Hawking-Page phase transition in BTZ black hole revisited

Myungseok Eune,\textsuperscript{a} Wontae Kim,\textsuperscript{a,b,c} and Sang-Heon Yi\textsuperscript{b}

\textsuperscript{a}Research Institute for Basic Science, Sogang University, Seoul, 121-742, Republic of Korea
\textsuperscript{b}Center for Quantum Spacetime, Sogang University, Seoul 121-742, Republic of Korea
\textsuperscript{c}Department of Physics, Sogang University, Seoul 121-742, Republic of Korea

E-mail: younms@sogang.ac.kr, wtkim@sogang.ac.kr, shyi@sogang.ac.kr

ABSTRACT: We consider the Hawking-Page phase transition between the BTZ black hole of $M \geq 0$ and the thermal soliton of $M = -1$. In this system, there exists a mass gap so that there does not seem to exist a continuous thermodynamic phase transition. We consistently construct the off-shell free energies of the black hole and the soliton by properly taking into account the conical space. And then, the continuous off-shell free energy to describe tunneling effect can be realized through non-equilibrium solitons.

KEYWORDS: Black Hole, Thermodynamics
1 Introduction

Since it has been claimed that a black hole has an entropy [1], thermodynamics of black holes has been one of the most important arena in black hole physics [2–4]. And then, there have been extensive studies for thermodynamics and phase transitions in various black holes [5–21]. For instance, in Einstein gravity, the hot flat space in the cavity is more probable than the large black hole below a critical temperature while the large black hole is more probable than the hot flat space above the critical temperature [4]. Such a phase transition can be read off from the on-shell free energy and the heat capacity.

In particular, for the Bañados-Teitelboim-Zanelli (BTZ) black hole system [6], there are two distinct solutions of the BTZ black hole of $M \geq 0$ and the thermal soliton of the global AdS$_3$ whose mass is $M = -1$ [22–24]. The Hawking-Page (HP) phase transition has been intensively studied by using the on-shell and off-shell free energies as well as conical singularity considerations in Refs. [25–33]. Recently, the Euclidean path integral has been calculated by taking into account the conical ensemble [33], where the metrics are not differential but continuous at the horizon because of the Riemannian geometry with the conical singularity [34].

Generically, it is possible to elaborate thermodynamic evolutions by the use of the off-shell free energy which plays a role of the potential at the given temperature of heat reservoir [3, 4]. The black hole below the critical temperature decays to the thermal vacuum while the thermal vacuum tends to tunnel to the black hole above the critical temperature. The tunneling effect in the phase transition is crucial since the black hole state can not avoid the barrier of the off-shell free energy in order to arrive at the vacuum state and vice versa. The extrema of the off-shell free energy are equilibrium states, and the others correspond to non-equilibrium states. All these states are connected with each other so that the off-shell free energy should be continuous everywhere. However, in the BTZ black hole, the mass spectrum is discontinuous so that at first glance the continuous evolution seems to be impossible in the presence of the mass gap.

In this work, we would like to reconsider the HP phase transition between the BTZ black hole and the thermal soliton by using the off-shell free energy. In fact, it would be
interesting to answer how to interpolate the states between the BTZ black hole \((M \geq 0)\) and the thermal soliton \((M = -1)\) in order to elaborate the tunneling effect which is essential in the HP phase transition. For this purpose, we will construct the off-shell free energy so that the black hole state and the soliton state can be interpolated continuously.

In section 2, we shall recapitulate the on-shell free energy which is just the free energy without considering conical singularities. In other words, one can define the free energy from the Euclidean path-integral, and it turns out to be the on-shell if the conical singularity contributions are neglected. As expected, the on-shell free energy describes equilibrium states of the black hole and the soliton. Next, in section 3, in order to construct the off-shell free energy, we will calculate the free energy of the black hole by taking into account the conical singularity at the event horizon. Then, it can be shown that it is the same with the off-shell free energy constructed from the conventional definition of \(F_{bh}^{\text{off}} = E - TS\), where \(E\) and \(S\) are the thermodynamic energy and the entropy, and \(T\) is an arbitrary temperature. Thus, the free energy reflecting the conical singularity can be identified with the off-shell free energy. Of course, the on-shell points can be obtained from the extrema of the off-shell free energy, where it amounts to removing the conical singularity of the metric. Then, we will apply this notion to the side of the soliton in such a way that we can obtain the off-shell states by taking into account the conical singularity arising from the soliton. After all, the continuous off-shell free energy to describe tunneling effect can be realized through non-equilibrium solitons. Finally, summary and discussion are given in section 4.

