Matrix Models vs. Matrix Integrals

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In a brief review, we discuss interrelations between arbitrary solutions of the loop equations that describe Hermitean one-matrix model and particular (multi-cut) solutions that describe concrete matrix integrals. These latter ones enjoy a series of specific properties and, in particular, are described in terms of Seiberg-Witten-Whitham theory. The simplest example of ordinary integral is considered in detail.

1 Introduction

Recent interest in matrix models and especially in their multi-cut solutions was inspired by the studies in $\mathcal{N} = 1$ SUSY gauge theories due to Cachazo, Intrilligator and Vafa [1] and by the proposal of Dijkgraaf and Vafa [2] to calculate the low energy superpotentials, using the partition function of multi-cut solutions. The solutions themselves are well-known already for a long time (see, e.g., [3]) with a new vim due to the paper by Bonnet, David and Eynard [4].

The matrix model under consideration is the Hermitean one matrix model. Its partition function is given by the integral over $N \times N$ Hermitean matrix

$$Z_N(t) \equiv \frac{1}{\text{Vol}_{U(N)}} \int DM \exp \left( \text{tr} \sum_k t_k M^k \right) \quad (1)$$

Here $DM$ is the invariant (Haar) measure on Hermitean matrices, which is just the flat measure and $\text{Vol}_{U(N)}$ is the volume of the unitary group $U(N)$ [5]. Since the integrand in (1) is invariant w.r.t. matrix conjugation, one can reduce formula (1) to an integral over eigenvalues of the matrix $M$ integrating out the angular variables. It can be done using formulas from [6], and the result reads

$$Z_N(T) = \frac{1}{N!} \int \prod_i dx_i \Delta^2(x) \exp \left( \sum_{k,i} t_k x_i^k \right) \quad (2)$$

where $\Delta(x)$ is the Vandermonde determinant, $\Delta(x) \equiv \det_{i,j} x_i^j - 1 = \prod_{i>j}(x_j - x_i)$.

The integrals (1) and (2) still need to be defined. Indeed, one has to fix the integration contours in these multiple integrals. Moreover, these contours can differ from each other. Saying this, one definitely forgets that we originally started from Hermitean matrices. However, it is in no way important for the properties of (1) and (2). In fact, the word “Hermitean” has no other sense but fixing the integration contour. What we really need to fix in (1) is the flat measure. Say, this is enough to obtain (2) from (1).

In order to define (1) and (2) one can substitute them with their saddle point approximations [3, 7]. The other possibility [3, 11, 12] is to observe that they satisfy an infinite set of loop (Virasoro) equations (=Schwinger-Dyson equations, =Ward identities) [11, 12]. Indeed, this is the consequence of only the flatness of measure and invariance of integrals w.r.t. the change of variables.

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The second possibility means that one calls the matrix model partition function any solution to the loop equations. Then, the partition function is not a function but a formal $D$-module, i.e. the entire collection of power series (in $t$-variables), satisfying a system of consistent linear equations. Solution to the equations does not need to be unique, however, an appropriate analytic continuation in $t$-variables transforms one solution to another, and, on a large enough moduli space (of coupling constants $t$), the whole entity can be considered, at least formally, as a single object: this is what we call the partition function. Naively different solutions are interpreted as different branches of the partition function, associated with different phases of the theory. Further, solutions to the linear differential equations can be often represented as integrals (over spectral varieties), but integration “contours” remain unspecified: they can be generic chains with complex coefficients (in the case of integer coefficients this is often described in terms of monodromies, but in the case of partition functions the coefficients are not required to be integer).

Therefore, in further consideration we distinguish between matrix model, which is a set of solutions to the loop equations, and matrix integral which is an integral with a specified integration contour. Note that the matrix model partition function is an arbitrary linear combination of a proper set of basis matrix integrals.

The natural choice for this basis is given by the multi-cut (or Dijkgraaf-Vafa) solutions mentioned above. They are distinguished by a special property of isomonodromy that allows one to associate a Seiberg-Witten-Whitham system with them [13, 14], the corresponding partition function having a multi-matrix model integral representation [4, 7].

2 1×1 matrix case: a toy example

We start with the simplest example of the 1×1 matrix, i.e. the matrix integral is just an ordinary one-fold integral. Let us start first from the integral with potential of the very general form and see what can be done.

