On an open problem of Skiba*

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Abstract

Let $\sigma = \{\sigma_i | i \in I\}$ be some partition of the set $P$ of all primes, that is, $P = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$. Let $G$ be a finite group. A set $\mathcal{H}$ of subgroups of $G$ is said to be a complete Hall $\sigma$-set of $G$ if every non-identity member of $\mathcal{H}$ is a Hall $\sigma_i$-subgroup of $G$ and $\mathcal{H}$ contains exactly one Hall $\sigma_i$-subgroup of $G$ for every $\sigma_i \in \sigma(G)$. $G$ is said to be a $\sigma$-group if it possesses a complete Hall $\sigma$-set. A $\sigma$-group $G$ is said to be $\sigma$-dispersive provided $G$ has a normal series $1 = G_1 < G_2 < \cdots < G_t < G_{t+1} = G$ and a complete Hall $\sigma$-set $\{H_1, H_2, \cdots, H_t\}$ such that $G_i H_i = G_{i+1}$ for all $i = 1, 2, \ldots, t$. In this paper, we give a characterizations of $\sigma$-dispersive group, which give a positive answer to an open problem of Skiba in the paper [1].

1 Introduction

Throughout this paper, all groups are finite and $G$ always denotes a finite group. Moreover, $n$ is an integer, $P$ is the set of all primes. The symbol $\pi(n)$ denotes the set of all primes dividing $n$ and $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of $G$.

In what follows, $\sigma = \{\sigma_i | i \in I\}$ is some partition of $P$, that is, $P = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$. $\Pi$ is always supposed to be a non-empty subset of the set $\sigma$ and $\Pi' = \sigma \setminus \Pi$. We write $\sigma(n) = \{\sigma_i | \sigma_i \cap \pi(n) \neq \emptyset\}$ and $\sigma(G) = \sigma(|G|)$.

Following [1–3], $G$ is said to be $\sigma$-primary if $G = 1$ or $|\sigma(G)| = 1$; $n$ is a $\Pi$-number if $\pi(n) \subseteq \bigcup_{\sigma_i \in \Pi} \sigma_i$; a subgroup $H$ of $G$ is called a $\Pi$-subgroup of $G$ if $|H|$ is a $\Pi$-number; a

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subgroup $H$ of $G$ is called a Hall $\Pi$-subgroup of $G$ if $H$ is a $\Pi$-subgroup of $G$ and $|G : H|$ is a $\Pi'$-number. A subgroup $H$ is said to be a $\sigma$-Hall subgroup of $G$ if $H$ is a Hall $\Pi$-subgroup of $G$ for some subset $\Pi$ of the set $\sigma$. A set $\mathcal{H}$ of subgroups of $G$ is said to be a complete Hall $\sigma$-set of $G$ if every non-identity member of $\mathcal{H}$ is a Hall $\sigma_i$-subgroup of $G$ for some $\sigma_i \in \sigma(G)$ and $\mathcal{H}$ contains exactly one Hall $\sigma_i$-subgroup of $G$ for every $\sigma_i \in \sigma(G)$.

If $G$ has a complete Hall $\sigma$-set $\mathcal{H} = \{H_1, \ldots, H_t\}$ such that $H_iH_j = H_jH_i$ for all $i, j$, then $\{H_1, \ldots, H_t\}$ is said to be a $\sigma$-basis of $G$. $G$ is said to be a $\sigma$-group if $G$ possesses a complete Hall $\sigma$-set; $G$ is called $\sigma$-soluble if every chief factor of $G$ is $\sigma$-primary; $G$ is called $\sigma$-nilpotent if every Hall $\sigma_i$-subgroup of $G$ is normal. As usual, we use $\mathfrak{S}_\sigma$ and $\mathfrak{N}_\sigma$ to denote the class of all $\sigma$-soluble groups and the class of all $\sigma$-nilpotent groups, respectively.

**Definition 1.1.** [1] A $\sigma$-group $G$ is said to be $\sigma$-dispersive if $G$ has a normal series

$$1 = G_1 < G_2 < \cdots < G_t < G_{t+1} = G$$

and a complete Hall $\sigma$-set $\mathcal{H} = \{H_1, \ldots, H_t\}$ such that $G_iH_i = G_{i+1}$ for all $i = 1, 2, \ldots, t$.

It is clear that when $|\sigma(G)| = |\pi(G)|$, then a $\sigma$-dispersive group $G$ is just a $\varphi$-dispersive group for some linear ordering $\varphi$ of primes (see [4, p. 6]).

