Symbol Error Rate Performance of Box-Relaxation Decoders in Massive MIMO

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Abstract—The maximum-likelihood (ML) decoder for symbol detection in large multiple-input multiple-output wireless communication systems is typically computationally prohibitive. In this paper, we study a popular and practical alternative, the box-relaxation optimization (BRO) decoder, which is a natural convex relaxation of the ML. For independent identically distributed real Gaussian channels with additive Gaussian noise, we obtain exact asymptotic expressions for the symbol error rate (SER) of the BRO. The formulas are particularly simple, they yield useful insights, and they allow accurate comparisons to the matched-filter bound (MFB) and to linear decoders, such as zero-forcing and linear minimum mean square error. For binary phase-shift keying signals, the SER performance of the BRO is within 3 dB of the MFB for square systems, and it approaches the MFB as the number of receive antennas grows large compared to the number of transmit antennas. Our analysis further characterizes the empirical density function of the solution of the BRO, and shows that error events for any fixed number of symbols are asymptotically independent. The fundamental tool behind the analysis is the convex Gaussian min–max theorem.

Index Terms—Massive MIMO, nonlinear decoders, convex relaxation, asymptotic performance, Gaussian processes.

I. INTRODUCTION

The problem of recovering an unknown vector of symbols that belong to a finite constellation from a set of noise corrupted linearly related measurements arises in numerous applications, and in particular in multiple-input multiple output (MIMO) wireless communication systems [1]–[4]. As a result, a large host of exact and heuristic optimization algorithms have been proposed over the years. Exact algorithms, such as sphere decoding and its variants, become computationally prohibitive as the problem dimension grows, a scenario that is typical in modern massive MIMO systems. Heuristic algorithms such as zero-forcing (ZF) and linear minimum mean square error (LMMSE) have inferior performances [5], and others such as local neighborhood search-based methods [6] and lattice reduction-aided (LRA) decoders [7], [8] are often difficult to precisely characterize; also, see [2] and references therein. In this paper, we study the so called box-relaxation optimization decoder, which is a natural convex relaxation of the maximum-likelihood (ML) decoder, and which allows one to recover the signal via efficient convex optimization followed by hard thresholding, e.g., [9]–[11]. Despite its popularity, very little is known analytically about the decoding performance of this method. In this paper, we close this gap by deriving asymptotic error-rate characterizations under the assumption of real Gaussian wireless channel and additive Gaussian noise.

A. Problem Formulation

We consider the problem of recovering an unknown vector \( \mathbf{x}_0 \) of \( n \) transmitted symbols each belonging to a finite constellation from the noisy multiple-input multiple-output relation, \( \mathbf{y} = \mathbf{A}\mathbf{x}_0 + \mathbf{z} \in \mathbb{R}^m \), where \( \mathbf{A} \in \mathbb{R}^{m \times n} \) is the MIMO channel matrix (assumed to be known) and \( \mathbf{z} \in \mathbb{R}^m \) is the noise vector. We assume independent and identically distributed (iid) real Gaussian channel with additive Gaussian noise. In particular, \( \mathbf{A} \) has entries iid \( \mathcal{N}(0, 1/n) \) and \( \mathbf{z} \) has entries iid \( \mathcal{N}(0, \sigma^2) \). The normalization is such that the signal-to-noise ratio (SNR) varies inversely proportional to the noise variance \( \sigma^2 \). We are interested in the large-system limit, where both the number \( n \) of transmit antennas and the number \( m \) of receive antennas are large. For simplicity of exposition we assume, for the most part of the paper, that \( \mathbf{x}_0 \) is a binary phase-shift keying (BPSK) vector, i.e., \( \mathbf{x}_0 \in \{\pm 1\}^n \). Extensions to M-ary constellations are also provided.

Maximum-Likelihood decoder: The ML decoder for BPSK signal recovery, which minimizes the block error probability (assuming the \( \mathbf{x}_{0,i} \) are equally likely), is given by \( \min_{\mathbf{x} \in \{\pm 1\}^n} \| \mathbf{y} - \mathbf{A}\mathbf{x} \|_2 \). Solving for the exact ML solution is often computationally intractable, especially when \( n \) is large, and therefore a variety of heuristics have been proposed (zero-forcing, mmse, decision-feedback, etc.) [5], [12].

Box-relaxation optimization decoder: The heuristic we consider in this paper is the box-relaxation optimization (BRO) decoder [9]–[11]. It consists of two steps. The first one involves solving a convex relaxation of the ML decoder, where
$x \in \{\pm 1\}^n$ is relaxed to $x \in [-1, 1]^n$. The output of the optimization is hard-thresholded in the second step to produce the final binary estimate. Formally, the algorithm outputs an estimate $\hat{x}$ of $x_0$ given as

$$\hat{x} = \arg \min_{-1 \leq x_i \leq 1, i \in [n]} \|y - Ax\|_2,$$  

(1a)$$x^* = \text{sign}(\hat{x}),$$  

(1b)

where $[n] = \{1, \ldots, n\}$, and the $\text{sign}(\cdot)$ function returns the sign of its input and acts element-wise on input vectors. The BRO decoder naturally extends to the case of recovering signals from higher-order constellations; see Section III.

**Symbol error rate:** We evaluate the performance of the decoder by the symbol error rate (SER), defined as

$$\text{SER} := \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}\{x_i^* \neq x_{0,i}\},$$

(2)

with $\mathbb{I}(\cdot)$ used to denote the indicator function. A closely related quantity that is also of interest is the symbol-error probability $P_e$, which is defined as the expectation of the SER averaged over the noise, over the channel, and over the constellation. Formally,

$$P_e := \mathbb{E}[\text{SER}] = \frac{1}{n} \sum_{i=1}^{n} \Pr(x_i^* \neq x_{0,i}).$$

(3)

**B. Contribution and Related Work**

In this paper, we derive the first precise characterization of the SER for the BRO decoder in the large-system limit, where the numbers $m$ and $n$ of receive and transmit antennas grow proportionally large at a fixed rate $\delta = m/n$. We complement the precise error formulas with closed-form, tight, upper and lower bounds that are simple functions of the SNR and of $\delta$. These bounds allow useful insights on the decoding performance of the BRO, and they allow a quantitative comparison to the matched-filter bound (MFB) and to conventional linear decoders. As a concrete example, if $\delta > 1/2$, then for BPSK signals the SER of the BRO at high-SNR is $Q(\sqrt{(\delta - 1/2)\text{SNR}})$, where the $Q$-function is the tail probability of the standard normal distribution. This value is within 3 dB of the MFB for square systems, and it approaches the MFB as $m$ approaches $n$. Finally, we evaluate the large-system empirical distribution of the output of the BRO, and we show that error events for any fixed number of symbol-errors are asymptotically independent.

To the best of our knowledge, a precise formula for the SER was unknown for the BRO. We remark that the replica method developed in statistical mechanics can be used to give formulas for the SER of various detectors in multiuser detection for code-division multiple access (CDMA) or massive MIMO systems. However, the replica method involves a set of conjectured assumptions that remain mostly unverified by rigorous means; please see [5], [13], [14] and references therein. In contrast, our analysis is rigorous, and the techniques used are fundamentally different. They are based on recent advances in comparison inequalities for Gaussian processes; in particular, the convex Gaussian min-max theorem (CGMT) [15], [16].

**A. Precise SER Performance**

We consider a large-system limit in which $m, n \to +\infty$ at a proportional (constant) rate $\delta > 0$. The SNR is assumed

$1$The analysis framework that we present here is general and can be used to analyze the performance of other decoders as well. For example, see our recent papers with co-authors [18], [19], which build upon the framework of this work.
constant; in particular, it does not scale with \( n \). Note that for \( x_0 \in \{ \pm 1 \}^n \), \( \text{SNR} = 1/\sigma^2 \).

We use standard notation \( \text{plim}_{n \to \infty} X_n = X \) to denote that a sequence of random variables \( X_n \) converges in probability towards a constant \( X \). All limits will be taken in the regime \( m, n \to +\infty, m/n = \delta \); to keep notation short we simply write \( n \to \infty \). Finally, we use \( Q(\cdot) \) denote the \( Q \)-function associated with the standard normal density \( p(h) = \frac{1}{\sqrt{2\pi}} e^{-h^2/2} \).

**Theorem II.1 (SER for BPSK signals):** Assume that the channel matrix \( A \in \mathbb{R}^{m \times n} \) has entries iid real Gaussians with zero mean and variance \( 1/n \). Also, assume iid Gaussian noise \( z \in \mathbb{R}^m \) of zero mean and variance \( 1/\text{SNR} \). Let \( E \) denote the symbol-error-rate of the box-relaxation optimization decoder in (1), for some fixed but unknown BPSK signal \( x_0 \in \{ \pm 1 \}^n \). Fix a constant SNR and a constant \( \delta \in \left( \frac{1}{4}, +\infty \right) \). Then, in the limit of \( m, n \to +\infty, m/n = \delta \), it holds:

\[
\text{plim}_{n \to \infty} \text{SER} = \mathcal{Q} \left( \frac{1}{\tau^*} \right),
\]

where \( \tau^* \) is the unique positive minimizer of the strictly convex function \( F : (0, +\infty) \to \mathbb{R} \) defined as:

\[
F(\tau) := \tau \left( \delta - \frac{1}{2} \right) + \frac{1/\text{SNR}}{\tau} + \left( \tau + \frac{4}{\tau} \right) \mathcal{Q} \left( \frac{2}{\tau} \right) - \sqrt{\frac{2}{\pi}} e^{-\frac{1}{\tau^2}}. \tag{4}
\]

The high-probability limit in the statement of the theorem is over the randomness of \( A \) and \( z \). The function \( F(\tau) \) in (4) is deterministic, strictly convex, and is parametrized by the value of the SNR and by the proportionality factor \( \delta \). The proof of the theorem uses the convex Gaussian min-max theorem (CGMT) [15], [16]. Beyond the result of Theorem II.1, we use the CGMT to prove a number of even stronger statements regarding the error performance of the BRO. We:

i) establish the large-system error performance of the BRO for a wide class of performance metrics; this class includes the squared-error and the SER as special cases.

ii) explicitly characterize the limiting empirical distribution of the output \( x \) of (1a).

iii) show that error events for any fixed number of bits are asymptotically independent.

