EQUATIONS FOR ABELIAN SUBVARIETIES

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Abstract. Given a finite group $G$ and an abelian variety $A$ acted on by $G$, to any subgroup $H$ of $G$, we associate an abelian subvariety $A_H$ on which the associated Hecke algebra $\mathcal{H}_H$ for $H$ in $G$ acts. Any irreducible rational representation $\tilde{W}$ of $\mathcal{H}_H$ induces an abelian subvariety of $A_H$ in a natural way. In this paper we give equations for this abelian subvariety. In a special case these equations become much easier. We work out some examples.

1. Introduction

Let $A$ be a complex abelian variety acted on by a finite group $G$. This induces an algebra homomorphism of the rational group ring $\mathbb{Q}[G]$ into the rational endomorphism ring $\text{End}_\mathbb{Q}(A) = \text{End}(A) \otimes \mathbb{Q}$,

$$\rho : \mathbb{Q}[G] \rightarrow \text{End}_\mathbb{Q}(A).$$

For any element $\alpha \in \mathbb{Q}[G]$ we define its image in $A$ by

$$\text{Im}(\alpha) := \text{Im}(n\alpha)$$

where $n$ is any positive integer such that $n\alpha$ is an endomorphism. Since multiplication by a non-zero integer on $A$ is an isogeny, the definition does not depend on the chosen integer $n$.

Now for any subgroup $H$ of $G$ the element $p_H := \frac{1}{|H|} \sum_{h \in H} h$ is an idempotent in $\mathbb{Q}[G]$ which defines an abelian subvariety of $A$,

$$A_H := \text{Im}(p_H).$$

The action of $G$ on $A$ induces an action of the Hecke algebra $\mathcal{H}_H$ on the abelian subvariety $A_H$. The aim of this note is to study this action and use it to find equations for the abelian subvarieties $A_{H,\tilde{W}}$ of $A_H$ which are given by the rational representations $\tilde{W}$ of $\mathcal{H}_H$. Here “equation” means to express $A_H$ as the connected component containing 0 of the zero-set of an endomorphism of $A_H$. Since this is fairly complicated, we do not repeat the result here, but refer to Theorem 3.6 below. However, we get an easy and important consequence: If the group $G$ acts on an abelian variety $A$ and $H$ is any subgroup of $G$, then Corollary 3.3 describes the complement of the abelian subvariety $A_G$ in $A_H$. This generalizes [9, Corollary 3.5, p.10] where it is proven for the Prym varieties $P(C/H, C/G)$ for any curve $C$ acted on by $G$ and any subgroup $H$ of $G$.

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In a special case the result becomes much simpler, namely let $V$ be an irreducible complex representation with $\dim V^H = 1$ and rational character. To this a one-dimensional rational representation $\widetilde{W}$ of the Hecke algebra $\mathcal{H}_H$ is associated in a natural way and one can find an explicit basis $q_1, \ldots, q_s$ of the algebra $\mathcal{H}_H$ for which we have (see Theorem 4.3) for the isotypical component $A_{H,\widetilde{W}}$ corresponding to $\widetilde{W}$ in $A_H$.

**Theorem 1.1.** With these assumptions the abelian subvariety $A_{H,\widetilde{W}}$ is

$$A_{H,\widetilde{W}} = \{ z \in A_H \mid q_i(z) = \chi_V(q_i)z \text{ for } 1 \leq i \leq s \}_0,$$

where the index 0 mean the connected component containing 0.

Several examples for this theorem will be given.

Section 2 contains some preliminaries. In particular we recall from [4] the decompositions of $A_H$ induced by the Hecke algebra $\mathcal{H}_H$. Section 3 contains the proof of the general theorem (Theorem 3.6). In Section 4 we prove the above mentioned theorem. Finally Section 5 contains some examples.

## 2. Preliminaries

Let $G$ be a finite group acting on an abelian variety $A$ of dimension $g$ over the field of complex numbers and let $H \leq G$ be a subgroup of $G$. The Hecke algebra for $H$ in $G$ acts in a natural way on the abelian variety $A$. To be more precise: the element

$$p_H := \frac{1}{|H|} \sum_{h \in H} h$$

is an idempotent in the group algebra $\mathbb{Q}[G]$.

The (rational) Hecke algebra for $H$ in $G$ is defined as the subalgebra

$$\mathcal{H}_H := p_H \mathbb{Q}[G]p_H = \mathbb{Q}[H \backslash G / H]$$

of the rational group group algebra $\mathbb{Q}[G]$. If we consider $\mathbb{Q}[G]$ as the algebra of functions $G \to \mathbb{Q}$ with multiplication the convolution product (see [5] §11), then $\mathbb{Q}[H \backslash G / H]$ is the subalgebra of functions which are constant on each double coset $HgH$, and $p_H$ is the unit in this algebra.

The action of $G$ on $A$ induces an action of $\mathbb{Q}[G]$ on $A$ in a natural way, giving an algebra homomorphism

$$\mathbb{Q}[G] \to \text{End}_\mathbb{Q}(A).$$

Since this homomorphism is canonical, we denote the elements of $\mathbb{Q}[G]$ and their images by the same letter. For any element $\alpha \in \mathbb{Q}[G]$ we define its image in $A$ by

$$\text{Im}(\alpha) := \text{Im}(n\alpha) \subset A$$

where $n$ is any positive integer such that $n\alpha$ is in $\text{End}(A)$. It is an abelian subvariety of $A$ which does not depend on the chosen integer $n$.

