Nonlocal Flow of Convex Plane Curves and Isoperimetric Inequalities

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Abstract

In the first part of the paper we survey some nonlocal flows of convex plane curves ever studied so far and discuss properties of the flows related to enclosed area and length, especially the isoperimetric ratio and the isoperimetric difference. We also study a new nonlocal flow of convex plane curves and discuss its evolution behavior.

In the second part of the paper we discuss necessary and sufficient conditions (in terms of the (mixed) isoperimetric ratio or (mixed) isoperimetric difference) for two convex closed curves to be homothetic or parallel.

1 Introduction

Recently there has been some interest in the nonlocal flow of convex closed plane curves. See the papers by Gage [GA2], Jiang-Pan [JP], Pan-Yang [PY], Ma-Cheng [MC] and Ma-Zhu [MZ]. All of the above papers deal with the evolution of a given convex embedded closed plane curve \( \gamma_0 \). The general form of the equation is given by

\[
\begin{cases}
\frac{\partial X}{\partial t}(\varphi, t) = [F(k(\varphi, t)) - \lambda(t)]N_{in}(\varphi, t) \\
X(\varphi, 0) = X_0(\varphi), \quad \varphi \in S^1,
\end{cases}
\]

which is a parabolic initial value problem. Here \( X_0(\varphi) : S^1 \rightarrow \gamma_0 \) is a smooth parametrization of \( \gamma_0 \); \( k(\varphi, t) \) is the curvature of the evolving curve \( \gamma_t = \gamma(\cdot, t) \) (parametrized by \( X(\varphi, t) \)) at the point \( \varphi \); and \( N_{in}(\varphi, t) \) is the inward normal of \( \gamma_t \). As for the speed, \( F(k) \) is a given function of curvature satisfying the parabolic condition \( F'(z) > 0 \) for all \( z \) in its domain and \( \lambda(t) \) is a function of time, which may depend on certain global quantities of \( \gamma_t \), say length \( L(t) \), enclosed area \( A(t) \), or others (see [A] and [B]). Note that if \( \lambda(t) \) depends on \( \gamma_t \), then it is not known beforehand.

The results claimed in each of the above mentioned papers are more or less the same: the flows preserve the convexity of a given initial curve \( \gamma_0 \) and evolve it to a round circle (or round point) in \( C^\infty \) sense as \( t \to \infty \).

For \( k \)-type flows, the following three flows:

\[
F(k) - \lambda(t) = k - \frac{2\pi}{L(t)} \quad \text{(area-preserving, gradient flow of } L^2 - 4\pi A) \tag{2}
\]

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\[\dagger\text{In this paper, } "\text{convex}" \text{ always means } "\text{strictly convex}". A convex closed plane curve has positive curvature everywhere. Also, for simplicity, all curves considered in this paper are smooth with positive orientation.\]
and
\[ F(k) - \lambda(t) = k - \frac{L(t)}{2A(t)} \] (gradient flow of \( L^2/4\pi A \)) \hspace{1cm} (3)

and
\[ F(k) - \lambda(t) = k - \frac{1}{2\pi} \int_0^{L(t)} k^2 ds \] (length-preserving) \hspace{1cm} (4)

are studied by Gage [GA2], Jiang-Pan [JP], and Ma-Zhu [MZ] respectively.

For 1/k-type flows, the following two flows:
\[ F(k) - \lambda(t) = k - \frac{1}{L(t)} \int_0^{L(t)} k ds \] (area-preserving) \hspace{1cm} (5)

and
\[ F(k) - \lambda(t) = \frac{L(t)}{2\pi} - \frac{1}{k} \] (length-preserving) \hspace{1cm} (6)

are studied by Ma-Cheng [MC] and Pan-Yang [PY] respectively.

We know that (see [GA2]) for a family of time-dependent simple closed curves \( X(\varphi, t) : S^1 \times [0, T] \rightarrow \mathbb{R}^2 \) with time variation
\[ \frac{\partial X}{\partial t}(\varphi, t) = W(\varphi, t) \in \mathbb{R}^2, \] \hspace{1cm} (7)

its length \( L(t) \) and enclosed area \( A(t) \) satisfy the following:
\[ \frac{dL}{dt}(t) = -\int_{\gamma_t} \langle W, kN_m \rangle ds, \quad \frac{dA}{dt}(t) = -\int_{\gamma_t} \langle W, N_m \rangle ds. \] \hspace{1cm} (8)

This says that the well-known curve shortening flow (with \( F(k) = k, \lambda = 0 \) in (1)) is the gradient flow of the length functional. See Gage-Hamilton [GH]. Also the unit-speed inward normal flow (with \( F(k) = 0, \lambda = -1 \) in (1)) is the gradient flow of the area functional.

In particular, for convex plane curves evolution, we can check that flows (2), (5) are area-preserving and flows (4), (6) are length-preserving.

As for flow (3), it is length-decreasing due to Gage’s inequality for convex closed curves (see [GA3]):
\[ \frac{dL}{dt}(t) = -\int_{\gamma_t} k^2 ds + \frac{\pi L(t)}{A(t)} \leq 0. \] \hspace{1cm} (9)

It is also area-increasing due to (note that \( \int_{\gamma_t} k ds = 2\pi \)):
\[ \frac{dA}{dt}(t) = -2\pi + \frac{L^2(t)}{2A(t)} \geq 0. \]

Thus this flow is the most efficient in evolving a convex curve to a round circle. As we shall see in Lemma 6 below, Jiang-Pan’s flow (3) is the gradient flow of the isoperimetric ratio functional.

As a comparison, for \( k \)-type flows with speed of the form \([k - p(t)]N_m\), we have
\[ \frac{dL}{dt}(t) = -\int_{\gamma_t} k^2 ds + 2\pi p(t), \quad \frac{dA}{dt}(t) = -2\pi + p(t)L \] \hspace{1cm} (10)

and for \( 1/k \)-type flows with speed of the form \([q(t) - 1/k]N_m\), we have
\[ \frac{dL}{dt}(t) = -2\pi q(t) + L, \quad \frac{dA}{dt}(t) = -q(t)L + \int_{\gamma_t} \frac{1}{k} ds. \] \hspace{1cm} (11)
It is interesting to observe that when \( q(t) = 1/p(t) \), there is a "dual relation" between (10) and (11), i.e.,

\[
\frac{1}{q(t)} \frac{dL}{dt}(t) \quad \text{(for 1/k-type flows)} = \frac{dA}{dt}(t) \quad \text{(for k-type flows).} \tag{12}
\]

Hence in the above, flows (2) and (6) are dual. We shall consider the dual flow of (3) in Section 3.

In the first part of the paper, we observe some interesting behavior of a general nonlocal flow (1), especially the properties related to the isoperimetric differences \( L^2 - 4\pi A \) and isoperimetric ratio \( L^2/4\pi A \). We also discuss certain difficulty in dealing with the flow (1) and (5), especially the possibility of curvature blowing up in finite time.

In the second part, we discuss certain necessary and sufficient conditions (in terms of the mixed isoperimetric ratio \( L_1L_2/4\pi A_{12} \) and mixed isoperimetric difference \( L_1L_2 - 4\pi A_{12} \)) for two convex closed plane curves \( \gamma_1, \gamma_2 \) to be homothetic or parallel. Here \( A_{12} \) is the mixed area determined by \( A_1, A_2 \).

For simplicity, throughout the rest of the paper, we shall use the following two abbreviations:

\[
\text{IPR} = \text{isoperimetric ratio}, \quad \text{IPD} = \text{isoperimetric difference}. \tag{13}
\]

### 2 The decreasing of the IPD

In this section we first prove an interesting property of the flow (1). It says that the IPD \( L^2 - 4\pi A \) is always non-increasing. To explain this, we need the following nice inequality due to Andrews (see p. 341 of [A]). One can view it as a generalization of the classical Hölder inequality.

**Lemma 1** (Andrews’s inequality) Let \( M \) be a compact Riemannian manifold with a volume form \( d\mu \), and let \( \xi \) be a continuous function on \( M \). Then for any increasing continuous function \( F : \mathbb{R} \to \mathbb{R} \), we have

\[
\int_M \xi d\mu \int_M F(\xi) d\mu \leq \int_M d\mu \int_M \xi F(\xi) d\mu. \tag{14}
\]

If \( F \) is strictly increasing, then equality holds if and only if \( \xi \) is a constant function on \( M \). Similarly, if \( F : \mathbb{R} \to \mathbb{R} \) is a decreasing function, then we replace \( \geq \) by \( \leq \) in (14).

**Remark 2** The sign of \( F \) plays no role in (14). In the case when \( M = S^1 \), it is easy to obtain (14) by Fubini theorem:

\[
\int_0^{2\pi} d\theta \int_0^{2\pi} \xi(\theta) F(\xi(\theta)) d\theta = \int_0^{2\pi} \xi(\theta) d\theta \int_0^{2\pi} F(\xi(\theta)) d\theta
\]

\[
= \frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} (F(\xi(x)) - F(\xi(y))) |\xi(x) - \xi(y)| dx dy \geq 0.
\]

With the help of Andrews’s inequality, we have (see p. 341 of [A] also):

**Lemma 3** (monotonicity of the IPD) Under the general parabolic flow (it could be contracting, expanding or a mixture of both)

\[
\frac{\partial X}{\partial t}(\varphi, t) = [F(k(\varphi, t)) - \lambda(t)] N_{in}(\varphi, t), \tag{15}
\]

where \( \lambda(t) \) is a time function which may depend on the global geometry \( \gamma_t \) of the flow. If the flow is well-defined on \([0, T)\) and the evolving curves \( \gamma_t \) stays embedded on \([0, T)\), then the IPD \( L^2(t) - 4\pi A(t) \) for \( \gamma_t \) is decreasing on \([0, T)\).
Proof. By (8), we have
\[
\frac{d}{dt} (L^2 - 4\pi A) = 2L \left[ -\int_{\gamma_t} F (k) k ds + 2\pi \lambda (t) \right] - 4\pi \left[ -\int_{\gamma_t} F (k) ds + \lambda (t) L \right] = 2 \left[ \int_{\gamma_t} k ds \int_{\gamma_t} F (k) ds - \int_{\gamma_t} ds \int_{\gamma_t} F (k) k ds \right] \leq 0.
\]
(16)

In particular, we note that the function \( \lambda (t) \) has been cancelled. The proof is done. □

As a consequence, we obtain the following:

**Corollary 4** Under the assumption of Lemma 3, if flow (15) is area-preserving, then it must be length-decreasing. On the other hand, if it is length-preserving, then it must be area-increasing. In particular, if it preserves either area or length, then the IPR \( L^2/4\pi A \) is decreasing.

