Bulk viscosity of $\mathcal{N} = 2^*$ plasma

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Abstract
We use gauge theory/string theory correspondence to study the bulk viscosity of strongly coupled, mass deformed $SU(N_c) \mathcal{N} = 4$ supersymmetric Yang-Mills plasma, also known as $\mathcal{N} = 2^*$ gauge theory. For a wide range of masses we confirm the bulk viscosity bound proposed in [1]. For a certain choice of masses, the theory undergoes a phase transition with divergent specific heat $c_V \sim |1 - T_c/T|^{-1/2}$. We show that, although bulk viscosity rapidly grows as $T \to T_c$, it remains finite in the vicinity of the critical point.

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1 Introduction

In [2] Maldacena proposed that $SU(N_c) \, \mathcal{N} = 4$ superconformal Yang-Mills (SYM) theory on $R^{3,1}$ is dual to type IIB string theory on (a Poincare patch of) $AdS_5 \times S^5$ with $N_c$ units of the Ramond-Ramond five-form flux through the $S^5$. Assuming that such a duality holds exactly at a superconformal fixed point, it must also hold for any relevant (in the infrared) deformation of a fixed point. Specifically, we might generate non-conformal examples of gauge theory/string theory correspondence by simply mapping the mass deformation on the SYM side to the string theory side. For infinitesimal supersymmetric mass deformations this was done in [3] using the operator/state correspondence in AdS/CFT [4, 5]. Extending an infinitesimal mass deformation to a finite one proved to be extremely difficult. In fact, only one particular deformation was shown to be integrable in the ’t Hooft limit $N_c \to \infty$, $g_{YM} \to 0$ (with $\lambda \equiv N_c g_{YM}^2$ kept constant), and for large ’t Hooft coupling $\lambda \gg 1$ [6] (PW). In [7] it was shown\(^1\) that PW massive deformation is dual to giving the same mass to two chiral multiplets of the parent $\mathcal{N} = 4$ SYM at a specific point on its Coulomb branch\(^2\). Such a mass deformation breaks supersymmetry down to $\mathcal{N} = 2$. The resulting gauge theory, commonly referred to as $\mathcal{N} = 2^\ast$ gauge theory, can be solved nonperturbatively [9]. The agreement [7] between the dual gravitational description of the massive $\mathcal{N} = 2$ gauge theory [6] and its exact field-theoretic solution [9] provides a highly nontrivial check of gauge theory/string theory correspondence in a non-conformal setting.

Once the duality is established for a supersymmetric ground state, following [10] it can be extended to correspondence involving a thermal equilibrium state of the gauge theory: a thermally equilibrium state of a gauge theory is dual to a Schwarzschild black brane solution in the supergravity description, with the temperature $T$ given by the Hawking temperature of the black brane. In the context of $\mathcal{N} = 2^\ast$ theory there is an interesting subtlety: the supersymmetric mass deformation involves deformation of the parent SYM by operators of different canonical dimensions, a dimension-two operator for the bosonic components of massive chiral multiplets and a dimension-three operator for the fermionic components of massive chiral multiplets. Such operators are mapped to different scalar gravitational modes of the effective five-dimensional gravitational description [6]. The coefficients of the non-normalizable modes of these scalars encode

\(^1\)Related discussion appeared in [8].
\(^2\)Extending the mass deformation duality to all Coulomb branch of the $\mathcal{N} = 4$ SYM is a difficult open problem.
the bosonic $m_b$ and fermionic $m_f$ masses respectively [3]. Although the vacuum state supersymmetry requires $m_b = m_f$, a thermal state breaks the supersymmetry anyway; thus, we can study a phase diagram of $\mathcal{N} = 2^*$ gauge theory with $m_b \neq m_f$ [11]. A full ten-dimensional type IIB supergravity solution dual to $\mathcal{N} = 2^*$ gauge theory for generic $(T, m_b, m_f)$ was constructed in [11]. It was shown there (see also [12]) that any such state (including small fluctuations about it) can be described within effective five-dimensional gauged supergravity presented in [6].

