Cooperation risk and Nash equilibrium: quantitative description for realistic players

G. M. Nakamura, G. S. Contesini and A. S. Martinez
Faculdade de Filosofia, Ciências e Letras de Ribeirão Preto (FFCLRP)
Universidade de São Paulo (USP), 14040-901 Ribeirão Preto, Brazil

The emergence of cooperation figures among the main goal of game theory in competitive-cooperative environments. Potential games have long been hinted as viable alternatives to study realistic player behavior. Here, we expand the potential games approach by taking into account the inherent risks of cooperation. We show the Public Goods game reduce to a Hamiltonian with one-body operators, with the correct Nash Equilibrium as the ground state. The inclusion of punishments to the Public Goods game reduces the cooperation risk, creating two-body interaction with a rich phase diagram, where phase transitions segregates the cooperative from competitive regimes.

I. INTRODUCTION

Since its initial conception, Nash equilibrium (NE) has been an iconic aspect in Game Theory [1]. It occurs in a game whenever a player cannot improve her own outcome by changing her current strategy, under the assumption the remaining players maintain their respective strategies. This must occur for all the players. Within a limited set of rules, NE allows for a consistent analysis of competitive-cooperative scenarios, taking into account the weight of individual rather than collective performance. The sharp contrast between NE and cooperative solutions has been the central point in several studies [2]. Indeed, it has been shown that cooperation may emerge as a result of spatial inhomogeneity between players, modelled with network theory, or a consequence of more general rules [3, 4].

Players’ behavior figures among the relevant aspects governing the game outcomes. NE assumes players always adopt the best strategy available to them, reflecting an extensive amount of rational thinking. Mixed strategies have been successful to express players interactions through iterated games. However, Game Theory lacks an exact model. The formalism of potential games overcome this issue [5] bby allowing players to adopt sub-optimal strategies with a chance governed by a single parameter \( \beta \). The formalism shares a striking resemblance with Statistical Physics. This similarity allows one to borrow tools, interpretations and results from this discipline and use them into Game Theory analysis [5].

Despite the major advances put forward by potential games, global rather than individual outcomes play a major role in the quantitative analysis. While global variables and their minimization are sensible for physical systems, mostly due to a minimization principle, the same cannot be said for Game Theory. This rationale suggests the inherent risks undertaken by players with cooperative behavior are underestimated. Hence, the resulting equilibrium may deviates from the NE.

II. COOPERATION RISK

Here, we address the role of NE in the potential game formalism, and provide an analytical expression for the risks associated with cooperation. Both are intertwined and necessary to produce a more realistic description of outcomes in Game Theory. The paper is organized as follows. We start employing the Public Goods (PG) game as our toy model and develop the quantitative description of cooperation risk. Next, we extend our analysis to include the effect of punishments. Phase transitions explains the conditions necessary to create cooperation among players.

In the Public Goods (PG) games \([4, 10, 11]\), players may forfeit the cost \( c \) from their own assets to a public resource (collaborator), or keep \( c \) (defector). The public resource is then increased by the factor \( b/c \) and, afterward, redistributed equally among all players (see Fig. 1). Since players still receive their share regardless of their own contributions, defectors lower the returns of collaborators while increasing their own earnings, thus establishing the cooperative-competitive scenario.

![Public goods (PG) game with \( N \) players.](Image)

**FIG. 1.** Public goods (PG) game with \( N \) players. Each player is assigned an unique identifier \( k = 0, 1, \ldots, N - 1 \). Collaborators (white) increase the size of public resource by bearing the entry cost \( c \) and then receive their shares in return (bidirectional arrow). Defector (black) keep their assets but still receive share (one-directional arrow).

