SIMPLICIAL VOLUME OF CLOSED LOCALLY SYMMETRIC SPACES OF NON-COMPACT TYPE

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Abstract. We show that compact, locally symmetric spaces of non-compact type have positive simplicial volume. This gives a positive answer to a question that was first raised by Gromov [Gr1] in 1982. We provide a summary of results that are known to follow from positivity of the simplicial volume.

1. Introduction

In his paper [Gr1], Gromov introduced the notion of the simplicial volume of a connected, closed, and orientable manifold $M$. This homotopy invariant is denoted by $||M|| \in [0, \infty)$ and measures how efficiently the fundamental class of $M$ may be represented using real cycles.

In the same paper, the question was raised as to whether the simplicial volume of a closed locally symmetric space of non-compact type is positive (pg. 11 in [Gr1]). Since then, this question has been mentioned in a variety of different sources ([Gr2], [L], [Sa], [CF1]), and has become a well-known “folk conjecture.” The purpose of this paper is to answer this conjecture in the affirmative. Namely, we obtain:

Main Theorem: If $M^n$ is a closed locally symmetric space of non-compact type, then $||M^n|| > 0$.

The approach we use is due to Thurston [Th] and bounds the simplicial volume $||M||$ from below by the proportion between the volume of $M$ and the maximal volume of suitably defined straightened top dimensional simplices in $M$. Technically, however, we are indebted to Besson, Courtois, and Gallot for their pioneering work around the use of the barycenter method in proving the rank one entropy rigidity conjecture for locally symmetric spaces [BCG] and to Connell and Farb for their subsequent development of the technique in higher rank spaces (see [CF1] for an extensive survey).

The main contribution of this paper lies in the idea of using the barycenter method in order to define the straightened simplices that are central to Thurston’s argument. This allows us to obtain control over the straightening process from estimates (similar to those) in [CF2].
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2. Background

In this section we collect together the relevant definitions and results that are used in the proof of the main theorem.

2.1. Simplicial volume. We begin with the definition of the simplicial volume and the important proportionality principle.

**Definition.** Let $M$ be a topological space, $C^0(\Delta^k, M)$ be the set of singular $k$-simplices, and let $c = \sum_{i=1}^j a_i \cdot f_i$ with each $a_i \in \mathbb{R}$ and $f_i \in C^0(\Delta^k, M)$ be a singular real chain. The $L^1$ norm of $c$ is defined by $||c||_{L^1} = \sum_i |a_i|$. The $L^1$ norm of a real singular homology class $[\alpha] \in H_{\text{sing}}^k(M, \mathbb{R})$ is defined by $||[\alpha]||_{L^1} = \inf \{||c||_{L^1} | \partial(c) = 0, [c] = [\alpha]\}$.

**Definition.** Let $M^n$ be an oriented closed connected $n$-manifold with fundamental class $[M^n]$. The simplicial volume of $M^n$ is defined as $||M^n|| = ||i([M^n])||_{L^1}$, where $i : H_n(M, \mathbb{Z}) \to H_n(M, \mathbb{R})$ is the change of coefficients homomorphism, and $[M^n]$ is the fundamental class arising from the orientation of $M^n$.

The proportionality principle ([Gr1], [Th], [St]) for simplicial volume is expressed in the following:

**Theorem 2.1.** Let $M$ and $M'$ be two closed Riemannian manifolds with isometric universal covers. Then

$$\frac{||M||}{\text{Vol}(M)} = \frac{||M'||}{\text{Vol}(M')}.$$  

In addition, the simplicial volume is particularly well behaved with respect to products and connected sums. Namely, the following relationships hold:

**Theorem 2.2.** For a pair of closed manifolds $M_1, M_2$, we have that:

$$C \cdot ||M_1|| \cdot ||M_2|| \geq ||M_1 \times M_2|| \geq ||M_1|| \cdot ||M_2||$$

where $C > 1$ is a constant that depends only on the dimension of $M_1 \times M_2$.

**Theorem 2.3.** For $n \geq 3$, the connected sums of a pair of $n$-dimensional manifolds $M_1$ and $M_2$ satisfy:

$$||M_1 \# M_2|| = ||M_1|| + ||M_2||$$

The proof of these last two results can be found in Sections 1.1 and 3.5 of [Gr1] respectively (for Theorem 2.2, see also Section F.2 in [BP]).
2.2. Thurston’s approach. Our proof will follow Thurston’s method in [Th] of showing positivity of the simplicial volume of $M^n$. Our version of Thurston’s method can be summarized in the following:

**Theorem 2.4.** Let us denote by $\tilde{M}^n$ the universal cover of $M^n$, $\Gamma$ the fundamental group of $M^n$, and $C^0(\Delta^k, \tilde{M}^n)$ be the set of singular $k$-simplices in $\tilde{M}^n$, where $\Delta^k$ is assumed to be equipped with a fixed Riemannian metric. Assume that we are given a collection of maps $st_k : C^0(\Delta^k, \tilde{M}^n) \to C^0(\Delta^k, \tilde{M}^n)$ where $C^0_{st}(\Delta^k, \tilde{M}^n) \subset C^0(\Delta^k, \tilde{M}^n)$. We will say this collection of maps is a straightening provided it satisfies the following four formal properties:

1. the maps $st_k$ are $\Gamma$-equivariant,
2. the maps $st_*$ induce a chain map $st_* : C^*_{\text{sing}}(\tilde{M}^n, \mathbb{R}) \to C^*_{\text{sing}}(\tilde{M}^n, \mathbb{R})$ which is $\Gamma$-equivariantly chain homotopic to the identity,
3. the image of $st_n$ lies in $C^1(\Delta^n, \tilde{M}^n)$, i.e. straightened top-dimensional simplices are $C^1$,
4. there exists a constant $C > 0$, depending solely on $\tilde{M}^n$ and the chosen Riemannian metric on $\Delta^n$, such that for any $f \in C^0(\Delta^n, M^n)$, and corresponding straightened simplex $st_n(f) : \Delta^n \to \tilde{M}^n$, there is a uniform upper bound on the Jacobian of $st_n(f)$:

$$|\text{Jac}(st_n(f))(\sigma)| \leq C$$

where $\sigma \in \Delta^n$ is arbitrary, and the Jacobian is computed relative to the fixed Riemannian metric on $\Delta^n$.

