Existence of symplectic surfaces

Tian-Jun Li
School of Mathematics
University of Minnesota
Minneapolis, MN 55455
tjli@math.umn.edu

Abstract. In this paper we show that every degree 2 homology class of a 2n-dimensional symplectic manifold is represented by an immersed symplectic surface if it has positive symplectic area. Moreover, the symplectic surface can be chosen to be embedded if 2n is at least 6. We also analyze the additional conditions under which embedded symplectic representatives exist in dimension 4.

1 Introduction

In this paper we will address the question which 2 dimensional homology class of a symplectic manifold \((M, \omega)\) can be represented by an embedded symplectic submanifold. To study this problem, we first explore Gromov’s beautiful h-principles on symplectic embeddings and immersions. We are able to give a complete answer in dimensions 6 and above and derive some interesting consequences. Clearly, a necessary condition for a class \(A \in H_2(M; \mathbb{Z})\) to be represented by an embedded (or an immersed) symplectic surface is that \(\omega(A) > 0\). We call such a class a \(\omega\)–positive class. Our observation is that every \(\omega\)–positive class is in fact effective.

**Theorem 1.1** Suppose \((M, \omega)\) is a symplectic manifold of dimension 2n. Let \(A\) be any \(\omega\)–positive class in \(H_2(M; \mathbb{Z})\). Then
1. \(A\) is represented by a connected embedded \(\omega\)–symplectic surface if \(2n \geq 6\).
2. \(A\) is represented by a connected immersed \(\omega\)–symplectic surface if \(2n \geq 4\).

Here, an embedding \(\Sigma \subset M\) is called symplectic if \(\omega\) restricts to a symplectic form on \(\Sigma\). Similarly, an immersion \(i : \sigma \to M\) is called symplectic if \(i^*\omega\) is a symplectic form on \(\Sigma\). This result should also be known to some experts, e.g. see the recent preprint [15].

We also analyze in dimension 4 when the immersed symplectic surface can be made embedded. In view of the normal connected sum operation in [8] and [26], it is a very important question. We will analyze some obstructions and also describe several constructions.

1991 Mathematics Subject Classification. Primary 57R57, 58D15.

The author is supported in part by NSF grant 0435099 and the McKnight fellowship.

©0000 American Mathematical Society
The author wishes to thank L. Ein, C. Leung, B. H. Li and Y. Ruan for very helpful discussions. The author also wishes to thank the referee for the careful readings and pointing out the reference [15].

2 Symplectic surfaces via h-principles

2.1 Smoothly embedded surfaces, monomorphisms and h-principles.
In this section $\Sigma$ is a closed oriented surface. We first collect some basic and well-known facts of representing a homology class of degree 2 by $\Sigma$.

**Lemma 2.1** (see Hopf [10]) Let $M$ be a connected oriented smooth manifold of dimension $2n$ and $A$ a homology class of degree 2. Then $A$ is represented by a continuous map from an oriented surface $\Sigma$ to $M$.

**Lemma 2.2** Let $M, A$ be as in Lemma 2.1 and suppose that $A$ is represented by a continuous map $f$ from an oriented surface $\Sigma$ to $M$. Then
1. When $2n \geq 6$, $f : \Sigma \longrightarrow M$ is homotopic to an embedding of $\Sigma$ into $M$.
2. When $2n = 4$, $f : \Sigma \longrightarrow M$ is homotopic to an immersion $g : \Sigma \longrightarrow M$; and the double points of the immersed surface $g(\Sigma)$ can be eliminated to get an embedded surface of higher genus still representing $A$.

Lemma 2.2 follows from the standard transversality theory: the continuous map $f$ can be first perturbed to an immersion when $2n \geq 2 \cdot 2$; furthermore the immersion can be approximated by an embedding when $2n \geq 2 \cdot 2 + 1$.

When $2n \geq 4$, any disconnected embedded (or immersed) surface can be tubed to get a connected embedded (or immersed) surface representing the same class. Thus, following from Lemmas 2.1 and 2.2, we have

**Proposition 2.3** (see Thom [36]) Let $M$ be a connected oriented smooth manifold of dimension $2n \geq 4$ and $A$ a homology class of degree 2. Then
1. $A$ is represented by a connected embedded surface.
2. If $A$ is a spherical class, then $A$ is represented by an embedded sphere if $2n \geq 6$ and an immersed sphere if $2n = 4$.

Here, a class is said to be spherical if it is represented by a continuous map from $S^2$ to $M$. In other words, it is in the image of the Hurewicz homomorphism $\pi_2(M) \longrightarrow H_2(M; \mathbb{Z})$.

Now we study the the space of real and complex monomorphisms between bundles over $\Sigma$.

**Lemma 2.4** Let $\Sigma$ be a closed oriented surface and $E$ and $F$ be complex vector bundles over $\Sigma$ with real dimensions 2 and $2n \geq 4$ respectively. Then the space of complex monomorphisms $\text{Mono}_c(E, F)$ is non-empty and connected. And the space of real monomorphisms $\text{Mono}_r(E, F)$ is non-empty and connected if $2n \geq 6$.

**Proof** Let us first consider the bundle $\text{HOM}_c(E, F)$ of complex homomorphisms. The stratified subbundle of singular homomorphisms has complex codimension $n$. By transversality, if $2n$ is bigger than the real dimension of the base which is 2, then a generic complex homomorphism is a complex monomorphism. And if $2n > 2 + 1$, then every generic complex homomorphism over $\Sigma \times [0, 1]$ is a complex monomorphism. Thus the claim about $\text{Mono}_c(E, F)$ is proved.

Let us now consider the bundle $\text{HOM}_r(E, F)$ of real homomorphisms. The stratified subbundle of singular homomorphisms now has real codimension $[2 - (2 - 1)]/[2n - (2 - 1)] = 2n - 2 + 1 = 2n - 1$. By transversality, if $2 < 2n - 1$, i.e. $2n \geq 4$
then a generic homomorphism is a monomorphism. Furthermore, if $2 + 1 < 2n - 1$, then a generic path of homomorphisms consists of monomorphisms only. The proof is now complete. 

**Remark 2.5** When $2n = 4$, a simple obstruction theory argument can be used to show that the connected components of the real monomorphisms $\text{Mono}_R(E, F)$ are indexed by the Euler number of the quotient bundle, which can be any integer congruent to $c_1(F) - c_1(E)$ modulo $2$. For a complex monomorphism, the Euler number of the quotient bundle is of course equal to $c_1(F) - c_1(E)$. For more on complex and real monomorphisms see [13].