Recall that if there is a subgroup chain $M_n < M_{n-1} < \cdots < M_1 < M_0 = G$ such that $M_i$ is a maximal subgroup of $M_{i-1}$, $i = 1, 2, \ldots, n$, then the chain is said to be a maximal chain of $G$ of length $n$ and $M_n$ is said to be an $n$-maximal subgroup of $G$.

**Definition 1.2.** [1] A subgroup $A$ of $G$ is called $\sigma$-subnormal in $G$ if there is a subgroup chain

$$A = A_0 \leq A_1 \leq \cdots \leq A_t = G$$

such that either $A_{i-1}$ is normal in $A_i$ or $A_i/(A_i - A_{i-1})$ is $\sigma$-primary for all $i = 1, 2, \cdots, t$.

If each $n$-maximal subgroup of $G$ is $\sigma$-subnormal in $G$ but, in the case $n > 1$, some $(n - 1)$-maximal subgroup is not $\sigma$-subnormal in $G$, then we write $m_\sigma(G) = n$ (see [5]). If $G$ is a soluble group, the rank $r(G)$ of $G$ is the maximal integer $k$ such that $G$ has a $G$-chief factor of order $p^k$ for some prime $p$ (see [6, p. 685]).

The relations between $n$-maximal subgroups (for $n > 1$) of $G$ and the structure of $G$ was studied by many authors (see, for example, [7–12] and Chapter 4 in the book [4]). One of the earliest results in this direction were obtained by Huppert [13], who proved that if every 2-maximal subgroup of $G$ is normal, then $G$ is supersoluble; if every 3-maximal subgroup of $G$ is normal in $G$, then $G$ is a soluble group of rank($G$) at most two. The first of these two results was generalized by Agrawal [14]. In fact, Agrawal proved that if every 2-maximal subgroup of $G$ is $S$-quasinormal in $G$, then $G$ is supersoluble. Mann [7] proved that if all $n$-maximal subgroups of a soluble group $G$ are subnormal and $|\pi(G)| \geq n + 1$, then $G$ is

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nilpotent; but if $|\pi(G)| \geq n - 1$, then $G$ is $\varphi$-dispersive for some ordering $\varphi$ of the set of all primes. In [1], Skiba studied the structure of a $\sigma$-soluble group $G$ by using the $\sigma$-subnormality of some $\sigma$-subnormal subgroup of $G$. It is natural to ask: what is the structure of a $\sigma$-soluble group $G$ if $|\sigma(G)| = n$ and every $(n + 1)$-maximal subgroups are $\sigma$-subnormal? In particular, Skiba posed the following open problem:

**Problem** [1, Question 4.8]. Let $G$ be a $\sigma$-soluble group and $|\sigma(G)| = n$. Assume that every $(n + 1)$-maximal subgroup of $G$ is $\sigma$-subnormal. Is it true then that $G$ is $\sigma$-dispersive?

In this paper, we give a positive answer to the above problem. In fact, we obtain the following theorem:

**Theorem 1.3.** Let $G$ be a $\sigma$-soluble group and $|\sigma(G)| = n$. Assume that every $(n + 1)$-maximal subgroup of $G$ is $\sigma$-subnormal. Then $G$ is $\sigma$-dispersive.

**Corollary 1.4.** Let $G$ be a $\sigma$-soluble group and $|\sigma(G)| \geq n$. Assume that every $(n + 1)$-maximal subgroup of $G$ is $\sigma$-subnormal. Then $G$ is $\sigma$-dispersive.

Note that in the case when $\sigma$ is the smallest partition of $\mathbb{P}$, that is, $\sigma = \{2, 3, \cdots\}$, we get from Corollary 1.4 the following known result.

**Corollary 1.5.** (See Mann [7]) Let each $n$-maximal subgroup of a soluble group $G$ be subnormal. If $|\pi(G)| \geq n - 1$, then $G$ has a Sylow tower.

All unexplained terminologies and notations are standard, as in [4], [15] and [16].

