CONFORMAL BLOCKS, VERLINDE FORMULA AND DIAGRAM AUTOMORPHISMS

JIUZU HONG

Abstract. The Verlinde formula computes the dimension of conformal blocks associated to simple Lie algebras and stable pointed curves. If a simply-laced simple Lie algebra admits a nontrivial diagram automorphism, then this automorphism acts on the space of conformal blocks naturally. We prove an analogue of Verlinde formula for the trace of the diagram automorphism on the space of conformal blocks. A closely related non simply-laced Lie algebra appears in the formula.

Contents

1. Introduction 2
2. The root systems and affine Weyl group of orbit Lie algebras 6
   2.1. Root systems 6
   2.2. Affine Weyl groups and diagram automorphisms 9
3. Conformal blocks 11
   3.1. Affine Lie algebra 12
   3.2. Affine Weyl groups and Weyl groups of affine Kac-Moody algebras 12
   3.3. Diagram automorphisms as intertwining operators of representations 13
   3.4. Conformal blocks and diagram automorphisms 16
   3.5. $\sigma$-twisted fusion ring 20
4. Sign problems 21
   4.1. Borel-Weil-Bott theorem on the affine flag variety 22
   4.2. Borel-Weil-Bott theorem on affine Grassmannian 26
   4.3. Affine analogues of BBG resolution and Kostant homology 29
5. $\sigma$-twisted representation ring and fusion ring 31
   5.1. $\sigma$-twisted representation ring 31
   5.2. A new definition of $\sigma$-twisted fusion ring via Borel-Weil-Bott theory 34
   5.3. Ring homomorphism from $\sigma$-twisted representation ring to $\sigma$-twisted fusion ring 36
   5.4. Characters of the $\sigma$-twisted fusion ring 39
   5.5. Proof of Theorem 1.2 42
   5.6. A corollary of Theorem 1.2 43
References 43

Key words and phrases. affine Lie algebra, affine Weyl group, conformal blocks, diagram automorphism, fusion ring, twining formula, Verlinde formula.
The Verlinde formula computes the dimension of conformal blocks. It is fundamentally important in conformal field theory and algebraic geometry. The formula was originally conjectured by Verlinde [V] in conformal field theory. It was mathematically derived by combining the efforts of mathematicians including Tsuchiya-Ueno-Yamada [TUY], Faltings [Fa] and Teleman [Te], etc. It was proved by Beauville-Laszlo [BL], Kumar-Narasimhan-Ramanathan [KNR], Faltings [Fa] that conformal blocks can be identified with the spaces of generalized theta functions on the moduli stack of parabolic $G$-bundles on projective curves where $G$ is a simply-connected simple algebraic group. Therefore Verlinde formula also computes the dimension of the spaces of generalized theta functions. For a survey on Verlinde formula, see Sorger’s Bourbaki talk [So].

Let $(C, \vec{p})$ be a stable $k$-pointed curve. Let $g$ be a simple Lie algebra over $\mathbb{C}$. Let $\ell$ be a positive integer. Given a tuple of dominant weights $\vec{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ such that $\langle \lambda_i, \check{\theta} \rangle \leq \ell$ for each $i$, where $\theta$ is the highest root of $g$ and $\check{\theta}$ is the coroot of $\theta$. We can attach the space $V_{g, \ell, \vec{\lambda}}(C, \vec{p})$ of conformal blocks of level $\ell$ to $(C, \vec{p})$ and $\vec{\lambda}$. The space $V_{g, \ell, \vec{\lambda}}(C, \vec{p})$ does not depend on the choice of points, and in fact it even does not depend on the complex structure on $C$. We will recall the definition of conformal blocks in Section 3.4.

Let $\sigma$ be a diagram automorphism on a simple Lie algebra $g$. One can associate another simple Lie algebra $g_{\sigma}$ as the orbit Lie algebra of $g$ (see Section 2 for details). If $\sigma$ is trivial, then $g = g_{\sigma}$. If $\sigma$ is non-trivial, then $g$ must be simply-laced and $g_{\sigma}$ is non simply-laced. Let $\Phi$ (resp. $\Phi_{\sigma}$) be the set of roots of $G$ (resp. $G_{\sigma}$). We put

$$\Delta = \prod_{\alpha \in \Phi} (e^\alpha - 1), \quad \Delta_{\sigma} = \prod_{\alpha \in \Phi_{\sigma}} (e^\alpha - 1).$$

There is a natural correspondence between $\sigma$-invariant weights (resp. dominant weights) of $g$ and weights (resp. dominant weights) of $g_{\sigma}$. For any dominant weight $\lambda$ of $g$ (resp. $g_{\sigma}$), we denote $V_{\lambda}$ (resp. $W_{\lambda}$) the irreducible representation of $g$ (resp. $g_{\sigma}$) of highest weight $\lambda$. Let $\check{h}$ (resp. $\check{h}_{\sigma}$) be the dual Coxeter number of $g$ (resp. $g_{\sigma}$).

Let $G$ (resp. $G_{\sigma}$) be the associated simply-connected simple algebraic group of $g$ (resp. $g_{\sigma}$). Let $T$ (resp. $T_{\sigma}$) be a maximal torus of $G$ (resp. $G_{\sigma}$). Let $W$ (resp. $W_{\sigma}$) denote the Weyl group of $G$ (resp. $G_{\sigma}$).

Throughout this paper, we denote by $\text{tr}(A|V)$ the trace of an operator $A$ on a finite dimensional vector space $V$.

The following is the celebrated Verlinde formula.

**Theorem 1.1 (Verlinde formula).** Let $(C, \vec{p})$ be a stable $k$-pointed curve of genus $g$. Given any tuple $\vec{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ of dominant weights of $g$ such that $\langle \lambda_i, \check{\theta} \rangle \leq \ell$ for each $i$, we have

$$\dim V_{g, \ell, \vec{\lambda}}(C, \vec{p}) = |T_i|^{|\gamma - 1|} \sum_{t \in T_{\ell}^{reg}/W} \frac{\text{tr}(t|V_{\vec{\lambda}})}{\Delta(t)^{\gamma - 1}},$$

where $T_i$ is a maximal torus of $G_i$. 

where $V_\vec{\lambda}$ denotes the tensor product $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_k}$ of representations of $g$ and 
\[ T_\ell = \{ t \in T \mid e^\alpha(t) = 1, \alpha \in (\ell + \hat{h})Q_\ell \} \]
is a finite abelian subgroup in the maximal torus $T$, $T_\ell^{\text{reg}}$ denotes the regular elements in $T_\ell$ and $T_\ell^{\text{reg}}/W$ is the set of $W$-orbits in $T_\ell^{\text{reg}}$. Here $Q_\ell$ denotes the lattice spanned by long roots of $g$, and for any $\alpha \in Q_\ell$, $e^\alpha$ is the corresponding character of $T$.

From now on we always assume $\sigma$ is nontrivial. When the tuple $\vec{\lambda}$ of dominant weights of $g$ is $\sigma$-invariant, one can define a natural operator on the conformal block $V_{g,\ell,\vec{\lambda}}(C, \vec{p})$ which we still denote by $\sigma$, see Section 3.4. A natural question is how to compute the trace of $\sigma$ as operators on the conformal blocks. In this paper, we derive a formula for the trace of $\sigma$, which is very similar to Verlinde formula for the dimension of conformal blocks. Very surprisingly, in the formula the role of $g$ is replaced by $g_\sigma$.

The following is the main theorem of this paper.

**Theorem 1.2.** Let $(C, \vec{p})$ be a stable $k$-pointed curve of genus $g$. Given a tuple $\vec{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ of $\sigma$-invariant dominant weights of $g$ such that for each $i$, $(\lambda_i, \theta) \leq \ell$, we have the following formula
\[ \text{tr}(\sigma|_{V_{g,\ell,\vec{\lambda}}(C, \vec{p})}) = \frac{1}{2} \text{dim} V_{g,\ell,\vec{\lambda}}(C, \vec{p}) + \text{tr}(\sigma|_{V_{g,\ell,\vec{\lambda}}(C, \vec{p})}), \]
where $V_{g,\ell,\vec{\lambda}}(C, \vec{p})$ denotes the tensor product $W_{\vec{\lambda}} := W_{\lambda_1} \otimes \cdots \otimes W_{\lambda_k}$ of representations of $g_\sigma$ and
\[ T_{\sigma,\ell} = \{ t \in T_\sigma \mid e^\alpha(t) = 1, \alpha \in (\ell + \hat{h})Q_\sigma \}. \]
Here $T_{\sigma,\ell}^{\text{reg}}$ denotes the set of regular elements in $T_{\sigma,\ell}$, and $T_{\sigma,\ell}^{\text{reg}}/W_\sigma$ denotes the set of $W$-orbits in $T_{\sigma,\ell}^{\text{reg}}$ and
\[ Q_\sigma = \begin{cases} \text{root lattice of } g_\sigma & \text{if } g \neq A_{2n} \\ \text{weight lattice of } g_\sigma & \text{if } g = A_{2n}. \end{cases} \]

Since the space of conformal blocks can be identified with the space of generalized theta functions, Theorem 1.2 implies the same formula for the trace of diagram automorphisms on the space of generalized theta functions.

**Remark 1.3.** We have the following formula
\[ \text{dim} V_{g,\ell,\vec{\lambda}}(C, \vec{p})^\sigma = \frac{1}{2} (\text{dim} V_{g,\ell,\vec{\lambda}}(C, \vec{p}) + \text{tr}(\sigma|_{V_{g,\ell,\vec{\lambda}}(C, \vec{p})})), \]
where $V_{g,\ell,\vec{\lambda}}(C, \vec{p})^\sigma$ denotes the space of $\sigma$-invariants in $V_{g,\ell,\vec{\lambda}}(C, \vec{p})$. Combining Theorem 1.2 and Theorem 1.3, we immediately get a formula for the dimension of $V_{g,\ell,\vec{\lambda}}(C, \vec{p})^\sigma$.

The proof of Theorem 1.2 will be completed in Section 5.5. Our proof closely follows [Fa, Be, Ku2] for the derivation of usual Verlinde formula, where the fusion ring plays essential role. In the standard approach on Verlinde formula for
general stable pointed curves, the factorization rules for conformal blocks and degeneration of projective smooth curves allow a reduction to projective line with three points case. Our basic idea is that we replace the dimension by the trace of the diagram automorphism everywhere. In our taking trace setting, we explain in Section 3.4 that factorization rules for conformal blocks and degeneration of curves are compatible well with the trace operation on the space of conformal blocks.

By replacing the dimension by the trace, we introduce $\sigma$-twisted fusion rings $R_{\ell}(\mathfrak{g}, \sigma)$ in Section 3.5. Similarly we introduce the $\sigma$-twisted representation ring $R(\mathfrak{g}, \sigma)$ of $\mathfrak{g}$ (see Section 3.1). For the usual fusion ring $R_{\ell}(\mathfrak{g})$ and the representation ring $R(\mathfrak{g})$, it is important to establish a ring homomorphism from $R(\mathfrak{g})$ to $R_{\ell}(\mathfrak{g})$. Similarly we establish a ring homomorphism from $R(\mathfrak{g}, \sigma)$ to $R_{\ell}(\mathfrak{g}, \sigma)$ in Section 5.3. One of the important technical tools is that we interpret $\sigma$-twisted fusion product via affine analogue of Borel-Weil-Bott theorem, where the new product is introduced in Section 5.2. A vanishing theorem of Lie algebra cohomology by Teleman [Te] plays a key role in our arguments as in the dimension setting (cf. [Ku2, Chapter 4]). We describe all characters of the ring $R_{\ell}(\mathfrak{g}, \sigma)$ in Section 5.4. Then the Verlinde formula for the trace of diagram automorphism will be a consequence of the characterization of the ring $R_{\ell}(\mathfrak{g}, \sigma)$ and the determination of the Casimir element in $R_{\ell}(\mathfrak{g}, \sigma)$. As a byproduct we obtain an analogue of Kac-Watson formula (Theorem 5.11) in Section 5.2.

In the process of proving the coincidence of the two products in the ring $R_{\ell}(\mathfrak{g}, \sigma)$ and establishing the ring homomorphism from $R(\mathfrak{g}, \sigma)$ to $R_{\ell}(\mathfrak{g}, \sigma)$, some interesting sign problems occur on the higher cohomology groups of vector bundles on affine Grassmannian and affine flag variety, also in affine BBG-resolution and affine Kostant homologies. The resolution of these sign problems is very crucial for the characterization of the ring $R_{\ell}(\mathfrak{g}, \sigma)$.

Let $L_{\ell}(V_\lambda)$ be the vector bundle on the affine Grassmannian $\text{Gr}_G$ of $G$ associated to the level $\ell$ and the representation $V_\lambda$ of $G$. By affine Borel-Weil-Bott theorem (cf. [Ku1]) there is only one nonzero cohomology $H^i(\text{Gr}_G, L_{\ell}(V_\lambda))$ and the restricted dual $H^i(\text{Gr}_G, L_{\ell}(V_\lambda))^\vee$ is the irreducible integral representation $\mathcal{H}_\lambda$ of the affine Lie algebra $\hat{\mathfrak{g}}$ of level $\ell$. The action of $\sigma$ on the highest weight vectors of $H^i(\text{Gr}_G, L_{\ell}(V_\lambda))^\vee$ is determined in Section 4.1 and Section 4.2. This problem is closely related to similar problem on the cohomology of line bundle on affine flag variety. The answer is very similar to the finite-dimensional situation which is due to S. Naito [N1] where Lefschetz fixed point formula is used. In affine setting, we don’t know how to apply Lefschetz fixed point formula since the affine Grassmannian and affine flag variety are infinite-dimensional. Instead our method is inspired by J. Lurie’s short proof of Borel-Weil-Bott theorem [Lu]. Our method should be applicable to similar problems of general symmetrizable Kac-Moody groups. Similar sign problems also appear in BGG resolution and the Kostant homology for affine Lie algebras. They are discussed in Section 4.3.

Our starting point of this work is the Jantzen’s twining formula (cf. [Ja, Ho, FSS, KLP, N1, N2]) relating representations of $\mathfrak{g}$ and $\mathfrak{g}_\sigma$, where the term “twining” is coined by Fuchs-Schellekens-Schweigert [FSS]. Given a $\sigma$-invariant dominant
weight \( \lambda \) of \( g \) where \( \sigma \) is the diagram automorphism as above. There is a unique operator \( \sigma \) on \( V_\lambda \) such that \( \sigma \) preserves the highest weight vector \( v_\lambda \in V_\lambda \) and for any \( u \in g \) and \( v \in V_\lambda \), \( \sigma(u \cdot v) = \sigma(u) \cdot \sigma(v) \).

**Theorem 1.4** (Jantzen). For any \( \sigma \)-invariant weight \( \mu \), we have

\[
\text{tr}(\sigma|V_\lambda(\mu)) = \dim W_\lambda(\mu).
\]

Given a tuple \( \vec{\lambda} \) of \( \sigma \)-invariant dominant weights of \( g \). Let \( V^g_\lambda \) (resp. \( W^g_\lambda \)) be the tensor invariant space of \( g \) (resp. \( g_\sigma \)). Induced from the action of \( \sigma \) on each \( V_\lambda, \sigma \) acts on \( V^g_\lambda \) diagonally. Shen and the author obtained the following twining formula in the setting of tensor invariant spaces in [HS].

\[
(4) \quad \text{tr}(\sigma|V^g_\lambda) = \dim W^g_\lambda.
\]

We use geometric Satake correspondence (cf. [MV]) and tropical parametrization of Satake cycles as basis in the space of tensor invariants to obtain the above formula (cf. [GS]). In the same paper and by the same method, we reduce the saturation problems for non-simply laced groups to simply laced groups. In particular we show that the saturation factor of the spin group \( \text{Spin}_{2n+1} \) is 2. Formula (4) will be crucially used in the proof of Theorem 1.4. A consequence of Formula (4) is that the \( \sigma \)-twisted representation ring \( R(g, \sigma) \) of \( g \) is isomorphic to the representation ring \( R(g_\sigma) \) of \( g_\sigma \). This is the reason why we are able to express the trace of \( \sigma \) on the space of conformal blocks by the data associated to \( g_\sigma \). In Section 5.1 we give a simple proof of Formula (4) directly using Theorem 1.4.

It is well-known that given a tuple \( \vec{\lambda} \) of dominant weights of \( g \), the space \( V_{g, \ell, \vec{\lambda}}(\mathbb{P}^1, \vec{p}) \) of conformal blocks on \((\mathbb{P}^1, \vec{p})\) stabilizes to the tensor co-invariant space \( (V_\vec{\lambda})^c_g \) when the level \( \ell \) increases. From Formula (4), it is natural to hope that the conformal blocks associated to \( g \) and the conformal blocks associated to \( g_\sigma \) are related and have a twining formula with a fixed level. Unfortunately it is not the case. We found the following counter-example using [Sw] (joint with P. Belkale).

**Example 1.5.** We have

\[
\dim V_{sl_6, 4, \lambda, \mu, \nu}(\mathbb{P}^1, 0, 1, \infty) = 4,
\]

where \( \lambda = \omega_2 + \omega_3 + \omega_4, \mu = \omega_1 + \omega_3 + \omega_5 \) and \( \nu = \omega_1 + 2\omega_3 + \omega_5 \). Here \( \omega_1, \omega_2, \omega_3, \omega_4, \omega_5 \) denote the fundamental weights of \( sl_6 \). Since the order of \( \sigma \) on \( sl_6 \) is 2, it forces that the trace \( \text{tr}(\sigma|V_{sl_6, 4, \lambda, \mu, \nu}(\mathbb{P}^1, 0, 1, \infty)) \) can not be equal to 3. On the other hand, we have

\[
\dim V_{so_7, 4, \lambda, \mu, \nu}(\mathbb{P}^1, 0, 1, \infty) = 3,
\]

where \( \lambda = \omega_{\sigma, 2} + \omega_{\sigma, 3}, \mu = \omega_{\sigma, 1} + \omega_{\sigma, 3} \) and \( \nu = \omega_{\sigma, 1} + 2\omega_{\sigma, 3} \). Here \( \omega_{\sigma, 1}, \omega_{\sigma, 2}, \omega_{\sigma, 3} \) denotes the fundamental weights of \( so_7 \).

