Application of Geometric Calculus in Numerical Analysis and Difference Sequence Spaces

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Abstract. The main purpose of this paper is to introduce the geometric difference sequence space \( l^G_\infty(\Delta^G) \) and prove that \( l^G_\infty(\Delta^G) \) is a Banach space with respect to the norm \( \| \cdot \|_{\Delta^G}^G \). Also we compute the \( \alpha \)-dual, \( \beta \)-dual and \( \gamma \)-dual spaces. Finally we obtain the Geometric Newton-Gregory interpolation formulae.

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1. Introduction and Notations

In 1967 Robert Katz and Michael Grossman created the first system of non-Newtonian calculus, which we call the geometric calculus. In 1970 they had created an infinite family of non-Newtonian calculi, each of which differs markedly from the classical calculus of Newton and Leibniz. Among other things, each non-Newtonian calculus possesses four operators: a gradient (i.e. an average rate of change), a derivative, an average, and an integral. For each non-Newtonian calculus there is a characteristic class of functions having a constant derivative.

In view of pioneering work carried out in this area by Grossman and Katz [7] we will call this calculus as multiplicative calculus, although the term of exponential calculus can also be used. The operations of multiplicative calculus will be called as multiplicative derivative and multiplicative integral. We refer to Grossman and Katz [7], Stanley [15], Bashirov et al. [2, 3], Grossman [6] for elements of multiplicative calculus and its applications. An extension of multiplicative calculus to functions of complex variables is handled in Bashirov and Riza [1], Uzer [18], Bashirov et al. [3], Çakmak and Başar [4], Tekin and Başar [16], Türkmen and Başar [17]. In [10], Kadak and Özük studied the generalized Runge-Kutta method with respect to non-Newtonian calculus. Kadak et al [8, 9] studied certain new types of sequence spaces over the Non-Newtonian Complex Field.

Geometric calculus is an alternative to the usual calculus of Newton and Leibniz. It provides differentiation and integration tools based on multiplication instead of addition. Every property in Newtonian calculus has an analog in multiplicative calculus. Generally speaking multiplicative calculus is a methodology that allows one to have a different look at problems which can be investigated via calculus. In some cases, for example for growth related problems, the use of multiplicative calculus is advocated instead of a traditional Newtonian one.

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The main aim of this paper is to construct the difference sequence space $l^G_\infty (\Delta_G)$ over geometric complex numbers which forms a Banach space with the norm defined on it and obtain the Geometric Newton-Gregory interpolation formulae which are more useful than Newton-Gregory interpolation formulae.

We should know that all concepts in classical arithmetic have natural counterparts in $\alpha - \text{arithmetic}$. Consider any generator $\alpha$ with range $A \subseteq \mathbb{C}$. By $\alpha - \text{arithmetic}$, we mean the arithmetic whose domain is $A$ and operations are defined as follows. For $x, y \in A$ and any generator $\alpha$,

\begin{align*}
\alpha - \text{addition} & : x + y = \alpha[\alpha^{-1}(x) + \alpha^{-1}(y)] \\
\alpha - \text{subtraction} & : x - y = \alpha[\alpha^{-1}(x) - \alpha^{-1}(y)] \\
\alpha - \text{multiplication} & : x \times y = \alpha[\alpha^{-1}(x) \times \alpha^{-1}(y)] \\
\alpha - \text{division} & : x/y = \alpha[\alpha^{-1}(x)/\alpha^{-1}(y)] \\
\alpha - \text{order} & : x < y \iff \alpha^{-1}(x) < \alpha^{-1}(y).
\end{align*}

If we choose $exp$ as an $\alpha - \text{generator}$ defined by $\alpha(z) = e^z$ for $z \in \mathbb{C}$ then $\alpha^{-1}(z) = \ln z$ and $\alpha - \text{arithmetic}$ turns out to Geometric arithmetic.

\begin{align*}
\alpha - \text{addition} & : x \oplus y = \alpha[\alpha^{-1}(x) + \alpha^{-1}(y)] = e^{\ln x + \ln y} = x.y \text{ geometric addition} \\
\alpha - \text{subtraction} & : x \ominus y = \alpha[\alpha^{-1}(x) - \alpha^{-1}(y)] = e^{\ln x - \ln y} = x/y, y \neq 0 \text{ geometric subtraction} \\
\alpha - \text{multiplication} & : x \odot y = \alpha[\alpha^{-1}(x) \times \alpha^{-1}(y)] = e^{\ln x \times \ln y} = x^{\ln y} \text{ geometric multiplication} \\
\alpha - \text{division} & : x \oslash y = \alpha[\alpha^{-1}(x)/\alpha^{-1}(y)] = e^{\ln x/\ln y} = x^{1/y}, y \neq 1 \text{ geometric division}.
\end{align*}

In [17] defined the geometric complex numbers $\mathbb{C}(G)$ as follows:

$$
\mathbb{C}(G) := \{e^z : z \in \mathbb{C}\} = \mathbb{C}\{0\}.
$$

Then $(\mathbb{C}(G), \oplus, \odot)$ is a field with geometric zero $1$ and geometric identity $e$.

Then for all $x, y \in \mathbb{C}(G)$

- $x \oplus y = xy$
- $x \ominus y = x/y$
- $x \odot y = x^{\ln y} = y^{\ln x}$
- $x \odot y$ or $x \odot y = x^{\ln y}, y \neq 1$
- $x^{2G} = x \odot x = x^{\ln x}$
- $x^{pG} = x^{\ln^{p-1}x}$
- $\sqrt{x^G} = e^{(\ln x)^{1/2}}$
- $x^{-1G} = e^{\ln^{-1}x}$
- $x \odot e = x$ and $x \oplus 1 = x$
- $e^x \odot x = x^n = x \odot x \odot \ldots \ldots$ (upto $n$ number of $x$)

Thus $|x|^G \geq 1$.

- $\sqrt{x^{2G}} = |x|^G$
- $|e^y|^G = e^{|y|}$
- $|x \odot y|^G = |x|^G \odot |y|^G$
For the convenience, in this paper we denote $l_\infty, c$ and $c_0$ be the linear spaces of complex bounded, convergent and null sequences, respectively, normed by
\[ ||x||_\infty = \sup_k |x_k|. \]

Türkmen and Başar [17] have proved that
\[ \omega(G) = \{(x_k) : x_k \in \mathbb{C}(G) \text{ for all } k \in \mathbb{N}\} \]
is a vector space over $\mathbb{C}(G)$ with respect to the algebraic operations $\oplus$ addition and $\odot$ multiplication
\[ \oplus : \omega(G) \times \omega(G) \rightarrow \omega(G) \]
\[ (x, y) \rightarrow x \oplus y = (x_k) \oplus (y_k) = (x_k y_k) \]
\[ \odot : \mathbb{C}(G) \times \omega(G) \rightarrow \omega(G) \]
\[ (\alpha, y) \rightarrow \alpha \odot y = \alpha \odot (y_k) = (\alpha^{\ln(y_k)}) \]
where $x = (x_k), y = (y_k) \in \omega(G)$ and $\alpha \in \mathbb{C}(G)$. Then
\[ l_\infty(G) = \{x = (x_k) \in \omega(G) : \sup_{k\in\mathbb{N}} |x_k|^G < \infty\} \]
\[ c(G) = \{x = (x_k) \in \omega(G) : G \lim_{k \rightarrow \infty} |x_k \odot l|^G = 1\} \]
\[ c_0(G) = \{x = (x_k) \in \omega(G) : G \lim_{k \rightarrow \infty} x_k = 1\}, \text{ where } G \lim \text{ is the geometric limit} \]
\[ l_p(G) = \{x = (x_k) \in \omega(G) : G \sum_{k=0}^{\infty} (|x_k|^G)^{pG} < \infty\}, \text{ where } G \sum \text{ is the geometric sum,} \]
are classical sequence spaces over the field $\mathbb{C}(G)$. Also it is shown that $l_\infty(G), c(G)$ and $c_0(G)$ are Banach spaces with the norm
\[ ||x||^G = \sup_k |x_k|^G, x = (x_1, x_2, x_3, \ldots) \in \lambda(G), \lambda \in \{l_\infty, c, c_0\}. \]

For the convenience, in this paper we denote $l_\infty(G), c(G), c_0(G)$, respectively as $l^G_\infty, c^G, c^G_0$.