## 2 Preliminaries

**Lemma 2.1.** (See [1, Lemma 2.6]) Let $A, K$ and $N$ be subgroups of $G$. Suppose that $A$ is $\sigma$-subnormal in $G$ and $N$ is normal in $G$. Then:

1. $A \cap K$ is $\sigma$-subnormal in $K$.
2. If $K$ is a $\sigma$-subnormal subgroup of $A$, then $K$ is $\sigma$-subnormal in $G$.
3. If $K$ is $\sigma$-subnormal in $G$, then $A \cap K$ and $\langle A, K \rangle$ are $\sigma$-subnormal in $G$.
4. $AN/N$ is $\sigma$-subnormal in $G/N$.
5. If $N \leq K$ and $K/N$ is $\sigma$-subnormal in $G/N$, then $K$ is $\sigma$-subnormal in $G$.
6. If $H \neq 1$ is a Hall $\Pi$-subgroup of $G$ and $A$ is not a $\Pi'$-group, then $A \cap H \neq 1$ is a Hall $\Pi$-subgroup of $A$.
7. If $A$ is a $\sigma$-Hall subgroup of $G$, then $A$ is normal in $G$.

**Lemma 2.2.** If $|\pi(G)| = |\sigma(G)|$ and $H$ is a $\sigma$-subnormal subgroup of $G$, then $H$ is subnormal in $G$.  

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Proof. By hypothesis, there exists a subgroup chain $H = H_0 \leq H_1 \leq \cdots \leq H_{t-1} \leq H_t = G$ such that either $H_{i-1}$ is normal in $H_i$ or $H_i/(H_{i-1})H_i$ is $\sigma$-primary for all $i = 1, \cdots, t$. We show that $H_{i-1} \triangleleft \triangleleft H_i$ for all $i = 1, \cdots, t$. Since $|\pi(G)| = |\sigma(G)|$, we have that $\sigma_i \cap \pi(G) = \{p_i\}$ for some prime $p_i$ and every $\sigma_i$ such that $\sigma_i \cap \pi(G) \neq \emptyset$. If $H_{i-1}$ is not normal in $H_i$, then $H_i/(H_{i-1})H_i$ is a $p$-group for some prime $p$ dividing $|G|$. Hence $H_{i-1}/(H_{i-1})H_i \triangleleft \triangleleft H_i/(H_{i-1})H_i$. Consequently $H_{i-1} \triangleleft \triangleleft H_i$ for all $i = 1, \cdots, t$. Thus $H$ is subnormal in $G$. □

Lemma 2.3. (See [5, Lemma 4.5]) The following statements hold:

1. If each $n$-maximal subgroup of $G$ is $\sigma$-subnormal and $n > 1$, then each $(n-1)$-maximal subgroup is $\sigma$-nilpotent.

2. If each $n$-maximal subgroup of $G$ is $\sigma$-subnormal, then each $(n+1)$-maximal subgroup is $\sigma$-subnormal.

Let $A$ and $B$ be subgroups of $G$. Following [5], we say that $A$ forms an irreducible pair with $B$ if $AB = BA$ and $A$ is a maximal subgroup of $AB$.

Lemma 2.4. (See [5, Lemma 6.1]) Suppose that $G$ is $\sigma$-soluble and let $\{H_1, H_2, \cdots, H_t\}$ be a $\sigma$-basis of $G$. If $H_i$ forms an irreducible pair with $H_j$, then $H_j$ is an elementary abelian Sylow subgroup of $G$.

Lemma 2.5. (See [17, Lemma 2.1]) (1) The class $\mathfrak{S}_\sigma$ is closed under taking direct products, homomorphic images and subgroups. Moreover, any extension of the $\sigma$-soluble group by a $\sigma$-soluble group is a $\sigma$-soluble group as well.

(2) If $M$ is a maximal subgroup of a $\sigma$-soluble group $G$, then $|G : M|$ is $\sigma$-primary.

(3) If $G$ is a $\sigma$-soluble group, then for any $i$ such that $\sigma_i \cap \pi(G) \neq \emptyset$, $G$ has a maximal subgroup $M$ such that $|G : M|$ is a $\sigma_i$-number.

Recall that a class of groups $\mathcal{F}$ is called a formation if it is closed under taking homomorphic images and subdirect products. A formation $\mathcal{F}$ is called saturated if $G \in \mathcal{F}$ whenever $G/\Phi(G) \in \mathcal{F}$ (see, for example, [4]). The $\mathcal{F}$-residual of $G$, denoted by $G^\mathcal{F}$, is the smallest normal subgroup of $G$ with quotient in $\mathcal{F}$.

Lemma 2.6. (See [18, p. 35]) For any ordering $\varphi$ of $\mathbb{P}$ the class of all $\varphi$-dispersive groups is a saturated formation.

Lemma 2.7. (See [7, Theorem 9]) Let $G$ be a soluble group and each $n$-maximal subgroup of $G$ be subnormal. If $|\pi(G)| \geq n - 1$, then each Sylow subgroup of $G$ is either normal or of one of the following types:

(i) Cyclic.