Supergravity equations of motion derived in [11] describing a thermal state of $\mathcal{N} = 2^*$ gauge theory are too difficult to solve analytically. If fact, in [11] these equations were solved only in high-temperature limit $m_b/T \ll 1$ and $m_f/T \ll 1$. Additional analysis of the $\mathcal{N} = 2^*$ phase diagram required numerical work. For two special cases, i.e., (susy) $\equiv \{m_b = m_f\}$ and (bosonic) $\equiv \{m_f = 0, m_b \neq 0\}$, such numerical analysis were performed in [13]. Using the holographic renormalization of the theory [14], the energy density $\mathcal{E}$, the free energy density $\mathcal{F}$ and the entropy density $s$ of $\mathcal{N} = 2^*$ strongly coupled plasma was computed. It was shown in [13] that the basic thermodynamic relation $\mathcal{F} = \mathcal{E} - Ts$ is satisfied exactly, while the first law of thermodynamics $d\mathcal{E} = Tds$ is exactly satisfied for the high temperature analytic solution of [11], and is satisfied numerically for (susy) and (bosonic) thermal states with an accuracy of $\sim 0.1\%$ and $\sim 0.01\%$ correspondingly. The latter provides a highly nontrivial check on our identification of the bosonic and fermionic masses in dual supergravity (see [13] for details).

An interesting critical point was found in the numerical analysis of the (bosonic) thermal state of the $\mathcal{N} = 2^*$ plasma [13]: for $T < T_c \approx m_b/2.29(9)$ $\mathcal{N} = 2^*$ plasma becomes unstable with respect to energy density fluctuations. Specifically, precisely at $T = T_c$ the speed of sound waves squared $c_s^2$ vanishes. A perturbative instability of this type is a defining feature of a second order phase transition. Since $c_s^2 \propto (T - T_c)^{1/2}$, the specific heat $c_V$ diverges as $|1 - T_c/T|^{-1/2}$, suggesting that such a critical point is in the universality class of the mean-field tricritical point. Physically, the existence of perturbative instability in $\mathcal{N} = 2^*$ plasma at low temperatures is not surprising. Indeed, once $m_b \neq m_f$ the supersymmetry is broken, and the theory is guaranteed to be stable only at high temperatures. It was conjectured in [13] that $\mathcal{N} = 2^*$ plasma would have a critical point

$$T_c = T_c \left( \nu \equiv \frac{m_f^2}{m_b^2} \right),$$

as long as $\nu < 1$. 

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Shear viscosity of $\mathcal{N} = 2^*$ plasma was computed in [15, 14]; it was shown to satisfy the universal bound [15, 16, 17]

$$\frac{\eta}{s} = \frac{1}{4\pi}. \quad (1.2)$$

Transport properties of $\mathcal{N} = 2^*$ plasma were further studied in [18]:
- equations of motion describing sound quasinormal modes of $\mathcal{N} = 2^*$ black brane with arbitrary momentum $q \equiv |\vec{q}|/(2\pi T)$ and frequency $\omega \equiv \omega/(2\pi T)$ were obtained;
- these equations were solved analytically in the hydrodynamic limit $q \ll 1, \omega \ll 1$, and for high temperatures $T \gg \{m_b, m_f\}$;
- following [19], the dispersion relation for the lowest quasinormal mode of the $\mathcal{N} = 2^*$ black brane was identified with the dispersion relation of the sound waves in strongly coupled $\mathcal{N} = 2^*$ plasma:

$$\omega = c_s q - 2\pi i \frac{\eta}{s} \left(\frac{2}{3} + \frac{\zeta}{2\eta}\right) q^2 + \mathcal{O}(q^3), \quad (1.3)$$

where $\zeta$ is the plasma bulk viscosity;
- to leading order in $\{m_f/T, m_b/T\}$ it was found that

$$c_s = \frac{1}{\sqrt{3}} \left(1 - \frac{1}{3} \frac{\Gamma(\frac{4}{3})^4}{3\pi^4} \left(\frac{m_f}{T}\right)^2 - \frac{1}{18\pi^4} \left(\frac{m_b}{T}\right)^4 + \cdots\right), \quad (1.4)$$

in precise agreement with the speed of sound obtained from the analytic equation of state of $\mathcal{N} = 2^*$ plasma at high temperature$^3$

