The Role of Correlation in Quantum and Classical Games

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Abstract

We use the example of playing a 2-player game with entangled quantum objects to investigate the effect of quantum correlation. We find that for simple game scenarios it is classical correlation that is the central feature and that these simple quantum games are not sensitive to the quantum part of the correlation. In these games played with quantum objects it is possible to transform a game such as Prisoner’s Dilemma into the game of Chicken. We show that this behaviour, and the associated enhanced equilibrium payoff over playing the game with quantum objects in non-entangled states, is entirely due to the classical part of the correlation.

Generalizing these games to the pure strategy 2-player quantum game where the players have finite strategy sets and a projective joint measurement is made on the output state produced by the players, we show that a given quantum game of this form can always be reproduced by a classical model, such as a communication channel. Where entanglement is a feature of these 2-player quantum games the matrix of expected outcomes for the players can be reproduced by a classical channel with correlated noise.

1 Introduction

The field of computer science has been revolutionised by the realisation that computers are physical objects obeying physical laws. Allowing computational devices to access the features of quantum mechanics, and entanglement in particular, has resulted in the potential for quantum devices that can perform certain computations significantly faster than their classical counterparts, lowering the complexity class of the associated problems. It was an innovative and groundbreaking step to ask the same question of classical game theory. Could the introduction of quantum objects and operations to the theory of games also result in a similar revolution allowing resolutions and enhancements not available within a classical treatment?
Since the seminal work of Eisert, Wilkens and Lewenstein (EWL) and Meyer [1,2] quantum games have been the subject of much work and controversy. The question of whether a game played with quantum objects can be considered to be quantum mechanical at all has been raised (see, in particular [3]) and the necessity of comparing like with like within the context of games has been beautifully formulated by Bleiler [4]. Much of the work has focused, naturally, on the use of entangled states (see [4-21] for a small selection of the extensive literature). In this paper we focus on 2-player non-cooperative games in which a single projective quantum measurement is performed to generate the measurement results over which the players have preferences. We address the question of whether such games can access the quantum nature of any correlation and also question to what extent such games can be considered to be truly quantum mechanical, requiring quantum objects in order to achieve a given result or outcome for the players.

We begin by considering some very simple, and restricted, examples of games played with quantum objects in order to gain an insight into the role of correlation in these systems. It is shown that, for these games, the results depend only on the classical component of the correlation. For these games, although we may begin with preferences over measurement results that are those of one game form, the actual game the players play is a different game altogether. In our first simple example we examine a game played with quantum objects that would appear to be a version of Prisoner’s Dilemma in the first instance. Upon closer inspection, however, it can be seen that the players are actually playing the classical game of Chicken\textsuperscript{1} (for an excellent text on game theory see, for example, [22]).

We show that the same game transformation can be achieved by a classical game in which the players’ choices are communicated over a noisy channel with a classical correlated noise process. Thus the properties of the quantum game are not dependent upon the quantum part of the correlation. By extending the classical game to the mixed case we see that a classical correlated noise process can lead to similar enhancements of equilibrium payoffs as that claimed for the quantum game with a full quantum strategy set. The role of the classical correlations is further highlighted by consideration of a simple quantum game in which the players communicate their choice by the transmission of their respective particles over some noisy channel such that the quantum interference terms in the density matrix are suppressed. In this scenario we see that an enhanced equilibrium and the transformation of the game is also obtained. These results strongly suggest that the enhancements and the game transformation are due to classical correlations in these 2-player games rather than any specifically quantum mechanical feature of the correlation.

We develop a general approach to simple 2-player games played with quantum objects that allows the analysis of a wider class of games than our initial examples. Thus we adopt the perspective that games can be played with quan-

\textsuperscript{1}Of course, ‘changing’ a game by considering an appropriate extension of it is nothing new within game theory. The physics of playable games [13] is highlighting this here in a rather dramatic fashion.
tum objects (we *game* the quantum [23-25]) rather than worry about whether such games are proper extensions of some underlying classical game. We show that such games can always be thought of as being *equivalent* to a classical game played with classical coins in the sense that the players analyse the game as if they were playing the equivalent classical game with a potentially different set of preferences to those of the initial quantum game. In this way we can see that a given classical game may sometimes be thought of as a decomposition consisting of a different quantum game with different preferences. Our simple example shows that 2 players can play the game of Chicken by playing a version of Quantum Prisoner’s Dilemma [23].

By focusing on the notion of *playable* games, that is there is an implementation of the game with physical objects, we describe the general features of any game whether played with classical or quantum objects. The requirement of playability allows us to develop a model in which the elements necessary for proper comparison of quantum vs classical behaviour are made clear. We believe that this approach, grounded in the physics of the game objects and mechanisms, gives a perspective on quantum games that helps to clarify the issue of just what is quantum mechanical in a quantum game.

2 Turning Prisoners into Chickens

Let us consider an attempt to implement the game of Prisoner’s Dilemma (PD) using quantum objects and operations. We shall assume the players (Alice and Bob) are each given a spin-1/2 particle upon which to operate. The players are each allowed only two operations on their own particles; flip or don’t-flip, with respect to the spin-$z$ direction. We shall label these operations as $F$ and $I$, respectively. There will be, in general, 4 possible output states that the players can produce, characterized by the choices $(I, I), (I, F), (F, I)$ and $(F, F)$ where the first element refers to the choice of Alice and the second to that of Bob. The output state is subject to a measurement as follows; the spin in the $z$-direction of Alice’s particle is measured and the spin in the $z$-direction of Bob’s particle is measured. The possible measurement results are listed as a tuple $(0, 0), (0, 1), (1, 0)$ or $(1, 1)$ where the ‘0’ result indicates spin-down. The measurement results are mapped to an outcome tuple for the players so that

\[
(0, 0) \rightarrow (3, 3) \\
(0, 1) \rightarrow (0, 5) \\
(1, 0) \rightarrow (5, 0) \\
(1, 1) \rightarrow (1, 1)
\]
The preferences of the players are thus encapsulated by the assignment of a numerical value as an outcome. If the initial state of the spin-1/2 particles is given, in the measurement basis, by $|0\rangle_A \otimes |0\rangle_B$ which we shall write as $|00\rangle$ then the payoff matrix becomes

\[
\begin{array}{c|cc}
A & I & F \\
\hline
I & (3, 3) & (0, 5) \\
F & (5, 0) & (1, 1) \\
\end{array}
\]

Table 1: the outcome matrix for an implementation of classical PD using quantum objects

This is nothing more than the standard description of classical Prisoner’s Dilemma [22].

