Magnetization induced dynamics of a Josephson junction coupled to a nanomagnet

Roopayan Ghosh(1), Moitri Maiti(2), Yury M. Shukrinov(2,3) and K. Sengupta(1)

(1) Theoretical Physics Department, Indian Association for the Cultivation of Science, Jadaupur, Kolkata-700032, India.
(2) BLTP, JINR, Dubna, Moscow region, 141980, Russia
(3) Dubna State University, Dubna, Russian Federation.

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We study the superconducting current of a Josephson junction (JJ) coupled to an external nanomagnet driven by a time dependent magnetic field both without and in the presence of an external AC drive. We provide an analytic, albeit perturbative, solution for the Landau-Lifshitz (LL) equations governing the coupled JJ-nanomagnet system in the presence of a magnetic field with arbitrary time-dependence oriented along the easy axis of the nanomagnet’s magnetization and in the limit of weak dimensionless coupling $\epsilon_0$ between the JJ and the nanomagnet. We show the existence of Shapiro-like steps in the I-V characteristics of the JJ subjected to a voltage bias for a constant or periodically varying magnetic field and explore the effect of rotation of the magnetic field and the presence of an external AC drive on these steps. We support our analytic results with exact numerical solution of the LL equations. We also extend our results to dissipative nanomagnets by providing a perturbative solution to the Landau-Lifshitz-Gilbert (LLG) equations for weak dissipation.

We study the fate of magnetization-induced Shapiro steps in the presence of dissipation both from our analytical results and via numerical solution of the coupled LLG equations. We discuss experiments which can test our theory.

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I. INTRODUCTION

The physics of Josephson junctions (JJs) has been the subject of intense theoretical and experimental endeavor for decades. The interest in the physics of such JJs has received renewed attention in recent years in the context of Majorana modes in unconventional superconductors. Indeed, it has been theoretically predicted and experimentally observed that such junctions may serve as a test bed for detection of Majorana end modes in unconventional superconductors. It has been shown that the presence of such end modes lead to fractional Josephson effect and results in the absence of odd Shapiro steps when such junctions are subjected to an external AC drive.

Recently molecular nanomagnets have been studied as potential candidates for qubit realization owing to their long magnetization relaxation times at low temperatures. Such a realization is expected to play a central role in several aspects of quantum information processing and spintronics using molecular magnets. These systems have potential for high-density information storage and are also excellent examples of finite-size spin systems which are promising test-beds for addressing several phenomena in quantum dynamics viz. quantum-tunneling of the magnetization, quantum information, entanglement, etc. The study of the spin dynamics of the nanomagnets is a crucial aspect of all such studies. One way to probe such dynamics is to investigate the spin response in bulk magnets using inelastic neutron scattering and subsequent finite-size extrapolation to obtain the inelastic neutron scattering spectra for a single molecule. Other, more direct, methods include determination of the real-space dynamical two-spin correlations in high-quality crystals.
theoretical studies were complemented by experimental work on these systems. More recently, magnetization reversal of a single spin using a JJ subjected to a static field and a weak linearly polarized microwave radiation has also been demonstrated in Ref. 34. However, to the best of our knowledge, these studies do not provide any analytic treatment of the coupled JJ-nanomagnet system even at a classical level where they are known to be governed by the Landau-Lifshitz-Gilbert (LLG) equations. Moreover, the current-voltage (I-V) characteristics of a JJ in the presence of a nanomagnet with time-dependent magnetic fields and in the presence of external AC drive has not been studied systematically so far.

In this work, we study a JJ coupled to a nanomagnet with a fixed easy-axis anisotropy direction (chosen to be \( \hat{y} \)) in the presence of an arbitrary time-dependent external magnetic field along \( \hat{y} \). For nanomagnets with weak anisotropy, we find an analytic perturbative solution to the coupled Landau-Lifshitz (LL) equations in the limit of weak coupling between the nanomagnet and the JJ. Using this solution, we show that a finite DC component of the supercurrent, leading to Shapiro-like steps in the I-V characteristics of a voltage-biased JJ, can occur, in the absence of any external radiation, for either a constant or a periodically time-varying magnetic field. Our theoretical analysis provides exact analytic results for the position of such steps. We study the stability of these steps against change in the direction of the applied magnetic field and increase of the dimensionless coupling strength \( \epsilon_0 \) between the JJ and the nanomagnet. We also provide a detailed analysis of the fate of this phenomenon in the presence of an external AC drive and demonstrate that the presence of such a drive leads to several new fractions (ratio between the applied DC voltage and the drive frequency) at which the supercurrent develops a finite DC component leading to Shapiro-like steps in the I-V characteristics. We support our analytical results with exact numerics and discuss details of Shapiro-step like features in the presence of a constant or periodic magnetic field. Finally, we chart out our main results, discuss experiments which can test our theory, and conclude in Sec. IV.

II. FORMALISM AND ANALYTICAL SOLUTION

In this section, we obtain analytic solution to the LL equations for the weakly coupled JJ-nanomagnet system. We shall sketch the general derivation of our result for arbitrary time-dependent magnetic field in Sec. II A and then apply these results to demonstrate the existence of Shapiro-like steps for constant or periodic magnetic fields in Sec. II B. The extension of these results for dissipative magnets will be charted out in Sec. II C.

A. Perturbative solution

The coupled JJ-nanomagnet system is schematically shown in Fig. 1. In what follows we consider a JJ along...
\[ \hat{\mathbf{B}}(t) = \frac{e V_0(t)}{\hbar} dt' \]

where \( E = E_1 + E_2 \), \( E_1 = -KM_0^2 - M_y B(t) \), \( E_2 = -E_J \cos \gamma \), \( (1) \)

where \( K > 0 \) denotes the magnetization anisotropy constant, \( \hat{B}(t) \parallel \hat{y} \) is the external magnetic field which can have arbitrary time dependence, and \( E_J \) is the Josephson energy of the junction. The phase difference \( \gamma \) across the junction is given by

\[ \gamma(t) = \gamma_0(t) + \gamma_1(t), \]

\[ \gamma_0(t) = \frac{2\pi}{\Phi_0} \int \hat{B} \cdot \hat{A}(\hat{r}), \]

\[ \gamma_1(t) = -\frac{2\pi}{\Phi_0} \int \hat{d}l \cdot \hat{A}(\hat{r}), \]

where \( \gamma_0 \) is the intrinsic DC phase of the JJ, \( \gamma_0 \) is the phase generated by the external voltage, \( V_0(t) = V_0 g(t) \) is the applied voltage across it, \( \omega_0 = 2eV_0/\hbar \) is the Josephson frequency of the junction, \( g(t) \) is the dimensionless function specifying the time dependence of the applied voltage, \( \Phi_0 = \hbar c/2e \) is the flux quantum, \( \hbar = 2\pi \hbar \) with \( \hbar \) being the Planck constant, \( e \) is the charge of an electron, and \( \epsilon \) is the speed of light. The vector potential \( \hat{A}(\hat{r}) \) is given by

\[ \hat{A}(\hat{r}, t) = \mu_0(\hat{M}(t) \times \hat{r})/(4\pi|\hat{r}|^3). \]

