LOW REGULARITY GLOBAL WELL-POSEDNESS FOR THE KLEIN-GORDON-SCHRÖDINGER SYSTEM WITH THE HIGHER ORDER YUKAWA COUPLING

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Abstract
In this paper, we consider the Klein-Gordon-Schrödinger system with the higher order Yukawa coupling in $\mathbb{R}^{1+1}$, and prove the local and global wellposedness in $L^2 \times H^{1/2}$. The method to be used is adapted from the scheme originally by Colliander J., Holmer J., Tzirakis N. [8] to use the available $L^2$ conservation law of $u$ and control the growth of $n$ via the estimates in the local theory.

Key Words: Cauchy problem, Global solution, Klein-Gordon-Schrödinger system, Strichartz estimates.

1 Introduction

The Cauchy problem
\[
\begin{cases}
  iu_t + \Delta u = -nu, & x \in \mathbb{R}^d, t \in \mathbb{R}; \\
  n_{tt} + (1 - \Delta)n = |u|^2, & x \in \mathbb{R}^d, t \in \mathbb{R}; \\
u(0) = u_0, & n(0) = n_0, n_t(0) = n_1.
\end{cases}
\]

have been considered in [3, 10, 11, 12]. Here $u : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{C}$ is the nucleon field and $n : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ is the meson field. H. Pecher [19] considered the system (1.1) in $\mathbb{R}^{3+1}$ by Fourier truncation method [6]. N. Tzirakis [21] consider the same system in one, two,
three dimension by I-method [15]. They also obtained a polynomial in time bound for the
growth of the norms. Recently, using the available $L^2$ conservation law of $u$ and controlling
the growth of $n$ via the estimate in the local theory, J. Colliander et al. [8] obtained the
optimal global well-posedness of (1.1) in $\mathbb{R}^{3+1}$. It is also applicable to 1D and 2D case.

Just as in [4, 7], the system (1.1) is naturally generalized to the following system

\[
\begin{aligned}
&\text{for } x \in \mathbb{R}, t \in \mathbb{R}; \\
&\quad \partial_x^2 u = -m|u|^{2(m-1)}u, \\
&\quad n = |u|^{2m},
\end{aligned}
\]

(1.2)

The restricted case $F(s) = s^m$ in GKLS will be called KLS$_m$. In this paper, we only consider
the case $1 \leq m < 2, d = 1$, that is

\[
\begin{aligned}
&\text{for } x \in \mathbb{R}, t \in \mathbb{R}; \\
&\quad \partial_x^2 u = -mn|u|^{2(m-1)}u, \\
&\quad n = |u|^{2m},
\end{aligned}
\]

(1.2)

The reason that the higher order powers are introduced into the physically relevant
dispersive PDEs is to adjust the strength of the nonlinearity relative to the dispersion to
work toward understanding the balance between the two effects. We give the similar scaling
analyses in next section.

It is well known that the following conservation laws hold for (1.2):

\[
\begin{aligned}
&\text{for } x \in \mathbb{R}, t \in \mathbb{R}; \\
&\quad \partial_x^2 u = -mn|u|^{2(m-1)}u, \\
&\quad n = |u|^{2m},
\end{aligned}
\]

(1.2)

The idea is to use the available $L^2$ conservation law of $u$ and control the growth of $n$ via
the estimates in the local theory.

Our main result is the following theorem

**Theorem 1.1.** Let $1 \leq m < 2$, then the KLS$_m$ in dimension $d = 1$ is global well-
posedness for $(u_0, n_0, n_1) \in L^2 \times H^{1/2} \times H^{-1/2}$. More precisely, the solution
$(u, n) \in C(\mathbb{R}; L^2) \times C(\mathbb{R}; H^{1/2})$ satisfies for $t \in \mathbb{R}$,

\[
\|u(t)\|_{L^2} = \|u_0\|_{L^2}
\]

and

\[
\|n(t)\|_{H^{1/2}} + \|\partial_t n(t)\|_{H^{-1/2}} \lesssim \exp(c|t|\|u_0\|^{4m-2}_{L^2}) \max(\|n_0\|_{H^{1/2}} + \|n_1\|_{H^{-1/2}}, \|u_0\|^{2m}_{L^2}).
\]
The paper is organized as follows.

