BRILL-NOETHER THEORY FOR CYCLIC COVERS

IRENE SCHWARZ

ABSTRACT. We recall that the Brill-Noether Theorem gives necessary and sufficient conditions for the existence of a $g^r_d$. Here we consider a general $n$-fold, étale, cyclic cover $p : \tilde{C} \to C$ of a curve $C$ of genus $g$ and investigate for which numbers $r, d$ a $g^r_d$ exists on $\tilde{C}$. For $r = 1$ this is asking for the gonality of $\tilde{C}$. Using degeneration to a special singular example (containing a Castelnuovo canonical curve) and the theory of limit linear series for tree-like curves we show that the Plücker formula yields a necessary condition for the existence of a $g^r_d$ which is only slightly weaker than the sufficient condition given by the results of Laksov and Kleiman [KL], for all $n, r, d$.

1. INTRODUCTION AND MAIN RESULTS

We recall the Brill-Noether theorem for curves (which in this paper are always assumed to be complete reduced algebraic over $\mathbb{C}$). If the Brill-Noether number

$$\rho(g, r, d) := g - (r + 1)(g - d + r) \quad (1.1)$$

is non-negative, then any smooth curve $C$ of genus $g$ carries a $g^r_d$, i.e. a linear series of dimension $r$ and degree $d$. This existence result is due to [K], [KL], see also [FL]. For general curves this sufficient condition is also necessary. This is the content of the classical Brill-Noether theorem, first proved rigorously in [GH], and subsequently in e.g. [EH1], [L].

In this paper we show a Brill-Noether-like theorem for cyclic covers $p : \tilde{C} \to C$, where more precisely $p$ is a finite, flat, unramified morphism between smooth curves which has degree $n$ and a cyclic Galois group. We recall that, if $C$ has genus $g$, such covers form an irreducible moduli space $R_{g,n}$ whose birational geometry has been studied recently, see [CEFS]. A general point in this space is a cyclic cover of a general curve $C$ of genus $g$. We obtain the following non-existence result.

**Theorem 1.1.** Let $p : \tilde{C} \to C$ be a general cyclic cover of a curve of genus $g$. If the Brill-Noether number satisfies the inequality

$$\rho(\tilde{g}, r, d) < -r, \quad (1.2)$$

where $\tilde{g} = n(g - 1) + 1$ is the genus of $\tilde{C}$, then $\tilde{C}$ carries no $g^r_d$.

This theorem immediately yields the following useful results.

**Theorem 1.2.** Let $p : \tilde{C} \to C$ be as above and assume that $r + 1$ divides $\tilde{g}$. Then a general $\tilde{C}$ has a $g^r_d$ if and only if

$$\rho(\tilde{g}, r, d) \geq 0. \quad (1.3)$$
In fact, there exists a $d$ such that $\rho(\tilde{g}, r, d) = 0$, if and only if $r + 1$ divides $\tilde{g}$. For this $d$ we have $\rho(\tilde{g}, r, d - k) = -k(r + 1)$. So by the existence result of [K], [KL], there will be a $g^1_{\tilde{g}}$ on $\tilde{C}$, but - by Theorem 1.1 - no $g^1_{\tilde{g} - k}$ for positive $k$. Since $\rho(\tilde{g}, r, d)$ is strictly monotone in $d$, we are done.

In the case $r = 1$ the Brill-Noether theorem gives an estimate for the gonality of $\tilde{C}$, defined as

$$\text{gon}(\tilde{C}) := \min\{d \in \mathbb{N}; \text{ there exists a } g^1_d \text{ on } \tilde{C}\}.$$ 

We recall from the discussion of the Brill-Noether theorem that in the classical language of Riemann surfaces $\text{gon}(C)$ is the minimal number of sheets needed in a representation of $C$ as a branched cover of $\mathbb{P}^1$ which is

$$d = \left\lfloor \frac{g + 1}{2} \right\rfloor + 1.$$

In our case of cyclic covers we have the following result:

**Theorem 1.3.** Let $p: \tilde{C} \to C$ be a general cover as above. Then the following estimate holds

$$\frac{\tilde{g} + 1}{2} = \frac{n(g - 1)}{2} + 1 \leq \text{gon}(\tilde{C}) \leq \frac{n(g - 1)}{2} + 2 = \frac{\tilde{g} + 3}{2}. \quad (1.4)$$

In fact, for $r = 1$ the smallest $d$ such that $\tilde{C}$ has a $g^1_d$ will satisfy $-1 \leq \rho(\tilde{g}, 1, d) \leq 1$ (using Theorem 1.1 for the lower and the existence result of [K], [KL] for the upper bound). This is equivalent to the estimate (1.4).