2 On-shell free energy

Let us begin with the Einstein-Hilbert action with a negative cosmological constant in the three-dimensional spacetime \(\mathcal{M}\), which is given by

\[
I_M = \frac{1}{16\pi G} \int_{\mathcal{M}} d^3x \sqrt{-g} (R - 2\Lambda),
\]  

(2.1)

where \(G\) is the gravitational constant, and the cosmological constant \(\Lambda\) is related to the radius \(\ell\) by \(\Lambda \equiv -1/\ell^2\). The Gibbons-Hawking term and the counterterm on the boundary \(\partial\mathcal{M}\) are written as [6, 35, 36]

\[
I_{\text{GH}} = -\frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^2x \sqrt{|\gamma|} K, \\
I_{\text{ct}} = -\frac{1}{8\pi G \ell} \int_{\partial\mathcal{M}} d^2x \sqrt{|\gamma|},
\]  

(2.2)

(2.3)

where \(\gamma_{ij}\) and \(K\) are the induced metric and the extrinsic curvature scalar on the boundary \(\partial\mathcal{M}\), respectively. Then, the total action is given by

\[
I = I_M + I_{\text{GH}} + I_{\text{ct}}.
\]  

(2.4)

Varying the action (2.4), we obtain the equations of motion as

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{1}{\ell^2} g_{\mu\nu} = 0.
\]  

(2.5)
The nonrotating BTZ solution to satisfy Eq. (2.5) is written as [6]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\phi^2,$$  \hspace{1cm} (2.6)

where \(f(r) = -M + \frac{r^2}{\ell^2}\). The line element (2.6) describes the geometry of the BTZ black hole, and the event horizon is well defined at \(r_H = \ell\sqrt{M}\). The integration constant \(M\) is the ADM mass which is assumed to be \(M > 0\). Moreover, the Hawking temperature and the Bekenstein-Hawking entropy of the BTZ black hole are given by

$$T = \frac{\kappa}{2\pi} = \frac{\sqrt{M}}{2\pi\ell},$$ \hspace{1cm} (2.7)

$$S_{BH} = \frac{A_H}{4G} = \frac{\pi\ell\sqrt{M}}{2G},$$ \hspace{1cm} (2.8)

where \(\kappa\) is the surface gravity and \(A_H = 2\pi r_H^2 = 2\pi\ell\sqrt{M}\). Note that the geometry governed by Eq. (2.6) becomes the AdS soliton which is just the global AdS\(_3\) spacetime if \(M = -1\) [6, 22, 37, 38].

In order to study thermodynamics of the system, we need to consider the Euclidean geometry, and thus from now on all quantities will be given in the Euclidean ones. The line element (2.6) becomes

$$ds^2_E = f(r)d\tau^2 + \frac{dr^2}{f(r)} + r^2d\phi^2,$$ \hspace{1cm} (2.9)

using \(\tau = it\). Then, the action (2.1), the Gibbons-Hawking boundary term (2.2), and the counterterm (2.3) become

$$I_g = -\frac{1}{16\pi G} \int_{\partial M} d^3x \sqrt{g} (R - 2\Lambda),$$ \hspace{1cm} (2.10)

$$I_{GH} = \frac{1}{8\pi G} \int_{\partial M} d^2x \sqrt{\gamma} K,$$ \hspace{1cm} (2.11)

$$I_{ct} = \frac{1}{8\pi G\ell} \int_{\partial M} d^2x \sqrt{\gamma},$$ \hspace{1cm} (2.12)

where the boundary \(\partial M\) is the hyperspace given by the constant \(r\). For the first case of the BTZ black hole with \(M > 0\), the actions are calculated as

$$I_g = \frac{\beta}{4G} \left( \frac{r^2}{\ell^2} - \frac{r_H^2}{\ell^2} \right),$$ \hspace{1cm} (2.13)