**Remark 5** We see that flows (2)-(6) are all IPR decreasing.

What happens to the IPR? We can compute
\[
\frac{d}{dt} \left( \frac{L^2}{4\pi A} \right) = I + II,
\]
where
\[
I = \frac{2}{(4\pi A)^2} \cdot \Phi, \\
\Phi = \frac{L^2}{4\pi} \left( \int_{\gamma_t} k ds \right) \left( \int_{\gamma_t} F (k) ds \right) - 4\pi A \left( \int_{\gamma_t} ds \right) \left( \int_{\gamma_t} F (k) k ds \right)
\]
and
\[
II = -\frac{4\pi \lambda (t) L}{(4\pi A)^2} (L^2 - 4\pi A) = \begin{cases} 
\leq 0, & \text{if } \lambda (t) \geq 0 \\
\geq 0, & \text{if } \lambda (t) \leq 0.
\end{cases}
\]
(17)

In (17), there is a competition between Andrews's inequality and the isoperimetric inequality \( L^2 \geq 4\pi A \). Hence it has no definite sign in general.

By (18) we also notice that, roughly speaking, the flow is better-behaved if it tends to expand more (\( \lambda (t) \geq 0 \)), and worse-behaved if it tends to contract more (\( \lambda (t) \leq 0 \)).

To explain the gradient flow of the IPR and IPD, by (7) and (8), we have
\[
\frac{d}{dt} (L^2 - 4\pi A) (t) = -2L \int_{\gamma_t} \left\langle W, \left( k - \frac{2\pi}{L} \right) N_{in} \right\rangle ds
\]
(19)

and
\[
\frac{d}{dt} \left( \frac{L^2}{4\pi A} \right) (t) = -\frac{L}{2\pi A} \int_{\gamma_t} \left\langle W, \left( k - \frac{L}{2A} \right) N_{in} \right\rangle ds,
\]
(20)

where \( W (\varphi, t) = (\partial \gamma/\partial t) (\varphi, t) \) is the speed vector of the flow. By (19), the nonlocal flow
\[
\frac{\partial X}{\partial t} (\varphi, t) = 2L (t) \left[ k (\varphi, t) - \frac{2\pi}{L (t)} \right] N_{in} (\varphi, t)
\]
(21)
is the gradient flow of the IPD functional. It only differs from Gage’s flow (2) by a time factor $2L(t)$. If $X(\varphi,t)$ is a solution to Gage’s flow (2), the function

$$\tilde{X}(\varphi,\tau) = X(\varphi,t(\tau))$$

(22)

will then be a solution to the flow (21) if we choose $t(\tau)$ to satisfy the identity

$$\frac{dt}{d\tau} = 2L(t), \quad L(t) = \text{length of } X(\varphi,t).$$

(23)

To see this, by the chain rule

$$\frac{\partial \tilde{X}}{\partial \tau}(\varphi,\tau) = \frac{dt}{d\tau} \frac{\partial X}{\partial t}(\varphi,t) = 2L \left( k - \frac{2\pi}{L} \right) N_{in}$$

and the relation

$$\tilde{L}(\tau) = L(t), \quad \tilde{k}(\varphi,\tau) = k(\varphi,t), \quad \tilde{N}_{in}(\varphi,\tau) = N_{in}(\varphi,t), \quad t = t(\tau)$$

we see that

$$\frac{\partial \tilde{X}}{\partial \tau}(\varphi,\tau) = 2\tilde{L}(\tau) \left[ \tilde{k}(\varphi,\tau) - \frac{2\pi}{\tilde{L}(\tau)} \right] \tilde{N}_{in}(\varphi,\tau).$$

(24)

Thus we can say that the two flows (2) and (21) are equivalent.

Similarly, by (20), the nonlocal flow

$$\frac{\partial X}{\partial t}(\varphi,t) = \frac{L(t)}{2\pi A(t)} \left[ k(\varphi,t) - \frac{L(t)}{2A(t)} \right] N_{in}(\varphi,t)$$

(25)

is the gradient flow of the IPR functional. Again, it only differs from Jiang-Pan’s flow (3) by a time factor $L(t)/2\pi A(t)$. Thus the two flows (3) and (25) are equivalent.

We summarize the following:

**Lemma 6** (gradient flow of the IPR and IPD) Gage’s nonlocal flow (2) is the gradient flow of the IPD functional, and Jiang-Pan’s nonlocal flow (3) is the gradient flow of the IPR functional. Both flows decrease the IPR and IPD.

Regarding the isoperimetric behavior of a flow, another important observation is the following: if we have $W = N_{in}$ in (19) and (20), then

$$\frac{d}{dt} \left( L^2 - 4\pi A \right)(t) = 0, \quad \frac{d}{dt} \left( \frac{L^2}{4\pi A} \right)(t) = \frac{L}{4\pi A^2} (L^2 - 4\pi A) \geq 0$$

(26)

and if we have $W = u N_{in}$, where $u = \langle \gamma(\cdot,t) \rangle$, $N_{out}$ is the support function of $\gamma(\cdot,t)$, then we have

$$\frac{d}{dt} \left( L^2 - 4\pi A \right)(t) = -2 \left( L^2 - 4\pi A \right) \leq 0, \quad \frac{d}{dt} \left( \frac{L^2}{4\pi A} \right)(t) = 0$$

(27)

due to the identities (see (44) also)

$$\int_{\gamma_t} u k ds = L, \quad \frac{1}{2} \int_{\gamma_t} u ds = A.$$

(28)

Hence if one replace the flow

$$\frac{\partial X}{\partial t}(\varphi,t) = F(k(\varphi,t)) N_{in}(\varphi,t)$$

(29)
\[
\frac{\partial X}{\partial t}(\varphi, t) = [F(k(\varphi, t)) - \lambda(t)]N_{in}(\varphi, t) \tag{30}
\]
then the IPD \(L^2 - 4\pi A\) is unaffected. Similarly if one replaces (29) by
\[
\frac{\partial X}{\partial t}(\varphi, t) = [F(k(\varphi, t)) - \lambda(t)u(\varphi, t)]N_{in}(\varphi, t), \tag{31}
\]
where \(u(\varphi, t)\) is the support function of \(\gamma_t\) at the point \(\varphi\), then the IPR \(L^2/4\pi A\) is unaffected.

Now we explain why the IPR \(L^2/4\pi A\) is unchanged by the extra term \(-\lambda uN_{in}\). If \(X(\varphi, t)\) is a solution to (29) and let \(\tilde{X}(\varphi, t) = \sigma(t)X(\varphi, t)\), \(\sigma(t) > 0\), which is a time-dependent dilation of \(X(\varphi, t)\), then the dilated \(\tilde{X}(\varphi, t)\) satisfies
\[
\frac{\partial \tilde{X}}{\partial t}(\varphi, t) = \begin{cases} 
\sigma(t)F\left(\sigma(t)\tilde{k}(\varphi, t)\right)N_{in}(\varphi, t) \\
+ \frac{\sigma'(t)}{\sigma(t)}\left<\tilde{X}(\varphi, t), \tilde{N}_{out}(\varphi, t)\right>N_{out}(\varphi, t) + \frac{\sigma'(t)}{\sigma(t)}\left<\tilde{X}(\varphi, t), \tilde{T}(\varphi, t)\right>T(\varphi, t)\nonumber \end{cases} \tag{32}
\]
Thus, up to a tangential component (it is known that a tangential component can be removed by a further change of variable \(\phi = \phi(\varphi, t)\) in parametrizing the dilated curve \(\tilde{\gamma}\)), we obtain
\[
\frac{\partial \tilde{X}}{\partial t}(\varphi, t) = \left[\sigma F\left(\sigma k\right) - \frac{\sigma'}{\sigma}u\right]N_{in}, \tag{33}
\]
where the support function \(\tilde{u}\) appears naturally. Since a dilation will not change the IPR, the extra term \(-\sigma^{-1}\sigma'\tilde{u}\) must have no effect at all.

