Lyapunov exponent for Lipschitz maps

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Abstract

It is well-known that the Lyapunov exponent plays a fundamental role in dynamical systems. In this note, we propose a definition of Lyapunov exponent for Lipschitz maps, which are not necessarily differentiable. Additionally, we show that the main results which are valid to discrete standard dynamical systems are also true when considering Lipschitz maps instead of considering differentiable maps. Therefore, this novel approach expands the theory of dynamical systems.

1 Introduction

Theory of dynamical systems is extensively investigated in the literature [10, 9, 16, 12, 4, 15, 7]. Lyapunov, in his fabulous work [10], made several important contributions in the investigation of the stability of motion. In fact, the Lyapunov exponent strongly characterizes the behavior of the system. In fact, all the papers mentioned above have dealt with investigations and computations of the Lyapunov exponent of the corresponding systems in order to characterize them.

The main contributions of this note are to propose a definition of Lyapunov exponent for Lipschitz maps as well as to show that the results which are valid to discrete standard dynamical systems also hold when considering Lipschitz maps instead of considering differentiable maps. Moreover, since a Lipschitz map is not necessarily differentiable (recall that a Lipschitz map \( f : \mathbb{R} \to \mathbb{R} \) satisfies that following condition: for all \( x, y \in \mathbb{R} \) with \( x \neq y \), one has \( \frac{|f(x) - f(y)|}{|x - y|} \leq c \), for some \( c \in \mathbb{R}, c > 0 \); the existence of the limit is not guaranteed), this novel approach expands the theory of discrete dynamical systems.

Another advantage of considering Lipschitz maps instead of differentiable ones is that it is not necessary to compute the derivative of the map in order to find fixed points which are sinks or sources. In fact, only by the feature of the map (Lipschitz or reverse Lipschitz) one can find which fixed points are sinks

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or sources. Thus, our method is easier to be applied when compared to the standard method (which only deals with differentiable maps).

Generalizations of Lyapunov exponent defined over continuous maps were presented in the literature \cite{8, 3}. In \cite{8}, the author considered Lyapunov exponent for non-smooth systems in order to apply to a pendulum with dry friction. In \cite{3}, the authors defined Lyapunov exponent for continuous maps (see Subsection 3.1). However, such definition is a little complex to be applied in practice.

Otherwise, our new approach is simple to be applied and it is complete in the sense that it characterizes the concept of sources, sinks, Lyapunov exponent, Lyapunov number and all the standard concepts and results of discrete dynamical systems in a natural way. More precisely, the results which hold for differentiable maps also hold in our new context, that is, also hold for Lipschitz maps.

This note is organized as follows. Section 2 presents the main concepts that will be utilized in this work. In Section 3, we present the contributions of the paper. More precisely, we propose a definition of Lyapunov exponent for Lipschitz maps which are not necessarily differentiable. Additionally, we show that the main results which are valid to discrete standard dynamical systems are also true when considering Lipschitz maps instead of considering differentiable maps. Finally, in Section 4 a brief summary of this work is drawn.

2 Preliminaries

In this section, we present the known results and concepts for the development of this work. Throughout this note, we denote by \( \mathbb{R} \) the field of real numbers and \( \mathbb{R}^m \) is the \( m \)-dimensional vector space over \( \mathbb{R} \). We only consider discrete dynamical systems.

As usual, a function whose domain is equal to its range is called \textit{map}. Let \( f : A \to A \) be a map and \( x \in A \). The orbit \( O_x \) of \( x \) under \( f \) is the set of points \( O_x = \{ x, f(x), f^2(x), \ldots \} \), where \( f^2(x) = f(f(x)) \) and so on. The point \( x \) is said to be the initial value of the orbit. If there exists a point \( p \) in the domain of \( f \) such that \( f(p) = p \) then \( p \) is called a \textit{fixed point} of \( f \).

Let \( f : \mathbb{R} \to \mathbb{R} \) be a map. Recall that \( f \) is said to be \textit{Lipschitz} if there exists a constant \( c \in \mathbb{R} \), \( c > 0 \) (called Lipschitz constant of \( f \)), such that \( \forall x, y \in \mathbb{R} \implies |f(x) - f(y)| \leq c|x - y| \), where \( |\cdot| \) denotes the absolute value function on \( \mathbb{R} \). In other words, if \( x \neq y \) then \( \frac{|f(x) - f(y)|}{|x - y|} \leq c \), i.e., the quotient is bounded. If \( \forall x, y \in \mathbb{R} \implies |f(x) - f(y)| < c|x - y| \), then \( f \) is called \textit{strictly Lipschitz}.