Here, \( N \) players partake in a single PG game, with a
single resource pool. Each player is labeled by a unique identifier \( k = 0, 1, \ldots, N - 1 \) and interacts with \( N - 1 \) different players. Individual player strategies (cooperator, defector) are mapped into two-level systems in the Dirac’s vector notation, namely, \( |1\) or \( |0\). The combined strategies of \( N \) players creates a configuration vector \( |s\rangle = |n_0 n_1 \cdots n_{N-1}\rangle \), where either \( n_k = 1 \) (cooperator) or vanishes otherwise (defector). Thus, the number of distinct configurations is \( 2^N \), with \( |00 \cdots 0\rangle \) being the configuration with defectors (see Ref. [12]).

As usual, player payoffs are crucial to the mathematical description of the game. They are written in terms of payoff matrices or operators. Payoff matrices are a common and convenient way to express players’ earnings and costs whenever the number of players is small, usually \( N = 2 \). For instance, see the payoff matrix of the Prisoner’s Dilemma in Ref. [13] or [14]. However, as \( N \) increases, the matrix representation becomes prohibitive. Further insights can be obtained using operators, which are defined by their action over the configuration vectors, remaining tractable even for large \( N \). In what follows we select the operatorial description to assess the PG with \( N \) players, employing the hat notation to distinguish operators from numbers.

Our main concern is the action of operator \( \hat{n}_k \) over an arbitrary configuration vector, \( \hat{n}_k |n_0 n_1 \cdots n_{N-1}\rangle = n_k |n_0 n_1 \cdots n_{N-1}\rangle \), which extracts the strategy of the \( k\)-th player from the configuration vector. The operators \( \hat{n}_k \) \((k = 0, 1, \ldots, N - 1)\) hold additional properties, namely, their eigenvalues are 0 and 1; and they are nil-potent \( \hat{n}_k^2 = \hat{n}_k \), making them suitable building blocks to describe earnings from the various strategies available to players. More explicitly, the operators that evaluate the earning of the \( k\)-th player using cooperative strategies and defective strategies are, respectively,

\[
\Pi_k^{(C)} = \left[ \frac{b}{N} \sum_{\ell=0}^{N-1} \hat{n}_\ell - c \hat{n}_k \right] \hat{n}_k, \quad (1a)
\]
\[
\Pi_k^{(D)} = \left[ \frac{b}{N} \sum_{\ell=0}^{N-1} \hat{n}_\ell - c \hat{n}_k \right] (1 - \hat{n}_k). \quad (1b)
\]

Since there are only two strategies per player, the total payoff operator regarding player \( k \) is

\[
\hat{\varepsilon}_k = \Pi_k^{(C)} + \Pi_k^{(D)} = \frac{b}{N} \sum_{\ell=0}^{N-1} \hat{n}_\ell - c \hat{n}_k. \quad (2)
\]

Notice that unlike Eqs. (1a) and (1b), Eq. (2) lacks products between operators, \( \hat{n}_k \hat{n}_\ell \), the so-called two-body operators. Instead, Eq. (2) holds only one-body operators and, thus, lacks interactions between different players.

In the formalism of potential games, the Hamiltonian \( \hat{H} = -\sum_k \hat{\varepsilon}_k \) dictates the likelihood of each configuration according to the Boltzmann distribution \[Z = \text{Tr}(e^{-\beta \hat{H}})\]. One of the key elements of the Boltzmann distribution is the partition function \( Z = \text{Tr}(e^{-\beta \hat{H}}) \), which depends on the parameter \( \beta \). In Statistical Physics, \( \beta \) is inversely proportional to temperature. In the context of potential games, \( \beta \in \mathbb{R}^+ \) serves as a scale that models the adoption of sub-optimal strategy by players. With \( \beta = 0 \), players tend to randomly adopt strategies, whereas \( \beta \to \infty \) means players tend to adopt the optimal strategy (rational players). Moreover, by introducing sub-optimal strategies, the potential games formalism replaces mixed strategies to describe the player dynamics. We reinforce that \( \beta \) should be a representative value for a pool of players much greater than \( N \).

