EXISTENCE AND REGULARITY OF INVARIANT GRAPHS FOR COCYCLES IN BUNDLES: PARTIAL HYPERBOLICITY CASE

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Abstract. We study the existence and regularity of the invariant graphs for bundle maps (or bundle correspondences with generating bundle maps motivated by ill-posed differential equations) having some relatively partial hyperbolicity in non-trivial bundles without local compactness. The regularity includes (uniformly) $C^0$ continuity, Hölder continuity and smoothness. A number of applications to both well-posed and ill-posed semi-linear differential equations and the abstract infinite-dimensional dynamical systems are given to illustrate its power, such as the existence and regularity of different types of invariant foliations (laminations) including strong stable laminations and fake invariant foliations, the existence and regularity of holonomies for cocycles, $C^{k,\alpha}$ section theorem and decoupling theorem, etc, in more general settings.

Contents

1. Introduction ................................................................. 2
   1.1. motivation .......................................................... 2
   1.2. nontechnical overviews of main results, proofs and applications ...... 4
   1.3. structure of this paper ............................................ 8
2. Basic Notions: Bundle, Correspondence, Generating Map ............. 9
   2.1. bundle with metric fibers, bundle map ................................ 9
   2.2. definition: discrete case .......................................... 10
   2.3. dual correspondence ............................................... 12
3. Hyperbolicity and (A)(B) condition .................................... 12
   3.1. definitions .......................................................... 12
   3.2. relation between (A)(B) condition and (A')(B') condition: Lipschitz case .... 13
   3.3. relation between (A)(B) condition and classical cone condition: $C^1$ case .... 17
4. Existence of Invariant Graphs .......................................... 20
   4.1. invariant graph: statements of the discrete case ...................... 20
   4.2. graph transform .................................................. 23
   4.3. invariant graph: proofs .......................................... 24
   4.4. more characterizations ........................................... 28
   4.5. corollaries ....................................................... 30
5. Uniform Property of Bundle, Bundle Map, Manifold and Foliation ..... 37
   5.1. locally metrizable space .......................................... 37
   5.2. fiber (or leaf) topology .......................................... 39
   5.3. connection: mostly review ....................................... 40

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1. Introduction

1.1. motivation. Invariant manifold theory is one of the central topics in the theory of dynamical systems with some hyperbolicity. Classical contributions go back to A. Lyapunov [Lia47] (originally published in Russian, 1892), J. Hadamard [Had01], É. Cotton [Cot11], O. Perron [Per29], V. A. Pliss [Pli64], A. Kelly [Kel67], N. Fenichel [Fen71], M. W. Hirsch, C. C. Pugh, M. Shub [HPS77], J. B. Pesin [Pes77], and others. For new developments in infinite-dimensional dynamical systems see e.g. [Rue82, Maña83, LL10] (non-uniformly hyperbolic case), [Hen81, CL88, BJ89, MPS88, DPL88, MR09] (different types of semi-linear differential equations), [BLZ98, BLZ99, BLZ00, BLZ08] (normally hyperbolic manifolds case), too numerous to list here. The invariant manifolds can exist for some ill-posed differential equations; see e.g. [EW91, Gal93, SS99, dIL09, EIB12], though they do not generate semiflows. For significant applications of invariant manifold theory in ergodic theory, see e.g. [PS97, Via08, AV10, BW10, BY17b].

This paper is part of a program to expand the scope of invariant manifold theory, following the work of Hirsch, Pugh, Shub [HPS77, Chapter 5-6] with a view towards making it applicable to both well-posed and ill-posed differential equations and abstract dynamical systems in non-compact spaces (including non-locally compact spaces). Here, we try to extend the classical theory to more general settings at least in the following directions with development of a number of new ideas and techniques.
about the dynamical systems. We focus on the dynamical systems on bundles, i.e. cocycles. Related works on invariant manifolds for cocycles are e.g. \cite{Hal61, Yi93, CY94, CL97}; but see also \cite[Chapter 6]{HPS77} and \cite{AV10, ASV13}. In terms of differential equations, there is a natural context we need to study cocycles. That is, when we look into the dynamical properties of a differential equation in a neighborhood of its an invariant set, we usually study the linearization of the equation along this set and consider the equation as a linearized equation plus a ‘small’ perturbation. In order to make our results applicable to ill-posed differential equations, we extend the notion of cocycle to *bundle correspondence with generating bundle map* (see Section 2 for definitions) originally due to Chaperon \cite{Cha08}. We notice that by using this notion, the difficulty that cocycles may be non-invertible (especially in the infinite-dimensional setting) can be overcome without strain.

about the bundles. We do not require the bundles to be trivial or the base spaces to be (locally) compact. Note that for the infinite-dimensional dynamical systems, this situation happens naturally. For the existence results (see Section 4), the assumption on bundles is only that their fibers are complete metric spaces which makes the results as general as possible. For the regularity results (see Section 6), the situation becomes more complicated which is one of major difficulties arising in this paper; see Section 5.4 for how we deal with the non-trivial bundle and Section 5.5 for how we extend compact spaces or Banach spaces to non-compact and non-linear spaces in the infinite dimension; this treatment is essentially due to e.g. \cite{HPS77, PSW12, BLZ99, BLZ08, Cha04, Eld13, Ama15}.

about the hyperbolicity. We consider the dynamical systems having some relatively partial hyperbolicity. There are many ways to describe the hyperbolicity in different settings. Here we adopt (A) (B) condition (see Section 3.1 for definitions) to describe the hyperbolicity originally due to \cite{MPS88} (see also \cite{LYZ13, Zel14}). Our definition of (A) (B) condition is about the ‘dynamical systems’ themselves and it also can make sense in the setting of metric spaces (in the spirit of \cite{Cha04, Cha08, Via08, AV10}). However, see Lemma 3.10 for the relation between (A) (B) condition and classical cone condition, where the latter only works for the smooth dynamical systems on smooth manifolds. As a first step, in this paper we only concentrate relatively partial hyperbolicity in the *uniform* sense (especially the regularity results); our future work (see \cite{Che18a}) will address the non-uniformly partial hyperbolicity in detail.

Using (A) (B) condition has its advantage that we can give a unified approach to establish some results about the invariant manifold theory in different hyperbolicity settings. A more important thing is that some classical hyperbolicity assumptions (or the spectral gap conditions) are not satisfied by some dynamics generated by some differential equations but the (A) (B) condition can still be verified. Consult the important paper \cite{MPS88}, where Mallet-Paret and Sell could successfully verify the (uniform) cone condition based on the principle of spatial average (see also \cite{Zel14}).

Combining with above, it seems that we first investigate *the existence and regularity of invariant manifolds (or especially the invariant graphs) for bundle maps (or bundle correspondences with generating bundle maps) on the non-trivial and non-compact bundles, having some (relatively) partial hyperbolicity described by (A) (B) condition*. Due to the general settings, our results have a number of applications to, for instance, not only the classical dynamics generated by well-posed differential equations such as some parabolic PDEs, delay equations, age structured models, etc, but also the ill-posed differential equations such as elliptic PDEs on the cylinders, mixed delay equations, and the spatial dynamics generated by some PDEs; see \cite{Che18d} for more details. Meanwhile, our results give a very unified approach to the existence and regularity of different types of invariant manifolds and invariant foliations (laminations) including strong (un)stable laminations and fake invariant foliations, the existence and regularity of holonomies for cocycles, and others; see e.g. Section 7.2.

about the differential equations. In \cite{Che18d}, we studied some classes of semi-linear differential equations that can generate cocycles for the well-posed case or cocycle correspondences with generating cocycles for the ill-posed case which can satisfy (A) (B) condition (see Section 3.1 for a definition), so that our main results in Section 4 and Section 6 (as well as Section 7.2) can be applied.
Our work (combining with [Che18d]) is a direct generalization of [Yi93, CY94, CL97] and the invariant manifolds of the equilibrium for the equations considered in [MPS88, BJ89, DPL88, MR09, EIB12] etc.

1.2. nontechnical overviews of main results, proofs and applications. Let us give some nontechnical overviews of main results, proofs and applications for reader’s convenience.

existence of invariant graphs. Heuristically, our main results about the existence of the invariant graphs for bundle maps (or bundle correspondences with generating bundle maps) may be summarized by the following (in a special case).

**Theorem A** (Existence). Let \( H : X \times Y \to X \times Y \) be a bundle map (or a bundle correspondence with generating bundle map \((F, G)\)) over a map \( u \) where \( X, Y \) are two bundles with fibers being complete metric spaces and the base space \( M \), and \( u : M \to M \). Let \( i = (i_X, i_Y) \) be a section of \( X \times Y \) which is invariant about \( H \). Assume \( H \) has some relatively partial hyperbolicity described by \((A) (B)\) condition. If the spectral condition holds, then the following hold.

1. **There is a** \( C^{0,1} \)-fiber bundle map \( f : X \to Y \) over \( \text{id} \) such that the graph of \( f \) is invariant under \( H \), i.e. \( \text{Graph} f_m \subset H_m^{-1} \text{Graph} f(u(m)) \) for \( m \in M \) (or \( f \) satisfies (6.1)), and \( i \in \text{Graph} f \). In some sense, \( f \) is unique.
2. The graph of \( f \) can be characterized by some asymptotic behaviors of \( H \).

Similar results in the ‘unstable direction’ \( Y \to X \) for \( H^{-1} \) also hold.

For more specific and general statements about our three existence results, see **Theorem 4.1**, **Theorem 4.3** and **Theorem 4.6** in Section 4.1, where the section \( i \) may be in other cases (originally motivated by [Cha08]). The characterization of the invariant graphs is given in Section 4.4. For the ‘hyperbolic dichotomy’ and ‘hyperbolic trichotomy’ cases, see **Theorem 4.16** and **Theorem 4.17** respectively. The former can be applied to e.g. Anosov dynamical systems (or more generally the restrictions of Axiom A diffeomorphisms to hyperbolic basic sets), the two-sided shifts of finite type or more general ones uniformly hyperbolic homeomorphisms introduced in [Via08] (see also [AV10]). The latter can be applied to e.g. the (partially) normally hyperbolic invariant manifolds in a particular context that the normal bundle of the (partially) normally hyperbolic manifold can embed into a trivial bundle and the dynamic also can extend to this trivial bundle maintaining the ‘hyperbolic trichotomy’ (see [HPS77, Eld13] and Remark 4.18).

In the ‘hyperbolic trichotomy’ case, if \( H \) is an invertible bundle map, then \( H \) can decompose into three parts: a decoupled center part with two stable and unstable parts depending on the center part; see **Corollary 4.19**. This theorem can be regarded as a generalization of Hartman-Grossman Theorem; see also **Remark 4.20**.

Except the description of partial hyperbolicity, our existence results we mention above are more general than the ones in classical literatures including e.g. [Sta99, Cha04, Cha08, MPS88, CY94, CL97, LYZ13] and [HPS77, Chapter 5] (about plaque families). No assumption on the base space \( M \) makes the existence results applicable to both deterministic and random dynamical systems. The continuity and measurability problems will be regarded as the regularity problems; the measurability problems will not be considered in this paper (see [Che18a]) and the \((\text{uniformly}) C^0 \) continuity, Hölder continuity and smoothness problems will be investigated in Section 6.

Besides above, our existence results not only work for \( H \) being a cocycle having some hyperbolicity but also \( H \) being a bundle correspondence with generating bundle map \((F, G)\) (see Section 2.2). Loosely speaking, one in fact needs a subset of \( X \times Y \) (the graph of \( H \): \( \text{Graph} H \)), then the ‘dynamical behavior’ in some sense is the ‘iteration’ of \( \text{Graph} H \). So giving a way to describe \( \text{Graph} H \) is important; in classical \( \text{Graph} H \) is given by the cocycle (bundle map) and in our context it is given by the generating map. We will introduce the precise notions about correspondences and generating maps in Section 2, which we learned from [Cha08]. We need these notions at least for two reasons.
• There are some differential equations (see e.g. [EW91, Gal93, SS99, dILL09, EIB12]), which are ill-posed and therefore can not generate semiflows so that the classical theory of dynamical systems (especially the invariant manifold theory) can not be applied to these equations directly, might define cocycle correspondences with generating cocycles (see [Che18d]). So our general results can be applied to more types of differential equations; see [Che18d] for details.

• Unlike the finite-dimensional dynamics, the dynamics generated by the differential equations in Banach spaces, usually are not invertible, so one can not use the proven ‘stable results’ to deduce the ‘unstable results’. Since the ‘hyperbolic systems’ (no matter invertible or not) can define correspondences with generating maps, so using the dual correspondences (see Section 2.3), one only needs to prove ‘one side results’. This idea was used in [Cha08] to prove (pseudo) (un-)stable manifolds (foliations) with a very unified approach.

regularity of invariant graphs. We will show \( f \) obtained in Theorem A has higher regularities including continuity, Hölderness and smoothness, once (i) the more regularity properties of the bundle \( X \times Y \) (and \( M \)), (ii) the more regularity properties of the maps \( u, i, F, G \), (iii) the spectral gap condition and (iv) additionally technical assumption on the (almost) continuity of the functions in (A) (B) condition are fulfilled. Roughly, our main results about the regularity of the invariant graphs for cocycles (or bundle correspondences with generating bundle maps) may be stated as follows.

**Theorem B** (Regularity). Under Theorem A, we have the following statements.

1. If the fibers of \( X \times Y \) are Banach spaces and \( F_m(\cdot) \), \( G_m(\cdot) \) \( \in C^1 \) for each \( m \), then \( f_m(\cdot) \) \( \in C^1 \). Moreover, there is a \( K^1 \in L(T^X \times T^Y \cdot \cdot \cdot) \) (see (5.1)) over \( f \) satisfying (6.2) and \( D^0 f = K^1 \). The case \( f_m(\cdot) \) \( \in C^{\alpha, \alpha} \) is very similar. See Lemma 6.7, Lemma 6.11 and Lemma 6.12.

   Furthermore under some continuity properties of the functions in the definitions of (A) (B) condition being assumed as well as the corresponding spectral gap conditions (which differ item by item in the following), we have the following regularities about \( f \). Here we omit specific assumptions on \( X, Y, M \) (in order to make sense of the different uniform properties of \( F, G, u \)).

2. Let \( F, G, i, u \in C^0 \). Then \( f \in C^0 \). See Lemma 6.25.

3. Under (2), if the fiber derivatives \( D^0 F, D^0 G \in C^0 \), one has \( K^1 \in C^0 \). See Lemma 6.26.

4. If \( F, G \in C^{0,1} \) and \( u \in C^{0,1} \), then \( m \mapsto f_m \) is uniformly (locally) Hölder. See Lemma 6.13.

   In addition, assume \( u \) is a 0-section and \( u \in C^{0,1} \), then we further have the following hold.

5. If \( F, G \in C^{1,1} \), and a ‘better’ spectral gap condition holds (than (4)), then \( m \mapsto f_m \) is also uniformly (locally) Hölder (in a better way). See also Lemma 6.13.

6. If \( m \mapsto Df_m(\cdot) \), \( DG_m(\cdot) \) are uniformly (locally) Hölder, where \( \cdot(\cdot) \) is \( = (|X|, |Y|, |u(m)|) \), then the distribution \( Df_m(\cdot) \) depends in a uniformly (locally) Hölder fashion on the base points \( m \). See Lemma 6.16.

7. If \( F, G, D^0 F, D^0 G \in C^{0,1} \), then \( m \mapsto Df_m(\cdot) \) is uniformly (locally) Hölder. See Lemma 6.17.

8. Let \( u \in C^1 \) and \( F, g \in C^{1,1} \). Then \( f \in C^1 \). Moreover, there is a unique \( K \in L(T^X \times T^Y \cdot \cdot \cdot) \) (see (5.1)) over \( f \) which is \( C^0 \) and satisfies (6.3) (or more precisely (6.22)) and \( \nabla f = K \) where \( \nabla f \) is the covariant derivative of \( f \). See Lemma 6.18.

9. Under (8), \( x \mapsto \nabla f_m(\cdot) \) is locally Hölder uniform for \( m \in M \). See Lemma 6.22.

10. Under (8) and \( u \in C^{1,1} \), then \( m \mapsto \nabla f_m(\cdot) \) is uniformly (locally) Hölder. See Lemma 6.23 (1).

11. Under (8), if \( F, G \in C^{2,1} \), \( u \in C^{1,1} \), and a ‘better’ spectral gap condition (than (10)) holds, one also has \( m \mapsto \nabla f_m(\cdot) \) is uniformly locally Hölder (in a better way). See Lemma 6.23 (2).

For more specific and general statements about regularity results, see the lemmas we list above (i.e. Section 6) where some uniform conditions on \( F, G, u, i \) and \( X, Y, M \) can be only around \( u(M) \) (i.e. the inflowing case); see also Theorem 6.2. In Section 6.10, we also give the corresponding regularity results for the case when \( i \) is a ‘bounded’ section without proofs (due to the same method as proving Theorem B) which was also studied in [CY94], while this case can be taken as a supplement of Theorem B. In fact, the regularity results do not depend on the existence results, see Section 6.11.
Our regularity results are very unified and recover many classical ones. The fiber smoothness of the invariant graphs, i.e. item (1) in Theorem B, is well known in different settings including deterministic and random dynamical systems (see e.g. [HPS77,CL97,FdlLM06,LL110]). Reducing to the case that $M$ consists of one element, it gives the spectral result about invariant ((strong) stable, center, pseudo-stable, etc) manifolds of an equilibrium (cf. e.g. [HPS77, Irw80,dILW95,VvG87]). Beside item (2) and item (3), more continuity results about $f$ are given in Section 6.9; see also Remark 6.30 for a simple application. The Hölder continuity of $f$ like item (4), only requiring the Lipschitz continuity of the dynamic, was also discussed in a very special setting in [Cha08,Sta99] (the metric space case) and [Wil13,Corollary 5.3] (the smooth space case). The result like item (5) is well known in different contexts; see e.g. [HPS77,PSW97,LYZ13]. The result like item (6) was reproved by many authors which should go back to D. V. Anosov for hyperbolic systems on compact sets (see e.g. [PSW12]); see also [Has97,HW99]. The conclusions like items (7)~(11) were studied e.g. [HPS77] (the normal hyperbolicity case), [PSW97] (the partial hyperbolicity case; see also Theorem 7.6) and [CY94] (for the cocycles in trivial bundles with base space being compact Riemannian manifolds). Very recently, in [ZZJ14, Lemma 3], the authors obtained the result like item (10) in a particular context; but see also [Sta99,Theorem 1.3]. The higher order smoothness of $f$ is not concluded in the present paper.

For the precise spectral gap conditions (or bunching condition ([PSW97]) or fiber bunching condition ([AV10,ASV13])) in items (2)~(11), which change with different regularity assumptions on $F,G$, look into the lemmas we list in Theorem B. The spectral gap conditions we give are general (due to the general assumptions on $F,G$), and in some cases they are also new. As is well known, the spectral gap conditions in some sense are essential for higher regularity of $f$, while the sharpness of spectral gap conditions in some cases was also studied in e.g. [HW99] (the Hölderness of the distribution in item (6)) and [Zel14] (in the different equations (see also [Che18d])).

The precise assumptions for $X,Y,M,u,i$ are given in Section 6.2. As a simple illustration, consider $X\times Y$ as a trivial bundle $M\times X_0\times Y_0$ with $M,X_0,Y_0$ being Banach spaces or regard $X,Y$ as smooth vector bundles with the base space $M$ being a smooth compact Riemannian manifold. Basically, we try to generalize the base space $M$ to the setting of non-compactness (and also non-metrizability), and bundles $X,Y$ to non-trivialization. See Appendix C for some relations (or examples) between our assumptions on $X,Y,M$ and bounded geometry ([Eic91,Ama15,Eld13]) as well as the manifolds studied in [BLZ99,BLZ08]. The assumptions for $X,Y,M$ are in order to make sense of the uniformly $C^k$-a continuity of $F,G,f,u$. Our way to describe the uniform properties of a bundle map, which in some sense is highly classical (see e.g. [HPS77, Chapter 6], [PSW12, Section 8], [Cha04, Section 2] and [Eld13,Ama15]), is by using the (particular choice of) local representations of the bundle map with respect to preferred atlases; see Section 5.4 and Section 5.5. The uniform property of a bundle map is said to be the uniform property of local representations. Here we mention that since our local representations are specially selected (see Definition 5.18), the conclusions (and also conditions) in Section 6.3 to Section 6.7 are weak in form, but these will be strengthened as in e.g. [HPS77,PSW12] once certain uniform properties of the transition maps respecting preferred atlases are assumed which again indicate higher regularity of the bundles; see Section 6.8 and Section 5.4 for details. The ideas how we extend the bundles with base spaces to be in more general settings are presented in Section 5.

The most hard obstacle for applying our existence and regularity results is that how to verify the hyperbolicity condition which is not our purpose to study in detail in this paper; but we refer the readers to see [CL99] for the detailed study about the spectral theory of linear cocycles in the infinite-dimensional setting; see also [LL10,SS01,LP08] in the cocycle or evolution case, and [LZ17,Zel14,NP00] in the ‘equilibrium’ case. We have given in Section 3.2 and Section 3.3 some relations between (A) (B) condition and some classical hyperbolicity conditions; see also [Che18d] for a discussion about dichotomy and (A) (B) condition in the context of differential equations.

**a nontechnical overview of proofs.** In the theory of invariant manifolds, there are two fundamental methods named Hadamard graph transform method (due to [Had01]) and Lyapunov-Perron method (due to [Lia47,Cot11,Per29]), where the former is more geometric and the latter more analytic. These
two methods have been demonstrated as a very successful approach to establish the existence and regularity of invariant manifolds and in many cases both two methods can work. We refer the readers to see e.g. [VvG87,CL88,CY94,CL97,Cha04,Cha08,MR09,LL10] where the Lyapunov-Perron method was applied and [HPS77, MPS88, PSW97, BLZ98, Sta99, LYZ13, Wil13] where the graph transform method was used.

In this paper, we employ the graph transform method to prove all our existence and regularity results. Intuitively, this method is to construct a \textit{graph transform} such that for a given graph, say \( G_1 \), define a unique graph \( G_2 \) such that \( G_2 \subset H^{-1} G_1 \), taking the proof of Theorem A as an example; see also Section 4.2. Unfortunately, this usually can not be done for general \( H \) and graphs. In our circumstance, \( H \) needs some hyperbolicity and the graphs, from our purpose, should be expressed as functions of \( X \to Y \). Now for the invariant graph \( \mathcal{G} \) of \( H \), one gets an equation like (6.1). Applying the Banach Fixed Point Theorem to the graph transform on some appropriately chosen space, one can get the desired invariant graph. It seems that our approach to Theorem A (i.e Theorem 4.1) based on the graph transform method is unified, elemental and concise.

A more challenging task is to give the higher regularity of \( f \). Our strategy of proving the Hölderness and smoothness of \( f \) is step by step from ‘fiber-regularity’ to ‘base-regularity’ and from ‘low-regularity’ to ‘high-regularity’, that is to say, following the sequence as in the items (1) \sim (11) in Theorem B. (It is not so easy to prove the \( C^1 \) continuity of \( f \) if one does not obtain items (1) to (7) in Theorem B at first.) For the Hölder regularity of \( f \), we use an argument which at least in some special cases is classical (cf. e.g. Example A.4 as motivation), but very \textit{unified} (see Remark A.10). However, due to our general settings (especially no uniform boundedness of the fibers being assumed and the hyperbolicity being described in a relative sense), the expression of spectral gap conditions is a little complicated (see Remark 6.1) and some preliminaries are needed to give probably in some sense sharp spectral gap conditions which are given in Appendix A. To prove the smoothness of \( f \), we employ a popular argument (see e.g. Lemma D.3 as an illustration); that is one needs first to find the ‘variant equation’ (see e.g. (6.2) or ‘(6.3)’) satisfied by the ‘derivative’ of \( f \) and then tries to show the solution of this ‘variant equation’ is indeed the ‘derivative’ of \( f \). Again the graph transform method is used to solve the ‘variant equation’. For the fiber-smoothness of \( f \) (i.e. Theorem B (1)), the ‘variant equation’ is easy to find (i.e. (6.2) or (†) in the proof of Lemma 6.12 for the higher order fiber-smoothness), whereas for the base-smoothness of \( f \) (i.e. Theorem B (8)) the situation becomes intricate. We introduce an additional structure of a bundle named \textit{connection} (see Section 5.3 for an overview) to give a derivative of a \( C^1 \) bundle map respecting base points, i.e. the \textit{covariant derivative} of \( f \). (Now ‘(6.3)’ has a precise meaning i.e. (6.22).) This approach to prove smooth regularity of \( f \) might be new and a light different from the classical way (e.g. [HPS77, PSW97]). Also, the proof given in this paper in some sense simplifies the classical one.

\textbf{Applications I: abstract dynamical systems.} In Section 7.1 and Section 7.2, we give some applications of our existence and regularity results to the abstract dynamical systems; more applications will appear in our future work (see e.g. [Che18b, Che18c]). A direct application of our main results is the \( C^{k,a} \) section theorem ([HPS77, Chapter 3, Chapter 6], [PSW97, Theorem 3.2] and [PSW12, Theorem 10]); see Theorem 7.1 and Theorem 7.3 where we generalize it to be in a more general setting. A second application is some results about invariant foliations contained in Section 7.2, where the infinite-dimensional and non-compact settings as well as the dynamics being not necessarily invertible are set up. The global version of the invariant foliations for bundle maps, i.e. Theorem 7.4, was reproved by many authors (see e.g. [HPS77, Sta99, Cha04, Cha08]). A basic application of different types of invariant foliations is to decouple the systems, see Corollary 4.19. However, the local version of the invariant foliations like fake invariant foliations which was first introduced in [BW10] (see also [Wil13]) is not so well known. In Section 7.2.3, we give the fake invariant foliations in the infinite-dimensional setting; see Theorem 7.12 and Theorem 7.13. We mention that unlike the center foliations for the partially hyperbolic systems which might not exist, the fake invariant foliations always exist but only are locally invariant.
The strong (un)stable lamination (and foliation) was extensively studied by e.g. [HPS77, Fen77, BLZ00, BLZ08] (for the normally hyperbolic case) and [PSW97] (for the partially hyperbolic case). In Section 7.2.2, we study the existence and the Hölder continuity of strong stable laminations in the setting of metric spaces; see Theorem 7.9 and Corollary 7.10, where they also give the \( s \)-lamination (resp. \( u \)-lamination) for maps (resp. invertible map) (see e.g. [AV10, Section 4.1]). The corresponding result in the smooth spaces setting is given in Corollary 7.11. In general, one can not expect the strong stable foliation is \( C^1 \), but it would be if restricting it inside each leaf of the center-stable foliation (if existing) and some bunching condition being assumed; see e.g. [PSW97, PSW00]. There is a difficulty that center-stable foliation usually again is not smooth, which was dealt with in [PSW97, PSW00]. We reprove this result by using our regularity results; see Theorem 7.6 for details. Similar argument is also used in the proof of regularity of fake invariant foliations (i.e. Theorem 7.13) and strong stable foliations in center-stable manifolds (see [Che18b]).

The existence and regularity of holonomies (over a lamination) for cocycles (see Definition 7.15) are also discussed in Section 7.2.4 which are mostly direct consequences of our main results; see Theorem 7.16 and Corollary 7.18. The holonomies for cocycles were studied in [Via08, Section 2] and [AV10, Section 5]; see also [ASV13]. While we give a generalization in more general settings (with a very unified approach) so that one could apply it to the infinite dimensional dynamical systems.

**applications II: differential equations.** To apply our existence and regularity results in Section 4 and Section 6 (as well as Section 7.2) to semi-linear differential equations, from the abstract view, one needs to show the differential equations can generate cocycles (for the well-posed case) or cocycle correspondences with generating maps (for the ill-posed case) satisfying \( (A) \ (B) \) condition. In [Che18d], we dealt with relationship between the dichotomy (or more precisely the exponential dichotomy of differential equations) and (A) (B) condition. See [Che18d] for the applications of our main results (as well as [Che18b, Che18c]) to both well-posed and ill-posed differential equations which are not included in this paper.

1.3. **structure of this paper.** The related notions about correspondences having generating maps are introduced in Section 2 and the \( (A) \ (B) \) condition with its relevance is given in Section 3. In Section 4, we give our existence results with their corollaries. Section 5 contains some preliminaries related with bundles and manifolds such as the description of uniformly \( C^{k,\alpha} \) continuity of a bundle map on appropriate types of bundles, casting them in a light suitable for our purpose to set up the regularity results. In Section 6, we give the regularity results with their proofs. Some applications of our main results are given in Section 7. Appendix A contains a key argument in the proof of our regularity results. Some miscellaneous topics such as a fixed point theorem under minimal conditions, Finsler manifold in the sense of Neeb-Upmeier, Lipschitz characterization in length spaces and bump function (and blid map), are provided in Appendix D for the convenience of readers. Appendix B continues Section 5.4. Appendix C provides some examples related with our (uniform) assumptions about manifolds and bundles.

**Guide to Notation.** The following is a guide to the notation used throughout this paper, included for the reader’s convenience.

- \( \text{Lip} f \): the Lipschitz constant of \( f \). \( \text{Hol}_\alpha f \): the \( \alpha \)-Hölder constant of \( f \).
- \( \mathbb{R}_+ \triangleq \{ x \in \mathbb{R} : x \geq 0 \} \).
- \( X(r) \triangleq B_r = \{ x \in X : |x| < r \} \), if \( X \) is a Banach space.
- For a correspondence \( H : X \to Y \) (defined in Section 2.2),
  - \( H(x) \triangleq \{ y : \exists (x, y) \in \text{Graph} H \} \).
  - \( A \subset H^{-1}(B) \), if \( \forall x \in A \), \( \exists y \in B \) such that \( y \in H(x) \).
  - \( \text{Graph} H \), the graph of the correspondence.
  - \( H^{-1} : Y \to X \), the inversion of \( H \) defined by \( (y, x) \in \text{Graph} H^{-1} \iff (x, y) \in \text{Graph} H \).
  - \( f(A) \triangleq \{ f(x) : x \in A \} \), if \( f \) is a map; \( \text{Graph} f \triangleq \{(x, f(x)) : x \in X \} \).
  - \( \text{diam} A \triangleq \sup\{ d(m, m') : m, m' \in A \} \): the diameter of \( A \), where \( A \) is a subset of a metric space.
• $X \otimes u Y, X \times Y$: the Whitney sum of $X, Y$ through $u$, defined in Section 2.1.

• $\lambda^{(k)}(m) \triangleq \lambda(u^{k-1}(m))\lambda(u^{k-2}(m)) \cdots \lambda(m)$: if $\lambda : M \to \mathbb{R}$ over $u$, defined in Section 5.4.1.

• $L_u(X, Y) \triangleq \bigcup_{m \in M} \{L(X_m, Y_{u(m)})\}$ if $u : M \to N$ is a map and $X, Y$ are vector bundles over $M, N$ respectively. Write $K \in L(X, Y)$ (over $u$) if $K \in L_u(X, Y)$. See Section 5.4.2.

• $\nabla$ and $\tilde{\nabla}$: see (5.1). If $K \in L(T^H_X, T^V_X)$ or $L(T^V_X, T^H_X)$ over some map, we write $K_m(x) = K_{(m,x)}$.

• $D_{f_m}(x) = D_x f_m(x)$: the derivative of $f_m(x)$ with respect to $x$: $D_1 f_m(x, y) = D_x f_m(x, y)$, $D_2 f_m(x, y) = D_y f_m(x, y)$: the derivatives of $f_m(x)$ with respect to $x, y$ respectively.

• $\nabla^\nu f$: the fiber derivative of a bundle map $f$ defined in Section 5.4.6.

• $\nabla f$: the covariant derivative of a bundle map $f$ defined in Section 5.3.3. $\nabla_m f_m(x)$: the covariant derivative of a bundle map $f$ at $(m, x)$ defined in Section 5.3.3.

• $f \in C^{k, \alpha}$: if $D^i f$ is bounded $i = 1, 2, \ldots, k$ and $D^k f \in C^{0, \alpha}$, i.e. globally $\alpha$-Hölder.

• $\mathcal{G}(X) = \bigsqcup_{m \in M} \mathcal{G}(X_m)$ is the $(C^k)$ Grassmann manifold of $X$ (see [AMR88] for a definition of a Grassmann manifold of a Banach space), where $X$ is a $C^k$ vector bundle.

• $\bar{d}(A, z) \triangleq \sup_{z \in A} d(z, z)$, if $A$ is a subset of a metric space defined in Section 4.1.

• $\sum_{\lambda} (X, Y) \triangleq \{\varphi : X \to Y : \text{Lip} \varphi \leq \lambda\}$, if $X, Y$ are metric spaces, defined in Section 4.2.

• $a_n \leq b_n, n \to \infty (a_n \geq 0, b_n > 0)$ means that $\sup_{n \geq 0} b_n^{-1} a_n < \infty$, defined in page 28.

• $\mathcal{E}(u)$: defined in Definition A.1.

• $\lambda \theta$, max{\lambda, \theta}: defined by $(\lambda \theta)(m) = \lambda(m) \theta(m)$, max{\lambda, \theta}(m) = max{\lambda(m), \theta(m)}$.

• $\lambda^{a} \theta < 1, \theta < 1$ if $\lambda, \theta : M \to \mathbb{R}_+$: see Remark 6.1 and Appendix A.

• $[a]$: the largest integer less than $a$ where $a \in \mathbb{R}$.

• $r_\alpha(x)$: the radial retraction see (3.1).

• For the convenience of writing, we usually write the metric $d(x, y) \triangleq |x - y|$.

All the metric spaces appeared in this paper are assumed to be complete, unless where mentioned.

2. Basic Notions: Bundle, Correspondence, Generating Map

In this section, we collect some basic notions for the convenience of readers; particularly, we will generalize the notion of correspondence having generating map in [Cha08] to different contexts from our purpose.

2.1. bundle with metric fibers, bundle map. $(X, M, \pi)$ is called a (set) bundle, if $\pi : X \to M$ is a surjection, where $X, M$ are sets. If $M, \pi$ are not emphasized, we also call $X$ a bundle. $X_m \triangleq \pi^{-1}(m)$ is called a fiber, and $M, X, \pi$ are called a base space, a total space, and a projection, respectively. The element of $X$ is sometimes written as $(m, x)$, where $x \in X_m$, in order to emphasis that $x$ belongs to the fiber $X_m$. The most important and simplest bundle may be the trivial bundle, i.e. $(M \times X, M, \pi)$, where $\pi(m, x) = m$.

If every fiber $X_m$ is a complete metric space with metric $d_m$, we say $(X, M, \pi)$ is a (set) bundle with metric fibers. Throughout this paper, the bundles are always assumed to be the bundle with metric fibers. Although the metric $d_m$ of $X_m$ may differ from each other, we will use the same symbol $d$ to indicate them. For the convenience of writing, we write $d(x, y) \triangleq |x - y|$.

If $(X, M, \pi_1), (Y, M, \pi_2)$ are bundles with metric fibers, then $(X \times Y, M, \pi)$ is also a bundle with metric fibers, where $\pi((m, x), (m, y)) = (m, x) \in X_m, y \in Y_m$. The fibers are $X_m \times Y_m, m \in M$ with the product metric, i.e.

\[
\begin{align*}
\delta_p((x, y), (x_1, y_1)) &= \begin{cases} 
(d(x, x_1)^p + d(y, y_1)^p)^{\frac{1}{p}}, & 1 \leq p < \infty, \\
\max\{d(x, x_1), d(y, y_1)\}, & p = \infty.
\end{cases}
\end{align*}
\]

Let $u : M \to N$ be a map. The Whitney sum of $X, Y$ through $u$, denoted by $X \otimes_u Y$, is defined by

\[
X \otimes_u Y \triangleq \{(m, x, y) : x \in X_m, y \in Y_{u(m)}, m \in M\}.
\]

This is a bundle with base space $M$ and fibers $X_m \times Y_{u(m)}, m \in M$. If $M = N$ and $u = \text{id}$, we also use the standard notation $X \times Y = X \otimes_{\text{id}} Y$. 

If \((X, M_1, \pi_1), (Y, M_2, \pi_2)\) are bundles, and \(f : X \to Y\) is a map satisfying \(\pi_2 f = u \pi_1\), where \(u : M_1 \to M_2\) is a map, we call \(f\) a bundle map over \(u\). We write \(f((m, x)) = (u(m), f_m(x)), x \in X_m, f_m(x) \in Y_{u(m)}\), and call \(f_m : X_m \to Y_{u(m)}\) a fiber map. Let \(M_1 = M_2 = M\) and \(X = Y\). The \(k\)th composition of \(f\) is a bundle map over \(u^k\). We write \(f^k(m, x) = (u^k(m), f_m^k(x))\). Usually, we use the notation \(f^k\) rather than \(f^k\) to stand for the \(k\)th composition of \(f\).

For example, let \(\lambda : M \to \mathbb{R}\). Consider \(\lambda\) as a bundle map over \(u\), i.e., \(\lambda(m, x) = (u(m), \lambda(m)x) : M \times \mathbb{R} \to \mathbb{R}\). From this view, we often call \(\lambda\) a function of \(M \to \mathbb{R}\) over \(u\), and for this case, the notation \(\lambda^{(k)}(m)\) hereafter means that

\[
\lambda^{(k)}(m) \triangleq \lambda_m^{(k)}(1) = \lambda(u^{k-1}(m))\lambda(u^{k-2}(m)) \cdots \lambda(m).
\]

2.2. definition: discrete case. Let \(X, Y\) be sets. \(H : X \to Y\) is said to be a correspondence (see \([Cha08]\)), if there is a non-empty subset of \(X \times Y\) called the graph of \(H\) and denoted by \(\text{Graph}H\). There are some operations between the correspondences we list in the following.

(a) (inversion) If \(H : X \to Y\) is a correspondence, define its inversion \(H^{-1} : Y \to X\) by \((y, x) \in \text{Graph}H^{-1}\) if and only if \((x, y) \in \text{Graph}H\).

(b) (composition) Let \(H_1 : X \to Y, H_2 : Y \to Z\) be correspondences. Define \(H_2 \circ H_1 : X \to Z\) by

\[
\text{Graph}H_2 \circ H_1 = \{(x, z) : \exists y \in Y, \text{such that } (x, y) \in \text{Graph}H_1, (y, z) \in \text{Graph}H_2\}.
\]

If \(H : X \to X\), as usual, \(H^{(n)} \triangleq H \circ \cdots \circ H (n\text{ times})\).

(c) (linear operation) If \(X, Y\) are vector spaces, and \(H_1, H_2 : X \to Y\) are correspondences, then \(H_1 - H_2 : X \to Y\) is defined by \(\text{Graph}(H_1 - H_2) = \{(x, y) : \exists (x, y_1) \in \text{Graph}H_1, \text{such that } y = y_1 - y_2\}\). In particular, if \(H : X \to Y\) is a correspondence, then \(H_m \triangleq H(m + \cdot - \tilde{m}) : X \to Y\) can make sense, i.e., \(\text{Graph}H_m = \{(x, y - \tilde{m}) : \exists (x + m, y) \in \text{Graph}H\}\).

The following notations for a correspondence \(H : X \to Y\) will be used frequently: (i)

\[
H(x) \triangleq \{y \in Y : \exists (x, y) \in \text{Graph}H\}, H(A) \triangleq \bigcup_{x \in A} H(x),
\]

if \(A \subseteq X\); we allow \(H(x) = \emptyset\); if \(H(x) = \{y\}\), write \(H(x) = y\). So \(A \subseteq H^{-1}(B)\) means that \(\forall x \in A, \exists y \in B\) such that \(y \in H(x)\) (i.e. \(x \in H^{-1}(y)\)). If \(X = Y\), we say \(A \subseteq X\) is invariant under \(H\) if \(A \subseteq H^{-1}(A)\). If \(A \subseteq H^{-1}(B)\), then \(H : A \to B\) can be regarded as a correspondence \(H|_A \to B\) defined by \((x, y) \in \text{Graph}H|_A \to B \iff y \in H(x) \cap B\); sometimes we say \(H : A \to B\) (or \(H|_A \to B\)) induces (or defines) a map (also denoted by \(H|_A \to B\)), if \(\forall x \in A, H(x) \cap B\) consists of only one element; if \(A = B\), we write \(H|_A = H|_A \to B\). In some sense, \(x \mapsto H(x)\) can be considered as a ‘multiple-valued map’, but it is useless from our purpose, for we only focus on the description of \(\text{Graph}H\).

We say a correspondence \(H : X_1 \times Y_1 \to X_2 \times Y_2\) has a generating map \((F, G)\), which is denoted by \(H \sim (F, G)\), if there are maps \(F : X_1 \times Y_2 \to X_2, G : X_1 \times Y_2 \to Y_1\), such that \((x_1, y_2) \in H(x_1, y_1) \iff y_1 = G(x_1, y_2), x_2 = F(x_1, y_2)\).

Example 2.1. (a) A map always induces a correspondence by using its graph, but it would not have a generating map. The following type of maps induce correspondences with generating maps. Let \(H = (f, g) : X_1 \times Y_1 \to X_2 \times Y_2\) be a map. Suppose for every \(x_1 \in X_1, g_{x_1}(\cdot) \triangleq g(x_1, \cdot) : Y_1 \to Y_2\) is a bijection. Let \(G(y_1, y_2) = g_{x_1}^{-1}(y_2), F(x_1, y_2) = f(x_1, G(x_1, y_2))\). Then we have \(H \sim (F, G)\).

A map having some hyperbolicity will be in this case, no matter if the map is bijective or not. This is an important observation for studying the invariant manifolds for non-invertible maps. By transferring the maps to correspondences having generating maps, M. Chaperon in \([Cha08]\) gave a unified method to prove the existence of different types of invariant manifolds.

(b) Let \(X, Y\) be two Banach spaces. Let \(T(t) : X \to X, S(-t) : Y \to Y, t \geq 0,\) be \(C_0\) semigroups with generators \(A, -B\) respectively and \(|T(t)| \leq e^{\mu t}, |S(-t)| \leq e^{-\mu t}, \forall t \geq 0\). Take \(F_1 : X \times Y \to X, F_2 : X \times Y \to Y\) to be Lipschitz with \(\text{Lip}F_i \leq \epsilon\). Then the time-one mild solutions of following
equation induce a correspondence $H : X \times Y \rightarrow X \times Y$ with generating map $(F, G)$,

\[
\begin{cases}
\dot{x} = Ax + F_1(x, y), \\
\dot{y} = By + F_2(x, y),
\end{cases}
\]

or equivalently,

\[
\begin{cases}
x(t) = T(t)x_1 + \int_0^t T(t-s)F_1(x(s), y(s)) \, ds, \\
y(t) = S(t-1)y_2 - \int_0^t S(t-s)F_2(x(s), y(s)) \, ds,
\end{cases} \quad 0 \leq t \leq 1.
\]

Define maps $F, G$ as follows. The above equation always exists a $C^0$ solution $(x(t), y(t)), 0 \leq t \leq 1$, with $x(0) = x_1, y(1) = y_2$ (see e.g. [ElB12, Che18d]); now let $F(x_1, y_2) = x(1), G(x_1, y_2) = y(0)$. Using $F, G$, one can define a correspondence $H \sim (F, G)$. Moreover, if $\mu_\alpha - \mu_\beta - 2\epsilon > 0$, then there are constants $\alpha, \beta, \lambda = e^{\mu_\alpha + \epsilon}, \lambda_\alpha = e^{-\mu_\alpha + \epsilon}$ such that $\alpha \beta < 1$ and $H$ satisfies (A)($\alpha, \lambda_\alpha$) (B)($\beta, \lambda$) condition (see Definition 3.1 for a definition). For details, see [Che18d]. Equation (*) is usually ill-posed, meaning that for arbitrarily given $(x_0, y_0) \in X \times Y$, there might be no (mild) solution $(x(t), y(t))$ satisfies (*) with $x(0) = x_0, y(0) = y_0$.

Let $(X, M, \pi_1), (Y, N, \pi_2)$ be two bundles, and $u : M \rightarrow N$ a map. Suppose $H_m : X_m \times Y_m \rightarrow X_{u(m)} \times Y_{u(m)}$ is a correspondence for every $m \in M$. Using $H_m$, one can determine a correspondence $H : X \times Y \rightarrow X \times Y$, by Graph$H \triangleq \bigcup_{m \in M} (m, \text{Graph}H_m)$, i.e. $(u(m), x_{u(m)}, y_{u(m)}) \in H(m, x_m, y_m) \Leftrightarrow (x_{u(m)}, y_{u(m)}) \in H_m(x_m, y_m)$. We call $H$ a bundle correspondence over a map $u$. Now, $H^{-1}$, the inversion of $H$, means $H_m^{-1} \triangleq (H_m)^{-1} : X_{u(m)} \times Y_{u(m)} \rightarrow X_m \times Y_m, m \in M$. If $H_m$ has a generating map $(F_m, G_m)$ for every $m \in M$, where $F_m : X_m \times Y_m \rightarrow X_{u(m)}, G_m : X_m \times Y_{u(m)} \rightarrow Y_m$ are maps, we say $H$ has a generating bundle map $(F, G)$, which is denoted by $H \sim (F, G)$. Let $M = N$. We use the notation $H^{(k)}$, the $k$th composition of $H$, which is defined by $H^{(k)}_m = H_{u^{(k)}(m)} \circ H_{u^{(k-2)}(m)} \circ \cdots \circ H_m$. This is a bundle correspondence over $u^k$.

**Example 2.2.** (a) Let $H : X \times Y \rightarrow X \times Y$ be a bundle map over $u$, where $(X, M, \pi_1), (Y, M, \pi_2)$ are bundles, and $H_m = (f_m, g_m) : X_m \times Y_m \rightarrow X_{u(m)} \times Y_{u(m)}$. Suppose for every $m \in M$, $x \in X_m$, $g_{m, x} \triangleq g_m(x, \cdot) : Y_m \rightarrow Y_{u(m)}$ is a bijection. Let $G_m(x, y) = g_{m, x}^{-1}(y) : X_m \times Y_{u(m)} \rightarrow Y_m, F_m(x, y) = f_m(x, G_m(x, y)) : X_m \times Y_{u(m)} \rightarrow X_{u(m)}$. Then we have $H_m \sim (F_m, G_m)$ and $H \sim (F, G)$.

(b) Take $A, B$ as the operators in Example 2.1 (b). Let $Z = X \times Y$ and $C = A \oplus B : Z \rightarrow Z$. Let $M$ be a topology space and $r : M \rightarrow M : \omega \mapsto tw$ a continuous semiflow. Let $L : M \rightarrow L(Z, Z)$ be strongly continuous (i.e. $(\omega, z) \mapsto L(\omega, z)$ is continuous) and $f(\cdot)(\cdot) : M \times Z \rightarrow Z$ continuous. Assume for every $\omega \in M, \sup_{t \geq 0} |L(t, \omega)| = \tau(\omega) < \infty$ and $\sup_{t \geq 0} \text{Lip } f(t, \omega)(\cdot) < \infty$. Consider the following equation,

\[
\dot{z}(t) = Cz(t) + L(t, t)z(t) + f(t, \omega)z(t).
\]

For the above different equation (which usually is ill-posed), the time-one mild solutions induce a bundle correspondence $H$ over $u(\omega) = 1 : \omega$ with generating bundle map. Moreover, under some uniform dichotomy assumption (see e.g. [LP08]), $H$ will satisfy (A) (B) condition (defined in Section 3.1.2). See [Che18d].

**Lemma 2.3.** (1) Let $H_i : X_i \times Y_i \rightarrow X_{i+1} \times Y_{i+1}, i = 1, 2$ be correspondences with generating maps $(F_i, G_i), i = 1, 2$, respectively. Assume that $\sup_y \text{Lip } G_2(\cdot, y) \sup_x \text{Lip } F_1(x, \cdot) < 1$. Then $H_2 \circ H_1$ also has a generating map $(F, G)$.

(2) Let $(X, M, \pi_1), (Y, M, \pi_2)$ be two bundles with metric fibers, $u : M \rightarrow M$ a map and $H : X \times Y \rightarrow X \times Y$ a bundle correspondence over $u$. Assume $H \sim (F, G)$ and

\[
\sup \text{Lip } G_{u(m)}(\cdot, y) \sup \text{Lip } F_m(x, \cdot) < 1, \quad m \in M.
\]

Then $H^{(k)}$ also has generating bundle map.
Proof. The result in (2) is a direct consequence of (1) and the proof of (1) is simple. Let \( x_1 \in X_1 \), \( y_1 \in Y_1 \). Since \( \text{Lip sup}_x G_2(\cdot, y) \sup_x F_1(x, \cdot) < 1 \), we see that there is a unique \( y_2 = y_2(x_1, y_1) \in Y_2 \) such that \( F_2 = G_2(F_1(x_1, y_2), y_2) \). Let \( x_2 = x_2(x_1, y_1) = F_1(x_1, y_2) \). Set \( G(x_1, y_3) = G_1(x_1, y_2) \), \( F(x_1, y_3) = F_2(x_2, y_3) \). Now \( F, G \) are what we desire. \( \square \)

2.3. dual correspondence. Let \( H : X_1 \times Y_1 \to X_2 \times Y_2 \) be a correspondence with a generating map \((F, G)\). The dual correspondence of \( H \), which is denoted by \( \tilde{H} \), is defined as follows. Set \( \tilde{X}_1 = Y_2 \), \( \tilde{X}_2 = Y_1 \), \( \tilde{Y}_1 = X_2 \), \( \tilde{Y}_2 = X_1 \) and
\[
\tilde{F}(\tilde{x}_1, \tilde{y}_2) = G(\tilde{y}_2, \tilde{x}_1), \quad \tilde{G}(\tilde{x}_1, \tilde{y}_2) = F(\tilde{y}_2, \tilde{x}_1).
\]
Now \( \tilde{H} : \tilde{X}_1 \times \tilde{Y}_1 \to \tilde{X}_2 \times \tilde{Y}_2 \) is uniquely determined by \( (\tilde{x}_2, \tilde{y}_2) \in \tilde{H}(\tilde{x}_1, \tilde{y}_1) \Leftrightarrow \tilde{y}_1 = \tilde{G}(\tilde{x}_1, \tilde{y}_2), \tilde{x}_2 = \tilde{F}(\tilde{x}_1, \tilde{y}_2) \), i.e. \( \tilde{H} \sim (\tilde{F}, \tilde{G}) \). One can similarly define the dual bundle correspondence \( \tilde{H} \) of bundle correspondence \( H \) over \( u \) if \( u \) is invertible; \( \tilde{H} \) now is over \( u^{-1} \).

\( \tilde{H} \) and \( H \) have some duality in the sense that \( \tilde{H} \) can reflect some properties of \( H^{-1} \). For example, if \( H \) satisfies \((A)(\alpha; \alpha'), (B)(\beta; \beta', \lambda)\) condition (see Definition 3.1 below), then \( \tilde{H} \) satisfies \((A)\beta; \beta', \lambda) \) condition; if one obtains the ‘stable results’ of \( H \) (see Section 4.1), then one can obtain the ‘unstable results’ of \( \tilde{H} \) through the ‘stable results’ of \( H \). In this paper, we only state the ‘stable results’, leaving the corresponding statements of ‘unstable results’ for readers.

3. Hyperbolicity and \((A)(B)\) condition

We focus on the dynamical systems having some hyperbolicity described by the so called \((A)(B)\) condition (see Section 3.1). Our definition of \((A)(B)\) condition is about the ‘dynamical system’ itself and is related to the cone condition in the version of non-linearity, which is motivated by [MPS88] (see also [LYZ13, Zel14] and Section 3.3). The related definitions about \((A)(B)\) condition are given in Section 3.1 and some relations between \((A)(B)\) condition and some classical hyperbolicity conditions are given in Section 3.2 and Section 3.3. See also [Che18d] for a discussion about the relation between (exponential) dichotomy and \((A)(B)\) condition in some classes of well-posed and ill-posed differential equations.

3.1. definitions.

3.1.1. \((A)(B)\) condition for correspondence. Let \( X_i, Y_i, i = 1, 2 \) be metric spaces. For the convenience of writing, we write the metrics \( d(x, y) \triangleq |x - y| \).

Definition 3.1. We say a correspondence \( H : X_1 \times Y_1 \to X_2 \times Y_2 \) satisfies \((A)(B)\) condition, or \((A)\alpha; \alpha', \lambda) \) \((B)\beta; \beta', \lambda) \) condition, if the following conditions hold. \( \forall (x_1, y_1) \times (x_2, y_2), (x'_1, y'_1) \times (x'_2, y'_2) \in \text{Graph} H \), \((A)\)
\[(A1) \text{ if } |x_1 - x'_1| \leq \alpha |y_1 - y'_1|, \text{ then } |x_2 - x'_2| \leq \alpha' |y_2 - y'_2|; \]
\[(A2) \text{ if } |x_1 - x'_1| \leq \alpha |y_1 - y'_1|, \text{ then } |y_1 - y'_1| \leq \lambda |x_2 - x'_2|; \]
\[(B) \]
\[(B1) \text{ if } |y_2 - y_2'| \leq \beta |x_2 - x'_2|, \text{ then } |y_1 - y'_1| \leq \beta' |x_1 - x'_1|; \]
\[(B2) \text{ if } |y_2 - y_2'| \leq \beta |x_2 - x'_2|, \text{ then } |x_2 - x'_2| \leq \lambda |x_1 - x'_1|. \]

If \( \alpha = \alpha' \), \( \beta = \beta' \), we also use notation \((A)\alpha, \lambda) \) \((B)\beta, \lambda) \) condition.

In particular, if \( H \sim (F, G) \), then the maps \( F, G \) satisfy the following Lipschitz conditions.
\[(A') \text{ (A'1) sup}_x \text{ Lip } F(x, \cdot) \leq \alpha', \text{ (A'2) sup}_x \text{ Lip } G(x, \cdot) \leq \lambda. \]
\[(B') \text{ (B'1) sup}_y \text{ Lip } G(\cdot, y) \leq \beta', \text{ (B'2) sup}_y \text{ Lip } F(\cdot, y) \leq \lambda. \]

If \( F, G \) satisfy the above Lipschitz conditions, then we say \( H \) satisfies \((A')\alpha, \lambda) \) \((B')\beta, \lambda) \) condition, or \((A') \) \((B') \) condition. Similarly, we can define \((A') \) \((B') \) condition, or \((A) \) \((B) \) condition; or \((A) \) \((B) \) condition, etc, if \( H \) only satisfies \((A), \alpha' \), etc, respectively.
3.1.2. (A) (B) condition for bundle correspondence. Let \((X, M, \pi_1), (Y, M, \pi_2)\) be two bundles with metric fibers (i.e., every fiber is a metric space) and \(u : M \to M\) a map. Let \(H : X \times Y \to X \times Y\) be a bundle correspondence over \(u\). We say \(H\) satisfies (A) (B) condition, or \((\alpha; \alpha', \lambda_u) (B; \beta', \lambda_s)\) condition, if every \(H_m \sim (F_m, G_m)\) satisfies \((\alpha)(\sigma(m); \alpha'(m), \lambda_u(m)) \sim (B)(\beta'(m); \beta'(m), \lambda_s(m))\) condition, where \(\alpha, \alpha', \lambda_u, \beta, \beta', \lambda_s\) are functions of \(M \to \mathbb{R}_+\). Also, if \(\alpha \equiv \alpha'\) and \(\beta \equiv \beta'\), then we use the notation \((\alpha)(\lambda_u) (B)(\beta, \lambda_s)\) condition. It is similar to define \((A')(B')\) condition, \((A')(B)\) condition, or \((A)(B')\) condition for bundle correspondences as the case for correspondences.

In linear dynamical systems, the spectral theory is well developed, see e.g., [KH95, SS94, CL99, LP08, LZ17, Rue82, Mañ83, LL10]. The main objective of this theory is to characterize asymptotic properties of linear dynamical systems. Lyapunov numbers play an important role which measure the average rate of separation of orbits starting from nearby initial points and describe the local stability properties of linear dynamical systems. The functions \(\lambda_s, \lambda_u\) appeared in our definition of (A) (B) condition are related with the Lyapunov numbers. The spectral spaces are the spaces that are invariant under the dynamical system and that every orbit starting from these spaces has the same asymptotic behavior. The fibers of the bundle \(X, Y\) in our setting are related with these spaces and the functions \(\alpha, \beta\) in the definition of (A) (B) condition describe how the fibers are approximately invariant. The robustness of (A) (B) condition is not clear but since the Lipschitz condition \((A') (B')\) implies (A) (B) in some contexts (see Lemma 3.5 and Corollary 3.7), the (A) (B) condition has some ‘open condition’ property. We do not give the corresponding condition in a geometry way associated with tangent fields, however, see [MPS88, Zel14] for some such results.

Remark 3.2. More generally, for every \(k\), let \(H^{(k)}\) be the \(k^{th}\) composition of \(H\). We say \(H\) satisfies \((A_1) (B_1)\) condition, or \((A_1)(\alpha; \lambda_s; c) (B_1)(\beta; \lambda_s; c)\) condition, if every \(H^{(k)}_m \sim (F^{(k)}_m, G^{(k)}_m)\) satisfies \((A)(\sigma(m); c(m), \lambda^k_s(m)) \sim (B)(\beta(m); c(m), \lambda^k_s(m))\) condition. In general, (A) (B) condition does not imply \((A_1) (B_1)\) condition. However, when \(\lambda_s, \lambda_u\) are orbitally bounded about \(u\), i.e.,
\[
\sup_{N \geq 0} \lambda_s(u^N(m)) < \infty, \quad \sup_{N \geq 0} \lambda_u(u^N(m)) < \infty,
\]
then \((A) (B)\) condition implies \((A_1) (B_1)\) condition. Indeed, one can use the \(\sup\) Lyapunov numbers of \(\{\lambda^k_s(m)\}, \{\lambda^k_u(m)\}\) (see Definition A.1).

We will explain how to verify the (A) (B) condition in the following.

3.2. relation between (A) (B) condition and (A') (B') condition: Lipschitz case. For simplicity, we only consider the case of correspondences. In the following, assume that \(X_i, Y_i, \ i = 1, 2\) are metric spaces, and \(H : X_1 \times Y_1 \to X_2 \times Y_2\) is a correspondence with a generating map \((F, G)\). The following lemma is just a consequence of the definition.

Lemma 3.3. (1) If \(H\) satisfies \((A)(\alpha; \alpha', \lambda_u) (B; \beta', \lambda_s)\) condition, then \(\bar{H}\), the dual correspondence of \(H\) (see Section 2.3), satisfies \((A')(\beta; \beta', \lambda_s) (B; \alpha; \alpha', \lambda_u)\) condition.
(2) If for all \(n, H_n : X_n \times Y_n \to X_{n+1} \times Y_{n+1}\) satisfies \((A)(\alpha, \lambda_{n,u}) (B)(\beta, \lambda_{n,s})\) condition, then \(H_n \circ \cdots \circ H_1\) satisfies \((A)(\alpha, \lambda_{1,u} \cdots \lambda_{n,u}) (B)(\beta, \lambda_{1,s} \cdots \lambda_{n,s})\) condition.

The following lemma’s condition was also used in [Cha08] to obtain the hyperbolicity. We will show that this condition implies the (A) (B) condition, so our main results recover [Cha08].

Lemma 3.4 (Lipschitz in \(d_\infty\)). If for all \(x_1, x'_1 \in X_1, \ y_2, y'_2 \in Y_2\)
\[
|F(x_1, y_2) - F(x'_1, y'_2)| \leq \max \{\lambda_s|x_1 - x'_1|, |y_2 - y'_2|\},
|G(x_1, y_2) - G(x'_1, y'_2)| \leq \max \{\lambda_u|x_1 - x'_1|, |y_2 - y'_2|\},
\]
and \(\alpha \beta < 1, \lambda_s \lambda_u < 1\), then \(H\) satisfies \((A)(\alpha, \lambda_u) (B)(\beta, \lambda_s)\) condition. In addition, if \(\alpha \beta < \lambda_s \lambda_u\), then \(H\) satisfies \((A)(\alpha^{-1}; \alpha, \lambda_u) (B)(\beta^{-1}; \beta, \lambda_s)\) condition where \(c = \lambda_s \lambda_u < 1\).
Proof. Let \((x_1, y_1) \times (x_2, y_2), (x'_1, y'_1) \times (x'_2, y'_2) \in \text{Graph}H\). If \(|x_1 - x'_1| \leq \alpha |y_1 - y'_1|\), then

\[
|y_1 - y'_1| \leq \max \{\beta |x_1 - x'_1|, \lambda_u |y_2 - y'_2|\} \leq \max \{\alpha \beta |y_1 - y'_1|, \lambda_u |y_2 - y'_2|\}
\]

\[
\leq \lambda_u |y_2 - y'_2| \quad (\text{since } \alpha \beta < 1),
\]

and

\[
|x_2 - x'_2| \leq \max \{\lambda_u |x_1 - x'_1|, \alpha |y_2 - y'_2|\} \leq \max \{\lambda_u \alpha |y_1 - y'_1|, \alpha |y_2 - y'_2|\}
\]

\[
\leq \max \{\lambda_u \lambda_u |y_2 - y'_2|, \alpha |y_2 - y'_2|\} = \alpha |y_2 - y'_2| \quad (\text{since } \lambda_u \lambda_u < 1).
\]

The (A) condition is satisfied by \(H\). Similar for (B) condition and the last statement. \(\square\)

Lemma 3.5 (Lipschitz in \(d_1\)). If \(H\) satisfies \((A')(\tilde{\alpha}, \tilde{\lambda}_u) (B')(\tilde{\beta}, \tilde{\lambda}_s)\) condition, and \(\tilde{\lambda}_s \tilde{\lambda}_u < c^2, \tilde{\alpha} \tilde{\beta} < (c - \sqrt{\tilde{\lambda}_s \tilde{\lambda}_u})^2\), where \(0 < c \leq 1\), then \(H\) satisfies \((A)(\alpha; c\alpha, \lambda_u) (B)(\beta; c\beta, \lambda_s)\) condition, where

\[
\alpha = \frac{b - \sqrt{b^2 - 4c\tilde{\alpha}\tilde{\beta}}}{2\tilde{\beta}}, \quad \beta = \frac{b - \sqrt{b^2 - 4c\tilde{\alpha}\tilde{\beta}}}{2\alpha}, \quad \lambda_s = \frac{\tilde{\lambda}_s}{1 - c\alpha}, \quad \lambda_u = \frac{\tilde{\lambda}_u}{1 - c\beta}, \quad b = c - \tilde{\lambda}_s \tilde{\lambda}_u + \tilde{\alpha} \tilde{\beta}.
\]

Furthermore, \(\alpha \beta < 1, \lambda_s \lambda_u < 1\).

Proof. Let \((x_1, y_1) \times (x_2, y_2), (x'_1, y'_1) \times (x'_2, y'_2) \in \text{Graph}H\). The \((A')(\tilde{\alpha}, \tilde{\lambda}_u) (B')(\tilde{\beta}, \tilde{\lambda}_s)\) condition says

\[
|x_2 - x'_2| \leq \tilde{\lambda}_s |x_1 - x'_1| + \tilde{\alpha} |y_2 - y'_2|, \quad |y_1 - y'_1| \leq \tilde{\beta} |x_1 - x'_1| + \tilde{\lambda}_u |y_2 - y'_2|.
\]

Using this, if \(|x_1 - x'_1| \leq \alpha |y_1 - y'_1|\), then \(|x_1 - x'_1| \leq \frac{\alpha \lambda_u \lambda_u}{1 - c\alpha} |y_2 - y'_2|\). So

\[
|x_2 - x'_2| \leq \frac{\alpha \lambda_s \lambda_u}{1 - c\alpha} |y_2 - y'_2|, \quad |y_1 - y'_1| \leq \frac{\lambda_u}{1 - c\beta} |y_2 - y'_2|.
\]

\(\alpha\) should satisfy \(\frac{\alpha \lambda_s \lambda_u}{1 - c\alpha} + \tilde{\alpha} \leq c\alpha, \quad \alpha \beta < 1\), or equivalently,

\[
\alpha^2 c\beta - ab + \tilde{\alpha} \leq 0, \quad \alpha \beta < 1,
\]

where \(b = c - \tilde{\lambda}_s \tilde{\lambda}_u + \tilde{\alpha} \tilde{\beta}\). This can be satisfied if

\[
b^2 \geq 4c\tilde{\alpha}\tilde{\beta}, \quad b + \frac{\sqrt{b^2 - 4c\tilde{\alpha}\tilde{\beta}}}{2\alpha} < \frac{1}{\alpha}, \quad b - \frac{\sqrt{b^2 - 4c\tilde{\alpha}\tilde{\beta}}}{2\alpha} \leq \alpha \leq \frac{b + \sqrt{b^2 - 4c\tilde{\alpha}\tilde{\beta}}}{2\alpha}.
\]

\(b^2 \geq 4c\tilde{\alpha}\tilde{\beta}\) if and only if

\[
x^2 - 2(c + \tilde{\lambda}_s \tilde{\lambda}_u)x + (c - \tilde{\lambda}_s \tilde{\lambda}_u)^2 \geq 0,
\]

where \(x = \tilde{\alpha} \tilde{\beta}\). So we need

\[
x < \frac{2(c + \tilde{\lambda}_s \tilde{\lambda}_u) - 4(c + \tilde{\lambda}_s \tilde{\lambda}_u)^2 - 4(c - \tilde{\lambda}_s \tilde{\lambda}_u)^2}{2} = (c - \sqrt{\tilde{\lambda}_s \tilde{\lambda}_u})^2.
\]

This is certainly the condition given in the lemma. \((b + \sqrt{b^2 - 4c\tilde{\alpha}\tilde{\beta}})/(2\alpha) < 1/\alpha\) is automatically satisfied by a simple computation. The same argument can be applied to the case of (B) condition.

At the end, take \(\alpha, \beta, \lambda_s, \lambda_u\) as in the lemma. All we need to show is \(\alpha \beta < 1, \lambda_s \lambda_u < 1\), which is through a simple computation. The proof is complete. \(\square\)

Lemma 3.6. Let \(H\) satisfy \((A')(\tilde{\alpha}, \tilde{\lambda}_u) (B')(\tilde{\beta}, \tilde{\lambda}_s)\) condition, and \(0 \neq \tilde{\lambda}_s \tilde{\lambda}_u < 1\). Choose \(\rho_1, \rho_2, \) such that

\[
\tilde{\lambda}_s \tilde{\lambda}_u < \rho_2 < (\tilde{\lambda}_s \tilde{\lambda}_u)^{1/2}, \quad 1 - \rho_2^{-1} \tilde{\lambda}_s \tilde{\lambda}_u < \rho_1 < 1 - \rho_2.
\]
Let $\rho = \rho_1 + \rho_2 (< 1)$. Suppose $\alpha \beta \leq \frac{1-\rho_1^{-1}\lambda_s\lambda_u}{\rho_1^{-1}} ( < 1)$, then $H$ satisfies $(A)(\alpha; \rho_1, \lambda_u) \ (B; \rho_2, \lambda_s)$ condition, where

$$\alpha = \rho_1^{-1}\alpha, \ \beta = \rho_2^{-1}\beta, \ \lambda_s = \frac{\lambda_s}{1-\alpha\beta}, \ \lambda_u = \frac{\lambda_u}{1-\beta\alpha}.$$ 

Furthermore $\alpha\beta < 1$, $\lambda_s\lambda_u < 1$.

Proof. Let $\alpha, \beta, \lambda_s, \lambda_u$ be as in the lemma, then $\alpha\beta = \beta\alpha = \rho_1^{-1}\alpha\beta \leq 1 - \rho_2^{-1}\lambda_s\lambda_u$, and

$$\alpha\beta = \rho_1^{-2}\alpha\beta \leq \rho_1^{-1}(1 - \rho_2^{-1}\lambda_s\lambda_u) < 1, \ \lambda_s\lambda_u = \frac{\lambda_s\lambda_u}{(1-\alpha\beta)(1-\beta\alpha)} \leq \frac{\rho_2^2}{1-\alpha\beta} < 1.$$

Let $(x_1, y_1) \times (x_2, y_2), \ (x'_1, y'_1) \times (x'_2, y'_2) \in \text{Graph} H$. Now, by $(A')(B')$ condition, if $|x_1-x'_1| \leq \alpha|y_1-y'_1|$, then

(A1) $|x_2-x'_2| \leq (\frac{\lambda_s\lambda_u}{1-\alpha\beta} + \alpha)|y_2-y'_2| \leq (\rho_2\alpha + \rho_1\alpha)|y_2-y'_2| = \rho\alpha|y_2-y'_2|,$

(A2) $|y_1-y'_1| \leq \frac{\lambda_u}{1-\alpha\beta}|y_2-y'_2| = \lambda_s|y_2-y'_2|.$

The same argument gives the $(B)$ condition.

Corollary 3.7. If for all $n$, $H_n : X_n \times Y_n \to X_{n+1} \times Y_{n+1}$ satisfies $(A)(\alpha, \lambda_u)$ $(B)(\beta, \lambda_s)$ condition, and $\lambda_s\lambda_u < 1, \alpha\beta < \frac{1}{2}$, then for large $n \in \mathbb{N}$, there exist $\beta_1', \lambda_s', \rho < \frac{1}{2}$, such that $H_n \circ \cdots \circ H_1$ satisfies $(A)(\alpha, \lambda_u^n) \ (B)(\beta; \rho\beta_1, \lambda_s^n)$ condition. Furthermore, $\alpha\beta_1 < 1$, $\lambda_s'\lambda_u < 1$. If $\lambda_s < 1$, then we also can take $\lambda_s' < 1$.

Proof. Note that $H_n \circ \cdots \circ H_1$ satisfies $(A)(\alpha, \lambda_u^n) \ (B)(\beta, \lambda_s^n)$ condition by Lemma 3.3, and $(\lambda_s\lambda_u)^n \to 0$, as $n \to \infty$. Since $\alpha\beta < \frac{1}{4}$, so $1 - (\alpha\beta)^{1/2} < 1 - 2\alpha\beta$. Choose large $n$ such that

$$\lambda_s\lambda_u^n < \min\{1 - (\alpha\beta)^{1/2}, 1/2\}.$$ 

Let $\rho_2$ satisfy

$$\frac{(\lambda_s\lambda_u)^n}{1-2\alpha\beta} < \rho_2 < \frac{(\lambda_s\lambda_u)^n}{1-(\alpha\beta)^{1/2}} ( < (\lambda_s\lambda_u)^{n/2}).$$

Choose $\rho_1$ such that

$$\frac{1-\rho_2^{-1}(\lambda_s\lambda_u)^n}{1-\rho_2^{-1}(\lambda_s\lambda_u)^n} < \rho_1 < \frac{1}{2} ( < 1 - \rho_2).$$

Now $\rho = \rho_1 + \rho_2 < \frac{1}{2}$ can be satisfied if $n$ is large. Then apply Lemma 3.6 to finish the proof. 

Next we give some maps which can satisfy $(A) (B)$ condition. Let $H = (f, g) : X_1 \times Y_1 \to X_2 \times Y_2$ be a map, and $g_\lambda(\cdot) \overset{\Delta}{=} g(\cdot, \cdot) : Y_2 \to Y_1$ is invertible. Then $H \sim (F, G)$, where $G(x, y) = g^{-1}_x(y)$, $F(x, y) = f(x, G(x, y))$. Note that if $\sup_{x} \text{Lip} g(\cdot, y) < \infty$, and $\sup_{y} \text{Lip} G(x, \cdot) < \infty$, then

$$\sup_{x} \text{Lip} G(\cdot, y) \leq \sup_{y} \text{Lip} g(\cdot, y) \text{sup} \text{Lip} G(x, \cdot).$$

The following lemma is a direct consequence of Lemma 3.5.

Lemma 3.8. Let $H = (f, g) : X_1 \times Y_1 \to X_2 \times Y_2$ be a map, and $g_\lambda(\cdot) \overset{\Delta}{=} g(\cdot, \cdot) : Y_1 \to Y_2$ is invertible. Suppose

(a) $\sup_{x} \text{Lip} f(\cdot, \cdot) \leq \epsilon_1$, $\sup_{x} \text{Lip} g_\lambda^{-1}(\cdot) \leq \lambda'_u$, 
(b) $\sup_{y} \text{Lip} f(\cdot, \cdot) \leq \lambda'_s$, $\sup_{y} \text{Lip} g(\cdot, \cdot) \leq \epsilon_2$. 

Then $H$ satisfies $(A')(\alpha, \lambda_u) (B'(\beta, \lambda_s)$ condition, where $\alpha = \varepsilon_1 \lambda_u'$, $\beta = \varepsilon_2 \lambda_u'$, $\lambda_s = \lambda_s' + \varepsilon_1 \lambda_u'$, $\lambda_u = \lambda_u'$. 

In particular, if $\lambda_s' \lambda_u' < 1$, $\varepsilon_1 < \frac{1 - \lambda_s' \lambda_u'}{(\lambda_u')^2}$, $\varepsilon_2 \leq \frac{(1 - \sqrt{1 + \varepsilon_3})^2}{1 - \lambda_s' \lambda_u'}$, then there exist $\alpha, \beta, \lambda_s, \lambda_u$, such that $H$ satisfies $(A)(\alpha, \lambda_u) (B(\beta, \lambda_s)$ condition, and $\alpha \beta < 1$, $\lambda_s \lambda_u < 1$. Furthermore $\alpha, \beta \rightarrow 0$, $\lambda_s \rightarrow \lambda_s'$, $\lambda_u \rightarrow \lambda_u'$, as $\varepsilon_1, \varepsilon_2 \rightarrow 0$.

Consider a special case. Assume $H = (f, g) : X \times Y \rightarrow X \times Y$, $f(x, y) = A \mu(x) + f'(x, y)$, $g(x, y) = A \mu(y) + g'(x, y)$, where $X, Y$ are Banach spaces, $A_\mu : X \rightarrow X$ is Lipschitz, and $A_\mu : Y \rightarrow Y$ is invertible. If $\text{Lip}(A_\mu) \text{Lip}(A_{\mu}^{-1}) \leq 1$, and $\text{Lip} f'$, $\text{Lip} g'$ are small, then $H$ satisfies $(A)(\alpha, \lambda_u) (B(\beta, \lambda_s)$ condition, for some $\alpha, \beta, \lambda_s, \lambda_u$, such that $\alpha \beta < 1$, $\lambda_s \lambda_u < 1$. Furthermore $\lambda_s \rightarrow \lambda_s(A_\mu), \lambda_u \rightarrow \text{Lip}(A_{\mu}^{-1})$, as $\text{Lip} f'$, $\text{Lip} g'$ $\rightarrow 0$.

The following lemma’s condition was used in [Cha04] to obtain the hyperbolicity. We will show that this condition certainly implies the (A) (B) condition, so our main results will recover [Cha04].

**Lemma 3.9.** Let $H = (f, g) : X_1 \times Y_1 \rightarrow X_2 \times Y_2$ be a map, and $g_\lambda(\cdot) \triangleq g(x, \cdot) : Y_1 \rightarrow Y_2$ is invertible. Assume $\text{Lip} f \leq k_0$ in $d_\infty$, i.e.

$$\sum_{x \in X_1 \times Y_1} [f(x_1, y_1) - f(x_2, y_2)] \leq k_0 \sum_{x \in X_1 \times Y_1} |x_1 - x_2|, \quad \forall (x_1, y_1), (x_2, y_2) \in X_1 \times Y_1,$$

and $\sup \text{Lip} g^{-1}(\cdot) \leq \mu_0$. If $\mu_0 + v_0 k_0 < 1$, then $H$ satisfies $(A)(\alpha, \lambda_u) (B(\beta, \lambda_s)$ condition, and $\lambda_s \lambda_u < 1$, where

$$\alpha = \frac{2v_0 k_0}{1 + \sqrt{1 - 4k_0 v_0 \mu_0}} (\leq 1), \quad \beta = \frac{\mu_0}{1 - k_0 v_0} (\leq 1), \quad \lambda_u = k_0, \quad \lambda_u' = \frac{v_0}{1 - \alpha \mu_0}.$$

Moreover, if $v_0 + \mu_0 < 1$, then $\lambda_u < 1$; if $k_0 < 1$, then $\lambda_s < 1$.

**Proof.** Let $(x_1, y_1) \in X_1 \times Y_1$, $(x_2, y_2) \in X_2 \times Y_2$ \in Graph$H$. The condition says

$$|x_2 - x_2' - y_2| \leq k_0 \max(|x_1 - x_1'|, |y_1 - y_1'|), \quad |y_1 - y_1'| \leq v_0|y_2 - y_2'| + \mu_0|x_1 - x_1'|.$$

(B) condition. Let $\beta = \frac{\mu_0}{1 - k_0 v_0}$. Note that $\beta < 1$, since $\mu_0 + v_0 k_0 < 1$. Let $|y_2 - y_2'| \leq \beta|x_2 - x_2'|$, then

$$|y_2 - y_2'| \leq \beta k_0 \max(|x_1 - x_1'|, |y_1 - y_1'|)$$

$$\leq \left\{
\begin{aligned}
\beta k_0 |x_1 - x_1'|, & \quad \text{if } |y_1 - y_1'| \leq |x_1 - x_1'|,
\beta k_0 \max(|x_1 - x_1'|, v_0|y_2 - y_2'| + \mu_0|x_1 - x_1'|) \leq \frac{\beta k_0 v_0}{1 - \beta k_0 v_0}|x_1 - x_1'|, & \quad \text{if } |x_1 - x_1'| \leq |y_1 - y_1'|.
\end{aligned}
\right.$$
α should satisfy $\frac{k_{\alpha \beta}}{1-\alpha \mu_0} \leq \alpha$, $\frac{k_{\alpha \beta}}{1-\alpha \mu_0} \leq \alpha$. Let us choose an appropriate value for α. Under the condition $\mu_0 + v_0 k_0 < 1$, we have $2v_0 k_0 < 1 + (1 - 2\mu_0)$ and $(1 - 2\mu_0)^2 \leq 1 - 4k_0 v_0 \mu_0$. We choose 

$$\alpha = \frac{1 - \sqrt{1 - 4k_0 v_0 k_0}}{2\mu_0} = \frac{2v_0 k_0}{1 + \sqrt{1 - 4k_0 v_0 k_0}} \leq \frac{2v_0 k_0}{1 + (1 - 2\mu_0)} < 1.$$ 

Therefore, $\frac{k_{\alpha v_0}}{1-\alpha \mu_0} = \alpha < 1$ and the proof is complete. □

3.3. relation between (A)(B) condition and classical cone condition: $C^1$ case. The cone condition was widely used in the invariant manifold theory in classical references, e.g. [BJ89, BLZ98, MPS88, LYZ13, Zel14]. Some ideas in the proof of the next lemma are motivated by [LYZ13]. Since we have (b) $\Rightarrow$ (a) in the next lemma, our main results recover the results about the invariant foliations obtained in [LYZ13]. And the abstract results about the existence and regularity of inertial manifolds in [MPS88] are also consequences of our main results.

**Lemma 3.10.** Assume that $X_i, Y_i$, $i = 1, 2$ are Banach spaces, $H \sim (F, G) : X_1 \times Y_1 \to X_2 \times Y_2$ is a correspondence. Moreover, $F, G \in C^1$, and $\alpha \beta < 1$. Then the followings are equivalent.

(a) $H$ satisfies $(A)(\alpha; \alpha', \lambda_u)$ condition and $\sup_y \lip G(\cdot, y) \leq \beta$.

(b) For every $(x_1, y_2) \in X_1 \times Y_2$, $(DF(x_1, y_2), DG(x_1, y_2))$ satisfies $(A)(\alpha; \alpha', \lambda_u)$ condition and $|D_1 G(x_1, y_2)| \leq \beta$.

**Proof.** First note that $\sup_y \lip G(\cdot, y) \leq \beta$ if and only if $\sup_{(x_1, y_2)} |D_1 G(x_1, y_2)| \leq \beta$. (See also Appendix D.3 for a general result.)

(a) $\Rightarrow$ (b). Let $(x_1, y_1) \times (x_2, y_2) \in \text{Graph}H$. Let $y_0 = DG(x_1, y_2)(x_0, y'_0)$, $x'_0 = DF(x_1, y_2)(x_0, y'_0)$, and $|x_0| \leq \alpha |y_0|$. There is no loss of generality in assuming $y_0 \neq 0$. We need to show $(A1) |y_0| \leq \lambda_u |y'_0|$ and $(A2) |x'_0| \leq \alpha |y'_0|$.

Take a linear $L : Y_1 \to X_1$ such that $L y_0 = x_0$, $|L| \leq \alpha$. Take further $L \in C^1(Y_1 \to X_1)$ such that $L(y_1) = x_1$, $D L(y_1) = L$ and $\lip L \leq \alpha$. For example, let $y^* \in (Y_1)^*$ such that $y^*(y_0) = |y_0|$, $|y^*| \leq 1$ and set 

$$L y = \frac{y^*(y)}{|y_0|} x_0, \quad \tilde{L} y = \frac{y^*(y - y_1)}{|y_0|} x_0 + x_1 = x_1 + L(y - y_1).$$

Since $\alpha \beta < 1$, there exists a unique $\tilde{y}(\cdot)$ such that

(*)

$$G(\tilde{L} \tilde{y}(y), y) = \tilde{y}(y), \quad F(\tilde{L} \tilde{y}(y), y) \triangleq g(y).$$

Furthermore, by (A) condition, we have $\lip \tilde{y} \leq \lambda_u (\Rightarrow \sup_y |D \tilde{y}(y)| \leq \lambda_u)$, $\lip g \leq \alpha' (\Rightarrow \sup_y |D g(y)| \leq \alpha')$, and $\tilde{y}(y_2) = y_1, g(y_2) = x_2$.

Since $F, G, \tilde{L} \in C^1$, we have (taking the derivative of (*) at $y_2$)

$$DG(x_1, y_2)(L D \tilde{y}(y_2), id) = D \tilde{y}(y_2), \quad DF(x_1, y_2)(L D \tilde{y}(y_2), id) = D g(y_2).$$

Since $y_0 = DG(x_1, y_2)(x_0, y'_0)$, $x'_0 = DF(x_1, y_2)(x_0, y'_0)$, $L y_0 = x_0$, we see $D \tilde{y}(y_2)y'_0 = y_0$ and $D g(y_2)y'_0 = x'_0$. Thus, $|y_0| \leq \lambda_u |y'_0|, |x'_0| \leq \alpha' |y'_0|$.

(b) $\Rightarrow$ (a). Let $(x_1, y_1) \times (x_2, y_2), (x'_1, y'_1) \times (x'_2, y'_2) \in \text{Graph}H$ and $|x_1 - x'_1| \leq \alpha |y_1 - y'_1|$. We need to show $|x_2 - x'_2| \leq \alpha' |y_2 - y'_2|$, and $|y_1 - y'_1| \leq \lambda_u |y_2 - y'_2|$. Without loss of generality, $|y_1 - y'_1| \neq 0$.

Take $\tilde{L} \in C^1(Y_1 \to X_1)$ such that $\lip \tilde{L} \leq \alpha, \tilde{L} y_1 = x_1$ and $\tilde{L} y'_1 = x'_1$. For example, let $y^* \in (Y_1)^*$ such that $y^*(y_1 - y'_1) = |y_1 - y'_1|, |y^*| \leq 1$ and set

$$\tilde{L} y = \frac{y^*(y - y'_1)}{|y_1 - y'_1|} (x_1 - x'_1) + x'_1.$$
Furthermore, \( \tilde{y} \) is differentiable (see e.g. Lemma D.3). Thus taking the derivative of (**) at \( y \), we obtain

\[
DG(x, y)(x_0, y_0') = D\tilde{y}(y)y_0', \quad DF(x, y)(x_0, y_0') = D\tilde{y}(y)y_0',
\]

where \( x_0 \triangleq D\tilde{L}(\tilde{y}(y))D\tilde{y}(y)y_0' \) and \( \tilde{y}(y) \). Since \( |x_0| \leq \alpha|D\tilde{y}(y)y_0'| \), by (A) condition, we have

\[
|D\tilde{y}(y)y_0'| = |DF(x, y)(x_0, y_0')| \leq \alpha'|y_0'| \quad \text{and} \quad |D\tilde{y}(y)y_0'| = |DG(x, y)(x_0, y_0')| \leq \lambda_u|y_0'| \quad \text{for any} \quad y \in Y_2.
\]

Thus, \( \text{Lip } \tilde{y} \leq \lambda_u \) and \( \text{Lip } \tilde{g} \leq \alpha' \). Since \( \tilde{L}_1 y_1 = x_1, \tilde{L}_1 y_1' = x_1' \), we further have \( \tilde{y}(y_2) = y_1, \tilde{y}(y_2') = y_1' \), \( \tilde{g}(y_2) = x_2, \tilde{g}(y_2') = x_2' \). So (A) condition holds and the proof is complete. \( \square \)

If \( X \) is a Banach space, we use the notation \( X(r) \triangleq \{ x \in X : |x| < r \} \). Consider some local results. First note that if \( f : X(r) \to Y \) is Lipschitz, \( Y \) is a complete metric space, and \( r < \infty \), then there is a unique \( \tilde{f} : X(r) \to Y \), such that \( \tilde{f}|_{X(r)} = f \). Moreover, \( \tilde{f} \) is Lipschitz.

**Lemma 3.11.** Let \( H \sim (F, G) : X_1(r_1) \times Y_1(r_1') \to X_2(r_2) \times Y_2(r_2') \) be a correspondence.

(a) Suppose \((F, G)\) satisfies (A)(\( \alpha, \alpha', \lambda_u \)) condition and \( \sup_{y_2} \text{Lip } G(\cdot, y_2) \leq \beta \). If \( \alpha \beta < \frac{1}{2} \), then \((DF(x_1, y_2), DG(x_1, y_2))\) also satisfies (A)(\( \alpha, \alpha', \lambda_u \)) condition and \( |Df(1)G(1, y_2)| \leq \beta \) for any \((x_1, y_2) \in X_1(r_1) \times Y_2(r_2) \).

(b) Suppose \((DF(x_1, y_2), DG(x_1, y_2))\) satisfies (A)(\( \alpha, \alpha', \lambda_u \)) condition and \( |Df(1)G(1, y_2)| \leq \beta \) for any \((x_1, y_2) \in X_1(r_1) \times Y_2(r_2) \). Then \((F, G)\) also satisfies (A)(\( \alpha, \alpha', \lambda_u \)) condition and \( \sup_{y_2} \text{Lip } G(\cdot, y_2) \leq \beta \).

**Proof.** In the proof of Lemma 3.10 (a) \( \Rightarrow \) (b), the major construction is the map \( \tilde{L} \) with differential at \( y_1 \). Thus, some truncation is needed. Consider \( \tilde{L}(y) = x_1 + L(r_5(y - y_1)) \), where \( x_1 \in X_1(r_1), \ y_1 \in Y_1(r_1') \). \( \varepsilon \) is small such that \( |L| \varepsilon < r_1 - |x_1| \), and \( r_\varepsilon \) is the radial retraction, i.e.,

\[
r_\varepsilon(x) = \begin{cases} x, & \text{if } |x| \leq \varepsilon, \\ \varepsilon x/|x|, & \text{if } |x| \geq \varepsilon. \end{cases}
\]

Now \( \tilde{L} : Y_1(r_1') \to X_1(r_1) \), and it is differentiable at \( y_1 \). But \( \text{Lip } \tilde{L} \leq 2 \alpha \). That is why we need \( \alpha \beta < \frac{1}{2} \).

Using the same argument line by line in the proof of Lemma 3.10 (a) \( \Rightarrow \) (b), one shows (a) holds.

In the proof of Lemma 3.10 (b) \( \Rightarrow \) (a), a key construction is the existence of \( \tilde{L} \), which also makes sense in the local case. So the proof is complete. \( \square \)

Using the (A') (B') condition, one can easily give more local results. The proof of the following lemma is straightforward (by using Lemma 3.5), so we omit it.

**Lemma 3.12.** Let \((X, M, \pi_1), (Y, M, \pi_2)\) be two bundles with the fibers being Banach spaces, and \( u : M \to M \) a map. Let \( H : X \times Y \to X \times Y \) be a bundle correspondence over \( u \) with a generating bundle map \((F, G)\). Assume that \( F_m(\cdot), G_m(\cdot) \in C^1 \), and that the following uniform conditions hold.

(a) \( \sup_m |Df_m(0, 0)| \leq \varepsilon_1, \sup_m |D^2G_m(0, 0)| \leq \lambda_\varepsilon(\cdot) \).

(b) \( \sup_m |D^2F_m(0, 0)| \leq \lambda_\varepsilon(\cdot), \sup_m |D^1G_m(0, 0)| \leq \varepsilon_2 \).

(c) \( DF_m, DG_m \) are almost continuous at \( (0, 0) \) uniform for \( m \) in the following sense,

\[
\sup_m \sup_{|x| \leq r_1, |y| \leq r_2} |DF_m(x, y) - DF_m(0, 0)| \leq \varepsilon'_1, \quad \sup_m \sup_{|x| \leq r_1, |y| \leq r_2'} |DG_m(x, y) - DG_m(0, 0)| \leq \varepsilon'_2,
\]

and \( \varepsilon'_1, \varepsilon'_2 \) are sufficiently small, as \( r_1, r_2' \to 0 \).

(d) \( \sup_m |F_m(0, 0)| \leq \eta, \sup_m |G_m(0, 0)| \leq \eta \).

If \( \sup_m \lambda_\varepsilon(\cdot) \lambda_\varepsilon(\cdot) < 1 \), and \( \eta \) is small, then there exist constants \( \alpha, \beta, r_1, r_2' \), \( i = 1, 2 \), such that

\[
H_m \sim (F_m, G_m) : X_m(r_1) \times Y_m(r_1') \to X_{u(m)}(r_2) \times Y_{u(m)}(r_2'),
\]

satisfies (A)(\( \alpha, \lambda_\varepsilon(\cdot) \)) (B)(\( \beta, \lambda_\varepsilon(\cdot) \)) condition, where \( \lambda_\varepsilon(\cdot) = \lambda_\varepsilon(\cdot) + \delta, \lambda_\varepsilon(\cdot) = \lambda_\varepsilon(\cdot) + \phi, \phi > 0 \) is sufficiently small, and \( \alpha, \beta \to 0 \) as \( \varepsilon_1, \varepsilon_2, r_1, r_2', \eta \to 0 \).
Lemma 3.13. Let $H : X \to X$ and $u : M \to M$ be maps, where $M \subset X$ and $X$ is a Banach space.

Let $\Pi^s_m, m \in M$, be projections and $\Pi^u_m = \text{id} - \Pi^s_m : X^s_m = R(\Pi^s_m), m \in M, \kappa = s, u$. Suppose for every $m \in M$, $DH(m) : X^s_m \oplus X^u_m \to X^s_{u(m)} \oplus X^u_{u(m)}$ satisfies the following:

(a) $\Pi^u_{u(m)}DH(m) : X^u_m \to X^u_{u(m)}$ is invertible; write

$$||\Pi^u_{u(m)}DH(m)||_{X^u_m} = \lambda'_u(m), \|\Pi^u_{u(m)}DH(m)\|^{-1} = \lambda''_u(m);$$

(b) $||\Pi^s_{u(m)}DH(m)\Pi^s_{u(m)}|| \leq \xi, \kappa_1, \kappa_2 \in \{s,u\};$

(c) $\sup_{m \in M} ||\Pi^s_m|| < \infty, \sup_{m \notin M} \lambda'_u(m) < \infty;$

(d) $DH$ is almost uniformly continuous around $M$, meaning that the amplitude $A(e)$ of $DH$ in

$$\mathbb{B}_e(M) = \{x : d(x, M) < e\}$$

can be sufficiently small when $e \to 0$.

(e) $|H(m) - u(m)| \leq \eta.$

If $\eta, \xi$ are small and $\sup_{m \in M} \lambda'_u(m) \lambda''_u(m) < 1$, then

(1) there are small $r_1, r', i = 1, 2$, and two maps $F_m, G_m$, such that

$$H_m = H(m + \cdot) - u(m) \sim (F_m, G_m) : X^s_{r_1} \times X^u_{r_1} \to X^s_{r_2} \times X^u_{r_2},$$

satisfies $(A(\alpha, \lambda_u(m))) \,(B(\beta, \lambda_s(m)))$ condition, where $\lambda_u(m) = \lambda'_u(m) + \zeta, \lambda_u(m) = \lambda''_u(m) + \zeta, \zeta > 0$ is sufficiently small, and $\alpha, \beta, \kappa \to 0$ as $r_1, r_2, \eta, \xi \to 0$ and $A(e) \to 0$;

(2) in addition, $|F_m(0,0)| \leq K_1 \eta$ and $|G_m(0,0)| \leq K_1 \eta$ for some $K_1 > 0$ independent of $m, \eta$;

(3) there are maps $F_m, G_m$ defined in all $X^s_m \times X^u_m$, $m \in M$ such that $F_m(x, y) + G_m(x, y) = H(m)$.

