SYMMETRY ANALYSIS, PERSISTENCE PROPERTIES AND UNIQUE CONTINUATION FOR THE CROSS-COUPLED CAMASSA-HOLM SYSTEM

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ABSTRACT. In this paper, we study symmetry analysis, persistence properties and unique continuation for the cross-coupled Camassa-Holm system. Lie symmetry analysis and similarity reductions are performed, some invariant solutions are also discussed. Then prove that the strong solutions of the system maintain corresponding properties at infinity within its lifespan provided the initial data decay exponentially and algebraically, respectively. Furthermore, we show that the system exhibits unique continuation if the initial momentum \( m_0 \) and \( n_0 \) are positive.

1. Introduction. In this paper, we consider the following cross-coupled Camassa-Holm system:

\[
\begin{align*}
    m_t + 2v_x m + vm_x &= 0, \\
    n_t + 2u_x n + un_x &= 0,
\end{align*}
\]

where \( m = u - u_{xx} \) and \( n = v - v_{xx} \). The system (1) was proposed in [16], from a variational principle by using the Euler-Poincaré theory for symmetry reduction of the right-invariant Lagrangians on the tangent space of a Lie group. From the system (1), we find that the momentum \( m \) (resp. \( n \)) is transported solely by the opposite induced velocity \( v \) (resp. \( u \)), and there are obviously no terms with self-interactions. Hence the system (1) is called as the cross-coupled system. Recently, the well-posedness of the system (1) has been studied in the Sobolev spaces [42] and in the Besov spaces [36]. A precise blow-up scenario for strong solutions was presented in [36]. The persistence of compact support for solutions of the system (1) was analyzed in [25].

Obviously, for \( v = u \), the system (1) is reduced to the Camassa-Holm (CH) equation

\[
m_t + 2u_x m + um_x = 0, \quad m = u - u_{xx},
\]

which was first derived formally by Fokas and Fuchssteiner in [20], and later derived as a model for unidirectional propagation of shallow water over a flat bottom by Camassa and Holm in [4]. Moreover, the CH equation could be also derived as

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a model for the propagation of axially symmetric waves in hyperelastic rods [17]. The CH equation has a bi-Hamilton structure [20] and is completely integrable [4, 5]. In particular, it possesses an infinity of conservation laws and is solvable by its corresponding inverse scattering transform. After the birth of the CH equation, many works have been carried out to it. For example, the CH equation has traveling wave solutions of the form $ce^{-|x-ct|}$, called peakons, which describes an essential feature of the traveling waves of largest amplitude [6, 7, 8, 9, 37]. Local well-posedness for the Cauchy problem of CH equation was established in [18, 19, 35]. It is shown in [10] that the blow-up occurs in the form of breaking waves, namely, the solution remains bounded but its slope becomes unbounded infinite time. Moreover, the CH equation has global conservative solutions [2] and dissipative solutions [3]. The orbital stability of solitary waves and the stability of peakons for CH equation are considered by Constantin and Strauss [11, 12]. For other methods to study the CH equation and other related equations, the reader is referred to [21, 22, 23, 24, 32, 33, 34, 15, 39, 43, 13] and the references therein.

The goal of the present paper is to study symmetry analysis, persistence properties and unique continuation for the system (1). We firstly we perform Lie symmetry analysis for the system (1), and based on the optimal system, all the symmetry reductions and invariant solutions to the system (1) are obtained. Then we introduce different weigh functions to prove the solution of the system maintain corresponding properties at infinity within its lifespan provided the initial data decay exponentially and algebraically, respectively. Finally, we demonstrate that the system exhibits unique continuation if the initial momentum $m_0$ and $n_0$ are positive.

This paper is organized as follows. In Section 2, Lie symmetry analysis and similarity reductions are performed, some invariant solutions of the system (1) are also discussed. Two persistence properties of the strong solutions to the system (1) are given in Section 3. In Section 4, we investigate unique continuation of solutions of the system (1).

2. Symmetry analysis. In this section, we shall perform Lie symmetry analysis for the system (1) and derive symmetry reductions and invariant solutions. The method of determining the Lie symmetry for a partial differential equation is standard which is described in [40, 1, 41].

First of all, let us consider a one-parameter group of infinitesimal transformation:

$$
\begin{align*}
\tilde{t} &= t + \epsilon \tau(t, x, u, v) + O(\epsilon^2), \\
\tilde{x} &= x + \epsilon \xi(t, x, u, v) + O(\epsilon^2), \\
\tilde{u} &= u + \epsilon \eta(t, x, u, v) + O(\epsilon^2), \\
\tilde{v} &= v + \epsilon \phi(t, x, u, v) + O(\epsilon^2),
\end{align*}
$$

where $\epsilon \ll 1$ is a group parameter. The vector field associated with the above group of transformations can be written as

$$
V = \tau(t, x, u, v) \frac{\partial}{\partial t} + \xi(t, x, u, v) \frac{\partial}{\partial x} + \eta(t, x, u, v) \frac{\partial}{\partial u} + \phi(t, x, u, v) \frac{\partial}{\partial v},
$$

where $\tau(t, x, u, v)$, $\xi(t, x, u, v)$, $\eta(t, x, u, v)$ and $\phi(t, x, u, v)$ are coefficient functions of the infinitesimal generators to be determined.

