Heat transfer attributes of MoS₂/Al₂O₃ hybrid nanomaterial flow through converging/diverging channels with shape factor effect

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Abstract
The energy transport for hybrid nanofluids flow through non-parallel surfaces with converging/diverging nature is becoming important engineering topics because of its occurrence in biomedicine, cavity flow model and flow through canals, etc. Therefore, this work attempted to study the momentum and heat transport for MHD Jeffery-Hamel flow of hybrid nanofluids through converging/diverging surfaces. This analysis further evaluates the heat transport features subject to thermal radiation and nanoparticles shape factor impacts. A mathematical formulation under single phase nanofluid model with modified thermophysical properties has been carried out. The leading equations are transmuted into dimensionless form with the implementation of appropriate scaling parameters. The collocated numerical procedure coded in MATLAB is employed to acquire the numerical solutions for governing coupled non-linear differential problem. Multiple branches (first and second) are simulated for flow and temperature fields with varying values of involved physical parameters in case of convergent channel. The studies revealed that there is a significant rise in fluid velocity for higher magnetic parameter in case of divergent channel. The findings reveal that the skin-friction coefficient (drag) significantly reduces with higher Reynolds number. In addition, the heat transfer rate enhances with channel angle as well as nanoparticles volume fraction in upper branches.

Keywords
Heat transport, hybrid nanofluids, thermal radiation, multiple solutions, converging/diverging surfaces, nanoparticles shape factor

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Introduction
In the contemporary generation of technology, innovation and science, the study of nanofluids has secured a miraculous consideration because they have unlimited spectrum of utilisations in almost all fields of science and technology. Accordingly, the nanofluids are being used in numerous fields, for example in biomedicine, fuel cells, hybrid-power engines, domestic refrigerator, chiller, polymer coating, aerospace technology, nuclear systems cooling, thermal storage, engine cooling, etc. Various investigations have proven that this new class

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of fluids acquires intensified thermo-physical properties, like, thermal conductivity, thermal diffusivity, convective heat transfer coefficients and viscosity in comparison to the base liquids. A stable and uniform suspension of ultra-fine solid particles (mostly of metals and their oxides) in base liquids, is termed as nanofluid. Choi\(^1\) is truly considered as a pioneer who introduced the term ‘nanofluid’ in the year 1995. In this investigation, he debated that the thermal transport properties of base liquids in the presence of nanoparticles have increased dramatically. The most significant point is the ultra-fine size of these nanoparticles which ranges from 1 to 100 nanometre, in this way nanofluids look like to perform more alike a single-stage liquid than a solid-liquid amalgamation. Eastman et al.\(^2\) have predicted an enhancement about (60%) of thermal conductivity of nanofluid for Al\(_2\)O\(_3\), CuO and Cu nanoparticles for different base fluid and taking only (5%) volume frication of nanoparticles. In regard of these characteristic, many researchers have shown their interest to examine different features of nanofluids flow along with their heat transport analysis. Later, the enhanced heat exchange properties of nanofluids were addressed by Buongiorno\(^3\) by considering Brownian motion and thermophoresis slip mechanics. In another study, Tiwari and Das\(^4\) proposed a different model to analyse the nanofluids heat transport characteristics by considering the fraction of the solid volume. Kuznetso\(\)v and Nield\(^5\) establishes an analytical solution for natural convection flow of nanofluid over a vertical plat with two slip mechanisms. A comprehensive review concerning nanofluids flow and heat transport is given by several researchers, see references.\(^6\)\(^–\)\(^10\)