Proof. Set $\lambda'_u = \sup_{m \in M} \lambda'_u(m), C_1 = \sup_{m \in M} (||\Pi^s_m||, ||\Pi^u_m||), A_m = \sup_{m \in M} \|DH(m)\|_{X^s_m},$ and

$$(f_m(x, y), g_m(x, y)) = \left(H(m + x + y) - u(m) : X^s_m \oplus X^u_m \to X^s_{u(m)} \oplus X^u_{u(m)} \right)(x, y) \in X^s_m \oplus X^u_m.$$ Here $g_m(x, y) = \Pi^u_{u(m)}H(m + x + y) - u(m)).$ We first show there is a small $r_1 > 0$ (independent of $m$) such that $g_m^{-1}(x, \cdot)|_{X^u_{u(m)}}$ exists for $|x| \leq r_1$ which is very standard by using e.g. Banach fixed point theorem. By (d), we have

$$A(r) = \sup\{|DH(m_1) - DH(m_2)| : |m_1 - m_2| \leq r, m_1, m_2 \in \mathbb{B}_r(M)\} \to 0,$$

as $r \to 0$. So there is a small $r > 0$ such that for all $|x| \leq r, |y| \leq r, m \in M$,

$$|D_2g_m(x, y) - D_2g_m(0,0)| \leq \sqrt{\|DH(m + x + y) - DH(m)\|} \leq \sqrt{\lambda''_u}^{-1}/4.$$

Let $\eta \leq (C_1 \lambda''_u)^{-1} \cdot \eta/4$. Consider the following function:

$$h_m(x, y, z) = -A_m^{-1}g_m(x, y) + y + z, y, z \in X^u_m.$$ By (*), we then see for $|x| \leq r$ and $y_1, y_2 \in X^u_m$,

$$|h_m(x, y_1, z) - h_m(x, y_2, z)| = |h_m(x, y_1, z) - h_m(x, y_2, z)| = |A_m^{-1} \int_0^1 \{D_2g_m(x, s y_1 + y_1 - 1, s y_2 - y_2) - D_2g_m(0,0)\} \, ds (y_1 - y_2)| \leq 1/4|y_1 - y_2|,$$

and for $|x|, |y| \leq r$ and $|z| \leq r/2$,

$$|h_m(x, y, z)| \leq |A_m^{-1} \Pi^u_{u(m)}\{H(m + x + y) - H(m) - u(m)\} + y + z| \leq |A_m^{-1} \Pi^u_{u(m)}\{H(m) - u(m)\}| + |A_m^{-1} \int_0^1 \{D_2g_m(x, s y) - D_2g_m(0,0)\} \, ds y| + |z| \leq r/4 + r/4 + |z| \leq r.$$
Therefore, for each \(x, z\) such that \(|x| \leq r\) and \(|z| \leq r/2\), there is a unique \(y(x, z) \in X^u_{\eta}(r)\) such that \(h_{\eta}(x, y(x, z), z) = y(x, z)\). Define \(G_m(x, z) = y(x, A^{-1}_m z)\), \(z \in X^u_{\eta}(\rho(r)) = X^u_{\eta}(r)\), and \(r_1 = \min\{r, (\lambda^*_u)^{-1}/2 \cdot r\}\). We obtain \(g^{-1}_m(x, \cdot)|_{X^u_{\eta}(r)}\) exists, and for \(|x| \leq r_1\),
\[
g_m(x, G_m(x, z)) = z, z \in X^u_{\eta}(r_1).
\]

Let \(F_m(x, z) = f_m(x, G_m(x, z))\). Then for some \(r_2 > 0\), we have
\[
H_m \sim (F_m, G_m) : X^u_{\eta}(r_1) \oplus X^u_{\eta}(r) \rightarrow X^u_{\eta}(r_2) \oplus X^u_{\eta}(r_1).
\]

Moreover, we have \(|G_m(0, 0)| \leq 2\lambda^*_u C_1 \eta\), and
\[
\text{Lip}
\frac{G_m(\cdot, z)}{z} \leq (\epsilon + \xi)(1 - \epsilon)^{-1}, \quad \text{Lip}
\frac{G_m(x, \cdot)}{x} \leq (\epsilon + \lambda_u(m))(1 - \epsilon)^{-1},
\]
where \(\epsilon = \lambda^*_u C_1 \eta \text{lip}(r)\). Similar for \(F_m(\cdot, z) \leq \lambda_u(m) + \epsilon_1\) and \(F_m(x, \cdot) \leq 2\lambda^*_u \xi + \epsilon_1\) for some \(\epsilon_1 > 0\) such that \(\epsilon_1 \rightarrow 0\) as \(r \rightarrow 0\); also \(|F_m(0, 0)| \leq K_1 \eta\) for some \(K_1 > 0\) (independent of \(\eta, m\)).

Define
\[
\bar{F}_m(x, y) = D F_m(0, 0)(x, y) + F_m(0, 0) + (F_m(\cdot, \cdot) - F_m(0, 0) - D F_m(0, 0)) \circ (r_1(x), r_1(y)),
\]
and similar for \(\bar{G}_m(x, y)\), where \(r_\alpha(\cdot)\) is the radial retraction given by (3.1); let \(\bar{H}_m \sim (\bar{F}_m, \bar{G}_m)\). Using Lemma 3.5, we have the (A) (B) condition of \(H_m\) and \(\bar{H}_m\) as given in the lemma. The proof is complete.

The above lemma shows us how this paper’s main results can be applied to some classical settings; see e.g. Section 7.2. Here \(X\) can be a Riemannian manifold having bounded geometry (see e.g. Definition C.6), or a uniformly regular Riemannian manifold at \(M\) (see e.g. Definition C.7) including the smooth compact Riemannian manifold with \(M\) being far away from the boundary, or more generally, \(X\) is \(C^{0,1}\)-uniform around \(M\) (see assumption (■) in page 49).

To apply our existence results in Section 4.1 for the local case like the one given in the above lemmas, one usually needs using the radial retraction (3.1) to truncate the generating maps so that the conditions become globality (e.g. Lemma 3.13 (3)); and to apply the regularity results in Section 6, instead one needs using smooth bump functions (or blid maps (see Appendix D.5)). If \(\sup_m \lambda_u(m) < 1\) (or \(\sup_m \lambda_u(m) < 1\), Theorem 4.6 and Theorem 6.39 can be applied directly in some contexts.

4. Existence of Invariant Graphs

Our existence results are stated in Section 4.1 and the proofs are presented in the following Section 4.2 and Section 4.3. More characterizations and corollaries are given in Section 4.4 and Section 4.5 respectively.

4.1. Invariant graph: statements of the discrete case. For the convenience of writing, we write the metrics \(d(x, y) \triangleq |x - y|\). A bundle with metric fibers means each fiber is a complete metric space.

**Theorem 4.1** (First existence theorem). Let \((X, M, \pi_1), (Y, M, \pi_2)\) be two bundles with metric fibers and \(u : M \rightarrow M\) a map. Let \(H : X \times Y \rightarrow X \times Y\) a bundle correspondence over \(u\) with a generating bundle map \((F, G)\). Take \(\varepsilon_1(\cdot) : M \rightarrow \mathbb{R}_+\). Assume that the following hold.

(i) \((\varepsilon\text{-pseudo-stable section}) i = (i_X, i_Y) : M \rightarrow X \times Y\ is an \(\varepsilon\text{-pseudo-stable section}\) of \(H\), i.e.

\[|i_X(u(m)) - F_m(i_X(m), i_Y(u(m))))| \leq \eta(u(m)), |i_Y(m) - G_m(i_X(m), i_Y(u(m)))| \leq \eta(m),\]

where \(\eta : M \rightarrow \mathbb{R}_+, with \eta(u(m)) \leq \varepsilon(m)\eta(m)\) and \(0 \leq \varepsilon(m) \leq \varepsilon_1(m), \forall m \in M\).

(ii) \(H\) satisfies \((A'))(\alpha, \lambda_u) (B')(\beta', \lambda_s)\) condition, where \(\alpha, \beta, \beta', \lambda_u, \lambda_s\) are functions of \(M \rightarrow \mathbb{R}_+\). In addition,

(a) (angle condition) \(\sup_m \alpha(m)\beta'(u(m)) < 1, \beta'(u(m)) \leq \beta(m), \forall m \in M, \sup_m \{\alpha(m), \beta(m)\} < \infty,\)

(b) (spectral condition) \(\sup_m \frac{\lambda_u(m)\lambda_s(m) + \lambda_u(m)\varepsilon_1(m)}{1 - \alpha(m)\beta'(u(m))} < 1\).
The invariant-stable case: $\text{Lip}_m f \leq \beta'(m)$, $\sup_m \alpha'(m) \beta'(u(m)) < 1$, $\alpha'(m) \leq \alpha(u(m))$, $\beta'(u(m)) \leq \beta(m)$, $\forall m \in M$.

Then there is a unique bundle map $f : X \to Y$ over id such that the following (1) (2) hold.

1. $\text{Lip}_m f \leq \beta'(m)$, $|f_m(i_x(m)) - i_y(m)| \leq K \eta(m)$, for some constant $K \geq 0$.
2. $\text{Graph} f \subset H^{-1} \text{Graph} f$. More precisely, $(x_m(x), f_m(x)) \in H_m(x, f_m(x)), \forall x \in X_m$, where $x_m(\cdot) : X_m \to X_{u(m)}$ such that $\text{Lip}_m x_m(\cdot) \leq \lambda_x(m)$, $|x_m(i_x(m)) - i_y(u(m))| \leq K_0 \eta(u(m)), \forall m \in M$, for some constant $K_0$.

Remark 4.2. There are two cases which frequently occur in the practical applications.

1. The invariant-stable case: $\varepsilon_1 = 0$. More particularly $i$ is an invariant section of $H$ (i.e. $\eta \equiv 0$ or $i(u(m)) \in H_m(i(m))$, $m \in M$). We usually apply this case to obtain some invariant foliations; see e.g. Section 7.2.
2. The bounded-stable case: $\varepsilon_1 = 1$, i.e. $\eta$ is orbitally bounded about $u$. For instance, if $M$ is a single point set, then all the constant sections belong to this case. Note that for this case, $\eta$ might not be a bounded function. More particularly $\sup_m \eta(m) < \infty$.

We use the notation $\tilde{d}(A, z) \triangleq \sup_{z \in A} d(z, z)$, if $A$ is a subset of a metric space.

Theorem 4.3 (Second existence theorem). Let $(X, M, \pi_1), (Y, M, \pi_2)$ be two bundles with metric fibers and $u : M \to M$ a map. Let $H : X \times Y \to X \times Y$ be a bundle correspondence over $u$ with a generating bundle map $(F, G)$. Take $\varepsilon_1(\cdot) : M \to \mathbb{R}_+$. Assume that the following hold:

1. For-Y-bounded-section $i = (i_X, i_Y) : M \to X \times Y$ is a For-Y-bounded-section of $H$, i.e.

   $$
   \tilde{d}(G_m(X_m, i_Y(u(m))), i_Y(m)) \leq \eta(m),
   $$

   where $\eta : M \to \mathbb{R}_+$, with $\eta(u(m)) \leq \varepsilon(m) \eta(m)$ and $0 \leq \varepsilon(m) \leq \varepsilon_1(m), \forall m \in M$.

2. $H$ satisfies (A')$(\alpha, \lambda_u)$ (B')$(\beta, \lambda_x)$ condition, where $\alpha, \lambda_u, \beta, \lambda_x$ are functions of $M \to \mathbb{R}_+$.

   In addition,

   a. (angle condition) $\sup_m \alpha(m) \beta'(u(m)) < 1$, $\beta'(u(m)) \leq \beta(m)$, $\forall m \in M$, $\sup_m \{\alpha(m), \beta(m)\} < \infty$.

   b. (spectral condition) $\sup_m \frac{\lambda_u(m) \varepsilon_1(m)}{1 - \alpha'(m) \beta'(u(m))} < 1$.

3. $H$ satisfies (A)$\alpha, \lambda_u)$ (B)$\beta, \lambda_x)$ condition, where $\alpha, \beta, \lambda_u, \lambda_x$ are functions of $M \to \mathbb{R}_+$.

   In addition,

   a. (angle condition) $\sup_m \alpha'(m) \beta'(u(m)) < 1$, $\alpha'(m) \leq \alpha(u(m))$, $\beta'(u(m)) \leq \beta(m)$, $\forall m \in M$, $\sup_m \{\alpha(m), \beta(m)\} < \infty$.

   b. (spectral condition) $\sup_m \lambda_u(m) \varepsilon_1(m) < 1$.

Then there is a unique bundle map $f : X \to Y$ over id such that the following (1) (2) hold.

1. $\text{Lip}_m f \leq \beta'(m)$, $\tilde{d}(f_m(X_m), i_Y(m)) \leq K \eta(m)$, $m \in M$, where $K \geq 0$ is a constant.
2. $\text{Graph} f \subset H^{-1} \text{Graph} f$. More precisely, $(x_m(x), f_m(x)) \in H_m(x, f_m(x)), \forall x \in X_m$, where $x_m(\cdot) : X_m \to X_{u(m)}$ such that $\text{Lip}_m x_m(\cdot) \leq \lambda_x(m), \forall m \in M$. In addition, if

   $$
   \tilde{d}(F_m(X_m, i_Y(u(m))), i_Y(u(m))) \leq \eta(u(m)), \forall m \in M,
   $$

   then $\tilde{d}(x_m(X_m), i_Y(u(m))) \leq K_0 \eta(u(m))$, $\forall m \in M$, for some constant $K_0$. Moreover, we have

   a. $f$ does not depend on the choice of $\eta' = \eta$, as long as $\eta'$ satisfies that $\eta'(u(m)) \leq \varepsilon'(m) \eta'(m)$, $\varepsilon'(m) \leq \varepsilon_1(m)$ and $\eta(m) \leq \eta'(m)$, for all $m \in M$. 


Remark 4.4. A very special case of Theorem 4.3 is that sup$_m \eta(m) < \infty$ (for this case one can take $\varepsilon_1 \equiv 1$). Now $f$ is unique in the following sense: if $i'$ is a section such that sup$_m d(i_y(m), i'_y(m)) < \infty$, and for $i'$ there is a bundle map $f' : X \to Y$ over id satisfying (a) Lip $f'_m \leq \beta'(m)$, sup$_m d(f'_m(X_m), i'_y(m)) < \infty$, and (b) Graph$f' \subset H^{-1}\text{Graph}f'$, then $f' = f$.

Suppose $H$ satisfies the condition (ii) or (ii') in Theorem 4.3 with $\varepsilon_1 \equiv 1$, and one of the following situations holds:

1. sup$_m \text{diam}X_m < \infty$, and $i = (i_X, i_Y)$ a section of $X \times Y$ such that sup$_m |G_m(i_X(m), i_Y(u(m))) - i_Y(m)| < \infty$;
2. sup$_m \text{diam}Y_m < \infty$, and $i = (i_X, i_Y)$ any section of $X \times Y$;
3. $X_m, Y_m$ are Banach spaces, $G_m(x, y) = B_m y + g'_m(x, y)$, where $B_m : Y_u(m) \to Y_m$ is a linear operator, and $g'_m$ is a bounded function uniform for $m \in M$. Let $i = 0$ be the 0-section. This case was also studied in [CY94] and [CL97].

Then we have the conclusions in Theorem 4.3 hold. For the case (2) or (3), it is obvious that condition (i) in Theorem 4.3 is satisfied. And for the case (1), use the fact that sup$_m \text{sup}_y \text{Lip} G_m(\cdot, y) < \infty$.

Remark 4.5. It is instructive to see what the difference is between Theorem 4.1 and Theorem 4.3 in simple settings where $M$ consisting of one element and $H$ is the time-one solution map of the following equation in $\mathbb{R}^n$,

$$\dot{z} = Az + h(z),$$

where $A \in \mathbb{R}^{n \times n}$ with $\sigma(A) \cap i\mathbb{R} \neq \emptyset$ and Lip $h$ is sufficient small. In this case, (a) if $h(0) = 0$, then the existence of the stable, center-stable or pseudo-stable manifold of 0 for $H$ is a direct consequence of Theorem 4.1, and (b) if $h$ is a bounded function (but $H$ might be no equilibrium), then the existence of the center-stable (or pseudo-stable) manifold for $H$ can be deduced from Theorem 4.3 (or Theorem 4.1 for $\varepsilon = 1$ case); in the latter case, if there is no equilibrium, $H$ might have no stable manifold. While, using Theorem 4.3, one might obtain some non-resonant manifolds due to no spectral gap condition being needed (see also [Cha02, dL97] in some sense for ‘higher order’ case).

We state a local version of the existence result for the strong stable case. In this case, we do not need to truncate of the system, but the thresholds in angle condition and spectral condition are a little different from Theorem 4.1 and Theorem 4.3.

Theorem 4.6 (Third existence theorem). Let $(X, M, \pi_1), (Y, M, \pi_2)$ be two bundles with fibers being Banach spaces and $u : M \to M$ a map. Let $\varepsilon_1(\cdot) : M \to \mathbb{R}_+$ and $i = 0 : M \to X \times Y$. For every $m \in M$, suppose that $H_m \sim (F_m, G_m) : X_m(r_1) \times Y_m(r'_1) \to X_{u(m)}(r_2) \times Y_{u(m)}(r'_2)$ is a correspondence satisfying (a) $\text{sup} G_m(a(m), \lambda_u(m), \lambda_u(m)) \leq \beta'(m)$, (b) $\beta'(m)$, $\lambda_u(m)$ condition, or (c) $\text{sup} (A)(a(m), \alpha'(m), \lambda_u(m)) \leq \beta'(m)$, $\beta'(m)$, $\lambda_u(m)$ condition, where $r_1, r'_1, i = 1, 2, 1, 2$ are independent of $m \in M$, and

$$F_m : X_m(r_1) \times Y_{u(m)}(r'_2) \to X_{u(m)}(r_2), G_m : X_m(r_1) \times Y_{u(m)}(r'_2) \to Y_{u(m)}(r'_2).$$

(i) $(\varepsilon)$-pseudo-stable section $i$ is an $\varepsilon$-pseudo-stable section of $H$, i.e.

$$|F_m(0, 0)| \leq \eta(u(m)), |G_m(0, 0)| \leq \eta(m),$$

where $\eta : M \to \mathbb{R}_+$, with $\eta(u(m)) \leq \varepsilon(m)\eta(m)$ and $0 \leq \varepsilon(m) \leq \varepsilon_1(m)$, $\forall m \in M$.

(ii) If the case (b) holds, assume

(a) $(\text{angle condition})$ sup$_m \alpha(m)\beta'(u(m)) < 1/2$, $\beta'(u(m)) \leq \beta(m), \forall m \in M, \sup_m \{\alpha(m), \beta(m)\} < \infty$,

(b) $(\text{spectral condition})$ sup$_m \frac{\lambda_u(m)\lambda_u(m) + \lambda_u(m)\varepsilon_1(m)}{1 - \alpha(m)\beta'(u(m))} < 1$, sup$_m \lambda_u(m) < 1$.

Or if the case (c) holds, assume

(a') $(\text{angle condition})$ sup$_m \alpha'(m)\beta'(u(m)) < 1/2$, $\alpha'(m) \leq \alpha(u(m)), \beta'(u(m)) \leq \beta(m), \forall m \in M, \sup_m \{\alpha(m), \beta(m)\} < \infty$,

(b') $(\text{spectral condition})$ sup$_m \lambda_u(m)\lambda_u(m) < 1$, sup$_m \lambda_u(m)\varepsilon_1(m) < 1$, sup$_m \lambda_u(m) < 1$. If $\eta_0 > 0$ is small and sup$_m \eta(m) \leq \eta_0$, then there is a small $\sigma_0 > 0$ such that there are maps $f_m : X_m(\sigma_0) \to Y_m, m \in M$, uniquely satisfying the following (1) (2).
Then we have the following estimates.

1. \( \text{Lip}_f \leq \beta'(m), |f_m(0)| \leq K\eta(m), \) for some constant \( K \geq 0 \).
2. \( \text{Graph} f \subset H^{-1}\text{Graph} f. \) More precisely, \((x_m(x), f_m(x)) \in H_m(x, f_m(x)), \forall x \in X_m, \) where \( x_m(\cdot) : X_m(\sigma_0) \rightarrow X_{m(m)}(\sigma_0) \) such that \( \text{Lip}_m(x_m) \leq \lambda_s(m), |x_m(0)| \leq K_0\eta(u(m)), \forall m \in M, \) for some constant \( K_0. \) Moreover, we have
3. \( f \) does not depend on the choice of \( \eta' = \eta, \) as long as \( \eta' \) satisfies that \( \eta'(u(m)) \leq \varepsilon'(m)\eta'(m), \) \( \varepsilon'(m) \leq \varepsilon_1(m) \) and \( \eta(m) \leq \eta'(m) \leq \eta_0, \) for all \( m \in M. \)

A similar local version of Theorem 4.3 can be stated, which is omitted.

4.2. \textbf{Graph transform.} We use the symbol \( \Sigma_i(X, Y) \triangleq \{ \varphi : X \rightarrow Y : \text{Lip} \varphi \leq \lambda \}, \) if \( X, Y \) are metric spaces. \( \text{Graph} f \triangleq \{(x, f(x)) : x \in X\} \), if \( f : X \rightarrow Y. \) In the following, assume that \( X_i, Y_i, i = 1, 2 \) are metric spaces, and \( H : X_1 \times Y_1 \rightarrow X_2 \times Y_2 \) is a correspondence with a generating map \((F, G)\) satisfying \((A')(\alpha, \lambda_m) (B) (\beta'; \beta', \lambda_s)\) condition.

**Lemma 4.7.** Let \( f_2 \in \Sigma_{\beta'}(X_2, Y_2), \) and \( \alpha \beta < 1, \beta \leq \beta'. \) Then there exist unique \( f_1 \in \Sigma_{\beta'}(X_1, Y_1) \) and \( x_1(\cdot) \in \Sigma_{\lambda_s}(X_1, X_2), \) such that \( (x_1(x), f_2(x_1(x))) \in H(x, f_1(x)), \) \( x \in X_1, i.e. \)
\[
F(x, f_2(x_1(x))) = x_1(x), \quad G(x, f_2(x_1(x))) = f_1(x). \]

**Proof.** Consider the fixed equation, \( F(x, f_2(x)) = \hat{x}. \) Because of \( \alpha \beta < 1, \) there is a unique \( \hat{x} \triangleq x_1(x) \) satisfies the fixed equation. Since \( \beta \leq \beta', \) by \((B)\) condition, we have \( f_1 \in \Sigma_{\beta'}(X_1, Y_1), \) and \( x_1(\cdot) \in \Sigma_{\lambda_s}(X_1, X_2). \)

**Lemma 4.8.** \textbf{Under Lemma 4.7,} let \( (\hat{x}_i, \hat{y}_i) \in X_i \times Y_i, i = 1, 2, \) satisfy
\[
|F(\hat{x}_1, \hat{y}_1) - \hat{x}_2| \leq \eta_1, |G(\hat{x}_1, \hat{y}_2) - \hat{y}_1| \leq \eta_2, |f_2(\hat{x}_2) - \hat{y}_2| \leq C_2.
\]
Then we have the following estimates.
\[
|f_1(\hat{x}_1) - \hat{y}_1| \leq \lambda_s \frac{\hat{\beta} \eta_1 + C_2}{1 - \alpha \beta} + \eta_2, \quad |x_1(\hat{x}_1) - \hat{x}_2| \leq \frac{\alpha C_2 + \eta_1}{1 - \alpha \beta}.
\]

Particularly, if \((x_2, y_2) \in H(x_1, y_1), \) and \( y_2 = f_2(x_2), \) then \( y_1 = f_1(x_1), x_2 = x_1(x_1). \)

**Proof.** This follows from the following computations.
\[
|x_1(\hat{x}_1) - \hat{x}_2| \leq |F(\hat{x}_1, f_2(x_1(\hat{x}_2))) - F(\hat{x}_1, \hat{y}_2)| + |F(\hat{x}_1, \hat{y}_2) - \hat{x}_2| \leq \alpha |f_2(x_1(\hat{x}_2)) - \hat{y}_2| + \eta_1,
\]
and
\[
|f_2(x_1(\hat{x}_2)) - \hat{y}_2| \leq |f_2(x_1(\hat{x}_2)) - f_2(\hat{x}_2)| + |f_2(\hat{x}_2) - \hat{y}_2| \leq \hat{\beta}|x_1(\hat{x}_1) - \hat{x}_2| + C_2 \\
\leq \alpha \beta |x_1(\hat{x}_1) - \hat{x}_2| + C_2.
\]
Thus we have \( |f_2(x_1(\hat{x}_2)) - \hat{y}_2| \leq \frac{\hat{\beta} \eta_1 + C_2}{1 - \alpha \beta}. \) Furthermore,
\[
|f_1(\hat{x}_1) - \hat{y}_1| \leq |G(\hat{x}_1, f_2(x_1(\hat{x}_2))) - G(\hat{x}_1, \hat{y}_2)| + |G(\hat{x}_1, \hat{y}_2) - \hat{y}_1| \leq \lambda_s |f_2(x_1(\hat{x}_2)) - \hat{y}_2| + \eta_2 \\
\leq \lambda_s \frac{\hat{\beta} \eta_1 + C_2}{1 - \alpha \beta} + \eta_2,
\]
\[
|x_1(\hat{x}_1) - \hat{x}_2| \leq \alpha \frac{\hat{\beta} \eta_1 + C_2}{1 - \alpha \beta} + \eta_1 = \frac{\alpha C_2 + \eta_1}{1 - \alpha \beta}.
\]
The proof is complete. \( \square \)

The following estimates are straightforward.
Lemma 4.9. Assume $\alpha \beta < 1$, $\hat{\beta} \leq \beta$. Let $f_2, f_2' \in \sum_{(X_2, Y_2)}$, and $x_1(\cdot) \in \sum_{(X_1, X_2)}$ such that $\text{Graph} f_1 \subset H^{-1} \text{Graph} f_2$, $\text{Graph} f_1' \subset H^{-1} \text{Graph} f_2'$. Then we have the following.

\[
|f_1(x) - f_1'(x)| \leq \lambda |f_2(x_1(x)) - f_2'(x_1'(x))|,
\]

\[
|f_2(x_1(x)) - f_2'(x_1'(x))| \leq |f_2(x_1(x)) - f_2'(x_1(x))| + \hat{\beta} |x_1(x) - x_1'(x)|,
\]

\[
|f_2(x_1(x)) - f_2'(x_1'(x))| \leq \frac{1}{1 - \alpha \beta} |f_2(x_1(x)) - f_2'(x_1(x))|,
\]

\[
|f_1(x) - f_1'(x)| \leq \lambda \alpha |f_2(x_1(x)) - f_2'(x_1'(x))| \leq \frac{\lambda \alpha}{1 - \alpha \beta} |f_2(x_1(x)) - f_2'(x_1(x))|.
\]

4.3. **Invariant graph: proofs.**

4.3.1. **Proof of Theorem 4.1.** Define a metric space as

\[
E_\infty \triangleq \{ f : X \to Y \text{ is a bundle map over id} : \text{Lip} f_m \leq \beta'(m), |f_m(i_X(m)) - i_Y(m)| \leq C_2(m), \forall m \in M \}.
\]

The metric is defined by

\[
\hat{d}(f, f') \triangleq \sup_{m \in M} \sup_{x \in X_m} \frac{d(f_m(x), f'_m(x))}{\max\{d(x, i_X(m)), C_1(m)\}},
\]

where $C_1(m) = K_2 \eta(m)$, $C_2(m) = K_1 \eta(m)$. The constants $K_1, K_2$ satisfy

\[
K_1 \geq \frac{\hat{\alpha} \beta + 1}{1 - \hat{\lambda}}, \quad K_2 \triangleq \frac{\hat{\alpha} K_1 + 1}{1 - \sup_m \alpha(m) \beta'(u(m))},
\]

where $\hat{\alpha} \triangleq \sup_{m} \frac{\alpha(m) \eta(m)}{1 - \alpha(m) \beta'(u(m))} < 1$, $\sup_m \alpha(m) \leq \hat{\alpha} < \infty$, $\sup_m \beta(m) \leq \hat{\beta} < \infty$. Note that the constant bundle map $f_m(x) = i_Y(m), x \in X_m$ belongs to $E_\infty$. First we show the metric is well defined.

**Sublemma 4.10.** The metric $\hat{d}$ is well defined, and the space $E_\infty$ is complete under the metric $\hat{d}$.

**Proof.** It suffices to show $\forall f, f' \in E_\infty, \hat{d}(f, f') < \infty$. This is easily from

\[
|f_m(x) - f'_m(x)| \leq |f_m(x) - f_m(i_X(m))| + |f_m(i_X(m)) - f'_m(i_X(m))| + |f'_m(i_X(m)) - f'_m(x)| \leq \hat{\beta} |x - i_X(m)| + 2 C_2(m),
\]

and $\sup_m \frac{C_2(m)}{C_1(m)} < \infty$. One can use the standard argument to show the completion of $E_\infty$, so we omit it.

\[ \square \]

In the following, we first assume $H$ satisfies condition (ii) in **Theorem 4.1**.

For every $f \in E_\infty$, since $f_{i_X(m)} \in \sum_{\beta'(u(m))}(X_{i_X(m)}, Y_{i_X(m)})$, $\sup_m \alpha(m) \beta'(u(m)) < 1, \beta'(u(m)) \leq \beta(m)$, by **Lemma 4.7**, there are unique $\tilde{f}_m \in \sum_{\beta'(u(m))}(X_m, Y_m)$ and $x_{i_X(m)} \in \sum_{\lambda \alpha'(m)}(X_m, X_{i_X(m)})$ such that

\[
(x_m(x), f_{i_X(m)}(x_{i_X(m)})) \in H_m(x, \tilde{f}_m(x)), \quad x \in X_m.
\]

Since $|f_{i_X(m)}(i_X(u(m))) - i_Y(u(m))| \leq C_2(u(m))$, and $|i_X(u(m)) - F_m(i_X(m), i_Y(u(m)))| \leq \eta(u(m))$, $|y(u(m)) - G_m(i_X(m), i_Y(u(m)))| \leq \eta(u(m))$, by **Lemma 4.8**, we have

\[
|\tilde{f}_m(i_X(m)) - i_Y(m)| \leq \tilde{\alpha} \beta'(u(m)) \eta(u(m)) + C_2(u(m)) \leq \tilde{\lambda} \beta \eta(m) + \tilde{\lambda} C_2(m) + \eta(m) \leq C_2(m),
\]

and

\[
|x_{i_X(m)}(i_X(m)) - i_Y(u(m))| \leq \frac{\hat{\alpha} C_2(u(m)) + \eta(u(m))}{1 - \sup_m \alpha(m) \beta'(u(m))} \leq C_1(u(m)).
\]
Let \( \bar{f}(m, x) = (m, \tilde{f}_m(x)) \). Then \( \bar{f} \in E_{\infty} \). Define the graph transform
\[
\Gamma : E_{\infty} \to E_{\infty}, \ f \mapsto \bar{f}.
\]
Note that Graph \( \bar{f} \subset H^{-1}\text{Graph} \ f \).

**Sublemma 4.11.** The graph transform \( \Gamma \) is Lipschitz under the metric \( \hat{d} \), and
\[
\text{Lip} \ \Gamma \leq \sup_m \frac{\lambda_u(m) + \lambda_s(m) + \lambda_u(m)\varepsilon_1(m)}{1 - \alpha(m)\beta'(u(m))} < 1.
\]

**Proof.** Let \( f, f' \in E_{\infty}, \bar{f} = \Gamma f, \bar{f}' = \Gamma f' \), with \( x_m(\cdot), x'_m(\cdot) \). Note first that by Lemma 4.9, we get
\[
|\tilde{f}_m(x) - \tilde{f}'_m(x)| \leq \frac{\lambda_u(m)}{1 - \alpha(m)\beta'(u(m))}|f_{\ell}(x) - f'_{\ell}(x)|,
\]
and
\[
|x_m(x) - ix(u(m))| \leq |x_m(x) - x_m(i\varepsilon(u(m)))| + |x_m(i\varepsilon(u(m)) - ix(u(m)))| \\
\leq \lambda_s(m)|x - i\varepsilon(u)| + C_1(u(m)).
\]
Thus,
\[
\frac{|\tilde{f}_m(x) - \tilde{f}'_m(x)|}{\max\{|x - i\varepsilon(u)|, C_1(u(m))\}} \leq \frac{\lambda_u(m)}{1 - \alpha(m)\beta'(u(m))} \max\{|x_m(x) - x_m(i\varepsilon(u(m)))|, C_1(u(m))\}/\max\{|x - i\varepsilon(u)|, C_1(u(m))\}
\]
and the proof is complete. \( \square \)

Since \( \Gamma \) is contractive, we see there is a unique fixed point of \( \Gamma \) in \( E_{\infty} \), denoted by \( f \). By the construction of \( \Gamma \), we have Graph \( f \subset H^{-1}\text{Graph} \ f \), showing the conclusions (1) (2) in Theorem 4.1. The constant \( K \) in (2) can be taken as \( K = \frac{\lambda_s + 1}{1 - \hat{d}} \).

1. \( f \) is unique in the sense that if \( f' : X \to Y \) is any bundle map over \( id \) such that it satisfies

   (a0) Lip \( f'_m \leq \beta'(\text{m}), |f'_m(i\varepsilon(\text{m})) - iv(\text{m})| \leq K'\eta'(\text{m}), \) where \( K' \geq 0 \) is a constant,

   (b0) and Graph \( f' \subset H^{-1}\text{Graph} \ f' \),

   then \( f' = f \), where \( \eta' \) satisfies that \( \eta'(\text{m}) \leq c'(\text{m)}\eta'(\text{m}), c'(\text{m)} \leq c(\text{m)} \leq \eta(\text{m)} \leq \eta'(\text{m)} \), for all \( m \in M \). This is easily from the unique fixed point of \( \Gamma \) in \( E_{\infty} \). Without lose of generality, we can assume \( K' \leq K_1 \). The construction of \( E_{\infty} \triangleq E_{\infty}(\eta) \) depends on \( \eta \). However, \( E_{\infty}(\eta) \subset E_{\infty}(\eta') \). So \( f, f' \in E_{\infty}(\eta') \). The uniqueness of the fixed point of \( \Gamma \) in \( E_{\infty}(\eta) \) shows \( f = f' \).

Next, we show that if \( H \) satisfies condition (ii) in Theorem 4.1, then the conclusions also hold. Consider \( H^{(k)} \), the \( k \)-th composition of \( H \).

**Sublemma 4.12.** \( H^{(k)} \) is well defined, which is a bundle correspondence over \( u^k \) with generating map \( (F^{(k)} , G^{(k)}) \), satisfying \((A)(\alpha'; \alpha_{k}^{(k)}, \lambda_{u}^{(k)})(B)(\beta_{k} ; \beta', \lambda_{s}^{(k)})\) condition, where \( \alpha_{k}^{(k)}(m) = \alpha'(u^{k-1}(m)), \beta_{k}(m) = \beta(u^{k-1}(m)), \lambda_{u}^{(k)}(m) = \lambda_{u}(u^{k-1}(m)) \cdots \lambda_{u}(u^{k-2}(m)) \cdots \lambda_{u}(u^{k-1}(m)), \lambda_{s}^{(k)}(m) = \lambda_{s}(u^{k-1}(m)) \cdots \lambda_{s}(u^{k-2}(m)) \cdots \lambda_{s}(u^{k-1}(m)). \)

Moreover, \( H_{u^k(m)} \circ H_{m}^{(k)} = H_{m}^{(k+1)} \), \( k = 1, 2, \cdots \).

**Proof.** Since \( H_{m} \sim (F_{m}, G_{m}) : X_{m} \times Y_{m} \to X_{u^2(m)} \times Y_{u^2(m)} \), and \( \alpha'(m)\beta'(u(m)) < 1 \), by Lemma 2.3 and Lemma 3.3, we know that \( H_{m}^{(2)} \triangleq H_{u^2(m)} \circ H_{m} \sim (F_{m}^{(2)} , G_{m}^{(2)}) : X_{m} \times Y_{m} \to X_{u^2(m)} \times Y_{u^2(m)} \) satisfies \((A)(\alpha; \alpha_{2}^{(2)}, \lambda_{u}^{(2)})(B)(\beta_{2} ; \beta', \lambda_{s}^{(2)})\) condition, and \( \alpha_{2}^{(2)}(m)\beta'(u^{2}(m)) = \alpha'(u(m))\beta'(u^{2}(m)) < 1 \). Using \( H_{m}^{(2)} \), one can define \( H_{r}^{(2)} : X \times Y \to X \times Y \) a bundle correspondence over \( u^2 \) with generating
bundle map \((F(2), G(2))\), i.e. \(\text{Graph} H^{(2)} = \bigcup_{m \in M}(m, \text{Graph} H^{(2)}_m)\). \(H^{(k)}\) can be defined inductively by using the composition of \(H^{(k-1)}\) and \(H\). Obviously \(H_m^{k}(m) \circ H_m^{1} = H_m^{(k+1)}\), \(k = 1, 2, \cdots\), by our construction.

Fix any \(m_0 \in M\). Let \(\tilde{M}_{m_0} = \{u^n(m_0) : n \geq 0\}\). \(H^{(k)}\) can be regarded as a bundle correspondence \(X|_{\tilde{M}_{m_0}} \times Y|_{\tilde{M}_{m_0}} \to X|_{\tilde{M}_{m_0}} \times Y|_{\tilde{M}_{m_0}}\) over \(u^k\), denoted by \(H^{(k)}|_{\tilde{M}_{m_0}}\). Define a function \(\hat{e}_1(\cdot)\) over \(u\) as

\[
\hat{e}_1(m) = \max\{\varepsilon(m), \lambda_s(m)\}, \quad m \in M.
\]

Set

\[
\lambda \triangleq \sup_{m} \lambda_s(m)\lambda_u(m), \quad \lambda_1 \triangleq \sup_{m} \varepsilon(m)\lambda_u(m), \quad \gamma \triangleq \sup_{m} \alpha'(m)\beta'(u(m)).
\]

**Sublemma 4.13.** The section \(i = (i_X, i_Y)\) now is an \(\hat{e}_1^{(k)}\)-pseudo-stable section of \(H^{(k)}|_{\tilde{M}_{m_0}}\), i.e.

\[
|i_X(u^k(m)) - F_m^{(k)}(i_X(m), i_Y(u^k(m)))| \leq \eta^{(m_0)}_k(u^k(m)),
\]

\[
|i_Y(m) - G_m^{(k)}(i_X(m), i_Y(u^k(m)))| \leq \eta^{(m_0)}_k(u^k(m)),
\]

\(m \in \tilde{M}_{m_0}\), where \(\eta^{(m_0)}_k(u^k(m_0)) = c_k \hat{e}_1^{(k)}(m_0)\eta(m_0), i \geq 0,\) and \(c_k \geq 1\) is a constant independent of \(m_0\).

**Proof.** This is a direct consequence of **Lemma 4.8**. We only consider the case \(k = 2\). Let \(\hat{y}_m\) be the unique point satisfying \(\hat{y}_m = G_u(m)(F_m(i_X(m), \hat{y}_m), i_Y(u^2(m)))\). Then

\[
|\hat{y}_m - i_Y(u(m))| \leq \frac{(\beta'(u(m)) + 1)\eta(u(m))}{1 - \beta'(u(m))\alpha'(m)},
\]

and

\[
|G_m^{(2)}(i_X(m), i_Y(u^2(m))) - i_Y(m)| = |G_m(i_X(m), \hat{y}_m) - i_Y(m)|
\leq \lambda_u(m)|\hat{y}_m - i_Y(u^2(m))| + \eta(m)
\leq \lambda_u(m)\left(\beta'(u(m)) + 1\right)\eta(u(m))
\leq \frac{\lambda_u(m)\left(\beta'(u(m)) + 1\right)\eta(u(m))}{1 - \beta'(u(m))\alpha'(m)} + \eta(m).
\]

Also,

\[
|F_m^{(2)}(i_X(m), i_Y(u^2(m))) - i_X(u^2(m))| = |F_u(m)(F_m(i_X(m), \hat{y}_m), i_Y(u^2(m))) - i_X(u^2(m))|
\leq \lambda_s(m)\frac{\alpha'(m) + 1}{1 - \beta'(u(m))\alpha'(m)} + \eta(u^2(m)).
\]

If \(m = u'(m_0)\), then \(\eta(m) \leq \hat{e}_1^{(k)}(m_0)\eta(m_0)\). So we can choose

\[
c_2 = \max\left\{\frac{\lambda_1(\hat{y} + 1)}{1 - \gamma} + 1, \frac{\lambda_1(\hat{y} + 1)}{1 - \gamma} + 1\right\},
\]

completing the proof.

Since \(\lambda < 1, \lambda_1 < 1, \gamma < 1\), we can choose large \(k\) (independent of \(m_0\)) such that \(2\lambda^k + \lambda_1^k < 1 - \gamma\). So

\[
\sup_{m} \frac{\lambda_s^{(k)}(m)\lambda_u^{(k)}(m) + \hat{e}_1^{(k)}(m)\lambda_u^{(k)}(m)}{1 - \alpha_u^{(k)}(m)\beta'(u^k(m))} < 1.
\]

So far, we have shown \(H^{(k)}|_{\tilde{M}_{m_0}}\) satisfies condition (ii) with the \(\hat{e}_1^{(k)}\)-pseudo-stable section \(i\). Using the first part what we have proven, we obtain a unique bundle map \(f^{k,(m_0)} : X|_{\tilde{M}_{m_0}} \to Y|_{\tilde{M}_{m_0}}\) over \(id\), such that

(a1) \(\text{Lip}_{f^{k,(m_0)}} \leq \beta'(u^k), |f^{k,(m_0)}(i_X(m)) - i_Y(m)| \leq K'_1\eta_k^{(m_0)}(m),\) where \(K'_1 \geq 0\) (independent of \(m_0\)),

(b1) \(\text{Graph}_{f^{k,(m_0)}} \subset (H^{(k)}_m)^{-1}\text{Graph}_{f^{k,(m_0)}}\), \(m \in \tilde{M}_{m_0}\).
(c1) Also, \( f^{k,(m_0)} \) does not depend on the choice of \( \eta^{0}(\cdot) = \eta^{(m_0)}_{k}(\cdot) \) as long as it satisfies \( \eta^{0}(u^{k}(m)) \leq \hat{e}^{1}_{1}(m) \eta^{0}(m), \eta^{(m_0)}_{k}(m) \leq \eta^{0}(m) \), for all \( m \in \bar{M}_{m_0} \).

Note that for \( H^{(k)} \), its corresponding graph transform is \( I^{k} = I \circ \cdots \circ I \) (\( k \) times). Since \( f^{k,(m_0)} \) is the unique fixed point of \( I^{k} \), we have \( I^{k}f^{k,(m_0)} = f^{k,(m_0)} \), i.e. \( \text{Graph} f^{k,(m_0)} \subset H^{-1}_{m} \text{Graph} f^{k,(m_0)} \), \( m \in \bar{M}_{m_0} \).

Set \( f_{m_0} = f^{k,(m_0)}, m_0 \in M \). Note that \( f_{u(m_0)} = f^{k,(u(m_0))} \). Indeed, \( \{f_{m_0}^{k,(m_0)} : m \in \bar{M}_{u(m_0)} \} \) also fulfills (a1) (b1) for the case \( m_0 \) is replaced by \( u(m_0) \), with \( \eta^{0}(\cdot) = \eta^{(m_0)}_{k}(\cdot)|_{\bar{M}_{u(m_0)}} \) instead of \( \eta^{k,(u(m_0))}(\cdot) \).

Also, note that \( |f^{k,(m_0)}_{m_0}(x(m_0)) - f^{k,(m_0)}_{u(m_0)}(y(m_0))| \leq K_{1}c_{k}\eta(m_0) \). This shows \( f \) satisfies conclusions (1) (2) in Theorem 4.1. The uniqueness of \( f \) follows from the uniqueness of \( f : X|_{\bar{M}_{m_0}} \to Y|_{\bar{M}_{m_0}} \) for \( H^{(k)}|_{\bar{M}_{m_0}} \).

□

4.3.2. Proof of Theorem 4.3. The proof of Theorem 4.3 is essentially the same as Theorem 4.1. We sketch the main steps. The metric space now is defined by

\[
E_{\infty}^{b} \triangleq \{f : X \to Y \text{ is a bundle map over id : } \text{Lip} f_{m} \leq \beta'(m),
\]

\[
\tilde{d}(f_{m}(X_{m}), i_{Y}(m)) \leq K_{1}\eta(m), \forall m \in M,
\]

with its metric

\[
\tilde{d}(f, f') = \sup_{m \in M, \eta(m) \neq 0} \sup_{x \in X_{m}} \frac{d(f_{m}(x), f'_{m}(x))}{\eta(m)},
\]

for some constant \( K_{1} > 0 \). The metric \( \tilde{d} \) is well defined, and \( E_{\infty}^{b} \) is complete under this metric.

Assume \( H \) satisfies condition (ii) in Theorem 4.3. The graph transform \( I \) now also has \( \Gamma E_{\infty}^{b} \subset E_{\infty}^{b} \), by simple computations. It is also Lipschitz with Lipschitz constant less than \( \sup_{m} \frac{\lambda_{u,m}(m)}{1-\alpha(m)\beta'(u(m))} < 1 \).

From these, we certainly obtain the results. For the case when \( H \) satisfies condition (ii)' in Theorem 4.3, consider \( H^{(k)} \) instead of \( H \) for large \( k \), then we have \( \text{Lip} I^{k} < 1 \), competing the proof.

\( \square \)

4.3.3. Proof of Theorem 4.6. The proof is almost identical to Theorem 4.1. We only consider the case (●), i.e. (A') (B) condition holds. The metric space now is defined by

\[
E_{0}^{0} \triangleq \{f : X(\sigma_{0}) \to Y \text{ is a bundle map over id : Lip} f_{m} \leq \beta'(m),
\]

\[
|f_{m}(0)| \leq K_{1}\eta(m), \forall m \in M,
\]

where \( \sigma_{0} > 0 \), and \( K_{1} > 0 \) will be chosen later. The metric is the same as in \( E_{\infty} \). Take the following constants.

\[
\gamma = \sup_{m} \alpha(m)\beta'(u(m)) < 1/2, \quad \overline{\lambda} = \sup_{m} \frac{\lambda_{u,m}(m)E_{1}(m)}{1-\alpha(m)\beta'(u(m))} < 1, \quad \hat{\alpha} = \sup_{m} \alpha(m), \quad \hat{\beta} = \sup_{m} \beta'(m),
\]

\[
K_{1} = \frac{\overline{\lambda}_{1}\hat{\alpha} + 1}{1 - \overline{\lambda}_{1}}, \quad K_{2} = \frac{\hat{\alpha}K_{1} + 1}{1 - 2\gamma}, \quad \overline{\lambda}_{s} = \sup_{m} \lambda_{s}(m) < 1, \quad r < \min\{r_{i}, r'_{i} : i = 1, 2\}/2.
\]

Let \( \eta_{0} = \sup_{m} \eta(m) \) be small and choose \( \sigma_{0} > 0 \) such that

\[
\frac{K_{2}\eta_{0}}{1 - \overline{\lambda}_{s}} \leq \sigma_{0} < r, \quad \hat{\beta}\sigma_{0} + K_{1}\eta_{0} < r.
\]

So \( \overline{\lambda}_{s}\sigma + K_{2}\eta_{0} < \sigma_{0} \), if \( \sigma < \sigma_{0} \); and \( f_{m}(X_{m}(\sigma_{0})) \subset Y_{m}(r) \), if \( f \in E_{0}^{0} \).

Let \( f \in E_{0}^{0} \) and \( f_{m}^{1}(x) = f_{m}(r_{x}(0)(x)) \), where \( r_{x}(\cdot) \) is the radial retraction (see (3.1)). Note that \( \text{Lip} f_{m}^{1} \leq 2\beta'(m) \). Consider the following fixed point equation for each \( x \in X_{m}(\sigma_{0}) \):

\[
F_{m}(x, f_{u(m)}^{1}(y)) = y, \quad y \in X_{u(m)}.
\]
Since $\alpha(m)\beta'(u(m)) < 1/2$, for each $x \in X_u(m(\sigma_0))$, the above equation has a fixed point $y = x_u(m(\sigma_0))$. As $|f^k_{u(m)}(0)| = |f_{u(m)}(0)| \leq K_1\eta_0$ and $|F_m(0,0)| \leq \eta_0$, we see that $|x_m(0)| \leq K_2\eta_0$. Let us show

**Sublemma 4.14.** Lip $x_m(\cdot)|_{X_u(m(\sigma_0))} \leq \lambda_s(m)$.

**Proof.** First note that Lip $x_m(\cdot)|_{X_u(m(\sigma_0))} \leq \frac{7}{1-2\lambda}$. So we can choose an $\varepsilon_0 > 0$ small such that

$$\frac{7}{1-2\lambda}\varepsilon_0 + K_2\eta_0 \leq \sigma_0,$$

yielding $x_m(X_u(m(\varepsilon_0))) \subset X_u(m)(\sigma_0)$ and $F_m(x,f_{u(m)}(x_m(x))) = x_m(x)$. By (B) condition, we know Lip $x_m(\cdot)|_{X_u(m(\varepsilon_0))} \leq \lambda_s(m)$. Set

$$\sigma_1 = \sup(\sigma \leq \sigma_0 : \text{Lip } x_m(\cdot)|_{X_u(m(\sigma))} \leq \lambda_s(m)).$$

If $\sigma_1 < \sigma_0$, then Lip $x_m(\cdot)|_{X_u(m(\sigma))} \leq \lambda_s(m)$ for any $\sigma < \sigma_1$, which again shows Lip $x_m(\cdot)|_{X_u(m(\sigma_1))} \leq \lambda_s(m)$. Thus, we can choose an $\varepsilon > 0$ small such that

$$\bar{\lambda}_s\sigma_1 + K_2\eta_0 < \sigma_0,$$

which implies that $x_m(X_u(m(\sigma_1 + \varepsilon))) \subset X_u(m(\sigma_0))$ and Lip $x_m(\cdot)|_{X_u(m(\sigma_1 + \varepsilon))} \leq \lambda_s(m)$, contradicting the definition of $\sigma_1$. So $\sigma_1 = \sigma_0$, showing the result. □

Particularly, by the above sublemma, $x_m(X_u(m(\sigma_0))) \subset X_u(m(\sigma_0))$. Now we can define

$$\bar{f}_m(x) = G_m(x,f_{u(m)}(x_m(x))), \text{ } x \in X_u(m(\sigma_0)),$$

and $\bar{f}(m,x) = (m,\bar{f}_m(x))$. Next, following the same steps in the proof of Theorem 4.1, one can see that $\bar{f} \in E_{\infty}$ and $\Gamma : f \mapsto \bar{f}$ is contractive. The proof is complete. □

### 4.4. more characterizations.

Let $H \sim (F,G) : X \times Y \to X \times Y$, $i : M \to X \times Y$ be as in Theorem 4.1. The bundle map $f$ obtained in Theorem 4.1 has more properties.

Below, we use the following notations. $W^s(i) \cong W^s \cong \text{Graph } f$, $W^u(i(m)) \cong W^u(m) \cong \text{Graph } f_m$. $h^s_m \triangleq x_m(\cdot) : X_m \to X_u(m)$. This gives a bundle map $h^s : X \to X$ over $u$ such that $h(m,x) = (m,h^s_m(x))$. Also, note that Lip $h^s_m \leq \lambda_s(m)$. $\{z_n = (x_n,y_n) : n = m,u(m),u^2(m),\cdots\}$ is called a **forward orbit** of $H$ from $m$, if $z_{i-1}(m) \in H_{u^{i-1}(m)}(z_{i-1}(m))$, $i = 1,2,3,\ldots$. Similar for the backward orbit. Let $P^X(m,x,y) = (m,P^X_m(x,y)) = (m,x)$, if $(x,y) \in X_m \times Y_m$; similarly, $P^Y, P^Y_m$.

Also we adopt the notation: $a_n \lesssim b_n, n \to \infty (a_n \geq 0,b_n > 0)$, if $\sup_{n \geq 0} b_n/a_n < \infty$.

1. $H_m : W^s(m) \to W^u(m)$ defines a Lipschitz bundle map $H^s_m$. This gives a bundle map $H^s : X \times Y \to X \times Y$ over $u$, such that $H^s(m,z) = (m,H^s_m(z))$. We have $P^{X,s} \circ H^s = h^s \circ P^X$.

2. Let $z'_m = (x'_m,y'_m) \in W^s(m)$, then there is a unique forward orbit $\{z_n = (x_n,y_n) = m,u(m),u^2(m),\cdots\}$ of $H$ from $m$, such that $z_m = z'_m$. Furthermore,

$$|i_X(u^k(m)) - x'_{u^k(m)}| \leq K_0 \sum_{i=1}^{k} \eta(u^{-1}(m))\lambda_{s}^{(k-i)}(u^i(m)) + \lambda_s^{(k)}(m)|i_X(m) - x'_m|.$$

Therefore,

$$|i(u^k(m)) - z_{u^k(m)}| \leq \bar{K} \left( \sum_{i=1}^{k} \eta(u^{-1}(m))\lambda_{s}^{(k-i)}(u^i(m)) + \lambda_s^{(k)}(m)|i_X(m) - x'_m| \right),$$

for some constant $\bar{K} > 0$.

**Proof.** Compute

$$|i_X(u^k(m)) - x'_{u^k(m)}|$$

$$= |i_X(u^k(m)) - x_{u^{k-1}(m)}(z'_{u^{k-1}(m)})|$$

$$\leq |i_X(u^k(m)) - x_{u^{k-1}(m)}(i_X(u^{k-1}(m)))|$$

$$\leq |i_X(u^k(m)) - x_{u^{k-1}(m)}(i_X(u^{k-1}(m)))|.$$
In particular,
\[ \text{where } \square \sim \text{showing (4.1).} \]

First consider the case condition (ii) in Theorem 4.1 holds. For this case, we have
\[ \text{Compute } \lambda \sim (\lambda, \eta, \theta) \text{ from (4.2), Theorem 4.1 and (4.1), and where } \theta(m) = (1 - \alpha(m)\beta'(u(m)))^{-1} \text{ otherwise), then } z_n \in W^s(n), n = m, u(m), u^2(m), \ldots. \]

**Proof.** First consider the case condition (ii') in Theorem 4.1 holds. Let \( \{z'_n = (x'_n, y'_n) \in W^s(n), n = m, u(m), u^2(m), \ldots \} \) be a forward orbit of \( H \) from \( m \) such that \( z'_n = (x_m, f_m(x_m)) \in W^s(m) \).

Letting \( k \to \infty \), we have \( y_m = f_m(x_m) \).

Next consider the case condition (ii) in Theorem 4.1 holds. For this case, we have
\[ \text{where } \overline{\Lambda}_u(m) = \lambda_u(m)\theta(m) \text{ (over } u). \]

Letting \( k \to \infty \), we see that \( y_m = f_m(x_m) \), completing the proof.

(4) In particular,
\[ \{j : M \to X \times Y : j \text{ is an invariant section of } H \text{ such that } \|j(m) - i(m)\| \leq K'\eta'(m), \] for some constant \( K' > 0, \forall m \in M \} \subset W^s, \]

where \( \eta'(u(m)) \leq e'(m)\eta'(m), \forall m \in M, \text{ and } \sup_m e'(m)\lambda(m) < 1. \]
We have the following characterization of \( W^s(m) \).

\[
W^s(m) = \{ z_m = (x_m, y_m) \in X_m \times Y_m : \text{there is a forward orbit } \{ z_k \}_{k \geq 0} \text{ of } H \text{ such that } z_0 = z_m, \sup_{k \geq 0} |\hat{\epsilon}^{(k)}(m)|^{-1} |z_k - i(u^k(m))| < \infty \},
\]

where \( \hat{\epsilon}(m) \geq \epsilon(m) \), \( \lambda_s(m) + \varsigma < \hat{\epsilon}(m) < (\lambda_u(m) \vartheta(m))^{-1} - \varsigma \) (for sufficiently small \( \varsigma > 0 \)), where \( \vartheta(\cdot) \) is given in (3). This follows from (2) (3).

Similar results hold for the bundle map \( f \) obtained in Theorem 4.3 under the conditions (i) (ii)’. The ‘unstable results’ (in the direction \( Y \to X \)) can be obtained through the dual bundle correspondence of \( H \) (see Section 2.3).

4.5. \textbf{corollaries.}

4.5.1. \textbf{hyperbolic dichotomy.}

**Definition 4.15.** Let \( H \sim (F, G) : X \times Y \to X \times Y \) be a bundle correspondence over \( u \). Let \( i = (i_X, i_Y) : M \to X \times Y \) satisfy

\[
|X(u(m)) - F_m(i_X(m), i_Y(u(m)))| \leq \eta(u(m)), |i_Y(m) - G_m(i_X(m), i_Y(u(m)))| \leq \eta(m),
\]

where \( \eta : M \to \mathbb{R}_+ \). Take \( \epsilon(\cdot) : M \to \mathbb{R}_+ \). \( i \) is called an \( \epsilon \)-pseudo-stable section of \( H \), if \( \epsilon(u(m)) \leq \epsilon(m) \eta(m) \), \( \forall m \in M \); an \( \epsilon \)-pseudo-unstable section of \( H \), if \( \epsilon(m) \leq \epsilon(\cdot) \eta(u(m)) \), \( \forall m \in M \); and an \( \epsilon \)-pseudo-invariant section of \( H \), if it is both \( \epsilon \)-pseudo-stable and \( \epsilon \)-pseudo-unstable section of \( H \). In particular, 0-pseudo-invariant section is an invariant section of \( H \), i.e. \( i(u(m)) \in H_m(i(m)) \) for all \( m \in M \).

Note that if \( u \) is invertible, then \( i \) is an \( \epsilon \)-pseudo-unstable section of \( H \) if and if only \( i \) is an \( \epsilon \circ u^{-1} \)-pseudo-stable section of the dual bundle correspondence \( \tilde{H} \) of \( H \) (see Section 2.3). The following theorem is a restatement of the previous results.

**Theorem 4.16.** Let \( (X, M, \pi_1), (Y, M, \pi_2) \) be two bundles with metric fibers and \( u : M \to M \) an invertible map. Let \( H : X \times Y \to X \times Y \) be a bundle correspondence over \( u \) with a generating bundle map \( (F, G) \). Assume \( i = (i_X, i_Y) \), \( j = (j_X, j_Y) \): \( M \to X \times Y \) are \( \epsilon_1 \)-pseudo-stable section and \( \epsilon_2 \)-pseudo-unstable section of \( H \), respectively, where \( \epsilon_1(\cdot), \epsilon_2(\cdot) : M \to \mathbb{R}_+ \) (over \( u \) and \( u^{-1} \) respectively). \( H \) satisfies (A)(\( \alpha; \alpha', \lambda_u \))(B)(\( \beta; \beta', \lambda_s \)) condition, where \( \alpha, \beta, \alpha', \beta', \lambda_u, \lambda_s \) are bounded functions of \( M \to \mathbb{R}_+ \). In addition,

(a) (angle condition) \( \sup_m \alpha'(m) \beta'(u(m)) < 1 \), \( \alpha'(m) \leq \alpha(u(m)) \), \( \beta'(u(m)) \leq \beta(m) \), \( \forall m \in M \),

(b) (spectral condition) \( \sup_m \lambda_u(m) \lambda_s(m) \eta(m) < 1 \), \( \sup_m \lambda_u(m) \vartheta(m) \leq 1 \), \( \sup_m \lambda_s(m) \vartheta(m) \leq 1 \).

Then there are \( W^s_m(i), W^u_m(j) \), \( m \in M \) such that the following hold.

1. \( W^s_m(i), W^u_m(j) \) are represented as Lipschitz graphs of \( X_m \to Y_m \) and \( Y_m \to X_m \) respectively, \( m \in M \), such that \( W^s_m(i) \subset H_m^{-1}W^s_{u(m)}(i) \) and \( W^u_m(j) \subset H_m^{-1}(m)W^u_{u^{-1}(m)}(j) \).

2. \( W^s_m(i) \cap W^u_m(j) = \{ k(m) \} \) and \( W^s(i) \cap W^u(j) = \{ k \} \). Therefore, \( k \) is an invariant section of \( H \), i.e. \( k(u(m)) \in H_m(k(m)) \).

3. (shadowing section) We have the following estimates,

\[
|k(u^k(m)) - i(u^k(m))| \lesssim \hat{\epsilon}_1^{(k)}(m), \quad |k(u^{-k}(m)) - j(u^{-k}(m))| \lesssim \hat{\epsilon}_2^{(k)}(m), \quad k \to \infty,
\]

where \( \hat{\epsilon}_i(\cdot), i = 1, 2 \), are functions of \( M \to \mathbb{R}_+ \) over \( u \) and \( u^{-1} \) respectively, such that \( \epsilon_1(m) \leq \hat{\epsilon}_1(m), \epsilon_2(m) \leq \hat{\epsilon}_2(m), \lambda_1(m) + \varsigma < \epsilon_1(m) < \lambda_u^{-1}(m) - \varsigma, \lambda_u(m) + \varsigma < \epsilon_2(m) < \lambda_s^{-1}(m) - \varsigma, \quad m \in M, \varsigma > 0 \) is sufficiently small.

4. An invariant section \( s \) of \( H \) belongs to \( W^s(i) \) if and only if for every \( m \in M \), \( |s(u^k(m)) - i(u^k(m))| \lesssim \hat{\epsilon}_1^{(k)}(m), \quad k \to \infty \). Similarly, \( s \) belongs to \( W^u(j) \) if and only if for every \( m \in M \), \( |s(u^{-k}(m)) - i(u^{-k}(m))| \lesssim \hat{\epsilon}_2^{(k)}(m), \quad k \to \infty \), \( \hat{\epsilon}_i(\cdot), i = 1, 2 \), are the functions in (3).

In applications, we usually take \( i, j \) as invariant sections of \( H \) or \( 1 \)-pseudo-invariant sections.
4.5.2. hyperbolic trichotomy. Let us consider the ‘hyperbolic trichotomy’ which can be applied to (partially) normal hyperbolicity in the trivial bundle case. See also Remark 4.18.

(HT) Let \( u : M \to M \) be an invertible map. Let \((X^s, M, \pi)\), \(\kappa = s, c, u\) be three (set) bundles with metric fibers. Let \( X = X^s \times X^c \times X^u \) and \( H : X \to X \) a bundle correspondence over \( u \) with generating bundle maps \((F^{cs}, G^c), (F^c, G^u)\), where

\[
F^{cs} : X^{cs}_m \times X^u_{u(m)} \to X^{cs}_{u(m)}, \quad G^c : X^{cs}_m \times X^u_{u(m)} \to X^u_{u(m)}
\]

\[
F^c : X^c_m \times X^u_{u(m)} \to X^c_{u(m)}, \quad G^u : X^c_m \times X^u_{u(m)} \to X^u_{u(m)}.
\]

Assume \( H \sim (F^{cs}, G^c) \) satisfies (A)(\( \sigma_u, \lambda_u \)) (B)(\( \beta_{cs}, \lambda_{cs} \)) condition, and \( H \sim (F^c, G^u) \) satisfies (A)(\( \sigma_{cu}, \lambda_{cu} \)) (B)(\( \beta_{cs}, \lambda_{cs} \)) condition where \( \alpha_{k1}, \lambda_{k1}, \beta_{k2}, \lambda_{k2}, \kappa_1 = u, cu, k_2 = s, cs, k \) are bounded functions of \( M \to \mathbb{R}_+ \) in addition with the following properties,

(a) (angle condition) \( \sup_m \alpha_{k1}(m) \beta_{cs}(u(m)) < 1 \), \( \sup_m \alpha_{cu}(m) \beta_{cs}(u(m)) < 1 \); \( \alpha_{k1}(m) \leq \alpha_{cu}(u(m)), \beta_{cu}(u(m)) \leq \beta_{cs}(m), \forall m \in M, \kappa_1 = u, cu, k_2 = s, cs, \)

(b) (spectral condition) \( \sup_m \alpha_{cu}(m) \lambda_{cs}(m) < 1 \), \( \sup_m \alpha_{cu}(m) \lambda_{cs}(m) < 1 \); \( \sup_m \alpha_{cu}(m) \lambda_{cs}(m) < 1 \), \( \sup_m \alpha_{cu}(m) \lambda_{cs}(m) < 1 \).

Let \( X^{k1:k2} = X^{k1}_i \times X^{k2}_i \) with fibers being equipped with \( d_m \) metrics (see (2.1)) where \( k_1, k_2 \in \{s, c, u\}, k_1 \neq k_2 \). Let \( P^k : X \to X^k \) be the natural bundle projection, \( k \in \{s, c, u, cs, cu\} \).

**Theorem 4.17.** Let (HT) hold. Assume there is a section \( i = (i_s, i_c, i_u) : M \to X^s \times X^c \times X^u \) which is a 1-pseudo-invariant section of \( H \) (see Definition 4.15). Let \( \zeta > 0 \) be any sufficiently small. Choose any functions \( \bar{\epsilon}_1(\cdot), i, 1 \leq 2 \), over \( u \) and \( u^{-1} \) respectively, such that \( \lambda_{cs}(m) + \zeta < \bar{\epsilon}_1(1) \lambda^{-1}(1) \lambda_{cu}(m) - \zeta \), and \( \lambda_{cu}(m) + \zeta < \bar{\epsilon}_2(1) \lambda^{-1}(1) \lambda_{cu}(m) - \zeta < \bar{\epsilon}_2(1) \lambda^{-1}(1) \lambda_{cu}(m) - \zeta < \bar{\epsilon}_2(1) \lambda^{-1}(1) \lambda_{cu}(m) - \zeta \), \( m \in M \). Then the following hold.

1. For any \( m \in M \), there is a Lipschitz graph \( W^{cs}_m \) (resp. \( W^{cu}_m \) of \( X^{cs}_m \to X^s_m \) (resp. \( X^{cu}_m \to X^u_m \)) with Lipschitz constant less than \( \beta_{cs}(m) \) (resp. \( \alpha_{cu}(m) \), such that \( W^{cs}_m \subset H^{-1} W^{cs}_{u(m)} \) (resp. \( W^{cu}_m \subset H^{-1} W^{cu}_{u(m)} \)). Moreover,

\[
W^{cs}_m = \{z_m \in X_m : \text{there is a forward orbit } \{z_k : k = 0, 1, 2, \ldots \} \text{ of } H \text{ such that } \]

\[
z_0 = z_m, \sup_{k \geq 0} (\bar{\epsilon}_1(k))^{-1}|z_k - i(u^k(m))| < \infty, \]

\[
W^{cu}_m = \{z_m \in X_m : \text{there is a backward orbit } \{z_k : k = 0, 1, 2, \ldots \} \text{ of } H \text{ such that } \]

\[
z_0 = z_m, \sup_{k \geq 0} (\bar{\epsilon}_2(k))^{-1}|z_k - i(u^k(m))| < \infty, \]

2. \( W^c_m = W^{cs}_m \cap W^{cu}_m \) is a Lipschitz graph of \( X^c_m \to X^s_m \times X^u_m \) such that \( W^c_m \subset H^{-1} W^{cu}_{u(m)} \) \( W^c_m \subset H^{-1} W^{cs}_{u(m)} \), \( m \in M \). By using \( W^c \), one has more characterizations about \( W^{cs}, W^{cu} \), that is,

\[
W^{cs}_m = \{z \in X_m : \exists \{z_k \}_{k \geq 0} \in H^{-1}(m)(z_k), z_0 = z, d(z_k, W^{cs}_{u(m)}) \to 0, k \to \infty \}
\]

\[
\{z \in X_m : \exists \{z_k \}_{k \geq 0} \in H^{-1}(m)(z_k), z_0 = z, \sup_{k \geq 0} d(z_k, W^{cs}_{u(m)}) < \infty \},
\]

\[
W^{cu}_m = \{z \in X_m : \exists \{z_k \}_{k \geq 0} \in H^{-1}(m)(z_k), z_0 = z, d(z_k, W^{cu}_{u(m)}) \to 0, k \to \infty \}
\]

\[
\{z \in X_m : \exists \{z_k \}_{k \geq 0} \in H^{-1}(m)(z_k), z_0 = z, \sup_{k \geq 0} d(z_k, W^{cu}_{u(m)}) < \infty \}.
\]

3. (exponential tracking and shadowing orbits) If \( \{z_k \}_{k \geq 0} \) is a forward orbit of \( H \) from \( m \), then there is a forward orbit \( \{\tilde{z}_k \}_{k \geq 0} \subset W^{cu}_m \) \( \tilde{z}_0 \in W^{cu}_m \) such that \( |z_k - \tilde{z}_k| \leq \bar{\epsilon}_1(k)(m), k \to \infty \); similarly if \( \{\tilde{z}_k \}_{k \geq 0} \) is a backward orbit of \( H \) from \( m \), then there is a backward orbit \( \{z_k \}_{k \geq 0} \subset W^{cs}_m \) \( \tilde{z}_0 \in W^{cs}_m \) such that \( |z_k - \tilde{z}_k| \leq \bar{\epsilon}_2(k)(m), k \to \infty \).
(4) Let $H^{cs} = H|_{W^{cs}}$, $H^{-cu} = H^{-1}|_{W^{cu}}$, $H^c = H|_{W^c}$ be the bundle maps over $u, u^{-1}, u$ induced by $H : W^{cs} \to W^{cs}, H^{-1} : W^{cu} \to W^{cu}$, and $H : W^c \to W^c$, respectively.

There are bundle maps $h^{cs} : X^{cs} \to X^{cs}$, $h^{cu} : X^{cu} \to X^{cu}$, $h^c : X^c \to X^c$ over $u, u^{-1}, u$ respectively with $\text{Lip}(h^{cs}) \leq \lambda_{cs}(m)$ and $\text{Lip}(h^{cu}) \leq \lambda_{cu}(m)$ such that the following commutative diagrams hold.

$$
\begin{array}{ccccccc}
W^{cs} & \xrightarrow{H^{cs}} & W^{cs} & \xrightarrow{H^{-cu}} & W^{cu} & \xrightarrow{W^c} & W^c \\
p^{cs} \downarrow & & p^{cs} \downarrow & & p^{cu} \downarrow & & p \downarrow & \downarrow \\
X^{cs} & \xrightarrow{h^{cs}} & X^{cs} & \xrightarrow{h^{-cu}} & X^{cu} & \xrightarrow{X^c} & X^c \\
\pi \downarrow & & \pi \downarrow & & \pi \downarrow & \downarrow & \downarrow \\
M & \xrightarrow{u} & M & \xrightarrow{u^{-1}} & M & \xrightarrow{u} & M \\
\end{array}
$$

(5) There are Lipschitz graphs $W^s_z$ (resp. $W^{uu}_z$), $z \in W^s_z$ (resp. $z \in W^{uu}_z$) with Lipschitz constants less than $\beta_s(m)$ (resp. $\alpha_u(m)$) such that $z \in W^{cs}_z$ (resp. $z \in W^{uu}_z$) and $H^{cs} z \subseteq W^{cs}_z$ (resp. $H^{-cu} z \subseteq W^{uu}_z$), $m \in M$.

Let $z \in W^{cs}_m$, then $z' \in W^{cs}_z$ if and only if $|(H^{cs})^k_m(z') - (H^{cs})^k_z(z)| \leq \epsilon_1^k(m), k \to \infty$; similarly let $z \in W^{cu}_m$, then $z' \in W^{cu}_z$ if and only if $|(H^{-cu})^k_m(z') - (H^{-cu})^k_z(z)| \leq \epsilon_2^k(m), k \to \infty$.

Now $W^{cs}_m, W^{cu}_m$ can be regarded as bundles over $W^{cs}_m, W^{uu}_z, z \in W^{uu}_z$, respectively. So are $W^{cs}, W^{cu}, W^{cu} \times W^{cu} \cong X^s \times X^c \times X^u$ over $W^c$.

For the regularity of $W^{cs}, W^{cu}$, and $W^{uu}_z$, $z \in W^{cs}, W^{uu}_z, z \in W^{uu}_z$, see Section 6 and Theorem 7.4.

**Proof.** (I). Since $i = (i_s, i_c, i_u) : M \to X^s \times X^c \times X^u$ is a 1-pseudo-invariant section of $H$, by Theorem 4.1, there are invariant Lipschitz graphs $W^{cs} = \text{Graph} f^{cs}, W^{cu} = \text{Graph} f^{cu}$ of $H$ such that $W^{cs} \subset HW^{cs}$ and $W^{cu} \subset H^{-1}W^{cu}$, where $f^{cs} : X^{cs} \to X^u$ over id with $\text{Lip}(f^{cs}) \leq \beta_{cs}(m)$ and $f^{cu} : X^{cu} \to X^s$ over id with $\text{Lip}(f^{cu}) \leq \alpha_{cu}(m)$. Note that

$$
(4.3)\quad H^{cs} = H|_{W^{cs}} : W^{cs} \to W^{cs}, (m, x^{cs}, f_m^{cs}(x^{cs})) \mapsto (u(m), h^{cs}_m(x^{cs}), f_m^{cu}(h_m^{cs}(x^{cs})),
$$

is a bundle map over $u$ and similarly,

$$
(4.4)\quad H^{-cu} = H^{-1}|_{W^{cu}} : W^{cu} \to W^{cu}, (m, x^{cu}, f_m^{cu}(x^{cu})) \mapsto (u^{-1}(m), h_m^{-cu}(x^{cu}), f_m^{cu}(h_m^{-cu}(x^{cu}))),
$$

is a bundle map over $u^{-1}$. Note that $\text{Lip}(h_m^{cu}) \leq \lambda_{cu}(m)$ and $\text{Lip}(h_m^{cu}) \leq \lambda_{cu}(m)$. Combining with the characterizations in Section 4.4, we know conclusions (1) and (4) hold.

(II) Under $\sup_m \alpha_{cu}(m)\beta_{cs}(m) < 1$, one has $h^{cs} \sim (\tilde{F}^s, \tilde{G}^s) : X^s \times X^c \to X^s \times X^c$ and $h^{-cu} \sim (\tilde{F}^u, \tilde{G}^u) : X^s \times X^u \to X^s \times X^u$. Take $h^{-cu}$ as an example. Let $F_m^{cs} = (F_m^{cs}, F_m^{cs})$, where $F_m^{cs} : X^s_m \times X^u_{u(m)} \to X^s_{u(m)}$ and $F_m^{cs} : X^s_m \times X^u_{u(m)} \to X^s_{u(m)}$. For any $(y_1, z_2) \in X^s_m \times X^u_{u(m)}$, we see there is only one $z_1 \in X^u_{u(m)}$ so that

$$
z_1 = G^{cs}_{m} (f_m^{cu}(y_1, z_1), z_2) \triangleq \tilde{G}^{cs}_{m}(y_1, z_2);
$$

defining $\tilde{F}^{cu}_{m}(y_1, z_2) \triangleq F_m^{cu}(y_1, z_1, z_2)$. Moreover, the following hold.

(a) If $\alpha_{cu}(m) \leq 1, \beta_{cs}(m) \leq 1, \beta_{cu}(m) \leq \beta_s(m), m \in M$, then $h^{cs} \sim (\tilde{F}^{s}, \tilde{G}^{s})$ satisfies (A)$\alpha_{cu}, \beta_{cu}$ (B)$\beta_s, \lambda_s$ condition;

(b) If $\alpha_{cu}(m) \leq 1, \beta_{cs}(m) \leq 1, \alpha_{cu}(m) \leq \alpha_u(m), m \in M$, then $h^{-cu} \sim (\tilde{F}^{s}, \tilde{G}^{s})$ satisfies (A)$\alpha_u, \lambda_u$ (B)$\beta_{cs}, \lambda_{cs}$ condition.
The proof is direct, so we omit it. Now for $h^{cs}$, by Theorem 4.1 (or Theorem 7.4), there are Lipschitz maps $f^{cs}_m : X^c_m \to X^c_m$, $z_m \in X^c_x \times X^c_m$ with $\text{Lip} f^{cs}_m \leq \beta_c(m)$ such that $z_m \in \text{Graph } f^{cs}_m$ and $h^{cs}_m \text{Graph } f^{cs}_m \subset \text{Graph } h^{cs}_m(z_m)$. Note that by the characterization (see Section 4.4), $\text{Graph } f^{cs}_m \cap \text{Graph } h^{cs}_m = \emptyset$ or $\text{Graph } f^{cs}_m = \text{Graph } h^{cs}_m$; particularly $\bigcup_{z_m \in X^c_x \times X^c_m} \text{Graph } f^{cs}_m \text{foliates } X^c_x \times X^c_m$. Now using $f^{cs}_m$, one obtains $W^{cs}_z$, $z \in W^{cs}_m \triangleq \text{Graph } f^{cs}_m$ such that $z \in W^s_z$, $H^{cs}_m W^{cs}_z \subset W^{ss}_H(z)$ and $W^{cs}_z$ is a Lipschitz graph of $X^s_m \to X^{cu}_m$. $\bigcup_{z \in W^{cs}_m} \text{Graph } W^{cs}_z \text{foliates } W^{cu}_m$. Similarly, one gets $W^{uu}_m$, $z \in W^{uu}_m \triangleq \text{Graph } f^{cu}_m$ such that $z \in W^{uu}_m$, $H_{cu}^{-1} W^{uu}_z \subset W^{uu}_m$ and $W^{uu}_z$ is a Lipschitz graph of $X^u_m \to X^{cu}_m$. $\bigcup_{z \in W^{uu}_m} \text{Graph } W^{uu}_z \text{foliates } W^{cu}_m$.

Since $\sup_m \alpha_{cu}(m) \beta_{cs}(m) < 1$, one has $W^c \triangleq W^{cs} \cap W^{cu} = \text{Graph } f^c$, where

$$f^c_m = (f^{1c}_m, f^{2c}_m) : X^c_m \to X^c_x \times X^u_m,$$

with $\text{Lip } f^{1c}_m \leq \alpha_{cu}(m)$, $\text{Lip } f^{2c}_m \leq \beta_{cs}(m)$. Note that $W_c \subset H^{cs} W^c$, particularly

$$(4.5) \quad H^c = H|_{W^c} : W^c \to W^c, (m, f^{1c}_m(x^c), x^c, f^{2c}_m(x^c)) \mapsto (u(m), f^{1c}_m(h^c(x^c)), h^c_m(x^c), f^{2c}_m(h^c_m(x^c))),$$

where $h^c_m : X^c_m \to X^c_{u(m)}$ is a bi-Lipschitz map with $\text{Lip } h^c_m \leq \lambda_{cs}(m), \text{Lip } (h^{-1}_m) \leq \lambda_{cu}(m)$. This also shows that if $X^c_m, m \in M$ are all more than one point, then $\lambda_{cs}(m) \lambda_{cu}(m) \geq 1$; in particular, we can assume

$$\lambda_{cs}(m) < \lambda^{-1}_{cs}(m) \leq \lambda_{cs}(m) < \lambda^{-1}_{cu}(m).$$

Note that $\sup_m \alpha_{au}(m) \beta_{su}(m) < 1$. We show in fact $W^{cs}, W^{cu}$ are bundles over $W^c$ with fibers $W^{ss}_z$, $W^{uu}_z$, $z \in W^c$ respectively. Let $W^c_m = W^{cs}_m \cap W^{cu}_m$. Due to $\sup_m \alpha_{au}(m) \beta_{su}(m) < 1$, one can define $\pi^u_z(m, z)$ as the unique point in $W^{ss}_z \cap W^c_m$, where $z \in W^{ss}_z$. So $(W^c, W^c, \pi^c_z)$ is a bundle, called the strong unstable fiber bundle. Similarly, one defines $\pi^u_z$, and $(W^{cu}, W^c, \pi^u_z)$ is a bundle called the strong unstable fiber bundle.

We have shown conclusion (5) of the first part of conclusion (2) hold.

(III). Finally, we show the characterization of $W^{cs}, W^{cu}$ by using $W^c$ and conclusion (3) hold. Let $\{z_k\}_{k \geq 0}$ be a forward orbit of $H$ from $m$, and $M_1 = \{u^k(m) : k \in \mathbb{N}\}$. Applying Theorem 4.1 to $X^1 \triangleq X|_{M_1}$, $H|_{X^1}, M_1$ with an invariant section $\tilde{t}$ defined by $\tilde{t}(u^k(m)) = z_k$, one gets a Lipschitz graph $W^{uu}_{u^k(m)}(\tilde{t})$ of $X^{uu}_{u^k(m)} \to X^{cu}_{u^k(m)}$ with Lipschitz constant less than $\beta_{su}(u^k(m))$ which is invariant under $H$. Let $\{z_k\} = W^{uu}_{u^k(m)}(\tilde{t}) \cap W^{cu}_{u^k(m)}$. One can easily check that $\{z_k\}_{k \geq 0}$ is a forward orbit of $H$ and $|z_k - z_{k-1}| \leq e_1(k(m))$, $k \to \infty$ (see e.g. Section 4.5.1). We have proved conclusion (3). Let us show $W^{cs}_m$ has the characterization in conclusion (2).

(i) If $z_m \in W^{cs}_m$, then there is a forward orbit $\{z_k : k = 0, 1, 2, \cdots\}$ of $H$ such that $z_0 = z_m$. For this orbit $\{z_k\}_{k \geq 0}$, one has $\{z_k\}_{k \geq 0} \subset W^{cu}$ such that $|z_k - z_{k-1}| \to 0$. This also shows that by the characterization of $W^{cs}, \{z_k\} \subset W^c$. Hence, $\{z_k\} \subset W^c$, which gives $d(z_k, W^{uu}_{u^k(m)}) \leq |z_k - z_{k-1}| \to 0$.

(ii) If $\{z_k\}_{k \geq 0}$ is a forward orbit of $H$ such that $z_0 = z_m \in X_m$ and $\sup_k d(z_k, W^{uu}_{u^k(m)}) < \infty$, then we need to show $z_m \in W^{cs}_m$. For this forward orbit $\{z_k\}$, there is a forward orbit $\{z_k\}_{k \geq 0} \subset W^{cu}$ such that $\sup_k |z_k - z_{k-1}| < \infty$. Write $\tilde{z}_k = (x^c_k, x^u_k)$. First we show $\sup_k |x^u_k - f^{2c}_{u^k(m)}(x^c_k)| < \infty$.

Take $\tilde{z}_k = (f^{1c}_{u^k(m)}(x^c_k), x^c_k, f^{2c}_{u^k(m)}(x^c_k)) \subset W^{cu}_{u^k(m)}$. Since $\sup_k d(z_k, W^{uu}_{u^k(m)}) < \infty$, there is $\tilde{z}_k = (f^{1c}_{u^k(m)}(x^c_k), x^c_k, f^{2c}_{u^k(m)}(x^c_k)) \subset W^{cu}$ such that $\sup_k |z_k - \tilde{z}_k| < \infty$. So $\sup_k |z_k - \tilde{z}_k| < \infty$, particularly $\sup_k |x^u_k - \tilde{x}_k^u| < \infty$. Thus, $\sup_k |z_k - \tilde{z}_k| < \infty$, which yields

$$\sup_{k \geq 0} |z_k - \tilde{z}_k| \leq \sup_{k \geq 0} |z_k - \tilde{z}_k| + \sup_{k \geq 0} |\tilde{z}_k - \tilde{z}_k| < \infty,$$
and so $\sup_{k \geq 0} |x_k^u - f_{u^k(m)}^{2c}(x_k^c)| < \infty$. Set

$$(H^c)^{(k)}(z_0) = (\tilde{x}_k^c', x_k^c'. x_k^u').$$

Then by (A) condition for $h^{-cu}$, one has for $k \geq 1$,

$$|x_k^c' - x_k^c'| \leq \alpha_u(u^{-1}(m))|x_k^{u'} - x_k^u| \leq \alpha_u(u^{-1}(m))|x_k^{u'} - f_{u^k(m)}^{2c}(x_k^c)| + \alpha_u(u^{-1}(m))|f_{u^k(m)}^{2c}(x_k^c) - x_k^u|,$$

yielding $|x_k^c' - x_k^c| \leq \frac{\alpha_u(u^{-1}(m))}{1 - \alpha_u(u^{-1}(m))}\sup_{\alpha_u(u^{-1}(m))|f_{u^k(m)}^{2c}(x_k^c) - x_k^u|}$. Thus, $\sup_{k \geq 0} |x_k^c' - x_k^c| < \infty$. It implies that $\sup_{k \geq 0} \sup_{|i(u^k(m)) - z_k|} x_k^c < \infty$, giving $z_m \in W^{cs}_m (by (1)).$ The proof is complete. □

**Remark 4.18** (partially normal hyperbolicity). Let $H : Z \rightarrow Z$ be a smooth map and $M \subset Z$, $H(\tilde{M}) \subset M$, where $Z$ is a smooth Finsler manifold and $M$ is a submanifold of $Z$. Let us assume $M$ is invariant and normally hyperbolic with respect to $H$; see [HPS77,Fen71]. Then we have the center-(un)stable manifolds $W^{cs}(M)$, $W^{cu}(M)$ of $H$ for $M$, and $M = W^{cs}(M) \cap W^{cu}(M)$. A small $C^1$ perturbation $\tilde{H}$ of $H$ corresponds $\tilde{W}^{cs}, \tilde{W}^{cu}$ and $\tilde{M} = \tilde{W}^{cs} \cap \tilde{W}^{cu}$. Now $\tilde{M}$ is a persistence of $M$ which is diffeomorphic to $M$. In general, Theorem 4.17 can not be used to deduce the existence of above invariant manifolds. However, there is a situation so that our results can be applied; that is, when the normal bundle $N$ of $M$ in $Z$ can embed into a trivial bundle $M \times Y$ where $Y$ is a Banach space, and $H$ can extend to $M \times Y$, denoted by $\tilde{H}$, so that $M$ is also normally hyperbolic for $\tilde{H}$. Now applying Theorem 4.17 to $\tilde{H}, M \times Y$, one gets the invariant manifolds of $\tilde{H}$, then pulls back all these manifolds to $Z$ to give the desired manifolds for $H$. This can be done for example when $Z$ is a smooth compact Riemannian manifold and $M$ is a compact submanifold of $Z$ (see e.g. [HPS77]), or $Z$ is a smooth Riemannian manifold having bounded geometry (see Definition C.6) and $M$ is a complete immersed submanifold of $Z$ (see e.g. [Eld13]). The trivialization of vector bundles is important in [HPS77,Eld13] to study the normal hyperbolicity. Also, note that the $C^0$ trivialization is inadequate, for one needs persistence of the normal hyperbolicity in the new trivial vector bundle.

A hyperbolicity between the normal hyperbolicity and partial hyperbolicity, namely partially normal hyperbolicity we called in [Che18c] which is of importance due to its both theoretical and practical application (see e.g. [BC16] and [LLSY16]), can have some similar properties with the normal hyperbolicity (see also [CLY00b]). The partially normal hyperbolicity can make sense for sets, see [CLY00a,BC16]. Analogously, if the normal bundle of the partially normally hyperbolic manifold can embed into a trivial bundle and the dynamic can extend to the trivial bundle maintaining the hyperbolic trichotomy, then Theorem 4.17 can be applied directly.

In [Che18b,Che18c], we extend the associated results (see e.g. [HPS77,Fen71,Eld13,BLZ98,BLZ08,CLY00b,CLY00a,BC16]) about normal hyperbolicity and partially normal hyperbolicity of the immersed submanifolds to correspondences in Banach spaces (therefore can be applied to some ill-posed differential equations).

In the following, we show how we use the invariant foliations to decouple the bundle map $H$ when $H$ is invertible.

**Corollary 4.19** (Decoupling Theorem). Let Theorem 4.17 hold. Assume $H$ is invertible (i.e. for very $m \in M$, $H_m$ is an invertible map). $H$ can be decoupled as $h^s \times h^c \times h^u$ with bundle maps $h^s : X^cs \rightarrow X^s$, $h^u : X^cu \rightarrow X^u$ and $h^c : X^c \rightarrow X^c$ over $u$, i.e. there is an invertible bundle map $\Pi : X \rightarrow X$ over id such that the following commutative diagram holds.

$$
\begin{array}{ccc}
X^s \times X^c \times X^u & \xrightarrow{H} & X^s \times X^c \times X^u \\
\Pi \downarrow & & \Pi \downarrow \\
X^s \times X^c \times X^u & \xrightarrow{h^s \times h^c \times h^u} & X^s \times X^c \times X^u
\end{array}
$$

**Proof.** By Theorem 7.4, we have the following different invariant foliations: $W^c(\kappa), \kappa \in X, \kappa_1 \in \{s,u,cs, cu\}$, such that
(a) $W^\kappa(z)$ is a Lipschitz graph of $X^\kappa_m \to X^\kappa_{cs}$ with Lipschitz constant $\sigma_{k_1}$, if $\pi(z) = m$, where $\sigma_{k_1} = \beta_k(m)$ if $k_1 = s, cs$, and $\sigma_{k_1} = \alpha_k(m)$ if $k_1 = u, cu$, and where $k_2 = scu - \kappa_1$ (meaning the letter by deleting the letter $k_2$ from $scu$; e.g. if $k_1 = s$, then $k_2 = cu$).

(b) $z \in W^\kappa(z), H W^\kappa(z) = W^\kappa(H(z))$, where $\kappa_1 \in \{s, u, cs, cu\}$.

We also need all the invariant foliations obtained in Theorem 4.17, i.e. $W^x$ and $W^{x'}$, $z \in W^x$, $\kappa = s, u$. Note that if $z \in W^cs$, then $W^c_{uu} = W^c(z)$; similarly if $z \in W^cu$, then $W^{cu} = W^u(z)$. Also, $W^s$ (resp. $W^u$) subfoliates each leaf of $\bigcup_{z \in X} W^cs(z)$ (resp. $\bigcup_{z \in X} W^cu(z)$) (which means that e.g. $\{W^s(z') : z' \in W^c\}(z)$ is a 'subfoliation' of $W^c\kappa(z)$ for any $z \in X$). These follow from the characterizations of those invariant foliations (see Section 4.4).

Define the following maps. For $m_c \in W^c$,

$$
\pi^s_{m_c} : z \mapsto W^c(z) \cap W^s(m_c), \quad \pi^u_{m_c} : z \mapsto W^c(z) \cap W^u(m_c),
$$

$$
\pi^c(m_c, z) = (m_c, \pi^s_{m_c}(z)), \quad \pi^u(m_c, z) = (m_c, \pi^u_{m_c}(z)).
$$

Consider $W^{cs}$ and $W^{cu}$ as bundles over $W^c$. Let $H^s$ be another representation of $H^{cs}$ as a bundle map over $H^c$, i.e.

$$
H^s : (m_c, z) \mapsto (H^c(m_c), H^{cs}(z)), \quad z \in W^{cs}_{m_c}, \quad m_c \in W^c.
$$

Similarly,

$$
H^u : (m_c, z) \mapsto (H^c(m_c), H^{cu}(z)), \quad z \in W^{cu}_{m_c}, \quad m_c \in W^c,
$$

where $H^{cu} = (H^{-cu})^{-1}$. By the invariance of the foliations, we have the following:

$$
\pi^c(H(m_c), H(z)) = (H(m_c), \pi^c_{H(m_c)}(H(z))) = H^c(m_c, \pi^c_{m_c}(z)), \quad \kappa = s, u.
$$

In particular, $\pi^c \circ (H^c, H) = H^c \circ \pi^c, \kappa = s, u$.

Let us give the precise meaning of $X^s \times X^c \times X^u = X \cong W^{cs} \times W^{cu}$. The most important thing is that we need to find a way to track the base points which can be done as follows. Let

$$
\pi_c : X \to W^c, \quad z \mapsto W^u(W^s(z) \cap W^{cu}) \cap W^c,
$$

and

$$
\tilde{\pi}(z) = (m_c, \pi^s_{m_c}(z), \pi^u_{m_c}(z)) : X \to W^{cs} \times W^{cu},
$$

where $m_c = \pi_c(z)$. By the invariance of those foliations, we see $\tilde{\pi}(H(z)) = H^c(\pi_c(z))$. What we have shown is the following commutative diagram holds.

$$
\begin{array}{ccc}
X^s \times X^c \times X^u & \xrightarrow{H} & X^s \times X^c \times X^u \\
\tilde{\pi} \downarrow & & \tilde{\pi} \downarrow \\
W^{cs} \times W^{cu} & \xrightarrow{H^s \times H^u} & W^{cs} \times W^{cu} \\
\pi_c \downarrow & & \pi_c \downarrow \\
W^c & \xrightarrow{H^c} & W^c
\end{array}
$$

where $\pi_c$ is the projection of the bundle $(W^{cs} \times W^{cu}, W^c, \pi_c)$.

Represent $H^{cs}$, $H^{cu}$ in $X$. Let $m_c = (m, z_c) \in W^c$, $z_c \in W^c$. Note that $P^c_m : W^c \cong X^c_m$. Since $H^{cs} : W^{cs}_{m_c} \to W^{ss}_{H(m_c)}$ and $W^{ss}_{m_c} \cong X^s_m$, through $P^c_m$, we have a map $h^s_m(P^{cs}_{m_z}, \cdot) : X^{s_m} \to X^{s_m}$ induced by $H^{cs}$ (similar as (4.3)). Also, we have another map $h^u_m(P^{cu}_{m_z}, \cdot) : X^{u_m} \to X^{u_m}$ induced by $H^{cu}$ (similar as (4.4)). By our construction, we get

$$
P^{s}_{u(m)}H^{s}_{m}(z) = h^s_m(P^{cs}_{m_z}, P^s_{m_z}), \quad z \in W^{s}_{m_c}, \quad m_c = (m, z_c) \in W^c;
$$

analogously,

$$
P^{u}_{u(m)}H^{cu}_{m}(z) = h^u_m(P^{cu}_{m_z}, P^u_{m_z}), \quad z \in W^{u}_{m_c}, \quad m_c = (m, z_c) \in W^c.
$$
Now we have the following commutative diagram holds.

\[ W^{c,s} \times W^{c,u} \xrightarrow{H^s \times H^u} W^{c,s} \times W^{c,u} \]

\[ \tilde{\pi} \downarrow \quad \tilde{\pi} \downarrow \]

\[ X^s \times X^c \times X^u \xrightarrow{h^s \times h^c \times h^u} X^s \times X^c \times X^u \]

\[ \pi^e \downarrow \quad \pi^e \downarrow \]

\[ X^c \xrightarrow{h^c} X^c \]

where \( h^c \) is defined in (4.5), and

\[ \tilde{\pi}(m_c, z_s, z_u) = (P^c_{m_c}, P^s_{m_c}, P^u_{m_u}) : W^{c,s} \times W^{c,u} \rightarrow X, \]

where \((z_s, z_u) \in W^{ss}_{m_c} \times W^{uu}_{m_c}, m_c = (m, z_c) \in W^c\). Combining the above two commutative diagrams, we have

\[ X^s \times X^c \times X^u \xrightarrow{H} X^s \times X^c \times X^u \]

\[ \tilde{\pi} \downarrow \quad \tilde{\pi} \downarrow \]

\[ W^{c,s} \times W^{c,u} \xrightarrow{H^s \times H^u} W^{c,s} \times W^{c,u} \]

\[ \tilde{\pi} \downarrow \quad \tilde{\pi} \downarrow \]

\[ X^s \times X^c \times X^u \xrightarrow{h^s \times h^c \times h^u} X^s \times X^c \times X^u \]

Therefore we decouple \( H \) as \( h^s \times h^c \times h^u \) with \( h^s : X^{c,s} \rightarrow X^s, h^u : X^{c,u} \rightarrow X^u \) and \( h^c : X^c \rightarrow X^c \). We show \( \tilde{\pi} \) and \( \tilde{\pi} \) are invertible.

