Heavy Arc Orientations of Gammoids

Immanuel Albrecht

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Abstract

In this work, we introduce a purely combinatorial way to obtain realizable orientations of a gammoid from a total order on the arc set of the digraph representing it, without first obtaining a matrix representing the gammoid over the reals.

Keywords. gammoids, oriented matroids, cascade matroids, directed graphs

This work is structured into two parts. First we develop a combinatorial method of obtaining an orientation of a cascade matroid — i.e. of a gammoid that may be represented using an acyclic digraph. Then we introduce the method of lifting cycles in order to deal with gammoid representations that involve cycles.

1 Preliminaries

In this work, we consider matroids to be pairs $M = (E, \mathcal{I})$ where $E$ is a finite set and $\mathcal{I}$ is a system of independent subsets of $E$ subject to the usual axioms ([8], Sec. 1.1). The family of circuits of $M$ shall be denoted by $\mathcal{C}(M)$. If $M = (E, \mathcal{I})$ is a matroid and $X \subseteq E$, then the restriction of $M$ to $X$ shall be denoted by $M|X$.
(8), Sec. 1.3), and the contraction of $M$ to $X$ shall be denoted by $M.X$ (8, Sec. 3.1). The dual matroid of $M$ shall be denoted by $M^*$.

A signed subset of $E$ shall be a map $X : E \mapsto \{-1, 0, +1\}$, furthermore the positive elements of $X$ shall be $X_+ = \{x \in E \mid X(x) = 1\}$, the negative elements of $X$ shall be $X_- = \{x \in E \mid X(x) = -1\}$, the support of $X$ shall be $X_\pm = \{x \in E \mid X(x) \neq 0\}$, and the zero-set of $X$ shall be $X_0 = E \setminus X_\pm$. The negation of $X$ shall be the signed subset $-X$ where $-X : E \mapsto \{-1, 0, 1\}$, $e \mapsto -X(e)$.

Oriented matroids are considered triples $\mathcal{O} = (E, C, C^*)$ where $E$ is a finite set, $C$ is a family of signed circuits and $C^*$ is a family of signed cocircuits subject to the axioms of oriented matroids (2, Ch. 3). Every oriented matroid $\mathcal{O}$ has a uniquely determined underlying matroid defined on the ground set $E$, which we shall denote by $M(\mathcal{O})$. A matroid shall be orientable, if there is an oriented matroid $\mathcal{O}$ such that $M = M(\mathcal{O})$.

The notion of a digraph shall be synonymous with what is described more precisely as finite simple directed graph that may have some loops, i.e. a digraph is a pair $D = (V,A)$ where $V$ is a finite set and $A \subseteq V \times V$ – thus $|A| < \infty$. All standard notions related to digraphs in this work are in accordance with the definitions found in [1]. A walk in $D = (V,A)$ is a non-empty sequence $w = w_1w_2 \ldots w_n$ of vertices $w_i \in V$ such that for each $1 \leq i < n$, $(w_i, w_{i+1}) \in A$. By convention, we shall denote $w_n$ by $w_{-1}$. Furthermore, the set of vertices traversed by a walk $w$ shall be denoted by $|w| = \{w_1, w_2, \ldots, w_n\}$ and the set of all walks in $D$ shall be denoted by $W(D)$. Furthermore, the set of arcs traversed by $w$ shall be denoted by $|w|_A = \{(w_1, w_2), (w_2, w_3), \ldots, (w_{n-1}, w_n)\}$. If $u, v \in W(D)$ with $u_{-1} = v_1$, then $u.v = u_1v_2 \ldots u_nv_{v_3} \ldots v_m$, i.e. $u.v$ is the walk that traverses the arcs of $u$ and then the arcs of $v$. A path in $D = (V,A)$ is a walk $p = p_1p_2 \ldots p_n$ such that $p_i = p_j$ implies $i = j$. The set of all paths in $D$ shall be denoted by $P(D)$. For $S, T \subseteq V$, an $S-T$-separator in $D$ is a set $X \subseteq V$ such that every path $p \in P(D)$ from $s \in S$ to $t \in T$ has $|p| \cap V \neq \emptyset$. A cycle is a walk $c_1c_2 \ldots c_n$ such that $n > 1$, $c_1 = c_n$, and $c_1c_2 \ldots c_{n-1}$ is a path. An $S-T$-connector shall be a routing $R : S' \rightarrow T$ with $S' \subseteq S$.

**Definition 1.1.** Let $D = (V,A)$ be a digraph, and $X, Y \subseteq V$. A routing from $X$ to $Y$ in $D$ is a family of paths $R \subseteq P(D)$ such that

(i) for each $x \in X$ there is some $p \in R$ with $p_1 = x$,

(ii) for all $p \in R$ the end vertex $p_{-1} \in Y$, and

(iii) for all $p, q \in R$, either $p = q$ or $|p| \cap |q| = \emptyset$.  

