Quantum mechanical perspectives and generalization of the fractional Fourier transformation∗

Jun-hua Chen1,2 † and Hong-yi Fan1
1 Department of Material Science and Engineering, University of Science and Technology of China, Hefei, Anhui, 230026, China
2 CAS Key Laboratory of Materials for Energy Conversion Hefei, Anhui, 230026, China

May 11, 2014

Abstract

Fourier and fractional-Fourier transformations are widely used in theoretical physics. In this paper we make quantum perspectives and generalization for the fractional Fourier transformation (FrFT). By virtue of quantum mechanical representation transformation and the method of integration within normal ordered product (IWOP) of operators, we find the key point for composing FrFT, and reveal the structure of FrFT. Following this procedure, a full family of generalized fractional transformations are discovered with the usual FrFT as one special case. The eigen-functions of arbitrary GFrT are derived explicitly.

Keywords: fractional Fourier transformation; additivity; Abelian Lie group; Parseval; eigen-function

1 Introduction

The fractional Fourier transformation (FrFT) is a very useful tool in Fourier optics and information optics, especially in optical communication, image manipulations and signal analysis [1–7]. The concept of the FrFT was originally described by Condon [3] and was later introduced for signal processing in 1980

∗Work supported by the National Natural Science Foundation of China under grant No. 11105133, the Fundamental Research Funds for the Central Universities under grant WK2060140013, and National Basic Research Program of China (973 Program, 2012CB922001)

†To whom correspondence should be addressed. Email: cjh@ustc.edu.cn
by Namias as a Fourier transform of fractional order. Sumiyoshi et al. also made an interesting generalization on FrFT in 1994. But FrFT did not have significant impact on optics until FrFT was defined physically based on propagation in quadratic graded-index media (GRIN media). Mendlovic and Ozaktas defined the $\alpha$-th FrFT as follows: let the original function be input from one side of quadratic GRIN medium, at $z = 0$, then the light distribution observed at the plane $z = z_0$ corresponds to the $\alpha$ equal to the $(z_0/L)$-th fractional Fourier transform of the input function, where $L \equiv (\pi/2)(n_1/n_2)^{1/2}$ is a characteristic distance, $n_1, n_2$ are medium’s physical parameters involved in the refractive index $n(r) = n_1 - n_2 r^2/2$, $r$ is the radial distance from the optical $z$ axis). For real parameter $\alpha$, the 1-dimensional $\alpha$-angle FrFT of a function $f$ is denoted by $F_\alpha[f]$ and defined by

$$F_\alpha[f](p) = \frac{e^{i(\alpha/2-\pi/4)}}{\sqrt{2\pi \sin \alpha}} \int_{-\infty}^{\infty} \exp \left[ i \frac{p^2 + x^2}{\tan \alpha} - \frac{2px}{\sin \alpha} \right] f(x) \, dx \quad (1)$$

The conventional Fourier transform is simply $F_{\pi/2}$. The composition $F_\alpha \circ F_\beta$ of two FrFT’s with parameters $\alpha$ and $\beta$ is defined by

$$(F_\alpha \circ F_\beta)[f] = F_\alpha[F_\beta[f]]. \quad (2)$$

$F$ is additive under definition Eq. (2), i.e.,

$$F_\alpha \circ F_\beta = F_{\alpha+\beta}. \quad (3)$$

In the context of quantum mechanics, function $f$ turns to quantum state $|f\rangle$, the value $f(x)$ of $f$ at given point $x$ turns to the matrix element $\langle x | f \rangle$. The usual Fourier transform is simply changing of basis,

$$\hat{f}(p) = \langle p | f \rangle = \int_{-\infty}^{\infty} \langle p | x \rangle \langle x | f \rangle \, dx = \int_{-\infty}^{\infty} \frac{e^{-ipx}}{\sqrt{2\pi}} f(x) \, dx. \quad (4)$$