Consider the abelian subvariety of $A$ given by

$$A_H := \text{Im}(p_H).$$
Restricting (2.1) to \( \mathcal{H}_H \) gives an algebra homomorphism
\[
\mathcal{H}_H \to \text{End}_\mathbb{Q}(A_H).
\]

The aim of this section is to recall from \([4]\) the isotypical and Hecke algebra decompositions of \( A_H \) with respect to this action of \( \mathcal{H}_H \).

Let \( \{W_1, \ldots, W_r\} \) denote the irreducible rational representations of \( G \). To any \( W_i \) there corresponds an irreducible complex representations \( V_i \), uniquely determined up to an element of the Galois group of \( K_i \) over \( \mathbb{Q} \), where \( K_i \) is the field obtained by adjoining to \( \mathbb{Q} \) the values of the character \( \chi_{V_i} \) of \( V_i \). The representations \( W_i \) and \( V_i \) are said to be \textit{Galois associated}.

To each \( W_i \) we can associate a central idempotent \( e_{W_i} \) of \( \mathbb{Q}[G] \) by
\[
e_{W_i} = \dim_{\mathbb{C}}(V_i) |G| \sum_{g \in G} \text{tr}_{K_i/\mathbb{Q}}(\chi_{V_i}(g^{-1})) g.
\]

Let \( \rho_H \) denote the representation of \( G \) induced by the trivial representation of \( H \). It decomposes as
\[
\rho_H \simeq \sum_{i=1}^r a_i W_i,
\]
with \( a_i = \frac{1}{s_i} \dim_{\mathbb{C}}(V_i^H) \) and \( s_i \) the Schur index of \( V_i \). Renumbering if necessary, let \( \{W_1, \ldots, W_t\} \) denote the set of all irreducible rational representations of \( G \) such that \( a_i \neq 0 \). Then there is a bijection from this set to the set \( \{\tilde{W}_1, \ldots, \tilde{W}_t\} \) of all irreducible rational representations of the algebra \( \mathcal{H}_H \). An analogous statement holds for the irreducible complex representations of \( G \) and of \( \mathbb{C}[H \backslash G/H] \). Let \( \tilde{V} \) denote the representation of \( \mathbb{C}[H \backslash G/H] \) associated to the complex irreducible representation \( V \) of \( G \).

According to \([4\text{, equation (2.4)}]\) and \([3\text{, p. 331]}\) the dimension of \( \tilde{W}_i \) given by
\[
\dim_{\mathbb{Q}}(\tilde{W}_i) = \dim_{\mathbb{Q}}(W_i^H) = [L_i : \mathbb{Q}] \dim_{\mathbb{C}}(V_i^H).
\]
where \( L_i \) denotes the field of definition of the representation \( V_i \). Recall that the index \( s_i = [L_i : K_i] \) is the Schur index of \( V_i \).

For \( i = 1, \ldots, t \) consider the central idempotents of \( \mathcal{H}_H \),
\[
f_{H,\tilde{W}_i} := p_H e_{W_i} = e_{W_i} p_H.
\]
Then \( p_H \) decomposes as
\[
p_H = \sum_{i=1}^t f_{H,\tilde{W}_i}.
\]
Defining for \( i = 1, \ldots, t \) the abelian subvarieties
\[
A_{H,\tilde{W}_i} := \text{Im}(f_{H,\tilde{W}_i}),
\]
one obtains the following isogeny decomposition of \( A_H \), given by the addition map
\[
+ : A_{H,\tilde{W}_1} \times A_{H,\tilde{W}_2} \times \cdots \times A_{H,\tilde{W}_t} \to A_H.
\]
It is uniquely determined by \( H \) and the action of \( G \) and called the \textit{isotypical decomposition} of \( A_H \).
If \( a_i \geq 2 \) in (2.3), the subvarieties \( A_{H,\overline{W}_i} \) can be decomposed further. In fact, given \( \overline{W}_i \), explicit orthogonal primitive idempotents \( f_{i,j}, 1 \leq j \leq n_i := \frac{1}{s_i} \dim V_i \) may be found such that
\[
e_{W_i} = f_{i,1} + \cdots + f_{i,n_i}.
\]
Multiplying by \( p_H \) gives
\[
f_{H,\overline{W}_i} = f_{i,1}p_H + \cdots + f_{i,n_i}p_H.
\]
We label the \( f_{i,j}p_H \) in such a way that for the first \( a_i = \frac{1}{s_i} \dim(V^H_i) \) of them, the minimal left ideals of the simple algebra \( Q[G]f_{H,\overline{W}_i} \)
\[
J_{i,j} := Q[G]f_{i,j}p_H
\]
and different among themselves; that is, such that
\[
Q[G]f_{H,\overline{W}_i} = \bigoplus_{j=1}^{a_i} J_{i,j}.
\]
Then there exist primitive idempotents \( v_{i,1}, \ldots, v_{i,a_i} \) in \( Q[G]f_{H,\overline{W}_i} \), each \( v_{i,j} \) generating the corresponding ideal \( J_{i,j} \), such that
\[
f_{H,\overline{W}_i} = v_{i,1} + \cdots + v_{i,a_i}.
\]
Note that by construction the \( v_{i,j} \) are orthogonal idempotents, not uniquely determined, since the \( f_{i,j} \) are not uniquely determined. Defining for \( j = 1, \ldots, a_i \),
\[
B_{H,\overline{W}_i,j} := \text{Im}(v_{i,j}),
\]
equation (2.6) induces the following isogeny
\[
+: B_{H,\overline{W}_1,j} \times B_{H,\overline{W}_2,j} \times \cdots \times B_{H,\overline{W}_{a_i,j}} \to A_{H,\overline{W}_i}.
\]
Here the subvarieties \( B_{H,\overline{W}_1,j}, \ldots, B_{H,\overline{W}_{a_i,j}} \) are pairwise isogenous. Hence combining with the isogeny (2.5), we get the following \( H \)-equivariant isogeny
\[
B_{a_1}^{H,\overline{W}_1,1} \times B_{a_2}^{H,\overline{W}_2,1} \times \cdots \times B_{a_i}^{H,\overline{W}_{a_i,1}} \to A_H,
\]
called the Hecke algebra decomposition of \( A_H \) with respect to the action of \( G \) and the subgroup \( H \).

