SUBMANIFOLD ALGEBRAS

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ABSTRACT. We review the notion of submanifold algebra, as introduced by T. Masson, and discuss some properties and examples. A submanifold algebra of an associative algebra $A$ is a quotient algebra $B$ such that all derivations of $B$ can be lifted to $A$. We will argue that in the case of smooth functions on manifolds every quotient algebra is a submanifold algebra, derive a topological obstruction when the algebras are deformation quantizations of symplectic manifolds, present some (commutative and noncommutative) examples and counterexamples.

1. INTRODUCTION

Motivated by the derivation-based differential calculus of M. Dubois-Violette and P.W. Michor [14, 15] (reviewed e.g. in [16, 17]), and inspired by the example of closed embedded submanifold of a smooth manifold, T. Masson introduced in [28] a notion of (noncommutative) submanifold algebra. The starting point is a short exact sequence of associative algebras (over a field $K$):

$$0 \to J \to A \xrightarrow{\pi} B \to 0 .$$

(1)

The sequence (1) induces an exact sequence of complexes

$$0 \to \text{Hom}(A^\otimes n, J) \to \text{Hom}_\pi(A^\otimes n, A) \to \text{Hom}(B^\otimes n, B)$$

(2)

where the first one is the Hochschild complex of $A$ with coefficients in the $A$-bimodule $J$, the last one is the Hochschild complex of $B$, and the one in the middle is the sub-complex of the Hochschild complex of $A$ given by $K$-linear maps $A^\otimes n \to A$ with image in $J$ if one of the arguments is in $J$.

Last arrow in (2) is, in general, not surjective. It is for $n = 0$, since (2) reduces to (1), and for $n = 1$ if restricted to Hochschild coboundaries, i.e. inner derivations. The restriction to Hochschild 1-cocycles, i.e. derivations, already gives a map

$$\pi_* : \text{Der}_\pi(A) \to \text{Der}(B)$$

(3)

that may not be surjective (cf. Examples 7, 12 and 16).
If $M$ is a smooth manifold and $S \subset M$ a closed embedded smooth submanifold, it is well known that the pullback of the inclusion $\iota : S \to M$ is surjective, and one has an exact sequence

$$0 \to J \to C^\infty(M) \xrightarrow{\iota^*} C^\infty(S) \to 0$$

like in (1), with $J$ the ideal of smooth functions on $M$ that are zero on $S$. The induced map (3) on vector fields (derivations) is surjective as well [27].

With this example in mind, whenever we have a sequence (1) such that (3) is surjective, we will call $B$ a submanifold algebra of $A$ [28] (see [2] for another possible approach).

This paper is mainly an attempt to understand what is the meaning of submanifold algebras from the point of view of noncommutative geometry. The plan is the following. In §2 we fix the notations and provide some algebraic background. In §3 we investigate the case of smooth manifolds (and closed subsets), exhibit examples of commutative submanifold algebras that are not algebras of functions on smooth manifolds, examples of quotient algebras that are not submanifold algebras, and argue that when both $A$ and $B$ in (1) are algebras of smooth functions the condition on $\pi_*$ is in fact redundant (Theorem 8). In §4 we discuss some examples of submanifold algebras of noncommutative algebras, including “almost commutative” spaces. In §5 we study deformation quantizations of Poisson manifolds and discuss some topological obstructions for a quotient algebra to be a submanifold algebra (Lemma 27). In the case of a star product on a symplectic manifold $M$ that is tangential to a closed symplectic submanifold $S$, the quotient map

$$A := (C^\infty(M)[[\hbar]], \star) \to B := (C^\infty(S)[[\hbar]], \star)$$

induces a linear map $H^1(M, \mathbb{R}) \to H^1(S, \mathbb{R})$ that is surjective when $B$ is a submanifold algebra of $A$. In §6 we study some general properties of submanifold algebras and elaborate on the categorical notion of (co)embedding.

2. Preliminaries and notations

2.1. Notations. Throughout the following, and unless stated otherwise, $A$ will denote an associative algebra over a field $K$, $M$ a smooth manifolds (without boundary), $C^\infty(M)$ the algebra of real-valued smooth function on
$M$, $C_0(M)$ the $C^*$-algebra of complex-valued continuous functions vanishing at infinity. For $S \subset M$ we define
\[ I(S) := \{ f \in C_0(M) : f|_S = 0 \} \]
and call both $I(S)$ and $I(S) \cap C^\infty(M)$ the vanishing ideal of $S$ (it will be clear from the context which one we are referring to). Recall that a derivation $D$ of an algebra $A$ is a $K$-linear map $A \to A$ satisfying the Leibniz rule
\[ D(ab) = D(a)b + aD(b), \quad a, b \in A. \tag{4} \]
The set of all derivations of $A$ will be denoted by $\text{Der}(A)$. A derivation is inner if it is of the form $a \mapsto [x, a]$ for some fixed $x \in A$. The set of all inner derivations will be denoted by $\text{Inn}(A)$.

### 2.2. Submanifold algebras.

Let $\pi : A \to B$ be a surjective homomorphism of associative algebras over a field $K$ and $J := \ker \pi$. Let
\[ \text{Der}_\pi(A) := \{ D \in \text{Der}(A) : D(a) \in J \forall a \in J \}. \]

**Lemma 1.** For every $D \in \text{Der}_\pi(A)$ there is a unique derivation $\tilde{D} \in \text{Der}(B)$ such that $\tilde{D} \circ \pi = \pi \circ D$.

**Proof.** Let $a, b \in A$ with $a - b \in J$. Then $D(a) - D(b) \in J$ and $\pi(D(a)) \in B$ depends only on the class $x := \pi(a)$ of $a$. We define $\tilde{D}(x) := \pi(D(a))$. Now if $x = \pi(a)$ and $y = \pi(b)$, then $xy = \pi(ab)$, and
\[ \tilde{D}(xy) = \pi(D(ab)) = \pi(D(a)b + aD(b)) = \tilde{D}(x)y + x\tilde{D}(y). \]
Similarly one proves that $\tilde{D}$ is linear. $\blacksquare$

The assignment $D \mapsto \tilde{D}$ gives a Lie algebra map:
\[ \pi_* : \text{Der}_\pi(A) \to \text{Der}(B). \]

**Definition 2.** If $\pi_*$ is surjective, we will call $\pi : A \to B$ a coembedding and $B$ a submanifold algebra of $A$.

Surjectivity of (3) guarantees, among other things, that there is a surjective homomorphism $\Omega_{\text{Der}}(A) \to \Omega_{\text{Der}}(B)$ of “minimal” derivation based differential calculi [28, Prop. IV.1].

It is straightforward to check that, since $J$ is a two-sided ideal, $\text{Inn}(A)$ is a Lie subalgebra of $\text{Der}_\pi(A)$ and $\pi_*$ maps $\text{Inn}(A)$ surjectively to $\text{Inn}(B)$.

\footnote{Since in the motivating example of smooth manifolds it is somehow dual to an embedding.}
Remark 3. If all derivations of $B$ are inner, then $B$ is a submanifold algebra of $A$.

In the next section we will study the case of commutative algebras, where no non-zero derivation is inner.

3. Function algebras

3.1. Submanifold algebras of commutative $C^*$-algebras. Let us start with topological spaces. Suppose that (1) is an exact sequence of commutative complex $C^*$-algebras. It is well known that, by Gelfand duality, $A \simeq C_0(M)$ and $B \simeq C_0(N)$ where $M$ and $N$ are locally compact Hausdorff spaces. Every $*$-homomorphism $\pi : C_0(M) \to C_0(N)$ is the pullback of a proper continuous map $F : N \to M$, and it is surjective only if $F$ is injective: it is then a topological embedding of $N$ as a closed subset of $M$. Up to isomorphisms, every exact sequence (1) of commutative $C^*$-algebras is then of the form

$$0 \to I(S) \to C_0(M) \xrightarrow{\pi} C_0(S) \to 0$$

with $S \subseteq M$ a closed topological subspace and $\pi(f) := f|_S \forall f \in C_0(M)$ (and every inclusion $S \hookrightarrow M$ of a closed subset gives rise to such a short exact sequence). We refer to Chapter 1 of [22] for the details.

Since commutative $C^*$-algebras have no non-zero derivations, the associated map $\pi_*$ is surjective as well.

3.2. Submanifold algebras of $C^\infty$-algebras. The crucial point in the discussion in previous section is that the functor $C_0$ is an equivalence between the category of locally compact Hausdorff topological spaces, with proper continuous functions, and the (opposite) category of commutative $C^*$-algebras with $*$-homomorphisms (see e.g. [5, II.2.2.7]).

Things with smooth manifolds become more involved. We can associate to every smooth manifold (without boundary) $M$ the algebra $C^\infty(M)$, and to any smooth map $F : M \to N$ the homomorphism $F^* : C^\infty(N) \to C^\infty(M)$. This gives a functor from the category of smooth manifolds to the category of commutative (real unital associative) algebras, which obviously fails to be surjective on objects. Nevertheless it is a full and faithful functor (see [26, Cor. 35.10] or [31, Theorem 2.8]), and this is enough for our purposes.

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2In fact, a stronger statement is due to Kadison [25, Thm. 2]: each derivation of a (possibly non-commutative) $C^*$-algebra annihilates its center.

