Dynamics on Hyperspaces

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Abstract

Given a compact metric space \((X, \rho)\) and a continuous function \(f: X \to X\), we study the dynamics of the induced map \(\bar{f}\) on the hyperspace of the compact subsets of \(X\). We show how the chain recurrent set of \(f\) and its components are related with the one of the induced map. The main result of the paper proves that, under mild conditions, the numbers of chain components of \(\bar{f}\) is greater than the ones of \(f\). Showing the richness in the dynamics of \(\bar{f}\) which cannot be perceived by \(f\).

Key words: Hyperspaces, chain recurrence, chain components

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1 Introduction

Let \((X, \rho)\) be a compact metric space and \(f: X \to X\) a continuous function. Our main goal here is to analyze the dynamic behavior of \(f\). However, in real life, the knowledge of the system \(f\) is not alway exact. In fact, each measurement of data carries an uncertainty and it is much convenient to consider a neighborhood of the state to be measured. In order to do that, we study the induced function \(\bar{f}: \mathcal{K}(X) \to \mathcal{K}(X)\) on the hyperspace of nonempty compact subsets of \(X\) endowed with the Hausdorff metric. According to Bauer et al [4] “The elements of \(\mathcal{K}(X)\) can be viewed as statistical states, representing imperfect knowledge of the system. The elements of \(X\) are imbedded in \(\mathcal{K}(X)\) as the pure states (see also [12]).

In this work we investigate some topological dynamics properties of \(f\) and \(\bar{f}\). Precisely, the chain recurrence set of \(\bar{f}\), its chain recurrence components and the relationship with the chain recurrent set of \(f\).

There are many works describing the dynamic behavior of the map \(f\). Some of them analyze the relations between the dynamic of \(f\) and those of \(\bar{f}\), [3], [10] and [12]. In the paper Chain transitivity in hyperspaces, [7], the authors explore which of these topological properties of \(f\) are inherited by \(\bar{f}\), and reciprocally, especially those related to chain transitivity. Our approach is novel and it based on the Conley theory and the functor \(\mathcal{K}\), which works inside of the category of compact spaces and continuous functions. As a matter of fact, several

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objects and properties of $\bar{f}$ can be recovered by this functor through the decomposition of the recurrent set $\mathcal{C}$ of $f$ by its attractors. In fact, if $A$ is an attractor, the dual $A^*$ of $A$ defined by

$$A^* = \{ x \in X : \mathcal{O}(x,f) \cap A = \emptyset \}$$

is a repellor and $(A, A^*)$ is called an attractor-repellor pair for $f$. We prove that $(\mathcal{K}(A), \mathcal{K}(A^*))$ is an attractor-repellor pair for the extension $\bar{f}$. And, we do not know if each $\bar{f}$-attractor comes from an $f$-attractor.

Through the Conley relationship between the recurrence set $\mathcal{C}$ of the original function $f$ and its attractors given by

$$\mathcal{C} = \cap \{ A \cup A^* : A \text{ is an attractor for } f \},$$

we show that $\mathcal{K}(\mathcal{C}) = \bar{\mathcal{C}}$. Furthermore, we analyze how the chain recurrent components of $\mathcal{C}$ and $\bar{\mathcal{C}}$ are related.

It turns out that under mild conditions the numbers of chain components of $\bar{\mathcal{C}}$ is bigger than the ones of $\mathcal{C}$. This fact shows a richness in the dynamics of $\bar{f}$ which cannot be perceived by the dynamic of $f$. But, what are the relations between the chain recurrent components of $\mathcal{C}$ and those of $\bar{\mathcal{C}}$? Let us denote by $\mathcal{B}$ the set of chain components of $\mathcal{C}$. In order to approach the chain recurrent components of $\bar{\mathcal{C}}$ we introduce the sets

$$\mathcal{C}_J := \left\{ A \in \mathcal{K}(X) : A \subset \bigcup_{P \in J} P \text{ and } A \cap P \neq \emptyset, P \in J \right\},$$

when $J \in \mathcal{K}(\mathcal{B})$. Then, we prove that each $\mathcal{C}_J$ is $\bar{f}$-invariant, compact and the disjoint union of these sets over $\mathcal{K}(\mathcal{B})$ cover $\bar{\mathcal{C}}$, i.e.,

$$\bar{\mathcal{C}} = \bigcup_{J \in \mathcal{K}(\mathcal{B})} \mathcal{C}_J.$$

This fact allow to prove our main results.

1.1 Theorem: The set $\mathcal{C}_J$ is a chain component of $\bar{f}$ if $\mathcal{K}(P)$ is chain transitive for any $P \in \mathcal{J}$. In particular, the family $\{ \mathcal{C}_J : J \in \mathcal{K}(\mathcal{B}) \}$ coincide with the chain components of $\bar{f}$ if $\mathcal{K}(P)$ is chain transitive for any $P \in \mathcal{B}$.

In particular, if $\mathcal{K}(P)$ is chain transitive for any $P \in \mathcal{B}$ then

$$|\mathcal{B}| < \infty \implies |\bar{\mathcal{B}}| = 2^{|\mathcal{B}|} - 1.$$

It is important to notice that the chain transitivity of $\mathcal{K}(P)$ does not follows directly from the chain transitivity of $P$. Actually, in [7] the authors give several equivalences for the transitivity of $\mathcal{K}(P)$.

The paper is structured as follows. In Section 2, we mention all the prerequisites needed to state and prove the main results of the paper. Its include the chain recurrent set of $f$ and its components, the notion of an attractor-repellor pair, some topological properties of functions on hyperspaces and the relationship between extension and the canonical projection $\pi : \mathcal{K}(X) \to X$. In Section 3, we state our main results about the chain recurrent components of $\bar{f}$. Finally, we give a couple of examples. One of them shows that $\bar{\mathcal{B}}$ is not numerable despite the fact that $\mathcal{B}$ it is.