Applying the third prolongation $P^3_V$ to the system (1), we find that the coefficient functions $\tau(t, x, u, v)$, $\xi(t, x, u, v)$, $\eta(t, x, u, v)$ and $\phi(t, x, u, v)$ must satisfy an over determined system of equations as follows:
\[ \tau_{tt} = \tau_x = \tau_u = \tau_v = 0, \]
\[ \xi_t = \xi_x = \xi_u = \xi_v = 0, \]
\[ \eta + u\tau_t = 0, \]
\[ \phi + v\tau_t = 0. \]

Solving above Eqs.(3), one get
\[ \tau(t, x, u, v) = C_1 t + C_2, \]
\[ \xi(t, x, u, v) = C_3, \]
\[ \eta(t, x, u, v) = -C_1 u, \]
\[ \phi(t, x, u, v) = -C_1 v, \]
where \( C_1, C_2 \) and \( C_3 \) are arbitrary constants. Hence the infinitesimal generators of the system (1) is spanned by the following vector fields:
\[ V_1 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}, \]
\[ V_2 = \frac{\partial}{\partial t}, \]
\[ V_3 = \frac{\partial}{\partial x}. \]

Then, all of the infinitesimal generators of the system (1) can be expressed as
\[ V = C_1 V_1 + C_2 V_2 + C_3 V_3. \]

The commutation relations of Lie algebra determined by \( V_1, V_2, V_3 \) are shown in Table 1. It is obvious that \( \{ V_1, V_2, V_3 \} \) is commute under the Lie bracket.

| \([V_i, V_j]\) | \(V_1\) | \(V_2\) | \(V_3\) |
|----------------|-------|-------|-------|
| \(V_1\)       | 0     | \(-V_2\) | 0     |
| \(V_2\)       | \(V_2\) | 0     | 0     |
| \(V_3\)       | 0     | 0     | 0     |

To get symmetry groups, we should solve the following ordinary differential equations (ODEs) with initial problems:
\[ \frac{dt}{d\epsilon} = \tau(\tilde{t}, \tilde{x}, \tilde{u}, \tilde{v}), \quad \frac{d\tilde{t}}{d\epsilon} = \tilde{t}|_{\epsilon=0} = t, \]
\[ \frac{dx}{d\epsilon} = \xi(\tilde{t}, \tilde{x}, \tilde{u}, \tilde{v}), \quad \tilde{x}|_{\epsilon=0} = x, \]
\[ \frac{du}{d\epsilon} = \eta(\tilde{t}, \tilde{x}, \tilde{u}, \tilde{v}), \quad \tilde{u}|_{\epsilon=0} = u, \]
\[ \frac{dv}{d\epsilon} = \phi(\tilde{t}, \tilde{x}, \tilde{u}, \tilde{v}), \quad \tilde{v}|_{\epsilon=0} = v, \]

then we obtain one-parameter symmetry groups \( g_i : (t, x, u, v) \to (\tilde{t}, \tilde{x}, \tilde{u}, \tilde{v}) \) of the infinitesimal generators \( V_i (i = 1, 2, 3) \) are given as follows:
\[ g_1 : (t, x, u, v) \to (te^\epsilon, x, ue^{-\epsilon}, ve^{-\epsilon}), \]
\[ g_2 : (t, x, u, v) \to (t + \epsilon, x, u, v), \]
\[ g_3 : (t, x, u, v) \to (t, x + \epsilon, u, v), \]

where \( g_1 \) is a scaling transformation, \( g_2 \) is a time translation and \( g_3 \) is a space translation. Thus the following theorem holds:

**Theorem 2.1.** If \( u = f(t, x), v = g(t, x) \) is a known solution of the system (1), then by using the above groups \( g_i (i = 1, 2, 3) \), the corresponding new solutions \( u_i, v_i (i = 1, 2, 3) \) can be obtained respectively as follows:
\[ u_1 = e^{-\epsilon}f(te^{-\epsilon}, x), \]
\[ v_1 = e^{-\epsilon}g(te^{-\epsilon}, x), \]
Using the Table 1 and the following Lie series
\[ \text{Ad}(\exp(\epsilon V_i))V_j = V_j - \epsilon [V_i, V_j] + \frac{\epsilon^2}{2!} [V_i, [V_i, V_j]] - \frac{\epsilon^3}{3!} [V_i, [V_i, [V_i, V_j]]] + \cdots, \]
we obtain the adjoint representation in Table 2.

