Coherent Fading Channels Driven by Arbitrary Inputs: Asymptotic Characterization of the Constrained Capacity and Related Information- and Estimation-Theoretic Quantities

Alberto Gil C. P. Ramos, Student Member, IEEE, and Miguel R. D. Rodrigues, Member, IEEE

Abstract

We consider the characterization of the asymptotic behavior of the average minimum mean-squared error (MMSE) and the average mutual information in scalar and vector fading coherent channels, where the receiver knows the exact fading channel state but the transmitter knows only the fading channel distribution, driven by a range of inputs. We construct low-snr and – at the heart of the novelty of the contribution – high-snr asymptotic expansions for the average MMSE and the average mutual information for coherent channels subject to Rayleigh fading, Ricean fading or Nakagami fading and driven by discrete inputs (with finite support) or various continuous inputs. We reveal the role that the so-called canonical MMSE in a standard additive white Gaussian noise (AWGN) channel plays in the characterization of the asymptotic behavior of the average MMSE and the average mutual information in a fading coherent channel: in the regime of low-snr, the derivatives of the canonical MMSE define the expansions of the estimation- and information-theoretic quantities; in contrast, in the regime of high-snr, the Mellin transform of the canonical MMSE define the expansions of the quantities. We thus also provide numerically and – whenever possible – analytically the Mellin transform of the canonical MMSE for the most common input distributions. We also reveal connections to and generalizations of the MMSE dimension. The most relevant element that enables the construction of these non-trivial expansions is the realization that the integral representation of the estimation- and information-theoretic quantities can be seen as an $h$-transform of a kernel with a monotonic argument: this enables the use of a novel asymptotic expansion of integrals technique – the Mellin transform method – that leads immediately to not only the high-snr but also the low-snr expansions of the average MMSE and – via the I-MMSE relationship – to expansions of the average mutual information. We conclude with applications of the results to the characterization
and optimization of the constrained capacity of a bank of parallel independent coherent fading channels driven by arbitrary discrete inputs.

**Index Terms**

Capacity, Constrained Capacity, Fading Channels, AWGN Channels, MMSE, Mutual Information, Average MMSE, Average Mutual Information, Asymptotic Expansions, Mellin Transform.

I. INTRODUCTION

The characterization of the constrained capacity of fading coherent channels, where the receiver knows the exact fading channel realization but the transmitter knows only the fading channel distribution, driven by arbitrary inputs is a problem with significant practical interest. The importance relates to the fact that this quantity defines the highest information transmission rate between the transmitter and the receiver with a message error probability that approaches zero in systems that employ arbitrary and practical signalling schemes, such as $m$-phase shift keying (PSK), $m$-pulse amplitude modulation (PAM) or $m$-quadrature amplitude modulation (QAM) constellations, rather than the capacity-achieving Gaussian one, thereby establishing a benchmark to the design and optimization of practical communications systems.

Unfortunately, the characterization of the constrained capacity, which – assuming that the fading channel variation over time is stationary and ergodic – is given by the average over the fading channel gain distribution of the mutual information between the input and the output of the channel conditioned on the channel gain, is complex due to the absence of closed-form expressions for the mutual information as well as the average mutual information.

An innovative approach that also leads to a characterization – in asymptotic regimes – of the constrained capacity of key communications channels was put forth in [1], [2]. The approach capitalizes on connections between mutual information and minimum mean-squared error (MMSE) [3], [4], key quantities in information theory and estimation theory, in order to express the mutual information or bounds to the mutual information in terms of the MMSE or bounds to the MMSE, respectively. Other applications of the relation between mutual information and MMSE can be found in [5], [6], [7], [8], [9], [10], [11], [12], [13].

In this paper, we leverage connections between the average value of the mutual information and the average value of the MMSE in a coherent fading channel to obtain characterizations of the average mutual information from characterizations of the average MMSE. We pursue the characterizations exclusively in the asymptotic regime of low signal-to-noise ratio (SNR) and, at the heart of the novelty of the contribution, in the asymptotic regime of high SNR. The asymptotic analysis, which bypasses the difficulty associated with the construction of general non-asymptotic results, often applies to a variety of practical scenarios leading to considerable insight.

We also use, in addition to the connections between the average mutual information and the average MMSE, key techniques that lead to the asymptotic expansions of the quantities not only in the regime of high SNR but also low SNR. In particular, by recognizing that the quantities can be seen as an $h$-transform with a kernel of monotonic argument [14], we capitalize on expansions of integrals techniques, such as Mellin transform based methods or
integration by part methods, in order to construct the asymptotic expansions of the average MMSE and – via the
connections between estimation and information theory – the average mutual information.

The original contributions of the article include:

- The identification of a comprehensive framework to construct asymptotic expansions of the average MMSE
  and the average mutual information in a single-input–single-output fading coherent channel driven by arbitrary
  inputs. The framework enables the construction of low- and – more importantly – high-SNR asymptotic
  expansions of the quantities in a unified way.

- The construction of novel asymptotic expansions for the average MMSE and the average mutual information
  in a single-input–single-output fading coherent channel driven by arbitrary inputs. We conceive expansions
  applicable to arbitrary fading channels, which are specialized to Rayleigh fading channels, Ricean fading
  channels as well as Nakagami fading channels. We also conceive expansions applicable to discrete inputs with
  finite support and continuous inputs distributed according to ∞-PSK, ∞-PAM, ∞-QAM and standard complex
  Gaussian distributions.

- The generalization of the asymptotic results from single-input–single-output to single-input–multiple-output
  fading coherent channels driven by arbitrary inputs.

- The application of the results in key communications problems. In particular, inspired by the ubiquitous use
  of Orthogonal Frequency-Division Multiplexing (OFDM) and multi-user OFDM based systems, we consider
  the determination of the power allocation policy that maximizes the constrained capacity of a bank of parallel
  independent fading coherent channels driven by arbitrary discrete inputs in the asymptotic regime of high SNR.

This paper is organized as follows: Section II describes the channel model and the main estimation-theoretic
and information-theoretic quantities. Sections III and IV, which contain the main contributions, describe in detail
the approach used to construct the asymptotic expansions of the estimation-theoretic and the information-theoretic
measures. Section V describes generalizations of the approach. Section VI includes a series of analytic and numerical
Mellin transform results useful for the construction of the asymptotic expansions. Section VII presents a series of
simulations that confirm the accuracy of the asymptotic expansions. Applications of the approach are covered in
Section VIII. Finally, we conclude with various remarks in Section IX.

A. Notation

We use the following notation: Italic lower case letters denote scalars, boldface lower case letters denote column
vectors and boldface upper case letters denote matrices, e.g., c, e and C, respectively. (·)*, (·)T and (·)† denote
the complex conjugate, transpose, and complex conjugate transpose operators, respectively. The norm of a vector v
is defined as ||v|| := √vTv. R(·) and I(·) denote the real and imaginary part operators, respectively. I_k denotes
the k × k identity matrix. We denote the probability mass/density function of a discrete/continuous random variable
x by f_x(·) and the expectation of a function g(·) with respect to the random variable x by E_x{g(x)}. Given
σ > 0, we say that a complex random variable x is distributed according to CN(μ, 2σ^2I_k) if [R(x)^T, I(x)^T]^T ∼
N([R(μ)^T, I(μ)^T]^T, σ^2I_{2k}). log(·) denotes the natural logarithm throughout the paper.
We also use the following asymptotic notation as in [14]: We write
\[ f(x) = O(g(x)), \quad x \to x_0 \]
if there exists a positive real number \( k \), and a neighborhood of \( x_0 \), \( N_{x_0} \), such that, \( \forall x \in N_{x_0}, |f(x)| \leq k|g(x)| \).
We also write
\[ f(x) = o(g(x)), \quad x \to x_0 \]
if for any positive real number \( \epsilon \), there exists a neighborhood of \( x_0 \), \( N_{x_0} \), such that, \( \forall x \in N_{x_0}, |f(x)| \leq \epsilon|g(x)| \).
In addition, if \( \{\phi_n(x)\}, n = 0, 1, 2, \ldots \), is a sequence of continuous functions such that, \( \forall n \in \mathbb{Z}^+ \),
\[ \phi_{n+1}(x) = o(\phi_n(x)), \quad x \to x_0 \]
then we write
\[ f(x) \sim \sum_{n=0}^{+\infty} a_n \phi_n(x), \quad x \to x_0 \]
where the formal series in the right-hand-side may or may not converge, if
\[ f(x) = \sum_{n=0}^{m} a_n \phi_n(x) + O(\phi_{m+1}(x)), \quad x \to x_0 \]
holds \( \forall m \in \mathbb{Z}_0^+ \) and we write
\[ f(x) \sim \sum_{n=0}^{N-1} a_n \phi_n(x), \quad x \to x_0 \]
if
\[ f(x) = \sum_{n=0}^{m} a_n \phi_n(x) + O(\phi_{m+1}(x)), \quad x \to x_0 \]
holds only for \( m \in \{0, 1, \ldots, N - 1\} \).

II. MODEL AND DEFINITIONS

We consider a standard frequency-flat fading channel, which for a single time instant, can be modeled as follows:
\[ y = \sqrt{\text{snr}}hx + n \tag{II.1} \]
where \( y \in \mathbb{C} \) represents the channel output, \( x \in \mathbb{C} \) represents the channel input, \( h \) is a complex scalar random variable (with support \( \mathbb{C} \) or \( \mathbb{C} \setminus \{0\} \)) such that \( E_h \{|h|^2\} < +\infty \) which represents the random channel fading gain between the input and the output of the channel, and \( n \in \mathbb{C} \) is a circularly symmetric complex scalar Gaussian random variable with zero mean and unit variance which represents standard noise. The scaling factor \( \text{snr} \in \mathbb{R}^+ \) relates to the signal-to-noise ratio. We assume that \( x, h \) and \( n \) are independent random variables. We also assume that the receiver knows the exact realization of the channel gain but the transmitter knows only the distribution of the channel gain.
In particular, we consider three conventional fading models: (i) the Rayleigh fading model; (ii) the Ricean fading model; and (iii) the Nakagami fading model. In the Rayleigh fading model, the channel gain $h \sim \mathcal{CN}(0, 2\sigma^2)$, where $\sigma > 0$, and hence $|h| \sim \text{Rayleigh}(\sigma)$ [15], i.e.,

$$f_{|h|}(r) = \frac{r}{\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right). \tag{II.2}$$

In the Ricean fading model, the channel gain $h \sim \mathcal{CN}(\mu, 2\sigma^2)$, where $\mu \in \mathbb{C} \setminus \{0\}$ and $\sigma > 0$, and hence $|h| \sim \text{Rice}(|\mu|, \sigma)$ [15], i.e.,

$$f_{|h|}(r) = \frac{r}{\sigma^2} \exp\left(-\frac{r^2 + |\mu|^2}{2\sigma^2}\right) I_0\left(\frac{r|\mu|}{\sigma^2}\right) \tag{II.3}$$

where $I_0(\cdot)$ is the modified Bessel function of the first kind with order zero [16, Equation 10.25.2]. In the Nakagami fading model, $|h| \sim \text{Nakagami}(\mu, w)$, where $\mu \geq \frac{1}{2}$ and $w > 0$ [15], i.e.,

$$f_{|h|}(r) = \frac{2\mu^\mu}{\Gamma(\mu) w^\mu} r^{2\mu-1} \exp\left(-\frac{\mu r^2}{w}\right)^w. \tag{II.4}$$

where $\Gamma(\cdot)$ is the Gamma function [16, Equation 5.2.1].

We now introduce a series of quantities which will be used throughout the paper. We define the conditional MMSE given that the channel gain $h = h_0$ associated with the estimation of the noiseless output given the noisy output of the channel model in (II.1) as

$$\text{mmse}_{h_0}(\text{snr}) := \mathbb{E}_{x,y,h_0}\left\{ |h_0x - E_{x|h_0} \{h_0x|y,h_0\}|^2 \bigg| h = h_0 \right\}$$

and the conditional mutual information given that the channel gain $h = h_0$ between the input and the output of the channel model in (II.1) as

$$I_{h_0}(\text{snr}) := \mathbb{E}_{x,y,h_0}\left\{ \log \left( \frac{f_{x,y|h_0}(x,y|h_0)}{f_{x|h_0}(x|h_0)f_{y|h_0}(y|h_0)} \right) \bigg| h = h_0 \right\}$$

We also define the average value of the MMSE and the average value of the mutual information as follows:

$$\overline{\text{mmse}}(\text{snr}) := \mathbb{E}_h \{ \text{mmse}_h(\text{snr}) \} \tag{II.5}$$

$$\overline{I}(\text{snr}) := \mathbb{E}_h \{ I_{h}(\text{snr}) \} \tag{II.6}$$

It will also be relevant to define the MMSE and the mutual information associated with the canonical additive white Gaussian noise (AWGN) channel model given by:

$$y = \sqrt{\text{snr}}x + n \tag{II.7}$$

where $y \in \mathbb{C}$ represents the channel output, $x \in \mathbb{C}$ represents the channel input and $n \in \mathbb{C}$ is a circularly symmetric complex scalar Gaussian random variable with zero mean and unit variance which represents standard noise. The scaling factor $\text{snr} \in \mathbb{R^+}$ also relates to the signal-to-noise ratio. We assume that $x$ and $n$ are independent random variables.

Now, the MMSE associated with the estimation of the input given the output of the canonical AWGN channel model in (II.7) is defined as:

$$\text{mmse}(\text{snr}) := \mathbb{E}_{x,y}\{ |x - E_{x|h_0} \{x|y\}|^2 \} = \text{mmse}_1(\text{snr}) \tag{II.8}$$
where \( \text{mmse}_1 (\text{snr}) \) denotes the conditional MMSE given that the channel gain \( h = 1 \) associated with the estimation of the noiseless output given the noisy output of the channel model in (II.1). The mutual information between the input and the output of the canonical AWGN channel model in (II.7) is defined as:

\[
I (\text{snr}) := E_{x,y} \left\{ \log \left( \frac{f_{x,y}(x,y)}{f_x(x) f_y(y)} \right) \right\} = I_1 (\text{snr}) \tag{II.9}
\]

where \( I_1 (\text{snr}) \) denotes the conditional mutual information given that the channel gain \( h = 1 \) between the input and the output of the channel model in (II.1). Therefore, it is very simple to express the average MMSE in (II.5) in terms of the canonical MMSE in (II.8) as:

\[
\text{mmse} (\text{snr}) = E_{|h|} \left\{ |h|^2 \text{mmse} (\text{snr}|h|^2) \right\} \tag{II.10}
\]

and the average mutual information in (II.6) in terms of the canonical mutual information in (II.9) as:

\[
I (\text{snr}) = E_{|h|} \left\{ I (\text{snr}|h|^2) \right\} \tag{II.11}
\]

The objective is to characterize the asymptotic behavior, as \( \text{snr} \to \infty \) or as \( \text{snr} \to 0^+ \), of the average MMSE and the average mutual information, which, when the channel variation over time is stationary and ergodic, leads to the constrained capacity of a fading coherent channel driven by a specific input distribution. We adopt a two-step procedure: i) We first obtain, via Mellin transform expansion techniques or via integration by parts expansion techniques, a characterization of the asymptotic behavior of the average MMSE in (II.5); ii) We then obtain a characterization of the asymptotic behavior of the average mutual information in (II.6) by capitalizing on the now well-known relation between average MMSE and average mutual information given by [4]:

\[
\frac{dI (\text{snr})}{d\text{snr}} = \text{mmse} (\text{snr}) \tag{II.12}
\]

We note that the Mellin transform of a function \( f(\cdot) \), which is defined as follows [14]:

\[
M [f; 1 + z] := \int_0^{+\infty} t^z f(t) \, dt,
\]

plays a key role in the definition of the asymptotic expansions.

III. HIGH-\text{SNR} REGIME

We now consider the construction of high-\text{snr} asymptotic expansions of the average MMSE and the average mutual information in a fading coherent channel driven by inputs that conform to either arbitrary discrete distributions (with finite support) or \( \infty \)-PSK, \( \infty \)-PAM, \( \infty \)-QAM, and standard complex Gaussian continuous distributions. The study of the first three continuous distributions, which represent the limit as \( m \to +\infty \) of the well-known equiprobable \( m \)-PSK, \( m \)-PAM or \( m \)-QAM discrete distributions, is justified by the fact that it is often analytically convenient to approximate discrete constellations using continuous distributions over a suitable region on the complex plane, in order to define meaningful asymptotic notions such as the shaping gain [17].

We proceed by obtaining a high-\text{snr} asymptotic expansion of the average MMSE and then – via the connection between average MMSE and average mutual information in (II.12) – a high-\text{snr} asymptotic expansion of the average...
mutual information. The construction of the high-snr expansions capitalizes on the fact that the average MMSE, which can be written as

\[ \text{MMSE}(\text{snr}) = E_{|h|} \left\{ |h|^2 \text{mmse}(\text{snr}|h|^2) \right\} = \int_0^{+\infty} \frac{\sqrt{t} f_{|h|} (\sqrt{t})}{2} \text{mmse}(\text{snr} \cdot t) \, dt, \]  

(III.1)
can be seen as an \( h \)-transform with a kernel of monotonic argument \[14\] Chapter 4. This enables the use of a key asymptotic expansion of integrals technique, which exploits Mellin transforms \[14\] Section 4.4, that leads to a dissection of the asymptotic behavior of the quantities in a large variety of scenarios. In fact, the construction of the high-snr expansions of the integral representation in (III.1) can be effected by exploiting a range of techniques, such as the Mellin transform method \[14\] Section 4.4 or the integration by parts methods \[14\] Chapter 3. It is important to emphasize though that the Mellin transform technique – when compared to the integration by parts technique – is able to produce expansions for a wider range of fading and input distributions.

A. Discrete Inputs

We consider arbitrary discrete inputs with finite support \( \{x_1, \ldots, x_m\} \) with \( m \in \mathbb{Z}^+ \) and \( x_1, \ldots, x_m \in \mathbb{C} \). The construction of the high-snr expansions of the average MMSE associated with arbitrary discrete inputs with finite support capitalizes on the characterization of the decay of the canonical MMSE in the regime of high-snr in [1].

The following Theorem defines the asymptotic expansion of the average MMSE in fading coherent channels driven by arbitrary (but fixed) discrete inputs with finite support, by capitalizing on the Mellin transform method \[14\] Section 4.4.

