Exact thermodynamics and Luttinger liquid properties of the integrable $t - J$ model

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Abstract

A Trotter-Suzuki mapping is used to calculate the finite-temperature properties of the one-dimensional supersymmetric $t - J$ model. This approach allows for the exact calculation of various thermodynamical properties by means of the quantum transfer matrix (QTM). The free energy and other interesting quantities are obtained such as the specific heat and compressibility. For the largest eigenvalue of the QTM leading to the free energy a set of just two non-linear integral equations is presented. These equations are studied analytically and numerically for different particle densities and temperatures. The structure of the specific heat is discussed in terms of the elementary charge as well as spin excitations. Special emphasis is placed on the study of the low-temperature behavior confirming scaling predictions by conformal field theory and Luttinger liquid theory. To our knowledge this is the first complete investigation of a strongly correlated electron system on a lattice at finite temperature.

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1 Introduction

Strongly correlated electron systems have attracted considerable interest in recent years in view of mechanisms for high-$T_c$ superconductivity [1, 2]. The $t-J$ model arises in various contexts and represents one of the most fundamental systems. The model describes the nearest-neighbor hopping of electrons with spin-exchange interaction. The effect of a strong repulsive on-site Coulomb interaction is modeled by the restriction of the Hilbert space to states without doubly occupied lattice sites. The one-dimensional Hamiltonian reads

$$H = -t \sum_{j,\sigma} P(c_{j,\sigma}c_{j+1,\sigma} + c_{j+1,\sigma}^\dagger c_{j,\sigma}^\dagger)P + J \sum_j (S_j S_{j+1} - n_j n_{j+1}/4), \quad (1)$$

where the projector $P = \prod_j (1 - n_j \uparrow n_j \downarrow)$ ensures that double occupancies of sites are forbidden. At the supersymmetric point $2t = J$ the system was shown to be integrable [3, 4] by the well-known Bethe ansatz [5, 6]. The ground state and excitation spectrum were investigated [7] and critical exponents calculated by finite-size scaling and conformal field theory studies [8, 9]. Here we will discuss the model at finite temperatures.

According to the seminal work [10, 11, 12] thermodynamical properties of general integrable systems are described by an infinite set of coupled integral equations reflecting the existence of infinitely many different rapidity patterns. For the $t-J$ model such integral equations were formulated in [13, 14, 15]. The method consists of a direct evaluation of the partition function by taking into account all excited states of the Hamiltonian. The excitations are derived on the basis of the so-called string conjecture which describes the solutions to the Bethe ansatz equations (for details see [16]). In the evaluation of finite-size quantities at zero temperature the validity of the string conjecture is controversial [17, 18, 19, 20, 21, 22]. Sometimes, a simplistic application is known to lead to erroneous results [23]. On the other hand, the string approach to thermodynamics apparently yields exact equations. (See for instance [24] where the traditional thermodynamics of the quantum RSOS chains were recovered by an independent approach utilizing the fusion hierarchy analysis.) For a nice illustration of the successes of the traditional approach to thermodynamic properties the reader is referred to the review [25] on the Kondo problem.

The general problem to be faced within the traditional string approach to thermodynamics is often the necessity to deal with infinitely many integral equations. In view of practical computations this requires a truncation scheme which is difficult to control. Also, and more fundamentally, the calculation of quantities other than the free energy, for instance correlation lengths at finite temperature, is not possible in the traditional approach.

Our approach takes a different route to overcome these problems. Quite generally, $d$-dimensional quantum systems at finite temperatures can be mapped onto classical systems on $d + 1$-dimensional inhomogeneous lattices by a Trotter-Suzuki mapping [26]. We therefore employ a convenient mapping to a two-dimensional classical model. The classical system corresponding to the integrable $t-J$ chain [27] is the Perk-Schultz model [28, 29, 30, 31] which is a well-known multi-component generalization of the six-vertex
model. The interesting thermodynamical quantities are expressed by eigenvalues of an appropriately defined operator, the so-called quantum transfer matrix (QTM). The largest eigenvalue, as usual, directly yields the free energy. The further knowledge of the next-largest eigenvalues provides the correlation lengths. This represents the important advantage [26, 32, 33, 34, 35] compared to the traditional thermodynamical Bethe ansatz requiring all eigenvalues of the Hamiltonian.

In addition to our general strategy we will incorporate another important ingredient. The most common structure of solvable models is the existence of a family of commuting matrices comprising the physical operator (Hamiltonian or transfer matrix). We therefore utilize a mapping to a classical system where the QTM is apparently embedded in such a family. While the general Trotter arguments respect the first conserved quantity only, our choice implies the existence of infinitely many conserved quantities. This assures the integrability of the QTM from the very beginning. The new approach has been applied to several quantum systems [33, 46, 24, 53, 22, 10, 11] which essentially reduce to scalar field theories. In particular a finite set of integral equations has been derived for the spin 1/2 Heisenberg chain and related models in [24, 33, 38, 11]. The characteristic feature of these equations is the compact formulation originating from the very closely related calculation of finite-size eigenvalues of transfer matrices [22]. The investigations of highly correlated electron systems, such as the Hubbard model, have just started in [43] along the novel strategy.

Here we adopt the most sophisticated method [38]. The QTM is of the “staggered” type and is labeled by two spectral parameters. The dependence on the first parameter leads to the commuting family and thereby playing an essential role in the diagonalization procedure. The second parameter, on the other side, intertwines the finite temperature system with the finite-size geometry of the underlying lattice. This spectral parameter represents a spatial anisotropy of Boltzmann weights in the two-dimensional model. Thus our procedure can be regarded as a finite temperature extension of the usual Hamiltonian limit. However, we must be careful in taking the limit to obtain finite temperature properties: a fine tuning between the parameter and the system size is necessary. This is the price to pay for dealing with non-vanishing temperatures. We will overcome the last problem by adopting the method developed in the finite-size correction problem [12]. All information of the Bethe ansatz equations will be transformed into finitely many coupled nonlinear integral equations, valid for any system sizes. The Trotter limit then can be taken analytically. The resultant equations yield the exact thermodynamical properties of the model.

The study of the thermodynamics of the \( t - J \) model using the QTM method has been introduced in a short letter [44]. The present paper is devoted to a more comprehensive analytical investigation of this system. This work constitutes the first approach allowing for analytical and high precision numerical treatments of a multicomponent Luttinger liquid. In fact, we confirm the general Luttinger liquid picture of interacting fermions analytically in the low-temperature limit. Furthermore, in this limit our equations reduce to linear integral equations thus making direct contact with the dressed energy formalism for the groundstate investigation of the Hamiltonian [15, 16, 17].
The paper is organized as follows. In Section 2 we derive the quantum transfer matrix based on the Perk-Schulz model by keeping the integrability structure (see also [48]). In Section 3 the eigenvalue equations of the QTM are derived by an algebraic Bethe ansatz. In Section 4 the eigenvalue equations are transformed into non-linear integral equations and general results for physical quantities are presented. Section 5 deals with several limiting cases which are studied analytically. Section 6 contains the conclusion of this work. Several more technical, however important aspects of the investigation are deferred to two appendices.

2 Quantum transfer matrix

The (classical) Perk-Schultz model is considered on a square lattice with periodic boundary conditions. Each bond of the lattice is occupied by a variable taking on values $1, \ldots, q$. The appropriate non-zero Boltzmann weights $R_{\alpha \beta}^{\mu \nu}$ associated with a vertex configuration $\alpha, \beta, \mu, \nu$ on the lower, upper, left, and right bond

$$R_{\alpha \alpha}^{\alpha \alpha}(v) = \frac{\sinh(\eta + \epsilon_\alpha v)}{\sinh \eta},$$
$$R_{\alpha \beta}^{\mu \mu}(v) = \epsilon_\alpha \epsilon_\mu \frac{\sinh(v)}{\sinh \eta},$$
$$R_{\mu \alpha}^{\alpha \mu}(v) = \exp(\text{sign}(\alpha - \mu) v),$$

(2)

satisfy the Yang-Baxter equation [28] ($\alpha, \mu = 1, \ldots, q$). The parameters $\epsilon_\alpha$ take only discrete values $\pm 1$ corresponding to bosonic or fermionic statistics of the state $|\alpha\rangle$. The Yang-Baxter equation implies the commutation of all row-to-row transfer matrices for arbitrary spectral parameters $u, v$: $T(u)T(v) = T(v)T(u)$ with

$$T_\alpha^\beta(v) = \sum_\mu \prod_{i=1}^N R_{\alpha_i \beta_i}^{\mu_i \mu_{i+1}}(v),$$

