunately, we were not able to produce the desired information using our analytic solutions, but the recursive solutions do lead to sensible quantities which we need. Combined with an expansion scheme which allows us to verify the conjecture at each order in $\lambda$, we thus use our recursive solutions to check the conjecture (1) for $\mathbb{P}^1$ up to order $\lambda^2$. Already at this order, we need to expand the differential operators in (1) up to $\lambda^6$ and need to consider up to genus-3 free energy in the $\tau$-functions, and the computations quickly become cumbersome with increasing order. We have managed to re-express the conjecture for this case into a form which resembles the Hirota-bilinear relations, but at this point, we have no insights into a general proof. It is nevertheless curious how the numbers work out, and we hope that our results would provide a humble support for Givental’s master equation.

Many confusions still remain – for instance, the discrepancy between our analytic and recursive solutions. As mentioned above, Givental’s conjecture for $\mathbb{P}^1$ can be re-written in a form which resembles the Hirota-bilinear relations for the KdV hierarchies (see [52]). It would thus be interesting to speculate a possible relation between his conjecture and the conjectural Toda hierarchy for $\mathbb{P}^1$.

We have organized our notes as follows: in §2 we review the canonical coordinates for $\mathbb{P}^1$, to be followed by our solutions to the flat-section equations in §3. We conclude by presenting our checks in §4.

2 Canonical Coordinates for $\mathbb{P}^1$.

We here review the canonical coordinates $\{u_\pm\}$ for $\mathbb{P}^1$ [1, 2, 4]. Recall that a Frobenius structure on $H^*(\mathbb{P}^1, \mathbb{Q})$ carries a flat pseudo-Riemannian metric $\langle \cdot, \cdot \rangle$ defined by the Poincaré intersection pairing.
The canonical coordinates are defined by the property that they form the basis of idempotents of the quantum cup-product, denoted in the present note by $\circ$. The flat metric $\langle \cdot, \cdot \rangle$ is diagonal in the canonical coordinates, and following Givental’s notation, we define $\Delta_{\pm} := 1/\langle \partial_{u_{\pm}}, \partial_{u_{\pm}} \rangle$.

Let $\{t^\alpha\}, \alpha \in \{0, 1\}$ be the flat coordinates of the metric and let $\partial_{\alpha} := \partial/\partial t^\alpha$. The quantum cohomology of $\mathbb{P}^1$ is

$$\partial_0 \circ \partial_0 = \partial_0 \quad \text{and} \quad \partial_1 \circ \partial_1 = e^{t_1} \partial_0.$$ 

The eigenvalues and eigenvectors of $\partial_1 \circ$ are

$$\pm e^{t_1/2} \quad \text{and} \quad (\pm e^{t_1/4} \partial_0 + e^{-t_1/4} \partial_1),$$

respectively. So, we have

$$(\pm e^{t_1/4} \partial_0 + e^{-t_1/4} \partial_1) \circ (\pm e^{t_1/4} \partial_0 + e^{-t_1/4} \partial_1) = \pm 2 e^{t_1/4} (\pm e^{t_1/4} \partial_0 + e^{-t_1/4} \partial_1),$$

which implies that

$$\frac{\partial}{\partial u_{\pm}} = \frac{\partial_0 \pm e^{-t_1/2} \partial_1}{2},$$

such that

$$\partial_{u_{\pm}} \circ \partial_{u_{\pm}} = \partial_{u_{\pm}} \quad \text{and} \quad \partial_{u_{\pm}} \circ \partial_{u_{\mp}} = 0.$$ 

We can solve for $u_{\pm}$ up to constants as

$$u_{\pm} = t^0 \pm 2 e^{t_1/2}. \quad (2)$$

To compute $\Delta_{\pm}$, note that

$$\frac{1}{\Delta_{\pm}} := \langle \partial_{u_{\pm}}, \partial_{u_{\pm}} \rangle = \pm \frac{1}{2 e^{t_1/2}}.$$ 

The two bases are related by

$$\partial_0 = \partial_{u_+} + \partial_{u_-} \quad \text{and} \quad \partial_1 = e^{t_1/2} (\partial_{u_+} - \partial_{u_-}).$$ 