If such a straightening exists, then $||M^n|| > 0$.

In the Theorem, one could replace properties (3) and (4) by the more general condition that the volume of the images of straightened top-dimensional simplices are uniformly bounded above. This more general approach was used in [Th] and [IY] to give a proof that closed negatively curved manifolds have positive simplicial volume. A different straightening procedure was developed by Savage [Sa] to show positivity of the simplicial volume for cocompact quotients of $SL_n(\mathbb{R})/SO_n(\mathbb{R})$. Our proof of the main theorem will involve a new straightening procedure for locally symmetric spaces of non-compact type which do not have any local $\mathbb{H}^2$ or $SL_3(\mathbb{R})/SO_3(\mathbb{R})$ factors. Our formulation isolates properties (3) and (4) because the barycenter method we will use is particularly well-adapted to establishing these properties.

We finish with a brief explanation of how Theorem 2.4 is proved. We first note that property (1) implies that the straightening procedure descends to a straightening procedure on the compact quotient $M^n$. Property (2) ensures that the homology of $M^n$ obtained via the complex of straightened chains coincides with the ordinary singular homology of $M^n$. Furthermore, since the straightening procedure is a projection operator on the level of chains, it is contracting in the $L^1$-norm. In particular, if $\sum a_i f_i$
is a real $n$-chain representing the fundamental class of $M^n$, then so is $\sum a_i st(f_i)$, and we have the inequality $\| \sum a_i f_i \|_{L^1} \geq \| \sum a_i st(f_i) \|_{L^1}$.

As a consequence, in order to show that the simplicial volume of $M^n$ is positive, it is sufficient to give a lower bound for the $L^1$-norm of a straightened chain representing the fundamental class. But now observe that, by property (3), the straightened chain is $C^1$, and hence we can compute the volume of $M^n$ by:

$$Vol(M^n) = \int \sum a_i st(f_i) \, dV_{M^n} = \sum a_i \int_{st(f_i)} dV_{M^n},$$

where $dV_{M^n}$ is the volume form on $M^n$. On the other hand, we have the bound:

$$\sum a_i \int_{st(f_i)} dV_{M^n} \leq \sum |a_i| \int_{\Delta^n} |Jac(st(f_i))| dV_{\Delta^n},$$

where $dV_{\Delta^n}$ is the volume form for the fixed Riemannian metric on $\Delta^n$. Now by property (4) the Jacobian of straightened simplices is bounded uniformly from above, and hence we have a uniform upper bound:

$$\int_{\Delta^n} |Jac(st(f_i))| dV_{\Delta^n} \leq K$$

where $K > 0$ depends solely on $M^n$. This yields the inequality:

$$Vol(M^n) \leq K \cdot \sum |a_i|$$

which upon dividing, and passing to the infimum over all straightened chains, provides the positive lower bound $||M^n|| \geq Vol(M^n)/K > 0$.

2.3. Locally symmetric spaces. The principle technical tool in our argument is the use of the barycenter method (see [BCG] for a history as well as a survey of applications). We will use this method in order to define our straightenings. The technique ensures that straightened simplices are $C^1$ and also gives a pointwise upper bound for their Jacobian (formal properties (3) and (4) in Theorem 2.4).

For details on locally symmetric spaces of non-compact type, we refer the reader to the summary given in [CF2]. Let $(M^n, g_0)$ denote a closed locally symmetric space of non-compact type. We let $(X, g)$ be the symmetric universal covering space of $M$ and fix a basepoint $p \in X$. We will use the following notation:

- $G = \text{Isom}(X)^0$, $K = \text{Stab}_G(p)$, so that $X \cong G/K$, and $P$ denotes a minimal parabolic subgroup of $G$.
- The visual boundary and Furstenberg boundaries are denoted by $\partial X$ and by $\partial_F X \cong G/P$, respectively.
- The $\nu(g_0)$-conformal density given by the family of Patterson-Sullivan measures is denoted by $\nu : X \to \mathcal{M}(\partial X)$, where $\mathcal{M}(\partial X)$ denotes the space of atomless probability measures on the visual boundary of $X$. 

• The map \( \nu \) is \( \Gamma \)-equivariant, in the sense that \( \nu(\gamma x) \) coincides with \( \gamma_*\nu(x) \), the pushforward of the measure \( \nu(x) \) under the \( \gamma \) action on \( \partial X \).

• \( B_p : X \times \partial X \to \mathbb{R} \) denotes the Busemann function based at the point \( p \in X \).

In higher rank, the family of Patterson-Sullivan measures were constructed in [K], [Al] and are \( \pi_1(M) \)-equivariant, atomless, and fully supported on \( \partial F X \). Additionally, the Busemann functions satisfy \( B_p(\cdot, \cdot) = B_{\gamma p}(\gamma \cdot, \gamma \cdot) \) for each \( \gamma \in \pi_1(M) \) and \( p \in X \).

