Vector Boson Scattering in the Standard Model –
an Overview of Formulae

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Abstract

Tree-level scattering amplitudes of longitudinally polarized electroweak vector bosons in
the Standard Model are calculated using Mathematica package Feyncalc. The modifications
of low-energy theorems for longitudinally polarized $W$ and $Z$ in the Standard Model are
discussed.

1 Introduction

One of the open questions in high energy physics is the mechanism of electroweak symmetry
breaking (EWSB). The physics that breaks electroweak symmetry is responsible for giving the $W$
and $Z$ their masses. Since a massive spin-one particle has three polarizations, rather than the
two of a massless mode, the new physics must supply degrees of freedom to be swallowed by the
$W$ and $Z$. These new degrees of freedom are the longitudinal polarizations $W_L$ and $Z_L$ of
the vector bosons. Therefore the interactions of the longitudinal components of the vector bosons
could provide a good way to probe the interactions of the symmetry breaking sector.

The interactions of $W_L$ and $Z_L$ in high energy processes are usually studied by means of different
production mechanisms of vector bosons followed by their purely leptonic decays e.g. $W^\pm \rightarrow l^\pm \nu$
and $Z \rightarrow l^+l^-$ ($l = e, \mu$), referred to as gold-plated channels. One of the production mechanism
is through light fermion anti-fermion i.e. $q\bar{q}$ or $e^+e^-$ annihilation. This yields vector boson pairs
that are mostly transversely polarized and is usually a background to the other processes. The
important exception is the production of longitudinally polarized vector bosons through new vector
resonance.

A second mechanism for producing longitudinal vector boson pairs in hadron colliders is gluon
fusion. The initial gluons turn into two vector bosons via an intermediate state that couples to both
 gluons and electroweak vector bosons like the top quark or new colored particles of a technicolor
model.

Finally, there is the vector-boson fusion process when vector bosons are radiated by colliding
fermions and then rescattered. When the fermions are quarks then the process of vector boson
scattering is considered as a subprocess of subprocess in hadron collision. Sensitivity of different
types of colliders to the above mentioned processes has been discussed in a series of articles.

In this paper I calculate exact tree-level scattering amplitudes of longitudinally polarized elec-
drawek vector bosons in the Standard Model. The aim of the present paper is to check indepen-
dently the existing results, in particular because I think there has been a minor error in

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an earlier paper [5]. As a consistency test I use the high-energy \((E \gg m_W)\) behaviour of a pure gauge amplitudes. Their quadratic growth \((\sim E^2)\) should be canceled by introducing Standard Model Higgs boson.

It has been shown that some universal low-energy theorems (LET) for the scattering of longitudinally polarized \(W\) and \(Z\) hold [8]. These theorems are valid below the scale \(\Lambda_{SB} \sim 1\text{ TeV}\), provided that the symmetry breaking sector contains no particles much lighter than \(\Lambda_{SB}\). The derivation in [8] shows that in this case the LET are given by \(SU(2)_L \times U(1)_Y\) gauge interactions of vector bosons alone. The particle which modifies pure gauge amplitudes (and LET) in the SM is the Higgs boson. To see these modifications, I plot the complete amplitudes for different values of Higgs boson mass and compare them with the pure gauge contributions.

## 2 Scattering Amplitudes

Calculation of the scattering amplitudes of the gauge bosons by hand is a tedious task. The use of function `SpecificPolarization` of the Mathematica package FeynCalc [9] has substantially reduced algebraical manipulations.

Longitudinal polarization picks up a specific direction in space. Thus the amplitudes written in terms of Mandelstam variables \(s, t, u\) do not describe scattering of longitudinally polarized particles in all Lorentz frames. For the process \(V(p_1) + V(p_2) \rightarrow V(k_1) + V(k_2)\) the function `SpecificPolarization` uses the following representation of the longitudinal polarization vectors \(\varepsilon(p, L)\)

\[
\begin{align*}
\varepsilon^\mu(p_1, L) &= F^\mu_L(p_1, k_1, k_2) \\
\varepsilon^\mu(p_2, L) &= F^\mu_L(p_2, k_1, k_2) \\
\varepsilon^\mu(k_1, L) &= F^\mu_L(k_1, p_1, p_2) \\
\varepsilon^\mu(k_2, L) &= F^\mu_L(k_2, p_1, p_2)
\end{align*}
\]

where

\[
F^\mu_L(r, a, b) = \frac{r^\mu (b \cdot r + a \cdot r) - (a + b)^\mu r^2}{\sqrt{r^2((b \cdot r + a \cdot r)^2 - r^2(b + a)^2)}}
\]

Using this function, we get the longitudinal polarization only in the CM system, where \(p_1 + p_2 = k_1 + k_2 = 0\).

Table 1 summarizes tree-level contributions (contact graphs are not listed) to the scattering amplitudes of the gauge bosons in the SM in the \(U\)-gauge. Contributions to the process \#n from graph with \(V\)-boson exchange in the \(k\)-channel are denoted as

\[
\mathcal{M}^{(n)}_{kV} = -\frac{g_1 g_2}{k - m_V^2} A^{(n)}_{k}
\]

and from the contact graph

\[
\mathcal{M}^{(n)}_c = g A^{(n)}_c
\]

| process # | process | \(s\) | \(t\) | \(u\) | references |
|-----------|----------|-------|-------|-------|------------|
| 1         | \(W_1^+W_2^- \rightarrow Z_3Z_4\) | \(H\) | \(W\) | \(W\) | [4, 5]     |
| 2         | \(W_1^+Z_2 \rightarrow Z_3W_4^+\) | \(W\) | \(W\) | \(H\) | [5]        |
| 3         | \(W_1^+W_2^+ \rightarrow W_3^+W_4^+\) | \(Z, \gamma, H\) | \(Z, \gamma, H\) | [6] |
| 4         | \(W_1^+W_2^- \rightarrow W_3^+W_4^-\) | \(Z, \gamma, H\) | \(Z, \gamma, H\) | [7] |

Table 1: Individual processes together with exchanged particles in different channels \((s, t, u)\). \(W, Z, \gamma\) are electroweak gauge bosons and photon and \(H\) is the SM Higgs boson. Contact graphs contribute to each of the process and are not listed.
with $g_1$, $g_2$ and $g$ denoting relevant coupling constants. Amplitudes given by the low-energy theorems are denoted as $\mathcal{M}^{(n)}_{\text{LET}}$. Note that in the paper of Bento and Llewellyn Smith the $WZ \rightarrow WZ$ amplitude is claimed to be (in the current notation)

$$\mathcal{M}^{(2)} = \mathcal{M}^{(2)}_c + \mathcal{M}^{(2)}_{uH} + \mathcal{M}^{(2)}_{uW} + \mathcal{M}^{(2)}_{tW}$$

(wrong)

and is also calculated in this way. This is wrong, because $W$ is exchanged in $s$ and $t$ (or $u$) channel.

