A PERIODIC SOLUTION OF PERIOD TWO OF A DELAY DIFFERENTIAL EQUATION

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Abstract. In this paper we prove that the following delay differential equation
\[ \frac{dx(t)}{dt} = rx(t) \left( 1 - \int_{0}^{1} x(t-s)ds \right), \]
has a periodic solution of period two for \( r > \frac{\pi}{2} \) (when the steady state, \( x = 1 \), is unstable). In order to find the periodic solution, we study an integrable system of ordinary differential equations, following the idea by Kaplan and Yorke [14]. The periodic solution is expressed in terms of the Jacobi elliptic functions.

1. Introduction

The delay differential equation
\[ \frac{dz(t)}{dt} = f(z(t-1)), \] where \( f : \mathbb{R} \to \mathbb{R} \) is a continuous function, has been extensively studied in the literature. For a special case, \( f(z) = r(1 - e^z) \), \( r > 0 \), the equation (1.1) is referred to as the Wright equation, named after the paper [32]. Jones investigated the existence of a periodic solution of the Wright equation in [12, 13] by the fixed-point theorem. Nussbaum then established a general fixed-point theorem and study the existence of periodic solutions for a class of functional differential equations in [20]. See also [18, 11] and references therein for the recent progress by a computer assisted approach.

Assuming that \( f \) is an odd function, in the paper [14], Kaplan and Yorke constructed a periodic solution of the equation (1.1) via a Hamiltonian system of ordinary differential equations. The idea is used to investigate a periodic solution of the equation (1.1) with a particular nonlinear function \( f \) in [5] and for a system of differential equations with distributed delay in [1]. We refer the readers to the survey paper [31] and the references therein. See also Chapter XV of [4]. In this paper we follow the approach by Kaplan and Yorke [14]: we find a periodic solution of a differential equation with distributed delay, considering a system of ordinary differential equations.

The following mathematical model for a single species population is known as the Hutchinson equation and as a delayed logistic equation
\[ \frac{dx(t)}{dt} = rx(t) \left( 1 - x(t-1) \right). \]

Key words and phrases. Elliptic integrals, Jacobi elliptic functions, Delay differential equation, Periodic solution, Hopf bifurcation, Integrable ordinary differential equations,
A periodic solution of period two of a delay differential equation

The equation (1.2) can be derived from the Wright equation by the transformation $z(t) = \ln x(t)$. Many extension of the Hutchinson equation (1.2) has been investigated, see [8, 20, 9] and references therein. Nevertheless, the Hutchinson-Wright equation still poses a mathematical challenges [28].

In this paper we study the existence of a periodic solution of the following delay differential equation

$$(1.3) \quad \frac{d}{dt} x(t) = rx(t) \left( 1 - \int_0^1 x(t-s)ds \right),$$

where $r$ is a positive parameter, $r > 0$. The delay differential equation (1.3) can be seen as a variant of the Hutchinson equation (1.2). The author’s motivation to study (1.3) is that the equation appears as a limiting case of an infectious disease model with temporary immunity (see Appendix). For the equation (1.3), the existence of a periodic solutions does not seem to be well understood. The periodicity, which may explain the recurrent disease dynamics, is a trigger of this study. Differently from the discrete delay case, the distributed delay is an obstacle, when one tries to construct a suitable Poincare map to find a periodic solution, but see [15, 29, 30].

In this paper we prove the following theorem.

**Theorem 1.** Let $r > \frac{\pi^2}{4}$. Then the delay differential equation (1.3) has a periodic solution of period 2, i.e., $x(t) = x(t-2)$, $t \in \mathbb{R}$, satisfying

$$x(t)x(t-1) = \text{Const}, \quad \int_0^2 x(t-s)ds = 2$$

for any $t \in \mathbb{R}$.

In the main text we provide more information of the periodic solution. We prove the existence of the periodic solution, solving a corresponding ordinary differential equation, which turns out to be equivalent to the Duffing equation. The periodic solution, explicitly expressed in terms of the Jacobi elliptic functions, appears at $r = \frac{\pi^2}{4}$, as the positive equilibrium ($x = 1$) loses stability via Hopf bifurcation.

This paper is organized as follows. In Section 2, we first study stability of the positive equilibrium, applying the principle of linearized stability. We then derive a system of ordinary differential equations (2.5) that generates the periodic solution of the original delay differential equation (1.3). In Section 3, the system of ordinary differential equations (2.5) is reduced to a scalar differential equation (3.4) that turns out to be the Duffing equation. The equation is explicitly solved using the Jacobi elliptic functions. In Section 4, we consider a fixed point problem to find a parameter such that the period of the solution becomes two.

