Boson-boson effective nonrelativistic potential for higher-derivative electromagnetic theories in D dimensions

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The problem of computing the effective nonrelativistic potential $U_D$ for the interaction of charged scalar bosons within the context of D-dimensional electromagnetism with a cutoff, is reduced to quadratures. It is shown that $U_3$ cannot bind a pair of identical charged scalar bosons; nevertheless, numerical calculations indicate that boson-boson bound states do exist in the framework of three-dimensional higher-derivative electromagnetism augmented by a topological Chern-Simons term.

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I. INTRODUCTION

We consider in this Brief Report the problem of determining the effective charged-scalar-boson—charged-scalar-boson low energy potential $U_D$ arising from D-dimensional electromagnetism with a cutoff $a$. The Lagrangian concerning this theory can be written as

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{a^2}{2} \partial_{\nu} F^{\mu\nu} \partial^\lambda F_{\mu\lambda},$$

(1)

where $F_{\mu\nu} = \partial_{\nu} A_{\mu} - \partial_{\mu} A_{\nu}$ is the usual electromagnetic tensor field. Lagrangian (1) is gauge and Lorentz invariant; in addition, it leads to local field equations which are linear in the field quantities. At distances much larger than the cutoff, the fields described by Eq. (1) become essentially equivalent to the Maxwell fields.

It worth mentioning that Lagrangian (1) was proposed a long time ago, in 3+1 dimensions, by Podolsky and Schwed [1], in a rather different context. The main reason for investigating this theory is that recently it was shown that in its framework the electromagnetic mass of a point charge occurs in the equation of motion in a form consistent with special relativity; moreover, the exact equation of motion does not exhibit runaway solutions or non-causal behavior, when the cutoff is larger than half of the classical radius of the electron [2, 3]. Massive fermions, in turn, have their helicity flipped on account of their interaction with an electromagnetic field described by Podolsky’s generalized electrodynamics, while massless ones seem to be unaffected by the electromagnetic field as far as their helicity is concerned [4]. Now, as is well known, if one starts from a renormalizable theory and calculates the effective action one obtains typically several operators of dimension 6 (or higher) and not just Podolsky’s term [5]. This raises the question: Would these latter results change significantly if one takes all these terms into account? Since the amount of spin-flip is determined by the vector nature of the electromagnetic field but not by its spatial distribution, the aforementioned results will not change at all. What about the finiteness of the electromagnetic mass? Since the evaluation of the classical self-force acting on the point charge depends on the cutoff $a$ [2, 3], one is led to conjecture that the finiteness of the electromagnetic mass is a very specific property of Podolsky’s electrodynamics and not a general feature of higher-derivative theories. It is interesting that the same route that leads to Maxwell’s electrodynamics leads also to Podolsky’s electrodynamics provided we start from Podolsky’s electrostatic force instead of the usual Coulomb’s law [6, 7]. Unlike Maxwell’s electrodynamics, Podolsky’s generalized electrodynamics implies a finite value for the energy of a point charge in the whole space [8, 9].

We are motivated by two quite similar developments: In the first, we investigate whether $U_3$ can form “Cooper pairs”.

Our second topic is related to three-dimensional Podolsky-Chern-Simons theory. Based on the interesting discussions from Jackiw [10] about the consistency of the nonrelativistic limit of certain relativistically invariant quantum field theories, it can be shown that the Chern-Simons term alone is unable to form boson-boson bound states [11]. Nonetheless, numerical calculations indicate that the Podolsky term provides an stabilizing mechanism allowing for the existence of “Cooper pairs”.

We use natural units throughout; our signature is $(+,−,−,\cdots,−)$.

II. EFFECTIVE CHARGED-SCALAR-BOSON—CHARGED-SCALAR-BOSON LOW ENERGY POTENTIAL

We begin by describing a method for computing the effective nonrelativistic potential for the interaction of two charged spinless bosons of equal masses via a “Podolskian photon” exchange. The prescription is based on the marriage of nonrelativistic quantum mechanics and quantum field theory in the nonrelativistic limit. An algorithm for calculating the propagator is then presented.
The recipe is used afterward to get the propagator for higher-derivative electromagnetism in the Lorentz gauge. Finally, we reduce the problem of computing the effective nonrelativistic potential to quadratures.