Explicit formulae for the scattering amplitudes are somewhat difficult to read and are postponed to the appendix. Table 2 summarizes relations among different $A$ parts of gauge amplitudes as defined in (3) and (4). These relation are almost obvious, nevertheless they can be used as a check. Besides these relations we have $A_{1W}^{(1)} |_{m_Z=m_W} = -A_{1W}^{(2)} |_{m_Z=m_W}$ but $A_{1W}^{(2)} \neq -A_{1W}^{(1)}$. Note that the

$$A^{(n)}_1 = A^{(n)}_2$$

because the longitudinal term in propagator ($\sim q^2 q^2$) does not contribute to the amplitudes with exchange of neutral gauge boson.

### 2.1 $W^+(k_1) + W^-(k_2) \rightarrow Z(k_3) + Z(k_4)$

On the tree-level the pure gauge contributions are from $W$-exchange in $t$ and $u$ channels and the contact graph

$$\mathcal{M}^{(1)}_{kW} = \frac{-g^2 \cos^2 \theta_W}{k - m_{W}^2} A^{(1)}_{kW} \quad k = t, u$$

$$\mathcal{M}^{(1)}_c = -g^2 \cos^2 \theta_W A^{(1)}_c$$

(5) (6)

Complete gauge amplitude grows linearly with $s$

$$\mathcal{M}^{(1)}_{\text{gauge}} = \mathcal{M}^{(1)}_{tW} + \mathcal{M}^{(1)}_{uW} + \mathcal{M}^{(1)}_c$$

$$= \mathcal{M}^{(1)}_{\text{LET}} - \frac{g^2}{2 \rho^2 \cos^2 \theta_W} \frac{(1 - x^2) - 2 \cos^2 \theta_W \rho (1 + x^2)}{} + O \left( \frac{m^2_W}{s} \right)$$

(7)

where

$$\mathcal{M}^{(1)}_{\text{LET}} = \frac{g^2}{4 \rho m^2_W}, \quad \rho = \frac{m^2_Z}{m^2_W \cos^2 \theta_W}$$

and $x \equiv \cos \theta_{cm}$.

with $\theta_{cm}$ an angle between $k_1$ and $k_3$ in CMS.

In the SM this growth is canceled by exchange of the Higgs boson in the $s$-channel

$$\mathcal{M}^{(1)}_{sH} = -\frac{g^2 m_W m_Z}{\cos \theta_W} \frac{(\varepsilon_1 \cdot \varepsilon_2) (\varepsilon_3 \cdot \varepsilon_4)}{s - m^2_H + i m_H \Gamma_H}$$

(8)

The amplitude $\mathcal{M}^{(1)}_{sH}$ has, for longitudinal polarizations, the high-energy ($E \gg m_H, m_W$) expansion (for exact formulae see appendix)

$$\mathcal{M}^{(1)}_{sH} = -\frac{g^2 m_W m_Z}{\cos \theta_W} \left[ \frac{s}{4 m^2_Z m^2_W} + O(s^0) \right] = -\frac{g^2 \sqrt{\rho}}{4 m^2_W} s + O(s^0)$$

There is the interchange $t \leftrightarrow u$ in (3) in comparison with this paper.
so the cancellation occurs for $\rho = 1$.

Figure 1 shows dependence of the complete amplitude $M^{(1)} = M^{(1)}_{\text{gauge}} + M^{(1)}_{\text{LET}}$ on the Higgs boson mass, $m_H$, in the region $m_W < \sqrt{s} < 1$ TeV and compares it with $M^{(1)}_{\text{gauge}}$ and $M^{(1)}_{\text{LET}}$. Note that $M^{(1)}_{\text{gauge}}$ differs from $M^{(1)}_{\text{LET}}$ in the limit $s \to \infty$ by a constant term. Numerical values of all parameters are the same throughout the paper.

2.2 $W^+(k_1) + Z(k_2) \rightarrow Z(k_3) + W^+(k_4)$

On the tree-level the pure gauge contributions are from $W$-exchange in $s$ and $t$ channels and the contact graph

$$M^{(2)}_{kW} = -\frac{g^2 \cos^2 \theta_W}{k - m_W^2} A^{(2)}_{kW}, \quad k = s, t \tag{9}$$

$$M^{(2)}_c = -g^2 \cos^2 \theta_W A^{(2)}_c \tag{10}$$

The asymptotic behaviour of the complete tree-level gauge amplitude is

$$M^{(2)}_{\text{gauge}} = M^{(2)}_{sW} + M^{(2)}_{tW} + M^{(2)}_c$$

$$= M^{(2)}_{\text{LET}} - \frac{g^2 (2x(1-x) + \rho \cos^2 \theta_W (3 + 2x - x^2))}{4\rho^2 \cos^2 \theta_W (1-x)} + O\left(\frac{m_W^2}{s}\right) \tag{11}$$

where a low-energy amplitude is usually defined as

$$M^{(2)}_{\text{LET}} = \frac{g^2 u}{4\rho m_W^2} = -\frac{g^2 s}{8\rho m_W^2} (1 + \cos \theta_{cm}) + O(s^0).$$

The exchange of the Higgs boson in the $u$-channel

$$M^{(2)}_{uH} = -\frac{g^2 m_W m_Z (\varepsilon_1 \cdot \varepsilon^*_4) (\varepsilon_2 \cdot \varepsilon^*_3)}{u - M_H^2} \tag{12}$$

has for longitudinal polarizations the high-energy expansion

$$M^{(2)}_{uH} = \frac{g^2 \sqrt{\rho}}{8m_W^2} (1 + \cos \theta_{cm}) s + O(s^0) = -\frac{g^2 \sqrt{\rho}}{4m_W^2} u + O(s^0).$$
Figure 2 shows complete tree-level amplitude $\mathcal{M}^{(2)}$ for different $m_H$ and compares it with $\mathcal{M}^{(2)}_{\text{gauge}}$ and $\mathcal{M}^{(2)}_{\text{LET}}$.