Very recently, another innovative class of heat transfer fluid recognized as ‘hybrid nanofluid’ acts as a heat transporter in heat transfer applications. Hybrid nanofluids are processed by the amalgamation of two or more distinct nanoparticles into a base fluid. It is assumed that hybrid nanofluids can increase or changed the thermal conductivity and heat transfer execution. Engineers, researchers and scientists are paying a lot of attention to hybrid nanofluid because of its wide range of commercial, scientific and technological applications, for instance, microfluidics, medical lubrication, transportation, acoustics, naval structures and solar heating and many more. Theoretical and numerical studies about hybrid nanofluid flow and heat transfer analysis have been conducted over the recent years to better understand their rheological and heat transfer properties. The pioneer work studying thermo-physical transport properties of hybrid nanofluids was prepared by Jana et al.\(^11\) Some reviews on the applications, preparation and thermophysical properties of hybrid nanofluids have been conducted by Sarkarn et al.\(^12\) and Akilu et al.\(^13\) Recently, Devi and Devi\(^14\) revised the thermophysical correlations for hybrid nanofluids and applied them to stretching flow problems, along with Tiwari and Das’s single phase nanofluid mathematical model. Mackolil and Mahanthesh\(^15\) discussed the radiative heat transport analysis for Cu – Al\(_2\)O\(_3\) – H\(_2\)O hybrid nanofluid flow past a vertical flat plate. Some other remarkable studies describing the hybrid nanofluid heat transport phenomenon with their applications is mentioned in works of Mayson and Mahanthesh,\(^16\) Ashlin and Mahanthesh,\(^17\) Thriveni and Mahanthesh,\(^18,\)\(^19\) and Mahanthesh et al.\(^20\)

Recently the topic of thermal radiations seems quite interesting field of research. The reasons behind this valuable attraction are that the radiative effect has a pivotal role in the fields of space technology, geophysics, polymer industry and engineering. In industry the radiative heat transport plays a key role in regulating heat transfer because the standard of final product directly relates with heat controlling factors. Moreover, heat transfer process in the presence of non-linear radiations has central role in renewable systems and in the investigation and warm recuperation of oil. Many researchers have focused their attention on thermal radiation effect since it has a wide range of physical, manufacturing and engineering applications. In this regard, mixed convection flow of nanofluid past a vertical wedge in the existence of thermal radiation was investigated by Chamkha et al.\(^21\) Later, Khan et al.\(^22\) reported a three-dimensional flow of Burgers nanofluid with heat transport analysis by considering non-linear thermal radiation and convective boundary conditions. Alam et al.\(^23\) investigated the entropy generation during radiative heat transfer analysis in a thin viscous nanofluid flow through a parallel channel within porous media. Waqas et al.\(^24\) elaborated the influence of nonlinear radiation on stretching flow of ferrofluids with magnetic dipole. Some other related works can be found in the studies.\(^25\)\(^–\)\(^27\)

After a careful review of above literatures, we have decided to present a numerical study which will focus on the following issues related to Jeffery-Hamel flow of hybrid nanofluids:

- The first one is to model the momentum and heat transport equations for molybdenum disulphide (MoS\(_2\))– aluminium-oxide (Al\(_2\)O\(_3\)) hybrid nanofluids flowing between two non-parallel walls.
- The impacts of non-linear thermal radiation and shape factor effects of nanoparticles are incorporated to investigate the flow phenomenon.
- The second novel objective is seeking branches (multiple solutions) for flow and heat transport fields in case of convergent channel.
- At last, the physical illustration of obtained numerical solutions for various key parameters with the help of graphical results.
Mathematical formulation and flow configuration

Description of physical problem

At present, a steady incompressible flow of an electrically conducting hybrid nanofluids flowing in the channel formed with converging/diverging nature is considered. The flow is due to the presence of a source or sink between walls restricting the flow which intersect an angle of $\alpha$:

The channel is taken to be convergent if $\alpha < 0$ whereas it is divergent if $\alpha > 0$. The hybrid nanofluids are composed of $\text{MoS}_2$ and $\text{Al}_2\text{O}_3$, while the water is chosen as base fluid. Impacts of thermal radiation as well as the nanoparticles shape factor are considered for heat transport analysis. We have considered cylindrical polar coordinate $r$, $\theta$, $z$, as shown in Figure 1. In addition, $z$– axis is coinciding with the intersection line of these walls and flow is purely radial with velocity $v_r(r, \theta)$. In literature, such type of flow is usually termed as the Jeffery-Hamel flow problem. A magnetic field of variable strength $B(r) = B_0$ acts transversely to the flow. The hybrid nanofluid temperature is denoted by $T$, while the surface temperature is $T_w$.