Now let us consider playing the game with a different input state of the particles, but keeping everything else the same. The players have the same preferences over the measurement results. We choose an input state that is not mapped onto an eigenstate of the measurement operator by the actions of the players. This means that the results of the measurement will be distributed according to some probability distribution. Let us choose the following initial state

\[
|\psi_0\rangle = \sqrt{\frac{3}{5}} |00\rangle + \sqrt{\frac{2}{5}} |11\rangle
\]

The four possible output states are given by

\[
|\psi\rangle_{II} = \sqrt{\frac{3}{5}} |00\rangle + \sqrt{\frac{2}{5}} |11\rangle
\]

\[
|\psi\rangle_{IF} = \sqrt{\frac{3}{5}} |01\rangle + \sqrt{\frac{2}{5}} |10\rangle
\]

\[
|\psi\rangle_{FI} = \sqrt{\frac{3}{5}} |10\rangle + \sqrt{\frac{2}{5}} |01\rangle
\]

\[
|\psi\rangle_{FF} = \sqrt{\frac{3}{5}} |11\rangle + \sqrt{\frac{2}{5}} |00\rangle
\]

Let us suppose the players choose the operation tuple $(I, I)$. We can see that if they choose these operations then they will obtain the output tuple $(3, 3)$ with probability $3/5$ and the output tuple $(1, 1)$ with probability $2/5$. Thus, for this
choice of operations they will obtain an expected outcome tuple of \((11/5, 11/5)\).

Doing a similar calculation for the other possible output states we obtain the matrix for the expected outcome tuples as

\[
\begin{array}{c|cc}
A & I & F \\
\hline
I & (11/5, 11/5) & (2, 3) \\
F & (3, 2) & (9/5, 9/5) \\
\end{array}
\]

Table 2: the outcome matrix for a game played with quantum objects having preferences over the measurement results in accord with Prisoner’s Dilemma, but in which the input quantum state is entangled according to equation (2)

The players will use this new matrix to determine their choice of strategy. Their actual choice of play is thus determined by this matrix of expected outcomes. However, the matrix of expected outcomes is nothing more than a numerical encapsulation of the preferences of the game of Chicken. The game of Chicken \cite{22} can be described by the preferences

\[
\begin{align*}
\text{Alice} : & O_{FI} > O_{II} > O_{IF} > O_{FF} \\
\text{Bob} : & O_{IF} > O_{II} > O_{FI} > O_{FF}
\end{align*}
\]

where \(O_{FI}\), for example, describes the outcome tuple when Alice plays \(F\) and Bob plays \(I\). In the game of Chicken the usual scenario is to imagine two somewhat irresponsible youths hurtling towards one another in their cars. The winner of the game is the one who doesn’t swerve \(O_{FI}\) in which Alice doesn’t swerve, but Bob does, is Alice’s most preferred outcome. If neither swerve \(O_{FF}\) then they crash and this is the least preferred outcome for both players. If both swerve then they are both ‘chickens’, which is more preferable than crashing, but not as preferable as winning. If Alice swerves, but Bob doesn’t, then this is preferable to crashing, but Alice has lost face (she is the chicken and Bob isn’t) and so is not as preferable to her as when they both swerve.

Thus for our quantum game with this entangled state input, despite initially having preferences over the measurement results in accordance with those of Prisoner’s Dilemma, these are transformed into preferences over the expected outcomes that are in accordance with the game of Chicken. The game that the players actually play is the classical game of Chicken, despite setting up the game as a quantum version of Prisoner’s Dilemma. The players analyze their choices in terms of this matrix of expected outcomes. At this point it is irrelevant to the players, as far as their game objectives are concerned, whether this matrix has been generated by some complicated quantum process or whether it has been generated by some game in which classical coins are mapped to the outcomes of
Chicken. This feature of game transformation has been examined in a general case by consideration of measurement of the Schmidt observables in entangled quantum games [10,11].

This simple example, based on the quantum game scenario of Marinatto and Weber (MW) [21], illustrates that care must be taken in ascribing quantum behaviour to a game scenario in which the objects used to implement a game are quantum mechanical in nature. Whilst the game scenario of MW raises legitimate concerns about whether it constitutes a proper extension of an underlying classical game [4] it is, nevertheless, a perfectly acceptable example of a game played with quantum objects. If we imagine the players are given some black box with dials for their strategy choices, then if they know the payoff function for the possible choices, they will base their final choice upon the analysis of this payoff function. Their transformed preferences over these expected payoffs define the actual game they are playing, despite the possibility that their initial preferences over the measurement results were those of another game form. In the example we have discussed we can see that, as far as the players are concerned, the same game can be implemented with either quantum objects and operations or with classical coins. In other words the players cannot tell whether the objects inside their black box are classical or quantum mechanical.

