Critical point equation on almost $f$-cosymplectic manifolds

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Abstract

**Purpose** – Besse first conjectured that the solution of the critical point equation (CPE) must be Einstein. The CPE conjecture on some other types of Riemannian manifolds, for instance, odd-dimensional Riemannian manifolds has considered by many geometers. Hence, it deserves special attention to consider the CPE on a certain class of almost contact metric manifolds. In this direction, the authors considered CPE on almost $f$-cosymplectic manifolds.

**Design/methodology/approach** – The paper opted the tensor calculus on manifolds to find the solution of the CPE.

**Findings** – In this paper, in particular, the authors obtained that a connected $f$-cosymplectic manifold satisfying CPE with $\lambda = \tilde{f}$ is Einstein. Next, the authors find that a three dimensional almost $f$-cosymplectic manifold satisfying the CPE is either Einstein or its scalar curvature vanishes identically if its Ricci tensor is pseudo anti-commuting.

**Originality/value** – The paper proved that the CPE conjecture is true for almost $f$-cosymplectic manifolds.

**Keywords** Critical point equation, Almost $f$-cosymplectic manifold, Cosymplectic manifold, Einstein manifold

**Paper type** Research paper

1. Introduction

One of the natural ways of finding canonical Riemannian metric, that is, Riemannian metrics with constant curvature in various form on a smooth manifold is to look for metrics which are critical points of a natural functional on the space of all metrics on a given manifold. In this context, it is very interesting to investigate the critical points of total scalar curvature functional $S : \mathcal{M} \rightarrow \mathbb{R}$ given by

$$S(g) = \int_{\mathcal{M}} r_g dv_g,$$

(1.1)

defined on a compact orientable Riemannian $n$-manifold $(\mathcal{M}, g)$, where $\mathcal{M}$ denotes set of all Riemannian metrics on $(\mathcal{M}, g)$ of unit volume, $r_g$ is the scalar curvature and $dv_g$ is the volume form. The functional $S$ in Eqn (1.1) restricted over $\mathcal{M}$ is known as Einstein–Hilbert functional and its critical points are the Einstein metric (see chapter 2 in [1]).

**JEL Classification** — 53C15, 53C20, 553C21

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Let $C \subset M$ be the subset of metrics with constant scalar curvature. If we consider the functional in Eqn (1.1) restricted to $C$, then it is not difficult to see that the Euler–Lagrangian equation is given by,

$$\text{Hess}_g \lambda - (\Delta_g \lambda) - \lambda \text{Ric}_g = \text{Ric}_g - \frac{r}{n} g,$$

for some smooth function $\lambda$ on $M$. Here $\text{Hess}$, $\Delta$ and $r$ stands for the Hessian form, the Laplacian, the Ricci tensor and the scalar curvature on $M$, respectively. Moreover, taking trace in Eqn (1.2), we obtain

$$\Delta_g \lambda + \frac{r\lambda}{n - 1} = 0.$$

We notice that if $\lambda$ is constant in Eqn (1.2), then $\lambda = 0$ and $g$ becomes Einstein. Therefore, we have the following definition:

**Definition 1.1.** A compact Riemannian manifold $(M, g)$ of dimension $n > 3$ with constant scalar curvature and unit volume together with a smooth potential function $\lambda$ satisfying (Eqn 1.2), is called critical point equation (shortly, CPE).

Besse first conjectured that the solution of the CPE must be Einstein [1]. Since then, we find many articles regarding the solution of the CPE. In [2], Barros and Ribeiro proved that the CPE conjecture is true under the assumption of half conformally flat spaces. Recently, Hwang [3] proved that the CPE conjecture is also true under certain condition on the bounds of the potential function $\lambda$. A necessary and sufficient condition for the norm of the gradient of the potential function for a CPE metric to be the Einstein metric was obtained by Neto [4].

It is very interesting to consider the CPE on odd-dimensional Riemannian manifolds. In this direction, Ghosh and Patra considered the $K$-contact metrics that satisfy the CPE [5], and proved that the CPE conjecture is true for this class of metric. Patra et al. in [6], and De and Mandal in [7] independently considered an almost Kenmotsu manifold with CPE. Recently, present authors in [8], and Blaga and Dey in [9] studied CPE on cosymplectic manifold and three dimensional $\alpha$-cosymplectic manifold, respectively.