Please refer to Theorem IV.1 and to Corollary IV.1 for the formal statements of these results. The detailed proof of Theorem II.1 is also deferred to Section IV.

Some further remarks on Theorem II.1 are given below.

1) **On \( \delta > \frac{1}{4} \):** The theorem holds as long as the ratio of proportionality \( \delta \) is (strictly) greater than \( 1/2 \). To begin with, note that this allows for the number of receive antennas to be less than the number of transmit antennas, and as low as (almost) half of them. When \( \delta < 1 \) the system of linear equations \( y = Ax \) is underdetermined; hence, recovering the true solution is generally ill-posed even in the absence of noise. However, in the problem of interest it is a-priori known that the true solution \( x_0 \) only takes values \( \{ \pm 1 \}^n \). The BRO decoder uses that information by enforcing an \( \ell_\infty \)-norm constraint in (1a). Of course, this idea of using convex optimization with (typically non-smooth) constraints that promote the particular structure of the unknown signal \( x_0 \) to solve underdetermined system of equations, is one of the core ideas that emerged from the Compressed Sensing literature (e.g. [27]). In fact, it is by now well-understood that in the noiseless case the program in (1a) successfully recovers the true \( x_0 \in \{ \pm 1 \}^n \) with high probability over the randomness of \( A \) if and only if \( \delta > 1/2 \) ([27], [28]). The same necessary condition naturally arises out of our proof of Theorem II.1.

2) **Probability of Error:** Recall from (3) that the symbol-error probability is given as \( P_e = E[\text{SER}] \). Also, the SER is bounded between 0 and 1. Thus, using Theorem II.1 we show in Appendix A1 that \( P_e \) converges (deterministically) to the same value \( Q(1/\tau^*) \).

**Corollary II.1 \((P_e)\):** Under the setting of Theorem II.1, let \( P_e \) denote the symbol-error probability of the BRO and \( \tau^* \) be the minimizer of (4). Then,

\[
\lim_{n \to \infty} P_e = Q \left( \frac{1}{\tau^*} \right).
\]

3) **Solving for \( \tau^* \):** In order to evaluate the large-system limit of the SER, one needs to compute the unique positive minimizer of \( F(\tau) \) in (4). The function \( F \) is strictly convex, hence this can be done numerically in an efficient way. Due to convexity, \( \tau^* \) can also be described as the unique solution to the first order optimality conditions of the minimization program (see Lemma A.2). By further analyzing the properties of \( \tau^* \), we derive in Section II-B simple closed-form (upper and lower) bounds on the quantity of interest, namely \( Q(1/\tau^*) \).

4) **Numerical Illustration:** Figure 1 illustrates the accuracy of the prediction of Theorem II.1, by comparing the asymptotic formula of the theorem to numerical simulations for \( \delta = 0.7 \) and \( \delta = 1 \). The simulation results are averages over 1000 realizations of \( A \) and \( z \) for three different values of \( n \) : 128, 256, and 512. For larger values of \( n \) the match between the numerical averages and the theoretical curve improves. Overall, note that although the theorem requires \( n \to \infty \), the prediction is already accurate for \( n \) on the scale of a few hundreds.
C. Comparison to the Matched Filter Bound

Here, we compare the performance of the BRO to an idealistic case, where all \( n - 1 \), but 1, bits of \( x_0 \) are known to us. As is customary in the field, we refer to the symbol error probability of this case as the matched filter bound (MFB) and denote it by \( P_{e}^{\text{MFB}} \). The MFB corresponds to the probability of error in detecting (say) \( x_{0,n} \in \{ \pm 1 \} \) from: \( \hat{y} = x_{0,n} a_n + z \), where \( \hat{y} = y - \sum_{i=1}^{n-1} x_{0,i} a_i \) is assumed known, and, \( a_i \) denotes the \( i \)th column of \( A \). (This can be equivalently thought of as the error probability of an isolated transmission of only the last bit over the channel.) The ML estimate is equal to the sign of the projection of the vector \( \hat{y} \) to the direction of \( a_n \). Without loss of generality we assume that \( x_{0,n} = +1 \). Then, the output of the matched filter becomes \( \text{sign}(\hat{X}) \), where

\[
\hat{X} = \| a_n \|^2 + \sigma^2 \tilde{z}_n,
\]
and \( \tilde{z}_n \sim \mathcal{N}(0, 1) \). Recall that the entries of the \( m \)-dimensional vector \( a_n \) are iid \( \mathcal{N}(0, 1/n) \), so it holds \( \lim_{n \to \infty} |a_n| = \sqrt{\delta} \). Hence,

\[
\lim_{n \to \infty} P_{e}^{\text{MFB}} = \lim_{n \to \infty} P(\hat{X} < 0) = Q(\sqrt{\delta} \cdot \text{SNR}). \quad (7)
\]

First, observe that this formula coincides with the lower bound on the probability of error of the BRO derived in Theorem II.2. Combined, they establish formally that the MFB is (strictly) better than the BRO. Of course, this is naturally expected since the former is an idealistic scheme.

Next, when compared to the upper-bound on the probability of error of the BRO derived in Theorem II.2, the formula in (7), leads to the following conclusion:

The BRO achieves a desired symbol-error probability at a higher SNR value by at most \( 10 \log_{10} \frac{\delta}{\delta - 1/2} \) dB than that predicted by the MFB.

In particular, in the square case (\( \delta = 1 \)), where the number of receive and transmit antennas are the same, the BRO is 3 dB off the MFB (cf., Figure 2). When the number of receive antennas is much larger, i.e., when \( \delta \gg 1 \), then the performance of the BRO approaches the MFB.

D. Box-Relaxation vs Zero-Forcing vs LMMSE

In this section, we use Theorem II.1 to compare the performance of the BRO to two widely used decoders, namely the zero-forcing (ZF) and the linear MMSE (LMMSE) decoders.

The ZF decoder obtains an estimate \( \hat{x}_Z \) of \( x_0 \) as follows

\[
\hat{x}_Z = \arg \min_{x \in \mathbb{R}^n} \| y - Ax \|_2, \quad (8a)
\]

\[
\hat{x}_Z = \text{sign}(\hat{x}_Z). \quad (8b)
\]

Observe that this is very similar to the BRO, only that in (8a) the minimization is unconstrained. Therefore, in contrast to the BRO, for the ZF decoder we require \( \delta > 1 \), i.e., the number of receive antennas be larger than the number of transmit antennas. When this is the case and \( n \) is large, \( A \) is full column-rank with probability one, and (8a) has a unique closed-form solution:

\[
\hat{x}_Z = (A^T A)^{-1} A^T y. \quad (9)
\]
Using standard tools from random matrix theory (RMT) it is known how to derive the symbol-error probability of the ZF decoder (e.g., [29]). For convenience of the reader, we briefly summarize the main idea here. Without loss of generality, consider the last bit \( x_n \) of \( x \). Further let \( A = QR \) be the QR decomposition of \( A \), such that \( Q \in \mathbb{R}^{m \times n} \) is a matrix with orthogonal columns and \( R \in \mathbb{R}^{n \times n} \) is upper triangular. Define \( \tilde{y} := Q^T y \) and \( \tilde{z} := Q^T z \) and note that

\[
\tilde{y}_n = R_{nn} x_n + \tilde{z}_n,
\]

where \( R_{nn} \) is the \( n \)th diagonal element of \( R \). From the rotational invariance of the Gaussian distribution, it holds \( \tilde{z}_n \sim \mathcal{N}(0, \sigma^2) \).

Next, we use the following well-known facts, e.g., [29, Lem. 1]: (i) \( Q \) and \( R \) are independent matrices. Hence, \( \tilde{z}_n \) is independent of \( R_{nn} \); (ii) \( R_{nn} \) is such that \( nR_{nn}^2 \) is \( \chi^2 \) random variable with \( (m - n + 1) \) degrees of freedom. Thus, by the corresponding formula for BPSK single-input single-output (SISO) Gaussian channel, the symbol-error probability of the zero-forcing decoder is

\[
P_e^{ZF} = E_{\gamma_1, \ldots, \gamma_{n-1}} \left[ Q \left( \sqrt{\frac{1}{\sigma^2} \sum_{j=1}^{n-1} \gamma_j^2} \right) \right],
\]

where \( \gamma_j \)'s are iid standard Gaussians \( \mathcal{N}(0, 1) \). But, \( \lim_{n \to \infty} \sum_{j=1}^{n-1} \gamma_j^2 = (\delta - 1) \), giving

\[
\lim_{n \to \infty} P_e^{ZF} = Q(\sqrt{(\delta - 1) \cdot \text{SNR}}).
\]

Comparing this formula to the upper bound on the probability of error of the BRO derived in Theorem II.2, we formally quantify the superiority of the BRO over the ZF decoder:

The BRO achieves the same performance as the ZF decoder at a lower SNR value by at least \( 10 \log_{10} \left( \frac{1 - \frac{1}{\delta}}{\frac{1}{\delta}} \right) \) dB.