2 Preliminaries

In this section we state all the prerequisites needed in order to understand the main results of the paper.

2.1 The chain recurrent set and its components

Let $(X, d)$ be a compact metric space and $f : X \to X$ a continuous function. For any $n > 0$ we denote by $f^n$ the $n$-th iterated of $f$ and $f^0 = \text{id}_X$. The forward orbit of $x \in X$ is the sequence $f^n(x), n \geq 0$ and is denoted
by \( O(x, f) := \{ f^n(x), \ n \geq 0 \} \). If \( A \subset X \) is nonempty, then its omega-limit set under \( f \) is
\[
\omega(A, f) := \bigcap_{k} \bigcup_{n \geq k} f^n(A).
\]

If \( U \) is an open neighborhood of \( \omega(A, f) \) then \( f^n(A) \subset U \) for \( n \) sufficiently large. Actually the omega-limit set is the smallest close set with this property. If \( A = \{ x \} \) is a singleton, we denote its omega-limit set only by \( \omega(x, f) \). A subset \( A \) is called forward invariant under the map \( f \) if \( f(A) \subset A \) and invariant if \( f(A) = A \). Note that \( \omega(A, f) \) is the largest invariant subset of \( f \) contained in \( O(A, f) \).

Our investigation concerns Conley’s chain recurrent set, whose definition follows: Given \( \varepsilon > 0 \) an \( \varepsilon \)-chain \( \xi \) is a finite subset \( \xi = \{ x_0, x_1, \ldots, x_n \} \) with \( n > 0 \) such that \( \rho(f(x_{j-1}), x_j) < \varepsilon \) for \( 0 < j < n \). We say that the \( \varepsilon \)-chain starts at \( x_0 \) and ends at \( x_n \), and the integer \( n > 0 \) is its length. We denote the set of points that are ends of \( \varepsilon \)-chains beginning at \( x \) by \( C(x, f) \), and define
\[
C(x, f) := \bigcap_{\varepsilon > 0} C_\varepsilon(x, f).
\]

If \( 0 < \delta < \varepsilon \) then \( \overline{C_\delta(x, f)} \subset C_\varepsilon(x, f) \) and therefore \( C(x, f) \) is a closed set. A point \( x \in X \) is said to be chain recurrent if \( x \in C(x, f) \). The set of the chain recurrent points, denoted by \( C(f) \), is a nonempty, compact invariant subset of \( X \). In \( C(f) \) one defines the relation \( \sim \) by
\[
x \sim y \quad \text{if} \quad x \in C(y, f) \quad \text{and} \quad y \in C(x, f).
\]
The associated equivalence classes are called the chain components for \( f \). Each of such components is closed and invariant for \( f \).

Let us denote by \( B_f \) the set of chain components of \( C(f) \), that is,
\[
B_f := \{ P \in K(X), \ P \text{ is a chain component of } C(f) \}.
\]
The quotient map from \( C(f) \) to \( B_f = C(f)/\sim \) induces upon the latter the structure of a compact metrizable space.

We finish this section with a technical lemma which will be needed ahead.

**2.1 Lemma:** It holds:

1. Let \( x_n, y_n \in X \) and assume that \( x_n \sim y_n \). If \( x_n \to x \) and \( y_n \to y \) then \( x \sim y \);
2. Let \( x, y \in C \) be distinct elements. Then, there exists an \( \varepsilon \)-chain between \( x \) and \( y \) contained in \( C \) if and only if there exists an \( \varepsilon \)-chain between \( x \) and \( y \) contained in \( C \).

**Proof:** 1. Let \( \varepsilon > 0 \) and consider \( \delta \in (0, \varepsilon/2) \) such that
\[
\rho(x, y) < \delta \quad \Rightarrow \quad \rho(f(x), f(y)) < \frac{\varepsilon}{2}.
\]
Let \( n \in \mathbb{N} \) such that \( \rho(x_n, x) < \delta \) and \( \rho(y_n, y) < \delta \) and let \( \xi = \{ z_0, \ldots, z_k \} \) to be an \( \varepsilon/2 \)-chain between \( x_n \) and \( y_n \). Since \( x_n = z_0 \) and \( y_n = z_k \) we have that
\[
\rho(f(x), z_1) \leq \rho(f(x), f(x_n)) + \rho(f(z_0), z_1) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]
and
\[
\rho(f(z_{n-1}), y) \leq \rho(f(z_{n-1}), z_k) + \rho(y_n, y) < \frac{\varepsilon}{2} + \delta < \varepsilon.
\]
and hence \( \xi' = \{ x, z_1, \ldots, z_{n-1}, y \} \) is an \( \varepsilon \)-chain between \( x \) and \( y \) implying \( y \in C_\varepsilon(x, f) \). By the arbitrariness of \( \varepsilon \) we have \( y \in C(x, f) \). Analogously we show that \( x \in C(y, f) \) and therefore \( x \sim y \) as stated.
2. Let \( x, y \in C \) and assume that there exists an \( \epsilon \)-chain between \( x \) and \( y \) contained in \( C \). Since any chain component is \( f \)-invariant there are distinct chain components \( P_1, \ldots, P_n \) with \( y \in P_1, x \in P_n \) and such that \( g(P_i, P_{i+1}) < \epsilon \) for \( i = 1, \ldots, n-1 \).

For \( j = 1, \ldots, n-1 \) let us consider \( x_i, z_i \in P_i \) and \( y_i \in P_{i+1} \) be such that \( g(x_i, y_i) < \epsilon \) and \( f(z_i) = x_i \). By setting \( y_0 = y \) and \( z_n = x \) let \( \xi \) be an \( \epsilon \)-chain in \( P_i \) between \( y \) and \( x \) as stated (see Figure 1).

\[ \square \]

Figure 1: \( \epsilon \)-chain between \( y \) and \( x \).

