PSEUDODIFFERENTIAL OPERATORS WITH ROUGH SYMBOLS

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Abstract. In this work, we develop \(L^p\) boundedness theory for pseudodifferential operators with rough (not even continuous in general) symbols in the \(x\) variable. Moreover, the \(B(L^p)\) operator norms are estimated explicitly in terms of scale invariant quantities involving the symbols. All the estimates are shown to be sharp with respect to the required smoothness in the \(\xi\) variable. As a corollary, we obtain \(L^p\) bounds for (smoothed out versions of) the maximal directional Hilbert transform and the Carleson operator.

1. Introduction

In this paper, we are concerned with the \(L^p\) mapping properties of the pseudodifferential operators in the form

\[
T_\sigma f(x) = \int_{\mathbb{R}^n} \sigma(x, \xi) e^{2\pi i \xi x} \hat{f}(\xi) d\xi.
\]

The operators \(T_\sigma\) have been subject of continuous interest since the sixties. We should mention that their usefulness in the study of partial differential equations have been realized much earlier, but it seems that their systematic study began with the fundamental works of Kohn and Nirenberg, [10] and Hörmander, [9].

To describe the results obtained in these early papers, define the Hörmander’s class \(S^m\), which consists of all functions \(\sigma(x, \xi)\), so that

\[
|D_\xi^\beta D_x^\alpha \sigma(x, \xi)| \leq C_{\alpha, \beta}(1 + |\xi|)^{m-|\alpha|},
\]

for all multiindices \(\alpha, \beta\). A classical theorem in [9] then states that \(Op(\sigma) : H^{s+m,p} \to H^{s,p}\) for all \(s \geq 0\) and \(1 < p < \infty\). In particular, \(Op(\sigma) : L^p \to L^p\), \(1 < p < \infty\), whenever the symbol \(\sigma \in S^m\). Subsequent improvements of these methods established the boundedness of \(Op(\sigma)\) (basically under the assumption \(\sigma \in S^m\) for appropriate \(m\)) to various related function spaces, like Besov, Triebel-Lizorkin spaces to name a few, but we will not review those here, since they fall outside of the scope of this paper.

It is worth mentioning however, that the simple to verify condition (2) is the one arising in many applications. The \(L^2\) boundedness plays special role in the theory and that is why we discuss it separately.

The class of symbols \(S_{\rho, \delta}^m\), defined via

\[
|D_\xi^\beta D_x^\alpha \sigma(x, \xi)| \leq C_{\alpha, \beta}(1 + |\xi|)^{m-\rho|\alpha|+\delta|\beta|},
\]

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\]
represents a larger set of symbols than $S^m = S^m_{1,0}$, which has subsequently found applications in local solvability for linear PDE’s.\[1\]

Here, we have to mention the celebrated result of Calderón-Vaillancourt, \[3\], \[4\] which states that $L^2$ boundedness for $T_{\sigma}$ holds, whenever $\sigma \in S^0_{\rho,\rho}$, $0 \leq \rho < 1$, whereas $S^0_{1,1}$ is a “forbidden” class, in the sense that there are symbols in that class, which give rise to unbounded on $L^2$ operators. We should mention here the work of Cordes \[6\], who improved the result for $S^0_{0,0}$ by requiring that $3$ holds only for $|\alpha|, |\beta| \leq [n/2] + 1$.

Regarding less regular in $x$ symbols, for any modulus of continuity $\omega : R_+ \to R_+$ (that is, an increasing and continuous function), define the space $C^\omega$ of all uniformly continuous and bounded functions $u : R^n \to C$, satisfying

$$|u(x + y) - u(x)| \leq \omega(|y|).$$

The following class of symbols was introduced and studied by Coifman-Meyer, \[5\]. More precisely, let $\sigma(x, \xi) \in C^\omega S^0_{1,0} \cap C^\omega S^0_{1,0}$, which means it satisfies

$$\sup_{x,y,\xi} < |\xi| > |\alpha| |D_\xi^\alpha \sigma(x + y, \xi) - \sigma(x, \xi)| \leq C_\omega \omega(|y|)$$

and assume that $\sum_{j=0}^\infty \omega(2^{-j})^2 < \infty$. Then for all $1 < p < \infty$, $Op(\sigma) : L^p \to L^p$. The condition $\sum_{j=0}^\infty \omega(2^{-j})^2 < \infty$ is clearly very mild continuity assumption for the function $x \to \sigma(x, \xi)$. In particular, one sees that $\cup_{\gamma \geq 0} C^\gamma S^0 \subset C^\omega S^0_{1,0}$. Related results can be found in the work of M. Taylor, \[23\] (see Proposition 2.4, p. 23) and J. Marschall, \[11\], where the spaces $C^\omega$ are replaced by $H^{\varepsilon,p}$ spaces with $p$ as large as one wish and $0 < \varepsilon = \varepsilon(p) << 1$ (see also \[23\], p. 61).

One of the purposes of this work is to get away from the continuity requirements on $x \to \sigma(x, \xi)$. Even more importantly, we would like to replace the pointwise conditions on the derivatives of $\xi$ by averaged ones. This particular point has not been thoroughly explored appropriately in the literature in the author’s opinion, see Theorem\[1\] below.

On the other hand, a particular motivation for such considerations is provided by the recent papers of Rodnianski-Tao \[14\] and the author \[17\], where concrete parametrices (i.e. pseudodifferential operators, representing approximate solutions to certain PDE’s) were constructed for the solutions of certain first order perturbation of the wave and Schrödinger equations. A very quick inspection of these examples shows that they do not obey pointwise conditions on the derivatives of the symbols and thus, these methods fail to imply $L^2$ bounds for these (and related problems). Moreover, one often times has to deal with the situation, where the maps $\xi \to \sigma(x, \xi)$ are not smooth in a pointwise sense. On the other hand, one may still be able to control averaged quantities like

$$\sup_x \|\sigma(x, \xi)\|_{H^{n/2}_\xi} < \infty.$$ 

This will be our treshold condition for $L^2$ boundedness, which we try to achieve.

Heuristically at least, $4$ must be “enough” in some sense, since if we had simple symbols like $\sigma(x, \xi) = \sigma_1(x) \sigma_2(\xi)$, then the $L^2$ boundedness of $Op(\sigma)$ is equivalent to $\|\sigma_1\|_{L^\infty_x} < \infty$.

\[\text{footnote 1}\]: Most readers are likely to have their own fairly long list with favorite examples, for which the Hörmander condition fails.
and in fact there is the endpoint estimate (4), but on the other hand, the quantity in (4) is controlled by the appropriate Besov space $B^{n/2}_{2,1}$ norm.

A final motivation for the current study is to achieve a scale invariant condition, which gives an estimate of the $L^2 \to L^2 (L^p \to L^p)$ norm of $Op(\sigma)$ in terms of a scale invariant quantity, that is, we aim at showing an estimate,

$$\|Op(\sigma)\|_{L^p \to L^p} \leq C \|\sigma\|_Y \|f\|_{L^p},$$

where for every $\lambda \neq 0$, one has $\|\sigma(\lambda \cdot, \lambda^{-1} \cdot)\|_Y = \|\sigma\|_Y$.

In that regard, note that the condition (which is one of the requirements of the Hörmander class $S^0$)

$$\sup_{x} |D^\alpha_x \sigma(x, \xi)| \leq C_\alpha |\xi|^{-|\alpha|}$$

is scale invariant in the sense described above. Moreover, by the standard Calderón-Zygmund theory (see [21]), the pointwise condition (5) together with $\|T_\sigma\|_{L^2 \to L^2} < \infty$ implies

$$T_\sigma f(x) = \int K(x, x - y) f(y) dy,$$

where $K(x, \cdot)$ satisfies the Hörmander-Mihlin conditions, namely $|K(x, z)| \leq C |z|^{-n}$ and $|\nabla_z K(x, z)| \leq C |z|^{-n-1}$, where the constant $C$ depends on the constants $C_\alpha : |\alpha| < \lfloor n/2 \rfloor + 1$ in (5). This in turn is enough to conclude that $T_\sigma : L^p \to L^p$ for all $1 < p \leq 2$ and in fact there is the endpoint estimate $T_\sigma : L^1 \to L^{1,\infty}$.

1.1. $L^p$ estimates for PDO with rough symbols - statement of results. We start now with our main theorems, which concern the $L^2$ and the $L^p$ boundedness for pseudodifferential operators $Op(\sigma)$ with rough symbols. Our first result establishes that a Besov space version of (4) is enough for $L^2$ boundedness and the result is sharp.

**Theorem 1.** ($L^2$ bounds) Let $\sigma(x, \xi) : \mathbb{R}^n \times \mathbb{R}^n \to C$ and $T_\sigma$ is the corresponding pseudodifferential operator. Then

$$\|T_\sigma\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \leq C \left( \sum_i 2^{ln/2} \sup_x \|P_i^{\xi} \sigma(x, \cdot)\|_{L^2(\mathbb{R}^n)} \right),$$

where $P_i^{\xi}$ is the Littlewood-Paley operator in the $\xi$ variable.

Moreover, the result is sharp in the following sense: for every $p > 2$, there exists $\sigma(x, \xi)$ so that $\sup_x |D^\alpha_x \sigma(x, \xi)| \leq C_\alpha |\xi|^{-|\alpha|}$ and $\sup_x \|\sigma(x, \cdot)\|_{W^{p, n/p}} < \infty$, but $T_\sigma$ fails to be bounded on $L^2(\mathbb{R}^n)$.

**Remark:**

1. Note that the estimate on $T_\sigma$ is scale invariant.

2. The sharpness claim of the theorem, roughly speaking, shows that in the scale of space $W^{p, n/p}$, $\infty \geq p \geq 2$, one may not require anything less than $W^{2, n/2} = H^{n/2}$ of the symbol in order to ensure $L^2$ boundedness.

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2Note that these spaces scale the same and moreover by Sobolev embedding these are strictly decreasing sequence, at least for $2 \leq p < \infty$. 
(3) The counterexample to which we refer in Theorem 1 is a simple variation of the well-known example of $\sigma \in S^0_{1,1}$, the “forbidden class”, which fails to be $L^2$ bounded, see [21], p. 272 and Section 6 below.

Our next result concerns $L^p$ boundedness for $T_\sigma$.

**Theorem 2.** ($L^p$ bounds) For the pseudodifferential operator $T_\sigma$ there is the estimate for all $2 < p \leq \infty$,

$$
\|T_\sigma\|_{L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)} \leq C \left( \sum_l 2^{ln/2} \sup_x \|P_l^\xi \sigma(x, \cdot)\|_{L^2(\mathbb{R}^n)} \right),
$$

For the range $1 < p < 2$ and indeed for the weak type $(1, 1)$, there is

$$
\|T_\sigma\|_{L^p \to L^p} + \|T_\sigma\|_{L^1 \to L^{1,\infty}} \leq C \left( \sum_l 2^{ln} \sup_x \|P_l^\xi \sigma(x, \cdot)\|_{L^1(\mathbb{R}^n)} \right).
$$

Alternatively, if one assumes the $L^2$ bound, together with (5), one still gets $L^p \to L^p$, $1 < p \leq 2$, and in fact weak type $(1, 1)$ bounds. Moreover,

$$
\|T_\sigma\|_{L^p \to L^p} \leq C \left( \sum_l 2^{ln/2} \sup_x \|P_l^\xi \sigma(x, \cdot)\|_{L^2(\mathbb{R}^n)} + \sup_{|\alpha| < [n/2]+1} \sup_x |\xi|^\alpha |D_\xi^\alpha \sigma(x, \xi)| \right).
$$

As we pointed out in Theorem 1, the estimates are essentially sharp for $L^p$, $2 \leq p < \infty$ boundedness. The following corollary gives even more precise condition under which a symbol $\sigma$ will give rise to a bounded operator on $L^q$ in the case of a given $1 < q < 2$.