We will denote by:

\[
\begin{align*}
\text{dB}_{(x, \theta)}(\cdot) : T_xX &\to \mathbb{R} \\
D\text{dB}_{(x, \theta)}(\cdot, \cdot) : T_xX \otimes T_xX &\to \mathbb{R}
\end{align*}
\]

the 1-form and 2-form obtained by differentiating the Busemann function (based at the point \( p \in X \)) corresponding to the direction \( \theta \in \partial X \) at the point \( x \in X \). Note that, while the Busemann functions depend on the chosen basepoint \( p \in X \), the 1-form and 2-form defined above do not. This justifies the omission of the basepoint \( p \in X \) in our notation for these forms.

For a measure \( \mu \in \mathcal{M}(\partial X) \), let

\[
g_\mu(\cdot) = \int_{\partial X} B_p(\cdot, \theta) d(\mu)(\theta).
\]

When \( g_\mu : X \to \mathbb{R} \) has a unique minimum, the barycenter of \( \mu \), denoted by \( \text{bar}(\mu) \in X \), is defined to be the unique point where \( g_\mu \) is minimized. In [CF2], the proof of their proposition 3.1 shows that measures fully supported on the Furstenberg boundary have well defined barycenters and that they are independent of the chosen basepoint \( p \in X \) used to define \( g_\mu \).

Finally, we recall the following result of Connell and Farb (see Section 4 in [CF2]):

**Theorem 2.5.** Let \( M \) be a closed locally symmetric space of non-compact type with no local direct factors locally isometric to \( \mathbb{H}^2 \) or \( SL_3(\mathbb{R})/SO_3(\mathbb{R}) \), and let \( X \) be its universal cover. Let \( \mu \in \mathcal{M}(\partial X) \) be a probability measure fully supported on \( \partial F X \) and let \( x \in X \). Consider the endomorphisms \( K_x(\mu), H_x(\mu) \), defined on \( T_xX \) by:

\[
\langle K_x(\mu)u, u \rangle = \int_{\partial F X} \text{DdB}_{(x, \theta)}(u, u) d(\mu)(\theta)
\]

and

\[
\langle H_x(\mu)u, u \rangle = \int_{\partial F X} \text{dB}^2_{(x, \theta)}(u) d(\mu)(\theta).
\]

Then \( \det(K_x(\mu)) > 0 \) and there is a positive constant \( C := C(X) > 0 \) depending only on \( X \) such that:

\[
J_x(\mu) := \frac{\det(H_x(\mu))^{1/2}}{\det(K_x(\mu))} \leq C.
\]

Furthermore, the constant \( C \) is explicitly computable.
In the proof of our main theorem, this result of Connell and Farb will be applied in the very specific case where $X$ is an irreducible higher rank locally symmetric space, distinct from $SL_3(\mathbb{R})/SO_3(\mathbb{R})$, and the measure $\mu$ is a weighted sum of Patterson-Sullivan measures.

3. Proof of the Main Theorem

We start out by providing a reduction, showing that it is sufficient to prove the Main Theorem for irreducible closed locally symmetric spaces of non-compact type. In order to see this, let $M$ be an arbitrary locally symmetric space of non-compact type, with universal cover $X$. We observe that by the proportionality principle in Theorem 2.1, in order to show that $||M|| > 0$, it is sufficient to show that $||M'|| > 0$ for some locally symmetric space of non-compact type whose universal cover is $X$.

Now let $G$ denote the identity component of Isom$(X)$ and $G = G_1 \times \cdots \times G_k$ be the product decomposition of $G$ into simple groups corresponding to the product decomposition of $X$ into irreducible symmetric spaces. By a result of Borel [Bo], there are cocompact lattices $\Gamma_i \subset G_i$ for each $i \in \{1, \ldots, k\}$. Take $M'$ to be the product locally symmetric space $M_1 \times \cdots \times M_k$ obtained from the product lattice $\Gamma_1 \times \cdots \times \Gamma_k$. From Theorem 2.2, the inequality $||M_1 \times \cdots \times M_k|| \geq \prod_{i=1}^k ||M_i||$ holds. Hence, if one has the Main Theorem for irreducible locally symmetric spaces of non-compact type, one obtains the Main Theorem for all locally symmetric spaces of non-compact type.

Next we observe that for the irreducible locally symmetric spaces modelled on $H^n$, $CH^n$, $HH^n$, and $Cay\mathbb{H}^2$ (the rank one cases), positivity of the simplicial volume follows from [Th] and [IY]. Furthermore, for $SL_3(\mathbb{R})/SO_3(\mathbb{R})$, positivity of the simplicial volume follows from [Sa]. The Main Theorem will then follow from:

**Claim:** If the manifold $M^n$ is a compact quotient of an irreducible higher rank symmetric space of non-compact type $X$, and $X \neq SL_3(\mathbb{R})/SO_3(\mathbb{R})$, then the simplicial volume satisfies $||M^n|| > 0$.

