CHERN SUBRINGS
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Abstract. Let $p$ be an odd prime. We show that for a simply-connected semisimple complex linear algebraic group, if its integral homology has $p$-torsion, the Chern classes do not generate the Chow ring of its classifying space.

1. Introduction

Let $p$ be an odd prime. Let $h^*(-)$ be one of the mod $p$ cohomology $H\mathbb{Z}/p$, the cohomology $H\mathbb{Z}(p)$ with coefficient $\mathbb{Z}(p)$ and the Brown-Peterson cohomology $BP$ with $BP_* = \mathbb{Z}(p)[v_1, v_2, \ldots]$.

Let $G$ be a compact connected Lie group and $G(\mathbb{C})$ its complexification, that is, $G(\mathbb{C})$ is a complex linear algebraic group which is homotopy equivalent to the compact connected Lie group $G$. Considering a finite dimensional complex representation $\rho : G \to GL_m(\mathbb{C})$, we have Chern classes $c_i(\rho)$ in the cohomology $h^*(BG)$ of classifying space and the Chern subring $Ch(G) \subset h^*(BG)$, a subalgebra over $h_*$ generated by Chern classes, where $\rho$ ranges over all finite dimensional representations. If $G$ is one of classical groups $SU(n)$, Spin($n$) and $Sp(n)$, the cohomology $h^*(BG)$ is generated by Chern classes and $h^*(BG) = Ch(G)$ for arbitrary odd prime $p$.

The case of the Brown-Peterson cohomology is particularly interesting in conjunction with the study of Chow rings of classifying spaces of complex linear algebraic groups defined by Totaro. In [To], Totaro considered the classifying space of the linear algebraic group $G(\mathbb{C})$ as a limit of algebraic varieties, defined the Chow ring for it and showed that the cycle map factors through the Brown-Peterson cohomology,

$$CH^*(BG(\mathbb{C}))(p) \to BP^*(BG) \otimes_{BP_*} \mathbb{Z}(p) \to H_{even}^*(BG; \mathbb{Z}(p)),$$

where $H_{even}^*(BG; \mathbb{Z}(p))$ is the direct sum of $H^i(BG; \mathbb{Z}(p))$ ($i \geq 0$). He also conjectured that the left homomorphism $CH^*(BG(\mathbb{C}))(p) \to BP^*(BG) \otimes_{BP_*} \mathbb{Z}(p)$ is an isomorphism. We may consider a Chern subring for the Chow ring $CH^*(BG(\mathbb{C}))$ as in the case of the above $Ch(G)$.

In [Ka-Ya] and [V1], the Chow ring $CH^*(BPGL_p(\mathbb{C}))(p)$ of the complex linear algebraic group $BPGL_p(\mathbb{C})$, which is the complexification of the projective unitary group $PU(p)$, and related cohomology theories were computed and it was shown that

$$CH^*(BPGL_p(\mathbb{C}))(p) = BP^*(BPU(p)) \otimes_{BP_*} \mathbb{Z}(p) = H_{even}^*(BPU(p); \mathbb{Z}(p))$$

through the cycle map above. In [Ka-Ya] Proposition 5.7, we showed similar results for $(p, G) = (3, F_4)$, $(5, E_6)$. For $p = 3$, the computation of the Brown-Peterson cohomology was done by Kono and Yagita in [Ko-Ya] and Kono and Yagita showed that $x^a$ is not in the Chern subring unless $a$ is divisible by 2. In [T], Targa showed...
that \( x_{2n+2}^a \) in \( CH^*(BPGL_p(\mathbb{C}))(p) \), where \( a \leq p - 2 \), is not in the Chern subring for arbitrary odd prime \( p \).

In this paper, we prove the following and generalize the above computation of Kono, Yagita and Targa. Let \( Q_i \) be the Milnor operations of degree \( 2p^i - 1 \) which acts on the mod \( p \) cohomology of a space.

**Theorem 1.1.** For \( (p, G) = (p, PU(p)) \), let \( x = Q_0Q_1x_2 \) where \( x_2 \) is the generator of \( H^2(BG; \mathbb{Z}/p) = \mathbb{Z}/p \). For \( (p, G) = (3, E_4), (3, E_6), (3, E_7), (3, E_8) \), \( (5, E_8) \), let \( x = Q_1Q_2x_4 \) where \( x_4 \) is the generator of \( H^4(BG; \mathbb{Z}/p) = \mathbb{Z}/p \). Then, \( x^a \) is not in the Chern subring \( CH_{H\mathbb{Z}/p}(G) \) unless \( a \) is divisible by \( p - 1 \).

