On the global unique solvability of a problem for one-dimensional equations of dynamics of mixtures of viscous compressible heat-conducting fluids

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Abstract. One-dimensional equations of the multi-velocity model of the dynamics of mixtures of viscous compressible heat-conducting fluids and their approximation are considered. The existence and uniqueness of a strong solution to the initial-boundary value problem is proved globally in time.

1. The formulation of differential conservation laws

Movement of a mixture which consists of $N \geq 2$ viscous compressible heat-conducting fluids, in a restricted Lipschitz domain $\Omega \subset \mathbb{R}^3$ over time $t \in [0, T]$, $T = \text{const} > 0$, is described by the following system of partial differential equations \cite{1,2}:

\begin{equation}
\frac{\partial \rho_i}{\partial t} + \text{div}(\rho_i u_i) = 0, \quad i = 1, \ldots, N,
\end{equation}

\begin{equation}
\frac{\partial (\rho_i u_i)}{\partial t} + \text{div}(\rho_i u_i \otimes u_i) + \nabla p_i = \text{div} \mathbf{S}_i + \mathbf{J}_i + \rho_i f_i, \quad i = 1, \ldots, N,
\end{equation}

\begin{equation}
\frac{\partial \mathcal{E}_i}{\partial t} + \text{div} \left( \sum_{i=1}^{N} \mathcal{E}_i u_i \right) + \text{div} \left( q - \sum_{i=1}^{N} \mathbf{S}_i u_i + \sum_{i=1}^{N} p_i u_i \right) = \sum_{i=1}^{N} \rho_i f_i \cdot u_i + \rho g.
\end{equation}

The equations (1)–(3) are respectively mathematical formulations of the laws of conservation of mass for each component, the laws of conservation of momentum for each component, and the law of conservation of total energy for the mixture. Here $\rho_i \geq 0$ is the density of the $i$-th component; $\rho = \sum_{i=1}^{N} \rho_i$ is the total density of the mixture; $u_i$ is the velocity of the $i$-th constituent; $\mathcal{E}_i = \frac{\rho_i |u_i|^2}{2} + \rho_i e_i$ is the total energy of the $i$-th component, where $e_i$ is the internal
specific energy of the $i$-th component; $\mathcal{E} = \sum_{i=1}^{N} \mathcal{E}_i$ is the total energy of the mixture; $p_i$ is the pressure in the $i$-th constituent;

$$S_i = \sum_{j=1}^{N} \left( (\lambda_{ij} \text{div} u_j) I + 2\mu_{ij} \mathbb{D}(u_j) \right) = \sum_{j=1}^{N} \left( (\eta_{ij} \text{div} u_j) I + 2\mu_{ij} \left( \mathbb{D}(u_j) - \frac{1}{3} (\text{div} u_j) I \right) \right)$$

is the viscous part of the stress tensor in the $i$-th component, where $I$ is the identity tensor, and $\mathbb{D}(v) = \frac{1}{2} ((\nabla \otimes v) + (\nabla \otimes v)^*)$ is the rate of deformation tensor of the vector field $v$ (the superscript $*$ means transposition), and the numerical viscosity coefficients $\lambda_{ij}$, $\mu_{ij}$ and $\eta_{ij}$ form the following matrices:

$$M = \{ \mu_{ij} \}_{i,j=1}^{N} > 0, \quad A = \{ \lambda_{ij} \}_{i,j=1}^{N}, \quad H = \{ \eta_{ij} \}_{i,j=1}^{N} = A + \frac{2}{3} M \geq 0,$$

which in particular leads to

$$N = \{ \nu_{ij} \}_{i,j=1}^{N} = A + 2M > 0;$$

further,

$$J_i = \sum_{j=1}^{N} a_{ij} (u_j - u_i), \quad a_{ij} = a_{ji}, \quad i, j = 1, \ldots, N$$

is the momentum income into the $i$-th component from the remaining components; $f_i$ is the density of external mass forces acting from the external environment on the $i$-th component;

$$q = -k(\theta) \nabla \theta$$

is the total heat flux, where $\theta > 0$ is the temperature of the mixture, $k$ is the thermal conductivity; finally, $g$ is the density of heat sources of the external environment.