$$c_s^2 = -\frac{\partial F}{\partial E}; \quad (1.5)$$

- using (1.2), (1.4), to leading order in $\left(\frac{1}{3} - c_s^2\right)$ it was found that

$$\frac{\zeta}{\eta}\bigg|_{m_f=0} = \frac{\pi^2 \beta_b^\Gamma}{16} \left(\frac{1}{3} - c_s^2\right) + \mathcal{O} \left(\left[\frac{1}{3} - c_s^2\right]^2\right), \quad (1.6)$$

where $\beta_b^\Gamma \approx 8.001$;

$$\frac{\zeta}{\eta}\bigg|_{m_b=0} = \frac{3\pi \beta_f^\Gamma}{2} \left(\frac{1}{3} - c_s^2\right) + \mathcal{O} \left(\left[\frac{1}{3} - c_s^2\right]^2\right), \quad (1.7)$$

where $\beta_f^\Gamma \approx 0.66666$ [20].

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$^3$This provides a highly nontrivial consistency check on our analysis of the quasinormal modes.
In this paper we extend analysis of [18, 1] to general mass deformations, i.e., apart from cases of (bosonic) \( \leftrightarrow \nu = 0 \) and (susy) \( \leftrightarrow \nu = 1 \) thermal states of \( \mathcal{N} = 2^* \) plasma, and for wide range of temperatures. Our goal is twofold:

- first, we would like to test the bulk viscosity bound conjecture of [1] in more general setting:
  \[ \frac{\zeta}{\eta} \geq 2 \left( \frac{1}{3} - c_s^2 \right) ; \]  
  (1.8)

- second, we would like to compute
  \[ \frac{\zeta}{\eta}_{\text{critical}} \equiv \frac{\zeta}{\eta}_{|T=T_c(\nu)} , \]
  (1.9)

for \( 0 < \nu < 1 \).

The paper is organized as follows. In the next section we outline the equations of motion and the boundary conditions for the \( \mathcal{N} = 2^* \) black brane background and its hydrodynamic quasinormal mode. The results of our extensive numerical analysis are presented in section 3. We conclude in section 4.

Most technical details are omitted due to their complexity. All equations of motion, their analytic asymptotic solutions, as well as raw numerical data is available from the authors upon request. We use numerical techniques developed in [21].

2 \( \mathcal{N} = 2^* \) black brane and its hydrodynamic quasinormal mode

We closely follow [13] in discussion of \( \mathcal{N} = 2^* \) black brane background, and we (mostly) follow notations of [18] in discussion of its quasinormal modes.

2.1 Effective action

The effective action of the five-dimensional gauged supergravity describing \( \mathcal{N} = 2^* \) black brane thermodynamics/hydrodynamics is given by

\[ S = \int_{\mathcal{M}_5} d\xi^5 \sqrt{-g} \, \mathcal{L}_5 \]

\[ = \frac{1}{4\pi G_5} \int_{\mathcal{M}_5} d\xi^5 \sqrt{-g} \left[ \frac{1}{4} R - 3(\partial \alpha)^2 - (\partial \chi)^2 - P \right] , \]  
(2.1)
where the potential\(^4\)

\[
P = \frac{1}{16} \left[ \frac{1}{3} \left( \frac{\partial W}{\partial \alpha} \right)^2 + \left( \frac{\partial W}{\partial \chi} \right)^2 \right] - \frac{1}{3} W^2
\]  

(2.2)
is a function of \(\alpha\) and \(\chi\), and is determined by the superpotential

\[
W = -e^{-2\alpha} - \frac{1}{2} e^{4\alpha} \cosh(2\chi)
\]

(2.3)

In our conventions, the five-dimensional Newton’s constant is

\[
G_5 \equiv G_{10} \frac{\text{vol}_{S^5}}{2^5} = \frac{4\pi}{N_c^2}.
\]

(2.4)

### 2.2 \(\mathcal{N} = 2^*\) black brane background

We parameterize the background metric of the \(\mathcal{N} = 2^*\) black brane as

\[
ds_5^2 = c_2(x)^2 \left( -(1 - x)^2 dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + g_{xx}(x) \, dx^2 \right),
\]