**Theorem III.1.** Consider the fading coherent channel in (II.1) driven by an arbitrary discrete input with finite support and define

\[ f(t) := \frac{\sqrt{t} f_{|h|} (\sqrt{t})}{2} \]

which relates to the distribution of the fading process and

\[ \gamma := \inf \left\{ \gamma^* : f(t) = O \left(t^{-\gamma^*}\right), t \to 0^+ \right\}, \]
\[ \delta := \sup \left\{ \delta^* : f(t) = O \left(t^{-\delta^*}\right), t \to +\infty \right\}, \]
\[ C := \{0, +\infty] \cap 1 - \delta, 1 - \gamma[. \]

If

\[ f(t) \sim \exp \left(-qt^{-\mu}\right) \sum_{m=0}^{+\infty} \sum_{n=0}^{N(m)} p_{mn} t^m (\log(t))^n, \quad t \to 0^+ \]  

(III.2)

where \( \mathcal{R}(q) \geq 0, \mu > 0, \mathcal{N}(m) \) is finite for each \( m, p_{mn} \in \mathbb{C} \) and \( \mathcal{R}(a_m) \uparrow +\infty, \)

\[ \gamma < \delta \]  

(III.3)

\[ C \neq \emptyset \]  

(III.4)

\[ \exists c \in C : \forall x \in [c, +\infty[, M[f; 1 - x - iy] = O \left(|y|^{-2}\right), \quad |y| \to +\infty \]  

(III.5)

May 3, 2014 DRAFT
where \( M [f; 1 - x - iy] \) is to be understood as the analytic continuation of \( M [f; 1 - x - iy] \) from \( \{ x + iy : 1 - \delta < x < 1 - \gamma \} \) to \( \{ x + iy : x > 1 - \delta \} \) \cite{18}, then

1) If \( q = 0 \) it follows that

\[
\text{mmse} (\text{snr}) \sim \\
\sim \sum_{m=0}^{+\infty} \text{snr}^{-1-a_m} \sum_{n=0}^{\infty} n^n \sum_{j=0}^{m} (-\log (\text{snr}))^j M^{(n-j)} [\text{mmse}; z] \bigg|_{z=1+a_m}, \quad \text{snr} \to +\infty \tag{III.6}
\]

2) If \( q \neq 0 \) it follows that

\[
\forall R \in \mathbb{R}^+, \text{mmse} (\text{snr}) = o (\text{snr}^{-R}), \quad \text{snr} \to +\infty \tag{III.7}
\]

**Proof:** See Appendix A.

Theorem III.1 provides a dissection of the asymptotic behavior as \( \text{snr} \to +\infty \) of the average MMSE in a fading coherent channel driven by arbitrary discrete inputs (with finite support) under very general fading conditions. In particular, the Theorem requires that the asymptotic expansion as \( t \to 0^+ \) of \( f (t) \), which relates to the probability density function of the fading random variable, behaves as in (III.2) and that the Mellin transform of the function \( f (\cdot) \) exists, is holomorphic in a certain vertical strip in the complex plane and decays along vertical lines as in (III.5). If the fading model does not accommodate for exponential decay of \( f (t) \) as \( t \to 0^+ \), i.e., \( q = 0 \), then the average MMSE behaves as in (III.6); otherwise, if the fading model accommodates for exponential decay of \( f (t) \) as \( t \to 0^+ \), i.e., \( q \neq 0 \), then the average MMSE behaves as in (III.7). The Theorem also requires the existence of the Mellin transform of the canonical MMSE and their higher-order derivatives, which is guaranteed by the fact that the input distribution is discrete without accumulation points.

Theorem III.1 reveals that in the important scenario which accommodates for the most common fading models (i.e., \( q = 0 \)) the asymptotic behavior as \( \text{snr} \to +\infty \) of the average MMSE depends mainly on the asymptotic behavior as \( t \to 0^+ \) of \( f (t) \). Interestingly, the fact that the behavior of some quantities around zero determine the behavior of other quantities in the infinity has already been pointed out in e.g. \cite{19}, \cite{20}, \cite{21}. Theorem III.1 also reveals that the asymptotic behavior as \( \text{snr} \to +\infty \) of the average MMSE depends on the input distribution via the Mellin transform of the canonical MMSE.

Theorem III.1 can also now be immediately specialized to the most common fading distributions, including: \( i \) Rayleigh fading; \( ii \) Ricean fading; and \( iii \) Nakagami fading.

**Corollary III.2.** Consider a Rayleigh or Ricean fading coherent channel as in (II.1) driven by an arbitrary discrete input with finite support, where (II.2) or (II.3) hold, respectively. Then, in the regime of high-snr the average MMSE behaves as follows:

\[
\text{mmse} (\text{snr}) \sim \\
\sim \frac{\exp \left( -\frac{|\mu|^2}{2\sigma^2} \right)}{\text{snr}^2} \sum_{m=0}^{+\infty} \left( \sum_{n=0}^{m} \frac{(-1)^{m-n}}{(m-n)!} \frac{|\mu|^{2n}}{(2\sigma^2)^{m+n+1}} M [\text{mmse}; 2 + m] \right) \frac{1}{\text{snr}^m}, \quad \text{snr} \to +\infty
\]
Corollary III.3. Consider a Nakagami fading coherent channel as in (II.1) driven by an arbitrary discrete input with finite support, where (II.4) holds. Then, in the regime of high-$\text{snr}$ the average MMSE behaves as follows:

$$\frac{\text{mmse} (\text{snr})}{\text{snr}^{\mu}} \sim \frac{\mu^\mu}{\Gamma (\mu)} \sum_{m=0}^{+\infty} \left( \frac{(-1)^m \mu^m}{m!w^m} M [\text{mmse}; 1 + \mu + m] \right) \frac{1}{\text{snr}^m}, \quad \text{snr} \to +\infty$$

Proof: See Appendix C.

Corollary III.4. Consider a Rayleigh or Ricean fading coherent channel as in (II.1) driven by an arbitrary discrete input with finite support, where (II.2) or (II.3) hold, respectively. Then, in the regime of high-$\text{snr}$ the average mutual information obeys the expansion:

$$\frac{\text{I} (\text{snr})}{\text{snr}} \sim \log (m) - \exp \left( -\frac{\| \mu \|^2}{2\sigma^2} \right) \sum_{m=0}^{+\infty} \left( \sum_{n=0}^{m} \frac{(-1)^{m-n}}{(m+1)(m-n)!(2\sigma^2)^{m+n+1}} \frac{|\mu|^2_2}{(n!)^2} M [\text{mmse}; 2 + m] \right) \frac{1}{\text{snr}^m}, \quad \text{snr} \to +\infty$$

Proof of Corollaries III.4 and III.5. The Corollaries follow immediately from Corollaries III.2 or III.3 and the relationship in (III.8), together with the fact that an order relation can be integrated with respect to the independent variable [14].

It is now relevant to reflect on the nature of the asymptotic expansions embodied in Corollaries III.2, III.3, III.4 and III.5. The expansions expose a dissection of the asymptotic behavior as $\text{snr} \to +\infty$ of the quantities. For Rayleigh and Ricean fading the $(m+1)$-th term in the average MMSE expansion is $O \left( \text{snr}^{-m-2} \right)$ and the
\( (m + 1) \)-th term in the average mutual information expansion is \( O \left( \frac{\text{snr}^{-m-1}}{\text{snr}} \right) \); in contrast, for Nakagami fading the \( (m + 1) \)-th terms in the average MMSE and the average mutual information expansions are \( O \left( \frac{\text{snr}^{-m-1-\mu}}{\text{snr}^{1+\mu}} \right) \) and \( O \left( \frac{\text{snr}^{-m-\mu}}{\text{snr}^{1+\mu}} \right) \), respectively. In Rayleigh or Ricean fading channels, a first-order high-\( \text{snr} \) expansion of the quantities can be written as follows:

\[
\overline{\text{mmse}}(\text{snr}) = \epsilon \cdot \frac{1}{\text{snr}^2} + O \left( \frac{1}{\text{snr}^3} \right)
\]

and

\[
\overline{I}(\text{snr}) \sim \log (m) - \epsilon \cdot \frac{1}{\text{snr}^2} + O \left( \frac{1}{\text{snr}^3} \right)
\]

where

\[
\epsilon = \lim_{\text{snr} \to +\infty} \text{snr}^2 \cdot \overline{\text{mmse}}(\text{snr}) = \frac{\exp \left( \frac{-|\mu|^2}{2\sigma^2} \right)}{2\sigma^2} M \left[ \text{mmse}; 2 \right]
\]

In contrast, in Nakagami fading channels, a first-order high-\( \text{snr} \) expansion of the average MMSE and the average mutual information can be written as follows:

\[
\overline{\text{mmse}}(\text{snr}) = \epsilon \cdot \frac{1}{\text{snr}^{1+\mu}} + O \left( \frac{1}{\text{snr}^{2+\mu}} \right)
\]

and

\[
\overline{I}(\text{snr}) \sim \log (m) - \epsilon \cdot \frac{1}{\text{snr}^{1+\mu}} + O \left( \frac{1}{\text{snr}^{2+\mu}} \right)
\]

where

\[
\epsilon = \lim_{\text{snr} \to +\infty} \text{snr}^{1+\mu} \cdot \overline{\text{mmse}}(\text{snr}) = \frac{\mu^\mu}{\Gamma(\mu) \left[ \text{w}^\mu \right] M \left[ \text{mmse}; 1 + \mu \right]}
\]

This shows that the rate (on a \( \log(\text{snr}) \) scale) at which the average MMSE or the average mutual information tend to the infinite-\( \text{snr} \) value is given by:

\[
- \lim_{\text{snr} \to +\infty} \frac{\log \left( \overline{\text{mmse}}(\text{snr}) \right)}{\log (\text{snr})} = 2 \quad \text{(III.9)}
\]

and

\[
- \lim_{\text{snr} \to +\infty} \frac{\log (m) - \overline{I}(\text{snr})}{\log (\text{snr})} = 1 \quad \text{(III.10)}
\]

respectively, both for Rayleigh and Ricean fading channels, and is equal to

\[
- \lim_{\text{snr} \to +\infty} \frac{\log \left( \overline{\text{mmse}}(\text{snr}) \right)}{\log (\text{snr})} = 1 + \mu
\]

and

\[
- \lim_{\text{snr} \to +\infty} \frac{\log (m) - \overline{I}(\text{snr})}{\log (\text{snr})} = \mu
\]

respectively, for Nakagami fading channels. The parameter \( \epsilon \), which is akin to the MMSE dimension put forth in [22], provides a more refined representation of the high-\( \text{snr} \) asymptotics. The incorporation of a higher number of terms in the expansions provides a more accurate approximation of the behavior of the quantities not only at high-\( \text{snr} \) but also medium-\( \text{snr} \).

Finally, it is also relevant to note that one can also construct asymptotic expansions by using other asymptotic expansions of integrals techniques. The following Theorem defines the asymptotic expansion of the average MMSE.
in fading coherent channels driven by arbitrary (but fixed) discrete inputs with finite support, which follows from an integration by parts method [14, Chapter 3] rather than the Mellin transform method [14, Section 4.4].

**Theorem III.6.** Consider the fading coherent channel in (II.1) driven by an arbitrary discrete input with finite support and define

\[ f(t) := \frac{\sqrt{tf|h|}(\sqrt{t})}{2} \]

which relates to the distribution of the fading process. If

\[ f \in C^\infty(R^+) \]

\[ \forall n \in Z_0^+, \exists \lim_{t \to 0^+} f^{(n)}(t) < +\infty \]

\[ \forall n \in Z_0^+, f^{(n)}(t) = O(t^{-2}), \quad t \to +\infty \]

then

\[ \text{mmse}(\text{snr}) \sim \sum_{m=0}^{+\infty} \frac{(-1)^{m+1}}{\text{snr}^{m+1}} f^{(m)}(0) \text{mmse}^{(-m-1)}(0), \quad \text{snr} \to +\infty \]

where

\[ \forall m \in Z_0^+, \text{mmse}^{(-m-1)}(x) := \int_{+\infty}^{x} \int_{+\infty}^{t_1} \cdots \int_{+\infty}^{t_m-1} \int_{+\infty}^{t_m} \text{mmse}(t_{m+1}) dt_{m+1} dt_m \cdots dt_2 dt_1 \]

**Proof:** See Appendix D.

We note that the asymptotic expansion of the average MMSE provided in Theorem [III.1] is defined via the Mellin transform of the canonical MMSE whereas the asymptotic expansion of the average MMSE provided in Theorem [III.6] is defined in terms of repeated integrals of the canonical MMSE. It is possible though to reconcile the asymptotic expansion in Theorem [III.1] with the asymptotic expansion in Theorem [III.6] in some key scenarios. In particular, if the requirements of Theorems [III.1] and [III.6] hold and

\[ (\forall m \in Z_0^+, (N(m) = 0 \land a_m = m)) \land q = 0 \]

then the asymptotic expansions [III.6] and [III.14] coincide because

\[ \text{snr}^{-1-a_m} \sum_{n=0}^{N(m)} p_m \sum_{j=0}^{n} \binom{n}{j} \left( -\log(\text{snr}) \right)^j M^{(n-j)} [\text{mmse}; z] \bigg|_{z=1+a_m} = \text{snr}^{-1-m} p_m M [\text{mmse}; 1 + m] \]

\[ = \text{snr}^{-1-m} p_m (-1)^{m+1} m! \text{mmse}^{(-m-1)}(0) \]

\[ = \frac{(-1)^{m+1}}{\text{snr}^{m+1}} f^{(m)}(0) \text{mmse}^{(-m-1)}(0) \]

where the first and the last equalities are due to [III.16] and the second equality is due to

\[ \text{mmse}^{(-m-1)}(0) = \frac{(-1)^{m+1}}{m!} M[\text{mmse}; m + 1] \]
in view of (III.15) and [16, Equation 1.4.31].

There are various scenarios of relevance in practice where both Theorems produce the same expansions, such as the Rayleigh and Ricean fading coherent channel driven by arbitrary discrete inputs with finite support, because the Theorem requirements hold and (III.16) also holds. However, there are also scenarios where the Mellin transform method yields an expansion but the integration by parts method does not. One important case relates to the definition of the asymptotic expansions in a fading coherent channel driven by arbitrary discrete inputs with finite support, subject to Nakagami fading (where \(|h| \sim \text{Nakagami}(\mu, w)\)). The Mellin transform method allows us to define the asymptotic expansions when the fading parameter \(\mu \in \left[\frac{1}{2}, +\infty\right]\). In contrast, the integration by parts method only allows us to define the asymptotic expansions when the fading parameter \(\mu \in \left[\frac{1}{2}, +\infty\right) \setminus \mathbb{Z}^+\). This is due to the fact that such a method requires stronger smoothness properties of the integrand functions. The other important case, which is the subject of the next subsection, relates to the definition of the asymptotic expansions in a fading coherent channel driven by continuous inputs. Note that in such a setting the repeated integrals of the canonical MMSE – which are necessary to conceive the asymptotic expansions via the integration by parts method – do not often exist.

B. Continuous Inputs

We now consider the continuous inputs: i) unit-power \(\infty\)-PSK, where \(x\) is uniformly distributed over the circle \(\{z \in \mathbb{C} : |z| = 1\}\) on the complex plane; ii) unit-power \(\infty\)-PAM, where \(x\) is uniformly distributed over the line \([-\sqrt{3}, \sqrt{3}]\) on the complex plane; iii) unit-power \(\infty\)-QAM, where \(x\) is uniformly distributed over the square \([-\sqrt{2}, \sqrt{2}] \times [-\sqrt{2}, \sqrt{2}]\) on the complex plane; and iv) standard complex Gaussian inputs, where \(x\) is distributed according to \(\mathcal{CN}(0, 1)\).

The construction of the high-\(\text{snr}\) expansions of the average MMSE associated with \(\infty\)-PSK, \(\infty\)-PAM and \(\infty\)-QAM uses the high-\(\text{snr}\) expansions of the canonical MMSEs given by [1]:

\[
\text{mmse}^{\infty-\text{PSK}}(\text{snr}) = \frac{1}{2\text{snr}} + O\left(\frac{1}{\text{snr}^2}\right), \quad \text{snr} \to +\infty \quad (\text{III.17})
\]

\[
\text{mmse}^{\infty-\text{PAM}}(\text{snr}) = \frac{1}{2\text{snr}} + O\left(\frac{1}{\text{snr}^2}\right), \quad \text{snr} \to +\infty \quad (\text{III.18})
\]

\[
\text{mmse}^{\infty-\text{QAM}}(\text{snr}) = \frac{1}{\text{snr}} + O\left(\frac{1}{\text{snr}^2}\right), \quad \text{snr} \to +\infty \quad (\text{III.19})
\]

respectively. It is important to note that this one-term expansions of the canonical MMSE lead to one-term expansions for the average MMSE; higher-order expansions for the canonical MMSE would lead to higher-order expansions for the average MMSE, by exploiting identical techniques. In contrast, the construction of the high-\(\text{snr}\) expansions of the average MMSE associated with standard complex Gaussian inputs uses the high-\(\text{snr}\) expansion of the canonical MMSE given by [1]:

\[
\text{mmse}^G(\text{snr}) = \frac{1}{1 + \text{snr}} = \frac{1}{\text{snr}} + \frac{1}{\text{snr}^2} + \sum_{m=0}^{+\infty} (-1)^m \text{snr}^{-m+1}, \quad \text{snr} \to +\infty \quad (\text{III.20})
\]
The following Theorem defines the asymptotic expansions of the average MMSE in a fading coherent channel driven by $\infty$-PSK, $\infty$-PAM, $\infty$-QAM or standard complex Gaussian continuous input distributions. It is relevant to note that the proof relies on the use of the more powerful Mellin transform asymptotic expansion of integrals technique.