That \( \tilde{\pi} \) is invertible is easy since

\[ \tilde{\pi}^{-1}(x^s, x^c, x^u) = (z_c, z_s, z_u), \]

where \( z_c = (f^1_{m_c}(x^c), x^c, f^2_{m_c}(x^c)), x^{c,s} = (x^s, f^s_{m_c}(z_c)(x^s)), z_s = (x^{c,s}, f^s_{m_c}(x^{c,s})), \) and similar for \( z_u \).

Next we show \( \tilde{\pi} \) is invertible. Let \((z_s, z_u) \in W^{ss}_{m_c} \times W^{uu}_{m_c}, m_c = (m, z_c) \in W^c\). We need to find \( a \in X_m \) such that

\[ W^u(W^s(a) \cap W^{cu}) \cap W^c = z_c, \quad W^{cu}(a) \cap W^{ss}_{m_c} = z_s, \quad W^{cs}(a) \cap W^{uu}_{m_c} = z_u. \]

Set \( b = W^{cs}(z_u) \cap W^{uu}_{m_c}, a = W^{cu}(z_s) \cap W^{s}(b) \). We show \( a \) is the desired choice. (Note that here we identify \((m, x) = x, \) if \( x \in X_m\)).

(i) Since \( a \in W^{cu}(z_s) \), one has \( W^{cu}(a) \cap W^{ss}_{m_c} = z_s \). (ii) Since \( b \in W^{cs}(z_u) \), by the subfoliation property, one gets \( W^s(b) \subset W^{cs}(z_u) \). Thus, \( a \in W^s(b) \subset W^{cs}(z_u) \), which yields \( W^{cs}(a) \cap W^{uu}_{m_c} = z_u \).

(iii) As \( a \in W^s(b) \), one has \( W^s(a) = W^s(b) \). And \( b \in W^{cu} \) (note that \( W^{cu} \subset W^{cu}_{m_c} \)), so \( W^s(a) \cap W^{cu}_{m_c} = W^s(b) \cap W^{cu}_{m_c} = b \in W^{uu}_{m_c} \). Therefore, \( z_c = W^u(W^s(a) \cap W^{cu}) \cap W^c \). The proof is finished. \( \Box \)

**Remark 4.20.**

(a) The regularity of \( \Pi \) in Corollary 4.19 relies on the regularity of the foliations \( W^s(z), z \in X, \kappa \in \{s,u,cs, cu\} \), and \( W^{cs}, W^{cu} \), which we will study in Section 6. A simple result is that if \( X^s, \kappa \in \{s,c,u\} \), are \( C^0 \) bundles and \( H \) are continuous in addition with all the functions in the (A) (B) condition are (almost) continuous, then \( \Pi \) is homeomorphic. If we do not care the regularity respecting the base points \( M \), then under \( H \) being bi-Lipschitz in the fiber topology, \( \Pi \) at least is bi-Hölder in the fiber topology (see e.g. **Theorem 7.4**). We do not give a detailed statement of the regularity of \( \Pi \) here.

(b) The choice of \( W^{cs}, W^{cu} \) is not unique. Different choices of \( W^{cs}, W^{cu} \) give different decoupled systems \( h^s \times h^c \times h^u \). This is important for us to give further refined decoupled system about \( H \). The existence of the 1-pseudo-invariant section of \( H \) and the condition \( \lambda_s < 1, \lambda_u < 1 \) are in order to ensure the existence of \( W^{cs}, W^{cu} \). Instead, if there is an invariant section of \( H \), the condition \( \lambda_s < 1, \)
$\lambda_0 < 1$ can be removed. All we need is using Theorem 4.1 (or Theorem 4.3) to produce invariant graphs $W^{cs}, W^{cu}$ such that $W^{cs} \times W^{cu}$ is a bundle over $W^{cs} \cap W^{cu}$.

(c) The existence of \( \Pi \) heavily relies on the existence of the foliations $W^\kappa(z), z \in X, \kappa \in \{s, u, cs, cu\}$, which might not always exist if \( H \) is not invertible. In general, one can not expect the Decompling Theorem holds for a bundle correspondence or a non-invertible bundle map. However, see \cite{Lu91} where the reader can find such result also holds for some special non-invertible maps.

(d) The corresponding result for flows is the same which is omitted here. Except the linearization of $h^{cs}, h^{cu}$, Decompling Theorem was also obtained by many authors. The most famous result is Hartman-Grossman Theorem (see e.g. \cite{KH95}) for a hyperbolic equilibrium with its generalization in Banach spaces. For a non-hyperbolic equilibrium, it might be first proved by F. Taken in \cite{Tak71}; K. Palmer also proved this case for the nonautonomous continuous systems (the corresponding nonautonomous discrete systems are the case $M = \mathbb{Z}, u(n) = n + 1$), see \cite{KP90} and the references therein. See also \cite{Irw70,Tak71} (hyperbolic periodic orbit), \cite{dM73} (hyperbolic compact set), and \cite{PS70} (normally hyperbolic compact invariant manifold).

5. Uniform Property of Bundle, Bundle Map, Manifold and Foliation

To set up our regularity results in Section 6, some definitions and notations are needed which we gather them in the following. We hope that in doing so, we can make clear how we can deal with the spaces being lack of compactness, high smoothness and boundedness, where the idea extends e.g. \cite{HPS77,PSW12,BLZ99,BLZ08,Cha04,Eld13,Ama15}. Meanwhile, we try to clarify certain concepts, facts and also the connections between the hypotheses in Section 6.2 and the classical reference e.g. \cite{HPS77}.

A purpose of this part is to describe the uniformly $C^{k, \alpha}$ ($k = 0, 1, 0 \leq \alpha \leq 1$) continuity of \( f, \) a bundle map between two bundles \( X, Y \), that respects fiber and base points in appropriate \( X, Y \) with uniform properties. This is done by an extremely natural way, that is, represent the bundle map in local bundle charts belonging to preferred bundle atlases. The specially chosen bundle atlas also gives the uniform properties of bundle (and so the base space). We attempt to discuss these concepts needed in this paper and also our forthcoming papers in a weaker sense. In order to do this, (uniformly) locally metrizable space is introduced in Section 5.1 to make sense of uniformly $C^{0, \alpha}$ continuity. A quick review of fiber (or leaf) topology and connection is given in Section 5.2 and Section 5.3 respectively. Different types of continuity of bundle maps associated with different classes of bundles are introduced in Section 5.4 (and Appendix B). The relevant uniform notions about manifold and foliation are contained in Section 5.5. Some examples related with our (uniform) assumptions on bundles and manifolds with uniform properties are given in Appendix C.

5.1. locally metrizable space. Let us consider a type of topology space which has a uniform topology structure but might be not globally metrizable, i.e. it is locally metrizable. That the space is not assumed to admit a metric sometimes is important for us when we deal with e.g. the immersed manifolds (see Appendix C.0.1), foliations with leaf topology or bundles with fiber topology (these manifolds might not be metrizable or non-separable, see Section 7.2.3), or non-manifolds (for instance the laminate with leaf topology, see Section 7.2.4). To set up our regularity results in Section 6, some definitions and notations are needed which we gather them in the following. We hope that in doing so, we can make clear how we can deal with the spaces being lack of compactness, high smoothness and boundedness, where the idea extends e.g. \cite{HPS77,PSW12,BLZ99,BLZ08,Cha04,Eld13,Ama15}. Meanwhile, we try to clarify certain concepts, facts and also the connections between the hypotheses in Section 6.2 and the classical reference e.g. \cite{HPS77}.

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**Definition 5.1.** Let $M$ be a Hausdorff topology space. $M$ is called a locally metrizable space (associated with an open cover $\{U_m : m \in M\}$) if the following hold. (a) $U_m$ is open and $m \in U_m$, $m \in M$. (b) Every $U_m$ is a metric space with metric $d_m$ (might be incomplete). Here in $U_m$ the metric topology is the same as the subspace topology induced from $M$. Write $U_m(\epsilon) \triangleq \{m' \in U_m : d_m(m', m) < \epsilon\}$. For convenience, set $U_m(\infty) = U_m$. Also, we denote the $\epsilon$-neighborhood of $M_1$ by $M_1^\epsilon = \bigcup_{m \in M_1} U_\epsilon(m)$, where $M_1 \subset M$.

**Example 5.2.** (a) Any metric space is a locally metrizable space.

(b) Any $C^k$ ($k = 0, 1, 2, \ldots, \infty, \omega$) manifold $M$ is a locally metrizable space. For example, take an atlas $\{(U_\alpha, \phi^\alpha) : \alpha \in \Lambda\}$ of $M$ where $\phi^\alpha : U_\alpha \to X^\alpha$ is a $C^k$ homeomorphism and $X^\alpha$ is a Banach
space with norm $| \cdot |_{\alpha}$. For any $m \in M$, choose an $\alpha(m) \in \Lambda$ such that $m \in U_{\alpha(m)} \triangleq U_m$. The metric $d_m$ in $U_m$ is defined by $d_m(x, y) = |\varphi^{\alpha(m)}(x) - \varphi^{\alpha(m)}(y)|_{\alpha(m)}$. In particular, any $C^k$ foliation (see e.g. [HPS77, AMR88]) with leaf topology is a locally metrizable space which usually is not separable and metrizable.

(c) Any $C^0$ topology bundle (see Example 5.17 (b)) over a locally metrizable space is a locally metrizable space. The local product structure will make the bundle to be a locally metrizable space. The details are following. Take an open cover $\{U_m : m \in M\}$ of the base space making it a locally metrizable space. For every $m \in M$, one can choose small $\varepsilon_m > 0$ such that $\{((U_m(\varepsilon_m), \phi^{\alpha(m)}) : m \in M\}$ is a $C^0$ bundle atlas of the bundle. Let $V_m = \phi^{\alpha(m)}(U_m \times X_m)$, the metric in $V_m$ defined by $d((m_1, x), (m_2, y)) = d_m(m_1, m_2) + d_m((\phi^{\alpha(m)}_m)^{-1}x, (\phi^{\alpha(m)}_m)^{-1}y)$, where $d_m$ is the metric in $X_m$ or $U_m$. We will discuss this type of bundle detailedly in Section 5.4.

(d) Any $C^k$ ($k \geq 1$) Finsler manifold $M$ with Finsler metric in each component of $M$ (see Appendix D.2) is a locally metrizable space. Denote its components by $V_{\alpha}$, $\alpha \in \Lambda$. Note that $V_{\alpha}$ is open in $M$ (as $M$ is locally connected). The metric in $V_{\alpha}$ is the Finsler metric.

(e) Analogously, any (set) bundle with metric fibers (see Section 2.1) equipped with fiber topology (see Section 5.2) or any foliation with metric leaves (see Section 7.2.4) endowed with leaf topology is a locally metrizable space.

See also Example C.2 for immersed manifolds in Banach spaces. See Section 5.2 for a discussion about leaf (or fiber) topology. We will talk about some properties of maps between two locally metrizable spaces.

In Definition 5.3 and Definition 5.4, let $M, N$ be two locally metrizable spaces associated with open covers $\{U_m : m \in M\}$ and $\{V_n : n \in N\}$ respectively.

**Definition 5.3** (uniform continuity). Let $g : M \to N$, $M_1 \subset M$. Define the amplitude of $g$ around $M_1$ (with respect to these two open covers $\{U_m\}$ and $\{V_n\}$) as

$$\mathcal{U}_{M_1}(\sigma) = \sup\{d_{g(m_0)}(g(m), g(m_0)) : m \in U_{m_0}(\sigma), m_0 \in M_1\},$$

where $d_{g(m_0)}$ is the metric in $V_{g(m_0)}$. Here for convenient, if $g(m) \notin V_{g(m_0)}$, let $d_{g(m_0)}(g(m), g(m_0)) = \infty$. We say $g$ is (a) uniformly continuous around $M_1$ if $\mathcal{U}_{M_1}(\sigma) \to 0$, as $\sigma \to 0$; (b) $\varepsilon$-almost uniformly continuous around $M_1$ if $\mathcal{U}_{M_1}(\sigma) \leq \varepsilon$, when $\sigma \to 0$; (c) continuous (or $C^0$) if for every $m_0 \in M$, $\mathcal{U}_{m_0}(\sigma) \to 0$ as $\sigma \to 0$; (d) $\varepsilon$-almost continuous if for every $m_0 \in M$, $\mathcal{U}_{m_0}(\sigma) \leq \varepsilon$ as $\sigma \to 0$. If $M_1 = M$, the words ‘around $M_1$’ usually are omitted.

Obviously, $g$ is $C^0$ in the sense of the above definition if and only if $g : M \to N$ is $C^0$ when $M, N$ are considered as topology spaces. Note that $g$ being $\varepsilon$-almost (uniformly) continuous even is not continuous. In Section 6 (also Section 7), we will frequently need some functions are $\varepsilon$-almost (uniformly) continuous around $M_1$. Consider a situation which can illustrate why almost (uniform) continuity is needed (see also [BLZ08]): $g : X \to Y$ is $C^0$ with $X, Y$ being infinite-dimensional Banach spaces, $M_1 \subset X$ is compact. Usually $g$ might not be uniform continuous in any $U_{\varepsilon}(M_1) = \{m' : d(m', M_1) < \varepsilon\}$ but the amplitude of $g$ in $U_{\varepsilon}(M_1)$ can be sufficiently small if $\varepsilon$ is small.

**Definition 5.4** (Hölder continuity). Let $g : M \to N$, $M_1 \subset M$. Define the uniformly (locally) $\theta$-Hölder constant of $g$ around $M_1$ (with respect to these two open covers $\{U_m\}$ and $\{V_n\}$) as

$$\text{Hol}_{\theta, M_1}(\sigma) \triangleq \text{Hol}_{\theta, M_1, \sigma}(g) = \sup \left\{ \frac{d_{g(m_0)}(g(m), g(m_0))}{d_{m_0}(m, m_0)} : m \in U_{m_0}(\sigma), m_0 \in M_1 \right\},$$

where $d_{m_0}$ and $d_{g(m_0)}$ are the metrics in $U_{m_0}$ and $V_{g(m_0)}$ respectively. We say $g$ is uniformly (locally) $\theta$-Hölder around $M_1$ if $\text{Hol}_{\theta, M_1}(\sigma) < \infty$, when $\sigma \to 0$. If for every $m_0 \in M_1$, $\text{Hol}_{\theta, m_0}(\sigma) < \infty$ as $\sigma \to 0$, then we say $g$ is (locally) $\theta$-Hölder around $M_1$. If $M_1 = M$, the words ‘around $M_1$’ usually are omitted. Often, 1-Hölder = Lipschitz.
Obviously, if \( g \) is uniformly (locally) Hölder around \( M_1 \), then it is uniformly continuous around \( M_1 \). Even if \( M, N \) are metric space, the above definition of (local) \( \theta \)-Hölderness in general does not imply \( g \) is actually Hölder. (For example, a \( C^1 \) map in some set has 0 derivatives but it might not be a constant.) However, if the metrics are length metrics, then it turns out to be true; see Appendix D.3 for details, where the term thereof uses upper scalar uniform \( \alpha \)-Hölder constant instead of uniformly (locally) \( \theta \)-Hölder constant.

The reader might wonder how about \( g \) in \( U_{m_0}(\epsilon_1) \rightarrow V_{\hat{m}_0}(\epsilon_2) \subset U_{g(m_0)}(\epsilon_2) \) if we know the property of \( g \) in \( U_{m_0}(\epsilon_1) \rightarrow V_{g(m_0)}(\epsilon_2) \). This requires some compatible property of the metrics in locally metrizable spaces, which is essentially the same as the definition of Finsler structure in the sense of Neeb-Upmeier (see Appendix D.2).

**Definition 5.5.** We say the metrics in a locally metrizable space \( M \) with an open cover \( \{U_m\} \) are **uniformly compatible** at \( M_1 \subset M \) if there are a constant \( \sigma > 0 \) (might be \( \sigma = \infty \)) and a constant \( \Xi \geq 1 \) such that if \( m, m' \in U_{m_1}(\sigma) \cap U_{m_2}(\sigma) \neq \emptyset, m_1, m_2 \in M_1 \), then

\[
\Xi^{-1}d_{m_1}(m, m') \leq d_m(m, m') \leq \Xi d_{m_2}(m, m'),
\]

where \( d_m \) is the metric in \( U_m \). If \( M_1 = M \), we also say \( M \) is a uniformly locally metrizable space.

From this, we know if \( M, N \) are uniformly locally metrizable spaces and \( g : M \rightarrow N \) is uniformly (locally) \( \theta \)-Hölder, then for small \( \epsilon_0 > 0 \), there is a constant \( L > 0 \) such that if \( m_1, m_2 \in U_{m_0}(\epsilon_0), \hat{m}_0 \in U_{g(m_0)}(\epsilon_0) \), one has

\[
d_{m_0}(g(m_1), g(m_2)) \leq L d_{m_0}(m_1, m_2).
\]

So for this reason, the definitions of uniform continuity and (uniformly) Hölder continuity of \( g \) are about \( g : U_m(\sigma) \rightarrow V_{g(\sigma)}(\epsilon) \). Uniformly locally metrizable space appears naturally in application. For instance, in Example 5.2 (a) (e), they are already uniformly locally metrizable spaces.

**Example 5.6.** (a) **Continue Example 5.2 (b).** We assume (i) each component of \( M \) is modeled on a Banach space \( X_\gamma \) with a norm \( \| \cdot \|_\gamma \), and (ii) there is an atlas \( \{(U_\alpha, \phi^\alpha)\} \) in each component satisfying \( \phi^\alpha(U_\alpha) = X_\gamma(1) = \{x \in X_\gamma : |x|_\gamma < 1\} \) and sup\( X_\gamma \cap U_\alpha = \text{Lip} \phi^\alpha \circ (\phi^\gamma)^{-1} < \infty \) (see also Definition C.7), then this type of manifold is a uniformly locally metrizable space.

(b) **Continue Example 5.2 (c).** For short, let ‘u.l.m.s’ mean uniformly locally metrizable space. Any \( C^0 \) topology bundle over a u.l.m.s such that it has an \( \epsilon \)-almost local \( C^{0,1} \)-fiber trivialization (see Definition 5.23), then it is a u.l.m.s. In particular, any \( C^0 \) vector bundle over a u.l.m.s having \( C^0 \) uniform Finsler structure (see Appendix D.2 (b)) is a u.l.m.s, and so is the tangent bundle of a \( C^k \) Finsler manifold in the sense of Neeb-Upmeier weak uniform or Palais (see Appendix D.2 (d)) or a Riemannian manifold.

(c) For a locally metrizable space \( M \) with an open cover \( \{U_m\} \), if there are a constant \( C \geq 1 \) and a metric \( d \) in \( M \) such that \( C^{-1}d_m(m_1, m_2) \leq d(m_1, m_2) \leq Cd_m(m_1, m_2) \), where \( m_1, m_2 \in U_m \) and \( d_m \) is the metric in \( U_m \), then \( M \) is uniformly locally metrizable space. See also Section 5.5.1.

It might happen that \( U_m(\epsilon_1) = U_m(\epsilon_2) \) if \( \epsilon_1 > \epsilon_2 \). In order to give a characterization about this case, we need the following notion of local uniform size neighborhood which is important for applying our main results (see Section 7). A related notion is the injectivity radius of the exponential map in Riemannian manifolds; see also Section 5.5.1, Appendix C.0.2 and Example C.3.

**Definition 5.7.** Let \( M \) be a locally metrizable space associated with an open cover \( \{U_m : m \in M\} \).

We say \( M \) has a **local uniform size neighborhood** at \( M_1 \), if there is an \( \epsilon_1 > 0 \), such that \( U_{m}(\epsilon_1) \subset U_m \), for \( \forall m \in M_1 \), where the closure is taken in the topology of \( M \). Note that if there is a such \( \epsilon_1 > 0 \), then the above is also satisfied for all \( \epsilon < \epsilon_1 \).

5.2. **fiber (or leaf) topology.** Take a bundle \( X \) over \( M \). Assume the fibers of \( X \) are (Hausdorff) topology spaces. Let \( \mathcal{B}_s \triangleq \bigcup_{m \in M} \{X_m \cap V : V \text{ is open in } X_m\} \). Then the unique topology such that \( \mathcal{B}_s \) is a topology subbase, is called the **fiber topology** of \( X \).
Example 5.8. (a) \((C^k\) bundle and its fiber topology). Let \((X, M, \phi)\) be a \(C^k\) bundle (see Definition 5.9). Then it is a \(C^k\) manifold locally modeled on \(T_m M \times T_x X_m\). Consider its fiber topology, it also being a \(C^k\) manifold but locally modeled on \(T_x X_m\); the fiber topology might even not make \(X\) a \(C^0\) bundle.

(b) (leaf topology for foliation). Let \(M\) be a (Hausdorff) topology space. Assume there are mutually disjoint subsets of \(M, U_\alpha, \alpha \in \Lambda\), such that \(M = \bigcup_\alpha U_\alpha\). Consider \(M\) as a bundle over \(\Lambda\). The fiber topology in \(M\) is called a leaf topology of \(M\). Particularly, foliations (see e.g. Section 5.5.2) or laminations (see e.g. Section 7.2.4 or [HPS77]) have leaf topologies if the mutually disjoint subsets are taken as the leaves of foliations or laminations.

We will use fiber (or leaf) topology frequently when applying our main results; see e.g. Section 7.2.

5.3. connection: mostly review.

5.3.1. \(C^k\) bundle.

Definition 5.9. \((X, M, \pi)\) is called a \(C^k\) fiber bundle (or for short \(C^k\) bundle, \(k = 0, 1, 2, \ldots, \infty, \omega\)) if the following hold (see e.g. [AMR88]).

(a) There is a family of bundle charts of \(X\), which is called a \(C^k\) bundle structure and denoted by \(\mathcal{A}_1 = \{(U_\alpha, \varphi^\alpha) : \alpha \in \Lambda\}\), where \(\Lambda\) is an index, such that

1. (open cover) \(U_\alpha, \alpha \in \Lambda\) are open in \(M\), and cover \(M\),
2. (\(C^k\) transition maps) if \((U_\alpha, \varphi_\alpha), (U_\beta, \varphi_\beta) \in \mathcal{A}_1\) are bundle charts at \(m_0, m_1\) respectively, and \(U_\alpha \cap U_\beta \neq \emptyset\), then \(\varphi_\alpha \circ \varphi_\beta^{-1} : U_{\alpha \beta} \times X_m \to U_{\alpha \beta} \times X_m\) is \(C^k\).
3. (maximal) \(\mathcal{A}_1\) is maximal in the sense that it contains all possible pairs \((U_\alpha, \varphi^\alpha)\) satisfying the properties (1) (2).

Any \(\mathcal{A}_0\) satisfying (1) (2) is called a \(C^k\) bundle atlas of \(X\). A \(C^k\) bundle atlas of \(X\) gives a unique \(C^k\) bundle structure and a unique \(C^k\) structure on \(X\) to make \(X\) both a \(C^k\) bundle and a \(C^k\) Banach manifold, locally modeled on \(T_m M \times T_x X_m\), such that the given \(C^k\) bundle atlas becomes a subset of those structures. Note that \(M\) now is a submanifold of \(X\).

5.3.2. connection. In order to give a tangent representation of a \(C^1\) bundle map respecting base points, we need to know how \(T_{(m,x)} X\) represents as \(T_m M \times T_x X_m\). So an additional structure of \(X\) is needed, named the connection in \(X\). See [Kli95] in the setting of vector bundles.

From now on, we assume \(X\) is a \(C^k\) bundle with a \(C^k\) \((k \geq 1)\) bundle atlas \(\mathcal{A}_0\). There are two natural \(C^{k-1}\) vector bundles associated with \(X\). Let

\[
\Gamma^V_X \triangleq \bigcup_{(m,x) \in X} (m,x) \times T_x X_m, \quad \Gamma^H_X \triangleq \bigcup_{(m,x) \in X} (m,x) \times T_m M.
\]

\(\Gamma^V_X \triangleq \Gamma^H_X \times \Gamma^V_X\) is a \(C^{k-1}\) vector bundle with base space \(X\) and fibers \(T_m M \times T_x X_m\). Write the natural fiber projections \(\Pi^H_{(m,x)} : T_m M \times T_x X_m \to T_m M, \Pi^V_{(m,x)} : T_m M \times T_x X_m \to T_x X_m\), and associated bundle projections \(\Pi^H : \Gamma^V_X \to \Gamma^H_X, \Pi^V : \Gamma^V_X \to \Gamma^V_X\). Another vector bundle is the tangent bundle of \(X\), i.e. \(TX\). Note that locally \(T_{(m,x)} X \cong T_m M \times T_x X_m\).

In the following, we will identify \(T_m M = T^{-1}(\{0\})\) and \(T_x X_m = \{0\} \times T_x X_m\).

Definition 5.10. \(C : TX \to \Gamma_X\) is called a \(C^{k-1}\) connection of \(X\) if the following hold.

(a) (global representation) \(C\) is a \(C^{k-1}\) vector bundle isomorphism over \(id\), i.e. \(C\) is a \(C^{k-1}\) bundle map over \(id\) and for every \((m, x) \in X\), \(C_{(m,x)} : T_{(m,x)} X \to T_{(m,x)} M \times T_{(m,x)} X_m\) is a vector isomorphism. (b) (local representation) \(C\) satisfies the local representation

\[
C_{(m', \varphi_{m'}(x'))} D_{(m', x')} \varphi = \begin{pmatrix} id & 0 \\ 0 & D\varphi_{m'}(x') \end{pmatrix} : T_m' M \times T_{x'} X_m \to T_m' M \times T_{\varphi_{m'}(x')} X_{m'}.
\]
for every bundle chart \((U, \varphi) \in \mathcal{A}_0\) at \(m\), \((m', x') \in U \times X_m\). \(\Gamma(m', x') : T_{m'} M \to T_{\varphi(m'(x'))} X_m\) (or \(\Gamma(m', x') = (D\varphi_{m'}(x'))^{-1} \Gamma(m', x') : T_{m'} M \to T_{x'} X_m\)) is often called a Christoffel map (or Christoffel symbol) in the bundle chart \(\varphi\).

Denote the horizontal space of \(TX\) at \((m, x)\) by \(\mathcal{H}_{(m,x)} \triangleq C^{-1}_{(m,x)}(T_m M \times \{0\})\), and the vertical space of \(TX\) at \((m, x)\) by \(\mathcal{V}_{(m,x)} \triangleq C^{-1}_{(m,x)}(\{0\} \times T_x X_m)\). \(T_{(m,x)} X = \mathcal{H}_{(m,x)} \oplus \mathcal{V}_{(m,x)}\).

Furthermore, if \(X\) is a vector bundle and for every \(m \in M\), \(x \in X_m\), \(\mathcal{C}(\cdot, a) \triangleq \mathcal{C}(\cdot, a) : \mathcal{C} \to \mathcal{C}\) is linear in \(\mathcal{C}\). We use in this paper always stands for the covariant derivative of bundle chart \(\mathcal{C}\).

**Lemma 5.11.** \(\mathcal{V} = \ker D\pi\), i.e. \(\mathcal{V}_{(m,x)} = \ker D_{(m,x)} \pi\). In particular, if \(C^1, C^2\) are connections of \(X\), then \(\Pi^\mathcal{V} C^1 = \Pi^\mathcal{V} C^2\).

**Proof.** Choose a bundle chart \((U, \varphi) \in \mathcal{A}_0\) at \(m\). Since \(\pi \circ \varphi : (m, x) \mapsto m\), we have \(D_{(m,x)} \pi D_{(m,x')} \varphi = (\text{id}, 0)\), where \(x = \varphi_m(x')\). \(v \in \ker D_{(m,x)} \pi\), if and only if \((D_{(m,x')} \varphi)^{-1} v = (0, x_1) \in T_m M \times T_x X_m\), if and only if \(v = (D_{(m,x')} \varphi)(0, x_1)\) for some \(x_1 \in T_x X_m\). As \(C_{(m,x)} D_{(m,x')} \varphi(0) \times T_x X_m = \{0\} \times T_x X_m\), we have \(v \in \ker D_{(m,x)} \pi\) if and only if \(C_{(m,x)} v \in \{0\} \times T_x X_m\), i.e. \(v \in \mathcal{V}_{(m,x)}\).

From above lemma, we see that the connection is described how we continuously choose horizontal spaces, and this is what the classical definition of connection says; see [Kli95].

**Example 5.12.** (a) If \(M\) is a \(C^k\) manifold, then we can (and always) choose \(C = \text{id}\).

(b) There is a canonical connection \(C = \begin{pmatrix} \text{id} & 0 \\ 0 & \text{id} \end{pmatrix}\) for the trivial bundle.

(c) If \(X, Y\) are \(C^k\) bundles over the same base space with \(C^{k-1}\) connections \(C^X, C^Y\). Then \(X \times Y\) has a natural connection \(C^X \times C^Y : T(X \times Y) = TX \times TY \rightarrow T(X \times Y)\). We always use this product connection for the product bundle.

(d) If \(M\) is a smooth Riemannian manifold, then there is a connection of \(TM\) called the Levi-Civita connection (see [Kli95]).

(e) If \(X\) is a \(C^1\) bundle over a paracompact \(C^1\) manifold \(M\), then \(X\) always exists a \(C^0\) (and locally \(C^{0,1}\)) connection. The proof is the same as the construction of a linear connection of a vector bundle, see e.g. [Kli95, Theorem 1.5.15].

### 5.3.3. covariant derivative.

Let \((X, M, \pi_1), (Y, N, \pi_2)\) be \(C^k\) bundles, and \(f : X \to Y\) a bundle map over a map \(u : M \to N\). We use the notation \(f \in C^k(X, Y)\) if and only if \(f\) is \(C^k\) regarding \(X, Y\) as \(C^k\) manifolds, and in this case, one can easily see that \(u \in C^k(M, N)\).

**Definition 5.13.** Suppose that \(X, Y\) have \(C^{k-1}\) connections \(C^X, C^Y\), respectively, and that \(f : X \to Y\) is a \(C^1\) bundle map over a map \(u : M \to N\). Define its covariant derivative as

\[
\nabla_m f_{m(x)} := \Pi^\mathcal{Y}_{f_{m(x)}} C^Y_{f_{m(x)}} Df_{m(x)} C^X_{m(x)}^{-1} : T_m M = T_{m(x)} M \times \{0\} \to T_{f_{m(x)}} Y_{u(m)}.
\]

Hereafter, the symbol \(\nabla\) we use in this paper always stands for the covariant derivative of bundle maps. See [Kli95] for the covariant derivative of a section, or more specially, a vector field, and another equivalent definitions of covariant derivative. Note that from Lemma 5.11, the covariant derivative of \(f\) does not depend on the choice of connection in \(Y\); we give a connection in \(Y\) only for convenience of writing (in a global form).

Note that if a bundle map \(f \in C^k(X, Y)\), and the connections are \(C^{k-1}\), then

\[
\nabla_f : T^H_X \to T^V_Y, (m, x, v) \mapsto (f(m, x), \nabla_m f_{m(x)} v)
\]

is a \(C^{k-1}\) vector bundle map over \(f\), i.e. \(\nabla f \in L_f(T^H_X, T^V_Y)\). Here are some properties of the covariant derivative.

**Lemma 5.14.** Let \(X_i\) be \(C^k\) \((k \geq 1)\) bundles with connections \(C^i\), \(i = 1, 2, 3\), respectively. \(f : X_1 \to X_2\), \(g : X_2 \to X_3\) are \(C^1\) bundle maps over \(u_1, u_2\), respectively. Then we have
(1) $\Pi^h_{f(x)} C^2_{f(x)} Df(m,x)(C^1_{(m,x)})^{-1}|T_m M = D u_1(m)$.
(2) $\Pi^v_{f(x)} C^2_{f(x)} Df(m,x)(C^1_{(m,x)})^{-1}|T_x M = D f_m(x)$.
(3) **(chain rule)** $\nabla_m (g_{u_1(m)} \circ f_m)(x) = \nabla_{u_1(m)} g_{u_1(m)} (f_m(x)) Du_1(m) + D g_{u_1(m)} (f_m(x)) \nabla_m f_m(x)$.

**Proof.** (1) (2) are just the another saying of the local representation of connections, and (3) is a consequence of (1) (2). Here are details. Take bundle charts $(U_i, \varphi^i)$ of $X_i$ at $m_i, i = 1, 2, x \in X_{m_1}, x' = f_m(x), \varphi^1_{m_1}(x_1) = x, \varphi^2_{m_2}(x_2) = x', m' = u_1(m)$. Then

$$C^2_{f(x)} Df(m,x)(C^1_{(m,x)})^{-1}$$

$$= C^2_{(m',x')} D(m',x') \varphi^2_{(m,x)} D(m,x)(\varphi^2_1 \circ f_1)(D(m,x)(\varphi^1_1)^{-1}(C^1_{(m,x)})^{-1}$$

$$= \begin{pmatrix} * & D \varphi^2_{m_2}(x_2) & 0 \\ 0 & 0 & D((\varphi^2_{m_2})^{-1} \circ f_m \circ \varphi^1_{m_2})(x_1) \end{pmatrix} \begin{pmatrix} \id & 0 \\ 0 & 0 \end{pmatrix}$$

This gives (1) (2). For (3), let $u_1(m) = m_2, u_2(m_2) = m_3, f_m(x) = x'_2, g_{m_2}(x'_2) = x'_3$. Now we have

$$\nabla_m (g_{u_1(m)} \circ f_m)(x)$$

$$= \Pi^v_{(m_3,x'_3)} C^3_{(m_3,x'_3)} D(g \circ f)(m,x)(C^1_{(m,x)})^{-1}|T_m M$$

$$= \Pi^v_{(m_3,x'_3)} C^3_{(m_3,x'_3)} D g_{m_2}(x'_2) D(m,x)(C^1_{(m,x)})^{-1}|T_m M$$

$$= \Pi^v_{(m_3,x'_3)} C^3_{(m_3,x'_3)} D g_{m_2}(x'_2)(C^2_{(m_2,x'_2)})^{-1} \Pi^h_{(m_2,x'_2)} C^2_{(m_2,x'_2)} D f(m,x)(C^1_{(m,x)})^{-1}|T_m M$$

$$+ \Pi^v_{(m_3,x'_3)} C^3_{(m_3,x'_3)} D g_{m_2}(x'_2)(C^2_{(m_2,x'_2)})^{-1} \Pi^v_{(m_2,x'_2)} C^2_{(m_2,x'_2)} D f(m,x)(C^1_{(m,x)})^{-1}|T_m M$$

$$= \nabla_{u_1(m)} g_{u_1(m)} (f_m(x)) Du_1(m) + D g_{u_1(m)} (f_m(x)) \nabla_m f_m(x).$$

The proof is finished. \qed

### 5.3.4. normal bundle chart.

**Definition 5.15.** Let $X$ be a $C^1$ bundle with $C^0$ connection $C$. A $C^k$ ($k \geq 1$) bundle chart $(V, \psi)$ at $m$ is called a $C^k$ **normal bundle chart** (with respect to $C$) if $\nabla_m \psi_m(x) = 0$ for every $x \in X_m$. A bundle atlas $\mathcal{A}$ is **normal** (with respect to $C$) at $M_1$ if for every bundle chart $(V, \psi) \in \mathcal{A}$ at $m \in M_1$ is normal.

In this paper, we use the normal bundle chart only for simplifying our computation in Section 6 (see particularly in the proof of Lemma 6.18). Our assumptions in Section 6 are usually stated respecting $C^1$ normal bundle charts and the connections are assumed to be $C^0$. However, if one needs constructing normal bundle charts from the given bundle charts (see Appendix D.4 for a discussion), the higher regularity is needed.

**Lemma 5.16.** Suppose there is a section $i$ of $X$, which is a 0-section (see **Section 5.4.3** with respect to a family of $C^k$ normal bundle charts $\mathcal{A}_{i_0}$ at $M_1$.

(1) Then $i$ is $C^k$ in a neighborhood of $M_1$ and $\nabla_m i(m) = 0$ for $m \in M_1$.

(2) Any $C^1$ bundle map $g : X \to Y$ over $u$ such that $i$ is invariant under $g$ in $u^{-1}(M_1)$, then $\nabla_m g_m(i(m)) = 0$.

**Proof.** This is easy. (1) Since $\psi_m(i(m)) = i(m')$, where $(U, \varphi) \in \mathcal{A}_0$ is a normal bundle chart at $m$, then $i$ is $C^k$ in $U$, and $\nabla_m i(m) = \nabla_m \psi_m(i(m)) = 0$. (2) Note that $g_m(i(m)) = i(u(m))$. Thus, $\nabla_m g_m(i(m)) = \nabla_{m'} g_{m'}(i(m')) Du_m(m) = 0$, where $m' = u(m) \in M_1$. \qed

### 5.4. bundle and bundle map with uniform property: part I.
5.4.1. bundle chart. If \( m_0 \in U \subset M \) and \( \varphi: U \times X_{m_0} \to X, (m, x) \mapsto (m, \varphi_m(x)) \) such that \( \varphi_m(x) \in X_m \) and \( \varphi_m \) is a bijection, we call \((U, \varphi)\) a (set) bundle chart of \( X \) at \( m_0 \) and \( U \) the domain of the bundle chart \((U, \varphi)\). If we do not emphasize the domain \( U \), we also say \( \varphi \) is a bundle chart (at \( m_0 \)). If \( \mathcal{A} = \{(U_\alpha, \varphi^\alpha): \alpha \in \Lambda\} \) is a family of bundle charts of \( X \) such that \( \bigcup_\alpha U_\alpha = M \), where \( \Lambda \) is an index, we usually call \( \mathcal{A} \) a bundle atlas (or for short atlas) of \( X \). For the convenience, we also say that \( \mathcal{A} \) is a bundle atlas at \( M \) if for every \( m_0 \in M \), there is a bundle chart at \( m_0 \) belonging to \( \mathcal{A} \).

Take a bundle atlas \( \mathcal{A} = \{(U_\alpha, \varphi^\alpha): \alpha \in \Lambda\} \) of \( X \). Take any bundle charts \((U_\alpha, \varphi^\alpha), (U_\beta, \varphi^\beta)\) in \( \mathcal{A} \) at \( m_0, m_1 \) respectively such that \( U_\alpha \cap U_\beta \triangleq U_{\alpha\beta} \neq \emptyset \). Then we call the following map a transition map (at \( m_0, m_1 \) with respect to \( \mathcal{A} \))

\[
\varphi^{\alpha\beta} \triangleq (\varphi^\beta)^{-1} \circ \varphi^\alpha: U_{\alpha\beta} \times X_{m_0} \to U_{\alpha\beta} \times X_{m_1}.
\]

If every domain \( U_\alpha \) has property \( P_0 \) and every transition map has property \( P \), we say the bundle \( X \) has property \((P, P_0)\) with respect to \( \mathcal{A} \). Consider the following examples.

**Example 5.17.** (a) If \( \Lambda \) only contains one element, \( X \) usually is called a trivial bundle.

(b) If \( M \) is a topology space, let \( P_0 \) mean openness and \( P \) continuity, then \( X \) is called a \( C^0 \) topology bundle. For this case, the bundle atlas can be maximal like in the classical way. And for convenient, a bundle chart belonging to this maximal bundle atlas is called a \( C^0 \) bundle chart, and any bundle atlas in this maximal bundle atlas is called a \( C^0 \) bundle atlas.

(c) Assume each fiber of \( X \) is a Banach space and the bundle charts in \( \mathcal{A} \) are linear (i.e. \( x \mapsto \varphi^\alpha_m(x) \) is an invertible linear operator). Let \( P_0 \) mean openness and \( P \) continuity in \( U \to L(X_{m_0}, X_{m_1}) \), then \( X \) is called a \( C^0 \) vector bundle. In addition, if \( M \) is a \( C^k \) manifold, \( P \) means \( C^k \) \((k = 0, 1, 2, \ldots, \infty, \omega)\) in \( U \to L(X_{m_0}, X_{m_1}) \), then \( X \) is called a \( C^k \) vector bundle. See e.g. [AMR88]. We refer the readers to see the definition of \( C^r \)-uniform (Banach)-bundle in [HPS77, Chapter 6]. A slightly more general type of bundle than vector bundle with some uniform properties like \( C^r \)-uniform (Banach)-bundle are summarized in Remark B.10; see also Example C.10.

Let \((X, M, \pi_1), (Y, N, \pi_2)\) be two bundles and \( F: X \to Y \) a bundle map over \( U \). If every \((U, \varphi) \in \mathcal{A}\) at \( m_0 \), \((V, \psi) \in \mathcal{B}\) at \( m'_0 \) such that \( W \triangleq U \cap \varphi^{-1}(V) \neq \emptyset \), then we call the following map a local representation of \( F \) (at \( m_0, m'_0 \) with respect \( \mathcal{A}, \mathcal{B} \)),

\[
(\#) \quad \widehat{F}_{m_0, m'_0}(m, x) \triangleq \psi^{-1}_{u(m)} \circ F_m \circ \varphi_m(x): W \times X_{m_0} \to Y_{m'_0}.
\]

If every local representation of \( F \) has property \( P_1 \), we say \( F \) has property \( P_1 \) with respect to \( \mathcal{A}, \mathcal{B} \). For example, if \( X, Y \) are \( C^0 \) topology bundles, and \( P_1 \) means \( C^0 \) continuity, then we say \( F \) is \( C^0 \). Note that in this case since all the transition maps are \( C^0 \), one just needs to know \( u \) is \( C^0 \) and every map \( \widehat{F}_{m_0, u(m)} \) is continuous at \( m_0 \), then \( F \) is \( C^0 \). Take another example. If \( X, Y \) are \( C^r \)-uniform (Banach)-bundle with preferred \( C^r \)-uniform atalases \( \mathcal{A}, \mathcal{B} \) respectively (see [HPS77, Chapter 6]), and \( P_1 \) means \( C^r \)-uniform (i.e. all the local representations of \( F \) are uniformly \( C^r \) equicontinuous), then we say \( F \) is \( C^r \)-uniform with respect to \( \mathcal{A}, \mathcal{B} \). Also for this case, it suffices to know the maps \( \widehat{F}_{m_0, u(m)} \), \( m_0 \in M \), are uniformly \( C^r \) equicontinuous (i.e. \( |D^i \widehat{F}_{m_0, u(m)}(m, x') - D^i \widehat{F}_{m_0, u(m)}(m_0, x)| \to 0 \), \( 0 \leq i \leq r \), uniform for \(|x|, |x'| \leq 1, m_0 \in M \) as \((m, x') \to (m_0, x)) \), then \( F \) is \( C^r \)-uniform. This is an important observation which can make us extremely simplify our argument for the proof of regularity results in Section 6. Based on this motivation, we usually consider the local representation of \( F \) at \( m_0, u(m) \) (not for the arbitrary choice of \( m'_0 \in N \)). Moreover, for every \( m \), we usually require that there is at most one bundle chart \( \varphi \) at \( m \) (but might has different domains) belonging to the bundle atlas, and also without loss of generality, \( \varphi_m = \text{id} \).

**Definition 5.18.** A regular bundle atlas \( \mathcal{A} \) of a bundle \((X, M, \pi_1)\) at \( M_1 \) if \((U, \phi), (V, \psi)\) are bundle charts at \( m \in M_1 \) belonging to \( \mathcal{A} \), then \( \phi = \psi \) in \( U \cap V \) and \( \phi_m = \text{id} \). For convenience when \( M \) is a topology space, we also say a bundle atlas \( \mathcal{A} \) is open regular at \( M_1 \) if it is regular, and the domains of the bundle charts at \( m_0 \in M_1 \) belonging to \( \mathcal{A} \) form a neighborhood basis at \( m_0 \).
If \((X, M, \pi_1), (Y, N, \pi_2)\) have preferred regular bundle atlases \(\mathcal{A}\) and \(\mathcal{B}\) at \(M_1, u(M_1)\) respectively, where \(u : M \to N, M_1 \subset M\). The \textit{regular} local representations of a map \(F : X \to Y\) over \(u\) at \(M_1\) with respect to \(\mathcal{A}, \mathcal{B}\) are \(\hat{F}_{m_0, u(m_0)}\) (defined by \((\#)\)), \(m_0 \in M_1\). We usually write \(\hat{F}_{m_0} = \hat{F}_{m_0, u(m_0)}\) and call \(\hat{F}_{m_0}(\cdot, \cdot)\) a (regular) local representation or a (regular) \textit{vertical part} of \(F\) (at \(m_0\) with respect to \(\mathcal{A}, \mathcal{B}\)). If \(X = M = N, u = \text{id}\), then \(F\) is a section of \(M \to Y\); in this case \(\hat{F}_{m_0}(\cdot, \cdot)\) is also called a \textit{principal part} of \(F\) (at \(m_0\) with respect to \(\mathcal{B}\)).

The same terminologies are used for manifolds if they are considered as bundles with zero fibers. Except where noted, \textit{the bundle atlases and the local representations are taken to be regular in the present paper}. In \textbf{Section 6}, the conditions and conclusions are stated about the regular local representations with respect to preferred regular bundle atlases. Once the transition maps with respect to the regular bundle atlases have more regular property (which means that the bundles and manifolds are more regular), then conclusions, in a way, become classical as e.g. in [HPS77]. For instance, see \textbf{Section 6.8}.

**Definition 5.19.** A bundle \(X\) over a locally metrizable space \(M\) associated with an open cover \(\{U_m\}\) is said has a \textit{uniform size trivialization} at \(M_1 \subset M\) with respect to \(\mathcal{A}\), if there is a \(\delta > 0\) such that for every \(m_0 \in M_1\), there is a bundle chart \((U_{m_0}(\delta), \varphi^{m_0})\) of \(X\) at \(m_0\) (and \(\varphi^{m_0}_{m_0} = \text{id}\)). If a (regular) bundle atlas \(\mathcal{A}\) contains such above bundle charts, we also say \(\mathcal{A}\) at \(M_1\) (or the bundle charts in \(\mathcal{A}\) at \(M_1\)) has \textit{uniform size domains}.

In the following definitions of uniform property about bundle and bundle map, uniform size trivialization is assumed. While only when \(M\) has a local uniform size neighborhood at \(M_1\) (see \textbf{Definition 5.7}), the uniform property would make more sense. So a more meaningful definition of uniform size trivialization is that further assume \(M\) has a local uniform size neighborhood at \(M_1\). However, we do not assume this first. Such (regular) bundle atlas in the \textbf{Definition 5.19} in some sense plays a similar role as the ‘plaquation’ used in [HPS77, Chapter 6].

\textbf{5.4.2. vector bundle.} A bundle is called a \textit{vector bundle} if each fiber of the bundle is a Banach space. \(C^k\) vector bundle is defined in \textbf{Example 5.17 (c)} (see also [AMR88]). A bundle atlas of a vector bundle is called a \textit{vector bundle atlas} if each bundle chart \((U, \phi)\) belonging to this atlas is linear, i.e. \(x \mapsto \phi_m(x)\) is linear for each \(m \in U\). A bundle map between two vector bundles is called a \textit{vector bundle map} if each fiber map is linear.

Denote \(L_u(X, Y) \equiv \bigcup_{m \in M}(m, L(X_m, Y_{u(m)}))\) which is a vector bundle over \(M\), when \(X, Y\) are vector bundles over \(M, N\) respectively, and \(u : M \to N\). Note that if \(X, Y\) are \(C^k\) vector bundles and \(u\) is \(C^k\) \((k \geq 0)\), then \(L_u(X, Y)\) is also a \(C^k\) vector bundle. If \(M = N\) and \(u = \text{id}\), we use standard notation \(L(X, Y) = L_{\text{id}}L(X, Y)\). Without causing confusion, write \(K \in L(X, Y)\) (over \(u\)) if \(K \in L_u(X, Y)\).

\textbf{5.4.3. 0-section.}

**Definition 5.20.** A map \(i : M \to X\) is called a \textit{section} of a bundle \((X, M, \pi)\), if \(i(m) \in X_m, \forall m \in M\). We say \(i\) is a \textit{0-section} of \(X\) with respect to the bundle atlas \(\mathcal{A}\) of \(X\), if for every \((U, \varphi) \in \mathcal{A}\) at \(m_0\), \(\varphi(m, i(m_0)) = (m, i(m)), \forall m \in U\). When \(X\) has a 0-section \(i\), we will use the notation \(|x| \equiv d(x, i(m))\), if \(x \in X_m\).

For example, the vector bundle has a natural 0-section with respect to any vector bundle atlas. The notion of 0-section is important for us to give higher regularities of invariant graphs for the invariant section case (see \textbf{Section 6}); we thought this is why [Cha08] could not obtain the classical spectral gap condition for the regularity of invariant foliations in the trivial bundle case (see [Cha08, note in Page 1431] and \textbf{Lemma 6.13, Remark 6.14, Remark 6.15}). While for the bounded section case (see \textbf{Section 6.10}), we only need the bounded section satisfying a weaker assumption (B3-) in page 83.
5.4.4. fiber-regularity of bundle map. For a map \( f \) between two metric spaces, we use the notations: Lip \( f \), the Lipschitz constant of \( f \); Hol\( _\alpha \) \( f \), the \( \alpha \)-Hölder constant of \( f \). We also use the notation \( \text{diam} A = \sup \{d(m, m') : m, m' \in A\} \), the diameter of \( A \) belonging to a metric space.

Consider the description of the fiber-regularity of bundle maps. Let \((X, M, \pi_1)\), \((Y, N, \pi_2)\) be two bundles with metric fibers, \(u : M \to N\) a map and \( f : X \to Y\) a bundle map over \( u\). Since the fiber maps \( f_m : X_m \to Y_{u(m)} \) depend on \( m \in M\), there are at least two cases which are needed to distinguish, i.e. point-wise dependence or uniform dependence on \( m \in M\). We list some fiber-regularities in the following.

(a) (\( C^0 \)) case. \( f \) is said to be \( C^0\)-fiber (resp. uniformly continuous-fiber, or equicontinuous-fiber) if \( f_m(\cdot), m \in M\), are \( C^0 \) (resp. uniformly continuous, or equicontinuous). For the equicontinuous-fiber case, we usually say \( f_m(\cdot), m \in M\), are uniformly continuous uniform for \( m \in M\).

(b) (\( C^{0,\alpha} \)) case. \( f \) is said to be \( C^{0,\alpha}\)-fiber if \( f_m \in C^{0,\alpha}(X_m, Y_{u(m)}) \) for all \( m \in M\). \( f \) is said to be uniformly \( C^{0,\alpha}\)-fiber if \( \sup_m \text{Hol}_\alpha f_m(\cdot) < \infty \). For the uniformly \( C^{0,\alpha}\)-fiber case, we often say \( f_m(\cdot) \) are \((\alpha\cdot)\)Hölder uniform for \( m \in M\).

(c) (\( C^{k,\alpha} \)) \((k \geq 1)\). For simplicity, in this case, the fibers of \( X, Y \) are assumed to be Banach spaces (or open subsets of Banach spaces); see also Remark B.11. Similar as \( C^{0,\alpha}\) case, \( f \) is said to be \( C^{k,\alpha}\)-fiber if \( f_m \in C^{k,\alpha}(X_m, Y_{u(m)}) \) for all \( m \in M\). If the \( C^{k,\alpha}\) norm of \( f_m, m \in M\) are uniformly bounded, we say \( f \) is uniformly \( C^{k,\alpha}\)-fiber, or \( f_m(\cdot) \) are uniformly \( C^{k,\alpha}\) uniform for \( m \in M\). If for any fiber bounded set \( A \), i.e. \( \sup_{m \in M} \text{diam} A_m < \infty \), the \( C^{k,\alpha}\) norm of \( f_m(\cdot)|_{A_m}, m \in M \) are uniformly bounded, we say \( f \) is uniformly locally \( C^{k,\alpha}\)-fiber, or \( f_m(\cdot) \) are locally \( C^{k,\alpha}\) uniform for \( m \in M\).

If \( f \) is \( C^1\)-fiber, then we use \( Df_m(x)(D_x f_m(x)) \) to stand for the tangent of the fiber map \( f_m : X_m \to Y_{u(m)} \) (at \( x \in X_m\)). Denote the fiber derivative of \( f \) by \( D^V f \), which is a vector bundle map of \( T_X^V \to T_Y^V \) (see (5.1)) below over \( f \), and is defined by

\[
D^V f : (m, x, v) \mapsto (f(m, x), Df_m(x)v),
\]

where \( Df_m(x) : T_x X_m \to T_{f_m(x)} Y_{u(m)} \).

5.4.5. uniform \( C^{0,1}\)-fiber bundle. If a bundle \( X \) is a \( C^0\) topology bundle (see Example 5.17 (b)), one would usually like to know how the metrics in the fibers continuously change, i.e. the metrics in fibers are compatible with the topology in the bundle. Motivated by the definition of Finsler structure in vector (Banach)-bundle (see e.g. Appendix D.2), we give the following definitions.

**Definition 5.21** \( C^{0,\alpha}\)-fiber bundle. The metrics in the fibers of a \( C^0\) topology bundle \( X \) (see Example 5.17 (b)) are said to be \( C^0\), if for every \( C^0\) bundle chart \((U, \varphi)\) at \( m, U \times X_m \times X_m \to \mathbb{R}_{+}, (m', x, y) \mapsto d_m(\varphi_m(x), \varphi_m(y)) \) is \( C^0\), where \( d_m \) is the metric in \( X_m\). In this case, \( X \) is called a \( C^{0,\alpha}\)-fiber bundle.

**Definition 5.22** \( C^1\)-fiber bundle. Suppose \( X \) is a \( C^0\) topology bundle (see Example 5.17 (b)) with the fibers being (paracompact) Banach manifolds. If there is a \( C^0\) bundle atlas \( \mathcal{A} \) of \( X \) such that

- every bundle chart \( \varphi^{m_0}_m \) satisfies \( x \mapsto \varphi^{m_0}_m(x) \) is \( C^1\), and for every transition map \( \varphi^{m_0,m_1} = (\varphi^{m_1})^{-1} \circ \varphi^{m_0} \) with respect to \( \mathcal{A}, (m, x) \mapsto D\varphi^{m_0,m_1}_m(x) \) is continuous,

then we say \( X \) has a local \( C^1\)-fiber topology trivialization with respect to \( \mathcal{A} \) and \( \mathcal{A} \) is a \( C^1\)-fiber topology bundle; this type of bundle atlas can be maximal; for this reason we also call \( X \) a \( C^1\) topology bundle. In addition, if \( X \) is also a \( C^{0,\alpha}\)-fiber bundle, \( X \) is called a \( C^1\)-fiber bundle.

**Definition 5.23** \( C^{0,1}\)-fiber bundle. Let \( M \) be a locally metrizable space associated with an open cover \( \{U_m : m \in M\} \) (see Definition 5.1). Let \( X \) be a bundle over \( M \) with metric fibers. Let \( M_1 \subset M \). Assume there is a preferred (open regular) bundle atlas \( \mathcal{A} \) (see Definition 5.18) of \( X \) at \( M_1 \) with the following properties. Assume if \( (V_{m_0}, \varphi^{m_0}_m) \in \mathcal{A} \) at \( m_0 \in M_1 \), then

1. \( V_{m_0} \subset U_{m_0} \) and \( V_{m_0} \) is open; so without loss of generality, one can assume \( V_{m_0} = U_{m_0}(\epsilon_{m_0}) \) for some \( \epsilon_{m_0} > 0 \).
(ii) \( \text{Lip}(\varphi_m^{m_0})^{\pm 1}(\cdot) \leq \eta_{m_0}(d_{m_0}(m, m_0)) + 1, m \in V_{m_0}, \) where \( \eta_{m_0} : \mathbb{R}^+ \to \mathbb{R}^+ \), and \( d_{m_0} \) is the metric in
\( U_{m_0}. \)

(a) Let \( M_1 = M \). If every \( \eta_{m_0} \) satisfies \( \lim_{\delta \to 0} \sup \eta_{m_0}(\delta) \leq \varepsilon, \) then we say \( X \) has an \( \varepsilon \)-almost local \( C^{0,1} \)-fiber trivialization with respect to \( \mathcal{A} \). If \( X \) is a \( C^0 \) topology bundle with \( C^0 \) atlas \( \mathcal{A} \) and every \( \eta_{m_0} \) is continuous with \( \eta_{m_0}(0) = 0, \) then \( X \) is called a \( C^{0,1} \)-fiber bundle (with respect to \( \mathcal{A} \)).

(b) Assume \( X \) has a uniform size trivialization at \( M_1 \) with respect to \( \mathcal{A} \) (see Definition 5.19), and \( \eta_{m_0} = \eta \), where \( \eta : \mathbb{R}^+ \to \mathbb{R}^+. \) If \( \lim_{\delta \to 0} \sup \eta(\delta) \leq \varepsilon, \) we say \( X \) has an \( \varepsilon \)-almost uniform \( C^{0,1} \)-fiber trivialization at \( M_1 \) with respect to \( \mathcal{A} \). If \( \eta \) is continuous and \( \eta(0) = 0, \) we say \( X \) has a uniform \( C^{0,1} \)-fiber trivialization at \( M_1 \) with respect to \( \mathcal{A} \); in addition if \( M_1 = M \) and \( X \) is a \( C^0 \) topology bundle with \( C^0 \) atlas \( \mathcal{A} \), then we call \( X \) a uniform \( C^{0,1} \)-fiber bundle (with respect to \( \mathcal{A} \)).

**Definition 5.24** (\( C^{1,1} \)-fiber bundle). Following the Definition 5.23, in addition assume the fibers of \( X \) are Banach spaces (or open subsets of Banach spaces) for simplicity (see also Remark B.11) and the following hold:

(iii) \( x \mapsto \varphi_m^{m_0}(x) \) is \( C^1 \), and \( \text{Lip}(\varphi_m^{m_0})^{\pm 1}(\cdot) \leq \eta_{m_0}(d_{m_0}(m, m_0)), m \in V_{m_0}, \) where \( \eta_{m_0} : \mathbb{R}^+ \to \mathbb{R}^+. \)

(a) Let \( M_1 = M \). Assume every \( \eta_{m_0} \) satisfies \( \lim_{\delta \to 0} \sup \eta_{m_0}(\delta) < \infty, \) If \( X \) has an \( \varepsilon \)-almost local \( C^{0,1} \)-fiber trivialization with respect to \( \mathcal{A} \), then we say \( X \) has an \( \varepsilon \)-almost local \( C^{1,1} \)-fiber trivialization with respect to \( \mathcal{A} \). If \( X \) is a \( C^{0,1} \)-fiber and \( C^1 \)-fiber bundle, then \( X \) is called a \( C^{1,1} \)-fiber bundle (with respect to \( \mathcal{A} \)).

(b) Let \( \eta_{m_0} = \eta^\infty \) where \( \eta^\infty : \mathbb{R}^+ \to \mathbb{R}^+ \) and \( \lim_{\delta \to 0} \sup \eta^\infty(\delta) < \infty. \) If \( X \) has an \( \varepsilon \)-almost uniform \( C^{0,1} \)-fiber trivialization at \( M_1 \) with respect to \( \mathcal{A} \), then we say \( X \) has an \( \varepsilon \)-almost uniform \( C^{1,1} \)-fiber trivialization at \( M_1 \) with respect to \( \mathcal{A} \). If \( X \) has a uniform \( C^{0,1} \)-fiber trivialization at \( M_1 \) with respect to \( \mathcal{A} \), then we say \( X \) has a uniform \( C^{1,1} \)-fiber trivialization at \( M_1 \) with respect to \( \mathcal{A} \); in addition if \( M_1 = M \) and \( X \) is also a \( C^1 \)-fiber bundle, then \( X \) is called a uniform \( C^{1,1} \)-fiber bundle (with respect to \( \mathcal{A} \)).

Notation warning: \( C^k \) fiber bundle (or \( C^k \) bundle, see Definition 5.9) and \( C^k \)-fiber bundle. The latter has ‘-’ between words ‘\( C^k \)’ and ‘fiber’ to emphasis the \( C^k \) regularity about the fibers not the higher regularity about the base space.

Now we have the following relational graph, where \( a \to b \) means if \( X \) is \( a \), then \( X \) is \( b \).

\[
\text{uniform } C^{1,1} \text{-fiber bundle} \quad \longrightarrow \quad \text{uniform } C^{0,1} \text{-fiber bundle} \\
\downarrow \quad \downarrow \\
C^{1,1} \text{-fiber bundle} \quad \longrightarrow \quad C^{0,1} \text{-fiber bundle} \\
\downarrow \quad \downarrow \\
C^1 \text{-fiber bundle} \quad \longrightarrow \quad C^0 \text{-fiber bundle} \\
\downarrow \quad \downarrow \\
C^1 \text{ topology bundle} \quad \longrightarrow \quad C^0 \text{ topology bundle}
\]

**Example 5.25** (vector bundle case). Consider a \( C^0 \) vector bundle \( X \) over a Finsler manifold \( M \) with Finsler metrics in each component of \( M \), modeled on \( \mathbb{R} \times \mathbb{R} \). Let \( \mathcal{A} \) be a \( C^0 \) (regular) vector bundle atlas of \( X \). Now \( X \) is a \( C^1 \) topology bundle. Fix a \( C^0 \) Finsler structure in \( X \) (see Appendix D.2 (a)). Now the Finsler structure gives a norm in every fiber \( X_m \), and it is \( C^0 \) meaning \( X \) is a \( C^0 \)-fiber bundle (and so a \( C^1 \)-fiber bundle). For vector bundles, Definition 5.24 does not give any information as vector bundle charts are linear and \( \eta_{m_0} \equiv 0 \). Let’s consider what Definition 5.23 really says under this situation. That \( X \) has an \( \varepsilon \)-almost local \( C^{0,1} \)-fiber trivialization with respect to \( \mathcal{A} \), means the Finsler structure in \( X \) is \( C^0 \) uniform with constant \( \varepsilon = 1 + \varepsilon \) (see Appendix D.2 (b)). That \( X \) is a \( C^{0,1} \)-fiber bundle means the Finsler structure in \( X \) is uniformly \( C^0 \) (see Appendix D.2 (a)); particularly if \( X = TM \) this means \( TM \) is a Finsler manifold in the sense of Palais (see Appendix D.2 (d)).
Let $M_1 \subset M$. We assume the bundle charts in $\mathcal{A}$ at $M_1$ have uniform size domains (see Definition 5.19), that is there is a $\delta > 0$, such that $(U_{m_0}(\delta), \varphi_{m_0}) \in \mathcal{A}$, $m_0 \in M_1$, where $U_{m_0}(\delta) = \{ m' \in U_{m_0} : d(m', m_0) < \delta \}$. That $X$ has an $\varepsilon$-almost uniform $C^{0,1}$-fiber trivialization at $M_1$ with respect to $\mathcal{A}$ means that there is a $\delta' > 0$ ($\delta' < \delta$) such that for any $m_0 \in M_1$,

$$1 - \varepsilon < |\varphi_{m_0}^{m_0}| + 1 < 1 + \varepsilon,$$

provided $m \in U_{m_0}(\delta')$, where $|\varphi_{m_0}^{m_0}|$ is the norm of $\varphi_{m_0} : X_{m_0} \to X_m$. And that $X$ has a uniform $C^{0,1}$-fiber trivialization at $M_1$ with respect to $\mathcal{A}$ if $\lim_{\varepsilon \to 0} \sup_{m_0 \in M_1} \sup_{m \in U_{m_0}(\varepsilon)} |(\varphi_{m_0}^{m_0}) + 1| = 1$. The notion of uniform $C^{1,1}$-fiber bundle is related with vector bundle having bounded geometry (see Example C.10). The uniform property of the trivialization of $X$ at $M_1$ is vital for our Hölder regularity results (see Section 6), in order to overcome the difficulty about the lack of compactness and high smoothness.

The following Section 5.4.6 is not used which can be skipped at a first reading.

5.4.6. base-regularity of bundle map, $C^{0,1}$-uniform bundle: Hölder case. We give a description about the Hölder continuity respecting the base points for a bundle map $f$, i.e. $m \mapsto f_m(x)$. In general, it is meaningless to talk about the Hölder continuity of $m \mapsto f_m(x)$ if we do not know the Hölder continuity of the fiber maps $x \mapsto f_m(x)$. A natural way to describe the Hölder continuity of $m \mapsto f_m(x)$ is using the local representations of $f$ (see e.g. [HPS77,PSW12]). The local representations of $f$ might not be Hölder with respect to $x$ even if $f$ is Hölder-fiber, so a natural setting for the bundles are they have some local $C^{0,1}$-fiber trivializations. However, we do not assume these at first. We give the details as follows.

**Definition 5.26.** Let $X$ be a bundle with metric fibers over $M$ and $M_1 \subset M$. For a subset $A \subset X$, we write $A_m = A \cap X_m$. $A$ is said to be bounded-fiber at $M_1$ if $\sup_{m_0 \in M_1} \operatorname{diam}A_{m_0} < \infty$.

A function $c : X \to \mathbb{R}_+$ ($c_m : X_m \to \mathbb{R}_+$), is said to be bounded at $M_1$ on any bounded-fiber sets if for any bounded-fiber set $A$ at $M_1$, $\sup_{m_0 \in M_1} \sup_{x \in A_{m_0}} c_{m_0}(x) < \infty$. For example,

(a) a uniformly bounded function, i.e. $\sup_{m_0 \in M_1} \sup_{x \in X_{m_0}} c_{m_0}(x) < \infty$;

(b) $c_{m_0}(x) \leq c_0(|x|)$, where $c_0 : \mathbb{R}_+ \to \mathbb{R}_+$, $i_X : M \to X$ is a fixed section of $X$ and $|x| = |x|_{m_0} = d_{m_0}(x, i_X(m_0))$. A more concrete function of $c_0$ is like

$$c_{c,\gamma}(a) = M_0(c + a^\gamma), \quad a \in \mathbb{R}_+,$$

where $c = 0$ or 1, $\gamma \geq 0$, $M_0 > 0$.

**Definition 5.27.** Let $M, N$ be two locally metrizable spaces associated with open covers $\{ U_m : m \in M \}$ and $\{ V_n : n \in N \}$ respectively. The metrics in $U_m$ and $V_n$ are $d^1_m$ and $d^2_n$ respectively. Let $X, Y$ be two $C^0$ topology bundles over $M, N$ with $C^0$ (open regular) bundle atlases $\mathcal{A}, \mathcal{B}$ respectively (see Definition 5.18). Let $u : M \to N$ be $C^0$. Let $f : X \to Y$ be a bundle map over $u$ and $\tilde{f}_{m_0} : U_{m_0}(\varepsilon_{m_0}) \times X_{m_0} \to Y_{u(m_0)}$, a (regular) local representation of $f$ at $m_0$ with respect to $\mathcal{A}, \mathcal{B}$ (see Definition 5.18), where $\varepsilon_{m_0} > 0$ is small. Assume

(a) We say (the vertical part of) $f$ depends in a (locally) $C^{0,0}$ fashion on the base points around $M_1$, or $m \mapsto f_m(\cdot)$ is (locally) $C^{0,0}$ around $M_1$ with respect to $\mathcal{A}, \mathcal{B}$.

(b) Assume $u$ is uniformly continuous around $M_1$ (see Definition 5.4). Suppose $X, Y$ both have uniform size trivializations at $M_1$ with respect to $\mathcal{A}, \mathcal{B}$ respectively (see Definition 5.19), i.e. one can choose $\varepsilon_{m_0}$ such that $\sup_{m_0 \in M_1} \varepsilon_{m_0} > 0$. If $c_{m_0}, m_0 \in M_1$, are bounded at $M_1$ on any bounded-fiber sets (see Definition 5.26), then we say (the vertical part of) $f$ depends in a uniformly (locally) $C^{0,0}$ fashion on the base points around $M_1$ ‘uniform for bounded-fiber sets’, or a more intuitive saying
we prefer using \( m \mapsto f_m(\cdot) \) is uniformly (locally) \( \theta \)-Hölder around \( M_1 \) \( ' \)uniform for bounded-fiber sets\( ' \), with respect to \( \mathcal{A}, \mathcal{B} \). Usually, the words in \( ' \ldots ' \) are omitted especially when \( c_m \) is the class of functions in case (a) or (b) in Definition 5.26.

(c) (vector case) Under case (b), in addition let \( X, Y, \mathcal{A}, \mathcal{B}, f \) be vector and \( |c_m(x)| \leq M_0|x| \) (where \( M_0 \) is a constant independent of \( m_0 \)). Note that \( m \mapsto f_m \) can be regarded naturally as a section of \( M \to L_u(X, Y) \). We also say \( m \mapsto f_m \) is uniformly (locally) \( \theta \)-Hölder around \( M_1 \) (with respect to \( \mathcal{A}, \mathcal{B} \)).

(d) The words \( ' \)around \( M_1 \)' in (a) (b) often are omitted if \( M_1 = M \). Also \( \theta \)-Hölder = \( C^{0, \theta} \), \( 1 \)-Hölder = \( C^{0,1} \) = Lipschitz. We will omit the letter \( ' \theta ' \) if we do not emphasis the Hölder degree \( \theta \).

Furthermore, without causing confusion, the words \( ' \)with respect to \( \mathcal{A}, \mathcal{B} ' \) will be omitted if \( \mathcal{A}, \mathcal{B} \) are predetermined. The same terminologies for a section will be used.

Remark 5.28. (a) \( c_m \) is the class of functions in case (b) in Definition 5.26, for example \( X, Y \) are vector bundles with vector bundle atlases \( \mathcal{A}, \mathcal{B} \) and \( f \) is a vector bundle map (usually \( c_0 = c_{0,1} \) in Definition 5.26); see also Section 6. A special case that the fibers of \( X \) (or \( Y \) are uniformly bounded, i.e. \( \sup_{m \in M} \text{diam}X_m < \infty \), for example \( X = M \) or a disk bundle, makes \( c_m \) be the class of functions in case (a) in Definition 5.26; see also Section 7.1. For the latter case, in [PSW12, Section 8], \( f \) is also said to have \( \theta \)-bounded vertical shear (at \( M_1 \)) with respect to \( \mathcal{A} \).

(b) Note that

\[
\text{Lip}_\mathcal{A} \hat{f}_m(m, \cdot) \leq \varepsilon^d m_0(m, m_0) + \varepsilon^2 \text{Lip}_{\mathcal{A}}(u(m), u(m_0)) + \text{Lip} f_m(\cdot), m \in U_m(c_m),
\]

if in addition that the fiber maps of \( f \) are Lipschitz and \( X, Y \) both have \( \varepsilon \)-almost local \( C^{0,1} \)-fiber trivializations with respect to \( \mathcal{A}, \mathcal{B} \). This fact will be frequently used in Section 6.

(c) See Section 5.4.1 for a reason why it is unnecessary to consider like this

\[
|\hat{f}_m(m, x) - \hat{f}_m(m', x)| \leq c_m(x) d^\theta_m(m, m') \varepsilon, m, m' \in U_m(c_m);
\]

see also Section 6.8 in concrete settings and the following discussion.

Until now, we do not give a higher regularity assumption on the base space, except the point-wisely \( C^0 \) continuity.

Definition 5.29 \((C^{0,1}, \text{uniform bundle})\). Let \( M \) be a uniformly locally metrizable space associated with an open cover \( \{U_m\} \) (see Definition 5.5) and \( M_1 \subset M \). Let \( d_m \) be the metric in \( U_m \). Let \( X \) be a bundle over \( M \) with a (open regular) bundle atlas \( \mathcal{A} \). Assume \( X \) has a uniform size trivialization at \( M_1 \) with respect to \( \mathcal{A} \) (see Definition 5.19). \( \mathcal{A} \) is said to be \( C^{0,1} \)-uniform around \( M_1 \), if the following hold. Choose a small \( \delta > 0 \) (in Definition 5.19), and take any bundle charts \((U_m(\delta), \varphi_m(\delta)), (U_{m_1}(\delta), \varphi^{m_1}) \) at \( m_0, m_1 \in M_1 \) respectively belonging to \( \mathcal{A} \). The transition map \( \varphi^{m_0,m_1} \) with respect to \( \mathcal{A} \), i.e.

\[
\varphi^{m_0,m_1} = (\varphi^m)^{-1} \circ \varphi^m : (W_{m_0,m_1}(\delta), d_{m_0}) \times X_{m_0} \to (W_{m_0,m_1}(\delta), d_{m_1}) \times X_{m_1},
\]

where \( W_{m_0,m_1}(\delta) = U_{m_0}(\delta) \cap U_{m_1}(\delta) \neq \emptyset \), satisfies

\[
\text{Lip} \varphi^{m_0,m_1}(\cdot) \leq \varepsilon^1 m_0,m_1(x),
\]

and \( m_0,m_1(x), m_1 \in M_1 \), are bounded at \( M_1 \) on any bounded-fiber sets (see Definition 5.26).

If \( \mathcal{A} \) is \( C^{0,1} \)-uniform around \( M_1 \) and \( X \) has an \( \varepsilon \)-almost uniform \( C^{0,1} \)-fiber trivialization at \( M_1 \) with respect to \( \mathcal{A} \), then we say \( X \) has an \( \varepsilon \)-almost \( C^{0,1} \)-uniform trivialization at \( M_1 \) with respect to \( \mathcal{A} \); in addition, if \( M_1 = M \) and \( \varepsilon = 0 \), we also call \( X \) a \( C^{0,1} \)-uniform bundle (with respect to \( \mathcal{A} \)).

By a simple computation as in Section 6.8, under above, one can show the definition of uniformly \( \theta \)-Hölder continuity of vertical part respecting the base points given in Definition 5.27 will imply a stronger form of uniformly \( \theta \)-Hölder continuity as in e.g. [PSW12, Section 8].

Lemma 5.30. Assume \((X, M, \pi_1), (Y, N, \pi_2)\) have \( \varepsilon \)-almost \( C^{0,1} \)-uniform trivializations at \( M^0_1 \), \( u(M^1_1) \) with respect to preferred \( C^{0,1} \)-uniform bundle atlases \( \mathcal{A}, \mathcal{B} \) respectively (see Definition 5.29), where
**Existence and Regularity of Invariant Graphs**

Let $M_1 \subset M$ and $M_1^{\epsilon_1}$ is the $\epsilon_1$-neighborhood of $M_1$. Suppose $f : X \to Y$ is uniformly $C^{0,1}$-fiber (see Section 5.4.4) over a map $u$ which is uniformly (locally) Lipschitz around $M_1$ (see Definition 5.4). Assume $f$ depends in a uniformly (locally) $C^{0,\theta}$ fashion on the base points around $M_1^{\epsilon_1}$ uniform for bounded-fiber sets (see Definition 5.27). Then there is a (small) $\delta > 0$, such that for any bundle charts $(U_{m_0}(\delta), \varphi^{m_0}) \in \mathcal{A}$ at $m_0 \in M_1$, $(V_{m_0}(\delta), \phi^{m_0}) \in \mathcal{B}$ at $m'_0 \in u(M_1)$, $W_{m_0,m'_0}(\delta) \triangleq U_{m_0}(\delta) \cap u^{-1}(V_{m'_0}(\delta)) \neq \emptyset$, the local representation of $f$ at $(m_0, m'_0)$,

$$(\phi^{m_0})^{-1} \circ f \circ \varphi^{m_0}(m, x) = (u(m), \tilde{f}_{m_0,m'_0}(m, x)) : (W_{m_0,m'_0}(\delta), d_{m_0}(m, m')) \times X_{m_0} \to (V_{m'_0}(\delta), d_{m'_0}^2) \times Y_{m'_0},$$

where $d_{m_0}, d_{m'_0}$ are the metrics in $U_{m_0}(\delta), V_{m'_0}(\delta)$ respectively, satisfies

$$|\tilde{f}_{m_0,m'_0}(m, x) - \tilde{f}_{m_0,m'_0}(m', x)| \leq c_{m_0,m'_0}^2(x) d_{m_0}(m, m')^{\theta}, \ m, m' \in W_{m_0,m'_0}(\delta),$$

and $c_{m_0,m'_0}, m'_0 \in u(M_1)$, are bounded at $M_1$ on any bounded-fiber sets (see Definition 5.26). Moreover, if $c_{m_0}$ (in Definition 5.27) and $c_{m_0,m'_1}$ (in Definition 5.29) are the class of functions in case (a) or (b) in Definition 5.26, so is $c_{m_0,m'_0}^2$.

For other types of uniform properties of bundle map such as uniformly $C^0$ base-regularity of $C^0$ bundle map, base-regularity of $C^1$-fiber bundle map, base-regularity of $C^1$ bundle map, and some extensions, are presented in Appendix B, which can be discussed very analogously as before.

### 5.5. manifold and foliation

#### 5.5.1. uniform manifold

Let us introduce a class of uniform manifolds, which is a generalized notion of Banach space and compact Riemannian manifold, or a Banach-manifold-like version of Riemannian manifold having bounded geometry (see Definition C.6) and uniformly regular Riemannian manifold (see Definition C.7).

Let $M$ be a $C^1$ Finsler manifold (see Appendix D.2) with Finsler metric $d$ in each component of $M$. Let $M_1 \subset M$. Suppose $M$ has a local uniform size neighborhood at $M_1$ (see Definition 5.7), i.e. there is an $\epsilon_1 < \epsilon'$ such that $\overline{U_{m_0}(\epsilon_1)} \subset U_{m_0}(\epsilon')$, $m_0 \in M_1$, where the closure is taken in the topology of $M$ and $U_{m_0}(\epsilon) = \{m' : d(m', m_0) < \epsilon\}$. Assume there is a constant $\Xi = \Xi(\epsilon') \geq 1$ such that for every $m_0 \in M_1$, there is a $C^1$ local chart $\chi_{m_0} : U_{m_0}(\epsilon') \to T_{m_0}M$ with $\chi_{m_0}(m_0) = 0$ and $D\chi_{m_0}(m_0) = id$, satisfying

$$\sup_{m' \in U_{m_0}(\epsilon')} |D\chi_{m_0}(m')| \leq \Xi, \sup_{m' \in U_{m_0}(\epsilon')} |(D\chi_{m_0}^{-1})(\chi_{m_0}(m'))| \leq \Xi.$$  \hspace{1cm} (5.2)

That is $TM$ has a $(\Xi - 1)$-almost uniform $C^{0,1}$-fiber trivialization (see Definition 5.23) at $M_1$ with respect to a $C^{0}$-canonical bundle atlas $M$ of $TM$ given by

$$M = \{(U_{m_0}(\epsilon), (id \times D\chi_{m_0}^{-1})(\chi_{m_0}(\cdot))) : m_0 \in M_1, 0 < \epsilon < \epsilon'\}.$$  

Usually, we say $M$ with $M_1$ satisfies (■) or $M$ is $\Xi$-$C^{0,1}$-uniform around $M_1$ (with respect to $M$). In addition if $\Xi \to 1$ as $\epsilon' \to 0$, we say $M$ is $C^{0,1}$-uniform around $M_1$ (with respect to $M$). This type of manifolds will be used as a natural base space in Section 7.2.

**Lemma 5.31.** For a $C^1$ Finsler manifold, if there is a local chart $\chi_{m_0} : U_{m_0}(\epsilon') \to T_{m_0}M$ such that $\chi_{m_0}(m_0) = 0$ and (5.2) holds, then there is a $\delta_{m_0} > 0$ such that $T_{m_0}M(\delta_{m_0}) \subset \chi_{m_0}(U_{m_0}(\epsilon'))$ and

$$\Xi^{-1}|x_1 - x_2| \leq d(\chi_{m_0}^{-1}(x_1), \chi_{m_0}^{-1}(x_2)) \leq \Xi|x_1 - x_2|, \ x_1, x_2 \in T_{m_0}M(\delta_{m_0}).$$

Suppose there is an $\epsilon_1 < \epsilon'$ such that $\overline{U_{m_0}(\epsilon_1)} \subset U_{m_0}(\epsilon')$, where the closure is taken in the topology of $M$, then we can take $\delta_{m_0} = \frac{\epsilon_1}{\Xi}$, and vice versa.
Definition 5.32. Let (□) hold. Let \( u : M \rightarrow M \) and \( u(M) \subset M_1 \). We say \( u \) is uniformly (locally) \( C^{0, 1} \) (resp. uniformly \( C^0 \)) around \( M_1 \), if \( M \) is considered as a locally metrizable space and \( u \) is uniformly (locally) \( C^{0, 1} \) (resp. uniformly \( C^0 \)) around \( M_1 \) in the sense of Definition 5.4. We say \( u \) is uniformly (locally) \( C^{1, 1} \) around \( M_1 \) with respect to \( M \), if \( u \) is uniformly (locally) \( C^{0, 1} \) around \( M_1 \) and \( m \mapsto Du(m) \) is uniformly (locally) \( C^{0, 1} \) around \( M_1 \) with respect to \( M \), \( M \) in the sense of Definition 5.27 (c), that is, for

\[
Du_{m_0}(m)v \triangleq D_{u(m_0)}(u(m))Du(m)D\chi_{m_0}^{-1}(\chi_m(m))v, \quad (m, v) \in U_{m_0}(\epsilon') \times T_{m_0}M,
\]

one has \( \|Du_{m_0}(m)\| \leq Cd(m, m_0) \), \( m \in U_{m_0}(\epsilon_1) \) for some small \( \epsilon_1 > 0 \) and some constant \( C > 0 \) (independent of \( m_0 \), \( m_0 \in M_1 \)). We say \( Du \) is uniformly \( C^0 \) around \( M_1 \) with respect to \( M \), if \( m \mapsto Du(m) \) is uniformly \( C^0 \) around \( M_1 \) with respect to \( M \), \( M \) in the sense of Definition B.1 (c). Similarly, if \( M \) with \( M_1 \) and \( N \) with \( N_1 \) both satisfy (□), and \( u : M \rightarrow N \) with \( u(M_1) \subset N_1 \), then we talk about \( u \) being uniformly (locally) \( C^{1, 1} \) or \( C^{0, 1} \) around \( M_1 \).

In the above definition, the assumption that \( M \) has a local uniform size neighborhood at \( M_1 \) (see Definition 5.7) can be removed.

Definition 5.33 (\( C^{0, 1} \)-uniform and \( C^{1, 1} \)-uniform manifold). (i) If \( M \) with \( M_1 = M \) satisfies (□), then we say \( M \) is a \( \Xi \)-\( C^{0, 1} \)-uniform manifold, or \( C^{0, 1} \)-uniform manifold in addition with \( \Xi \rightarrow 1 \) as \( \epsilon' \rightarrow 0 \).

(ii) We say a \( C^1 \) Finsler manifold \( M \) is \( \Xi \)-\( C^{1, 1} \)-uniform around \( M_1 \) if (□) holds and \( M \) is \( C^{0, 1} \), uniform around \( M_1 \) in sense of Definition 5.29, where \( C^{0, 1}_{m_0, m_1} \) thereof satisfies \( C^{0, 1}_{m_0, m_1}(x) \leq C|x| \) for some \( C > 0 \) independent of \( m_0, m_1 \). If \( M_1 = M \) (resp. \( \Xi \rightarrow 1 \) as \( \epsilon' \rightarrow 0 \)), then the words ‘around \( M_1 \)’ (resp. ‘\( \Xi \)-’) will be omitted.

If \( M, N \) are \( C^{1, 1} \)-uniform manifold, then a map \( u : M \rightarrow N \) being uniformly (locally) \( C^{1, 1} \) in the sense of Definition 5.32 is also \( C^{1, 1} \) in the classical sense as Lemma 5.30; see also [HPS77, Eld13].

Here, we continue to give a light more general definition \( C^{0, 1} \)-uniform manifold than Definition 5.33. (A point-wise version of \( C^{0, 1} \) manifold was also defined in [Pal66].) This type of manifolds in some cases is important as some smooth (and Lipschitz) approximation can be made to weaken the requirement of high smooth regularity of the manifold. For example, any \( C^1 \) compact embedding submanifold of a smooth (finite-dimensional) Riemannian manifold satisfies the following definition;
see [BLZ08, Theorem 6.9] for a proof of the case when the $C^1$ compact embedding submanifold is in a Banach space.

**Definition 5.34.** Let $M$ be a $C^1$ Finsler manifold (see Appendix D.2) with Finsler metric $d$ in each component of $M$. Let $M_1 \subset M$. We say $M$ is $\Xi$-$C^{0,1}$-uniform (resp. strongly $\Xi$-$C^{0,1}$-uniform) around $M_1$ if the following (a) (b)(i) (c) (resp. (a) (b)(ii) (c)) hold, where $\Xi \geq 1$.

(a) (base space). Suppose $M$ has a local uniform size neighborhood at $M_1$ (see Definition 5.7), i.e. there is an $\varepsilon_1 < \varepsilon'$ such that $U_{m_0}(\varepsilon_1) \subset U_{m_0}(\varepsilon')$, $m_0 \in M_1$, where the closure is taken in the topology of $M$ and $U_{m_0}(\varepsilon) = \{m' : d(m', m_0) < \varepsilon\}$.

(b) (approximate tangent bundle). Let $X$ be a $C^0$ vector bundle over $M$ endowed with a $C^0$ Finsler structure (see Appendix D.2). Let $\mathcal{A}$ be a (regular) $C^0$ vector bundle atlas of $X$. (i) $X$ has a $(\Xi - 1)$-almost uniform $C^{0,1}$-fiber trivialization at $M_1$ with respect to $\mathcal{A}$ (see Definition 5.23) (resp. (ii) $X$ has a $(\Xi - 1)$-almost uniform $C^{0,1}$-fiber trivialization at $M_1$ with respect to $\mathcal{A}$ (see Definition 5.29)).

(c) (compatibility) For certain $\varepsilon > 0$ and each $m_0 \in M_1$, there is a bi-Lipschitz map (local chart) $\chi_{m_0} : U_{m_0}(\varepsilon) \to X_{m_0}$ such that $\chi_{m_0}(0) = 0$, $\text{Lip } \chi_{m_0}(\cdot)_{|U_{m_0}(\varepsilon)} \leq \varepsilon$ and $\text{Lip } \chi_{m_0}^{-1}(\cdot)_{|U_{m_0}(U_{m_0}(\varepsilon))} \leq \varepsilon(\varepsilon)$, where $\chi : \mathbb{R}_+ \to \mathbb{R}_+$ and $\chi(\varepsilon) \leq \Xi - \varepsilon < 0$.

If $M_1 = M$ (resp. $\Xi = 1$), then the words ‘around $M_1$’ (resp. ‘$\Xi$-’ will be omitted.

In some sense $TM \approx X$ (if $\chi(\varepsilon) - 1 > 0$ is small). In fact, the base space $M$ can be only a uniformly locally metrizable space (see Definition 5.5) with a local uniform size neighborhood at $M_1$ (see Definition 5.7); in this case, (c) also implies $M$ is a $C^{0,1}$ manifold in the sense of [Pal66]. Also, as Lemma 5.31, there is an $\varepsilon_1 > 0$ such that $X_{m_0}(\varepsilon_1) \subset X_{m_0}(U_{m_0}(\varepsilon)), m_0 \in M_1$.

5.5.2. $\Xi$-Hölder foliation. A $C^0$ foliation $\mathcal{F}$ of a $C^1$ manifold $M$, is a division of $M$ into disjoint (immersed) submanifolds called leaves of $\mathcal{F}$ with the following properties; see also [HPS77, AMR88].

(a) Each leaf of $\mathcal{F}$ is a connected $C^1$ injectively immersed submanifold of $M$. The unique leaf through $m$ is denoted by $\mathcal{L}_m$.

(b) There are two $C^0$ subbundles $E, F$ of $TM$ such that $E \oplus F = TM$. For each $m \in M$, there is a homomorphism $\varphi_m : U_m \to T_mM$, where $U_m$ is a neighborhood of $m$ in $M$, such that $\varphi_m(m) = 0$ and $\varphi_m(\mathcal{L}^c_m) \subset E_m + y'$, where $\phi(m') = (x', y') \in E_m \times F_m, m' \in U_m$, and $\mathcal{L}^c_m$ is the component of $U_m \cap \mathcal{L}^c_m$ containing $m'$.

Particularly $T_m\mathcal{L}_m = E_m$, so we denote $T_{\mathcal{L}} = E$. $E, F$ are called tangent bundle and normal bundle to the foliation $\mathcal{F}$, respectively. $\mathcal{L}^c_m$ is called a plaque of $\mathcal{F}$ at $m$ which is an embedding submanifold of $M$. Such $\varphi_m$ is called a foliation chart at $m$. Note that $\varphi_m$ gives a local chart of the plaque at $m$, i.e. $\varphi_m : \mathcal{L}^c_m \to T_m\mathcal{L}_m = E_m$. Sometimes, we also say $M$ is ($C^0$) foliated by $\mathcal{F}$. If for each $m$, there is a $C^1$ foliation chart at $m$, then we say $\mathcal{F}$ is a $C^1$ foliation. See also [AMR88] for more details about $C^1$ foliation; note that $C^1$ foliation is extremely different from $C^0$ foliation. Similarly, if the foliation charts can be chosen locally $C^{k,\theta}$, we say $\mathcal{L}$ is locally $C^{k,\theta}$ or $M$ is $C^{k,\theta}$ foliated by $\mathcal{L}$.

Let us give some descriptions about uniform properties of a foliation. Here we employ a way which is essentially the same as in [HPS77, PSW97], but in terms of application, the plaques are represented in approximate tangent bundle.

Let $M$ be a $C^1$ Finsler manifold and $\mathcal{M}_1 \subset \mathcal{M}$. Let $M$ with $M_{\varepsilon} = \bigcup_{m_0 \in \mathcal{M}_1} U_{m_0}(\varepsilon_1)$ (for some $\varepsilon_1 > 0$) satisfy Definition 5.34 (a) (b)(i) (c) where notations thereof will be used below. Assume $X = X^c \oplus X^h$ with $X^c, X^h$ being two $C^0$ (vector) subbundles of $X$. So we have projections $\Pi_{m}^{c}, m \in M$, such that $R(\Pi_{m}^{c}) = X^c, \kappa = c, h; \Pi_{m}^{c} + \Pi_{m}^{h} = \text{id}$ and $m \mapsto \Pi_{m}^{c}$ is $C^0$. We assume $m \mapsto |\Pi_{m}^{c}|$ is $\varepsilon_0$-almost uniformly continuous around $M_{\varepsilon}^c$ (see Definition 5.3) with small $\varepsilon_0 > 0$; a special case is that $m \mapsto \Pi_{m}^{c}$ is uniformly $C^0$ around $M_{\varepsilon}^c$ (see also Remark B.4). In this case, there are two natural (vector) bundle atlases $\mathcal{A}^c, \mathcal{A}^h$ of $X^c, X^h$ induced by $\mathcal{A}$, respectively, such that $X^c$ has a $\Xi_1$-almost uniform $C^{0,1}$-fiber trivialization at $M_{\varepsilon}^c$ with respect to $\mathcal{A}^c, \kappa = c, h$; i.e. for certain $\varepsilon > 0$, if $(U_{m}(\varepsilon), \varphi^m) \in \mathcal{A}^c, then (U_{m}(\varepsilon), \kappa \varphi^m) \in \mathcal{A}^h$, where $\kappa \varphi^m(m', x) = (m', \Pi_{m'}^c \varphi^m_{m'} \Pi_{m'}^c x)$. 
The plaque at $m$ in $U_m(\epsilon)$ is denoted by $\mathcal{L}_m(\epsilon)$. To characterize the (uniformly) Hölder continuity of $\mathcal{L}$ respecting base points around $M_1$, as before, we need $\mathcal{L}_m(\epsilon)$ has a local representation with uniform size domain and the representation depends on $m$ in a Hölder fashion. Suppose for every $m_0 \in M_1^\epsilon$, there is a $C^1$ map $f_{m_0} : X^c_{m_0}(\epsilon_{m_0}) \to X^h_{m_0}$, such that $\text{Graph}f_{m_0} \subset \chi_{m_0} (\mathcal{L}_m(\epsilon'))$ and $f_{m_0}(0) = 0$; this map always exists if in some sense $X^c \approx E$ and $X^h \approx F$. The following terms will be used in Section 7.2.