This holds for \( \delta > 1 \). However, Theorem II.1 further shows that the BRO can successfully decode even when \( \delta < 1 \), and in particular as low as 1/2.

Similarly to the above, we may use the result of Theorem II.1 to compare the performance of the BRO to the LMMSE decoder

\[
x_{LMMSE} = \text{sign}(\langle AT \mathbf{z} + \sigma^2 \mathbf{T} \rangle^{-1} A^T y).
\]

The asymptotic SER of the latter is known in the literature (e.g., [5, Sec. 2.2.2]). For ease of reference, we include the result translated to our setting: the large-system error probability of the LMMSE is given by

\[
Q \left( \sqrt{\frac{1}{2\sigma^2} (\delta - 1 - \sigma^2 + \sqrt{(\delta - 1 + \sigma^2)^2 + 4\sigma^2})} \right).
\]

In order to compare this with the formula of Theorem II.1, recall that \( \sigma^2 = 1/\text{SNR} \). For an illustration, in Figure 3 we have plotted the symbol-error probability of the two schemes as a function of the SNR for two values of \( \delta \). Observe that there is a regime of low SNR where the LMMSE performs slightly better than the BRO. However, this regime shrinks as the value of \( \delta \) increases. Moreover, for medium to high-SNR values, the BRO significantly outperforms the LMMSE. We mention in passing that the BRO can be appropriately modified by adding a ridge regularization term in the optimization in (1a) so that it outperforms the LMMSE decoder even for low values of SNR. With coauthors, we have analyzed this modification in [18] based on tools that we develop here; we refer the interested reader therein for further details.

Finally, we remark that alternatively to the existing derivations (10) and (11), which use tools from RMT, we can obtain the same formulas using the CGMT; the proof technique is very similar to that of Theorem II.1. The use of RMT tools for the analysis of the ZF and MMSE decoders is in large possible because \( x_{ZF} \) and \( x_{LMMSE} \) can be expressed in closed-form as a function of \( A \) and \( z \). On the contrary, this is not the case with the BRO decoder and the use of the CGMT is critical for establishing Theorem II.1.

### III. EXTENSION TO M-PAM CONSTELLATIONS

#### A. Setting

Each transmit antenna sends a symbol \( x_{0,i}, i \in [n] \) that takes values

\[
x_{0,i} \in C := \{ \pm 1, \pm 3, \ldots, \pm (M - 1) \},
\]

for some \( M = 2^b \) and \( b \) a positive integer. When each antenna transmits a single bit, i.e., \( b = 1 \), then \( x_0 \in \{ \pm 1 \}^n \) and the setting is the same as in Section II. As always, we assume additive Gaussian noise of variance \( \sigma^2 \).

The ML decoder is given by \( \min_{x \in C^n} \| y - Ax \|_2 \), but it is often computationally intractable for large number of receive/transmit antennas. We consider, the natural extension of the box-relaxation decoder for BPSK in (1). Specifically, for M-PAM symbol transmission, the BRO outputs an estimate \( \hat{x}' \) of \( x_0 \) as follows:

\[
\hat{x} = \arg \min_{x \in C} \| y - Ax \|_2,
\]

\[
\hat{x}' = \arg \min_{x \in C} \| x_i - c \|, \text{ for all } i \in [n].
\]

The optimization in (12a) is convex, and (12b) simply selects the symbol value \( c \) that is closest to the solution \( \hat{x} \) among a total of \( M \) choices: \( \{ \pm 1, \pm 3, \ldots, \pm (M - 1) \} \). Therefore, the proposed decoder is computationally efficient. In the next section, we evaluate its error-rate performance.
Appendix C. Most of the remarks that followed the statement of Theorem II.1 in Section II, are readily extended to general M-PAM constellations. The guarantees of Theorem III.1 hold as long as the ratio of transmit to receive antennas $\delta$ is larger than $1 - 1/M$.\footnote{It is worth noting that this coincides with the threshold of approximate message passing (AMP) methods that are analyzed in [22].} Thus, successful transmission is possible with fewer number of receive than transmit antennas. The minimum allowed ratio increases for higher-order constellations. Similar to Theorem II.2, we can show the following simple upper bound on probability of error $P_e$ for all values of SNR:

$$\lim_{n \to \infty} P_e \leq 2 \left( 1 - \frac{1}{M} \right) Q \left( \frac{\sqrt{3}}{M^2 - 1} \text{SNR} \right).$$

(16)

Moreover, the bound is tight at high-SNR. Of course, for $M = 2$, this coincides with the upper bound in (5).

IV. PROOF OF MAIN RESULT

This section includes the proof of Theorem II.1. In fact, towards proving the theorem, we obtain a more general result which is stated as Theorem IV.1 in Section IV-A. In Section IV-B we show how the latter can be used to prove Theorem II.1. Next, in Section IV-C we rely again on Theorem IV.1 to prove that error events for any fixed number of bits are asymptotically independent. Finally, Section IV-D is devoted to the proof of Theorem IV.1.

As in the previous sections, we assume throughout that $\mathbf{A}$ has entries iid $N(0, 1/n)$ and $\mathbf{z}$ has entries iid $N(0, \sigma^2)$. For simplicity, we make use of the following notation onwards. We say that an event $\mathcal{E}$ holds with probability approaching 1 (w.p.a.) if $\lim_{n \to \infty} \Pr(\mathcal{E}) = 1$. Also, we use the following shorthands: $X_n \xrightarrow{p} X$ to denote convergence in probability; $X \overset{d}{=} Y$ to denote that the random variables $X$ and $Y$ have the same distribution; and, $\| \cdot \|$ to denote the $n$-dimensional Euclidean norm.

A. Main Technical Result

As far as the performance is concerned, we can assume without loss of generality that $x_0 = +1$, $x \overset{d}{=} (1, 1, \ldots, 1)$. Also, it is convenient to re-write (1a) by changing the variable to the error vector $\mathbf{w} := \mathbf{x} - \mathbf{x}_0 = \mathbf{x} - 1$:

$$\mathbf{w} := \arg\min_{-2 \leq \mathbf{w}_i \leq 0, i \in [n]} ||\mathbf{z} - \mathbf{A}\mathbf{w}||.$$

(17)

Then, observe that the SER defined in (2) is written in terms of the error vector $\mathbf{w}$ as:

$$\text{SER} = \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}(\mathbf{w}_i \leq -1).$$

(18)

The following theorem characterizes the limit of the empirical distribution of the optimal solution $\mathbf{w}$ in (17), and yields Theorem II.1 as a corollary.

Theorem IV.1 (Lipschitz metrics and empirical distribution): Recall the definition of $\tau_*$ in Theorem II.1, and assume, without loss of generality, that $x_0 = +1$. Let $\hat{\mathbf{w}}$ be as in (17) and...
consider its (normalized) empirical density function
\[ \mu_\hat{w} := n^{-1} \sum_{i=1}^{n} \delta_{\hat{w}_i}. \]

Further, consider the function \( \theta : \mathbb{R} \to [-2, 0] \):
\[ \theta(\gamma) := \begin{cases} 
0, & \text{if } \gamma \geq 0, \\
\tau \gamma, & \text{if } -\frac{2}{\tau} \leq \gamma < 0, \\
-2, & \text{if } \gamma < -\frac{2}{\tau},
\end{cases} \]
and let \( \mu_W \) be the density measure of a random variable \( W \)
\[ W \overset{d}{=} \theta(N(0, 1)). \] (19)

The following are true:

a) \( \mu_\hat{w} \) converges weakly in probability to \( \mu_W \).

b) For all Lipschitz functions \( \psi : [-2, 0] \to \mathbb{R} \) with Lipschitz constant \( L \) (independent of \( n \)), it holds
\[ \frac{1}{n} \sum_{i=1}^{n} \psi(\hat{w}_i) \overset{P}{\to} \mathbb{E}_W[\psi(W)]. \] (20)

Next, we comment on the interpretation of the theorem. Note that \( \mu_\hat{w} \) defines a (sequence of) random probability measure(s), while \( \mu_W \) is a deterministic measure. In part (a) of the theorem, we use terminology that is standard in the theory of random matrices to say that the sequence of random measures \( \mu_\hat{w} \) converges weakly to the deterministic measure \( \mu_W \) if for every continuous compactly supported function \( \psi \) it holds:
\[ \int \psi \, d\mu_\hat{w} \overset{P}{\to} \int \psi \, d\mu_W \] (see for example [30, pg. 160]). Note here that the support of the random variable \( W \) in (19) is \([-2, 0]\) and, also, \( \hat{w}_i \in [-2, 0], i \in [n] \); thus, we only need to consider continuous functions compactly supported in \([-2, 0]\). Of course, we may also rewrite part (b) in terms of the empirical distribution. Specifically, (20) is equivalent to \[ \int \psi \, d\mu_\hat{w} \overset{P}{\to} \int \psi \, d\mu_W \] for all Lipschitz functions \( \psi : [-2, 0] \to \mathbb{R} \). In fact, parts (a) and (b) of the theorem are equivalent. Clearly, (a) \( \Rightarrow \) (b). But the reverse implication is also true, the reason being that any continuous compactly supported function can be uniformly approximated to arbitrary accuracy by some compactly supported Lipschitz continuous function (e.g., a polynomial from the Stone-Weierstrass theorem); please see Lemma A.3 in the appendix for a formal proof. For all the results in this paper, part (b) of Theorem IV.1 is sufficient (e.g., see Sections IV-B and IV-C), but we have also included part (a) for completeness. Figure 5 includes a numerical illustration of the convergence result. We prove part (b) of the theorem in Section IV-D.

