EXISTENCE OF SOLUTIONS FOR QUASILINEAR DIRICHLET PROBLEMS WITH GRADIENT TERMS

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Dedicated to Professor Vicentiu Radulescu on the occasion of his 60th birthday, with deep feelings of esteem and affection.

Abstract. In this paper we prove an existence theorem for positive solutions of a nonlinear Dirichlet problem involving the \( p \)-Laplacian operator on a smooth bounded domain when a nonlinearity depending on the gradient is considered. Our main theorem extends a previous result by Ruiz in [19], in which a slight modification of the celebrated blowup technique due to Gidas and Spruck, [11] and [12], is introduced.

1. Introduction. Motivated by [19], in this paper we prove an existence theorem for positive weak solutions of a nonlinear Dirichlet problem involving the \( p \)-Laplacian operator, namely

\[
\begin{cases}
\Delta_p u + f(x,u,\nabla u) = 0 , & x \in \Omega \\
 u(x) = 0 , & x \in \partial \Omega
\end{cases}
\]

where \( \Omega \subset \mathbb{R}^N, N > 1 \), is a bounded smooth domain, \( \Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u) \) is the \( p \)-Laplacian operator with \( 1 < p < N \) and \( f : \Omega \times \mathbb{R}_+^+ \times \mathbb{R}^N \to \mathbb{R} \) is a nonnegative continuous function such that

\[
(F) \quad u^{\delta} - Mu^{s}|\eta|^\vartheta \leq f(x,u,\eta) \leq c_0 u^{\delta} + Mu^{s}|\eta|^\vartheta,
\]

for all \((x,u,\eta) \in \Omega \times \mathbb{R}_+^+ \times \mathbb{R}^N\), where \( c_0 \) and \( M \) are positive constants, with \( c_0 \geq 1 \), while the exponents \( \delta, s, \) and \( \vartheta \) satisfy

\[
\delta \in (p-1, p_*) - 1 \quad \text{with} \quad p_* = \frac{p(N-1)}{N-p},
\]

\[
s \in [0, S) \quad \text{and} \quad \vartheta \in \left( p - s - 1, \frac{p(\delta - s)}{\delta + 1} \right)
\]

with \( S = \min \left\{ p - 1, \frac{\delta}{N}, \delta - \frac{p}{p}(\delta + 1) \right\} \).

The novelty we introduce, respect to [19], is to consider a nonlinearity \( f \) involving an explicit dependence on the solution \( u \) in the gradient term. We point out that the presence of a factor depending both on a power of \( u \) and of \( |\nabla u| \) makes the analysis fairly delicate. Ruiz in [19], considers the subcase of \((F)\) when \( s = 0 \).

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Observe that problem (1) does not have, in general, a variational structure because of the dependence on the gradient of the nonlinearity. Thus topological methods will be used to prove the existence of solutions.

Similar problems have been intensively studied in literature, especially when \( p = 2 \), for instance in the classical paper by Brezis and Turner [3], the authors assume conditions on \( f \) stronger than \((F)\) and while in the paper by De Figueiredo, Lions and Nussbaum [7], the nonlinearity \( f \) does not depend on the gradient and some other technical conditions are imposed. For further result relative to boundary Dirichlet problems when \( p = 2 \) and with a gradient term we refer to the pioneering paper by Ghergu and Radulescu [10], see also [8] where a competition between an anisotropic potential, a convection term \(|\nabla u|\) and a singular nonlinearity is taken under consideration, see also the recent paper [9] where also a diffusion term depending on \( u \) inside the divergence.

Concerning the \( p \)-Laplacian case, a first natural approach to solve problem (1) is to deal with radial solutions. This procedure has been carried out by Clément, Manásevich and Mitidieri in [4] for the case of systems, but without a dependence of the nonlinearity on the gradient. Later in [2], Azizieh and Clement, studied problem (1), under some additional conditions, indeed they consider the case \( 1 < p \leq 2 \) and a nonlinear function \( f \) not depending on \( x \) and on \( \nabla u \), and when \( \Omega \) is a convex domain. In [19], Ruiz removes all these conditions in order to treat problem (1), with a nonlinearity \( f \) satisfying \((F)\) in the subcase \( s = 0 \).

For further existence results for more general Dirichlet problems with gradient terms, we refer to [16] where Motreanu and Tanaka develop an approach based on approximate solutions and on a new strong maximum principle, and to the paper of Radulescu, Xiang and Zhang, [18], where existence of nonnegative solutions for a \( p \)-Kirchhoff type problem driven by a non-local integro-differential operator with homogeneous Dirichlet boundary data is investigated.

To prove the existence of positive solutions for (1), when \( s = 0 \), Ruiz uses a degree argument which was first utilized by Krasnoselskii (see [13]). The natural approach to do that is to find a priori \( L\infty \) estimates for positive solutions and then use the degree theory. The a priori estimates have been established by a blowup procedure, a method developed for the semilinear case by Gidas and Spruck in their celebrated papers, [11] and [12]. Roughly speaking, the blowup technique is based on suitable scaling arguments and allows to obtain existence of solutions of Dirichlet problems on bounded domains by combining a priori estimates and Liouville type theorems. Indeed, in [12], where general bounded domains \( \Omega \) are considered, not necessarily convex for instance, Gidas and Spruck reduce the question of the a priori bounds for positive solutions of \( \Delta u = f(x, u) \) in \( \Omega \), \( u = 0 \) in \( \partial \Omega \) to the nonexistence of positive classical solutions for the two problems

\[
\Delta u + u^\delta = 0 \quad \text{in } \mathbb{R}^N 
\]

and

\[
\begin{cases}
\Delta u + u^\delta = 0 & \text{in } H_+ \\
u = 0 & \text{on } \partial H_+
\end{cases}
\]

where the exponent \( \delta \) is related to a growth condition on \( f(x, u) \) with respect to \( u \) and \( H_+ = \{ x \in \mathbb{R}^N \mid x_N > 0 \} \) is a halfspace in \( \mathbb{R}^N \). The latter problem, when the \( p \)-Laplacian operator is involved, is very delicate and still nowadays Liouville type theorems in general cases, are not available in literature. For this reason, to
avoid problem (5) several methods in literature have been developed, for instance the convexity of the domain as in [4] and [2].

The technique used starts with the proof of a priori estimates on the pair \((u, \lambda)\), with \(\lambda \geq 0\), solution of the parametric problem

\[
\begin{cases}
\Delta_p u + f(x, u, \nabla u) + \lambda = 0, & x \in \Omega \\
u(x) = 0, & x \in \partial \Omega
\end{cases}
\]

when \(\lambda\) is small. Indeed, in Proposition 1, we prove that (6) has no solutions at all when \(\lambda \geq \lambda_0\), for a certain \(\lambda_0\) positive. Thus we need to obtain uniform estimates (in \(L^\infty\) sense) on the weak solutions of (6), only when \(\lambda \in [0, \lambda_0)\).

This uniform estimates are obtained by using suitable scaling arguments together with Liouville type theorems on the entire space \(\mathbb{R}^N\). The idea is to suppose, by contradiction, that there exists a divergent sequence of positive solutions \(u_n\) of (6), attaining their maxima in \(x_n \in \Omega\). This procedure yields the existence of a nontrivial positive solutions of the analogous limit problems (4) or (5) for the \(p\)-Laplacian. In particular, Gidas and Spruck, in [12], show that this blowup method produces a solution of (5) in a halfspace, when the points \(x_n\) approach sufficiently fast to the boundary of \(\Omega\). In order to avoid the boundary case, for instance in [2], the authors assume that \(\Omega\) is convex and that \(1 < p \leq 2\), this restriction is due to a simmetry result due to Damascelli and Pacella in [5] based on the moving plane method which guarantees that the sequence \(x_n\) cannot approach to the boundary.

In [19], Ruiz, to avoid problem (5), produces a slight modification of the blowup method using the same technique but centered on a certain fixed point \(y_0 \in \Omega\) instead of \(x_n\). In order to do this, he needs some Harnack type inequalities, due to Trudinger [23], Serrin [20] and Serrin and Zou [21]. Hence, to use the above new version of the blowup technique, we have extended these Harnack type inequalities, following mainly the arguments in [20] and [21] to include the new case \((F)\), where \(s \neq 0\). Thanks to this procedure, the corresponding limit problem will be defined in all of \(\mathbb{R}^N\) so that we obtain a contradiction by a classical Liouville theorem.

This paper is organized as follows. In Section 2, some Harnack type inequalities are proved together with some preliminaries, then Section 3 is devoted to prove the main a priori estimates for solution of the parametric problem (6). Finally, in Section 4 we present the main result of the paper, namely Theorem 4.2.

2. Harnack inequalities. Before stating and proving the main result, we need to give several preliminary lemmas, which extend the analogous ones in [19], given for the case \(s = 0\). In what follows \(C\) is a positive constant which may vary from one expression to another, but it is always independent to \(u\).