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We shall write \( R: X \Rightarrow Y \) in \( D \) as a shorthand for “\( R \) is a routing from \( X \) to \( Y \) in \( D \)”, and if no confusion is possible, we just write \( X \Rightarrow Y \) instead of \( R \) and \( R: X \Rightarrow Y \). A routing \( R \) is called \textit{linking} from \( X \) to \( Y \), if it is a routing onto \( Y \), i.e. whenever \( Y = \{ p_{-1} \mid p \in R \} \).

\[ \text{Definition 1.2.} \text{ Let } D = (V, A) \text{ be a digraph, } E \subseteq V, \text{ and } T \subseteq V. \text{ The \textit{gammoid} represented by } (D, T, E) \text{ is defined to be the matroid } \Gamma(D, T, E) = (E, I) \text{ where} \]

\[ I = \{ X \subseteq E \mid \text{there is a routing } X \Rightarrow T \text{ in } D \} . \]

The elements of \( T \) are usually called \textit{sinks} in this context, although they are not required to be actual sinks of the digraph \( D \). To avoid confusion, we shall call the elements of \( T \) \textit{targets} in this work. A matroid \( M' = (E', I') \) is called \textit{gammoid}, if there is a digraph \( D' = (V', A') \) and a set \( T' \subseteq V' \) such that \( M' = \Gamma(D', T', E') \). A gammoid \( M \) is called \textit{strict}, if there is a representation \( (D, T, E) \) of \( M \) with \( D = (V, A) \) where \( V = E \). Whenever \( t \in T \cap E \), we have \( \Gamma(D, T, E).E \setminus \{ t \} = \Gamma(D, T \setminus \{ t \}, E \setminus \{ t \}) \).

\[ \text{Definition 1.3.} \text{ Let } D = (V, A) \text{ be a digraph, } s \in V \text{ be a vertex of } D, \text{ and } r \in V \text{ be a vertex such that } (r, s) \in A \text{ is an arc of } D. \text{ The } r-s-\text{pivot of } D \text{ shall be the digraph } D_{r \leftarrow s} = (V, A_{r \leftarrow s}) \text{ where} \]

\[ A_{r \leftarrow s} = \{ (u, v) \in A \mid u \neq r \} \cup \{ (s, x) \mid (r, x) \in A, x \neq s \} . \]

For example, pivoting \((r, s)\) in \( \begin{array}{c}
p \\
\hline \\
q \\
\hline \\
s \\
\end{array} \)

yields \( \begin{array}{c}
p \\
\hline \\
q \\
\hline \\
s \\
\end{array} \)

\[ \text{Theorem 1.4 (\cite{7}, The Fundamental Theorem (4.1.1)). Let } D = (V, A) \text{ be a digraph, } T, E \subseteq V, \text{ and } s \in T \text{ which is sink in } D, \text{ and } r \in V \setminus T \text{ with } (r, s) \in A. \text{ Then} \]

\[ \Gamma(D, T, E) = \Gamma(D_{r \leftarrow s}, T \setminus \{ s \} \cup \{ r \}, E) . \]

For a proof, see \cite{7}.

\[ \text{Lemma 1.5.} \text{ Every gammoid } M = (E, I) \text{ is orientable.} \]

\[ \text{Proof.} \text{ The class of gammoids is characterized as the closure of the class of transversal matroids under duality and minors (\cite{7} Addendum from 21 Mar 1972; due to results from \cite{5}). Since every transversal matroid is representable over} \]

\[ \text{\hspace{1cm}} \]
the reals, there is a set $T$ with $|T| = \text{rk}_M(E)$ and there is a matrix $\mu \in \mathbb{R}^{E \times T}$, such that $M = M(\mu)$ (4, Sec. 1). Thus $M$ is orientable since every matroid representable by elements of a real vector space has a natural orientation corresponding to the sign-patterns of the minimal non-trivial linear combinations of the zero vector (2, Sec. 1.2 (a)).

**Definition 1.6.** Let $D = (V, A)$ be a digraph and $w: A \rightarrow \mathbb{R}$. Then $w$ shall be called **indeterminate weighting** of $D$ if $w$ is injective, and if there is no $r \in w[A]$ which may be expressed as $r = a_0 + a_1x_{1,1}x_{1,2} \cdots x_{1,n_1} + a_2x_{2,1}x_{2,2} \cdots x_{2,n_2} + \ldots + a_mx_{m,1}x_{m,2} \cdots x_{m,n_m}$ with $a_i \in \mathbb{Z}$ and $x_{i,j} \in w[A]\{r\}$.

Since the cardinality of the subset of the reals, which may be expressed as above, is countably infinite, a cardinality argument yields that every digraph $D = (V, A)$ has an indeterminate weighting. Furthermore, if $w$ is an indeterminate weighting of $D$ and $z \in (\mathbb{Z}\{0\})^A$, then $w'$ with $w'(a) = z(a) \cdot w(a)$ is an indeterminate weighting of $D$, too.

**Definition 1.7.** Let $D = (V, A)$ be a digraph, $w: A \rightarrow \mathbb{R}$ be a map, and let $q = (q_i)_{i=1}^n \in \mathcal{W}(D)$. We shall write $\prod q$ in order to denote $\prod_{i=1}^{n-1} w((q_i, q_{i+1}))$.