(2.5)

where the radial coordinate \(x \in [0, 1]\), with \(x \to 0_+\) corresponding to the \(AdS_5\) boundary and \(x \to 1_-\) corresponding to a regular Schwarzschild horizon of the black brane. The metric (2.5) is supported by nontrivial profiles of two scalar fields:

\[
\rho_6(x) \equiv e^{6\alpha}, \quad c(x) \equiv \cosh(2\chi).
\]

(2.6)

Notice that we redefined the gauged supergravity scalars as in (2.1) — this is done in order to speed up numerical integration. We further introduce

\[
c_2 \equiv e^A, \quad A \equiv \ln \hat{\delta}_3 - \frac{1}{4} \ln(2x - x^2) + a(x),
\]

(2.7)

where \(\hat{\delta}_3\) is related to the Hawking temperature of the black brane as follows \([13]\)

\[
T = \frac{\hat{\delta}_3}{2\pi} \lim_{x \to 1_-} e^{-3a(x)}.
\]

(2.8)

The asymptotic solution of \(\{\rho_6, c, a\}\) near the boundary, \(x \to 0_+\), takes the form \([13]\)

\[
\rho_6 = 1 + x^{1/2} (6\rho_{10} + 6\rho_{11} \ln x) + \mathcal{O}(x \ln^2 x),
\]

\[
c = 1 + 12\nu \ x^{1/2} \rho_{11} + 24\nu \rho_{11} \ x (\chi_{10} + \nu \rho_{11} + 2\nu \rho_{11} \ln x) + \mathcal{O}(x^{3/2} \ln^2 x),
\]

\[
a = -\frac{2}{3} \nu \rho_{11} x^{1/2} + \mathcal{O}(x \ln^2 x),
\]

(2.9)

\(^4\)We set the five-dimensional gauged supergravity coupling to one. This corresponds to setting the radius \(L\) of the five-dimensional sphere in the undeformed metric to 2.
and near the horizon, \( y \equiv 1 - x \rightarrow 0^+ \) \[13\]

\[ \rho_6 = \rho_0^6 + \mathcal{O}(y^2), \quad c = \frac{\chi_0^4 + 1}{2\chi_0^2} + \mathcal{O}(y^2), \quad a = a_0 + a_1 y^2 + \mathcal{O}(y^4). \quad (2.10) \]

In (2.9), (2.10) we indicated explicitly only terms necessary to unambiguously determine the asymptotic black brane geometry for a fixed set \( \{\mu, \nu\} \). \( \rho_{11} \) is related to \( \mu \equiv \frac{m_b}{T} \) by [13]

\[ \rho_{11} = \frac{\sqrt{2}}{24\pi^2} e^{-6a_0} \mu^2. \quad (2.11) \]

Notice that for a fixed set \( \{\mu, \nu\} \), \( \mathcal{N} = 2^* \) black brane geometry is specified by six parameters:

\[ \{\mu, \nu\} \implies \left\{ \rho_{10}, \chi_{10}, \rho_0, \chi_0, a_0, a_1 \right\}, \quad (2.12) \]

which is precisely the number of parameters needed to uniquely determine the solution of three second order equations of motion for \( \{\rho_6, c, a\} \). These parameters are functions of \( \{\mu, \nu\} \).

We use numerical techniques developed in [21] to generate data sets

\[ S_{\text{background}} \equiv \left\{ \nu; \mu; \rho_{10}, \chi_{10}, \rho_0, \chi_0, a_0, a_1 \right\}, \quad (2.13) \]

which can be used to study the thermodynamics of \( \mathcal{N} = 2^* \) black branes as detailed in [13]. The data sets \( S_{\text{background}} \) are available from the authors upon request. We obtain the following results.

- For \( \nu > 1 \), in agreement with the conjecture in [13], we did not find a critical point in \( \mathcal{N} = 2^* \) plasma down to \( \mu \sim 10 \), which corresponds to temperatures of order \( T \sim m_b/10 \). As for \( \nu = 1 \) in [13], the low temperature (\( \mu \sim 10 \)) thermodynamics of \( \mathcal{N} = 2^* \) black brane can be well approximated by the following equation of state

\[ \mathcal{F} \propto -N_c^2 T^4 e^{-\frac{m_{\text{eff}}}{T}}, \quad (2.14) \]

where \( m_{\text{eff}} = m_{\text{eff}}(\nu) \sim m_b/10 \).