**Lemma 2.1.** Let \(u\) be a positive weak \(C^1\) solution of the inequality

\[
-\Delta_p u \geq u^\delta - M u^s |\nabla u|^\vartheta
\]

in a domain \(\Omega \subset \mathbb{R}^N\), where \(\delta > p - 1\) and

\[
s \in \left[0, \min \left\{p - 1, \delta - \frac{\vartheta}{p} (\delta + 1) \right\} \right], \quad p - s - 1 < \vartheta < \frac{(\delta - s)p}{\delta + 1}.
\]

Take

\[
\gamma \in (0, \delta), \quad \sigma \in [0, \delta) \quad \text{and} \quad \mu \in \left(0, \frac{(\delta - \sigma)p}{\delta + 1}\right).
\]
Let $R_0 > 0$ be fixed, and $0 < R < R_0$. Denote by $B_R$ a ball of radius $R$ such that the corresponding ball $B_{2R}$ of radius $2R$ is contained in the domain $\Omega$. Then, there exists a positive constant $C = C(N, p, \delta, \gamma, R_0)$ such that
\[
\int_{B_R} u^\gamma \, dx \leq C R^{N - \frac{\mu + 1}{\mu + p} - \tau}.
\] (9)
for all $\gamma \in (0, \delta)$. Similarly, there exists a positive constant $C = C(N, p, \delta, \gamma, \sigma, \mu, R_0)$ such that
\[
\int_{B_R} u^\sigma |\nabla u|^\mu \, dx \leq C R^{N - \frac{\mu(d + 1) + \sigma}{\mu + p} - \mu - \sigma}.
\] (10)

**Proof.** The proof basically uses the same ideas of the proof of Lemma 2.1 of [19], but adapted to the new case with $s \neq 0$.

We can suppose that the ball $B_R$ is centered at zero. First we prove the integral estimate (9).

Let $\xi$ be a radially symmetric $C^2$ cut-off function such that

(i) $\xi = 1$ for $|x| < 1$;
(ii) $\xi$ has compact support in $B_2(0)$ and $0 \leq \xi \leq 1$ in $\mathbb{R}^N$;
(iii) $|\nabla \xi| \leq C$.

Let $d = \delta - \gamma > 0$ and take $\phi = \left[\xi\left(\frac{x}{\delta}\right)\right]^d u^{-d}$ as a test function for the weak formulation of inequality (7). This gives at once
\[
d\int_{\Omega} \xi^k u^{s-d-1} |\nabla u|^p \, dx + \int_{\Omega} \xi^k u^s \, dx \leq M \int_{\Omega} \xi^k u^{s-d} |\nabla u|^\sigma \, dx
+ k \int_{\Omega} \xi^{k-1} u^{-d} |\nabla u|^{|\sigma|} |\nabla \xi| \, dx.
\] (11)

We apply Young inequality to the first term on the right hand side of (11), with exponents $\mu$ and $\mu'$, and since
\[
\xi^k u^{s-d} |\nabla u|^\sigma = \varepsilon \xi^k \left(\frac{\sigma}{\sigma + \mu'}\right) u^{s-d - \frac{d+1}{\mu} + \frac{d+1}{\mu}} |\nabla u|^\sigma
\leq \varepsilon \mu' \xi^k u^{s-d-1} |\nabla u|^\sigma \mu' + \frac{1}{\varepsilon \mu} \xi^k u^{s-d+\frac{d+1}{\mu}}
\]
choosing $\mu' = p/\theta$, hence $\mu = p/(p-\theta)$, we have
\[
\int_{\Omega} \xi^k u^{s-d} |\nabla u|^\sigma \, dx \leq \varepsilon \frac{\mu}{\sigma + \mu'} \int_{\Omega} \xi^k u^{s-d-1} |\nabla u|^p \, dx
+ \frac{1}{\varepsilon \mu} \int_{\Omega} \xi^k u^{\frac{d+1}{\sigma + \mu'}}^{-1-d} \, dx.
\] (12)

Now apply again Young inequality to the second term on the right hand side of (11), with exponents $p$ and $p'$, and since
\[
\xi^{k-1} u^{-d} |\nabla u|^{p-1} |\nabla \xi| = \xi^{k-1} \left(\frac{1}{p} + \frac{1}{p'}\right) u^{-d - \frac{d+1}{\mu} + \frac{d+1}{p'}} |\nabla u|^{p-1} |\nabla \xi|
\leq \frac{d}{2} \xi^{k-1} u^{-d-1} |\nabla u|^{p} + \left(\frac{2}{d}\right)^{\frac{d}{p}} \xi^{k-p}(d+\frac{d+1}{p'}) |\nabla \xi|^{p}
\]
we have
\[
\int_{\Omega} \xi^{k-1} u^{-d} |\nabla u|^{p-1} |\nabla \xi| \, dx
\]
\[
\int_{\Omega} \xi^k u^{-d-1} |\nabla u|^p \, dx + \left( \frac{2}{d} \right)^{\frac{p}{d}} \int_{\Omega} \xi^{k-p} u^{p-1-d} |\nabla \xi|^p \, dx.
\]

Observe that by (iii) then \(|\nabla \xi (|x|/R)| \leq C/R\), therefore we have
\[
\int_{\Omega} \xi^{k-1} u^{-d} |\nabla u|^{p-1} |\nabla \xi| \, dx \leq \frac{d}{2} \int_{\Omega} \xi^k u^{-d-1} |\nabla u|^p \, dx
\]
and
\[
\int_{\Omega} \xi^{k-1} u^{-d} |\nabla u|^{p-1} |\nabla \xi| \, dx \leq \frac{d}{2} \int_{\Omega} \xi^k u^{-d-1} |\nabla u|^p \, dx
\]
\[
+ \frac{C}{R^p} \int_{A_R} \xi^{k-p} u^{p-1-d} \, dx,
\]
where \(A_R = B_{2R}(0) \setminus B_R(0) = supp|\nabla \xi|\). Substituting (12) and (13) in (11) we obtain
\[
\left( \frac{d}{2} - M\varepsilon \frac{p}{d} \right) \int_{\Omega} \xi^k u^{-d-1} |\nabla u|^p \, dx + \int_{\Omega} \xi^k u^\gamma \, dx
\]
\[
\leq M\varepsilon \frac{p}{d} \int_{\Omega} \xi^k u^{\frac{p}{d} + \frac{\gamma}{2}} \, dx + \frac{C}{R^p} \int_{A_R} \xi^{k-p} u^{\gamma-r} \, dx
\]
where \(r = \delta + 1 - p\).

Let us focus on the case \(\gamma > r\). Apply again Young inequality to the first integral of the right hand side, with exponents \(\nu\) and \(\nu'\), and we obtain
\[
\int_{\Omega} \xi^k u^{\frac{p}{d} + \frac{\gamma}{2} - 1 - d} \, dx \leq \tau^\nu \int_{\Omega} \xi^k u^{\frac{p(s+1)}{d} - 1 - d} \, dx + \frac{1}{\tau^{\nu'}} \int_{\Omega} \xi^k u \, dx.
\]
Now we choose \(\nu\) such that
\[
\left[ \frac{p(s+1)}{d} - 1 - d \right] \nu = \delta - d (= \gamma)
\]
thus
\[
\nu = \frac{\delta - d}{\frac{p(s+1)}{d} - 1 - d}, \quad \nu' = \frac{(p - \vartheta)(\delta - d)}{p(\delta - s) - \vartheta(\delta + 1)}.
\]
In particular, \(d < \delta\) from the choice of \(d\), while \(\gamma > r\) forces that
\[
\delta + 1 - \frac{p(s+1)}{d} < r,
\]
being \(\vartheta > p - s - 1\), so that
\[
\frac{p(s+1)}{d} - 1 - d > 0.
\]
Furthermore, \(\nu > 1\) since from \(\vartheta < \frac{\delta - s}{\delta + 1}\) it follows
\[
(\delta - d)(p - \vartheta) > p(s+1) - (d+1)(p - \vartheta).
\]
Consequently, it holds
\[
\int_{\Omega} \xi^k u^{\frac{p}{d} + \frac{\gamma}{2} - 1 - d} \, dx \leq \tau^\nu \int_{\Omega} \xi^k u \, dx + CR^N
\]
being \(0 \leq \xi \leq 1\) and \(supp \xi \) is in \(B_{2R}(0)\). Now apply Young inequality to the second integral of the right hand side on (14), with exponents \(\alpha\) and \(\alpha'\), and since
\[
\xi^{k-p} u^{\gamma-r} = \xi^k \left( \frac{1}{\alpha'} + \frac{1}{\alpha} \right) - p u^{-d+p-1 - \frac{\delta - d}{\alpha'} - \frac{\delta - d}{\alpha}}
\]
we have
\[
\int_{A_R} R^{-p} \xi^{k-p} u^{\gamma-r} \, dx \leq \frac{1}{2} \int_{A_R} \xi^k u^{\delta-d} \, dx + C \int_{A_R} \xi^{k-\alpha p} u^{(-d+p-1 - \frac{\delta - d}{\alpha})} R^{-p\alpha} \, dx,
\]
and we impose

\[-d + p - 1 = \frac{\delta - d}{\alpha'}\]

from which

\[\alpha' = \frac{\delta - d}{p - 1 - d}, \quad \alpha = \frac{\delta - d}{\delta - p + 1}\]

and \(\alpha > 1\) follows from \(\delta > p - 1\) and \(\gamma > r\), that is \(d < p - 1\). Consequently, from \(0 \leq \xi \leq 1\) in \(A_R\) and for \(k\) sufficiently large, say \(k > \alpha p\), and since \(\gamma = \delta - d\) we have

\[
\int_{A_R} R^{-\frac{p}{\varepsilon - \sigma}} |\nabla u|^p dx + C R^{N - \frac{d - \sigma}{\varepsilon - \sigma + \tau}}. \tag{16}
\]