And the $\alpha$-angle fractional Fourier transform is simply

$$F_\alpha[f](p) = \langle p | e^{i(\pi/2-\alpha)a_1^+a} | f \rangle = \int_{-\infty}^{\infty} K_\alpha(p, x) \ f(x) \, dx \quad (5)$$

where the kernel of transformation $K_\alpha(p, x)$ is

$$K_\alpha(p, x) = \langle p | K_\alpha | x \rangle = \langle p | e^{i(\pi/2-\alpha)a_1^+a} | x \rangle = \frac{e^{i(\alpha/2-\pi/4)}}{\sqrt{2\pi \sin \alpha}} \exp \left[ i \frac{p^2 + x^2}{\tan \alpha} - \frac{2px}{\sin \alpha} \right]. \quad (6)$$

I.e., the $\alpha$-angle fractional Fourier transform is the composite transformation of both the basis changing and unitary transformation generated by the operator $K_\alpha = e^{i(\pi/2-\alpha)a_1^+a}$. The key feature here is that the transformation is
compositable and additive, i.e., one can perform fractional Fourier transform repeatedly on given function $f$,

$$(F_\alpha \circ F_\beta)[f] \equiv F_\alpha[F_\beta[f]],$$

(7)

and $F$ is additive under definition Eq. (7), $F_\alpha \circ F_\beta = F_{\alpha+\beta}$.

Enlightened by the above analysis, we hope to find the criteria for constructing new generalized fractional transformation (GFrT), and point out the perspectives of FrFT. In other words, we want to generalize FrFT to all possible compositable and additive transformations which automatically exhibit fractional transform’s properties. We shall do this by virtue of quantum mechanical representation transformation [8] and the method of integration within normal ordered product (IWOP) of operators [9].

2 Analysis of the key point of GFrT

Let $|A\rangle$ and $|B\rangle$ denote two sets of basis. In order to perform their mutual transformation repeatedly, $|A\rangle$ and $|B\rangle$ must have matching parameterizations, i.e., each pair of $A_i$ and $B_i$ in

$$A = (A_1, \cdots, A_n)$$

(8)

$$B = (B_1, \cdots, B_n)$$

must be both integer/real/complex parameters with exactly the same ranges. As usual, we demand the completeness of $|A\rangle$ and $|B\rangle$

$$\int |A\rangle \langle A| d^n A = 1, \quad \int |B\rangle \langle B| d^n B = 1$$

(9)

where $\int d^n A \equiv \int \prod_i dA_i$, and

$$\int dA_i \equiv \begin{cases} \sum_{A_i}, \text{for integer } A_i \\ \int dA_i, \text{for real } A_i \\ \int \int d(ReA_i) d(ImA_i), \text{for complex } A_i \end{cases}$$

(10)

General transformation $F_K$ is defined by

$$F_K[f](B) = \langle B| K|f\rangle = \int \langle B| K|A\rangle \langle A| f\rangle d^n A$$

(11)

$$= \int K(B, A) \langle A| f\rangle d^n A,$$

where $K(B, A)$ is a composite

$$K(B, A) \equiv \langle B| K|A\rangle.$$
Since \( F_K \) is a transformation on functions, and functions \( f \) and \( F_K[f] \) have matching variables, it is natural to define the successive composite transformation \( F_{K_1} \circ F_{K_2} \) of \( F_{K_1} \) and \( F_{K_2} \) on function \( f \) by

\[
(F_{K_1} \circ F_{K_2})[f] = F_{K_1}[F_{K_2}[f]],
\]

i.e.

\[
(F_{K_1} \circ F_{K_2})[f](B) = \int \int K_1(B, A) K_2(A', A') f(A') d^n A' d^n A,
\]

where, according to Eq. (12),

\[
K_1(B, A) = \langle B | K_1 | A \rangle,
\]

and \( K_2(A, A') \) should be composite too,

\[
K_2(A, A') = \langle B' | B' = A K_2 | A' \rangle.
\]