Define an orthonormal basis by $f_i = \Delta_{i/2} \partial/\partial u_{\alpha}$. Then the transition matrix $\Psi$ from $\{\partial/\partial t^\alpha\}$ to $\{f_i\}$ is given by

$$\Psi^i_\alpha = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-t_1/4} & -i e^{-t_1/4} \\ i e^{t_1/4} & i e^{t_1/4} \end{pmatrix} = \begin{pmatrix} \Delta_{+}^{1/2} & \Delta_{-}^{1/2} \\ \frac{1}{2} \Delta_{+}^{1/2} & \frac{1}{2} \Delta_{-}^{1/2} \end{pmatrix}, \quad (3)$$

such that

$$\frac{\partial}{\partial t^\alpha} = \sum_i \Psi^i_\alpha f_i.$$ 

We will also need the inverse of (3):

$$\Psi^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{t_1/4} & e^{-t_1/4} \\ i e^{t_1/4} & -i e^{-t_1/4} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \Delta_{+}^{1/2} & \Delta_{-}^{1/2} \\ \frac{1}{2} \Delta_{+}^{1/2} & -\Delta_{-}^{1/2} \end{pmatrix}. \quad (4)$$
3 Solutions to the Flat-Section Equations

The relevant data $V_{ij}^{kl}, \Delta_j$ and $T_{il}^j$ are extracted from the solutions to the flat-section equations of the genus-0 Frobenius structure for $H^*(\mathbb{P}^1, \mathbb{Q})$. We here find two particular solutions. The analytic solution correctly encodes the two-point descendant GW-invariants, while the recursive solution is used in the next section to verify Givental’s conjecture.

3.1 Analytic Solution

The genus-0 free energy for $\mathbb{P}^1$ is

$$F_0 = \frac{1}{2}(t^0)^2 t^1 + e t^1.$$

Flat sections $S_\alpha$ of $TH^*(\mathbb{P}^1, \mathbb{Q})$ satisfy the equations

$$z \partial_\alpha S_\beta = F_{\alpha \beta \mu} g^{\mu \nu} S_\nu ,$$

(5)

where $z \neq 0$ is an arbitrary parameter and $F_{\alpha \beta \mu} := \partial^3 F/\partial t^\alpha \partial t^\beta \partial t^\mu$. The only non-vanishing components of $F_{\alpha \beta \mu}$ are $F_{001} = 1$ and $F_{111} = e t^1$. Hence, we find that the general solutions to the flat-section equations (5) are

$$S_0 = e^{t_0/z} \left[ c_1 I_0(2 e^{t_1/2}/z) - c_2 K_0(2 e^{t_1/2}/z) \right]$$

and

$$S_1 = e^{t_0/z} e^{t_1/2} \left[ c_1 I_1(2 e^{t_1/2}/z) + c_2 K_1(2 e^{t_1/2}/z) \right],$$

(6)

where $I_n(x)$ and $K_n(x)$ are modified Bessel functions, and $c_i$ are integration constants which may depend on $z$.

We would now like to find two particular solutions corresponding to the following Givental’s expression:

$$S_{\alpha\beta}(z) = g_{\alpha\beta} + \sum_{n \geq 0, (n,d) \neq (0,0)} \frac{1}{n!} \langle \phi_{\alpha} \cdot \phi_{\beta} \rangle (t_0^0 t_0^1 + t^1_0 + t^1_1)^n \rangle_d ,$$

(7)

where $S_{\alpha\beta}$ denotes the $\alpha$-th component of the $\beta$-th solution. Here, $\{\phi_{\alpha}\}$ is a homogeneous basis of $H^*(\mathbb{P}^1, \mathbb{Q})$, $g_{\alpha\beta}$ is the intersection paring $\int_{\mathbb{P}^1} \phi_{\alpha} \cup \phi_{\beta}$ and $\psi \in H^2(M_{0,n+2}(\mathbb{P}^1,d), \mathbb{Q})$ is the first Chern class of the universal cotangent line bundle over the moduli space $M_{0,n+2}(\mathbb{P}^1,d)$. In order to find the particular solutions, we compare our general solution (6) with the 0-th components of $S_{0\beta}$ in (7) at the origin of the phase space. The two-point functions appearing in (7) have been computed at the origin in (6) and have the following forms:

$$S_{00}|_{t^0 = 0} = - \sum_{m=1}^{\infty} \frac{1}{z^{2m+1}} \frac{2 d_m}{(m!)^2} , \text{ where } d_m = \sum_{k=1}^{m} 1/k ,$$