Actually from the formula (3) in Theorem 1.2 it is quite clear that \( \text{tr}(\sigma|V_{g, \ell, \vec{\lambda}}(\mathbb{P}^1, \vec{p})) \) should not be the same as \( \dim V_{g_\sigma, \ell, \vec{\lambda}}(\mathbb{P}^1, \vec{p}) \). Nevertheless for the special pair \((sl_{2n+1}, sp_{2n})\) we do have a twining formula where we need to take different levels on both sides.
Theorem 1.6. If $\ell$ is an odd positive integer, we have the following formula
\[
\text{tr}(\sigma|V_{sl_{2n+1},\ell,\vec{\lambda}}(C,\vec{p})) = \dim V_{sp_{2n},\ell^{-1},\vec{\lambda}}(C,\vec{p}).
\]

This theorem is a corollary of Theorem 1.2, and the proof will be given in Section 5.6. It has following interesting numerical consequences where $\ell$ is assumed to be odd.

- The trace $\text{tr}(\sigma|V_{sl_{2n+1},\ell,\vec{\lambda}}(C,\vec{p}))$ is non-negative.
- If $\dim V_{sl_{2n+1},\ell,\vec{\lambda}}(C,\vec{p})$ is 1, then $\dim V_{sp_{2n},\ell^{-1},\vec{\lambda}}(C,\vec{p})$ is also 1.
- If $V_{sp_{2n},\ell^{-1},\vec{\lambda}}(C,\vec{p})$ is nonempty, then $V_{sl_{2n+1},\ell,\vec{\lambda}}(C,\vec{p})$ is nonempty.

Theorem 1.6 establishes a bridge between the conformal blocks for $sl_{2n+1}$ and $sp_{2n}$. It would be interesting to understand the quantum saturation problems for $sp_{2n}$ via the quantum saturation theorem for $sl_{2n+1}$ which is due to Belkale [Bel].

The failure of the formula (5) in general is not really the end of the story. The combinatorial data appearing in the formula in Theorem 1.2 actually suggests a close connection with twisted affine Lie algebras. It is very natural from the point of view of the twining formula for affine Lie algebras by Fuchs-Schellekens-Schweigert [FSS]. Moreover the $\sigma$-twisted fusion ring $R_\ell(g,\sigma)$ defined in this paper is closely related to Kac-Peterson formula for S-matrices of twisted affine Lie algebras (cf. [Ka]). The analogue of Kac-Watson formula obtained in this paper is also a strong hint. In fact the connection on the trace of diagram automorphism on the space of conformal blocks and certain conformal field theory related to twisted affine Lie algebra has been predicted by Fuchs-Schweigert [FS]. It is therefore getting emergent and demanding to develop a mathematical theory of twisted conformal blocks. The work [FSZ] by Frenkel-Szcsesny is a first step toward this direction. A further theory is being developed by S. Kumar and the author [HK].

Acknowledgements The author would like to express his gratitude to P. Belkale for introducing him into the theory of conformal blocks, and for many helpful and stimulating discussions throughout this work. He would like to thank S. Kumar for helpful discussions and for his careful reading on the first draft of the paper, and also for sharing his unpublished book on Verlinde formula [Ku2]. He also wants to thank I. Cherednik for his interest and many helpful comments.

2. The root systems and affine Weyl group of orbit Lie algebras

In Section 2.1 we discuss the root systems of a simply-laced simple Lie algebra $g$ with a diagram automorphism $\sigma$ and its orbit Lie algebra $g_\sigma$. In Section 2.2 we discuss the relationships between the affine Weyl gorups of $g$ and $g_\sigma$, and also their alcoves.

2.1. Root systems. Let $g$ be a simple Lie algebra over $\mathbb{C}$. Let $I$ be the set of the vertices of the Dynkin diagram of $g$. For each $i \in I$, let $\alpha_i$ (resp. $\omega_i$) be the corresponding simple root (resp. fundamental weight). Let $P$ be the weight lattice of $g$ and let $P^+$ be the set of dominant weights of $g$. Let $\Phi$ (resp. $\Phi^*$) be the set of roots (resp. coroots) of $g$, and let $Q$ (resp. $\hat{Q}$) be the root lattice (resp. coroot
lattice) of \( g \). For each root \( \alpha \in \Phi \), let \( \check{\alpha} \in \check{\Phi} \) be the associated coroot of \( \alpha \). Let \( \langle , \rangle : P \times \check{Q} \rightarrow \mathbb{Z} \) be the perfect pairing between weight lattice and coroot lattices. Note that the matrix \( (\langle \alpha_i, \check{\alpha}_j \rangle) \) is the Cartan matrix of \( g \).

We denote by \( e_i, f_i, h_i \) the corresponding Chevalley generators in \( g \) for each \( i \in I \). Let \( \sigma \) be a nontrivial diagram automorphism of the Dynkin diagram of \( g \). Note that \( g \) can only be of types \( A_n, D_n, E_6 \) when \( \sigma \) is nontrivial. The automorphism \( \sigma \) acts on \( P \), such that \( \sigma(\alpha_i) = \alpha_{\sigma(i)} \) and \( \sigma(\omega_i) = \omega_{\sigma(i)} \) for each \( i \in I \). Clearly \( \sigma \) maps each dominant weight to another dominant weight.

The diagram automorphism \( \sigma \) defines an automorphism \( \sigma \) of the Lie algebra \( g \) such that \( \sigma(e_i) = e_{\sigma(i)}, \sigma(f_i) = f_{\sigma(i)}, \sigma(h_i) = h_{\sigma(i)} \) for each \( i \in I \). Here we use the same notation \( \sigma \) to denote these automorphisms if no confusion accrues.

Let \( I_\sigma \) be the set of orbits of \( \sigma \) on \( I \). There exists a unique simple Lie algebra \( g_\sigma \) over \( \mathbb{C} \) whose vertices of Dynkin diagram is indexed by \( I_\sigma \) (cf. [HS, Section 2.2]), and the Cartan matrix is given as follows,

\[
a_{ij} = \begin{cases} \frac{1}{|\iota|}a_{ij}, & \text{if } g \text{ is of type } A_{2n} \text{ and } \iota \text{ is disconnected} \\ |\iota|a_{ij}, & \text{otherwise} \end{cases}
\]

for any \( i \neq j \in I_\sigma \), where \( i \in \iota, j \in j \) and \( |\iota| \) is the cardinality of the \( \iota \). The Lie algebra is called the orbit Lie algebra of \( (g, \sigma) \) in literature.

Let \( \alpha_i \) (resp. \( \check{\alpha}_i \)) be the corresponding simple root (resp. simple coroot) for \( i \in I_\sigma \). Let \( P_\sigma \) be the weight lattice of \( g_\sigma \). There exists a bijection of lattices \( \iota : P_\sigma \simeq P_\sigma \) such that \( \iota^{-1}(\alpha_i) = \sum_{i \in \iota} \omega_j \) for each \( i \in I_\sigma \), where \( P_\sigma \) is the fixed point lattice of \( \sigma \) on \( P \). Let \( \rho \) (resp. \( \rho_\sigma \)) be the summation of fundamental weights of \( g \) (resp. \( g_\sigma \)). Note that \( \rho \in P_\sigma \) and \( \iota(\rho) = \rho_\sigma \). Moreover

\[
\iota^{-1}(\alpha_i) = \begin{cases} \sum_{i \in \iota} \alpha_i & \text{for any } i \neq j \in \iota, a_{ij} = 0 \\ 2(\alpha_i + \alpha_j) & \{i, j\}, a_{ij} = -1 \end{cases}
\]

Let \( Q_\sigma \) (resp. \( \check{Q}_\sigma \)) be the root lattice (resp. coroot lattice) of \( g_\sigma \). There is a projection map \( \iota : \check{Q} \rightarrow \check{Q}_\sigma \). Under this projection, we have

\[
\iota(\check{\alpha}_i) = \check{\alpha}_i, \quad \text{for any } i \in \iota.
\]

For any \( \lambda \in P_\sigma \) and \( x \in \check{Q}_\sigma \), we have the following compatibility

\[
\langle \iota(\lambda), \iota(x) \rangle = \langle \lambda, x \rangle_\sigma,
\]

where \( \langle , \rangle : P_\sigma \times \check{Q}_\sigma \rightarrow \mathbb{Z} \) is the perfect pairing between the weight lattice and dual root lattice for \( g_\sigma \).

Here is a table of \( g \) and \( g_\sigma \) for nontrivial \( \sigma \) (cf. [HS, Section 2.2] or [Lus, 6.4]):

1. If \( g = A_{2n-1} \) and \( \sigma \) is of order 2, then \( g_\sigma = B_n, n \geq 2 \).
2. If \( g = A_{2n} \) and \( \sigma \) is of order 2, then \( g_\sigma = C_n, n \geq 1 \).
3. If \( g = D_n \) and \( \sigma \) is of order 2, then \( g_\sigma = C_{n-1}, n \geq 4 \).
4. If \( g = D_4 \) and \( \sigma \) is of order 3, then \( g_\sigma = G_2 \).
5. If \( g = E_6 \) and \( \sigma \) is of order 2, then \( g_\sigma = F_4 \).

Let \( \theta \) be the highest root of \( g \). It is clear that \( \sigma(\theta) = \theta \).
Lemma 2.1. We have

\[ i(\theta) = \begin{cases} \theta_{\sigma,s} & (\mathfrak{g}, \mathfrak{g}_\sigma) \neq (A_{2n}, C_n) \\ \frac{1}{2} \theta_{\sigma} & (\mathfrak{g}, \mathfrak{g}_\sigma) = (A_{2n}, C_n) \end{cases} \]

where \( \theta_\sigma \) is the highest root of \( \mathfrak{g}_\sigma \) and \( \theta_{\sigma,s} \) is the highest short root of \( \mathfrak{g}_\sigma \), moreover

\[ i(\tilde{\theta}) = \begin{cases} \tilde{\theta}_\sigma & (\mathfrak{g}, \mathfrak{g}_\sigma) \neq (A_{2n}, C_n) \\ 2\tilde{\theta}_{\sigma,s} & (\mathfrak{g}, \mathfrak{g}_\sigma) = (A_{2n}, C_n) \end{cases} \]

where \( \tilde{\theta}_\sigma \) (resp. \( \tilde{\theta}_{\sigma,s} \)) is the highest coroot (highest short coroot) of \( \mathfrak{g}_\sigma \).

Proof. We first determine \( i(\tilde{\theta}) \). Let \( \tilde{\mathfrak{g}} \) be the Lie algebra with root system dual to the root system of \( \mathfrak{g} \). We still denote by \( \sigma \) the diagram automorphism on \( \tilde{\mathfrak{g}} \) which is compatible with the diagram automorphism \( \sigma \) on \( \mathfrak{g} \). It is well-known that the root system of \( \mathfrak{g}_\sigma \) is dual to the root system of the fixed Lie algebra \( \tilde{\mathfrak{g}}^\sigma \) (cf. [Ho] [HS]).

By [Ho] Lemma 4.3, \( \sigma \) acts on highest root subspace \( \tilde{\mathfrak{g}}_\delta \) by 1 if \( \mathfrak{g} \) is not of type \( A_{2n} \); otherwise \( \sigma \) acts on \( \tilde{\mathfrak{g}}_\delta \) by \(-1\). It implies that if \( \mathfrak{g} \) is not \( A_{2n} \), then \( \tilde{\mathfrak{g}}_\delta \) is the highest root subspace of the fixed point Lie subalgebra \( \tilde{\mathfrak{g}}^\sigma \). Hence in this case \( i(\tilde{\theta}) = \tilde{\theta}_\sigma \). When \( \mathfrak{g} \) is of type \( A_{2n} \), by [Ka] Prop. 8.3 \( i(\tilde{\theta}) = 2\tilde{\theta}_{\sigma,s} \).

Finally we can determine \( i(\theta) \) from \([\mathfrak{g}]\) and \([\text{Hu}2\] Table 2, p.88\), and we get the formula \([\mathfrak{g}]\).

Note that \( \tilde{\theta}_\sigma \) is the coroot of \( \theta_{\sigma,s} \) and \( \tilde{\theta}_{\sigma,s} \) is the coroot of \( \theta_\sigma \).

Lemma 2.2. Let \( I_k \) be the Dynkin diagram of type \( C_k \) where \( I_k \) consists of vertices \( i_1, i_2, \ldots, i_k \) such that the simple root \( \alpha_{i_j} \) is a long root. Then the long root lattice \( Q_l \) of \( C_k \) is spanned by \( \alpha_{i_1}, 2\alpha_{i_2}, \ldots, 2\alpha_{i_k} \).

Proof. For any \( k \geq 1 \), let \( I_k \) be the Dynkin diagram of \( C_k \) (where \( C_1 = A_1 \)), there exists natural embedding \( I_k \hookrightarrow I_{k+1} \). Assume \( I_k \) consists of vertices \( i_1, i_2, \ldots, i_k \), where the simple root \( \alpha_{i_j} \) is the long root. Let \( \theta_k \) be the highest long root of \( C_k \). Then \( \theta_{k+1} - \theta_k = 2\alpha_{k+1} \). Therefore the lattice of long roots of \( C_k \) for \( k \geq 2 \), is spanned by \( \alpha_{i_1}, 2\alpha_{i_2}, \ldots, 2\alpha_{i_k} \).

Let \( Q^\sigma \) denote the group of \( \sigma \)-invariant elements in the root lattice \( Q \) of \( \mathfrak{g} \).

Lemma 2.3. With respect to the isomorphism \( i : P_\sigma \simeq P^\sigma \), we have

\[ i(Q^\sigma) = \begin{cases} Q_\sigma & \text{if } G \text{ is not of type } A_{2n} \\ \frac{1}{2}Q_{\sigma,l} & \text{if } G \text{ is } A_{2n} \end{cases} \]

where \( Q_{\sigma,l} \) is the lattice spanned by the long roots of \( \mathfrak{g}_\sigma \).

Proof. It is clear that \( Q^\sigma \) has a basis \( \{ \sum_{\ell \in L} \alpha_\ell | \ell \in L \} \). In view of \([\mathfrak{g}]\), it is easy to see when \( \mathfrak{g} \) is not of type \( A_{2n} \), \( i(Q^\sigma) \) is the root lattice \( Q_\sigma \) of \( \mathfrak{g}_\sigma \).

Otherwise if \( \mathfrak{g} = A_{2n} \), then \( i(Q^\sigma) = \sum_{\ell \in L} a_\ell Z_{\alpha_\ell} \), where

\[ a_\ell = \begin{cases} 1 & \text{if } \ell \text{ is not connected} \\ 1/2 & \text{if } \ell \text{ is connected} \end{cases} \]
Let $\iota_0$ be the connected $\sigma$-orbit in $I$. Note that $\iota_0$ corresponds to the long root of $C_n$. By Lemma 2.2, our lemma follows. 

Throughout this paper, we denote by $G$ (resp. $G_\sigma$) the simply-connected and connected simple algebraic group associated to $\mathfrak{g}$ (resp. $\mathfrak{g}_\sigma$). The diagram automorphism $\sigma$ also defines automorphisms on the algebraic group $G$ and its Langlands dual $\hat{G}$. We denote by $T$ (resp. $T_\sigma$) the maximal torus of $G$ (resp. $G_\sigma$) with Lie algebra $t$ (resp. $t_\sigma$).

**Remark 2.4.** The algebraic group $G_\sigma$ is isomorphic to the Langlands dual of the connected component of the fixed point group $(\hat{G})^\sigma$ of the diagram automorphism $\sigma$ on $\hat{G}$, where $\hat{G}$ is the Langlands dual of $G$ (cf. [Ho, HS]).

### 2.2. Affine Weyl groups and diagram automorphisms

In this subsection, we refer to [Hu1] the basics of affine Weyl groups and alcoves.