**Theorem III.7.** Consider the fading coherent channel in (II.1) driven by an $\infty$-PSK, $\infty$-PAM, $\infty$-QAM or standard complex Gaussian continuous input and define

$$f(t) := \sqrt{tf|h|} \left(\sqrt{t}\right)$$

which relates to the distribution of the fading process and

$$\gamma := \inf \left\{ \gamma^* : f(t) = O\left(t^{-\gamma^*}\right), t \to 0^+ \right\}$$

$$\delta := \sup \left\{ \delta^* : f(t) = O\left(t^{-\delta^*}\right), t \to +\infty \right\}$$

$$C := [0, 1[ \cap [1 - \delta, 1 - \gamma]$$

If

$$f(t) \sim \exp\left(-qt^{-\mu}\right) \sum_{m=0}^{+\infty} \sum_{n=0}^{N(m)} p_{mn} t^m \left(\log(t)\right)^n, \quad t \to 0^+ \quad (III.21)$$

where $R(q) \geq 0, \mu > 0, N(m)$ is finite for each $m$ and $R(a_m) \uparrow +\infty$,

$$x \notin CN(0, 1) \land q = 0 \Rightarrow R(a_0) > 0 \quad (III.22)$$

$$\gamma < \delta \quad (III.23)$$

$$C \neq \emptyset \quad (III.24)$$

$$\exists c \in C : \forall x \in [c, +\infty], M[f; 1 - x - iy] = O\left(|y|^{-2}\right), \quad |y| \to +\infty \quad (III.25)$$

where $M[f; 1 - x - iy]$ is to be understood as the analytic continuation of $M[f; 1 - x - iy]$ from $\{x + iy : 1 - \delta < x < 1 - \gamma\}$ to $\{x + iy : x > 1 - \delta\}$ [18], then

1) If $q = 0$ then:

a) If $x$ is distributed according to $\infty$-PSK it follows that

$$\overline{\text{MMSE}} (\text{snr}) \sim \frac{M[f; 0]}{2\text{snr}} + O\left(\frac{1}{\text{snr}R}\right), \quad \text{snr} \to +\infty, \quad \forall R \in \left[1, \min\{R(a_0) + 1, 2\}\right]$$

b) If $x$ is distributed according to $\infty$-PAM it follows that

$$\overline{\text{MMSE}} (\text{snr}) \sim \frac{M[f; 0]}{2\text{snr}} + O\left(\frac{1}{\text{snr}R}\right), \quad \text{snr} \to +\infty, \quad \forall R \in \left[1, \min\{R(a_0) + 1, 3/2\}\right]$$

c) If $x$ is distributed according to $\infty$-QAM it follows that

$$\overline{\text{MMSE}} (\text{snr}) \sim \frac{M[f; 0]}{\text{snr}} + O\left(\frac{1}{\text{snr}R}\right), \quad \text{snr} \to +\infty, \quad \forall R \in \left[1, \min\{R(a_0) + 1, 3/2\}\right]$$

May 3, 2014 DRAFT
d) If \( x \) is distributed according to standard complex Gaussian it follows that

\[
\text{mmse} (\text{snr}) \sim \sum_{m \in R} (-1)^m M [f; -m] + \\
\sum_{m \in A} \frac{1}{\text{snr}^{1+a_m}} \sum_{n=0}^{N(m)} \sum_{j=0}^{(n-j)} \left( -1 \right)^j \left( \frac{d}{dz} \right)^{(n-j)} \left( \frac{\pi}{\sin(\pi z)} \right) \bigg|_{z=1+a_m} \\
+ \sum_{m \in RA} \frac{1}{\text{snr}^{m+1}} \sum_{n=0}^{(N(m)+1)} j! (N(m) + 1 - j)! \times \\
\left( \frac{d}{dz} \right)^{(N(m)+1-j)} \left( \frac{\pi}{\sin(\pi z)} \right) \bigg|_{z=m+1}
\]

where

\[
R := \{ m \in \mathbb{Z}^+_0 : \not\exists n \in \mathbb{Z}^+_0 : m = a_n \} \\
A := \{ n \in \mathbb{Z}^+_0 : \not\exists m \in \mathbb{Z}^+_0 : m = a_n \} \\
RA := \{ m \in \mathbb{Z}^+_0 : m+1 \in \{ z \in \mathbb{C} : \exists (p,q) \in \mathbb{Z}^+_0 \times \mathbb{Z}^+_0 : z = p+1 = a_q + 1 \} \}
\]

2) If \( q \neq 0 \) then:

a) If \( x \) is distributed according to \( \infty \)-PSK it follows that

\[
\text{mmse} (\text{snr}) \sim \frac{M [f; 0]}{2\text{snr}} + O \left( \frac{1}{\text{snr}^R} \right), \quad \text{snr} \to +\infty, \quad \forall R \in ]1, 2[
\]

b) If \( x \) is distributed according to \( \infty \)-PAM it follows that

\[
\text{mmse} (\text{snr}) \sim \frac{M [f; 0]}{2\text{snr}} + O \left( \frac{1}{\text{snr}^R} \right), \quad \text{snr} \to +\infty, \quad \forall R \in ]1, \frac{3}{2}[
\]

c) If \( x \) is distributed according to \( \infty \)-QAM it follows that

\[
\text{mmse} (\text{snr}) \sim \frac{M [f; 0]}{\text{snr}} + O \left( \frac{1}{\text{snr}^R} \right), \quad \text{snr} \to +\infty, \quad \forall R \in ]1, \frac{3}{2}[
\]

d) If \( x \) is distributed according to standard complex Gaussian it follows that

\[
\text{mmse} (\text{snr}) \sim \sum_{m=0}^{+\infty} (-1)^m \frac{M [f; -m]}{\text{snr}^{m+1}}, \quad \text{snr} \to +\infty
\]

**Proof:** See Appendix E.

Theorem [III.7] provides a simple asymptotic expansion as \( \text{snr} \to +\infty \) of the average MMSE in a fading coherent channel driven by \( \infty \)-PSK, \( \infty \)-PAM, \( \infty \)-QAM or standard complex Gaussian continuous inputs under very general fading conditions. In particular, this Theorem – akin to Theorem [III.1] – only requires that the asymptotic expansion as \( t \to 0^+ \) of \( f (t) \), which relates to the probability density function of the fading random variable, behaves as in [III.21] and that the Mellin transform of the function \( f (\cdot) \) exists, is holomorphic in a certain vertical strip in the complex plane and decays along vertical lines as in [III.25]. The error term in the asymptotic expansion is more or
less refined depending on whether the fading model does not \((q = 0)\) or does \((q \neq 0)\) accommodate for exponential decay of \(f(t)\) as \(t \to 0^+\).

This Theorem – akin to Theorem III.1 – reveals that the asymptotic behavior as \(snr \to +\infty\) of the average MMSE also depends on the asymptotic behavior as \(t \to 0^+\) of \(f(t)\). The proof of the Theorem also reveals that the canonical MMSE – via its Mellin transform – plays an important role in the construction of the asymptotic behavior of the average MMSE.

Theorem III.7 can also be immediately specialized to the most common fading distributions.

Corollary III.8. Consider a Rayleigh or Ricean fading coherent channel as in (II.1) driven by an \(\infty\)-PSK, \(\infty\)-PAM, \(\infty\)-QAM or standard complex Gaussian continuous input, where (II.2) or (II.3) hold, respectively. Then, in the regime of high-\(snr\) the average MMSE behaves as follows:

1) If \(x\) is distributed according to \(\infty\)-PSK then

\[
\text{mmse}(snr) \sim \frac{1}{2snr} + O\left(\frac{1}{snr^R}\right), \quad snr \to +\infty, \quad \forall R \in [1, 2[.
\]

2) If \(x\) is distributed according to \(\infty\)-PAM then

\[
\text{mmse}(snr) \sim \frac{1}{2snr} + O\left(\frac{1}{snr^R}\right), \quad snr \to +\infty, \quad \forall R \in \left]1, \frac{3}{2}\right[.
\]

3) If \(x\) is distributed according to \(\infty\)-QAM then

\[
\text{mmse}(snr) \sim \frac{1}{snr} + O\left(\frac{1}{snr^R}\right), \quad snr \to +\infty, \quad \forall R \in \left]1, \frac{3}{2}\right[.
\]

4) If \(x\) is distributed according to standard complex Gaussian it follows that

\[
\text{mmse}(snr) \sim \frac{1}{snr} + \frac{1}{snr^2} \sum_{m=0}^{\infty} \frac{1}{snr^m} \sum_{j=0}^{1} \frac{(-\log(snr))^j}{j!(1-j)!} \times
\]

\[
\times \left(\frac{d}{dz}\right)^{(1-j)} \left\{ \left(z - m - 2\right)^2 \pi \exp\left(-\frac{|\mu|^2}{2\sigma^2}\right) (2\sigma^2)^{1-z} \Gamma(2-z) I_1(2-z; 1; \frac{|\mu|^2}{2\sigma^2}) \right\}_{z=m+2}
\]

where \(I_1(a; b; c)\) is the confluent hypergeometric series [16, Equation 13.2.2].

Proof: The conditions required for this specialization have already been established in Subappendices B-A and B-B. Note also that \(R = \{0\}, A = \emptyset\) and \(RA = \mathbb{Z}^+\) in both Rayleigh and Ricean cases.

Corollary III.9. Consider a Nakagami fading coherent channel as in (II.1) driven by an \(\infty\)-PSK, \(\infty\)-PAM, \(\infty\)-QAM or standard complex Gaussian continuous input, where (II.4) holds. Then, in the regime of high-\(snr\) the average MMSE behaves as follows:

1) If \(x\) is distributed according to \(\infty\)-PSK then

\[
\text{mmse}(snr) \sim \frac{1}{2snr} + O\left(\frac{1}{snr^R}\right), \quad snr \to +\infty, \quad \forall R \in [1, \min\{\mu + 1, 2\}].
\]
2) If $x$ is distributed according to $\infty$-PAM then 
\[ \text{mmse}(\text{snr}) \sim \frac{1}{2\text{snr}} + O\left(\frac{1}{\text{snr}^R}\right), \quad \text{snr} \to +\infty, \quad \forall R \in \left[\frac{1}{2}, \frac{3}{2}\right] \]

3) If $x$ is distributed according to $\infty$-QAM then 
\[ \text{mmse}(\text{snr}) \sim \frac{1}{\text{snr}} + O\left(\frac{1}{\text{snr}^R}\right), \quad \text{snr} \to +\infty, \quad \forall R \in \left[\frac{1}{2}, \frac{3}{2}\right] \]

4) If $x$ is distributed according to standard complex Gaussian it follows that 
   a) If $\mu \in \left[\frac{1}{2}, +\infty\right] \cap \mathbb{Z}^+$ then:
   \[ \text{mmse}(\text{snr}) \sim \frac{1}{\text{snr}} \sum_{m=0}^{\mu-1} (-1)^m \frac{1}{\Gamma(\mu)} \left(\frac{w}{\text{snr}}\right)^{-m} \Gamma(\mu - m) \]
   \[ + \frac{1}{\text{snr}^{1+\mu}} \sum_{m=0}^{+\infty} \frac{1}{\text{snr}^m} \sum_{j=0}^{1} (-\log(\text{snr}))^j j!(1-j)! \times \]
   \[ \times \left(\frac{d}{dz}\right)^{(1-j)} \left\{ (z - m - 1 - \mu)^2 \frac{\Gamma(\mu + 1 - z)}{\sin(\pi z)} \right\} \bigg|_{z=m+1+\mu} \], \quad \text{snr} \to +\infty
   
   b) If $\mu \in \left[\frac{1}{2}, +\infty\right] \setminus \mathbb{Z}^+$ then:
   \[ \text{mmse}(\text{snr}) \sim \frac{1}{\text{snr}} \sum_{m=0}^{+\infty} (-1)^m \frac{1}{\Gamma(\mu)} \left(\frac{w}{\text{snr}}\right)^{-m} \Gamma(\mu - m) \]
   \[ + \frac{1}{\text{snr}^{1+\mu}} \sum_{m=0}^{+\infty} \frac{1}{\text{snr}^m} \mu^m (-1)^m \frac{\mu^m}{\Gamma(\mu + 1 + \mu + m)} \Gamma(\mu + 1 + \mu + m) \]
   \[ \quad \text{snr} \to +\infty \]

Proof: The conditions required for this specialization have already been established in Appendix C. Note also that

\[ R = \begin{cases} 
0, 1, 2, \ldots, \mu - 1 & \text{if } \mu \in \left[\frac{1}{2}, +\infty\right] \cap \mathbb{Z}^+, \\
\mathbb{Z}_0^+ & \text{if } \mu \in \left[\frac{1}{2}, +\infty\right] \setminus \mathbb{Z}^+,
\end{cases} \]

\[ A = \begin{cases} 
\emptyset & \text{if } \mu \in \left[\frac{1}{2}, +\infty\right] \cap \mathbb{Z}^+, \\
\mathbb{Z}_0^+ & \text{if } \mu \in \left[\frac{1}{2}, +\infty\right] \setminus \mathbb{Z}^+,
\end{cases} \]

\[ RA = \begin{cases} 
\{\mu, \mu + 1, \mu + 2, \ldots\} & \text{if } \mu \in \left[\frac{1}{2}, +\infty\right] \cap \mathbb{Z}^+, \\
\emptyset & \text{if } \mu \in \left[\frac{1}{2}, +\infty\right] \setminus \mathbb{Z}^+.
\end{cases} \]

in the Nakagami fading case.

It is evident that the high-$\text{snr}$ behavior of the average MMSE for the most common fading coherent channel models is rather distinct for discrete and continuous inputs. For systems driven by discrete inputs the leading term...
of the high-snr asymptotic expansion of the average MMSE is $O \left( \frac{1}{\text{snr}} \right)$ for Rayleigh and Ricean fading channels and $O \left( \frac{1}{\text{snr}^{1+\mu}} \right)$ for Nakagami fading channels; in contrast, for systems driven by $\infty$-PSK, $\infty$-PAM, $\infty$-QAM or standard complex Gaussian inputs the leading term in the asymptotic expansion of the average MMSE is $O \left( \frac{1}{\text{snr}} \right)$ for the four fading models. This is consistent with the fact that the high-snr average mutual information for systems with continuous inputs is higher than for systems with discrete inputs, i.e., it grows without bound for continuous inputs but it is bounded by the input cardinality for discrete inputs.

The high-snr asymptotic expansions of the average MMSE embodied in Corollaries III.8 and III.9 do not lead directly to a high-snr asymptotic expansion of the average mutual information, because it is not possible to explore a suitable integral representation of the connection between average MMSE and average mutual information in II.12.

IV. LOW-SNR REGIME

We now consider the construction of low-snr asymptotic expansions of the average MMSE and the average mutual information in a fading coherent channel driven by a range of inputs. The element of novelty, in view of the fact that examples of low-snr asymptotic expansions of a series of estimation- and information-theoretic quantities are plentiful (see e.g. [23], [24]), relates to the use of Mellin transform expansions techniques to study the behavior of the quantities in such an asymptotic regime. This distinct unconventional approach, as opposed to the common method that involves in some suitable integral representation of the quantities the substitution of the integrand or part of the integrand by a series and its integration term by term, also illustrates that with Mellin transform expansions techniques one can often derive with little effort the asymptotic expansions as $\text{snr} \to 0^+$ from asymptotic expansions as $\text{snr} \to +\infty$ and vice versa thereby coupling the regimes [14].

We also proceed by obtaining a low-snr asymptotic expansion of the average MMSE and then – via the connection between average MMSE and average mutual information in II.12 – a low-snr asymptotic expansion of the average mutual information.

A. Discrete and Continuous Inputs

We consider arbitrary discrete inputs with finite support and unit-power $\infty$-PSK, $\infty$-PAM, $\infty$-QAM or standard complex Gaussian continuous input[1]. The following Theorem, which applies to relatively general fading models, provides the asymptotic expansion of the average MMSE in fading coherent channels driven by a range of inputs.

Theorem IV.1. Consider the fading coherent channel in II.11 driven by inputs that conform to: i) a discrete distribution with finite support; or ii) a $\infty$-PSK, $\infty$-PAM, $\infty$-QAM or standard complex Gaussian continuous distribution. Define

$$h_1(t) := \frac{\sqrt{t} f_{|h|}(\sqrt{t})}{2}$$

[1]The expansions are also applicable to finite-power inputs, i.e., $E_x\{|x|^2\} < +\infty$, such that the canonical MMSE obeys $\text{mmse}(\text{snr}) = O \left( \frac{1}{\text{snr}} \right)$, $\text{snr} \to +\infty$. 

May 3, 2014 DRAFT
which relates to the distribution of the fading process and

\[ \alpha_1 := \inf \{ \alpha^*_1 : h_1(t) = O(t^{-\alpha^*_1}), t \to 0^+ \} \]

\[ \beta_1 := \sup \{ \beta^*_1 : h_1(t) = O(t^{-\beta^*_1}), t \to +\infty \} \]

\[ C_1 := \begin{cases} [\alpha_1, \beta_1[ - \infty, 1[ & \text{if } x \text{ is discrete}, \\ \alpha_1, \beta_1[0, 1[ & \text{if } x \text{ is continuous}. \end{cases} \]

If

\[ h_1(t) = O(\exp(-k_1 t^{v_1})), \quad t \to +\infty \] (IV.1)

where \( R(k_1) > 0 \) and \( v_1 > 0 \),

\[ \alpha_1 < \beta_1 \] (IV.2)

\[ C_1 \neq \emptyset \] (IV.3)

\[ \exists c_1 \in C_1 : \forall x \in [c_1, +\infty[, M[h_1; x + iy] = O(|y|^{-2}), \quad |y| \to +\infty \] (IV.4)

where \( M[h_1; x + iy] \) is to be understood as the analytic continuation of \( M[h_1; x + iy] \) from \( \{x + iy : \alpha_1 < x < \beta_1\} \) to \( \{x + iy : x > \alpha_1\} \) [18], then

\[ \operatorname{mmse}(\text{snr}) \sim \exp\left(-\frac{|\mu|^2}{2\sigma^2}\right) \sum_{m=0}^{+\infty} \frac{1}{m!} M[h_1; m + 1] \operatorname{mmse}^{(m)}(z) \bigg|_{z=0^+} \text{snr}^m, \quad \text{snr} \to 0^+ \]

**Proof:** See Appendix F.

Theorem IV.1 can also be immediately specialized to the most common fading distributions, including: i) Rayleigh fading; ii) Ricean fading; and iii) Nakagami fading.

**Corollary IV.2.** Consider a Rayleigh or Ricean fading coherent channel as in (II.1) driven by an arbitrary discrete input with finite support or by \( \infty \)-PSK, \( \infty \)-PAM, \( \infty \)-QAM or standard complex Gaussian continuous inputs, where (II.2) or (II.3) hold, respectively. Then, in the regime of low-snr the average MMSE behaves as follows:

\[ \operatorname{mmse}(\text{snr}) \sim \exp\left(-\frac{|\mu|^2}{2\sigma^2}\right) \sum_{m=0}^{+\infty} (m + 1) (2\sigma^2)^{m+1} \cdot \operatorname{F_1} \left( 2 + m; \frac{|\mu|^2}{2\sigma^2} \right) \operatorname{mmse}^{(m)}(z) \bigg|_{z=0^+} \text{snr}^m, \quad \text{snr} \to 0^+ \]

where \( \operatorname{F_1}(a; b; c) \) is the confluent hypergeometric series [16 Equation 13.2.2].

**Proof:** See Appendix G.

**Corollary IV.3.** Consider a Nakagami fading coherent channel as in (II.1) driven by an arbitrary discrete input with finite support or by \( \infty \)-PSK, \( \infty \)-PAM, \( \infty \)-QAM or standard complex Gaussian continuous inputs, where (II.4) holds. Then, in the regime of low-snr the average MMSE behaves as follows:

\[ \operatorname{mmse}(\text{snr}) \sim \sum_{m=0}^{+\infty} \frac{\Gamma(\mu + m + 1)}{\Gamma(\mu) \Gamma(m + 1)} \left( \frac{w}{\mu} \right)^{m+1} \operatorname{mmse}^{(m)}(z) \bigg|_{z=0^+} \text{snr}^m, \quad \text{snr} \to 0^+ \]

May 3, 2014 DRAFT
Proof: See Appendix [H] □

The low-snr asymptotic expansions of the average MMSE embodied in Corollaries [IV.2] and [IV.3] also lead immediately to a low-snr asymptotic expansion of the average mutual information, by leveraging the following integral representation of the connection between average MMSE and average mutual information in (II.12) given by:

\[ I(\text{snr}) = \int_0^{\text{snr}} \text{mmse}(\epsilon) \, d\epsilon. \quad (IV.5) \]

**Corollary IV.4.** Consider a Rayleigh or Ricean fading coherent channel as in (II.1) driven by an arbitrary discrete input with finite support or by ∞-PSK, ∞-PAM, ∞-QAM or standard complex Gaussian continuous inputs, where (II.2) or (II.3) hold, respectively. Then, the average mutual information obeys the low-snr expansion given by:

\[ I(\text{snr}) \sim \exp \left( -\frac{|\mu|}{2\sigma^2} \right) \sum_{m=0}^{+\infty} \frac{(2\sigma^2)^{m+1}}{m+1} \text{F}_1 \left( 2 + m; 1; \frac{|\mu|^2}{2\sigma^2} \right) \text{mmse}^{(m)}(z) \bigg|_{z=0^+} \text{snr}^{m+1}, \quad \text{snr} \to 0^+ \]

where \( \text{F}_1(a; b; c) \) is the confluent hypergeometric series [16, Equation 13.2.2].

