Scale invariance in galaxy structures: general concepts and application to the most recent data

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The debate on the correlation properties of galaxy structures has having an increasing interest during the last year. In this lecture we discuss the claims of different authors who have criticized our approach and results. In order to have a clear cut of the situation, we focus mainly on galaxy distribution in the intermediate range of distances $\sim 100 \div 200 h^{-1}$ Mpc. In particular we discuss: (i) the validity of the actual data and the concept of fair sample, (ii) the shift of $r_0$ with sample depth and luminosity bias, (iii) the value of the fractal dimension, (iv) the problem of the counts from a single point and the case of ESP, (v) uniformity of angular catalogs. The detection of fractal behavior up to $\sim 100 \div 200 h^{-1}$ Mpc is enough to rise serious problems to the usual statistical methods used for the characterization of galaxy correlations, standard interpretation of galaxy distribution and theoretical models developed. The clarification of the intermediate scale behavior is very instructive for the subsequent interpretation of the very large scale galaxy distribution.

1 Introduction

The presence of large scale structures (hereafter LSS) of galaxies and galaxy clusters is nowadays one of the most intriguing feature of the visible universe. The discovery of the inhomogeneous galaxy distribution began with the availability of intense redshift measurements in the eighties. The statistical characterization of such structures is however a matter of debate. While there is a general agreement that the inhomogeneities seen in the galaxy catalogs correspond to a correlated fractal distribution up to $10 \div 20 h^{-1}$ Mpc, at larger scales the correct statistical properties are still controversial. Moreover there is no agreement on the value of the fractal dimension even at small scales. The clarification of these facts is crucially important with respect to the theoretical interpretation of such structures. In fact, homogeneity of matter distribution is one of the cornerstone of the Hot Big Bang model as well as of the different models of galaxy formation, and then it is quite important to establish whether galaxy structures approach to a smooth distribution on large enough scales.

By the analysis of galaxy redshift samples one can study the structure

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of the Local Universe up to \( \sim 100 \div 200h^{-1}\text{Mpc} \). The recent availability of wide angle galaxy surveys such as CfA, SSRS, Perseus-Pisces, APM-Stromlo, IRAS and LEDA, has in fact allowed a statistical description of the nearby luminous matter distribution in a rather complete way. The fact that galaxy redshift catalogs are dominated by LSS and huge voids has lead several authors to define them *not fair* with respect to the *expected* statistical properties of the large scale matter distribution in the universe, i.e. homogeneity. By using the standard two-points correlation function it has been found that galaxy structures are characterized by having a very well defined "correlation length" that is established to be \( r_0 \approx 5h^{-1}\text{Mpc} \). The physical interpretation of such a scale being the distance above which the density fluctuations become of the same order of the average density. At twice this distance the fluctuations have small amplitude and the linear theory holds, i.e the distribution becomes homogeneous. The evidence in favor of this result coming from various different observations: the uniformity of galaxy angular catalogs, the number counts of galaxies as a function of apparent magnitude and the so-called "rescaling" of the amplitude of the angular correlation function. In addition, it has been found by several authors that the correlation length \( r_0 \) has a very qualitative dependence on galaxy luminosity. According to such an effect, called "luminosity bias", \( r_0 \) increases in the samples which contain the brightest galaxies. In summary, the most popular point of view, is that galaxy distribution exhibits indeed fractal properties with dimension \( D \approx 1.3 \) at small scale (i.e. up to \( \sim 10 \div 20h^{-1}\text{Mpc} \)), and at larger scales there is an overwhelming evidence in favor of homogeneity. We will refer to this *picture of galaxy clustering as the standard one*. The main problems that this picture rises are:

- The problem of the statistical validity of different redshift samples.
- The presence of LSS and the detection of a small correlation length.
- The mismatch galaxy-cluster (different correlation lengths of galaxies and galaxy clusters).
- The luminosity bias and the clustering properties of different galaxy types.
- The possible "*reconstruction*" of the three dimensional properties from angular catalogs.

This problematic situation has lead us to reconsider the basic assumption of the usual analysis of galaxy clustering. In this respect, a completely different
interpretation of galaxy correlations has been suggested by our group. By analyzing the different galaxy samples with the statistical tools suitable and appropriate to characterize highly irregular distributions, as well as regular ones, it has been found that

- galaxy and cluster distributions exhibit very well defined scale invariant properties with fractal dimension $D \approx 2$ up to the limits of the analyzed sample ($R_s \approx 150 h^{-1} \text{Mpc}$),

- the different catalogs show almost the same statistical properties and are in a reasonable agreement with each other,

- the standard picture of galaxy correlations has been derived by an analysis which assumes a priori homogeneity of matter distribution. Such a statistical method is not suitable for the characterization of self-similar structures. In particular, both the fractal dimension ($D \approx 1.3$) and the "correlation length" $r_0 \approx 5 h^{-1} \text{Mpc}$ are artifact of the data analysis, and do not correspond to the correct statistical properties of galaxy distribution. The standard correlation analysis is one of the statistics usually used, which are not suitable to detect the properties of irregular (self-similar) structures. Others are the power spectrum, the density contrast as a function of scale and quantities related (see Sylos Labini et al. 1998 - hereafter Paper 1).

- There are various evidences, which are statistically weaker, and which support the continuation of the fractal behavior with $D \approx 2$ up to the deepest scale investigated so far.

Mandelbrot was the first to propose the relevance of fractal structures in respect with the problem of galaxy clustering. He has introduced the concept of "Fractal Geometry" which has opened a new perspective on natural phenomena and of which we will make extensive use in what follows. However up to the eighties the discussion was based mainly on the angular data, as the redshift measurements were very sparse. The debate on galaxy correlations has had an important impulse in 1987 with the work of L.P. Then in 1996 in the Conference "Critical Dialogues in Cosmology" there has been a debate between Prof. Marco Davis and one of us (L.P.) The point of view of Prof. Davis was that $r_0$ is indeed a real correlation length and hence a characteristic scale of galaxy distribution, while L.P. has argued in favour of a fractal behavior of galaxy distribution in the available samples. In the last few months a wide debate on this subject is in progress and different authors have expressed different points of view, which are in
general more articulated with respect to the one of Davis. (See the web page [http://www.phys.uniroma1.it/DOCS/PIL/pil.html](http://www.phys.uniroma1.it/DOCS/PIL/pil.html) where all these materials have been collected). However different authors supporting the homogeneity of large scale galaxy distribution are often in contradiction with each other. In this lecture, after reviewing our main results, we address the most controversial points in this debate, which can be summarized as follows:

- Validity of the actual data and the concept of "fair sample.
- The shift of $r_0$ with sample depth and luminosity bias.
- Value of the fractal dimension.
- Problems of counts from a single point: radial counts in the European Slide Project (ESP) redshift survey.
- Uniformity of the angular catalogs.