**Corollary 1.** Let $1 < q < 2$. Then

$$
\|T_\sigma\|_{L^q \to L^q} \leq C \left( \sum_l 2^{ln/q} \sup_x \|P_l^\xi \sigma(x, \cdot)\|_{L^q(\mathbb{R}^n)} \right).
$$

Clearly the proof follows by interpolation from the $L^q$ estimates in Theorem 1 and the weak type $(1, 1)$ estimates of Theorem 2.

1.2. PDO’s with homogeneous of degree zero symbols - statement of results. Regarding symbols that are homogeneous of degree zero, i.e. $\sigma(x, \xi) = q(x, \xi/|\xi|)$, where $q : \mathbb{R}^n \times S^{n-1} \to \mathcal{C}$, we obtain more precise results in terms of the smoothness of $q$.

Note that the classical Hörmander condition requires pointwise smoothness of the function $q$ in both variables. Our result on the other hand requires much less than that.

**Theorem 3.** (Lp bounds for homogeneous of degree zero symbols)

Let $q : \mathbb{R}^n \times S^{n-1} \to \mathcal{C}$. Let

$$
T_q f(x) = \int_{\mathbb{R}^n} q(x, \xi/|\xi|) e^{2\pi i \xi x} \hat{f}(\xi) d\xi.
$$

Then $T_q : L^2 \to L^2$, if $\sum_l 2^{l(n-1)/2} \|P_l^\xi q\|_{L^2(S^{n-1})} < \infty$ and in fact

$$
\|T_q\|_{L^2 \to L^2} \leq C \sum_l 2^{l(n-1)/2} \sup_x \|P_l^\xi q(x, \cdot)\|_{L^2(S^{n-1})}.
$$
Concerning $L^p$ bounds, we have for every $2 \leq p \leq \infty$.

(10) \[ \|T_q\|_{B_0^{p,1} \rightarrow L^p} \leq C_n \left( \sum_l 2^{(n-1)/p'} \sup_x \|P_l^{\xi/|\xi|} q\|_{L^{p'}(\mathbb{S}^{n-1})} \right). \]

Note that in (10), the constant $C_n$ is independent of $p, r$.

Remark:

(1) It would be interesting to see whether the usual $L^p \rightarrow L^p$ boundedness holds true.

(2) Note that there is no weak type $(1, 1)$ statement in Theorem 3. This is a difficult issue even for multipliers.

The sharpness statement associated with Theorem 3 is Proposition 1. For every $N > 1$, there exists a homogeneous of degree zero symbol $\sigma(x, \xi) : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$, so that $\sup_{x, \xi} |\sigma(x, \xi)| < \infty$ and $\sup_x \|\sigma(x, \xi)\|_{W^{1,1}(\mathbb{S}^1)} < \infty$, and so that $\|T_\sigma\|_{L^2 \rightarrow L^2} > N$.

1.3. PDO's with radial symbols. Finally, we consider the case of radial symbols. That is for $\rho : \mathbb{R}^n \times \mathbb{R}^1_+ \rightarrow \mathcal{C}$ and

$$T_\rho f(x) = \int_{\mathbb{R}^n} \rho(x, |\xi|) e^{2\pi i \xi \cdot x} \hat{f}(\xi) d\xi.$$ 

Theorem 4. The operator $T_\rho : L^2 \rightarrow L^2$, if $\sum_l 2^{l/2} \sup_x \|P_l^{\xi/|\xi|} \rho(x, \cdot)\|_{L^2(\mathbb{R}^1)} < \infty$. In fact, \[
\|T_\rho\|_{L^2 \rightarrow L^2} \leq C \sum_l 2^{l/2} \sup_x \|P_l^{\xi/|\xi|} \rho(x, \cdot)\|_{L^2(\mathbb{R}^1)}.
\]

Clearly, establishing $L^p, p \neq 2$ bounds for simple radial symbols is already a notoriously difficult problem. One only needs to point out to the Bochner-Riesz multiplier $(1 - |\xi|^2)^\delta_+$ (which satisfy $L^p$ bounds only in certain range of $p'$s, depending on the dimension and $\delta$) or even the simpler “thin annulus” multiplier $\varphi(2^m(1 - |\xi|^2))$ to understand the difficulty of the problem in general.
2. Applications

In this section, we demonstrate the effectiveness of the $L^p$ boundedness theorems for rough PDO’s. We will mostly concentrate on application to maximal functions and operators. Some of our examples will be well-known results for maximal operators, while others will be a higher dimensional extensions of such results.

2.1. Almost everywhere convergence for Cesaro sums of $L^p$ functions in 1 D. We start with Cesaro’s sum for Fourier series in one space dimension. For any $\delta > 0$, define

$$C_\delta f(x) = \sup_{u>0} \int_{\mathbb{R}} (1 - \xi^2/u^2)^{\delta/2} e^{2\pi i \xi x} \hat{f}(\xi) d\xi.$$ 

Clearly, as a limit as $\delta \to 0$, we get the Carleson’s operator. Unfortunately, one cannot conclude that $\sup_{\delta>0} \|C_\delta\|_{L^p} < \infty$, for that would imply the famous Carleson-Hunt theorem. On the other hand, define the maximal “thin interval operator”

$$T_m f(x) = \sup_{u>0} \int_{\mathbb{R}} \varphi(2^m (1 - \xi^2/u^2)) e^{2\pi i \xi x} \hat{f}(\xi) d\xi.$$ 

A simple argument based on (the proof of) Theorem 2 yields

**Proposition 2.** For any $\varepsilon > 0$, $1 < p < \infty$, there exists $C_{p,\varepsilon}$, so that

$$(11) \quad \sup_m \|T_m\|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})} \leq C_p.$$ 

In fact, there is the more general pointwise bound $\sup_m |T_m f(x)| \leq CM(\sup_k |P_k f|(x))$, which implies (11) as well as

$$(12) \quad \|C_\delta\|_{L^p \to L^p} + \|C_\delta\|_{F^{0,1}_1(\mathbb{R}) \to L^{1,\infty}(\mathbb{R})} \leq C_{\delta,p}.$$ 

**Remark:**

- Note that this result, while clearly inferior to the Carleson-Hunt theorem still implies a.e. convergence for any Cesaro summability method, when applied to $L^p(\mathbb{R})$ functions, and in fact for the larger class of $F^{0,1}_1$ functions.
- Using the method of proof here, one may actually prove $L^2$ estimates\(^3\) for the *maximal Bochner-Riesz operator*

$$BR_\delta f(x) = \sup_{u>0} \int_{\mathbb{R}^n} (1 - |\xi|^2/u^2)^{\delta/2} e^{2\pi i \xi x} \hat{f}(\xi) d\xi.$$ 

in any dimension.

**Proof.** It clearly suffices to show the pointwise estimate $|T_m f(x)| \leq CM(\sup_k P_k f)(x)$ for any $k$. The statements about $B^{0,1}_{p,1} \to L^p$ bounds follow by elementary Littlewood-Paley theory and the $l^p$ bounds for the Hardy-Littlewood maximal function. The restricted-to-weak estimate $F^{0,1}_{1,\infty} \to L^{1,\infty}$ for $C_\delta$ follows by summing an exponentially decaying series in the quasi-Banach space $L^{1,\infty}$.

\(^3\)In addition, the author has also identified several applications to bilinear/multilinear operators of importance to certain dispersive PDE’s, which will be addressed in a future publication.

\(^4\)And in fact $L^p \to L^p$ estimates for $p = p(\delta)$ close to 2.
By support considerations, it is clear that
\[
T_m f(x) = \sum_k \sup_{u > 0} \int_{\mathbb{R}^1} \phi(2^m (1 - \xi^2 / u^2)) e^{2\pi i \xi x} \varphi(2^{-k} \xi) \hat{f}(\xi) d\xi =
\]
\[
= \sum_k \sup_{u \in (2^{k-2}, 2^{k+2})} \int_{\mathbb{R}^1} \phi(2^m (1 - \xi^2 / u^2)) e^{2\pi i \xi x} \varphi(2^{-k} \xi) \hat{f}(\xi) d\xi =
\]
\[
= \sum_k T_{m,u(\cdot) \in (2^{k-2}, 2^{k+2})} f_k.
\]
Clearly, the requirement \( u \in (2^{k-2}, 2^{k+2}) \) creates (almost) disjointness in the \( x \) support, whence
\[
|T_m f(x)| \leq C \sup_k |T_{m,u(\cdot) \in (2^{k-2}, 2^{k+2})} f_k(x)|.
\]
Our basic claim is that
\[
|T_{m,u(\cdot) \in (2^{k-2}, 2^{k+2})} f_k(x)| \leq CM(f_k).
\]
Clearly (13) and (14) imply \( \sup_m |T_m f(x)| \leq CM(\sup_k |f_k|) \), whence the Proposition 2.

By scale invariance, (14) reduces to the case \( k = 0 \), that is we need to show
\[
|T_{m,u(x) \in (1/4, 4)} P_0 f(x)| \leq CM(P_0 f)(x).
\]
for any Schwartz function \( f \) and any \( m \gg 1 \). By (31) (in the proof of Theorem 2 below), it will suffice to show
\[
\sum_l 2^l \sup_x \| P_1^x [\varphi(2^m (1 - \xi^2 / u(x)^2)) \varphi(\xi)] \|_{L^1(\mathbb{R})} \lesssim 1.
\]
for any measurable function \( u \), which takes its values in (1/4, 4).

For (15), we have
\[
\sum_{l \leq m} 2^l \sup_x \| P_1^x [\varphi(2^m (1 - \xi^2 / u(x)^2)) \varphi(\xi)] \|_{L^1(\mathbb{R})}
\]
\[
\lesssim \sum_{l < m} 2^l \sup_{u \in (1/4, 4)} \int \| \varphi(2^m (1 - \xi^2 / u^2)) \varphi(\xi) \| d\xi \lesssim \sum_{l < m} 2^{l-m} \lesssim 1,
\]
while
\[
\sum_{l \geq m} 2^l \sup_x \| P_1^x [\varphi(2^m (1 - \xi^2 / u(x)^2)) \varphi(\xi)] \|_{L^1(\mathbb{R})}
\]
\[
\lesssim \sum_{l \geq m} 2^{-l} \sup_{u \in (1/4, 4)} \int \frac{d^2}{d\xi^2} \varphi(2^m (1 - \xi^2 / u^2)) \varphi(\xi) \| d\xi \leq C \sum_{l \geq m} 2^{-l+m} \lesssim 1.
\]
\[\square\]
2.2. Maximal directional Hilbert transforms and the Kakeya maximal function. Another interesting application is provided by the directional Hilbert transform in dimensions $n \geq 2$. Namely, take

$$H_\delta^* = \sup_{u \in S^{n-1}} \int \langle (u, \xi/|\xi|) \rangle^\delta e^{2\pi i \xi x} \hat{f}(\xi) \varphi(\xi) d\xi,$$

where $\text{supp} \varphi \subset \{1/2 < |\xi| < 2\}$.

As $\delta \to 0$, we obtain the operator $f \to \sup_{u \in S^{n-1}} \int \{\langle u, \xi \rangle > 0\} e^{2\pi i \xi x} \hat{f}(\xi) d\xi$, which is closely related to the maximal directional Hilbert transform

$$H_s f(x) = \sup_u |H_u f(x)| = \sup_u \left| \int \text{sgn}(u, \xi) e^{2\pi i \xi x} \hat{f}(\xi) \varphi(\xi) d\xi \right|.$$

$H_s$ was of course shown to be $L^p(\mathbb{R}^2), p > 2$ bounded by Lacey and Li, [12] by very sophisticated time-frequency analysis methods.