As a further consequence of Theorem 1.2, this estimate can in certain cases be sharpened.

**Corollary 1.4.** For $p: \tilde{C} \to C$ as above, with $n$ odd and $g$ even, we have

$$\text{gon}(\tilde{C}) = \frac{n(g - 1) + 3}{2}. \quad (1.5)$$

The upper bound on $\text{gon}(C)$ - for each smooth genus $g$ curve - was the first rigorous result on the Brill-Noether problem (see [M] with its analytic proof). The lower bound has until now only been proven by methods from algebraic geometry. Our Theorem 1.3 extends this result to curves which are general in the class of cyclic covers.

Concerning previous results on Brill-Noether theory for cyclic covers, we recall [AF], which treats the case $r = 1$ and $n = 2$, i.e. gonality for cyclic double covers, and obtains a sharper result computing $\text{gon}(\tilde{C})$ in each case: For $g$ even it is $g$, for $g$ odd it is $g + 1$. Here [AF] uses Proposition 1.4.1 in [F] on 2-pointed elliptic curves. We do not see how this argument could be mimicked even in the case $n = 2, r > 1$. This motivated our search for a different approach which finally led to this work using Plücker’s formula instead.

The outline of this paper is as follows. In Section 2 we present the preliminaries needed to prove our theorems: the moduli space $R_{g,n}$, limit linear series and Castelnuovo canonical curves. All this (except for the use of Plücker’s formula and some further results on moduli spaces of cyclic covers) is very close to the approach in [EH1], which simplifies the original rigorous proof of the Brill-Noether theorem given in [GH], all based on degeneration to a singular curve. In Section 3 we prove Theorem 1.1.
2. Preliminaries

2.1. The moduli space $R_{g,n}$. In this paper we will work in the moduli space $R_{g,n}$ of cyclic covers as explained in the introduction.

In the more traditional language of complex analysis a cyclic cover may be considered as a principal $\mathbb{Z}_n$-bundle, where the action is the action of the monodromy group.

By the Hurwitz formula for any cyclic cover $p: \tilde{C} \to C$ (which is unramified by definition), the genus of $\tilde{C}$ will be $\tilde{g} = n(g - 1) + 1$, if the genus of $C$ is $g$. From now on, we keep this relation between $\tilde{g}, g, n$ fixed and we use only $g, n$ as the free variables in our problem.

There is a bijection between cyclic covers and level $n$ curves. See [H] chapter IV exercise 2.7 for the case $n = 2$, and, for general $n$, [S] or [BHPV] for a brief discussion. We need:

Definition 2.1. A level $n$ curve (of genus $g$) or a curve with level $n$ structure is a pair $(C, L)$ consisting of a smooth curve $C$ (of genus $g$) together with an $n$-torsion point $L$ of its Jacobian, i.e $L \in \text{Pic}^0(C)$ a point of order $n$ in the group $\text{Pic}^0(C)$.

We recall that (e.g. based on the theory of stacks) one obtains a coarse moduli space $R_{g,n}$ of level $n$ curves, see [CF]. We shall need the following result:

Theorem 2.2. $R_{g,n}$ is irreducible.

The proof in the seminal paper [DM] is given for the moduli space of curves with full level $n$ structure (which is smooth). But the intermediate space $R_{g,n}$ arises as a finite quotient and thus inherits irreducibility, see [B].

In this paper we will also require a compactification $\overline{R}_{g,n}$ of $R_{g,n}$. It turns out that $\overline{R}_{g,n}$ arises as the coarse moduli space of twisted level $n$ curves, defined as a connected component of $\overline{M}_g(B\mathbb{Z}_n)$, which itself arises as the coarse moduli space of the category $\overline{M}_g(B\mathbb{Z}_n)$ of twisted $n$th roots, which forms a smooth and proper Deligne-Mumford stack. For our notation see e.g. [CEFS], using results from [ACV] and [AV].