In view of the above, one can keep dilating a solution \(\gamma_t\) to the flow (29) so that its length or area is independent of time. The flow equation for the dilated solution will have an extra term involving the support function. There are two of them for the \(k\)-type flow with \(F(k) = k\), and two of them for the \(1/k\)-type flow with \(F(k) = -1/k\). In conclusion, we have the following four flows:
\[
\frac{\partial X}{\partial t}(\varphi, t) = \left(k - \frac{\pi}{A}u\right)N_{in} \quad \text{(area-preserving)} \tag{34}
\]
and
\[
\frac{\partial X}{\partial t}(\varphi, t) = \left[k - \left(\frac{1}{L}\int_0^L k^2 ds\right)u\right]N_{in} \quad \text{(length-preserving)} \tag{35}
\]
and
\[
\frac{\partial X}{\partial t}(\varphi, t) = \left[\left(\frac{1}{2A}\int_0^L \frac{1}{k} ds\right)u - \frac{1}{k}\right]N_{in} \quad \text{(area-preserving)} \tag{36}
\]
and
\[
\frac{\partial X}{\partial t}(\varphi, t) = \left(u - \frac{1}{k}\right)N_{in} \quad \text{(length-preserving)}, \tag{37}
\]
where \(u(\varphi, t)\) is the support function of the curve \(\gamma_t\). For example, for \(F(k) = k\), (33) becomes
\[
\frac{\partial X}{\partial t}(\varphi, t) = \sigma^2\left(k - \frac{\sigma'}{\sigma^3}u\right)N_{in}
\]
and a change in time variable can get rid of the coefficient \(\sigma^2\). Hence we may assume
\[
\frac{\partial X}{\partial t}(\varphi, t) = \left(k - \frac{\sigma'}{\sigma^3}u\right)N_{in}. \tag{38}
\]
If we want to keep the enclosed area fixed, by (8) we need to require
\[
\frac{dA}{dt}(t) = - \int_{\gamma_t} \left( k - \frac{\sigma'}{\sigma^3} u \right) ds = 0,
\]
which implies $\sigma'(t) / \sigma^3(t) = \pi / A(t)$ and the flow (38) becomes
\[
\frac{\partial X}{\partial t}(\phi, t) = \left( k - \frac{\pi}{A} u \right) N_m. \tag{39}
\]
The same argument can be applied to the other three flows.

We find that if we replace $u$ by $2A/L$ in the area-preserving flows (34) and (36), we get Gage’s flow (2) and Ma-Cheng’s flow (5). Also if we replace $u$ by $L/2\pi$ in the length-preserving flows (35) and (37), we get Ma-Zhu’s flow (4) and Pan-Yang’s flow (6). This is due to formula (44) below.

### 3 The dual flow of (3) and an improved isoperimetric inequality

We already know that flows (2) and (6) are dual to each other. Motivated by it, we can also consider the dual flow of (3), which is probably the only remaining interesting case not dealt with among those nonlocal flows (2)-(6). It has the form
\[
\begin{align*}
\frac{\partial X}{\partial t}(\phi, t) &= \left[ \frac{2A(t)}{L(t)} - \frac{1}{k(\phi, t)} \right] N_m(\phi, t), \\
X(\phi, 0) &= X_0(\phi), \quad \phi \in S^1,
\end{align*} \tag{40}
\]
where $X_0(\phi)$ is the parametrization of a given smooth convex closed curve $\gamma_0$.

In $k$-type flows, the Gage’s inequality for a convex closed curve $\gamma$:
\[
\int_{\gamma} k^2(s) ds = \int_{0}^{2\pi} k(\theta) d\theta \geq \frac{\pi L}{A}, \tag{41}
\]
where $s$ is arc length parameter, plays an important role. In contrast, in the $1/k$-type flows, we need to use the Pan-Yang’s isoperimetric inequality for a convex closed curve $\gamma$:
\[
\int_{\gamma} \frac{1}{k(s)} ds \geq \frac{L^2 - 2\pi A}{\pi}, \tag{42}
\]
where the equality holds if and only if $\gamma$ is a circle.

(42) is proved in [PY] using methods established in Green-Osher [GO]. This inequality seems to be new as it had not appeared in any book or reference before. Here we can use Fourier series expansion to give an alternative proof and, at the same time, improves it also.

In the book by Courant-John [CJ], they used support function and Fourier series method to prove the isoperimetric inequality $L^2 \geq 4\pi A$ for a closed plane curve (see p. 366 of [CJ]). Our method is motivated by theirs. Let $C$ be a convex closed plane curve. One can use its outward normal angle $\theta \in [0, 2\pi]$ to parametrize it. In doing so, the inequality (42) becomes
\[
\int_{0}^{2\pi} \frac{1}{k^2(\theta)} d\theta \geq \frac{L^2 - 2\pi A}{\pi}. \tag{43}
\]
It is also known that one can use the support function $u(\theta), \theta \in [0, 2\pi]$, of $C$ to express its curvature, enclosed area and length (see the book by Schneider [S]). We have
\[
\begin{align*}
\frac{1}{k(\theta)} &= u_{\theta\theta}(\theta) + u(\theta), \quad L = \int_{0}^{2\pi} u(\theta) d\theta \\
A &= \frac{1}{2} \int_{C} u ds = \frac{1}{2} \int_{0}^{2\pi} u(\theta) [u_{\theta\theta}(\theta) + u(\theta)] d\theta. \tag{44}
\end{align*}
\]
We can state our result in the following:

**Lemma 7** (refined Pan-Yang’s isoperimetric inequality) For any convex closed plane curve \( C \) there holds the inequality

\[
\int_C \frac{1}{k(s)} ds = \int_0^{2\pi} \frac{1}{k^2(\theta)} d\theta \geq \frac{2}{\pi} \left( L^2 - 4\pi A \right) + 2A, \tag{45}
\]

where the equality holds if and only if the support function of \( C \) has the form

\[
u(\theta) = a_0 + a_1 \cos \theta + b_1 \sin \theta + a_2 \cos 2\theta + b_2 \sin 2\theta, \quad \theta \in [0, 2\pi]
\tag{46}
for some constants \( a_0, a_1, b_1, a_2, b_2 \) satisfying

\[
u_{\theta\theta}(\theta) + \nu(\theta) = a_0 - 3a_2 \cos 2\theta - 3b_2 \sin 2\theta > 0 \quad \text{for all} \quad \theta \in [0, 2\pi]. \tag{47}

Here the variable \( \theta \) is the outward normal angle of \( C \).

**Remark 8** Since

\[
\frac{2}{\pi} (L^2 - 4\pi A) + 2A = \frac{L^2 - 4\pi A}{\pi} + \frac{L^2 - 2\pi A}{\pi} \geq \frac{L^2 - 2\pi A}{\pi},
\tag{45}
\]

(45) is an improvement of (42).

**Proof.** Using Fourier series, one can express the \( 2\pi \)-periodic smooth support function \( \nu(\theta) \) of \( C \) as

\[
u(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta). \tag{48}
\]

Then

\[
L^2 = \left( \int_0^{2\pi} \nu d\theta \right)^2 = (\pi a_0)^2 \tag{49}
\]
and

\[
4\pi A = (\pi a_0)^2 + 2\pi^2 \left[ \sum_{n=1}^{\infty} (1 - n^2) \left( a_n^2 + b_n^2 \right) \right] = L^2 + 2\pi^2 \left[ \sum_{n=2}^{\infty} (1 - n^2) \left( a_n^2 + b_n^2 \right) \right], \tag{50}
\]
which gives the classical isoperimetric inequality \( L^2 \geq 4\pi A \). Also we have

\[
\int_0^{2\pi} \nu_{\theta\theta}(\nu_{\theta\theta} + \nu) d\theta = \sum_{n=2}^{\infty} n^2 (n^2 - 1) \pi \left( a_n^2 + b_n^2 \right)
\]
and therefore

\[
\int_0^{2\pi} \frac{1}{k^2(\theta)} d\theta = \int_0^{2\pi} \nu_{\theta\theta}(\nu_{\theta\theta} + \nu) d\theta + \int_0^{2\pi} \nu (\nu_{\theta\theta} + \nu) d\theta = \sum_{n=2}^{\infty} n^2 (n^2 - 1) \pi \left( a_n^2 + b_n^2 \right) + 2A.
\]
To prove (45), it suffices to show that

\[
\sum_{n=2}^{\infty} n^2 (n^2 - 1) \pi \left( a_n^2 + b_n^2 \right) \geq \frac{2}{\pi} \left( L^2 - 4\pi A \right),
\]
where by (50) the right hand side is
\[ \frac{2}{\pi} (L^2 - 4\pi A) = 4\pi \left[ \sum_{n=2}^{\infty} (n^2 - 1) \left( a_n^2 + b_n^2 \right) \right]. \]

We clearly have
\[ \sum_{n=2}^{\infty} n^2 (n^2 - 1) \pi (a_n^2 + b_n^2) \geq 4\pi \left[ \sum_{n=2}^{\infty} (n^2 - 1) \left( a_n^2 + b_n^2 \right) \right] \]
and the equality holds if and only if \( a_n = b_n = 0 \) for all \( n \geq 3 \). That is, if and only if \( u(\theta) \) has the form
\[ u(\theta) = c + a_1 \cos \theta + b_1 \sin \theta + a_2 \cos 2\theta + b_2 \sin 2\theta, \quad \theta \in [0, 2\pi] \]
for some constants \( c, a_1, b_1, a_2, b_2 \). The proof is done. \( \square \)

**Remark 9** It is very unlikely to use Fourier series expansion method to prove Gage’s inequality (41) because the integrand \( k(\theta) = [u_{\theta\theta}(\theta) + u(\theta)]^{-1} \) is not of the right form.

Regarding flows (2)-(6) and (40), I think one can use the result in section 2 of Gage-Hamilton [GH] (Nash-Moser inverse function theorem) to prove that for any initial curve, the nonlocal flow considered has a solution for short time. However, since the Nash-Moser inverse function theorem is itself hard to understand, one would prefer to invoke a more straightforward or elementary theory. As the dual flow (40) has more of a linear structure (for the support function or for the inverse of curvature), it is possible to prove short time existence directly using Fourier series.