(ii) A direct product of a cyclic group and a group of prime order.
(iii) The group \( (a, b|a^{p^{m-1}} = b^p = 1, b^{-1}ab = a^{1+p^{m-2}}) \), \( p \) a prime.
(iv) The quaternion group.

Lemma 2.8. (See [13, Satz 14]) If \( r(G) = 2 \), then a Sylow subgroup corresponding the maximal prime divisor of the order of the group is invariant (=normal) under the condition that this prime divisor is greater than 3. In particular, if \( 2 \nmid \text{maximal prime divisor of the order of the group} \), then \( G \) satisfies Sylow tower property (see [19, p. 5]).

3 Proof of Theorem 1.3 and Corollary 1.4

Proof of Theorem 1.3. Suppose that this theorem is false and let \( G \) be a counterexample with \( |G| + |\sigma(G)| \) minimal. Since \( G \) is \( \sigma \)-soluble, by [17, Theorem A] there exists a \( \sigma \)-basis of \( G \), \( \{H_1, H_2, \ldots, H_n\} \) say. Without loss of generality, we may assume that \( H_i \) is a \( \sigma_i \)-group and \( p \in \sigma_1 \) where \( p \) is the smallest prime dividing \( |G| \). Clearly, \( n > 1 \). We now proceed by the following steps.

(1) \( G \) has no normal Hall \( \sigma_i \)-subgroup for any \( i \in \{1, \ldots, n\} \).

Assume that \( G \) has a normal Hall \( \sigma_i \)-subgroup \( H_i \) of \( G \) for some \( i \in \{1, \ldots, n\} \). Then \( H_i \) has a complement \( M \) in \( G \) such that \( G = H_i \rtimes M \) by Schur-Zassenhaus Theorem. Clearly, \( |\sigma(M)| = n - 1 \) and every \( n \)-maximal subgroup of \( M \) is at least \( (n + 1) \)-maximal of \( G \). By Lemma 2.3(2) and the hypothesis, every \( n \)-maximal subgroup of \( M \) is \( \sigma \)-subnormal in \( G \), so it is \( \sigma \)-subnormal in \( M \) by Lemma 2.1(1). This shows that \( M \) satisfies the hypothesis. The choice of \( G \) and Lemma 2.5(1) imply that \( G/H_i \cong M \) is \( \sigma \)-dispersive. It follows that \( G \) is \( \sigma \)-dispersive, a contradiction. Hence (1) holds.

(2) \( H_1 \) is \( (n - 1) \)-maximal in \( G \) for every maximal chain containing \( H_1 \).

By Lemma 2.5, \( H_1 \) is at least \( (n - 1) \)-maximal in \( G \). But by (1) and Lemma 2.1(7), \( H_1 \) is not \( \sigma \)-subnormal in \( G \). Hence by Lemma 2.3(2) and the hypothesis, \( H_1 \) is \( k \)-maximal in \( G \) where \( k = n - 1 \) or \( k = n \). If \( H_1 \) is \( n \)-maximal in \( G \), then every maximal subgroup of \( H_1 \) is \( (n + 1) \)-maximal in \( G \) and so it is \( \sigma \)-subnormal in \( G \). But as \( H_1 \) is not \( \sigma \)-subnormal, by Lemma 2.1(3), \( H_1 \) has only one maximal subgroup. Hence \( H_1 \) is a cyclic subgroup of prime power order, which means that \( H_1 = G_p \) is a cyclic Sylow \( p \)-subgroup of \( G \) and \( \sigma_1 \cap \pi(G) = \{p\} \). Hence \( G \) is \( p \)-nilpotent by [6, IV, Theorem 2.8], which implies that \( G \) has a normal Hall \( \sigma_1 \)-subgroup \( E \). Consequently \( |\sigma(E)| = n - 1 \) and every \( n \)-maximal subgroup of \( E \) is at least \( (n + 1) \)-maximal in \( G \). By Lemma 2.3(2) and the hypothesis, every \( n \)-maximal subgroup of \( E \) is \( \sigma \)-subnormal in \( G \), and so it is \( \sigma \)-subnormal in \( E \) by Lemma 2.1(1). This shows that \( E \) satisfies the hypothesis. The choice of \( G \) implies that \( E \) is \( \sigma \)-dispersive. So \( E \) has a normal series \( 1 = E_1 < E_2 < \cdots < E_{n-1} < E_n = E \) and a complete Hall \( \sigma \)-set \( K = \{K_2, K_3, \ldots, K_n\} \).
such that $E_{i+1} = E_i K_{i+1}$ for all $i = 1, 2, \ldots, n - 1$. Since $E$ is a Hall $\sigma_1'$-subgroup of $G$, we have that $K_i$ is a Hall $\sigma_i'$-subgroup of $G$ for all $i = 2, \cdots, n$. But as $E_{i+1} = E_i K_{i+1}$, we see that $E_{i+1}$ is also a Hall subgroup of $E$. Hence $E_{i+1}$ is characteristic in $E$, and so $E_{i+1}$ is normal in $G$ for all $i = 1, 2, \cdots, n - 1$. Consequently $G$ has a normal series

