Sums of fourth powers of Fibonacci and Lucas numbers∗

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Abstract
We obtain closed-form expressions for all sums of the form \( \sum_{k=1}^{n} F_{mk}^4 \) and \( \sum_{k=1}^{n} L_{mk}^4 \) and their alternating versions, where \( F_i \) and \( L_i \) denote Fibonacci and Lucas numbers respectively. Our results complement those of Melham who studied the alternating sums.

1 Introduction

The Fibonacci numbers, \( F_n \), and Lucas numbers, \( L_n \), are defined, for \( n \in \mathbb{Z} \), as usual, through the recurrence relations \( F_n = F_{n-1} + F_{n-2}, \) \( F_0 = 0, \) \( F_1 = 1 \) and \( L_n = L_{n-1} + L_{n-2}, \) \( L_0 = 2, \) \( L_1 = 1 \), with \( F_n = (-1)^{n-1} F_n \) and \( L_n = (-1)^n L_n \).

About two decades ago, motivated by the results of Clary and Hemenway [1] who obtained factored closed-form expressions for sums of the form \( \sum_{k=1}^{n} F_{mk}^3 \), Melham [2] obtained factored closed-form expressions for alternating sums of the form \( \sum_{k=1}^{n} (-1)^{k-1} F_{mk}^4 \).

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Since no evaluations were reported in the Melham paper for the non alternating sums, we have attempted to fill that gap in this paper. Our main results are the following, valid for integers $m$ and $n$, with $m$ not equal to zero:

$$25 \sum_{k=1}^{n} F_{mk}^4 = \frac{F_{2mn+m}(L_{2mn+m} + 4(-1)^{mn-1}L_m)}{F_{2m}} + 6n + 3$$

and

$$\sum_{k=1}^{n} L_{mk}^4 = \frac{F_{2mn+m}(L_{2mn+m} + 4(-1)^{mn}L_m)}{F_{2m}} + 6n - 5.$$

We also re-derived the alternating sums, in slightly different but equivalent forms to the results contained in [2]:

$$\sum_{k=1}^{n} (-1)^{k-1} F_{mk}^4 = F_{mn}F_{mn+m} \left\{ (-1)^{n-1}L_{mn}L_{mn+m} + (-1)^{n(m-1)}4L_{2m} \right\} 5L_mL_{2m}$$

and

$$\sum_{k=1}^{n} (-1)^{k-1} L_{mk}^4 = (-1)^{n-1}5F_{mn}F_{mn+m} \left\{ L_mL_{mn}L_{mn+m} + (-1)^{nm}4L_{2m} \right\},$$

valid for all integers $m$ and $n$.

## 2 Required identities and preliminary results

### 2.1 Telescoping summation identities

The following telescoping summation identities are special cases of the more general identities proved in [3].

**Lemma 2.1.** If $f(k)$ is a real sequence and $m$ and $n$ are positive integers, then

$$\sum_{k=1}^{n} [f(mk + m) - f(mk)] = f(mn + m) - f(m).$$

**Lemma 2.2.** If $f(k)$ is a real sequence and $m$ and $n$ are positive integers, then

$$\sum_{k=1}^{n} (-1)^{k-1} [f(mk + m) + f(mk)]$$

$$= (-1)^{n-1} f(mn + m) + f(m).$$
2.2 First-order Lucas summation identities

Lemma 2.3. If \( m \) and \( n \) are integers, then

\[
F_m \sum_{k=1}^{n} (-1)^{mk-1} L_{2mk} = (-1)^{mn-1} F_{mn} L_{mn+m}.
\]

Proof. Setting \( v = m \) and \( u = 2mk \) in the identity

\[
F_{u+v} - (-1)^v F_{u-v} = F_v L_u,
\]

gives

\[
F_{2mk+m} - F_{2mk-m} = F_m L_{2mk}, \quad m \text{ even},
\]
and

\[
F_{2mk+m} + F_{2mk-m} = F_m L_{2mk}, \quad m \text{ odd}.
\]

Using identity (2.2) in Lemma 2.1 with \( f(k) = F_{2k-m} \), it is established that

\[
F_m \sum_{k=1}^{n} L_{2mk} = F_{m+2mn} - F_m
= F_{m+mn+mn} - F_{m+mn-mn}
= F_{mn} L_{mn+m}, \quad m \text{ even},
\]

on account of identity (2.1).