Note that in our chosen geometry, as shown in Fig. [1], \( \hat{d}l \parallel \hat{x} \) and \( \hat{r} \) lies in the \( x-y \) plane, so that

\[ \gamma_1(t) = -k_0 M_z \sin(\gamma), \]

\[ k_0 = \mu_0 M_0 l/(2\Phi_0 a \sqrt{R^2 + a^2}), \]

where the geometrical factor \( k_0 \) can be tuned by tuning the distance \( a \) between the JJ with the nanomagnet (Fig. [1]). Moreover, in this geometry, the orbital effect of the magnetic field do not affect the phase of the JJ since \( \hat{d}l \cdot \hat{A}_B \sim \hat{d}l \cdot (\hat{B} \times \hat{r}) = 0. \) In this geometry, the LL equations for the nanomagnet reads

\[ \frac{d\hat{M}}{dt} = \gamma(\hat{M} \times \hat{B}_{\text{eff}}) \]

\[ \hat{B}_{\text{eff}} = -\delta E \delta \hat{M} = B(M_y)\hat{y} + \frac{E_J k_0}{|\hat{M}|} \sin(\gamma_0(t) + \gamma_1(t)) \hat{z} \]

where \( B(M_y) = KM_y + B(t) \) and \( \gamma_0 \) is the gyromagnetic ratio. These LL equations are to be solved along with the constraint of constant \( |\hat{M}| \); in what follows we shall set \( |\hat{M}| = M_0 \). We note that Eq. [5] do not include dissipation which shall be treated in Sec. [11C]. Thus the solutions obtained in this section can be treated as limiting case of very weakly dissipating nanomagnets. We also note that our analysis do not take into account the change in \( I_s \) arising from the spin-flip scattering induced by the coupling of the JJ with the nanomagnet.\[ \hat{M} = \frac{M_0}{M_0} \cos(\theta_0) + \hat{M}_1 \]

This can be justified by the fact that in our geometry, the nanomagnet does not reside atop the junction and thus we expect the spin-flip scattering matrix elements to be small. Further, even with a significant contribution from spin-flip scattering, such effects become important when the Larmor frequency of the magnetization \( \omega_L \geq \Delta_0/\hbar \) which is not the regime that we focus on. This issue is discussed further in Sec. [IV].

Eq. [5] represents a set of non-linear equations which, in most cases, need to be solved numerically. Here we identify a limit in which these equations admits an analytic, albeit perturbative, solution for arbitrary \( B(t) \). To this end we define the following dimensionless quantities

\[ \vec{m} = \frac{\vec{M}}{M_0} = (\sin \theta \cos \phi, \cos \theta, \sin \theta \sin \phi) \]

\[ \omega_B(t) = B(M_y)/B_1, \quad \epsilon_0 = k_0 E_J/(B_1 M_0) \]

\[ B(t) = B_1 f(\tau), \quad \tau = \gamma B_1 t, \quad \omega_0 = \omega_0/(\gamma_B B_1) \]

where \( f(t) \) is a dimensionless function specifying the time dependence of the magnetic field, \( \omega_0 \) is the dimensionless Josephson frequency (scaled with the frequency associated with the magnetic field \( B_1 \)), and \( B_1 \) is the amplitude of the external magnetic field. In what follows we shall seek perturbative solution for \( \vec{m} \) in the weak coupling and weak anisotropy limit (for which \( \epsilon_0, K M_0/B_1 \ll 1 \) and \( k_0 \leq 1 \) to first order in \( \epsilon_0 \) and \( K \). In terms of the scaled variables, the LL equations (Eq. [5]) can be written in terms of \( \theta \) and \( \phi \) as

\[ \frac{d\phi}{d\tau} = \omega_B(\tau) - \epsilon_0 \cot \theta \sin \phi \sin(\gamma(\tau) - k_0 \sin \theta \sin \phi) \]

\[ \frac{d\theta}{d\tau} = \epsilon_0 \cos \phi \sin(\gamma(\tau) - k_0 \sin \theta \sin \phi). \]

with the initial condition \( \phi(\tau = 0) = 0 \) and \( \theta(\tau = 0) = \theta_0 \). We note that the choice of this initial condition for \( \theta \) and \( \phi \) amounts to choosing the initial magnetization of the nanomagnet in the \( x-y \) plane: \( \vec{M} = (M_1, M_2, 0) \) where \( \cos \theta_0 = M_2/M_0 \), and \( M_1^2 + M_2^2 = M_0^2 \). We choose \( \theta_0 \) such that \( \cot \theta_0 < 1 \) and the perturbative solutions that we present remains valid as long as \( \epsilon_0 \cot(\theta_0) \ll 1 \). We have checked that this limit is satisfied in all our numerical simulations described in Sec. [III].

The perturbative solutions of Eq. [7] can be obtained by writing

\[ \theta(\tau) = \delta \theta(\tau), \quad \phi(\tau) = z(\tau) + \delta \phi(\tau) \]

\[ z(\tau) = K \cos(\theta_0) M_0 \tau / B_1 + \int_0^\tau d\tau f(\tau) \]

where \( \delta \theta(\tau) \) and \( \delta \phi(\tau) \) satisfies, to first order in \( \epsilon_0 \) and
\[ K \text{ [i.e., neglecting terms } O(\epsilon_0^2), O(K\epsilon_0) \text{ and } O(k_0\epsilon_0)]}, \]
\[
\frac{d\delta \phi}{d\tau} = -\epsilon_0 \cot(\theta_0) \sin(z(\tau)) \\
\times \sin(\gamma_0(\tau) - k_0 \sin(\theta_0) \sin(z(\tau))) \tag{9}
\]
\[
\frac{d\delta \theta}{d\tau} = \epsilon_0 \cos(z(\tau)) \sin(\gamma_0(\tau) - k_0 \sin(\theta_0) \sin(z(\tau))).
\]