Recall the notion of the normal Euler number of an immersion of an oriented surface in an oriented 4-manifold. Given an immersion $f : \Sigma \to M$, the differential $Df$ is a monomorphism from $T\Sigma$ to the pull-back bundle $f^*TM$. The normal bundle $N(f)$ of $f$ is then the quotient bundle of $f^*TM$ by $Df(T\Sigma)$, and the Euler number of $N(f)$ is called the normal Euler number of $f$. Suppose $A$ is represented by a continuous map from $\Sigma$. We can apply Remark 2.5 and Hirsch’s fundamental result on immersions in [9] (which can be considered as the h-principle for smooth immersions) to realize $A$ by an immersion from $\Sigma$ to $M$ with any normal Euler number as long as it is congruent to $A \cdot A$ modulo $2$.

Now we state the relevant h-principles on symplectic embeddings and immersions (see [4]). We first need to introduce several definitions. Let $(M, \omega_M)$ and $(V, \omega_V)$ be symplectic manifolds of dimensions $2n$ and $2q$ respectively. Let $f : V \to M$ be a continuous map and $F : TV \to TM$ be a real homomorphism which covers $f$. The map $f$ is called an isosymplectic immersion if $f^*\omega_M = \omega_V$, and it is called a symplectic embedding if it is also a smooth embedding. Notice that a symplectic immersion is automatically a smooth immersion. The homomorphism $F$ is called symplectic if $F^*\omega_M$ is non-degenerate and $F^*[\omega_M] = [\omega_V]$, and $F$ is called isosymplectic if it is symplectic and $F^*\omega_M = \omega_V$. Notice that symplectic and isosymplectic homomorphisms are necessarily monomorphisms.

We begin with the embedded version.

**Theorem 2.6** (Gromov’s isosymplectic embedding theorem) Let $(M, \omega_M)$ and $(V, \omega_V)$ be symplectic manifolds of dimensions $2n$ and $2q$ respectively. Suppose that an embedding $f_0 : V \to M$ satisfies the cohomological condition $f_0^*[\omega_M] = [\omega_V]$, and that the differential $F_0 = Df_0$ is homotopic via a homotopy of monomorphisms $F_t : TV \to TM$ covering $f_0$ to an isosymplectic homomorphism $F_1 : TV \to TM$. If $V$ is closed and $2n \geq 2q + 4$, then there exists an isotopy $f_t : V \to M$ such that the embedding $f_1 : V \to M$ is isosymplectic and the differential $Df_1$ is homotopic to $F_1$ through isosymplectic homomorphisms (with varying base maps). Moreover, one can choose the isotopy $f_t$ to be arbitrarily $C^0$-close to $f_0$.

Next is the immersed version.

**Theorem 2.7** (Gromov’s isosymplectic immersion theorem) Let $(M, \omega_M)$ and $(V, \omega_V)$ be symplectic manifolds of dimensions $2n$ and $2q$ respectively. Suppose a continuous map $f_0 : V \to M$ satisfies the cohomological condition $f_0^*[\omega_M] = [\omega_V]$, and that $f_0$ is covered by an isosymplectic homomorphism $F : TV \to TM$. If $V$ is closed and $2n \geq 2q + 2$, then there exists a homotopy $f_t : V \to M$ such that $f_1 : V \to M$ is an isosymplectic immersion, i.e. $f_1^*\omega_M = \omega_V$, and the differential $Df_1$ is homotopic to $F$ through isosymplectic homomorphisms.
Notice that, unlike the embedded case, \( f_0 \) is not assumed to be an immersion. And even when \( f_0 \) is an immersion, \( F \) is not necessarily homotopic to \( Df_0 \) via a homotopy of real monomorphisms.

In general, it is not easy to verify the conditions required in these two theorems. However, for embeddings and immersions of symplectic surfaces, we can effectively use almost complex structures to achieve these conditions.

### 2.2 Existence and some consequences.

We first briefly review how almost complex structures come into play. Given a non-degenerate 2–form \( \tau \) and an almost complex structure \( J \) on \( M \), we say \( \tau \) and \( J \) are tamed by each other if \( \tau(v,Jv) > 0 \) for any non-zero tangent vector \( v \). Notice that a closed 2–form on \( M \) is a symplectic form if and only if it is tamed by \( J \in \mathcal{J}_\omega \). Let \( \omega \) be a symplectic structure on \( M \), and \( \mathcal{J}_\omega \) be the space of the almost complex structures on \( M \) tamed by \( \omega \). Then for any \( J \in \mathcal{J}_\omega \), an embedded or immersed \( J \)–holomorphic submanifold is \( \omega \)–symplectic. Conversely, any embedded symplectic submanifold is \( J \)–holomorphic for some \( J \in \mathcal{J}_\omega \).

**Proposition 2.8** Suppose \((M,\omega)\) is a symplectic manifold of dimension \( 2n \) with symplectic canonical class \( K_\omega \). Let \( A \) be a homology class in \( H_2(M;\mathbb{Z}) \) with \( \omega(A) > 0 \), \( \Sigma \) be a connected oriented surface of genus \( g \), and \( f_0 : \Sigma \longrightarrow M \) an embedding representing \( A \). Then

1. There exists an embedding \( f_1 : \Sigma \longrightarrow M \) such that \( f_1(\Sigma) \) is a symplectic surface representing \( A \) if \( 2n \geq 6 \).
2. There exists an immersion \( f_1 : \Sigma \longrightarrow M \) such that \( f_1(\Sigma) \) is a symplectic surface representing \( A \) if \( 2n = 4 \). Moreover the normal Euler number of the immersion \( f_1 \) is given by the adjunction formula \( 2g - 2 - K_\omega(A) \).

**Proof** We first assume \( 2n \geq 6 \) so that the codimension 4 condition in Theorem 2.6 is satisfied.

Let \( j \) be a complex structure on \( \Sigma \) and \( J \) be an almost complex structure on \( M \) tamed by \( \omega_M \). By Lemma 2.4, we can find a complex monomorphism \( F_1 \) which is homotopic to the real monomorphism \( F_0 = Df_0 \) via a homotopy of real monomorphisms covering \( f_0 \).

To find an isosymplectic homomorphisms required in Theorem 2.6, we need the following two lemmas.

**Lemma 2.9** For any complex monomorphism \( F : (T\Sigma,j) \longrightarrow (TM,J) \) covering \( f_0 : \Sigma \longrightarrow M \), \( F^*\omega_M \) is non-degenerate on \( T\Sigma \) and is a closed 2–form with \( \int_{\Sigma} F^*\omega_M > 0 \).