As the generalization of almost Kenmotsu and almost cosymplectic manifolds, the results obtained in [6–9] motivates us to consider almost $f$-cosymplectic manifolds. In this paper, we classify an almost $f$-cosymplectic manifold which satisfies CPE.

**2. Preliminaries**

Let $M$ be a smooth differentiable manifold of dimension $2n + 1$ equipped with a triple $(\varphi, \xi, \eta)$, where $\varphi$ is a $(1, 1)$-tensor field, $\xi$ is a Reeb vector and $\eta$ is a one-form such that

$$\varphi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1,$$

(2.1)

which implies $\varphi(\xi) = 0$, $\eta(\varphi) = 0$ and $\text{rank}(\varphi) = 2n$. If $M$ admits a Riemannian metric $g$ such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X),$$

for any vector fields $X, Y$, then $M$ is said to have an almost contact metric structure $(\varphi, \xi, \eta, g)$. On such a manifold, the fundamental two-form $\Phi$ of $M$ is defined by

$$\Phi(X, Y) = g(\varphi X, Y),$$

for any vector field $X$ and $Y$ on $M$. One can define an almost complex structure $J$ on $M \times \mathbb{R}$ by
where $t$ is the coordinate of $R$ and $u$ is a smooth function. If the aforesaid structure $J$ is integrable, then we call an almost contact structure as normal, and this is equivalent to require

$$[\varphi, \varphi] = -2d\eta \otimes \xi,$$

where $[\varphi, \varphi]$ indicates the Nijenhuis tensor of $\varphi$.

An almost contact metric manifold $M$ is said to be almost cosymplectic if $d\eta = 0$ and $d\Phi = 0$, where $d$ is the exterior differential operator, and it is said to be cosymplectic if in addition the almost contact structure is normal. An almost $\alpha$-Kenmotsu manifold is an almost contact metric manifold, in which $d\eta = 0$ and $d\Phi = 2\alpha\eta \wedge \Phi$, for a nonzero constant $\alpha$. More generally, if the constant $\alpha$ is any real number, then almost contact structure is said to be almost $\alpha$-cosymplectic [10]. Moreover, the authors in [11] generalizes the almost $\alpha$-cosymplectic manifold by allowing the real number $\alpha$ to any smooth function $f$, and it is called as an almost $f$-cosymplectic manifold, which is an almost contact metric manifold $M$ such that $d\Phi = 2f\eta \wedge \Phi$ and $d\eta = 0$ for a smooth function $f$ satisfying $df \wedge \eta = 0$. In addition, a normal almost $\alpha$-cosymplectic manifold is said to be $f$-cosymplectic manifold. In particular, $M$ is an almost cosymplectic manifold under the condition $f(\text{constant}) = 0$ and an almost $\alpha$-Kenmotsu manifold if ($\alpha = f \neq 1$).

Besides, we recall that there is an operator $h = \frac{1}{2} \xi \cdot \varphi$, which is a self-dual operator. We denote by $R$ and $Ric$ the Riemannian curvature tensor and Ricci tensor, respectively. For an almost $f$-cosymplectic manifold $M$, the following equations were proved [11]

$$\nabla_X \xi = -f \varphi^2 X - \varphi h X, \quad \text{trace}(\varphi h) = 0,$$

$$R(X, \xi) \xi - \varphi R(\varphi X, \xi) \xi = 2(\tilde{f} \varphi^2 X - h^2 X),$$

$$Ric(\xi, \xi) = -2n\tilde{f} - \text{trace}(h^2),$$

$$R(X, \xi) \xi = \tilde{f} \varphi^2 X + 2\tilde{f} \varphi h X - h^2 X + \varphi(\nabla_\xi h)X,$$

for any vector fields $X, Y$ on $M$, where $\tilde{f} = \xi(f) + f^2$.

3. CPE on normal almost $f$-cosymplectic manifolds

In this section, we aim to study CPE on normal almost $f$-cosymplectic manifold. We are aware that if almost contact metric manifold is normal then $h = 0$. Hence, as a result of Proposition 9 and Proposition 10 of [11] we have the following identities, which are valid on $f$-cosymplectic manifolds;

$$\nabla_X \xi = -f \varphi^2 X,$$

$$Q \xi = -2n\tilde{f} \xi,$$
where $Q$ is the Ricci operator of $M$.

Now, we will give some properties, which will be used in the proof of our results.

