Principal Minor Assignment, Isometries of Hilbert Spaces, Volumes of Parallelepipeds and Rescaling of Sesqui-holomorphic Functions

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Abstract

In this article we consider the following equivalence relation on the class of all functions of two variables on a set $X$: we will say that $L, M : X \times X \to \mathbb{C}$ are rescalings if there are non-vanishing functions $f, g$ on $X$ such that $M(x, y) = f(x) g(y) L(x, y)$, for any $x, y \in X$. We give criteria for being rescalings when $X$ is a topological space, and $L$ and $M$ are separately continuous, or when $X$ is a domain in $\mathbb{C}^n$ and $L$ and $M$ are sesqui-holomorphic.

A special case of interest is when $L$ and $M$ are symmetric, and $f = g$ only has values $\pm 1$. This relation between $M$ and $L$ in the case when $X$ is finite (and so $L$ and $M$ are square matrices) is known to be characterized by the equality of the principal minors of these matrices. We extend this result for the case when $X$ is infinite. As an application we get the following theorem.

Theorem. Let $H$ be a real Hilbert space and let $B \subset H$ be linearly independent set which is connected in the weak topology of $H$ and such that $\text{span} \ B = H$. Let $\Phi : B \to H$ be continuous with respect to the weak topology and such that for any distinct $v_1, \ldots, v_n \in B$ the parallelepiped spanned by $v_1, \ldots, v_n$ has the same volume as the parallelepiped spanned by $\Phi(v_1), \ldots, \Phi(v_n)$. Then there is an isometry $T$ on $H$ such that $T|_B = \Phi$.

Keywords: Reproducing Kernel Hilbert Spaces; rescalings of sesqui-holomorphic functions; Principal Minor; Volumes of Parallelepipeds; Isometries of Hilbert Spaces.

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1 Introduction

The following well-known open problem in algebra is called Principal Minor Assignment Problem.

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Question 1.1. What are the necessary and sufficient conditions for a collection of $2^n$ complex numbers to be the set of the principal minors of an $n \times n$ matrix?

For some recent developments in relation to this question see e.g. [10] and [11]. An adjacent to the Principal Minor Assignment Problem is the following easier question.

Question 1.2. What is the relationship between two $n \times n$ matrices having equal corresponding principal minors of all orders?

It turns out that the answer very much depends on certain additional assumption about the matrices (see e.g. [8]). In particular, in the class of complex symmetric matrices the following characterization holds.

Theorem 1.3 ([6]). If two complex symmetric $n \times n$ matrices $L$ and $M$ have equal corresponding principal minors of all orders, then there is a diagonal matrix $D$ of order $n$ with diagonal entries in $\{-1, 1\}$ such that $L = DMD^{-1}$.

The proof of this fact given in [6] and [11] is mostly combinatorial and heavily exploits an auxiliary object – the graph of a matrix. Namely, the graph of a symmetric $n \times n$ matrix $L = [l_{ij}]_{i,j=1}^n$ is a graph with vertices in $\{1, \ldots, n\}$ and $(i, j)$ being an edge if $l_{ij} \neq 0$. Hence, the indices $i$ and $j$ are treated as points of a certain space, rather than numbers. Also, the proof does not rely much on finiteness of $\{1, \ldots, n\}$, which motivates us to consider the following type of object.

Let $X$ be a set. We will call complex-valued functions defined on $X \times X$ bi-functions. Note that a bi-function on a finite set $X = \{1, \ldots, n\}$ is a square $n \times n$ matrix. One can introduce the notion of the graph of a bi-function in an analogous way to the graph of a matrix. In order to state an analogue of Theorem 1.3 we need to find the concepts corresponding to the principal minors and the diagonal similarity. Namely, for a bi-function $L$ on $X$ and $x_1, \ldots, x_n \in X$ denote $\det_L(x_1, \ldots, x_n) = \det [L(x_i, x_j)]_{i,j=1}^n$. A concept analogous to the diagonal similarity is the following. We will say that a non-vanishing function $f$ on $X$ reciprocally rescales a bi-function $L$ on $X$ to a bi-function $M$, if $M(x, y) = f(x) \frac{1}{f(y)}L(x, y)$, for any $x, y \in X$. It is easy to see that this definition generalizes the condition that appears in Theorem 1.3. In fact, the theorem holds in this infinite context verbatim.

Theorem 1.4. Symmetric bi-functions $L$ and $M$ on a set $X$ are reciprocal rescalings of each other if and only if $\det_L = \det_M$.

More generally, for non-vanishing functions $f, g : X \to \mathbb{C}$ we will say that the pair $(f, g)$ rescales $L$ to $M$ if $M(x, y) = f(x) \frac{1}{f(y)}g(y)L(x, y)$, for any $x, y \in X$. Reciprocal rescaling corresponds to the case when $g = \frac{1}{f}$, but we can choose other relations between $f$ and $g$. In particular, if $g = f$ (or $g = \overline{f}$) we will say that $f$ symmetrically (or Hermiteanly) rescales $L$ to $M$. Hermitean rescalings of positive semi-definite kernels play a role in studying Multiplication Operators on Reproducing Kernel Hilbert Spaces (see e.g. [2]).
One can ask if there is a way to ascertain if two bi-functions \( L \) and \( M \) are rescalings without referring to \( f \) and \( g \). It is easy to see that if \( L \) and \( M \) are rescalings, then
\[
M(x, y)M(y, z)L(x, z)L(y, y) = L(x, y)L(y, z)M(x, z)M(y, y),
\]
for every \( x, y, z \) in \( X \). It is also easy to see that the converse holds if there is \( y \in X \) such that \( L(\cdot, y) \) and \( L(y, \cdot) \) do not vanish. However, in general the converse does not hold.

Since the underlying set \( X \) is not confined to the finite world, one can add structure on \( X \) into consideration. In this article we deal with two examples of such structures. We first assume that \( X \) is a topological space and consider bi-functions that satisfy some continuity conditions. We study how the topological properties of \( X \) impact the graph theoretical properties of the graphs of such bi-functions, and under some additional restrictions we show that the equality above is a sufficient condition for \( L \) and \( M \) to be rescalings (see Theorem 5.17).

We also consider the case when \( X \) is a domain in \( \mathbb{C}^n \) and study holomorphic and sesqui-holomorphic bi-functions on \( X \). Due to rigidity of the complex structure, criterions for holomorphic and sesqui-holomorphic bi-functions to be rescalings are much more succinct (see Proposition 6.5 and Theorem 6.7).

Another aspect of Theorem 1.3 that we would like to point out is its geometric interpretation in the case when we restrict to the real positive definite matrices. These matrices are the Gram matrices of a certain collection of vectors in \( \mathbb{R}^n \) and the determinant of the Gram matrix is the square of the volume of the parallelepiped spanned by that collection. Hence, a direct consequence of Theorem 1.3 is the following fact.

**Corollary 1.5.** Two parallelepipeds with equal volumes of the corresponding faces are isometric.

Finally, joining the topological and geometrical approaches described above we arrive at the following result.

**Theorem 1.6.** Let \( H \) be a real Hilbert space and let \( B \subset H \) be linearly independent set which is connected in the weak topology of \( H \) and such that \( \overline{\text{span}}B = H \). Let \( \Phi : B \to H \) be continuous with respect to the weak topology and such that for any distinct \( v_1, \ldots, v_n \in B \) the parallelepiped spanned by \( v_1, \ldots, v_n \) has the same volume as the parallelepiped spanned by \( \Phi(v_1), \ldots, \Phi(v_n) \). Then there is an isometry \( T \) on \( H \) such that \( T |_B = \Phi \).

Let us describe the content of the article. In Section 2 we study the notion of rescaling and its relation to the graph of a bi-functions. In Section 3 we consider the minors of bi-functions and in particular prove Theorem 1.4. In Section 4 we focus on the geometric interpretation of this theorem in both finite and infinite cases. In particular, we give a mostly geometric proof of Corollary 1.5 independent of Theorem 1.3. In Section 5 we deal with separately continuous bi-functions on topological spaces. This includes proving Theorem 5.17 and showing that the conditions of this theorem...
are essential (see Example \ref{example:5.19}). Section \ref{section:6} is dedicated to holomorphic and sesqui-holomorphic bi-functions on domains in $\mathbb{C}^n$.

**Some notations and conventions.** Let $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ and let $\mathbb{T} = \{\lambda \in \mathbb{C}, |\lambda| = 1\}$. We will also denote the imaginary unit number by $i$ in order to reserve the letter $i$ for indexation (typically by integers). If $X$ is a set, we will view the Kronecker’s $\delta$ as a function defined on $X \times X$, and for a fixed $x \in X$ $\delta_x$ will be viewed as a function on $X$. Finally, $\text{Fin}(X)$ will be the set of all finite subsets of $X$. If $\mu : \text{Fin}(X) \to \mathbb{C}$, we will adopt the following abuse of notations: for $x_1, \ldots, x_n \in X$, we will denote $\mu (x_1, \ldots, x_n) = \mu (\{x_1, \ldots, x_n\})$, if $x_1, \ldots, x_n$ are all distinct, and $\mu (x_1, \ldots, x_n) = 0$ otherwise.

## 2 Bi-functions

The main subject of this article are the (abstract) functions of two variables. Let $X$ be a set. We will call functions defined on $X \times X$ bi-functions and usually use capital letters to denote them, in order to distinguish from the “usual” function, i.e. scalar-valued functions defined on $X$. Note that a bi-function on a finite set $X = \{1, \ldots, n\}$ is a $n \times n$ matrix. Consider the simplest examples of bi-functions on a general set: if $f, g : X \to \mathbb{C}$, define $f \otimes g : X \times X \to \mathbb{C}$ by $(f \otimes g)(x, y) = f(x)g(y)$ and $\text{diag}f : X \times X \to \mathbb{C}$ by $\text{diag}f(x, y) = \delta_{x,y}f(x)$. Note that $f_1 \otimes g_1 \cdot f_2 \otimes g_2 = f_1f_2 \otimes g_1g_2$ and $\text{diag}f_1 \text{diag}f_2 = \text{diag}(f_1f_2)$. Consider also a transition from a bi-function to a function: if $L : X \times X \to \mathbb{C}$, define the diagonal function $\hat{L} : X \to \mathbb{C}$ by $\hat{L}(x) = L(x, x)$, for $x \in X$. We will say that $L$ is non-degenerate if $\hat{L}$ does not vanish. Although purely technical, this condition is very useful in our considerations. See Remark \ref{remark:4.2} for a justification of the term “non-degenerate”.

For a bi-function $L$ on $X$ define bi-functions $L'$ and $L^*$ by $L'(x, y) = L(y, x)$, for $x, y \in X$, and $L^* = \overline{L}$. We will say that $L$ is symmetric (Hermitean) if $L = L'$ ($L = L^*$). Clearly, $f \otimes f$ and $\text{diag}f$ are symmetric and $f \otimes \overline{f}$ is Hermitean, for any $f : X \to \mathbb{C}$. Note that if $L$ is Hermitean, then $\hat{L}$ is real-valued.

**Rescalings.** We will say that a pair $(f, g)$ of non-vanishing functions on $X$ rescales a bi-function $L$ on $X$ to a bi-function $M$, if $M = f \otimes g L$. Note that if in this case $L$ is non-degenerate then $fg = \frac{M}{L}$. We will say that bi-functions $L$ and $M$ on $X$ are rescalings, if there are functions $f, g : X \to \mathbb{C}^\times$ such that that $(f, g)$ rescales $L$ to $M$.

It is easy to see that if $(f, g)$ rescales $L$ to $M$, then $(g, f)$ rescales $L'$ to $M'$, $(\overline{f}, \overline{g})$ rescales $\overline{L}$ to $\overline{M}$, $(g, f)$ rescales $L^*$ to $M^*$, and $\left(\frac{i}{\lambda}, \frac{i}{\mu}\right)$ rescales $M$ to $L$. Also, $(\alpha f, \beta g)$ rescales $L$ to $\alpha \beta M$, for any $\alpha, \beta \in \mathbb{C}^\times$, and in particular $(\lambda f, \lambda^{-1}g)$ rescales $L$ to $M$ for any $\lambda \in \mathbb{C}^\times$. Hence, $f$ and $g$ are not uniquely determined by $L$ and $M$.

If $(f_1, g_1)$ rescales $K$ to $L$ and $(f_2, g_2)$ rescales $L$ to $M$, then $(f_1f_2, g_1g_2)$ rescales $K$ to $M$. Hence, the relation of being rescalings is an equivalence relation. If $(f_i, g_i)$ rescales $L_i$ to $M_i$, then $\left(\prod_i f_i, \prod_i g_i\right)$ rescales $\prod_i L_i$ to $\prod_i M_i$. 
Let \( Y, Z \) be subsets of \( X \) such that \( Y \subset Z \subset X \) and let \( f, g : Z \to \mathbb{C}^x \). We will say that \((f, g)\) rescales \( L \) to \( M \) on \( Y \) if \( f \) and \( g \) rescale \( L|_{Y \times Y} \) to \( M|_{Y \times Y} \); in this case we will say that \( L \) and \( M \) are rescalings on \( Y \). Being rescalings on \( Y \) is an equivalence relation. Let us now introduce equivalence relations between bi-functions, which are finer than being rescalings.

We will say that \( f \) symmetrically rescales \( L \) to \( M \) if \((f, f)\) rescales \( L \) to \( M \). In this case we will say that \( L \) and \( M \) are symmetric rescalings. We will say that \( f \) Hermiteanly rescales \( L \) to \( M \) if \((f, \overline{f})\) rescales \( L \) to \( M \). In this case we will say that \( L \) and \( M \) are Hermitean rescalings. Note that then \( \hat{M} \hat{L} \geq 0 \). If \( f \) symmetrically (or Hermiteanly) rescales \( L \) to \( M \) and \( L \) is non-degenerate, then \( f^2 = \frac{\hat{M}}{L} \) (or \(|f|^2 = \frac{\hat{M}}{L}\)). We will say that \( f \) reciprocally rescales \( L \) to \( M \) if \( \left((f, \frac{f}{L})\right) \) rescales \( L \) to \( M \). In this case we will say that \( L \) and \( M \) are reciprocal rescalings. Note that this concept is the analogue of the diagonal similarity of matrices. If \( L \) and \( M \) are reciprocal rescalings, then \( \hat{L} = \hat{M} \); conversely if \( L \) and \( M \) are rescalings, \( L \) is non-degenerate and \( \hat{L} = \hat{M} \), then \( L \) and \( M \) are reciprocal rescalings. For any \( h : X \to \mathbb{C} \) any reciprocal rescaling of \( \text{diag}h \) is \( \text{diag}h \) itself. Hence, if \( f \) reciprocally rescales \( L \) to \( M \), then \( f \) also reciprocally rescales \( L + \text{diag}h \) to \( M + \text{diag}h \). Finally, we will say that \( L \) and \( M \) are \( \pm 1 \) symmetric rescalings, if there is \( f : X \to \{-1, 1\} \) that symmetrically rescales \( L \) to \( M \). If is easy to see that if \( L \) and \( M \) are \( \pm 1 \) symmetric rescalings, then \( \hat{L} = \hat{M} \) and \( M^2 = M^2 \). Conversely, if \( L \) and \( M \) are non-degenerate symmetric rescalings with \( \hat{L} = \hat{M} \), then they are \( \pm 1 \) symmetric rescalings (see more similar properties in Proposition 2.3).