2. NEW GEOMETRIC SEQUENCE SPACE

In 1981, Kizmaz [11] introduced the notion of difference sequence spaces using forward difference operator $\Delta$ and studied the classical difference sequence spaces $l_\infty(\Delta), c(\Delta), c_0(\Delta)$. In this section we define the following new geometric sequence space
\[ l^G_\infty(\Delta_G) = \{x = (x_k) \in \omega(G) : \Delta_G x \in l^G_\infty\}, \text{ where } \Delta_G x = x_k \oplus x_{k+1}. \]

**Theorem 2.1.** The space $l^G_\infty(\Delta_G)$ is a normed linear space w.r.t. the norm
\[ ||x||^G_{\Delta_G} = ||x_1|^G \oplus ||\Delta_G x||^G. \]
Proof. For \( x = (x_k), y = (y_k) \in I_1^G (\Delta_G), \)

\[
N1. \quad \|x\|_G = |x_1|^G \oplus \|\Delta_G x\|_G
\]
\[
= |x_1|^G \cdot \sup_k |x_k \oplus x_{k+1}|^G
\]
\[
\geq 1, \quad \text{since } |x_1|^G \geq 1 \text{ and } |x_k \oplus x_{k+1}|^G \geq 1.
\]

\[
N2. \quad \|x\|_G = 1 \iff |x_1|^G \oplus \|\Delta_G x\|_G = 1
\]
\[
\iff |x_1|^G \cdot \sup_k |x_k \oplus x_{k+1}|^G = 1 \forall k
\]
\[
\iff |x_1|^G = 1 \text{ and } |x_k \oplus x_{k+1}|^G = 1
\]
\[
\iff x_1 = 1 \text{ and } x_k \oplus x_{k+1} = 1 \forall k
\]
\[
\iff x_1 = 1 \text{ and } x_k / x_{k+1} = 1 \forall k
\]
\[
\iff x_1 = 1 \text{ and } x_k = x_{k+1} \forall k
\]
\[
\iff x_k = 1 \forall k
\]
\[
\iff x = (1, 1, 1, 1, \ldots) = 0_G.
\]

\[
N3. \quad \|x \oplus y\|_G \leq |x_1 \oplus y_1|^G \oplus \|\Delta_G (x_k \oplus y_k)\|_G
\]
\[
= |x_1 \oplus y_1|^G \oplus \|\Delta_G (x_k y_k)\|_G
\]
\[
= |x_1 \oplus y_1|^G \oplus \sup_k |x_k y_k \oplus x_{k+1} y_{k+1}|^G
\]
\[
= |x_1 \oplus y_1|^G \oplus \sup_k \left| \frac{x_k y_k}{x_{k+1} y_{k+1}} \right|^G
\]
\[
= |x_1 \oplus y_1|^G \oplus \sup_k \left| \frac{x_k}{x_{k+1}} \oplus \frac{y_k}{y_{k+1}} \right|^G
\]
\[
\leq |x_1 \oplus y_1|^G \oplus \sup_k \left\{ \left| \frac{x_k}{x_{k+1}} \right|^G \oplus \frac{y_k}{y_{k+1}} \right\}^G
\]
\[
= |x_1 \oplus y_1|^G \oplus \sup_k \left\{ |x_k \oplus x_{k+1}|^G \oplus |y_k \oplus y_{k+1}|^G \right\}
\]
\[
= |x_1 \oplus y_1|^G \oplus \sup_k \left\{ |\Delta_G x|^G \oplus |\Delta_G y|^G \right\}
\]
\[
\leq |x_1|^G \oplus |y_1|^G \oplus \sup_k \left\{ |\Delta_G x|^G \right\} \oplus \sup_k \left\{ |\Delta_G y|^G \right\}
\]
\[
= \left[ |x_1|^G \oplus \sup_k \left\{ |\Delta_G x|^G \right\} \right] \oplus \left[ |y_1|^G \oplus \sup_k \left\{ |\Delta_G y|^G \right\} \right]
\]
\[
= \|x\|_{\Delta_G} \oplus \|y\|_{\Delta_G}.
\]
Theorem 2.2. The space $l^G_\infty (\Delta_G)$ is a Banach space w.r.t. the norm $\| \cdot \|_{\Delta_G}^G$.

Proof. Let $(x_n)$ be a Cauchy sequence in $l^G_\infty (\Delta_G)$, where $x_n = \left( x_k^{(n)} \right) = \left( x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, \ldots \right)$, $\forall n \in \mathbb{N}, x_k^{(n)}$ is the $k^{th}$ coordinate of $x_n$. Then

$$\| x_n \ominus x_m \|_{\Delta_G}^G = \left| x_1^{(n)} \ominus x_1^{(m)} \right| G + \| \Delta_G x_n \ominus \Delta_G x_m \|_{\Delta_G}^G \to 1 \text{ as } m, n \to \infty$$

This implies that $\left| x_k^{(n)} \ominus x_k^{(m)} \right| G \to 1$ as $n, m \to \infty \forall k \in \mathbb{N}$, since $\left| x_k^{(n)} \ominus x_k^{(m)} \right| G \geq 1$.

Therefore for fixed $k$, $k^{th}$ co-ordinates of all sequences form a Cauchy sequence in $\mathbb{C}(G)$ i.e. $x_k^{(n)} = \left( x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, x_k^{(4)}, \ldots \right)$ is a Cauchy sequence. Then by the completeness of $\mathbb{C}(G), (x_k^{(n)})$ converges to $x_k$ (say) as follows:

\[
\begin{align*}
x_1 & = ( x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \ldots, x_k^{(1)}, \ldots ) \\
x_2 & = ( x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, \ldots, x_k^{(2)}, \ldots ) \\
x_3 & = ( x_1^{(3)}, x_2^{(3)}, x_3^{(3)}, \ldots, x_k^{(3)}, \ldots ) \\
& \vdots \\
x_m & = ( x_1^{(m)}, x_2^{(m)}, x_3^{(m)}, \ldots, x_k^{(m)}, \ldots ) \\
& \vdots \\
x_n & = ( x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, \ldots, x_k^{(n)}, \ldots ) \\
& \vdots \\
\downarrow & \downarrow \downarrow \downarrow \downarrow \\
x & = ( x_1, x_2, x_3, \ldots, x_k, \ldots )
\end{align*}
\]

i.e.