Since coarse moduli spaces are sufficient for our purpose in this paper, we shall simply take for granted the existence of $\overline{R}_{g,n}$ as a compactification of $R_{g,n}$, without amplifying what kind of geometric objects are parametrized by this space. For our purpose it is sufficient to look at the closely related moduli space Adm$_g(\mathbb{Z}_n)$ of admissible $\mathbb{Z}_n$-covers, which are geometrically more accessible, and describe this space in geometric terms. We will not go into detail on the geometry of these two moduli spaces and their relation to each other. For this paper we only need to know that there is a surjective morphism $\overline{R}_{g,n} \to \text{Adm}_g(\mathbb{Z}_n)$; see the Diagram 2.1.

Definition 2.3. An admissible $\mathbb{Z}_n$-cover is a morphism $p: \tilde{C} \to C$ of stable curves together with a $p$-invariant (i.e. fibre preserving) action of $\mathbb{Z}_n$ on $\tilde{C}$ such that:

1. every node of $\tilde{C}$ maps to a node of $C$
2. away from the nodes $p$ is a principal $\mathbb{Z}_n$-bundle
3. for any node $x \in \tilde{C}$ there is an integer $r > 0$ and a local equation $\xi \eta = 0$ of $\tilde{C}$ at $x$, such that any element of the stabilizer of $x$ under the action of $\mathbb{Z}_n$ acts as $(\xi, \eta) \mapsto (\zeta \xi, \zeta^{-1} \eta)$, where $\zeta$ is an $r$th root of unity.
The integer \( r \) is called the **index** of \( x \).

By conjugation it easily follows that every node of \( \tilde{C} \) lying above the same node of \( C \) has the same index. This index can be considered as a ramification index. So an admissible \( \mathbb{Z}_n \)-cover will be unramified, if and only if every node has index 1.

It is clear from the definition that for an admissible \( \mathbb{Z}_n \)-cover \( p: \tilde{C} \to C \) of a smooth curve \( C \), \( \tilde{C} \) is also smooth and \( p \) is simply a principal \( \mathbb{Z}_n \)-bundle, which corresponds to a cyclic cover. We shall denote the subset (in fact subscheme) of \( \text{Adm}_g(\mathbb{Z}_n) \) of such isomorphism classes of principal \( \mathbb{Z}_n \)-bundles as \( \mathcal{P}\mathbb{Z}_n \).

For the notion of families of admissible \( \mathbb{Z}_n \)-covers and an explicit construction of the coarse moduli space \( \text{Adm}_g(\mathbb{Z}_n) \) of admissible \( \mathbb{Z}_n \)-covers see [ACG]. This construction uses Kuranishi families of admissible \( \mathbb{Z}_n \) covers and largely resembles the construction of the moduli space \( \overline{\mathcal{M}}_g \), avoiding the use of algebraic (or Deligne-Mumford) stacks. Thus it is both more elementary and geometrically accessible, but, if one so wishes, this moduli space can be identified with a moduli space arising in the algebraically more powerful theory of stacks. In the notation of [CEFS], the Deligne-Mumford stack \( \text{Root}_{g,n} \) of quasi-stable \( n \)th roots (see also [J1], [J2]) is relevant. We also refer to Remark 1.6 in [CF] for the relation between stacks and admissible covers.

There are two forgetful maps (which are morphisms of schemes) \( \text{Adm}_g(\mathbb{Z}_n) \to \overline{\mathcal{M}}_g \) and \( \text{Adm}_g(\mathbb{Z}_n) \to \overline{\mathcal{M}}_{\tilde{g}} \) that map the isomorphism class of an admissible \( \mathbb{Z}_n \)-cover \( p: \tilde{C} \to C \) to the isomorphism class of \( C \) or \( \tilde{C} \). Our proof only needs the map \( \text{Adm}_g(\mathbb{Z}_n) \to \overline{\mathcal{M}}_{\tilde{g}} \).

The crucial point of this very brief review is that one obtains the following commutative diagram

\[
\begin{array}{ccc}
\overline{\mathcal{R}}_{g,n} & \longrightarrow & \text{Adm}_g(\mathbb{Z}_n) \longrightarrow \overline{\mathcal{M}}_{\tilde{g}} \\
\downarrow & & \downarrow \\
\mathcal{R}_{g,n} & \longrightarrow & \mathcal{P}\mathbb{Z}_n \longrightarrow \overline{\mathcal{M}}_{\tilde{g}}
\end{array}
\]

(2.1)

where \( \longrightarrow \) is a surjective morphism and \( \hookrightarrow \) an inclusion. This is implicit in the discussion of [CEFS], combining the results of Section 1.2 and Section 1.3 (generalizing from \( n \) prime to arbitrary natural numbers \( n \); for arbitrary \( n \) there are additional singularities due to the factorization). This diagram is essential for our proof. On the left hand side we have irreducibility due to [DM], via stacks, and in the middle we have a geometric description which will fit with the theory of limit linear series in the following section.