3 Playable Games and Quantum/Classical Comparisons

Any game that is actually playable must have an implementation in the physical world. The strategy choices represent some manipulation of physical entities be those entities classical coins or quantum particles. For any playable game, quantum or classical, the following elements must be present

- some physical objects prepared in an initial state
- a set of manipulations that can be performed on these objects. The manipulations that are possible are the available strategy choices of the players
- some measurement of the state of these objects after the manipulations of the players have been carried out
- a mapping of the measurement results to some outcomes over which the players have different preferences

In a simple classical game, such as normal form Prisoner’s Dilemma, the initial state and measurement elements are implicit since there is a one-to-one correspondence between the elements, and the strategy choices can be directly related to the measurement results. If we wish to play a game using quantum objects and operations these elements must be made explicit and there is no longer a direct one-to-one correspondence between the strategy choices and the measurement results, in general.
This general description of any playable game, quantum or classical, allows us to make correct comparisons between different game versions. For example, in our simple example of the previous section we assume an entangled input state of the particles. The players can perform local operations on their respective particles, which in this example is restricted to just flip or no-flip. However, the operations performed by the players affect the *entire* quantum state which must be considered to be a single entity. Thus the correct quantum classical comparison to draw is between the quantum game and classical games in which the players have some ability to affect each other’s coins in some way. As we shall see, this can be achieved by the simple expedient of assuming a *classical correlated noise* on the communication of the strategy choices of the players to the device that measures this communication and assigns the payoffs.

Similarly, in comparing classical and quantum games we must, as a minimum condition, give the players strategy sets of the same size. It makes little sense, for example, to compare a classical game in which the players have only 2 strategy choices each with a quantum game in which the players have 4 available strategies each. At best we could describe this as a possible quantum extension of the classical game, but then we must compare the quantum extension with the relevant classical extension of the game in order to draw a direct comparison [4]. Thus, if we are to consider entangled quantum games we must compare these to classical games in which correlation features in some way, otherwise the comparison is essentially meaningless. In other words we must compare the relevant extensions of the game in the classical and quantum domains.

Let us consider 2-player games in which the players, Alice and Bob, have the respective available operations \( \{\alpha_1, \alpha_2, \ldots, \alpha_p\} \) and \( \{\beta_1, \beta_2, \ldots, \beta_q\} \). We shall assume that the physical objects to be manipulated are prepared in some initial state \( \psi_{in} \). The output is therefore a state of the form \( \psi_{out} = \beta_k \alpha_j \psi_{in} \) where this description is applicable to both quantum and classical game scenarios. This operational perspective also highlights the possibility that certain games can be non-commutative so that the order of play matters [23]. There is then some measurement on the output state which yields a set of possible measurement results. These measurement results are the input to the payoff function.

It is customary in entangled quantum games to consider that the players perform independent local operations on their respective particles and to identify a strategy choice with a quantum spin state (or the associated unitary operator that generates this from some specified initial state). This, however, is something of an illusion. If we consider a game of the form considered by EWL [1] then the output state is given by

\[
\psi_{out} = E^{-1} \beta_k \alpha_j E \psi_{in} = E^{-1} \beta_k E E^{-1} \alpha_j E \psi_{in} \quad (5)
\]

Here we note that \( E \) acts on the entire input state and should not be confused with the local operations that are usually assumed in treatments of quantum games. When \( E \) is the entanglement operator that produces a maximally entangled state from the ‘ground’ state of two spin-1/2 particles then the strategy
sets of the players are equivalent to the sets

\( \{ \tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_p \} \) and \( \{ \tilde{\beta}_1, \tilde{\beta}_2, \ldots, \tilde{\beta}_q \} \) \hspace{1cm} (6)

where

\[ \tilde{\alpha}_j = E^{-1} \alpha_j E \quad \text{and} \quad \tilde{\beta}_k = E^{-1} \beta_k E \] \hspace{1cm} (7)

and so the possible manipulations of the players involve directly interacting with the spin of their opponent, even if the sets \( \{ \alpha_1, \alpha_2, \ldots, \alpha_p \} \) and \( \{ \beta_1, \beta_2, \ldots, \beta_q \} \) represent strictly local operations. The identification of a so-called quantum strategy with a spin state of a single particle is, therefore, nothing more than a convenient illusion for entangled quantum games; a game involving entangled states is formally equivalent to a game in which we allow the players entanglement operations as part of their strategy sets.

If we are to make a sensible comparison between quantum and classical games in an attempt to elucidate genuine quantum behaviour we must compare like with like. For games of the EWL type where \( E \) is an entanglement operator, therefore, we must compare with the extension of the classical game to include correlation. It is critical, therefore, that the role of correlation is understood in both classical and quantum games. In the following simple game examples we demonstrate that features that may be initially considered to arise from the quantum-mechanical nature of a correlation actually arise from only the classical component of the correlation in an entangled state.

## 4 Preservation of Preferences

The simple example discussed above shows that our original preferences (over the individual measurement results) in a quantum game may be transformed by the measurement process into different preferences over the expected outcomes. It is the probabilistic mapping induced, in general, by the quantum measurement that forces the final analysis of the game in terms of a matrix of expected outcomes, and these expected outcomes can generate tuples that do not correspond to the original ordering of the measurement tuples as expressed by the players’ preferences over the individual measurement results. In our simple example it is the change of input state, whilst keeping all other elements of the physical game unchanged, that ultimately leads to this possibility of the transformation of the preferences. A general input state can be expanded in the basis of the measurement so that we have \( |\psi_0\rangle = a |00\rangle + b |01\rangle + c |10\rangle + d |11\rangle \).

It is natural to ask the question as to what are the conditions on the input state, whilst keeping all other elements unchanged, that will strictly preserve the original preferences? Before attempting to give a general perspective on this we consider some special cases.

### 4.1 \( |\psi_0\rangle = a |00\rangle + d |11\rangle \)

Let us generalize the input state given in equation (2) and ask when this preserves the original preferences. We shall not be fully general in this approach.
because we maintain the specific numerical weightings of PD to express the original preferences, but nevertheless it provides us with an insight into the way the original game can be transformed with different inputs. In fact the transformation (or otherwise) of the preferences depends upon the specific values chosen to represent those preferences. It is possible to choose numerical weightings that respect the preferences such that those preferences are also reflected in the matrix of expected outcomes for any input state, including entangled states. We shall choose, however, the usual initial preferences of PD expressed by the weightings $5, 3, 1, 0$ in order to allow some form of comparison.