Given \( x \in \mathbb{R} \), the \textit{epsilon neighborhood} \( N_\epsilon(x) \) of \( x \) is defined as \( N_\epsilon(x) = \{ y \in \mathbb{R} : |x - y| < \epsilon \} \). Let \( f : \mathbb{R} \to \mathbb{R} \) be a map and \( x \in \mathbb{R} \). We say that \( f \) is \textit{locally Lipschitz at} \( x \) if there exists a constant \( c \in \mathbb{R} \), \( c > 0 \) (called Lipschitz constant of \( f \)), such that \( \forall x, y \in \mathbb{R} \implies |f(x) - f(y)| \leq c|x - y| \), where \( |\cdot| \) denotes the absolute value function on \( \mathbb{R} \). In other words, if \( x \neq y \) then \( \frac{|f(x) - f(y)|}{|x - y|} \leq c \), i.e., the quotient is bounded. If \( \forall x, y \in \mathbb{R} \implies |f(x) - f(y)| < c|x - y| \), then \( f \) is called \textit{strictly Lipschitz}.

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Here, we introduce in the literature the new concept of \textit{reverse Lipschitz map}. This concept will be utilized in order to characterize sources (see Definition 2.2 in the following).
Definition 2.1 Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a map. We say that \( f \) is reverse Lipschitz (RL) if there exists a constant \( c \in \mathbb{R}, c > 0 \) (called reverse Lipschitz constant of \( f \)) such that, \( \forall x, y \in \mathbb{R} \implies |f(x) - f(y)| \geq c|x - y| \). Similarly, \( f \) is called locally reverse Lipschitz at \( x \) if there exists an \( \epsilon \)-neighborhood \( N_{\epsilon}(x) \) of \( x \) such that \( f \) restricted to \( N_{\epsilon}(x) \) is reverse Lipschitz.

Definition 2.2 Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a map and \( p \) be a fixed point of \( f \). One says that \( p \) is a sink (or attracting fixed point) if there exists an \( \epsilon > 0 \) such that, for all \( x \in N_{\epsilon}(p) \), \( \lim_{k \to \infty} f^k(x) = p \). On the other hand, if all points sufficiently close to \( p \) are repelled from \( p \), then \( p \) is called a source. In other words, \( p \) is a source if there exists an epsilon neighborhood \( N_{\epsilon}(p) \) such that, for every \( x \in N_{\epsilon}(p), x \neq p \), there exists a positive integer \( k \) with \( |f^k(x) - p| \geq \epsilon \).

3 The Results

In this section, we present the contributions of this paper. We divide the section into four subsections: the stability of fixed points in \( \mathbb{R} \), stability of periodic orbits, stability of maps on the Euclidean space \( \mathbb{R}^n \), and a new definition of Lyapunov exponent for Lipschitz maps.

3.1 Stability of fixed points in \( \mathbb{R} \)

We begin this subsection by recalling a well-known result shown in the literature.

Theorem 3.1 \( \square \) [11] Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a smooth map and let \( p \) be a fixed point of \( f \). If \( |f'(p)| < 1 \), then \( p \) is a sink. On the other hand, if \( |f'(p)| > 1 \), then \( p \) is a source.

From here to the end of the paper, we show that exchanging the differentiability condition to the Lipschitz condition, the results which hold for standard discrete dynamical systems are also true in this new context. Since a Lipschitz map do not need to be differentiable (remember that for \( x \neq y \), it follows that \( \frac{|f(x) - f(y)|}{|x - y|} \leq c \); the existence of the limit is not guaranteed), we can utilize this new approach for a wider class of maps.

Theorem 3.2 shown in the following, is the first contribution of this paper.

Theorem 3.2 (Stability test for fixed points) Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a map and \( p \in \mathbb{R} \) a fixed point of \( f \).

1. If \( f \) is strictly locally Lipschitz map at \( p \), with Lipschitz constant \( c < 1 \), then \( p \) is a sink.

2. If \( f \) is locally reverse Lipschitz map at \( p \), with constant \( r > 1 \), then \( p \) is a source.
Proof: To show Item 1-), let $f$ be a strictly locally Lipschitz map at $p$ with Lipschitz constant $c < 1$. Then there exists an $\epsilon$-neighborhood $N_\epsilon(p)$ of $p$ such that $|f(x) - f(y)| < c|x - y|$ for all $x \in N_\epsilon(p)$. Therefore, if $x \in N_\epsilon(p)$ then $|f(x) - f(p)| = |f(x) - f(p) - (f(x) - f(y)) + (f(x) - f(y))| < |f(x) - f(y)| < \epsilon$, i.e., $f(x) \in N_\epsilon(p)$. Applying the same argument, it follows that $f^2(x), f^3(x), \ldots, f^n(x), \ldots$ also belong to $N_\epsilon(p)$.