**Proposition 3.** For the “thin big circle” multiplier

$$T_m f(x) = \sup_{u \in S^{n-1}} \left| \int_{\mathbb{R}^n} \varphi(2^m \langle u, \xi/|\xi| \rangle) e^{2\pi i \xi x} \hat{f}(\xi) \varphi(\xi) d\xi \right|,$$

we have

$$\|T_m f\|_{L^2 \to L^2} \leq C \xi^2 2^{m(n/2 - 1)}$$

In particular

$$\left\| H_\delta^* \right\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} \leq C_{p, \xi, \delta} 2^{n/2 - 1}.$$

**Remark:**

- We believe that the operator $T_m (m \gg 1)$ has a particular connection to the Kakeya maximal function and the corresponding Kakeya problem. Indeed, the kernel of the corresponding singular integral behaves like a $(L^1$ normalized) characteristic function of a rectangle with long side along $u$ of length $2^m$ and $(n - 1)$ short sides of length 1 in the transverse directions!
- In relation to that, one expects the conjectured Kakeya bounds

$$\|T_m f\|_{L^p \to L^p} \leq C \xi^2 2^{m(n/p - 1)}$$

for $p \leq n$ to hold, while one only gets

$$\|T_m f\|_{L^p \to L^p} \leq C \xi^2 2^{m(n/p - 2/p)}$$

as a consequence of Theorem 3. Nevertheless, the two match when $p = 2$. So it seems that (16), at least in principle, captures the Kakeya conjecture for $p = 2$ in general and in particular the full Kakeya conjecture in two dimensions.

Since our estimates do not seem to contribute much toward the resolution of any new Kakeya estimates, we do not pursue here the exact relationship between $T_m$ and the Kakeya maximal operator, although from our heuristic arguments above it should be clear that it is a close one.
Proof. We proceed as in the proof of Proposition 2. We need only show

\[ \sum_{l} 2^{l(n-1)/2} x \sup_{\xi} \left\| F_l \phi(2^{m} \langle u(x), \xi \rangle) \right\|_{L^2(S^{n-1})} \lesssim 1. \]  

We have

\[ \sum_{l<m} 2^{l(n-1)/2} x \sup_{\xi} \left\| F_l \phi(2^{m} \langle u(x), \xi \rangle) \right\|_{L^2(S^{n-1})} \]

\[ \leq C \sum_{l<m} 2^{l(n-1)/2} \sup_{\xi} \left\| F_l \phi(2^{m} \langle u(x), \xi \rangle) \right\|_{L^2(S^{n-1})} \]

\[ \leq C \sum_{l<m} 2^{l(n-1)/2} 2^{-m/2} \lesssim 2^{m(n/2-1)}. \]

\[ \sum_{l\geq m} 2^{l(n-1)/2} x \sup_{\xi} \left\| F_l \phi(2^{m} \langle u(x), \xi \rangle) \right\|_{L^2(S^{n-1})} \]

\[ C \leq \sum_{l\geq m} 2^{-l(n-1)/2} \sup_{\xi} \left\| F_l \phi(2^{m} \langle u(x), \xi \rangle) \right\|_{L^2(S^{n-1})} \]

\[ \leq C \sum_{l\geq m} 2^{-l(n-1)/2} 2^{m(n-1)} 2^{-m/2} \lesssim 2^{m(n/2-1)}. \]

\[ \square \]

2.3. Estimates on \( T_{\sigma, \sigma_2}, T_{e^s} \) etc. We now present a result, which allows us to treat pseudodifferential operators, whose symbols are products, exponentials (or more generally entropy functions) of symbols, which satisfy the requirements in Theorems 1, 2, 3. We would like to point out that similar in spirit (by essentially requiring \( n/2 + \varepsilon \) derivatives in \( L^2 \), but in a more general setting) functional calculus type result was obtained in [16].

Proposition 4. Let \( \sigma, \sigma_1, \sigma_2 : \mathbb{R}^n \times \mathbb{R}^n \rightarrow C \), so that \( T_{\sigma_j}, j = 1, 2 \) define \( L^p \) bounded operators, as in [6], [8]. Then for every \( 1 < p < \infty \), \( T_{\sigma_1, \sigma_2} \) is \( L^p \) bounded and

\[ \left\| T_{\sigma_1, \sigma_2} \right\|_{L^2(R^n) \rightarrow L^2(R^n)} \leq C \prod_{j=1}^{2} \left( \sum_{l} 2^{ln/2} \sup_{x} \left\| P_l \sigma_j(x, \cdot) \right\|_{L^2(R^n)} \right), \]

\[ \left\| T_{\sigma_1, \sigma_2} \right\|_{L^p(R^n) \rightarrow L^p(R^n)} \leq C \prod_{j=1}^{2} \left( \sum_{l} 2^{ln} \sup_{x} \left\| P_l \sigma_j(x, \cdot) \right\|_{L^1(R^n)} \right). \]

In the same spirit, \( T_{e^s} \) is also \( L^p \) bounded and there is

\[ \left\| T_{e^s} \right\|_{L^p(R^n) \rightarrow L^p(R^n)} \leq C \exp \left( \sum_{l} 2^{ln} \sup_{x} \left\| P_l \sigma(x, \cdot) \right\|_{L^1(R^n)} \right) \]
Similar statements can be made for homogeneous of degree zero symbols \( \mu_1(x, \xi/|\xi|), \mu_1(x, \xi/|\xi|) \).

\[
\|T_{\mu_1\mu_2}\|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)} \leq C \prod_{l=1}^{2} \left( \sum_{l} 2^{l(n-1)/2} \sup_{x} \|P^l_{l} \mu_{l}(x, \cdot)\|_{L^1(\mathbb{R}^n)} \right),
\]

\[
\|T_{v}\|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)} \leq C \exp\left( \sum_{l} 2^{l(n-1)} \sup_{x} \|P^l_{l} \mu(x, \cdot)\|_{L^1(\mathbb{R}^n)} \right).
\]

The proof of Proposition 4 is based on the corresponding Theorem for \( L^p \) boundedness, combined with the fact that our requirements form a Banach algebra under the multiplication. Take for example (18). By Theorem 1, we have

\[
\|T_{\sigma_1\sigma_2}\|_{L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)} \leq C \left( \prod_{l=1}^{2} \left( \sum_{l} 2^{ln/2} \sup_{x} \|P^l_{l} \mu(x, \cdot)\|_{L^2(\mathbb{R}^n)} \right) \right).
\]

We finish by invoking the estimate

\[
\sum_{l} 2^{ln/2} \sup_{x} \|P^l_{l} \sigma_1\sigma_2(x, \cdot)\|_{L^2(\mathbb{R}^n)} \leq C \prod_{l=1}^{2} \left( \sum_{l} 2^{ln/2} \sup_{x} \|P^l_{l} \sigma_j(x, \cdot)\|_{L^2(\mathbb{R}^n)} \right).
\]

where this last inequality essentially means that \( B^{n/2}_{2,1} \) is a Banach algebra of functions.

The argument above can be performed for the proof of (19). For (20) (and more generally for any symbols of the form \( g(\sigma) \), where \( g \) is entire function), one iterates the product estimate (23) to

\[
\sum_{l} 2^{ln/2} \sup_{x} \|P^l_{l} \epsilon(x, \cdot)\|_{L^2(\mathbb{R}^n)} \leq C \exp\left( \sum_{l} 2^{ln/2} \sup_{x} \|P^l_{l} \sigma(x, \cdot)\|_{L^2(\mathbb{R}^n)} \right).
\]

For the proof of (21), (22), one has to use the fact that \( B^{(n-1)/2}_{2,1}(S^{n-1}) \) is a Banach algebra as well, whence one gets an estimate similar to (23) and (24).

3. Preliminaries

We start by introducing some basic concepts in Fourier analysis.

3.1. Fourier analysis on \( \mathbb{R}^n \). First, define the Fourier transform and its inverse

\[
\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x \cdot \xi} dx,
\]

\[
f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi)e^{2\pi i x \cdot \xi} d\xi.
\]

For a positive, smooth and even function \( \chi : \mathbb{R}^n \rightarrow \mathbb{R}_+ \), supported in \( \{ \xi : |\xi| \leq 1 \} \) and so that \( \chi(\xi) = 1 \) for all \( |\xi| \leq 1 \). Define \( \varphi(\xi) = \chi(\xi) - \chi(2\xi) \), which is supported in the annulus \( 1/2 \leq |\xi| \leq 2 \). Clearly \( \sum_{k \in \mathbb{Z}} \varphi(2^{-k} \xi) = 1 \) for all \( \xi \neq 0 \).

\[\text{This is well-known, but can be verified easily by means of the Kato-Ponce estimate } \|\partial^{n/2}v\|_{L^2} \leq C(\|\partial^{n/2}u\|_{L^2} + \|\partial^{n/2}v\|_{L^2}) \text{, the embedding } B^{n/2}_{2,1} \hookrightarrow L^\infty \text{ and some Littlewood-Paley theory.}\]

\[\text{The discussion henceforth will be for } \mathbb{R}^n, \text{ unless explicitly specified otherwise.}\]
The $k^{th}$ Littlewood-Paley projection is given by $\widehat{P_k f}(\xi) = \varphi(2^{-k}\xi)\hat{f}(\xi)$. Note that the kernel of $P_k$ is integrable uniformly in $k$ and thus $P_k : L^p \to L^p$ for $1 \leq p \leq \infty$ and $\|P_k\|_{L^p \to L^p} \leq C_n \|\hat{\chi}\|_{L^1}$. In particular, the bounds are independent of $k$.

It is a standard observation that $\nabla P_k = P_k \nabla = 2^k \tilde{P}_k$, where $\tilde{P}_k$ is a multiplier type operator similar to $P_k$ and thus $\|P_k \nabla \psi\|_{L^p} \sim 2^k \|P_k \psi\|_{L^p}$. In what follows, we will use the notation $P_x l f(x,\xi)$ to denote Littlewood-Paley operator acting on the variable $x$, and $P_\xi l f(x,\xi)$ will be a Littlewood-Paley operator in the variable $\xi$. That is

$$P_x l f(x,\xi) = 2^n \int \hat{\varphi}(2^l(x - y))f(y,\xi)dy,$$

$$P_\xi l f(x,\xi) = 2^n \int \hat{\varphi}(2^l(\xi - \eta))f(x,\eta)d\eta.$$

The Bernstein inequality takes the form

$$\|P_l f\|_{L^q} \leq C_n 2^{n(1/p - 1/q)} \|P_l f\|_{L^p},$$

for $1 \leq p < q \leq \infty$.

The (uncentered) Hardy-Littlewood maximal function is

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|dy.$$

It is well-known that $M : L^p \to L^p$ for all $1 < p \leq \infty$ and is of weak type $(1, 1)$. It is also convenient to use the pointwise bound

$$\sup_{t > 0} t^{-n} |f * \Phi(t^{-1} \cdot)(x)| \leq C \|\Phi\|_{L^1} Mf(x),$$

for a radially dominated function $\Phi$. For integer values of $s$, we may define $W^{p,s}$ to be the Sobolev space with $s$ derivatives in $L^p$, $1 \leq p \leq \infty$, with the corresponding norm

$$\|f\|_{W^{p,s}} := \sum_{|\alpha| \leq s} \|D^\alpha f\|_{L^p}.$$

Equivalently, and for noninteger values of $s$, define

$$\|f\|_{W^{p,s}} := \|f\|_{L^p} + \left(\sum_{l=0}^{\infty} 2^{ls} |P_l f|^2\right)^{1/2} \|f\|_{L^p}.$$

and its homogeneous analogue

$$\|f\|_{\dot{W}^{p,s}} := \left(\sum_{l=-\infty}^{\infty} 2^{ls} |P_l f|^2\right)^{1/2} \|f\|_{L^p}.$$

Note $W^{p,s} = \dot{W}^{p,s} \cap L^p$.

The (homogeneous) Besov spaces $\dot{B}^{p,q}_s$, which scale like $\dot{W}^{p,s}$, are defined as follows

$$\|f\|_{\dot{B}^{p,q}_s} := \left(\sum_{l \in \mathbb{Z}} 2^{lsq} \|P_l f\|_{L^p}^q\right)^{1/q}.$$
The Triebel-Lizorkin spaces are defined via
\[ \|f\|_{\dot{F}^{s,q}_{p}} := \|\left( \sum_{l \in \mathbb{Z}} 2^{lsq} |P_{l} f|^{q} \right)^{1/q} \|_{L^{p}}. \]

3.2. **Fourier analysis on** $S^{n-1}$. In this section, we define the Sobolev and Besov spaces for functions $q$ defined on $S^{n-1}$. For that, the standard approach is to fix the basis of the spherical harmonics and define the Littlewood-Paley operators by projecting over the corresponding set of the harmonics within the fixed frequency.