B. Proof of Theorem II.1

On the one hand, by (18), it suffices to prove that \[ \frac{1}{n} \sum_{i=1}^{n} I_{\{w_i \leq -1\}} \overset{P}{\to} Q(1/\tau_*). \] On the other hand, it is easily checked that \( \mathbb{E}_W[I_{\{W \leq -1\}}] = \mathbb{E}_{\gamma \sim N(0, 1)}[I_{\{\gamma \leq -1/\tau_*\}}] = Q(1/\tau_*). \) Note that the indicator function \( I_{\{W \leq -1\}} \) is not Lipschitz, so we cannot directly apply Theorem IV.1(b). However, since the discontinuity point (i.e., \(-1\)) of the indicator function has \( \mu_W \)-measure zero, and also \( W \) is a continuous random variable, one can appropriately approximate the indicator with Lipschitz functions and conclude the desired based on Theorem IV.1(b). This is a somewhat standard argument, but we reproduce a detailed proof of the claim in Lemma A.4 in Appendix B for completeness.

C. Independence of Error Events

Here, we obtain as a corollary of Theorem IV.1(b) that error events for any fixed number of bits are asymptotically independent. We defer the proof of the corollary to Appendix B3.

Corollary IV.1 (Independence of error events): Under the notation and definition of Theorem IV.1, let \( \psi_i : [-2, 0] \to \mathbb{R}, i = 1, \ldots, k \) be bounded Lipschitz functions for fixed \( k \geq 2 \). Then, it holds
\[ n^{-k} \sum_{1 \leq i_1, \ldots, i_k \leq n} \psi_1(\hat{w}_{i_1}) \cdots \psi_k(\hat{w}_{i_k}) \overset{P}{\to} \prod_{\ell=1}^{k} \mathbb{E}[\psi_\ell(W_\ell)], \]
where the expectations of the right-hand side are with respect to \( W_1, \ldots, W_k \) that are iid random variables distributed as \( \theta(N(0, 1)) \). Moreover, it holds
\[ n^{-k} \sum_{1 \leq i_1, \ldots, i_k \leq n} I_{\{\hat{w}_{i_1} \leq -1, \ldots, \hat{w}_{i_k} \leq -1\}} \overset{P}{\to} (Q(1/\tau_*))^k. \]

D. Proof of Theorem IV.1

1) The Fundamental Tool: The CGMT associates with a primary optimization (PO) problem a simplified auxiliary optimization (AO) problem from which we can tightly infer properties of the original (PO), such as the optimal cost, the optimal solution, etc. In particular, the (PO) and (AO) problems are defined respectively as follows:

\[ \Phi(G) := \min_{u \in S_w} \max_{w \in S_w} u^T Gw + \psi(w, u), \] (21a)
\[ \phi(g, h) := \min_{w \in S_w} \max_{w \in S_w} \|w\|^T u - \|u\| h^T w + \psi(w, u), \] (21b)
where $G \in \mathbb{R}^{m \times n}$, $g \in \mathbb{R}^m$, $h \in \mathbb{R}^n$, $S_w \subset \mathbb{R}^n$, $S_u \subset \mathbb{R}^m$ and $\psi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$. Denote $w_S := w_S(G)$ and $w_\phi := w_\phi(g, h)$ any optimal minimizers in (21a) and (21b), respectively. Further let $S_w, S_u$ be convex and compact sets, $\psi$ be continuous and convex-concave on $S_w \times S_u$, and, $G, g, h$ all have entries iid standard normal.

Theorem IV.2 (CGMT 16): Let $S$ be an arbitrary open subset of $S_w$ and $S^e = S_w \setminus S$. Denote $\phi_S(g, h)$ the optimal cost of the optimization in (21b), when the minimization over $w$ is now constrained over $w \in S^e$. Suppose there exist constants $\phi$ and $\eta > 0$ such that in the limit of $n \to +\infty$ it holds w.p.a.1: (i) $\phi_S(g, h) \leq \phi + \eta$, and, (ii) $\phi_S(g, h) \geq \phi + 2\eta$. Then, $\lim_{n \to \infty} \text{Pr}(w_S \in S) = 1$.

It is not hard to argue that the conditions (i) and (ii) regarding the optimal cost of the (AO) imply the following for its solution: $w_\phi \in S$ w.p.a.1. The non-trivial and powerful part of the theorem is that the same conclusion is true for the optimal solution $w_\phi$ of the (PO) as well.

2) Strategy: As mentioned the proof is based on the use of the CGMT. The first step is to identify the (PO) and the (AO), such that $\bar{w}$ is optimal for the (PO). Then, our goal is to apply Theorem IV.2 to the following set

$$S_\epsilon := \left\{ v : \left| n^{-1} \sum_{i=1}^n \psi(v_i) - E_W[\psi(W)] \right| < \epsilon \right\}, \quad (22)$$

where $\epsilon > 0$ is arbitrary and $\psi$ is a Lipschitz function defined in $[-2, 0]$. To see that this is desired note that if for all $\epsilon > 0$ it holds $w_\phi \in S_\epsilon$ (in the notation) w.p.a.1, then $n^{-1} \sum_{i=1}^n \psi(w_i) \overset{D}{\to} E_W[\psi(W)]$. Thus, the bulk of the proof amounts to checking that the conditions of Theorem IV.2 are satisfied for $S_\epsilon$ in (22).

To aid the readers, we have partitioned the proof in six steps, each presented in one of the upcoming Sections IV-D3–IV-D8; please also refer to Figure 6.

In what follows, we fix $\epsilon > 0$ and denote $S := S_\epsilon$, for convenience.

3) Identifying the (PO) and the (AO): Using the CGMT requires as a first step expressing the optimization in (1a) in the form of a (PO) as it appears in (21a). It is easy to see that (17) is equivalent to

$$\frac{1}{\sqrt{n}} \min_{-2 \leq w_i, 0 \leq n} \max_{u \leq 1} \mathbf{u}^T A w - \mathbf{u}^T z. \quad (23)$$

Observe that the constraint sets above are both convex and compact; also, the objective function is convex in $w$ and concave in $u$. These are consistent with the requirements of the CGMT. The corresponding (AO) problem becomes:

$$\phi(g, h) := \frac{1}{n} \min_{-2 \leq w_i, 0 \leq n} \max_{u \leq 1} \left( \|w\|g - \sqrt{n}z\right)^T u - \|u\|h^T w. \quad (24)$$

Note the normalization to account for the variance of the entries of $A$. Onwards, we refer to the optimization in (24) as the (AO) problem.

4) Simplifying the (AO): We begin by simplifying the (AO) problem as it appears in (24). First, since both $g$ and $z$ have entries iid Gaussian, then, the vector $\|w\|g - \sqrt{n}z$ has entries iid $\mathcal{N}(0, \sqrt{\frac{\|w\|^2}{n} + n\sigma^2})$. Hence, for our purposes and using some abuse of notation so that $g$ continues to denote a vector with iid standard normal entries, the first term in (24) can be treated as $\sqrt{\|w\|^2 + n\sigma^2} g^T u$, instead. As a next step, fix the norm of $u$ to say $\|u\| = \beta$. Optimizing over its direction is now straightforward, and gives

$$\min_{-2 \leq w_i, 0 \leq n} \max_{0 \leq \beta \leq 1} \frac{\beta}{n} \left( \sqrt{\|w\|^2 + n\sigma^2} |g| - h^T w \right).$$

In fact, it is easy to now further optimize over $\beta$ as well: its optimal value is 1 if the term in the parenthesis is non-negative, and, is 0 otherwise. With this, the (AO) simplifies to the following:

$$\phi(g, h) = \min_{-2 \leq w_i, 0 \leq n} \left( \sqrt{\frac{\|w\|^2}{n} + \sigma^2} \frac{|g|}{\sqrt{n}} - \frac{1}{n} h^T w \right), \quad (25)$$

where we used the notation $(\cdot)_+ := \max(\cdot, 0)$.

In order to perform the optimization over $w$, we will express the “square-root term” $\sqrt{\|w\|^2/n + \sigma^2}$ in a variational form using the following easily checked identity:

$$\sqrt{\|w\|^2/n + \sigma^2} = \min_{\chi > 0} \frac{\chi}{2} + \frac{\|w\|^2/n + \sigma^2}{2\chi}.$$

With this trick, the minimization over the entries of $w$ becomes separable as follows:

$$\min_{0 \leq \chi < \infty} \frac{\chi}{2} + \frac{\sigma^2}{2\chi} + \frac{1}{n} \sum_{i=1}^n \min_{-2 \leq w_i, 0} \left\{ \frac{\|g_i\|^2}{2\chi} w_i^2 - h_i w_i \right\}. \quad (26)$$

