THE GROUP OF CONTACTOMORPHISMS OF THE SPHERE
FIXING AN OVERTWISTED DISK

Katarzyna Dymara*
Institute of Mathematics
Wrocław University

April 2005

Abstract
We calculate the weak homotopy type of the group of contactomorphisms
of the three-sphere which coincide with the identity on (a neighborhood of) an
overtwisted disk.

1 Preliminaries. Known results

A contact structure on a 3–dimensional manifold is a field of planes defined (at least locally)
as the kernel of a 1–form $\alpha$ such that $\alpha \wedge d\alpha$ nowhere vanishes.

We say that two contact structures $\zeta_0$ and $\zeta_1$ are isotopic if there exist a smooth family
of contact structures $\{\zeta_t, t \in [0,1]\}$.

We say that $\zeta_0$ and $\zeta_1$ are contactomorphic if there exists a diffeomorphism $f : M \to M$
such that $f_*(\zeta_0) = \zeta_1$; equivalently, if $\zeta_0 = \ker \alpha_0$ and $\zeta_1 = \ker \alpha_1$, then $(f^{-1})^*(\alpha_1) = t\alpha_0$
for some non-zero function $t : M \to \mathbb{R}$. Such $f$ is referred to as a contactomorphism.

Gray proved in [3] that on a closed manifold two contact structures are contactomorphic if and only if they are isotopic. His proof (which uses a vector field whose flow consists of the desired diffeomorphisms, constructed locally and glued together by means of a partition of unity) can be applied—without essential changes—to a relative situation, either in the sense of considering contact structures on a manifold modulo a compact set, or fixing the contact structure along a (contractible) subset of the parameter space. Here we will simply state Gray’s theorem in necessary generality.

Theorem 1.1 (Gray’s Theorem). Let $\{\zeta_t, t \in D^n\}$ be a smooth family of contact
structures on a closed manifold $M$. Assume that $\zeta_t|_A = \zeta_0|_A$ for a compact set $A \subset M$
and for all $t \in D^n$. Moreover, let $\zeta_t = \zeta_{t_0}$ for all $t \in D'$, where $D'$ is a contractible subset
of $D^n$. Then there exists a family $\{\phi_t, t \in D^n\}$ of diffeomorphisms $\phi_t : M \to M$ such that
for all $t$, $\phi_t \zeta_t = \zeta_{t_0}$, for all $t \in D^n$ $\phi_t|_A = \text{Id}_A$ and for all $t \in D'$ $\phi_t = \text{Id}_M$.

A contact structure is called overtwisted if it contains a 2-dimensional disk which
is tangent to the contact structure along boundary. For example, consider the contact

* Partially supported by KBN grant 2 P03A 017 25.
structure $\xi$ on $\mathbb{R}^3$ defined as the kernel of the 1-form (written in cylindrical coordinates $r, \theta, z$)
\[
\cos r \, dz - \varrho(r) \sin r \, d\theta,
\]
where $\varrho : \mathbb{R}_+ \to \mathbb{R}_+$ is a smooth function such that $\varrho(0) = 0$, $\varrho'(0) > 0$, $\varrho'(r) \geq 0$ for all $r \in \mathbb{R}$ (we introduce the function $\varrho$ simply to make sure that the form is well-defined and smooth at $r = 0$).

The disk $\Delta = \{(r, \theta, z) : r \leq \pi, z = 0\}$ is indeed tangent to the contact structure along boundary. Actually, every overtwisted contact structure contains a contactomorphic copy of (a neighborhood of) $\Delta$; this copy is referred to as an overtwisted disk.

A contact structure which is not overtwisted is called tight.

**Theorem 1.2 (Eliashberg’s Theorem).** For $\Delta$ a 2-dimensional disk in an arbitrary 3-manifold $M$, let $\xi_\Delta$ be a contact structure on a neighborhood of $\Delta$ for which $\Delta$ is an overtwisted disk. Denote by $\text{Cont}(M \text{ rel } \Delta)$ the space of contact structures on $M$ which coincide with $\xi_\Delta$ on a neighborhood of $\Delta$. Moreover, let $\text{Distr}(M \text{ rel } \Delta)$ be the space of all plane distributions on $M$ which coincide with $\xi_\Delta$ on a neighborhood of $\Delta$. Then the natural embedding $\text{Cont}(M \text{ rel } \Delta) \to \text{Distr}(M \text{ rel } \Delta)$ is a weak homotopy equivalence.