$$I_{GH} = \frac{\beta}{4G} \left( M - 2\frac{r^2}{\ell^2} \right),$$ \hspace{1cm} (2.14)

$$I_{ct} = \frac{\beta r}{4G\ell} \sqrt{f},$$ \hspace{1cm} (2.15)

respectively. The total Euclidean action for the BTZ black hole becomes

$$I_{bh} = I_g + I_{GH} + I_{ct}$$

$$= \frac{\beta r}{4G\ell} \left( \sqrt{f} - \frac{r}{\ell} \right)$$

$$= -\beta \left[ \frac{M}{8G} + O \left( \frac{1}{r^2} \right) \right].$$ \hspace{1cm} (2.16)
We now take the period of Euclidean time $\beta$ as the inverse of the Hawking temperature $\beta(2.7)$, then $I_{\text{bh}} = I_{\text{on}}^{\text{bh}}$. At the infinite boundary of $r \to \infty$, the thermodynamic energy, the entropy, and the free energy are obtained as

$$E = \frac{\partial I_{\text{on}}^{\text{bh}}}{\partial \beta} = \frac{M}{8G}, \quad (2.17)$$

$$S = \beta E - I_{\text{on}}^{\text{bh}} = \frac{\pi \ell \sqrt{M}}{2G}, \quad (2.18)$$

$$F_{\text{bh}}^{\text{on}} = \beta^{-1} I_{\text{on}}^{\text{bh}} = -\frac{M}{8G}, \quad (2.19)$$

where $A_H = 2\pi \ell \sqrt{M}$ is the area of the horizon. Note that the entropy (2.18) agrees with the Bekenstein-Hawking entropy (2.8). The heat capacity is given by

$$C = \frac{\partial E}{\partial T} = \frac{\pi \ell \sqrt{M}}{2G}, \quad (2.20)$$

where the BTZ black hole is stable since the heat capacity is always positive. Note that we have considered thermodynamic quantities obtained form the Euclidean metric without conical singularities.

Next, for the soliton of $M = -1$, the actions are calculated as

$$I_g = \frac{\beta}{4G} \frac{r^2}{\ell^2}, \quad (2.21)$$

$$I_{GH} = \frac{\beta}{4G} \left(-1 - \frac{2r^2}{\ell^2}\right), \quad (2.22)$$

$$I_{ct} = \frac{\beta}{4G \ell} \sqrt{1 + \frac{r^2}{\ell^2}}, \quad (2.23)$$

where the period of the Euclidean time has been chosen as $\beta$, so that the total action of the AdS soliton is obtained as

$$I_{\text{sol}}^{\text{on}} = I_g + I_{GH} + I_{ct} = -\frac{\beta}{8G} \left[1 + O\left(\frac{1}{r^2}\right)\right]. \quad (2.24)$$

With the infinite boundary, the free energy of the thermal soliton is given by

$$F_{\text{sol}}^{\text{on}} = \beta^{-1} I_{\text{sol}}^{\text{on}} = -\frac{1}{8G}. \quad (2.25)$$

Actually, the temperature $T = \beta^{-1}$ is the temperature of the heat reservoir, and the soliton has the same temperature with that of the reservoir.

The free energies with respect to the temperature are shown in Fig. 1, which shows that the phase transition occurs at $T_c = 1/(2\pi \ell)$. As is well-known, the thermal soliton is more probable than the black hole below the critical temperature while the black hole is more probable than the thermal soliton above the critical temperature.
Figure 1. The solid and the dashed lines show the on-shell free energies of the BTZ black hole and the thermal soliton, respectively.

Figure 2. The solid lines show the off-shell free energies of the BTZ black hole depending on given temperatures, and the thick dot at $M = -1$ is the on-shell free energy of the thermal soliton. There are some missing off-shell states for $M < 0$.