**Table 2. The adjoint representation**

| \( \text{Ad}(\exp(\epsilon V_i))V_j \) | \( V_1 \) | \( V_2 \) | \( V_3 \) |
|---------------------------------------|-----|-----|-----|
| \( V_1 \)                           | \( V_1 \) | \( V_2 \) | \( V_3 \) |
| \( V_2 \)                           | \( V_1 - \epsilon V_2 \) | \( V_2 \) | \( V_3 \) |
| \( V_3 \)                           | \( V_1 \) | \( V_2 \) | \( V_3 \) |

Based on the adjoint representation, we have the following theorem:

**Theorem 2.2.** The optimal system of one-dimensional subalgebras of the Lie algebra spanned by \( V_1, V_2, V_3 \) of the system (1) given by
\[ V_1, V_3, V_1 + aV_3, V_2 + cV_3 (c \neq 0). \]

Next, making use of optimal system in Theorem 2.2, we shall derive symmetry reductions and invariant solutions.

**Case 1.** \( V_1 = t \frac{\partial}{\partial t} - u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v} \).

The invariants are
\[ \xi = x, \quad f(\xi) = tu, \quad g(\xi) = tv, \]
then the system (1) can be reduced to the following ODEs:
\[ \begin{cases} -f + f'' + 2gf'f - 2g'f'' + f'g - gf'' = 0, \\ -g + g'' + 2f'g' - 2f'g'' + g'f - fg'' = 0. \end{cases} \] (4)

The nontrivial solutions of above ODEs (4) yield similarity solutions of the system (1) as follows:
\[ u(t, x) = \frac{f(x)}{t}, \quad v(t, x) = \frac{g(x)}{t}. \] (5)

**Case 2.** \( V_3 = \frac{\partial}{\partial x} \).

The invariants are
\[ \xi = t, \quad f(\xi) = u, \quad g(\xi) = v, \]
then the system (1) can be reduced to the following ODEs:
\[ \begin{cases} f' = 0, \\ g' = 0. \end{cases} \] (6)

Obviously, this leads to the trivial solution \( u(t, x) = C_1, v(t, x) = C_2 \), where \( C_1 \) and \( C_2 \) are constants.

**Case 3.** \( V_1 + aV_3 = t \frac{\partial}{\partial t} - u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v} + a \frac{\partial}{\partial x} \).

For simplicity, setting \( a = 1 \), the invariants are
\[ \xi = te^{-x}, \quad f(\xi) = ue^x, \quad g(\xi) = ve^x, \]
then the system (1) can be reduced to the following ODEs:

\[
\begin{aligned}
-3f' - 5\xi f'' - \xi^2 f''' + 12\xi_g f' + 8\xi^2_g f' + 6\xi^2 g' f' + 2\xi^3 g' f'' + 3\xi^3 g f''' &= 0, \\
-3g' - 5\xi g'' - \xi^2 g''' + 12\xi_f g' + 8\xi^2 f g' + 6\xi^2 f' g' + 2\xi^3 f' g'' + \xi^3 f g''' &= 0.
\end{aligned}
\]  

(7)

The nontrivial solutions of above ODEs (7) yield group-invariant solutions of the system (1) as follows:

\[
u(t, x) = \frac{f(te^{-x})}{e^x}, \quad v(t, x) = \frac{g(te^{-x})}{e^x}.
\]

**Case 4.** \(V_2 + cV_3 = \frac{\partial}{\partial t} + c \frac{\partial}{\partial x}\). The invariants are

\(\xi = x - ct, \quad f(\xi) = u, \quad g(\xi) = v\).

This case the invariant solution corresponds to the travelling wave solutions \(u = f(x - ct), v = g(x - ct)\). In Ref. [16], we know the system (1) admits single peaked traveling wave solution of the form:

\[
u(t, x) = -ce^{-|x-ct|}, \quad v(t, x) = -ce^{-|x-ct|}.
\]

**Remark 1.** The similarity solutions (5) will exhibit decay \(u, v \to 0\) as \(t \to \infty\) and blow up \(u, v \to \infty\) at \(t = 0\). Applying a a time translation, one can get new solutions \(u = (t - t_0)^{-1} f(x), v = (t - t_0)^{-1} g(x), t_0 = \text{const}\), which still decays to 0 for large \(t\) but no blow up for \(t \geq 0\) when \(t_0 < 0\).

3. **Persistence properties.** In this section, we present two persistence properties based on the work for CH equation [33, 39, 43, 14, 30, 31].

Note that \(p(x) = \frac{1}{2} e^{-|x|}\), we have \((1 - \partial_x^2)^{-1} f = p * f\) for all \(f \in L^2(\mathbb{R}), p * m = u\) and \(p * n = v\), where we denote by * the convolution. Then we set up the Cauchy problem of the system (1) as follows:

\[
\begin{aligned}
ut + vu_x + p * F_1(u, v) &= 0, & t > 0, & x \in \mathbb{R}, \\
v_t + uv_x + p * F_2(u, v) &= 0, & t > 0, & x \in \mathbb{R}, \\
u(0, x) &= u_0(x), & x \in \mathbb{R}, \\
v(0, x) &= v_0(x), & x \in \mathbb{R},
\end{aligned}
\]  

(8)

where \(F_1(u, v) = 2uv_x + u_x v_{xx}\) and \(F_2(u, v) = 2vu_x + v_x u_{xx}\).