**The graph of a bi-function.** The main idea that we borrow from \([6]\) and \([11]\) is the introduction of the following object. For a bi-function \( L \) on \( X \) let \( X_L \) be the (undirected) graph with the set of vertices equal to \( X \) and \((x, y) \in X \times X \) being an edge if either \( L(x, y) \neq 0 \) or \( L(y, x) \neq 0 \). Note that if \( L \) and \( M \) are rescalings, then \( X_L = X_M \). The (graph-theoretical) components of \( X_L \) are the equivalence classes of the minimal equivalence relation on \( X \) that contains pairs \((x, y) \in X \times X \) such that \( L(x, y) \neq 0 \). Also, the components of \( X_L \) can be viewed as “degrees of freedom” of rescaling of \( L \), as the following proposition shows.

**Proposition 2.1.** Bi-function \( L \) and \( M \) on a set \( X \) are rescalings if and only if \( X_L = X_M \) and \( L \) and \( M \) are rescalings on every component of this graph. In particular if \( f, g : X \to \mathbb{C}^x \), then \((f, g)\) rescales \( L \) to \( M \) if and only if \( (f|_Y, g|_Y) \) rescales \( L \) to \( M \) on every component \( Y \) of \( X_L \).

**Proof.** Since \( X_L = X_M \) we will view \( X \) endowed with a fixed graph structure. Let \( X = \bigcup_{j \in I} X_j \), where \( X_j \) are the graph components of \( X \). Let \( f_j, g_j : X_j \to \mathbb{C}^x \) and let \( f, g : X \to \mathbb{C}^x \) be defined by \( f(x) = f_j(x) \) and \( g(x) = g_j(x) \), where \( x \in X_j \).

We only need to show that if \( M|_{X_j \times X_j} = f_j \otimes g_j L|_{X_j \times X_j} \), for every \( j \in I \), then \( M = f \otimes g \). Let \( x, y \in X \). If there is \( j \in I \) such that \( x, y \in X_j \), we have \( f(x) g(y) L(x, y) = f_j(x) g_j(y) L(x, y) = M(x, y) \). Otherwise, \( f(x) g(y) L(x, y) = 0 = M(x, y) \). \( \square \)

**Corollary 2.2.** Let \( L \) be a bi-function on a set \( X \).
(i) \( f : X \to \mathbb{C}^\times \) reciprocally rescales \( L \) to itself if and only if \( f \) is constant on every component of \( X_L \).

(ii) If \( L \) is non-degenerate then for \( f_1, g_1, f_2, g_2 : X \to \mathbb{C}^\times \) we have that \( f_1 \otimes g_1 L = f_2 \otimes g_2 L \) if and only if \( \frac{f_1}{f_2} = \frac{g_2}{g_1} \) is constant on every component of \( X_L \).

Proof. (i): Sufficiency follows from the preceding proposition. Let us prove necessity. Observe that if \( L(x, y) \neq 0 \), then \( f(x) \frac{1}{f(y)} L(x, y) = L(x, y) \), and so \( f(x) = f(y) \).

Hence, \( f(x) = f(y) \) is an equivalence relation on \( X \) that contains all pairs \((x, y) \in X \times X \) such that \( L(x, y) \neq 0 \). Thus, \( f \) is constant on every component of \( X \).

Part (ii) follows from part (i) applied to \( h = \frac{f_1}{f_2} \), since if \( L \) is non-degenerate, we have that \( \frac{g_2}{g_1} = \frac{1}{h} \).

Remark 2.3. The condition that \( L \) is non-degenerate is essential in part (ii). Indeed, let \( X =\{1, 2, 3, 4\} \) and consider \( L : X \times X \to \mathbb{C} \) and \( f : X \to \mathbb{C} \) defined by \( L(i, j) = 1 - (-1)^{i+j} \) and \( f(i) = \exp\left((-1)^i\right), i, j \in X \). Then \( X_L \) is connected, but a non-constant function \( f \) symmetrically rescales \( L \) to itself.

Corollary 2.4. Let \( L \) be a bi-function on a set \( X \), such that \( X_L \) is connected. Then:

(i) \( f : X \to \mathbb{C}^\times \) reciprocally rescales \( L \) to itself if and only if \( f \) is constant.

(ii) Assume that \( L \) is non-degenerate, let \( M \) be a bi-function on \( X \) and let \( x \in X \). If \( L \) and \( M \) are rescalings, then there are unique \( f, g : X \to \mathbb{C}^\times \) such that \((f, g)\) rescales \( L \) to \( M \) with \( f(x) = 1 \) (or \( g(x) = 1 \)). If \( L \) and \( M \) are Hermitean rescalings, then there is a unique \( f : X \to \mathbb{C}^\times \) that Hermiteanly rescales \( L \) to \( M \) with \( f(x) > 0 \). If \( L \) and \( M \) are symmetric rescalings and \( f, g : X \to \mathbb{C}^\times \) both symmetrically rescale \( L \) to \( M \), then either \( f = g \) or \( f = -g \).

The preceding results allow us to translate properties of rescalings \( L \) and \( M \) on properties of \( f, g \) such that \((f, g)\) rescales \( L \) to \( M \).

Proposition 2.5. Let \( L, M : X \times X \to \mathbb{C} \) be rescalings. Then:

(i) If \( L \) and \( M \) are symmetric, then they are symmetric rescalings. If moreover, they are non-degenerate, and \( L^2 = M^2 \), then \( L \) is a \( \pm 1 \) symmetric rescaling of either \( M \) or \(-M\).

(ii) If \( L \) and \( M \) are Hermitean and non-degenerate, then \( L \) is a Hermitean rescaling of either \( M \) or \(-M\).

(iii) If \( L \) and \( M \) are non-degenerate and real-valued, then there are \( f, g : X \to \mathbb{R}^\times \) such that \((f, g)\) rescales \( L \) to \( M \). If moreover, \( L \) and \( M \) are symmetric, then there is \( f : X \to \mathbb{R}^\times \) such that \( f \) symmetrically rescales \( L \) either to \( M \), or \(-M\).

(iv) If \( |L| = |M| \), and \( L \) and \( M \) are non-degenerate, then there are \( f, g : X \to \mathbb{T} \) such that \((f, g)\) rescales \( L \) to \( M \). If moreover, \( L \) and \( M \) are symmetric (Hermitean), then there is \( f : X \to \mathbb{T} \) such that \( f \) symmetrically rescales \( L \) to \( M \) (Hermiteanly rescales \( L \) either to \( M \), or \(-M\)).
Proof. In the light of part (i) of Proposition 2.1, without loss of generality we may assume that $X_L = X_M$ is connected. Let $f, g : X \to \mathbb{C}^\times$ be such that $(f, g)$ rescales $L$ to $M$.

(i): Since $L$ and $M$ are symmetric, $(g, f)$ also rescales $L$ to $M$. We have that $e = \frac{L}{g}$ reciprocally rescales $L$ to itself, and so from part (i) of the preceding corollary, $e$ is constant. Then $h = \frac{f}{e} = \sqrt{eg}$ symmetrically rescales $L$ to $M$, and so $L$ and $M$ are symmetric rescalings.

Assume now that $L$ and $M$ are non-degenerate, and $L^2 = M^2$. Then, $h^4 \equiv 1$. If $x, y \in X$ are such that $L(x, y) \neq 0$, then $h^2(x)h^2(y) = 1$, and so $h^2(x) = \frac{1}{h^2(y)} = h^2(y)$. Hence, $h^2(x) = h^2(y)$ is an equivalence relation on $X$ that contains all pairs $(x, y) \in X \times X$ such that $L(x, y) \neq 0$, and since $X_L$ is connected, we conclude that $h^2$ is a constant, which can be only $\pm 1$. Hence, either $h(x) = \pm 1$, for every $x \in X$, or $h(x) = \pm i$, for every $x \in X$. In the first case we conclude that $L$ and $M$ are $\pm 1$ symmetric rescaling; in the latter $ih$ has range in $\{-1, 1\}$, and symmetrically rescales $L$ to $-M$.

(ii): Since $L$ and $M$ are Hermitean and non-degenerate, $(\overline{g}, \overline{f})$ rescale $L$ to $M$, and $fg$ is real-valued. Then $h = \frac{f}{g} = \frac{\overline{g}}{\overline{f}}$ is also real-valued. Since $h$ and $\frac{1}{h}$ rescale $L$ to itself, $h$ is a real constant. If $h > 0$, then $\frac{f}{\sqrt{h}} = \sqrt{h}g$ Hermiteanly rescales $L$ to $M$. If $h < 0$, then $-\frac{f}{\sqrt{-h}} = \sqrt{-h}g$ Hermiteanly rescales $L$ to $-M$.

(iii): Fix $z \in X$. From part (ii) of the preceding corollary, without loss of generality we may assume $f(z) = 1$. Since $L$ and $M$ are non-degenerate and real-valued we have that $fg$ is real-valued. If $x, y \in X$ are such that $L(x, y) \neq 0$, then $f(x)g(y) \in \mathbb{R}^\times$, and so $\frac{f(x)}{f(y)} = \frac{f(x)g(y)}{f(y)g(y)} \in \mathbb{R}^\times$. Since $\frac{f(x)}{f(y)} \in \mathbb{R}^\times$ is an equivalence relation and $X_L$ is connected, we conclude that $\frac{f(x)}{f(y)} \in \mathbb{R}^\times$ for every $x, y \in X$. In particular, $f(x) = \frac{f(x)}{f(z)} \in \mathbb{R}^\times$. Finally, $g = \frac{f}{f}$ is also real-valued.

The second claim is proven by applying the argument from the proof of part (i) to $f$ and $g$.

(iv): Fix $z \in X$. From part (ii) of the preceding corollary, without loss of generality we may assume $f(z) = 1$. Since $L$ and $M$ are non-degenerate and $|L| = |M|$, it follows that $|fg| \equiv 1$, and so for any $x, y \in X$ such that $L(x, y) \neq 0$ we have $1 = |f(x)||g(y)| = \frac{|f(x)|}{|f(y)|}$. Arguing as in the proof of part (iii) we get that both $|f|$ and $|g|$ are constant functions, which are equal to 1, since $f(z) = 1$ and $|fg| \equiv 1$.

The second claim is proven by applying the argument from the proof of part (i) to $f$ and $g$. \hfill \Box

3 Minors of a bi-function

Since a bi-function is a generalization of a square matrix, it is natural to introduce a concept related to determinants. Namely, for a bi-function $L$ on $X$ and $x_1, \ldots, x_n \in X$
denote $\det_L (x_1, ..., x_n) = \det [L(x_i, x_j)]_{i,j=1}^n$. Any renumeration of $x_1, ..., x_n$ does not affect $\det_L (x_1, ..., x_n)$, since swapping $x_i$ with $x_j$ corresponds to swapping the $i$-th and $j$-th columns as well as $i$-th and $j$-th rows of the corresponding matrix, and so the determinant gets multiplied with $(-1)^2 = 1$. Analogously, if $x_i = x_j$ for some $i, j \in \overline{1, n}$, then $\det_L (x_1, ..., x_n) = 0$. Hence, $\det_L$ may be viewed as a scalar function defined on the collection $Fin(X)$ of finite subsets of $X$. In particular, $\det_L (x) = \hat{L} (x)$ and $\det_L (x, y) = \hat{L} (x) \hat{L} (y) - L(x,y) L(y,x)$, for $x, y \in X$. Also, note that $\det_L = \det_{\hat{L}}$ and $\det_{\hat{L}} = \det_{L^*} = \overline{\det_L}$.

Clearly, if $f : X \rightarrow \mathbb{C}$, then $\det_{\text{diag}} f (x_1, ..., x_n) = \prod_{i=1}^n f(x_i)$, for any distinct $x_1, ..., x_n \in X$. Also, it is easy to see that if $f, g : X \rightarrow \mathbb{C}$, then $\det_{f \otimes g}$ vanishes on subsets of $X$ of cardinality higher than 1. Finally, if $L$ is symmetric (Hermitean), then $\hat{L} (x) \hat{L} (y) = \det_L (x, y)$ is equal to $L(x,y)^2 (|L(x,y)|^2)$, for $x, y \in X$.

It is natural to ask how certain relations between two bi-functions $L$ and $M$ reflect on the relations between $\det_L$ and $\det_M$. First, observe, that if $(f, g)$ rescales $L$ to $M$, then $\det_M = \det_{\text{diag}} f g \det_L$. Indeed, for any $x_1, ..., x_n \in X$ the matrix $[M(x_i,x_j)]_{i,j=1}^n$ is obtained from the matrix $[L(x_i,x_j)]_{i,j=1}^n$ by multiplication $i$-th row with $f(x_i)$ and $i$-th column with $g(x_i)$, for every $i \in \overline{1, n}$. Hence,

$$\det_M (x_1, ..., x_n) = \prod_{i=1}^n f(x_i) g(x_i) \det_L (x_1, ..., x_n). \quad (*)$$

On the other hand, it would be interesting to see what are the relations between $L$ and $M$ for which $\det_L = \det_M$. We immediately get $\hat{L} = \hat{M}$, and so $L$ and $M$ coincide on the diagonal. More can be said about bi-functions that posses certain symmetry. First, if $L$ and $M$ are symmetric or Hermitean, then $\det_L = \det_M$ imply $X_L = X_M$. Also, it is easy to see that if $L$ and $M$ are symmetric then $\det_L = \det_M$ on sets of cardinality 1 and 2 if and only if $\hat{L} = \hat{M}$ and $L^2 = M^2$, i.e. $M(x,y) = \pm L(x,y)$, for any $x, y \in X$. Furthermore, if $\det_L = \det_M$ on sets of cardinality 1, 2 and 3 then

$$L(x,y)L(y,z)L(z,x) = M(x,y)M(y,z)M(z,x), \quad (**)$$

for any $x, y, z \in X$, and the converse holds, if $L$ and $M$ are non-degenerate. Indeed, substituting $x = y = z$ we get that $\hat{L} = \hat{M}$, while inputting $x = z$ gives us $L^2 = M^2$.