This theorem implies that if \( x \) comes from the Chow ring through the cycle map, then the Chow ring is not generated by Chern classes. Recall that motivic cohomology \( H_*^{mot}(BG(\mathbb{C}), \mathbb{Z}/p) \) contains \( CH^*(BG(\mathbb{C}))/p \) as

\[
CH^*(BG(\mathbb{C}))/p = H^{2*+*}(BG(\mathbb{C}), \mathbb{Z}/p).
\]

Moreover, the motivic cohomology has the action of Milnor operations \( Q_i \) where the degree of \( Q_i \) is \( (2p^i - 1, p^i - 1) \). If there exists an element \( x_{4,3} \) in \( H^4BG(\mathbb{C}); \mathbb{Z}/p) \) corresponding to \( x_4 \) in \( H^4BG(\mathbb{C}); \mathbb{Z}/p) \), then \( x = Q_1Q_2(x_{4,3}) \) is in the Chow ring

\[
CH^p+q+1(BG(\mathbb{C}))/p = H^{2p^2+2p+2}BG(\mathbb{C}); \mathbb{Z}/p).
\]

and through the cycle map it maps to \( x \) in Theorem 1.1. In [Yagita, Lemma 9.6, Yagita proved that if \( px_4 \in H^4BG; \mathbb{Z}(p) \) is a Chern class of some representation, then the element \( x_{4,3} \) above exists. In [Sc-Yagita], Schuster and Yagita showed that for \( (p, G) = (3, E_4) \), \( 3x_4 \) is the Chern class of the complexification of the irreducible representation of \( E_4 \). In this paper, by computing the Chern class of the adjoint representation of \( E_8 \), we prove the following proposition.

**Proposition 1.2.** For \( (p, G) = (3, E_4), (3, E_6), (3, E_7), (3, E_8) \) and \( (5, E_8) \), there exists a complex representation \( \alpha \) of \( G \) and \( \gamma \in \mathbb{Z}(p) \) such that the element \( \gamma px_4 \in H^4BG; \mathbb{Z}(p) \) is a Chern class \( c_2(\alpha) \)

Thus, we have the following result on Chern subrings of Chow rings.

**Theorem 1.3.** For \( (p, G) = (p, PU(p)), (3, E_4), (3, E_6), (3, E_7) \) and \( (5, E_8) \), the Chow ring \( CH^*(BG(\mathbb{C}))(p) \) is not generated by Chern classes.

In §2, we consider Chern classes of elementary abelian \( p \)-groups. In §3, we prove Theorem 1.1. In §4, we prove Proposition 1.2. We thank François-Xavier Deho for informing us of the work of Targa.

2. Chern classes of elementary abelian \( p \)-groups

In this section, we investigate the total Chern class of finite dimensional complex representation \( \rho : A_n \to GL_n(\mathbb{C}) \) of elementary abelian \( p \)-group \( A_n \) of rank \( n \).

Firstly, we recall the cohomology of \( BA_n \). The mod \( p \) cohomology of elementary abelian \( p \)-group is a polynomial tensor exterior algebra

\[
\mathbb{Z}/p[t_1, \ldots, t_n] \otimes \Lambda(dt_1, \ldots, dt_n).
\]

The elements \( dt_1, \ldots, dt_n \in H^1(BA_n; \mathbb{Z}/p) \) correspond to the dual of the basis of \( \pi_1(BA_n) = H_1(BA_n; \mathbb{Z}/p) \). The elements \( t_1, \ldots, t_n \) are obtained from \( dt_1, \ldots, dt_n \) by applying the Milnor operation \( Q_0 \). For the mod \( p \) cohomology of a space, there exists an action of Milnor operations \( Q_0, Q_1, Q_2, \ldots \) and reduced power operations
The action of reduced power operations is given by $GL$ linear groups. The action of Milnor operations commutes with the action of general elementary abelian $\mathbb{P}^\mu$ sake of notational simplicity, we write $V$. We write $\psi^0 = 1, \psi^1, \psi^2, \ldots$. The action of Milnor operations on the mod $p$ cohomology of elementary abelian $p$-group is given by

$$Q_i(dt_k) = t_k^{p^i}, \quad Q_i = 0, \quad Q_i(x \cdot y) = Q_i(x) \cdot y + (-1)^{\deg x} x \cdot Q_i(y).$$