**Lemma 3.1.** An $f$-cosymplectic manifold $M$ of dimension $2n + 1$ satisfies
\[
(\nabla_X Q)\xi = -f QX - 2n(X\tilde{f})\xi - 2n\tilde{f}f X,
\]
(3.4)
\[
(\nabla_\xi Q)X = -2f QX - (2n - 1)(X\tilde{f})\xi - (\xi\tilde{f})X - 4n\tilde{f}f X.
\]
(3.5)

**Proof.** Differentiation of Eqn (3.2), and utilization of first term of Eqn (3.1) provides Eqn (3.4). Now differentiating Eqn (3.3) along $Z$ leads to
\[
(\nabla_Z R)(X, Y)\xi = (Z\tilde{f})\{\eta(X)Y - \eta(Y)X\} + \tilde{f}f \{g(X, Z)Y - g(Y, Z)X\} - fR(X, Y)Z.
\]
Taking $X = Z = E_i$ in the above equation and then summing over $i$ shows that
\[
\sum_{i=1}^{2n+1} g((\nabla_{E_i}R)(E_i, Y)\xi, Z) = (\xi\tilde{f})g(Y, Z) - (Z\tilde{f})\eta(Y) + 2n\tilde{f}f g(Y, Z) + f \text{Ric}(Y, Z).
\]
(3.6)

Feeding Eqn (3.7) into Eqn (3.6) and with the help of Eqn (3.4), we obtain
\[
g((\nabla_\xi Q)Z, Y) = -2f \text{Ric}(Z, Y) - (2n - 1)(X\tilde{f})\eta(Y)
- (\xi\tilde{f})g(Z, Y) - 4n\tilde{f}f g(Z, Y),
\]
which proves Eqn (3.5).

**Theorem 3.3.** Let $(g, \lambda)$ be a nontrivial solution of the CPE (Eqn 1.2) on $n$-dimensional Riemannian manifold $M$. Then the curvature tensor $R$ can be expressed as
\[
R(X, Y)D\lambda = (X\lambda)QY - (Y\lambda)QX + (\nu+1)((V_XQ)Y - (V_YQ)X) + (X\nu)Y - (Y\nu)X,
\]
(3.8)

for any vector fields $X, Y$ on $M$, where $\nu = -\frac{2r}{n-1} + \frac{1}{r}$.

In the following, we will consider an $f$-cosymplectic manifold $M$ satisfying a CPE and assume that the function $f$ satisfies $\xi(\tilde{f}) = 0$.

**Theorem 3.3.** Let $M$ be an $f$-cosymplectic manifold of dimension $2n + 1$ with $\xi(\tilde{f}) = 0$. If $(g, \lambda)$ is a solution of the CPE (Eqn 1.2), then one of the following statement holds:

(1) $M$ is Einstein
(2) \( M \) is locally the product of a Kähler manifold and an interval or unit circle \( S^1 \).

**Proof.** Taking scalar product of Eqn (3.8) with \( \xi \) and making use of Eqns (3.2) and (3.3), we obtain
\[
-(2n+1)\tilde{f}\{(Y\lambda)\eta(X)-(X\lambda)\eta(Y)\} = 2n(\lambda+1)\{\eta(X)(Y\tilde{f}) - \eta(Y)(X\tilde{f})\} + (X\nu)\eta(Y) - (Y\nu)\eta(X).
\]

Replacing \( X \) by \( \varphi X \) and \( Y \) by \( \xi \) in above relation, we get
\[
(2n+1)\tilde{f}\varphi D\lambda + 2n(\lambda+1)\varphi D\tilde{f} - \varphi D\nu = 0. \tag{3.9}
\]

According to Proposition 2.1 of Chen [12], it is know that if \((\xi\tilde{f}) = 0\), then \( \tilde{f} \) is constant. So that, Eqn (3.9) implies
\[
(2n+1)\tilde{f}\varphi D\lambda = \varphi D\nu. \tag{3.10}
\]

The scalar curvature \( r \) of \( g \) is constant (as \( (g, \lambda) \) is a solution of the CPE). For a \((2n+1)\)-dimensional \( f \)-cosymplectic manifold, we have \( \nu = -r(\frac{\lambda}{2n} + \frac{1}{2n+1}) \), therefore from Eqn (3.10) it appears that
\[
((2n+1)\tilde{f} + \frac{r}{2n})\varphi D\lambda = 0. \tag{3.11}
\]

From Eqn (3.11), we have either \( r = -2n(2n+1) \) or \( \varphi D\lambda = 0 \).