On $\pi_0$ level this means that isotopy classes of contact structures overtwisted along a fixed disk (which, since embeddings of a disk into a connected manifold are all isotopic, actually exhaust all isotopy classes of overtwisted contact structures) remain in a one-to-one correspondence with homotopy classes of plane fields.

To understand better this space, let us fix a parallelization of the manifold $M$, i.e. a triple of vector fields $(u, v, w)$ such that $(u(x), v(x)w(x))$ forms a basis of $T_xM$. Using this parallelization, we can identify a co-oriented plane field $\zeta$ with its normal Gauss map $G_\zeta : M \to S^2$ as follows: let $n_\zeta(x)$ be the unit normal vector to $\zeta$ at $x$, then $G_\zeta(x) = (n_1, n_2, n_3)$ for $n_\zeta(x) = n_1 u(x) + n_2 v(x) + n_3 w(x)$. For $S^3$, which is our main object of interest, the space of homotopy classes of co-oriented plane fields (and hence also the space of isotopy classes of overtwisted contact structures) is parameterized by $\pi_0(\text{Map}(S^3 \to S^2)) = \pi_3 S^2 \simeq \mathbb{Z}$.

Moreover, we may consider a parallelized plane field $\zeta$, i.e. one equipped with its own trivialization as a bundle. This yields another Gauss map, $\tilde G_\xi : M \to SO(3)$, cf. [1], p. 301. Parallelization and co-orientation of $\zeta$ form together a trivialization of the whole tangent bundle for which both Gauss maps $G_\zeta$ and $\tilde G_\xi$ are constant.

### 2 The Group of Contactomorphisms

For a manifold $M$ and a compact subset $K \subset M$, let $\text{Diff}(M \text{ rel } K)$ denote the group of diffeomorphisms which become identity when restricted to $K$, i.e.
\[
\text{Diff}(M \text{ rel } K) = \{ f \in \text{Diff}(M) : f|_K = \text{Id}_K \}.
\]

1 Of course, all plane fields on $S^3$ (or any manifold with trivial $H_1$) are co-orientable; as a matter of convenience, we regard them as pre-equipped with one of the two possible co-orientations.
Let $\xi$ be a contact structure on the manifold $M$. Denote by $\text{Diff}_\xi(M \text{ rel } K)$ the group of contactomorphisms of $(M, \xi)$ coinciding with the identity on $K$:

$$\text{Diff}_\xi(M \text{ rel } K) = \{ f \in \text{Diff}(M \text{ rel } K) : f_*\xi = \xi \}. $$

In [1] we have proved that the group $\text{Diff}_\xi(S^3 \text{ rel } K)$ for $K$ being a small (closed) ball containing an overtwisted disk is not connected; in fact, that $\pi_0$ of this group has exactly two elements. In the present paper we refine our argument so as to obtain the complete knowledge of the weak homotopy type of $\text{Diff}_\xi(S^3 \text{ rel } \Delta)$.

**Theorem 2.1.** Let $\xi$ be an overtwisted contact structure on $S^3$, $\Delta$ an overtwisted disk for $\xi$, $K \supset \Delta$ a small closed ball containing the overtwisted disk. Then the group $\text{Diff}_\xi(S^3 \text{ rel } K)$ is weakly homotopy equivalent to $\Omega^4 S^2$.

In order to prove this theorem, we first introduce a number of definitions and lemmas.

Let $\widetilde{\text{Diff}}_\xi(M \text{ rel } K)$ be the space of paths in $\text{Diff}(M \text{ rel } K)$ beginning at identity and ending at a contactomorphism, i.e.

$$\widetilde{\text{Diff}}_\xi(M \text{ rel } K) = \{ \{ f_t, t \in [0, 1] \} : f_t \in \text{Diff}(M \text{ rel } K) \text{ for all } t, \text{ } f_0 = \text{Id}, f_1 \in \text{Diff}_\xi(M \text{ rel } K) \}. $$

Define the map $\tau : \widetilde{\text{Diff}}_\xi(M \text{ rel } K) \to \text{Diff}_\xi(M \text{ rel } K)$ as “taking the endpoint of a path”, i.e. $\tau(\{ f_t \}) = f_1$.