The solution of Eq. 9 is straightforward and can be written as
\[
\delta \theta(\tau) = \epsilon_0 \int_0^\tau \cos(z(t')) \sin[\gamma_0(t')] dt' \\
- k_0 \sin(\theta_0) \sin(z(t'))] \]
\[
\delta \phi(\tau) = -\epsilon_0 \cot \theta_0 \int_0^\tau \sin(z(t')) \sin[\gamma_0(t')] dt' \\
- k_0 \sin(\theta_0) \sin(z(t'))] \tag{10}
\]

Eqs. 8, 9, and 10 constitute the central result of this work. These equations describe the dynamics of a nanomagnet in the presence of weak coupling with a JJ. We note that in obtaining these results, we have neglected the normal state resistance of the JJ which can be safely done since they allow for Shapiro-like features in the magnet field at all times; we shall discuss this domain in the context of specific drives in Sec. III. We now use these solutions to study the behavior of the supercurrent of the JJ given by
\[
I_s = I_c \sin[\gamma_0(\tau) - k_0 \sin(\phi(\tau)) \sin(\theta(\tau))] \tag{11}
\]
for several possible magnetic field profiles. Here \(I_c\) is the critical current of the JJ. Although Eq. 11 yields \(I_s\) for any magnetic field profile, in what follows we shall concentrate on constant and periodically varying magnetic fields since they allow for Shapiro-step like features in the I-V characteristics of a voltage biased JJ.

Before ending this subsection, we note that the solutions for \(\dot{M}\) is stable against small fluctuations of the direction of the applied magnetic field. To see this, we write the external magnetic field \(\vec{B}\) is applied in an arbitrary direction in the \(x-y\) plane: \(\vec{B} = B_1 f(t) (\sin(\alpha_0), \cos(\epsilon_0), 0)\) with \(K\alpha_0 \ll K\). Next, we move to a rotated coordinate frame for which the magnetization \(\vec{m}'\) is related to \(\vec{m}\) by
\[
\begin{pmatrix}
m'_x \\
m'_y \\
m'_z
\end{pmatrix} = \begin{pmatrix}
\cos \alpha_0 & -\sin \alpha_0 & 0 \\
\sin \alpha_0 & \cos \alpha_0 & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
m_x \\
m_y \\
m_z
\end{pmatrix} \tag{12}
\]
We proceed by using the parametrization \(\vec{m}' = (\sin \theta' \cos \phi', \cos \theta', \sin \theta' \sin \phi')\). In this representation, the initial values of \(\vec{m}'\) are given by
\[
m'_x = \sin(\theta_0 - \alpha_0), \ n'_y = \cos(\theta_0 - \alpha_0), \ n'_z = 0 \tag{13}
\]
where \(\theta_0\) and \(\phi_0 = 0\) depicts the initial condition for \(\vec{m}\).

Next, repeating the same algebraic steps as outlined earlier in the section, one finds that the equations governing \(\theta'\) and \(\phi'\) are given by
\[
\frac{d\theta'}{d\tau} = \epsilon_0 \cos(\phi') \sin(\gamma_0(\tau) - k \sin(\theta') \sin(\phi')) \tag{14}
\]
\[
\frac{d\phi'}{d\tau} = \omega B(\tau) - \epsilon_0 \cot(\theta') \sin(\phi') \\
\times \sin(\gamma_0(\tau) - k_0 \sin(\theta') \sin(\phi')) \tag{15}
\]
\[
\omega_B(\tau) = K(\cos(\alpha_0) \cos(\phi') + \sin(\alpha_0) \sin(\phi') \sin(\phi''))/B_1 + f(\tau) \approx \omega B(\tau) + O(K\alpha_0)
\]

Note that the analytic solution to Eq. 15 can only be obtained when terms \(O(K\alpha_0)\) can be neglected. In this case, the perturbative solution to Eq. 15 can be obtained in the same way as done before in this section. The result is
\[
\theta'(\tau) = \delta \theta'(\tau), \quad \phi'(\tau) = z(\tau) + \delta \phi'(\tau)
\]
\[
\frac{d\delta \theta}{d\tau} = \epsilon_0 \int_0^\tau \cos(z(t')) \sin[\gamma_0(t')] dt' \\
- k_0 \sin(\theta_0 - \alpha_0) \sin(z(t'))] \]
\[
\frac{d\delta \phi}{d\tau} = -\epsilon_0 \cot \theta_0 \int_0^\tau \sin(z(t')) \sin[\gamma_0(t')] dt' \\
- k_0 \sin(\theta_0 - \alpha_0) \sin(z(t'))] \tag{16}
\]

The behavior of these solutions shall be checked against exact numeric in Sec. III.

B. Constant and Periodically varying magnetic fields

In this section, we apply our perturbative results on constant and periodically time-varying magnetic fields for which the I-V characteristics of the JJ may have Shapiro-like steps. While this effect has been discussed, using a somewhat different geometry, in Ref. 27 for constant magnetic field, we demonstrate its presence for periodic magnetic fields.

Constant magnetic field: This case was studied in Ref. 27. For an external constant voltage, \(g(t) = 1\) and one has \(\gamma_0 = \omega_0 \tau + \gamma_{00}\), where \(\gamma_{00}\) is the intrinsic phase difference across the JJ at \(t = 0\). Further, in this case, \(f(t) = 1\), and \(z(\tau) = \omega_c \tau\) where \(\omega_c = 1 + KM_2/B_1\). Thus the supercurrent to the leading order and for \(\epsilon_c, K \ll 1\), is given by
\[
I_s \simeq I_c \sin(\omega_0 \tau + \gamma_{00} - k_0 \sin(\theta_0) \sin(\omega_c \tau)) \\
= I_c \sum_n J_n [k_0 \sin(\theta_0)] \sin[(\omega_0 - n\omega_c) \tau + \gamma_{00}] \tag{17}
\]
which indicates the presence of a finite DC component of \(I_s\) leading to Shapiro steps in the I-V characteristics of the JJ-nanomagnet system at
\[
\omega_0 = n_0 \omega_c. \tag{18}
\]
To study the stability of these steps we consider the solution to \(O(\epsilon_0)\). For constant magnetic field, the \(O(\epsilon_0)\)
The behavior of the DC component of $I_s$ in the presence of these corrections is charted out in Sec. III. The supercurrent to first order in $\epsilon_0$ and $K$ is thus given by

$$I_s \simeq I_c \sin(\omega_0 \tau - k_0 \sin(\theta_0 + \delta \theta(\tau)) \sin(\omega_c \tau + \delta \phi(\tau)))$$

(20)

The behavior of the DC component of $I_s$ in the presence of these corrections is charted out in Sec. III.