2.3 $W^+(k_1) + W^+(k_2) \to W^+(k_3) + W^+(k_4)$

On the tree-level graphs with $\gamma$ and $Z$ exchange in $t$ and $u$ channels and contact graph contribute. In the Standard Model, Higgs boson exchange in $t$ and $u$ channels gives the desired high energy behaviour.

$$\mathcal{M}^{(3)}_{kZ} = -g^2 \cos^2 \theta_W A^{(3)}_{kZ} \quad \mathcal{M}^{(3)}_{k\gamma} = \frac{-e^2}{k} A^{(3)}_{k\gamma} \quad k = t, u$$

Because longitudinal term in the $Z$ boson propagator does not contribute we have

$$A^{(3)}_{k\gamma} = A^{(3)}_{kZ}.$$  

Contact graph amplitude for longitudinally polarized gauge bosons has the form

$$\mathcal{M}^{(3)}_c = g^2 A^{(3)}_c = \frac{g^2 s}{8m_W^2} (-8m_W^2 + 3s - s \cos^2 \theta_{cm})$$  \hspace{1cm} (13)

The gauge amplitude can be written as

$$\mathcal{M}_{\text{gauge}}^{(3)} = -g^2 \cos^2 \theta_W \left[ \frac{A^{(3)}_{kZ}}{t - m_Z^2} + \frac{A^{(3)}_{uZ}}{u - m_Z^2} \right] - g^2 \sin^2 \theta_W \left[ \frac{A^{(3)}_{k\gamma}}{t} + \frac{A^{(3)}_{u\gamma}}{u} \right] + \mathcal{M}^{(3)}_c$$  \hspace{1cm} (14)

Expanding this expression in powers of $s$ gives

$$\mathcal{M}_{\text{gauge}}^{(3)} = -g^2 \cos^2 \theta_W \left[ \frac{3 - \cos^2 \theta_{cm}}{8m_W^2} s^2 - \frac{3m_Z^2}{4m_W^2} s + O(s^0) \right]$$

$$- g^2 \sin^2 \theta_W \left[ \frac{3 - \cos^2 \theta_{cm}}{8m_W^2} s^2 + O(s^0) \right]$$

$$+ \left[ \frac{3 - \cos^2 \theta_{cm}}{8m_W^2} s^2 - \frac{s}{m_W^2} \right]$$  \hspace{1cm} (15)
Figure 3: Tree-level amplitudes of the process $W^+ W^+ \to W^+ W^+$. 

Quadratic (in $s$) divergencies are canceled and including constant terms of the order $O(s^0)$ we get 

$$
\mathcal{M}^{(3)}_{\text{gauge}} = \mathcal{M}^{(3)}_{\text{LET}} - g^2 \left( \frac{3 + x^2 - \rho \cos^2 \theta_W (6 - 4 \rho + 10 x^2 - 12 \rho x^2)}{2 \rho^2 (1 - x^2) \cos^2 \theta_W^2} + O\left(\frac{m_W^2}{s}\right) \right)
$$

where 

$$
\mathcal{M}^{(3)}_{\text{LET}} = -\frac{g^2 s}{4 m_W^2} \left(4 - \frac{3}{\rho}\right).
$$

This linear divergence should be canceled by Higgs exchange in $t$ and $u$ channels 

$$
\mathcal{M}^{(3)}_H = -g^2 m_W^2 \left[ \frac{(\varepsilon_1 \cdot \varepsilon_4^*) (\varepsilon_2 \cdot \varepsilon_4^*)}{t - m_H^2} + \frac{(\varepsilon_1 \cdot \varepsilon_4^*) (\varepsilon_2 \cdot \varepsilon_4^*)}{u - m_H^2} \right]
$$

with high-energy expansion 

$$
\mathcal{M}^{(3)}_H = \frac{g^2 s}{4 m_W^2} + O(s^0).
$$

Comparing (17) and (19) we see that $\rho = 1$ ensures desired cancellation.

2.4 $W^+(k_1) + W^-(k_2) \to W^+(k_3) + W^-(k_4)$

In this case we have to consider $Z$, $\gamma$ and Higgs boson exchange in $s$ and $t$ channels and contact graph. As in the previous sections we write 

$$
\mathcal{M}^{(4)}_{kZ} = -\frac{g^2 \cos^2 \theta_W}{k - m_Z^2} A^{(4)}_{kZ} \quad \mathcal{M}^{(4)}_{k\gamma} = -\frac{e^2}{k} A^{(4)}_{k\gamma} \quad k = s, t
$$

The results for the $A$ parts of the amplitude can be obtained directly from corresponding formulae for the process $W^+ Z \to W^+ Z$ by setting $m_Z = m_W$ 

$$
A^{(4)}_{(s, t)Z} = -A^{(2)}_{(s, t)W} \big|_{m_Z=m_W}
$$

or we can also notice that $A^{(4)}_{tZ} = -A^{(3)}_{tZ}$. Again we have $A^{(4)}_{kZ} = A^{(4)}_{k\gamma}$. In the case of longitudinally polarized gauge bosons 

$$
\mathcal{M}^{(4)}_{kZ} = g^2 A^{(4)}_{kZ} = \frac{g^2 s}{16 m_W^2} \left(8 m_W^2 - 3 s - 24 m_W^2 x + 6 s x + s x^2 \right)
$$