**Notations.** For simplicity, we firstly introduce the following notations

\[M = \sup \| (u, v) \|_{H^s \times H^s} = \sup (\| u \|_{H^s} + \| v \|_{H^s}).\]

\[|f(x)| \sim O(|g(x)|) \text{ as } x \uparrow \infty \text{ if } \lim_{x \to \infty} \frac{|f(x)|}{|g(x)|} = L,\]

where \(L\) is a nonnegative constant.

**Theorem 3.1.** Let \((u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})\) with \(s > \frac{3}{2}\) and \(T\) be the maximal existence time of the strong solution \((u, v)\) to the Cauchy problem (8). If there exists some \(\theta \in (0, 1)\) such that

\[|u_0(x)|, |u_0(x)|, |v_0(x)|, |v_0(x)| \sim O(e^{-\theta x}) \text{, as } x \uparrow \infty,\]

then

\[|u(t, x)|, |u_x(t, x)|, |v(t, x)|, |v_x(t, x)| \sim O(e^{-\theta x}) \text{, as } x \uparrow \infty \text{ uniformly in the time interval } [0, T].\]
Proof. First, we introduce the following weigh function:

\[ \varphi_N(x) = \begin{cases} 
1, & x \leq 0, \\
\theta^x, & x \in (0, N), \\
\theta^N, & x \geq N, 
\end{cases} \]

where \( \theta \in (0, 1) \) and \( N \in \mathbb{N}^+ \). Note that for all \( N \), we have

\[ 0 \leq \varphi_N(x) \leq \varphi_N(x), \text{ a.e. } x \in \mathbb{R}. \]

Multiplying the first equation of the system (8) by \((u\varphi_N)^{2n-1}\varphi_N\) with \( n \in \mathbb{N}^+ \) and integrating the result in \( x \)-variable, we get

\[
\int_{\mathbb{R}} (u\varphi_N)^{2n-1}\partial_t(u\varphi_N)dx + \int_{\mathbb{R}} (u\varphi_N)^{2n-1}\varphi_Nv u_x dx + \int_{\mathbb{R}} (u\varphi_N)^{2n-1}\varphi_Np \ast F_1(u,v) dx = 0. \tag{9}
\]

Note that

\[
\int_{\mathbb{R}} (u\varphi_N)^{2n-1}\partial_t(u\varphi_N)dx = \frac{1}{2n} \frac{d}{dt} \| u\varphi_N \|_{L^{2n}}^{2n-1} \frac{d}{dt} \| u\varphi_N \|_{L^{2n}}, \tag{10}
\]

\[
\left| \int_{\mathbb{R}} (u\varphi_N)^{2n-1}\varphi_Nv u_x dx \right| = \| v \|_{L^\infty} \left| \int_{\mathbb{R}} (u\varphi_N)^{2n-1}(u\varphi_N)_x - (u\varphi_N)^{2n-1}u\varphi'_N dx \right| \\
\leq C \| v \|_{L^\infty} \| u\varphi_N \|_{L^{2n}}^{2n-1} \\
\leq CM \| u\varphi_N \|_{L^{2n}}^{2n}, \tag{11}
\]

and

\[
\left| \int_{\mathbb{R}} (u\varphi_N)^{2n-1}\varphi_Np \ast F_1(u,v) dx \right| \leq \| u\varphi_N \|_{L^{2n}}^{2n-1} \| \varphi_Np \ast F_1(u,v) \|_{L^{2n}}. \tag{12}
\]

Substituting above estimates (10)-(12) to (9) yields

\[
\frac{d}{dt} \| u\varphi_N \|_{L^{2n}} \leq CM \| u\varphi_N \|_{L^{2n}} + \| \varphi_Np \ast F_1(u,v) \|_{L^{2n}}.
\]

By Gronwall’s inequality, we have

\[
\| u\varphi_N \|_{L^{2n}} \leq \left( \| u_0\varphi_N \|_{L^{2n}} + \int_0^t \| \varphi_Np \ast F_1(u,v)(\tau) \|_{L^{2n}}d\tau \right) e^{CMt}. \tag{13}
\]

Differentiating the first equation of the system (8) with respect to \( x \), then multiplying by \((u_x\varphi_N)^{2n-1}\varphi_N\) with \( n \in \mathbb{N}^+ \) and integrating the result in \( x \)-variable, we get

\[
\int_{\mathbb{R}} (u_x\varphi_N)^{2n-1}(u_x\varphi_N)_x dx + \int_{\mathbb{R}} (u_x\varphi_N)^{2n}v_x dx + \int_{\mathbb{R}} (u_x\varphi_N)^{2n-1}\varphi_N v u_{xx} dx \\
+ \int_{\mathbb{R}} (u_x\varphi_N)^{2n-1}\varphi_N \theta_x p \ast F_1(u,v) dx = 0. \tag{14}
\]