**Lemma 1.8 (Lindström [6]).** Let $D = (V, A)$ be an acyclic digraph, $n \in \mathbb{N}$, $S = \{s_1, s_2, \ldots, s_n\} \neq \subseteq V$ and $T = \{t_1, t_2, \ldots, t_n\} \neq \subseteq V$ be equicardinal, and $w: A \rightarrow \mathbb{R}$ be an indeterminate weighting of $D$. Furthermore, $\mu \in \mathbb{R}^{V \times V}$ shall be the matrix defined by the equation

\[
\mu(u, v) = \sum_{p \in \mathcal{P}(D; u, v)} \prod p,
\]

where $\mathcal{P}(D; u, v) = \{p \in \mathcal{P}(D) \mid p_1 = u \text{ and } p_{-1} = v\}$. Then

\[
\det(\mu| S \times T) = \sum_{L: S \supseteq T} \left(\text{sgn}(L) \prod_{p \in L} \prod p\right)
\]

where the sum ranges over all linkings $L$ that route $S$ to $T$ in $D$; and where $\text{sgn}(L) = \text{sgn}(\sigma)$ for the unique permutation $\sigma \in \mathfrak{S}_n$ with the property that for every $i \in \{1, 2, \ldots, n\}$ there is a path $p \in L$ with $p_1 = s_i$ and $p_{-1} = t_{\sigma(i)}$.

Furthermore,

\[
\det(\mu| S \times T) = 0
\]

if and only if there is no linking from $S$ to $T$ in $D$.

I.M. Gessel and X.G. Viennot gave a nice bijective proof in [3].
2 Heavy Arcs and Routings

Lemma 2.1. Let $E$ and $T$ be finite sets, and let $\mu \in \mathbb{R}^{E \times T}$ be a matrix, and $M = M(\mu)$ be the matroid represented by $\mu$ over $\mathbb{R}$. Further, let $\mathcal{O} = (E, \mathcal{C}, \mathcal{C}^*) = \mathcal{O}(\mu)$ be the oriented matroid obtained from $\mu$, let $C \in \mathcal{C}(M)$ and $c \in C$. Let $T_0 \subseteq T$ such that $\det(\mu|(C \setminus \{c\}) \times T_0) \neq 0$. Consider the signed subset $C_c$ of $E$ with

$$C_c(e) = \begin{cases} 0 & \text{if } e \notin C, \\ -1 & \text{if } e = c, \\ \text{sgn} \left( \frac{\det(\nu_e)}{\det(\mu|(C \setminus \{c\}) \times T_0)} \right) & \text{otherwise} \end{cases}$$

where

$$\nu_e : C \setminus \{c\} \times T_0 \rightarrow \mathbb{R}, \quad (x, t) \mapsto \begin{cases} \mu(c, t) & \text{if } x = e, \\ \mu(x, t) & \text{otherwise.} \end{cases}$$

Then $C_c \in \mathcal{C}$.

Proof. By Cramer’s rule we obtain that

$$\mu_c = \sum_{e \in C \setminus \{c\}} \frac{\det(\nu_e)}{\det(\mu|(C \setminus \{c\}) \times T_0)} \cdot \mu_e$$

where $\mu_i$ denotes the row of $\mu$ with index $i$, i.e. $\mu_i = \mu(i, \bullet)$. Therefore,

$$-\mu_c + \sum_{e \in C \setminus \{c\}} \frac{\det(\nu_e)}{\det(\mu|(C \setminus \{c\}) \times T_0)} \cdot \mu_e = 0$$

is a non-trivial linear combination of the zero vector. Clearly $C_c$ consists of the signs of the corresponding coefficients and therefore $C_c \in \mathcal{C}$ is an orientation of $C$ with respect to $\mathcal{O}(\mu)$.

Definition 2.2. Let $D = (V, A)$ be a digraph, let $\sigma : A \rightarrow \{0, 1\}$ be a map and let $\ll$ be a binary relation on $A$. We shall call $(\sigma, \ll)$ a heavy arc signature of $D$, if $\ll$ is a linear order on $A$.

Definition 2.3. Let $D = (V, A)$ be a digraph and $(\sigma, \ll)$ be a heavy arc signature of $D$. The $(\sigma, \ll)$-induced routing order of $D$ shall be the linear order $\ll$ on the family of routings of $D$, where $Q \ll R$ holds if and only if the $\ll$-maximal element $x$ of the symmetric difference $Q_A \triangle R_A$ has the property $x \in R_A$, where $Q_A = \bigcup_{p \in Q} |p|_A$ and $R_A = \bigcup_{p \in R} |p|_A$. 

Clearly, ≪ is a linear order on all routings in \( D \), because every routing \( R \) in \( D \) is uniquely determined by its set of traversed arcs \( R_A \).