(20)
Let us examine high-energy expansion of gauge amplitude

$$M^{(4)}_{\text{gauge}} = -g^2 \cos^2 \theta_W \left[ \frac{A^{(4)}_Z}{s-m^2_Z} + \frac{A^{(4)}_t}{t-m^2_Z} \right] - g^2 \sin^2 \theta_W \left[ \frac{A^{(4)}_s}{s} + \frac{A^{(4)}_t}{t} \right] + M^{(4)}_c. \quad (21)$$

$$M^{(4)}_{\text{gauge}} = -g^2 \cos^2 \theta_W \left( \frac{x}{4m_W^2} \frac{s^2 + m_Z^2 x}{4m_W^4} s + \frac{x^2 + 2x - 3}{16m_W^4} s^2 + \frac{3m_Z^2 - 16m_W^2 x + m_Z^2 x}{8m_W^4} s \right)$$

$$- g^2 \sin^2 \theta_W \left( \frac{x}{4m_W^2} \frac{s^2 + 2x - 3}{16m_W^4} s^2 - \frac{2x}{m_Z^2} s \right) + O(s^0)$$

$$+ g^2 \left( \frac{x^2 + 6x - 3}{16m_W^4} s^2 + \frac{1 - 3x}{2m_W^4} s \right) \quad (22)$$

After simplification and including terms of order $O(s^0)$ we get

$$M^{(4)}_{\text{gauge}} = M^{(4)}_{\text{LET}}$$

$$+ g^2 \left( \frac{3 + x^2 - \rho \cos^2 \theta_W (12 - 12 \rho + 12 x - 16 \rho x - 8 x^2 + 12 \rho x^2)}{4 \rho^2 (1-x) \cos^2 \theta_W} \right)$$

$$+ O\left( \frac{m_W^4}{s} \right)$$

where as in the case of the process #1 I denote

$$M^{(4)}_{\text{LET}} = -\frac{g^2 u}{4m_W^2} \left( 4 - \frac{3}{\rho} \right)$$

in accordance with $[8]$. High-energy expansion of the Higgs boson contribution

$$M^{(4)}_H = -g^2 M^2_W \left[ \frac{\varepsilon_1 \cdot \varepsilon_2}{s - m^2_H} + \frac{\varepsilon_1 \cdot \varepsilon_3 (\varepsilon_2^* \cdot \varepsilon_4^*)}{t - m^2_H} \right] \quad (23)$$

is

$$M^{(4)}_H = -\frac{g^2 u}{8m_W^2} (1 + \cos \theta_{cm}) + O(s^0) = \frac{g^2 u}{4m_W^2} + O(s^0)$$
A Exact formulae

The following Feynman rules and notation is used (all momenta are outgoing) \[10\]

\[\begin{align*}
\gamma(Z), \rho & \rightarrow i e (\gamma \cos \theta_W) V_{\mu\rho}(k, p, q) \\
W^+, \mu & \rightarrow W^-, \nu \\
W^+, \nu & \rightarrow W^-, \lambda \\
W^+, \lambda & \rightarrow W^-, \nu
\end{align*}\]

where

\[
V_{\lambda\mu\nu}(k, p, q) = (k - p)\nu \ g_{\lambda\mu} + (p - q)\lambda \ g_{\mu\nu} + (q - k)\mu \ g_{\lambda\nu}, \quad k + p + q = 0
\]  

(24)

Propagators of the gauge bosons are

\[
D_\alpha\beta(V(q)) = -g_\alpha\beta + \frac{q_\alpha q_\beta}{m_V^2} = P^{\alpha\beta}_V(q), \quad D^{\alpha\beta}_\gamma = \frac{-g^{\alpha\beta}}{q^2} = \frac{P^{\alpha\beta}_V}{q^2}.
\]

Mandelstam variables are defined as

\[
\begin{align*}
 s &= (k_1 + k_2)^2 = (k_3 + k_4)^2 \\
t &= (k_1 - k_3)^2 = (k_4 - k_2)^2 \\
u &= (k_1 - k_4)^2 = (k_3 - k_2)^2
\end{align*}
\]

The expression containing polarization vectors has in all amplitudes the form

\[
E^{\mu\rho\sigma} = \varepsilon_1^{\mu} \varepsilon_2^{\nu} \varepsilon_3^{\rho} \varepsilon_4^{\sigma}
\]

where \(\varepsilon_i = \varepsilon(k_i)\). I use the abbreviation \(x = \cos \theta_{cm}\), where \(\theta_{cm}\) is the angle between \(k_1\) a \(k_3\) in CMS.

**A.1** \(W^+(k_1) + W^-(k_2) \rightarrow Z(k_3) + Z(k_4)\)

\[
\mathcal{M}^{(1)} = -g^2 \cos^2 \theta_W \left[ \frac{A_W^{(1)}}{t - m_W^2} + \frac{A_{WW}^{(1)}}{u - m_W^2} + A_c^{(1)} \right] + \mathcal{M}_{sH}^{(1)}
\]
\[ A_{W}^{(1)} = V_{\mu\rho}(-k_1, q, k_3) V_{\beta\sigma}(-q, -k_2, k_4) P_{W}^{\alpha\beta}(q) \mathcal{E}_{\mu\nu\rho\sigma} \]
\[ A_{uW}^{(1)} = V_{\mu\sigma}(-k_1, q, k_4) V_{\beta\nu}(q, -k_2, k_3) P_{W}^{\alpha\beta}(q) \mathcal{E}_{\mu\nu\rho\sigma} \]
\[ A_{c}^{(1)} = V_{\rho\sigma \mu \nu} \mathcal{E}_{\mu\nu\rho\sigma} \]