In Section 2, we first give some scaling analyses of the criticality, then give the linear and nonlinear estimates along with Ginibre, Tsutsumi and Velo \[13\] in the $X^{s,b}$ spaces, which was introduced by Bourgain \[5\], Kenig, Ponce and Vega \[13\], Klainerman and Machedon \[17, 18\]. We also can refer to Foschi \[9\], Grünrock \[14\] and Selberg \[20\].

For the free dispersive equation of the form

$$iu_t + \varphi(D_x)u = 0, \quad D_x = -i\partial_x,$$

where $\varphi$ is a measurable function, let $X^{s,b}_\varphi$ be the completion of $S(\mathbb{R})$ with respect to

$$\|f\|_{X^{s,b}_\varphi} := \|\langle \xi \rangle ^s \langle \tau \rangle ^b \mathcal{F}(e^{-it\varphi(D_x)}f(x,t))\|_{L^2_{\xi,\tau}}.$$

In general, we use the notation $X^{s.b}_\pm$ for $\varphi(\xi) = \pm \xi > 0$ and $X^{s,b}$ for $\varphi(\xi) = -|\xi|^2$ without confusion. For a given time interval $I$, we define

$$\|f\|_{X^{s,b}(I)} = \inf_{\tilde{f} = f} \|\tilde{f}\|_{X^{s,b}} \quad \text{where} \quad \tilde{f} \in X^{s,b};$$

$$\|f\|_{X^{s,b}_{\pm}(I)} = \inf_{\tilde{f} = f} \|\tilde{f}\|_{X^{s,b}_{\pm}} \quad \text{where} \quad \tilde{f} \in X^{s,b}_{\pm}.$$ 

In Section 3, we transform the KLS$_m$ \[12\] into an equivalent system of first order in $t$ in the usual way, then make use of Strichartz type estimates to give the local well-posedness in the $X^{0,b}([0, \delta]) \times X^{1/2,b}_{\pm}([0, \delta])$ spaces for some $0 < b < 1/2$, which is useful for the iteration procedure. In general, we can obtain the local well-posedness for $b \geq \frac{1}{2}$, but in order to get the global well-posedness, we use $0 < b < 1/2$ to obtain some gains.

In Section 4, we show that the local result can be iterated to get a solution on any time interval $[0, T]$. We first can construct the solution step by step on some time intervals, which is only dependent of $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$. Then we can repeat this entire procedure to get the desired time $T$, each time advancing a time of length $\sim 1/\|u_0\|^{4m-2}_{L^2}$ (independent of $\|\partial_t u(t), \partial_t u(t)\|_{H^{1/2}, H^{-1/2}}$).

We use the following standard facts about the space $X^{s,b}_\varphi$ \[14].

Let $\psi \in C_0^\infty(\mathbb{R})$ and satisfy $\text{supp}\{\psi\} \subset (-2, 2); \psi|_{[-1,1]} = 1; \psi(t) = \psi(-t), \psi \geq 0$. For $0 < \lambda \leq 1$, define $\psi_\lambda(t) = \psi(\lambda t)$.

For $s \in \mathbb{R}, b \geq 0$, we have the following homogeneous estimate

$$\|\psi_\lambda e^{it\varphi(D_x)}f(x)\|_{X^{s,b}_{\varphi}} \leq c\delta^{\frac{1}{2} - b}\|f\|_{H^s};$$

$$\|e^{it\varphi(D_x)}f(x)\|_{C(R, H^s)} = \|f\|_{H^s};$$

For $b' + 1 \geq b \geq 0 \geq b' > -\frac{1}{2}$, we have the following inhomogeneous estimates

$$\|\psi_\lambda \int_0^t e^{i(t-s)\varphi(D_x)}F(s)ds\|_{X^{s,b}_{\varphi}} \leq c\delta^{1+b'-b}\|F\|_{X^{s,b'}_{\varphi}};$$
\[\left\| \int_0^t e^{i(t-s)\phi(D_x)} F(s) ds \right\|_{C([0,\delta], H^s_x)} \leq c\delta^{\frac{1}{2} + \nu} \| F \|_{X^{s,\nu}}; \quad (1.8)\]

For \(1 < p \leq 2, b \leq \frac{1}{2} - \frac{1}{p}\), we have the following Sobolev inequality
\[\| f \|_{X^{s,b}} \leq c\| f \|_{L^p_t(\mathbb{R}, H^{s,b})}. \quad (1.9)\]

Last, we introduce the following notation: For \(\lambda \in \mathbb{R}\), Japanese symbol \(<\lambda>\) denotes 
\((1 + |\lambda|^2)^{1/2}\); \(a+\) (resp. \(a-\)) denotes a number slightly larger (resp. smaller) than \(a\).

