CERTAIN LOCALLY NILPOTENT VARIETIES OF GROUPS

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Abstract. Let \( c \geq 0, d \geq 2 \) be integers and \( \mathcal{N}^{(d)}_c \) be the variety of groups in which every \( d \)-generator subgroup is nilpotent of class at most \( c \). N.D. Gupta posed this question that for what values of \( c \) and \( d \) it is true that \( \mathcal{N}^{(d)}_c \) is locally nilpotent? We prove that if \( c \leq 2^d + 2^{d-1} - 3 \) then the variety \( \mathcal{N}^{(d)}_c \) is locally nilpotent and we reduce the question of Gupta about the periodic groups in \( \mathcal{N}^{(d)}_c \) to the prime power finite exponent groups in this variety.

1. Introduction and results

Let \( c \geq 0, d \geq 2 \) be integers and \( \mathcal{N}_c \) be the variety of nilpotent groups of class at most \( c \). We denote by \( \mathcal{N}^{(d)}_c \) the variety of groups in which every \( d \)-generator subgroup is in \( \mathcal{N}_c \). In [2], Gupta posed the following question:

For what values of \( c \) and \( d \) it is true that \( \mathcal{N}^{(d)}_c \) is locally nilpotent?

Then he proved that for \( c \leq (d^2 + 2d - 3)/4 \), the variety \( \mathcal{N}^{(d)}_c \) is locally nilpotent. In [1], Endimioni improved the latter result where he proved that for \( c \leq 2^d - 2 \), the variety \( \mathcal{N}^{(d)}_c \) is locally nilpotent. Here we improve the number \( 2^d - 2 \) to \( 2^d + 2^{d-1} - 3 \). In fact we prove:

**Theorem 1.1.**

1. For \( c \leq 2^d + 2^{d-1} - 3 \), the variety \( \mathcal{N}^{(d)}_c \) is locally nilpotent.

2. For \( c \leq 2^d + 2^{d-1} + 2^{d-2} - 3 \), every \( p \)-group in the variety \( \mathcal{N}^{(d)}_c \) is locally nilpotent, where \( p \in \{2, 3, 5\} \).

Note that the variety \( \mathcal{N}^{(2)}_c \) is contained in the variety of \( c \)-Engel groups and it is yet unknown whether every \( c \)-Engel group is locally nilpotent, even, so far there is no published example of a non-locally nilpotent group in the variety \( \mathcal{N}^{(2)}_c \). In the last section of this paper, by considering the problem of locally nilpotency of the variety \( \mathcal{N}^{(2)}_c \), we study periodic groups in this variety. Note that since every two generator subgroup of a group in \( \mathcal{N}^{(2)}_c \) is nilpotent, every periodic group in \( \mathcal{N}^{(2)}_c \) is a direct product of \( p \)-groups (\( p \) prime). We reduce the question of Gupta for periodic groups in \( \mathcal{N}^{(d)}_c \) to the locally nilpotency of \( p \)-groups of finite exponent in this variety where the exponent depends only on the numbers \( p \) and \( c \). In fact we prove that

**Theorem 1.2.** Let \( p \) be a prime, \( c > 1 \) an integer and \( r = r(c, p) \) be the integer such that \( p^{r-1} < c - 1 \leq p^r \). Then the following are equivalent:

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(1) every $p$-group in $\mathcal{N}_c^{(2)}$ is locally nilpotent.
(2) if $p$ is odd, every $p$-group of exponent dividing $p^r$ in $\mathcal{N}_c^{(2)}$ is locally nilpotent, and if $p = 2$, every $2$-group of exponent dividing $2^{q+1}$ in $\mathcal{N}_c^{(2)}$ is locally nilpotent.