Governing flow equations

The momentum and energy transport within porous media are modelled using above mentioned assumptions and basic conservation laws as follows:

1. \[
\frac{\partial v_r}{\partial r} + \frac{1}{r} \frac{\partial}{\partial \theta} (r v_r) = 0, \tag{1}
\]

2. \[
\frac{1}{\rho_{\text{nf}}} \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{\mu_{\text{nf}}}{r^2} \frac{\partial v_r}{\partial \theta} \right) = 0, \tag{2}
\]

3. \[
\frac{1}{\rho_{\text{nf}}} \frac{1}{r} \frac{\partial p}{\partial \theta} - \frac{\mu_{\text{nf}}}{r^2} \frac{2}{\partial \theta} \left( \frac{\partial v_r}{\partial \theta} \right) = 0, \tag{3}
\]

4. \[
\frac{1}{\rho_{\text{nf}}} \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{\partial T}{\partial \theta} \right) = 0, \tag{4}
\]

In the above expressions, $p$ denotes fluid pressure, $B_0$ is the magnitude of applied magnetic field, $\rho_{\text{nf}}$ depicts the density, $\mu_{\text{nf}}$ signifies the viscosity, $(c_p)_{\text{nf}}$ represents the specific heat, $k_{\text{nf}}$ denotes the thermal conductivity and $\sigma_{\text{nf}}$ means the electric conductivity of the hybrid nanofluids. Furthermore, the radiative heat fluxes along $(r, \theta)$– direction are denoted by $q_{r,\text{rad}}$ and $q_{\theta,\text{rad}}$, respectively. The Rosseland approximation is accounted, and the radiative heat fluxes expressed in the following form:

5. \[
q_{r,\text{rad}} = -\frac{16\sigma^* T_r^3}{3k^*} \frac{\partial T}{\partial r} \quad \text{and} \quad q_{\theta,\text{rad}} = -\frac{16\sigma^* T_r^3}{3k^*} \frac{\partial T}{\partial \theta}, \tag{5}
\]

where $\sigma^*$ is the Stefan-Boltzmann constant and $k^*$ is the coefficient of mean absorption.
Substituting equations (5) into (4), the energy equation subject to thermal radiation takes the form:

\[ \frac{\partial T}{\partial r} = \frac{k_{hnf}}{(pc_p)_{hnf}} \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} \right) \]

\[ + \frac{16\sigma \alpha^2 T^3}{3k^* (pc_p)_{hnf}} \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} \right) \]  

(6)

**Boundary conditions**

The momentum and energy transport are added by the following boundary conditions:

\[ \nu_r = \frac{U}{r}, \quad \frac{\partial v_r}{\partial \theta} = 0, \quad \frac{\partial T}{\partial \theta} = 0, \text{ at } \theta = 0, \]

\[ \nu_r \rightarrow v_w = \frac{s}{r} \frac{T}{T_m}, \text{ as } \theta \rightarrow \alpha, \]  

(8)

**Thermo-physical properties of hybrid nanofluids**

In the present study, the thermophysical properties of water and hybrid nanofluids with platelet, spherical and brick nanoparticles \((\mu_{hnf}, \rho_{hnf}, \sigma_{hnf}, k_{hnf}, (pc_p)_{hnf})\) are calculated using following relations:

\[ \mu_{hnf} = (1 - \phi_1)^{-2.5} (1 - \phi_2)^{-2.5}, \]  

(9)

\[ \frac{\rho_{hnf}}{\rho_f} = \left( 1 - \phi_2 \right) \left( 1 - \phi_1 + \frac{\phi_1 \rho_1}{\rho_f} \right) + \frac{\phi_2 \rho_2}{\rho_f}, \]  