(8)

and

$$S_{01}|_{t^0 = 0} = 1 + \sum_{m=1}^{\infty} \frac{1}{z^{2m}} \frac{1}{(m!)^2}.$$
Using the standard expansion of the modified Bessel function $K_0$, we can evaluate (6) at the origin of the phase space to be

\[ c_1 I_0 \left( \frac{2}{z} \right) - c_2 K_0 \left( \frac{2}{z} \right) = c_1 I_0 \left( \frac{2}{z} \right) - c_2 \left[ - (\log(z) + \gamma_E) I_0 \left( \frac{2}{z} \right) + \sum_{m=1}^{\infty} \frac{e_m}{z^{2m}(m!)^2} \right], \]

where $\gamma_E$ is Euler's constant. Now matching (10) with (8) gives

\[ c_1 = -c_2 \log(1/z) - c_2 \gamma_E \quad \text{and} \quad c_2 = \frac{2}{z}, \]

while noticing that (9) is precisely the expansion of $I_0(2/z)$ and demanding that our general solution coincides with (9) at the origin yields

\[ c_1 = 1 \quad \text{and} \quad c_2 = 0. \]

To recapitulate, we have found

\[
S_{00} = -\frac{2e^{t_0/Z}}{z} \left[ (\gamma_E - \log(z)) I_0 \left( \frac{2e^{t_1/2}}{z} \right) + K_0 \left( \frac{2e^{t_1/2}}{z} \right) \right],
\]

\[
S_{10} = \frac{2e^{t_0/Z}e^{t_1/2}}{z} \left[ K_1 \left( \frac{2e^{t_1/2}}{z} \right) - (\gamma_E - \log(z)) I_1 \left( \frac{2e^{t_1/2}}{z} \right) \right],
\]

\[
S_{01} = e^{t_0/Z} I_0 \left( \frac{2e^{t_1/2}}{z} \right),
\]

\[
S_{11} = e^{t_0/Z} e^{t_1/2} I_1 \left( \frac{2e^{t_1/2}}{z} \right).
\]

We have checked that these solutions correctly reproduce the corresponding descendant Gromov-Witten invariants obtained in [6].

If the inverse transition matrix in (4) is used to relate the matrix elements $S_{\alpha \beta}^i$ to $S_{\alpha \beta}$ as $S_{\alpha \beta}^i = S_{\alpha \beta}(\psi^{-1})_j^i \delta^{ji}$, then we should have

\[
S_{\alpha}^{\pm} = \sqrt{\pm 2} e^{t_1/4} \left( \frac{1}{2} S_{00} \pm e^{-t_1/2} S_{01} \right). \tag{11}
\]

### 3.2 Recursive Solution

In [4, 5], Givental has shown that near a semi-simple point, the flat-section equations (5) have a fundamental solution given by

\[
S_{\alpha}^i = \Psi_{\alpha}^j \left( (R_0 + zR_1 + z^2R_2 + \cdots + z^n R_n + \cdots)_{jk} [\exp(U/z)]^{ki} \right),
\]

where $R_n = (R_n)_{jk}$, $R_0 = \delta_{jk}$ and $U$ is the diagonal matrix of canonical coordinates. The matrix $R_1$ satisfies the relations