- We say $\mathcal{L}$ is uniformly (locally) $(\theta)$-Hölder (respecting base points) around $M_1$ or $m \mapsto \mathcal{L}_m(\epsilon)$ is uniformly (locally) $C^{0,\theta}$ around $M_1$ (in $C^0$-topology in bounded sets), if $f_{m_0}(\cdot)$ are $C^{0,\theta}$ uniformly (locally) $C^{1}$-uniform for $m_0 \in M_1^\epsilon$ (see Section 5.4.4), $\sup_{m_0 \in M_1^\epsilon} \epsilon_{m_0} > \epsilon$, and $m_0 \mapsto f_{m_0}$ is uniformly (locally) $(\theta)$-Hölder around $M_1$ with respect to $\mathcal{A}^c, \mathcal{A}^h$ in the sense of Definition 5.27 (b). In addition if $M_1 = M$, then we say $\mathcal{L}$ is a uniformly (locally) $(\theta)$-Hölder foliation.

Since in the infinite-dimensional setting, the leaves of the ‘foliation’ might not be $C^1$, but above terminologies also make sense. We allow each leaf of $\mathcal{L}$ is only $C^{0,1}$ and the map $f_{m_0}$, the representation of the plaque at $m_0$, is only $C^{0,1}$.

- Assume each map in Definition 5.34 (c) is $C^1$. We say $m \mapsto \mathcal{L}_m(\epsilon)$ is $C^0$ (resp. uniformly $C^0$), uniformly (locally) $C^{0,\theta}$ around $M_1$ in $C^1$-topology in bounded sets if (i) $f_{m_0}(\cdot)$ are $C^{1,\theta}$ uniform for $m_0 \in M_1^\epsilon$ (see Section 5.4.4), and (ii) $m \mapsto f_{m}$ is $C^0$ (resp. uniformly $C^0$), uniformly (locally) $C^{0,\theta}$ around $M_1$ in $C^1$-topology in bounded sets (with respect to $\mathcal{A}^c, \mathcal{A}^h$) (see Definition B.5 (d)). Particularly, in this case, if we take $X = TM$ and $M$ is $C^{0,1}$-uniform (or $C^{1,1}$-uniform) around $M_1$ in the sense of Definition 5.33 with $T\mathcal{L} = E = X^c, F = X^h$, then $m \mapsto E_m : M \to \mathbb{G}(X)$ is $C^0$ (resp. uniformly $C^0$, uniformly (locally) $C^{0,\theta}$) around $M_1$.

- Assume $X$ is a $C^1$ bundle with a $C^0$ connection $C$ and also $X^c, X^h$ are $C^1$ subbundles of $X$. (So $X^c, X^h$ have natural $C^0$ connections induced by $C$). And assume the maps in Definition 5.34 (c) are $C^{1,1}$ uniform for $m_0 \in M_1^\epsilon$. We say $\mathcal{L}$ is uniformly (locally) $C^{1,\theta}$ (respecting base points) around $M_1$ if (i) it is $C^1$ and $m \mapsto \mathcal{L}_m(\epsilon)$ is uniformly (locally) $C^{0,1}$ around $M_1$ in $C^1$-topology in bounded sets, and (ii) $x \mapsto \nabla_{m_0} f_{m_0}(x)$ is $C^{0,1}$ uniform for $m \in M_1^\epsilon$ (see Section 5.4.4) and $m_0 \mapsto \nabla_{m_0} f_{m_0}(\cdot)$ is uniformly $(\theta)$-Hölder around $M_1$ with respect to $\mathcal{A}^c, \mathcal{A}^h$ in the sense of Definition B.7. In addition, if $M_1 = M$, $\mathcal{L}$ is called a uniformly (locally) $C^{1,\theta}$ foliation.

If $M$ is a $C^{1,1}$-uniform manifold (see Definition 5.33) with $TM$ being $C^{1,1}$-uniform (see Definition B.9) (resp. a strongly $C^{0,1}$-uniform manifold (see Definition 5.34)), then the definition of uniformly $C^{1,\theta}$ (resp. $C^{0,\theta}$) continuity of $\mathcal{L}$ respecting base points is classical as in e.g. [Fen77, HPS77, PSSW97]. We refer the reader to see [PSW97] for more discussions about the regularity of foliation.

Let $\mathcal{L}, \mathcal{F}$ be two $C^0$ foliations of $M$. $\mathcal{L}$ is called a subfoliation of $\mathcal{F}$ (or $\mathcal{L}$ subfoliates $\mathcal{F}$), if $\mathcal{L}_m \subset \mathcal{F}_m$ for every $m \in M$ and $T\mathcal{L}$ is a subbundle of $T\mathcal{F}$. This also means $\mathcal{F}_m$ is $(C^0)$ foliated by $\mathcal{L} \cap \mathcal{F}_m$. For convenience of writing, if $\mathcal{F}_m$ is $C^{k,\theta}$ foliated by $\mathcal{L} \cap \mathcal{F}_m$ for each $m \in M$, then we also say each leaf of $\mathcal{F}$ is $C^{k,\theta}$ foliated by $\mathcal{L}$, or $\mathcal{L}$ is a $C^{k,\theta}$ foliation inside each leaf of $\mathcal{F}$; in this case, it is equivalent to say $\mathcal{L} C^{k,\theta}$ foliates $\mathcal{F}$ when $\mathcal{F}$ is endowed with leaf topology (see Section 5.2), which now becomes a $C^{k,\theta}$ manifold locally modeled on $T_m \mathcal{F}_m$.

### 6. Regularity of Invariant Graphs

The overviews of our regularity results and the settings are given in Section 6.1 and Section 6.2 respectively, and the detailed statements of Hölderness and smoothness regularities with their proofs are presented in Section 6.3 to Section 6.7. In Section 6.8, we give a more classical way to describe the Lipschitz (or Hölder) results respecting base points if stronger assumptions are assumed. The continuity regularities of the invariant graphs are given in Section 6.9. The corresponding results for the bounded section case are stated in Section 6.10 without proofs. At the end, some generalized regularity results in a local version is given in Section 6.11.
6.1. statements of the results: invariant section case. Throughout Section 6.3 to Section 6.7, we make the following basic setting.

- Let \((X, M, \pi_1), (Y, M, \pi_2), u : M \to M, H \sim (F, G), i : M \to X \times Y\) be as in Theorem 4.1 and \(f\) the bundle map obtained in Theorem 4.1 under the conditions (ii) or (ii)' and \(i\) is an invariant section of \(H\). Since condition (ii) has already recovered condition (ii)' (if we consider \(H^{(n)}\) for large \(n\) instead of \(H\)), so in the following proofs, we only deal with the case that condition (ii) holds.
- Since we only focus on the partial hyperbolicity in the uniform sense, we add an additional assumption on the spectral condition, i.e. the functions in \((A')\) or \((A)\) (B) condition are bounded (except Lemma 6.7 which is not used).

In the following, we will show \(f\) has higher regularities, once (a) the more regularity properties of the bundle \(X \times Y\), (b) the more regularity properties of the maps \(u, i, F, G\), (c) the spectral gap condition and also (d) additionally technical assumption on the continuity of functions in \((A)\) or \((A')\) (B) condition are satisfied. We will consider the following regularity properties of \(f\): (a) the continuity of \((m, x) \mapsto f_m(x)\) (Lemma 6.25); (b) the smoothness of \(x \mapsto f_m(x)\) (Lemma 6.7); (c) the Hölder continuity of \(m \mapsto f_m(x)\) (Lemma 6.13); (d) the continuity of \((m, x) \mapsto D f_m(x)\) (Lemma 6.26 and Lemma 6.27); (e) the Hölder continuity of \(x \mapsto D f_m(x)\) (Lemma 6.11 and Lemma 6.12); (f) the Hölder continuity of \(m \mapsto D f_m(x)\) (Lemma 6.16 and Lemma 6.17); (g) the smoothness of \(m \mapsto f_m(x)\) (and so \((m, x) \mapsto f_m(x)\) (Lemma 6.18); (h) the continuity of \((m, x) \mapsto \nabla_m f_m(x)\) (Lemma 6.28 and Lemma 6.29); (i) the Hölder continuity of \(x \mapsto \nabla_m f_m(x)\) (Lemma 6.22); and finally (j) the Hölder continuity of \(m \mapsto \nabla_m f_m(x)\) (Lemma 6.23).

Since \(Graph f \subset H^{-1}Graph f\), we have the following equation,

\[
\begin{align*}
F_m(x, f_{u(m)}(x_m(x))) = x_m(x), \\
G_m(x, f_{u(m)}(x_m(x))) = f_m(x).
\end{align*}
\]

Consider its variation equations in some reasonable sense,

\[
\begin{align*}
DF_m(x, f_{u(m)}(x_m(x)))(\text{id}, K_{u(m)}^1(x_m(x)))R_m^1(x) &= R_m^1(x), \\
DG_m(x, f_{u(m)}(x_m(x)))(\text{id}, K_{u(m)}^1(x_m(x)))R_m^1(x) &= K_m^1(x),
\end{align*}
\]

and

\[
\begin{align*}
D_m F_m(x, y) + D_2 F_m(x, y)(K_{u(m)}(x_m(x))Du(m) + D f_{u(m)}(x_m(x))R_m(x)) &= R_m(x), \\
D_m G_m(x, y) + D_2 G_m(x, y)(K_{u(m)}(x_m(x))Du(m) + D f_{u(m)}(x_m(x))R_m(x)) &= K_m(x),
\end{align*}
\]

where \(y = f_{u(m)}(x_m(x))\); see (6.22) for the explicit meaning of (6.3).

**Remark 6.1** (abbreviation of spectral gap condition). To simplify our writing, we make the following abbreviations. Set \(\theta(m) = (1 - \alpha(m)\beta'(u(m)))^{-1}\) if under the condition (ii) of Theorem 4.1, and \(\theta(m) = 1\) if under the condition (ii)' of Theorem 4.1.

Let \(\lambda : M \to \mathbb{R}_+\). The notation \(\lambda < 1\) means that \(\sup_M \theta(m)\lambda(m) < 1\). Moreover, if \(\theta : M \to \mathbb{R}_+\) and \(\theta < 1\), we write \(\lambda^{\alpha\theta} < 1\) which stands for the following meanings in different settings.

1. \(\sup_m \lambda^{\alpha\theta}(m)\theta(m)\theta(m) < 1\) or \(\sup_m (\lambda^{\alpha\theta})'(m) < 1\) if \(x_m(\cdot)\) is a bounded function uniform for \(m \in M\) (or particularly when \(x_m\) is bounded uniform for \(m \in M\) in Lemma 6.11, 6.22, and with additional assumption that \(u\) is a bounded function (particularly when \(M\) is bounded) in Lemma 6.13, 6.16, 6.17 and 6.23;
2. \(\sup_m \lambda^{\alpha\theta}(m)\theta(m)\theta(m) < 1\) or \(\sup_m (\lambda^{\alpha\theta})'(m) < 1\) if \(\theta \in \mathcal{E}(u)\) or \(\alpha = 1\);
3. \(\sup_m \lambda^{\alpha\theta}(m)\theta(m)\theta(m) < 1\) otherwise.

See Definition A.1 for the meaning of the notations \(\lambda^*\) (sup Lyapunov numbers of \(\{\lambda^{(n)}\}\)) and \(\mathcal{E}(u)\).

**Additional notations:** \(A\theta, \max\{A, \theta\}\) are defined by

\[
(\lambda\theta)(m) = \lambda(m)\theta(m), \max\{A, \theta\}(m) = \max\{A(m), \theta(m)\}.
\]
Suggestion: At the first reading, the readers may think the notation $\lambda^\alpha\theta < 1$ means
\[
\sup_m \sup_{N \geq 0} \lambda(u^N(m)) \sup_{N \geq 0} \theta(u^N(m))\theta(u^N(m)) < 1;
\]
or for simplicity, assume all functions in spectral gap condition are constants.

Let’s take a quick glimpse of what our whole regularity results say in a not very sharp and general setting.

For the meaning of $C^{\alpha,\beta}$ regularity of bundle maps, see Section 5.4, where we use the local representations of bundle maps in given local bundle charts, i.e. $M, \mathcal{A}, \mathcal{B}$ in the following assumptions.

(E1) (about $M$) Let $M$ be a $C^1$ Finsler manifold and $M_1^0 \subset M$. Let $M_1$ be the $\varepsilon$-neighborhood of $M_1^0$ ($\varepsilon > 0$). Take a $C^1$ atlas $N$ of $M$. Let $M$ be the canonical bundle atlas of TM induced by $N$.

Assume $M$ is $(1 + \frac{\alpha}{\beta})C^{1,1}$-uniform around $M_1$ with respect to $M$ (see Definition 5.3) where $\lambda > 0$; see also Appendix C.0.2 for examples.

(E2) (about $X \times Y$) $(X, M, \pi_1), (Y, M, \pi_2)$ are $C^1$ bundles endowed with $C^0$ connections $C^X, C^Y$ which are uniformly (locally) Lipschitz around $M_1$ (see Definition B.8 or Remark 6.24). Take $C^{1}$ normal bundle atlases $\mathcal{A}, \mathcal{B}$ of $X, Y$ respectively. Assume $(X, M, \pi_1), (Y, M, \pi_2)$ both have $\lambda$-almost $C^{1,1}$-uniform trivializations at $M_1$ with respect to $\mathcal{A}, \mathcal{B}$ (and $M$), respectively; see Definition B.9 and also Appendix C.0.2 for examples. The fibers $X_m, Y_m$ are Banach spaces (see also Remark 6.4).

(E3) (about $i$) The section $i$ is a 0-section of $X \times Y$ with respect to $\mathcal{A} \times \mathcal{B}$, see Section 5.4.3.

(E4) (about $u$) $u : M \to M$ is a $C^1$ map such that (i) $Du(m)$ lies in $C^X, C^Y$ which are uniformly (locally) Lipschitz around $M_1$ and $\sup_{m \in M_1} |Du(m)| < \infty$; (ii) $m \mapsto |Du(m)|$ is $\lambda$-almost uniformly continuous around $M_1$ (see Definition 5.3); (iii) $u(M) \subset M_1^0$.

(E5) The functions in $(A')$ (or $(A')$ $(B)$ condition are $\lambda$-almost uniformly continuous around $M_1$ and $\lambda$-almost continuous; see Definition 5.3.

Theorem 6.2. Assume (E1) ~ (E5) hold with $\lambda > 0$ small depending on the following spectral gap conditions. Let $H : X \times Y \to X \times Y$ be a bundle correspondence over $u$ with a generating bundle map $(F, G)$. Let $f$ be given in the Theorem 4.1 when $i$ is an invariant section of $H$. Then we have the following results about the regularity of $f$. (In the following, $0 < \alpha, \beta \leq 1$.)

(1) $f_m(i_X(m)) = i_Y(m)$. If $F, G$ are continuous, so is $f$.

(2) Assume for every $m \in M, F_m(\cdot), G_m(\cdot)$ are $C^1$. Then so is $f_m(\cdot)$. Moreover, if the fiber derivatives satisfy (see Section 5.4.4) $D^0 F, D^0 G \in C^1$, so is $D^0 f$. Also, there is a unique $K^1 \in L(T^V_X, T^V_Y)$ over $f$ satisfying (6.2) and $D^0 f = K^1$.

(3) Under (2), in addition, suppose (i) $DF_m(\cdot), DG_m(\cdot)$ are $C^{0,\alpha}$ uniform for $m$, and (ii) $\lambda^\alpha\lambda^\beta \lambda^\alpha\lambda^\beta < 1$. Then $D^0 f_m(\cdot)$ is $C^{0,\gamma\alpha}$ uniform for $m$.

(4) Suppose (i) $F, G$ are uniformly (locally) $C^{0,1}$ around $M_1$ (i.e. (6.5) holds), (ii) $(\max \{\lambda^{-1}, 1\})^{\alpha\beta} \lambda^\alpha\lambda^\beta < 1$. Or suppose (i') $F, G$ are uniformly (locally) $C^{1,1}$ around $M_1$ (see Remark 6.6), (ii') $(\frac{\max \{\lambda^{-1}, 1\}}{\lambda})^{\alpha\beta} \lambda^\alpha\lambda^\beta < 1$. Then $m \mapsto f_m(x)$ is uniformly (locally) $\alpha$-Hölder around $M_1$.

(5) Suppose (i) $m \mapsto DF_m(i_X(m), i_Y(u(m)))$, $m \mapsto DG_m(i_X(m), i_Y(u(m)))$ are uniformly (locally) $C^{0,\gamma\alpha}$ around $M_1$, (ii) $\mu^{\gamma\alpha} \lambda^\alpha\lambda^\beta < 1$. Then $m \mapsto D^0 f_m(i_X(m))$ is uniformly (locally) $C^{0,\gamma\alpha}$ around $M_1$.

(6) Suppose (i) $D^0 F, D^0 G$ are uniformly (locally) $C^{0,1}$ around $M_1$ (i.e. estimates (6.6) hold), (ii) $\lambda^\beta \lambda^\alpha\mu^{\alpha\beta} < 1, \lambda^\beta \lambda^\alpha\mu^{\alpha\beta} < 1, \mu^{\alpha\beta} \lambda^\alpha\lambda^\beta < 1$. Then $m \mapsto D^0 f_m(x)$ is uniformly (locally) $\alpha \beta$-Hölder around $M_1$.

(7) Suppose (i) $F, G$ are $C^{1,1}$ around $M_1$ (see Remark 6.6) and $C^1$ in $X \times Y$, (ii) $\lambda^\beta \lambda^\alpha\lambda^\alpha\lambda^\beta < 1$. Then $f$ is $C^1$, $\nabla_m f_m(i_X(x)) = 0$ for all $m \in M_1$ and there is a constant $C$ such that $|\nabla_m f_m(x)| \leq C|x|$ for all $x \in X_m, m \in M_1$. Also, there is a unique $K \in L(T^V_X, T^V_Y)$ over $f$ satisfying (6.3) (or more precisely (6.22)) and $\nabla f = K$ (the covariant derivative of $f$, see Definition 5.13).

Moreover, if an additional gap condition holds: $\lambda^\beta \lambda^\alpha\lambda^\alpha\lambda^\beta < 1$ and $\max \{\lambda, \lambda^\alpha\lambda^\beta \lambda^\alpha\lambda^\beta \alpha\beta < 1$, then $\nabla_m f_m(\cdot)$ is locally $\alpha \beta$-Hölder uniform for $m \in M$. 


(8) Under (7), assume \(u\) is uniformly (locally) \(C^{1,1}\) around \(M_1\) (see Definition 5.32). Suppose \(\lambda_0^a \lambda_a \mu < 1, \lambda_0^a \lambda_a \mu \leq 1\). Or suppose (\(i'\)) \(TM\) has a \(0\)-almost \(C^{1,1}\), uniform trivialization at \(M_1\) with respect to \(M\), (\(ii'\)) \(F, G\) are uniformly (locally) \(C^{2,1}\) around \(M_1\), \(\lambda_0^a \lambda_a \mu < 1, \mu^{a0} \lambda_a \mu \leq 1\). Then \(m \mapsto \nabla_m f_m(x)\) is uniformly (locally) \(\alpha\)-Hölder around \(M_1\).

**Proof.** See Lemmas 6.25, 6.7, 6.11, 6.26, 6.13, 6.16, 6.17, 6.18, 6.22, 6.23, and Section 6.8. \(\square\)

### 6.2. settings for the regularity of invariant graphs: an overview

We require some basic settings about the base space \(M\) and the bundles \(X, Y\) for studying the regularity of invariant graphs. The settings will change lemma by lemma according to different regularities of \(f\). The assumptions for the invariant section \(i\) and the map \(u\) are also given here.

(H1) (about \(M\)) (i) Let \(M\) be a locally metrizable space (associated with an open cover \(\{U_m : m \in M\}\)); see Definition 5.1. The metric in \(U_m\) is denoted by \(d_m(\cdot, \cdot) = | \cdot - \cdot |\). Let \(M_1 \subset M\).
(ii) \(M\) is a \(C^1\) Finsler manifold (see Appendix D.2). The norm of \(T_M M\) will be denoted by \(| \cdot |_{T_m}\).

(H1a) Let (H1) (i) hold.

(H1b) (uniform compatibility of (i) (ii)) Let (H1) hold. There exist two constants \(\Theta_0, \Theta_2 > 0\), such that for every \(m_0 \in M_1\), there is a constant \(\epsilon_{m_0} > 0\), and a \(C^1\) local chart \(\chi_{m_0} : U_{m_0}(\epsilon_{m_0}) \to T_{m_0} M, \chi_{m_0}(m_0) = 0\), such that the norm \(| \cdot |_{m_0}\) of \(T_{m_0} M\) satisfying \(D\chi_{m_0}(m_0) = \text{id}\) and

\[
\Theta_1 d_{m_0}(m_1, m_2) \leq |\chi_{m_0}(m_1) - \chi_{m_0}(m_2)|_{m_0} \leq \Theta_2 d_{m_0}(m_1, m_2),
\]

for all \(m_1, m_2 \in U_{m_0}(\epsilon_{m_0})\).

(H1c) Let \(M\) be a \(C^1\) Finsler manifold with Finsler metric \(d\) in its components. Assume there are constants \(\Xi, \epsilon' > 0\) and \(C^1\) local charts \(\chi_{m_0} : U_{m_0}(\epsilon') \to T_{m_0} M\) at \(m_0 \in M_1\), satisfying \(D\chi_{m_0}(m_0) = \text{id}\),

\[
\sup_{m' \in U_{m_0}(\epsilon')} |D\chi_{m_0}(m')| \leq \Xi, \quad \sup_{m' \in U_{m_0}(\epsilon')} |D\chi_{m_0}^{-1}(\chi_{m_0}(m'))| \leq \Xi.
\]

Note that now (H1b) holds with \(d_m = d\) (with \(\Theta_1 = \Xi^{-1}, \Theta_2 = \Xi\)) and \(\sup_{m_0 \in M_1} \epsilon_{m_0}' > 0\).

(H2) (about \(X \times Y\)) Let \(\epsilon > 0\) (might be \(\epsilon_2 = \infty\)) and

\[
\mathcal{A} = \{(U_{m_0}(\epsilon), \varphi_{m_0}) : \varphi_{m_0} = \text{id}, m_0 \in M_1, 0 < \epsilon \leq \epsilon_2\}, \mathcal{B} = \{(U_{m_0}(\epsilon), \phi_{m_0}) : \phi_{m_0} = \text{id}, m_0 \in M_1, 0 < \epsilon \leq \epsilon_2\},
\]

where \(\mathcal{A}', \mathcal{B}'\) are bundle atlases of \(X, Y\) at \(M_1\) respectively.

(i) The fibers \(X_m, Y_m\) of the bundles \(X, Y\) are Banach spaces (see also Remark 6.4).

(ii) \(X, Y\) are \(C^1\) bundles with \(C^0\) connections \(C^X, C^Y\) respectively (see also Section 5.3), \(\mathcal{A}', \mathcal{B}'\) are \(C^1\) and \(\mathcal{A}, \mathcal{B}\) are normal with respect to \(C^X, C^Y\) respectively (see Definition 5.15).

(iii) \(X, Y\) are \(C^0\) topology bundles with \(C^0\) bundle atlases \(\mathcal{A}', \mathcal{B}'\) (see e.g. Example 5.17 (b)).

(iv) \(X, Y\) are \(C^1\) topology bundles with \(C^1\)-fiber bundle atlases \(\mathcal{A}', \mathcal{B}'\); see Definition 5.22.

(v) Let \(\epsilon > 0\) be small (depending on the following spectral gap conditions). (1) \(X, Y\) both have \(\epsilon\)-almost uniform \(C^{0,1}\)-fiber trivializations at \(M_1\) with respect to \(\mathcal{A}, \mathcal{B}\), respectively; (2) \(X, Y\) both have \(\epsilon\)-almost uniform \(C^{1,1}\)-fiber trivializations at \(M_1\) with respect to \(\mathcal{A}, \mathcal{B}\), respectively; (1') \(X, Y\) both have \(\epsilon\)-almost local \(C^{0,1}\)-fiber trivializations with respect to \(\mathcal{A}', \mathcal{B}'\), respectively; (2') \(X, Y\) both have \(\epsilon\)-almost local \(C^{1,1}\)-fiber trivializations with respect to \(\mathcal{A}', \mathcal{B}'\), respectively. See Section 5.4.5.

(H2a) Let (H2) (v)(1) hold.

(H2b) Let (H2) (i) (v)(2) hold.

(H2c) Let (H2) (i) (ii) (v)(1) hold.

(H2d) Let (H2) (i) (ii) (v)(2) hold.
Remark 6.3. (a) For (H1a), see also Section 5.1; we emphasize again no requirement that (H4b) is a Banach space which loses some generalities, is needed more explanations. This can be weakened in (H3') i : M → X × Y is continuous.

(H4) (about u) Assume u(M) ⊂ M₁, u : M → M is continuous.

(H4') u : M → M is uniformly continuous around M₁; see Definition 5.3.

(H4a) (Lipschitz continuity of u) Let (H4) and (H1a) hold. There exist an ε₁ > 0 (might be ε₁ = ∞) and a function μ : M → ℝ₊ such that supₘ₀∈M₁ μ(m₀) ≡ μ̃ < ∞ and for every m₀ ∈ M₁, u(μₘ⁻¹(m₀)ε₁) ⊂ Uₘ₀(μₘ⁻¹(m₀)ε₁),

\[ d_{μₘ₀}(u(m₀), u(m₀)) ≤ \mu(m₀)d_{m₀}(m₀, m₀), \quad m ∈ Uₘ₀(μₘ⁻¹(m₀)ε₁). \]

(H4b) (smoothness of u) Let (H4a) and (H1) hold. Assume u is C¹ and |Du(m₀)| ≤ μ(m₀) for every \( m₀ ∈ M₁ \), where |Du(m)| is defined by

\[ |Du(m)| = \sup\{|Du(m)x|_{[u(m)]} : x ∈ TₘM, |x|ₘ ≤ 1\}. \]

(H4c) (Lipschitz continuity of Du) Let (H4b) and (H1c) hold. Du is uniformly (locally) Lipschitz around M₁ in the sense of Definition 5.32; that is, for the local representation

\[ \overline{Du}_{m₀}(m)v \triangleq D_X{u}_{m₀}(u(m))Du(m)D_{X{m₀}}(χ_{m₀}(m))v, \quad (m, v) ∈ Uₘ₀(ε₁') × TₘM, \]

one has \[ ||\overline{Du}_{m₀}(m) - Du(m)|| ≤ C₀d(m₀, m₀), \quad m ∈ Uₘ₀(ε₁'), \quad m₀ ∈ M₁, \]

for some small \( ε₁' < \mũ⁻¹ε₁ \) and some constant \( C₀ > 0 \) (independent of \( m₀ \)).

(H5) The functions in (A') or (A) (B) condition are \( ε \)-almost uniformly continuous around \( M₁ \) and \( ε \)-almost continuous in \( M \), where \( ε > 0 \) is small (depending on the following spectral gap conditions); see Definition 5.3.

(H5') The functions in (A') or (A) (B) condition are \( ε \)-almost continuous in \( M \), where \( ε > 0 \) is small (depending on the following spectral gap conditions); see Definition 5.3.

We give some comments about above settings.

Remark 6.3. (a) For (H1a), see also Section 5.1; we emphasis again no requirement that \( M \) admits a metric is made. For (H1c), see also Section 5.5.1 and Appendix C for some examples; here unlike the assumption (■) in page 49, we do not assume \( M \) has a local uniform size neighborhood at \( M₁ \) (see Definition 5.7).

(b) For (H2), \( X \times Y \) has a natural bundle atlas at \( M₁ \), i.e. \( \mathcal{A} × \mathcal{B} \triangleq \{(Uₘ₀(ε), φₘ₀ × ϕₘ₀) : m₀ ∈ M₁, 0 < ε ≤ ε₂\} \). If (H2a) (resp. (H2b)) holds, then \( X \times Y \) also has an \( ε \)-almost uniform \( C¹⁻¹ \), resp. \( C¹⁻¹ \)-fiber trivialization at \( M₁ \) with respect to \( \mathcal{A} × \mathcal{B} ; \) so is \( X \otimes_u Y \) (see (2.2)) with respect to \( \mathcal{A} \otimes_u \mathcal{B} \triangleq \{(Uₘ₀(ε), φₘ₀ × ϕₘ₀) : m₀ ∈ M₁, 0 < ε ≤ ε₂\} \) if in addition (H4a) holds. If (H2) (ii) holds and \( X \times Y \) is equipped product connection \( C^X × C^Y \), then \( \mathcal{A} × \mathcal{B} \) is also normal at \( M₁ \) with respect \( C^X × C^Y \).

(c) For (H4b), note that from the assumptions (H1b) and (H4a), we have \[ |Du(m₀)| ≤ \frac{ε₁}{ε₂}μ(m₀). \]

So (H4b) really only says \( u ∈ C¹ \) if we use \( \frac{ε₁}{ε₂}μ \) instead of \( μ \) to characterize the ‘spectral gap condition’ in the following subsections. If one chooses a ‘better’ metric in \( Uₘ₀ \), then \( \frac{ε₁}{ε₂} \) could be sufficiently close to 1 (or equal to 1). For example, in many applications, we take \[ d_{m₀}(m₁, m₂) = |χₘ₀(m₁) - χₘ₀(m₂)|_{m₀}. \] But this is not a ‘global’ defined metric. Another choice is the Finsler metric (see Appendix D.3). Furthermore, if \( M \) is a Finsler manifold in the sense of Palais (see Appendix D.2), then the Finsler metric in each component of \( M \) is a length metric (see Appendix D.3) and \[ |Du(m₀)| ≤ μ(m₀). \]

(d) For (H5) (H5'), we mention that the functions might be even not continuous.

Remark 6.4 (about (H2) (i)). The assumption (H2) (i), i.e. the fibers \( Xₘ₀Yₘ₀ \) of the bundles \( X \), \( Y \) are Banach spaces which loses some generalities, is needed more explanations. This can be weakened in
some sense but can not be just that the fibers of $X$, $Y$ being paracompact $C^1$ Banach manifolds. We need this assumption at least for two purposes. (i) We need that $(DF(z), DG(z))$ satisfies (A) (or (A')) (B) condition which is implied by $(F, G)$ satisfies (A) (or (A')) (B) condition (see Lemma 3.10). (ii) Some bounded geometry property of the fibers (see e.g. Appendix C.0.2) are needed in order to give some uniform estimates.

For (i), due to the local result Lemma 3.11, one evidently has that $(DF(z), DG(z))$ satisfies (A) (or (A')) (B) condition once $(F, G)$ satisfies (A) (or (A')) (B) condition with a restriction on the angle condition, and $X_m, Y_m$ are complete $C^1$ Finsler manifolds in the sense of Palais (see Appendix D.2) with Finsler metrics (note the in this case Lemma D.7 holds which we used in the proof of Lemma 3.10). If the metrics in the fibers are not length metrics, one can assume directly that $(DF(z), DG(z))$ satisfies (A) (or (A')) (B) condition or use a stronger Lipschitz condition instead of (A) (or (A')) (B) condition (see e.g. Lemma 3.4 and Lemma 3.5), which is similar as we do for $u$ (by separate assumptions (H4a) and (H4b)). For (ii), if $X_m, Y_m$ are complete Riemannian manifolds having bounded geometry (see Appendix C.0.2), then the results in this section really hold (but the statements are more delicate); another generalization to the Banach-manifold-like setting is as we do for $M$ (for example $X_m, Y_m$ are complete connected $C^1$ Finsler manifolds which are $C^{1,1}$-uniform, see Definition 5.33).

In this paper, we do not give a detailed analysis of the generalization of (H2) (i). All the main ideas have been given here as we deal with the generalization of the base space.

**Remark 6.5** (a few different generalizations). (a) There is a simple case about the Hölder continuity of $m \mapsto f_m(x), K^1_m(x), K_m(x)$ with ‘better’ spectral gap conditions, that is, there is an $\varepsilon_1 > 0$ (included $\varepsilon_1 = \infty$) such that for any $m_0 \in M_1, (U_{m_0}(\varepsilon_1), \varphi^{m_0}) \in \mathcal{A}, (U_{m_0}(\varepsilon_1), \phi^{m_0}) \in \mathcal{B}$ (in (H2)) and $u(U_{m_0}(\varepsilon_1)) \subset U_{u(m_0)}(\varepsilon_1)$ (in (H4a)); see Remark A.11. This situation will appear, e.g. in the trivial bundles; see also Theorem 7.4 and Theorem 7.12, Theorem 7.13.

(b) In fact the regularity results do not depend on the existence results; see Section 6.11.

**Remark 6.6** (uniformly $C^{k,\gamma}$ continuity of $F, G$). Let $\tilde{F}_{m_0}(\cdot, \cdot), \tilde{G}_{m_0}(\cdot, \cdot)$ be the (regular) local representations of $F, G$ at $m_0 \in M_1$ with respect to $\mathcal{A}, \mathcal{B}$ under (H1a) (H2a) (H4a); see (6.9) or Definition 5.18. Also write $\tilde{D}_m F_{m_0}(\cdot, \cdot), \tilde{D}_m G_{m_0}(\cdot, \cdot)$ as the (regular) local representations of $\nabla F, \nabla G$ at $m_0 \in M_1$ with respect to $\mathcal{A}, \mathcal{B}, M$ under (H1c) (H2d) (H4b) and $F, G$ being $C^1$; see (6.40) or Appendix B.0.3. For convenience of our writing, we use the following notions. Consider the estimates below:

\[
\max \left\{ |\tilde{F}_{m_0}(m_1, z) - \tilde{F}_{m_0}(m_0, z)|, |\tilde{G}_{m_0}(m_1, z) - \tilde{G}_{m_0}(m_0, z)| \right\} \leq M_0 |m_1 - m_0|, 
\]

\[
\max \left\{ |D_z \tilde{F}_{m_0}(m_1, z) - D_z \tilde{F}_{m_0}(m_0, z)|, |D_z \tilde{G}_{m_0}(m_1, z) - D_z \tilde{G}_{m_0}(m_0, z)| \right\} \leq M_0 |m_1 - m_0|, 
\]

\[
\max \left\{ |DF(m_1) - DF(z_2)|, |DG(m_1) - DG(z_2)| \right\} \leq M_0 |z_1 - z_2|, 
\]

for all $m_1 \in U_{m_0}(\varepsilon'), z \in X_{m_0} \times Y_{m_0}(m), m_0 \in M_1, m \in M$, where $\varepsilon' > 0$ is small and $M_0 > 0$ independent of $m_0 \in M_1$. Note that we have assumed $F, G$ are uniformly $C^{0,1}$-fiber (see Section 5.4.4). (i) Under (H1a) (H2a) (H4a), we say $F, G$ are uniformly (locally) $C^{0,1}$ around $M_1$ if estimate (6.5) holds. (ii) Under (H1a) (H2b) (H4a), we say $D^* F, D^* G$ are uniformly (locally) $C^{0,1}$ around $M_1$ if estimates (6.6) hold. (iii) Under (H1c) (H2d) (H4b), we say $\nabla F, \nabla G$ are uniformly (locally) $C^{0,1}$ around $M_1$ if estimates (6.7) hold. (iv) Under (H1c) (H2d) (H4b), we say $F, G$ are uniformly (locally) $C^{1,1}$ around $M_1$ if estimates (6.5) ~ (6.7) hold. (v) Similarly, $C^{0,\gamma}$ and $C^{1,\gamma}$ can be defined.

In the following, the constants $M_0, C$ are independent of $m \in M$ or $m_0 \in M_1$. Also in the proofs, the constant $\tilde{C}$ will differ line by line, but it is independent of $m \in M$ or $m_0 \in M_1$.
6.3. **smooth leaf: $C^{k,a}$ continuity of $x \mapsto f_m(x)$.**

6.3.1. **$C^1$ leaf: smoothness of $x \mapsto f_m(x)$.**

**Lemma 6.7.** Assume the following conditions hold.

(a) The fibers of the bundles $X,Y$ are Banach spaces (see also Remark 6.4).
(b) For every $m \in M$, $F_m(\cdot), G_m(\cdot)$ are $C^1$.
(c) (spectral gap condition) $\lambda_s \lambda_u < 1$; see Remark 6.1.

Then we have the following conclusions.

1. There exists a unique vector bundle map $K^1 \in L(Y^V, Y^V)$ (see (5.1)) over $f$, such that $|K^1_m(x)| \leq \beta'(m)$, $\forall m \in M$ and it satisfies (6.2).
2. ($C^1$ smooth leaf) For every $m \in M$, $f_m(\cdot) : X_m \to Y_m$ is $C^1$, and $Df_m(x) = K^1_m(x)$ (i.e. $D^v f = K^1$).

Hereafter for $K^1 \in L_f(Y^V, Y^V)$, we write $K^1_m(x) = K^1_{(m,x)}$.

**Proof.** Define a metric space

$$E^L_1 \triangleq \{ K^1 \in L(Y^V, Y^V) \text{ is a vector bundle map over } f : |K^1_m(x)| \leq \beta'(m), \forall x \in X_m, x \mapsto K^1_m(x) \text{ is continuous, } m \in M \},$$

with a metric

$$d_1(K^1, K^1') \triangleq \sup_{m \in M} \sup_{x \in X_m} |K^1_m(x) - K^1'_m(x)|.$$

Note that $\sup_m \beta(m) \leq \hat{\beta}$. So $d_1$ is well defined and $(E^L_1, d_1) \neq \emptyset$ is complete.

Also, note that $\forall (x,y) \in X_m \times Y_{u(m)}$, $(Df_m(x,y), DG_m(x,y))$ satisfies ($A'(\alpha(m), \lambda_u(m))$, (B) ($\beta'(u(m)); \beta'(m), \lambda_s(m)$) condition), when $F_m, G_m$ satisfies ($A'(\alpha(m), \lambda_u(m))$ (B) ($\beta'(u(m)); \beta'(m), \lambda_s(m)$) condition (here $\beta'(u(m)) \leq \beta(m)$); see e.g. Lemma 3.10.

Since $\alpha(m) \beta'(u(m)) < 1$, given a $K^1 \in E^L_1$, there is a unique $R^1_m(x) \in L(X_m, X_{u(m)})$ satisfying

$$DF_m(x, f_{lm}(x_m))(\text{id}, K^1_{u(m)}(x_m))R^1_m(x) = R^1_m(x),$$

and $|R^1_m(x)| \leq \lambda_s(m)$ (by (B) condition). Moreover, $x \mapsto R^1_m(x)$ is continuous; see e.g. Lemma D.2.

Now let

$$\overline{K}^1_m(x) \triangleq DG_m(x, f_{lm}(x_m))(\text{id}, K^1_{u(m)}(x_m))R^1_m(x).$$

By (B) condition, we see $\overline{K}^1 \in E^L_1$.

Consider the graph transform $\Gamma^1 : E^L_1 \to E^L_1, \overline{K}^1 \mapsto \overline{K}^1$.

**Sublemma 6.8.** $\Gamma^1$ is Lipschitz, and $\text{Lip} \Gamma^1 \leq \sup_m \frac{\lambda_s(m) \lambda_u(m)}{1 - \alpha(m) \beta'(u(m))} < 1$.

**Proof.** Let $\overline{K}^1 = \Gamma^1 K^1$, $\overline{K}^1' = \Gamma^1 K^1'$. By Lemma 4.9, we have

$$|\overline{K}^1_m(x) - \overline{K}^1'_m(x)| \leq \frac{\lambda_u(m)}{1 - \alpha(m) \beta'(u(m))} |K^1_{u(m)}(x_m)R^1_m(x) - K^1'_{u(m)}(x_m)R^1_m(x)|$$

$$\leq \frac{\lambda_s(m) \lambda_u(m)}{1 - \alpha(m) \beta'(u(m))} |K^1_{u(m)}(x_m) - K^1'_{u(m)}(x_m)|,$$

completing the proof.

Therefore there is a unique $K^1 \in E^L_1$ satisfying (6.2). In the following, we will show $Df_m(x) = K^1_m(x)$. Set

$$Q(m,x',x) \triangleq f_m(x') - f_m(x) - K^1_m(x)(x' - x).$$
Sublemma 6.9. \(\sup_m \sup_{x \in X_m} \limsup_{x' \to x} \frac{|Q(m, x', x)|}{|x' - x|} < \infty\), and
\[
\sup_{x \in X_m} \limsup_{x' \to x} \frac{|Q(m, x', x)|}{|x' - x|} \leq \frac{\lambda_s(m)\lambda_u(m)}{1 - \alpha(m)\beta'(u(m))} \sup_{x \in X_{m}(i_u(m))} \limsup_{x' \to x} \frac{|Q(u(m), x', x)|}{|x' - x|}.
\]

Proof. The first inequality is apparently true. Let us consider the second inequality.
We use the notation \(|g_m(x', x)| \leq O_m(1)\), if \(\limsup_{x' \to x} |g_m(x', x)| = 0\). Compute
\[
|x_m(x') - x_m(x) - R^1_m(x)(x' - x)| = |F_m(x', f_{i_u(m)}(x_m(x'))) - F_m(x, f_{i_u(m)}(x_m(x))) - DF_m(x, f_{i_u(m)}(x_m(x)))| \left| (x' - x, f_{i_u(m)}(x_m(x'))) - f_{i_u(m)}(x_m(x)) \right| + |D^2 F_m(x, f_{i_u(m)}(x_m(x)))| \left| (f_{i_u(m)}(x_m(x')) - f_{i_u(m)}(x_m(x)) - R^1_{i_u(m)}(x_m(x))) - R^1_m(x)(x' - x) \right|
\leq O_m(1)|x' - x| + \alpha(m)|Q(u(m), x_m(x'), x_m(x))| + \alpha(m)\beta'(u(m))|x_m(x') - x_m(x) - R^1_m(x)(x' - x)|.
\]
Similarly,
\[
|Q(m, x', x)| \leq O_m(1)|x' - x| + \lambda_u(m)|Q(u(m), x_m(x'), x_m(x))| + \lambda_u(m)\beta'(u(m))|x_m(x') - x_m(x) - R^1_m(x)(x' - x)| \leq O_m(1)|x' - x| + \frac{\lambda_u(m)}{1 - \alpha(m)\beta'(u(m))} |Q(u(m), x_m(x'), x_m(x))|.
\]
Thus,
\[
\sup_{x \in X_m} \limsup_{x' \to x} \frac{|Q(m, x', x)|}{|x' - x|} \leq \frac{\lambda_u(m)}{1 - \alpha(m)\beta'(u(m))} \sup_{x \in X_m} \limsup_{x' \to x} \frac{|Q(u(m), x_m(x'), x_m(x))|}{|x_m(x') - x_m(x)|} \frac{|x_m(x') - x_m(x)|}{|x' - x|} \leq \frac{\lambda_u(m)}{1 - \alpha(m)\beta'(u(m))} \sup_{x \in X_{m}(i_u(m))} \limsup_{x' \to x} \frac{|Q(u(m), x', x)|}{|x' - x|}.
\]
The proof is complete. \(\square\)

Using the above sublemma and \(\theta \triangleq \sup_m \frac{\lambda_s(m)\lambda_u(m)}{1 - \alpha(m)\beta'(u(m))} < 1\), we have
\[
\sup_m \sup_{x \in X_m} \limsup_{x' \to x} \frac{|Q(m, x', x)|}{|x' - x|} \leq \sup_m \frac{\lambda_s(m)\lambda_u(m)}{1 - \alpha(m)\beta'(u(m))} \sup_{x \in X_{m}(i_u(m))} \limsup_{x' \to x} \frac{|Q(u(m), x', x)|}{|x' - x|} \leq \theta \sup_m \sup_{x \in X_m} \limsup_{x' \to x} \frac{|Q(m, x', x)|}{|x' - x|} < \infty,
\]
yielding \(\lim_{x' \to x} \frac{|Q(m, x', x)|}{|x' - x|} = 0\). The proof is finished. \(\square\)

Remark 6.10. A careful examination of the above proof, we also have that if \(F_m, G_m\) are only differentiable at \((i_X, m), i_Y(u(m)))\), then \(f_m\) is differentiable at \(i_X(m)\); see also Section 6.11.

6.3.2. \(C^{1, \alpha}\) leaf: Hölderiness of \(x \mapsto DF_m(x) = K_m(x)\).

Lemma 6.11. Assume the conditions in Lemma 6.7 hold. In addition,
(a) \(DF_m(\cdot), DG_m(\cdot)\) are \(C^{0, \gamma}\) uniform for \(m \in M\), i.e.
\[
|DF_m(z_1) - DF_m(z_2)| \leq M_0|z_1 - z_2|^\gamma, \ |DG_m(z_1) - DG_m(z_2)| \leq M_0|z_1 - z_2|^\gamma,
\]
for all \(z_1, z_2 \in X_m \times Y_{i_u(m)}\) and \(m \in M\), where \(0 < \gamma \leq 1\).
(b) (spectral gap condition) \(\lambda_s, \lambda_u < 1, \lambda_s \alpha \lambda_u < 1\), where \(0 < \alpha \leq 1\); see Remark 6.1.

Then we have \(K'_m(\cdot)\) is \(C^{0, \gamma} \alpha\) uniform for \(m\), i.e. \(|K^1_m(x_1) - K^1_m(x_2)| \leq C|x_1 - x_2|^\gamma\alpha\), \(\forall x_1, x_2 \in X_m\). In particular, if \(F_m(\cdot), G_m(\cdot) \in C^{0,1}\) uniform for \(m\), \(\lambda_s, \lambda_u < 1, \lambda_s^2 \lambda_u < 1\), then \(K^1_m(\cdot) \in C^{0,1}\) uniform for \(m\).

Proof. Since \(K^1\) satisfies (6.2), we have

\[
|K^1_m(x_1) - K^1_m(x_2)| \leq |DG_m(x_1, f_{i(m)}(x_m(x_1)))(id, K^1_{i(m)}(x_m(x_1)))R^1_m(x_1)) - DG_m(x_2, f_{i(m)}(x_m(x_2)))(id, K^1_{i(m)}(x_m(x_2)))R^1_m(x_2))|
\]

Thus,

\[
|K^1_m(x_1) - K^1_m(x_2)| \leq C|x_1 - x_2|^\gamma + \alpha(m)|K^1_{i(m)}(x_m(x_1)) - K^1_{i(m)}(x_m(x_2))| + \alpha(m)\beta(u(m))|R^1_m(x_1) - R^1_m(x_2)|.
\]

Similarly,

\[
|R^1_m(x_1) - R^1_m(x_2)| \leq C|x_1 - x_2|^\gamma + \alpha(m)|K^1_{i(m)}(x_m(x_1)) - K^1_{i(m)}(x_m(x_2))| + \alpha(m)\beta(u(m))|R^1_m(x_1) - R^1_m(x_2)|.
\]

Thus,

\[
(6.8) \quad |K^1_m(x_1) - K^1_m(x_2)| \leq C|x_1 - x_2|^\gamma + \frac{\lambda_s(m)\lambda_u(m)}{1 - \alpha(m)\beta(u(m))}|K^1_{i(m)}(x_m(x_1)) - K^1_{i(m)}(x_m(x_2))|.
\]

Now using the argument in Appendix A (see Remark A.10 (a)), we obtain the result. \(\square\)

6.3.3. \(C^k\) leaf: higher order smoothness of \(x \mapsto f_m(x)\).

Lemma 6.12. Under Lemma 6.7 and further assume \(F, G\) are \(C^k\)-fiber and uniformly \(C^{k-1,1}\)-fiber (see Section 5.4.6) and \(\lambda^k_s\lambda_u < 1\), then \(f\) is \(C^k\)-fiber and uniformly \(C^{k-1,1}\)-fiber.

Proof. (Sketch.) The proof is essentially the same as the \(C^1\)-fiber case by induction (note that we also have \(\lambda^k_i\lambda_u < 1\), \(i = 1, 2, \cdots, k\)). We give a sketch here. We use the notation \(L^k(Z_1, Z_2) \triangleq L(Z_1 \times \cdots \times Z_1, Z_2)\), if \(Z_i\) are vector bundles over \(M_i\), \(i = 1, 2\). Assume Lemma 6.12 holds for \(k - 1\)

\((k \geq 2)\). Taking \(k\)-th order derivative of (6.1) with respect to \(x\) informally, one has

\[
\begin{align*}
W^k_{1, m}(x) + D_2 F_m(x, y) \left( K^k_{i(m)}(x_m(x))(R^k_m(x))(x_m(x))R^k_m(x) \right) &= R^k_m(x), \\
W^k_{2, m}(x) + D_2 G_m(x, y) \left( K^k_{i(m)}(x_m(x))(R^k_m(x))(x_m(x))R^k_m(x) \right) &= R^k_m(x),
\end{align*}
\]

where \(y = f_{i(m)}(x_m(x))\), \((R^k_m(x)) = (R^k_m(x), \cdots, R^k_m(x))\) \((k\) components\), \(K^k \in L^k(T^u_X, T^u_Y)\) over \(f\), \(R^k \in L^k(T^u_X, T^u_Y)\) over \(x_{i,j}(\cdot)\). \(W^k_{i, m}, i = 1, 2\), consist of a finite sum of terms which can be explicitly calculated with the help of Faà Di Bruno formula (see e.g. [MR09, FdlLM06]); the non-constant factors are \(D^j F_m, D^j G_m\) \((1 \leq j \leq k)\), \(D^j f_m, D^j g_m\) \((1 \leq j \leq k)\). Note that \(W^k_{i, m}, i = 1, 2\), are bounded uniform for \(m \in M\) by induction. Since \(\lambda^k_s\lambda_u < 1\), there exists a unique \(K^k \in L^k(T^u_X, T^u_Y)\) over \(f\) satisfying (\(\dagger\)) (see Sublemma 6.8 for a similar proof). One can further show \(D^k f_m(x) = K^k_m(x)\) by an analogous argument as in Sublemma 6.9. \(\square\)
6.4. H"older vertical part: H"olderness of $m \mapsto f_m(x)$. In order to make sense of the H"older continuity of $m \mapsto f_m(x)$, we need some (uniform) assumptions on the bundle $X \times Y$ and the base space $M$. The natural settings are (H1) (H2a) (H3) (H4a) (H5).

Regard $F : X \otimes_u Y \to X$ as a bundle map over $u$ and $G : X \otimes_u Y \to Y$ as a bundle map over id; also $f : X \to Y$ over id, $x_i(\cdot) : X \to X$ over $u$.

Consider the local representations of $F, G, f, x_i(\cdot)$ at $m_0 \in M_1$ with respect to $\mathcal{A}, \mathcal{B}$, i.e.:

\[
\begin{align*}
\widehat{F}_{m_0}(m, x, y) &\triangleq (\phi^{u(m_0)}_{\mu_0})^{-1} \circ F_m \circ (\phi^m_{\mu_0}(x), \phi^m_{\mu_0}(y)) : \\
\widehat{G}_{m_0}(m, x, y) &\triangleq (\phi^{m_0}_{\mu_0})^{-1} \circ G_m \circ (\phi^m_{\mu_0}(x), \phi^m_{\mu_0}(y)) : \\
\widehat{f}_{m_0}(m, x) &\triangleq (\phi^{m_0}_{\mu_0})^{-1} \circ f_m \circ \phi^m_{\mu_0}(x) : U_{m_0}(\mu^{-1}(m_0) e_1) \times X_{m_0} \to Y_{m_0}, \\
\widehat{x}_{m_0}(m, x) &\triangleq (\phi^{u(m_0)}_{\mu_0})^{-1} \circ x_m \circ \phi^m_{\mu_0}(x) : U_{m_0}(\mu^{-1}(m_0) e_1) \times X_{m_0} \to X_{u(m_0)}.
\end{align*}
\]

Then we have

\[
\begin{align*}
\widehat{F}_{m_0}(m, x, \widehat{f}_{u(m_0)}(u(m), \widehat{x}_{m_0}(m, x))) = \widehat{x}_{m_0}(m, x), \\
\widehat{G}_{m_0}(m, x, \widehat{f}_{u(m_0)}(u(m), \widehat{x}_{m_0}(m, x))) = \widehat{f}_{m_0}(m, x),
\end{align*}
\]

for $m \in U_{m_0}(\mu^{-1} e_1), x \in X_{m_0}, m_0 \in M_1$.

If $i$ is an invariant section of $H$ and also a 0-section of $X \times Y$, then

\[
\begin{align*}
\widehat{F}_{m_0}(m_0, i_X(m_0)) = i_Y(m_0), \\
\widehat{x}_{m_0}(m_0, i_X(m_0)) = i_X(u(m_0)),
\end{align*}
\]

and also

\[
\begin{align*}
\widehat{F}_{m_0}(m_0, i_X(m_0), i_Y(u(m_0))) = i_Y(u(m_0)), \\
\widehat{G}_{m_0}(m_0, i_X(m_0), i_Y(u(m_0))) = i_Y(m_0).
\end{align*}
\]

Hereafter if $m_1 \in U_{m_0}$, then $|m_1 - m_0|$ means the metric between $m_1, m_0$ in $U_{m_0}$, i.e. $|m_1 - m_0| = d_{m_0}(m_1, m_0)$, where $d_{m_0}$ is the metric in $U_{m_0}$ (see (H1) (i)).

**Lemma 6.13.** Assume the following conditions hold.

(a) Let (H1a) (H2a) (H3) (H4a) (H5) hold.

(b) (about $F, G$) $F, G$ satisfy the following estimates:

\[
\begin{align*}
|\widehat{F}_{m_0}(m_1, z) - \widehat{F}_{m_0}(m_0, z)| &\leq M_0 |m_1 - m_0|^\gamma |z|^{\zeta}, \\
|\widehat{G}_{m_0}(m_1, z) - \widehat{G}_{m_0}(m_0, z)| &\leq M_0 |m_1 - m_0|^\gamma |z|^{\zeta},
\end{align*}
\]

for all $m_1 \in U_{m_0}(\mu^{-1}(m_0) e_1), z \in X_{m_0} \times Y_{u(m_0)}, m_0 \in M_1$, where $0 < \gamma \leq 1$, $\zeta \geq 0$.

(c) (spectral gap condition) $\lambda_s \lambda_u < 1$, $(\max\{\lambda^{-1}_{s-1}, 1\})^{\mu} \lambda_s \lambda_u < 1$, where $0 < \alpha \leq 1$; see Remark 6.1.

If $e_1' \leq \mu^{-1} e_1$ is small, then we have the following.

\[
|\widehat{f}_{m_0}(m_1, x) - \widehat{f}_{m_0}(m_0, x)| \leq C |m_1 - m_0|^\gamma |x|^{\zeta + 1 - \alpha},
\]

for every $m_1 \in U_{m_0}(e_1'), x \in X_{m_0}, m_0 \in M_1$ under $|m_1 - m_0|^\gamma |x|^{\zeta} \leq \hat{\gamma} \min\{|x|, |x|^{\zeta_c^{1-c-1}}\}$, where the constant $C$ depends on the constant $\hat{\gamma} > 0$ but not $m_0 \in M_1, c > 1$ and $\hat{\gamma}$ does not depend on $m_0 \in M_1$.

In particular, if $\gamma = \zeta = 1$, and $\lambda_s \lambda_u < 1, \lambda_s \lambda_u \mu < 1$, then

\[
|\widehat{f}_{m_0}(m_1, x) - \widehat{f}_{m_0}(m_0, x)| \leq C |m_1 - m_0| |x|.
\]

See also Section 6.8 and Remark 5.28 (c).

**Remark 6.14.** (about the condition on $F, G$.) Consider some cases that the condition on $F, G$ can be satisfied. First note that under (H1c) (H2d) (H3), if (6.5) and the following estimates hold,

\[
|\nabla_m \phi_{m_0}^m(x)| \leq M_0 |x|^{\zeta}, |\nabla_m \phi_{m_0}^{u(m)}(x)| \leq M_0 |x|^{\zeta}, \text{ and } |\nabla_m F_m(z)| \leq M_0 |z|^{\zeta}, |\nabla_m G_m(z)| \leq M_0 |z|^{\zeta},
\]

...
Proof of Lemma 6.13. Now under the assumptions of Lemma 6.13 with (6.24) (see below) is used.

(a) If $F, G$ are uniformly (locally) $C^0,1$ (i.e. estimates (6.5) hold), then (6.11) is satisfied for $\gamma = 1$, $\zeta = 0$. If $\lambda < 1$, then the spectral gap condition is read as: $\lambda_0 \lambda < 1$, $(\frac{\mu}{\lambda})^{\alpha} \lambda_0 \lambda < 1$. This case was also discussed in e.g. [Sta99, Cha08, Wil13].

(b) A well known case is the following. If $D_m F_m(x, \cdot), D_m G_m(x, \cdot) \in C^{0,\beta}$ uniform for $m_0, m$, then estimates (6.11) hold for $\gamma = 1$, $\zeta = \beta$. Particularly, for this case, in addition with $\beta = 1$ and the spectral gap condition: $\lambda_0 \lambda < 1$, $\mu^{\alpha} \lambda_0 \lambda < 1$, the uniformly (locally) $\alpha$-Hölder continuity of $m \mapsto f_m(x)$ is obtained. We mention that if $\lambda < 1$, the spectral gap condition in this case is better than (a); that is to say the higher regularity of $F, G$, the better spectral gap condition.

Remark 6.15. When $\zeta = 0$, the assumption (H3) does not need. Instead, we need the following Lipschitz continuity of $i$.

(B3) $i$ is Lipschitz around $M_1$ with respect to $\mathcal{A} \times \mathcal{B}$ in the following sense. Let

$$
\tilde{i}^{m_0}(m) = (\tilde{i}_X^{m_0}(m), \tilde{i}_Y^{m_0}(m)) = ((\tilde{\varphi}_m^{m_0})^{-1}(i_X(m)), (\tilde{\varphi}_m^{m_0})^{-1}(i_Y(m)),
$$

Then $m \mapsto \tilde{i}^{m_0}(m) : U_{m_0}(e_0) \to X_{m_0} \times Y_{m_0}$ is Lipschitz with Lipschitz constant less than $c_0$ for a fixed $e_0$ and every $m_0 \in M_1$, where $c_0$ is independent of $m_0$.

The assumption (H3) actually means that $i$ is Lipschitz with Lipschitz constant 0, i.e. locally constant. Now under the assumptions of Lemma 6.13 with (H3) replaced by (B3) and $\zeta = 0$, we also have $f$ is Hölder in the base points. For a proof, see Remark A.10 (b). See also [Cha08] for the same result in the trivial bundles case.

Proof of Lemma 6.13. Take $m_0 \in M_1$. First note that by (H5) (H2a), if $m \in U_{m_0}(\hat{\mu}^{-2} e_1)$ and $e_1$ is small, then we can choose $\alpha''$, $\beta''$, $\lambda''$, $\lambda''$, such that $1 - \alpha''(m) \beta''(u(m)) > 0$, and $\lambda'' \lambda'' < 1$, $(\max((\lambda''), \beta''(u(m))) \mu^{\alpha})^{\alpha} \lambda'' \lambda'' < 1$, and the following inequalities hold:

$$
(6.13) \begin{align*}
\text{Lip} \tilde{F}_m(m, x, \cdot) &\leq (1 + \eta(d(u(m), u(m_0)))) \lambda(m)(1 + \eta(d(u(m), u(m_0)))) \leq \alpha''(m_0), \\
\text{Lip} \tilde{G}_m(m, x, \cdot) &\leq (1 + \eta(d(u(m, m_0)))) \lambda(m)(1 + \eta(d(u(m), u(m_0)))) < \lambda''(m_0), \\
\text{Lip} \tilde{\varphi}_m(m, y) &\leq (1 + \eta(d(m, m_0))) \beta'(m)(1 + \eta(d(m, m_0))) \leq \beta''(m_0), \\
\text{Lip} \tilde{\varphi}_m(m, y) &\leq (1 + \eta(d(u(m, u(m_0)))) \lambda(m)(1 + \eta(d(m, m_0))) \lambda''(m_0), \\
\text{Lip} \tilde{\varphi}_m(m, y) &\leq (1 + \eta(d(u(m, u(m_0)))) \lambda(m)(1 + \eta(d(m, m_0))) \leq \lambda''(m_0), \\
\text{Lip} \tilde{\varphi}_m(m, y) &\leq (1 + \eta(d(u(m, u(m_0)))) \lambda(m)(1 + \eta(d(m, m_0))) \leq \lambda''(m_0), \\
\text{Lip} \tilde{\varphi}_m(m, y) &\leq (1 + \eta(d(u(m, u(m_0)))) \lambda(m)(1 + \eta(d(m, m_0))) \leq \lambda''(m_0), \\
\text{Lip} \tilde{\varphi}_m(m, y) &\leq (1 + \eta(d(u(m, u(m_0)))) \lambda(m)(1 + \eta(d(m, m_0))) \leq \lambda''(m_0),
\end{align*}
$$

where $\eta(\cdot)$ is the function in the definition of uniform $C^{0,1}$-fiber trivialization (for both $X, Y$) (see Definition 5.23).

Note that as $\tilde{f}_{m_0}(m, i(m_0)) = i(m_0)$, $\tilde{\varphi}_{m_0}(m, i(m_0)) = i(u(m_0)))$, we have

$$
|\tilde{f}_{m_0}(m, x)| \leq \beta''(m_0)|x|, \quad |\tilde{\varphi}_{m_0}(m, x)| \leq \lambda''(m_0)|x|
$$

Since $\tilde{f}_{m_0}(m, x)$ satisfies (6.10), we have

$$
|\tilde{f}_{m_0}(m_1, x) - \tilde{f}_{m_0}(m_0, x)|
$$
Thus,

\[ \{ \hat{G}_{m_0}(m_1, x, \hat{f}_{i}(m_0)(u(m_1), \hat{\gamma}_{m_0}(m_1, x))) - \hat{G}_{m_0}(m_0, x, \hat{f}_{i}(m_0)(u(m_1), \hat{\gamma}_{m_0}(m_1, x))) \}
+ \{ \hat{G}_{m_0}(m_0, x, \hat{f}_{i}(m_0)(u(m_1), \hat{\gamma}_{m_0}(m_1, x))) - \hat{G}_{m_0}(m_0, x, \hat{f}_{i}(m_0)(u(m_0), \hat{\gamma}_{m_0}(m_0, x))) \}
\leq \hat{C}|m_1 - m_0|^\gamma|x|^\xi + \lambda''(m_0)|\hat{f}_{i}(m_0)(u(m_1), \hat{\gamma}_{m_0}(m_1, x)) - \hat{f}_{i}(m_0)(u(m_0), \hat{\gamma}_{m_0}(m_0, x))|
\leq \hat{C}|m_1 - m_0|^\gamma|x|^\xi + \lambda''(m_0)|\hat{f}_{i}(m_0)(u(m_1), \hat{\gamma}_{m_0}(m_1, x)) - \hat{f}_{i}(m_0)(u(m_0), \hat{\gamma}_{m_0}(m_0, x))|
+ \lambda''(m_0)|\hat{f}_{i}(m_0)|\hat{\gamma}_{m_0}(m_1, x) - \hat{\gamma}_{m_0}(m_0, x)|.
\]

Similarly,

\[ |\hat{\gamma}_{m_0}(m_1, x) - \hat{\gamma}_{m_0}(m_0, x)| \leq \hat{C}|m_1 - m_0|^\gamma|x|^\xi + \alpha''(m_0)|\hat{f}_{i}(m_0)(u(m_1), \hat{\gamma}_{m_0}(m_1, x)) - \hat{f}_{i}(m_0)(u(m_0), \hat{\gamma}_{m_0}(m_0, x))| + \alpha''(m_0)|\hat{f}_{i}(m_0)|\hat{\gamma}_{m_0}(m_1, x) - \hat{\gamma}_{m_0}(m_0, x)|.
\]

Thus, (6.14)

\[ |\hat{f}_{i}(m_1, x) - \hat{f}_{i}(m_0, x)| \leq \hat{C}|m_1 - m_0|^\gamma|x|^\xi + \frac{\lambda''(m_0)}{1 - \alpha''(m_0)|\hat{f}_{i}(m_0)|\hat{\gamma}_{m_0}(m_1, x) - \hat{\gamma}_{m_0}(m_0, x)|}|\hat{f}_{i}(m_0)(u(m_1), \hat{\gamma}_{m_0}(m_1, x)) - \hat{f}_{i}(m_0)(u(m_0), \hat{\gamma}_{m_0}(m_0, x))|.
\]

Using the argument in Appendix A (see Remark A.10 (b)), we deduce the results.

6.5. Hölder distribution: Hölderness of \( m \mapsto Df_{m}(x) = K_{m}^{1}(x) \). (Based on the Hölder continuity of \( m \mapsto f_{m}(x) \) and \( x \mapsto K_{m}^{1}(x) \))

Let \( K^{1} \in L(T_{X}^{1}, T_{Y}^{V}) \) be the bundle map over \( f \) obtained in Lemma 6.7. In order to make sense of the Hölder continuity of \( m \mapsto Df_{m}(x) = K_{m}^{1}(x) \), we need assumption (H2b) instead of (H2a). Now taking the derivative of (6.10) with respect to \( x \), we have

\[
\begin{align*}
D_{z}(\hat{F}_{m_0}(m, z) & \mapsto \hat{R}_{m_0}(m, z)) = \hat{R}_{m_0}(m, z), \\
D_{z}(\hat{G}_{m_0}(m, z) & \mapsto \hat{R}_{m_0}(m, z)) = \hat{K}_{m_0}(m, z),
\end{align*}
\]

where \( z = (x, \hat{f}_{i}(m_0)(u(m), \hat{\gamma}_{m_0}(m, x))) \), and \( \hat{K}, \hat{R}^{1} \) are the local representations of \( K^{1}, R^{1} \) with respect to \( A, B \), i.e.

\[
\begin{align*}
\hat{K}_{m_0}^{1}(m, x) & \triangleq (D\phi_{m_0})^{-1}(f_{m}(\phi_{m_0}(x)))K_{m_0}^{1}(\phi_{m_0}(x))D\phi_{m_0}(x), \\
\hat{R}_{m_0}^{1}(m, x) & \triangleq (D\phi_{m_0})^{-1}(x_{m}(\phi_{m_0}(x)))R_{m_0}^{1}(\phi_{m_0}(x))D\phi_{m_0}(x),
\end{align*}
\]

Note that \( \hat{K}_{m_0}^{1}(m, x) = D_{x}\hat{f}_{m_0}(m, x) \), \( \hat{R}_{m_0}^{1}(m, x) = D_{x}\hat{\gamma}_{m_0}(m, x) \).

First consider a special case, i.e. \( m \mapsto K_{m}^{1}(i_{x}(m)) \).

Lemma 6.16. Assume the conditions in Lemma 6.7 hold. In addition, assume the following hold.

(a) Let (H1a) (H2b) (H3) (H4a) (H5) hold.

(b) (about \( F, G \)) \( Df_{m}(i_{X}(m), i_{Y}(u(m))) \), \( DG_{m}(i_{X}(m), i_{Y}(u(m))) \) are uniformly (locally) \( \gamma \)-Hölder around \( M_{1} \) in the following sense:

\[
\begin{align*}
|D_{z}\hat{F}_{m_0}(m_1, i_{X}(m_0), i_{Y}(u(m_0))) - D_{z}\hat{F}_{m_0}(m_0, i_{X}(m_0), i_{Y}(u(m_0)))| & \leq M_{0}|m_1 - m_0|^{\gamma}, \\
|D_{z}\hat{G}_{m_0}(m_1, i_{X}(m_0), i_{Y}(u(m_0))) - D_{z}\hat{G}_{m_0}(m_0, i_{X}(m_0), i_{Y}(u(m_0)))| & \leq M_{0}|m_1 - m_0|^{\gamma},
\end{align*}
\]

for all \( m_{1} \in U_{m_0}(\mu^{-1}(m_0)E_{1}), m_0 \in M_{1}, 0 < \gamma \leq 1 \).

(c) (spectral gap condition) \( \lambda_{\alpha} \lambda_{\mu} < 1, \mu^{\gamma\alpha} \lambda_{\alpha} \lambda_{\mu} < 1 \), where \( 0 < \alpha \leq 1 \); see Remark 6.1.
When \( e_1^* \leq \tilde{\mu}^{-2} e_1 \) is small, we have
\[
(6.17) \quad |\tilde{K}_{m_0}(m_1, iX(m_0)) - \tilde{K}_{m_0}^1(m_0, iX(m_0))| \leq C|m_1 - m_0|^{\alpha},
\]
for every \( m_1 \in U_{m_0}(e_1^*) \), \( m_0 \in M_1 \).

**Lemma 6.17.** Assume the conditions in Lemma 6.7 hold. In addition, assume the following hold.
(a) Let (H1a) (H2b) (H3) (H4a) (H5) hold.
(b) (about \( F, G \)) DF, DG are \( C^{0,1} \) around \( M_1 \), i.e. the estimates (6.6) hold. Moreover, estimates (6.11) in Lemma 6.13 hold for \( \gamma = \xi = 1 \); see also Remark 6.14.
(c) (spectral gap condition) \( \lambda_s \lambda_u < 1 \), \( \lambda_s^2 \lambda_u \mu^\alpha < 1 \), \( \lambda_s^2 \lambda_u \mu^\beta < 1 \), \( \mu^\alpha \lambda_s \lambda_u < 1 \), \( \mu^\alpha \lambda_s \lambda_u < 1 \), where \( 0 < \alpha, \beta \leq 1 \);
see Remark 6.1.

If \( e_1^* \leq \tilde{\mu}^{-2} e_1 \) is small, then we have
\[
(6.18) \quad |\tilde{K}_{m_0}^1(m_1, x) - \tilde{K}_{m_0}^1(m_0, x)| \leq C \{ |m_1 - m_0|^\alpha (|x| + 1) + (|m_1 - m_0|^\alpha |x|)^\beta \},
\]
for every \( m_1 \in U_{m_0}(e_1^*) \), \( x \in X_{m_0} \), \( m_0 \in M_1 \), where the constant \( C \) depends on the constant \( e_1^* \) but not \( m \in M \).

One can discuss the case when \( \tilde{F}_{m_0}(\cdot), \tilde{G}_{m_0}(\cdot) \in C^{1,\gamma} \), which is more complicated.

**Proof of Lemma 6.16 and Lemma 6.17.** As in the proof of Lemma 6.13, when let \( e_1 \) be small and \( m \in U_{m_0}(\tilde{\mu}^{-2} e_1) \), we can choose \( \alpha''', \beta''', \lambda'''_s, \lambda'''_u \), such that the following spectral gap condition holds (see Remark 6.1 for the actual meaning, where \( \alpha'', \beta'' \) is instead of \( \alpha, \beta \'))
\[
\lambda'''_s \lambda'''_u < 1, (\lambda'''_s)^2 \lambda'''_u \mu^\alpha < 1, (\lambda'''_s)^\beta \lambda'''_u \mu^\beta < 1, \mu^\alpha \lambda'''_s \lambda'''_u < 1
\]
(6.13) holds, and in addition the following two inequalities hold.

\[
(6.19) \quad \begin{cases}
\sup_{x \in X_m} |\tilde{K}_{m_0}^1(m, x) - \tilde{K}_{m_0}^1(m_0, x)| \\
\sup_{x \in X_m} |\tilde{K}_{m_0}^1(m, x)| 
\end{cases}
\leq (1 + \eta (d(m, m_0))) \beta''(m)(1 + \eta (d(m, m_0))) \leq \beta''(m_0),
\]
Let \( y_1 = \tilde{f}_{u(m_0)}(u(m_1), \tilde{x}_{m_0}(m_1, x)), y_2 = \tilde{f}_{u(m_0)}(u(m_0), \tilde{x}_{m_0}(m_0, x)) \), \( \Delta_1 = |y_1 - y_2| \). By (6.15), it yields
\[
|\tilde{K}_{m_0}^1(m_1, x) - \tilde{K}_{m_0}^1(m_0, x)| \\
\leq |\tilde{D}_z \tilde{G}_{m_0}(m_1, x, y_1) - \tilde{D}_z \tilde{G}_{m_0}(m_1, x, y_2)|(\tilde{K}_{u(m_0)}^1(m_1), \tilde{x}_{m_0}(m_1, x))| \tilde{K}_{m_0}^1(m_1, x) | \\
+ |\tilde{D}_z \tilde{G}_{m_0}(m_0, x, y_2) - \tilde{D}_z \tilde{G}_{m_0}(m_0, x, y_2)|(\tilde{K}_{u(m_0)}^1(m_1), \tilde{x}_{m_0}(m_1, x))| \tilde{K}_{m_0}^1(m_1, x) | \\
+ |\tilde{D}_z \tilde{G}_{m_0}(m_0, x, y_2)| (\tilde{K}_{u(m_0)}^1(m_1), \tilde{x}_{m_0}(m_1, x))| \tilde{K}_{m_0}^1(m_1, x) | \\
- \tilde{K}_{u(m_0)}^1(u(m_0), \tilde{x}_{m_0}(m_0, x))| \tilde{K}_{m_0}^1(m_0, x) | \\
\leq \tilde{C} \Delta_1 + \tilde{C} |m_1 - m_0| + \lambda'''_s(m_0) \lambda'''_u(m_0)| \tilde{K}_{u(m_0)}^1(u(m_1), \tilde{x}_{m_0}(m_1, x)) | \\
- \tilde{K}_{u(m_0)}^1(u(m_0), \tilde{x}_{m_0}(m_0, x)) | + \lambda'''_u(m_0) \beta''(u(m_0))| \tilde{K}_{m_0}^1(m_1, x) - \tilde{K}_{m_0}^1(m_0, x) |.
\]
Similarly,
\[
|\tilde{K}_{m_0}^1(m_1, x) - \tilde{K}_{m_0}^1(m_0, x) | \leq \tilde{C} \Delta_1 + \tilde{C} |m_1 - m_0| + \lambda'''_s(m_0) \alpha'''(m_0) \tilde{K}_{u(m_0)}^1(u(m_1), \tilde{x}_{m_0}(m_1, x)) \\
- \tilde{K}_{u(m_0)}^1(u(m_0), \tilde{x}_{m_0}(m_0, x)) | + \alpha'''(m_0) \beta''(u(m_0))| \tilde{K}_{m_0}^1(m_1, x) - \tilde{K}_{m_0}^1(m_0, x) |.
\]
Now we have
\[
|\tilde{K}_{m_0}^1(m_1, x) - \tilde{K}_{m_0}^1(m_0, x) | \leq \tilde{C} \Delta_1 + \tilde{C} |m_1 - m_0| \\
+ \theta''(m_0)| \tilde{K}_{u(m_0)}^1(u(m_1), \tilde{x}_{m_0}(m_1, x)) - \tilde{K}_{u(m_0)}^1(u(m_0), \tilde{x}_{m_0}(m_0, x)) |,
\]
where \( \theta''(m_0) = \frac{x''(m_0)\gamma(m_0)}{1-\alpha''(m_0)\beta(m_0)} < 1. \)

For the proof of Lemma 6.16, the estimate becomes

\[
(6.20) \quad |\tilde{K}_{m_0}^1(m_1, i_X(m_0)) - \tilde{K}_{m_0}^1(m_0, i_X(m_0))| \leq C|m_1 - m_0|\gamma + \theta''(m_0)|\tilde{K}_{u(m_0)}^1(u(m_1), i_X(m_0)) - \tilde{K}_{u(m_0)}^1(u(m_0), i_X(m_0))|.
\]

Using the argument in Appendix A (see Remark A.10 (c)), one can obtain Lemma 6.16.

For the general case, we need the H"older continuity of \( m \mapsto f_m(x) \) and \( x \mapsto K_1^m(x) \), i.e., Lemma 6.13 and Lemma 6.11. There is a constant \( \epsilon'_1 > 0 \), such that the following (a) (b) hold.

(a) By Lemma 6.11 and (H2b), we have

\[
|\tilde{K}_{m_0}^1(m_1, x) - \tilde{K}_{m_0}^1(m_0, x)| \leq \tilde{C}|x_1 - x_2|^{\beta}, \quad |\tilde{K}_{m_0}^1(m_1, x) - \tilde{K}_{m_0}^1(m_0, x)| \leq \tilde{C}|x_1 - x_2|^{\beta},
\]

for \( m \in U_{m_0}(\epsilon'_1) \), \( x \in X_{m_0} \), where the constant \( \tilde{C} \) only depends on \( \epsilon'_1 \).

(b) Meanwhile, by Lemma 6.13, we have

\[
|\tilde{f}_{m_0}(m_1, x) - \tilde{f}_{m_0}(m_0, x)| \leq \tilde{C}|m_1 - m_0|^{\alpha}|x|, \quad |\tilde{x}_{m_0}(m_1, x) - \tilde{x}_{m_0}(m_0, x)| \leq \tilde{C}|m_1 - m_0|^{\alpha}|x|,
\]

for every \( m \in U_{m_0}(\epsilon'_1), x \in X_{m_0} \), where the constant \( \tilde{C} \) only depends on the choice of \( \epsilon'_1 \). Now we see that

\[
(6.21) \quad |\tilde{K}_{m_0}^1(m_1, x) - \tilde{K}_{m_0}^1(m_0, x)| \leq \tilde{C}\left\{ |m_1 - m_0|^{\alpha}|x| + (|m_1 - m_0|^{\alpha}|x|)^{\beta} \right\} + \tilde{C}|m_1 - m_0|
\]

Apply the argument in Appendix A (see Remark A.10 (c)) to obtain the result. \(\square\)

### 6. smoothness of \( m \mapsto f_m(x) \) and Hölderness of \( x \mapsto \nabla_m f_m(x) = K_m(x) \).

#### 6.1. smoothness of \( m \mapsto f_m(x) \).

Based on the \( C^1 \) smoothness of \( f_m(\cdot) \) and the Lipschitz continuity of \( m \mapsto f_m(x) \).

In the following, we need some smooth properties of the bundles \( X, Y \) in order to make sense of the smoothness of \( m \mapsto f_m(x) \). The natural assumption is that \( X, Y \) are \( C^1 \) bundles with \( C^0 \) connections (see Section 5.3). Now we can interpret (6.3) precisely by using the notion of covariant derivative (see Definition 15.13), i.e.

\[
(6.22) \quad \begin{cases}
\nabla_m f_m(x, y) + D_2 F_m(x, y) (K_{u(m)}(x_0(m))) Du(m) + D f_u(m)(x_0(m)) R_u(x) = R_m(x), \\
\nabla_m G_m(x, y) + D_2 G_m(x, y) (K_{u(m)}(x_0(m))) Du(m) + D f_u(m)(x_0(m)) R_u(x) = K_m(x),
\end{cases}
\]

where \( y = f_u(m)(x_0(m)) \) and where \( K \in L(T_X^H, T_Y^V) \) is a vector bundle map over \( f \), and \( R \in L(T_X^H, T_Y^V) \) is a vector bundle map over \( x_{i,j}(\cdot) \). Hereafter for \( K \in L(T_X^H, T_Y^V) \) (over \( x_{i,j}(\cdot) \)), we write \( K_m(x) = K_{(m,x)} \).

**Lemma 6.18.** Assume the following conditions hold.

(a) Let (H1b) (H2c) (H3) (H4b) (H5) hold.

(b) (about \( F, G \)) \( F, G \) are \( C^1 \) and satisfy the following estimates:

\[
(6.23) \quad \begin{cases}
|\tilde{F}_{m_0}(m_1, z) - \tilde{F}_{m_0}(m_0, z)| \leq M_0|m_1 - m_0||z|, \\
|\tilde{G}_{m_0}(m_1, z) - \tilde{G}_{m_0}(m_0, z)| \leq M_0|m_1 - m_0||z|,
\end{cases}
\]

for all \( m_1 \in U_{m_0}(\rho^{-1}(m_0)\epsilon_1), z \in X_{m_0} \times Y_{u(m_0)}, m_0 \in M_1 \).

(c) (spectral gap condition) \( \lambda_s \lambda_u < 1, \lambda_s \lambda_d \mu < 1 \); see also Remark 6.1.

Then the following hold.
(1) There exists a unique $C^0$ vector bundle map $K \in L(\mathcal{Y}^H_X, \mathcal{Y}^V_Y)$ over $f$, such that

$$K_m(i_X(m)) = 0, \quad m \in M_1, \quad \sup_{m \in M_1} \sup_{x \neq i_X(m)} \left| \frac{K_m(x)}{x} \right| < \infty,$$

and it satisfies (6.22). In addition, if

$$\nabla_m F_m(i_X(m), i_Y(u(m))) = 0, \quad \nabla_m G_m(i_X(m), i_Y(u(m))) = 0,$$

for all $m \in M$, then $K_m(i_X(m)) = 0$ for all $m \in M$.

(2) $f$ is $C^1$ and $\nabla_m f_m(x) = K_m(x)$ for $m \in M, x \in X_m$.

**Remark 6.19** (about the condition on $F, G$). (a) Since $\mathcal{A}, \mathcal{B}$ consist of normal bundle charts, by the chain rule (see Lemma 5.14), we have

$$D_m \tilde{F}_{m_0}(m, z)|_{m = m_0} = \nabla_u\phi^{u(m_0)}_{u(m_0)}(x) Du(m_0) + D\phi^{u(m_0)}_{u(m_0)}(x) D\phi^{u(m_0)}_{u(m_0)}(y) Du(m_0)) = \nabla_m \tilde{F}_{m_0}(z'),$$

where $z = (x, y), z' = (\phi^{m_0}_m(x), \phi^{u(m_0)}_m(y)), x'' = F_m(z')$. Similarly,

$$D_m \tilde{G}_{m_0}(m, z)|_{m = m_0} = \nabla_m \tilde{G}_{m_0}(z').$$

(b) One can replace (6.23) by the following estimates:

$$|\tilde{F}_{m_0}(m_1, z) - \tilde{F}_{m_0}(m_0, z)| \leq M_0|m_1 - m_0||z|^\gamma, \quad |\tilde{G}_{m_0}(m_1, z) - \tilde{G}_{m_0}(m_0, z)| \leq M_0|m_1 - m_0||z|^\gamma,$$

(see also Remark 6.14) and the spectral gap condition by $\lambda_\alpha \lambda_\mu < 1, \lambda_\gamma \lambda_\mu < 1, \lambda_s < 1$, where $0 < \gamma \leq 1, (\lambda_\alpha \lambda_\mu < 1, \max\{\lambda_\gamma^{-1}, 1\} \lambda_\alpha \lambda_\mu < 1)$. Then one also gets $f \in C^1$ but

$$\sup_{m \in M_1} \sup_{x \neq i_X(m)} \left| \frac{\nabla_m f_m(x)}{x} \right| < \infty.$$

The proof is very similar with the case $\gamma = 1$. Note that under $\lambda_\alpha \lambda_\mu < 1, \lambda_s < 1$, by using Lemma 6.13, one gets

$$\sup_{m \in M_1} \sup_{x \in X_{m_0}} \lim_{m_0 \to m_0} \left| \frac{f_{m_0}(m_1, x) - \tilde{F}_{m_0}(m_0, x)}{m_1 - m_0} \right| \leq C.$$

Now we can use the same step in the proof of Lemma 6.18 with some naturally minor modifications.

**Proof of Lemma 6.18.** First note that the covariant derivatives of $F, G$ satisfy

$$|\nabla_m F_{m_0}(z)| \leq C_0|z|, \quad |\nabla_m G_{m_0}(z)| \leq C_0|z|,$$

for all $z \in X_{m_0} \times Y_{m_0}$ and $m_0 \in M_1$. This is quite simple as the following shown. Let $m_0 \in M_1, X_{m_0}(m_1) = x_1 \in T_{m_0} M_1$. Then

$$\frac{|\tilde{F}_{m_0}(x_1), z) - \tilde{F}_{m_0}(x_1), 0)|}{|x_1|} \leq \Theta^{-1}_1 |M_0| |z|,$$

and so $|\nabla_m F_{m_0}(z)| = |D_m \tilde{F}_{m_0}(m, z)| \leq M_0 |x| \leq \Theta^{-1}_1 |M_0| |z|$. Similar for $|\nabla_m G_{m_0}(z)|$.

(1). Let $e : M_1 \to M$ be the inclusion map, i.e. $e(m) = m$. Consider the pull-back bundles $e^*X, e^*Y$ of $X, Y$ through $e$. Through two natural inclusion maps of $e^*X \to X, e^*Y \to Y$, which are induced by $e$, there are two natural pull-back vector bundles of $TX, TY$, denoted by $e^*TX, e^*TY$. Using $e^*X, e^*Y$ and $e^*TX, e^*TY$ instead of $X, Y, TX, TY$, we can, without loss of the generality, assume $M_1 = M$. So first let $M_1 = M$. The existence of $K$ suffices under the assumptions in Lemma 6.28 below. So in this step, we in fact prove the conclusion (1) in Lemma 6.28.

Define a metric space

$$E_0^0 \triangleq \{K \in L(\mathcal{Y}^H_X, \mathcal{Y}^V_Y) \text{ is } C^0 \text{ a vector bundle map over } f : |K_m(x)| \leq C_1|x| \text{ if } m \in M_1\},$$
with a metric

\[ d_0(K, K') \triangleq \sup_m \sup_{x \neq i(x(m))} \frac{|K_m(x) - K'_m(x)|}{|x|}, \]

where

\[ C_1 = \frac{C_0}{1 - \theta_0}, \quad \theta_0 = \sup_{m \in M_1} \frac{\lambda_u(m)\mu(m)}{1 - \alpha(m)\beta'(u(m))} < 1, \]

and \( C_0 \) is given by (6.29) in the following. Note that in our notation \( |x| = d_m(x, i(x(m))) \), if \( x \in \mathcal{X}_m \).

The metric space \( (E_0^L, d_0) \) is complete. (Note that in the local representation, the convergence in \( E_0^L \) is uniform for \( m \) and \( x \) belonging to any bounded-fiber set.)

Since \( \alpha(m)\beta'(u(m)) < 1 \), given a \( K \in E_0^L \), there is a unique \( R \in L(\mathcal{T}_X^H, \mathcal{T}_Y^V) \) over \( x \), satisfying

\[ \nabla_m F_m(x, y) + D_2 F_m(x, y)(K_{u(m)}(x(m))Du(m) + Df_{u(m)}(x(m))R_m(x)) = R_m(x), \]

where \( y = f_{u(m)}(x(m)) \); and define \( \widetilde{K} \) by

\[ \widetilde{K}_m(x) \triangleq \nabla_m G_m(x, y) + D_2 G_m(x, y)(K_{u(m)}(x(m))Du(m) + Df_{u(m)}(x(m))R_m(x)). \]

Consider the graph transform \( \Gamma^0 : K \mapsto \widetilde{K} \).

**Sublemma 6.20.** \( \Gamma^0 \) is a Lipschitz map of \( E_0^L \to E_0^L \) with \( \mathrm{Lip} \Gamma^0 \leq \sup_{m \in M_1} \frac{\lambda_u(m)\mu(m)}{1 - \alpha(m)\beta'(u(m))} < 1. \)

**Proof.** Let \( \Gamma^0(K) = \widetilde{K}, \Gamma^0(K') = \widetilde{K}' \). Since

\[ |\widetilde{K}_m(x) - \widetilde{K}_m'(x)| \leq \left| \{K_{u(m)}(x(m)) - K'_{u(m)}(x(m))\}Du(m)\right| + \alpha(m)\beta'(u(m))|R_m(x) - R'_m(x)|, \]

and

\[ |R_m(x) - R'_m(x)| \leq \alpha(m)|\{K_{u(m)}(x(m)) - K'_{u(m)}(x(m))\}Du(m)| + \alpha(m)\beta'(u(m))|R_m(x) - R'_m(x)|, \]

we have

\[ |\widetilde{K}_m(x) - \widetilde{K}_m'(x)| \leq \frac{\lambda_u(m)\mu(m)}{1 - \alpha(m)\beta'(u(m))}|K_{u(m)}(x(m)) - K'_{u(m)}(x(m))|. \]

If \( K \in E_0^L \), then (a) \( \widetilde{K} \in L(\mathcal{T}_X^H, \mathcal{T}_Y^V) \) by the construction, (b) \( \widetilde{K} \) is \( C^0 \), (c) |\( \widetilde{K}_m(x) | \leq C_1 |x|. \) Thus, \( \widetilde{K} \in E_0^L \).

To show (b), using a suitable local representation of (6.27) (see e.g. (6.34) in the following), the fact that \( f, K' \) are continuous proved in Lemma 6.25 and Lemma 6.26 below, and the assumption (H5'), one can easily see that \( \widetilde{K} \) is \( C^0 \) (see also the proof of \( K \) is \( C^0 \) in \( L(\mathcal{T}_X^H, \mathcal{T}_Y^V) \)) in the following.

We show (c) as follows. Letting \( K' = 0 \), by (6.26), we have

\[ |\Gamma^0(0)| \leq M_0 |z| + \lambda_u(m)\beta'(u(m))|R'_m(x)| \]

\[ \leq M_0 |z| + \lambda_u(m)\beta'(u(m))(1 - \alpha(m)\beta'(u(m)))^{-1}M_0 |z| \leq C_0 |x|, \]

where \( z = (x, f_{u(m)}(x(m))), C_0 \) is a constant independent of \( m \). Then by (6.28), it holds that

\[ |\widetilde{K}_m(x)| \leq |\Gamma^0(0)| + \frac{\lambda_u(m)\mu(m)}{1 - \alpha(m)\beta'(u(m))}|K_{u(m)}(x(m))| \leq C_0 |x| + \theta_0 C_1 |x| \leq C_1 |x|. \]

Inequality (6.28) also gives the estimate of the Lipschitz constant of \( \Gamma^0 \), and the proof is complete. \( \square \)

Therefore, there is a unique \( K \in E_0^L \) satisfying (6.22) when \( M_1 = M \). Next since \( u(M) \subset M_1 \), using (6.22), one can uniquely define \( K \) in all \( X \) (not only in \( e^* X \).)

We show \( K : \mathcal{T}_X^H \to \mathcal{T}_Y^V \) is a \( C^0 \) vector bundle map. This is direct from the local representation of (6.27). We give details as follows. Let \( m_0 \in M \). Choose \( C^1 \) local charts of \( M \) at \( m_0, u(m_0) \), e.g. \( \xi_{m_0} : U_0 \to T_{m_0}M \) and \( \xi_{u(m_0)} : V \to T_{u(m_0)}M \), such that \( u(U_0) \subset V \subset u(U_{m_0}) \). Note that \( u(m_0) \in M_1 \). Moreover, take \( C^1 \) bundle charts \( (U_0, \varphi^0), (U_0, \phi^0) \) of \( X, Y \) at \( m_0 \) respectively, as
well as \((U_{u(m_0)}, \phi^{u(m_0)}) \in A, (U_{u(m_0)}, \phi^{u(m_0)}) \in B\). Using the above charts, one can give the local representations of \(K, R, I, e\).

\[
\begin{align*}
(6.30) & \quad \begin{cases}
\hat{K}_{u(m_0)}(m, x) \triangleq D(\phi_{u(m_0)}^{-1}(f_m(\phi_m^0(x))))K_m(\phi_m^0(x))D\xi_{m_0}(\xi_m(m)) : U \times X_{m_0} \to L(T_{m_0}M, Y_{m_0}), \\
\hat{R}_{u(m_0)}(m, x) \triangleq D(\phi_{u(m_0)}^{-1}(f_m(\phi_m^0(x))))\hat{K}_m(\phi_m^0(x))D\xi^{-1}_{u(m_0)}(\xi_{u(m_0)}(m)) : \\
\hat{R}_{u(m_0)}(m, x) \triangleq D(\phi_{u(m_0)}^{-1}(x_m(\phi_m^0(x))))\hat{R}_m(\phi_m^0(x))D\xi^{-1}_{m_0}(\xi_{m_0}(m)) : \\
U(m_0) \times X_{u(m_0)} \to L(T_{u(m_0)}M, Y_{u(m_0)}),
\end{cases} \\
\end{align*}
\]

as well as \(\nabla F, \nabla G, G, x, (\cdot), Du, i.e.

\[
\begin{align*}
\hat{D}_m F_{m_0}(m, x, y) & \triangleq D(\phi_{u(m_0)}^{-1}(F_m(\phi_m^0(x), \phi_{u(m_0)}(y)))) \\
(6.31) & \quad \begin{cases}
\nabla_m F_m(\phi_m^0(x), \phi_{u(m_0)}(y))D\xi_{m_0}(\xi_{m_0}(m)) : U \times X_{m_0} \times X_{u(m_0)} \to L(T_{m_0}M, X_{u(m_0)}), \\
\nabla_m G_m(\phi_m^0(x), \phi_{u(m_0)}(y))D\xi_{m_0}(\xi_{m_0}(m)) : U \times X_{m_0} \times X_{u(m_0)} \to L(T_{m_0}M, X_{u(m_0)}),
\end{cases} \\
\end{align*}
\]

Then by a simple computation, we know they satisfy

\[
\begin{align*}
(6.32) & \quad \begin{cases}
\hat{F}_m^*(m, x, y') & \triangleq (\phi_{u(m_0)}^{-1} \circ \Phi^0_{u(m_0)}(m, x)) : U \times X_{m_0} \times X_{u(m_0)} \to X_{u(m_0)}, \\
\hat{G}_m^*(m, x, y) & \triangleq (\phi_{u(m_0)}^{-1} \circ \Phi^0_{u(m_0)}(m, x)) : U \times X_{m_0} \times X_{u(m_0)} \to X_{u(m_0)}, \\
\hat{X}_m^*(m, x, y) & \triangleq (\phi_{u(m_0)}^{-1} \circ \Phi^0_{u(m_0)}(m, x)) : U \times X_{m_0} \times X_{u(m_0)} \to X_{u(m_0)}, \\
\hat{D}_m F_{m_0}(m, x, y) & \triangleq D\xi_{m_0}((\phi_{m_0}^0(\Phi_{u(m_0)}(m, x))))Du(m)D\xi_{m_0}^{-1}(\xi_{m_0}(m)) : U \times T_{m_0}M \to T_{u(m_0)}M.
\end{cases}
\end{align*}
\]

where \(\hat{x}' = \hat{K}_{u(m_0)}(m, x, \hat{y}') = \hat{f}_{u(m_0)}(m, \hat{y}')\). Note that \(f, K^1\) are continuous proved in Lemma 6.25 and Lemma 6.26 below, so \(\hat{D}_m F_{m_0}, \hat{D}_m G_{m_0}, \hat{D}_m \hat{F}_{m_0}, \hat{D}_m \hat{G}_{m_0}\), and \(\hat{K}_{m_0}(m, x) = D_x \hat{f}_{u(m_0)}(m, x)\) are all continuous for every \(m_0 \in M\).