3We could, however, consider the category of $C^\omega$-rings, for which there are characterizations of those objects that are isomorphic to $C^\infty(M)$ for some $M$. See e.g. [30].
Let us consider then a short exact sequence
\[ 0 \rightarrow J \rightarrow C^\infty(M) \xrightarrow{\pi} B \rightarrow 0, \] (5)
where we know a priori that the one in the middle is the algebra of smooth functions on some smooth manifold \( M \). We will assume that \( \dim(M) \geq 1 \) and use the standard identification of \( \text{Der}(C^\infty(M)) \) with the set \( \mathfrak{X}(M) \) of smooth global vector fields on \( M \).

There are several natural questions we may ask:

(i) Is there a commutative example where \( \pi \) is surjective but \( \pi_* \) is not? (Are these two conditions independent from each other or is every quotient algebra of \( C^\infty(M) \) a submanifold algebra?)

(ii) Is there an example where \( \pi \) and \( \pi_* \) are both surjective (\( B \) is a submanifold algebra of \( C^\infty(M) \)), but \( B \not\cong C^\infty(N) \) for any smooth manifold \( N \)?

(iii) If we known that \( J \) is the vanishing ideal of a subset \( S \subset M \) and \( B \cong C^\infty(M)/J \) a submanifold algebra, can we conclude that \( S \) is a submanifold of \( M \)?

(iv) What can we conclude under the assumption that \( B = C^\infty(N) \)?

The first two examples give a positive answer to question (ii).

**Example 4** (Dual numbers). Let \( M = \mathbb{R} \) and \( B \) be the subalgebra of \( M_2(\mathbb{R}) \) spanned by the identity matrix and the matrix
\[ \varepsilon := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \]
Up to an isomorphism, \( B \cong \mathbb{R}[\varepsilon]/(\varepsilon^2) \) is the algebra of dual numbers. Let \( \pi : C^\infty(\mathbb{R}) \rightarrow B \) be the homomorphism (the identity matrix is omitted):
\[ \pi(f) := f(0) + f'(0)\varepsilon \quad \forall f \in C^\infty(\mathbb{R}). \]
It is not difficult to check that both \( \pi \) and \( \pi_* \) are surjective. One has
\[ \text{Der}(C^\infty(\mathbb{R})) = \{ f \mapsto \phi f' : \phi \in C^\infty(\mathbb{R}) \}, \]
\[ \text{Der}_{\pi}(C^\infty(\mathbb{R})) = \{ f \mapsto \phi f' : \phi \in C^\infty(\mathbb{R}), \phi(0) = 0 \}, \]
\[ \text{Der}(B) = \{ \lambda \varepsilon \frac{d}{d\varepsilon} : \lambda \in \mathbb{R} \}. \]
If \( Df := \phi f' \) with \( \phi(0) = 0 \), then \( \pi_*(D) = \phi'(0)\varepsilon \frac{d}{d\varepsilon} \). Since \( \phi'(0) \) can be any real number, the map \( \pi_* \) is surjective.

We saw in previous example that: the algebra of dual numbers is a submanifold algebra of \( C^\infty(\mathbb{R}) \); it is not isomorphic to any algebra of smooth
functions on a manifold, since it has a non-zero nilpotent element \( \varepsilon \); the kernel of \( \pi \) is not the vanishing ideal of any subset of \( \mathbb{R} \), even if \( \text{Der}_\pi(C^\infty(\mathbb{R})) \) is the set of vector fields on \( \mathbb{R} \) vanishing on the subset \( S = \{0\} \).

More generally, for arbitrary \( M \), given a point \( p \in M \) and a non-zero tangent vector \( v \in T_pM \), we can construct a surjective homomorphism
\[
\pi : C^\infty(M) \to \mathbb{R}[\varepsilon]/(\varepsilon^2), \quad \pi(f) := f(p) + v(f)\varepsilon,
\]
and prove (using local coordinates) that \( \pi_* \) is surjective as well.

**Example 5** (Germs of smooth functions). Fix \( p \in M \), let \( B := C^\infty_p(M) \) be the algebra of all germs of smooth functions at \( p \), and let
\[
\pi : C^\infty(M) \to C^\infty_p(M)
\]
be the map sending a function into its germ at \( p \). Such a map is surjective (every germ can be represented by a global smooth function). Its kernel \( J \) is the set of all functions that are zero near \( p \); since derivations are local, \( X(J) \subset J \) for every \( X \in \text{Der}(C^\infty(M)) \) and \( \text{Der}_\pi(C^\infty(M)) \equiv \text{Der}(C^\infty(M)) \). The map
\[
\pi_* : \mathfrak{x}(M) \to T_pM
\]
is just the evaluation of vector fields at \( p \). It is well known that such a map is surjective [27, Prop 8.7]. Stalks of germs of smooth functions on \( M \) are then examples of submanifold algebras of \( C^\infty(M) \) not isomorphic to \( C^\infty(N) \) for any smooth manifold \( N \).\(^4\)

We want now to answer to question (iii). If \( S \subset M \) is a closed subset of a smooth manifold \( M \), we denote by \( C^\infty(S)^\circ \) the algebra of real-valued functions on \( S \) that admit a smooth prolongation to an open neighbourhood of \( S \) in \( M \) (and then to the whole \( M \), cf. [27, Lemma 2.26]). If \( S \) is an embedded submanifold, then these are just smooth functions with respect to the smooth structure of \( S \). (However, it should be noted that if \( S \) is an immersed submanifold this is not true.)

**Theorem 6** (Closed subsets). \( C^\infty(S)^\circ \) is a submanifold algebra of \( C^\infty(M) \).

**Proof.** By construction, elements of \( C^\infty(S)^\circ \) are functions on \( S \) that are in the image of the restriction map \( \pi : C^\infty(M) \to C^\infty(S)^\circ \), \( \pi(f) := f|_S \). We need to prove surjectivity of \( \pi_* \).

\(^4\)By contraddiction, assume that \( C^\infty_p(M) \cong C^\infty(N) \) for some smooth manifold \( N \). Since \( \mathfrak{x}(N) \cong T_pM \) is finite-dimensional, \( N \) must be 0-dimensional. But then \( \mathfrak{x}(N) \) must be \( \{0\} \), that implies \( \dim(M) = \dim(T_pM) = 0 \).
Any derivation of $C^\infty(S)^\circ$ composed with the evaluation at a point $p \in S$ gives a derivation of $C^\infty(M)$ at $p$, i.e. a vector $X_p \in T_p M$. We need to prove that the map $X : S \to TM, p \mapsto X_p$, is the restriction to $S$ of a global smooth vector field on $M$.

Let $p \in S$, let $(U, \varphi = (x^1, \ldots, x^n))$ be a chart on $M$ centered at $p$ and $B$ a coordinate ball with $p \in B$ and $\overline{B} \subset U$. For all $q \in U \cap S$ we can write

$$X_q = \sum_{i=1}^n v^i(q) \frac{\partial}{\partial x^i}|_{q}$$

for some (unique) $v^i(q) \in \mathbb{R}$. Choose smooth functions $\tilde{x}^i \in C^\infty(M)$ such that $\tilde{x}^i$ coincide with $x^i$ on $\overline{B}$ [27, Lemma 2.26]. For every $i = 1, \ldots, n$, since $X(\tilde{x}^i|_S) \in C^\infty(S)^\circ$ and $\pi$ is surjective, we can choose $f^i \in C^\infty(M)$ such that

$$X(\tilde{x}^i|_S) = \pi(f^i).$$

For all $q \in \overline{B} \cap S$, one has $f^i(q) = X_q(\tilde{x}^i) = X_q(x^i) = v^i(q)$. As a consequence, the smooth vector field $Y \in \mathfrak{X}(U)$ given by

$$Y := f^i \frac{\partial}{\partial x^i}$$

satisfies $Y|_B = X|_B$. The local vector field $X$ is locally the restriction of a local smooth vector field on $M$ (it is a smooth vector field along $S$ according to the terminology of [27]). It follows from [27, Lemma 8.6] that $X$ is the restriction to $S$ of a global smooth vector field on $M$. ■

Thus, for any closed subset $S$ of a smooth manifold $M$, $C^\infty(S)^\circ$ is a submanifold algebra of $C^\infty(M)$ (with $\pi$ the pullback of the inclusion $S \hookrightarrow M$ and $J$ the vanishing ideal of $S$).

We can now answer to question (i) and give a commutative example of quotient algebra that is not a submanifold algebra.

**Example 7** (The cross). Let $S := \{(x, y) \in \mathbb{R}^2 : x y = 0\}$, let $\iota : \mathbb{R} \to S$ be the map $x \mapsto (x, 0)$, and $\pi = \iota^* : C^\infty(S)^\circ \to C^\infty(\mathbb{R})$ its pullback. It follows from Theorem 6 that a derivation $X$ of $C^\infty(S)^\circ$ is a smooth vector field along $S$ of the form

$$X_p = \sum_{i=1}^2 v^i(p) \frac{\partial}{\partial x^i}|_p .$$

Let $f(x, y) := x y$. The condition that $X_p(f) = 0 \ \forall \ p \in S$ implies that $v^1(0, y) = 0 \ \forall \ y \neq 0$ and $v^2(x, 0) = 0 \ \forall \ x \neq 0$. By continuity, $X_{p=0} = 0$. The image $\pi_* (X)$ is a vector field on $\mathbb{R}$ vanishing at $0$. The vector field $\partial / \partial x^1$ is not in the image of $\pi_*$, which is then not surjective. ◊
Consider now an exact sequence (5), with $B = \mathcal{C}^\infty(N)$ the algebra of smooth function on a second smooth manifold $N$ (question (iv)). The following theorem holds:

**Theorem 8.** Let $M$ and $N$ be smooth manifolds and $\pi : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(N)$ a surjective homomorphism. Then $\pi = F^*$ is the pullback of a proper embedding $F : N \to M$ and $S := F(N)$ is a closed embedded submanifold of $M$.