2. Groups in the variety $\mathcal{N}_c^{(d)}$

Let $F_\infty$ be the free group of infinite countable rank on the set $\{x_1, x_2, \ldots\}$, we define inductively the following words in $F_\infty$:

\[
W_1 = W_1(x_1, x_2) = [x_1, x_2, x_1, x_2], \\
W_n = W_n(x_1, x_2, \ldots, x_{n+1}) = [W_{n-1}, x_{n+1}, W_{n-1}, x_{n+1}] \quad n > 1; \\
V_1 = V_1(x_1, x_2, x_3) = [[x_2, x_1, x_2, x_2, x_1, x_1, x_1], x_3, [x_2, x_1, x_1, x_1, x_1, x_1, x_1], x_3], \\
V_n = V_n(x_1, x_2, x_3, \ldots, x_{n+2}) = [V_{n-1}, x_{n+2}, V_{n-1}, x_{n+2}] \quad n > 1.
\]

For a group $G$ and a subgroup $H$ of $G$, we denote by $HP(G)$ the Hirsch-Plotkin radical of $G$ and $H^G$ the normal closure of $H$ in $G$. We use the following result due to Hirsch (see Lemma 8 of [3] and see Lemma 2 of [5] for the left-normed version).

**Lemma 2.1.** Let $G$ be a group and $g$ an element in $G$ such that $[g, x, g, x] = 1$ for all $x \in G$. Then the normal closure of $\langle g \rangle$ in $\langle g \rangle^G$ is abelian. In particular, $g \in HP(G)$.

**Lemma 2.2.** Let $G$ be a group satisfying the law $W_n = 1$ for some integer $n \geq 1$. Then, $G$ has a normal series $1 = G_n \vartriangleleft G_{n-1} \vartriangleleft \cdots \vartriangleleft G_1 = G$ in which each factor $G_i/G_{i+1}$ is locally nilpotent ($i = 1, 2, \ldots, n - 1$).

**Proof.** We argue by induction on $n$. If $n = 1$, then Lemma 2.1 yields that $x_1 \in HP(G)$ for all $x_1 \in G$ and so $G$ is locally nilpotent. Now suppose that the lemma is true for $n$ and $G$ satisfies the law $W_{n+1} = 1$. By Lemma 2.1, we have $W_n(x_1, \ldots, x_{n+1}) \in HP(G)$ for all $x_1, \ldots, x_{n+1} \in G$ and so $G/HP(G)$ satisfies the law $W_n = 1$. Thus by induction hypothesis $G/HP(G)$ has a normal series of length $n$ with locally nilpotent factors, it completes the proof.

**Lemma 2.3.** Let $G$ be a $p$-group satisfying the law $V_n = 1$ for some integer $n \geq 1$ where $p \in \{2, 3, 5\}$. Thus $G$ has a normal series $1 = G_{n+1} \vartriangleleft G_n \vartriangleleft \cdots \vartriangleleft G_1 = G$ in which each factor $G_i/G_{i+1}$ is locally nilpotent ($i = 1, 2, \ldots, n$).

**Proof.** We argue by induction on $n$. If $n = 1$, then by Lemma 2.1 $[x_2, x_1, x_1, x_1] \in HP(G)$ for all $x_1, x_2 \in G$. Thus $G/HP(G)$ is a 4-Engel group and since every 4-Engel $p$-group is locally nilpotent where $p \in \{2, 3, 5\}$ (see Traustason [3] and Vaughan-Lee [4]), so $G/HP(G)$ is locally nilpotent. Now suppose that $G$ satisfies the law $V_{n+1} = 1$. By Lemma 2.1, $V_n(x_1, \ldots, x_{n+2}) \in HP(G)$ for all $x_1, \ldots, x_{n+2} \in G$, so $G/HP(G)$ satisfies the law $V_n = 1$. Thus by induction hypothesis, $G/HP(G)$ has a normal series of length $n + 1$ with locally nilpotent factors, it completes the proof.