$$G \lim_{n \to \infty} x_k^{(n)} = x_k \forall k \in \mathbb{N}.$$ 

Further for each $\varepsilon > 1$, $\exists N = N(\varepsilon)$ s.t. $\forall n, m \geq N$ we have

$$\left| x_1^{(n)} \ominus x_1^{(m)} \right| G < \varepsilon, \left| x_k^{(n)} \ominus x_k^{(m)} \ominus (x_k^{(n)} \ominus x_k^{(m)}) \right| G < \varepsilon$$
and
\[
G \lim_{m \to \infty} |x_1^{(n)} \ominus x_1^{(m)}|^G = |x_1^{(n)} \ominus x_1|^G < \varepsilon.
\]

This implies
\[
G \lim_{m \to \infty} |(x_{k+1}^{(n)} \ominus x_{k+1}^{(m)}) \ominus (x_k^{(n)} \ominus x_k^{(m)})|^G = |(x_{k+1}^{(n)} \ominus x_{k+1}) \ominus (x_k^{(n)} \ominus x_k)|^G < \varepsilon \quad \forall \ n \geq N.
\]

Since \(\varepsilon\) is independent of \(k\),
\[
\sup_k |(x_{k+1}^{(n)} \ominus x_{k+1}) \ominus (x_k^{(n)} \ominus x_k)|^G < \varepsilon.
\]
\[
\Rightarrow \sup_k |(x_{k+1}^{(n)} \ominus x_k^{(n)}) \ominus (x_{k+1} \ominus x_k)|^G = \|\Delta_G x_n \ominus \Delta_G x\|^G_{\infty} < \varepsilon.
\]

Consequently we have \(\|x_n \ominus x\|^G_{\Delta_G} = |x_1^{(n)} \ominus x_1|^G + \|\Delta_G x_n \ominus \Delta_G x\|^G_{\infty} < \varepsilon^2 \quad \forall \ n \geq N.
\]
Hence we obtain \(x_n \to x\) as \(n \to \infty\).

Now we must show that \(x \in l^G_{\infty}(\Delta_G)\). We have
\[
|x_k \ominus x_{k+1}|^G = |x_k \ominus x_k^N \ominus x_{k+1}^N \ominus x_{k+1} \ominus x_{k+1}|^G
\leq |x_k^N \ominus x_{k+1}|^G + \|x^N \ominus x\|^G_{\Delta_G} = O(\varepsilon).
\]

This implies \(x = (x_k) \in l^G_{\infty}(\Delta_G)\). \(\square\)

Furthermore since \(l^G_{\infty}(\Delta_G)\) is a Banach space with continuous coordinates (that is \(\|x_n \ominus x\|^\infty_{\Delta_G} \to 1\) implies \(|x_k^{(n)} \ominus x_k|^G \to 1\) for each \(k \in \mathbb{N}\), as \(n \to \infty\) it is a BK-space.

**Remark 2.1.** The spaces
(a) \(c^G(\Delta_G) = \{(x_k) \in w(G) : \Delta_G x_k \in c^G\}\)
(b) \(c^G(\Delta_G) = \{(x_k) \in w(G) : \Delta_G x_k \in c^G_0\}\)

are Banach spaces with respect to the norm \(||\cdot||^G_{\Delta_G}\). Also these spaces are BK-space.

Now we define \(s : l^G_{\infty}(\Delta_G) \to l^G_{\infty}(\Delta_G), x \to sx = y = (1, x_2, x_3, \ldots)\). It is clear that \(s\) is a bounded linear operator on \(l^G_{\infty}(\Delta_G)\) and \(||s||^G_{\infty} = e\). Also
\[
s \left[ l^G_{\infty}(\Delta_G) \right] = sl^G_{\infty}(\Delta_G) = \{x = (x_k) : x \in l^G_{\infty}(\Delta_G), x_1 = 1\} \subset l^G_{\infty}(\Delta_G)
\]

is a subspace of \(l^G_{\infty}(\Delta_G)\) and as \(|x_1|^G = 1\) for \(x_1 = 1\) we have
\[
||x||^G_{\Delta_G} = ||\Delta_G x||^G_{\infty} \quad \text{in} \quad sl^G_{\infty}(\Delta_G).
\]

On the other hand we can show that
\[
\Delta_G : sl^G_{\infty}(\Delta_G) \to l^G_{\infty}
\]
\[
x = (x_k) \to y = (yk) = (x_k \ominus x_{k+1})
\]
is a linear homomorphism. So \(sl^G_{\infty}(\Delta_G)\) and \(l^G_{\infty}\) are equivalent as topological space. \(\Delta_G\) and \(\Delta_G^{-1}\) are norm preserving and \(||\Delta_G||^G_{\infty} = ||\Delta_G^{-1}||^G_{\infty} = e\).

Let \([sl^G_{\infty}(\Delta_G)]^*\) and \([l^G_{\infty}]^*\) denote the continuous duals of \(sl^G_{\infty}(\Delta_G)\) and \(l^G_{\infty}\), respectively. We can prove that
\[
T : [sl^G_{\infty}(\Delta_G)]^* \to [l^G_{\infty}]^*, \ f_{\Delta_G} \to f = f_{\Delta_G} \circ \Delta_G^{-1}
\]
is a linear isometry. Thus \([sl^G_{\infty}(\Delta_G)]^*\) is equivalent to \([l^G_{\infty}]^*\). In the same way we can show that \(sc^G(\Delta_G)\) and \(c^G_0(\Delta_G)\) and \(c^G_0\) are equivalent as topological spaces and \([sc^G(\Delta_G)]^* = [sc^G(\Delta_G)]^* = l^G_1\) (the space of geometric absolutely convergent series).
3. DUAL SPACES OF $l^G_\infty(\Delta_G)$

**Lemma 3.1.** The following conditions (a) and (b) are equivalent:

(a) $\sup_k |x_k \oplus x_{k+1}|^G < \infty$ i.e. $\sup_k |\Delta_G x_k|^G < \infty$;

(b)(i) $\sup_k e^{k-1} \odot |x_k|^G < \infty$ and

(ii) $\sup_k |x_k \odot e^{k(k+1)-1} \odot x_{k+1}|^G < \infty$.

**Proof.** Let (a) be true i.e. $\sup_k |x_k \oplus x_{k+1}|^G < \infty$.

Now $|x_1 \oplus x_{k+1}|^G = \left| \sum_{v=1}^{k} (x_v \oplus x_{v+1}) \right|^G$

$= \left| \sum_{v=1}^{k} \Delta_G x_v \right|^G$

$\leq \sum_{v=1}^{k} |\Delta_G x_v|^G = O(e^k)$

and $|x_k|^G = |x_1 \oplus x_1 \oplus x_{k+1} \oplus x_k \oplus x_{k+1}|^G$

$\leq |x_1|^G + |x_1 \oplus x_{k+1}|^G + |x_k \oplus x_{k+1}|^G = O(e^k)$.

This implies that $\sup_k e^{k-1} \odot |x_k|^G < \infty$. This completes the proof of b(i).