2.1 Loop equations

Thus, we start from the integral

$$Z(\{t_k\}) = \int_C dx \exp\left(\sum_k t_k x^k\right)$$

(3)

At the moment we do not specify the integration contour $C$, but assume it is closed, or, at least, the integrand is canceled at the ends of the contour with any monomial $x^k$ of arbitrary degree. Then, one can immediately obtain an infinite set of equations satisfied by this integral. To obtain it, one suffices to consider the integral of the full derivative

$$\int_C dx \frac{\partial}{\partial x}\left[x^{n+1}\exp\left(\sum_k t_k x^k\right)\right], \quad n \geq -1$$

(4)

which is zero. (4) gives rise to the set of constraints

$$\sum_k kt_k \frac{\partial^{k+n} Z}{\partial t_1^{k+n}} + (n + 1) \frac{\partial Z}{\partial t_n} = 0, \quad n \geq -1$$

(5)

where $\frac{\partial Z}{\partial t_0} \equiv Z$. Note that

$$\frac{\partial Z}{\partial t_k} = \frac{\partial^{k} Z}{\partial t_1^{k}}$$

(6)

These equations are called loop equations, Virasoro constraints, Schwinger-Dyson equations, Ward identities, see [14].
and, therefore, the first terms of the sum in (5) can be rewritten as the linear derivatives w.r.t. the couplings t_{k+n}.

Let us try to solve (5) as a power series in t_k’s,

\[ Z = c^{(0)} + \sum_k c^{(1)}_k t_k + \sum_{k,l} c^{(2)}_{kl} t_k t_l + \ldots \]  

Then, from the constraint with n=0 we immediately obtain c^{(0)} = 0, similarly, c^{(1)}_k = 0 etc. This means the solution is trivial: Z = 0. In fact, this should not come as a surprise, since the dimension of coupling t_k is equal to \( k \), and the dimension of c^{(l)} should be negative. However, we have no quantities with any negative dimensions at hands.

In order to get a non-trivial solution, we need to allow, at least some couplings t_k to be in the denominator. Let us fix a few (say, p) first couplings not to be small, i.e. we shift these couplings \( t_k \rightarrow T_k + t_k, \ k = 1, \ldots, p \) and consider Z as a power series in t_k’s but not in T_k’s. Then (5) has non-trivial solutions.

In other words, we fix a polynomial \( V(x) \equiv \sum_k T_k x^k \) and consider the integral in (3) as taken perturbatively w.r.t. to t_k’s,

\[ Z(\{t_k\}) = \int_C dx \exp \left( V(x) + \sum_k t_k x^k \right) \]  

i.e. one calculates the moments

\[ \int_C dx x^k e^{V(x)} \]  

The generating function (resolvent) for these moments is

\[ G(z) \equiv \int_C dx \frac{e^{V(x)}}{z - x} = \sum_k \frac{1}{z^{k+1}} \int_C dx x^k e^{V(x)} \]  

where integral is understood as the principal value integral.

### 2.2 “Matrix” model

Below we consider the simple example of the cubic potential \( V(x) \). By the shift of x one can erase the quadratic term in the potential and, by rescaling x, make \( T_3 \) equal to \(-1/3\). Therefore, we are left with the only essential variable, which we choose at the moment to be \( T_1 \). Constraints (5) taken at zero couplings reduce then to the single equation

\[ \frac{d^2 Z}{dT_1^2} + T_1 Z = 0 \]  

which is nothing but the Airy equation. It has two solutions

\[ Z(T_1) = \int_C dx \exp \left( -\frac{x^3}{3} - T_1 x \right) \]  

don that correspond to two basically different choices of the integration contour \( C \) in (8). The contour has to be chosen so that the integrand vanishes at its ends, i.e. the contour should go to infinity where \( \Re x^3 < 0 \). One of the possibilities is to choose the imaginary axis as \( C \). This gives the standard Airy function

\[ \text{Ai}(T_1) = \int_0^\infty \cos \left( \frac{x^3}{3} - T_1 x \right) dx \]
Another independent solution to the Airy equation should be associated with a contour that connects other infinities. Say, one can choose the contour that goes along the imaginary axis from $+\infty$ to zero and then goes along the positive ray of the real axis. We call the corresponding function $\textrm{Ai}_2(T_1)$.\footnote{The difference $\textrm{Ai}_2(T_1) - \textrm{Ai}(T_1)$ is usually denoted $\text{Bi}(T_1)$, see [15], formula (10.4.33).} Then, the “matrix” model (general) solution is, in our case,

$$Z(T_1) = \xi \textrm{Ai}(T_1) + \zeta \textrm{Ai}_2(T_1)$$  \hspace{1cm} (14)

where $\xi$ and $\zeta$ are arbitrary constants. Now one can use equations [5] to generate, recursively and unambiguously, $Z$ as a power series in $t_k$’s.