**Corollary IV.5.** Consider a Nakagami fading coherent channel as in (II.1) driven by an arbitrary discrete input with finite support or by ∞-PSK, ∞-PAM, ∞-QAM or standard complex Gaussian continuous inputs, where (II.4) holds. Then, the average mutual information obeys the low-snr expansion given by:

\[ I(\text{snr}) \sim \sum_{m=0}^{+\infty} \frac{1}{m+1} \frac{\Gamma(\mu+m+1)}{\Gamma(\mu)\Gamma(m+1)} \left( \frac{w}{\mu} \right)^{m+1} \text{mmse}^{(m)}(z) \bigg|_{z=0^+} \text{snr}^{m+1}, \quad \text{snr} \to 0^+ \]

Proof of Corollaries [IV.4] and [IV.5] The Corollaries follow immediately from Corollaries [IV.2] or [IV.3] and the relationship in (IV.5), together with the fact that an order relation can be integrated with respect to the independent variable [14]. □

It is interesting to note the role that the canonical MMSE plays in the definition of the asymptotic expansions of the average MMSE and the average mutual information as \( \text{snr} \to 0^+ \) and \( \text{snr} \to +\infty \). The high-snr behavior of the quantities (for discrete inputs) is dictated via the Mellin transform and the higher-order derivatives of the Mellin transform of the canonical MMSE or, equivalently, the repeated integrals of the canonical MMSE. In contrast, the low-snr behavior of the quantities is dictated via the derivatives of the canonical MMSE.

## V. Generalizations

We now consider a more general frequency-flat fading channel, which for a single time instant, can be modeled as follows:

\[ y = \sqrt{\text{snr}}hx + n \quad (V.1) \]
where $y = [y_1, \ldots, y_k]^T \in \mathbb{C}^k$ represents the channel output, $x \in \mathbb{C}$ represents the channel input, $h = [h_1, \ldots, h_k]^T$ is a complex random vector such that $E_h \{\|h\|^2\} < +\infty$, which represents the random channel fading gains between the input and the outputs of the channel, and $n = [n_1, \ldots, n_k]^T \in \mathbb{C}^k$ is a circularly symmetric complex Gaussian random vector with zero mean and identity covariance matrix which represents standard noise. The scaling factor $\text{snr} \in \mathbb{R}^+$ also relates to the signal-to-noise ratio. We assume that $x$, $h$ and $n$ are independent random variables/vectors. We also assume that the receiver knows the exact realization of the channel gains but the transmitter knows only the distribution of the channel gains. Note that the model in (V.1) represents a generalization of the model in (II.1), that captures various communications scenarios such as the use of multiple antennas at the receiver.

We write the average MMSE associated with the model in (V.1) as follows:

$$\overline{\text{mmse}}(\text{snr}) := E_h \{\text{mmse}_h(\text{snr})\} \quad (V.2)$$

where

$$\text{mmse}_{h_0}(\text{snr}) := E_{x,y|h} \left\{\|h_0 x - E_{x|h} \{h_0 x|y, h_0\}\|^2 \mid h = h_0\right\}$$

represents the conditional MMSE given that the channel gains $h = h_0$ associated with the estimation of the noiseless output given the noisy output of the channel model in (V.1). We also write the average mutual information associated with the model in (V.1) as follows:

$$I(\text{snr}) := E_h \{I_h(\text{snr})\} \quad (V.3)$$

where

$$I_{h_0}(\text{snr}) := E_{x,y|h} \left\{\log \left(\frac{f_{x,y|h}(x,y|h_0)}{f_{x|h}(x|h_0)f_{y|h}(y|h_0)}\right) \mid h = h_0\right\}$$

represents the conditional mutual information given that the channel gains $h = h_0$ between the input and the output of the channel model in (V.1). We can also immediately write the average MMSE in (V.2) in terms of the canonical MMSE in (II.8) as:

$$\overline{\text{mmse}}(\text{snr}) = E_{\|h\|} \left\{\|h\|^2 \text{mmse}(\text{snr}\|h\|^2)\right\} \quad (V.4)$$

and the average mutual information in (V.3) in terms of the canonical mutual information in (II.9) as:

$$I(\text{snr}) = E_{\|h\|} \left\{I(\text{snr}\|h\|^2)\right\} \quad (V.5)$$

The fact that the form of (V.4) and (V.5) is identical to the form of (II.10) and (II.11), respectively, enables the use of the previous machinery to characterize the asymptotic behavior, as $\text{snr} \rightarrow \infty$ or as $\text{snr} \rightarrow 0^+$, of the quantities.$^2$

$^2$It is important to note that, whilst the generalization of the techniques is immediate from single-input–single-output to single-input–multiple-output channel models, such generalization does not seem possible to multiple-input–single-output or multiple-input–multiple-output channel models.
A. High-snr Regime

1) Discrete Inputs: We now concentrate on the generalization of the characterizations of the asymptotic behavior as $\text{snr} \to +\infty$ of the average MMSE and the average mutual information for vector Rayleigh and Ricean fading coherent channels driven by arbitrary discrete inputs with finite support.

**Corollary V.1.** Consider a vector fading coherent channel as in (V.1) driven by an arbitrary discrete input with finite support, where $h \sim \mathcal{CN} (\mu, 2\sigma^2 I_k)$ with $\mu \in \mathbb{C}^{k \times 1}$, and $\sigma > 0$. Then, in the regime of high-snr the average MMSE obeys:

$$\text{mmse} (\text{snr}) \sim \exp \left( -\frac{\|\mu\|^2}{2\sigma^2} \right) \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} \frac{(-1)^{m-n} \|\mu\|^{2n}}{(m-n)! (2\sigma^2)^{m+n+k} n! \Gamma (n+k)} M \left[ \text{mmse}; k + 1 + m \right] \right) \frac{1}{\text{snr}^m}, \quad \text{snr} \to +\infty$$

*Proof:* See Appendix I.

**Corollary V.2.** Consider a vector fading coherent channel as in (V.1) driven by an arbitrary discrete input with finite support, where $h \sim \mathcal{CN} (\mu, 2\sigma^2 I_k)$ with $\mu \in \mathbb{C}^{k \times 1}$, and $\sigma > 0$. Then, in the regime of high-snr the average mutual information obeys the expansion:

$$I (\text{snr}) \sim \log (m) - \exp \left( -\frac{\|\mu\|^2}{2\sigma^2} \right) \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} \frac{(-1)^{m-n} \|\mu\|^{2n}}{(m+n+k)! \Gamma (n+k)} M \left[ \text{mmse}; k + 1 + m \right] \right) \frac{1}{\text{snr}^m}, \quad \text{snr} \to +\infty$$

*Proof:* The expansion follows immediately from the expansion in Corollary V.1 and the relationship in (III.8), together with the fact that an order relation can be integrated with respect to the independent variable [14].

Note that the rate at which the average MMSE and the average mutual information tend to their infinite-snr values in the scalar channel model in (I.I) is equal to 2 and 1, respectively, whereas the rate at which such quantities tend to their infinite-snr values in the vector channel model in (V.1) is equal to $k+1$ and $k$, respectively. This indicts, as expected, that the availability of multiple receive dimensions leads to a lower value for the average MMSE and a higher value for the average mutual information for a certain signal-to-noise ratio.

2) Continuous Inputs: We now focus on the generalization of the characterizations of the asymptotic behavior as $\text{snr} \to +\infty$ of the average MMSE and the average mutual information for vector Rayleigh and Ricean fading coherent channels driven by continuous inputs.

**Corollary V.3.** Consider a vector fading coherent channel as in (V.1) driven by an $\infty$-PSK, $\infty$-PAM, $\infty$-QAM or standard complex Gaussian continuous input, where $h \sim \mathcal{CN} (\mu, 2\sigma^2 I_k)$ with $\mu \in \mathbb{C}^{k \times 1}$, and $\sigma > 0$. Then, in the regime of high-snr the average MMSE obeys:
1) If \(x\) is distributed according to \(\infty\)-PSK then 
\[
\text{mmse}(\text{snr}) \sim \frac{1}{2\text{snr}} + O\left(\frac{1}{\text{snr}R}\right), \quad \text{snr} \to +\infty, \quad \forall R \in ]1, 2[
\]

2) If \(x\) is distributed according to \(\infty\)-PAM then 
\[
\text{mmse}(\text{snr}) \sim \frac{1}{2\text{snr}} + O\left(\frac{1}{\text{snr}R}\right), \quad \text{snr} \to +\infty, \quad \forall R \in \left]\frac{3}{2}, 3\right[ 
\]

3) If \(x\) is distributed according to \(\infty\)-QAM then 
\[
\text{mmse}(\text{snr}) \sim \frac{1}{\text{snr}} + O\left(\frac{1}{\text{snr}R}\right), \quad \text{snr} \to +\infty, \quad \forall R \in \left]1, \frac{3}{2}\right[ 
\]

4) If \(x\) is distributed according to standard complex Gaussian then 
\[
\text{mmse}(\text{snr}) \sim \sum_{m=0}^{k-1} (-1)^m \exp \left(-\frac{\|\mu\|^2}{2\sigma^2}\right) \left(2\sigma^2\right)^{-m} \frac{\Gamma(-m+k)}{(k-1)!} \frac{1}{\text{snr}^{m+1}} + \frac{1}{\text{snr}^{k+1}} \sum_{m=0}^{+\infty} \frac{(-\log(\text{snr}))^j}{j!(1-j)!} \times \left(\frac{d}{dz}\right)^{(1-j)} \left\{ \left(\frac{\pi}{\sin(\pi z)}\right) \left(2\sigma^2\right)^{1-z} \frac{\Gamma(1-z+k)}{(k-1)!} \frac{1}{\text{snr}^{m+1}} \right\}_{z=m+k+1} 
\]

where \(1F_1(a;b;c)\) is the confluent hypergeometric series [16, Equation 13.2.2].

**Proof:** The proof for the case \(\mu = 0\) follows the steps in Appendix B-A with some minor modifications that have been reported in Appendix I-A. Similarly, the proof for the case \(\mu \neq 0\) follows the steps in Appendix B-B with some minor modifications that have been reported in Appendix I-B. Note also that \(R = \{0, 1, \ldots, k-1\}\), \(A = \emptyset\) and \(RA = \mathbb{Z}_0^+ \setminus \{0, 1, \ldots, k-1\}\) in both vector Rayleigh and vector Ricean cases.

Note that the rates at which the average MMSE and the average mutual information tend to their infinite-\text{snr} values in the scalar channel model in (II.1) and the vector channel model in (V.1) are identical. In fact, the leading first-order term in the expansions of the quantities is identical but subsequent higher-order terms in the expansions may differ.

**B. Low-snr Regime**

1) **Discrete and Continuous Inputs:** The generalization of the characterizations of the asymptotic behavior as \(\text{snr} \to 0^+\) of the average MMSE and the average mutual information for vector Rayleigh and Ricean fading coherent channels is also immediate.

**Corollary V.4.** Consider a vector fading coherent channel as in (V.1) driven by an arbitrary discrete input with finite support or by \(\infty\)-PSK, \(\infty\)-PAM, \(\infty\)-QAM or standard complex Gaussian continuous inputs, where \(h \sim\)
\( \mathcal{CN}(\mu, 2\sigma^2 I_k) \) with \( \mu \in \mathbb{C}^{k \times 1} \), and \( \sigma > 0 \). Then, in the regime of low-snr the average MMSE obeys:

\[
\operatorname{mmse}(\text{snr}) \sim 
\exp \left( -\frac{\|\mu\|^2}{2\sigma^2} \right) \sum_{m=0}^{\infty} \frac{1}{m!} \left( 2\sigma^2 \right)^{m+1} \frac{\Gamma(m+1+k)}{(k-1)!} \right] \right|_{z=0^+} \operatorname{snr}^m, \quad \text{snr} \to 0^+
\]

where \( \text{F}_1(a; b; c) \) is the confluent hypergeometric series [16, Equation 13.2.2].

**Proof:** The proof for the case \( \mu = 0 \) follows the steps in Appendix G-A with some minor modifications that have been reported in Appendix I-A. Similarly, the proof for the case \( \mu \neq 0 \) follows the steps in Appendix G-B with some minor modifications that have been reported in Appendix I-B.

**Corollary V.5.** Consider a vector fading coherent channel as in (V.1) driven by an arbitrary discrete input with finite support or by \( \infty \)-PSK, \( \infty \)-PAM, \( \infty \)-QAM or standard complex Gaussian continuous inputs, where \( h \sim \mathcal{CN}(\mu, 2\sigma^2 I_k) \) with \( \mu \in \mathbb{C}^{k \times 1} \), and \( \sigma > 0 \). Then, in the regime of low-snr the average mutual information obeys the expansion:

\[
I(\text{snr}) \sim 
\exp \left( -\frac{\|\mu\|^2}{2\sigma^2} \right) \sum_{m=0}^{\infty} \frac{1}{m+1} \frac{1}{(m+1)!} \left( 2\sigma^2 \right)^{m+1} \frac{\Gamma(m+1+k)}{(k-1)!} \right] \right|_{z=0^+} \operatorname{snr}^{m+1}, \quad \text{snr} \to 0^+
\]

where \( \text{F}_1(a; b; c) \) is the confluent hypergeometric series [16, Equation 13.2.2].

**Proof:** The expansion follows immediately from the expansion in Corollary V.4 and the relationship in (III.8), together with the fact that an order relation can be integrated with respect to the independent variable [14].

**VI. SOME MMSE MELLIN TRANSFORM RESULTS**

It has been established that the Mellin transform of the canonical MMSE given by:

\[
M[\operatorname{mmse}; 1 + z] := \int_0^{+\infty} t^z \operatorname{mmse}(t) \, dt,
\]

plays an important role in the definition of the high-snr behavior of the average MMSE and the average mutual information in fading coherent channels driven by a range of inputs. The objective now is to compute either analytically or numerically such a quantity for the most common input distributions, which when used in conjunction with the previous results, leads to concrete asymptotic expansions.

**A. BPSK and QPSK Inputs**

The Mellin transform of the canonical MMSE associated with binary phase shift keying (BPSK) and quadrature phase shift keying (QPSK) inputs can be obtained analytically. These results capitalize on the following Theorem.
Theorem VI.1. Consider the canonical AWGN channel model in (II.7), where the input \( x \) is uniformly distributed over \( \{-d, d\} \) with \( d > 0 \). Then, \( \forall z \in \mathbb{C} : \Re (z) > 0 \),

\[
M_{\text{MMSE}}; 1 + z = \frac{d^2}{2} \left( \frac{2}{d} \right)^{2(1+z)} \frac{\Gamma \left( \frac{3}{2} + z \right)}{\sqrt{\pi} (1 + z)} + 2d^{2-2(1+z)} \frac{\Gamma (2 + 2z)}{\Gamma (2 + z)} \sum_{l=1}^{\infty} (-1)^l \frac{2F_1 \left( 1, \frac{1}{2}; 2 + z; 1 - \frac{1}{(1+2l)^2} \right)}{1 + 2l}
\]

where \( 2F_1 (a, b; c; d) \) is the Gauss hypergeometric series \([16, Equation 15.2.1]\).

**Proof:** See Appendix J.

This Mellin transform can be immediately specialized for BPSK and QPSK inputs.

Theorem VI.2. Consider the canonical AWGN channel model in (II.7) driven by a standard unit-power BPSK input. Then, \( \forall z \in \mathbb{C} : \Re (z) > 0 \),

\[
M_{\text{MMSE}}; 1 + z = 2 \left( \frac{\Gamma \left( \frac{3}{2} + z \right)}{\sqrt{\pi} (1 + z)} + 2^{-1-2z} \frac{\Gamma (2 + 2z)}{\Gamma (2 + z)} \sum_{l=1}^{\infty} (-1)^l \frac{2F_1 \left( 1, \frac{1}{2}; 2 + z; 1 - \frac{1}{(1+2l)^2} \right)}{1 + 2l} \right)
\]

**Proof:** The result follows from Theorem VI.1 and the fact that the input \( x \) is uniformly distributed over \( \{-1, 1\} \).

Theorem VI.3. Consider the canonical AWGN channel model in (II.7) driven by a standard unit-power QPSK input. Then, \( \forall z \in \mathbb{C} : \Re (z) > 0 \),

\[
M_{\text{MMSE}}; 1 + z = 2^{2+z} \left( \frac{\Gamma \left( \frac{3}{2} + z \right)}{\sqrt{\pi} (1 + z)} + 2^{-1-2z} \frac{\Gamma (2 + 2z)}{\Gamma (2 + z)} \sum_{l=1}^{\infty} (-1)^l \frac{2F_1 \left( 1, \frac{1}{2}; 2 + z; 1 - \frac{1}{(1+2l)^2} \right)}{1 + 2l} \right)
\]

**Proof:** The result follows from Theorem VI.2 and the fact that the input \( x \) is uniformly distributed over \( \left\{ -\frac{1}{\sqrt{2}} - j \frac{1}{\sqrt{2}}, j \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} + j \frac{1}{\sqrt{2}}, j \frac{1}{\sqrt{2}} + j \frac{1}{\sqrt{2}} \right\} \), which implies that

\[
M_{\text{MMSE}_{\text{QPSK}}}; 1 + z = \int_0^{+\infty} t^2 \text{mmse}_{\text{QPSK}} (t) \, dt = \int_0^{+\infty} t^2 \text{mmse}_{\text{BPSK}} \left( \frac{t}{2} \right) \, dt = 2^{1+z} M_{\text{MMSE}_{\text{BPSK}}}; 1 + z
\]

B. Other Inputs

The Mellin transform of the canonical MMSE associated with 4-PAM, 16-QAM, 8-PAM and 64-QAM inputs, for points \( z \in \left\{ \frac{1}{2} + \frac{w}{\pi} : w \in \{0, 1, \ldots, 20\} \right\} \) has been obtained by numerical evaluation of (VI.1). These results are summarized in Table II.