In Pan-Yang’s length-preserving flow (6), they pointed out that if the flow has a smooth convex solution on short time interval \([0, T)\), then the function \((\text{in below the variable } \theta \text{ represents outward normal angle of the convex curve } \gamma_t)\)
\[ w(\theta, t) := e^{-t} \left( \frac{1}{k(\theta, t)} - \frac{L(t)}{2\pi} \right), \quad L(t) = L(0), \quad (\theta, t) \in S^1 \times [0, T) \]
will satisfy a standard linear heat equation \( w_t(\theta, t) = w_{\theta\theta}(\theta, t) \) on \( S^1 \) with initial condition \( w_0(\theta) = 1/k_0(\theta) - L(0)/2\pi \), \( k_0(\theta) > 0 \). Conversely one can use the linear heat equation to establish short time existence of a solution to the nonlocal flow (3) because if one knows \( w(\theta, t) \) then one can know the curvature \( k(\theta, t) \) (here we need to use the fact \( L(t) = L(0) \) is preserved), and then we use curvature to construct a solution to the nonlocal flow (6).

For the dual flow (40), the situation is different. If it has a smooth convex solution on \([0, T)\) for short time \( T > 0 \), then the curvature and length satisfy the following evolution equations:
\[ \frac{\partial}{\partial t} \left( \frac{1}{k(\theta, t)} - \frac{L(t)}{2\pi} \right) = \left( \frac{1}{k(\theta, t)} \right)_{\theta\theta} + \frac{1}{k(\theta, t)} - \frac{2A(t)}{L(t)} \]
and
\[ \frac{d}{dt} \left( \frac{L(t)}{2\pi} \right) = -\frac{1}{2\pi} \int_{\gamma_t} \left( \frac{2A(t)}{L(t)} - \frac{1}{k} \right) kds = \frac{L(t)}{2\pi} - \frac{2A(t)}{L(t)} \geq 0. \]
Hence we have the nice-looking equation
\[ \frac{\partial}{\partial t} \left( \frac{1}{k(\theta, t)} - \frac{L(t)}{2\pi} \right) = \left( \frac{1}{k(\theta, t)} - \frac{L(t)}{2\pi} \right)_{\theta\theta} + \left( \frac{1}{k(\theta, t)} - \frac{L(t)}{2\pi} \right) \]
and the function
\[ w(\theta, t) := e^{-t} \left( \frac{1}{k(\theta, t)} - \frac{L(t)}{2\pi} \right), \quad \frac{L(t)}{2\pi} = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{1}{k(\theta, t)} d\theta \]
satisfies a linear heat equation

\[ w_t (\theta, t) = w_{\theta\theta} (\theta, t), \quad w (\theta, 0) = \frac{1}{k_0 (\theta)} - \frac{L (0)}{2\pi}. \]  

(57)

Unfortunately, if we go in the reverse direction and solve \( w (\theta, t) \) from the heat equation (57), we are not able to recover the curvature \( k (\theta, t) \) since from an identity of the form

\[ \frac{1}{k (\theta, t)} - \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{k (\theta, t)} d\theta = e^t w (\theta, t) \]  

(58)

we cannot determine \( k (\theta, t) \) uniquely (note that if \( 1/k (\theta, t) \) satisfies (58), so is \( 1/k (\theta, t) + g (t) \) for any function of time \( g (t) \)). In addition, even we find one \( k (\theta, t) \) satisfies (58), we do not know if it satisfies the evolution equation (53). The same difficulty also happens in Ma-Cheng’s flow (5). See Theorem 16 of [MC].

**Remark 10** Note that in Pan-Yang’s flow (6) we have \( L (t) = L (0) \) for all \( t \). Hence if \( w (\theta, t) \) is known from (57), one can determine \( k (\theta, t) \) and its evolution equation, and the short time existence of a solution to the flow is established.

To overcome the above difficulty, we can use Fourier series method again. The evolution of the support function \( u (\theta, t) \) under flow (10) on \([0, T)\) is given by

\[ \frac{\partial u}{\partial t} (\theta, t) = u_{\theta\theta} (\theta, t) + u (\theta, t) - \int_0^{2\pi} u (\theta, t) (u_{\theta\theta} (\theta, t) + u (\theta, t)) d\theta, \quad (\theta, t) \in S^1 \times [0, T) \]  

(59)

where \( u (\theta, 0) = u_0 (\theta) \), \( u''_0 (\theta) + u_0 (\theta) > 0 \), and \( u_0 (\theta) \) is the support function of the initial convex curve \( \gamma_0 \). One can use relation (14) to derive the equation (59). We want to use (59), instead of equation (57), to prove the existence of a nonlocal flow solution (the fact is that if we have a support function solution to (59), one can use it to construct a flow solution to (10)).

Expand \( u_0 (\theta) \) and \( u (\theta, t) \) as

\[
\begin{align*}
  u_0 (\theta) &= \frac{a_0 (0)}{2} + \sum_{n=1}^{\infty} (a_n (0) \cos n\theta + b_n (0) \sin n\theta) \\
  u (\theta, t) &= \frac{a_0 (t)}{2} + \sum_{n=1}^{\infty} (a_n (t) \cos n\theta + b_n (t) \sin n\theta), \quad u (\theta, 0) = u_0 (\theta)
\end{align*}
\]

and compute

\[
\begin{align*}
  \frac{\partial}{\partial t} u (\theta, t) &= \frac{a_0' (t)}{2} + \sum_{n=1}^{\infty} (a_n' (t) \cos n\theta + b_n' (t) \sin n\theta) \\
  u_{\theta\theta} (\theta, t) + u (\theta, t) &= \frac{a_0 (t)}{2} + \sum_{n=1}^{\infty} (1 - n^2) (a_n (t) \cos n\theta + b_n (t) \sin n\theta)
\end{align*}
\]

(60)

and

\[
\frac{\int_0^{2\pi} u (\theta, t) (u_{\theta\theta} (\theta, t) + u (\theta, t)) d\theta}{\int_0^{2\pi} u (\theta, t) d\theta} = \frac{\pi a_0^2 (t) + \pi \sum_{n=1}^{\infty} (1 - n^2) (a_n^2 (t) + b_n^2 (t))}{\pi a_0 (t)}.
\]

Hence, by comparing the coefficients, we want \( a_0 (t) \), \( a_n (t) \), \( b_n (t) \) to satisfy

\[
\begin{align*}
  a_0' (t) &= \frac{\sum_{n=1}^{\infty} (n^2 - 1) (a_n^2 (t) + b_n^2 (t))}{a_0 (t)} \geq 0, \quad a_0 (0) > 0 \\
  a_n' (t) &= (1 - n^2) a_n (t), \quad b_n' (t) = (1 - n^2) b_n (t).
\end{align*}
\]

(61)
This ODE system can be easily solved for all $t \in [0, \infty)$ to get

$$
\begin{aligned}
\left\{
\begin{array}{l}
a_n (t) = a_n (0) e^{(1-n^2)t}, \\
b_n (t) = b_n (0) e^{(1-n^2)t}, \\
a_0 (t) = \sqrt{a_0^2 (0) + 2 \sum_{n=1}^{\infty} (1 - e^{2(1-n^2)t}) (a_n^2 (0) + b_n^2 (0))}.
\end{array}
\right.
\end{aligned}
$$

(62)

Finally we note that as $a_0 (t), a_n (t), b_n (t)$ are all exponentially decay, the above computations can all be justified by Fourier series theory.

In view of the above, we can conclude the following:

**Lemma 11** (short time existence of the dual flow \([40]\)) There is a smooth convex solution $X (\varphi, t)$ to the nonlocal flow \([40]\) for short time interval $[0, T), T > 0$.

A major difficulty in studying the $1/k$-type flows \([3], [5], [40]\) is the possibility of developing a singularity in finite time with curvature $k = \infty$ somewhere. Although this seems quite unlikely to happen, we are not able to rule it out mathematically. Note that the flow \([40]\) is equivalent to the support function equation \((59)\) only under the condition

$$
0 < u_{\theta\theta} (\theta, t) + u (\theta, t) < \infty
$$

(63)

since the curvature is given by $k = 1 / (u_{\theta\theta} + u)$. From equation \((59)\) we see that $u_{\theta\theta} (\theta, t) + u (\theta, t)$ will not blow up in finite time, but it may be possible that $u_{\theta\theta} (\theta_0, t_0) + u (\theta_0, t_0) = 0$ at some finite time $t_0$ for some $\theta_0 \in S^1$ (note that the initial condition satisfies $u''_0 (\theta) + u_0 (\theta) > 0$ everywhere). Even we have an explicit Fourier series expansion for $u_{\theta\theta} (\theta, t) + u (\theta, t)$, we do not know how to exclude the possibility.

**Remark 12** In both \([PY]\) and \([MC]\), although they claim that the flow they studied will converge to a round circle as $t \to \infty$, the possibility of $k$ becoming infinity in finite time is not discussed. However, this should not diminish their contributions to the study of nonlocal flows. In both papers, they derived the estimate (see \([PY]\), p. 481 and \([MC]\), p. 8)

$$
\left| \frac{1}{k (\theta, t)} - L (t) \right| \leq M e^{T^*} \quad \text{for any } T^* > 0.
$$

(64)

This can exclude a finite time extinction ($k = 0$) of the curvature, but it cannot exclude a finite time blow-up ($k = \infty$) of the curvature.

For the dual flow \([40]\), if we only look at equation \((59)\), we can easily obtain the following convergence. As mentioned above, it can give information of the flow only when \((63)\) is satisfied.