The system (1)–(3) describes quite arbitrary movements of mixtures, including the cases when the components of the mixture are poorly mixed, and therefore their velocities, concentrations and pressures differ significantly from the equilibrium values. The only assumption of equilibrium already imposed in the above model is the hypothesis of equal phase temperatures, so there is no need to write the energy equations for each component, and one can deal with one equation (3).

### 2. Approximate differential conservation laws

Assume that

(i) $\text{div}(\rho_i (u_i - v)) = 0$ in the equations (1), where $v = \sum_{j=1}^{N} \beta_j u_j$ is the weighted average velocity,

$$\beta_i = \text{const} \in (0,1), \quad \sum_{j=1}^{N} \beta_j = 1;$$

1 In the framework of the above assumptions, the mixture motion is mainly characterized by averaged and/or total characteristics (velocity $v$, density $\rho$, pressure $p$ and temperature $\theta$), however, we are also interested in the individual densities $\rho_i$ and velocities $u_i$ of the components.
(ii) \( \text{div}(\rho_i(u_i - v) \otimes u_i) - J_i + \nabla((p_i - \beta_i p)) = 0 \) in the equations (2), where \( p = \sum_{j=1}^{N} p_j \) is the total pressure;

(iii) \( \text{div} \left( \sum_{i=1}^{N} \mathcal{E}_i(u_i - v) + \sum_{i=1}^{N} (p_i - \beta_i p)u_i \right) = 0 \) in the equations (3).

As a result, we arrive at the following equations

\[
\frac{\partial \rho_i}{\partial t} + \text{div} (\rho_i v) = 0, \quad i = 1, \ldots, N, \tag{9}
\]

\[
\frac{\partial (\rho_i u_i)}{\partial t} + \text{div} (\rho_i v \otimes u_i) + \beta_i \nabla p = \text{div} S_i + \rho_i f_i, \quad i = 1, \ldots, N, \tag{10}
\]

\[
\frac{\partial \mathcal{E}}{\partial t} + \text{div} (\mathcal{E} v) + \text{div} \left( q - \sum_{i=1}^{N} S_i u_i + pv \right) = \sum_{i=1}^{N} \rho_i f_i \cdot u_i + \rho g \tag{11}
\]

for \( N + 1 \) scalar-valued \((\rho_i, i = 1, \ldots, N, \text{and } \theta)\) and \( N \) vector-valued \((u_i, i = 1, \ldots, N)\) (i.e. totally \( 4N + 1 \) scalar-valued) unknown functions, provided that \( p \) and \( e_i, i = 1, \ldots, N, \) are somehow defined as functions of \( \rho_1, \ldots, \rho_N, \theta. \)

**Remark 1.** The equations (9)–(11) admit an equivalent form

\[
\frac{\partial \rho_i}{\partial t} + v \cdot \nabla \rho_i + \rho_i \text{div} v = 0, \quad i = 1, \ldots, N, \tag{12}
\]

\[
\rho_i \frac{\partial u_i}{\partial t} + \rho_i (\nabla \otimes u_i)^* v + \beta_i \nabla p = \text{div} S_i + \rho_i f_i, \quad i = 1, \ldots, N, \tag{13}
\]

\[
\sum_{i=1}^{N} \left( \rho_i \frac{\partial}{\partial t} \left( \frac{|u_i|^2}{2} + e_i \right) \right) + \rho_i v \cdot \nabla \left( \frac{|u_i|^2}{2} + e_i \right) + \text{div} \left( q - \sum_{i=1}^{N} S_i u_i + pv \right) = \sum_{i=1}^{N} \rho_i f_i \cdot u_i + \rho g \tag{14}
\]

( here we can note that \((\nabla \otimes u_i)^* v = (v \cdot \nabla)u_i \), in which the material derivative operator \( \frac{d}{dt} = \frac{\partial}{\partial t} + v \cdot \nabla, \) common to all equations of the system, is visible.

