Typically, self-organization is defined as a spontaneous formation of spatial, temporal, spatiotemporal structures or functions in a system comprising multiple interacting components. Importantly, a self-organizing process is assumed to be developing in the absence of specific external controls, as pointed out by Haken [1]:

a system is self-organizing if it acquires a spatial, temporal or functional structure without specific interference from the outside. By “specific” we mean that the structure or functioning is not impressed on the system, but that the system is acted upon from the outside in a non-specific fashion. For instance, the fluid which forms hexagons is heated from below in an entirely uniform fashion, and it acquires its specific structure by self-organization.

To explain structures that spontaneously self-organize when energy or matter flows into a system typically describable by many variables, Haken employed the notion of order parameters (degrees of freedom) and control parameters [1, 2]: slowly varying a relevant control parameter, such as temperature of a ferromagnetic material, may induce an abrupt change, a phase transition, in an observable order parameter, such as the net magnetization. The emergence of global order in complex systems is most striking at criticality, where small changes in control parameters result in a sudden global re-organization. We introduce a measure of thermodynamic efficiency of interactions in self-organizing systems, which quantifies the change in the system’s order per unit work carried out on (or extracted from) the system. We analytically derive the thermodynamic efficiency of interactions for the case of quasi-static variations of control parameters in the exactly solvable Curie-Weiss (fully connected) Ising model, and demonstrate that this quantity diverges at the critical point of a second order phase transition. This divergence is shown for quasi-static perturbations in both control parameters, the external field and the coupling strength. Our analysis formalizes an intuitive understanding of thermodynamic efficiency across diverse self-organizing dynamics in physical, biological and social domains.

The emergence of global order in complex systems with locally interacting components is most striking at criticality, where small changes in control parameters result in a sudden global re-organization. We introduce a measure of thermodynamic efficiency of interactions in self-organizing systems, which quantifies the change in the system’s order per unit work carried out on (or extracted from) the system. We analytically derive the thermodynamic efficiency of interactions for the case of quasi-static variations of control parameters in the exactly solvable Curie-Weiss (fully connected) Ising model, and demonstrate that this quantity diverges at the critical point of a second order phase transition. This divergence is shown for quasi-static perturbations in both control parameters, the external field and the coupling strength. Our analysis formalizes an intuitive understanding of thermodynamic efficiency across diverse self-organizing dynamics in physical, biological and social domains.

These definitions concur with many other approaches to formalize self-organization, highlighting three important aspects [3–6]: (i) a system dynamically advances to a more organized state, while exchanging energy, matter and/or information with the environment, but without a specific external ordering influence; (ii) the interacting system components have only local information, and so exchange only local information, but exhibit long-range correlations; (iii) the increase in organization can be observed as a more coherent global behavior.

In general, as the state of a complex system evolves, its configurational entropy changes. The reduction (or increase) in the configurational entropy occurs at the expense of work extracted or carried out on the system, and the heat exported to the environment. Thus, a thermodynamic analysis of the interactions in self-organizing systems aims to quantify the work, heat and energy exchange between the system and the environment. One can reasonably expect that self-organization is most thermodynamically efficient in the vicinity of the critical points, i.e., at criticality one may expect that a smaller amount of work extracted/done on a system can result in a larger change of the configurational entropy. Indeed, it has been conjectured before that a system in a self-organized low-dimensional phase with fewer available fluid molecules or crystal ions, while the interactions within a biological organism may evolve over generations under environmental selection pressures, bring survival benefits. The role of locally interacting particles contributing to self-organizing pattern formation in biological systems has been captured in a definition offered by Camazine et al. [3]:

Self-organization is a process in which pattern at the global level of a system emerges solely from numerous interactions among the lower-level components of the system. Moreover, the rules specifying interactions among the system’s components are executed using only local information, without reference to the global pattern.
configurations (i.e., describable by just a few order parameters and exhibiting macroscopic stability) may be more efficient than the system in a high-dimensional disorganized phase with more configurations.