**Definition 2.4.** Let \( D = (V, A) \) be a digraph, and let \( (\sigma, \ll) \) be a heavy arc signature of \( D \). Let \( R: X \Rightarrow Y \) be a routing in \( D \) where \( X = \{x_1, x_2, \ldots, x_n\} \neq \emptyset \) and \( Y = \{y_1, y_2, \ldots, y_m\} \neq \emptyset \) are implicitly ordered. The sign of \( R \) with respect to \( (\sigma, \ll) \) shall be

\[
\text{sgn}_\sigma(R) = \text{sgn} (\varphi) \cdot \left( \prod_{p \in R, a \in |p|_A} \sigma (a) \right)
\]

where \( \varphi: \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, m\} \) is the unique map such that for all \( i \in \{1, 2, \ldots, n\} \) there is a path \( p \in R \) with \( p_1 = x_i \) and \( p_{-1} = y_{\varphi(x)} \); and where

\[
\text{sgn} (\varphi) = (-1)^{\left| \{(i, j) \mid i, j \in \{1, 2, \ldots, n\} \text{ and } i < j \text{ and } \varphi(i) > \varphi(j) \} \right|}.
\]

**Definition 2.5.** Let \( D = (V, A) \) be a digraph such that \( V = \{v_1, v_2, \ldots, v_n\} \neq \emptyset \) is implicitly ordered, \( (\sigma, \ll) \) be a heavy arc signature of \( D \), and let \( T, E \subseteq V \) be subsets that inherit the implicit order of \( V \). Furthermore, let \( M = \Gamma(D, T, E) \) be the corresponding gammoid, and let \( C \in C(M) \) be a circuit of \( M \) such that \( C = \{c_1, c_2, \ldots, c_m\} \neq \emptyset \) inherits its implicit order from \( V \); and let \( i \in \{1, 2, \ldots, m\} \).

The signature of \( C \) with respect to \( M \), \( i \), and \( (\sigma, \ll) \) shall be the signed subset \( C^{(i)}_{(\sigma, \ll)} \) of \( E \) where

\[
C^{(i)}_{(\sigma, \ll)} (e) = \begin{cases} 
0 & \text{if } e \notin C, \\
-\text{sgn}_\sigma(R_i) & \text{if } e = c_i, \\
(-1)^{i-j+1} \cdot \text{sgn}_\sigma(R_j) & \text{if } e = c_j \neq c_i,
\end{cases}
\]

and where for all \( k \in \{1, 2, \ldots, m\} \)

\[
R_k = \max_{\ll} \{ R \mid R: C \setminus \{c_k\} \Rightarrow T \text{ in } D \}
\]

denotes the unique \( \ll \)-maximal routing from \( C \setminus \{c_k\} \) to \( T \) in \( D \).

Note that the factors \((-1)^{i-j+1}\) in Definition 2.5 do not appear explicitly in Lemma 2.1 where \( \nu_e \) is obtained from the restriction \( \mu| (C \setminus \{e\}) \times T_0 \) by replacing the values in row \( e \) with the values of \( \mu_e \). We have to account for the number of row transpositions that are needed to turn \( \nu_e \) into the restriction \( \mu| (C \setminus \{e\}) \times T_0 \), which depends on the position of \( e = c_j \) relative to \( c = c_i \) with respect to the implicit order of \( V \).
**Definition 2.6.** Let $D = (V, A)$ be a digraph and $(\sigma, \ll)$ a heavy arc signature of $D$, and let $w: A \to \mathbb{R}$ be an indeterminate weighting of $D$. We say that $w$ is a $(\sigma, \ll)$-weighting of $D$ if, for all $a \in A$, the inequality $|w(a)| \geq 1$, the strict inequality
\[
\sum_{L \subseteq \{x \in A \mid x \ll a, x \neq a\}} \left( \prod_{x \in L} |w(x)| \right) < |w(a)|,
\]
and the equality $\text{sgn}(w(a)) = \sigma(a)$ hold.

**Lemma 2.7.** Let $D = (V, A)$ be a digraph and $(\sigma, \ll)$ be a heavy arc signature of $D$. There is a $(\sigma, \ll)$-weighting of $D$.

**Proof.** Let $w: A \to \mathbb{R}$ be an indeterminate weighting of $D$. For every $\zeta \in \mathbb{Z}^A$ and every $\tau \in \{-1, 1\}^A$, the map $w_{\zeta, \tau}: A \to \mathbb{R}$, which is defined by the equation
\[
w_{\zeta, \tau}(a) = \tau(a) \cdot \frac{w(a)}{\text{sgn}(w(a))} + \tau(a) \cdot \zeta(a)
\]
is an indeterminate weighting of $D$, too. Now, let $\zeta \in \mathbb{Z}^A$, such that for all $a \in A$ we have the following recurrence relation
\[
\zeta(a) = \left[ \sum_{L \subseteq \{x \in A \mid x \ll a, x \neq a\}} \left( \prod_{x \in L} (|w(x)| + \zeta(x)) \right) \right].
\]
The map $\zeta$ is well-defined by this recurrence relation because $|A| < \infty$ and therefore there is a $\ll$-minimal element $a_0$ in $A$. In particular, we have the equation $\zeta(a_0) = \prod_{x \in L = \emptyset} (|w(x)| + \zeta(x)) = 1$. Thus $w_{\zeta, \sigma}$ is a $(\sigma, \ll)$-weighting of $D$. Clearly,
\[
\text{sgn} (w_{\zeta, \sigma}(a)) = \text{sgn} \left( \sigma(a) \cdot \frac{w(a)}{\text{sgn}(w(a))} + \sigma(a) \cdot \zeta(a) \right)
\]
\[
= \text{sgn} (\sigma(a)) \cdot \text{sgn} \left( \frac{w(a)}{\text{sgn}(w(a))} + \zeta(a) \right)
\]
\[
= \sigma(a) \cdot 1 = \sigma(a)
\]
Lemma 2.9. Let $(\sigma, \ll)$ be a heavy arc weighting of $D, E, T \subseteq V, C \in \mathcal{C}(\Gamma(D,T,E))$, be a circuit in the corresponding gammoid, and let $c, d \in C$. Furthermore, let $R_c : C \setminus \{c\} \triangleleft T$ and $R_d : C \setminus \{d\} \triangleleft T$ be the $\ll$-maximal routings in $D$. Then \( \{p_{-1} \mid p \in R_c\} = \{p_{-1} \mid p \in R_d\} \) holds.