2. PRELIMINARY

For the delay differential equation (1.3) the natural phase space is $C = C([-1,0], \mathbb{R})$ equipped with the supremum norm (1.9). We consider the following initial condition

$$x(\theta) = \phi(\theta), \quad \theta \in [-1,0]$$

where $\phi \in C$ with $\phi(0) > 0$, so that positive solutions are generated.
Observe that (1.3) is equivalent to the following system of delay differential equations

\begin{align}
\frac{d}{dt}x(t) &= -rx(t)y(t), \\
\frac{d}{dt}y(t) &= x(t) - x(t - 1)
\end{align}

with the following initial condition

\begin{align*}
    x(\theta) &= \phi(\theta), \quad \theta \in [-1, 0] \\
    y(0) &= \int_{0}^{1} \phi(-s)ds - 1.
\end{align*}

It is easy to see that \( x = 1 \) is the unique positive equilibrium for the equation (1.3). One obtains the following result for stability of the positive equilibrium (see also Theorem 4.1 of [22]).

**Proposition 2.** The positive equilibrium \( x = 1 \) is asymptotically stable for \( 0 < r < \frac{\pi}{2} \) and unstable for \( r > \frac{\pi}{2} \). Hopf bifurcation occurs at \( r = \frac{\pi}{2} \) and a periodic solution appears.

**Proof.** We deduce the following characteristic equation ([4, 9])

\begin{equation}
\lambda = -r \int_{0}^{1} e^{-\lambda s} ds, \quad \lambda \in \mathbb{C}.
\end{equation}

Let \( \lambda = \mu + i\omega, \quad (\mu, \omega \in \mathbb{R}) \) to obtain the following two equations

\begin{align}
\mu &= -r \int_{0}^{1} e^{-\mu s} \cos(\omega s) ds, \\
\omega &= r \int_{0}^{1} e^{-\mu s} \sin(\omega s) ds.
\end{align}

First one sees that if \( \text{Re}\lambda > 0 \) then

\begin{equation}
|\lambda| = \sqrt{\mu^2 + \omega^2} \leq r.
\end{equation}

Assume that there is a root in the right half complex plane (i.e., \( \mu > 0 \)) for sufficiently small \( r > 0 \). One sees \( \int_{0}^{1} e^{-\mu s} \cos(\omega s) ds > 0 \) from the estimation (2.4), thus, if \( r > 0 \) is sufficiently small, from (2.3a) all roots of the characteristic equation (2.2) are in the left half complex plane.

Suppose now that for some \( r > 0 \) purely imaginary roots exist. Substituting \( \mu = 0 \) into the equation (2.3a), one sees that for \( r = \frac{1}{2}((2n + 1)\pi)^2 \) the characteristic equation (2.2) has purely imaginary roots \( \lambda = \pm i\omega = \pm i(2n + 1)\pi \) for \( n = 0, 1, 2, \ldots \). We show that, for \( n = 0, 1, 2, \ldots \), the purely imaginary roots \( \lambda = \pm i(2n + 1)\pi \) cross the imaginary axis transversally from left to right as \( r \) increases in the neighborhood of \( r = \frac{1}{2}((2n + 1)\pi)^2 \). Applying the implicit function theorem to the equation (2.2), one has

\begin{equation}
\lambda'(r) \left( 1 - r \int_{0}^{1} s e^{-\lambda s} ds \right) + \int_{0}^{1} e^{-\lambda s} ds = 0.
\end{equation}
One sees that
\[
\int_0^1 se^{-\lambda s} ds = -\frac{e^{-\lambda}}{\lambda} + \frac{1}{\lambda} \int_0^1 e^{-\lambda s} ds = -\frac{e^{-\lambda}}{\lambda} - \frac{1}{r}.
\]
\[
\int_0^1 e^{-\lambda s} ds = -\frac{\lambda}{r}.
\]
Therefore, at \( r = \frac{1}{2} (2n + 1) \pi \) and \( \lambda = i\omega = i (2n + 1) \pi \), it follows that
\[
\lambda'(r) \bigg|_{r=\frac{1}{2}((2n+1)\pi)^2} = \frac{i\omega}{1 + 1 + i\omega} \implies \Re\lambda'(r) \bigg|_{r=\frac{1}{2}((2n+1)\pi)^2} = \frac{1}{4 + (\frac{\omega}{\pi})^2} > 0.
\]
From the Hopf bifurcation theorem, we obtain the conclusion. \( \square \)

Thus a periodic solution of period 2 emerges at \( r = \frac{\pi^2}{4} \) and the positive equilibrium is unstable for \( r > \frac{\pi^2}{4} \).