Let $W$ be the Weyl group of $g$. The group $W$ acts on the weight lattice $P$. Let $P_\mathbb{R}$ be the space $P_\mathbb{Z} \otimes \mathbb{R}$. For each root $\alpha \in \Phi$, let $s_\alpha$ be the corresponding reflection in $W$, i.e. for any $\lambda \in P_\mathbb{R}$,

$$s_\alpha(\lambda) = \lambda - \langle \lambda, \check{\alpha} \rangle \alpha.$$ 

Let $W_\ell$ be the affine Weyl group $W \ltimes \ell \mathbb{Q}$ for any $\ell \in \mathbb{Q}$. Since $g$ is simply-laced, the Coxeter number is equal to the dual Coxeter number, moreover all roots have the same length. For any $\ell \in \mathbb{N}$, $W_\ell$ is the affine Weyl group corresponding to the affine Lie algebra $\hat{g}$ of level $\ell$. Let $s_0$ be the affine reflection $s_{\theta,1}$, i.e

$$(10) \quad s_{\theta,1}(\lambda) = \lambda - (\langle \lambda, \check{\theta} \rangle - 1)\theta,$$

where $\theta$ is the highest root of $g$. The affine Weyl group $W_\ell$ is a Coxeter group generated by $\{s_i \mid i \in \hat{I}\}$. For any $\alpha \in \Phi$ and $n \in \mathbb{Z}$, the hyperplane

$$H_{\alpha,n} = \{ \lambda \in P_\mathbb{R} \mid \langle \lambda, \check{\alpha} \rangle = n \}$$

is an affine wall. Every component of the complements of affine walls in $P_\mathbb{R}$ is an alcove. The affine Weyl group $W_\ell$ acts on the set of alcoves simply and transitively. Let $A_0$ be the fundamental alcove, and it can be described as follows,

$$\{ \lambda \in P_\mathbb{R} \mid \langle \lambda, \check{\alpha}_i \rangle > 0, \text{ for any } i \in I, \text{ and } \langle \lambda, \check{\alpha} \rangle < \ell \}.$$

The diagram automorphism $\sigma$ acts on $W$. Let $W^\sigma$ be the fixed point group of $\sigma$ on $W$. Let $W_\sigma$ be the Weyl group of $g_\sigma$ with simple reflections $\{s_i \mid i \in I_\sigma\}$. There exists an isomorphism $\iota : W^\sigma \simeq W_\sigma$ such that for any $i \in I_\sigma$,

$$(11) \quad \iota^{-1}(s_i) = \begin{cases} \prod_{i \in \xi} s_i & \text{any } i \neq j \in \xi, a_{ij} = 0 \\ s_is_js_i & \text{if } i = \{i,j\} \text{ and } a_{ij} = -1 \end{cases}.$$

The following lemma is obvious.

**Lemma 2.5.** The $W_\sigma$ act on $P_\sigma$ and $W^\sigma$ act on $P^\sigma$ is compatible with respect to the isomorphisms $\iota : P^\sigma \simeq P_\sigma$ and $\iota : W^\sigma \simeq W_\sigma$. 


The diagram automorphism $\sigma$ also acts naturally on $W_\ell$. Let $W_\ell^\sigma$ denote the fixed point group of $\sigma$ on $W_\ell$. It is easy to see

$$W_\ell^\sigma = W^\sigma \ltimes \ell Q^\sigma.$$ 

Let $\hat{I}_\sigma$ be the set $I_\sigma \sqcup \{0\}$.

**Lemma 2.6.** $W_\ell^\sigma$ is a Coxeter group generated by $\{i^{-1}(s_i) \mid i \in \hat{I}_\sigma\}$.

**Proof.** cf. [ESS, Section 5.2]. □

The group $W_\ell^\sigma$ naturally acts on $P_\sigma^\ell \otimes \mathbb{R}$. Let $A$ be the set of alcoves of $W_\ell$ in $P_\sigma^\ell$. Let $A^\sigma$ be the set of $\sigma$-stable alcoves.

**Lemma 2.7.**

1. For any $A \in A^\sigma$, $A^\sigma$ is not empty.
2. For any two $\sigma$-invariant alcoves $A$ and $A'$ in $A$, there exists a unique $w \in W_\ell^\sigma$ such that $w(A^\sigma) = A'^\sigma$.

**Proof.** We first prove (1). For any $\lambda \in A$, $\lambda, \sigma(\lambda), \cdots, \sigma^{r-1}(\lambda) \in A$, where $r$ is the order of $\sigma$. By the convexity of $A$,

$$\frac{\lambda + \sigma(\lambda) + \cdots + \sigma^{r-1}(\lambda)}{r} \in A,$$

which is $\sigma$-invariant.

Now we prove (2). The affine Weyl group $W_\ell$ acts simply and transitively on $A$ (cf. [Hu1, §4.5]). Hence given any two elements $A, A' \in A^\sigma$, there exists a unique $w \in W_\ell$ such that $w(A) = A'$. In particular, we have

$$\sigma(w)(A) = \sigma w \sigma^{-1}(A) = \sigma w(A) = \sigma(A') = A' = w(A).$$

By the uniqueness of $w$, we have $\sigma(w) = w$. □

Let $P_{\sigma,\mathbb{R}}$ be the space $P_\sigma \otimes \mathbb{R}$. We still denote by $\iota : W_\ell^\sigma \simeq W_\ell \ltimes \iota(\ell Q^\sigma)$ the natural isomorphism of groups. By Lemma 2.3, $W_\sigma \ltimes \iota(\ell Q^\sigma)$ is an affine Weyl group. In view of Lemma 2.1 and Lemma 2.3,

$$A_{0,\sigma} = \{ \lambda \in P_{\sigma,\mathbb{R}} \mid \langle \lambda, \check{\alpha}_i \rangle_\sigma > 0 \text{ for any } i \in I_\sigma, \text{ and } \langle \lambda, \check{\iota} \check{\theta} \rangle_\sigma < \ell \}$$

is the fundamental alcove of $W_\sigma \ltimes \iota(\ell Q^\sigma)$.

Let $A_\sigma$ be the set of alcoves of $W_\sigma \ltimes \iota(\ell Q^\sigma)$ on $P_{\sigma,\mathbb{R}}$.

**Proposition 2.8.**

1. The isomorphism $\iota : P_{\sigma,\mathbb{R}}^\sigma \simeq P_{\sigma,\mathbb{R}} \ltimes \iota(\ell Q^\sigma)$ induces a bijection $\iota : (A_{0})^\sigma \simeq A_{0,\sigma}$.
2. There exists a bijection $A^\sigma \simeq A_\sigma$ with the map given by

$$A \mapsto \iota(A^\sigma).$$

3. For any $\lambda \in P_{\sigma,\mathbb{R}}^\sigma$, $\lambda$ is in an affine wall of $W_\ell$ if and only if $\iota(\lambda) \in P_{\sigma,\mathbb{R}}$ is in an affine wall of $W_\sigma \ltimes \iota(\ell Q^\sigma)$.
Proof. We first prove (1). For any $\lambda \in P^*_R$, $\lambda \in (A_0)^*$ if and only if $i(\lambda) \in (A_0)^*$, since
\[ \langle \lambda, \hat{\alpha}_i \rangle = \langle i(\lambda), i(\hat{\alpha}_i) \rangle = \langle i(\lambda), \hat{\alpha}_i \rangle > 0, \]
for any $i \in I_\sigma$ and $i \in i$, and
\[ \langle \lambda, \bar{\theta} \rangle = \langle i(\lambda), i(\bar{\theta}) \rangle < \ell. \]

The second part (2) of proposition follows from Lemma 2.7 and the first part of the proposition. The third part (3) of the proposition follows from the first and second part of proposition. \qed

Let $\ell_\sigma : W^* \rightarrow \mathbb{N}$ denote the length function on the Coxeter group $W^*_i$. For any $\lambda \in (Q^*)$, let $\tau_\lambda$ be the translation on $P^*_R$ by $\lambda$. The following lemma will be used in the proof of Proposition 5.15 in Section 5.3.

Lemma 2.9. The length $\ell_\sigma(\tau_\lambda)$ is even.

Proof. If $g$ is not $A_{2n}$, by Lemma 2.3 $i(Q^*) = Q_\sigma$. Hence the affine group $W^*_i$ is isomorphic to $W_\sigma \ltimes Q_\sigma$. The problem is reduced to that for any $\lambda \in Q_\sigma$, $\tau_\lambda$ has even length in $W_\sigma \ltimes Q_\sigma$.

For any dominant weight $\lambda$, we have (cf. [IM])
\[ \ell_\sigma(\tau_\lambda) = \langle \lambda, 2\tilde{\rho}_\sigma \rangle, \]
where $\tilde{\rho}_\sigma$ is the sum of all fundamental coweights of $g_\sigma$. For any $\lambda \in Q_\sigma$, $\lambda$ can be expressed as $\lambda^+_i - \lambda^+_j$ where $\lambda^+_i$ is a dominant weight for each $i$. Hence
\[ (-1)^{\ell_\sigma(\tau_\lambda)} = (-1)^{\ell_\sigma(\tau^+_i + \lambda^+_j)}. \]
Then $\lambda^+_i + \lambda^+_j = \lambda + 2\lambda^+_2$. Note that the fundamental group $P_\sigma/Q_\sigma$ of $g_\sigma$ is either trivial or isomorphic to $\mathbb{Z}/2\mathbb{Z}$ (cf. [Hu2, §13.1]), since $g_\sigma$ is non simply-laced. It follows that $2\lambda^+_2 \in Q_\sigma$, and hence $\lambda^+_i + \lambda^+_j$ is dominant and is also an element in $Q_\sigma$. By Formula (12), the length of $\tau^+_i + \lambda^+_j$ is even. Thus $\ell_\sigma(\tau_\lambda)$ is even.

When $g$ is of type $A_{2n}$, by Lemma 2.3 $i(Q^*) = \frac{2}{Z}Q_\sigma$. The normalized Killing form on $g_\sigma$ can identify $\frac{2}{Z}Q_\sigma$ with $\frac{2}{Z}Q_\sigma$ (cf. [Be, Proof of Lemma 9.3 (b)]), where $Q_\sigma$ is the coroot lattice of $g_\sigma$. This identification is compatible with the action of $W_\sigma$. Hence $W^*_i$ is isomorphic to $W_\sigma \ltimes \hat{Q}_\sigma$. Note that in this case $\hat{P}_\sigma/\hat{Q}_\sigma \simeq \mathbb{Z}/2\mathbb{Z}$. Then by the same argument as above, the length $\ell_\sigma(\tau_\lambda)$ is also even. \qed

3. Conformal blocks

In Section 3.1 we introduce the affine Lie algebra $\hat{g}$ of $g$. In Section 3.3 we define the intertwining operators of representations of $g$ and $\hat{g}$ from the diagram automorphism $\sigma$. In Section 3.4 we define operators on conformal blocks from $\sigma$, and we prove that these operators compatible well with propulsion of vacua, factorization properties of conformal blocks, and the trace takes constant values on conformal blocks along a family of stable pointed curves. In Section 3.5 we show that the trace of $\sigma$ on conformal blocks on $\mathbb{P}^1$ gives a non-degenerate fusion rule, hence it defines a fusion ring, which we call a $\sigma$-twisted fusion ring.
3.1. **Affine Lie algebra.** Let $\mathfrak{g}$ be a semisimple Lie algebra. Let $\mathbb{C}((t))$ be the field of Laurent series. Let $\hat{\mathfrak{g}}$ be the associated affine Kac-Moody algebra $\hat{\mathfrak{g}}((t))\oplus \mathbb{C}\oplus \mathbb{C}d$, where $\mathfrak{g}((t))$ denotes the loop Lie algebra $\mathfrak{g} \otimes \mathbb{C} [t, t^{-1}]$. The Lie bracket $[,]$ on $\hat{\mathfrak{g}}$ is given by

$$[u \otimes f, v \otimes g] := [u, v] \otimes fg + (u|v) \text{Res}_{t=0} \frac{df}{dt} gc,$$

and

$$[u \otimes t^n, d] = nu \otimes t^n, \quad [d, c] = 0, \quad [u \otimes f, c] = 0$$

for any $u, v \in \mathfrak{g}$ and $f, g \in \mathbb{C}((t))$, where $[u, v]$ is the Lie bracket on $\mathfrak{g}$ and $(\cdot | \cdot)$ is the normalized invariant bilinear form on $\mathfrak{g}$. For convenience, we identify $u \otimes 1$ with $u$ for any $u \in \mathfrak{g}$, and hence $\mathfrak{g}$ is naturally a Lie subalgebra of $\hat{\mathfrak{g}}$.

Let $\delta$ be the linear functional on $\hat{\mathfrak{g}}$ such that

$$\delta|_{\mathfrak{g}} = 0, \quad \delta(c) = 0, \quad \delta(d) = 1.$$

Let $\alpha_0 = -\theta + \delta$, where $\theta$ is the highest root of $\mathfrak{g}$. Then $\{\alpha_i | i \in \hat{I}\}$ is the set of simple roots of $\hat{\mathfrak{g}}$. The set of roots is given by

$$\hat{\Phi} := \{\alpha + k\delta | \alpha \in \Phi \cup \{0\}, k \in \mathbb{Z}\} \setminus \{0\}.$$

Among these roots, the set $\hat{\Phi}^+$ of positive roots consists of

$$\{\alpha + k\delta | \alpha \in \Phi^+, k \geq 0\} \cup \{\alpha + k\delta | \alpha \in \Phi^- \cup \{0\}, k > 0\}.$$

Similarly the set $\hat{\Phi}^-$ of negative roots consists of

$$\{\alpha + k\delta | \alpha \in \Phi^-, k \leq 0\} \cup \{\alpha + k\delta | \alpha \in \Phi^+ \cup \{0\}, k < 0\}.$$

We denote by $\{\Lambda_i | i \in \hat{I}\}$ the set of fundamental weights of $\hat{\mathfrak{g}}$. In particular $\Lambda_0$ is given by the linear functional on $\hat{\mathfrak{g}}$ such that

$$\Lambda_0|_{\mathfrak{g}} = 0, \quad \Lambda_0(c) = 1, \quad \Lambda_0(d) = 0.$$

3.2. **Affine Weyl groups and Weyl groups of affine Kac-Moody algebras.**

In the following we describe the relationship between the affine Weyl groups of simple Lie algebras and the Weyl groups of affine Kac-Moody algebras. For more details, one can refer to [Ka, §6]. These two different point of views are both crucial in this paper.

Let $\hat{W}$ be the Weyl group of the affine Kac-Moody algebra $\hat{\mathfrak{g}}$ (cf. [Ka, §3.7]). Set

$$\hat{t}_R := P_R + \mathbb{R}\Lambda_0 + \mathbb{R}\delta.$$

The Weyl group $\hat{W}$ acts on $\hat{t}_R$. Note that $\hat{W}$ keep $\delta$ invariant (cf. [Ka, §6.5]). Hence the Weyl group $\hat{W}$ acts on $\hat{P}_{R,\ell}$ for any $\ell \in \mathbb{R}$, where

$$\hat{P}_{R,\ell} := \{x \in \hat{t}_R | \langle x, c \rangle = \ell \}/\mathbb{R}\delta.$$

With respect to the isomorphism $P_R \simeq \hat{P}_{R,\ell}$ given by $\lambda \mapsto \lambda + \ell \Lambda_0$, we have the following lemma.
Lemma 3.1. There exists an isomorphism \( \text{af} : \hat{W} \cong W_\ell \) of groups such that for any \( \Lambda = \lambda + \ell \Lambda_0 \in \hat{P}_{\mathbb{R},\ell} \) and \( w \in \hat{W} \), the following formula holds,

\[
w \cdot \Lambda = \text{af}(w) \cdot \lambda + \ell \Lambda_0 \text{ in } \hat{P}_{\mathbb{R},\ell},
\]

where \( \text{af}(w) \in W_\ell \).

Proof. The isomorphism \( \text{af} : \hat{W} \cong W_\ell \) follows from [Ka, §6.5,§6.6].

Note that \( \langle w \cdot \Lambda, c \rangle = \langle \Lambda, w \cdot c \rangle = \langle \Lambda, c \rangle = \ell \). In view of the formula [Ka, §6.2.7], we have

\[
w \cdot \Lambda = \text{af}(w) \cdot \lambda + \langle w \cdot \Lambda, c \rangle \Lambda_0 = \text{af}(w) \cdot \lambda + \ell \Lambda_0 \in \hat{P}_{\mathbb{R},\ell}.
\]

Hence the lemma follows. \( \square \)

Let \( \hat{\rho} \) be the summation \( \sum_{i \in I} \Lambda_i \) of all fundamental weights of \( \hat{g} \). By [Ka, 6.2.8], \( \hat{\rho} = \rho + \hat{h} \Lambda_0 \) where \( \rho \) is the sum \( \sum_{i \in I} \omega_i \) of all fundamental weights of \( g \), and \( \hat{h} \) is the dual Coxeter number of \( g \).

We define \( \ast \) action of \( \hat{W} \) on \( \hat{P}_{\mathbb{R},\ell} \) as follows,

\[
w \ast \Lambda = w \cdot (\Lambda + \hat{\rho}) - \hat{\rho}, \quad w \in \hat{W}, \Lambda \in \hat{P}_{\mathbb{R},\ell}.
\]

Similarly without confusion of notations we also denote by \( \ast \) the following action of \( W_\ell \) on \( P_{\mathbb{R}} \),

\[
w \ast \lambda = w \cdot (\lambda + \rho) - \rho, \quad w \in W_\ell, \lambda \in P_{\mathbb{R}}.
\]

Lemma 3.2. Given \( \Lambda = \lambda + \ell \Lambda_0 \in \hat{P}_{\mathbb{R},\ell} \) and \( w \in \hat{W} \), we have

\[
w \ast \Lambda = \text{af}(w) \ast \lambda + \ell \Lambda_0,
\]

where \( \text{af}(w) \in W_{\ell + \hat{h}} \).

Proof. It follows from Lemma 3.1 and the formula \( \hat{\rho} = \rho + \hat{h} \Lambda_0 \). \( \square \)

3.3. Diagram automorphisms as intertwining operators of representations. We denote by \( V_\lambda \) the irreducible representation of \( g \) of highest weight \( \lambda \) for each \( \lambda \in P^+ \). From now on we always fix a highest weight vector \( v_\lambda \in V_\lambda \) for each \( \lambda \). There exists a unique operator \( \sigma : V_\lambda \to V_{\sigma(\lambda)} \) such that

1. \( \sigma(v_\lambda) = v_{\sigma(\lambda)} \),
2. \( \sigma(u \cdot v) = \sigma(u) \cdot \sigma(v) \) for any \( u \in g \) and \( v \in V_\lambda \).