Again

$\sup_k |x_k \odot e^{k(k+1)-1} \odot x_{k+1}|^G = \left| \left\{ e^{(k+1)-1} \right\} \odot x_k \odot e^{k(k+1)-1} \odot x_{k+1} \right|^G$

$= \left| \left\{ e^{k} \oplus e \right\} \odot e^{(k+1)-1} \odot x_k \odot e^{k(k+1)-1} \odot x_{k+1} \right|^G$

$= \left| \left\{ e^{k(k+1)-1} \right. \odot x_k \odot e^{(k+1)-1} \left. \odot x_k \right\} \odot e^{k(k+1)-1} \odot x_{k+1} \right|^G$

$= \left| \left\{ e^{k(k+1)-1} \right. \odot (x_k \odot x_{k+1}) \left. \right\} \odot e^{(k+1)-1} \odot x_k \right|^G$

$\leq e^{k(k+1)-1} \odot |x_k \odot x_{k+1}|^G + e^{(k+1)-1} \odot |x_k|^G$

$= O(e)$. 

Therefore $\sup_k |x_k \odot e^{k(k+1)-1} \odot x_{k+1}|^G < \infty$. This completes the proof of b(ii).

Conversely let (b) be true. Then

$|x_k \odot e^{k(k+1)-1} \odot x_{k+1}|^G = \left| e^{(k+1)(k+1)-1} \odot x_k \odot e^{k(k+1)-1} \odot x_{k+1} \right|^G$

$\geq e^{k(k+1)-1} \odot |x_k \odot x_{k+1}|^G + e^{(k+1)-1} \odot |x_k|^G$

i.e. $e^{k(k+1)-1} \odot |x_k \odot x_{k+1}|^G \leq e^{(k+1)-1} \odot |x_k|^G + |x_k \odot e^{k(k+1)-1} \odot x_{k+1}|^G$.

Thus $\sup_k |x_k \odot x_{k+1}|^G < \infty$ as $b(i)$ and $b(ii)$ hold. \qed
Geometric form of Abel’s partial summation formula: Abel’s partial summation formula states that if \((a_k)\) and \((b_k)\) are sequences, then

\[
\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} S_k (b_k - b_{k+1}) + S_n b_{n+1},
\]

where \(S_k = \sum_{i=1}^{k} a_i\). Then

\[
\sum_{k=1}^{\infty} a_k b_k = \sum_{k=1}^{\infty} S_k (b_k - b_{k+1}) + \lim_{n \to \infty} S_n b_{n+1}
\]

\[
\sum_{k=1}^{\infty} a_k b_k = \sum_{k=1}^{\infty} S_k (b_k - b_{k+1}), \text{ if } (b_k) \text{ monotonically decreases to zero.}
\]

Similarly as \(\odot\) is distributive over \(\oplus\) we have

\[
G \sum_{k=1}^{\infty} a_k \odot b_k = G \sum_{k=1}^{\infty} S_k \odot (b_k \ominus b_{k+1}), \text{ where } S_k = G \sum_{i=1}^{k} a_i.
\]

In particular, if \((b_k) = (e^{-k})\), then \((b_k)\) monotonically decreases to zero. Then

\[
G \sum_{k=1}^{\infty} a_k \odot e^{-k} = G \sum_{k=1}^{\infty} S_k \odot (e^{-k} \ominus e^{-(k+1)})
\]

\[
= G \sum_{k=1}^{\infty} S_k \odot e = G \sum_{k=1}^{\infty} S_k.
\]

Let \((p_n)\) be a sequence of geometric positive numbers monotonically increasing to infinity. Then \((\frac{G}{p_n})\) is a sequence monotonically decreasing to zero (i.e. to 1).

**Lemma 3.2.**

\[
\text{If } \sup_n \left| G \sum_{v=1}^{n} c_v \right|^G \leq \infty \text{ then } \sup_n \left( p_n \odot \left| G \sum_{k=1}^{\infty} \frac{c_{n+k-1}}{p_{n+k}} G \right|^G \right) < \infty.
\]

**Proof.** Using this Abel’s partial summation formula to \((c_v)\) and \((\frac{G}{p_n})\) we get

\[
G \sum_{k=1}^{\infty} \frac{c_{n+k-1}}{p_{n+k}} G = G \sum_{k=1}^{\infty} \left( G \sum_{v=1}^{k} c_{n+v-1} \right) \odot \left( \frac{e}{p_{n+k}} G \ominus \frac{e}{p_{n+k+1}} G \right)
\]

and

\[
p_n \odot \left| G \sum_{k=1}^{\infty} \frac{c_{n+k-1}}{p_{n+k}} G \right|^G = O(e).
\]

\(\square\)

**Lemma 3.3.** If the series \(\sum_{k=1}^{\infty} c_k\) is convergent then

\[
\lim_n \left( p_n \odot G \sum_{k=1}^{\infty} \frac{c_{n+k-1}}{p_{n+k}} G \right) = 1.
\]
Proof. Since

\[ |G \sum_{v=1}^{n} p_n \ominus v a_v| G = |G \sum_{v=n}^{n+k-1} c_v| G = O(e) \]

for every \( k \in \mathbb{N} \). Using (3.1) we get

\[ p_n \ominus G \sum_{k=1}^{n} c_{n+k-1} G = O(e). \]

\[ \square \]

Corollary 3.4. Let \( (p_n) \) be monotonically increasing. If

\[ \sup_n |G \sum_{v=1}^{n} p_n \ominus v a_v| G < \infty \textnormal{ then } \sup_n |p_n \ominus G \sum_{k=n+1}^{\infty} a_k| G < \infty. \]

Proof. We put \( p_{k+1} \ominus a_{k+1} \) instead of \( c_k \) in Lemma 3.2 we get

\[ p_n \ominus G \sum_{k=1}^{\infty} p_{n+k} \ominus a_{n+k} G = p_n \ominus G \sum_{k=1}^{\infty} a_{n+k} \]

\[ = p_n \ominus G \sum_{k=n+1}^{\infty} a_k = O(e). \]

\[ \square \]

Corollary 3.5.

If \( G \sum_{k=1}^{\infty} p_k \ominus a_k \) is convergent then \( \lim_n p_n \ominus G \sum_{k=n+1}^{\infty} a_k = 1. \)

Proof. We put \( p_{k+1} \ominus a_{k+1} \) instead of \( c_k \) in Lemma 3.3.

\[ \square \]

Corollary 3.6.

\( G \sum_{k=1}^{\infty} e^k \ominus a_k \) is convergent iff \( G \sum_{k=1}^{\infty} R_k \) is convergent with \( e^n \ominus R_n = O(e) \), where

\[ R_n = G \sum_{k=n+1}^{\infty} a_k. \]

Proof. Let \( p_n = e^n \). Then it is monotonically increasing to infinity. Then

\[ G \sum_{k=1}^{n} e^k \ominus a_{k+1} = e \ominus a_2 \oplus e^2 \ominus a_3 \oplus e^3 \ominus a_4 \oplus \ldots \oplus e^n \ominus a_{n+1} \]

\[ = (a_2 \ominus a_3 \ominus \ldots \ominus a_{n+1}) \oplus (a_3 \ominus a_4 \ominus \ldots \ominus a_{n+1}) \]

\[ \oplus \ldots \oplus (a_n \ominus a_{n+1}) \oplus (a_{n+1}) \]

\[ = (R_1 \ominus R_{n+1}) \oplus (R_2 \ominus R_{n+1}) \oplus \ldots \oplus (R_{n-1} \ominus R_{n+1}) \oplus (R_n \ominus R_{n+1}) \]

\[ = G \sum_{k=1}^{n} R_k \ominus \{e^n \ominus R_{n+1}\}. \]
Therefore as \( e^n \odot R_n = O(e) \), so \( e^n \odot R_{n+1} = O(e) \). This implies
\[
G \sum_{k=1}^{n} e^k \odot a_{k+1} \text{ is convergent if } G \sum_{k=1}^{n} R_k \text{ is convergent and vice versa.}
\]

\[\square\]

4. \( \alpha-, \beta-, \gamma- \) Duals

**Definition 4.1.** \([5, 12, 13, 14]\) If \( X \) is a sequence space, we define

(i) \( X^\alpha = \{ a = (a_k) : \sum_{k=1}^{\infty} |a_k x_k| < \infty, \text{ for each } x \in X \} \);

(ii) \( X^\beta = \{ a = (a_k) : \sum_{k=1}^{\infty} a_k x_k \text{ is convergent, for each } x \in X \} \);

(iii) \( X^\gamma = \{ a = (a_k) : \sup_n |\sum_{k=1}^{n} a_k x_k| < \infty, \text{ for each } x \in X \} \).