We use in the sequel the following special case of this well-known fact due to Plotkin [3] that every Engel radical group is locally nilpotent. (see also Lemma 2.2 of [5])

**Lemma 2.4.** Let $H$ be a normal subgroup of an Engel group $G$. If $H$ and $G/H$ are locally nilpotent, then $G$ is locally nilpotent.
Proof of Theorem 1.1. One can see that $W_n$ and $V_n$ are, respectively, in the $(2^{n+1}+2^n-2)$th term and $(2^{n+2}+2^{n+1}+2^n-2)$th term of the lower central series of $F_\infty$, for all integers $n \geq 1$.

1. every group $G$ in the variety $N_c^{(d)}$ satisfies the law $W_{d-1} = 1$ and so it follows from Lemmas 2.3 and 2.4, that $G$ is locally nilpotent.

2. if $d = 2$ then $c \leq 4$ and so every group in the variety $N_c^{(d)}$ is 4-Engel. But as it is mentioned in the proof of Lemma 2.3, every 4-Engel $p$-group is locally nilpotent, where $p \in \{2, 3, 5\}$. Now assume that $d \geq 3$, then every group $G$ in the variety $N_c^{(d)}$ satisfies the law $V_{d-2} = 1$ and so it follows from Lemmas 2.3 and 2.4, that $G$ is locally nilpotent. □

3. $p$-groups in the variety $N_c^{(2)}$

Lemma 3.1. Let $c \geq 1$ be an integer, $p$ a prime number and $G$ a finite $c$-Engel $p$-group. Suppose that $x, y \in G$ such that $x^{p^n} = y^{p^n} = 1$ for some integer $n > 0$. Let $r$ be the integer such that $p^{r-1} < c \leq p^r$. Then:

(a) if $p$ is odd and $n > r$ then $[x^{p^{n-1}}, y^{p^{n-1}}] = 1$.

(b) if $p = 2$ and $n > r + 1$ then $[x^{2^{n-1}}, y^{2^{n-1}}] = 1$.

Proof. Suppose that $K \leq H$ are two normal subgroups of $G$ such that $H/K$ is elementary abelian and $a$ is an arbitrary element of $G$. Put $t = aK$ and $V = H/K$. Since $[V, t] = 1$, we have that $[V_{p^r}, t] = 1$ and $0 = (t-1)^{p^r} - t^{p^r} - 1$ in $\text{End}(V)$. Thus $[H, a^{p^r}] \leq K$ for all $a \in G$. Now let $N = \langle x^{p^n}, y^{p^n} \rangle$. Then $[H, N] \leq K$ and since $K, H$ are normal in $G$; $[H, M] \leq K$ where $M = N^G$ the normal closure of $N$ in $G$. Thus $M$ is a normal subgroup of $G$ centralizing every abelian normal section of $G$. By a result of Shalev [8]: if $p$ is odd then $M$ is powerful and if $p = 2$ then $M^2$ as well as all subgroups of $M^2$ which are normal in $G$, are powerful. Suppose that $p$ is odd. Since $M$ is generated by $\{(x^q)^{p^r}, (y^q)^{p^r} \mid q \in G\}$ and $M$ is powerful, by Corollary 1.9 of [4], $M^{p^{n-r}}$ is generated by $\{(x^q)^{p^{n-r}}, (y^q)^{p^{n-r}} \mid q \in G\}$ and so $M^{p^{n-r}} = 1$. On the other hand $M^{p^{n-r-1}}$ is powerful by Corollary 1.2 of [4]. Thus $[M^{p^{n-r-1}}, M^{p^{n-r-1}}] \leq (M^{p^{n-r}})^{p^r}$. Now by Theorem 1.3 of [4], we have $(M^{p^{n-r}})^{p^r} = M^{p^{n-r}} = 1$. Thus $M^{p^{n-r-1}}$ is abelian and the part (a) has been proved. Now assume that $p = 2$. As it is mentioned in the above, since the subgroup $\langle x^{2^{n+1}}, y^{2^{n+1}} \rangle^G$ of $M$ is normal in $G$, it is also powerful and the rest of the proof is similar to the latter case. □

Lemma 3.2. Let $c > 1$ be an integer, $p$ a prime number and $G$ a $p$-group in the variety $N_c^{(2)}$ and let $r$ be the integer satisfying $p^{r-1} < c - 1 \leq p^r$.