However, we argue \( \hat{H} \) fails to correctly describe the system. Consider the simplest case with \( N = 2 \), which is formally equivalent to a particular instance of the Prisoner’s Dilemma (PD). In this case,

\[
\hat{H}^{(PD)} = -\Delta (\hat{n}_0 + \hat{n}_1), \quad \Delta = b - c, \quad (3)
\]

where \( \Delta \) is the net profit assuming both players cooperates. Accordingly, the partition function is \( Z = (1 + e^{\beta \Delta})^2 \), producing the average strategy per player \( \langle n \rangle = (1/2)|1 + \tanh(\beta \Delta)| \). Notice that \( \beta \gg 1 \) and \( \Delta > 0 \) produce \( \langle n \rangle = 1 \), i.e., rational players would cooperate regardless of net profit as long as \( \Delta > 0 \). This result is incompatible with the expected Nash Equilibrium (NE) for \( N = 2 \). Therefore, \( \hat{H} \) requires further corrections to take into account the inherent risks associated with cooperation.

Luckily, the NE requirements can also be used to model the risk. The condition states the NE occurs whenever a player cannot improve her own earnings by changing her current strategy, regardless of the strategies of the remaining players. We also note that the NE condition implicitly assumes the various player strategies are uncorrelated, to accommodate the assumption of independent strategy variations. Let \( \langle \varepsilon_0 \rangle = (b/2)\langle n_0 \rangle + c\langle n_0 \rangle \) be the average earnings of player \( k = 0 \) in a single round PG game with two players, with \( \langle n_{0,1} \rangle \in [0,1] \). The NE condition reads \( \partial \langle \varepsilon_0 \rangle \partial \langle n_0 \rangle = (b/2) - c \), so that increasing cooperation incurs into additional costs unless \( c < b/2 \), with an analogous result for the other player. Thus, the addition of linear operators \( \hat{n}_{0,1} \) with coupling constants \( \mu = \mu_0 = \mu_1 = c - (b/2) \) incorporates the NE requirements into the desired PG Hamiltonian with \( N = 2 \):

\[
\hat{H}^{(PD)} = -(\Delta - \mu) (\hat{n}_0 + \hat{n}_1). \quad (4)
\]

It is worth noting that even though \( \Delta > 0 \), the coupling \( \Delta - \mu \) might acquire negative values. Hence, cooperation becomes a viable strategy for rational players only if the net return \( \Delta \) overcomes the inherent cost \( \mu \), associated with cooperation. Therefore, we define \( \mu_k \) as the cooperation risk of player \( k \), and \( \mu_k \hat{n}_k \) as the cooperation risk operator.

The NE as the ground state of Eq. (4) can be generalized for arbitrary \( N \). From Eq. (2), we evaluate the cooperation risk \( \mu_k = -\partial \langle \varepsilon_k \rangle / \partial \langle n_k \rangle = c - b/N \). Due to player translational invariance, \( \mu_k \equiv \mu \) and the cooperation risk equals to the net difference between investment.
and minimum returns. Players in PG game aim for increasing returns while avoiding risks, and are described by the PG Hamiltonian
\[ \hat{H} = -\sum_{k=0}^{N-1} (\hat{\varepsilon}_k + \mu_k \hat{n}_k) = - (\Delta - \mu) \sum_{k=0}^{N-1} \hat{n}_k, \] (5)
with \( \mu = c - b/N \).

With Eq. (5) in hands, the partition function \( Z_0 = \prod_k [1 + e^{\beta(\Delta - \mu)}] \) provides the average density of cooperators:
\[ \langle n_k \rangle \equiv \bar{n} = \frac{1}{1 + e^{-\beta(\Delta - \mu)}}. \] (6)

Hence, \( \Delta > \mu \) favors cooperation for large values of \( \beta \). Conversely, \( \Delta < \mu \) inhibits cooperation as players become aware of risks. Regardless, \( \langle n_1 \hat{n}_k \rangle - \langle n_2 \rangle \langle n_k \rangle = 0 \), there is no correlation between players’ strategies.