The CGMT builds upon a classical result due to Gordon [31]. Gordon’s original result is classically used to establish non-asymptotic probabilistic lower bounds on the minimum singular value of Gaussian matrices [32], and has a number of other applications in high-dimensional convex geometry [33, 34]. The idea of combining the GMT with convexity is attributed to Stojnic [35]. Thrampoulidis et. al. built and significantly extended on this idea arriving at the CGMT as it appears in [16, Thm. 6.1].
In particular, the optimal $\hat{w}_i := \hat{w}_i(g, h)$ of (24) satisfies for all $i = 1, \ldots, n$:

$$
\hat{w}_i(g, h) = \begin{cases} 
0, & \text{if } h_i \geq 0, \\
\frac{\bar{\chi}_i \sqrt{n}}{\|g\| |h_i|}, & \text{if } \frac{-2 \|g\|}{\bar{\chi}_i \sqrt{n}} \leq h_i < 0, \\
-2, & \text{if } h_i < \frac{-2 \|g\|}{\bar{\chi}_i \sqrt{n}},
\end{cases}
$$

where, $\bar{\chi}_n := \bar{\chi}_n(g, h)$ is the solution to the following:

$$
\phi(g, h) = \left( \min_{\chi > 0} \frac{\chi \|g\|}{2 \sqrt{n}} + \frac{\sigma^2 \|g\|}{2 \chi \sqrt{n}} + \frac{1}{n} \sum_{i=1}^n \left( \chi \sqrt{n} \left| \frac{h_i}{\|g\|} \right| \right) \right)_+,
$$

with, $v_n :=$

$$
v_n(\alpha; h) := \begin{cases} 
0, & \text{if } h \geq 0, \\
-\frac{\alpha}{2} h^2, & \text{if } -\frac{\alpha}{2} \leq h < 0, \\
\frac{\alpha}{2} + 2h, & \text{if } h \leq -\frac{\alpha}{2},
\end{cases}
$$

for all $\alpha > 0$ and $h \in \mathbb{R}$. In the sequel, it is convenient to rescale the optimization variable $\chi$ in (28) to $\tau := \chi / \sqrt{n}$. With this, we write (28) equivalently as

$$
\phi(g, h) = \left( \min_{\tau > 0} F_n(\tau; g, h) \right)_+,
$$

where, for all $\tau > 0$ we have defined

$$
F_n(\tau; g, h) := \frac{\tau \sqrt{n} \|g\|}{2} + \frac{\sigma^2 \|g\|}{2 \tau \sqrt{n}} + \frac{1}{n} \sum_{i=1}^n \left( \tau \sqrt{n} \left| \frac{h_i}{\|g\|} \right| \right).
$$

We remark that $F_n(\tau; g, h)$ is a convex function of $\tau$ in $(0, \infty)$. The easiest way to see this is noting that the objective function in (26) is jointly convex in $w$ and $\tau$.

**5) Convergence Properties of the (AO):** Now that the (AO) is simplified as in (29), we study here its behavior in the limit of $m, n \to \infty$ with $m/n = \delta$.

First, we compute the point-wise (in $\tau$) limit of the objective function $F_n(\tau; g, h)$ in (29). Clearly,

$$
\|g\| / \sqrt{n} \xrightarrow{p} \sqrt{\delta}.
$$

Also, conditioned on the value of $n^{-1/2}\|g\|$, the random variable $\sum_{i=1}^n v_n(\tau \sqrt{n} \|g\| / \|h_i\|)h_i$ is a sum of absolutely integrable iid random variables. Hence, combining the weak law of large numbers (WLLN) with (31) it follows that, for all $\tau > 0$,

$$
\frac{1}{n} \sum_{i=1}^n v_n \left( \frac{\tau \sqrt{n} \|g\|}{\|g\|} ; h_i \right) \xrightarrow{p} Y(\tau),
$$

where,

$$
Y(\alpha) := -\frac{\alpha}{2} \int_{-\infty}^{\infty} \frac{h^2}{2} p(h) dh + \frac{\alpha}{2} Q \left( \frac{2}{\alpha} \right) - 2 \int_{-\infty}^{\infty} h p(h) dh
$$

$$
= -\frac{\alpha}{4} + \frac{\alpha}{2} \int_{-\infty}^{\infty} \left( h - \frac{2}{\alpha} \right)^2 p(h) dh.
$$

Recall, that $p(h) = \frac{1}{\sqrt{2\pi}} e^{-h^2 / 2}$. Combined, we have shown that $F_n(\tau; g, h)$ converges pointwise for all $\tau > 0$ to

$$
\hat{F}(\tau) := \frac{\tau \delta}{2} + \frac{\sigma^2}{2\tau} + Y(\tau).
$$

Expanding the square in the second summand in (32) and applying integration by parts, it can be checked that $\hat{F}(\tau) = 2F(\tau)$, where $F(\tau)$ is defined in (4).

Next, the function $\hat{F}(\tau)$ is strictly convex in the optimization variable $\tau$. Its convexity follows directly as it is the point-wise limit of convex functions in (29), which is known to be convex. Alternatively, and to further check strict convexity, it can be shown that the second derivative is positive. Furthermore, we show in Lemma A.2 that $\hat{F}(\tau)$ attains its unique minimizer at some $\tau_\star > 0$. With this information we can establish uniform convergence of $F_n(\tau; g, h)$ to $\hat{F}(\tau)$, which leads to Lemma IV.1 below. The proof of the lemma is deferred to Appendix B4. Essentially, it follows from [36, Cor. II.1], which is known in the literature of estimation theory as the convexity lemma: point wise convergence of convex functions implies uniform convergence in compact subsets (see also [37, Lem. 7.75]).

**Lemma IV.1 (Uniform convergence):** For $\tau > 0$, consider $F_n(\tau; g, h)$ and $\hat{F}(\tau)$, as defined in (30) and (33), respectively. Let $\tau_\star > 0$ be the unique minimizer of $\hat{F}(\tau)$ and

$$
\bar{\tau} := \min_{\tau > 0} \hat{F}(\tau).
$$

The following are true:

a) $\min_{\tau > 0} F_n(\tau; g, h) \xrightarrow{p} \bar{\tau}$,

b) Let $\tilde{\tau}_n(g, h)$ be a minimizer of $F_n(\tau; g, h)$. Then,

$$
\tilde{\tau}_n(g, h) \xrightarrow{p} \tau_\star.
$$

When $\delta > 1/2$, it is easily checked that $\hat{F}(\tau) \geq 0$ for all $\tau > 0$. Thus, $\bar{\tau} \geq 0$, and consequently from Lemma IV.1(a) and (29),

$$
\phi(g, h) \xrightarrow{p} \bar{\tau}.
$$

**6) The Optimal Solution of the (AO):** We now have all the tools necessary to study the properties of the optimal solution $\tilde{w}$ of the (AO). The lemma below establishes that for Lipschitz functions, $\tilde{w} \in S$ (recall the definition of $S$ in (22)).

**Lemma IV.2 (Lipschitz convergence of the (AO)):** Let $\psi : [-2, 0] \to \mathbb{R}$ be $L$-Lipschitz, $\tilde{w} = \tilde{w}(g, h)$ as in (27), and random variable $W$ as in the statement of Theorem IV.1. It holds, $\frac{1}{n} \sum_{i=1}^n \psi(\tilde{w}_i) \xrightarrow{p} \mathbb{E}_W[\psi(W)]$.

**Proof:** For $i = 1, \ldots, n$, define $v_i := \theta(h_i)$ (recall the definition of $\theta(\cdot)$ in the statement of Theorem IV.1). The WLLN gives

$$
n^{-1} \sum_{i=1}^n \psi(v_i) \xrightarrow{p} \mathbb{E}_{\gamma \sim \mathcal{N}(0, 1)}[\psi(\theta(\gamma))] = \mathbb{E}_W[\psi(W)],
$$

where we also used the Gaussianity of $h_i$ and (19). Hence, it will suffice for the proof to show that $|n^{-1} \sum_{i=1}^n (\psi(\tilde{w}_i) - \psi(v_i))| \xrightarrow{p} 0$. We show this using the Lipschitz assumption and
(35). First, by the Lipschitz property:
\[ |\psi(\hat{w}_i) - \psi(v_i)| \leq L|\hat{w}_i - v_i|. \tag{38} \]
Next, the expression of $\hat{w}$ in (27), along with (31) and with (35),
they can be used to show that the RHS in (38) is appropriately small. Formally,
writing $\xi := \xi(g, h) = \frac{\lambda_n}{|g|_2^2}$ for simplicity, it follows from the
continuous mapping property that for some $\eta > 0$ (the value of which to be chosen later) we have
w.p.a.1: $|\xi - \tau_*| \leq \eta$, and, $|\tilde{\eta} - \frac{1}{\tau_*}| \leq \eta$. Hence, w.p.a.1:
\[ |\hat{w}_i - v_i| \leq \max \left\{|\tau_* - \xi||h_i|_2 \max \{-2/\tau, -2/\xi\}, \right. \]
\[ |\tau_* + 2||h_i|_2 \max \{-2/\tau, -2/\xi\}, \right\} \]
\[ \leq \eta(\eta + 2/\tau_* + \eta + \eta(\eta + \tau_*). \]
For any $\zeta > 0$, choose $\eta = \min\left\{\frac{\sqrt{\tau_*}}{\zeta}, \frac{1}{\tau_*} + \frac{1 + \tau_*}{2}\right\}$. With
this choice,
\[ |\hat{w}_i - v_i| \leq 2\eta^2 + 2\eta\left(\frac{1}{\tau_*} + \frac{1 + \tau_*}{2}\right) \leq \zeta, \]
such that in view of (38) $|\psi(\hat{w}_i) - \psi(v_i)| \leq L\zeta$, which completes
the proof.

7) Satisfying the Conditions of the CGMT: The following result uses
Lemma IV.2 and strong-convexity of the (AO) to show that the optimal cost of the (AO) strictly increases when
the optimization is constrained outside the set $S$ defined in (22). The
proof is deferred to Appendix B5.

Lemma IV.3 (Strong convexity of the (AO)): Let $\psi : [-2, 0] \rightarrow \mathbb{R}$ be L-Lipschitz, $W$ a random variable as in the statement of
Theorem IV.1, and $S := S$, the set defined in (22). Finally, denote $f(w) := f(w; g, h)$ the objective function in (25). There
exists constant $C > 0$, such that the following statement holds
w.p.a.1,
\[ \min_{w \in [-2, 0]^n} f(w; g, h) \geq \phi(g, h) + \frac{C}{L}. \]

The lemma above essentially verifies conditions (i) and (ii)
of the CGMT Theorem IV.2. To be specific, let $C$ as in the
statement of Lemma IV.3, $\phi$ as in (34), and, choose $\eta := \frac{C}{2L}$. From
(36) it holds w.p.a.1: $|\phi(g, h) - \bar{\phi}| \leq \eta$. Combine this
with Lemma IV.3 to conclude that $\phi(S, g, h) \geq \bar{\phi} + 2\eta$ w.p.a.1, as desired.