In our opinion the fact that the clarification of the statistical properties of the local universe (i.e. up to $\sim 100h^{-1}\text{Mpc}$) is a fundamental fact with respect to the interpretation of the far away galaxy distribution. (We note that in this range of scale the data are statistically very robust.) Instead of reviewing the different arguments for scales larger than $\sim 100h^{-1}\text{Mpc}$ (see Paper 1), we will focus our discussion on smaller scales. Moreover we briefly discuss the fact that the CDM-like models of galaxy formation are unable to reproduce the clustering properties of real galaxy samples. The problem here is twofold. From one side the theoretical models have been developed to explain the incorrect properties deduced from an inconsistent analysis of galaxy surveys. From the other side the simulations are analyzed, as well as real galaxy surveys, with statistical methods which are unable to recover the properties of irregular (self-similar) distributions. From this confusing situation several concepts have arisen, like the bias galaxy formation, the luminosity bias and others, which are not appropriate to describe the scale-invariant structures observed in the data.

A interesting theoretical attempt in the new perspective of scaling exponents have been developed very recently the group of Prof. N. Sanchez and Prof. H. De Vega. We stress below the importance of the change of theoretical perspective related to the use of the proper statistical methods.

## 2 Review of main results and statistical validity of galaxy data

The proper methods to characterize irregular as well as regular distributions have been discussed in Coleman & Pietronero and Sylos Labini et al. in a
detailed and exhaustive way. The basic point is that, as far as a system shows power law correlations, the usual \( \xi(r) \) analysis gives an incorrect result, since it is based on the a-priori assumption of homogeneity. In order to check whether homogeneity is present in a given sample one has to use the conditional density \( \Gamma(r) \) defined as

\[
\Gamma(r) = \frac{\langle n(r_s)n(r_s + r) \rangle}{\langle n \rangle} = \frac{BD}{4\pi r^{D-3}}
\]  

(1)

where the last equality holds in the case of a fractal distribution with dimension \( D \) and prefactor \( B \). In the case of an homogenous distribution (\( D = 3 \)) the conditional density equals the average density in the sample. Hence the conditional density is the suitable statistical tool to identify fractal properties (i.e. power law correlations with codimension \( \gamma = 3 - D \)) as well as homogeneous ones (constant density with sample size). If there exists a transition scale \( \lambda_0 \) towards homogenization, we should find \( \Gamma(r) \) constant for scales \( r \gtrsim \lambda_0 \).

It is simple to show that in the case of a fractal distribution the usual \( \xi(r) \) function in a spherical sample of radius \( R_s \) is

\[
\xi(r) = \frac{D}{3} \left( \frac{r}{R_s} \right)^{D-3} - 1.
\]  

(2)

From Eq.2 we can see two main problems of the \( \xi(r) \) function: its amplitude depends on the sample size \( R_s \) (and the so-called correlation length \( r_0 \), defined as \( \xi(r_0) \equiv 1 \), linearly depends on \( R_s \)) and it has not a power law behavior. Rather the power law behavior is present only at scales \( r \ll r_0 \), and then it is followed by a sharp break in the log-log plot as soon as \( \xi(r) \lesssim 1 \). Such a behavior does not correspond to any real change of the correlation properties of the system (that is scale-invariant by definition) and it makes extremely difficult the estimation of the correct fractal dimension as it is shown in Fig.1. In particular if the sample size is not large enough with respect to the actual value of \( r_0 \), the codimension estimated by the \( \xi(r) \) function (\( \gamma \approx 1.7 \)) is systematically larger than \( 3 - D \) (\( \gamma \approx 1 \)).

Given this situation it is clear that the \( \xi(r) \) analysis is not suitable to be applied unless a clear cut-off towards homogenization is present in the samples analyzed. As this is not the case, it is appropriate and convenient to use \( \Gamma(r) \) instead of \( \xi(r) \). We have discussed in detail in Paper 1 that the use of the correct statistical methods is complementary to a change of perspective from a theoretical point of view. In Tab.1 we report the characteristics of the various catalogs we have analyzed by using the methods previously illustrated. Let us consider in more detail some recent catalogs and discuss the disagreement of Tab.1 with the analyses reported by several other authors.

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Figure 1: Behaviour of the tangent to $\xi(r)$ for a mathematical fractal with dimension $D = 2$ in the log-log plot. The sample depth $R_s$ is such that $r_0 = 5h^{-1}\text{Mpc}$. We may see that at $r = r_0$ the slope is $-2\gamma$ and then it rapidly decays to $-\infty$. 

$\xi(r)$ log-derivative for a fractal (D=2)

$\xi(r)$ log-derivative for a fractal (D=2)
Table 1: The volume limited catalogues are characterized by the following parameters: - \( R_d(h^{-1}\text{Mpc}) \) is the depth of the catalogue - \( \Omega \) is the solid angle - \( R_s(h^{-1}\text{Mpc}) \) is the radius of the largest sphere that can be contained in the catalogue volume. This gives the limit of statistical validity of the sample. - \( r_0(h^{-1}\text{Mpc}) \) is the length at which \( \xi(r) \equiv 1 \). - \( \lambda_0 \) is the eventual real crossover to a homogeneous distribution that is actually never observed. The value of \( r_0 \) is the one obtained in the deepest VL sample. (Distances are expressed in \( h^{-1}\text{Mpc} \).

