NORMAL FORMS FOR SUB-LORENTZIAN METRICS SUPPORTED ON ENGEL TYPE DISTRIBUTIONS.

MAREK GROCHOWSKI

Abstract. We construct normal forms for Lorentzian metrics on Engel distributions under the assumption that abnormal curves are timelike future directed Hamiltonian geodesics. Then we indicate some cases in which the abnormal timelike future directed curve initiating at the origin is geometrically optimal. We also give certain estimates for reachable sets from a point.

1. Introduction

1.1. Preliminaries. In the series of papers [8], [9], [11] we studied (germs of) contact sub-Lorentzian structures on $\mathbb{R}^3$. In turn, in the series [12], [13], [14] some classes of non-contact sub-Lorentzian structures on $\mathbb{R}^3$ were studied (in all cases the underlying distribution is of rank 2). The next reasonable step is to study sub-Lorentzian structures again supported by rank 2 distributions but on $\mathbb{R}^n$, $n \geq 4$. In this paper we begin studies in this direction, namely we examine the simplest such case, i.e. one supported by the so-called Engel distribution. Before giving precise definition we will first present basis notions and facts from the sub-Lorentzian geometry that will be needed to state the results.

For all details and proofs the reader is referred to [10] (and to other papers by the author; see also [15], [18]). Let $M$ be a smooth manifold, and let $H$ be a smooth distribution on $M$ of constant rank. For a point $q \in M$ and an integer $i$ let us define $H^i_q$ to be the linear subspace in $T_q M$ generated by all vectors of the form $[X_1, [X_2, ..., [X_{k-1}, X_k]...,]](q)$, where $X_1, ..., X_k$ are smooth (local) sections of $H$ defined near $q$, and $k \leq i$. We say that $H$ is bracket generating if for every $q \in M$ there exists a positive integer $i = i(q)$ such that $H^i_q = T_q M$. Now, by a sub-Lorentzian structure (or metric) on $M$ we mean a pair $(H, g)$ made up of a smooth bracket generating distribution $H$ of constant rank and a smooth Lorentzian metric on $H$. A triple $(M, H, g)$ is called a sub-Lorentzian manifold.

Up to the end of this subsection we fix a sub-Lorentzian manifold $(M, H, g)$. A vector $v \in H_q$ is called timelike if $g(v, v) < 0$, is called nonspacelike if $g(v, v) \leq 0$ and $v \neq 0$, is null if $g(v, v) = 0$ and $v \neq 0$, finally is spacelike if $g(v, v) > 0$ or $v = 0$. By a time orientation of $(H, g)$ we mean a continuous timelike vector field on $M$. Suppose that $X$ is a time orientation of $(M, H, g)$. Then a nonspacelike $v \in H_q$ is said to be future directed if $g(v, X(q)) < 0$, and is past directed if $g(v, X(q)) > 0$. An absolutely continuous curve $\gamma : [a, b] \longrightarrow M$ is called horizontal if $\gamma(t) \in H_{\gamma(t)}$ a.e. on $[a, b]$. A horizontal curve is nonspacelike (resp. timelike, null, nonspacelike future directed etc.) if so is $\dot{\gamma}(t)$ a.e.

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Below we will need a notion of Hamiltonian geodesics. Let $\mathcal{H} : T^*M \to \mathbb{R}$ be the so-called \textit{geodesic (or metric)} Hamiltonian associated with our structure $(H, g)$. A global definition of $\mathcal{H}$ is given for instance in [10]. Locally $\mathcal{H}$ looks as follows. Take an orthonormal basis $X_0, \ldots, X_k$ for $H$ defined on an open set $U \subset M$, where $X_0$ is timelike. Then the restriction of $\mathcal{H}$ to $T^*U$ is given by $\mathcal{H}(q, p) = -\frac{1}{2} (p, X_0(q))^2 + \frac{1}{2} \sum_{j=1}^k (p, X_j(q))^2$. Denote by $\tilde{\mathcal{H}}$ the Hamiltonian vector field corresponding to the function $\mathcal{H}$. A horizontal curve is called a \textit{Hamiltonian geodesic} if it can be represented in the form $\gamma(t) = \pi \circ \lambda(t)$, where $\lambda = \tilde{\mathcal{H}}$ and $\pi : T^*M \to M$ is the canonical projection. $\lambda(t)$ is called a Hamiltonian lift of $\gamma(t)$. It is immediate from the very definition that if $\gamma : [a, b] \to M$ is a Hamiltonian geodesic and $\dot{\gamma}(t_0)$ is a nonspacelike (resp. timelike, null, nonspacelike future directed etc.) vector, then so is $\dot{\gamma}(t)$ for every $t \in [a, b]$.

Before going further, it seems sensible to clarify why we use the word 'geodesic'. So, first of all, if $\gamma : [a, b] \to M$ is a nonspacelike curve then we define its \textit{sub-Lorentzian length} by formula

$$L(\gamma) = \int_a^b \sqrt{-g(\dot{\gamma}(t), \dot{\gamma}(t))} dt.$$ 

Next, for an open subset $U \subset M$ and any pair of points $q_1, q_2 \in U$, denote by $\Omega_{q_1, q_2}^{\text{nspc}}(U)$ the set of all nonspacelike future directed curves contained in $U$ and joining $q_1$ to $q_2$. Now we say that a nonspacelike future directed curve $\gamma : [a, b] \to U$ is a \textit{maximizing $U$-geodesic} or simply a $U$-\textit{maximizer} if

$$L(\gamma) = \max \left\{ L(\eta) : \eta \in \Omega_{\gamma(a), \gamma(b)}^{\text{nspc}}(U) \right\}.$$ 

By a $U$-\textit{geodesic} we mean a curve in $U$ whose every sufficiently small subarc is a $U$-maximizer (such an approach follows the ideas elaborated in the Lorentzian case - see e.g. [1], [19] or [16]). It turns out [10] that for every nonspacelike Hamiltonian geodesic $\gamma : [a, b] \to M$ and for every $t \in (a, b)$ there exists a neighbourhood $U$ of $\gamma(t)$ such that $U \cap \gamma$ is a $U$-maximizer. Note that in the Lorentzian (or Riemannian) geometry every geodesic is Hamiltonian. It is known that in the sub-Lorentzian (or sub-Riemannian) geometry there are maximizers (minimizers) that are not Hamiltonian geodesics - see e.g. [9] and remark 1.1 below for examples in the sub-Lorentzian case (and [20], [22] for the sub-Riemannian situation).

Denote by $\Phi_t$ the (local) flow of the field $\tilde{\mathcal{H}}$. For a fixed point $q_0 \in M$ let us define $\mathcal{D}_{q_0}$ to be the set of all $\lambda \in T^*_{q_0}M$ such that the curve $t \mapsto \Phi_t(\lambda)$ is defined on the whole interval $[0, 1]$. $\mathcal{D}_{q_0}$ is an open subset in $T^*M$. Now we define the \textit{exponential mapping} with the pole at $q_0$

$$\exp_{q_0} : \mathcal{D}_{q_0} \to M, \quad \exp_{q_0}(\lambda) = \pi \circ \Phi_1(\lambda).$$ 

Using properties of Hamiltonian equations it is easy to see that the Hamiltonian geodesic with initial conditions $(q_0, \lambda)$ can be written as $\gamma(t) = \exp_{q_0}(t\lambda)$. It can also be observed that if $\gamma(t)$ is a Hamiltonian geodesic with a Hamiltonian lift $\lambda(t) = \Phi_t(\lambda)$ then, from the definition of the geodesic Hamiltonian (see [10] for more details), it follows that for any $v \in H_{\gamma(t)}$ we have

$$g(\dot{\gamma}(t), v) = \langle \Phi_t(\lambda), v \rangle. \quad (1.1)$$
At the end let us recall the notion of abnormal curves (cf. e.g. [20]). So an absolutely continuous curve \( \lambda : [a, b] \rightarrow T^* M \) is called an abnormal biextremal if \( \lambda([a, b]) \subset H^+ \), \( \lambda \) never intersects the zero section, and moreover \( \Omega_{\lambda(t)}(\dot{\lambda}(t), \zeta) = 0 \) for almost every \( t \in [a, b] \) and every \( \zeta \in T_{\lambda(t)}H^+ \); here \( H^+ \) is the annihilator of \( H \), and \( \Omega \) denotes the restriction of \( H^+ \) of the standard symplectic form on \( T^* M \). A horizontal curve \( \gamma : [a, b] \rightarrow M \) is said to be abnormal if there exists an abnormal biextremal \( \lambda : [a, b] \rightarrow T^* M \) such that \( \gamma = \pi \circ \lambda \).

Throughout the paper we will use the following abbreviations: "t." for "time-like", "nspc." for "nonspacelike", and "f.d." for "future directed". Moreover, unless otherwise stated, we assume all curves and vectors to be horizontal. Thus e.g. a t.f.d. curve is a horizontal curve whose tangent is t.f.d. a.e.