**Proof.** By [26, Cor. 35.10], every homomorphism $\pi : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(N)$ is the pullback of a smooth map $F : N \to M$. Assuming that $\pi = F^*$ is surjective, we will prove first that $F$ is injective, then that it is an immersion, and finally that it is a proper embedding.

**$F$ is injective:** by contradiction, suppose that $F$ is not injective: i.e. $F(p) = F(q)$ for some $p \neq q$ in $N$. Then, every function $f \in \mathcal{C}^\infty(N)$ in the image of $F^*$ has $f(p) = f(q)$. Since (for every $p \neq q$) there exists a smooth function with $f(p) \neq f(q)$ (closed disjoint subsets of a smooth manifold can be separated by a smooth function), $F^*$ is not surjective.

**$F$ is an immersion:** under the usual identification $T_p \mathbb{R} \cong \mathbb{R}$ and $T_p f = df_p$ for a scalar function, for all $f \in \mathcal{C}^\infty(N)$ and all $p \in N$ one has

$$df_p = dg_{F(p)} \circ T_p F,$$

where $g \in \mathcal{C}^\infty(M)$ is any function satisfying $F^*(g) = f$. It follows from (6) that covectors $df_p \in T^*_p N$ vanish on $\ker T_p F$ for all $f$, which implies that $\ker T_p F = \{0\}$ and $F$ is an immersion at $p$.

**$F$ is a proper embedding:** we proved that $S := F(N)$ is an immersed submanifold of $M$. It is well known that the restriction map $\mathcal{C}^\infty(M) \to \mathcal{C}^\infty(S)$, $f \mapsto f|_S$, to a submanifold is surjective if and only if $S$ is properly embedded (see e.g. [27, Ex. 5-18(b)]), and in particular closed in $M$. $\blacksquare$

An immediate corollary of previous theorem is that, for smooth function on manifolds, the condition of surjectivity of $\pi_*$ is redundant: if $\mathcal{C}^\infty(N)$ is a quotient algebra of $\mathcal{C}^\infty(M)$, then it is also a submanifold algebra.

### 3.3. Polynomial algebras.

For commutative algebras an equivalent formulation of the condition of submanifold algebra is via Kähler differentials. Let $A$ be a commutative algebra over a field $K$, $M$ an $A$-module and $(\Omega_{A/K}, d)$ the module of Kähler differentials. Then, the map

$$\Hom_A(\Omega_{A/K}, M) \xrightarrow{\sim} \text{Der}_K(A, M)$$

$$f \mapsto f \circ d$$
sending \(\Lambda\)-linear maps \(\Omega_{\Lambda/K} \to M\) to \(M\)-valued derivations of \(\Lambda\) (\(K\)-linear maps \(\Lambda \to M\) satisfying (4)) is a bijection [18, Cap. 16]. The universal derivation will be denoted always by \(d\), whatever is the algebra considered.

If \(\pi : \Lambda \to B\) is a homomorphism of commutative algebras, since \(d \circ \pi : \Lambda \to \Omega_{B/K}\) is a derivation, by the universal property of Kähler differentials there exists \(\pi_{**} \in \text{Hom}_\Lambda(\Omega_{\Lambda/K}, \Omega_{B/K})\) (where we think of \(B\)-modules as \(A\)-modules via \(\pi\)) that makes the following diagram commute:

\[
\begin{array}{ccc}
A & \xrightarrow{d} & \Omega_{\Lambda/K} \\
\pi \downarrow & & \downarrow \pi_{**} \\
B & \xrightarrow{d} & \Omega_{B/K}
\end{array}
\]

Explicitly \(\pi_{**}(adb) = \pi(a)d\pi(b) \quad \forall \ a, b \in \Lambda.\) Note that if \(\pi\) is surjective, \(\pi_{**}\) is surjective as well. We can now rephrase the definition of submanifold algebra in terms of Kähler differential.

**Proposition 9.** Let \(\pi : \Lambda \to B\) be a surjective homomorphism of commutative algebras. Then \(B\) is a submanifold algebra of \(\Lambda\) if and only if for every \(f \in \text{Hom}_B(\Omega_{B/K}, B)\) there exists \(\tilde{f} \in \text{Hom}_\Lambda(\Omega_{\Lambda/K}, \Lambda)\) that makes the following diagram commute:

\[
\begin{array}{ccc}
\Omega_{\Lambda/K} & \xrightarrow{\tilde{f}} & A \\
\downarrow \pi_{**} & & \downarrow \pi \\
\Omega_{B/K} & \xrightarrow{f} & B
\end{array}
\]

**Proof.** The situation is illustrated in the following diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{d} & \Omega_{\Lambda/K} & \xrightarrow{\tilde{f}} & A \\
\pi \downarrow & & \downarrow \pi_{**} & \downarrow \pi \\
B & \xrightarrow{d} & \Omega_{B/K} & \xrightarrow{f} & B
\end{array}
\]

Suppose every \(D \in \text{Der}(B)\) admits a lift \(\tilde{D}\) making the outer diagram commutative (automatically \(\tilde{D}(J) \subset J\) and \(\tilde{D} \in \text{Der}_\pi(\Lambda)\)). Then for every \(f \in \text{Hom}_B(\Omega_{B/K}, B)\) we can lift the derivation \(D = f \circ d\) to a derivation...
of \( A \). By universality \( \tilde{D} = \tilde{f} \circ d \) for some \( \tilde{f} \in \text{Hom}_A(\Omega_{A/K}, A) \), and \( \tilde{f} \) is the lift of \( f \) we are looking for.

Vice versa, if every \( f \) admits a lift \( \tilde{f} \), given any \( D \in \text{Der}(B) \) and given \( f \) such that \( f \circ d = D \), we can lift \( f \) to \( \tilde{f} \) and form the derivation \( \tilde{D} = \tilde{f} \circ d \). Automatically \( \pi_* (\tilde{D}) = D \), so that \( \pi_* \) is surjective.

**Proposition 10.** If \( A \) is commutative and \( \Omega_{A/K} \) is a free \( A \)-module, then every quotient algebra of \( A \) is a submanifold algebra.

**Proof.** It follows from Prop. 9. Let \( S \) be a free generating set for \( \Omega_{A/K} \). Then, every Kähler differential can be written in a unique way as a finite sum \( \sum_{\xi \in S} a_{\xi} \cdot \xi \) with \( a_{\xi} \in A \). Let \( f \in \text{Hom}_B(\Omega_{B/K}, B) \). For every \( \xi \in S \) there exists (by surjectivity of \( \pi \)) an element \( \tilde{\xi} \in A \) such that \( \pi(\tilde{\xi}) = f \circ \pi_*(\xi) \). Let \( \tilde{f} \) be defined by

\[
\tilde{f}(\sum_{\xi \in S} a_{\xi} \cdot \xi) := \sum_{\xi \in S} a_{\xi} \cdot \tilde{\xi}.
\]

By construction \( \tilde{f} \in \text{Hom}_A(\Omega_{A/K}, A) \) and \( \pi \circ \tilde{f} = f \circ \pi_* \) as requested. \( \blacksquare \)

**Example 11** (Affine sets). Let \( A := K[x_1, \ldots, x_n] \), \( S \subset K^n \) an affine algebraic set, \( J \) the radical ideal of \( S \), \( B := A/J \) its coordinate ring and \( \pi : A \to B \) the quotient map. Since \( \Omega_{A/K} \) is a free \( A \)-module [18, Prop. 16.1], \( B \) is a submanifold algebra of \( A \).

Similarly to the case of smooth functions, the coordinate algebra \( B \) of \( S \) in previous example is a submanifold algebra even when \( S \) is not a smooth manifold. In the case of affine varieties there is however a simple criteria to know if \( S \) is smooth by looking at derivations: if \( K \) is algebraically closed and \( S \) is irreducible, then \( S \) is a smooth submanifold of \( K^n \) if and only if the Lie algebra \( \text{Der}(B) \) is simple (see e.g. \[4\]).

We close this section with an example of quotient of a polynomial algebra that is not a submanifold algebra (the polynomial version of Example 7).

**Example 12** (The ordinary double point). Let \( A := K[x, y]/(xy) \), \( B := K[x] \) and \( \pi \) the surjective homomorphism defined by \( \pi(y) = 0 \). A simple computation shows that: \( \text{Der}(A) \) is generated by the two derivations \( x \frac{\partial}{\partial x} \) and \( y \frac{\partial}{\partial y} \); \( \text{Der}_\pi(A) = \text{Der}(A) \); the image of \( \pi_* \) is freely generated (as a \( B \)-module) by \( x \frac{\partial}{\partial x} \); \( \text{Der}(B) \) is freely generated by \( \frac{\partial}{\partial x} \). The derivation \( \frac{\partial}{\partial x} \) is not in the image of \( \pi_* \), that is then not surjective. \( \blacksquare \)
4. NONCOMMUTATIVE SUBMANIFOLDS: EXAMPLES

In parallel with previous section, let’s start with (separable complex) C*-algebras. In this case, it is well known that any derivation of a quotient algebra can be lifted [33]. Thus:

**Example 13** (C*-algebras). If π : A → B is a surjective morphism between separable complex C*-algebras, then π* is surjective.