(3)

Consequently, the Hamiltonian [4] with $t = 1$ is obtained as the logarithmic derivative at $v = 0$ (where $T(0)$ reduces to the right-shift operator $T_R$)

$$\mathcal{H} = \frac{d}{dv} \ln T(v) \bigg|_{v=0} = \sum_{i=1}^N h_i,$$

(4)

and turns out to be integrable. For the isotropic limit $\eta \to 0$ (with a rescaling $v \to \eta v$) the non-vanishing matrix elements of the local operators $h_i$ read

$$(h_i)_{\alpha \alpha}^{\alpha \alpha} = \epsilon_\alpha, \quad (h_i)_{\mu \alpha}^{\alpha \mu} = \epsilon_\alpha \epsilon_\mu,$$

yielding the $t - J$ model in the case $q = 3$ and $\{\epsilon_1, \epsilon_2, \epsilon_3\} = \{+ + -\}$. For $q = 3$ and $\{\epsilon_1, \epsilon_2, \epsilon_3\} = \{+ + +\}$ one obtains the Uimin-Sutherland model [19, 3] and for $q = 4$ and $\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\} = \{+ + - -\}$ the Essler-Korepin-Schoutens model [50], respectively. For more details about “generalized $t - J$ systems” see [31, 27].
For further algebraic manipulations we introduce the Boltzmann weights $\mathcal{R}$ and $\tilde{\mathcal{R}}$ of two models related to (2) by anticlockwise and clockwise $90^\circ$ rotations

$$
\mathcal{R}^\alpha_\beta(v) = \mathcal{R}^\alpha_\beta(v), \quad \tilde{\mathcal{R}}^\mu_\nu(v) = \mathcal{R}^\beta_\alpha(-v).
$$

According to (4) we can write

$$
T(v) = T_R e^{\beta H + O(v^2)}, \quad \tilde{T}(v) = T_L e^{\beta H + O(v^2)},
$$

(5)

where $\tilde{T}$ is defined in analogy to (3) and $T_R, T_L$ are the right- and left-shift operators, respectively. By means of the substitution

$$
u = -\beta/N,
$$

(6)

where $\beta$ denotes the inverse temperature and $N$ is a large integer “Trotter” number we find

$$
T(u) \tilde{T}(u) \overset{N/2}{=} e^{-\beta H + O(1/N)}.
$$

(7)

The partition function of the quantum system

$$
Z = \lim_{N \to \infty} \text{Tr} \left( T(u) \tilde{T}(u) \right)^{N/2},
$$

(8)

is identical to the partition function of an inhomogeneous Perk-Schulz model with alternating rows [48]. The technically more convenient column-to-column transfer matrix of such a system is often referred to as the quantum transfer matrix (QTM). Obviously, it is a member of the following family of matrices

$$
T^{QTM}(v) = \sum_\mu \prod_{i=1}^{N/2} R^{\mu_{2i-1}}_{\alpha_{2i-1}} R^{\mu_{2i}}_{\beta_{2i}} (v + u), \quad \tilde{T}^{QTM}(v) = \sum_\mu \prod_{i=1}^{N/2} \tilde{R}^{\mu_{2i}}_{\alpha_{2i+1}} R^{\mu_{2i+1}}_{\beta_{2i+1}} (v - u),
$$

(9)

at $v = 0$. The remarkable property of this family of matrices is its commutativity, as the $R$ and $\tilde{R}$ operators possess the same intertwiner which can be derived immediately from the Yang-Baxter equation solely for $\mathcal{R}$ [52].

The free energy $f$ per lattice site is obtained from the largest eigenvalue of $T^{QTM}(0)$

$$
f = -\frac{1}{\beta} \lim_{L \to \infty} \ln \left( \frac{Z}{L} \right) = -\frac{1}{\beta} \lim_{N \to \infty} \ln \Lambda_{\text{max}},
$$

(10)

Furthermore, the next-leading eigenvalues yield the correlation lengths $\xi$ of the static correlation functions

$$
\frac{1}{\xi} = -\lim_{N \to \infty} \ln \left| \frac{\Lambda}{\Lambda_{\text{max}}} \right|.
$$

(11)
Although leaving the evaluation of the finite-temperature correlation length of the $t-J$ model as a future problem, we stress that there is, in principle, no obstacle in obtaining it in contrast to the traditional string approach.

Concluding this section we would like to give some comments on the treatment of the thermodynamics of Hamiltonian (4) for different particle densities and magnetizations. As usual, this is achieved most conveniently by introducing appropriate external fields (chemical potential $\mu$, magnetic field $h$). This in turn modifies (8) by an additional factor under the trace typically of the kind $\exp(\beta \mu \sum_i n_i + \beta h \sum_i s_i)$, where $n_i$ and $s_i$ are the particle number and spin operators at site $i$. The definition of the associated quantum transfer matrix (4) is modified only by a boundary term [38] depending on the variable $\mu_{N+1}$.

\section{Algebraic Bethe ansatz}

As previously mentioned the largest eigenvalue and the next-leading eigenvalues of the quantum transfer matrix yield the thermodynamical quantities of the quantum chain. In [48] these eigenvalues were obtained by an application of the algebraic Bethe ansatz. The monodromy matrix for this case is defined by an alternating product

$$L^X_\lambda(v) = R^{\lambda \mu_2}_{\alpha_1 \beta_1}(v + u) \tilde{R}^{\mu_2 \mu_3}_{\alpha_2 \beta_2}(v - u) \ldots R^{\mu_{N-1} \mu_N}_{\alpha_{N-1} \beta_{N-1}}(v + u) \tilde{R}^{\mu_N \lambda'}_{\alpha_N \beta_N}(v - u),$$

which is related to the quantum transfer matrix (4) by taking the trace over the auxiliary space

$$T^{\text{QTM}}(v) = \text{Tr}_{\text{aux}} L(v) \equiv \sum_\lambda L^\lambda(v).$$

The monodromy matrix $L(v)$ satisfies the Yang-Baxter equation

$$R^{\lambda \beta}_{\mu \nu}(v - w) L^X_\beta(v) L^\mu_\nu(w) = L^\beta_\mu(w) L^\lambda_\nu(v) R^{\beta X}_\mu \nu(v - w).$$

Using the “Néel state” $|\Omega\rangle = |1, 2, 1, 2, 1, 2, \ldots\rangle$ as a reference state we can construct states $|\Psi\rangle$ by applications of creation operators $L^1_i(v_i)$ and $L^2_3(w_i)$ with different rapidities $v_i$ and $w_i$. These states are shown to be eigenstates [48] of the quantum transfer matrix with eigenvalues

$$\Lambda(v) = \lambda_-(v) + \lambda_+(v) + \lambda_0(v),$$

where

$$\lambda_-(v) = \prod_j \frac{v - w_j + i \epsilon_1}{v - w_j} \left[ (v - i u - i \epsilon_1)(v + i u) \right]^{N/2} e^{\beta \mu_1},$$

$$\lambda_+(v) = \prod_k \frac{v - v_k - i \epsilon_2}{v - v_k} \left[ (v + i u + i \epsilon_2)(v - i u) \right]^{N/2} e^{\beta \mu_2},$$

$$\lambda_0(v) = \prod_j \frac{v - w_j - i \epsilon_3}{v - w_j} (v + i u)^{N/2} \prod_k \frac{v - v_k + i \epsilon_3}{v - v_k} (v - i u)^{N/2} e^{\beta \mu_3},$$

6
and for further convenience we have replaced \( v \) by \( i v \). The quantities \( \beta \) and \( \mu_j \) denote the inverse temperature and external fields coupling to the three different quantum states.