After contraction
\[ A_{W}^{(1)} = \frac{1}{32 m_{W}^{2} m_{Z}^{2}} [-96 m_{W}^{4} m_{Z}^{4} + 32 m_{W}^{2} m_{Z}^{2} s + 8 m_{W}^{4} m_{Z}^{2} s + 16 m_{W}^{2} m_{Z}^{2} s] \]
\[ = 8 m_{W}^{4} s - 4 m_{W}^{2} m_{Z}^{2} s^{2} + 10 m_{W}^{2} m_{Z}^{2} s^{2} + 2 m_{Z}^{4} s^{2} + 3 m_{W}^{2} s^{3} \]
\[ + 16 m_{W}^{4} m_{Z}^{2} \beta_{W} \beta_{Z} s x + 12 m_{W}^{4} m_{Z}^{2} \beta_{W} \beta_{Z} s^{2} x + 24 m_{W}^{4} m_{Z}^{2} \beta_{W} \beta_{Z} s^{3} x + 32 m_{W}^{4} m_{Z}^{2} \beta_{W} \beta_{Z} s^{4} \]
\[ - 4 m_{Z}^{2} \beta_{W} \beta_{Z} s^{2} x + 5 m_{W}^{4} \beta_{W} \beta_{Z} s^{3} x + 32 m_{W}^{4} s x^{2} \]
\[ + 96 m_{W}^{4} m_{Z}^{2} s x^{2} + 32 m_{W}^{4} m_{Z}^{2} s x^{2} - 16 m_{W}^{4} m_{Z}^{2} s x^{2} \]
\[ - 22 m_{W}^{4} m_{Z}^{2} s x^{2} + 2 m_{W}^{2} s^{2} x^{2} + m_{W}^{2} s^{3} x^{2} + m_{W}^{2} \beta_{W} \beta_{Z} s^{3} x^{3} \]

Replacing all polarization vectors \( \varepsilon_i \) by \( \varepsilon_i(L) \) given in [1] we get amplitudes for longitudinally polarized gauge bosons in CMS

\[ A_{uW}^{(1)}(s, x) = A_{W}^{(1)}(s, -x) \]

and

\[ A_{c}^{(1)} = \frac{s (-4 m_{W}^{2} m_{Z}^{2} s + 3 s - s x^{2})}{8 m_{W}^{2} m_{Z}^{2}} \]

Kinematical variables are related by

\[ t = m_{W}^{2} + m_{Z}^{2} - \frac{s}{2} + \frac{s}{2} \beta_{W} \beta_{Z} \cos \theta_{cm} \]
\[ u = m_{W}^{2} + m_{Z}^{2} - \frac{s}{2} - \frac{s}{2} \beta_{W} \beta_{Z} \cos \theta_{cm} \]

where

\[ \beta_{W} = \sqrt{1 - \frac{4 m_{W}^{2}}{s}} \quad \beta_{Z} = \sqrt{1 - \frac{4 m_{Z}^{2}}{s}} \]

\[ M_{gauge}^{(1)} = \frac{g^{2} m_{Z}^{2} \cos \theta_{W}^{2}}{4 m_{W}^{2}} \frac{C^{(1)}}{J^{(1)}} \]

\[ C^{(1)} = 96 m_{W}^{4} m_{Z}^{2} - 32 m_{W}^{2} m_{Z}^{4} - 48 m_{W}^{4} m_{Z}^{2} s + 8 m_{W}^{2} m_{Z}^{2} s \]
\[ + 8 m_{W}^{4} s^{2} + 4 m_{W}^{2} s^{2} - 6 m_{Z}^{2} s^{2} + s^{3} + 128 m_{W}^{2} x^{2} + 32 m_{W}^{4} s x^{2} \]
\[ - 64 m_{W}^{2} m_{Z}^{2} s x^{2} + 8 m_{W}^{2} s^{2} x^{2} + 6 m_{Z}^{2} s^{2} x^{2} - s^{3} x^{2} \]
\[ J^{(1)} = 4 m_{W}^{4} - 4 m_{Z}^{2} s + s^{2} + 16 m_{W}^{2} m_{Z}^{2} x^{2} \]
\[ + 4 m_{W}^{2} s x^{2} + 4 m_{W}^{2} s x^{2} - s^{2} x^{2} \]

For the Higgs boson amplitude \( \mathcal{M}_{sH}^{(1)} \) we have

\[ \mathcal{M}_{sH}^{(1)} = \frac{g^{2} m_{W} m_{Z}}{4 m_{W}^{2} m_{Z}^{2}} \frac{(2 m_{W}^{2} - s)(2 m_{Z}^{2} - s)}{(s - m_{H}^{2} + i m_{H} \Gamma_{H})} \]
A.2 \( W^+(k_1) + Z(k_2) \rightarrow Z(k_3) + W^+(k_4) \)

\[
\mathcal{M}^{(2)} = -g^2 \cos^2 \theta_W \left[ \frac{A^{(2)}_{W}}{s - m^2_W} + \frac{A^{(2)}_{W}}{t - m^2_W} + A^{(2)}_{c} \right] + \mathcal{M}_{s_H}^{(2)}
\]

\[
A^{(2)}_{W} = V_{\mu\nu}(-k_1, q, -k_2) V_{\beta\rho}(q, k_4, k_3) P^{\mu\beta}_{W}(q) \mathcal{E}^{\nu\rho}
\]

\[
A^{(2)}_{W} = V_{\mu\nu}(-k_1, q, k_3) V_{\beta\rho}(q, k_4, -k_2) P^{\mu\beta}_{W}(q) \mathcal{E}^{\nu\rho}
\]

\[
A^{(2)}_{c} = \mathcal{V}_{\mu\nu\rho} \mathcal{E}^{\nu\rho}
\]