To formalize this conjecture, Kauffman proposed a succinct principle behind the higher efficiency of self-organized systems — the generation of constraints during the release of energy — the constrained release channels energy to perform some useful work, which can propagate and be used again to create more constraints, releasing further energy and so on [7]. Following a similar characterization, Carteret et al. [8] have shown that available power efficiency is maximized at critical Boolean networks. The question of thermodynamic efficiency has also been proposed and studied in the context of the cellular information-processing, from the perspective of how close life has evolved to approach maximally efficient computation [9] [10]. Furthermore, a recent thermodynamic analysis of a model of active matter demonstrated how close life has evolved to approach maximally efficient computation [11]. Furthermore, recent thermodynamic analysis of a model of active matter demonstrated that the efficiency of the collective motion diverges at the transition between disordered and coherent collective motion [11]. However, the precise nature of the divergence of the efficiency of collective motion, and its relation to the critical exponents describing the system behavior in the vicinity of the phase transitions remained unclear, due to the lack of analytical expressions for the corresponding configurational probability distributions.

In this study, we offer a generic measure of thermodynamic efficiency of interactions within self-organizing systems, aiming to clearly differentiate between phases of system dynamics, and identify the regimes when the efficiency is maximal. This measure is expressed by contrasting (i) the change of organization attained within the system (i.e., change in the created order or predictability) with (ii) the thermodynamic work involved in driving such a change. We demonstrate that the maximal efficiency is indeed achieved at the critical regime, i.e., during the phase transition, rather than at the macroscopically stable low-dimensional phase per se. The reasons for the maximal efficiency exhibited by systems during self-organization, i.e., at a critical regime, are articulated precisely in terms of the increased order (or the reduction of Shannon entropy) related to the amount of the work carried out during the transition. This measure is defined for specific configurational changes (perturbations), rather than states or regimes — in line with the point made by Carteret et al. [8] that the maximization of power efficiency occurs at a finite displacement from equilibrium.

In developing our approach we build up on information-theoretic and statistical-mechanical methods, interpreting the process of self-organization as a thermodynamic phenomenon, while considering the interactions within the system as distributed information processing or distributed computation [8] [13]. Our aim is to develop a common understanding of thermodynamic efficiency across multiple examples of self-organizing dynamics in physical, biological and social domains. These phenomena include transitions from disordered to coherent collective motion [11] [14] [19], phase transitions in spin systems and active matter [20] [22], chaos-to-order transitions in genetic regulatory networks modeled as random Boolean networks [8] [23], synchronization in networks of coupled oscillators near the edge of chaos [24] [26], transitions across epidemic thresholds during contagions and cascading failures [27] [31], critical dynamics of urban evolution [32] [33], among many others. Self-organizing criticality (SOC) [35] is a related but distinct phenomenon, as we are not attempting to reveal the mechanisms of self-organization towards critical regimes, focusing instead on defining and determining the thermodynamic efficiency of interactions in a representative self-organizing system.

In this work, we select an abstract statistical-mechanical model (Curie-Weiss model of interacting spins in a fully connected graph) — one of the simplest model exhibiting a second-order phase transition — from the widely applicable mean-field universality class. We analytically evaluate dynamics of this model in the vicinity of a phase transition, and prove that the thermodynamic efficiency has a power law divergence at the critical point, and compute its critical exponent.

I. FRAMEWORK

Consider a statistical mechanical system in thermodynamic equilibrium with $X = \{X_1, ..., X_n\}$ global control parameters, which can be changed externally (e.g., magnetic field). A perturbation in the control parameter, $X \rightarrow X + \delta X$, will result in a change in thermodynamic potentials in the system including its entropy and energy. We define the thermodynamic efficiency of interactions as

$$\eta(X; \delta X) = \frac{1}{k_B} \frac{\delta S}{\delta W},$$

where $\delta S$ and $\delta W$ are the change in entropy and the work done/extracted on the system due to the perturbation $\delta X$. Entropy $S$ is a configurational entropy, and thus $\eta(X; \delta X)$ quantifies the reduction (increase) of uncertainty in the state of the system that we gain per unit of work done. A high value of $\eta$ signifies that it is energetically easy to create order (reduce the configurational uncertainty) in the system by changing a control parameter, whereas a low value of $\eta$ indicates that a lot of work is needed to change the order in the system.