Next, we will prove by induction that the inequality $|f^k(x) - p| < c^k|x - p|$, $\forall x \in N_\epsilon(p)$, holds for all $k \geq 1$. It is clear that for $k = 1$ the inequality holds. Assume that the inequality is true for $k$: $|f^k(x) - p| < c^k|x - p|$. We must prove that $|f^{k+1}(x) - p| < c^{k+1}|x - p|$ is also true. As $f$ is strictly locally Lipschitz in $N_\epsilon(p)$ and since $f^k(x) \in N_\epsilon(p)$ we know that $|f^{k+1}(x) - p| < c|f^k(x) - p|$. From induction hypothesis one has $|f^{k+1}(x) - p| < c^{k+1}|x - p|$ and the result follows. Since $c < 1$ it follows that $\lim_{k \to \infty} c^{k+1}|x - p| = 0$. Thus $\lim_{k \to \infty} f^k(x) = p$, i.e., $p$ is a sink, as desired.

In order to prove Item 2-). From hypothesis, we know that there exists an $\epsilon$-neighborhood $N_\epsilon(p)$ of $p$ such that $|f(x) - p| \geq r|x - p|$ for all $x \in N_\epsilon(p)$. Fix $x \in N_\epsilon(p), x \neq p$. If $|f(x) - p| \geq \epsilon$, then the result follows. Otherwise, $|f(x) - p| < \epsilon$, which implies that $f(x) \in N_\epsilon(p)$. Applying again the fact that $f$ is locally reverse Lipschitz in $N_\epsilon(p)$, one has $|f^2(x) - p| \geq r|f(x) - p| \geq r^2|x - p|$. If $|f^2(x) - p| \geq \epsilon$, the result holds. Otherwise, $|f^2(x) - p| < \epsilon$, i.e., $f^2(x) \in N_\epsilon(p)$. Because $f^2(x) \in N_\epsilon(p)$ and since $f$ is locally reverse Lipschitz in $N_\epsilon(p)$, it follows that $|f^3(x) - p| \geq r|f^2(x) - p| \geq r^3|x - p|$. If $|f^3(x) - p| \geq \epsilon$, then the result is true. Otherwise, we proceed similarly as above. Applying repeatedly this reasoning, it follows that there exists an integer $k^* \geq 1$ such that $|f^{k^*}(x) - p| \geq r^{k^*}|x - p| \geq \epsilon$, i.e., $|f^{k^*}(x) - p| \geq \epsilon$. More precisely, because $r > 1$ and since $|x - p|$ is a fixed positive real number, there exists a sufficiently large positive integer $k^*$ such that $r^{k^*}|x - p| \geq \epsilon$, which implies that $|f^{k^*}(x) - p| \geq \epsilon$ holds. Therefore, $p$ is a source. The proof is complete. \hfill \Box

Corollary 3.3 Let $f : \mathbb{R} \to \mathbb{R}$ be a map.

1- If $f$ is strictly Lipschitz, with constant $c < 1$, then there exists only one fixed point $p$ which is a sink.

2- If $f$ is reverse Lipschitz with constant $r > 1$, then all fixed point $p$ is a source.

Proof: Item 1-) is the well-known Banach contraction theorem on the real line (see [Thm. 5.2.1]).

To show Item 2-), it suffices to utilize an analogous proof of Theorem 3.2 in the following way. From hypothesis, we know that $|f(x) - p| \geq r|x - p|$ for all $x \in \mathbb{R}$. We fix $x \in \mathbb{R}, x \neq p$. If $|f(x) - p| \geq \epsilon$, then the result follows. Otherwise, applying the fact that $f$ is reverse Lipschitz at $p$, one has $|f^2(x) - p| \geq r|f(x) - p| \geq r^2|x - p|$. If $|f^2(x) - p| \geq \epsilon$, the result holds. Otherwise, $|f^2(x) - p| < \epsilon$. Since $f$ is reverse Lipschitz at $p$, it follows that $|f^3(x) - p| \geq r|f^2(x) - p| \geq r^3|x - p|$. If $|f^3(x) - p| \geq \epsilon$, then one has the result. Otherwise, applying repeatedly this reasoning, there will be an integer $k_0 \geq 1$...
such that $|f^{k_0}(x) - p| \geq \epsilon$. □

3.2 Stability of periodic points in $\mathbb{R}$

In this subsection, we deal with the stability of periodic points.