With this initial entangled input state and the standard PD weightings and noting that $|a|^2 + |d|^2 = 1$ we find the matrix of expected outcomes

| A \ B | I \ F |
|-------|-------|
| I     | $(1 + 2|a|^2, 1 + 2|a|^2)$ \ $(5 - 5|a|^2, 5|a|^2)$ |
| F     | $(5|a|^2, 5 - 5|a|^2)$ \ $(3 - 2|a|^2, 3 - 2|a|^2)$ |

Table 3: the outcome matrix for the simple 2-player game of section 2 with the input state $|\psi_0\rangle = a|00\rangle + d|11\rangle$

If we write a general outcome matrix in the following way

| A \ B | I \ F |
|-------|-------|
| I     | $O_1$ \ $O_2$ |
| F     | $O_3$ \ $O_4$ |

Table 4: general form of the outcome matrix of the pure strategy 2-player game in which players have 2 choices of action each.

then we can see that the game of Prisoner’s Dilemma occurs when we have the preferences

\[ P_A : O_3 > O_1 > O_4 > O_2 \]

\[ P_B : O_2 > O_1 > O_4 > O_3 \]  

(8)

In order to strictly preserve these preferences for our given input state we therefore require that

\[ 5|a|^2 > 1 + 2|a|^2 > 3 - 2|a|^2 > 5 - 5|a|^2 \]  

(9)
which give the conditions under which the preferences of both players are preserved for this input state. We have plotted the expected outcomes for Alice as a function of \( p = |a|^2 \) in Figure 1 below. We can see that there are 3 regions with each region giving a different preference ordering for the outcomes.

![Figure 1: expected outcomes for Alice as a function of \( p = |a|^2 \).](image)

The different regions can be determined by consideration of the inequalities in equation (3) and we obtain the regions (where we exclude the boundary points)
The preferences for the players for these regions are given in the table below

|       | Alice | Bob |
|-------|-------|-----|
| Region I | $O_3 > O_2 > O_1 > O_4$ | $O_2 > O_3 > O_1 > O_4$ |
| Region II | $O_3 > O_1 > O_2 > O_4$ | $O_2 > O_1 > O_3 > O_4$ |
| Region III | $O_3 > O_1 > O_4 > O_2$ | $O_2 > O_1 > O_4 > O_3$ |

*Table 5: the preferences of the players over the expected outcomes expressed as preference relations for the different entanglement regions where the input state is given by $|\psi_0\rangle = a|00\rangle + d|11\rangle$*

In Region III we can see that the players play Prisoner’s Dilemma, but in Region II they play the game of Chicken.

We can see that for $|a|^2 > \frac{2}{3}$ (and by symmetry for $|a|^2 < \frac{1}{3}$ with a switch in the interpretation of cooperate and defect) the players just play Prisoner’s Dilemma, which might suggest that for these values of the parameter $|a|^2$ the input state is not ‘quantum’ enough to change the game. We must, however, be careful in making such a claim. Is this changing of the game by inducing a probability distribution over the measurements really a non-classical effect? Let us look at the singlet-type state as input next.

### 4.2 $|\psi_0\rangle = b|01\rangle + c|01\rangle$

Once again we maintain the numerical weightings for the original PD game and note that $|b|^2 + |c|^2 = 1$. With this initial state and the available actions of the players the expected payoff matrix is given by

|       | $I$                  | $F$                  |
|-------|----------------------|----------------------|
| $A \setminus B$ | \(5 - 5|b|^2, 5|b|^2\) | \(1 + 2|b|^2, 1 + 2|b|^2\) |
| $I$   | \(3 - 2|b|^2, 3 - 2|b|^2\) | \(5|b|^2, 5 - 5|b|^2\) |
Table 6: the outcome matrix for the simple 2-player game of section 2 with the input state $|\psi_0\rangle = b|01\rangle + c|01\rangle$

The expected outcomes for Alice are plotted in Figure 2 below as a function of $p = |b|^2$.

![Figure 2: expected outcomes for Alice as a function of $p = |b|^2$.](image)

Of course, these are the same lines as before for the input of section 4.1 but the expected outcomes they represent are different entries in the expected payoff matrix. We note from Figure 2 (and analysis of the conditions for strict preservation of the preferences) that this input state always changes the preference relations from the original with none of the new preferences over the expected payoff matrix being equivalent to a game of Prisoner’s Dilemma. The regions where the preferences over the expected outcomes change from one ordering to another are just as before and the preferences for the players for these regions are given in the table below.
It would not be surprising in quantum PD with this input state, or indeed with the input state of the previous section with $\frac{1}{2} < |a|^2 < \frac{2}{3}$, that the equilibrium payoff might be different to that of standard PD as we are no longer actually playing Prisoner’s Dilemma! The entangled states are often taken to be the most ‘non-classical’ states possible. Accordingly, it is always tempting to ascribe any unusual result when working with these states to a ‘quantum’ behaviour. However, as the analysis of Bell’s Theorem shows, pinning down non-classicality is often surprisingly subtle. In Bell’s Theorem for 2 spin-1/2 particles we need to examine correlations between sets of measurements in different spin directions in order to reveal behaviour that can be directly attributed to quantum mechanics in the sense that a ‘classical’ local hidden variable description cannot predict the correct correlations. Determining what is ‘quantum’ in a quantum game is, in our opinion, not a trivial issue.

### 4.3 A Classical Model

The feature that the quantum measurement introduces is that the measurement maps the output state of the players onto an eigenstate of the measurement with a probability distribution determined by the amplitudes of the eigenstates in the expansion of the output state in the measurement basis. We can view this as a noise process. The players are trying to communicate a particular choice, but noise on the channel gives rise to an error rate. If the players are aware of the noise and its characteristics then they can build this knowledge into their strategy. This is exactly what we have in the quantum situation. So let’s model the transmission of the players’ choices in a classical game as a communication over a noisy channel in which the players are aware of the noise characteristics and can tailor their choices accordingly. As in the quantum case we will have to deal with a matrix of expected payoffs which could lead to the playing of a different game by transformation of the preferences. Can we achieve this transformation of preferences with such a classical game over a noisy channel?