**Lemma 2.2.** For $K$ a three-ball in $S^3$ the map $\tau : \widetilde{\text{Diff}}_\xi(S^3 \text{ rel } K) \to \text{Diff}_\xi(S^3 \text{ rel } K)$ is a weak homotopy equivalence.

**Proof:** It suffices to check that the fiber $\tau^{-1}(f)$ is contractible for any point $f$ in the space $\text{Diff}_\xi(S^3 \text{ rel } K)$. The whole $\text{Diff}(S^3 \text{ rel } K)$ is contractible (Smale conjecture, [4]). Since $\tau^{-1}(f)$ consists of all paths in $\text{Diff}(S^3 \text{ rel } K)$ with fixed endpoints (joining the identity with $f$), it is contractible as well. \( \square \)

For a contact structure $\xi_K$ on a compact subset $K$ of a manifold $M$, denote by $\text{Cont}(M \text{ rel } K, \xi_K)$ the space of all contact structures on $M$ which coincide with $\xi_K$ on $K$. Let $\xi_*$ be the map $\text{Diff}(M \text{ rel } K) \to (\text{Cont}(M \text{ rel } K), \xi_K)$ defined as $\xi_*(f) = f_*(\xi)$.

Note that for $\{ f_t \} \in \widetilde{\text{Diff}}_\xi(M \text{ rel } K)$ the family $\{ \xi_*(f_t) \}$ is actually an element of the space $\Omega(\text{Cont}(M \text{ rel } K), \xi_K)$, because both $f_0$ and $f_1$ are contactomorphisms. Let us call thus induced map $\Omega \xi_* : \widetilde{\text{Diff}}_\xi(M \text{ rel } K) \to \Omega \text{Cont}(M \text{ rel } K)$.

The following lemma is a corollary of Gray’s theorem (Theorem 1.1).

**Lemma 2.3.** For any manifold $M$ with a contact structure $\xi$, and any compact subset $K$, the map $\Omega \xi_* : \widetilde{\text{Diff}}_\xi(M \text{ rel } K) \to \Omega \text{Cont}(M \text{ rel } K, \xi_K)$ is a weak homotopy equivalence.

**Proof:** We wish to show that the maps induced by $\Omega \xi_*$ on the homotopy groups are isomorphisms.
Consider a $k$-sphere mapped into $\Omega \text{Cont}$, i.e. a family of contact structures $\{\xi^z_t \mid z \in S^k, t \in [0, 1]\}$, such that

- $\xi^z_0 = \xi^z_1 = \text{Id}$ for all $z \in S^k$;
- for $N$ the north pole of $S^k$, $\xi^N_t = \text{Id}$ for all $t$ (this condition, as well as any analogous condition formulated below, means simply taking the sphere with a fixed base point, in compliance with the very definition of homotopy groups).

This family is parameterized by $S^k \times [0, 1]$, which is not contractible; therefore Gray’s theorem does not apply. Extend the family of contact structures to the $k + 1$-dimensional disk $D = S^k \times [0, 1] \cup D^{k+1} \times \{0\}$, setting $\xi^z_0 = \xi^z$ for all $z \in D^{k+1}$. Now the subset of $D$ where we require that the contact structure coincides with $\xi$ is the heavily shaded area of Figure 2.1 (including the thick lines).

![Figure 2.1: The parameter space for the extended family of contact structures.](image)

By Gray’s theorem (1.1) the extended family has a trivialization, i.e. a family of projections $f^z_t$ on a fixed fiber. The relative version of Gray’s theorem (1.1) allows to assume that the desired family of projections extends a certain given family of diffeomorphisms parameterized by a contractible subset of $D$; we choose the contractible subset consisting of the disk $D^{k+1} \times \{0\}$ and the line segment $N \times [0, 1]$, where the projections can be assumed to be the identity. Thus we get the family $\{f^z_t \mid (z, t) \in D\}$ such that $(f^z_t)_* \xi = \xi^z_t$ for all $z \in S^k$ and all $t$ and at the same time satisfying the following conditions:

- $f^z_0 = \text{Id}$ for all $z \in D^{k+1}$;
- $f^N_t = \text{Id}$ for all $t \in [0, 1]$.