First suppose that \( r = -2n(2n+1) \tilde{f} \), then we have \( D\nu = -(2n+1)\tilde{f} D\lambda \). Plugging \( X = \xi \) in Eqn (3.8) and calling back Lemma 3.1, we aimed at obtaining
\[
R(X, \xi)D\lambda = -2n\tilde{f}(X\lambda) - (\xi\lambda)QX + (\lambda+1)\{f QX + 2n\tilde{f} fX\} + (X\nu) - (\xi\nu)X.
\]

From Eqn (3.3), we deduce \( R(X, \xi)Y = \tilde{f}\{g(X, Y)\xi - \eta(Y)X\} \), by virtue of this the foregoing equation reduces to
\[
-(2n+1)\tilde{f}(X\lambda)\xi + (\tilde{f}(\xi\lambda) - (\xi\nu) + 2n\tilde{f}(\lambda+1))X + (f(\lambda+1) - (\xi\lambda))QX + (X\nu)\xi = 0. \tag{3.12}
\]

Making use of \( D\nu = -(2n+1)\tilde{f} D\lambda \) in Eqn (3.12) we reach at
\[
(f(\lambda+1) - (\xi\lambda))(QX + 2n\tilde{f}X) = 0.
\]

Since \( \nabla_\xi \xi = 0 \) and \( (\xi\lambda) = g(\xi, D\lambda) \), taking into account \( \nabla_X D\lambda = (\lambda + 1)QX + \nu X \), we deduce \( \xi(\xi\lambda) = f\lambda \). If possible, let \( (\xi\lambda) = f(\lambda + 1) \) in some open set \( \mathcal{O} \) of \( M \) then we have \( f\lambda = ((\xi\lambda) + f^2)(\lambda + 1) \). By virtue of \( \tilde{f} = (\xi\lambda) + f^2 \), one can see \( \lambda = \lambda + 1 \), that is, \( 1 = 0 \), which is absurd. Hence \( QX = -2n\tilde{f}X \) and \( M \) is Einstein.

Next we assume \( r \neq -2n(2n+1) \tilde{f} \), then from Eqn (3.11) we have \( \varphi D\lambda = 0 \). Action of \( \varphi \) on this equation gives \( D\lambda = (\xi\lambda)\xi \). Differentiating this along \( X \), calling back Eqn (3.1) furnishes
\[
\nabla_X D\lambda = X(\xi\lambda)\xi - f(\xi\lambda)\varphi^2 X. \tag{3.13}
\]

On the other hand, from Eqn (1.2) we can easily find that
\[
\nabla_X D\lambda = (\lambda + 1)QX + \left(\Delta \lambda - \frac{r}{2n+1}\right)X. \tag{3.14}
\]
Comparing aforementioned equation with Eqn (3.13), we get

$$(\lambda + 1)QX + \left(\Delta \lambda - \frac{r}{2n+1}\right)X = X(\xi \lambda)\xi - f(\xi \lambda)\phi^2 X.$$ 

Taking $X = \xi$ in the above equation and making use of Eqns (3.2) and (2.1), we obtain

$$\xi(\xi \lambda) = \left(\Delta \lambda - \frac{r}{2n+1}\right) - 2nf(\lambda + 1).$$  \hspace{1cm} (3.15)$$

Contraction of Eqn (3.13) with respect to $X$ brings into view

$$\Delta \lambda = \xi(\xi \lambda) + 2nf(\xi \lambda).$$  \hspace{1cm} (3.16)$$

Unifying this with Eqn (3.15) implies

$$2nf(\xi \lambda) - \frac{r}{2n+1} - 2n\tilde{f}(\lambda + 1) = 0.$$  \hspace{1cm} (3.17)$$

Differentiating Eqn (3.17) along $\xi$, keeping in mind that $\tilde{f}$ and $r$ are constants, we obtain

$$\xi(\xi \lambda)f + (\xi \lambda)(\xi f) = \tilde{f}(\xi \lambda),$$

and further, it implies

$$\xi(\xi \lambda) = f^2(\xi \lambda),$$  \hspace{1cm} (3.18)$$

where we used $\tilde{f} = (\xi f) + f^2$.