### 2.2. Limit linear series.

Our proof will rely largely on degenerating a family of linear series on smooth curves to a singular curve. For this we need the theory of limit linear series.

For the theory of families of linear series we refer to [ACG], chapter XXI.

We recall that (using the determinantal description of varieties of linear series) the existence of a \( g_0^r \) is a closed condition, i.e. the set \( \mathcal{G}_d(\mathcal{M}_g) := \{ C \in \mathcal{M}_g \mid C \text{ carries a } g_0^r \} \) is a closed subscheme of \( \mathcal{M}_g \).
To prepare our discussion of limit linear series on a (single) tree-like curve (where the irreducible components might be nodal) we need to characterize how a single \( g^r_d \) on a single, possibly singular, curve behaves at a smooth point.

**Definition 2.4.** For a curve \( C \), a smooth point \( x \in C \) and a linear series \( \ell = (L, V) \in G^r_d(C) \) we define the **vanishing sequence** of \( \ell \) at \( x \), 
\[
\{ a_i(\ell, x) \mid i = 0, \ldots, d \} = \{ \text{ord}_p(\sigma) \mid \sigma \in V, \sigma \neq 0 \}.
\]

The **ramification sequence** of \( \ell \) at \( x \), 
\[
\{ \alpha_i(\ell, x) \mid i = 0, \ldots, d \} = \{ a_i(\ell, x) - i \}.
\]

The **weight** or **ramification index** of \( \ell \) at \( x \) is 
\[
w^{\ell}(x) := \sum_{i=0}^{r} \alpha_i(\ell, x) = \sum_{i=0}^{r} a_i(\ell, x) - \frac{1}{2} r (r + 1).
\]

Finally, we recall the Plücker formula, which in the following version is a crucial ingredient of our proof, see e.g. [HM]:

**Theorem 2.5.** If \( \ell \) is any \( g^r_d \) on a smooth and irreducible curve \( C \) of genus \( g \), then the weights will satisfy 
\[
\sum_{x \in C} w^{\ell}(x) = (r + 1)(d + r(g - 1)).
\]
of smooth curves. The original theory is due to Eisenbud and Harris (see [EH2]) and was originally only developed for curves of compact type. Since then it has been extended to tree-like curves (in [EH3]).

**Definition 2.6.** Let $C$ be a tree-like curve. A limit linear series (or limit $g_r^d$) $\ell$ of degree $d$ and dimension $r$ assigns to every irreducible component $X$ of $C$ a linear series $\ell_X \in G^d_r(X)$ called the aspect of $\ell$ on $X$, such that for every pair of distinct irreducible components $X, Y \subset C$ meeting in the node $x$ the aspects satisfy for all $i$

$$a_i(\ell_X, x) + a_{r-i}(\ell_Y, x) \geq d. \quad (2.6)$$

If equality holds everywhere the limit linear series $\ell$ is called refined.

We recall that in [EH2] our limit linear series are called crude. The important thing about limit linear series is that they arise as limits of linear series. To make this more precise consider the following situation: Let $B$ be a regular, integral scheme of dimension 1, $b_0 \in B$ a closed point and $U = B \setminus \{b_0\}$. For example $B$ might be a smooth curve or the spectrum of a discrete valuation ring. Consider a family of stable curves $\phi : X \rightarrow B$ such that the fibre $C_0$ over $b_0$ is tree-like and the family is smooth everywhere else, i.e. $\phi|_U$ is a family of smooth curves.

Now let $(\mathcal{L}_U, \mathcal{H}_U)$ in the notation of [ACG] be a family of $g^d_r$s on $\phi|_U$.

Then for every irreducible component of $C_0$ the line bundle $\mathcal{L}_U$ on $\phi^{-1}(U) \subset X$ can be uniquely extended to a line bundle $\mathcal{L}$ on $X$, such that $\mathcal{L}|_{C_0}$ has degree $d$ on this component and degree 0 on all other components. Then $\mathcal{H}_U$ will induce $g^d_r$s on the irreducible components of $C_0$. These will turn out to form a unique limit $g^d_r$ on $C_0$.