\( X^\alpha, X^\beta, \text{ and } X^\gamma \) are called \( \alpha- \) (or Köthe-Toeplitz), \( \beta- \) (or generalised Köthe-Toeplitz), and \( \gamma- \) dual spaces of \( X \). We can show that \( X^\alpha \subseteq X^\beta \subseteq X^\gamma \). If \( X \subseteq Y \), then \( Y^\dagger \subseteq X^\dagger \), for \( \dagger = \alpha, \beta \) or \( \gamma \).

**Theorem 4.1.**

(i) If \( D_1 = \left\{ a = (a_k) : G \sum_{k=1}^{\infty} e^k \odot |a_k|^G < \infty \right\} \) then \( (sl^G_\infty(\Delta_G))^\alpha = D_1 \).

(ii) If \( D_2 = \left\{ a = (a_k) : G \sum_{k=1}^{\infty} e^k \odot a_k \text{ is convergent with } G \sum_{k=1}^{\infty} |R_k|^G < \infty \right\} \).

Then \( (sl^G_\infty(\Delta_G))^\beta = D_2 \).

(iii) If \( D_3 = \left\{ a = (a_k) : \sup_n |G \sum_{k=1}^{n} e^k \odot a_k|^G < \infty, G \sum_{k=1}^{\infty} |R_k|^G < \infty \right\} \).

Then \( (sl^G_\infty(\Delta_G))^\gamma = D_3 \).

**Proof.** (i) Let \( a \in D_1 \). Then for each \( x \in sl^G_\infty(\Delta_G) \) we have
\[
G \sum_{k=1}^{\infty} |a_k \odot x_k|^G = G \sum_{k=1}^{\infty} \left( e^k \odot |a_k|^G \right) \odot \left( e^{k-1} \odot |x_k|^G \right) < \infty \quad \text{by using Lemma 3.1}
\]

This implies that \( a \in (sl^G_\infty(\Delta_G))^\alpha \). Therefore
\[
(4.1) \quad D_1 \subseteq (sl^G_\infty(\Delta_G))^\alpha.
\]

Again let \( a \in (sl^G_\infty(\Delta_G))^\alpha \). Then \( G \sum_{k=1}^{\infty} |a_k \odot x_k|^G < \infty \) (by definition of \( \alpha \)-dual) for each \( x \in sl^G_\infty(\Delta_G) \). So we take
\[
x_k = \begin{cases} 1, & \text{if } k = 1; \\ e^k, & \text{if } k \geq 2,
\end{cases}
\]
then \( x = (1, e^2, e^3, \ldots) \in sl^G_\infty(\Delta_G) \). Therefore
\[
G \sum_{k=1}^{\infty} e^k \odot |a_k|^G = |a_1|^G \oplus G \sum_{k=2}^{\infty} e^k \odot |a_k|^G
\]
\[
= |a_1|^G \oplus G \sum_{k=1}^{\infty} |a_k \odot x_k|^G < \infty \quad \text{as } a_1 \odot x_1 = 1.
\]
Therefore $a \in D_1$. This implies that
\[(4.2) \quad (sl^G_\infty(\Delta G))^\alpha \subseteq D_1.\]
Therefore from (4.1) and (4.2) we get
\[(4.2) \quad (sl^G_\infty(\Delta G))^\alpha = D_1.\]

(ii) Let $a \in D_2$. If $x \in sl^G_\infty(\Delta G)$ then there exists one and only one $y = (y_k) \in l^G_\infty$ such that (see 2.1)
\[
x_k = \bigoplus_G \sum_{v=1}^{k} y_{v-1}, \quad y_0 = 1
\]
Therefore $x_1 = \bigoplus_G \sum_{v=1}^{1} y_{v-1} = \bigoplus y_0 = 1$
\[
x_2 = \bigoplus_G \sum_{v=1}^{2} y_{v-1} = \bigoplus y_1
\]
\[
x_3 = \bigoplus_G \sum_{v=1}^{3} y_{v-1} = \bigoplus y_1 \oplus y_2
\]
\[
x_4 = \bigoplus_G \sum_{v=1}^{4} y_{v-1} = \bigoplus y_1 \oplus y_2 \oplus y_3
\]
\[
..............
\]
Then $G \sum_{k=1}^{n} a_k \bigcirc x_k = a_1 \bigcirc x_1 \bigoplus a_2 \bigcirc x_2 \bigoplus a_3 \bigcirc x_3 \bigoplus \ldots \bigoplus a_n \bigcirc x_n$
\[
= a_1 \bigcirc 1 \bigoplus a_2 \bigcirc y_1 \bigoplus a_3 \bigcirc (y_1 \bigoplus y_2) \bigoplus a_4 \bigcirc (y_1 \bigoplus y_2 \bigoplus y_3) \bigoplus \ldots \bigoplus a_n \bigcirc (y_1 \bigoplus y_2 \bigoplus \ldots \bigoplus y_{n-1})
\]
\[
= \bigoplus (a_2 \bigoplus a_3 \bigoplus \ldots \bigoplus a_n) \bigcirc y_1
\]
\[
\bigoplus (a_3 \bigoplus a_4 \bigoplus \ldots \bigoplus a_n) \bigcirc y_2 \bigoplus \ldots \bigoplus a_n \bigcirc y_{n-1}
\]
\[
= (\bigoplus R_1 \bigcirc y_1 \bigoplus R_n \bigcirc y_1) \bigoplus (\bigoplus R_2 \bigcirc y_2 \bigoplus R_n \bigcirc y_2) \bigoplus \ldots
\]
\[
\bigoplus (\bigoplus R_{n-1} \bigcirc y_{n-1} \bigoplus R_n \bigcirc y_{n-1})
\]
\[
= \bigoplus_G \sum_{k=1}^{n} R_k \bigcirc y_k \bigoplus R_n \bigcirc_G \sum_{k=1}^{n-1} y_k.
\]
\[(4.3) \quad G \sum_{k=1}^{n} a_k \bigcirc x_k = \bigoplus_G \sum_{k=1}^{n-1} R_k \bigcirc y_k \bigoplus R_n \bigcirc G \sum_{k=1}^{n-1} y_k.
\]

Since $G \sum_{k=1}^{\infty} R_k \bigcirc y_k$ is absolutely convergent and $R_n \bigcirc_G \sum_{k=1}^{n-1} y_k \to 1$ as $n \to \infty$ (Corollary [3.6]),
the series $G \sum_{k=1}^{n} a_k \bigcirc x_k$ is convergent for each $x \in sl^G_\infty(\Delta G)$. This yields $a \in (sl^G_\infty(\Delta G))^\beta$. 
Therefore $D_2 \subseteq (sl^G(\Delta G))^\beta$.