If $f \neq 0$, then we can assume $f \neq 0$ on some neighborhood $O$ of $M$. Thus, Eqn (3.18) implies $\xi(\xi \lambda) = (\xi \lambda)f$ on $O$. Inserting this into Eqn (3.16), we find $\Delta \lambda = (2n + 1)f(\xi \lambda)$. Moreover, applying Eqn (3.17) in the previous relation shows that

$$\Delta \lambda = (2n + 1)\left\{\tilde{f}(\lambda + 1) + \frac{r}{2n+1}\right\}.$$  \hspace{1cm} (3.19)$$

Taking trace of CPE (1.2), we obtain $2n\Delta \lambda = -\lambda r$, and this together with Eqn (3.19) gives that $r = -2n(2n + 1)f$, which is contradictory to our assumption. Hence $f \neq 0$, and so $M$ is cosymplectic. According to Blair’s [13] result, we can easily conclude that $M$ is locally the product of a Kähler manifold and an interval or unit circle $S^1$. This finishes the proof. \hfill $\square$

In particular, when dimension of $M$ is three, due to Theorem 3.3 we have the following outcome:

**Corollary 3.4.** Let $M$ be an $f$-cosymplectic manifold of dimension three satisfying CPE Eqn (1.2). If $(\xi \lambda) = 0$, then $M$ is either locally the product of a Kähler manifold and an interval or unit circle $S^1$ or $M$ has constant negative sectional curvature $-\tilde{f}$.

It is known that an $\alpha$-cosymplectic manifold is actually an $f$-cosymplectic manifold with $f$ constant. By the reason of this, we obtain the following conclusion from Theorem 3.3.

**Corollary 3.5.** Let $M$ be an $\alpha$-cosymplectic manifold of dimension $2n + 1$ with $\xi(\tilde{f}) = 0$. If $(g, \lambda)$ is a solution of the CPE Eqn (1.2), then $M$ is either Einstein or locally the product of a Kähler manifold and an interval or unit circle $S^1$.

Now we consider CPE with $\lambda = \tilde{f}$, and obtain the following result.

**Theorem 3.6.** If a connected $f$-cosymplectic manifold $M$ satisfying CPE Eqn (1.2) with $\lambda = \tilde{f}$, then $M$ is Einstein.
Proof. One can easily obtain from Eqn (3.9) that

\[ \{(4n + 1)\lambda + 2n\} \varphi D\lambda = \varphi D\nu, \]

where we applied our assumption \( \lambda = \tilde{f} \). Uptaking \( \nu = -r \left( \frac{\lambda}{2n + 1} + \frac{1}{2n + 3} \right) \) in the above relation implies

\[ \{(4n + 1)\lambda + 2n + \frac{r}{2n + 1} \} \varphi D\lambda = 0. \]

Suppose that \( (4n + 1)\lambda + 2n + \frac{r}{2n + 1} \neq 0 \). Due to constancy of \( r \), we see that \( \lambda \) is constant. Next, we assume that \( (4n + 1)\lambda + 2n + \frac{r}{2n + 1} \neq 0 \) in a neighborhood \( O \) of \( M \). Consequently, one can get \( \varphi D\lambda = 0 \). Applying \( \varphi \) to this equation implies \( D\lambda = (\xi\lambda)\xi \). In this context (3.13) holds, from which we can get

\[ 2nf(\xi\lambda) - \frac{r}{2n + 1} - 2n\lambda(\lambda + 1) = 0. \]  \hspace{1cm} (3.20)

Differentiating this along \( \xi \) gives \( (2\lambda + 1)(\xi\lambda) = \xi(\xi\lambda)f + (\xi\lambda)(\xi f) \), due to our assumption \( \lambda = \tilde{f} = (\xi f) + f^2 \) which further implies

\[ (\lambda + f^2 + 1)(\xi\lambda) = \xi(\xi\lambda)f. \]  \hspace{1cm} (3.21)

Suppose that \( f = 0 \), then from Eqn (3.20), we have \( \lambda(\lambda + 1) + \frac{r}{2n + 1} = 0 \), which means that \( \lambda \) is constant. In the following we suppose \( f \neq 0 \), then as a result of Eqns (3.16), (3.20) and (3.21), we find

\[ \Delta\lambda = \{(\lambda + 1) + f^2(2n + 1)\} \frac{(\xi\lambda)}{f}. \]

Substitute this into \( 2n\Delta\lambda = -\lambda r \) to obtain

\[ -2n \frac{(\xi\lambda)}{f} - 2n(2n + 1)\lambda = r. \]

Differentiating the aforesaid relation along \( \xi \), remembering \( r \) is constant and applying Eqn (3.21), we reach at

\[ \left(2n + 3 + \frac{1}{f^2}\right) (\xi\lambda) = 0. \]