Replacing \((15)\) and \((16)\) in \((14)\) and using that \(\gamma = \delta - d\) we obtain

\[
\left(\frac{d}{2} - M\varepsilon^\frac{\gamma}{\sigma}\right) \int |\nabla u|^p dx + \left(1 - \frac{M\tau}{\varepsilon - \sigma}\right) \int |u|^{\delta - p} dx \leq \frac{MC}{\varepsilon - \sigma} R^N + C R^{N - \frac{d - \sigma}{\varepsilon - \sigma + \tau}} \tag{17}
\]

with \(C_1, C_2 > 0\) and where we have chosen \(\varepsilon\) and \(\tau\) such that

\[
\frac{d}{2} - M\varepsilon^\frac{\gamma}{\sigma} = \frac{d}{4}, \quad 1 - \frac{M\tau}{\varepsilon - \sigma} = \frac{1}{4},
\]

namely

\[
\varepsilon = \left(\frac{d}{4M}\right)^{\sigma/p}, \quad \tau = \left[\frac{1}{4M} \left(\frac{d}{4M}\right)^{-\frac{\sigma}{\sigma - \sigma + \tau}}\right]^{\frac{p(\sigma + 1) - (d + 1)(\sigma - \sigma + \tau)}{(\sigma - \sigma + \sigma - \sigma + \tau)}}. \tag{18}
\]

Taking into account that

\[
\int |\nabla u|^p dx \geq 0
\]

and since \(C_1 + C_2 R^{\frac{d - \sigma}{\varepsilon - \sigma + \tau}} \leq C = C(R_0)\), where \(R_0\) is given in the statement of the theorem, from \((17)\), being \(0 \leq \xi \leq 1\) in \(B_{2R}\) with \(\xi = 1\) on \(B_R\), we obtain

\[
\int_{B_R} |u|^{\delta - p} dx = \int_{B_R} |\nabla u|^p dx \leq \int |\nabla u|^p dx \leq C R^{N - \frac{d - \sigma}{\varepsilon - \sigma + \tau}}
\]

namely the estimate \((9)\) holds for \(\gamma > r\).

Now consider the case \(\gamma = r\). Observe that we have

\[
\frac{p(s + 1)}{p - \vartheta} - 1 - d = \frac{p(s + 1 - p + \vartheta)}{p - \vartheta} := \varphi > 0 \tag{19}
\]

since \(\vartheta > p - 1 - s\) and furthermore \(\varphi < \gamma\), indeed

\[
\frac{p(s + 1)}{p - \vartheta} - 1 - d < 0 \quad \text{if and only if} \quad \vartheta < \frac{p(\delta - s)}{\delta + 1}.
\]
Thus replacing $\gamma = r$ in (14), with $\varepsilon$ given in (18), we have
\[
d\frac{d}{4} \int_{\Omega} \xi^{k} u^{-d-1} |\nabla u|^{p} \, dx + \int_{\Omega} \xi^{k} u^{r} \, dx \leq M \varepsilon^{-\frac{p}{r-p}} \int_{\Omega} \xi^{k} u^{\varrho} \, dx + \frac{C}{R^{p}} \int_{A_{R}} \xi^{k-p} \, dx \]
which, as above, yields
\[
\int_{\Omega} \xi^{k} u^{r} \, dx \leq C \int_{\Omega} \xi^{k} u^{\varrho} \, dx + CR^{N-p}.
\]
From $\varrho < r$ we can apply Young inequality to the first integral on the right hand side, with exponents $r/\varrho$ and $r/(r-\varrho)$,
\[
\int_{\Omega} \xi^{k} u^{\varrho} \, dx \leq \frac{1}{2} \int_{\Omega} \xi^{k} u^{r} \, dx + \int_{\Omega} \xi^{k} u^{\varrho} \, dx \leq \frac{1}{2} \int_{\Omega} \xi^{k} u^{r} \, dx + CR^{N},
\]
so that, thank to (21), we arrive to
\[
\int_{B_{R}} u^{\gamma} \, dx \leq R^{N-p} \left( C_{1} R^{p} + C_{2} \right) \leq CR^{N-p}
\]
where $C = C_{1} R^{p} + C_{2}$. Thus also for $\gamma = r$ inequality (9) holds.
Now consider the case $\gamma < r$. We apply Hölder inequality, with exponents $\frac{\varrho}{\gamma}$ and $r/(r-\gamma)$, to obtain
\[
\int_{B_{R}} u^{\gamma} \, dx \leq \left( \int_{B_{R}} u^{r} \, dx \right)^{\frac{\varrho}{r}} \left( \int_{B_{R}} u^{\gamma} \, dx \right)^{\frac{r}{\gamma}}
\]
from which
\[
\left( \int_{B_{R}} u^{r} \, dx \right)^{\frac{\varrho}{r}} \leq CR^{N} \left( \frac{\varrho}{\gamma} \right) \int_{B_{R}} u^{r} \, dx
\]
and applying (9) with $\gamma = r$, we have
\[
\int_{B_{R}} u^{\gamma} \, dx \leq CR^{N} \left( \frac{\varrho}{\gamma} \right) + N-p \right)^{\frac{\gamma}{\gamma-r}} = CR^{N-\frac{\varrho}{r}}.
\]
This gives the required conclusion (9), since $r = \delta + 1 - p$.

Finally we can prove the integral estimate (10). Note that from (9) it results that $\mu < p$ and write
\[
ua^{\sigma} |\nabla u|^{\mu} = u^{\frac{\gamma-\delta-1}{p} + \sigma - \frac{\gamma-\delta-1}{p}} |\nabla u|^{\mu},
\]
for some $\gamma > r$. By Hölder inequality, with exponents $p/\mu$ and $p/(p-\mu)$, to get
\[
\int_{B_{R}} u^{\sigma} |\nabla u|^{\mu} \, dx \leq \left( \int_{B_{R}} u^{\gamma-\delta-1} |\nabla u|^{p} \, dx \right)^{\frac{\mu}{p}} \left( \int_{B_{R}} u^{\tau} \, dx \right)^{1-\frac{\mu}{p}}
\]
where $\tau = \left[ \sigma - \frac{\gamma-\delta-1}{p} \right] \frac{p}{p-\mu}$. To apply the integral estimate (9) to the second integral in the right hand side we need $\tau < \delta$, which holds if and only if $\mu < p(\delta - \sigma)/(2\delta + 1 - \gamma)$. Since
\[
\lim_{\gamma \to \delta} \frac{p(\delta - \sigma)/(2\delta + 1 - \gamma) = p(\delta - \sigma)/(\delta + 1)}
\]
and being $\mu < p(\delta - \sigma)/\delta + 1$ by (8), thus we can choose $\gamma$ suitably close to $\delta$ so that $\gamma < \delta$. From (17), since $-d - 1 = \gamma - \delta - 1$ and having chosen $\gamma > r$, we obtain
\[\int_{B_R} u^{q-\delta-1} |\nabla u|^p \, dx \leq \int_{\Omega} \xi^k u^{-d-1} |\nabla u|^p \, dx \leq CR^{N-p+1} \gamma^{-\delta}.\]
Inserting this inequality in (22) and using (9) with $\gamma = \gamma$, we arrive to
\[\int_{B_R} u^q |\nabla u|^p \, dx \leq CR^{N-p+1} \gamma^{1-\gamma} \gamma^1 \gamma^{-\gamma} \gamma^{-\gamma} = CR^{N-p+1} \gamma^{-\gamma}, (23)\]
that is the required estimate (10).

**Remark 1.** Lemma 2.1, when $s = 0$ in (7) and $\sigma = 0$ in (10) reduces to Lemma 2.1 in [19] and furthermore to Lemma 2.4 in [21] when $M = M_s = 0$, $s = 0$, $\vartheta = \vartheta = 0$ and $\sigma = 0$.

The following lemma is a result of Harnack type due to Serrin, see also Lemma 4.2 in [21].

**Lemma 2.2.** (Theorem 5, [20]) Let $u$ be a nonnegative weak solution in a domain $\Omega$ of
\[|\Delta_p u| \leq c(x)|\nabla u|^{p-1} + d(x)u^{p-1} + f(x),\]
where $c \in L^{q'}(\Omega)$, $d \in L^q(\Omega)$, $q' > N$.

Then, for every $R$ such that $B_{2R} \subset \Omega$, there exists constant $C$ depending on
\[N, p, q, R^{1-N/q'}c||L^{q'}, R^{q-N/q}d||L^q\]
such that
\[\sup_{B_R} u \leq C \left( \inf_{B_R} u + R^{q-N/q}||f||L^{q'} \right).\]

Note that assumption $q' > N$ implies $p > N/q$, so that all the exponents of $R$ in (24) are positive.