\[
\Psi^{-1} \frac{\partial \Psi}{\partial t^1} = [\frac{\partial U}{\partial t^1}, R_1]. \tag{12}
\]
and
\[ \left[ \frac{\partial R_1}{\partial t^1} + \Psi^{-1} \left( \frac{\partial \Psi}{\partial t^1} \right) R_1 \right]_{\pm\pm} = 0, \tag{13} \]
which we use to find its expression. From the transition matrix given in (3) we see that
\[ \Psi^{-1} \frac{\partial \Psi}{\partial t^1} = \frac{1}{4} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \]
while taking the (+−) component of the relation (12) gives
\[ \frac{i}{4} = \frac{\partial U_{++}}{\partial t^1} (R_1)_{++} - (R_1)_{+-} \frac{\partial U_{--}}{\partial t^1} = 2e^{t^1/2}(R_1)_{+-}, \]
where in the last step we have used the definition (2) of canonical coordinates. We therefore have
\[ (R_1)_{+-} = \frac{i}{8} e^{-t^1/2}, \]
and similarly considering the (−+) component of (12) gives
\[ (R_1)_{-+} = \frac{i}{8} e^{-t^1/2}. \]
The diagonal components of \( R_1 \) can be obtained from (13), which implies that
\[ \frac{\partial (R_1)_{++}}{\partial t^1} = (R_1)_{+-} \frac{\partial U_{--}}{\partial t^1} (R_1)_{++} - (R_1)_{-+} \frac{\partial U_{--}}{\partial t^1} (R_1)_{+-} = \frac{\exp(-t^1/2)}{32} = -\frac{\partial (R_1)_{--}}{\partial t^1}. \]
Hence, \( (R_1)_{++} = -\exp(-t^1/2)/16 = -(R_1)_{--} \) and the matrix \( R_1 \) can be written as
\[ (R_1)_{jk} = \frac{1}{16} e^{-t^1/2} \begin{pmatrix} -1 & 2i \\ 2i & 1 \end{pmatrix}. \tag{14} \]
In general, the matrices \( R_n \) satisfy the recursion relations (4)
\[ \left( d + \Psi^{-1}d\Psi \right) R_n = [dU, R_{n+1}], \]
which, for our case, imply the following set of equations:
\[ \frac{\partial R_n}{\partial t^0} = 0, \tag{15} \]
\[ \frac{\partial (R_n)_{++}}{\partial t^1} = -\frac{i}{4} (R_n)_{++}, \tag{16} \]
\[ (R_{n+1})_{--} = \frac{1}{2} e^{-t^1/2} \left[ \frac{\partial (R_n)_{--}}{\partial t^1} + \frac{i}{4} (R_n)_{++} \right], \tag{17} \]
\[ \frac{\partial (R_n)_{--}}{\partial t^1} = \frac{i}{4} (R_n)_{+-}, \tag{18} \]
\[ (R_{n+1})_{+-} = \frac{1}{2} e^{-t^1/2} \left[ \frac{\partial (R_n)_{+-}}{\partial t^1} + \frac{i}{4} (R_n)_{--} \right]. \tag{19} \]
Lemma 3.1 For $n \geq 1$, the matrices $R_n$ in the fundamental solution are given by

$$ \left( R_n \right)_{ij} = \frac{(-1)^n}{(2n-1)} \frac{\alpha_n}{2^n} e^{-nt^2/2} \left( \begin{array}{cc} -1 & (-1)^{n+1} 2ni \\ 2ni & (-1)^n \end{array} \right), \tag{20} $$

where

$$ \alpha_n = (-1)^n \frac{1}{8^n n!} \prod_{\ell=1}^{n} (2\ell - 1)^2, \quad \alpha_0 = 1. $$

These solutions satisfy the unitarity condition

$$ R(z)R^t(-z) := (1 + zR_1 + z^2R_2 + \cdots + z^nR_n + \cdots)(1 - zR_1^t + z^2R_2^t + \cdots + (-1)^n z^nR_n^t + \cdots) = 1 $$

and the homogeneity condition and, thus, are unique.

Proof: For $n = 1$, $\alpha_1 = -1/8$ and (20) is equal to the correct solution (14). The proof now follows by an induction on $n$. Assume that (20) holds true up to and including $n = m$. Using the fact that

$$ \alpha_{m+1} = \frac{-(2m+1)^2}{8(m+1)} \alpha_m, $$

we can show that $R_{m+1}$ in (20) satisfies the relations (16)–(19) as well as (15).

To check unitarity, consider the $z^k$-term $P_k := \sum_{\ell=0}^{k} (-1)^{\ell} R_{k-\ell} R_{\ell}^t$ in $R(z)R^t(-z) = \sum_{k=0} P_k z^k$. As shown by Givental, the equations satisfied by the matrices $R_n$ imply that the off-diagonal entries of $P_k$ vanish. As a result, combined with the anti-symmetry of $P_k$ for odd $k$, we see that $P_k$ vanishes for all odd $k$. Hence, we only need to show that for our solution, $P_k$ vanishes for all positive even $k$ as well. To this end, we note that Givental has also deduced from the equation $dP_k + [\Psi^{-1}d\Psi, P_k] = [dU, P_{k+1}]$ that the diagonal entries of $P_k$ are constant. The expansion of $P_{2k}$ is

$$ P_{2k} = R_{2k} + R_{2k}^t + \cdots, $$

where the remaining terms are products of $R_{\ell}$, for $\ell < 2k$. Now, we proceed inductively. We first note that $R_1$ and $R_2$ given in (20) satisfy the condition $P_2 = 0$, and assume that $R_{\ell}$’s in (20) for $\ell < 2k$ satisfy $P_\ell = 0$. Then, since the off-diagonal entries of $P_n$ vanish for all $n$, the expansion of $P_{2k}$ is of the form

$$ P_{2k} = Ae^{-2k t^2/2} + B, $$

where $A$ is a constant diagonal matrix resulting from substituting our solution (20) and $B$ is a possible diagonal matrix of integration constants for $R_{2k}$. But, since the diagonal entries of $P_n$ are constant for all $n$, we know that $A = 0$. We finally choose the integration constants to be zero so that $B = 0$, yielding $P_{2k} = 0$. Hence, the matrices in our solution (20) satisfy the unitarity condition and are manifestly homogeneous. It then follows by the proposition in [5] that our solutions $R_n$ are unique.