3. Equations for the Abelian Subvarieties \( A_{H,\overline{W}_i} \)

In this section we describe the abelian subvarieties \( A_{H,\overline{W}_i} \). The same method works for the abelian subvarieties \( B_{H,\overline{W}_i,j} \), which we omit however. The method relies on the following proposition.

Let \( G \) be a finite group acting on the abelian variety \( A \). For any idempotent \( \iota \) of \( Q[G] \) there are two abelian subvarieties of \( A \), namely
\[
A_\iota := \text{Im} \iota \subset A \quad \text{and} \quad C_\iota := \text{Im}(1_G - \iota).
\]

**Proposition 3.1.**

1. The addition map gives an isogeny
\[
+: A_\iota \times C_\iota \to A.
\]
(2) \( A_\iota = \ker(1_G - \iota)_0 \) and \( C_\iota = \ker(\iota)_0 \),
where the index 0 means the connected component containing 0.

To be more precise, for an element \( \alpha \in \text{End}_{\mathbb{Q}}(A) \) we denote by \( \ker(\alpha)_0 \) the connected component containing 0 of some positive multiple \( n\alpha \) which is an endomorphism. This does not depend on the chosen \( n \).

Given a polarization of \( A \), there is an analogous result for any abelian subvariety using norm-endomorphisms (see [1, Section 5.3]). One could give also a proof of the proposition introducing polarizations, but we prefer to give a direct proof.

Proof. (1): Since \( \iota + (1_G - \iota) = 1_G \), the addition map \( + : A_\iota \times C_\iota \to A \) is surjective. To see that is is an isogeny, it suffices to show that on the level of tangent spaces it is an isomorphism. So suppose \( A = V/\Lambda \) with a \( \mathbb{C} \)-vector space \( V \) and a lattice \( \Lambda \). we may choose the basis of \( V \) such that the analytic representation \( \rho_\alpha(\iota) \) is given by the diagonal matrix \( \text{diag}(1, \ldots, 1, 0, \ldots, 0) \) with the number of 1's equal to \( \dim A_\iota \). Hence \( \rho_\alpha(1_G - \iota) = \text{diag}(0, \ldots, 0, 1, \ldots 1) \) with the number of 0's equal to \( \dim A_\iota \). This implies \( \dim C_\iota = \dim A - \dim A_\iota \) and thus the assertion.

(2): Choose a positive integer \( n \) such that \( n(1_G - \iota) \in \text{End}_{\mathbb{Q}}(A) \) and consider the exact sequence

\[ 0 \to \ker(n(1_G - \iota)) \to A \xrightarrow{n(1_G - \iota)} C_\iota \to 0. \]

Certainly \( A_\iota \subset \ker(n(1_G - \iota)) \). Since both have the same dimension, this implies the assertion. \( \square \)

Now let \( H \) be a subgroup of \( G \) and \( \iota \) be an idempotent of the corresponding Hecke algebra \( \mathcal{H}_H \). It induces two abelian subvarieties of the abelian variety \( A_H \), namely

\[ A_{H,\iota} := \text{Im}(\iota) \quad \text{and} \quad P_{H,\iota} := \text{Im}(p_H - \iota). \]

Notice that \( p_H - \iota : A_H \to A_H \) is also an idempotent of \( \mathcal{H}_H \), since \( p_H \) is the unit element of the algebra \( \mathcal{H}_H \) and hence its image in \( \text{End}_{\mathbb{Q}}(A_H) \) is the identity on \( A_H \). If we denote by \( \iota \) also the corresponding idempotent of \( \text{End}_{\mathbb{Q}}(A_H) \), this implies that \( p_H - \iota \) is an idempotent of \( \text{End}_{\mathbb{Q}}(A_H) \).

**Proposition 3.2.**

(1) The addition map gives an isogeny

\[ + : A_{H,\iota} \times C_{H,\iota} \to A_H. \]

(2) \( A_{H,\iota} = \ker(p_H - \iota)_0 \) and \( P_{H,\iota} = \ker(\iota)_0 \).

The proof is essentially the same as the proof of Proposition 3.1. We want to use the proposition to describe the isotypical components of \( A_H \) as fixed-point sets of particular endomorphisms of \( A_H \).