We first recall some known results concerning such topic. For more details, we refer the reader to [1, 11]. Let $f : \mathbb{R} \to \mathbb{R}$ be a map and $p \in \mathbb{R}$. Recall that $p$ is said to be a periodic point of period $k$ (or $k$-periodic point) if $f^k(p) = p$ and if $k$ is the smallest such positive integer. The orbit of $p$ (which consists of $k$ points) is called a periodic orbit of period $k$ (or $k$-periodic orbit). We will denote the $k$-periodic orbit of $p$ by $O^k_p$.

If $f : \mathbb{R} \to \mathbb{R}$ is a map and $p$ is a $k$-periodic point, then the orbit $O^k_p$ of $p$ is called a periodic sink if $p$ is a sink for the map $f^k$. Analogously, $O^k_p$ is a periodic source if $p$ is a source for $f^k$.

Let us recall the stability criteria for periodic orbits.

**Theorem 3.4** [1, 11] Let $f : \mathbb{R} \to \mathbb{R}$ be a map. If $|f'(p_1) \cdots f'(p_k)| < 1$ then the $k$-periodic orbit $O^k_p = \{p_1, \ldots, p_k\}$ is a sink; if $|f'(p_1) \cdots f'(p_k)| > 1$ then $O^k_p$ is a source.

The following result is new version of Theorem 3.4 for (reverse) Lipschitz maps.

**Theorem 3.5** (Stability test for periodic orbits) Let $g = f^k : \mathbb{R} \to \mathbb{R}$ be a map and $p \in \mathbb{R}$ a fixed point of $g$.

1- If $g$ is strictly locally Lipschitz map at $p$, with Lipschitz constant $c < 1$, then $O^k_p$ is a periodic sink.

2- If $g$ is locally reverse Lipschitz map at $p$, with constant $r > 1$, then $O^k_p$ is a periodic source.

**Proof:** 1-) Assume that $g$ is strictly locally Lipschitz map at $p$. Then there exists an $\epsilon$-neighborhood $N_\epsilon(p)$ in which $g$ is strictly Lipschitz. Furthermore, we have shown in the proof of Theorem 3.2 that $|g'(x) - p| < c|x - p|$ for all $x \in N_\epsilon(p)$. From Theorem 3.2 $p$ is a sink for the map $g = f^k$, i.e., $O^k_p$ is a periodic sink.

2-) The proof is the same to that of Item 2-) of Theorem 3.2 considering the map $g = f^k$. □

3.3 Stability of fixed points in $\mathbb{R}^m$

As usual, we denote vectors in $\mathbb{R}^m$ and maps on $\mathbb{R}^m$ by boldface letters. Let us consider the $m$-dimensional real vector space $\mathbb{R}^m$ endowed with a norm $\| \cdot \|$ (in particular, the Euclidean norm). Let $p = (p_1, \ldots, p_m), v = (v_1, \ldots, v_m) \in \mathbb{R}^m$
be two points (vectors). The $\epsilon$-neighborhood $N_{\epsilon}(p)$ of $p$ is defined by $N_{\epsilon}(p) = \{ v \in \mathbb{R}^m : \| v - p \| < \epsilon \}$. Let $f : \mathbb{R}^m \rightarrow \mathbb{R}^m$ be a map and let $p \in \mathbb{R}^m$ be a fixed point of $f$, i.e., $f(p) = p$. If there exists an $\epsilon$-neighborhood $N_{\epsilon}(p)$ of $p$ such that $\forall v \in N_{\epsilon}(p)$, \( \lim_{k \rightarrow \infty} f^k(v) = p \), then $p$ is called a sink (or attracting fixed point). If there exists an $N_{\epsilon}(p)$ such that $\forall v \in N_{\epsilon}(p)$, except for $p$ itself, eventually maps outside of $N_{\epsilon}(p)$, then $p$ is called a source (or repeller).