The explicit relation to the chemical potential \( \mu \) and the external magnetic field \( h \) for the \( t-J \) model \( \{\epsilon_1, \epsilon_2, \epsilon_3\} = \{+ + -\} \) reads

\[
\begin{align*}
\mu_1 &= \mu + h/2, \\
\mu_2 &= \mu - h/2, \\
\mu_3 &= 0.
\end{align*}
\]

(17)

The particular set of spectral parameters \( \{w_j, v_k\} \) – often referred to as roots or rapidities – is determined by the condition that the ‘unwanted terms’ cancel in the eigenvalue problem for \( |\Psi\rangle \). This provides an eigenvector with eigenvalue \( \Lambda(v) \). Using the Yang-Baxter equation (14) it turns out [48] that the parameters \( \{w_j, v_k\} \) have to satisfy a system of coupled equations. These Bethe ansatz equations read

\[
\begin{align*}
\left( \frac{v_i + i u + i \epsilon_2}{v_i + i u} \right)^{N/2} &= -\prod_j \frac{v_i - w_j - i \epsilon_3}{v_i - w_j} \prod_k \frac{v_i - v_k + i \epsilon_3}{v_i - v_k - i \epsilon_2} e^{\beta (\mu_3 - \mu_2)}, \\
\left( \frac{w_i - i u - i \epsilon_1}{w_i - i u} \right)^{N/2} &= -\prod_k \frac{w_i - v_k + i \epsilon_3}{w_i - v_k} \prod_j \frac{w_i - w_j - i \epsilon_3}{w_i - w_j + i \epsilon_1} e^{\beta (\mu_3 - \mu_1)}.
\end{align*}
\]

(18)

It is convenient to use an alternative approach to the Bethe ansatz: The defining relations for the rapidities – the Bethe ansatz equations – are equivalent to the analyticity of \( \Lambda(v) \) as a function of \( v \), i.e. the absence of poles in (15). In this sense the denominators in (15) require immediately the Bethe ansatz equations (18). We like to point out that any appropriate treatment of the eigenvalue equations rendering \( \Lambda(v) \) analytic may be considered as an implicit determination of the unknown roots. This idea is used in the next section in a very essential way.

Next, some general properties of the Bethe ansatz roots are discussed. Consider the three different cases \( \{\epsilon_1, \epsilon_2, \epsilon_3\} = \{+ + -\} \) and \( \{+ - +\} \) and \( \{- + +\} \) for which the general solutions \( \{w_j\} \) and \( \{v_k\} \) to (18) are quite different. The eigenvalues \( \Lambda(v) \), however, remain the same, because each set \( \{\epsilon_k\} \) describes the same physical system – the \( t-J \) model. Thus, we may confine ourselves to the case \( \{+ + -\} \) which implies analytical simplifications. The largest eigenvalue of the quantum transfer matrix is characterized by \( N/2 \) roots for each set \( \{w_j\} \) and \( \{v_k\} \). Due to the structure of the Bethe ansatz equations (18) it is natural to assume (at least for vanishing magnetic field \( h \)) that the two sets \( \{w_j\} \) and \( \{v_k\} \) for the largest eigenvalue are symmetric with respect to complex conjugation

\[
w_j = \overline{v}_j, \quad j = 1, \ldots, N/2, \quad h = 0.
\]

(19)

We are led to this conjecture by the typical situation in other Bethe ansatz systems, e.g. the groundstate of the Heisenberg model where the roots are symmetrically arranged in the complex plane.

Using (19) the two different sets of Bethe ansatz equations (with \( h = 0 \)) reduce to

\[
\left( \frac{v_i + i u + i \epsilon_2}{v_i + i u} \right)^{N/2} = -\prod_{j=1}^{N/2} \frac{v_i - \overline{v}_j + i \epsilon_3}{v_i - \overline{v}_j} e^{-\beta \mu}.
\]

(20)
Due to the denominator on the right-hand side no real root is allowed, i.e. all roots must possess non-vanishing imaginary parts. Eventually, we are interested in the Trotter limit $N \to \infty$ in order to calculate thermodynamical properties. It turns out that for $N \to \infty$ the roots accumulate at the origin with vanishing imaginary (as well as real) parts. This delicate point makes it difficult to analyze the limit $N \to \infty$ directly on the basis of the Bethe ansatz equations either analytically or numerically. One can overcome this problem by introducing well adapted integral equations which is the topic of the next section. A detailed investigation of properties of the Bethe ansatz roots is presented in appendix A.

4 Non-linear integral equations

The eigenvalues $\Lambda(v)$ of $T^{QT\!M}(v)$ are analytic functions of the spectral parameter $v$. We use this analyticity to determine the largest eigenvalue with explicit representation (15) by a finite set of non-linear integral equations [48, 44]. This approach also allows for taking the limit $N \to \infty$ analytically. We found that the following combinations of $\lambda_{\pm}(v)$ and $\lambda_0(v)$ (see (16))

$$
\begin{align*}
\mathcal{b} & := \frac{\lambda_{-}}{\lambda_{+} + \lambda_{0}}, & \mathcal{B} & := 1 + \mathcal{b}, \\
\overline{\mathcal{b}} & := \frac{\lambda_{+}}{\lambda_{-} + \lambda_{0}}, & \overline{\mathcal{B}} & := 1 + \overline{\mathcal{b}}, \\
\mathcal{c} & := \frac{\lambda_{-}\lambda_{+}}{\lambda_{0}(\lambda_{-} + \lambda_{+} + \lambda_{0})}, & \mathcal{C} & := 1 + \mathcal{c}.
\end{align*}
$$

(21) (22) (23)

define useful auxiliary functions which satisfy a closed system of functional equations (cf. Appendix B). The function $\mathcal{b}$ ($\overline{\mathcal{b}}$) is an analytic complex function along a finite strip in the upper (lower) half plane, while $\mathcal{c}$ is an analytic complex function on the real axis. The analyticity is due to a cancellations of singularities among the $\lambda$’s like in the case of $\Lambda(v)$. In this sense, the identities between these auxiliary functions encode the information on the Bethe ansatz equations as pointed out in the previous section.

After some lengthy calculations which are described in appendix B we obtain the following relations for the auxiliary functions

$$
\begin{align*}
\log \mathcal{b}(x) & = -2\pi\beta \Psi_{\mathcal{b}}(x + i\gamma) + \beta(\mu + h/2) \\
& - \Psi_{\mathcal{b}} \ast \log(1 + \overline{\mathcal{b}})|_{x+2i\gamma} - \Psi_{\mathcal{b}} \ast \log(1 + \mathcal{c})|_{x+i\gamma}, \\
\log \overline{\mathcal{b}}(x) & = -2\pi\beta \Psi_{\overline{\mathcal{b}}}(x - i\gamma) + \beta(\mu - h/2) \\
& - \Psi_{\overline{\mathcal{b}}} \ast \log(1 + \mathcal{b})|_{x-2i\gamma} - \Psi_{\overline{\mathcal{b}}} \ast \log(1 + \mathcal{c})|_{x-i\gamma}, \\
\log \mathcal{c}(x) & = -2\pi\beta \Psi_{\mathcal{c}}(x) + 2\beta\mu \\
& - \Psi_{\mathcal{b}} \ast \log(1 + \overline{\mathcal{b}})|_{x+i\gamma} - \Psi_{\overline{\mathcal{b}}} \ast \log(1 + \mathcal{b})|_{x-i\gamma} - \Psi_{\mathcal{c}} \ast \log(1 + \mathcal{c})|_{x},
\end{align*}
$$

(24)

with the driving terms (and kernels)

$$
2\pi \Psi_{\mathcal{b}}(x) = \frac{1}{x(x - i)}, \quad 2\pi \Psi_{\overline{\mathcal{b}}}(x) = \frac{1}{x(x + i)}, \quad \text{and} \quad 2\pi \Psi_{\mathcal{c}}(x) = \frac{2}{x^2 + 1},
$$

8
where \( \ast \) denotes the usual convolution \( f \ast g |_x = \int f(x-y)g(y)\,dy \) taken at the indicated arguments \( x, x \pm i \gamma \) and \( x \pm 2i \gamma \) with arbitrary but fixed \( 0 < \gamma < 1 \). The solution of the integral equations provides the largest eigenvalue via:

\[
\log \Lambda = -\log c(0) + 2\beta \mu. \tag{25}
\]

For vanishing magnetic field \( (h = 0) \) the functions \( b(x), \overline{b}(x) \) are related by complex conjugation which leads to a reduction to two nonlinear integral equations. For many applications we need only this case to study.

The integral equations (24) can be solved by iteration (for the time being \( h = 0 \))

\[
b^{(k+1)} = \exp \left( -2\pi \beta \Psi_b + \beta \mu - \Psi_b * \log(1 + \overline{b}^{(k)}) - \Psi_b * \log(1 + c^{(k)}) \right),
\]

\[
c^{(k+1)} = \exp \left( -2\pi \beta \Psi_c + 2\beta \mu - 2\Re(\Psi_b * \log(1 + \overline{b}^{(k)})) - \Psi_c * \log(1 + c^{(k)}) \right).
\]

Choosing appropriate initial functions (see next section) the series \( \{b^{(k)}, c^{(k)}\} \) with \( k = 0, 1, 2, \ldots \) converges rapidly. In practice only a few steps are necessary to reach a high-precision result. Moreover, using the known finite fast Fourier transform algorithm we can compute the convolutions very efficiently.