A. The method

Nonrelativistic quantum mechanics tells us that in the first Born approximation the cross section for the scattering of two indistinguishable massive particles, in the center-of-mass frame (CM), is given by

$$\frac{d\sigma}{d\Omega} = \frac{1}{16\pi m^2} |\mathcal{M}|^2.$$  \hspace{1cm} (2)

On the other hand, from quantum field theory we know that the cross section, in the CM, for the scattering of two identical massive charged bosons by an electromagnetic field, can be written as

$$\frac{d\sigma}{d\Omega} = \left|\frac{m}{4\pi} \int r V(r) e^{ik \cdot r} d^{D-1} r \right|^2,$$  \hspace{1cm} (3)

where $k = p' - p$ is the transfer momentum.

From Eqs. (2) and (3) we come to the conclusion that the expression that enables us to compute the D-dimensional effective nonrelativistic potential has the form

$$V(r) = \frac{1}{4m^2 (2\pi)^{D-1}} \int d^{D-1} k \mathcal{M}_{N.R.} e^{-ik \cdot r},$$  \hspace{1cm} (4)

which clearly shows how the potential from quantum mechanics and the Feynman amplitude obtained via quantum field theory are related to each other.

Now, in the Lorentz gauge Podolsky’s scalar QED is described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{a^2}{2} \partial_\mu \phi \partial^\mu \phi^* F_{\mu\nu} - \frac{m^2}{2} \left( \partial_\nu A^\nu \right)^2 + (D_\mu \phi)^* D^\mu \phi - m^2 \phi^* \phi,$$  \hspace{1cm} (5)

where $D_\mu \equiv \partial_\mu + iQA_\mu$. Therefore, the interaction Lagrangian to order $Q$ for the process $S + S \rightarrow S + S$, where $S$ denotes a spinless boson of mass $m$ and charge $Q$, is $\mathcal{L}_{int} = iQ A^\mu (\partial_\mu \phi^* - \phi^* \partial_\mu \phi)$. The Feynman rule for the elementary vertex is shown in Fig. 1. Accordingly, the Feynman amplitude for the interaction of two charged spinless bosons of equal mass via a “Podolskian photon” exchange (see Fig. 2) is

$$\mathcal{M} = V^\mu(p,p') D_{\mu\nu}(k) V^\nu(q,q'),$$  \hspace{1cm} (6)

where $D_{\mu\nu}(k)$ designates the “Podolski photon” propagator.

B. The propagator

We propose now an algorithm for computing the propagator for electromagnetic theories with higher-derivatives, based on the usual transverse and longitudinal vector projector operators, namely $\theta_{\mu\nu} = \eta_{\mu\nu} - \partial_\mu \partial_\nu$, $\omega_{\mu\nu} = \frac{1}{2} \eta_{\mu\nu}$, which satisfy the relations $\theta_{\mu\nu} \eta_{\rho\sigma} = \delta_{\mu\rho} \delta_{\nu\sigma}$, $\theta_{\mu\nu} \omega_{\rho\sigma} = \theta_{\mu\rho} \omega_{\nu\sigma} + \omega_{\mu\rho} \theta_{\nu\sigma} = 0$, $\eta_{\mu\nu} \omega_{\rho\sigma} = 0$, where $\eta_{\mu\nu}$ is the Minkowski metric. The set of operators $\{\theta, \omega\}$ is a complete set of projector operators for rank-one tensors. Indeed, they are idempotent, mutually orthogonal and satisfy the completeness relation $[\theta + \omega]_{\mu\nu} = \eta_{\mu\nu} = I_{\mu\nu}$.