(10)

\[ \sigma_{hnf} = \left\{ \begin{array}{l}
\sigma_2 (1 + 2\phi_2) + 2\sigma_2 (1 - \phi_2) \\
\sigma_1 (1 + 2\phi_1) + 2\sigma_1 (1 - \phi_1)
\end{array} \right\}, \]  

\[ \sigma_f = \left\{ \begin{array}{l}
\sigma_2 (1 - \phi_2) + \sigma_2 (2 + \phi_2) \\
\sigma_1 (1 - \phi_1) + \sigma_1 (2 + \phi_1)
\end{array} \right\}, \]  

(11)

\[ \frac{(pc_p)_{hnf}}{(pc_p)_f} = \left( \begin{array}{l}
(1 - \phi_2) \left( 1 - \phi_1 + \frac{\phi_1 (pc_p)_s}{(pc_p)_f} \right) \\
+ \frac{\phi_2 (pc_p)_s}{(pc_p)_f}
\end{array} \right), \]  

(12)

\[ k_{hnf} = \frac{k_2 + (m - 1) k_{bf} - (m - 1) \phi_2 (k_{bf} - k_2)}{k_2 + (m - 1) k_{bf} + \phi_2 (k_{bf} - k_2)} k_{bf}, \]

(13)

\[ k_{bf} = \frac{k_1 + (m - 1) k_{bf} - (m - 1) \phi_1 (k_{bf} - k_1)}{k_1 + (m - 1) k_{bf} + \phi_1 (k_{bf} - k_1)} k_{bf}. \]

The numerical values of water - MoS\(_2\)/Al\(_2\)O\(_3\) hybrid nanofluid employed in this analysis are displayed in Table 1.

**Table 1. Thermo-physical properties of the base fluid and different nanoparticles.**

| Physical properties | Base fluid | Nanoparticles |
|---------------------|------------|---------------|
| \(\rho\) \((kg/m^3)\) | 997.1 | 5060 | 3970 |
| \(\sigma_f\) \((\mu\)kgK\) | 4179 | 397.2 | 765 |
| \(k\) \((W/mK)\) | 0.613 | 904.4 | 40 |
| \(\alpha\) \((5/m)\) | 5.5 \(\times 10^{-6}\) | 2.09 \(\times 10^{4}\) | 1 \(\times 10^{-10}\) |

**Non-dimensional analysis**

To make the prevailing system dimensionless, we introduce the following dimensionless parameters:

\[ \xi = \frac{\theta}{\alpha}, \quad f(\xi) = \frac{F(\theta)}{U}, \quad F(\theta) = rv_r(r, \theta), \quad T = \frac{T_m}{r^2} \Phi(\theta), \]  

(14)

where \(U\) represents the centreline velocity.

Eliminating pressure term from equations (2) and (3) and incorporating equation (14), subsequent system of governing ordinary differential equations become:

\[ \frac{N_1}{N_2} (f'' + 4\alpha^2 f') + 2\alpha \Re f'' - \frac{N_3}{N_2} \alpha^2 M f' = 0, \]

(15)

\[ (N_5 + Rd)(\Phi'' + 4\alpha^2 \Phi) + 2N_5 \alpha^2 \Pr f \Phi = 0, \]  

(16)

with boundary constraints

\[ f(0) = 1, \quad f'(0) = 0, \quad \Phi'(0) = 0, \]

(17)

\[ f(1) = \lambda, \quad \Phi(1) = 1, \]  

(18)

where,

\[ \frac{\mu_{hnf}}{\mu_f} = N_1, \quad \frac{\rho_{hnf}}{\rho_f} = N_2, \quad \frac{\sigma_{hnf}}{\sigma_f} = N_3, \]

\[ \frac{(pc_p)_{hnf}}{(pc_p)_f} = N_4, \quad \frac{k_{hnf}}{k_f} = N_5. \]  