Similar to above, we have

\[
\int_{\mathbb{R}} (u_x\varphi_N)^{2n-1}(\varphi_N u_x) dx = \| u_x\varphi_N \|_{L^{2n}}^{2n-1} \frac{d}{dt} \| u_x\varphi_N \|_{L^{2n}}, \tag{15}
\]

\[
\left| \int_{\mathbb{R}} (u_x\varphi_N)^{2n}v_x dx \right| \leq \| v_x \|_{L^\infty} \| u_x\varphi_N \|_{L^{2n}}^{2n} \\
\leq CM \| u_x\varphi_N \|_{L^{2n}}^{2n}, \tag{16}
\]

\[
\frac{d}{dt} \| u_x\varphi_N \|_{L^{2n}} \leq \| v_x \|_{L^\infty} \| u_x\varphi_N \|_{L^{2n}}^{2n}.
\]
By Gronwall's inequality, we have

\[
\left| \int \! \left( u_x \varphi \right)^{2n-1} \varphi \, u_x \, dx \right| \leq \|v\|_{L^\infty} \left| \int \! \left( u_x \varphi \right)^{2n-1} \left( u_x \varphi \right)_{x} - \left( u_x \varphi \right)^{2n-1} u_x \varphi' \, dx \right| 
\leq C \|v\|_{L^\infty} \|u_x \varphi\|_{L^{2n}^{\infty}}^{2n} 
\leq CM \|u_x \varphi\|_{L^{2n}^{\infty}},
\]

and

\[
\left| \int \! \left( u_x \varphi \right)^{2n-1} \varphi \varphi_{x} * F_1 (u, v) \, dx \right| \leq \|u_x \varphi\|_{L^{2n}^{\infty}} \|\varphi \varphi_{x} * F_1 (u, v)\|_{L^{2n}^{\infty}}. \tag{17}
\]

Substituting above estimates (15)-(18) to (14) yields

\[
\frac{d}{dt} \|u_x \varphi\|_{L^{2n}^{\infty}} \leq CM \|u_x \varphi\|_{L^{2n}^{\infty}} + \|\varphi \varphi_{x} * F_1 (u, v)\|_{L^{2n}^{\infty}}.
\]

By Gronwall's inequality, we have

\[
\|u_x \varphi\|_{L^{2n}^{\infty}} \leq \left( \|u_{x \varphi}\|_{L^{2n}^{\infty}} + \int_0^t \|\varphi \varphi_{x} * F_1 (u, v)\|_{L^{2n}^{\infty}} \, d\tau \right) e^{CMt}. \tag{19}
\]

By adding (13) and (19) yields

\[
\|u_x \varphi\|_{L^{2n}^{\infty}} + \|u_{x \varphi}\|_{L^{2n}^{\infty}} 
\leq e^{CMt} \left( \|u_{0 \varphi}\|_{L^{2n}^{\infty}} + \|u_{0 x \varphi}\|_{L^{2n}^{\infty}} \right) 
+ e^{CMt} \int_0^t \left( \|\varphi \varphi_{x} * F_1 (u, v)\|_{L^{2n}^{\infty}} + \|\varphi \varphi_{x} * F_1 (u, v)\|_{L^{2n}^{\infty}} \right) \, d\tau. \tag{20}
\]

Since for any function \( f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) implies \( \lim_{n \to \infty} \|f\|_{L^n} = \|f\|_{L^\infty} \), taking the limit in (20), we get

\[
\|u_x \varphi\|_{L^{\infty}} + \|u_{x \varphi}\|_{L^{\infty}} 
\leq e^{CMt} \left( \|u_{0 \varphi}\|_{L^{\infty}} + \|u_{0 x \varphi}\|_{L^{\infty}} \right) 
+ e^{CMt} \int_0^t \left( \|\varphi \varphi_{x} * F_1 (u, v)\|_{L^{\infty}} + \|\varphi \varphi_{x} * F_1 (u, v)\|_{L^{\infty}} \right) \, d\tau. \tag{21}
\]

A simple calculation shows that there exist \( C_0 > 0 \), depending only on \( \theta \in (0, 1) \) such that

\[
\varphi_{N} (x) \int_{\mathbb{R}} e^{-|x-y|} \frac{1}{\varphi_{N} (y)} \, dy \leq \frac{4}{1-\theta} = C_0.
\]

Thus, for any function \( f, g \in L^\infty \), we have

\[
|\varphi_{N} * (fg) (x)| = \left| \frac{1}{2} \varphi_{N} (x) \int_{\mathbb{R}} e^{-|x-y|} (fg) (y) \, dy \right| 
\leq \frac{1}{2} \left( \varphi_{N} (x) \int_{\mathbb{R}} e^{-|x-y|} \frac{1}{\varphi_{N} (y)} \, dy \right) \|f\|_{L^{\infty}} \|g \varphi_{N}\|_{L^{\infty}} \leq C_0 \|f\|_{L^{\infty}} \|g \varphi_{N}\|_{L^{\infty}}, \tag{22}
\]

\[
|\varphi_{N} \varphi_{x} * (fg) (x)| = \left| \frac{1}{2} \varphi_{N} (x) \int_{\mathbb{R}} \text{sgn} (x-y) e^{-|x-y|} (fg) (y) \, dy \right| 
\leq \frac{1}{2} \left( \varphi_{N} (x) \int_{\mathbb{R}} e^{-|x-y|} \frac{1}{\varphi_{N} (y)} \, dy \right) \|f\|_{L^{\infty}} \|g \varphi_{N}\|_{L^{\infty}} \leq C_0 \|f\|_{L^{\infty}} \|g \varphi_{N}\|_{L^{\infty}}, \tag{23}
\]
ψ enlightens us to consider a slower decay rate. If we introduce another weigh function to initial date with exponential decay at infinity will be asymptotically in the

\[ \tilde{C} \left( \| u_0 \varphi_N \|_{L^\infty} + \| u_{0x} \varphi_N \|_{L^\infty} + \int_0^t (\| u(\tau) \varphi_N \|_{L^\infty} + \| u_{x} (\tau) \varphi_N \|_{L^\infty}) d\tau \right). \] (24)