Let $\mathcal{L}$ be the Lagrangian for electromagnetism with higher-derivatives. Since $\mathcal{L}$ is a gauge-invariant Lagrangian, we add to it a gauge-fixing Lagrangian $\mathcal{L}_{gf}$, which implies that $\mathcal{L} \equiv \mathcal{L} + \mathcal{L}_{gf}$ can be written as $\mathcal{L} = \frac{1}{4} A^\mu O_{\mu\nu} A^\nu$. Expanding $O$ in the basis $\{\theta, \omega\}$, yields $O = x_1 \theta + x_2 \omega$. Accordingly, $O^{-1} = y_1 \theta + y_2 \omega$, where $O^{-1}$ is the propagator and $y_1$ and $y_2$ are parameters to be determined. Now, taking into account that $OO^{-1} = I$, we promptly obtain

$$O^{-1} = \frac{1}{x_1} \theta + \frac{1}{x_2} \omega.$$  \hspace{1cm} (7)

FIG. 1: The relevant vertex for boson-boson interaction.

FIG. 2: One-Podolskian photon-exchange contribution to the scattering of two identical massive charged boson.
where we are supposing that both \( x_1 \) and \( x_2 \) are non-vanishing. Note that the procedure we have just outlined is quite straightforward: on the one hand it reduces the work of calculating the propagator to a trivial algebraic exercise; on the other hand it great simplifies calculations involving the contraction of conserved currents \((\partial_\mu J^\mu = 0)\) with the propagator since in this case the alluded contraction simply gives

\[
O^{-1}\mu\nu J^\mu = \frac{J^\nu}{x_1}.
\]  

(7)

From the above we find that the propagator for Podolsky’s electrodynamics in the Lorentz gauge assumes the form

\[
D_{\mu\nu}(k) = \frac{M^2}{k^2(k^2 - M^2)} \theta_{\mu\nu} - \frac{\lambda}{k^2} \omega_{\mu\nu},
\]

(8)

where \( M^2 \equiv \frac{1}{\alpha^2} \).

C. The potential

From Eqs. (6), (7) and (8), we get immediately \( \mathcal{M} = \frac{M^2 Q^2 (2p - k)(2q + k)}{k^2(k^2 - M^2)} \), which implies

\[
\mathcal{M}_{N.R.} = \frac{4\alpha^2 M^2 Q^2}{k^2(k^2 + M^2)}.
\]

(9)

Inserting Eq. (9) into Eq. (4), we obtain

\[
V(r) = \int_0^\infty f(|\mathbf{k}|)|\mathbf{k}|^{n-1} d|\mathbf{k}| \int_0^{2\pi} d\theta_1 \times \int_0^\pi \sin \theta_2 d\theta_2 \int_0^\pi \sin^2 \theta_3 d\theta_3 
\times \int_0^\pi e^{-i|\mathbf{k}| r \cos \theta_{n-1}} \sin^{n-2} \theta_{n-1} d\theta_{n-1},
\]

where \( 2 < n = D - 1 \) and \( f(|\mathbf{k}|) = \frac{|\mathbf{k}|^2}{(2\pi)^n} \left( \frac{1}{k^2} - \frac{1}{k^2 - M^2} \right) \).

Now, taking into account that

\[
\int_0^\pi \sin^n \theta d\theta = \frac{\sqrt{\pi} \Gamma \left(\frac{m-1}{2}\right)}{\Gamma \left(\frac{m+2}{2}\right)}.
\]

\[
\int_0^\pi e^{-i|\mathbf{k}| r \cos \theta_{n-1}} \sin^{n-2} \theta_{n-1} d\theta_{n-1} = \frac{2^{\frac{m-2}{2}} \Gamma \left(\frac{1}{2}\right)}{|\mathbf{k}|^{\frac{m-2}{2}}} \times \Gamma \left(\frac{n-1}{2}\right) \times J_{\frac{n-2}{2}} (|\mathbf{k}| r),
\]

where \( J \) denotes the Bessel function, we arrive at the following expression for the potential

\[
U_D(r) = \frac{Q}{(2\pi)^{\frac{D-1}{2}}} \frac{4\alpha^2 M^2 Q^2}{k^2(k^2 + M^2)} \int_0^\infty \left( \frac{1}{k^2} - \frac{1}{k^2 - M^2} \right) |\mathbf{k}|^\frac{D-1}{2} J_{\frac{D-1}{2}} (|\mathbf{k}| r) d|\mathbf{k}|,
\]

(10)

which is just the same result as that obtained in Podolsky’s electromagnetic theory [1].