Introduce the angular differentiation operators $\Omega_{ij} = x_{j} \partial_{i} - x_{i} \partial_{j}$. It is well-known that $\{\Omega_{ij}\}_{i \neq j}$ generate the algebra of all differential operators, acting on $C^{\infty}(S^{n-1})$. The spherical Laplacian is defined via
\[ \Delta_{sph} = \sum_{i<j} \Omega_{ij}^{2}. \]

The spherical harmonics $\{Y^{n}_{l,k}\}_{k \in A_{n}^{l}}$ are eigenfunctions of $\Delta_{sph}$, so that $\Delta_{sph}Y^{n}_{l,k} = -l(l - 2 + l)Y^{n}_{l,k}$, where $l \geq 0$, $k$ varies in a finite set $A_{n}^{l}$. An equivalent way to define them is to take all the homogeneous of degree $l$ polynomials that are solutions to
\[ (\partial_{r}^{2} + r^{-1} \partial_{r} + r^{-2} \Delta_{sph}) Y^{l} = 0 \]

It turns out that (26) has \( n + l - 1 \choose l \) linearly independent solutions $\{Y^{n}_{l,k}\}_{k \in A_{n}^{l}}$. Another important property of the family $\{Y^{n}_{l,k}\}$ is that it forms an orthonormal basis for $L^{2}(S^{n-1})$.

Let $f : S^{n-1} \to \mathbb{C}$ be a smooth function. One can then define the expansion in spherical harmonics in the usual way
\[ f(\theta) = \sum_{l,k \in A_{n}^{l}} c^{n}_{l,k} Y^{n}_{l,k}(\theta), \]
where $c^{n}_{l,k} = \langle f, Y^{n}_{l,k}\rangle_{L^{2}(S^{n-1})}$. The Littlewood-Paley operators may be defined via
\[ P_{m}^{\xi/|\xi|} f = \sum_{l,k \in A_{n}^{l}} c^{n}_{l,k} \varphi(2^{-m} l) Y^{n}_{l,k}(\theta), \]
and there is the equivalence for all $1 < p < \infty$, due to Strichartz [22]
\[ \|f\|_{L^{p}(S^{n-1})} \sim \|\left( \sum_{k=-\infty}^{\infty} |P_{k}^{\xi/|\xi|} f|^{2} \right)^{1/2} \|_{L^{p}(S^{n-1})}. \]

As a simple consequence, one has for all $1 < p < \infty$, $\|P_{k}^{\xi/|\xi|}\|_{L^{p} \to L^{p}} \leq C_{n,p}$. Such an inequality actually extends (as in the case of $\mathbb{R}^{n}$) to the endpoint cases $p = 1, p = \infty$, see \footnote{The constant of equivalence here depends only on $p$ and the cutoff function $\varphi$.}
One can also define the Sobolev spaces \( W^{p,s}(S^{n-1}) : 1 < p < \infty \) via
\[
\|f\|_{W^{p,s}(S^{n-1})} = \sum_{|\alpha| \leq s} \|\Omega^{\alpha} f\|_{L^p(S^{n-1})}
\]
where \( \Omega = \sqrt{-\Delta_{\text{sph}}} \). These last two formulas give equivalent definitions for the case of integer \( s \). The (homogeneous) Besov spaces are defined in the usual manner as follows
\[
\|f\|_{B^{p,q}_{s}(S^{n-1})} = \left( \sum_{k=-\infty}^{\infty} 2^{ks} \|P^k f\|_{L^p(S^{n-1})}^q \right)^{1/q}.
\]
It is worth mentioning at this point that a variant of Bernstein inequality holds\(^8\) in this context, see [19], p. 201. This together with the Littlewood-Paley theory outlined above implies Sobolev embedding for \( L^p(S^{n-1}) \) spaces. For future reference, let us record this estimate
\[
\tag{27} \|f\|_{L^q(S^{n-1})} \leq C_{p,q,n} \|\Omega^{(n-1)(1/p-1/q)} f\|_{L^p(S^{n-1})},
\]
which holds whenever for \( 1 < p < q < \infty \).

If \( q \) is a finite sum of harmonics, it is actually an analytic function (it is a fact a restriction of a polynomial to the unit sphere) and one may write
\[
\tag{28} q(\theta) = \sum_{\alpha > 0} \frac{D_{\xi}^{\alpha} q(\theta_0)}{\alpha!} (\theta - \theta_0)^\alpha.
\]

Here, \( D_{\xi}^{\alpha} q(\theta_0) \) should be understood as taking \( \alpha \) derivatives of the corresponding homogeneous polynomial and evaluating at \( \theta_0 \). The following lemma is standard, but since we need a specific dependence of our estimates upon the parameter \( \alpha \), we state it here for completeness.

**Lemma 1.** Let \( q : S^{n-1} \to C \) and \( q_m = P_m^{\xi/|\xi|} q \). Then, there is a constant \( C_n \), so that for every \( 1 \leq p \leq \infty \), there is a the estimate
\[
\|D_{\xi}^{\alpha} q_m\|_{L^p(S^{n-1})} \leq C_n |\alpha| 2^{|\alpha|} \|q_m\|_{L^p(S^{n-1})}.
\]

The proof of Lemma 1 is standard. One way to proceed is to note that if we extend the function \( q_m \) off \( S^{n-1} \) to some annulus, say via \( Q_m(\xi) = \varphi(|\xi|) q_m(\xi/|\xi|) \), and then
\[
\|D_{\xi}^{\alpha} q_m\|_{L^p(S^{n-1})} \lesssim \|D_{\xi}^{\alpha} Q_m\|_{L^p(\mathbb{R}^n)}
\]

\[4. \text{ } L^p \text{ estimates for } \text{PDO with rough symbols}
\]

We start with the \( L^2 \) estimate to illustrate the main ideas in the proof.\(^8\)

\(^8\)The proof is simply that there are \( \sim N^{n-1} \) spherical harmonics at frequency \( N \), just as for the Bernstein inequality one uses that the volume of \( \{ \xi \in \mathbb{R}^n : |\xi| \sim N \} \) is \( \sim N^n \).
4.1. $L^2$ estimates: Proof of Theorem 1. Our first remark is that we will for convenience consider only real-valued symbols $\sigma$, since of course the general case follows from splitting into a real and imaginary part.

To show $L^2$ estimates for $T_\sigma$, it is equivalent to show $L^2$ estimates for the adjoint operator, which takes the form

$$T_\sigma^* g(x) = \int e^{2\pi i \xi \cdot x} \int e^{-2\pi i \xi \cdot y} \hat{g}(y) \sigma(y, \xi) dy \, d\xi.$$ 

Our next task is to decompose $T_\sigma^* g$ and we start by taking a Littlewood-Paley partition of unity in the $\xi$ variable for $g$. We have

$$T_\sigma^* g(x) = \sum_{l \in \mathbb{Z}} \int e^{2\pi i \xi \cdot x} \int e^{-2\pi i \xi \cdot y} \hat{g}(y) P_l^\xi \sigma(y, \xi) dy \, d\xi$$

Now that the function $g$ is frequency localized at frequency $2^l$, we introduce further decomposition in the $\xi$ integration.

For the $L^2$ estimates, because of the orthogonality, we only need rough partitions, so for each fixed $l$, take a tiling of $\mathbb{R}^n$ composed of cubes $\{Q\}$ with diameter $2^{-l}$. Denote the characteristic functions of $Q$ by $\chi_Q$. We have

$$T_\sigma^* g(x) = \sum_{l \in \mathbb{Z}} \sum_{Q \ni \xi} \int e^{2\pi i \xi \cdot x} \chi_Q(\xi) \int e^{-2\pi i \xi \cdot y} \hat{g}(y) P_l^\xi \sigma(y, \xi) dy \, d\xi$$

The main point of our next decompositions is that the function $P_l^\xi \sigma$ is essentially constant in $\xi$ over any fixed cube $Q$. We exploit that by observing that $\xi \to P_l^\xi \sigma(x, \xi)$ is an entire function and there is the expansion

$$P_l^\xi \sigma(y, \xi) \chi_Q(\xi) = [P_l^\xi \sigma(y, \xi_Q) + \sum_{|\alpha| > 0} D_{\xi}^{\alpha} P_l^\xi \sigma(y, \xi_Q) \frac{(\xi - \xi_Q)^\alpha}{\alpha!}] \chi_Q(\xi)$$

for any fixed $y$ and for any $\xi_Q \in Q$. Note that $D_{\xi}^{\alpha} P_l^\xi \sigma(y, \xi_Q) \sim 2^{l|\alpha|} P_l^\xi \sigma(y, \xi_Q)$ and $||(\xi - \xi_Q)^\alpha| \leq 2^{-l|\alpha|}$, by support consideration (recall $d(Q) = 2^{-l}$). On a heuristic level, by the presence of $\alpha!$, one should think that the series above behave like $P_l^\xi \sigma(y, \xi_Q)$ plus exponential tail.

Going back to $D_\xi^{\alpha} P_l^\xi$, as we have mentioned in Section 3, we can write $D_\xi^{\alpha} P_l^\xi = 2^{l|\alpha|} P_{l,\alpha}^\xi$, where $P_{l,\alpha}^\xi$ is given by the multiplier $\varphi(2^{-l} \xi)(2^{-l} \xi)^\alpha$. It is clear that $\|P_{l,\alpha}^\xi f\|_{L^2(\mathbb{R}^n)} \leq C_{n,|\alpha|} \|P_l^\xi f\|_{L^2(\mathbb{R}^n)}$.

\footnote{There is the small technical problem that the $\xi$ integral does not converge absolutely. This can be resolved by judicious placement of cutoffs $\chi(\xi/N)$, after which, one may subsume that part in $\hat{f}(\xi)$. In the end we let $N \to \infty$ and all the estimates will be independent of the cutoff constant $N$.}
Thus, we have arrived at

\[ T^*_\sigma g(x) = \sum_{|\alpha| \geq 0} \sum_{l \in \mathbb{Z}} \sum_{Q: d(Q) = 2^{-t}} \int e^{2\pi i \xi \cdot x} \chi_Q(\xi) \frac{2^{|\alpha|}(\xi - \xi_Q)^\alpha}{\alpha!} \times \]

\[ \times \left( \int e^{-2\pi i y \cdot [g(y) P^x_{l,\alpha} \sigma(y, \xi_Q)]} dy \right) d\xi = \sum_{l,\alpha} (\alpha!)^{-1} \sum_{Q: d(Q) = 2^{-t}} P_{Q, l, \alpha} [g(\cdot) P^x_{l,\alpha} \sigma(\cdot, \xi_Q)], \]

where \( P_{Q, l, \alpha} \) acts via \( \widehat{P_{Q, l, \alpha} f}(\xi) = \chi_Q(\xi) 2^{|\alpha|}(\xi - \xi_Q)^\alpha \hat{f}(\xi) \). Note

\[ \| P_{Q, l, \alpha} \|_{L^2 \to L^2} = \sup_{\xi} |\chi_Q(\xi) 2^{|\alpha|}(\xi - \xi_Q)^\alpha| \leq 1. \]

For fixed \( l, \alpha \), take \( L^2 \) norm. Using the orthogonality of \( P_{Q, l, \alpha} \) and its boundedness on \( L^2 \), we obtain

\[ \| \sum_{Q: d(Q) = 2^{-t}} P_{Q, l, \alpha} [g(\cdot) P^x_{l,\alpha} \sigma(\cdot, \xi_Q)] \|_{L^2}^2 = \sum_{Q: d(Q) = 2^{-t}} \| P_{Q, l, \alpha} [g(\cdot) P^x_{l,\alpha} \sigma(\cdot, \xi_Q)] \|_{L^2}^2 \leq \sum_{Q: d(Q) = 2^{-t}} \| g(\cdot) P^x_{l,\alpha} \sigma(\cdot, \xi_Q) \|_{L^2}^2 = \int |g(y)|^2 \left( \sum_{Q} |P^x_{l,\alpha} \sigma(y, \xi_Q)|^2 \right) dy. \]