The action of reduced power operations is given by

$$\psi^i dt_k = 0, \quad \psi^i t_k = \begin{cases} t_k^{p^i} & (i = 1), \\ 0 & (i \geq 2), \end{cases} \quad \psi^i(x \cdot y) = \sum_{i=0}^j \psi^{i-j} x \cdot \psi^j y.$$

Secondly, we recall the invariant theory of finite general linear groups and special linear groups. The action of Milnor operations commutes with the action of general linear group $GL_n(\mathbb{Z}/p)$ since the action of the general linear group on the mod $p$ cohomology comes from the one on the elementary abelian $p$-group $A_n$. For the sake of notational simplicity, we write $V_n$ for the subspace spanned by $t_1, \ldots, t_n$,

$$V_n = \mathbb{Z}/p[t_1, \ldots, t_n].$$

We write $SM_n$, $M_n$ for M"{u}i invariants

$$H^*(BA_n; \mathbb{Z}/p)^{SL_n(\mathbb{Z}/p)}, \quad H^*(BA_n; \mathbb{Z}/p)^{GL_n(\mathbb{Z}/p)},$$

respectively. We also write $SD_n$, $D_n$ for Dickson invariants

$$\mathbb{Z}/p[t_1, \ldots, t_n]^{SL_n(\mathbb{Z}/p)}, \quad \mathbb{Z}/p[t_1, \ldots, t_n]^{GL_n(\mathbb{Z}/p)},$$

respectively. Kameko and Mimura [Ka-Mi] gave a simpler description for $SM_n$, $M_n$ using Milnor operations. For Dickson invariants and M"{u}i invariants, we refer the reader to [Ka-Mi] and its references. Let us define $c_{n,i}$ for $n = 1, \ldots, n - 1$ as follows: Consider the polynomial

$$f_n(X) = \prod_{v \in V_n} (X + v)$$

in $\mathbb{Z}/p[t_1, \ldots, t_n] [X]$. We define $(-1)^{n-i} c_{n,i}$ to be the coefficient of $X^{p^{n-i}}$ in $f_n(X)$. We define $e_n$ by $e_n = Q_0 \cdots Q_{n-1}(dt_1 \cdots dt_n)$. Then, we have the following. For a ring $R$ and for a finite set $\{a_1, \ldots, a_r\}$, we denote by $R\{a_1, \ldots, a_r\}$ a free $R$-module with the basis $\{a_1, \ldots, a_r\}$.

**Proposition 2.1.** There hold the following:

1. $c_{n,0} = e_n^{p-1}$.
2. $f_n(X) = X^{p^n} - c_{n,n-1} X^{p^{n-1}} + \cdots + (-1)^n c_{n,0} X$.
3. $SD_n$ is a polynomial algebra $\mathbb{Z}/p[e_n, c_{n,n-1}, \ldots, c_{n,1}]$.
4. $D_n$ is also a polynomial algebra $\mathbb{Z}/p[e_{n-1}, \ldots, c_{n,1}, c_{n,0}]$.
5. $M_n$ is a free $D_n$-module

$$D_n\{1, e_n^{p^2} dt_1 \cdots dt_n, e_n^{p^2} Q_{i_1} \cdots Q_{i_r} (dt_1 \cdots dt_n)\}$$

and

6. $SM_n$ is a free $SD_n$-module

$$SD_n\{1, dt_1 \cdots dt_n, Q_{i_1} \cdots Q_{i_r} (dt_1 \cdots dt_n)\},$$

where $0 \leq i_1 < \cdots < i_r \leq n - 1, 1 \leq r \leq n - 1$.