Now assume that there exists a periodic solution of period 2 for (1.3). Denote by \( x^*(t) \) the periodic solution of (1.3), i.e., \( x^*(t) = x^*(t - 2) \). We let
\[
x_1(t) = x^*(t), \quad y_1(t) = \int_0^1 x^*(t-s)ds - 1,
\]
\[
x_2(t) = x^*(t-1), \quad y_2(t) = \int_1^2 x^*(t-s)ds - 1.
\]
We are also interested in the positive periodic solution. The periodic solution satisfies the following system of ordinary differential equations
\begin{align*}
(2.5a) & \quad \frac{d}{dt}x_1(t) = -rx_1(t)y_1(t), \\
(2.5b) & \quad \frac{d}{dt}y_1(t) = x_1(t) - x_2(t), \\
(2.5c) & \quad \frac{d}{dt}x_2(t) = -rx_2(t)y_2(t), \\
(2.5d) & \quad \frac{d}{dt}y_2(t) = x_2(t) - x_1(t).
\end{align*}
The initial condition is
\begin{align*}
(2.6a) & \quad x_1(0) = a > 0, \quad x_2(0) = b > 0, \\
(2.6b) & \quad y_1(0) = y_2(0) = 0,
\end{align*}
where \( a \) and \( b \) will be determined later (\( a = x^*(0) = x^*(2) \), \( b = x^*(-1) = x^*(1) \)) in Section 4, so that \( x_1(t) = x_1(t + 2) \) holds.

From (2.5) one sees that
\begin{align*}
(2.7a) & \quad y_1(t) + y_2(t) = 0, \\
(2.7b) & \quad x_1(t)x_2(t) = ab
\end{align*}
hold for any \( t \geq 0 \). Thus one sees that the periodic solution satisfies the following properties
\begin{align*}
(2.8) & \quad \int_0^2 x^*(t-s)ds = 2, \quad x^*(t)x^*(t-1) = \text{Const, } t \in \mathbb{R}.
\end{align*}
3. Integrable Ordinary differential equations

We construct an initial function $\phi$ for the delay differential equation (1.3) that generates a periodic solution of period 2. From (2.5) and (2.7), the system (2.5) is reduced to the following system of ordinary differential equations

\begin{align}
\frac{d}{dt} x(t) &= -rx(t)y(t), \\
\frac{d}{dt} y(t) &= x(t) - ab \frac{1}{x(t)},
\end{align}

dropping the indices from $x_1$ and $y_1$ (cf. (2.1)). From (2.6) the initial condition of (3.1) is

\begin{align}
x(0) &= a, \\
y(0) &= 0.
\end{align}

We show that the system (3.1) has a conservative quantity.

**Proposition 3.** It holds that

\begin{equation}
x(t) + ab \frac{1}{x(t)} + r \frac{1}{2} y^2(t) = a + b, \quad t \in \mathbb{R}
\end{equation}

for the solution of the equation (3.1) with the initial condition (3.2).

**Proof.** Differentiating the left hand side of (3.3), we obtain

\begin{align*}
\frac{d}{dt} \left( x(t) + ab \frac{1}{x(t)} + r \frac{1}{2} y^2(t) \right) &= \left( 1 - ab \frac{1}{x^2(t)} \right) x'(t) + ry(t)y'(t) \\
&= \left( 1 - ab \frac{1}{x^2(t)} \right) (-rx(t)y(t)) + ry(t) \left( x(t) - ab \frac{1}{x(t)} \right) \\
&= 0.
\end{align*}

From (3.2), it then follows that

\begin{equation*}
x(t) + ab \frac{1}{x(t)} + r \frac{1}{2} y^2(t) = x(0) + ab \frac{1}{x(0)} + r \frac{1}{2} y^2(0) = a + b
\end{equation*}

for $t \in \mathbb{R}$. \hfill \Box

Differentiating the both sides of the equation (3.1b), we obtain

\begin{align*}
\frac{d^2}{dt^2} y(t) &= - \left( 1 + ab \frac{1}{x^2(t)} \right) rx(t)y(t) \\
&= -ry(t) \left( x(t) + ab \frac{1}{x(t)} \right).
\end{align*}