Finally, a simple calculation shows that under the assumption that $\hat{L} = \hat{M}$ and $L^2 = M^2$, $(**)$ is equivalent to $\det_L (x, y, z) = \det_M (x, y, z)$. If $L$ and $M$ are Hermitean, then $\det_L = \det_M$ on sets of cardinality 1 and 2 if and only if $\hat{L} = \hat{M}$ and $|L| = |M|$. Slightly less obvious property are given in the following proposition.

**Proposition 3.1.** Let $L$ and $M$ be bi-functions on $X$. Then if $\det_L = \det_M$, then $\det_{L+\text{diag} h} = \det_{M+\text{diag} h}$, for any $h : X \rightarrow \mathbb{C}$.

In order to prove the proposition, we will need the following lemma.
Lemma 3.2. Let $M$ be a $n \times n$ complex matrix, let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{C}^n$ be a row and let $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{C}^n$ be a column. Then for any $a, b \in \mathbb{C}$ we have
\[
\det \begin{bmatrix} a & \mu \\ \nu & M \end{bmatrix} - \det \begin{bmatrix} b & \mu \\ \nu & M \end{bmatrix} = (a - b) \det M.
\]

Proof. Using the fact that determinant is a polylinear functional we get
\[
\det \begin{bmatrix} a & \mu \\ \nu & M \end{bmatrix} - \det \begin{bmatrix} b & \mu \\ \nu & M \end{bmatrix} = \det \begin{bmatrix} a - b & 0 \nu \\ \nu & M \end{bmatrix} = (a - b) \det M,
\]
where the last equality follows from the Laplace expansion over the first row. \qed

Proof of Proposition 3.1. Let $L$ and $M$ be bi-functions on $X$ such that $\det_L = \det_M$. Let us start with showing that for any $x \in X$ and $\alpha \in \mathbb{C}$ we have $\det_{L_1} = \det_{M_1}$, where $L_1 = L + \alpha \text{diag} \delta_x$ and $M_1 = M + \alpha \text{diag} \delta_x$. Let $x_1, \ldots, x_n \in X$ be distinct. If this collection of points does not contain $x$, then $L_1|_{\{x_1, \ldots, x_n\}} = L|_{\{x_1, \ldots, x_n\}}$ and $M_1|_{\{x_1, \ldots, x_n\}} = M|_{\{x_1, \ldots, x_n\}}$. Hence,
\[
\det_{L_1} (x_1, \ldots, x_n) = \det_L (x_1, \ldots, x_n) = \det_M (x_1, \ldots, x_n) = \det_{M_1} (x_1, \ldots, x_n).
\]
If $x$ is present among $x_1, \ldots, x_n$, without loss of generality we may assume that $x_1 = x$. In this case it follows from lemma that
\[
\det_{L_1} (x_1, \ldots, x_n) - \det_L (x_1, \ldots, x_n) = \alpha \det_L (x_2, \ldots, x_n)
\]
\[
= \alpha \det_M (x_2, \ldots, x_n) = \det_{M_1} (x_1, \ldots, x_n) - \det_M (x_1, \ldots, x_n),
\]
from where $\det_{L_1} (x_1, \ldots, x_n) = \det_{M_1} (x_1, \ldots, x_n)$.

Now, let us show that for any distinct $x_1, \ldots, x_n \in X$ we have
\[
\det_{L + \text{diag} h} (x_1, \ldots, x_n) = \det_{M + \text{diag} h} (x_1, \ldots, x_n).
\]
Recursively, define a sequence $\{L_i\}_{i=0}^n$ of bi-functions on $X$ by $L_0 = L$ and $L_i = L_{i-1} + h(x_i) \delta_{x_i}$, for every $i \in \{1, \ldots, n\}$. Define $\{M_i\}_{i=0}^n$ analogously. Using induction and the previous step we get that $\det_{L_i} = \det_{M_i}$, for every $i \in \{1, \ldots, n\}$. Also, note that $L_n|_{\{x_1, \ldots, x_n\}} = [L + \text{diag} h]|_{\{x_1, \ldots, x_n\}}$ and $M_n|_{\{x_1, \ldots, x_n\}} = [M + \text{diag} h]|_{\{x_1, \ldots, x_n\}}$, and so
\[
\det_{L + \text{diag} h} (x_1, \ldots, x_n) = \det_{L_n} (x_1, \ldots, x_n) = \det_{M_n} (x_1, \ldots, x_n) = \det_{M + \text{diag} h} (x_1, \ldots, x_n).
\]
Since $x_1, \ldots, x_n$ were chosen arbitrarily, we conclude that $\det_{L + \text{diag} h} = \det_{M + \text{diag} h}$. \qed

From (4) it follows that if $M$ to $L$ are reciprocal rescalings, then $\det_L = \det_M$. It turns out that in the class of symmetric bi-functions the last assertion can be reversed.

Theorem 3.3. Symmetric bi-functions $L$ and $M$ on a set $X$ are $\pm 1$ symmetric rescalings if and only if $\det_L = \det_M$.
This result for the case when $X$ is finite was first proven in [6] and the in [10] in relation to Principal Minor Assignment Problem. In [11] the algorithmical side of the problem was considered. In the next section we adapt combinatorial proof from [6] and [11] to the infinite case. However, in Section 4 we will present a geometric interpretation of the theorem and give an alternative geometric proof under some restrictions. Note that since $\det_L = \det_{L'}$ the result is specific to the case of symmetric bi-functions, although under some additional assumptions $\det_L = \det_M$ implies that either $L$ and $M$, or $L'$ and $M$ are $\pm 1$ symmetric rescalings (see e.g. [4] which discusses the skew-symmetric case, and also [9]). An example of Hermitean bi-functions $L$ and $M$ on $\{1, 2, 3, 4\}$ such that $\det_L = \det_M$, but neither $M$ nor $M'$ is a $\pm 1$ symmetric rescaling of $L$ are given by the matrices

$$
\begin{bmatrix}
4 & e^{i\pi} & 1 & 1 \\
e^{-i\pi} & 4 & 1 & e^{i\pi} \\
1 & 1 & 4 & e^{i\pi} \\
1 & e^{-i\pi} & e^{-i\pi} & 4
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
4 & e^{i\pi} & 1 \\
e^{-i\pi} & 4 & 1 & e^{i\pi} \\
1 & 1 & 4 & e^{i\pi} \\
1 & e^{-i\pi} & e^{-i\pi} & 4
\end{bmatrix}.
$$

**Proof of the theorem.** In order to prove the theorem we will need the following technical lemma.

**Lemma 3.4.** Let $n \geq 3$ and let $a_1, ..., a_n, b_1, ..., b_n$ and $c_1, ..., c_n$ be complex numbers, such that $b_i = \pm c_i$, for every $i \in \overline{1, n}$. Then $\prod_{i=1}^{n} b_i = \prod_{i=1}^{n} c_i$ if and only if

$$
\begin{vmatrix}
a_1 & b_1 & 0 & \cdots & 0 & b_n \\
b_1 & a_2 & b_2 & \ddots & \vdots & \vdots \\
0 & b_2 & \ddots & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & b_{n-2} & 0 \\
b_n & 0 & \cdots & 0 & b_{n-1} & a_n
\end{vmatrix}
= \det
\begin{vmatrix}
a_1 & c_1 & 0 & \cdots & 0 & a_n \\
c_1 & a_2 & c_2 & \ddots & \vdots & \vdots \\
0 & c_2 & \ddots & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & c_{n-2} & 0 \\
c_n & 0 & \cdots & 0 & c_{n-1} & a_n
\end{vmatrix}.
$$

**Proof.** Denote the determinants in the statement by $d_n (a_1, ..., a_n, b_1, ..., b_n)$ and $d_n (a_1, ..., a_n, c_1, ..., c_n)$. For $a_1, a_2, b \in \mathbb{C}$ define $d_2 (a_1, a_2, b, 0) = a_1a_2 - b^2$. It is clear that $d_2 (a_1, a_2, b, 0) = d_2 (a_1, a_2, -b, 0)$.

The proof is done by induction. Here we only provide a sketch. The case when $n = 3$ is a simple computation. For $n > 3$ expanding the determinant by the first row, and then by the first columns of the (second and third) obtained matrices we get the recursive formula

$$
d_n (a_1, ..., a_n, b_1, ..., b_n) + 2 (-1)^n \prod_{i=1}^{n} b_i = a_1d_{n-1} (a_2, ..., a_n, b_2, ..., b_{n-1}, 0)
-b_1^2d_{n-2} (a_3, ..., a_n, b_3, ..., b_{n-1}, 0) - b_2^2d_{n-2} (a_2, ..., a_{n-1}, b_2, ..., b_{n-2}, 0).
$$

Using this formula and the hypothesis of induction the result follows. \qed
Recall that an induced cycle in a graph is a cycle that does not contain edges between non-adjacent vertices. Theorem 3.3 follows immediately from the following result, essentially proven in [6].

**Proposition 3.5.** Let $L$ be a symmetric bi-function on a set $X$ such that the length of the induced cycles in $X_L$ is less than $l \in \mathbb{N} \cup \{\infty\}$. Then any symmetric bi-function $M$ on $X$ is a $\pm 1$ symmetric rescaling of $L$ whenever $\det_L = \det_M$ on all of the subsets of $X$ of cardinality less than $l$.

**Proof.** Since $\det_L = \det_M$ on all of the subsets of $X$ of cardinality less than $l > 2$, it follows that $L = M$ and $L^2 = M^2$. In the light of Proposition 2.1 we can assume that $X_L = X_M$ is connected. Let us start with showing that if $x_0, x_1, \ldots, x_n = x_0$, $n > 2$, is a cycle in $X_L$, then $\prod_{i=0}^{n-1} M(x_i, x_{i+1}) = 1$. In the case when it is an induced cycle we have $n < l$ and the claim follows from the Lemma 3.4 applied to the numbers $a_i = L(x_i, x_i) = M(x_i, x_i)$, $b_i = L(x_i, x_{i+1})$ and $c_i = M(x_i, x_{i+1})$, $i \in \overline{0, n-1}$.

Now argue by induction. For $n = 3$ any cycle is induced, and so the equality holds. Assume that the claim is true for all $k \in \overline{3, n}$ and let $x_0, x_1, \ldots, x_n, x_{n+1} = x_0$ be a cycle. Since we only need to consider non-induced cycles, without loss of generality we may assume that there is $k \in \overline{2, n-1}$ such that $x_0$ and $x_k$ are joined with an edge. Then, from the hypothesis of induction applied to the cycles $x_0, x_1, \ldots, x_k, x_0$ and $x_0, x_k, x_{k+1}, \ldots, x_n, x_{n+1} = x_0$ we have that

$$\frac{M(x_0, x_k)^2}{L(x_0, x_k)^2} \prod_{i=0}^{n-1} \frac{M(x_i, x_{i+1})}{L(x_i, x_{i+1})} = 1.$$ 

Hence, the claim follows since $L(x_0, x_k)^2 = M(x_0, x_k)^2 \neq 0$.

Fix $z \in X$. Define $f : X \to \{-1, 1\}$ by $f(x) = \prod_{i=0}^{n-1} \frac{M(x_i, x_{i+1})}{L(x_i, x_{i+1})}$, where $z = x_0, x_1, \ldots, x_n = x$ is a path in $X_L$ from $z$ to $x$ (this product is always equal to $\pm 1$ as $L^2 = M^2$). The function is well-defined, since if $z = x_0, x_1, \ldots, x_n = x$ and $z = y_0, y_1, \ldots, y_m = x$ are two paths, then $z = x_0, x_1, \ldots, x_n = y_m, \ldots, y_0 = z$ is a cycle, and so from the claim above

$$\prod_{i=0}^{n-1} \frac{M(x_i, x_{i+1})}{L(x_i, x_{i+1})} = \left(\prod_{i=0}^{m-1} \frac{M(y_i, y_{i+1})}{L(y_i, y_{i+1})}\right)^{-1} = \prod_{i=0}^{m-1} \frac{M(y_i, y_{i+1})}{L(y_i, y_{i+1})}.$$ 

Now let us show that $M = f \otimes f L$. If $x, y \in X$ are not joined by an edge in $X_L$, then $M(x, y) = 0 = f(x)f(y)L(x, y)$. Assume that $x, y \in X$ are joined by an edge. Let $z = x_0, x_1, \ldots, x_n = x$ be a path from $z$ to $x$. Then, $z = x_0, x_1, \ldots, x_n = x, x_{n+1} = y$ is a path from $z$ to $y$. Hence, $f(y) = \frac{M(x, y)}{L(x, y)} f(x)$, and since $f(x) = \pm 1 = \frac{1}{f(x)}$ we conclude that $M(x, y) = f(x)f(y)L(x, y)$. \hfill \Box
One can ask what other characteristics of the graph \( X_L \) can be used to reduce the size of sets on which \( \det_L = \det_M \) have to be tested in order to conclude that symmetric \( L \) and \( M \) are \( \pm 1 \) symmetric rescalings. Such characteristic is the diameter, i.e. the supremum of the length of the shortest path between two vertices.

**Proposition 3.6.** Let \( L \) be a symmetric bi-function on a set \( X \) such that the the diameter of every component in \( X_L \) is less than \( l \in \mathbb{N} \cup \{\infty\} \). Then any symmetric bi-function \( M \) on \( X \) is a \( \pm 1 \) symmetric rescaling of \( L \) whenever \( \det_L = \det_M \) on all of the subsets of \( X \) of cardinality less than \( 2l \).

**Proof.** We will provide a sketch of the proof. Again, we only need to consider the case when \( X_L \) is connected.

In the same way as above one can show that if \( x_0, x_1, ..., x_n = x_0, 2l > n > 2 \), is a cycle in \( X \), then \( \prod_{i=0}^{n-1} \frac{M(x_i, x_{i+1})}{L(x_i, x_{i+1})} = 1 \). Fix some \( z \in X \) and define \( f : X \to \{-1, 1\} \) by

\[
f(x) = \prod_{i=0}^{n-1} \frac{M(x_i, x_{i+1})}{L(x_i, x_{i+1})},
\]

where \( z = x_0, x_1, ..., x_n = x \) is a shortest path in \( X_L \) from \( z \) to \( x \) (and so \( n < l \)). Analogously to the previous proof we can show that \( f \) is well-defined.