### 2.2 Attractors-repellor pairs

There is an important connection between the structure of the set of chain components of \( f \) and the set of attractors of \( f \). In order to describe this, we first define a closed subset \( U \subset X \) to be a trapping region for \( f \) if

\[ f(U) \subset U. \]

For any trapping region \( U \) for \( f \), the associated attractor \( A \) for \( f \), is defined by \( \omega(U, f) \), which is nothing more than \( \cap_{n \geq 1} f^n(U) \). Thus, an attractor \( A \) for \( f \) is a closed, invariant set which is the limit of the decreasing sequence of iterates \( \{ f^n(U) \} \), \( n \geq 1 \) for some trapping region \( U \). Moreover, the numbers of attractors for \( f \) in \( X \) is at most countable (see [2]).

Let us notice that if \( U \) is a trapping region for \( f \) with attractor \( A \), then any chain recurrent point that is contained in \( U \) must actually lie in \( A \).

Suppose now that \( U \) is inward for \( f \) with attractor \( A \). The repellor dual of \( A \) is the set

\[ A^* := \{ x \in X; \mathcal{O}(x, f) \cap U = \emptyset \}. \]

The pair \( (A, A^*) \) is called an attractor-repellor pair. The relationship between attractor-repellor pairs and \( \mathcal{C}(f) \) comes from the fact that (see [6], Chap. II, 6.2.A.)

\[ \mathcal{C}(f) = \bigcap \{ A \cup A^*; A \text{ is an attractor for } f \}. \]  

### 2.3 Functions on hyperspaces

Let \( (X, g) \) be a compact metric space. For any subset \( A \subset X \) we denote by

\[ \mathcal{K}(A) := \{ B \text{ is a compact subset of } X \text{ and } B \subset A \}, \]
the set of compact subsets of $A$. We also consider the decomposition of $\mathcal{K}(A)$ in subsets with finite cardinality as follows,

$$ F(A) := \bigcup_{n \geq 1} F_n(A) \quad \text{with} \quad F_n(A) := \{ B \in \mathcal{K}(A) ; |B| = n \} . $$

It is a standard fact that $F(A)$ is dense in $\mathcal{K}(A)$ when $A$ is a compact subset of $X$.

It is well known that $(\mathcal{K}(X), \rho_H)$ is a metric space, where $\rho_H$ is the Hausdorff distance given by

$$ \rho_H(A, B) := \inf \{ \epsilon > 0 ; A \subset N_\epsilon(B) \text{ and } B \subset N_\epsilon(A) \} , $$

where $N_\epsilon(A)$ stands for the $\epsilon$-neighborhood of $A$.

The finite topology (or Vietoris topology) in $\mathcal{K}(X)$ is the topology generated by the open sets of the form

$$ \langle U_1, \ldots, U_m \rangle := \left\{ A \in \mathcal{K}(X) ; A \subset \bigcup_{i=1}^m U_i \text{ and } A \cap U_i \neq \emptyset, \text{ for all } i = 1, \ldots, n \right\} . $$

It’s a well known fact that the finite topology and the topology induced by the Hausdorff metric are equivalents when $X$ is a compact metric space (see for instance [9]).

Let $f : X \to X$ be a continuous function. The extension $\bar{f}$ of $f$ from $X$ to $\mathcal{K}(X)$ is the continuous function

$$ A \in \mathcal{K}(X) \mapsto \bar{f}(A) := f(A) \in \mathcal{K}(X). $$

The next lemma states several topological properties of hyperspaces that will be needed ahead.

2.2 **Lemma:** It holds:

1. Let $\{A_\alpha\}_{\alpha \in \Delta}$ be a family of subsets of $X$. Then

$$ \mathcal{K}\left( \bigcap_{\alpha \in \Delta} A_\alpha \right) = \bigcap_{\alpha \in \Delta} \mathcal{K}(A_\alpha) \quad \text{and} \quad \mathcal{K}\left( \bigcup_{\alpha \in \Delta} A_\alpha \right) = \bigcup_{\alpha \in \Delta} \mathcal{K}(A_\alpha). $$

2. $\mathcal{K}(X \setminus U) = \mathcal{K}(X) \setminus \mathcal{K}(U)$;

3. $\mathcal{K}(U) = \mathcal{K}(\bar{U})$;

4. $\bar{f}(\mathcal{K}(U)) \subset \mathcal{K}(f(U))$ for any subset $U \subset X$.

**Proof:** 1.

$$ B \in \mathcal{K}\left( \bigcap_{\alpha \in \Delta} A_\alpha \right) \iff B \subset \bigcap_{\alpha \in \Delta} A_\alpha \iff B \subset A_\alpha, \text{ for all } \alpha \in \Delta $$

$$ \iff B \in \mathcal{K}(A_\alpha), \text{ for all } \alpha \in \Delta \iff B \in \bigcap_{\alpha \in \Delta} \mathcal{K}(A_\alpha) $$

$$ B \in \mathcal{K}\left( \bigcup_{\alpha \in \Delta} A_\alpha \right) \iff B \subset \bigcup_{\alpha \in \Delta} A_\alpha \iff B \subset A_\alpha, \text{ for some } \alpha \in \Delta $$

$$ \iff B \in \mathcal{K}(A_\alpha), \text{ for some } \alpha \in \Delta \iff B \in \bigcup_{\alpha \in \Delta} \mathcal{K}(A_\alpha) $$

2.

$$ B \in \mathcal{K}(X \setminus U) \iff B \subset X \setminus U \iff B \not\subset U \iff B \in \mathcal{K}(X) \setminus \mathcal{K}(U) $$
3. It is straightforward to see that $\mathcal{K}(\overline{U})$ is a closed subset of $\mathcal{K}(X)$ and hence $\overline{\mathcal{K}(U)} \subset \mathcal{K}(U)$. Reciprocally, let $B \in \mathcal{K}(X) \setminus \overline{\mathcal{K}(U)}$. There exists a neighborhood $V$ of $B$ such that $\langle V \rangle \cap \mathcal{K}(U) = \emptyset$. Therefore,

$$\langle V \rangle \cap \mathcal{K}(U) = \mathcal{K}(V) \cap \mathcal{K}(U) = \mathcal{K}(V \cap U)$$

implying that $V \cap U = \emptyset$ and consequently that $B \in \mathcal{K}(X) \setminus \overline{\mathcal{K}(U)}$.