May 3, 2014 DRAFT
TABLE I
NUMERICAL APPROXIMATION OF THE MELLIN TRANSFORM OF THE CANONICAL MMSE [VI] ASSOCIATED WITH 4-PAM, 16-QAM, 8-PAM AND 64-QAM INPUTS (ENTRIES WITH – HAVE NOT BEEN COMPUTED)

| z        | 4-PAM     | 16-QAM    | 8-PAM     | 64-QAM    |
|----------|-----------|-----------|-----------|-----------|
| 1/2      | 2.04943   | 5.79667   | 5.30675   | 1.50097 × 10^1 |
| 1/2 + 1/4 | 2.88309   | 9.69751   | 1.03121 × 10^1 | 3.46857 × 10^1 |
| 1        | 4.34356   | 1.73742 × 10^1 | 2.18091 × 10^1 | 8.72366 × 10^1 |
| 3/2      | 6.91253   | 3.25817 × 10^1 | 4.91577 × 10^1 | 2.38385 × 10^2 |
| 3/2 + 1/4 | 1.15073 × 10^4 | 6.50951 × 10^1 | 1.16461 × 10^2 | 6.588 × 10^2 |
| 3/2      | 1.98962 × 10^4 | 1.38345 × 10^4 | 2.87314 × 10^2 | 1.93281 × 10^3 |
| 2        | 3.55419 × 10^4 | 2.84336 × 10^2 | 7.3338 × 10^2 | 5.86704 × 10^3 |
| 3/2      | 6.53372 × 10^4 | 6.21506 × 10^2 | 1.92794 × 10^3 | 1.83418 × 10^4 |
| 3        | 1.23221 × 10^2 | 1.39409 × 10^3 | 5.20186 × 10^3 | 5.88523 × 10^4 |
| 3/2 + 1/4 | 2.37821 × 10^2 | 3.19972 × 10^3 | 1.43672 × 10^4 | 1.93302 × 10^5 |
| 3        | 4.67874 × 10^2 | 7.50071 × 10^3 | 4.05343 × 10^4 | 6.48549 × 10^5 |
| 3/2      | 9.42243 × 10^2 | 1.79284 × 10^4 | 1.16616 × 10^5 | 2.21889 × 10^6 |
| 7/2      | 1.92833 × 10^4 | 4.36331 × 10^4 | 3.41629 × 10^5 | 7.73017 × 10^6 |
| 7/2 + 1/4 | 4.01341 × 10^3 | 1.07996 × 10^5 | 1.01784 × 10^6 | 2.73886 × 10^7 |
| 4        | 8.48601 × 10^3 | 2.71552 × 10^5 | 3.08083 × 10^6 | 9.85865 × 10^7 |
| 7/2      | 1.82117 × 10^4 | 6.93041 × 10^5 | - | - |
| 9/2      | 3.96371 × 10^4 | 1.79377 × 10^6 | - | - |
| 9/2 + 1/4 | 8.7425 × 10^4 | 4.70498 × 10^6 | - | - |
| 5        | 1.95284 × 10^5 | 1.24982 × 10^7 | - | - |
| 9/2      | 4.41507 × 10^5 | 3.36027 × 10^7 | - | - |
| 11/2     | 1.00974 × 10^6 | 9.13912 × 10^7 | - | - |

VII. Numerical Results

We illustrate the accuracy of the results by comparing the high-snr asymptotic expansions of the quantities to the numerical approximation, in a range of scenarios.

Figures 1–5 consider the average MMSE and the average mutual information in Rayleigh, Ricean and Nakagami fading coherent channels driven by QPSK inputs. We observe that the expansions capture very well the high-snr behavior of the quantities. In particular, one concludes that it is possible to approximate the behavior of the quantities over a wider snr range by using several terms in the asymptotic expansions. We also observe that a single term expansion is sufficient to approximate well the high-snr behavior of the average MMSE and the average mutual information in channels subject to Rayleigh and Nakagami fading. However, expansions with a higher number of terms are necessary to approximate the high-snr behavior of the average MMSE and the average mutual information in channels subject to Ricean fading. This phenomenon, which is specially pronounced in the regime |μ| ≫ σ, is due to the fact that in such a scenario the behavior of the fading channel approaches the behavior of an AWGN.
channel, where the average MMSE and the average mutual information tend to their infinite-

\textit{SNR} values at rates greater than 2 and 1, respectively (see also (III.9) and (III.10)). This can only be captured by incorporating more than a single term in the asymptotic expansions.

Fig. 1. Average MMSE in a Rayleigh fading coherent channel driven by a QPSK input \((\sigma = \frac{1}{\sqrt{2}})\).

Fig. 2. Average mutual information in a Rayleigh fading coherent channel driven by a QPSK \((\sigma = \frac{1}{\sqrt{2}})\).

Fig. 3. Average MMSE in a Ricean fading coherent channel driven by a QPSK input \((|\mu| = \sqrt{\frac{9}{10}} \text{ and } \sigma = \frac{1}{2\sqrt{2}})\).

Fig. 4. Average mutual information in a Ricean fading coherent channel driven by a QPSK \((|\mu| = \sqrt{\frac{9}{10}} \text{ and } \sigma = \frac{1}{2\sqrt{2}})\).

Fig. 5. Average MMSE in a Nakagami fading coherent channel driven by a QPSK input \((\mu = \frac{1}{2} \text{ and } \omega = 1)\).

Fig. 6. Average mutual information in a Nakagami fading coherent channel driven by a QPSK input \((\mu = \frac{1}{2} \text{ and } \omega = 1)\).
Figures 7–9 consider the average MMSE in Rayleigh, Ricean and Nakagami fading coherent channels driven by $\infty$-PSK inputs. We also observe that the single-term expansions capture well the high-$\text{snr}$ behavior of the quantities.

Fig. 7. Average MMSE in a Rayleigh fading coherent channel driven by $\infty$-PSK input ($\sigma = \frac{1}{\sqrt{2}}$).

Fig. 8. Average MMSE in a Ricean fading coherent channel driven by $\infty$-PSK input ($|\mu| = \sqrt{\frac{9}{10}}$ and $\sigma = \frac{1}{\sqrt{5}}$).

Fig. 9. Average MMSE in a Nakagami fading coherent channel driven by $\infty$-PSK ($\mu = \frac{1}{2}$ and $\varpi = 1$).

VIII. PRACTICAL APPLICATIONS

We conclude by considering a problem of optimal power allocation in a bank of $k$ parallel independent fading coherent channels driven by arbitrary discrete inputs, in order to showcase the application of the results. The channel model is given by:

$$y_i = \sqrt{\text{snr}} h_i \sqrt{p_i} x_i + n_i, \quad i = 1, \ldots, k$$ (VIII.1)

where $y_i \in \mathbb{C}$ represents the $i$-th sub-channel output, $x_i \in \mathbb{C}$ represents the $i$-th sub-channel input, $h_i$ is a complex scalar random variable with support $\mathbb{C}$ or $\mathbb{C} \setminus \{0\}$ such that $E_{h_i} \{|h_i|^2\} < +\infty$ which represents the random channel fading gain between the input and the output of the $i$-th sub-channel, and $n_i \in \mathbb{C}$ is a circularly symmetric complex scalar Gaussian random variable with zero mean and unit variance which represents standard noise. The scaling factor $p_i \in \mathbb{R}^+_0$ represents the power injected into sub-channel $i$. The scaling factor $\text{snr} \in \mathbb{R}^+$ relates to the signal-to-noise ratio. We assume that $x_i, i = 1, \ldots, k$, $h_i, i = 1, \ldots, k$ and $n_i, i = 1, \ldots, k$ are independent random variables. We also assume that the receiver knows the exact realization of the sub-channel gains but the transmitter knows only the distribution of the sub-channel gains. This channel model is applicable to a OFDM and multi-user OFDM communications system [1], [2].

We denote the average MMSE and the canonical MMSE of sub-channel $i$ in the model in (VIII.1) as $\text{mmse}_i (\cdot)$ and $\text{mmse}_{ci} (\cdot)$, respectively. We also denote the average mutual information and the canonical mutual information of sub-channel $i$ in the model in (VIII.1) as $I_i (\cdot)$ and $I_{ci} (\cdot)$, respectively.

The objective is to determine the power allocation policy that maximizes the constrained capacity given by:

$$\bar{T} (\text{snr}; p_1, \ldots, p_k) = \sum_{i=1}^k \bar{T}_i (\text{snr} \cdot p_i)$$
subject to a total power constraint:

\[ \sum_{i=1}^{k} p_i \leq P \]

and \( p_i \geq 0, i = 1, \ldots, k \). The following Theorem, which is based on the asymptotic expansions put forth in the previous sections, defines the optimal power allocation policy in the asymptotic regime of high SNR for Rayleigh and Ricean fading models.

**Theorem VIII.1.** Consider a bank of \( k \) parallel independent Rayleigh or Ricean fading coherent channels as in (VIII.1) driven by arbitrary discrete inputs with finite support, where \( h_i \sim \mathcal{CN}(\mu_i, 2\sigma_i^2) \) with \( \mu_i = 0 \) or \( \mu_i \neq 0 \), respectively, and \( \sigma_i > 0, i = 1, \ldots, k \). Then, in the regime of high SNR the optimal power allocation policy obeys:

\[ p_i^* = \sqrt{\exp\left(-\frac{|\mu_i|^2}{2\sigma_i^2}\right) \cdot \frac{M[\text{mmse}_i; 2]}{2\sigma_i^2} \cdot \frac{1}{\lambda_{SNR}} + O\left(\frac{1}{SNR}\right)}, \quad SNR \to +\infty, \quad i = 1, \ldots, k \]

where \( \lambda \) is such that \( \sum_{i=1}^{k} p_i^* = P \).

**Proof:** See Appendix K. \qed

Theorem [VIII.1] reveals the impact of the nature of the fading distribution and the input distribution on the high-SNR optimal power allocation policy. In Rayleigh fading channels, given equal sub-channel inputs, it can be seen that the higher the average sub-channel strength (i.e., the higher \( 2\sigma_i^2 \)) then the lower the allocated power. In Ricean fading channels, it can also be seen that the presence of line-of-sight components affects dramatically the power allocation policy. It is interesting to note that, as expected, the nature of the inputs affects the optimal power allocation policy via the Mellin transform of the canonical MMSE. It is also interesting to note that the power allocation policies embodied in Theorem [VIII.1] in fact represent a generalization of the power allocation policy put forth in [2], in the sense that – in the single-user setting – it applies to Ricean fading in addition to Rayleigh fading and to scenarios where the different input signals conform to different discrete constellations. Figures 10 and 11 confirm that the optimal power allocation rapidly converges to the high-SNR power allocation uncovered by Theorem [VIII.1] for a bank of two parallel independent fading coherent channels.
IX. CONCLUSIONS

Motivated by the need to understand the behavior of the constrained capacity of fading channels, we have unveiled asymptotic expansions of key estimation- and information-theoretic measures in scalar and vector fading coherent channels, where the receiver knows the exact fading channel state but the transmitter knows only the fading channel distribution, driven by a range of inputs. In particular, we have constructed low-\textit{snr} and – at the heart of the novelty of the contribution – high-\textit{snr} asymptotic expansions for the average minimum mean-squared error and the average mutual information for coherent channels subject to Rayleigh fading, Ricean fading or Nakagami fading and driven by arbitrary discrete inputs (with finite support) or by $\infty$-PSK, $\infty$-PAM, $\infty$-QAM, and standard complex Gaussian continuous inputs. The most relevant element for the construction of the asymptotic expansions is the realization that the integral representation of the measures can be seen as an $h$-transform of a kernel with a monotonic argument. This paves the way to the use of a range of expansion of integrals techniques, most notably, Mellin transform methods, that yield the asymptotic expansions for the average minimum mean-squared error and – via the now well-known I-MMSE relationship – for the average mutual information.

We have also considered as a case study a standard power allocation problem over a bank of parallel independent fading coherent channels driven by arbitrary discrete inputs, a scenario representative of orthogonal frequency division multiplexing communications systems. In particular, we have illustrated how to determine the power allocation policy that maximizes the constrained capacity of the bank of parallel independent fading channels in key asymptotic regimes.

APPENDIX A

PROOF OF THEOREM III.1

Let

$$f(t) := \frac{\sqrt{t}f_{|h|}(\sqrt{t})}{2}$$
Then
\[ \text{mmse} (\text{snr}) = E_{[h]} \left\{ |h|^2 \text{mmse} (\text{snr}|h|^2) \right\} = \int_0^{+\infty} f (u) h (snru) \, du \quad (A.1) \]

We note that (A.1) is an $h$-transform with Kernel of Monotonic Argument, so that we can capitalize on the method of Mellin transforms [14, Section 4.4] to obtain the asymptotic expansion of $\text{mmse} (\text{snr})$ as $\text{snr} \to +\infty$ via [14, Theorem 4.4]. The application of [14, Theorem 4.4] requires that:

1) Both $h (\cdot)$ and $f (\cdot)$ are locally integrable functions on $\mathbb{R}^+$;
2) The following holds true
   a) The function $h (\cdot)$ decays as
   \[ h (t) \sim \exp (-kt^v) \sum_{m=0}^{+\infty} \sum_{n=0}^{N(m)} c_{mn}t^{-r_m} (\log (t))^n, \quad t \to +\infty \quad (A.2) \]
   where $\mathcal{R} (k) \geq 0$, $v > 0$, $c_{mn} \in \mathbb{C}$, $\mathcal{R} (r_m) \uparrow +\infty$ and $N (m)$ is finite for each $m$.
We note that in the case $k \neq 0$ a small modification of the proof of [14, Lemma 4.3.1] also yields the conclusions in [14, Theorem 4.4] if instead of (A.2) we have
\[ h (t) = O (\exp (-kt^v)), \quad t \to +\infty \]
where $\mathcal{R} (k) > 0$ and $v > 0$.
We also note that in the case $k = 0$, if it suffices to obtain an asymptotic expansion with a finite number of terms then the proof of [14, Lemma 4.3.3] also reveals that we obtain the same conclusions in [14, Theorem 4.4] if instead of (A.2) we have
\[ h (t) = \sum_{m=0}^{M_h} \sum_{n=0}^{N(m)} c_{mn}t^{-r_m} (\log (t))^n + O \left( t^{-M_h+1} (\log (t))^{N(M_h+1)} \right), \quad t \to +\infty \quad (A.3) \]
where $M_h < +\infty$ and $N (m)$ is finite for each $m$.

b) The function $f (\cdot)$ decays as
\[ f (t) \sim \exp (-qt^{-\mu}) \sum_{m=0}^{+\infty} \sum_{n=0}^{N(m)} p_{mn}t^{a_m} (\log (t))^n, \quad t \to 0^+ \quad (A.4) \]
where $\mathcal{R} (q) \geq 0$, $\mu > 0$, $p_{mn} \in \mathbb{C}$, $\mathcal{R} (a_m) \uparrow +\infty$ and $N (m)$ is finite for each $m$.
We note that in the case $q \neq 0$ a small modification of the proof of [14, Lemma 4.3.4] also yields the conclusions in [14, Theorem 4.4] if instead of (A.4) we have
\[ f (t) = O (\exp (-qt^{-\mu})), \quad t \to 0^+ \]
where $\mathcal{R} (q) > 0$ and $\mu > 0$.
We also note that in the case $q = 0$, if it suffices to obtain an asymptotic expansion with a finite number of terms then the proof of [14, Lemma 4.3.6] also reveals that we obtain the same conclusions in [14, Theorem 4.4] if instead of (A.4) we have
\[ f (t) = \sum_{m=0}^{M_h} \sum_{n=0}^{N(m)} p_{mn}t^{a_m} (\log (t))^n + O \left( t^{-M_h+1} (\log (t))^{N(M_h+1)} \right), \quad t \to 0^+ \]
Theorem 4.4] if instead of (A.4) we have
\[
f(t) = \sum_{m=0}^{\infty} \sum_{n=0}^{N(m)} p_{mn} t^{m} (\log(t))^n + O \left( t^{M_f+1} (\log(t))^N(M_f+1) \right), \quad t \to 0^+
\] (A.5)
where \( M_f < +\infty \) and \( N(m) \) is finite for each \( m \).

3) Let
\[
\alpha := \inf \left\{ \alpha^* : h(t) = O \left( t^{-\alpha^*} \right), t \to 0^+ \right\}
\]
\[
\beta := \sup \left\{ \beta^* : h(t) = O \left( t^{-\beta^*} \right), t \to +\infty \right\}
\]
\[
\gamma := \inf \left\{ \gamma^* : f(t) = O \left( t^{-\gamma^*} \right), t \to 0^+ \right\}
\]
\[
\delta := \sup \left\{ \delta^* : f(t) = O \left( t^{-\delta^*} \right), t \to +\infty \right\}
\]

We require that \( \alpha < \beta \) and \( \gamma < \delta \), so that \( M[h; z] \) and \( M[f; z] \) are holomorphic in the strips \( \alpha < R(z) < \beta \) and \( \gamma < R(z) < \delta \), respectively, \([14] \) p. 106.

4) Let
\[
C := \{ z \in C : (\alpha < R(z) < \beta) \land (1 - \delta < R(z) < 1 - \gamma) \}
\]
We require that
\[
C \neq \emptyset
\]
so that
\[
G(z) := M[h; z] M[f; 1-z]
\]
is holomorphic in \( C \) (which is then continued to the right as a meromorphic function at worst \([14] \) p. 118); 5) The following holds true
\[
\exists c \in C \cap \mathbb{R} : \left( \int_{0}^{+\infty} f(t) h(t) dt = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} G(z) dz \land \forall x \in [c, +\infty[, \lim_{|y| \to +\infty} G(x + iy) = 0 \right) \] (A.6)

6) The following holds true
\( a \) If \( k \neq 0 \land q \neq 0 \), then we require that there exists a real sequence \( u_n \) such that \( u_n \uparrow +\infty \) and
\[
\forall n \in \mathbb{Z}^+, \int_{-\infty}^{+\infty} |G(u_n + iy)| dy < +\infty;
\]
\( b \) If \( k \neq 0 \land q \neq 0 \), let \( U := \{ R(a_m) + 1 : m \in \mathbb{Z}^+ \} \) and let \( u_n \) be the real sequence such that \( u_n \uparrow +\infty \) and \( U = \{ u_n : n \in \mathbb{Z}^+ \} \). Then we require that \( \forall n \in \mathbb{Z}^+, \exists x \in ]u_n, u_{n+1}[: \int_{-\infty}^{+\infty} |G(x + iy)| dy < +\infty; \)
\( c \) If \( k = 0 \land q \neq 0 \), let \( V := \{ R(r_m) : m \in \mathbb{Z}^+ \} \) and let \( u_n \) be the real sequence such that \( u_n \uparrow +\infty \) and \( V = \{ u_n : n \in \mathbb{Z}^+ \} \). Then we require that \( \forall n \in \mathbb{Z}^+, \exists x \in ]u_n, u_{n+1}[: \int_{-\infty}^{+\infty} |G(x + iy)| dy < +\infty; \)
\( d \) If \( k = 0 \land q = 0 \), let \( W := \{ R(a_m) + 1 : m \in \mathbb{Z}^+ \} \cup \{ R(r_m) : m \in \mathbb{Z}^+ \} \) and let \( u_n \) be the real sequence such that \( u_n \uparrow +\infty \) and \( W = \{ u_n : n \in \mathbb{Z}^+ \} \). Then we require that \( \forall n \in \mathbb{Z}^+, \exists x \in ]u_n, u_{n+1}[: \int_{-\infty}^{+\infty} |G(x + iy)| dy < +\infty; \)

We note that in the case (A.3) and/or (A.5) the real sequence \( u_n \) must be replaced by the finite list \( \{ u_n : u_n < \min\{ R \left( m_{M_f+1} \right), R \left( a_{M_f+1} \right) + 1 \} \} \).
The asymptotic expansion of \([A.1]\) as \(snr \to +\infty\) is then given by \([14]\) Theorem 4.4:
\[
\int_0^{+\infty} f(u) h(snru) \, du = - \sum_{c \in R(z)} \text{res}\{snr^{-z}G(z)\}
\]
(A.7)

(where \(c\) is as in \(A.6\) and does not need to be unique, and \(\text{res}\{snr^{-z}G(z)\}\) denotes the residue of the meromorphic function \(snr^{-z}G(\cdot)\) at \(z\) \([18]\) which can be written more explicitly by using \(N(m), c_{mn}, r_m, \overline{N}(m), p_{mn}\) and \(a_m\) as well as by identifying the appropriate scenario, i.e., \(k \neq 0 \land q \neq 0, k \neq 0 \land q = 0, k = 0 \land q \neq 0\) or \(k = 0 \land q = 0\). We note that in the case \((A.3)\) and/or \((A.5)\) the sum in \((A.7)\) will be taken with respect to \(c < R(z) < \min\{R(r_{M+1}), R(a_{M+1}) + 1\}\) instead of with respect to \(c < R(z)\).