After contraction

\[
A^{(2)}_{s_W} = 4 (k_1 \cdot \epsilon_2) (k_3 \cdot \epsilon^*_4) (\epsilon_1 \cdot \epsilon^*_3) - 4 (k_1 \cdot \epsilon_2) (k_4 \cdot \epsilon^*_3) (\epsilon_1 \cdot \epsilon^*_4)
\]

\[
A^{(2)}_{s_W} = 4 (k_2 \cdot \epsilon_1) (k_3 \cdot \epsilon^*_4) (\epsilon_2 \cdot \epsilon^*_3) + 4 (k_2 \cdot \epsilon_1) (k_4 \cdot \epsilon^*_3) (\epsilon_2 \cdot \epsilon^*_4)
\]

\[
A^{(2)}_{s_W} = 2 (k_3 \cdot \epsilon^*_1) (\epsilon_1 \cdot \epsilon_2) ((k_1 - k_2) \cdot \epsilon^*_3) + 2 (k_4 \cdot \epsilon^*_1) (\epsilon_1 \cdot \epsilon_2) ((k_1 - k_2) \cdot \epsilon^*_4)
\]

\[
A^{(2)}_{s_W} = 2 (k_1 \cdot \epsilon_2) (\epsilon^*_1 \cdot \epsilon^*_4) ((k_3 - k_4) \cdot \epsilon_1) + 2 (k_2 \cdot \epsilon_1) (\epsilon^*_3 \cdot \epsilon^*_4) ((k_3 - k_4) \cdot \epsilon_2)
\]

\[
A^{(2)}_{s_W} = \frac{(m^2_W - m^2_Z)^2}{m^2_W} (\epsilon_1 \cdot \epsilon_2) (\epsilon^*_3 \cdot \epsilon^*_4) + (u - t) (\epsilon_1 \cdot \epsilon_2) (\epsilon^*_3 \cdot \epsilon^*_4)
\]

\[
A^{(2)}_{t_W} = -A^{(2)}_{s_W} (k_2 \leftrightarrow -k_3, \epsilon_2 \leftrightarrow \epsilon^*_3)
\]

\[
A^{(2)}_{c} = 2 (\epsilon_1 \cdot \epsilon^*_1) (\epsilon_2 \cdot \epsilon^*_3) - (\epsilon_1 \cdot \epsilon^*_3) (\epsilon_2 \cdot \epsilon^*_1) - (\epsilon_1 \cdot \epsilon_2) (\epsilon^*_3 \cdot \epsilon^*_4)
\]

\[
t = m^2_W + m^2_Z - \frac{s}{2} + \frac{(m^2_Z - m^2_W)^2}{2s} + 2 k^2 \cos \theta_{cm}
\]

\[
u = -2k^2 (1 + \cos \theta_{cm})
\]

where in CMS

\[
k^2 = \frac{1}{4s} \left[ s^2 + (m^2_W - m^2_Z)^2 - 2s (m^2_W + m^2_Z) \right] = |k_i|^2 \quad i = 1, 2, 3, 4
\]

\[
\mathcal{M}_{\text{gauge}}^{(2)} = \frac{g^2 m^2_Z \cos \theta_W^2}{8 m^4_W s (s - m^2_W)} C^{(2)} / \mathcal{J}^{(2)}
\]

\[
C^{(2)} = 3 m^4_W - 12 m^8_W m^4_Z + 18 m^6_W m^4_Z + 12 m^4_m^6_m^2 + 3 m^2_m^6_m^2
\]

\[
+ 17 m^8_s - 32 m^6_s m^2_s + 10 m^4_s m^4_s + 8 m^2_s m^6_s
\]

\[
- 3 m^8_s s + 26 m^6_s s^2 + 32 m^2_s m^2_s s^2 - 30 m^2_s m^4_s s^2
\]

\[
+ 4 m^6_s s - 50 m^6_s s^3 + 16 m^2_s m^2_s s^3 + 3 m^2_s m^4_s s^3 + 1 m^2_s s^4
\]

\[
- 4 m^4_s s + s^5 + 6 m^10_w x - 24 m^8_w m^2_w x + 36 m^6_w m^4_w x
\]

\[
- 24 m^8_w m^2_w x + 6 m^8_w m^2_w x + 12 m^6_w m^4_w x + 20 m^4_w m^2_w x
\]

\[
- 28 m^8_w m^2_w x + 12 m^6_w m^4_w x - 4 m^8_w m^2_w x + 20 m^6_w m^4_w x
\]

\[
- 44 m^4_w m^2_w s^2 x - 20 m^6_w m^4_w s^2 x + 12 m^8_w s^2 x - 16 m^4_w s^3 x
\]

\[
- 4 m^2_w m^2_w s^3 x - 12 m^4_w s^3 x + 6 m^6_w s^4 x + 3 m^8_w s^4 x
\]

\[
- 12 m^8_w m^2_w x^2 + 18 m^6_w m^4_w x^2 - 12 m^4_w m^6_w x^2 + 3 m^2_w m^8_w x^2
\]

\[
- 33 m^8_w m^2_w x^2 + 68 m^6_w m^4_w x^2 - 38 m^4_w m^6_w x^2 + 4 m^2_w m^8_w x^2
\]

\[
- m^8_w x + 122 m^6_w s^2 x^2 + 12 m^4_w m^2_s^2 x^2 - 6 m^2_w m^4_s^2 x^2
\]

\[
+ 34 m^4_w s^2 x^2 - 4 m^2_w m^2_s^2 x^2 + 2 m^2_s^3 x^2 + 3 m^2_s^2 x^2 + 3 m^2_s^2 x^2
\]

\[
\mathcal{J}^{(2)} = m^4_w - 2 m^2_w m^2_w + m^4_w + 2 m^2_s s - s^2 + m^4_w x - 2 m^2_w m^2_w x
\]

\[
+ m^4_w x - 2 m^2_w s x + 2 m^2_s s x + s^2 x
\]

10
Higgs boson amplitude \textsuperscript{(12)}

\[
\mathcal{M}_{uH}^{(2)} = \frac{g^2}{8 m_W m_Z \cos \theta_W s} \frac{C_{uH}^{(2)}}{J_{uH}^{(2)}}
\]

\[
C_{uH}^{(2)} = m_W^8 - 4 m_W^6 m_Z^2 + 6 m_W^4 m_Z^4 - 4 m_W^2 m_Z^6 + m^8_Z - 4 m^6_W s + 4 m^4_W m_Z^2 s
+ 4 m^2_W m_Z^6 s - 4 m_Z^8 s + 4 m^4_W m_Z^2 s^2 + 4 m_Z^6 s^2 - 4 m_Z^8 s^3
- 4 m_Z^2 s^3 + s^4 + 2 m_Z^2 x - 8 m^6_W x Z^2 + 12 m^4_W m_Z^2 x
- 8 m^4_W m_Z^2 x - 4 m^2_W m_Z^2 x^2 + 4 m^2_W m_Z^4 s x - 4 m^4_W s x
+ 4 m^4_W s^2 x - 8 m^2_W m_Z^2 s^2 x + 4 m_Z^2 s^4 x
- 4 m_Z^4 s^3 x + 2 s^4 x + m_W^4 x^2 - 4 m_W^2 m_Z^2 x^2 + 6 m_W^4 m_Z^2 x^2
- 4 m_W^8 m_Z^2 x^2 + m_Z^2 x^2 - 2 m^4_W s^2 x^2 + 4 m^2_W m_Z^2 s^2 x^2 - 2 m^4_W s^2 x^2 + s^4 x^2.
\]

