Equation of state of imbalanced cold matter from chiral perturbation theory

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We study the thermodynamic properties of matter at vanishing temperature for non-extreme values of the isospin chemical potential and of the strange quark chemical potential. From the leading order pressure obtained by maximizing the static chiral Lagrangian density we derive a simple expression for the equation of state in the pion condensed phase and in the kaon condensed phase. We find an analytical expression for the maximum of the ratio between the energy density and the Stefan-Boltzmann energy density as well as for the isospin chemical potential at the peak both in good agreement with lattice simulations of quantum chromodynamics. We speculate on the location of the crossover from the Bose-Einstein condensate state to the Bardeen-Cooper-Schrieffer state by a simple analysis of the thermodynamic properties of the system. For $\mu_I \gtrsim 2m_\pi$ the leading order chiral perturbation theory breaks down; as an example it underestimates the energy density of the system and leads to a wrong asymptotic behavior.

I. INTRODUCTION

Nuclear matter at sufficiently high density can be described as an imbalanced system having non-vanishing isospin density and, in some circumstances, strangeness. The most remarkable and extreme example is matter in the interior of compact stars, which is believed to be neutron-rich with a small proton fraction [1]. Eventually, in the core of compact stars hyperonic nucleation can take place, leading to a non-vanishing strangeness density. In any case, understanding the properties of matter at nonzero isospin and strangeness density allows us to explore quantum chromodynamics (QCD) in a regime in which various methods can be employed with significant overlap.

The isospin chemical potential, $\mu_I$, and the strange quark chemical potential, $\mu_S$, have several effects on matter. Confining ourselves to the mesonic sector, the obvious effect is a Zeeman-like mass splitting within multiplets. A nontrivial effect is a rotation of the vacuum in flavor space. Indeed, sufficiently large values of $\mu_I$, and/or of $\mu_S$, are able to tilt the chiral condensate leading to meson condensation. The grand canonical phase diagram of matter at vanishing temperature as a function of $\mu_I$ and $\mu_S$ has been firmly established by chiral perturbation theory (\chiPT), see [2]. Varying $\mu_I$ and $\mu_S$, three different phases can be realized: the normal phase, the pion condensed ($\pi c$) phase and the kaon condensed ($Kc$) phase.

Very powerful methods for the analysis of imbalanced quark matter have been developed in lattice QCD (LQCD) [3–8]. Remarkably, LQCD simulations are feasible at nonzero $\mu_I$ and $\mu_S$. Unfortunately these simulations are not easy to perform for physical meson masses and/or large $\mu_I$ and $\mu_S$. Moreover, the interpretation of the LQCD results requires a careful understanding of the physics, which is certainly more transparent when using effective field theories or perturbative QCD (pQCD).

The meson condensed phases have also been studied by Nambu-Jona Lasinio (NJL) models in [9–15] and by random matrix models in [16–17]. In [10–14] the $\mu_I$-$T$ phase diagram was obtained by NJL models. Finite temperature effects were also considered in \chiPT [18–20]. In particular, in [18, 19] the $\mu_I$-$T$ phase diagram has been obtained.

The thermodynamic properties of the meson condensed phases have been studied by LQCD simulations in [3, 6]. Previously, various results on the $\pi c$ phase were derived in [14] by a NJL model, including an expression of the equation of state (EoS). Recently in [21–23] a perturbative analysis of QCD at large isospin density has been presented. Those pQCD results are consistent with LQCD for $\mu_I \gtrsim 3m_\pi$ [23], where $m_\pi$ is the pion mass. At smaller values of $\mu_I$ it seems that pQCD underestimates the energy density and is not able to capture the condensation mechanism. However, for small values of $\mu_I$ (and $\mu_S$), \chiPT can be used. Although the \chiPT isospin density of the system has been determined several years ago [24], to the best of our knowledge a careful study of the EoS of imbalanced matter has never been derived within this framework.

By the present manuscript we fill this gap, analyzing the thermodynamic properties of cold matter both in the $\pi c$ phase and in the $Kc$ phase at non-extreme values of $\mu_I$ and/or $\mu_S$. We use a realization of \chiPT that includes only the pseudoscalar mesons [22, 29]. Thus our results are valid for $|\mu_I| < 770$ MeV, $|\mu_S| < 550$ MeV and for baryonic chemical potentials below the nucleon mass, see for example the discussion in [2] and in [31]. We derive a remarkable simple expression for the leading order (LO) EoS. As we will see, the comparison with the LQCD results of [3] will clarify some interesting aspects of the energy density in the $\pi c$ phase. Moreover, we estimate the value of $\mu_I$ corresponding to the crossover from the Bose-Einstein condensate (BEC) state to the Bardeen-Cooper-Schrieffer (BCS) state.