Thus, we can define the “matrix” model partition function $Z(\{t_k\})$ as a solution to the defining set of equations, [5]. Then, the entire freedom that we have in our “matrix” model is due to the constants $\xi$, $\zeta$. Choosing these constants, that is, choosing a formal sum of the distinct integration contours $C$, fixes the “matrix” model solution uniquely.

### 2.3 “Matrix” integral

In contrast to the “matrix” model, the “matrix” integral is defined by an integral. The freedom one then has is in choosing the integration contour. Therefore, one can take a basis in the space of all solutions by choosing some basis contours. This gives us a basis of “matrix” integrals, or the “matrix” integral provided with an index associated with the set of basis contours. Then, constructing linear combinations of these integrals, one obtains the general solution to the “matrix” model.

To perform effective calculations, one has to make a clever choice of the basis of integration contours. They should be naturally associated with the asymptotical expansion of the integrals. Say, dealing with cubic potential, one would better choose the basis contours corresponding to the solutions of the Airy equation controlled by different quasiclassical expansions. Indeed, let us make an asymptotic expansion of [13] at large $T_1$. Then, the saddle point equation has the two solutions $x = \pm \sqrt{T_1}$. Depending on the choice of the integration contour $C$, one should choose one or the other solution and expand the integral around this solution to obtain the asymptotic expansion.

Generally, there are $p - 1$ solution to the saddle point equation $V'(x) = 0$, and exactly that many different quasiclassical expansions and basis contours.

Now, in order to obtain $Z$ as a function of all $t_k$, one may use two different strategies. First of all, one can just iteratively solve equations [5]. The other possibility is to calculate moments [9] using properties of the corresponding “matrix” integrals, i.e. in the cubic case, those of the Airy functions.

There is also a more tricky possibility which is in no way technically simpler, but will be of great use later. That is, let us consider a possibility of immediate calculating integral [8]. This integral is typically not a power series in the higher couplings. Indeed, since the degree of potential is now much higher than $p$, there are much more possibilities of choosing the integration contours (=quasiclassical regimes, =solutions to the saddle point equation). We have to fix the integration contours so that to have a smooth (power) behaviour upon bringing higher couplings to zero, while the behaviour of the integral w.r.t. the first $p$ couplings remains arbitrary. This leaves us with the freedom of exactly $p$ different integration contours.

Let us see how it works in the cubic case. Here we have two possibilities of choosing the integration contours. Note that the Airy equation [11] can be reduced to the Bessel equation and, therefore, its solution is expressed via the cylindric functions. More concretely, the function $\sqrt{T_1}Z_{1/3}(\frac{2}{3}T_1^{3/2})$ solves the Airy equation, [15], where $Z_{1/3}(z)$ is any cylindric function of order 1/3. Now let the two basis cylindric functions be the Hankel functions of the first and second kinds $H^{(1,2)}(z)$ (this choice corresponds not to $\text{Ai}(T_1)$ or $\text{Ai}_2(T_1)$, but to their linear combinations). We also restore the
dependence on all the three couplings (and slightly rescale them for the sake of convenience). Then,

\[ Z(T_1, T_2, T_3) = \int dx \exp \left( iT_3 x^3 - T_2 x^2 + iT_1 x \right) = \sqrt{\eta} Z_{1/3}(\frac{2}{3} \eta^{3/2}), \quad \eta \equiv \frac{T_2}{(3T_3)^{4/3}} - \frac{T_1}{(3T_3)^{1/3}} \] (15)

and, using the asymptotic expansion of the Hankel functions, we find that the smooth behaviour under \( T_3 \to 0 \), i.e. \( \eta \to \infty \) is celebrated with \( H^{(2)}(z) \) (see [16], formula (7.13.2)),

\[ H^{(2)}_{1/3}(z) \sim \sqrt{\frac{2}{\pi z}} \exp \left( -i \frac{4z - 2\pi \nu - \pi}{4} \right) \sum_{m=0} \frac{(1/3, m)}{(2i z)^m}, \quad \frac{1}{3}, m \equiv \frac{\Gamma \left( \frac{5}{6} + m \right)}{m! \Gamma \left( \frac{5}{6} - m \right)} \] (16)