III. PLANAR QUADRATIC ELECTROMAGNETISM

For \( D = 3 \), Eq. (10) yields

\[
U_3(r) = -\frac{Q}{2\pi} \left[ \ln \frac{r}{r_0} + K_0(Mr) \right],
\]

(11)

where \( r_0 \) is an infrared regulator and \( K \) is the modified Bessel function.

We discuss now the existence of boson-boson bound states in the context of planar quadratic electromagnetism. The corresponding time-independent Schrödinger equation can be written as
\[ H_r R_{nl} = \frac{1}{m} \left( \frac{d^2}{dt^2} R_{nl} + \frac{1}{d \sqrt{r}} \frac{d}{dr} R_{nl} \right) + V^{\text{eff}}_r R_{nl}, \]
\[ = E_{nl} R_{nl}, \]
\[ V^{\text{eff}}_r = \frac{i^2}{m r^2} + Q U_3(r) \]
\[ = \frac{i^2}{m r^2} - \frac{Q^2}{2 \pi} \ln \frac{r}{r_0} + K_0(Mr), \]
where \( R_{nl} \) is the \( n \)th normalizable eigenfunction of the radial Hamiltonian \( H_r \) whose corresponding eigenvalue is \( E_{nl} \) and \( V^{\text{eff}}_r \) is the \( l \)th partial wave effective potential. On the other hand,
\[ \frac{d}{dr} V^{\text{eff}}_r = - \frac{2i^2}{m r^3} - \frac{Q^2}{2 \pi r} + \frac{Q^2 M}{2 \pi} K_1(Mr), \]
which allows us to conclude that \( \frac{d}{dr} V^{\text{eff}}_r < 0 \) in the interval \( 0 < r < \infty \), implying that \( V^{\text{eff}}_r \) is strictly decreasing in this interval. Consequently, in the framework of planar quadratic electromagnetism, no bound state concerning the two charged scalar bosons system exists.

IV. PODOLSKY-CHERN-SIMONS PLANAR ELECTROMAGNETISM

Since boson-boson bound states do not show up in Podolsky planar electromagnetism, we investigate here whether the effective boson-boson low energy potential related to Podolsky-Chern-Simons (PCS) planar theory can bind a pair of identical charged scalar bosons. The Lagrangian for PCS scalar QED, in the Lorentz gauge, can be written as
\[ \mathcal{L} = - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{a^2}{2} \partial_{\mu} F^{\mu \nu} \partial_{\nu} F^{\mu \nu} - \frac{1}{2 \lambda} (\partial_{\mu} A^\nu)^2 + (D_\mu A^\nu)^* D^\nu \phi - m^2 \phi^* \phi + \frac{s}{2} \varepsilon_{\mu \nu \rho \sigma} A^\mu \partial^\nu A^\rho, \quad (12) \]
where \( s > 0 \) is the topological mass.

In the basis \( \{ \theta, \omega, S \} \), where \( S_{\mu \nu} \equiv \varepsilon_{\mu \nu \rho \sigma} \partial^\rho \), the propagator assumes the form
\[ O^{-1} = \frac{A^2 k^4 - k^2}{(a^2 k^4 - k^2)^2 - s^2 k^2} \frac{\lambda \omega}{k^2} \frac{s S}{(a^2 k^4 - k^2)^2 - s^2 k^2}. \]