4. $B \in \overline{\mathcal{K}(U)}$ implies that $B \neq \emptyset$. Since $\mathcal{K}(X)$ is dense in $\mathcal{K}(X)$, it follows that $B$ is dense in $\mathcal{K}(X)$.

In the sequel we define a projection from the space of subsets $\mathcal{K}(X)$ to $X$ that still preserve several topological properties. For any subset $\mathcal{U} \subset \mathcal{K}(X)$ we can define its $X$-projection by

$$\pi(\mathcal{U}) := \{ x \in X; \exists B \in \mathcal{U} \text{ with } x \in B \} \subset X.$$ The next proposition gives us the main topological relation between $\mathcal{U}$ and its $X$-projection.

**2.3 Proposition:** It holds:

1. If $\mathcal{U}$ is open in $\mathcal{K}(X)$ then $\pi(\mathcal{U})$ is an open $X$;

2. $\overline{\pi(\mathcal{U})} = \pi(\overline{\mathcal{U}})$;

3. If $\mathcal{U}_1 \subset \mathcal{U}_2$ then $\pi(\mathcal{U}_1) \subset \pi(\mathcal{U}_2)$;

4. $f(\pi(\mathcal{U})) = \pi(f(\mathcal{U}))$;

5. If $A \in \mathcal{K}(X)$ then $\pi(\mathcal{K}(A)) = A$.

**Proof:**

1. Let $x \in \pi(\mathcal{U})$ and consider $B \in \mathcal{U}$ such that $x \in B$. Since $\mathcal{U}$ is open, there exists a compact neighborhood $V$ of $B$ such that $\langle V \rangle \subset \mathcal{U}$. In particular, $V \in \mathcal{U}$ implying that $x \in B \subset V \subset \pi(\mathcal{U})$ and showing that $x$ is an interior point of $\pi(\mathcal{U})$.

2. $\pi(\mathcal{U})$ is closed: Let $x_n \in \pi(\mathcal{U})$ such that $x_n \to x$ and consider $B_n \in \overline{\mathcal{U}}$ with $x_n \in B_n$. By the compacity of $\mathcal{K}(X)$, there exists $B_{n_k}, k \in \mathbb{N}$ such that $B_{n_k} \to B$. Since $\overline{\mathcal{U}}$ is closed, we must have $B \in \overline{\mathcal{U}}$ which implies $x \in B$ and therefore $x \in \pi(\mathcal{U})$.

3. $\pi(\mathcal{U})$ is dense in $\pi(\overline{\mathcal{U}})$: Let $x \in \pi(\overline{\mathcal{U}})$ and $B \in \overline{\mathcal{U}}$ with $x \in B$. For any $\varepsilon > 0$ there exists $B' \in \mathcal{U}$ with $d_H(B', B) < \varepsilon$. Therefore, there exists $x' \in B$ such that $d(x', x) < \varepsilon$. In particular, it holds $x' \in \pi(\mathcal{U})$ and consequently $\pi(\mathcal{U})$ is dense in $\pi(\overline{\mathcal{U}})$.

4. It holds that $x \in \pi(\mathcal{U}_1) \iff \exists B \subset \mathcal{U}_1; x \in B$. Since $\mathcal{U}_1 \subset \mathcal{U}_2$ we get that $B \in \mathcal{U}_2$ implying that $x \in \pi(\mathcal{U}_2)$.

5. Reciprocally, for any $x \in \pi(\mathcal{K}(A))$ there is $B \in \mathcal{K}(A)$ with $x \in B$. However, $B \in \mathcal{K}(A)$ if and only if $B \subset A$ implying that $x \in A$ and concluding the proof.

Now, we are able to show the relationship between the attractors and repellers of $f$ with those of $\tilde{f}$. 

6
2.4 Proposition: If \( A \subset X \) is an attractor, then \( K(A) \) is an attractor. Reciprocally, if \( A \subset K(X) \) is an attractor, then \( \pi(A) \) is an attractor.

Proof: Let \( U \) be a trapping region for \( f \) with attractor \( A \). From Lemma 2.2 \( K(U) \) is closed and
\[
\overline{f(U)} = \overline{f(K(U))} \subset \overline{f(f(U))} = K(f(U)) \subset K(U) = U
\]
which implies that \( K(U) \) is a trapping region for \( \overline{f} \). Moreover, since \( A \) is \( f \)-invariant, we have that \( K(A) \) is \( f \)-invariant and so \( K(A) \subset \bigcap_{n \in \mathbb{N}} f^n(U) \). On the other hand, let \( B \in \bigcap_{n \in \mathbb{N}} f^n(U) \). For any \( n \in \mathbb{N} \) there exists \( B_n \subset U \) such that \( B = f^n(B_n) \). Therefore, for all \( n \in \mathbb{N} \) we get
\[
B = f^n(B_n) \subset f^n(U) \implies B \subset \bigcap_{n \in \mathbb{N}} f^n(U) = A \implies B \in K(A)
\]
showing that \( K(A) = \bigcap_{n \in \mathbb{N}} f^n(U) \) and consequently \( K(A) \) is an attractor.