Multiplying the second equation of the system (8) by \((v \varphi_N)^{2n-1} \varphi_N\) with \(n \in \mathbb{N}^+\) and integrating result in the \(x\)-variable, then differentiating the the second equation of the system (8) with respect to \(x\), multiplying by \((v \varphi_N)^{2n-1} \varphi_N\) and integrating the result in the \(x\)-variable yields, using the similar steps above, we have

\[ \| v \varphi_N \|_{L^\infty} + \| v_{x} \varphi_N \|_{L^\infty} \leq \tilde{C} \left( \| v_0 \varphi_N \|_{L^\infty} + \| v_{0x} \varphi_N \|_{L^\infty} + \int_0^t (\| v(\tau) \varphi_N \|_{L^\infty} + \| v_{x} (\tau) \varphi_N \|_{L^\infty}) d\tau \right). \] (25)

Adding (24) and (25) yields

\[ \| u \varphi_N \|_{L^\infty} + \| u_{x} \varphi_N \|_{L^\infty} + \| v \varphi_N \|_{L^\infty} + \| v_{x} \varphi_N \|_{L^\infty} \leq \tilde{C} \left( \| u_0 \varphi_N \|_{L^\infty} + \| u_{0x} \varphi_N \|_{L^\infty} + \| v_0 \varphi_N \|_{L^\infty} + \| v_{0x} \varphi_N \|_{L^\infty} \right) + \int_0^t (\| u(\tau) \varphi_N \|_{L^\infty} + \| u_{x} (\tau) \varphi_N \|_{L^\infty} + \| v(\tau) \varphi_N \|_{L^\infty} + \| v_{x} (\tau) \varphi_N \|_{L^\infty}) d\tau. \] (26)

Hence, for any \(N \in \mathbb{N}^+\) and any \(t \in [0, T]\), by Gronwall’s inequality, we have

\[ \| u \varphi_N \|_{L^\infty} + \| u_{x} \varphi_N \|_{L^\infty} + \| v \varphi_N \|_{L^\infty} + \| v_{x} \varphi_N \|_{L^\infty} \leq \tilde{C} (\| u_0 \varphi_N \|_{L^\infty} + \| u_{0x} \varphi_N \|_{L^\infty} \right) + \| v_0 \varphi_N \|_{L^\infty} + \| v_{0x} \varphi_N \|_{L^\infty}), \] (27)

Finally, taking the limit as \(N\) goes to infinity in (27), we find that for any \(t \in [0, T]\)

\[ \| u(t, x)e^{\theta x} \|_{L^\infty} + \| u_{x} (t, x)e^{\theta x} \|_{L^\infty} + \| v(t, x)e^{\theta x} \|_{L^\infty} + \| v_{x} (t, x)e^{\theta x} \|_{L^\infty} \leq \tilde{C} (\| u_0 \max\{1, e^{\theta x}\} \|_{L^\infty} + \| u_{0x} \max\{1, e^{\theta x}\} \|_{L^\infty} \right) + \| v_0 \max\{1, e^{\theta x}\} \|_{L^\infty} + \| v_{0x} \max\{1, e^{\theta x}\} \|_{L^\infty}), \]

which implies the desired result. This completes the proof of the theorem. \( \Box \)

Theorem 3.1 tells us that the strong solution of the system (8) corresponding to initial date with exponential decay at infinity will be asymptotically in the \(x\)-variable at infinity in its lifespan. However, exponential decay is too fast, which enlightens us to consider a slower decay rate. If we introduce another weigh function \(\psi_N(x)\) as follows:

\[ \psi_N(x) = \begin{cases} 
1, & x \leq 0, \\
(1 + x)^\alpha, & x \in (0, N), \\
(1 + N)^\alpha, & x \geq N,
\end{cases} \]

where \(\alpha \in (0, 1]\) and \(N \in \mathbb{N}^+\). Note that for all \(N\), we have

\[ 0 \leq \psi_N(x) \leq \psi_N(x), \text{ a.e. } x \in \mathbb{R}, \]

and there exist \(C_\alpha > 0\), depending only on \(\alpha \in (0, 1]\) such that

\[ \psi_N(x) \int_{\mathbb{R}} e^{-|x-y|} \frac{1}{\psi_N(y)} dy \leq 3 + (1 + \alpha)^2 = C_\alpha. \]
Lemma 4.1. The following results are very crucial in study the unique continuation of strong solutions. We will need two ingredients in order to provide a sufficient answer: How will strong solutions behave at infinity when given compactly supported initial data? It is essentially infinite speed of propagation of its support. Therefore, it is natural to ask the question: How will strong solutions behave at infinity when given compactly supported initial data? We will need two ingredients in order to provide a sufficient answer. These considerations were first made for CH in [26, 14, 27, 29, 38].