(19)

The physical parameters governing the flow and heat transport phenomena are listed as follows:

The Reynolds number \(Re = \frac{\alpha U}{v_f}\), the Prandtl number \(Pr = \frac{U (pc_p)_f}{k_f}\), the stretching/shrinking parameter \(\lambda = \frac{s}{r}\), the magnetic parameter \(M = \frac{\alpha \beta_0^2}{\mu_f}\) and radiation parameter \(Rd = \frac{16\sigma \alpha^2 T^3}{3k^* k_f}\).
Engineering quantities

The skin-friction coefficient at the wall of the channel is represented by the mathematical expression $C_f$ as

$$C_f = \frac{\tau_w}{\rho \left( \frac{y}{\alpha} \right)^2}, \text{ where } C_f = \text{Re}C_f = N_1f'(1). \quad (20)$$

The Nusselt number has been computed at the wall of the channel which measures the heat transfer rate, and it is given by $Nu$ as:

$$Nu = \frac{q_w}{k_f T_w}, \text{ where } Nu = \alpha^2 Nu = -(N_5 + Rd) \Phi'(1). \quad (21)$$

In equations (20) and (21), the wall shear stress and heat flux are defined by the relation

$$\tau_w = \mu \left[ \frac{1}{r} \left( \frac{\partial v_r}{\partial \theta} \right) \right]_{\theta = \alpha},$$

$$q_w = -\left( k_{nf} + \frac{16 \alpha^2 T_w^3}{3 k_e} \right) \left( \frac{1}{r} \frac{\partial T}{\partial \theta} \right)_{\theta = \alpha}. \quad (22)$$

Computational scheme

In this work, a numerical technique based on finite difference scheme, namely, bvp4c solver in MATLAB is implemented for numerical simulations. Therefore, the two-dimensional flow and heat transport equations governing the current model (equations (15) and (16)) with auxiliary boundary conditions (equations (17) and (18)) are numerically tackled with the said technique. The most fundamental step of this procedure is to transmute the given higher order boundary value problem into that of the boundary value problems of order one. For this purpose, we introduce the new variables $y_1(\xi), y_2(\xi), y_3(\xi), y_4(\xi)$ and $y_5(\xi)$. By utilizing these new variables, our governing problem equations (15–18) is modified into the following problem:

$$\begin{align*}
 f &= y_1, f' &= y_2, f'' &= y_3, \quad \Phi = y_4, \quad \Phi' = y_5, \quad y_1(0) = 0, y_2(0) = 0, y_3(0) = 0, \quad y_1(1) = \lambda, y_4(1) = 1.
\end{align*} \quad (23)$$

The bvp4c routine is executed by giving suitable values to involved physical parameters and appropriate initial guesses at the mesh points. As, the multiple branches (upper and lower solutions) exist for our problem, we need two good initial guesses in this routine which fulfil all the boundary conditions.

Results and validation

Code validation

The comparison of simulated results in terms of skin-friction coefficient $f'(1)$ and Nusselt number $-\Phi'(1)$ with already published works of Turkyilmazoglu\textsuperscript{30} is presented in Table 2. The current outcomes reveal a satisfactory agreement with published data, justifying the reliability of our numerical data.

Discussion

In this section, we provide the numerical outcomes in the form of non-dimensional velocity and temperature profiles of hybrid nanofluids by taking different variation in involved physical parameters, like, magnetic parameter $M$, Reynolds number $Re$, stretching/shrinking parameter $\Lambda$, channel angle $\alpha$, nanoparticles volume fraction $\varphi_1$ and $\varphi_2$, radiation parameter $Rd$, Prandtl number $Pr$ shape factor $n$. The purpose of this research would then be to predict the multiple numerical solutions (first and second solutions) that occur in the case of convergent channel ($\alpha < 0$). The numerical simulation in present study is conducted by imposing fixed values for the parameters involved: $M = 1$, $Re = 10$, $\varphi_1 = 0 = \varphi_2 = M = Rd$.