If \( (\xi\lambda) = 0 \), then we have \( D\lambda = 0 \), which means \( \lambda \) is constant. Suppose \( (\xi\lambda) \neq 0 \), then we get \( f^2 = \frac{1}{2n + 3} \). Due to \( f \neq 0 \), which shows \( (\xi f) = 0 \). This together with \( \lambda = \tilde{f} = (\xi f) + f^2 \) yields \( \lambda = \tilde{f}^2 \), showing \( \lambda \) constant. In a word, we have proved that \( \lambda \) is always constant in the neighborhood \( O \) of \( M \), thus \( \lambda = \text{constant} \) in \( M \). Hence, the proof completes from Eqn (1.2).

4. CPE on non-normal almost \( f \)-cosymplectic manifolds

Here, we consider a three dimensional almost \( f \)-cosymplectic manifold \( M \) with pseudo anti-commuting Ricci tensor, that is,

\[ \varphi Q + Q\varphi = 2\kappa \varphi, \quad \kappa \text{ is constant.} \]

This notion was introduced by Jeong and Suh [14], and they made use of this condition to classify a real hypersurface of complex two-plane Grassmannians.

At first, we have the following lemma:
Lemma 4.1. [15] For a three dimensional almost $f$-cosymplectic manifold with pseudo anticommuting Ricci tensor the following formula holds:

$$r - 2\kappa = a,$$

where $a = g(Q\xi, \xi)$.

Let $\mathcal{U}$ be the open subset where the tensor $h \neq 0$ and $\mathcal{U}'$ be the open subset such that $h$ is identically zero. Thus, $\mathcal{U} \cup \mathcal{U}'$ is open dense in $M$. There exists a local orthonormal frame field $E = \{\xi, e, \varphi e\}$ such that $he = \mu e$ and $h\varphi e = -\mu \varphi e$, where $\mu$ is a positive nonvanishing smooth function of $M$. The following proposition is obtained from Proposition 12 and Proposition 14 of Öztürk et al. [10]:

Proposition 4.2. For a three dimensional almost $f$-cosymplectic manifold, the following relations hold:

$$h^2 - f^2 \varphi^2 = \frac{a}{2} \varphi^2, \quad (4.1)$$

$$\nabla_\xi h = 2bh\varphi + (\xi \mu)s, \quad (4.2)$$

where $b$ is a function defined by $b = g(\nabla_\xi \varphi e, e)$ and $s$ is a $(1,1)$ tensor field defined by $se = e$, $s\varphi e = -\varphi e$ and $s\xi = 0$.

From this onwards, we assume that a three dimensional almost $f$-cosymplectic manifold $M$ satisfies the CPE Eqn (1.2), then its scalar curvature $r$ is constant. As a result of Lemma 4.1, it can be seen that $a$ is constant. Chen in [15] obtained the relation $(r - 2\kappa - 2\tilde{f} - 2a)fX = 2\varphi h^2X$.

From this we have $(-2\tilde{f} - a)fX = 2\varphi h^2X$, that is, $2h^2X = (2\tilde{f} + a)f^2X$ as $h\xi = 0$. Further, from Eqn (4.1) we find $(\xi f) = 0$, due to $df \wedge \eta = 0$ we obtain $f$ is constant. Moreover, we can also find $a = -2(f^2 + \mu^2)$ from Eqn (4.1). Thus $\mu$ is constant. In view of Lemma 2 of [10], we have the following (also see [15]):

Lemma 4.3. Let $M$ be a three dimensional almost $f$-almost cosymplectic manifold satisfying the CPE Eqn (1.2). If the Ricci tensor is pseudo anticommuting, then with respect to $E$ the Levi-Civita connection $V$ is given by

$$\nabla_\xi e = -b\varphi e, \quad \nabla_\xi \varphi e = be, \quad \nabla_\xi \xi = 0,$$

$$\nabla_{\varphi e} \xi = fe - \mu \varphi e, \quad \nabla_{\varphi e} e = -f \xi, \quad \nabla_{\varphi e} \varphi e = \mu e,$$

$$\nabla_{\varphi e} e = -\mu e + f \varphi e, \quad \nabla_{\varphi e} \varphi e = -f \xi, \quad \nabla_{\varphi e} e = \mu \xi. \quad (4.3)$$