Periodic Magnetic fields: In this case, we choose a periodic magnetic field so that $f(\tau) = \cos(\omega_1 \tau)$, where $\omega_1$ is the external drive frequency measured in units of $\gamma_0 B_1^{18}$. For this choice, one has $z(\tau) = \omega_2 \tau + \sin(\omega_1 \tau)/\omega_1$, where $\omega_2 = \gamma_0 K M_2/B_1$. Thus the zeroth order solution for the

$$\delta \theta_p = -\frac{\epsilon_0}{2} \sum_{n_1,n_2,n_3} J_{n_1}(1/\omega_1) J_{n_2}(k_0 \sin(\theta_0)) J_{n_3}(n_2/\omega_1) \sum_{s=\pm 1} \frac{\cos(\gamma_0 + (n_3 + s n_1 - (n_2 + s) \omega_2) \tau) - \cos(\gamma_0)}{\omega_0 - (n_3 + s n_1 - (n_2 + s) \omega_2)}$$

(24)

$$\delta \phi_p = \frac{\epsilon_0}{2} \cot(\theta_0) \sum_{n_1,n_2,n_3} J_{n_1}(1/\omega_1) J_{n_2}(k_0 \sin(\theta_0)) J_{n_3}(n_2/\omega_1) \sum_{s=\pm 1} \frac{\sin(\gamma_0 + (n_3 + s n_1 - (n_2 + s) \omega_2) \tau) - \sin(\gamma_0)}{\omega_0 - (n_3 + s n_1 - (n_2 + s) \omega_2)}$$

(25)

for $s = \pm 1$. The perturbation theory thus remain valid for $\tau \leq T_p$ so that

$$\epsilon_0 T_p J_{n_1}(1/\omega_1) J_{n_2}(k_0 \sin(\theta_0)) J_{n_3}(n_2/\omega_1) \leq 1.$$
using Eqs. 8 and 11 one obtains, to zeroth order in $\epsilon_0$

$$I_s \simeq I_c \sin \left[ \omega_0 \tau + A \sin(\omega_A \tau) / \omega_A \right]$$

$$- k_0 \sin(\theta_0) \sin \omega_c \tau + \gamma_{00} \right]$$

$$\simeq I_c \sum_{n_1,n_2} J_{n_1}(A / \omega_A) J_{n_2}(k_0 \sin(\theta_0))$$

$$\times \sin(\gamma_{00} + (\omega_0 + n_1 \omega_A - n_2 \omega_c) \tau \right)$$

Thus the presence of the steps now occurs for a set of integers $(n_1^0, n_2^0, n_3^0)$ which satisfies

$$\omega_0 + n_1^0 \omega_A - n_2^0 \omega_c = 0.$$ (28)

The condition of occurrence of these peaks mimics those for periodic magnetic field in the absence of external AC drive and the peak amplitude depends on the product of two Bessel functions. We note that the resultant Shapiro steps may occur for low AC drive frequencies and thus could, in principle, be amenable to easier experimental realization.

Next, we consider a periodically varying magnetic field in the presence of external radiation. In this case, one has, $f(\tau) = \omega_2 + \cos(\omega_1 \tau)$. Using Eq. 8 one has $z(\tau) = \omega_2 \tau + \sin(\omega_1 \tau) / \omega_1$ which leads to (Eq. 11)

$$I_s \simeq I_c \sin \left[ \omega_0 \tau + A \sin(\omega_A \tau) / \omega_A - k_0 \sin(\theta_0) \times$$

$$\sin(\omega_2 \tau + \sin(\omega_1 \tau)) / \omega_1 + \gamma_{00} \right]$$

$$\simeq I_c \sum_{n_1,n_2,n_3} J_{n_1}(A / \omega_A) J_{n_2}(k_0 \sin(\theta_0)) J_{n_3}(n_2 / \omega_2)$$

$$\times \sin(\gamma_{00} + (\omega_0 + n_1 \omega_A - n_2 \omega_c - n_3 \omega_1) \tau \right).$$ (29)

Thus the presence of the steps now occurs for a set of integers $(n_1^0, n_2^0, n_3^0)$ which satisfies

$$\omega_0 + n_1^0 \omega_A - n_2^0 \omega_c - n_3^0 \omega_1 = 0.$$ (30)

The perturbative $O(\epsilon_0)$ corrections to the above solutions can be carried out in similar manner to that outlined above.

C. Dissipative nanomagnets

In this section, we include the dissipative (Gilbert) term in the LLG equations to model dissipative nanomagnets and seek a solution to these equations in the limit weak dissipation and weak coupling between the JJ and the nanomagnets. The resultant LLG equations are given by

$$\frac{d\vec{M}}{dt} = \gamma_{\parallel} (\dot{\vec{M}} \times \vec{B}_{\text{eff}}) - \eta \frac{\eta}{M_0} \vec{M} \times \dot{\vec{M}}$$ (31)

where $\eta$ is a dimensionless constant specifying the strength of the dissipative term. Following the same parametrization as in Eqs. 6 we find that one can express the LLG equations, in terms of $\theta$ and $\phi$ as

$$\frac{d\phi}{d\tau} = \omega_B(\tau) - \epsilon_0 \cot \theta \sin(\gamma_0(\tau) - k_0 \sin(\theta_0) \sin(\phi))$$

$$\frac{d\theta}{d\tau} = \epsilon_0 \cos(\phi) \sin(\gamma_0(\tau) - k_0 \sin(\theta(\tau) \sin(\phi))$$

$$- \eta \omega_B(\tau) \sin(\theta).$$ (32)

where we have neglected terms $O(\epsilon_0 \eta)$ and $O(\eta^2)$. We note that the effect of the dissipative term manifests itself in $\theta$ but not in $\phi$; this fact can be understood as a consequence of the fact that to leading order $\dot{M} \times (\dot{M} \times B_{\text{eff}})$ lies along $\hat{y}$ and hence only effects the dynamics of $M_y$ which depend only on $\theta$. For small $\epsilon_0$ and $\eta$, Eq. 32 therefore admits a perturbative solution

$$\phi(\tau) = z(\tau) + \delta \phi(\tau), \quad \theta(\tau) = \delta \theta(\tau)$$

$$\delta \theta(\tau) = 2 \arctan[\tan(\theta_0 / 2) e^{-n z(\tau)}] + \delta \theta(\tau).$$ (33)

where $z(\tau), \delta \theta(\tau), \text{and} \delta \phi(\tau)$ are given by Eq. 8 and we have neglected terms $O(\epsilon_0 \eta)$. The supercurrent, in the presence of the dissipative term is given by

$$I_s = I_c \sin[\gamma_0(\tau) - k_0 \sin(\theta_0) \sin(\phi)]$$ (34)