In particular when \( \sigma(\lambda) = \lambda \), \( \sigma \) acts on \( V_\lambda \). Given any \( \sigma \)-invariant dominant weight of \( g \) and any \( r \)-th root of unity \( \xi \in \mathbb{C} \) where \( r \) is the order of \( \sigma \), we denote by \( V_{\lambda,\xi} \) the representation of \( g \rtimes \langle \sigma \rangle \), i.e., it consists of \( V_\lambda \) as representation of \( g \) and an operator \( \sigma : V_\lambda \to V_\lambda \) such that

1. \( \sigma \) acts on \( v_\lambda \) by \( \xi \),
2. \( \sigma(u \cdot v) = \sigma(u) \cdot \sigma(v) \).

Given a tuple \( \bar{\lambda} = (\lambda_1, \cdots, \lambda_k) \) of dominant weights of \( g \). We denote by \( V_{\bar{\lambda}} \) the tensor product \( V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_k} \). Denote by \( V_{\bar{\lambda}}^{\lambda} \) the the invariant space of \( g \) on \( V_{\bar{\lambda}} \).

The collection of operators \( \{ \sigma : V_\lambda \to V_{\sigma(\lambda)} \} \) induce

\[
\sigma : V_{\bar{\lambda}} \to V_{\sigma(\bar{\lambda})}, \quad \sigma : V_{\bar{\lambda}}^{\lambda} \to V_{\sigma(\bar{\lambda})}^{\lambda}.
\]
where $\sigma(\tilde{\lambda}) = (\sigma(\lambda_1), \ldots, \sigma(\lambda_k))$.

The following lemma is well-known (cf. [Ka §12.4]).

**Lemma 3.3.** For any $\lambda \in P$ and $\ell \in \mathbb{N}$, $\lambda + \ell \Lambda_0$ is a dominant weight of $\hat{g}$ if and only if $\lambda$ is a dominant weight of $g$ and $\langle \lambda, \tilde{\theta} \rangle \leq \ell$.

Put

$$P_{\ell} = \{ \lambda \in P^+ \mid \langle \lambda, \tilde{\theta} \rangle \leq \ell \}.$$

Let $\hat{M}(V_\lambda)$ be the generalized Verma module $U(\hat{g}) \otimes_{U(\hat{p})} V_\lambda$ of $\hat{g}$, where $\hat{p} = g[[t]] \oplus \mathbb{C} \cdot c$ acts on $V_\lambda$ by evaluating $t = 0$ and $c$ acts by $\ell$. Then the unique maximal irreducible quotient $\mathcal{H}_\lambda$ is an irreducible integrable representation of $\hat{g}$ of level $\ell$. The representation $\mathcal{H}_\lambda$ of $\hat{g}$ extends uniquely to the irreducible integrable representation of $\hat{g}$ of highest weight $\lambda + \ell \Lambda_0$ by declaring $d$ acting trivially on the highest weight vectors.

From the construction of $\mathcal{H}_\lambda$, there exists a natural inclusion $V_\lambda \rightarrow \mathcal{H}_\lambda$. We denote by $\tilde{v}_\lambda$ the image of $v_\lambda \in V_\lambda$ in $\mathcal{H}_\lambda$, which is again a highest weight vector in $\mathcal{H}_\lambda$.

The diagram automorphism $\sigma : g \rightarrow g$ extends to an automorphism on $\hat{g}$ (by abuse of notation we still denote by $\sigma$) such that $\sigma(u \otimes f) = \sigma(u) \otimes f$ for any $u \in g$ and $f \in \mathbb{C}((t))$, and $\sigma(c) = c$.

As in the case of $V_\lambda$, there exists a unique operator $\sigma : \mathcal{H}_\lambda \rightarrow \mathcal{H}_{\lambda(\lambda)}$ such that

1. $\sigma(\tilde{v}_\lambda) = v_{\sigma(\lambda)}$,
2. $\sigma(X \cdot v) = \sigma(X) \sigma(v)$ for any $X \in \hat{g}$ and $v \in \mathcal{H}_\lambda$.

In particular $\sigma$ acts on $\mathcal{H}_\lambda$ when $\sigma(\lambda) = \lambda$. As in the case of $V_\lambda$, for any $\sigma$-invariant dominant weight $\lambda$ of $g$ and for any $r$-th root of unity $\xi$, we denote by $\mathcal{H}_{\lambda, \xi}$ the representation of $\mathcal{H}_{\lambda, \xi}$ which satisfies similar conditions as $V_{\lambda, \xi}$.

Given a tuple $\tilde{\lambda}$ of dominant weights, denote by $\mathcal{H}_{\tilde{\lambda}}$ the tensor product $\mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_k}$. The operators $\{ \sigma : \mathcal{H}_\lambda \rightarrow \mathcal{H}_{\lambda(\lambda)} \}$ induce the operator $\sigma : \mathcal{H}_{\tilde{\lambda}} \rightarrow \mathcal{H}_{\sigma(\tilde{\lambda})}$ such that

$$\sigma(v_1 \otimes \cdots \otimes v_k) = \sigma(v_1) \otimes \cdots \otimes \sigma(v_k),$$

for any $v_i \in \mathcal{H}_{\lambda_i}$, $i = 1, \ldots, k$.

The inclusion $V_\lambda \rightarrow \mathcal{H}_\lambda$ is compatible with the diagram automorphism, i.e.

$$\begin{array}{ccc}
V_\lambda & \longrightarrow & \mathcal{H}_\lambda \\
\sigma \downarrow & & \sigma \downarrow \\
V_{\sigma(\lambda)} & \longrightarrow & \mathcal{H}_{\sigma(\lambda)}
\end{array}$$

Let $\hat{g}^\circ$ denote the Lie subalgebra $t^{-1}g[t^{-1}]$. We denote by $(\mathcal{H}_\lambda)_{\hat{g}^\circ}$ the coinvariant space of $\mathcal{H}_\lambda$ with respect to the action of $\hat{g}^\circ$. The Lie algebra $g$ acts naturally on $(\mathcal{H}_\lambda)_{\hat{g}^\circ}$. The following lemma is well-known.
Lemma 3.4. As representations of $\mathfrak{g}$, we have a natural isomorphism $V_\lambda \simeq (\mathcal{H}_\lambda)_{\hat{g}^-}$. Moreover the following diagram commutes

\[ \begin{array}{ccc}
V_\lambda & \longrightarrow & (\mathcal{H}_\lambda)_{\hat{g}^-} \\
\sigma & \downarrow & \sigma \\
V_{\sigma(\lambda)} & \longrightarrow & (\mathcal{H}_{\sigma(\lambda)})_{\hat{g}^-}
\end{array} \]

Let $\tau$ be the Cartan involution of $\mathfrak{g}$ such that $\tau(e_i) = -f_i, \tau(f_i) = -e_i, \tau(h_i) = -h_i$, where $e_i, f_i, h_i, i \in I$, are Chevalley generators of $\mathfrak{g}$. Then $\tau$ is an automorphism on $\mathfrak{g}$. For any finite dimensional representation $V$ of $\mathfrak{g}$. By composing $\tau$, we can redefine a new representation structure on $V$,

\[ X \ast v := \tau(X) \cdot v, \]

for any $X \in \mathfrak{g}$ and $v \in V$. We denote by $V^\tau$ this $\tau$-twisted representation.

For any dominant weight $\lambda$, let $\lambda^*$ be another dominant weight $-\omega_0(\lambda)$ where $\omega_0$ is the longest element in the Weyl group $W$. The space $V^\lambda_{\mathfrak{g}^*}$ is isomorphic to $V_{\lambda^*}$ as representation of $\mathfrak{g}$.

The Cartan involution $\tau$ on $\mathfrak{g}$ extends to an automorphism on $\hat{\mathfrak{g}}$ (by abuse of notation we still denote by $\tau$) such that $\tau(u \otimes f) = \tau(u) \otimes f$ and $\tau(c) = c$ for any $u \in \mathfrak{g}, f \in \mathbb{C}(t))$. Denote by $\mathcal{H}_\lambda^\tau$ the representation of $\hat{\mathfrak{g}}$ by composing the automorphism $\tau : \hat{\mathfrak{g}} \rightarrow \hat{\mathfrak{g}}$. Then $\mathcal{H}_\lambda^\tau \simeq \mathcal{H}_{\lambda^*}$.

Summarize the above discussions, we have the following lemma.

Lemma 3.5. (1) There exists a unique $\mathbb{C}$-linear isomorphism $\tau_\lambda : V_\lambda \rightarrow V_{\lambda^*}$ such that

\[ \tau_\lambda(v_\lambda) = v_{\lambda^*}, \quad \tau_\lambda(u \cdot v) = \tau(u) \cdot \tau_\lambda(v), \]

for any $u \in \mathfrak{g}$ and $v \in V_\lambda$.

(2) There exists a unique $\mathbb{C}$-linear isomorphism $\tau_\lambda : \mathcal{H}_\lambda \rightarrow \mathcal{H}_{\lambda^*}$ such that

\[ \tau_\lambda(v_\lambda) = v_{\lambda^*}, \quad \tau_\lambda(X \cdot v) = \tau(X) \cdot \tau_\lambda(v), \]

for any $X \in \hat{\mathfrak{g}}$ and $v \in \mathcal{H}_\lambda$.

The isomorphism $\tau_\lambda : V_\lambda \rightarrow V_{\lambda^*}$ for each $\lambda$ induces an isomorphism $\tau^\lambda : V^g_\lambda \rightarrow V^g_{\lambda^*}$ for any tuple of dominant weights $\tilde{\lambda}$. Since for any weight $\lambda$, we have $\sigma(\lambda^*) = \sigma(\lambda)^*$, and $\sigma \circ \tau = \tau \circ \sigma$. We have the following lemma.

Lemma 3.6. The following diagram commutes:

\[ \begin{array}{ccc}
V^g_\lambda & \overset{\tau}{\longrightarrow} & V^g_{\lambda^*} \\
\sigma & \downarrow & \sigma \\
V^g_{\sigma(\lambda)} & \overset{\tau}{\longrightarrow} & V^g_{\sigma(\lambda)^*}
\end{array} \]

where $\tilde{\lambda}^* = (\lambda_1^*, \ldots, \lambda_k^*)$.  

\[ \begin{array}{ccc}
\]
3.4. Conformal blocks and diagram automorphisms. A $k$-pointed projective curve consists a projective curve $C$ and $k$-distinct smooth points $\mathbf{p} = (p_1, \ldots, p_k)$ in $C$. Given a $k$-pointed projective curve $(C, \mathbf{p})$, on each point $p_i$ we associate a dominant weight $\lambda_i \in P_\ell$. Let $g(C \setminus \mathbf{p})$ be the space of $g$-valued regular functions on $C \setminus \mathbf{p}$. The space $g(C \setminus \mathbf{p})$ is naturally a Lie algebra induced from $g$. By Residue formula, there exists a Lie algebra action of $\mathfrak{g}(C \setminus \mathbf{p})$ on $\mathcal{H}_\lambda$ defined as follows: for any $X \in g(C \setminus \mathbf{p})$ and $v_i \in \mathcal{H}_{\lambda_i}$, $i = 1, \ldots, k$,

$$X \cdot (v_1 \otimes \cdots \otimes v_k) = \sum_{i=1}^{k} v_1 \otimes \cdots \otimes X_i \cdot v_i \otimes \cdots \otimes v_k,$$

where $X_i$ is the Laurent expansion of $X$ at $p_i$, and hence an element in $g((t_i)) \subset \mathfrak{g}((t_i)) + \mathbb{C}$. Here $t_i$ is a formal parameter of $C$ at $p_i$.

We define the space of conformal block $V_{g,\ell,\lambda}(C, \mathbf{p})$ associated to $\mathbf{p}$ and $\lambda$ as follows:

$$V_{g,\ell,\lambda}(C, \mathbf{p}) := (\mathcal{H}_\lambda)_{g(C \setminus \mathbf{p})} = \mathcal{H}_\lambda / g(C \setminus \mathbf{p}) \mathcal{H}_\lambda.$$

The space $V_{g,\ell,\lambda}(C, \mathbf{p})$ is independent of the choice of $\mathbf{p}$.

Let $\tau_\lambda : \mathcal{H}_\lambda \to \mathcal{H}_\lambda$ be the $\mathbb{C}$-linear isomorphism $\tau_\lambda \otimes \cdots \otimes \tau_{\lambda_k}$. The map $\tau_\lambda$ descends to an isomorphism on the space of conformal blocks

$$\tau_\lambda : V_{g,\ell,\lambda}(C, \mathbf{p}) \to V_{g,\ell,\lambda}(C, \mathbf{p}).$$

**Lemma 3.7.** We have the following commutative diagram:

$$
\begin{array}{ccc}
V_{g,\ell,\lambda}(C, \mathbf{p}) & \xrightarrow{\tau_\lambda} & V_{g,\ell,\lambda}(C, \mathbf{p}) \\
\downarrow{\sigma} & & \downarrow{\sigma} \\
V_{g,\ell,\sigma(\lambda)}(C, \mathbf{p}) & \xrightarrow{\tau_\lambda} & V_{g,\ell,\sigma(\lambda)}(C, \mathbf{p})
\end{array}
$$

**Proof.** The automorphism $\sigma$ commutes with the automorphism $\tau$ on $g$, i.e. $\tau \circ \sigma = \sigma \circ \tau$. Then commutativity easily follows. \qed 

**Proposition 3.8.** Let $\mathbf{p} = \{p_1, p_2, \ldots, p_s\}$, $\mathbf{q} = \{q_1, q_2, \ldots, q_t\}$ be two finite nonempty subsets smooth points of $C$, without common points; let $\lambda_1, \ldots, \lambda_s, \mu_1, \ldots, \mu_t$ be elements in $P_\ell$. We let $g(C \setminus \mathbf{p})$ act on $V_{g,\ell}$ through the evaluation map $X \otimes f \mapsto f(q_j)X$. The inclusions $V_{g,\ell} \hookrightarrow \mathcal{H}_{\mu_j}$ induce an isomorphism

$$\mathcal{H}_{\lambda} \otimes V_{g,\ell} \mid_{g(C \setminus \mathbf{p})} \simeq \mathcal{H}_{\lambda} \otimes \mathcal{H}_{\mu} \mid_{g(C \setminus \mathbf{p}, \mathbf{q})} = V_{g,\ell,\lambda,\mu}(C, \mathbf{p}, \mathbf{q}),$$

and this isomorphism is compatible with the diagram automorphism $\sigma$, i.e. the following diagram commutes

$$
\begin{array}{ccc}
(H_{\lambda} \otimes V_{g,\ell}) \mid_{g(C \setminus \mathbf{p})} & \xrightarrow{\sim} & V_{g,\ell,\lambda,\mu}(C, \mathbf{p}, \mathbf{q}) \\
\downarrow{\sigma} & & \downarrow{\sigma} \\
(H_{\sigma(\lambda)} \otimes V_{g,\ell}(\mu)) \mid_{g(C \setminus \mathbf{p})} & \xrightarrow{\sim} & V_{g,\ell,\sigma(\lambda),\sigma(\mu)}(C, \mathbf{p}, \mathbf{q})
\end{array}
$$

(15)
Proof. The isomorphism (15) is a well-known theorem (cf. [Be, Proposition 2.3]).
The commutativity (16) follows from the commutativity (14). □

When \( \tilde{q} = q \) and \( \mu = 0 \). The isomorphism (15) is the so-called “propogation of vacua”. Proposition 3.8 shows that the propagation of vacua is compatible with the action of the diagram automorphism.

**Lemma 3.9.**

(1) For any \( p \in \mathbb{P}^1 \), one has \( V_{g,\ell}([P^1]) \simeq V_{g,\ell,0}([P^1],p) \simeq \mathbb{C} \) by 1, and the automorphism \( \sigma \) acts on \( V_{g,\ell}([P^1]) \) and \( V_{g,\ell,0}([P^1],p) \) by 1.

(2) For any \( p \neq q \) in \( \mathbb{P}^1 \), one has \( V_{g,\ell,\lambda,\lambda'}([P^1],p,q) \simeq (V_\lambda \otimes V_{\lambda'})_g = \mathbb{C} \), the automorphism \( \sigma \) acts on \( V_{g,\ell,\lambda,\lambda'}([P^1],p,q) \) by 1 for any \( \sigma \)-invariant dominant weight \( \lambda \).

Proof. By Proposition 3.8 there exists a map \( \mathbb{C} \to V_{p,\ell}(p;0) \) compatible with the action of \( \sigma \) where \( \mathbb{C} \) is viewed as a trivial representation of \( g \) and \( \sigma \) acts on \( \mathbb{C} \) trivially. By [Be Corollary 4.4], this map is an isomorphism. By Proposition 3.8 again, \( V_{g,\ell}([P^1]) \simeq V_{g,\ell,0}([P^1],p) \simeq \mathbb{C} \) and this isomorphism is also compatible with the action of \( \sigma \). Hence \( \sigma \) acts on \( V_{g,\ell}([P^1]) \) and \( V_{g,\ell,0}([P^1],p) \) by 1. This proves (1).

Similarly by Proposition 3.8 there exists a map \( (V_\lambda \otimes V_{\lambda'})_g \to V_{g,\ell,\lambda,\lambda'}([P^1],p,q) \) which is compatible with the action of \( \sigma \). This map is an isomorphism in view of [Be Corollary 4.4]. On the other hand it is easy to see that \( \sigma \) acts on \( (V_\lambda \otimes V_{\lambda'})_g \) by 1. Hence it also acts on \( V_{g,\ell,\lambda,\lambda'}([P^1],p,q) \) by 1. □

Now we recall the definition of stable \( k \)-pointed curve (cf. [Ku2, Def.2.1.1] or [Ue, Def.1.16]).