1.2. **Statement of the results.** Let \( H \) be a rank 2 distribution of constant rank on a 4-dimensional manifold \( M \). We say that \( H \) is an Engel (or Engel type) distribution if \( H^2 \) is of constant rank 3, and \( H^3 \) is of constant rank 4, i.e. \( H^3 = TM \). The remarkable property of Engel distributions is the fact that they are topologically stable, see e.g. [17] (note that apart from Engel case, the only stable distributions are rank 1 distributions, and also contact and pseudo-contact distributions). On the other hand, if one slightly perturbs any given rank 2 distribution on a 4-manifold it becomes Engel on an open and dense subset. All this gives rise to the importance of Engel distributions. But Engel distributions are important also because of another reason, namely they appear in applications. For instance our flat case (see example below) serves as a model for a motion of a car with a single trailer (cf. e.g. [7]).

Using for instance [20] one makes sure that if \( H \) is an Engel distribution on \( M \) then through each point \( q \in M \) there passes exactly one unparameterized abnormal curve. Moreover the abnormal curves are all (at least locally) trajectories of a single smooth vector field.

Let \( H \) be an Engel type distribution and let \( g \) be a Lorentzian metric on \( H \). A couple \((H, g)\) is called an Engel sub-Lorentzian structure (or metric) if the abnormal curves for \( H \) are timelike. If moreover the abnormal curves are, possibly after reparameterization, t.f.d. Hamiltonian geodesics then \((H, g)\) will be called Engel sub-Lorentzian structure of Hamiltonian type.

**Example 1.1.** As a model example of an Engel sub-Lorentzian structure of Hamiltonian type we use the following one. Let \( H = \text{Span} \{X, Y\} \) with \( X = \frac{\partial}{\partial \sigma} + \frac{1}{2}y \frac{\partial}{\partial \tau} + \frac{1}{2}y^2 \frac{\partial}{\partial \nu}, Y = \frac{\partial}{\partial \nu} - \frac{1}{2}x \frac{\partial}{\partial \tau} - \frac{1}{2}xy \frac{\partial}{\partial \sigma} \). Clearly \( [X, Y] = -\frac{\partial}{\partial \tau} - \frac{3}{2}y \frac{\partial}{\partial \nu}, [X, [X, Y]] = 0, [Y, [X, Y]] = \frac{3}{2} \frac{\partial}{\partial \sigma} \), so indeed \( H^2 \) is of constant rank 3 while \( H^3 \) is of constant rank 4. The trajectories of \( X \) are the curves \( t \rightarrow (x_0 + t, y_0, z_0 + \frac{1}{2}y_0 t, w_0 + \frac{1}{2}y_0^2 t) \), and these curve are easily checked to be abnormal. Now we define a metric by declaring \( X, Y \) to be an orthonormal basis with a time orientation \( X \), and we make sure that a curve \( t \rightarrow (x_0 + t, y_0, z_0 + \frac{1}{2}y_0 t, w_0 + \frac{1}{2}y_0^2 t, -1, 0, 0, 0) \) represents a Hamiltonian lift for the corresponding trajectory of \( X \).

The structure just described will be called the flat Engel sub-Lorentzian structure. The reason for such a name is the same as in the previous papers by the author (see for instance [12], [13]) - any Engel structure of Hamiltonian type may be viewed as a perturbation of the flat structure.

**Remark 1.1.** It is easy to construct Engel sub-Lorentzian structures \((H, g)\) which are not of Hamiltonian type. The idea may be taken from Sussmann who gives in
a simple recipe how to contract Riemannian metrics on Engel distributions  
with respect to which abnormal curves are strictly abnormal, i.e. are not Hamiltonian  
geo desics. The construction goes without any changes in the case of Lorentzian  
metrics. So it is enough to find two fields \( V, W \) spanning \( H \) such that  
(i) \( V \wedge W \wedge [V, W] \wedge [W, [V, W]] \neq 0 \), (ii) \( [V, [V, W]] = fV + gW + h [V, W] \),  
with \( f, g, h \) being smooth functions such that \( f \) vanishes nowhere. Now define a Lorentzian  
metric on \( H \) by declaring \( V, W \) to be an orthonormal basis with time orientation \( V \). The abnormal curves  
for just defined structure \( (H, g) \) (which all are trajectories of \( V \)) are not Hamiltonian geodesics (For the convenience of the reader we present  
the argument. Suppose that a trajectory \( \gamma \) of \( V \) is a t.f.d. Hamiltonian geodesic. Let \( \lambda(t) \) be its  
Hamiltonian lift; then by (1.7) it follows that \( \langle \lambda(t), V \rangle = -1 \) and \( \langle \lambda(t), W \rangle = 0 \) for every \( t \). Now, the successive  
differentiations of the second equation give: \( \langle \lambda(t), [V, W] \rangle = 0 \), and \( 0 = \langle \lambda(t), [V, [V, W]] \rangle = f \langle \lambda(t), V \rangle + g \langle \lambda(t), W \rangle + h \langle \lambda(t), [V, W] \rangle = -f \) which is a contradiction with the assumption imposed on \( f \).

The main objective of this paper is to prove the following normal form theorem  
(cf. [8], [12], [13], and also [2]).

**Theorem 1.1.** Let \((H, g)\) be a smooth time-oriented Engel sub-Lorentzian structure of  
Hamiltonian type defined in a neighbourhood of a point \( q_0 \) on a 4-manifold. Then there are  
coordinates \( x, y, z, w \) around \( q_0 \), \( x(q_0) = = w(q_0) = 0 \), in which \((H, g)\) has an orthonormal frame in the normal form

\[
\begin{align*}
X &= \frac{\partial}{\partial x} + y \varphi \left( y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right) + \frac{1}{2} y (1 + \psi_1) \frac{\partial}{\partial z} + \frac{1}{2} y^2 (1 + \psi_2) \frac{\partial}{\partial w} \\
Y &= \frac{\partial}{\partial y} - x \varphi \left( y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right) - \frac{1}{2} x (1 + \psi_1) \frac{\partial}{\partial z} - \frac{1}{2} x y (1 + \psi_2) \frac{\partial}{\partial w}
\end{align*}
\]

(1.2)

where \( \varphi, \psi_1, \psi_2 \) are smooth functions satisfying \( \psi_1(0, 0, z, w) = \psi_2(0, 0, 0, w) = 0 \),  
and \( X \) is a time orientation whose trajectories contained in \( \{ y = 0 \} \) are abnormal curves for \( H \).

Theorem 1.1 is a starting point to the investigation of Engel sub-Lorentzian  
structures. By the way we obtain the following partial result. Let \( H \) be such a  
rank two bracket generating distribution on a 4-manifold \( M \) that \( H^2 \) is everywhere of rank 3  
(in particular, the situation \( H^4 \neq TM \) is allowed now, so one may need more Lie brackets  
to generate the whole tangent space). Suppose moreover that through each point of \( M \) there passes exactly one abnormal curve and besides all abnormal curves are trajectories of a single smooth vector field. Now let \( g \) be a Lorentzian metric on \( H \) such that all abnormal curves are t.f.d. Hamiltonian  
geo desics. Then we can prove

**Proposition 1.1.** Let \((H, g)\) be a germ at \( q_0 \in M \) of a time-oriented sub-Lorentzian  
structure defined above. Then, around \( q_0 \), there exist coordinates \( x, y, z, w \), \( x(q_0) = = w(q_0) = 0 \), in which \((H, g)\) admits an orthonormal frame in the form

\[
\begin{align*}
X &= \frac{\partial}{\partial x} + y \varphi \left( y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right) + \frac{1}{2} y (1 + \psi_1) \frac{\partial}{\partial z} - y A_2 \frac{\partial}{\partial w} \\
Y &= \frac{\partial}{\partial y} - x \varphi \left( y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right) - \frac{1}{2} x (1 + \psi_1) \frac{\partial}{\partial z} + x A_2 \frac{\partial}{\partial w}
\end{align*}
\]

where \( \varphi, \psi_1, A_2 \) are smooth functions, \( \psi_1(0, 0, z, w) = 0 \), and \( X \) is a time orienta-  
tion whose trajectories contained in \( \{ y = 0 \} \) are abnormal curves for \( H \).
Using theorem 1.1, in further parts of the paper, we attempt to describe reachable sets from the origin for Engel sub-Lorentzian structures.

If \((M, H, g)\) is a sub-Lorentzian manifold, \(q_0\) is a fixed point in \(M\), and \(U\) is a neighbourhood of \(q_0\), then by the (future) nonspacelike reachable set from \(q_0\) we mean the set of all points \(q \in U\) such that \(q\) can be reached from \(q_0\) by nspc.f.d. curve contained in \(U\); this set will be denoted by \(J^+(q_0, U)\). Replacing nspc.f.d. curves with t.f.d. and null f.d. curves we obtain the definition of the (future) timelike reachable set \(I^+(q_0, U)\), and the (future) null reachable set \(N^+(q_0, U)\), respectively.