The fate of the DC component of $I_s$ leading to Shapiro-like steps in the presence of the dissipative term shall be checked numerically in Sec. III

III. NUMERICAL RESULTS

In this section, we analyze the coupled JJ-nanomagnet system both without and in the presence of dissipation and compare these results, wherever applicable, to the theoretical results obtained in Sec. II A. In what follows, we focus on cases of constant or periodically varying magnetic fields since Shapiro-step like features are expected to appear in the I-V characteristics of the JJ only for these protocols. The LLG equations for magnetization solved numerically to generate the data for the plots are given by

$$\frac{dm_x}{d\tau} = [-\beta_1(1 + \eta^2 m_y^2) - \beta_2 \eta(m_z + \eta m_x m_y)]$$

$$+ \beta_3 \eta(m_y - \eta m_x m_y)] / (1 + \eta^2)$$

$$\frac{dm_y}{d\tau} = [-\beta_2(1 + \eta^2 m_y^2) - \beta_1 \eta(m_x + \eta m_z m_y)]$$

$$+ \beta_3 \eta(m_z - \eta m_x m_y)] / (1 + \eta^2)$$

$$\frac{dm_z}{d\tau} = [-\beta_3(1 + \eta^2 m_z^2) - \beta_1 \eta(m_y + \eta m_x m_z)]$$

$$+ \beta_2 \eta(m_x - \eta m_y m_x)] / (1 + \eta^2)$$ (35)

where

$$\beta_1 = -\left(f(\tau) \cos \alpha_0 + K' m_y m_z \right)$$

$$+ \epsilon_0 \eta m_y \sin(\gamma(\tau) - k_0 m_z)$$ (36)

$$\beta_2 = -\epsilon_0 m_x \sin(\gamma(\tau) - k_0 m_z) + \sin(\alpha_0) f(\tau) m_z$$

$$\beta_3 = (f(\tau) \cos(\alpha_0) + K' m_y m_x - \sin(\alpha_0) f(\tau) m_y$$
In these equations \( f(\tau) = 1 \) for constant and \( f(\tau) = \cos(\omega(\tau)) \) for the periodically varying magnetic fields, \( \alpha_0 = 0 \) indicates an applied magnetic field along \( \hat{y} \), we have set \( \theta_0 = \pi/3 \) and \( \gamma_0 = \pi/2 \) for all simulations, and \( K' = K_0/M_0/B_0 \). Note that Eq. 35 reduces to the usual LL equations for \( \eta = 0 \). The supercurrent is then computed using the values of \( m_\pm \) obtained from Eq. 33:

\[
I_s = I_c \sin(\gamma_0(\tau) - k_0 m_z).
\]

**FIG. 2:** (a) Comparison between theoretical (red dots) and numerical (black solid line) values of \( I_s(\tau)/I_c \) as a function of time at late times \( \tau \gg 2 \times 10^4 \) for a constant magnetic field \( \omega_B = 0.5 \) along \( \hat{y} \). Other parameters are \( \epsilon_0 = 10^{-4} \), \( \eta = 0 \), \( k_0 = 0.05 \), \( K = 0.0001 \) and \( \omega_0 = 0.5 \). (b) Plot of time \( T' \) after which the theoretical and analytic results for \( I_s(t) \) deviates by more than 1% at the peak position as a function of \( \epsilon_0 \).

To compare the theoretical results with exact numerics, we first compare the values \( I_s(\tau)/I_c \) computed theoretically (Eq. 20) with exact numerical result. For comparing the two results, we have fixed the external voltage \( \omega_0 = \omega_B \) which leads to a Shapiro step in the IV characteristics of the JJ with \( n^0 = 1 \). As discussed in Sec. II B, one expects one of the perturbative terms to grow linearly in time in this case; the presence of this linear term is expected to invalidate the perturbative theoretical results for \( \tau > T' \sim \epsilon_0^{-1} \). In Fig. 2(a), we show the comparison between theoretical and numerical values of \( I_s(\tau)/I_s \) at \( \tau > 2 \times 10^4 \) for \( \epsilon_0 = 10^{-4} \); we find that the numerical and analytical results differ by less than 5% even at late times (\( t \approx 2T' \)). In Fig. 2(b), we plot \( T' \), which is the minimum time at which the deviation between theoretical and numerical values of \( I_s(\tau)/I_c \) reaches 1% near the peak position, as a function of \( \epsilon_0 \); the result shows the expected decrease of \( T' \sim 1/\epsilon_0 \) as \( \epsilon_0 \) increases. In Fig. 3(a), we carry out a similar comparison for dissipative nanomagnets with \( \eta = 0.0001 \); we find that \( T' \) decreases with \( \epsilon_0 \) in a qualitatively similar manner to the non-dissipative case. However, we note that the value of \( T' \) with finite \( \eta \) (Fig. 3(a)) is larger than its \( \eta = 0 \) counterpart (Fig. 2(b)); this feature is a consequence of opposite signs of the correction terms due to \( \epsilon_0 \) (Eq. 19) and \( \eta \) (Eq. 33). For small \( \eta \) and \( \epsilon_0 \), these corrections tend to mutually cancel leading to a better stability of the zeroth order result which results in higher values of \( T' \). Finally, in Fig. 3(b), we plot \( T'_p \) for a periodically varying magnetic field with \( \omega_1 = 1.2 \). As expected from Eq. 20, \( T'_p \sim 100\epsilon_0^{-1} \gg \epsilon_0^{-1} \) which implies much better stability for the Shapiro-like steps for periodic magnetic field compared to their constant field counterparts.