Similarly, using identity (2.3) in Lemma 2.2 with \( f(k) = F_{2k-m} \), we have

\[
F_m \sum_{k=1}^{n} (-1)^{k-1} L_{2mk} = (-1)^{n-1} F_{m+2mn} + F_m
= (-1)^{n-1} (F_{m+mn+mn} - (-1)^n F_{m+mn-mn})
= (-1)^{n-1} (F_{m+mn+mn} - (-1)^m F_{m+mn-mn}), \quad \text{since } m \text{ is odd}
= (-1)^{n-1} F_{mn} L_{mn+m}, \quad m \text{ odd}.
\]

Identities (2.4) and (2.5) combine to give Lemma (2.3). \( \square \)

Lemma 2.4. If \( m \) and \( n \) are integers, then

\[
L_m \sum_{k=1}^{n} (-1)^{k(m-1)} L_{2mk} = (-1)^{n(m-1)} L_{2mn+m} - L_m.
\]
Proof. Setting $v = m$ and $u = 2mk$ in the identity
\[ L_u + v + (-1)^v L_{u-v} = L_v L_u, \tag{2.6} \]
gives
\[ L_{2mk+m} - L_{2mk-m} = L_m L_{2mk}, \quad m \text{ odd}, \tag{2.7} \]
and
\[ L_{2mk+m} + L_{2mk-m} = L_m L_{2mk}, \quad m \text{ even}. \tag{2.8} \]
Using (2.7) in Lemma 2.1 with $f(k) = L_{2k-m}$, we have
\[ L_m \sum_{k=1}^{n} L_{2mk} = L_{m+2mn} - L_m, \quad m \text{ odd}. \tag{2.9} \]
Similarly, using (2.8) in Lemma 2.2 with $f(k) = L_{2k-m}$, we have
\[ L_m \sum_{k=1}^{n} (-1)^{k-1} L_{2mk} = (-1)^{n-1} L_{m+2mn} + L_m \quad m \text{ even}. \tag{2.10} \]
Identities (2.9) and (2.10) combine to give Lemma (2.4). \qed

3 Main results

3.1 Non alternating sums

Theorem 3.1. If $m$ is a non-zero integer and $n$ is any integer, then
\[ 25 \sum_{k=1}^{n} F_{mk}^4 = \frac{F_{2mn+m}(L_{2mn+m} + 4(-1)^{mn-1}L_m)}{F_{2m}} + 6n + 3. \]

Proof. By squaring the identity
\[ 5F_u^2 = L_{2u} - (-1)^u 2, \quad u \in \mathbb{Z}, \tag{3.1} \]
and making use of the identity
\[ L_v^2 = L_{2v} + (-1)^v 2, \quad v \in \mathbb{Z}, \tag{3.2} \]
and finally setting $u = mk$, it is established that
\[ 25F_{mk}^4 = L_{4mk} + (-1)^{mk-1} 4L_{2mk} + 6. \tag{3.3} \]
By summing both sides of identity (3.3), using Lemma 2.3 to sum each of the first two terms on the right hand side, we have

$$25 \sum_{k=1}^{n} F_{mk}^4 = \frac{(F_{2mn}L_{2mn+2m} + 4(-1)^{mn-1}L_mF_{mn}L_{mn+m})}{F_{2m}} + 6n \ , \quad (3.4)$$

Using the identity (2.1) we can write

$$F_{2mn}L_{2mn+2m} = F_{4mn+2m} - F_{2m} = F_{2mn+m}L_{2mn+m} - F_{2m} \quad (3.5)$$

and

$$L_mF_{mn}L_{mn+m} = L_m(F_{2mn+m} - (-1)^{mn}F_m) = L_mF_{2mn+m} + (-1)^{mn-1}F_{2m} \ . \quad (3.6)$$

Substituting (3.5) and (3.6) into (3.4) proves Theorem 3.1.

Corollary 3.2. If \( n \) is an integer, then

$$25 \sum_{k=1}^{n} F_{k}^4 = F_{2n+1}L_{n-1}L_{n+2} + 6n + 3 \ .$$

Proof. From Theorem 3.1 we have

$$25 \sum_{k=1}^{n} F_{k}^4 = F_{2n+1}(L_{2n+1} + 4(-1)^{n-1}) + 6n + 3 \ . \quad (3.7)$$

From identity (2.6) with \( u = n + 2 \) and \( v = n - 1 \) we have

$$L_{2n+1} + 4(-1)^{n-1} = L_{2n+1} + (-1)^{n-1}L_{3} = L_{n-1}L_{n+2} \ , \quad (3.8)$$

and the result follows.