To obtain the Claim, we use the Thurston approach. From Theorem 2.4, it is sufficient to define a straightening for $M^n$. Before proceeding to do this, we fix some notation. Recall that a singular $k$-simplex in $M^n$ is a continuous map $f : \Delta^k \to M^n$, where $\Delta^k$ is the standard Euclidean $k$-simplex realized as the convex hull of the standard unit basis vectors in $\mathbb{R}^{k+1}$. For our purpose it is more convenient to work with the spherical $k$-simplex $\Delta^k_s = \{(a_1, \ldots, a_{k+1})| a_i \geq 0, \sum_{i=1}^{k+1} a_i^2 = 1\} \subset \mathbb{R}^{k+1}$, equipped with the Riemannian metric induced from $\mathbb{R}^{k+1}$. We will denote by $e_i$ ($1 \leq i \leq k+1$) the standard basis vectors for $\mathbb{R}^{k+1}$. Finally, we will denote by $\Gamma$ the fundamental group of $M^n$. We now define the straightening procedure we will use:
Definition. Given a singular $k$-simplex $f \in C^0(\Delta^k, X)$, with vertices $x_i := f(e_i)$, define $st_k(f) \in C^0(\Delta^k, X)$ by $st_k(f)(\sum_i a_i e_i) = \text{bar}(\sum_i a_i^2 \nu(x_i))$.

The fact that the simplex $st_k(f)$ is well defined follows from the comments in Section 2.3. Moreover, observe that $st_k(f)$ depends only on the vertices of the original simplex $f$, i.e. on the $(k + 1)$-tuple of points $V := (x_1, \ldots, x_{k+1})$. Let $V$ denote the collection of vertices of $f$. As $st_k(f)$ depends only on $V$, we set $st_V(\sigma) := st_k(f)(\sigma)$. We now proceed to verify that this straightening procedure satisfies the four formal properties needed. For the convenience of the reader, we restate each property prior to proving it.

Property (1): The maps $st_k$ are $\Gamma$-equivariant.

Proof. Fix a point $\sigma = \sum_i a_i e_i \in \Delta^k$. Then for any $\gamma \in \Gamma$, $st_{\gamma V}(\sigma)$ is defined as the unique minimizer of the function $g_\sigma(\cdot) = \int_{\partial_F X} B_p(\cdot, \theta) d(\sum_i a_i^2 \nu(\gamma x_i))(\theta)$. Since

$$
\int_{\partial_F X} B_p(\cdot, \theta) d(\sum_i a_i^2 \nu(\gamma x_i))(\theta) =
$$

$$
\int_{\partial_F X} B_p(\cdot, \theta) d(\sum_i a_i^2 \gamma_i \nu(x_i))(\theta) =
$$

$$
\int_{\partial_F X} B_p(\cdot, \gamma^{-1} \theta) d(\sum_i a_i^2 \nu(x_i))(\theta) =
$$

$$
\int_{\partial_F X} B_{\gamma^{-1} \gamma p}(\gamma^{-1} \gamma \cdot, \gamma^{-1} \theta) d(\sum_i a_i^2 \nu(x_i))(\theta) =
$$

$$
\int_{\partial_F X} B_{\gamma p}(\gamma \cdot, \theta) d(\sum_i a_i^2 \nu(x_i))(\theta),
$$

and since $B_{\gamma p}(\cdot, \cdot)$ and $B_p(\cdot, \cdot)$ differ by a function $k(\theta)$ of $\theta$, it follows that the unique minimizer of $g_\sigma(\cdot)$ is also the unique minimizer of the function:

$$
h_\sigma(\cdot) = \int_{\partial_F X} B_p(\gamma \cdot, \theta) d(\sum_i a_i^2 \nu(x_i))(\theta).
$$

Indeed, we have that the difference of the two functions is:

$$
g_\sigma(\cdot) - h_\sigma(\cdot) = \int_{\partial_F X} k(\theta) d(\sum_i a_i^2 \nu(x_i))(\theta)
$$

which is a constant function on $X$. But if $x \in X$ is the unique minimizer of $h_\sigma(\cdot)$, then $\gamma^{-1} x = st_V(\sigma)$. It follows that $st_{\gamma V}(\sigma) = \gamma st_V(\sigma)$, completing the proof of Property (1).
Property (2): The maps $st_*$ induce a chain map $st_* : C^\text{sing}_*(X, \mathbb{R}) \to C^\text{sing}_*(X, \mathbb{R})$ which is $\Gamma$-equivariantly chain homotopic to the identity.

Proof. The fact that $st_k$ commutes with the boundary operator follows from the fact that $st_k(f)$ depends solely on the vertices of the singular simplex $f$, along with the fact that $st_k(f)$ restricted to a face of $\Delta^k$ coincides with the straightening of that face.

To see that $st$ is chain homotopic to the identity, first note that the uniqueness of geodesics in $X$ gives rise to a well defined $\Gamma$-equivariant straight line homotopy between any simplex $f$ and its straightening $st(f)$. Hence there are canonically defined homotopies between simplices and their straightenings in $X$. Moreover, these homotopies when restricted to lower dimensional faces agree with the homotopies canonically defined on those faces. Appropriately ($\Gamma$-equivariantly) subdividing these homotopies defines the required chain homotopy, concluding the proof of Property (2).

Property (3): The image of $st_n$ lies in $C^1(\Delta^n, X)$, i.e. straightened top-dimensional simplices are $C^1$.

Proof. Notice that for any simplex $f \in C^0(\Delta^n, X)$ and any $\sigma = \sum_i a_i e_i \in \Delta^n$, we have an implicit characterisation of the point $st_n(f)(\sigma) = st_V(\sigma)$ via the 1-form equation:

$$0 \equiv d(g_\sigma)_{st_V(\sigma)}(\cdot) = \int_{\partial_F X} dB_{st_V(\sigma), \theta}(\cdot) d(\sum_i a_i^2 \nu(x_i))(\theta).$$

Indeed, the fact that $st_V(\sigma) = \text{bar}(\sum_i a_i^2 \nu(x_i))$ is defined as the unique minimum of the function:

$$g_\sigma(\cdot) = \int_{\partial_F X} B(p, \cdot, \theta) d(\sum_i a_i^2 \nu(x_i))(\theta),$$

yields equation (1) upon differentiating.