Proof. Let $S$ be a $C$-$T$-separator of minimal cardinality in $D$, i.e. a $C$-$T$-separator with $|S| = |C| - 1$ (Menger’s Theorem). Since $R_c$ and $R_d$ are both $C$-$T$-connectors with maximal cardinality, we obtain that for every $s \in S$ there is a path $p^s_c \in R_c$ and a path $p^s_d \in R_d$ such that $s \in |p^s_c|$ and $s \in |p^s_d|$, thus there are paths $l^s_c, l^s_d, r^s_c, r^s_d \in \mathcal{P}(D)$ such that $p^s_c = l^s_c.r^s_c$ and $p^s_d = l^s_d.r^s_d$ with $(r^s_c)_1 = (r^s_d)_1 = s$. Now let $R^S_c = \{r^s_c \mid s \in S\}$ and $R^S_d = \{r^s_d \mid s \in S\}$, clearly both $R^S_c$ and $R^S_d$ are routings from $S$ to $T$ in $D$. Assume that $R^S_c \neq R^S_d$, then we have $R^S_c \ll R^S_d$ — without loss of generality, by possibly switching names for $c$ and $d$. Then $Q = \{l^s_c.r^s_d \mid s \in S\}$ is a routing from $C \setminus \{c\}$ to $T$ in $D$. But for the symmetric differences we have the equality

\[
\left( \bigcup_{p \in Q} |p|_A \right) \triangle \left( \bigcup_{p \in R_c} |p|_A \right) = \left( \bigcup_{p \in R^S_c} |p|_A \right) \triangle \left( \bigcup_{p \in R^S_d} |p|_A \right),
\]

which implies $R_c \ll Q$, a contradiction to the assumption that $R_c$ is the $\ll$-maximal routing from $C \setminus \{c\}$ to $T$. Thus $R^S_c = R^S_d$ and the claim of the lemma follows. \hfill \Box

Lemma 2.8. Let $D = (V, A)$ be a digraph, $(\sigma, \ll)$ be a heavy arc weighting of $D, E, T \subseteq V, C \in \mathcal{C}(\Gamma(D,T,E))$, be a circuit in the corresponding gammoid, and let $c, d \in C$. Furthermore, let $R_c : C \setminus \{c\} \triangleleft T$ and $R_d : C \setminus \{d\} \triangleleft T$ be the $\ll$-maximal routings in $D$. Then \( \{p_{-1} \mid p \in R_c\} = \{p_{-1} \mid p \in R_d\} \) holds.

Proof. Let $S$ be a $C$-$T$-separator of minimal cardinality in $D$, i.e. a $C$-$T$-separator with $|S| = |C| - 1$ (Menger’s Theorem). Since $R_c$ and $R_d$ are both $C$-$T$-connectors with maximal cardinality, we obtain that for every $s \in S$ there is a path $p^s_c \in R_c$ and a path $p^s_d \in R_d$ such that $s \in |p^s_c|$ and $s \in |p^s_d|$, thus there are paths $l^s_c, l^s_d, r^s_c, r^s_d \in \mathcal{P}(D)$ such that $p^s_c = l^s_c.r^s_c$ and $p^s_d = l^s_d.r^s_d$ with $(r^s_c)_1 = (r^s_d)_1 = s$. Now let $R^S_c = \{r^s_c \mid s \in S\}$ and $R^S_d = \{r^s_d \mid s \in S\}$, clearly both $R^S_c$ and $R^S_d$ are routings from $S$ to $T$ in $D$. Assume that $R^S_c \neq R^S_d$, then we have $R^S_c \ll R^S_d$ — without loss of generality, by possibly switching names for $c$ and $d$. Then $Q = \{l^s_c.r^s_d \mid s \in S\}$ is a routing from $C \setminus \{c\}$ to $T$ in $D$. But for the symmetric differences we have the equality

\[
\left( \bigcup_{p \in Q} |p|_A \right) \triangle \left( \bigcup_{p \in R_c} |p|_A \right) = \left( \bigcup_{p \in R^S_c} |p|_A \right) \triangle \left( \bigcup_{p \in R^S_d} |p|_A \right),
\]

which implies $R_c \ll Q$, a contradiction to the assumption that $R_c$ is the $\ll$-maximal routing from $C \setminus \{c\}$ to $T$. Thus $R^S_c = R^S_d$ and the claim of the lemma follows. \hfill \Box