**Definition 3.10.** A stable \( k \)-pointed curve consists of a pair \( (C,\bar{p}) \) where \( C \) is a reduced projective curve with at worst only nodal singularity and \( \bar{p} = (p_1, \ldots, p_k) \) are \( k \) distinct non-singular points on \( C \), moreover the following conditions are satisfied,

- each irreducible component of \( C \) contains at least one point \( p_i \);
- the automorphism group of \( C \) keeping \( \bar{p} \) invariant is finite.

Recall that \( g_C := \text{dim} \; H^1(C, \mathcal{O}_C) \) is the genus of \( C \), where \( \mathcal{O}_C \) is the structure sheaf of \( C \).

Given a stable \( k \)-pointed curve \( (C,\bar{p}) \). Assume that \( q \in C \) is a nodal point in \( C \). Let \( \pi : \tilde{C} \to C \) be the normalization of \( C \) at \( q \). Denote by \( \{q_+, q_-\} \) the preimage of \( q \) via \( \pi \). Without confusion we will still denote by \( p_1, \ldots, p_k \) the preimages of \( p_1, p_2, \ldots, p_k \in C \) in \( \tilde{C} \).

We choose a system of \( g \)-equivariant maps \( C \to V_\mu \otimes V_{\mu}^* \) for \( \mu \in P^+ \) such that the following diagram commutes

\[
\begin{array}{ccc}
C & \xrightarrow{\kappa_\mu} & V_\mu \otimes V_{\mu}^* \\
& \xrightarrow{\kappa_{\sigma(\mu)}} & \sigma \\
& & V_{\sigma(\mu)} \otimes V_{\sigma(\mu)^*}
\end{array}
\]
for any dominant weight $\mu$. Note that the map $\kappa_\mu$ induces the following map

$$\hat{\kappa}_\mu : V_{g,\ell,\bar{\lambda}}(C, \bar{p}) \simeq V_{g,\ell,\bar{\lambda},0}(C, \bar{p}, q) \to V_{g,\ell,\bar{\lambda},\mu,\mu^*}(C, \bar{p}, q_+, q_-).$$

Moreover it is easy to see that the following diagram commutes

$$(17) \quad \begin{array}{c}
V_{g,\ell,\bar{\lambda}}(C, \bar{p}) \xrightarrow{\hat{\kappa}_\mu} V_{g,\ell,\bar{\lambda},\mu,\mu^*}(C, \bar{p}, q_+, q_-) \\
\downarrow \sigma \quad \quad \quad \quad \downarrow \sigma
\end{array}$$

**Theorem 3.11.** The map

$$(18) \quad \begin{array}{c}
V_{g,\ell,\bar{\lambda}}(C, \bar{p}) \xrightarrow{\hat{\kappa}_\mu} \bigoplus_{\mu \in \mathcal{P}_\ell} V_{g,\ell,\bar{\lambda},\mu,\mu^*}(\tilde{C}, \bar{p}, q_+, q_-)
\end{array}$$

is an isomorphism, moreover the following diagram commutes,

$$(19) \quad \begin{array}{c}
V_{g,\ell,\bar{\lambda}}(C, \bar{p}) \xrightarrow{\hat{\kappa}_\mu} \bigoplus_{\mu \in \mathcal{P}_\ell} V_{g,\ell,\bar{\lambda},\mu,\mu^*}(\tilde{C}, \bar{p}, q_+, q_-) \\
\downarrow \sigma \quad \quad \quad \quad \downarrow \sigma
\end{array}$$

**Proof.** The isomorphism (18) is the well-known factorization theorem of conformal blocks (cf. [Ue, Theorem 3.19]), and the commutativity (19) follows from the commutativity (17). □

**Corollary 3.12.** With the same setup as above. If $\sigma(\bar{\lambda}) = \bar{\lambda}$, then the following equality holds

$$\text{tr}(\sigma|V_{g,\ell,\bar{\lambda}}(C, \bar{p})) = \sum_{\mu \in \mathcal{P}_\ell} \text{tr}(\sigma|V_{g,\ell,\bar{\lambda},\mu,\mu^*}(\tilde{C}, \bar{p}, q_+, q_-)).$$

**Proof.** It is an immediate consequence of Theorem 3.11. □

Given a family $(\pi : C \to X, \bar{p})$ of stable $k$-pointed curves where $\pi$ is a family of projective curves with at most nodal singularities over a smooth variety $X$ and $\bar{p} = (p_1, \ldots, p_k)$ is a collection of sections $p_i : X \to C$ with disjoint images such that $p_i(x)$ is a smooth point in $C_x := \pi^{-1}(x)$ for each $i$ and $x \in X$, one can associate a sheaf of conformal blocks $V_{g,\ell,\bar{\lambda}}(C, \bar{p})$ over $X$ which is locally free and of finite rank, see [Loo, Ts] for the coordinate-free approach on the space of conformal blocks. For each $x \in X$, the fiber $V_{g,\ell,\bar{\lambda}}(C, \bar{p})|_x$ is the space of conformal blocks $V_{g,\ell,\bar{\lambda}}(C, \bar{p}(x))$, where $\bar{p}(x) = (p_1(x), \ldots, p_k(x))$ are the $k$-distinct smooth points in $C_x$ as the image of $x$ via $\bar{p}$.

From the construction the sheaf of conformal blocks in [Loo, Ts], one can see the diagram automorphism $\sigma$ acts algebraically on $V_{g,\ell,\bar{\lambda}}(C, \bar{p})$. Denote by $\langle \sigma \rangle$ the cyclic group generated by $\sigma$. Then the group $\langle \sigma \rangle$ is isomorphic to $\mathbb{Z}/r\mathbb{Z}$, where $r$ is the order of $\sigma$.  

18
Lemma 3.13. For any family \((\pi : C \to X, \vec{p})\) of stable pointed curves, the function \(x \in X \mapsto \text{tr}(\sigma|V_{g,\ell,\vec{\lambda}}(C_x, \vec{p}(x)))\) is constant.

Proof. Given any irreducible representation \(\rho\) of \(\langle \sigma \rangle\), we denote by \(\text{ch}(\rho)\) and \(\text{ch}(V_{g,\ell,\vec{\lambda}}(C_x, \vec{p}(x)))\) the characters of \(\rho\) and \(V_{g,\ell,\vec{\lambda}}(C_x, \vec{p}(x))\) as representations of \(\langle \sigma \rangle\). For any two functions \(\phi, \psi\) on \(\langle \sigma \rangle\), we define the bilinear form

\[
(\phi, \psi) = \frac{1}{r} \sum_{i=0}^{r-1} \phi(\sigma^i) \psi(\sigma^{-i}),
\]

where \(r\) is the order of \(\sigma\).

For any \(x \in X\), let \(m_\rho(x)\) be the multiplicity of \(\rho\) appearing in \(V_{g,\ell,\vec{\lambda}}(C_x, \vec{p}(x))\).

By representation theory of finite groups, we have

\[m_\rho(x) = (\text{ch}_\rho, \text{ch}(V_{g,\ell,\vec{\lambda}}(C_x, \vec{p}(x))))\]

It is a continuous function on \(X\) and it is an integer. It is forced to be constant.

Hence

\[
\text{tr}(\sigma|V_{g,\ell,\vec{\lambda}}(C_x, \vec{p}(x))) = \sum m_\rho(x) \text{tr}(\sigma|\rho)
\]

is constant along \(x \in X\). \(\square\)

The following theorem shows that the trace of the diagram automorphism on the space of conformal blocks satisfies factorization properties.

Theorem 3.14. (1) For any stable \(k\)-pointed curve \((C, \vec{p})\), the number \(V_{g,\ell,\vec{\lambda}}(C, \vec{p})\)

only depends on \(\vec{\lambda}\) and the genus of \(C\).

(2) Given a stable \(k\)-pointed curve \((C, \vec{p})\) of genus \(g \geq 1\) and a stable \((k+2)\)

-pointed curve \((C', \vec{q})\) of genus \(g-1\), we have the following formula

\[
\text{tr}(\sigma|V_{g,\ell,\vec{\lambda}}(C, \vec{p})) = \sum_{\rho \in P_\ell} \text{tr}(\sigma|V_{g,\ell,\vec{\lambda},\mu}^*(C', \vec{q})).
\]

(3) Given any tuples of dominant weights \(\vec{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_s)\) and \(\vec{\mu} = (\mu_1, \cdots, \mu_t)\)

in \(P_\ell\) where \(s, t \geq 2\), we have the following equality

\[
\text{tr}(\sigma|V_{g,\ell,\vec{\lambda},\vec{\mu}}(\mathbb{P}^1, \vec{p}_1)) = \bigoplus_{\nu \in P_\ell} \text{tr}(\sigma|V_{g,\ell,\vec{\lambda},\nu}^*(\mathbb{P}^1, \vec{p}_2)) \text{tr}(\sigma|V_{g,\ell,\vec{\mu},\nu}^*(\mathbb{P}^1, \vec{p}_3)),
\]

where \(\vec{p}_1\) is any tuple of \(s+t\) distinct points, \(\vec{p}_2\) is any tuple of \(s+1\) distinct points and \(\vec{p}_3\) is any tuple of \(t+1\) distinct points in \(\mathbb{P}^1\).

Proof. We first prove (1). By standard theory of moduli of curves (cf. [HM]), there exists a chain of families of stable \(k\)-pointed curves over smooth bases connecting any two stable \(k\)-pointed curves with the same genus. In view of Lemma 3.13, (1) follows.

From the theory of moduli of curves (cf. [HM]), we know that any smooth pointed stable curve can be degenerated to an irreducible stable pointed curve with only one nodal point. Then (2) follows from (1) and Corollary 3.12.

We now proceed to prove (3). Let \(C\) be the union of two projective lines \(C = C_1 \cup C_2\) where \(C_1\) and \(C_2\) intersect at one point \(z\). Let \(\vec{p} = (p_1, \cdots, p_s)\) be a set of \(s\)
distinct points in $C_1 \setminus \{ z \}$ and $q = \{ q_1, \cdots, q_t \}$ be another set of $t$ distinct points in $C_2 \setminus \{ z \}$ where $s, t \geq 2$. Clearly $(C, \vec{p} \cup \vec{q})$ is a stable $(s + t)$-pointed curve of genus zero. Again by the theory of moduli of curves, there exists a family $\pi : C \to X$ of stable $(s + t)$-pointed curves over a smooth variety $X$ such that $C_{x_0} = C$ with $\vec{p} \cup \vec{q}$ and any other fiber is a projective line with $s + t$ points $\vec{p}_1$. By Lemma 3.13

$$\text{tr}(\sigma|_{P_{g, \ell, \vec{p}, \vec{q}}}(C, \vec{p}, \vec{q})) = \text{tr}(\sigma|_{P_{g, \ell, \vec{p}, \vec{q}}}(\mathbb{P}^1, \vec{p}_1)).$$

Let $\pi : \tilde{C} \to C$ be the normalization of $C$ at $z$ with the preimage $(z_+, z_-)$ of $z$. The pointed curve $(\tilde{C}, \vec{p}, \vec{q}, z_+, z_-) = (\mathbb{P}^1, \vec{p}, z_+) \cup (\mathbb{P}^1, \vec{q}, z_-)$ is a disjoint union of a $(s + 1)$-points projective line and a $(t + 1)$-pointed projective line. Then (3) follows from Corollary 3.12 and Lemma 3.13.

**Remark 3.15.** By Theorem 3.14, the computation of the trace of the diagram automorphism on the space of conformal blocks can be reduced to the trace of the diagram automorphism on the space of conformal blocks on the pointed curve $(\mathbb{P}^1, (0, 1, \infty))$.

### 3.5. $\sigma$-twisted fusion ring

Let $J$ be a finite set with an involution $\lambda \mapsto \lambda^*$. We denote by $\mathbb{N}^J$ the free commutative monoid generated by $J$, that is, the set of sums $\sum_{\lambda \in J} n_\lambda \lambda$ with $n_\lambda \in \mathbb{N}$. The involution of $J$ extends by linearity to an involution $x \mapsto x^*$ of $\mathbb{N}^J$. First of all, we recall the definition of fusion rule (cf. [Be, Section 5]).

**Definition 3.16.** A fusion rule on $J$ is a map $N : \mathbb{N}^J \to \mathbb{Z}$ satisfying the following conditions:

1. One has $N(0) = 1$, and $N(\lambda) > 0$ for for some $\lambda \in J$;
2. $N(x^*) = N(x)$ for every $x \in \mathbb{N}^J$;
3. For $x, y \in \mathbb{N}^J$, one has $N(x + y) = \sum_\lambda N(x + \lambda)N(y + \lambda^*)$.

The kernel of a fusion rule $N$ by definition is the set of elements $\lambda \in J$ such that $N(\lambda + x) = 0$ for all $x \in \mathbb{N}^J$. A fusion rule on $J$ is called non-degenerate if the kernel is empty.

**Lemma 3.17.** If $\sigma(\bar{\lambda}) = \bar{\lambda}$, then the trace $\text{tr}(\sigma|_{P_{g, \ell, \vec{p}}}(C, \vec{p}))$ is an integer.

**Proof.** When the order of $\sigma$ is 2, it is obvious. In general it follows from Theorem 5.11 Formula (1) in the introduction and part (3) of Theorem 3.14.

**Theorem 3.18.** The map $\text{tr}_\sigma : P_{g, \ell}^* \to \mathbb{Z}$ given by

$$\sum_\lambda n_\lambda \mapsto \text{tr}(\sigma|_{P_{g, \ell, \vec{p}}}(\mathbb{P}^1, \vec{p}))$$

where $\bar{\lambda} = (\lambda_1, \cdots, \lambda_k)$ and $\vec{p} = (p_1, \cdots, p_k)$ is the set of any $k$-distinct points in $\mathbb{P}^1$, is a non-degenerate fusion rule. Here the set $P_{g, \ell}^*$ is equipped with the involution $\lambda \mapsto \lambda^* := -w_0(\lambda)$, where $w_0$ is the longest element in the Weyl group $W$.

**Proof.** By Lemma 3.17 the trace map $\text{tr}_\sigma$ indeed always takes integer values.

Condition (1) of Definition 3.16 follows from part (1) of Lemma 3.9. Condition (2) follows from Lemma 3.7. Condition (3) follows from the part (3) of Theorem 3.14.
The non-degeneracy follows from the part (2) of Lemma 3.9.

Let \( R_{\ell}(g, \sigma) \) be a free abelian group with the set \( P_{\sigma}^{\ell} \) as a basis. As a consequence of Theorem 3.18 and [Be, Proposition 5.3], we can define a ring structure on \( R_{\ell}(g, \sigma) \) by putting

\[
\lambda \cdot \mu := \sum_{\nu \in P_{\sigma}^{\ell}} \text{tr}(\sigma|_{V_{g,\lambda,\mu,\nu}(\mathbb{P}^1, 0, 1, \infty)}) \nu,
\]

for any \( \lambda, \mu \in P_{\sigma}^{\ell} \).

Let \( S_{\sigma} \) be the set of characters (i.e. ring homomorphisms) of \( R_{\ell}(g, \sigma) \) into \( \mathbb{C} \). The following proposition is a consequence of general fact on fusion ring by Beauville [Be, Corollary 6.2].

**Proposition 3.19.**
\begin{enumerate}
  \item \( R_{\ell}(g, \sigma) \otimes \mathbb{C} \) is a reduced commutative ring.
  \item The map \( R_{\ell}(G, \sigma) \otimes \mathbb{C} \to \mathbb{C}^{S_{\sigma}} \) given by \( x \mapsto (\chi(x))_{x \in S_{\sigma}} \) is an isomorphism of \( \mathbb{C} \)-algebra.
  \item one has \( \chi(x^*) = \overline{\chi(x)} \) where \( \overline{\chi(x)} \) denotes the complex conjugation of \( \chi(x) \) for any \( \chi \in S_{\sigma} \) and \( x \in R_{\ell}(g, \sigma) \).
\end{enumerate}

Let \( \omega_{\sigma} \) be the Casimir element in \( R_{\ell}(g, \sigma) \) defined as follows

\[
\omega_{\sigma} = \sum_{\lambda \in P_{\sigma}^{\ell}} \lambda \cdot \lambda^*.
\]

**Proposition 3.20.** For any \( k \)-pointed stable curve \( (C, \vec{p}) \) and for any \( \sigma \)-invariant tuple \( \vec{\lambda} \) of dominant weights in \( P_{\ell}^{\sigma} \), we have the following formula

\[
\text{tr}(\sigma|_{V_{g,\ell,\vec{\lambda}}(C, \vec{p})}) = \sum_{\chi \in S_{\sigma}} \chi(\lambda_1) \cdots \chi(\lambda_k) \chi(\omega_{\sigma})^g, \]

where \( g \) is the genus of \( C \) and \( \chi(\omega_{\sigma}) = \sum_{\lambda \in P_{\sigma}^{\ell}} |\chi(\lambda)|^2 \).

**Proof.** It is a consequence of the part (2) of Theorem 3.14 and [Be, Proposition 6.3].

From this proposition, if we can determine the set \( S_{\sigma} \) and the value \( \chi(\omega_{\sigma}) \) for each \( \chi \in S_{\sigma} \), then the trace \( \text{tr}(\sigma|_{V_{g,\ell,\vec{\lambda}}(C, \vec{p})}) \) is known.

