Abstract

Description of the compatibility matrices method for solution of systems of Boolean equations. The method is applied to determine whether SAT instances have clauses in their disjunctive normal form.
A Polynomial Time Algorithm for SAT

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1 Introduction

SAT [1, 2, 3, 4, 5] is a problem to determine whether a given logical formula, written in the conjunctive normal form, is satisfiable:

\[ f = c_1 \land c_2 \land \ldots \land c_m, \quad (1) \]

- where clauses \( c_k, \ k = 1, 2, \ldots, m, \) are disjunctions of literals over a set of \( n \) Boolean variables

\[ B = \{ b_1, b_2, \ldots, b_n \}. \]

In other words, given formula (1), it is required to determine whether there exists a truth assignment

\[ \tau : B \to \{ false, true \}, \]

- which satisfies that formula:

\[ f(\tau(B)) = true. \]

Using reductions [4, 5, or other], an efficient algorithm for the problem can be derived from the algorithms described in [7, 8, 9]. The goal of this work was to develop an efficient algorithm, custom for SAT.

The algorithm uses the method of compatibility matrices, described below, and detects whether there are members in disjunctive normal form of formula (1). The algorithm’s computational complexity is \( O(m^3t_1t_2) \), where \( m \) is the number of clauses in formula (1) and \( t_1, t_2 \) are the lengths of the two longest clauses.

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2 Compatibility matrices method

Method of compatibility matrices was developed to solve systems of Boolean equations. In [8, 9] it was applied to 3-SAT. Here, the method is adapted to detect whether a system of Boolean equations is compatible/has solutions/satisfiable.

Let’s have the following system of Boolean equations:

\[ f_i(b_{i1}, b_{i2}, \ldots, b_{in_i}) = true, \quad i = 1, 2, \ldots, m \]  \hspace{1cm} (2)

- where \( f_i, i = 1, 2, \ldots, m \) are logical formulas; \( n_i \) is the number of variables in formula \( f_i \); and \( b_{ij}, j = 1, 2, \ldots, n_i \) are Boolean variables in formula \( f_i \).

The problem is to detect whether system (2) has solutions. The method’s idea is to detect whether the truth assignments satisfying separate equations (2) are compatible.

Let’s calculate truth-table \( T_i \) for each of the formulas \( f_i, i = 1, 2, \ldots, m \):

| \#  | \( b_{i1} \) | \( b_{i2} \) | \ldots | \( b_{in_i} \) | \( f_i \) |
|-----|-------------|-------------|--------|---------------|-------|
| 1   | false       | false       | \ldots | false         | \( f_i(false, false, \ldots, false) \) |
| 2   | false       | false       | \ldots | true          | \( f_i(false, false, \ldots, true) \) |
|     |             |             |        | \ldots        |        |
| \( 2^{n_i} \) | true       | true       | \ldots | true          | \( f_i(true, true, \ldots, true) \) |

Let’s arbitrarily enumerate strings in each of the tables. Let’s take \( \alpha \)-th string from truth-table \( T_i \) and \( \beta \)-th string from truth table \( T_j \). Let’s call them compatible iff

1). True assignments in the \( \alpha \)-th string from truth-table \( T_i \) and in the \( \beta \)-th string from truth-table \( T_j \) are compatible. That means that any variable common for these two strings has the same value in them;

2). The formulas \( f_i \) and \( f_j \) both have value \( true \) in these strings.

Due to the definition, if \( f_i = false \) in a string, then that string cannot be compatible with any other string.

Let’s take all 2-combinations of formulas \( f_i, i = 1, 2, \ldots, m \):

\[
(f_1, f_2) \quad (f_1, f_3) \quad \ldots \quad (f_1, f_m) \\
(f_2, f_3) \quad \ldots \quad (f_2, f_m) \\
\vdots \\
(f_{m-1}, f_m)
\]

For each of the combinations, let’s build a compatibility matrix. The compatibility matrix for combination \( (f_i, f_j) \) is a \( 2^{n_i} \times 2^{n_j} \) matrix \( F_{ij} \), whose
elements are true or false - a Boolean matrix. An element \( e_{\alpha\beta} \) of matrix \( F_{ij} \) is true iff the \( \alpha \)-th string of truth-table \( T_i \) and the \( \beta \)-th string of truth-table \( T_j \) are compatible.