Reciprocally, let \( U \) be a trapping region for \( \overline{f} \) with attractor \( \overline{A} \) and consider \( U = \pi(U) \). By Proposition 2.3 we obtain that \( U \) is closed and
\[
\overline{f(U)} = \overline{f(\pi(U))} = \overline{f(f(U))} = \pi \left( \overline{f(U)} \right) \subset \pi(U) = U
\]
showing that \( U \) is a trapping region for \( f \). Moreover, since \( \overline{A} \) is \( \overline{f} \)-invariant, we have that \( \pi(A) \) is \( f \)-invariant and so \( \pi(A) \subset \bigcap_{n \in \mathbb{N}} f^n(U) \). On the other hand, if \( x \in \bigcap_{n \in \mathbb{N}} f^n(U) \), for each \( n \in \mathbb{N} \) there exists \( B_n \in U \) such that \( x \in f^n(B_n) \). Furthermore, for some subsequence \( \{B_{n_k}\} \) we have that \( f^{n_k}(B_{n_k}) \to B \) for some \( B \in K(X) \). In particular, \( B \in \omega(U, \overline{f}) = \overline{A} \) which implies that \( B \subset \pi(A) \). Since \( x \in f^{n_k}(B_{n_k}) \) we get \( x \in B \subset \pi(A) \) implying that \( \bigcap_{n \in \mathbb{N}} f^n(U) = \pi(A) \) and ending the proof. \( \square \)

Concerning the dual repellors we have the following result.

2.5 Proposition: Let \( A \subset X \) and consider its associated repeller \( A^* \). Then
\[
K(A)^* = K(A^*)
\]
Proof: As before, if \( U \) is a trapping region for \( f \) with attractor \( A \) then \( U = K(U) \) is a trapping region for \( \overline{f} \) with attractor \( K(A) \). Therefore,
\[
\mathcal{O}(B, \overline{f}) \cap U = \emptyset \iff \forall n \geq 0, \ \overline{f}^n(B) \in K(X) \setminus U \iff \forall n \geq 0, \ f^n(B) \subset X \setminus U \iff \forall n \geq 0, \ f^n(B) \cap U = \emptyset \iff \mathcal{O}(x, f) \cap U = \emptyset, \ \text{for all} \ x \in B.
\]
We obtain,
\[
B \in K(A)^* \iff \mathcal{O}(B, \overline{f}) \cap U = \emptyset \iff \mathcal{O}(x, f) \cap U = \emptyset, \ \text{for all} \ x \in B
\]
\[
\iff B \subset A^* \iff B \in K(A^*)
\]
concluding the proof. \( \square \)

3 Chain recurrence on Hyperspaces

In this section we prove our main results relating the chain recurrent sets of \( f \) and its extension \( \overline{f} \). We also show that, although there is a close relation between the chain components of the recurrent sets of \( f \) and \( \overline{f} \), however their grows exponentially.
3.1 The chain recurrent set

From here on we denote only by $\mathcal{C}$ and $\bar{\mathcal{C}}$ the chain recurrent sets of $f$ and $\bar{f}$, respectively. The next theorem relates $\mathcal{C}$ and $\bar{\mathcal{C}}$.

3.1 Theorem: It holds that $\mathcal{K}(\mathcal{C}) \supset \bar{\mathcal{C}}$.

Proof: We start to prove that $\mathcal{K}(\mathcal{C}) \subset \bar{\mathcal{C}}$. Since $F(\mathcal{C})$ is dense in $\mathcal{K}(\mathcal{C})$ and $\bar{\mathcal{C}}$ is closed, it is enough to show that $F(\mathcal{C}) \subset \bar{\mathcal{C}}$. Moreover, the fact that $F(\mathcal{C}) = \bigcup_{n \in \mathbb{N}} F_n(\mathcal{C})$ implies that is enough to show that $F_n(\mathcal{C}) \subset \bar{\mathcal{C}}$ for each $n \in \mathbb{N}$, which we prove by induction on $n \in \mathbb{N}$.

It is easy to see that $F_1(\mathcal{C})$ is chain recurrent. Let us then assume that $F_n(\mathcal{C})$ is chain recurrent and consider $A = \{x_1, \ldots, x_{n+1}\} \in F_{n+1}(\mathcal{C})$. By the induction hypothesis, for a given $\epsilon > 0$ there exists a chain $\xi_1$ in $F_n(\mathcal{C})$, $B_{i_0}^0, B_{i_1}^0 \in F_n(\mathcal{C})$ with $B_{i_0}^0 = B_{i_1}^0 = \{x_1, \ldots, x_n\}$ and $g_H(f(B_i), B_{i+1}) < \epsilon$ for $i = 0, \ldots, k_1$ and a chain $\xi_2$ in $F_1(\mathcal{C})$, $\{a_0\}, \ldots, \{a_{k_2}\} \subset X$ with $a_0 = a_{k_2} = x_{n+1}$ and $g(f(a_i), a_{i+1}) < \epsilon$ for $i = 0, \ldots, k_2$. By concatenating $k_2$-times $\xi_1$ and $k_1$-times $\xi_2$ we can assume that $k = k_1 = k_2$. Thus, $A_i = B_i \cup \{a_i\}, i = 0, \ldots, n + 1$ is an $\epsilon$-chain from $A$ to itself in $F_{n+1}(\mathcal{C})$. Since $\epsilon$ was arbitrary we get that $F_{n+1}(\mathcal{C}) \subset \bar{\mathcal{C}}$ implying that $\mathcal{K}(\mathcal{C}) \subset \bar{\mathcal{C}}$.

