Note on the magnetic moments of the nucleon

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Abstract

The Goldberger-Treiman relation \( M = \frac{2\pi}{\sqrt{3}} f^\text{cl}_\pi \) where \( M \) is the constituent quark mass in the chiral limit (cl) and \( f^\text{cl}_\pi \) the pion decay constant in the chiral limit predicts constituent quark masses of \( m_u = 328.8 \pm 1.1 \text{ MeV} \) and \( m_d = 332.3 \pm 1.1 \text{ MeV} \) for the up and down quark, respectively, when \( f^\text{cl}_\pi = 89.8 \pm 0.3 \text{ MeV} \) is adopted. Treating the constituent quarks as bare Dirac particles the following zero order values \( \mu_p^{(0)} = 2.850 \pm 0.009 \) and \( \mu_n^{(0)} = -1.889 \pm 0.006 \) are obtained for the proton and neutron magnetic moments, leading to deviations from the experimental data of 2.0% and 1.3%, respectively. These unavoidable deviations are discussed in terms of contributions to the magnetic moments proposed in previous work.

1 Introduction

The prediction of the magnetic moments of octet baryons in a constituent quark model obeying \( SU(6) \) spin-flavor symmetry has attracted many researchers over a long period of time (see [1–11] and references therein). The overall success of these investigations has become one of the main supports for the validity of the constituent quark model. Furthermore, theoretical evidence has been presented that the constituent quarks behave like bare Dirac particles [12]. Remaining discrepancies showing up in previous work have been removed in the latest of this series of papers [11] where a general agreement was achieved between the experimental data and the predictions. One very remarkable result of this latter investigation is that the general agreement is obtained by a proper determination of the constituent quark masses, showing that other effects on the magnetic moments are of minor importance. However, by adjusting the predictions to the experimental magnetic moment of the proton [11] a discrepancy between theory and experiment of 3.3% is obtained in case of the neutron. This discrepancy shows that at a few-percent level of precision the proper choice of the constituent quark mass is not sufficient for obtaining a complete agreement between theory and experiment for both nucleons. Researches on possible additional contributions to the octet baryon magnetic moments were carried out in [3–10]. The following additional contributions were discussed:

(i) relativistic effects [4, 8, 10],
(ii) configuration mixing in the ground state wave functions [4],
(iii) loop corrections [9],
(iv) loop and vertex corrections [13], and
(v) pion exchange currents between constituent quarks [3, 5–8].

The present investigation is motivated by the fact that two other fundamental structure constants of the nucleon, \( \alpha \) and \( \beta \), have been successfully predicted on an absolute scale by treating them as composites of the nucleon structure (or \( s \)-channel) parts \( \alpha^s \) and \( \beta^s \) and the \( t \)-channel parts \( \alpha^t \) and \( \beta^t \), where the \( t \)-channel parts could be quantitatively predicted [14–17] on the basis of the Goldberger-Treiman relation on the quark level \( M = \frac{2\pi}{\sqrt{3}} f^\text{cl}_\pi \) derived by Delbourgo and Scadron [19] where \( M \) is the mass of the constituent quark in the chiral limit and \( f^\text{cl}_\pi \) the pion decay constant in the chiral limit. The Goldberger-Treiman relation has been derived in a model which the authors [19]...
name the dynamically generated linear sigma model (LσM) on the quark level. In our previous work [14–17] and in the present work no use is made of properties of the LσM, except for the Goldberger-Treiman relation which has been derived from it. Our attitude is to use the Goldberger-Treiman relation in the form \( M = 2\pi / \sqrt{3} f_{\pi}^{cl} \) independent of the special method of its derivation and to find experimental arguments which support its usefulness and validity. The pion decay constant in the chiral limit \( f_{\pi}^{cl} \) has been derived from the experimental pion decay constant \( f_{\pi} = (92.42 \pm 0.26) \text{ MeV} \) through a small correction given in [20]. Therefore, the statement is allowed that the \( t \)-channel parts of the electromagnetic polarizabilities are predicted on an absolute scale [14–17] with the experimentally known pion decay as the only input. A second available case is the prediction of the two-photon width \( \Gamma(\sigma \rightarrow \gamma\gamma) \) of the \( \sigma \) meson [17, 18]. However, in this latter case the experimental value to compare with is not very precise. As a further result the Goldberger-Treiman relation leads to predictions for the constituent quark masses on an absolute scale and we consider it very interesting to investigate to what level of precision the magnetic moments of the nucleon can be predicted on this basis.