In §3 we studied examples with no non-zero inner derivations. On the other side of the spectrum, any quotient algebra with only inner derivations is a submanifold algebra (Remark 3). This includes the universal enveloping algebra of a semisimple finite-dimensional Lie algebra (over a field K of characteristic 0), any central simple algebra (as a corollary of Skolem-Noether theorem, cf. [13, Pag. 80 Ex. 4c]), Weyl algebras [12, Lemma 4.6.8], all von Neumann algebras (Sakai-Kadison theorem [25, 34]) and in particular finite-dimensional complex C*-algebras (since they coincide with their weak closure). It is not difficult to extend the latter result to the real case. Let us record the result for future use.

**Lemma 14.** Let B be a finite-dimensional (real or complex) C*-algebra. Then all derivations of B are inner.

*Proof.* We need to prove the statement in the real case. Assume then that B is real, so that $B_\mathbb{C} = B + iB$ is a complex finite-dimensional C*-algebra. Every $\mathbb{R}$-linear endomorphism $\phi$ of the vector space underlying $B$ can be extended to a $\mathbb{C}$-linear endomorphism $\phi_\mathbb{C}$ of $B_\mathbb{C}$ by

$$\phi_\mathbb{C}(a + ib) = \phi(a) + i\phi(b), \quad \forall \ a, b \in B.$$ 

It is straightforward to check that if $\phi \in \text{Der}(B)$ then $\phi_\mathbb{C} \in \text{Der}(B_\mathbb{C})$. Since $B_\mathbb{C}$ has only inner derivations, $\phi_\mathbb{C} = [x + iy, \cdot]$ for some fixed $x, y \in B$. But then for all $a \in B$:

$$\phi(a) = \phi_\mathbb{C}(a) = [x, a] + i[y, a].$$

Since $\phi_\mathbb{C}$ is the complexification of a real map $\phi$, $y$ must be central, and $\phi(a) = [x, a]$ for all $a \in B$. ■

As a corollary:

**Example 15.** Let $\pi : A \to B$ be a surjective homomorphism with $B$ a finite-dimensional real or complex C*-algebra (and $A$ any real or complex algebra). Then $B$ is a submanifold algebra of $A$. ◊
In this list of sporadic examples, the next is a noncommutative example of quotient algebra that is not a submanifold algebra.

**Example 16.** Let $A := U(su(2, \mathbb{R}))$ be the universal enveloping algebra of the Lie algebra spanned by two elements $H$ and $E$ with relation $[H, E] = E$. Let $B := \mathbb{R}[x]$ be the algebra of polynomials in an indeterminate $x$. By Peter-Weyl theorem $A$ has basis $(H^jE^k)_{j \geq 0, k \geq 1}$. The map $\pi : A \to B$ defined by

$$\pi(H^jE^k) = x^j\delta_{k,0}$$

is a surjective homomorphism. The kernel $J$ is generated by $E$. From

$$[H, \frac{1}{k}H^jE^k] = H^jE^k \quad \forall \ j \geq 0, k \geq 1$$

we deduce that $J$ is contained in the commutator ideal, and since $B \simeq A/J$ is commutative $J = [A, A]$ must be exactly the commutator ideal.

Let us prove that the map $\pi_*$ is not surjective. Let $\tilde{D} \in \text{Der}(B)$ be the derivation $\tilde{D} = x\frac{d}{dx}$ and suppose $\tilde{D} = \pi_*(D)$ for some $D$. Set

$$D(H) = \sum_{j,k \geq 0} a_{j,k}H^jE^k \quad \text{and} \quad D(E) = \sum_{j,k \geq 0} b_{j,k}H^jE^k$$

with $a_{j,k}, b_{j,k} \in \mathbb{R}$. From $\tilde{D} = \pi_*(D)$ we deduce that $a_{j,0} = 1$ and $a_{j,0} = 0$ for all $j \neq 1$. From $D(E) \in J$ we deduce that $b_{j,0} = 0 \ \forall \ j \geq 0$. From the Leibniz rule we get

$$0 = D([H, E] - E) = [D(H), E] + [H, D(E)] - D(E)$$

$$= \left[ H + \sum_{j \geq 0, k \geq 1} a_{j,k}H^jE^k, E \right] + \left[ H, \sum_{j \geq 0, k \geq 1} b_{j,k}H^jE^k \right] - \sum_{j \geq 0, k \geq 1} b_{j,k}H^jE^k$$

$$= E + \sum_{j \geq 0, k \geq 1} a_{j,k}[H, E]E^k + \sum_{j \geq 0, k \geq 1} b_{j,k}(k - 1)H^jE^k.$$

Since $[H, E] \in J$, we get $0 = E + (\ldots)E^2$, which is not zero whatever is the element that multiplies $E^2$. We arrived at a contradiction, proving that $\tilde{D} \notin \text{Im}(\pi_*)$. \hfill \Box

Example 16 has a “folkloristic” interpretation: (the complexification of) the algebra $A$ can be interpreted as the coordinate algebra of $\kappa$-Minkowski space [29] in $1 + 1$ dimensions, that is generated by a time operator $t$ and position operator $q$ subject to the relation $[t, q] = i\kappa^{-1}q$ (with $\kappa \neq 0$ a constant). The algebra $B$ can be interpreted as the coordinate algebra on the time “axis”. Such an axis is not a submanifold of $\kappa$-Minkowski in the sense of Def. 2.
4.1. **Free algebras.** Any homomorphism $\pi : A \to B$ maps the center $Z(A)$ of $A$ into the center $Z(B)$ of $B$. If $\pi$ is surjective, $\pi_* : \text{Der}(A) \to \text{Der}(B)$ is a module map that covers the map $\pi|_{Z(A)} : Z(A) \to Z(B)$, meaning that

$$\pi_*(zD) = \pi(z)\pi_*(D) \quad \forall D \in \text{Der}(A) \text{ and } z \in Z(A).$$

In Example 16 the map $\pi|_{Z(A)} : Z(A) \to Z(B)$ is not surjective (even if $\pi$ is). One may think that this condition has something to do with non-surjectivity of $\pi_*$, but next example shows that this is not the case.

**Example 17 (Free algebras).** Let $A = K\langle x, y \rangle$ be the free algebra with two generators $x$ and $y$, $B = K[x, y]$, $J = [A, A]$ the commutator ideal. Note that every derivation $D$ sends the commutator ideal into itself, since $D([a, b]) = [D(a), b] + [a, D(b)] \forall a, b \in A$.

Thus $\text{Der}(A) = \text{Der}(B)$. Since $Z(A) = K$ is the set of constant polynomials and $Z(B) = B$, clearly $\pi|_{Z(A)} : Z(A) \to Z(B)$ is not surjective.

If we pick up two elements $a, b \in A$, there exists a unique derivation $D$ such that $D(x) = a$ and $D(y) = b$ (by universality, extended linearly and via the Leibniz rule). Thus $\text{Der}(A) \simeq A^2$ as a vector space. For such a $D$, $\pi_*(D) = \pi(a)\frac{\partial}{\partial x} + \pi(b)\frac{\partial}{\partial y}$, hence $\pi_*$ is surjective. ♦

In fact, any quotient algebra of a free algebra is a submanifold algebra. The tensor algebra example in [28] is a special case of Prop. 19. We need first a lemma whose proof is straightforward.

**Lemma 18.** Let $A$ be an associative algebra. A map $\partial : A \to A$ is a derivation if and only if the map

$$f_\partial : A \to M_2(A), \quad f_\partial(a) := \begin{pmatrix} a & \partial a \\ 0 & a \end{pmatrix}$$

is a homomorphism (of associative algebras).

**Proposition 19.** Let $S$ be a set, $A := K(S)$ the free algebra generated by $S$ and $\pi : A \to B$ a surjective homomorphism to a second associative algebra $B$. Then $B$ is a submanifold algebra of $A$.

**Proof.** By the universal property, for every map $f : S \to C$ from $S$ to an associative algebra $C$ there exists a unique homomorphism $\tilde{f} : A \to C$ such that $\tilde{f}|_S = f$. Let $D \in \text{Der}(B)$. For all $x \in S$ there exists $\delta_\epsilon \in A$ such that
\[ \pi(\delta_x) = D(\pi(x)) \] (by surjectivity of \( \pi \)). Let \( C \subset M_2(A) \) be the subalgebra of matrices of the form
\[
\begin{pmatrix}
a & b \\
0 & a
\end{pmatrix}
a, b \in A.
\]

Let \( f : S \to C \) be the map given by
\[ f(x) := \begin{pmatrix} x & \delta_x \\ 0 & x \end{pmatrix} \]
and \( \tilde{f} : A \to C \) the corresponding algebra morphism. For \( i = 1, 2 \) denote by \( \text{pr}_i : C \to A \) the projection on the matrix element in position \((i, i)\). Since \( \text{pr}_i \circ \tilde{f} : A \to A \) is a homomorphism given on \( S \) by \( \text{pr}_i \circ \tilde{f} = \text{Id}_S \), by unicity of the lift it must be \( \text{pr}_i \circ \tilde{f} = \text{Id}_A \). Therefore
\[ \tilde{f}(a) = \begin{pmatrix} a & \tilde{D}(a) \\ 0 & a \end{pmatrix} \]
for some map \( \tilde{D} : A \to A \). Since \( \tilde{f} \) is a homomorphism, \( \tilde{D} \) is a derivation.