| $\alpha$ | $\lambda$ | $f'(1)$ (Turkyilmazoglu) | $f'(1)$ (Present study) | $-\Phi'(1)$ (Turkyilmazoglu) | $-\Phi'(1)$ (Present study) |
|---|---|---|---|---|---|
| 0 | 2.8339514330 | -2.83395146063 | 0.0421517243 | 0.0421517243404 |
| -1 | -4.6521591354 | -4.6521759323 | 0.0373226368 | 0.037322660848 |
| -2 | -5.1309222926 | -5.13092207485 | 0.0315761821 | 0.0315761817798 |
| -3 | -3.4941140854 | -3.4941140854 | 0.0240026913 | 0.0240026911171 |
| 0 | -2.8339514330 | -2.83395146063 | 0.0421517243 | 0.0421517243404 |
| 1 | 0.0000000000 | 0.0000000000 | 0.0464015106 | 0.0464013881367 |
| 2 | 3.6697111853 | 3.66970969206 | 0.0502423154 | 0.0502423172341 |

Table 2. Comparison of the results of present study and existing works for various values of $\lambda$ when $Re = 50$, $Pr = 1$, $\varphi_1 = 0 = \varphi_1 = M = Rd$. 

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\[
\alpha = -20^\circ, \quad \lambda = -0.5, \quad \varphi_1 = 0.1 = \varphi_2, \quad \text{Pr} = 6.2, \\
Rd = 0.2 \text{ and } m = 3.7.
\]

**Velocity fields**

Figure 2(a) and 2(b) illustrates the effect of magnetic parameter \( M \) on non-dimensional velocity distributions \( f(\zeta) \) for both convergent \((\alpha < 0)\) and divergent \((\alpha > 0)\) channels, respectively. We can clearly observe dual solutions (first and second) for fluid velocity for varying values of magnetic parameter in case of convergent channel \((\alpha < 0)\), as displayed in Figure 2(a). It is interesting to note that the existence of dual branches (first and second) highly depends upon the angle between nonparallel walls \( \alpha \), that is, the wall angle plays a primary role in determining the dual nature of solutions. Thus, dual solutions exist for our modelled problem in case of convergent channel \((\alpha < 0)\). From geometrical point of view, the gradual narrowing of the flow region for negative \( \alpha \) may be the reason for the existence of multiple solutions. Currently, the external force suddenly exerts a disturbance on the fluid, resulting in the existence of dual possibilities of fluid response. It is quite clear from this figure that velocity profiles depict an increasing behaviour with higher magnetic parameter for first solution and simultaneously reduction in velocity profile is observed for second solution. From physical point of view, this happens due to the Lorentz force, a drag like force, which, inflect decreases the fluid velocity. On the other hand, we can see in Figure 2(b) that only single solutions are possible for flow through divergent channel for both stretching and shrinking case. We have plotted the velocity curves in case of stretching \((\lambda = 0.5)\) and shrinking \((\lambda = -0.5)\) walls for different values of magnetic parameter. The simulated results for flow velocity \( f(\zeta) \) are manifested against the dimensionless coordinate \( \zeta \) through Figure 3(a-c) to expose the influence of stretching/shrinking parameter \( \lambda \) in case of convergent and divergent channels. The comparison of figures suggest that multiple solutions are possible for the flow through convergent channel either the walls are stretching or shrinking. On the other hand, it can be depicted from Figure 3(c) that unique solution exists for velocity fields in case of flow through divergent channel both for stretching and shrinking channel walls. As shown in Figure 3(a) and 3(b), the velocity curves for upper branch increases significantly when the shrinking or stretching parameters are increased and all the curves fulfil the boundary conditions. However, these curves switched their behaviour for lower branch and display a reduction fluid velocity near the wall. On contrary to velocity behaviour in above figures, Figure 3(c) reveals that non-dimensional velocity profiles rises directly with higher values of stretching/shrinking parameter in case of divergent channel. The effects of different Reynolds number \( \text{Re} \) on the velocity \( f(\zeta) \) is illustrated through Figure 4(a) and 4(b) for convergent and divergent channels. Physically, Reynolds number depict the ratio of viscous forces to momentum forces. As is noted, the dual solutions for velocity distribution exist for certain value of walls angle, that is, \( \alpha = -20^\circ \) for varying Reynolds number. We observed that as the Reynolds number increases, the radial component velocity \( f(\zeta) \) increases for both solutions in case of convergent channel. It can be seen from these two graphs that the first solution is very stable and does not change much with the change of Reynolds number. From physical perspective, this increasing behaviour of fluid velocity for higher Reynolds number is well justified by the definition of
Reynolds number. That is, a higher Reynolds number means that the momentum forces are strong enough to move the fluid quickly and as a result its velocity enhances. While a counter-current of the velocity has been produced in case of divergent channel, which can be explained as the impact exerted by the sudden expansion of the flow region. As a result, the uplifting values of Reynolds number significantly reduce the velocity curves.