Since the map $st_V$ is given implicitly, one can apply the implicit function theorem: in order for $st_V$ to be $C^1$, one needs to check the non-degeneracy condition. But this merely requires that for the endomorphism $K$ defined by:

$$\langle K(u), u \rangle := \int_{\partial_F X} DdB_{st_V(\sigma), \theta}(u, u) d(\sum_i a_i^2 \nu(x_i))(\theta),$$

defined on the tangent space $T_{st_V(\sigma)} M^n$, the determinant be non-zero. Note however that in the notation of Theorem 2.5, the determinant of this matrix is precisely $\det(K_{st_n(\sigma)}(\sum_i a_i^2 \nu(x_i)))$, and hence must be non-zero as the measure $\sum_i a_i^2 \nu(x_i)$ has full support on the Furstenberg boundary. This completes the proof of property (3).
Property (4): There exists a constant $C > 0$, depending solely on $X$, such that for any $f \in C^0(\Delta^n_s, X)$, and corresponding straightened simplex $st_n(f) : \Delta_s^n \to X$, there is a uniform upper bound on the Jacobian of $st_n(f)$:

$$|Jac(st_n(f))(\sigma)| \leq C$$

where $\sigma = \sum_i a_i e_i \in \Delta^n_s$ is arbitrary, and the Jacobian is computed relative to the Riemannian metric on the spherical simplex $\Delta^n_s$ induced from $\mathbb{R}^{n+1}$.

Proof. Differentiating the implicit 1-form equation with respect to directions in $T_{st}(\Delta^n_s)$, one obtains the two form equation

$$0 \equiv D_{\sigma}d(g_{stV}(\sigma))(\cdot, \cdot) = \sum_i 2a_i \langle \cdot, e_i \rangle_{\sigma} \int_{\partial_F X} dB_{(stV, \theta)}(\cdot) d(\nu(x_i))(\theta)$$

$$+ \int_{\partial_F X} Dd_{(stV, \theta)}(D(stV)_{\sigma}(\cdot), \cdot) d(\sum_i a_i^2 \nu(x_i))(\theta).$$

defined on $T_{\sigma}(\Delta^n_s) \otimes T_{stV(\sigma)}(X)$. Now define symmetric endomorphisms $H_{\sigma}$ and $K_{\sigma}$ of $T_{stV(\sigma)}(X)$ by

$$\langle H_{\sigma}(u), u \rangle_{stV(\sigma)} = \int_{\partial_F X} Dd_{(stV, \theta)}(D(stV)_{\sigma}(\cdot), \cdot) d(\sum_i a_i^2 \nu(x_i))(\theta),$$

and

$$\langle K_{\sigma}(u), u \rangle_{stV(\sigma)} = \int_{\partial_F X} Dd_{(stV, \theta)}(u, u) d(\sum_i a_i^2 \nu(x_i))(\theta).$$

The fact that $K_{\sigma}$ is positive definite follows from Theorem 2.5. Let $\{v_j\}_{j=1}^n$ be an orthonormal eigenbasis of $T_{stV(\sigma)}(X)$ for $H_{\sigma}$. At points $\sigma \in \Delta^n_s$ where the Jacobian of $stV$ is nonzero, let $\{\tilde{u}_j\}$ be the basis of $T_{\sigma}(\Delta^n_s)$ obtained by pulling back the $\{v_j\}$ basis by $K_{\sigma} \circ D(stV)_{\sigma}$, and $\{u_j\}$ be the orthonormal basis of $T_{\sigma}(\Delta^n_s)$ obtained from the $\{\tilde{u}_j\}$ basis by applying the Gram-Schmidt algorithm. We now have the sequence of equations (which we will justify in the next paragraph):

$$\det(K_{\sigma}) \cdot |Jac(stV)(\sigma)| = |det(K_{\sigma} \circ D(stV)_{\sigma})|$$

$$= \prod_{j=1}^n |\langle K_{\sigma} \circ D(stV)_{\sigma}(u_j), v_j \rangle_{stV(\sigma)}|$$

$$= \prod_{j=1}^n \left| \sum_{i=1}^{n+1} \langle u_j, e_i \rangle_{\sigma} \cdot 2a_i \int_{\partial_F X} dB_{(stV, \theta)}(v_j) d(\nu(x_i))(\theta) \right|$$

(3) \hspace{1cm} (4) \hspace{1cm} (5)
\[
\prod_{j=1}^{n} \left[ \sum_{i=1}^{n+1} \langle u_j, e_i \rangle^2 \right]^{1/2} \left[ \sum_{i=1}^{n+1} a_i^2 \left( \int_{\partial F X} dB_{(stV(\sigma), \theta)}(v_j) d(\nu(x))(\theta) \right)^2 \right]^{1/2}
\]

\[
\leq 2^n \prod_{j=1}^{n} \left[ \sum_{i=1}^{n+1} a_i^2 \right]^{1/2} \left[ \sum_{i=1}^{n+1} \int_{\partial F X} dB_{(stV(\sigma), \theta)}(v_j) d(\nu(x))(\theta) \right]^{1/2}
\]

\[
= 2^n \prod_{j=1}^{n} \langle H_\sigma(v_j), v_j \rangle_{stV(\sigma)}^{1/2} = 2^n \det(H_\sigma)^{1/2}.
\]

We now justify each step in the previous list of equations. Equation (3) follows from the definition of the Jacobian, along with the fact that \( \det(AB) = \det(A) \cdot \det(B) \). Equation (4) follows from the fact that, with respect to the \( \{u_j\} \) and \( \{v_j\} \) bases, \( K_\sigma \circ D_{stV(\sigma)} \) is upper triangular, and hence the determinant is the product of the diagonal entries. Equation (5) follows from equations (4) and (2). Inequalities (6) and (7) follow from the Cauchy-Schwartz inequality applied in \( \mathbb{R}^{n+1} \) and the spaces \( L^2(\partial F X, \nu(x)) \), respectively, along with the fact that the \( u_j \) are unit vectors in \( T_{\sigma}(\Delta^n) \subset T_\sigma(\mathbb{R}^{n+1}) \). The two equalities in (8) follow from the definition of \( H_\sigma \), and the fact that the \( \{v_j\}_{j=1}^{n} \) is an orthonormal eigenbasis for \( H_\sigma \).