For general \(U\) we know that the three reachable sets have the same interiors (which are nonempty) and closures relative to \(U\). In order to be able to obtain more precise results we need to impose certain assumptions on \(U\). To this end notice that if \(U\) is sufficiently small then the metric \(g\) can be extended to a Lorentzian metric \(\tilde{g}\) defined on a neighbourhood of \(U\). Now, \(U\) is called a normal neighbourhood of \(q_0\) if it is a convex normal neighbourhood of \(q_0\) with respect to \(\tilde{g}\) (see e.g. [19]) and its closure is contained in some other convex normal neighbourhood of \(q_0\) with respect to \(\tilde{g}\). Recall [10] in this place that if \(U\) is a normal neighbourhood of \(q_0\) then

\[
J^+(q_0, U) = \text{cl}_U \left( \text{int} \ I^+(q_0, U) \right) = \text{cl}_U \left( \text{int} \ N^+(q_0, U) \right),
\]

where \(\text{cl}_U\) is the closure with respect to \(U\). The basic objects, when studying reachable sets, are the so-called geometrically optimal curves. A nspc.f.d. curve \(\gamma : [0, T] \to U\) is called geometrically optimal if \(\gamma ([0, T]) \subset \partial_U J^+(\gamma(0), U)\), where \(\partial_U\) stands for the boundary operator taken with respect to \(U\). It is a standard fact that geometrically optimal curves satisfy Pontriagin maximum principle - see eg. [1].

First of all we prove the following

**Proposition 1.2.** Let \((H, g)\) be given by an orthonormal frame in the normal form \([1,2]\), where \(\varphi = \varphi(x, y, w), \psi_2 = \psi_2(x, y, w)\), i.e. \(\varphi, \psi_2\) do not depend on \(z\). Then the abnormal curve starting from zero (which is t.f.d) is geometrically optimal.

The proof uses the observation that lifts of geometrically optimal curves are again geometrically optimal. Remark here that timelike abnormal curves always satisfy necessary conditions for optimality from Pontriagin maximum principle, and in general it is not a trivial thing to determine if a given timelike abnormal curve is geometrically optimal or not (cf. [5] and note that timelike abnormal curves correspond to singular trajectories of affine control systems - see [10]). Examples of timelike abnormal curves which are not geometrically optimal can be found in [13].

Using proposition 1.2 we come to the investigation of reachable sets. In papers [12], [13], the author managed to give a precise description of reachable sets from the origin for sub-Lorentzian structures, where the abnormal t.f.d. curves fill a hypersurface passing through the origin. As it is noticed above, Engel case is much harder to study since here abnormal t.f.d. curves pass through every point. From this reason the methods developed earlier by the author do not work (or at least require serious modifications), and therefore we obtain only certain estimates on reachable sets - propositions 3.2, 3.3.

**The organization of the paper.**

In section 2 we prove theorem 1.1 and proposition 1.1. In section 3 we prove propositions 1.2, 3.2, and 3.3.
2. Normal Forms

Let \((H,g)\) be a time-oriented Engel sub-Lorentzian structure of Hamiltonian type. Without loss of generality we can suppose it to be defined in a neighbourhood \(U\) of \(0 \in \mathbb{R}^4\). Throughout this section \(U\) will be supposed to be as small as it is needed to justify various statements that are made below.

2.1. Normal coordinates. Let \(\tilde{X}, \tilde{Y}\) be an orthonormal frame for \((H,g)\) such that \(\tilde{X}\) is a time orientation and the trajectories of \(\tilde{X}\) are exactly the abnormal curves for \(H\). Such a field exists by \([20, 22]\), and we can assume that these trajectories are t.f.d. Hamiltonian geodesics (if it were not the case we change parameterization).

Choose a curve \(\Gamma\) transverse to \(H\). Denote by \(g^\tilde{X}_t\) (resp. \(g^\tilde{Y}_t\)) the (local) flow of \(\tilde{X}\) (resp. \(\tilde{Y}\)) defined on \(U\), and let \(P = \bigcup_{t, s} g^\tilde{X}_s \circ g^\tilde{X}_t(\tilde{X}, \tilde{Y})\); clearly \(P\) is a smooth hypersurface.

**Lemma 2.1.** There are coordinates \(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{w}\) on \(U\) such that
(i) \(P = \{\tilde{y} = 0\}\), \(\Gamma = \{\tilde{x} = \tilde{y} = \tilde{z} = 0\}\);
(ii) \(\frac{\partial}{\partial \tilde{z}} |_P, \frac{\partial}{\partial \tilde{y}} |_P\) is an orthonormal frame for \((H,g)\), and \(\frac{\partial}{\partial \tilde{z}} |_P\) is a time orientation.

**Proof.** Let \(\sigma(\tilde{w})\), \(\sigma(0) = 0\), be a parameterization of \(\Gamma\). Then the coordinates we look for are given by the diffeomorphism \((\tilde{x}, \tilde{y}, \tilde{z}, \tilde{w}) \mapsto \tilde{g}^\tilde{X}_\tilde{y} \circ \tilde{g}^\tilde{X}_\tilde{z} \circ \tilde{g}^\tilde{X}_\tilde{y} \sigma(\tilde{w})\).

Since \(\tilde{X} = \frac{\partial}{\partial \tilde{z}}\) on \(P\), the curves \(t \mapsto (t, 0, z_0, w_0)\) (written in just constructed coordinates \(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{w}\)) are abnormal. Denote by \(p_{\tilde{x}}, p_{\tilde{y}}, p_{\tilde{z}}, p_{\tilde{w}}\) the dual coordinates to \(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{w}\). Then \(\mathcal{H}|_{T_p \mathbb{R}^4} = -\frac{1}{2}p^2_{\tilde{x}} + \frac{1}{2}p^2_{\tilde{y}}\) where \(T_p \mathbb{R}^4 = \bigcup_{t, s} T_s \mathbb{R}^4\), and since, by assumption, the curves \(t \mapsto (t, 0, z_0, w_0)\) are abnormal t.f.d. Hamiltonian geodesics we obtain
\[
\mathcal{H}(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{w}, p_{\tilde{x}}, p_{\tilde{y}}, p_{\tilde{z}}, p_{\tilde{w}}) = -\frac{1}{2}p^2_{\tilde{x}} + \frac{1}{2}p^2_{\tilde{y}} + \tilde{y}^2 \mathcal{G}(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{w}, p_{\tilde{x}}, p_{\tilde{y}}, p_{\tilde{z}}, p_{\tilde{w}})
\]
for a smooth function \(\mathcal{G}\).

Let
\[A_T = \{(0, 0, \tilde{z}, \tilde{w}, p_{\tilde{x}}, p_{\tilde{y}}, 0, 0) : |\tilde{z}|, |\tilde{w}| < \varepsilon\}\,.
\]
Consider the mapping \(\mu : A_T \to U\) given by
\[\mu(\tilde{z}, \tilde{w}, p_{\tilde{x}}, p_{\tilde{y}}) = \pi \circ \Phi_1 (0, 0, \tilde{z}, \tilde{w}, -p_{\tilde{x}}, p_{\tilde{y}}, 0, 0)\,,\]
i.e. in terms of the exponential mapping
\[\mu(\tilde{z}, \tilde{w}, p_{\tilde{x}}, p_{\tilde{y}}) = \exp_{(0,0,\tilde{z},\tilde{w})} (-p_{\tilde{x}}, p_{\tilde{y}}, 0, 0)\,.
\]
If \(N\) is a sufficiently small neighbourhood of the set \(\{(0, 0, \tilde{z}, \tilde{w}, 0, 0, 0, 0) : |\tilde{z}|, |\tilde{w}| < \varepsilon\}\) in \(A_T\) then \(\mu : N \to \mu(N)\) is a diffeomorphism. Therefore we can write \(\mu(N) = U\), and now we are ready to define normal coordinates \(x, y, z, w\) on \(U\). These are coordinates given by the mapping
\[U \xrightarrow{u^{-1}} N \xrightarrow{(\tilde{z}, \tilde{w}, -p_{\tilde{x}}, p_{\tilde{y}}) \to \mathbb{R}^4} \mathbb{R}^4 \xrightarrow{\alpha} \mathbb{R}^4\]
where $\alpha(a, b, c, d) = (c, d, a, b)$. To be more precise, a point $q \in U$ has normal coordinates $(x, y, z, w)$ if and only if $q = \exp_{(0,0,z,0)}(-x, y, 0, 0)$. It follows that the lines $t \mapsto (at, bt, z_0, w_0)$ are Hamiltonian geodesics and that $P = \{y = 0\}$.

Let us define four sets

\begin{align*}
S_1^+ &= \{q \in U : |y| < |x|, x > 0\} \\
S_1^- &= \{q \in U : |y| < |x|, x < 0\} \\
S_2^+ &= \{q \in U : |y| > |x|, y > 0\} \\
S_2^- &= \{q \in U : |y| > |x|, y < 0\}.
\end{align*}

Then let us put $S_1 = S_1^+ \cup S_1^-$, $S_2 = S_2^+ \cup S_2^-$. Moreover let

\begin{align*}
R_1 &= \left\{ \sqrt{x^2 - y^2} \text{ on } S_1^+, \right. \\
&\quad \left. -\sqrt{x^2 - y^2} \text{ on } S_1^-. \right\} \\
R_2 &= \left\{ \sqrt{y^2 - x^2} \text{ on } S_2^+, \right. \\
&\quad \left. -\sqrt{y^2 - x^2} \text{ on } S_2^- \right\}.
\end{align*}

Now we can introduce hyperbolic cylindrical coordinates $R, \varphi, z, w$ on $S_1$: $x = R_1 \cosh \varphi$, $y = R_1 \sinh \varphi$, and on $S_2$ and $x = R_2 \sinh \varphi$, $y = R_2 \cosh \varphi$. Clearly

$$\frac{\partial}{\partial R_1} = \frac{x}{R_1} \frac{\partial}{\partial x} + \frac{y}{R_1} \frac{\partial}{\partial y}$$

is unit t.f.d. on $S_1^+$ since it is the velocity vector of the geodesic $s \mapsto (s \cosh \varphi, s \sinh \varphi, z_0, w_0)$. From similar reasons it is unit t. past directed on $S_1^-$. Also, for instance on $S_1^+$

$$\frac{\partial}{\partial \varphi} = y \frac{\partial}{\partial x} + x \frac{\partial}{\partial x}$$

however $\frac{\partial}{\partial \varphi}$ is not horizontal in general.