The element \( e_{\alpha\beta} \) of matrix \( F_{ij} \) shows whether the \( \alpha \)-th string of truth-table \( T_i \) and the \( \beta \)-th string of truth-table \( T_j \) are a solution of system of two Boolean equations

\[
f_i = f_j = \text{true}.
\]

(3)

Due to the definition, if \( f_i = \text{false} \) in \( \alpha \)-th string of truth-table \( T_i \), then \( \alpha \)-th string in all compatibility matrices \( F_{ij} \) and \( \alpha \)-th column in all compatibility matrices \( F_{ji} \) are filled with false.

The method starts with its initialization - Step 0. In this step, the triangular box matrix is written down

\[
S_0 = \begin{bmatrix}
F_{12} & F_{13} & \ldots & F_{1m} \\
F_{23} & \ldots & F_{2m} \\
\vdots & \ddots & \vdots \\
F_{m-1m} & \ldots & \ldots & \ldots
\end{bmatrix}
\]

There are \( m(m - 1)/2 \) boxes in matrix \( S_0 \). Let’s mark them with a third index, whose value is 0:

\[
F_{ij0} = F_{ij}, \ 0 < i < j \leq m.
\]

Any true-element \( e_{\alpha\beta} \) (an element which equals true) of box \( F_{ij0} \), \( 1 \leq i < j \leq m \), indicates compatibility of system (3), i.e. string \( \alpha \) from truth-table \( T_i \) and string \( \beta \) from truth-table \( T_j \) are compatible.

Next, the method performs \( m - 2 \) iterations - Step 1 trough Step \( m - 2 \). On its Step \( k \), \( 1 \leq k \leq m - 2 \), the method recalculates box matrix \( S_{k-1} \) - a result of Step \( k - 1 \). Let box matrix \( S_k \) be the result of those calculations. Let \( F_{ijk} \) be boxes of \( S_k \) and \( F_{ijk,k-1} \) be boxes of \( S_{k-1} \). Then:

\[
F_{ijk} = \begin{cases} 
F_{ijk,k-1}; & i \leq k, \ i < j \leq m \\
(F_{ki,k-1}^T \times F_{kj,k-1}) \land F_{ijk,k-1}; & k < i < j \leq m
\end{cases}, \quad (4)
\]

Formula (4) involves certain matrix operations with Boolean matrices. Here, let’s define them as follows.

Let \( X = (x_{ij})_{a \times b} \) and \( Y = (y_{ij})_{b \times c} \) be Boolean matrices. Then,

\[
X^T = (x_{ji})_{b \times a}, \ X \times Y = (\bigvee_k x_{ik} \land y_{kj})_{a \times c}, \ X \land Y = (x_{ij} \land y_{ij})_{a \times b}.
\]

Sure, the last definition requires the sizes of \( X \) and \( Y \) to be the same.
**Theorem 1.** Box $F_{ijk}$, $0 \leq k < i < j \leq m$, has a true-element iff the following Boolean system is compatible:

\[
f_1 = f_2 = \ldots = f_k = f_i = f_j = true. \tag{5}
\]

*Proof.* Let $k = 0$. That is a case of Step 0. System (5) in this case is just system (3). Obviously, the theorem is correct in this case.

Let $k = 1$. That is a case of Step 1. Due to (4),

\[
F_{ij1} = (F_{1i0}^T \times F_{1j0}) \land F_{ij0}.
\]

Let

\[
F_{1i0} = (x_{\alpha \beta})_{axb}, \quad F_{1j0} = (y_{\alpha \beta})_{axc}, \quad F_{ij0} = (z_{\alpha \beta})_{bxc}.
\]

Then:

\[
F_{ij1} = (\bigvee_{\gamma=1}^{\gamma=a} x_{\gamma\alpha} \land y_{\gamma\beta} \land z_{\alpha\beta})_{bxc}.
\]

Thus, box $F_{ij1}$ contains a true-element iff

\[
\exists \alpha, \beta, \gamma : x_{\gamma\alpha} \land y_{\gamma\beta} \land z_{\alpha\beta} = true.
\]

Or, in other words, box $F_{ij1}$ contains a true-element iff the following system is compatible:

\[
\begin{cases}
x_{\gamma\alpha} = true \\
y_{\gamma\beta} = true \\
z_{\alpha\beta} = true
\end{cases}
\]

Due to (3), the following system must be compatible:

\[
\begin{cases}
x_{\gamma\alpha} = true \Rightarrow f_1 = f_i = true \\
y_{\gamma\beta} = true \Rightarrow f_1 = f_j = true \\
z_{\alpha\beta} = true \Rightarrow f_i = f_j = true
\end{cases}
\]

Because $\alpha$ and $\beta$ are the same in all three of the system’s subsystems, box $F_{ij1}$ contains a true-element iff there is a string $\gamma$ in truth-table $T_1$ which is compatible with solution $(\alpha, \beta)$ of the last subsystem. That means compatibility of system

\[
f_1 = f_i = f_j = true
\]

That proves the theorem for $k = 1$.