Theorem 3.3. If \( m \) is a non-zero integer and \( n \) is any integer, then

$$\sum_{k=1}^{n} L_{mk}^4 = \frac{F_{2mn+m}(L_{2mn+m} + 4(-1)^{mn}L_m)}{F_{2m}} + 6n - 5 \ .$$
Proof. The theorem is proved by summing both sides of the following identity,
\[ L_{mk}^4 = L_{4mk} - (-1)^{mk-1}4L_{2mk} + 6, \]  
(3.9)
applying Lemma 2.3 to sum each of the first two terms on the right hand side. Identity (3.9) is obtained by squaring identity (3.2) and finally setting \( v = mk \).

Corollary 3.4. If \( n \) is an integer, then
\[
\sum_{k=1}^{n} L_k^4 = 5F_{2n+1}F_{n-1}F_{n+2} + 6n - 5.
\]

Proof. From Theorem 3.3 we have
\[
\sum_{k=1}^{n} L_k^4 = F_{2n+1}(L_{2n+1} - 4(-1)^{n-1}) + 6n - 5. \tag{3.10}
\]
From identity (3.15) with \( u = n + 2 \) and \( v = n - 1 \) we have
\[
L_{2n+1} - 4(-1)^{n-1} = L_{2n+1} - (-1)^{n-1}L_3 = 5F_{n-1}F_{n+2}, \tag{3.11}
\]
and the result follows. \( \square \)

3.2 Alternating sums

Theorem 3.5. If \( m \) and \( n \) are integers, then
\[
\sum_{k=1}^{n} (-1)^{k-1} F_{mk}^4 = \frac{F_{mn}F_{mn+m}\{(−1)^{n−1}L_mL_{mn}L_{mn+m}+(−1)^{n(m−1)}4L_{2m}\}}{5L_mL_{2m}}.
\]

Proof. Multiplying through identity (3.3) by \((-1)^{k-1}\) and summing over \( k \), we have the identity
\[
25\sum_{k=1}^{n} (-1)^{k-1} F_{mk}^4 = \sum_{k=1}^{n} (-1)^{k-1}L_{4mk}
+ 4\sum_{k=1}^{n} (-1)^{k(m-1)}L_{2mk} + 3((-1)^{n-1} + 1). \tag{3.12}
\]
When Lemma 2.4 is used to evaluate the sums on the right hand side we have

\[
25 \sum_{k=1}^{n} (-1)^{k-1} F_{mk}^4 = \frac{(-1)^{n-1} L_{4mn+2m} + L_{2m}}{L_{2m}}
\]

\[
+ 4 \left\{ (-1)^{n(m-1)} L_{2mn+m} - L_{m} \right\} \frac{L_{m}}{L_{2m}}
\]

\[
+ 3 \left\{ (-1)^{n-1} + 1 \right\} ,
\]

that is,

\[
25 \sum_{k=1}^{n} (-1)^{k-1} F_{mk}^4 = \frac{(-1)^{n-1} L_{4mn+2m} + 4(-1)^{n(m-1)} L_{2mn+m}}{L_{2m}} + 3(-1)^{n-1}
\]

\[
\frac{L_{2m}}{L_{2m}} \left\{ (-1)^{n-1} L_{4mn+2m} - L_{2m} \right\}
\]

\[
+ 4\left\{ 4(-1)^{n(m-1)} \left\{ L_{2mn+m} - (-1)^{mn} L_{m} \right\} \right\} \frac{L_{m}}{L_{2m}} ,
\]

(3.13)

Theorem 3.5 then follows when the identities

\[
L_{u+v} - (-1)^v L_{u-v} = 5 F_v F_u
\]

(3.15)

and

\[
F_{2u} = F_u L_u
\]

(3.16)

are used to write the right hand side of (3.14).

\[
\]

Corollary 3.6. If \( n \) is an integer, then

\[
\sum_{k=1}^{n} (-1)^{k-1} F_k^4 = \frac{(-1)^{n-1}}{3} F_n F_{n+1} F_{n-2} F_{n+3} .
\]