Let $R := (R_0 + zR_1 + z^2R_2 + \cdots + z^nR_n + \cdots)$. Then, we can use the matrices $R_n$ from Lemma 3.1 to find

$$ S_0^+ = (R_{++} - i R_{+-}) \exp(u_+/z) \sqrt{\Delta_+}. $$
If we define

\[ S_i = \left( R_{ii} + i R_{i+} \right) \frac{\exp(u_+/z)}{\sqrt{\Delta_+}} \]

then after some algebraic manipulations we obtain

\[ S_0^- = (R_{--} + i R_{-+}) \frac{\exp(u_-/z)}{\sqrt{\Delta_-}} \]

\[ S_1^+ = (R_{++} + i R_{+-}) \frac{\sqrt{\Delta_+}}{2} \exp(u_+/z) \]

\[ S_1^- = (R_{--} - i R_{-+}) \frac{\sqrt{\Delta_-}}{2} \exp(u_-/z) \]

Using the above expressions for \( S_\alpha^i(z) \), we can also find \( V^{ij}(z, w) \), which is given by the expression

\[ V^{ij}(z, w) := \frac{1}{z + w} \left[ S_\mu^i(w) \right]^I \left[ g^{\mu\nu} \right] \left( S_\nu^j(z) \right). \]

If we define

\[ A_{p,q} := \frac{(4pq - 1)}{(2p - 1)(2q - 1)} \frac{\alpha_p \alpha_q \exp(-(\nu \nu))}{2^{p+q}} \]

and

\[ B_{p,q} := \frac{2(p - q)}{(2p - 1)(2q - 1)} \frac{\alpha_p \alpha_q \exp(-(\nu \nu))}{2^{p+q}}, \]

then after some algebraic manipulations we obtain

\[ V^{++}(z, w) = e^{u_+/w+u_+/z} \left\{ \frac{1}{z + w} + \sum_{k,l=0}^{\infty} \left[ \sum_{n=0}^{k} (-1)^n A_{l+n+1,k-n} \right] (-1)^{k+l} w^{k} z^{l} \right\}, \]

\[ V^{--}(z, w) = e^{u_-/w+u_-/z} \left\{ \frac{1}{z + w} - \sum_{k,l=0}^{\infty} \left[ (-1)^{k+l} \sum_{n=0}^{k} (-1)^n A_{l+n+1,k-n} \right] (-1)^{k+l} w^{k} z^{l} \right\}, \]

\[ V^{+-}(z, w) = e^{u_+/w+u_-/z} \left\{ \sum_{k,l=0}^{\infty} \left[ i (-1)^l \sum_{n=0}^{k} B_{l+n+1,k-n} \right] (-1)^{k+l} w^{k} z^{l} \right\}, \]

\[ V^{-+}(z, w) = e^{u_-/w+u_+/z} \left\{ \sum_{k,l=0}^{\infty} \left[ i (-1)^k \sum_{n=0}^{k} B_{l+n+1,k-n} \right] (-1)^{k+l} w^{k} z^{l} \right\}. \]

### 3.3 A Puzzle

Incidentally, we note that in the asymptotic limit \( z \to 0 \),

\[ S_0^+ = \Re \left[ \frac{2\pi}{z} e^{v/z} I_0 \left( \frac{2e^{v/2}}{z} \right) \right] \]
and

\[ S_0^- = -i \sqrt{\frac{2}{\pi z}} e^{z/2} K_0 \left( \frac{2e^{z/2}}{z} \right) \]

reproduce the expansions in (21) and (22). This is in contrast to what was expected from the discussion leading to (11). Despaired of matching the two expressions, it seems to us that the analytic correlation functions obtained in §3.1 do not encode the right information that appear in Givental’s conjecture. In the following section, we will use the recursive solutions from §3.2 to check Givental’s conjectural formula at low genera.