Let the theorem be correct for some $k < m - 2$. Then, due to (4):

\[
F_{ij,k+1} = (F_{k+1,ik}^T \times F_{k+1,jk}) \land F_{ijk}.
\]
Let
\[ F_{k+1,j} = (x_{\alpha \beta})_{a \times b}, \quad F_{k+1,j0} = (y_{\alpha \beta})_{a \times c}, \quad F_{ij,k} = (z_{\alpha \beta})_{b \times c}. \]
Then:
\[ F_{ijk} = (\bigvee_{\gamma=1}^{a} x_{\gamma \alpha} \land y_{\gamma \beta} \land z_{\alpha \beta})_{b \times c}. \]
Thus, box \( F_{ij,k} \) contains a true-element iff
\[ \exists \alpha, \beta, \gamma : \ x_{\gamma \alpha} \land y_{\gamma \beta} \land z_{\alpha \beta} = \text{true}. \]
Or, in other words, box \( F_{ij,k} \) contains a true-element iff the following system is compatible:
\[
\begin{align*}
x_{\gamma \alpha} &= \text{true} \\
y_{\gamma \beta} &= \text{true} \\
z_{\alpha \beta} &= \text{true}
\end{align*}
\]
Due to the induction’s hypothesis, the following system must be compatible:
\[
\begin{align*}
x_{\gamma \alpha} &= \text{true} \Rightarrow f_1 = f_2 = \ldots = f_k = f_{k+1} = f_i = \text{true} \\
y_{\gamma \beta} &= \text{true} \Rightarrow f_1 = f_2 = \ldots = f_k = f_{k+1} = f_j = \text{true} \\
z_{\alpha \beta} &= \text{true} \Rightarrow f_1 = f_2 = \ldots = f_k = f_i = f_j = \text{true}
\end{align*}
\]
Because \( \alpha \) and \( \beta \) are the same in all three of the system’s subsystems, box \( F_{ij,k+1} \) contains a true-element iff there is a string \( \gamma \) in truth-table \( T_{k+1} \) which is compatible with solution \( (\alpha, \beta) \) of the last subsystem. That means compatibility of system
\[ f_1 = f_2 = \ldots = f_k = f_{k+1} = f_i = f_j = \text{true}. \]
That concludes the induction. \( \square \)

The following theorem is a simple subsequence of Theorem 1.

**Theorem 2.** System (2) is compatible/satisfiable iff box \( F_{m-1,m,m-2} \) contains true-elements.

The method of compatibility matrices is a computation of box \( F_{m-1,m,m-2} \) using formulas (4). Examples of how it works can be found in [9].

The method may be seen as an application of dynamic programming [6]. Besides iteration schema (4), the method allows other schemes as well. Schema (4) resembles the Gauss’ exclusions method the most.
3 Computational simplification

For system (2) with \( m \) equations, the method performs \( m - 2 \) steps at most. Each step is a Boolean transformation of less than \( m^2 \) matrices of size \( 2^{n_1} \times 2^{n_2} \) or less, where \( n_1 \) and \( n_2 \) are the biggest numbers of variables in equations (2). Thus, the computational complexity of the method is

\[
O(2^{n_1+n_2}m^3).
\]

That means that the direct application of this method to 3-SAT (\( n_1, n_2 \leq 3 \)) will produce a polynomial time algorithm. The time is

\[
O(64m^3) = O(m^3).
\]

And that was used in [8].

But let’s mention that there is no need to use the entire truth-tables of equations (2). Only those strings in which \( f_i = \text{true} \) matter. So, if only the true-strings are used to calculate the compatibility matrices, then the boxes’ sizes will be reduced to \( t_1 \times t_2 \) or less, where \( t_1 \) and \( t_2 \) are the two biggest quantities of true-strings in the truth-tables. The computational complexity of such simplified method is

\[
O(t_1t_2m^3).
\]

The simplified method directly checks for compatibility of those true assignments which satisfy the separate equations of system (2).