Eq. (14) then reads

\[
(F_{K_1} \circ F_{K_2})[f](B) = \int \int \langle B | K_1 | A \rangle \langle B' = A K_2 | A' \rangle \langle A' | f \rangle d^n A' d^n A
\]

\[
= \langle B | K_1 M K_2 | f \rangle = F_{K_1 M K_2}[f](B),
\]

where an operator \( M \) emerged

\[
M = \int |A \rangle \langle B' = A | d^n A.
\]

This is an essence for the successive composite transformations. Note that the rule of composition of transformations is

\[
F_{K_1} \circ F_{K_2} = F_{K_1 M K_2} \neq F_{K_1 K_2},
\]

i.e., the mapping \( K \rightarrow F_K \) is not homomorphic. The key point of GF\( \hat{\text{Y}} \)T is to determine all allowed operators \( K \) so that the transformations are compositable and additive.

3 Determination of allowed \( K \)

We determine the allowed operator \( K \) by two criteria, the first is the additivity of the transformations, the second is that the transformations must also satisfy the Parseval theorem as the conventional Fourier transformation obeys.
3.1 Additivity

Since the fundamental property of FrFT is the additivity, \( F_\alpha \circ F_\beta = F_{\alpha+\beta} \), we demand that the additivity still holds for GFrT \( F_K \), therefore there must be a way of parameterization \( K_\alpha \) for \( K \) such that

\[
K_\alpha \mathcal{M} K_\beta = K_{\alpha+\beta}.
\]

This equation is necessary and sufficient for \( F_K \) to be additive,

\[
F_{K_\alpha} \circ F_{K_\beta} = F_{K_\alpha \mathcal{M} K_\beta} = F_{K_{\alpha+\beta}}.
\]

In this case \( F_{K_\alpha} \) can be considered as the natural generalization of the FrFT.

By defining

\[
\tilde{K}_\alpha = K_\alpha \mathcal{M},
\]

then equation \( K_\alpha \mathcal{M} K_\beta = K_{\alpha+\beta} \) becomes

\[
\tilde{K}_\alpha \tilde{K}_\beta = \tilde{K}_{\alpha+\beta},
\]

i.e. the allowed \( \tilde{K}_\alpha \)'s form an Abelian Lie group \( \tilde{K} \) (or the sub group of an Abelian Lie group, in the case that some components of \( \alpha \) take discrete values), \( K_\alpha \)'s form the right coset \( \mathcal{M}^{-1} \) of \( \tilde{K} \). Conversely, if we have an Abelian Lie group \( \tilde{K} \) and two sets of basis \( |A\rangle \) and \( |B\rangle \) with matching parameterization, then we can define a GFrT by

\[
F_{K_\alpha} [f](B) = \langle B | K_\alpha | f \rangle = \int K_\alpha(A) \langle A | f \rangle d^n A
\]

where \( K_\alpha = \tilde{K}_\alpha \mathcal{M}^{-1} \).

3.2 Parseval’s Theorem

Further, we demand some sort of Parseval’s theorem for the new transformation, i.e.

\[
\int |F_K[f](B)|^2 d^n B = \int |f(A)|^2 d^n A
\]

\[
\int \langle f | K^\dagger | B \rangle \langle B | K | f \rangle d^n B = \int \langle f | A \rangle \langle A | f \rangle d^n A
\]

\[
\langle f | K^\dagger K | f \rangle = \langle f | f \rangle
\]

therefore \( K \) must be unitary, \( K^\dagger K = 1 \).

Parseval’s theorem demands that all the allowed \( K' \)'s must be unitary, therefore \( \mathcal{M} \mathcal{M} K_2 \) must be unitary if we wish to define \( F_{K_1} \circ F_{K_2} \) properly

\[
(K_1 \mathcal{M} K_2)^\dagger K_1 \mathcal{M} K_2 = K_2^\dagger \mathcal{M}^\dagger K_1^\dagger \mathcal{M} K_1 K_2 = K_2^\dagger \mathcal{M}^\dagger \mathcal{M} K_2 = 1
\]