3 Off-shell free energy

Let us first consider the space $\hat{\mathcal{M}}$ with a conical singularity, and denote the singular set by $\Sigma$ and the conical deficit angle by $\Delta_\Sigma$. Then, the Ricci tensor and the curvature scalar in the conical space are given by \[34\]
\[
\hat{R}_{\mu\nu} = R_{\mu\nu} + (n_{\mu}n_{\nu})\Delta_\Sigma \delta_\Sigma, \quad (3.1)
\]
\[
\hat{R} = R + 2\Delta_\Sigma \delta_\Sigma, \quad (3.2)
\]
where $\delta_\Sigma$ is the delta function defined by $\int_{\hat{\mathcal{M}}} f \delta_\Sigma = \int_\Sigma f$ for any function $f$ and $n^k = n^k_\mu dx^\mu$ ($k = 1, 2$) are two unit vectors orthogonal to $\Sigma$, $(n_{\mu}, n_{\nu}) = \delta_{ij}n_{\mu}^i n_{\nu}^j$. And $R_{\mu\nu}$ and $R$ are the Ricci tensor and the curvature scalar calculated in the regular points $\mathcal{M}/\Sigma$ by the standard method, respectively. From Eq. (3.2), the integration of the curvature scalar $\hat{R}$
for the conical space is related to the curvature scalar $R$ at the ordinary space as follows

$$\int_{\mathcal{M}} \sqrt{g} \hat{R} = \int_{\mathcal{M}/\Sigma} \sqrt{g} R + 2 \Delta_{\Sigma} A_{\Sigma},$$  \(3.3\)

where $A_{\Sigma} = \int_{\Sigma}$ is the area of the space $\Sigma$ for the fixed cone.

For the case of the BTZ black hole of $M > 0$, the conical singularity appears from the Euclidean time at the event horizon. For the arbitrary period of $\beta$, the conical deficit angle and the area of $\Sigma$ are calculated as

$$\Delta_{\Sigma} = 2\pi - \beta r_H/\ell^2 = 2\pi - \beta \sqrt{M}/\ell, \quad (3.4)$$

$$A_{\Sigma} = 2\pi r_H = 2\pi \ell \sqrt{M}. \quad (3.5)$$

Then, the total action (2.4) implemented by Eq. (3.3) is written as

$$I_{\text{off}} = I_{\text{bh}} + I_{\text{sing}}$$

$$= \frac{\beta M}{8G} - \frac{\pi \ell \sqrt{M}}{2G}, \quad (3.6)$$

where $I_{\text{sing}} = \beta M/4G - \pi \ell \sqrt{M}/(2G)$ from the second term in Eq. (3.3). Thus, we can define the off-shell action of the black hole which consists of the regular contribution from Eq. (2.16) and the singular contribution from Eq. (3.3). Then it is natural to obtain the off-shell free energy as

$$F_{\text{off}} = \beta^{-1} I_{\text{off}} = \frac{M}{8G} - \frac{\pi \ell \sqrt{M}}{2G \beta}. \quad (3.7)$$

In this section, we define $\beta$ as the arbitrary temperature of the heat reservoir, and $\beta_H$ as the inverse of the Hawking temperature presented in Eq. (2.7) for convenience. Note that the off-shell free energy (3.7) is reduced to the on-shell free energy (2.19), $F_{\text{off}}^{\text{on}}|_{\beta=\beta_H} = -M/8G$ by identifying $\beta = \beta_H$. This condition amounts to taking extrema of the off-shell free energy since the extrema can be guaranteed by the thermodynamic first law which satisfies $\beta = \beta_H$.

However, for the case of $\beta \neq \beta_H$, the free energy is still in the off-shell, which means that the thermodynamic system is in non-equilibrium. It is interesting to note that the off-shell free energy (3.7) can be written in the form of the well-known Helmholtz free energy,

$$F_{\text{bh}} = E - TS, \quad (3.8)$$

once we identify the on-shell quantities of the energy $E$ and the entropy $S$ with Eqs. (2.17) and (2.18), respectively while the temperature $T$ is treated as a variable.