Below we will need the following observations. If we define $C = \mathcal{H}^{-1}(\frac{1}{2})$ then obviously $\Phi_s(C) \subset C$. Also, if $\alpha$ is the Liouville form then $\alpha$ restricted to $C$ is preserved by the flow $\Phi_s$. Denote by $(R_0, \varphi_0, z_0, w_0)$ the hyperbolic cylindrical coordinates on $A_1$. Then evidently

$$\frac{\partial}{\partial \varphi} = (\pi \circ \Phi_s)_* \frac{\partial}{\partial \varphi_0}, \quad \frac{\partial}{\partial z} = \pi_* \frac{\partial}{\partial z_0}, \quad \frac{\partial}{\partial w} = \pi_* \frac{\partial}{\partial w_0}.$$

Moreover the fields $\frac{\partial}{\partial \varphi_0}, \frac{\partial}{\partial z_0}, \frac{\partial}{\partial w_0}$ are tangent to $C$, and in addition $\frac{\partial}{\partial \varphi_0}$ is tangent to the sets $C \cap T_{(0,0,z_0,w_0)}^* \mathbb{H}^4$.

In what follows we will use the notion of the horizontal gradient, so now we recall the definition. Let $f : U \rightarrow \mathbb{R}$ be a smooth function defined on an open subset $U$ of the sub-Lorentzian manifold $(M, H, g)$; the horizontal gradient of the function $f$ is the (horizontal) vector field $\nabla_H f$ determined by the condition $d_q f(v) = g(\nabla_H f(q), v)$ for every $v \in H_q$, $q \in U$. If $X_0, \ldots, X_k$ is an orthonormal frame for $(H, g)$ on $U$ with $X_0$ timelike, then $\nabla_H f = -X_0(f)X_0 + X_1(f)X_1 + \ldots + X_k(f)X_k$.

Now we can prove the following

**Lemma 2.2.** $\nabla_H R_1 = -\frac{\partial}{\partial R_1}$ on $S_1^+$; in particular $\nabla_H R_1$ is unit timelike past directed.
Proof. Fix a point \( q \in S^1 \); then \( q = \pi \circ \Phi_s(0, 0, z_0, w_0, -\cosh \varphi, \sinh \varphi, 0, 0) = \exp(0, 0, z_0, w_0) s \) for suitable \( \varphi \) and \( s > 0 \). If \( \lambda = (0, 0, z_0, w_0, -\cosh \varphi, \sinh \varphi, 0, 0) \) then \( s \rightarrow \Phi_s(\lambda) \) is a Hamiltonian lift of the geodesic \( \gamma(s) = \exp(0, 0, z_0, w_0)(s\lambda) \). We have a sequence of equalities
\[
\left\langle \Phi_s(\lambda), \frac{\partial}{\partial \varphi} \right\rangle = \left\langle \Phi_s(\lambda), \pi_s \circ \Phi_s, \pi_s \circ \frac{\partial}{\partial \varphi} \right\rangle = \left\langle \alpha(\Phi_s(\lambda)), \Phi_s, \frac{\partial}{\partial \varphi} \right\rangle = \left\langle \alpha(\lambda), \frac{\partial}{\partial \varphi} \right\rangle
\]
and the result follows from the definition of the horizontal gradient.

\[ \blacksquare \]

Corollary 2.1. Geodesics \( s \rightarrow (s \cosh \varphi, s \sinh \varphi, z_0, w_0) \) are unique \( U \)-maximizers.

Proof. As it was mentioned in the previous papers by the author, every trajectory of a t.f.d. field of the form \( \nabla_H f \), where \( f \) is a smooth function defined on an open set \( U \) and such that \( g(\nabla_H f, \nabla_H f) = \text{const} \) on \( U \), is a unique \( U \)-maximizer.

2.2. Construction of normal forms. Using what we have said in the proof of lemma 2.2, there exists an orthonormal frame \( F, G \) for \( (H, g) \), defined on \( U \setminus \{ R_i = 0 \} = S_1 \cup S_2 \) with \( F \) being a timelike field, which is of the form
\[
(2.1) \quad F = \frac{x}{R_1} \frac{\partial}{\partial x} + \frac{y}{R_1} \frac{\partial}{\partial y} + a_{11} \frac{\partial}{\partial z} + a_{21} \frac{\partial}{\partial w} + \left( b_1 + \frac{1}{R_1} \right) \left( y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right)
\]
on \( S_1 \), and
\[
(2.2) \quad F = a_{12} \frac{\partial}{\partial z} + a_{22} \frac{\partial}{\partial w} + \left( b_2 + \frac{1}{R_2} \right) \left( y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right), \quad G = \frac{x}{R_2} \frac{\partial}{\partial x} + \frac{y}{R_2} \frac{\partial}{\partial y}
\]
on \( S_2 \), where \( a_{ji}, b_i \) are smooth on \( S_i \), \( i, j = 1, 2 \). Indeed, first let us remark here that although all calculations in lemma 2.2 were carried out on \( S_1^+ \), it is not difficult to extend them to \( S_1^- \). For instance, since \( \gamma(s) = (-s \cosh \varphi, -s \sinh \varphi, z_0, w_0) \in S_1^- \) is a timelike geodesic, the vector \( \left( \frac{\partial}{\partial R_1} \right)_{\gamma(s)} = \frac{\partial}{\partial R_1} + \frac{\partial}{\partial y} = \hat{\gamma}(s) \) is also timelike on \( S_1^- \). Secondly, the formula (2.2) on \( S_2 \) follows from (2.1) valid on \( S_1 \) by replacing the metric \( (H, g) \) with \( (H, -g) \).

Below we prove the following

Proposition 2.1. There exist functions \( A_1, A_2 \in C^\infty(U) \) such that
\[
A_1 = \begin{cases} \frac{\partial}{\partial R_1} \text{ on } S_1 \quad \text{and} \quad A_2 = \frac{\partial}{\partial R_2} \text{ on } S_2 \end{cases} \text{ and } A_2 = \begin{cases} \frac{\partial}{\partial R_1} \text{ on } S_1 \quad \text{and} \quad A_2 = \frac{\partial}{\partial R_2} \text{ on } S_2 \end{cases}.
\]

The proof is similar to the proof of analogous result in [8]. On each of the sets \( S_1, S_2 \) we will write the Hamiltonian \( H \), which is the smooth function on the whole \( U \). So on \( S_1 \) we have
\[
2H(x, y, z, w, px, py, pz, pw) = -\left( \frac{x}{R_1} px + \frac{y}{R_1} py \right)^2 + \left( a_{11}pz + a_{21}pw + b_1 + \frac{1}{R_1} \right) (ypx + xpy)^2 =
\]
Let \( \tilde{X}, \tilde{Y} \) be a frame of \( G \) such that the expression under the root does not vanish. Now if we set \( p_x = x, p_y = y, p_z = z, p_w = w \), while on \( S_2 \) we can write
\[
2\mathcal{H}(x, y, z, w, p_x, p_y, p_z, p_w) = -\left(a_{12}p_z + a_{22}p_w + \frac{b_2}{R_2}(b_1R_1 + 2)(yp_x + xp_y)\right)^2 + \left(\frac{x}{R_2}p_x + \frac{y}{R_2}p_y\right)^2 = -p_x^2 + p_y^2 - (a_{12}p_z + a_{22}p_w)^2 - \frac{b_2}{R_2}(b_1R_1 + 2)(yp_x + xp_y)^2 - \frac{2}{R_2}(b_2R_2 + 1)(yp_x + xp_y)(a_{12}p_z + a_{22}p_w).
\]

**Lemma 2.3.** The exist smooth function \( \tilde{a}_1, \tilde{a}_2 : U \rightarrow \mathbb{R} \) such that
\[
\tilde{a}_1 = \begin{cases} a_{11} \text{ on } S_1 \\ -a_{12} \text{ on } S_2 \end{cases} \quad \text{and} \quad \tilde{a}_2 = \begin{cases} a_{21} \text{ on } S_1 \\ -a_{22} \text{ on } S_2 \end{cases}
\]
In particular, \( \tilde{a}_{1|R_i=0} = \tilde{a}_{2|R_i=0} = 0 \).