We first consider the following ordinary differential equations:

$$\begin{cases}
\frac{dp(t,x)}{dt} = v(t,p(t,x)), & (t,x) \in [0,T] \times \mathbb{R}, \\
p(0,x) = x, & x \in \mathbb{R},
\end{cases}$$

and

$$\begin{cases}
\frac{dq(t,x)}{dt} = u(t,q(t,x)), & (t,x) \in [0,T] \times \mathbb{R}, \\
q(0,x) = x, & x \in \mathbb{R}.
\end{cases}$$

The following results are very crucial in study the unique continuation of strong solutions.

Lemma 4.1. [36] Let $\left(u_0, v_0\right) \in H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$ with $s > \frac{5}{2}$ and $T > 0$ be the maximal existence time of the corresponding solutions $(u, v)$ to the Cauchy problem (8). Then Eq. (28) and Eq. (29) have unique solutions $p \in C^1([0,T] \times \mathbb{R}; \mathbb{R})$ and $q \in C^1([0,T] \times \mathbb{R}; \mathbb{R})$, respectively. Moreover, the maps $p(t, \cdot)$ and $q(t, \cdot)$ are increasing diffeomorphisms of $\mathbb{R}$ with

$$p_x(t,x) = \exp \left( \int_0^t v_x(s,p(s,x))ds \right) > 0,$$

and

$$q_x(t,x) = \exp \left( \int_0^t u_x(s,p(s,x))ds \right) > 0,$$

for all $(t,x) \in [0,T] \times \mathbb{R}$.

Lemma 4.2. [36]. Let $\left(u_0, v_0\right) \in H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$ with $s > \frac{5}{2}$ and $T > 0$ be the maximal existence time of the corresponding solutions $(u, v)$ to the Cauchy problem (8). Then we have

$$m(t,p(t,x))p_x^2(t,x) = m_0(x),$$

and

$$n(t,q(t,x))q_x^2(t,x) = n_0(x),$$

for all $(t,x) \in [0,T] \times \mathbb{R}$.

Theorem 3.2. Let $\left(u_0, v_0\right) \in H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$ with $s > \frac{5}{2}$ and $T > 0$ be the maximal existence time of the corresponding solutions $(u, v)$ to the Cauchy problem (8). If there exists some $\alpha \in (0,1]$ such that

$$|u_0(x)|, |u_0(x)|, |v_0(x)|, |v_0(x)| \sim O((1 + x)^{-\alpha}) \text{ as } x \uparrow \infty,$$

then

$$|u(t,x)|, |u_x(t,x)|, |v(t,x)|, |v_x(t,x)| \sim O((1 + x)^{-\alpha}) \text{ as } x \uparrow \infty$$

uniformly in the time interval $[0,T]$.

Proof. The proof of the theorem is similar to the proof of Theorem 3.1, so we omit it here. □

4. Compactly supported initial data. In this section, we reflect on the property of unique continuation which we have just shown the Cauchy problem for the system (8) to exhibit. In the case of compactly supported initial data unique continuation is essentially infinite speed of propagation of its support. Therefore, it is natural to ask the question: How will strong solutions behave at infinity when given compactly supported initial data? We will need two ingredients in order to provide a sufficient answer. These considerations were first made for CH in [26, 14, 27, 29, 38].

Theorem 3.2. Let $\left(u_0, v_0\right) \in H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$ with $s > \frac{5}{2}$ and $T > 0$ be the maximal existence time of the corresponding solutions $(u, v)$ to the Cauchy problem (8). If there exists some $\alpha \in (0,1]$ such that

$$|u_0(x)|, |u_0(x)|, |v_0(x)|, |v_0(x)| \sim O((1 + x)^{-\alpha}) \text{ as } x \uparrow \infty,$$

then

$$|u(t,x)|, |u_x(t,x)|, |v(t,x)|, |v_x(t,x)| \sim O((1 + x)^{-\alpha}) \text{ as } x \uparrow \infty$$

uniformly in the time interval $[0,T]$.

Proof. The proof of the theorem is similar to the proof of Theorem 3.1, so we omit it here. □
Theorem 4.3. Let \((u, v) \in C([0, T); H^s(\mathbb{R})) \times C([0, T); H^s(\mathbb{R}))\), \(s > \frac{2}{5}\) be a non-trivial solution of the Cauchy problem (8), with maximal time of existence time \(T > 0\), which is initially compactly supported on an interval \([\alpha, \beta]\). Then we have

\[
E_+(t) = \int_{p(t, \alpha)}^{p(t, \beta)} e^y m(t, y) dy, \quad E_-(t) = \int_{p(t, \alpha)}^{p(t, \beta)} e^{-y} m(t, y) dy,
\]

\[
F_+(t) = \int_{q(t, \alpha)}^{q(t, \beta)} e^y n(t, y) dy, \quad F_-(t) = \int_{q(t, \alpha)}^{q(t, \beta)} e^{-y} n(t, y) dy.
\]

Moreover, \(E_+, E_-, F_+\) and \(F_-\) are continuous non-vanishing functions with \(E_+(0) = E_-(0) = F_+(0) = F_-(0)\). If \(m_0\) and \(n_0\) are positive, then \(E_+(t), F_+(t)\) strictly increasing and \(E_-(t), F_-(t)\) strictly decreasing for \(t \in [0, T]\).