In order to calculate derivatives of the thermodynamical potential one can avoid numerical differentiations by utilizing similar integral equations guaranteeing the same numerical accuracy as for the free energy. The idea is as follows. Consider the function

\[
lc_\beta := \frac{\partial}{\partial \beta} \log c \quad \text{with} \quad \frac{\partial}{\partial \beta} \log(1 + c) = \frac{1}{1 + c} \frac{\partial c}{\partial \beta} = \frac{c}{1 + c} lc_\beta,
\]

we have

\[
lc_\beta = -\frac{\partial}{\partial \beta} (2\pi \beta \Psi_c) - \Psi_c \ast \left( \frac{c}{1 + c} lc_\beta \right) - \ldots, \tag{26}
\]

which is a linear integral equation for \( lc_\beta \). The equations for \( lb_\beta \) and \( lc_\beta \) (and similar ones such as for \( lb_\mu, \ldots \) ) are also solvable by iteration after the computation of \( b \) and \( c \).

Now we consider various thermodynamical quantities at intermediate temperatures by solving the integral equations (24) numerically. Using the iteration described we obtain the grand canonical potential \( f = f(T, \mu) \) by (24) and its derivative by (26), e.g. the entropy \( S \), particle density \( n \) and the specific heat \( C \) are calculated by means of

\[
S = -\left( \frac{\partial f}{\partial T} \right)_\mu, \quad n = -\left( \frac{\partial f}{\partial \mu} \right)_T, \quad C = T \left( \frac{\partial S}{\partial T} \right)_n.
\]

As we are interested in the thermodynamical quantities with fixed particle density \( n \) we have to allow for a temperature dependent chemical potential \( \mu = \mu(T) \) which follows from

\[
\left( \frac{\partial \mu}{\partial T} \right)_n = -\left( \frac{\partial n}{\partial T} \right)_\mu \left( \frac{\partial n}{\partial \mu} \right)_T^{-1}, \quad \left( \frac{\partial S}{\partial T} \right)_n = \left( \frac{\partial S}{\partial T} \right)_\mu - \left( \frac{\partial n}{\partial T} \right)_\mu \left( \frac{\partial n}{\partial \mu} \right)_T^{-2} \left( \frac{\partial n}{\partial \mu} \right)_T^{-1}.
\]
Figure 1: Specific heat as function of $T$ (with $h = 0$) for different particle densities $n$ with $n \leq 0.6$ and $n \geq 0.6$.

From the second equation the specific heat for fixed particle density $n$ is obtained within the grandcanonical ensemble. For numerical results compare Figure 1. First of all, we note a linear temperature dependence at low $T$. According to conformal field theory the coefficient is given by $\pi(1/v_s + 1/v_c)/3$ where $v_s$ and $v_c$ are the velocities of the elementary spin and charge excitations (see also (31) in section 5.1). Our numerical data are consistent with this expression. (For a completely analytical argument deriving the Luttinger liquid properties in the low-temperature limit the reader is referred to section 5.3). Furthermore, we observe two maxima with changing dominance for increasing particle density $n$. The nature of this structure can be understood from the elementary excitations of the system. In the groundstate the particles are bound in singlet pairs with binding energies varying from zero to some density dependent value. There are two types of excitations. First, there are charge excitations due to energy-momentum transfer onto individual pairs. Second, there are excitations due to the breaking of pairs. The latter excitation is of spin type at lower excitation energies, but changes the character at higher (density dependent) energies to charge type as it describes the motion of single particles. Therefore, the first and second maximum at lower densities (figure 1.a) are caused by charge excitations due to pairs and single particles, respectively. At higher densities (figure 1.b) the maximum at lower temperatures is dominated by excitations of pairs whereas the second one at higher temperatures is caused by spin excitations. For increasing concentration the spin contribution becomes dominant as the charge excitations freeze out. This is in accordance with the limiting case $n = 1$ leading to the spin-1/2 Heisenberg chain. We will come back to this point in section 5.2. The missing spin structure in the specific heat at low and intermediate densities is found at quite low temperatures shown in figure 2.

It is worthwhile to compare these results with the findings for the Hubbard model
investigated by the traditional thermodynamical Bethe ansatz [53]. The structure found in the specific heat is explained by spin and charge excitations which do not change their character in contrast to the $t-J$ model. For certain densities a low-temperature charge peak was found which is caused by single particle excitations. A charge peak at higher temperatures appears because of excitations due to doubly occupied lattice sites with energies of the order $U$ for large Coulomb interaction.

To conclude our investigation we present numerical results for other thermodynamical quantities. Figure 3 presents the entropy and the compressibility $\kappa = \partial n / \partial \mu$ for various particle densities. Note the divergent low-temperature compressibility for particle densities $n \to 0$ and $n \to 1$.

5 Analytical solutions of the integral equations

5.1 Low-density regime

In the previous sections we have applied the algebraic Bethe ansatz to the quantum transfer matrix and derived non-linear integral equations (24) and (25) for the largest eigenvalue which is directly related to the free energy (10) of the quantum system at finite temperature $T = 1/\beta$. This section is devoted to the low-density limit and analytical solutions to the integral equations.

Let us consider the low-temperature limit for small $\mu$, i.e. the case $\beta \gg 1$ with $\mu \ll 1$. Because of $2\pi\beta \Psi_{b,c} \gg 1$ for small values $x$ in (24), we see that the functions $b$ and $c$ are almost zero. Therefore, the essential contribution in (24) is caused by $x \gg 1$. As a first
approximation we can assume that $b$ and $c$ do not vary much for which the equations read $(h = 0)$

\[
\log b \simeq -\frac{\beta}{x^2} + \beta \mu - \log(1 + b) - \log(1 + c),
\]
\[
\log c \simeq -2\frac{\beta}{x^2} + 2\beta \mu - 2\log(1 + b) - \log(1 + c),
\]

where the function $b$ becomes real for $\gamma = 1/2$. As a solution we immediately have

\[
b = \frac{e^{-\beta/x^2 + \beta \mu}}{1 + e^{-\beta/x^2 + \beta \mu}} \quad \text{and} \quad c = \frac{e^{-2\beta/x^2 + 2\beta \mu}}{1 + 2e^{-\beta/x^2 + \beta \mu}},
\] (27)

which is inserted in (24, 25) again to obtain an approximation for the largest eigenvalue (25). The analytical expression for the grand canonical potential $f = -(\log \Lambda)/\beta$ yields (after the change of variable $x \to \sqrt{\beta/x}$)

\[
f = -\int_{-\infty}^{\infty} \frac{dx}{\pi \beta^{3/2}} \log \left(1 + e^{-x^2 + \beta \mu}\right),
\] (28)

\[
= -\frac{4}{3\pi} \mu^{3/2} - \frac{\pi}{6\sqrt{\mu}} T^2 + o(\mu^{3/2}, T^2/\sqrt{\mu}),
\] (29)

in agreement with the numerical results. This implies for the particle density $n = -\partial f/\partial \mu$, entropy $S = -\partial f/\partial T$ and specific heat $C = T\partial S/\partial T$:

\[
n = \frac{2}{\pi} \sqrt{\mu}, \quad S = \frac{2}{3n} T, \quad \text{and} \quad C = \frac{2}{3n} T.
\] (30)

These values represent thermodynamical quantities of the $t - J$ model (with $h = 0$) for small particle densities $n \ll 1$ in the low-temperature limit $T \ll 1$ such that $(T/n) \ll 1$. 

Figure 3: Entropy $S$ and compressibility $\kappa$ versus $T$ for different densities $n$. 

---

\[\text{Figure 3: Entropy } S \text{ and compressibility } \kappa \text{ versus } T \text{ for different densities } n.\]
Figure 4: Comparison of exact (numerical) calculations (denoted by the solid line) with the analytical approximation (denoted by the dotted line) according to (35). Entropy $S$ (a) and specific heat $C$ (b) as function of $T$ for different particle densities $n$.

The low-temperature behavior of the $t-J$ model described by (30) can be compared with independent calculations by means of conformal field theory [54, 55]. Due to predictions of conformal field theory we expect for the low-temperature asymptotics

$$f = f_0 - \left( \frac{\pi c_s}{6 v_s} + \frac{\pi c_c}{6 v_c} \right) T^2,$$

(31)

where $v_{s,c}$ and $c_{s,c}$ are the velocities and central charges ($c_{s,c} = 1$) for the elementary spin and charge excitations. For small particle densities we have $v_{s,c} = \pi n [7]$, thus (29) and (31) are consistent.