| Sample         | \( \Omega \) (sr) | \( R_d \) | \( R_s \) | \( r_0 \) | \( D \) | \( \lambda_0 \) |
|----------------|-------------------|-----------|-----------|---------|------|-------------|
| CfA1           | 1.83              | 80        | 20        | 6       | 1.7 ± 0.2 | > 80        |
| CfA2South      | 1.23              | 130       | 30        | 10      | 2.0 ± 0.1 | > 120       |
| PP             | 0.9               | 130       | 30        | 10      | 2.0 ± 0.1 | > 130       |
| SSRS1          | 1.75              | 120       | 35        | 12      | 2.0 ± 0.1 | > 120       |
| SSRS2          | 1.13              | 150       | 50        | 15      | 2.0 ± 0.1 | > 130       |
| Stromlo-APM    | 1.3               | 100       | 35        | 12      | 2.2 ± 0.1 | > 150       |
| LEDA           | \( \sim 5 \)      | 300       | 150       | 45      | 2.1 ± 0.2 | > 150       |
| LCRS           | 0.12              | 500       | 18        | 6       | 1.8 ± 0.2 | > 500       |
| IRAS2 Jy       | \( \sim 5 \)      | 60        | 20        | 5       | 2.0 ± 0.1 | > 50        |
| IRAS1.2 Jy     | \( \sim 5 \)      | 80        | 30        | 8       | 2.0 ± 0.1 | > 50        |
| ESP            | 0.006             | 700       | 8         | 3       | 2.0 ± 0.2 | > 700       |
• ESP. In this case the estimation of $R_s$ slightly differs from that of Guzzo, probably because we have not used the relativistic corrections (see below). Also the value of $r_0$ is slightly different ($r_0 \sim 3h^{-1}Mpc$ instead of the measured $r_0 \sim 4.5h^{-1}Mpc$), and this is probably due to the fact that ESP does not cover a continue solid angle in the sky, as it is a collection of pencil beams. Such a situation necessarily requires the introduction of spurious treatments of the boundary conditions: we limit the analysis of the conditional density to a depth $R_s$ corresponding to the radius of the maximum sphere fully contained in the sample volume.

• LCRS. This survey has the peculiar property of being limited by two limits in apparent magnitude (a lower and an upper one). In order to construct a VL sample in this case, one has to impose two limits in distances and correspondingly two in absolute magnitude. This is the origin of a smaller $R_s$ in our Table 1 than this reported by Guzzo ($R_s \sim 32h^{-1}Mpc$). This implies a smaller $r_0$, much closer to the measured one.

• Stromlo/APM. We have extensively analyzed this catalog and the value of $r_0$ is reported in Tab.1. Due to the sparse sampling strategy adopted to construct this catalog, we are able to measure the correlation properties up to $R_s \sim 40h^{-1}Mpc$ and not $83h^{-1}Mpc$ as reported by Guzzo. The disagreement with the work of Loveday et al. ($r_0 \approx 12h^{-1}Mpc$ rather than $r_0 \approx 5h^{-1}Mpc$) is probably due to the treatment of the boundary conditions and their use of ML samples rather than VL ones (i.e. they used weighting schemes with the luminosity selection function). In any case, we stress again, the proper test is to check whether the conditional density has a power behavior.

We show in Fig.2 the results of the conditional density determinations in various redshift surveys. All the available data are consistent with each other and show fractal correlations with dimension $D = 2.0 \pm 0.2$ up to the deepest scale probed up to now by the available redshift surveys, i.e. $\sim 150h^{-1}Mpc$. A similar result has been obtained by the analysis of galaxy cluster catalogs.

3 Shift of $r_0$ and luminosity bias

Several authors based a part of their arguments in favour of homogeneity on the fact that a luminosity bias is responsible of the shift of $r_0$ with sample size. In this respect, we would like to remark that the authors who have addressed this concept have never presented any quantitative argument that
Figure 2: The spatial density $\Gamma(r)$ computed in some VL samples of CfA1, PP, LEDA, APM, ESP, LCRS, SSRS1, IRAS and ESP (from Sylos Labini et al., 1998).
explains the shift of \( r_0 \) with sample size. An exception is represented by the paper of Davis et al. in which the authors claim that \( r_0 \) behaves as \( R_0^{0.5} \). However for what concerns luminosity segregation, the meaningful parameter must be the absolute magnitude limit of the volume limited (VL) sample considered rather than the depth \( (R_s) \). Having fixed the limiting apparent magnitude of the catalog, at each \( R_s \) would correspond a well defined absolute magnitude. The brightest galaxies are present yet in samples like CfA1, and hence according to the luminosity segregation paradigm, there is no reason one should expect that in deeper sample (like CfA2 or SSRS2) \( r_0 \) is increased. However this is actually the case.

We have discussed in detail in that the observation that the giant galaxies are more clustered than the dwarf ones, i.e. that the massive elliptical galaxies lie in the peaks of the density field, is a consequence of the self-similar behavior of the whole matter distribution. The increasing of the correlation length of the \( \xi(r) \) has nothing to do with this effect, rather it is related to the sample size. It may be useful to mention that the view presented here actually redefines the notion of bias. In standard biasing scenarios the distribution of galaxies of different luminosity are Gaussian but with different amplitudes. Here we consider the fact that galaxies of different luminosities have different correlation properties and hence different fractal dimensions (the cluster distribution being a coarse graining of the galaxy one, and hence having the same fractal dimension \( D \approx 2 \) of that of galaxies). The more detailed picture is just that the fractal dimension of a set of density fluctuations depends on the threshold. This tendency is actually indicated by present observations, which show a slight increase of the fractal dimension with decreasing absolute luminosity of the galaxies in the sample (however, still with relatively modest statistics). Such a scenario is naturally formulated within the framework of multi-fractals.

It remains open the question whether the distribution of dark matter continues the multi-fractal behaviour found in the galaxies. In other words, it would be possible that the dark matter fills homogeneously the space: in this case the fractal dimension of the distribution of dark matter would be \( D = 3 \). Such a scenario has been studied by Durrer & Sylos Labini and Baryshev et al. in more detail.

4 Value of the fractal dimension

On the actual value of the fractal dimension at small scale (\(< 10 h^{-1} Mpc\) there are, at least, three different points of view. The first one is the canonical point of view of Peebles, who obtains from the standard correlation function \( \xi(r) \)
a value of $D \simeq 1.3$ in the range of scale $0.1h^{-1}Mpc \lesssim r \lesssim 10h^{-1}Mpc$. The second is due to Guzzo who claims that $\Gamma(r)$ is a power law with $D \sim 1.2$ up to $r \sim 3.5h^{-1}Mpc$, then it shows a different scaling between $\sim 3h^{-1}Mpc$ and $30h^{-1}Mpc$ with fractal dimension $D \approx 2$. Finally we claim that $D \approx 2$ in the whole range of scale $0.5h^{-1}Mpc \lesssim r \lesssim 150h^{-1}Mpc$ (This result has been recently confirmed by Cappi et al. by the analysis of the SSRS2 galaxy sample.) The fact that the first two determinations are in contradiction has never been discussed by the authors. The first point has been discussed in Sec.2 and here we have nothing new to add. We focus on the determination of the fractal dimension at very small scale $r < 10h^{-1}Mpc$ by the conditional density analysis, clarifying the difference with our result with that of Guzzo and coworkers.