III. COOPERATION RISK IN ASYMMETRIC GAMES

Punishments are socio-economic measures input upon players who disobey agreements. Punishments are special because they are asymmetric, only affecting a specific subset of players (defectors). They can be understood as adjustment of rules to enforce cooperation. In what follows, we explore the PG games with punishment (PGP) to create asymmetric payoff operators, introducing correlations among players.

Let us quantify punishment as the reduction of defectors’ earnings by the factor \( 0 \leq \gamma \leq 1 \). More specifically, the defector payoff operator in Eq. (1b) is modified according to
\[ \hat{\Pi}^{(P)}_k = (1 - \gamma) \hat{\Pi}^{(D)}_k, \] (7)
which inhibits non-cooperative strategies by decreasing their effectiveness. The consequence from Eq. (7) appears in the earning operator regarding the \( k \)-th player,
\[ \hat{\varepsilon}_k = -c \hat{n}_k + \frac{b}{N} (1 - \gamma + \gamma \hat{n}_k) \sum_{\ell=0}^{N-1} \hat{n}_\ell. \] (8)

Due to punishment \( \gamma \), \( \hat{\varepsilon}_k \) acquires two-body operators \( \hat{n}_k \hat{n}_\ell \) with coupling constant proportional to \( \gamma \). Using the property \( \hat{n}_k^2 = \hat{n}_k \) and the guidelines used in the previous sections, we evaluate the cooperation risk \( \mu'_k = \mu_k - \gamma (b/N) \sum_{\ell \neq k} \langle n_\ell \rangle \) for PGP. Thus, the PGP Hamiltonian with \( N \) players reads
\[ \hat{H}' = -\frac{\gamma b}{N} \sum_{\ell,k=0}^{N-1} \hat{n}_k \hat{n}_\ell - \sum_{k=0}^{N-1} (h'_k - \mu'_k) \hat{n}_k, \] (9)
where the one-body coupling \( h'_k \equiv h' = \Delta - \gamma b \) differs by \( -\gamma b \) from its counterpart in the PG.

Equation (9) supports two remarkable properties. First, punishment always decreases the risk associated with cooperation: \( \mu'_k - \mu = -\gamma (b/N) \sum_{\ell \neq k} \langle n_\ell \rangle \leq 0 \). Lower risks favor cooperation among players, so that one may conclude that punishments favor cooperation. However, punishments also lower the actual value of \( b'_k \), which is a primary component of players’ earnings. Payoff decrements \( \delta \varepsilon \) due to punishment can be estimated using meanfield approximation: \( \delta \varepsilon \approx -\gamma b \langle n \rangle [1 - \langle n \rangle] \), where \( \langle n \rangle \in [0,1] \) describes the mean cooperation density of players. Therefore, at the same time that punishment produces a bias towards cooperations, payoffs decrease by amounts proportional to \( \gamma \). Thus, this quantitative result recovers some findings first reported in Refs. 16–18 for iterated games.

We can learn additional insights about the cooperation risk \( \mu'_k \) by replacing the local average \( \langle n_k \rangle \) with the global average, i.e., \( \langle n_k \rangle \approx \langle n \rangle \). Under the above approximation
\[ \mu'_k \approx \mu_k - \gamma b \langle n \rangle (N - 1)/N. \] (10)

In fact, the approximation becomes exact for translational invariant systems as players become equivalent to each other: \( \mu' = \mu - \gamma b \langle n \rangle (N - 1)/N \). Alternatively, for the sake of practical applications, one may replace \( \langle n \rangle \) in Eq. (9) by \( \bar{n} \), yielding
\[ \mu' = \mu - \frac{(N - 1)}{N} \frac{\gamma b}{1 + e^{-\beta(\Delta - \mu)}} + o(\gamma^2). \] (11)

The second property of Eq. (9) concerns two-body operators \( \hat{n}_k \hat{n}_\ell \). In general, the overall contribution attributed to two-body operators depends on the punishment parameter \( \gamma \) and on the local density of cooperators \( \langle n_k \rangle \). Eventually, cooperative strategies become competitive against the inherent risk associated with cooperation.