Thirdly, we consider Chern classes. It is well-known that any finite dimensional complex representation of an abelian group is a direct sum of 1-dimensional complex representations. Therefore, the total Chern class $c(\rho)$ is a product of $c(\lambda)$’s where $c(\lambda) = 1 + v, v \in V_n$. Thus, the Chern classes are in $\mathbb{Z}/p[t_1, \ldots, t_n]$ instead of $H^*(BA_n; \mathbb{Z}/p)$. Let us consider the total Chern class $c(\text{reg})$ of the regular
representation \( reg : A_n \to \text{GL}_{p^n}(\mathbb{C}) \). It is clear that \( \text{GL}_n(\mathbb{Z}/p) \) acts on \( A_n \) and \( c(\text{reg}) \in M_n \).

**Proposition 2.2.** There holds

\[ c(\text{reg}) = \prod_{v \in V_n \setminus \{0\}} (1 + v) = 1 - c_{n,n-1} + \cdots + (-1)^n c_{n,0} \in D_n. \]

For a group \( W \) acting \( V_n \setminus \{0\} \), we say the action of \( W \) is transitive on \( V_n \setminus \{0\} \) if and only if for each \( u, v \) in \( V_n \setminus \{0\} \), there exists \( w \in W \) such that \( wu = v \). We investigate the total Chern class \( c(\rho) \) when the image of the induced homomorphism \( B \rho^* : H^*(\text{BGL}_n(\mathbb{C}); \mathbb{Z}/(0)) \to \mathbb{Z}/p[t_1, \ldots, t_n] \) is invariant under certain group action.

**Lemma 2.3.** Let \( \rho : A_n \to \text{GL}_n(\mathbb{C}) \) be a complex representation of elementary abelian \( p \)-group \( A_n \) of rank \( n \). Suppose that a subgroup \( W \) of \( \text{GL}_n(\mathbb{Z}/p) \) acts on \( A_n \) in the obvious manner. Suppose that the total Chern class \( c(\rho) \) is in \( \mathbb{Z}/p[t_1, \ldots, t_n]^W \) and suppose that the action of \( W \) on \( V_n \setminus \{0\} \) is transitive. Then, \( c(\rho) = c(\text{reg})^\alpha \) for some \( \alpha \geq 0 \).

**Proof.** Suppose that

\[ c(\rho) = \prod_{v \in V_n \setminus \{0\}} (1 + v)^{\mu(v)}. \]

The non-negative integer \( \mu(v) \) is the divisibility of \( c(\rho) \) by \( 1 + v \). In other words, \( c(\rho) \) is divisible by \( (1 + v)^{\mu(v)} \) but not divisible by \( (1 + v)^{\mu(v)+1} \). In order to prove the lemma, it suffices to show that \( \mu(v) \) is a constant function of \( v \in V_n \setminus \{0\} \). Suppose that \( \mu(u) < \mu(v) \) for some \( u, v \in V_n \setminus \{0\} \). Let \( w \in W \) be an element such that \( uw = u \). Then, since \( w \) acts trivially on \( c(\rho) \), we have

\[ c(\rho) = wc(\rho) = \prod_{v \in V_n \setminus \{0\}} (w(1 + v'))^{\mu(v')} = \left( \prod_{v' \in V_n \setminus \{0,v\}} (1 + uv')^{\mu(v')} \right) (1 + u)^{\mu(v)}. \]

This implies that \( \mu(u) \geq \mu(v) \). It is a contradiction. Hence, we have the desired result.

By Proposition 2.2 and Lemma 2.3, we have the following result:

**Proposition 2.4.** Let \( G \) be a compact connected Lie group and let \( A_n \) be an elementary abelian \( p \)-subgroup of \( G \). Suppose that the Weyl group of \( A_n \), that is the quotient of the normalizer of \( A_n \) in \( G \) by the centralizer of \( A_n \) in \( G \), acts transitively on \( V_n \setminus \{0\} \). Then, \( B\eta^*(\text{Ch}_{HZ/p}(G)) \subset D_n \), where \( \eta : A_n \to G \) be the inclusion of \( A_n \) into \( G \).

We end this section by recalling the following fact:

**Proposition 2.5.** The action of \( \text{SL}_n(\mathbb{Z}/p) \) on \( V_n \setminus \{0\} \) is transitive for \( n \geq 2 \).