Extend \( \pi \) to \( M_2(A) \) in the obvious way. By construction
\[ \pi(\tilde{f}(x)) = \begin{pmatrix} \pi(x) & \pi(\delta_x) \\ 0 & \pi(x) \end{pmatrix} = \begin{pmatrix} \pi(x) & D(\pi(x)) \\ 0 & \pi(x) \end{pmatrix} = f(\pi(x)) \]
for all \( x \in S \). Since \( \pi \circ \tilde{f} \) and \( f \circ \pi \) are homomorphism that coincide on generators, they must be equal, which means \( \pi \circ \tilde{D} = D \circ \pi \). The latter automatically implies that \( \tilde{D} \in \text{Der}_\pi(A) \).

4.2. **Almost commutative spaces.** One of the main applications of vector bundles in physics is to Yang-Mills theories: here \( L^2 \) sections of a complex smooth Hermitian vector bundle \( \pi : E \to M \) describe the physical state of a particle (or several particles) “living” in the manifold \( M \); vectors in a fiber describe the “internal” degrees of freedom of such a particle. In the celebrated Dirac equation, for example, \( M \) is a 4-dimensional Riemannian spin manifold, \( \pi : E \to M \) the spinor bundle and one looks for solutions of the equation – among smooth sections of such a bundle – describing the state of a couple particle-antiparticle with spin \( 1/2 \).

Inspired by Kaluza-Klein theories, where one derives a 4-dimensional Yang-Mills theory coupled with gravity from a purely gravitational theory on some auxiliary 5-dimensional manifold, A. Connes suggested to replace the unobserved extra dimension of Kaluza-Klein by a 0-dimensional non-commutative space. The starting point is the tensor product \( C^\infty(M) \otimes F \) of
smooth functions on a manifold and a finite-dimensional real algebra $F$, describing some kind of virtual 0-dimensional noncommutative space. Starting from a purely geometric theory on such a product, one is able to derive the complicated Lagrangian of the Standard Model of particle physics coupled with gravity\(^5\) (see e.g. [9] or [36] and references therein).

Following the point of view of Connes, an “almost commutative space” is something described by the tensor product of the algebra of smooth functions on a manifold and some finite-dimensional algebra or, more generally, by an algebra bundle over a manifold.

Let $K = \mathbb{R}$ or $\mathbb{C}$ and $F$ be a finite-dimensional $K$-algebra. A smooth algebra bundle over a (real) smooth manifold $M$, with typical fiber $F$, is a smooth vector bundle $\pi: E \to M$ whose fibers are $K$-algebras and whose local trivializations give maps $E_p \to \{p\} \times F$ ($\forall p \in M$) that are not only isomorphisms of $K$-vector spaces, but of $K$-algebras as well [23, Pag. 377].

Given any smooth vector bundle $\pi: E \to M$, it is well known that $M$ is a submanifold of $E$ (via the zero section); moreover if $S \subset M$ is a submanifold, then $\pi^{-1}(S) \subset E$ is a submanifold (inverse image of a submanifold by means of a submersion) and in particular all fibers $E_p = \pi^{-1}(p)$ are submanifolds of $E$. Having at our disposal an algebraic notion of submanifold, we wonder if analogous properties hold for algebra bundles.

If $\pi: E \to M$ is an algebra bundle, the module $\Gamma^\infty(\pi)$ of global smooth sections is a $K$-algebra with pointwise product. For $\xi, \eta \in \Gamma^\infty(\pi)$ we define

$$(\xi \cdot \eta)(p) := \xi(p)\eta(p) \forall p \in M,$$

where the one on the right is the product in the fiber $E_p$. By construction for any $p \in M$, the evaluation at $p$ gives a homomorphism $ev_p : \Gamma^\infty(\pi) \to E_p$.

If $E = M \times F$ and $\pi$ is the projection on the first factor, then $\Gamma^\infty(\pi) \simeq C^\infty(M) \otimes F$ similarly to the Standard Model example.

**Proposition 20.** Let $\pi: E \to M$ be an algebra bundle with typical fiber a finite-dimensional real or complex $C^*$-algebra.\(^6\) Then, for every $p \in M$, the map

$ev_p : \Gamma^\infty(\pi) \to E_p$

is a coembedding.

---

\(^5\)This is of course an oversimplification: the full story is beyond the scope of this paper. The interested reader can consult the books [9, 36].

\(^6\)For example $F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ in the case of the Standard Model, where $\mathbb{H}$ is the (real) algebra of quaternions.
Proof. It is true for every vector bundle that any vector in a fiber can be extended to a global smooth section. The map \( ev_p \) is then surjective. Since every derivation of \( E_p \) is inner (Lemma 14), the induced map on derivations is surjective as well (Remark 3).

In the example of the Standard Model, previous proposition can be interpreted by saying that the 0-dimensional noncommutative space encoding the internal degrees of freedom of particles is a “noncommutative submanifold” of the product space. One may wonder if \( M \) is a “noncommutative submanifold”: it is difficult to answer such a question in general, since a homomorphism \( \Gamma^\infty(\pi) \rightarrow C^\infty(M) \) may not even exist. We will investigate this question for trivial algebra bundles, i.e. tensor products of algebras, cf. Example 21.

Another example covered by Prop. 20 is the rational noncommutative torus. Let \( \theta = p/q \in \mathbb{Q} \) be a rational number, with \( p \) and \( q \) coprime. The algebra of “complex-valued smooth functions” on the noncommutative torus \( T_\theta \) is isomorphic to \( \Gamma^\infty(\pi) \) with \( \pi : E \rightarrow \mathbb{T}^2 \) a suitable algebra bundle over the (ordinary) 2-torus [22]. The typical fiber is the algebra \( M_q(\mathbb{C}) \) of all \( q \times q \) complex matrices. The spectral triple of the rational noncommutative torus was recently studied from the point of view of algebra bundles in [7].

Example 21 (Tensor products). Let \( K \) be any field and \( A := A_1 \otimes A_2 \) a tensor product of two associative \( K \)-algebras. Suppose \( \varepsilon : A_1 \rightarrow K \) is a non-zero augmentation. Then \( \pi := \varepsilon \otimes \text{Id} : A \rightarrow A_2 \) is a surjective homomorphism. For every \( D \in \text{Der}(A_2) \), the formula

\[
\tilde{D}(a_1 \otimes a_2) := a_1 \otimes D(a_2), \quad \forall \ a_1 \in A_1, a_2 \in A_2,
\]

defines a derivation \( \tilde{D} \in \text{Der}_\pi(A) \) satisfying by construction \( \pi_*(\tilde{D}) = D \). Thus, \( A_2 \) is a submanifold algebra of \( A \).

5. Formal deformations

A rich source of examples of “noncommutative spaces” is from deformation quantizations of Poisson manifolds. Only in this section, \( C^\infty(M) \) will denote complex-valued smooth functions on a (real) smooth manifold \( M \).

Definition 22. A star product on a Poisson manifold \( (M, \{,\}) \) is a \( \mathbb{C}[[\hbar]] \)-bilinear associative binary operation \( * \) on \( C^\infty(M)[[\hbar]] \) of the form:

\[
f * g = \sum_{k=0}^{\infty} \hbar^k C_k(f, g), \quad \forall \ f, g \in C^\infty(M),
\]
where each \( C_k : C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M) \) is a bi-differential operator and for all \( f, g \in C^\infty(M) \):

\[
\begin{align*}
C_0(f, g) &= fg & \text{is the pointwise multiplication,} \\
C_1(f, g) - C_1(g, f) &= 2i\{f, g\} & \text{is the Poisson bracket,} \\
C_k(1, f) &= 0 \quad \forall \ k \geq 1 & (1 \text{ is a neutral element for } \star).
\end{align*}
\]

From now on will always assume that \( C_1 \) is antisymmetric,\(^7\) so that \( C_1(f, g) = i\{f, g\} \).

If we stop the sum (7) at order \( r \geq 1 \) and work over the ring \( \mathbb{C}[\hbar]/(\hbar^{r+1}) \) we get the notion of order \( r \) deformation of a Poisson manifold.

In the framework of deformation quantization, we can consider the problem of star products that are tangential to submanifolds, and investigate under what conditions every derivation of the star product on the submanifold admits a prolongation (cf. §5.2). For 1st order deformations this becomes a problem of prolongation of Poisson vector fields, i.e. vector fields \( X \in \mathfrak{X}(M) \) satisfying

\[
X((f, g)) = \{X(f), g\} + \{f, X(g)\}, \quad \forall \ f, g \in C^\infty(M).
\]

In §5.3 we will consider short exact sequences of formal deformations coming from coisotropic reduction of a Poisson manifold.

5.1. \( \hbar \)-linear derivations. Let \( A := (C^\infty(M)[[\hbar]], \star) \) be a deformation quantization of a Poisson manifold \( M \). When dealing with star products, we will only consider derivations of star products that are \( \hbar \)-linear. An element of \( \text{Der}(A) \) will be then a formal power series

\[
D = \sum_{k=0}^{\infty} \hbar^k D_k
\]

of differential operators on \( M \) satisfying the Leibniz rule:

\[
D(f \star g) = (Df) \star g + f \star (Dg), \quad \forall \ f, g \in C^\infty(M).
\]

At order 0 in \( \hbar \) this means that \( D_0 \) is a derivation of \( C^\infty(M) \), i.e. a vector field. At order 1 we get

\[
D_1(fg) - (D_1f)g - f(D_1g) = C_1(D_0f, g) + C_1(f, D_0g) - D_0C_1(f, g).
\]

\(^7\)This can be done without loss of generality: any star product is equivalent to one with \( C_1 \) antisymmetric [24, Prop. 2.23].
Since in previous equality the left hand side is symmetric and the right hand side antisymmetric, we deduce that they must both vanish. Thus, $D_1$ must be a vector field as well, and $D_0$ must be a Poisson vector field. If we are interested in first order deformations, this completely characterizes derivations.