Using the identity (3.3) in Proposition 3, we derive the Duffing equation:

\begin{equation}
\frac{d^2}{dt^2} y(t) = -ry(t) \left( a + b - r \frac{1}{2} y^2(t) \right)
\end{equation}
with the following initial condition
\[(3.5a)\quad y(0) = 0,\]
\[(3.5b)\quad \frac{d}{dt} y(0) = x(0) - ab \frac{1}{x(0)} = a - b.\]

Denote by \(s_n\) the Jacobi elliptic sine function \([3, 19]\). It is known that the solution of the Duffing equation \((3.4)\) is given by
\[(3.6)\quad y(t) = \alpha s_n(\beta t, k),\]
where \(\alpha, \beta\) and \(k\) are functions of \(a\) and \(b\) defined by
\[(3.7)\quad \alpha(a, b) = \sqrt{\frac{2}{r}} \left( \sqrt{a} - \sqrt{b} \right), \quad \beta(a, b) = \sqrt{\frac{r}{2}} \left( \sqrt{a} + \sqrt{b} \right),\]
\[(3.8)\quad k(a, b) = \frac{\sqrt{a} - \sqrt{b}}{\sqrt{a} + \sqrt{b}}.\]

To simplify the notation, we occasionally drop \((a,b)\) from \(\alpha, \beta\) and \(k\).

We then obtain the exact solution of the system \((3.1)\) with the initial condition \((3.2)\).

**Proposition 4.** The solution of the equations \((3.1)\) with the initial condition \((3.2)\) is expressed as
\[
(3.9)\quad x(t) = a \left( \frac{1 - k}{dn(\beta t, k) - kcn(\beta t, k)} \right)^2 = a \left( \frac{dn(\beta t, k) + kcn(\beta t, k)}{1 + k} \right)^2,
\]
\[
(3.10)\quad y(t) = \alpha s_n(\beta t, k),
\]
where \(\alpha, \beta\) and \(k\) are defined in \((3.7)\) and \((3.8)\).

**Proof.** Since \((3.10)\) is given in \((3.6)\), we show the equality in \((3.9)\), integrating the equation \((3.1a)\). We get
\[
x(t) = ae^{-r \int_0^t y(s) \, ds}.
\]
Using \((3.10)\) we compute
\[
r \int_0^t y(s) \, ds = r \alpha \int_0^t s_n(\beta u, k) \, du
\]
\[
= \frac{r \alpha}{\beta k} \left[ \ln \left( dn(s, k) - kcn(s, k) \right) \right]_0^{\beta t}
\]
\[
= \frac{r \alpha}{\beta k} \ln \left( \frac{dn(\beta t, k) - kcn(\beta t, k)}{1 - k} \right).
\]
Note that \(\frac{r \alpha}{\beta k} = 2\) holds from the definitions in \((3.7)\) and \((3.8)\). We then get
\[
r \int_0^t y(s) \, ds = 2 \ln \left( \frac{dn(\beta t, k) - kcn(\beta t, k)}{1 - k} \right),
\]
from which the first equality in \((3.9)\) follows.

Using the properties of the elliptic functions, it holds that
\[
(dn - kcn) (dn + kcn) = dn^2 - k^2 cn^2 = 1 - k^2.
\]
Therefore, we obtain the following equality
\[
\left( \frac{1 - k}{\text{dn}(\beta t, k) - k \text{cn}(\beta t, k)} \right)^2 = \left( \frac{\text{dn}(\beta t, k) + k \text{cn}(\beta t, k)}{1 + k} \right)^2.
\]

\[\square\]

4. Periodic solution of period 2

In this section we will determine $a$, the initial value for the $x$ component of the system (3.1), so that, for the solution given in Proposition 4, the period is 2 and the integral constant becomes $-1$. The periodic solution finally solves the delay differential equation (1.3).

Let us introduce the complete elliptic integrals of the first kind and of the second kind [3, 19]. Those are respectively given as
\[
K(k) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2 \theta}} d\theta,
\]
\[
E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta} d\theta
\]
for $0 \leq k < 1$. In the following theorem we derive the two conditions for that the period of the solution given in Proposition 4 is 2.