$$1 = E_1 < E_2 < \cdots < E_{n-1} < E_n = E < E_{n+1} = E H_1 = G$$

and a complete Hall $\sigma$-set $\{K_2, K_3, \ldots, K_n, H_1\}$ such that $E_{i+1} = E_i K_{i+1}$ for all $i = 1, 2, \cdots, n-1$ and $E_{n+1} = E_n H_1$. This means that $G$ is $\sigma$-dispersive, a contradiction. Hence $H_1$ is $(n-1)$-maximal in $G$ for every maximal chain containing $H_1$.

(3) $H_i$ is an $n$-maximal subgroup of $G$ and $H_i$ is a cyclic group of prime order, for $i = 2, \cdots, n$.

Since $\{H_1, H_2, \cdots, H_n\}$ is a $\sigma$-basis of $G$, $H_i H_j = H_j H_i$ for all $i, j$. But as $H_1$ is $(n-1)$-maximal in $G$ for every maximal chain containing $H_1$ by (2), we see that $H_1$ is a maximal subgroup of $H_1 H_i$. This shows that $H_1$ forms an irreducible pair with $H_i$, where $i = 2, \cdots, n$. Hence $H_i$ ($i = 2, \cdots, n$) is an elementary abelian Sylow subgroup of $G$ by Lemma 2.4. By the same discussion as (2), $H_i$ is at least $k$-maximal in $G$, where $k = n - 1$ or $k = n$. Assume that, for some $i > 1$, $H_i$ is a $(n-1)$-maximal subgroup of $G$ for every maximal chain containing $H_i$. Then with a similar argument as above, we have that $H_i$ forms an irreducible pair with $H_1$. So $H_1$ is an elementary abelian Sylow subgroup of $G$ by Lemma 2.4, and so $|\pi(G)| = |\sigma(G)|$.

Without loss of generality, we may assume that $H_1, \cdots, H_r$ is $(n-1)$-maximal in $G$ for every maximal chain containing $H_i$, where $i = 1, \ldots, r$, and $H_{r+1}, \ldots, H_n$ is $n$-maximal in $G$, where $r > 1$. Then for every $j \in \{r + 1, \cdots, n\}$, every maximal subgroup of $H_j$ is an $(n+1)$-maximal subgroup of $G$, so it is $\sigma$-subnormal in $G$ by the hypothesis. But by (1) and Lemma 2.1(7), $H_j$ is not $\sigma$-subnormal in $G$. It follows from Lemma 2.1(3) that $H_j$ has only one maximal subgroup, which implies that $H_j$ is a cyclic subgroup of prime power order. But as above, we know that $H_j$ is an elementary abelian Sylow subgroup of $G$, so $H_j$ is a cyclic subgroup of prime order. Since $G$ is $\sigma$-soluble and $|\pi(G)| = |\sigma(G)|$, it is easy to see that $G$ is soluble. Let $R$ be a minimal normal subgroup of $G$. Then $R$ is an elementary abelian $q$-group for some prime $q$. If $q \in \sigma_j$, for some $j \in \{r+1, \cdots, n\}$, then as $H_j$ is a cyclic subgroup of prime order, we have that $H_j = R$ is normal in $G$, which contradicts (1). Hence $R \leq H_i$ for $i \leq r$. Assume that $R \leq H_1$. Since $H_2$ is $(n-1)$-maximal in $G$ for every maximal chain containing $H_2$, with a similar argument as above, we have that $H_2$ forms an irreducible pair with $H_1$, which means that $H_2$ is a maximal subgroup of $H_1 H_2$. But as $H_2 < RH_2 \leq H_1 H_2$, we have that $H_1 = R$ is normal in $G$, a contradiction. Hence for every $i \in \{2, \cdots, n\}$, we have that $H_i$ is an $n$-maximal subgroup of $G$. By the same discussion as above, we have that $H_i$ is a cyclic subgroup of prime order for $i \in \{2, \cdots, n\}$. So we have (3).
(4) \(|\pi(H_1)| \leq 2\).