First we consider the trivial representation \( W_0 \) of \( G \) and any subgroup \( H \) of \( G \). Note that

\[ e_{W_0} := p_G = \frac{1}{|G|} \sum_{g \in G} g \]

is the central idempotent in \( \mathbb{Q}[G] \) corresponding to \( W_0 \). Moreover, choose a set of representatives \( g_1, \ldots, g_s \) for both the right and left cosets of \( H \) in \( G \). Such a set exists according to [3, Theorem 5.1.7]. Using this, we get as a special case,
Corollary 3.3. With the above notations we have for the abelian subvariety $A_G = (A^G)_0$ and the complement $P(A_H, A_G)$ of $A_G$ in $A_H = \text{Im}(p_H)$,

(i) $A_G = \{ z \in A_H : \sum_{j=1}^s g_j(z) = z \}_0$,

(ii) $P(A_H, A_G) = \{ z \in A_H : \sum_{j=1}^s g_j(z) = 0 \}_0$.

The notation $P(A_H, A_G)$ comes from the fact that in the case of a Galois cover of curves $C \to C/G$ and any subgroup $H$ of $G$ this is just the Prym variety $P(C/H, C/G)$. In fact, Corollary 3.3 generalizes [9, Corollary 3.5, p.10] where it is proven for these Prym varieties $P(C/H, C/G)$.

Proof. We may assume that $H \leq G$. Then we have

$$p_H = (p_H - eW_0) + eW_0.$$  

Observe that because of the special choice of the $g_i$,

$$[G : H] eW_0 = p_H \left( \sum_{j=1}^s g_s \right) = \left( \sum_{j=1}^s g_j \right) p_H.$$  

Hence $eW_0$ is in $\mathcal{H}_H$ and furthermore $\text{Im}(eW_0) = A_G$. The assertion now follows from Proposition 3.2 since

$$A_G = \text{Ker}(p_H - eW_0)_0 = \{ z \in A_H : \sum_{j=1}^s g_j(z) = z \}_0$$  

and

$$P(A_H, A_G) = \text{Ker}(eW_0)_0 = \{ z \in A_H : \sum_{j=1}^s g_j(z) = 0 \}_0$$  

and $A_H = (A^H)_0$. □

Let $\tilde{W}$ be the irreducible rational representation of the Hecke algebra $\mathcal{H}_H$ associated to the irreducible rational representation $W$ of $G$ and

$$f_{H,\tilde{W}} = p_H eW = eW p_H$$  

the corresponding central idempotent of $\mathcal{H}_H$. In order to find a more convenient expression of $f_{H,\tilde{W}}$, consider the decomposition of $G$ into double cosets of $H$ in $G$,

$$G = H_1 \cup H_2 \cup \cdots \cup H_s$$  

with $H_i = H x_i H$ and $x_1 = 1$, i.e. $H_1 = H$. A basis for the Hecke algebra $\mathcal{H}_H$ is given by the elements

$$q_i := \frac{|H|}{|H \cap x_i H x_i^{-1}|} p_{H x_i H} = \frac{1}{|H|} \sum_{h_i \in H_i} h_i$$  

for $i = 1, \ldots, s$ (see [5, Proposition 11.30(i)]). For the last equation use that

$$\frac{|H|}{|H \cap x_i H x_i^{-1}|} = |H : H \cap x_i H x_i^{-1}| = \frac{|H x_i H|}{|H|} = \frac{|H_i|}{|H|}$$
Let \( \{ g_{i,j}, j = 1, \ldots, d_i \} = \{ H : x_i H x_i^{-1} \} \) denote a set of simultaneous representatives for the right and left cosets for \( H \) in \( H_i \). Such a set exists again by \cite[Theorem 5.1.7]{6}. Then we have,

**Lemma 3.4.** Considering the elements \( q_i \) as elements of \( \text{End}_Q(A_H) \), we have: \( q_i \) is an endomorphism of \( A_H \) and as such

\[
q_i(z) = \sum_{j=1}^{d_i} g_{i,j}(z)
\]

for each \( z \in A_H \).

**Proof.** Since \( H_i = \bigcup_{j=1}^{d_i} H g_{i,j} = \bigcup_{j=1}^{d_i} g_{i,j} H \), we have,

\[
q_i = \frac{1}{|H|} \sum_{x \in H_i} x = p_H \left( \sum_{j=1}^{d_i} g_{i,j} \right) = \left( \sum_{j=1}^{d_i} g_{i,j} \right) p_H.
\]

This gives the assertion, since \( p_H \) is the identity on \( A_H \). \( \square \)

**Lemma 3.5.** The following equality is valid in the Hecke algebra \( \mathcal{H}_H \):

\[
f_{H,w} = \text{dim } V = \sum_{g \in G} \frac{|H \cap x_i H x_i^{-1}| \text{tr}_{K/Q}(\chi_V(g))}{|H|} q_i
\]