Combining Lemma 2.1 and Lemma 2.2 it follows another Harnack type inequality, which extends Theorem 2.3 in [19], which corresponds to the case $s = 0$.

**Theorem 2.3.** Let $\lambda > 0$. Suppose that $u$ is a positive weak solution in $\Omega$ of the inequality
\[u^\delta - Mu^s|\nabla u|^\sigma \leq -\Delta_p u \leq c_0 u^\delta + Mu^s|\nabla u|^\vartheta + \lambda\]
with $\delta, s$ and $\vartheta$ satisfy (2) and (3). For every $R \geq 0$ such that $B_{2R} \subset \Omega$, then there exists a positive constant $C = C(N, p, \delta, \vartheta, s, R_0, M)$ such that
\[\sup_{B_R} u \leq C \left( \inf_{B_R} u + R^{q-N/q}||f||L^{q'} \right).\]

**Remark 2.** As noted in [19] that Theorem 2.3 is a version of Theorem 4.1(b) in [21], where only the case $\vartheta = p - 1$ and $s = 0$ is considered.

**Proof.** We use Lemma 2.2 with
\[c(x) = Mu(x)^s|\nabla u(x)|^{\vartheta+1-p}, \quad d(x) = c_0 u(x)^\delta-p+1, \quad f(x) = \lambda,\]
thus we have to verify that $c \in L^{q'}(B_{2R})$, for some $q' > N$, and $d, f \in L^q(B_{2R})$ so that
\[R^{1-N/q'}||c||L^{q'}(B_{2R}) < C, \quad R^{q-N/q}||d||L^q(B_{2R}) < C\]
with for certain $C > 0$. Consequently, the estimate (25) follows at once since $||f||L^q(B_{2R}) = \lambda CR^{N/q}$ from which $R^{q-N/q}||f||L^q(B_{2R}) = \lambda CR^p$. 

To obtain the first estimate in (26) it is enough to apply inequality (10) in Lemma 2.1 with 
\[ \mu = q'(\vartheta + 1 - p) \quad \text{and} \quad \sigma = sq', \]
yielding 
\[ \|c\|^q \|_{L^q(B_{2R})} = M \int_{B_R} u^{q'} |\nabla u|^{q'(\vartheta + 1 - p)} \, dx \leq CR^\frac{1}{q} \left[ N - \frac{\mu(\vartheta + 1 - p)}{\sigma} \right] \]  
for some \( q' > N \). Thus we have to verify \( \sigma < \vartheta \) and \( \mu < p(\vartheta - \sigma)/(\delta + 1) \). The first inequality follows choosing \( q' > N \) close enough to \( N \) since \( sN < \vartheta \) by assumption.

To obtain the second inequality it is enough to prove \( \mu < p(\vartheta - \sigma)/(\delta + 1) \). Indeed, denoting with \( l_1 \) and \( l_2 \)
\[ \lim_{q' \to N^+} q'(\vartheta + 1 - p) = N(\vartheta + 1 - p) := l_1 \]
and 
\[ \lim_{q' \to N^+} \frac{p(\vartheta - \sigma)}{\delta + 1} = \frac{p(\vartheta - \sigma)}{\delta + 1} := l_2 \]
we have that for \( q' \) close to \( N, q' > N \), then
\[ l_1 - \varepsilon < q'(\vartheta + 1 - p) < l_1 + \varepsilon \quad \text{and} \quad l_2 - \tau < \frac{p(\vartheta - \sigma)}{\delta + 1} < l_2 + \tau, \]
thus choosing \( \varepsilon \) and \( \tau \) small enough such that \( l_1 + \varepsilon < l_2 - \tau \) the inequality follows. This obviously occurs if \( l_1 < l_2 \), that is
\[ N(\vartheta - \vartheta + p + 1) < \frac{p(\vartheta - \sigma)}{\delta + 1}. \]
By (3)_2, the last inequality is a consequence of
\[ N\left( \frac{p(\vartheta - \sigma)}{\delta + 1} - p + 1 \right) < \frac{p(\vartheta - \sigma)}{\delta + 1} \]
which holds since \( \delta < p_* - 1 \). From (27), to conclude the proof of (26) we write
\[ R_{1 - \frac{\vartheta}{q}}^{\frac{1}{q}} \|c\|_{L^q(B_{2R})} \leq CR^{\frac{1}{q}} \left[ N - \frac{\mu(\vartheta + 1 - p)}{\sigma} \right] \]
We just need to show that \( 1 - \frac{p(\vartheta + 1 - p)}{s(\vartheta - \vartheta + p + 1)} \geq 0 \) to have the boundness of the right hand side when \( R \leq R_0 \). This occurs if and only if
\[ \vartheta \leq \frac{s(1 - 2p) + \delta p}{\delta + 1 - s} \]
which is implied since \( \vartheta < p(\vartheta - \vartheta + p + 1) \) by (3)_2, we just need to verify that
\[ \frac{p(\vartheta - \sigma)}{\delta + 1} < \frac{s(1 - 2p) + \delta p}{\delta + 1 - s} \]  
by virtue of (3)_2. Finally, inequality (28) is equivalent to
\[ s < \frac{\delta + 1 - p}{p} \]
which follows from \( s < \delta - \vartheta/p(\delta + 1) \), being
\[ \delta - \frac{\vartheta}{p}(\delta + 1) < \frac{\delta + 1 - p}{p} \]  
which holds since \( p - \delta - 1 < 0 \).
It remains to prove the second inequality in (26). We have
\[ \|d\|_{L^q(B_{2R})} = c_0 \left[ \int_{B_{2R}} u^{\gamma} \, dx \right]^{1/q} \leq CR^{(N-\frac{p\gamma}{p'})\frac{1}{q}} = CR^{\frac{N}{q} - \frac{p}{p'}} \]
where \( \gamma = (\delta + 1 - p)q \). In the above expression we have used inequality (9) in Lemma 2.1 and to do that we need to verify that \( \gamma < \delta \). Again, as above since \( q' > N \) implies that \( q > N/p \), thus taking \( q > N/p \) as close as necessary to \( N/p \), it suffices to show that
\[ (\delta + 1 - p)\frac{N}{p} < \delta \]
and this holds since \( \delta < p_* - 1 \). Hence (26) is proved.

In the proof of the main theorem we will also make use of the following weak Harnack inequality, due to Trudinger, [23].

**Theorem 2.4.** Let \( u \geq 0 \) be a weak solution of the inequality
\[ \Delta_p u \leq 0 \quad \text{in} \quad \Omega. \]
Take \( \gamma \in [1, p_* - 1) \) and \( R > 0 \) such that \( B_{2R} \subset \Omega \).
Then there exists \( C = C(N, p, \gamma) \) (independent of \( R \)) such that
\[ \inf_{B_R} u \geq CR^{-N/\gamma}\|u\|_{L^\gamma(B_{2R})}. \]

3. **A priori estimates.** In order to prove the main result of the existence of positive solutions for (1), we will use a degree argument. First we need to obtain a priori bound for the parameter \( \lambda \) of the parametric problem (6) to find this bound, as in [19], we use the following consequence of Picone identity for the \( p \)-Laplacian, see also Theorem 1.1 in [1].

**Lemma 3.1.** Let \( u \) be a positive solution of the problem
\[ \begin{cases} -\Delta_p u = h(x) \geq 0, & x \in \Omega \\ u(x) = 0, & x \in \partial \Omega. \end{cases} \]
Then
\[ \int_{\Omega} h(x) \frac{\phi_1^p}{u^{p-1}} \, dx \leq \lambda_1 \int_{\Omega} \phi_1^p \, dx \]
where \( \lambda_1 \) is the first eigenvalue of the \( p \)-Laplacian with Dirichlet boundary conditions, and \( \phi_1 \) is the associated eigenfunction.
Moreover, the equality holds if and only if \( u(x) = a\phi_1(x) \) for some \( a > 0 \).

**Remark 3.** Note that \( \frac{\phi_1^p}{u^{p-1}} \) belongs to \( W_0^{1,p}(\Omega) \) since \( u \) is positive in \( \Omega \) and has nonzero outward derivative on the boundary because of the Hopf lemma (see [24]).