On the other hand, we can compute the high-temperature limit $T \to \infty$, i.e. $\beta \to 0$. The solutions to the integral equations (24) are constants. One obtains

$$b = \frac{e^{\beta \mu}}{1 + e^{\beta \mu}}, \quad c = \frac{e^{2\beta \mu}}{1 + 2e^{\beta \mu}} \quad \text{and} \quad f = -\beta^{-1} \log(1 + 2e^{\beta \mu}),$$

(32)

leading to

$$S = n \log \frac{2(1-n)}{n} - \log(1-n) \quad \text{with} \quad \mu = T \log \frac{n}{2(1-n)},$$

(33)

as expected by counting the degrees of freedom per lattice site. The simple high-temperature limit for finite $\mu$ immediately leads to $S = \ln 3$ and $n = 2/3$.

The analytical benchmarks for $T \to 0$ and $T \to \infty$ imply the consistency of our approach. The integral equations for intermediate temperatures have to be treated numerically. However, we have found an excellent analytical ansatz for $0 \leq T < \infty$ valid in the
case of small particle densities $n \ll 1$ (cf. figure 4). One can extend (27) to finite $\beta$ (with $\gamma = 1/2$) by

$$b_0(x) = \frac{e^{-\beta/(x^2+1/4)+\beta\mu}}{1 + e^{-\beta/(x^2+1/4)+\beta\mu}} \quad \text{and} \quad c_0(x) = \frac{e^{-2\beta/(x^2+1)+2\beta\mu}}{1 + 2e^{-\beta/(x^2+1)+\beta\mu}}, \quad (34)$$

yielding (25)

$$f_o = - \int_{-\infty}^{\infty} \frac{dx}{\pi\beta} \left( \frac{1}{x^2 + 1/4} \log (1 + b_0(x)) + \frac{1}{x^2 + 1} \log (1 + c_0(x)) \right). \quad (35)$$

It turns out that this function describes the thermodynamical properties of the $t-J$ model very well. In figure 4 the entropy and specific heat (derived from (35)) are depicted as a function of $T$ and $n$. They are compared with the exact quantities calculated numerically. In particular, for small particle densities $n \ll 1$ the deviation from the exact values is negligible. Furthermore, the low-temperature limit provides the integral (28) with the corresponding asymptotics (29).

### 5.2 High-density regime

Next, we turn to the study of the $t-J$ model in the limit of large chemical potential $\mu$ while keeping the temperature finite. With a glance to (24) we see that for large $\mu$ the following scaling behavior sets in

$$\log b = O(1), \quad \log \bar{b} = O(1), \quad \log c = \beta\mu + O(1).$$

In this limit the approximation $\log c \simeq \log(1 + c)$ holds up to exponentially small corrections. Hence, $\log c$ can be solved from the last equation in (24) in terms of the $\log(1 + b)$ and $\log(1 + \bar{b})$ functions. The actual calculation is done most conveniently in Fourier space. For details we refer to Appendix B. The resulting relations are

$$\log b(x) = -2\pi\beta \Phi(x + i\gamma) + \beta h/2$$

$$+ k \ast \log(1 + b)|_{x} - k \ast \log(1 + \bar{b})|_{x+2i\gamma},$$

$$\log \bar{b}(x) = +2\pi\beta \Phi(x - i\gamma) - \beta h/2$$

$$+ k \ast \log(1 + \bar{b})|_{x} - k \ast \log(1 + b)|_{x-2i\gamma},$$

$$\log c(x) = -2\pi\beta k(x) + \beta\mu$$

$$+ \Phi \ast \log(1 + b)|_{x-i\gamma} - \Phi \ast \log(1 + \bar{b})|_{x+i\gamma}, \quad (36)$$

with the driving term and kernel

$$\Phi(x) = -\frac{i}{2} \frac{1}{\sinh \pi x}, \quad k(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ikx}}{1 + e^{ikl}} dk.$$
Finally, the corresponding eigenvalue of the quantum transfer matrix is given by
\[
\log \Lambda = 2\beta \log 2 + \beta \mu + \frac{i}{2} \int_{-\infty}^{\infty} \log\left(1 + b(x + i \gamma)\right) \frac{dx}{\sinh \pi(x + i \gamma)} - \frac{i}{2} \int_{-\infty}^{\infty} \log\left(1 + \overline{b}(x - i \gamma)\right) \frac{dx}{\sinh \pi(x - i \gamma)}.
\]

According to this result the particle density in the limit of large chemical potential is \( n = 1 \) which in fact is the largest possible value. By comparison of (36) with (38) we find the nonlinear integral equations of the isotropic antiferromagnetic spin-1/2 Heisenberg chain. This was expected as the \( t - J \) model in the limit of “half-filling” reduces to the Heisenberg model.

5.3 Low-temperature asymptotics

Let us consider the model at low temperatures but at arbitrary filling. We will employ an approximation to obtain the free energy in the low-temperature regime up to explicit \( O(1/\beta) \) terms. The approximation yields a direct relation between the truncated nonlinear integral equations and the so-called dressed energy formalism \([13, 49]\) for the groundstate properties of quantum chains. This correspondence itself is novel, and serves as a consistency check of our results from the nonlinear integral approach on one side and the dressed energy formalism and conformal field theory on the other side.

Consider the system in an external magnetic field. Then \( b(x) \) and \( \overline{b}(x) \) are no longer complex conjugate to each other. The latter function can be neglected as \( b \sim O(e^{-\beta h}) \) for positive \( h \). We choose \( \gamma = 1/2 \) so that \( b(x) \) is a real-valued function on the real axis. Then the approximated nonlinear integral equations read
\[
\log b(x) = -\beta \epsilon_0^b(x) - \Psi_b \ast \log(1 + c)|_{x+i/2},
\]
\[
\log c(x) = -\beta \epsilon_0^c(x) - \Psi_b \ast \log(1 + b)|_{x+i/2} - \Psi_c \ast \log(1 + b)|_x,
\]
where
\[
\epsilon_0^b(x) = 2\pi \Psi_b(x + i/2) - (\mu + h/2) \quad \text{and} \quad \epsilon_0^c(x) = 2\pi \Psi_c(x) - 2\mu.
\]
Numerically, we have verified that \( b(x) \) and \( c(x) \) exhibit a crossover behavior from \( b(x), c(x) \ll 1 \) to \( b(x), c(x) > 1 \). Due to the driving term in (37) of order \( O(\beta) \) the crossover becomes very pronounced in the low-temperature limit. We define “Fermi surfaces” by \( b(\pm \Lambda_b) = c(\pm \Lambda_c) = 1 \). Keeping these remarks in mind, we split the contribution from the first integral term in (37) into three pieces
\[
\Psi_b \ast \log(1 + c)|_{x+i/2} \rightarrow \frac{1}{2\pi} \int_{|x'| > \Lambda_c} T_{s,c}(x - x') \log c(x') \, dx' + 2\pi \int_{|x'| > \Lambda_c} T_{s,c}(x - x') \log(1 + \frac{1}{c(x')}) \, dx'
\]
\[
\left. + \frac{1}{2\pi} \int_{-\Lambda_c}^{\Lambda_c} T_{s,c}(x - x') \log(1 + c(x')) \, dx' \right. \ 
\]
(38)
where

\begin{align*}
T_{s,c}(x) &= 2\pi \Psi_b(x + i/2) = 1/(x^2 + 1/4), \\
T_{c,c}(x) &= 2\pi \Psi_c(x) = 2/(x^2 + 1). 
\end{align*}

As the slope is sufficiently steep at low temperatures, we can approximate \(c(x)(|x| < \Lambda_c)\)
and \(1/\epsilon(x)(|x| > \Lambda_c)\) by \(\exp(-\log c)'(\Lambda_c)|x \pm \Lambda_c|\) in the vicinity of the “Fermi” surfaces.