**Proof.** Since \( e_W \) is central and \( p_H^2 = p_H \), we have

\[
f_{H,w} = p_H e_W = p_H e_W p_H = \frac{\text{dim } V}{|G|} \sum_{g \in G} \text{tr}_{K/Q}(\chi_V(g^{-1}))(p_H g p_H)
\]

\[
= \frac{\text{dim } V}{|G|} \sum_{i=1}^{s} \sum_{g \in H_i} \text{tr}_{K/Q}(\chi_V(g^{-1})) p_H g p_H
\]

\[
= \frac{\text{dim } V}{|G|} \sum_{i=1}^{s} \left( \sum_{g \in H_i} \text{tr}_{K/Q}(\chi_V(g^{-1})) p_H g p_H \right)
\]

\[
= \frac{\text{dim } V}{|G|} \sum_{i=1}^{s} \left( \sum_{g \in H_i} \frac{|H \cap x_i H x_i^{-1}|}{|H|} \text{tr}_{K/Q}(\chi_V(g^{-1})) \right) \frac{|H|}{|H \cap x_i H x_i^{-1}|} p_H x_i p_H.
\]

But for any \( g \in G \) we have \( \chi_V(g^{-1}) = \overline{\chi_V(g)} \), and therefore

\[
\sum_{g \in H_i} \text{tr}_{K/Q}(\chi_V(g^{-1})) = \sum_{g \in H_i} \text{tr}_{K/Q}(\chi_V(g)) = |H| \text{tr}_{K/Q}(\chi_V(q_i)).
\]

So equation (3.2) gives the assertion. \( \square \)

Recall that we consider any element of \( \mathbb{Q}[G] \) also as an element of \( \text{End}_Q(A) \). We denote for any element \( \alpha \) of \( \mathcal{H}_H \subset \mathbb{Q}[G] \) by

\[
\text{Ker}(\alpha)_0
\]
the connected component of \( \operatorname{Ker}(n\alpha) \subset A_H \) containing 0, where \( n \) is any positive integer such that \( n\alpha \) is actually an endomorphism. This does not depend of the chosen \( n \). Then we get the following equation for the abelian subvariety \( A_{H,\tilde{W}} \) of \( A_H \).

**Theorem 3.6.** The isotypical component \( A_{H,\tilde{W}} \) of \( A_H \) is given by

\[
A_{H,\tilde{W}} = \operatorname{Ker} \left( p_H - \frac{\dim V}{|G|} \sum_{i=1}^{s} (|H \cap x_i H x_i^{-1}| \text{tr}_{K/Q}(\chi_V(q_i))) q_i \right). 
\]

**Proof.** By definition of \( A_{H,\tilde{W}} \) and Proposition 3.2 we have

\[
A_{H,\tilde{W}} = \text{Im}(f_{H,\tilde{W}}) = \operatorname{Ker} (p_H - f_{H,\tilde{W}}) \cap 0.
\]

So Lemma 3.5 gives the assertion. \( \square \)

In the next section we will use the following orthogonality relations,

**Lemma 3.7.** Let \( \tilde{U}, \tilde{V} \) be complex irreducible representations of \( C[H \backslash G/H] \) associated to the irreducible representations \( U, V \) of \( G \). Then, with the notation of above,

\[
\sum_{j=1}^{s} \frac{|H|}{|H \cap x_j H x_j^{-1}|} \chi_{\tilde{U}}(q_j^{-1}) \chi_{\tilde{V}}(q_j) = \begin{cases} 0 & \text{if } \tilde{U} \neq \tilde{V}; \\ \frac{|G:H|}{\dim U} \dim V & \text{if } \tilde{U} = \tilde{V}, \end{cases}
\]

where \( q_j^{-1} := \frac{1}{|H|} \sum_{y \in H_j} y^{-1} \).

**Proof.** The orthogonality relations [5, Theorem 11.32 (ii)] say, in the special case that \( \psi \) is the trivial representation,

\[
\sum_{j=1}^{s} \frac{1}{|H : H \cap x_j H x_j^{-1}|} \chi_{\tilde{U}} \left( \frac{1}{|H|} \sum_{y \in H_j} y^{-1} \right) \chi_{\tilde{V}} \left( \frac{1}{|H|} \sum_{x \in H_j} x \right) = \begin{cases} 0, & \text{if } \tilde{U} \neq \tilde{V}; \\ \frac{|G:H|}{\dim U} \dim V, & \text{if } \tilde{U} = \tilde{V}. \end{cases}
\]

This implies the assertion. \( \square \)

The following lemma is proven in [5, p.282, l. -4].

**Lemma 3.8.** Let \( \tilde{V} \) be a complex irreducible representation of \( C[H \backslash G/H] \) associated to the irreducible representation \( V \) of \( G \). Then their characters satisfy

\[
\chi_{\tilde{V}}(x) = \chi_V(x) \quad \text{for all} \quad x \in C[H \backslash G/H].
\]

4. A SPECIAL CASE

The equation of Theorem 3.6 seems fairly complicated. In a special case we can describe the subvariety \( A_{H,\tilde{W}} \) in a simpler way. Let the notation be as above, but assume in addition that

\[
\dim V^H = 1 \quad \text{and} \quad K = \mathbb{Q}.
\]

According to [7, Corollary 10.2], this implies that also the Schur index \( s(V) \) of \( V \) is equal to 1. Then we have, according to equation (2.4),

\[
\dim \tilde{W} = \dim W^H = 1.
\]
This implies that the complex representation \( \tilde{V} \) of \( \mathbb{C}[H\backslash G/H] \) is rational of dimension 1 with \( \tilde{W} = \tilde{V} \otimes \mathbb{C} \), and the complex representation \( V \) of \( G \) is rational with \( W = V \otimes \mathbb{C} \).