In practice, to evaluate $\eta(X; \delta X)$ we need to specify the perturbation protocol. A change in control parameters moves the system out of thermal equilibrium, and we need to compute the amount of work done/extracted, $\delta W$, as the system relaxes back to its equilibrium state. Thus, $\eta(X; \delta X)$ depends on how we perturb the system, and on the master equation that describes the relaxation of the system back to its equilibrium state. In what follows we will consider the case of a quasi-static perturbation protocol, i.e., we assume that the perturbation is...
sufficiently slow that the system effectively adjusts instantaneously to its new equilibrium state.

Helmholtz free energy, \( F(\theta, X) \), is the most useful thermodynamic potential for analyzing the quasi-static protocols at constant temperature. Helmholtz free energy is related to the internal energy \( U \) and entropy \( S \) via equation

\[
U(\theta, X) = \theta S(\theta, X) + F(\theta, X),
\]

where \( \theta \equiv k_B T \). To a first order in \( \delta X \) the change in internal energy, entropy and free energy induced by varying the control parameters are \( \delta U = \delta X \cdot \nabla U|_X \), \( \delta S = \delta X \cdot \nabla S|_X \) and \( \delta F = \delta X \cdot \nabla F|_X \). In a quasi-static process the change in free energy can be identified with the work done on the system, \( \delta F = \delta W \), and the entropy change in the system balances the entropy exported to the environment, \( \delta S = -\delta S^{\text{exp}} \). Thus, for a quasi-static protocol, the thermodynamic efficiency reduces to

\[
\eta(X; \delta X) = \frac{1}{k_B} \frac{\delta X \cdot \nabla S|_X}{\delta F} \equiv \frac{1}{k_B} \frac{\delta S}{\delta F}. \tag{2}
\]

In the case when the variation of order parameter is one-dimensional \( X = x \), equation (2) simplifies to

\[
\eta(x, \delta x) = \frac{1}{k_B} \frac{\delta x \cdot \nabla S|x}{\delta F} \equiv \frac{1}{k_B} \frac{\delta S}{\delta F}. \tag{3}
\]

All of the thermodynamic quantities in (3), expressed in terms of Gibbs probability distribution, have a clear information-theoretic interpretation. Entropy \( S \) is directly proportional to the Shannon entropy \( H \), \( S = -k_B \sum x p(x) \log p(x) = k_B H \). The free energy \( F \) is related to the Fisher information \( I \) via equation \( I = \partial F^2 / \partial^2 X \), with the Fisher information quantifying the sensitivity of the probability distribution to the change in the control parameter, \( I = \sum_x (\partial \log p(x) / \partial X)^2 p(x) \).

There are several interpretations of the Fisher information relevant to critical dynamics and scale dependence: \( I \) is equivalent to the thermodynamic metric tensor, a curvature of which diverges at phase transitions; \( I \) measures the size of the fluctuations in the collective variables around equilibrium; also, \( I \) is proportional to the derivatives of the corresponding order parameters with respect to the collective variables. \( \partial^2 F / \partial X \) gives

\[
\eta(X) = \frac{\partial H / \partial X}{\int I dX}. \tag{4}
\]

Equation (4) expresses the thermodynamic efficiency of interaction during configurational perturbations in terms of information-theoretic quantities of entropy and Fisher information.

Equation (4) was derived and used in [11] in the thermodynamic analysis of collective motion (swarming) exhibiting a kinetic phase transition. Crosato et al. [11] computed \( \eta \) from the distribution \( p(x) \) estimated via sampling from numerical simulations of the model, consequently yielding estimates of \( H \) and \( I \). It was then demonstrated that \( \eta \) diverges around the critical point at which the swarm transitions from disordered to coherent motion.

In this paper we will consider an analytically solvable model exhibiting a second-order phase transition, and demonstrate that indeed the thermodynamic efficiency of interactions peaks at the critical point when the system attains maximally efficient self-organization.