When comparing our results with those obtained by different methods we use consistent values of $m_\pi$ and of

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the pion decay constant, \( f_\pi \). In particular, the LQCD simulations of [3] and the pQCD results of [23-24] are obtained for \( m_\pi = 390 \text{ MeV} \). Given the rather large value of the pion mass, we take \( f_\pi = 110 \text{ MeV} \), see for example [30].

The present paper is organized as follows. In Sec. II, we briefly review the meson condensed phases. In Sec. III, we derive the expression of the equation of state and compare our result for the energy density with that obtained by LQCD simulations and by pQCD. In Sec. IV, we draw our conclusions.

**II. MESON CONDENSATION**

The LO Lagrangian density describing the in-medium pseudoscalar mesons can be written as [25]

\[
\mathcal{L} = \frac{f_\pi^2}{4} \text{Tr}(D_{\mu} \Sigma D^{\mu} \Sigma^\dagger) + \frac{f_\pi^2}{4} \text{Tr}(X \Sigma^\dagger + \Sigma X^\dagger),
\]

where \( \Sigma \) corresponds to the meson fields, \( X = 2BM \) with \( M = \text{diag}(m, m, m) \) and the trace is in flavor space.

\[
\Sigma = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \beta & -\sin \beta \\
0 & \sin \beta & \cos \beta
\end{pmatrix}
\begin{pmatrix}
\cos \alpha & 0 \\
\sin \alpha & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \beta & \sin \beta \\
0 & -\sin \beta & \cos \beta
\end{pmatrix},
\]

with \( \alpha, \beta \in (0, \pi/2) \). The explicit form of the ground state is found maximizing the static Lagrangian

\[
\mathcal{L}_0 = -\frac{f_\pi^2}{4} \text{Tr} \left[ \mu, \Sigma \right] \left[ \mu, \Sigma^\dagger \right] + \frac{f_\pi^2B}{2} \text{Tr} \left[ M(\Sigma + \Sigma^\dagger) \right],
\]

with respect to the angles \( \alpha \) and \( \beta \). The maximum of the static Lagrangian corresponds to the the tree-level pressure of the system at vanishing temperature.

In this picture the isospin and strange quark chemical potentials can lead to an increase of the pressure if they are able to tilt the vacuum in a direction in \( SU(N_f)_V \) space. When this happens, the system undergoes a phase transition to a meson condensed phase. More in detail, by varying \( \mu_I \) and \( \mu_S \) it is possible to show that three different phases can be realized: the normal phase (which at zero temperature corresponds to the vacuum), the \( \pi c \) phase and the \( Kc \) phase, characterized by a non-vanishing pion and kaon condensate, respectively [2]. We will normalize the pressure of the vacuum to zero by subtracting to the tree-level results the pressure \( p^0 = \mathcal{L}_0^N \) corresponding to the maximum of the static Lagrangian in the normal phase.

The pion and the kaon masses can be expressed in terms of the quark masses by the usual relations \( m_\pi^2 = 2Bm \) and \( m_K^2 = B(m + m_s) \), see [23-24]. The in-medium effects can be introduced by means of external currents in the covariant derivative [25]. For the case of interest we define

\[
D_\mu \Sigma = \partial_\mu \Sigma - \frac{i}{2} [\nu_\mu, \Sigma], \quad \nu_\mu = -2\mu \delta_{\mu 0},
\]

with \( \mu = \text{diag}(\mu_I/2, -\mu_I/2, -\mu_S) \). Since we focus on the zero-temperature thermodynamics, we neglect fluctuations and replace \( \Sigma \rightarrow \bar{\Sigma} \), where \( \bar{\Sigma} \) is the vacuum expectation value of the mesonic fields. In the normal phase \( \bar{\Sigma} = I \), but a generic vacuum depends on \( N_f^2 - 1 \) parameters, corresponding to the possible orientations in \( SU(N_f)_V \) space. However, it can be shown that many vacua are degenerate, see [2] and the discussion in [31]. The \( N_f = 2 \) vacuum Lagrangian depends only on one angle, \( \alpha \), and the three-flavor vacuum on two angles, \( \alpha \) and \( \beta \). As in [2] we consider the three-flavor parametrization

\[1\] We do not include the baryonic chemical potential because mesons have zero baryonic charge.