Assume that \(|\pi(H_1)| \geq 3\). Let \(R\) be a minimal normal subgroup of \(G\). Since \(G\) is \(\sigma\)-soluble, \(R\) is a \(\sigma_i\)-group for some \(\sigma_i \in \sigma(G)\). But by (1) and (3), we know that \(R\) is a \(\sigma_1\)-group, so \(R \leq H_1\). First suppose that \(R\) is an abelian group. Then \(R\) is a \(r\)-group for some prime \(r \in \sigma_1\). Let \(Q\) be a Sylow \(q\)-subgroup of \(H_1\), where \(q \neq r\) and let \(E = H_2H_3\cdots H_n\). Then by [17, Theorem A(ii)], there exists some \(x \in G\) such that \(EQ^x = Q^xE\). Since \(|\pi(H_1)| \geq 3\), we have that \(EQ^xR < G\). Hence \(G\) has a subgroup chain

\[H_2 < H_2H_3 < \cdots < H_2\cdots H_n = E < EQ^x < EQ^xR < G.\]

This shows that \(H_2\) is at least \((n + 1)\)-maximal in \(G\), so \(H_2\) is \(\sigma\)-subnormal in \(G\) by the hypothesis and Lemma 2.3(2). It follows that \(H_2\) is normal in \(G\) by Lemma 2.1(7), which contradicts (1). Hence \(R\) is not an abelian group. Then for any odd prime \(q\) dividing \(|R|\), \(R\) is not \(q\)-nilpotent. By the Glauberman-Thompson normal \(q\)-complement Theorem (see [20, p. 280, Theorem 3.1]), we have that \(R_q < N_R(Z(J(R_q))) < R\), where \(R_q\) is a Sylow \(q\)-subgroup of \(R\). By Frattini argument, \(G = R\mathcal{N}_G(R_q)\). Since \(R \leq H_1\), \(H_2\) normalizes some Sylow \(q\)-subgroup of \(R\), say \(R_q\). Hence \(H_2 \leq N_G(Z(J(R_q)))\). But then we have the following subgroup chain

\[H_2 < H_2R_q < H_2N_R(Z(J(R_q))) < H_2R \leq H_2H_1 < H_2H_1H_3 < \cdots < H_1\cdots H_n = G,\]

which means that \(H_2\) is at least \((n + 1)\)-maximal in \(G\). Then with a similar argument as above, we have that \(H_2\) is normal in \(G\), which contradicts (1). Hence \(|\pi(H_1)| \leq 2\).

(5) \(|\pi(H_1)| = 2\).

Assume that this is false. Then by (4), we have that \(|\pi(H_1)| = 1\), so \(|\pi(G)| = |\sigma(G)| = n\) by (3). Hence \(\{H_1, H_2, \ldots, H_n\} = \{P_1, P_2, \ldots, P_n\}\) is a Sylow basis of \(G\), where \(H_i = P_i\) is a Sylow subgroup of \(G\) and \(P_2, \ldots, P_n\) is a cyclic subgroup of prime order. Since \(G\) is \(\sigma\)-soluble and \(|\pi(G)| = |\sigma(G)| = n\), we have that \(G\) is soluble.

Let \(R\) be a minimal normal subgroup of \(G\), then \(R\) is an elementary abelian group and \(R \leq P_i\) for some \(i\). But \(P_2, \ldots, P_n\) is a cyclic subgroup of prime order and not normal in \(G\), so \(R\) is an elementary abelian \(p\)-group and \(R \leq P_1\). By (1), \(R < P_1\), so \(|\sigma(G/R)| = |\pi(G/R)| = n\) and all \((n + 1)\)-maximal subgroup of \(G/R\) is \(\sigma\)-subnormal in \(G/R\) by Lemma 2.1(4) and hypothesis. This shows that \(G/R\) satisfies the hypothesis. The choice of \(G\) implies that \(G/R\) is a \(\varphi\)-dispersive group for some ordering \(\varphi\) of the set of all primes. Then by Lemma 2.6, \(R \notin \Phi(G)\). Since \(G/R\) is a \(\varphi\)-dispersive group, \(P_2R/R \leq G/R\) for some \(i \geq 2\) by (1). Without loss of generality, we can assume that \(P_2R/R \leq G/R\). Let \(H = P_2R\). Then \(H \leq G\). Clearly, \(R\) is not cyclic and so \(|R| > p\). Indeed, if \(R\) is cyclic, then \(H = P_2R\) is supersoluble. But since \(p\) is the smallest prime dividing \(|G|\), \(P_2 \leq G\), which contradicts (1). By Lemma 2.7, \(P_1\) has a
cyclic maximal subgroup $V$. If $R \leq V$, then $R$ is cyclic, a contradiction. Hence $R \not\leq V$. So $P_1 = RV$ and $|P_1 : V| = p = |R : R \cap V|$. Since $V$ is cyclic, $|R \cap V| = p$. It follows that $|R| = p^2$.