\[
\mathcal{M}^\dagger \mathcal{M} = 1
\]
must be unitary. The unitarity of \( M \) is guaranteed if either \(|A\rangle\)'s or \(|B\rangle\)'s are orthogonal. In fact, when 
\[
\langle A' | A \rangle = \delta^{(n)} (A' - A),
\]
then 
\[
\begin{align*}
M^\dagger M &= \int |B\rangle_{B' = A'} \langle A'| d^n A' \int |A\rangle \langle B|_{B = A} d^n A \\
&= \int |B\rangle_{B' = A'} \langle B|_{B = A} \delta^{(n)} (A' - A) d^n A' d^n A \\
&= \int |B\rangle_{B' = A'} \langle B|_{B = A} d^n A = 1.
\end{align*}
\]
If \( \langle B' | B \rangle = \delta^{(n)} (B' - B) \), then we also have 
\[
\begin{align*}
M^\dagger = \int |A\rangle \langle B'|_{B' = A} d^n A \int |B\rangle_{B' = A'} \langle A'| d^n A' \\
&= \int |A\rangle \langle A'| \delta^{(n)} (B' - B) |B' = A, B' = A' d^n A' d^n A \\
&= \int |A\rangle \langle A| d^n A = 1.
\end{align*}
\]

4 The construction and the eigen-problem of GFrT

For the generalized transformation satisfying Parseval’s Theorem, we need a unitary Abel Lie group \( \tilde{\mathcal{K}} \). The structure of such group \( \tilde{\mathcal{K}} \) is simple, each element takes the form 
\[
\tilde{\mathcal{K}}_\alpha = \exp \left[ i \sum_j \alpha_j O_j \right],
\]
where \( \alpha_j \) are real parameters and \( O_j \) are Hermitian operators that commute with each other, \( [O_j, O_k] = 0 \). Conversely, given a set of commuting Hermitian operators \( O_j \) and two sets of basis \( |A\rangle \) and \( |B\rangle \) with matching parameterization, we can construct the corresponding generalized fractional transform \( F_{K,\alpha} \) with 
\[
K_\alpha = \tilde{\mathcal{K}}_\alpha M^\dagger = \exp \left[ i \sum_j \alpha_j O_j \right] M^\dagger.
\]

As usual, the eigen-functions \( f \) of classical GFrT must satisfy 
\[
\hat{f}(B) = \int K(B, A) f(A) d^n A = \lambda f(B).
\]
The quantum version of Eq. (30) is 
\[
\langle B | \hat{f} \rangle = \langle B | K | f \rangle = \lambda \langle A | f \rangle_{A = B},
\]
which is equivalent to

\[ \int d^n B \langle B | K | f \rangle = \lambda \int d^n B \langle A | f \rangle_{A=B} \]  \hspace{1cm} (32) \]

\[ \exp \left[ \sum_{j=1}^{s} i \alpha_j O_j \right] \mathcal{M}^\dagger | f \rangle = \lambda \mathcal{M}^\dagger | f \rangle. \]  \hspace{1cm} (33) \]

Since \( O_j \)'s commute with each other, Eq. (33) can be decomposed as equations

\[ O_j \mathcal{M}^\dagger | f \rangle = \theta_{j,m} \mathcal{M}^\dagger | f \rangle, \]  \hspace{1cm} (34) \]

i.e., \( \mathcal{M}^\dagger | f \rangle \) is the common eigenstate of the commutating Hermitian operators \( O_j \)'s. Let \( | \varphi_m \rangle \) be the common eigenstate of \( O_j \)'s

\[ O_j | \varphi_m \rangle = \theta_{j,m} | \varphi_m \rangle \]  \hspace{1cm} (35) \]

(\( | \varphi_m \rangle \)'s form a complete set, \( \sum_m | \varphi_m \rangle \langle \varphi_m | = 1 \), and \( | \varphi_m \rangle \) can be chosen to be orthogonal, \( \langle \varphi_m | \varphi_{m'} | = \delta_{m,m'} \)), then

\[ | f_m \rangle = \mathcal{M} | \varphi_m \rangle. \]  \hspace{1cm} (36) \]