II. EXAMPLE: CURIE-WEISS MODEL

We will proceed by computing \( \eta \) in one of the simplest exactly solvable spin models — the Curie-Weiss (CW) model. The CW model is a model of ferromagnetism, where each spin interacts with all other spins via pairwise interactions and for this reason it is also known as the fully connected Ising model. This model exhibits a second-order phase transition at a finite critical temperature \( T_c \). In the vicinity of the critical point, the analytic expression to all of the thermodynamic quantities are known, which enables the derivation of the analytic expression for \( \eta \). The phase transition from ferromagnetic to paramagnetic states in the Curie-Weiss model belongs to the mean field universality class.

Let \( N \) spins \( \sigma_i \in \{ \pm 1 \} \) be assigned to sites \( i \in \{1...N\} \). A configuration of the system is given by \( \sigma = (\sigma_1, ..., \sigma_N) \). The energy function for the system containing pairwise interactions between spins and in the presence of an external magnetic field \( B \) is given by

\[
E(\sigma) = -\frac{J}{N} \sum_{(i,j)} \sigma_i \sigma_j - \mu B \sum_{i=1}^{N} \sigma_i, \tag{5}
\]

where \( J \) is the exchange interaction strength, \( \mu \) is the spin magnetic moment, and the sum over \( (i,j) \) runs over all of the \( N(N-1)/2 \) pairs of spins. Here, the \( 1/N \) scaling in front of the spin-spin interaction term is to yield an extensive free energy.

The probability of finding the system in configuration \( \sigma \) is given by the Gibbs measure

\[
p(\sigma; T, B, J) = \frac{e^{-E(\sigma)/\theta}}{Z_N(\theta, B, J)}, \tag{6}
\]

where \( \theta \equiv k_B T \) and \( Z_N \) is a partition function for the \( N \)-spin system. The free energy of the \( N \) spin system is given by \( F_N(\theta, B) = \ln Z_N(\theta, B) \). The thermodynamic limit is obtained by taking \( N \to \infty \). In the thermodynamic limit the free energy density \( f(\theta, B) = \lim_{N \to \infty} F_N(\theta, B)/N \) can have the following analytic expression [32]:

\[
f(\theta, B) = -\theta \ln 2 - \theta \ln(\Phi(\theta, B)) \tag{7}
\]
\[ \Phi(\theta, B) = e^{-\frac{J y^2}{2}} \cosh \left( \frac{J y + B}{\theta} \right) \]  

(8)

Here \( y \) is defined as solution to the equation

\[ y = \tanh \left( \frac{J y + \mu B}{\theta} \right). \]  

(9)

The average magnetization per spin is the order parameter of the system, defined as \( m = -\langle \partial f/\partial B \rangle_\theta = \mu y \), and thus the equation of state is \( m = \mu \tan(J m/\theta) \). The phase diagram can be constructed by analyzing the equation of state. The critical point of a second-order phase transition occurs at \( B = 0 \) and \( \theta_c = J \). When \( B = 0 \) and \( \theta > J \) there is only one stable solution of the equation of state, which is \( m = 0 \). When \( B = 0 \) and \( \theta < J \), there are three solutions: one unstable solution \( m = 0 \) and two stable solution \( m = \pm m^* \) where \( m^* \) is found by numerically solving the equation \( m = \mu \tan(J m/\theta) \). Thus at \( B = 0 \) and at the critical temperature \( \theta_c = J \), we transition from a paramagnetic disordered state where \( m = 0 \) to a ferromagnetic ordered state where \( m = \pm m^* \). This transition is of second order, since the second derivatives of \( f \) with respect to both \( B \) and \( \theta \) can easily be shown to be discontinuous at \( \theta_c \).

Having reviewed the phase change behavior of the CW model, we will now evaluate the thermodynamic efficiency \( \eta \) associated with varying the magnetic field \( B \) along a quasi-static protocol. The entropy density is related to the free energy density via equation

\[ s = -\frac{\partial f(y(\theta, B), \theta, B)}{\partial \theta}. \]  

(10)

The equations (7)-(10) are sufficient to evaluate both \( \eta(\theta; \delta B) \) and \( \eta(\theta; \delta J) \) using equation (3).