We now again use \( P^x_{l,\alpha} \sigma(y, \xi_Q) \sim P^x_{l,\alpha} \sigma(y, \eta) \) for any \( \eta \in Q \), this time to estimate the contribution of \( \sum_Q |P^x_{l,\alpha} \sigma(y, \xi_Q)|^2 \). This is done as follows. Expand

\[ P^x_{l,\alpha} \sigma(y, \xi_Q) = \sum_{\beta : |\beta| \geq 0} \frac{D^n_{\beta} P^x_{l,\alpha} \sigma(y, \eta)}{\beta!} (\xi_Q - \eta)^\beta, \]

(29)

to be used for \( \eta \in Q \). Thus, if we average over \( Q \),

\[ |P^x_{l,\alpha} \sigma(y, \xi_Q)| = \left( |Q|^{-1} \int_Q | P^x_{l,\alpha} \sigma(y, \eta) \right) \left( \beta! \sum_{\beta : |\beta| \geq 0} \frac{D^n_{\beta} P^x_{l,\alpha} \sigma(\eta)}{\beta!} (\xi_Q - \eta)^\beta |^2 d\eta \right)^{1/2} \leq \]

\[ \leq |Q|^{-1/2} \sum_{\beta : |\beta| \geq 0} \frac{C_{\beta} 2^{-|\beta|}}{\beta!} \left( \int_Q |D^n_{\beta} P^x_{l,\alpha} \sigma(\eta)|^2 d\eta \right)^{1/2}. \]

and so (recalling \( |Q| \sim 2^{-ln} \))

\[ \sum_Q |P^x_{l,\alpha} \sigma(y, \xi_Q)|^2 \leq C2^{ln/2} \sum_{\beta} \frac{C_{\beta} 2^{-|\beta|}}{\beta!} \left( \int_Q |D^n_{\beta} P^x_{l,\alpha} \sigma(y, \cdot)|^2 d\eta \right)^{1/2} \leq \]

\[ \leq C2^{ln/2} \| P^x_{l,\alpha} \sigma(y, \cdot) \|_{L^2}. \]
Thus,

\[ \|T_\sigma^* g\|_{L^2} \lesssim \sum_{l,\alpha} 2^{ln/2}(\alpha!)^{-1} \left( \int |g(y)|^2 \|P_{l,\alpha}^\varepsilon \sigma(y, \cdot)\|_{L^2}^2 dy \right)^{1/2} \lesssim \sum_{l,\alpha} 2^{ln/2}(\alpha!)^{-1} \|g\|_{L^2} \sup_y \|P_{l,\alpha}^\varepsilon \sigma(y, \cdot)\|_{L^2}. \]

Furthermore,

\[ \sup_y \|P_{l,\alpha}^\varepsilon \sigma(y, \cdot)\|_{L^2} \leq C_n \sup_y \|P_l^\varepsilon \sigma(y, \cdot)\|_{L^2}. \]

Put everything together

\[ \|T_\sigma^* g\|_{L^2} \leq C_n \|g\|_{L^2} \sum_\alpha (\alpha!)^{-1} C_n^\alpha \sum_l 2^{ln/2} \sup_y \|P_l^\varepsilon \sigma(y, \cdot)\|_{L^2} \leq D_n \|g\|_{L^2} \sum_l 2^{ln/2} \sup_y \|P_l^\varepsilon \sigma(y, \cdot)\|_{L^2}, \]

as desired.

4.2. \( L^p \) estimates: \( 2 < p \leq \infty \). The result announced in Theorem 2 follows by interpolation between the \( L^2 \) estimate just proved and the boundedness of \( T_\sigma : L^\infty \to L^\infty \), which we need to show next.

We do that by showing that the adjoint \( T_\sigma^* : L^1 \to L^1 \). This is relatively easy, since one can reduce to showing that

\[ \sup_{a \in \mathbb{R}^n} \|T_\sigma^* \delta(a)\|_{L^1(\mathbb{R}^n)} < C \sum_l 2^{ln/2} \sup_{a} \|P_l^\varepsilon \sigma(a, \cdot)\|_{L^2}. \] (30)

This is standard, since one can embed \( L^1 \) into the space of all Borel measures \( M(\mathbb{R}^n) \). The next observation is that by Krein-Milman’s theorem, the convex combinations of the set of Dirac masses \( \{\delta_a : a \in \mathbb{R}^n\} \) are weak* dense in the unit ball of \( M(\mathbb{R}^n) \).

Fix \( a \in \mathbb{R}^n \). We have

\[ T_\sigma^* \delta_a(x) = \int e^{2\pi i \xi \cdot (x-a)} \sigma(a, \xi) d\xi = \mathcal{F}_\xi(\sigma(a, \cdot))(x). \]

where \( \mathcal{F}_\xi \) signifies the Fourier transform in the \( \xi \) variable. Denote \( g_a(z) = \mathcal{F}_\xi(\sigma(a, \cdot))(z) \).

We have by Cauchy-Schwartz and the Plancherel’s theorem

\[ \|T_\sigma^* \delta\|_{L^1} = \int |g_a(a-x)| dx = \sum_{l \in \mathbb{Z}} \int_{|x-a| \sim 2^l} |g_a(a-x)| dx \leq C_n \sum_{l \in \mathbb{Z}} 2^{ln/2} \int_{|z| \sim 2^l} |g(z)|^2 dz \] \[ \leq C_n \sum_{l \in \mathbb{Z}} 2^{ln/2} \|P_l^\varepsilon \sigma(a, \cdot)\|_{L^2} \]

This is (30), whence (7).
4.3. \textbf{\$L^p\$ estimates:} \textit{1 < \(p\) \leq 2.} We take slightly different approach than in the case of \(L^2\) estimates. Namely, we will show that \(T_\sigma\) is of weak type \((1, 1)\) operator, whence, by interpolation with the \(L^2\) estimate, one gets the full range \(1 < p \leq 2\). Note that the \(L^2\) estimate comes with
\[
\|T_\sigma\|_{L^2 \to L^2} \leq C_n \sum_l 2^{ln/2} \sup_y \|P^\xi_l \sigma(y, \cdot)\|_{L^2} \leq C_n \sum_l 2^{ln} \sup_y \|P^\xi_l \sigma(y, \cdot)\|_{L^1},
\]
where in the last inequality, we have used the Bernstein inequality
\[
\|P^\xi_l \sigma(a, \cdot)\|_{L^2} \leq C_n 2^{ln/2} \|P^\xi_l \sigma(a, \cdot)\|_{L^1}.
\]
Thus, it remains to show weak type \((1, 1)\) bounds for \(T_\sigma\). We proceed by performing a decomposition for \(T_\sigma\), inspired by the \(L^2\) bounds. Our goal is to show the pointwise estimate
\[
|T_\sigma f(x)| \leq C_n \left( \sum_l 2^{ln} \sup_y \|P^\xi_l \sigma(y, \cdot)\|_{L^1} \right) M f(x),
\]
which implies the desired weak type bounds since \(M : L^1 \to L^{1, \infty}\).

To achieve that, we have to be a bit more careful than in the \(L^2\) case, since the rough cutoffs in the \(\xi\) variable will be insufficient to show (31).

For any integer \(l\), introduce smooth partition of unity, which is adapted to the cover \(R^n = \bigcup_{Q: \text{diam}(Q)=2^{-l}Q} \), that is a family of functions \(\{\psi_{l,Q}\}\), with \(\text{supp} \psi_{l,Q} \subset Q^*\) and
\[
|D^\alpha \psi_{l,Q}(\xi)| \leq C \alpha 2^{l|\alpha|}
\]
for every multiindex \(\alpha\). Choose and fix a family of arbitrary points \(\xi_Q \in Q\). By rescaling, one can choose \(\psi_{l,Q} := \psi(x, \xi_Q)\), where \(\text{supp} \psi_{l,Q} \subset \{|\xi| < 2\}\) and
\[
\sum_Q \psi_{l,Q}(2^l(\xi - \xi_Q)) = 1
\]
Write
\[
T_\sigma f(x) = \sum_{l \in Z} R^n \int P^\xi_l \sigma(x, \xi) e^{2\pi i \xi x} \hat{f}(\xi) d\xi = 
\]

\[
= \sum_{l \in Z} \sum_Q R^n \int P^\xi_l \sigma(x, \xi) \psi_{l,Q}(2^l(\xi - \xi_Q)) e^{2\pi i \xi x} \hat{f}(\xi) d\xi
\]

We now expand the \(P^\xi_l \sigma\) around \(\xi_Q\). We have
\[
P^\xi_l \sigma(x, \xi) \psi_{l,Q}(2^l(\xi - \xi_Q)) = \left( \sum_{\alpha: |\alpha| \geq 0} \frac{D^\xi_l P^\xi_l \sigma(x, \xi_Q)}{\alpha!} (\xi - \xi_Q)^\alpha \psi_{l,Q}(2^l(\xi - \xi_Q)) \right)
\]
Plugging that in the formula for \(T_\sigma f\) yields
\[
T_\sigma f(x) = \sum_{l, \alpha} (\alpha!)^{-1} \sum_Q 2^{-|\alpha|} D^\xi_l P^\xi_l \sigma(x, \xi_Q) \bar{Z}_{l,Q}^\alpha f(x).
\]
where
\[
\bar{Z}_{l,Q}^\alpha \hat{f}(\xi) = \psi_{l,Q}(2^l(\xi - \xi_Q))(2^l(\xi - \xi_Q))^\alpha \hat{f}(\xi) = \psi_{l,Q}^\alpha(2^l(\xi - \xi_Q)) \hat{f}(\xi), \text{i.e.} \psi_{l,Q}^\alpha(z) = \psi_{l,Q}(z) z^\alpha.
\]
By (25), we get
\[
|Z_{l,Q}^\alpha f(x)| \leq C n \|\psi_{l,Q}^\alpha\|_{L^1} M f(x).
\]
By the elementary properties of the Fourier transform
\[ \| \hat{\psi}_{l,Q}^\alpha \|_{L^1} = C_n |\alpha| \int |D_n^\alpha[\hat{\psi}_{l,Q}(\eta)]|d\eta \leq C_n |\alpha| \sum_{k=-\infty}^{\infty} 2^{|\alpha|k} \int |P_k[\hat{\psi}_{l,Q}](\eta)|d\eta \]

But by support considerations, \( P_k^\alpha[\hat{\psi}_{l,Q}] = 0 \) if \( k > 3 \). Also since \( P_k : L^1 \to L^1 \), we get
\[ \| \hat{\psi}_{l,Q}^\alpha \|_{L^1} \leq C_n |\alpha| \| P_k[\hat{\psi}_{l,Q}] \|_{L^1} \leq C_n |\alpha| \| \hat{\psi}_{l,Q} \|_{L^1} \leq C_n |\alpha| . \]

Thus, it remains to show for every \( k \) and contains the main ideas for the Littlewood-Paley partition of unity
\[ \text{5.1. ESTIMATES FOR HOMOGENEOUS OF DEGREE ZERO SYMBOLS} \]

We start with the Littlewood-Paley partition of unity \( P_t^{\xi/|\xi|} \). We have
\[ T^* g(x) = \sum_{l=0}^{\infty} \int e^{2\pi i \xi \cdot x} \left( \int e^{-2\pi i \xi \cdot y} [g(y) P_t^{\xi/|\xi|} q(y, \xi/|\xi|)] dy \right) d\xi \]

For every \( l \geq 0 \), introduce a partition of unity on \( S^{n-1} \), say \( \{K\} \), which consists of disjoint sets of diameter comparable to \( 2^{-l} \). One may form \( \{K\} \) by introducing a \( 2^{-l} \) net on \( S^{n-1} \), say \( \xi_l^m \), form the conic sets \( H_l^m = \{ \xi \in \mathbb{R}^n : |\xi/|\xi| - \xi_l^m | \leq 2^{-l} \} \) and construct \( K_l^m = H_l^m \setminus \bigcup_{j=0}^{m-1} H_j^l \). We have
\[ T^*_l g(x) = \sum_{l=0}^{\infty} \sum_m \int e^{2\pi i \xi \cdot x} \chi_{K_l^m}(\xi) \left( \int e^{-2\pi i \xi \cdot y} [g(y) P_t^{\xi/|\xi|} q(y, \xi/|\xi|)] dy \right) d\xi \]