4 Checks of the Conjecture at Low Genera

The \( T_n^i \) that appear in Givental’s formula (1) are defined by the equations [5]

\[ S_0^\pm := \left[ 1 - \sum_{n=0}^{\infty} T_n^\pm (-z)^n \right] \exp(u_{\pm}/z) \sqrt{\Delta x}. \]

From the computations of \( S_0^+ \) and \( S_0^- \) in (21) and (22), respectively, one can extract \( T_n^i \) to be

\[
T_n^+ = \begin{cases} 
0, & n = 0, 1, \\
-\frac{\alpha_{n-1}}{2n-1} \exp \left[ \frac{-(n-1)t^1}{2} \right], & n \geq 2,
\end{cases}
\]

\[
T_n^- = \begin{cases} 
0, & n = 0, 1, \\
(-1)^{n-1} \frac{\alpha_{n-1}}{2n-1} \exp \left[ \frac{-(n-1)t^1}{2} \right], & n \geq 2.
\end{cases}
\]

Notice that

\[ T_n^- = (-1)^{n-1} T_n^+ . \tag{27} \]

The functions \( V_{kl}^{ij} \) are defined [1] by the expansion [5]

\[ V^{ij}(z,w) = e^{u_{i}/w+u_{j}/z} \left[ \frac{\delta^{ij}}{z+w} + \sum_{k,l=0}^{\infty} (-1)^{k+l} V_{kl}^{ij} w^k z^l \right], \]

and from (25) and (26) we see that

\[
V_{kl}^{++} = \sum_{n=0}^{k} (-1)^n A_{l+n+1,k-n} = \sum_{n=0}^{k} \frac{(-1)^n(4(l + n + 1)(k - n) - 1)}{(2l + 2n + 1)(2k - 2n - 1)} T_{l+n+2}^+ T_{k-n+1}^+ , \]

\[
V_{kl}^{+-} = i(-1)^l \sum_{n=0}^{k} B_{l+n+1,k-n} = i(-1)^l \sum_{n=0}^{k} \frac{2(l + 2n + 1 - k)}{(2l + 2n + 1)(2k - 2n - 1)} T_{l+n+2}^+ T_{k-n+1}^+ .
\]

Now, the \( \tau \)-function for the intersection theory on the Deligne-Mumford moduli space \( \overline{M}_{g,n} \) of stable curves is defined by

\[ \tau(\lambda; \{ q_k \}) = \exp \left( \sum_{g=0}^{\infty} \lambda^{g-1} F^g_{\overline{M}}(\{ q_k \}) \right) \]

1There seems to be a misprint in the original formula for \( V_{kl}^{ij} \) in [5], i.e. we believe that \( w \) and \( z \) should be exchanged, as in our expression here.
and has the following nice scaling invariance: consider the scaling of the phase-space variables $q_k$ given by
\[ q_k \mapsto s^{k-1} q_k \] (28)
for some constant $s$. Then, since a non-vanishing intersection number $\langle \tau_{k_1} \cdots \tau_{k_n} \rangle$ must satisfy
\[ \sum_{i=1}^n (k_i - 1) = \text{dim}(\mathcal{M}_{g,n}) - n = 3g - 3, \]
we see that under the transformation (28), the genus-$g$ generating function $\mathcal{F}_g^{\nu \tau}$ must behave as
\[ \mathcal{F}_g^{\nu \tau}(\{ s^{k-1} q_k \}) = (s^3)^{g-1} \mathcal{F}_g^{\nu \tau}(\{ q_k \}). \]
Hence, upon scaling the “string coupling constant” $\lambda$ to $s^{-3} \lambda$, we see that
\[ \tau(s^{-3} \lambda; \{ s^{k-1} q_k \}) = \tau(\lambda; \{ q_k \}). \] (29)

Now, consider the function
\[ F(\{ q_n^+ \}, \{ q_n^- \}) := \left[ e^{\frac{1}{2} \sum_{k,t \geq 0} \sum_{i,j(\pm)} V_k^{ij} \sqrt{\Delta_i} \sqrt{\Delta_j} \frac{\partial}{\partial q_k^i} \frac{\partial}{\partial q_t^j} \tau(\lambda \Delta_+; \{ q_n^+ \}) \tau(\lambda \Delta_-; \{ q_n^- \})} \right]. \] (30)