A. Varying external field, \( B \)

Since equation (9) does not have a closed form solution for \( y(\theta, B) \), it has to be solved numerically. Consequently, in general, \( \eta(\theta; \delta B) \) needs to be evaluated numerically. However, in the vicinity of the critical point, equation (9) has an analytic solution, which allows us to evaluate all thermodynamic quantities including \( \eta \) analytically and study their scaling behaviors. Since the average magnetization \( y \) is zero in the paramagnetic phase and remains very small just below the critical point, we can Taylor-expand equation (9) in the vicinity of the critical point, only keeping several low powers in \( y \). Keeping up to \( O(y^3) \) the equation of state is

\[ K^3 y^3 - 3y(K - 1) - 3h = 0, \]  

(11)

where \( K \equiv J/\theta = \theta_c/\theta \), \( h = \mu B/\theta \). In the case of zero magnetic field, \( h = 0 \), the solution of (11) is

\[ y = 0, \quad \text{for } t \geq 0, \]

(12)

\[ = \pm \sqrt{\frac{3(K - 1)}{K^3}} \sim \sqrt{3}(-t)^{1/2}, \quad \text{for } t < 0 \]

where \( t \) is the reduced temperature \( t \equiv (\theta - \theta_c)/\theta_c \) and \( h \equiv \mu B/\theta \). Equation (11) produces the well-known mean field scaling law for magnetization \( m \sim (-t)^{\beta} \) for \( t < 0 \), with the critical exponent \( \beta = 1/2 \). To evaluate \( \eta(\theta, \delta B) \), we need to compute \( partial f/partial B \) and \( partial s/partial B \), which can now be done using the analytic expression for \( y \) near the vicinity of the critical point.

The magnetization per spin, \( y \), is related to the free energy density via \( y = -\frac{\partial f}{\partial B} \) \( B=0 \). In the ferromagnetic case, \( t < 0 \), we have \( partial B f = -\sqrt{3}(-t)^{1/2} \). Using the equation (10), the derivative of the entropy density with respect to \( \theta \) is \( partial s/partial B = -partial^2 f/partial Bpartial \theta \). This can be evaluated by swapping the order of differentiation, \( partial s/partial B = -partial^2 f/partial \theta partial B = partial(\sqrt{3}(-t))/partial \theta = \sqrt{3}/2(-t)^{-1/2} \).

In the paramagnetic case \( t \geq 0 \), \( y = 0 \), and both \( partial f/partial B \) and \( partial s/partial B \) are zero. The plot of the derivatives of free energy and entropy densities are shown in Figure 1. The plots were constructed by numerically solving equation (9) to find \( y \), and numerically evaluating the derivatives \( partial f/partial B \) and \( partial s/partial B \). The thermodynamic efficiency is the ratio of these two derivatives, \( \eta = partial s/partial f partial B \). For the mean-field model \( \eta \) is undefined for \( t \geq 0 \) since in the paramagnetic phase both derivatives are zero. However, if there is a small bias external field \( B_0 \) then both \( partial f/partial B \) and \( partial s/partial B \) become continuous and non-zero for all \( t \), and \( \eta \) can be evaluated for \( t > 0 \). For the case of small non-zero \( B_0 \), the scaling of \( partial f/partial B \) and \( partial s/partial B \) with \( t \) can be evaluated away from the critical point \( (t \gg 0) \). For \( t \gg 0 \), the equation (11) simplifies to \( y(1 - K) - h = 0 \), since the term \( K^3 y^3 \) is negligible and thus \( y \sim h/(1 - K) = \mu B/(\theta - \theta_c) \) and \( f \) can now be evaluated using equations (7) and (8). From \( f \) we can compute \( partial f/partial B \) and \( partial s/partial B \) and evaluate the scal-
Figure 2. Thermodynamic efficiency $\eta(\theta, \delta B)$ as a function of $\theta$ at several small values of $B$. For $\theta > 1.0$, $\eta$ is undefined at $B = 0$. The solid lines $-1/2t^{\mp}$ for $t < 0$ and $t^{\pm}$ for $t > 0$ are analytic expressions for $\eta$ in the vicinity of the criticality.