Lemma 2.9. Let $D = (V, A)$ be an acyclic digraph where $V$ is implicitly ordered, $(\sigma, \ll)$ be a heavy arc signature of $D$, and $T, E \subseteq V$. Then there is a unique oriented matroid $\mathcal{O} = (E, \mathcal{C}, \mathcal{C}^*)$ where

\[
\mathcal{C} = \left\{ \pm C^{(1)}_{(\sigma, \ll)} \mid C \in \mathcal{C}(\Gamma(D, T, E)) \right\}.
\]
Proof. Let $M = \Gamma(D, T, E)$, and let $w: A \rightarrow \mathbb{R}$ be a $(\sigma, \ll)$-weighting of $D$. Furthermore, let $\mu \in \mathbb{R}^{E \times T}$ be the matrix defined as in the Lindström Lemma 1.8 with respect to the $(\sigma, \ll)$-weighting $w$ and the implicit order on $V$. The second statement of the Lindström Lemma yields $M = M(\mu)$. Let $\mathcal{O} = \mathcal{O}(\mu) = (E, C_\mu, C_\mu^*)$ be the oriented matroid that arises from $\mu$, thus $M(\mathcal{O}) = M(\mu)$. It suffices to prove that for all $C \in \mathcal{C}(M)$, all $D \in \mathcal{C}_\mu$ with $D_\pm = C$, and all $D' \in \mathcal{C}$ with $D_\pm = C$ we have $D \in \{D', -D'\}$. Now, let $C \in \mathcal{C}(M)$ and let $C = \{c_1, c_2, \ldots, c_k\} \neq$ implicitly ordered respecting the implicit order of $V$. The claim follows if $D(c_1)D(c_j) = D'(c_1)D'(c_j)$ holds for all $j \in \{2, 3, \ldots, k\}$. Let $T_0 \subseteq T$ be the target vertices onto which the $\ll$-maximal and $|\cdot|$-maximal $C$-$T$-connectors link in $D$ (Lemma 2.8). From Lemma 2.7 we obtain that

$$D(c_1)D(c_j) = -1 \cdot \text{sgn} \left( \frac{\det(\nu_j)}{\det(\mu|\{c_1\} \times T_0)} \right)$$

$$= -\text{sgn}(\det(\nu_j)) \cdot \text{sgn}(\mu|\{c_1\} \times T_0))$$

where

$$\nu_j: C \setminus \{c_1\} \times T_0 \rightarrow \mathbb{R}, \ (x, t) \mapsto \begin{cases} \mu(c_1, t) & \text{if } x = c_j, \\ \mu(x, t) & \text{otherwise.} \end{cases}$$

Observe that $\nu_j$ arises from the restriction $\mu|\{c_1\} \times T_0$ by a row-permutation, which has at most one non-trivial cycle, and this cycle then has length $j - 1$, therefore $\det(\nu_j) = (-1)^{j-2} \det(\mu|\{c_1\} \times T_0)$ holds, so we get $D(c_1)D(c_j) = (-1)^{j-1} \text{sgn}(\det(\mu|\{c_1\} \times T_0)) \cdot \text{sgn}(\mu|\{c_1\} \times T_0))$. We further have $D'(c_1)D'(c_j) = (-1)^{j+1} \cdot \text{sgn}(R_1) \cdot \text{sgn}(R_j)$ where for all $i \in \{1, 2, \ldots, k\}$ the symbol $R_i$ denotes the unique $\ll$-maximal routing from $C \setminus \{c_i\}$ to $T$ in $D$. By the Lindström Lemma 1.8 we obtain that for all $i \in \{1, 2, \ldots, k\}$ the equation

$$\det(\mu|\{c_i\} \times T_0) = \sum_{R: C \setminus \{c_i\} \rightarrow T_0} \left( \text{sgn}(R) \prod_{p \in R} \left( \prod_{a \in \{a_i\}} w(a) \right) \right)$$

holds, where $\text{sgn}(R)$ is the sign of the permutation implicitly given by the start and end vertices of the paths in $R$, both with respect to the implicit order on $V$. Since $w$ is a $(\sigma, \ll)$-weighting, we have
where \( a_i \in \bigcup_{p \in R_i} |p|_A \) is the \( \ll \)-maximal arc in the \( \ll \)-maximal routing \( R_i \) from \( C \setminus \{c_i\} \) to \( T_0 \) in \( D \). Therefore the sign of \( \det (\mu|C \setminus \{c_i\} \times T_0) \) is determined by the sign of the summand that contains \( w(a_i) \) as a factor, which is the summand that corresponds to \( R = R_i \). Therefore

\[
\text{sgn} (\det (\mu|C \setminus \{c_i\} \times T_0)) = \text{sgn}(R_i) \prod_{p \in R_i, a \in |p|_A} \text{sgn}(w(a))
\]

So we obtain

\[
D(c_1)D(c_j) = (-1)^{1-j} \text{sgn}_\sigma(R_1) \cdot \text{sgn}_\sigma(R_j) = D'(c_1)D'(c_j).
\]

**Example 2.10.** We consider the digraph \( D = (V, A) \) with the implicitly ordered vertex set \( V = \{a, b, c, d, e, f, g, h, i, x, y\} \neq \emptyset \), and \( A \) as depicted on the right. Let \( T = \{a, b, c, d\} \). Clearly, \( W(D) \) contains the cycle walk \( ghig \). Let \((\sigma, \ll)\) be the heavy arc signature of \( D \) where \( \sigma(a) = 1 \) for all \( a \in A \), and where \( a_1 \ll a_2 \) if the tuple \( a_1 \) is less than the tuple \( a_2 \) with respect to the lexicographic order on \( V \times V \) derived from the implicit order of the vertex set. Let \( C_1 = \{f, g, i\} \), \( C_2 = \{d, e, f, i\} \), \( C_f = \{d, e, g, i\} \). Clearly \( C_1, C_2, C_f \in C(\Gamma(D, T, E)) \). The following routings are \( \ll \)-maximal among all routings in \( D \) with the same set of initial vertices and with targets in \( T \).