**Theorem 13** Let $u_0 (\theta)$ be a smooth function on $S^1$ satisfying $u''_0 (\theta) + u_0 (\theta) > 0$ everywhere. Then the solution to the equation \((59)\) with initial condition $u (\theta, 0) = u_0 (\theta), \theta \in S^1$, is defined on $S^1 \times [0, \infty)$ with

$$
\lim_{t \to \infty} \|u (\theta, t) - (c + a_1 (0) \cos \theta + b_1 (0) \sin \theta)\|_{C^k (S^1)} = 0 \quad \text{for any } k \in \mathbb{N}
$$

(65)

where $c > 0$ is a constant given by

$$
c = \frac{1}{2} \sqrt{a_0^2 (0) + 2 \sum_{n=2}^{\infty} (a_n^2 (0) + b_n^2 (0))}
$$

$$
= \frac{1}{\sqrt{2\pi}} \left( \int_0^{2\pi} [u_0 (\theta) - a_1 (0) \cos \theta - b_1 (0) \sin \theta]^2 d\theta \right)^{1/2}.
$$

(66)

Here $a_0 (0), a_n (0), b_n (0)$ are the Fourier coefficients of the function $u_0 (\theta)$.
Proof. This is obvious from Fourier series expansion. Perhaps one would worry about the denominator in (59) being zero in finite time, but this will not happen from the Fourier series expansion of $a_0(t)$, or one can compute (note that $\int_0^{2\pi} u_0(\theta) \, d\theta > 0$)

$$\frac{d}{dt} \int_0^{2\pi} u(\theta, t) \, d\theta = \int_0^{2\pi} u(\theta, t) \, d\theta - \frac{2\pi \int_0^{2\pi} u(\theta, t) (u_0(\theta, t) + u(\theta, t)) \, d\theta}{\int_0^{2\pi} u(\theta, t) \, d\theta} \geq 0$$

due to the Poincaré inequality for 2π-periodic functions (see p. 179 of the book [BCH]):

$$\int_0^{2\pi} dx \int_0^{2\pi} f \left( \frac{d^2 f}{dx^2} + f \right) \, dx \leq \left( \int_0^{2\pi} f(x) \, dx \right)^2. \quad (67)$$

We now assume that the dual flow (40) will not develop a singularity ($k = \infty$) in finite time. Then the flow must be defined on the infinite time interval $[0, \infty)$, with each $\gamma_t$ remaining smooth and convex, and then we can look at its asymptotic geometry. The convexity of $\gamma_t$ can be seen from equation (53) since $2A(t)/L(t)$ is uniformly bounded (see below) and by the maximum principle, $1/k(\theta, t)$ will not blow up in finite time, which implies that $k(\theta, t) > 0$ will not become zero in finite time.

In below we can quickly prove the convergence of the flow again without relying on the support function and its Fourier series expansion. The evolution of the length $L(t)$ is known by (54). As for area $A(t)$, we have

$$\frac{dA}{dt} = -2A + \int_{\gamma_t} \frac{1}{k} \, ds \geq -2A + \frac{L^2 - 2\pi A}{\pi} = \frac{L^2 - 4\pi A}{\pi} \geq 0, \quad (68)$$

where we have used the Pan-Yang’s isoperimetric inequality (42) in (68). Hence in flow (40), both length and area are increasing. In particular

$$\frac{d}{dt} \left( \frac{A}{L} \right) \geq \frac{(L^2 - 4\pi A)(L^2 - \pi A)}{\pi L^3} \geq 0.$$ 

As a consequence, we obtain

$$\frac{d}{dt} (L^2 - 4\pi A) \leq 2L \frac{L^2 - 4\pi A}{L} - 4\pi \frac{L^2 - 4\pi A}{\pi} = -2 (L^2 - 4\pi A) \leq 0, \quad (69)$$

and derive the exponential decay of the IPD

$$0 \leq L^2(t) - 4\pi A(t) \leq e^{-2t} (L^2(0) - 4\pi A(0)) \to 0 \text{ as } t \to \infty. \quad (70)$$

In particular, the IPR

$$1 \leq \frac{L^2}{4\pi A} = \frac{L^2 - 4\pi A}{4\pi A} + 1 \leq \frac{e^{-2t} (L^2(0) - 4\pi A(0))}{4\pi A(0)} + 1 \to 1 \text{ as } t \to \infty. \quad (71)$$

is exponentially decaying to 1 since $A(t)$ is increasing.

Due to the exponential decay of $L^2(t) - 4\pi A(t)$, the increasing $L(t)$ and $A(t)$ actually converge as $t \to \infty$. By

$$0 \leq \frac{d}{dt} \left( \frac{L^2}{2} \right) = L \frac{dL}{dt} = L^2 - 4\pi A \leq e^{-2t} (L^2(0) - 4\pi A(0)) \quad (72)$$

2It is interesting to know that, in fact, the Poincaré inequality, the classical isoperimetric inequality $L^2 \geq 4\pi A$, and the Minkowski mixed area inequality $\sqrt{A_1 A_2} \leq A_{12}$ (see 116 below) are all equivalent.
we have
\[ 0 \leq \frac{L^2(t)}{2} - \frac{L^2(0)}{2} \leq \frac{1}{2} \left[ L^2(0) - 4\pi A(0) \right] \left( 1 - e^{-2t} \right). \]

In particular, the limit \( \lim_{t \to \infty} L(t) = L(\infty) > 0 \) exists (note that \( L(t) \) is increasing), where
\[
L(0) \leq L(\infty) \leq \sqrt{L^2(0) + [L^2(0) - 4\pi A(0)]}. \tag{73}
\]
As for \( A(t) \), we have \( \lim_{t \to \infty} A(t) = A(\infty) = L^2(\infty)/4\pi \). We also have \( \lim_{t \to \infty} (2A(t)/L(t)) = 2A(\infty)/L(\infty) \) in [53].

**Remark 14** Note that if we do not use Pan-Yang’s isoperimetric inequality (42) in (69), by Hölder inequality we would only obtain
\[
\frac{d}{dt} \left( L^2 - 4\pi A \right) = 2 \left( L^2 - 2\pi \int_0^1 ds \right) \leq 0,
\]
which is not good enough to imply the convergence of \( L(t) \) as \( t \to \infty \).

To show the convergence of \( 1/k \), we can apply the following simple result (see [LT2], p. 2625) to the linear equation (55):

**Lemma 15** Let \( \sigma(\theta, t) \) be a smooth solution to the linear equation
\[
\frac{\partial \sigma}{\partial t}(\theta, t) = \sigma_{\theta\theta}(\theta, t) + \sigma(\theta, t), \quad \sigma(\theta, 0) = \sigma_0(\theta) \tag{74}
\]
on \( S^1 \times [0, \infty) \). Then \( \sigma(\theta, t) \) is uniformly bounded on \( S^1 \times [0, \infty) \) if and only if \( \int_0^{2\pi} \sigma_0(\theta) d\theta = 0 \). Moreover, if \( \int_0^{2\pi} \sigma_0(\theta) d\theta = 0 \), then \( \sigma(\theta, t) \) converges uniformly to the following function
\[
\lim_{t \to \infty} \sigma(\theta, t) = \left( \frac{1}{\pi} \int_0^{2\pi} \sigma_0(\theta) \cos \theta d\theta \right) \cos \theta + \left( \frac{1}{\pi} \int_0^{2\pi} \sigma_0(\theta) \sin \theta d\theta \right) \sin \theta, \quad \theta \in S^1. \tag{75}
\]

Applying Lemma 15 to (55), we obtain the uniformly convergence
\[
\lim_{t \to \infty} \left( \frac{1}{k(\theta, t)} - \frac{L(t)}{2\pi} \right) = \left[ \frac{1}{\pi} \int_0^{2\pi} \left( \frac{1}{k_0(\theta)} - \frac{L(0)}{2\pi} \right) \cos \theta d\theta \right] \cos \theta + \left[ \frac{1}{\pi} \int_0^{2\pi} \left( \frac{1}{k_0(\theta)} - \frac{L(0)}{2\pi} \right) \sin \theta d\theta \right] \sin \theta = 0
\]
where the last identity is due to the identity (see [GH], p. 79)
\[
\int_0^{2\pi} \frac{\cos \theta}{k_0(\theta)} d\theta = \int_0^{2\pi} \frac{\sin \theta}{k_0(\theta)} d\theta = 0. \tag{76}
\]

Hence
\[
\lim_{t \to \infty} \frac{1}{k(\theta, t)} = \frac{L(\infty)}{2\pi} = \frac{2A(\infty)}{L(\infty)}, \quad \text{uniformly in } \theta \in S^1. \tag{77}
\]

With the above \( C^0 \) convergence, together with the equation (55), we can obtain \( C^\infty \) convergence of \( 1/k(\theta, t) \) to the number \( L(\infty)/2\pi = 2A(\infty)/L(\infty) \). Therefore we conclude that, if no singularity \( (k = \infty) \) forming in finite time, the flow (40) converges to a round circle with radius \( L(\infty)/2\pi \) in \( C^\infty \) sense.

It is interesting to know that, with the help of Fourier series expansion (see Theorem 13), the length of the flow (40) begins with
\[
L(0) = \int_0^{2\pi} u_0(\theta) d\theta \left( = \int_0^{2\pi} [u_0(\theta) - a_1(0) \cos \theta - b_1(0) \sin \theta] d\theta \right) \tag{78}
\]
and increases asymptotically to

\[ L(\infty) = \sqrt{2\pi} \left( \int_{0}^{2\pi} \left[ u_0(\theta) - a_1(0) \cos \theta - b_1(0) \sin \theta \right]^2 d\theta \right)^{1/2}. \tag{79} \]

One additional property worth mentioning is that the center (average position vector) of the evolving curve \( \gamma_t \) is fixed. In fact, all \( 1/k \)-type flows have this property. The center of the evolving curve \( \gamma_t \) is given by (see \cite{LT2}, p. 2621)

\[ \frac{1}{\pi} \int_{0}^{2\pi} u(\theta, t) (\cos \theta, \sin \theta) d\theta, \tag{80} \]

where \( u(\theta, t) \) is the support function of \( \gamma_t \) and it satisfies the evolution equation (59). This is independent of time because it is obvious that

\[ \frac{d}{dt} \left( \frac{1}{\pi} \int_{0}^{2\pi} u(\theta, t) (\cos \theta, \sin \theta) d\theta \right) = 0. \tag{81} \]

At this moment, the asymptotic behavior of the nonlocal flow (40) is well-understood (if no singularity forming in finite time).