In [EH2], Eisenbud and Harris construct the limit $g^d_r$ for the case that $B$ is the spectrum of a discrete valuation ring and the special fibre is of compact type. In [EH3] they extend this to a more general situation, but without giving a detailed proof. In particular, they claim:

**Lemma 2.7.** Let $B$ be a regular, integral scheme of dimension 1, $b_0 \in B$ a closed point and $U = B \setminus \{b_0\}$. Let $\phi : X \rightarrow B$ be a family of stable curves such that the fibre $C_0$ over $b_0$ is tree-like and the family is smooth everywhere else, i.e. $\phi|_U$ is a family of smooth curves. If $(\mathcal{L}_U, \mathcal{H}_U)$ is a family of $g^d_r$s on $\phi|_U$, then there exists a unique crude limit $g^d_r$ on $C_0$ which arises as the limit of $(\mathcal{L}_U, \mathcal{H}_U)$.

We will need the following result, which (if $B$ consists of more than 2 points) is slightly stronger than the contra-position of the statement in Lemma 2.7. The easy special case mentioned above is the important case of $B$ being a discrete valuation ring consisting of one open and one closed point. We also could have reduced our proof to this special case, but we feel that the more general case is more geometrical and fits more naturally in the discussion of moduli spaces. Since we need the generalization of [EH2] to tree-like curves anyway, we have formulated a general result.

**Proposition 2.8.** In the setting of Lemma 2.7, if $C_0$ does not carry a crude limit $g^d_r$, then the general smooth fibre $C_b$ of $\phi$ does not carry a $g^d_r$. 
The idea of a proof is as follows. Similar to $G^r_d(M_g)$ in $M_g$, one considers families of limit linear series and shows that the space

$$G^r_d(M_g) := \{ C \in \overline{M}_g \mid C \text{ carries a limit-}g^r_d \}$$

is closed in $\overline{M}_g$.

Some of the most important applications of limit linear series are in using them to prove theorems on general smooth curves. A typical example (needed in our proof) is:

**Lemma 2.9.** A general pointed curve $(C, x)$ of genus $g$ possesses a $g^r_d$ with ramification sequence $0 \leq \alpha_0(\ell, x) \leq \alpha_1(\ell, x) \leq \cdots \leq \alpha_r(\ell, x) \leq d - r$ at $x$, if and only if

$$\sum_{i=0}^{r} \max\{\alpha_i + g - d + r, 0\} \leq g. \quad (2.7)$$

For a proof see [EH3], Proposition 1.2. The proof of sufficiency of (2.7) uses the smoothing out result for limit linear series, the proof of necessity uses Lemma 2.7. Only this part is needed for our proof.

2.3. **Castelnuovo canonical curves.** We shall introduce a special class of singular curves (tree-like, but not of compact type) which will play an essential role in our proof of Theorem 1.1. These Castelnuovo canonical curves go back to Castelnuovo’s paper [C], see [GH] for a modern reference. They have topological genus 0, but arithmetical genus $g$.

**Definition 2.10.** An irreducible curve of arithmetical genus $g$ is called a **Castelnuovo canonical curve** if and only if it has precisely $g$ simple nodes.

Obviously such a curve is tree-like, since its dual graph consists of one vertex with $g$ distinct loops attached. Its normalization is $\mathbb{P}^1$.

Conversely, a Castelnuovo canonical curve is most easily constructed from its normalization by selecting $2g$ different points $x_j, y_j$ for $j = 1, \ldots, g$ and identifying the points $x_j$ with $y_j$, which is also well defined for reducible curves. This gives the normalization map

$$N : \mathbb{P}^1 \to \mathbb{P}^1/x_1 \sim y_1, \ldots, x_g \sim y_g.$$ 

Since Castelnuovo canonical curves are irreducible, it makes sense to consider $g^r_d$s defined on them. Any such $g^r_d$ $\ell$ can be pulled back to a $g^r_d$ on $\mathbb{P}^1$, and one has:

**Lemma 2.11.** The weights in smooth points coincide, i.e. if $C$ is a Castelnuovo canonical curve with normalization map $N : \mathbb{P}^1 \to C$ and $\ell$ is a $g^r_d$ on $C$, then

$$w^{N*\ell}(x) = w^{\ell}(N(x))$$

for any $x \in \mathbb{P}^1$ with $N(x)$ smooth.
3. Proof of Theorem 1.1

We have to show that for fixed \( n, r, g, d \) and \( \tilde{g} = n(g - 1) + 1 \) with

\[
\rho(\tilde{g}, r, d) < -r,
\]

(3.1)

there is an open and dense set \( V \) in \( \mathcal{R}_{g,n} \) such that for any cyclic cover \( p : \tilde{C} \to C \) corresponding to a point in \( V \) no \( g^r_d \) exists on \( \tilde{C} \).