**Temperature fields**

The dimensionless temperature distributions $\Phi(\zeta)$ of hybrid nanofluids for various physical parameters are plotted versus the dimensionless coordinate $\zeta$ in case of convergent and divergent channels through Figure 5–10. Influence of magnetic parameter $M$ on dimensionless temperature field $\Phi(\zeta)$ has been demonstrated through Figure 5(a) and 5(b) for convergent channel ($\alpha = -20^\circ$) and divergent channel ($\alpha = 20^\circ$), respectively. It is worth mentioning that the angle we take in current analysis is small enough such that its absolute maximum value is $20^\circ$. More interestingly, we see that multiple solutions for temperature distributions are possible corresponding to such small angle of the convergent channel. The curve connected to $\Phi(\zeta)$ in Figure 5(a) shows that higher magnetic parameter enhances the temperature profiles in first solution while it reduces the temperature profiles for second solutions. The temperature curves increase rather slightly with the increase of $M$ for upper branch solutions. However, the temperature values are much
higher for upper branch as compared to lower branch. On the other hand, the favourable impact of higher magnetic parameter \( M \) in case of divergent channel for both stretching and shrinking walls is producing higher and higher hybrid nanofluids temperature distributions. Figure 6(a)–(c) shows the effect of stretching or shrinking parameter \( \lambda \) on the variation of the dimensionless temperature for flow through convergent and divergent channels. The profiles are drawn at fixed values of all other involved parameters and taking a particular wall angle \( \alpha = -20^\circ \) and \( \alpha = 20^\circ \), respectively. The plotted curves depict that with increasing \( \lambda \), the dimensional temperature profiles for convergent channel rises substantially in case of upper branch solution, while, the behaviour is quite opposite for lower branch solutions, as seen through Figure 6(a) and (b). Moreover, Figure 6(c) depicts that temperature profiles increases with higher stretching/shrinking parameter \( \lambda \) for both stretching and shrinking divergent channels. Figures 7(a) and (b) and 8(a) and (b) demonstrate the most important results of this paper, that is, the influence of nano-particle volume fraction on temperature field. Overall, the effects of \( \text{MoS}_2 \) and \( \text{Al}_2\text{O}_3 \) nanoparticles volume fraction on dual temperature fields are quite similar. The influence of nanoparticles volume fraction on the second solution of the temperature field is to enhance it and for first solution it reduces in case of convergent channel. However, for divergent case, higher volume fraction reduces the temperature field. In physical point of view, by adding more nanoparticles to the base fluid results in higher nanofluid density and it became denser. Therefore, it has more and more difficulty in moving through the channel. The temperature
distributions for different radiation parameter $Rd$ in case of convergent and divergent channels are presented through Figure 9(a) and 9(b). On one hand, the results presented in Figure 9(a) reveal that dual solution exist in this case too and the effects are minor for second solution as compare to first solution. In addition, distribution of temperature only increases in case of lower branch and decreases for upper branch solutions as well as in case of divergent channel when walls are stretching and shrinking. Figure 10(a) and 10(b) elucidate the impacts of nanoparticles shape factor $m$ on non-dimensional profiles of temperature for both the cases. Interestingly, we noticed that the fluid temperature is much higher in case when the nanoparticles are of sphere shape ($m = 3$) for upper branch solution and is much lower for lower branch solutions.