Therefore, we finally have (here we put \( T_1 = 0 \) for the sake of simplicity)

\[ Z(T_1, T_2, T_3) = \frac{1}{\sqrt{T_2}} \sum_{m=0} (-)^m \left( \frac{1}{3}, m \right) \left( \frac{27T_3^2}{4T_2^3} \right)^m \] (17)

i.e., with this choice of solution, \( Z \) is indeed a power series in \( T_3 \). One can easily check that the corresponding power coefficients coincide with the corresponding moments of the Gaussian integral. Therefore, one may really calculate moments of the Gaussian integral in this tricky way.

### 2.4 Resolvent and loop equation

To conclude our discussion of the toy example, we comment on the properties of the resolvent \[10\]. We start from the simplest Gaussian potential. Then, the resolvent

\[ G(z) \equiv \int_{-\infty}^{\infty} dx \frac{e^{-T_2 x^2}}{z - x} = 2z \int_{-\infty}^{\infty} dx \frac{e^{-T_2 x^2}}{z^2 - x^2} = w(\sqrt{T_2} z), \quad w(z) \equiv \int_{-\infty}^{\infty} dx \frac{e^{-x^2}}{z - x} \] (18)

can be calculated in two different ways. First of all, one can use the formula

\[ \frac{1}{z - x} = \int_{0}^{\infty} du e^{-u(z-x)} \] (19)

and further calculating the Gaussian integral to express the resolvent through the error function\(^3\),

\[ w(z) = e^{-z^2} \left( -i \pi + 2\sqrt{\pi} \int_{0}^{\infty} e^{x^2} dx \right) = e^{-z^2} e r f c(-iz), \quad e r f c(z) \equiv -i2\sqrt{\pi} \int_{z}^{\infty} e^{-x^2} dx \] (20)

The other possibility is to find the differential equation for \( w(z) \). To this end, let us again use vanishing of the integral of the full derivative,

\[ 0 = \int_{-\infty}^{\infty} dx \frac{\partial}{\partial x} \left( \frac{e^{-x^2}}{z - x} \right) \] (21)

This leads to the equation

\[ w'(z) = 2\sqrt{\pi} - 2zw(z) \] (22)

Solving this equation, one can arrive at (20) again.

In the case of more complicated potentials \( V(x) \), one can not manifestly calculate the integral for \( G(z) \). Moreover, the differential equation is also quite complicated, with order increasing as the degree of the polynomial \( V(x) \) increases. However, there is a universal form of the equation for the resolvent that we discuss now. To obtain it, let us consider the next simple example of the cubic potential. Then, we can write, as above,

\[ 0 = \int_{C} dx \frac{\partial}{\partial x} \left( \frac{e^{\frac{3}{x} - T_1 x}}{z - x} \right) \] (23)

\(^3\)One should be careful with integrating around the point \( z = x \).
which is equivalent to
\[ G'(z) = V'(z)G(z) - \int_C dxe^{\frac{a}{x} - T_1x} - 4z \int_C dxe^{\frac{a}{x} - T_1x} \] (24)

In contrast to the Gaussian integral, one cannot manifestly calculate moments of \( e^{\frac{a}{x} - T_1x} \). Moreover, these moments contain an ambiguity related to the choice of the integration contour.

Note, however, that the uncalculable part is a (linear) polynomial. Therefore, one obtains that the following equation is correct
\[ G'(z) = \left[ V'(z)G(z) \right]_- \] (25)
where \([...]'_-\) denotes the projector onto negative powers of \( z \), and we took into account that \([G'(z)]_- = G'(z)\). Looking at (22), one observes that it satisfies (25) too. Moreover, repeating the derivation for the general polynomial potential, one comes to the same universal result (25).

3 \( N \times N \) matrix case

Now we consider the true matrix integral following mainly the line discussed in the previous section.