**Proof of Theorem III.1:** We now establish the requirements \([1]\)–\([6]\) for the application of the Mellin transform method.

The function \(f(\cdot)\) is locally integrable on \(\mathbb{R}^+\) because of the hypothesis \(E_h\) \(|h|^2\) \(< +\infty\) which ensures that
\[
a < b \Rightarrow \int_a^b f(t) \, dt = \int_a^b \frac{\sqrt{t}f_{|h|}(\sqrt{t})}{2} \, dt = \int_{\sqrt{a}}^{\sqrt{b}} u^2 f_{|h|}(u) \, du \leq \int_0^{+\infty} u^2 f_{|h|}(u) \, du = E_h\) \(|h|^2\) \(< +\infty\)

The Mellin transform \(M[f; z]\) converges absolutely and is holomorphic in the strip \(\gamma < R(z) < \delta\) because of hypothesis \((III.3)\) \([14]\) p. 106).

The function \(h(\cdot)\) is locally integrable on \(\mathbb{R}^+\) and the Mellin transform \(M[h; z]\) converges absolutely and is holomorphic in the strip \(R(z) > 0\) because of
\[
\begin{align*}
  h(0) &< +\infty & \text{(A.8)} \\
  \forall t \in \mathbb{R}^+, \ h(t) &> 0 & \text{(A.9)} \\
  \forall t \in \mathbb{R}_0^+, \ h'(t) &< 0 & \text{(A.10)} \\
  h(t) &= O\left(\exp\left(-\frac{d^2}{4}t\right)\right), \quad t \to +\infty & \text{(A.11)}
\end{align*}
\]
where \(d > 0\) denotes the minimum distance between the elements of the support of the input distribution \([7]\) \([1]\) Theorem 4]. Hence, requirements \([1]\) and \([2]\) are satisfied.

Requirement \([3]\) is ensured by hypothesis \((III.2)\) and \((A.11)\).

Requirement \([4]\) is ensured by hypothesis \((III.4)\).

Combining
\[
M[f; 1 - c - iy] \in L^1(-\infty < y < +\infty)
\]
(which is true due to hypothesis \((III.5)\) and the fact that \(M[f; 1 - c - iy]\) is holomorphic in the line \([c-i\infty, c+i\infty]\) with
\[
t^c h(t) \in L^1(0 \leq t < +\infty)
\]
(which is true due to \((A.8), (A.9), (A.10)\) and \(\emptyset \neq C \subseteq \mathbb{R}^+\)) it is clear \([14]\) p. 108] that
\[
\int_0^{+\infty} f(t) h(t) \, dt = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} G(z) \, dz
\]
Combining hypothesis [III.5] with
\[\forall x \in \mathbb{R}^+, M [h; x + iy] = o (1), \quad |y| \to +\infty\]
(which is true because \(M [h; z]\) is holomorphic in the strip \(\mathcal{R} (z) > 0\) yields
\[\forall x \in [c, +\infty[, G (x + iy) = O (|y|^{-2}), \quad |y| \to +\infty\]
which in turn implies
\[\forall x \in [c, +\infty[, \lim_{|y| \to +\infty} G (x + iy) = 0\]
as well as (note that [A.11] implies that \(k \neq 0\))
- If \(k \neq 0 \land q \neq 0\), there exists a real sequence \(u_n\) such that \(u_n \uparrow +\infty\) and because \(\forall n \in \mathbb{Z}^+, G (u_n + iy)\) is holomorphic in the line \([u_n - i\infty, u_n + i\infty[, - \forall n \in \mathbb{Z}^+, \int_{-\infty}^{\infty} |G (u_n + iy)| dy < +\infty\),
- If \(k \neq 0 \land q = 0\), let \(U := \{\mathcal{R} (a_m) + 1 : m \in \mathbb{Z}^+\}\) and let \(u_n\) be the real sequence such that \(u_n \uparrow +\infty\) and \(U = \{u_n : n \in \mathbb{Z}^+\}\). Since \(\forall x \in [u_n, u_n+1[, G (x + iy)\) is holomorphic in the line \([x - i\infty, x + i\infty[,\) we have that \(\forall n \in \mathbb{Z}^+, \exists x \in ]u_n, u_n+1[, \int_{-\infty}^{\infty} |G (x + iy)| dy < +\infty\).

Hence, requirements [5] and [6] are satisfied.

These established requirements lead immediately – via [A.7] – to the expansions:
- If \(q = 0\) then
  \[
  \text{mmse} (\text{snr}) \sim \sum_{m=0}^{+\infty} \text{snr}^{-1-a_m} \sum_{n=0}^{N(m)} \sum_{j=0}^{n} p_{mn} (-1)^{j} \frac{\log (\text{snr})}{n} M^{(n-j)} [\text{mmse}; z] \bigg|_{z=1+a_m}, \quad \text{snr} \to +\infty
  \]
- If \(q \neq 0\) then
  \[
  \forall R \in \mathbb{R}^+, \text{mmse} (\text{snr}) = o \left(\text{snr}^{-R}\right), \quad \text{snr} \to +\infty
  \]

**APPENDIX B**

**PROOF OF COROLLARY [III.2]**

**A. Case \(\mu = 0\)**

Since, by Taylor’s Theorem [13],
\[
f (t) := \sqrt{f_h (\sqrt{t})} = \frac{t}{2\sigma^2} \exp \left( - \frac{t}{2\sigma^2} \right) \sim \sum_{m=0}^{+\infty} \frac{(-1)^m}{m! (2\sigma^2)^m+1} t^{m+1}, \quad t \to 0^+
\]
we have that requirement [III.2] holds with
\[
q = 0, \quad N (m) = 0, \quad p_{m0} = \frac{(-1)^m}{m! (2\sigma^2)^m+1}, \quad a_m = m + 1
\]

Since, the Mellin transform of \(f (\cdot)\)
\[
M [f; z] = \int_{0}^{+\infty} t^{z-1} \frac{t}{2\sigma^2} \exp \left( - \frac{t}{2\sigma^2} \right) dt = (2\sigma^2)^z \Gamma (z + 1) < +\infty
\]

May 3, 2014

DRAFT
converges absolutely and is holomorphic in the strip $\mathcal{R}(z) > -1$ [16 Equation 5.2.1], we have that $\gamma = -1$ and $\delta = +\infty$ which satisfy $\gamma < \delta$ and $[0, +\infty]\{1 - \delta, 1 - \gamma\}[0, +\infty]\{1 - \delta, 1 - \gamma\}[0, +\infty]\{1 - \delta, 1 - \gamma\}[0, +\infty]\{1 - \delta, 1 - \gamma\}[0, +\infty]\{1 - \delta, 1 - \gamma\}[0, +\infty]\{1 - \delta, 1 - \gamma\}[0, +\infty]\{1 - \delta, 1 - \gamma\}[0, +\infty]\{1 - \delta, 1 - \gamma\}[0, +\infty]\{1 - \delta, 1 - \gamma\}[0, +\infty]\{1 - \delta, 1 - \gamma\}[0, +\infty]\{1 - \delta, 1 - \gamma\}[0, +\infty]\{1 - \delta, 1 - \gamma\}[0, +\infty]\{1 - \delta, 1 - \gamma\} 2 \not= 0$, i.e., requirements [III.3] and [III.4] are satisfied. Also, since [14] p. 138

$$\forall x \in \mathbb{R}, \Gamma (x + iy) = O \left( \exp \left( -\frac{\pi}{2} - \epsilon \right) |y| \right), \quad |y| \to +\infty$$

where $\epsilon$ is any small positive real number, we also have that

$$\forall x \in \left[ \frac{1}{2}, +\infty \right], M [f; 1 - x - iy] = O \left( |y|^{-2} \right), \quad |y| \to +\infty$$

i.e., we also have requirement [III.5].

The result now follows from Theorem [III.1]

B. Case $\mu \not= 0$

Since

$$f(t) = \frac{\sqrt{t}f_{|k|}(\sqrt{t})}{2}$$

$$= \frac{t}{2\sigma^2} \exp \left( -\frac{|\mu|^2}{2\sigma^2} \right) \exp \left( -\frac{t}{2\sigma^2} I_0 \left( \frac{\sqrt{t}|\mu|}{\sigma^2} \right) \right)$$

$$= \frac{t}{2\sigma^2} \exp \left( -\frac{|\mu|^2}{2\sigma^2} \right) \sum_{n=0}^{+\infty} \frac{(-1)^n}{n! (2\sigma^2)^n} t^n \sum_{m=0}^{+\infty} \frac{|\mu|^{2m}}{(2\sigma^2)^{2m}} t^m$$

$$= \frac{t}{2\sigma^2} \exp \left( -\frac{|\mu|^2}{2\sigma^2} \right) \sum_{k=0}^{+\infty} \sum_{l=0}^{k} \frac{(-1)^{k-l}}{(k-l)! (2\sigma^2)^{k+l+1}} \frac{|\mu|^{2l}}{(l!)^2} t^{k+l+1}$$

(B.1)

where the second equality is due to the definition of the modified Bessel function of the first kind [16 Equation 10.25.2], we have that requirement [III.2] holds with

$$q = 0, \quad \mathcal{N}(n) = 0, \quad p_{k0} = \exp \left( -\frac{|\mu|^2}{2\sigma^2} \right) \sum_{i=0}^{k} \frac{(-1)^{k-l}}{(k-l)! (2\sigma^2)^{k+l+1}} \frac{|\mu|^{2l}}{(l!)^2}, \quad a_k = k + 1$$

Since, the Mellin transform of $f(\cdot)$

$$M[f; z] = \int_{0}^{+\infty} t^{z-1} \frac{t}{2\sigma^2} \exp \left( -\frac{|\mu|^2}{2\sigma^2} \right) \exp \left( -\frac{t}{2\sigma^2} I_0 \left( \frac{\sqrt{t}|\mu|}{\sigma^2} \right) \right) dt$$

$$= \exp \left( -\frac{|\mu|^2}{2\sigma^2} \right) \frac{(2\sigma^2)^z}{z} \int_{0}^{+\infty} u^z \exp (-u) I_0 \left( 2 \sqrt{u \frac{|\mu|^2}{2\sigma^2}} \right) du$$

$$= \exp \left( -\frac{|\mu|^2}{2\sigma^2} \right) \frac{(2\sigma^2)^z}{z} \sum_{n=0}^{+\infty} \frac{1}{(n!)^2} u^n \exp (-u) \sum_{n=0}^{+\infty} \frac{1}{(n!)^2} u^n \left( \frac{|\mu|^2}{2\sigma^2} \right)$$

$$= \exp \left( -\frac{|\mu|^2}{2\sigma^2} \right) \frac{(2\sigma^2)^z}{z} \sum_{n=0}^{+\infty} \frac{1}{(n!)^2} u^{z+n} \exp (-u) du \frac{1}{(n!)^2} \left( \frac{|\mu|^2}{2\sigma^2} \right)^n$$

$$= \exp \left( -\frac{|\mu|^2}{2\sigma^2} \right) \frac{(2\sigma^2)^z}{z} \Gamma(z+1) \frac{\Gamma(1)}{\Gamma(z+1)} \sum_{n=0}^{+\infty} \frac{1}{(n!)^2} \Gamma(z+n+1) \frac{\Gamma(1)}{\Gamma(1+n)} \frac{1}{(n!)^2} \left( \frac{|\mu|^2}{2\sigma^2} \right)^n$$

May 3, 2014 DRAFT
\[
\exp \left( -\frac{|\mu|^2}{2\sigma^2} \right) \left( 2\sigma^2 \right)^{z+1} \Gamma (z+1) \, _{1}F_{1} \left( z+1;1;\frac{|\mu|^2}{2\sigma^2} \right)
\]

< +\infty

where the third equality is due to the definition of the modified Bessel function of the first kind [16 Equation 10.25.2], the fourth equality is due to Fubini Theorem [25 Theorem 6.5], the fifth equality is due to \( R(z) > -1 \) which implies \( \forall n \in \mathbb{Z}_0^+, R(z) + n > -1 \) and the sixth equality is due to the definition of the Confluent hypergeometric series [16 Equation 13.2.2], converges absolutely and is holomorphic in the strip \( R(z) > -1 \), we have that \( \gamma = -1 \) and \( \delta = +\infty \) which satisfy \( \gamma < \delta \) and \( 0, +\infty[\gamma]1 - \delta, 1 - \gamma[=]0, +\infty[\gamma] - \infty, 2[=]0, 2[\neq]0, \) i.e., we satisfy requirements (III.3) and (III.4).

It is now important to examine the asymptotic behavior of the Mellin transform of \( f(\cdot) \). We can conclude from [B.1] that \( f(\cdot) \) is infinitely continuously differentiable on \( \mathbb{R} \). We can thus also conclude – in view of the fact that the power series [B.1] has infinite radius of convergence – that \( f^{(j)}(\cdot), j = 1, 2, \ldots \) is given by term-by-term differentiation of [B.1] [18 p. 74]. This leads to the fact that

\[
g(t,x,p) := \left( t \left( \frac{d}{dt} \right) \right)^p (t^x f(t))
\]

is a finite sum where the terms are given by a constant, times a power of \( t \) (namely, \( t^y \) with \( y \in \mathbb{R} \)), times \( \exp \left( -\frac{t}{2\sigma^2} \right) \) and times \( I_0 \left( \frac{\sqrt{t}|\mu|}{\sigma^2} \right) \) and/or a derivative of \( I_0(t) \) evaluated at \( t = \frac{\sqrt{t}|\mu|}{\sigma^2} \), and together with [16 Equation 10.29.5]

\[
I_v^{(k)}(z) = \frac{1}{2^k} \left( I_{v-k}(z) + \frac{k}{1} I_{v-k+2}(z) + \frac{k}{2} I_{v-k+4}(z) + \cdots + I_{v+k}(z) \right)
\]

and [16 Equation 40.1]

\[
I_v(z) \sim \frac{\exp(z)}{(2\pi z)^{\frac{1}{2}}} \sum_{k=0}^{+\infty} (-1)^k \frac{a_k(v)}{z^k}, \quad z \to +\infty
\]

where [16 Equation 17.1]

\[
a_0(v) = 1 \quad k \geq 1 \Rightarrow a_k(v) = \frac{(4v^2 - 1^2) (4v^2 - 3^2) \cdots (4v^2 - (2k - 1)^2)}{k!8^k}
\]

to the fact that

\[
g(t,x,p) = O \left( \exp(-k(p)t) \right), \quad t \to +\infty
\]

where \( \forall p \in \mathbb{Z}_0^+, k(p) > 0 \).

We have now established that \( f(t) \) is infinitely continuously differentiable on \( \mathbb{R}^+ \), that

\[
f(t) \sim \sum_{m=0}^{+\infty} p_m t^{a_m}, \quad t \to 0^+
\]

where \( R(a_m) \uparrow +\infty \), that the asymptotic expansion of \( f^{(j)}(t), j = 1, 2, \ldots \) as \( t \to 0^+ \) is obtained from the asymptotic expansion of \( f(\cdot) \) by successively differentiating term-by-term, and that \( g(t,x,p) \) vanishes as \( t \to +\infty \) for \( p = 0, 1, \ldots \) and \( x > -R(a_0) \). This implies [14 Corollary 6.2.3] that

\[
\forall R \in \mathbb{R}^+, \forall x \in \mathbb{R}, M[f; x + iy] = O \left( |y|^{-R} \right), \quad |y| \to +\infty
\]
where \( M[f; x + iy] \) is to be understood as the analytic continuation of \( M[f; x + iy] \) from \( \{x + iy : 1 - \delta < x < 1 - \gamma\} \) to the entire \( z \) plane \([18\text{ p. 181}]\) and hence that

\[
\forall x \in [\frac{1}{2}, +\infty[, M[f; 1 - x - iy] = O\left(||y||^{-2}\right), \quad ||y|| \to +\infty
\]

i.e., requirement (III.5).