\[
J_{uH}^{(2)} = m_W^4 - 2 m_W^2 m_Z^2 + m_Z^4 + 2 m_Z^2 s - 2 m_W^2 s - 2 m^2_Z s + s^2 + m^4_W x
- 2 m^2_W m^2_Z x + m_Z^2 x - 2 m_W^4 s x - 2 m^2_Z s x + s^2 x.
\]

A.3 \quad W^+(k_1) + W^+(k_2) \rightarrow W^+(k_3) + W^+(k_4)

\[
\mathcal{M}^{(3)} = -g^2 \cos^2 \theta_W \left[ \frac{A_{tZ}^{(3)}}{t - m_Z^2} + \frac{A_{uZ}^{(3)}}{u - m_Z^2} \right] - g^2 \sin^2 \theta_W \left[ \frac{A_{tZ}^{(3)}}{t} + \frac{A_{uZ}^{(3)}}{u} \right] + g^2 A_c^{(3)} + \mathcal{M}_{tH}^{(3)} + \mathcal{M}_{uH}^{(3)}.
\]

\[
A_{tZ,\gamma}^{(3)} = V_{\mu\rho \alpha} (-k_1, k_3, q) V_{\nu\sigma \beta} (-k_2, k_4, -q) P_{\gamma Z,\gamma}^{\alpha \beta} (q) \mathcal{E}^{\mu \nu \rho \sigma}
\]

\[
A_{uZ,\gamma}^{(3)} = V_{\mu \rho \alpha} (-k_1, k_3, q) V_{\nu \sigma \beta} (-k_2, k_4, -q) P_{\gamma Z,\gamma}^{\alpha \beta} (q) \mathcal{E}^{\mu \nu \rho \sigma}
\]

\[
A_c^{(3)} = V_{\mu \nu \rho \sigma} \mathcal{E}^{\mu \nu \rho \sigma}.
\]

Kinematical variables are given by

\[
t = 2 \left( m_W^2 - \frac{s}{4} \right) \left( 1 - x \right) \quad u = 2 \left( m_W^2 - \frac{s}{4} \right) \left( 1 + x \right)
\]

\[
A_{tZ}^{(3)} = 2 (k_2 \cdot \varepsilon^*_1) ((k_1 \cdot \varepsilon_2) + (k_3 \cdot \varepsilon_3)) (\varepsilon_1 \cdot \varepsilon_3) - 4 (k_1 \cdot \varepsilon_3^*) (k_2 \cdot \varepsilon_1^*) (\varepsilon_1 \cdot \varepsilon_2)
+ 2 ((k_1 \cdot \varepsilon_4^*) + (k_3 \cdot \varepsilon_4^*)) (k_2 \cdot \varepsilon_2) (\varepsilon_1 \cdot \varepsilon_3) - 4 (k_1 \cdot \varepsilon_3^*) (k_4 \cdot \varepsilon_2) (\varepsilon_1 \cdot \varepsilon_4^*)
+ 2 (k_1 \cdot \varepsilon_3^*) ((k_2 \cdot \varepsilon_1) + (k_4 \cdot \varepsilon_1)) (\varepsilon_2 \cdot \varepsilon_4^*) - 4 (k_2 \cdot \varepsilon_1^*) (k_3 \cdot \varepsilon_1) (\varepsilon_2 \cdot \varepsilon_3)
+ 2 (k_3 \cdot \varepsilon_1) ((k_2 \cdot \varepsilon_3^*) + (k_4 \cdot \varepsilon_3^*)) (\varepsilon_2 \cdot \varepsilon_4^*)
- (s - u) (\varepsilon_1 \cdot \varepsilon_3^*) (\varepsilon_2 \cdot \varepsilon_4^*) - 4 (k_3 \cdot \varepsilon_1) (k_4 \cdot \varepsilon_2) (\varepsilon_3^* \cdot \varepsilon_4^*).
\]

\[
A_{uZ}^{(3)} = A_{tZ}^{(3)} (3 \leftrightarrow 4)
\]

\[
A_{(t,u)\gamma}^{(3)} = A_{(t,u)Z}^{(3)}
\]

For relations among different \(A\) see table 2 and below.

\[
A_{tZ}^{(3)}(s, x) = \frac{1}{(32 m_W^2)} \left( 64 m_W^6 - 16 m_W^4 s + 12 m_W^2 s^2 - 3 s^3 + 64 m_W^6 x + 112 m_W^4 s x - 52 m_W^2 s^2 x + 5 s^3 x - 160 m_W^4 s x^2 + 36 m_W^2 s^2 x^2 - s^3 x^2 + 4 m_W^4 s^2 x^3 - 3 s^3 x^3 \right)
\]

\[
A_{uZ}^{(3)}(s, x) = A_{tZ}^{(3)}(s, -x)
\]
\[ M_{\text{gauge}}^{(3)} = \frac{g^2 s}{8 m_W^2} \left( -8 m_W^2 + 3 s - s x^2 \right) + \frac{g^2 \cos^2 \theta_W}{8 m_W^2} C_Z^{(3)} - g^2 \sin^2 \theta_W C_\gamma^{(3)} \]