3.1 Loop equations

We start now with integral (11), again not specifying integration contours. (Since this is the multiple integral, we have freedom to choose different contour for each integration.) Let us first obtain the set of equations satisfied by this integral. As before, we consider the integral of the total derivative
\[ \int DM \text{tr} \left[ \frac{\partial}{\partial M} \left( M^{n+1} \exp \left( \text{tr} V(M) + \sum_k t_k \text{tr} M^k \right) \right) \right] = 0 \] (26)
where \( M^t \) is the transpose matrix. These equations lead to the following set of constraints (=Schwinger-Dyson equations, =Ward identities) (12)
\[ \hat{L}_m Z_N(t) = 0, \quad m \geq -1 \hat{L}_m = \sum_{k \geq 0} k (t_k) \frac{\partial}{\partial t_{k+m}} + \sum_{a+b=m} \frac{\partial^2}{\partial t_a \partial t_b} \] (27)
where \( \frac{\partial Z_N}{\partial t_0} \equiv NZ_N \). Note that this time we cannot replace derivatives w.r.t. higher couplings with higher order derivatives w.r.t. the first coupling. As before, in order to have non-trivial solutions to these equations, we need to shift first \( p \) couplings: \( t_k \to T_k + t_k, \quad k = 1, \ldots, p \) and then deal with the partition function (11) as a power series in the couplings \( t_k \)'s. Then, one also needs to add to constraints (27) the conditions
\[ \frac{\partial Z_N}{\partial T_k} = \frac{\partial Z_N}{\partial t_k} \quad \forall k = 0, \ldots, n + 1 \] (28)

However, since now we have the set of equations w.r.t. various couplings, they cannot longer be reduced to a single equation w.r.t., say, \( T_1 \). Therefore, the freedom in solving equations (27) is much larger.

3.2 Matrix model

How many solutions to equations (27) do we expect now? To understand this, let us again study an oversimplified example. Namely, look at the model similar to (2), but without the Vandermonde determinant. Then, it reduces to the product of independent factors
\[ Z_N(T|t) = \prod_i \int_{C_i} dx_i \exp \left( \sum_k t_k x_i^k \right) \] (29)
Each factor here is a solution to the corresponding ordinary differential equation. Say, in the cubic case, one has
\[ Z_N(T|t) = \prod_{i}^{N}[\zeta_i \text{Ai}(\eta) + \zeta_i \text{Ai}_2(\eta)] = \sum_{k} \Xi_k [\text{Ai}(\eta)]^k [\text{Ai}_2(\eta)]^{N-k} \]  
(30)
where \( \Xi_k \) are the coefficients constructed from products of \( \zeta_i \)'s and \( \zeta_i \)'s which determine the freedom in the matrix model partition function. Since \( \Xi_k \) are arbitrary coefficients, one may interpret them as counterparts of Fourier series coefficients of an arbitrary function \( Z_N(\eta) \) of \( \eta \), the Fourier exponentials being substituted with the combination of the Airy functions.

Generally, as we discussed in the previous section, for the polynomial potential of degree \( p \), there are \( p - 1 \) independent basis functions, i.e. the freedom in the matrix model partition function is an arbitrary function of \( p - 1 \) variables. The same counting certainly remains valid for the model with the Vandermonde determinant present, (2).

Let us now understand the origin of the arbitrary function of \( p - 1 \) variables in terms of constraints (27). To this end, note that if one solves them recurrently, expanding the partition function into the power series in couplings \( t_k \)'s (like (7)), only the first two constraints of (27) are really restrictive for \( Z_N(T) \), while all other constraints just allow one to restore recurrently the dependence on the couplings \( t_k \)'s. As we already explained, (27) are less restrictive than (5) since they can not be reduced to an ordinary differential equation in \( T_1 \).

The first two constraints of (27) are linear in derivatives and, therefore, we consistently truncate them to \( t = 0 \) and then express two derivatives, say, \( \partial Z/\partial t_{p+1} \) and \( \partial Z/\partial t_n \) through \( \partial Z/\partial T_l \) with \( l = 0, \ldots, p - 1 \). As a corollary, the partition function can be represented as
\[ Z_N(T) = \int dkz(k; \eta_2, \ldots, \eta_p)e^{\frac{k}{2}(kz^2 - k^2w)} \]  
(31)
with an arbitrary function \( z \) of \( p \) arguments \( (k, \eta_2, \ldots, \eta_p) \). Here the following variables invariant w.r.t. the first constraint are used,
\[ w = \frac{1}{p + 1} \log T_{p+1}, \quad x = T_0 + \ldots + T_{p+1} \]  
(32)
\[ \eta_k = \left( \frac{T_k^p + k(k - 2)}{p!} \sum_{l=1}^{k-1} \frac{(-1)}{(p + 1)! (p - l)! (k - l - 1)!} T_{p-l} T_{p-l}^{k-l-1} T_{p+1}^{l} \right) T_{p+1}^{\frac{k}{p+1}} \]  
(33)
We discussed a particular case of these formulas in s.2.3.