**Remark 2.** From (9) and (10), the kinetic energy balance equation for each component of the mixture follows

\[
\frac{\partial}{\partial t} \frac{\rho_i |u_i|^2}{2} + \text{div} \left( \frac{\rho_i |u_i|^2}{2} v \right) + \beta_i u_i \cdot \nabla p = u_i \cdot \text{div} S_i + \rho_i u_i \cdot f_i, \quad i = 1, \ldots, N. \tag{15}
\]

Therefore, the energy equation (11) can be written in the following equivalent form

\[
\frac{\partial}{\partial t} \sum_{i=1}^{N} \rho_i e_i + \text{div} \left( \sum_{i=1}^{N} \rho_i e_i v \right) + pd\text{div}v + \text{div}q = \sum_{i=1}^{N} S_i : \mathbb{D}(u_i) + \rho g. \tag{16}
\]
3. Constitutive relations and equations for temperature and entropy
Assume that \( p_i = p_i(\rho_i, \theta) \), \( e_i = e_i(\rho_i, \theta) \) and \( s_i = s_i(\rho_i, \theta) \), \( i = 1, \ldots, N \), where \( s_i \) is the specific entropy of the \( i \)-th component. The constitutive equations relating thermodynamic parameters to each other must satisfy certain restrictions, in particular, the Gibbs relations

\[
\theta ds_i = de_i + p_i d\left( \frac{1}{\rho_i} \right), \quad i = 1, \ldots, N, \tag{17}
\]

which are equivalent to the Maxwell relations

\[
\rho_i^2 \frac{\partial e_i}{\partial \rho_i} = p_i - \theta \frac{\partial p_i}{\partial \theta}, \quad i = 1, \ldots, N. \tag{18}
\]

In addition, the conditions of thermodynamic stability must be satisfied

\[
\frac{\partial p_i}{\partial \rho_i} > 0, \quad \frac{\partial e_i}{\partial \theta} > 0, \quad i = 1, \ldots, N. \tag{19}
\]

Then the equation (11)=(16) can be written in the following equivalent forms

\[
\sum_{i=1}^{N} \frac{\partial e_i}{\partial \theta} \left( \frac{\partial (\rho_i \theta)}{\partial t} + \text{div}(\rho_i \theta \mathbf{v}) \right) + \text{div} \mathbf{q} = \sum_{i=1}^{N} S_i : \mathbb{D}(u_i) - \theta \frac{\partial \rho_i}{\partial \theta} \text{div} \mathbf{v} + \rho g, \tag{20}
\]

\[
\frac{\partial}{\partial t} \sum_{i=1}^{N} \rho_i s_i + \text{div} \left( \sum_{i=1}^{N} \rho_i s_i \mathbf{v} \right) + \text{div} \left( \frac{\mathbf{q}}{\theta} \right) = \frac{1}{\theta} \sum_{i=1}^{N} S_i : \mathbb{D}(u_i) - \frac{\mathbf{q} \cdot \nabla \theta}{\theta^2} + \frac{\rho g}{\theta}. \tag{21}
\]