Step 1: One singular example suffices

First note that \( \mathcal{R}_{g,n} \) is irreducible by Theorem 2.2. Therefore every non-empty open set is already dense. Now consider the Diagram 2.1 from Section 2.1.

Since the set \( \mathcal{G}^r_d(\mathcal{M}_{\tilde{g}}) = \{ C \in \mathcal{M}_{\tilde{g}} \mid C \text{ carries a } g^r_d \} \) is closed, its preimage in \( \mathcal{R}_{g,n} \) is also closed. Therefore, its complement \( V \) is open, and we are reduced to showing that it is non-empty.

The map \( \mathcal{R}_{g,n} \to \mathcal{PZ}_n \) is surjective and it therefore suffices to exhibit a single principal \( \mathbb{Z}_n \)-bundle \( p : \tilde{C} \to C \) over \( \mathcal{PZ}_n \), such that there is no \( g^r_d \) on \( \tilde{C} \). Unfortunately, it seems impossible to directly construct such a smooth example.

It is, however, possible to construct an admissible \( \mathbb{Z}_n \)-cover \( p_0 : \tilde{C}_0 \to C_0 \), such that there is no limit \( g^r_d \) on \( \tilde{C}_0 \). We claim that this proves the existence of some smooth example \( p : \tilde{C} \to C \), such that there is no \( g^r_d \) on \( \tilde{C} \).

In fact, choose a family of admissible \( \mathbb{Z}_n \)-covers over a regular one-dimensional integral base \( B \), such that the special fibre over some closed point \( b_0 \in B \) corresponds to our singular example \( p_0 : \tilde{C}_0 \to C_0 \) and the family is smooth over \( B \setminus \{ b_0 \} \).

This family will induce a family \( \tilde{\phi} : X \to B \) of stable curves of genus \( \tilde{g} \), which satisfies the conditions of Lemma 2.7. If there is no limit \( g^r_d \) on \( \tilde{C}_0 \) then by Proposition 2.8 there is some smooth fibre \( \tilde{\phi}^{-1}(b) = \tilde{C} \) which carries no \( g^r_d \).

By the construction of \( \tilde{\phi} \) it is clear that this fibre came from some principal \( \mathbb{Z}_n \)-bundle \( p : \tilde{C} \to C \). This proves our claim.

Step 2: Singular examples and a preliminary estimate

We shall now construct singular examples.

Fix a general 1-pointed curve \( (C_0, x) \in \mathcal{M}_{g-1,1} \) and a (possibly tree-like) elliptic 1-pointed curve \( (E, y) \in \overline{\mathcal{M}}_{1,1} \), where \( y \) is a smooth point of \( E \).

Fix also an admissible \( \mathbb{Z}_n \)-cover \( p_E : \tilde{E} \to E \) over \( E \) and set \( \{ y_1, \cdots, y_n \} := p_E^{-1}(y) \). By the Hurwitz formula \( \tilde{E} \) is also an elliptic curve.

Now choose \( n \) identical copies \( (C_1, x_1), \cdots (C_n, x_n) \) of \( (C_0, x) \) and define the tree-like curves

\[
C := C_0 \cup E/x \sim y, \quad \tilde{C} := \tilde{E} \cup C_1 \cup \cdots \cup C_n/(x_1 \sim y_1, \cdots, x_n \sim y_n).
\]

(3.2)

Clearly this induces an admissible \( \mathbb{Z}_n \)-cover \( p : \tilde{C} \to C \) (with index \( r = 1 \) at each node).
Assume that $\ell = \{\ell_{E}, \ell_{C_1}, ..., \ell_{C_n}\}$ is a limit $g_{d}^{r}$ on $\hat{C}$. Denote by
\[ a_{i} : 0 \leq a_{i,0} < a_{i,1} < \cdots < a_{i,r} \leq d \]
the vanishing sequence of $\ell_{E}$ at $x_i = y_i$, by
\[ b_{i} : 0 \leq b_{i,0} < b_{i,1} < \cdots < b_{i,r} \leq d \]
the vanishing sequence of $\ell_{C_i}$ at $x_i$ and by
\[ w(x_i) := w^{E}(x_i) = \sum_{j=0}^{r} a_{i,j} - j \]
the weight of $\ell_{E}$ at $x_i$.