The normal phase is stable for \( \mu_I < m_\pi \) and \( \mu_S < m_K - 1/2 \mu_I \). In this case \( \bar{\Sigma}_N = \text{diag}(1, 1, 1) \) and the vacuum pressure is given by \( p_0^I = f_\pi^2m_\pi^2 \) for \( N_f = 2 \) and \( p_0^S = f_\pi^2m_\pi^2 \left( \frac{1}{2} + \frac{m_K}{m_\pi} \right) \) for \( N_f = 3 \).

The \( \pi c \) phase is stable for \( \mu_I > m_\pi \) and \( \mu_S < (-m_\pi^2 + \sqrt{(m_\pi^2 - m_B^2)^2 + 4m_\pi^2m_B^2})/(2\mu_I) \), and is characterized by \( \cos \alpha_\pi = m_\pi^2/m_B^2 \) and \( \beta_\pi = 0 \). In this phase the LO normalized pressure is given by [2] [24]

\[
p^\pi_0 = \frac{f_\pi^2m_B^2}{2} \left( 1 - \frac{m_\pi^2}{m_B^2} \right)^2,
\]

which does not depend on the kaon mass and on the strange quark chemical potential. The reason is that this pressure is determined by the condensation of pions, and therefore it can only depend on the properties of pions. This expression is valid for both \( N_f = 2 \) and \( N_f = 3 \).

The \( Kc \) phase is stable for \( \mu_I > 2(m_K - \mu_S) \) and \( \mu_S > (-m_\pi^2 + \sqrt{(m_\pi^2 - m_B^2)^2 + 4m_\pi^2m_B^2})/(2\mu_I) \), and is characterized by \( \cos \alpha_K = \left( \frac{m_K}{m_B + \mu_B} \right)^2 \) and \( \beta_K = \pi/2 \). The LO normalized pressure is analogous to Eq. (4) with the replacement \( m_\pi \rightarrow m_K \) and \( \mu_I \rightarrow \mu_K = \mu_I/2 + \mu_S \), related to the fact that kaons have isospin 1/2 and strangeness 1.
III. EQUATION OF STATE

The energy density of the system can be obtained by
\[
\epsilon = \mu_I n_I + \mu_S n_S - p, \tag{6}
\]
where \(n_I\) and \(n_S\) are the isospin and strange number densities, respectively. Given that the considered phases are characterized by the condensation of pions or of kaons, we can rewrite the energy density as follows
\[
\epsilon^{\pi c} = \mu_I n_I^{\pi c} - p^{\pi c}, \tag{7}
\]
for the \(\pi c\) phase (since \(n_S^{\pi c} = 0\)), and
\[
\epsilon^{K c} = \mu_K n_K^{K c} - p^{K c}, \tag{8}
\]
for the \(K c\) phase. The LO isospin number density in the \(\pi c\) phase can be calculated from
\[
n_I^{\pi c, LO} = \frac{\partial \rho_{I, LO}^{\pi c}}{\partial \mu_I} = f^2_{\pi} \mu_I \left( 1 - \frac{m_I^4}{\mu_I^4} \right), \tag{9}
\]
which agrees with the result obtained in [24]. The kaon number density in the \(K c\) phase has an analogous expression, with \(\mu = m_K\) and \(\mu_I \to \mu_K\), see [6].

Upon substituting Eq. (9) in Eq. (7) we obtain the LO energy density in the \(\pi c\) phase
\[
\epsilon^{\pi c}_{LO} = \frac{f^2_{\pi} m^2_{\pi}}{2} \left( 1 + 2 \frac{m_I^2}{\mu_I^2} - 3 \frac{m_I^4}{\mu_I^4} \right), \tag{10}
\]
Inverting Eq. (4) we get \(\mu_I(p)\), that allows us to obtain the EoS
\[
\epsilon^{\pi c}_{LO}(p) = 2 \sqrt{\frac{p}{2 f^2_{\pi} m^2_{\pi} + p}} - p, \tag{11}
\]
which is an increasing function of \(p\) and vanishes for \(p = 0\). The expression of the energy density in the \(K c\) phase can be obtained from Eq. (11) with the replacement \(m_{\pi} \to m_K\). In Fig. 1 we report a plot of the EoS for the \(\pi c\) phase and we compare the result with the pQCD findings of [23] and with the ideal gas case. From this figure it is clear that the agreement with pQCD is rather poor. It is interesting to note that the conformal relation \(\epsilon = 3p\) is satisfied for
\[
\mu_I = \sqrt{3} m_{\pi}, \tag{12}
\]
corresponding as well to the point at which the energy density convexity changes. This coincidence is suggestive of a possible change from two different regimes. Since the conformal limit separates the BEC state from the BCS state, see [32, 33] for a non-relativistic discussion, it is tempting to interpret \(\mu_I\) as the chemical potential corresponding to the BEC-BCS crossover. Although our result is consistent with the NJL findings of [15], a more careful analysis is necessary for substantiating this conjecture.