By the assumption (H5'), we get for \(m \in U_0\),

\[
|D_y \hat{F}_m^*(m, x, y')D_x \hat{f}_{u(m_0)}(u(m), \hat{x}')| \approx a(m_0)b'(u(m_0)) < 1,
\]

where \(U_0\) might be smaller if necessary. Since \(\hat{K}_{u(m_0)}(u(\cdot), \cdot)\) is continuous at \((m_0, x)\) for all \(x \in X_{u(m_0)}\), we see \(\hat{K}_{m_0}(\cdot, \cdot)\) is continuous at \((m_0, x)\), and so is \(\hat{R}_{m_0}(\cdot, \cdot)\), which gives that \(K = C^0\).

(II). In the following, we need to show \(f\) is \(C^1\) and \(\nabla_m f_{m_0}(x) = K_{m_0}(x)\). Working in the bundle charts, let us consider (6.10) and the local representation of (6.22). Let \(\hat{K}_{m_0}(x) = D(\phi_{m_0}^{-1}(y')K_{m_0}(\phi_{m_0}^0(x))) = K_{m_0}(x), \hat{R}_{m_0}(x) = D(\phi_{u(m_0)}^{-1}(x')(R_{m_0}(\phi_{m_0}^0(x))) = R_{m_0}(x),\)

where \(y' = f_{m_0}(\phi_{m_0}^0(x)), x' = x_{m_0}(\phi_{m_0}^0(x)), m_0 \in M_1\). Using (6.24) and (6.25), we know they satisfy

\[
\begin{align*}
\hat{D}_m \hat{F}_{m_0}(m, x, y)|_{m=m_0} + D_y \hat{F}_{m_0}(m_0, x, y) \\
\hat{D}_m \hat{G}_{m_0}(m, x, y)|_{m=m_0} + D_x \hat{G}_{m_0}(m_0, x, y) \\
\hat{D}_m \hat{X}_{m_0}(m, x, y)|_{m=m_0} + D_x \hat{X}_{m_0}(m_0, x, y) \\
\end{align*}
\]
where $y = \hat{f}_{u(m_0)}(u(m_0), \hat{x})$, $\hat{x} = \hat{x}_{m_0}(m_0, x)$. Define

$$Q^0(m', m_0, x) \equiv \hat{f}_{m_0}(m', x) - \hat{f}_{m_0}(m_0, x) - \hat{K}_{m_0}(x)(m' - m),$$

where $m' \in U_{\hat{m}_0}(\hat{\mu}^{-2}e_1)$, $m_0 \in M_1$. $e_1$ is taken from the proof of Lemma 6.13, as well as $\alpha'', \beta''$, $\lambda''$, $\mu''$, such that $1 - \alpha''(m)\beta''(u(m)) > 0$,

$$\lambda'' < 1, \mu \lambda'' < 1,$

and (6.13), (6.19) hold.

**Convention.** Here, we identify $U_{\hat{m}_0}(\hat{\mu}^{-2}e_1)$ as the Banach space $T_{m_0}M$ by the local chart $\chi_{m_0} : U_{\hat{m}_0}(\hat{\mu}^{-2}e_1) \rightarrow T_{m_0}M$ given in (H1b); that is we identify $x'$ as $m'$, if $\chi_{m_0}(m') = x'$. For example, the local representation of $\hat{f}_{m_0}(m', x)$ by $\chi_{m_0} : U_{m_0}(e'_{m_0}) \rightarrow T_{m_0}M$ is $\hat{f}_{m_0}(\chi^{-1}(x), x)$, but we also write it as $\hat{f}_{m_0}(m', x)$. The metric in $T_{m_0}M$ induces from $d_{m_0}$, i.e. $|x_1 - x_2| = d_{m_0}(\chi^{-1}(x_1), \chi^{-1}(x_2))$. $e_1$ may be taken small and even may depend on $m$. This does not cause any problem since at last we will let $e_1 \rightarrow 0$, i.e. $U_{\hat{m}_0} \ni m' \rightarrow m_0$.

We will use the notation $|g(m', m_0, x)| \leq O_\chi(|m' - m_0|)$ below, if $\limsup_{m' \rightarrow m_0} \frac{|g(m', m_0, x)|}{|m' - m_0|} = 0$.

**Sublemma 6.21.** \[\sup_{m_0 \in M_1} \sup_{x \in X_{m_0}} \limsup_{m \rightarrow m_0} \frac{|Q^0(m', m_0, x)|}{|m' - m_0||x|} < \infty \] and

$$|Q^0(m', m_0, x)| \leq O_\chi(|m' - m_0|) + \frac{\lambda''(m_0)}{1 - \alpha''(m_0)\beta''(u(m_0))}|Q^0(u(m'), u(m_0), \hat{x}_{m_0}(m', x))|.$$

**Proof.** Note that under these conditions given in the lemma, by Lemma 6.13, we have

$$\sup_{m_0 \in M_1} \sup_{x \in X_{m_0}} \limsup_{m \rightarrow m_0} \frac{|\hat{x}_{m_0}(m_1, x) - \hat{x}_{m_0}(m_0, x)|}{|m_1 - m_0||x|} \leq C,$$

$$\sup_{m_0 \in M_1} \sup_{x \in X_{m_0}} \limsup_{m \rightarrow m_0} \frac{|\hat{f}_{m_0}(m_1, x) - \hat{f}_{m_0}(m_0, x)|}{|m_1 - m_0||x|} \leq C.$$

Now the first inequality follows from (6.35) (6.4) and the above construction of $K$. (That the constants $\Theta_1, \Theta_2$ in (H1b) are independent of $m_0$ is exactly used here.) Next, let us consider the second inequality. Compute

$$|Q^0(m', m_0, x)|$$

$$\leq \left| \int_0^1 D_y \hat{G}_{m_0}(m', x, y) \left\{ \hat{f}_{(m_0)}(u(m'), \hat{x}_{m_0}(m', x)) - \hat{G}_{m_0}(m_0, x, y)m = m_0(m' - m_0) \right\} \right|$$

$$\leq \int_0^1 D_y \hat{G}_{m_0}(m', x, y) \left\{ \hat{f}_{(m_0)}(u(m'), \hat{x}_{m_0}(m', x)) - \hat{G}_{m_0}(m_0, x, y)m = m_0(m' - m_0) \right\}$$

Note that $|\boxed{2}| \leq O_\chi(|m' - m_0|)$.
Thus, have continuity of where

\[ m \leq D_{x} \hat{f}_{u}(m', \hat{x}) \]

The same computation gives

\[ m \leq \hat{f}_{u}(m, \hat{x}) \]

Note that by (6.35), and the Lipschitz continuity of \( u(\cdot), f_{m}(\cdot, \cdot), \tilde{x}_{m}(\cdot, \cdot) \), we have

\[ \left| Q \right| \triangleq \left| \int_{0}^{1} \{ D_{y} \hat{G}_{m_{0}}(m', x, y) - D_{x} \hat{G}_{m_{0}}(m_{0}, x, y) \} \, dt \left\{ \hat{f}_{u}(m_{0})((u(m'), \tilde{x}_{m}(m', x)) - \hat{f}_{u}(m_{0}),(\hat{x}) \right\} \]

Consider

\[ Q \triangleq \hat{f}_{u}(m_{0})(u(m'), \tilde{x}_{m}(m', x)) - \hat{f}_{u}(m_{0})(u(m_{0}), \hat{x}) \]

\[ - \hat{K}_{u}(m_{0})(\hat{x})Du(m_{0})(m' - m_{0}) - D_{x}\hat{f}_{u}(m_{0})(u(m_{0}), \hat{x})\tilde{x}_{m_{0}}(x)(m' - m_{0}) \]

Since \((m, x) \to \tilde{x}_{m_{0}}(m, x)\) is continuous (see e.g. Lemma 6.25 below), \( f_{m}(\cdot) \) is \( C^{1} \), and (6.35), we have

\[ \left| \tilde{Q} \right| \triangleq \left| \hat{f}_{u}(m_{0})(u(m_{0}), \tilde{x}_{m_{0}}(m', x)) - \hat{f}_{u}(m_{0})(u(m_{0}), \hat{x}) \right| \]

\[ - D_{x}\hat{f}_{u}(m_{0})(u(m_{0}), \hat{x})\{ \tilde{x}_{m_{0}}(m', x) - \tilde{x}_{m_{0}}(m_{0}, x) \} \]

Due to that \( \tilde{x}_{m_{0}}(\cdot) \) is continuous, \( K \) is \( C^{0} \), and \( u \) is Lipschitz, we have

\[ \left| \tilde{Q} \right| \triangleq \left| \tilde{K}_{u}(m_{0})(\tilde{x}_{m_{0}}(m', x)) - \tilde{K}_{u}(m_{0})(\hat{x}) \right| (u(m') - u(m_{0})) \leq O_{\lambda}(m' - m_{0}) \]

As \( u \) is \( C^{1} \) and (6.4), we have

\[ \left| Q \right| \triangleq \left| \tilde{K}_{u}(m_{0})(\hat{x}) \{ u(m') - u(m_{0}) - Du(m_{0})(m' - m_{0}) \} \right| \leq O_{\lambda}(m' - m_{0}) \]

Thus,

\[
\begin{align*}
|Q^{0}(m', m_{0}, x)| & \leq O_{\lambda}(m' - m_{0}) + \lambda''(m_{0})|Q^{0}(u(m'), u(m_{0}), \tilde{x}_{m_{0}}(m', x))| + \lambda''(m_{0})\beta''(u(m_{0}))|\tilde{Q}|.
\end{align*}
\]

The same computation gives

\[ |\tilde{Q}| \triangleq |\tilde{x}_{m_{0}}(m', x) - \tilde{x}_{m_{0}}(m_{0}, x) - \tilde{R}_{m_{0}}(x)(m' - m_{0})| \]

\[ \triangleq \hat{f}_{u}(m_{0})(u(m'), \tilde{x}_{m_{0}}(m', x)) - \hat{f}_{u}(m_{0})(u(m_{0}), \hat{x}) \]

\[ - \hat{K}_{u}(m_{0})(\hat{x})Du(m_{0})(m' - m_{0}) - D_{x}\hat{f}_{u}(m_{0})(u(m_{0}), \hat{x})\tilde{x}_{m_{0}}(x)(m' - m_{0}) \]
\[ \leq O_x(|m' - m_0|) + \alpha''(m_0)|Q^0(u(m'), u(m_0), \tilde{\phi}_{m_0}(m', x))| + \alpha''(m_0)\beta''(u(m_0))|\delta|. \]

Therefore,

\[ |Q^0(m', m_0, x)| \leq O_x(|m' - m_0|) + \frac{\lambda''(m_0)}{1 - \alpha''(m_0)\beta''(u(m_0))}|Q^0(u(m'), u(m_0), \tilde{\phi}_{m_0}(m', x))|. \]

The proof is thus finished. \( \Box \)

Using the above sublemma, we obtain

\[ \sup_{m_0 \in M_1} \sup_{x \in \mathbb{X}} \limsup_{m' \to m_0} \frac{|Q^0(m', m_0, x)|}{|m' - m_0|} \leq \theta''_0 \sup_{m_0 \in M_1} \sup_{x \in \mathbb{X}} \limsup_{m' \to m_0} \frac{|Q^0(m', m_0, x)|}{|m' - m_0|}, \]

where \( \theta''_0 = \sup_{m_0 \in M_1} \frac{\lambda''(m_0)}{1 - \alpha''(m_0)\beta''(u(m_0))} \) < 1, which yields \( \limsup_{m' \to m_0} \frac{|Q^0(m', m_0, x)|}{|m' - m_0|} = 0 \), i.e. \( \tilde{f}_{m_0}(\cdot, x) \) is differentiable at \( m_0 \in M_1 \) and

(6.36)

\[ D_{m_0}\tilde{f}_{m_0}(m, x)|_{m=m_0} = \tilde{K}_{m_0}(x). \]

Finally, we need to show \( f \) is \( C^1 \) everywhere. Let \( m', m_0 \in M \), and choose \( C^1 \) bundle charts \( (U_0, \varphi^0), (U_0, \varphi^0) \) of \( X \), \( Y \) at \( m' \) respectively, such that \( u(U_0) \subset U_{u(m_0)}(\mu^{-1}(u(m_0))\xi_1) \). Note that \( u(m_0) \in M_1 \). Consider \( F, G, f, x_\gamma(\cdot) \) in local bundle charts \( \varphi^0 : U_0 \times X \to X, \varphi^0 : U_0 \times Y \to Y, \varphi(m) : U_u(m_0)(\xi_2) \times X_u(m_0) \to X, \varphi(m) : U_u(m_0)(\xi_2) \times Y_u(m_0) \to Y \). That is

(6.37)

\[ \begin{aligned}
\tilde{F}_{m_0}(m, x, y) &\triangleq (\varphi(m))^{-1} \circ F_m \circ (\varphi(m))(x, \varphi(m)(y)) : U_0 \times X \times Y \to X_u(m_0), \\
\tilde{G}_{m_0}(m, x, y) &\triangleq (\varphi(m))^{-1} \circ G_m \circ (\varphi(m))(x, \varphi(m)(y)) : U_0 \times X \times Y \to Y_u(m_0), \\
\tilde{f}_{m_0}(m, x) &\triangleq (\varphi(m))^{-1} \circ f_m \circ \varphi(m)(x) : U_0 \times X \to X_u(m_0), \\
\tilde{\phi}_{m_0}(m, x) &\triangleq (\varphi(m))^{-1} \circ \varphi(x) : U_0 \times X \to X_u(m_0). 
\end{aligned} \]

Then we have (see also (6.10))

(6.38)

\[ \begin{aligned}
\tilde{F}_{m_0}(m, x, \tilde{f}_{m_0}(u(m), \tilde{\phi}_{m_0}(m, x))) = \tilde{\phi}_{m_0}(m, x), \\
\tilde{G}_{m_0}(m, x, \tilde{f}_{m_0}(u(m), \tilde{\phi}_{m_0}(m, x))) = \tilde{f}_{m_0}(m, x). 
\end{aligned} \]

Due to (H5), we can let \( U_0 \) be much smaller such that if \( m \in U_0 \), then

\[ \text{Lip } \tilde{F}_{m_0}(m, x, \cdot) \leq (1 + \eta(d(u(m), u(m_0))))\alpha(m) + \eta(d(u(m), u(m_0)))) \leq \alpha''(m_0), \]

and \( \alpha''(m_0)\beta''(u(m_0)) < 1 \), where \( \beta''(u(m_0)) \) is given by

\[ \text{Lip } \tilde{f}_{m_0}(u(m), \cdot) \leq (1 + \eta(d(u(m), u(m_0))))\beta'(u(m))(1 + \eta(d(u(m), u(m_0)))) \leq \beta''(u(m_0)). \]

Due to \( \text{Lip } \tilde{F}_{m_0}(m, x, \tilde{f}_{u(m)}(u(m), \cdot) \leq \alpha''(m_0)\beta''(u(m_0)) < 1 \) and the differential of \( \tilde{f}_{u(m)}(\cdot, x) \) at \( u(m_0) \), using Lemma D.3, we know that \( \tilde{f}_{m_0}(\cdot, x) \) is differentiable at \( m_0 \). Since \( m_0 \) can be taken in a small neighborhood of \( m' \), \( \tilde{f}_{m_0}(\cdot, x) \) is differentiable in a neighborhood of \( m' \) and then \( (m, x) \mapsto D_{m_0}\tilde{f}_{m_0}(m, x) \) is continuous (as \( K \) is \( C^0 \) and (6.36)). Moreover, \( (m, x) \mapsto D_x\tilde{f}_{m_0}(m, x) \) is continuous, hence \( f \) is differentiable and consequently \( C^1 \).

Now we have \( \nabla_m f_{m_0}(x) = K_{m_0}(x) \) if \( m \in M_1 \), as \( \varphi^m, \varphi^m \) are normal bundle charts and (6.36). For \( m \in M \setminus M_1 \), this follows from (6.22), \( \nabla_{u(m)} f_{u(m)}(x) = K_{u(m)}(x) \), and taking the covariant derivative of (6.1). The proof is thus finished. \( \Box \)
6.6.2. Hölderness of $x \mapsto \nabla_m f_m(x) = K_m(x)$. (Based on the Hölder continuity of $x \mapsto Df_m(x) = K^1_m(x)$)

It suffices to consider the unique vector bundle map $K$ satisfying (6.22). Since we do not need $\nabla_m f_m(x) = K_m(x)$, some assumptions in Lemma 6.18 can be weakened; see the proof of Lemma 6.18.

**Lemma 6.22.** Assume the following conditions hold.

(a) Let (H1) (H2) (i) (ii) (v) (1') (H3) (H4b) (H5') hold.

(b) (about $\mathcal{F}$) (i) $F, G \in C^1$ and (ii) satisfy

$$|\nabla_m F_m(z)| \leq M_0|z|, \ |\nabla_m G_m(z)| \leq M_0|z|,$$

for all $m \in M_1$ and $z \in X \times Y \times \mathcal{M},$ (iii) Moreover, $\nabla_m F_m(\cdot), \nabla_m G_m(\cdot), D F_m(\cdot), D G_m(\cdot)$ are $C^{0, \gamma}$ uniform for $m$, i.e.

$$\begin{cases}
|\nabla_m F_m(z_1) - \nabla_m F_m(z_2)| \leq M_0|z_1 - z_2|^{\gamma}, & |\nabla_m G_m(z_1) - \nabla_m G_m(z_2)| \leq M_0|z_1 - z_2|^{\gamma}, \\
|DF_m(z_1) - DF_m(z_2)| \leq M_0|z_1 - z_2|^{\gamma}, & |DG_m(z_1) - DG_m(z_2)| \leq M_0|z_1 - z_2|^{\gamma},
\end{cases}$$

for all $z_1, z_2 \in X \times Y \times \mathcal{M}$, $m \in M$, where $0 < \gamma \leq 1$.

(c) (spectral gap condition) $\lambda_x \lambda_x < 1, \lambda_x \lambda_x < 1, \alpha^2 \lambda_x \lambda_x < 1,$ max$\{\alpha^2 \lambda_x, \alpha^2 \lambda_x \lambda_x \} < 1$, where $\gamma \geq \beta$; see also Remark 6.1.

Then the $C^0$ vector bundle map $K \in L(T^H_X, Y^V_Y)$ over $f$ satisfying (6.22) has the following Hölder property:

$$|K_m(x_1) - K_m(x_2)| \leq C \left\{ (|x_1 - x_2|^\gamma (1 + |x_1|) + (|x_1 - x_2|^\beta |x_1|)^\gamma (1 + |x_2|)^{1-\alpha}, \right.$$

where the constant $C$ depends on the constant $\hat{r}$ but not $m \in M$.

**Proof.** First note that under these conditions, by Lemma 6.11, we have $Df_m(x) = K^1_m(x)$ and $|K^1_m(x_1) - K^1_m(x_2)| \leq \overline{C}|x_1 - x_2|^\beta$. Also, $\sup_{m \in M_1} \sup_{x \neq x(m)} |K^1_m(x)|/|x| < \infty$ by Lemma 6.18 (1). Using (6.22), one gets

$$K_m(x_1) - K_m(x_2) = \nabla_m G_m(x_1, f(m)(x_1(m)) - \nabla_m G_m(x_2, f(m)(x_2(m))) + D G_m(x_1, f(m)(x_1(m)))Df_m(m) + D G_m(x_2, f(m)(x_2(m)))Df_m(m) - K_{u(m)}(x(m))|u(m)|R_m(x_1) - R_m(x_2)|.$$ 

Now we have for $m \in M$,

$$|K_m(x_1) - K_m(x_2)| \leq \overline{C}|x_1 - x_2|^\gamma (1 + |x_1|) + \overline{C}|x_1 - x_2|^\beta |x_1| + \lambda_x(m)|u(m)|R_m(x_1) - R_m(x_2)|,$$

and similarly,

$$|R_m(x_1) - R_m(x_2)| \leq \overline{C}|x_1 - x_2|^\gamma (1 + |x_1|) + \overline{C}|x_1 - x_2|^\beta |x_1| + \alpha(m)|u(m)|R_m(x_1) - R_m(x_2)|.$$ 

Hence,

$$|K_m(x_1) - K_m(x_2)| \leq \overline{C}|x_1 - x_2|^\gamma (1 + |x_1|) + \overline{C}|x_1 - x_2|^\beta |x_1| + \lambda_x(m)|u(m)|R_m(x_1) - R_m(x_2)|.$$ 

Apply the argument in Appendix A to conclude the proof (see Remark A.10 (d)).
Under the conditions in Lemma 6.18, we know that if \( F, G \) are uniformly \( C^{1,1} \), then from this lemma, \( \nabla_{m} f_{m}(\cdot) \) at least is locally Hölder uniform for \( m \in M \).

### 6.7. Hölderness of \( m \mapsto \nabla_{m} f_{m}(x) = K_{m}(x) \)

(Based on all the previous lemmas)

Let \( K \in L(T_{X}^{H}, T_{X}^{V}) \) over \( f \) obtained Lemma 6.18. In particular, \( K \) and \( R \in L(T_{X}^{H}, T_{X}^{V}) \) over \( x_{i}(\cdot) \) satisfy (6.22). Write \( \tilde{D}_{m} F_{m}(\cdot, \cdot), \tilde{D}_{m} G_{m}(\cdot, \cdot), \tilde{K}_{m}(\cdot, \cdot), \tilde{R}_{m}(\cdot, \cdot) \) as the local representations of \( \nabla_{m} F_{m}, \nabla_{m} G_{m}, K, R \) at \( m_{0} \in M_{1} \) with respect to \( A, B, M \) (see e.g. (6.31) (6.32) (6.30) but using the local bundle charts belonging to \( A, B, M \)). We rewrite them explicitly here for convenience of readers; see also Appendix B.0.3 and Appendix B.0.4.

\[
\begin{align*}
\tilde{D}_{m} F_{m}(m, x, y) & \triangleq D(\varphi_{u(m)}^{i(m)})^{-1}(F_{m}(z)) \nabla_{m} F_{m}(z) D\chi_{m_{0}}^{-1}(\chi_{m_{0}}(m)) : \quad U_{m_{0}}(\epsilon') \times X_{m_{0}} \times Y_{u(m_{0})} \rightarrow L(T_{m_{0}}M, Y_{u(m_{0}))}, \\
\tilde{D}_{m} G_{m_{0}}(m, x, y) & \triangleq D(\varphi_{u(m)}^{m_{0}})^{-1}(G_{m}(z)) \nabla_{m} G_{m}(z) D\chi_{m_{0}}^{-1}(\chi_{m_{0}}(m)) : \quad U_{m_{0}}(\epsilon') \times X_{m_{0}} \times Y_{u(m_{0})} \rightarrow L(T_{m_{0}}M, Y_{u(m_{0}))},
\end{align*}
\]

where \( z = (\varphi_{m_{0}}^{m_{0}}(x), \varphi_{u(m_{0})}^{m_{0}}(y)) \);

\[
\begin{align*}
\tilde{K}_{m_{0}}(m, x) & \triangleq D(\varphi_{u(m)}^{m_{0}})^{-1}(f_{m}(\varphi_{m}^{m_{0}}(x))) K_{m}(\varphi_{m_{0}}^{m_{0}}(x)) D\chi_{m_{0}}^{-1}(\chi_{m_{0}}(m)) : \quad U_{m_{0}}(\epsilon') \times X_{m_{0}} \rightarrow L(T_{m_{0}}M, X_{u(m_{0})}), \\
\tilde{R}_{m_{0}}(m, x) & \triangleq D(\varphi_{u(m)}^{m_{0}})^{-1}(x_{m}(\varphi_{m}^{m_{0}}(x))) R_{m}(\varphi_{m_{0}}^{m_{0}}(x)) D\chi_{m_{0}}^{-1}(\chi_{m_{0}}(m)) : \quad U_{m_{0}}(\epsilon') \times X_{m_{0}} \rightarrow L(T_{m_{0}}M, X_{u(m_{0})}).
\end{align*}
\]

By analogy with (6.34), we know they satisfy

\[
\begin{align*}
\tilde{D}_{m} F_{m_{0}}(m, x, \hat{y}) + D_{y} \tilde{F}_{m_{0}}(m, x, \hat{y})
\cdot \left\{ \tilde{K}_{u(m_{0})}(u(m), \hat{x}) \tilde{D}u_{m_{0}}(m) + \tilde{R}_{u(m_{0})}(u(m), \hat{x}) \tilde{R}_{m_{0}}(m, x) \right\} = \tilde{R}_{m_{0}}(m, x),
\end{align*}
\]

\[
\begin{align*}
\tilde{D}_{m} G_{m_{0}}(m, x, \hat{y}) + D_{y} \tilde{G}_{m_{0}}(m, x, \hat{y})
\cdot \left\{ \tilde{K}_{u(m_{0})}(u(m), \hat{x}) \tilde{D}u_{m_{0}}(m) + \tilde{R}_{u(m_{0})}(u(m), \hat{x}) \tilde{R}_{m_{0}}(m, x) \right\} = \tilde{K}_{m_{0}}(m, x),
\end{align*}
\]

where \( \hat{x} = \hat{x}_{m_{0}}(m, x), \hat{y} = \hat{u}_{m_{0}}(u(m), \hat{x}), \tilde{K}_{u(m_{0})}(u(m), \hat{x}) = D_{x} \hat{u}_{m_{0}}(u(m), \hat{x}). \)

**Lemma 6.23.** Assume the following conditions hold.

(a) Let (H1c) (H2d) (H3) (H4c) (H5) hold.

(b) (about \( F, G \)) \( F, G \) are \( C^{1} \) and uniformly (locally) \( C^{1,1} \) around \( M_{1} \) in the following sense. \( F, G \) are \( C^{1} \), and (6.23) (see also Remark 6.14), (6.6) \( (D'F, D'G \in C^{0,1}) \) and (6.7) \( (\nabla F, \nabla G \in C^{0,1}) \) hold.

(c) (spectral gap condition) \( \lambda_{s} \lambda_{u} < 1, \lambda_{s}^{2} \lambda_{u} < 1, \lambda_{s} \lambda_{u} \mu < 1, \lambda_{s}^{2} \lambda_{u} \mu < 1, \max\{\frac{\mu}{\lambda_{s}}, \mu\}^{a} \lambda_{s} \lambda_{u} \mu < 1 \).

\( 0 < \alpha \leq 1. \) See Remark 6.1.

Then there exists \( \epsilon_{1} > 0 \) small such that the following hold.

(1) We have

\[
|\tilde{K}_{m_{0}}(m_{1}, x) - \tilde{K}_{m_{0}}(m_{0}, x)| \leq C|m_{1} - m_{0}|^{\alpha}(|x| + 1)^{\alpha} |x|^{1-\alpha},
\]

for all \( m_{1} \in U_{m_{0}}(\epsilon'_{1}), x \in X_{m_{0}}, m_{0} \in M_{1} \). The constant \( C \) depends on the constant \( \epsilon'_{1} > 0 \) but not \( m_{0} \in M_{1} \).

(2) Suppose \( \lambda_{s} < 1 \) and \( M \) is \( C^{1,1} \)-uniform around \( M_{1} \) (see Definition 5.33). If \( F, G \) satisfy additional estimates,

\[
|\tilde{D}_{m} F_{m}(m_{1}, z) - \tilde{D}_{m} F_{m}(m_{0}, z)| \leq M_{0}|m_{1} - m_{0}| |z|^{\epsilon'},
\]

\[
|\tilde{D}_{m} G_{m}(m_{1}, z) - \tilde{D}_{m} G_{m}(m_{0}, z)| \leq M_{0}|m_{1} - m_{0}| |z|^{\epsilon'},
\]
for all \( m_1 \in U_{m_0}(e) \), \( z_1, z_2 \in X_{m_0} \times Y_{u(m_0)} \), \( m_0 \in M_1 \), where \( 0 < \zeta \leq 1 \), and (a ‘better’ spectral gap condition) \( \lambda_s \lambda_u < 1, \lambda_s \lambda_u \mu < 1, (\frac{\mu}{\lambda_s})^{a_s} \lambda_s \lambda_u \mu < 1 \) hold, where \( 0 < \alpha \leq 1 \), then

\[
|\tilde{K}_{m_0}(m_1, x) - \tilde{K}_{m_0}(m_0, x)| \leq C|m_1 - m_0|^\alpha (|x| + |x|^\zeta)^{a_s} |x|^{1-\alpha},
\]

for all \( m_1 \in U_{m_0}(e) \), \( m_0 \in M_1 \), \( x \in X_{m_0} \). The case \( \zeta = 1 \) is satisfied, e.g. \( F_i, G_i \in C^{1,1}, \lambda_s \lambda_u < 1, \lambda_s \lambda_u \mu < 1, 0 < \alpha \leq 1 \), then

\[
|\tilde{K}_{m_0}(m_1, x) - \tilde{K}_{m_0}(m_0, x)| \leq C|m_1 - m_0|^\alpha |x|.
\]

The result like (1) was also obtained in [ZZJ14, Lemma 3, Page 82] for the trivial bundle and invertible map, where the authors used the Perron method; see also [Stu99, Theorem 1.3]. The result like (2) is well known in different settings, see e.g. [PSW97].

**Proof of Lemma 6.23.** We need the desired Lipschitz continuity of \( m \mapsto f_m(x) \), \( x \mapsto K^1_m(x) \), \( K_m(x) \) and \( m \mapsto K^1_m(x) \).

(a) From Lemma 6.11 and Lemma 6.22, under the spectral gap condition: \( \lambda_s \lambda_u < 1, \lambda_s \lambda_u \mu < 1, \lambda_s \lambda_u \mu < 1 \), we see

\[
|K^1_m(x_1) - K^1_m(x_2)| \leq C_0|x_1 - x_2|, \quad |K_m(x_1) - K_m(x_2)| \leq C_0|x_1 - x_2|(|x_1| + 1).
\]

(b) From Lemma 6.13, under (6.23) and the spectral gap condition: \( \lambda_s \lambda_u < 1, \lambda_s \lambda_u \mu < 1 \), we know

\[
|\tilde{f}_{m_0}(m_1, x) - \tilde{f}_{m_0}(m_0, x)| \leq C_0|m_1 - m_0||x|,
\]

for every \( m_1 \in U_{m_0}(e) \), where \( e \) is small and independent of \( m_0 \in M_1 \).

c) Similarly, from Lemma 6.17, under spectral gap condition: \( \lambda_s \lambda_u < 1, \lambda_s \lambda_u \mu < 1, \lambda_s \lambda_u \mu < 1 \), we have

\[
|\tilde{K}^1_{m_0}(m_1, x) - \tilde{K}^1_{m_0}(m_0, x)| \leq C_0|m_1 - m_0|(|x| + 1),
\]

for \( m_1 \in U_{m_0}(e) \).

First observe that by (6.42),

\[
\tilde{K}_{m_0}(m_1, x) - \tilde{K}_{m_0}(m_0, x) = \tilde{D}_m F_{m_0}(m_1, x, \hat{y}_1) - \tilde{D}_m F_{m_0}(m_0, x, \hat{y}_0) + \left\{ D_y \tilde{G}_{m_0}(m_1, x, \hat{y}_1) - D_y \tilde{G}_{m_0}(m_0, x, \hat{y}_0) \right\}
\]

\[
\times \left\{ \tilde{K}_{u(m_0)}(u(m_1), \hat{x}_1) \tilde{D}_{u(m_0)}(m_1) + \tilde{K}_{u(m_0)}(u(m_1), \hat{x}_1) \tilde{R}_{m_0}(m_1, x) \right\}
\]

\[
+ D_y \tilde{G}_{m_0}(m_0, x, \hat{y}_0) \left\{ \tilde{K}_{u(m_0)}(u(m_1), \hat{x}_1) \tilde{D}_{u(m_0)}(m_1) + \tilde{K}_{u(m_0)}(u(m_1), \hat{x}_1) \tilde{R}_{m_0}(m_1, x) \right\}
\]

\[
- \tilde{K}_{u(m_0)}(u(m_0), \hat{x}_0) \tilde{D}_{u(m_0)}(m_0) - \tilde{K}_{u(m_0)}(u(m_0), \hat{x}_0) \tilde{R}_{m_0}(m_0, x),
\]

for \( m_1 \in U_{m_0}(e) \), where \( \hat{x}_i = \tilde{x}_{u(m_0)}(m_i, x), \hat{y}_i = \tilde{u}_{u(m_0)}(u(m_i), \hat{x}_i), i = 0, 1 \). \( e \) is taken further smaller, such that the functions \( \alpha', \beta', \lambda''', \lambda''' \) in the proof of Lemma 6.13, satisfy (6.13) (6.19), and the function \( \mu'' \) satisfies (H4c), \( |\tilde{D}_{u(m_0)}(m_1)| \leq \mu''(m_0) \) for \( m_1 \in U_{m_0}(e) \), and in addition these functions fulfill the following spectral gap condition,

\[
\lambda_s \lambda_u \mu < 1, \lambda_s^2 \lambda_u < 1, \lambda'' \lambda_u \mu'' < 1, \lambda_s^2 \lambda_u \mu'' < 1, \max\{\frac{\mu''}{\lambda_s'}, \mu''\}^{a_s} \lambda'' \lambda_u \mu'' < 1.
\]

Compute

\[
|\tilde{K}_{m_0}(m_1, x) - \tilde{K}_{m_0}(m_0, x)|
\]

\[
\leq C|m_1 - m_0||1 + |x| + \lambda''(m_0)| \tilde{K}_{u(m_0)}(u(m_1), \hat{x}_1) \tilde{D}_{u(m_0)}(m_1) - \tilde{K}_{u(m_0)}(u(m_0), \hat{x}_0) \tilde{D}_{u(m_0)}(m_0)|
\]

\[
+ \lambda''(m_0) \tilde{K}_{u(m_0)}(u(m_1), \hat{x}_1) \tilde{R}_{m_0}(m_1, x) - \tilde{K}_{u(m_0)}(u(m_0), \hat{x}_0) \tilde{R}_{m_0}(m_0, x)|
\]

\[
\leq C|m_1 - m_0||1 + |x| + \lambda''(m_0)\mu''(m_0)| \tilde{K}_{u(m_0)}(u(m_1), \hat{x}_1) - \tilde{K}_{u(m_0)}(u(m_0), \hat{x}_1)|
\]
Similarly,
\[ |\tilde{R}_{m_0}(m_1, x) - \tilde{R}_{m_0}(m_0, x)| \]
\[ \leq \tilde{C}|m_1 - m_0|(1 + |x|) + \alpha''(m_0)\mu''(m_0)|\tilde{K}_{u(m_0)}(u(m_1), \hat{x}_1) - \tilde{K}_{u(m_0)}(u(m_0), \hat{x}_1)| + \alpha''(m_0)\beta''(u(m_0))|\tilde{R}_{m_0}(m_1, x) - \tilde{R}_{m_0}(m_0, x)|. \]

Thus,
\[ |\tilde{R}_{m_0}(m_1, x) - \tilde{R}_{m_0}(m_0, x)| \]
\[ \leq \tilde{C}|m_1 - m_0|(1 + |x|) + \frac{\lambda''(m_0)\mu''(m_0)}{1 - \alpha''(m_0)\beta''(u(m_0))}|\tilde{K}_{u(m_0)}(u(m_1), \hat{x}_1) - \tilde{K}_{u(m_0)}(u(m_0), \hat{x}_1)| + \lambda''(m_0)\beta''(u(m_0))|\tilde{R}_{m_0}(m_1, x) - \tilde{R}_{m_0}(m_0, x)|. \]

Now we can apply the argument in Appendix A to conclude the proof of (1) (see Remark A.10 (e)). The proof of (2) is the same as (1), where the following estimate is needed.
\[ |\tilde{R}_{m_0}(m_1, x) - \tilde{R}_{m_0}(m_0, x)| \]
\[ \leq \tilde{C}|m_1 - m_0|(1 + |x|) + \lambda''(m_0)\mu''(m_0)|\tilde{K}_{u(m_0)}(u(m_1), \hat{x}_1) - \tilde{K}_{u(m_0)}(u(m_0), \hat{x}_1)| + \lambda''(m_0)\beta''(u(m_0))|\tilde{R}_{m_0}(m_1, x) - \tilde{R}_{m_0}(m_0, x)|. \]

The proof is thus finished. \[\square\]

Remark 6.24 (Lipschitz continuity of connection). How about the $C^{1,\gamma}$ continuity of $f$? We need the Lipschitz continuity of the connections $C^X, C^Y$ (see also Definition B.8). Let $(U_{m_0}(e'), \varphi_{m_0}) \in \mathcal{A}$, and
\[ \tilde{D}_m \varphi_{m_0}(m, x) \triangleq (D \varphi_{m_0})^{-1}(\varphi_{m_0}(x))\nabla_m \varphi_{m_0}(x)D \chi^{-1}_m(\chi_{m_0}(m)) \]
\[ = \tilde{\Gamma}_{m_0}^{\text{(m,x)}}(\chi_{m_0}(m)) : U_{m_0}(e'_m) \times X_{m_0} \rightarrow L(T_{m_0}M, X_{m_0}), \]
where $\tilde{\Gamma}_{m_0}^{\text{(m,x)}}$ is the Christoffel map in the bundle chart $\varphi_{m_0}$ (see Definition 5.10). We say the connection $C^X$ is uniformly (locally) Lipschitz around $M_1$ with respect to $\mathcal{A}, M$, if
\[ \sup_{m_0 \in M} \sup_{m \in U_{m_0}(e')} \mathop{\text{Lip}}\limits_{x} \tilde{D}_m \varphi_{m_0}(m_1, x) \leq C|x||m_1 - m_0|, \quad m_1 \in U_{m_0}(e'_m), \]
where $C$. (Note that what this means for the linear connection.) We have
\[ D_m \tilde{f}_{m_0}(m, x) = \nabla_m (\phi_m^{m_0} - 1)(x') + D(\phi_m^{m_0} - 1)(x') \left\{ \nabla_{m_0} f_m(x') + Df_m(x') \nabla_m \varphi_{m_0}(x) \right\}, \]
where $x' = \varphi_m^{m_0}(x)$, $x'' = f_m(x')$. From this, one can deduce the Hölder continuity of $m \mapsto D_m \tilde{f}_{m_0}(m, x)$, i.e.
\[ m \mapsto D_m \tilde{f}_{m_0}(m, x)D \chi^{-1}_m(\chi_{m_0}(m)), \]
is Hölder, and in addition $x \mapsto D_m \tilde{f}_{m_0}(m, x)$ is (uniformly) Lipschitz from the $C^{0,1}$ continuity of $x \mapsto \nabla_m f_{m_0}(x), Df_{m_0}(x)$. Thus, through using Hölder continuity of $(m, x) \mapsto Df_m(x), \nabla_m f_m(x)$ (and the uniformly Lipschitz continuity of $C^X, C^Y$), the $C^{1,\gamma}$ continuity of $f$ is well understood.

6.8. appendix. Lipschitz continuity respecting base points. One can give a more classical way to describe the Hölder continuity of $m \mapsto f_m(x), K_m^1(x), K_m(x)$, if some Lipschitz property of the transition maps is assumed. Let $M_1^\varepsilon = \bigcup_{m_0 \in M_1} U_{m_0}(\varepsilon)$. Use $M_1^{\varepsilon_1}$ instead of $M_1$ in all assertions in Section 6.4 to Section 6.7 for some small $\varepsilon_1 > 0$.

(*) When (H1a) or (H1b) holds, we assume $M$ is a uniformly locally metrizable space (see Definition 5.5); note that if (H1c) holds, then $M$ already is a uniformly locally metrizable space. Moreover, assume $X, Y$ have $\varepsilon$-almost $C^{0,1}$-uniform trivializations (i.e. (R) below holds) if (H1a) holds, or $\varepsilon$-almost $C^{1,1}$-fiber-uniform trivializations (i.e. (R) and (S) below hold) if (H1b) or
Let $\phi^{m_0,m_1} = (\varphi^m)_{1}^{-1} \circ \varphi^m : (W_{e_0}, d_{m_0}) \times X_{m_0} \to (W_{e_0}, d_{m_1}) \times X_{m_1}$,

$\phi^{m_0,m_1} = (\varphi^{m_1})_{1}^{-1} \circ \varphi^{m_0} : (W_{e_0}, d_{m_0}) \times Y_{m_0} \to (W_{e_0}, d_{m_1}) \times Y_{m_1}$,

$\chi^{m_0,m_1}(m') = D\chi^m(m') (D\chi^m_0(m))^{-1} : W_{e_{m_0}} \to L(T_{m_0}M, T_{m_1}M)$,

where $W_{e_0} = U_{m_0}(e_0) \land U_{m_1}(e_0) \neq \emptyset$, satisfy the following Lipschitz conditions,

($\mathcal{R}$) $\text{Lip} \phi^{m_0,m_1}(x) \leq \hat{h}_{m_0}^{X_0}(x)$, $\text{Lip} \phi^{m_0,m_1}(y) \leq \hat{h}_{m_0}^{Y_0}(y)$,

($\mathcal{S}$) $\text{Lip} D_x \phi^{m_0,m_1}(x) \leq \hat{h}_{m_0}^{X_1}(x)$, $\text{Lip} D_x \phi^{m_0,m_1}(y) \leq \hat{h}_{m_0}^{Y_1}(y)$,

($\mathcal{N}$) $\text{Lip} \chi^{m_0,m_1}(\cdot) \leq M_0$,

where the functions $\hat{h}_{m_0}^{X_0}(\cdot)$, $\hat{h}_{m_0}^{Y_0}(\cdot) : X_{m_0} \to \mathbb{R}_+$, $i = 0, 1$, satisfy

$\hat{h}_{m_0}^{X_0}(x) \leq M_0|x|$, $\hat{h}_{m_0}^{Y_0}(x) \leq M_0, \hat{h}_{m_0}^{X_0}(y) \leq M_0|y|, \hat{h}_{m_0}^{Y_1}(y) \leq M_0,$

where $M_0 > 0$ is a constant independent of $m_0 \in M_1^{e_1}, x, y$.

Take small $\varepsilon'_2 < \varepsilon_1$. Now we have the following statements.

(a) In Lemma 6.13, (6.12) can be changed as

$$|\widehat{f}_{m_0}(m_1, x) - \widehat{f}_{m_0}(m_2, x)| \leq C|m_1 - m_2|^{\gamma|a|x}|x|^\zeta a + 1 - a,$$

for every $m_1, m_2 \in U_{m_0}(e_1^e), x \in X_{m_0}, m_0 \in M_1^{e_1}$, under $|m_i - m_0|^\gamma |x| \leq \hat{f} \min \{|x|, |x|^\zeta c - (c - 1)\}$, $i = 1, 2$.

(b) In Lemma 6.16, (6.17) can be taken as

$$|\widehat{K}_{m_0}(m_1, iX(m_0)) - \widehat{K}_{m_0}(m_2, iX(m_0))| \leq C|m_1 - m_2|^{\gamma|a|x},$$

for every $m_1, m_2 \in U_{m_0}(e_1^e), m_0 \in M_1^{e_1}$.

In Lemma 6.17, (6.18) is taken as

$$|\widehat{K}_{m_0}(m_1, x) - \widehat{K}_{m_0}(m_2, x)| \leq C \left\{ |m_1 - m_2|^\alpha (|x| + 1) + (|m_1 - m_2|^\alpha |x|)^\beta \right\},$$

for every $m_1, m_2 \in U_{m_0}(e_1^e), x \in X_{m_0}, m_0 \in M_1^{e_2}$.

(c) In Lemma 6.23, (6.43) now is strengthened as

$$|\widehat{K}_{m_0}(m_1, x) - \widehat{K}_{m_0}(m_2, x)| \leq C|m_1 - m_2|^{\alpha (|x| + 1)^\alpha |x|} 1 - a,$$

for all $m_1, m_2 \in U_{m_0}(e_1^e), x \in X_{m_0}, m_0 \in M_1^{e_2}$.

Here $|m_1 - m_2| = d_{m_0}(m_1, m_2)$, where $d_{m_0}$ is the metric in $U_{m_0}$.

Proof. As an example, we only consider the case $m \mapsto f(m) \in X$ addition with that in Lemma 6.13, $\alpha = 1, \gamma = 1, \zeta = 1$. Let $m'_0 \in U_{m_0}(e_1), m_0 \in M_1^{e_2}, e_1 < e_2$, where $e_1, e'_1 < e_2$ are chosen such that $U_{m_0}(e_1) \subset M_1^{e_2}$ for all $m_0 \in M_1^{e_2}$. Note that we have the following local change of coordination,

$$\widehat{f}_{m_0}(m, x) = \phi_{m_0}^{m_0, m_0} \circ \widehat{f}_{m_0}(m, \varphi_{m, m_0, m_0}^{m_0}(x)).$$

Let $m_1, m_0' \in U_{m_0}(e_1)$. Let us compute

$$|\widehat{f}_{m_0}(m_1, x) - \widehat{f}_{m_0}(m_0', x)| = \left| \phi_{m_0}^{m_0, m_0} \circ \widehat{f}_{m_0}(m_1, \varphi_{m_1, m_0, m_0}^{m_0}(x)) - \phi_{m_0}^{m_0, m_0} \circ \widehat{f}_{m_0}(m_0', \varphi_{m_0', m_0, m_0}^{m_0}(x)) \right|$$

$$\leq \left| \phi_{m_0}^{m_0, m_0} \circ \widehat{f}_{m_0}(m_1, \varphi_{m_1, m_0, m_0}^{m_0}(x)) - \phi_{m_0}^{m_0, m_0} \circ \widehat{f}_{m_0}(m_1, \varphi_{m_1, m_0, m_0}^{m_0}(x)) \right|$$

$$+ \left| \phi_{m_0}^{m_0, m_0} \circ \widehat{f}_{m_0}(m_0', \varphi_{m_0, m_0, m_0}^{m_0}(x)) - \phi_{m_0}^{m_0, m_0} \circ \widehat{f}_{m_0}(m_0', \varphi_{m_0, m_0, m_0}^{m_0}(x)) \right|$$
\[ \begin{align*}
\leq & \frac{M_0}{|f_{m_0}(m_1, \varphi_{m_1}^{m_0,m_0'}(m_1, x))|} |m_1 - m_0'|_{m_0'} + \text{Lip } \phi_{m_0'}^{m_0,m_0'}(\cdot) \left| f_{m_0}(m_1, \varphi_{m_1}^{m_0,m_0'}(x)) - f_{m_0}(m_0', \varphi_{m_0'}^{m_0,m_0'}(x)) \right| \\
\leq & C|m_1 - m_0'|_{m_0}|x| + \text{Lip } \phi_{m_0'}^{m_0,m_0'}(\cdot) \\
& \cdot \left\{ \left| f_{m_0}(m_1, \varphi_{m_1}^{m_0,m_0'}(x)) - f_{m_0}(m_0', \varphi_{m_1}^{m_0,m_0'}(x)) \right| + \left| f_{m_0}(m_0', \varphi_{m_1}^{m_0,m_0'}(x)) - f_{m_0}(m_0', \varphi_{m_0'}^{m_0,m_0'}(x)) \right| \right\} \\
\leq & C_0|m_1 - m_0'|_{m_0}|x|.
\end{align*} \]

for \( m_0, m_0' \in U_{m_0}(\epsilon) \), and some constant \( C_0 \) being independent of \( m_0 \in M_1 \), where \( \epsilon \), is taken smaller than \( \epsilon', \epsilon'\), independent of \( m_0 \).

6.9. \textbf{Continuity properties of } \( f \). For a bundle map \( g : X \to Y \), one can talk about its fiber-continuity (i.e. \( x \mapsto g_m(x) \)) and base-continuity (i.e. \( m \mapsto g_m(x) \)). The \( C^0 \) continuity of \( m \mapsto g_m(x) \) or \( (m, x) \mapsto g_m(x) \) is easy. Let us consider the continuity in uniform sense. The following different types of continuity usually arise in applications; see also Appendix B.0.1 and Appendix B.0.2.

(a) The uniform \( C^0 \)-fiber case: (i) for each \( m \in M \), \( x \mapsto g_m(x) \) is uniformly continuous; (ii) \( x \mapsto g_m(x) \) is uniformly continuous uniform for \( m \), i.e. \( x \mapsto g_m(x) \), \( m \in M \), are equicontinuous.

(b) The uniform \( C^0 \)-base case. (i) \( m \mapsto g_m(\cdot) \) is continuous or uniformly continuous in \( C^0 \)-topology in bounded sets or in the whole space. This means for the local representation \( \tilde{g} \) of \( g \) with respect to the bundle atlases of \( A, B \), one has \( L[\tilde{g}_{m_0}(m', x) - \tilde{g}_{m_0}(m_0, x)] = 0 \), where \( L \) stands for the following four different limits respectively:

\[ \begin{align*}
L_{1,\text{base}}^r : \lim_{r \to 0} \sup_{m_0 \in M} \sup_{m' \to m_0} \sup_{x \in X_{m_0}} \sup_{|x| \leq r} |m_0 - m'|_{m_0} \\
L_{2,\text{base}}^r : \lim_{r \to 0} \sup_{m_0 \in M} \sup_{x \in X_{m_0}} \sup_{|x| \leq r} |m_0 - m'|_{m_0}.
\end{align*} \]

(ii) \( m \mapsto g_m(\cdot) \) is continuous or uniformly continuous in \( C^1 \)-topology in bounded sets or in all space. This means that \( L[\tilde{g}_{m_0}(m', x) - \tilde{g}_{m_0}(m_0, x)] = 0 \) and \( L[D_x g_m(m', x) - D_x g_{m_0}(m_0, x)] = 0 \), where \( D_x g_m \) is the local representation of the fiber derivative \( D_x g \), i.e. \( D_x \tilde{g}_{m_0} = D_x \tilde{g}_{m_0} \).

In the following, we will consider the continuity of \( (m, x) \mapsto f_{m_0}(x), K_{m_0}^{m_0'}(x) \) at \( m_0 \in M_1 \) with respect to \( A, B \); see (6.9).

We say \( m' \to m_0 \in M_1 \) is in the \textit{uniform} sense meaning that the limit \( \lim_{m' \to m_0} \) is replaced by \( \sup_{m \to m_0} f_{m_0}(x), K_{m_0}^{m_0'}(x) \).

\textbf{Lemma 6.25} \textbf{(continuity of } } \( f \). \textbf{Assume the following hold.}

(a) \textbf{Let} (H1a) (H2) (iii) (v) (1’) (H3’) (H4) (H5’) \textbf{hold.}

(b) \textbf{(about } \( F, G \)) \textbf{Assume } \( F, G \) \text{ are continuous. (spectral gap condition) } \lambda_x \lambda_u < 1; \text{ see Remark 6.1.}

(1) \textbf{Then } \( f : X \to Y \) \text{ is continuous.}

Moreover, the stronger continuity of \( F, G \), the stronger that of \( f \), i.e. the following hold.

(2) \textbf{If } \( F, G \) \text{ satisfy for any } \( r > 0 \)

\[ \begin{align*}
\lim_{m' \to m_0} \sup_{0 \not= |z - i_{m_0}^1| \leq r} \frac{|\tilde{F}_{m_0}(m', z) - \tilde{F}_{m_0}(m_0, z)|}{|z - i_{m_0}^1|} &= 0, \\
\lim_{m' \to m_0} \sup_{0 \not= |z - i_{m_0}^1| \leq r} \frac{|\tilde{G}_{m_0}(m', z) - \tilde{G}_{m_0}(m_0, z)|}{|z - i_{m_0}^1|} &= 0,
\end{align*} \]

where \( i_{m_0}^1 = (i_X(m_0), i_Y(u(m_0))) \), \( z \in X_{m_0} \times Y_{u(m_0)} \), \( m_0 \in M_1 \), then

\[ \lim_{m' \to m_0} \sup_{0 \not= |x - i_X(m_0)| \leq r} \frac{|\tilde{F}_{m_0}(m', x) - \tilde{F}_{m_0}(m_0, x)|}{|x - i_X(m_0)|} = 0. \]
If \( F, G \) satisfy (6.46) for \( r = \infty \), so is \( f \).

(3) Assume that \( i : M \to X \times Y \) is uniformly continuous around \( M_1 \) (see Definition 5.3) and (H3’’) (H4’) hold; if in (6.46) \( m' \to m_0 \in M_1 \) is in the uniform sense, so is \( f \).

Proof. We first show \( f \) is continuous at \((m_0, x), m_0 \in M_1, x \in X_{m_0}\). It suffices to show for fixed \( x, m \mapsto \tilde{f}_{m_0}(m, x) \) is continuous at \( m_0 \), as \( x \mapsto \tilde{f}_{m_0}(m, x) \) is Lipschitz locally uniform for \( m \). By (H5’), we have \( \lim \sup_{m' \to m_0} |\kappa(m') - \kappa(m_0)| \leq \varepsilon \) where \( \kappa(x) \) is taken as the functions in (A’) (or (A)) (B) condition; set \( \kappa''(m_0) = \kappa(m_0) + \varepsilon \). From the computation in the proof Lemma 6.13, we have

\[
\limsup_{m' \to m_0} \left| \tilde{f}_{m_0}(m', x) - \tilde{f}_{m_0}(m_0, x) \right| \leq \frac{\lambda''(m_0)}{1 - \alpha''(m_0) / \beta''(u(m_0))} \limsup_{m' \to m_0} \left| \tilde{f}_{u(m_0)}(u(m'), \tilde{x}_{m_0}(m_0, x)) - \tilde{f}_{u(m_0)}(u(m_0), \tilde{x}_{m_0}(m_0, x)) \right|.
\]

Set \( \theta = \sup_{m_0 \in M_1} \frac{\lambda''(m_0) / \beta''(u(m_0))}{1 - \alpha''(m_0) / \beta''(u(m_0))} < 1 \) (by taking \( \varepsilon \) small). Then

\[
\sup_{m_0 \in M_1} \sup_{x \in X_{m_0}} \limsup_{m' \to m_0} \frac{\left| \tilde{f}_{m_0}(m', x) - \tilde{f}_{m_0}(m_0, x) \right|}{\left| x - iX(m_0) \right|} \leq \theta \sup_{m_0 \in M_1} \sup_{x \in X_{m_0}} \limsup_{m' \to m_0} \frac{\left| \tilde{f}_{u(m_0)}(u(m'), \tilde{x}_{m_0}(m_0, x)) - \tilde{f}_{u(m_0)}(u(m_0), \tilde{x}_{m_0}(m_0, x)) \right|}{\left| \tilde{x}_{m_0}(m_0, x) - iX(u(m_0)) \right|}.
\]

Note that

\[
\sup_{m_0 \in M_1} \sup_{x \in X_{m_0}} \limsup_{m' \to m_0} \frac{\left| \tilde{f}_{m_0}(m', x) - \tilde{f}_{m_0}(m_0, x) \right|}{\left| x - iX(m_0) \right|} < \infty,
\]

which follows from

\[
\left| \tilde{f}_{m_0}(m', x) - \tilde{f}_{m_0}(m_0, x) \right| \leq \left| \tilde{f}_{m_0}(m', x) - iY(m_0) \right| + \left| iY(m_0) - \tilde{f}_{m_0}(m_0, x) \right|,
\]

and

\[
\limsup_{m' \to m_0} \left| \tilde{f}_{m_0}(m', x) - iY(m_0) \right| \leq \limsup_{m' \to m_0} \left| \tilde{f}_{m_0}(m', x) - \tilde{f}_{m_0}(m', \tilde{i}_X(m')) \right| + \left| \tilde{f}_{m_0}(m', \tilde{i}_X(m')) - iY(m_0) \right| \leq \beta'(m_0) \limsup_{m' \to m_0} \left| x - \tilde{i}_X(m') \right| + \limsup_{m' \to m_0} \left| (\hat{\varphi}_m)_{m'}^{-1}(iY(m')) - iY(m_0) \right| \leq \beta'(m_0) \left| x - iX(m_0) \right|,
\]

where \( \tilde{i}_X(m') = (\varphi_{m'})_{m'}^{-1}(iX(m')) \). Since \( \theta < 1 \), we have

\[
\lim_{m' \to m_0} \left| \tilde{f}_{m_0}(m', x) - \tilde{f}_{m_0}(m_0, x) \right| = 0.
\]

Now we have shown \( f \) is continuous at \((m_0, x), m_0 \in M_1 \). We need to show \( f \) is continuous everywhere. This follows by the same argument as in the last part proof of Lemma 6.18.

Let \( m_0 \in M \), and choose \( C^0 \) bundle charts \((U_0, \varphi^0), (U_0, \phi^0)\) at \( m_0 \) of \( X, Y \), respectively, such that \( u(U_0) \subset V_{u(m_0)} \subset U_{u(m_0)} \). Note that \( u(m_0) \in M_1 \). Consider \( F, G, f, \iota, i, \cdot \) in local bundle charts \( \varphi^0 : U_0 \times X_{m_0} \to X, \phi^0 : U_0 \times Y_{m_0} \to Y, \varphi^m(m_0) : V_{u(m_0)} \times X_{u(m_0)} \to X, \phi^m(m_0) : V_{u(m_0)} \times Y_{u(m_0)} \to Y \).
That is (see also (6.37))
\[ \tilde{F}_m^r(m, x, y) \triangleq \left( \varphi_{u(m)}^{(m)} \right)^{-1} \circ F_m \circ (\varphi_0(x), \varphi_{u(m)}^{(m)}(y)) : U_0 \times X_{m_0} \times Y_{u(m_0)} \to X_{u(m_0)}, \]
\[ \tilde{G}_m^r(m, x, y) \triangleq \left( \varphi_{u(m)}^{(m)} \right)^{-1} \circ G_m \circ (\varphi_0(x), \varphi_{u(m)}^{(m)}(y)) : U_0 \times X_{m_0} \times Y_{u(m_0)} \to Y_{m_0}, \]
\[ \tilde{f}_0^r(m, x) \triangleq \left( \varphi_{u(m)}^{(m)} \right)^{-1} \circ f_m \circ (\varphi_0(x)) : U_0 \times X_{m_0} \to Y_{m_0}, \]
\[ \tilde{f}_{u(m)}^r(m, x) \triangleq \left( \varphi_{u(m)}^{(m)} \right)^{-1} \circ f_{u(m)} \circ (\varphi_{u(m)}(x)) : U_0 \times X_{u(m_0)} \to Y_{u(m_0)}, \]
\[ \tilde{x}_m^r(m, x) \triangleq \varphi_{u(m)}^{(m)} \circ x_m(\varphi_0(x)) : U_0 \times X_{m_0} \to X_{u(m_0)}. \]

Then we have (6.38) holds.

By (H5'), we can take \( U_0 \) smaller such that if \( m \in U_0 \), then

\[ \text{Lip} \tilde{F}_m^r(m, x, \tilde{f}_{u(m)}(u(m), \cdot)) < 1. \]

As \( \tilde{f}_{u(m)}(\cdot, \cdot) \) is continuous at \( (u(m_0), x) \), by Lemma D.2 and (6.38), \( \tilde{f}_0^r(\cdot, \cdot) \) is continuous at \( (m_0, x) \), and thus the proof of (1) is complete.

For the proof of (2), consider following limits (the uniformity of \( m' \to m_0 \) is the same):

\[ \limsup_{m_0 \in M_1} \limsup_{m' \to m_0} \sup_{|x - \tilde{x}_m(m_0)| \leq r} \frac{|\tilde{f}_m(m', x) - \tilde{f}_m(m_0, x)|}{|x - iX(m_0)|}, \]

\[ \sup_{m_0 \in M_1} \limsup_{m' \to m_0} \sup_{x \in X_{m_0}} \frac{|\tilde{f}_m(m', x) - \tilde{f}_m(m_0, x)|}{|x - iX(m_0)|}. \]

Also, note that by (6.46),

\[ \limsup_{m_0 \in M_1} \sup_{m' \to m_0} \sup_{x \in X_{m_0}} \frac{|\tilde{f}_m(m', x) - \tilde{f}_m(m_0, x)|}{|x - iX(m_0)|} < \infty. \]

Using the same argument as (1), one gives the proof. \( \square \)

We write \( \tilde{K}_m^1(\cdot, \cdot), \tilde{R}_m^1(\cdot, \cdot) \), the local representations of \( K^1, R^1 \) at \( m_0 \in M_1 \) with respect to \( \mathcal{A}, \mathcal{B}; \) see (6.16).

**Lemma 6.26** (continuity of \( K^1_m(\cdot) \)). Assume the following hold.

(a) Let (H1a) (H2) (i) (iv) (i') (H3') (H4) (H5') hold.

(b) (about \( F, G \)) Assume \( F, G \) are \( C^0 \) and \( C^1 \)-fiber with their fiber derivatives \( D^vF, D^vG \) being continuous. (spectral gap condition) \( \lambda_3 \lambda_u < 1 \); see Remark 6.1.

(1) Then the bundle map \( K^1 \in L(f_Y^V, f_Y^V) \) obtained in Lemma 6.7 is \( C^0 \).

(2) If for every \( m \in M, DF_m(\cdot), DG_m(\cdot) \) are uniformly continuous, then so is \( K^1_m(\cdot) \). If \( DF_m(\cdot), DG_m(\cdot), m \in M, \) are equicontinuous, then so are \( K^1_m(\cdot) \).

**Proof.** (1). That \( K^1 \in L(f_Y^V, f_Y^V) \) is \( C^0 \) means \( \tilde{K}_m^1(\cdot, \cdot) \), which is the local representation of \( K^1 \) at \( m_0 \) (see e.g. (6.16)), is continuous for at \( (m_0, x) \) for \( m_0 \in M, x \in X_{m_0} \). Let

\[ E_{1-c}^L = \{ K^1 \in E_{1}^L : K^1 \text{ is } C^0 \}. \]

Then one can easily verify that \( E_{1-c}^L \) is closed in \( E_{1}^L \) (using the fact that the convergence in \( E_{1}^L \) is uniform) and \( f^1 \), the graph transform defined in the proof of Lemma 6.7, satisfies \( fE_{1-c}^L \subset E_{1-c}^L \) (using (6.15), Lemma D.2 and the continuity of \( f \)). Now we can deduce that \( K^1 \in E_{1-c}^L \).

(2). The proof is essentially the same as in the proof of Lemma 6.25, by using the computation in the proof of Lemma 6.11 and considering the following limits respectively:

\[ \sup_{m \in M} \limsup_{t \to 0^+} \sup_{|x_1 - x_2| \leq r} |K_m^1(x_1) - K_m^1(x_2)|, \limsup_{r \to 0^+} \sup_{m \in M} \sup_{|x_1 - x_2| \leq r} |K_m^1(x_1) - K_m^1(x_2)|. \]

Note that \( \sup_{m \in M} \sup_{x_1, x_2} |K_m^1(x_1) - K_m^1(x_2)| < \infty \). So the above limits are all finite, and finally equal to 0. \( \square \)
Lemma 6.27 (continuity of $m \mapsto K_m^1(x)$). Assume the following hold.

(a) Let (H1a) (H2) (i) (iv) (v) (2') (H3) (H4) (H5') hold.
(b) (about $F, G$) Assume $F, G$ are $C^1$, (spectral gap condition) $\lambda_x \lambda_u < 1$; see also Remark 6.1.

Then we have following.

1) Suppose $DF_m(\cdot), DG_m(\cdot), m \in M$, are uniformly continuous (resp. equicontinuous, and assume (H2)(v)(2')(H4') (H5) hold). If $m \mapsto DF_m(\cdot), DG_m(\cdot)$ are continuous (resp. uniformly continuous) around $M_1$ in $C^0$-topology in bounded sets (or in the whole space), so is $m \mapsto K_m^1(\cdot)$; see Definition B.5.

2) Assume (H4') (H5) instead of (H4) (H5'). If $m \mapsto DF_m(i^1(m)), DG_m(i^1(m))$ are uniformly continuous around $M_1$, so is $m \mapsto K_m^1(i_x(m))$, where $i_m = (i_x(m), i_y(u(m)))$.

Proof. Note that we have assumed $i$ is the 0-section. We first show that, under the case (1), (6.46) holds. This is easy from that

$$
\bar{F}_{m_0}(m', z) - \bar{F}_{m_0}(0, z) = \int_0^1 \left\{ D_1 \bar{F}_{m_0}(m', z_t) - D_1 \bar{F}_{m_0}(0, z_t) \right\} dt (z - i^1(m)),
$$

where $z_t = tz + (1 - t)i^1(m_0)$, $i_1^1(m_0) = (i_X(m_0), i_Y(u(m_0)))$, and similarly for $\bar{G}_{m_0}$. So by Lemma 6.25, $\bar{m_0}$ is continuous at $(m_0, x)$ uniform for $x$ belonging to any bounded set of $X_{m_0}$. Using the above fact, Lemma 6.26 (by the assumption (b) on $F, G$), and the computation in the proof of Lemma 6.17 (by (6.15)), we see that

$$
\limsup_{m' \to m_0} \sup_{|x| \leq r} |\bar{K}_{m_0}^1(m', x) - \bar{K}_{m_0}^1(m_0, x)|
\leq \frac{\lambda_x (m_0) \lambda_u (m_0)}{1 - \alpha(m_0) \beta'(u(m_0))} \limsup_{m' \to m_0} \sup_{|x| \leq r} |\bar{K}_{u(m_0)}^1(m', \bar{x}_{m_0}(m_0, x)) - \bar{K}_{u(m_0)}^1(m_0, \bar{x}_{m_0}(m_0, x))|.
$$

Thus,

$$
\limsup_{r \to \infty} \sup_{m_0 \in M_1} \limsup_{m' \to m_0} \sup_{|x| \leq r} |\bar{K}_{m_0}^1(m', x) - \bar{K}_{m_0}^1(m_0, x)|
\leq \theta \limsup_{r \to \infty} \sup_{m_0 \in M_1} \limsup_{m' \to m_0} \sup_{|x| \leq r} |\bar{K}_{m_0}^1(m', x) - \bar{K}_{m_0}^1(m_0, x)|
$$

where $\theta = \sup_{m_0 \in M_1} \frac{\lambda_x (m_0) \lambda_u (m_0)}{1 - \alpha(m_0) \beta'(u(m_0))} < 1$, which gives for any $r > 0,$

$$
\limsup_{m' \to m_0} \sup_{|x| \leq r} |\bar{K}_{m_0}^1(m', x) - \bar{K}_{m_0}^1(m_0, x)| = 0.
$$

The proof of the case $\mathcal{L}_{m_0}^1 = \mathcal{X}_{m_0}^i \times Y_{u(m_0)}$ is similar by using $\mathcal{L}_{m_0}^1$, instead of $\mathcal{L}_{m_0}^1$; for the uniform case, use the limit $\mathcal{L}_{m_0}^{1, u}, \mathcal{L}_{m_0}^{2, u}$ instead of $\mathcal{L}_{m_0}^{1, u}, \mathcal{L}_{m_0}^{2, u}$ (see (6.45)). The proof of case (2) is much easier than case (1) by taking $x = i_x(m_0)$ in the above analysis. □

Lemma 6.28 (continuity of $K_m(\cdot)$). Assume the following hold.

(a) Let (H1) (H2) (i) (ii) (v) (1') (H3) (H4) (H5') hold.
(b) (about $F, G$) Let $F, G$ be $C^1$ and satisfy

$$
|\nabla_m F_{m_0}(z)| \leq M_0 |z|, \quad |\nabla_m G_{m_0}(z)| \leq M_0 |z|,
$$

for all $m_0 \in M_1$ and $z \in \mathcal{X}_{m_0} \times Y_{u(m_0)}$, where $M_0$ is a constant independent of $m$.

(c) (spectral gap condition) $\lambda_x \lambda_u < 1, \lambda_x \lambda_{u, x} < 1$; see also Remark 6.1.

Then the following hold.

1) There exists a unique $C^0$ vector bundle map $K \in L_f(\gamma_X^H, \gamma_Y^V)$ such that $\sup_{m \in M} \sup_{x} \frac{|K_m(x)|}{|x|} < \infty$ and it satisfies (6.22).
(2) If $DF_m, DG_m, \nabla_m F_m, \nabla_m G_m$ satisfy
\begin{equation}
 L \frac{|DF_m(z_1) - DF_m(z_2)|}{|z_1| + |z_2|} = 0, \quad L \frac{|DG_m(z_1) - DG_m(z_2)|}{|z_1| + |z_2|} = 0,
\end{equation}
(6.47)\[L \frac{|\nabla_m F_m(z_1) - \nabla_m F_m(z_2)|}{|z_1| + |z_2|} = 0, \quad L \frac{|\nabla_m G_m(z_1) - \nabla_m G_m(z_2)|}{|z_1| + |z_2|} = 0,
\end{equation}
(6.48) then
\begin{equation}
 L \frac{|K_m(x_1) - K_m(x_2)|}{|x_1| + |x_2|} = 0,
\end{equation}
(6.49) where $L$ stands for the two different types of limits, $\sup$, $\lim_{m \to 0^+}$, $\sup_{m \in M}$, $\lim_{m \to 0^+}$. \(r \to 0^+\), $m \in M$, $|z_1 - z_2| \leq r$, respectively. (For $K$, use $x_1, x_2$ instead of $z_1, z_2$ in the supremum.) In particular, this means $K_m(\cdot)$, $m \in M$, is uniformly continuous (or equicontinuous).

**Proof.** Conclusion (1) has already been proved in Lemma 6.18. To prove (2), we can use almost the same strategy as proving Lemma 6.26. Consider the following limits,
\begin{equation}
 L \frac{|K_m(x_1) - K_m(x_2)|}{|x_1| + |x_2|} .
\end{equation}
(6.50) Using the continuity of $K^1$ (by Lemma 6.26 and noting (6.47) implies the condition on $DF_m, DG_m$ in (2) in Lemma 6.26), the continuity of $f$ (by Lemma 6.25) and the computation in the proof of Lemma 6.22 (by (6.22)), one can deduce that
\begin{equation}
 L \frac{|K_m(x_1) - K_m(x_2)|}{|x_1| + |x_2|} \leq \theta_0 L \frac{|K_m(x_1) - K_m(x_2)|}{|x_1| + |x_2|},
\end{equation}
(6.51) where $\theta_0 = \sup_{m \in M} \frac{A_{\lambda(m), \mu(m), \rho(m)}(m)}{1 - \alpha(m) \beta(u(m))} < 1$. Observe that $\sup_{m \in M} \sup_{x_1, x_2} \frac{|K_m(x_1) - K_m(x_2)|}{|x_1| + |x_2|} < \infty$, giving the results.

We write $\widetilde{D}_m, F_m, \widetilde{G}_m, K_m, R$ as the local representations of $\nabla F, \nabla G, K, R$ (at $m_0 \in M_1$) with respect to $\mathcal{A}, \mathcal{B},$ and any $C^1$ local charts $\xi_{m_0}, \zeta_{u(m_0)}$ of $M$ at $m_0, u(m_0)$, such that $D\xi_{m_0}(m_0) = id, D\zeta_{u(m_0)}(u(m_0)) = id$. See e.g. (6.31), (6.32), (6.33), (6.30). If (H1c) is assumed, then $\xi_{m_0}, \zeta_{u(m_0)}$ are replaced by $\lambda_{m_0}, \zeta_{u(m_0)}$, see (6.40) and (6.41).

**Lemma 6.29** (continuity of $m \mapsto K_m(x)$). Under the assumptions in Lemma 6.28 with (H2) (v) (1’) replaced by (H2) (v) (2’). Furthermore, (6.47) and (6.48) are satisfied for $L = \sup_{m \in M} \lim_{m \to 0^+} \sup_{|z_1 - z_2| \leq r}$. Then the following hold.

(1) If $\nabla F, \nabla G$ satisfy for any $m_0 \in M_1$, $r > 0$,
\begin{equation}
 \lim_{m \to m_0} \sup_{|z| \leq r} \frac{|\widetilde{D}_m F_{m_0}(m', z) - \widetilde{D}_{m_0} F_{m_0}(m_0, z)|}{|z|} = 0,
\end{equation}
(6.49)\[\lim_{m \to m_0} \sup_{|z| \leq r} \frac{|\widetilde{D}_m G_{m_0}(m', z) - \widetilde{D}_{m_0} G_{m_0}(m_0, z)|}{|z|} = 0,
\end{equation}
(6.50) and the $C^0$ continuity in bounded-fiber sets case in Lemma 6.27 (1) is satisfied, then
(2) If $\nabla F, \nabla G$ satisfy (6.49) for $r = \infty$, and the $C^0$ continuity in the whole space case in Lemma 6.27 (1) is satisfied, then $\lambda$ also satisfies (6.50) for $r = \infty$. **Proof.** See Lemma 6.29.
(3) Suppose that (H1c) (H2d) (H5) hold and $Du$ is uniformly continuous around $M_1$, that (6.47) and (6.48) are satisfied for $\lim_{r \to 0^+} \sup_{m \in M_1} |z_1 - z_2| \leq r$ (resp. in the whole space) case in Lemma 6.27 (1) is satisfied. If in (6.49) $m' \to m_0 \in M_1$ is in the uniform sense, so is (6.50) for any $r > 0$ (resp. $r = \infty$).

Proof. Only consider case (1); others are similar. Under the assumptions of this lemma, we can use Lemma 6.25 (2), Lemma 6.26 (2), Lemma 6.27 (1), Lemma 6.28 (2) to obtain the desired continuity of $f, K^1, K_m(\cdot)$. Consider the following limit:

$$\lim_{r \to \infty} \sup_{m_0 \in M_1} \sup_{m' \to m_0} \frac{|\hat{K}_{m_0}(m', x) - \hat{K}_{m_0}(m_0, x)|}{|x|}.$$ 

Through the similar computation in the proof of Lemma 6.23 (but this time using (6.34)), one concludes that

$$\lim_{r \to \infty} \sup_{m_0 \in M_1} \sup_{m' \to m_0} \frac{|\hat{K}_{m_0}(m', x) - \hat{K}_{m_0}(m_0, x)|}{|x|} \leq \theta_0 \lim_{r \to \infty} \sup_{m_0 \in M_1} \sup_{m' \to m_0} \frac{|\hat{K}_{m_0}(m', x) - \hat{K}_{m_0}(m_0, x)|}{|x|},$$

where $\theta_0 = \sup_{m \in M_1} \frac{\lambda_x(m)\lambda_y(m)\lambda_z(m)}{Lip_2(u(m))} < 1$. And note that sup $\sup_{m_0 \in M_1} \frac{|K_{m_0}(m_0)|}{|x|} < \infty$. Now combine all the above facts to give (6.50). \qed

The condition on $F, G$ in Lemma 6.29 can be satisfied if $F, G$ are $C^{1,1}$. This is actually used in some classical results; see e.g. [HPS77]. Also, note that the assumptions in the $C^0$ continuity results are much weaker than the Hölder continuity results. Now we have extended the results in [HPS77, Chapter 5 (about the plaque families), Chapter 6] in our general settings.

**Remark 6.30.** A basic application of the continuity results is the study of dynamical systems with parameters. Consider a special case. Let $u = id : M \to M, X = M \times X_0, Y = M \times Y_0$ (the trivial bundle case), where $M, X_0, Y_0$ are all Banach spaces. Now $m \mapsto H_m \sim (F_m, G_m)$ can be viewed as parameter-dependent correspondences. The associated invariant graphs $f_m, m \in M$, obtained in Theorem 4.1, depend on the parameter $m$. Lemma 6.25 (2) for $r = \infty$ and Lemma 6.27 (1) tell us the continuous dependence of $m \mapsto f_m(\cdot)$ and $m \mapsto Df_m(\cdot)$, i.e. $\lim_{m \to m_0} \|f_m - f_{m_0}\|_{C^1_b(X_0, Y_0)} = 0$. Also, Lemma 6.29 (2) gives the smooth dependence of $m \mapsto f_m(\cdot)$. To see this, consider

$$\lim_{m \to m_0} \frac{\|f_m - f_{m_0} - K_{m_0}(m - m_0)\|}{|m - m_0|} = \lim_{m \to m_0} \sup_x \frac{|f_m(x) - f_{m_0}(x) - K_{m_0}(x)(m - m_0)|}{|m - m_0|} \leq \int_0^1 \lim_{m \to m_0} \sup_x |K_{m_1}(x) - K_{m_0}(x)| \, dr,$$

where $m_1 = tm + (1 - t)m_0$. In the above argument we technically assume $F_m, G_m, m \in M$, are bounded in order to let $f_m \in C_b(X_0, Y_0)$.

We also can take $\text{Lip}_0(X_0, Y_0) = \{g : X_0 \to Y_0 \text{ is lipschitz} : g(0) = 0\}$ instead of $C_b(X_0, Y_0)$ (assuming the section $i = 0$). However, in this case the results do not give the smooth dependence of $m \mapsto f_m(\cdot)$ in $\text{Lip}_0(X_0, Y_0)$. This can be done if we consider the higher order differential of $f$.

### 6.10. Regularity of invariant graph: bounded section case.

Throughout this appendix, we make the following assumptions.

- Let $(X, M, \pi_1), (Y, M, \pi_2), u : M \to M, H \sim (F, G), i : M \to X \times Y$ be as in Theorem 4.1, and $f$ be the bundle map obtained in Theorem 4.1 under the conditions (ii) or (ii)' and $i$ being a $1$-pseudo-stable section of $H$ with $\sup_m \eta(m) < \infty$.

In general, the uniform assumption $\sup_m \eta(m) < \infty$ sometimes can be weakened, for example we can only assume for every orbit of $\eta(\cdot)$ is bounded and $\eta(\cdot)$ is locally bounded. However, in this...
case, the conditions on $F, G$ should point-wisely depend on $m$. In this paper, we do not consider this situation.

- We add additional assumption that the functions in (A) (or (A')) (B) condition are bounded. Note that for the 1-pseudo-stable section case, the spectral condition in Theorem 4.1 is $\lambda_s \lambda_u < 1, \lambda_u < 1$ (we use the abbreviation; see Remark 6.1).

Like the case $i$ is an invariant section of $H$, one can also show that $f$ might have more regularities. Since the proof of results are very similar with those in Section 6.3 to Section 6.7, we only give the statements of the results and omit all the proofs. The conditions on $M, X, Y$, and $u$ are the same as in the invariant case section. The main different assumptions are about $F, G$ and spectral gap condition. Also, the section $i$ should not be a 0-section, instead of being assumed to be uniformly (locally) bounded (see assumption (B3-) below).

We use the same notations about abbreviation of spectral gap condition (see Remark 6.1). Also, the constants $M_0, C, \alpha$ appearing in the following, are independent of $m \in M$ or $m_0 \in M_1$.

**Smooth leaf:** $C^{k, \alpha}$ continuity of $x \mapsto f_m(x)$. The same results of Lemma 6.7, Lemma 6.11 and Lemma 6.12 also hold when $i$ is a 1-pseudo-stable section of $H$. But there is no result like Remark 6.10.

**Hölder vertical part:** Hölderness of $m \mapsto f_m(x)$. Use the notations in Section 6.4, $\hat{F}_{m_0}, \hat{G}_{m_0}, \hat{f}_{m_0}$, $\hat{x}_{m_0}$, i.e. the local representations of $F, G, \hat{f}, \hat{x}(\cdot)$ at $m_0 \in M_1$ with respect to $A, B$; see (6.9). Also, let

$$\hat{F}_{m_0}(m) = (i_{\hat{x}}^{m_0}(m), i_{\hat{y}}^{m_0}(m)) = ((\varphi_m^{m_0})^{-1}(i_X(m), (\psi_m^{m_0})^{-1}(i_Y(m))) : U_{m_0}(\epsilon) \to X_{m_0} \times Y_{m_0},$$

be the local representation of $i$ (at $m_0$) with respect to $A \times B$. For brevity, we also use the notation $|x| = d(x, i_X(m))$ if $x \in X_{m_0}, |y| = d(y, i_Y(m))$ if $y \in Y_{m_0}$, and $|z| = |x, y| = d((x, y), (i_X(m), i_Y(m))$ if $z = (x, y) \in X_{m_0} \times Y_{m_0}$. Note that $i_{\hat{x}}^{m_0}(m_0) = i_X(m_0), i_{\hat{y}}^{m_0}(m_0) = i_Y(m_0).$ The assumption on $i$ is the following.

(B3-) There are $\epsilon_0, \hat{c}_1 > 0$ such that $|\hat{F}_{m_0}(m) - \hat{F}_{m_0}(m_0)| \leq \hat{c}_1, m \in U_{m_0}(\epsilon_0), \text{for all } m_0 \in M_1$.

**Lemma 6.31.** Assume the following conditions hold.
(a) Let (H1a) (H2a) (B3-) (H4a) (H5) hold.
(b) (about $F, G$) $F, G$ is uniformly (locally) $\gamma$-Hölder around $M_1$ in the following sense.

\begin{align*}
|\hat{F}_{m_0}(m_1, z) - \hat{F}_{m_0}(m_0, z)| &\leq M_0|m_1 - m_0|^\gamma(|z|^{\gamma} + \hat{c}_0), \\
|\hat{G}_{m_0}(m_1, z) - \hat{G}_{m_0}(m_0, z)| &\leq M_0|m_1 - m_0|^\gamma(|z|^{\gamma} + \hat{c}_0),
\end{align*}

for all $m_1 \in U_{m_0}(\mu^{-1}(m_0)e_1), z \in X_{m_0} \times Y_{m_0}, m_0 \in M_1$, where $0 < \gamma \leq 1, \hat{c}_0 \geq 0, \hat{c}_0 > 0$.

(c) (spectral gap condition) $\lambda_u < 1, \lambda_s \lambda_u < 1, (\max\{1, \lambda_s^{\frac{1}{2}}\})^\alpha \lambda_u < 1$, and

\begin{align*}
(\max\{1, 1/\lambda_s, 1/\lambda_s^{1-\hat{\gamma}}\})^\alpha \lambda_s \lambda_u < 1,
\end{align*}

where $0 < \alpha \leq 1$; see also Remark 6.1.

If $\epsilon_1^* \leq \mu^{-2} \epsilon_1$ is small, then we have

$$|\hat{F}_{m_0}(m_1, x) - \hat{F}_{m_0}(m_0, x)| \leq C|m_1 - m_0|^\gamma(|x|^{\gamma} + 1)^{\alpha}(|x| + 1)^{1-\alpha},$$

for every $m_1 \in U_{m_0}(\epsilon_1^*), x \in X_{m_0}, m_0 \in M_1$ under $|m_1 - m_0|^\gamma(|x|^{\gamma} + 1) \leq \hat{r} \min\{|x| + 1, (|x|^{\gamma} + 1)^{\epsilon}(|x| + 1)^{-(\epsilon-1)}\}$, where the constant $C$ depends on the constant $\hat{r} > 0$ but not $m_0 \in M_1, c > 1$ and $\hat{r}$ does not depend on $m_0 \in M_1$.

**Remark 6.32.** (a) Note that $\hat{c}_0 = 0$ is the case that $i$ is invariant. So we only consider $\hat{c}_0 > 0$, and in this case we can assume $\hat{c}_0 = 1$. Two especial cases, $\gamma = 0$ and $\gamma = 1$, are important. For the case $\gamma = 1, \zeta = 1$, under spectral gap condition $\lambda_u < 1, \lambda_s \lambda_u < 1, \lambda_u \mu < 1, \lambda_s \lambda_u \mu < 1$, we have for all $m_1 \in U_{m_0}(\epsilon_1^*)$,

$$|\hat{F}_{m_0}(m_1, x) - \hat{F}_{m_0}(m_0, x)| \leq C|m_1 - m_0|(|x| + 1).$$
(b) Under the conditions (a) (b) in Lemma 6.31, with \( \sup_{m} d(f_m(X_m), i\gamma(m)) < \infty \) and a ‘better’ spectral gap condition, i.e. \( \lambda_{\mu} < 1, \lambda_{s} \lambda_{\mu} < 1, (\max\{\lambda_{s}^{\alpha}, \lambda_{\mu}^{\alpha}\})^{\alpha} \lambda_{\mu} < 1 \), then we have

\[
|\hat{f}_{m_0}(m_1, x) - \hat{f}_{m_0}(m_0, x)| \leq C|m_1 - m_0|^{\alpha}(|x|^\gamma + 1)^{\alpha},
\]

for every \( m_1 \in U_{m_0}(e_1^*) \), \( x \in X_{m_0}, m_0 \in M_1 \) under \( |m_1 - m_0|^{\gamma}(|x|^\gamma + 1) \leq \hat{\ell} \), where the constant \( C \) depends on the constant \( \hat{\ell} > 0 \) but not \( m_0 \in M_1 \) and \( \hat{\ell} \) does not depend on \( m_0 \in M_1 \).

(c) Consider the trivial bundle \( X \times Y = M \times X_0 \times Y_0 \) case with \( X_0, Y_0 \) being Banach spaces. A special condition such that \( F, G \) satisfy (b) is the following. Let

\[
F_m(x, y) = A(m)x + f(m, x, y), \quad G_m(x, y) = B(m)y + g(m, x, y),
\]

where \( A(m), B(m) \in L(X_0, Y_0), f : M \times X_0 \times Y_0 \to X_0, g : M \times X_0 \times Y_0 \to Y_0, \) Assume \( A(\cdot), B(\cdot) \in C^{0,1}, f(\cdot, x, y), g(\cdot, x, y) \) are \( C^{0,1} \) uniform for \( x, y \), and \( \sup_{m, x, y} |f(m, 0, y)| \leq \eta_0, \sup_{m, x} |g(m, x, 0)| \leq \eta_0 \) for some constant \( \eta_0 > 0 \). Take the section \( i = 0 \). See also the similar results in [CY94] for the cocycles in the finite dimensional setting.

**Hölder distribution: Hölderness of** \( m \mapsto Df_m(x) = K^1_m(x) \). There is no result like Lemma 6.16.

**Lemma 6.33.** Assume the conditions in Lemma 6.17 hold with (H3) and (6.11) in Lemma 6.13 replaced by (B3-) and (6.51) (6.52) in Lemma 6.31 (for \( \gamma = 1, \zeta = 1 \), respectively. In addition, we need the following spectral gap condition (c) instead of (c).

\[
(\text{c'}) \quad (\text{spectral gap condition}) \lambda_{\mu} < 1, \lambda_{s} \lambda_{\mu} < 1, \mu^{\alpha} \lambda_{s}^{\alpha} \lambda_{\mu} < 1, \lambda_{s}^{1+\beta} \lambda_{\mu} < 1, \lambda_{s} \mu^{\alpha} < 1, \lambda_{s} \lambda_{\mu} \mu^{\alpha} < 1.
\]

If \( \varepsilon_1^* \leq \hat{\mu}^{-2} \varepsilon_1 \) is small, then we have

\[
|\hat{K}_{m_0}^1(m_1, x) - \hat{K}_{m_0}^1(m_0, x)| \leq C|m_1 - m_0|^{\alpha \beta}, \quad \forall m_1 \in U_{m_0}(e_1^*), \ x \in X_{m_0}, m_0 \in M_1.
\]

**smootheness of** \( m \mapsto f_m(x) \) and Hölderness of \( x \mapsto \nabla_m f_m(x) = K_m(x) \).

**Lemma 6.35.** Assume the conditions in Lemma 6.18 hold, but replace (H3) by (B3-), and the conditions (b) (c) by the following conditions (b’) (c’) or (b’’) (c’’).

\[
(\text{b'}) \quad (\text{about} \ F, G) \ (i) \ F, G \in C^1, \quad (ii) \ (6.51) \ (6.52) \text{ hold for} \ \gamma = 1, \ \zeta = 1.
\]

\[
(\text{c'}) \quad (\text{spectral gap condition}) \lambda_{\mu} < 1, \lambda_{s} \lambda_{\mu} < 1, \lambda_{s} \mu^{\alpha} < 1, \lambda_{s} \lambda_{s} \mu^{\alpha} < 1; \text{ see also Remark 6.1}.
\]

(b’’) (about \( F, G \)) (i) \( F, G \in C^1 \), (ii) (6.51) (6.52) hold for \( \gamma = 1, \ \zeta = 0 \). (iii) In addition, \( \sup_{m} d(f_m(X_m), i\gamma(m)) < \infty \).

\[
(\text{c’’}) \quad (\text{spectral gap condition}) \lambda_{\mu} < 1, \lambda_{s} \lambda_{\mu} < 1, \lambda_{s} \mu^{\alpha} < 1; \text{ see also Remark 6.1}.
\]

Then the following hold.

(1) There exists a unique \( C^0 \) vector bundle map \( K \in L(\gamma^H_X, \gamma^V_Y) \) over \( f \) such that it satisfies (6.22), and if (b’) (c’) hold, \( |K_m(x)| \leq C(|x| + 1) \), or if (b’’) (c’’) hold, \( |K_m(x)| \leq C, \text{ for all} \ x \in X_m, m \in M_1 \).

(2) \( f \) is \( C^1 \) and \( \nabla f = K \).

**Lemma 6.36.** Assume the conditions in Lemma 6.22 hold, but replace (H3) by (B3-), and the conditions (b) (c) by the following (b’’) (c’) (b’’) (c’’).

\[
(b’’) \quad (\text{ii}) \ |\nabla_m f_m(z)| \leq M_0(|z| + 1), \ |\nabla_m G_m(z)| \leq M_0(|z| + 1).
\]
(c') (spectral gap condition) Let the condition (c') in Lemma 6.35 hold and $\lambda^{\beta}_s \lambda_s \lambda_u < 1$, $(\lambda^{\alpha}_s)^{\alpha} \lambda_s \lambda_u < 1$; see also Remark 6.1.

(b)'' (ii) $|\nabla_m F_m(z)| \leq M_0$, $|\nabla_m G_m(z)| \leq M_0$.

(c)'' (spectral gap condition) Let the condition (c'') in Lemma 6.35 hold and $\lambda^{\beta}_s \lambda_s \lambda_u < 1$, $(\lambda^{\alpha}_s)^{\alpha} \lambda_s \lambda_u < 1$; see also Remark 6.1.

If $0 < \alpha \leq 1$, then the $C^0$ vector bundle map $K \in L(\Gamma^{TH}_X, \Gamma^{TV}_Y)$ over $f$ satisfying (6.22) has the following Hölder property.

(1) If (b)'' (c') hold, then we have

$$|K_m(x_1) - K_m(x_2)| \leq C \{(|x_1 - x_2|^\gamma(1 + |x_1|)^\alpha + (|x_1 - x_2|^\beta(1 + |x_1|)^\alpha)\} (|x_1| + |x_2| + 1)^{1-\alpha},$$

for all $m \in M_1$, under $(|x_1 - x_2|^\gamma + |x_1 - x_2|^\beta(1 + |x_1|)) \leq \tilde{r}(|x_1| + |x_2| + 1)$, where the constant $C$ depends on the constant $\tilde{r} > 0$ but not $m \in M$.

(2) If (b)'' (c'') hold, then we have

$$|K_m(x_1) - K_m(x_2)| \leq C(|x_1 - x_2|^\alpha + |x_1 - x_2|^{\beta\alpha}).$$

Hölderness of $m \mapsto \nabla_m f_m(x) = K_m(x)$.

**Lemma 6.37.** Let (a) in Lemma 6.23 hold with (H3) replaced by (B3-). In addition, assume the following conditions (b)' (c') or (b)'' (c'') hold.

(b)' (about $F, G$) $F, G$ are $C^1$. Furthermore, let (6.51) (6.52) hold for $\gamma = 1, \zeta = 1$, as well as (6.6) and (6.7).

(c)' (spectral gap condition) $\lambda_u < 1, \lambda_s \lambda_u < 1, \lambda^2 \lambda_u < 1, \lambda_s \lambda_u < 1, \lambda_s \lambda_u < 1$; see also Remark 6.1.

(b)'' (about $F, G$) $F, G$ are $C^1$. (6.51) (6.52) hold for $\gamma = 1, \zeta = 1$; (6.6) $(DF, DG \in C^{0,1})$ and (6.7) $(\nabla F, \nabla G \in C^{0,1})$ hold. Also, $\sup_m \tilde{d}(f_m(X_m), \gamma_m(x)) < \infty$.

(c)'' (spectral gap condition) $\lambda_u < 1, \lambda_s \lambda_u < 1, \lambda^2 \lambda_u < 1, \lambda_s \lambda_u < 1, \lambda_s \lambda_u < 1, \mu^{\alpha} \lambda_s \lambda_u < 1$; see also Remark 6.1.

If $0 < \alpha \leq 1$, then there exists a small $\epsilon^*_1 > 0$ such that the following hold.

(1) Under the conditions (b)' (c'), we have

$$|\tilde{K}_{m_0}(m_1, x) - \tilde{K}_{m_0}(m_0, x)| \leq C|m_1 - m_0|^\alpha(|x| + 1),$$

for all $m_1 \in U_{m_0}(\epsilon^*_1), m_0 \in M_1$. The constant $C$ depends on the constant $\epsilon^*_1$ but not $m_0 \in M_1$.

(2) Under the conditions (b)'' (c''), we have

$$|\tilde{K}_{m_0}(m_1, x) - \tilde{K}_{m_0}(m_0, x)| \leq C|m_1 - m_0|^\alpha,$$

for all $m_1 \in U_{m_0}(\epsilon^*_1), m_0 \in M_1$. The constant $C$ depends on the constant $\epsilon^*_1$ but not $m_0 \in M_1$.