Then, by construction, each $(C_i, x_i)$ is a general 1-pointed curve of genus $g - 1$. In particular, it satisfies the hypothesis in Proposition 2.9. Thus the inequality (2.7) holds, which implies
\[ \sum_{j=0}^{r} b_{i,j} \leq (r + 1)d - r(g - 1) - \frac{1}{2}r(r + 1). \tag{3.3} \]

By definition of a limit $g_{d}^{r}$ we have for all $i, j$
\[ a_{i,j} + b_{i,r-j} \geq d. \tag{3.4} \]
Adding these inequalities over all $j$ and subtracting (3.3) gives for all $i$
\[ w(x_i) \geq r(g - 1). \tag{3.5} \]
Now assume that $E$ and $\tilde{E}$ are smooth. Then we can apply the the Plücker formula (2.5) to $\tilde{E}$ to get

\[(r + 1)d = \sum_{x \in \tilde{E}} w(x) \geq \sum_{i=1}^{n} w(x_i) \geq \sum_{i=1}^{n} r(g - 1) = nr(g - 1) = r(\tilde{g} - 1).\]  

(3.6)

So a $g^d_\tilde{E}$ on $\tilde{C}$ can only exist if $d \geq \frac{r}{r + 1} (\tilde{g} - 1)$. This is equivalent to $\rho(\tilde{g}, r, d) \geq -r(r + 2)$.

Step 3: Improved estimate

It is clear from Step 1 that the estimate can be optimized by choosing $E$ and $\tilde{E}$ in a special way. Thus we are led to optimizing the lower bound for $d$ such that a $g^d_\tilde{E}$ with given weights $w(y_i) \geq r(g - 1)$ exists on a specific elliptic curve $\tilde{E}$. Therefore we shall now choose $E$ and $\tilde{E}$ as Castelnuovo canonical curves.

This will allow us to lower the left hand side of (3.6) by pinching away the hole of the elliptic curve and using the Plücker formula for $g = 0$.

We choose $E$ and $\tilde{E}$ as the projective line with the points 0 and $\infty$ identified and denote its normalisation by $N : \mathbb{P}^1 \rightarrow \mathbb{P}^1 / 0 \sim \infty = E = \tilde{E}$.

The map $z \mapsto z^n$ on $\mathbb{P}^1$ induces an admissible $\mathbb{Z}_n$-cover $p_E : \tilde{E} \rightarrow E$ (with index $r = n$ at the node). As the smooth point $y$ on $E$ and its preimages $\{y_1, \cdots, y_n\} := p_E^{-1}(y)$, we choose $y = N(1)$ and $y_i = N(\zeta^i)$, where $\zeta$ is a primitive $n$th root of unity.

We now have to investigate for which $g, r, d$ there exists a $g^d_\tilde{E}$ on $\tilde{E}$ with given weights $w(y_i) \geq r(g - 1)$.

Note that by Lemma 2.11 any $g^d_\tilde{E}$ $\ell$ on $\tilde{E}$ can be pulled back to a $g^d_E$ $\ell$ on $\mathbb{P}^1$ via the normalization $N : \mathbb{P}^1 \rightarrow \tilde{E}$. Under this correspondence the weight in any smooth point is preserved, i.e. $w^\ell(N(x)) = w^\ell(x)$ for any point $x \in \mathbb{P}^1$ with $N(x)$ smooth.

Thus applying the Plücker formula (2.5) to $\ell$ gives the following estimate:

\[(r + 1)(d - r) = \sum_{x \in \mathbb{P}^1} w^\ell(x) \geq \sum_{i=1}^{n} w^\ell(\zeta^i) = \sum_{i=1}^{n} w^\ell(y_i) \geq \sum_{i=1}^{n} r(g - 1) = rn(g - 1) = r(\tilde{g} - 1).\]  

(3.7)

This is equivalent to $d \geq \frac{r}{r + 1} (\tilde{g} - 1) + r$ and thus to $\rho(\tilde{g}, r, d) \geq -r$, proving Theorem 1.1.