2 Predictions based on the Goldberger-Treiman relation on the quark level

In the dynamically generated LσM on the quark level which is related to the bosonized Nambu–Jona-Lasinio (NJL) model the gap parameter or constituent quark mass \( M \) in the chiral limit and the pion decay constant in the chiral limit \( f_{\pi}^{cl} \) are related to each other through the relations [14,19]

\[
f_{\pi}^{cl} = -4iN_c gM \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 - M^2)^2},
\]

\[
M = -\frac{8iN_c g^2}{(m_{\sigma}^{cl})^2} \int \frac{d^4p}{(2\pi)^4} \frac{M}{p^2 - M^2},
\]

where \( N_c = 3 \) is the number of colors, \( m_{\sigma}^{cl} \) the mass of the \( \sigma \) meson in the chiral limit and \( g \) the Yukawa coupling constant which is related to the quantities \( f_{\pi}^{cl} \) and \( M \) through the Goldberger-Treiman relation for the chiral limit:

\[
gf_{\pi}^{cl} = M. \tag{3}
\]

Applying dimensional regularization in (1) and (2) and using (3) and \( m_{\sigma}^{cl} = 2M \), we arrive at

\[
g = g_{\pi qq} = g_{\sigma qq} = 2\pi / \sqrt{N_c} = 3.63. \tag{4}
\]

The pion decay constant in the chiral limit is [20] \( f_{\pi}^{cl} = 89.8 \pm 0.3 \text{ MeV} \). Using this value and applying (3) and (4) the following value for the constituent quark mass in the chiral limit is obtained:

\[
M = 325.8 \pm 1.1 \text{ MeV}. \tag{5}
\]

According to the PDG [21] the presently accepted values of the current quark masses are

\[
m_u^{\text{curr.}} = 3.0 \text{ MeV}, \tag{6}
\]

\[
m_d^{\text{curr.}} = 6.5 \text{ MeV}. \tag{7}
\]

This leads to the predicted constituent quark masses

\[
m_u = M + m_u^{\text{curr.}} = 328.8 \pm 1.1 \text{ MeV}, \tag{8}
\]

\[
m_d = M + m_d^{\text{curr.}} = 332.3 \pm 1.1 \text{ MeV}. \tag{9}
\]
The argument leading to (8) and (9) may be found in Eq. (4.19) of [22] where a gap equation is formulated for the constituent quark mass \( m^* \) including the effects of explicit symmetry breaking. This gap equation Eq. (4.19) shows that (8) and (9) are valid except for a very small and, therefore, negligible correction.

In principle the quantities \( g \) and \( M \) in Eqs. (4) and (5) may depend on the regularization scheme. However, as has been shown already in a previous work [14] this dependence on the regularization scheme apparently is marginal. The argument was as follows. When we calculate the mass of the \( \sigma \) meson according to

\[
    m_\sigma = (4M^2 + m_\pi^2)^{1/2}
\]

(see e.g. [22]) we arrive at

\[
    m_\sigma = 666.0 \text{ MeV}.
\]

An independent calculation [23] (see also the discussion in [14]) in terms of the four-fermion version of the NJL model with regularization through a cut-off parameters \( \Lambda \) has led to

\[
    m_\sigma \simeq 668 \text{ MeV}.
\]

The good agreement of the numbers in (11) and (12) gives us confidence that the dependence of the quantity \( M \) and consequently also of the constituent quark masses \( m_u \) and \( m_d \) given in (8) and (9) on the regularization scheme is very small.

The spin-dependent part of the nucleon wave function may be given in the form

\[
    |p\rangle = \sqrt{\frac{2}{3}} \chi(1,1) \phi(1/2, -1/2) - \sqrt{\frac{1}{3}} \chi(1,0) \phi(1/2, 1/2),
\]

\[
    |n\rangle = \sqrt{\frac{2}{3}} \phi(1,1) \chi(1/2, -1/2) - \sqrt{\frac{1}{3}} \phi(1,0) \chi(1/2, 1/2),
\]

where \( \chi(J, M) \) represents the up quarks and \( \phi(J, M) \) the down quarks. This leads to the magnetic moments

\[
    \mu_p = \frac{2}{3} (2\mu_u - \mu_d) + \frac{1}{3} \mu_d = \frac{4}{3} \mu_u - \frac{1}{3} \mu_d,
\]

\[
    \mu_n = \frac{2}{3} (2\mu_d - \mu_u) + \frac{1}{3} \mu_u = \frac{4}{3} \mu_d - \frac{1}{3} \mu_u,
\]

in units of the nuclear magneton \( \mu_N = e\hbar/2m_p \) [21]. Constituent quark masses enter through the relations

\[
    \mu_u = \frac{2}{3} \frac{m_p}{m_u}, \quad \mu_d = -\frac{1}{3} \frac{m_p}{m_d}.
\]