See Section 6.8 for how we can give the Lipschitz continuity of $m \mapsto f_m(x)$, $K^m_{m}(x), K_m(x)$ in classical sense.

**continuity properties of $f$.** The results about the continuity of $f$ are essentially the same as in Section 6.9 with minor changes. And in order to do not add length of the paper, the statements are all omitted and left to the readers.

**Remark 6.38.** We have already considered the regularities of $f$ obtained in Theorem 4.1 for the two extreme cases, i.e. $\varepsilon = 0$ and $\varepsilon = 1$. The same results of Lemma 6.7 and Lemma 6.11 also hold for all $0 \leq \varepsilon \leq 1$ without changing the proofs. Similar results such as Hölder continuity and smoothness respecting base points should also hold for the case when $i$ is an $\varepsilon$-pseudo-stable section, where $0 < \varepsilon < 1$. However, we omit all statements and proof and leave for reader’s exercises. The readers can also consider the regularity of $f$ obtained in Theorem 4.3, which are very similar with results listed in this appendix for the case when $\sup_m \tilde{d}(f_m(X_m), \gamma_m(x)) < \infty$. The measurable dependence respecting base points will be studied in our future work; see [Che18a].
6.11. a local version of the regularity results. Generally speaking, the regularity results do not depend on the existence results, that is if there is a graph which is (locally) invariant under \( H \), then one can investigate directly the regularities of this graph; our proofs given in Section 6.3 to Section 6.7 have indicated this. For a simple motivation, see also Lemma D.2 and Lemma D.3; this idea was also used in [BLZ99] to prove the smoothness of the invariant manifolds. In the following, we will present detailed statements of conditions such that the regularity results hold in a local version included for our convenient application; see Theorem 7.12, Theorem 7.13, and [Che18b, Che18c] as well. In this way, one may comprehend what we really need in our proofs.

Take a bundle correspondence \( H \sim (F, G) : X \times Y \to X \times Y \) over \( u \). Let \( X_\sharp^2 \subset X_\sharp^1 \subset X_m \). Assume there are functions \( f_m : X_m \to Y_m \), and \( x_m(\cdot) : X_m \to X_{u(m)}, m \in M \), such that

\[
\text{Graph} f_m|_{X_m^1} \subset H_m^{-1}\text{Graph} f_u(m)|_{X_{u(m)}^2},
\]

i.e. \( x_m(X_m^1) \subset X_{u(m)}^2 \), and for all \( x \in X_m^1 \),

\[
\begin{align*}
F_m(x, f_u(m)(x_m(x))) &= x_m(x), \\
G_m(x, f_u(m)(x_m(x))) &= f_m(x).
\end{align*}
\]

Take \( Y_m^1 \subset Y_m \) such that \( f_m(X_m^1) \subset Y_m^1 \).

(I) Assume \( H_m \sim (F_m, G_m) : X_m^1 \times Y_m^1 \to X_m^{1'} \times Y_m^{1'} \) satisfies (A') (\( \alpha(m), \lambda_x(m) \)) (B') (\( \beta'(m), \lambda_u(m) \)) condition.

(II) For each \( m \in M \) and each \( z \in X_m^1 \times Y_m^1 \), \( (DF_m(z), DG_m(z)) : X_m \times Y_m \to X_{u(m)} \times Y_{u(m)} \) satisfies (A') (\( \alpha(m), \lambda_x(m) \)) (B) (\( \beta'(u(m)), \beta''(m), \lambda_u(m) \)) condition.

(III) For every \( m \in M \), there is a neighborhood \( X_m^1 \) of \( X_m^1 \) in \( X_m \) such that

\[
\text{Lip} f_m|_{X_m^1} \leq \beta'(m), \quad \text{Lip} x_m|_{X_m^1} \leq \lambda_x(m).
\]

If (H3) is assumed, then we also suppose that \( |f_m(x)| \leq \beta'(m)|x| \) and \( |x_m(x)| \leq \lambda_x(m)|x| \) for all \( x \in X_m^1, m \in M_1 \).

(IV) For \( m \in M \), there are \( \epsilon''_{m'} > 0, X_m^0 \times Y_m^0 \), such that \( X_m^2 \subset X_m^1 \subset X_m^0 \), \( f_m(X_m^1) \subset Y_m^0 \), and moreover

\[
X_m^2 \subset \varphi_m^\prime(X_m^0), \quad X_m^1 \subset \varphi_m^\prime(X_m^0), \quad X_m^1 \subset Y_m^0, \quad m \in U_{m_0}(\epsilon''_{m_0}), \quad m_0 \in M,
\]

where \( (U_{m_0}(\epsilon''_{m_0}), \varphi_m^\prime) \in \mathcal{A}', (U_{m_0}(\epsilon''_{m_0}), \varphi_m^\prime) \in \mathcal{B}' \) (in (H2)), and \( \sup_{m_0 \in M} \epsilon''_{m_0} > 0 \).

Let \( X^1 = \{ (m, x) : x \in X_m^1, m \in M \}, Y^1 = \{ (m, y) : y \in Y_m^1, m \in M \} \) be the subbundles of \( X \) and \( Y \) respectively. For simplicity, write (see (5.1))

\[
Y^V_{X^1} \triangleq \bigcup_{(m, x) \in X^1} (m, x) \times X_m, \quad Y^H_{X^1} \triangleq \bigcup_{(m, x) \in X^1} (m, x) \times T_m M.
\]

Then the results in Section 6.3 ~ Section 6.9 also hold for this case with some modifications, i.e. in the statement of the conditions and the conclusions, \( X_m \times Y_{u(m)} \) is replaced by \( X_m^0 \times Y_{u(m)}^0 \), \( X_m \times Y_m \) by \( X_m \times X_m^1 \), \( X_m^1 \times Y_m^1 \), and \( X_m \) by \( X_m^1 \), in the following Theorem 6.39 we makes this convention.

Theorem 6.39. (a) \( (x \mapsto f_m(x)) \) Under (I) ~ (III) and the same conditions in Lemma 6.7 with (b) replaced by (b'): \( F_m, G_m \) are differentiable at \( X_m^1 \times Y_m^1 \) and \( X_m^1 \times Y_m^1 \) \( \ni z \mapsto DF_m(z), DG_m(z) \) are \( C^0 \), then \( f_m \) is differentiable at \( X_m^1 \). Moreover, there exists a unique vector bundle map \( K^1 \in L(T_m^V, T_m^V) \) over \( f \), such that \( \|K_m^1(x)\| \leq \beta'(m), \forall m \in M, X_m^1 \ni x \mapsto K_m^1(x) \) is \( C^0 \), and it satisfies (6.2) with \( Df_m(x) = K^1_m(x), x \in X_m^1 \).

(b) \( (x \mapsto K_m^1(x)) \) Let (I) ~ (III) hold. (i) Then Lemma 6.11 holds.
(i) Suppose F, G are uniformly $C^{k-1,1}$-fiber in a neighborhood of $X^1 \otimes_u Y^1$ (see Section 5.4.6) and $\sup_m \text{Lip } D^{k-1} f_m(x) |x|^{-1} < \infty$. If $\lambda_s \lambda_u < 1$, $\lambda^u_s \lambda_u < 1$, and $F, G$ are $C^k$-fiber at $X^1 \otimes_u Y^1$, then $f$ is $C^k$-fiber at $X^1$. (Lemma 6.12)

(c) $(m \mapsto f_m(x))$ Let (I) (III) (IV) hold. Then Lemma 6.13 holds.

(d) $(m \mapsto K^i_m(x))$ Let (I) (IV) hold. Then Lemma 6.17 holds.

(e) $(C^1)$ smoothness of $f$ (i) Under (I) (III) and the same conditions in Lemma 6.28 but assuming $F, G$ are $C^1$ in a neighborhood of $X^1 \otimes_u Y^1$ instead of in $X \otimes_u Y$, there exists a unique $C^0$ vector bundle map $K \in L(T^H_{X^1}, T^V_{Y^1})$ (over $f$) such that $\sup_{m \in M} \text{sup}_x \frac{|K^i_m(x)|}{|x|} < \infty$ and it satisfies (6.22) for all $x \in X^m$, $m \in M$.

(ii) Let (I) (IV) hold. Assume all the conditions in Lemma 6.18 hold but $F, G$ are $C^1$ in a neighborhood of $X^1 \otimes_u Y^1$ instead of in $X \otimes_u Y$, then $f$ is differential at $X^2 = \bigcup_{m \in M} X^2$ with $\nabla_m f_m(x) = K^i_m(x)$ for $x \in X^m$ and $m \in M$, where $K$ is given by (i).

(f) $(x \mapsto K^i_m(x))$ Let (I) (III) hold. Assume $F, G$ are $C^1$ in a neighborhood of $X^1 \otimes_u Y^1$ instead of condition (b) (i) in Lemma 6.22. Then Lemma 6.22 holds.

(g) $(m \mapsto K^i_m(x))$ Let (I) (IV) hold. Assume $F, G$ are $C^1$ in a neighborhood of $X^1 \otimes_u Y^1$ instead of condition (b) (i) in Lemma 6.23. Then Lemma 6.23 holds. Similar result of Lemma 6.23 (2) also holds.

(h) We leave the similar modifications about the continuities of $f$ in Section 6.9 for readers, and the results in Section 6.10 as well.

**Proof.** Without any changes except some notations (e.g. $X^i_m \times Y^i_{u(m)}$ replaced by $X^0_{m0} \times Y^0_{u(m)}$, $X^i_{m0}$ by $X^0_{m0}$, $X^i_m \times Y^i_{u(m)}$ by $X^i_m \times Y^i_{u(m)}$, and $X_m$ by $X^i_m$), the proofs given in Section 6.3 to Section 6.7 also show this theorem holds. □

**Remark 6.40.** In Section 6.3 ~ Section 6.7, we in fact take $X^i_m = X^i_m$, $i = 0, 1, 2$, and $Y^i_m = Y^i_m$, $i = 0, 1$. In this case, (IV) is redundant. The existence conditions in Theorem 4.1 imply (I) (III), and also (II) if $F_m(\cdot), G_m(\cdot) \in C^1$ (see Remark 3.10).

In many cases, (i) we might take $X^i_m = \{iX^i(m)\}$ if $i : M \to X \times Y$ is an invariant section of $H$. For this case $Y^i_m = \{iY^i(m)\}$. Now (IV) holds if (H3) is assumed, i.e. $i$ is a 0-section of $X \times Y$ with respect to $\mathcal{A} \times \mathcal{B}$. For example, see Remark 6.10, Lemma 6.16 and Lemma 6.27 (2).

(ii) Since in some situations, we use a bump function or a bilinear map (see Appendix D.5) to truncate the fibers of $X, Y$ usually are Banach spaces (i.e. (H2) (ii) holds). In this case, we generally take $X^i_m = X^i_m(\delta^i)$ ($r_2 < r_0 < r_1$) and $Y^i_m = Y^i_m(\delta^i)$ ($\delta_0 < \delta_1$). Now (IV) is guaranteed by $X, Y$ having $\varepsilon$-almost uniform $C^{0,1}$-fiber trivialisations at $M_1$ with respect to $\mathcal{A}, \mathcal{B}$. If $\sup_m \lambda^u_s \lambda_u < 1$ and $i = 0$ is an invariant section of $H$, then $x_m(X^i_m) \subseteq X^2_{u(m)}$ is satisfied. Thus, we can apply Theorem 6.39 to deduce the regularities of the invariant graph obtained in Theorem 4.6 (also note that the existence conditions of Theorem 4.6 have implied (I) (III) hold; for (II), see also Lemma 3.11). Here it’s worth noting that although we use the radial retraction (see (3.1)) so that the systems would be non-smooth in all $X \times Y$ but they are smooth in $X^1 \times Y^1$. In general Banach spaces, the radial retraction always exists but the smooth and Lipschitz bump functions might not, unlike in $\mathbb{R}^m$ or separable Hilbert spaces; Theorem 6.39 sometimes is important for the infinite-dimensional dynamical systems. We refer readers to see Section 7.2 and [Che18b] for more applications of those two cases.

At last, when we work in the Riemannian manifolds, the local bundle charts in $\mathcal{A}$ and $\mathcal{B}$ usually are taken as the parallel translations along the geodesics, so they are isometric (see also [Kli95, Section 1.8] for more results about the local isometric trivialization, particularly Theorem 1.8.23 thereof). For (IV), in this situation it might happen $X^2_m = X^0_m = X^i_m(r), Y^i_m = Y^i_m = Y^i_m(\delta)$.

7. Some Applications

In the following, we give some applications of our main results, i.e. $C^{k,\alpha}$ section theorem (Section 7.1), the existence and regularity of (un)strong stable foliations, fake invariant foliations and
the holonomy over a lamination for a bundle map (Section 7.2). For the application to differential equations, see [Che18d].

7.1. $C^{k,\alpha}$ section theorem. The $C^{k,\alpha}$ section theorem, which was appeared in [HPS77, Chapter 3, Chapter 6], [PSW97, Theorem 3.2] and [PSW12, Theorem 10] in different settings, has a wide application in invariant manifold theory, e.g. the theorem is used to prove the regularity of the invariant manifolds (and also the invariant foliations). As a first application of our paper’s main results obtained in Section 4.1 and Section 6, we will give a $C^{k,\alpha}$ section theorem in more general settings.

(HH1) Let $M$ be a locally metrizable space (associated with an open cover $\{U_m : m \in M\}$); see Definition 5.1. The metric in $U_m$ is denoted by $d_m$. Let $M_1 \subset M$ and $\epsilon > 0$ a sufficiently small constant.

(HH2) Let $(X, M, \pi)$ be a $C^0$ topology bundle (see e.g. Example 5.17 (b)) with metric fibers. Let

$$\mathcal{A} = \{(U_{m_0}(\epsilon), \varphi_{m_0}^0) : \text{a } C^0 \text{ bundle chart of } X \text{ at } m_0 : \varphi_{m_0}^0 = \text{id}, m_0 \in M_1, 0 < \epsilon < \epsilon'\},$$

be a bundle atlas of $X$ at $M_1$. $(X, M, \pi)$ has an $\epsilon$-almost uniform $C^{0,1}$-fiber trivialization at $M_1$ with respect to $\mathcal{A}$, where $\epsilon > 0$ is small depending on bunching condition; see Section 5.4.5.

(HH3) (Lipschitz continuity of $h^{-1}$) Let $h : M_1 \to M$ be a $C^0$ invertible map. Assume there exist an $\epsilon_1 > 0$ and a function $\mu : M \to \mathbb{R}_+$ such that for every $m_0 \in M_1$,

(i) $h^{-1}(U_{m_0}(\mu^{-1}(m_0)\epsilon_1)) \subset U_{h^{-1}(m_0)}(h^{-1}(m), h^{-1}(m_0)) \leq \mu(m_0)d_{m_0}(m, m_0)$, $m \in U_{m_0}(\mu^{-1}(m_0)\epsilon_1)$, and

(ii) $\sup_{m_0 \in M_1} \mu(m_0) < \infty$.

Theorem 7.1 ($C^{k,\alpha}$ section theorem I). Let (HH1) (HH2) (HH3) hold. Let $f : X \to X$ be a $C^0$ bundle map over $h$. Assume that the fibers of $X$ are uniformly bounded, i.e. $\sup_m \text{diam } X_m < \infty$, and that $\text{Lip } f_m \leq \lambda(m)$, where $\lambda : M \to \mathbb{R}_+$ is $\epsilon$-almost uniformly continuous around $M_1$ and $\epsilon$-almost continuous (see Definition 5.3), and $\sup_m \lambda(m) < 1$. Then the following hold.

(1) There is a unique section $\sigma : M \to X$ being invariant under $f$. Also $\sigma$ is $C^0$.

(2) In addition, if $f_m$ depends in a uniformly Lipschitz fashion on the base point $m$ (around $M_1$ with respect to $\mathcal{A}$) and the following bunching condition holds, $\sup_m \mu^\theta(m)\lambda(m) < 1$, then $\sigma$ is uniformly (locally) $C^{0,\theta}$ (around $M_1$).

Here we explain some notions.

(a) That $f_m$ depends in a uniformly Lipschitz fashion on the base point $m$ around $M_1$ (see also [AV10, section 2.5] using this way without the word ‘uniformly’) means that $\tilde{f}_m$, the local representation of $f$ at $m_0 \in M_1$ with respect to $\mathcal{A}$, i.e.

$$\tilde{f}_m(m, x) \triangleq (\varphi_m^{m_0})^{-1} \circ f_{h^{-1}(m_0)} \circ (\varphi^{h^{-1}(m_0)}_{h^{-1}(m)}(x)) : U_{m_0}(\mu^{-1}(m_0)\epsilon_1) \times X_{h^{-1}(m_0)} \to X_{m_0},$$

where $\varphi_m^{m_0}, \varphi^{h^{-1}(m_0)}$ are two bundle charts at $m_0, h^{-1}(m_0)$ respectively belonging to $\mathcal{A}$, satisfies

$$|\tilde{f}_m(m, x) - \tilde{f}_m(m_0, x)| \leq M_0d_{m_0}(m, m_0),$$

where the constant $M_0$ does not depend on $m_0 \in M_1$ (see Definition 5.27 and Remark 5.28); see also [PSW12].

(b) Similarly, that $\sigma$ is uniformly (locally) $C^{0,\theta}$ (around $M_1$) means that $\tilde{\sigma}$, the local representation of $\sigma$ at $m_0 \in M_1$ with respect to $\mathcal{A}$, i.e.

$$\tilde{\sigma}_m(m) \triangleq (\varphi_m^{m_0})^{-1} \circ \sigma(m) : U_{m_0}(\mu^{-1}(m_0)\epsilon_1) \to X_{m_0},$$

satisfies

$$|\tilde{\sigma}_m(m) - \tilde{\sigma}_m(m_0)| \leq C_0d_{m_0}(m, m_0)^\theta,$$

where the constant $C_0 > 0$ does not depend on $m_0 \in M_1$. 
Remark 7.2. The assumption that the fibers of $X$ are uniformly bounded is only for simplicity. One can replace the assumption by that (a) there is a section $i : M \to X$ such that $d(f_m(i(m)), i(h(m))) < \infty$, where $d$ is the metric in $X_m$. The existence result now follows from Theorem 4.1, and others are the same; the regularity results would need that (b) $i$ satisfies assumption (B3) in page 83. The uniqueness now means in the sense that if there is a invariant section $\sigma'$ : $M \to X$ of $f$ such that $\sup_m d(\sigma'(m), i(m)) < \infty$, then $\sigma' = \sigma$. Note that no requirement that $X$ has a continuous (or H"older) section is made.

Proof of Theorem 7.1. Let $\overline{M} = M, \overline{X}_m = \{m\}, \overline{Y}_m = X_m, u = h^{-1}$, and

$$G_m(x, y) = f_{h^{-1}(m)}(y) : \overline{X}_m \times \overline{Y}_u(m) \to \overline{Y}_m, F_m(x, y) = u(m) : \overline{X}_m \times \overline{Y}_u(m) \to \overline{X}_u(m).$$

Then $H \sim (F, G)$. Note that $H$ satisfies (A)($\alpha, \lambda$) (B)(0, 0) condition for any positive constant $\alpha < 1$. Now Theorem 4.1 or Theorem 4.3 gives (1); that $\sigma$ is $C^0$ can be proved similarly as Lemma 6.25. (2) is a direct consequence of Lemma 6.31. \(\square\)

(HE1) (about $M$) Let $M$ be a $C^1$ Finsler manifold with Finsler metric $d$ in its components, which satisfies (H1c) in Section 6.2; see also Appendix C for some examples.

(HE2) (about $X$) $(X, M, \pi)$ are $C^1$ bundles with $C^0$ connection $CX$. Take a $C^1$ (regular) normal bundle atlas $\mathcal{A}$ of $X$. Assume $(X, M, \pi)$ has an $\epsilon$-almost uniform $C^{1,1}$-fiber trivialization at $M_1$ with respect to $\mathcal{A}$ (see Section 4.5.4), where $\epsilon > 0$ is small depending on bunching condition. The fibers of $X$ are Banach spaces (see also Remark 6.4).

(HE3) (about $h$) $h : M_1 \to M$ is a $C^1$ invertible map. Assume $h$ satisfies the following.

(i) There is a function $\mu : M \to \mathbb{R}_+$ such that $|Dh^{-1}(m)| \leq \mu(m)$.

(ii) $m \mapsto |Dh^{-1}(m)|$ is $\epsilon$-almost uniformly continuous around $M_1$; see Definition 5.3.

(iii) $\sup_{m \in M_1} \mu(m) < \infty$.

Theorem 7.3 ($C^{k,\alpha}$ section theorem II). Let the assumptions in Theorem 7.1 hold with (HH1) (HH2) (HH3) replaced by (HE1) (HE2) (HE3) and the assumption that the fibers of $X$ are uniformly bounded is replaced by as Remark 7.2 (a) (b). Also assume $f$ is $C^1$. Let $\sigma$ be the unique invariant section of $f$. Then the following hold.

(1) If $f_m$ depends in a uniformly Lipschitz fashion on the base points (around $M_1$ with respect to $\mathcal{A}$) and the following bunching condition holds, $\sup_m \mu(m)\lambda(m) < 1$, then $\sigma$ is $C^1$ (and uniformly (locally) $C^{0,1}$ around $M_1$).

(2) In addition, assume that $D^0 f, \nabla f$ are uniformly (locally) $C^{0,1}$ around $M_1$, that $h^{-1}$ is uniformly (locally) $C^{0,1}$ around $M_1$ and that the following bunching condition holds, $\sup_m \mu(m)\lambda(m) < 1$, then $m \mapsto \nabla_m \sigma(m)$ is uniformly (locally) $C^{0,0}$ (around $M_1$).

Some explanations about the conditions in above theorem are needed. Use the notations in assumption (\(\square\)) (in page 49).

(a) For the meaning that $D^0 f, \nabla f$ are uniformly (locally) $C^{0,1}$ around $M_1$, see e.g. Remark 6.6. That is $Df_m(\cdot), \nabla_m f_m(\cdot) \in C^{0,1}$ uniform for $m \in M$ and for the local representations $D_x \tilde{f}_m, \tilde{D}_m f_m$ of $D^0 f, \nabla f$ with respect to $\mathcal{A}, M$, one has

$$\max \{|D_x \tilde{f}_m(m_1, x) - D_x \tilde{f}_m(m_0, x)|, |\tilde{D}_m f_m(m_1, x) - \tilde{D}_m f_m(m_0, x)|\} \leq M_0 d(m_1, m_0),$$

for all $m_1 \in U_m(\mu^{-1}(m_0)\epsilon_1), x \in X_m, m_0 \in M_1$, where $M_0$ is independent of $m_0 \in M_1$.

(b) That $h^{-1}$ is uniformly (locally) $C^{1,1}$ around $M_1$ means that in local representation

$$Dh^{-1} m_0(m) \nu = D_X h^{-1}(m_0) h^{-1}(m) Dh^{-1}(m) D_X h^{-1}(m_0) \nu, (m, \nu) \in U_m(\epsilon') \times T_m M,$$

one has $|Dh^{-1} m_0(m) - Dh^{-1}(m_0)| \leq C_0 d(m, m_0), m \in U_m(\epsilon'_1), m_0 \in M_1$, for some small $\epsilon'_1 > 0$ and some constant $C_0 > 0$ (independent of $m_0$); see also Definition 5.32.
(c) Similarly, that \( m \mapsto \nabla_m \sigma(m) \) is uniformly (locally) \( C^{0,\theta} \) (around \( M_1 \)) means that for the local representations \( \hat{D}_m \sigma_{m_0} \) of \( \nabla \sigma \) with respect to \( \mathcal{A}, M \), one has

\[
|\hat{D}_m \sigma_{m_0}(m_1) - \nabla_m \sigma(m_0)| \leq C_0 \alpha(d(m_1, m_0), m_1 \in U_{m_0}(\epsilon''), m_0 \in M_1,
\]

for some small \( \epsilon'' > 0 \) and some constant \( C_0 \) does not depend on \( m_0 \in M_1 \).

(d) See also Section 6.8 (with e.g. [HPS77, Sta99, Cha04, Cha08] and [CHT97, CLL91, Hal61, Yi93, CY94, CL97]).

\[\text{Proof.}\] Using the notations in the proof of Theorem 7.1 and considering \( \overline{X}_m \) as a zero space, now from Lemma 6.35 and Lemma 6.37 we obtain the results. \( \square \)

7.2. **Invariant foliations.** Invariant foliations have been investigated by many authors, see e.g. [HPS77, Fen71, PSW97, PSW12, CY94] in finite dimensional setting, [CLL91, BLZ00, BLZ99, Rue82, LL10] in the infinite dimensional setting, [Cha08] even in metric spaces and many others in different settings, too numerous to list here. The settings include the different hyperbolicity assumptions (uniform hyperbolicity, non-uniform hyperbolicity, partial hyperbolicity, etc), the different types of ‘dynamics’ (maps, semiflows, cocycles, random dynamical systems, etc), the different state spaces (compact Riemannian manifolds, Riemannian manifolds having bounded geometry, \( \mathbb{R}^n \), Banach spaces, even metric spaces). Invariant foliations (with invariant manifolds) are extremely useful in the study of qualitative properties of a dynamical system nearby invariant sets. And they also provide some coordinates such that the dynamical system can be decoupled (see e.g. Corollary 4.19 and [Lu91]). Roughly, the strong (un)stable foliations exist in partially hyperbolic systems, but the center foliations do not (see e.g. [RHRHTU12]). The notion of dynamical coherence is important for further studying center foliations (see [HPS77]). To avoid assuming the dynamical coherence, in their paper [BW10], Burns and Wilkinson introduced the ‘fake invariant foliations’ (see also Section 7.2.3 below) which are locally invariant.

The regularity of invariant foliations is of crucial importance. For example, the Hölder and smooth regularity of the foliations (or more precisely the holonomy maps) are not only vital in the study of stably ergodic behavior (see e.g. [PS97]) but also important to give higher smooth linearization (see e.g. [ZZJ14] and Remark 4.20). Usually, the smoothness of the leaves of the strong (un-)stable foliations is the same as the dynamical system (see e.g. [HPS77]). However, the foliations may only be Hölder. Absolute continuity of the invariant foliations, which is another topic we do not address here, usually needs the Hölder continuity of the foliations as preparations; see e.g. [LYZ13, BY17a] in the infinite dimension setting.

As a second application of our results, we give some results about strong (un)stable foliations, fake invariant foliations and the holonomy over a lamination for a bundle map which are extremely useful when studying the partially hyperbolic dynamics. **For some notions related with foliation we will use in the following, see Section 5.5.2.**

7.2.1. **Invariant foliations for bundle maps.** The following result was reproved by many authors, see e.g. [HPS77, Sta99, Cha04, Cha08] and [CHT97, CLL91, Hal61, Yi93, CY94, CL97].

**Theorem 7.4** (invariant foliations: global version). Let \((X, M, \pi_1), (Y, M, \pi_2)\) be bundles with metric fibers. Let \( u : M \to M \) be a map. Let \( H \sim (F, G) : X \times Y \to X \times Y \) be a bundle map over \( u \) satisfying (i) \((A')(\alpha, \lambda_u)(B)(\beta; \beta', \lambda_u)\) condition (or (i') \((A)\alpha(\lambda_u)(B)(\beta; \beta', \lambda_u)\) condition), where \( \alpha, \lambda_u, \beta, \beta', \lambda_u \) are bounded functions of \( M \to \mathbb{R}_+ \). Set \( \vartheta(m) = (1 - \alpha(m)\beta'(u(m)))^{-1} \) if (i) holds, and \( \vartheta(m) = 1 \) if (i') holds. In addition,

(a) (angle condition) \( \sup_m \alpha(m)\beta'(u(m)) < 1, \beta'(u(m)) \leq \beta(m), \forall m \in M, \)
(b) (spectral condition) \( \sup_m \lambda_u(m)\lambda_u(m)\vartheta(m) < 1. \)

**Existence.** There exist unique \( W^s_m \triangleq \bigsqcup_{q \in X_M \times Y_M} W^s_m(q), m \in M, \) such that (1) (2) hold.
(1) $q \in W^s_m(q)$ and each $W^s_m(q)$ is a Lipschitz graph of $X_m \to Y_m$ with Lipschitz constant less than $\beta'(m);$  
(2) $H_m W^s_m(q) \subset W^s_{u(m)}(H_m(q)), q \in X_m \times Y_m, m \in M.$  
In addition, we have  
(3) $W^s_m(q) = \{ p \in X_m \times Y_m : d_{u(m)}(H_m^n(p), H_m^n(q)) \leq \varepsilon(n), n \to \infty \},$ where $\varepsilon(\cdot)$ is a function of $M \to \mathbb{R}_+$ satisfying $\lambda_s(m) + \zeta < \varepsilon(m) < (\lambda_s(m)\theta(m))^{-1} - \zeta$ for all $m \in M,$ where $\zeta$ is small and $d_m$ is the metric in $X_m \times Y_m.$  
(4) for each $m \in M,$ $W^s_m$ is a $C^1$ ‘foliation’ of $X_m \times Y_m$; in particular, if $W^s_m(q_1) \cap W^s_m(q_2) \neq \emptyset,$ then $q_1 \in W^s_m(q_2).$

**Regularity.**

(5) If $\sup_m \operatorname{Lip} H_m(\cdot) < \infty,$ then $W^s_m$ is a uniformly (locally) Hölder ‘foliation’ (uniform for $m$).  
(6) Moreover, suppose $X_m, Y_m, m \in M,$ are Banach spaces (see also Remark 6.4).  
(i) If $F_m(\cdot), G_m(\cdot)$ are $C^1$ for all $m \in M,$ then every $W^s_m(q)$ is a $C^1$ graph. If $F_m(\cdot), G_m(\cdot)$ are $C^{1,\gamma}$ ($\gamma > 0$) uniform for $m,$ then every $W^s_m(q)$ is a $C^{1,\zeta}$ graph (uniform for $m, q$) where $0 < \zeta \leq \gamma;$ If in addition $\sup_m \operatorname{Lip} H_m(\cdot) < \infty,$ then the tangent distribution $T_q W^s_m(q)$ depends in a uniformly (locally) Hölder fashion on $q \in X_m \times Y_m,$ where the Hölder constants are independent of $m;$  
(ii) If $\sup_m \operatorname{Lip} H_m(\cdot) < \infty,$ $F_m(\cdot), G_m(\cdot) \in C^{1,1}$ uniform for $m,$ and the following bunching condition holds:  
$$\sup_m \operatorname{Lip} H_m(\cdot) \lambda_s(m) \lambda_u(m) \theta(m) < 1,$$

then $W^s_m$ is a $C^1$ foliation of $X_m \times Y_m;$ in addition (a) $H_m(\cdot) \in C^{1,1}$ uniform for $m,$ and (b) $\sup_m \lambda_s(m) < 1,$ or (b')  
$$\sup_m \lambda_s^2(m) \lambda_u(m) \theta(m) < 1, \quad \sup_m \operatorname{Lip} H_m(\cdot) \lambda_s^2(m) \lambda_u(m) \theta(m) < 1,$$

then the foliation $W^s_m$ is uniformly (locally) $C^{1,\zeta}$ for some $\zeta > 0$ (uniform for $m$).

**Remark 7.5.** (a) The Hölder exponent $\theta$ in item (5) can be chosen as $((\mu/\lambda_s)^\theta) \lambda_s \lambda_u < 1,$ where $\mu(m) = \operatorname{Lip} H_m(\cdot).$ If $\sup_m \mathrm{diam} H_m(X_m \times Y_m) < \infty$ or $\theta = 1,$ $((\mu/\lambda_s)^\theta) \lambda_s \lambda_u \theta < 1$ could be read as  
$$\sup_m (\mu(m)/\lambda_s(m))^\theta \lambda_s(m) \lambda_u(m) \theta(m) < 1.$$

(b) Let $W^s_m(q) = \operatorname{Graph} f_{(m,q)},$ where $f_{(m,q)} : X_m \to Y_m.$ That the foliation $W^s_m$ is uniformly (locally) $C^{k,\gamma}$ ($\gamma \geq 0$) uniform for $m$ means that  
$$(q, x) \mapsto f_{(m,q)}(x) : (X_m \times Y_m) \times X_m \to Y_m,$$
is (locally) $C^{k,\gamma},$ and if $\gamma > 0,$ the (locally) $C^{k,\gamma}$ constant is independent of $m.$ $W^s_m(q)$ is a $C^{k,\gamma}$ graph (uniform for $m, q$) if $f_{(m,q)}(\cdot) \in C^{k,\gamma}$ with $\sup_{p_{m,q}} |f_{(m,q)}(\cdot)|^{C^{k,\gamma}} < \infty$ (if $\gamma > 0$) and $$(q, x) \mapsto D_j f_{(m,q)}(x)$$ being $C^0, i = 0, 1, \ldots, k.$  
(c) In the first statement of item (6) (i), suppose further that $DF_m(\cdot), DG_m(\cdot), m \in M,$ are uniformly continuous, then $q \mapsto f_{(m,q)}(\cdot)$ is uniformly continuous in $C^1$-topology in any bounded subset of $X_m,$ i.e. for any bounded subset $A \subset X_m,$ one has  
$$\sup_{x \in A} |f_{(m,q)}(x) - f_{(m,q)}(x)| + \sup_{x \in A} ||D f_{(m,q)}(x) - D f_{(m,q)}(x)|| \to 0,$$
as $q' \to p$ uniform for $p.$ This follows from Lemma 6.25 and Lemma 6.27.  
(d) We do not consider the continuity and smoothness of $m \mapsto W^s_m(q).$ This is what our general results deal with, so see Section 6 for a detailed study. A simple result is that if $M$ is topology space, $X, Y$ are $C^{0,1}$-fiber bundle (see Definition 5.23), and $u, H$ (also $\alpha, \beta, \beta', \lambda_s$) are continuous, then $(m, q) \mapsto W^s_m(q)$ is continuous point-wisely, i.e. $(m, q, x) \mapsto f_{(m,q)}(x)$ is continuous (as a bundle map $(X \times Y) \times X \to Y$ over id).
Proof of Theorem 7.4. Let \( \hat{M} = X \times Y \) be a bundle over \( M \), where the topology of \( \hat{M} \) is given by the fiber topology (see Section 5.2). Define the bundle \( \hat{X} \times \hat{Y} \) over \( \hat{M} \) as \((X \times Y) \times (X \times Y)\). Consider the bundle map \( \hat{H} \) over \( H \), which is defined by \( \hat{H}(m, p, q) = (H(m, p), H_m(q)) \), where \((m, p) \in \hat{M}, q \in \hat{X}(m, p) \times \hat{X}(p, q) = X_m \times X_p \). Set \( i(m, p) = p : \hat{M} \rightarrow \hat{X} \times \hat{Y} \). Then \( i \) is an invariant section of \( \hat{H} \).

Now the existence of \( W_{m}^s \) such that (1) (2) hold follows from Theorem 4.1 by applying to \( \hat{H}, \hat{X} \times \hat{Y}, H, \hat{M}, i \). (3) follows from Section 4.4 and (4) is a consequence of Lemma 6.25 and (3). (5) follows from Remark 6.15.

The first and second statements in item (6) (i) follow from Lemma 6.7 and Lemma 6.11 respectively. The third statement in item (6) (i) follows from Lemma 6.16. The results in item (6) (ii) are consequences of Lemma 6.18 and Lemma 6.23. Here note that \( F_m(\cdot), G_m(\cdot) \in C^1 \) implies \( H_m(\cdot) \in C^1 \).

For a local version, one can use Lemma 3.12 and radial retraction (or smooth bump function or bid map (see Appendix D.5) if one needs smooth results). See also Theorem 7.12 below for a local version for maps. The parallel results for continuous cocycles, which we do not give here, can be deduced from the bundle map case; see also [Che18d].

7.2.2. strong stable laminations. We give a restatement of Theorem 4.1 and the results in Section 4.4 for maps (might not be invertible) in a local version. The parallel results for semiflows also hold which we omit the detailed statements. The results can hold for Lipschitz maps in metric spaces which include the two-sided shifts of finite type and the restrictions of Axiom A diffeomorphisms to hyperbolic basic sets, or smooth maps in Finsler(-Banach) manifolds. First consider a classical result in the finite dimensional setting; we reprove this result to show how our main results can be applied.

(Settings): Let \( f : M \rightarrow M \) be a \( C^{1,1} \) map. For simplicity, assume \( M \) is a smooth compact Riemannian manifold without boundary and \( f \) is invertible. Assume \( f \) is partially hyperbolic with \( TM = X^s \oplus X^c \oplus X^u \) where \( X^s, X^c, X^u \) are three continuous vector subbundles of \( TM \) and invariant under \( Df \). \( Df \) in \( X^s \) and \( X^u \) uniformly contracts and expands respectively. Let \( T^k f = Df|_{X^s}, \kappa = s, c, u, \) then

\[
\|T^k f\| < 1, \quad \|(T^k f)^{-1}\| < 1, \quad \|T_m^s f\| \cdot \|(T_m^c f)^{-1}\| < 1, \quad \|T_m^c f\| \cdot \|(T_m^u f)^{-1}\| < 1, \quad m \in M.
\]

The strong stable (resp. unstable) foliation \( W^s \) (resp. \( W^u \)) exists uniquely such that \( T_m W^s_m = X^s_m \) (resp. \( T_m W^u_m = X^u_m \)). Assume \( W^s \) (resp. \( W^u_m \)) is the leaf of \( W^s \) (resp. \( W^u_m \)) through \( m \); see [HPS77] or Corollary 7.11 below. Assume there are \( (f\text{-invariant}) \) foliations \( W^{cs} \) and \( W^{cu} \) such that \( TW^{cs} = X^s \oplus X^c, TW^{cu} = X^c \oplus X^u \). Under this context, \( H \) is said to be dynamically coherent (see [HPS77]) and it also yields the integrability of \( X^c \). By the characterization of \( W^s \), we see \( W^s \) subordinates \( W^{cs} \).

In [PSW97, PSW00], Pugh, Shub and Wilkinson showed the following.

Theorem 7.6 ([PSW97, PSW00]). Under above settings, if the following center bunching condition holds,

\[
\|T^k_m f\| \cdot \|T^s_m f\| \cdot \|(T^c_m f)^{-1}\| < 1, \quad m \in M,
\]

then \( W^s \) is a \( C^1 \) foliation inside each leaf of \( W^{cs} \). In fact this is also \( C^{1,\zeta} \) (\( \zeta > 0 \)) under additional assumption \( \|T^k_m f\|^2 \cdot \|(T^u_m f)^{-1}\| < 1, \quad m \in M. \) (Similar fact about \( W^u, W^{cu} \) holds.)

Proof. Since \( M \) is a smooth compact boundaryless Riemannian manifold, there is a \( \delta > 0 \), such that for every \( k \in \mathbb{N} \), the Riemannian metric up to \( k \)-th order derivatives and the Christoffel symbols in the normal coordinates up to \( k \)-th order derivatives are all uniformly bounded in normal coordinates of radius \( \delta \) around each \( m \in M \) with the bounds independent of \( m \in M \). In particular, for some small \( \delta > 0 \), the normal coordinate or exponential map \( \exp_m \) at \( m \),

\[
\exp : T_m M(\delta) \rightarrow U_m(\delta) = \{m' \in M, d(m', m) < \delta\},
\]

satisfies (i) \( |D \exp^s_m(\cdot)| \leq C \) (so \( |\exp^{-1}_m(m_1)| \leq C d(m, m_1) \)), and (ii)

\[
|\exp^{-1}_m \exp_m(z_0 + z) - P^m_{m_1} z| \leq C d(m_1, m)|z|,
\]

(7.1)
for some constant $C > 0$ independent of $m, m_1$, where $z_0 = \exp^{-1}_m m_1, m_1 \in U_m(\delta/4), z \in T_m M(\delta/4)$, and $P^m_{m_1} : T_m M \to T_m M$ is the parallel translation along the geodesic connecting $m, m_1$. For convenience of the reader, we show (7.1) as follows. Here what we really need is that $M$ has 3-th order bounded geometry (see Definition C.6).

**Proof of (7.1).** For the Christoffel symbol $\Gamma^m_i(z) \in L(T_m M, T_m M), z \in T_m M(\delta/4)$ in the local chart $\exp^{-1}_m$ (see also Definition 5.10), one has $z \mapsto \Gamma^m_i(z) \in C^{1,1}$ uniform for $m_1 \in M$. The constructions of geodesic and parallel translation are through solving the following two ODEs respectively,

$$u'' = \Gamma^m_i(u) (u', u'), \ y' = \Gamma^m_i(x_a)(x'_a, y),$$

where $x_a(t), 0 \leq t \leq 1$, is the representation of the geodesic connecting $\exp_m a, m_1$ with $x_a(0) = \exp_m a$ in the local chart $\exp^{-1}_m$. Denote the two solutions associated with above ODEs by $u(t)(z, v)$, $y(t)(a)v$ respectively, where $u(0)(z, v) = z, u'(0)(z, v) = v, y(0)(a)v = v$. Then

$$u(1)(\exp^{-1}_m m, D \exp^{-1}_m(m)(z_0 + z)) = \exp^{-1}_m \exp_m(z_0 + z),$$

$$y(1)(\exp^{-1}_m m, D \exp^{-1}_m(m)z) = P^m_{m_1} z.$$

Since $z \mapsto \Gamma^m_i(z)$ is $C^{1,1}$, we see $(z, v) \mapsto u(1)(z, v)$ is $C^{1,1}$, and so $a \mapsto x_a(\cdot)$ and $a \mapsto y(1)(a) \in L(T_m M, T_m M)$ are $C^0,1$, uniform for $m_1$. Let $z_1 = \exp^{-1}_m m$. Note that $y(1)(0, v) = v, u(1)(0, v) = v$, and $u(1)(z_1, D \exp^{-1}_m(m)(z_0)) = 0$, so

$$u(1)(z_1, D \exp^{-1}_m(m)z_0 + v) - y(1)(z_1)v$$

$$= \int_0^1 D_2 u(1)(z_1, D \exp^{-1}_m(m)z_0 + tv)v - y(1)(z_1)v \ dt$$

$$\leq \int_0^1 |D_2 u(1)(z_1, D \exp^{-1}_m(m)z_0 + tv)v - D_2 u(1)(0, D \exp^{-1}_m(m)z_0 + tv)v|$$

$$+ |y(1)(z_1)v - y(1)(0)v| \ dt \leq C|z_1||v|.$$

Letting $v = D \exp^{-1}_m(m)z$, we finish the proof. □

Let $\mathbb{G}(TM)$ be the Grassmann manifold of $TM$. Let $\Pi^k_m \in L(T_m M, T_m M)$ be the projections such that $R(\Pi^k_m) = X^s_m$ and $\Pi^c_m + \Pi^u_m = \Pi_m = id, m \in M, k = s, c, u$. Since $m \mapsto X^s_m \in \mathbb{G}(TM)$ is $C^0$, by smooth approximation of $X^s$ in $\mathbb{G}(TM)$, there are projections $\Pi^k_m \in L(T_m M, T_m M), R(\Pi^k_m) = \hat{X}^k_m, k = s, c, u$, such that $m \mapsto \hat{X}^k_m$ is $C^1$ (so $C^{1,1}$) and $sup_m \Pi^c_m - \hat{\Pi}^c_m \leq \varepsilon$ for sufficiently small $\varepsilon > 0$.

(I). (base space). Let $\tilde{M} (= M)$ be $W^{cs}$ endowed with leaf topology (see Section 5.2). Since each leaf of $W^{cs}$ is $C^1$, boundaryless and injectively immersed submanifold of the compact manifold $M$, we see that the $C^1$ Finsler manifold $\tilde{M}$ satisfies (H1b) (in Section 6.2); the details are given as follows. (Here note that in general $\tilde{M}$ is non-separable and so is not metrizable, but each leaf of $W^{cs}$ is a Riemannian manifold.) The open cover of $\tilde{M}$ is taken as $\tilde{U}_m = W^{cs}_m, m \in M$. There is an $\varepsilon_0 > 0$, such that the small plaque at $m$ with radius $\varepsilon_0$, denoted by $\tilde{U}_m(\varepsilon_0)$ (the component of $W^{cs}_m \cap U_m(\varepsilon_0)$ containing $m$), is a $C^1$ embedding submanifold of $M$ which can be represented as $\exp^{-1}_m \tilde{U}_m(\varepsilon_0) = \mathrm{Graph}_{g_m}$, where $g_m : \tilde{X}^{cs}_m \to \tilde{X}^c_m$ with $\sup_m \mathrm{Lip}_m g_m|\tilde{X}^{cs}_m(\varepsilon_0)$ being (uniformly) small and $x \mapsto Dg_m(x), m \in M$, being equicontinuous. See also [HPS77, Chapter 6 ~ 7]. So

$$\chi_m : \tilde{U}_m(\varepsilon_0) \to \tilde{X}^{cs}_m \simeq X^{cs}_m,$$

defined by $\chi^{-1}_m(x) = \exp_m(x + g_m(x)), satisifies $sup_m sup|D\chi^{-1}_m(\cdot)| < \infty$, and $m' \mapsto D\chi^{-1}_m(m'), m \in M$, are equicontinuous; through the map $\Pi^{cs}_m : \tilde{X}^{cs}_m \to X^{cs}_m$, we can assume $D\chi_m(m) = id$.

(II). (base map). Since $W^{cs}$ is invariant under $f$, i.e. $f(W^{cs}_m) \subset W^{cs}_{f(m)}$, where $W^{cs}_m$ is the leaf of $W^{cs}$ through $m$, we see that $f$ induces a map in $\tilde{M}$, also denoted by $f : \tilde{U}_m \to \tilde{U}_{f(m)}$, which is $C^1$ with $\mathrm{Lip}_f|\tilde{U}_m(\varepsilon_0) \triangleq \mu(m)$ approximately less than $\|T^{cs}_f\| = \|T^{c}_f\|$ if $\varepsilon_0$ is small.
(III). (bundles). \( \mathring{X}^s \) can be considered naturally as a bundle over \( \mathring{M} \) which is \( C^1 \). The natural bundle atlas \( \mathcal{B}^s \) for \( \mathring{X}^s \) can be taken as

\[
\kappa^\varphi_{m^0} : \mathring{U}_{m_0}(\delta) \times \mathring{X}^s_{m_0} \rightarrow \mathring{X}^s, \quad (m, x) \mapsto (m, \Pi_m^s P^m_{m^0} x).
\]

Since \( M \rightarrow \mathcal{G}(TM) \), \( m \mapsto \mathring{\Pi}_m^s \), \( m \mapsto P^m_{m^0} \) are \( C^2 \), there exists a natural \( C^0 \) connection \( \mathring{C}^s \) for \( \mathring{X}^s \) such that \( C^s D(\kappa^\varphi_{m^0})(m_0, x) = \text{id} \), i.e. \( \kappa^\varphi_{m^0} \) is normal with respect to \( \mathring{C}^s \). (Here also note that \( \nabla_{m^0} P^m_{m^0} = 0 \) and what’s more this connection \( \mathring{C}^s \) is uniformly Lipschitz in the sense of Definition B.8.) That is \( \mathring{X}^s \times \mathring{X}^{cu} \) satisfies (H2c) (in Section 6.2).

(IV). (bundle map). Let

\[
\mathring{f}_m = \exp_{f(m)}^{-1} \circ f \circ \exp_m : T_M(\delta) \rightarrow T_{f(m)} M.
\]

Since \( D\mathring{f}_m(0) = T_m f \), we see \( \mathring{f}_m \sim (f, G_m) \), where \( f_m : \mathring{X}^s_{m}(r_1) \oplus \mathring{X}^{cu}_{f(m)}(r_2) \rightarrow X^s_{f(m)}(r_2) \) and \( G_m : \mathring{X}^s_{m}(r_1) \oplus \mathring{X}^{cu}_{f(m)}(r_2) \rightarrow X^m_{f(m)}(r_2) \), satisfies \((A(\alpha, \lambda_\mu(m)) (B(\beta, \lambda_\mu(m)) \text{ condition (see Lemma 3.12)}, \)

where \( \lambda_s(m) = \| T_m f \| + \zeta, \lambda_\mu(m) = \| (T^c_m f)^{-1} \| + \zeta \), and \( \alpha, \beta \rightarrow 0 \), as \( r_1, r_2 \rightarrow 0 \).

By using smooth bump function, one has \( \mathring{F}_m, \mathring{G}_m \) defined in all \( \mathring{X}^s \oplus \mathring{X}^{cu} \), such that

\[
\mathring{F}_m|\mathring{X}^s_{m}(r) \oplus \mathring{X}^{cu}_{f(m)}(r) = f_m, \quad \mathring{G}_m|\mathring{X}^s_{m}(r) \oplus \mathring{X}^{cu}_{f(m)}(r) = G_m.
\]

\[
\mathring{F}_m(\mathring{X}^s_{m}(r) \oplus \mathring{X}^{cu}_{f(m)}(r)) \subset \mathring{X}^s_{f(m)}(r_2), \quad \mathring{G}_m(\mathring{X}^s_{m}(r) \oplus \mathring{X}^{cu}_{f(m)}(r)) \subset \mathring{X}^{cu}_{f(m)}(r_1),
\]

and \( \mathring{H} \sim (\mathring{F}_m, \mathring{G}_m) : \mathring{X}^s \oplus \mathring{X}^{cu} \rightarrow \mathring{X}^s \) and \( \mathring{G} : \mathring{X}^s \oplus \mathring{X}^{cu} \rightarrow \mathring{X}^{cu} \) are \( C^1 \) and \( \mathring{F}_m(\cdot), \mathring{G}_m(\cdot) \) are \( C^{1,1} \) uniform for \( m \) as \( f \in C^{1,1} \). In general, \( \mathring{F}_m, \mathring{G}_m \) are not \( C^{1,1} \) since \( \mathring{M} \) is only \( C^1 \). Let us show

Sublemma 7.7. the local representations \( \mathring{F}_m(\cdot, \cdot), \mathring{G}_m(\cdot, \cdot) \) satisfy (6.23) in Lemma 6.18.

Proof. The local representations \( \mathring{F}_m(\cdot, \cdot), \mathring{G}_m(\cdot, \cdot) \) are written as

\[
\begin{align*}
\mathring{F}_m(m_0(m), x, y) &\triangleq (\varphi_{f(m)}^{-1}) \circ \mathring{F}_m \circ (\varphi_{m^0}^{-1}(x, \varphi_{f(m)}(y)) : \mathring{U}_{m_0}(e_1) \times \mathring{X}^s_{m_0}(\hat{r}) \times \mathring{X}^{cu}_{f(m_0)}(\hat{r}) \rightarrow \mathring{X}^s_{m_0}(e_1), \\
\mathring{G}_m(m_0(m), x, y) &\triangleq (\varphi_{m^0})^{-1} \circ \mathring{G}_m \circ (\varphi_{m^0}(x, \varphi_{f(m)}(y)) : \mathring{U}_{m_0}(e_1) \times \mathring{X}^s_{m_0}(\hat{r}) \times \mathring{X}^{cu}_{f(m_0)}(\hat{r}) \rightarrow \mathring{X}^{cu}_{m_0}(e_1),
\end{align*}
\]

where \( e_1 < (sup_{m_0}(\mu(m_0)))^{-1} \) is small and \( \hat{r} < \delta / 4 \) is small. Let \( m_0 \in M, m_1 \in \mathring{U}_{m_0}(e_1), \) and

\[
\begin{align*}
\exp^{-1}_{m_0} m_1 &\triangleq x^s_0 + x^{cu}_0, \quad \exp^{-1}_{f(m_0)} f(m_1) = x^s_1 + x^{cu}_1, \quad x_s \in \mathring{X}^s_{m_0}, \quad x^{cu} \in \mathring{X}^{cu}_{f(m_0)}, \quad \kappa = s, cu.
\end{align*}
\]

Here note that, since \( e_1 \) is small and (7.2), we see \( \mathring{F}_{m_0}(x^s_0, x^{cu}_0) = x^s_1, \mathring{G}_{m_0}(x^s_0, x^{cu}_1) = x^{cu}_0 \).

Let \( z = (x, y) \in X^s_{m_0}(\hat{r}) \times X^{cu}_{f(m_0)}(\hat{r}) \), and

\[
\begin{align*}
\hat{y} &\triangleq \mathring{G}_{m_0}(x, y), \quad y^t_1 \triangleq \mathring{G}_{m_0}(x + x^s_0, y + x^{cu}_0) - x^s_0 - x^{cu}_0, \\
\hat{x} &\triangleq \mathring{F}_{m_0}(x, y), \quad x^t_2 \triangleq \mathring{F}_{m_0}(x + x^s_0, y + x^{cu}_0) - x^s_1, \\
y_1 &\triangleq \mathring{G}_{m_1}(x^s_0, y, \varphi_{f(m_1)}(y)), \quad x_2 \triangleq \mathring{F}_{m_1}(x^s_0, y, \varphi_{f(m_1)}(y)).
\end{align*}
\]

We need to show \( |\hat{x} - (\varphi_{f(m)}^{-1}) x^s_1 | + |\hat{y} - (\varphi_{m^0}^{-1}) y_1 | \leq \mathring{C} d(m_1, m_0) |z| \) for some constant \( \mathring{C} \) independent of \( m_0, m_1 \). (In the following, \( \mathring{C} \) might be different line by line. As \( \mathring{F}(\cdot), \mathring{G}(\cdot) \) are \( C^{1,1} \), we get

\[
|\hat{y} - y^t_1 | + |\hat{x} - x^t_2 | \leq \mathring{C} (|x^s_0| + |x^{cu}_0|) |z| \leq \mathring{C} d(m_1, m_0) |z|.
\]

Set

\[
\begin{align*}
\exp^{-1}_{m_0} \exp^{-1}_{m_0} (x + x^s_0 + y^t_1 + x^{cu}_0) &= \mathring{x}_1 + \mathring{x}^{cu}, \\
\exp^{-1}_{f(m_1)} \exp^{-1}_{f(m_1)} (x^s_2 + x^s_1 + y + x^{cu}_1) &= \mathring{x}^{cu}_2 + \mathring{x}^{cu}_1.
\end{align*}
\]
where $\bar{x}^1 \in \bar{X}^{s}_{m_1}$, $\bar{x}^2 \in \bar{X}^{s}_{f(m_1)}$, $\kappa = s, cu$. Then by (7.1), we have

$$|\bar{x}^1 + \bar{x}^{cu} - s \varphi^{m_1}_{f(m_1)}(x) - cu \varphi^{m_1}_{m_1}(y')| \leq \tilde{C}d(m_1, m_0)(|x| + |y'|) \leq \tilde{C}d(m_1, m_0)(|x| + |y|),$$

$$|\bar{x}^2 + \bar{x}^{cu} - s \varphi^{m_1}_{f(m_1)}(x'_2) - cu \varphi^{m_1}_{f(m_1)}(y)| \leq \tilde{C}d(m_1, m_0)(|x'_2| + |y|) \leq \tilde{C}d(m_1, m_0)(|x| + |y|),$$

yielding

$$|\bar{x}^1 - y_1| = |\bar{x}^1 - y_1 - \bar{G}_{m_1}(\bar{x}^1, \bar{x}^{cu}) + \bar{G}_{m_1}(s \varphi^{m_1}_{m_1}(x), cu \varphi^{m_1}_{f(m_1)}(y))|$$

$$\leq \tilde{C}|(\bar{x}^1 - s \varphi^{m_1}_{m_1}(x) + \bar{x}^{cu} - cu \varphi^{m_1}_{f(m_1)}(y))| \leq \tilde{C}d(m_1, m_0)(|x| + |y|),$$

and similarly, $|\bar{x}^2 - x_2| \leq \tilde{C}d(m_1, m_0)(|x| + |y|)$. Therefore,

$$|\hat{x} - (s \varphi^{f(m_1)}_{f(m_1)} - 1)x_2| \leq |\hat{x} - x'_2| + |x'_2 - (s \varphi^{f(m_1)}_{f(m_1)} - 1)x_2|$$

$$\leq \tilde{C}d(m_1, m_0)(|x| + |y|) + |x - (s \varphi^{f(m_1)}_{f(m_1)} - 1)x|$$

$$\leq \tilde{C}d(m_1, m_0)(|x| + |y|) + \tilde{C}(|x - (s \varphi^{f(m_1)}_{f(m_1)} - 1)x|)$$

and analogously, $|\hat{y} - (cu \varphi^{m_1}_{m_1} - 1)y_1| \leq \tilde{C}d(m_1, m_0)|z|$. This completes the proof. \qed

Now, all the assumptions in Lemma 6.18 are satisfied for $\tilde{H}$, $\tilde{X}^s$ $\tilde{X}^{cu}$, $\tilde{f}$, $\tilde{M}$ and the natural $\sigma$-section of $\tilde{X}^s \oplus \tilde{X}^{cu}$, which yields the unique invariant graph $\tilde{W}^s$ of $\tilde{X}^s$ such that $\tilde{H}\tilde{W}^s \subset \tilde{W}^s$ is $C^1$. Let $\tilde{W}^s_m(\sigma) = \tilde{W}^s \cap (\tilde{X}^s_m(\sigma) \oplus \tilde{X}^{cu}_m)$. From $\sup \|T^s_m f\| < 1$, $\tilde{H}_{\tilde{m}}|_{\tilde{W}^s_m(\sigma)} = \tilde{f}_{\tilde{m}}|_{\tilde{W}^s_m(\sigma)}$ and the characterization about the strong stable foliation $W^s$ of $f$, if $\sigma$ is small, one gets $\exp^{-1}_{m_1} \tilde{W}^s_m(\sigma)$ is open in $W^s_m$, the leaf of $W^s$ through $m$. The $C^1$ dependence of $W^c_{m_1} \ni m \mapsto \exp^{-1}_{m_1} \tilde{W}^s_m(\sigma)$ shows that $W^s$ is a $C^1$ foliation inside each leaf of $W^{cu}_{m_1}$.

For the last conclusion, by noting that under $\|T^s_m f\|^2 \cdot \|T^c_{f(m)} f\| < 1$, $m \in M$, each leaf of $W^{cu}_{m_1}$ is $C^2$ and consequently $\tilde{F}, \tilde{G} \in C^{1,1}$ (i.e. $\tilde{F}_m(\cdot, \cdot, \cdot), \tilde{G}_m(\cdot, \cdot, \cdot) \in C^{1,1}$ uniform for $m$). Now it follows from Lemma 6.23 (1). \qed

**Remark 7.8.** (a) If $f \in C^{1,\gamma}$ ($\gamma > 0$) and center bunching condition is replaced by the following

$$\|T^s_m f\| \cdot \|T^c_{f(m)} f\|^\gamma \cdot \|(T^c_{f(m)} f)^{-1}\| < 1, m \in M,$$

then one also gets $W^s$ is a $C^1$ foliation inside each leaf of $W^{cu}$; see Remark 6.19.

(b) Assume $TM$ has a $DF$-invariant continuous splitting $TM = X^s \oplus X^{cu}$ with the following inequality,

$$\|T^s_{f(m)} f\| < 1, \|T^c_{f(m)} f\| \cdot \|(T^c_{f(m)} f)^{-1}\| < 1, m \in M.$$

Let $X^{c_1}$ be a $DF$-invariant continuous subbundle of $TM$ such that $X^{c_1} \subset X^{c}$. Assume there is a $C^0$ foliation $W^{c_1}$ tangent to $X^s \oplus X^{c_1}$ which is invariant under $\phi$, i.e. $f(W^{c_1}_{m}) \subset W^{c_1}_{f(m)}$, where $W^{c_1}_{m}$ is the $C^1$ leaf of $W^{c_1}$ through $m_1$. Suppose the following bunching condition holds,

$$\|T^c_{f(m)} f\| \cdot \|T^s_{f(m)} f\| \cdot \|(T^s_{f(m)} f)^{-1}\| < 1, m \in M.$$

Then each leaf of $W^{c_1}$ is $C^1$-foliated by the strong stable foliation $W^s$ if $f \in C^{1,1}$. In the above, we use the notation $T^s f = DF_{|X^s}$, $\kappa = s, c, c_1$. The proof is the same as Theorem 7.6 (by using $W^{c_1}$ instead of $W^{cu}$).

(c) This argument given in Theorem 7.6 which we think is highly general, was also used in our proof of a center-stable manifold of a normally hyperbolic invariant manifold being $C^1$ foliated by the strong stable foliation in the Banach space setting under some circumstance; see [Che18b].
**Theorem 7.9** (strong stable laminations I: Lipschitz case). Let $M$ be a metric space, $u : M \rightarrow M$ continuous, and $\Lambda \subset M$ a subset such that $u(\Lambda) \subset \Lambda$. Assume that there is an $\epsilon > 0$ such that, for every $m \in \Lambda$, there is an open ball $U_m(\epsilon') \subset M$, where $U_m(\epsilon) = \{m' \in M : d(m', m) < \epsilon\}$, such that $m \in U_m(\epsilon') \subset X_m \times Y_m$, where $X_m, Y_m$ are complete metric spaces. (This means that there is $\phi_m : U_m(\epsilon') \rightarrow \phi_m(U_m(\epsilon')) \subset X_m \times Y_m$ which is bi-Lipschitz with Lipschitz constants less than for a fixed constant. The metric in $U_m(\epsilon')$ now induces from $U_m(\epsilon')$ through $\phi_m$.)

For each $m \in \Lambda$, there is a correspondence $\tilde{u}_m : X_m \times Y_m \rightarrow X_{u(m)} \times Y_{u(m)}$ satisfying (A$'$)(\alpha, $\lambda_u(m)$) (B)(\beta, $\lambda_s(m)$) condition, such that $u_m|U_m(\epsilon') = \tilde{u}_m(U_m(\epsilon'))$, where $\alpha, \beta$ are constants (for simplicity) and $\lambda_s, \lambda_u$ are bounded functions of $\Lambda \rightarrow \mathbb{R}_+$. Assume $\alpha \beta < 1$ and sup$_m(\lambda_s(m) + \epsilon) \lambda_u(m) < 1 - \alpha \beta$, where $\epsilon > 0$.

Then there are unique $\Lambda^s_{\text{loc}}(m, \epsilon') \subset U_m(\epsilon')$, $m \in \Lambda$ such that (1) (2) hold.

1. $m \in \Lambda^s_{\text{loc}}(m, \epsilon')$ and every $\Lambda^s_{\text{loc}}(m, \epsilon')$ is a Lipschitz graph of $X_m \rightarrow Y_m$ with Lipschitz constant less than $\beta$.

2. $u(\Lambda^s_{\text{loc}}(m, \epsilon')) \subset U_m(\epsilon') \subset C(\Lambda^s_{\text{loc}}(u(m), \epsilon'))$.

Moreover, $\Lambda^s_{\text{loc}}(m, \epsilon'), m \in \Lambda$ have the following properties.

3. $\{x \in U_m(\epsilon') : u^n(\phi_m)(x) \in U_m(m), |u^n(x) - u^n(m)| \lesssim \epsilon(m)(n \rightarrow \infty) \subset \Lambda^s_{\text{loc}}(m, \epsilon'),$ where $\varepsilon : \Lambda \rightarrow \mathbb{R}_+$ such that $\lambda_s(x) + \varepsilon < \varepsilon(m) < \lambda^{-1}_u(1-\alpha \beta)$ if $u(\Lambda^s_{\text{loc}}(m, \epsilon')) \subset U_m(\epsilon')$ for all $m \in \Lambda$, then the two sets are equal.

4. Suppose that (i) sup$_m \lambda_s(m) + \epsilon < 1$, and that (ii) if $m_1 \in \Lambda, i = 1, 2$ and $d(m_1, m_2) < \delta$, then $|\lambda_s(m_1) - \lambda_s(m_2)| < \epsilon$. Write $\Lambda^s_{\text{loc}}(m_1, \epsilon_1) = \Lambda^s_{\text{loc}}(m_1, \epsilon') \cap U_m(\epsilon_1)$. If $\Lambda^s_{\text{loc}}(m_1, \epsilon_1) \cap \Lambda^s_{\text{loc}}(m_2, \epsilon_1) = 0$, then $\Lambda^s_{\text{loc}}(m_1, \epsilon_1) = \Lambda^s_{\text{loc}}(m_2, \epsilon_1)$, where $\epsilon_1 \leq \min(\delta, \epsilon')/4$. In particular, letting

$$\Lambda^s := \bigcup_{n \geq 0} u^{-n}(\Lambda^s_{\text{loc}}(u^n(m), \epsilon_1)) = \{x \in M : |u^n(x) - u^n(m)| \lesssim \epsilon_1(m, n \rightarrow \infty),$$

then $\Lambda^s := \{\Lambda^s_{\text{loc}} : m \in \Lambda\}$ is a (u-invariant) lamination called a (global) **strong stable lamination** of $\Lambda$ for $u$, where $\lambda_s(m) + \varepsilon < \varepsilon_1(m) < \min\{1, \lambda^{-1}_u(1-\alpha \beta)\}$, and $\varepsilon_1$ is sufficiently small.

We also call $\Lambda^s_{\text{loc}}(\epsilon_1) := \{\Lambda^s_{\text{loc}}(m, \epsilon_1) : m \in \Lambda\}$ the **local strong stable lamination** of $\Lambda$ for $u$. $\Lambda^s_{\text{loc}}(\epsilon_1)$ is called the strong stable manifold (resp. local strong stable manifold) of $m \\in \Lambda$.

**Proof.** Let $X \times Y = \{(m, x, y) : (x, y) \in X_m \times Y_m, m \in \Lambda\}$ be a bundle over $\Lambda$. Let $H(m, z) = (u(m), \tilde{u}_m(z)) : X \times Y \rightarrow X \times Y$ be a bundle correspondence over $u$. Now the existence of $\Lambda^s_{\text{loc}}(m, \epsilon)$ and (1) (2) follow from Theorem 4.1 by applying to $H, X \times Y, u, \Lambda$. (3) follows from Section 4.4. To show (4), note that sup$_m \lambda_s(m) + \epsilon < 1$, so $u(\Lambda^s_{\text{loc}}(m, \epsilon')) \subset U_m(\epsilon')$. If $x \in \Lambda^s_{\text{loc}}(m_1, \epsilon_1) \cap \Lambda^s_{\text{loc}}(m_2, \epsilon_1) \neq 0$, then $u^n(x) \in \Lambda^s_{\text{loc}}(u^n(m_1), \epsilon_1) \cap \Lambda^s_{\text{loc}}(u^n(m_2), \epsilon_1) \subset U_m(\epsilon')$. So by (3), one gets $m_1 \in \Lambda^s_{\text{loc}}(m_2, \epsilon_1)$. If $z \in \Lambda^s_{\text{loc}}(m_1, \epsilon_1)$, then $u^n(z) \in \Lambda^s_{\text{loc}}(u^n(m_1), \epsilon_1) \subset U_m(\epsilon')$, which yields $\Lambda^s_{\text{loc}}(m_1, \epsilon_1) \subset \Lambda^s_{\text{loc}}(m_2, \epsilon_1)$. The proof is complete. \[\square\]

**Corollary 7.10** (holonomy map). Let Theorem 7.9 (4) hold. Assume the following about uniform continuity of $m \mapsto X_m, Y_m$ holds.

(UC) Let $(\phi^{m_1, m_2}_{m_1}, \phi^{m_2, m_1}_{m_2}) : U_m(\epsilon) \subset X_m \times Y_m \rightarrow U_m(\epsilon') \subset X_m \times Y_m$, $m \in \Lambda$, where $\epsilon < \epsilon'/2$. Then for any $\eta > 0$, there is a $\rho > 0$ such that Lip $\phi^{m_1, m_2}_{m_1} < 1 + \eta, i = 1, 2$, provided $d(m_1, m) < \rho < \epsilon'/2$.

Let $\tilde{\Lambda} \subset U_m(\epsilon_1) \cap \Lambda$. Consider $\Lambda^s_{\text{loc}} = \bigcup_{m \in \tilde{\Lambda}} \Lambda^s_{\text{loc}}(m', \epsilon_1) \subset U_m(\epsilon')$, a stack of leaves (i.e. the plaque saturate) over $\tilde{\Lambda}$. For any Lipschitz graphs $\Sigma_1, \Sigma_2 \subset U_m(\epsilon')$ of $Y_m \rightarrow X_m$ with Lipschitz constants less than $\beta$, we have, for small $\epsilon_1 > 0$.

1. For any $z \in \Lambda^s_{\text{loc}} \cap \Sigma_1$, there is a unique $m' = m'(z) \in \tilde{\Lambda}$, such that $\Lambda^s_{\text{loc}}(m', \epsilon_1) \subset \Sigma_1 = \{z\}$.

2. Define $p(z)$ by the unique point of $\Lambda^s_{\text{loc}}(m'(z), \epsilon_1) \cap \Sigma_2$, then $z \mapsto p(z)$ is continuous. The map $p(z)$ is called a **holonomy map** for the strong stable lamination $\Lambda^s$.

3. If $u$ is Lipschitz on $\Lambda^s_{\text{loc}} \cap \Sigma_1 = \{m : d(m, \Lambda) < \epsilon'/2\}$, then $z \mapsto p(z)$ is also Hölder.
Proof. Choose $\epsilon_1$ small, then by the assumption (UC) we know that $m' \mapsto \Lambda_{loc}^s (m', \epsilon_1)$ is continuous point-wisely (Lemma 6.25), and that for any $m' \in \Lambda$, $\Lambda_{loc}^s (m', \epsilon_1) \cap \Sigma_i$ consists of one point $z_i(m')$, $i = 1, 2$, and $m' \mapsto z_i(m')$ is continuous (see e.g. Lemma D.2). By the lamination property, $m' \mapsto z_i(m')$ is injective. But one can not expect that in general that $z \mapsto z_i^{-1}(z)$ is also continuous (in where they can be defined) except that $\Lambda$ is compact. In contract, the map $p(z) = z_2(z_1^{-1}(z))$ is continuous. This can be argued as follows (see also the proof of [PSW97, Corollary 4.4]). Note that $u(\Lambda_{loc}^s (\epsilon_2)) \subset \Lambda_{loc}^s (\epsilon_2)$ and the assumptions for $\Lambda$ are also satisfied by $\Lambda_{loc}^s (\epsilon_2)$, where $\epsilon_2 < \epsilon_1$. So there also exists the local strong stable lamination $\{\Lambda_{loc}^s (m', \epsilon_2) : m' \in \Lambda_{loc}^s (\epsilon_2)\}$ of $\Lambda_{loc}^s (\epsilon_2)$. By characterization the strong stable lamination, if taking $\epsilon_2$ small, one has $\Lambda_{loc}^s (m', \epsilon_2) < \Lambda_{loc}^s (m, \epsilon_1)$ if $m' \in \Lambda_{loc}^s (m, \epsilon_2)$. Also, note that $m' \mapsto \Lambda_{loc}^s (m', \epsilon_2)$ is continuous point-wisely. Now

$$\{p(z) = \Lambda_{loc}^s (m'(z), \epsilon_2) \cap \Sigma_2 = \Lambda_{loc}^s (z_1(z), \epsilon_2) \cap \Sigma_1, \text{if } z \in \Lambda_{loc}^s (z_1(z), \epsilon_2) \cap \Sigma_1 \subset \Lambda_{loc}^s (\epsilon_2), \text{ Taking } \epsilon_2 \text{ small instead of } \epsilon_1, \text{ we finish the proof of (1) (2).}

To show (3), by Lemma 6.13, if $u$ is Lipschitz on $\Lambda_{\epsilon'/2}$, then one also gets $m' \mapsto \Lambda_{loc}^s (m', \epsilon_1)$ is Hölder if taking $\epsilon_1$ small, which yields the Hölder continuity of $m' \mapsto z_i(m')$ (see e.g. Lemma D.2). Thus, we obtain the Höldererness of $p(\cdot)$, completing the proof. \(\square\)

If $\Lambda = M$, Theorem 7.9 means $u$ admits an $s$-lamination $\{\Lambda^s_m : m \in M\}$ (see e.g. [AV10]). In addition, if $u$ is a homeomorphism and $\tilde{u}_m : X_m \times Y_m \to \tilde{X}_{u(m)} \times \tilde{Y}_{u(m)}$ satisfies (A)($\alpha$, $\lambda_u(m)$) ($\beta$, $\lambda_s(m)$) condition with $\sup_{u,m} \lambda_s(m) + \epsilon < 1$ and $\sup_{u,m} \lambda_u(m) + \epsilon < 1$, then $u$ admits an $u$-lamination. Above Corollary 7.10 also gives that the homeomorphism $u$ is (uniformly) hyperbolic (see also [Via08, AV10]).

SS1: Let $X$ be a $C^1$ Finsler manifold with Finsler metric $d$ in its components, which is $C^{0,1}$-uniform (i.e. assumption (iii) in page 49 holds for $X = M = M_1$); see also Appendix C for some examples. $\chi_m : U_m(\epsilon') \to T_m X$ is the local chart given by the assumption (iii). $\tilde{M}$ is the canonical bundle atlas of $TX$ induced by $\chi_m$, $m_0 \in X$.

Let $u : X \to X$ be a $C^1$ map, $\Lambda \subset X$ and $u(\Lambda) \subset \Lambda$.

SS2: Let $T_m X = X^s_m \oplus X^u_m$ with projections $\Pi^s_m$, so that $\Pi^s_m + \Pi^u_m = \text{id}$, $R(\Pi^u_m) = X^u_m$, $\kappa = s, u$, $m \in \Lambda$. Suppose $\sup_{u,m} \|\Pi^s_m\| < \infty$ and $m \mapsto X^s_m : X \to \overline{\mathcal{G}(TX)}$ is uniformly continuous around $\Lambda$, $\kappa = s, u$, where $\mathcal{G}(TX)$ is the Grassmann manifold of $TX$; see Remark B.4.

SS3: Assume that $Du(m) : X^s_m \oplus X^u_m \to X^s_{u(m)} \oplus X^u_{u(m)}$, $m \in \Lambda$, satisfy

(i) $\|\Pi^s_{u(m)} Du(m)\|_{X^s_m} = \lambda_s(m),$

(ii) $\Pi^s_{u(m)} Du(m) : X^s_{u(m)} \to X^s_{u(u(m))}$ is invertible, $\|\Pi^s_{u(m)} Du(m)\|_{X^s_{u(m)}}^{-1}\| = \lambda_s(m),$

(iii) $\Pi^s_{u(m)} Du(m)Du(m)\|_{X^s_{u(m)}} \leq \eta$, $k_1 \neq k_2$, $k_1, k_2 \in \{s, u\},$

(iv) $\sup_{u,m} \lambda_s(m), \lambda_u(m) < 1$, $\lim_{m \to \infty} \lambda_u(m) < \infty$, $\lambda_u(m) < \infty,$

(v) $Du$ is uniformly continuous around $\Lambda$ with respect to $M$; see Definition 5.32.

Corollary 7.11 (strong stable laminations II: smooth case). Let (SS1) (SS2) (SS3) hold. If $\eta > 0$ is small, then there are $\epsilon' > 0$ and $\eta_1 > \eta$ small, such that there are unique $\Lambda^s_{loc}(m, \epsilon') \subset U_m(\epsilon')$, $m \in \Lambda$, such that (1) (2) hold.

(1) $m \in \Lambda^s_{loc}(m, \epsilon')$ and every $\Lambda^s_{loc}(m, \epsilon')$ is a Lipschitz graph of $X^s_m \to X^u_m$ (through $\chi_m$) with Lipschitz constant less than $\eta_1$. (2) $u(\Lambda^s_{loc}(m, \epsilon')) \subset \Lambda^s_{loc}(u(m), \epsilon').$

Moreover, $\Lambda^s_{loc}(m, \epsilon'), m \in \Lambda,$ have the following properties.

(3) $\{x \in U_m(\epsilon') : u^s(x) \in U_{u(u(m))(\epsilon')}, ||u^s(x) - u^s(m)|| \leq \epsilon(m), n \to \infty\} = \Lambda^s_{loc}(m, \epsilon'),$ where $s : \Lambda \to \mathbb{R}_+$ such that $\lambda_s(m) + \epsilon < \epsilon(m) < \lambda_u^s(m) - \epsilon$ (for small $\epsilon > 0$).

(4) Each $\Lambda^s_{loc}(m, \epsilon')$ is a $C^1$ graph. If $\eta = 0$, then $T_m \Lambda^s_{loc}(m, \epsilon') = X^s_m$. Moreover, if $u \in C^{k,\gamma}$, then $\Lambda^s_{loc}(m, \epsilon')$ is a $C^{k,\gamma}$ graph. Furthermore, $m \mapsto \Lambda^s_{loc}(m, \epsilon')$ is continuous in $C^1$ topology in any bounded subset.
If $\Lambda^s_{loc}(m_1, \epsilon') \cap \Lambda^s_{loc}(m_2, \epsilon') \neq \emptyset$, then $\Lambda^s_{loc}(m_1, \epsilon') = \Lambda^s_{loc}(m_2, \epsilon')$. In particular, let

$$\Lambda^s_m \triangleq \bigcup_{n \geq 0} u^{-n}(\Lambda^s_{loc}(u^n(m), \epsilon')) = \{ x \in X : |u^n(x) - u^m(x)| \leq \epsilon_1(n)(m), n \to \infty \},$$

then $\Lambda^s \triangleq \{ \Lambda^s_m : m \in \Lambda \}$ is a (u-invariant) lamination called a strong stable lamination of $\Lambda$ for $u$, where $\lambda_s(m) + \varsigma < \kappa_1(m) \leq \min(1, \lambda^{-1}_s(m)) - \varsigma$ (for small $\varsigma > 0$).

(6) The holonomy maps for $\Lambda^s$ are uniformly (locally) Hölder if $\sup_{m \in \Lambda} |Du(m)| < \infty$.

Proof. One can reduce the smooth case to the Lipschitz case. Let $m \in \Lambda$. Consider

$$\tilde{u}_m = \chi_m \circ u_m \circ \chi^{-1}_m : X^s_m(r_1) \oplus X^u_m(r_2') \to X^s_{u(m)}(r_1) \oplus X^u_{u(m)}(r_2'),$$

where $r_1, r_2'$ are chosen such that the maps are well defined. By taking $r_1, r_2'$ further smaller, we can assume that it satisfies $(A')(\alpha, \lambda'_u(m))$ condition, where $\lambda'_u(m) = \lambda_u(m) + \epsilon$, $\lambda'_s(m) = \lambda_s(m) + \epsilon$, $\epsilon > 0$ (sufficiently small), and $\alpha, \beta \to 0$ as $r_1, r_2', r_2, \eta \to 0$. Using the radial retractions in $X^s_m, X^u_m$ one gets

$$\tilde{u}_m \sim (F_m, G_m) : X^s_m \oplus X^u_m \to X^s_{u(m)} \oplus X^u_{u(m)}$$

such that $\tilde{u}_m|_{\tilde{u}_m(e')} = \tilde{u}_m$ and it satisfies $(A')(\alpha, \lambda'_u(m))$ condition (see e.g. Lemma 3.12 and Lemma 3.13). Now (1) (2) (3) and (5) are consequences of Theorem 7.9. Note that $\sup_{m \in \Lambda} |Du(m)| < \infty$ implies that $|Du(m)|$ is bounded in a neighborhood of $\Lambda$ for $Du(m)$ is uniformly continuous around $\Lambda$. So (6) follows from Corollary 7.10. The regularity results in (4) are consequences of Lemma 6.7, Lemma 6.11 and Lemma 6.25, Lemma 6.27 (by taking $\epsilon'$ further smaller); see also Theorem 6.39.

If $\Lambda = X$, we also call $\Lambda^s$, obtained in Corollary 7.11, the strong stable foliation for $u$.

7.2.3. fake invariant foliations. In the following, we give a result about the existence and regularity of fake invariant foliations in a little more general settings than [BW10] (see also [Wil13]). Also, with a little effort, we can deal with a smooth map which might not be invertible in a Finsler manifold. For the general case in a global version, see Theorem 7.4.

Let $X$ be a Banach space, compact Riemannian manifold without boundary, Riemannian manifold having bounded geometry, or satisfy the following general setting; see e.g. Appendix C.

(HF1) Let $X$ be a $C^1$ Finsler manifold with Finsler metric $d$ in its components, modeled on a Banach space $\mathbb{B}$, which is $C^{1,1}$-uniform (i.e. assumption (■) in page 49 holds for $X = M = M_1$).

$$\chi_m : U_m(e') \to T_mX$$

is the local chart given by the assumption (■). $\mathcal{M}$ is the canonical bundle atlas of $TX$ induced by $\chi_m$, $m_0 \in X$.

Let $u : X \to X$ be a $C^1$ map with Lip $u < \infty$.

(HF2) Let $T_mX = X^s_m \oplus X^c_m \oplus X^u_m$ with projections $\Pi^s_m, \Pi^c_m, \Pi^u_m = id$, $R(\Pi^c_m) = X^s_m \oplus X^u_m$. Set $\Pi^s_{m_i} = \Pi^c_{m_i} + \Pi^u_{m_i}$ and $X^s_{m_i} = X^c_{m_i} + X^u_{m_i}$. Suppose $\sup_m |\Pi^c_m| < \infty$, and $m \mapsto X^s_m : X \to \mathbb{G}(TX)$ is uniformly continuous, $\kappa = c, s, u$, where $\mathbb{G}(TX)$ is the Grassmann manifold of $TX$; see Remark B.4.

(HF3) Assume that $Du(m) : X^c_{m_i} \oplus X^u_{m_i} \to X^c_{u(m)} \oplus X^u_{u(m)}$, $m \in X$, satisfy

(i) $||\Pi^s_{u(m)} Du(m)||_{X^c_m} = \lambda'_s(m), ||\Pi^c_{u(m)} Du(m)||_{X^c_m} = \lambda'_c(m), \Pi^u_{u(m)} Du(m) : X^u_m \to X^u_{u(m)}$

is invertible, $||\Pi^s_{u(m)} Du(m)||_{X^c_m} = \lambda'_s(m), \Pi^c_{u(m)} Du(m) : X^c_m \to X^c_{u(m)}$

(ii) $||\Pi^u_{u(m)} Du(m)||_{X^c_m} \leq \eta, \kappa \neq \kappa_1, \kappa_2, \kappa \in \{ s, u \}$

(iii) $\sup_m \lambda'_s(m, \lambda'_c(m) < 1, \sup_m \lambda'_s(m, \lambda'_c(m) < \infty, \sup_m \lambda'_s(m, \lambda'_c(m) < \infty$.

(iv) $Du$ is uniformly continuous with respect to $\mathcal{M}$; see Definition 5.32.

Theorem 7.12 (fake invariant foliations I). Let (HF1) (HF2) (HF3) hold. Then the following hold.

Existence. For any sufficiently small $\epsilon > 0$, if $\eta$ is small enough, then there are $r > r_0 > 0$ such that the following hold. There exist (in general not unique) Lipschitz graphs $W^c_{m}(q, r)$ in a neighborhood of $m \in X$ with Lipschitz constants small (independent of $m, q$), which can be parameterized as Lipschitz graphs of $X^c_m(r) \to X^u_{u(m)}$ through $\chi_m$ (i.e. the sets $\chi_m(W^c_{m}(q, r))$, such that
(1) (Local invariance) If \( q \in U_m(r_0) \), then \( u(W^{cs}_m(q, r_0)) \subseteq W^{cs}_{u(m)}(u(q), r) \), \( q \in W^{cs}_m(q, r_0) \subseteq U_m(\varepsilon') \). Moreover, if \( \operatorname{sup}_m \lambda'_{cs}(m) < 1 \), then \( u(W^{cs}_m(q, r_0)) \subseteq W^{cs}_{u(m)}(u(q), r_0) \) for \( q \in U_m(r_0) \).

(2) (Tangency) For each \( m \in X \), \( W^{cs}_m(m, r) \) is differentiable at \( m \) with \( T_mW^{cs}_m(m, r) \) a (linear) graph of \( X^{cs}_m \rightarrow X^u_m \). Moreover, if for all \( m \in X \), \( Du(m)X^{cs}_m \subseteq X^{cs}_{u(m)} \), then \( T_mW^{cs}_m(m, r) = X^{cs}_m \).

(3) (Exponential growth bounds) Let \( \eta_j = u'(p), j = u'(q), m_j = u'(m), j = 1, 2, \ldots, n \). If \( q_j \in U_m(r_0) \), and \( p_j \in W^{cs}_m(q_j, r_0) \), \( j = 1, 2, \ldots, n - 1 \), then \( p_n \in W^{cs}_m(q_n, r) \) and \( d(p_n, q_n) \leq \lambda'_{cs}(m)d(p, q) \), where \( \lambda'_{cs}(m) \leq \lambda_{cs}(m) + \varepsilon \).

(4) If \( \operatorname{sup}_m \lambda'_{cs}(m) < 1 \), then for \( m \in X \), \( W^{cs}_m(m, r) \), is the (local) strong stable manifold of \( m \) for \( u \).

**Regularity.**

(1) (Hölder Regularity) \( W^{cs}_m(r) \triangleq \bigsqcup_{q \in U_m(r)} W^{cs}_m(q, r), m \in X \), are (uniformly) Hölder ‘foliations’.

(2) (\( C^1 \) leaves) Assume that \( \mathcal{E} \) exists a \( C^1 \cap C^{0,1} \) bump function, then for all \( m \in X \), all leaves \( W^s_m(q, r_0), q \in U_m(r_0), m \in X \), are \( C^1 \) graphs (so \( W^{cs}_m(r) \) indeed is a foliation in \( U_m(r) \)). Moreover, the tangent distribution \( T_qW^{cs}_m(q, r) \) depends in a uniformly (locally) Hölder fashion on \( q \in U_m(r_0) \) if \( u \) is uniformly (locally) \( C^{1,\gamma} \) (\( \gamma > 0 \)) and in addition the bump function is \( C^{0,1} \); the Hölder constants are independent of \( m \).

If one of the following holds, (i) \( \operatorname{sup}_m \lambda'_{cs}(m) < 1 \), or (ii) every \( X^u_m \) admits a \( C^1 \cap C^{0,1} \) bump function and \( \operatorname{sup}_m \lambda'_{cs}(m) < 1 \), then for every \( m \in X \), the leaf \( W^{cs}_m(m, r_0) \) is a \( C^1 \) graph.

(\( HF3' \)) Further, assume that

(i) \( \Pi^1_{u(m)}Du(m) : X^u_m \rightarrow X^u_{u(m)} \) is invertible, \( ||(\Pi^1_{u(m)}Du(m))X^u_m||^{-1} = (\lambda'_{cs}(m))^{-1} \),

(ii) \( \operatorname{sup}_m \lambda'_{cs}(m) \lambda'_{cs}(m) < 1 \), \( \operatorname{sup}_m \lambda'_{cs}(m) < \infty \).

**Theorem 7.13** (fake invariant foliations II). Let \( HF1 \) (HF2) (HF3) (HF3') hold. Let \( W^{cs}_m(q, r), \ W^s_m(q, r), q \in U_m(r_0), m \in X \), be obtained in **Theorem 7.12** such that they are parameterized as Lipschitz graphs of \( X^{cs}_m(r) \rightarrow X^u_m \) and \( X^s_m(r) \rightarrow X^{cs}_m \) through \( X^s_m \) respectively. Let \( W^{cs}_m(r) \triangleq \bigsqcup_{q \in U_m(r)} W^{cs}_m(q, r), m \in X \), \( k = cs, s \). Then we have the following.

(1) (Coherence) Every leaf \( W^{cs}_m(q, r) \) is ‘foliated’ by the leaves of \( W^s_m(r), m \in X \), (ii) \( W^{cs}_m(r) \) if \( p \in W^{cs}_m(q, r_0) \). So also \( W^s_m(r) \) ‘foliates’ \( W^{cs}_m(r) \).

(2) (Regularity) Moreover suppose that \( \mathcal{E} \) exists a \( C^1 \cap C^{0,1} \) bump function, that \( u \) is \( C^{1,\gamma} \), and that the center bounding condition (spectral gap condition) holds, \( \operatorname{sup}_m \lambda'_{cs}(m) \lambda'_{cs}(m) < 1 \). Then each leaf of \( W^{cs}_m(r) \) is \( C^1 \) foliated by \( W^s_m(r), m \in X \).

**Proof of Theorem 7.12 and Theorem 7.13.** We use the notation \( a \ll b \) if \( a \) is sufficiently smaller than \( b \), where \( a, b > 0 \). **Theorem 7.12** is a restatement of **Theorem 7.4** in a local version. We also give a direct proof.

(1) Let \( \varepsilon > 0 \) be any sufficiently small. Usually the norm in \( X_1 \oplus X_2 \) will be taken as maximal norm, i.e., \( |(x_1, x_2)| \triangleq \max \{|x_1|, |x_2|\} \), where \( x_i \in X_i, i = 1, 2 \). Let

\[ H_m(x) = \chi_{u(m)} \circ u \circ X^{-1}(x) : X^{cs}_m(r_1) \oplus X^u_m(r_1') \rightarrow X^{cs}_{u(m)}(r_2) \oplus X^u_m(r_2'), \]

which satisfies (A)(\( a, \lambda_u(m) \)) (B)(\( \beta, \lambda_{cs}(m) \)) condition by (HF3) (see **Lemma 3.5** and **Lemma 3.12**), where \( \lambda_{cs}(m) = \lambda'_{cs}(m) + \varepsilon, \lambda_u(m) = \lambda'_u(m) + \varepsilon \), and \( a, \beta \rightarrow 0 \) as \( r_1, r_1' \rightarrow 0 \). The \( r_1, r_1' \) are chosen, which are independent of \( m \) due to (HF2) (ii) and (HF3) (iv), such that \( X^{cs}_m(r_1) \oplus X^u_m(r_1') \subset X_m(\varepsilon') \) and \( u \circ X^{-1}(X^{cs}_m(r_1) \oplus X^u_m(r_1')) \subset \chi_{u(m)}(U_m(\varepsilon')) \) for all \( m \in X \).

Take \( 0 < \varepsilon_0 \ll r' \ll r \ll \min \{r_i, r_i': i = 1, 2\} / 2 \). Using the radial retractions in \( X^s_m \) (see (3.1)), one gets

\[ \tilde{H}_m : T_mX \rightarrow T_{u(m)}X, \]

such that \( \tilde{H}_m|X(\varepsilon_0) = H_m \) and \( \tilde{H}_m|T_mX \subset T_{u(m)}X(r') \); here \( T_mX(\varepsilon_0) = X^s_m(\varepsilon_0) \oplus X^u_m(\varepsilon_0) \).

Let \( H_m = (F_m, G_m) \), where \( F_m : X^{cs}_m(r_1) \oplus X^u_{u(m)}(r_2') \rightarrow X^{cs}_{u(m)}(r_2) \) and \( G_m : X^u_m(r_1) \oplus X^u_{u(m)}(r_2') \rightarrow X^u_m(r_1) \).

As before, using the radial retractions in \( X^s_m \) (see (3.1)), one gets \( F_m, G_m \) are defined in all
\(X^c_m \oplus X^u_{u(m)}\) such that \(\widetilde{F}_m |_{X^c_m(r') \oplus X^u_{u(m)}(r')} = \widetilde{F}_m, \widetilde{G}_m |_{X^c_m(r') \oplus X^u_{u(m)}(r')} = G_m\) (see e.g. (3.2)). Set
\[
\tilde{H}_m \sim (\widetilde{F}_m, \widetilde{G}_m) : X^c_m \oplus X^u \to X^c_{u(m)} \oplus X^u_{u(m)}.
\]

Define
\[
\tilde{M} = \{(m, q) : q \in \tilde{U}_m(r), m \in X\}, \tilde{M}_1 = \{(m, q) : q \in \tilde{U}_m(r), m \in X\},
\]
and
\[
\tilde{U}_m(\epsilon) = \chi^{-1}_m(X^c_m(\epsilon) \oplus X^u_m(\epsilon) \oplus X^u_{u(m)}(\epsilon)), (0 < \epsilon \leq \rho),
\]
with metric \(d_m(p_1, p_2) = |\chi_m(p_1) - \chi_m(p_2)|_m \approx d(p_1, p_2)\), where \(d\) is the Finsler metric in each component of \(X\). Let
\[
H_m = \chi_{u(m)} \circ \tilde{H}_m \circ \chi_m,
\]
and
\[
\tilde{H}_{(m),q}(x) = \tilde{H}_m(x + \chi_m(q)) - \tilde{H}_m(\chi_m(q)), q \in \tilde{U}_m(r).
\]

Note that \(H_m|_{\tilde{U}_m(\epsilon_0)} = u\) and \(H_m|_{\tilde{U}_m(\rho)} \subset \tilde{U}_u(\rho)\).

Let \(\bar{X}, \bar{Y}\) be bundles over \(\tilde{M}\) with fibers \(\bar{X}_{(m),q} = X^c_m, \bar{Y}_{(m),q} = X^u_m\). \(\tilde{M} \equiv X \times X\) indeed is a \(C^1\) bundle (over \(X\)), however \(\bar{X}, \bar{Y}\) are even not \(C^1\) by our construction. We can give another topology in \(\tilde{M}\) such that \(\bar{X}, \bar{Y}\) are \(C^1\). This can be done if we consider the leaf topology in \(\tilde{M}\) (see Section 5.2). Then \(\tilde{M}\) (with \(\tilde{M}_1\)) trivially satisfies (H1c) (in Section 6.2). Now \(\bar{X}_{\tilde{M}}\), \(\bar{Y}_{\tilde{M}}\) are trivial, so \(\bar{X}, \bar{Y}\) are \(C^1\) and satisfy (H2c) (in Section 6.2). Let
\[
g(m, q) = (u(m), H_m^{-}(q)) : \tilde{M} \to \tilde{M}_1 \subset \tilde{M},
\]
and
\[
\bar{H}(m, q, x) = (g(m, q), \tilde{H}_{(m),q}(x)) : \bar{X} \oplus \bar{Y} \to \bar{X} \oplus \bar{Y}.
\]

Now
\[
\bar{H}_{(m),q} \sim (\tilde{F}_{(m),q}, \tilde{G}_{(m),q}) : \bar{X}_{(m),q} \oplus \bar{Y}_{(m),q} \to \bar{X}_{(g(m),q)} \oplus \bar{Y}_{(g(m),q)},
\]
satisfies (A)(\(\alpha, \lambda_u(m)\)) (B)(\(\beta, \lambda_{cs}(m)\)) condition for all \(q \in \tilde{U}_m(r)\).