**Proof** \( F^*\omega_M \) is closed on \( \Sigma \) since it is a 2–form on a 2–manifold. Since \( J \) is tamed by \( \omega_M \), we have \( \omega_M(v,Jv) > 0 \) for any \( x \) in \( M \) and any non-zero \( v \in T_xM \). Since \( F \) is a complex monomorphism, we have \( F(v) \neq 0 \) in \( T_{f_0(x)}M \), and

\[
F^*\omega_M(v,jv) = \omega_M(F(v),F(jv)) = \omega_M(F(v),J(F(v))) > 0,
\]

for any non-zero \( v \in T_x\Sigma \). \( \square \)

**Lemma 2.10** Let \( G_0 \) be a complex monomorphism from \((T\Sigma,j)\) to \((TM,J)\), and \( R \) be any positive real number. Then \( G_0 \) is homotopic via a homotopy of complex monomorphisms to a complex monomorphism \( G_1 \) such that

\[
\int_{\Sigma} G_1^*\omega_M = R.
\]
Existence of symplectic surfaces

**Proof** By Lemma 2.9, \( r = \int_\Sigma G_0^* \omega_M > 0 \). Since, for any positive real number \( t \), \( tG \) is still a complex monomorphism with \( (tG)^* \omega = t(G^* \omega) \), we find that
\[
G_t = [(1-t) + t \frac{R}{r}] G_0
\]
is a required homotopy. \( \square \)

Applying Lemma 2.10 to \( G_0 = F_{\frac{1}{2}} \) and \( R = \omega_M(A) \), we obtain a complex monomorphism \( F_1 = G_1 \) which is homotopic to \( F_{\frac{1}{2}} \) via a homotopy of (complex) monomorphisms covering \( f_0 \) and satisfies
\[
\int_\Sigma F_1^* \omega_M = \omega_M(A).
\]
Notice that the monomorphism \( F_1 \) is homotopic to \( F_0 = Df_0 \) via a homotopy of real monomorphisms covering \( f_0 \) since \( F_{\frac{1}{2}} \) is assumed to be so. Now if we let \( \omega_{\Sigma} = F_1^* \omega_M \), then \( [\omega_{\Sigma}] = f_0^* [\omega_M] \), and moreover \( \omega_{\Sigma} \) is a symplectic form on \( \Sigma \) by Lemma 2.9.

We have shown that \( F_1 \) is the required homotopy, therefore we can apply Theorem 2.6 to \( (\Sigma, \omega_{\Sigma}), (M, \omega_M), f_0 \) and \( F_1 \) to conclude that \( f_0 \) is isotopic to an embedding \( f_1 \) such that \( f_1^* \omega_M = \omega_{\Sigma} \). In particular, \( f_1(\Sigma) \) is an immersed \( \omega_M \)-symplectic surface representing the class \( A \).

Now we assume that \( 2n = 4 \). Since the isosymplectic immersion theorem only requires codimension 2, it can be applied to this case.

By Lemma 2.4, there exists a complex monomorphism \( G : (T\Sigma, j) \longrightarrow (TM, J) \) covering \( f_0 \) whose quotient complex line bundle has Euler number
\[
\int_\Sigma f_0^* c_1(TM, J) - \int_\Sigma c_1(T\Sigma, j) = -K_\omega(A) - (2 - 2g).
\]
Applying Lemma 2.10 to \( G_0 = G \) and \( R = \omega_M(A) \), we obtain a complex monomorphism \( F = G_1 \) which satisfies
\[
\int_\Sigma F^* \omega_M = \omega_M(A).
\]
Now let \( \omega_{\Sigma} = F^* \omega_M \), then \( [\omega_{\Sigma}] = f_0^* [\omega_M] \), and \( \omega_{\Sigma} \) is a symplectic form on \( \Sigma \) by Lemma 2.9.

We have shown that \( F \) is the required isosymplectic homomorphism, and therefore we can apply Theorem 2.7 to \( (\Sigma, \omega_{\Sigma}), (M, \omega_M), f_0 \) and \( F \) to conclude that \( f_0 \) is homotopic to an immersion \( f_1 \) such that \( f_1^* \omega_M = \omega_{\Sigma} \). In particular, \( f_1(\Sigma) \) is an immersed \( \omega_M \)-symplectic surface representing the class \( A \). The proof of Proposition 2.8 is now complete. \( \square \)

**Proof** of Theorem 1. It immediately follows from Lemma 2.1, Proposition 2.8 and the first part of Proposition 2.3. \( \square \)

**Remark 2.11** The same method can be used to show that the symplectic surfaces in Theorem 1.1 are symplectically isotopic.

As we will see in §3.2, there are other constructions of symplectic submanifolds. For an integral symplectic manifold \((M, \omega)\), Donaldson [3] uses an approximately holomorphic technique to construct codimension 2 symplectic submanifolds representing the Poincaré dual to high multiples of the \([\omega]\). In fact, Donaldson is able to show that any closed symplectic manifold contains symplectic submanifolds of any even codimension. There have been various generalizations (see e.g. [1], [24], [28]).
However, the constructed symplectic submanifolds only represent homology classes close to the Poincaré duals to the powers of $[\omega]$. For 2-dimensional symplectic submanifolds, the h-principle applies to any class of degree 2. And we have better control of the genera of the symplectic surfaces. In particular, for manifolds which are not symplectically aspherical, we can find symplectic spheres. Recall that a symplectic manifold $(M, \omega)$ is said to be symplectically aspherical if $[\omega]$ vanishes on the image of the Hurewicz homomorphism $\pi_2(M) \to H_2(M; \mathbb{Z})$. The following result is a direct consequence of Proposition 2.8 and the second part of Proposition 2.3.

**Corollary 2.12** Suppose $(M, \omega)$ is a symplectic manifold of dimension $2n$. Let $A$ be a spherical homology class in $H_2(M; \mathbb{Z})$ with $\omega(A) > 0$. Then $A$ is represented by an immersed $\omega$-symplectic 2-sphere; and if $2n \geq 6$, $A$ is represented by an embedded $\omega$-symplectic 2-sphere. In particular there are symplectic spheres in $(M, \omega)$ if and only if it is not symplectically aspherical.

Specifically, when $M$ is simply connected, by the Hurewicz Theorem, every class of degree 2 is spherical, and hence represented by an embedded symplectic sphere if $2n \geq 6$ and an immersed symplectic sphere if $2n = 4$.

Since an embedded symplectic submanifold is $J$-holomorphic for some $J \in \mathcal{J}_\omega$, we also have the existence of embedded pseudo-holomorphic curves.

**Corollary 2.13** Let $(M, \omega)$ be a symplectic manifold of dimension at least 6, and $A$ a 2 dimensional homology class with $\omega(A) > 0$. Then $A$ is represented by an embedded $J$-holomorphic curve for some $J \in \mathcal{J}_\omega$. And if $M$ is simply connected, $A$ is represented by an embedded $J$-holomorphic sphere for some $J \in \mathcal{J}_\omega$.

However, in dimension 4, the immersed symplectic surfaces obtained in Proposition 2.8 cannot always be chosen to be pseudo-holomorphic. In fact, an immersed symplectic surface is $J$-holomorphic for some $J$ if and only if it is positively immersed, i.e. all the double points are positive double points. On the other hand, we will see from Lemma 3.16 that, for a fixed class $A$, the existence of a simple pseudo-holomorphic curve (not necessarily embedded) is equivalent to the existence of an embedded symplectic surface.