Since $\xi^z_1 = \xi$ for all $z \in S^k$, the diffeomorphisms $f^z_t$ are contactomorphisms. Therefore for any $z \in S^k$ the family $F^z = \{f^z_t \mid t \in [0, 1]\}$ is an element of the group $\widetilde{\text{Diff}}_\xi(M \text{ rel } K)$, so that any sphere in $\Omega \text{Cont}(M \text{ rel } K, \xi)$ is indeed the image (under $\Omega \xi_*$) of a sphere in $\widetilde{\text{Diff}}(M \text{ rel } K, \xi)$. This proves the surjectivity of the homomorphisms of homotopy groups.

In order to prove their injectivity it is enough to show that if the image of a sphere is homotopically trivial in $\Omega \text{Cont}$ (i.e. the family of loops of contactomorphisms parameterized by a sphere can be extended to the disk bounded by the sphere) then the same is true for the sphere in $\widetilde{\text{Diff}}$ itself. It turns out that a construction analogous to that employed
in the first part of the proof allows to pull back to $\widetilde{\text{Diff}}$ the disk spanned by the sphere. Indeed, consider a family of diffeomorphisms $F = \{f^z_t \in \widetilde{\text{Diff}}_x(M \text{ rel } K) \mid z \in S^k, t \in [0, 1]\}$, such that

- $f^z_0 = \text{Id}$ for all $z \in S^k$;
- $f^z_1$ is a contactomorphism for all $z \in S^k$;
- for $N$ the north pole of $S^k$, $f^N_t = \text{Id}$ for all $t$.

Then $\Omega_k(F) = \{\xi^z_t \mid z \in S^k, t \in [0, 1]\}$ is a sphere in $\Omega \text{Cont}$, as described in the first part of the proof. In particular,

- $\xi^z_0 = \xi$ for all $z \in S^k$ (because $f^z_0 = \text{Id}$);
- $\xi^z_1 = \xi$ for all $z \in S^k$ (because $f^z_1$ is a contactomorphism);
- for $N$ the north pole of $S^k$, $\xi^N_t = \xi$ for all $t$.

The assumption that $\Omega_{\xi^*}(F)$ is homotopically trivial in $\Omega \text{Cont}$ means that it extends to a family of contact structures $\{\xi^z_t \mid z \in D^{k+1}, t \in [0, 1]\}$ such that $\xi^z_0 = \xi^z_1 = \xi$ for all $z \in D^{k+1}$. The subset $D = S^k \times [0, 1] \cup D^{k+1} \times \{0\} \subset D^{k+1} \times [0, 1]$ is contractible, thus again we can apply Gray’s theorem (1.1) and obtain a family of diffeomorphisms $\mathcal{F} = \{f^z_t \in \widetilde{\text{Diff}}_x(M \text{ rel } K) \mid z \in D^{k+1}, t \in [0, 1]\}$ such that

- $f^z_0 = f^z_1 = \text{Id}$ for all $z \in D^{k+1}$;
- $f^z_t = f^z_t$ for all $(z, t) \in D$.

Therefore $\mathcal{F}$ gives a $(k + 1)$-disk in $\widetilde{\text{Diff}}_x(M \text{ rel } K)$ spanned by the original sphere $F$.

This concludes the proof. \hfill $\square$

Now, fix a plane field $\zeta_K$ on a compact subset $K \subset M$. Let $\text{Distr}(M \text{ rel } K, \zeta_K)$ denote the space of all distributions on $M$ coinciding with $\zeta_K$ on $M$.

The following two lemmas are easy observations.

**Lemma 2.4.** For any co-oriented, parallelized distribution $\zeta$ there is a homeomorphism

$$\text{Distr}(M \text{ rel } K, \zeta|_K) \to \text{Map}((M, K) \to (S^2, N) : \xi \mapsto G_\xi),$$

where the map $G_\xi$ is defined using the parallelization of $M$ induced by that of $\zeta$.

