OPTION PRICING WITH PADÉ APPROXIMATIONS

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Abstract. In this paper, Padé approximations are applied Black-Scholes model which reduces to heat equation. This paper shows various Padé approximations to obtain an effective and accurate solution to the Black-Scholes equation for a European put/call option pricing problem. At the end of the paper, results of closed-form solution of Black-Scholes problem, solution of Crank-Nicolson approach and the solution of \((1, 1), (1, 2), (2, 0), (2, 1), (2, 2)\) Padé approximations are given at a table.

1. Introduction

In recent years, studies of solutions of Black-Scholes partial differential equations have increased. Although in 1970’s Merton [1,2] and Black and Scholes [3] has formulated Black-Scholes model according to stochastic differential equations, nowadays the model has been solved both stochastic and numerical solutions. Especially, in books of Seydel [6], Ugur [7] and Brandimarte [5] the results of examples solved by applying finite differences. In this paper, we will give a new approach for solving the Black-Scholes model to reduced heat equation. Firstly, as implementing the finite difference algorithms to the diffusion equation, the equation will be transformed the system of ordinary differential equation. Then the system will be solved with Padé approximations \((1, 1), (1, 2), (2, 0), (2, 1), (2, 2)\) and the results obtained will be compared with results of Crank-Nicolson solution, Closed-Form solution and Matlab solution.

2. Padé Approximations

Black-Scholes equation for European option \(V(S,t)\):

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - \delta)S \frac{\partial V}{\partial S} - rV = 0
\]  

(2.1)

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is parabolic partial differential equation in domain 
\[ D_V = \{(S, t) : S > 0, 0 \leq t \leq T\} \]. Black-Scholes equation with appropriate variable transformation is equating to heat equation:

\[ \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \quad (2.2) \]

Thus, domain of Black-Scholes equation is change with domain 
\[ D_u = \{(x, \tau) : -\infty < x < \infty, 0 \leq \tau \leq \frac{T^2}{2}\} \] [6]. If \( x \) derivative of eq.(2.2) is changed with following finite difference formula

\[ \frac{d u(x)}{d \tau} = \frac{1}{h^2} \{u(x-h, \tau) - 2u(x, \tau) + u(x+h, \tau)\} + O(h^2), \]

then eq.(2.2) can be written as

\[ \frac{d V(\tau)}{d \tau} = \frac{1}{h^2} \left[ \begin{array}{cccc}
  V_1 & 1 & -2 & 1 \\
  V_2 & 1 & -2 & 1 \\
  \vdots & \ddots & \ddots & \ddots \\
  V_{N-2} & 1 & -2 & 1 \\
  V_{N-1} & 1 & -2 & 1 \\
\end{array} \right]
\]

\[ \left[ \begin{array}{c}
  V_1 \\
  V_2 \\
  \vdots \\
  V_{N-2} \\
  V_{N-1} \\
\end{array} \right] + \frac{1}{h^2} \left[ \begin{array}{c}
  0 \\
  0 \\
  \vdots \\
  0 \\
  V_N \\
\end{array} \right] \quad (2.4) \]

That is, above matrix form can be shown as

\[ \frac{d V(\tau)}{d \tau} = AV(\tau) + b \quad (2.5) \]

where \( V(\tau) = [V_1, V_2, \ldots, V_{N-1}]^T \) is approximation of \( u \), \( b \) is a column vector which has zeros and known boundary values and

\[ A = \frac{1}{h^2} \left[ \begin{array}{cccc}
  -2 & 1 & \ddots & 1 \\
  1 & -2 & \ddots & \ddots \\
  \vdots & & \ddots & \ddots \\
  1 & -2 & 1 & -2 \\
\end{array} \right] \]

is \((N - 1)\) order matrix. Solution of ordinary differential equation \( \frac{d V}{d \tau} = AV + b \) such that \( V(0) = [g_1, g_2, \ldots, g_{N-1}]^T = g \) initial condition is

\[ V(\tau) = -A^{-1}b + \exp(\tau A)(g + A^{-1}b) \quad (2.6) \]

In step \((\tau + k)\), eq.(2.6) can be written as

\[ V(\tau + k) = -A^{-1}b + \exp(kA)V(\tau + A^{-1}b) \]
In this paper, we have made approximation to \( \exp(kA) \) with Padé approximations.

(1, 1) Padé approximation as matrix form:

\[
(I - \frac{1}{2}kA)V(\tau + k) = (I + \frac{1}{2}kA)V(\tau) + kb
\]

(1, 2) Padé approximation as matrix form:

\[
(I - \frac{1}{3}kA)V(\tau + k) = (I + \frac{2}{3}kA + \frac{1}{6}k^2A^2)V(\tau) + (I + \frac{1}{6}kA)b
\]

(2, 0) Padé approximation as matrix form:

\[
(I - kA + \frac{1}{2}k^2A^2)(V(\tau + k) = V(\tau) + (kb - \frac{1}{2}k^2Ab)
\]

(2, 1) Padé approximation as matrix form:

\[
(I - \frac{2}{3}kA + \frac{1}{6}k^2A^2)V(\tau + k) = (I + \frac{1}{3}kA)V(\tau) + (I - \frac{1}{6}kA)kb
\]

(2, 2) Padé approximation as matrix form:

\[
(I - \frac{1}{2}kA + \frac{1}{12}k^2A^2)V(\tau + k) = (I + \frac{1}{2}kA + \frac{1}{12}k^2A^2)V(\tau) + kb
\]

Padé approximations given above form following systems of linear equations:

\[ CV^{(j+1)} = BV^{(j)} + b^{(j)} \quad (2.7) \]