## 2 Linear and Nonlinear Estimates

In this section, we first transform the Klein-Gordon-Schrödinger system into an equivalent system of first order in \(t\) in the usual way to discuss the notion of criticality for the system \((1.2)\). Later, we give some useful linear and nonlinear estimates.

First, for the notion of criticality, we define
\[n_\pm := \frac{1}{2}(n \pm \frac{1}{iA} n_t), \quad A = (I - \partial^2_x)^{\frac{1}{2}}.\]

Then we have
\[n = n_+ + n_-, \quad n_t = iA(n_+ - n_-), \quad n_+ = \overline{n}_-.

and the equivalent system is
\[
\begin{cases}
iu_t + \partial^2_x u = - m(n_+ + n_-)|u|^{2(m-1)}u \\
i\partial_t n_\pm \pm An_\pm = \pm \frac{1}{2} A^{-1}(|u|^{2m}) \\
u(0) = u_0 \in H^k_x, \quad n_\pm(0) = \frac{1}{2}(n_0 \pm \frac{1}{iA} n_1) \in H^l_x.
\end{cases}
\quad (2.1)

We follow with Ginibre et al. [13] to discuss the criticality through scaling. Consider the following similar system
\[
\begin{cases}
i\partial_t u + \partial^2_x u = - m(n_+ + n_-)|u|^{2(m-1)}u \\
i\partial_t n_\pm \pm (-\partial^2_x)^{1/2} n_\pm = \pm \frac{1}{2} (-\partial^2_x)^{-1/2}(|u|^{2m}).
\end{cases}
\quad (2.2)

If there were not the term \(\partial^2_x u\) in the LHS of the first equation in \((2.2)\), then the system \((2.2)\) would be invariant under the dilation
\[u \rightarrow u_\lambda = \lambda^{3/(4m-2)} u(\lambda t, \lambda x), \quad n \rightarrow n_\lambda = \lambda^{(2-m)/(2m-1)} n(\lambda t, \lambda x)

and the system \((2.2)\) would be critical for \((u_0, n_\pm(0)) \in H^k_x \times H^l_x\) for \(k = \frac{d}{2} - \frac{3}{4m-2}, l = \frac{d}{2} - \frac{2m}{2m-1}\). Hence it is \(L^2_x \times H^{1/2}_x\)-subcritical case for \(d = 1, 1 \leq m < 2\).
If there were not the term $i\partial_t u$ in the LHS of the first equation in (2.2), then the system (2.2) would be invariant under the dilation
\[ u \rightarrow u_\lambda = \lambda^{2/(2m-1)}u(\lambda t, \lambda x) \]
\[ n \rightarrow n_\lambda = \lambda^{2/(2m-1)}n(\lambda t, \lambda x) \]
and the system (2.2) would be critical for $(u_0, n_\pm(0)) \in H^k_x \times H^l_x$ for $k = \frac{d}{2} - \frac{2}{2m-1}$, $l = \frac{d}{2} - \frac{2}{2m-1}$. Hence it is $L^2_x \times L^2_x$-subcritical case for $d = 1, 1 \leq m < \frac{5}{2}$.

If there were not the term $\pm (-\partial_x^2)^{1/2}n_\pm$ in the LHS of the second equation in (2.2), then the system (2.2) would be invariant under the dilation
\[ u \rightarrow u_\lambda = \lambda^{5/(4m-2)}u(\lambda^2 t, \lambda x) \]
\[ n \rightarrow n_\lambda = \lambda^{(3-m)/(2m-1)}n(\lambda^2 t, \lambda x) \]
and the system (2.2) would be critical for $(u_0, n_\pm(0)) \in H^k_x \times H^l_x$ for $k = \frac{d}{2} - \frac{5}{4m-2}$, $l = \frac{d}{2} - \frac{3m-2}{2m-1}$. Hence it is $L^2_x \times H^{1/2}_x$-subcritical case for $d = 1, 1 \leq m < 3$.