Reciprocally, we need to show that $\mathcal{K}(\mathcal{C}) \supset \bar{\mathcal{C}}$. Using the relation $[1]$ we get

$$\mathcal{K}(\mathcal{C}) = \mathcal{K}\left(\bigcap\{A \cup A^*, A \text{ is an attractor for } f\}\right)$$

$$\bigcap\{\mathcal{K}(A) \cup \mathcal{K}(A^*), A \text{ is an attractor for } f\} =$$

$$\bigcap\{\mathcal{K}(A) \cup \mathcal{K}(A^*), A \text{ is an attractor for } f\} \supset$$

$$\bigcap\{A \cup A^*, A \text{ is an attractor for } \bar{f}\} = \bar{\mathcal{C}}$$

which ends the proof.

As a direct corollary we have the following:

3.2 Corollary: Let $X$ be a compact metric space and $f : X \to X$ a continuous function. Then, $f$ is chain recurrent if and only if $\bar{f}$ is chain recurrent.

3.3 Remark: The previous corollary shows that the chain recurrence of $f$ and $\bar{f}$ are equivalent. However, the same does not holds for the chain transitivity as stated in [7].

3.2 Chain recurrent components

In this section we analyze the relation between the chain recurrent components of $\mathcal{C}$ and of $\bar{\mathcal{C}}$. We show that under mild conditions the numbers of chain components of $\bar{\mathcal{C}}$ is much greater than the one of $\mathcal{C}$ which implies a richness in the dynamics of $\bar{f}$ which cannot be perceived by $f$.

Let us denote by $\mathcal{B}$ the chain components of $f$ and by $\mathcal{K}(\mathcal{B})$ the set of its compact subsets. For any $\mathcal{J} \in \mathcal{K}(\mathcal{B})$ we define the sets

$$\mathcal{C}_\mathcal{J} := \left\{ A \in \mathcal{K}(X); A \subset \bigcup_{P \in \mathcal{J}} P \text{ and } A \cap P \neq \emptyset, P \in \mathcal{J} \right\}.$$  

Our aim is to relate the sets $\mathcal{C}_\mathcal{J}$ with the chain recurrent components of $\bar{\mathcal{C}}$. The next proposition goes in this direction.
3.4 Proposition: The sets $C_J$ are $\bar{f}$-invariant, compact and satisfy

$$\bar{C} = \bigcup_{J \in \mathcal{K}(\mathcal{B})} C_J.$$ 

Proof: The $f$-invariance of $P \in \mathcal{B}$ implies that

$$f(A) \subset f\left(\bigcup_{P \in J} P\right) = \bigcup_{P \in J} f(P) \subset \bigcup_{P \in J} P$$

and hence $\bar{f}(C_J) \subset C_J$. The disjointness follows from the fact that if $A \in C_{J_1} \cap C_{J_2}$ then

$$A \cap P_1 \subset \bigcup_{P_2 \in J_2} P_2 \implies P_1 \cap P_2$$

for some $P_2 \in J_2 \implies J_1 \subset J_2$

and analogously $J_2 \subset J_1$ implying that $C_{J_1} = C_{J_2}$.

By Theorem 3.1 if $A \in \bar{C}$ then $A \subset C = \bigcup_{P \in \mathcal{B}} P$ implying that

$$A = \bigcup_{P \in J} A \cap P, \quad \text{where} \quad J := \{P \in \mathcal{B}; \ A \cap P \neq \emptyset\}.$$ 

Moreover, if $P_n \in J$ is such that $P_n \to P$ in $\mathcal{B}$, the fact that $P_n \cap B \neq \emptyset$ implies the existence of $x_n \in P_n \cap B$. Since $A$ is compact we have that $x_{n_k} \to x \in A$ and hence $x \in P$ implying $P \cap A \neq \emptyset$ and consequently $P \in J$. In particular, $J \in \mathcal{K}(\mathcal{B})$ and

$$\bar{C} = \bigcup_{J \in \mathcal{K}(\mathcal{B})} C_J.$$ 

In order to show the compactness of $C_J$ it is enough to show that it is closed. If $B_n \in C_J$ is such that $B_n \to B$. We have

1. $B \cap P \neq \emptyset$ for any $P \in J$;
   In fact, there exists $x_n \in B_n \cap P$ for any $P \in J$ and by compactness $x_{n_k} \to x \in B \cap P$ implying the assertion.

2. $B \subset \bigcup_{P \in J} P$;
   In fact, by the definition of the Hausdorff metric we have that

$$B \subset N_{\epsilon}\left(\bigcup_{P \in J} P\right), \quad \text{for any} \quad \epsilon > 0.$$ 

Since $\bigcap_{\epsilon > 0} N_{\epsilon}\left(\bigcup_{P \in J} P\right) = \bigcup_{P \in J} P$ we have the assertion.

By assertions 1. and 2. above, the closedness of $C_J$ is equivalent to the closedness of $\bigcup_{P \in J} P$.

Let then $x_n \in \bigcup_{P \in J} P$ and assume that $x_n \to x$ in $\mathcal{C}$. For any $n \in \mathbb{N}$ let $P_n \in J$ with $x_n \in P_n$. Since $x_n \to x$ in $\mathcal{C}$ we have that $P_n \to P$ in $\mathcal{B}$, where $x \in P$. Being that $J \in \mathcal{K}(\mathcal{B})$ we must have $P \in J$ and consequently $x \in P \subset \bigcup_{P \in J} P$ showing that $\bigcup_{P \in J} P$ is closed and concluding the proof.

3.5 Remark: The above proof shows, that

$$\bigcup_{P \in J} P \in C_J, \quad \text{for any} \quad J \in \mathcal{K}(\mathcal{B}) \implies C_J \neq \emptyset.$$ 

In particular, each $C_J$ admits a fixed point of $\bar{f}$. 

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The next example shows that such sets can be much greater than one expects.