To end this section, we would like to say something about the Ma-Cheng’s flow (5). If there is a convex solution to the flow (5) on \([0, T)\), the support function \( u(\theta, t) \) satisfies

\[ \frac{\partial u}{\partial t}(\theta, t) = u_{\theta\theta}(\theta, t) + u(\theta, t) - \frac{\int_{0}^{2\pi} [u_{\theta\theta}(\theta, t) + u(\theta, t)]^2 d\theta}{\int_{0}^{2\pi} u(\theta, t) d\theta}, \quad (\theta, t) \in S^1 \times [0, T) \tag{82} \]

with \( u(\theta, 0) = u_0(\theta), \ u''_0(\theta) + u_0(\theta) > 0 \). On the other hand, similar to equation (59), one can use Fourier series expansion to prove the existence of a solution \( u(\theta, t) \) to (82) for short time. This implies the existence of a convex solution to the flow (5) for short time also.

We can check that if

\[ u(\theta, t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} (a_n(t) \cos n\theta + b_n(t) \sin n\theta), \quad u(\theta, 0) = u_0(\theta) \]

where \( a_n(t) = a_n(0) e^{(1-n^2)t}, \ b_n(t) = b_n(0) e^{(1-n^2)t}, \) and \( a_0(t) \) satisfies

\[ \frac{a'_0(t)}{2} = -\sum_{n=2}^{\infty} \frac{(n^2 - 1)^2 [a_n^2(t) + b_n^2(t)]}{a_0^2(t)} \leq 0, \quad a_0(0) > 0 \]

then \( u(\theta, t) \) will be a solution to (82). We find that

\[ a_0(t) = \sqrt{a_0^2(0) - 2 \sum_{n=2}^{\infty} (n^2 - 1) \left( 1 - e^{2(1-n^2)t} \right) (a_n^2(0) + b_n^2(0))}. \]

Thus (82) has a solution for short time and maintains the inequality (63). Therefore the flow (5) also has a convex solution for short time. Again there is a possibility that \( u_{\theta\theta}(\theta_t, t_0) + u(\theta_t, t_0) = 0 \) at finite time \( t_0 \) for some \( \theta_0 \in S^1 \), which is bad and we are not able to exclude it.

As \( t \to \infty \), we get

\[ u(\theta, t) \to \frac{1}{2} \sqrt{a_0^2(0) - 2 \sum_{n=2}^{\infty} (n^2 - 1) (a_n^2(0) + b_n^2(0)) + a_1(0) \cos \theta + b_1(0) \sin \theta} \tag{83} \]
Note that
\[2A(0) = \int_0^{2\pi} u_0(\theta) [(u_0)_{\theta\theta}(\theta) + u_0(\theta)] d\theta = \frac{\pi}{2} \left( a_0^2(0) - 2 \sum_{n=2}^{\infty} (n^2 - 1) \left[ a_n^2(0) + b_n^2(0) \right] \right)\]
and so
\[u(\theta, t) \to \sqrt{A(0)} + a_1(0) \cos \theta + b_1(0) \sin \theta \text{ \ as \ } t \to \infty. \tag{84}\]
Hence the flow converges to a circle with radius \(\sqrt{A(0)/\pi}\) centered at \((a_1(0), b_1(0))\). Its enclosed area is same as the initial area \(A(0)\). This matches with the fact that the flow is area-preserving. Again, the center of the evolving curve \(\gamma_t\) is fixed. See Ma-Cheng [MC] for more detailed discussion of the flow.

Remark 16 To prove asymptotic convergence of the flow, in p. 10, Theorem 21 of Ma-Cheng [MC] they quote an estimate established in Gage-Hamilton [GH], which says that \(k(\theta, t) r_{in}(t) (r_{in}(t) is the inradius of \(\gamma_t\)) converges uniformly to 1 when the IPD \(L^2(t) - 4\pi A(t) \to 0\) (see Theorem 5.4, p. 89 of [GH]). We can not understand why this result can be applied to their flow since these two flows (curve shortening flow and flow (5)) are very different. For example, in Corollary 5.2, p. 88 of [GH] we have Harnack-type estimate \(k(\theta, t) \geq (1 - \varepsilon) k_{\text{max}}(t)\) near maximum point. But in [MC] there is no such estimate at all.

4 Determine Parallel Relation Using Mixed Isoperimetric Difference

In classical differential geometry, it is known that if two simple closed curves \(\alpha, \beta\) are parallel with distance \(r > 0\) apart (assume \(\beta\) is an outer parallel of the fixed curve \(\alpha\)), their length, enclosed area, and curvature are related by (see Do Carmo [D], p. 47)
\[L_\beta = L_\alpha + 2\pi r, \quad A_\beta = A_\alpha + rL_\alpha + \pi r^2, \quad k_\beta(s) = \frac{k_\alpha(s)}{1 + rk_\alpha(s)}, \tag{85}\]
where \(s\) is arc length parameter of \(\alpha\). As a consequence, we have the infinitesimal identities
\[\frac{dL_\beta}{dr} = 2\pi, \quad \frac{dA_\beta}{dr} = L_\beta, \quad \frac{dk_\beta}{dr}(s) = -k_\beta^2(s) \tag{86}\]
and
\[\frac{d}{dr} \left( L_\beta^2 - 4\pi A_\beta \right) = 0, \quad \frac{d}{dr} \left( \frac{L_\beta^2}{4\pi A_\beta} \right) = -\frac{L_\beta}{4\pi A_\beta^2} \left( L_\beta^2 - 4\pi A_\beta \right) \leq 0 \tag{87}\]
for all \(r > 0\) small (as long as the denominator of (85) is not zero).

By the derivative formulas in (87), we clearly have:

Lemma 17 (parallel-invariance of the IPD) If two simple closed curves \(\alpha, \beta\) are parallel, then they have the same IPD, that is
\[L_\beta^2 - 4\pi A_\beta = L_\alpha^2 - 4\pi A_\alpha. \tag{88}\]
Moreover, a curve’s inner (outer) parallels increase (decrease) its IPR.

For the convex case, there is an additional invariance, which is
\[\int_0^{2\pi} \frac{1}{k_\beta^2(\theta)} d\theta - 2A_\beta \quad \left( \text{or} \int_0^{2\pi} \frac{1}{k_\beta^2(\theta)} d\theta - \frac{L_\beta^2}{2\pi} \right) \tag{89}\]
due to
\[
\int_0^{2\pi} \frac{1}{k^2_\beta(\theta)} d\theta - 2A_\beta
= \int_0^{2\pi} \frac{1}{k^2_\beta(\theta)} d\theta + 2rL_\alpha + 2\pi r^2 - 2(A_\alpha + rL_\alpha + \pi r^2) = \int_0^{2\pi} \frac{1}{k^2_\alpha(\theta)} d\theta - 2A_\alpha
\]
where by the refined isoperimetric inequality (45) we know
\[
\int_0^{2\pi} \frac{1}{k^2_\beta(\theta)} d\theta - 2A_\beta \geq \frac{2}{\pi} (L^2_\alpha - 4\pi A_\alpha) \geq 0.
\]

Combining the simple identities in (86) and using Gage’s inequality (41) for convex curves, we can obtain the interesting *monotonicity formula for convex parallel curves* (or call it *entropy estimate*):

**Lemma 18** (monotonicity formula for convex parallel curves) There holds the inequality
\[
\frac{d}{dr} \int_0^{2\pi} \log \left( k_\beta(\theta) \sqrt{\frac{A_\beta}{\pi}} \right) d\theta \leq 0, \quad \forall \ r \in [0, \infty).
\]

Hence the integral
\[
\int_0^{2\pi} \log \left( k_\beta(\theta) \sqrt{\frac{A_\beta}{\pi}} \right) d\theta \geq 0
\]
is a decreasing function of $r \in [0, \infty)$, which will converge to 0 as $r \to \infty$.

**Remark 19** One can view Lemma 18 as the integration of Gage’s inequality (under parallel evolution). This provides a clear explanation of Theorem 0.6 in p. 661 of Green-Osher [GO], which has been described by them as ”physically intriguing”.

**Remark 20** By (92) we have the *entropy estimate* for a convex closed curves $\gamma$ (also see Theorem 0.2 of [GO]):
\[
\int_0^{2\pi} \log \left( k(\theta) \sqrt{\frac{A}{\pi}} \right) d\theta \geq 0, \quad A = \text{enclosed area of } \gamma
\]
where $k(\theta)$ is the curvature of $\gamma$. This is already known as a consequence of the fact that under the normalized curve shortening flow, the entropy is decreasing to 0 as $t \to \infty$ (see p. 10 of the book by Zhu [Z]). Obviously if $\gamma$ is a circle, then the equality holds. What is not so clear is the converse. If we have equality in (93), then by the above lemma we must have equalities in both (91) and (92). In particular we have
\[
\int_0^{2\pi} k(\theta) d\theta = \frac{\pi L}{A}, \quad L = \text{length of } \gamma.
\]
This is the equality case of Gage’s inequality, which is not discussed in Gage’s paper [GA3] either. But according to a recent communication with Professor Gage, he asserted that (94) implies $\gamma$ is a circle. Hence the equality holds in (93) if and only if $\gamma$ is a circle.