Suppose, for simplicity, we have a spherical sample of volume $V$ in which there are $N$ points, and we want to measure the conditional density. It is possible to compute the average distance between neighbor galaxies $\langle \Lambda \rangle$, in a fractal distribution with dimension $D$, and the result is

$$\langle \Lambda \rangle = \left( \frac{1}{B} \right)^{\frac{D}{D}} \Gamma \left( 1 + \frac{1}{D} \right)$$

(3)

where $\Gamma$ is the Euler’s gamma-function. (Note that the prefactor $B$ is dependent on the luminosity selection function of the VL chosen). Clearly this quantity is related to the lower cut-off of the distribution $B$ (eq.8) and to the fractal dimension $D$. If we measure the conditional density at distances $r \lesssim \langle \Lambda \rangle$, we are affected by a finite size effect. In fact, due the depletion of points at these distances we underestimate the real conditional density finding an higher value for the correlation exponent (and hence a lower value for the fractal dimension). In the limiting case, for distances $r \ll \langle \Lambda \rangle$, we find almost no points and the slope is $\gamma = -3 \ (D = 0)$. In general, when one measures $\Gamma(r)$ at distances which correspond to a fraction of $\langle \Lambda \rangle$, one finds systematically an higher value of the conditional density exponent. Such a trend is completely spurious and due to the depletion of points at such distances. It is worth to notice that this effect gives rise to a curved behavior of $\Gamma^*(r)$ (the integral of $\Gamma(r)$) at small distances, because of its integral nature. This is exactly the case of the deepest VL of Perseus-Pisces which Guzzo et a. considered in their analysis, and for which $\langle \Lambda \rangle \sim 8h^{-1}Mpc$. Note that $\langle \Lambda \rangle$ is of order $1h^{-1}Mpc$ for a VL sample with $M_{lim} \approx -18$ while it growths up to $\sim 10 \div 15h^{-1}Mpc$ for VL samples with $M = 20 \div -21$. This is due to the depletion of points in VL samples with brighter magnitudes.

A clarifying test in this respect would be to check whether this change of slope is actually present also in the others VL samples of the same survey,
which have a larger number of points (and hence a lower $\langle \Lambda \rangle$). This test has been performed by our group and the conclusion is that the change of slope is due a finite size effect rather being an intrinsic property of galaxy distribution.

5 Problems of counts from a single point and the case of ESP

In a recent paper Scaramella et al. (hereafter S&C) have applied the same statistical analysis to a deep survey of large scale structure -the ESP galaxy survey- as performed by us (Paper 1). Despite the adoption of the same method, they have reached very different conclusions: That there is no evidence for a scale invariant distribution of galaxies with dimensionality $D \approx 2$, as argued in Paper 1, and that their results favour the “canonical” value of $D = 3$ (corresponding to a homogeneous universe). In Joyce et al. we have discussed in more detail this matter: here we review the main results.

As already mentioned the length scale for analysis with the conditional density, is fixed by the maximum radius of a sphere which can be fitted inside the sample volume of the corresponding surveys, since the statistic $\Gamma(r)$ can only be computed up to such a distance. To extract information from surveys about correlations at length scales greater than this, one needs to consider other statistics. The most simple one is the radial count $N(< r)$ from the origin in whatever solid angle is covered by the survey. Depending on the survey geometry the difference between the length scales to which we can calculate $\Gamma(r)$ and $N(< r)$ varies greatly. The number count from the origin is obviously a much less powerful statistic since it doesn’t involve the average. It is intrinsically a more fluctuating quantity. Such fluctuations are, however, about the average behaviour and, by sampling a sufficiently large range of scale, one should be able to recover the average behaviour of the number count $\sim r^D$.

What one means by “sufficiently large range of scale” depends strongly on the underlying nature of the fluctuations. In particular (see section 6.1 of Paper 1) the cases of a homogeneous distribution with Poissonian type fluctuations and a fractal structure with scale invariant fluctuations are quite different. In the former case one predicts a very rapid approach to perfect $D = 3$ behaviour at a few times the scale characterizing the fluctuations; in the latter $N(< r)$ has intrinsic fluctuations on all scales (which average out in $\Gamma(r)$), in addition to the statistical sampling Poisson noise in $N(< r)$ which dominates up to a scale which depends on the sample (on the number of points, and therefore on the solid angle of the survey).

We now turn to the ESP survey, which is a very deep survey extending to red-shifts $z \sim 0.3$ in a very narrow solid angle $\Omega \approx 0.006\ sr$. Because of this geometry the analysis with $\Gamma(r)$ only extends to $R_s \approx 10 \div 12$; in this regime
it shows a clear $D \approx 2$ behaviour consistent with the other surveys analyzed in Paper 1. With the radial counts from the origin $N(< r)$, however, the analysis can extend to distances almost two orders of magnitude greater. The results of the latter analysis can be summarized as follows:

- Up to a scale of $\sim 300 \, h^{-1} Mpc$ the number counts are highly fluctuating;
- Beyond this scale the counts can be well fitted by a fairly stable average behaviour $N(< r) \sim r^D$. The value obtained for the dimension of the number count $D$ in this range depends (i) on the assumptions about the cosmology and (ii) on the so-called K corrections. The uncorrected data (i.e. with the euclidean distance relation and without K corrections) gives a slope of $D \approx 2$, while both corrections lead to an increase in the slope. In particular S&C apply the corrections in a way which produces values $D \approx 3$ which they argue to reflect the true behaviour of the galaxy distribution.

Consider first what conclusions may be drawn from the fluctuating regime. If the universe is homogeneous on large scales, the data clearly show that the scale characterizing such homogeneity is at least $\sim 100 \div 300 \, h^{-1} Mpc$. The implication from our discussion above is that any standard analysis using the correlation function is inappropriate for characterizing the properties of the fluctuations. In particular the authors claim in various papers that in the ESP galaxy sample, $r_0 \approx 4h^{-1} Mpc$. The origin of this length scale can also be inferred from the discussion above: In a fractal distribution (with $D = 2$) we see from (2) that $r_0 \approx R_s/3$, and as noted above $R_s \approx 10\div 12$ in the ESP survey. It is related not to a physical property of the galaxy distribution but to the specific geometry of the survey it has been determined from. The more general implication of this observation of large fluctuations up to $300 \, h^{-1} Mpc$ for all standard analyses of existing catalogues is also clear. If, on the other hand, the true underlying behaviour is fractal, the transition from a highly fluctuating to a more stable behaviour can be understood as the transition between large statistical fluctuations and much smaller scale-invariant intrinsic fluctuations.