\[ C_Z^{(3)} = 256 m_W^6 - 128 m_W^6 m_Z^2 - 128 m_W^6 s + 32 m_W^4 m_Z^2 s + 64 m_W^4 s^2 - 24 m_W^2 m_Z^2 s^2 - 24 m_W^2 s^3 + 6 m_W^2 s^3 + 3 s^4 + 256 m_W^2 x^2 - 256 m_W^2 s x^2 + 320 m_W^4 m_Z^2 s x^2 - 16 m_W^4 s^2 x^2 - 72 m_W^2 m_Z^2 s^2 x^2 + 32 m_Z^2 s^3 x^2 + 2 m_Z^2 s^3 x^2 - 4 s^4 x^2 + 16 m_W^4 s^2 x^4 - 8 m_W^2 s^3 x^4 + 4 s^4 x^4 \]

\[ J_Z^{(3)} = (4 m_W^2 - 2 m_Z^2 - s + 4 m_W^2 x + s x) \left( -4 m_W^2 + 2 m_Z^2 + s - 4 m_W^2 x + s x \right) \]

\[ C_\gamma^{(3)} = 64 m_W^6 - 16 m_W^4 s + 12 m_W^2 s^2 - 3 s^3 + 64 m_W^6 x^2 - 48 m_W^4 s x^2 - 16 m_W^2 s^2 x^2 + 4 s^3 x^2 + 4 m_W^2 s^2 x^4 - s^4 x^4 \]

\[ J_{\gamma}^{(3)} = (s - 4 m_W^2) (x^2 - 1) \]

Higgs boson amplitude

\[ M_{H}^{(3)} = g^2 \frac{C_H^{(3)}}{J_{H}^{(3)}} \]

\[ C_H^{(3)} = 32 m_W^2 m_W^4 - 64 m_W^6 - 16 m_W^4 m_W^2 s + 48 m_W^4 s + 2 m_W^4 s^2 - 12 m_W^2 s^2 + s^3 - 32 m_W^4 s x^2 + 2 m_H^2 s^2 x^2 + 12 m_W^2 s^2 x^2 - s^3 x^2 \]

\[ J_{H}^{(3)} = 4 m_W^2 (2 m_W^3 - 4 m_W^2 + s + 4 m_W^2 x - s x) (2 m_H^2 - 4 m_W^2 + s - 4 m_W^2 x + s x) \]

A.4 \( W^+ (k_1) + W^- (k_2) \rightarrow W^+ (k_3) + W^- (k_4) \)

\[ M^{(4)} = -g^2 \cos^2 \theta_W \left[ \frac{A_{\gamma}^{(4)}}{s - m_Z^2} + \frac{A_\gamma^{(4)}}{l - m_Z^2} \right] - g^2 \sin^2 \theta_W \left[ \frac{A_s^{(4)}}{s} + \frac{A_\gamma^{(4)}}{l} \right] + g^2 A_\gamma^{(4)} + M_{sH}^{(4)} + M_{tH}^{(4)} \]

\[ A_{\gamma}^{(4)} = V_{\mu \alpha} (-k_1, -k_2, q) V_{\nu \beta} (k_4, k_3, -q, q) P_{Z, \gamma}^{\alpha \beta} (q) \epsilon_{\mu \nu \rho \sigma} \]

\[ A_t^{(4)} = V_{\mu \alpha} (-k_1, k_3, q) V_{\nu \beta} (k_4, -k_2, -q) P_{Z, \gamma}^{\alpha \beta} (q) \epsilon_{\mu \nu \rho \sigma} \]

\[ A_\gamma^{(4)} = g^2 V_{\mu \rho \nu \omega} \epsilon_{\mu \nu \rho \sigma} \]

\[ M_{\text{gauge}}^{(4)} = \frac{g^2 s \left( 8 m_W^2 - 3 s - 24 m_W^2 x + 6 s x + s x^2 \right)}{16 m_W^2} \]

\[ + M_{sZ}^{(4)} + M_{s\gamma}^{(4)} - g^2 \cos^2 \theta_W C_{Z}^{(4)} J_{Z}^{(4)} - g^2 \sin^2 \theta_W C_{\gamma}^{(4)} J_{\gamma}^{(4)} \]

\[ M_{sZ}^{(4)} = -g^2 \cos^2 \theta_W \frac{\left( 4 m_W^2 - s \right) \left( 2 m_W^2 + s \right)^2 x}{4 m_W^2 \left( m_Z^2 - s \right)} \]

\[ M_{s\gamma}^{(4)} = -g^2 \sin^2 \theta_W \left( -3 x - \frac{4 m_W^2 x}{s} + \frac{s^2 x}{4 m_W^2} \right) \]

\[ C_{Z}^{(4)} = -64 m_W^6 + 16 m_W^4 s - 12 m_W^2 s^2 + 3 s^3 - 64 m_W^6 x - 112 m_W^4 s x + 52 m_W^2 s^2 x - 5 s^3 x + 160 m_W^4 s x^2 - 36 m_W^2 s^2 x^2 + 3 s^3 x^2 - 4 m_W^2 s^2 x^3 + s^4 x^3 \]
\[ J_{t\gamma}^{(4)} = 16 m_W^4 \left( 4 m_W^2 - 2 m_Z^2 - s - 4 m_W^2 x + s x \right) \]
\[ C_{t\gamma}^{(4)} = J_{t\gamma}^{(4)} \quad J_{t\gamma}^{(4)} = J_{tZ}^{(4)} \bigg|_{m_Z=0} \]

Higgs boson amplitude \[\mathcal{M}_H^{(4)} = g^2 \frac{C_{H}^{(4)}}{J_{H}^{(4)}}\]
\[ C_{H}^{(4)} = 32 m_H^2 m_W^4 - 32 m_W^6 - 24 m_H^2 m_W^2 s + 24 m_W^4 s \]
\[ + 5 m_H^2 s^2 - 8 m_W^2 s^2 + s^3 + 32 m_W^6 + 8 m_H^2 m_W^2 s x \]
\[ - 40 m_W s x - 2 m_H^2 s^2 x + 8 m_W^2 s^2 x + m_H^2 s^2 x - s^3 x^2 \]
\[ J_{H}^{(4)} = 8 m_W^2 \left( m_H^2 - s \right) \left( 2 m_H^2 - 4 m_W^2 + s + 4 m_W^2 x - s x \right) \]

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