Using the constituent quark masses given in (8) and (9) the zero-order values of the magnetic moments of the nucleon can be calculated via

\[
    \mu_p^{(0)} = \frac{1}{3} \left( 4\mu_u^{(0)} - \mu_d^{(0)} \right), \quad \mu_n^{(0)} = \frac{1}{3} \left( 4\mu_d^{(0)} - \mu_u^{(0)} \right),
\]

with

\[
    \mu_u^{(0)} = \frac{2}{3} \frac{m_p}{m_u} = 1.902, \quad \mu_d^{(0)} = -\frac{1}{3} \frac{m_p}{m_d} = -0.941.
\]

This leads to the results given in Table I. In addition to the data for the proton and the neutron also the isoscalar \( \mu_S = \frac{1}{2}(\mu_p + \mu_n) \) and isovector magnetic moments \( \mu_V = \frac{1}{2}(\mu_p - \mu_n) \) are given. It is interesting to note that the difference between the zero-order values \( \mu^{(0)} \) and experimental values \( \mu_{\exp} \) amounts to only 2.0% for the proton and 1.3% for the neutron. This difference is mainly isoscalar.
Table 1: Predicted magnetic moments of the nucleon in zero-order approximation $\mu^{(0)}$ compared with experimental data. The quantities $\mu^{(0)}$ have been calculated from Eqs. (18) and (19). The corrections $\mu^{\text{corr.}(0)}$ are the differences between the experimental values $\mu^{\exp.}$ and the zero-order values $\mu^{(0)}$.

|           | proton      | neutron     | isoscalar    | isovector    |
|-----------|-------------|-------------|--------------|--------------|
| $\mu^{(0)}$ | $+2.856 \pm 0.009$ | $-1.889 \pm 0.006$ | $+0.480$     | $+2.370$     |
| $\mu^{\exp.}$ | $+2.793$     | $-1.913$    | $+0.440$     | $+2.353$     |
| $\mu^{\text{corr.}(0)}$ | $-0.057 \pm 0.009$ | $-0.024 \pm 0.006$ | $-0.040$     | $-0.017$     |

It may be of interest to compare the predictions of the Goldberger-Treiman relation with an approach where the constituent quark masses are adjusted to the magnetic moments of the nucleon using (15), (16) and (17) as has been done in previous work [11, 24]. Then the quantity $M$ entering into (8) and (9) is an adjustable parameter which may be denoted by $M(p, n)$. In this case the constituent-quark masses are

$$m_{u}^{(p)} = 335.6 \text{ MeV}, \quad m_{d}^{(p)} = 339.1 \text{ MeV}, \quad (20)$$

when adjusted to the magnetic moment of the proton and

$$m_{u}^{(n)} = 324.6 \text{ MeV}, \quad m_{d}^{(n)} = 328.1 \text{ MeV}, \quad (21)$$

when adjusted to the magnetic moment of the neutron. An interesting feature of this prediction of the constituent quark masses is that the arithmetic averages

$$\frac{1}{2}(m_{u}^{(p)} + m_{u}^{(n)}) = 330.1 \text{ MeV} \quad \text{and} \quad \frac{1}{2}(m_{d}^{(p)} + m_{d}^{(n)}) = 333.6 \text{ MeV} \quad (22)$$

both are larger than the corresponding predictions of the Goldberger-Treiman relation in (8) and (9) by $+0.4\%$. Therefore, this difference between the results in (8) and (9) and in (22) may be interpreted in terms of the uncertainty of the pion decay constant $f^{\text{cl}}_{\pi}$. First we notice that by inserting $M(p, n) = 327.1 \text{ MeV}$ instead of $M = 325.8 \text{ MeV}$ into (8) and (9) we exactly arrive at the numbers given (22). This means that the use of $M(p, n) = 327.1 \text{ MeV}$ instead of $M = 325.8 \text{ MeV}$ may be understood in terms of a shift

$$f^{\text{cl}}_{\pi} = 89.8 \text{ MeV} \implies f^{\text{cl}}_{\pi} = 90.1 \text{ MeV} \quad (23)$$

of the pion decay constant. This shift by $\sim 0.3\%$ is within the error of the quantity $f^{\text{cl}}_{\pi}$.