In Paper 1 it is shown how, using the dimension and normalization of the radial density from the analysis of the other surveys analyzed with the $\Gamma(r)$ statistic, one obtains a simple estimate for this scale in ESP which agrees with the observation that it is $\sim 300 \, h^{-1} Mpc$. The observed fluctuating regime is therefore consistent with the continuation of the fractal behaviour seen at smaller scales. In fig.3 it is shown the distribution of an homogeneous sample with the same geometry, number of points and luminosity selection effects of the ESP survey, while in fig.4 it is shown the real ESP survey. The fact that the survey is not homogeneous, or that it does not become homogenous at some tens $h^{-1} Mpc$, is evident by the simple visual inspection.

Now let us move onto the conclusions which can be drawn from the smoother
Figure 3: Distribution of an homogeneous sample with the same geometry, number of points and luminosity selection effects of the ESP survey in galactic coordinates.

Figure 4: The same as Fig. 3 but for the real ESP survey.
regime at larger scales. In Paper 1 it was noted that from $\sim 300h^{-1}Mpc$ the number counts show very approximately a dimension $D \approx 2$, and therefore provide weak evidence for the continuation of the behaviour seen at smaller scales. In S&C the authors have looked at the precise effect which certain corrections can have on the data in the regime beyond $300h^{-1}Mpc$ in much greater detail, and drawn the strong conclusion that the $D \approx 2$ behaviour is ruled out and $D = 3$ behaviour clearly favoured. The difference of interpretation centers on the K corrections, and the strength of any conclusion on the confidence with which they can be made. We will now see that some checking shows that the K corrections as they have been applied to obtain this result are in fact not just uncertain to an important extent, but actually clearly inconsistent with the conclusion of an underlying $D = 3$ dimensionality.

In order to construct VL limited catalogues we need to determine the absolute magnitude $M$ of a galaxy at red-shift $z$ from its observed apparent magnitude $m$, and it is here that the K correction enters:

$$M = m - 5\log(d_L) - 25 - K(z).$$

Here the luminosity distance $d_L = r(1 + z)$ in FRW models where $r$ is the comoving distance, and $d_L = (c/H_0)z$ in the euclidean case. The K correction corrects for the fact that when a galaxy is red-shifted we observe it in a redder part of its spectrum where it may be brighter or fainter. It can in principle be determined from observations of the spectral properties of galaxies, and has been calculated for various galaxy types as a function of red-shift. When applying them to a red-shift survey like ESP, we need to make various assumptions since we lack information about various factors (galaxy types, spectral information etc. as a function of red-shift and magnitude).

It is not difficult to understand how K corrections - whether themselves correct or incorrect - can change the number counts systematically. The number count in a volume limited (VL) sample corresponding to a magnitude limit of $M_{lim}$ can be written schematically as

$$N(< R) = \int_0^R \rho(r)d^3r \int_{-\infty}^{M_{lim}} \phi(M, r)dM$$

where $\rho(r)$ is the real galaxy density, and $\phi(M, r)$ the appropriately normalized luminosity function (LF) in the radial shell at $r$. If the latter integral is independent of the spatial coordinate (i.e. if the fraction of galaxies brighter than a given absolute magnitude is independent of $r$), it is just an overall normalization and the exponent of the number counts in the VL limited sample show the behaviour of the true number count, with $N(< R) \sim R^D$ corresponding to
the average behaviour $\rho \sim r^{D-3}$ as we have assumed. What we do when we apply K corrections or other red-shift dependent alterations to the relation (3) is effectively change the LF $\phi(M,r)$. Using the incorrect relation will induce $r$ dependence in this function, which in turn will distort the relation between radial dependence of the number counts and the density.

When one applies a K correction and observes a significant change in the number counts, there are thus two possible interpretations - that one has applied the physical correction required to recover the underlying behaviour for the galaxy number density, or that one has distorted the LF to produce a radial dependence unrelated to the underlying density. How can one check which interpretation is correct? Consider the effect of applying too large a K correction of the type applied by S&C. To a good approximation the net effect is a linear shift in the magnitude $M$ with red-shift, so that $M \rightarrow M - kz$. The second integral in (5) can in this case be written

$$
\int_{-\infty}^{M_{1,m}} \phi_p(M + kz)dM = \int_{-\infty}^{M_{1,m} + kz} \phi_p(M)dM
$$

(6)

where $\phi_p$ denotes the physical, r-independent LF. This is a function whose shape is well known to be fitted by a very flat power-law with an exponential cut-off at the bright end. If the upper cut-off of the integral is in the former range, one can see from (3), taking $z \sim r$, that the number count picks up an additional contribution going as $R^{D+1}$, so that we expect the slope to increase by one at some sufficiently large scale. As we go to the brighter end of the luminosity function, where it turns over, the fractional number of galaxies being added by the correction is even greater and we expect to see a growing effect on the slope with depth of sample. On the scales over which the ESP data are analysed, we should be able to clearly distinguish the case of a spurious K correction from the case of a real underlying $D = 3$ behaviour, which should be relatively stable as a function of the absolute magnitude limit of the VL sample. In Figure 2 of their paper S&C show the number counts for a few VL samples, and conclude that there is evidence for a real sample slope of $D = 3$, the variation being ascribed to random errors. In Fig. 3 we show precisely the same figure with additional VL samples at greater depth, which have been omitted for no apparent reason by S&C. The conclusions one can draw from these two figures are clear: The $D \approx 3$ behaviour observed by S&C in their K corrected data is clearly not stable as would be the case if it represented the real underlying behaviour of the density. On the contrary the “corrected” data are in fact clearly better interpreted as indicating an underlying distribution with $D \approx 2$ which has been subjected to an unphysically large K correction in the relevant range of red-shift $z \sim 0.1 - 0.3$. 
Figure 5: The figure corresponds to Fig.2 of S&C ($q_0 = 0.5$ with exactly the K correction given in Zucca et al.1997), but including deeper VL samples. The systematic growth of the exponent of the number counts as a function of the absolute magnitude limit of the sample is clearly seen.
In this case we have adopted a linear K-correction \( k = 1 \). We find a stable slope \((D \approx 2.2)\) in euclidean coordinates.