The eigenfunctions of the classical GFrT is

\[ f_m (A) = \langle A | f_m \rangle = \langle A | \mathcal{M} | \varphi_m \rangle \]  \hspace{1cm} (37) \]

with eigenvalue \( \exp \left[ \sum_{j=1}^{s} i \alpha_j \theta_{j,m} \right] \).

\[ F_{K_{\alpha}} | f_m \rangle (B) = \exp \left[ \sum_{j=1}^{s} i \alpha_j \theta_{j,m} \right] f_m (B) \]  \hspace{1cm} (38) \]

The eigen-functions \( f_m (A) \)'s are orthogonal

\[ \int f_m^\ast (A) f_{m'} (A) d^n A = \int \langle \varphi_m | \mathcal{M}^\dagger | A \rangle \langle A | \mathcal{M} | \varphi_{m'} \rangle d^n A \]  \hspace{1cm} (39) \]

\[ = \langle \varphi_m | \mathcal{M}^\dagger \mathcal{M} | \varphi_{m'} \rangle = \delta_{m,m'}. \]

And \( f_m (A) \)'s are complete too, any "good" functions \( g (A) \) can be written as the linear combination of \( f_m (A) \)'s,

\[ g (A) = \langle A | g \rangle = \sum_m \langle A | \mathcal{M} | \varphi_m \rangle \langle \varphi_m | \mathcal{M}^\dagger | g \rangle \]  \hspace{1cm} (40) \]

\[ = \sum_m \langle f_m | g \rangle \langle A | f_m \rangle = \sum_m C_m f_m (A). \]
5 Some remarkable examples

5.1 Example 1

Let $|A\rangle = |x\rangle$, the coordinate representation, and $|B\rangle = |p\rangle$, the momentum representation, in Fock space they are expressed as

\[
|A\rangle = |x\rangle = \frac{1}{\pi^{1/4}} \exp \left[ -\frac{1}{2} x^2 + \sqrt{2} x a^\dagger - \frac{1}{2} a^\dagger 2 \right] |0\rangle , \tag{41}
\]

\[
|B\rangle = |p\rangle = \frac{1}{\pi^{1/4}} \exp \left[ -\frac{1}{2} p^2 + i \sqrt{2} p a^\dagger + \frac{1}{2} a^\dagger 2 \right] |0\rangle
\]

where $[a, a^\dagger] = 1$, and $|0\rangle$ is the vacuum state annihilated by $a$, $a |0\rangle = 0$.

Using the method of integration within normal ordered product of operators and $|0\rangle \langle 0| = : \exp \left[ -a^\dagger a \right] :$, we can perform the following integration (constructed according to Eq. (18))

\[
\mathfrak{M} = \int_{-\infty}^{\infty} dx |x\rangle \langle p|_{p=x}
\]

\[
= \int_{-\infty}^{\infty} dx \sqrt{\frac{2}{\pi}} \exp \left[ -x^2 + \sqrt{2} x a^\dagger - \frac{1}{2} a^\dagger 2 - a^\dagger a - i \sqrt{2} x a + \frac{1}{2} a^\dagger 2 \right] : \exp \left[ -(1 + i) a^\dagger a \right] : \exp \left[ -\frac{i \pi}{2} a^\dagger a \right],
\]

obviously $K_\alpha = e^{i \left( \frac{\pi}{2} - \alpha \right) a^\dagger a}$ in Eq. (40) obeys $K_\alpha \mathfrak{M} K_\beta = K_{\alpha + \beta}$, no wonder Eqs. (40)-(43) can embody the characters of fractional Fourier transform. And we now understand better why the kernel $K_\alpha (p, x)$ should be defined in the seemingly unnatural way $\langle p | e^{i \left( \frac{\pi}{2} - \alpha \right) a^\dagger a} | x \rangle$ instead of naturally $\langle p | e^{-i \alpha a^\dagger a} | x \rangle$, this is because $K_\alpha = K_\alpha^\dagger = e^{-i \alpha a^\dagger a} e^{i \frac{\pi}{2} a^\dagger a} = e^{i \left( \frac{\pi}{2} - \alpha \right) a^\dagger a}$. And we see that the eigenfunctions of FrFT are