Then, since the Gromov-Witten potentials of $\mathbb{P}^1$ for $g \geq 2$ all vanish, Givental’s conjectural formula for $\mathbb{P}^1$ is
\[ F(\{ T_n^+ \}, \{ T_n^- \}) = 1, \]
where it is understood that one sets $q_k^i = T_k^i$ after taking the derivatives with respect to $T_k^i$. Since $T_n^+$ and $T_n^-$ are related by (27), let us rescale $q_n^- \mapsto (-1)^{k-1} q_n^-$ in (30). Then, since $\Delta_+ = -\Delta_-$, we observe from (29) that
\[ F(\{ T_n^+ \}, \{ T_n^- \}) = \left\{ \exp \left[ \frac{\lambda}{2} \Delta_+ \sum_{k,t \geq 0} \left( V_k^{++} \frac{\partial}{\partial q_k^i} \frac{\partial}{\partial q_t^j} + i(-1)^{t-1} V_k^{++} \frac{\partial}{\partial q_k^i} \frac{\partial}{\partial q_t^j} \right) \right] \tau(\lambda \Delta_+; \{ q_n^+ \}) \tau(\lambda \Delta_-; \{ q_n^- \}) \right\}_{q_n^+,q_n^- = T_n^+}. \]

But, the $V_k^{ij}$ satisfy the relations $V_k^{+-} = -(-1)^{k+l} V_k^{++}$ and $V_k^{-+} = V_k^{--}$, so
\[ F(\{ T_n^+ \}, \{ T_n^- \}) = \left\{ \exp \left[ \frac{\lambda}{2} \Delta_+ \sum_{k,t \geq 0} \left( V_k^{++} \frac{\partial}{\partial q_k^i} \frac{\partial}{\partial q_t^j} + \frac{\partial}{\partial q_k^i} \frac{\partial}{\partial q_t^j} \right) \right] \tau(\lambda \Delta_+; \{ q_n^+ \}) \tau(\lambda \Delta_-; \{ q_n^- \}) \right\}_{q_n^+,q_n^- = T_n^+}. \] (31)

Now, consider the following transformations of the variables:
\[ q_k^+ = x_k + y_k \quad \text{and} \quad q_k^- = x_k - y_k \]
so that

\[ \frac{\partial q_k}{\partial x} = \frac{1}{2} (\partial x_k + \partial y_k) \quad \text{and} \quad \frac{\partial q_k}{\partial y} = \frac{1}{2} (\partial x_k - \partial y_k). \]

Then, in these new coordinates, (31) becomes

\[ \mathcal{F}(\{T^+_n\}, \{T^-_n\}) = \mathcal{G}(\{T^+_n\}, \{0\}), \]

where the new function \( \mathcal{G}(\{x_k\}, \{y_k\}) \) is defined\(^2\) by

\[ \mathcal{G}(\{x_n\}, \{y_n\}) = \exp \left[ \frac{\lambda}{4} \Delta_+ \sum_{k,l \geq 0} (V_{kl} \partial x_k \partial x_l + W_{kl} \partial y_k \partial y_l) \right] \tau(\lambda \Delta_+; \{x_n + y_n\}) \tau(\lambda \Delta_+; \{x_n - y_n\}), \tag{32} \]

where

\[ V_{kl} := V^{++}_{kl} + i(-1)^{l-1} V^{-+}_{kl}, \]
\[ W_{kl} := V^{++}_{kl} - i(-1)^{l-1} V^{-+}_{kl}. \]

**Remark:** The conjecture expressed in terms of (32), i.e. that \( \mathcal{G}(\{T^+_k\}, \{0\}) = 1 \), is now in a form which resembles the Hirota bilinear relations, which might be indicating some kind of an integrable hierarchy, perhaps of Toda-type.

Because the tau-functions are exponential functions, upon acting on them by the differential operators, we can factor them out in the expression of \( \mathcal{G}(\{x_k\}, \{y_k\}) \). We thus define

**Definition 4.1** \( \mathcal{P}(\lambda \Delta_+, \{x_k\}, \{y_k\}) \) is a formal power series in the variables \( \lambda \Delta_+, \{x_k\} \) and \( \{y_k\} \) such that

\[ \mathcal{G}(\{x_k\}, \{y_k\}) = \mathcal{P}(\lambda \Delta_+, \{x_k\}, \{y_k\}) \tau(\lambda \Delta_+; \{x_n + y_n\}) \tau(\lambda \Delta_+; \{x_n - y_n\}). \]

Hence, Givental’s conjecture for \( \mathbb{P}^1 \) can be restated as

**Conjecture 4.2 (Givental)** The generating function \( \mathcal{G}(\{T^+_k\}, \{0\}) = 1 \), or equivalently

\[ \mathcal{P}(\lambda \Delta_+, \{T^+_k\}, \{0\}) = \frac{1}{\tau(\lambda \Delta_+, \{T^+_k\})^2}. \tag{33} \]

This conjecture can be verified order by order\(^3\) in \( \lambda \).