To simplify the notation, let the PGP Hamiltonian be written as \( \hat{H} = -\alpha_2 N^2 - \alpha_1 N \), with \( N = \sum_k \hat{n}_k \), and couplings \( \alpha_2 = \gamma b/N \) and \( \alpha_1 = (\Delta - c + b/N) - \gamma b (1 - \bar{n}q/N) \). The corresponding partition function reads
\[ Z = Z_0(x) \sum_{k=0}^{\infty} \frac{(\beta \alpha_2)^k}{k!} \left[ \frac{1}{Z_0(x)} \frac{\partial^{2k}}{\partial x^{2k}} Z_0(x) \right], \] (12)
where \( x = \beta \alpha_1 \) and \( Z_0(x) = (1 + e^x)^N \). A more useful formulation for operators \( (\partial/\partial x)^{2k} \) is obtained after the variable change \( u = 1 + e^x \), so that \( (\partial/\partial x)^{2k} = \sum_{\ell=0}^{2k} \left( \begin{array}{c} 2k \\ell \end{array} \right) u^{(1-\ell)}(\partial/\partial u)^\ell \). The symbol \( \left( \begin{array}{c} 2k \\ell \end{array} \right) \) refers to the Stirling numbers of second kind for \( \ell \leq 2k \), or 0 otherwise [19]. By applying this expression for \( (\partial/\partial x)^{2k} \) to the term inside the square brackets in Eq. (12), one derives the polynomial \( P_{2k}(\xi) = (\partial/\partial x)^{2k} \ln Z_0 \), where \( \xi = (1 + e^{-x})^{-1} \). More specifically,
\[ P_{2k}(\xi) = \sum_{\ell=1}^{N} \left\{ \begin{array}{c} 2k \\ell \end{array} \right\} \frac{N!}{(N-\ell)!} \xi^\ell. \] (13)
Further algebraic manipulations of Eq. (12), with \( y \equiv \beta \alpha_2 \), produce

\[
Z = Z_0(x) \left[ 1 + G(x, y) \right],
\]

\[
G = \sum_{\ell=1}^{N} \frac{N!}{(N-\ell)!} C_\ell(y) \xi^\ell,
\]

\[
C_\ell \equiv \sum_{k=1}^{\infty} \frac{(2k)!}{k!} \left\{ \ell \right\}_k.
\]

In the special case \( \ell = 1 \), \( C_1 = (e^y - 1) \) grows as the exponential function. Under the asymptotic approximation \( \left\{ z_\ell \right\} \approx \ell^{\ell}/\ell! \), the functions \( C_\ell \approx (e^{y^\ell} - 1)/\ell! \) acquire a much more tractable form. Note that in both cases, variations in \( y \) create the same behavior \( \delta C_\ell = C_\ell + (e^{y^\ell}/\ell!) \delta y + o(\delta y^2) \).

Now, we turn our attention to the average density of collaborators \( \langle n \rangle \). By accelerating player compromise rate either via collaboration or defection, depending on the values of \( \gamma \) and \( c/b \), for increasing player optimal strategy adoption \( \beta \); b) punishment \( \gamma \) promotes cooperation among players for scenarios that would be otherwise dominated by non-cooperative behavior. From the analytical point of view, \( \langle n \rangle = \bar{n} \) for vanishing \( \gamma \), by construction. In addition, the density \( \langle n \rangle \) satisfies the following inequality:

\[
\langle n \rangle = \xi + \frac{1}{N+1+\bar{G}} \sum_{\ell=1}^{N} \ell N! C_\ell \xi^\ell \leq \frac{\bar{G}}{1+\bar{G}} \left( 1 + \frac{\xi}{\bar{G}} \right).
\]

Thus, the density of collaborators meets an upper bound which depends on \( \xi \) and the function \( \mathcal{G} \).