(a) if $p$ is odd then $G^{p^r}$ is locally nilpotent.

(b) if $p = 2$ then $G^{2^{r+1}}$ is locally nilpotent.

Proof. Note that $G$ is a $c$-Engel group. Suppose $p$ is odd. First we prove that if $x$ is an element of $G$ such that $x^{p^n} = 1$ for some $n > r$, then $[a, x^{p^{n-1}}, x^{p^{n-1}}] = 1$ for all $a \in G$. Let $y = (x^{-1})^a$, then it is enough to show that $[y^{p^{n-1}}, x^{p^{n-1}}] = 1$. Since $\langle x, a \rangle \in N_c$ then $\langle y, x \rangle \in N_{c-1}$; therefore by Lemma 3.1 we have that $[y^{p^{n-1}}, x^{p^{n-1}}] = 1$. Now let $z$ be an arbitrary element of $G$ such that $z^{p^n} = 1$ for some positive integer $n$. We prove by induction on $n$ that $z^{p^n} \in HP(G)$ and so it completes the proof for the case $p$ odd. If $n \leq r$, then $z^{p^n} = 1 \in HP(G)$. Assume that $n > r$ then by induction hypothesis, $z^{p^{n+1}} \in HP(G)$. Now by the first part of the proof, $[a, z^{p^n}, z^{p^n}] = 1$
mod $HP(G)$, for all $a \in G$. So the normal closure of $z^{p^r}HP(G)$ in $G/HP(G)$ is abelian. Therefore $\langle z^{p^r} \rangle^G$ is (locally nilpotent)-by-abelian, hence Lemma 2.4 implies that $z^{p^r} \in HP(G)$.

The case $p = 2$ is similar. □

Proof of Theorem 1.2. Suppose that $p$ is odd and every $p$-group of exponent dividing $p^r$ in $\mathcal{N}^{(2)}_5$ is locally nilpotent. Let $G$ be a $p$-group in $\mathcal{N}^{(2)}_5$, then by Lemma 3.2(a), $G^{p^r}$ is locally nilpotent. By assumption, $G/G^{p^r}$ is locally nilpotent and since $G$ is a $c$-Engel group, it follows from Lemma 2.4 that $G$ is locally nilpotent. The case $p = 2$ is similar and the converse is obvious. □

Now we use this result to some special cases. In fact we prove:

Proposition 3.3. Every 2-group or 3-group in the variety $\mathcal{N}^{(2)}_5$ is locally nilpotent.

Proof. By Theorem 1.2, we must prove that every 2-group of exponent dividing 8 and every 3-group of exponent dividing 9 in $\mathcal{N}^{(2)}_5$ is locally nilpotent. Suppose that $G$ is a 2-group of exponent 8 in $\mathcal{N}^{(2)}_5$. Let $x, y \in G$, then $\langle x, y \rangle \in \mathcal{N}_5$ and of exponent 8. It is easy to see that $[x^4, y, x^4, y] = 1$. So by Lemma 2.1, $x^4 \in HP(G)$ for all $x \in G$. Therefore $G/HP(G)$ is of exponent dividing 4. By a famous result of Sanov (see 14.2.4 of [7]), $G/HP(G)$ is locally nilpotent and so by Lemma 2.4, $G$ is locally nilpotent.

Now suppose that $G$ is a 3-group of exponent 9 in the variety $\mathcal{N}^{(2)}_5$. Let $x, y \in G$, it is easy to see that $[x^3, y, x^3, y] = 1$. The rest of the proof is similar to latter case, but we may use this well-known result that every group of exponent 3 is nilpotent (see 12.3.5 and 12.3.6 of [7]). □

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