Let \(i(m, q) = 0\) be the 0-section of \(\bar{X} \times \bar{Y}\). Then \(i\) is invariant under \(\bar{H}\). By applying Theorem 4.1 to \(\bar{H}, \bar{X} \times \bar{Y}, g, \tilde{M}, i\), we have \(f^c_{cs}(m) : \bar{X}_{(m),q} \to \bar{Y}_{(m),q}\) with Lip \(f^c_{cs}(m) \leq \bar{\beta}' < 1\), and Graph \(f^c_{cs}(m) \subset \bar{H}^{-1}_{(m),q}\) Graph \(f^c_{cs}(u(m),H_m^{-}(q))\), or more especially,
\[
(h^c_{cs}(p), f^c_{cs}(u(m),H_m^{-}(q))(h^c_{cs}(p))) \in \bar{H}_{(m),q}(p, f^c_{cs}(u(m),H_m^{-}(q))(p)), p \in X^c_m,
\]
where \(h^c_{cs}(p) : X^c_m \to X^c_{u(m)}\) with Lip \(h^c_{cs}(m, q) \leq \chi^{-1}_m(m, q)\). Set
\[
f^c_{cs}(\chi^{-1}_m(m, q) + f^c_{cs}(\chi^{-1}_m(m, q))) + \chi_m(q), W_m(q, \varepsilon) = f^c_{cs}(\tilde{U}_m(\varepsilon)).
\]

- If we take \(r_0 \ll \varepsilon_0, q \in \tilde{U}_m(r_0)\), then \(W_m(q, r_0) \subset \tilde{U}_m(\varepsilon_0)\). So \(u(W_m(q, r_0)) \subset W_m(u(m), u(q), r)\).
- \(W_m(m, r)\) is differentiable at \(m\), i.e. \(f^c_{cs}(u(m))\) is differentiable at \(m\) (see Remark 6.10). If \(Du(m)X^c_m \subset X^c_{u(m)}\), then \(Du(m)X^c_m \subset X^c_{u(m)}\), and \(Df^c_{cs}(u(m))(0) = 0\) and so \(T_mW_m(q, r) = X^c_m\).
- By our construction, for every \(m \in X, W_m(q, r), q \in \tilde{U}_m(r_0), \) follows from the characterization of Graph \(f^c_{cs}(m)\) (see e.g. Section 4.4). Also under sup \(m \lambda'_{cs}(m) \ll 1\), that for every \(m \in X, W_m(m, r)\) is the (local) strong stable manifold of \(m\) for \(u\) follows from the characterization of (local) strong stable foliation for \(u\); see also Corollary 7.11.
- The characterization of the exponential growth bounds (Theorem 7.12 (3)) follows directly from (A) (B) condition.
Let us consider the regularity of the leaves. Since the radial retractions might not be smooth, so is $\overline{H}(m,q)(\cdot)$. Thus, Lemma 6.7 can not apply. However, if $\exists$ has a $C^1 \cap C^{0,1}$ bump function, then we can choose a suitable $C^1 \cap C^{0,1}$ bump function instead of the radial retractions so that $\overline{H}(m,q)(\cdot)$ is $C^1$. Now apply Lemma 6.7 to get that all leaves of $W^{cs}_{m}(r)$ are $C^1$. Moreover, by Lemma 6.16, the tangent distribution $T_{q}W^{cs}_{m}(q,r)$ depends in a uniformly (locally) Hölder fashion on $q \in U_{m}(r)$ if $u$ is $C^{1,\gamma}$ ($\gamma > 0$) and in addition the bump function is $C^{1,1}$; the Hölder constants are independent of $m$.

Set
$$W^{cs}_{(m,q)}(e) = \{(x^{s},x^{c},f^{cs}_{(m,q)}(x^{s},x^{c})) : (x^{s},x^{c}) \in X^{s}_{m}(e) \oplus X^{c}_{m}(e)\}.$$ Suppose that every $X^{i}_{m}$ exists a $C^1 \cap C^{0,1}$ bump function. Now we can choose a suitable $C^1 \cap C^{0,1}$ bump function instead of the radial retraction in $X^{i}_{m}$. Since $\sup_{m} \lambda_{s}(m) < 1$, now $\overline{H}(m,m)W^{cs}_{(m,m)}(e_{0}) \subset W^{cs}_{(m,m)}(e_{0})$. Since $\overline{H}_{(m,m)} = \overline{H}$ and $\overline{F}_{m}(\cdot),\overline{G}_{m}(\cdot)\text{ now are }C^1$ at $W^{cs}_{(m,m)}(e_{0})$, by Theorem 6.39, $W^{cs}_{(m,m)}(e_{0})$ is $C^{1}$ and so is $W^{cs}_{m}(e_{0})$. If $\sup_{m} \lambda_{c}(m) < 1$, we can reduce this case to $X^{i}_{m} = \{0\}$.

This completes the proof of Theorem 7.12.

(II). If (HF3') holds, then for every $m \in X$, $q \in \overline{U}_{m}(r)$, we have $W^{s}_{m}(q,r)$, which is parameterized as a Lipschitz graph of $X^{s}_{m}(r) \rightarrow X^{cu}_{m}$ through $\chi_{m}$.

By the characterization (for $\overline{H}$), one has $W^{s}_{m}(p,p) \subset W^{s}_{0}(q,r)$, if $p \in W^{cs}_{m}(q,r,0)$, i.e. Theorem 7.13 (1) holds. Next we will show for every $m \in X$, $W^{s}_{m}(r)$ is a $C^1$ foliation inside every leaf of $W^{cs}_{m}(r)$.

Let
$$\tilde{M}^{cs}_{m} = \{(m,p,q) : p \in W^{cs}_{m}(q,r), q \in \overline{U}_{m}(r), m \in X\},$$
$$\tilde{M}^{cs}_{1} = \{(m,p,q) : p \in W^{cs}_{m}(q,r,0), q \in \overline{U}_{m}(r), m \in X\},$$
and
$$g^{cs}(m,p,q) = (g(m,q),H^{-}_{m}(p)) : \tilde{M}^{cs}_{m} \rightarrow \tilde{M}^{cs}_{m} \subset \tilde{M}^{cs}_{m}.$$ Consider $\tilde{X},\tilde{Y}$ as bundles over $\tilde{M}^{cs}_{m}$ with fibers $\tilde{X}^{s}_{m} = X^{s}_{m}, \tilde{X}^{cu}_{(m,q,p)} = X^{cu}_{m}$, respectively. Set
$$\overline{H}^{cs}_{m}(m,q,p,x) = (g^{cs}(m,p,q),\overline{H}_{(m,p)}(x)) : \tilde{X} \oplus \tilde{Y} \rightarrow \tilde{X} \oplus \tilde{Y}.$$ Note that the unique invariant graph $\tilde{X} \rightarrow \tilde{Y}$ of $\overline{H}^{cs}_{m}$ contains $\cup_{m,p} \chi_{m}(W^{s}_{m}(p,r))$, and $\text{Lip } H_{m}^{s}|W^{cs}_{m}(q,r) \leq \tilde{\lambda}^{cs}_{m}(m)$. If $\exists$ exists a $C^1 \cap C^{0,1}$ bump function, then we can assume $\tilde{H}_{m}(\cdot),\tilde{F}_{m}(\cdot),\tilde{G}_{m}(\cdot)$ are $C^1$, so are $H^{s}_{m}(\cdot),W^{s}_{m}(p,r)$ for every $p$, and $(q,x) \mapsto \tilde{F}_{(m,q)}(x),\tilde{G}_{(m,q)}(x)$. Thus we can give the $C^1$ leaf topology in $\tilde{M}^{cs}_{m}$ (by considering it as a bundle over $\tilde{M}$, see Section 5.2), making it satisfy (H1b) (in Section 6.2). (Note that $\tilde{M}^{cs}_{m}$ is not a $C^1$ bundle if we consider it locally as $X \times X \times W^{s}_{m}(p,r)$.) Since $\tilde{X}_{|W^{c}_{m}(r)},\tilde{Y}_{|W^{c}_{m}(r)}$ are trivial, so $\tilde{X},\tilde{Y}$ are $C^1$ and satisfy (H2c) (in Section 6.2). If $u$ is $C^{1,1}$ (i.e. $H_{m}(\cdot) \in C^{1,1}$ uniform for $m$) and the center bunching condition holds:
$$\sup_{m} \lambda(m)_{1,c}(m) \lambda^{cs}_{c}(m) < 1,$$ then all the assumptions in Lemma 6.18 now are satisfied for $\overline{H}^{cs}_{m}$, $\tilde{X} \oplus \tilde{Y}$, $g^{cs}_{m}$, $\tilde{M}^{cs}_{m}$ and the natural 0-section, so by Lemma 6.18, $W^{s}_{m}(r)$ is a $C^1$ foliation inside every leaf of $W^{cs}_{m}(r)$. Here note that in general $(p,x) \mapsto \overline{H}^{cs}_{m}(m,q,p,x) = \tilde{H}_{(m,p)}(x)$ is not $C^{1,1}$ (as $W^{cs}_{m}(p,r)$ only is $C^1$) but satisfies (6.23) (since $H_{m}(\cdot) \in C^{1,1}$ uniform for $m$).

Here note that the fake invariant foliations are in general not unique. Our proof of the existence of fake invariant foliations depends on the particular choice of the bump function. Under some circumstance, one can further discuss how the ‘fake’ invariant foliations depend on choice of the bump functions which are omitted in this paper. See Appendix D.5 for a review of bump functions (and blid maps); in fact we can use blid maps instead of bump functions; note that the smooth blid maps exist in $C[0, 1]$. 

7.2.4. holonomy over a lamination for a bundle map. Let $M$ be a set. $M$ is called a lamination with metric leaves (or for short a lamination) if for every $m \in M$, there is a set $\mathcal{F}_m \subset M$ with following properties.

(a) $m \in \mathcal{F}_m$. If $\mathcal{F}_{m_1} \cap \mathcal{F}_{m_2} \neq \emptyset$, then $\mathcal{F}_{m_1} = \mathcal{F}_{m_2}$. $\mathcal{F}_m$ is called a leaf of $M$.

(b) Each leaf $\mathcal{F}_m$ is a complete metric space with metric $d_m$. $\mathcal{F}_m(r) \triangleq\{m' \in \mathcal{F}_m : d_m(m', m) < r\}$ is called a plaque of $M$ (at $m$).

$M/\mathcal{F}$ denotes the leaf space of $M$ which is the set of all leaves of $M$. Using $M/\mathcal{F}$, one can define a leaf topology in $M$, i.e. consider $M$ as a bundle over $M/\mathcal{F}$ with fibers the leaves of $M$ and the projection $\pi$ sending $m \in M$ to $\mathcal{F}_m$. A $C^1$ regularity of lamination $M$ can be defined as follows. Let $M$ be a subset of a $C^1$ Riemannian manifold. $M$ is a $C^1$ lamination if the disjoint leaves are $C^1$-injectively (and connected) immersed submanifolds and $\mathcal{F}_m, T_m \mathcal{F}_m$ are $C^0$ respecting to $m$ (see [HPS77]).

Let $f : M_1 \to M_2$ be a map, where $M_1, M_2$ are two laminations. $f$ is called a lamination map if $f$ sends the leaves of $M_1$ to the leaves of $M_2$, i.e. $f : (M_1, \mathcal{F}_1, \pi_1) \to (M_2, \mathcal{F}_2, \pi_2)$ is a bundle map. $M$ is called an $f$-invariant lamination if $f$ is a lamination map of $M \to M$.

Example 7.14. A typical lamination is the strong stable lamination of a $C^1$ map on a Finsler manifold which is partially hyperbolic on a compact set; for a more general setting, see Theorem 7.9.

Definition 7.15. Let $M$ be a lamination and $E$ a bundle over $M$ with metric fibers. Let $f : M \to M$ be a lamination map and $H : E \to E$ a bundle map over $f$. A family $\mathcal{H}$ of $C^0$ maps $h_{x,y} : E_x \to E_y$ for all $x, y$ in the same leaves of $M$ is called a holonomy over lamination $M$ for $H$ if the following hold.

(hl1) (transition) $h_{x,x} = \text{id}$, $h_{y,z} \circ h_{x,y} = h_{x,z}$ for all $x, y, z$ in the same leaves of $M$. So $h_{x,y}$ is a homeomorphism.

(hl2) (invariance) $h_x \circ h_{x,y} = h_{f(x),f(y)} \circ h_y$.

(hl3) (regularity) $(x, y, \xi) \mapsto h_{x,y}(\xi)$ is continuous when $x, y \in \mathcal{F}_m(e)$ if $E$ is a $C^{0,1}$-fiber bundle (see Section 5.4.5).

See also [Via08, AV10] for the definitions of $x$-holonomy and $u$-holonomy which are special cases of Definition 7.15 (by taking $M$ as strong (un)stable laminations). Let us consider the existence of the holonomy over a lamination for a bundle map. Here we give such results in a more general setting so that one could apply them to infinite dimensional dynamical systems. We need the following assumptions.

(HS1) Let $M$ be a lamination with plaques $\{\mathcal{F}_m(e) : m \in M, e > 0\}$. Let $\phi : M \to M$ be a lamination map. Assume for small $e < e'$, $\phi_m \triangleq \phi : \mathcal{F}_m(e) \to \mathcal{F}_{\phi(m)}(e)$ with $\text{Lip} \phi_m \leq \lambda(e, m)$.

$$\text{sup}_m, \lambda_m(e, m) < \infty.$$ (HS2) Let $E$ be a uniform $C^{0,1}$-fiber bundle (see Definition 5.23) over $M$ with respect to bundle atlas $\mathcal{A}$, where

$$\mathcal{A} = \{(\mathcal{F}_m(e), \varphi_m) \text{ is a bundle chart of } E \text{ at } m \}.$$

(HS3) Let $H : E \to E$ be a bundle map over $\phi$. Assume the following hold.

(i) The fiber maps $H_m(\cdot)$, $m \in M$, have left invertible maps $H^{-1}_m(\cdot)$, i.e. $H^{-1}_m \circ H_m = \text{id}$, and $H_m(\mathcal{E}_m)$ is open.

(ii) $\text{sup}_m, \text{Lip} H^{-1}_m(\cdot) < \infty$. $m \mapsto H^{-1}_m$ depends in a uniformly (locally) $C^{0,1}$ fashion on the base points (in each small plaque $\mathcal{F}_m(e')$), i.e. $\text{sup}_m, \text{sup}_x, \text{Lip} H^{-1}_m(\cdot, x) < \infty$, where $\text{Lip} H^{-1}_m(\cdot, x)$ is the Lipschitz constant of the vertical part $H^{-1}$ at $m$ in local representations with respect to $\mathcal{A}$ (see Definition 5.27).

(iii) $\text{sup}_m, \text{Lip} H^{-1}_m(\cdot) \lambda_m(e, m) < 1$ (bunching condition) and $m' \mapsto \lambda_m(m') \text{Lip} H^{-1}_m(\cdot)$ are $C^{0}$ (or $e'\text{-almost }C^{0}$ with $e' > 0$ is small (see Definition 5.3)) (in each $\mathcal{F}_m(e)$).

Theorem 7.16. Let (HS1) (HS2) (HS3) hold with sufficiently small $e$ (in (HS1)). Then there is a unique (local) partition $W_e = \{W_e(m, x) : (m, x) \in E\}$ of $E$ such that
Let $\hat{\mathcal{W}}_e$ be a bundle over $\hat{\Sigma}$, where $\hat{\Sigma}$ is a bundle chart in $\mathcal{Q}$, and a bundle map $\hat{\Sigma} \to \hat{\Sigma}$, we define the bundle charts in $\hat{\Sigma}$, one can assume $h_{m, \lambda}(\cdot) \in \mathcal{W}_e(m_1, m_2)$, $m_2 \in \mathcal{F}_m(e)$, $x \in \mathcal{E}_m$, where $C$ is a constant independent of $m_1, x$. If $\sup_m \text{Lip } h_m(\cdot) < \infty$, then $h_{m, \lambda}$ is also a Hölder map with Hölder constant uniform for $m_1, m_2 \in \mathcal{F}_m(e)$, when the fibers of $\mathcal{E}$ are uniformly bounded, the Hölder exponent $\theta > 0$ can be chosen uniformly such that

$$\sup_m (\text{Lip } h_m(\cdot))^\theta \text{ Lip } h_m^{-1}(\cdot) \lambda_s(m) < 1.$$ 

$(4)$ $\mathcal{W}_e = \{ h_{m, \lambda} : m_1 \in \mathcal{F}_m(e), m_2 \in M \}$ is a local holonomy over $M$ for $H$, i.e. $\mathcal{W}_e$ satisfies (hl1), (hl2), (hl3) in Definition 7.15;

$(5)$ in addition, if $\sup_m \lambda_s(m) < 1$, define $\mathcal{W}(m, x) = \bigcup_{n \geq 0} H^{-n} \mathcal{W}_e(H^n(m, x))$ and $\mathcal{W} = \{ \mathcal{W}(m, x) : (m, x) \in \mathcal{E} \}$. Then $\mathcal{W}$ is a partition of $\mathcal{E}$ which is invariant under $H$. Using $\mathcal{W}$, the maps $h_{m, \lambda} : \mathcal{E}_m \to \mathcal{E}_m$ can be defined for all $m_1, m_2$ in the same leaves, and they give a holonomy over $M$ for $H$.

**Proof.** Let

$$\tilde{\mathcal{E}} = \{ (m, x) : x \in \mathcal{E}_m, m \in \mathcal{F}_m(e) \},$$

which is a bundle over $\mathcal{F}_m(e) = \{ m' \in \mathcal{F}_m : d_m(m', m) \leq \epsilon \}$. By the assumption (HS2), using the bundle charts in $\mathcal{A}$, one can assume $\tilde{\mathcal{E}} \cong \mathcal{F}_m(e) \times \mathcal{E}_m$. Set

$$\tilde{M} = \{ (\tilde{m}, m) : m \in \mathcal{F}_m(e), \tilde{m} \in M \}.$$

Define the bundle $\tilde{\mathcal{E}}$ as

$$\tilde{\mathcal{E}} = \{ (\tilde{m}, m, x) : x \in \mathcal{E}_m, (\tilde{m}, m) \in \tilde{M} \},$$

where the topology in $\tilde{\mathcal{E}}$ is the leaf topology; here the bundle $\tilde{\mathcal{E}}$ will be considered as over $\tilde{M}$ or $M$. (We define the bundle $\tilde{\mathcal{E}}$ only in order to give another immersed topology in $\tilde{\mathcal{E}}$.) The maps $H$ and $\phi$ have natural extensions to $\tilde{\mathcal{E}}$ and $\tilde{M}$, respectively, i.e.

$$\phi^-(\tilde{m}, m) = (\phi(\tilde{m}), \phi_m(m)) : \tilde{M} \to \tilde{M},$$

$$H^-(\tilde{m}, m, x) = (\phi^-(\tilde{m}, m), H_m(x)) = (\phi(\tilde{m}), H(m, x)) : \tilde{\mathcal{E}} \to \tilde{\mathcal{E}}.$$

Define a bundle $\Sigma$ over $\tilde{\mathcal{E}}$ as

$$\Sigma = \{ (\tilde{m}, \overline{m}, \overline{x}, m, x) : (m, x) \in \tilde{\mathcal{E}}, (\tilde{m}, \overline{m}, \overline{x}) \in \tilde{\mathcal{E}} \},$$

and a bundle map $H : \Sigma \to \Sigma$ over $H^-$ with fiber maps as

$$H_{\tilde{m}, \overline{m}, \overline{x}}(m, x) = (\phi_{\tilde{m}}(m), H_m(x)) = H(m, x), (m, x) \in \tilde{\mathcal{E}}.$$

Through the bundle charts in $\mathcal{A}$, we assume

$$H_{\tilde{m}, \overline{m}, \overline{x}}(m, x) = (\phi_{\tilde{m}}(m), \tilde{H}_{\tilde{m}}(m, x)) : \tilde{\mathcal{F}}_m(e) \times \mathcal{E}_m \cong \tilde{\mathcal{F}}_m(e) \times \mathcal{E}_m.$$ 

where $\tilde{H}_{\tilde{m}}(\cdot, \cdot)$ is the local representation of $H$ at $\tilde{m}$ with respect to $\mathcal{A}$. Let $\tilde{H}_{\tilde{m}}^{-1}(m, \cdot)$ be the local representation of $H_{\tilde{m}}^{-1}(\cdot, \cdot)$ at $\tilde{m}$ with respect to $\mathcal{A}$.

Now by Lemma 3.5, $\tilde{H}_{\tilde{m}, \overline{m}, \overline{x}} = (\phi_{\tilde{m}}(\cdot), \tilde{H}_{\tilde{m}}^{-1}(\cdot, \cdot))$ satisfies (A)(0), $\lambda_s(\overline{m})$ (B)(β, $\lambda_a(\overline{m})$) condition, where $\lambda_a(\overline{m}) : \text{Lip } H_{\tilde{m}}^{-1}(\cdot, \cdot)$ as $\epsilon \to 0$, and $\beta$ is a large number depending on $\sup_m \sup_x \text{Lip } H_{\tilde{m}}^{-1}(\cdot, x)$, $\sup_m \lambda_s(m)$, $\sup_m \lambda_a(m)$. Let $i(\tilde{m}, \overline{m}, \overline{x}) = (\tilde{m}, \overline{m}, \overline{x})$ be a section of $\Sigma$ which is invariant under $H$. The existence of $\mathcal{W}_e$ now is a consequence of Theorem 4.1 by applying to $\Sigma, \tilde{\mathcal{E}}, \tilde{H}, H^-, i$. More precisely,
take $\epsilon$ small, then there are $f_{(\bar{m}, \bar{x})} = f_{(\bar{m}, \bar{x})}$ : $\mathcal{F}_m(\epsilon) \to \mathcal{E}_m$, $(\bar{m}, \bar{x}) \in \mathcal{E}$ with $\operatorname{Lip} f_{(\bar{m}, \bar{x})} \leq \beta$ such that $f_{(\bar{m}, \bar{x})}(\bar{m}) = \bar{x}$ and
\[ f_{(\bar{m}, \bar{x})}(\mathcal{F}_{\phi_{(\bar{m})}}(\epsilon)) \subset \mathcal{H}_m(\mathcal{F}_m(\epsilon') \times \mathcal{E}_m), \]

$H_{(\bar{m}, \bar{x})}\operatorname{Graph} f_{(\bar{m}, \bar{x})} \subset \operatorname{Graph} f_{(\bar{m}, \bar{x})}$;

here we use that $H_m(\mathcal{E}_m)$ is open ((HS3) (i)). Let $\mathcal{W}_e(\bar{m}, \bar{x}) = \operatorname{Graph} f_{(\bar{m}, \bar{x})}$, which is a Lipschitz graph of $\mathcal{F}_{\bar{m}}(\epsilon) \to \mathcal{E}_{\bar{m}}$. Thus, we obtain (1) (2). Note that $f_{(m_1, m_2)}(x) = f_{(m_1, x)}(m_2)$. Since $f_{(m_1, x)}(m_1) = x$ and $(m_2, x') \in \mathcal{W}_e(m_1, x)$ if and only if $(m_1, x) \in \mathcal{W}_e(m_2, x')$, we have (hl1). The invariance of $\mathcal{W}_e$ gives (hl2). (7.3) follows from the Lipschitz continuity of $f_{(\bar{m}, \bar{x})}$. By Lemma 6.25, $(m_1, m_2, x) \mapsto h_{m_1, m_2}(x)$ is $C^0$ by regarding $\mathcal{E}$ as a bundle over $M$ (here the continuity of $m' \mapsto \lambda_s(m')$, $\operatorname{Lip} H_{m'}^{-1}(\cdot)$ being used). By Lemma 6.13, we know $h_{m_1, m_2}$ is Hölder uniform for $m_1, m_2$, under the condition $\sup_m \operatorname{Lip} H_m(\cdot) < \infty$ (considering $\mathcal{E}$ as a bundle over $\hat{M}$). Thus, we obtain the conclusion (3) (4), (5) follows from (1)–(4) and the fact that every $m_1 \in \mathcal{F}_m$, there is an integral $N$ such that $\phi^N(m_1) \in \mathcal{F}_{\phi^N(\bar{m})}(\epsilon)$. The proof is complete. \hfill \Box

Remark 7.17. (a) The lamination $M$ is usually taken as the strong stable lamination of a partial hyperbolic (compact) set for the Lipschitz (or smooth) map $\phi$ (see also [AV10, section 4] and [Via08]); so in this case $\sup_m \lambda_s(m) < 1$. For applications, the bundle $\mathcal{E}$ is usually the center-stable foliation for $\phi$.

(b) If for each $m \in M$, $H_m$ is invertible, then the Lipschitz continuity of $m \mapsto H_m$ and $\sup_m \operatorname{Lip} H_m^{-1}(\cdot) < \infty$ will imply the Lipschitz continuity of $m \mapsto H_m^{-1}$; using this fact, Theorem 7.16 recovers the existence of $s(\cdot)$-holonomies for cocycles given in [AV10, ASV13].

(c) If one only focuses on the existence of $\mathcal{W}$, the Lipschitz continuity of $m \mapsto H_m^{-1}$ can be replaced by that $m \mapsto H_m^{-1}$ depends in a uniformly (locally) Hölder fashion on the base points, i.e. $\sup_m \sup_x \operatorname{Hol}_x H_x^{-1}(\cdot, x) < \infty$. But for this case the ‘bunching condition’ is said to be as $\sup_m \operatorname{Lip} H_m^{-1}(\cdot, x)\lambda_s^x(m) < 1$. Indeed, if we redefine the metric in $\mathcal{F}_m(\epsilon)$ as $d'(m, x, y) = d(m, x, y)'$, then in this new metric the ‘bunching condition’ is $\sup_m \operatorname{Lip} H_m^{-1}(\cdot, x)\lambda_s^x(m) < 1$. So one can apply Theorem 7.16. See also [AV13, section 3].

(d) Since $h_{m_1, m_2}$ satisfies (7.3), an easy computation shows that $h_{m_1, m_3}(x) = \lim_{m_2 \to \infty} (H^{-1})_{m_2} \circ H_{m_1}(x)$, where the convergence is uniform for $x, m_1, m_2$. Using this fact, one can give more regularities about $x \mapsto h_{m_1, m_2}(x)$ (or $(m_1, m_2, x) \mapsto h_{m_1, m_2}(x)$). See also [AV13, section 3] and [AV10, section 5]. Moreover, if the bundle map in (HS3) depends on an external parameter $\lambda$, i.e. $\lambda \mapsto H$, then the partition of $\mathcal{E}$ obtained in Theorem 7.16 also depends on $\lambda$, i.e. $\lambda \mapsto \mathcal{W}$. One can consider the continuity of $\lambda \mapsto \mathcal{W}$ by using the results in Section 6.9, left to readers.

Corollary 7.18 ([Via08, ASV13]). Assume (HS1) and following (HS3’) hold.

(HS3’) Let $\mathcal{E} = M \times X$, where $X$ is a Banach space. Suppose $m \mapsto A(m) \in GL(X, X)$ (the all invertible continuous linear maps in $X$) is uniformly $\nu$-Hölder in each small plaque of $M$, i.e.
\[ \|A(m_1) - A(m_2)\| \leq C_0 d(m_1, m_2)\nu, \]
\[ m_1, m_2 \in \mathcal{F}_m(\epsilon), \]
where $C_0 > 0$ is a constant independent of $m$ and $0 < \nu \leq 1$, and $\sup_m \|A^{\pm 1}(m)\| < \infty$.
Suppose the following fiber bunching condition holds:
\[ \sup_m \|A(m)\| \cdot \|A^{-1}(m)\| \lambda^x_s(m) < 1. \]

Let $H(m, x) = (\phi(m), A(m)x)$ be a bundle map on $\mathcal{E}$. Then there is a unique holonomy over $M$ for $H$ satisfying (hl1), (hl2) in Definition 7.15. In fact, $h_{m_1, m_2}$ is a linear map. Moreover, if $m' \mapsto \lambda_s(m')$ is continuous in each small $\mathcal{F}_m(\epsilon)$, then the holonomy over $M$ for $H$ also satisfies (hl3) in Definition 7.15.

Proof. Without loss of generality, we assume $\nu = 1$. Note that $\operatorname{Hol}_x H(\cdot, x) \leq C_0 |x|$, so in general $\sup_x \operatorname{Hol}_x H(\cdot, x) = \infty$ which is not the case in Theorem 7.16. In order to deal with this case, let us
Definition A.1. (a) Let $E$ be a set, $u : M \rightarrow M$ and $\theta : M \rightarrow \mathbb{R}_+$. We use the notation $\theta^{(k)}(m) = \theta(m)\theta(u(m))\cdots\theta(u^{(k-1)}(m))$ when we say $\theta$ is over $u$.

(b) A function sequence $\theta_n : M \rightarrow \mathbb{R}_+$, $n = 1, 2, 3, \ldots$, is ln-subadditive (about $u$) if $\theta_n(m) > 0$ and $\ln \theta_{n_1+n_2}(m) \leq \ln \theta_{n_1}(m) + \ln \theta_{n_2}(u^{n_1}(m))$, for all $n_1, n_2 \in \mathbb{N}$ and $m \in M$.

(c) Denoted by $\{\theta_n\} \in E(u)$ if $\{\theta_n\}$ is ln-subadditive (about $u$) and $\overline{\theta}(m) \triangleq \lim_{n \to \infty} (\theta_n(m))^{1/n}$ exists for every $m \in M$. In this case, $\overline{\theta}(m)$ (or $\ln \overline{\theta}(m)$) is called a (exact) Lyapunov number of $\{\theta_n\}$ (at $m$). Note that in general $\overline{\theta}(m) \leq \overline{\theta}(u(m))$ and $\overline{\theta}(m) = \overline{\theta}(u(m))$ if $u$ is invertible. Kingman Subadditive Ergodic Theorem ([Kin68]) can ensure that some function sequences belong to $E(u)$.

(d) A function $\theta : M \rightarrow \mathbb{R}_+$ (over $u$) is orbitally decreased (resp. orbitally increased) (about $u$) if $\theta(u(m)) \leq \theta(m)$ (resp. $\theta(u(m)) \geq \theta(m)$). $\theta$ is (positively) orbitally bounded (about $u$) if $\sup_{N \geq 0} \theta(u^N(m)) < \infty$ for every $m \in M$.

(e) Denoted by $\theta \in E(u)$ if $\{\theta^{(n)}\} \in E(u)$. Note that in this case, $\overline{\theta}(u(m)) = \overline{\theta}(m)$. If $u$ is a period function or $\theta$ is orbitally decreased, then $\theta \in E(u)$.

(f) Given an ln-subadditive (about $u$) and strictly positive function sequence $\{\theta_n\}$, assume $\theta_1$ is orbitally bounded about $u$. Note that by ln-subadditivity, $\theta_{n_1, \ldots, n_k}(m) \leq \theta_{n_1}(u^{n_1}(m))\cdots\theta_{n_k}(u^{n_k}(m))$, for all $n_1, \ldots, n_k \in \mathbb{N}$. We use the notation $\overline{\theta}_{n_1, \ldots, n_k}(m) = \sup_{N \geq 0} \theta_{n_1, \ldots, n_k}(u^{(N)}(m))$. Then $\{\overline{\theta}_{n_1, \ldots, n_k}(m)\}$ is also ln-subadditive (about $u$). Since $\overline{\theta}_{n_1, \ldots, n_k}(u^{(m)}) \leq (\overline{\theta}_{n_1, \ldots, n_k}(m))^{1/n}$, we know that $\{\overline{\theta}_{n_1, \ldots, n_k}(m)\} \in E(u)$. Let $\theta^{*}(m), m \in M$ be its (exact) Lyapunov numbers. We call them the sup Lyapunov numbers of $\{\theta_n\}$. Note that $\theta^{*}(u(m)) = \theta^{*}(m)$ and $\theta^{*}(u(m)) \leq \theta^{*}(m)$. Also, $\theta^{*}(m) = \inf_{N \geq 1} \sup_{0 \leq k} \theta_{n_1, \ldots, n_k}(u^k(m))^{1/n}$. The above notations will be used for the function $\theta$ if we consider the function sequence $\{\theta^{(n)}\}$.

Example A.2. (a) Let $A$ be a bounded linear operator from a Banach space into itself. Let $\theta_n = \|A^n\|$, then $\overline{\theta} = r(A)$, the spectral radius of $A$. (Gelfand Theorem.)

(b) Let $M$ be a probability space with probability measure $P$, and $u : M \rightarrow M$ be an invertible function preserving $P$. Let $A : M \rightarrow L(X, X)$ be strongly measurable (over $u$) where $X$ is a Banach space. Set $\theta^{*}(m) = \|A^{(m)}(0)\|$. Assume $\ln \|A^{(m)}(0)\| \in L^1(P)$. Then $\theta^{*}(m)$ exists for $m \in M \in \overline{M}$, where $P(M_1) = 1$ and $u(M_1) = M_1$. Moreover, if $u$ is ergodic, then $\overline{\theta}$ is a constant function. This is a direct consequence of Kingman Subadditive Ergodic Theorem ([Kin68]). $\overline{\theta}(m), m \in M_1$, consider the map

$$H_1(m, x) = (\phi(m), \|A(m)\|^{-1}A(m)x) : M \times \mathbb{B}_1 \rightarrow M \times \mathbb{B}_1,$$

where $\mathbb{B}_1$ is the closed unit ball of $X$. Let $r_1$ be the radial retraction (see (3.1)). $H_1(m, \cdot)$ has a natural left invertible map $H^{-1}_1(m, x) = r_1(\|A(m)\|^{-1}A(m)x)$.

We can assume $\sup_m \|A(m)\| \cdot \|A^{-1}(m)\| < 1/2$ if we consider $H^{-1}_1(n)$ instead of $H_1$ for large $n$. Note that $\operatorname{Lip} H^{-1}_1(m, \cdot) \leq 2\|A(m)\| \cdot \|A^{-1}(m)\|$, sup$_{x \in \mathbb{B}_1} \operatorname{Hol}_n H_1(x, \cdot) < \infty$; furthermore $H_1(m, \mathbb{B}_1)$ is open (as $A$ is invertible), and

$$\|A_{m_1}^{-1} - A_{m_2}^{-1}\| = \|A_{m_1}A_{m_2}^{-1} - A_{m_1}A_{m_2}^{-1}\| \leq \|A_{m_1}^{-1}\| \cdot \|A_{m_1} - A_{m_2}\| \cdot \|A_{m_2}^{-1}\|.$$
are the largest Lyapunov numbers. See [CL99, section 8.1] for more characterizations about $\theta^*$. See [LL10] for new important development about Multiplicative Ergodic Theorem in the infinite dimensional setting.

(c) Let $\overline{X}$ be a bundle with base space $\overline{M}$ and fibers $\overline{X}_m, m \in \overline{M}$ being metric spaces. Let $w : \overline{M} \to \overline{M}$ and $f : \overline{X} \to \overline{X}$ over $w$. Assume that $\text{Lip}_w f^m(\cdot) = \mu_n(m)$ and $\sup_{N \geq 0} \mu_1(u^N(m)) < \infty$. Note that $\{\mu_n\}$ is in-subadditive. Consider the sup Lyapunov numbers of $\{\mu_1^{(m)}\}$ and $\{\mu_n\}$ respectively, which are denoted by $\mu^\ast_1, \mu_1^\ast$ respectively. Then $\mu^\ast_1(m) \leq \mu^\ast_1(m)$. Note that $\mu^\ast_1, \mu^\ast_1$ do not depend on the choice of the uniform equivalent metrics in fibers of $\overline{X}$, i.e. any $\overline{d}_m$ in $\overline{X}_m$ satisfies $C^{-1}d_m(x,y) \leq \overline{d}_m(x,y) \leq Cd_m(x,y)$ for all $(x,y) \in \overline{X}_m$, where $C$ is a constant independent of $m \in \overline{M}$ and $d_m$ is the original metric in $\overline{X}_m$. For this reason, one can say $\mu^\ast_1(m)$ is the (non-linear) spectral radius of $f$ along the orbit from $m$.

**Lemma A.3.** Given two strictly positive function sequences $\{\mu_n\}, \{\theta_n\}$ (over $u$) which are in-subadditive and orbitally bounded about $u$, i.e. $\sup_{N \geq 0} \mu_1(u^N(m)) < \infty$, $\sup_{N \geq 0} \theta_1(u^N(m)) < \infty$. Let us describe some different types of ‘spectral gap condition’:

(i) $\sup_{r \geq 0} \mu_r(1) \sup_{N \geq 0} \theta_1(u^N(m)) < 1.$ (i) $\sup_{r \geq 0} \mu_r(m) \sup_{N \geq 0} \theta_1(u^N(m)) < 1.$

(ii) $\sup_{N \geq 0} \mu_1(1) \theta^\ast_1(m) < 1.$ (ii) $\sup_{N \geq 0} \mu_1(m) \theta^\ast_1(m) < 1.$

(iii) $\sup_{N \geq 0} \mu_1(m) \theta^\ast_1(m) < 1.$

(iv) $\sup_{N \geq 0} \mu_1(1) \theta_1(m) < 1.$ (iv) $\sup_{N \geq 0} \mu_1(m) \theta_1(m) < 1.$

(v) $\sup_{N \geq 0} \mu_1(1) \theta_1^\ast(m) < 1.$ (v) $\sup_{N \geq 0} \mu_1(m) \theta_1^\ast(m) < 1.$

Then we have (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), (iii) $\Rightarrow$ (iv), (iv) $\Rightarrow$ (v), (v) $\Rightarrow$ (vi), (vi) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iv), (iv) $\Rightarrow$ (v).

The proof is easy, so we omit it.

In the literatures, each of the above types of the ‘spectral gap conditions’ (or ‘bunching conditions’) have been used, e.g. [PSW97, PSW12, BLZ99, BLZ08] use (iv) (so (v) (vi)), and [Fen73, Has97, BLZ98, BLZ00, LYZ13] use (iii) In our general settings, we can’t succeed to use (iv) (so (v) (vi)) type of spectral gap condition to give our regularity results. However, our results will also recover some classical results when using (iv) (so (v) (vi)) type of spectral gap condition. This is not surprising since in classical setting the fibers are assumed to be uniformly bounded; now the case (ii) in (R3) (see below) can be applied. Note that for this case, what we actually need is that the Lipschitz functions are also Hölder.

First, consider a special example as a motivation.

**Example A.4.** Let

$$(A.1) \quad f(y) \leq xa^t + yb^t, \forall t > 0,$$

where $f(y) \geq 0$, $x, y > 0$, $a, b > 0$.

(1) If $0 < a, b < 1$, then $f(y) = 0$. If $a < 1 = b$, then $f(y) \leq y$.

(2) Assume $a < 1 < b$. Take

$$t_0(y) = \frac{1}{\ln b - \ln a} \ln \left( \frac{-\ln a}{y \ln b} \right),$$

substituting in (A.1) for $t$, then we see

$$(A.2) \quad f(y) \leq Cx^{1-\alpha}y^\alpha, \text{ if } y \leq rx,$$

where $\alpha = \frac{r}{r-1} < 1$, $r = \frac{-\ln a}{\ln b} > 0$, and $C = \sup_{r \geq 0} \left( e^{-\frac{\ln r}{r}} + e^\frac{\ln r}{r} \right)$.

If (A.1) holds only for $t \in \mathbb{N}$, letting $t = [t_0] + 1$ in (A.1), then we also have (A.2) for a different constant $C$. If (A.1) further holds for $t \in \mathbb{R}$, the restriction $y \leq rx$ is not needed.
Let us give the following setting.

(R1) Assume $\overline{X}, \overline{Y}, \hat{Y}$ are bundles with base space $\overline{M}$ and fibers $\overline{X}_m, \overline{Y}_m, \hat{Y}_m, m \in \overline{M}$, being metric spaces. We write all the metrics $d(z, z') = |z - z'|$. Let $r : \overline{M} \to \overline{Y}$, $j : \overline{M} \to \overline{X}$, $t_0 : \overline{M} \to \hat{Y}$ be fixed sections. We also write $|y| = d(y, \epsilon(m))$ if $y \in \overline{Y}_m$; $|x| = d(x, j(m))$ if $x \in \overline{X}_m$; $|y_0| = d(y_0, t_0(m))$ if $y_0 \in \overline{Y}_m$.

(R2) Let $w : \overline{M} \to \overline{M}$. $g : \overline{X} \times \overline{Y} \to \hat{Y}$ is a bundle map over $w$. Let $u : \overline{X}(\epsilon_1) \to \overline{X}$, $v : \overline{X} \times \overline{Y} \to \hat{Y}$ be two bundle maps over $w$ satisfying

\begin{equation}
|u_m(x_1) - u_m(x_2)| \leq \mu(m)|x_1 - x_2|, \quad x_1, x_2 \in \overline{X}_m(\epsilon_1),
\end{equation}

\begin{equation}
|v_m(x, y)| \leq \kappa(m)|y|, \quad \forall (x, y) \in \overline{X}_m \times \overline{Y}_m, \quad u_m(j(m)) = j(w(m)), \quad m \in \overline{M},
\end{equation}

where $\overline{X}(\epsilon)$ is defined by

\[ \overline{X}(\epsilon) = \{ (m, x) : x \in \overline{X}_m(m), m \in \overline{M} \}, \overline{X}_m(\epsilon) = \{ x \in \overline{X}_m : |x - j(m)| \leq \epsilon \}. \]

Assume

\begin{equation}
|g_m(x_1, y) - g_m(x_2, y)| \leq C_1|x_1 - x_2|^\gamma_1|y|^\gamma_2 + C_2|x_1 - x_2|^\gamma_2|y|^\gamma_2 + \theta(m)|g_m(u_m(x_1), v_m(x_1, y)) - g_m(u_m(x_2), v_m(x_1, y))|,
\end{equation}

where $0 \leq \gamma_i, \gamma_2 \leq 1$, $C_i \geq 0$, $i = 1, 2$.

(R3) We use the notation $(k^{\vee_2}; \mu^{\vee_2})^{\alpha_2} \theta < 1$ which means the followings in different settings. Similar for $(k^{\vee_2}; \mu^{\vee_2})^{\alpha_2} \theta < 1$.

(i) $C_2 = 0$, delete the condition.

(ii) $\sup_m(k^{\vee_2}(m)\mu^{\vee_2}(m)\theta(m)) < 1$ or $\sup_m((k^{\vee_2}(m)\mu^{\vee_2}(m)\theta(m)) < 1$ if $u_m(\cdot)|_{X_m(\epsilon_1)}$, $v_m(\cdot, \cdot)$ are bounded functions uniform for $m \in \overline{M}$; particularly when $\epsilon_1 \leq \infty$, and $\overline{Y}_m$ is bounded uniform for $m \in \overline{M}$ if $\gamma_2 > 0$.

(iii) $\sup_m(k^{\vee_2}(m)\mu^{\vee_2}(m)\theta(m)) < 1$ or $\sup_m((k^{\vee_2}(m)\mu^{\vee_2}(m)\theta(m)) < 1$ if $\theta \in \mathcal{E}(w)$.

(iv) $\sup_m(k^{\vee_2}(m)\mu^{\vee_2}(m)\theta(m)) < 1$ or $\sup_m((k^{\vee_2}(m)\mu^{\vee_2}(m)\theta(m)) < 1$ if $\alpha = 1$.

(v) $\sup_m(k^{\vee_2}(m)\mu^{\vee_2}(m)\theta(m)) < 1$ otherwise.

Remark A.5. (a) Let $\text{Lip}_m(x) = \mu_n(m)$ and $\kappa_n(m) = \inf \{ k' : |v_m(n, x, y)| \leq k'|y| \}$. One can use $\{ \mu_n \}$, $\{ \kappa_n \}$ instead of $\{ \mu(n) \}$, $\{ \kappa(n) \}$ to give a better ‘spectral gap condition’. The proof is essentially the same.

(b) In Lemma A.6 andLemma A.7, the condition $\sup_m k^{\vee_2}(m)\mu^{\vee_2}(m)\theta(m) < 1$ can be replaced by $\sup_m((k^{\vee_2}(m)\mu^{\vee_2}(m)\theta(m)) < 1$.

First, consider the following two cases under $\epsilon_1 = \infty$ or $\sup_m \mu_n \leq 1$.

Lemma A.6. Let (R1) (R2) (R3) hold. Assume $\epsilon_1 = \infty$, or $\sup_m |u_m| \leq 1$, or $u_m(X_m(\epsilon_1)) \subset X_n(\epsilon_1)$ for all $m$. $\mu, \kappa, \theta$ are bounded functions. (A.3) and (A.4) hold for all $x_1, x_2 \in \overline{X}(\epsilon_1)$, or $x_1 \in \overline{X}(\epsilon_1)$, $x_2 = j(m)$.

(a) Suppose

(i) $g$ is bounded in the sense that $\sup_{(x, y) \in \overline{X}_m \times \overline{Y}_m} g_m(x, y) < \infty$.

(ii) $\sup_m \theta(m) < 1$, $\sup_m k^{\vee_2}(m)\mu^{\vee_2}(m)\theta(m) < 1$, $(k^{\vee_2}(m)\mu^{\vee_2}(m)\theta(m)) < 1$.

Then

\[ |g_m(x_1, y) - g_m(x_2, y)| \leq CC_1|x_1 - x_2|^\gamma_1|y|^\gamma_2 + CC_2|x_1 - x_2|^\gamma_2|y|^\gamma_2, \]

if $\alpha < 1$, under a restriction $|x_1 - x_2|^\gamma_2|y|^\gamma_2 \leq \hat{r}$, and if $\alpha = 1$, for $y \in \overline{Y}_m, m \in \overline{M}$, where $C$ is a constant depending on the constant $\hat{r} > 0$, but not $m \in \overline{M}$.

(b) Suppose

(i) $|g_m(x, y)| \leq M_0|y|$ for all $x \in \overline{X}_m$, $y \in \overline{Y}_m$, where $M_0$ is a constant independent of $m, x$.

(ii) $\sup_m \theta(m) \kappa(m) < 1$, $\sup_m k^{\vee_2}(m)\mu^{\vee_2}(m)\theta(m) < 1$, $(k^{\vee_2}(m)\mu^{\vee_2}(m)\theta(m)) < 1$.\]
Then
\[ |g_m(x_1, y) - g_m(x_2, y)| \leq CC_1 |x_1 - x_2|^{\gamma_1} |y|^{\vec{C}} + CC_2 (|x_1 - x_2|^{\gamma_2} |y|^{\vec{C}})^{\alpha} |y|^{1-\alpha}, \]
if \( \alpha < 1 \), under a restriction \( |x_1 - x_2|^{\gamma_2} |y|^{\vec{C}} < \vec{r} \), and if \( \alpha = 1 \), for \( y \in \overline{Y}_m \), \( m \in \overline{M} \), where \( C \) is a constant depending on the constant \( \vec{r} > 0 \), but not \( m \in \overline{M} \).

**Proof.** We only give the proof of (a). The case (b) is the same as (a). First, assume \( \alpha < 1 \). Set
\[ v_1 = \sup_m k^{\vec{C}} (m) \mu^{\gamma_1} (m) \theta (m), \quad v_2 (k) (m) = (k^{\vec{C}} (m) \mu^{\gamma_1} (m))^{\gamma_2} \theta (k) (m). \]
Take \( \theta_0 \) such that \( \sup_m \theta (m) < \theta_0 < 1 \). Note that \( (k^{\vec{C}} (m))^{\gamma_1} (\mu^{\gamma_1} (m))^{\gamma_2} \theta (k) (m) \leq v_1 \).

First, we consider the case when \( (k^{\vec{C}} \mu^{\gamma_2})^{\alpha} \theta < 1 \) means \( \sup_m ((k^{\vec{C}} \mu^{\gamma_2})^{\alpha} \theta) (m) < 1 \) under \( \theta \in \mathcal{E}(w) \).

Choose \( \hat{\alpha} \) such that
\[ \max \left\{ 1, \hat{C}_2 \gamma_2 \sup_k \sup_{m \to \infty} \frac{\ln k^{\vec{C}} (m) \ln \mu^{\gamma_1} (m)}{-\ln \theta (k) (m)} \right\} < \hat{\alpha} - 1 < \alpha^{-1}. \]
Since \( \theta \in \mathcal{E}(w) \), we have \( \overline{\theta} : \overline{M} \to \mathbb{R}_+ \), such that
\[ (a') (1 + \varepsilon (m))^{-k} \overline{\theta} (m) \leq \theta (k) (m), \quad (b') \theta (k) (m) \leq (1 + \varepsilon (m))^{k^{\vec{C}}} \overline{\theta} (m), \]
for \( k \geq N \geq N (m) \). As \( \sup_m ((k^{\vec{C}} \mu^{\gamma_2})^{\alpha} \theta) (m) < 1 \), we can further get
\[ (c') (k^{\vec{C}} (m))^{\gamma_2} (\mu^{\gamma_1} (m))^{\gamma_2} \theta (k) (m) < 1, \]
for \( k \geq N \geq N (m) \). The function \( \varepsilon : M \to \mathbb{R}_+ \) we choose satisfies \((1 + \varepsilon (m))\overline{\theta} (m) = 1\), where \( \beta = \frac{\hat{\alpha} - \alpha}{\alpha + \hat{\alpha} - 2 \alpha} > 0 \). Set \( \overline{\theta}_1 (m) = \left( \frac{\overline{\theta} (m)}{1 + \varepsilon (m)} \right)^{1-1/\hat{\alpha}} > 1 \) and \( \overline{\theta}_2 (m) = (1 + \varepsilon (m)) \overline{\theta} (m) \). Note that \( \overline{\theta}_2 (m) < 1 \) and \( \overline{\theta}_1 (m) = \overline{\theta} (m)^{1-1/\hat{\alpha}} (m) \). Also, we can assume that \( \sup_m \overline{\theta}_2 (m) < \theta_0 \) and that \( \overline{\theta}_2 (k) (m) \leq (\theta (k) (m))^{1-1/\hat{\alpha}} \)
for \( k \geq N \geq N (m) \). Assume \( C_2 > 0 \). Note that \( v_2 (k) (m) \leq (\theta (k) (m))^{1-1/\hat{\alpha}} \leq \overline{\theta}_1 (m) \), if \( k \geq N \geq N (m) \).

By iterating (A.4) \( kN \) times, \( k \geq 1 \), we have
\[ |g_m (x_1, y) - g_m (x_2, y)| \leq \left\{ \begin{array}{l}
C_1 \left\{ 1 + v_1 + \cdots + v_1^{kN-1} \right\} |x_1 - x_2|^{\gamma_1} |y|^{\vec{C}} \\
+ C_2 \left\{ 1 + v_2 (1) (m) + \cdots + v_2^{kN-1} (m) \right\} |x_1 - x_2|^{\gamma_2} |y|^{\vec{C}} \\
+ \theta^{kN} (m) g_{w^k \kappa N} (u_{m}^{kN} (x_1), v_{m}^{kN} (x_1), y) - g_{w^k \kappa N} (u_{m}^{kN} (x_2), v_{m}^{kN} (x_1), y)) \leq \overline{C}_1 |x_1 - x_2|^{\gamma_1} |y|^{\vec{C}} + \overline{C}_2 (m, k, N) |x_1 - x_2|^{\gamma_2} |y|^{\vec{C}}.
\end{array} \right. \]  
(A.5)

where
\[ \overline{C}_1 = \frac{C_1}{1 - v_1}, \quad \overline{C}_2 (m, k, N) = C_2 V (m) + C_2 \frac{\theta_1^k (m) - \theta_1 (m)}{\theta_1^{kN} (m) - \theta_1 (m)} , \]
and \( V (m) = 1 + v_2 (1) (m) + \cdots + v_2^{(N-1)} (m) \).

Consider \( g \) as a bundle map over \( w^{kN} \). By iterating the above inequality (A.5) \( n - 1 \) times, we get
\[ |g_m (x_1, y) - g_m (x_2, y)| \leq \overline{C}_1 \left\{ 1 + v_1^{kN} + \cdots + v_1^{(n-1)kN} \right\} |x_1 - x_2|^{\gamma_1} |y|^{\vec{C}} + \overline{C}_2 (m, k, N) \left\{ 1 + \theta_1 (m) + \cdots + \theta_1^{(n-1)kN} (m) \right\} |x_1 - x_2|^{\gamma_2} |y|^{\vec{C}} + C_0 (\overline{\theta}_2 (m))^n \]
\[ \leq \overline{C}_1 |x_1 - x_2|^{\gamma_1} |y|^{\vec{C}} + \overline{C}_2 (m, k, N) \left\{ 1 + \theta_1 (m) + \cdots + \theta_1^{(n-1)kN} (m) \right\} |x_1 - x_2|^{\gamma_2} |y|^{\vec{C}} + C_0 (\overline{\theta}_2 (m))^n \]
Lemma A.7. Let (R1) (R2) (R3) (R3') hold. Assume $\epsilon_1 < \infty$ and $\sup_m \mu(m) > 1$. $\mu, \kappa, \theta$ are bounded functions. (A.3) and (A.4) hold for all $x_1 \in \overline{X}(e_1), x_2 = j(m)$. Also, $C_2 > 0$.

(a) Suppose

(i) $g$ is bounded in the sense that $\sup_{(x,y) \in \overline{X} \times Y} |g_m(x,y)| < \infty$,

(ii) $\sup_m \theta(m) < 1$, $\sup_m \kappa^\alpha(m) \mu^\alpha(m) \theta(m) < 1$, $(\kappa^\alpha \mu^\alpha)^\alpha \theta < 1$, $((\mu^\alpha)^\alpha)^\alpha \theta < 1$.

If $0 < \alpha < 1$, then we have,

$$|g_m(x_1, y) - g_m(j(m), y)| \leq CC_1 |x_1 - j(m)| |y|^{\tilde{c}_1} + CC_2 |x_1 - j(m)| |y|^{\tilde{c}_2},$$

under $|x_1 - j(m)| |y|^{\tilde{c}_2} \leq \tilde{r} \min\{c \gamma^{c-1}, |y|\}, y \in \overline{Y}_m, m \in \overline{M}$, and some constants $\tilde{r} > 0, c > 1$, where $C$ is a constant depending on the constant $\tilde{r} > 0$ but independent of $m$.

(b) Suppose

(i) $|g_m(x,y)| \leq M_0 |y|$ for all $x \in \overline{X}_m, y \in \overline{Y}_m$, where $M_0$ is a constant independent of $m, x$.

(ii) $\sup_m \kappa^\alpha(m) \mu^\alpha(m) \theta(m) < 1$, $(\kappa^\alpha \mu^\alpha)^\alpha \theta < 1$, $(\mu^\alpha)^\alpha \theta < 1$.

If $0 < \alpha < 1$, then we have,

$$|g_m(x_1, y) - g_m(j(m), y)| \leq CC_1 |x_1 - j(m)| |y|^{\tilde{c}_1} + CC_2 |x_1 - j(m)| |y|^{\tilde{c}_2},$$

under $|x_1 - j(m)| |y|^{\tilde{c}_2} \leq \tilde{r} \min\{c \gamma^{c-1}, |y|\}, y \in \overline{Y}_m, m \in \overline{M}$, and some constant $\tilde{r} > 0, c > 1$, where $C$ is a constant depending on the constant $\tilde{r} > 0$ but not $m \in \overline{M}$.
Remark A.8. If \( C_2 = 0 \), then we can reduce the case to \( C_1 = 0 \) and \( \alpha = 1 \). The corresponding ‘spectral gap condition’ is \( \sup_m \theta(m) < 1 \), \( \sup_m k^{\xi_2}(m)\mu^{\tau^2}(m)\theta(m) < 1 \), \( \sup_m \mu^{\tau^2}(m)\theta(m) < 1 \) for case (a), and \( \sup_m \theta(m)\kappa(m) < 1 \), \( \sup_m k^{\xi_2}(m)\mu^{\tau^2}(m)\theta(m) < 1 \), \( \sup_m \mu^{\tau^2}(m)\theta(m)\kappa(m) < 1 \) for case (b), respectively.

**Proof.** We only give the proof of (a). The case (b) is the same as (a).

Case \( \alpha < 1 \). We first consider the case when \( (k^{\xi_2} \mu^{\tau^2})^{-\alpha} \theta < 1 \), \( (\mu^{\tau^2})^{-\alpha} \theta < 1 \) mean \( \sup_m ((k^{\xi_2} \mu^{\tau^2})^{-\alpha} \theta)^*(m) < 1 \), \( \sup_m (\mu^{\tau^2})^{-\alpha} \theta)^*(m) < 1 \) respectively, under \( \theta \in \mathcal{E}(w) \). Let \( v_1, v_2^{(k)}(m), \theta_0 \) as in the proof of Lemma A.6.

Since \( \theta \in \mathcal{E}(w) \), we have \( \tilde{\theta} : \overline{\mathcal{M}} \rightarrow \mathbb{R}_+ \), such that

\[
\begin{align*}
(a)' & (1 + \varepsilon(m))^{-\alpha} \tilde{\theta}^k(m) \leq \theta^{(n)}(m), \\
(b)' & \theta^{(k)}(m) \leq (1 + \varepsilon(m))^{k} \tilde{\theta}^k(m),
\end{align*}
\]

for \( k \geq N \geq N(m) \). As \( \sup_m ((k^{\xi_2} \mu^{\tau^2})^{-\alpha} \theta)^*(m) < 1 \) and \( \sup_m (\mu^{\tau^2})^{-\alpha} \theta)^*(m) < 1 \), we can further get

\[
\begin{align*}
(c)' & (k^{\xi_2}(m))^\gamma \alpha' \theta^{(k)}(m) < 1, \\
(d)' & (\mu^{\xi_2}(m))^{\gamma \alpha'} (1 + \varepsilon(m))^{k} \tilde{\theta}^k(m) < 1,
\end{align*}
\]

for \( k \geq N \geq N(m) \), where \( \alpha' \) is chosen so that

\[
\max \left \{ 1, \xi_2 \gamma \sup \limsup_{m \to \infty} \frac{\ln k^{\xi_2}(m) \ln \mu^{\xi_2}(m)}{-\ln \theta^{(k)}(m)}, \gamma \sup \limsup_{m \to \infty} \frac{\ln \mu^{\xi_2}(m)}{-\ln \theta^{(k)}(m)} \right \} < \alpha'^{-1} < \alpha^{-1}.
\]

The function \( \varepsilon : \mathcal{M} \rightarrow \mathbb{R}_+ \) is chosen so that it satisfies \((1 + \varepsilon(m))^{\tilde{\theta}^k} = 1\), where \( \beta = \frac{\alpha' - \alpha}{\alpha + \alpha' - 2\alpha} > 0 \) is sufficiently small. Set \( \tilde{\theta}_1(m) = \left( \frac{\underline{\theta}(m)}{1 + \varepsilon(m)} \right)^{1/1 - \alpha'} > 1 \) and \( \tilde{\theta}_2(m) = (1 + \varepsilon(m))\tilde{\theta}(m) \). Note that \( \tilde{\theta}_2(m) < 1 \) and \( \tilde{\theta}_1(m) = \tilde{\theta}^{-1/\alpha}(m) \).

When \( |x_1 - j(m)| \leq (\mu^{(k+1)}(m))^{-1} \varepsilon_1 \), by iterating (A.4) \( kN - 1 \) times, we have (A.5) holds for \( x_2 = j(m) \). Consider \( \gamma \) as a bundle map over \( w^k \). Under \( |x_1 - j(m)| \leq (\mu^{(n-k)}(m))^{-1} \varepsilon_1 \), by iterating (A.5) \( n - 1 \) times, we also get (A.6) holds for \( x_2 = j(m) \).

Let

\[
n(m) \equiv \frac{-\alpha}{\ln \tilde{\theta}_2(m)} \ln \left( \frac{\alpha}{1 - \alpha} \frac{C_0 / \tilde{\mathcal{C}}}{|x_1 - x_2|^\gamma \xi_2} \right),
\]

and \( n = [n(m)] + 1 \). Now \( |x_1 - j(m)| \leq (\mu^{(n-k)}(m))^{-1} \varepsilon_1 \) can be satisfied if

\[
|x_1 - j(m)| \leq (\tilde{\theta}_2(m))^{n(m)kN/\gamma^2 \alpha' \varepsilon_1},
\]

or equivalently,

\[
|x_1 - j(m)|^{\gamma} |y|^{\xi_2} \leq \left( \frac{\alpha}{1 - \alpha} \right)^{-\gamma (1 - \alpha')} |y|^{\xi_2/((1 - \alpha') \overline{M}_0)},
\]

where \( \tilde{\alpha} = \alpha' / \alpha' < 1 \), \( \overline{M}_0 = (\tilde{\mathcal{C}} / C_0)^{1/(1 - \alpha')} \varepsilon_1^{\gamma (1 - \alpha')} \), then

\[
|g_n(x_1, y) - g_m(j(m), y)| \leq 2 \tilde{\mathcal{C}} |x_1 - j(m)|^{\gamma} |y|^{\xi_2} + C \xi_2 (|x_1 - j(m)|^{\gamma} |y|^{\xi_2})^\alpha,
\]

for some constant \( C \) independent of \( m \) (and \( \alpha \)).

Case \( \alpha = 1 \). In this case, the condition is given by \( \sup m k^{\xi_2}(m) \mu^{\tau^2}(m)\theta(m) < 1 \), \( \sup_m \mu^{\tau^2}(m)\theta(m) < 1 \). (Similarly \( \sup_m k^{\xi_2}(m)\mu^{\tau^2} \theta)^*(m) < 1 \) and \( \sup_m (\mu^{\tau^2})^{-\alpha} \theta)^*(m) < 1 \). Let \( v_2 = \sup_m k^{\xi_2}(m)\mu^{\tau^2}(m)\theta(m) \) (\( < 1 \)). Now we have \( v_2^{(k)}(m) \leq v_2^{(k)} \). Without lost of the generality, assume \( C_1 = 0 \). Since \( \sup_m \mu^{\tau^2}(m)\theta(m) < 1 \), we can choose \( \alpha'' > 1 \) such that \( \sup_m (\mu^{\tau^2})^{-\alpha''} \theta(m) < 1 \). When \( |x_1 - j(m)| \leq (\mu^{(n-1)}(m))^{-1} \varepsilon_1 \), by iterating (A.4) \( n - 1 \) times, we get

\[
|g_m(x_1, y) - g_m(j(m), y)| \leq \tilde{\mathcal{C}} |x_1 - j(m)|^{\gamma} |y|^{\xi_2} + C_0 \theta^{(n)}(m),
\]

where \( \tilde{\mathcal{C}} = C_2 / (1 - v_2) \). The above inequality can be satisfied if \( |x_1 - j(m)| \leq (\theta^{(n)}(m))^{1/(\gamma^2 \alpha'')} \varepsilon_1 \). If \( |x_1 - j(m)|^{\gamma} |y|^{\xi_2} \neq 0 \), choose \( n \) so that

\[
C_0 \theta^{(n)}(m) \leq \tilde{\mathcal{C}} |x_1 - j(m)|^{\gamma} |y|^{\xi_2}.
\]
In order to do this, we need

$$|x_1 - j(m)| \leq (\theta^{(n)}(m))^{1/(2\gamma \alpha''')} \epsilon_1 \leq (C_0^{-1} \widetilde{C}_3 |x_1 - j(m)|^{\gamma_2} |y|^{\gamma_2} \alpha''')^{1/(2\gamma \alpha''')} \epsilon_1,$$

i.e.

$$|x_1 - j(m)|^{\gamma_2} |y|^{\gamma_2} \leq \widetilde{M}_1 |y|^{\gamma_2 \alpha''/(\alpha''')},$$

where $\widetilde{M}_1 = (C_0^{-1} \widetilde{C}_3)^{\gamma_2/(\alpha''')-1} \epsilon_1^{\gamma_2/(\alpha''')}$. Let us give more details that under

(\heartsuit) $$|x_1 - j(m)|^{\gamma_2} |y|^{\gamma_2} \leq \min\{\widetilde{M}_1 |y|^{\gamma_2 \alpha''/(\alpha''')}, \hat{r}\},$$

where $\hat{r} > 0$ is small, one can always choose $n$ such that

(\spadesuit) $$|x_1 - j(m)| \leq (\mu^{(n+1)})^{-1} \epsilon_1, \quad C_0 \theta^{(n)}(m) \leq \widetilde{C}_3 |x_1 - j(m)|^{\gamma_2} |y|^{\gamma_2}.$$

Since we have assumed $\mu$ is a bounded function, so without loss of generality, let $\inf_n \theta(m) \geq \tilde{\theta} > 0$ (otherwise taking $\theta(m) + \zeta$ instead of $\theta(m)$ where $\zeta$ is sufficiently small). Reselect $\alpha''$ such that $\sup_n (\mu^{\gamma_2}(m))^\rho \theta(m) < 1$ for all $\rho \in [1, \rho \alpha''']$, where $\tau > 1$ satisfying $\theta_0 < \theta_1^{1/\tau}$. Then there exists a $\delta > 0$ such that

$$(0, \delta) \subset \bigcup_{n \geq 0} \{(\theta^{(n)}(m))^{1/(2\gamma \alpha'''), (\theta^{(n)}(m))^{1/(2\gamma \alpha''')}\}.$$ 

Indeed, since $\theta^{(n)} \leq \tilde{\theta}^{(n+1)/\tau}$ for large $n$, we have $a_n \leq b_{n+1}$ where $a_n = (\theta^{(n)}(m))^{1/(2\gamma \alpha''')}, b_n = (\theta^{(n)}(m))^{1/(2\gamma \alpha''')}$. If $\hat{r}$ is small and (\heartsuit) holds, then there is an $\epsilon$ such that

$$|x_1 - j(m)| \leq (\theta^{(n)}(m))^{1/(2\gamma \rho(n))} \epsilon_1 \leq (C_0^{-1} \widetilde{C}_3 |x_1 - j(m)|^{\gamma_2} |y|^{\gamma_2})^{1/(2\gamma \alpha''')} \epsilon_1,$$

where $\rho(n) \in [\alpha'''', \rho \alpha''']$, which yields (\spadesuit) holds.

By using $\theta''$ instead of $\theta$, one can give the case (v) in (R3) or (iv) in (R3'). The final case is (ii) in (R3) and (i) in (R3'), which can be reduced to the case $\alpha = 1$. Thus, the proof is complete. \hfill \Box

**Remark A.9.** The readers can consider more general case when (A.4) is replaced by

$$|g_m(x_1, y) - g_m(x_2, y)| \leq C_1 |x_1 - x_2|^\gamma |y|^{\gamma_1} \epsilon_1 + C_2 |x_1 - x_2|^\gamma |y|^{\gamma_2} + \epsilon_2 + \theta(m)|g_m(x_1, y) - g_m(x_2, y)|,$$

where $c_1 = 0$ or 1, and give the corresponding results. The proof is only needed a minor change, so we omit it.

**Remark A.10.** In the following, we show how the results obtained in this appendix can be applied to prove the regularity results in Lemmas 6.11, 6.13, 6.16, 6.17, 6.22, 6.23.

(a) ($x \mapsto K_m^{(1)}(x)$) In the proof of Lemma 6.11, we need to consider (6.8). Let $\overline{M} = M, \overline{X}_m = X_m, \overline{Y}_m = \{m\}, \overline{X}_m = L(X_m, Y_m), g_m(x, y) = K_m^{(1)}(x), w(m) = u(m), u_m(x) = x_m(x), j = j_0, t_0 = 0. Then apply Lemma A.6 (a).

(b) ($m \mapsto f_m(x)$) In the proof of Lemma 6.13, we have (6.14). Let $\overline{M} = M, \overline{X}_m = U_m, \overline{Y}_m = Y_m, g_m(x, m) = \overline{f}_m(x, m), w(m) = u(m), u_m(m) = u(m), \nu_m(m, x) = \overline{x}_m(x), j = j_0, t = t_0 = t_1. Then apply Lemma A.7 (b).

Let us consider the proof of Remark 6.15. For this case, now (6.14) becomes

(A.7) $$|\overline{f}_m(m_1, x) - \overline{f}_m(m_0, x)| \leq \overline{C} |m_1 - m_0|^\gamma + \theta_1(m_0) |\overline{u}(m_1, x) - \overline{u}(m_0, x)|,$$

where $\theta_1(m_0) = \frac{\overline{K}_m^{(1)}(m_0)}{1 - \alpha''(m_0, \hat{M}_1^{(1)}(m_0))}$. Although $\overline{f}_m(m_1, x)$ now does not satisfy the condition (i) in Lemma A.7 (b), the same argument can also apply. Note that

$$|\overline{f}_m(m_1, x) - \overline{f}_m(m_0, x)|$$

$$\leq |\overline{f}_m(m_1, x) - \overline{f}_m(m_1, \overline{x}_m^{(m_0)}(m_1))| + |\overline{f}_m(m_1, \overline{x}_m^{(m_0)}(m_1)) - \overline{f}_m(m_0, \overline{x}_m^{(m_0)}(m_0))|$$
Remark A.11. Assume there is an $\hat{\beta}'(m_0) |x - \hat{m}_0(m_1)| + |x - \hat{m}_0(m_0)| + |\hat{f}_X(m_1) - \hat{f}_X(m_0)|$

\begin{align}
\leq & \beta''(m_0) |x - \hat{m}_0(m_1)| + |x - \hat{m}_0(m_0)| + \epsilon_0 |m_1 - m_0|.
\end{align}

(A.8)

Iterating (A.7) by $n$ times, we have

$$\left| \hat{f}_{m_0}(m_1, x) - \hat{f}_{m_0}(m_0, x) \right| \leq \hat{C} \left\{ 1 + (\mu(m_0))^r \theta_1(m_0) + \cdots + (\mu^{(n-1)}(m_0))^r \theta_1^{(n-1)}(m_0) \right\} |m_1 - m_0|^r
\leq \hat{C} \left\{ 1 + (\mu(m_0))^r \theta_1(m_0) + \cdots + (\mu^{(n-1)}(m_0))^r \theta_1^{(n-1)}(m_0) \right\} |m_1 - m_0|^r
+ \beta''(m_0) \psi_1^{(n)}(m_0) \alpha_0^{(n)}(m_0) \left| x - \hat{m}_0(m_1) \right| + \left| x - \hat{m}_0(m_0) \right|
+ c_0 \mu^{(n)}(m_0) \psi_1^{(n)}(m_0) |m_1 - m_0|.
$$

where $\hat{\alpha}^{(n)}$ is the $n$th composition of $\hat{\alpha}$. From (A.8), we see

$$\left| \hat{f}_{m_0}(m_1, x) - \hat{f}_{m_0}(m_0, x) \right| \leq \hat{C} |m_1 - m_0|^{\gamma_0},$$

if $|x - \hat{m}_0(m_1)| + |x - \hat{m}_0(m_0)| \leq r_1$, and $|m_1 - m_0| \leq r_1$, for some suitable constant $\hat{r}(r_1)$ depending on $r_1$, where $\hat{C}$ is a constant depending on $r_1$, but not $m_0 \in M_1$, and $r_1$ is chosen arbitrarily.

(c) (m $\mapsto K_m^{(c)}$) To obtain Lemma 6.16, consider (6.20). Let $M = M_1$, $\bar{X}_m = U_{m_0}(\hat{m}^{-2} \epsilon_1)$, $\bar{Y}_m = \{iX(m_0), \bar{Y}_m = L(X_{m_0}, Y_{m_0}), g_{m_0}(m, y) = \hat{K}_m(m, iX(m_0)), w(m) = u(m), u_{m_0}(m) = v(m), j = id, t_0 = 0$. Then apply Lemma A.7 (a).

To give the final step of the proof of Lemma 6.17, we need to consider (6.21). Let $M = M_1$, $\bar{X}_m = U_{m_0}(\hat{m}^{-2} \epsilon_1)$, $\bar{Y}_m = \{iX(m_0), \bar{Y}_m = L(X_{m_0}, Y_{m_0}), g_{m_0}(m, m) = \hat{K}_m(m, x), w(m) = u(m), u_{m_0}(m) = v(m), v_{m_0}(m, x) = \bar{X}_m(m, x), j = id, t_0 = 0$. Apply Lemma A.7 (a).

Now apply Lemma A.6 (b).

(c) (m $\mapsto K_m^{(c)}$) To prove Lemma 6.23, we need to consider (6.44). Let $M = M_1$, $\bar{X}_m = U_{m_0}(\epsilon_1)$, $\bar{Y}_m = X_{m_0}, \bar{Y}_m = L(T_{m_0}, Y_{m_0}), g_{m_0}(m, x) = \hat{K}_m(m, x), w(m) = u(m), u_{m_0}(m) = v(m), v_{m_0}(m, x) = \bar{X}_m(m, x), j = id, t = iX, \bar{Y}_m = 0$. Apply Lemma A.7 (b) and Remark A.9.

Remark A.11. Assume there is an $\epsilon_1 > 0$ (included $\epsilon_1 = \infty$) such that for any $m_0 \in M_1$, $(U_{m_0}(\epsilon_1), \varphi^{m_0}) \in A$, $(U_{m_0}(\epsilon_1), \varphi^{m_0}) \in B$ (in (H2)) and $u(U_{m_0}(\epsilon_1)) < U_{m_0}(\epsilon_1)$ (in (H4a)).

(a) Under the same conditions in Lemma 6.13 except the spectral gap condition (c) replaced by $\lambda_s \lambda_u < 1, (\lambda_s^{-1} \mu) \alpha_0 \lambda_u < 1$, where $0 < \alpha \leq 1$, then (6.12) also holds for $m_1 \in U_{m_0}(\epsilon_1)$ and if $\alpha < 1$ under $|m_1 - m_0|^r |x|^{\gamma_0} \leq \hat{r}$. (By Lemma A.6 (b).)

(b) The conditions and conclusions in Lemma 6.16 and Lemma 6.17 are the same but the conclusions hold for $m_1 \in U_{m_0}(\epsilon_1)$, if $\alpha < 1$, $\epsilon_1^*$ is small, and otherwise $\epsilon_1^* = \epsilon_1$. (By Lemma A.6 (a).)

(c) In addition, if for all $m_0 \in M_1$, $(U_{m_0}(\epsilon_1), (dx \tilde{X}_{m_0}(\chi_{m_0}))) \in M$ (in (H1c)), then under the same conditions in Lemma 6.13 except in the spectral gap condition (c), max $\frac{\mu^{(n)}(m_0)}{\lambda_s^{(n)}(m_0)} \lambda_s \lambda_u < 1$ is replaced by $\frac{\mu^{(n)}(m_0)}{\lambda_s^{(n)}(m_0)} \lambda_s \lambda_u < 1$, one has the conclusion (1) in Lemma 6.13 holds for $m_1 \in U_{m_0}(\epsilon_1)$, if $\alpha < 1$, $\epsilon_1^*$ is small, and otherwise $\epsilon_1^* = \epsilon_1$. (By Lemma A.6 (b).)
**B. Appendix. Bundle and Bundle Map with Uniform Property: Part II**

This is an appendix that expands Section 5.4. We only give the definitions to express our main ideas.

**B.0.1. base-regularity of bundle map, $C^0$-uniform bundle: $C^0$ case.** We give a description about the $C^0$ continuity respecting the base points for a bundle map $f$ in the uniform sense: $m \mapsto f_m(\cdot)$ is continuous or uniformly continuous in $C^0$-topology in bounded sets or in the whole space. Write the following four different limits respectively:

$$L_{1}^{\text{base}} : \sup_{m_{0} \in M_{1}} \limsup_{m' \to m_{0}} \sup_{x \in A_{m_{0}}} \sup_{m_{0} \in M_{1}} \limsup_{m' \to m_{0}} \sup_{x \in A_{m_{0}}} \sup_{m' \to m_{0}}$$

$$L_{2}^{\text{base}} : \limsup_{m' \to m_{0}} \sup_{x \in A_{m_{0}}} \sup_{m_{0} \in M_{1}} \limsup_{m' \to m_{0}} \sup_{x \in A_{m_{0}}} \sup_{m' \to m_{0}}$$

$$L_{3}^{\text{base}} : \sup_{m_{0} \in M_{1}} \limsup_{m' \to m_{0}} \sup_{x \in X_{m_{0}}} \sup_{m_{0} \in M_{1}} \limsup_{m' \to m_{0}} \sup_{x \in X_{m_{0}}} \sup_{m' \to m_{0}}$$

where for $L_{i}^{\text{base}}$, $A_{m_{0}}$ is any bounded set of $X_{m_{0}}$, and for $L_{i}^{\text{base}}$, $\sup_{m_{0} \in M_{1}} \text{diam}A_{m_{0}} < \infty$, $i = 1, 2$.

**Definition B.1.** Let $X, Y$ be $C^0$ topology bundles over $M, N$ with $C^0$ (open regular) bundle atlases $\mathcal{A}, \mathcal{B}$ respectively, $u : M \to N$ be $C^0$, and $M_{1} \subset M$. Let $f : X \to Y$ be a bundle map over $u$.

(a) If every (regular) local representation $f_{m_{0}}$ of $f$ at $m_{0} \in M$ with respect to $\mathcal{A}, \mathcal{B}$, satisfies $L_{L_{i}^{\text{base}}}(\tilde{f}_{m_{0}}(m', x) - \tilde{f}_{m_{0}}(m_{0}, x)) = 0$, where $L = L_{1}^{\text{base}}$ (resp. $L = L_{2}^{\text{base}}$), then we say $m \mapsto f_{m}$ is continuous around $M_{1}$ in $C^0$-topology in bounded sets (resp. in the whole space) with respect to $\mathcal{A}, \mathcal{B}$.

(b) In addition, let $M, N$ be two locally metrizable spaces associated with open covers $\{U_{m_{0}} : m_{0} \in M\}$ and $\{V_{n} : n \in N\}$ respectively, and $X, Y$ both have uniform size trivialization at $M_{1}$ with respect to $\mathcal{A}, \mathcal{B}$. $u$ is uniformly continuous around $M_{1}$. If $L_{L_{i}^{\text{base}}}(\tilde{f}_{m_{0}}(m', x) - \tilde{f}_{m_{0}}(m_{0}, x)) = 0$, where $L = L_{1}^{\text{base}}$ (resp. $L = L_{2}^{\text{base}}$), then we say $m \mapsto f_{m}$ is uniformly continuous around $M_{1}$ in $C^0$-topology in bounded sets (resp. in the whole space) with respect to $\mathcal{A}, \mathcal{B}$.

(c) (vector case) Under (b), let $X, Y, \mathcal{A}, \mathcal{B}, f$ be $C^0$ vector. We also say $m \mapsto f_{m} : M \to L_{u}(X, Y)$ is uniformly continuous around $M_{1}$ (with respect to $\mathcal{A}, \mathcal{B}$) for short, if $m \mapsto f_{m}$ is uniformly continuous around $M_{1}$ in $C^0$-topology in bounded sets (with respect to $\mathcal{A}, \mathcal{B}$).

As usual, if $M_{1} = M$, the words ‘around $M_{1}$’ will be omitted. Also, the words ‘with respect to $\mathcal{A}, \mathcal{B}$’ will be omitted if $\mathcal{A}, \mathcal{B}$ are predetermined.

In analogy with Definition 5.29 and Lemma 5.30, we have the following.

**Definition B.2 (C$^0$-uniform bundle).** Follows Definition 5.29. $\mathcal{A}$ is said to be $C^0$-uniform around $M_{1}$, if the transition maps $\varphi_{m_{0}, m_{1}}^{m_{0}, m_{1}}$, $m_{0}, m_{1} \in M_{1}$, are equicontinuous on bounded-fiber sets, i.e.

$$\varphi_{m_{0}, m_{1}}^{m_{0}, m_{1}}(x) \Rightarrow \varphi_{m_{0}, m_{1}}^{m_{0}, m_{1}}(x), \text{ as } m' \to m_{0},$$

uniform for $x \in A_{m_{0}, m_{0}, m_{1} \in M_{1}}$, where $\sup_{m_{0} \in M_{1}} \text{diam}A_{m_{0}} < \infty$. If $\mathcal{A}$ is $C^0$-uniform around $M_{1}$ and $X$ has an $\varepsilon$-almost uniform $C^0$-fiber trivialization at $M_{1}$ with respect to $\mathcal{A}$, then we say $X$ has an $\varepsilon$-almost $C^0$-uniform trivialization at $M_{1}$ with respect to $\mathcal{A}$; in addition, if $M_{1} = M$ and $\varepsilon = 0$, we also call $X$ a $C^0$-uniform bundle (with respect to $\mathcal{A}$).

**Lemma B.3.** Assume $(X, M, \pi_{1}), (Y, N, \pi_{2})$ have $\varepsilon$-almost $C^0$-uniform trivializations at $M_{1}^{\varepsilon_{1}}$, $u(M_{1}^{\varepsilon_{1}})$ with respect to preferred $C^0$-uniform bundle atlases $\mathcal{A}, \mathcal{B}$ respectively (see Definition B.2), where $M_{1} \subset M$ and $M_{1}^{\varepsilon_{1}}$ is the $\varepsilon_{1}$-neighborhood of $M_{1}$. Suppose $f : X \to Y$ is uniformly continuous-fiber (resp. equicontinuous-fiber) (see Section 5.4.4) over a map $u$ which is uniformly continuous around $M_{1}^{\varepsilon_{1}}$ (see Definition 5.4). If (i) $m \mapsto f_{m}$ is continuous (resp. uniformly continuous) around $M_{1}^{\varepsilon_{1}}$ in $C^0$-topology in bounded sets (see Definition B.1), and (ii) for each $m \in M_{1}^{\varepsilon_{1}}$, $f_{m}$ maps bounded sets of $X_{m}$ into bounded sets of $X_{u(m)}$ (resp. $f$ maps bounded-fiber sets at $M_{1}^{\varepsilon_{1}}$ into bounded-fiber sets at $u(M_{1}^{\varepsilon_{1}})$), then the local representation $\tilde{f}_{m_{0}}(m)^{m_{0}}$, of $f$ at $m_{0}, m'_{0}$ (see Lemma 5.30) satisfies...
Let $\mathcal{L}[\hat{f}_{m_0,m_0'}(m',x) - \hat{f}_{m_0,m_0'}(m_0,x)] = 0$, where $\mathcal{L} = \mathcal{L}_1^{\text{base}}$ (resp. $\mathcal{L} = \mathcal{L}_2^{\text{base}}$). Condition (ii) can be removed if the fibers of $X,Y$ are length spaces (see Appendix D.3 and Lemma D.9).

If $m \mapsto f_m$ is continuous (resp. uniformly continuous) around $M_1^1$ in $C^0$-topology in the whole space (see Definition B.1), then $\mathcal{L}[\hat{f}_{m_0,m_0'}(m',x) - \hat{f}_{m_0,m_0}(m_0,x)] = 0$, where $\mathcal{L} = \mathcal{L}_2^{\text{base}}$ (resp. $\mathcal{L} = \mathcal{L}_2^{\text{base}}$).

Remark B.4.5 (uniform property of vector bundle map and subbundle). In Definition 5.27 (c) and Definition B.1 (c), we in fact give a description about Hölder continuity and uniformly $C^0$ continuity of a vector bundle map respecting base points. Let $X,Y$ be vector bundles with vector bundle atlases $\mathcal{A}, \mathcal{B}$. Here any vector bundle map $L \in L(X,Y)$ over $u$ can be considered as a section $m \mapsto L_m : M \to L_u(X,Y)$. A particular situation is $L = Du \in L_u(TM, TN)$ if $u : M \to N$ is $C^1$ and $M, N$ are $C^1$ Finsler manifolds. Note that $m \mapsto L_m$ is $C^0$ in the $C^0$-topology in bounded sets now meaning it is a $C^0$ vector bundle map if $X,Y, \mathcal{A}, \mathcal{B}$ are $C^0$.

Consider a special case. Suppose each $\Pi_m \in L(X_m, X_m)$ is a projection (i.e. $\Pi_m^2 = \Pi_m$). Let $X_m = R(\Pi_m), X_m^0 = R(\text{Id} - \Pi_m)$. We say $m \mapsto X_m, \kappa = c, h,$ are uniformly $C^0$ (resp. uniformly (locally) $C^0$) around $M_1$, if $m \mapsto \Pi_m$ is uniformly $C^0$ (resp. uniformly (locally) $C^0$) around $M_1$. This is equivalent to say $m \mapsto X_m : M \to \mathbb{G}(X), \kappa = c, h$, are uniformly $C^0$ or uniformly (locally) $C^0$ around $M_1$, where $\mathbb{G}(X) = \bigsqcup_m \mathbb{G}(X_m)$ is the Grassmann manifold of $X$ (see [AMR88] for a definition of a Grassmann manifold of a Banach space). These concepts will be used frequently in Section 7.2. We also write $X' = \bigsqcup_m X_m', \kappa = c, h$, which are subbundles of $X$, and $X' \oplus X'' = X$.

B.0.2. base-regularity of $C^1$-fiber bundle map. For a $C^1$-fiber bundle map $f : X \to Y$ over $u$, $D^v f \in L(T_X^v, T_Y^v)$ (see (5.1)) over $f$; see Section 5.4.4. So a description about the Hölder continuity of $m \mapsto K_m^1(x)$, where $K^1 \in L_f(T_X^v, T_Y^v)$ (see Section 5.4.2) and we write $K^1_{m}(x) = K^1(x_{\cdot},x)$, will give that of $m \mapsto Df_m(x)$. Assume the fibers of $X,Y$ are Banach spaces (or open subsets of Banach spaces) for simplicity (see also Remark B.11).

Take a bundle atlas $\mathcal{A}$ for bundle $X$. If each bundle chart belonging to $\mathcal{A}$ is $C^1$-fiber, then there is a canonical bundle atlas $\mathcal{A}_1$ (with respect to $\mathcal{A}$) for $T_X^v$ (see (5.1)), i.e.

$$U_{m_0} \times X_{m_0} \times X_{m_0} \to T_X^v, ((m, \varphi_m(m)), D\varphi_{m_0}(m)),$$

where $(U_{m_0}, \varphi_{m_0}) \in \mathcal{A}$ at $m_0$. Or $(\varphi_{m_0}(U_{m_0} \times X_{m_0}), (D^v f_{m_0}) \circ (\varphi_{m_0})^{-1}) \in \mathcal{A}_1$. Particularly, if $X$ is a $C^1$ topology bundle (see Definition 5.22), then $T_X^v$ is a $C^0$ vector bundle over $X$. Thus, for a vector bundle map $K' \in L_f(T_X^v, T_Y^v)$, the canonical local representation of $K^1$ with respect to bundle atlases $\mathcal{A}, \mathcal{B}$ can be taken as

$$\hat{K}_{m_0}^1(m,x) \triangleq (D\varphi_{u(m)}^1)^{-1}(f_m(\varphi_{m_0}(m)))(D^v \varphi_{m_0}(m)) : U_{m_0} \times X_{m_0} \to L(X_{m_0}, Y_{u(m_0)}),$$

where $(U_{m_0}, \varphi_{m_0}) \in \mathcal{A}$ at $m_0$ and $(V_{u(m_0)}, \varphi_{u(m_0)}) \in \mathcal{B}$ at $u(m_0)$. So if $\hat{f}_{m_0}$ is a (regular) local representation of $f$ at $m_0$ with respect to $\mathcal{A}, \mathcal{B}$, the canonical (regular) local representation $\hat{D}_x f_{m_0}$ of $D^v f$ can be taken as

$$\hat{D}_x f_{m_0}(m,x) \triangleq (D\varphi_{u(m)}^1)^{-1}(f_m(\varphi_{m_0}(m)))(D^v \varphi_{m_0}(m)) : U_{m_0} \times X_{m_0} \to L(X_{m_0}, Y_{u(m_0)}),$$

i.e. $\hat{D}_x f_{m_0}(m,x) = D_x \hat{f}_{m_0}(m,x)$. Now we have the following definition similar as Definition B.1 and Definition 5.27.

Definition B.5. Let $X, Y$ be $C^1$ topology bundles over $M, N$ with (open regular) $C^1$-fiber bundle atlases $\mathcal{A}, \mathcal{B}$ respectively (see Definition 5.22), $u : M \to N$ be $C^0$, and $M_1 \subset M$. Let $f : X \to Y$ be a $C^1$-fiber and $C^0$ bundle map over $u$. Assume the fibers of $X,Y$ are Banach spaces (or open subsets of Banach spaces) for simplicity. Take a $K^1 \in L_f(T_X^v, T_Y^v)$. Let $\hat{f}_{m_0}$ be a (regular) local representation of $f$, and $\hat{K}_{m_0}$ a canonical local representation $K^1$, at $m_0 \in M$ with respect to $\mathcal{A}, \mathcal{B}$.
(a) If $\mathcal{L}[\hat{K}_m^1(m', x) - \hat{K}_m^1(m_0, x)] = 0$, where $\mathcal{L} = \mathcal{L}_1^{\text{base}}$ (resp. $\mathcal{L} = \mathcal{L}_2^{\text{base}}$), then we say $m \mapsto K_m^1(\cdot)$ is continuous around $M_1$ in $C^0$-topology in bounded sets (resp. in the whole space) with respect to $\mathcal{A}, \mathcal{B}$. In addition, if $Df_m(x) = K_m^1(x)$ and $\mathcal{L}[\hat{f}_m(m', x) - \hat{f}_m(m_0, x)] = 0$, then we say $m \mapsto f_m$ is continuous around $M_1$ in $C^1$-topology in bounded sets (resp. in the whole space) with respect to $\mathcal{A}, \mathcal{B}$.

Suppose that (i) $M, N$ are two locally metrizable spaces associated with open covers $\{U_m : m \in M\}$ and $\{V_n : n \in N\}$ respectively, that (ii) $\mathcal{A}, \mathcal{B}$ have uniform size domains (see Definition 5.19), and that (iii) $m$ be uniformly continuous around $M_1$ (see Definition 5.4).

(b) Under (i) ~ (iii), if $\mathcal{L}[\hat{K}_m^1(m', x) - \hat{K}_m^1(m_0, x)] = 0$, where $\mathcal{L} = \mathcal{L}_1^{\text{base}}$ (resp. $\mathcal{L} = \mathcal{L}_2^{\text{base}}$), then we say $m \mapsto K_m^1(\cdot)$ is uniformly continuous around $M_1$ in $C^0$-topology in bounded sets (resp. in the whole space) with respect to $\mathcal{A}, \mathcal{B}$. In addition, if $Df_m(x) = K_m^1(x)$ and $\mathcal{L}[\hat{f}_m(m', x) - \hat{f}_m(m_0, x)] = 0$, then we say $m \mapsto f_m$ is uniformly continuous around $M_1$ in $C^1$-topology in bounded sets (resp. in the whole space) with respect to $\mathcal{A}, \mathcal{B}$.

(c) Under (i), suppose $\hat{K}_m^1$ satisfies

$$\|\hat{K}_m^1(m, x) - \hat{K}_m^1(m_0, x)\| \leq c_m(x)d_m^0(m, m_0)^\theta, \ m \in U_{m_0}(\varepsilon_{m_0}),$$

where $\varepsilon_{m_0} : X_{m_0} \to \mathbb{R}^+$, $m_0 \in M_1$. Then we say $K^1$ depends in a (locally) $C^{0, \theta}$ fashion on the base points around $M_1$, or $m \mapsto K_m^1(\cdot)$ is (locally) $C^{0, \theta}$ around $M_1$, with respect to $\mathcal{A}, \mathcal{B}$.

(d) Under (i) ~ (iii) (particularly, one can choose $\varepsilon_{m_0}$ such that $\sup_{m \in M_1} \varepsilon_{m_0} > 0$), if $\varepsilon_{m_0}^3$, $m_0 \in M_1$, are bounded at $M_1$ on any bounded-fiber sets (see Definition 5.26), then we say $K^1$ depends in a uniformly (locally) $C^{0, \theta}$ fashion on the base points around $M_1$ "uniform for bounded-fiber sets", or $m \mapsto K_m^1(\cdot)$ is uniformly (locally) $\theta$-Hölder around $M_1$ "uniform for bounded-fiber sets", with respect to $\mathcal{A}, \mathcal{B}$. In addition, if $Df_m(x) = K_m^1(x)$ and $f$ depends in a uniformly (locally) $C^{0, \theta}$ fashion on the base points around $M_1$ uniform for bounded-fiber sets, then we say $m \mapsto f_m$ is uniformly (locally) Hölder around $M_1$ in $C^1$-topology in bounded sets. Usually, the words in ‘...’ are omitted especially when $\varepsilon_{m_0}^3$ is the class of functions in case (a) or (b) in Definition 5.26. As usual, if $M_1 = M$, the words ‘around $M_1$’ will be omitted.