Consider the regime of high rationality and low returns, \( i.e., \beta \gg 1 \) and vanishing \( \xi \). According to Eq. (15), there exist three possible outcomes for \( \langle n \rangle \), namely, \( \langle n \rangle \rightarrow 0 \), if \( \mathcal{G} \rightarrow 0 \); \( \langle n \rangle \rightarrow 1 \), if \( \mathcal{G} \gg 1 \); and \( \langle n \rangle \) converges to a finite number in the interval \([0, 1]\) if \( \mathcal{G} \) converges to a finite value. These conditions are evident if we consider the largest term in Eq. (15), \( i.e., \mathcal{G} \propto \exp[\beta \ell (\gamma \ell/N + \alpha_1)] \). Since \( \alpha_1 \) can take negative (positive) values then \( \mathcal{G} \) may vanish (diverge) for large values of \( \beta \). Therefore, a non-trivial relationship between

FIG. 2. Effects of punishment \( \gamma \) on \( N = 1024 \) players in a single public goods game. a) For \( b > 2c \), positive values of \( \gamma \) accelerate the adoption rate of cooperative strategies. The special case \( b = 2c \) states the equivalence between cooperative and non-cooperative strategies, without punishment. The addition of punishments to the game dynamics, however, shifts players towards cooperative strategies. b) Punishment retains its efficacy only for a short interval of values \( 2c > b \). In the graphic, for \( c/b = 0.75 \), punishment increases the adoption rate of non-collaborative strategies, surpassing the case without punishment. The special point \( \beta^* = 1.386 \) marks the point where the density of collaborators with punishment equals its counterpart without punishment. Monte Carlo simulations are performed using Metropolis algorithm (\( 10^7 \) realizations). Errors bars are estimated using integrated correlation time [12].

FIG. 3. Equivalence between density of collaborators between \( \gamma = 1 \) and \( \gamma = 0 \) for \( N = 1024 \) players in PGP. The solid lines represent \( \langle n \rangle \) with \( \gamma = 1 \), whereas \( \gamma = 0 \) for dashed lines. Vertical dashed lines indicate crossing between curves with same cost \( c \) but different punishment parameters \( \gamma \).
which punishment drives cooperative behavior. However, the new cost suffers a small increment over the previous one. \( \langle n \rangle \) and \( \gamma \) must emerge under the assumption that the thermodynamic limit exists for PGP. Indeed, the largest contribution in Eq. (12) provides the desired expression:

\[
2\beta_0 \alpha_2 N \langle n \rangle + \beta \alpha_1 = \Psi(N \langle n \rangle + 1) - \Psi(N - N \langle n \rangle + 1),
\]

where \( \Psi(z) \) is the Digamma function. Turns out that for fixed \( \gamma \), there exists a cost threshold \( c' \equiv c'(\gamma) \) below which punishment drives cooperative behavior. However, \( c > c' \) accelerates the defection rate, leading to the crossing between the curves \( \langle n \rangle \) and \( \bar{n} \). In fact, the crossing occurs at the inverse temperature

\[
\beta^* = \frac{\ln 2}{2c - b},
\]

obtained from Eq. (12), as shown in Fig. 3.

Finally, there is the question concerning the value \( c' \). Fig. 4 depicts the behavior of \( \langle n \rangle \) for \( c_0 = 0.664b \) and \( c_1 = 0.665b \), with \( N = 2^{10} \) and punishment parameter \( \gamma = 1 \). In the first case, punishment tilts the tendency of players toward cooperation. More importantly, the cooperators density \( \langle n \rangle \) increases monotonically and continuously for increasing values of \( \beta \). In the other case, the new cost suffers a small increment over the previous one, \( c_1 = c_0 + \delta c \) with \( \delta c = 10^{-3} \). However, the behavior of \( \langle n \rangle \) changes rapidly after \( \beta > 2.08 \). In fact, numerical data in Fig. 4 suggests \( \langle n \rangle \) converges to unity for large \( \beta \). The exact nature of the transition and whether it occurs as single critical points or rather critical lines is not entirely clear at this point, being well beyond the scope of this paper.