The simplest case of noise we could consider would be to model the communication as two independent channels with the same error rate $\varepsilon$. A simple calculation shows that such a case preserves the preference relations (or flips

|          | Region I          | Region II         | Region III         |
|----------|-------------------|-------------------|--------------------|
| Alice    | $O_4 > O_1 > O_2 > O_3$ | $O_1 > O_4 > O_2 > O_3$ | $O_4 > O_2 > O_3 > O_1$ |
| Bob      | $O_1 > O_4 > O_2 > O_3$ | $O_1 > O_2 > O_4 > O_3$ | $O_1 > O_2 > O_3 > O_4$ |

Table 7: the preferences of the players over the expected outcomes expressed as preference relations for the different entanglement regions where the input state is given by $|\psi_0\rangle = b|01\rangle + c|01\rangle$. 
them when $\varepsilon > \frac{1}{2}$ but the original PD is recovered with an interchange of the choices cooperate and defect). It is of course formally equivalent to the extension to a mixed game in which the players choose the same probability. A more interesting case occurs when we consider a correlated noise such that both channels, or neither, experience an error for a given symbol with a rate $\varepsilon$. Such a correlated noise is, of course, a form of classical noise. We could imagine the players’ signals sent over the same channel and experiencing the same noise, for example. With this kind of noise, if the players send a pair of symbols then either both are correct with probability $1 - \varepsilon$ or both are flipped with probability $\varepsilon$.

The expected outcomes for Alice and Bob when they communicate their choices over such a channel are

$$
A \backslash B & I & F \\
I & (3 - 2\varepsilon, 3 - 2\varepsilon) & (5\varepsilon, 5 - 5\varepsilon) \\
F & (5 - 5\varepsilon, 5\varepsilon) & (1 + 2\varepsilon, 1 + 2\varepsilon)
$$

Table 8: the matrix of expected outcomes for classical PD in which the players’ strategy choices are communicated over a channel with correlated noise such that both bits, or neither, are flipped. which is just the same as the expected outcome matrix for the input state $|\psi_0\rangle = a|00\rangle + d|11\rangle$ considered in section 4.1 for the quantum PD where $\varepsilon = 1 - |a|^2$.

So we can see that a game of PD played over channels with this kind of correlated noise will also change the preferences of the players and in Region II the players will be playing Chicken rather than PD. Thus, there is nothing particularly quantum mechanical in nature about the transformation of preferences we obtain for the entangled quantum games considered above.

### 4.4 Mixing and Correlated Noise

Whilst we have not considered the mixed game at all so far, it is instructive to examine the effect of having a classical correlated noise when we extend a (classical) game by mixing. As before, we begin with classical PD but now assume the players will adopt a probabilistic strategy so that Alice chooses to flip (defect) with probability $p$ and Bob chooses to flip with probability $q$. As in the previous section they attempt to communicate their choice over a channel that experiences a correlated noise so that either both bits representing the players’ choice are transmitted error free, or both are flipped. If we assume the error rate is $\varepsilon$ as before then the joint probabilities for obtaining the measured results $I$ and $F$ are as follows:
The expected outcomes for Alice and Bob now become

\[
\langle O_A(\varepsilon) \rangle = [3 + 2p - 3q - pq](1 - \varepsilon) + [1 - p + 4q - pq] \varepsilon
\]

\[
\langle O_B(\varepsilon) \rangle = [3 + 2q - 3p - pq](1 - \varepsilon) + [1 - q + 4p - pq] \varepsilon
\]

The (1 - \varepsilon) part of this expected outcome is just the usual expected outcome from the mixed PD without any noise. In this noiseless case the players are forced to the equilibrium position (F, F) just as the non-mixed game and the expected outcome is (1, 1). However, the noise term now changes this expected outcome. If \( \varepsilon = 1 \) then the players would play the equilibrium position (I, I) with an expected outcome of (1, 1). The actual choice of probability the players make is a function of the error rate \( \varepsilon \) and we can see that their best response is given by the choice \( p = q = 1 - \varepsilon \). This yields the expected outcomes for the players

\[
\langle O_A(\varepsilon) \rangle = \langle O_B(\varepsilon) \rangle = 1 + 5\varepsilon - 5\varepsilon^2
\]

This expected outcome is plotted above in Figure 3 and we can see that the maximum value is obtained when \( \varepsilon = \frac{1}{2} \) and this gives an expected outcome for the players of \( \frac{9}{4} \) which is an improvement on their equilibrium output in the noise-free case (or the all-noise case). This is, of course, nothing more than a uniform distribution of the possible outcomes and the same result is obtained for quantum games in which there is maximal decoherence [19, 20]. In this case, it is the correlated, but classical, noise that is giving an enhanced equilibrium payoff for the players in all regions except the boundary points (noise-free or all-noise).

### 4.5 Noise in the Quantum Game

Now let us consider the case where the players play their version of Prisoner’s Dilemma with the input state \( |\psi_0\rangle = a |00\rangle + d |11\rangle \). We shall consider that they attempt to communicate their choice by sending their respective particles over some channel to be measured. Thus we now have a quantum channel. We shall
Figure 3: Expected outcomes for Alice as a function of the error rate $\epsilon$.

suppose that there is some noise source on this channel. In this case we shall not be too concerned with the details of the noise, but merely suppose that it is sufficient to rapidly suppress the off diagonal coherences in the density matrix. Such a suppression of off-diagonal coherences is, of course, a general feature of open quantum systems [26-28]. If we restrict the available operations, as before, to this binary choice of whether to flip or not in the measurement basis, then after this decohering noise process the density matrix description of the state that arrives at the measurement apparatus is given, for each of the possible choices of the players, as
The expected outcomes for the players are given by the expected outcome matrix

\[
\begin{array}{c|cc}
A \setminus B & I & F \\
\hline
I & \left( 1 + 2 |a|^2, 1 + 2 |a|^2 \right) & \left( 5 - 5 |a|^2, 5 |a|^2 \right) \\
F & \left( 5 |a|^2, 5 - 5 |a|^2 \right) & \left( 3 - 2 |a|^2, 3 - 2 |a|^2 \right)
\end{array}
\]

Table 9: the matrix of expected outcomes for the 2-player game with input state \( |\psi_0\rangle = a |00\rangle + d |11\rangle \) in which the players particles are sent over a noisy quantum channel that leads to suppression of the off-diagonal coherences.