Now, that the symbol is frequency localized around frequencies \( \sim 2^l \) and the sets \( K_l^m \cap S^{n-1} \) have diameters less than \( 2^{-l} \), we expand \( q(y, \xi/|\xi|) \) around an arbitrary point \( \theta_l^m \in K_l^m \). According to \( (28) \), we have for all \( \xi \in K_l^m \),
\[ q(y, \xi/|\xi|) = \sum_{\alpha \geq 0} \frac{D_2^\alpha q(y, \theta_l^m)}{\alpha!} (\xi/|\xi| - \theta_l^m)^\alpha . \]
We proceed to further bound the expression in (33) yields

$$T_\sigma^* g(x) = \sum_{l=0}^{\infty} \sum_{m} \sum_{\alpha} (\alpha!)^{-1} \int e^{2\pi i \xi \cdot x} \chi_{K_m^l} (\eta / |\xi| - \theta_m^l)^\alpha \times$$

$$\times \left( \int e^{-2\pi i \xi y} [g(y) D^{\alpha}_\xi P_{\xi}^{\ell / |\xi|} q(y, \theta_m^l)]dy \right) d\xi =$$

$$= \sum_{l=0}^{\infty} \sum_{m} \sum_{\alpha} (\alpha!)^{-1} Z_{l,m}[g(\cdot)2^{-l|\alpha|} D^{\alpha}_\xi P_{\xi}^{\ell / |\xi|} q(\cdot, \theta_m^l)],$$

where $Z_{l,m}^\alpha$ is given by the multiplier $\chi_{K_m^l} (\eta / |\xi| - \theta_m^l)^\alpha$. Note the disjoint support of the multipliers $\{Z_{l,m}^\alpha\}$ and $\|Z_{l,m}^\alpha\|_{L^2 \to L^2} = \sup_\xi |\chi_{K_m^l} (\eta / |\xi| - \theta_m^l)^\alpha| \leq 4^{|\alpha|}$. Take $L^2$ norm of $T_\sigma^* g$.

$$\|T_\sigma^* g\|_{L^2(\mathbb{R}^n)} \lesssim \sum_{\alpha} \left( \sum_{m} \left\| Z_{l,m}^\alpha [g(\cdot)2^{-l|\alpha|} D^{\alpha}_\xi P_{\xi}^{\ell / |\xi|} q(\cdot, \theta_m^l)] \right\|^2_{L^2} \right)^{1/2} \leq$$

$$\leq 4^{|\alpha|} \sum_{\alpha} \left( \sum_{m} \left\| g(\cdot)2^{-l|\alpha|} D^{\alpha}_\xi P_{\xi}^{\ell / |\xi|} q(\cdot, \theta_m^l) \right\|^2_{L^2} \right)^{1/2}.$$  

We proceed to further bound the expression in the $m$ sum. Since

$$\sum_{m} \left\| g(\cdot)2^{-l|\alpha|} D^{\alpha}_\xi P_{\xi}^{\ell / |\xi|} q(\cdot, \theta_m^l) \right\|^2_{L^2} = 2^{-2l|\alpha|} \int |g(y)|^2 \left( \sum_{m} \left\| D^{\alpha}_\xi P_{\xi}^{\ell / |\xi|} q(y, \theta_m^l) \right\|^2 \right) dy,$$

matters reduce to a good estimate for $\sum_{m} \left| D^{\alpha}_\xi P_{\xi}^{\ell / |\xi|} q(y, \theta_m^l) \right|^2$. We proceed as before. By (28), we get for all $\eta \in K_m^l \cap S^{n-1},$

$$D^{\alpha}_\xi P_{\xi}^{\ell / |\xi|} q(y, \theta_m^l) = \sum_{\beta \geq 0} \frac{D^{\alpha+\beta}_\xi P_{\eta/|\eta|}^{\beta} q(y, \eta)}{\beta!} (\theta_m^l - \eta)^\beta.$$  

Averaging over $K_m^l \cap S^{n-1}$ and taking into account $|K_m^l \cap S^{n-1}| \sim 2^{l(n-1)}$ yields

$$(\sum_{m} \left| D^{\alpha}_\xi P_{\xi}^{\ell / |\xi|} q(y, \theta_m^l) \right|^2)^{1/2} \lesssim$$

$$\lesssim \sum_{\beta} \frac{2^{-l|\beta|}}{\beta!} \left( \sum_{m} |K_m^l \cap S^{n-1}|^{-1} \int_{K_m^l \cap S^{n-1}} \left| D^{\alpha+\beta}_\xi P_{\eta/|\eta|}^{\beta} q(y, \eta) \right|^2 d\eta \right)^{1/2}$$

$$\lesssim 2^{l(n-1)/2} \sum_{\beta} \frac{2^{-l|\beta|}}{\beta!} \| D^{\alpha+\beta}_\xi P_{\eta/|\eta|}^{\beta} q(y, \cdot) \|_{L^2}$$

$$\lesssim 2^{l(n-1)/2+|\alpha|} \sum_{\beta} \frac{C_n^{\alpha+|\beta|}}{\beta!} \| P_{\eta/|\eta|}^{\beta} q(y, \cdot) \|_{L^2(S^{n-1})}$$

$$\leq C_n^{\alpha+2l(n-1)/2} \| P_{\eta/|\eta|}^{\beta} q(y, \cdot) \|_{L^2(S^{n-1})}.$$
Putting this back into the estimate for \( \|T^*_\sigma g\|_{L^2(\mathbb{R}^n)} \) implies
\[
\|T^*_\sigma g\|_{L^2(\mathbb{R}^n)} \lesssim \|g\|_{L^2} \sum_l 2^{l(n-1)/2} \sup_y \|P_l^{\eta/|\eta|} q(y, \cdot)\|_{L^2(\mathbb{S}^{n-1})}
\]
as desired.

5.2. \( L^p \) estimates for homogeneous of degree zero multipliers. Fix \( p : 2 \leq p < \infty \). To verify the estimate \( \|T\|_{L^p_{\ell,1} \rightarrow L^p} \), it will suffice to fix \( k \) and show
\begin{equation}
\|T(P_k f)\|_{L^p} \leq C\|f\|_{L^p}.
\end{equation}
Furthermore, by the scale invariance of the quantity \( \sum_l 2^{l(n-1)} \sup_y \|P_l^{\xi/|\xi|} q(y, \cdot)\|_{L^1(\mathbb{S}^{n-1})} \) this is equivalent to verifying (35) only for \( k = 0 \). That is, it suffices to establish the \( L^p \), \( p \geq 2 \) boundedness of the operator
\[
G f(x) = \int_{\mathbb{R}^n} q(x, \xi/|\xi|) e^{2\pi i \xi x/\varphi(|\xi|)} \hat{f}(\xi) d\xi.
\]
provided the multiplier \( m \) satisfies
\[
\sum_l 2^{l(n-1)} \sup_y \|P_l^{\xi/|\xi|} q(y, \cdot)\|_{L^1(\mathbb{S}^{n-1})} < \infty.
\]

Next, we make the angular decomposition as in the case of the \( L^2 \) estimates for the adjoint operator \( G^* \). However, this time we will have to be more careful and instead of the rough cutoffs \( \chi_{K_m^l} \), we shall use a smoothed out versions of them. Fix \( l \). Choose and fix a \( 2^{-l} \) net \( \theta_m^l \in K_m^l \cap \mathbb{S}^{n-1} \), so that the family \( \{ \theta \in \mathbb{S}^{n-1} : |\theta^l_m - \theta| \leq 2^{-l}\} \) has the finite intersection property. Introduce a family of functions \( \varphi_{l,m} : \mathbb{R}^n \to [0, 1] \), so that for every \( \xi \in \mathbb{R}^n \),
\begin{equation}
\sum_m \varphi_{l,m}(2^l(\xi/|\xi| - \theta^l_m)) = 1
\end{equation}
\[
\sup_{\eta} |D_{\eta}^\beta \varphi_{l,m}(\eta)| \leq C_\beta.
\]
In other words, the family of functions \( \{\varphi_{l,m}\} \) provides a smooth partition of unity, subordinated to the cover \( \{K_m^l\} \).

As before, write
\[
G^* g(x) = \sum_{l \geq 0} \int_{\mathbb{R}^n} e^{2\pi i \xi x} \varphi(|\xi|) \int e^{-2\pi i |\xi| y} [g(y) P_l^{\xi/|\xi|} q(y, \xi/|\xi|)] dy d\xi.
\]
Inserting the partition of unity discussed above into the \( (l) \) term of the last formula for \( G^* \) yields
\[
G^* g(x) = \sum_{l \geq 0} \sum_m \int e^{2\pi i \xi (x-y)} \varphi_{l,m}(2^l(\xi/|\xi| - \theta^l_m)) \varphi(|\xi|) [g(y) P_l^{\xi/|\xi|} q(y, \xi/|\xi|)] dy d\xi.
\]
Following the same strategy as before, we expand \( q(y, \xi/|\xi|) \) around \( \theta^l_m \in K_m^l \). According to (28), we have
\[
P_l^{\xi/|\xi|} q(y, \xi/|\xi|) = \sum_{\alpha \geq 0} \frac{D_\xi^\alpha P_l^{\xi/|\xi|} q(y, \theta^l_m)}{\alpha!} (\xi/|\xi| - \theta^l_m)^\alpha.
\]
Of course, the last formula is useful only when \(|\xi/|\xi| - \theta^l_m| \lesssim 2^{-l}\), in particular on the support of \(\varphi_{l,m}(2^l(\xi/|\xi| - \theta^l_m))\). This gives us the representation

\[
G^* g = \sum_{l \geq 0} \sum_m \sum_{|\alpha| \geq 0} (\alpha!)^{-1} \int e^{2\pi i \xi y} \varphi_{l,m}(2^l(\xi/|\xi| - \theta^l_m))(\xi/|\xi| - \theta^l_m)\alpha \varphi(|\xi|) \times
\]

\[
\times \int e^{2\pi i y} g(y) P^\xi_{l,|\xi|} D^\xi_{l,|\xi|} q(y, \theta^l_m) dy d\xi =
\]

\[
= \sum_{l \geq 0} \sum_m \sum_{|\alpha| \geq 0} (\alpha!)^{-1} Z^\alpha_{l,m} [g(\cdot) 2^{-l|\alpha|} D^\alpha_{l,|\xi|} P^\xi_{l,|\xi|} q(\cdot, \theta^l_m)]
\]

where

\[
\tilde{Z}^\alpha_{l,m} f(\xi) = \varphi_{l,m}(2^l(\xi/|\xi| - \theta^l_m)) 2^{-l|\alpha|} (\xi/|\xi| - \theta^l_m)\alpha \varphi(|\xi|) \tilde{f}(\xi) = \varphi_{l,m}(\xi/|\xi| - \theta^l_m) \varphi(|\xi|) \tilde{f}(\xi).
\]

Taking \(L^p\) norm of \(G^* g\), we get

\[
\|G^* g\|_{L^p} \leq \sum_{l \geq 0} \sum_m \sum_{|\alpha| \geq 0} (\alpha!)^{-1} \|Z^\alpha_{l,m} [g(\cdot) 2^{-l|\alpha|} D^\alpha_{l,|\xi|} P^\xi_{l,|\xi|} q(\cdot, \theta^l_m)]\|_{L^p}
\]