In connection with the conical singularity, we would like to mention compatibility of off-shell structures with the Einstein equation (2.5) in Euclidean gravity. For this purpose, let us first consider Eq. (2.5) in the absence of any conical singularities, then it gives on-shell solutions corresponding to classical trajectories. However, if we allow the conical singularity in space, then the equation of motion should be replaced by

$$\hat{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \hat{R} - \frac{1}{\ell^2} g_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{1}{\ell^2} g_{\mu\nu} + [(n_{\mu} n_{\nu}) - g_{\mu\nu}] \Delta_{\Sigma} \delta_{\Sigma}, \quad (3.9)$$
since the Ricci tensor (3.1) and the curvature scalar (3.2) for the conical space consist of the contributions from the ordinary space and the conical defects. So, any conical singularity deforms the Einstein equation and it is responsible for the off-shell solutions. The off-shell free energy (3.7) also follows a similar content in thermodynamics. Thus, the off-shell solutions are not the solutions of the Einstein equation, and they will be used to interpolate the two on-shell states of the black hole and the soliton.

In Fig. 2, one can see that the end point \( M = 0 \) of the free energy of the black hole is not connected with the free energy of the soliton which is the thermal vacuum. As is shown in section II, the BTZ black hole is more probable below the critical temperature while the thermal soliton is more probable above the critical temperature. To understand this Hawking-Page phase transition in terms of the tunneling process, we have to obtain the continuous off-shell free energy. We expect there exist some missing off-shell states in the region of the negative \( M \). So, we should consider additional non-equilibrium states which can be characterized by the conical space by allowing an arbitrary negative value of \( M \).

For \( M < 0 \), we now consider the space described by the metric (2.9) including a conical singularity at \( r = 0 \), which appears at the cone defined by the coordinate \( r \) and \( \phi \). The deficit angle and the area of \( \Sigma \) are calculated as

\[
\Delta \Sigma = 2\pi (1 - \sqrt{-M}),
\]

\[
A_{\Sigma} = \int d\tau \sqrt{g_{\tau\tau}} = \beta \sqrt{-M},
\]

when the period of the Euclidean time is \( \beta \). Then, Eqs. (2.10), (2.11), and (2.12) are calculated as

\[
I_g = \frac{\beta}{4G} \left( \frac{r^2}{\ell^2} - M - \sqrt{-M} \right),
\]

\[
I_{GH} = \frac{\beta}{4G} \left( M - 2\frac{r^2}{\ell^2} \right),
\]

\[
I_{ct} = \frac{\beta}{4G} \frac{r}{\ell} \sqrt{f(r)},
\]

by taking into account the conical space, which yields

\[
F_{\text{off sol}}^\text{eff} = \frac{\beta}{4G} \left( \frac{r}{\ell} \sqrt{f(r)} - \frac{r^2}{\ell^2} - \sqrt{-M} \right) = \frac{\beta}{8G} \left[ -M - 2\sqrt{-M} + O \left( \frac{1}{r^2} \right) \right].
\]

Taking the infinite boundary, the off-shell free energy for the soliton can be obtained as

\[
F_{\text{sol}}^\text{off} = \frac{1}{8G} \left( -M - 2\sqrt{-M} \right).
\]

Note that it recovers the on-shell free energy (2.25) for \( M = -1 \). It is independent of the temperature in contrast to the case of the black hole (3.7). In fact, the black hole
Figure 3. The three solid lines for $M > 0$ show the off-shell free energies of the BTZ black hole depending on temperature of the heat reservoir, and the single solid line for $M < 0$ shows the off-shell free energy of the soliton. The four minima consist of the three black hole states and one soliton state in each equilibrium. Note that $F_{\text{off}}(0) = 0$, which is finite.

The behavior of the off-shell free energy of the BTZ black hole and the off-shell free energy of the soliton is shown in Fig. 3. The off-shell free energy (3.16) is connected with the off-shell free energy of the BTZ black hole continuously at $M = 0$. Note that $F_{\text{off}}$ is zero at $M = 0$. One can easily see that the dot at $M = -1$ is just the thermal soliton of the global AdS$_3$ which is clearly stable. Furthermore, from the off-shell structure, it can be shown that the BTZ black hole decays into the soliton below the critical temperature through the quantum tunneling whereas the soliton can tunnel to the black hole above the critical temperature. The tunneling probability for this to occur will be of the form [3]

$$\Gamma = Ae^{-B},$$

(3.17)

where $A$ is some determinant and $B$ is the absolute value of each action of the BTZ black hole for $T < T_c$ or the soliton for $T > T_c$.