Now, in the nonrelativistic limit the Feynman amplitude for the process shown in Fig. 2 reduces to
\[ \mathcal{M}_{NR} = \left[ \frac{8 i s m Q^2}{a^4} k \wedge \mathbf{P} \right] \left[ \sum_{j=1}^{3} \frac{B_j}{k^2 - x_j} + \frac{a^4}{8 s^2 k^2} \right] + \frac{4 Q^2 m^2}{a^4} \sum_{j=1}^{3} \frac{A_j}{k^2 - x_j}, \]
where \( x_1, x_2, x_3 \) are the roots of Eq. (13) and
\[ A_1 \equiv \frac{(x_1 - x_2)(x_1 - x_3)}{x_1 - x_2}, \quad A_2 \equiv \frac{(x_1 - x_1)(x_2 - x_3)}{x_1 - x_2}, \quad A_3 \equiv \frac{(x_2 - x_3)(x_3 - x_1)}{x_2 - x_3}, \quad B_1 \equiv \frac{-(1 + a^2 x_1)}{s^2 (x_1 - x_2)(x_1 - x_3)}, \quad B_2 \equiv \frac{-(1 + a^2 x_2)}{s^2 (x_2 - x_1)(x_2 - x_3)}, \quad B_3 \equiv \frac{-(1 + a^2 x_3)}{s^2 (x_3 - x_1)(x_3 - x_2)}.
\]

It follows that the effective nonrelativistic potential can be calculated from the expression
\[ U_3(r) = \frac{i s Q}{\pi m a^4} \left[ \frac{a^4}{s^2} \lim_{\sigma \to 0} \int_0^\infty \frac{(k \wedge \mathbf{P}) J_0(|k| r) |k| d|k|}{k^2 + \sigma^2} \right] + \sum_j \int_0^\infty \frac{(k \wedge \mathbf{P}) B_j J_0(|k| r) |k| d|k|}{k^2 - x_j} + \frac{Q}{2 \pi a^2} \sum_j \int_0^\infty \frac{A_j}{k^2 - x_j} J_0(|k| r) |k| d|k|. \]

Performing the computations, we obtain
\[ U_3(r) = - \frac{s Q}{\pi m a^4} \left[ \frac{a^4}{s^2} r^2 + \frac{1}{r} \sum_j B_j \sqrt{|x_j|} K_1(\sqrt{|x_j|} r) \right] L + \frac{Q}{2 \pi a^2} \left[ \sum_j A_j K_0(\sqrt{|x_j|} r) \right], \quad (14) \]
where \( \mathbf{L} \equiv \mathbf{r} \wedge \mathbf{P} \) is the orbital angular momentum.

Using Jackiw’s arguments \([10]\), one can show that the topological term alone is unable to bind the charged scalars bosons \([11]\).

We return now to the problem of probing whether “Cooper pairs” exist in the framework of PCS scalar QED. In this case the radial Schrödinger equation is

\[
\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right] R_{nl} + m \left[ E_{nl} - V_l^{\text{eff}} \right] R_{nl} = 0, \tag{15}
\]

where

\[
V_l^{\text{eff}}(r) = -\frac{sQ^2}{\pi ma^4} \left[ \frac{a^4}{s^2 r^2} + \frac{1}{r} \sum_j B_j \sqrt{|x_j|} K_1(\sqrt{|x_j|} r) \right] + \frac{Q^2}{2\pi a^4} \sum_j A_j K_0(\sqrt{|x_j|} r) + \frac{l^2}{mr^2}.
\]

Employing the dimensionless parameters \( y \equiv sr, \alpha \equiv \frac{Q^2}{\pi^2}, b_j \equiv \frac{2}{\pi} B_j, X_j \equiv \frac{2}{\pi^2}, \beta \equiv \frac{Q^2}{\pi^2}, a_j \equiv \frac{2}{\pi^2} \) and \( E_{nl} \equiv \frac{mE_{nl}}{s^2} \), we can rewrite Eq. (15) as

\[
\left[ \frac{d^2}{dy^2} + \frac{1}{y} \frac{d}{dy} \right] R_{nl} + \left[ \tilde{E}_{nl} - \tilde{V}_l^{\text{eff}} \right] R_{nl} = 0, \tag{16}
\]

with

\[
\tilde{V}_l^{\text{eff}} = -\frac{l(\alpha - l)}{y^2} + \frac{\alpha \beta}{2} \sum_j a_j K_0(X_j y) - \frac{\alpha l}{y} \sum_j b_j X_j K_1(X_j y).
\]