We end section 2.2 by showing that Theorem 1.1 implies a duality between surface cones and symplectic cones over $\mathbb{Q}$. We first need to introduce several definitions. Let $V$ be a vector space over $\mathbb{Q}$ or $\mathbb{R}$ and $V^*$ be the dual space of $V$. If $U$ is a subset of $V$, the dual of $U$ is the subset $U^*$ in $V^*$ given by $U^* = \{ \alpha \in V^* | \alpha(v) > 0 \text{ for any } v \in U \}$.

In the following, $V$ would either be $H^2(M; \mathbb{R})$ or $H^2(M; \mathbb{Q})$.

Define the $\omega$-effective set $\mathcal{A}_\omega$ by $\mathcal{A}_\omega = \{ A \in H_2(M; \mathbb{Z}) | A \text{ is represented by an embedded } \omega \text{-symplectic surface} \}$.

In terms of this definition, Theorem 1.1 is then simply the following statement.

**Corollary 2.14** If $2n \geq 6$, then we have $\mathcal{A}_\omega = [\omega]^* \cap H_2(M; \mathbb{Z})$, where $[\omega]^*$ is the real dual of $[\omega]$.

If we define the rational $\omega$-surface cone $\mathcal{S}_\omega^\mathbb{Q}$ (real $\omega$-surface cone $\mathcal{S}_\omega^\mathbb{R}$) as the convex cone in $H_2(M; \mathbb{Q})$ ($H_2(M; \mathbb{R})$) generated by $\mathcal{A}_\omega$, then we have the following weaker version of Theorem 1.1.
The $\omega$–surface cones are given by
\[ S^Q_\omega = [\omega]^* \cap H_2(M; \mathbb{Q}), \quad S^R_\omega = [\omega]^*. \]

**Proof** It is clear from Theorem 1.1 that a rational point in $[\omega]^*$ is in $S^Q_\omega$. Since any class in $A_\omega$ has positive $\omega$–area we always have $S^R_\omega \subset [\omega]^*$. Thus we obtain the first equality. To further show that $S^R_\omega = [\omega]^*$, we notice that $[\omega]^*$ is an open subset of $H_2(M; \mathbb{R})$. Hence, as observed in [2], every class $v \in [\omega]^*$ can be written as $v = \sum_i v_i$ with each $v_i$ on a rational ray in $[\omega]^*$. Thus $v \in S^R_\omega$ as each $v_i$ is in $S^Q_\omega$.

We can generalize the duality to a family of symplectic forms as follows. Let now $M$ be an oriented smooth manifold and $\Omega(M)$ be the space of orientation-compatible symplectic structures on $M$. There is a natural $\mathbb{R}^+$ action on $\Omega(M)$ obtained from multiplying a symplectic form by a positive real number. Let $W$ be any subset of $\Omega(M)$ invariant under the $\mathbb{R}^+$ action. Then the $W$–symplectic cone is defined as
\[ C_W = \{ [\omega] \in H^2(M; \mathbb{R}) | \omega \in W \}, \]
and the intersection of $C_W$ with $H^2(M; \mathbb{Z})$ is called the rational $W$–symplectic cone. The subset of $H^2(M; \mathbb{Z})$, $A_W = \cap_{\omega \in W} A_\omega$, is called the $W$–effective set. We similarly define the rational $W$–surface cone $S^Q_W$ as the convex cone in $H^2(M; \mathbb{Q})$ generated by $A_W$. We can similarly define $A^\text{imm}_W$ and immersed versions of other concepts.

Now we can state the duality between the rational $W$–surface cone and the rational $W$–symplectic cone.

**Corollary 2.16** Suppose $M$ is a closed, oriented manifold of dimension $2n$ admitting symplectic structures, i.e. $\Omega(M)$ is nonempty. If $2n \geq 6$, then for any subset $W \subset \Omega(M)$, we have
\[ A_W = \{ A \in H_2(M; \mathbb{Z}) | \omega(A) > 0 \text{ for any } \omega \in W \}. \]
Consequently, the rational $W$–surface cone is dual to the rational $W$–symplectic cone over $\mathbb{Q}$.

**Remark 2.17** Notice that we do not in general have the duality over $\mathbb{R}$. This is because when the quotient of $C_W$ under $\mathbb{R}^+$ is non-compact, the real dual of $C_W$ may have boundaries containing irrational rays.

When $2n = 4$, the conclusion in Corollary 2.16 is still true for the immersed rational $W$–surface cone. However, as can be seen from the next section, for the embedded surface cone, we can only expect the duality to hold for minimal 4-manifolds with $b^+ = 1$. Such a duality has been verified in [13] for several classes of such manifolds in the case $W$ is $\Omega_K(M) = \{ \omega \in \Omega(M) | K_\omega = K \}$ for some $K \in H^2(M; \mathbb{Z})$.

### 3 Embedded symplectic surfaces in 4-manifolds

Let $(M, \omega)$ be a symplectic manifold of dimension 4. One distinctive feature of embedded symplectic surfaces in this dimension is the adjunction formula: Given a $\omega$–positive class $A$, if it is represented by a connected symplectic surface, then the genus $g$ of such a surface is uniquely determined by the adjunction formula,
\[ 2g - 2 = K_\omega(A) + A \cdot A. \]
We call it the $\omega$–symplectic genus of $A$ and denote it by $g_\omega(A)$.
In fact, the generalized Thom conjecture ([14], [25], [17], and [27]) asserts that $g_\omega(A)$ is smaller than or equal to the genus of any smoothly embedded connected surface representing $A$. We should point out that the adjunction formula also applies to a surface $\Sigma$ with several components $\Sigma_1, \ldots, \Sigma_l$, provided that we define the genus to be $\sum_{i=1}^l g(\Sigma_i) - (l-1)$.

We know from Theorem 1.1 that every $\omega$--positive class is represented by a connected immersed symplectic surface. We would like to know which $\omega$--positive class is represented by an embedded symplectic surface (not necessarily connected).

In this section we will sometimes identify a cohomology class in $H^2(M; \mathbb{Z})$ with its Poincaré dual in $H^2(M; \mathbb{Z})$. For instance when we say a cohomology class $e$ is realized by a surface $\Sigma$, it means that the Poincaré dual to $e$ is represented by $\Sigma$.

We remark that, for the isotopy problem of embedded symplectic surfaces in this dimension, there are both uniqueness and non-uniqueness results (see [7] and [32]).

3.1 Obstructions. There are several obstructions to represent a $\omega$--positive class by a connected embedded symplectic surface. We begin with the elementary constraint from the $\omega$--symplectic genus.

Lemma 3.1 If $g_\omega(A)$ is negative, then $A$ cannot be represented by a connected embedded symplectic surface. Thus, if $A$ has negative square and $k$ is a sufficiently large integer, then $kA$ cannot be represented by a connected embedded symplectic surface.