**Theorem 5.** Assume that the following two conditions hold
\[
\sqrt{\frac{r}{2}} \left( \sqrt{a} + \sqrt{b} \right) = 2K(k),
\]
\[
\left( \sqrt{a} + \sqrt{b} \right) \sqrt{\frac{2}{r} E(k)} - \sqrt{ab} = 1.
\]
Then, for the solution of the equation (3.1) with the initial condition (3.2), it holds that
\[
(x(t), y(t)) = (x(t + 2), y(t + 2))
\]
and that
\[
y(t) = \int_{t-1}^{t} x(s) ds - 1
\]
for any $t \in \mathbb{R}$.

**Proof.** From (4.1), we have $2\beta = 4K(k)$. Since the Jacobi elliptic functions, sn, cn and dn, have a period $4K(k)$, one has
\[
\text{sn} (\beta t, k) = \text{sn}(\beta (t + 2), k),
\]
\[
\text{cn} (\beta t, k) = \text{cn}(\beta (t + 2), k),
\]
\[
\text{dn} (\beta t, k) = \text{dn}(\beta (t + 2), k).
\]
Then it is easy to see that (4.3) follows from (3.9) and (3.10). Next we show that (4.4) holds. From the symmetry of the Jacobi elliptic functions, we have
\[
\text{cn} (\beta (t - 1), k) = -\text{cn}(\beta t, k),
\]
\[
\text{dn} (\beta (t - 1), k) = \text{dn}(\beta t, k).
\]
Thus from (3.9) we obtain
\[ x(t-1) = a \left( \frac{\text{dn}(\beta (t-1), k) + k \text{cn}(\beta (t-1), k)}{1+k} \right)^2 = a \left( \frac{\text{dn}(\beta t, k) - k \text{cn}(\beta t, k)}{1+k} \right)^2 \]
and \( x(t)x(t-1) = a^2 \left( \frac{1-k}{1+k} \right)^2 = ab \) follows. Then from (3.11), for the solution of the equation (3.1), we have the following equality
\[ \frac{d}{dt} y(t) = x(t) - x(t-1), \]
implying that
\[ y(t) = \int_0^1 x(t-s)ds + \text{const.} \]
From (4.1) (i.e., \( 2\beta = 4K(k) \)) and (3.10) we have
\[ y(0) = y(1) = 0. \]
Now we show that
\[ \int_0^1 x(t)dt = \left( \sqrt{a} + \sqrt{b} \right) \sqrt{\frac{1}{2}} E(k) - \sqrt{ab}. \]
Using the properties of the Jacobi elliptic functions [3], we compute
\[ (\text{dn} + k \text{cn})^2 = \text{dn}^2 + 2k \text{dncn} + k^2 \text{cn}^2 = k^2 - 1 + 2 \text{dn}^2 + 2k \text{dncn}. \]
From the following computations
\[ \int_0^1 \text{dn}^2(\beta t, k)dt = \frac{1}{b} \int_0^{2K(k)} \text{dn}^2(u, k)dt = \frac{2}{b} E(k), \]
\[ \int_0^1 \text{dn}(\beta t, k)\text{cn}(\beta t, k)dt = 0, \]
one sees that
\[ \int_0^1 x(t)dt = a \frac{k^2 - 1 + \frac{4}{b} E(k)}{(1+k)^2}. \]
by (3.9). Then we obtain (4.6) from (3.7) and (3.8). from the condition (4.2) the integral constant in (4.5) becomes \(-1\), for the solution of the equation (3.1). □

The conditions (4.1) and (4.2) ensure the existence of a periodic solution of period 2 for the system of ordinary differential equations (3.1), satisfying (3.3). The periodic solution obtained in Theorem 5 is also a periodic solution of the delay differential equation (1.3). Our remaining task is to interpret the conditions (4.1) and (4.2) in terms of the parameter \( r \) in the equation (1.3).