**Lemma 23.** An $\varepsilon$-linear derivation of $(\mathcal{C}^\infty(M)[\varepsilon]/(\varepsilon^2), \ast)$ is a sum

$$D_0 + \varepsilon D_1$$

of a Poisson vector field $D_0$ and an arbitrary vector field $D_1$ on $M$.

On a symplectic manifold, the correspondence between derivations of a star product and Poisson (in this case symplectic) vector fields holds at any order in $h$. Suppose $M$ is a symplectic manifold and denote by $\mathfrak{X}^{\text{sym}}(M)[[h]]$ the space of formal symplectic vector fields on $M$. Elements of $\mathfrak{X}^{\text{sym}}(M)[[h]]$ are formal power series

$$X = \sum_{k=0}^{\infty} h^k X_k$$

with $X_k$ a symplectic vector field on $M$ for all $k \geq 0$. Every symplectic vector field on a contractible open set is Hamiltonian. We can then cover $M$ by contractible open subsets, and on each $U$ of this cover find functions $f^U_k \in \mathcal{C}^\infty(U)$ such that

$$X_k(g)|_U = \{f^U_k, g\}$$

for all $g \in \mathcal{C}^\infty(M)$ and all $k \geq 0$. These functions are determined by $X$ up to an additive constant ($U$ is connected). We can then define a new function $Dg \in \mathcal{C}^\infty(M)$ given, on each set $U$ of this cover, by

$$Dg|_U = \frac{1}{h} \sum_{k \geq 0} h^k (f^U_k \ast g - g \ast f^U_k) .$$

For all $g$ we get a well-defined global smooth function $Dg$, and a well define derivation $D$ of the star product. Such a derivation depends only on $X$. A simple argument by induction shows that every derivation of the star product is in fact of this form:

**Theorem 24 ([24, Prop. 3.5]).** Every $h$-linear derivation of a star product on a symplectic manifold $M$ corresponds to a formal symplectic vector field via the construction above.
If $H^1(M, \mathbb{R}) = 0$ every symplectic vector field is Hamiltonian, and every derivation of the star product is essentially inner, given by $\frac{1}{\hbar} [f, \cdot]_\ast$ for some $f \in C^\infty(M)[[\hbar]]$.

5.2. **Deformation of Poisson submanifolds.** Suppose
\[
\pi : (C^\infty(M)[[\hbar]], \ast) \to (C^\infty(N)[[\hbar]], \ast)
\] is a homomorphism between deformation quantizations of two Poisson manifolds $M$ and $N$. We will assume $\pi$ is $\hbar$-linear, i.e. of the form
\[
\pi = \sum_{k=0}^\infty \hbar^k \pi_k
\] where each $\pi_k$ maps $C^\infty(M)$ into $C^\infty(N)$ and is extended to formal power series by $\mathbb{C}[[\hbar]]$-linearity.

If we look at the condition $\pi(f \ast g) = \pi(f) \ast \pi(g)$ at order 0 we get that $\pi_0$ must be a homomorphism between the commutative algebras $C^\infty(M)$ and $C^\infty(N)$, hence the pullback of a smooth map $\varphi_0 : N \to M$ [26, Cor. 35.10]. At order 1 we get:
\[
\varphi_0^\ast ([f, g]) - \{\varphi_0^\ast f, \varphi_0^\ast g\} = \pi_1(f)\pi_1(g) - \pi_1(fg)
\]
Since the symmetric and antisymmetric part must both vanish, we deduce that $\varphi_0$ is a Poisson map and $\pi_1$ a homomorphism, hence the pullback of a smooth map $\varphi_1 : N \to M$.

**Lemma 25.** The map (9) is surjective if and only if $\pi_0 : C^\infty(M) \to C^\infty(N)$ is surjective.

**Proof.** “$\Rightarrow$” If $\pi$ is surjective, for all $g = \sum_{k=0}^\infty \hbar^k g_k$, with $g_k \in C^\infty(N)$, there exists $f = \sum_{k=0}^\infty \hbar^k f_k$, with $f_k \in C^\infty(M)$, such that
\[
\pi(f) = \pi_0(f_0) + O(\hbar) = g = g_0 + O(\hbar)
\]
Thus $\pi_0(f_0) = g_0$ and $\pi_0$ is surjective.

“$\Leftarrow$” Suppose $\pi_0$ is surjective and let $g = \sum_{k=0}^\infty \hbar^k g_k$, with $g_k \in C^\infty(N)$. It follows from surjectivity of $\pi_0$ that the recursive equation
\[
\pi_0(f_k) = g_k - \sum_{j=1}^k \pi_j(f_{k-j})
\]
admits a solution $f = \sum_{k=0}^\infty \hbar^k f_k$, with $f_k \in C^\infty(M)$. Such a formal power series satisfies by construction $\pi(f) = g$, hence $\pi$ is surjective. 

\[8\text{Inner except for the factor } \frac{1}{\hbar} \text{ in front.} \]
If $\pi_0$ is surjective, it follows from Theorem 8 that $\varphi_0(N)$ is a closed embedded Poisson submanifold of $M$.

Conversely, suppose $S$ is an embedded closed submanifold of a Poisson manifold $M$ and $J$ is the vanishing ideal of $S$. If $J$ is a Poisson ideal, i.e. $\{J, f\} \subset J$ for all $f \in C^\infty(M)$, the Poisson structure of $M$ induces one on $S$ and $S$ becomes a Poisson submanifold of $M$.\footnote{Note that $J$ being a Poisson ideal means that all Hamiltonian vector fields $X_f := \{f, \cdot\}$ belong to $\text{Der}_{\cdot \cdot}(C^\infty(M))$.}

A star product (7) on $M$ is tangential to $S$ if $C^k(J, f) \subset J$ for all $k \geq 1$ and for all $f \in C^\infty(M)$. If such a condition is satisfied, we get by restriction a star product on $S$. In such a situation, the pullback of the inclusion extends by $C[h]$-linearity to a surjective homomorphism

$$t^*: A := (C^\infty(M)[[h]], \star) \to B := (C^\infty(S)[[h]], \star).$$

This is an instance of surjective homomorphism (8) where $\pi = \pi_0 = t^*$ has no higher order terms. Among the examples in this class we find regular coadjoint orbits of compact Lie groups.

**Example 26** (Regular coadjoint orbits). Let $G$ be a compact connected Lie group, $\iota: O \hookrightarrow \mathfrak{g}^*$ a regular coadjoint orbit, and equip $C^\infty(\mathfrak{g}^*)[[h]]$ with BCH star product $\star_{\text{BCH}}$. Then there exists a star product $\star$ on $O$ and a series of $\mathfrak{g}$-invariant differential operators $T = \text{Id} + \sum_{k=1}^\infty h^k T_k$ on $\mathfrak{g}^*$ such that

$$t^* \circ T : (C^\infty(\mathfrak{g}^*)[[h]], \star_{\text{BCH}}) \to (C^\infty(O)[[h]], \star)$$

is a homomorphism [3, Thm. 5.2]. If we define a new star product $\star$ on $\mathfrak{g}^*$ (equivariantly equivalent to $\star_{\text{BCH}}$) by

$$f \star g := T^{-1} f \star_{\text{BCH}} T^{-1} g,$$

then the new star product is tangential to $O$, and

$$\pi := t^* : (C^\infty(\mathfrak{g}^*)[[h]], \star) \to (C^\infty(O)[[h]], \star)$$

is a surjective homomorphism. The coadjoint orbits of a compact connected Lie group are simply connected [20, Thm. 2.3.7], thus $H^1(O, \mathbb{R}) = 0$, derivations of $(C^\infty(O)[[h]], \star)$ are essentially inner and $\pi_\star$ is surjective. \hfill \Box

One could conjecture that (10) is always a coembedding if $S$ is symplectic, since at least locally the derivations are essentially inner (Theorem 24). Unfortunately this is not the case, as shown by the next lemma.
Lemma 27 (Obstructions). Consider a surjective homomorphism like in (10).

(a) If there is a Poisson vector field on $S$ that cannot be extended to a Poisson vector field on $M$, then (10) is not a coembedding.

Assume that both $M$ and $S$ are symplectic. Then:

(b) the morphism (10) induces a linear map:
$$H^1(M, \mathbb{R}) \rightarrow H^1(S, \mathbb{R}) .$$

(11)

If (10) is a coembedding, the map (11) is surjective.

(c) If $H^1(M, \mathbb{R}) = 0$, then (10) is a coembedding if and only if $H^1(S, \mathbb{R}) = 0$.

Proof. (a) If $D = D_0 + O(h)$ and $\tilde{D} = \tilde{D}_0 + O(h)$ are derivations of the star products on $M$ and $S$, respectively, related by $\pi^* (D) = \tilde{\tilde{D}}$

(with $\pi = \imath^*$ the pullback of the inclusion), then $\tilde{D}_0$ must be the restriction to $S$ of the vector field $D_0$ on $M$. Recall that $D_0$ and $\tilde{D}_0$ are Poisson vector fields. Now, for any Poisson vector field $\tilde{D}_0$ on $S$ there is a derivation of the star product such that $\tilde{\tilde{D}} = \tilde{D}_0 + O(h)$ [35, Prop. 3.1]. If $\pi_*$ is surjective on derivations, it must be surjective on Poisson vector fields.