**Proposition 4.1.** With the assumption (4.1) we have

\[
f_{H,\tilde{W}} = \dim V \frac{\sum_{i=1}^{s} |H \cap x_i H x_i^{-1}| \chi_V(q_i) q_i}{|G|}.
\]

**Proof.** This is a direct consequence of Lemma 3.5. \( \square \)

Now consider all elements of \( H \) as elements of \( \text{End}_Q(A_H) \). According to Lemma 3.4, \( q_i \) is an endomorphism of \( A_H \) for all \( i \) and we can express \( A_{H,\tilde{W}} \) as the kernel of an actual endomorphism. In fact, we get as a direct consequence of Proposition 4.1 and Theorem 3.6,

**Corollary 4.2.** With the assumption (4.1) let \( n := |G| \). Then \( nf_{H,\tilde{W}} \) is an endomorphism of \( A_H \) and we have

\[
A_{H,\tilde{W}} = \text{Ker} \left( nA_H - \dim V \sum_{i=1}^{s} |H \cap x_i H x_i^{-1}| \chi_V(q_i) q_i \right)_0.
\]

Define the abelian subvariety \( B_{H,\tilde{W}} \) by

\[
B_{H,\tilde{W}} := \{ z \in A_H \mid q_i(z) = \chi_V(q_i) z \text{ for } i = 2, \ldots, s \}.
\]

Note that the condition \( q_i(z) = \chi_V(q_i) z \) (\( = z \)) is hidden the assumption \( z \in A_H \). So one could equivalently write

\[
B_{H,\tilde{W}} = \{ z \in A \mid q_i(z) = \chi_V(q_i) z \text{ for } i = 1, \ldots, s \}.
\]

The aim of this section is the proof of the following theorem.

**Theorem 4.3.** Under the assumptions (4.1) we have

\[
A_{H,\tilde{W}} = B_{H,\tilde{W}}.
\]

Recall that \( \{f_{H,\tilde{W}}\}_{\tilde{W}} \) are the central primitive idempotents in \( H_H \) and the isotypical decomposition of \( H_H \) is

\[
H_H = \oplus_{\tilde{W}} H_H f_{H,\tilde{W}}
\]

where \( \tilde{W} \) acts on the simple subalgebra \( H_H f_{H,\tilde{W}} \) and by 0 on the other components. Recall moreover that the \( q_i, i = 1, \ldots, s \) as defined in (3.2) a basis of \( H_H \).

**Lemma 4.4.** Under the assumption (4.1) the action of the Hecke algebra \( H_H \) on the abelian variety \( A_{H,\tilde{W}} \) is given by

\[
q_i(z) = \chi_V(q_i) z \text{ for all } z \in A_{H,\tilde{W}} \text{ and for all } i = 1, \ldots, s.
\]

**Proof.** Note first that the left hand side of the equation makes sense, since \( q_i \) is an endomorphism on \( A_H \) by Lemma 3.4. In order to see that also the right hand side makes sense, we have to show that \( \chi_V(q_i) \) is an integer. But [8, Lemma 7.1] says that for any subgroup \( H \) of any finite group \( G \) and any complex representation \( V \) of \( G \) the numbers \( \chi_V(q_i) \) are algebraic integers. Since in our case \( \chi_V \) has rational values, this means \( \chi_V(q_i) \) is an integer.
For the proof of the lemma, it suffices to show that the analogous equation is valid for
the action of $H_H$ on $H_H f_{H,\tilde{W}}$ by $\tilde{W}$, i.e. to show
$$q_i(x) = \chi_V(q_i)x \quad \text{for all} \quad x \in H_H f_{H,\tilde{W}}.$$  
Since $\tilde{W}$ is of dimension one, we have $\tilde{V} = \tilde{W} \otimes \mathbb{C}$ and $H_H f_{H,\tilde{W}}$ is a simple algebra of
dimension 1, hence equal to $H_H f_{H,\tilde{W}} = Q f_{H,\tilde{W}}$. But the action of a one-dimensional
complex representation is given by multiplication by the character (which equals the
representation). This implies in particular $q_i(x) = \chi_{\tilde{V}}(q_i)x$ for all $i$ and all $x \in H_H f_{H,\tilde{W}}$.
Since $\chi_{\tilde{V}} = \chi_V$ on $H_H$ by Lemma 3.8, this gives the assertion. This completes the proof
of the theorem. \hfill \Box

Proof of Theorem 4.3. Recall that
$$A_{H,\tilde{W}} = \text{Im}(|G|f_{H,\tilde{W}}) = \{z \in A_H \mid |G|f_{H,\tilde{W}}(z) = |G|z\}_0$$
First we show $A_{H,\tilde{W}} \subset B_{H,\tilde{W}}$: Suppose that $z \in A_{H,\tilde{W}}$. Since $q_i = q_i f_{H,\tilde{W}}$, we get for all $i$,
$$|G|q_i(z) = |G|q_i f_{H,\tilde{W}}(z) = \chi_V(q_i)|G|f_{H,\tilde{W}}(z) \quad \text{(by Lemma 4.4)}$$
$$= \chi_V(q_i)|G|(z).$$
So
$$z \in \{z \in A_H \mid |G|q_i(z) = \chi_V(q_i)|G|z\}_0 = \{z \in A_H \mid q_i(z) = \chi_V(q_i)z\}_0$$
i.e. $z \in B_{H,\tilde{W}}$.
Finally we show $B_{H,\tilde{W}} \subset A_{H,\tilde{W}}$: So suppose $z \in B_{H,\tilde{W}}$, i.e.
$$q_i(z) = \chi_V(q_i)z \quad \text{for all} \quad i = 1, \ldots, s.$$  
Since in our case $\chi_V(g^{-1}) = \chi_V(g)$ for all $g$ and $\chi_{\tilde{V}} = \chi_V|_{\mathfrak{h}_n}$ by Lemma 3.8, part of the
orthogonality relations 3.7 become
$$\dim V \sum_{i=1}^s |H \cap x_i H x_i^{-1}| \chi_V(q_i) \chi_V(q_i) = |G|.$$  
From this we get
$$\dim V \sum_{i=1}^s |H \cap x_i H x_i^{-1}| \chi_V(q_i)q_i(z) = \dim V \sum_{i=1}^s |H \cap x_i H x_i^{-1}| \chi_V(q_i) \chi_V(q_i)(z)$$
$$= |G|z.$$  
This means that $z \in A_{H,\tilde{W}}$. \hfill \Box