4. Sign problems

By Borel-Bott-Weil theorem for affine Kac-Moody groups, the dual of certain higher cohomology of vector bundles on affine Grassmannian or affine flag variety realize irreducible integral representations of affine Lie algebras. The action of the diagram automorphism on this space is determined in Section 4.1 and 4.2. A similar sign problem on affine BGG resolution and affine Kostant homology is determined in Section 4.3.
4.1. **Borel-Weil-Bott theorem on the affine flag variety.** Let $G$ be a connected and simply-connected simple algebraic group associated to a simple Lie algebra $\mathfrak{g}$. Let $G((t))$ be the loop group of $G$. Let $\hat{G}$ be the nontrivial central extension of $G((t))$ by the center $\mathbb{C}^\times$. Then the Lie algebra of $\hat{G}$ is the affine Lie algebra $\mathfrak{g}$. Let $\hat{G}$ be the group $\hat{G} = \hat{G} \rtimes \mathbb{C}^\times$ whose Lie algebra is the affine Kac-Moody algebra $\mathfrak{g}$.

Let $I$ be the Iwahori subgroup of $G((t))$, i.e. $I = ev_0^{-1}(B)$, where $B$ is the Borel subgroup of $G$. Let $Fl_G$ be the affine flag variety $G((t))/I$ of $G$. Let $\hat{I}$ be the group $I \rtimes \mathbb{C}^\times$, where $\mathbb{C}^\times$ is the center of $\hat{G}$. Let $\hat{\mathcal{F}}$ be the product $\hat{\mathcal{F}} \rtimes \mathbb{C}^\times$ as subgroup of $\hat{G}$. Then we have

$$Fl_G \simeq \hat{G}/\hat{I} \simeq \hat{G}/\hat{I}.$$  

Given any representation $V$ of $\hat{\mathcal{F}}$, we can attach a $\hat{G}$-equivariant vector bundle $\mathcal{L}(V)$ on $Fl_G$ as follows

$$\mathcal{L}(V) = \hat{G} \times_{\hat{\mathcal{F}}} V^*,$$

where $V^*$ is the dual representation of $\hat{\mathcal{F}}$. Let $\Lambda$ be a character of $\hat{\mathcal{F}}$ and let $\mathbb{C}_\Lambda$ be the associated 1-dimensional representation of $\hat{\mathcal{F}}$. We denote by $\mathcal{L}(\Lambda)$ the $\hat{G}$-equivariant line bundle $\mathcal{L}(\mathbb{C}_\Lambda)$ on $Fl_G$.

For any ind-scheme $X$ and any vector bundle $\mathcal{F}$ on $X$, the cohomology groups $H^\ast(X, \mathcal{F})$ carry a topology. We put $H^\ast(X, \mathcal{F})^\vee$ the restricted dual of $H^\ast(X, \mathcal{F})$, i.e. $H^\ast(X, \mathcal{F})^\vee$ consists of continuous functional on $H^\ast(X, \mathcal{F})$ where we take discrete topology on $\mathbb{C}$. The affine flag variety $Fl_G$ is an ind-scheme of ind-finite type. We refer the reader to [Ku1] for the foundation of flag varieties of Kac-Moody groups.

Recall the following affine analogue of Borel-Weil-Bott theorem (cf. [Ku1, Theorem 8.3.11]).

**Theorem 4.1.** Given any dominant weight $\Lambda$ of $\hat{G}$ and any $w \in \hat{W}$, the space $H^{\ell(w)}(Fl_G, \mathcal{L}(w \ast \Lambda))^\vee$ is naturally the integral irreducible representation $\mathcal{H}_\Lambda$ of $\mathfrak{g}$ of highest weight $\Lambda$, where $w \ast \Lambda = w \cdot (\Lambda + \hat{\rho}) - \hat{\rho}$ and $H^{\ell(w)}(Fl_G, \mathcal{L}(w \ast \Lambda))$ is the cohomology of the line bundle $\mathcal{L}(w \ast \Lambda)$ on $Fl_G$. Moreover $H^i(Fl_G, \mathcal{L}(w \ast \Lambda)) = 0$ if $i \neq \ell(w)$.

Let $\sigma$ be a diagram automorphism on $G$. Note that $\sigma$ preserves $\hat{\mathcal{F}}$. For any $\sigma$-invariant character $\Lambda$ of $\hat{\mathcal{F}}$, we have a natural $\sigma$-equivariant structure on $\mathcal{L}(\Lambda)$, since

$$\hat{G} \times \langle \sigma \rangle \times_{\hat{\mathcal{F}} \times \langle \sigma \rangle} (\mathbb{C}_\Lambda)^* \simeq \hat{G} \times_{\hat{\mathcal{F}}} (\mathbb{C}_\Lambda)^*,$$

where we declare the action of $\sigma$ on $\mathbb{C}_\Lambda$ by $1$. Let $\xi$ be an $r$-th root of unity, where $r$ is the order of $\sigma$. We denote by $\mathcal{L}(\Lambda, \xi)$ the $\hat{G} \times \langle \sigma \rangle$-equivariant line bundle,

$$\mathcal{L}(\Lambda, \xi) := \hat{G} \times \langle \sigma \rangle \times_{\hat{\mathcal{F}} \times \langle \sigma \rangle} (\mathbb{C}_{\Lambda, \xi})^*$$

where $\hat{\mathcal{F}}$ acts on $\mathbb{C}_{\Lambda, \xi}$ by $\Lambda$ and $\sigma$ acts on $\mathbb{C}_{\Lambda, \xi}$ by $\xi$. Hence by this convention the natural $\hat{G} \times \langle \sigma \rangle$-equivariant structure on $\mathcal{L}(\Lambda)$ is isomorphic to $\mathcal{L}(\Lambda, 1)$.

For any $\sigma$-orbit $\iota$ in the affine Dynkin diagram $\hat{I}$, let $G_\iota$ be the the simply-connected algebraic group associated to the the sub-diagram $\iota$ and let $B_\iota$ be the
Borel subgroup of $G_i$. We have the following all possibilities

\[
G_i = \begin{cases} 
\text{SL}_2 & i = \{i\} \\
\text{SL}_2 \times \text{SL}_2 & i = \{i, j\} \text{ and } i, j \text{ are not connected} \\
\text{SL}_2 \times \text{SL}_2 \times \text{SL}_2 & i = \{i, j, k\} \text{ and } i, j, k \text{ are not connected} \\
\text{SL}_3 & i = \{i, j\} \text{ and } i, j \text{ are connected}
\end{cases}
\]

We still denote by $\sigma$ the diagram automorphism on $G_i$ which preserves $B_i$. For any $\sigma$-invariant weight $\lambda$ of $G_i$, it can be written as $n\rho_i$ for some integer $n \in \mathbb{Z}$, where $\rho_i$ is the sum of all fundamental weights of $G_i$. Let $\mathcal{B}_i := G_i/B_i$ be the flag variety of $G_i$. Put $d_i = \dim G_i/B_i$.

As in the affine case for any $r$-th root of unity and any $\sigma$-invariant character $\lambda$ of $B_i$, we set

\[
\mathcal{L}(\lambda, \xi) = G_i \times_{B_i} (\mathbb{C}_{\lambda, \xi})^*\]

the $G_i \rtimes \langle \sigma \rangle$-equivariant line bundle on $\mathcal{B}_i$. Let $\Omega_i$ be the canonical bundle of $\mathcal{B}_i$. Note that the canonical bundle $\Omega_i$ is naturally a $G_i \rtimes \langle \sigma \rangle$-equivariant line bundle.

**Lemma 4.2.** We have the following isomorphism of $G_i \rtimes \langle \sigma \rangle$-equivariant line bundles

\[
\Omega_{\mathcal{B}_i} \simeq \mathcal{L}(-2\rho_i, \epsilon_i),
\]

where $\epsilon_i = (-1)^{d_i-1}$.

**Proof.** The canonical bundle $\Omega_i$ is naturally isomorphic to $G_i \times_{B_i} (\wedge^d(\mathfrak{g}_i/\mathfrak{b}_i))^*$, where $\mathfrak{g}_i$ (resp. $\mathfrak{b}_i$) is the Lie algebra of $G_i$ (resp $B_i$). Hence it suffices to determine the action of $T_i$ and $\sigma$ on $\wedge^d(\mathfrak{g}_i/\mathfrak{b}_i)$, where $T_i$ is the maximal torus of $G_i$, contained in $B_i$. Note that

\[
\wedge^d(\mathfrak{g}_i/\mathfrak{b}_i) \simeq \wedge^d n_i^-,
\]

where $n_i^-$ is the nilpotent radical of the negative Borel subalgebra of $\mathfrak{g}_i$. Hence as 1-dimensional representation of $T_i$ it isomorphic to $-2\rho_i$, and by case-by-case analysis it is easy to check $\sigma$ acts on it exactly by $\epsilon_i$. It finishes the proof of the lemma. \qed

Only when $i$ consists of two vertices and $i = \{i, j\}$ is not connected, $\epsilon_i = -1$; otherwise $\epsilon_i = 1$.

**Lemma 4.3.** Given any $n \in \mathbb{Z}$ and any $r$-th root of unity $\xi$, there exists a unique isomorphism up to a scalar

\[
H^d(\mathcal{B}_i, \mathcal{L}(n\rho_i, \xi)) \simeq H^0(\mathcal{B}_i, \mathcal{L}((-n-2)\rho_i, \epsilon_i \cdot \xi)),
\]

as representations of $G_i \rtimes \langle \sigma \rangle$. Moreover

\[
H^i(\mathcal{B}_i, \mathcal{L}(n\rho_i, \xi)) = 0 \quad \text{if } i \neq 0, d_i.
\]

**Proof.** By Borel-Weil-Bott theorem we have the following isomorphism of representations of $G_i \rtimes \langle \sigma \rangle$

\[
H^0(\mathcal{B}_i, \mathcal{L}(n\rho_i, \xi))^* = \begin{cases} 
V_{n\rho_i, \xi} & n \geq 0 \\
0 & n < 0
\end{cases},
\]

(22)
for any \( n \in \mathbb{Z} \) and \( r \)-th root of unity \( \xi \), where \( V_{n \rho, \xi} \) is the irreducible representation of \( G \), of highest weight \( n \rho \), with the compatible action of \( \sigma \) which acts on the highest weight vectors by \( \xi \).

By Serre duality we have the following canonical isomorphism

\[
H^d_i(B, L(n\rho_i, \xi)) \simeq H^0(B, L(-n\rho_i, \xi^{-1}) \otimes \Omega_{B_i})^*
\]
as representations of \( G_i \rtimes \langle \sigma \rangle \). In view of Lemma 4.2

\[
H^0(B, L(-n\rho_i, \xi^{-1}) \otimes \Omega_{B_i}) \simeq H^0(B, L((n-2)\rho_i, \epsilon_i \cdot \xi^{-1})).
\]

In view of (22), by Schur lemma there exists a unique isomorphism up to a scalar

\[
H^0(B, L((-n-2)\rho_i, \epsilon_i \cdot \xi^{-1}))^* \simeq H^0(B, L((-n-2)\rho_i, \epsilon_i \cdot \xi))
\]
as representations of \( G_i \rtimes \langle \sigma \rangle \). Therefore we have an isomorphism

\[
H^d_i(B, L(n\rho_i, \xi)) \simeq H^0(B, L((-n-2)\rho_i, \epsilon_i \cdot \xi))
\]
as representations of \( G_i \rtimes \langle \sigma \rangle \).

Now we prove the second part of the Lemma. When \( n \geq 0 \), \( n\rho_i \) is dominant, then Borel-Weil-Bott theorem implies that \( H^i(B, L(n\rho_i, \xi)) = 0 \) unless \( i = 0 \). In view of the isomorphism (24), when \( n \leq -2 \), \( H^i(B, L(n\rho_i, \xi)) = 0 \) unless \( i = d_i \). When \( n = -1 \), it is easy to check that \( s_i \cdot \rho_i = \rho_i \). Hence \( H^1(B, L(-\rho_i, \xi)) = 0 \) for any \( i \).

Let \( \tilde{P}_i \) be the parabolic subgroup of \( \tilde{G} \) containing \( \tilde{I} \) and \( G_i \). We have an isomorphism of varieties \( \tilde{P}_i / \tilde{I} \simeq B_i \). Let \( \pi_i : Fl_G \rightarrow \tilde{G} / \tilde{P}_i \) be the projection map. The fiber is isomorphic to \( B_i \). There exists the following natural isomorphism as \( \tilde{G} \rtimes \langle \sigma \rangle \)-equivariant ind-schemes

\[
\tilde{G} \rtimes \langle \sigma \rangle \times \tilde{P}_i \times \langle \sigma \rangle B_i \simeq Fl_G.
\]

Let \( R^i(\pi_i)_* \) be the \( i \)-th derived functor of the pushforward functor \( (\pi_i)_* \). From the \( G_i \rtimes \langle \sigma \rangle \)-equivariant line bundle \( L(n\rho_i, \xi) \) on \( B_i \), by descent theory one can attach a \( \tilde{G} \rtimes \langle \sigma \rangle \)-equivariant line bundle \( L_{\pi_i}(n\rho_i, \xi) \) on \( Fl_G \), i.e.

\[
L_{\pi_i}(n\rho_i, \xi) := \tilde{G} \rtimes \langle \sigma \rangle \times \tilde{P}_i \times \langle \sigma \rangle L(n\rho_i, \xi),
\]

where the action of \( \tilde{P}_i \times \langle \sigma \rangle \) on \( L(n\rho_i, \xi) \) factors through \( G_i \rtimes \langle \sigma \rangle \). Let \( \Omega_{\pi_i} \) be the relative canonical line bundle of \( Fl_G \) over \( \tilde{G} / \tilde{P}_i \). By Lemma 4.2 it is clear that as a \( \tilde{G} \rtimes \langle \sigma \rangle \)-equivariant bundle, we have

\[
\Omega_{\pi_i} \simeq L_{\pi_i}(-2\rho_i, \epsilon_i).
\]

**Lemma 4.4.** There exits a natural isomorphism of \( \tilde{G} \rtimes \langle \sigma \rangle \)-equivariant vector bundles

\[
R^d_i(\pi_i)_*(L_{\pi_i}(n\rho_i, \xi)) \simeq (\pi_i)_*(L_{\pi_i}((-n-2)\rho_i, \xi \cdot \epsilon_i)).
\]

**Proof.** This lemma follows from relative Serre duality on the morphism \( \pi_i : Fl_G \rightarrow \tilde{G} / \tilde{P}_i \), descent theory and Lemma 1.3.

By Lemma 2.6 the affine Weyl group \( \hat{W}^\sigma \) consists of simple reflections \( \{ s_i \mid i \in \hat{I}_\sigma \} \).
Proposition 4.6. For any $\sigma$-invariant weight $\Lambda$ and for any $\sigma$-orbit in $\hat{I}$, we have

$$s_i \cdot \Lambda = \begin{cases} 
\Lambda - \langle \Lambda, \tilde{\alpha}_i \rangle \sum_{i \in I} \alpha_i & \text{if } i \text{ is not connected} \\
\Lambda - 2\langle \Lambda, \alpha_i \rangle (\alpha_i + \alpha_j) & \text{if } i = \{i, j\} \text{ is connected}
\end{cases}.$$

Proof. For any $\sigma$-orbit $i$ in $I$, it is routine to check, in particular we use the formula (26). When $i = \{0\}$, it is simply the definition of $s_0$. \hfill \square

Proposition 4.6. For any $\sigma$-invariant weight $\Lambda$, and for any $\sigma$-orbit in the affine Dynkin diagram $\hat{I}$ and any $r$-th root of unity $\xi$, we have the following isomorphism

$$H^{i+d}(\text{Fl}_G, L(s_i \cdot \Lambda, \xi)) \simeq H^i(\text{Fl}_G, L(\Lambda, \epsilon_i \cdot \xi))$$

of representations of $\tilde{G} \rtimes \langle \sigma \rangle$.

Proof. Note that the restriction $L(\Lambda, \xi)|_{B_i}$ of the $\tilde{G} \rtimes \langle \sigma \rangle$-equivariant line bundle $L(\Lambda, \xi)$ to the fiber $B_i$, is isomorphic to $L(\Lambda, \tilde{\alpha}_i)\rho_i, \xi)$ as a $G_i \rtimes \langle \sigma \rangle$-equivariant line bundle for any $i \in I$. Note that for any $i, j \in I$, $\langle \Lambda, \tilde{\alpha}_i \rangle = \langle \Lambda, \tilde{\alpha}_j \rangle$. In view of Lemma 4.5, we have

$$s_i \cdot \Lambda = \begin{cases} 
\Lambda - (\langle \Lambda, \tilde{\alpha}_i \rangle + 1) \sum_{i \in I} \alpha_i & \text{if } i \text{ is not connected} \\
\Lambda - 2(\langle \Lambda, \alpha_i \rangle + 1)(\alpha_i + \alpha_j) & \text{if } i = \{i, j\} \text{ is connected}
\end{cases}.$$

Hence for any $\sigma$-orbit $i$ in $\hat{I}$ and $i \in I$, we have

$$\langle s_i \cdot \Lambda, \tilde{\alpha}_i \rangle = -\langle \Lambda, \tilde{\alpha}_i \rangle - 2.$$

It follows that

$L(s_i \cdot \Lambda, \xi)|_{B_i} = L(-\langle \Lambda, \tilde{\alpha}_i \rangle + 2)\rho_i, \xi).$ By Lemma 4.3, we have the following natural isomorphism of $\tilde{G} \rtimes \langle \sigma \rangle$-equivariant vector bundles

$$R^d(\pi_\ast(L(\Lambda, \xi))) \simeq \pi_\ast(L(s_i \cdot \Lambda, \epsilon_i \cdot \xi)).$$

By Lemma 4.3 we have

$$R^i(\pi_\ast(L(\Lambda, \xi))) = 0 \quad \text{if } i \neq 0, d_i.$$

In view of (25) and (26), Leray’s spectral sequence implies that

$$H^{i+d}(\text{Fl}_G, L(\Lambda, \xi)) \simeq H^i(\text{Fl}_G, L(\Lambda, \epsilon_i \cdot \xi))$$

as representations of $\tilde{G} \rtimes \langle \sigma \rangle$. \hfill \square

For any $w \in (\hat{W})^\sigma$, put $\epsilon_w = (-1)^{\ell(w) - \ell_0(w)}$. For any reduced expression $w = s_{i_k} s_{i_{k-1}} \cdots s_{i_1}$ of $w$ in the Coxeter group $\hat{W}^\sigma$ where $i_1, \cdots, i_k$ are $\sigma$-orbits in $\hat{I}$ and each $s_i$ is defined in (11) for any $i \in I_\sigma$ and $s_{\{0\}} = s_0$, we have

$$\epsilon_w = \epsilon_{i_k} \cdots \epsilon_{i_1},$$

where $\epsilon_i$ is introduced in Lemma 4.2.