### 3.3 Matrix integral

Let us discuss now what could be a choice of basis matrix integrals. We basically repeat the procedure we applied in the ordinary integral case, i.e. use the saddle point approximation. Different saddle points \( M = M_0 \) are given by the equation \( V'(M_0) = 0 \). If the polynomial
\[ V'(x) = \prod_{i=1}^{p} (x - \alpha_i) \]  
(34)
has roots \( \alpha_i \), then, since \( M_0 \) are matrices defined modulo \( U(N) \)-conjugations (which allow one to diagonalize any matrix and permute its eigenvalues), the different saddle points are represented by
\[ M_0 = \text{diag}(\alpha_1, \ldots, \alpha_1; \alpha_2, \ldots, \alpha_2; \ldots; \alpha_p, \ldots, \alpha_p) \]  
(35)
with \( \alpha_i \) appearing \( N_i \) times, \( \sum_{i=1}^{p} N_i = N \). In fact, there is no need to keep these \( N_i \) non-negative integers: in final expressions they can be replaced by any complex numbers. Moreover, \( N_i \) can depend on \( T_k \) (i.e. on the shape of \( V(M) \)).

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Now, using at the intermediate stage the eigenvalue representation of matrix integrals, one can rewrite \([4, 7]\) the matrix integral (??) over \(N \times N\) matrix \(M\) as a \(p\)-matrix integral over \(N_i \times N_i\) matrices \(M_i\) (each obtained with the shift by \(\alpha_i\): just changing variables in the matrix integral (??)), which is nothing but the multi-cut solutions \([2]\)

\[
Z_V(t|M_0) \sim \int \prod_{i=1}^{p} DM_i \exp \left( \sum_{i,k} \text{tr} t_k^{(i)} M^{k} \right) \prod_{i<j}^{2N_iN_j} \times 
\times \exp \left( 2 \sum_{k,l=0}^{\infty} \frac{(-)^k (k + l - 1)!}{\alpha_{i,j}^{k+l} k! l!} \text{tr} M_k \text{tr} M_l \right)
\]

The variables \(t_k^{(i)}\) are given by the relation

\[
\sum_{k=0}^{\infty} t_k \left( \sum_{i=1}^{p} \text{tr} (\alpha_i + M_i)^k \right) = \sum_{i=1}^{p} \left( \sum_{k=0}^{\infty} t_k^{(i)} \text{tr} M_i^k \right)
\]

with arbitrary \(N_i \times N_i\) matrices \(M_i\).

Thus, we finally have \(p-1\) basis functions that can be described by the proper choices of the integration contours, and are associated with different solutions to the saddle point equation.

### 3.4 Resolvent and loop equation

Another form of constraints \([27]\) is produced by rewriting the infinite set through the unique generating function of all single trace correlators

\[
\rho^{(1)}(z|t) \equiv \hat{\nabla}(z) \mathcal{F}, \quad \hat{\nabla}(z) \equiv \sum_{k \geq 0} \frac{1}{z^{k+1}} \frac{\partial}{\partial t_k}, \quad \mathcal{F} \equiv g^2 \log Z_V
\]

Introducing the notation \(v(z)\) for \(\sum_k t_k z^k\), one obtains the loop equation \([\overline{11}]\)

\[
\left[ V'(z) \rho^{(1)}(z|t) \right]_- = V'(z) \rho^{(1)}(z|t) - \left[ V'(z) \rho^{(1)}(z|t) \right]_+ =
\]

\[
\left( \rho^{(1)}(z|t) \right)^2 + \left[ v'(z) \rho^{(1)}(z|t) \right]_- + g^2 \hat{\nabla}(z) \rho^{(1)}(z|t)
\]

In order to consider (connected) multi-trace correlators, one needs to introduce higher generating functions (also named loop mean, resolvent etc)

\[
\rho^{(m)}(z_1, ..., z_m|t) \equiv \hat{\nabla}(z_1)...\hat{\nabla}(z_m)\mathcal{F}
\]