\[
f_m (x) = \langle x | \mathfrak{M} | m \rangle = \langle x | \exp \left[ -\frac{i \pi}{2} a^\dagger a \right] | m \rangle
\]

\[
= \frac{1}{i^m} \langle x | m \rangle = \frac{1}{i^m \sqrt{2^m m!}} H_m (x) e^{-x^2/2}
\]

with eigenvalues $e^{-i \alpha a}$ by Eqn. (37).

5.2 Example 2

Let $|A\rangle = |x\rangle$, $|B\rangle = |p\rangle$, and

\[
\tilde{K}_\alpha = \exp \left[ -\frac{i \alpha}{2} \left( a^2 e^{i \theta} + e^{-i \theta} a^\dagger 2 \right) \right], \tag{44}
\]

8
\( \tilde{K} \) is Abelian with respect to the parameter \( \alpha \). The disentangling of \( \tilde{K}_\alpha \) is

\[
\tilde{K}_\alpha = \frac{1}{\sqrt{\cosh \alpha}} : \exp \left[ \frac{i}{2} a^2 e^{-i\theta} \tanh \alpha + a^\dagger a \left( \frac{1}{\cosh \alpha} - 1 \right) - \frac{i}{2} a^2 e^{i\theta} \tanh \alpha \right] : .
\]

Using the completeness relation of the coherent state

\[
\int \frac{d^2z}{\pi} \ket{z}\bra{z} = 1, \quad |z| = e^{-|z|^2/2} a^\dagger |0\rangle
\]

and knowing \( \mathfrak{M} \) in Eq. (42) having the property \( \mathfrak{M}^\dagger |x\rangle = |p'\rangle |p'=x\rangle \), we have

\[
\langle p| \tilde{K}_\alpha \mathfrak{M}^\dagger |x\rangle = \frac{1}{\sqrt{\cosh \alpha}} \int \frac{d^2z_1}{\pi} \int \frac{d^2z_2}{\pi} \langle p| z_1 \rangle \langle z_1| \tilde{K}_\alpha | z_2 \rangle \langle z_2| p' \rangle |p'=x\rangle .
\]

Then using the overlap

\[
\langle p| z \rangle = \frac{1}{\pi^{1/4}} \exp \left[ -\frac{p^2}{2} - |z|^2 + \frac{1}{2} v^2 \right]
\]

and the integration formula

\[
\int \frac{d^2z}{\pi} \exp \left[ \zeta |z|^2 + f z^2 + g z x^2 + \xi z + \eta z^* \right] = \frac{1}{\sqrt{\zeta^2 - 4fg}} \exp \left[ -\zeta \xi \eta + f \eta^2 + g \xi^2 \right],
\]

we see

\[
\langle p| \tilde{K}_\alpha \mathfrak{M}^\dagger |x\rangle = \frac{1}{\sqrt{\cosh \alpha}} \int \frac{d^2z_1}{\pi} \int \frac{d^2z_2}{\pi} \exp \left[ -|z_1|^2 + \frac{z_1^2}{2} - \frac{z_1^2}{2} e^{-i\theta} \tanh \alpha - \sqrt{2pz_1} + \frac{1}{\cosh \alpha} z_1^* z_2 - \frac{z_2^2}{2} - \frac{z_2^2}{2} e^{i\theta} \tanh \alpha - |z_2|^2 + \frac{z_2^2}{2} + \sqrt{2iz_1 z_2} \right] .
\]