**Definition B.6** ($C^1$-fiber-uniform bundle, $C^{1,1}$-fiber-uniform).

Follows Definition 5.29. $\mathcal{A}$ is said to be $C^1$-fiber-uniform around $M_1$, if the transition maps $\varphi^{m_0, m_1}$, $m_0, m_1 \in M_1$, are $C^1$-fiber equicontinuous on bounded-fiber sets, i.e.

$$\varphi^{m_0, m_1}(x) \equiv \varphi^{m_0, m_1}(x), \ D\varphi^{m_0, m_1}(x) \equiv D\varphi^{m_0, m_1}(x), \text{ as } m' \to m_0,$$

uniform for $x \in A_{m_0}, m_0, m_1 \in M_1$, where $\sup_{m \in M_1} \dim A_{m_0} < \infty$. Assume the fibers of $X, Y$ are Banach spaces (or open subsets of Banach spaces) for simplicity (see also Remark B.11). $\mathcal{A}$ is said to be $C^{1,1}$-fiber-uniform around $M_1$, if it is $C^{0,1}$-uniform around $M_1$ and the transition map $\varphi^{m_0, m_1}$ satisfies further

$$\sup_{x \in X_{m_0}} \text{Lip } D_x \varphi^{m_0, m_1}(x) \leq C,$$

where $C > 0$ is a constant independent of $m_0, m_1 \in M_1$. If $\mathcal{A}$ is $C^1$-fiber-uniform (resp. $C^{1,1}$-fiber-uniform) around $M_1$ and $X$ has an $\varepsilon$-almost uniform $C^{1,1}$-fiber trivialization (see Definition 5.24) at $M_1$ with respect to $\mathcal{A}$, then we say $X$ has an $\varepsilon$-almost $C^1$-fiber-uniform trivialization (resp. $\varepsilon$-almost $C^{1,1}$-fiber-uniform trivialization) at $M_1$ with respect to $\mathcal{A}$; in addition, if $M_1 = M$ and $X$ is uniform $C^{1,1}$-fiber (see Definition 5.24), then we also call $X$ a $C^1$-fiber-uniform bundle (resp. $C^{1,1}$-fiber-uniform bundle) with respect to $\mathcal{A}$.

Using this definition, one can obtain similar results about the regularity of $(m, x) \mapsto Df_m(x)$ and $(m, x) \mapsto K_m^1(x)$ like Lemma B.3 and Lemma 5.30 which are omitted here.
B.0.3. **base-regularity of $C^1$ bundle map, $C^{1,1}$-uniform bundle, $C^1$-uniform bundle.** As usual, for a $C^1$ manifold $M$ and a $C^1$ atlas $\mathcal{A}_0$ of $M$, $\mathcal{A}_0$ will induce a natural $C^0$ bundle atlas $\mathcal{M}$ for $TM$, called a **canonical** bundle atlas of $TM$. That is for every local chart $\chi_m: U_m \to T_m M$ with $\chi_m(m) = 0$ and $D\chi_m(m) = \text{id}$, let $\psi^m_\ast (x) = D\chi_m^\ast (\chi_m(m'))x$, then $(U_m, \psi^m)$ is a **(canonical) bundle chart of $TM$.**

Let $M, N$ be two $C^1$ Finsler manifolds with $C^1$ (regular) atlases $\mathcal{A}_0, \mathcal{B}_0$. Let $(X, M, \pi_1), (Y, N, \pi_2)$ be two $C^1$ bundles (see **Definition 5.9**) and $f: X \to Y$ a $C^1$ bundle map over $u$. Consider the Hölder continuity of $Df$. As before, we will consider it in the local representations. Take $C^1$ (regular) bundle atlases $\mathcal{A}, \mathcal{B}$ of $X, Y$ respectively. Here we assume the charts belonging to $\mathcal{A}_0$ share the same domains as the one belonging to $\mathcal{A}$; similar for $\mathcal{B}_0, \mathcal{B}$.

For a precise presentation of $Df$ respecting the base points, one needs the *connection* structures in $X, Y$ and the **covariant derivative** of $f$, see **Section 5.3** for a quick review of these two notions. If $f: X \to Y$ is $C^1$, now we have $\nabla f \in L_f(\nabla^X f, \nabla^Y f)$ (see (5.1) and **Section 5.3.3**). So a description about the Hölder continuity of $m \mapsto K_m(x)$, where $K \in L_f(\nabla^X f, \nabla^Y f)$ (see **Section 5.4.2**) and we write $K_m(x) = K_{m, x}$, will give that of $m \mapsto \nabla f_{m, f}(x)$. Assume the fibers of $X, Y$ are Banach spaces (or open subsets of Banach spaces) for simplicity (see also Remark B.11).

Take a $C^0$ (regular) bundle atlas $\mathcal{M}$ for $TM$ and a $C^1$ (regular) bundle atlas $\mathcal{A}$ for $C^1$ bundle $X$ (over $M$); in many cases, $M$ is the canonical bundle atlas induced from $\mathcal{A}_0$, the $C^1$ (regular) atlas of $M$. Then there is a **canonical bundle atlas** $\mathcal{A}_2$ (with respect to $M$ (and $\mathcal{A}$)) for $\nabla^X f$ (see (5.1)), i.e.

$$\nabla f_{m, f}(x) \in \mathcal{A}_2 \quad \text{at} \quad m \mapsto K_m(x), \quad \nabla f_{m, f}(x) \in \mathcal{A}_2$$

where $(U_{m_0}, \varphi^{m_0} ) \in \mathcal{A}$ at $m_0$ and $(U_m, \psi^m ) \in \mathcal{M}$ at $m_0$. Or $(\varphi^m_\ast (U_{m_0} \cap X_{m_0}), \id \times \psi^m_\ast ) \in \mathcal{A}_2$.

Particularly, $\nabla^X f$ is a $C^0$ vector bundle over $X$. Thus, for a vector bundle map $K \in L_f(\nabla^X f, \nabla^Y f)$, the canonical local representation of $K$ with respect to bundle atlases $\mathcal{A}, \mathcal{B}, M$ can be taken as

$$\hat{K}_{m_0}(x) \triangleq (D^X f_{m_0})^{-1}(f_m(\varphi^{m_0}_m(x)))K_m(\varphi^{m_0}_m(x))\psi^m_\ast : U_{m_0} \times X_{m_0} \to L(T_{m_0} M, Y_{m_0})$$

where $(U_{m_0}, \varphi^{m_0} ) \in \mathcal{A}$ at $m_0$ and $(V_{m_0}, \psi^{m_0} ) \in \mathcal{B}$ at $u(m_0)$. So the canonical (regular) local representation $D_{m_0}f_{m_0}$ of $\nabla f$ (at $m_0$) can be taken as

$$D_{m_0}f_{m_0}(m, x) \triangleq (D^X f_{m_0})^{-1}(f_m(\varphi^{m_0}_m(x)))\nabla f_{m_0}(\varphi^{m_0}_m(x))\psi^m_\ast : U_{m_0} \times X_{m_0} \to L(T_{m_0} M, Y_{m_0})$$

In analogy with **Definition B.5**, one can give precise definitions about the Hölder continuity of $m \mapsto K_m(x)$.

**Definition B.7.** Take a vector bundle map $K \in L(\nabla^X f, \nabla^Y f)$ over $f$. We say $K$ depends in a *(locally) $C^{0,0}$* fashion on the base points around $M_1$ with respect to $\mathcal{A}, \mathcal{B}, M$, if the canonical local representation $\hat{K}_{m_0}$ of $K$ satisfies

$$|\hat{K}_{m_0}(m_1, x) - \hat{K}_{m_0}(m_0, x)| \leq c_{m_0}^4 d(m_1, m_0), \quad m_1 \in U_{m_0}(x),$$

where $c_{m_0}^4 : X_{m_0} \to \mathbb{R}_+$, $m_0 \in M_1$, and $d$ is the Finsler metric in $M$.

Assume $u$ is uniformly continuous around $M_1$ (see **Definition 5.4**). Suppose $X, Y$ both have uniform size trivializations at $M_1$ with respect to $\mathcal{A}, \mathcal{B}$ respectively (see **Definition 5.19**), (and so $TM$ with respect to $M_1$, i.e. one can choose $\varphi_{m_0}$ such that $\sup_{m_0 \in M_1} c_{m_0} > 0$. If $c_{m_0}^4, m_0 \in M_1$, are bounded at $M_1$ on any bounded-fiber sets (see **Definition 5.26**), then we say $K$ depends in a uniformly (locally) $C^{0,0}$ fashion on the base points around $M_1$ ‘uniform for bounded-fiber sets’, or $m \mapsto K_m(x)$ is uniformly (locally) $\theta$-Hölder around $M_1$ ‘uniform for bounded-fiber sets’, with respect to $\mathcal{A}, \mathcal{B}, M$. Usually, the words in ‘...’ are omitted especially when $c_{m_0}^4$ is the class of functions in case (a) or (b) in **Definition 5.26**.

Under the assumptions that $X, Y$ have $\varepsilon$-almost $C^1$-fiber-uniform (resp. $C^{1,1}$-fiber-uniform) trivializations (see **Definition B.6**) and $TM$ has an $\varepsilon$-almost $C^0$-uniform (resp. $C^{0,1}$-uniform) trivialization at $M_1$ (see **Definition B.2** and **Definition 5.29**), one can obtain similar results about the regularity of $(m, x) \mapsto \nabla f_{m, f}(x)$ and $(m, x) \mapsto K_m(x)$ like **Lemma B.3** (resp. **Lemma 5.30**) which are omitted.
here. Next let’s focus on the $C^{1,\gamma}$ continuity of $f$. See Section 5.3 for a quick review of some notions related with connections.

**Definition B.8** (uniform property of connection). Let $C^X$ be a $C^0$ connection of $C^1$ bundle $X$ over $C^1$ Finsler manifold $M$. Let $\mathcal{A}$ and $\mathcal{M}$ be $C^1$ bundle atlases of $X$ and $TM$, respectively. Let $M_1 \subset M$. Take $(U_m(\epsilon_m), \varphi^m) \in \mathcal{A}$ and $(U_m(\epsilon'_m), \psi^m) \in \mathcal{M}$ at $m_0$. Let

$$
\hat{D}_m \varphi^m_0(m, x) = (D(\varphi^m_0)^{-1}(\varphi^m_0(x)))\nabla_m \varphi^m_0(x) \psi^m \in \mathcal{M},
$$

where $\hat{T}^m_0$ is the Christoffel map in the bundle chart $\varphi^m_0$ (see Definition 5.10). We call $\hat{D}_m \varphi^m_0$ a local representation of $C^X$ at $m_0$ with respect to $\mathcal{A}, \mathcal{M}$. We say the connection $C^X$ is locally Lipschitz (with respect to $\mathcal{A}, \mathcal{M}$), if

$$
\sup_{m \in U_m(\epsilon_m)} \text{Lip} \left( \hat{T}^m_0 \right) < C_{m_0},
$$

where $\epsilon_{m_0} : X_{m_0} \to \mathbb{R}_+ > 0$ and $C_{m_0} > 0$. The connection $C^X$ is said to be uniformly (locally) Lipschitz around $M_1$ (with respect to $\mathcal{A}, \mathcal{M}$), if $\inf_{m \in M_1} \epsilon_m > 0$, $\sup_{m \in M_1} C_{m_0} < \infty$ and $\epsilon_{m_0}$ are bounded at any bounded-fiber sets (see Definition 5.26). We say $C^X$ is uniformly $C^0$ around $M_1$ (with respect to $\mathcal{A}, \mathcal{M}$), if $\inf_{m \in M_1} \epsilon'_m > 0$, and

$$
\hat{D}_m \varphi^m_0(m', x') \Rightarrow \hat{D}_m \varphi^m_0(m_0, x), \quad as \ (m', x') \rightarrow (m_0, x),
$$

uniform for $x \in \mathcal{A}_{m_0}, m_0 \in M_1$, where $\sup_{m \in M_1} \text{dim} \mathcal{A}_{m_0} < \infty$.

**Definition B.9** ($C^1$-uniform bundle, $C^{1,1}$-uniform bundle). Let $M$ be a $C^1$ Finsler manifold with a $C^1$ (regular) atlas $\mathcal{A}_0$. $\mathcal{M}$ is the canonical bundle atlas of $TM$ induced by $\mathcal{A}_0$. Let $X$ be a $C^1$ bundle over $M$ with $C^1$ (regular) bundle atlas $\mathcal{A}$ and $C^0$ connection $C^X$. Let $M_1 \subset M$. The charts belonging to $\mathcal{A}$ and $\mathcal{A}_0$ share the same domains and both have uniform size domains at $M_1$. Suppose that $TM$ has an $\varepsilon$-almost $C^0$-uniform trivialization at $M_1$ with respect to $\mathcal{M}$ (see Definition B.2). $\mathcal{A}$ (with $\mathcal{M}$) is said to be $C^1$-uniform around $M_1$, if the transition maps $\varphi^{m_0,m_1}$ (with respect to $\mathcal{A}$), $m_0, m_1 \in M_1$, are $C^1$ equicontinuous on bounded-fiber sets, i.e.

$$
D_m \varphi^{m_0,m_1}_m(m', x') : D_m \varphi^{m_0,m_1}_m(x) \Rightarrow D_x \varphi^{m_0,m_1}_m(x), \quad as \ (m', x') \rightarrow (m_0, x),
$$

uniform for $x \in \mathcal{A}_{m_0}, m_0, m_1 \in M_1$, where $\sup_{m_0 \in M_1} \text{dim} \mathcal{A}_{m_0} < \infty$. If $\mathcal{A}$ (with $\mathcal{M}$) is $C^1$-uniform around $M_1$, $X$ has an $\varepsilon$-almost uniform $C^{1,1}$-fiber trivialization at $M_1$ with respect to $\mathcal{A}$ and $C^X$ is uniformly $C^0$ around $M_1$ (Definition B.8), then we say $X$ has an $\varepsilon$-almost $C^{1,1}$-uniform trivialization at $M_1$ with respect to $\mathcal{A}$, $\mathcal{M}$; in addition, if $M_1 = M$ and $\varepsilon = 0$, we also call $X$ a $C^1$-uniform bundle (with respect to $\mathcal{A}$, $\mathcal{M}$).

Assume that (i) $TM$ has an $\varepsilon$-almost $C^{0,1}$-uniform trivialization (see Definition 5.29) at $M_1$ with respect to $\mathcal{M}$ (where in Definition 5.29 we need $\epsilon_{m_0,m_1}^1$ thereof satisfies $\epsilon_{m_0,m_1}^1(x) \leq C|x|$ for some constant $C > 0$ independent of $m_0, m_1$), and (ii) that the fibers of $X, Y$ are Banach spaces (or open subsets of Banach spaces) for simplicity (see also Remark B.11). We say $\mathcal{A}$ (with $\mathcal{M}$) is $C^{1,1}$-uniform around $M_1$, if $X$ has an $\varepsilon$-almost $C^{1,1}$-fiber-uniform trivialization (see Definition B.6) at $M_1$ with respect to $\mathcal{A}$, and for the transition map $\varphi^{m_0,m_1}$ with respect to $\mathcal{A}$ (see Definition 5.29), $m_0, m_1 \in M_1$, and the local chart $\chi_{m_0} \in \mathcal{A}_0$ at $m_0$, they satisfy

$$
\text{Lip}(D_m \varphi^{m_0,m_1}_m(x)) \leq C, \quad \text{Lip}(D_m \varphi^{m_0,m_1}_m(x)) D_{X_1}^{-1}(\chi_{m_0}(m)) \leq \frac{C}{m_0,m_1}(x),
$$

where $\frac{C}{m_0,m_1}(x), m_1 \in M_1$, are bounded at $M_1$ on any bounded-fiber sets (see Definition 5.26) and $C > 0$ is a constant independent of $m_0, m_1$. We say $X$ has an $\varepsilon$-almost $C^{1,1}$-uniform trivialization at $M_1$ with respect to $\mathcal{A}, \mathcal{M}$, if $\mathcal{A}$ with $\mathcal{M}$ is $C^{1,1}$-uniform and $C^X$ is uniformly (locally) Lipschitz.
around $M_1$ (see Definition B.8); in addition, if $M_1 = M$ and $\varepsilon = 0$, then we call $X$ a $C^{1,1}$-uniform bundle (with respect to $\mathcal{A}$, $M$).

**Remark B.10** (vector bundle). For the vector bundle case, we require the bundle atlas $\mathcal{A}$ is vector, and $c^1_{m_0, m_1}$ (in Definition 5.29) and $c^5_{m_0, m_1}$ (in Definition B.9) satisfy $c^1_{m_0, m_1}(x) \leq C|x|$ and $c^5_{m_0, m_1}(x) \leq C|x|$ where $C > 0$ independent of $m_0, m_1$. Now we have different classes of vector bundles with local uniform property (about the base space), named $C^0$-uniform (Definition B.2), $C^{0,1}$-uniform (Definition 5.29), $C^{1,1}$-uniform (Definition B.9), and $C^{1,1}$-uniform (Definition B.9) vector bundle.

\[ \begin{array}{ccc} C^{1,1}\text{-uniform vector bundle} & \longrightarrow & C^{1}\text{-uniform vector bundle} \\
C^{0,1}\text{-uniform vector bundle} & \longrightarrow & C^{0}\text{-uniform vector bundle} \end{array} \]

Moreover, for the local uniform property about fiber space, there is a class of vector bundle called uniform $C^0,1$-fiber bundle (Definition 5.23). Note that in the setting of vector bundle, uniform $C^0,1$-fiber = uniform $C^{1,1}$-fiber, $C^{1,1}$-uniform (Definition B.6) = $C^0$-uniform, and $C^{1,1}$-fiber-uniform (Definition B.6) = $C^{0,1}$-uniform. The notion of $C^{1,1}$-uniform vector bundle is the same as [HPS77, Chapter 6]. The above notions are related with vector bundles having bounded geometry, see Example C.10.

Or more generally, if the bundle $X$ has a 0-section $i$ with respect to the bundle atlas $\mathcal{A}$ (see Section 5.4.3), we also require $c^1_{m_0, m_1}$ (in Definition 5.29) and $c^5_{m_0, m_1}$ (in Definition B.9) satisfy $c^1_{m_0, m_1}(x) \leq C|x|$ and $c^5_{m_0, m_1}(x) \leq C|x|$ where $C > 0$ independent of $m_0, m_1$. Here $|x| = d(x, i(m))$, if $x \in X_m$.

Here note that for the (regular) local representation $f_{m_0}$ of $f$ at $m_0$ with respect to $\mathcal{A}$, $\mathcal{B}$, one has

\[ D_m f_{m_0}(m, x) = \nabla_{u(m)}(\phi_{u(m)}^{m_0})^{-1}(x''')D u(m) + D(\phi_{u(m)}^{m_0})^{-1}(x''') \{ \nabla_m f_{m}(x') + D_{f_m}(x')\nabla_m \phi_{m}^{m_0}(x) \}, \]

where $x' = \phi_{m_0}^{m}(x)$, $x''' = f_m(x')$. Once we know the (uniformly) Lipschitz continuity of the connections in $X, Y$ (see Definition B.8), $m \mapsto D u(m)$ (see Definition 5.32), $m \mapsto f_{m}(\cdot)$ and $x \mapsto \nabla_m f_{m}(x), D_{f_m}(x)$, then the Hölder continuity of $m \mapsto D_m f_{m_0}(m, x)$ is equivalent to the Hölder continuity of $m \mapsto \nabla_m f_{m}(x)$. We hold the opinion that a natural setting to discuss the uniformly $C^{1,1}$-continuity of bundle map $f$ in classical sense is that the bundles are $C^{1,1}$-uniform. Through the study of $(m, x) \mapsto D f_{m}(x)$ and $(m, x) \mapsto \nabla_m f_{m}(x)$ to understand the properties of $D f$, this is an important idea we used in Section 6.

B.0.4. summary and extension. Let $X, Y$ be two bundles with base spaces $M, N$ respectively. Assume the fibers of $X, Y$ are Banach spaces (or open subsets of Banach spaces) for simplicity (see also Remark B.11). Take (regular) bundle atlases $\mathcal{A}$ and $\mathcal{B}$ of $X, Y$ respectively.

Let’s consider a special type of vector bundle over $X$. Suppose $\Theta$ is a vector bundle over $X$ with a (regular) vector bundle atlas $\mathcal{A}_1$. The fibers of $\Theta$ are $\Theta_{(m_0, x_0)} = \Theta_{m_0, (m_0, x_0)} \in X$. Every bundle chart at $(m_0, x_0)$ belonging to bundle atlas $\mathcal{A}_1$ for $\Theta$ satisfies

\[ U_{m_0} \times X_{m_0} \times \Theta_{m_0} \to \mathcal{A}_1, \quad ((m, x), v) \mapsto ((m, \phi_{m_0}^{m_0}(x), \psi_{m_0}^{m_0}(x)v), \]

and $\psi_{m_0}^{m_0}(x)$ is id for all $x \in X_{m_0}$, where $(U_{m_0}, \phi_{m_0}^{m_0}) \in \mathcal{A}$ at $m_0 \in M$. That is $(\phi_{m_0}^{m_0}(U_{m_0} \times X_{m_0}), (id \times\psi_{m_0}^{m_0}) \in \mathcal{A}_1$, where $id \times \psi_{m_0}^{m_0}(m, x, v) = ((m, x), \psi_{m_0}^{m_0}(x)v)$. We call $\Theta$ a pre-tensor bundle over $X$ (with respect to $\mathcal{A}_1$). For example $T^X_M$ and $T^Y_M$ are all pre-tensor bundles (see Appendix B.0.3 and Appendix B.0.2). If $\Theta_1$, $\Theta_2$ are pre-tensor bundles over $X$, then so is $\Theta_1 \times \Theta_2$.

Take a bundle map $f : X \to Y$ over $u$. Let $\Theta, \Omega$ be two pre-tensor bundles over $X, Y$ with respect to bundle atlases $\mathcal{A}_1, \mathcal{B}_1$ respectively. Take a vector bundle map $E \in L(\Theta, \Omega)$ over $f$; we write $E_{m}(x) = E_{m}(m, x) \in L(\Theta_{m}, \Omega_{m}(u(m)))$ and consider it as

\[ (m, x) \mapsto E_{m}(x) : X \to L_f(\Theta, \Omega). \]
• (fiber-regularity). Similar as in the bundle map case, the same terminologies in Section 5.4.4, e.g. \(C^0\)-fiber, uniformly continuous-fiber, equicontinuous-fiber, \(C^{k,\theta}\)-fiber, uniformly \(C^{k,\theta}\)-fiber, fiber derivative (i.e. \(D^\nu E\)), etc, can be used to talk about the \(C^{k,\theta}\)-fiber continuity of \(x \mapsto E_m(x) : X_m \rightarrow L(\Theta_m, \Omega_{m(\cdot)})\).

The (regular) canonical local representation of \(E\) at \(m_0\) with respect to \(\mathcal{A}_1, \mathcal{B}_1\) is taken as

\[
\hat{E}_{m_0}(m, x)\nu = (\phi^{u(m_0)}_m)^{-1}(f_m(\phi^{m_0}_m(x)))E_m(\phi^{m_0}_m(x))\psi^{m_0}_m(x)\nu,
\]

where \(\phi^{u(m_0)} \in \mathcal{B}_1, \psi^{m_0}_m \in \mathcal{A}\) and \(\psi^{m_0}_m \in \mathcal{A}_1\).

• (base-regularity: Hölder case). Similar as in Definition 5.27, Definition B.5, Definition B.7, one can define the Hölder continuity of \(m \mapsto E_m(\cdot)\) by using the canonical local representation \(\hat{E}_{m_0}\).

The terminologies that \(E\) depends in a (locally) \(C^{0,\theta}\) fashion on the base points around \(M_1\), or \(m \mapsto E_m(\cdot)\) is (locally) \(C^{0,\theta}\) around \(M_1\) with respect to \(\mathcal{A}_1, \mathcal{B}_2\), and \(E\) depends in a uniformly (locally) \(C^{0,\theta}\) fashion on the base points around \(M_1\) ‘uniform for bounded-fiber sets’, or \(m \mapsto E_m(\cdot)\) is uniformly (locally) \(\theta\)-Hölder around \(M_1\) ‘uniform for bounded-fiber sets’ with respect to \(\mathcal{A}_1, \mathcal{B}_1\), will be used.

• (base-regularity: \(C^0\) case). Also, one can discuss the \(C^0\) continuity of \(m \mapsto E_m(\cdot)\) as in Appendix B.0.1 and Appendix B.0.2; the terminologies that \(m \mapsto E_m(\cdot)\) is continuous (resp. uniformly continuous) around \(M_1\) in \(C^0\)-topology (resp. \(C^1\)-topology) in bounded sets (resp. in the whole space) with respect to \(\mathcal{A}_1, \mathcal{B}_1\), will be used.

Until now, we have given a way to describe the (uniformly) Hölder continuity and \(C^0\) continuity (about fiber or base space) of \(f\) and \(DF\) in appropriate bundles \(X, Y\). For the base-regularity, this is done by using regular local representations of \(f\) with respect to preferred bundle atlases. While the regularity of regular local representations can yield the classical description of regularity in some sense under stronger regularity of the bundles; see Lemma B.3 and Lemma 5.30. The regularity of local representations in fact does not depend on the choice of bundle atlases having the same properties i.e. the equivalent bundle atlases; this is simple which we do not give details in the present paper and refer the reader to see [Ama15] for a discussion in the manifold setting.

Using pre-tensor bundles, one can further study the high order derivative of \(f\). For example, assume \(X, Y\) are paracompact \(C^k\) bundle with \(C^{k-1}\) connections \(\nabla_X, \nabla_Y\) respectively, and \(f \in C^k(X, Y)\). Now \(TX \equiv \mathcal{T}_X = \mathcal{T}^H_X \times \mathcal{T}^V_X\), so we can write \(DF = (\nabla f, D^\nu f)\) and

\[
(m, x) \mapsto DF(m, x) = (\nabla_m f_m(x), D_f m(x)) : X \rightarrow L_f(TX, \mathcal{T}^V_Y) \cong L_f(\mathcal{T}_X, \mathcal{T}^V_Y),
\]

where we ignore the base map of \(f\) (i.e. \(u\)). \(L_f(\mathcal{T}_X, \mathcal{T}^V_Y)\) is a \(C^{k-1}\) pre-tensor bundle over \(X\) (and so we can give a connection in it). Taking the derivative of \(DF\), we have

\[
D^2 f = \begin{pmatrix}
\nabla \nabla f \\
\nabla D^\nu f
\end{pmatrix}
\begin{pmatrix}
D^\nu \nabla f \\
D^\nu D^\nu f
\end{pmatrix}
\in L_f(\mathcal{T}_X \times \mathcal{T}_X, \mathcal{T}^V_Y).
\]

Similarly, \(D^i f \in L^i_f(\mathcal{T}_X, \mathcal{T}^V_Y), i = 1, 2, \ldots, k\). Here \(L^i_f(Z_1, Z_2) \triangleq L_u(Z_1 \times \cdots \times Z_i, Z_2)\), if \(Z_i\) is a vector bundle over \(M_i, i = 1, 2, \) and \(u : M_1 \rightarrow M_2\). Although our regularity results only concern \(C^{k,\alpha}\) continuity of \(f, k = 0, 1, 0 \leq \alpha \leq 1\), by using almost the same strategy in Section 6 with induction and above idea, one can obtain the higher order smoothness of \(f\) (but the statements are more complicated due to the non-triviality of \(X, Y\)); the details are omitted in this paper (see also [HPS77]).

Remark B.11. In order to discuss the \(C^{k,\theta}\)-fiber continuity \((k \geq 1)\) of a bundle map, in general, we assume the fibers of \(X, Y\) are Banach spaces (or open subsets of Banach spaces), i.e. each fiber can be represented by a single chart. In most cases, it suffices for us to apply our regularity results. However, under above preliminaries, one can further extend it to the fibers being general (connected) Finsler manifolds with local uniform properties, as we do for the extension of the base space in more general settings. A model for these Finsler manifolds is e.g. the Riemannian manifolds having
bounded geometry (see e.g. Definition C.6), or more general, the Banach-manifold-like manifolds in Definition 5.33. Note that for this case, if \( k \geq 2 \), the \( k \)-th order (covariant) derivative of \( f_m(\cdot) \) is considered as an element of the pre-tensor bundle over \( X_m, m \in M \). (A little modification about the definition of pre-tensor bundle is needed, left to the readers.)

C. Appendix. some examples related with manifolds and bundles

In the appendix, we give some examples related with our (uniform) assumptions about manifolds and bundles, i.e. the immersed manifolds in Banach spaces studied in [BLZ99, BLZ08] and the bounded geometry of Riemannian manifolds introduced in e.g. [Ama15, Eic91].

C.0.1. immersed manifold.

Example C.1 (trivial example). Let \( M \) be any open subset of a Banach space and \( M_1 \subset \{ m \in M : d(m, \partial M) > \epsilon \} (\epsilon > 0) \), then \( M \) is trivially \( C^{1,1} \)-uniform around \( M_1 \) (see Definition 5.33).

Example C.2 (immersed manifolds in Banach spaces I). Let \( M \subset X \) be an immersed manifold where \( X \) is a Banach space for simplicity. That is there are a \( C^1 \) manifold \( \hat{M} \) and an immersion \( \phi : \hat{M} \to X \) with \( \phi(\hat{M}) = M \); the latter means that \( \phi \) is \( C^1 \) with \( T_m \phi \) injective and (not necessarily) \( R(T_m \phi) \) closed splitting in \( X \) (i.e. \( R(T_m \phi) \oplus X_m = X \) with \( R(T_m \phi) \) closed for all \( m \in \hat{M} \); see e.g. [AMR88]. (We do not assume \( \phi \) is injective here.) First assume \( \hat{M} \) is boundaryless. Let \( \Pi_m^c \) be the projection associated with \( R(T_m \phi) \oplus X_m = X \) and \( R(\Pi_m^c) = R(T_m \phi) \). Let \( \Lambda_m = \phi^{-1}(m) \) and \( X_m^c = R(T_m \phi), m \in M \). Let \( U_\epsilon \) be the canonical bundle atlas of \( \hat{M} \) in \( X \). Take \( U_m(\epsilon_m) \) as the component of the set \( \phi^{-1}(B_m(\epsilon_m)) \) containing \( m \), where \( m \in \Lambda_m \) and \( B_m(\epsilon_m) = \{ m' : |m' - m| < \epsilon_m \} \). Let

\[
\sup_{m_1, m_2 \in U_m(\epsilon_m)} \frac{\|\phi(m_1) - \phi(m_2) - \Pi_m^c(\phi(m_1) - \phi(m_2))\|}{\|\phi(m_1) - \phi(m_2)\|} \leq r_m.
\]

(C.1)

Set \( U_m^c(\epsilon_m) = \phi(U_m(\epsilon_m)) \) with the metric \( d(m_1, m_2) = d_{m_1}(m_1, m_2) = |m_1 - m_2| \). Since \( r_m \to 0 \) as \( \epsilon_m \to 0 \), we know

\[
\chi_m : U_m^c(\epsilon_m) \to X_m^c, m' \mapsto \Pi_m^c(m' - m),
\]

is a \( C^1 \) diffeomorphism and \( X_m^c(\epsilon'_m) \subset \chi_m(U_m^c(\epsilon_m)) \) for some small \( \epsilon'_m > 0 \). Now \( \{ U_m^c(\epsilon_m) : m \in \Lambda_m, m \in M \} \) is an open cover of \( M \) (endowed with immersed topology), and so \( M \) is a uniformly locally metrizable space (see Definition 5.5). Note that if \( m_i \in U_m(\epsilon_m), m_i = m, i = 1, 2 \), in general we have no metric between \( m_1, m_2 \) though in \( X, |m_1 - m_2| < \epsilon_m \). \( \{ U_m^c(\epsilon_m), \chi_m \} \) gives a \( C^1 \) atlas for \( M \). Let \( M \) be the canonical bundle atlas of \( TM \) induced by \( M \).

Also, a set \( M \in X \) might have different immersed representations; see e.g. Figure 1, where (a) (b) are non-injectively immersed representations with local uniform size neighborhoods at themselves (see Definition 5.7), and (c) is an injectively immersed representation but in this case it does not have a local uniform size neighborhood at itself. If \( \hat{M} \) has boundary \( \partial \hat{M} \), one can consider \( \hat{M}/\partial \hat{M} \) and \( \partial \hat{M} \) separately. But unlike the boundaryless case, \( M \) would look like very non-smooth; e.g. the closure of a homoclinic orbit. Also, note that for a map \( u : M \to M \), even \( u \) is Lipschitz considered as a map in \( X \), it might not be Lipschitz when \( M \) is endowed with the immersed topology.

Example C.3 (immersed manifolds in Banach spaces II). Continue Example C.2. We will identify \( M = \hat{M} \) if we endow the immersed smooth structure for \( M \). Note that the norms \( | \cdot |_m, m \in M \) give the natural Finsler structure for \( TM \). Furthermore, if \( m' \mapsto \Pi_m^c, m' \in U_m^c(\epsilon_m) \) is continuous for each \( m \in \Lambda_m, m \in M \), then \( M \) is a Finsler manifold in the sense of Palais (see Appendix D.2 (d)); in particular now \( M \) satisfies the assumption (H1b) in page 55.

Let \( M_1 \subset M \). Consider the following assumptions.
Let \( \inf_{m_0 \in M_1} \epsilon_{m_0} > 0 \) such that (C.2) holds. This is characterized in [BLZ08] by the following.

There is an \( r_0 > 0 \) such that (a) \( \phi(U_{m_0}(r_0)) \) is closed in \( X \) for all \( m_0 \in \Lambda_{m_0} \), \( m_0 \in M_1 \), and (b) \( \sup_{m_0 \in M_1} \epsilon_{m_0} < 1/2 \) (see (C.1)) if \( \sup_{m_0 \in M_1} \epsilon_{m_0} < r_0 \). That is \( M \) has a local uniform size neighborhood around \( M_1 \) (see Definition 5.7). See also Lemma 5.31.

(2) (almost uniform continuity case). Let (1) hold. There are an \( \varepsilon > 0 \) and a \( \delta > 0 \) such that for each \( m_c \in \Lambda_{m_0} \), \( m_0 \in M_1 \), \( ||\Pi_{m_1} - \Pi_{m_2}|| \leq C_0|m_1 - m_2| \), \( m_1, m_2 \in U_{m_0}(\delta) \). Now \( TM \) has an \( \varepsilon \)-almost uniform \( C^{0,1} \)-fiber trivialization at \( M_1 \) with respect to \( M \) (see Definition 5.23), and \( M \) is \( C^{0,1} \)-uniform around \( M_1 \) (see Definition 5.33).

(3) (Lipschitz continuity case). Let (1) hold. Assume \( m \mapsto \Pi_{m}^c \) is uniformly Lipschitz in the immersed topology in the following sense. There is a constant \( C_0 > 0 \) such that for each \( m_c \in \Lambda_{m_0} \), \( m_0 \in M_1 \), \( ||\Pi_{m_1} - \Pi_{m_2}|| \leq C_0|m_1 - m_2| \), \( m_1, m_2 \in U_{m_0}(\delta) \), (i.e. the assumption (H2) in [BLZ08]). Then \( TM \) has a 0-almost \( C^{0,1} \)-uniform trivialization at \( M_1 \) (see Definition 5.29), and \( M \) is \( C^{1,1} \)-uniform around \( M_1 \) (see Definition 5.33).

For instance, the immersed representations (b) (c) in Figure 1 give \( M \) a \( C^{1,1} \)-uniform property but the immersed representation (d) does not.

**Example C.4 (C^2 compact submanifold).** We assume that in Example C.2 \( \phi \) is a \( C^2 \) injective map and \( M \) is a compact subset of \( X \), i.e. \( M \) is a \( C^2 \) (closed) compact submanifold of \( X \). Then the assumption (3) in Example C.3 is satisfied, which is shown as follows.

Here note that since \( M \) is \( C^2 \) and compact, \( m \mapsto T_m M \in G(X) \) is \( C^1 \); through \( C^1 \) partitions of unity, one can further construct the \( C^1 \) normal bundle over \( M \), i.e. one has \( m \mapsto X^h_m \in G(X) \) is \( C^1 \) such that \( X^h_m \oplus T_m M = X \). Now we have projections \( \Pi_{m_1}^c \), \( m \in M \), associated with \( X^h_m \oplus T_m M = X \) such that \( R(\Pi_{m_1}^c) = T_m M \), and moreover, \( m \mapsto \Pi_{m_1}^c \in L(X, X) \) is \( C^1 \). Then it follows from [BLZ98] that assumption (3) in Example C.3 holds.

\( \phi \in C^2 \) can be relaxed as \( C^1 \) by the smooth approximation (see e.g. [BLZ08, Theorem 6.9] for details), but now \( M \) is strongly \( C^{0,1} \)-uniform in the sense of Definition 5.34.

**Example C.5 (vector bundles over immersed manifolds).** Continue to consider the Example C.3 with (3) hold. Let \( \Pi_m^h \), \( m \in M \) be projections of \( X \) (i.e. \( \Pi_m^h \circ \Pi_m^h = \Pi_m^h \) and \( X^h_m \oplus R(\Pi_m^h) \). Let us consider the following bundle over \( M \),

\[ X^h = \{(m, x) : x \in X^h_m, m \in M \}. \]

The natural Finsler structure in \( X^h \) is given by \( |x|_m = |x|, x \in X^h_m \). The natural bundle atlas \( \mathcal{A} \) for \( X^h \) are given by (formally) \( \varphi^m(m', x) = (m', \Pi_m^h x), m' \in U_{m'}(\epsilon_m), x \in X^h_m, m \in \Lambda_m \); under the following (a), it’s indeed a bundle atlas.
(a) If \( m \mapsto \Pi^h_m \), \( m \in M \) is continuous in the immersed topology of \( M \), i.e. for each \( m \in M \), \( m_0 \in \Lambda_m, m' \mapsto \Pi^h_m, m' \in U^c_{m_0}(e_m) \), is continuous (particularly when \( m \mapsto \Pi^h_m \) is continuous in the topology induced from \( X \) the assumption holds), then \( X^h \) is a \( C^0 \) vector bundle with strongly \( C^0 \) Finsler structure (see Appendix D.2 (a)).

(b) If \( m \mapsto \Pi^h_m \) is uniformly continuous (resp. \( \epsilon \)-almost uniformly continuous) around \( M_1 \) (see Definition 5.4) in the immersed topology of \( M \), then \( X^h \) is \( C^0 \)-uniform around \( M_1 \) (see Definition B.2) (resp. \( X^h \) has an \( \epsilon \)-almost uniform \( C^{1,1} \)-fiber trivialization at \( M_1 \) (see Definition 5.23)) with respect to \( \mathcal{A} \). Similar as (●3) in Example C.3, if \( m \mapsto \Pi^h_m \) is uniformly Lipschitz in the sense of (●3), then \( X^h \) has a \( 0 \)-almost \( C^{0,1} \)-uniform trivialization at \( M_1 \) (see Definition 5.29) with respect to \( \mathcal{A} \).

(c) Suppose for every \( m \in M, m_c \in \Lambda_m, \Pi^h_m|U^c_{m_c}(e_m) \) is differentiable at \( m \), and let \( C_{m_c} = D_m\Pi^h_m|U^c_{m_c}(e_m) \in L(X^c_m \times X^h_m; X^h_m) \). (Note that since \( X^c_m, X^h_m \) are all closed splitting, we can consider \( C_{m_c} \in L(X \times X; X) \).) Assume \( m' \mapsto C_{m'} \), \( m' \in U^c_{m_c}(e_m) \), is continuous, then \( X^h \) is a \( C^1 \) bundle. The readers should notice that \( m \mapsto \Pi^h_m \) might not be differentiable in \( X \) and \( C_{m_c} \neq C_{m'_c} \) when \( m_c, m'_c \in \Lambda_m \) for \( M \) only being immersed. Moreover, \( \{C_{m_c}\} \) gives a natural \( C^0 \) linear connection \( C \) of \( X^h \) such that \( \varphi^{m_c} \) is a normal bundle chart at \( m_c \) with respect to \( C \), i.e. \( \partial \varphi^{m_c}(m_c, x) = \text{id} \).

In addition, if \( m'' \mapsto \Pi^c_m, m' \mapsto \Pi^h_m \) and \( m'' \mapsto C_{m''} \) are both uniformly continuous (see Definition 5.4) in the immersed topology, then \( X^h \) is \( C^1 \)-uniform vector bundle (see Definition B.9); what’s more, if \( m'' \mapsto C_{m''} \) is uniformly (locally) Lipschitz in the immersed topology, then \( X^h \) is \( C^{1,1} \)-uniform vector bundle (see Definition B.9).

C.0.2. bounded geometry. Let \( (M, g) \) be a connected Riemannian manifold modeled on a Hilbert space \( \mathbb{H} \) and equipped with the Levi-Civita connection where \( g \) is a Riemannian metric. Let \( R \) be the curvature tensor induced by the Levi-Civita connection and \( r(m) \) the injectivity radius at \( m \in M \), i.e. the supremum radius for which the exponential map at \( m \) is a diffeomorphism. See e.g. [Kli95] for these basic conceptions.

Definition C.6 (bounded geometry). A \( C^{k+2} \) complete Riemannian manifold \( (M, g) \) has \( k \)-th order bounded geometry if the following hold:

(i) the injectivity radius of \( M \) is positive, i.e. \( r(M) = \inf_{m \in M} r(m) > 0 \);

(ii) the covariant derivatives of \( R \) up to \( k \)-th order and \( R \) are uniformly bounded, i.e.

\[
\sup_{m \in M} \|\nabla^i_m R(m)\| < \infty, \quad 0 \leq i \leq k.
\]

See also [Ama15, DDS16, Eld13, Eic91] for details where the readers can find many concrete examples of Riemannian manifolds having bounded geometry and look into some motivations coming from differential equations and dynamical systems. Note that every \( C^{k+2} \) compact Riemannian manifold has \( k \)-th order bounded geometry. The global definition of bounded geometry does not give a very clear relation with our assumptions. In [Ama15], the author introduced the uniformly regular Riemannian manifolds, and proved that a Riemannian manifold admitting bounded geometry is uniformly regular.

Follow the definition in [Ama15] to introduce the uniformly regular Riemannian manifolds. Take a \( C^k \) atlas \( M = \{(U_\gamma, \varphi_\gamma) : \gamma \in \Lambda \} \) of the \( C^k \) manifold \( M \) where \( \Lambda \) is an index. Take \( S \subset M \) and \( \Lambda_S = \{\gamma \in \Lambda : U_\gamma \cap S \neq \emptyset\} \). We say (i) that \( M \) is normalized on \( S \) if \( \varphi_\gamma(U_\gamma) = \mathbb{B}_1, \gamma \in \Lambda_S \) where \( \mathbb{B}_1 \) is the unit (open) ball of \( \mathbb{H} \), (ii) that \( M \) has finite multiplicity on \( S \) if there is an \( N \in \mathbb{N} \) such that any intersection of \( N + 1 \) elements of \( \{U_\gamma : \gamma \in \Lambda_S\} \) is empty, and (iii) that \( M \) is uniformly shrinkable on \( S \) if there is a \( r \in (0, 1) \) such that \( \{\varphi_\gamma^{-1}(r\mathbb{B}_1) : \gamma \in \Lambda_S\} \) is also an open cover of \( S \).

Definition C.7 (uniformly regular Riemannian manifolds). We say a \( C^{k+1} \) Riemannian manifold \( (M, g) \) is a \( k \)-th order uniformly regular Riemannian manifold on \( S \) if there is a \( C^{k+1} \) atlas \( M = \{(U_\gamma, \varphi_\gamma) : \gamma \in \Lambda \} \) such that the following hold:
(UR1) \( M \) is normalized and uniformly shrinkable and has finite multiplicity on \( S \);
(UR2) \( |\varphi_1 \circ \varphi_\gamma|_{k+1} \leq C(k+1) \), \( \gamma \in \Lambda_S \) where \( C(k+1) \) is a constant;
(UR3) \((\varphi_\gamma^{-1})^*g \sim dH\), i.e. there is \( C \geq 1 \) such that for all \( \gamma \in \Lambda_S \), \( m \in \mathbb{B}_1 \), \( x \in \mathbb{H}, \ C^{-1}|x|^2 \leq ((\varphi_\gamma^{-1})^*g)(m)(x,x) \leq C|x|^2 \), where \((\varphi_\gamma^{-1})^*g \) denotes the pull-back metric of \( g \) by \( \varphi_\gamma^{-1} \).
(UR4) \(|(\varphi_\gamma^{-1})^*g|_k \leq C_1(k) \) for all \( \gamma \in \Lambda_S \) where \( C_1(k) \) is a constant.
If \( S = M \), we also say \( M \) is a \( k \)-th order uniformly regular Riemannian manifold. Here \(|u|_k = \sup_{0 \leq i \leq k} \sup_x |\partial_i^k u(x)|\).

**Example C.8** (0-th and 1-th order uniformly regular Riemannian manifolds). If \( M \) is 0-th order uniformly regular, then \( M \) is a \( C^{0,1} \)-uniform manifold. Indeed, the atlas \( M \) in Definition B.7 provides us the desired atlas needed in assumption (\( \blacksquare \)) (in page 49). By the uniformly shrinkable condition of \( M \), one has an \( r \in (0,1) \) such that \( \{\varphi_\gamma^{-1}(r\mathbb{B}_1) : \gamma \in \Lambda_\} \) is also an open cover of \( M \), and combining (UR3) one gets that there is a \( c \geq 1 \) such that
\[
U_m(\delta/c) \subset \varphi_\gamma^{-1}(\mathbb{B}(x,\delta)) \subset U_m(c\delta),
\]
where \( x = \varphi_\gamma(m) \in r\mathbb{B}_1, \ \delta < 1 - r, \ \mathbb{B}(x,\delta) = x + \delta\mathbb{B}_1; \) see e.g. [DSS16] and Lemma 5.31. Now for any \( m \in M \), choose a map \( \varphi_\gamma \) such that \( \varphi_\gamma(m) = x \in r\mathbb{B}_1 \), and let \( \chi_m(m') = (D\varphi_\gamma(m'))^{-1}(\varphi_\gamma(m') - x) \). (Note that \( \sup, \sup_{x \in \mathbb{B}_1} |D\varphi_\gamma(x)| < \infty \) by (UR3).) By choosing further smaller \( \delta > 0 \), one gets the atlas \( M_1 = \{(U_m(\delta/c), \chi_m) : m \in M \} \) satisfies (\( \blacksquare \)) (in page 49) (with \( M_1 = M \)). Similarly, if \( M \) is a 0-th order uniformly regular Riemannian manifold on \( M_1 \), then \( M \) is \( C^{0,1} \)-uniform around \( M_1 \) (see (\( \blacksquare \)) in page 49) and if \( M \) is a 1-th order uniformly regular Riemannian manifold on \( M_1 \), then \( M \) is \( C^{1,1} \)-uniform around \( M_1 \) (see Definition 5.33); for the latter case, the Levi-Civita connection is uniformly (locally) Lipschitz in the sense of Definition B.8.

**Example C.9** (2-th order bounded geometry). When \( M \) has 2-th order bounded geometry, then \( M \) is a \( C^{0,1} \)-uniform manifold (see Definition 5.33).

The most important thing from the implication \( M \) having \( k \)-th bounded geometry is that there is an \( r_0 > 0 \) such that the Riemannian metric up to \( k \)-th order derivatives and the Christoffel maps in the normal charts (i.e. normal coordinates) up to \((k-1)\)-th order derivatives are all uniformly bounded in normal charts of radius \( r_0 \) around each \( m \in M \) with the bounds independent of \( m \in M \). See e.g. [Eic91] and note that the proof given there also makes sense in the infinite-dimensional setting. Let
\[
\chi_m = \exp^{-1}_m : U_m(r_0) \to \mathbb{B}_{r_0} = T_m(M(r_0) \subset \mathbb{H},
\]
be the normal chart at \( m \in M \), where \( U_m(r_0) = \{m' \in M : d(m,m') < r_0\} \). Now the atlas \( M' \) given by the normal charts satisfies (UR3) (UR4) (for \( k = 2 \)) and (UR2) (for \( \gamma_1 = m', \gamma = m \in \Lambda = M \) with \( d(m',m) < r_0 \) and for \( k = 0 \)). (See also [Eld13, Chapter 2].) So the assumption (\( \blacksquare \)) (in page 49) holds.

The readers can find in [Ama15, DSS16] the equivalence that a \( C^\infty \) Riemannian manifold has any order bounded geometry if and only if it is any order uniformly regular in the setting of \( \mathbb{H} = \mathbb{R}^n \).

**Example C.10** (vector bundles having bounded geometry). Consider a special class of vector bundles, i.e. vector bundle having \( k \)-th order bounded geometry which was introduced in e.g. [Eld13, Page 45] (for finite \( k \)) and [Shu92, Page 65] (for all \( k \in \mathbb{N} \)). Let \( M \) be a Riemannian manifold having \( k \)-th bounded geometry and \( X \) a vector bundle over \( M \). We say \( X \) has \( k \)-th bounded geometry if there exist preferred bundle charts \( \varphi^{m_0} : U_{m_0}(\delta) \times X_{m_0} \to X (\varphi^{m_0}_{m_0} = \text{id}) \), \( m_0 \in M \), such that all the transition maps \( \varphi^{m_0,m_1} (m) = (\varphi^{m_1}_{m_1})^{-1} \circ \varphi^{m_0}_{m} \in L(X_{m_0},X_{m_1}), U_{m_0}(\delta) \cap U_{m_1}(\delta) \neq \emptyset \), satisfy \( \|\varphi^{m_0,m_1}\|_k \leq C_0(k) \), where \( C_0(k) > 0 \) is a constant independent of \( m_0, m_1 \in M \). In particular, \( TM \) has \((k-2)\)-th order bounded geometry if \( M \) has \( k \)-th order bounded geometry. Now vector bundle having \((k,2)\)-th order bounded geometry is \( C^{0,1} \)-uniform (resp. \( C^{1,1} \)-uniform) by definition; see Definition 5.29 and Definition B.9.
We refer the readers to see some similar assumptions of the manifolds and bundles in [Cha04, Section 2.2], [PSW12, Section 8] and [HPS77, Chapter 6]. It seems that J. Elderfirst introduced the bounded geometry to investigate the normal hyperbolicity theory for flows (see [Eld13]); also, the author provided a detailed exposition of bounded geometry. Example C.3 and Example C.5 are essentially due to [BLZ99, BLZ08], where the authors studied the normal hyperbolicity theory in the setting like Example C.3 and Example C.5 for semiflows. No attempt has been made in this paper to develop some intrinsic characterizations (like the one in the definition of bounded geometry) of our assumptions about manifolds and bundles.

D. Appendix. miscellaneous

D.1. a parameter-dependent fixed point theorem. The following well known results are frequently used in this paper. We write here (also the very simple proof) for the convenience of the readers. Some redundant conditions are removed in order for us to apply.

Let $(X, d)$ be a metric space with metric $d$, $Y$ be a topology space, and $G : X \times Y \to X$. We also write $d(x, y) = |x - y|$.

1. Existence of $x(\cdot)$

Lemma D.1. Suppose $\text{Lip}(G, \cdot, y) < 1$, for all $y \in Y$, and $X$ is complete. Then by Banach Fixed Point Theorem, there is a unique $x(y) \in X$, such that

\begin{equation}
G(x(y), y) = x(y).
\end{equation}

2. Continuity of $x(\cdot)$

Lemma D.2. Let $\text{Lip}(G, \cdot, y) < 1$ and $x(\cdot)$ satisfy (D.1). Without assuming $X$ is complete. Then the following hold.

1. If $G(x_2, y_2) = x_2$, $y \mapsto G(x_2, y)$ is continuous at $y_2$, and there is a neighborhood $U_{y_2}$ of $y_2$ in $Y$ such that $\sup_{y \in U_{y_2}} \text{Lip}(G, \cdot, y) < 1$, then $x(\cdot)$ is continuous at $x_2$.

   In particular, if $\sup_{y \in U_{y_2}} \text{Lip}(G, \cdot, y) < 1$ (or $y \mapsto \text{Lip}(G, \cdot, y)$ is continuous) and $G$ is continuous, then $x(\cdot)$ is continuous.

2. $G(x, \cdot)$ is continuous uniform for $x$ in the following sense: $\forall y > 0$, $\forall \epsilon > 0$, $\exists U_y$ a neighborhood of $y$ in $Y$, such that $|G(x, y') - G(x, y)| < \epsilon$ for all $x \in X$, provided $y' \in U_y$. Then $x(\cdot)$ is continuous.

   Note that in this case, we also have $y \mapsto \text{Lip}(G, \cdot, y)$ and $G$ are continuous.

   Similarly, if $G(x, \cdot)$, $x \in X$ are equicontinuous, then $x(\cdot)$ is uniformly continuous, where $Y$ is a uniform space.

3. Let $Y$ be a metric space. If $\text{Lip}(G, \cdot, y) \leq \alpha(x) < \infty$, then $x(\cdot)$ is locally Lipschitz. Furthermore, suppose $\text{Lip}(G, \cdot, y) \leq \beta < 1$ and $\sup_x \alpha(x) < \alpha$, then $|x(y_1) - x(y_2)| \leq \frac{\alpha}{1 - \beta}|y_1 - y_2|$, which yields $x(\cdot)$ is global Lipschitz.

4. Let $Y$ be a metric space. If $\sup_x \text{Lip}(G, \cdot, y) < 1$ and $G(x, \cdot)$ is $\alpha$-Hölder uniform for $x$, i.e. $\sup_x \text{Hol}_a G(x, \cdot) < \infty$, then $x(\cdot)$ is $\alpha$-Hölder.

   Similar case for $G(x, \cdot)$ is $\alpha$-Hölder (or locally $\alpha$-Hölder) uniform for $x \in X$ (or $x$ belonging to any bounded sets of $X$), one can obtain $x(\cdot)$ satisfies one of the following situations: $\alpha$-Hölder, locally $\alpha$-Hölder, or $\alpha$-Hölder on any bounded sets of $Y$.

Proof. Consider

$$|x(y_1) - x(y_2)| \leq |G(x(y_1), y_1) - G(x(y_2), y_1)| + |G(x(y_2), y_1) - G(x(y_2), y_2)|$$

$$\leq |G(\cdot, y_1)(x(y_1) - x(y_2))| + |G(x(y_2), y_1) - G(x(y_2), y_2)|$$

which yields

$$|x(y_1) - x(y_2)| \leq \frac{|G(x(y_2), y_1) - G(x(y_2), y_2)|}{1 - \text{Lip}(\cdot, y_1)}.$$

Then all the results follow from the above inequality. $\square$
3. Differential of $x(\cdot)$

**Lemma D.3.** In order to make sense of the differential of $x(\cdot)$, we first assume $X,Y$ are open sets of normed spaces. If $x(\cdot)$ satisfying (D.1) in a neighborhood of $y$ is continuous at $y$, $G(\cdot,y)$ is differentiable at $(x(y),y)$, and $|D_1G(x(y),y)| < 1$, then $x(\cdot)$ is differentiable at $y$ with $Dx(y) = A(y)$, where

$$A(y) = (I - D_1G(x(y),y))^{-1}D_2G(x(y),y).$$

**Proof.** The proof is standard. Note that $|D_1G(x(y),y)| < 1$, so $A(y)$ is well defined. By the differential of $G$ at $(x(y),y)$, for every $\epsilon > 0$, there is a $\delta' > 0$, such that

$$|G(x',y') - G(x(y),y) - DG(x,y)(x' - x(y),y' - y)| \leq \epsilon(|x' - x(y)| + |y' - y|),$$

provided $|x' - x(y)| + |y' - y| < \delta'$. Moreover, by the continuity of $x(\cdot)$ at $y$, there is a $\delta > 0$, such that $|x(y + a) - x(y)| \leq \delta$, if $|a| \leq \delta$. Thus, if $\epsilon < \frac{1}{2}(1 - |D_1G(x(y),y)|)$ and $|a| \leq \delta$, we have

$$|x(y + a) - x(y) - A(y)a| = |G(x(y + a),y + a) - G(x(y),y) - DG(x,y)y)(A(y)a)|$$

$$\leq |DG(x,y)||x(y + a) - x(y),a) - DG(x,y)y)(A(y)a,a)| + \epsilon(|x(y + a) - x(y)| + |a|)$$

$$\leq |D_1G(x,y)||x(y + a) - x(y) - A(y)a| + \epsilon(|x(y + a) - x(y)| + |a|),$$

which yields

$$|x(y + a) - x(y) - A(y)a| \leq \frac{\epsilon}{1 - |D_1G(x(y),y)|}(|x(y + a) - x(y)| + |a|),$$

and

$$|x(y + a) - x(y)| \leq 2(|A(y)| + 1)|a|.$$

Therefore,

$$\frac{|x(y + a) - x(y) - A(y)a|}{|a|} \leq \frac{\epsilon}{1 - |D_1G(x(y),y)|}(2|A(y)| + 3),$$

which shows $Dx(y) = A(y)$.

Unlike the classical situation, $Y$ does not need to be complete (which is important in Section 6), and $G \in C^1$ is weakened as $G$ being only differentiable at $(x(y),y)$. We apply Lemma D.2 and Lemma D.3 in Section 6 usually in the following way: $x(\cdot)$ is known by global construction and the regularity of $x(\cdot)$ is deduced from locality.

Note that if $\text{Lip}G(\cdot,y) < 1$, then $|D_1G(x(y),y)| < 1$, if $X$ is an open set of a normed space.

**Corollary D.4.** Under $\text{Lip}G(\cdot,y) < 1$ for $y \in Y$ and $X,Y$ being normed spaces, if $G \in C^k (k = 1,2,\cdots, \infty, \omega)$, and $x(\cdot)$ satisfying (D.1) is continuous, then $x(\cdot) \in C^k$. Consider the general case.

**Corollary D.5.** Assume that $X$ is a $C^k$ Finsler manifold (might not be complete) and $Y$ is a $C^k$ Banach manifold. Also suppose $G \in C^k(X \times Y,X)$, $|D_1G(x(y),y)| < 1$. If $x(\cdot)$ satisfying (D.1) is continuous, then it is also $C^k$.

**Proof.** Choose $C^k$ local charts $\varphi : U' \subset U_{x(y)} \to T_{x(y)}X$ and $\phi : V_y \to T_yY$, such that $\varphi(x(y)) = 0$, $\phi(y) = 0$, $G(U',V_y) \subset U_{x(y)}$, and $D\varphi(x(y)) = \text{id}$. Set

$$G'(x',y') = \varphi \circ G(\varphi^{-1}(x'),\phi^{-1}(y')) : \varphi(U') \times \phi(V_y) \to \varphi(U_{x(y)}).$$

Note that $D_1G'(0,0) = D_1G(x(y),y) : T_{x(y)}X \to T_{x(y)}X$. So $|D_1G'(0,0)| < 1$. Let $x'(y') = \varphi \circ x \circ \phi^{-1}(y')$, then

$$G'(x'(y'),y') = x'(y').$$

Therefore, $x(\cdot)$ is $C^k$. □
Corollary D.6. Assume that \( X \) is a \( C^k \) Finsler manifold with a compatible metric \( d \) satisfying for every \( f \in C^1(X, X) \), \( \|Df\| \leq K \operatorname{Lip} f \), where \( K \) is some constant and \( Y \) is a \( C^k \) Banach manifold. Suppose \( G \in C^k(X \times Y, X) \) and \( \operatorname{Lip} G(\cdot, y) < 1 \) for all \( y \in Y \). If \( x(\cdot) \) satisfying (D.1) is continuous, then it is also \( C^k \).

See Appendix D.2 for a situation: \( f \in C^1(X, X) \Rightarrow \|Df\| \leq K \operatorname{Lip} f \).

D.2. Finsler structure in the sense of Neeb-Upmeier. We give some basic definitions of Finsler structure for the convenience of readers. In the infinite-dimensional setting, the following definitions are not equivalent.

(a) We say a \( C^0 \) vector bundle \((X, M, \pi)\) has a \( C^0 \) Finsler structure if there is a \( C^0 \) map \(|(\cdot, \cdot)| : X \to \mathbb{R}_+\), such that \((x, \|x\|) \) is a norm on \( X_m \). Moreover, for every \( m \in M \), there is a \( C^0 \) (vector) bundle chart \((U, \psi)\) at \( m \), such that \( \sup_{m \in U} \|\psi_m\| < \infty \), where the norm of \( \psi_m \) is defined by

\[
\|\psi_m\| = \sup\{\|\psi_m x_m\| : x \in X_m, |x|_m \leq 1\}.
\]

Here \(|(\cdot, \cdot)| \) is in the following sense: for every \( m \in M \) and every \( C^0 \) (vector) bundle chart \((U, \psi)\) at \( m \), \( (m', x) \mapsto |\psi_m x_m| \) is \( C^0 \). We say the Finsler structure is uniformly \( C^0 \) (or \( X \) has a uniformly \( C^0 \) Finsler structure) if for every \( m \in M \) and every \( C^0 \) (vector) bundle chart \((U, \psi)\) at \( m \) such that \( \sup_{m \in U} \|\psi_m\| \leq \Xi, \sup_{m \in U} |\psi_m| \leq \Xi \).

(b) We say a \( C^0 \) vector bundle \((X, M, \pi)\) has a \( C^0 \) uniform Finsler structure (with constant \( \Xi \)) if there is a \( C^0 \) map \(|(\cdot, \cdot)| : X \to \mathbb{R}_+\) such that \((i) x \mapsto |x| \approx |x|_m \) is a norm on \( X_m \), and \( (ii) \) there exists a constant \( \Xi > 0 \) such that for every \( m \in M \), there is a \( C^0 \) (vector) bundle chart \((U, \psi)\) at \( m \) such that \( \sup_{m \in U} \|\psi_m\| \leq \Xi, \sup_{m \in U} |\psi_m| \leq \Xi \).

(c) Let \( X \) be a \( C^0 \) vector bundle \((X, M, \pi)\) with fibers being modeled on \( \mathbb{F} \), i.e. every \( X_m \) is isomorphic to \( \mathbb{F} \) with isomorphism \( x_m \to \mathbb{F} \). We say \( X \) has a \( C^0 \) strong uniform Finsler structure (with constant \( \Xi \)) if there is a \( C^0 \) map \(|(\cdot, \cdot)| : X \to \mathbb{R}_+\) such that \((i) x \mapsto |x| \approx |x|_m \) is a norm on \( X_m \), and \( (ii) \) there exists a constant \( \Xi > 0 \) such that for every \( m \in M \), there is a \( C^0 \) (vector) bundle chart \((U, \psi)\) at \( m \) such that \( \sup_{m \in U} \|\psi_m\| \leq \Xi, \sup_{m \in U} |\psi_m| \leq \Xi \).

(d) A \( C^k \) Banach manifold \( M \) is called a \( C^k \) Finsler-(Banach) manifold (in the sense of Neeb-Upmeier) if there is a \( C^0 \) Finsler structure in \( TM \). Moreover it is called a \( C^k \) Finsler manifold in the sense of Neeb-Upmeier \( \Xi \)-weak uniform, if there is a \( C^0 \) uniform Finsler structure (with constant \( \Xi \)) in \( TM \).

Finally, it is called a \( C^k \) Finsler manifold in the sense of Palais (see [Pal66]) if the \( C^0 \) Finsler structure is uniformly \( C^0 \); or equivalently, for any constant \( \Xi > 1 \), and for every \( m \in M \), there is a \( C^0 \) (vector) bundle chart \((U, \psi)\) at \( m \), such that \( \sup_{m \in U} \|\psi_m\| \leq \Xi |\psi_m|, \sup_{m \in U} |\psi_m| \leq \Xi |\psi_m| \). Note that any Riemannian manifolds belong to this case.

(e) Let \( M \) be a \( C^k \) Banach manifold modeled on \( \mathbb{E} \). Note that in this case, \( TM \) is modeled on \( \mathbb{E} \times \mathbb{E} \). \( M \) is called a \( C^k \) Finsler manifold in the sense of Neeb-Upmeier \( \Xi \)-uniform if there is a \( C^0 \) strong uniform Finsler structure (with constant \( \Xi \)) in \( TM \).

(f) The Finsler metric in Finsler manifolds can be defined as usual way; see e.g. [Upm85]. Let \( M \) be a \( C^1 \) Finsler manifold. The associated Finsler metric \( d_M \) in each component of \( M \) is defined by

\[
d_M(p, q) = \inf \left\{ \int_0^t |c'(t)|_{c(t)} \, dt : c \in C^1[0, 1] \to M \text{ such that } c(0) = p, c(1) = q \right\}.
\]

For any \( f \in C^1(M, N) \), note that the following basic facts hold. If \( M, N \) are Finsler manifolds in the sense of Palais, then \( \sup_m |Df(m)| = \operatorname{Lip} f \) (see e.g. [GJ07]); if \( M, N \) are Finsler manifolds in the sense of Neeb-Upmeier, then one always has \( \operatorname{Lip} f \leq \sup_m |Df(m)| \); if \( M, N \) are Finsler manifolds in the sense of Neeb-Upmeier \( \Xi_1 \)-weak uniform and \( \Xi_2 \)-weak uniform respectively, then \( \sup_m |Df(m)| \leq \Xi_1 \Xi_2 \operatorname{Lip} f \); if \( M, N \) are Finsler manifolds in the sense of Neeb-Upmeier \( \Xi_1 \)-uniform and \( \Xi_2 \)-uniform respectively, then \( \sup_m |Df(m)| \leq \Xi_1^2 \Xi_2 \operatorname{Lip} f \). For a proof of the latter three results, see [JSSG11].
D.3. **length metric and Lipschitz continuity.** In this appendix, we discuss some conditions such that ‘local Lipschitz (Hölder) continuity’ might imply ‘global Lipschitz (Hölder) continuity’.

Let $M$ be a metric space with metric $d$. A path $s$ from $x$ to $y$ in $M$ is a continuous map $s : [a, b] \to M$ such that $s(a) = x$, $s(b) = y$, and its length is defined by

$$ l(s) = \sup \left\{ \sum_{i=1}^{k} d(s(t_{i-1}), s(t_i)) : a = t_0 < t_1 < \cdots < t_k = b, k = 1, 2, 3, \cdots \right\}. $$

If $l(s) < \infty$, then $s$ is called a rectifiable path (or bounded variation path). If for every $x, y \in M$,

$$ d(x, y) = \inf \{l(s) : s \text{ is a path from } x \text{ to } y \}, $$

then $M$ is said to be a length space and $d$ a length metric. Some examples of length space are following (see [GJ07] for details): (a) normed spaces, and any convex sets of normed spaces; (b) connected Finsler manifolds in the sense of Palais (see Appendix D.2) with Finsler metrics, in particular, connected Riemannian manifolds with Riemannian metrics; (c) geodesic spaces.

Let $M, N$ be two metric spaces, and $f : M \to N$. Define the **upper scalar derivative** of $f$ at $x$ by (see [GJ07])

$$(D)f_{x} = \limsup_{y \to x} \frac{d(f(y), f(x))}{d(y, x)}.$$  

If $f : M \to N$ is continuous, $s$ is a rectifiable path in $M$, and $t = f \circ s$, then (see [GJ07, Proposition 3.8])

$$ l(t) \leq \sup_{s \in \text{Im}(s)} D_{x}^{+} f \cdot l(s). $$

Particularly, we have the following property.

**Lemma D.7.** If $M, N$ are two length spaces, $\sup_{x} D_{x}^{+} f < \infty$, then $f$ is Lipschitz with $\text{Lip } f = \sup_{x} D_{x}^{+} f$.

See [GJ07] for more properties of upper scalar derivatives. Let us consider the Hölder case. Define the **upper scalar uniform $\alpha$-Hölder constant** of $f$ by

$$ D_{x}^{\alpha} f = \limsup_{r \to 0} \sup_{d(z, x) < r} \frac{d(f(z), f(x))}{d(z, x)^{\alpha}}, $$

where $0 < \alpha \leq 1$. Now we have the following.

**Lemma D.8.** If $M, N$ are length spaces, $D_{x}^{\alpha} f < \infty$, then for any given bounded set $A$ of $X$,

$$ d(f(x), f(y)) \leq \Omega(\text{diam}(A))(D_{x}^{\alpha} f + 1)d(x, y)^{\alpha}, \quad \forall x, y \in A, $$

where $\Omega : \mathbb{R}_{+} \to \mathbb{R}_{+}$.

In contrast to $D_{x}^{+} f$, $D_{x}^{\alpha} f$ is defined uniformly, so the proof is simpler than the Lipschitz case.

**Proof.** Since $D_{x}^{\alpha} f \triangleq M_0 < \infty$, there is a $\delta > 0$, such that $d(f(z), f(x)) \leq (M_0 + 1)d(z, x)^{\alpha}$, provided $d(z, x) \leq \delta$. Let $x, y \in A$, $n = [\frac{\text{diam}(A)}{\delta}] + 1$. For any given $\varepsilon > 0$, there is a path $s : [a, b] \to X$ from $x$ to $y$ such that $l(s) < d(x, y) + \varepsilon$ and $d(x, y) + \varepsilon \leq n\delta$. Now one can choose $a = t_0 < t_1 < \cdots < t_m = b$, $m \leq n$, such that $d(s(t_{i-1}), s(t_i)) = \delta$, $i = 1, 2, \cdots, m - 1$, and $d(s(t_{m-1}), s(t_m)) \leq \delta$. Let $r = f \circ s$. Then $d(r(t_{i-1}), r(t_i)) \leq (M_0 + 1)d(s(t_{i-1}), s(t_i))^{\alpha}$. Hence,

$$ d(f(x), f(y)) \leq \sum_{i=1}^{m} d(r(t_{i-1}), r(t_i)) \leq (M_0 + 1) \sum_{i=1}^{m} d(s(t_{i-1}), s(t_i))^{\alpha} \leq (M_0 + 1)n^{1-\alpha}(d(x, y) + \varepsilon)^{\alpha}, $$

Note that in the finite-dimensional setting, Finsler structure being $C^0$ is equivalent to being uniformly $C^0$; particularly, the definitions given in (a) ~ (c) are identical. The definitions of Finsler manifolds in the sense of Neeb-Upmeier are taken from [JSSG11]; see also [Pal66, Upm85, Nee02].
where Hölder inequality is used. Since $\epsilon$ is arbitrary, we have $d(f(x), f(y)) \leq (M_0 + 1)n^{1-\alpha}d(x, y)^\alpha$.

One can define the point-wise version of upper scalar $\alpha$-Hölder constant of $f$ at $x$ by

$$D^+_{\alpha,x}f = \limsup_{z \to x} \frac{d(f(z), f(x))}{d(z, x)\alpha}.$$  

However, $\sup_x D^+_{\alpha,x}f < \infty$ does not imply the $\alpha$-Hölderness of $f$ even in compact metric spaces, as the following example shown. Let $f(x) = x \sin^{-1}x$, if $x \in (0, 1]$, and $f(0) = 0$. One can easily verify that for any $0 < \alpha < 1$, $D^+_{\alpha,x}f = 0$ for all $x \in [0, 1]$. However, $\text{H}o\text{l}_{\alpha} f\|_{[0,1]} = \infty$, if $\alpha > \frac{1}{2}$. To see this, consider $x_n = \frac{1}{2n\pi}, y_n = \frac{1}{2n\pi + \frac{\pi}{2}}$.

**Lemma D.9.** If $M, N$ are length spaces and $f : M \to N$ is uniformly continuous, then for any given bounded set $A$ of $X$, we have $\text{diam}(f(A)) \leq \Omega(\text{diam}(A))$, where $\Omega : \mathbb{R_+} \to \mathbb{R_+}$.

**Proof.** Since $f : M \to N$ is uniformly continuous, there is a $\delta > 0$, such that $d(f(z), f(x)) \leq 1$, provided $d(z, x) \leq \delta$. Let $x, y \in A$, $n = \lceil \frac{\text{diam}(A)}{\delta} \rceil + 1$. The same argument in Lemma D.8 shows that $d(f(x), f(y)) \leq n$.

**D.4. construction normal bundle charts from given bundle charts.** Consider the construction of a normal bundle chart from a given bundle chart $(V, \phi)$ at $m_0$. Locally, this is quite easy as the following shown in formal:

$$\psi_m(x) = \phi_m(x) - \nabla_{m_0} \phi_{m_0}(x)(m - m_0).$$

We need to give the explicit meaning of above expression. It is essentially the same as the construction of parallel translation for the vector bundles, which we give briefly in the following.

Let $M$ be a $C^1$ manifold, and $(Y, N, \pi_2)$ a $C^1$ bundle with $C^0$ connection $C$. Let $f : M \to Y$ be a $C^1$ bundle map over $u$, i.e. $f(m) \in Y_{u(m)}$. Take $\phi : U \times Y_{m_0} \to Y$ as a $C^1$ bundle chart and $\varphi = \phi^{-1}$. We say $f$ is parallel (with respect to $C$) if $\nabla_{du} f(m) = 0$ for all $m \in M$. Consider its meaning in the local bundle chart. Let $\Gamma_{(m', x')}$ be the Christoffel map in the local bundle chart $\phi$. Set $g(m) = \varphi_{u(m)} f(m) : M \to Y_{m_0}$. Now by the chain rule, we have

$$Dg(m) = \nabla_{u(m)} \varphi_{u(m)}(f(m)) Du(m) = -D\varphi_{u(m)} g(m)) \Gamma_{(u(m), g(m))} Du(m),$$

Set $\tilde{\Gamma}_{(m, x)} = D\varphi_m(x) \Gamma_{(m, x)}$.

Let us construct the normal bundle chart from $\phi$. The construction will lose the smoothness. Consider a special case. Using the local chart of $N$ at $n_0$, we can identify $U = T_{n_0} N$. Assume $M = [-2, 2], u(t) = (1 - t)n_0 + tn, n \in U$. For any $x \in Y_{n_0}, n \in U$, we will show that there is a unique $f(t) \in Y_{u(t)}$ such that $f(0) = x$, $f'(t) = \nabla_{u(t)} f(t) u(t) = 0, t \in M$, once we assume $x \mapsto \tilde{\Gamma}_{(m, x)}$ is locally Lipschitz. (Note that $U$ might be smaller around $n_0$ depending on $x$ due to $\tilde{\Gamma}_{(m, x)}$ being locally Lipschitz). For this case $g(t) = \varphi_{u(t)}(f(t))$. Now $f'(t) = 0$ is equivalent to

$$g'(t) = -\tilde{\Gamma}_{(u(t), g(t))}(n - n_0).$$

The above differential equation in $Y_{n_0}$ has a unique solution $g$ with $g(0) = x$. Take

$$\psi_n(x) = f(1) : U \times Y_{n_0} \to Y_n.$$  

Next we show $\psi_n$ is the desired normal bundle chart. Note that $f(t) = \psi_{u(t)}(x)$. Since $f'(t) = 0$, in particular, one gets $\nabla_{u(t)} \psi_{u(t)}(x)(n - n_0) = 0$ for all $n \in U$. For $t = 0$, it follows $\nabla_{n_0} \psi_{n_0}(x)(n - n_0) = 0, \text{'yielding'} \nabla_{n_0} \psi_{n_0}(x) = 0$. Since $C$ is only locally Lipschitz, $x \mapsto \psi_n(x)$ is also only locally Lipschitz. In general, if $\phi \in C^{k+1}$ and $C \in C^k$, then $\psi \in C^k$. By the construction, if $Y$ is a vector bundle, $\phi$ is a vector bundle chart and $C$ is a linear connection, then $\psi$ is also a vector bundle chart. Combining the above argument, we obtain the following.
Lemma D.10 (Existence of the normal bundle chart). Let $M$ be a $C^2$ manifold and $M_1 \subset M$. Let $(X, M, \pi)$ be a $C^2$ bundle with $C^2$ bundle atlas $\mathcal{A}$ and $C^1$ connection $C$. (We can further assume each fiber of $X$ is a Banach space or open subset of a Banach space for simplicity.)

1. If $X$ is a vector bundle, and $C$ is a linear connection, and $(U, \phi) \in \mathcal{A}$ at $m_0$ is a vector bundle chart, then there is an open $V \subset U$, and $\psi : V \times X_{m_0} \to X$ such that $(V, \psi)$ is a $C^1$ vector normal bundle chart at $m_0$.

2. Take $(U_{m_0}, \phi^{m_0}) \in \mathcal{A}$ at $m_0$, $m_0 \in M_1$. Suppose $\chi^{m_0} : U_{m_0} \to T_{m_0}M$ is a $C^2$ chart with $\chi(m_0) = 0$ and $V_{m_0} = X^{m_0}(T_{m_0}M(\epsilon_{m_0})) \subset U_{m_0}$, where $\epsilon_{m_0} > 0$. Assume the connection $C$ is uniformly Lipschitz around $M_1$ in the following sense:

$$\text{Lip} \chi_{m_0}^{m_0} \leq C_{m_0} < \infty, \ m \in V_{m_0},$$

where $C_{m_0} > 0$ is a constant independent of $m \in V_{m_0}$. $\chi_{m_0}^{m_0}$ is the Christoffel map of $C$ in $\phi^{m_0}$. Now there are $C^1$, $\psi^{m_0} : V_{m_0} \times X_{m_0} \to X$, $m_0 \in M_1$ such that the $C^1$ normal bundle charts at $M_1$, denoted by $\mathcal{A}_{m_0}$. Moreover, if $X$ has an $\epsilon$-almost uniform $C^{0,1}$ fiber trivialization at $M_1$ with respect to $\mathcal{A}$ (see Section 5.4.5), and $\sup_{m_0 \in M_1} C_{m_0} < \infty$, $\sup_{m_0 \in M_1} \epsilon_{m_0} > 0$, then $X$ also has a $C^\epsilon$-almost uniform $C^{0,1}$ fiber trivialization at $M_1$ with respect to $\mathcal{A}_{m_0}$ where $C > 0$.

Proof. All we need is to consider (D.3). Also, note that since $C$ is $C^1$, $\phi$ is $C^2$, one indeed has that $\psi$ is $C^3$, so the covariant derivative of $\psi$ exists.

D.5. bump function and blid map. A bump function on a Banach space $X$ is a function with bounded nonempty support on $X$. The existence of a special bump function will indicate a special geometrical property of $X$. As usual, one uses the bump function to truncate a map. The Lipschitz bump function always exists in every Banach space. In order to apply our regularity results in Section 6 to the local case, a $C^{0,1} \cap C^1$ or $C^{1,1}$ bump function is needed (see e.g. Theorem 7.12 and Theorem 7.13). In general Banach spaces, such bump functions may not exist, but some Banach spaces may have which we list in the following for sake of the readers and refer the readers to see [DZ93] for details.

- If a Banach space has a $C^1$ (resp. $C^{1,1}$) norm away from the origin, then it admits a $C^{0,1} \cap C^1$ (resp. $C^{1,1}$) bump function. For example (i) any Banach space $X$ with separable dual have a $C^1$ norm away from the origin; (ii) the Hilbertian norm in each Hilbert space is $C^\infty \cap C^{0,1}$ norm away from the origin; (iii) the canonical norms on the functions spaces $L^p(\mathbb{R}^n)$ (or $L^p(\mathbb{R}^n)$), $1 \leq p < \infty$, are $C^\infty \cap C^{0,1}$ away from 0 if $p$ is even, $C^{p-1,1}$ if $p$ is odd, and $C^{[p],p-[p]}$ if $p$ is not an integer.

The smooth bump functions do not exist in the continuous space $C[0,1]$, but this space sometimes is important in the study of differential equations such as delay equations and reaction-diffusion equations. In order to avoid using bump function, follow the definition in [BR17] to introduce the blid map.

- A $C^{k,\alpha}$ blid map for a Banach space $X$ is a global bounded local identity at zero $C^{k,\alpha}$ map $b_x : X \to X$ where $b_x(x) = x$ if $x \in X(e)$.

Obviously, (i) any Banach space admitting $C^{k,\alpha}$ bump functions also possesses $C^{k,\alpha}$ blid map. (ii) $C(\Omega)$ admits $C^{k,1}$ blid map ($k = 0, 1, \ldots, \infty$), where $\Omega$ is a compact Hausdorff space; this is constructed in [BR17] as $b_x(x)(t) = h(x(t))x(t), \ t \in \Omega$

where $h : [0,1] \to C(\Omega)$ is a $C^\infty$ bump function in $\mathbb{R}$ with $h(t) = t$ if $|t| \leq e$. (iii) If $X$ is a complemented in $C(\Omega)$ or is an ideal of $C(\Omega)$, then $X$ has a $C^{k,1}$ blid map.

The $C^{k,\alpha}$ blid maps play the same role as bump functions to truncate a map; for more details, see [BR17]. So in this paper, one can use $C^{k,\alpha}$ blid maps instead of $C^{k,\alpha}$ bump functions where needed (particularly Theorem 7.12 and Theorem 7.13), which makes our results also hold in the space $X$ listed in (i) $\sim$ (iii).
REFERENCES

[Ama15] H. Amann, Uniformly regular and singular Riemannian manifolds, Elliptic and parabolic equations, 2015, pp. 1–43. MR3375165

[AR88] R. Abraham, J. E. Marsden, and T. Ratiu, Manifolds, tensor analysis, and applications, Second, Applied Mathematical Sciences, vol. 75, Springer-Verlag, New York, 1988. MR960687

[ASV13] A. Avila, J. Santamaria, and M. Viana, Holonomy invariance: rough regularity and applications to Lyapunov exponents, Astérisque \textbf{358} (2013), 13–74. MR3203216

[AV10] A. Avila and M. Viana, Extremal Lyapunov exponents: an invariance principle and applications, Invent. Math. \textbf{181} (2010), no. 1, 115–189. MR2651382

[BC16] C. Bonatti and S. Crovisier, Center manifolds for partially hyperbolic sets without strong unstable connections, J. Inst. Math. Jussieu \textbf{15} (2016), no. 4, 785–828. MR3569077

[BJ89] P. W. Bates and C. K. R. T. Jones, Invariant manifolds for semilinear partial differential equations, Dynamics reported, Vol. 2, 1989, pp. 1–38. MR1009074

[BLZ00] P. W. Bates, K. Lu, and C. Zeng, Invariant foliations near normally hyperbolic invariant manifolds for semiflows, Trans. Amer. Math. Soc. \textbf{352} (2000), no. 10, 4641–4676. MR1675237

[BLZ08] P. W. Bates, K. Lu, and C. Zeng, Approximately invariant manifolds and global dynamics of spike states, Invent. Math. \textbf{174} (2008), no. 2, 355–433. MR2439610

[BLZ98] P. W. Bates, K. Lu, and C. Zeng, Persistence of overflowing manifolds for semiflow, Comm. Pure Appl. Math. \textbf{52} (1999), no. 8, 983–1046. MR1686965

[BR17] G. Belitskii and V. Rayskin, A New Method of Extension of Local Maps of Banach Spaces. Applications and Examples, ArXiv e-prints (June 2017), available at arXiv:1706.08513.

[BW10] K. Burns and A. Wilkinson, On the ergodicity of partially hyperbolic systems, Ann. of Math. (2) \textbf{171} (2010), no. 1, 451–489. MR2630044

[BY17a] A. Blumenthal and L.-S. Young, Absolute continuity of stable foliations for mappings of Banach spaces, Comm. Math. Phys. \textbf{354} (2017), no. 2, 591–619. MR3663618

[BY17b] A. Blumenthal and L.-S. Young, Entropy, volume growth and SRB measures for Banach space mappings, Invent. Math. \textbf{207} (2017), no. 2, 833–893. MR3595337

[Cha02] M. Chaperon, Invariant manifolds revisited, Tr. Mat. Inst. Steklova \textbf{236} (2002), no. Diff. Uravn. i Din. Sist., 428–446. MR1931043

[Cha04] M. Chaperon, Stable manifolds and the Perron-Irwin method, Ergodic Theory Dynam. Systems \textbf{24} (2004), no. 5, 1359–1394. MR2104589

[Cha08] M. Chaperon, The Lipschitzian core of some invariant manifold theorems, Ergodic Theory Dynam. Systems \textbf{28} (2008), no. 5, 1419–1441. MR2449535

[Che18a] D. Chen, Existence and regularity of invariant graphs for cocycles in bundles: non-uniformly partial hyperbolicity case, 2018, in preparation.

[Che18b] D. Chen, Invariant manifolds of approximately normally hyperbolic manifolds in Banach spaces, 2018. submitted.

[Che18c] D. Chen, Invariant manifolds of partially normally hyperbolic invariant manifolds in Banach spaces, 2018. in preparation.

[Che18d] D. Chen, The exponential dichotomy and invariant manifolds for some classes of differential equations, 2018. submitted.

[CHT97] X.-Y. Chen, J. K. Hale, and B. Tan, Invariant foliations for $C^1$ semigroups in Banach spaces, J. Differential Equations \textbf{139} (1997), no. 2, 283–318. MR1472350

[CL88] S.-N. Chow and K. Lu, Invariant manifolds for flows in Banach spaces, J. Differential Equations \textbf{74} (1988), no. 2, 285–317. MR952900

[CL97] C. Chicone and Y. Latushkin, Center manifolds for infinite-dimensional nonautonomous differential equations, J. Differential Equations \textbf{141} (1997), no. 2, 356–399. MR1488358

[CL99] C. Chicone and Y. Latushkin, Evolution semigroups in dynamical systems and differential equations, Mathematical Surveys and Monographs, vol. 70, American Mathematical Society, Providence, RI, 1999. MR1707332

[CL98] S.-N. Chow, X.-B. Lin, and K. Lu, Smooth invariant foliations in infinite-dimensional spaces, J. Differential Equations \textbf{94} (1991), no. 2, 266–291. MR1137616

[CLY00a] S.-N. Chow, W. Liu, and Y. Yi, Center manifolds for invariant sets, J. Differential Equations \textbf{168} (2000), no. 2, 355–385. Special issue in celebration of Jack K. Hale’s 70th birthday, Part 2 (Atlanta, GA/Lisbon, 1998). MR1808454

[CLY00b] S.-N. Chow, W. Liu, and Y. Yi, Center manifolds for smooth invariant manifolds, Trans. Amer. Math. Soc. \textbf{352} (2000), no. 11, 5179–5211. MR1650077

[Cot11] É. Cotton, Sur les solutions asymptotiques des équations différentielles, Ann. Sci. École Norm. Sup. (3) \textbf{28} (1911), 473–521. MR1509144
[Kin68] J. F. C. Kingman, *The ergodic theory of subadditive stochastic processes*, J. Roy. Statist. Soc. Ser. B 30 (1968), 499–510. MR0254907

[Kli95] W. P. A. Klingenberg, *Riemannian geometry*, Second, De Gruyter Studies in Mathematics, vol. 1, Walter de Gruyter & Co., Berlin, 1995. MR1330918

[KP90] U. Kirchgraber and K. J. Palmer, *Geometry in the neighborhood of invariant manifolds of maps and flows and linearization*, Pitman Research Notes in Mathematics Series, vol. 233, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1990. MR1068954

[Lia47] A. Liapounoff, *Problème Général de la Stabilité du Mouvement*, Annals of Mathematics Studies, no. 17, Princeton University Press, Princeton, N. J.; Oxford University Press, London, 1947. MR0021186

[LL10] Z. Lian and K. Lu, *Lyapunov exponents and invariant manifolds for random dynamical systems in a Banach space*, Mem. Amer. Math. Soc. 206 (2010), no. 967, vi+106. MR2674952

[LLSY16] M. Y. Li, W. Liu, C. Shan, and Y. Yi, *Turning points and relaxation oscillation cycles in simple epidemic models*, SIAM J. Appl. Math. 76 (2016), no. 2, 663–687. MR3477765

[LP08] Y. Latushkin and A. Pogan, *The dichotomy theorem for evolution bi-families*, J. Differential Equations 245 (2008), no. 2, 2267–2306. MR2446192

[Lu91] K. Lu, A *Hartman-Grobman theorem for scalar reaction-diffusion equations*, J. Differential Equations 93 (1991), no. 2, 364–394. MR1125224

[LYZ13] Z. Lin, L.-S. Young, and C. Zeng, *Absolute continuity of stable foliations for systems on Banach spaces*, J. Differential Equations 254 (2013), no. 1, 283–308. MR2983052

[Li81] R. Mañé, *Lyapounov exponents and stable manifolds for compact transformations*, Geometric dynamics (Rio de Janeiro, 1981), 1983, pp. 522–577. MR730286

[MPS88] J. Mallet-Paret and G. R. Sell, *Inertial manifolds for reaction diffusion equations in higher space dimensions*, J. Amer. Math. Soc. 1 (1988), no. 4, 805–866. MR943276

[MR09] P. Magal and S. Ruan, *Center manifolds for semilinear equations with non-dense domain and applications to Hopf bifurcation in age structured models*, Mem. Amer. Math. Soc. 202 (2009), no. 951, vi+71. MR2559965

[Ne02] K.-H. Neeb, *A Cartan-Hadamard theorem for Banach-Finsler manifolds*, Proceedings of the Conference on Geometric and Combinatorial Group Theory, Part II (Haifa, 2000), 2002, pp. 115–156. MR1950888

[NP00] R. Nagel and J. Poland, *The critical spectrum of a strongly continuous semigroup*, Adv. Math. 152 (2000), no. 1, 120–133. MR1762122

[Pal66] R. S. Palais, *Lusternik-Schnirelman theory on Banach manifolds*, Topology 5 (1966), 115–132. MR0259955

[Per29] O. Perron, *über Stabilität und asymptotisches Verhalten der Integrale von Differentialgleichungssystemen*, Math. Z. 29 (1929), no. 1, 129–160. MR1544998

[Pe77] J. Besicovitch, *Characteristic Ljapunov exponents, and smooth ergodic theory*, Uspehi Mat. Nauk 32 (1977), no. 4 (196), 55–112, 287. MR0466791

[Pl64] V. A. Pliss, *A reduction principle in the theory of stability of motion*, Izv. Akad. Nauk SSSR Ser. Mat. 28 (1964), 1297–1324. MR0190449

[PS70] C. Pugh and M. Shub, *Linearization of normally hyperbolic diffeomorphisms and flows*, Invent. Math. 10 (1970), 187–198. MR0283825

[PS97] C. Pugh and M. Shub, *Stably ergodic dynamical systems and partial hyperbolicity*, J. Complexity 13 (1997), no. 1, 125–179. MR1446975

[PSW00] C. Pugh, M. Shub, and A. Wilkinson, *Correction to: “Hölder foliations” [Duke Math. J. 86 (1997), no. 3, 517–546; MR1432307 (97m:58155)], Duke Math. J. 105 (2000), no. 1, 105–106. MR1788044

[PSW12] C. Pugh, M. Shub, and A. Wilkinson, *Hölder foliations, revisited*, J. Mod. Dyn. 6 (2012), no. 1, 79–120. MR2929134

[PSW97] C. Pugh, M. Shub, and A. Wilkinson, *Hölder foliations*, Duke Math. J. 86 (1997), no. 3, 517–546. MR1432307

[RHRHTU12] F. Rodriguez Hertz, M. A. Rodriguez Hertz, A. Tahzibi, and R. Ures, *Maximizing measures for partially hyperbolic systems with compact center leaves*, Ergodic Theory Dynam. Systems 32 (2012), no. 2, 825–839. MR2901373

[Rue82] D. Ruelle, *Characteristic exponents and invariant manifolds in Hilbert space*, Ann. of Math. (2) 115 (1982), no. 2, 234–290. MR647807

[Shu92] M. A. Shubin, *Spectral theory of elliptic operators on noncompact manifolds*, Astérisque 207 (1992), 5, 35–108. Méthodes semi-classiques, Vol. 1 (Nantes, 1991). MR1205177

[SS01] B. Sandstede and A. Scheel, *On the structure of spectra of modulated travelling waves*, Math. Nachr. 232 (2001), 39–93. MR1871473

[SS94] R. J. Sacker and G. R. Sell, *Dichotomies for linear evolutionary equations in Banach spaces*, J. Differential Equations 113 (1994), no. 1, 17–67. MR1296160

[SS99] B. Sandstede and A. Scheel, *Essential instability of pulses and bifurcations to modulated travelling waves*, Proc. Roy. Soc. Edinburgh Sect. A 129 (1999), no. 6, 1263–1290. MR1728529
[Sta99] J. Stark, Regularity of invariant graphs for forced systems, Ergodic Theory Dynam. Systems 19 (1999), no. 1, 155–199. MR1677161

[Tak71] F. Takens, Partially hyperbolic fixed points, Topology 10 (1971), 133–147. MR0307279

[Upm85] H. Upmeier, Symmetric Banach manifolds and Jordan $C^*$-algebras, North-Holland Mathematics Studies, vol. 104, North-Holland Publishing Co., Amsterdam, 1985. Notas de Matemática [Mathematical Notes], 96. MR776786

[Via08] M. Viana, Almost all cocycles over any hyperbolic system have nonvanishing Lyapunov exponents, Ann. of Math. (2) 167 (2008), no. 2, 643–680. MR2415384

[VvG87] A. Vanderbauwhede and S. A. van Gils, Center manifolds and contractions on a scale of Banach spaces, J. Funct. Anal. 72 (1987), no. 2, 209–224. MR886811

[Wil13] A. Wilkinson, The cohomological equation for partially hyperbolic diffeomorphisms, Astérisque 358 (2013), 75–165. MR3203217

[Yi93] Y. Yi, A generalized integral manifold theorem, J. Differential Equations 102 (1993), no. 1, 153–187. MR1209981

[Zel14] S. Zelik, Inertial manifolds and finite-dimensional reduction for dissipative PDEs, Proc. Roy. Soc. Edinburgh Sect. A 144 (2014), no. 6, 1245–1327. MR3283067

[ZZJ14] W. Zhang, W. Zhang, and W. Jarczyk, Sharp regularity of linearization for $C^{1,1}$ hyperbolic diffeomorphisms, Math. Ann. 358 (2014), no. 1-2, 69–113. MR3157992

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