8) Applying the CGMT: At the end of last section we showed
that the conditions of the CGMT Theorem IV.2 are satisfied.
Hence, its application yields that any minimizer $\hat{w}$ of the (PO)
in (17) satisfies $\hat{w} \in S$, w.p.a.1. This completes the proof of
Theorem IV.1.

V. DISCUSSION AND FUTURE WORK
In this paper we have used the recently developed CGMT framework in [15, 16] to precisely compute the large-system
error-rate performance of the popular box-relaxation optimization
method for recovering signals from M-ary constellations,
when the channel matrix and additive noise are both iid real
Gaussians. The derived formulas were previously unknown.
Also, the CGMT was previously only used to analyze squared-
error performance; here, we illustrate for the first time its use
to analyze the error-rate performance of convex optimization-
based massive MIMO decoders.
In future work, we seek to extend the analysis to complex
Gaussian channels with symbols originating from complex-
valued constellations. At its core, this task requires extending
the CGMT to complex-valued Gaussian matrices, an extension that is
currently unavailable; thus, it poses a challenging, yet practically
important, research direction. What appears more accessible is
establishing the universality of our results for iid channels beyond
Gaussians. We believe that this is possible by combining the ideas of our paper for extended use of the CGMT
with the techniques in [38], where the universality property has
been proven for the squared-error (rather than for the symbol-
error-rate).

For BPSK signal recovery using the BRO, we proved in
Corollary IV.1 that error events for any fixed number of bits
in the solution of the BRO are iid. This fact has potentially signif-
cantly consequences to be explored. For example, it implies
that, when a block of data is in error, only a few of its bits are.
This means that the output of the BRO can be used by various
local methods to further reduce the SER. We are planning to
explore such implications further in future work.

Finally, we remark that the analysis framework that is de-
veloped in this paper is general and can be applied to ob-
tain the asymptotic error rates of other convex-based detectors.
For one such example, we refer the reader to our recent work
in [19] with co-authors. Therein, we consider constellations
that appear in MIMO systems with spatial modulation where
$x_i \in \{-1, 0, +1\}, i \in [n]$. To account for the induced sparsity
of the transmitted signal, the BRO decoder is modified such that
the optimization in (1a) involves an additional $\ell_1$-regularization
term. For that modified detector and constellation, we apply the
framework presented in this paper and obtain the corresponding
asymptotic performance.

APPENDIX

A. Supplementary Proofs for Section II

1) Corollary II.1: The corollary follows from Theorem II.1
when combined with the following statement, which we prove
here: “If SER(A, z) $\overset{P}{\rightarrow} c$, for some deterministic constant c,
then, $P_e \rightarrow c$.”

For convenience, let us define the random variable $X := X(A, z) := \text{SER}(A, z)$. With this notation, $P_e = \mathbb{E}_{A, z}[X]$. Thus, for any $\epsilon > 0$,
\[ P_e \leq \mathbb{E}[X | |X - c| \leq \epsilon] \]
\[ + \mathbb{E}[X | |X - c| > \epsilon] \cdot P(|X - c| > \epsilon). \]
\[ \leq (c + \epsilon) + P(|X - c| > \epsilon). \]
where in the second inequality we used the fact that $X \leq 1$. Notice that $(c + \epsilon) + P(|X - c| > \epsilon) \rightarrow (c + \epsilon)$ as $n \rightarrow \infty$, since
X \xrightarrow{\text{f}} c, by assumption. In a similar vein,
\[ P_e \geq \mathbb{E} \left[ |X| |X - c| \leq \epsilon \right] \cdot \mathbb{P} \left( |X - c| \leq \epsilon \right) \]
\[ \geq (c - \epsilon) \cdot \mathbb{P} \left( |X - c| > \epsilon \right), \]
where, again, \((c - \epsilon)\mathbb{P} \left( |X - c| \leq \epsilon \right) \rightarrow 0\) as \(n \to \infty\), since \(X \xrightarrow{p} c\). Since the above holds for all \(\epsilon\), we have shown that \(P_e \rightarrow c\, c\), as desired.

2) Proof of Theorem II.2: Here, we prove the first part of the theorem, namely the lower and upper bounds on \(Q(1/\tau)\).

The tightness of the upper bound at high-SNR is shown later in Section A3. Due to the decreasing nature of the function \(Q\), it suffices to prove that
\[ \sqrt{(\delta - 1/2)} \cdot \text{SNR} < \tau_s^{-1} < \sqrt{\delta} \cdot \text{SNR}. \]  
(39)

This is shown in Lemma A.2(b) below. The proof of the lemma builds on understanding the behavior of the function \(F(\tau)\) in (4). The function \(F\) is composed of 4 additive terms. The first is linear in \(\tau\) and the second is simply \(\tau\) and the third is decreasing and \(\tau\) is strictly increasing by strict convexity.

For the limit \(\lim_{\tau \to \infty} F(\tau)\), as follows:
\[ F(u) := \sqrt{(2/\pi)} u e^{-2u^2} + (1 - 4u^2) \cdot Q(2u). \]  
(44)

Theorem A.1 (High-SNR regime):
\[ H(u) := F(\tau). \]

In particular, properties of \(G\) to be used later in the proof follow from Lemma A.1.

Starting with (42) and using Lemma A.1(a) and (45):
\[ H(u) := \delta - \frac{1}{2} u^2 - \frac{u^2}{\text{SNR}} + \sqrt{\frac{2}{\pi}} u e^{-2u^2} + (1 - 4u^2) Q(2u). \]

This proves the first statement. Moreover, since \(F(\tau)\) is strictly convex, we have that \(F'(\tau)\) is strictly increasing, and equivalently that \(H(u)\) is a decreasing function of \(u\).

Next, we prove that,
\[ \tau_s^{-1} = \sqrt{(\delta - 1/2)} \cdot \text{SNR} := \tau_0^{-1}. \]  
(46)

From Lemma A.1(c) and (45),
\[ G(\tau_0) > 0, \text{ for all } \tau > 0. \]

Hence, \(H(\tau_0^{-1}) = G(\tau_0^{-1}) > 0\). But, \(H(u)\) is decreasing and \(\tau_s^{-1}\) is its unique zero, from which (46) follows.

Finally, we show that
\[ \tau_s^{-1} < \sqrt{\delta} \cdot \text{SNR} := \tau_1^{-1}. \]  
(47)

Note that,
\[ H(\tau_1^{-1}) = -\frac{1}{2} + G(\tau_1^{-1}). \]

Again, from Lemma A.1(c) and (45), it follows that \(G(\tau) < 1/2\). Therefore, \(H(\tau_1^{-1}) < 0\). Combine this with the fact that \(H(\tau)\) is decreasing and \(\tau_s^{-1}\) is its unique zero, to conclude with (47), as desired.

3) High-SNR Regime: Theorem A.1 below formalizes and proves (6).

Theorem A.1 (High-SNR regime): As in the statement of Theorem II.1, fix \(\delta \in (\frac{1}{2}, \infty)\) and let SER denote the bit error probability of the detection scheme in (1) for some fixed but unknown BPSK signal \(x_0 \in \{\pm1\}^\ell\). For any \(\epsilon > 0\), there exists
constant $\text{SNR} := \overline{\text{SNR}}(\epsilon)$ such that for all values $\text{SNR} > \text{SNR}$, it holds
\[
\lim_{m, n \to \infty} P \left( \left| \frac{\overline{\text{SNR}}(\epsilon)}{Q(\sqrt{\delta - 1/2} \text{SNR})} - 1 \right| > \epsilon \right) = 0.
\]

**Proof:** Fix any $\epsilon > 0$. Recall $\tau_* := \tau_*(\text{SNR})$, the minimizer of (4), and define for convenience:
\[
\tau_0 := \tau_0(\text{SNR}) = \left( \sqrt{\delta - 1/2} \text{SNR} \right)^{-1}.
\]

We will prove that there exists $\text{SNR}(\epsilon)$, such that
\[
\frac{Q(\tau_0^{-1})}{Q(\tau_0^{-1})} - 1 \leq \frac{\epsilon}{2},
\]
for all $\text{SNR} \geq \overline{\text{SNR}}(\epsilon)$. This would suffice to complete the proof of the theorem. To see this, write
\[
\left| \frac{\text{SER}}{Q(\tau_0^{-1})} - 1 \right| = \left| \frac{\text{SER} - Q(\tau_0^{-1})}{Q(\tau_0^{-1})} + \frac{Q(\tau_0^{-1})}{Q(\tau_0^{-1})} - 1 \right|
\]
and observe the following. (a) The last term above is further upper bounded by $\epsilon/2$ using (49) for large enough $\text{SNR} > \overline{\text{SNR}}(\epsilon)$. (b) From Theorem II.1, for all values of $\text{SNR}$, there exist large enough $m, n$ such that the nominator of the first term is upper bounded by $(\epsilon/2)Q(\tau_0^{-1})$ with probability $1$.

In what follows, we show (49), which is a deterministic statement about the minimizer $\tau_* := \tau_*(\text{SNR})$ of (4). We use Lemma A.2.