3 Discussion

The surprising feature of the numbers in Table 1 is that the correction terms $\mu^{\text{corr.}(0)}$ are so small. This is quite satisfactory because it gives us a further good example that predictions obtained on the basis of the Goldberger-Treiman relation on the quark level are valid to a high level of precision. Nevertheless it appears justified to ask for reasons that these correction terms exist. For this purpose we discuss one of the previous proposals which at first sight appears to us especially relevant and which remains within the present ansatz of a bare Dirac particle. This is the configuration mixing.
In the $SU(6)$ harmonic oscillator basis the ground state of the nucleon may be given in the form
\[ |P_{11}(939)\rangle = a_S |N^2S_{1/2}\rangle_S + a'_S |N^2S'_{1/2}\rangle_S + a_M |N^2S_{1/2}\rangle_M + a_D |N^4D_{1/2}\rangle_M, \] (24)
where the coefficients have been determined [25] to be
\[ a_S = 0.931, \quad a'_S = -0.274, \quad a_M = -0.233, \quad a_D = -0.067. \] (25)
The first two terms on the r.h.s. of (24) differ by the oscillator quantum number $N$, being $N = 0$ and $N = 2$ respectively, but have the same $SU(6)$ structure otherwise. The $D$ wave admixture represented by the last term enters with a coefficient of $P_D = a_D^2 = 0.0045$ into the expression for the magnetic moment and therefore may be disregarded. This justifies to treat the nucleon ground state as a linear combination of only $^2S_S$ and $^2S_M$ components, so that the magnetic moments can be expressed as follows [4]:
\[ \mu_p^{\text{conf.}} = \frac{1}{3} (4\mu_u - \mu_d) \cos^2 \phi_S^N + \frac{1}{3} (2\mu_u + \mu_d) \sin^2 \phi_S^N, \] (26)
\[ \mu_n^{\text{conf.}} = \frac{1}{3} (4\mu_d - \mu_u) \cos^2 \phi_S^N + \frac{1}{3} (2\mu_d + \mu_u) \sin^2 \phi_S^N. \] (27)
The component $^2S_S$ corresponds to the quark structure given in (13) and (14) or to $[56, 0^+]$ states in $SU(6)$ notation whereas the impurity $^2S_M$ corresponds to $[70, 0^+]$ states in $SU(6)$ notation. These impurities have been introduced as a consequence of color hyperfine interactions. In [4] the mixing amplitude is given as $\sin \phi_S^N = -0.27$ in close agreement with $a_M = -0.233$. Using $\sin \phi_S^N = -0.27$ and the zero-order predictions for the quark magnetic moments given in (19) we arrive at corrections due to configuration mixing as given in Table 2. In Table 2 the discrepancy between experiment and prediction is 3% for the proton and 9.3% for the neutron. Apparently, the discrepancies are much larger when the configuration mixing is included than in case of the zero-order predictions $\mu^{(0)}$. This means that when introducing corrections due to configuration mixing it would be necessary to simultaneously find another sizable effect which compensates for the configuration mixing effects. Without going into details here the same conclusions can be drawn for the other corrections proposed in the literature.

### 4 Summary and conclusions

It has been shown that the magnetic moments of the nucleon can be calculated with a high level of precision of 1–2% on an absolute scale using the constituent quark masses predicted on the basis of Goldberger-Treiman relation $M = 2\pi/\sqrt{3} f_\pi^{cl}$ with $f_\pi^{cl} = 89.8 \pm 0.3$ MeV derived from...
the experimentally known pion decay constant $f_\pi = (92.42 \pm 0.26)$ MeV as the only input. The importance of the present finding is that in addition to the $t$-channel parts of the electromagnetic polarizabilities and the two-photon width $\Gamma(\sigma \to \gamma\gamma)$ of the $\sigma$ meson [17, 18] we now have a further example where this relation is successful in predicting the correct results. This leads us to the conclusion that the Goldberger-Treiman relation $M = \frac{2\pi}{\sqrt{3}} f_\pi^{cl}$ predicts the mass $M$ of the constituent quark in the chiral limit with a high level of precision. Furthermore, since this prediction is successful in connection with three different experimentally known observables it may be concluded that this success cannot be fortuitous but may be considered as a proof for the general validity of the Goldberger-Treiman relation.

Another important result is that none of the available predictions of possible deviations from the constituent quark approach in zero-order approximation with the constituent quarks treated as bare Dirac particles leads to an explanation of the corrections terms $\mu^{corr.(0)}$ given in Table 1. This has been explicitly shown for configuration mixing but is also true for the other cases listed in the introduction. Therefore, the explanation of this residual discrepancy remains a problem for future work.

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