5. Examples

In this section we work out the equations of Theorem 4.3 in some cases. For the
computations we used the computer program GAP.
5.1. **Example 1.** Let $G = S_4$ and $W = V$ the standard representation of degree 3:
\[
\chi_W(1) = 3, \chi_W(\ldots) = 1, \chi_W((\ldots)(\ldots)) = -1, \chi_W(\ldots) = 0, \chi_W(\ldots\ldots) = -1
\]
where $n$ dots mean any cycle of length $n$.

**Case 1:** $H = \langle (12), (34) \rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ (not normal in $G$).

Clearly the assumptions (4.1) are satisfied and
\[
G = H \cup (H(23)H) \cup (H(13)(24)H) =: H_1 \cup H_2 \cup H_3.
\]

One checks
\[
q_1 = \frac{1}{4} \sum_{h \in H_1} h = p_H, \quad q_2 = \frac{1}{4} \sum_{k \in H_2} k, \quad q_3 = \frac{1}{4} \sum_{k \in H_3} k
\]
and
\[
\chi_W(q_1) = 1, \quad \chi_W(q_2) = 0, \quad \chi_W(q_3) = -1.
\]
We get from Theorem 4.3 for any abelian variety $A$ with $G$-action,
\[
A_{H,\overline{W}} = \{ z \in A_H : q_2(z) = 0, q_3(z) = -z \}_0.
\]

Now observe that $H_3 = \{(13)(24), (14)(23), (1324), (1423)\} = D_4 - H$
where $D_4$ denotes the dihedral subgroup of $S_4$. This implies
\[
A_{\mathbb{Z}_2 \times \mathbb{Z}_2,\overline{W}} = P(A_{\mathbb{Z}_2 \times \mathbb{Z}_2}/A_{D_4})
\]
where $P(A_{\mathbb{Z}_2 \times \mathbb{Z}_2}/A_{D_4})$ denotes the complement of $A_{D_4}$ in $A_{\mathbb{Z}_2 \times \mathbb{Z}_2}$.

**Case 2:** $H = \langle (34), (243) \rangle \simeq S_3$.

Again the assumptions (4.1) are satisfied. Here we have
\[
G = H \cup (H(14)H) =: H_1 \cup H_2.
\]

One checks
\[
q_1 = \frac{1}{6} \sum_{h \in H} h = p_H, \quad q_2 = \frac{1}{6} \sum_{k \in H_2} k
\]
and
\[
\chi_W(q_1) = 1, \quad \chi_W(q_2) = -1.
\]
We get from Theorem 4.3 for any abelian variety $A$ with $G$-action,
\[
A_{H,\overline{W}} = \{ z \in A_H : q_2(z) = -z \}_0.
\]

Now observe that $H_2 = S_4 - H$, which implies, as in Case 1,
\[
A_{H,\overline{W}} = P(A_{S_3}/A_{S_4}).
\]
5.2. **Example 2.** Let \( G = N \times P \simeq \mathbb{Z}_2^4 \times \mathbb{Z}_5 \) denote the subgroup of order 80 of \( S_{10} \) (which occurred in [2]) that is generated by the permutations \( s_i := (i \ i + 1)(5 + i \ 6 + i) \) for \( i = 1, \ldots, 4 \) and \( \sigma := (1 \ 2 \ 3 \ 4 \ 5)(6 \ 7 \ 8 \ 9 \ 10) \).

For \( i = 1, 2, 3 \) consider the rational irreducible representation \( W_i \) defined as follows: The group \( P \) acts on the character group \( \mathcal{N} \) of \( N \) in the usual way. Apart from the trivial representation \( \chi_0 \) there are 3 orbits of the action of \( P \) on \( \mathcal{N} \). Let \( \chi_i, \ i = 1, 2, 3 \) be representatives of them. Then, if \( \rho \) denotes the representation of \( G \) induced by the trivial representation of \( N \), the rational irreducible representation \( W_i \) of degree five is defined as

\[
W_i = \rho \otimes \chi_i \quad \text{for} \quad i = 1, 2, 3.
\]