In contrast with the conclusion of S&C that the \( D \approx 2 \) result of Paper 1 is only tenable by “both unphysically ignoring the galaxy K-correction and using euclidean rather than FRW cosmological distances”, we conclude therefore that the alternative \( D = 3 \) result is arrived at only by applying an unphysical K correction and taking a quite specific FRW cosmology.

It is perhaps interesting to note that any approximately linear K correction will not produce a stable dimension near \( D = 3 \) for these data in these models. Clearly the physical K correction in this range of red-shift must be a very non-trivial function quite different from that used by S&C if the ESP data arises from an underlying homogeneous distribution. In the absence of a clearly consistent and well understood way of applying such corrections, it makes little sense to draw conclusions which depend so strongly on them. By contrast the interpretation of a continuation of fractal behaviour with \( D \approx 2 \) (and a relatively unimportant role for K corrections) is a consistent interpretation, supported also by the behaviour seen in the fluctuating regime. With a smaller linear K correction with \( k = 1 \), for example, we find a stable slope in euclidean coordinates around \( D = 2.2 \) (see fig.\( \PageIndex{6} \)).
6 Uniformity of angular catalogs

One of the most important elements in the discussion about galaxy correlations, is the analysis of angular distributions. Angular catalogs are qualitatively inferior to three dimensional ones because they correspond to the angular projection and do not contain any information on the third coordinate. However, the fact that they contain more galaxies than the 3-d catalogs has led some authors to assign an excessive importance to these catalogs and they are supposed to represent a clear evidence for homogeneity. Actually the interpretation of angular catalogs is quite delicate and ambiguous for a variety of reasons which are usually neglected. It is important to stress that the existence of large scale structures of galaxies has been found only in redshift surveys, while angular catalogs are relatively uniform. The reconstruction of 3-d properties of the galaxy distribution from angular ones, is based on a series of assumptions that must be tested in real data. We show that the usual hypotheses used so far contradict the behavior found in the data analysis of angular correlations free of any a priori assumption.

Usually the analysis of angular correlations of galaxies is performed through the two point correlation function \( \omega(\theta) \). This allows one to determine a well defined characteristic scale in the angular distribution (defined by \( \omega(\theta_0) = 1 \)), and the correlation exponent at small angular separation is found to be \( \gamma_a = 0.7 \). This value of the power law exponent is claimed to be compatible with the value \( \gamma = 1.7 \) found in 3-d samples, by the \( \xi(r) \) analysis. In particular, the standard method used to analyze angular catalogs, is based on the assumption that galaxies are correlated only at small distances. In such a way the effect of the large spatial inhomogeneities is not considered at all. Under this assumption, which is not supported by any observational evidence, it is possible to derive Limber’s equation. In practice, the angular analysis is performed by computing the two point correlation function

\[
\omega(\theta) = \frac{\langle n(\theta_0)n(\theta_0 + \theta) \rangle}{\langle n \rangle} - 1 \tag{7}
\]

where \( \langle n \rangle \) is the average density in the survey. This function is the analog of \( \xi(r) \) for the 3-d analysis. The results of such an analysis are quite similar to the three dimensional ones. In particular, it has been obtained that, in the limit of small angles,

\[
\omega(\theta) \sim \theta^{-\gamma+1} \tag{8}
\]

with \( \gamma \approx 1.7 \) (i.e. \( \gamma_a = \gamma - 1 = 0.7 \)).

We now study the case of a self-similar angular distribution so that, if such properties are present in real catalogs, we are able to recognize them correctly.
Of course, if the distribution is homogenous, we are able to reproduce the same results obtained by the $\omega(\theta)$ analysis. Hereafter we consider the case of small angles ($\theta \ll 1$), that is quite reasonable for the catalogs investigated so far. In this case the number of points within a cone of opening angle $\theta$ scales as

$$N(\theta) = B_a \theta^{D_a} \quad (9)$$

where $D_a$ is the fractal dimension corresponding to the angular projection and $B_a$ is related to the lower cut-off of the distribution. Eq. (9) holds from every occupied point, and in the case of an homogenous distribution we have $D_a = 2$.

Following Coleman & Pietronero $^8$ we define the conditional average density as

$$\Gamma(\theta) = \frac{1}{S(\theta)} \frac{dN(\theta)}{d\theta} = \frac{BD_a}{2\pi} \theta^{-\gamma_a} \quad (10)$$

where $S(\theta)$ is the differential solid angle element ($S(\theta) \approx 2\pi \theta$ for $\theta \ll 1$) and $\gamma_a = 2 - D_a$ is the angular correlation exponent (angular codimension). The last equality holds in the limit $\theta \ll 1$. From the very definition of $\Gamma(\theta)$ we conclude that

$$\omega(\theta) = \frac{\Gamma(\theta)}{\langle n \rangle} - 1 \quad (11)$$

A first important consequence of Eq. (11) is that if $\Gamma(\theta)$ has a power law behavior, and $\omega(\theta)$ is a power law minus one. This corresponds to a break in the log-log plot for angular scales with $\omega(\theta) \ll 1$. We show in Fig. 7 the behaviour of such a quantity.

The codimension found by fitting $\omega(\theta)$ with a power law function is higher than the real one. This is an important effect which has never been considered before. In Fig.refangle we show also the $\omega(\theta)$ for the APM Bright Galaxies catalog (see below), that is fitted quite well by Eq. (11) with $D = 1.92$. The second important point is that the break of $\omega(\theta)$ in the log-log plot is clearly artificial and does not correspond to any characteristic scale of the original distribution. The basic problem is that in the case of a scale-invariant distribution the average density in eq. (1) is not well defined, as it depends on the sample size.

Before we proceed, it is useful to recall the theorem for orthogonal projection of fractal sets. Orthogonal projections preserve the sizes of objects. If an object of fractal dimension $D$, embedded in a space of dimension $d = 3$, is projected on a plane (of dimension $d' = 2$) it is possible to show that the projection has dimension $D'$ with

$$D' = D \text{ if } D < d' = 2; \quad D' = d' \text{ if } D \geq d' = 2. \quad (12)$$
Figure 7: In this figure we show the behaviour of $\omega(\theta)$ (dotted line) is the case of a fractal structure with $D_a = 1.92$ ($\gamma_a = 0.08$). It can be seen that the exponent obtained by fitting this function with a power law behavior (solid line) is higher than the real one ($\gamma = -0.7$). Also the break in the power law behaviour is completely artificial. The amplitude has been matched to the one of APM-BG with $m_{lim} = 16.44$ (filled circles).
This explains, for example, why clouds which have fractal dimension $D \approx 2.5$, give rise to a compact shadow of dimension $D' = 2$. The angular projection represents a more complex problem due to the mix of different length scales. Nevertheless the theorem given by Eq.12 can be extended to the case of angular projections in the limit of small angles ($\theta < 1$). Therefore according to Eq.(6) we have $D' = D_a$

We have analyzed the angular properties of the following catalogs: CfA1, SSRS1, Perseus-Pisces, Zwicky and APM-Bright galaxies (APM-BG). The results are shown in Fig.8. It turns out that all the catalogs show consistent correlation properties. The angular fractal dimension is $D_a \approx 0.1 \pm 0.1$ (depending on the sample analyzed). No characteristic angular scale is present in any of the analyzed catalogs.