In view of Eqn (4.2), the relation Eqn (2.5) implies

$$R(X, \xi)\xi = -\frac{a}{2} \varphi^2 + 2f \varphi hX + 2bhX. \quad (4.4)$$

By virtue of Eqn (3.8), we obtain

$$g(R(X, \xi)D\lambda, \xi) = a(X\lambda) - a(\xi \lambda)\eta(X) + (X\nu) - (\xi \nu)\eta(X)$$

$$= \left( a - \frac{r}{2} \right) ((X\lambda) - (\xi \lambda)\eta(X)).$$
Substituting the above relation into Eqn (4.4), we obtain
\[
\frac{a}{2}g(\varphi^2X, D\lambda) - 2fg(\varphi hX, D\lambda) - 2bg(hX, D\lambda) = \left(a - \frac{r}{2}\right)((X\lambda) - (\xi\lambda)\eta(X)).
\] (4.6)

Employing \(X\) by \(\varphi X\) in Eqn (4.6) we reach at
\[
\left(-\frac{3a - r}{2}\right)\varphi D\lambda = 2fhD\lambda + 2bh\varphi D\lambda.
\] (4.7)

In orthonormal frame field \(E\), the gradient vector field \(D\lambda\) can be written as
\[
D\lambda = (e\lambda)e + (\varphi e\lambda)\varphi e + (\xi\lambda)\xi.
\] (4.8)

Thus from Eqn (4.7), one can obtain
\[
-\left(-\frac{3a - r}{2}\right)(\varphi e\lambda) = 2f\mu(e\lambda) - 2bh(\varphi e\lambda),
\] (4.9)

and
\[
\left(-\frac{3a - r}{2}\right)(e\lambda) = -2f\mu(\varphi e\lambda) + 2bh(e\lambda).
\] (4.10)

First we assume \(f \neq 0\), because of \(f\) is constant and we shall divide this discussion into two cases:

**Case 1.** If \((e\lambda) = 0\), then from Eqn (4.10) we can observe \((\varphi e\lambda) = 0\). This together with (4.8) yields \(D\lambda = (\xi\lambda)\xi\). Differentiating this along \(X\), using Eqn (2.2) gives
\[
\nabla_X D\lambda = X((\xi\lambda) - (\xi\lambda)(f\varphi^2X + \varphi hX)).
\] (4.11)

Employing \(X = \xi\) in the above equation and remembering \(\nabla_{\xi} D\lambda = (\lambda + 1)Q\xi + (\Delta\lambda - \frac{r}{3})\xi\), we aimed at obtaining
\[
\xi((\xi\lambda)) = (\lambda + 1)a + \Delta\lambda - \frac{r}{3}.
\] (4.12)

One can find from Eqn (4.11) and second term of Eqn (2.2) that
\[
\Delta\lambda = \xi((\xi\lambda)) + 2f(\xi\lambda).
\] (4.13)

By virtue of the foregoing relation, Eqn (4.12) transforms into \((\xi\lambda) = \frac{r}{2f} - \frac{(\lambda + 1)a}{2f}\), which further gives
\[
\xi((\xi\lambda)) = \frac{(\xi\lambda)a}{2f} = -\frac{ra}{12f^2} + \frac{(\lambda + 1)a^2}{4f^2},
\]
where we applied \(f\) and \(a\) are constants. Also, it is know that \(\Delta\lambda = -\frac{r}{2f}\). Thus Eqn (4.12) transforms into
\[
x((\xi\lambda)) = \frac{a^2}{4f^2} + \frac{r}{2} - a = \frac{ra}{12f^2} - \frac{r}{3} - \frac{a^2}{4f^2} + a.
\] (4.14)
If \( a^2 + \frac{e^2}{C_1} - a = 0 \), then we have \( \frac{e^2}{C_1} - a - \frac{a^2}{C_1} + a = 0 \), which further shows that \( e \left( \frac{a}{C_1} + 1 \right) = 0 \). The former case implies that the scalar curvature of \( M \) vanishes identically. In the latter case, we have \( a = -2f^2 \), which together with \( a = -2(f^2 + \mu^2) \) implies \( \mu = 0 \), which is not possible.

Next suppose \( \frac{e^2}{C_1} + \frac{e^2}{C_1} - a \neq 0 \), then from Eqn (4.14) it can easily conclude that \( \lambda \) is constant.