Upon dividing, we now obtain the inequality:

\[
|Jac(stV(\sigma))| \leq 2^n \frac{\det(H_\sigma)^{1/2}}{\det(K_\sigma)}
\]

But now note that, in the notation of Theorem 2.5, the expression \( \det(H_\sigma)^{1/2}/\det(K_\sigma) \) is exactly \( J_{stV(\sigma)}(\sum a_i^2 \nu(x_i)) \). Since the measure \( \sum a_i^2 \nu(x_i) \) has full support in the Furstenberg boundary, Theorem 2.5 now yields a uniform constant \( C' \), depending solely on \( X \), with the property that:

\[
|Jac(stV(\sigma))| \leq 2^n J_{stV(\sigma)}(\sum a_i^2 \nu(x_i)) \leq 2^n C' =: C
\]

This completes the proof of Property (4).

Having verified Properties (1)-(4) in the definition of straightening, we now conclude that the Claim holds, completing the proof of the Main Theorem.

4. Applications

In this section, we summarize the known consequences of positivity of simplicial volume. Most of the applications we mention can be found in Gromov’s original paper [Gr1]. We also refer the reader to Pansu’s article [Pa] and to Chapter 14 in Lück’s book [L].

For the applications we give, we point out that:
those discussed in Sections 4.5-4.8, and 4.11 were previously unknown for higher rank locally symmetric spaces of non-compact type.

- the result in Section 4.1 was unknown for higher rank locally symmetric spaces of non-compact type that contain local $\mathbb{H}^2$ or $SL_3(\mathbb{R})/SO_3(\mathbb{R})$ factors.

- the estimates in Sections 4.9 and 4.10 can be explicitly computed, as our procedure gives a computable bound for the simplicial volume.

- let $\mathcal{M}$ be the smallest class of topological manifolds that (1) contains all closed locally symmetric spaces of non-compact type, (2) is closed under connected sums with arbitrary closed manifolds of dimension $\geq 3$, (3) is closed under products, and (4) is closed under fiber extensions by surfaces of genus $\geq 2$ (i.e. if $M \in \mathcal{M}$, and $M'\xrightarrow{f} M$ with fiber a surface $S_g$ of genus $\geq 2$, then $M' \in \mathcal{M}$). Then combining our Main Theorem, Theorems 2.2 and 2.3 from the introduction, along with a result of Hoster and Kotschick [HK], one obtains that for every manifold $M \in \mathcal{M}$, $|\|M\|| > 0$. For the manifolds in $\mathcal{M}$, we obtain all the applications given in Sections 4.1 through 4.8. We also point out that by a result of Kapovich and Leeb [KL], there exist surface bundles over surfaces (both of genus $\geq 2$, and hence in the class $\mathcal{M}$) that do not support metrics of non-positive curvature. More generally, manifolds in the class $\mathcal{M}$ that arise from a connected sum will fail to be aspherical, and hence cannot support metrics of non-positive curvature.

We now list out the applications.

4.1. Degree theorem. We provide a new application of positivity of the simplicial volume:

**Lemma 4.1.** Suppose that $(N^n, g_N)$ and $(M^n, g_M)$ are connected, closed, and orientable Riemannian manifolds and that $||M^n|| > 0$. If $||N^n|| = 0$, then there are no continuous maps $f : N^n \rightarrow M^n$ of positive degree. Otherwise, there is a constant $C := C(g_M, g_N) > 0$ depending on the Riemannian universal coverings of $M$ and $N$ such that

$$\deg(f) \leq C \frac{\text{Vol}(N)}{\text{Vol}(M)}.$$ 

**Proof.** Assume $||M|| > 0$, and that $f : N \rightarrow M$ is a continuous map. It is easy to see that $||N|| \geq \deg(f)||M||$ (pg. 8 in [Gr1]) and since $||M|| > 0$, this gives the inequality $\deg(f) \leq ||N||/||M||$. This immediately implies that if $||N|| = 0$, then $\deg(f) = 0$.

On the other hand, if $||N|| > 0$, then by Theorem 2.1, there are constants $\delta_{g_M}, \delta_{g_N} > 0$ such that $||M|| = \delta_{g_M} \text{Vol}(M)$ and $||N|| = \delta_{g_N} \text{Vol}(N)$. Letting $C := \delta_{g_N}/\delta_{g_M}$, we immediately obtain:

$$\deg(f) \leq \frac{||N||}{||M||} = \frac{\delta_{g_N} \text{Vol}(N)}{\delta_{g_M} \text{Vol}(M)} = C \frac{\text{Vol}(N)}{\text{Vol}(M)}.$$
concluding the proof of the Lemma.

This yields a degree theorem (upper bound on the degree in terms of the volume ratio between the codomain and domain) whenever the codomain manifold has positive simplicial volume. The question of whether the degree theorem could be obtained from positivity of the simplicial volume was brought up in the survey paper [CF1].