A comparison of the LO expression of the pressure and energy with the NJL results has been done in [14]. In agreement with [14] we obtain that the LO \(\chi PT\) expression of the energy density and of the pressure are consistent with the corresponding NJL quantities for \(\mu_I \lesssim 2 m_{\pi}\).

For comparing the \(\chi PT\) energy density with the results obtained by LQCD simulations in [2] and by pQCD in [23], we divide it by the Stefan-Boltzmann limit, which has been defined in [5] as \(\epsilon_{SB} = 9 \mu_I^4 / (4 \pi^2)\). In Fig. 2 we report our ratio \(\epsilon^{\pi c}_{LO}/\epsilon_{SB}\) and the results of [5] and [23]. We immediately notice that the \(\chi PT\) curve perfectly captures the peak structure at low \(\mu_I\), while it begins to depart from the LQCD results after \(\mu_I \sim 2 m_{\pi}\), indicating the breakdown of the LO approximation. An interesting result is that within our framework we can obtain an analytic expression for the position of the peak in this ratio, which for the \(\pi c\) phase is given by
\[
\mu_I^{\text{peak}} = \left( \sqrt{13} - 2 \right)^{1/2} m_{\pi} \simeq 1.276 m_{\pi}, \tag{13}
\]
and is independent of \(f_\pi\). This result is very close to the LQCD results obtained in [5], where the values \(\mu_I^{\text{peak}} = \{1.20, 1.25, 1.275\} m_{\pi}\) have been obtained considering different spatial volumes \(L^3\) with side \(L = \{16, 20, 24\}\), respectively. The continuum-linearly-extrapolated value for the peak position is \(\mu_I^{\text{peak}} = 1.30 m_{\pi}\).

In [5] the top of the peak is interpreted as the point where the pion condensation sets in. However, in our case the pion condensate has already reached a significant value at that point, almost approaching its asymptotic value. We are therefore more inclined to interpret this peak structure as a consequence of the filling of the condensate, rather than of the onset of its formation, which at zero-temperature should occur at \(\mu_I = m_{\pi}\). We also obtain an analytic expression for the ratio at the peak
\[
\frac{\epsilon}{\epsilon_{SB}} \bigg|_{\text{peak}} = \frac{4(\sqrt{13} - 5) \pi^2 f^2_{\pi}}{9(-2 + \sqrt{13}) m^2_{\pi}}, \tag{14}
\]
which would give information on the \(f_\pi/m_{\pi}\) scaling if precise LQCD data were available. It is worth men-
density. The basic reason is that for large breaks down, resulting in an underestimate of the energy from Eq. (10) that predicts a term proportional to \( \chi PT \). It is possible to show that the pQCD results of \([23]\) are in good agreement with the LQCD results, paving the way for a more quantitative comparison between \( \chi PT \) and LQCD, if precise lattice data will be available. The behavior of the LQCD data for \( \mu_I \geq 3m_\pi \) can be used to constrain some NLO constants, as in Eq. (15). The results obtained in the \( \pi c \) phase can be easily extended to the \( Kc \) phase.

We have observed that the energy density to pressure ratio reaches the conformal limit at \( \bar{\mu}_I \) given in Eq. (12), which corresponds as well to the point of vanishing curvature of the energy density, suggestive of a BEC-BCS crossover. Whether the \( \chi PT \) can capture some aspects of the BEC-BCS crossover is a tantalizing possibility, which should be further explored with a detailed analysis of the system, see for example \([36]\).

There are several directions for improving the present work. One possibility is to systematically analyze all NLO temperature and chemical potential corrections for a precise matching with LQCD data. Ultimately, a realistic description of dense matter in compact stars will require the inclusion of baryonic degrees of freedom, for example by means of the heavy-baryon effective theory, see \([28]\).

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