The angular distribution of galaxies turns out to exhibit marginal scale invariance with angular fractal dimension $D_a = 0.1 \pm 0.1$. Such a result, in
view of the theorem for orthogonal projections of fractal sets is fully compatible with the existence of a three dimensional fractal structure with dimension $D = 1.9 \pm 0.1$ which we have obtained in the analysis of the redshift samples (Paper 1). This result alone is marginally compatible with an homogenous distribution in real space, because if $D > 2$ than we have $D_a = 2$. It results therefore that the angular analysis alone cannot be a strong evidence in favor of either a homogeneous or a fractal distribution in space with dimension 2. However, we stress again that the result $\gamma_a = 0.7$ is just an artefact due to an inconsistent data analysis.

It is useful to discuss briefly the angular fluctuations expected in the case of fractal dimension $D$. It is possible to show that the mean square fluctuations of the counts in two field of angular size $\theta$, with centers separated by angular distances $\Theta \gg \theta$ is given by

$$\langle (N_1 - N_2)^2 \rangle \sim \langle N \rangle^2 (\theta^{-\gamma_a} - \Theta^{-\gamma_a})$$

where $\langle N \rangle$ is the number of points over the whole sky (it depends on the apparent magnitude limit of the sample). If the value of the fractal dimension approaches two, then $\gamma_a \to 0$ and the angular mean square fluctuations $\langle (N_1 - N_2)^2 \rangle \to 0$. As we find $\gamma = 0.1 \pm 0.1$, this is compatible with a fractal distribution in space with $D \approx 2$, and explains the uniform distribution of angular maps. A fractal distribution in space is characterized by having strong inhomogeneities on all scales, while the angular projection can be quite uniform and isotropic if $D \gtrsim 2$, and exactly this appears to be the case in the galaxy catalogs (redshift and angular) available up to now.

Dogterom & Pietronero studied the surprising and subtle properties of the angular projection of a fractal distribution. They find that the angular projection produces an artificial crossover towards homogenization with respect to the angular density. This crossover is artificial (just due to the projection) as it does not correspond to any physical features of the three dimensional distribution. Moreover they showed that there is an explicit dependence of the angular two point correlation function $\omega(\theta)$ on $\theta_M$ the sample angle: this effect has never been taken into account in the discussion of real angular catalogs. These arguments show that it is very dangerous to make any definite conclusion just from the knowledge of the angular distribution. By the way this is the reason why one has to measure redshifts, which, of course, is not an easy task.

Let us now consider the much debated angular projection of an artificial fractal. Some authors (e.g. Peebles) pointed out that a fractal with dimension $D$ significantly less than three cannot approximate the observed isotropic angular distributions of deep samples. In particular Peebles showed that a fractal with dimension $D \approx 2$ has large-scale angular fluctuations which are not
compatible with the observed angular maps. We stress that there are various problems, which are usually neglected in constructing an artificial distribution with the properties of the real one:

- The first point is that in generating an artificial fractal structure a very important role is played by \textit{lacunarity}: even if the fractal dimension is fixed, one can have very different angular distributions depending on the value of the lacunarity. Mandelbrot has insisted from a long time on this point. In fact, if lacunarity is large, and the sample is characterized by having voids of the order of the sample size, it is clear that the angular distribution is highly inhomogeneous. On the other hand, if lacunarity is small (with respect to the sample size) one can obtain more uniform angular projections. A low value of the lacunarity should therefore be used for the reproducing galaxy distribution, because the real galaxy distribution has indeed a low value of the lacunarity: in the available samples, the dimension voids is smaller than the survey volume. A more detailed and realistic study of this problem is now in progress.

- The second important point which should be considered is that the real angular distributions are \textit{magnitude limited ones}, i.e. contains all the galaxies with apparent magnitudes brighter than a certain limit \(m_{\lim}\). This implies a mixing of length scales due to the fact that galaxies have a very spread luminosity function, and their absolute luminosity can change of more than a factor ten. For example suppose that \(m_{\lim} = 14\), then one gets contributions from galaxies in the range of distances \(\sim 1.5 \div 50h^{-1} \text{Mpc}\). However if, for example, \(m_{\lim} = 17\) then the range of distances from which one has important contributions to the angular distribution, rapidly grows. This implies that galaxies in angular catalogs correspond to extremely different distances. Correspondingly the projection is a complex convolution of various luminosities and distances. This is another important reason why if one looks an angular map limited at galaxies with apparent magnitude brighter than 14 one sees large scale fluctuations. On the other hand a catalog (as the angular APM catalog) limited at \(m_{\lim} > 18\), is quite smoother, because the mixing of length scales is large.

- An important point, recently pointed out by Durrer \textit{et al.}, is the following: The angular lacunarity of a 3-d fractal set with \(D \approx 2\) can be very small, even if there are large voids in the space distribution. In fact, it can be shown that the angular lacunarity depends on whether the galaxies are projected with apparent or fixes size, being much more uniform in the latter case.
7 Comparison of N-body simulations with data

It is simple to see that a fractal behavior of galaxy distribution with dimension $D \approx 2$ up to, at least, $\sim 50h^{-1}Mpc$ is not compatible with standard CDM models. In fig.9 we show the behavior of the fractal dimension versus distance in three Cold Dark Matter models of power spectra with shape and normalized parameters (from Wu et al., 1998). We may see that a fractal dimension of $D \approx 2$ at $\sim 40 \div 50h^{-1}Mpc$ is incompatible with all the models. Probably by varying the parameters of the simulation (or the mixture of Hot and Cold Dark Matter) one may hope to obtain a better agreement. Any new survey has required a new adjustment of the parameters and this alone shows the internal
problems of the standard models of galaxy formation.