Finally we are now ready to prove the following theorem.
Theorem 4.7. For any \( w \in \hat{W}^\sigma \) and for any \( \sigma \)-invariant dominant weight \( \Lambda \) of \( \hat{G} \). We have the following isomorphism
\[
H^{\ell(w)}(\text{Fl}_G, \mathcal{L}(w \ast \Lambda, \xi)) \simeq H^0(\text{Fl}_G, \mathcal{L}(\Lambda, \epsilon_w \cdot \xi))
\]
as representations of \( \hat{G} \rtimes \langle \sigma \rangle \).

Proof. We can write \( w = s_{i_k} s_{i_{k-1}} \cdots s_{i_1} \) as a reduced expression in the Coxeter group \( \hat{W}^\sigma \), where \( i_1, \cdots, i_k \) are \( \sigma \)-orbits in \( \hat{I} \). Then
\[
\Lambda, s_{i_1} \ast \Lambda, (s_{i_2} s_{i_1}) \ast \Lambda, \cdots, w \ast \Lambda
\]
are all \( \sigma \)-invariant weights of \( \hat{G} \).

Note that as an element in \( \hat{W} \), the length \( \ell(w) \) of \( w \) is equal to \( \sum_{i=1}^k d_{i_i} \).

In view of Proposition 4.6, we get a chain of isomorphisms of \( \hat{G} \rtimes \langle \sigma \rangle \)-representations
\[
H^{\ell(w)}(\text{Fl}_G, \mathcal{L}(w \ast \Lambda, \xi)) \simeq H^{\ell(w)-d_{i_1}}(\text{Fl}_G, \mathcal{L}((s_{i_1} w) \ast \Lambda, \epsilon_{i_1} \xi))
\]
\[
\simeq H^{\ell(w)-d_{i_1}-d_{i_2}}(\text{Fl}_G, \mathcal{L}((s_{i_2} s_{i_1} w) \ast \Lambda, \epsilon_{i_2} \epsilon_{i_1} \xi))
\]
\[
\cdots \cdots
\]
\[
\simeq H^0(\text{Fl}_G, \mathcal{L}(\Lambda, \epsilon_w \cdot \xi)).
\]

It finishes the proof of the theorem. \( \square \)

For any dominant weight \( \Lambda \) of \( \hat{g} \) and an \( r \)-th root of unity, as always we denote by \( H_{\Lambda, \xi} \) the irreducible integral representation of \( \hat{g} \) of highest weight \( \Lambda \) together with a compatible action of \( \sigma \) which acts on the highest weight vectors of \( H_{\Lambda, \xi} \) by \( \xi \).

Corollary 4.8. In the same setting as in Theorem 4.7, we have the following isomorphism
\[
H^{\ell(w)}(\text{Fl}_G, \mathcal{L}(w \ast \Lambda, \xi))^\vee \simeq H_{\Lambda, \epsilon_w \cdot \xi}
\]
of representations of \( \hat{g} \rtimes \langle \sigma \rangle \).

Proof. It is an immediate consequence of Theorem 4.1 and Theorem 4.7. \( \square \)

Remark 4.9. For any \( \sigma \)-invariant weight \( \lambda \) of \( G \), let \( \mathcal{L}(\lambda) \) be the associated line bundle on \( G/B \). By Borel-Weil-Bott theorem, \( H^i(G/B, \mathcal{L}(\lambda)) \) carries the action of the diagram automorphism. The action was determined by Naito. Theorem 4.7 and Theorem 4.13 are the affine analogues of the results of Naito [N].

4.2. Borel-Weil-Bott theorem on affine Grassmannian. For any weight \( \lambda \) of \( G \), let \( \mathcal{L}_\ell(\lambda) \) be the \( \hat{G} \)-equivariant line bundle on \( \text{Fl}_G \) defined as follows,
\[
\mathcal{L}_\ell(\lambda) := \hat{G} \times \hat{I} I_\ell(\mathbb{C}_\lambda)^*,
\]
where \( I_\ell(\mathbb{C}_\lambda) \) is the 1-dimensional representation of \( \hat{I} \) such that \( \mathcal{I} \) factors through the character \( \lambda : B \to \mathbb{C}^\times \) and the center \( \mathbb{C}^\times \) acts by \( t \mapsto t^\ell \), and \( I_\ell(\lambda)^* \) is the dual of \( I_\ell(\lambda) \) as the representation of \( \hat{I} \).

For any character \( \Lambda \) of \( \hat{I} \), if \( \Lambda = \lambda + \ell \Lambda_0 \) where \( \Lambda \) is a weight of \( \hat{G} \) and \( \lambda \) is a weight of \( G \), then as \( \hat{G} \)-equivariant line bundles \( \mathcal{L}(\Lambda) = \mathcal{L}_\ell(\lambda) \).
If \( \lambda \) is \( \sigma \)-invariant, then \( \mathcal{L}_t(\lambda) \) has a natural \( \sigma \)-equivariant structure as in the case of \( \mathcal{L}(\Lambda) \). Similarly for a \( r \)-th root of unity where \( r \) is the order of \( \sigma \), we can associate a \( \hat{G} \times \langle \sigma \rangle \)-equivariant line bundle \( \mathcal{L}_t(\lambda, \xi) \). If \( \Lambda = \lambda + \ell \Lambda_0 \) where \( \lambda \in \mathcal{P}^\sigma \), then \( \mathcal{L}(\Lambda, \xi) = \mathcal{L}_t(\lambda, \xi) \) as \( \hat{G} \times \langle \sigma \rangle \)-equivariant line bundles.

Recall from Lemma 3.3 the weight \( \Lambda = \lambda + \ell \Lambda_0 \) is dominant for \( \tilde{G} \) if and only if \( \lambda \) is dominant for \( G \) and \( (\lambda, \tilde{\theta}) \leq \ell \). Recall the affine Weyl group \( \mathcal{W}_{\ell+h} \) discussed in Section 2.2, the action of \( \mathcal{W}_{\ell+h} \) on the weight lattice \( \mathcal{P} \) of \( G \) is compatible with the action of \( \tilde{W} \) on the space of weights of \( \tilde{G} \) of level \( \ell \), see Lemma 3.1 and Lemma 3.2. Therefore we can translate Theorem 4.7 into the following equivalent theorem.

**Theorem 4.10.** For any \( w \in \mathcal{W}_{\ell+h} \) such that \( \sigma(w) = w \) and for any \( \sigma \)-invariant dominant weight \( \lambda \in \mathcal{P}_t \), we have the following isomorphism

\[
H^\ell(w)(\text{Fl}_G, \mathcal{L}_t(w \ast \lambda, \xi)) \simeq H^0(\text{Fl}_G, \mathcal{L}_t(\lambda, \epsilon_w \cdot \xi))
\]

as representations of \( \hat{G} \times \langle \sigma \rangle \).

Let \( \tilde{\mathcal{P}} \) be the subgroup \( G[[t]] \times \mathbb{C}^\times \) of \( \hat{G} \) where \( \mathbb{C}^\times \) is the center torus. The affine Grassmannian \( \tilde{\text{Gr}}_G := G((t))/G[[t]] \) is isomorphic to the partial flag variety \( \tilde{G} / \tilde{\mathcal{P}} \). For any finite dimensional representation \( V \) of \( G \), let \( I_t(V) \) be the representation of \( \tilde{\mathcal{P}} \) such that \( G[[t]] \) acts via the evaluation map \( \text{ev}_0 : G[[t]] \to G \) given by evaluating \( t = 0 \), and the center \( \mathbb{C}^\times \) acts by \( t \mapsto t^t \). Let \( \mathcal{L}_t(V) \) be the induced \( \hat{G} \)-equivariant vector bundle on \( \tilde{\text{Gr}}_G \), i.e. \( \mathcal{L}_t(V) := \hat{G} \times_{\tilde{\mathcal{P}}} I_t(V)^* \), where \( I_t(V)^* \) is the dual of \( I_t(V) \) as the representation of \( \tilde{\mathcal{P}} \).

The diagram automorphism \( \sigma \) on \( G \) induces an automorphism on \( \hat{G} \) and it preserves \( \tilde{\mathcal{P}} \). For any \( \lambda \in (\mathcal{P}_t)^\sigma \), the vector bundle \( \mathcal{L}_t(\lambda) \) is naturally equipped with a \( \sigma \)-equivariant structure, since

\[
\hat{G} \times \langle \sigma \rangle \times_{\tilde{\mathcal{P}} \times \langle \sigma \rangle} I_t(\lambda)^* \simeq \hat{G} \times_{\tilde{\mathcal{P}}} I_t(\lambda)^*.
\]

Similarly for any \( r \)-th root of unity \( \xi \), we have the \( \hat{G} \times \langle \sigma \rangle \)-equivariant vector bundle \( \mathcal{L}_t(\lambda, \xi) \) on \( \tilde{\text{Gr}}_G \).

The following lemma is well-known.

**Lemma 4.11.** Let \( H_1 \) be a linear algebraic group and \( H_2 \) be a subgroup of \( H_1 \). Let \( V_1 \) be a finite dimensional representation of \( H_1 \) and let \( V_2 \) be a finite dimensional representation of \( H_2 \). Then we have an isomorphism of \( H_1 \)-equivariant vector bundles

\[
H_1 \times_{H_2} (V_2 \otimes V_1|_{H_2}) \simeq (H_1 \times_{H_2} V_2) \otimes V_1,
\]

given by

\[
(h_1, v_2 \otimes v_1) \mapsto (h_1, v_2) \otimes h_1 \cdot v_1,
\]

where \( h_1 \in H_1 \), \( v_1 \in V_1 \) and \( v_2 \in V_2 \).

**Lemma 4.12.** Let \( \lambda \) be a \( \sigma \)-invariant dominant weight of \( G \), and let \( V \) be a finite dimensional representation of \( G \times \sigma \). There is an isomorphism of \( \hat{G} \times \langle \sigma \rangle \)-representations

\[
H^1(\text{Gr}_G, \mathcal{L}_t(V_{\lambda, \xi} \otimes V)) \simeq H^1(\text{Fl}_G, \mathcal{L}_t(\mathcal{C}_\lambda \xi \otimes V|_{B \times \langle \sigma \rangle})).
\]
for any $i \geq 0$ and $\xi$ an $r$-th root of unity.

**Proof.** We have the following isomorphisms of $\hat{G} \rtimes \langle \sigma \rangle$-equivariant vector bundles
\[
\mathcal{L}_\ell(C_{\lambda,\xi} \otimes V|_{\mathcal{B} \rtimes \langle \sigma \rangle}) \simeq \hat{G} \rtimes \langle \sigma \rangle \rtimes \hat{P} \rtimes \langle \sigma \rangle I_{\ell}(C_{\lambda,\xi} \otimes V|_{\mathcal{B} \rtimes \langle \sigma \rangle}))^*.
\]
The last isomorphism follows from Lemma 4.11.

It is a $\hat{G} \rtimes \langle \sigma \rangle$-equivariant vector bundle on $Fl_G$. By Borel-Weil-Bott theorem for finite type algebraic group, we have
\[
R^i\pi_*\mathcal{L}_\ell(C_{\lambda,\xi} \otimes V|_{\mathcal{B} \rtimes \langle \sigma \rangle}) \simeq \begin{cases} 0 & i > 0 \\ \mathcal{L}_\ell(V_{\lambda,\xi} \otimes V) & i = 0 \end{cases}.
\]
By Leray’s spectral sequence, our lemma follows. □

Let $W_{\ell+\hat{h}}^\dagger$ denote the set of the minimal representatives of the left cosets of $W$ in $W_{\ell+\hat{h}}$, then for any $w_1 \in W$ and $w_2 \in W_{\ell+\hat{h}}^\dagger$, we have
\[
\ell(w_1w_2) = \ell(w_1) + \ell(w_2).
\]
Moreover for any $w \in W_{\ell+\hat{h}}$ and $\lambda \in P_{\ell}$
\[
(27) \quad w * \lambda \in P^+ \text{ if and only if } w \in W_{\ell+\hat{h}}^\dagger,
\]
see [Ko, Remark 1.3]. Since $P_{\ell}$ is the set of integral points in the fundamental alcove of the affine Weyl group $W_{\ell+\hat{h}}$, for any dominant weight $\lambda \in P^+$, there exists a unique $w \in W_{\ell+\hat{h}}^\dagger$ such that $w^{-1} * \lambda \in P_{\ell}$. By Lemma 2.7 for any $\sigma$-invariant dominant weight $\lambda \in P^+$, there exists a unique $w \in (W_{\ell+\hat{h}}^\dagger)^\sigma$ such that $w^{-1} * \lambda \in P_{\ell}^\sigma$.

Recall that we defined in Section 3.3 the representation $V_{\lambda,\xi}$ of $\mathfrak{g} \rtimes \langle \sigma \rangle$ as the representation $V_{\lambda}$ of $\mathfrak{g}$ together with an operator $\sigma$ such that $\sigma$ acts on the highest weight vectors by $\xi$, where $\lambda \in (P^+)_{\sigma}$ and $\xi$ is an $r$-th root of unity. Similarly the representation $H_{\lambda,\xi}$ is the representation $H_{\lambda}$ of $\hat{\mathfrak{g}} \rtimes \langle \sigma \rangle$ of level $\ell$ together with an operator $\sigma$ such that $\sigma$ acts on the highest weight vectors by $\xi$. We have the following theorem

**Theorem 4.13.** For any $w \in W_{\ell+\hat{h}}^\dagger$ such that $\sigma(w) = w$ and for any $\lambda \in P_{\ell}^\sigma$, we have the following isomorphism
\[
H^0(Gr_G, \mathcal{L}_\ell(V_{w*\lambda,\xi})) \simeq H^\ell(Gr_G, \mathcal{L}_\ell(V_{\lambda,\xi}))
\]
as representations of $\hat{G} \rtimes \langle \sigma \rangle$.

**Proof.** It follows from Theorem 4.10 and Lemma 4.12. □

**Corollary 4.14.** With the same assumption as in Theorem 4.13.
(1) There exists an isomorphism
\[ H^\ell(w)(\text{Gr}_G, \mathcal{L}_\ell(V_\lambda))^\vee \simeq \mathcal{H}_{\lambda, \varepsilon_w} \]
as representations of \( \hat{\mathfrak{g}} \times \langle \sigma \rangle \).

(2) There exists an isomorphism
\[ (H^\ell(w)(\text{Gr}_G, \mathcal{L}_\ell(V_{w^*\lambda})))^\vee_{\hat{\mathfrak{g}}^-} \simeq V_{v, \varepsilon_w} \]
as representations of \( \mathfrak{g} \times \langle \sigma \rangle \), where \( \hat{\mathfrak{g}}^- = t^{-1}\mathfrak{g}[t^{-1}] \).

Proof. It follows from Theorem 4.13, combining with Corollary 4.8, Lemma 4.12 and Lemma 3.4.

4.3. Affine analogues of BBG resolution and Kostant homology. We first recall the construction of BGG resolution in the setting of affine Lie algebra, we refer the reader to [Ku1, Section 9.1] for more details, in particular Theorem 9.1.3 therein. There exists a Koszul resolution of the trivial representation \( C \) of \( \hat{\mathfrak{g}} \),
\[ \cdots \to X_p \overset{\delta_p}{\to} \cdots \overset{\delta_1}{\to} X_0 \overset{\delta_0}{\to} C, \]
where
\[ X_p = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{p}})} \Lambda^p(\hat{\mathfrak{g}}/\hat{\mathfrak{p}}). \]

By the construction, this complex is \( \hat{\mathfrak{g}} \times \langle \sigma \rangle \)-equivariant. Given a \( \sigma \)-invariant dominant weight \( \lambda \in P_\ell \). Set \( X_{\lambda, \bullet} := U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{p}})} (\Lambda^p(\hat{\mathfrak{g}}/\hat{\mathfrak{p}}) \otimes \mathcal{H}_\lambda) \). The complex \( X_{\lambda, \bullet} \) is a resolution of \( \mathcal{H}_\lambda \).

Set
\[ F_{\lambda, p} := \bigoplus_{w \in W_{\ell_+}^{1, \ell(w)} = p} \hat{M}(V_{w^*\lambda}), \]
where \( \hat{M}(V_{w^*\lambda}) \) is the generalized Verma module introduced in Section 3.3. In fact \( F_{\lambda, \bullet} \) is a \( \sigma \)-stable subcomplex of \( X_{\lambda, \bullet} \), and moreover \( X_{\lambda, \bullet} \) is quasi-isomorphic to \( F_{\lambda, \bullet} \). Hence \( F_{\lambda, \bullet} \) is a resolution of \( \mathcal{H}_\lambda \).