To take full advantage of estimation (6) and make the method purely polynomial, system (2) shall be transformed in such a way, that number of the resulting equations and number of true-strings in their truth-tables would be polynomial over initial input. Sometimes, it can be achieved with XOR operations.

4 Application to SAT

Theoretically, it is possible to use distributive laws and rewrite (1) in disjunctive form:

\[
f = d_1 \lor d_2 \lor \ldots \lor d_p, \ p \leq t_1^m
\]

- where \( t_1 \) is the maximal length of clauses in formula (1); and \( d_i \) are conjunctions of \( m \) literals - one per clause \( c_k, \ k = 1, 2, \ldots, m; \)

\[
d_i = l_1 \land l_2 \land \ldots \land l_m, \ i = 1, 2, \ldots, p
\]
- where \( l_k \) is a literal from clause \( c_k \).

There is a generator for conjunctions (7):

\[
g = \bigwedge_{k=1}^{m} (\xi_{k1} \oplus \xi_{k2} \oplus \cdots \oplus \xi_{kn_k}),
\]

(8)

- where operation \( \oplus \) is XOR; \( n_k \) is the length of clause \( c_k \); and \( \xi_{k\mu} \) is a Boolean variable, which is true iff \( \mu \)-th literal of clause \( c_k \) enters in (7).

To exclude from (7) those conjunctions with complimentary literals, let’s add to (8) another formula

\[
h = \bigwedge_{\alpha,\beta} (\bar{\xi}_{\alpha\mu_\alpha} \lor \bar{\xi}_{\beta\mu_\beta}),
\]

(9)

- where \( \xi_{\alpha\mu_\alpha} \) and \( \xi_{\beta\mu_\beta} \) are the variables \( \xi \) for complimentary literals allocated in clauses \( c_\alpha \) and \( c_\beta \) appropriately.\(^1\) The number of clauses in (9) is no bigger than \( t_1 t_2 C_m^2 \),

- where \( t_1 \) and \( t_2 \) are the sizes of two longest clauses in formula (1). Thus, equation

\[
g \land h = true
\]

(10) has a polynomial size. Solutions of this equation represent those conjunctions (7) which enter in disjunctive normal form of formula (1). Thus, formula (1) is satisfiable iff formula (10) is satisfiable.

But equation (10) produces system (2) with polynomial number of equations, whose truth-tables have no more than \( t_1 \) true-strings, where \( t_1 \) is the size of the longest clause in formula (1). Thus, due to estimation (6), application of the simplified method to system/equation (10) will produce a polynomial time algorithm for SAT. Because of size of formula (9), the algorithm’s computational complexity can be grossly estimated as

\[
O(t_1^4 t_2^4 m^6).
\]

The last estimation is very safe. Also, there is no need to combine formulas (8) and (9) in one formula (10). The effect of formula (9) can be accounted for when calculating the compatibility matrices of formula (8). That can be done by including an additional step in the method - Step 0.5, where compatibility matrices of formula (8) are depleted in accordance with formula (9). Computational complexity of such modernized method is

\[
O(t_1 t_2 m^3).
\]

Let us illustrate this approach.

\(^1\)In certain sense, formula (9) may be seen as a reduction of formula (1) to 2-SAT. Author plans to research this in a separate article.
5 Example

Let’s illustrate the approach with one simple example:

\[ f = \bar{p} \land \bar{q} \land (p \lor q). \]

Equation (10) for the given 2-SAT instance:

\[ \alpha \land \beta \land (\gamma \oplus \delta) \land (\bar{\alpha} \lor \bar{\gamma}) \land (\bar{\beta} \lor \bar{\delta}) = true. \]

System (2) for the example:

\[
\begin{align*}
\alpha &= true \\
\beta &= true \\
\gamma \oplus \delta &= true \\
\bar{\alpha} \lor \bar{\gamma} &= true \\
\bar{\beta} \lor \bar{\delta} &= true
\end{align*}
\]