**Remark 21** Is there a proof of the entropy estimate (93) without using a flow method?

**Proof.** By Gage’s inequality for convex curves (Gage’s inequality is not true for non-convex curves) we have
\[
\frac{d}{dr} \int_0^{2\pi} \left[ \log k_\beta(\theta) + \frac{1}{2} \log \left( \frac{A_\beta}{\pi} \right) \right] d\theta = -\int_0^{2\pi} k_\beta(\theta) d\theta + \frac{\pi L_\beta}{A_\beta} \leq 0.
\]
Also note that
\[
\lim_{r \to \infty} \int_0^{2\pi} \log \left( k_\beta (\theta) \sqrt{\frac{A_\beta}{\pi}} \right) d\theta = \lim_{r \to \infty} \int_0^{2\pi} \log \left( \frac{k_\alpha (s)}{1 + rk_\alpha (s)} \sqrt{\frac{A_\alpha + rL_\alpha + \pi r^2}{\pi}} \right) d\theta = 0.
\]

The proof is done.

The converse of Lemma 17 is clearly not true. For two convex curves \( C_1 \) and \( C_2 \), we will show that if their IPD and \textit{mixed IPD} are all the same, then they must be parallel. This is motivated by the fact that if their IPR and \textit{mixed IPR} are all the same, then they must be homothetic. See Lemma 23 and Lemma 27 below.

Recall that two simple closed curves \( \alpha (s), \beta (s) : I \to \mathbb{R}^2 \), are said to be homothetic if there exist some constant \( \lambda > 0 \) and some point \( (a, b) \in \mathbb{R}^2 \) such that \( \beta (s) = \lambda \alpha (s) + (a, b) \) for all \( s \in I \). Clearly two homothetic curves have the same IPR. This is similar to the property that two parallel curves have the same IPD.

If two simple closed curves \( \alpha, \beta \) have the same IPR and IPD, then
\[
(4\pi A_\beta - 4\pi A_\alpha) \left( \frac{L_\beta^2}{4\pi A_\beta} - 1 \right) = 0.
\]
Thus if \( \beta \) is not a circle, then \( A_\alpha = A_\beta \) and also \( L_\alpha = L_\beta \). But if \( \beta \) is a circle, so is \( \alpha \). Thus unless they are both circles, they must have the same length and the same enclosed area.

Note that even for the convex case, two different curves \( \alpha, \beta \) may have the same length and enclosed area but without other significant relations at all. Let \( u_\alpha \) be the support function of the convex curve \( \alpha \) with Fourier series
\[
\begin{align*}
u_\alpha (\theta) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta). \tag{96}
\end{align*}
\]
If we replace the coefficients \( a_n, b_n \) by \( -a_n, -b_n \) in (96) for some \( n \), where \( n \) is large enough to maintain the convexity condition \( u_{\theta\theta} (\theta) + u (\theta) > 0 \), then the new convex curve \( \beta \) (with the new support function) will have the same length and area as \( \alpha \) due to formulas (49) and (50). The curve \( \beta \) is just a small perturbation of the curve \( \alpha \).

We now ask the following two interesting converse questions:

(A). If two curves have the same IPR, under what conditions are they homothetic?

(B). If two curves have the same IPD, under what conditions are they parallel?

When the curves are \textit{convex} (with positive curvature everywhere), we can answer these questions. For general case of simple closed curves, we still do not know the answer.

We now confine to the convex case. The advantage is that any convex curve can be parametrized by its outward normal angle \( \theta \in [0, 2\pi] \) and its support function \( u (\theta) \) has domain \( \theta \in [0, 2\pi] \). Moreover, \( u (\theta) \) behaves well with respect to either homothetic or parallel relation. If two convex curves \( C_1, C_2 \) are homothetic, their support functions \( u_1 (\theta), u_2 (\theta) \) satisfy the identity
\[
u_2 (\theta) = \lambda u_1 (\theta) + a \cos \theta + b \sin \theta, \quad \forall \theta \in [0, 2\pi] \tag{97}
\]
for some constants \( \lambda > 0, a, b \in \mathbb{R} \). If they are parallel, then \( u_2 (\theta) = u_1 (\theta) + r \) for some constant \( r \in \mathbb{R} \).

It is also known that if a \( 2\pi \)-periodic function \( u (\theta) \) satisfies the inequality
\[
u_{\theta\theta} (\theta) + u (\theta) > 0, \quad \forall \theta \in [0, 2\pi] \tag{98}
\]
then it becomes the support function of a simple convex closed curve $C$ in the plane (see [LT2]). The parametrization of $C$ is given by

$$X (\theta) = u (\theta) (\cos \theta, \sin \theta) + u_\theta (\theta) (- \sin \theta, \cos \theta), \quad \theta \in [0, 2\pi].$$

(99)

and its curvature is given by $k (\theta) = 1 / [u_{\theta\theta} (\theta) + u (\theta)] > 0$, $\theta \in [0, 2\pi]$.

Another nice property for the support functions is related to the sum and mixed area of the regions enclosed by convex curves $C_1$, $C_2$. Let $\Omega_1$, $\Omega_2$ be two strictly convex plane regions enclosed by $C_1$, $C_2$. The sum (vector sum in $\mathbb{R}^2$) of $\Omega_1$ and $\Omega_2$ is defined by

$$\Omega := \Omega_1 + \Omega_2 = \{ a + b : a \in \Omega_1 \subset \mathbb{R}^2, b \in \Omega_2 \subset \mathbb{R}^2 \} \subset \mathbb{R}^2.$$

(100)

It is easy to check that $\Omega$ is also a convex region in $\mathbb{R}^2$. Moreover, in classical convex geometry, it is proved that the boundary of $\Omega$ is a convex closed curve with support function $u (\theta)$ satisfying $u (\theta) = u_1 (\theta) + u_2 (\theta)$ for all $\theta \in [0, 2\pi]$. The unique boundary point of $\Omega$ with outward normal angle $\theta$ comes from the sum of the unique point on $C_1$ with outward normal angle $\theta$ and the unique point on $C_2$ with outward normal angle $\theta$, and no others.

By (100), we can define the mixed area $A (\Omega_1, \Omega_2)$ of $\Omega_1$ and $\Omega_2$, which is through the identity

$$A (\Omega_1 + \Omega_2) = A (\Omega_1) + 2A (\Omega_1, \Omega_2) + A (\Omega_2).$$

(101)

Using the support functions $u_1 (\theta), u_2 (\theta)$, we have

$$A (\Omega_1 + \Omega_2) = \frac{1}{2} \int_0^{2\pi} (u_1 (\theta) + u_2 (\theta)) [(u''_1 (\theta) + u_1 (\theta)) + (u''_2 (\theta) + u_2 (\theta))] \, d\theta.$$

(102)

Hence the mixed area $A (\Omega_1, \Omega_2)$ is given by

$$2A (\Omega_1, \Omega_2)$$

$$= \frac{1}{2} \int_0^{2\pi} u_1 (\theta) (u''_2 (\theta) + u_2 (\theta)) \, d\theta + \frac{1}{2} \int_0^{2\pi} u_2 (\theta) (u''_1 (\theta) + u_1 (\theta)) \, d\theta := A_{12} + A_{21},$$

(103)

where, by integration by parts, we actually have $A_{12} = A_{21}$. From now on, we shall denote the mixed area $A (\Omega_1, \Omega_2)$ by $A_{12}$.

Note that the values of the integrals in (102) and (103) are invariant under the transformation

$$u_1 (\theta) \rightarrow u_1 (\theta) + a_1 \cos \theta + b_1 \sin \theta, \quad u_2 (\theta) \rightarrow u_2 (\theta) + a_2 \cos \theta + b_2 \sin \theta,$$

(104)

where $a_1, b_1, a_2, b_2$ are constants. Geometrically, this says that the areas $A (\Omega_1 + \Omega_2)$ and $A (\Omega_1, \Omega_2)$ are invariant under translations of $\Omega_1$ and $\Omega_2$ in $\mathbb{R}^2$.

With the mixed area $A_{12}$, we can define the mixed IPD as $L_1 L_2 - 4\pi A_{12}$, where $L_1$, $L_2$ are the length of $C_1$, $C_2$ respectively. Unlike the usual IPD $L^2 - 4\pi A \geq 0$, the mixed IPD $L_1 L_2 - 4\pi A_{12}$ can be positive or negative (see Theorem 19 below). In particular, if one of $C_1$, $C_2$ is a circle, say $C_1$ is a circle with radius $r$, then $A_{12} = A_{21} = r L_2/2$ and the mixed IPD disappears with $L_1 L_2 - 4\pi A_{12} = 2\pi r L_2 - 2\pi r L_2 = 0$. The mixed IPD is related to the IPD of $\Omega_1 \Omega_2$ via the identity

$$L^2 - 4\pi A \quad (\text{for } \Omega_1 + \Omega_2) = (L_1^2 - 4\pi A_1) + (L_2^2 - 4\pi A_2) + 2 (L_1 L_2 - 4\pi A_{12}).$$

(105)

By (105) and Lemma 17, the mixed IPD $L_1 L_2 - 4\pi A_{12}$ is clearly invariant under parallel relations also. That is:

**Lemma 22** (parallel-invariance of the mixed IPD) If we replace $C_1$ by a parallel curve $\tilde{C}_1$ and $C_2$ by another parallel curve $\tilde{C}_2$, then the mixed IPD for the pair $\tilde{C}_1$, $\tilde{C}_2$ is the same as that for the pair $C_1$, $C_2$. 

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The most important property for mixed area is the following well-known Minkowski mixed area inequality (see the encyclopedic book by R. Schneider [S]):

\[ \sqrt{A_1 A_2} \leq A_{12}, \]  

(106)

where the equality holds if and only if \( C_1 \) and \( C_2 \) are homothetic.

With the above theorem, our answer to question (A) comes immediately:

Lemma 23 \((\text{characterization of homothetic relation})\) Two convex closed curves \( C_1, C_2 \) are homothetic if and only if

\[ \frac{L_1^2}{4\pi A_1} = \frac{L_2^2}{4\pi A_2} = \frac{L_1 L_2}{4\pi A_{12}}. \]

(107)

That is, all IPR, including the mixed one, are the same.