Which is precisely the same as that for the game played in the noiseless case considered in section (4.1). In other words, the off-diagonal components in the density matrix in the noiseless case are not contributing to the determination of the expected outcomes. This is only to be expected since any single pair of measurements on the separate systems can, at most, only access half the information contained within the quantum correlation [29,30]. If we expressed our entangled state in the Schmidt basis, and made measurements of the Schmidt observables (this is considered within the context of quantum games in [10,11]), then we would access precisely half the information contained within the quantum correlation [29,30]. In other words, the enhanced equilibrium obtained for quantum games in which a single measurement is made in each the subspaces is a result of a classical correlation because the off-diagonal interference terms are not being accessed in such a measurement.

We note that a similar problem has been studied in more generality by Shimamura et. al. [18] in which they consider the difference between an entangled state input and its classical counterpart in games of the EWL type (see also Chen et. al. [19] who consider a decoherence protocol for Quantum Prisoner’s Dilemma). Both [18] and [19] are different to the situation we envisage here in which the decoherence occurs during the transmission of the quantum states.
to the measurement device (or referee). In [18] the referee employs the disen-
tangling transformation before measurement and the full space of local unitary
operations is allowed by each player. The formal correspondence discussed in
section 3 above is no longer applicable because the decoherence destroys the
symmetry and the placement of the decohering process in the chain of events
becomes significant. Furthermore in both [18] and [19] the referee performs a dis-
entangling operation before measurement, which amounts to a re-entanglement
in the case of a separable mixed state input.

Our purpose here is not to examine the effect of decoherence in general (see
for example, [20]) but to provide another illustration that, for simple 2-player
games of the form consider here, it is correlation, and not quantum correlation,
that is the interesting feature. Of course a more general treatment is required to
determine the game types and conditions under which quantum correlations do
become significant. We shall examine the ramifications of this, and the results
of the previous sections, for the interpretation of games such as EWL [1] and
MW [21] elsewhere.

5 A General Approach

In the preceding sections we examined some very simple, and restricted, quan-
tum games in order to gain some insight into the role of correlation in quantum
games. An obvious question is whether the results obtained depend in some
way on the nature of the restriction imposed (the choices cooperate or defect
being the only operations available to the players).

Consider a game played with quantum mechanical objects and operations
that obey the laws of quantum mechanics. The game consists of the following
[23-25]:

- An input state $|\psi_0\rangle$ that is assumed to be known by the players
- Actions available to player $A$ described by a finite set of unitary operators
  $\{\hat{\alpha}_1, \ldots, \hat{\alpha}_i, \ldots, \hat{\alpha}_n\}$
- Actions available to player $B$ described by a finite set of unitary operators
  $\{\hat{\beta}_1, \ldots, \hat{\beta}_j, \ldots, \hat{\beta}_m\}$
- The actions of the players on the input state produce some output state
  $|\psi_{ij}\rangle$ that is characterized by the choice of $\hat{\alpha}_i$ and $\hat{\beta}_j$ by players $A$ and $B$,
  respectively. There are $n \times m$ possible output states from this game for a
given input state $|\psi_0\rangle$
- A projective measurement $\hat{M}$ on the output state that produces an eigen-
  state $|m_i\rangle$ of the measurement operator where there are $r$ such eigenstates.
  We assume non-degenerate eigenvalues so that each measurement result
  can be unambiguously identified with a measurement eigenstate.
• The players each have a different preference relation over the measurement eigenstates. Accordingly, we shall use the terms preference basis and measurement basis interchangeably. The preference relations therefore induce a preference relation for each player over the set of possible output states $|\psi_{ij}\rangle$.

• We shall encapsulate the notion of preference by assigning a numerical value to each measurement eigenstate for each player such that a higher numerical value indicates a greater preference for that player.

• We shall assume that (nominally) each player has some object upon which to act so that the Hilbert space is described by $H = H_A \otimes H_B$. Note that this does not, therefore, imply that the unitary operations available to the players act only in their respective subspaces.

The players, as noted above, have some preference over the measurement eigenstates so that the output state produced can be expressed in this measurement, or preference, basis as follows

$$|\psi_{ij}\rangle = \sum_{i=1}^{r} a_i |\psi_{ij}\rangle |m_i\rangle$$

(15)

Upon measurement, the result $|m_l\rangle$ is mapped to a numerical value in accordance with the preference relations as

$$(|m_l\rangle) \rightarrow (\omega^A_l, \omega^B_l)$$

(16)

where $\omega^A_l$ is the outcome for player A if the result of the measurement yields the eigenvalue $|m_l\rangle$. We can formally combine the measurement and assignment of outcomes into the single Hermitian ‘outcome’ operators

$$\hat{\omega}_A = \sum_{i=1}^{r} \omega^A_i |m_i\rangle \langle m_i|$$

$$\hat{\omega}_B = \sum_{i=1}^{r} \omega^B_i |m_i\rangle \langle m_i|$$

(17)

In general, the output state will be a superposition over the preference bases and will not be an eigenstate of the measurement operator (or equivalently the outcome operators). There will therefore be a distribution over the output tuples for any given choice of $\hat{\alpha}_j$ and $\hat{\beta}_k$. Thus for each choice of $\hat{\alpha}_j$ and $\hat{\beta}_k$ there will be an average outcome tuple expressed as the expected value of the outcome operators

$$(\langle \hat{\omega}_A \rangle_{jk}, \langle \hat{\omega}_B \rangle_{jk}) = \left( \sum_{i=1}^{r} \omega^A_i |\langle m_i | \psi_{jk}\rangle|^2 , \sum_{i=1}^{r} \omega^B_i |\langle m_i | \psi_{jk}\rangle|^2 \right)$$

(18)
Each expected outcome tuple can therefore be thought of as an entry in an \( n \times m \) matrix of outcome tuples, just as we would describe any 2-player game. Thus the quantum mechanical game is entirely equivalent, as far as the players are concerned, to a classical game in which the outcomes relating to the choices of the players are described by this matrix. The fact that these outcomes are derived from a quantum measurement and the resultant probabilities is utterly irrelevant. The players play the game according to the outcomes expressed in this expected payoff matrix. The game is defined not by their original preferences over the measurement results, but by the induced preferences as expressed by the matrix of expected outcomes. In effect, the quantum mechanical measurement has the potential to change the players’ preferences to those expressed in the expected payoff matrix. So although the players start off with a set of preferences over the results of the measurement they act as if they had a new set of preferences, given by the expected outcomes. The players choose their strategies according to this new matrix. It is this matrix which defines the actual game they are playing.