If $f$ is a smooth map and $p \in \mathbb{R}^m$, we represent $f$ in terms of its coordinates functions $f = (f_1, \ldots, f_m)$. Let $Df(p)$ be the Jacobian matrix of $f$ at $p$. With this notation in mind, we can state the following well-known result:

**Theorem 3.6 (Thm. 2.11)** Let $f : \mathbb{R}^m \rightarrow \mathbb{R}^m$ be a map and assume that $p \in \mathbb{R}^m$ is a fixed point of $f$. If the magnitude of each eigenvalue of $Df(p)$ is less than 1, then $p$ is a sink; if the magnitude of each eigenvalue of $Df(p)$ is greater than 1, then $p$ is a source.

On the vector space $\mathbb{R}^m$, the concept of Lipschitz map reads as follows.

Let $f : \mathbb{R}^m \rightarrow \mathbb{R}^m$ be a map. We say that $f$ is Lipschitz if there exists a constant $c \in \mathbb{R}$, $c > 0$ such that $\forall v, w \in \mathbb{R}^m \Rightarrow \| f(v) - f(w) \| \leq c \| v - w \|$, where $\| \cdot \|$ denotes a norm over $\mathbb{R}^m$. If $\forall v, w \in \mathbb{R}^m \Rightarrow \| f(v) - f(w) \| < c \| v - w \|$, we say that $f$ is strictly Lipschitz.

The next result is a natural generalization of Theorem 3.2 to the Euclidean space $\mathbb{R}^m$.

**Theorem 3.7 (Stability test for fixed points on $\mathbb{R}^m$)** Let $f : \mathbb{R}^m \rightarrow \mathbb{R}^m$ be a map and let $p \in \mathbb{R}^m$ a fixed point of $f$.

1. If $f$ is strictly locally Lipschitz map at $p$, with Lipschitz constant $c < 1$, then $p$ is a sink.

2. If $f$ is locally reverse Lipschitz map at $p$, with constant $r > 1$, then $p$ is a source.

**Proof:** The proofs of both items are the same to that of Theorem 3.2 only changing the absolute value function $\| \cdot \|$ on $\mathbb{R}$ by a norm $\| \cdot \|$ on $\mathbb{R}^m$.

Item 1) We know there exists an $\epsilon$-neighborhood $N_{\epsilon}(p)$ of $p$ such that $\| f(x) - f(p) \| < c \| x - p \|$ for all $x \in N_{\epsilon}(p)$. Therefore, if $x \in N_{\epsilon}(p)$ then $\| f(x) - f(p) \| < \epsilon$, i.e., $f(x) \in N_{\epsilon}(p)$. By the same argument, it follows that $f^k(x), \ldots, f^m(x), \ldots$ also belong to $N_{\epsilon}(p)$. Applying induction, we can show that $\| f^k(x) - p \| < c^k \| x - p \|$, $\forall x \in N_{\epsilon}(p)$, holds for all $k \geq 1$. Since $c < 1$ it follows that $\lim_{k \rightarrow \infty} c^{k+1} = 0$, so $p$ is a sink.

Item 2-) There exists an $\epsilon$-neighborhood $N_{\epsilon}(p)$ of $p$ such that $\| f(x) - p \| \geq r \| x - p \|$ for all $x \in N_{\epsilon}(p)$. Let us consider $x \in N_{\epsilon}(p)$, $x \neq p$. If $\| f(x) - p \| \geq \epsilon$, there is nothing to prove. Otherwise, $\| f(x) - p \| < \epsilon$, which implies that $f(x) \in N_{\epsilon}(p)$. Because $f$ is locally reverse Lipschitz in $N_{\epsilon}(p)$, one has $\| f^2(x) - p \| \geq r^2 \| x - p \|$. If $\| f^2(x) - p \| \geq \epsilon$, the result holds. Otherwise, $\| f^2(x) - p \| < \epsilon$, i.e., $f^2(x) \in N_{\epsilon}(p)$. Proceeding similarly as in the proof of
Theorem 3.2 the results follows.

\[\square\]

Corollary 3.8 Let \( f : \mathbb{R}^m \rightarrow \mathbb{R}^m \) be a map.

1. If \( f \) is strictly Lipschitz map with Lipschitz constant \( c < 1 \), then there is only one fixed point \( p \) which is a sink.