\[
\langle n \rangle \text{ decays as } |\beta - \beta_0|^{-\omega_1} \exp(-\omega_2 \beta), \text{ with } \\
\beta_0 = 2.175, \omega_1 = 0.140 \pm 0.002, \text{ and } \omega_2 = 1.400 \pm 0.005 \approx 10 \omega_1.
\]

More importantly, the fluctuation \( \langle n^2 \rangle - \langle n \rangle^2 \) displays the well-known shape of \( \lambda \)-transitions in log-log scale, as Fig. 5 depicts, with a peak around \( \beta \approx 2.2 \). The maximum occurs around \( \beta = 2.2 \) for \( N = 1024 \), \( c = 0.665 \), \( b = 1 \) and \( \gamma = 1 \), recreating the shape of \( \lambda \) letter, the hallmark of \( \lambda \)-transitions. Error bars omitted for clarity.

\[
\langle n^2 \rangle - \langle n \rangle^2 \text{ decays as } |\beta - \beta_0|^{-\omega_1} \exp(-\omega_2 \beta), \text{ with } \\
\beta_0 = 2.175, \omega_1 = 0.140 \pm 0.002, \text{ and } \omega_2 = 1.400 \pm 0.005 \approx 10 \omega_1.
\]

\[
\text{FIG. 4. Phase transition in the public goods game with } N = 1024 \text{ players, punishment parameter } \gamma = 1 \text{ and } b = 1. \text{ a) } \langle n \rangle \text{ converges continuously to unity with inverse temperature } \beta, \text{ for } c = 0.664 \text{ (triangles). After a small cost increment, player’s behavior change towards non-cooperation (full circles). T he gap } \Lambda_c \equiv \Lambda_c(\beta) \text{ converges to unity for large } \beta. 50\% \text{ of data omitted for clarity. b) } \langle n \rangle \text{ decays as } |\beta - \beta_0|^{-0.14} \exp(-1.40 \beta), \text{ with } \beta_0 = 2.175 \text{ (solid line).}
\]

\[
\text{FIG. 5. Variance of cooperation density in log-log scale. The maximum occurs around } \beta = 2.2 \text{ for } N = 1024, \ c = 0.665, \ b = 1 \text{ and } \gamma = 1, \text{ recreating the shape of } \lambda \text{ letter, the hallmark of } \lambda \text{-transitions. Error bars omitted for clarity.}
\]

\[
\langle n^2 \rangle - \langle n \rangle^2 \text{ decays as } |\beta - \beta_0|^{-\omega_1} \exp(-\omega_2 \beta), \text{ with } \\
\beta_0 = 2.175, \omega_1 = 0.140 \pm 0.002, \text{ and } \omega_2 = 1.400 \pm 0.005 \approx 10 \omega_1.
\]

\[
\text{FIG. 5. Variance of cooperation density in log-log scale. The maximum occurs around } \beta = 2.2 \text{ for } N = 1024, \ c = 0.665, \ b = 1 \text{ and } \gamma = 1, \text{ recreating the shape of } \lambda \text{ letter, the hallmark of } \lambda \text{-transitions. Error bars omitted for clarity.}
\]

\[
\langle n^2 \rangle - \langle n \rangle^2 \text{ decays as } |\beta - \beta_0|^{-\omega_1} \exp(-\omega_2 \beta), \text{ with } \\
\beta_0 = 2.175, \omega_1 = 0.140 \pm 0.002, \text{ and } \omega_2 = 1.400 \pm 0.005 \approx 10 \omega_1.
\]

\[
\text{IV. CONCLUSION}
\]

In this paper, a quantitative formulation of cooperation risk is introduced to the analytical machinery of Game Theory. Cooperation risk operators are one-body
interactions and compete against payoff operators, and ultimately provide the individuality component required by the NE. Our numerical results show the PGP develops a first-order phase transition, separating a cooperative phase from a non-cooperative phase. Another phase transition is hinted along the rationality parameter $\beta$ for specific value of $c/b$ and $\gamma$. However, the classification of the phase transition and the whole range of parameters in which it occurs is still under study. In closing, our findings lay out the groundwork for the investigation of more complex games with competitive-cooperative dynamics, while also taking into account the individual aspect required by the NE. We plan to expand this study and evaluate the threshold cost $c'$ for arbitrary $\gamma$.