Lemma\[ in the Appendix allows us to treat expressions of the type \(\|\sum_m Z^\alpha_{l,m} g^\alpha_m\|_{L^p}\). Indeed, according to (45), we have

\[
\|\sum_m Z^\alpha_{l,m} [g(\cdot) 2^{-l|\alpha|} D^\alpha_{l,|\xi|} P^\xi_{l,|\xi|} q(\cdot, \theta^l_m)]\|_{L^p} \lesssim \left(\sum_m \|g(\cdot) 2^{-l|\alpha|} D^\alpha_{l,|\xi|} P^\xi_{l,|\xi|} q(\cdot, \theta^l_m)\|^p_{L^p}\right)^{1/p}
\]

\[
= 2^{-l|\alpha|} \left(\int |g(y)|^p \left(\sum_m |D^\alpha_{l,|\xi|} P^\xi_{l,|\xi|} q(y, \theta^l_m)|^p dy\right)^{1/p}\right)
\]

By virtue of (34), we get

\[
D^\alpha_{l,|\xi|} P^\xi_{l,|\xi|} q(y, \theta^l_m) = \sum_{\beta \geq 0} \frac{D_{l}^{\alpha+\beta} P_{l}^{\eta/|\eta|} q(y, \eta)}{\beta!} (\theta^l_m - \eta)^\beta.
\]

whence by averaging\[ over \(K_{m}^{l} \cap S^{n-1}\),

\[
\left(\sum_m |D^\alpha_{l,|\xi|} P^\xi_{l,|\xi|} q(y, \theta^l_m)|^p\right)^{1/p} \leq
\]

\[
\lesssim \sum_{\beta} \frac{2^{-l|\beta|}}{\beta!} \left(\sum_m |K_{m}^{l} \cap S^{n-1}|^{-1} \int_{K_{m}^{l} \cap S^{n-1}} |D_{l}^{\alpha+\beta} P_{l}^{\eta/|\eta|} q(y, \eta)|^p d\eta\right)^{1/p}
\]

\[
\lesssim 2^{((n-1)/p) - |\alpha|} \sum_{\beta} \frac{2^{-l|\beta|}}{\beta!} \|D_{l}^{\alpha+\beta} P_{l}^{\eta/|\eta|} q(y, \cdot)\|_{L^p(S^{n-1})}
\]

\[
\lesssim 2^{((n-1)/p + |\alpha|)} \sum_{\beta} \frac{C_{n}^{l|\alpha|+|\beta|}}{\beta!} \|P_{l}^{\eta/|\eta|} q(y, \cdot)\|_{L^p(S^{n-1})}
\]

\[
\leq C_{n}^{l|\alpha|+|\beta|} 2^{((n-1)/p + |\alpha|)} \|P_{l}^{\eta/|\eta|} q(y, \cdot)\|_{L^p(S^{n-1})}.
\]

\[this step is identical to the one performed earlier for the \(L^2\) bounds, except that now the \(l^2\) sums are replaced by \(l^p\) sums.
All in all, 

\[ \|G^*g\|_{L^p} \leq C_n \|g\|_{L^p} \sum_{l \geq 0} \sum_{|\alpha| \geq 0} (\alpha!)^{-1} C_n^{|\alpha|} 2^{|(n-1)/p|} \sup_y \|P_{\xi}^{y/|\xi|} q(y, \cdot)\|_{L^p(S^{n-1})} \]

\[ \leq C_n \|g\|_{L^p} \sum_l 2^{l(n-1)/p} \sup_y \|P_{\xi}^{y/|\xi|} q(y, \cdot)\|_{L^p(S^{n-1})}, \]

as desired.

6. **Counterexamples**

6.1. **Theorem 1 is sharp.** Given \( p > 2 \), we will construct an explicit symbol \( \sigma(x, \xi) \), so that the corresponding PDO \( T_\sigma \) is not bounded on \( L^2(\mathbb{R}^n) \), but which satisfies 

\[ \sup_x |D^\alpha \sigma(x, \xi)| \leq C_n |\xi|^{-|\alpha|} \]

and \( \sup_{x \neq y} \|\sigma(x, \cdot)\|_{1,p,1/p} < \infty \). The construction is a minor modification of the standard example of a symbol in \( S^0_{1,1} \), which is not bounded on \( L^2 \), see for example [21], page 272. We carry out the construction in \( n = 1 \), but this can be easily generalized to higher dimensions.

For the given \( p > 2 \), fix small \( 0 < \delta < 1/2 \), so that \( 2 + 4\delta/(1 - 2\delta) < p \). Define

\[ \sigma(x, \xi) := \sum_{j=8}^{\infty} \frac{e^{-2\pi i 2^j x}}{j^{1/2-\delta}} \varphi(2^{-j} \xi), \]

where the function \( \varphi \) is \( C^\infty \), -supported in \( 1/2 \leq |\xi| \leq 3/2 \), and \( \varphi(\eta) = 1 \) for all \( 3/4 \leq |\eta| \leq 5/4 \).

To show the unboundedness of \( T_\sigma \) on \( L^2 \), let us test it against the function

\[ f_N(x) = \sum_{j=8}^{N} \frac{e^{-2\pi i 2^j x}}{j^{1/2-\delta}} f_0(x), \]

where \( f_0 \) is a Schwartz function, whose Fourier transform is supported in \( \{\xi : |\xi| \leq 1/10\} \).

Clearly \( \|f_N\|_{L^2} = (\sum_{j=8}^{N} j^{1+2\delta})^{1/2} \|f_0\|_{L^2} \leq (\sum_{j=1}^{\infty} j^{1+2\delta})^{1/2} \|f_0\|_{L^2} = C_8 \|f_0\|_{L^2} \), while

\[ T_\sigma f_N(x) = \sum_{j_1 \geq 8, N \geq j_2 \geq 8} \int \frac{e^{-2\pi i 2^{j_1} x}}{j_1^{1/2-\delta}} \frac{\varphi(2^{-j_2} \xi)}{j_2^{1/2+\delta}} \hat{f}_0(\xi - 2^{j_2}) e^{2\pi i \xi x} d\xi. \]

Clearly, by Fourier support considerations the terms \( j_1 \neq j_2 \) disappear and we get

\[ T_\sigma f_N(x) = (\sum_{N \geq j \geq 8} j^{-1}) f_0(x), \]

whence \( \|T_\sigma\|_{L^2 \to L^2} \geq \ln(N) \), whence \( T_\sigma \) is not bounded on \( L^2 \).

On the other hand, it is clear that for \( |\xi| > 1 \),

\[ \sup_x |D^\alpha \sigma(x, \xi)| \sim |\xi|^{-|\alpha|} \ln^{1/2}(|\xi|) \leq |\xi|^{-|\alpha|}. \]

\(^{11}\)The reason for this choice of \( \delta \) will become apparent in the proof below.
Finally, to estimate $\sup_x \|\sigma(x, \cdot)\|_{W^{p,1/p}}$, write
\[
\sigma = \sum_{s=3}^{\infty} \sum_{j=2^s}^{2^{s+1}} e^{-2\pi j 2^j x} \varphi(2^{-j} \xi) = \sum_{s=3}^{\infty} \sigma^s,
\]
By the convexity of the norms, we have with $\theta : 1/p = \theta/2$,
\[
\|\sigma^s(x, \cdot)\|_{W^{p,1/p}} \leq \|\sigma^s(x, \cdot)\|_{H^{1/2}}^{\theta} \|\sigma^s(x, \cdot)\|_{L^\infty}^{(1-\theta)}
\]
It is now easy to compute the norms on the right hand side. We have
\[
\sup_x \|\sigma^s(x, \cdot)\|_{H^{1/2}} \sim \left(\sum_{j=2^s}^{2^{s+1}} \frac{1}{j^{1/2-\delta}}\right)^{1/2} \sim 2^{\delta s}.
\]
On the other hand,
\[
\|\sigma^s(x, \cdot)\|_{L^\infty} \sim 2^{-s(1/2-\delta)},
\]
whence $\sup_x \|\sigma^s(x, \cdot)\|_{W^{p,1/p}} \leq 2^{s(\delta\theta - (1/2-\delta)(1-\theta))}$. Clearly, such an expression dyadically sums in $s \geq 3$, provided $\delta\theta < (1/2 - \delta)(1 - \theta)$ or equivalently $p > 2 + 4\delta/(1 - 2\delta)$.

6.2. Proposition 1: Theorem 3 is sharp.

Proof. (Proposition 1) We construct a sequence of symbols $\sigma_\delta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^1$, so that for a fixed Schwartz function $f$
\[
\lim_{\delta \to 0^+} |T_{\sigma_\delta}f| = |H_\ast f(x)| = \sup_{u \in S^1} |H_u f(x)|
\]
Since we already know, [12], that $H_\ast$ is unbounded on $L^2(\mathbb{R}^2)$, we should have
\[
(37) \quad \lim_{\delta \to 0^+} \|T_{\sigma_\delta}\|_{L^2 \to L^2} = \infty.
\]
In our construction $\sigma_\delta$ will depend on $f$, but it is still clear that one can achieve (37). Namely, take a sequence $f_N : \|f_N\|_{L^2(\mathbb{R}^2)} = 1$, so that $\|H_\ast(f_N)\|_{L^2(\mathbb{R}^2)} \geq N$. Then construct $\sigma_{N,\delta}$, so that $\lim_{\delta \to 0^+} |T_{\sigma_{N,\delta}}f_N| = H_\ast f_N$. Then clearly,
\[
\lim_{N \to \infty, \delta \to 0^+} \|T_{\sigma_{N,\delta}}\|_{L^2 \to L^2} = \infty.
\]
Now, from the $L^2$ boundedness results of Theorem 3 (or rather the lack thereof), we must have
\[
(38) \quad \lim_{\delta \to 0^+} \|P_{\xi/|\xi|} \sigma_\delta(x, \cdot)\|_{L^2(S^1)} = \infty.
\]
On the other hand, we will see that $\sup_{x,\xi,\delta} |\sigma_\delta(x, \xi)| \leq 1$ and
\[
(39) \quad \sup_{\delta, x} \|\sigma_\delta(x, \cdot)\|_{W^{1,1}(S^1)} < \infty.
\]
Note in contrast that (at least heuristically) (38) states
\[
\lim_{\delta \to 0^+} \sup_x \|\sigma_\delta(x, \cdot)\|_{B^{1/2}_{2,1}} = \infty
\]
and by the Sobolev embedding estimate on the sphere (27) (and up to the usual Besov spaces adjustments at the endpoints), one should have that the quantity in (39) (at least in principle) controls (38). Having both (38) and (39) for a concrete example suggests that the conditions imposed in Theorem 3 are extremely tight.
Let us now describe the construction of $\sigma_\delta$. First of all,
\[
H_\ast f(x) = \sup_{u \in S^1} |H_u f(x)| = \sup_{u \in S^1} |\int \text{sgn}(u \cdot \xi/|\xi|) \hat{f}(\xi)e^{2\pi i x \cdot \xi} d\xi| = \\
= |\int \text{sgn}(u(x) \cdot \xi/|\xi|) \hat{f}(\xi)e^{2\pi i x \cdot \xi} d\xi|,
\]
for some measurable function $u(x) : R^1 \to S^1$. Clearly $u(x)$ will depend on the function $f$, see the remarks above after (37).

Introduce a function $\psi : \psi \in C^\infty, -1 \leq \psi(x) \leq 1$, and so that $\psi(z) = -1 : z \in (-\infty, -1], \psi(z) = 1 : z \in [1, \infty)$. Clearly
\[
H_\ast f(x) = \lim_{\delta \to 0^+} T_{\sigma_\delta} f(x) = \lim_{\delta \to 0^+} |\int \psi \left( \frac{u(x) \cdot \xi/|\xi|}{\delta} \right) \hat{f}(\xi)e^{2\pi i x \cdot \xi} d\xi|,
\]
that is $\sigma_\delta(x, \xi/|\xi|) = \psi(\frac{u(x) \cdot \xi/|\xi|}{\delta})$, for which we will verify (39), while it is clearly bounded in absolute value by one.

We pause for a second to comment on the particular form of $T_{\sigma_\delta}$. Note that the function $u(x)$ in general will not be smooth\footnote{Note that under some extra smoothness assumptions on $u$, Lacey and Li have managed to prove $L^2$ boundedness!} and therefore will not fall under the scope of any standard boundedness theory for PDO. Also, note that while the map $\xi \to \sigma_\delta(x, \xi)$ is definitely smooth, its derivatives are quite large and blow up at the important limit $\delta \to 0$. This shows that in order to treat maximal operators, build upon singular multipliers (as is the case here), one needs the full strength of Theorems 1, 3 and beyond.