3.6 Example: Let us consider $X = \{a, b, c\}$ endowed with the zero-one metric and define $f : X \to X$ by the relations

$$f(a) = b, \quad f(b) = a \quad \text{and} \quad f(c) = a.$$  

For such metric, there are no chains from $a$ or $b$ to $c$ for any $\epsilon < 1$ and therefore, $\mathcal{C} = \{a, b\}$. On the other hand, the Hausdorff metric induced in $\mathcal{K}(X)$ is also the zero-one metric and therefore $\check{\mathcal{C}} = \{(a), \{b\}, \{a, b\}\} = \mathcal{K}(\mathcal{C})$.

Since $\mathcal{B} = \{\mathcal{C}\}$ we have that $\mathcal{C}_\mathcal{B} = \check{\mathcal{C}}$. On the other hand, the chain recurrent components of $\check{\mathcal{C}}$ are given by

$$\{\{a\},\{b\}\} \quad \text{and} \quad \{\{a\}\}$$

showing that the sets $\mathcal{C}_\mathcal{J}$ do not need to be chain transitive.

In fact, as in the above example, the sets $\mathcal{C}_\mathcal{J}$ are an “outer” approximation for the chain components of $\check{f}$.

3.7 Proposition: Let $\mathcal{A}_i \in \mathcal{C}_\mathcal{J}_i$, $\mathcal{J}_i \in \mathcal{K}(\mathcal{B})$ for $i = 1, 2$. Then

$$\mathcal{A}_1 \sim \mathcal{A}_2 \implies \mathcal{C}_\mathcal{J}_1 = \mathcal{C}_\mathcal{J}_2.$$  

In particular, $\mathcal{C}_\mathcal{J}$ is a disjoint union of chain components of $\check{f}$.

Proof: Since $\mathcal{A}_1, \mathcal{A}_2 \in \check{\mathcal{C}}$ and $A_1 \sim A_2$ we have that $\mathcal{A}_1$ and $\mathcal{A}_2$ are in the same chain component of $\check{f}$. Therefore, for any $\epsilon > 0$ there exists a chain between $A_2$ and $A_1$ contained in $\check{\mathcal{C}}$.[1] Let then $P \in \mathcal{J}_1$ and consider $x \in A_1 \cap P$ and a 1/n-chain between $A_2$ and $A_1$. There exists $x_n \in P_n \in \mathcal{J}_2$ and a 1/n-chain component between $x_n$ and $x$ contained in $\mathcal{C}$. By Lemma 2.1 there exists an 1/n-chain between $x$ and $x_n$ contained in $\mathcal{C}$. Being that $\mathcal{C}$ is compact, we can assume w.l.o.g. that $x_n \to x'$ and hence $x' \sim x$ implying that $x' \in P$. Therefore, $P_n \to P$ in $\mathcal{B}$ and since $\mathcal{J}_2$ is closed we must have $P \in \mathcal{J}_2$. Since $P \in \mathcal{J}_1$ is arbitrary we get $\mathcal{J}_1 \subset \mathcal{J}_2$. Analogously we have $\mathcal{J}_2 \subset \mathcal{J}_1$ and hence $\mathcal{C}_{\mathcal{J}_1} = \mathcal{C}_{\mathcal{J}_2}$ as stated. $\square$

The next step is to analyze when the sets $\mathcal{C}_\mathcal{J}$ are chain recurrent. If $\mathcal{J} = \{P\}$ is an unitary set, we have that $\mathcal{C}_\mathcal{J} = \mathcal{K}(P)$ and by invariance $\overline{\mathcal{f}P} = \mathcal{f}|_{\mathcal{K}(P)}$. For the unitary case, there are several equivalences for the chain transitivity of the induced map (see Theorem 3 of [1]). Our aim is to show that, in fact, the knowledge of the chain components of $\check{f}$ are well-known if one have transitivity of the maps restricted to $\mathcal{K}(P)$ for any chain component $P \in \mathcal{B}$.

Let then $P \in \mathcal{B}$ be a chain component and assume that $\mathcal{K}(P)$ is chain transitive. In this case, for any $A, B \in \mathcal{K}(P)$ there exists $n \in \mathcal{N}$ such that for any $m \geq n$ there is a chain $\xi$ connecting $A$ and $B$ with exactly length $m$.

In fact, since $P$ is $f$-invariant we have that, as an element of $\mathcal{K}(P)$, $P$ is a fixed point of $\check{f}$. For any given $A, B \in \mathcal{K}(P)$ there exists chains $\xi_1, \xi_2$ connecting $A$ to $P$ and $P$ to $B$ with length $n_1, n_2$, respectively. If $n = n_1 + n_2$, for any $m \geq n$ we can “insert” $m - n$ points in the chain $\xi_1$ equals to $P$ and concatenate with the chain $\xi_2$ in order to obtain a chain from $A$ to $B$ with length $m$ as stated.

Now we are able to show our main result relating the chain components of $f$ and $\check{f}$.

3.8 Theorem: The set $\mathcal{C}_\mathcal{J}$ is a chain component of $\check{f}$ if $\mathcal{K}(P)$ is chain transitive for any $P \in \mathcal{J}$. In particular, $\mathcal{C}_\mathcal{J}$ are the chain components of $f$ if $\mathcal{K}(P)$ is chain transitive for any $P \in \mathcal{B}$.

Proof: Let us assume first that $\mathcal{J} = \{P_1, \ldots, P_n\}$ and that $\mathcal{K}(P_i)$ is chain transitive for $i = 1, \ldots, n$. If $A, B \in \mathcal{C}_\mathcal{J}$ we have that

$$A = \bigcup A_i \quad \text{and} \quad B = \bigcup B_i \quad \text{where} \quad A_i = A \cap P_i \quad \text{and} \quad B_i = B \cap P_i, \quad i = 1, \ldots, n.$$  

By the previous discussion and the assumption on the chain transitivity of $\mathcal{K}(P_i)$ there exist $\epsilon$-chains $\xi_i$ from $A_i$ to $B_i$ with same length $n \in \mathcal{N}$ for $i = 1, \ldots, n$. The chain $\xi$ given by $\xi_1 \cup \cdots \cup \xi_n$ is then an $\epsilon$-chain from $A$ to $B$ showing, by arbitrariness, that $\mathcal{C}_\mathcal{J}$ is chain transitive.