The result now follows from Theorem (III.1)

**APPENDIX C**

**PROOF OF COROLLARY (III.3)**

Since, by Taylor’s Theorem \([18]\),

\[
f(t) := \frac{\sqrt{f_{|x|}}(\sqrt{t})}{2} = \frac{\mu^\mu}{\Gamma(\mu) w^\mu} t^\mu \exp\left(-\frac{\mu^\mu}{w^\mu} t\right) \sim \sum_{m=0}^{+\infty} \frac{\mu^\mu}{\Gamma(\mu) w^\mu} \frac{(-1)^m \mu^m}{m! w^m} t^{m+\mu}, \quad t \to 0^+
\]

we have that requirement (III.2) holds with

\[
q = 0, \quad N(m) = 0, \quad p_{m0} = \frac{\mu^\mu}{\Gamma(\mu) w^\mu} \frac{(-1)^m \mu^m}{m! w^m}, \quad a_m = m + \mu
\]

Since, the Mellin transform of \( f(\cdot) \)

\[
M[f; z] = \int_0^{+\infty} t^{z-1} \frac{\mu^\mu}{\Gamma(\mu) w^\mu} t^\mu \exp\left(-\frac{\mu^\mu}{w^\mu} t\right) dt = \frac{1}{\Gamma(\mu)} \left(\frac{w}{\mu}\right)^z \Gamma(z + \mu) < +\infty
\]

converges absolutely and is holomorphic in the strip \( \Re(z) > -\mu \) \([16\text{ Equation 5.2.1}]\), we have that \( \gamma = -\mu \) and \( \delta = +\infty \) which satisfy (in view of the fact that \( \mu \geq \frac{1}{2} \)) \( \gamma < \delta \) and \( 0, +\infty \] \( 1 - \delta, 1 - \gamma \] \( 0, +\infty \] \( -\infty, 0 ] \) \( 0, +\infty \] \( 0, +\infty \] \( \emptyset \), i.e., requirements (III.3) and (III.4) are satisfied. Also, since \([14\text{ p. 138}]\)

\[
\forall x \in \mathbb{R}, \Gamma(x + iy) = O\left(\exp\left(-\left(\frac{\pi}{2} - \epsilon\right) ||y||\right)\right), \quad ||y|| \to +\infty
\]

where \( \epsilon \) is any small positive real number, we also have that

\[
\forall x \in [\frac{1}{2}, +\infty[, M[f; 1 - x - iy] = O\left(||y||^{-2}\right), \quad ||y|| \to +\infty
\]

i.e., we also have requirement (III.5).

The result now follows from Theorem (III.1)

**APPENDIX D**

**PROOF OF THEOREM (III.6)**

Let

\[
f(t) := \frac{\sqrt{f_{|x|}}(\sqrt{t})}{2}
\]

\[
h(t) := mmse(t)
\]
Let also \( n, M \in \mathbb{Z}_0^+ \) and \( x \in \mathbb{R}_0^+ \). We have that
\[
h^{(-n-1)}(x) = \int_0^x \int_{t_1}^{t_n} \cdots \int_{t_n}^{t_{n-1}} \int_{t_n}^{t_{n-1}} h(t_{n+1}) \, dt_{n+1} \, dt_1 \, \cdots \, dt_2 \, dt_1
= \frac{(-1)^{n+1}}{n!} \int_x^{+\infty} h(t) (t-x)^n \, dt
= \frac{(-1)^{n+1}}{n!} \int_0^{+\infty} h(t+x) t^n \, dt
\]
where in the first equality we use (III.15) and in the second equality we use [16, Equation 1.4.31] and hence that
\[
\left| h^{(-n-1)}(x) \right| = \left| \frac{(-1)^{n+1}}{n!} \int_0^{+\infty} h(t+x) t^n \, dt \right|
= \frac{1}{n!} \int_0^{+\infty} h(t+x) t^n \, dt
\leq \frac{1}{n!} \int_0^{+\infty} h(t) t^n \, dt
= \frac{1}{n!} M [h; n+1]
< +\infty
\]
where the second equality and the first inequality are due to (A.8), (A.9) and (A.10) and the second inequality has been justified in Appendix A. We also have that
\[
\lim_{x \to 0^+} h^{(-n-1)}(x) = \frac{(-1)^{n+1}}{n!} \lim_{x \to 0^+} \int_0^{+\infty} h(t+x) t^n \, dt
= \frac{(-1)^{n+1}}{n!} \int_0^{+\infty} \lim_{x \to 0^+} (h(t+x) t^n) \, dt
= \frac{(-1)^{n+1}}{n!} \int_0^{+\infty} h(t) t^n \, dt
= h^{(-n-1)}(0)
\]
where the first equality is due to (D.1) and the second equality is due to (D.2) and to Lebesgue Dominated Convergence Theorem [25, Theorem 5.8]. One can now use (D.2) and (D.3) with the set of hypotheses (III.11), (III.12) and (III.13) to conclude that
\[
\forall snr \in \mathbb{R}^+, \exists \lim_{t \to +\infty} f^{(n)}(t) h^{(-n-1)}(snrt) = 0
\forall snr \in \mathbb{R}^+, \exists \lim_{t \to 0^+} f^{(n)}(t) h^{(-n-1)}(snrt) < +\infty
\exists \int_0^{+\infty} f^{(n+1)}(t) h^{(-n-1)}(snrt) \, dt = O(1), \quad snr \to +\infty
\]
This leads immediately – via integration by parts – to the asymptotic expansion given by

\[ \text{mmse} (\text{snr}) = \]

\[ = E_{|h|} \left\{ |h|^2 \text{mmse} (\text{snr}|h|^2) \right\} \]

\[ = \int_0^{+\infty} f (u) h (\text{snru}) du \]

\[ = \sum_{m=0}^{M} \frac{(-1)^m}{\text{snr}^{m+1}} \left[ f^{(m)} (t) h^{(-m-1)} (\text{snrt}) \right]_0^{+\infty} + \frac{(-1)^{M+1}}{\text{snr}^{M+1}} \int_0^{+\infty} f^{(M+1)} (t) h^{(-M-1)} (\text{snrt}) dt \]

\[ = \sum_{m=0}^{M} \frac{(-1)^m}{\text{snr}^{m+1}} \left[ f^{(m)} (t) h^{(-m-1)} (\text{snrt}) \right]_0^{+\infty} + \]

\[ + \frac{(-1)^{M+1}}{\text{snr}^{M+1}} \left( \left[ f^{(M+1)} (t) h^{(-M-2)} (\text{snrt}) \right]_0^{+\infty} - \int_0^{+\infty} f^{(M+2)} (t) h^{(-M-2)} (\text{snrt}) dt \right) \]

\[ = \sum_{m=0}^{M} \frac{(-1)^{m+1}}{\text{snr}^{m+1}} f^{(m)} (0) h^{(-m-1)} (0) + \]

\[ + \frac{(-1)^{M+2}}{\text{snr}^{M+2}} \left( f^{(M+1)} (0) h^{(-M-2)} (0) + \int_0^{+\infty} f^{(M+2)} (t) h^{(-M-2)} (\text{snrt}) dt \right) \]

\[ = \sum_{m=0}^{M} \frac{(-1)^{m+1}}{\text{snr}^{m+1}} f^{(m)} (0) h^{(-m-1)} (0) + O \left( \frac{1}{\text{snr}^{M+2}} \right), \quad \text{snr} \to +\infty \]

\[ \sim \sum_{m=0}^{+\infty} \frac{(-1)^{m+1}}{\text{snr}^{m+1}} f^{(m)} (0) h^{(-m-1)} (0), \quad \text{snr} \to +\infty \]

**APPENDIX E**

**PROOF OF THEOREM III.7**

Let

\[ f (t) := \frac{\sqrt{t} f_{|h|} (\sqrt{t})}{2} \]

\[ h (t) := \text{mmse} (t) \]

Then

\[ \text{mmse} (\text{snr}) = E_{|h|} \left\{ |h|^2 \text{mmse} (\text{snr}|h|^2) \right\} = \int_0^{+\infty} f (u) h (\text{snru}) du \quad (E.1) \]

We note that (E.1) is also an \( h \)-transform with Kernel of Monotonic Argument, so that we can capitalize on the method of Mellin transforms \[14\, Section 4.4\] to obtain the asymptotic expansion of \( \text{mmse} (\text{snr}) \) as \( \text{snr} \to +\infty \) via \[14\, Theorem 4.4\].

We now establish the requirements for the application of the Mellin transforms method.

**A. Case: \( x \sim \infty-\text{PSK} \), \( x \sim \infty-\text{PAM} \) or \( x \sim \infty-\text{QAM} \)**

We can establish that \( f (\cdot) \) is locally integrable on \( \mathbb{R}^+ \) and that \( M [f; z] \) converges absolutely and is holomorphic in the strip \( \gamma < R (z) < \delta \) by capitalizing on \( E_h \{ |h|^2 \} < +\infty \) and hypothesis (III.23), respectively, as we did in Appendix A. We now extend \( M [f; z] \) to the left of the strip: consider the two cases \( q = 0 \) and \( q \neq 0 \):
1) Case \( q = 0 \): In this case we have [14, Lemma 4.3.6] that \( M[f; 1 - z] \) can be analytically continued as a meromorphic function from \( 1 - \delta < \mathcal{R}(z) < 1 - \gamma \) into \( \mathcal{R}(z) > 1 - \delta \) with poles at \( z = a_m + 1 \) for every \( m \in \mathbb{Z}_0^+ \).

2) Case \( q \neq 0 \): In this case we have [14, Lemma 4.3.4] that \( M[f; 1 - z] \) can be analytically continued as a meromorphic function from \( 1 - \delta < \mathcal{R}(z) < 1 - \gamma \) into \( \mathcal{R}(z) > 1 - \delta \).

We can also establish that \( h(t) \) is locally integrable on \( \mathbb{R}^+ \) because \( h(\cdot) \) is continuous on \( \mathbb{R}_0^+ \) and that \( M[h; z] \) converges absolutely and is holomorphic in the strip \( 0 < \mathcal{R}(z) < 1 \) for \( -\infty \)-PSK, \( -\infty \)-PAM and \( -\infty \)-QAM because of (A.8), (A.9) and (A.10), and of (III.17), (III.18) and (III.19), i.e., of

\[
h(t) = \zeta t^{-r_0} + O\left(t^{-r_1}\right), \quad t \to +\infty
\]

where

\[
\zeta := \begin{cases}
\frac{1}{2} & \Leftrightarrow -\infty \text{-PSK} \\
\frac{1}{4} & \Leftrightarrow -\infty \text{-PAM} \\
1 & \Leftrightarrow -\infty \text{-QAM}
\end{cases}
\]

\[
r_0 := 1 \\
r_1 := \begin{cases}
\frac{3}{4} & \Leftrightarrow -\infty \text{-PAM} \\
\frac{3}{2} & \Leftrightarrow -\infty \text{-QAM}
\end{cases}
\]

We now extend \( M[h; z] \) to the right of the strip: indeed, it can be [14, Lemma 4.3.3] analytically continued as a meromorphic function into \( 0 < \mathcal{R}(z) < r_1 \) with a pole at \( z = r_0 \).

Note that hypothesis (III.21) and (E.2) ensure a “correct” type of decay.

Note also that hypothesis (III.24) ensures that the function \( G(\cdot) := M[h; \cdot] M[f; 1 - \cdot] \) is holomorphic in \( C := [0, 1][1 - \delta, 1 - \gamma][\neq \emptyset] \). Note also that \( G(\cdot) \) can be analytically continued: consider the two cases \( q = 0 \) and \( q \neq 0 \):

1) Case \( q = 0 \): The function \( G(z) \) can be analytically continued as a meromorphic function from \( C \) into \( \max\{0, 1 - \delta\} < \mathcal{R}(z) < \min\{\mathcal{R}(a_0) + 1, r_1\} \) because \( r_0 = 1 \) and by hypothesis (III.22) and \( \mathcal{R}(a_m) \uparrow +\infty \) implies that \( \mathcal{R}(a_m) + 1 > r_0 \).

2) Case \( q \neq 0 \): The function \( G(z) \) can be analytically continued as a meromorphic function from \( C \) into \( \max\{0, 1 - \delta\} < \mathcal{R}(z) < r_1 \).

Combining

\[
M[f; 1 - c - iy] \in L^1(-\infty < y < +\infty)
\]

(which is true due to hypothesis (III.25) and the fact that \( M[f; 1 - c - iy] \) is holomorphic in the line \( ]c-i\infty, c+i\infty[ \)) with

\[
t^{-1}h(t) \in L^1(0 \leq t < +\infty)
\]

(which is true due to (A.8), (A.9), (A.10), (E.2) and \( \emptyset \neq C \subseteq [0, 1] \)) it is clear that [14, p. 108]

\[
\int_{0}^{+\infty} f(t) h(t) dt = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} G(z) dz
\]

Combining hypothesis (III.25) with the fact that
1) Case \( q = 0 \):

\[
\forall x \in [c, \min \{ R(a_0) + 1, r_1 \} ], M[h; x + iy] = o(1), \quad |y| \to +\infty
\]

we conclude – due to \( \emptyset \neq C \subseteq [0, 1] \) – that

\[
\forall x \in [c, \min \{ R(a_0) + 1, r_1 \} ], G(x + iy) = O\left(|y|^{-2}\right), \quad |y| \to +\infty
\]

which in turn implies that

\[
\forall x \in [c, \min \{ R(a_0) + 1, r_1 \} ], \lim_{|y| \to +\infty} G(x + iy) = 0
\]

and that

\[
\forall x \in [r_0, \min \{ R(a_0) + 1, r_1 \} ], \int_{-\infty}^{+\infty} |G(x + iy)| \, dy < +\infty
\]

The fact that the requirements for the application of the Mellin transform method are met leads to the expansions \([14, \text{Theorem 4.4}]\):

\[
\text{mmse}(\text{snr}) \sim \frac{\zeta M[f; 0]}{\text{snr}} + O\left(\frac{1}{\text{snr} R}\right), \quad \text{snr} \to +\infty, \quad \forall R \in [1, \min \{ R(a_0) + 1, r_1 \} ]
\]

2) Case \( q \neq 0 \):

\[
\forall x \in [c, r_1], M[h; x + iy] = o(1), \quad |y| \to +\infty
\]

we conclude – due to \( \emptyset \neq C \subseteq [0, 1] \) – that

\[
\forall x \in [c, r_1], G(x + iy) = O\left(|y|^{-2}\right), \quad |y| \to +\infty
\]

which in turn implies that

\[
\forall x \in [c, r_1], \lim_{|y| \to +\infty} G(x + iy) = 0
\]

and that

\[
\forall x \in [r_0, r_1], \int_{-\infty}^{+\infty} |G(x + iy)| \, dy < +\infty
\]

The fact that the requirements for the application of the Mellin transform method are met leads to the expansions \([14, \text{Theorem 4.4}]\):

\[
\text{mmse}(\text{snr}) \sim \frac{\zeta M[f; 0]}{\text{snr}} + O\left(\frac{1}{\text{snr} R}\right), \quad \text{snr} \to +\infty, \quad \forall R \in [1, r_1]
\]

B. Case: \( x \sim CN(0, 1) \)

This proof follows the steps of the previous proof. The difference is that we use \([III.20]\)

\[
h(t) = \frac{1}{1 + t} = \frac{1}{t} \frac{1}{1 + 1} \sim \sum_{m=0}^{+\infty} (-1)^m t^{-(m+1)}, \quad t \to +\infty
\]

and \([14, \text{p. 123}]\)

\[
M[h; z] = \frac{\pi}{\sin(\pi z)} \quad \text{ (E.4)}
\]

Hence, the Mellin transform \( M[h; z] \) converges absolutely and is holomorphic in the strip \( 0 < R(z) < 1 \), and can be analytically continued from \( 0 < R(z) < 1 \) to the entire complex plane via \((E.4)\).

We note that, in this case, we have capitalized not only on \([14, \text{Theorem 4.4}]\), but also on \([14, \text{Exercise 4.16}]\).
APPENDIX F

PROOF OF THEOREM IV.1

Let

\[ f_1(t) := \text{mmse}(t) \]
\[ h_1(t) := \frac{\sqrt{t} |f_1(t)|}{2} \]
\[ \lambda := \frac{1}{\text{snr}} \]

Then

\[ \text{mmse}(\text{snr}) = E_{|h|^2} \left\{ |h|^2 \text{mmse}(\text{snr}^{|h|^2}) \right\} = \int_0^{+\infty} h_1(u) f_1(\text{snr}u) du = \lambda \int_0^{+\infty} f_1(u) h_1(\lambda u) du \quad (F.1) \]

\[ \text{snr} \to 0^+ \Rightarrow \lambda \to +\infty \]

We also note that (F.1) is an \( h \)-transform with Kernel of Monotonic Argument, so that we can also capitalize on the method of Mellin transforms [14, Section 4.4] to obtain the asymptotic expansion of \( \text{mmse}(\text{snr}) \) as \( \text{snr} \to 0^+ \) or \( \lambda \to +\infty \).

We now establish the requirements for the application of the Mellin transforms method.

We showed in Appendix A (for discrete inputs) and in Appendix E (for the continuous inputs) that \( h_1(t) \) and \( f_1(t) \) are locally integrable functions on \( \mathbb{R}^+ \). We also showed in Appendix A and in Appendix E that \( M[f_1; z] \) converges absolutely and is holomorphic in the strip \( \mathcal{R}(z) > 0 \) (for discrete inputs) and in the strip \( 0 < \mathcal{R}(z) < 1 \) (for the continuous inputs) and hence that \( M[f_1; 1 - z] \) converges absolutely and is holomorphic in the strip \( \mathcal{R}(z) < 1 \) (for discrete inputs) and in the strip \( 0 < \mathcal{R}(z) < 1 \) (for the continuous inputs). The Mellin transform \( M[h_1; z] \) also converges absolutely and is holomorphic in the strip \( \alpha_1 < \mathcal{R}(z) < \beta_1 \) because of hypothesis [IV.2] [14, p. 106].

Note hypothesis [IV.1] and

\[ f_1(t) = \sum_{m=0}^{+\infty} \frac{1}{m!} \text{mmse}^{(m)}(z) \bigg|_{t=0^+} t^m, \quad t \to 0^+ \]
\[ \sim \sum_{m=0}^{+\infty} p_m t^m, \quad t \to 0^+ \quad (F.2) \]

(which is true because the fact that the input has finite moments implies that the function \( f_1(t) \) is infinitely right differentiable at \( t = 0 \) [7, Proposition 7] which enables a straightforward application of Taylor’s Theorem [18]) which ensures a “correct” type of decay.

Note hypothesis [IV.3] which ensures that the function \( G_1(\cdot) := M[h_1; \cdot] M[f_1; 1 - \cdot] \) is holomorphic in \( C_1 \neq \emptyset \). Combining

\[ M[h_1; c_1 + iy] \in L^1(-\infty < y < +\infty) \]

(which is true due to hypothesis [IV.4] and the fact that \( M[h_1; c_1 + iy] \) is holomorphic in the line \( |c_1 - i\infty, c_1 + i\infty| \)) with

\[ t^{-c_1} f_1(t) \in L^1(0 \leq t < +\infty) \]
which in turn implies holomorphic in the line $k$ as well as (note that (IV.1) implies that (A.9), (A.10) and (A.11) (for discrete inputs) or (E.2) and (E.3) (for the continuous inputs)) it is clear that (A.3) p. 108]

$$
\int_0^{+\infty} f_1(t) h_1(t) \, dt = \frac{1}{2\pi i} \int_{c_1-i\infty}^{c_1+i\infty} G_1(z) \, dz
$$

Combining hypothesis (IV.4) with

$$
M[f_1; 1-x-iy] = o(1), \quad |y| \to +\infty
$$

(this holds $\forall x \in \mathbb{R}$ (for discrete inputs) or $\forall x \in \mathbb{R}^+$ (for the continuous inputs) [14, Lemma 4.3.6]) where $M[f_1; 1-x-iy]$ is to be understood as the analytic continuation of $M[f_1; 1-x-iy]$ from $\{x + iy : x < 1\}$ (for discrete inputs) or from $\{x + iy : 0 < x < 1\}$ (for the continuous inputs) to the entire $z$ plane (for discrete inputs) or to the strip $\{x + iy : x > 0\}$ (for the continuous inputs) [18, p. 181] yields

$$
\forall x \in [c_1, +\infty[, G(x + iy) = O(|y|^{-2}), \quad |y| \to +\infty
$$

which in turn implies

$$
\forall x \in [c_1, +\infty[, \lim_{|y| \to +\infty} G(x + iy) = 0
$$

as well as (note that (IV.1) implies that $k_1 \neq 0$ and that (IV.2) implies that $q_1 = 0$) if $U := \{\mathcal{R}(a_n) + 1 : m \in \mathbb{Z}^+\}$ and $u_n$ is the real sequence such that $u_n \uparrow +\infty$ and $U = \{u_n : n \in \mathbb{Z}^+\}$ then, since $\forall x \in [u_n, u_{n+1}], G(x + iy)$ is holomorphic in the line $|x - i\infty, x + i\infty|$, we have that $\forall n \in \mathbb{Z}^+, \exists x \in [u_n, u_{n+1}]: \int_{-\infty}^{+\infty} |G(x + iy)| \, dy < +\infty$.