4. A particular form of the constitutive equations
We assume that the equations of state of an ideal gas are satisfied

\[
p_i(\rho_i, \theta) = R_i \rho_i \theta, \quad e_i(\rho_i, \theta) = c_i \theta, \quad i = 1, \ldots, N, \tag{22}
\]

where \( R_i = \frac{R}{M_i} > 0 \) is the gas constant of the \( i \)-th component (\( R \) is the universal gas constant, \( M_i \) is the molar mass of the \( i \)-th component), \( c_i = \frac{\nu_i}{2} R_i > 0 \) is the specific heat capacity of the \( i \)-th constituent (\( \nu_i \) is the number of degrees of freedom of a molecule in the \( i \)-th constituent). Then the conditions (18), (19) are clearly satisfied. From (17) we find

\[
s_i(\rho_i, \theta) = \ln \left( \frac{\rho_i^{c_i}}{\theta^{R_i}} \right) + s_{0i}, \quad i = 1, \ldots, N, \tag{23}
\]

where \( s_{0i} \) are arbitrary constants. The equation (20) then takes the form

\[
\frac{\partial (\rho_i \theta)}{\partial t} + \text{div} (\rho_i \theta \mathbf{v}) + \text{div} \mathbf{q} + \theta \rho_R \text{div} \mathbf{v} = \sum_{i=1}^{N} S_i : \mathbb{D}(u_i) + \rho g, \tag{24}
\]

where \( \rho_c = \sum_{j=1}^{N} c_j \rho_j \) and \( \rho_R = \sum_{j=1}^{N} R_j \rho_j \).
Remark 3. Both from the physical and mathematical point of view, it is necessary to ensure
the non-negativity of the production of entropy. Total entropy of the system
\[ S = \sum_{i=1}^{N} \int_{\Omega} \rho_i s_i \, dx \]  
must be non-decreasing in time in the case of the thermodynamic isolation (i.e. when \( g = 0 \)).
From (8), (21) and the conditions \( u_i|_{\partial \Omega} = 0, \ q \cdot n|_{\partial \Omega} = 0, \ i = 1, \ldots, N \) (where \( \partial \Omega \) is the boundary of the domain \( \Omega \), and \( n \) is the outer unit normal for \( \Omega \)), we obtain
\[ \frac{dS}{dt} = \int_{\Omega} \frac{1}{\theta} \sum_{i=1}^{N} S_i : \mathbb{D}(u_i) \, dx + \int_{\Omega} \frac{k(\theta)|\nabla \theta|^2}{\theta^2} \, dx + \int_{\Omega} \rho g \, \theta \, dx. \]  

Thus, it is sufficient to require the fulfillment of the condition
\[ k \geq 0, \]  
as well as the following condition for viscous stress tensors
\[ \sum_{i=1}^{N} S_i : \mathbb{D}(u_i) \geq 0. \]  
However, under the conditions for the viscosity matrices listed in (5), the validness of (28) is
obvious due to the equality
\[ \sum_{i=1}^{N} S_i : \mathbb{D}(u_i) = \sum_{i,j=1}^{N} \left( \eta_{ij}(\text{div } u_i)(\text{div } u_j) + 2\mu_{ij} \left( \mathbb{D}(u_i) - \frac{1}{3}(\text{div } u_i)I \right) : \left( \mathbb{D}(u_j) - \frac{1}{3}(\text{div } u_j)I \right) \right). \]  
Moreover, the conditions (5) (see (6)), in view of \( u_i|_{\partial \Omega} = 0, \ i = 1, \ldots, N \), imply the following
important inequality
\[ \sum_{i=1}^{N} \int_{\Omega} S_i : (\nabla \otimes u_i) \, dx \geq B_0 \sum_{i=1}^{N} \int_{\Omega} |\nabla \otimes u_i|^2 \, dx \]  
with some positive constant \( B_0 = B_0(\Lambda, M) \).