Let us check (33) up to order \( \lambda^2 \), for which we need to consider up to \( \lambda^6 \) expansions in the differential operators acting on the \( \tau \)-functions. Let \( \hbar = \lambda \Delta_+ \). The low-genus free energies for a point target space can be easily computed using the KdV hierarchy and topological axioms; they can also be verified using Faber’s program\(^\text{[3]}\). The terms relevant to our computation are

\(^2\)We have simplified the expression by noting that the mixed derivative terms cancel because of the identity \( V^{++}_{kl} = (-1)^{k-l} V^{+-}_{kl} \).

\(^3\)This procedure is possible because when \( q_0 = q_1 = 0 \), only a finite number of terms in the free-energies and their derivatives are non-vanishing. In particular, the genus-0 and genus-1 free energies vanish when \( q_0 = q_1 = 0 \).
\[ \frac{F_0^{\text{pt}}}{h} + F_1^{\text{pt}} + h F_2^{\text{pt}} = \frac{1}{h} \left[ \frac{(q_0)^3}{3!} + \frac{(q_0)^3 q_1}{3!} + \frac{2! (q_0)^3 (q_1)^2}{3! 2!} + \frac{3! (q_0)^3 (q_1)^3}{3! 3!} + \frac{(q_0)^4 q_2}{4!} + \frac{3 (q_0)^4 q_1 q_2}{4!} + \cdots \right] + \frac{12 (q_0)^4 (q_1)^2 q_2}{4! 2!} + \frac{(q_0)^5 q_3}{5!} + \frac{4 (q_0)^5 q_1 q_3}{5!} + \frac{6 (q_0)^5 (q_2)^2}{5! 2!} + \frac{30 (q_0)^5 q_1 (q_2)^2}{5! 2!} + \cdots \]

This expression gives the necessary expansion of \( \tau(\lambda \Delta_+; \{ x_k \pm y_k \}) \) for our consideration, and upon evaluating \( G(\{ T^+_k \}, \{ 0 \}) \), we find

\[
P(h, \{ T^+_k \}, \{ 0 \}) = 1 - \frac{17}{2359296} e^{-3t^1/2h} + \frac{41045}{695784701952} e^{-3t^1 h^2} + O(h^3). \quad (34)
\]

At this order, the expansion of the right-hand-side of \( \tau \) is

\[
\tau(h, \{ T^+_k \})^{-2} = 1 - 2 F_2^{\text{pt}} h + 2 \left[ (F_2^{\text{pt}})^2 - F_3^{\text{pt}} \right] h^2 + O(h^3).
\]

At \( q_n = T_n^+ \), \( \forall n \), the genus-2 free energy is precisely given by

\[
F_2^{\text{pt}} = \frac{1}{1152} T_4 + \frac{29}{5760} T_3 T_2 + \frac{7}{240} T_2^3 = 17 \frac{1}{4718592} e^{-3t^1/2} h^2
\]

and the genus-3 free energy is

\[
F_3^{\text{pt}} = \frac{1}{82944} T_7 + \frac{77}{414720} T_2 T_6 + \frac{503}{1451520} T_3 T_5 + \frac{17}{11520} (T_2)^2 T_5 + \frac{607}{2903040} (T_4)^2
\]
\[
\begin{align*}
&+ \frac{1121}{241920} T_2 T_3 T_4 + \frac{53}{6912} (T_2)^3 T_4 + \frac{583}{580608} (T_3)^3 + \frac{205}{13824} (T_2)^2 (T_3)^2 \\
&+ \frac{193}{6912} (T_2)^4 T_3 + \frac{245}{20736} (T_2)^6 \\
&= -\frac{656431}{2265110462464} e^{-3t_1}.
\end{align*}
\]

Thus, we have
\[
\tau(h, \{T_k^+\})^{-2} = 1 - \frac{17}{2359296} e^{-3t_1/2} h + \frac{41045}{695784701952} e^{-3t_1} h^2 + \mathcal{O}(h^3),
\]
which agrees with our computation of \(P(\lambda, \{T_k^+\}, \{0\})\) in (33).

It would be very interesting if one could actually prove Givental’s conjecture, but even our particular example remains elusive and verifying its validity to all orders seems intractable using our method.

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