Note that \((\log c)'(\Lambda_c)\) is of order \(\beta\). We can thus justify this linearization over the total integral range for the last two integrals in (38). Hence the last term in (38) reduces to

\begin{equation}
\sim \frac{1}{2\pi} (T_{s,c}(x - \Lambda_c) + T_{s,c}(x + \Lambda_c)) \times \int_0^\infty \log(1 + e^{-(\log c)'(\Lambda_c)} x) \, dx \tag{40}
= (T_{s,c}(x - \Lambda_c) + T_{s,c}(x + \Lambda_c)) \frac{\pi}{24(\log c)'(\Lambda_c)}.
\end{equation}

The second term in (38) can be treated in the same way, and turns out to be identical to the result in (40). Similarly, we can linearize the integral equation for \(c(x)\). The resultant equations are now given by linear integral equations over finite integration intervals.

\begin{align*}
\log b(x) &= -\beta \epsilon_b^0(x) - (T_{s,c}(x - \Lambda_c) + T_{s,c}(x + \Lambda_c)) \frac{\pi}{12(\log c)'(\Lambda_c)} \\
&\quad - \frac{1}{2\pi} \int_{|x'| > \Lambda_c} T_{s,c}(x - x') \log c(x') \, dx', \\
\log c(x) &= -\beta \epsilon_c^0(x) - (T_{s,c}(x - \Lambda_b) + T_{s,c}(x + \Lambda_b)) \frac{\pi}{12(\log b)'(\Lambda_b)} \\
&\quad - (T_{c,c}(x - \Lambda_c) + T_{c,c}(x + \Lambda_c)) \frac{\pi}{12(\log c)'(\Lambda_c)} \\
&\quad - \frac{1}{2\pi} \int_{|x'| > \Lambda_b} T_{s,c}(x - x') \log b(x') \, dx' - \frac{1}{2\pi} \int_{|x'| > \Lambda_c} T_{c,c}(x - x') \log c(x') \, dx'. 
\end{align*}

Let us introduce the short-hand notation

\begin{equation}
\begin{pmatrix} 1 \\ T_{s,c} \\ 1 + T_{c,c} \end{pmatrix} \ast \begin{pmatrix} \log b \\ \log c \end{pmatrix} = \begin{pmatrix} \phi_b \\ \phi_c \end{pmatrix}, \tag{42}
\end{equation}

with

\begin{align*}
\phi_b(x) &= -\beta \epsilon_b^0(x) - (T_{s,c}(x - \Lambda_c) + T_{s,c}(x + \Lambda_c)) \frac{\pi}{12(\log c)'(\Lambda_c)}, \\
\phi_c(x) &= -\beta \epsilon_c^0(x) - (T_{s,c}(x - \Lambda_b) + T_{s,c}(x + \Lambda_b)) \frac{\pi}{12(\log b)'(\Lambda_b)} \\
&\quad - (T_{c,c}(x - \Lambda_c) + T_{c,c}(x + \Lambda_c)) \frac{\pi}{12(\log c)'(\Lambda_c)} \tag{43}.
\end{align*}

We are interested in the evaluation of \(\log c(0)\). Apparently, it consists of two contributions, \(O(\beta)\) and \(O(1/\beta)\) which are contained in the integral equation (41). We will separate these
terms by introducing two further ‘dressed’ functions $\xi_b(x), \xi_c(x)$ satisfying

$$
\begin{pmatrix}
1 \\
T_{s,c}\frac{T_{s,c}}{1 + T_{s,c}}
\end{pmatrix}
\begin{pmatrix}
\xi_b \\
\xi_c
\end{pmatrix}
= 
\begin{pmatrix}
T_{s,c} \\
T_{c,c}
\end{pmatrix},
$$

(44)

where we have adopted the same abbreviation as in (12). After changes in the order of integrations, all integrands are given by products of $\xi_b, \xi_c$ and known functions. As all functions appearing here are even, we arrive at the expression:

$$
\log c(0) \simeq 2(\mu - 1)\beta + \frac{\beta}{2\pi} \int_{|x| > \Lambda_b} \epsilon_b^0(x) \xi_b(x) \, dx + \frac{\beta}{2\pi} \int_{|x| > \Lambda_c} \epsilon_c^0(x) \xi_c(x) \, dx
$$

$$
- \frac{\pi}{6(\log c)'(\Lambda_c)} \left( T_{c,c}(\Lambda_c) - \frac{1}{2\pi} \int_{|x| > \Lambda_c} T_{s,c}(x - \Lambda_c) \xi_b(x) \, dx - \frac{1}{2\pi} \int_{|x| > \Lambda_c} T_{c,c}(x - \Lambda_c) \xi_c(x) \, dx \right)
$$

$$
- \frac{\pi}{6(\log b)'(\Lambda_b)} \left( T_{s,c}(\Lambda_b) - \frac{1}{2\pi} \int_{|x| > \Lambda_b} T_{s,c}(x - \Lambda_b) \xi_c(x) \, dx \right).
$$

(45)

By definition (12) the contents of the last brackets are nothing but $\xi_c(\Lambda_c), \xi_b(\Lambda_b)$ themselves. We thus obtain the free energy at finite filling as

$$
f = \frac{1}{2\pi} \int_{|x| > \Lambda_b} \epsilon_b^0(x) \xi_b(x) \, dx + \frac{1}{2\pi} \int_{|x| > \Lambda_c} \epsilon_c^0(x) \xi_c(x) \, dx
$$

$$
- \frac{\pi \xi_b(\Lambda_b)}{6\beta(\log b)'(\Lambda_b)} - \frac{\pi \xi_c(\Lambda_c)}{6\beta(\log c)'(\Lambda_c)} + O(1/\beta^2).
$$

(46)

Now we compare our results with those found within the dressed energy formalism. We quote the result by Kawakami and Yang [17]. Immediately seen from their equation (2.26), the dressed energy functions for spinon ($\epsilon_s$) and holon ($\epsilon_c$) satisfy the same integral equations (12) except that the right-hand side should be replaced by ($-\epsilon_b^0, -\epsilon_c^0$). Therefore, we have the connecting relations

$$
\log b(x) = \beta \epsilon_s(x) + O(1/\beta), \quad \log c(x) = \beta \epsilon_c(x) + O(1/\beta),
$$

(47)

within the $O(1/\beta)$ approximation. Now that we want to evaluate $\log b(x), \log c(x)$ and $\beta f$ including all $O(1/\beta)$ corrections we can replace ($\log b)'(\Lambda_b)$ and ($\log c)'(\Lambda_c)$ in the denominators of (46) by $\beta \epsilon_s'(\Lambda_b)$ and $\beta \epsilon_c'(\Lambda_c)$, respectively. Moreover, we find that our $\xi$ functions are proportional to bulk density functions of spinons and holons: $\xi_{b,c}(x) = 2\pi \rho_{s,c}(x)$. To this end compare equation (44) with (2.20) in [17]. Hence the integrands of the first two terms in equation (46) are products of bare energy functions times bulk density functions. Therefore, they give “bulk” (or zero temperature) contributions, as expected. The remaining terms, which represent finite temperature contributions, can be neatly written down by adopting the expressions for the sound velocities of the elementary excitations in the dressed energy approach. Namely, let the sound velocities be $v_\alpha = \epsilon_\alpha' / 2\pi \rho_\alpha |_{\Lambda_\alpha}$ (with $\alpha = s, c$) then the $O(1/\beta^2)$ terms in the free energy read

$$
- \frac{\pi}{6\beta^2} \left( \frac{1}{v_s} + \frac{1}{v_c} \right).
$$

(48)
We thus observe various consistency relations. The free energy obeys the prediction by conformal invariance and Luttinger liquid theory for arbitrary filling. The sound velocities coincide with those obtained from the dressed energy calculations. We remark that there should be some subtleties in treating the model for vanishing external fields in the same approximation scheme, while the original non-linear integral equations are valid for all cases. Even for the linear integral equations obtained in this section analytic solutions cannot be found for general filling. Therefore, one has to resort to numerical treatments. Here however, we restrict ourselves to the already presented numerical results for the original nonlinear integral equations which are not limited to the low-temperature regime.

6 Discussion

We have derived eigenvalue equations for the quantum transfer matrix of the $t-J$ model. This approach permits the exact analytical as well as numerical calculation of thermodynamical quantities. Instead of solving an infinite set of integral equations – as is necessary in the traditional thermodynamical Bethe ansatz – we have to solve integral equations for only two functions (in the case of vanishing magnetic field). We have considered analytically certain low- and high-temperature limits verifying the values predicted by conformal field theory. Moreover, the case of intermediate temperatures was treated numerically. As shown, the specific heat and compressibility display an interesting behavior in dependence on particle density and temperature. Also, the reduction to the Heisenberg model in the limit of ‘half-filling’ has been proved in a straightforward way. A direct relation has been established to the dressed energy formulation concerning the groundstate properties. This has been easily achieved by a linearization scheme in the vicinity of the “Fermi surface”.