One can easily check that this new transformations satisfy

\[
(F_\alpha \circ F_\beta) [f] (p) = \frac{1}{\sqrt{2\pi \alpha \cos \theta \sinh \beta}} \exp \left[ i \left( \frac{x^2 + q^2}{2 \tanh \beta \cos \theta} + (x^2 - p^2) \tan \theta - \frac{2xp}{\sinh \beta \cos \theta} \right) \right] f(x),
\]

so the transformations are additive.
If we choose $\theta = \frac{\pi}{2}$ in $\widetilde{K}_\alpha$, $\widetilde{K}_\alpha$ becomes the single-mode squeezing operator
\[ \exp \left[ \frac{\alpha}{2} \left( a^2 - a^\dagger{}^2 \right) \right] = \int_{-\infty}^{\infty} e^{a^\dagger \alpha/2 dp} |pe^\alpha\rangle \langle p| = \int_{-\infty}^{\infty} e^{-a^\alpha/2 dx} |xe^{-a}\rangle \langle x|, \]
the corresponding transformation becomes the Hadamard transformation of continuum variables \[10\], i.e.,
\[ K_\alpha = \exp \left[ \frac{\alpha}{2} (a^2 - a^\dagger{}^2) \right] \exp \left[ \frac{i\pi}{2} a^\dagger a \right] \] (52)
we have
\[ \lim_{\theta \to \frac{\pi}{2}} K_\alpha (p, x) = e^{-\alpha/2} \delta \left( x - pe^{-a} \right) \] (53)

Hadamard transform is not only an important tool in classical signal processing, but also is of great importance for quantum computation applications.

If we choose $\theta = 0$ in $\widetilde{K}_\alpha$, $\widetilde{K}_\alpha$ becomes \[ \exp \left[ -\frac{i\alpha}{2} (a^2 + a^\dagger{}^2) \right] \], which is still sort of one-mode squeezing operator. The kernel of the new transformation is
\[ \langle p| \widetilde{K}_\alpha \mathcal{M}^\dagger \langle x| = \exp \left[ \frac{i}{2} \left( \frac{x^2 + p^2}{\tanh \alpha} - \frac{2xp}{\sinh \alpha} \right) \right] \frac{1}{\sqrt{2\pi i \sinh \alpha}}, \] (54)

note the similarity and difference between $(\tanh \alpha, \sinh \alpha)$ here in Eq. (54) and $(\tan \alpha, \sin \alpha)$ in Eq. (1).

In summary, by virtue of quantum mechanical representation transformation and the method of integration within normal ordered product (IWOP) of operators we have found the key point of composing FrFT, and reveal the structure for constructing new GFrT, in so doing a full family of GFrT is discovered. The eigen-functions of arbitrary GFrT are derived explicitly also.

References

[1] E. U. Condon, "Immersion of the Fourier transform in a continuous group of functional transformations", Proc. Nat. Acad. Sci. USA 23, (1937) 158–164
[2] V. Namias, J. Inst. Math. Its Appl. 25 (1961) 241
[3] D. Mendlovic and H. M. Ozaktas, Fractional Fourier transforms and their optical implementation, I, J. Opt. Soc. Am. A 10 (1993) 1875-1881
[4] D. Mendlovic, H. M. Ozaktas and A. W. Lohmann, Graded-index fiber, Wigner-distribution functions, and the fractional Fourier transform, Appl. Opt. 33 (1994) 6188-6193
[5] A. C. McBride and F. H. Kerr, IMA J. Appl. Math. 39 (1987) 159
[6] A. W. Lohmann, Image rotation, Wigner rotation and fractional Fourier transform, J. Opt. Soc. Am. A 10 (1993) 2181-2186
[7] L. Bernardo and O. D. D. Soares, Fractional Fourier transform and optical systems, Opt. Commun. 110 (1994) 517-522

[8] P. A. M. Dirac, The Principle of Quantum Mechanics, 3rd Claredon Press, Oxford 1958

[9] Hong-yi Fan, Hai-liang Lu and Yue Fan, Ann. Phys. 321 (2006) 480

[10] Fan Hong-yi and Guo Qin, Commun. Theor. Phys. 49 (2008) 859

[11] Sumiyoshi Abet and John T Sheridant, J. Phys. A: Math. Gen. 27 (1994) 4179-4187