Simulating 4D Simplicial Gravity including Degenerate Triangulations

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We extend simulations of simplicial gravity in four dimensions to include degenerate triangulations and demonstrate that using this ensemble the geometric finite-size effects are much reduced. We provide strong numerical evidence for the existence of an exponential bound on the entropy of the model and establish that the phase structure is identical to that of a corresponding model restricted to an ensemble of combinatorial triangulations.

1. INTRODUCTION

In discretized models of four-dimensional Euclidean quantum gravity, known as simplicial gravity, the integration over metrics is replaced by summations over an ensemble of triangulations constructed by all possible gluings of equilateral 4-simplexes into closed (piece-wise linear) simplicial manifolds (see e.g. Ref. [1]). The regularized Euclidean Einstein-Hilbert action is particularly simple in this approach; it can be taken to depend on only two coupling constants, $\kappa$ and $\mu$, related to the inverse Newton’s and cosmological constants. The regularized grand-canonical partition function thus becomes:

$$Z(\mu, \kappa) = \sum_{T \in T} \frac{1}{C_T} e^{-\mu N_4 + \kappa N_2}. \tag{1}$$

The sum is over all distinct triangulations $T \in T$, $N_i$ is the number of $i$-simplexes in the triangulation $T$ and $C_T$ denotes its symmetry factor — the number of equivalent labelings of the vertexes.

Extensive numerical simulations have established that the model Eq. (1) has a strong-coupling (small $\kappa$) crumpled phase and a weak-coupling (large $\kappa$) elongated phase, separated by a discontinuous phase transition. In the crumpled phase the geometry is dominated by a singular structure; two singular vertexes connected to an extensive fraction of the total volume, joined by a sub-singular edge. The elongated phase, on the other hand, is dominated by essentially one-dimensional (tree-like) triangulations — branched polymers.

2. DEGENERATE TRIANGULATIONS

In Eq. (1), $T$ denotes a suitable ensemble of triangulations included in the partition function. Different ensembles are defined by imposing various restriction on how the simplexes are glued together. Provided this leads to a well-defined partition function, and as long as this difference is only at the level of discretization, one expects different choices of $T$ to lead to the same continuum theory in the thermodynamic limit. This is known to be true in two dimensions where models of simplicial gravity corresponding to different choices of $T$ are soluble as matrix models [2].

All simulations of four-dimensional simplicial gravity have, as of today, used an ensemble of combinatorial triangulations $T_C$. In a combinatorial triangulation every 4-simplex is uniquely defined by a set of five distinct vertexes — it is said to be combinatorially unique. Here we report on simulations of the model Eq. (1) defined with a larger ensemble $T_D$ including degenerate triangulations. We relax the above constraint and allow distinct simplexes to be defined by the same set of vertexes. We do, however, retain the restriction that every 4-simplex is defined by a set of distinct vertexes, i.e. we exclude degenerate simplexes. Clearly $T_C \subset T_D$.

The benefit of using a larger ensemble of triangulations is well known from simulations of simplicial gravity in two [3] and three [4] dimensions. It is established that less constrained the triangulations are the smaller the geometric finite-size effects are. Simulations of four-dimensional simpli-
cational gravity are notoriously time-consuming, primarily due to the large volumes needed to observe any "true" infinite volume behavior, hence any reduction in the finite-size effects is of great practical importance.

3. RESULTS

We have simulated the model Eq. (1) using degenerate triangulations on volumes up to 25,600 4-simplexes using Monte Carlo methods. As customary we work in a quasi-canonical ensemble of spherical manifolds with almost fixed $N_4$:

$$Z(\mu, \kappa; \bar{N}_4) = \sum_{N_4} e^{-\mu N_4 - \delta (N_4 - \bar{N}_4)^2} \Omega_{N_4}(\kappa),$$

where $\Omega_{N_4}(\kappa) = \sum_{T \in \mathcal{T}(N_4)} \exp(\kappa N_2)$ is the canonical partition function. As there do not exist ergodic volume conserving geometric moves, the canonical ensemble cannot be simulated directly, we must allow the volume to fluctuate. The quadratic potential term added to the action ensures, for an appropriate choice of $\delta$, that these fluctuations are small.