Next, we study the presence of a finite DC component of \( I_s \) in the case of a constant applied magnetic field along \( \hat{y} \) (\( f(\tau) = 1 \) and \( \alpha_0 = 0 \)) in the absence of dissipation (\( \eta = 0 \)) and external AC voltage (\( \gamma_0(\tau) = \omega_B \tau \)). The results of our study is shown in Fig. 4 where we plot \( I_s^{DC}/I_c \), with \( I_s^{DC} \) given by

\[
I_s^{DC} = \frac{1}{T_{\text{max}}} \int_0^{T_{\text{max}}} I_s(t')dt' = I_s(\omega = 0),
\]

as a function of \( \omega_0 \) for a fixed constant \( \omega_B \). Here \( T_{\text{max}} = 40,000 \) represents the maximum time up to which we average \( I_s(\tau) \). Note that \( I_s(\tau) \) is chosen so that increasing it any further does not lead to a change in the peak height for \( \epsilon_0 = 0 \). As shown in Fig. 4(a), (b) and (c), we find that for \( \epsilon_0 \ll 1 \), \( I_s^{DC} \) shows sharp peaks at \( \omega_0 = \omega_B \), \( 2\omega_B \) corresponding to \( n^0 = 1, 2 \) in Eq. 18 the position of this peaks match exactly with our theoretical results. However, the peak heights turn out to be smaller than that predicted by theory and they rapidly decrease with increasing \( \epsilon_0 \). This mismatch between theoretical and numerical results is a consequence of the linearly growing perturbative terms \( \sim \epsilon_0 \) in expression for \( \delta\theta(\tau) \) and \( \delta\phi(\tau) \) (Eq. 19) which invalidate the theoretical result for \( T' \sim \epsilon_0^{-1} \). Thus for constant magnetic field and moderate \( \epsilon_0 > 0.01 \), the step-like feature predicted in Eq. 18 disappears. In Fig. 4(d), we study the behavior of the peak with variation of \( \alpha_0 \). We find that the height of the peak increases with \( \alpha_0 \) for small \( \alpha_0 \) in accordance with the theoretical prediction of Sec. II A. For larger \( \alpha_0 > \alpha_0^{\text{max}} \), the peak height starts to decrease and the peak height becomes almost half of its maximum for \( \alpha_0 = \pi/2 \) when \( \dot{\theta} = \dot{\phi} \).

Next, we study the characteristics of the peaks in \( I_s^{DC} \) for periodically varying magnetic field for which \( f(\tau) = \cos(\omega(\tau)) \). In Figs. 5(a), (b) and (c), we plot \( I_s^{DC}/I_c \) as a function of \( \omega_1 \) for a fixed \( \omega_0 = 1.2 \), \( \alpha_0 = \omega_2 = \eta = 0 \), and for several values of \( \epsilon_0 \). We find that the position of the peaks corresponds to integer values of \( n_2 \) (as indicated in the caption of Fig. 5) in complete accordance with Eq. 22 with \( \omega_2 = 0 \). Moreover, in contrast to the constant magnetic field case, the peaks of \( I_s^{DC} \) are much more stable against increasing \( \epsilon_0 \). This features of the peaks can be understood as follows. For periodic
magnetic field with $\omega_2 = 0$, the zeroth order solution is given by $z(\tau) = \sin(\omega_1 \tau)/\omega_1$; thus the perturbative terms $\delta \theta(\tau)$ and $\delta \phi(\tau)$ (Eq. [2] involve product of Bessel functions. This renders the effective perturbative parameter to be $\epsilon_0^{\text{eff}} \approx \epsilon_0 J_n(1/s)J_n(k_0 \sin(\theta_0)), J_n(\omega_0/\omega_1)$ (Eqs. [25] and [26]). Consequently, the effect of the perturbative correction to the weak coupling solution is drastically reduced in this case leading to a better stability of peak height with increasing $\epsilon_0$. Thus periodic magnetic fields are expected to lead to enhanced stability of Shapiro steps compared to their constant field counterparts. Finally in Fig. 5(d), we show the variation of the peak height of $I_s^{\text{DC}}/I_c$ as a function of $\alpha_0$. We again find similar non-monotonic behavior of the peak height as a function of $\alpha_0$; the reason for this is similar to that already discussed in the context of constant magnetic field case. However, in the present case, the correction terms are much smaller and the peak height is accurately predicted by the zeroth order perturbative results: $I_s^{\text{DC}}/I_c \sim 2J_2(k_0 \sin(\theta_0 - \alpha_0))J_2(\omega_0/\omega_1)$. This is most easily checked by noting that the peak height vanishes for $\alpha_0 = \theta_0 = \pi/3$ for which $J_2(\omega_0/\omega_1) = \delta_{n,0}$ leading to vanishing of the peak for $n = 1$.

Next, we study the behavior of the system in the presence of an applied AC field of amplitude $A$ and frequency $\omega_A$. In the presence of such a field $\omega(\tau) = \omega_0 + A \sin(\omega_A \tau)/\omega_A$. In Fig. 6(a), we show the behavior of the peaks of $I_s^{\text{DC}}$ as a function of $\omega_0$ for a fixed $\omega_A = 0.2$ and $A = 0.1$ in the presence of a constant magnetic field. The peaks in $I_s^{\text{DC}}$ occur at $\omega_0 = 0.4, 0.6, 0.8, 1$ (from left to right); each of these correspond to two sets of $(n_0^0, n_0^0) = (3, 1)$ and $(-2, 0), (2, 1)$ and $(-3, 0), (1, 1)$ and $(-4, 0)$ and $(0, 1)$ and $(-5, 0)$ respectively as predicted in Eq. [28]. In Fig. 6(b), we investigate the behavior for $I_s^{\text{DC}}$ for a periodically varying magnetic field as a function of $\omega_1$ for $\omega_0 = 1.2$ and for same amplitude and frequency of the AC field. We find several peaks in $I_s^{\text{DC}}$; each of these peaks corresponds to a fixed set of integers $(n_0^0, n_0^0)$ (Eq. [30] with $\omega_2 = 0$) as shown in Table I.

Next, we study the effect of dissipation on these peaks by plotting $I_s^{\text{DC}}$ as a function of $\omega_0$ in Fig. 7(a) (for constant magnetic field) and as a function of $\omega_1$ in Fig. 7(a) (periodic magnetic field) for $\eta = 0.0001$. As seen in both cases, the position of the peaks remain same as that for $\eta = 0$ in accordance with the analysis of Sec. II C. The variation of the peak height as a function of $\log \epsilon_0$ and

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**FIG. 4:** Plot $I_s(\omega_0)/I_c \equiv I_s^{\text{DC}}/I_c$ as a function of the Josephson frequency $\omega_0 = 2eV_0/(\hbar \gamma B_1)$ for a constant magnetic field $\omega_B \approx 1$ with $K = 0.0001$, $k_0 = 0.1$ and (a) $\epsilon_0 = 0.0001$ (b) $\epsilon_0 = 0.001$ and (c) $\epsilon_0 = 0.01$. The position of the peaks corresponds to $n^0 = 1$ and $n^0 = 2$ as predicted by theoretical analysis. (d) Plot of the peak height for the $n^0 = 1$ peak as a function of the angle $\alpha_0$ made by $B$ with $\hat{y}$ for $\epsilon_0 = 10K = 0.0001$ and $k_0 = 0.1$. The red dots correspond to results from perturbative theoretical analysis near $\alpha_0 = 0$.