Going back to the proof of (39), compute
\[
\frac{\partial \sigma}{\partial \xi_1} = \frac{\psi'(u(x)\xi/|\xi|)}{\delta|\xi|^3} \left( u_1(x)\xi_2^2 - u_2(x)\xi_1\xi_2 \right) = \frac{\psi'(u(x)\xi/|\xi|)}{\delta|\xi|^3} u(x) \cdot \left( \frac{\xi}{|\xi|} \right)^{\perp}
\]
\[
\frac{\partial \sigma}{\partial \xi_2} = \frac{\psi'(u(x)\xi/|\xi|)}{\delta|\xi|^3} \left( u_1(x)\xi_1^2 - u_2(x)\xi_1\xi_2 \right) = \frac{\psi'(u(x)\xi/|\xi|)}{\delta|\xi|^3} u(x) \cdot \left( \frac{\xi}{|\xi|} \right)^{\perp}
\]
Clearly, the supports of both derivatives are in $\xi : |u(x) \cdot \xi/|\xi|| \leq \delta << 1$. Also, on their support, $|\nabla \sigma(x, \xi/|\xi|)| \sim C\delta^{-1}$. It follows
\[
\|\sigma_\delta(x, \cdot)\|_{W^{1,1}(S^1)} \leq \int_{\xi \in S^1 : |u(x) \cdot \xi/|\xi| \leq \delta} |\nabla \sigma_\delta(x, \xi)| d\xi \leq C,
\]
where $C$ is independent of $\delta$. This was the claim in (39).

7. Appendix

7.1. Estimates for Fourier transforms of functions supported on small spherical caps.

In this section, we present a pointwise estimate for the kernels of multipliers that restrict the Fourier transform to a small spherical cap.
Lemma 2. Let $\theta \in \mathbb{S}^{n-1}$ and $\varphi$ is a $C^\infty$ function with $\text{supp} \varphi \subset \{\xi : 1/2 \leq |\xi| \leq 2\}$. Let also $l > 0$ be any integer. Define $K_{l,\theta}$ to be the inverse Fourier transform of $\varphi(2^l(\xi/|\xi| - \theta))\varphi(|\xi|)$, that is

$$K_{l,\theta}(x) = \int \varphi(2^l(\xi/|\xi| - \theta))\varphi(|\xi|)e^{2\pi i x \cdot \xi} d\xi.$$  

Then, for every $N > 0$, there exists $C_N$, so that

$$|K_{l,\theta}(x)| \leq C_N 2^{-l(n-1)}(1 + |\langle x, \theta \rangle|)^{-N} (1 + 2^{-l}|x - \langle x, \theta \rangle\theta|)^{-N}.\quad (40)$$

That is, in the direction of $\theta$, the function has any polynomial decay, while in the directions transversal to $\theta$, one has decay like $(2^{-l} < x' >)^{-N}$, where $x = \langle x, \theta \rangle\theta + x'$. In particular,

$$\sup_{\theta, l} \int |K_{l,\theta}(x)| dx \leq C_n < \infty,\quad (41)$$

where the constant $C_n$ depends on $\|\varphi\|_{L^\infty}$ and the smoothness properties of $\varphi$.

Proof. By rotation symmetry, we can assume without loss of generality that $\theta = e_1 = (1, 0, \ldots, 0)$. Fix $l$ and drop the subindices for notational convenience. We will need to show that for every $x = x_1 e_1 + x'$,

$$|K(x)| \leq C_N 2^{-l(n-1)} < x_1 >^{-N} < 2^{-l}x' >^{-N} \quad (42)$$

First of all, by support considerations, one has $|K(x)| \leq C_n 2^{-l(n-1)}$. Next, we will show that integration by parts in the variable $\xi_1$ yields

$$K(x) = x_1^{-1}\tilde{K}(x),\quad (43)$$

whereas integration by parts in each of the variables $\xi_j : j = 2, \ldots, n$ yield

$$K(x) = (2^{-l}x_j)^{-1}\tilde{K}(x),\quad (44)$$

where $\tilde{K}(x)$ is different in each instance, but it has the form

$$\tilde{K}(x) = \int \varphi_1(2^l(\xi/|\xi| - e_1))\varphi_2(|\xi|)e^{2\pi i x \cdot \xi} d\xi,$$

for some $C^\infty$ functions $\varphi_1, \varphi_2$ with $\text{supp} \varphi_k \subset \{\xi : 1/2 \leq |\xi| \leq 2\}, k = 1, 2$.

That is enough to deduce $(42)$ and thus Lemma 2. Indeed, by iterating $(43)$ and $(44)$, one gets the formula

$$K(x) = x_1^{-N_1}(\prod_{j=2}^n (2^{-l}x_j)^{N_j})^{-1}\tilde{K}_{N_1,\ldots,N_n}(x),$$

for any $n$ tuple of integers $(N_1, N_2, \ldots, N_n)$. Combining this representation with the estimate $|\tilde{K}_{N_1,\ldots,N_n}(x)| \leq C_{n,N_1,\ldots,N_n}2^{-l(n-1)}$, one deduces $(42)$.

---

13As we shall see the functions $\varphi_1, \varphi_2$ are obtained in a specific way from $\varphi$ via the operations differentiation and multiplication by monomial.
For (43), integration by parts yields
\[
K(x) = -\frac{1}{2\pi i x_1} \int \partial_{\xi_1} \left[ \varphi(2^l (\xi/|\xi| - e_1)) \varphi(|\xi|) \right] e^{2\pi i x \xi} d\xi = \\
= -\frac{1}{2\pi i x_1} \int \sum_{j=2}^{n} 2^l \frac{\xi_j^2}{|\xi|^2} \partial_j \varphi(2^l (\xi/|\xi| - e_1)) \varphi(|\xi|)|\xi|^{-1} e^{2\pi i x \xi} d\xi + \\
+ \frac{1}{2\pi i x_1} \int \sum_{j=2}^{n} 2^l \frac{\xi_j \xi_1}{|\xi|^2} \partial_j \varphi(2^l (\xi/|\xi| - e_1)) \varphi(|\xi|)|\xi|^{-1} e^{2\pi i x \xi} d\xi + \\
- \frac{1}{2\pi i x_1} \int \varphi(2^l (\xi/|\xi| - e_1)) \varphi'(|\xi|) \frac{\xi_1}{|\xi|} e^{2\pi i x \xi} d\xi
\]

The third term is clearly in the form \( x^{-1} \tilde{K}(x) \), by taking into account that \( \text{supp} \varphi \subset \{ \xi : 1/2 \leq |\xi| \leq 2 \} \).

The second term above can be rewritten in the form
\[
\frac{1}{2\pi i x_1} \int \varphi(2^l (\xi/|\xi| - e_1)) \frac{\xi_1}{|\xi|} \varphi(|\xi|)|\xi|^{-1} e^{2\pi i x \xi} d\xi = \\
= \frac{1}{2\pi i x_1} \int \left[ \varphi(2^l (\xi/|\xi| - e_1)) + 2^{-l} \varphi(2^l (\xi/|\xi| - e_1)) \right] \varphi(|\xi|)|\xi|^{-1} e^{2\pi i x \xi} d\xi = \\
= x^{-1} \tilde{K}(x)
\]

where \( \varphi_1(\eta) = \sum_{j=2}^{n} \eta_j \partial_{\eta_j} \varphi(\eta) \) and \( \tilde{\varphi}_1(\eta) = \eta_1 \varphi_1(\eta) \).

Analogously, one can rewrite the first term of \( K(x) \) in the form \( 2^{-l} x^{-1} \tilde{K}(x) \), i.e. it has an extra decay factor of \( 2^{-l} \). This establishes (43).

For (44), we obtain by integration by parts in \( \xi_j, 2 \leq j \leq n \),
\[
K(x) = \frac{1}{2\pi i x_j} \int \partial_j \varphi(2^l (\xi/|\xi| - e_1)) 2^l \frac{\xi_j \xi_1}{|\xi|^3} \varphi(|\xi|) e^{2\pi i x \xi} d\xi + \\
+ \frac{1}{2\pi i x_j} \sum_{k \neq j, k=2}^{n} \int \partial_k \varphi(2^l (\xi/|\xi| - e_1)) 2^l \frac{\xi_k \xi_j}{|\xi|^3} \varphi(|\xi|) e^{2\pi i x \xi} d\xi + \\
- \frac{1}{2\pi i x_j} \int \partial_j \varphi(2^l (\xi/|\xi| - e_1)) 2^l \left( \sum_{k \neq j, k=1}^{n} \frac{\xi_k^2}{|\xi|^2} \right) \varphi(|\xi|) e^{2\pi i x \xi} d\xi + \\
- \frac{1}{2\pi i x_j} \int \varphi(2^l (\xi/|\xi| - e_1)) \varphi'(|\xi|) \frac{\xi_j}{|\xi|} e^{2\pi i x \xi} d\xi
\]

By performing similar analysis as in the proof of (43), we easily see that the first term above is in the form \( x^{-1} \tilde{K}(x) \), the second and the fourth terms are in fact even better, since they are in the form \( 2^{-l} x^{-1} \tilde{K}(x) \). The third term has two types of terms. Clearly,
\[
\frac{1}{2\pi i x_j} \int \partial_j \varphi(2^l (\xi/|\xi| - e_1)) 2^l \left( \sum_{k \neq j, k=2}^{n} \frac{\xi_k^2}{|\xi|^2} \right) \varphi(|\xi|) e^{2\pi i x \xi} d\xi + 
\]
is of the form $2^{-l}x_j^{-1}\tilde{K}(x)$, while lastly,
\[
\frac{1}{2\pi i x_j} \int \partial_j \varphi(2^l(\xi/|\xi| - e_1)) 2^l \frac{\xi_j^2}{|\xi|^2} \varphi(|\xi|) e^{2\pi i x \cdot \xi} \, d\xi
\]
is of the form $2^l x_j^{-1}\tilde{K}(x)$, as is the statement of (44). □

7.2. $l^p$ functions of cone multipliers. In this section, we discuss a simple extension of Lemma 2, which is concerned with appropriate $L^p$ bounds for $l^p$ functions of such cone multipliers.

**Lemma 3.** Let $l >> 1$ and $\{\theta^l_m\}_m$ be a $2^{-l}$ net in $S^{n-1}$, so that the family $\{\theta^l \in S^{n-1} : |\theta^l_m - \theta| \leq 2^{-l}\}_m$ has the finite intersection property. Define
\[
\hat{P}_m f(\xi) = \varphi_{l,m}(2^l(\xi/|\xi| - \theta^l_m)) \varphi(|\xi|) \hat{f}(\xi).
\]
where $\varphi_{l,m}$ are as in (36). Then one has
\[
\sum_m \|P_m g_m\|_{L^p(\mathbb{R}^n)} \leq C \left( \sum_m \|g_m\|_{L^p}^p \right)^{1/p} \quad \text{if } 1 \leq p \leq 2
\]
\[
\left( \sum_m \|P_m g\|_{L^q(\mathbb{R}^n)}^q \right)^{1/q} \leq C \|f\|_{L^q} \quad \text{if } 2 \leq q \leq \infty.
\]

**Proof.** Since (45) and (46) are dual, it will suffice to check (46). Next, the $L^2$ estimate is trivial by the Plancherel’s theorem and the finite intersection property of the supports of $\varphi_{l,m}(2^l(\xi/|\xi| - \theta^l_m))$. Thus, by interpolation it suffices to check
\[
\sup_m \|P_m g\|_{L^\infty} \leq C \|g\|_{L^\infty}.
\]
But $P_m g(x) = \hat{K}_{l,\theta^l_m} \ast g(x)$ and so
\[
\|P_m g\|_{L^\infty} \leq \|K_{l,\theta^l_m}\|_{L^1} \|g\|_{L^\infty} \leq C \|g\|_{L^\infty}.
\]
where the last inequality follows from (41). □

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