Proof. From Theorem 3.5

\[
\sum_{k=1}^{n} (-1)^{k-1} F_k^4 = \frac{F_n F_{n+1}((-1)^{n-1} L_n L_{n+1} + L_2 L_3)}{15} .
\]

(3.17)

From identity (3.15)

\[
L_n L_{n+1} = L_{2n+1} - (-1)^{n-1} , \quad L_2 L_3 = L_5 + 1 .
\]

(3.18)
We therefore have
\[
\sum_{k=1}^{n} (-1)^{k-1} F_k^4 = \frac{(-1)^{n-1} F_n F_{n+1}(L_{2n+1} + (-1)^{n-1} L_5)}{15} - \frac{(-1)^{n-1} F_n F_{n+1}(L_{2n+1} - (-1)^{n-2} L_5)}{15} = \frac{(-1)^{n-1}}{3} F_n F_{n+1} F_{n-2} F_{n+3}, \quad \text{by identity (3.15).}
\]

\[\Box\]

**Theorem 3.7.** If \(m\) and \(n\) are integers, then
\[
\sum_{k=(1+(-1)^n)/2}^{n} (-1)^{k-1} L_{mk}^4 = \frac{(-1)^{n-1} 5 F_{mn} F_{mn+m} \{L_m L_{mn} L_{mn+m} + (-1)^{nm} 4 L_{2m}\}}{L_m L_{2m}}.
\]

**Proof.** Multiplying through identity (3.9) by \((-1)^{k-1}\) and summing over \(k\), we have the identity
\[
\sum_{k=1}^{n} (-1)^{k-1} L_{mk}^4 = \sum_{k=1}^{n} (-1)^{k-1} L_{4mk} - 4 \sum_{k=1}^{n} (-1)^{k(m-1)} L_{2mk} + 3((-1)^{n-1} + 1),
\]
which by the use of Lemma 2.4 gives
\[
\sum_{k=1}^{n} (-1)^{k-1} L_{mk}^4 = \frac{(-1)^{n-1} L_{4mn+2m} - 4(-1)^{n(m-1)} L_{2mn+m} + 3(-1)^{n-1} + 8}{L_m L_{2m}}.
\]
so that if \(n\) is even we have
\[
\sum_{k=1}^{n} (-1)^{k-1} L_{mk}^4 = -\frac{(L_{4mn+2m} - L_{2m})}{L_{2m}} - \frac{4(L_{2mn+m} - L_m)}{L_m},
\]
while if \(n\) is odd we have
\[
\sum_{k=1}^{n} (-1)^{k-1} L_{mk}^4 = -\frac{(L_{4mn+2m} - L_{2m})}{L_{2m}} - \frac{4(-1)^{m-1}(L_{2mn+m} - (-1)^m L_m)}{L_m} + 16.
\]
that is,

\[
\sum_{k=0}^{n} (-1)^{k-1} L_{mk}^4 = \frac{L_{4mn+2m} - L_{2m}}{L_{2m}} - 4(-1)^{m-1}(L_{2mn+m} - (-1)^m L_m) \tag{3.21}
\]

Using identities (3.15) and (3.16) to write the right side of identities (3.20) and (3.21) and combining the results we obtain the statement of Theorem 3.7.

\[\square\]

**Corollary 3.8.** If \(n\) is an integer, then

\[
\sum_{k=(1+(-1)^n)/2}^{n} (-1)^{k-1} L_k^4 = (-1)^{n-\frac{1}{3}} \frac{5}{3} F_n F_{n+1} (L_{n+2} L_{n+3} + (-1)^{n+2}).
\]

**References**

[1] S. CLARY and P. D. HEMENWAY (1993), On sums of cubes of Fibonacci numbers, *in Applications of Fibonacci Numbers, Kluwer Academic Publishers, Dordrecht, The Netherlands* 5:123–136.

[2] R. S. MELHAM (2000), Alternating sums of fourth powers of Fibonacci and Lucas numbers, *The Fibonacci Quarterly* 38 (3):254–259.

[3] K. ADEGOKE (2017), Generalizations for reciprocal Fibonacci-Lucas sums of Brousseau, [arXiv:1702.08321](https://arxiv.org/abs/1702.08321)