Again let $a \in (sl^G(\Delta G))^\beta$ then $G \sum_{k=1}^{\infty} a_k \odot x_k$ is convergent for each $x \in sl^G(\Delta G)$. We take

$$x_k = \begin{cases} 1, & \text{if } k = 1; \\
 e^k, & \text{if } k \geq 2. 
\end{cases}$$

Thus $G \sum_{k=1}^{\infty} e^k \odot x_k$ is convergent. This implies $e^n \odot R_n = O(e)$ (Corollary 3.6).

Using (4.3) we get $G \sum_{k=1}^{\infty} a_k \odot x_k = \oplus G \sum_{k=1}^{\infty} R_k \odot y_k$ converges for all $y \in l^G$. So we have

$$G \sum_{k=1}^{\infty} |R_k|^G < \infty \quad \text{and} \quad a \in D_2.$$

Therefore

$$(sl^G(\Delta G))^\beta = D_2.$$

(iii) The proof of this part is same as above. \hfill \Box

5. SOME APPLICATIONS OF GEOMETRIC DIFFERENCE

In this section we find the Geometric Newton-Gregory interpolation formulae and solve some numerical problems using these new formulae.

**Geometric Factorial:** Let us define geometric factorial notation $!_G$ as

$$n!_G = e^n \odot e^{n-1} \odot e^{n-2} \odot \cdots \odot e^2 \odot e = e^n.$$

For example,

$$0!_G = e^0 = 1,$$

$$1!_G = e^1 = e = 2.71828,$$

$$2!_G = e^2 = e^2 = 7.38906,$$

$$3!_G = e^3 = e^6 = 4.03429 \times 10^2,$$

$$4!_G = e^4 = e^{24} = 2.64891 \times 10^{10},$$

$$5!_G = e^5 = e^{120} = 1.30418 \times 10^{52} \quad \text{etc.}$$

**Generalized Geometric Forward Difference Operator:** Let

$$\Delta_G^0 f(a) = f(a \oplus h) \ominus f(a).$$

$$\Delta_G^2 f(a) = \Delta_G f(a \oplus h) \odot \Delta_G f(a)$$

$$= \{f(a \oplus e^2 \odot h) \ominus f(a \oplus h)\} \odot \{f(a \oplus h) \ominus f(a)\}$$

$$= f(a \oplus e^2 \odot h) \ominus e^2 \odot f(a \oplus h) \ominus f(a).$$

$$\Delta_G^3 f(a) = \Delta_G^2 f(a \oplus h) \odot \Delta_G^2 f(a)$$

$$= \{f(a \oplus e^3 \odot h) \ominus e^3 \odot f(a \oplus e^2 \odot h) \ominus f(a \oplus h)\}$$

$$\odot \{f(a \oplus e^2 \odot h) \ominus e^2 \odot f(a \oplus h) \ominus f(a)\}$$

$$= f(a \oplus e^3 \odot h) \ominus e^3 \odot f(a \oplus e^2 \odot h) \ominus e^3 \odot f(a \oplus h) \ominus f(a).$$
Thus, $n^{th}$ geometric forward difference is
\[
\Delta_G^n f(a) = G \sum_{k=0}^{n} (\otimes e)^k e^{(\delta)} e(f(a) + e^{-k} \otimes h), \text{ with } (\otimes e)^0_G = e.
\]

**Generalized Geometric Backward Difference Operator:** Let
\[
\nabla_G f(a) = f(a) \otimes f(a \oplus h).
\]

\[
\nabla_G^2 f(a) = \nabla_G f(a) \otimes \nabla_G f(a \oplus h)
= \{f(a) \otimes f(a \oplus h)\} \otimes \{f(a \oplus h) \otimes f(a \oplus e^2 \oplus h)\}
= f(a) \otimes e^2 \otimes f(a \oplus h) \otimes f(a \oplus e^2 \oplus h).
\]

\[
\nabla_G^3 f(a) = \nabla_G^2 f(a) \otimes \nabla_G^2 f(a - h)
= \{f(a) \otimes e^2 \otimes f(a \oplus h) \otimes f(a \oplus e^2 \oplus h)\}
\otimes \{f(a \oplus h) \otimes e^2 \otimes f(a \oplus e^2 \oplus h) \otimes f(a \oplus e^3 \oplus h)\}
= f(a) \otimes e^3 \otimes f(a \oplus h) \otimes e^3 \otimes f(a \oplus e^2 \oplus h) \otimes f(a \oplus e^3 \oplus h).
\]

Thus, $n^{th}$ geometric backward difference is
\[
\nabla_G^n f(a) = G \sum_{k=0}^{n} (\otimes e)^k e^{(\delta)} e(f(a) \oplus e^k \otimes h).
\]

**Factorial Function:** The product of $n$ consecutive factors each at a constant geometric difference, $h$, the first factor being $x$ is called a factorial function of degree $n$ and is denoted by $x^{(nc)}$. Thus
\[
x^{(nc)} = x \otimes (x \oplus e \otimes h) \otimes (x \oplus e^2 \oplus h) \otimes (x \oplus e^3 \oplus h) \otimes \cdots \otimes (x \oplus e^{n-1} \otimes h).
\]

In particular, for $h = e$,
\[
x^{(nc)} = x \otimes (x \oplus e) \otimes (x \oplus e^2) \otimes (x \oplus e^3) \otimes \cdots \otimes (x \oplus e^{n-1}).
\]

**Geometric Newton-Gregory Forward Interpolation Formula:** Let $y = f(x)$ be a function which takes the values $f(a), f(a \oplus h), f(a \oplus e^2 \oplus h), f(a \oplus e^3 \oplus h), \ldots, f(a \oplus e^n \oplus h)$ for the $n + 1$ geometrically equidistant values (which form a Geometric Progression in ordinary sense) $a, a \oplus h, a \oplus e^2 \oplus h, a \oplus e^3 \oplus h, \ldots, a \oplus e^n \oplus h$ of the independent variable $x$ and let $P_n(x)$ be a geometric polynomial in $x$ of degree $n$ defined as:
\[
P_n(x) = A_0 \oplus A_1 \otimes (x \oplus a) \otimes A_2 \otimes (x \oplus a) \otimes (x \oplus a \oplus h)
\]
\[
\oplus A_3 \otimes (x \oplus a) \otimes (x \oplus a \oplus h) \otimes (x \oplus a \oplus e^2 \oplus h) \oplus \cdots
\]
\[
\oplus A_n \otimes (x \oplus a) \otimes (x \oplus a \oplus h) \otimes \cdots \otimes (x \oplus a \oplus e^{n-1} \otimes h).
\]

We choose the coefficients $A_0, A_1, A_2, \ldots, A_n$ such that
\[
P_n(a) = f(a), P_n(a \oplus h) = f(a \oplus h), P_n(a \oplus e^2 \oplus h) = f(a \oplus e^2 \oplus h), \ldots, P_n(a \oplus e^n \oplus h) = f(a \oplus e^n \oplus h).
\]