For \( x, y \in X \) we need to show that if \( (x, y) \) is an edge in \( X_L \), then \( M(x, y) = f(x) f(y) L(x, y) \). Let \( z = x_0, x_1, ..., x_n = x \) and \( z = y_0, y_1, ..., y_m = y \) be shortest paths from \( z \) to \( x \) and \( y \). Then using the fact that \( z = x_0, x_1, ..., x_n = x, y = y_m, ..., y_1, y_0 = z \) is a cycle of length at most \( 2l - 1 \), from the claim above we can deduce the required equality. \( \square \)

**Remark 3.7.** It is possible for a graph to have an infinite diameter but contain only small induced cycles. Indeed, the graph of a bi-function \( L \) from Example \( 3.9 \) below has an infinite diameter and no induced cycles. It is also possible for a graph of a small diameter to contain arbitrarily large induced cycles. Indeed, consider a disconnected union of the cycles of all possible length (starting with 3), choose a vertex in the first cycle and join that vertex with every other vertex of the union. The obtained graph contains induced cycles of all lengths, but its diameter is 2. \( \square \)

In the case when \( L \) and \( M \) do not vanish, i.e. \( X_L = X_M \) is complete, the diameter of \( X \) is 1. Hence, being \( \pm 1 \) symmetric rescalings is equivalent to the fact that \( \det_L = \det_M \) on sets of cardinality up to 3, which gives us the following criterion.

**Corollary 3.8.** Let \( L \) and \( M \) be symmetric bi-functions on a set \( X \) which do not vanish and such that \( (***) \) folds for any \( x, y, z \in X \). Then \( L \) and \( M \) are \( \pm 1 \) symmetric rescalings.

The following example shows that in general we cannot conclude that \( L \) and \( M \) are \( \pm 1 \) symmetric rescalings when \( \det_L = \det_M \) on small sets. In particular, if \( X \) is infinite, in order to decide whether \( L \) and \( M \) are \( \pm 1 \) symmetric rescalings we have to guarantee \( \det_L = \det_M \) on sets of arbitrary size.

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Example 3.9. For every $n > 2$ consider the $n \times n$ matrices

\[
L^+_n = \begin{bmatrix}
4 & 1 & 0 & \cdots & 0 & 1 \\
1 & 4 & 1 & \ddots & \ddots & 0 \\
0 & 1 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & 1 & 4 & 1 & 0 \\
0 & \ddots & \ddots & 1 & 4 & 1 \\
1 & 0 & \cdots & 0 & 1 & 4 \\
\end{bmatrix}
\quad \text{and} \quad
L^-_n = \begin{bmatrix}
4 & 1 & 0 & \cdots & 0 & -1 \\
1 & 4 & 1 & \ddots & \ddots & 0 \\
0 & 1 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & 1 & 4 & 1 & 0 \\
0 & \ddots & \ddots & 1 & 4 & 1 \\
-1 & 0 & \cdots & 0 & 1 & 4 \\
\end{bmatrix}.
\]

Applying Lemma 3.4 one can show that all proper principal minors of these matrices coincide, but $\det L^+_n \neq \det L^-_n$, and so $L^+_n$ and $L^-_n$ are not $\pm 1$ symmetric rescalings.

Consider a “basal” bi-function $L$ on $X = \mathbb{N}$ defined by

\[
L(m, n) = \begin{cases}
4 & m = n \\
1 & |m - n| = 1 \\
0 & |m - n| > 1
\end{cases}
\]

The graph $X_L$ is the infinite path $1, 2, 3, \ldots$. By adding extra edges to this graph we will increase its complexity. Namely, for any $A \subset \{2, 3, 4, \ldots\}$ define $L_A$ by

\[
L_A(m, n) = \begin{cases}
L(m, n) + 1 & m + n = (m - n)^2 \text{ and } |m - n| \in A \\
L(m, n) - 1 & m + n = (m - n)^2 \text{ and } |m - n| \in \mathbb{N} \setminus A \\
L(m, n) & \text{otherwise}
\end{cases}
\]

Note that $m + n = (m - n)^2$ means that $m = \frac{(m-n)^2 + (m-n)}{2}$ and $n = \frac{(m-n)^2 - (m-n)}{2}$. Therefore, the graph $X_{L_A}$ consists of the infinite path $1, 2, 3, \ldots$ with additional edges $\left(\frac{k^2 - k}{2}, \frac{k^2 + k}{2}\right)$, for $k > 1$. Hence, for every $k \in A$ we have $[L_A(m, n)]_{m,n=k^2-k}^{k^2+k} = L^+_k$ and for $k > 1, k \notin A$ we have $[L_A(m, n)]_{m,n=k^2-k}^{k^2+k} = L^-_k$. Thus, $L_A$ and $L_B$ are $\pm 1$ symmetric rescalings, for $A, B \subset \{2, 3, 4, \ldots\}$ if and only if $A = B$, and in order to guarantee that we have to check that $\det L_A = \det L_B$ on sets of arbitrary size.

4 Geometric interpretation

In this section we will consider geometric consequences of Theorem 3.3. While general bi-functions are devoid of geometric meaning, we will focus on a more special class of them.

Positive semi-definite matrices and kernels. By a Hilbert space we will mean either finite- or infinite-dimensional complete inner product space over either $\mathbb{R}$ or $\mathbb{C}$. However, we will sometimes focus on the real Hilbert spaces only. In both cases we will use the term unitary operator for a surjective isometric operator.
Recall that a \( n \times n \) complex matrix \( M \) is called positive (semi-) definite if for every row \( (\mu_1, ..., \mu_n) \in \mathbb{C}^n \setminus \{0\} \) we have that \( \mu M \mu^* \) is positive (non-negative), where \( \mu^* \) is the column \( (\overline{\mu_1}, ..., \overline{\mu_n}) \). By Sylvester’s criterion (see [7, Theorem 7.2.5]) a matrix is positive (semi-) definite if and only if all of its principal minors are positive (non-negative). Hence, the diagonal entries of a positive (semi-) definite matrix are positive (non-negative). On the other hand, every Hermitian strictly diagonally dominant matrix with positive diagonal elements is positive definite (see [7, Corollary 7.2.3]). A diagonal matrix is positive (semi-) definite if and only if its entries are positive (non-negative).

It is easy to see that any positive semi-definite matrix is Hermitian. In particular, a \( n \times n \) real matrix \( M \) is positive (semi-) definite if and only if it is symmetric and for every row \( (\mu_1, ..., \mu_n) \in \mathbb{R}^n \setminus \{0\} \) we have that \( \mu M \mu^* \) is positive (non-negative). However, the latter condition alone is not sufficient for positive semi-definiteness.

If \( M \) and \( L \) are positive semi-definite, then \( \alpha M + \beta L \) is positive semi-definite, for any \( \alpha, \beta > 0 \); this sum is in fact positive definite if either \( L \) or \( M \) is positive definite.

A bi-function \( K \) on \( X \) is called positive (semi-) definite if for any \( x_1, ..., x_n \in X \), the matrix \( [K(x_i, x_j)]_{i,j=1}^n \) is positive (semi-) definite. Note that positive (semi-) definite bi-functions are traditionally called positive (semi-) definite kernels. From the properties of positive (semi-) definite matrices discussed above, we have the following list of properties:

- \( K : X \times X \to \mathbb{C} \) is positive (semi-) definite if and only if \( \det_K \) is a positive (non-negative) function on \( \text{Fin}(X) \); in this case \( K \) is Hermitian, and \( K' = \overline{K} \) is also positive (semi-) definite, as well as \( \text{Re} \ K \) and \( \alpha K \), for any \( \alpha > 0 \).

- A sum and a product of positive (semi-) definite kernels are positive (semi-) definite; a sum of a positive semi-definite and a positive definite kernels is positive definite.

- A pointwise limit of positive semi-definite kernels is positive semi-definite.

It also follows from \([\ast]\) that any Hermitian rescaling of a positive (semi-) definite kernel is positive (semi-) definite; in particular, \( \text{diag}f \) is positive (semi-) definite for every \( f : X \to (0, +\infty) \) \((f : X \to [0, +\infty))\), and \( f \otimes \overline{f} \) is positive semi-definite for any \( f : X \to \mathbb{C} \).

The importance of the established class of bi-functions is revealed by the following theorem (see [1]).

**Theorem 4.1** (Moore-Aronszajn). Let \( X \) be a set. A complex-valued (real-valued) bi-function \( K \) in \( X \) is positive semi-definite if and only if there is a complex (real) Hilbert
space $H$ and a map $\kappa : X \to H$, such that $\text{span } \kappa(X) = H$ and $K(x, y) = \langle \kappa(x), \kappa(y) \rangle$. Furthermore, the pair of $H$ and $\kappa$ is unique up to the unitary equivalence, i.e. if $H_1$ and $\kappa_1$ satisfy the conditions above, for $i = 1, 2$, then there is a unitary operator $T : H_1 \to H_2$ such that $T\kappa_1 = \kappa_2$.

**Remark 4.2.** In the notations of the theorem, $K$ is non-degenerate if and only if $0_H \notin \kappa(X)$, and positive definite if and only $\kappa(X)$ is a linearly independent set.

If $K$ is positive definite on $X$, then $\det_K$ is a positive function of finite subsets of $X$. Then, if $L$ is a rescaling of $K$, it follows from [3.3] that $\det_L$ does not vanish and $\det_L \det_L^{-1}$ is a multiplicative function on $\text{Fin}(X)$. In fact, the converse is also true if we assume that $L$ is symmetric.

**Proposition 4.3.** Let $K$ be a positive definite real-valued kernel on $X$ and let $L$ be a non-degenerate symmetric bi-function on $X$. Then $L$ is a rescaling of $K$ if and only if $\det_L \det_L^{-1}$ is a multiplicative function on $\text{Fin}(X)$.

**Proof.** We only need to prove sufficiency. Assume that $\det_L \det_L^{-1}$ is a multiplicative function on $\text{Fin}(X)$, and so there is $f : X \to \mathbb{C}$ such that for any $x_1, \ldots, x_n \in X$ we have $\det_L(x_1, \ldots, x_n) = \prod_{i=1}^n f(x_i) \det_K(x_1, \ldots, x_n)$. Since $L$ is non-degenerate, it follows that $f$ does not vanish. Let $g : X \to \mathbb{C}^\times$ be such that $g^2 = f$. Then $M = g \otimes gK$ is a symmetric bi-function such that $\det_M = \det_L$. From Theorem 3.3 there is $h : X \to \{-1, 1\}$ such that $L = h \otimes hM$. Hence, $L = gh \otimes ghK$ is a rescaling of $K$.

**Parallelepipeds.** For a finite subset $B = \{v_1, \ldots, v_n\}$ of a real Hilbert space, we adopt the notations $-B = \{-v_1, \ldots, -v_n\}$ and $\sum B = \sum v_i$. Also, for $v \in H$ define $v_B$ to be the projection of $v$ onto $\text{span}B$. Then $\|v - v_B\|$ is the distance from $v$ to the closed subspace $\text{span}B$ of $H$. Note that if $A \subset B$, then $\langle v_B \rangle_A = v_A$, for every $v \in H$. Assume that $B = \{v_1, \ldots, v_n\} \subset H$ is linearly independent. The *parallelepiped* defined by $B$ is a set

$$P(v_1, \ldots, v_n) = P(B) = \left\{ \sum_{i=1}^n \alpha_i v_i \middle| \alpha_i \in [0, 1] \right\}$$

(we will also adopt a convention that $P(\emptyset) = \{0_H\}$). Then $P(B)$ is convex, and its extreme points (vertices) are $\{\sum A \left| A \subset B \right\}$. Hence, there are $2^n$ vertices in total. Note that $\sum A$ is the vertex, located on the opposite “end” to the origin in the “subparallelepiped” $P(A)$, which is a face of $P(B)$. Hence, we have a correspondence between vertices and faces of $P(B)$. For example, $0_H$ corresponds to the face $P(\emptyset)$ and $P(B)$ itself corresponds to the vertex $\sum B$.

The set of edges of $P(B)$ that radiate from $\sum A$ are $-A \cup B\setminus A$. Hence, $P(-A \cup B\setminus A) = P(B) - \sum A$. In essence, the shift $-\sum A$ accounts for change of the point of $H$ which we call the origin.

While the collection $B$ is reconstructible from $P(B)$ as a subset of $H$, it is not reconstructible from $P(B)$ as a geometric figure. Indeed, $P(-A \cup B\setminus A)$ is a parallel
translation of \( P(B) \), for any \( A \subset B \), and if \( T : H \to H \) is an isometric operator, then \( P(TB) = TP(B) \) is isometric to \( P(B) \). Below we will show that isometric parallelepipeds can be transformed into each other by a combination of a unitary operator on \( H \) and a parallel translation.

Define \( V(v_1, ..., v_n) = V(B) = V_n(P(B)) \), where \( V_n \) is the \( n \)-dimensional volume (we will also adopt a convention that \( V(B) = 0 \) if \( B \) is linearly dependant). Then

\[
V(B)^2 = \det [\langle v_i, v_j \rangle]_{i,j=1}^n.
\]

In particular, \( V(v) = \|v\| \) and \( V(v, w) = \sqrt{\|v\|^2\|w\|^2 - |\langle v, w \rangle|^2} \). Also note that for any \( v \in H \) we have

\[
\|v - v_B\| = \frac{V(B \cup \{v\})}{V(B)}.
\]

**Theorem 3.3** for real-valued positive semi-definite kernels. Let \( K \) and \( L \) be non-degenerate positive semi-definite kernels over \( X \) and let \( \kappa : X \to H \setminus \{0_H\} \) and \( \lambda : X \to E \setminus \{0_E\} \) be the maps into Hilbert spaces provided by the Moore-Aronszajn theorem. We can describe the realtion of being a rescaling in terms of \( \kappa \) and \( \lambda \). Namely, \( L = f \otimes T \kappa \), for \( f : X \to \mathbb{C}^\times \), if and only if there is an unitary operator \( T : H \to E \) such that \( \lambda = f \cdot T \kappa \).