2. If \( f \) is reverse Lipschitz map with constant \( r > 1 \), then all fixed point \( p \) is a source.

Proof: Item 1-) is the well-known Banach contraction theorem (see for example [11, Thm. 5.2.1]). The proof of Item 2) is similar to that of Corollary 3.3. We present it here for completeness. Since \( f \) is locally reverse Lipschitz map at \( p \), it follows that \( \| f(x) - p \| \geq r \| x - p \| \) for all \( x \in \mathbb{R}^m \). Let us consider a fixed vector \( x \in \mathbb{R}^m \), \( x \neq p \). If \( \| f(x) - p \| \geq \epsilon \), then the result follows. Otherwise, because \( f \) is reverse Lipschitz at \( p \), it follows that \( \| f^i(x) - p \| \geq r \| f(x) - p \| \geq r^2 \| x - p \| \). If \( \| f^i(x) - p \| \geq \epsilon \), the result holds. On the other hand, \( \| f^i(x) - p \| < \epsilon \). Applying again the fact that \( f \) is reverse Lipschitz at \( p \) one has \( \| f^i(x) - p \| \geq r \| f^i(x) - p \| \geq r \| x - p \| \). If \( \| f^i(x) - p \| \geq \epsilon \), then the result follows. Otherwise, applying repeatedly this procedure, there will be an integer \( k_0 \geq 1 \) such that \( \| f^{k_0}(x) - p \| \geq \epsilon \). \[\square\]

Remark 3.9 Note that the procedure utilized in Subsection 3.2 can be easily adapted to generate analogous results for the stability of periodic orbits of maps defined over \( \mathbb{R}^m \). Since both proofs are similar, we do not present the last one here.

3.4 Lyapunov exponent

In this subsection, we introduce in the literature the Lyapunov number and the Lyapunov exponent for Lipschitz maps. We only consider the case of maps defined over \( \mathbb{R} \) (or over any subset of \( \mathbb{R} \)), since the procedure for maps on \( \mathbb{R}^n \) (or over any subset of \( \mathbb{R} \)) is quite similar.

We denote by \( \mathcal{O}_{x_1} = \{x_1, x_2, x_3, \ldots \} \) an arbitrary orbit with initial point \( x_1 \in \mathbb{R} \), where \( x_2 = f(x_1), x_3 = f^2(x_1), x_4 = f^3(x_1), \ldots \). Assume that \( f \) is a smooth map on \( \mathbb{R} \) and \( x_1 \in \mathbb{R} \). Recall that the Lyapunov number \( L(x_1) \) of the orbit \( \mathcal{O}_{x_1} = \{x_1, x_2, x_3, \ldots \} \) is defined as \( L(x_1) = \lim_{n \to \infty} (\prod |f'(x_1)| \cdots |f'(x_n)|)^{1/n} \), if the limit exists. The Lyapunov exponent \( h(x_1) \) is defined as \( h(x_1) = \lim_{n \to \infty} (1/n) \log |f'(x_1)| + \cdots + \log |f'(x_n)| \), if the limit exists.

Let \( \mathcal{O}_{y_1} = \{y_1, y_2, \ldots, y_n, \ldots \} \) be an orbit and let \( \mathcal{O}_{y_1}^k = \{y_1, y_2, \ldots, y_k\} \) be a \( k \)-periodic orbit (see beginning of Subsection 3.2). We say that the orbit \( \mathcal{O}_{y_1} \) is asymptotically periodic (see for instance [1] Definition 3.3) if it converges to a periodic orbit \( \mathcal{O}_{y_1}^k \) for some integer \( k \geq 1 \) and \( y_1 \in \mathbb{R} \), when
In other words, there exists a periodic orbit \( \{y_1, y_2, \ldots, y_k\} = \{y_1, y_2, \ldots, y_k, y_1, y_2, \ldots, y_k, \ldots\} \) such that \( \lim_{n \to \infty} |x_n - y_n| = 0 \).

Until now in this subsection, we recall the concepts of Lyapunov number and Lyapunov exponent for smooth maps, which are well-known in the literature. Now, we will propose the definition of Lyapunov number and Lyapunov exponent for Lipschitz maps, which are not necessarily differentiable. The first result that will be utilized to this goal is due to Rademacher:

**Theorem 3.10 (Rademacher’s theorem)** [Thm. 3.1.6.][5] (see also [14]) If \( f : \mathbb{R}^m \to \mathbb{R}^n \) is a Lipschitz map, then \( f \) is differentiable at Lebesgue almost all points of \( \mathbb{R}^m \).

A variant of this result is given below.

**Theorem 3.11** [6, Thm. 3.1] Let \( \Omega \subset \mathbb{R}^m \) be an open set, and let \( f : \Omega \to \mathbb{R}^n \) be a Lipschitz map. Then \( f \) is differentiable at almost every point (Lebesgue) in \( \Omega \).