4 Discussion

We have studied the phase transition of the BTZ black hole in terms of the off-shell free energy incorporated with the conical singularity contributions. In particular, thanks to the non-trivial off-shell structure of the soliton, the isolated soliton state can be continuously connected with the black hole state, so that the phase transition which is associated with the tunneling effect can be easily seen.

We have identified the off-shell free energy with a free energy in the presence of the conical singularity whereas the on-shell free energy is a free energy in the absence of the conical singularity. In other words, the criteria whether the free energy (or the action) is on-shell or off-shell relies on the conical singularity contributions. The reason for this is that if one defines an arbitrary temperature, then it can cause conical singularities in
the metric, so that they affect the action through the modification of the curvature scalar.
Actually, the nonrotating BTZ solution does not satisfy the modified equation of motion given by Eq. (3.9) when the conical singularity in space is allowed. Then, the system is in off-shell which is corresponding to non-equilibrium state thermodynamically. Eventually, the off-shell free energy (3.7) evaluated from the off-shell action by taking into account the conical singularity turns out to be compatible with the off-shell free energy (3.8) derived from the conventional definition of the off-shell free energy. Of course, setting $\beta = \beta_H$, the conical singularity can be removed so that the system is in equilibrium. As for the soliton case of $M < 0$ which is a little bit different from the case of the black hole in that there is no intrinsic temperature of the soliton, the temperature of the soliton can be always the same with that of the heat reservoir. In fact, there are two states at any temperatures, one is the soliton in equilibrium satisfying the on-shell condition at $M = -1$, and the others are in non-equilibrium. In this respect, to determine universally whether the system is in equilibrium or in non-equilibrium, it is more plausible to justify the states in terms of the conical singularity contributions to the free energy instead of the temperature argument in our model.

Finally, we discuss two aspects on the cusp of the off-shell free energy at $M = 0$ which is not differentiable although it is continuous. In particular, $F_{\text{off}}(0) = 0$ which is not singular but finite as seen from Eqs. (3.7) and (3.16) along with Fig. 3. First, the cusp is a local maximum and it seems to be a locally unstable state. To clarify this point, let us regard the off-shell free energy of $F_{\text{off}} = E - TS$ as an effective potential in finite temperature field theory as $Z = e^{-\beta F_{\text{off}}} = e^{-\beta V_{\text{eff}}}$. As is well-known, the extrema of the effective potential are on-shell states which are physical, while the other states are virtual. Similarly, the extrema from the off-shell free energy are regarded as equilibrium states thermodynamically, whereas the others correspond to non-equilibrium states because the only on-shell states satisfying $(dF_{\text{off}}/dM)_T = 0$ give the thermodynamic first law of $0 = dF_{\text{off}} = dE - TdS$ where $T$ is the given temperature of the heat reservoir. So, the on-shell states are physical from viewpoint of the effective potential and they are also in equilibrium from thermodynamical point of view. Now one can assume that some quantum corrections may make the sharp free energy smooth, which yields a smooth extremum. Eventually, we can regard this as a massless black hole state mentioned in Ref. [6]. But it is unstable as seen from the profile of the off-shell free energy in Fig. 3. Note that we can not justify the stability of the massless black hole in terms of the heat capacity (2.20) since it vanishes at $M = 0$. So there are three on-shell states: stable black hole, stable soliton, unstable massless black hole. From the off-shell structure, we can see the massless black hole state corresponding to the cusp is unstable. Secondly, as for phase transitions, the free energy is not singular but finite at $M = 0$ since $F_{\text{off}}(0) = 0$, so that tunneling is possible obeying the finite tunneling probability $\Gamma = Ae^{-\beta F}$ [3] where $F$ is the larger free energy among stable states. For instance, for the case of $T < T_c$, the BTZ black hole undergoes a quantum tunneling and then decays thermodynamically into the stable soliton whereas the soliton can tunnel to the black hole for $T > T_c$. This fact can be also understood from the analogous role of the effective potential in finite temperature field theory. By the way, one might ask whether any instanton solution connecting two states is possible or not in connection with the tunneling...
effect. We can expect that it might be a BTZ type solution which asymptotically goes
to the BTZ solution at two end states. Unfortunately, we do not know what the explicit
solution is, which is beyond the scope the present work. We hope this interesting issue
would be addressed elsewhere.

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