(b) Recall that on a symplectic manifold $M$ the 1st de Rham cohomology is isomorphic to the quotient of symplectic by Hamiltonian vector fields:
$$H^1(M, \mathbb{R}) \simeq \mathcal{X}^{\text{sym}}(M) / \mathcal{X}^{\text{ham}}(M) .$$

The map $\pi_*$ sends symplectic vector fields into symplectic vector fields (onto if it is surjective on derivations) and Hamiltonian into Hamiltonian. The only thing we have to prove is that every symplectic vector field $Y$ is in the domain of $\pi_*$, i.e. satisfies $Y(J) \subset J$ where $J$ is the vanishing ideal of $S$. In fact, we are going to prove that this is true for any vector field.

Let $\omega$ be the symplectic form on $M$ and $Y \in \mathcal{X}(M)$. Any 1-form on $M$ can be written as a finite sum $\sum f_i d g_i$ for some $f_i, g_i \in C^\infty(M)$. Thus
$$\omega(Y, \cdot) = \sum f_i d g_i = \sum f_i \omega(X_{g_i}, \cdot) = \sum f_i \omega(f_i X_{g_i}, \cdot) ,$$

for some $f_i, g_i$ and where $X_{g_i}$ denotes the Hamiltonian vector field of $g_i$.

By non-degeneracy of $\omega$:
$$Y = \sum f_i X_{g_i} .$$

In other words, $\mathcal{X}^{\text{ham}}(M)$ generates $\mathcal{X}(M)$ as a $C^\infty(M)$-module.
Now, $X_{g_i}(J) \subset J$ since $J$ is a Lie ideal, and $f_i X_{g_i}(J) \subset J$ since $J$ is an associative ideal, hence the thesis: $Y(J) \subset J$.

(c) is a simple corollary of (b). If $H^1(S, \mathbb{R}) = 0$, then up to a factor $1/\hbar$ every derivation of $B$ is inner and $\pi_*$ is surjective on inner derivations. If $H^1(S, \mathbb{R}) \neq 0$, then (11) cannot be surjective and (10) cannot be a coembedding.

Examples of pairs of symplectic manifolds $M, S$ with $S$ (closed embedded) symplectic submanifold of $M$, $H^1(M, \mathbb{R}) = 0$ and $H^1(S, \mathbb{R}) \neq 0$ can be constructed as follows. Take $M = \mathbb{C}P^n$ a complex projective space with standard symplectic structure and $S$ any complex submanifold: $\mathbb{C}P^n$ has vanishing cohomology in odd degree, and complex submanifolds are symplectic submanifolds. Now it is not difficult to find examples where $S$ is not simply connected: take any Riemann surface with genus $\geq 1$.

An easier and explicit example of quotient map (10) that is not a coembedding is the following.

**Example 28.** On $M := \mathbb{R}^2$ with Cartesian coordinates $(x, y)$ consider the commuting vector fields

$$X := \frac{\partial}{\partial x}, \quad Y := y \frac{\partial}{\partial y},$$

the Poisson structure given by the bivector field $X \wedge Y$, and the associated Weyl-type star product

$$f \star g := \mu \circ e^{\hbar(X \otimes Y - Y \otimes X)}(f \otimes g), \quad \forall f, g \in C^\infty(\mathbb{R}^2),$$

where $\mu$ is the pointwise multiplication map.

Embed $S := \mathbb{R}$ in $\mathbb{R}^2$ as horizontal axis. Then $\star$ is tangential to $S$ and we have a surjective homomorphism

$$\pi : A := (C^\infty(\mathbb{R}^2)[[\hbar]], \star) \to B := C^\infty(\mathbb{R})[[\hbar]]$$

where the product on the right is the $\mathbb{C}[[\hbar]]$-linear extension of the pointwise product. A vector field

$$D_0 = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y}, \quad a, b \in C^\infty(\mathbb{R}^2),$$

is a Poisson vector field on $\mathbb{R}^2$ if and only if

$$X(a) + Y(b) = 0.$$
This implies $\frac{\partial a}{\partial x}|_{y=0} = 0$, so $a(x, 0) = c$ is constant on the horizontal axis and
\[ \pi_*(D) = c \frac{\partial}{\partial x} + O(h). \]
The (Poisson) vector field $x \frac{\partial}{\partial x}$ on $\mathbb{R}$ is not the restriction of any Poisson vector field on $\mathbb{R}^2$. Using Lemma 27(c) we conclude that the map (13) is not a coembedding.

**Remark 29.** Note that, similarly to Example 16, in Example 28 one has
\[ x \star y - y \star x = 2i \hbar y. \]
Replacing the formal parameter by a non-zero real number, $\mathbb{C}[x, y]$ with star product (12) becomes an algebra isomorphic to the complexification of $\mathfrak{u}(\mathfrak{sb}(2, \mathbb{R}))$.

5.3. **Coisotropic reduction.** A more general class of examples of surjective homomorphisms of formal deformations comes from coisotropic reduction. Let us review the classical notion of phase space reduction from an algebraic point of view [19]. Suppose $M$ is a Poisson manifold, that we interpret as a phase space of a physical system, and imagine that the system is constrained to move in a (closed embedded) submanifold $S$ of $M$. To obtain a phase space which represents in some sense the true degrees of freedom of the system, we can then perform phase space reduction. Recall that $S$ is a *coisotropic* submanifold of $M$ if the vanishing ideal
\[ J := I(S) = \{ f \in C^\infty(M) : f|_S = 0 \} \]
is closed under the Poisson bracket. Let
\[ A := \{ f \in C^\infty(M) : \{ f, J \} \subset J \} \]
be the *normalizer* of $J$. Since $J$ is closed under Poisson bracket, $J \subset A$. Since $f \mapsto -X_f := -\{ f, \cdot \}$ is a Lie algebra morphism, $A$ is closed under Poisson bracket; since $X_{fg} = fX_g + gX_f$ and $J$ is an ideal, $A$ is closed under pointwise product: it is a Poisson subalgebra of $(C^\infty(M), \{ , \})$. Finally, by construction $J$ is a Poisson ideal in $A$.\(^{10}\) We thus have an exact sequence (1) of *Poisson algebra maps*, where the quotient algebra $B$ is interpreted as algebra of smooth functions on what we would call geometrically the reduced phase space $M_{\text{red}}$.\(^{11}\)

---

\(^{10}\) The normalizer is indeed the largest Poisson subalgebra of $(C^\infty(M), \{ , \})$ containing $J$ as a Lie ideal.

\(^{11}\) Note that if $S$ is a symplectic submanifold, then $A = C^\infty(M)$ and $M_{\text{red}} \simeq S$. 

23
“Good” star products on $M$ induce formal deformations of the constrained ideal, normalizer and reduced phase space, fitting an exact sequence:

$$0 \to (J[[h]], \star) \to (A[[h]], \star) \overset{\pi_*}{\to} (C^\infty(M_{\text{red}})[[h]], \star) \to 0.$$  \quad (14)

A procedure that always works when $S$ has codimension 1 in $M$ is in [21]. For a general discussion of the problem one can see the review [6] and references therein. A more recent “categorical” approach is in [11].

Since symplectic submanifolds are special examples of this construction, we cannot expect (14) to be always a coembedding.

As a concrete example one can take $M := \mathbb{C}^{n+1} \setminus \{0\}$ with standard symplectic structure. The submanifold $S := S^{2n+1}$ is then coisotropic. Let $\partial_0$ be the vector field on $M$ (tangent to $S$) generating the obvious $U(1)$ action given by multiplication of all complex coordinates by the same phase. The normalizer $J$ is given by those functions $f \in C^\infty(M)$ such that $\partial_0 f$ vanishes on $S$. The quotient algebra is isomorphic to the algebra of smooth functions on $S^{2n+1}$ that are $U(1)$-invariant, that we identify with smooth functions on $M_{\text{red}} = \mathbb{CP}^n$. It is shown in [21] that Wick star product on $M$ can be reduced to $\mathbb{CP}^n$ and one has a sequence (14) of formal deformations. In this example $\pi_*$ is surjective again for trivial reasons: $M_{\text{red}}$ is simply connected and all derivations are essentially inner.

6. CATEGORICAL PROPERTIES

6.1. COMPOSITION OF COEMBEDDINGS. In this section we show that coembeddings (in the sense of Def. 2) are closed under composition and discuss some of their categorical properties. Notice that given a sequence $A \overset{f}{\to} B \overset{g}{\to} C$ of coembeddings, in general $g_* \circ f_*$ and $(g \circ f)_*$ are not equal (they don’t even have the same domain). It is then not obvious that the composition $g \circ f$ is still a coembedding.

It is convenient to rephrase properties of derivations in terms of the associated morphisms given by Lemma 18. Let us introduce the following terminology.

**Definition 30.** If $A$ is an associative algebra, a homomorphism $A \to M_2(A)$ will be called admissible if it is of the form

$$a \mapsto \begin{pmatrix} a & \partial a \\ 0 & a \end{pmatrix}$$

for some $\partial \in \text{Der}(A)$. 

24
Given a map $f : A \to B$, we will denote by $F : M_2(A) \to M_2(B)$ the map obtained by applying $f$ to each matrix element. The definition of coembedding can be restated in the following form.

**Lemma 31.** A surjective homomorphism $f : A \to B$ is a coembedding if and only if for every admissible homomorphism $\xi : B \to M_2(B)$ there exists an admissible homomorphism $\eta : A \to M_2(A)$ making the following diagram commute

\[
\begin{array}{ccc}
M_2(A) & \xrightarrow{F} & M_2(B) \\
\eta \uparrow & & \xi \uparrow \\
A & \xrightarrow{f} & B
\end{array}
\]

It is now easy to prove the following:

**Proposition 32.** Consider a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of maps between associative algebras. If $f$ and $g$ are coembeddings, then $g \circ f$ is a coembedding.