From Rademacher’s theorem, we can guarantee that a Lipschitz map \( f : \mathbb{R} \to \mathbb{R} \) is differentiable in a set \( X = \mathbb{R} - Y \), where the set \( Y \) has zero Lebesgue measure. In this new context, we can define the Lyapunov number and the Lyapunov exponent as well as the concept of asymptotically periodic orbit for Lipschitz maps.

**Definition 3.1** Let \( f : \mathbb{R} \to \mathbb{R} \) be a Lipschitz map and assume that \( O_{x_1} \subset X \). Then the Lyapunov number \( L(x_1) \) of the orbit \( O_{x_1} = \{x_1, x_2, x_3, \ldots\} \) is defined as

\[
L(x_1) = \lim_{n \to \infty} (|f'(x_1)| \cdots |f'(x_n)|)^{1/n},
\]

if the limit exists.

The Lyapunov exponent \( h(x_1) \) is defined as

\[
h(x_1) = \lim_{n \to \infty} (1/n)[\ln |f'(x_1)| + \cdots + \ln |f'(x_n)|],
\]

if the limit exists.

Recall that a map \( f : \mathbb{R} \to \mathbb{R} \) is said to be **locally Lipschitz** on an open interval \((a, b) \subset \mathbb{R}\) if \( f \) restricted to \((a, b)\) is Lipschitz. In terms of locally Lipschitz maps, we have the following variant to Definition 3.1.

**Definition 3.2** Let \( f : \mathbb{R} \to \mathbb{R} \) be a locally Lipschitz map in a (nondegenerate) open interval \((a, b)\), and assume that \( O_{x_1} \subset (a, b) \cap X \). Then the Lyapunov number \( L(x_1) \) of the orbit \( O_{x_1} = \{x_1, x_2, x_3, \ldots\} \) is defined as

\[
L(x_1) = \lim_{n \to \infty} (|f'(x_1)| \cdots |f'(x_n)|)^{1/n},
\]

if the limit exists.
The Lyapunov exponent $h(x_1)$ is defined as
\begin{equation}
    h(x_1) = \lim_{n \to \infty} (1/n)[\ln|f'(x_1)| + \cdots + \ln|f'(x_n)|],
\end{equation}
if the limit exists.

Let us reformulate the concept of asymptotically periodic orbit in terms of Lipschitz maps.

**Definition 3.3** Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz map. An orbit $\{x_1, x_2, \ldots, x_n, \ldots\}$ is called asymptotically periodic if it converges to a periodic orbit when $n \to \infty$. In other words, there exists a periodic orbit $\{y_1, y_2, \ldots, y_k, y_1, y_2, \ldots, y_k, \ldots\}$ such that $\lim_{n \to \infty} |x_n - y_n| = 0$.

The following result is a variant of [1, Theorem 3.4] based on Lipschitz maps.

**Theorem 3.12** Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz map with first derivative continuous in the set $X$. Assume that the orbit $O_{x_1} = \{x_1, x_2, \ldots, x_n, \ldots\} \subset X$ satisfies $f'(x_i) \neq 0$ for all $i = 1, 2, \ldots$. If $O_{x_1}$ is asymptotically periodic to the periodic orbit $O_{y_1} = \{y_1, y_2, \ldots, y_k, y_1, y_2, \ldots, y_k, \ldots\}$, then $h(x_1) = h(y_1)$, if both Lyapunov exponent exist.

**Proof:** Since $f$ is a Lipschitz map, applying Rademacher's theorem with $n = m = 1$, it follows that $f$ has derivative in the set $X$. As $O_{x_1} \subset X$, we can guarantee the derivative of all point of $O_{x_1}$. Although from here the proof is similar to the proof of [1, Theorem 3.4], we even present it here for completeness.

Assume that $k = 1$; then $\lim_{n \to \infty} x_n = y_1$. Since the derivative is continuous, it follows that $\lim_{n \to \infty} f'(x_n) = f'(y_1)$. Moreover, one has $\lim_{n \to \infty} \ln|f'(x_n)| = \ln|f'(y_1)|$. Therefore, $h(x_1) = \lim_{n \to \infty} 1/n \sum_{i=1}^{n} \ln|f'(x_i)| = \ln|f'(y_1)| = h(x_1)$. If $k > 1$, we know that $y_1$ is a fixed point of $f^k$ and $O_{x_1}$ is asymptotically periodic under $f^k$ to $O_{y_1}$. Applying the reasoning above to $x_1$ and $f^k$ it follows that $h(x_1) = \ln|(f^k)'(y_1)|$. It is known that if $L$ is the Lyapunov number of $O_{x_1}$ under the map $f$, then the Lyapunov number of $O_{x_1}$ under the map $f^k$ is $L^k$ (see [1, Ex T3.1]). Then the Lyapunov exponent $h(x_1)$ of $x_1$ under $f$ equals $h(x_1) = 1/k \ln|(f^k)'(y_1)| = h(y_1)$. The proof is complete.