Let us emphasize again the advantage of the novel method. In the traditional approach one has to deal with infinitely many coupled integral equations. Except for special cases, the set of equations defies further analysis because of its complexity. Indeed, it was almost 20 years after the derivation of equations [56] that the numerical calculation has been done for several thermodynamic quantities of the Hubbard model [53]. Still, the explicit calculation allowed for only 2 (!) bound charge rapidities and 15-30 spin rapidities, while the original equations contain $\infty \times \infty$ rapidities. Although it was claimed that such truncation works for very low temperatures, it might be rather inaccurate at finite temperatures unless the Coulomb interaction takes very large values.

Our treatment, on the other hand, deals with only two integral equations which clearly is advantageous in practical calculations. In view of this, the presented work might be considered as the first approach yielding explicit and concrete results for a lattice system with interacting spin-1/2 fermions at all temperatures. We actually performed numerical computations over a wide range of densities and temperatures. The results are of high accuracy ($10^{-6}$) in the entire range of parameters.

The application of the present approach is not limited to the study of 1D quantum chain problems but also to those of deformed conformal field theories. At the moment the main tool in this field is the so-called TBA method which originates from the traditional
string approach. As remarked by several authors \cite{39,57}, the novel approach is an efficient alternative for scalar models (the sine-Gordon model, spin 1/2 Heisenberg chain). The necessity of a generalization of this method to the multicomponent case (Luttinger liquid) has been stated as an important problem in \cite{58}. This program has been carried out in the present paper. It was shown explicitly for the $t-J$ model. The success of our approach is certainly not accidental, i.e. the applicability is not model dependent, but rather universal as shown by the general embedding of the QTM into a commuting family of matrices. We expect to report in the near future further research on other models, e.g. on a generalized Hubbard model \cite{59}, as well as on additional thermodynamical quantities such as correlation lengths.

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\textbf{A Properties of Bethe ansatz roots}

We introduce the function $\alpha(v)$

$$
\alpha(v) = \left(\frac{v + i u + i}{v + i u - 1}\right)^{N/2} \prod_{j=1}^{N/2} \frac{v - v_j}{v - v_j + i} e^{\beta \mu}.
$$

(49)

According to (27) a root $v_k$ yields

$$
\alpha(v_k) = -1.
$$

(50)

It is useful to understand the principal behavior of the roots determining the largest eigenvalue. This can be done analytically for the case of finite $N$ for $\mu = h = 0$ and large $\beta$. Within this limit the integral equation (see appendix B, equations (73) and (74)) can be used to obtain

$$
\log \alpha(v_k) = (2k + 1)\pi i \approx \Phi_\beta^{(N)}(v_k),
$$

providing

$$
v_k \simeq -\frac{i}{2} + i \sqrt{1/4 + u^2 + i u \cot((2k + 1)\pi/N)}, \quad k = 0, \pm 1, \pm 2, \ldots
$$

(51)

which will be used as initial values for $\mu$ and $\beta$ finite ($u = -\beta/N$). Numerically, these values are shown to approximate the actual roots rather well even for finite $\beta$. In the limit
Figure 5: Distribution of roots $v_k$ (full circles) and holes $v^h_k$ (open circles) in the complex plane for $N = 32$, $\beta = 8$, $h = 0$ and for (a) $\mu = 0$ and (b) $\mu = 0.5$.

$N \to \infty$ the equation (51) reads

$$v_k \simeq -\frac{i}{2} + i \sqrt{1/4 - i \beta /((2k + 1)\pi)}.$$  \hspace{1cm} (52)

Here one can see easily the accumulation of roots for $|k| \gg 1$.

In figure 5 the distribution of roots in the complex plane is depicted for two different chemical potentials $\mu$. The exact values are computed numerically by the Newton method using the initial values (51). Also shown are the so-called holes $v^h_k$ satisfying

$$a(v^h_k) = -1,$$ \hspace{1cm} (53)

i.e. (50), but not coinciding with any roots. As seen from figure 5 the imaginary parts of the roots become smaller for increasing $\mu$. Simultaneously, the largest holes converge to the real axis.
B Derivation of the integral equations

In this appendix we describe the detailed derivation of the integral equations introduced in section 4. We define the following auxiliary functions

\[ a := \frac{\lambda_0}{\lambda_+}, \quad \mathfrak{A} := 1 + a = \frac{\lambda_+ + \lambda_0}{\lambda_+}, \]  

(54)

\[ b := \frac{\lambda_-}{\lambda_+ + \lambda_0}, \quad \mathfrak{B} := 1 + b = \frac{\lambda_- + \lambda_+ + \lambda_0}{\lambda_+ + \lambda_0}, \]  

(55)

\[ c := \frac{\lambda_- \lambda_+}{\lambda_0 (\lambda_- + \lambda_+ + \lambda_0)}, \quad \mathfrak{C} := 1 + c = \frac{(\lambda_- + \lambda_0)(\lambda_+ + \lambda_0)}{\lambda_0 (\lambda_- + \lambda_+ + \lambda_0)}, \]  

(56)

where \( \lambda_i \) are given in (16). Note that the rescaled eigenvalue (corresponding to zero groundstate energy for vanishing chemical potential and magnetic field)

\[ \log \Lambda \to \log \Lambda - N \log(1 - u), \]

(57)

(with \( u = -\beta/N \)) of the QTM is given by

\[ \log \Lambda = N \log(1 + u) - N \log(1 - u) - \log(c(0)) + 2\beta \mu. \]

For further convenience we write

\[ \Phi_{\pm}(v) = (v \pm i u)^{N/2}, \quad q_-(v) = \prod_i (v - w_i), \quad q_+(v) = \prod_i (v - v_i), \]  

(58)

which provide the following relations

\[ \lambda_- = \frac{q_-(v + i)}{q_-(v)} \Phi_+(v) \Phi_-(v - i), \]  

(59)

\[ \lambda_+ = \frac{q_+(v - i)}{q_+(v)} \Phi_-(v) \Phi_+(v + i), \]  

(60)

\[ \lambda_0 = \frac{q_- (v + i)}{q_-(v)} \frac{q_+ (v - i)}{q_+(v)} \Phi_+(v) \Phi_-(v). \]  

(61)

Introducing \( \mathfrak{a}, \mathfrak{b}, \mathfrak{A}, \mathfrak{B} \) as in (54-56) with all subscripts + and − interchanged we find

\[ \lambda_- + \lambda_0 = \lambda_0 \mathfrak{C} \mathfrak{B} = \lambda_+/\mathfrak{b}, \]  

(62)

\[ \lambda_+ + \lambda_0 = \lambda_0 \mathfrak{C} \overline{\mathfrak{B}} = \lambda_-/\mathfrak{b}, \]  

(63)

\[ (\lambda_- + \lambda_0)(\lambda_+ + \lambda_0) = \lambda_- \lambda_+ \mathfrak{C}/\mathfrak{c}. \]  

(64)

Defining the auxiliary function

\[ \mathfrak{D} := \frac{1}{q_+(v)} \left( q_-(v) \Phi_+(v + i) + q_- (v + i) \Phi_+(v) \right), \]  

\[ = (\lambda_+(v) + \lambda_0(v)) \frac{q_-(v)}{q_+(v - i) \Phi_-(v)} = \frac{q_-(v)}{q_+(v)} \Phi_+(v + i) \mathfrak{A}(v), \]  

(65)
we can show that $\mathcal{D}$ is analytic and non-zero in the upper complex plane $\mathbb{C}^+$ including the real axis. The zeros of the denominator $q_+(v)$ cancel with the zeros of $\mathfrak{A}$ because these values are identical to the roots $v_i$. Therefore, the only zeros of $\mathcal{D}$ are caused by the holes in $\mathbb{C}^-$ (cf. figure 3, appendix A). It is useful to rewrite $\mathcal{D}$ by means of (63)

$$
\mathcal{D} = \frac{q_-(v+i)}{q_+(v)} \Phi_+(v) \mathfrak{C}(v) \bar{\mathfrak{B}}(v),
$$

which yields (64)

$$
\mathcal{D} \mathfrak{D} = \Phi_-(v-i) \Phi_+(v+i) \mathfrak{C}/\mathfrak{C},
$$

Now we use the Fourier transform $\widehat{f}^\pm$ of the logarithmic derivative of $f(v)$

$$
\widehat{f}^\pm = \int_{L^\pm} \frac{dv}{2\pi} \left[ \log f(v) \right]' e^{-iv},
$$

where the integration contour $L^\pm$ is a straight line near the real axis taken in the upper and lower half plane such that the contour is parameterized by $v = x \pm i \gamma$ with $v \in L^\pm$, $x \in \mathbb{R}$ and $\gamma > |u|$. $\mathfrak{D}^\pm$ is calculated for expression (63) and $\mathfrak{D}^{(\pm)}$ for expression (67).