The expectation values defined in equation (18) are those considered by Cheon and Tsutsui \[17\], in which they show that each \( \langle \omega_A \rangle_{jk} \) and \( \langle \omega_B \rangle_{jk} \) can be considered to arise from a pseudo-classical part and quantum interference terms. So each element in our matrix of expected payoffs can be thought to arise partly from some quantum interference term in which a different quantum interference term is obtained for each choice of operation (strategy) by Alice and Bob. So whilst each separate entry into the payoff matrix may be thought of in this manner, the entire matrix is just a set of classical probabilities that can, as we argue below, be reproduced by modelling the game classically as a communication channel in which we allow the possibility of classical correlated noise. Once again, we emphasize that it is the matrix of expected outcomes that defines the game the players actually play, and not whatever complicated physical mechanism we have used to produce this matrix.

It is at this point we must ask what is quantum mechanical about games of this type? It is irrelevant to the players whether the expected payoff matrix that defines the game they are playing has been generated by some quantum process, or whether the entries in the matrix are assigned to measurement of classical coins, just as in any standard classical 2-player game. There is nothing particularly quantum-mechanical about a matrix of tuples and any quantum game of this form can be implemented entirely by classical objects with a given functional mapping of measurement to outcomes. The specific functional decomposition of the game that has generated the final game function is irrelevant to the players; the quantity they analyze is the matrix of expected outcomes. We believe that van Enk and Pike \[3\] were right to be uncomfortable about the ‘quantum’ claim for games of this type.

A matrix of expected outcome tuples is generated in any pure strategy game, quantum or classical, where there is some probability distribution over the measurement results. In the quantum case the measurement induces this distribution, but we can similarly imagine a classical game in which the measurement process is imperfect. The example of the implementation of a classical game...
as a communication over a noisy channel considered above is just one way of realizing such a distribution of measurement results in the classical case. For a classical game with a distribution over \( r \) possible measurement results, \( m_i \), we have the conditional distribution \( P(m_i \mid \alpha_j, \beta_k) \) which gives the probability of obtaining the measurement result \( m_i \) given that the players choose the operations \( \alpha_j \) and \( \beta_k \). In this case the expected outcomes for Alice are

\[
\langle \omega_A \rangle_{jk} = \sum_{i=1}^{r} P(m_i \mid \alpha_j, \beta_k) \omega_i^A
\]

with a similar expression for Bob’s outcomes. By modelling the classical game as a communication of a binary number representing the strategy choice of the players we can see that a classical noise process on the channel such that the channel transition probabilities are

\[
P(m_i \mid \alpha_j, \beta_k) = |\langle m_i \mid \psi_{jk} \rangle|^2
\]

will reproduce the results of the quantum game. Thus by assuming a channel with a classical correlated noise we can reproduce the results of a quantum entangled game where we assume a single projective measurement is performed on the resultant output state. The game, classical or quantum, can be thought of as a communication channel where the input symbols are the \( n \times m \) strategy choices \( \alpha_j \beta_k \) and the \( r \) output symbols are the measurement results \( m_i \). We are free to model the noise on such a classical channel with any legitimate set of conditional probabilities and these probabilities represent a classical noise (although we may need some peculiar classical noise process to generate the results of a particular quantum game, it is still classical).

The modelling of a pure strategy game as a communication channel in which the input symbol is chosen according to preferences over the output symbols is instructive. For convenience we shall assume the players have strategy sets of equal size where \( n = m = 2^\mu \) so that a classical game can be represented as a channel over which the players each communicate an \( \mu \)-bit binary string. We can implement this classical game using spin-1/2 particles prepared in the state \(|00...0\rangle\) in some spin basis where the players can perform a flip or a no-flip operation in this basis and the measurement is performed on each particle in this basis. This can be considered to be an expensive quantum implementation of the classical game. The expensive quantum implementation can also be thought of as a quantum communication channel over which qubits are transmitted.

The expensive quantum implementation can now be altered in 3 obvious ways:

- the initial state is prepared as \(|\bar{0}\bar{0}...\bar{0}\rangle\) in some other basis
- the initial state is \(|00...0\rangle\) but the players are given flip and no-flip in some basis aligned at some angle to the basis of the input states
- the initial state is \(|00...0\rangle\) and the players are given flip and no-flip in this basis, but the measurement of the qubits is now performed in some other basis aligned at some angle to the input basis
Of course we can imagine any combination of these things, or consider different bases for each qubit, for example. The point here is that we have changed the quantum implementation so that the output state produced by the players is no longer an eigenstate of the measurement operator and this induces a distribution over the measurement results, which in turn induces preferences over the expected outcomes that can lead to a different game form than that described by the preferences over the individual measurement results. In effect, the measurement induces noise on the quantum communication channel.

This changing of the quantum implementation of the game can be thought of as a kind of game extension in which the original game pertains when we adjust the alignments to yield the zero noise case. In order to draw a sensible quantum/classical comparison, therefore, one must compare the noisy quantum channel with a noisy classical channel in the context of the application to the description of a game. It is clear that in the unentangled case the expensive quantum implementation can be modelled as a classical game where $2\mu$ bits replace the $2\mu$ qubits such that the noise characteristics of the classical channel reproduce the measurement-induced noise characteristics of the quantum channel.