4.2. Co-Hopf property. The co-Hopf property for a group $G$ states that every monomorphism $G \hookrightarrow G$ is in fact an isomorphism. Note that the co-Hopf property for $\pi_1(M^n)$ follows immediately from the Degree theorem, provided that $M^n$ is aspherical. The co-Hopf property for lattices was first shown by Prasad in [Pr], and also follows from Margulis’ superrigidity theorem in the higher rank case.

4.3. Positivity of MinVol. The minimal volume of a smooth manifold $M$, denoted by $\text{MinVol}(M)\in [0,\infty)$, is defined as the infimum of $\text{Vol}(M, g)$ as $g$ varies through complete Riemannian metrics with $|K(g)| \leq 1$. It was shown by Gromov (pgs. 35-37 in [Gr1]) that positive simplicial volume implies positive MinVol.

4.4. Positivity of Minimal Entropy. The minimal entropy of a smooth manifold $M$ is defined to be the infimum of the topological entropies of the geodesic flow over all complete Riemannian metrics of unit volume on $M$. There is the following inequality between simplicial volume and the minimal entropy $h$ (see pg. 37 in [Gr1]):

$$C \cdot ||M|| \leq h(M)^n$$

where $C$ is a uniform constant, depending only on the dimension $n$ of $M$. Hence positivity of simplicial norm implies positivity of the minimal entropy.

4.5. Non-collapsing. We say that $M$ collapses provided that there exists a sequence of Riemannian metrics $g_i$ on $M$, satisfying $|K(g_i)| \leq 1$, and having the property that at every point $p \in M$, the injectivity radius with respect to the metric $g_i$ is $< 1/i$. Gromov showed that manifolds with positive simplicial volume do not collapse (pgs. 67-68 in [Gr1]).

4.6. Non-existence of F-structures. Loosely speaking, a positive rank $F$-structure on a manifold $M^n$ consists of a finite open cover $\mathcal{U} := \{U_i\}$, along with effective torus actions $T^{k_i}$ (of dimension $k_i \geq 1$) on each $U_i$, with the property that the torus actions commute on all the various intersections (we refer the reader to the survey article by Fukaya [F] for the precise definition as well as applications). A fundamental result of Cheeger and Gromov [CG] is that existence of positive rank F-structures is equivalent to collapsing. Hence if the simplicial volume of a manifold is positive, it does not support any F-structure. We point out that a smooth, fixed point free $S^1$-action is a special case of an F-structure. In particular, positivity of the simplicial volume implies that the manifold cannot support any such $S^1$-actions (see also the paper by Yano [Y]).
4.7. Non-vanishing of top-dimensional bounded cohomology. Bounded cohomology $\hat{H}^*(M^n)$ is defined in Gromov [Gr1], where it is shown (pgs. 16-17) that $M^n$ has positive simplicial volume if and only if the map induced by inclusion of chains $i^n : \hat{H}^n(M^n) \to H^n_{\text{sing}}(M^n, \mathbb{R})$ is non-zero. This immediately implies that the $n$-dimensional bounded cohomology of our manifolds $M^n$ is in fact non-zero.

In fact, Gromov has shown (pgs. 46-47 in [Gr1]) the following more general statement: if $f : X \to Y$ is a continuous map between path-connected spaces, with the property that the induced map $f_* : \pi_1(X) \to \pi_1(Y)$ is surjective with amenable kernel, then the induced map $\hat{f}^* : \hat{H}^*(Y) \to \hat{H}^*(X)$ on bounded cohomology is an isometric isomorphism. In view of the previous paragraph, one obtains that the $n$-dimensional bounded cohomology $\hat{H}^n(X)$ is non-zero for any space $X$ which has a map $f : X \to M^n$ satisfying the hypotheses above.

4.8. Covering by amenable subsets. Let us call a subset $Y \subset M^n$ amenable provided that for every path component $Y' \subset Y$, with inclusion map $i : Y' \to M^n$, the image of the morphism $i_* : \pi_1(Y') \to \pi_1(M^n)$ is amenable. It is shown by Gromov (pgs. 47-48 in [Gr1]) that if a space $X$ has a cover $\{U_\alpha\}$ by amenable sets, with the property that every point $p \in X$ lies in $\leq k$ of the sets in the cover, then the bounded cohomology $\hat{H}^i(X) = 0$ for all $i \geq k$.

Hence the non-vanishing of the $n$-dimensional bounded cohomology of $M^n$ implies that for every covering $\{U_\alpha\}$ of $M^n$ by amenable sets, there exists a point $p \in M^n$ contained in at least $(n + 1)$ of the sets in our cover.

4.9. Bounds on Euler characteristics of flat bundles. Let $E \to M^n$ be an $n$-dimensional affine flat bundle over $M^n$, and $\chi$ the Euler number of $E$ (obtained by evaluating the Euler class of $E$ on the fundamental class of $M^n$). Then the inequality $|\chi| \leq 2^{-n}||M^n||$ holds.

This is due to an argument of Smillie, presented by Gromov (pgs. 21-23 in [Gr1]), improving on the earlier inequality $|\chi| \leq ||M^n||$ due to Milnor and Sullivan ([Mi], [Su]).

4.10. Bounds on the sum of Betti numbers. If $M$ is a connected sum of locally symmetric spaces of non-compact type, then there exists a constant $C$, depending solely on the dimension of $M$, with the property that:

$$\sum b_i(M) \leq C \cdot ||M||,$$

where the $b_i(M)$ are the Betti numbers of $M$ with any given coefficients. This result is announced on pg. 12 of [Gr1].