**ACKNOWLEDGMENTS**

We are grateful for F. Meloni comments during manuscript preparation and subsequent discussions. GMN holds grant CAPES 88887.136416/2017-00, ASM acknowledges grants CNPq 307948/2014-5. GSC thanks CAPES.

[1] Nash J (1950) Equilibrium points in n-person games. *Proceedings of the National Academy of Sciences* 36(1):48–49.
[2] Axelrod R, Hamilton W (1981) The evolution of cooperation. *Science* 211(4489):1390–1396.
[3] Szabó G, Fáth G (2007) Evolutionary games on graphs. *Physics Reports* 446(4):97 – 216.
[4] Wakano JY, Nowak MA, Hauert C (2009) Spatial dynamics of ecological public goods. *Proceedings of the National Academy of Sciences* 106(19):7910–7914.
[5] Szabó G, Bodó KS, Allen B, Nowak MA (2015) Four classes of interactions for evolutionary games. *Phys. Rev. E* 92(2):022820.
[6] Szabó G, Bodó KS, Allen B, Nowak MA (2014) Fourier decomposition of payoff matrix for symmetric three-strategy games. *Phys. Rev. E* 90(4):042811.
[7] Blume LE (1993) The statistical mechanics of strategic interaction. *Games and Economic Behavior* 5(3):387 – 424.
[8] Monderer D, Shapley LS (1996) Potential games. *Games and Economic Behavior* 14(1):124 – 143.
[9] Szabó G, Borsos I (2016) Evolutionary potential games on lattices. *Physics Reports* 624(Supplement C):1 – 60. Evolutionary potential games on lattices.
[10] Traulsen A, Hauert C, De Silva H, Nowak MA, Sigmund K (2009) Exploration dynamics in evolutionary games. *Proceedings of the National Academy of Sciences* 106(3):709–712.
[11] Wang Z, Szolnoki A, Perc M (2013) Interdependent network reciprocity in evolutionary games. *Scientific Reports* 3:1183.
[12] Nakamura GM, Monteiro ACP, Cardoso GC, Martinez AS (2017) Efficient method for comprehensive computation of agent-level epidemic dissemination in networks. *Scientific Reports* 7:40885.
[13] Doebeli M, Hauert C (2005) Models of cooperation based on the prisoner’s dilemma and the snowdrift game. *Ecology Letters* 8(7):748–766.
[14] Pereira MA, Martinez AS (2010) Pavlovian prisoner’s dilemma—analytical results, the quasi-regular phase and spatio-temporal patterns. *Journal of Theoretical Biology* 265(3):346 – 358.
[15] Amit DJ, Martin-Mayor V (2005) *Field Theory, the Renormalization Group, and Critical Phenomena: Graphs to Computers*. (World Scientific Press).
[16] Sigmund K, Hauert C, Nowak MA (2001) Reward and punishment. *Proceedings of the National Academy of Sciences* 98(19):10757–10762.
[17] Nowak MA, Sigmund K (2005) Evolution of indirect reciprocity. *Nature* 437(7063):1291–1298.
[18] Rand DG, Dreber A, Ellingsen T, Fudenberg D, Nowak MA (2000) Positive interactions promote public cooperation. *Science* 325(5945):1272–1275.
[19] Abramowitz M, Stegun IA (1964) *Handbook of mathematical functions: with formulas, graphs, and mathematical tables*, Applied Mathematics Series. (U.S. Department of Commerce, National Bureau of Standards) Vol. 53.