**Proposition 1.** Consider problem (6) and let \( f \) satisfy condition (F) with
\[ \delta > p - 1, \quad \vartheta \in \left( p - s - 1, \frac{p(\delta - s)}{\delta + 1} \right). \]
Then there exists \( \lambda_0 > 0 \) such that problem (6) has no positive solutions for any \( \lambda \geq \lambda_0 \).
Proof. Suppose that $u$ is a positive solution of (6). Applying Lemma 3.1 with $h(x) = f(x, u(x), \nabla u(x)) + \lambda$, we get

$$\int_\Omega [f(x, u(x), \nabla u(x)) + \lambda] \frac{\phi_1^p}{u^{p-1}} \, dx \leq \lambda \int_\Omega \phi_1^p \, dx. \quad (30)$$

Consider the function

$$\phi_\lambda(t) = \frac{\lambda + \ell \delta}{t^{p-1}}, \quad t > 0.$$ 

Since

$$\lim_{t \to 0^+} \phi_\lambda(t) = \lim_{t \to \infty} \phi_\lambda(t) = \infty$$

being $\delta > p - 1$ and

$$\phi'_\lambda(t) = t^{-p}[\delta - p + 1]t^{\delta - \lambda(p - 1)]$$

then $\phi'_\lambda(t) = 0$ if and only if $t = t^*$, with $t^* = [\lambda(p - 1)/(\delta - p + 1)]^{1/\delta}$, and $\phi_\lambda(t^*) > 0$. So it is well defined the function

$$\ell(\lambda) = \min_{R^N^+} \phi_\lambda(t) = \phi_\lambda(t^*) = C\lambda^{1 - \frac{p-1}{p}}.$$

Obviously, $\ell(\lambda) \geq 0$ for all $\lambda > 0$, and $\ell(\lambda) \to \infty$ if $\lambda \to \infty$ since $1 - (p-1)/\delta > 0$.

Using property (F), and the positivity of $\phi_1$ and $u$, we get

$$\int_\Omega [f(x, u, \nabla u) + \lambda] \frac{\phi_1^p}{u^{p-1}} \, dx \geq \int_\Omega \left[u^p - Mu^s|\nabla u|^\theta + \lambda\right] \frac{\phi_1^p}{u^{p-1}} \, dx$$

$$= -M \int_\Omega |\nabla u|^\theta \frac{\phi_1^p}{u^{p-s-1}} \, dx + \int_\Omega \phi_\lambda(t) \phi_1^p \, dx$$

$$\geq -M \int_\Omega |\nabla u|^\theta \frac{\phi_1^p}{u^{p-s-1}} \, dx + \ell(\lambda) \int_\Omega \phi_1^p \, dx,$$

where in the last inequality we have used the definition of $\ell(\lambda)$. Combining the previous inequality with (30) we have that

$$(\ell(\lambda) - \ell_1) \int_\Omega \phi_1^p \, dx \leq \int_\Omega M|\nabla u|^\theta \frac{\phi_1^p}{u^{p-s-1}} \, dx. \quad (31)$$

Obviously, the integral on the right hand side is bounded from below thanks to the positivity of $\ell(\lambda)$. Now we claim that it is bounded. If so, (31) provides a bound for $\ell(\lambda)$ from above and therefore the coercivity of $\ell(\lambda)$ forced the boundness of $\lambda$.

In order to prove the claim, we use the weak formulation of

$$-\Delta_p u \geq u^\delta - Mu^s|\nabla u|^\theta$$

with test function $\psi = \phi_1^p/u^{p-1}$ which belongs to $W_0^{1,p}(\Omega)$, as noted at the end of the statement of Lemma 3.1, namely

$$-(p-1) \int_\Omega \frac{\phi_1^p}{u^p} |\nabla u|^p \, dx + p \int_\Omega \phi_1^{p-1} |\nabla u|^{p-2} u^{1-p} \, dx < u^\delta, \nabla \phi_1 > dx$$

$$\geq \int_\Omega u^{\delta-p+1} \phi_1^p \, dx - M \int_\Omega |\nabla u|^\theta \frac{\phi_1^p}{u^{p-s-1}} \, dx$$

which gives

$$(p-1) \int_\Omega \frac{\phi_1^p}{u^p} |\nabla u|^p \, dx + \int_\Omega u^{\delta-p+1} \phi_1^p \, dx$$

$$\leq p \int_\Omega \phi_1^{p-1} u^{1-p} |\nabla u|^{p-1} \, dx + M \int_\Omega |\nabla u|^\theta \frac{\phi_1^p}{u^{p-s-1}} \, dx. \quad (32)$$
Now we estimate the first term on the right hand side of (32) by using Young inequality with exponents $p$ and $p'$ and we obtain
\[
\int_\Omega \phi_1^{p-1} u^{1-p} |\nabla u|^{p-1} |\nabla \phi_1| dx \leq \frac{\epsilon^{p'}}{p'} \int_\Omega \phi_1^{p} |\nabla u|^p dx + C_\epsilon,
\]
where $C_\epsilon = \|\nabla \phi_1\|_{L^p(\Omega)}/p \epsilon^p$. Substituting in (32) and choosing $\epsilon$ such that
\[
p - \frac{p \epsilon}{p'} = \frac{p - 1}{2}
\]
we have
\[
\frac{p - 1}{2} \int_\Omega \frac{\phi_1^{p}}{u^{p-s-1}} |\nabla u|^p dx + \int_\Omega \phi_1^{p} u^{\delta-p+1} dx \leq M \int_\Omega \frac{\phi_1^{p}}{u^{p-s-1}} |\nabla u|^\theta dx + C_p
\]
(33)
where $C_p = \|\nabla \phi_1\|_{L^p(\Omega)}/\epsilon^p$.

Now we apply Young inequality to the second integral on the right hand side, with exponents $\mu$ and $\mu' > 1$, and we have
\[
\frac{\phi_1^{p}}{u^{p-s-1}} |\nabla u|^\theta = \epsilon \frac{1}{\epsilon} \phi_1^{p(\frac{1}{p} + \frac{1}{\mu})} u^{1+s-p(\frac{1}{p} + \frac{1}{\mu})} |\nabla u|^\theta
\]
\[\leq \epsilon \mu \phi_1^{p} |\nabla u|^\theta + \frac{1}{\epsilon \mu} \phi_1^{p} u^{(1+s-p\frac{1}{\mu})\mu'}.
\]
Choosing $\mu$ such that $\mu = p$, that is $\mu = p/\vartheta$ and $\mu' = p/(p-\vartheta)$, this can be done since $\vartheta > 1$ from $\vartheta < p(\delta-s)/(\delta+1)$ so that $\vartheta < p$, we obtain
\[
\int_\Omega \frac{\phi_1^{p}}{u^{p-s-1}} |\nabla u|^\theta dx \leq \frac{\epsilon^{\frac{p}{\vartheta}}}{\vartheta} \int_\Omega \phi_1^{p} |\nabla u|^p dx + \frac{1}{\epsilon \vartheta} \int_\Omega \phi_1^{p} u^{\frac{p(\delta-s-p+1)}{p-s}} dx.
\]
(34)
Using again Young inequality to the last integral of the above inequality, with the exponents $\nu$ and $\nu'$, we obtain
\[
\int_\Omega \phi_1^{p} u^{\frac{p(\delta-s-p+1)}{p-s}} dx \leq \tau' \int_\Omega \phi_1^{p} u^{\nu} dx + \frac{1}{\tau'} \int_\Omega \phi_1^{p} dx,
\]
and inserting in (34) we arrive to
\[
\int_\Omega \frac{\phi_1^{p}}{u^{p-s-1}} |\nabla u|^\theta dx \leq \frac{\epsilon^{\frac{p}{\vartheta}}}{\vartheta} \int_\Omega \phi_1^{p} |\nabla u|^p dx + \frac{\tau'}{\vartheta} \int_\Omega \phi_1^{p} u^{\delta-p+1} dx + \frac{1}{\epsilon \vartheta \tau'} \int_\Omega \phi_1^{p} dx,
\]
(35)
having chosen $\nu$ such that
\[
p \frac{\vartheta - p + s + 1}{p - \vartheta} = \delta - p + 1,
\]
this is possible since $\vartheta > p - s - 1$ and $\delta > p - 1$, and thus
\[
\nu = \frac{(\delta - p + 1)(p - \vartheta)}{p(\vartheta + s - p + 1)},
\]
furthermore condition $\nu > 1$ holds since $\vartheta < p(\delta-s)/(\delta+1)$.

Consequently, (33) yields
\[
\left( \frac{p - 1}{2} - M \epsilon^{\frac{p}{\vartheta}} \right) \int_\Omega \frac{\phi_1^{p}}{u^{p}} |\nabla u|^p dx + \left( 1 - \frac{M \tau'}{\epsilon \vartheta} \right) \int_\Omega \phi_1^{p} u^{\delta-p+1} dx \leq \frac{M}{\epsilon \vartheta \tau'} \int_\Omega \phi_1^{p} dx + C_p
\]
which gives
\[ \frac{p-1}{4} \int_{\Omega} \frac{\partial u}{\partial y} |\nabla u|^p \, dx + \frac{1}{2} \int_{\Omega} \frac{\partial u}{\partial y} u^{\delta-1} \, dx \leq C, \] (36)
where we have chosen \( \varepsilon \) and \( \tau \) such that
\[ \frac{p-1}{2} - M \frac{\varepsilon}{\theta} = \frac{p-1}{4}, \quad 1 - \frac{M \tau'}{\varepsilon / \theta} = \frac{1}{2} \]
namely
\[ \varepsilon = \left( \frac{p-1}{4M} \right)^{\frac{2}{p}}, \quad \tau = \left[ \frac{1}{2M} \left( \frac{p-1}{4M} \right)^{\frac{2}{p-1}} \right]^{\frac{p(\delta+1-p)+1}{(p-1)\theta(p-\delta)}}. \]
We have so obtained
\[ \int_{\Omega} \frac{\partial u}{\partial y} |\nabla u|^\delta \, dx \leq C \] (37)
thanks to (35) and (36). The claim is proved.