**Proof.** It is an easy exercise of linear algebra. It suffices to show that for any \( a = (a_1, a_2, \ldots, a_n) \in V_n \setminus \{0\} \), there exists a matrix \( g \) in \( \text{SL}_n(\mathbb{Z}/p) \) such that

\[ g \left( \begin{array}{c} 1 \\ 0 \\ \vdots \\ 0 \end{array} \right) = \alpha a, \]
where $t^a$ is the transpose of $a$. If necessary, applying a permutation, without loss of generality, we may assume that $a_1 \neq 0$. We choose the first column vector of $g$ to be $t^a$ and the first row vector of $g$ to be $(a_1, 0, \ldots, 0)$ and we choose the rest of the entries in the matrix $g$ so that the matrix obtained from $g$ by removing the first column and the first row is a diagonal matrix whose $(i, i)$-entry is 1 for $i = 1, \ldots, n - 2$ and $(n - 1, n - 1)$-entry is $a_{i-1}^{-1}$. Then by computing the cofactor expansion along the first row, we see that the determinant of $g$ is 1 and so $g$ is in $SL_n(\mathbb{Z}/p)$. By definition, it is clear that $g$ satisfies the required equality. □

Thus, in order to prove Theorem 1.1, it suffices to show that there exists an elementary abelian $p$-subgroup $A_n$ whose Weyl group is $SL_n(\mathbb{Z}/p)$ and that $B\eta^*(x) \notin D_n$. This is what we do in the next section.

3. Chern subrings

In this section, we prove Theorem 1.1 by observing the cohomology of non-toral elementary abelian $p$-subgroup of $G$. There exist non-toral elementary abelian $p$-subgroups in a compact connected Lie group if the integral homology of the Lie group has $p$-torsion. These non-toral elementary abelian $p$-subgroups and their Weyl groups are known for $(p, G) = (p, PU(n)), (3, F_4), (3, E_6), (3, E_7), (3, E_8), (5, E_8)$. We refer the reader to Andersen et al. [A-G-M-V] and its references. In this paper, we use the following results for $(p, G) = (p, PU(p)), (3, F_4)$ and $(5, E_8)$ only:

**Proposition 3.1.** There hold the following:

1. For $(p, G) = (p, PU(p))$, there exists a non-toral elementary abelian $p$-subgroup $A_2$ of rank 2 such that its Weyl group in $G$ is the special linear group $SL_2(\mathbb{Z}/p)$.
2. For $(p, G) = (3, F_4), (5, E_8)$, there exists a non-toral elementary $p$-subgroup $A_3$ of rank 3 such that its Weyl group in $G$ is the special linear group $SL_3(\mathbb{Z}/p)$.

Let $\eta : A_n \to G$ be the inclusion of non-toral elementary abelian $p$-subgroup in $G$. In [Ka-Ya], we computed the image of the induced homomorphism

$$B\eta^* : H^*(BG; \mathbb{Z}/p) \to SM_n$$

for $(p, G) = (p, PU(p)), n = 2$ and for $(p, G) = (3, F_4), (3, E_6), (3, E_7), (5, E_8), n = 3$. Since we wish to include the case $(p, G) = (3, E_8)$ in Theorem 1.1 instead of making use of the computation of the image of $B\eta^*$, we use the following result, which is also used in the computation of the image of $B\eta^*$:

**Proposition 3.2.** There hold the following:

1. The induced homomorphism

$$H^2(BPU(p); \mathbb{Z}/p) \to SM_2^2 = \mathbb{Z}/p\{dt_1 dt_2\}$$

is an isomorphism.
2. For $(p, G) = (3, F_4)$ and $(5, E_8)$, the induced homomorphism

$$H^4(BG; \mathbb{Z}/p) \to SM^4 = \mathbb{Z}/p\{Q_0 dt_1 dt_2 dt_3\}$$

is an isomorphism.

Now, we prove Theorem 1.1 for $(p, G) = (3, E_8)$. As we mentioned at the end of the previous section, it suffices to show that $B\eta^*(x) \notin D_3$. There is a sequence of inclusions

$$F_4 \to E_6 \to E_7 \to E_8$$
are isomorphisms. Recall that we denote the generator of $H^4(\mathbb{Z}/3)$ by $x_4$. We define $x \in H^{26}(\mathbb{Z}/3)$ by $x = Q_1 Q_2(x_4)$. Since the induced homomorphism maps $x_4$ to $Q_0(dt_1 dt_2 dt_3)$ by Proposition 3.2, it maps $x$ to $e_3 = Q_0 Q_1 Q_2(dt_1 dt_2 dt_3)$ in $SD_3$. It is clear that $e_3^2$ is not in $D_3$ unless $a$ is divisible by $p - 1$. Thus, we have Theorem 1.1 for $(p, G) = (3, E_8)$. Theorem 1.1 for the other $(p, G)$'s can be proved in the same manner.