**Lemma 2.5.** For $K$ a three-ball in $S^3$ the space $\text{Map}((S^3, K) \to (S^2, N))$ is homeomorphic to $\Omega^3 S^2$.

We conclude this section with the proof of theorem formulated at its beginning.

**Proof of Theorem 2.1:**

The subset $K = \overline{U(\Delta)}$ is a three-ball; therefore by Lemma 2.2 $\text{Diff}_x(S^3 \text{ rel } K)$ is homotopy equivalent to $\widetilde{\text{Diff}}_x(S^3 \text{ rel } K)$.

By Lemma 2.3 with $M = S^3$, the space $\widetilde{\text{Diff}}_x(S^3 \text{ rel } K)$ is homotopy equivalent to $\Omega(\text{Cont}(S^3 \text{ rel } K), \xi|_K)$.

By Eliashberg’s theorem with $M = S^3$, the space $\text{Cont}(S^3 \text{ rel } K, \xi|_K)$ is homotopy equivalent to $\text{Distr}(S^3 \text{ rel } K, \xi|_K)$. The homotopy equivalence induces a homotopy equivalence on the loop spaces, therefore also $\Omega \text{Cont}(S^3 \text{ rel } K, \xi|_K)$ is homotopy equivalent to $\Omega \text{Distr}(S^3 \text{ rel } K, \xi|_K)$. 

5
By Lemma 2.4 with \( M = S^3 \), \( Distr(S^3 \text{ rel } K, \zeta|_K) \) is homeomorphic (therefore homotopy equivalent) to \( \text{Map}((S^3, K) \to (S^2, \mathcal{N})) \). Thus also \( \Omega Distr(S^3 \text{ rel } K, \zeta|_K) \) is homeomorphic to \( \Omega \text{Map}((S^3, K) \to (S^2, \mathcal{N})) \).

By Lemma 2.5, \( \Omega \text{Map}((S^3, K) \to (S^2, \mathcal{N})) \) is homeomorphic to \( \Omega(\Omega^3 S^2) = \Omega^4 S^2 \). \qed

### 3 Discussion

In the previous section we studied the group \( \text{Diff}_\xi(S^3 \text{ rel } K) \) of contactomorphisms fixing (pointwise) a neighborhood of an overtwisted disk, rather than the full group of contactomorphisms \( \text{Diff}_\xi(S^3) \). Such is the price we had to pay for the knowledge of the full homotopy type of the group of contactomorphisms (due to the restrictions placed by the assumptions of Theorem 1.2). The only known result about the full group \( \text{Diff}_\xi(S^3) \) (for \( \xi \) overtwisted) is Theorem 2.2.1 in [1]: it is not connected.

Fix an overtwisted disk \( \Delta \) and its small open neighborhood \( K \). By a contact embedding of \( K \) into the contact manifold \( (S^3, \xi) \) we will understand a map \( f: K \hookrightarrow S^3 \) such that \( f: K \to f(K) \) is a contactomorphism. Two such embeddings are contact isotopic (or simply isotopic), if they are homotopic through contact embeddings. We will say that two overtwisted disks are (contact) isotopic if they are images of \( \Delta \) by two isotopic embeddings of \( K \). Denote the space of contact embeddings of the neighborhood \( K \) of an overtwisted disk into \( S^3 \) by \( \text{Emb}_\xi(K \hookrightarrow S^3) \) and consider the fibration

\[
\text{Diff}_\xi(S^3 \text{ rel } K) \to \text{Diff}_\xi(S^3) \to \text{Emb}_\xi(K \hookrightarrow S^3).
\]

One can see that what we are lacking to grasp the structure of the full group of contactomorphisms is exactly understanding the space of contact embeddings of the overtwisted disk; Proposition 3.4 gives a partial result in this direction. Two lemmas are needed in the proof; the first one essentially has been contained in [1], but here for the first time we state it explicitly.