A version of Theorem 3.12 to locally Lipschitz maps is given below.

**Theorem 3.13** Let $f : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz map on the open interval $(a, b)$ with first derivative continuous in $(a, b)$. Assume that the orbit $O_{x_1} \subset (a, b) \cap X$ satisfies $f'(x_i) \neq 0$ for all $i = 1, 2, \ldots$. If $O_{x_1}$ is asymptotically periodic to the periodic orbit $O_{y_1} = \{y_1, y_2, \ldots, y_k, y_1, y_2, \ldots, y_k, \ldots\}$, then $h(x_1) = h(y_1)$ if both Lyapunov exponent exist.
Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a map, and let \( O_{x_1} \) be a bounded orbit of \( f \). Recall that the orbit is chaotic if \( O_{x_1} \) is not asymptotically periodic and if the Lyapunov exponent \( h(x_1) \) is greater than zero. In terms of Lipschitz maps one has the following new definition for chaotic orbits:

**Definition 3.4** Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be a Lipschitz map with first derivative continuous at \( X \), and let \( O_{x_1} \) be a bounded orbit of \( f \). We call the orbit chaotic if:

1. \( O_{x_1} \) is not asymptotically periodic;
2. the Lyapunov exponent \( h(x_1) \) is greater than zero.

**Remark 3.14** It is interesting to note that, since Lipschitz maps are differentiable almost everywhere (with respect to the Lebesgue measure) according to Theorems 3.10 and 3.11, then the numerical simulations are performed similarly to the standard case, i.e., the cases where the map is differentiable. Because of this fact, we do not present numerical simulations proofen in this paper.

To finish this section, we give three examples. The first one shows a map which is locally Lipschitz but it is not differentiable. The second presents a family of Lipschitz maps which are not differentiable; the third example consists in a family of logistic maps which are both Lipschitz and differentiable.

**Example 3.1** Let us consider the map \( f : \mathbb{R} \rightarrow \mathbb{R} \) given by

\[
x_{n+1} = f(x_n) = \begin{cases} 
2x_n, & x_n < 0 \\
0.5x_n + 0.5, & x_n \geq 1
\end{cases}
\]

Figure 1 shows the graphic of the map \( f \). Note that \( f \) is not differentiable at \( p = 0 \) and \( p = 1 \), but it is locally Lipschitz in the open intervals \((-\infty, 0), (0, 1) \) \((1, +\infty)\). Thus, our method can be applied.

**Example 3.2** Here, let us consider the family of Lipschitz maps \( g_{a,b} : \mathbb{R} \rightarrow \mathbb{R} \) given by

\[
x_{n+1} = g_{a,b}(x_n) = a|x_n| + b,
\]

where \( a, b \) are real numbers. For \( a = -2 \) and \( b = 1 \), the graphic of \( g_{-2,1} \) is shown in Figure 2. Note that the map \( g_{-2,1} \) is Lipschitz, so our method can be applied, but it is not differentiable. These maps are a type of tent maps.

**Example 3.3** Let \( f : [0, 1] \rightarrow [0, 1] \) be a map given by

\[
x_{n+1} = f(x_n) = ax_n(1-x_n),
\]

where \( 0 < a < 4 \) is a real number. These (family of) maps are well known in the literature as logistic maps. It is easy to see that \( f \) is a Lipschitz (and also differentiable) map. Figure 3 displays the graphic of the logistic map for \( a = 3 \).
Figure 1: Example of locally Lipschitz map which is not differentiable

![Graph of a locally Lipschitz map which is not differentiable.]

Figure 2: Example of Lipschitz map which is not differentiable

![Graph of a Lipschitz map which is not differentiable.]

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4 Final Remarks

We have proposed a definition of Lyapunov exponent for Lipschitz maps, which are not necessarily differentiable. Furthermore, we have shown that the results which are valid to discrete standard dynamical systems are also true in this new context. This novel approach expands the theory of dynamical systems and it is simple to be applied. As a future work, it will be interesting to investigate the possibility of defining Lyapunov exponents for other class of maps, such as Holder continuous maps.

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