Proof The first claim is obvious since the $\omega$--symplectic genus of a class $A$ is the genus of any connected embedded symplectic surface representing $A$ and a connected surface must have non-negative genus. The second claim follows from the domination of $K_\omega(kA)$ by $kA \cdot kA$ if $A \cdot A < 0$ and $k$ is large. 

Many classes with positive squares may also have negative $g_\omega(A)$. The following is an explicit example.

Example 3.2 Let $M = \mathbb{C}P^2 \# 2\overline{\mathbb{C}P}^2$ with a symplectic form $\omega$ in the class $PD(\lambda H - E_1 - E_2)$ for some $\lambda > 2$, where $H$ is the positive generator of $H_2(\mathbb{C}P^2; \mathbb{Z})$ and $E_1$ and $E_2$ are the positive generators of the $H_2$ of the $\overline{\mathbb{C}P}^2$. Consider the class $A = 3H - 2E_1 - 2E_2$. It is $\omega$--positive and has square 1. As the symplectic canonical class $K_\omega$ is the Poincaré dual to $-3H + E_1 + E_2$, we have $K_\omega(A) = -9 + 2 + 2 = -5$ and hence $g_\omega(A) = -1$. Thus $A$ has no connected symplectic representative.

The $\omega$--symplectic genus can also be used to show that some classes with disconnected symplectic representatives do not have connected representatives. The following is such an example.

Example 3.3 Let $M = S^2 \times T^2$ with a product symplectic form $\omega$. The class $A = 2[S^2]$ is represented by two disjoint parallel embedded symplectic spheres. These two symplectic spheres can be tubed to a smoothly embedded sphere $C$ representing $A$. However, $C$ cannot be a $\omega$--symplectic sphere. As a matter of fact, there is no connected symplectic representative of $A$ as $K_\omega([S^2]) = -2$ and hence the $\omega$--symplectic genus of $A$ is equal to $-1$.

The second obstruction is the following $H_1$--rank invariant $b(A)$ introduced by B. H. Li and the author.
Definition 3.4 Consider a map from a surface \( \Sigma \) (possibly disconnected) to a smooth 4-manifold \( M \) representing the homology class \( A \). The pull back of \( H^1(M; \mathbb{R}) \) is a subspace of \( H^1(\Sigma; \mathbb{R}) \). The rank of the skew-symmetric cup product restricted to this subspace is an even integer. Let \( b(A) \) be half of that integer.

It is not difficult to show that \( b(A) \) only depends on the class \( A \) using a bordism argument. In fact, W. Browder pointed out to us that \( b(A) \) is half the rank of the skew-symmetric pairing on \( H^1(M; \mathbb{R}) \) given by \( < a, b >_A = (a \cup b)(A) \). This is simply because, if \( f: \Sigma \to M \) is a map representing the class \( A \), then, for \( a, b \in H^1(M; \mathbb{R}) \), we have \( (f^* a \cup f^* b)(|\Sigma|) = (a \cup b)(A) \).

With this interpretation of \( b(A) \) we see that \( b(A) \leq b_1(M)/2 \). Notice also that if there is a map from a connected surface \( \Sigma \) to \( M \) representing \( A \), then \( b(A) \leq g(\Sigma) \). From the latter inequality we obtain the following constraint.

**Lemma 3.5** If \( g_\omega(A) < b(A) \), then \( A \) cannot be represented by a connected symplectic surface.

Here is an example where Lemma 3.5 can be applied.

**Example 3.6** Let \( M = \Sigma \times \Sigma' \) be the product of two surfaces of genus 2 with a product symplectic form \( \omega' \). Choose a pair of disjoint non-homologous circles \( \gamma_1 \) and \( \gamma_2 \) in \( \Sigma \) and similar pair of circles \( \gamma'_1 \) and \( \gamma'_2 \) in \( \Sigma' \). Then \( T_1 = \gamma_1 \times \gamma'_1 \) and \( T_2 = \gamma_2 \times \gamma'_2 \) are embedded Lagrangian tori in \( M \). By an observation of Gompf in [8], \( \omega' \) can be deformed to a symplectic form \( \omega \) making \( T_1 \) and \( T_2 \) \( \omega \)-symplectic. Let \( T \) be the union of \( T_1 \) and \( T_2 \) and \( A \) be the class \([T]\). Then \( A \cdot A = 0 \) and

\[
K_{\omega}(A) = K_{\omega'}(A) = (2[\Sigma] + 2[\Sigma']) \cdot ([T_1] + [T_2]) = 0,
\]

hence \( g_\omega(A) = 1 \). On the other hand, \( b(A) = 2 \), as \( H_1(T; \mathbb{R}) \) has rank 4 and injects into \( H_1(M; \mathbb{R}) \).

**Remark 3.7** We say a class \( A \) is big if \( A \cdot A > 0 \). We note that, for a big and \( \omega \)-positive class \( A \) and a large integer \( k \), the \( \omega \)-symplectic genus of \( kA \) is dominated by the positive term \( k^2A \cdot A \), while \( b(kA) \) is always bounded by \( b_1(M) \). Thus the constraints from \( g_\omega(A) \) and \( b(A) \) disappear for a sufficiently large multiple of a big and \( \omega \)-positive class.

From the remark above, to get a general existence result we should focus our attention to big and \( \omega \)-positive classes. It turns out there is yet another obstruction for big and \( \omega \)-positive classes, which comes from the stable classes.

**Definition 3.8** A homology class \( B \) is said to be stable if \( B \) is \( J \)-effective for any \( J \in \mathcal{J}_\omega \), i.e. it can be represented by a \( J \)-holomorphic curve.

In fact, it follows from the Gromov-Uhlenbeck compactness that \( B \) is stable if \( B \) is \( J \)-effective for a dense subset of \( J \in \mathcal{J}_\omega \). Moreover, it follows from the regularity theorem for pseudo-holomorphic curves (Proposition 7.1 in [34]), that a stable class of a minimal manifold must satisfy \(-K_\omega(B) + B \cdot B \geq 0 \).

**Lemma 3.9** Suppose \( A \) is realized by an embedded symplectic surface, each component with non-negative self-intersection. Then \( \alpha = PD(A) \) is non-negative on any stable class.

**Proof** This is based on the intersection property of pseudo-holomorphic curves. We can easily construct an \( \omega \)-compatible almost complex structure \( J \) such that any component \( C_i \) of \( C \) is a connected embedded \( J \)-holomorphic curve. By definition, a stable class \( S \in H_2(M; \mathbb{Z}) \) is represented by a union of irreducible \( J \)-holomorphic
curves $D = \bigcup_j D_j$, where each $D_j$ might be singular and has multiplicity $m_j > 0$. In any case, $C \cdot D = \sum_i m_i C_i \cdot D_j$. If $C_i \neq D_j$, then $C_i \cdot D_j \geq 0$ by the positivity of intersections of distinct irreducible pseudo-holomorphic curves in $[21]$. If $C_i = D_j$, then $C_i \cdot D_j = C_i \cdot C_i$, which is also non-negative by assumption. Thus we have shown that $\alpha(S) = A \cdot S = C \cdot D$ is non-negative.