**Remark 4.** Proposition 1 in the case \( s = 0 \) reduces to Proposition 3.2 in [19]. Furthermore, as a consequence of Proposition 1, we need to give a priori estimates on the solutions \( u \) of (6) only when \( \lambda \) is bounded.

To obtain the required estimates, Ruiz, in [19], uses a slight modification of the well known blowup technique due to Gidas and Spruck in [11], [12]. As described in the Introduction, usually the blowup technique is based on a contradiction argument, that is the construction of a divergent sequence \( \{u_n\}_n \) of solutions of (6) and on a sequence of points \( x_n \) in which \( u_n \) attain their maxima. When \( x_n \to x \in \Omega \), then a Liouville theorem in the entire \( \mathbb{R}^N \), produces the required contradiction. However, it may happen that \( d(x_n, \partial \Omega) \to 0 \) and in this case the blowup method provides a solution in a halfspace when the points \( x_n \) approach sufficiently fast (in comparison with the \( L^\infty \) norm of \( u_n \)) to the boundary of \( \Omega \). Thus in order to get a contradiction we need a Liouville-type theorem in the halfspace. In [19], Ruiz uses the same blowup technique but centered on a certain fixed point \( y_0 \in \Omega \), instead of \( x_n \).

**Proposition 2.** Assume that (F) holds and that \( \lambda < \lambda_0 \) for some \( \lambda_0 \) fixed.

Then, there exists \( C > 0 \) such that \( ||u|| \leq C \) for any \( C^1 \) solution \( u \) of (6), where \( || \cdot || \) denotes the uniform norm.

**Proof.** We argue by contradiction and we suppose that there exist a sequence of parameters \( \{\lambda_n\}_n \), \( \lambda_n < \lambda_0 \) for all \( n \in \mathbb{N} \) and a sequence of points \( \{x_n\}_n \) in \( \Omega \) such that, denoting with \( u_n \) a solution of (6) with \( \lambda \) substituted by \( \lambda_n \), we have
\[ S_n := ||u_n|| = \sup_{\Omega} u_n(x) = u_n(x_n) \to \infty \quad \text{as} \quad n \to \infty. \] (38)
We proceed in several steps.

**Step 1.** There exists \( c \), a positive constant independent of \( n \), such that
\[ c < \delta_n, S_n^{(\delta+1-p)/p}, \quad \delta_n := d(x_n, \partial \Omega). \] (39)
From now on, we use \( c \) to denote positive constants, which may vary from one expression to another, but always independent of \( n \). To obtain (39) we make a change of variable and define the so called blowup functions
\[ w_n(x) = \frac{u_n(y)}{S_n}, \quad y = M_n x + x_n, \quad M_n = S_n^{\frac{p-1-\delta}{p}}. \] (40)
The functions \( w_n \) are well defined at least in \( B(0, \frac{\delta}{M_n}) \) and \( w_n(0) = \|w_n\| = 1 \). By standard calculations, we have
\[
\nabla w_n(x) = S_n^{-1} M_n \nabla u_n(y)
\]
and
\[
\Delta_p w_n(x) = S_n^{1-p} M_n^p \Delta_p u_n(y).
\]
Hence replacing \( u_n \) in (6) we get
\[
-\Delta_p w_n(x) = S_n^{1-p} M_n^p \left[ f(y, u_n(y), \nabla u_n(y)) + \lambda_n \right] = S_n^{-\delta} \left[ f(M_n x + x_n, S_n w_n(x), S_n M_n^{-1} \nabla w_n(x)) + \lambda_n \right] = \varphi_n(x, w_n, \nabla w_n).
\]
By (40) and using condition (F), we obtain
\[
|\varphi_n(x, w_n, \nabla w_n)| \leq S_n^{-\delta} \left[ c_0 S_n^\vartheta |w_n(x)|^\vartheta + M S_n^\vartheta |w_n(x)|^s S_n^{\vartheta \frac{p+1}{p}} |\nabla w_n(x)|^\vartheta + \lambda_n \right] = c_0 |w_n|^\vartheta + M S_n^{\vartheta \frac{p+1}{p} + s - \delta} |w_n|^s |\nabla w_n|^\vartheta + \lambda_n S_n^{-\delta}
\]
and recalling that \( \|w_n\| = 1 \) we have
\[
|\varphi_n(x, w_n, \nabla w_n)| \leq c_0 |w_n|^\vartheta + M S_n^{\vartheta \frac{p+1}{p} + s - \delta} |\nabla w_n|^\vartheta + \lambda_n S_n^{-\delta}. \quad (43)
\]
Since \( \vartheta(\delta + 1)/p + s - \delta < 0 \), by assumptions on \( \vartheta \), we have
\[
M S_n^{\vartheta \frac{p+1}{p} + s - \delta} \to 0 \quad \text{as } n \to \infty
\]
and since \( \delta > 0 \)
\[
\lambda_n S_n^{-\delta} \to 0 \quad \text{as } n \to \infty,
\]
being \( S_n \to \infty \) by (38). Thus, for \( n \) large, we can assert
\[
|\varphi_n(x, w_n, \nabla w_n)| \leq c_0 |w_n|^\vartheta + |\nabla w_n|^\vartheta + 1. \quad (45)
\]
On the other hand, we can deduce from (F) that
\[
\varphi_n(x, w_n, \nabla w_n) \geq S_n^{-\delta} \left[ S_n^\vartheta |w_n|^\vartheta - M S_n^\vartheta |w_n|^s S_n M_n^{-1} |\nabla w_n|^\vartheta + \lambda_n \right] \geq |w_n|^\vartheta - M S_n^{\vartheta \frac{p+1}{p} + s - \delta} |\nabla w_n|^\vartheta + \lambda_n S_n^{-\delta}
\]
hence
\[
\varphi_n(x, w_n, \nabla w_n) - w_n^\delta \geq -M S_n^{\vartheta \frac{p+1}{p} + s - \delta} |\nabla w_n|^\vartheta + \lambda_n S_n^{-\delta} \quad (46)
\]
for \( x \in B(0, M_n^{-1} \delta_n) \). Now we use a \( C^{1,\gamma} \) regularity result up to the boundary due to Lieberman (Theorem 1, [14]) to conclude from (45) that
\[
|\nabla w_n| \leq C \quad (47)
\]
for certain \( C > 0 \) independent of \( n \).

Let now \( y_n \in \partial \Omega \) such that \( d(x_n, y_n) = \delta_n \). By using the mean value theorem we have
\[
1 = w_n(0) - w_n(M_n^{-1} (y_n - x_n)) \leq |\nabla w_n(\xi)| M_n^{-1} \delta_n \leq C M_n^{-1} \delta_n,
\]
where
\[
w_n(M_n^{-1} (y_n - x_n)) = S_n^{-1} u_n(y_n) = 0 \quad (48)
\]
since \( y_n \in \partial\Omega \) and \( u_n \) is a solution of (6). So we find a positive constant \( c \) such that
\[
c < S_n^{(\delta + 1) - p} \delta_n = M_n \delta_n
\]
hence the required bound from below is claimed and the proof of Step 1 is concluded.

Now we have two possibilities, either \( M_n \delta_n \) is unbounded or is bounded. In the first case thanks to (38), (44) and (47) we have
\[
M S_n^{\frac{\delta + 1}{p} - \delta} |\nabla w_n|^\delta + \lambda_n S_n^{-\delta} \to 0 \quad \text{as} \quad n \to \infty
\]
so that, by (42), (46) and the fact that \( B(0, M_n^{-1} \delta_n) \to \mathbb{R}^N \) as \( n \to \infty \), we would obtain, by standard theory, a weak \( C^1 \) positive solution for the problem
\[
\Delta p u + u^\delta \leq 0 \quad \text{in} \quad \mathbb{R}^N
\]
contradicting the Liouville result of [15].

If \( M_n^{-1} \delta_n \) is bounded we cannot proceed in the same way since we arrive to a contradiction. This is the reason why Ruiz develops a blowup argument around a fixed point \( y_0 \in \Omega \).

**Step 2.** For all \( \gamma > N(\delta + 1 - p)/p \) then
\[
\int_{B(x_n, \delta_n/2)} |u_n|^{\gamma} \to \infty \quad \text{as} \quad n \to \infty.
\]
We use the Harnack type inequality given in Theorem 2.3 with \( R = \delta_n/2 \), so that by (38) we have
\[
S_n = \sup_{B(x_n, \delta_n/2)} u_n \leq C \left[ \inf_{B(x_n, \delta_n/2)} u_n + \lambda_n \frac{\delta_n^p}{2p} \right].
\]
Since \( \lambda_n \) and \( \delta_n \) are bounded, we have that \( \inf_{B(x_n, \delta_n/2)} u_n \geq c S_n \) for certain \( c > 0 \) and for \( n \) sufficiently large thus for any positive \( \gamma \)
\[
\int_{B(x_n, \delta_n/2)} |u_n|^{\gamma} \geq c S_n^{\gamma N} \geq c S_n^{\gamma(p - 1 - \delta)N/p} = c S_n^{\gamma \frac{(\delta + 1 - p)N}{p}}
\]
where in the last inequality we have used (39). Step 2 follows immediately when \( \gamma > (\delta + 1 - p)N/p \), since \( S_n \to \infty \) as \( n \to \infty \).