Eliminating \( a \) and \( b \) from the conditions (4.1) and (4.2), we obtain the following equality
\[ r = L(k), \ 0 \leq k < 1, \]
where
\[ L(k) := 2K(k) \left( 2E(k) - K(k) \left( 1 - k^2 \right) \right). \]
For the derivation of (4.7), see the proof of Proposition 7 below. Now we show that the equation (4.7) has a unique root.
Lemma 6. The function $L$ is a strictly increasing function with
\[ L(0) = \frac{\pi^2}{2} < \lim_{k \to 1^-} L(k) = \infty. \]

Proof. From the definition of $L$, it is easy to see $L(0) = \frac{\pi^2}{2}$. By the straightforward calculation, we obtain
\[
\frac{d}{dk} \left( 2E(k) - K(k) \left( 1 - k^2 \right) \right) = 2 \frac{E(k) - K(k)}{k} - \frac{1}{k} \left( E(k) - (1 - k^2) K(k) \right) + 2kK(k) = (1 - k^2) K'(k) > 0,
\]
noting that
\[
K'(k) = \frac{1}{k} \left( \frac{E(k)}{1 - k^2} - K(k) \right),
E'(k) = \frac{1}{k} (E(k) - K(k)),
\]
see e.g. P. 282 of [3]. Since it can be shown that
\[
\lim_{k \to 1^-} K(k)(1 - k^2) = \lim_{k \to 1^-} \int_0^{\pi/2} \frac{1 - k^2}{\sqrt{1 - k^2 \sin^2 \theta}} d\theta = 0,
\]
$L$ is a strictly increasing function with $\lim_{k \to 1^-} L(k) = \infty$. \qed

Then, $a$ and $b$ are determined by the following Proposition.

Proposition 7. There exist $a > 0$ and $b > 0$ such that the two conditions (4.1) and (4.2) in Theorem 5 hold if and only if $r > \frac{\pi^2}{2}$. In particular, $a$ and $b$ are given as
\[
(4.8) \quad \begin{bmatrix} a \\ b \end{bmatrix} = \frac{K(k)}{2E(k) - K(k)(1 - k^2)} \begin{bmatrix} (1 + k)^2 \\ (1 - k)^2 \end{bmatrix},
\]
where $k = L^{-1}(r)$, $r > \frac{\pi^2}{2}$.

Proof. Consider $a > 0$ and $b > 0$ for the two equations (4.1) and (4.2). From the definition of $k$ in (3.8) we have
\[
(4.9) \quad \sqrt{b} = \frac{1 - k}{1 + k} \sqrt{a}.
\]
thus the two conditions (4.1) and (4.2) are expressed in terms of $a$ and $k$, namely
\[
(4.10) \quad \sqrt{a} = K(k) (1 + k) \sqrt{\frac{2}{r}}.
\]
Substituting (4.10) to (4.2), we arrive at the following equation
\[ L(k) = r, \ 0 \leq k < 1. \]
From Lemma 6 for $r > \frac{\pi^2}{2}$, we can find $k = L^{-1}(r) > 0$. From (4.9) and (4.10), $a$ and $b$ can be computed as in (4.8). \qed
Finally we obtain the following theorem

**Theorem 8.** Let $r > \frac{\pi^2}{2}$. Then the delay differential equation (1.3) has a periodic solution of period 2. The periodic solution is expressed as in (3.9), where $a$ and $b$ are determined in Proposition 7.

Denote by $x^*(t)$ the periodic solution of (1.3) with $x^*(0) = a$, which satisfies (2.8). It is easy to see that

$$\max_{t \in [-1, 1]} x^*(t) = x^*(0) = a, \quad \min_{t \in [-1, 1]} x^*(t) = x^*(-1) = x^*(1) = b.$$ 

Thus one sees that $x^*(t)x^*(t - 1) = ab$ for $t \in \mathbb{R}$. From (4.8) it can be shown that $\lim_{r \to \infty} (a, b) = (\infty, 0)$, thus the amplitude of the periodic solution tends to $\infty$ as $r \to \infty$. We also note that

$$\lim_{r \to \infty} ab = 0.$$ 

Finally, from the symmetry of the Jacobi elliptic functions, it follows that

$$x^*(2n + s) = x^*(2n - s), \quad n \in \mathbb{Z}, \quad s \in \mathbb{R}.$$ 

5. **Discussion**

In this paper we prove the existence of a periodic solution of the delay differential equation (1.3). The periodic solution satisfies the nonlinear ordinary differential equation (2.5). Since the system (2.5) has conservative quantities, the system (2.5) is reduced to the Duffing equation with surprise. Then we obtain the explicit solution in terms of the Jacobi elliptic functions.
Figure 4.2. Time profile of the periodic solution for $r = 5$ and $r = 10$.