**Lemma 3.1.** Let \( K, L \) be compact subsets of a closed contact manifold \( (M, \xi) \) such that \( M \setminus K \) is connected, \( \Delta \) an overtwisted disk in \( M \setminus (K \cup L) \). Let \( f: M \to M \) be a diffeomorphism which fixes a neighborhood of \( \Delta \), moves \( K \) onto \( L \), and becomes a contactomorphism when restricted to a neighborhood of \( K \). Assume that the contact structures \( \xi \) and \( f_\ast \xi \) are homotopic as plane fields on \( M \text{ rel}(K \cup \Delta) \). Then there is a contactomorphism \( F: M \to M \) moving \( K \) onto \( L \).

**Proof:** By Eliashberg’s theorem (1.2) \( \xi \) and \( f_\ast \xi \) are isotopic. By Gray’s theorem (1.1) they are also isomorphic, i.e. there is a diffeomorphism \( g \) such that \( g_\ast \xi = f_\ast \xi \), \( g|_K = \text{Id} \). Thus \( F = f \circ g^1 \) is a contactomorphism and \( F(K) = L \) as desired. \qed

The assumptions of Lemma 3.1 can be rephrased in the language of Gauss maps with respect to the parallelization induced by the parallelization of \( \xi \) as follows: there is a homotopy \( G_t: M \to S^2 \) such that \( G_0 = G_\xi \) is constant, \( G_t|_{K \cup \Delta} \) is constant for all \( t \) and \( G_1 = G_{f_\ast \xi} \).
Corollary 3.2. Let $K \subset \mathbb{R}^3$ be a contractible compact set, $\xi_K$ a contact structure on a neighborhood $U_K$ of $K$. Furthermore, let $j_1, j_2$ be two embeddings of $U_K$ into a contact manifold $M$ which are contactomorphisms onto their respective images. If there is an overtwisted disk $\Delta$ in $M \setminus (j_1(U_K) \cup j_2(U_K))$, then there exists a contactomorphism of $M$ moving $j_1(K)$ onto $j_2(K)$.

Proof: The map $j_2^{-1} \circ j_1$ can be extended to a diffeomorphism $J : M \to M$ coinciding with the identity outside an open ball $B \supset j_1(K) \cup j_2(K), B \cap \Delta = \emptyset$. Since the group $\text{Diff}(B \text{ rel } \partial B)$ is connected, $J$ must be isotopic to the identity; denote this isotopy by $J_t : M \to M$, so that we have $J_t \big|_{M \setminus B} = \text{Id}, J_0 = \text{Id}, J_1 = J$. Now, consider the homotopy of plane fields given by $(J_t)_* \xi$, and the associated homotopy of Gauss maps $G_t : M \to S^2$. We see that $G_0$, as well as $G_t \big|_{M \setminus B}$ and $G_1 \big|_{j_1(K)}$ are constant. Since $K$ is contractible, the map $\mathcal{G} : K \times [0,1] \to S^2 : (x,t) \mapsto G_t(x)$ is homotopic (rel $K \times \{0,1\}$) to the constant map. This homotopy can be extended to the whole $M$, yielding (by the $\xi \leftrightarrow G_\xi$ identification) the plane field homotopy satisfying the assumptions of Lemma 3.1.

Lemma 3.3. Let $L \subset S^3$ be a compact set, $f : S^3 \to S^3$ a contactomorphism coinciding with the identity on (a neighborhood of) an overtwisted disk $\Delta$ disjoint with $L$. Then $L$ and $f(L)$ are ambiently isotopic, i.e. there exists a family of contactomorphisms $\{f_t : t \in [0,1]\}$, $f_0 = \text{Id}, f_1(L) = f(L)$.

Proof: As before, let $K = \overline{U(\Delta)}$. The weak homotopy equivalence $\text{Diff}_\xi(S^3 \text{ rel } K) \to \Omega^4 S^2$ postulated by Theorem 2.1 induces a map $\mathcal{H} : \text{Diff}_\xi(S^3 \text{ rel } K) \to \pi^4 S^2 \simeq \mathbb{Z}^2$. Choose a contactomorphism $g$ coinciding with the identity outside a small ball disjoint with $K$ and $L$ and such that $\mathcal{H}(g) = \mathcal{H}(f)^{-1}$. Then $g \circ f$, since it lies in the kernel of $\mathcal{H}$, is isotopic to the identity through contactomorphisms. This isotopy provides the desired family of contactomorphisms.