- For \( 0 < \nu < 1 \), \( \mathcal{N} = 2^* \) brane thermodynamics has a critical point \( T_c = T_c(\nu) \), such that

\[ \frac{s}{c_V} = c_s^2 = -\frac{\partial \mathcal{F}}{\partial \mathcal{E}} \propto \pm (T - T_c)^{1/2}, \quad (T - T_c) \ll T_c. \quad (2.15) \]
2.3 Quasinormal sound mode

Sound quasinormal modes in the $N = 2^*$ black brane geometry involve coupled fluctuations of gauge-invariant metric fluctuations $Z_H$ and the gauge-invariant fluctuations of two scalar fields $\{Z_\alpha, Z_\chi\}$, see [18] for details. The spectrum of quasinormal modes is determined [19] by imposing on $\{Z_H, Z_\alpha, Z_\chi\}$ an incoming wave boundary condition at the horizon, and requiring vanishing of the non-normalizable modes for $\{Z_H, Z_\alpha, Z_\chi\}$ near the boundary. In the hydrodynamic limit $w \to 0$, $q \to 0$ with $\frac{w}{q}$ kept fixed, this leads to the following perturbative expansions

$$Z_H = (1 - x)^{-i\omega} \left( z_{H,0} + i q z_{H,1} + \mathcal{O}(q^2) \right),$$

$$Z_\alpha = (1 - x)^{-i\omega} \left( z_{\alpha,0} + i q z_{\alpha,1} + \mathcal{O}(q^2) \right),$$

$$Z_\chi = (1 - x)^{-i\omega} \left( z_{\chi,0} + i q z_{\chi,1} + \mathcal{O}(q^2) \right), \quad (2.16)$$

with the following boundary conditions on $\{z_{H,i}, z_{\alpha,i}, z_{\chi,i}\}$:

$$\lim_{x \to 1^-} z_{H,0} = 1, \quad \lim_{x \to 1^-} z_{H,1} = 0, \quad \lim_{x \to 1^-} z_{\alpha,i} = \lim_{x \to 1^-} z_{\chi,i} = \text{finite},$$

$$z_{H,i} = \mathcal{O}(x), \quad z_{\alpha,i} = \mathcal{O} \left( x^{1/2} \right), \quad z_{\chi,i} = \mathcal{O} \left( x^{3/4} \right), \quad \text{as} \quad x \to 0_+. \quad (2.17)$$

Additionally, we parameterize dispersion of the lowest quasinormal mode as

$$w = \frac{1}{\sqrt{3}} q \beta_1 - i \frac{\beta_2}{3} q^2 + \mathcal{O}(q^3), \quad (2.18)$$

where $\beta_i = \beta_i(\mu, \nu)$. In the conformal case, $\mu \to 0$ (with $\nu = 0$) we expect [22]

$$\lim_{\mu \to 0} \beta_i(\mu, 0) = 1. \quad (2.19)$$

Identifying (1.3) with (2.18) and using the universal result for the shear viscosity (1.2) we find

$$\left( \frac{1}{3} - c_s^2 \right) = \frac{1}{3} \left( 1 - \beta_1^2 \right), \quad \frac{\zeta}{\eta} = \frac{4}{3} (\beta_2 - 1). \quad (2.20)$$

2.3.1 Leading order in the hydrodynamic approximation

To leading order in the hydrodynamic approximation, wave functions of the gauge-invariant fluctuations $\{z_{H,0}, z_{\alpha,0}, z_{\chi,0}\}$ satisfy the following equations

$$0 = z_{H,0}'' + C_{101} z_{H,0}' + C_{102} z_{H,0} + C_{103} z_{\alpha,0} + C_{104} z_{\chi,0},$$

$$0 = z_{\alpha,0}'' + C_{201} z_{\alpha,0}' + C_{202} z_{\alpha,0} + C_{203} z_{H,0} + C_{204} z_{\alpha,0} + C_{205} z_{\chi,0},$$

$$0 = z_{\chi,0}'' + C_{301} z_{\chi,0}' + C_{302} z_{\chi,0} + C_{303} z_{H,0} + C_{304} z_{\alpha,0} + C_{305} z_{\chi,0}, \quad (2.21)$$
where connection coefficients $C_{i0j}$ are nonlinear functionals of $\{\rho_6, c, a\}$ with explicit dependence on $x$ and $\beta_{12} \equiv \beta_1^2$:

$$C_{i0j} = C_{i0j} \left[ \{\rho_6, c, a\}; \ x; \ \beta_{12} \right].$$

(2.22)