5. Statement of the problem
Let us consider the system of equations, in the domain \( QT = (0, T) \times (0, 1) \), designed to determine
the functions \( (\rho_1, \ldots, \rho_N, u_1, \ldots, u_N, \theta) \):
\[ \frac{\partial \rho_i}{\partial t} + \frac{\partial (\rho_i v)}{\partial x} = 0, \quad i = 1, \ldots, N, \quad v = \sum_{j=1}^{N} \beta_j u_j, \]  
\[ \frac{\partial (\rho_i u_i)}{\partial t} + \frac{\partial (\rho_i v u_i)}{\partial x} + \beta_i \frac{\partial (\rho R \theta)}{\partial x} = \sum_{j=1}^{N} \nu_{ij} \frac{\partial^2 u_j}{\partial x^2}, \quad i = 1, \ldots, N, \quad \rho_R = \sum_{j=1}^{N} R_j \rho_j. \]
\[
\frac{\partial (\rho_c \theta)}{\partial t} + \frac{\partial (\rho_c u \theta)}{\partial x} + \rho_R \theta \frac{\partial v}{\partial x} = k \frac{\partial^2 \theta}{\partial x^2} + \sum_{i,j=1}^{N} \nu_{ij} \left( \frac{\partial u_i}{\partial x} \right) \left( \frac{\partial u_j}{\partial x} \right), \quad \rho_c = \sum_{j=1}^{N} c_j \rho_j. \tag{33}
\]

in which the coefficients \(\beta_i, \nu_{ij}, R_i, c_i\) and \(k\) are constants, and

\[
\beta_i \in (0, 1), \quad \sum_{j=1}^{N} \beta_j = 1, \quad R_i > 0, \quad c_i > 0, \quad i = 1, \ldots, N, \tag{34}
\]

\[k > 0, \quad N = \{\nu_{ij}\}_{i,j=1}^{N} > 0, \quad N = N^*.\]

At the initial moment of time, let the distribution of densities, velocities and temperature

\[
\rho_i|_{t=0} = \rho_{0i}(x), \quad u_i|_{t=0} = u_{0i}(x), \quad i = 1, \ldots, N, \quad \theta|_{t=0} = \theta_0(x), \quad x \in [0, 1] \tag{35}
\]

be given. The conditions

\[
u_i|x=0 = u_i|x=1 = 0, \quad i = 1, \ldots, N, \quad \frac{\partial \theta}{\partial x} \bigg|_{x=0} = \frac{\partial \theta}{\partial x} \bigg|_{x=1} = 0 \tag{36}\]

are assumed at the boundary of \((0, 1)\). The initial data satisfy the conditions

\[
(\rho_{01}, \ldots, \rho_{0N}, u_{01}, \ldots, u_{0N}, \theta_0) \in W^1_2(0, 1), \quad \rho_{0i} > 0, \quad \theta_0 > 0, \quad i = 1, \ldots, N, \tag{37}
\]

\[u_{0i}(0) = u_{0i}(1) = 0, \quad i = 1, \ldots, N. \]

For the problem (31)–(37), an analysis is made of the existence and uniqueness of its solution in the class of functions \((i = 1, \ldots, N)\)

\[
\rho_i \in L_{\infty}(0, T; W^1_2(0, 1)), \quad \frac{\partial \rho_i}{\partial t} \in L_{\infty}(0, T; L_2(0, 1)), \tag{38}
\]

\[u_i, \theta \in L_{\infty}(0, T; W^1_2(0, 1)) \bigcap L_2(0, T; W^2_2(0, 1)), \quad \frac{\partial u_i}{\partial t}, \frac{\partial \theta}{\partial t} \in L_2(Q_T), \]

moreover, \(\rho_i\) and \(\theta\) are strictly positive. In the isothermal case, the existence and uniqueness of a strong solution to the corresponding initial-boundary value problem are shown in [3, 4].

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**References**

[1] Rajagopal K L and Tao L 1995 *Mechanics of Mixtures* (Singapore: World Scientific Publishing)
[2] Nigmatulin R I 1990 *Dynamics of Multiphase Media* vol 1 (New York: Hemisphere)
[3] Mamontov A E and Prokudin D A 2018 *Siberian Electr. Math. Reports* 15 631–49
[4] Mamontov A E and Prokudin D A 2019 *J. Math. Fluid Mech.* 21 9