We believe that the most important theoretical consequence of our results is that one may shift the attention of the study from correlation amplitudes to correlation exponents.\footnote{[10,22,23]}

\section{Conclusion}

We have tried to consider the main controversial points in the discussion about galaxy correlations. The distribution of galaxies is one of the most robust experimental fact in modern cosmology. The new surveys like Sloan and 2dF will explore a larger volume of the universe than the one now available, and hence it will be possible to make and intensive study of galaxy distributions up to $\langle z \rangle 0.1$. We summarize our opinion on this subject, the corresponding technical discussion can be found in Paper 1.

\begin{itemize}
  \item Until the seventies galaxy distributions were only known in terms of angular catalogues. These catalogues are limited by the apparent luminosity of galaxies. Since the intrinsic luminosity can vary over a range of the order of one million, the points of the angular catalogues correspond to extremely different distances and they are a complex convolution of these - not just a projection up to some radius, which is the only case you discuss in your letter. These angular distributions appeared rather smooth and, as such, they justified the usual statistical assumptions of large scale homogeneity.
  \item The redshift measurements permit to locate galaxies in 3-d space. They immediately showed a clumpy distribution with large clusters and large voids, in apparent contrast with the angular data. This finally led to 3-d catalogues from which one could make volume limited samples, which provide the best and most direct information for the correlation analysis. These 3-d data have been and are extensively analysed with the usual $\xi(r)$ method. The main result of this approach is that a characteristic length is derived $r_0 \approx 5h^{-1}Mpc$ which should mark the tendency towards homogenization. This is in apparent agreement with the structureless angular data but it is puzzling with respect to the structures of the 3-d data. From this perspective it seems that the presence or absence of structures in the data is irrelevant for the determination of $r_0$.

Puzzled by these results we have decided to reconsider the question of correlations from a broader and critical perspective. This allowed us to test homogeneity, instead of assuming it as in the $\xi(r)$ analysis, and to
produce a totally unbiased description of the correlation properties using the methods of modern Statistical Physics. The main results are:

- The correlation properties of all the 3-d catalogues that we could collect are **consistent with each other** and show fractal correlations up to the sample limits with dimension $D \approx 2$. Such a situation allows us to establish the statistical validity and robustness of the present redshift data. In this respect, we have also performed several different tests to check whether the different catalogs are statistically stable and compatible with each other. The result is that, a part several case (like the IRAS samples which are very dilute), the optical catalogs are rather good.

- The $\xi(r)$, with its a priori homogeneity assumption, appears therefore to be inconsistent for all these samples. The "correlation length" $r_0 \approx 5h^{-1}\text{Mpc}$ is also an artefact of this analysis and $r_0$ is just a fraction of the sample size. A consequence of these results, is that the various catalogues appear to be in contrast with each other if the $\xi(r)$ method is used, but this apparent discrepancy is just due to the inappropriate statistics.

- Also for the exponent (fractal dimension) the $\xi(r)$ analysis leads to a small value ($D \approx 1.2$ instead of the correct one $D \approx 2$) because of its artificial drop in a log-log plot. It is interesting to note that the same problem for the determination of the correct exponent is also present in the angular data where we get $\gamma \approx 0.1$ ($D \approx 1.9$) instead of the usual value for $\xi(r)$ $\gamma \approx 0.7$ ($D \approx 1.3$).

The clarification of the appropriate methodology for the analysis of the 3-d data is a very crucial point also in relation to the redshifts catalogues which will appear in the near future (2dF and SLOAN). Only after this point is clarified it makes sense to consider the question of the quality of the data. In this respect we have made extensive tests on the effects of partial incompleteness and our conclusion is that the available samples are rather stable. A strong support to our result comes also from the fact that the genuine correlation properties of the different catalogues are in good agreement with each other.

We have discussed at length various properties of the angular projections in previous papers. The problem of the angular data is that they are intrinsically ambiguous because one co-ordinate is missing and, in general, it is not possible to reconstruct (without assumptions) the properties of the real $3-d$ distribution from the angular catalogues. It is not a matter of number of points, it is a qualitative problem. In fact, if this would be possible, then there would be no need to measure the redshifts and to built $3-d$ catalogues. One can see
instead that all the large scale structures detected by redshift measurements, could never be predicted by the angular data.

So again, one cannot make any robust statement from the angular projection. In specific, for example, the projections depend on the value of the fractal dimension but also on a variety of other properties like lacunarity, morphology and pixel size. For example if the dimension is $D = 2 + \epsilon$ it is very easy to have compact projections. If one has instead $D = 2 - \epsilon$ this is harder but, in any case, everything will depend on many other elements. The simple example Peebles proposed is very far from a realistic analysis of these properties.

In addition we find the discussion on angular distribution rather confusing. For example, what has to do with our analysis the smooth projection of radio galaxies, that refers to sizes much larger than those we have ever considered? The discussion has to address separate questions:

1. Are our results correct from a methodological point of view? If yes, the previous results on the same data should be considered as incorrect on conceptual grounds.

2. The $3-d$ data available now may be incomplete or problematic. Then one has to wait for better data and analyse them anyhow with the methods we propose. The hope that the incompleteness of the data together with the conceptual mistake of the usual analysis may lead to correct results, should finally be abandoned. Also the idea that some puzzling properties of the angular projections may magically save the usual scenario, is untenable.

3. How far the fractal properties extend? This is an important question that has to be answered by new deep data. Our point was never to propose a model universe that is fractal all the way. We have instead proposed a new method of analysis that has disproved all the previously developed concepts ($r_0 = 5h^{-1}Mpc$ for galaxies, $r_0 = 25h^{-1}Mpc$ for clusters, etc.) up to $100 \div 200 h^{-1}Mpc$, namely the region covered by extensive data (see also Teerikorpi et al). If in the end the galaxy distribution will turn to be really homogenous at $\sim 2000 h^{-1}Mpc$ this will be detected with our methods and in no way will save the use and the results of the $\xi(r)$ approach.

4. The eventual fractal properties of galaxy distributions have nothing to do with the validity of General Relativity. The point is quite simple to define: we have experimental facts. These are the galaxies with their correlations, the background radiation, the Hubble law, and various other facts. One should try to define these properties correctly and then make
a theory that gives an understanding for the properties of these various elements. If galaxies are more fractal than one would like, this maybe problematic for the models and theories of structure formation. Well, one should look for better theories. As for General Relativity, it is possible that is the end, the constant density assumption of Friedmann solutions should be changed into something more complex. This would not be the first time in Physics. In Condensed Matter only the Hydrogen Atom can be solved exactly.

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