Now assume that \( K \) and \( L \) are real-valued (and so symmetric), and \( H \) and \( E \) are real Hilbert spaces. Then \( K \) and \( L \) are \( \pm 1 \) symmetric rescalings if and only if \( \lambda(x) = \pm T \kappa(x) \), for every \( x \in X \). On the other hand we have \( \det_K(x_1, ..., x_n) = V(\kappa(x_1), ..., \kappa(x_n))^2 \), for any \( x_1, ..., x_n \in X \). We can now restate Theorem 3.3 in a geometric form.

**Theorem 4.4.** Let \( H \) be a real Hilbert space and let \( \kappa, \lambda : X \to H \) be such that \( \kappa(X) \) and \( \lambda(X) \) are both linearly independent and \( \text{span} \, \kappa(X) = H \). Then the following are equivalent:

(i) There is an isometry \( T \) on \( H \) such that \( \lambda(x) = \pm T \kappa(x) \), for every \( x \in X \).

(ii) For any distinct \( x_1, ..., x_n \in X \) the parallelepipeds \( P(\kappa(x_1), ..., \kappa(x_n)) \) and \( P(\lambda(x_1), ..., \lambda(x_n)) \) are isometric.

(iii) For any distinct \( x_1, ..., x_n \in X \) the parallelepipeds \( P(\kappa(x_1), ..., \kappa(x_n)) \) and \( P(\lambda(x_1), ..., \lambda(x_n)) \) have the same volume.

**Proof.** (i)\( \Rightarrow \) (ii) follows from the fact that for any linearly independent \( v_1, ..., v_n \in H \) all parallelepipeds \( P(\pm v_1, ..., \pm v_n) \) are parallel translations of each other. (ii)\( \Rightarrow \) (iii) is obvious and (iii)\( \Rightarrow \) (i) follows from Theorem 3.3 and the discussion above (applied to \( H \) and \( E = \text{span} \lambda(X) \)). \( \square \)

**Remark 4.5.** Note that the counterexamples considered in Example 3.9 are all positive definite because symmetric strictly diagonally dominant matrices with non-negative diagonal entries are positive definite. In particular, \( L_3^\pm \) correspond to vectors \( u^\pm, v^\pm, w^\pm \in \).
\[ \mathbb{R}^3 \] of length 2 and such that the angles between \( u^\pm \) and \( w^\pm \) is \( \text{arccos} \frac{\pm 1}{4} \), while the angles between these vectors and \( v^\pm \) is \( \text{arccos} \frac{1}{4} \). Thus, we need to check the equality of volumes of parallelepipeds of all dimensions. \( \square \)

**Equality of parallelepipeds.** In the particular case when \( X \) is finite, Theorem 4.4 turns into a criterion for equality of parallelepipeds.

**Theorem 4.6.** Two parallelepipeds with equal volumes of the corresponding faces are isometric.

Since the result above is a purely geometric fact it is desirable to have a geometric proof for it. We will present a proof which roughly follows the pattern of the proof of Theorem 4.3 but operates with geometric objects instead of matrices and abstract bi-functions. Let \( K \) be a positive semi-definite kernel again and let \( H \) and \( \kappa : X \to H \) be given by Moore-Aronszajn theorem.

Consider the decomposition \( X = \bigcup_{j \in I} X_j \) into the components of the graph \( X_K \) from the geometric viewpoint. It is clear that if \( i \neq j \), then \( \kappa(X_i) \perp \kappa(X_j) \), and so \( H = \bigoplus_{j \in I} \text{span}_K(X_j) \). Therefore, for any collection \( \{\alpha_j\}_{j \in I} \subset T \) the operator \( T = \bigoplus_{j \in I} \alpha_j I_{\text{span}_K(X_j)} \) is a unitary operator on \( H \) such that \( \kappa(X_j) \) belongs to the eigenspace of \( T \) that corresponds to \( \alpha_j \). Conversely, if for some \( j \in I \) every element of \( \kappa(X_j) \) is an eigenvector of an isometry \( T : H \to H \), then the corresponding eigenvalues coincide. This observation is an analogue of Proposition 2.1.

Let us consider a special class of paths in \( X_K \). We will call a finite sequence \( v_1, ..., v_n \in H \) a chain (from \( v_1 \) to \( v_n \)) if \( v_i \perp v_j \) whenever \( |i - j| > 1 \) and \( v_i \notin \text{span} v_{i+1} \), for every \( i \in \{1, n-1\} \). It is clear that if \( x \) and \( y \) are connected by a path in \( X_K \), then \( \kappa(x) \) is connected with \( \kappa(y) \) via a chain of elements of \( \kappa(X) \), which corresponds to a minimal path from \( x \) to \( y \). This configuration has the following property.

**Lemma 4.7.** If \( v_1, ..., v_n \in H \) is a chain, then \( v_1, ..., v_{n-1} \) are linearly independent.

**Proof.** We will show that \( v_k \notin \text{span}\{v_1, ..., v_{k-1}\} \) for every \( 1 < k < n \). Assume that \( v_k = \alpha_1 v_1 + ... + \alpha_{k-1} v_{k-1} \), where \( 1 < k < n \) and \( \alpha_1, ..., \alpha_{k-1} \in \mathbb{C} \). Then each of \( v_1, v_2, ..., v_{k-1} \) are orthogonal to \( v_{k+1} \), and so \( v_k \perp v_{k+1} \), which contradicts the definition of chain. \( \square \)

**Remark 4.8.** It also follows that \( v_1 \notin \text{span}\{v_2, ..., v_{n-1}\} \). Indeed, if \( v_1 = \alpha_2 v_2 + ... + \alpha_k v_k \), and \( \alpha_k \neq 0 \), then \( v_k \in \text{span}\{v_1, ..., v_{k-1}\} \), and so \( k = n \). \( \square \)

Chains help to estimate a dimension corresponding to the components of \( X_K \), which is also utilized in the proof of the following lemma.

**Lemma 4.9.** Let \( B \subset H \) be finite and linearly independent. Let \( u, w \in \text{span}B \) be such that \( u_A \perp w_A \), for any \( A \subset B \). Then there are subsets \( B_u \) and \( B_w \) of \( B \) such that \( u \in \text{span}B_u, w \in \text{span}B_w \) and \( B_u, B_w \) and \( B \setminus (B_u \cup B_w) \) are mutually orthogonal.
Proof. Without loss of generality we may assume that span\(B = H\).

Define \(B_u\) to be the set of all \(v \in B\) such that there is a chain \(u_0 = u, u_1, ..., u_n = v\), where \(u_1, ..., u_n \in B\), and define \(B_u\) analogously. It is clear that \(u \perp B, B_u, w \perp B, B_u\) \(\cap B, B_u \perp B, B_u\). In order to prove the lemma it is enough to show that \(B_u \perp B, B_u\) since then we would have \(u \in (B, B_u)^\perp = \text{span}\(B_u\) and analogously \(w \in \text{span}\(B_w\).\)

We need to show that \(u_1, ..., u_n, w_1, ..., w_m \in B\) are such that \(u_0 = u, u_1, ..., u_n\) and \(w_0 = w, w_1, ..., w_m\) are chains, then \(u_n \perp w_m\). We will use the induction by \(m + n\).

When \(m + n = 0\) this follows from \(u_0 = u = u_B \perp w = w_0\).

Assume the claim holds for \(m + n\) and let \(A = \{u_1, ..., u_n, u_{n+1}, w_1, ..., w_m\} \subset B\) be such that \(u, u_1, ..., u_n, u_{n+1}\) and \(w, w_1, ..., w_m\) are chains. Let \(u_0 = u\) and \(w_0 = w\).

Then it is easy to see that \(u_i \perp w_j\), when \(i \leq n\), and so \(\{u_0, u_1, ..., u_n\} \perp \{w_0, w_1, ..., w_m\}\). All these \(m + n + 2\) vectors belong to \(\text{span}\(A\) whose dimension is \(m + n + 1\).

Since \(u_0, u_1, ..., u_n, w_0, w_1, ..., w_{m-1}\) are linearly independent, it follows that \(w_m \in \text{span}\{w_0, w_1, ..., w_{m-1}\}\). As all of the vectors in the latter span are orthogonal to \(u_{n+1}\), we conclude that \(u_{n+1} \perp w_m\).

Geometric proof of Theorem 4.6. Let \(\{v_1, ..., v_n\}\) and \(\{w_1, ..., w_n\}\) be two linearly independent sets in a real Hilbert space. We will denote the fact that \(V(v_1, ..., v_{i_k}) = V(w_{i_1}, ..., w_{i_k})\), for any \(\{i_1, ..., i_k\} \subset \{1, ..., n\}\) by \(\{v_1, ..., v_n\} \approx \{w_1, ..., w_n\}\). Note that \(\{v_1, ..., v_n\} \approx \{\pm v_1, ..., \pm v_n\}\), for any \(\{v_1, ..., v_n\}\), and any distribution of signs.

After all the preparatory work we have done, let us prove that if \(\{v_1, ..., v_n\} \approx \{w_1, ..., w_n\}\), then there is a unitary operator \(T : H \rightarrow H\) such that \(Tw_i = \pm v_i\), for every \(i \in \{1, n\}\). The proof is done by induction over \(n\). For \(n = 1\) the result holds trivially.

Assume that the claim is true for \(n\) and let \(\{v_0, v_1, ..., v_n\} \approx \{w_0, w_1, ..., w_n\}\). From the hypothesis of induction applied to \(\{v_1, ..., v_n\}\) and \(\{w_1, ..., w_n\}\) there is a unitary operator \(S : H \rightarrow H\) such that \(Sw_i = \pm v_i\) for \(i \in \{1, n\}\). Let \(v'_0 = Sw_0\). Then

\[\{v_0, v_1, ..., v_n\} \approx \{w_0, w_1, ..., w_n\} \approx \{Sw_0, Sw_1, ..., Sw_n\} \approx \{v'_0, v_1, ..., v_n\},\]

and so for any \(A \subset B = \{v_1, ..., v_n\}\) we have

\[\|v_0 - (v_0)_A\| = \|V(A)\| = \|V(A \cup \{v_0\}) = \|V(A \cup \{v'_0\}) = \|v'_0 - (v'_0)_A\| \|V(A).\]

Since we also have \(\|v_0\| = V(\{v_0\}) = V(\{v'_0\}) = \|v'_0\|\), by Pythagoras theorem we conclude that \(\|v_0\| = \|v'_0\|\).

Define \(2u = (v_0)_B + (v'_0)_B\) and \(2w = (v_0)_B - (v'_0)_B\). Then \((v_0)_A = u_A + w_A\) and \((v'_0)_A = u_A - w_A\), for any \(A \subset B\). Therefore, \(\|u_A + w_A\| = \|u_A - w_A\|\), and so \(u_A \perp w_A\). By Lemma 4.9 there are subsets \(B_u\) and \(B_w\) of \(B\) such that \(u \in \text{span}\(B_u\), \(w \in \text{span}\(B_w\) and \(B_u, B_w, B, \cap (B_u \cup B_w)\) are mutually orthogonal.
Since $v' - (v')_B$ and $v - (v)_B$ are both orthogonal to $\text{span}B$, there is a unitary operator $Q$ on $H$ such that $Qv = v$, when $v \in B_u$, $Qv = -v$, when $v \in B \setminus B_u$ and $Q(v' - (v')_B) = v_0 - (v)_B$. Thus, $T = QS$ is a unitary operator on $H$ such that $Tw_i = QS_i = Q(\pm v_i) = \pm v_i$, for $i \in \overline{1,n}$ and

$$Tw_0 = QS_0 = Qv' = T(v' - (v')_B + u - w) = v_0 - (v)_B + u + w = v_0.$$ 

**Remark 4.10.** One may wonder if during the proof of the step of induction we can slightly deform $S$ and find a unitary operator $T$ such that $Tw_i = Sw_i$, for $i \in \overline{1,n}$ and $Tw_0 = \pm v_0$. This is not always possible: consider $H = \mathbb{R}^3$ and let $v_1 = w_1 = [1,0,0]$, $v_2 = w_2 = [0,1,0]$, $v_3 = [1,1,1]$ and $w_3 = [1,-1,1]$. Then there is no isometry $T$ of $H$, which is identity on the first two coordinates and $Tw_3 = \pm v_3$. \qed

## 5 Rescaling on topological spaces

Until now we studied general bi-functions on sets without any additional structure. In this section we will consider the topological aspect of the topic. Let us start with the following question.

**Question 5.1.** If $X$ is a connected topological space and $K$ and $L$ are continuous positive definite and such that $K^2 = L^2$, does it follow that $K = L$?

If $K$ and $L$ are separately continuous non-vanishing bi-functions with $K^2 = L^2$, then either $L = K$ or $L = -K$. Indeed, for every $y$ we have $\frac{L(y)}{K(y)} = \pm 1$, and from the continuity we get that either $L(\cdot, y) = K(\cdot, y)$, or $L(\cdot, y) = -K(\cdot, y)$. Analogously, either $L(x, \cdot) = K(x, \cdot)$, or $L(x, \cdot) = -K(x, \cdot)$, for every $x \in X$. Combining these assertions we conclude that either $L = K$ or $L = -K$. If we also assume that both $K$ and $L$ are positive semi-definite then $K = L$, since $-K$ is not positive semi-definite (unless $K = L \equiv 0$).

It is clear that we heavily relied on the assumption that $K$ and $L$ do not vanish, and the answer is in fact negative without this additional condition. Hence, the properties of $X_K = X_L$ come into play in this question. In this sections we will investigate the connection between the topology of $X$ and the graph structure of $X_L$ when $L$ satisfies certain minimal assumptions of continuity.

**A graph of a continuous bi-function.** Recall that the (closed) neighborhood of a vertex in a graph is the set of all adjacent vertices (together with the original vertex). More generally, the (closed) $n$-th neighborhood of a vertex in a graph is the set of all vertices of distance $1, 2, \ldots, n$ $(0, 1, \ldots, n)$. The component that contains a vertex is the union of all of its closed neighborhoods.