From (39), we have that
\[
\lim_{\text{SNR} \to +\infty} \tau_*^{(\text{SNR})} = +\infty.
\]

Also, recall from (48) that $(\delta - 1/2) = \frac{\tau_0^2}{\text{SNR}}$. Substituting this in (43) we find that
\[
0 \leq \tau_*^{-2} - \tau_0^{-2} = \text{SNR} \cdot G(\tau_*^{-1})
\]
for $G$ as in (44) (also, recall (45)). The non-negativity above follows from the lower bound in (39). From Lemma A.1(c) and (45), $G$ is decreasing in $(0, \infty)$. Using this, and applying the lower bound in (39) once more, (51) leads to the following:
\[
0 \leq \tau_*^{-2} - \tau_0^{-2} \leq \text{SNR} \cdot G(\tau_0^{-1}) = \text{SNR} \cdot G(\sqrt{\delta - 1/2} \text{SNR}).
\]

But, from Lemma A.1(c) the limit of the right-hand side as $\text{SNR} \to +\infty$ is equal to $0$. Combining,
\[
\lim_{\text{SNR} \to +\infty} (\tau_*^{-2} - \tau_0^{-2}) = 0.
\]

Next, write $\tau_*^{-2} - \tau_0^{-2} = \tau_*^{-2} (1 - \frac{\tau_0^2}{\tau_*^2})$ and combine (50) with (53) to further show that
\[
\lim_{\text{SNR} \to +\infty} \frac{\tau_*}{\tau_0} = 1.
\]

We are now ready to prove (49). For simplicity, we write $f(x) \sim g(x)$ instead of $\lim_{x \to +\infty} \frac{f(x)}{g(x)} = 1$. It is well known that $Q(x) \sim p(x)/x$. Therefore,
\[
\frac{Q(\tau_0^{-1})}{Q(\tau_0^{-1})} \sim \frac{p(\tau_0^{-1})}{p(\tau_0^{-1})} \tau_* \frac{\tau_0}{\tau_*} \exp \left( -\frac{\tau_*^{-2} - \tau_0^{-2}}{2} \right) \sim 1,
\]
where the second line follows from (53) and (54).

**B. Supplementary Proofs for Section IV**

1) **Theorem IV.1:** (b)$\Rightarrow$(a): Lemma A.3 (From Lipschitz to continuous compactly supported functions): Let $\mu$ be a continuous measure supported on an interval $[a, b]$. Further let $\{\mu_n\}$ be a sequence of random measures supported on $[a, b]$ and indexed by $n$ such that as $n \to +\infty$, $\int \psi \, d\mu_n \overset{P}{\to} \int \psi \, d\mu$, for all Lipschitz functions $\psi : [a, b] \to \mathbb{R}$. Then, for all continuous compactly supported functions $f : [a, b] \to \mathbb{R}$ it also holds that $\int f \, d\mu_n \overset{P}{\to} \int f \, d\mu$.

**Proof:** Fix any $\epsilon > 0$.

First, from the Stone-Weierstrass theorem, there exists a polynomial $\psi : [a, b] \to \mathbb{R}$ such that
\[
|\psi(\epsilon) - \psi(x)| < \epsilon/3 \quad \text{for all} \quad x \in [a, b].
\]

Next, the polynomial $\psi$ is Lipschitz in the bounded interval $[a, b]$. Indeed, this follows easily by the fact that $\psi$ is continuously differentiable in $[a, b]$. Thus, the derivative $\psi'$ is bounded in $[a, b]$, i.e., there is a constant $L$ such that $|\psi'(x)| \leq L$. Then, the Lipschitz property of $\psi$ follows directly by the Mean value theorem. Consequently, from the statement of the lemma it holds that for sufficiently large $n$ the random variable
\[
\Delta_\psi := \left| \int \psi \, d\mu_n - \int \psi \, d\mu \right|
\]
satisfies
\[
P(\Delta_\psi < \epsilon/3) \overset{n \to \infty}{\to} 0.
\]

Finally, consider the random variable
\[
\Delta_f := \left| \int f \, d\mu_n - \int f \, d\mu \right|
\]
Then,
\[
\Delta_f \leq \Delta_\psi + \left| \int f \, d\mu_n - \int \psi \, d\mu_n \right| + \left| \int f \, d\mu - \int \psi \, d\mu \right|
\]
\[
\leq \Delta_\psi + \int |f - \psi| \, d\mu_n + \int |f - \psi| \, d\mu
\]
\[
\leq \Delta_\psi + \int \frac{\epsilon}{3} \, d\mu_n + \int \frac{\epsilon}{3} \, d\mu \leq \Delta_\psi + 2\epsilon/3,
\]
where the first inequality in the last line follows from (55). To conclude the proof combine the above display with (56) to find that
\[
P(\Delta_f < \epsilon) \overset{n \to \infty}{\to} 0,
\]
as desired.
2) From Lipschitz to the Indicator Function:
Lemma A.4 (Approximating the indicator): Let \( \mu \) be a continuous measure on the real line such that \( c \in \mathbb{R} \) is a point of measure zero. Further let \( \{\mu_n\} \) be a sequence of random measures indexed by \( n \) such that as \( n \to +\infty \), \( \int \psi d\mu_n \overset{P}{\to} \int \psi d\mu \), for all Lipschitz functions \( \psi : \mathbb{R} \to \mathbb{R} \). For the indicator function \( \chi_c(\alpha) := \mathbb{1}_{\{\alpha \leq c\}} \) it holds that, \( \int \chi_c d\mu_n \overset{P}{\to} \int \chi_c d\mu \).

Proof: Fix any \( \epsilon, \zeta > 0 \) and consider the random variable \( X = \int \chi_c d\mu_n - \int \chi_c d\mu \). Note that this is random since the measures \( \mu_n \) are random. It will suffice to show that there exists \( N_* \) such that for all \( n > N_* \): \( \mathbb{P}(X > \epsilon) \leq \zeta \).

Let \( \eta > 0 \), the exact value of which to be determined later, and, consider the following functions parametrized by \( \eta \):

\[
\tilde{\psi}_n(\alpha) := \begin{cases} 
1, & \alpha \leq c - \eta \\
1 - \frac{1}{n}(\alpha - c), & c - \eta \leq \alpha \leq c + \eta \\
0, & \alpha > c + \eta,
\end{cases}
\]

and

\[
\psi_n(\alpha) := \begin{cases} 
1, & \alpha \leq c - \eta \\
1 - \frac{1}{n}(\alpha - c), & c - \eta \leq \alpha \leq c + \eta \\
0, & \alpha > c + \eta.
\end{cases}
\]

These functions are both Lipschitz with Lipschitz constant \( 1/\eta \).

Define, the random variable \( Y_\eta \) as

\[
Y_\eta := \max \left\{ \int \tilde{\psi}_n d\mu_n - \int \tilde{\psi}_n d\mu, \int \psi_n d\mu_n - \int \psi_n d\mu \right\}.
\]

From the assumption of the lemma there is \( N(\epsilon, \zeta, \eta) \) such that for all \( n \geq N(\epsilon, \zeta, \eta) \):

\[
\mathbb{P}(Y_\eta > \epsilon/2) \leq \zeta.
\]

Moreover, \( \psi_n(\alpha) \leq \chi_c(\alpha) \leq \tilde{\psi}_n(\alpha) \). Thus,

\[
X \leq Y_\eta + \int |\tilde{\psi}_n - \psi_n| d\mu \leq Y_\eta + \mu\{\epsilon - \eta, c + \eta\}.
\]

where for the second inequality we further used the fact that \( |\tilde{\psi}_n - \psi_n| \) is upper bounded by \( 1 \) and has support \( \{\epsilon - \eta, c + \eta\} \).

Finally, from continuity of \( \mu \) and the fact that \( c \) is \( \mu \)-measure zero, we can choose \( \eta = \eta_1(\epsilon) \) such that

\[
\mu\{\epsilon - \eta, c + \eta\} \leq \epsilon/2.
\]

Combining, (58)–(60), we conclude, as desired, that there is \( N_* := N(\epsilon, \zeta, \eta_1(\epsilon)) \) such that for all \( n > N_* \), it holds

\[
\mathbb{P}(X > \epsilon) \leq \mathbb{P}(Y_\eta > \epsilon/2) \leq \zeta.
\]

3) Proof of Corollary IV.1: On the one hand, by Theorem IV.1(b), it holds for all \( \ell = 1, \ldots, k \) that

\[
\tilde{\psi}_n(\tilde{w}) := n^{-1} \sum_{i=1}^n \psi_i(\tilde{w}_i) \overset{P}{\to} \mathbb{E}_{W_i}[\psi_i(W_i)].
\]

On the other hand, for some constant \( C > 0 \)

\[
\prod_{\ell=1}^k \tilde{\psi}_n(\tilde{w}) - n^{-k} \sum_{1 \leq i_1, \ldots, i_k \leq n} \psi_1(\tilde{w}_{i_1}) \cdots \psi_k(\tilde{w}_{i_k}) \leq \frac{C}{n}.
\]

To see this, expand the product term on the left-hand side and use the boundedness of the functions \( \psi_i \).

Combining the above proves the first statement of the corollary. The second statement follows with the exact same argument starting from Theorem II.1 and observing that \( \mathbb{I}_{\{w_{i_1}, \ldots, w_{i_k} \leq -1\ldots, w_{i_k} \leq -1\}} = \prod_{\ell=1}^k \mathbb{I}_{\{w_{i_\ell} \leq -1\}} \).

4) Proof of Lemma IV.1: The proof of the lemma is almost identical to the proof of [39, Thm. 2.7], which is in turn based on the convexity lemma [36, Cor. II.1]. For completeness, we include the main arguments in the proof of [39, Thm. 2.7] translated to our setting. Also, for convenience, we suppress the dependence of \( F_n(\tau; g, h) \) and \( \tilde{r}_n(g, h) \) on \( g, h \) and simply write \( F_n(\tau) \) and \( \tilde{r}_n \), respectively.