Note that \(G(z)\) introduced in \([10]\) is equal to \(Z(T|t)\rho^{(1)}(z|t)\). Moreover, taken at all \(t_k = 0\), \((39)\) reduces to \([25]\). However, the quantity \(\rho^{(1)}(z|0)\) generates only correlators of single-trace operators (moments), which is not enough in the matrix case. Therefore, in this case one needs to know the whole quantity \(\rho^{(1)}(z|t)\). It can be, however, expressed through \(\rho^{(m)}(z_1, ..., z_m|0)\),

\[
\rho^{(1)}(z|t) = \sum_{m \geq 0} \frac{1}{m!} \int \cdots \int v(z_1) ... v(z_m) \rho^{(m+1)}(z, z_1, ..., z_m|0)
\]

### 4 Genus zero solution

One can solve the loop equations \([32]\) recurrently (see, e.g., \([8]\)) expanding them into sum of \(\rho^{(m)}(z_1, ..., z_m|0)\) and these latter into series in \(g\). This gives one the double recurrent relation (in \(m\) and order of \(g\)).
Note that the solution of these recurrent relations in the leading order can be immediately obtained. Indeed, omitting the last term from (39), one obtains that
\[ \rho^{(0|1)}(z) = \frac{V'(z) - y(z)}{2} \] (42)
(where the first superscript 0 refers to the leading in \( g \) approximation) with
\[ y^2(z) = (V'(z))^2 - 4P_{p-1}(z) \] (43)
where \( P_{p-1}(z) \equiv \left[ V'(z)\rho^{(1)}(z|0) \right]_+ \) is a polynomial of degree \( p - 1 \). Coefficients of this polynomial depend on \( T_k \)'s and first derivatives of the arbitrary function of \( p - 1 \) variables, which parameterize solutions to constraints [27] (=the loop equations).

Note that formula (43) gives a hyperelliptic Riemann surface of genus \( p - 1 \), since mapping (43) gives \( p \) cuts on the complex plane (this is the celebrated multi-cut solution we discussed in the introduction). It turns out that, at least with some specific fixing of the ambiguity in solutions of the loop equations, the (multi-)resolvents can be constructed as differentials given on this Riemann surface, [17, 14]. This specific fixing is given by the conditions
\[ \oint_{A_i} \rho^{(m+1)}(z, z_1, \ldots, z_m | 0) dz = 0 \] (44)
for all the (multi-)resolvents except \( \rho^{(0|1)}(z) \),
\[ \oint_{A_i} \rho^{(0|1)}(z) = S_i \] (45)
where \( S_i \) are arbitrary constants not depending on \( T_k \) and \( A_i \) are \( A \)-cycles on the Riemann surface [13]. The constants \( S_i \) can be, in fact, associated (up to the factor \( g \)) with \( N_i \) from [35]. Moreover, the basis functions [36] exactly lead to conditions (44)-(45). These conditions are also associated with Seiberg-Witten-Whitham system corresponding to the Riemann surface (43).

We do not go into further details here, just referring to [10] and [14] for the latest development and proper references. Let us only note that conditions (44)-(45) distinguish specific solutions that survive while smooth changing the number \( p \) of couplings \( T_k \), much similar to what we observed in s.2.3. Indeed, for any number \( q \) of non-zero \( T_k \)'s, one may require the curve to be of the form
\[ y^2(z) = (V'(z))^2 - 4P_{p-1}(z) = H_{q-p}^2(z)R_{2p}(z) \] (46)
where \( H_{q-p}^2(z) \) and \( R_{2p}(z) \) are polynomials. Therefore, independently of \( q \), one has the hyperelliptic curve \( \tilde{y}^2(z) = R_{2p}(z) \) of genus \( p - 1 \). Moreover, the freedom one has in matrix model in this case is dictated by \( p - 1 \) constants \( S_i \), see the details in [13]. The crucial difference from the case considered in s.2.3 is that, when one is dealing with the ordinary integral, one may calculate higher moments differentiating the integral in lower couplings several times, while, in the matrix case, it is not possible because of the traces. Therefore, technically this is very convenient to add traces of matrices in higher degrees directly into the matrix potential. One, however, has to be careful not to change the solution of the matrix model with this procedure. This is what exactly achieved with preserving the curve \( \tilde{y}(z) \), [16].

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