The matrix \( C \) can be written as \( LU \)-decomposition \( C = LU \), where \( L \) is a lower and \( U \) is an upper triangular matrix. The solution to the system of linear equations (2.7) can be written:

\[ V^{(j+1)} = U^{-1}L^{-1}(BV^j + b^j) \]

For each Padé approximations, the solution of Black-Scholes Model reduced to heat equation is given. The results can be seen at Table 1. Also, in this table the results of Crank-Nicolson solution of Black-Scholes Model reduced to heat equation is illustrated. For put option and call option, it has been taken \( S_0 = 10, K = 10, r = 0.25, \sigma = 0.6, \text{div} = 0.2 \) and maturity time \( T = 1 \). Values at Table 1 and Table 2 has been found respectively put and call options.
| S  | C-N | (1,1)   | (1,2)   | (2,0)   |
|----|-----|---------|---------|---------|
| 10 | 1.688723 | 1.688723 | -85914525509.448166 | 1.688514 |
| 20 | 0.332834  | 0.332834  | -19140576.582597 | 0.333149 |
| 30 | 0.084809  | 0.084809  | 0.248896 | 0.085082 |
| 40 | 0.024170  | 0.024170  | 0.024057 | 0.024190 |
| 50 | 0.009358  | 0.009358  | 0.009330 | 0.009283 |
| 60 | 0.003313  | 0.003313  | 0.003278 | 0.003216 |
| 70 | 0.001435  | 0.001435  | 0.001407 | 0.001351 |
| 80 | 0.000592  | 0.000592  | 0.000572 | 0.000530 |
| 90 | 0.000320  | 0.000320  | 0.000305 | 0.000273 |
|100| 0.000169  | 0.000169  | 0.000159 | 0.000135 |

| S  | (2,1) | (2,2) | Closed-Form Sol. | Matlab Sol. |
|----|-------|------|------------------|-------------|
| 10 | 1.688828 | 1.688824 | 1.593673 | 1.690363 |
| 20 | 0.332945  | 0.332948  | 0.284594 | 0.34044  |
| 30 | 0.084902  | 0.084902  | 0.066802 | 0.08533  |
| 40 | 0.024174  | 0.024173  | 0.017639 | 0.02559  |
| 50 | 0.009330  | 0.009329  | 0.006431 | 0.008799 |
| 60 | 0.003280  | 0.003279  | 0.002123 | 0.003366 |
| 70 | 0.001407  | 0.001407  | 0.000866 | 0.001401 |
| 80 | 0.000572  | 0.000572  | 0.000333 | 0.000625 |
| 90 | 0.000304  | 0.000305  | 0.000171 | 0.000295 |
|100| 0.000158  | 0.000158  | 0.000085 | 0.000147 |

Table 1.
S C-N (1,1) (1,2) (2,0) 
10 2.088070 2.088070 -85914525509.0493 2.087858 
20 8.983799 8.983799 19140567.883981 9.032449 
30 16.892984 16.892984 17.055754 16.893249 
40 25.437537 25.437537 25.439717 25.437547 
50 32.773602 32.773602 32.776947 32.773515 
60 41.745939 41.745939 41.751329 41.745827 
70 49.759815 49.759814 49.766811 49.759713 
80 55.811386 55.811386 55.822353 55.811297 
90 66.103337 66.103334 66.099704 66.103267 
100 73.874374 73.874366 73.874562 73.874313 

S (2,1) (2,2) Closed-Form Sol. Matlab Sol. 
10 2.088174 2.088170 1.992973 2.0897 
20 9.032248 9.032251 8.983799 8.9270 
30 16.893074 16.893074 16.874825 16.8592 
40 25.437538 25.437536 25.430801 24.9868 
50 32.773571 32.773569 32.770423 33.1573 
60 41.745900 41.745900 41.744440 41.3392 
70 49.759780 49.759780 49.758884 49.5245 
80 55.811356 55.811356 55.810648 57.7111 
90 66.103312 66.103312 66.102724 65.8981 
100 73.874374 73.874366 73.874562 73.874313 

Table 2.

3. Conclusion

In this study Black-Scholes equation for European put/call options model solved by using Padé approximations which applied to heat equation which is the classical reduced form of Black-Scholes model. Tables 1 and 2 show various estimations of Padé approximations along with Crank-Nicholson (C-N), closed form solutions and Matlab solution. The maturity times shown in Tables 1 and 2 illustrate how the closed form (exact) solutions as well as the approximate solutions obtained via Crank-Nicholson solution, various Padé approximations and Matlab solution behave. Although the discretizations used for spatial variables are uniform, the discretization for asset price is non-uniform, this generate the slight differences on the maturity times. Although, these slight differences insignificant, I think that it’s due to the transformation $S = e^x$. Hence, a shortcoming of these transformations perhaps that the resulting grid is not uniform for the asset price but it is inevitable in our case. Of course, one can get a uniform grid in the asset price by choosing constant step size in the asset price. However, the construction of non-uniform grids for the finite difference methods for the heat equation may not be as easy as the one over a uniform grid. Another difficulty is that after solving the equation...
numerically, a back transformation must be used to interpret the solution in terms of option values which is also a tricky task which we will confront, but this is our prospect study in the future.

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Özet: Bu makalede Padé yaklaşımları ısı denklemini indirgenen Black-Scholes modeline uygulanıyor. Makale Avrupa put ve call opsiyon problemi için Black-Scholes denkleminin etkili ve doğru çözümünü elde etmek için çeşitli Padé yaklaşımlarını gösteriyor. Makalenin sonunda tablo halinde Black-Scholes probleminin kapalı-cözümü, Crank-Nicolson çözümü ve (1, 1), (1, 2), (2, 0), (2, 1), (2, 2) Padé yaklaşımlarının çözümleri verilmektedir.

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