The characters of \( W_i \) for \( i = 1, 2, 3 \) are given by \( \chi_{W_1}(1_G) = 5, \chi_{W_2}(\sigma) = 0, \) and

\[
\begin{align*}
\chi_{W_1}(s_j) &= 1, \quad \chi_{W_1}(s_1 s_2) = 1, \quad \chi_{W_1}(s_1 s_3) = -3, \\
\chi_{W_2}(s_j) &= -3, \quad \chi_{W_2}(s_1 s_2) = 1, \quad \chi_{W_2}(s_1 s_3) = 1, \\
\chi_{W_3}(s_j) &= 1, \quad \chi_{W_3}(s_1 s_2) = -3, \quad \chi_{W_3}(s_1 s_3) = 1.
\end{align*}
\]

**Case 1:** \( H = P \).

For the representation \( \rho_H \) of \( G \) induced by the trivial representation of \( H \) we have \( \rho_H = \chi_0 \oplus W_1 \oplus W_2 \oplus W_3 \) which implies for \( i = 1, 2, 3 \),

\[
\dim \overline{W_i} = \dim W_i^H = \langle W_i, \rho_H \rangle = 1.
\]

Hence the assumptions (1.1) are satisfied for all \( W_i \). The double coset decomposition of \( G \) is

\[
G = H \cup (H s_1 H) \cup (H s_1 s_2 H) \cup (H s_1 s_3 H) =: H_1 \cup H_2 \cup H_3 \cup H_4.
\]

The double cosets \( H_2, H_3, H_4 \) contain each a complete conjugacy class of involutions of \( G \) and all other elements are of order 5. A basis of the (commutative) Hecke algebra \( \mathcal{H}_H \) is given by

\[
q_1 = p_H, \quad \text{and} \quad q_i = \frac{1}{5} \sum_{k_i \in H_i} k_i \quad \text{for} \quad i = 2, 3, 4.
\]

The multiplication of \( \mathcal{H}_H \) is given by

\[
q_1 q_j = q_j, \quad q_2^2 = 5 q_1 + 2 q_3 + 2 q_4, \quad q_3^2 = 5 q_1 + 2 q_2 + 2 q_4, \quad q_4^2 = 5 q_1 + 2 q_2 + 2 q_3, \\
q_2 q_3 = 2 q_2 + 2 q_3 + q_4, \quad q_2 q_4 = 2 q_2 + q_3 + 2 q_4, \quad q_3 q_4 = q_2 + 2 q_3 + 2 q_4.
\]

Since

\[
\chi_{W_1}(q_1) = \chi_{W_1}(q_2) = \chi_{W_1}(q_3) = \chi_{W_1}(q_4) = \chi_{W_2}(q_2) = \chi_{W_1}(q_3) = 1,
\]

and

\[
\chi_{W_1}(q_4) = \chi_{W_2}(q_2) = \chi_{W_1}(q_3) = -3,
\]

we get from Theorem 4.3 that for any abelian variety \( A \) with an action of the group \( G \),

\[
A_{H, \overline{W_1}} = \{ z \in A_H \mid q_2(z) = q_3(z) = z, \ q_4(z) = -3z \}_0,
A_{H, \overline{W_2}} = \{ z \in A_H \mid q_3(z) = q_4(z) = z, \ q_2(z) = -3z \}_0,
A_{H, \overline{W_3}} = \{ z \in A_H \mid q_2(z) = q_4(z) = z, \ q_3(z) = -3z \}_0.
\]

**Case 2:** \( H = \langle s_1, s_2, s_3 \rangle \simeq \mathbb{Z}_2^3 \).
For the representation $\rho_H$ of $G$ induced by the trivial representation of $H$ we have

$$\rho_H = \chi_0 \oplus W_3 \oplus \psi$$

with $\psi = \rho_1 \oplus \ldots \oplus \rho_4$ is the rational irreducible representation of degree four given by the sum of the linear complex irreducible characters of $G$ induced by each of the non-trivial characters of $P \simeq \mathbb{Z}_5$ by the projection $G \to P$. Hence the assumptions (4.1) are satisfied for the representation $W_3$.

The double coset decomposition of $G$ is

$$G = H \cup (Hs_4H) \cup (H\sigma^4H) \cup (H\sigma^3H) \cup (H\sigma^2H) \cup (H\sigma H) =: H_1 \cup \ldots \cup H_6,$$

where the first double coset is $H$, the second is $N - H$, and the other four consist of 16 elements of order five each.

A basis for the (commutative) Hecke algebra $H_H$ is given by

$$q_1 = p_H \text{ and } q_j = \frac{1}{8} \sum_{k_j \in H_j} k_j \text{ for } j = 2, \ldots, 6$$

and one checks

$$q_1q_j = q_j \text{ for all } j, \quad q_2q_j = 8q_j \text{ for } 3 \leq l \leq 6,$$

$$q_2^2 = 64q_1, \quad q_3^2 = 8q_3, \quad q_4^2 = 8q_6, \quad q_5^2 = 8q_3, \quad q_6^2 = 8q_5.$$

Since

$$\chi_{W_3}(q_1) = 1, \chi_{W_3}(q_2) = -1 \text{ and } \chi_{W_3}(q_i) = 0 \text{ for } i = 3, \ldots, 6,$$

we get from Theorem 4.3

$$A_{H,W_3} = \{ z \in A_H \mid q_2(z) = -z, q_i(z) = 0 \text{ for } i = 3, \ldots, 6 \} = P(A_H/A_N).$$

An analogous result can be proved for the representations $\tilde{W}_1$ and $\tilde{W}_2$.

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