$\eta(J, \delta J)$ is expected to peak near the critical point. Near the critical point there is a closed form expression for $\eta$, and thus we can derive the scaling relation between $\eta$ and the reduced coupling strength $j \equiv (J - J_c)/J_c$. For the ferromagnetic case, $J > J_c$, inserting equation (12) into the expressions for the free energy and entropy, taking derivatives with respect to $j$ and then Taylor-expanding in $j$ to the lowest orders yields

$$\partial_j f(j, \theta) = -3j \frac{\mu_j}{2\theta} \quad \text{for } j > 0 \quad (14)$$

$$\partial_j s(j, \theta) = \frac{2B^2 \mu^2}{j^3} \quad \text{for } j < 0. \quad (15)$$

In the paramagnetic case, $j < 0$, the magnetization is zero in the absence of the external magnetic field and the efficiency of interactions is undefined since $\partial_j f = 0$. However, in the presence of small bias magnetic field $B_0$, $\eta$ can be computed since in that case $y \sim h/(1 - K) = \mu B_0/(\theta - J)$. Taylor-expanding $\partial_j f$ and $\partial_j s$ computed with this expression for $y$ to the lowest orders in $j$ gives

$$\partial_j f(j, \theta) = -\frac{B^2 \mu^2}{j^3} \quad \text{for } j > 0 \quad (16)$$

$$\partial_j s(j, \theta) = \frac{B^2 \mu^2}{j^3} \quad \text{for } j < 0. \quad (17)$$

Using equations (14)-(17) we can compute the efficiency of interactions in the vicinity of the critical point:

$$\eta(J, \delta J) = \frac{1}{k_B} \partial_j s$$

$$= \left\{ \begin{array}{ll}
-\frac{1}{k_B^2} \frac{2(-j)^{-1}}{j} & \text{for } j < 0 \\
-\frac{1}{k_B^2} j^{-1} & \text{for } j > 0
\end{array} \right. \quad (18)$$

Figure 3 shows the plot of $\eta(J, \delta J)$ in the vicinity of the critical point for several small values of bias field $B_0$. The dotted curves were obtained by numerically solving for $y$ and numerically computing the derivative of $f$ and $s$. The solid black lines indicate the $|j|^{-1}$ scaling, which agrees very well with the numerical results.

III. CONCLUSIONS

The increasing interest in developing a comprehensive thermodynamic framework for studying complex systems, including the process of self-organization, is driven by several recent developments: theoretical advances in stochastic thermodynamics [33] which enable rigorous quantitative analysis of small and mesoscale systems; technological advances that enable measurement of thermodynamic quantities of such systems [34, 46]; a fusion of information-theoretic, computation-theoretic and statistical-mechanical approaches for analyzing energy-efficiency of information processing devices [47].

We have introduced a measure of thermodynamic efficiency of interactions in self-organizing systems, which
Figure 3. Thermodynamic efficiency $\eta(J, \delta J)$ as a function of $J$ at several small values of $B$ at $\theta = 1.0$. The critical point is at $J_c = 1.0$ or equivalently at $j \equiv (J - J_c)/J_c = 0$. For $j < 0.0$, $\eta$ is undefined at $B = 0$. The solid lines $-2j^{-1}$ for $j < 0$ and $j^{-1}$ for $j > 0$ are analytic expressions for $\eta$ in the vicinity of the criticality.

The solid lines $-2j^{-1}$ for $j < 0$ and $j^{-1}$ for $j > 0$ are analytic expressions for $\eta$ in the vicinity of the criticality.

Our work paves the way to a systematic thermodynamic study of self-organization in complex systems. In the future, it would be interesting to examine models exhibiting other types of phase transitions in a broad class of dynamical systems, including econo- and socio-computation [48–53], and determine whether they also maximally efficient at criticality. It will also be important to extend the analysis to the protocols that drive the system out-of-equilibrium. We believe that an approach to self-organization incorporating the thermodynamic efficiency will also help in clarifying the fundamental relationship between the structure of a complex system and its collective behavior and function [54], as well as support efforts to systematically control and guide the dynamics of complex systems [54].

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