Currently, the only way to tell whether a class is stable or not is to evaluate the Taubes-Witten invariants or the Gromov-Witten invariants of this class (see [11]). The Taubes-Witten invariants of a class $A$, like the more well-known Gromov-Witten invariants, also count pseudo-holomorphic curves of fixed genus representing $A$, however the curves are now allowed to be disconnected. A class is called a TW class if some Taubes-Witten invariant of this class is non-trivial. GW classes are defined in the same way. Certainly a TW class or a GW class is a stable class.

An important property of both the TW invariants and the GW invariants is that they are invariant under deformation of the symplectic forms. We use this property to give a different argument for the obstructions coming from the TW classes (the same argument works for the GW classes as well).

Suppose $W$ is a TW class and $A$ is represented by an embedded $\omega$-symplectic surface $C$ with each component having trivial or positive normal bundle. By the inflation process in $[20]$, for any $t \geq 0$, the class $[\omega] + t PD(A)$ is represented by a symplectic form $\omega_t$ deformation equivalent to $\omega$. Since TW classes only depend on the deformation equivalence class of symplectic forms, $W$ is a still a TW class for any $\omega_t$ and therefore we have $\omega_t(W) > 0$. However if $PD(A)(W) < 0$, then $[\omega] + t PD(A)$ is negative on $W$ for $t$ large. Thus we must have $PD(A)(W) \geq 0$ to begin with.

An immediate consequence of Lemma 3.9 is the following

**Corollary 3.10** Suppose $A$ is an $\omega$-positive class with $A \cdot A \geq 0$ and is represented by a connected embedded symplectic surface, then $PD(A)$ is positive on any stable class.

As observed in $[23]$, the class of a symplectic $-1$ sphere is a Gromov-Witten class. Therefore, if a $\omega$-positive homology class $A$ with $A \cdot A \geq 0$ has negative intersection with such a class, it cannot be represented by a connected embedded symplectic surface. Actually, if we notice that in Example 3.2, the class $H - E_1 - E_2$ is the class of symplectic $-1$ sphere and $A \cdot (H - E_1 - E_2) = -1$, we may reach the same conclusion by Corollary 3.10.

So far it is not clear to the author whether there are other obstructions: i.e. whether there is an $\omega$-positive class $A$ which satisfies $g_{\omega}(A) \geq b(A)$ and is positive on the stable classes but not representable by a connected symplectic surface. Vidussi $[33]$ showed that for fibered 3-manifolds $N$, there are obstructions for the symplectic 4-manifold $M = S^1 \times N$ coming from the Seiberg-Witten monopole classes of $N$. But from Taubes’ picture $[34]$ linking solutions to the Seiberg-Witten equations and the pseudo-holomorphic curves, it is possible these monopole classes of $N$ may give rise to stable classes of $M$.

**3.2 Constructions.** Having discussed some obstructions to the existence of connected embedded symplectic surfaces we now describe several constructions. In view of Remark 3.7, we will focus on the large multiples of big and $\omega$-positive classes whose Poincaré duals are non-negative on stable classes.

The prototype of a big and $\omega$-positive class which is positive on any stable class is the Poincaré dual to the class of an integral symplectic form. For such
classes we have the following beautiful result of Donaldson already mentioned in §2.

**Theorem 3.11** If $\text{PD}(A)$ is sufficiently close to the ray generated by $[\omega]$, then a sufficiently large multiple of $A$ is represented by a connected embedded symplectic surface. In particular, if $[\omega]$ is an integral class then sufficiently large multiples of $\text{PD}(\omega)$ are thus represented.

The starting observation for this theorem is that, in a symplectic vector space $(V, \omega)$, a small perturbation of a symplectic subspace remains symplectic. In particular, a subspace is symplectic if it is close to a complex subspace (for some $J \in \mathcal{J}_\omega(V)$). The following lemma in [3] makes it precise.

**Lemma 3.12** Let $f : \mathbb{C}^n \rightarrow \mathbb{C}$ be $\mathbb{R}$-linear with $f = a' + a''$, where $a'$ (respectively $a''$) is $\mathbb{C}$-linear (respectively $\mathbb{C}$-antilinear). If $|a''| < |a'|$, then $f$ is surjective and $\ker(f)$ is a symplectic subspace of $\mathbb{C}^n$.

To describe Donaldson’s construction we first assume that $[\omega]$ is an integral class. In this case there is a complex line bundle $L$ over $M$ with the $c_1(L) = [\omega]$, and the codimension 2 submanifolds are obtained as the zero sets $Q_s$ of suitable transverse sections $s$ of $L^\otimes k$ for large $k$. More precisely, fix a compatible almost complex structure $J$ on $M$ and choose a connection on the line bundle $L$ whose curvature is $\omega$. The connection on $L$ gives rise to operators $\bar{\partial}_J$ and $\partial_J$ on sections of $L$ and hence $L^\otimes k$. Given a section $s$ of $L^\otimes k$ and a point $x \in M$, $|\bar{\partial}_J s|$ and $|\partial_J s|$ when restricted to $x$ are respectively the holomorphic and anti-holomorphic parts of the differential of $s$ at $x$, viewed as a map between the complex vector spaces $T_x M$ and $T_{x,0} L^\otimes k$. Thus by Lemma 3.12 if there are sections such that $|\bar{\partial}_J s| < |\partial_J s|$ on $Q_s$ then $Q_s$ is symplectic. Such sections are called approximately holomorphic and are shown to exist by Donaldson. This construction clearly applies to symplectic manifolds with rational symplectic forms. As for a non-rational symplectic form, we can always approximate it by rational symplectic forms.

The following well-known result of Kähler surfaces suggests that we actually should be able to go far beyond the ray generated by $\text{PD}([\omega])$.

**Lemma 3.13** Suppose $\omega$ is a Kähler form on a projective surface $M$, and $A$ is a big class Poincaré dual to an integral cohomology class of type $(1, 1)$. If we further assume that $\text{PD}(A)$ is positive on all holomorphic curves, then a sufficiently large multiple of $A$ is represented by a connected embedded symplectic surface.

Notice that here $\text{PD}(A)$ is positive on all stable classes since it is assumed that it is positive on all holomorphic curves. The proof of the lemma goes as follows: $\text{PD}(A)$ is an ample class by the Nakai-Moishezon criterion, thus, by Kodaira’s embedding theorem, sufficiently large multiples of $A$ are Poincaré dual to very ample classes and the corresponding holomorphic line bundles have plenty of holomorphic sections whose zero loci are irreducible smooth holomorphic curves.

In view of the two preceding results, we would like to raise the following question.