**Proof.** Indeed, using above formulas we have
\[
2\mathcal{H}(x, y, z, w, 0, 0, 1, 0) = a_{11}^2, \quad 2\mathcal{H}(x, y, z, w, 0, 0, 0, 1) = a_{21}^2
\]
on \( S_1 \) and
\[
2\mathcal{H}(x, y, z, w, 0, 0, 1, 0) = -a_{12}^2, \quad 2\mathcal{H}(x, y, z, w, 0, 0, 0, 1) = -a_{22}^2
\]
on \( S_2 \). Thus it is enough to define \( \tilde{a}_1(x, y, z, w) = 2\mathcal{H}(x, y, z, w, 0, 0, 1, 0), \tilde{a}_2(x, y, z, w) = 2\mathcal{H}(x, y, z, w, 0, 0, 0, 1) \). 

Next let \( \langle \cdot, \cdot \rangle \) be the Minkowski scalar product on \( U \), i.e. the one induced by the Lorentzian metric on \( U \) defined by supposing the basis \( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial w} \) to be orthonormal with the time orientation \( \frac{\partial}{\partial x} \). For an orthonormal basis \( X, Y \) of \( (H, g) \) let \( G = \det \begin{pmatrix} \langle X, X \rangle & \langle X, Y \rangle \\ \langle X, Y \rangle & \langle Y, Y \rangle \end{pmatrix} \). Note that since the matrices from the Lorentz group have determinant equal to \( \pm 1 \), \( G \) is independent of the choice of the orthonormal frame \( X, Y \). Clearly \( G \) is a smooth function on \( U \); we will compute the values of \( G_i = G|_{S_i}, i = 1, 2 \). So
\[
G_1 = -a_{11}^2 - a_{21}^2 - (b_1R_1 + 1)^2 = -\tilde{a}_{1|S_1} - \tilde{a}_{2|S_1} - (b_1R_1 + 1)^2,
\]
and
\[
G_2 = a_{12}^2 + a_{22}^2 - (b_2R_2 + 1)^2 = -\tilde{a}_{1|S_2} - \tilde{a}_{2|S_2} - (b_2R_2 + 1)^2.
\]
In particular, for \( R_i = 0 \) we get, by lemma 2.3, \( G_i = -1 \) which means that \( G \) is negative on \( U \). Therefore
\[
b_1R_1 + 1 = \sqrt{-\tilde{a}_{1|S_1} - \tilde{a}_{2|S_1} - G_1}, \quad i = 1, 2.
\]
Let \( \tilde{d} = \begin{cases} b_1R_1 \text{ on } S_1 \\ b_2R_2 \text{ on } S_2 \end{cases} \). Then simply \( \tilde{d} = \sqrt{-\tilde{a}_1 - \tilde{a}_2 - G} - 1 \), so \( \tilde{d} \in C^\infty(U) \), since the expression under the root does not vanish. Now if we set \( H_i = H_i|_{T_{S_i}^*U}, i = 1, 2 \), then
\[
\frac{\partial H_1}{\partial p_z}|_{p_z=0} = p_z\tilde{a}_{1|S_1} + \frac{a_{11}}{R_1}(\tilde{d} + 1)(yp_x + xp_y),
\]
and similarly
\[
\frac{\partial H_2}{\partial p_z}|_{p_z=0} = p_z\tilde{a}_{1|S_2} - \frac{a_{12}}{R_2}(\tilde{d} + 1)(yp_x + xp_y).
\]
Now let $A_1$ be as in the hypotheses of proposition 2.1. Then (2.3) and (2.4) become
\[
\frac{\partial H_1}{\partial p_w}|_{p_w=0} = p_z \tilde{a}_1|_{S_1} + A_1 \left( \tilde{d} + 1 \right) (yp_x + xp_y)
\]
and
\[
\frac{\partial H_2}{\partial p_w}|_{p_w=0} = p_z \tilde{a}_1|_{S_2} + A_1 \left( \tilde{d} + 1 \right) (yp_x + xp_y)
\]
and all terms, apart from $A_1$ perhaps, are smooth on the whole $U$. Since $\tilde{d} + 1 \neq 0$, it follows that $A_1 (yp_x + xp_y)$ is smooth. Setting $p_x = 1, p_y = 0$ and then $p_x = 0, p_y = 1$ we arrive at $xA_1, yA_1 \in C^\infty(U)$. But this means $A_1 \in C^\infty(U)$ as it was stated.

Now let $A_2$ be defined as in the hypotheses of proposition 2.1. Considering this time derivatives $\frac{\partial H_1}{\partial p_w}|_{p_w=0}, i = 1, 2$, we are led to
\[
\frac{\partial H_1}{\partial p_w}|_{p_w=0} = p_w \tilde{a}_2|_{S_1} + A_2 \left( \tilde{d} + 1 \right) (yp_x + xp_y)
\]
and
\[
\frac{\partial H_2}{\partial p_w}|_{p_w=0} = p_w \tilde{a}_2|_{S_2} + A_2 \left( \tilde{d} + 1 \right) (yp_x + xp_y)
\]
which results in smoothness of $A_2$. The proof of proposition 2.1 is over.

**Proposition 2.2.** There exists a function $B \in C^\infty(U)$ such that $B = \left\{ \begin{array}{ll} \frac{b_1}{R_2} & \text{on } S_1 \\ -\frac{b_2}{R_2} & \text{on } S_2 \end{array} \right.$

**Proof.** Using the above formulas we can write on $S_1$
\[
2H (x, y, z, w, p_x, p_y, 1, 0) = -p_x^2 + p_y^2 + \tilde{a}_1|_{S_1} + B \left( \tilde{d} + 2 \right) (yp_x + xp_y)^2 + 2A_1 \left( \tilde{d} + 1 \right) (yp_x + xp_y)
\]
and
\[
2H (x, y, z, w, p_x, p_y, 1, 0) = -p_x^2 + p_y^2 + \tilde{a}_1|_{S_2} + B \left( \tilde{d} + 2 \right) (yp_x + xp_y)^2 + 2A_1 \left( \tilde{d} + 1 \right) (yp_x + xp_y)
\]
on $S_2$. Since all terms, perhaps apart from $B$, are smooth on $U$, and $\tilde{d} + 2 \neq 0$, we again arrive at $xB, yB \in C^\infty(U)$, which in turn gives $B \in C^\infty(U)$.

To conclude our considerations, similarly as in [8], we change our frame $F, G$ as follows:
\[
(2.5) \quad F \longrightarrow \frac{x}{R_1} F - \frac{y}{R_1} G, \quad G \longrightarrow -\frac{y}{R_1} F + \frac{x}{R_1} G
\]
on $S_1$ and
\[
(2.6) \quad F \longrightarrow \frac{y}{R_2} F - \frac{x}{R_2} G, \quad G \longrightarrow -\frac{x}{R_2} F + \frac{y}{R_2} G
\]
on $S_2$; note that both, the frame $F, G$ and our change are singular on $\{R_i = 0\}$. Carrying out calculations as indicated in (2.5) and (2.6), we obtain the following pre-normal form for our structure
\[
(2.7) \quad X = \frac{\partial}{\partial x} - yB \left( \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right) - yA_1 \frac{\partial}{\partial z} - yA_2 \frac{\partial}{\partial w}
\]
\[
Y = \frac{\partial}{\partial y} + xB \left( \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right) + xA_1 \frac{\partial}{\partial z} + xA_2 \frac{\partial}{\partial w}
\]
where $X$ is a time orientation and $A_1, A_2, B$ are smooth around the origin. The only thing that may require some explanations is the fact that $X$ is a time orientation.
However it is clear that for a timelike field $X$ to be a time orientation it suffices to be future directed at a single point, and surely $X = \frac{1}{\lambda} F - \frac{1}{\lambda} G$ is future directed on $S^+_t$, since $F$ is.

In order to be able to find some additional conditions that can be imposed on $A_1, A_2$ in (2.7) we have to use our assumptions. First let us note that, by construction, $\frac{\partial}{\partial w}(\sigma) \geq 0$ is transverse to $H^2$, so $\frac{\partial}{\partial w}(\sigma)$ is transverse to $H^2$ on $U$ (recall that $U$ is sufficiently small).

We compute the commutator of $X$ and $Y$ to be equal to

$$[X, Y] = I \frac{\partial}{\partial x} + II \frac{\partial}{\partial y} + III \frac{\partial}{\partial z} + IV \frac{\partial}{\partial w}$$

where

$I = y \left( 3B + x \frac{\partial B}{\partial x} + y \frac{\partial B}{\partial y} + (x^2 - y^2) B^2 \right)$,

$II = x \left( 3B + x \frac{\partial B}{\partial x} + y \frac{\partial B}{\partial y} + (x^2 - y^2) B^2 \right)$,

$III = 2A_1 + x \frac{\partial A_1}{\partial x} + y \frac{\partial A_1}{\partial y} + (x^2 - y^2) A_1 B$, and

$IV = 2A_2 + x \frac{\partial A_2}{\partial x} + y \frac{\partial A_2}{\partial y} + (x^2 - y^2) A_2 B$.