Using (2.9), (2.10), for each set $S_{\text{background}}$ we construct the asymptotic solution of (2.21). Near the boundary, $x \to 0_+$ we find

$$z_{H,0} = x \ z_{H,2,0}^{(0)} + O \left( x^2 \ln x \right),$$

$$z_{\alpha,0} = x^{1/2} \ z_{\alpha,1,0}^{(0)} + O \left( x \ln x \right), \quad z_{\chi,0} = x^{1/4} \left( x^{1/2} \ z_{\chi,1,0}^{(0)} + O \left( x \ln x \right) \right),$$

(2.23)

and near the horizon, $y \equiv 1 - x \to 0_+$

$$z_{H,0} = 1 + O \left( y^2 \right), \quad z_{\alpha,0} = q_0^{(0)} + O \left( y^2 \right), \quad z_{\chi,0} = x_0^{(0)} + O \left( y^2 \right).$$

(2.24)

Thus, altogether we have six new parameters:

$$S_{\text{background}} \quad \Longrightarrow \quad \left\{ \beta_{12}, z_{H,2,0}^{(0)}, z_{\alpha,1,0}^{(0)}, z_{\chi,1,0}^{(0)}, q_0^{(0)}, x_0^{(0)} \right\},$$

(2.25)

precisely what is necessary to construct a unique solution for $\{z_{H,0}, z_{\alpha,0}, z_{\chi,0}\}$ for a given $S_{\text{background}}$.

We use numerical techniques developed in [21] to general data sets

$$S_{\text{sound}} \equiv \left\{ S_{\text{background}}; \ \beta_{12}, z_{H,2,0}^{(0)}, z_{\alpha,1,0}^{(0)}, z_{\chi,1,0}^{(0)}, q_0^{(0)}, x_0^{(0)} \right\}. \quad (2.26)$$

The data sets $S_{\text{sound}}$ are available from the authors upon request.

2.3.2 The first subleading order in the hydrodynamic approximation

To the first subleading order in the hydrodynamic approximation, wave functions of the gauge-invariant fluctuations $\{z_{H,1}, z_{\alpha,1}, z_{\chi,1}\}$ satisfy the following equations

$$0 = z_{H,1}'' + C_{111} \ z_{H,1}'+ C_{112} \ z_{H,1} + C_{113} \ z_{\alpha,1} + C_{114} \ z_{\chi,1} + C_{115} \ z_{H,0}' + C_{116} \ z_{H,0},$$

$$+ C_{117} \ z_{\alpha,0} + C_{118} \ z_{\chi,0},$$

$$0 = z_{\alpha,1}'' + C_{211} \ z_{\alpha,1}'+ C_{212} \ z_{H,1} + C_{213} \ z_{H,1} + C_{214} \ z_{\alpha,1} + C_{215} \ z_{\chi,1} + C_{216} \ z_{\alpha,0}' + C_{217} \ z_{H,0}'$$

$$+ C_{218} \ z_{H,0} + C_{219} \ z_{\alpha,0} + C_{2110} \ z_{\chi,0},$$

$$0 = z_{\chi,1}'' + C_{311} \ z_{\chi,1}'+ C_{312} \ z_{H,1} + C_{313} \ z_{H,1} + C_{314} \ z_{\alpha,1} + C_{315} \ z_{\chi,1} + C_{316} \ z_{\alpha,0}' + C_{317} \ z_{H,0}'$$

$$+ C_{318} \ z_{H,0} + C_{319} \ z_{\alpha,0} + C_{3110} \ z_{\chi,0},$$

(2.27)
where connection coefficients $C_{i1j}$ are nonlinear functionals of $\{\rho_6, c, a\}$ with explicit dependence on $x$ and $\{\beta_1, \beta_2\}$:

$$C_{i1j} = C_{i1j}\left[\{\rho_6, c, a\}; x; \{\beta_1, \beta_2\}\right].$$

Using (2.9), (2.10), (2.23), (2.24), for each set $S_{\text{sound}}$ we construct the asymptotic solution of (2.27). Near the boundary, $x \to 0_+$ we find

$$z_{H,1} = x \left( z_{H,2,0}^{(1)} + O\left(x^2 \ln x\right)\right),$$

$$z_{\alpha,1} = x^{1/2} z_{\alpha,1,0}^{(1)} + O\left(x \ln x\right), \quad z_{\chi,1} = x^{1/4} \left(x^{1/2} z_{\chi,1,0}^{(1)} + O\left(x \ln x\right)\right),$$

and near the horizon, $y \equiv 1 - x \to 0_+$

$$z_{H,1} = 0 + O(y^2), \quad z_{\alpha,1} = q_0^{(1)} + O(y^2), \quad z_{\chi,1} = x_0^{(1)} + O(y^2).$$

Thus, altogether we have six new parameters:

$$S_{\text{sound}} \implies \left\{\beta_2, z_{H,2,0}^{(1)}, z_{\alpha,1,0}^{(1)}, z_{\chi,1,0}^{(1)}, q_0^{(1)}, x_0^{(1)}\right\},$$

precisely what is necessary to construct a unique solution for $\{z_{H,1}, z_{\alpha,1}, z_{\chi,1}\}$ for a given $S_{\text{sound}}$.

We use numerical techniques developed in [21] to general data sets

$$S_{\text{attenuation}} \equiv \left\{S_{\text{sound}}; \beta_2, z_{H,2,0}^{(1)}, z_{\alpha,1,0}^{(1)}, z_{\chi,1,0}^{(1)}, q_0^{(1)}, x_0^{(1)}\right\}.$$  

The data sets $S_{\text{attenuation}}$ are available from the authors upon request.

### 3 Bulk viscosity of $\mathcal{N} = 2^*$ plasma

Given $S_{\text{attenuation}}$ we have all the necessary data to study bulk viscosity of strongly coupled $\mathcal{N} = 2^*$ plasma. Of primary interest to us are the bulk viscosity bound conjecture [1], and the behaviour of bulk viscosity near the phase transition.

#### 3.1 Bulk viscosity bound

Fig. 1 represents the ratio of bulk to shear viscosities $\zeta/\eta$ as a function of $(\frac{1}{3} - c_s^2)$ for $\mathcal{N} = 2^*$ plasma with mass deformation parameter $\nu$ ranging from $\nu = 0.2$ to $\nu = 0.9$. 

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Figure 1: Ratio of viscosities $\frac{\zeta}{\eta}$ versus the speed of sound in $\mathcal{N} = 2^*$ gauge theory plasma with mass deformation parameter $\nu \equiv \frac{m_f^2}{m_b^2} \in [0.2, 0.9]$, with intervals $\Delta \nu = 0.05$. The solid line represents the bulk viscosity bound (1.8).

with intervals of $\Delta \nu = 0.05$. Different color sets of points represent different values of $\nu$. The solid blue line represents the bulk viscosity bound (1.8). We verified that in the vicinity of $c_s^2 = \frac{1}{3}$ (which corresponds to a high-temperature regime of $\mathcal{N} = 2^*$ plasma), the results are in excellent agreement with (1.6), (1.7). Although in this paper we truncated the plots to a near-conformal regime\(^5\), available data sets $S_{\text{attenuation}}$ allow us to study the behaviour of bulk viscosity near the critical point $T_c(\nu)$. Much like for the $\nu = 0$ case discussed in [1], for each value of $\nu < 1$ we observe a rapid, power-law like, growth of $\zeta/\eta$ in the vicinity of the critical point. Nonetheless, this ratio is finite precisely at $T = T_c$ (see Fig. 3 below). The results are qualitatively identical to the $\nu = 0$ case discussed in [1].