**Question 3.14** Let $A$ be a big and $\omega$-positive class whose Poincaré dual is non-negative on any stable class. Then, is a sufficiently high multiple of $A$ represented by a connected embedded symplectic surface?

To shed more light on this question let us presently make a general remark on the constructions of smooth 2-dimensional submanifolds. In general there are two ways to do so. The first method, which we call ‘mapping into’, starts with mapping.
a surface into $M$, and then deforming the map and possibly smoothing the image to an embedding. The second method, which we call ‘mapping out’, instead starts by choosing another manifold $N$ and a codimension 2 submanifold $S \subset N$, and then taking the inverse image of a map from $M$ into $N$ which is transversal to $S$. For the ‘mapping out’ constructions, $N$ is often taken to be a complex line bundle over $M$.

The constructions of 2-dimensional symplectic submanifolds also follow the same routes. In fact, the constructions in Theorem 3.11 and Lemma 3.13 are clearly of the second kind.

Let us presently turn to the ‘mapping into’ constructions. In §2 we start with a continuous map and then use transversality and the h-principles to deform it to a symplectic embedding. However, this method fails in dimension 4 and we can only obtain symplectic immersions with both positive and negative double points.

As we already mentioned, only positively immersed symplectic surfaces can be smoothed to an embedded symplectic surface. One way to obtain positively immersed symplectic surface is to start with a connected embedded symplectic surface with positive self-intersection. Since being a closed symplectic submanifold is an open condition, we can perturbe it to several nearby symplectic surfaces intersecting each other transversally and positively. By smoothing the double points, we obtain the following result.

**Lemma 3.15** If a big class $A$ is represented by a connected embedded symplectic surface, then for any positive integer $k$, the class $kA$ is also represented as such.

In fact, the same conclusion holds for a class with square zero by the circle sum construction (see [16] for details).

In dimension 4, the most effective ‘mapping into’ approach to obtain a positively immersed symplectic surface is to construct a simple pseudo-holomorphic curve. Here a pseudo-holomorphic curve $u : \Sigma \rightarrow M$ is said to be simple if the restriction of $u$ to any of its components is not multiply covered and no two components have the same image. The observation in [21] that any simple pseudo-holomorphic curve can be perturbed to an immersed pseudo-holomorphic curve $C$ of the same genus for some nearby almost complex structure readily leads to the following result.

**Lemma 3.16** If $A$ is represented by a simple $J$–holomorphic curve $u : (\Sigma, j) \rightarrow (M, J)$ for some $\omega$–tamed almost complex structure $J$, then $A$ is represented by an embedded symplectic surface.

In view of Lemma 3.16, we say a class $A$ is simple if it is represented by a simple $J$–holomorphic curve for some $J \in \mathcal{J}_\omega$.

There are abundant simple classes in a Kähler surface. Let us recall some relevant facts here. In a Kähler surface $(M, \omega, J)$, holomorphic curves arise as the zero loci of sections of holomorphic line bundles. If a holomorphic line bundle is globally generated off a finite set of points, Bertini’s Theorem will show that the generic section is smooth away from the base locus of the system. Since the zero locus is pure codimension one (locally defined by one equation), this shows then a generic divisor is a reduced curve. By the desingularization of curves, if $C$ is a reduced curve in an algebraic surface, then there exists a compact Riemann surface $\tilde{C}$ and a holomorphic map $\psi : \tilde{C} \rightarrow C$ that is one-to-one over smooth points of $C$. Thus we have the following criterion of a simple class.
Corollary 3.17 If $\mathcal{L}$ is a holomorphic line bundle globally generated off a finite set of points, then $\text{PD}(c_1(\mathcal{L}))$ is a simple class and hence can be represented by an embedded symplectic surface.

In particular, if a holomorphic line bundle is generated by global sections, then it is represented by connected embedded symplectic surfaces. Kawamata’s base-point-free theorem (12) provides many such line bundles which are sufficiently high powers of certain nef line bundles. Another general source is the following theorem of Reider (29): Let $M$ be a projective Kähler surface and $\mathcal{L}$ be an ample line bundle on $M$. If $c_1(\mathcal{L})^2 \geq 5$ and $c_1(\mathcal{L})(\Gamma) \geq 2$ for all irreducible curves $\Gamma \subset M$, then $K \otimes \mathcal{L}$ is globally generated. In particular, $K \otimes \mathcal{L}^3$ is globally generated for any ample line bundle $\mathcal{L}$.

Finally, we describe the construction originated from the Seiberg-Witten theory. On a symplectic 4-manifold, given a cohomology class $e$, there is an associated Spin$^c$ structure $\mathcal{L}_e$ whose $c_1$ is equal to $-K_\omega + 2e$. Witten observed that, on a Kähler surface, if the equations are deformed by positive multiples of Kähler forms, then the solutions correspond exactly to holomorphic sections of a holomorphic line bundle with $c_1 = e$, therefore giving rise to holomorphic curves representing $e$.

Taubes vastly generalizes this picture to symplectic 4-manifolds. He (34) starts by fixing a compatible almost complex structure on a symplectic 4-manifold $(M, \omega)$ and then uses the induced metric to define the Seiberg-Witten equations. He is able to prove that, if the Seiberg-Witten invariant is non-trivial, then the solutions to the Seiberg-Witten equations for the Spin$^c$ structure $\mathcal{L}_e$ deformed by a large multiple of the symplectic form gives rise to sections of the complex line bundle with $c_1$ equal to $e$. Moreover the zero loci are (possibly disconnected) pseudo-holomorphic subvarieties. He further shows that, by imposing $d(e) = K_\omega \cdot e + e \cdot e$ number of generic point constraints, for a generic choice of a compatible almost complex structure, the pseudo-holomorphic subvarieties satisfying the constraints are essentially embedded pseudo-holomorphic submanifolds. Motivated by this remarkable result, Taubes (35) defines a Gromov type invariant counting embedded pseudo-holomorphic curves in a fixed class (in the connected case, an earlier attempt was made in 30), which we call the Gromov-Taubes invariant. The final piece in this grand picture is that the Gromov-Taubes invariant of $\text{PD}(e)$ is the same as the Seiberg-Witten invariant for the Spin$^c$ structure $\mathcal{L}_e$ at least when $b^+ > 1$.

On a symplectic 4-manifold with $b^+ > 1$, one consequence of the Taubes-Seiberg-Witten theory is that the symplectic canonical class $K_\omega$ is a Gromov-Taubes class, hence is always realized by an embedded symplectic surface. Moreover, if $e$ is realized, so is $K_\omega - e$.

When $b^+ = 1$, together with the wall crossing formula in (18), we are able to prove in (19) that most big and $\omega$–positive classes in a minimal symplectic 4-manifold with $b^+ = 1$ are represented by a connected embedded symplectic surface. More precisely, we have

Proposition 3.18 Let $(M, \omega)$ be a minimal symplectic 4–manifold with $b^+ = 1$. Let $A$ be a big and $\omega$–positive class. If $A - \text{PD}(K_\omega)$ is also $\omega$–positive and has non-negative square, then $A$ is represented by a connected symplectic surface. In particular, for $N$ big, $NA$ is represented by a connected symplectic surface.