- \( R_{\{f,g\}} = \{fxb, gyc\} \quad \text{sgn}_\sigma(R_{\{f,g\}}) = +1 \)
- \( R_{\{f,i\}} = \{fxb, igyc\} \quad \text{sgn}_\sigma(R_{\{f,i\}}) = +1 \)
- \( R_{\{g,i\}} = \{gyc, ifxb\} \quad \text{sgn}_\sigma(R_{\{g,i\}}) = -1 \)
- \( R_{\{d,e,f\}} = \{dye, fxb\} \quad \text{sgn}_\sigma(R_{\{d,e,f\}}) = -1 \)
- \( R_{\{d,e,i\}} = \{dye, fxb\} \quad \text{sgn}_\sigma(R_{\{d,e,i\}}) = +1 \)
- \( R_{\{d,f,i\}} = \{dye, ifxb\} \quad \text{sgn}_\sigma(R_{\{d,f,i\}}) = +1 \)
- \( R_{\{e,f,i\}} = \{ifxb, fd, igyc\} \quad \text{sgn}_\sigma(R_{\{e,f,i\}}) = -1 \)

\( D(c_1)D(c_j) = (-1)^{1-j} \text{sgn}_\sigma(R_1) \cdot \text{sgn}_\sigma(R_j) = D'(c_1)D'(c_j). \)
Heavy Arc Orientations of Gammoids

\[ R_{\{d,e,g\}} = \{d, eyc, ghifxb\} \quad \text{sgn}_\sigma (R_{\{d,e,g\}}) = -1 \]
\[ R_{\{d,g,i\}} = \{d, gyc, ifxb\} \quad \text{sgn}_\sigma (R_{\{d,g,i\}}) = -1 \]
\[ R_{\{e,g,i\}} = \{exb, gyc, ifd\} \quad \text{sgn}_\sigma (R_{\{e,g,i\}}) = +1 \]

Now let us calculate the signatures of \( C_1 \), \( C_2 \), and \( C_f \) according to Definition 2.5. We obtain
\[ (C_1)_{(1,\prec)} = \{f, g, -i\} \]
\[ (C_2)_{(1,\prec)} = \{d, e, -f, -i\} \]
\[ (C_f)_{(1,\prec)} = \{-d, -e, -g, -i\} \]

This clearly violates the oriented strong circuit elimination ([2], Thm. 3.2.5): if we eliminate \( f \) from \( (C_1)_{(1,\prec)} \) and \( (C_2)_{(1,\prec)} \), then the resulting signed circuit must have opposite signs for \( d \) and \( i \), but \( d \) and \( i \) have the same sign with respect to \( (C_f)_{(1,\prec)} \). Therefore we see that the assumption, that \( D \) is acyclic, cannot be dropped from Lemma 2.9.

3 Dealing with Cycles in Digraphs

We can still use the construction involved in Lemma 2.9 to obtain an orientation for a representation \( (D, T, E) \) of a gammoid \( M \) where \( D \) is not acyclic, but we first have to construct something we call complete lifting of \( D \), which yields an acyclic representation of a co-extension \( M' \) of \( M \). Finally, we may obtain an orientation of \( M \) by contraction of a heavy arc orientation of \( M' \).

**Definition 3.1.** Let \( D = (V, A) \) be a digraph, \( x, t \not\in V \) be distinct new elements, and let \( c = (c_i)_{i=1}^n \in W(D) \) be a cycle in \( D \). The lifting of \( c \) in \( D \) by \( (x, t) \) is the digraph \( D'(x,t) = (V \cup \{x, t\}, A') \) where \( A' = A \setminus \{(c_1, c_2)\} \cup \{(c_1, t), (x, c_2), (x, t)\} \).

Observe that the cycle \( c \in W(D) \) is no longer a walk with respect to the lifting of \( c \) in \( D \) anymore.

Clearly, if \( c' = (c'_i)_{i=1}^n \in W(D') \) is a cycle walk in the lifting \( D' \) of the cycle \( c \) of \( D \), then \( c' \in W(D') \), too. Thus lifting of cycles strictly decreases the number of cycles in the digraph.