Let $L$ be a bi-function on a set $X$. For $y \in X$ define $U_y = \{x \in X | L(x, y) \neq 0\}$ and $U^y = \{x \in X | L(y, x) \neq 0\}$. Note that $x \in U_y \iff y \in U^x$ and $L(y, y) \neq 0 \iff y \in U_y \iff y \in U^y$. In fact, $U_y \cup U^y$ is either the neighborhood, or the closed neighborhood of $y$ in $X_L$, depending on whether $L(y, y) = 0$. \textit{19}
Assume that $M$ and $L$ are bi-functions on $X$ and $y \in X$ and $y \in Y \subset U_y$ are such that there are $f, g : X \to \mathbb{C}^\times$ are such that $f(x) g(z) L(x, z) = M(x, z)$, for every $x, z \in Y$. Then for every $x \in Y$ we have $f(x) = \frac{\lambda M(x, y)}{L(x, y)}$, for some $\lambda \in \mathbb{C}^\times$, and so $f$ is determined uniquely on $Y$ up to a constant multiple. Analogously, $g$ is determined uniquely on $Y$ up to a constant multiple in the case when $y \in Y \subset U_y$.

Now assume that $X$ is a topological space and $x, y \in X$ are such that $L(x, y) \neq 0$. If $L$ is continuous in the first variable at $(x, y)$, then $x \in \text{int} U_y$. Note that every point of $U_y$ is of distance at most 2 from $x$ in $X_L$, and so the closed second neighborhood of $x$ in the topological space $X$. In particular, if in this case $x = y$, then $y \in \text{int} U_y$, and so the closed neighborhood of $y$ in $X_L$ is a also a topological neighborhood. This observation leads to the following results.

**Proposition 5.2.** Let $X$ be a topological space and let $L$ be a bi-function on $X$. Then the components of $X_L$ are open once $L$ satisfies one of the following properties:

(i) $L$ is non-degenerate and separately continuous in the first variable at the points of the diagonal;

(i') $L$ is non-degenerate and separately continuous in the second variable at the points of the diagonal;

(ii) $L$ is separately continuous in the first variable and for every $x \in X$ there is $y \in X$ such that $L(x, y) \neq 0$.

(ii') $L$ is separately continuous in the second variable and for every $x \in X$ there is $y \in X$ such that $L(y, x) \neq 0$.

(iii) $L$ is separately continuous and the graph $X_L$ has no isolated vertices.

Another consequence of separate continuity of bi-functions is the continuity of the functions that rescale them.

**Proposition 5.3.** Let $X$ be a topological space and let $L$ and $M$ be bi-functions on $X$, which satisfy the same of the conditions (i), (ii) or (iii) of the preceding proposition. If $f, g : X \to \mathbb{C}^\times$ are such that $(f, g)$ rescales $L$ to $M$, then $f$ is continuous.

**Proof.** We will only provide a proof for the case when both $L$ and $M$ satisfy the condition (ii) of the preceding proposition, since the other proofs are similar. Fix $x \in X$. There is $y \in X$ such that $L(x, y) \neq 0$, and so $x \in \text{int} U_y$. Recall that on $U_y$ we have $f = \frac{\lambda M(\cdot, y)}{L(\cdot, y)}$, for some $\lambda$ that depends on $y$. Since both $M(\cdot, y)$ and $L(\cdot, y)$ are continuous, we conclude that $f$ is continuous at $x$. Since $x$ was chosen arbitrarily we conclude that $f$ is continuous on $X$. □

*Remark 5.4.* In the notation of the proposition, if $L$ and $M$ satisfy the same of the conditions (i'), (ii') or (iii) of Proposition 5.2 then $g$ is continuous. □

We can now show that if two bi-functions are “almost rescalings”, then they are rescalings.
Proposition 5.5. Let $X$ be a topological space and let $L$ and $M$ be separately continuous non-degenerate bi-functions on $X$. Assume that $L$ and $M$ are rescalings on a dense set $Y \subset X$, and also for any $x \in X$ there is a neighborhood $U$ of $x$ such that $L$ and $M$ are rescalings on $U$. Then $L$ and $M$ are rescalings on $X$.

Proof. First, note that we do not have $X_L = X_M$ yet, and so $U_x$ and $U^x$, for $x \in X$, might be different with respect to $L$ and $M$. However, since these bi-functions are separately continuous and non-degenerates, these sets are open neighborhoods of $x$. Let $V_x$ be the intersection of all these four sets ($U_x$ and $U^x$ with respect to $L$, and $U_x$ and $U^x$ with respect to $M$), which is an open neighborhood of $x$.

Let $f, g : X \to \mathbb{C}^\times$ be such that $(f, g)$ rescales $L$ to $M$ on $Y$. Let $x \in X$ and let $U$ be an open neighborhood of $x$ such that $L$ and $M$ are rescalings on $U$. Then $V_x \cap U$ is an open neighborhood of $x$, and since $Y$ is dense, there is $y \in Y \cap V_x \cap U$. Therefore, $x \in V_y$, and so $W_x = V_x \cap V_y \cap U$ is an open neighborhood of both $x$ and $y$. Let $f_x, g_x : U \to \mathbb{C}^\times$ such that $(f_x, g_x)$ rescales $L$ to $M$ on $U$ with $f_x(y) = f(y)$. Then both $(f, g)$ and $(f_x, g_x)$ rescale $L$ to $M$ on $W_x \cap Y$ with $f_x(y) = f(y)$, and $y \in W_x \cap Y \subset V_y$. Hence, $f_x|_{W_x \cap Y} = f|_{W_x \cap Y}$, and $g_x|_{W_x \cap Y} = g|_{W_x \cap Y}$.

Let $x, z \in X$ and let $W_x$ and $W_z$ be as constructed above. Then $f_x|_{W_x \cap W_z} = f|_{W_x \cap W_z}$, and both $f_x$ and $f_z$ are continuous due to Proposition 5.3. Since $W_x \cap Y \cap W_z$ is dense in $W_x \cap W_z$, we conclude that $f_x|_{W_x \cap W_z} = f_z|_{W_z \cap W_x}$.

Thus, the extensions of $f$ and $g$ on $X$ given by $f(x) = f_x(x)$ and $g(x) = g_x(x)$, for $x \in X$, are well defined and continuous. It is left to show that $(f, g)$ rescales $L$ to $M$ on $X$. We know that $f(x) g(y) L(x, y) = M(x, y)$, whenever $x, y \in Y$. Fix $y \in Y$. Since $L(\cdot, y)$, $M(\cdot, y)$ and $f$ are continuous, and $Y$ is dense, it follows that $f(x) g(y) L(x, y) = M(x, y)$, for every $x \in X$. Applying the same argument to the second variable, we conclude that $(f, g)$ rescales $L$ to $M$ on $X$. □

Remark 5.6. If in the statement of the proposition $L$ and $M$ were symmetric / Hermitian/ reciprocal rescalings on $Y$, then they are symmetric / Hermitian/ reciprocal rescalings on $X$. □

Rescaling and compactness. Consider how compactness of $X$ is reflected on the properties of $X_L$.

Proposition 5.7. Let $L$ be a bi-function on a compact Hausdorff space $X$ which satisfies one of the conditions of Proposition 5.2. Then $X_L$ has finite number of components and every component has finite diameter.

Proof. Since the components of $X_L$ are open disjoint sets, which cover $X$, it follows that there is a finite number of them. Let us prove the second claim. Without loss of generality we may assume that $X_L$ is connected.

For $x \in X$ and $n \in \mathbb{N}$ let $V_x^n$ be the closed $n$-th neighborhood of $x$ in $X_L$. Clearly, $x \in V_x^n \subset V_x^{n+1}$, and if $y \in V_x^n$, then $V_y^m \subset V_y^{m+n}$, for any $m \in \mathbb{N}$.

Recall that if either of the conditions of Proposition 5.2 hold, then $x \in \text{int} V_x^2$, and so $V_x^n \subset \text{int} V_x^{n+2}$. Therefore, \( \bigcup_{n \in \mathbb{N}} \text{int} V_x^n = \bigcup_{n \in \mathbb{N}} V_x^n = X \), where the last equality follows
from the fact that \( X_L \) is connected. Hence, \( \{\text{int}V^n_x\}_{n\in\mathbb{N}} \) is an increasing sequence of open sets, that cover \( X \). Since \( X \) is compact we conclude that there is \( n \in \mathbb{N} \) such that \( X = V^n_x \), and so the diameter of \( X_L \) is at most \( 2n \).

Combining this proposition with Proposition \( \ref{5.6} \) we obtain the following fact.

**Corollary 5.8.** Let \( L \) be a bi-function on a compact Hausdorff space \( X \) which satisfy one of the conditions of Proposition \( \ref{5.2} \). Then there is \( l \in \mathbb{N} \) such that any symmetric bi-function \( M \) on \( X \) is a \( \pm 1 \) symmetric rescaling of \( L \) whenever \( \det_L = \det_M \) on all of the subsets of \( X \) of cardinality less than \( l \).

**Proposition 5.9.** Let \( L \) be a continuous non-degenerate bi-function on a compact Hausdorff space \( X \). Then there is \( l \in \mathbb{N} \) such that \( X_L \) does not contain induced cycles of length exceeding \( l \).

**Proof.** Since \( L \) is continuous and non-degenerate, for every \( x \) there is an open neighborhood \( V^x \) of \( x \) such that \( L(y, z) \neq 0 \), for every \( y, z \in V^x \). Then \( V^x_L \) is a complete subgraph of \( X_L \). Since \( X \) is compact there are \( x_1, \ldots, x_n \in X \) such that \( X = \bigcup_{i=1}^{n} V_{x_i} \).

Assume that \( y_0, y_1, \ldots, y_m = y_0 \) is an induced cycle in \( X_L \). Let \( i \in 0, m - 1 \) and let \( y_i \in V_{x_k} \), for some \( k \in 1, n \). Then \( y_i \) is not joined by an edge with \( y_j \), unless \( j = i \pm 1 \) mod \( n \), and so the cardinality of the set \( \{ j \in 0, m - 1 | y_j \in V_k \} \) is at most \( 3 \). Hence, \( m \leq 3n \), and so the length of the induced cycles in \( X_L \) does not exceed \( l = 3n \). \( \square \)

The following example shows that continuity cannot be replaced by separate continuity in the statement of the preceding proposition.

**Example 5.10.** Let \( X = [0, 1] \). Let \( Y = X \times X \setminus \{(0, 0), (1, 1)\} \). Consider \( Z = \{(x, y) \in Y \mid y \leq x^2, x^2 + (1 - y)^2 \leq 1\} \cup \{(x, y) \in Y \mid x \leq y^2, y^2 + (1 - x)^2 \leq 1\} \), which is a closed subset of \( Y \). It is not difficult to construct a continuous function \( L : Y \to [0, 1] \) such that \( L^{-1}(0) = Z \), and

\[
L(x, 0) = L(x, 1) = L(0, x) = L(1, x) = L(x, x) = L(0, 1) = L(1, 0) = 1,
\]

for every \( x \in (0, 1) \). Extending \( L \) to be defined on \( X \times X \) by \( L(0, 0) = L(1, 1) = 1 \) we obtain a separately continuous non-degenerate bi-function on \([0, 1] \). We will show however that \( X_L \) contains induced cycles of arbitrary length.

Fix \( x \in (0, 1) \) and \( y \in (x^2, x) \). Consider a sequence \( x, y, x^2, y^2, x^4, y^4, \ldots, z \) where \( z \) is the first element of the sequence such that \( x^2 + (1 - z)^2 > 1 \). It is easy to see that we have obtained an induced cycle in \( X_L \), of length approximately equal to \( 2t \), where \( x^{2^t} = 1 - \sqrt{1 - x^2} \). Then \( 2t = \frac{\ln(1 - \sqrt{1 - x^2})}{\ln x} \), and since

\[
\lim_{x \to 1^-} \frac{\ln(1 - \sqrt{1 - x^2})}{\ln x} = \lim_{x \to 1^-} \frac{\frac{\sqrt{1 - x^2}}{x}}{-\frac{\sqrt{1 - x^2}}{x}} = \lim_{x \to 1^-} \frac{1}{\sqrt{1 - x^2}} = +\infty,
\]

we conclude that the length of the considered cycle grows infinitely, as we choose \( x \) closer to \( 1 \). \( \square \)
Rescaling and connectedness. Let us bring connectedness into the picture.

**Proposition 5.11.** Let \( L \) be a bi-function on a connected topological space \( X \) which satisfies one of the conditions of Proposition 5.2. Then:

(i) \( X_L \) is connected.

(ii) If \( M \) is a bi-function on \( X \) which satisfy one of the conditions of Proposition 5.2, and \( L \) and \( M \) are \( \pm 1 \) symmetric rescaling, then \( L = M \).

**Proof.**

(i): From Proposition 5.2 the components of \( X_L \) are open disjoint sets, which cover \( X \). Since \( X \) is connected it follows that there is just one of them.

(ii): Assume that \( f : X \to \{-1,1\} \) symmetrically rescales \( L \) to \( M \). Then \( f \) is continuous, according to Proposition 5.3. Since a continuous function on a connected space with values in \( \{-1,1\} \) has to be constant, we conclude that either \( M = 1^2 L \), or \( M = (-1)^2 L \).

Combining part (ii) with Corollary 5.8 we get stronger version of the latter.

**Corollary 5.12.** Let \( L \) be a bi-function on a connected compact Hausdorff space \( X \) which satisfy one of the conditions of Proposition 5.2. Then there is \( l \in \mathbb{N} \) such that any symmetric bi-function \( M \) on \( X \) is equal to \( L \) whenever \( \text{det}_L = \text{det}_M \) on all of the subsets of \( X \) of cardinality less than \( l \).

**Corollary 5.13.** Let \( L \) and \( M \) be non-degenerate bi-functions on a connected topological space \( X \), which are rescalings. Assume that \( L \) and \( M \) are both separately continuous in the first or second variable at the points of the diagonal. Then:

(i) For every \( x \in X \) there are unique \( f, g : X \to \mathbb{C}^\times \) such that \( (f, g) \) rescales \( L \) to \( M \) with \( f(x) = 1 \).

(ii) If \( L \) and \( M \) are real-valued positive semi-definite and \( |L| = |M| \), then \( L = M \).

**Proof.**

(i) follows from combining part (i) of the preceding proposition with part (ii) of Corollary 2.4.

(ii): We can assume that \( L \neq 0 \neq M \). Since \( L \) and \( M \) are real-valued positive semi-definite with \( |L| = |M| \), it follows that they are symmetric and \( L^2 = M^2 \). Hence, from part (i) of Proposition 2.5 either \( M \), or \( -M \) is a \( \pm 1 \) symmetric rescaling of \( L \). However, any rescaling of a positive semi-definite kernel is positive semi-definite, and since \( M \neq 0 \), we conclude that \( -M \) cannot be a \( \pm 1 \) symmetric rescaling of \( L \). Hence, \( L \) and \( M \) are \( \pm 1 \) symmetric rescalings, and so it follows from part (ii) of Proposition 5.11 that \( L = M \).