Now let $M/N$ be any chief factor of $G$. Then $M/N$ is an elementary abelian $q$-group for some prime $q$. If $q \neq p$, then $M/N \leq P_i/N \cong P_i/P_i \cap N$ for some $i \geq 2$, so $|M/N| = q$ by (3). Now assume that $q = p$. Then $M/N \leq P_1/N$. If $P_1/N = VN/N$, then $P_1/N$ is cyclic, so $|M/N| = p$. If $VN/N < P_1/N$, then $VN/N$ is a maximal subgroup of $P_1/N$. So $M/N \leq VN/N$ or $(M/N)(VN/N) = P_1/N$. In the former case, we have that $|M/N| = p$. In the latter case, $|P_1/N : VN/N| = p = |M/N : M/N \cap VN/N|$. Since $VN/N$ is cyclic, $|M/N \cap VN/N| \leq p$. Consequently $|M/N| \leq p^2$. Hence in any case we always have $|M/N| \leq p^2$. This shows that the rank of $G$ is at most 2. Since $G$ does not have a normal Sylow subgroup by (1), $|G| = 2^α3^β = 2^33$ by Lemma 2.8 and (3). This shows that $n = 2$, and for every minimal normal subgroup $N$ of $G$, we have that $N \leq P_i$ and $G/N$ is $\varphi$-dispersive, where $\varphi$ is the unique ordering of the set of primes $\{2, 3\}$. But $|G| = 2^α3$, we can let $\varphi$ be the unique ordering of the set of all primes. Hence by Lemma 2.6, we have that $R$ is the unique minimal normal subgroup of $G$. Since $n = 2$, we obtain that every 3-maximal subgroup of $G$ is subnormal in $G$ by the hypothesis and Lemma 2.2. As $R \not\leq \Phi(G)$ and $R$ is the unique minimal normal subgroup of $G$, there exists a maximal subgroup $M$ of $G$ such that $G = RM = R \times M$, and clearly $C_G(R) = R$. Then $P_1 = R(P_1 \cap M)$ and $1 \neq P_1 \cap M < M$. If $P_1 \cap M$ is not maximal in $M$, then $P_1 \cap M$ is at least 3-maximal in $G$, and so is subnormal in $G$. By [16, A, 14.3], $R \leq N_G(P_1 \cap M)$. Hence $R(P_1 \cap M) = R \times (P_1 \cap M)$, so $P_1 \cap M \leq C_G(R) = R$, which means that $P_1 \cap M = 1$, a contradiction. Hence $P_1 \cap M$ is a maximal subgroup of $M$. Let $W$ be a maximal subgroup of $P_1 \cap M$. Then $W$ is a 3-maximal subgroup of $G$, so $W$ is subnormal in $G$. By the same discussion as above, we have that $W \leq C_G(R) \cap (P_1 \cap M) = R \cap (P_1 \cap M) = 1$. This implies that $|P_1 \cap M| = p = 2$. Hence $|P_1| = |R||P_1 \cap M| = 2^3$ and so $|G| = 2^33$. Then $G$ is a group of order 24 possessing an elementary abelian normal subgroup $R$ of order 4 and $C_G(R) = R$, which implies that $G \simeq S_4$ by [6, II, Lemma 8.17]. But $S_4$ has a 3-maximal subgroup (of order 2) which are not subnormal, a contradiction. Hence $|\pi(H_1)| = 2$.

(6) Final contradiction.