Remark 24 \(\text{Note that the mixed IPR } L_1 L_2/4\pi A_{12} \text{ is invariant under dilations or translations of } C_1 \text{ and } C_2. \text{ Compare with the parallel-invariance property in Lemma 22.}\)

Proof. If \( C_1, C_2 \) are homothetic, we clearly have \( L_1^2/4\pi A_1 = L_2^2/4\pi A_2 \) (call this value \( \lambda \)), which gives \( L_1 = \sqrt{4\pi\lambda A_1}, L_2 = \sqrt{4\pi\lambda A_2} \). By Minkowski inequality, we also have \( A_{12} = \sqrt{A_1 A_2} \). Hence

\[ \frac{L_1 L_2}{4\pi A_{12}} = \frac{\sqrt{4\pi\lambda A_1} \sqrt{4\pi\lambda A_2}}{4\pi \sqrt{A_1 A_2}} = \lambda. \]

Conversely if (107) holds with common value \( \lambda \), then \( L_1 = \sqrt{4\pi A_1\lambda}, L_2 = \sqrt{4\pi A_2\lambda} \), and

\[ 4\pi A_{12} = \frac{L_1 L_2}{\lambda} = \frac{\sqrt{4\pi A_1\lambda} \sqrt{4\pi A_2\lambda}}{\lambda} = 4\pi \sqrt{A_1 A_2}, \]

which gives the equality \( A_{12} = \sqrt{A_1 A_2} \). By Minkowski theorem they are homothetic. The proof is done. \( \square \)

To answer question (B), one can not rely on the Minkowski inequality because it does not have the right form. Instead we can use the following, which concerns the IPD:

Theorem 25 \((\text{mixed IPD inequality})\) For any two convex closed curves \( C_1, C_2 \) in \( \mathbb{R}^2 \) with support functions \( u_1(\theta), u_2(\theta) \), there holds the estimate

\[ -\sqrt{I_1 \sqrt{I_2}} \leq L_1 L_2 - 4\pi A_{12} \leq \sqrt{I_1} \sqrt{I_2} \]

(108)

where \( I_1 = L_1^2 - 4\pi A_1, I_2 = L_2^2 - 4\pi A_2, \) and \( A_{12} \) is the mixed area of \( C_1 \) and \( C_2 \). The equality holds in the lower bound estimate if and only if

\[ \frac{\sqrt{I_2}}{k_1(\theta)} + \frac{\sqrt{I_1}}{k_2(\theta)} = c, \quad \forall \theta \in [0, 2\pi] \]

(109)

for some constant \( c > 0 \). Also, the equality holds in the upper bound estimate if and only if

\[ \frac{\sqrt{I_2}}{k_1(\theta)} - \frac{\sqrt{I_1}}{k_2(\theta)} = c, \quad \forall \theta \in [0, 2\pi] \]

(110)

for some constant \( c \). Here \( k_1(\theta), k_2(\theta) \) are the curvature of \( C_1, C_2 \) respectively.

Remark 26 \(\text{We actually proved inequality (108) by ourselves (just a simple observation), but later on Professor Schneider kindly told us that it had been proved by Favard in 1930 and also reappeared in (4), p. 105 of the book Bonnesen-Fenchel [BF]. A higher-dimensional version of (108) is inequality (6.4.9), p. 335 of his book [S]. However, since we do not see the equality results (109) and (110) in p. 105 of [BF] or [S], we still give a proof of Theorem 25 because it takes only a few lines.}\)
Proof. Without loss of generality, we may assume $I_1 > 0$, $I_2 > 0$, otherwise we would have $L_1 L_2 - 4\pi A_{12} = 0$ and (108), (109), (110) all hold. Let $u_1(\theta)$, $u_2(\theta)$ be the support functions of $C_1$, $C_2$ respectively. If $C_{\alpha\beta}$ is a convex closed curve with support function $u_{\alpha\beta}(\theta)$ given by the linear combination

$$u_{\alpha\beta}(\theta) = \alpha u_1(\theta) + \beta u_2(\theta), \quad \alpha, \beta \text{ are non-zero constant}$$

then the IPD of $C_{\alpha\beta}$ is given by

$$L_{\alpha\beta}^2 - 4\pi A_{\alpha\beta} = \alpha^2 I_1 + \beta^2 I_2 + 2\alpha\beta (L_1 L_2 - 4\pi A_{12}) \geq 0. \quad (111)$$

Thus if we choose $\alpha = \sqrt{T_2}$, $\beta = \sqrt{T_1}$ (this is the optimal choice), we would have

$$L_{\alpha\beta}^2 - 4\pi A_{\alpha\beta} = 2I_1 I_2 + 2\sqrt{T_1 T_2} (L_1 L_2 - 4\pi A_{12}) \geq 0,$$

which gives the lower bound and the equality holds if and only if $C_{\alpha\beta}$ is a circle with constant curvature. Hence (108) follows.

On the other hand, if we choose $\alpha = \sqrt{T_2}$, $\beta = -\sqrt{T_1}$, then $u_{\alpha\beta}(\theta)$ may not satisfy $u''_{\alpha\beta}(\theta) + u_{\alpha\beta}(\theta) > 0$. But we can modify it by considering

$$u_{\alpha\beta}(\theta) = \sqrt{T_2} u_1(\theta) - \sqrt{T_1} u_2(\theta) + c$$

for some large constant $c > 0$. Now $u_{\alpha\beta}(\theta)$ is the support function of some convex closed curve $C_{\alpha\beta}$ with (adding $c$ has no effect in $L_{\alpha\beta}^2 - 4\pi A_{\alpha\beta}$)

$$L_{\alpha\beta}^2 - 4\pi A_{\alpha\beta} = 2I_1 I_2 - 2\sqrt{T_1 T_2} (L_1 L_2 - 4\pi A_{12}) \geq 0,$$

which gives the upper bound. The equality holds if and only if $C_{\alpha\beta}$ is a circle and we have (110). □

Motivated by Lemma 23 and with the help of Theorem 25, our answer to question (B) is given by:

**Lemma 27** (characterization of parallel relation) Two convex curves $C_1$, $C_2$ are parallel (up to a translation) if and only if

$$L_1^2 - 4\pi A_1 = L_2^2 - 4\pi A_2 = L_1 L_2 - 4\pi A_{12}. \quad (112)$$

That is, all IPD, including the mixed one, are the same.

**Proof.** If $C_1$, $C_2$ are parallel, then $u_2(\theta) = u_1(\theta) + r + a \cos \theta + b \sin \theta$ for some constants $r$, $a$, $b$ and then $L_2 = L_1 + 2\pi r$. Hence

$$A_{12} = \frac{1}{2} \int_0^{2\pi} u_2(\theta) (u'_1(\theta) + u_1(\theta)) d\theta = A_1 + \frac{1}{2} r L_1$$

and

$$L_1 L_2 - 4\pi A_{12} = L_1 (L_1 + 2\pi r) - 4\pi \left( A_1 + \frac{1}{2} r L_1 \right) = L_1^2 - 4\pi A_1.$$

Hence (112) holds.

Conversely, if (112) holds, then by (110) we have

$$U''(\theta) + U(\theta) = c, \quad U(\theta) := u_2(\theta) - u_1(\theta), \quad \forall \theta \in [0, 2\pi]$$

for some constant $c$. If $D > 0$ is a large constant, the function $V(\theta) = U(\theta) + D$ will be the support function of some circle of $\mathbb{R}^2$. Hence

$$V(\theta) = \rho + a \cos \theta + b \sin \theta$$

for some constants $a$, $b$, $\rho \in \mathbb{R}$, $\rho > 0$. As a result, we get

$$u_2(\theta) - u_1(\theta) = r + a \cos \theta + b \sin \theta$$

for some constants $a$, $b$, $r \in \mathbb{R}$ and $C_1$, $C_2$ are parallel. The proof is done. □
Remark 28 Although Lemma 17 is valid for all simple closed curves, it is not clear how to generalize Lemma 23 and Lemma 27 to the non-convex case. For non-convex sets $\Omega_1, \Omega_2$ in $\mathbb{R}^2$ (bounded by simple closed curves $C_1, C_2$), one can still use identity (101) to define their mixed area $A(\Omega_1, \Omega_2) = A_{12}$. Hence we can still talk about their mixed IPR and IPD. However, it is not clear whether we have good results similar to the Minkowski inequality and Theorem 24.

In higher-dimensional space, say $\mathbb{R}^3$, parallel surfaces $S_1, S_2$ do not have, in general, the same IPD any more. The classical isoperimetric inequality for a compact connected closed surface $S$ is

$$A^3(S) \geq 36\pi V^2(\Omega)$$

(113)

where $A(S)$ is the surface area of $S$ and $\Omega$ is the domain enclosed by it with volume $V(\Omega)$. Moreover, the inequality holds if and only if $S$ is a sphere. In view of (113), the IPD quantity for a space surface is $A^3 - 36\pi V^2$.

If we assume that parallel surfaces have the same IPD, we would have the infinitesimal identity

$$\frac{d}{dr} [A^3(S_r) - 36\pi V^2(\Omega_r)] = 0, \quad S_0 = S,$$

(114)

where $S_r$ is parallel to $S$ (for small $r$) with enclosed domain $\Omega_r$. After computation, we would get (evaluated at $r = 0$)

$$6A^2(S) \int_S H(p) \, dp - 72\pi V(\Omega) A(S) = 0,$$

where $H$ is the mean curvature of $S$. The above is same as

$$A(S) \int_S H(p) \, dp = 12\pi V(\Omega).$$

(115)

However, we know that (115) does not hold for a general compact closed surface $S$.

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