**Remark 5.14.** Part (i) of the corollary can be extended by adding other claims from Corollary 2.4.

Combining part (ii) of Proposition 5.11 with Theorem 3.3 and Moore-Aronszajn theorem, we obtain a result, which is a continuous version of Theorem 4.4.
Theorem 5.15. Let $H$ be a real Hilbert space and let $B \subset H$ be linearly independent, connected in the weak topology of $H$ and such that $\text{span } B = H$. Let $\Phi : B \to H$ be continuous with respect to the weak topology and such that for any distinct $v_1, ..., v_n \in B$ the parallellepipeds $P(v_1, ..., v_n)$ and $P(\Phi(v_1), ..., \Phi(v_n))$ have the same volume. Then there is an isometry $T$ on $H$ such that $T|_B = \Phi$.

Proof. Let $K$ and $L$ be bi-functions on $B$ defined by $K(v, w) = \langle v, w \rangle$ and $L(v, w) = \langle \Phi(v), \Phi(w) \rangle$. Clearly, $K$ is positive definite, $L$ is positive semi-definite and both of them are symmetric. From the definition of the weak topology $K$ is separately continuous with respect to the weak topology on $B$, but the same is also true for $L$, since $\Phi$ is continuous with respect to the weak topology.

Since parallellepipeds $P(v_1, ..., v_n)$ and $P(\Phi(v_1), ..., \Phi(v_n))$ have the same volume, it follows that $L$ is also positive definite, and $\det_K = \det_L$. Hence, $K$ and $L$ are $\pm 1$ symmetric rescalings, due to Theorem 3.3. Since $B$ endowed with the weak topology is a connected topological space, and $K$ and $L$ are non-degenerate and separately continuous, from the part (ii) of Proposition 5.11 we conclude that $K = L$. Hence, according to Moore-Aronszajn theorem there is a unitary operator $T : H \to \text{span}\Phi(B)$, such that $Tv = \Phi(v)$, for every $v \in B$. \hfill \Box

A criterion for rescaling. Note that the definition of the fact that bi-functions $L$ and $M$ on a set $X$ are rescalings is extrinsic, i.e. it involves objects other than $L$ and $M$. Hence, it is desirable to be able to decide if given $L$ and $M$ are rescalings by examining a certain criterion. Proposition 4.3 provides with such a criterion, but it is only applicable in some specific cases, and its condition is difficult to verify. It turns out, that for separately continuous bi-functions one can find a criterion, which is more suitable for applications.

Let $Y$ stand for the class of all $Y$ of $X$, such that $Y_L = Y_M$ is connected, and $L$ and $M$ are rescalings on $Y$. We will need the following technical property of this family.

Lemma 5.16. If $M$ and $L$ are non-degenerate, then $Y$ satisfies the conditions of Zorn’s lemma, with respect to the partial order given by inclusion of sets.

Proof. Let $I$ be a linearly directed family and let $\{Y_i\}_{i \in I} \subset Y$ be increasing. Clearly if $Y = \bigcup_{i \in I} Y_i$, then $Y_L$ is connected. We will show that $Y \in Y$.

Fix some $z \in Y$. Without loss of generality we may assume that $z \in Y_i$, for any $i \in I$ (otherwise restrict $I$ so that it is true; this transition does not affect $Y$). Since $L$ and $M$ are rescalings on $Y_i$ for every $i \in I$, there are functions $f_i, g_i : Y_i \to \mathbb{C}^\times$ such that $(f_i, g_i)$ rescales $L$ to $M$ on $Y_i$ and such that $f_i(z) = 1$.

Let $i, j \in I$ and assume that $i \prec j$. Then $Y_i \subset Y_j$, and so $(f_j, g_j)$ rescales $L$ to $M$ on $Y_i$. Since $(Y_i)_L$ is connected, from part (ii) of Corollary 2.7 we have that $f_j|_{Y_i} = f_i$, and $g_j|_{Y_i} = g_i$. Thus, we can define functions $f, g : Y \to \mathbb{C}^\times$, such that $f_i = f|_{Y_i}$ and $g_i = g|_{Y_i}$, for every $i \in I$. Let $x, y \in Y$. There is $i \in I$, such that $x, y \in Y_i$, and so

\[ M(x, y) = f_i(x) g_i(y) L(x, y) = f(x) g(y) L(x, y). \]

Since $x$ and $y$ were chosen arbitrarily, we can conclude that $Y \in Y$. \hfill \Box
For bi-functions $L$ and $M$ on $X$ consider an equality
\[
M(x, y) M(y, z) L(x, z) L(y, y) = L(x, y) L(y, z) M(x, z) M(y, y), \tag{***}
\]
where $x, y, z \in X$. It is easy to see that if $L$ and $M$ are rescalings, then \(\text{[***]}\) holds for any $x, y, z$. Also note that if for some $y \in X$ such that $M(y, y) \neq 0$ the equality \(\text{[***]}\) is satisfied for any $x, z \in U_y \cap U^y$, then $(f_y, g_y)$ rescales $L$ to $M$ on $U_y \cap U^y$, where $f_y = \frac{M(\cdot, y)L(y, \cdot)}{L(\cdot, y)M(y, \cdot)}$ and $g_y = \frac{M(y, \cdot)}{L(y, \cdot)}$. Hence, in this case $U_y \cap U^y \in \mathcal{Y}$. Consequently, if there is $y \in X$ such that $U_y = U^y = X$ and \(\text{[***]}\) is satisfied for any $x, z \in X$, then $L$ and $M$ are rescalings.

It is easy to check that if $L$ and $M$ are symmetric non-degenerate and satisfy \(\text{[**]}\), for every $x, y, z$, then they also satisfy \(\text{[***]}\) for every $x, y, z$. In the continuous setting one can use the equality \(\text{[***]}\) as a criterion for rescaling of bi-functions.

**Theorem 5.17.** Let $X$ be a topological space and let $L$ and $M$ be non-degenerate bi-functions on $X$ such that \(\text{[***]}\) is satisfied for every $x, y, z \in X$. Then $L$ and $M$ are rescalings whenever one of the following conditions is satisfied:

(i) $L$ and $M$ are separately continuous, and there is $z \in X$ such that $\overline{U_z} = \overline{U^z} = X$ ($U_z$ and $U^z$ with respect to $L$);

(ii) $X$ is connected and locally connected, $L$ is separately continuous and $\overline{U_z} = \overline{U^z} = X$ for every $z \in X$.

**Proof.** First, note that (i) follows from Proposition 5.5, since if \(\text{[***]}\) is satisfied for every $x, y, z \in X$, then $L$ and $M$ are rescalings on $U_y \cap U^y$, for any $y \in X$. Hence, $L$ and $M$ are rescalings in a neighborhood of any point, and also, they are rescalings on a dense set $U_z \cap U^z$. Let us now prove (ii).

Let $\mathcal{U}$ stand for the class of all open connected subsets $U$ of $X$, such that $L$ and $M$ are rescalings on $U$. Combining the fact that the union of an increasing collection of open connected sets is open and connected with Lemma 5.16, we see that $\mathcal{U}$ satisfies the conditions of Zorn’s lemma.

For every $x \in X$ define $V_x = U_x \cap U^x$, which is an open neighborhood of $x$. Note also that from the discussion above $L$ and $M$ are rescalings on $V_x$, and so any component of $V_x$ belongs to $\mathcal{U}$ (since $X$ is locally connected any component of an open set is open). Since both $U_z$ and $U^z$ are dense, then $(X \setminus U_x) \cup (X \setminus U^x)$ is nowhere dense, and so $V_z$ is dense (and open).

Using Zorn’s Lemma we can choose a maximal element $U$ of $\mathcal{U}$. Since $X$ is connected, in order to prove that $U = X$ it is enough to show that $\partial U = \emptyset$. Assume that $w \in \partial U$. Then $V_w$ is an open neighborhood of $w \in \partial U \subset \overline{U}$, and so $U \cap V_w$ is a nonempty open set. Choose $v$ in the latter set; it follows that $v, w \in V_v$. Let $W$ be the component of $V_v$ that contains $w$. Then $W$ is a connected open neighborhood of $w$, and so it intersects with $U$. Since $U$ is connected, $U \cup W$ is also connected. If we show that $U \cup W \in \mathcal{U}$, we will reach a contradiction with the maximality of $U$, since $U \cup W$ contains $U$ and also contains $w \in \partial U \subset X \setminus U$. 

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Since $U \in \mathcal{U}$, there are $f, g : U \to \mathbb{C}^\times$ such that $(f, g)$ rescales $L$ to $M$ on $U$ with $f(v) = 1$; we also have that $(f_v, g_v)$ rescales $L$ to $M$ on $V_v \supset W$ and $f_v(v) = 1$. Note that $v \in U \cap W \subset V_v$, and since both pairs $(f, g)$ and $(f_v, g_v)$ rescale $L$ to $M$ on $U \cap W$ with $f(v) = f_v(v)$ it follows that $f, g$ and $f_v, g_v$ agree on $U \cap W$. Hence, there are common extensions of $f$ and $f_v$ and of $g$ and $g_v$ on $U \cup W$ (we will also denote it by $f$ and $g$).

Now, for $x, y \in U \cup W$ choose $z \in V_x \cap V_y \cap W \cup U$ (this set is nonempty since $V_x$ and $V_y$ are dense open sets and $U \cap W$ is open and nonempty). Then, each of $L(x, z), L(y, z), M(z, z)$ are not zero, and from \([**]*\) we have that

$$M(x, y) = \frac{M(x, z)M(z, y)}{L(x, z)L(z, y)} \frac{L(z, y)}{M(z, z)} L(x, y)$$

$$= \frac{f(x)g(z)f(y)g(z)}{f(z)g(z)} L(x, y) = f(x)g(y) L(x, y).$$

Since $x, y$ were chosen arbitrarily, we conclude that $(f, g)$ rescales $L$ to $M$ on $U \cup W$, and so we have reached the contradiction. \qed

For $x, y, z \in X$ consider equalities

$$M(x, y)M(z, y)L(x, z) L(y, y) = L(x, y)L(z, y) M(x, z)M(y, y), \quad (***')$$

$$M(x, y)\overline{M(z, y)}L(x, z)L(y, y) = L(x, y)|L(z, y)|M(x, z)M(y, y). \quad (**"**)$$

It is clear that if $L$ and $M$ are symmetric (Hermitean) rescalings, then \([**]*\) \((***')\) is satisfied for any $x, y, z \in X$. Also, note that applying the equations above to $L'$ and $M'$ or $\overline{L}$ and $\overline{M}$ we obtain other variations of these equations. Adopting the proof of the preceding theorem and using a suitable variation of Lemma 5.16 one can show the following criteria.

**Proposition 5.18.** Let $X$ be a topological space and let $L$ and $M$ be non-degenerate bi-functions on $X$ such that \([**]*\) \(/(***'')\) is satisfied for any $x, y, z \in X$. Then $L$ and $M$ are symmetric / Hermitean rescalings once one of the following conditions is satisfied:

(i) $L$ and $M$ are separately continuous, and there is $z \in X$ such that $\overline{U_z} = X$;

(ii) $X$ is connected and locally connected, $L$ is separately continuous and $\overline{U_z} = X$ for every $z \in X$.

Consider an example that shows that the existence of $z$ with $\overline{U_z} = X$ is essential. In fact, we will construct two real-valued positive definite functions on a closed interval, which satisfy \([**]*\) for every $x, y, z$, but are not rescalings.

**Example 5.19.** Let $K_0$ be an arbitrary continuous positive definite kernel on $(-1, 1)$, such that $\lim_{|x| \to 1} \overline{K_0}(x) = 0$. For example consider

$$K_0(x, y) = \text{Re} \left( \frac{1 - x^2)(1 - y^2)}{4 - e^{i\pi(x-y)}} \right) = \frac{1 - x^2}{2} \frac{1 - y^2}{2} \text{Re} \left( \frac{e^{i\pi x}}{2}, \frac{e^{i\pi y}}{2} \right).$$
depending on whether \( y \) is in \( M \). Aronszajn theorem. Then \( \lim_{|x| \to 1} \kappa_0 (x) = 0_H \), and so we can continuously extend \( \kappa_0 \) on \( \mathbb{R} \) with \( \kappa_0 (x) = 0_H \), when \( |x| \geq 1 \). For \( n \in \mathbb{Z} \) define \( \kappa_n : \mathbb{R} \to \mathbb{R} \) by \( \kappa_n (x) = \kappa_0 (x - n) \) and \( \kappa : \mathbb{R} \to \bigoplus n \in \mathbb{Z} H \) by \( \kappa = \bigoplus \kappa_n \). It is clear that locally the sum is of at most two non-zero summands, and so \( K : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) defined by \( K(x, y) = \langle \kappa(x), \kappa(y) \rangle \), for \( x, y \in \mathbb{R} \), is continuous. Also, it is easy to see that \( \kappa(\mathbb{R}) \) is a linearly independent set, and so \( K \) is positive definite.

Observe that if \( x \leq n \) and \( y \geq n + 1 \), for some \( n \), then \( K(x, y) = 0 \). Therefore, the set of points in \( \mathbb{R} \times \mathbb{R} \) where \( K \) is not zero is a “ladder” that may be viewed as a continuous analogue of \( L \) from Example 3.9. Adding extra regions where \( K \) does not vanish in a way similar to construction of \( L_A \) from that example will increase the complexity of the graph of \( K \). Here we will only construct an analogue of \( L^+_4 \) from the example, but it should be clear how to construct analogues of \( L^+_n \) or general \( L_A \).

Let \( f^\pm_4 : [0, 4] \to \mathbb{R} \) be a continuous function defined by

\[
f^\pm_4(x) = \begin{cases}
\pm (1 - 2x) & 0 \leq x \leq \frac{1}{2} \\
0 & \frac{1}{2} \leq x \leq \frac{7}{2} \\
2x - 7 & \frac{7}{2} \leq x \leq 4
\end{cases}
\]

Let \( M^\pm = K_{[0,4] \times [0,4]} + f^\pm_4 \otimes f^\pm_4 \). Observe that

\[
M^+ (x, y) = \begin{cases}
K(x, y) & \frac{1}{2} \leq x \leq \frac{7}{2} \text{ or } \frac{1}{2} \leq y \leq \frac{7}{2} \\
K(x, y) + f^+ (x) f^+ (y) & x, y \leq \frac{1}{2} \text{ or } x, y \geq \frac{7}{2} \\
\pm f^+ (x) f^+ (y) & x \leq \frac{1}{2} \text{ and } y \geq \frac{7}{2}, \text{ or } y \leq \frac{1}{2} \text{ and } x \geq \frac{7}{2}
\end{cases}
\]

Thus, \( M^+ \) and \( M^- \) are non-equal continuous positive definite real-valued kernels with \( |M^+| = |M^-| \). Hence, they are not rescalings, according to part (ii) of Corollary 5.13. However, we will show that \([**]\) holds for \( M^+ \), \( M^- \) and any \( x, y, z \in X \). Without loss of generality we may assume that \( x \leq y \leq z \).