4. PROOF OF PROPOSITION 1.2

In this section, we prove Proposition 1.2 by computing the second Chern class of the adjoint representation of the exceptional Lie group $\alpha : E_8 \to SO(248)$. Similar computation was done in [Sc-Ya] for the irreducible representation $F_4 \to SO(26)$.

Since the induced homomorphism

$$H^4(BF_4; \mathbb{Z}/p) \leftrightarrow H^4(BE_6; \mathbb{Z}/p) \leftrightarrow H^4(BE_7; \mathbb{Z}/p) \leftrightarrow H^4(BE_8; \mathbb{Z}/p) = \mathbb{Z}/p$$

are isomorphisms, if $3x_4$ in $H^4(BE_8; \mathbb{Z}/3)$ is a Chern class, so is in $H^4(BG; \mathbb{Z}/3)$ for $G = F_4, E_6, E_7$. So, it suffices to show the proposition for $G = E_8$.

Let $\alpha : E_8 \to SO(248)$ be the adjoint representation of $E_8$. By the construction of the exceptional Lie group $E_8$ in [Ad], there exists a homomorphism $\beta : \text{Spin}(16) \to E_8$ such that the induced representation $\alpha \circ \beta$ is the direct sum of $\lambda^2_{16} : \text{Spin}(16) \to SO(120)$ and $\Delta^+_1 : \text{Spin}(16) \to SO(128)$. See [Ad] Corollary 7.3] and [Mi-Ni] p. 143). Let $T^8$ be the maximal torus of $\text{Spin}(16)$. Let $T^1$ be the first factor of $T^8$ and $\eta : T^1 \to \text{Spin}(16)$ the inclusion of $T^1$ into $\text{Spin}(16)$. Denote by $R(G)$ the complex representation ring of $G$. The complexification of $\lambda^2_{16}$ corresponds to the second elementary symmetric function of $z_1^2 + z_2^2, \ldots, z_8^2 + z_8^2$ in $R(T^8)$ and the complexification of $\Delta^+_1$ corresponds to $\sum_{z_1 \cdot z_8 = 1} z_1^1 \cdot z_8^8$ in $R(T^8)$, where $\varepsilon_r = \pm 1$ for $r = 1, \ldots, 8$.

So, the restriction of the complexification of $\lambda^2_{16}$ to $T^1$ corresponds to

$$2^5 \begin{pmatrix} 7 \\ 2 \end{pmatrix} + 2 \begin{pmatrix} 7 \\ 1 \end{pmatrix} (z_1^2 + z_8^2) = 84 + 14(z_1^2 + z_8^2) \quad \text{in } R(T^1).$$

The restriction of the complexification of $\Delta^+_1$ to $T^1$ corresponds to

$$2^6(z_1 + z_8^{-1}) = 64(z_1 + z_8^{-1}) \quad \text{in } R(T^1).$$

Therefore, the total Chern class of the complexification of $\alpha \circ \beta \circ \eta$ is

$$\{(1 + 2u)(1 - 2u)\}^{14} \{(1 + u)(1 - u)\}^{64} = 1 - 120u^2 + \cdots \in \mathbb{Z}[u] = H^*(BT^1; \mathbb{Z}),$$

where $u$ is the generator of $H^2(BT^1; \mathbb{Z}) = \mathbb{Z}$. Since $120 = 2^3 \cdot 3 \cdot 5$, the Chern class $c_2^p(\alpha)$ represents $\gamma p x_4$ for $p = 3, 5$ in $H^4(BE_8; \mathbb{Z}(p))$, where $\gamma$ is a unit in $\mathbb{Z}(p)$, and $x_4$ is the generator of $H^4(BE_8; \mathbb{Z}(p)) = \mathbb{Z}(p)$. This completes the proof of Proposition 1.2.
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