If there were not the term $i\partial_t n_\pm$ in the LHS of the second equation in (2.2), then the system (2.2) would be invariant under the dilation
\[ u \rightarrow u_\lambda = \lambda^{2/(2m-1)}u(\lambda^2 t, \lambda x) \]
\[ n \rightarrow n_\lambda = \lambda^{2/(2m-1)}n(\lambda^2 t, \lambda x) \]
and the system (2.2) would be critical for $(u_0, n_\pm(0)) \in H^k_x \times H^l_x$ for $k = \frac{d}{2} - \frac{2}{2m-1}$, $l = \frac{d}{2} - \frac{2}{2m-1}$. Hence it is $L^2_x \times L^2_x$-subcritical case for $d = 1, 1 \leq m < \frac{5}{2}$.

That is the reason why we here focus on the local and global wellposedness of the system (2.1) in $L^2(\mathbb{R}) \times H^{1/2}(\mathbb{R})$ for $1 \leq m < 2$. We will take the other cases into account in the forthcoming papers.

Second, we give some known linear estimates. For the Schrödinger equation, we have

**Lemma 2.1 (Strichartz estimate).** [14] Assume that $4 \leq q \leq +\infty, 2 \leq r \leq +\infty$, $0 \leq \frac{q}{r} \leq \frac{1}{2} - \frac{1}{r}$, and
\[ s = \frac{1}{2} - \frac{1}{r} - \frac{2}{q}. \]
Then we have
\[ \|u\|_{L^q_t L^r_x(\mathbb{R})} \leq c \|u\|_{X^{s, \frac{1}{2}}}. \]
In particular, combining with the trivial equality $\|u\|_{L^2_{t,x}} = \|u\|_{X^{0,0}}$, we have

**Lemma 2.2.** [14] Assume that $0 < \frac{1}{q} \leq \frac{1}{2}, \frac{1}{2} - \frac{1}{r} \leq \frac{1}{q} < \frac{1}{2} + \frac{1}{r}$ and
\[ b > \frac{1}{2} - \frac{1}{q} + \frac{1}{2}(\frac{1}{2} - \frac{1}{r}). \]
Then the estimate
\[ \|u\|_{L^q_t L^r_x(\mathbb{R})} \leq c \|u\|_{X^{0,b}} \] (2.3)
holds true for all $u \in X^{0,b}$. 

For the Klein-Gordon equation, we will use the fact that
\[
\|n_\pm\|_{L_t^p H_x^\frac12(\mathbb{R})} \leq c\|n_\pm\|_{X^{s,b}}, \text{ for } 2 < p < \infty, b > \frac12 - \frac{1}{p}
\] (2.4)
which can be obtained from the interpolation between
\[
\|n_\pm\|_{L_t^p H_x^s} \leq c\|n_\pm\|_{X^{s,1}}
\]
and the trivial equality
\[
\|n_\pm\|_{L_t^1 H_x^0} = \|n_\pm\|_{X^{s,0}}.
\]
Finally, we give some useful nonlinear estimates, which are especially important to the iteration procedure.

**Lemma 2.3 (Nonlinear estimates).** Let \(1 \leq m < 2\), then there exists some \(0 < \epsilon < 1 - \frac{m}{2}\), such that the estimates
\[
\|n_\pm\|_{X^{s,b_2}} \leq c\|n_\pm\|_{X^{s,b_1}}
\]
hold for any \(n_\pm \in X^{s,b_2}\) and \(u \in X^{0,b_1}\) where \(b_1 = b_2 = \frac{2m-1}{4m} + \epsilon\), \(b_1' = b_2' = -\frac12 + 2m\epsilon\).

**Proof:** By (1.3), Hölder inequality, we have
\[
\|n_\pm\|_{X^{s,b_2}} \leq c\|n_\pm\|_{X^{s,b_1}}.
\]
From (2.4), we have
\[
\|n_\pm\|_{L_t^{4m} H_x^{1/2}} \leq c\|n_\pm\|_{X^{s,b_2}}.
\]
From Lemma 2.2 we have
\[
\|u\|_{L_t^{4m} L_x^{4m-2\theta}} \leq c\|u\|_{X^{0,b_1}},
\]
under the conditions
\[
\begin{aligned}
\theta + 2\epsilon &< 2 - m, \\
\theta - 4m\epsilon &< m + \frac{1}{2m} - 2,
\end{aligned}
\] (2.5)
which can be satisfied for \(1 \leq m < 2\).