If \( x \geq \frac{1}{2} \), or \( z \leq \frac{7}{2} \) we have \( M^+ (x, y) = M^- (x, y) \), \( M^+ (y, z) = M^- (y, z) \), \( M^+ (z, x) = M^- (z, x) \), and so \([**]\) holds. If \( y \leq \frac{1}{2} \) and \( z \geq \frac{7}{2} \) we have \( M^+ (x, y) = M^- (x, y) \), \( M^+ (y, z) = -M^- (y, z) \) and \( M^+ (z, x) = -M^- (z, x) \), and so \([**]\) holds. The case when \( y \geq \frac{7}{2} \) and \( x \leq \frac{1}{2} \) is analogous. Finally, if \( x < \frac{1}{2} \), \( \frac{1}{2} \leq y \leq \frac{7}{2} \) and \( z > \frac{7}{2} \), then \( M^+ (x, y) = K(x, y) \), \( M^+ (y, z) = K(y, z) \), and either \( K(x, y) = 0 \) or \( K(y, z) = 0 \) depending on whether \( y \geq 2 \) or \( y \leq 2 \). Hence, \([**]\) holds.

6 Rescaling of sesqui-holomorphic functions

Let \( X \) be a domain in \( \mathbb{C}^n \), i.e. an open connected set. In this section we perform an analogous investigation to the preceding section, but this time instead of continuity we will be considering holomorphicity. Let us start with the following fact (the proof is similar to the proof of Proposition 5.3):

\[
\]
Proposition 6.1. Let \(X\) be a domain in \(\mathbb{C}^n\) and let \(L\) and \(M\) be bi-functions on \(X\), which are (anti-) holomorphic in the first variable, and for every \(x \in X\) there is \(y \in X\) such that \(L(x, y) \neq 0\). If \(f, g : X \to \mathbb{C}^\times\) are such that \((f, g)\) rescales \(L\) to \(M\), then \(f\) is (anti-) holomorphic.

It is clear that the same result holds for the second variable and \(g\).

Recall that by Hartog’s theorem (see [3, VII.4, Theorem 4]) a bi-function \(L\) on \(X\) is holomorphic if and only if it is holomorphic in each variable. In this case \(\hat{L}\) is also holomorphic. Consider another class of bi-functions. A bi-function \(L : X \times X^* \to \mathbb{C}\) is sesqui-holomorphic if it is holomorphic in the first variable and anti-holomorphic in the second. Let \(X^* = \{\overline{x} \mid x \in X\} \subset \mathbb{C}^n\). By Hartog’s theorem \(L\) is sesqui-holomorphic if and only if the function \(M : X \times X^* \to \mathbb{C}\) defined by \(M(x, y) = L(x, \overline{y})\) is holomorphic. Hence, any sesqui-holomorphic bi-function is continuous.

In this section we will heavily rely on the Uniqueness Principle. Namely, recall that if two (anti-) holomorphic functions coincide on a somewhere dense (i.e. with a nonempty interior) subset of their domain, then they are equal (see [3, II.2, Theorem 4]). Hence, if \(L\) is a (sesqui-) holomorphic non-degenerate bi-function, then \(\overline{U}_y = \overline{U}_{\overline{y}} = X\), for any \(y \in X\). Indeed, if \(\overline{U}_y \neq X\) \((\overline{U}_{\overline{y}} \neq X)\), then the (anti-) holomorphic function \(L(\cdot, y) (L(y, \cdot))\) vanishes on an open set, and so it is identically zero, which contradicts \(L(y, y) \neq 0\). Another consequence is that being rescalings is determined by the behavior on a nonempty open set.

Proposition 6.2. Let \(L\) and \(M\) be (sesqui-) holomorphic non-degenerate bi-functions on a domain \(X \subset \mathbb{C}^n\). If \(L\) and \(M\) are rescalings on a somewhere dense subset of \(X\), then \(L\) and \(M\) are rescalings. Moreover, if there is a nonempty open connected \(U \subset X\) and \(f, g : U \to \mathbb{C}^\times\) such that \((f, g)\) rescales \(L\) to \(M\) on \(U\), then \(f\) and \(g\) can be extended on \(X\) so that \((f, g)\) rescales \(L\) to \(M\) on \(X\).

Proof. Let \(Y = X \times X\). Define a (sesqui-) holomorphic function \(N : Y \times Y \to \mathbb{C}\) by
\[
N(x, y, z, w) = M(x, w) M(y, z) L(x, z) L(y, w) - L(x, w) L(y, z) M(x, z) M(y, w).
\]
\(L\) and \(M\) are rescalings on a somewhere dense subset \(V\) of \(X\), a simple calculation shows that \(N\) vanishes on \(V \times V\). Therefore, from the Uniqueness Principle \(N \equiv 0\), and substituting \(y = w\) in this equality we get that \([*]\) holds for any \(x, y, z \in X\). Since \(L\) and \(M\) satisfy the conditions of Theorem 5.17, we conclude that \(L\) and \(M\) are rescalings.

Now assume that there is a nonempty open connected \(U \subset X\) and \(f, g : U \to \mathbb{C}^\times\) such that \((f, g)\) rescales \(L\) to \(M\) on \(U\). Then \(L\) and \(M\) are rescalings, and so there are \(f_1, g_1 : X \to \mathbb{C}^\times\) such that \(M = f_1 \otimes g_1 L\) and \(f_1(x) = f(x)\), for some \(x \in U\). Since \(M = f \otimes g L = f_1 \otimes g_1 L\) on \(U\), which is connected, and \(L\) and \(M\) are continuous, it follows from Corollary [5.13] that \(f = f_1\) and \(g = g_1\) on \(U\). Hence, \(f_1\) and \(g_1\) are extensions of \(f\) and \(g\) respectively. \(\square\)
Remark 6.3. If in the preceding proposition \( f = g \), then \( f_1 = g_1 \), due to Uniqueness Principle. Hence, if \( L \) and \( M \) are (sesqui-) holomorphic non-degenerate bi-functions on \( X \), which are symmetric rescalings on a nonempty open set \( U \subset X \), then they are symmetric rescalings. The analogous is also true for Hermitean and reciprocal rescalings.

We can now state a criterion for being symmetric rescalings in the class of holomorphic bi-functions.

**Proposition 6.4.** Let \( L \) and \( M \) be holomorphic non-degenerate bi-functions on a domain \( X \). Then \( L \) and \( M \) are symmetric rescalings if and only if there is a somewhere dense set \( U \subset X \) such that

\[
\frac{L(x,y)^2}{L(x,x)L(y,y)} = \frac{M(x,y)^2}{M(x,x)M(y,y)},
\]

for every \( x, y \in U \).

**Proof.** Necessity follows from substituting \( x = z \) into \([**]**\). Let us show sufficiency. Since both \( L \) and \( M \) are continuous and non-degenerate, by replacing \( U \) with \( \text{int}U \) we may assume that \( U \) is a nonempty open set.

Since \( M \) and \( L \) are non-degenerate, \( \hat{M} \hat{L} \) is a holomorphic non-vanishing function on \( X \). Fix \( z \in U \). Let \( V \subset U \) be a neighborhood of \( z \) such that \( M(x,y) \neq 0 \neq L(x,y) \), for any \( x, y \in V \).

There is an open neighborhood \( W \subset V \) of \( z \) and holomorphic \( f : W \to \mathbb{C}^\times \) such that \( \hat{M} \hat{L} = f^2 \) on \( W \). Then, for any \( x, y \in W \) we have \( \frac{f(x)^2 f(y)^2 L(x,y)}{M(x,y)} = 1 \), and so \( \frac{f \otimes f L}{M} \) is a holomorphic function on \( W \) with values in \( \{ -1, 1 \} \). Hence, either \( M = f \otimes f L \) on \( W \), or \( M = -f \otimes f L \). In the first case \( f \) symmetrically rescales \( L \) to \( M \) on \( W \), while in the second if symmetrically rescales \( L \) to \( M \) on \( W \). Thus, from the preceding remark, \( L \) and \( M \) are symmetric rescalings.

Hermitean rescalings of sesqui-holomorphic bi-functions admit more characterizations since the latter are subject of a special Uniqueness Principle (it follows from [3, II.4, Theorem 7]; see also [9]).

**Proposition 6.5.** Let \( L \) and \( M \) be sesqui-holomorphic bi-functions on a domain \( X \subset \mathbb{C}^n \). If \( \hat{L} = \hat{M} \) on a somewhere dense set \( U \subset X \), then \( L = M \).

Applying the proposition to \( L \) and \( L^* \), it follows that a sesqui-holomorphic bi-function \( L \) is Hermitean if and only if \( \hat{L} \) is real-valued on a somewhere dense set. This observation leads to the following result.

**Lemma 6.6.** Let \( L \) and \( M \) be non-degenerate sesqui-holomorphic bi-functions on a domain \( X \). If \( \frac{\hat{M}}{\hat{L}} \) is real-valued on a somewhere dense set, then either \( \frac{\hat{M}}{\hat{L}} > 0 \) on \( X \), or \( \frac{\hat{M}}{\hat{L}} < 0 \) on \( X \).

**Proof.** We may assume that there is a nonempty open set \( U \subset X \) such that \( \frac{\hat{M}}{\hat{L}} \) is real-valued on \( U \). Fix \( z \in U \). Since \( L(z,y) \neq 0 \neq M(z,y) \) and \( L \) and \( M \) are continuous, there is an open connected neighborhood \( V \subset U \) of \( z \) such that neither \( L \) nor \( M \) do not vanish on \( V \). Then, \( K = \frac{\hat{M}}{\hat{L}} \) is a sesqui-holomorphic bi-function on \( V \), such that \( \hat{K} \) is...
real-valued. From the discussion above it follows that $K$ is Hermitean, and so $ML^* = M^*L$ on $V$. Since both of these bi-functions are sesqui-holomorphic, from the (regular) Uniqueness Principle $ML^* = M^*L$ on $X$, and so $M(x, x)\hat{L}(x, x) = M(x, x)L(x, x)$, for every $x \in X$. Therefore, $\frac{M}{L}$ is real-valued on $X$. Since $X$ is connected, and $\frac{M}{L}$ is continuous and non-vanishing, it is either always positive, or always negative. 

Now we can state a criterion of rescaling of sesqui-holomorphic functions.

**Theorem 6.7.** Let $L$ and $M$ be sesqui-holomorphic non-degenerate bi-functions on a domain $X$. The following are equivalent:

(i) $L$ and $M$ are Hermitean rescalings (on a nonempty open subset of $X$);

(ii) There is a somewhere dense set $U \subset X$ and a holomorphic function $f : U \to \mathbb{C}^\times$, such that $\hat{M} = |f|^2\hat{L}$ on $U$;

(iii) There is a nonempty open set $U \subset X$ such that $\frac{\partial^2}{\partial x_j \partial \bar{x}_k} \log \hat{L} = \frac{\partial^2}{\partial x_j \partial \bar{x}_k} \log \hat{M}$ and $\hat{L}\hat{M} > 0$ on $U$;

(iv) There are $y, z \in X$ and a somewhere dense set $U \subset X$ such that $\frac{\overline{M(z, z)}}{L(z, z)} > 0$ and $\frac{|L(x, y)|^2}{L(x, x)L(y, y)} = \frac{|M(z, z)|^2}{M(x, x)M(y, y)}$, for every $x \in U$.

Note that the local equivalence of (i) and (iii) is known as Calabi Rigidity (see [5]). Also note that in (iv) we do not require $y, z \in U$.

**Proof.** From Remark 6.3 being a Hermitean rescaling is the same as being a Hermitean rescaling on a open nonempty subset of $X$. (i)$\Rightarrow$(ii) is trivial; both (i)$\Rightarrow$(iii) and (i)$\Rightarrow$(iv) follow from a simple calculation. In fact, (iv) is just a rearrangement of $(\ast \ast \ast'')$ for $x = z$. (ii)$\Rightarrow$(i) follows from Proposition 6.5 applied to $M$ and $f \otimes \overline{f} L$.

(iii)$\Rightarrow$(ii): The conditions of (ii) mean that $\log \frac{M}{L}$ is pluriharmonic on some open set $V$. Then there is a nonempty open set $W \subset V$ and a holomorphic function $f : V \to \mathbb{C}^\times$, such that $|f|^2 = \frac{M}{L}$ on $W$ (see [12, Theorem 4.4.4]).

(iv)$\Rightarrow$(ii): We may assume that $U$ is open. Since $U_y$ is dense in $X$, it follows that $V = U \cap U_y$ is a nonempty open set. Let $\alpha = \frac{M(y, y)}{L(y, y)} \neq 0$. For any $x \in V$ we have $\frac{\alpha M(x, x)}{L(x, x)} = \left|\frac{M(x, y)}{L(x, y)}\right|^2$, and so $\alpha \frac{M}{L}$ is positive on $V$. Therefore, from the preceding lemma $\frac{\alpha M}{L}$ is positive on $X$. In particular we have that $\alpha^2 = \frac{\alpha M(y, y)}{L(x, x)} > 0$ and so $\alpha \in \mathbb{R}^\times$. Also, $\alpha \frac{M(z, z)}{L(z, z)} > 0$ and since $\frac{M(z, z)}{L(z, z)} > 0$ it follows that $\alpha > 0$.

Define $f : V \to \mathbb{C}^\times$ by $f(x) = \alpha^{-\frac{1}{2}}\frac{M(x, y)}{L(x, y)}$. Then for $x \in V$ we have $|f(x)|^2 = \alpha^{-1}\left|\frac{M(x, y)}{L(x, y)}\right|^2 = \alpha^{-1} \frac{\alpha M(x, x)}{L(x, x)}$, and so $\hat{M} = |f|^2\hat{L}$ on $V$. 

**Remark 6.8.** It is easy to see that all the results in this section are valid when $X$ is a general complex manifold.
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