Fig. 2 represents the ratio of bulk and shear viscosities $\zeta/\eta$ as a function of $(\frac{1}{3} - c_s^2)$

\(^5\)The rapid growth of bulk viscosity indicates that if any violation of the bound (1.8) would occur, it would occur in a near-conformal regime.
Figure 2: Ratio of viscosities $\zeta/\eta$ versus the speed of sound in $\mathcal{N} = 2^*$ gauge theory plasma with mass deformation parameter $\nu \equiv \frac{m_b^2}{m_T^2} \in \{1, 1.05, [1.1, 2.1]_{\Delta \nu = 0.1}, [3, 6]_{\Delta \nu = 1}\}$. The solid line represents the bulk viscosity bound (1.8).

for $\mathcal{N} = 2^*$ plasma with mass deformation parameter $\nu = \{1, 1.05\}$, also in the range from $\nu = 1.1$ to $\nu = 2.1$ with intervals of $\Delta \nu = 0.1$, and in the range from $\nu = 3$ to $\nu = 6$ with intervals of $\Delta \nu = 1$. Different color sets of points represent different values of $\nu$. The solid blue line represents the bulk viscosity bound (1.8). Again, we verified that in the vicinity of $c_s^2 = \frac{1}{3}$ (which corresponds to a high-temperature regime of $\mathcal{N} = 2^*$ plasma), the results are in excellent agreement with (1.6), (1.7). Notice the accumulation of the points as data sets approach the bound. Much like for the $\nu = 1$ case discussed in [1], this regime corresponds to a low-temperature $\mathcal{N} = 2^*$ plasma regime. Specifically, there we have $m_b/T \sim 5 \cdots 10$. Much like in [1] we can extrapolate the speed of sound and the viscosity ratio to $T \to 0$. We find that the endpoints of such extrapolations land on (or slightly above) the bulk viscosity bound line.

It is very interesting to explore the thermodynamics of $\mathcal{N} = 2^*$ plasma (for $\nu > 1$) at
Figure 3: Ratio of viscosities $\frac{\zeta}{\eta}$ at $T = T_c$ versus the mass deformation parameter $\nu$ of $\mathcal{N} = 2^*$ plasma.

extremely low temperature\(^6\), ideally, as $T \to 0$. So far we have no indication that such a zero temperature limit would be singular. On the other hand, we can not exclude the presence of some exotic instabilities/phase transitions at very low temperatures, see [11] for more details.

3.2 Bulk viscosity at a critical point

We already mentioned that $\mathcal{N} = 2^*$ plasma with mass deformation parameter $0 \leq \nu < 1$ undergoes a phase transition, which appears to be in the universality class of the mean-field tricritical point. At such a phase transition, specific heat diverges as $c_V \sim |1 - T_c/T|^{-1/2}$. Fig. 3 represents the ratio of bulk to shear viscosities $\frac{\zeta}{\eta}$ as a function of $\nu \equiv \frac{m_f^2}{m_b^2}$ at the critical temperature $T = T_c(\nu)$. We took the $\nu = 0$ result from [1]. Notice the rapid growth of $\frac{\zeta}{\eta}\bigg|_{\text{critical}}$ as $\nu \to 1_-$. Actually, such a behavior is

\(^6\)Our current numerical algorithms become very slow as low temperatures.
expected if $\nu = 1$ is the end point at which transitions cease to occur, and for $\nu \geq 1$ the ratio of $\zeta/\eta$ is relatively close to the bound (1.8) down to rather low temperatures.

4 Conclusion

In this paper we study the bulk viscosity of $\mathcal{N} = 2^*$ gauge theory plasma at strong coupling as a function of temperature and for various masses, $\nu \equiv m_f^2/m_b^2$. In all cases, we find that the viscosity bound (1.8) is satisfied. Similar to a thermal state of $\mathcal{N} = 2^*$ plasma with $\nu = 0$, we find that, while the bulk viscosity of the $\mathcal{N} = 2^*$ plasma grows rapidly in the vicinity of the critical point for $0 < \nu < 1$, it is finite precisely at $T = T_c(\nu)$.

We did not discuss in this paper gauge theory/string theory duality-motivated phenomenological approaches to transport properties (see however [23]). We also did not discuss potential applications to RHIC/LHC physics (see however [24]).

In the future it would be interesting to study transport properties of the Klebanov-Strassler [25] cascading plasma. The preliminary work necessary for such analysis already appeared in the literature [26, 21, 27].

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