Let’s use the simplified method. The truth-tables for the equations:

| # | \(\alpha\) | \(\alpha\) | # | \(\beta\) | \(\beta\) |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 |

| # | \(\gamma\) | \(\delta\) | \(\gamma \oplus \delta\) |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 |
| 2 | 1 | 0 | 1 |
| 3 | 1 | 1 | 0 |

| # | \(\alpha\) | \(\gamma \lor \bar{\gamma}\) | # | \(\beta\) | \(\bar{\beta} \lor \bar{\delta}\) |
|---|---|---|---|---|---|
| 1 | 0 | 0 | 1 | 1 | 0 | 1 |
| 2 | 0 | 1 | 1 | 2 | 0 | 1 |
| 3 | 1 | 0 | 1 | 3 | 1 | 0 |
| 4 | 1 | 1 | 0 | 4 | 1 | 0 |

Step 0 - initialization. Box matrix \(S_0\):

\[
\begin{array}{cccc}
\alpha & \beta & \gamma \oplus \delta & \bar{\alpha} \lor \bar{\gamma} & \bar{\beta} \lor \bar{\delta} \\
1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
\end{array}
\]
Step 1. Box matrix $S_1$:

|     | $\beta$ | $\gamma \oplus \delta$ | $\bar{\alpha} \lor \bar{\gamma}$ | $\bar{\beta} \lor \bar{\delta}$ |
|-----|---------|-----------------|-----------------|-----------------|
| $\alpha$ | 1   | 1   | 1   | 0   | 0   | 1   |
| $\beta$ | 1   | 1   | 0   | 0   | 1   |
| $\gamma \oplus \delta$ | 0   | 0   | 1   | 0   | 1   |
| $\bar{\alpha} \lor \bar{\gamma}$ | 0   | 0   | 0   |
| $\bar{\beta} \lor \bar{\delta}$ | 0   | 0   |

Step 2. Box matrix $S_2$:

|     | $\beta$ | $\gamma \oplus \delta$ | $\bar{\alpha} \lor \bar{\gamma}$ | $\bar{\beta} \lor \bar{\delta}$ |
|-----|---------|-----------------|-----------------|-----------------|
| $\alpha$ | 1   | 1   | 1   | 0   | 0   | 1   |
| $\beta$ | 1   | 1   | 0   | 0   | 1   |
| $\gamma \oplus \delta$ | 0   | 0   | 1   | 0   | 0   |
| $\bar{\alpha} \lor \bar{\gamma}$ | 0   | 0   | 0   |
| $\bar{\beta} \lor \bar{\delta}$ | 0   | 0   |

Step 3. Box matrix $S_3$:

|     | $\beta$ | $\gamma \oplus \delta$ | $\bar{\alpha} \lor \bar{\gamma}$ | $\bar{\beta} \lor \bar{\delta}$ |
|-----|---------|-----------------|-----------------|-----------------|
| $\alpha$ | 1   | 1   | 1   | 0   | 0   | 1   |
| $\beta$ | 1   | 1   | 0   | 0   | 1   |
| $\gamma \oplus \delta$ | 0   | 0   | 1   | 0   | 0   |
| $\bar{\alpha} \lor \bar{\gamma}$ | 0   | 0   | 0   |
| $\bar{\beta} \lor \bar{\delta}$ | 0   | 0   | 0   |

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Thus, the given SAT instance is unsatisfiable.

Now, let’s use the modernized method. Equation (8) for the given formula $f$ looks:

$$\alpha \land \beta \land (\gamma \oplus \delta) = \text{true}.$$  

Step 0. Box matrix $S_0$ for the equation:

|   | $\beta$ | $\gamma \oplus \delta$ |
|---|---------|--------------------------|
| $\alpha$ | 1       | 1 1                       |
| $\beta$  | 1 1     |                           |

Step 0.5. Formula (9) for the example:

$$(\bar{\alpha} \lor \bar{\gamma}) \land (\bar{\beta} \lor \bar{\delta}) = \text{true}.$$  

Using the truth-tables above, let’s deplete the boxes in matrix $S_0$:

|   | $\beta$ | $\gamma \oplus \delta$ |
|---|---------|--------------------------|
| $\alpha$ | 1       | 1 0                       |
| $\beta$  | 0 1     |                           |

Step 1. Box matrix $S_1$:

|   | $\beta$ | $\gamma \oplus \delta$ |
|---|---------|--------------------------|
| $\alpha$ | 1       | 1 0                       |
| $\beta$  | 0 0     |                           |

Thus, the given SAT instance is unsatisfiable.

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