Note that ambiently isotopic overtwisted disks are necessarily isotopic.

Proposition 3.4. Two disjoint overtwisted disks are isotopic.

Proof: Let $\Delta_1$ and $\Delta_2$ be the disjoint overtwisted disks. A parallel copy of $\Delta_1$ (close enough to it) is an overtwisted disk disjoint with both $\Delta_1$ and $\Delta_2$. By Corollary 3.2 there is a contactomorphism moving $\Delta_1$ onto $\Delta_2$. Then Lemma 3.3 implies that $\Delta_1$ is isotopic to $\Delta_2$.

No actual example of two non-isotopic overtwisted disks is known to the author. One may, however, try contemplating the following construction.

Let $T$ be the solid torus

$$\{(r, \theta, z) : r < \pi + \varepsilon\}/(r, \theta, z) \sim (r, \theta, z + 2\pi)$$

with the contact structure

$$\xi = \ker \left( \cos r - \phi(r) \sin r \, d\theta \right),$$

as in Section 1. Identify two copies of $T$ along the subset

$$N = \{(r, \theta, z) : \pi/2 - \varepsilon < r < \pi + \varepsilon\}$$
via the diffeomorphism
\[ \phi : (r, \theta, z) \mapsto (3\pi/2 - r, z, \theta). \]

Topologically, the manifold we obtain is the three-sphere; one should think about the genus 1 Heegaard splitting of \( S^3 \) with both solid tori slightly enlarged, so that the identification, rather than along the boundary, is performed along its tubular neighborhood. We shall denote the two inclusions of \( T \) into \( S^3 \) by \( \psi_1 \) and \( \psi_2 \); on \( N \) we have \( \phi = \psi_2^{-1} \circ \psi_1 \).

Moreover, if we choose the function \( \rho \) so that \( \rho(r) = 1 \) for \( r \in (\pi/2 - \varepsilon, \pi + \varepsilon) \), then \( \phi \) is a contactomorphism, so the resulting sphere carries a contact structure pushed forward from the solid tori. This contact structure is overtwisted, with two obvious families of overtwisted disks of the form \( \psi_i \left( \{(r, \theta, z) : r \leq \pi, z = z_0 \} \right) \) for \( i = 1, 2 \). Note that the formula \( \Psi(x) = \begin{cases} \psi_2 \circ \psi_1^{-1}(x) \text{ for } x \in \psi_1(T), \\ \psi_1 \circ \psi_2^{-1}(x) \text{ for } x \in \psi_2(T) \end{cases} \) defines a contactomorphism of \( S^3 \) exchanging these two families.

Consider two disks, one of each of the families, say \( \Delta_1 = \psi_1 \left( \{(r, \theta, z) : r \leq \pi, z = 0 \} \right) \) and \( \Delta_2 = \psi_2 \left( \{(r, \theta, z) : r \leq \pi, z = 0 \} \right) \). It is easy to see that \( \Delta_1 \) and \( \Delta_2 \) intersect along an interval, and not much more difficult—that \( \xi \) restricted to \( S^3 \setminus (\Delta_1 \cup \Delta_2) \) is tight\(^2\); therefore assumptions of neither Corollary 3.2 nor Proposition 3.4 are satisfied.

**Question 3.5.** Are \( \Delta_1 \) and \( \Delta_2 \) isotopic?

### 4 References

[1] K. Dymara, *Legendrian knots in overtwisted contact structures on \( S^3 \)*, Ann. Global Anal. Geom., 19 (2001), pp. 293–305.

[2] K. Dymara, *Legendrian knots in overtwisted contact structures*. arXiv:math.GT/0410122, 2004.

[3] J. W. Gray, *Some global properties of contact structures*, Ann. of Math. (2), 69 (1959), pp. 421–450.

[4] A. Hatcher, *A proof of a Smale conjecture*, Diff(\( S^3 \)) \( \simeq \) O(4), Ann. of Math., 117 (1983), pp. 553–607.

---

\(^2\) For a detailed proof that even \( S^3 \setminus (\partial\Delta_1 \cup \partial\Delta_2) \) is tight see [2].