**FIG. 5:** Plot $I_s^{\text{DC}}/I_c$ for a periodically varying magnetic field $B = B_1 \sin(\omega_1 \tau)$ as a function of $\omega_0$ with $K = 0.0001$, $k_0 = 0.1$, $\omega_0 = 1.2$ and (a) $\epsilon_0 = 0.0001$ (b) $\epsilon_0 = 0.001$ and (c) $\epsilon_0 = 0.01$. The position of the peaks corresponds to $n^0 = 1, 2, 3, 4$ (from right to left) as predicted by theoretical analysis. (d) Plot of the peak height for the $n^0 = 1$ peak as a function of the angle $\alpha_0$ made by $B$ with $\hat{y}$ for $\epsilon_0 = K = 0.0001$ and $k_0 = 0.1$.

**FIG. 6:** (a) Plot of $I_s^{\text{DC}}/I_c$ in the presence of an AC field $\omega(\tau) = \omega_0 + A \sin(\omega_A \tau)/\omega_A$ with $A = 0.1$, $\omega_A = 0.2$, $k_0 = 0.1$ as a function of $\omega_0$ for constant magnetic field $\omega_B = 1$. (b) Similar plot as a function of $\omega_1$ for periodic magnetic field with $\omega_0 = 1.2$. All the peak positions conform to the theoretical prediction in Sec. II B.
| $\omega_1$ | $n^0_1$ | $n^0_3$ | $\omega_1$ | $n^0_1$ | $n^0_3$ |
|---|---|---|---|---|---|
| 1.2 | 0 | 1 | 0.3 | 0 | 4 |
| 1 | -1 | 1 | 0.28 | 1 | 5 |
| 0.8 | 2 | 2 | 0.24 | 0 | 5 |
| 0.6 | 0 | 2 | 0.2 | -3 | 3 |
| 0.4 | -2 | 2 | 0.2 | 0 | 6 |
| 0.4 | 0 | 3 | 0.08 | 0 | 15 |

TABLE I: Tabulated values of $n^0_1$ and $n^0_3$ for all the peaks that appear in Fig. 7(b) at specific $\omega_1$ values listed above. Note that $n^0_2$ does not appear in the table since the peaks correspond to $\omega_B = 0$ so that their position are independent of $n^0_2$ (Eq. 30).

FIG. 7: (a)Plot of $I^{DC}/I_s$ for a constant applied magnetic field as a function of $\omega_1$ with $\eta = 0.0001$. All other parameters are same in Fig. 3 a. (b) Variation of the peak height (for $n^0 = 1$) as a function of log $\epsilon_0$ and log $\eta$ showing the presence of a line in the $\epsilon_0 - \eta$ plane for which the peak height is maximal.

log $\eta$ is shown in Figs. 7(b) for a constant magnetic field. We find that the maximal peak-height occur along a line in the $\epsilon_0 - \eta$ plane. This can be seen more clearly by plotting $I^{DC}/I_s$ as a function of $\eta$ for a fixed $\epsilon_0$ as shown in Fig. 8(a); the figure displays a clear peak in $I^{DC}$ at $\epsilon_0 \approx \eta$. This can be understood from Eq. 33 and 19 as follows. For the constant magnetic field, $z(\tau) = \omega_c \tau$; consequently for small $\eta$, the correction to the zeroth order solution from the dissipative term varies linearly with $\eta$ (Eq. 33)

$$\delta \theta_d(\tau) \simeq \delta \theta(\tau) + \theta_0 + \sin(2\theta_0) \eta \omega \tau + ...$$

(38)

where the ellipsis indicate higher order terms in $\eta$. This correction has opposite sign to the $\tau$-linear correction terms (terms corresponding to $n = n^0 \mp 1$ in Eq. 19) arising due to a finite $\epsilon_0$ in $\delta \theta(\tau)$. The corrections from $\eta$ and $\epsilon_0$ with opposite signs cancel along some specific line $\epsilon_0 - \eta$ plane leading to enhanced better stability of the zeroth order solution and hence enhanced peak height. We note that the angle of this line depends on details of the relative magnitude of the correction terms. Thus we find that the presence of dissipation in a nanomagnet may lead to enhancement of the Shapiro-like steps for constant magnetic fields.

In contrast, as shown in 8(b), the peak height is almost independent of $\eta$ for small $\eta$ for periodically varying magnetic field. This can also be clearly seen from Fig. 9(b) where $I^{DC}$ is shown to be independent of $\eta$ for small $\eta$ at fixed $\epsilon_0$. For such fields, $z(\tau) = \omega_2 \tau + \sin(\omega_1 \tau)/\omega_1$, where $\omega_2 = \gamma g K M_2 / B_1 \ll \omega_1$ for our choice of parameters. In this case, one can write, for small $\eta$

$$\delta \theta_d(\tau) \simeq \delta \theta(\tau) + \theta_0 + \sin(2\theta_0) \eta \sin(\omega_1 \tau)/\omega_1 + ...$$

(39)

where the ellipsis indicate higher order term in $\eta$. Thus the correction term is bounded and provides an oscillatory contribution to $\theta(\tau)$. For small $\eta$, it is insignificant compared to the correction term from $\epsilon_0$ and hence the peak height stays almost independent of $\eta$. Thus we find that the role of dissipation is minimal for small $\eta$ in case of periodically varying magnetic fields. The oscillatory variation of the peak height as a function of $\epsilon_0$ for a fixed $\eta$ can be traced to its dependence on product of three Bessel functions as can be seen from Eq. 24.

Finally, we briefly study the effect of increasing $T_{\text{max}}$ in our numerical study. The relevance of this lies in the fact that for any finite $\epsilon_0$ and $\eta$, our analytical results hold till $\tau \sim T'$ (constant magnetic field) or $\tau \sim T_p'$ (periodic magnetic field) while the DC signal receives contribution from all $\tau$. Thus it is necessary to ensure that these deviations do not lead to qualitatively different results for the DC response. To this end, we plot the height of the peak value of $I^{DC}$ as a function of $1/T_{\text{max}}$ in Fig. 10. We find from Fig. 10(a) that for constant magnetic field, the peak height indeed extrapolates to zero indicating that

FIG. 8: (a)Plot of $I^{DC}/I_s$ for a periodically varying magnetic field as a function of $\omega_1$ with $\eta = 0.0001$. All other parameters are same in Fig. 3 b). (b) Variation of the peak height (for $n^0 = 1$) as a function of log $\epsilon_0$ and log $\eta$ showing the region of maximal peak height.