If we now extend the quantum implementation to allow entanglement, the above arguments show that, for the situation where a projective measurement is made on the output state in the quantum case, the game can be modelled as a classical communication channel in which we allow the possibility of correlated noise. In both cases the players transmit $2\mu$ bits or qubits over the channel. The single projective measurement is not sensitive enough to distinguish between classical and quantum correlations in these 2-player pure strategy finite games. This is essentially for the same reason that we require more than just a single joint probability distribution to distinguish between hidden variable models and quantum mechanics in tests of Bell’s inequality. In tests of local realism we need to establish the non-existence of the joint distribution $P(A, B, C)$ that correctly reproduces the marginal distributions $P(A, B)$ and $P(A, C)$, for example. Such a distribution only exists if the marginals satisfy the Bell inequality. In a sense, the classical communication over a noisy channel can be thought of as a hidden variable implementation of the quantum game and from this perspective it is not surprising that a 2-player quantum game of the form we have considered does not display non-classical behaviour.

Other authors have considered more general formulations of quantum games. Of particular note is the work of Lee and Johnson [16] who show that finite classical games are a strict subset of quantum games, as we would expect. In the context of communication channels this is expressing the fact that any classical communication channel with a finite input alphabet is a subset of the possible quantum channels. It is how these channels are exploited that determines whether they display quantum or classical characteristics. Lee and Johnson show that a given classical game can have a more efficient implementation using quantum objects in terms of the relative number of bits and qubits, respectively. This is reminiscent of the ability of quantum channels to transmit classical information using fewer qubits [31]. Here we are interested in a converse (and
more restricted) question: whether a quantum game can be modelled by a game played with classical objects in the context of the 2-player pure games with finite strategy sets in which a single projective measurement is made on the output. We believe that our analysis offers an insight into where we need to look for game properties that display necessarily quantum-mechanical features.

The more general form of the 2-player game we have considered assumes a finite strategy set for the players, and that a single projective measurement is made on the resultant output state. Furthermore we have assumed a pure strategy game. The simple example of the classical mixed game with correlated noise considered above shows that similar considerations may also apply in the more general mixed game case. We consider such situations elsewhere.

6 Discussion

There is no doubt that the pioneering work [1,2] that brought together game theory and quantum mechanics represented a new and original direction in both fields. There has been much work since on various quantum game scenarios, usually focusing on the use of entangled states in games [1-21] (we have referenced only a very small selection of the work that has been done). In our previous work [23-25] we argued that a game should be seen as something that can actually be played. In other words there is a physical implementation of a game with real objects that obey the laws of physics. With this perspective the necessary elements required to actually play a game can be identified. These elements are: preparation of some initial state, operations by the players on that state, a resulting output state that is subject to a measurement, and a mapping of the results of that measurement to given outcomes. In order to call such a thing a game, rather than just an experiment in physics, we require that the players have some preference over the outcomes, which ultimately determines which operation on the input state they will choose.

In the games we’ve discussed here we have assumed that a single measurement on the output state produced by the players is performed. This measurement remains fixed however many times we play the game. The simple restricted examples examined consider a measurement of spin in the $z$-direction for Alice’s particle and a measurement of spin, also in the $z$-direction, for Bob’s particle. Experimentally, therefore, we are only accessing information about the correlations between these two specific observables. This measurement cannot uncover the full richness of the quantum correlations inherent in an entangled state of 2 particles; it is only accessing some of the information about the quantum correlations [29,30]. The specific examples of games considered are only allowing us to probe correlations between the spin-$z$ and spin-$z$ measurements, and that can’t give us enough information to decide whether it’s quantum or classical behaviour we’re seeing. The information that can be recovered about the correlations from this kind of fixed measurement is not sufficient to distinguish between the classical and quantum nature of the correlation. The more general analysis for 2-player pure strategy games in which the players can choose
from finite strategy sets shows that a single measurement of the output is also insufficient to access the quantum nature of the correlation. In order to do that we need to compare correlations for multi-player games or in 2-player quantum games in which different measurement angles are selected, for example, just as we need to in establishing the experimental violation of Bell’s inequality [6-9].

The general analysis of these 2-player pure strategy games shows that the same game can be played either with quantum objects in which a single measurement is made, or as a classical communication channel in which the players know the noise characteristics. Where correlation features in the quantum game, when entanglement is introduced, the classical version of the game as a communication channel requires a classical correlated noise. The important point to note is that the single measurement in the quantum case reduces everything to a set of probabilities that can be achieved by an equivalent classical communication channel. We do not require quantum objects to play games of this form. In other words, given a set of operations and an associated payoff matrix, there is no way for the players to determine whether they are playing with quantum or classical objects in these games.

The reason we can reproduce these general 2-player pure strategy quantum games as a classical communication channel is that we assume a single (joint) measurement that produces the probability distribution over the results that are then mapped to the expected outcomes via some payoff function. With a single joint measurement the probabilities can always be reproduced by a classical system. Overall, the game is a function that takes some inputs and performs a computation on those inputs. If that computation can be achieved by classical systems then it seems to us that the underlying game is essentially classical even if implemented by quantum objects.

In mathematical terms, then, a 2-player game is nothing more than a function that takes a pair of inputs representing the choices of the players and maps these to outcomes. The physical elements required to play a game are nothing more than a particular functional decomposition of this overall game function. In these terms, therefore, we can see that for the initial example based on the MW protocol [21] the function that represents the game of Chicken can be decomposed into functional elements that look like a version of quantum Prisoner’s Dilemma, or it can be functionally decomposed as a game played over a classical communication channel with correlated noise. In general, therefore, if we are to observe genuine quantum behaviour in a game we must consider richer game structures that allow us to probe the quantum regime of the correlation, and effectively perform a quantum computation on the inputs in order to produce the outcomes. Games that allow us to do this are multi-player games [9,32] or games in which the final outcomes are determined from comparison of the results of a sequence of games in which different measurements are made.

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