Note that \( \tilde{V}_l^{\text{eff}} \) behaves as \( \frac{1}{y^2} \) at the origin and as \( \frac{l(\alpha - l)}{y^2} \) asymptotically. On the other hand, the derivative of this potential with respect to \( y \) is given by

\[
\frac{d}{dy} \tilde{V}_l^{\text{eff}} = \frac{2l(\alpha - l)}{y^3} + \alpha \sum_j \left[ \frac{2l}{y^2} b_j - \frac{\beta a_j}{2} \right] X_j K_1(X_j y) + \frac{\alpha l}{y} \sum_j b_j X_j^2 K_0(X_j y) - \frac{\alpha l}{y^2} K_0(y) - \frac{\alpha l}{y} K_1(y).
\]

In order to find out whether or not boson-boson bound states could be formed, we shall analyze how \( \frac{d}{dy} \tilde{V}_l^{\text{eff}} \) behaves for small values of the cutoff \( a \). Indeed, only if \( a \ll 1 \) will the well recognized properties of QED be preserved. In this limit, we get

\[
\frac{d}{dy} \tilde{V}_l^{\text{eff}} \sim \frac{2l(\alpha - l)}{y^3} - \frac{\alpha l}{y^2} X_1(y) - \frac{\alpha l}{y} K_0(y).
\]

We assume from now on \( a \ll l > 0 \), without any loss of generality. It is trivial to see that if \( l > \alpha \), the potential is strictly decreasing, which precludes the existence of bound states. The remaining possibility is \( l < \alpha \). In this interval \( \tilde{V}_l^{\text{eff}} \) approaches \(+\infty\) at the origin and \( 0^- \) for \( y \to +\infty \), which is indicative of a local minimum. Therefore, the existence of “Cooper pairs” is subordinated to the conditions \( a \ll 1 \) and \( 0 < l < \alpha \).

Of course, it is impossible to solve Eq. (16) analytically; however, it can be solved numerically. To do that, we rewrite beforehand the radial function as \( R_{nl} \equiv \frac{u_{nl}}{\sqrt{y}} \).

As a consequence, Eq. (16) takes the form

\[
\left[ \frac{d^2}{dy^2} + \frac{1}{4y^2} \right] u_{nl} + \left[ \tilde{E}_{nl} - \tilde{V}_l^{\text{eff}} \right] u_{nl} = 0. \tag{17}
\]

Using the Numerov algorithm \([12]\), we solved Eq. (17) numerically for several values of the parameters \( \alpha, \beta, \) and \( l \), keeping the cutoff \( a \) fixed. The latter was chosen equal to \( \frac{4l}{2l - a} = 9.52033 \times 10^3 \text{MeV}^{-1} \), where \( r_c \) is the four-dimensional classical radius of the electron. It is worth mentioning that the anomalous factor of \( \frac{4}{3} \) in the inertia related to the Abraham-Lorentz model for the electron does not show up if \( a > \frac{4}{3} r_c \) \([2,3]\).

In Fig. 3 we present our numerical results for the potential and the corresponding radial eigenfunctions concerning the first three bound states in the specific case of \( l = 4 \). The associated energies are

\[
E_{34} = -6.37501 \times 10^{-7} \text{MeV}, \quad E_{24} = -1.2536 \times 10^{-7} \text{MeV}, \quad E_{34} = -5.22341 \times 10^{-8} \text{MeV}.
\]

The graphs shown in Fig. 3 exhibit, in a sense, the generic features of the potential and of the radial eigenfunctions, although they have been composed using particular values of the parameters \( \alpha, \beta, \) and \( a \). A detailed study of the modifications of the effective potential induced by radioactive corrections, as well as the corresponding alterations to the eigenvalue structure, will be published elsewhere \([11]\).

To conclude, we remark that “Cooper pairs” exist in the context of PCS scalar QED if \( a \ll 1 \) and \( 0 < l < \alpha \).
FIG. 3: $V_{4}^{\text{eff}}$ with the lowest three allowed energies and the corresponding energy eigenfunctions. Here $[V_{4}^{\text{eff}}] = eV$, $[r] = MeV^{-1}$, $\alpha = 8$, $\beta = 2000$ and $a = 0.00952$ MeV$^{-1}$.

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