Theorem 4.3 tells us that as long as the solutions \(u(t, x)\) and \(v(t, x)\) exists, then it is positive at positive infinity and negative at negative infinity.

Proof. If \(u_0\) and \(v_0\) are initially supported on the compact interval \([\alpha, \beta]\) then so are \(m_0\) and \(n_0\), then we deduce from Lemma 4.1 that \(m(t, \cdot)\) has its support on the interval \([p(t, \alpha), p(t, \beta)]\) and \(n(t, \cdot)\) has its support on the interval \([q(t, \alpha), q(t, \beta)]\).

We now use the relation \(u = \frac{1}{2} e^{-|x|} * m\) and \(v = \frac{1}{2} e^{-|x|} * n\) to write

\[
u(t, x) = \frac{e^{-x}}{2} \int_{-\infty}^{x} e^y m(t, y) dy + \frac{e^x}{2} \int_{x}^{\infty} e^y m(t, y) dy,
\]

\[
u_x(t, x) = -\frac{e^{-x}}{2} \int_{-\infty}^{x} e^y m(t, y) dy + \frac{e^x}{2} \int_{x}^{\infty} e^y m(t, y) dy,
\]

and

\[
E_+(t) = \int_{p(t, \alpha)}^{p(t, \beta)} e^y m(t, y) dy, \quad E_-(t) = \int_{p(t, \alpha)}^{p(t, \beta)} e^{-y} m(t, y) dy,
\]

\[
F_+(t) = \int_{q(t, \alpha)}^{q(t, \beta)} e^y n(t, y) dy, \quad F_-(t) = \int_{q(t, \alpha)}^{q(t, \beta)} e^{-y} n(t, y) dy.
\]

We define our functions

\[
u(t, x) = \frac{e^{-x}}{2} E_+(t), \quad x > p(t, \beta),
\]

\[
u(t, x) = \frac{e^x}{2} E_-(t), \quad x < p(t, \alpha),
\]
\[ v(t, x) = \frac{e^{-x}}{2} F_+(t), \quad x > q(t, \beta), \]
\[ v(t, x) = \frac{e^{-x}}{2} F_-(t), \quad x < q(t, \alpha), \]

and therefore from differentiating (30) directly we get
\[ \frac{e^{-x}}{2} E_+(t) = u(t, x) = -u_x(t, x) = u_{xx}(t, x), \quad x > p(t, \beta), \]
\[ \frac{e^{x}}{2} E_-(t) = u(t, x) = u_x(t, x) = u_{xx}(t, x), \quad x < p(t, \alpha), \]
\[ \frac{e^{-x}}{2} F_+(t) = v(t, x) = -v_x(t, x) = v_{xx}(t, x), \quad x > q(t, \beta), \]
\[ \frac{e^{x}}{2} F_-(t) = v(t, x) = v_x(t, x) = v_{xx}(t, x), \quad x < q(t, \alpha). \]

Since \( u(0, \cdot) \) and \( v(0, \cdot) \) is supported in the interval \([\alpha, \beta]\) this immediately gives us \( E_+(0) = E_-(0) = F_+(0) = F_-(0). \)

Since \( m(t, \cdot) \) is supported in the interval \([p(t, \alpha), p(t, \beta)]\) and assume that \( m_0 \) and \( n_0 \) are positive, for \( t \in [0, T] \), we have
\[
\frac{dE_+(t)}{dt} = \int_{p(t, \alpha)}^{p(t, \beta)} e^y m_1(t, y) dy \\
= \int_{-\infty}^{\infty} e^y m_1(t, y) dy \\
= -2 \int_{-\infty}^{\infty} e^y v_y m dy - \int_{-\infty}^{\infty} e^y v m_y dy \\
= \int_{-\infty}^{\infty} (v - v_y) m e^y dy > 0,
\]
where the strict positivity of the relation above follows from our assumption that the solution is nontrivial. Similarly, we have
\[
\frac{dE_-(t)}{dt} = \int_{p(t, \alpha)}^{p(t, \beta)} e^{-y} m_1(t, y) dy \\
= \int_{-\infty}^{\infty} e^{-y} m_1(t, y) dy \\
= -2 \int_{-\infty}^{\infty} e^{-y} v_y m dy - \int_{-\infty}^{\infty} e^{-y} v m_y dy \\
= \int_{-\infty}^{\infty} (v + v_y) m e^{-y} dy < 0.
\]

Using the similar methods give the properties of \( F_+(t) \) and \( F_-(t) \). This concludes the proof of the theorem. \( \square \)

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