**Step 3.** There exists \( y_0 \in \Omega \) such that \( u_n(y_0) \to \infty \) as \( n \to \infty \).

We now use the smoothness of the boundary of \( \Omega \) (For instance \( C^2 \) regularity is enough). We can find \( \varepsilon > 0 \) and \( (y_n)_n \in \Omega \) such that
\[
d(y_n, \partial\Omega) = 2\varepsilon, \quad B(x_n, \delta_n/2) \subset B(y_n, 2\varepsilon) \quad \text{for all} \quad n \in \mathbb{N}.
\]
As noted in [19] the fact that \( \varepsilon \) can be chosen independent of \( n \) is due to the compactness and regularity of \( \partial\Omega \). We now use Theorem 2.4 and Step 2 with \( \gamma \) such that
\[
\frac{(\delta + 1 - p)N}{p} < \gamma < p_* - 1, \quad \gamma \geq 1
\]
(this is possible since \( \delta < p_* - 1 \) implies \( (\delta + 1 - p)N/p < p_* - 1 \)) to conclude
\[
\inf_{B(y_n, \varepsilon)} u_n \geq c \left( \int_{B(y_n, 2\varepsilon)} |u_n|^{\gamma} \right)^{1/\gamma} \geq c \left( \int_{B(x_n, \delta_n/2)} |u_n|^{\gamma} \right)^{1/\gamma} \to +\infty,
\]
being \( u_n \) a solution of (6) with \( f \) and \( \lambda \) nonnegative. If we define
\[
\Omega_\varepsilon = \{ x \in \Omega \mid d(x, \partial \Omega) \geq \varepsilon \}
\]
in particular we have \((y_n)_n \in \Omega_\varepsilon\). Taking a subsequence if necessary, we can assume that \( y_n \to y_0 \in \Omega \). Note that \( y_0 \in \Omega_\varepsilon = \overline{\Omega_\varepsilon} \subseteq \Omega \), hence in particular \( y_0 \in \Omega \).

Since \( y_n \to y_0 \) as \( n \to \infty \) then \( d(y_n, y_0) < \mu \) for all \( \mu \) and for \( n \) large, thus choosing \( \mu = \varepsilon \) we have \( y_0 \in B(y_n, \varepsilon) \) for \( n \) large. Hence, for all \( y \in B(y_n, \varepsilon) \) we have
\[
u_n(y_0) \geq \inf_{B(y_n, \varepsilon)} u_n \to \infty \quad \text{as} \quad n \to \infty.
\]

Thus the claim is proved. Observe also that by continuity of \( u_n(y) \) then \( u_n(y) \to +\infty \) when \( n \to \infty \), for any \( y \in B(y_n, \varepsilon/2) \).

We suppose there exist \( \overline{y} \in \Omega \) such that \( u_n(\overline{y}) \) is bounded for \( n \) large. By using Harnack inequality (Theorem 2.3) in a ball \( B_{2R}(\overline{y}) \subseteq \Omega \) such that \( B_{2R} \cap B(y_n, \varepsilon/2) \neq \emptyset \) we have
\[
\sup_{B_R} u_n(y) \leq c \left( \inf_{B_R} u_n(y) + R^\rho \lambda \right) \leq c(u_n(\overline{y}) + R^\rho \lambda) \leq cR
\]
and this is a contradiction since that \( \sup u_n(y) = \infty \). Iterating this procedure we arrive to \( u_n(y) \to \infty \) as \( n \to \infty \) for all \( y \in \Omega \).

Now we use the blowup technique given in [19] around \( y_0 \). Let
\[
\tilde{S}_n := u_n(y_0),
\]
clearly \( \tilde{S}_n \to \infty \) as \( n \to \infty \) thanks to Step 3. Now define
\[
\zeta_n(x) = \frac{u_n(y)}{\tilde{S}_n}, \quad y = \tilde{M}_n x + y_0, \quad \tilde{M}_n = \tilde{S}_n^{\frac{1-\rho}{\rho}}.
\]

We first need a uniform bound on the sequence \( \zeta_n \).

Here we have \( \zeta_n(0) = \tilde{S}_n^{-1} u_n = 1 \) by the definition of \( \tilde{S}_n \) but \( \| \zeta_n \| \geq 1 \), not exactly 1 as in Step 1. Take \( d_0 = d(y_0, \partial \Omega) > 0 \) and apply the Harnack inequality given in Theorem 2.3, to obtain
\[
\sup_{B(y_0, d_0/2)} u_n \leq C \left[ \inf_{B(y_0, d_0/2)} u_n + \lambda_n \left( \frac{d_0}{2} \right)^p \right].
\]
By the change of variable if \( y \in B(y_0, d_0/2) \), then \( x \in B(0, \tilde{M}_n^{-1} d_0/2) \). It follows that
\[
\sup_{B(0, \tilde{M}_n^{-1} d_0/2)} \zeta_n \leq C \quad \text{and} \quad \tilde{M}_n^{-1} d_0/2 \to +\infty,
\]
since \( \tilde{M}_n \to 0 \). Define then
\[
z_n(x) = \begin{cases}
\zeta_n(x), & x \in B(0, \tilde{M}_n^{-1} d_0/2) \\
0, & \text{otherwise}.
\end{cases}
\]

We can now apply the usual convergence argument (see [11], for instance) to the sequence \( z_n \) taking into account that \( z_n \leq C \) in \( \Omega \). Fixed a ball \( B(0, R) \subset \mathbb{R}^N \), we can take \( n \) large enough so that \( 2R < \tilde{M}_n^{-1}(d_0/2) \). We use the definition of \( v_n \) given in the Step 1, but with \( S_n \) and \( M_n \) replaced by \( \tilde{S}_n \) and \( \tilde{M}_n \), respectively. Note that the expressions (43), (45) and (46) are also valid here, with \( S_n \) replaced to \( \tilde{S}_n \). Hence we can use the regularity result [6], [22], which is possible thanks to (45), to conclude that \( \| z_n \|_{C^{1,r}} \leq C \) in \( B(0, R) \) for certain \( C \) independent of \( n \).
Therefore, $z_n$ converges in the $C^1$ norm, up to a subsequence, to a certain function $z_0$. Observe that $z_0(0) = 1$, being $\zeta_n(0) = 1$. Applying (46) with $|x| < M_n^{-1}d_0/2$, we obtain that
\[
\varphi_n(x, z_n, \nabla z_n) - z_0^\delta \geq M \tilde{S}_n^{-\delta} \tilde{S}_n^{\delta \frac{2+\gamma}{p}} C^\delta + \lambda_n \tilde{S}_n^{-\delta} \to 0 \quad n \to \infty,
\]
so that arguing as in (49), we obtain that $z_0$ is a nonnegative weak solution of the problem
\[
\Delta_p z_0 + z_0^\delta \leq 0, \quad \text{in } B(0, R).
\]

The strong maximum principle of Vázquez [24], see also for further generalizations [17], yields that $z_0$ is actually positive in that ball. Since $R$ is arbitrary, using a diagonal procedure, we can take a subsequence (still denoted by $z_n$) such that $z_n$ converge to $z_0$ in compact sets of $\mathbb{R}^n$ (in the norm $C^1$) where $z_0$ is now defined in all $\mathbb{R}^N$, namely $z_0$ solves $\Delta_p z_0 + z_0^\delta \leq 0$ in $\mathbb{R}^N$. Then we obtain that $z_0$ is a positive solution of problem (50) which is a contradiction with [15].

\[\Box\]

4. Existence result. In this section we are ready to prove the main result of the paper. For this purpose we use a particular version of a theorem of Krasnoselskii, precisely Theorem 4.1 stated in [19]. The a priori estimates obtained in the previous section play a key role in verifying the conditions required in that fixed point theorem which we state for completeness.

**Theorem 4.1.** Let $C$ be a cone in a Banach space and $K : C \to C$ a compact operator such that $K(0) = 0$. Assume that there exists $r > 0$, verifying
\[a) \quad u \neq tK(u) \quad \text{for all } \|u\| = r, \quad t \in [0, 1].\]
\[b) \quad K(u) = H(0, u) \quad \text{for all } u \in C,\]
\[b) \quad H(t, u) \neq u \quad \text{for any } \|u\| = R, \quad t \in [0, 1].\]

Let $D = \{u \in C \mid r < \|u\| < R\}$. Then $K$ has a fixed point in $D$.