This leads immediately to the expansions [14, Theorem 4.4., Case II):

$$
\text{mmse} \left(\text{snr} \right) \sim \sum_{m=0}^{+\infty} p_m M[h_1; a_m + 1] \lambda^{-a_m}, \quad \lambda \to +\infty
$$

$$
\sim \sum_{m=0}^{+\infty} \frac{1}{m!} M[h_1; m + 1] \text{mmse}^{(m)} (z) \bigg|_{z=0}^{snr^m}, \quad snr \to 0^+
$$

**Appendix G**

**Proof of Corollary (IV.2)**

**A. Case $\mu = 0$**

Since

$$
h_1(t) = \frac{\sqrt{t} f_1(t) \sqrt{t}}{2} = \frac{t}{2\sigma^2} \exp \left( -\frac{t}{2\sigma^2} \right) = O\left( \exp \left( -k_1 t e^1 \right) \right), \quad t \to +\infty
$$

where $\mathcal{R}(k_1) > 0$ and $e_1 > 0$, we have that requirement (IV.1) holds.

Since, we showed in Appendix B-A that the Mellin transform of $h_1(\cdot)$, which is given by

$$
M[h_1; z] = (2\sigma^2)^z \Gamma(z + 1),
$$
converges absolutely and is holomorphic in the strip $\mathcal{R}(z) > -1$, we have that $\alpha_1 = -1$ and $\beta_1 = +\infty$, which satisfy $\alpha_1 < \beta_1$, and

$$\left\{ \begin{array}{ll} |\alpha_1, \beta_1| \to -\infty, 1[\neg 1, +\infty[\to -\infty, 1[\neg 1, 1[\not= \emptyset & \text{if } x \text{ is discrete}, \\ |\alpha_1, \beta_1| \to 0, 1[\neg 1, +\infty[\to 0, 1[\not= \emptyset & \text{if } x \text{ is continuous}. \end{array} \right.$$ 

i.e., we satisfy requirements (IV.2) and (IV.3).

We also showed in Appendix B-A that

$$(\forall x \in \left[\frac{1}{2}, +\infty[\right], \forall t \to +\infty)$$

i.e., we satisfy requirement (IV.4).

The result now follows from Theorem IV.1.

B. Case $\mu \neq 0$

Since

$$h_1(t) = \frac{\sqrt{t}f_1(h_1(t))}{2}$$

$$= \frac{t}{2\sigma^2} \exp \left( -\frac{|\mu|^2}{2\sigma^2} \right) \exp \left( -\frac{t}{2\sigma^2} \right) I_0 \left( \frac{\sqrt{tv}}{\sigma} \right)$$

$$\sim \frac{t}{2\sigma^2} \exp \left( -\frac{|\mu|^2}{2\sigma^2} \right) \exp \left( -\frac{t}{2\sigma^2} \right) \exp \left( \frac{\sqrt{tv}}{\sigma^2} \right) \sum_{k=0}^{+\infty} \frac{\left( \frac{\sqrt{tv}}{\sigma^2} \right)^k}{(2\pi)^k}, \quad t \to +\infty$$

$$= O \left( \exp \left( -k_1 t^{v_1} \right) \right), \quad t \to +\infty$$

where the asymptotic expansion is due to [16, Equation 10.40.1], $\mathcal{R}(k_1) > 0$ and $v_1 > 0$, we have that requirement (IV.1) holds.

Since, we showed in Appendix B-B that the Mellin transform of $h_1(\cdot)$, which is given by

$$M[h_1; z] = \exp \left( -\frac{|\mu|^2}{2\sigma^2} \right) \left( \frac{2\sigma^2}{\sqrt{tv}} \right)^z \Gamma(z+1) \left( z + 1; 1, \frac{\mu^2}{2\sigma^2} \right),$$

converges absolutely and is holomorphic in the strip $\mathcal{R}(z) > -1$, we have that $\alpha_1 = -1$ and $\beta_1 = +\infty$, which satisfy $\alpha_1 < \beta_1$, and

$$\left\{ \begin{array}{ll} |\alpha_1, \beta_1| \to -\infty, 1[\not= 1, +\infty[\to -\infty, 1[\not= \emptyset & \text{if } x \text{ is discrete}, \\ |\alpha_1, \beta_1| \to 0, 1[\not= 1, +\infty[\to 0, 1[\not= \emptyset & \text{if } x \text{ is continuous}. \end{array} \right.$$ 

i.e., we satisfy requirements (IV.2) and (IV.3).

We also showed in Appendix B-B that

$$(\forall x \in \left[\frac{1}{2}, +\infty[\right], \forall t \to +\infty)$$

i.e., we satisfy requirement (IV.4).

The result now follows from Theorem IV.1.
APPENDIX H
PROOF OF COROLLARY [IV.3]

Since
\[ h_1 (t) := \frac{\sqrt{t} f_{|h|} (\sqrt{t})}{2} = \frac{\mu^n}{\Gamma (\mu) \mu^n} t^{\mu} \exp \left( -\frac{\mu}{w} t \right) = O \left( \exp \left( -k_1 t^{v_1} \right) \right), \quad t \to +\infty \]
where \( R (k_1) > 0 \) and \( v_1 > 0 \), we have that requirement [IV.1] holds.

Since, we showed in Appendix C that the Mellin transform of \( h_1 (\cdot) \), which is given by
\[ M[h_1; z] = \frac{1}{\Gamma (\mu)} \left( \frac{w}{\mu} \right)^z \Gamma (z + \mu), \]
converges absolutely and is holomorphic in the strip \( R (z) > -\mu \), we have that \( \alpha_1 = -\mu \) and \( \beta_1 = +\infty \) which satisfy \( \alpha_1 < \beta_1 \) because \( \mu \geq \frac{1}{2} \) and
\[
\begin{cases}
\alpha_1, \beta_1 |n| - \infty, 1[=] - \mu, +\infty|n| - \infty, 1[\geq] - \frac{1}{2}, +\infty|n| - \infty, 1[=] - \frac{1}{2}, 1[\neq \emptyset] & \text{if } x \text{ is discrete,} \\
\alpha_1, \beta_1 |n|0, 1[=] - \mu, +\infty|n|0, 1[\geq] - \frac{1}{2}, +\infty|n|0, 1[=]0, 1[\neq \emptyset] & \text{if } x \text{ is continuous.}
\end{cases}
\]
i.e., we satisfy requirements [IV.2] and [IV.3].

We also showed in Appendix C that
\[ \forall x \in \left[ \frac{1}{2}, +\infty \right], M [h_1; x + iy] = O \left( |y|^{-2} \right), \quad |y| \to +\infty \]
i.e., we satisfy requirement [IV.4].

The result now also follows from Theorem [IV.1]

APPENDIX I
PROOF OF COROLLARY [V.1]

A. Case \( \mu = 0 \)

The proof follows the steps of Appendix [B-A] taking into account that now:
\[ f_{||h||} (t) = \frac{2 t^{2k-1} \exp \left( -\frac{t^2}{2\sigma^2} \right)}{(2\sigma^2)^k (k-1)!} \]

By Taylor’s Theorem [18] we have that
\[ f (t) \sim \sum_{m=0}^{+\infty} \frac{(-1)^m}{(k-1)! m!} (2\sigma^2)^{m+k} t^{m+k}, \quad t \to 0^+ \]

We also have that
\[ M[f; z] = \frac{(2\sigma^2)^z \Gamma (z + k)}{(k-1)!} < +\infty \]
converges absolutely and is holomorphic in the strip \( R (z) > -k \) [16] Equation 5.2.1].
B. Case $\mu \neq 0$

The proof follows the steps of Appendix B-B taking into account that now:

$$ f_{\|h\|}(t) = \frac{t^k}{\|\mu\|^k \cdot 1_{\sigma^2}} \exp \left( -\frac{\|\mu\|^2}{2\sigma^2} \right) \exp \left( -\frac{t^2}{2\sigma^2} \right) I_{k-1} \left( \frac{t\|\mu\|}{\sigma^3} \right) $$

By Taylor’s Theorem [18] and [16, Equation 10.25.2] we have that

$$ f(t) \sim \exp \left( -\frac{\|\mu\|^2}{2\sigma^2} \right) + \infty \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \frac{(-1)^{a-b} \|\mu\|^{2b}}{(b-1)! (2\sigma^2)^{a+b+k}} b! \Gamma (b+k) t^a, t \to 0^+ $$

We also have that

$$ M[f; z] = \exp \left( -\frac{\|\mu\|^2}{2\sigma^2} \right) \frac{\Gamma (z+k)}{(k-1)!} 1_{F_1} \left( z+k; \frac{\|\mu\|^2}{2\sigma^2} \right) < +\infty $$

converges absolutely and is holomorphic in the strip $\mathcal{R}(z) > -k$ [16, Equation 5.2.1].

APPENDIX J

PROOF OF THEOREM [VI.1]

Note that

$$ M[\text{mmse}; 1+z] = \int_0^{+\infty} t^2 \text{mmse} (t) \, dt $$

$$ = \frac{d^2}{2} \int_0^{+\infty} t^2 \sum_{l=0}^{+\infty} (-1)^l \exp \left( t (l+1) \cdot t d^2 \right) \text{erfc} \left( (2l+1) \sqrt{\frac{d}{2}} \right) dt $$

$$ = \frac{d^2}{2} \sum_{l=0}^{+\infty} (-1)^l \int_0^{+\infty} t^2 \exp \left( t (l+1) \cdot t d^2 \right) \text{erfc} \left( (2l+1) \sqrt{\frac{d}{2}} \right) dt $$

$$ = \frac{d^2}{2} \left( \frac{2}{d} \right)^2 \frac{\Gamma \left( \frac{3}{2} + z \right)}{\sqrt{\pi} (1+z)} + 2d^{-2(1+z)} \frac{\Gamma \left( 2 + 2z \right)}{\Gamma (2 + z)} \sum_{l=1}^{+\infty} (-1)^l \frac{2F_1 \left( 1, \frac{3}{2}; 2 + z; 1 - \frac{1}{(1+2l)^2} \right)}{1 + 2l} $$

where the second equality is due to the characterization of the canonical MMSE associated with BPSK in [1], the third equality is due to uniform convergence and the fourth equality follows from algebraic manipulations.

APPENDIX K

PROOF OF THEOREM [VIII.1]

Consider the function:

$$ p : \mathbb{R}^+ \to \mathbb{R}^+_0 \times \cdots \times \mathbb{R}^+_0 $$

$$ \text{snr} \mapsto p(\text{snr}) = (p_1(\text{snr}), \ldots, p_k(\text{snr})) := \arg \max_{\forall i \in \{1, \ldots, k\}, \sum_{i=1}^k p_i \leq P} T(\text{snr}; p_1, \ldots, p_k) $$

It is possible to establish, from the KKT conditions associated with this optimization problem, that

$$ \exists \text{snr}_0 > 0 : \exists \epsilon > 0 : \forall i \in \{1, \ldots, k\}, \forall \text{snr} > \text{snr}_0, \left( \epsilon < p_i(\text{snr}) < \frac{1}{\epsilon} \wedge \text{mmse}_i(\text{snr} p_i(\text{snr})) \text{snr} = \lambda(\text{snr}) \right) $$
Therefore, due to the expansion embodied in Corollary [III.2] it follows, for \( i = 1, \ldots, k \), that

\[
\text{mmse}_i (\text{snr} p_i (\text{snr})) = \frac{\tau_i}{\text{snr}^2 p_i^2 (\text{snr})} + O \left( \frac{1}{\text{snr}^3} \right), \quad \text{snr} \to +\infty
\]

where

\[
\tau_i := \exp \left( -\frac{\mu_i^2}{2\sigma_i^2} \right) M \left[ \text{mmse}_i; 2 \right]/2\sigma_i^2
\]

which leads to

\[
\lambda (\text{snr}) \text{snr} = \frac{\tau_i}{p_i^2 (\text{snr})} + O \left( \frac{1}{\text{snr}} \right), \quad \text{snr} \to +\infty
\]

\[
\frac{1}{\lambda (\text{snr}) \text{snr}} = O \left( 1 \right), \quad \text{snr} \to +\infty
\]

and

\[
p_i (\text{snr}) = \sqrt{\frac{\tau_i}{\lambda (\text{snr}) \text{snr}}} \sqrt{1 + \frac{1}{\lambda (\text{snr}) \text{snr}} O \left( \frac{1}{\text{snr}} \right)}, \quad \text{snr} \to +\infty
\]

\[
= \sqrt{\frac{\tau_i}{\lambda (\text{snr}) \text{snr}}} \left( 1 + O \left( \frac{1}{\text{snr}} \right) \right), \quad \text{snr} \to +\infty
\]

\[
= \sqrt{\frac{\tau_i}{\lambda (\text{snr}) \text{snr}}} + O \left( \frac{1}{\text{snr}} \right), \quad \text{snr} \to +\infty
\]

REFERENCES

[1] A. Lozano, A. M. Tulino, and S. Verdú, “Optimum power allocation for parallel Gaussian channels with arbitrary input distributions,” *IEEE Trans. Inf. Theory*, vol. 52, no. 7, pp. 3033–3051, Jul. 2006.

[2] ——, “Optimum power allocation for multiuser OFDM with arbitrary signal constellations,” *IEEE Trans. Commun.*, vol. 56, no. 5, pp. 828–837, May 2008.

[3] D. Guo, S. Shamai (Shitz), and S. Verdú, “Mutual information and minimum mean-square error in Gaussian channels,” *IEEE Trans. Inf. Theory*, vol. 51, no. 4, pp. 1261–1282, Apr. 2005.

[4] D. P. Palomar and S. Verdú, “Gradient of mutual information in linear vector Gaussian channels,” *IEEE Trans. Inf. Theory*, vol. 52, no. 1, pp. 141–154, Jan. 2006.

[5] S. Verdú and D. Guo, “A simple proof of the entropy power inequality,” *IEEE Trans. Inf. Theory*, vol. 52, no. 5, pp. 2165–2166, May 2006.

[6] A. M. Tulino and S. Verdú, “Monotonic decrease of the non-Gaussianness of the sum of independent random variables: A simple proof,” *IEEE Trans. Inf. Theory*, vol. 52, no. 9, pp. 4295–4297, Sep. 2006.

[7] D. Guo, Y. Wu, S. Shamai (Shitz), and S. Verdú, “Estimation in Gaussian noise: Properties of the minimum mean-square error,” *IEEE Trans. Inf. Theory*, vol. 57, no. 4, pp. 2371–2385, Apr. 2011.

[8] F. Pérez-Cruz, M. R. D. Rodrigues, and S. Verdú, “MIMO Gaussian channels with arbitrary inputs: Optimal precoding and power allocation,” *IEEE Trans. Inf. Theory*, vol. 56, no. 3, pp. 1070–1084, Mar. 2010.

[9] M. Payaró and D. P. Palomar, “Hessian and concavity of mutual information, differential entropy, and entropy power in linear vector Gaussian channels,” *IEEE Trans. Inf. Theory*, vol. 55, no. 8, pp. 3613–3628, Aug. 2009.

[10] ——, “On optimal precoding in linear vector Gaussian channels with arbitrary input distribution,” in *IEEE International Symposium on Information Theory*, June-July 2009.

[11] M. Lamarca, “Linear precoding for mutual information maximization in MIMO systems,” in *International Symposium on Wireless Communications Systems*, Sep. 2009.

[12] C. Xiao, Y. R. Zheng, and Z. Ding, “Globally optimal linear precoders for finite alphabet signals over complex gaussian channels,” *IEEE Trans. Signal Process.*, vol. 59, pp. 3301–3314, Jul. 201.
[13] W. Zeng, C. Xiao, M. Wang, and J. Lu, “Linear precoding for finite-alphabet inputs over mimo fading channels with statistical csi,” IEEE Trans. Signal Process., vol. 60, pp. 3134–3148, Jun. 2012.
[14] N. Bleistein and R. A. Handelsman, Asymptotic Expansions of Integrals. New York: Dover, 1986.
[15] J. G. Proakis, Digital Communications, 3rd ed. McGraw-Hill, 1995.
[16] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, Eds., NIST Handbook of Mathematical Functions. Cambridge University Press, 2010.
[17] G. D. Forney and L.-F. Wei, “Multidimensional constellations-part i: Introduction, figures of merit, and generalized cross constellations,” IEEE J. Sel. Areas Commun., vol. 7, no. 6, pp. 877–892, Aug. 1989.
[18] H. A. Priestley, Introduction to Complex Analysis, 2nd ed. Oxford University Press, 2004.
[19] L. Zheng and D. N. C. Tse, “Diversity and multiplexing: A fundamental tradeoff in multiple-antenna channels,” IEEE Trans. Inf. Theory, vol. 49, no. 5, pp. 1073–1096, May 2003.
[20] M. R. D. Rodrigues, “On the constrained capacity of multi-antenna fading coherent channels with discrete inputs,” IEEE International Symposium on Information Theory, Jul. 2011.
[21] ——, “Characterization of the constrained capacity of multiple-antenna fading coherent channels driven by arbitrary inputs,” IEEE International Symposium on Information Theory, Jul. 2012.
[22] Y. Wu and S. Verdú, “MMSE dimension,” IEEE Trans. Inf. Theory, vol. 57, no. 8, pp. 4857–4879, Aug. 2011.
[23] S. Verdú, “Spectral efficiency in the wideband regime,” IEEE Trans. Inf. Theory, vol. 48, no. 6, pp. 1319–1343, Jun. 2002.
[24] V. V. Prelov and S. Verdú, “Second-order asymptotics of mutual information,” IEEE Trans. Inf. Theory, vol. 50, no. 8, pp. 1567–1580, Aug. 2004.
[25] J. F. C. Kingman and S. J. Taylor, Introduction to Measure and Probability. Cambridge University Press, 1973.