Putting $x = a, a \oplus h, a \oplus e^2 \oplus h, a \oplus e^3 \oplus h, \ldots, a \oplus e^n \oplus h$ in (5.1) and then also putting the values of $P_n(a), P_n(a \oplus h), \ldots, P_n(a \oplus e^n \oplus h)$, we get
\[
f(a \oplus h) = A_0 \oplus A_1 \otimes h \implies A_1 = \frac{f(a) \oplus f(a \oplus h)}{h} G = \frac{\Delta_G f(a)}{h} G.
\]
The geometric difference table for given data is as follows:

Solution.

\[ f(a \oplus e^2 \odot h) = A_0 \oplus e^2 \odot h \odot A_1 \oplus e^2 \odot h \odot A_2 \]
\[ \Rightarrow A_2 = \frac{f(a \oplus e^2 \odot h) \oplus e^2 \odot [f(a \oplus h) \odot f(a)] \odot f(a)}{e^2 \odot h^{2G}} \]
\[ = \frac{f(a \oplus e^2 \odot h) \oplus e^2 \odot f(a \oplus h) \odot f(a)}{2!G \odot h^{2G}} \]
\[ = \frac{\Delta^2_G f(a)}{2!G \odot h^{2G}} G. \]

Similarly \[ A_3 = \frac{\Delta^3_G f(a)}{3!G \odot h^{3G}} G \]
\[ \ldots \ldots \ldots \]
\[ A_n = \frac{\Delta^n_G f(a)}{n!G \odot h^{nG}} G. \]

Putting the values of \( A_0, A_1, A_2, \ldots, A_n \) found above in (5.1), we get
\[ P_n(x) = f(a) \oplus \frac{\Delta_G f(a)}{h} G \odot (x \odot a) \oplus \frac{\Delta^2_G f(a)}{2!G} G \odot (x \odot a) \odot (x \odot a \odot h) \]
\[ \oplus \frac{\Delta^3_G f(a)}{3!G} G \odot (x \odot a) \odot (x \odot a \odot h) \odot (x \odot a \odot e^2 \odot h) \oplus \ldots \]
\[ \oplus \frac{\Delta^n_G f(a)}{n!G} G \odot (x \odot a) \odot (x \odot a \odot h) \odot \ldots \odot (x \odot a \odot e^{n-1} \odot h). \]

This is the Geometric Newton-Gregory forward interpolation formula.

Putting \( \frac{x \odot a}{h} G = u \) or \( x = a \oplus h \odot u \), formula takes the form
\[ P_n(x) = f(a) \oplus u \odot \Delta_G f(a) \oplus \frac{u \odot (u \odot e)}{2!G} G \odot \Delta^2_G f(a) \]
\[ \oplus \frac{u \odot (u \odot e) \odot (u \odot e^2)}{3!G} G \odot \Delta^3_G f(a) \oplus \ldots \]
\[ \oplus \frac{u \odot (u \odot e) \odot (u \odot e^2) \odot \ldots \odot (u \odot e^{n-1})}{n!G} G \odot \Delta^n_G f(a). \]

The result (5.2) can be written as
\[ P_n(x) = P_n(a \oplus h \odot u) = f(a) \oplus u^{(1G)} \odot \Delta_G f(a) \oplus \frac{u^{(2G)}}{2!G} G \odot \Delta^2_G f(a) \oplus \frac{u^{(3G)}}{3!G} G \odot \Delta^3_G f(a) \oplus \ldots \]
\[ \ldots \oplus \frac{u^{(nG)}}{n!G} G \odot \Delta^n_G f(a). \]

where \( u^{(nG)} = u \odot (u \odot e) \odot (u \odot e^2) \odot \ldots \odot (u \odot e^{n-1}) \).

Example 5.1. Given \( f(x) = f(e^x) = \sin(e^x) \). From the following table, find \( \sin(e^{1.3}) \) using geometric forward interpolation formula.

| \( e \)  | \( e^{1.2} \)  | \( e^{1.4} \)  | \( e^{1.6} \)  |
|--------|--------|--------|--------|
| \( f(x) \) | 0.0474 | 0.0579 | 0.0707 | 0.0863 |

Solution. The geometric difference table for given data is as follows:
We have to calculate

\[ f(e^{1.3}) = f(a \oplus u \odot h), \text{ say}. \]

\[ \therefore \quad a \oplus u \odot h = e^{1.3} \]

\[ \Rightarrow \quad e \oplus u \odot e^{0.2} = e^{1.3}, \quad (\text{here } h = e^{1.2} \odot e = e^{0.2}) \]

\[ u = e^{1.3} \odot e \odot G \]

\[ = (e^{0.3})^{\frac{1}{3}} \]

\[ = e^{1.5} \]

By Geometric Newton-Gregory forward interpolation formula we get

\[ f(a \oplus u \odot h) = f(a) \oplus u \odot \Delta_G f(a) \oplus \frac{u \odot (u \odot e)}{e^2} G \odot \Delta_G^2 f(a) \]

\[ \oplus \frac{u \odot (u \odot e) \odot (u \odot e^2)}{e^6} G \odot \Delta_G^3 f(a) \]

\[ f(e^{1.3}) = f(e) \oplus \{ e^{1.5} \odot \Delta_G f(e) \} \oplus \{ \frac{e^{1.5} \odot (e^{1.5} \odot e)}{e^2} G \odot \Delta_G^2 f(e) \} \]

\[ \oplus \{ \frac{e^{1.5} \odot (e^{1.5} \odot e) \odot (e^{1.5} \odot e^2)}{e^6} G \odot \Delta_G^3 f(e) \} \]

\[ = 0.0474 \oplus \{ e^{1.5} \odot 1.2215 \} \oplus \{ \frac{e^{1.5} \odot e^{0.5}}{e^2} G \odot 0.9997 \} \]

\[ \oplus \{ \frac{e^{1.5} \odot e^{0.5} \odot e^{-0.5}}{e^6} G \odot 0.9999 \} \]

\[ = 0.0474 \oplus (1.2215)^{1.5} \oplus (0.9997)^{0.325} \oplus (0.9999)^{0.0625} \]

\[ = 0.0474 \oplus 1.3500 \oplus 0.9999 \oplus 0.9984 \]

\[ = 0.0474 \times 1.3500 \times 0.9999 \times 0.9984 \]

\[ = 0.0639 \]

Thus \( \sin(e^{1.3}) = 0.0639. \)