4.11. $L^2$-invariants. In Lück’s book, the following question is raised (Conjecture 14.1, pg. 485, in [L]): let $M$ be an aspherical closed orientable manifold of dimension
≥ 1, and suppose that ||M|| = 0. Does it follow that \(\tilde{M}\) is of determinant class, and satisfies \(b^{(2)}_p(\tilde{M}) = 0\) for all \(p \geq 0\), and \(\rho^{(2)}(\tilde{M}) = 0\)?

We observe that, in view of the computations on pg. 230 of [L], there exist locally symmetric spaces of non-compact type that do not satisfy the conclusion of this conjecture. By our main theorem, they also fail to satisfy the hypotheses of this conjecture, and hence do not provide counterexamples to this conjecture.

5. Concluding remarks

We conclude by pointing out some open questions related to our Main Theorem:

Conjecture: Let \(M^n\) be a closed Riemannian manifold, whose sectional curvatures are \(\leq 0\), and whose Ricci curvatures are \(< 0\). Then ||\(M^n|| > 0\).

This conjecture was attributed to Gromov in [Sa]. It seems plausible that a similar approach could be used to verify this conjecture. The main difficulty lies in obtaining formal property (4) for the analogous straightening procedure when the space \(M^n\) is locally irreducible and is not a locally symmetric space. We can also ask the:

Question: For a given closed locally symmetric space of non-compact type \(M^n\), what is the precise value of the ratio ||\(M^n||/\text{Vol}(M^n)||\)?

One of our applications is the non-vanishing of the top-dimensional bounded cohomology. We have the natural:

Question: What is the dimension of \(\hat{H}^n(M^n)\) for a locally symmetric space of non-compact type? In particular, is it finite dimensional?

References

[Al] P. Albuquerque. Patterson-Sullivan theory in higher rank symmetric spaces, *Geom. Funct. Anal.* (GAFA), 9 (1999), 1-28.

[BP] R. Benedetti and C. Petronio. *Lectures on hyperbolic geometry*. Universitext, Springer, 1992.

[BCG] G. Besson, G. Courtois, and S. Gallot. Minimal entropy and Mostow’s rigidity theorems. *Ergodic Theory and Dynamical Systems*, 16 (1996), no. 4, 623-649.

[Bo] A. Borel. Compact Clifford Klein forms of symmetric spaces. *Topology*, 2 (1963), 111-122.

[CG] J. Cheeger and M. Gromov. Collapsing Riemannian manifolds while keeping their curvature bounded. II. *J. Differential Geom.* 32 (1990), no. 1, 269-298.

[CF1] C. Connell and B. Farb. Some recent applications of the barycenter method in geometry. *Topology and geometry of manifolds (Athens, GA 2001)*, 19-50. Proc. Sympos. Pure Math., 71, Amer. Math. Soc.

[CF2] C. Connell and B. Farb. The degree theorem in higher rank. *J. Diff. Geom.* 65 (2003), no. 1, 19-59.

[F] K. Fukaya. Hausdorff convergence of Riemannian manifolds and its applications, in *Recent topics in differential and analytic geometry*, 143–238. Adv. Stud. Pure Math., 18-I, Academic Press, 1990.

[Gr1] M. Gromov. Volume and bounded cohomology, *Inst. Hautes Études Sci. Publ. Math.*, 56 (1982), 5-99.
[Gr2] M. Gromov. *Metric structures for Riemannian and non-Riemannian spaces*, Vol. 152 of *Progress in Mathematics*. Birkhäuser, 1999.

[HK] M. Hoster and D. Kotschick. On the simplicial volumes of fiber bundles. *Proc. Amer. Math. Soc.*, 129 (2000), no. 4, 1229-1232.

[IY] H. Inoue and K. Yano. The Gromov invariant of negatively curved manifolds. *Topology*, 21 (1982), 83-89.

[KL] M. Kapovich and B. Leeb. Actions of discrete groups on nonpositively curved spaces. *Mathematische Annalen*, 306 (1996), 341-352.

[K] G. Knieper. On the asymptotic geometry of nonpositively curved manifolds. *Geom. Funct. Anal. (GAFA)*, 7 (1997), 755-782.

[L] W. Lück. *L²-Invariants: Theory and Applications to Geometry and K-Theory*, Vol. 44 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. 3.Folge, Springer, 2002.

[Mi] J. Milnor. On the existence of a connection with curvature zero. *Comment. Math. Helv.*, 32 (1958), 215-223.

[Pa] P. Pansu. Effondrement des variétés riemanniennes, d’après J. Cheeger et M. Gromov. Seminar Bourbaki, Vol. 1983/84. *Astérisque*, No. 121-122 (1985), 63-82.

[Pr] G. Prasad. Discrete subgroups isomorphic to lattices in Lie groups. *Amer. J. Math.*, 98 (1976), no. 4, 853-863.

[Sa] R. Savage. The space of positive definite matrices and Gromov’s invariant, *Trans. Amer. Math. Soc.*, 274 (1982), no. 1, 241-261.

[St] C. Strohm. The Proportionality Principle of Simplicial Volume. (Diploma Thesis) Available at http://front.math.ucdavis.edu/math.AT/0504106

[Su] D. Sullivan. A generalization of Milnor’s inequality concerning affine foliations and affine manifolds. *Comment. Math. Helv.* 51 (1976), no. 2, 183-189.

[Th] W.P. Thurston. *Geometry and Topology of 3-Manifolds*. Lecture notes, Princeton, 1978.

[Y] K. Yano. Gromov invariant and $S^1$-actions. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* 29 (1982), no. 3, 493-501.

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