Therefore, we obtain
\[
\|n_\pm\|_{X^{s,b_2}} \leq c\|n_\pm\|_{X^{s,b_2}}.
\]
In addition, by (1.9) and Sobolev inequality and Lemma 2.2, we have
\[ \| u \|_{X^0, b_1}^2 \leq c \| u \|_{L_t^1 L_x^{2m}}^2 \]
and
\[ \| u \|_{L_t^1 L_x^{2m}}^2 \leq c \| u \|_{X^0, b_1}^2. \]

The proof is completed.

Remark 2.1. If \( 1 \leq m \leq 1 + \frac{\sqrt{2}}{2} \), we take the value of \( \theta \) and \( \epsilon \) in the region ABC of Figure 1; If \( 1 + \frac{\sqrt{2}}{2} \leq m \leq 1 + \frac{\sqrt{3}}{2} \), we take the value of \( \theta \) and \( \epsilon \) in the region ABoD of Figure 2 (see next page); If \( 1 + \frac{\sqrt{3}}{2} \leq m < 2 \), we take the value of \( \theta \) and \( \epsilon \) in the region ABo of Figure 3 (see next page). As we know, when \( m = 2 \), it is difficult to prove the nonlinear estimates in the above lemma for some \( b_1, b_2, b'_1, b'_2 \) satisfying \( 2m + b'_1 + b'_2 = (4m - 1)b_1 + b_2 \). Hence we cannot prove the global well-posedness in Theorem 1.1 for the endpoint case \( m = 2 \).

3 The Local Well-Posedness

In this section, we construct a solution of (2.1) in some time interval \([0, \delta]\) using the fixed point argument.

The KLS \( m \) (2.1) has the following equivalent integral equation formulation
\[ u(t) = U(t)u_0 + imU^*R[(n_+ + n_-)|u|^{2(m-1)}u](t); \]
\[ n_+ = W_+ n_+(0) + \frac{i}{2} W_+(A^{-1}|u|^{2m})(t). \]
Figure 2: $\theta - \epsilon$ parameter picture for $1 + \frac{\sqrt{3}}{2} \leq m \leq 1 + \frac{\sqrt{3}}{2}$.

Figure 3: $\theta - \epsilon$ parameter picture for $1 + \frac{\sqrt{3}}{2} \leq m < 2$. 
Define the metric in the set such that the above Cauchy problem

Remark 3.1. According to the value of \( n_+ \), we have

\[
\delta \leq c_1 \leq 1;
\]

\[
\delta \leq c_2 \leq 1;
\]

\[
\delta \leq c_3 \leq 1;
\]

such that the above Cauchy problem has a unique solution \( u(t, x) \in C([0, \delta], L^2) \) and \( n_+(t, x) \in C([0, \delta], H^{1/2}) \) with the property

\[
\| u \|_{L^2} \leq \delta^{1/2} \| u_0 \|_{L^2}, \quad \| n_+ \|_{H^{1/2}} \leq \delta^{1/2} \| n_+ \|_{H^{1/2}}.
\]

Remark 3.1. According to the value of \( b_1, b_2, b_1', b_2' \), we have

\[
m + \frac{1}{2} + b_2 - (2m - 1)b_1 - b_2 = m + \frac{1}{2} + b_1' - (2m - 1)b_1' - b_2
\]

\[
= m + \frac{1}{2} + b_2' - 2mb_1
\]

\[
= \frac{1}{2}.
\]

Proof: Define the closed set \( Y \) as

\[
Y = \left\{ \| u \|_{X_{b_1}^{0,1}([0,\delta])} \leq 2c\delta^{1/2} \| u_0 \|_{L^2}, \quad \| n_+ \|_{X_{b_2}^{1/2}([0,\delta])} \leq 2c\delta^{1/2} \| n_+ \|_{H^{1/2}} \right\}
\]

Define the metric in the set \( Y \) as

\[
d((u_1, n_{1+}), (u_2, n_{2+})) = \| u_1 - u_2 \|_{X_{b_1}^{0,1}([0,\delta])} + \| n_{1+} - n_{2+} \|_{X_{b_2}^{1/2}([0,\delta])}.
\]

First, we prove that \( M \) maps \( Y \) into itself under some conditions on \( \delta \).