**Note:** It is to be noted that \( e^x \odot e^y = e^{xy}, e^x \oplus e^y = e^{x+y}, x \odot e^y = x \frac{1}{y}. \)

**Geometric Newton-Gregory Backward Interpolation Formula:** Let \( y = f(x) \) be a function which takes the values \( f(a \oplus e^n \odot h), f(a \oplus e^{n-1} \odot h), f(a \oplus e^{n-2} \odot h), f(a \oplus e^{n-3} \odot h), \ldots, f(a) \) for the \( n + 1 \) geometrically equidistant values \( a \oplus e^n \odot h, a \oplus e^{n-1} \odot h, a \oplus e^{n-2} \odot h, a \oplus e^{n-3} \odot h, \ldots, a \) of the independent variable \( x \) and let \( P_n(x) \) be a
geometric polynomial in $x$ of degree $n$ defined as:
\[
P_n(x) = A_0 \oplus A_1 \odot (x \odot a \odot e^n \odot h) \oplus A_2 \odot (x \odot a \odot e^n \odot h) \odot (x \odot a \odot e^{n-1} \odot h) \\
\oplus A_3 \odot (x \odot a \odot e^n \odot h) \odot (x \odot a \odot e^{n-1} \odot h) \odot (x \odot a \odot e^{n-2} \odot h) \oplus \ldots \\
\oplus A_n \odot (x \odot a \odot e^n \odot h) \odot (x \odot a \odot e^{n-1} \odot h) \odot \ldots \odot (x \odot a \odot h).
\]
where $A_0, A_1, A_2, \ldots, A_n$ are constants which are to be determined so as to make
\[
P_n(a \oplus e^n \odot h) = f(a \oplus e^n \odot h), P_n(a \oplus e^{n-1} \odot h) = f(a \oplus e^{n-1} \odot h), \ldots, P_n(a) = f(a)
\]
Putting $x = a \oplus e^n \odot h, a \oplus e^{n-1} \odot h, \ldots$ in (5.3) and also putting $P_n(a \oplus e^n \odot h) = f(a \oplus e^n \odot h), \ldots$, we get
\[
A_0 = f(a \oplus e^n \odot h) \\
A_1 = \frac{\nabla_G f(a \oplus e^n \odot h)}{h} G \\
A_2 = \frac{\nabla^2_G f(a \oplus e^n \odot h)}{2!_G \odot h^{2_G}} G \\
A_3 = \frac{\nabla^3_G f(a \oplus e^n \odot h)}{3!_G \odot h^{3_G}} G \\
\ldots \hspace{1cm} \ldots \ldots \ldots \\
A_n = \frac{\nabla^n_G f(a \oplus e^n \odot h)}{n!_G \odot h^{n_G}} G
\]
Substituting the values of $A_0, A_1, A_2, \ldots$ in (5.3), we get
\[
P_n(x) = f(a \oplus e^n \odot h) \oplus \frac{\nabla_G f(a \oplus e^n \odot h)}{h} G \odot (x \odot a \odot e^n \odot h) \\
\oplus \frac{\nabla^2_G f(a \oplus e^n \odot h)}{2!_G \odot h^{2_G}} G \odot (x \odot a \odot e^n \odot h) \odot (x \odot a \odot e^{n-1} \odot h) \\
\oplus \frac{\nabla^3_G f(a \oplus e^n \odot h)}{3!_G \odot h^{3_G}} G \odot (x \odot a \odot e^n \odot h) \odot (x \odot a \odot e^{n-1} \odot h) \odot (x \odot a \odot e^{n-2} \odot h) \oplus \ldots \\
\oplus \frac{\nabla^n_G f(a \oplus e^n \odot h)}{n!_G \odot h^{n_G}} G \odot (x \odot a \odot e^n \odot h) \odot (x \odot a \odot e^{n-1} \odot h) \odot \ldots \odot (x \odot a \odot h).
\]
This is the Geometric Newton-Gregory backward interpolation formula.

Putting $u = \frac{x(a \odot e^n \odot h)}{h}$ or $x = a \oplus e^n \odot h \oplus u \odot h$, we get
\[
P_n(x) = P_n(a \oplus e^n \odot h \oplus u \odot h) = f(a \oplus e^n \odot h) \oplus u \odot \nabla_G f(a \oplus e^n \odot h) \\
\oplus \frac{u \odot (u \odot e)}{2!_G} G \odot \nabla^2_G f(a \oplus e^n \odot h) \\
\oplus \frac{u \odot (u \odot e) \odot (u \odot e^2)}{3!_G} G \odot \nabla^3_G f(a \oplus e^n \odot h) \oplus \ldots \\
\oplus \frac{u \odot (u \odot e) \odot (u \odot e^2) \odot \ldots \odot (u \odot e^{n-1})}{n!_G} G \odot \nabla^n_G f(a \oplus e^n \odot h).
\]

**Example 5.2.** Given, $f(x) = \ln(x)$. From the following table, find $\ln(22)$ using geometric backward interpolation formula.
Solution. The geometric difference table for given data is as follows:

| $x$  | 3    | 6    | 12   | 24   |
|------|------|------|------|------|
| $f(x)$ | 1.0986 | 1.7918 | 2.4849 | 3.1781 |

$$
\begin{array}{|c|c|c|c|}
\hline
x & f(x) & \nabla G f(x) & \nabla^2 G f(x) & \nabla^3 G f(x) \\
\hline
3 & 1.0986 & 1.6310 & 0.8503 & 1.0847 \\
6 & 1.7918 & 1.3868 & 0.9223 & \\
12 & 2.4849 & 1.2790 & & \\
24 & 3.1781 & 1.2790 & & \\
\hline
\end{array}
$$

We have to compute

$$f(22) = f(a \oplus e^n \odot h \oplus u \oplus h), \text{ say.}$$

$$\therefore a \oplus e^n \odot h \oplus u \oplus h = 22$$

$$\Rightarrow 24 \oplus u \odot h = 22, \text{ (here } h = 6 \oplus 3 = 2)$$

$$u = \frac{22 \oplus 24}{2}G$$

$$= (0.9167)^{\frac{1}{2}}$$

$$= 0.8820.$$  

By Geometric Newton-Gregory backward interpolation formula we get

$$f(22) = f(24) \oplus u \odot \nabla G f(24) \oplus \frac{u \odot (u \oplus e)}{2!G} G \odot \nabla^2 G f(24)$$

$$\oplus \frac{u \odot (u \oplus e) \odot (u \oplus e^2)}{3!G} G \odot \nabla^3 G f(24)$$

$$= 3.1781 \oplus \{0.8820 \odot 1.2790\} \oplus \{0.8820 \odot (0.8820 \oplus e)G \odot 0.9223\}$$

$$\oplus \{0.8820 \odot (0.8820 \oplus e) \odot (0.8820 \oplus e^2)G \odot 1.0847}\}$$

$$= 3.1781 \oplus 0.9696 \oplus \{0.9466 \odot 0.9223\} \oplus \{0.9663 \odot 1.0847\}$$

$$= 3.1781 \oplus 0.9696 \oplus 1.0045 \oplus 0.9972$$

$$= 3.0867.$$  

Therefore $\ln(22) = 3.0867$.  

Note: Since small change in $x$ results large change in $e^x$. So, for better accuracy, values should be taken up to maximum possible decimal places.

Advantages of Geometric Interpolation Formulae over Ordinary Interpolation Formulae: All the ordinary interpolation formulae are based upon the fundamental assumption that the data is expressible or can be expressed as a polynomial function with fair degree of accuracy. But geometric interpolation formulae have no such restriction. Because geometric interpolation formulae are based on geometric polynomials which are not polynomials in ordinary sense. So geometric interpolation formulae can be used to
generate transcendental functions, mainly to compute exponential and logarithmic functions. Also geometric forward and backward interpolation formulae are based on the values of the argument that are geometrically equidistant but need not be equidistant like classical interpolation formulae.

6. Conclusion

In this paper, we have defined geometric difference sequence space and obtained the Geometric Newton-Gregory interpolation formulae. Our main aim is to bring up geometric calculus to the attention of researchers in the branch of numerical analysis and to demonstrate its usefulness. We think that geometric calculus may especially be useful as a mathematical tool for economics, management and finance.

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