Since $|\pi(H_1)| = 2$ by (5), $H_1$ is soluble by the well known Burnside $p$-$q$ Theorem. But as $G$ is $\sigma$-soluble, we have that $G$ is soluble by (3). By (3) and (5), $|\pi(G)| = |\sigma(G)| + 1 = n + 1$. By (1), (3) and Lemma 2.1(7), $H_2$ is an $n$-maximal subgroup of $G$ and $H_2$ is not $\sigma$-subnormal in $G$. Hence $m_\sigma(G) = n + 1 = |\pi(G)|$. Then $G = D \times M$, where $D = G^{\text{ab}}_\sigma$ is an abelian Hall subgroup of $G$ by [5, Theorem 1.10]. If $q||D$ for some prime $q \in \sigma_i$, where $i \in \{2, 3, \cdots, n\}$, then $D_q$ is a normal Sylow $q$-subgroup of $G$, where $D_q$ is a Sylow $q$-subgroup of $D$. Hence $H_i = D_q$ by (3), which means that $H_i$ is normal in $G$, a contradiction. So $D$ is a $\sigma_1$-group, that
is, $D \leq H_1$. But since $G/D$ is $\sigma$-nilpotent, we have that $H_1$ is normal in $G$, which contradicts (1). The final contradiction completes the proof of the theorem.

**Proof of Corollary 1.4.** Assume that this corollary is false. Then by Theorem 1.3, we may assume that $|\sigma(G)| \geq n + 1$. Assume that $|\sigma(G)| = t > n + 1$. Since $G$ is $\sigma$-soluble, $G$ has a $\sigma$-basis $\{H_1, \ldots, H_t\}$ by [17, Theorem A]. Then we have a subgroup chain

$$H_i < H_1H_i < \cdots < H_1 \cdots H_{i-2}H_i < H_1 \cdots H_{i-1}H_i < \cdots < H_1 \cdots H_t = G,$$

for $i = 1, 2, \ldots, n$. Hence $H_i$ is at least $(t - 1)$-maximal in $G$, so $H_i$ is at least $(n + 1)$-maximal in $G$. Then by the hypothesis and Lemma 2.3(2), $H_i$ is $\sigma$-subnormal in $G$, so $H_i$ is normal in $G$ by Lemma 2.1(7). Hence $G$ is $\sigma$-nilpotent and thereby $G$ is $\sigma$-dispersive. This contradiction shows that $|\sigma(G)| = n + 1$.

Now we claim that $|\pi(G)| = |\sigma(G)| = n + 1$. In fact, if $|\pi(G)| > |\sigma(G)| = n + 1$, then there exists a Hall $\sigma$-subgroup $H_i$ with $|\pi(H_i)| \geq 2$. Let $P$ be a Sylow $p$-subgroup of $H_i$ and $E = H_1 \cdots H_{i-1}H_{i+1} \cdots H_{n+1}$. Then by [17, Theorem A(ii)], there exists some $x \in G$ such that $EP^x = P^xE$. Then for any $j \neq i$, we have the following subgroup chain

$$H_j < H_jH_1 < \cdots < E < EP^x < EH_i = G,$$

which means that $H_j$ is at least $(n + 1)$-maximal in $G$. Hence $H_j$ is normal in $G$ by the same discussion as above. For any Sylow subgroup $Q$ of $H_i$, we have $Q < H_i < H_iH_1 < \cdots < H_1 \cdots H_{n+1} = G$, so $Q$ is at least $(n + 1)$-maximal in $G$. Hence $Q$ is $\sigma$-subnormal in $G$ by Lemma 2.3(2). It follows from Lemma 2.1(3)(7) that $H_i$ is normal in $G$. Hence $G$ is $\sigma$-nilpotent and so $G$ is $\sigma$-dispersive, a contradiction. Therefore $|\pi(G)| = |\sigma(G)| = n + 1$. Hence $G$ is soluble and $\{H_1, \ldots, H_{n+1}\} = \{P_1, \ldots, P_{n+1}\}$ is a Sylow basis of $G$. Then for every Sylow subgroup $P_i$ of $G$, every maximal subgroup of $P_i$ is at least $(n + 1)$-maximal in $G$, so it is subnormal in $G$ by the hypothesis and Lemma 2.3(2) and Lemma 2.2. Hence $P_i$ is cyclic or $P_i \leq G$. If every $P_i$ ($i = 1, \ldots, n + 1$) is cyclic, then $G$ is supersoluble, and so $G$ is $\varphi$-dispersive for some ordering $\varphi$ of all primes. Consequently, $G$ is $\sigma$-dispersive, a contradiction. Hence there exists $i \in \{1, \ldots, n + 1\}$ such that $P_i$ is normal in $G$, say $P_1$. Then $|\sigma(G/P_1)| = n$ and all $(n + 1)$-maximal subgroup of $G/P_1$ is $\sigma$-subnormal in $G/P_1$ by the hypothesis and Lemma 2.1(4). Hence by Theorem 1.3, $G/P_1$ is $\sigma$-dispersive. It follows that $G$ is $\sigma$-dispersive. The final contradiction completes the proof.

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