Now, since $X|_{\Gamma} = \frac{\partial}{\partial y}, Y|_{\Gamma} = \frac{\partial}{\partial y}, \dim H^2_{(0,0,z,w)} = 3$, and as it was noticed $\frac{\partial}{\partial w}$ is transverse to $H^2$, it follows that $III|_{\Gamma} = 2A_1|_{\Gamma}$ does not vanish. We renormalize the $z$-axis by making the following change of coordinates: $(x, y, z, w) \rightarrow (x, y, \alpha(z, w), w)$, where $\alpha$ solves the equation ($w$ is a parameter here)

$$\alpha(z, w) + z \frac{d}{dz} \alpha(z, w) = \frac{1}{2A_1(0, 0, z, w)}.$$

In this way we keep the form (2.4) and, in the new coordinates, $A_1(0, 0, z, w) = -\frac{1}{z}$. Now setting $\psi_1 = -2A_1 - 1$ we obtain proposition 1.1.

Before we proceed we prove the following lemma.

**Lemma 2.4.** $\frac{\partial}{\partial x}|_{\Gamma}$ is tangent to $H^2$.

**Proof.** If we look closer at coordinates $\tilde{x}, \tilde{y}, \tilde{z}, \tilde{w}$ and $x, y, z, w$, it is seen that $\frac{\partial}{\partial x} = \frac{\partial}{\partial \tilde{x}}$. Indeed, $\frac{\partial}{\partial \tilde{x}} = \frac{\partial}{\partial \tilde{x}} + \frac{\partial}{\partial \tilde{y}} \frac{\partial \tilde{y}}{\partial \tilde{x}} + \frac{\partial}{\partial \tilde{z}} \frac{\partial \tilde{z}}{\partial \tilde{x}} + \frac{\partial}{\partial \tilde{w}} \frac{\partial \tilde{w}}{\partial \tilde{x}}$, and $z = \tilde{z}, w = \tilde{w}$. Therefore it is enough to carry out all computations in the first set of coordinates. To this end fix a point $q = (\tilde{x}, 0, \tilde{z}, \tilde{w}) = g^{\tilde{z}}_X \circ g^{\tilde{z}, \tilde{y}}_{(\tilde{X}, \tilde{Y})} \sigma(\tilde{w})$ belonging to $P$. Then

$$\left( \frac{\partial}{\partial \tilde{x}} \right)_q = \frac{d}{ds}|_{s=0} g^{\tilde{z}}_X \circ g^{\tilde{z}, \tilde{y}}_{(\tilde{X}, \tilde{Y})} \sigma(\tilde{w}) = (dg^{\tilde{z}}_X)_{(\tilde{X}, \tilde{Y})} \left( g^{\tilde{z}, \tilde{y}}_{(\tilde{X}, \tilde{Y})} \sigma(\tilde{w}) \right)$$

Now let $\gamma(t) = g^{\tilde{z}}_X \circ g^{\tilde{z}, \tilde{y}}_{(\tilde{X}, \tilde{Y})} \sigma(\tilde{w})$, i.e. $\gamma$ is the abnormal curve passing through $g^{\tilde{z}}_X \circ g^{\tilde{z}, \tilde{y}}_{(\tilde{X}, \tilde{Y})} \sigma(\tilde{w})$ at time $t = 0$. Let moreover $\lambda(t)$ be an abnormal lift of $\gamma$ satisfying PMP, Pontriagin maximum principle, that is to say $(\gamma(t), \lambda(t))$ is an abnormal biextremal. Then clearly $\lambda(t) \in (H^2_{(\gamma(t))})^\perp \subset T^*\mathbb{R}^4$ where the latter stands for the annihilator of $H^2_{(\gamma(t))}$ - cf. [30]. This in particular implies that $H^2_{(\gamma(t))} = \ker \lambda(t)$.

Further, from the proof of PMP - see [11] - it follows that $\lambda(t) = (dg^{\tilde{z}}_X)^\ast \lambda(0)$ for every $t$. Thus, taking all above-mentioned facts together we obtain

$$\left\langle (dg^{\tilde{z}}_X)^\ast \lambda(t), [\tilde{X}, \tilde{Y}] \left( g^{\tilde{z}, \tilde{y}}_{(\tilde{X}, \tilde{Y})} \sigma(\tilde{w}) \right) \right\rangle = 0.$$
which terminates the proof.

Now let us see what happens on $P$. So
\[ [X, Y]_P = x(3B + x \frac{\partial B}{\partial x} + x^2 B) \frac{\partial}{\partial y} + (2A_1 + x \frac{\partial A_1}{\partial x} + x^2 A_1 B) \frac{\partial}{\partial y} + (2A_2 + x \frac{\partial A_2}{\partial x} + x^2 A_2 B) \frac{\partial}{\partial w}, \]

$X|_P = \frac{\partial}{\partial x}$, and
\[ Y|_P = (1 + x^2 B) \frac{\partial}{\partial y} + x A_1 \frac{\partial}{\partial z} + x A_2 \frac{\partial}{\partial w}, \]

where $A_1$, $A_2$, $B$ are evaluated at $(x, 0, z, w)$. Calculations give
\[ [X, Y]|_P = \frac{x (3B + x \frac{\partial B}{\partial x} + x^2 B)}{1 + x^2 B} \frac{\partial}{\partial y} + \left( \frac{B^2 x^4 - B x^4 + 2}{B x^2 + 1} \right) \left( A_1 \frac{\partial}{\partial z} + A_2 \frac{\partial}{\partial w} \right) \]

from which it follows that $A_1 \frac{\partial}{\partial z} + A_2 \frac{\partial}{\partial w}$ is tangent to $H^2$ on $P$. But since $\frac{\partial}{\partial z}$ is also tangent to $H^2$ it follows that $A_2 \frac{\partial}{\partial w}$ is tangent to $H^2$ at point of $P$ which is possible only when $A_2(x, 0, z, w) = 0$ identically. This means that $A_2$ may be replaced by $y A_2$ for some other smooth function $A_2$. Thus we are led to
\[ X = \frac{\partial}{\partial x} - y B \left( y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right) - y A_1 \frac{\partial}{\partial z} - y^2 A_2 \frac{\partial}{\partial w} \]
\[ Y = \frac{\partial}{\partial y} + x B \left( y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right) + x A_1 \frac{\partial}{\partial z} + x y A_2 \frac{\partial}{\partial w} \]

with $A_1(0, 0, z, w) = \frac{1}{2}$.

Let $a d X, Y = [X, Y]$, and $a d^{k+1} X, Y = [X, a d^k X, Y]$, $k = 1, 2, ...$ Now we will extract some more information about the commutators of the fields $X, Y$.

**Lemma 2.5.** $a d^n X, Y$ is tangent to $H^2|_P$, $n = 0, 1, 2, ...$

**Proof.** We use similar considerations as in the proof of lemma 2.4. Consider the abnormal curve $\gamma(t)$ starting from a point $\gamma(0) = q = (x_0, 0, z_0, w_0)$. Let $\lambda(t)$ be the abnormal lift of $\gamma$ satisfying PMP; then clearly $H^2|_{\gamma(t)} = \ker \lambda(t)$ and, again by [1], $\lambda(t) = (dy_{X(t)}^{-1})^* \lambda(0)$ for every $t$. Now for each $t$, $|t|$ sufficiently small, and every integer $n$ we have
\[ 0 = \langle \lambda(t), Y_{\gamma(t)} \rangle = \langle \lambda(0), (dy_{X(t)}^{-1}) Y_q \rangle = \left( \lambda(0), \sum_{k=0}^{n} \frac{t^k}{k!} (a d^k X, Y)_q \right) + o(t^n), \]

and because $\langle \lambda(0), Y_q \rangle = 0$, the result follows by induction.

As a consequence of lemma 2.4 we know that $X, Y, [X, Y], [Y, [X, Y]]$ are linearly independent everywhere. We will examine the $w$-coordinate of $[Y, [X, Y]]$ on $\Gamma$. The most convenient way is to treat $[Y, [X, Y]]$ as an operator: $[Y, [X, Y]] = -Y^2 \circ X + 2 Y \circ X \circ Y - X \circ Y^2$. Only result on $\Gamma$ interests us, so it is enough to carry out computations as follows:
\[ Y^2 \circ X(w) = Y^2(-y^2 A_2) = -2(1 + x^2 B)^2 A_2 + O(y), \]
\[ Y \circ X \circ Y(w) = Y \circ X(xy A_2) = (1 + x^2 B)(1 - y^2 B)A_2 A_2 + x \frac{\partial A_2}{\partial x} + y A_2 + O(y) + O(x), \]
\[ X \circ Y^2(w) = X \circ Y(xy A_2) = (1 + x^2 B)(1 - y^2 B)(A_2 + \frac{\partial A_2}{\partial y}) + O(y) + O(x). \]

Consequently
\[ (2.9) \quad [Y, [X, Y]](w)|_\Gamma = 2A_2(0, 0, z, w). \]

But [2.9] does not vanish, so we renormalize the $w$-axis by making the change $(x, y, z, w) \rightarrow (x, y, z, \beta(w)w)$, where $\beta$ is a solution to the equation
\[ \beta(w) + w \frac{d}{d w} \beta(w) = \frac{1}{2A_2(0, 0, 0, w)}. \]
3.1. Geometric optimality of abnormal curves.

Notice that if \( \gamma \) is such that (3.1) by the frame in the normal form Lorentzian structure of Hamiltonian type which generated on an open set reachable sets. Therefore one must content oneself only with certain estimates on the in the present paper, however, is more complicated and the mentioned methods do not work. In particular, if we assign weights to coordinates in the following way weight \( x = weight(y) = 1, weight(z) = 2, weight(w) = 3 \), then the fields defining the flat structure are homogeneous of degree \(-1\).