**Proof.** The composition $g \circ f$ of two surjective homomorphisms is a surjective homomorphisms. Consider the diagram:

\[
\begin{array}{ccc}
M_2(A) & \xrightarrow{F} & M_2(B) & \xrightarrow{G} & M_2(C) \\
\eta \uparrow & & \xi \uparrow & & \phi \uparrow \\
A & \xrightarrow{f} & B & \xrightarrow{g} & C
\end{array}
\]

If $\phi$ is admissible, since $g$ is a coembedding there exists an admissible $\xi$ making the right square commute. But $f$ is a coembedding as well, so there exists an admissible $\eta$ making the left square – and then the outer rectangle – commute. Since for every $\phi$ there exists $\eta$ making the outer rectangle commute, $g \circ f$ is a coembedding. ■

Embeddings of smooth manifolds satisfy an additional property:

**Proposition 33.** Consider a sequence $N \xrightarrow{\alpha} S \xrightarrow{\beta} M$ of smooth maps between smooth manifolds. If $\beta \circ \alpha$ is a closed embedding, then so is $\alpha$.

The proof is straightforward if we rephrase it in terms of algebra morphisms. Recall that every morphism $C^\infty(M) \to C^\infty(S)$ is the pullback of a smooth map $\alpha : S \to M$ [26, Cor. 3.5.10], and it is surjective if and only if $\alpha$ is a closed embedding (Theorem 8). Proposition 33 is then equivalent to the next Prop. 34, whose proof is straightforward:
Proposition 34. Consider a sequence $C^\infty(M) \xrightarrow{f} C^\infty(S) \xrightarrow{g} C^\infty(N)$ of algebra morphisms. If $g \circ f$ is surjective, then so is $g$.

Proof. The image of $g \circ f$ is a subset of the image of $g$. ■

On the other hand, there are simple examples where $g \circ f$ and $g$ are both surjective but $f$ is not, or dually where $\beta \circ \alpha : N \to M$ and $\alpha : N \to S$ are both closed embeddings, but $\beta : S \to M$ is not. In categorical language, embeddings do not satisfy the 2-out-of-3 property.

Recall that in a given category, we say that a class $\mathcal{C}$ of morphisms satisfies the 2-out-of-3 property if for any sequence of morphisms $A \xrightarrow{f} B \xrightarrow{g} C$, whenever two of the three morphisms $f$, $g$ and $g \circ f$ are in $\mathcal{C}$, then so is the third one. Typical examples are quasi-isomorphisms of chain complexes, weak equivalences of topological spaces, or more generally weak equivalences in a model category.

Coembeddings clearly don’t satisfy such a property (since it is violated by embeddings) in the category of algebras with morphisms given by algebra maps. They don’t even satisfy it if we restrict morphisms to surjective algebra maps only. Furthermore, the analogue of Prop. 34 does not hold. Namely, there are sequences $A \xrightarrow{f} B \xrightarrow{g} C$ of surjective algebra maps such that $f$ and $g \circ f$ are coembeddings but $g$ is not. Consider for example the sequence $C^\infty(\mathbb{R}^2) \xrightarrow{f} C^\infty(S) \xrightarrow{g} C^\infty(\mathbb{R})$, where $S$ is the cross, $f$ is the pullback of the inclusion $S \hookrightarrow \mathbb{R}^2$ and $g$ the map in Example 7. By Theorem 6, the map $f$ is a coembedding, and $g \circ f$ is the pullback of the inclusion $\mathbb{R} \to \mathbb{R}^2$ as horizontal axis, so it is a coembedding as well. However $g$ is not a coembedding, cf. Example 7.

6.2. On the notion of noncommutative submanifold. Let us make a few final comments on whether Def. 2 is the “right” notion of submanifold in noncommutative geometry. Inspired by Gelfand duality, establishing that any commutative $C^*$-algebra is (isomorphic to) the algebra of continuous functions vanishing at infinity on a locally compact Hausdorff space, the point of view of noncommutative geometry is to regard any associative algebra (possibly with additional structure, e.g. a Dirac operator [10]) as describing some virtual “noncommutative” space. It is natural to wonder what is the correct notion of noncommutative space contained within another noncommutative spaces.

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\[\text{For example } \alpha : \mathbb{R} \to \mathbb{R}^2 \text{ given by } \alpha(x) = (x,0) \text{ and } \beta : \mathbb{R}^2 \to \mathbb{R} \text{ given by } \beta(x, y) = x.\]

Clearly $\alpha$ and $\beta \circ \alpha$ are (closed) embeddings, while $\beta$ is not.
For smooth manifolds, we saw that the dual of an embedding is simply a surjective homomorphism (of commutative algebras), which is automatically a coembedding by Theorem 8. For general algebras, however, not every surjective homomorphism \( \pi : A \to B \) is a coembedding, and there are properties that are desirable— if we want to study these algebras using tools adapted from differential geometry—and are guaranteed only by co-embeddings. For example, the possibility of pushing forward differential forms from \( A \) to \( B \) [28, Prop. IV.1].

There is a general notion of mathematical structure contained within another mathematical structure that is worth recalling.

In a concrete category \( C \) (over \( \text{Set} \)), a morphism \( f : A \to B \) is called initial if for every commutative triangle (15.1) of maps between underlying sets, if the diagonal arrow is a morphism in \( C \) then so is the vertical arrow; dually, a morphism \( f : A \to B \) is final (or terminal) if for every commutative triangle (15.2) of maps between underlying sets, if the diagonal arrow is a morphism in \( C \) then so is the vertical arrow. (See [1, §8].)

\[
\begin{align*}
\begin{tikzcd}
A \arrow[r, f] & B \\
C \arrow[u, g] \arrow[r, f \circ g] & \end{tikzcd} & \begin{tikzcd}
A \arrow[r, f] & B \\
C \arrow[u, g \circ f] \arrow[r, g] & \end{tikzcd}
\end{align*}
\]

(15.1) Initial morphism \hspace{1cm} (15.2) Final morphism

An embedding, in a given concrete category over \( \text{Set} \), is then defined as an injective initial morphism. Dually, a quotient morphism is defined as a surjective final morphism.

The categorical notions of embedding and quotient morphism agree with the usual notions in the case of vector spaces and algebras,\(^{13}\) but not in the case of smooth manifolds. Indeed, every smooth embedding is an embedding in the category of smooth manifolds,\(^{14}\) and every surjective smooth submersion is a quotient morphism (see e.g. Theorem 4.29 and Corollary 5.30 in [27]), but the converse is not true.

\(^{13}\)In the category of vector spaces resp. algebras, every injective homomorphism is an embedding and every surjective homomorphism is a quotient morphisms.

\(^{14}\)One may wonder if smooth embeddings are also final morphisms. For a simple counterexample showing that smooth embeddings are not terminal, take as \( f : \mathbb{R} \to \mathbb{R}^2 \) the embedding \( f(x) := (x, x^3) \), as \( g : \mathbb{R}^2 \to \mathbb{R} \) the map \( g(x, y) = \sqrt[3]{y} \). Then \( g \circ f = \text{Id}_{\mathbb{R}} \) is smooth, even if \( g \) is not.
Let us focus on embeddings and repeat the definition of initial morphism in the category of smooth manifolds. A smooth map $f : S \to M$ is initial if for every smooth manifold $N$ and every map $g : N \to S$, if $f \circ g : N \to M$ is smooth then $g$ is also smooth.

Categorical embeddings (injective initial morphisms) of smooth manifolds are more general than smooth embeddings, and even more than injective immersions. An example of categorical embedding that is not an immersion is the map $t \mapsto (t^p, t^q)$, for any coprime $p, q \in \mathbb{N}$ [26, Rem. 2.13].

A map that is both an injective immersion and an initial morphism is called weak embedding in [27], and its image is called initial submanifold. Weak embeddings are still more general than smooth embeddings, as shown by the next example.

**Example 35** (Kronecker’s foliation). Let $M := \mathbb{R}^2/\mathbb{Z}^2$ be the 2-torus, $S := \mathbb{R}$, $\theta \in \mathbb{R} \setminus \mathbb{Q}$ an irrational number and, for all $0 \leq x < 1$, let $f_x : S \to M$ be the map

$$f_x(t) := (x + t, \theta t) \mod \mathbb{Z}^2.$$

For each $x$, the image of $f_x$ is a leaf of Kronecker’s foliation of the 2-torus. Each $f_x$ is a weak embedding [27, Prob. 5-13], but not a smooth embedding. Note that the pullback $f_x^* : C^\infty(\mathbb{R}^2/\mathbb{Z}^2) \to C^\infty(\mathbb{R})$ is not a quotient morphism in the category of algebras (it is injective, rather than surjective, since the image of $f$ is dense in the 2-torus).

The importance of weak embeddings is often underestimated, since they appear in many places in differential geometry. Leaves of a (not necessarily regular) foliation are initial submanifolds [26, Thm. 3.22], and in particular symplectic leaves of a Poisson manifold are initial submanifolds.

It is interesting to notice that, while the categorical notion of embedding is more general than the one of smooth embedding, the dual notion of quotient morphism could be a good candidate to generalize (closed smooth) embedded submanifolds to the noncommutative realm. In the commutative case, quotient morphisms between algebras of smooth functions are in bijection with closed smooth embeddings.

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