Primarily, with Gabor Kiss and Gabriella Vas, the project has started with finding a periodic solution of the delay differential equation of the form

$$\frac{dx(t)}{dt} = rx(t) \left(1 - \alpha x(t) - \int_0^1 x(t-s) \, ds\right),$$

where $0 \leq \alpha < 1$. The delay differential equation (5.1) arises from a mathematical model for disease transmission dynamics, as it is explained in Appendix. Differently from the Wright equation, the estimation of the non-delay term, together with the distributed delay term, seems to be an obstacle, when one tries to construct a Poincare map to find a periodic solution (cf. [15, 29, 30]). Multiple periodic solutions seem to be possible for the SIRS model in Appendix with the demographic turn-over [27]. Multiplicity of the periodic solution is also shown for logistic equations with multiple delays [16].

In this paper we study the delay differential equation (1.3), setting $\alpha = 0$ in the equation (5.1) to simplify the problem. Numerically we observe the periodic solution of period 2 that attracts many positive solutions. Uniqueness and stability of the periodic solution is an open problem, which are left for a future work.

Acknowledgement. The work has been initiated at the discussion with Prof. Hans-Otto Walther, who kindly introduced his habilitation thesis to the author. The author is grateful for his hospitality at the University of Giessen in February 2016. The author thanks Gabriella Vas and Gabor Kiss for a lot of discussions on the periodic solutions of delay differential equations. The author also thanks Prof.
Benjamin Kennedy and Prof. Tibor Krisztine for the interest in the study. Finally the author would like to thank Prof. Emiko Ishiwata, who kindly introduced the area of integrable systems to the author. The author was supported by JSPS Grant-in-Aid for Young Scientists (B) 16K20976 of Japan Society for the Promotion of Science.

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The delay differential equation (1.3) can be related to an epidemic model that accounts for temporary immunity ([2, 10, 7, 27, 33]). Let us derive the delay differential equation (1.3) as a limiting case of the following SIRS type epidemic model with temporary immunity

\[
\begin{align*}
\frac{d}{dt}S(t) &= -\beta S(t)I(t) + \gamma I(t - \tau), \\
\frac{d}{dt}I(t) &= \beta S(t)I(t) - \gamma I(t), \\
\frac{d}{dt}R(t) &= \gamma I(t) - \gamma I(t - \tau).
\end{align*}
\]

where \(S(t), I(t)\) and \(R(t)\) respectively denote the fraction of susceptible, infective and recovered populations at time \(t\). The model describes transition of susceptible, infective and recovered populations. The model (A.1) has three parameters: transmission coefficient \(\beta > 0\), the recovery rate \(\gamma > 0\) and the immune period \(\tau > 0\). The initial condition is given as follows

\[
S(0) = S_0 > 0, \quad I(s) = \psi(s), \quad s \in [-\tau, 0], \quad R(0) = \gamma \int_{0}^{\tau} \psi(-s)ds,
\]

where \(\psi\) is a positive continuous function. We now require that

\[
S(0) + I(0) + R(0) = S_0 + \psi(0) + \gamma \int_{0}^{\tau} \psi(-s)ds = 1,
\]
so that

\[(A.2) \quad S(t) + I(t) + R(t) = 1, \quad t \geq 0\]

implying that the total population is constant. We also have the following identity

\[(A.3) \quad R(t) = \gamma \int_0^T I(t-s)ds, \quad t \geq 0.\]

From (A.2) and (A.3) we get

\[S(t) = 1 - I(t) - \gamma \int_0^T I(t-s)ds.\]

Then from (A.1b) we obtain the following scalar delay differential equation

\[(A.4) \quad \frac{d}{dt}I(t) = I(t) \left\{ \beta \left(1 - I(t) - \gamma \int_0^T I(t-s)ds\right) - \gamma \right\}.\]

We let \(x(t) = \frac{I(t)}{I_e}\), where \(I_e\) is a nontrivial equilibrium of (A.4) given as

\[I_e = \frac{1}{1 + \gamma \tau}.\]

It is assumed that \(\beta > \gamma\) to ensure \(I_e > 0\). Considering a nondimensional time so that the immune period is 1, we obtain

\[\frac{d}{dt}x(t) = (\beta - \gamma) x(t) \left(1 - \frac{x(t) + \gamma \tau \int_0^1 x(t-s)ds}{1 + \gamma \tau}\right).\]

We now fix \(r = \beta - \gamma\) and let \(\gamma \tau \to \infty\) to formally obtain the equation \((1.3)\). Local stability analysis for (A.4) can be found in [10, 7]. See also [23] for the application of the mathematical model to explain the periodic outbreak of a childhood disease.