**Definition 3.2.** Let \( D = (V, A) \) be a digraph. A complete lifting of \( D \) is an acyclic digraph \( D' = (V', A') \) for which there is a suitable \( n \in \mathbb{N} \) such that there is a set \( X = \{x_1, t_1, x_2, t_2, \ldots, x_n, t_n\} \neq \emptyset \) with \( X \cap V = \emptyset \), a family of digraphs
\[ D^{(i)} = (V^{(i)}, A^{(i)}) \] for \( i \in \{0, 1, \ldots, n\} \) where \( D' = D^{(n)}, D^{(0)} = D \), and for all \( i \in \{1, 2, \ldots, n\} \)

\[ D^{(i)} = \left( D^{(i-1)} \right)^{(c_i)}_{(x_i, t_i)} \]

with respect to a cycle walk \( c_i \in W \left( D^{(i-1)} \right) \).

**Lemma 3.3.** Let \( D = (V, A) \) be a digraph. Then \( D \) has a complete lifting.

**Proof.** By induction on the number of cycle walks in \( D \), lifting an arbitrarily chosen cycle walk yields a digraph with strictly less cycles, and every acyclic digraph is its own complete lifting. \( \square \)

**Lemma 3.4.** Let \( D = (V, A), E, T \subseteq V, c \in W(D) \) a cycle, \( x, t \notin V \), and let \( D' = D^{(c)}_{(x, t)} \) be the lifting of \( c \) in \( D \). Then \( \Gamma(D, T, E) = \Gamma(D', T \cup \{t\}, E \cup \{x\}) \).

**Proof.** Let \( M = \Gamma(D, T, V) \) be the strict gammoid induced by the representation \((D, T, V)\) of the gammoid \( \Gamma(D, T, E) \), and let \( M' = \Gamma(D', T \cup \{t\}, V') \) be the strict gammoid obtained from the lifting of \( c \). Then \( M'' = M'.V \cup \{t\} \) is a strict gammoid that is represented by \((D'', T, V \cup \{t\})\) where the digraph \( D'' = (V_0 \setminus \{x\}, A'') \) with \( A'' = A_0 \setminus (V_0 \times \{x\}) \) and where \( D'_{x \rightarrow t} = (V_0, A_0) \). It is easy to see from the involved constructions (Fig. 1), that \( A'' = (A \setminus \{(c_1, c_2)\}) \cup \{(c_1, t), (t, c_2)\} \). A routing \( R \) in \( D \) can have at most one path \( p \in R \) such that \( (c_1, c_2) \in [p]_A \), and since \( t \notin V \), we obtain a routing \( R' = (R \setminus \{p\}) \cup \{qtr\} \) for \( q, r \in P(D) \) such that \( p = qr \) with \( q_{-1} = c_1 \) and \( r_1 = c_2 \). Clearly, \( R' \) routes \( X \) to \( Y \) in \( D'' \) whenever \( R \) routes \( X \) to \( Y \) in \( D \). Conversely, let \( R' : X' \Rightarrow Y' \) be a routing in \( D'' \) with \( t \notin X' \). Then there is at most one \( p \in R' \) with \( t \in [p] \). We can invert the construction and let \( R'' = (R' \setminus \{p\}) \cup \{qr\} \) for the appropriate paths.

**Figure 1:** Constructions involved in Lemma 3.4.
Corollary 3.5. Let $M = (E, \mathcal{I})$ be a gammoid. Then there is an acyclic digraph $D = (V, A)$ and sets $T, E' \subseteq V$ such that $M = \Gamma(D, T, E')$. $E$ and $|T| = \text{rk}_M(E) + |E'|$.

Proof. Let $M = \Gamma(D', T', E)$ with $|T'| = \text{rk}_M(E)$ — such a representation may be obtained from any other representation by adding the appropriate amount of new targets to the digraph, and connecting every new target with every element from the old target set. Then let $D$ be a complete lifting of $D'$ (Lemma 3.3), and let $D^{(0)}, D^{(1)}, \ldots, D^{(n)}$ be the family of digraphs and $c_1, c_2, \ldots, c_n$ be the cycle walks that correspond to the complete lifting $D$ of $D'$ as required by Definition 3.2, and let $\{x_1, t_1, \ldots, x_n, t_n\} \neq \emptyset$ denote the new elements such that $D^{(i)} = (D^{(i-1)})^{(c_i)}_{(x_i, t_i)}$ holds for all $i \in \{1, 2, \ldots, n\}$. Induction on the index $i$ with Lemma 3.4 yields that $\Gamma(D', T, E) = \Gamma(D^{(i)}, T \cup \{t_1, t_2, \ldots, t_i\}, E \cup \{x_1, x_2, \ldots, x_i\}). E$ holds for all $i \in \{1, 2, \ldots, n\}$. Clearly, $|T \cup \{t_1, t_2, \ldots, t_n\}| = |T| + n = \text{rk}_M(E) + n = \text{rk}_M(E) + |\{x_1, x_2, \ldots, x_n\}|$.

Since the contraction $O.X$ (Prop. 3.3.2) of an oriented matroid $O$ is an orientation of the contraction $M(O).X$ of its underlying matroid, we are able to obtain heavy arc orientations of gammoids that cannot be represented without cycles in their digraphs through complete lifting.

Since every heavy arc orientation of a gammoid is representable, an open question that occurs naturally is, whether there is a similar combinatorial way that also yields non-representable orientations of gammoids. Furthermore, is there a way to refine the definition of heavy arc orientations that allows to circumvent the formation of a complete lifting?

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