[1]Here we used the fact that two points in the same chain component can be joined by a chain contained in the chain component.
Let us consider now the case $|J| = \infty$ and assume that $K(P)$ is chain transitive for any $P \in J$. For any given $\delta > 0$ and any $A \in \mathcal{C}_J$ we have that

$$A \subset N_\delta \left( \bigcup_{P \in J} A \cap P \right) = \bigcup_{P \in J} N_\delta(A \cap P).$$

By compactness, there exists $J_A = \{P_1, \ldots, P_n\} \subset B$ such that

$$A \subset \bigcup_{i=1}^n N_\delta(A \cap P_i) = \bigcup_{i=1}^n N_\delta \left( A \cap P_i \right). \quad (2)$$

Let then $A, B \in \mathcal{C}_J$ and consider as above $J_A, J_B \subset B$ satisfying (2). By setting $J_{A,B} = J_A \cup J_B$ we have that

$$A \subset N_\delta \left( \bigcup_{P \in J_A} A \cap P \right) \subset N_\delta \left( \bigcup_{P \in J_{A,B}} A \cap P \right) \text{ and } B \subset N_\delta \left( \bigcup_{P \in J_B} B \cap P \right) \subset N_\delta \left( \bigcup_{P \in J_{A,B}} B \cap P \right).$$

If

$$A_\delta = \bigcup_{P \in J_{A,B}} A \cap P \text{ and } B_\delta = \bigcup_{P \in J_{A,B}} B \cap P$$

we have that $A_\delta, B_\delta \in \mathcal{C}_{J_{A,B}}$ and, since $A_\delta \subset A$ and $B_\delta \subset B$, the above implies $\varrho_H(A, A_\delta) < \delta$ and $\varrho_H(B, B_\delta) < \delta$. Moreover, the hypothesis that $K(P)$ is chain transitive for any $P \in J$ implies by the finite case that $\mathcal{C}_{J_{A,B}}$ is chain transitive and hence $A_\delta \sim B_\delta$.

By the arbitrariness of $\delta > 0$ there exists $A_n \to A$ and $B_n \to B$ with $A_n \sim B_n$ which by Proposition 2.1 implies $A \sim B$ and hence $\mathcal{C}_J$ is chain transitive. □

A direct corollary of the previous results is the following.

3.9 Corollary: If $K(P)$ is chain transitive for any $P \in B$ then

$$|B| < \infty \implies |\overline{B}| = 2^{|B|} - 1.$$

Proof: In fact, under the assumption that $K(P)$ is chain transitive for any $P \in B$ we have that the chain components of $\overline{f}$ are parametrized by the nonempty compact subsets of $B$ and hence $|\overline{B}| = |K(\overline{B})| = 2^{|B|} - 1$. □

3.10 Remark: It is important to notice that the chain transitivity of $K(P)$ it is not direct from the chain transitivity of $P$. In fact, in [7] the authors give several equivalence for the transitivity of $K(P)$

Next we give an example where $\mathcal{B}$ is enumerable and $\overline{\mathcal{B}}$ is nonenumerable.

3.11 Example: Let $X = [0, 1]$ and consider

$$f : X \to X, \text{ given by } f(x) := \begin{cases} x \sin \left( \frac{x}{2} \right), & \text{if } x > 0 \\ 0, & \text{if } x = 0. \end{cases}$$

It is not hard to see that

$$A = \{0\} \cup \left\{ \frac{1}{2k + 1}, k \geq 0 \right\}$$

is the set of fixed points of $f$ (see Figure 2) and hence $A \subset \mathcal{C}(f)$. On the other hand, a simple calculation shows that $f^n(x) \to 0$ for any $x \notin A$ and hence

$$A_k := \left[ 0, \frac{1}{2k + 1} \right] \text{ and } A_k^* = \left\{ \frac{1}{2m + 1}, m \leq k \right\}.$$
is an attractor-repellor pair of $f$. Therefore,

$$ \mathcal{C}(f) \subset \bigcap_k A_k \cup A_k^* \subset A \implies \mathcal{C}(f) = A, $$

showing that $f$ has an enumerable number of chain components.

In particular, $\mathcal{B} = \{ \{x\}, \ x \in \mathcal{C}\}$ and since $\mathcal{K}(\{x\}) = \{x\}$ we have that $\mathcal{K}(\{x\})$ is chain transitive for any $x \in \mathcal{C}$. By the previous results we have that the chain components of the induced map $\bar{f}$ are of the form $\mathcal{C}_J$ with $J \in \mathcal{K}(\mathcal{B})$. However,

$$ \mathcal{B} \sim \mathcal{C} = \{0\} \cup \left\{ \frac{1}{2k+1}, \ k \geq 0 \right\} \implies \mathcal{K}(\mathcal{B}) = \{J \subset A; \ |J| < \infty\} \cup A $$

and hence $\bar{f}$ has an unenumerable numbers of chain components.

![Figure 2: Graphic of $f$.](image)

### 3.12 Remark:
Normally, the complexity of the total space $\mathcal{K}(X)$ is greater than the complexity of the base space $X$. For instance, respect to the topological entropy we know that

$$ \text{top}_e(\mathcal{K}(X), \bar{f}) \geq \text{top}_e(X, f), $$

(see [5]). However, in [10, 12] the author consider the extention of $f$ to $\bar{f}$ over the class of compact an convex subsets of a real interval $X$. In this case, the complexity of $\bar{f}$ decreases. We intend to continue to search on this topic by considering new extentions over distinguished subspaces of $\mathcal{K}(X)$.

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