Now take any \( (u, n_+) \in Y \). By (1.5), (1.7) and Lemma 2.8, we have

\[
\| \Lambda_S(u, n_+) \|_{X_{b_1}^{0,1}([0,\delta])} \leq c\delta^{1/2} \| u_0 \|_{L^2} + c\delta^{1/2} \| u_0 \|^{2m-1} \| u \|_{X_{b_1}^{0,1}}
\]

\[
\leq c\delta^{1/2} \| u_0 \|_{L^2} + c\delta^{1/2} \| u_0 \|^{2m-1} \| n_+ \|_{X_{b_2}^{1/2}}
\]

\[
\leq c\delta^{1/2} \| u_0 \|_{L^2} + c\delta^{1/2} \| u_0 \|^{2m-1} \| n_+ \|_{H^{1/2}} \leq c\delta^{1/2} \| u_0 \|_{L^2}^{2m-1}
\]

\[
\leq 2c\delta^{1/2} \| u_0 \|_{L^2}
\]
under the condition (3.2).

In addition, we have
\[ \| \Lambda_{W^\pm}(u, n_\pm) \|_{X^b_{\pm, b_2}([0, \delta])} \leq c_\delta \| n_\pm(0) \|_{H^{1/2}} + c_\delta |b_2 - b_1| \| u \|_{X^b_{\pm, b_2}} \]
\[ \leq c_\delta \| n_\pm(0) \|_{H^{1/2}} + c_\delta |b_2 - b_1| \| u \|_{X^b_{0, b_1}} \]
\[ \leq c_\delta \| n_\pm(0) \|_{H^{1/2}} + c_\delta |b_2 - b_1| (\delta^{1/2 - b_1} \| u_0 \|_{L^2})^{2m} \]
\[ \leq 2c_\delta \| n_\pm(0) \|_{H^{1/2}} \]
under the condition (3.3). Therefore, we prove that $M$ maps $Y$ into itself.

Second, we can prove that $M$ is a contraction map under another conditions on $\delta$. Take any $(u_1, n_{1\pm}), (u_2, n_{2\pm}) \in Y$, we have
\[ \| \Lambda_S(u_1, n_{1\pm}) - \Lambda_S(u_2, n_{2\pm}) \|_{X^b_{0, b_1}([0, \delta])} \]
\[ \leq c_\delta \| n_{1\pm} u_1 \|_{2(m-1)} - n_{2\pm} u_2 \|_{2(m-1)} \| u_1 - u_2 \|_{X^b_{0, b_1}} \]
\[ \leq c_\delta \| n_{1\pm} \|_{X^b_{0, b_1}} (\| u_1 \|_{X^b_{0, b_1}} + \| u_2 \|_{X^b_{0, b_1}}) \| u_1 - u_2 \|_{X^b_{0, b_1}} \]
\[ + \| u_1 \|_{2(m-1)} + \| u_2 \|_{2(m-1)} \| n_{1\pm} - n_{2\pm} \|_{X^b_{0, b_1}} \]
\[ \leq c_\delta \| n_{1\pm} \|_{X^b_{0, b_1}([0, \delta])} + \| n_{1\pm} - n_{2\pm} \|_{X^b_{0, b_2}([0, \delta])} \]
under the conditions (3.2) and
\[ \delta^{m + 1/2 + b_2 - 2m b_1} \| u_0 \|_{L^2}^{2m-1} \leq 1 \]
which is equivalent to (3.1) for $b_1 = b_2$ and $b'_1 = b'_2$.

In addition, we have
\[ \| \Lambda_{W^\pm}(u_1, n_{1\pm}) - \Lambda_{W^\pm}(u_2, n_{2\pm}) \|_{X^b_{\pm, b_2}([0, \delta])} \]
\[ \leq c_\delta \| n_{1\pm} u_1 \|_{2(m-1)} - n_{2\pm} u_2 \|_{2(m-1)} \| u_1 - u_2 \|_{X^b_{\pm, b_2}} \]
\[ \leq c_\delta \| n_{1\pm} \|_{X^b_{0, b_1}} (\| u_1 \|_{X^b_{0, b_1}} + \| u_2 \|_{X^b_{0, b_1}}) \| u_1 - u_2 \|_{X^b_{0, b_1}} \]
\[ \leq c_\delta \| n_{1\pm} \|_{X^b_{0, b_1}([0, \delta])} + \| n_{1\pm} - n_{2\pm} \|_{X^b_{\pm, b_2}([0, \delta])} \]
under the condition (3.3).