In Section IV-D5 we have shown the following facts, which we will use throughout the proof: (i) \( \tilde{F}(\tau) \) is strictly convex and is uniquely minimized at \( \tau_* > 0 \); (ii) \( F_n(\tau) \) is convex; (iii) \( F_n(\tau) \overset{P}{\to} \tilde{F}(\tau) \) for all \( \tau > 0 \).

Fix \( \epsilon > 0 \) such that the set \( \mathcal{C} := [\tau_* - 2\epsilon, \tau_* + 2\epsilon] \) satisfies \( C \in (0, \infty) \). Let \( B \) denote the boundary of \( \mathcal{C} \). Pointwise convergence of convex functions on a dense subset of an open set implies uniform convergence on any compact subset of the open set [36, Cor. II.1]. Thus, from (ii) and (iii), it follows that \( F_n(\tau) \) converges to \( \tilde{F}(\tau) \) in probability uniformly on \( \mathcal{C} \). With this, it only takes a standard argument (e.g., see [39, Thm. 2.1]) to further conclude that:

\[
\min_{\tau \in \mathcal{C}} F_n(\tau) \overset{P}{\to} \min_{\tau \in \mathcal{C}} \tilde{F}(\tau),
\]

and that any \( \tilde{\tau}_n^C \) that minimizes \( F_n(\tau) \) in \( \mathcal{C} \) satisfies

\[
\tilde{\tau}_n^C \overset{P}{\to} \tau_*.
\]

Note (by (i)) that \( \min_{\tau \in \mathcal{C}} \tilde{F}(\tau) = \min_{\tau > 0} \tilde{F}(\tau) = \tilde{\varnothing} \). Thus, to complete the proof of (a), it suffices from (61) that \( \min_{\tau \in \mathcal{C}} F_n(\tau) = \min_{\tau > 0} F_n(\tau) \) w.p.a. 1. We further use convexity to show this below.

From (62), the event that \( \tilde{\tau}_n^C \in (\tau_* - \epsilon, \tau_* + \epsilon) \), so that \( F_n(\tilde{\tau}_n^C) < \max_{\tau \in B} F_n(\tau) \), occurs w.p.a. 1. In this event, for any \( \tau \) outside \( \mathcal{C} \), there is a linear convex combination \( \lambda \tau + (1 - \lambda)\tilde{\tau}_n^C \) that lies in \( B \) (with \( \lambda > 0 \)), so that

\[
F_n(\tilde{\tau}_n^C) < F_n(\lambda \tau + (1 - \lambda)\tilde{\tau}_n^C)
\]

\[
\leq \lambda F_n(\tau) + (1 - \lambda) F_n(\tilde{\tau}_n^C).
\]

Rearranging the above display, \( F_n(\tilde{\tau}_n^C) < F_n(\tau) \) in the entire interval \( (0, \infty) \), as desired. The same argument, combined with (62), shows that \( \tilde{\tau}_n^C \) is minimizing \( F_n(\tau) \) in the entire interval \( (0, \infty) \), as desired.
In what follows we show that \( n \cdot f(w) \) is \( C \)-strongly convex for appropriate constant \( C > 0 \). In view of (63) and recalling \( \phi(g, h) = f(w) \), this will suffice to complete the proof.

It can be checked that the Hessian \( \nabla^2 f(w) \) satisfies \( n \nabla^2 f(w) \geq \frac{|g|}{\sqrt{n}} \frac{\sigma^2}{\sqrt{\frac{\alpha}{\sigma}}} I \). Further use the fact that \( \|g\|_2/\sqrt{n} \geq \sqrt{\frac{\alpha}{\delta}} \) w.p.a.1 and \( \|g\|^2 \leq 4n \), to conclude that w.p.a.1 \( F \) is \( \frac{\sigma^2}{\sqrt{\frac{\alpha}{\sigma}}} \)-strongly convex with \( C := \frac{\sigma^2}{\sqrt{\frac{\alpha}{\sigma}}} \), or \( f(w) \geq f(w^*) + \frac{C}{2\|w-w^*\|} \).

C. Proof of Theorem III.1

The proof of the theorem requires repeating, mutatis mutandis, the line of arguments detailed in Section IV for the proof of Theorem II.1. We omit most of the details for brevity, and only show the necessary calculations that yield to function \( F_M \) in (14). The idea is the same as in Section IV: thanks to the CGMT, it suffices to analyze a corresponding Auxiliary Optimization (AO) instead of the original optimization in (12a). Repeating the steps in Section IV-D4, the corresponding (AO) becomes (compare to Eqn. (26)):

\[
\min_{\tau \geq 0} \frac{\tau \|g\|^2}{2n} + \frac{\sigma^2 \|g\|^2}{2\sqrt{n}} + \frac{1}{n} \sum_{i=1}^n \min_{x_i \in [n]} \left\{ \frac{\|g\|}{2\sqrt{n}} w_i^2 - h_i w_i \right\},
\]

where, as always \( w = x_0 - x \) denotes the "error-vector" and we further defined

\[
X_{0,i} := -(M-1) - x_{0,i}, \quad X_{0,i}^+ := (M-1) - x_{0,i}^-.
\]

For simplicity in notation, further denote \( A = \|g\|_2/\sqrt{n} \). Then, the optimal \( w_i := \hat{w}_i(g, h, x_0) \) satisfies

\[
\hat{w}_i = \begin{cases} x_{0,i}, & \text{if } h_i < A x_{0,i}^- \wedge 0, \\ \frac{1}{\alpha} h_i, & \text{if } A x_{0,i}^- \leq h_i \leq A x_{0,i}^+ \\ x_{0,i}^+, & \text{if } h_i > A x_{0,i}^+ \end{cases}
\]

(64),

where \( \tau := \tau(g, h, x_0) \) is the solution to the following:

\[
\min_{\tau > 0} \frac{\tau \|g\|^2}{2n} + \frac{\sigma^2 \|g\|^2}{2\sqrt{n}} + \frac{1}{n} \sum_{i=1}^n v_n(\frac{\tau}{\sqrt{n}} h_i, x_{0,i}^+, x_{0,i}^-),
\]

(65)

with

\[
v_n(\alpha; h, u, u) := \begin{cases} \frac{1}{2\alpha} \ell^2 - h \ell, & \text{if } \alpha h < \ell, \\ -\frac{1}{2} \ell^2, & \text{if } \ell \leq \alpha h \leq u, \\ \frac{1}{2\alpha} u^2 - hu, & \text{if } \alpha h > u. \end{cases}
\]

This is of course very similar to Equation (29). Next, we follow the same steps as in Section IV-D5 and study the convergence of the (AO) in (65). For the first two summands in (65), we use the fact that \( \|g\|_2/\sqrt{n} \rightarrow \sqrt{\frac{\alpha}{\delta}} \). For the third summand, recall that each \( x_{0,i} \) takes values \( \pm 1, \pm 3, \ldots, \pm (M-1) \) with equal probability \( 1/M \). Let \( j = 1, 3, \ldots, M-1 \) and denote,

\[
\ell_j := (M-1) - j \quad \text{and} \quad u_j := (M-1) + j.
\]

Then, the pairs \( (x_{0,i}^-, x_{0,i}^+) \) take values \( (-u_j, \ell_j) \) and \( (-\ell_j, u_j) \) with equal probability \( 1/M \) each. With these, \( \frac{1}{n} \sum_{i=1}^n v_n(\frac{\tau}{\sqrt{n}} h_i, x_{0,i}^+, x_{0,i}^-) \rightarrow 2 \left( 1 - \frac{1}{M} \right) Q(\tau_{\alpha}^{-1}) \).

The objective function in (68) can be identified with the function \( F_M(\tau) \) in the statement of the theorem. From Lemma A.1(b) the second derivative of \( F_M(\tau) \) is strictly positive for \( \tau > 0 \), hence (68) has a unique minimizer, which we denote \( \tau_1 \). With arguments same as in the end of Section IV-D5, we can show that \( \sqrt{\frac{\delta}{\alpha}}(g, h, x_0) \rightarrow \frac{\tau_1}{\sqrt{\delta}} \).

Finally, we sketch how all these leads to the desired, namely:

\[
\min_{\tau > 0} \frac{\tau \delta}{2} + \frac{\sigma^2}{2\tau} + Y(\tau).
\]

First, consider the case: \( x_0,i \in \{ \pm 1, \pm 3, \ldots, \pm (M-3) \} \). Then, the thresholding rule (12b) implies that there is an error iff \( |\hat{w}_i| > 1 \). Equivalently, in view of (64), and noting that \( x_{0,i}^+ \geq 2 \), it follows that and error occurs iff \( |h_i| > A \). Next, consider the case(s) \( x_0,i = M-1 \) or \( x_0,i = -(M-1) \). Then the error event corresponds to \( \hat{w}_i < -1 \) (or \( \hat{w}_i > 1 \)), which in view of (64) translates to \( h_i < -A \) (or \( h_i > A \)).
and conditioning on the high-probability events \( \|g\|/\sqrt{n} \overset{P}{\to} \sqrt{\delta} \)
and \( \overset{P}{\tau} \to \tau_* \), we find that
\[
\frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{\arg \min_{s \in C} |x_{i0, s}| + \mathbb{1}_{\{s \neq x_{i0, s}\}} \}} \overset{P}{\to} \frac{2}{M} \left( (M - 2)Q(\tau_*^{1}) + Q(\tau_*^{1}) \right) = 2 \left( 1 - \frac{1}{M} \right) Q(\tau_*^{1}).
\]

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