FIG. 9: (a)Plot of $I^{DC}/I_s$ for a constant magnetic field $\omega_0 = 1.0$ as a function of $\epsilon_0$ with $\epsilon_0 = 0.0004$. (b) Similar plot for periodic magnetic field with $\omega_1 = 1.2$. All other parameters are same in Fig. 4.

FIG. 10: (a) The height of the peak value of $I^{DC}$ as a function of $1/T_{\text{max}}$ for a constant magnetic field. The peak height indeed extrapolates to zero indicating that
the Shapiro steps will be destabilized due to perturbative corrections if $I_s$ is averaged over very long time. However, we note from Fig. 10(c), $I_{s}^{DC}$ could retain a non-zero value in the presence of a finite dissipation parameter $\eta$. This could be understood since the effect of damping, as we note from Fig. 10(c), of the JJ to be much more stable for periodically varying magnetic fields. Thus we expect that the Shapiro-step like features in the I-V characteristics of the JJ to be much more stable for periodically varying magnetic fields.

Furthermore, from Figs. 10(b) and (d), we note that for the periodic magnetic fields the extrapolated value of $I_{s}^{DC}$ is a finite which is lead to finite Shapiro steps in the I-V characteristics of theses JJs. Thus we expect that the presence of such peaks in $I_{s}^{DC}$ has not been theoretically reported for periodically time-varying magnetic fields. Moreover, we show, both from our analytical results and by performing exact numerics which supports these results, that the peaks in $I_{s}^{DC}$ for periodically varying magnetic field are much more robust against increase of both $\epsilon_0$ and $\eta$ compared to their constant field counterparts; we therefore expect such peaks to be more experimentally accessible. We have also studied the behavior of such JJ-nanomagnet systems in the presence of external AC voltage. The presence of such a voltage leads to more peaks in $I_{s}^{DC}$ whose positions are accurately predicted by our theoretical analysis. We note that our analysis, which is carried out at zero temperature, is expected to be valid at low temperature where $k_BT \ll \Delta_0, g\mu_B B$ so that the presence of thermal noise can be neglected. However, we point out that the effect of such noise term in our formalism can be addressed by adding a (white) noise term in the Gilbert equations following standard procedure.

Our analysis could be easily extended to unconventional superconductors hosting Majorana end states. For these junctions, the current-voltage relation is $4\pi$ periodic and given by $I_s = I_c \sin(\gamma t)/2$. An analysis using this I-V relation immediately reveals that the Shapiro steps will be present for $\omega_0 = 2n\omega_c$ for constant magnetic field (Eq. 18) and $\omega_0 = 2(n_0\omega_1 + n_2\omega_2)$ for periodic magnetic fields (Eq. 22). The additional factor of 2 is a consequence of $4\pi$ periodicity mentioned above. Thus coupling such JJs with Majorana end modes to nanomagnets in the presence of a magnetic field may lead to new experimental signatures of such end modes.

The experimental verification our work would involve preparing a voltage biased JJ-nanomagnet system with sufficiently small values of $\epsilon_0$. The current in such a junction, assuming a resistive junction, is given by

$$I(t) = I_c \sin(\phi(t) - k_0 m_z) + V_0/R + \frac{h\omega_c}{2eR} dm_z/dt$$  \hspace{1cm} (40)$$

where $V_0$ is the bias voltage, $R$ is the resistance of the junction and $\phi(t) = 2eV_0 t/\hbar = \omega_0 t$. Thus the DC component of the current will show additional spikes when Eqs. 18 (constant magnetic field) or 22 (periodic magnetic field) is satisfied. We note that it is essential to have a voltage bias to observe these steps. This can be seen from the fact that for current-biased junctions, the phase $\phi(t)$ is not locked to a fixed value of $\omega_0 t$ but has to be obtained from the solution of

$$I = \frac{h}{2eR} \dot{\phi}(t) + I_c \sin[\phi(t) - k_0 m_z] - \frac{h\omega_c}{2eR} dm_z/dt$$ \hspace{1cm} (41)$$

where $I$ is the bias current. Thus $\phi$ becomes a function of $m_z$; we have checked both using perturbative analytic

![FIG. 10: Plot $I_{s}^{DC}/I_c$ as a function of $1/T_{max}$ for (a) constant magnetic field with $\epsilon_0 = 0.001$ and $\eta = 0$, (b) periodically varying magnetic field with $\epsilon_0 = 0.001$ and $\eta = 0$, (c) constant magnetic field with $\epsilon_0 = \eta = 0.001$ and (d) Periodically varying magnetic field with $\epsilon_0 = \eta = 0.001$. All other parameters are same as in Figs. 4 and 5.](image-url)
method[9] and exact numerics that in this case no steps exist. This situation is to be contrasted to the case of standard Shapiro steps induced via external radiation of amplitude $A$ where steps can be shown to exist for both current and voltage bias.[10] Thus the present system would require a voltage-biased junction for observation of Shapiro steps.

For experimental realization of such a system, this we envisage a 2D thin film superconducting junction in the $x-y$ plane coupled to the nanomagnet as shown in Fig. 1. We note that value of Josephson energy and superconducting gap in a typical niobium film are $E_J \sim 40K$ and $\Delta_0 \sim 3$ meV respectively. Thus for a typical magnetic field $\sim 100$ Gauss, for which $\gamma_B \Delta = 0.28GHz$, one could estimate an $e\gamma_B \sim 0.0005$ for $k_0 \sim 0.1$. The Larmor frequency $\omega_L$ associated with such magnetic field would be order of GHz while the spin-flip processes responsible for any change in the Josephson current in niobium junctions would require a voltage-bias of $0.1\mu V$ which is also well within present experimental capability. We also note here that such experiments should also be possible with 1D junctions which have been prepared experimentally in recent times using nanowires with spin-orbit coupling[9].

To conclude, we have provided a perturbative analytic results for supercurrent of a coupled JJ-nanomagnet system in the limit of weak coupling between them and in the presence of a time dependent field applied to the system. Using this analytic result and exact numerical solutions of the LL and the LLG equations, we predict existence of peaks in $I_{\text{DC}}$ for both constant and periodic magnetic fields which are expected to provide Shapiro-like steps in the I-V characteristics of a voltage-biased JJ without the presence of external AC drive. We have analyzed the effect of finite dissipation of the nanomagnet and the presence of external AC drive on these peaks and discussed experiments which can test our theory.

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