The main idea of this procedure consists in the implicit determination of the unknown roots, i.e. the zeros of $\mathfrak{A}$, by employing functional equations in general and the analyticity of $\mathcal{D}$ in $\mathbb{C}^+$ in particular. Provided a closed set of equations for $\mathfrak{b}$, $\mathfrak{b}$, and $\mathfrak{c}$ in terms of $\mathfrak{B}$, $\bar{\mathfrak{B}}$, and $\mathfrak{C}$ is found the eigenvalue $\Lambda$ of the QTM can be calculated. Employing (65), (66), and (67) we have

$$
\begin{align*}
\mathfrak{D}^\pm_{k>0} &= e^{-k} \tilde{q}_- + \tilde{c} + \mathfrak{B}, \\
\mathfrak{D}^\pm_{k<0} &= e^{k} \tilde{q}_+ - \tilde{c} + \mathfrak{B} = 0,
\end{align*}
$$

where we used the vanishing of the Fourier transform $\widehat{F}(k)$ for $k < 0$ ($k > 0$) for $F(v)$ analytic in $\mathbb{C}^+$ ($\mathbb{C}^-$). Similar relations can be obtained for $\mathfrak{D}$ reading

$$
\begin{align*}
\widehat{\mathfrak{D}}^\pm_{k>0} &= e^{-k} \widehat{\Phi}_+ - e^{-k} \widehat{q}_- - \mathfrak{b} = 0, \\
\widehat{\mathfrak{D}}^\pm_{k<0} &= e^{k} \widehat{q}_+ + \widehat{\Phi}_+ - \widehat{\Phi}_- - \mathfrak{b},
\end{align*}
$$

According to (68) we have

$$
\begin{align*}
\widehat{\mathfrak{D}}^\pm_{k>0} + \mathfrak{b} &= e^{-k} \widehat{\Phi}_+ + \mathfrak{c} - \widehat{\mathfrak{B}}, \\
\widehat{\mathfrak{D}}^\pm_{k<0} + \mathfrak{b} &= e^{k} \widehat{\Phi}_- + \mathfrak{c} - \widehat{\mathfrak{B}}.
\end{align*}
$$

Consider the case $k > 0$. With respect to $\widehat{\mathfrak{D}}^-$ the equations (70), (71) and (72) provide

$$
e^{-k} \widehat{q}_- + \mathfrak{b} = e^{-k} \widehat{\Phi}_+ - \widehat{\mathfrak{B}}, \\
\widehat{\mathfrak{c}} = \widehat{\mathfrak{b}} - \widehat{\mathfrak{B}},
$$

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After inserting the second relation (71) for $\tilde{S}^{(+)}$ it follows

$$
\hat{b} = e^{-k}(\hat{\Phi}_+ - \hat{\Phi}_-) - e^{-k}(\hat{B} + \hat{C}),
$$
$$
\hat{c} = e^{-k}(\hat{\Phi}_+ - \hat{\Phi}_-) - e^{-k}(\hat{B} + \hat{C}) - \hat{B}.
$$

By similar steps we obtain the full system of equations

$$
\hat{b} = \begin{cases} 
(\hat{\Phi}_+ - \hat{\Phi}_-) - (\hat{B} + \hat{C}) & : k > 0, \\
(e^k(\hat{\Phi}_+ - \hat{\Phi}_+) - e^k(\hat{B} + \hat{C}) & : k < 0,
\end{cases}
$$

$$
\hat{c} = \begin{cases} 
(e^{-k}(\hat{\Phi}_+ - \hat{\Phi}_-) - e^{-k}(\hat{B} + \hat{C})) & : k > 0, \\
(e^k(\hat{\Phi}_+ - \hat{\Phi}_+) - e^k(\hat{B} + \hat{C}) - \hat{B}) & : k < 0.
\end{cases}
$$

Furthermore, we obtain: $(\hat{\Phi}_+ - \hat{\Phi}_-) = iN \text{ sign}(k) \sinh(ku)$. Applying the inverse Fourier transform we are led to a system of non-linear integral equations (denoted by the convolution $*$)

$$
\log b = \Phi_b^{(N)} - \Psi_b * \log \hat{B} - \Psi_b * \log \hat{C} + \beta(\mu + h/2),
$$
$$
\log b = \Phi_b^{(N)} - \Psi_b * \log \hat{B} - \Psi_b * \log \hat{C} + \beta(\mu - h/2),
$$
$$
\log c = \Phi_c^{(N)} - \Psi_b * \log \hat{B} - \Psi_b * \log \hat{C} - \Psi_c * \log \hat{C} + 2\beta \mu,
$$

with

$$
\Phi_b^{(N)} = i N \left( \text{arctan} \frac{v - i}{u} - \text{arctan} \frac{v}{u} \right), \quad \Psi_b = \frac{1}{2\pi v(v - i)},
$$
$$
\Phi_b^{(N)} = -i N \left( \text{arctan} \frac{v + i}{u} - \text{arctan} \frac{v}{u} \right), \quad \Psi_b = \frac{1}{2\pi v(v + i)},
$$
$$
\Phi_c^{(N)} = i N \left( \text{arctan} \frac{v - i}{u} - \text{arctan} \frac{v + i}{u} \right), \quad \Psi_c = \frac{2}{2\pi(v^2 + 1)}.
$$

The additional terms $\beta(\mu \pm h/2)$ and $2\beta \mu$ in (73) are integration constants which follow from considering the limit $v \to \infty$

$$
b \to \frac{e^{\beta(\mu + h/2)}}{e^{\beta(\mu - h/2)} + 1}, \quad \bar{b} \to \frac{e^{\beta(\mu - h/2)}}{e^{\beta(\mu + h/2)} + 1}, \quad c \to \frac{e^{2\beta \mu}}{e^{\beta(\mu + h/2)} + e^{\beta(\mu - h/2)} + 1}.
$$

We observe that the Trotter-Suzuki number $N$ enters only as a simple parameter. In the limit $N \to \infty$ we have

$$
\Phi_b^{(N \to \infty)} = -2\pi \beta \Psi_b \bar{b} \quad \text{and} \quad \Phi_c^{(N \to \infty)} = -2\pi \beta \Psi_c.
$$
It turns out that the integration contour for $\Psi \ast \log \mathcal{C}$ can be moved to the real axis. Hence, the equation for $b, \tilde{b}$ is taken on the line $v = x \pm i\gamma$ and $c$ on the real axis $v = x$ leading to the set of equations (24) in section 4.

Lastly, we want to comment on the treatment of the large $\mu$ limit. In this case the Fourier coefficients $\hat{c}$ and $\hat{C}$ coincide up to exponentially small corrections. Hence the last equation of (74) can be solved in terms of $\hat{B}$ and $\tilde{\hat{B}}$

\[
\hat{b} = \text{sign}(k) \frac{1}{1 + e^{-k}} (\hat{\Phi}_+ - \hat{\Phi}_-) + \frac{1}{1 + e^{|k|}} (\hat{\mathcal{B}} - \tilde{\hat{\mathcal{B}}}),
\]

\[
\tilde{\hat{b}} = \text{sign}(k) \frac{1}{1 + e^{k}} (\hat{\Phi}_+ - \hat{\Phi}_-) - \frac{1}{1 + e^{|k|}} (\hat{\mathcal{B}} - \tilde{\hat{\mathcal{B}}}),
\]

\[
\hat{c} = \hat{C} = \text{sign}(k) \frac{1}{1 + e^{|k|}} (\hat{\Phi}_+ - \hat{\Phi}_-) - \frac{1}{1 + e^{k}} (\hat{\mathcal{B}} + e^k\tilde{\hat{\mathcal{B}}}).
\]

Applying the inverse Fourier transform and respecting the $v \to \infty$ limit

\[
b \to e^{\beta h}, \quad \tilde{b} \to e^{-\beta h}, \quad c \to \frac{e^{\beta \mu}}{e^{\beta h/2} + e^{-\beta h/2}},
\]

we arrive at the integral equations (36).
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