When $M$ is not minimal, the same is true if we take into account the obstructions coming from $\mathcal{E}_\omega$, which is the set of the exceptional classes represented by symplectic $-1$ spheres.
Proposition 3.19 Let \((M, \omega)\) be a symplectic 4-manifold with \(b^+ = 1\) and symplectic canonical class \(K_\omega\). Let \(A\) be a big and \(\omega\)-positive class. Assume that \(A - PD(K_\omega)\) is \(\omega\)-positive and has non-negative square. Further assume that \(A \cdot E \geq -1\) for all \(E \in \mathcal{E}_\omega\). Then \(A\) can be represented by an embedded symplectic surface. Furthermore, if \(A \cdot E \geq 0\) for all \(E \in \mathcal{E}_\omega\), then the symplectic surface is connected.

See also \([5]\) and \([31]\) for a purely symplectic approach to some of the consequences of the Taubes-Seiberg-Witten theory building on the existence of Lefschetz pencils by \([4]\).

Remark 3.20 With a combination of the constructions above it is shown in \([16]\) that, in fact every big and \(\omega\)-positive class of a symplectic \(S^2\)-bundle is represented by a connected embedded symplectic surface.

In summary the situations where Question 3.14 has an affirmative answer are the following:
1. \(b^+ = 1\).
2. \(A\) is already represented by a connected embedded symplectic surface.
3. \(PD(A)\) is close to the ray generated by \([\omega]\).
4. \(PD(A)\) is an ample class of a projective Kähler surface.

References

D. Auroux, Asymptotically holomorphic families of symplectic submanifolds, Geom. Funct. Anal. 7 (1997), 971-995.
P. Biran, Symplectic packings in dimension 4, Geom. Funct. Anal. 7 (1997), no.3. 420-437.
S. Donaldson, Symplectic submanifolds and almost-complex geometry, J. Differential Geom. 44 (1996), 666-705.
S. Donaldson, Lefschetz pencils on symplectic manifolds, J. Differential Geom. 53 (1999), 205-236.
S. Donaldson, I. Smith, Lefschetz pencils and the canonical class for symplectic 4-manifolds, Topology 42 (2003), 743-785.
Y. Eliashberg, N. Mishachev, Introduction to the h-principle, GSM 48, American Mathematical Society, 2002.
R. Fintushel, R. Stern, Symplectic surfaces in a fixed homology class, J. Differential Geom. 52 (1999), no. 2, 203-222.
R. Gompf, A new construction of symplectic manifolds, Ann. of Math. 142 (1995), 527-595.
M. Hirsch, Immersions of manifolds, Trans. Amer. Math. Soc. 93 (1959) 242-276.
H. Hopf, Fundamentalgruppe und zweite Bettische Gruppe, Comment. Math. Helv. 14 (1942), 257-309.
E. Ionel, T. Parker, Gromov-Witten invariants of symplectic sums, Math. Res. Letters 5 (1998), 563-576.
Y. Kawamata, A generalization of Kodaira-Ramanujam’s vanishing theorem, Math. Ann. 261 (1982), no. 1, 43–46.
U. Koschorke, Complex and real vector bundle monomorphism, Topology and its Applications 91 (1999), 259-271.
P. Kronheimer and T. Mrowka, The genus of embedded surfaces in the projective plane, Math. Res. Letters 1 (1994), 797-808.
H. V. Le, Realizing homology classes by symplectic submanifolds. MPI preprint Nr. 61/2004.
B. H. Li, T. J. Li, The symplectic circle sum, in preparation.
T. J. Li, A. K. Liu, Symplectic structures on ruled surfaces and a generalized adjunction inequality, Math. Res. Letters 2 (1995), 453-471.
T. J. Li, A. K. Liu, Uniqueness of symplectic canonical class, surface cone and symplectic cone of 4-manifolds with \(b^+ = 1\), J. Differential Geom. 58 (2001), 331-370.
T. J. Li, A. K. Liu, Family Seiberg-Witten invariants and wall crossing formulas, Comm. in Analysis and Geometry 9 (2001), no. 4, 777-823.
F. Lalonde, D. McDuff, The classification of ruled symplectic 4-manifolds, Math. Res. Letters 3 (1996), 769-778.
D. McDuff, *The structure of rational and ruled symplectic 4-manifolds*. J. Amer. Math. Soc. 3 (1990), no. 3, 679–712.

D. McDuff, *The local behavior of holomorphic curves in almost complex 4-manifolds*, J. Differential Geom. 34 (1991), 679–710.

D. McDuff, *Immersed spheres in symplectic 4–manifolds*, Ann. Inst. Fourier (Grenoble) 42 (1992), no.1-2, 369-392.

V. Muñoz, F. Presas, I. Sols, *Almost holomorphic embeddings in Grassmannians with applications to singular symplectic submanifolds*, J. Reine Angew. Math. 547 (2002), 149-189.

J. Morgan, Z. Szabo, C. Taubes, *A product formula and the generalized Thom conjecture*, J. Differential Geom. 44 (1996), 706-788.

J. McCarthy, J. Wolfson, *Symplectic normal connect sum*, Topology 33 (1994), no. 4, 729–764.

P. Ozsváth, Z. Szabo, *The symplectic Thom conjecture*, Ann. of Math. (2) 151 (2000), no. 1, 93-124.

R. Paoletti, *Symplectic subvarieties of projective fibrations over symplectic manifolds*, Ann. Inst. Fourier Grenoble 49 (1999), 1661-1672.

I. Reider, *Vector bundles of rank 2 and linear systems on algebraic surfaces*, Ann. of Math. (2) 127 (1988), no. 2, 309–316.

Y. Ruan, *Symplectic topology and complex surfaces*, Geometry and analysis on complex manifolds, 171–197, World Sci. Publishing, River Edge, NJ, 1994.

I. Smith, *Serre-Taubes Duality for pseudoholomorphic curves*, Topology 42 (2003), 931-979.

B. Siebert, G. Tian, *On the holomorphicity of genus two Lefschetz fibration*, preprint.

S. Vidussi, *Norms on the cohomology of a 3-manifold and Seiberg-Witten theory*, Pacific J. Math. 208 (2003), no.1, 169-186.

C. H. Taubes, *SW⇒Gr: From Seiberg-Witten equations to pseudo-holomorphic curves*, J. Amer. Math. Soc. 9 (1996) 845-918.

C. H. Taubes, *Counting pseudo-holomorphic submanifolds in dimension 4*, J. Differential Geom. 44 (1996) 818-893.

R. Thom, *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv. 28 (1954), 17-86.