In the simulations the triangulation-space is explored using a set of local geometric changes, the $(p, q)$–moves [5]. For combinatorial triangulations the $(p, q)$–moves are known to be ergodic for $D \leq 4$. To demonstrate that the same holds true for degenerate triangulations we observe that every set of combinatorially equivalent simplexes, or sub-simplexes, can be made distinct by a finite-sequence of the $(p, q)$–moves. Thus every degenerate triangulations can be reached from a combinatorial one. In addition, the local nature of the $(p, q)$–moves prohibits the creation of pseudo-manifolds in the simulations, i.e. triangulations containing vertexes with a neighborhood not homeomorphic to the 4-sphere.

A major benefit of including degenerate triangulations is the reduction of geometric finite-size effects. An example of this is the volume dependence of the pseudo-critical cosmological constant $\mu_c(N_4)$ shown in Figure 1. For comparison we show the corresponding values, $\mu_c^C(N_4)$, for combinatorial triangulations.

Table 1

| $N_4$ | $\mu$ | $\gamma$ | $\chi^2$/d.o.f. |
|-------|-------|----------|-----------------|
| Eq. (3) | 2.556(3) | 0.55(5) | 3.8            |
| Eq. (4) | 2.385(4) | 117     |                |

For comparison we show the corresponding values for combinatorial triangulations. For degenerate triangulations we observe a rapid convergence to an infinite volume value $\bar{\mu}$; this is quantified by fits to two different functional forms: a power-law convergence,

$$\mu_c(N_4) = \bar{\mu} + b \frac{N_4}{N_4^4},$$

and a logarithmic divergence,

$$\mu_c(N_4) = \bar{\mu} + b \log N_4.$$
volume. And, as \( T_C \in T_D \), this implies an exponential bound on the number of combinatorial triangulations as well.

As for combinatorial triangulations, in the crumpled phase the internal geometry of degenerate triangulations is dominated by a singular structure. The probability distribution of the vertex orders \( o_i \) — the number of 4-simplexes containing the vertex — contains an isolated peak in the tail, indicating singular vertexes (Figure 2). However, the distributions \( P(o) \) differ in two respects from the corresponding ones measured on combinatorial triangulations:

(i) The number of singular vertexes is larger than two and increases (logarithmically) with the volume, i.e. a gas of singular vertexes.

(ii) On each volume the distribution effectively separates in two, depending on whether the vertex orders are odd or even.

It is not clear though, how much significance should be attached to this observed difference. Due to the collapsed nature of the internal geometry it is unlikely that any sensible continuum limit exist in the crumpled phase, hence there is no reason to expect identical scaling behavior for the two different ensembles.

Additional evidence of a collapsed intrinsic geometry comes from the (absence of) volume scaling of the simplex-simplex correlation function, from which we conclude that \( d_H = \infty \).

We have also investigated the phase structure of the model for non-zero values of the inverse Newtons’s constant \( \kappa \). As for combinatorial triangulations we observe a phase transition at a value \( \kappa_c \approx 1.5 \). For \( \kappa > \kappa_c \) the model is in a branched polymer phase; this we establish by measuring the fractal dimensions \( d_H \) and the spectral dimension \( d_s \) for \( \kappa = 2 \). Including measurements on volumes, \( N_4 = 400 \) to 1600 we get \( d_H = 1.9(2) \) and \( d_s = 1.32(5) \), in excellent agreement with \( d_H = 2 \) and \( d_s = 4/3 \) expected for branched polymers.

Our results demonstrate that including degenerate triangulations in simulations of four-dimensional simplicial gravity has many potential advantages over a model restricted to combinatorial triangulations. This agrees with the same observation previously made in both two and three dimensions. The chief benefit is the reduction in geometric finite-size effects, mainly due to an enlarged ensemble — with a larger triangulation-space the infinite-volume fractal structure is more easily approximated on small volumes. Further work is needed to fully explore the nature of this ensemble, for now the most important result we want to emphasize is the strong numerical evidence for an exponential bound on the canonical ensemble \( \Omega_{N_4} \) of degenerate triangulations.

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