The standard fixed point arguments gives a unique solution in time interval $[0, \delta]$. According to (1.6) and (1.8), we can get that $u \in C([0, \delta], L^2)$ and $n_{\pm} \in C([0, \delta], H^{1/2})$. Summarizing, the proof is completed.
4 Global Well-posedness

In this section, we show that the process can be iterated to get a solution on any time interval \([0, T]\). We first can construct the solution step by step on some time intervals, which is only dependent of \(\|u(t)\|_{L^2} = \|u_0\|_{L^2}\). Then we can repeat this entire procedure to get the desired time \(T\).

According the mass conservation in (1.3), we conclude that \(\|u(t)\|_{L^2} = \|u(0)\|_{L^2}\). In order to iterate the local result to obtain the global well-posedness, we are only concerned with the growth in \(\|n_{\pm}(t)\|_{H^{1/2}}\) from one time step to the next step.

Suppose that after some number of iterations we reach a time where \(\|n_{\pm}(t)\|_{H^{1/2}} \gg \|u(t)\|_{L^2}^{2m} = \|u_0\|_{L^2}^{2m}\). Take this time position as the initial time \(t = 0\) so that \(\|u_0\|_{L^2} \ll \|n_{\pm}(0)\|_{H^{1/2}}\). Then (3.3) is automatically satisfied and by (3.2), we may select a time increment of size

\[
\delta \sim (\|u_0\|_{L^2}^{2m-2} \|n_{\pm}(0)\|_{H^{1/2}})^{-1/(m+\frac{1}{2}+b_1'(2m-1)b_1-b_2)} \quad (4.1)
\]

Since

\[n_{\pm}(t) = W_{\pm}n_{\pm}(0) + \frac{i}{2} W_{\pm}\partial_t(A^{-1}|u|^4)(t),\]

We can apply (1.6) and (1.7) and Proposition 3.1 to obtain

\[
\|n_{\pm}(\delta)\|_{H^{1/2}} \leq \|n_{\pm}(0)\|_{H^{1/2}} + c\delta^{\frac{1}{2}+b_1'}\|u\|_{X_{\pm}^{\frac{1}{2}+b_1'}}^{2m} \\
\leq \|n_{\pm}(0)\|_{H^{1/2}} + c\delta^{\frac{1}{2}+b_2'}\|u\|_{X_{\pm}^{\frac{1}{2}+b_1}}^{2m} \leq \|n_{\pm}(0)\|_{H^{1/2}} + c\delta^{m+\frac{1}{2}+b_2'-2mb_1}\|u_0\|_{L^2}^{2m}
\]

where \(c\) is some fixed constant. From this we can see that we can carry out \(N\) iterations on time intervals each of length \((4.2)\), where

\[
N \sim \frac{\|n_{\pm}(0)\|_{H^{1/2}}^{2m}}{\delta^{m+\frac{1}{2}+b_2'-2mb_1}\|u_0\|_{L^2}^{2m}} \quad (4.2)
\]

before the quantity \(\|n_{\pm}(t)\|_{H^{1/2}}\) doubles. The total time we advance after these \(N\) iterations, by (3.1) and (4.2) and \(2m + b_1' + b_2' = (4m-1)b_1 + b_2\), is

\[
N\delta \sim \frac{\|n_{\pm}(0)\|_{H^{1/2}}^{2m}}{\delta^{m+\frac{1}{2}+b_2'-2mb_1}\|u_0\|_{L^2}^{2m}} \sim \frac{1}{\delta^{2m+b_1'+b_2'-(4m-1)b_1-b_2}\|u_0\|_{L^2}^{4m-2}} \sim \frac{1}{\|u_0\|_{L^2}^{4m-2}},
\]

which is independent of \(\|n_{\pm}(t)\|_{H^{1/2}}\).

We can now repeat this entire procedure, each time advancing a time of length \(\sim 1/\|u_0\|_{L^2}^{4m-2}\). Upon each repetition, the size of \(\|n_{\pm}(t)\|_{H^{1/2}}\) will at most double, giving the exponential-in-time upper bound stated in Theorem 1.

**Acknowledgments:** We are deeply grateful to Prof. James Colliander for his valuable suggestions and discussions.
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