Moreover, a significant decline in upper branch solution and small enhancement in lower branch solution of temperature profiles for varying $m$.

**Engineering quantities**

The computational outcomes for the coefficient of skin-friction $N_{1f'}(1)$ and wall heat transfer (Nusselt number) $- (N_{a} + Rd) \Phi'(1)$ are manifested against Reynolds number $Re$ and magnetic parameter $M$ in the Figures 11(a) and (b) and 12(a) and (b) to expose the impacts of various parameters on these physical quantities pertaining to engineering applications. It is apparent from all these figures that dual solutions (first and second) are possible for the flow through convergent channel ($\alpha < 0$) within a limited range of involved physical

Figure 7. Performance of $\text{MoS}_2$ volume fraction $\varphi_1$ on temperature variation $\Phi(\zeta)$ for: (a) convergent and (b) divergent channels.

Figure 8. Performance of $\text{Al}_2\text{O}_3$ volume fraction $\varphi_2$ on temperature variation $\Phi(\zeta)$ for: (a) convergent and (b) divergent channels.
parameters. A quite interesting outcome from Figure 11(a) and (b) is discovered that the skin-friction coefficient reduces with decreasing values of channel angle in both the solutions. However, its values decrease as the Reynolds number becomes higher. Similarly, the coefficient of skin-friction reduces with higher magnetic parameter and nanoparticles volume fraction for second solution. On the other hand, Figure 12(a) and (b) respectively, exhibits the influence of channel angle and nanoparticle volume fraction on Nusselt number against the magnetic parameter. It is revealed that increments in nanoparticles volume fraction and magnetic parameter enhances the wall heat transfer rate for upper branch, but they recede the wall mass transfer rate in lower branch.

**Conclusions**

This theoretical and computational analysis is concerned with MHD Jeffery-Hamel flow of MoS$_2$/Al$_2$O$_3$ hybrid nanofluids in a non-parallel walls channel to the situation where the channel walls are stretching or shrinking. Moreover, the heat transport characteristics are explored subject to thermal radiation, viscous dissipation and considering three different shapes of nanoparticles, like, platelet, brick and sphere. The transformed two-dimensional differential formulation is tackled numerically with the help of MATLAB solver ‘bvp4c’. The key parameters are varied to anticipate their consequences on dimensionless profiles of hybrid nanofluids velocity as well as their temperature. The novelty of this work lies in the existence of multiple
branches (First and Second solutions) in case of convergent channel. The most important findings are outlined as follows:

- It is perceived that the coefficient of skin-friction enhances with higher nanoparticles volume fraction in upper solution.
- Increasing the magnetic field strength has rising influence on the rate of heat transfer for upper solutions.
- The Reynolds number provides a clear description of flow and heat transfer mechanism. Hence, an increase in Reynolds number leads to an enhancement in both first and second solutions of velocity field for convergent channel, while quite opposite is true for divergent channel.
- One the other hand, the progressing values of Reynolds number reduced the temperature profiles in both solutions.
- An investigation of thermal characteristics of hybrid nanofluids, we noticed more enhancement in temperature profiles with greater stretching or shrinking parameter for both solutions.

Figure 11. Combined performance of $M$, $\alpha$, $Re$ on skin friction coefficient $N_{f1}(1)$.

Figure 12. Combined performance of $M$, $\alpha$ and $\varphi_i$ on Nusselt number $-\Phi_i^1(1)$.
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