As in [19], we consider $C(\overline{\Omega})$ as a Banach space with the uniform norm $\|\cdot\|$ and $C^{1,\tau}(\overline{\Omega})$ with the Hölder norm $\|\cdot\|_{C^{1,\tau}}$. We define the continuous operator
\[T_v : C(\overline{\Omega}) \to C^{1,\tau}(\overline{\Omega})\]
such that $T_v(v) \in C^{1,\tau}(\Omega)$ denotes the unique weak solution $u$ of the problem
\[\Delta_p u + v = \Delta_p T_v(v) + v = 0\]
with zero Dirichlet conditions in the boundary of $\Omega$. Clearly $T_v$ maps bounded sets into bounded sets. We define also the compact operator
\[N : C^{1,\tau}(\overline{\Omega}) \to C(\overline{\Omega}), \quad N(u) = f(x, u, \nabla u)\]
and consider
\[K = T_v \circ N : C^{1,\tau}(\overline{\Omega}) \to C^{1,\tau}(\overline{\Omega})\]
which is also a compact operator. We apply Theorem 4.1 in the cone
\[C = \{u \in C^{1,\tau}(\overline{\Omega}) \mid u \text{ is nonnegative}\}.\] (52)

In particular, $K$ maps $C$ into $C$ because of the maximum principle, furthermore positive solution of (1) are obviously fixed points of $K$ in $C$.

**Theorem 4.2.** Let condition $(F)$ holds under $(2)$ and $(3)$.

Then, problem (1) has at least one positive solution.
Proof. It is enough to apply Theorem 4.1 with C and T given above. Let C be the set given in (52). Take \( u \in C \setminus \{0\} \) such that \( u = tK(u) \) for certain \( t \in (0,1] \), indeed \( t = 0 \) cannot occur by the choice of \( u \), so that (a) is trivially satisfied for \( t = 0 \). Consider now \( t \in (0,1] \) as above, using the notation below Theorem 4.1 we can rewrite \( u = tK(u) \) as \( u = tT_v(f(x,u,\nabla u)) \), so that by the homogeneity of \( \Delta_p \) we have \( -\Delta_p u = t^{p-1}f(x,u,\nabla u) \) with zero boundary conditions. Now multiplying by \( u \) and integrating we obtain

\[
\int_\Omega |\nabla u|^p \, dx = t^{p-1} \int_\Omega u f(x,u,\nabla u) \, dx \\
\leq t^{p-1} \left[ c_0 \int_\Omega u^{s+1} \, dx + M \int_\Omega u^{s+1} |\nabla u|^\sigma \, dx \right]
\]  

(53)

where in the last inequality we have used condition (F) and the fact that \( u > 0 \). Now consider separately the two terms on the right hand side. First by Hölder inequality, with exponents \( \mu \) and \( \mu' \) thanks to (2)

\[
\mu = \frac{p^*}{\delta + 1} \quad \text{and} \quad \mu' = \frac{p^*}{p^* - \delta - 1}, \quad \text{where} \quad p^* = \frac{Np}{N-p}
\]

we get, since \( t \in (0,1] \) and \( p > 1 \),

\[
c_0 t^{p-1} \int_\Omega u^{s+1} \, dx \leq C \left( \int_\Omega u^{p^*} \, dx \right)^{\frac{s+1}{p^*}}, \quad C = c_0 |\Omega|^{1/\mu'}.
\]  

(54)

Take now the second term on the right hand side of (53), applying twice Hölder inequality, first with exponents

\[
\mu = \frac{p}{\vartheta} \quad \text{and} \quad \mu' = \frac{p}{p-\vartheta},
\]

this is possible since \( \vartheta < p(\delta - s)/\delta + 1 \) and \( s \in [0,\delta) \) forces that \( \vartheta < p \), then with exponents

\[
\sigma = \frac{p^* (p - \vartheta)}{p(s+1)}, \quad \text{and} \quad \sigma' = \left( \frac{p^* (p - \vartheta)}{p(p^* - s - 1) - p^* \vartheta} \right),
\]

we obtain

\[
\int_\Omega u^{s+1} |\nabla u|^\sigma \, dx \leq \left( \int_\Omega |\nabla u|^\vartheta \, dx \right)^{\frac{s}{p}} \left( \int_\Omega u^{(s+1)p^{\vartheta}} \, dx \right)^{\frac{p-\vartheta}{p}} \\
\leq \left( \int_\Omega |\nabla u|^p \, dx \right)^{\frac{s}{p}} C \left( \int_\Omega u^{\frac{p(s+1)p^*(p-\vartheta)}{p(p^* - s - 1) - p^* \vartheta}} \, dx \right)^{\frac{p(s+1)p^*(p-\vartheta)}{p(p^* - s - 1) - p^* \vartheta}} \\
= \left( \int_\Omega |\nabla u|^p \, dx \right)^{\frac{s}{p}} |\Omega|^{1/\sigma'} \left( \int_\Omega u^{p^*} \, dx \right)^{\frac{s+1}{p}}.
\]  

(55)

We point out that \( \sigma = \frac{p^* (p - \vartheta)}{p(s+1)} > 1 \), indeed this is equivalent to

\[
s < \frac{p^*}{p} (p - \vartheta) - 1.
\]  

(56)

But from \( \delta < p^* - 1 \) it immediately follows that

\[
\delta - \frac{\vartheta(\delta + 1)}{p} < \frac{p^*}{p} (p - \vartheta) - 1.
\]
So that (56) follows directly from (2)$_1$. Inserting (54) and (55) in (53) we obtain
\[ \int_{\Omega} |\nabla u|^p \, dx \leq C \left[ \left( \int_{\Omega} u^r \, dx \right)^{\frac{\gamma + s}{p}} + \left( \int_{\Omega} |\nabla u|^p \, dx \right)^{\frac{\gamma + s}{p}} \right] \]
which gives, thanks to Sobolev inequality obtained extending \( u \equiv 0 \) outside \( \Omega \),
\[ \int_{\Omega} |\nabla u|^p \, dx \leq C \left[ \left( \int_{\Omega} |\nabla u|^p \, dx \right)^{\frac{\gamma + s}{p}} + \left( \int_{\Omega} |\nabla u|^p \, dx \right)^{\frac{\gamma + s + 1}{p}} \right] \]
\[ \leq C \left[ \left( \int_{\Omega} |\nabla u|^p \, dx \right)^{\frac{\gamma + s}{p}} + \left( \int_{\Omega} |\nabla u|^p \, dx \right)^{\frac{\gamma + s + 1}{p}} \right], \quad C > 0 \]
that is
\[ \left( \int_{\Omega} |\nabla u|^p \right)^{\frac{\gamma + s - r}{p}} + \left( \int_{\Omega} |\nabla u|^p \right)^{\frac{\gamma + s + 1 - r}{p}} \geq 1/C. \] (57)

Now, since \( \delta > p - 1 \) and \( \theta + s > p - 1 \) by (2) and (3), we deduce that there exists \( c > 0 \) such that
\[ \int_{\Omega} |\nabla u|^p \, dx > c. \] (58)

On the other hand, thanks to the compactness of \( K \) we can choose \( r < c \) such that \( \|u\|_{C^{1,\tau}} \leq r/|\Omega| \). Since \( \|u\|_{C^{1,\tau}} \geq \|\nabla u\|^p/|\Omega| \) we have \( \int_{\Omega} |\nabla u|^p \leq r \), hence \( \int_{\Omega} |\nabla u|^p < c \) which contradicts (58). Thus condition \( a \) of the Theorem 4.1 is proved.

Now, define the homotopy \( H : [0, 1] \times C \to C \) as \( H(t, u) = T_{\epsilon}[N(u) + t\lambda_0] \) where \( \lambda_0 \) is given in Proposition (1). Note that the image of \( H \) is contained in \( C \) because of \( T_{\epsilon} \). First note that if \( t = 0 \) then \( H \) coincides with \( K \) thus condition \( b1 \) of Theorem 4.1 is clearly verified. To prove \( b2 \) observe that for the case \( t = 1 \) condition \( b2 \) is trivially satisfied since by Proposition 1 problem (8) has not solutions for \( \lambda = \lambda_0 \). Consequently, we consider \( t \in [0, 1) \). In this case inequality \( H(t, u) = u \) is equivalent to \( T[N(u) + t\lambda_0] = u \) that is \( |f(x, u, \nabla u) + t\lambda_0| \) hence
\[ \Delta_p u + f(x, u, \nabla u) + t\lambda_0 = 0 \] (59)
with zero Dirichlet boundary conditions in \( \Omega \). By Proposition 2 we have proved that solutions of (59) are a priori bounded in their uniform norm since \( \lambda_0 t < \lambda_0 \), being \( t \in (0, 1) \). Thus making use again Theorem 1 in [14], there exist \( R > 0 \) such that all solutions of (59) verify the inequality \( \|u\|_{C^{1,\tau}} < R \). Hence, condition \( b2 \) is then verified for all \( t \in [0, 1] \). Finally, by Proposition 1, we have \( b3 \) holds since when \( \lambda = \lambda_0 \) we have no solutions i.e. \( u \neq K(u) \). Thus all the assumptions of Theorem 4.1 are satisfied, hence there exists a fixed point \( u \) for \( K \) in \( C \) satisfying \( \|u\|_{C^{1,\tau}} \in (r, R) \). By the definition of \( K \), \( u \) is a nontrivial solution of problem (1). Hence the proof is complete.

\[ \square \]

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