4. Acknowledgements

This paper is a restructured and substantially shortened version of my diploma thesis [Sch] written under the supervision of Prof. Farkas at the Humboldt University of Berlin. I want to thank my advisor G. Farkas both for suggesting this interesting subject which allowed me

\[1\text{If } E \text{ is singular, the usual Plücker formula for singular curves replaces the first equality in (3.6) by } \geq, \text{ leading to the same estimate.}\]
to work and think about a real problem in algebrailal geometry already during my diploma thesis and for contributing crucial ideas to the proof.

References

[ACG] E. Arbarello, M. Cornalba, P. A. Griffiths: Geometry of algebraic curves, vol. 2, Grundlehren der mathematischen Wissenschaften, Springer (2011)

[ACV] D. Abramovich, A. Corti, A. Vistoli: Twisted bundles and admissible covers, Comm. Algebra 31 (2003), 3547–3618

[AF] M. Aprodu, G. Farkas: Green’s conjecture for general covers, Compact Moduli Spaces and Vector Bundles: Conference on Compact Moduli and Vector Bundles, University of Georgia, Athens, Georgia, Vol. 564, (October 21-24, 2010) J. Amer. Math. Soc. (2012)

[AV] D. Abramovich, A. Vistoli: Compactifying the space of stable maps, J. Amer. Mat. Soc. 15 (2001), 27–75

[B] M. Bernstein: Moduli of curves with level structure, Dissertation, Harvard (1999)

[BHPV] W. P. Barth, K. Hulek, C. A. M. Peters, A. van de Ven: Compact complex surfaces, Springer (2004)

[C] G. Castelnuovo: Numero delle involutione rationali glancenti sopra una curva di dato genere, Rend. della R. Acad. Lincei, ser. 4, 5 (1889)

[CEFS] A. Chiodo, D. Eisenbud, G. Farkas, F. O. Schreyer: Syzygies of torsion bundles and the geometry of the level L modular variety over $\overline{M}_g$, Invent. math. 194 (2013), 73–118

[CF] A. Chiodo, G. Farkas: Singularities of the moduli space of level curves, arXiv:1205.0201v4 (2015)

[DM] P. Deligne, D. Mumford: The irreducibility of the space of curves of given genus, Publications Mathématiques de l’IHES 36 (1969), 75–109

[EH1] D. Eisenbud, J. Harris: On the Brill-Noether theorem, Algebraic geometry - open problems. Springer Berlin Heidelberg (1983) 131–137

[EH2] D. Eisenbud, J. Harris: Limit linear series: basic theory, Invent. math. 85 (1986), 337–371

[EH3] D. Eisenbud, J. Harris: The Kodaira dimension of the moduli space of curves of genus $\leq 23$, Invent. Math. 90 (1987), 359–387

[F] G. Farkas: The birational geometry of the moduli space of curves, Akademisch Proefschrift, Amsterdam (2000)

[FL] W. Fulton, R. Lazarsfeld: On the connectedness of degeneracy loci and special divisors, Acta Math. 146.1 (1981), 271–283

[GH] P. Griffiths, J. Harris: On the variety of special linear systems on a general algebraic curve, Duke Math. J. 47(1980), 233–272

[H] R. Hartshorne: Algebraic Geometry, Springer (1977)

[HM] J. Harris, I. Morrison: Moduli of Curves, Springer (1998)

[J1] T. J. Jarvis: Torsion-free sheaves and moduli of higher spin curves, Compositio Mathematica 110 (1998), 291–333

[J2] T. J. Jarvis: Geometry of the moduli of higher spin curves, Int. J. Math. 11 (2000), 23–47

[K] G. Kempf: On the geometry of a theorem of Riemann, Ann. of Math. 98 (1973), 178–185

[KL] S. Kleiman, D. Laksov: On the existence of special divisors, Amer. J. Math. 94 (1972), 431–436

[L] R. Lazarsfeld: Brill-Noether-Petri without degeneration, J. Diff. Geom. 23 (1986), 299–307

[M] Th. Meis: Die minimale Blätterzahl der Konkretisierungen einer kompakten Riemannschen Fläche, Schriftenreihe d. Math. Inst. d. Univ. Münster, H. 16 (1960)

[S] J. P. Serre: Sur la topologie des variétés algébriques en caractère p, Symposium de topologie algébrique, Mexico (1956), 24–53

[Sch] I. Schwarz: Brill-Noether theory for cyclic covers, diploma thesis, Berlin (2016)

Humboldt Universität Berlin, Institut für Mathematik