**Case 2.** If \( (e\lambda) \neq 0 \) on a neighborhood \( \mathcal{O} \) of \( M \), then from (4.9) and (4.10) we extract

\[
\left( \frac{3a - r}{2} \right)^2 = 4(f^2 + b^2)\mu^2. \tag{4.15}
\]

From the preceding equation, we can easily observe that \( b \) is constant because of \( \mu \) and \( f \) are constant. It is easy to seen from Eqns (4.9) and (4.10) that \( (e\lambda) \) and \( (\varphi e\lambda) \) are constants in \( \mathcal{O} \).

By the support of Eqn (4.3), we may easily compute that

\[
\nabla_v D\lambda = (-f(e\lambda) + \mu(\varphi e\lambda) + e(\xi\lambda))\xi + (\xi\lambda)(fe - \mu e),
\n\nabla_\varphi D\lambda = (\mu(e\lambda) - f(\varphi e\lambda) + \varphi e(\xi\lambda))\xi + (\xi\lambda)(-\mu e + \varphi e),
\n\nabla_\xi D\lambda = -b(e\lambda)\varphi e + b(\varphi e\lambda)e + \xi(\xi\lambda)\xi.
\]

Thus \( \Delta \lambda = 2(f(\xi\lambda) + \xi(e\lambda)) \). So, utilization of \( \nabla_\xi D\lambda = (\lambda + 1)\varphi e + (\Delta\lambda - \xi)\xi \) followed from (3.14), shows that

\[
(\xi\lambda) = \frac{r}{6f} - \frac{(\lambda + 1)a}{2f}.
\]

As followed by Case 1, we can conclude that either \( r \) vanishes or \( \lambda \) is constant.

Next we assume \( f = 0 \), then from Eqns (4.9) and (4.10) we find \( b(e\lambda)(\varphi e\lambda) = 0 \) as \( \mu > 0 \), which further implies either \( (e\lambda)(\varphi e\lambda) = 0 \) or \( b = 0 \). We shall also discuss this matter into two cases.

**Subcase i.** If \( b = 0 \), then from Eqn (4.10) we find \( r = 3a \). For three dimensional case, it is known that the Riemannian curvature is

\[
R(X, Y)Z = g(Y, Z)QX - g(X, Z)QY + g(QY, Z)X - g(QX, Z)Y
- \frac{r}{2} \{g(Y, Z)X - g(X, Z)Y\}.
\]

From this, we have

\[
R(X, \xi)\xi = QX - 2a\eta(X)\xi + aX - \frac{r}{2} \{X - \eta(X)\xi\}
= QX - a \frac{a}{2} \eta(X)\xi - a \frac{a}{2} X.
\]

This together with Eqn (4.4) yields \( QX = aX \), which means \( M \) is Einstein.

**Subcase ii.** If \( b \neq 0 \) on some neighborhood \( \mathcal{O} \) of \( M \), then we find \( (e\lambda)(\varphi e\lambda) = 0 \) on \( \mathcal{O} \). If possible, let \( e\lambda = 0 = \varphi e\lambda \), then from Eqn (4.8) we obtain \( D\lambda = (\xi\lambda)\xi \). From this it is not hard to see that Eqn (4.12) holds and Eqn (4.13) implies \( \Delta \lambda = \xi(\xi\lambda) \). Thus Eqn (4.12) transforms into \( (\lambda + 1)a - \xi = 0 \), which means \( \lambda \) is constant on \( \mathcal{O} \).

If \( (e\lambda) = 0 \) and \( (\varphi e\lambda) \neq 0 \), then from Eqns (4.3) and (4.8) we compute

\[
\nabla_\xi D\lambda = \xi(\varphi e\lambda)\varphi e + b(\varphi e\lambda)e + \xi(\xi\lambda)\xi.
\]
Utilization of above relation in $\nabla_\xi D\lambda = (\Delta\lambda - \frac{1}{2})\xi + (\lambda + 1)\alpha\xi$, we find $b(\varphi\xi\lambda) = 0$. Because of $b \neq 0$, we have $(\varphi\xi\lambda) = 0$, which is a contradiction. In a similar manner, we also come to contradiction if we consider $(\xi\lambda) \neq 0$ and $(\varphi\xi\lambda) = 0$.

From the above detailed discussion, we have concluded that $M$ is Einstein or its scalar curvature vanishes, and so we state following result:

**Theorem 4.4.** Let $M$ be a three dimensional almost $f$-almost cosymplectic manifold satisfying the CPE Eqn (1.2), if its Ricci tensor is pseudo anti-commuting, then $M$ is either Einstein or its scalar curvature vanishes identically.

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