2.3. Remarks. Having proved theorem 1.1 it is seen why the structure from example 1.1 is called flat: every structure in the normal form (3.2) can be regarded as a perturbation of the flat structure. Moreover (cf. 3) we see that the flat Engel structure is the nilpotent approximation for general Engel structures of Hamiltonian type given by (3.2). In particular, if we assign weights to coordinates in the following way weight \( x = 1, weight(y) = 2, weight(z) = 3, weight(w) = 4 \), then we obtain the mapping \( x \to y \). Indeed, if \( \gamma \) is an nspc.f.d. curve with respect to (3.1) is mapped by \( \gamma \).

3. Reachable sets.

In his previous papers the author managed to find a sort of algorithm allowing to compute functions describing reachable sets - see [11], [12], [13]. The case considered in the present paper, however, is more complicated and the mentioned methods do not work. Therefore one must content oneself only with certain estimates on the reachable sets.

In section 2 we recalled the definition of the horizontal gradient of a smooth function. Notice that if \( \gamma : [a, b] \to U \) is a nspc.f.d. curve and a smooth function \( f \) is such that \( \nabla_H f \) is null f.d. on \( U \), then \( t \to f(\gamma(t)) \) is nonincreasing.

3.1. Geometric optimality of abnormal curves. Let \( (H, g) \) be an Engel sub-Lorentzian structure of Hamiltonian type which generated on an open set \( U \subset \mathbb{R}^4 \) by the frame in the normal form

\[
\begin{align*}
X &= \frac{\partial}{\partial x} + y \varphi \left( y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right) + \frac{1}{2} y^2 (1 + \psi_1) \frac{\partial}{\partial z} + \frac{1}{2} y^2 (1 + \psi_2) \frac{\partial}{\partial w}, \\
Y &= \frac{\partial}{\partial y} - x \varphi \left( y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right) - \frac{1}{2} x (1 + \psi_1) \frac{\partial}{\partial z} - \frac{1}{2} x y (1 + \psi_2) \frac{\partial}{\partial w},
\end{align*}
\]

where we additionally suppose that \( \varphi = \varphi(x, y, w) \) and \( \psi_2 = \psi_2(x, y, w) \) i.e. \( \varphi \) and \( \psi_2 \) do not depend on \( z \). Consider now a projection \( p : \mathbb{R}^4 \to \mathbb{R}^3 \), \( p(x, y, z, w) = (x, y, w) \). (3.1) is mapped by \( p \) to the frame

\[
\begin{align*}
\tilde{X} &= \frac{\partial}{\partial x} + y \varphi \left( y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right) + \frac{1}{2} y^2 (1 + \psi_2) \frac{\partial}{\partial w}, \\
\tilde{Y} &= \frac{\partial}{\partial y} - x \varphi \left( y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right) - \frac{1}{2} x y (1 + \psi_2) \frac{\partial}{\partial w},
\end{align*}
\]

on the open set \( \tilde{U} = p(U) \subset \mathbb{R}^3 \). If \( \tilde{H} = \text{Span} \{ \tilde{X}, \tilde{Y} \} \) and \( \tilde{g} \) is a metric on \( \tilde{H} \) defined by assuming \( \tilde{X}, \tilde{Y} \) to be an orthonormal frame with a time orientation \( \tilde{X} \), then we obtain the mapping

\[
p : (U, H, g) \to (\tilde{U}, \tilde{H}, \tilde{g})
\]

of sub-Lorentzian manifolds with the property that \( d_{q} p_{|H_q} : H_q \to \tilde{H}_{p(q)} \) is an isometry for every \( q \in U \). Obviously, the image under \( p \) of a nspc.f.d. (t.f.d., null f.d.) curve with respect to \( (H, g) \) is a nspc.f.d. (t.f.d., null f.d.) curve with respect to \( (\tilde{H}, \tilde{g}) \). Conversely, if \( \tilde{\gamma}(t) = (x(t), y(t), w(t)) \) is a nspc.f.d. (t.f.d., null f.d.) curve on \( (\tilde{U}, \tilde{H}, \tilde{g}) \) and \( q_0 \in p^{-1}(\tilde{\gamma}(0)) \cap U \), then there exists exactly one nspc.f.d. (t.f.d., null f.d.) curve \( \gamma(t) \) on \( (U, H, g) \) such that \( p(\gamma(t)) = \tilde{\gamma}(t), \gamma(0) = q_0 \). Indeed,
the z-coordinate of γ is computed from \( \dot{z} = \frac{1}{2}(\dot{x} x - x \dot{y}) \). One of the immediate consequences of this reasoning is the relation

\[ p(J^+(q_0, U)) = \tilde{J}^+(p(q_0), \tilde{U}), \]

where by \( \tilde{J}^+(p(q_0), \tilde{U}) \) we denote the corresponding reachable set for the structure \((\tilde{H}, \tilde{g})\). The other is enclosed in the proposition below. Recall that \( \partial_U \) (resp. \( \partial_U^c \)) denotes the boundary with respect to \( U \) (resp. to \( \tilde{U} \)).

**Proposition 3.1.** Suppose that \( \tilde{\gamma} : [0, T] \rightarrow \tilde{U}, \tilde{\gamma}(0) = \tilde{q}_0, \) is geometrically optimal, i.e. \( \tilde{\gamma}([0, T]) \subset \partial_U J^+(\tilde{q}_0, \tilde{U}) \). Chose \( q_0 \in p^{-1}(\tilde{q}_0) \), and let \( \gamma : [0, T] \rightarrow U, \gamma(0) = q_0, \) be the lift described above. Then \( \gamma \) is also geometrically optimal, i.e. \( \gamma([0, T]) \subset \partial_U J^+(q_0, U) \).

**Proof.** Suppose that \( \tilde{\gamma} : [0, T] \rightarrow \tilde{U} \) is geometrically optimal but \( \gamma([0, T]) \subset \text{int } J^+(q_0, U) \). Take an open set \( V \) in \( U \) such that \( \gamma(T) \in V \) and \( V \subset J^+(q_0, U) \). Then \( \tilde{\gamma}(T) \in p(V) \) where the latter set is open and contained in \( \tilde{J}^+(\tilde{q}_0, \tilde{U}) \). This contradicts the geometric optimality of \( \tilde{\gamma} \) and the proof is over. \( \square \)

**Corollary 3.1.** The abnormal t.f.d. curve starting from the origin is geometrically optimal for \((U, H, g)\). Consequently, the set \( I^+(0, U) \) is not open, and \( N^+(0, U) \) is not closed.

**Proof.** It is enough to notice that the frame \((\tilde{U}, \tilde{H}, \tilde{g})\) is given in the normal form for Martinet sub-Lorentzian structures of Hamiltonian type considered in \textcite{12}. Thus, using \textcite{12}, we know that the abnormal curve for \((\tilde{H}, \tilde{g})\) initiating at the origin is geometrically optimal. Now proposition above applies. The second part is clear - cf. \textcite{10}. \( \square \)

In particular this proves proposition 1.2. As it was mentioned above, abnormal timelike curves always satisfy necessary conditions for optimality, so the presented method may prove to be useful in applications.

### 3.2. Some estimates in the flat case

Recall that the flat Engel sub-Lorentzian structure is, by definition, the structure defined by an orthonormal frame \( X = \frac{\partial}{\partial x} + \frac{1}{2} y \frac{\partial}{\partial z} + \frac{1}{2} y^2 \frac{\partial}{\partial w}, \) \( Y = \frac{\partial}{\partial y} - \frac{1}{2} x \frac{\partial}{\partial z} - \frac{1}{2} x y \frac{\partial}{\partial w} \) where \( X \) is a time orientation. We see that \( X(x, y, z, 0), Y(x, y, z, 0) \) determine, in the space \( \mathbb{R}^3(x, y, z) \), the Heisenberg sub-Lorentzian metric considered in \textcite{11}, while \( X(x, y, 0, w), Y(x, y, 0, w) \) stipulate, in the space \( \mathbb{R}^3(x, y, w) \), the flat Martinet sub-Lorentzian structure investigated in \textcite{12}. This leads us to considering the following Cauchy problems. Similarly as in the mentioned papers let \( \Gamma_1 \) be the hyperplane \( \{y = x\} \), and \( \Gamma_2 \) be the hyperplane \( \{y = -x\} \). Consider the following Cauchy problems (cf. \textcite{11}):

\[
\begin{align*}
(X - Y)(\eta) &= 0, \quad \eta_{|\Gamma_1} = z, \\
(X - Y)(\eta) &= 0, \quad \eta_{|\Gamma_2} = -z.
\end{align*}
\]

Their solutions are respectively

\[
\begin{align*}
\hat{f}_1(x, y, z, w) &= z - \frac{1}{4}(x^2 - y^2), \\
\hat{f}_2(x, y, z, w) &= -z - \frac{1}{4}(x^2 - y^2).
\end{align*}
\]

The horizontal gradients are computed to be

\[ \nabla_H \hat{f}_1 = \frac{1}{2}(x - y)(X - Y), \]
\[ \nabla_H \hat{f}_2 = \frac{1}{2}(x + y)(X + Y). \]

Note that \( \nabla_H \hat{f}_i \) is null f.d. on \( \{|y| < x\}, i = 1, 2 \).

Next consider the following Cauchy problems (cf. [12]):

(3.5) \[ (X - Y)(\eta) = 0, \quad \eta|_{\Gamma_1} = w, \]

(3.6) \[ (X + Y)(\eta) = 0, \quad \eta|_{\Gamma_2} = w, \]

(3.7) \[ (X + Y)(\eta) = 0, \quad \eta|_{y=0} = -w, \]

(3.8) \[ (X - Y)(\eta) = 0, \quad \eta|_{y=0} = -w. \]

Their solutions are:

\[ \hat{g}_1(x, y, z, w) = w - \frac{1}{16}(x^2 - y^2)(x + 3y), \]

\[ \hat{g}_2(x, y, z, w) = w - \frac{1}{16}(x^2 - y^2)(x - 3y), \]

\[ \hat{g}_3(x, y, z, w) = -w - \frac{1}{4}(xy^2 - y^3), \]

\[ \hat{g}_4(x, y, z, w) = -w - \frac{1}{4}(xy^2 + y^3), \]

respectively. Their horizontal gradients are:

\[ \nabla_H \hat{g}_1 = \frac{1}{16}(x - y)(x + 3y)(X - Y), \]

\[ \nabla_H \hat{g}_2 = \frac{1}{16}(x + y)(x - 3y)(X + Y), \]

\[ \nabla_H \hat{g}_3 = \frac{1}{2}y^2(X + Y), \]

\[ \nabla_H \hat{g}_4 = \frac{1}{2}y^2(X - Y). \]

It is easy to see [12] that all \( \nabla_H \hat{g}_i \) are null fields, and \( \nabla_H \hat{g}_1 \) is f.d. on the set \( \{-\frac{1}{4}x < y < x, x > 0\} \), \( \nabla_H \hat{g}_2 \) is f.d. on \( \{-x < y < \frac{1}{4}x, x > 0\} \), finally \( \nabla_H \hat{g}_3 \) and \( \nabla_H \hat{g}_4 \) are f.d. on \( \{y \neq 0, x > 0\} \). Let us define the following subsets of \( \mathbb{R}^4 \):

\[ A_{11} = \left\{ \hat{f}_1 \leq 0 \right\} \cap \{\hat{g}_1 \leq 0\} \cap \{x \geq 0, y \geq 0, z \geq 0, w \geq 0\}, \]

\[ A_{12} = \left\{ \hat{f}_1 \leq 0 \right\} \cap \{\hat{g}_2 \leq 0\} \cap \{x \geq 0, y \leq 0, z \geq 0, w \geq 0\}, \]

\[ A_{13} = \left\{ \hat{f}_1 \leq 0 \right\} \cap \{\hat{g}_3 \leq 0\} \cap \{x \geq 0, y \geq 0, z \geq 0, w \leq 0\}, \]

\[ A_{14} = \left\{ \hat{f}_1 \leq 0 \right\} \cap \{\hat{g}_4 \leq 0\} \cap \{x \geq 0, y \leq 0, z \geq 0, w \leq 0\}, \]

\[ A_{21} = \left\{ \hat{f}_2 \leq 0 \right\} \cap \{\hat{g}_1 \leq 0\} \cap \{x \geq 0, y \geq 0, z \leq 0, w \geq 0\}, \]

\[ A_{22} = \left\{ \hat{f}_2 \leq 0 \right\} \cap \{\hat{g}_2 \leq 0\} \cap \{x \geq 0, y \leq 0, z \leq 0, w \geq 0\}, \]

\[ A_{23} = \left\{ \hat{f}_2 \leq 0 \right\} \cap \{\hat{g}_3 \leq 0\} \cap \{x \geq 0, y \geq 0, z \leq 0, w \leq 0\}, \]

\[ A_{24} = \left\{ \hat{f}_2 \leq 0 \right\} \cap \{\hat{g}_4 \leq 0\} \cap \{x \geq 0, y \leq 0, z \leq 0, w \leq 0\}. \]

Using corollary 3.1, the remark on horizontal gradients from the beginning of this section, and computations made in [11], [12] we obtain

**Proposition 3.2.** Let \( J^+(0) \) be the reachable set from zero for the flat Engel structure. Then

\[ J^+(0) \subset \bigcup_{i=1,2} \bigcup_{j=1,...,4} A_{ij}. \]
Remark 3.1. It should be mentioned that our flat case was treated by Krener and Schattler in [21]. If $Z_1, Z_2, Z_3$ are vector fields, denote by $Z_1 Z_2 Z_3$ the curve starting from the origin which is a concatenation of a segment of the trajectory of $Z_1$ starting from the origin with a segment of a trajectory of $Z_2$ and a segment of a trajectory of $Z_3$. The authors observed that geometrically optimal curves are the following concatenations: $(X + Y) X (X + Y), (X + Y) X (X - Y), (X - Y) X (X - Y), (X - Y) X (X + Y)$, and also $(X + Y) (X - Y) (X + Y), (X - Y) (X + Y) (X - Y)$ where in the last series the following restriction on time applies: the time along the intermediate arc is greater than or equal to the sum of times along the first and the last arc.

3.3. Some estimates in the general case. Now consider a structure generated by the frame $X, Y$ as in (3.1), i.e. $\varphi = \varphi(x, y, w)$ and $\psi_2 = \psi_2(x, y, w)$ but additionally assume that all objects are real analytic. Fix a normal neighbourhood $U$ of the origin and consider in $U$ the Cauchy problems (3.3), ..., (3.8) (where $X, Y$ are as in (3.1)). Denote respective solutions by $f_i, i = 1, 2$, and $g_j, j = 1, ..., 4$. Again according to [11], [12] $f_i = f_i + O(r^4), i = 1, 2$, and $g_j = g_j + O(r^4), j = 1, ..., 4$, where $r = \sqrt{x^2 + y^2 + z^2 + w^2}$. Also the horizontal gradients keep the suitable signs, provided $U$ is sufficiently small. Now take a semi-analytic set $\Sigma$ from theorem 1.2 in [12]. Considering $\Sigma$ as a subset of $\mathbb{R}^4$, $\Sigma$ becomes a set of dimension 3, and hence $U \cap \{x \geq 0\} \setminus \Sigma$ has two connected components $\Sigma^+$ and $\Sigma^-$. Let us agree that $\Sigma^+$ contains the trajectory of $X + Y$ starting from 0. Now, if we define

$$A_{11} = \{f_1 \leq 0\} \cap \{g_1 \leq 0\} \cap \Sigma^+ \cap U \cap \{x \geq 0, z \geq 0, w \geq 0\},$$

$$A_{12} = \{f_1 \leq 0\} \cap \{g_2 \leq 0\} \cap \Sigma^- \cap U \cap \{x \geq 0, z \geq 0, w \geq 0\},$$

$$A_{13} = \{f_1 \leq 0\} \cap \{g_3 \leq 0\} \cap U \cap \{x \geq 0, y \geq 0, z \geq 0, w \leq 0\},$$

$$A_{14} = \{f_1 \leq 0\} \cap \{g_4 \leq 0\} \cap U \cap \{x \geq 0, y \leq 0, z \geq 0, w \leq 0\},$$

$$A_{21} = \{f_2 \leq 0\} \cap \{g_1 \leq 0\} \cap \Sigma^+ \cap U \cap \{x \geq 0, z \leq 0, w \geq 0\},$$

$$A_{22} = \{f_2 \leq 0\} \cap \{g_2 \leq 0\} \cap \Sigma^- \cap U \cap \{x \geq 0, z \leq 0, w \geq 0\},$$

$$A_{23} = \{f_2 \leq 0\} \cap \{g_3 \leq 0\} \cap U \cap \{x \geq 0, y \leq 0, z \leq 0, w \leq 0\},$$

$$A_{24} = \{f_2 \leq 0\} \cap \{g_4 \leq 0\} \cap U \cap \{x \geq 0, y \leq 0, z \leq 0, w \leq 0\},$$

then again using corollary 3.1 and computations from [11], [12] we get

Proposition 3.3. Let $J^+(0, U)$ be the reachable set from zero for analytic Engel structure as in (3.1). Then

$$J^+(0, U) \subset \bigcup_{i=1,2} \bigcup_{j=1,\ldots,4} A_{ij}.$$
3.4. **Final remarks.** First let us notice that for all Engel sub-Lorentzian structures the half-lines \( \{ y = \pm x, \ z = w = 0 \} \) (in coordinates from theorem 1.1) are geometrically optimal. Indeed, this follows from theorem proved in [10] and asserting that null f.d. Hamiltonian geodesics are geometrically optimal.

Secondly, in all cases treated in proposition 3.1, the curves \((X - Y) (X + Y), (X + Y) (X - Y), X (X - Y), X (X + Y)\) (we use here the notation from remark 3.1) are geometrically optimal - this follows from proposition 3.1 and the properties of Martinet sub-Lorentzian structures described in [12].

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Faculty of Mathematics and Science, Cardinal Stefan Wyszyński University, ul. Dewajtis 5, 01-815 Warszawa, Poland

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E-mail address: m.grochowski@uksw.edu.pl