Proposition 4.15. The complex \( F_{\lambda, \bullet} \) is a resolution of \( \mathcal{H}_\lambda \) as representations of \( \hat{\mathfrak{g}} \times \langle \sigma \rangle \), where \( \sigma \) maps \( \hat{M}(V_{w^*\lambda}) \) to \( \hat{M}(V_{\sigma(w^*\lambda)}) \). In particular when \( \sigma(w) = w \), \( \sigma \) acts on the highest weight vectors of \( \hat{M}(V_{w^*\lambda}) \) by \( \varepsilon_w \).

Proof. First of all, we note that \( \sigma \) maps \( \hat{M}(V_{w^*\lambda}) \) to \( \hat{M}(V_{\sigma(w^*\lambda)}) \) for any \( w \in W_{\ell_+}. \)
In particular if \( \sigma(w) = w \), \( \sigma \) acts on \( \hat{M}(w^*\lambda) \). We need to determine the action of \( \sigma \) on the highest weight vector \( m_{w^*\lambda} \) of \( \hat{M}(w^*\lambda) \). Assume it acts by the scalar \( \varepsilon_w' \).

The resolution \( F_{\lambda, \bullet} \) can be employed to compute the \( \hat{\mathfrak{g}}^- \)-homologies. We have the following isomorphism
\[ H_p(\hat{\mathfrak{g}}^-, \mathcal{H}_\lambda) \simeq \bigoplus_{w \in W_{\ell_+}^{1, \ell(w)} = p} V_{w^*\lambda} \]
for each \( p \), where \( \sigma \) acts on the highest weight vector \( v_{w^*\lambda} \) of \( V_{w^*\lambda} \) by \( \varepsilon_w' \) if \( \sigma(w) = w \).
Let \( n^- \) be the nilpotent radical of the negative Borel subalgebra \( b^- \) of \( \mathfrak{g} \). Put
\[
\hat{n}^- = \hat{g}^- \oplus n^-.
\]
Note that \( \hat{n}^- \) is exactly the nilpotent radical of the negative Iwahori subalgebra of \( \hat{g} \), and \( \hat{n}^- \) is \( \sigma \)-stable.

We have the following spectral sequence which is compatible with the action of \( \sigma \),
\[
H_i(n^-, H_j(\hat{g}^-, \mathcal{H}_\lambda)) \Rightarrow H_{i+j}(\hat{n}^-, \mathcal{H}_\lambda).
\]
From the left-hand side of (29) we have
\[
H_i(n^-, H_j(\hat{g}^-, \mathcal{H}_\lambda)) \simeq \bigoplus_{w \in W^{(\hat{g}^-)}} \bigoplus_{\ell(y)=i} \mathbb{C}_{y^*(\mu^s)} = \bigoplus_{w \in W^{(\hat{g}^-)}} \mathbb{C}_{\mu^*},
\]
where \( \sigma \) acts on \( \mathbb{C}_{\mu^*} \) again by \( \epsilon'_w \) for any \( w \in W^{(\hat{g}^-)} \) and \( \sigma(w) = w \). The scalar \( \epsilon'_w \) is exactly equal to \( \epsilon_w \) (due to S. Naito [N2, Proposition 3.2.1, 3.2.2]). This finishes the proof.

For any finite dimensional representation \( V \) of \( \mathfrak{g} \) and for any \( z \in \mathbb{C}^x \), we denote by \( V^z \) the representation of \( \hat{g}^- \) by evaluating \( t \) at \( z \). Let \( H_i(\hat{g}^-, \mathcal{H}_\lambda \otimes V^1_\mu) \) be the \( i \)-th \( \hat{g}^- \)-homology on \( \mathcal{H}_\lambda \otimes V^1_\mu \) where \( \hat{g}^- \) acts on \( \mathcal{H}_\lambda \otimes V^1_\mu \) diagonally.

**Theorem 4.16.** For any \( \lambda \in P^* \) and \( \mu \in (P^+)^\sigma \), The \( \hat{g}^- \)-homology groups \( H_*(\hat{g}^-, \mathcal{H}_\lambda \otimes V^1_\mu) \) can be computed by the cohomology groups of a complex of \( \mathfrak{g} \rtimes \langle \sigma \rangle \)-representations,
\[
\cdots \rightarrow D_p \xrightarrow{\delta^p} \cdots \xrightarrow{\delta^1} D_1 \xrightarrow{\delta^0} D_0 \xrightarrow{\delta^0} 0,
\]
where as representations of \( \mathfrak{g} \),
\[
D_p = \bigoplus_{w \in W^{(\hat{g}^-)} \ell(w)=p} V_{w^*\lambda} \otimes V^1_\mu,
\]
and \( \sigma \) maps \( V_{w^*\lambda} \otimes V^1_\mu \) to \( V_{w(w^*\lambda)} \otimes V^1_\mu \). In particular if \( \sigma(w) = w \), then \( \sigma \) acts on the highest weight vectors of \( V_{w^*\lambda} \) by \( \epsilon_w = (-1)^{\ell(w)-\ell(w)} \).

**Proof.** From the resolution \( F_{\lambda, \circ} \rightarrow \mathcal{H}_\lambda \), by tensoring with \( V^1_\mu \) we get a resolution of \( \mathcal{H}_\lambda \otimes V^1_\mu \) as representations of \( \mathfrak{g} \rtimes \langle \sigma \rangle \)
\[
\cdots \rightarrow F_{\lambda, p} \otimes V^1_\mu \xrightarrow{\delta^p} \cdots \xrightarrow{\delta^1} F_{\lambda, 1} \otimes V^1_\mu \xrightarrow{\delta^0} F_{\lambda, 0} \otimes V^1_\mu \xrightarrow{\delta^0} 0.
\]
As \( \mathfrak{g} \)-modules,
\[
(M(V_{w^*\lambda}) \otimes V^1_\mu)_{\hat{g}^-} \simeq (V_{w^*\lambda} \otimes U(\hat{g}^-)) \otimes_{U(\hat{g}^-)} V^1_\mu \simeq V_{w^*\lambda} \otimes V^1_\mu.
\]
Hence the complex
\[
\cdots \rightarrow (F_{\lambda, p} \otimes V^1_\mu)_{\hat{g}^-} \xrightarrow{\delta^p} \cdots \xrightarrow{\delta^1} (F_{\lambda, 1} \otimes V^1_\mu)_{\hat{g}^-} \xrightarrow{\delta^0} (F_{\lambda, 0} \otimes V^1_\mu)_{\hat{g}^-} \xrightarrow{\delta^0} 0
\]
is quasi-isomorphic to
\[
\cdots \rightarrow D_p \xrightarrow{\delta^p} \cdots \xrightarrow{\delta^1} D_1 \xrightarrow{\delta^0} D_0 \xrightarrow{\delta^0} 0.
\]
By Proposition 4.15, $\sigma$ maps $V_{w^*+\lambda} \otimes V_{\mu}$ to $V_{\sigma(w^*+\lambda)} \otimes V_{\mu}$. In particular if $\sigma(w) = w$, then $\sigma$ acts on the highest weight vectors of $V_{w^*+\lambda}$ by $\epsilon_w = (-1)^{\ell(w)-\ell_\sigma(w)}$. □

5. $\sigma$-twisted representation ring and fusion ring

In Section 5.1 we define the $\sigma$-twisted representation ring $R(\mathfrak{g}, \sigma)$ for the pair $(\mathfrak{g}, \sigma)$, and we show that $R(\mathfrak{g}, \sigma)$ is isomorphic to the representation ring $R(\mathfrak{g}_\sigma)$ of $\mathfrak{g}_\sigma$. In Section 5.2 we give a new definition of the $\sigma$-twisted fusion ring $R_\ell(\mathfrak{g}, \sigma)$ of $\mathfrak{g}$ by Borel-Weil-Bott theory of $\hat{\mathfrak{g}}$, and we show that two products on $R_\ell(\mathfrak{g}, \sigma)$ coincide. In Section 5.3 we construct and prove a ring homomorphism from $R(\mathfrak{g}, \sigma)$ to $R_\ell(\mathfrak{g}, \sigma)$. In Section 5.4 we determine all characters of $R_\ell(\mathfrak{g}, \sigma)$, and the proof of Theorem 1.2 is completed in Section 5.5. A corollary of Theorem 1.2 will be given in Section 5.6.

5.1. $\sigma$-twisted representation ring. Let $V$ be a finite dimensional representation of $\mathfrak{g}$. For any irreducible representation $V_\lambda$ of $\mathfrak{g}$ of highest weight $\lambda$, we denote by $\text{Hom}_\mathfrak{g}(V_\lambda, V)$ the multiplicity space of $V_\lambda$ in $V$. In particular we have the natural decomposition

$$V = \bigoplus_{\lambda \in P^+} \text{Hom}_\mathfrak{g}(V_\lambda, V) \otimes V_\lambda.$$

Let $R(\mathfrak{g}, \sigma)$ be the free abelian group with the symbols $[V_\lambda]_\sigma$ as a basis, where $\lambda \in (P^+)^\sigma$. Given any finite dimensional representation $V$ of $\mathfrak{g} \rtimes \langle \sigma \rangle$, $V$ can be decomposed as follows

$$V = \bigoplus_{\lambda \in (P^+)^\sigma} \text{Hom}_\mathfrak{g}(V_\lambda, V) \otimes V_\lambda \otimes \bigoplus_{\lambda \in (P^+)^\sigma} \text{Hom}_\mathfrak{g}(V_\lambda, V) \otimes V_\lambda,$$

as a representation of $\mathfrak{g}$. Put

$$[V]_\sigma := \sum_{\lambda \in (P^+)^\sigma} \text{tr}(\sigma|\text{Hom}_\mathfrak{g}(V_\lambda, V)) [V_\lambda]_\sigma \in R(\mathfrak{g}, \sigma).$$

Let $X$ be a finite dimensional representation of the cyclic group $\langle \sigma \rangle$, and for any representation $V$ of $\mathfrak{g} \rtimes \langle \sigma \rangle$, $X \otimes V$ is naturally a representation of $\mathfrak{g} \rtimes \langle \sigma \rangle$, which is defined as follows

$$(u, \sigma^i) \cdot x \otimes v = \sigma^i \cdot x \otimes (u, \sigma^i) \cdot v,$$

for any $u \in \mathfrak{g}, x \in X, v \in V$ and $i \in \mathbb{Z}$. Similarly $V \otimes X$ is also naturally a representation of $\mathfrak{g} \rtimes \langle \sigma \rangle$. The following lemma is clear.

Lemma 5.1. We have

$$[X \otimes V]_\sigma = \text{tr}(\sigma|X) [V]_\sigma \quad \text{and} \quad [V \otimes X]_\sigma = \text{tr}(\sigma|X) [V]_\sigma.$$

We define a multiplication $\otimes$ on $R(\mathfrak{g}, \sigma)$,

$$[V_\lambda]_\sigma \otimes [V_\mu]_\sigma := [V_\lambda \otimes V_\mu]_\sigma,$$

for any $\lambda, \mu \in (P^+)^\sigma$. By definition

$$[V_\lambda \otimes V_\mu]_\sigma = \sum_{\sigma(\nu) = \nu} \text{tr}(\sigma|\text{Hom}_\mathfrak{g}(V_\nu, V_\lambda \otimes V_\mu)) [V_\nu]_\sigma.$$
Proposition 5.2. \( R(\mathfrak{g}, \sigma) \) is a commutative ring with \([V_0]_\sigma\) as the unit.

Proof. The commutativity is clear. We first show that the product \( \otimes \) on \( R(\mathfrak{g}, \sigma) \) satisfies the associativity, i.e. for any \( \lambda, \mu, \nu \in (P^+)_\sigma \),
\[
([V_\lambda]_\sigma \otimes [V_\mu]_\sigma) \otimes [V_\nu]_\sigma = [V_\lambda]_\sigma \otimes ([V_\mu]_\sigma \otimes [V_\nu]_\sigma).
\]
It suffices to show that for any \( \lambda \in (P^+)_\sigma \) and any representation \( V \) of \( \mathfrak{g} \rtimes \langle \sigma \rangle \),
\[
[V_\lambda]_\sigma \otimes [V]_\sigma = [V_\lambda \otimes V]_\sigma, \quad \text{and} \quad [V]_\sigma \otimes [V_\lambda]_\sigma = [V \otimes V_\lambda]_\sigma.
\]
We have the following equalities
\[
[V_\lambda]_\sigma \otimes [V]_\sigma = \sum_{\sigma(\mu) = \mu} \text{tr}(\sigma|\text{Hom}_\mathfrak{g}(V_\mu, V))([V_\lambda]_\sigma \otimes [V_\mu]_\sigma)
= \sum_{\sigma(\mu) = \mu} \text{tr}(\sigma|\text{Hom}_\mathfrak{g}(V_\mu, V))[V_\lambda \otimes V_\mu]_\sigma
= \sum_{\sigma(\mu) = \mu} [V_\lambda \otimes V_\mu \otimes \text{Hom}_\mathfrak{g}(V_\mu, V)]_\sigma
= \bigoplus_{\mu} V_\lambda \otimes V_\mu \otimes \text{Hom}_\mathfrak{g}(V_\mu, V)_\sigma
= [V_\lambda \otimes V]_\sigma,
\]

where the third isomorphism follows from Lemma 5.1 and others follows from definition of the multiplication \( \otimes \). The equality \([V]_\sigma \otimes [V_\lambda]_\sigma = [V \otimes V_\lambda]_\sigma\) can be proved similarly.

In the end \([V_0]_\sigma\) is the unit since for any \( \lambda \in (P^+)_\sigma \),
\[
[V_\lambda]_\sigma \otimes [V_0]_\sigma = [V_\lambda \otimes V_0]_\sigma, \quad [V_0]_\sigma \otimes [V_\lambda]_\sigma = [V_\lambda \otimes V_0]_\sigma = [V_\lambda]_\sigma.
\]
\[\square\]

Theorem 5.3 (Jantzen). Let \( \lambda \in (P^+)_\sigma \) and \( \mu \in P^\sigma \). The following formula holds
\[
\text{tr}(\sigma|V_\lambda(\mu)) = \dim W_\lambda(\mu).
\]

For any finite dimensional representation \( V \) of \( \mathfrak{g} \rtimes \langle \sigma \rangle \), we define the \( \sigma \)-twisted character \( \text{ch}_\sigma(V) \) of \( V \) as follows
\[
\text{ch}_\sigma(V) := \sum_{\mu \in P^\sigma} \text{tr}(\sigma|V(\mu))e^\mu,
\]
where \( V(\mu) \) denotes the \( \mu \)-weight space of \( V \). The following lemma is obvious.

Lemma 5.4. For any two finite dimensional \( \mathfrak{g} \rtimes \langle \sigma \rangle \)-representations \( V, V' \), we have
\[
\text{ch}_\sigma(V \otimes V') = \text{ch}_\sigma(V) \text{ch}_\sigma(V').
\]

Lemma 5.5. Let \( \tilde{\lambda} \) be a tuple of \( \sigma \)-invariant dominant weights of \( \mathfrak{g} \) and let \( \nu \) be another \( \sigma \)-invariant dominant weight of \( \mathfrak{g} \). The following equality holds
\[
\text{tr}(\sigma|\text{Hom}_\mathfrak{g}(V_\nu, V_{\tilde{\lambda}})) = \text{tr}(\sigma|V_{\tilde{\lambda}} \otimes V_{\nu^*})^\sigma.
\]
Proof. Let $w_0$ be the longest element in the Weyl group $W$ of $g$. There exists a representative $\bar{w}_0$ of $w_0$ in $G$ such that $\sigma(\bar{w}_0) = \bar{w}_0$ (see [HS, Section 2.3]). Hence $\sigma(\bar{w}_0 \cdot v_\nu) = \bar{w}_0 \cdot v_\nu$, where $v_\nu \in V_\nu$ is the highest weight vector. The vector $\bar{w}_0 \cdot v_\nu$ is of the lowest weight $w_0(\nu)$. Let $V_\nu^*$ be the dual representation of $V_\nu$. Denote by $\sigma^*$ the action on $V_\nu^*$ induced by the action $\sigma$ on $V_\nu$. Then $\sigma^*$ keeps the highest weight vectors in $V_\nu^*$ invariant.

As representations of $g$, there is an isomorphism $V_\nu^* \cong V_{-w_0(\nu)} = V_{\nu^*}$ unique up to a scalar. It intertwines the action of $\sigma^*$ on $V_\nu^*$ and $\sigma$ on $V_\nu$. Note that there is a natural isomorphism $\text{Hom}_g(V_\nu, V_{\vec{\lambda}}) \cong (V_{\vec{\lambda}} \otimes V_{\nu^*})^g$, which is $\sigma$-equivariant. It finishes the proof. □

The following theorem was proved in [HS]. We give a simple proof here using Jantzen formula directly.

Theorem 5.6 ([HS]). Let $\lambda \in (P^+)^\sigma$ and $\mu \in P^\sigma$. The following formula holds
\[ \text{tr}(\sigma|V_{\vec{\lambda}}^g) = \dim W_{\vec{\lambda}}^g. \]

Proof. On one hand from the decomposition
\[ V_{\vec{\lambda}} \cong \bigoplus_{\nu \in P^+} \text{Hom}_g(V_\mu, V_{\vec{\lambda}}) \otimes V_\mu, \]
we have
\[ \text{ch}_\sigma(V_{\vec{\lambda}}) = \sum_{\mu \in (P^+)^\sigma} \text{tr}(\sigma|\text{Hom}_g(V_\mu, V_{\vec{\lambda}})) \text{ch}_\sigma(V_\mu). \]

On the other hand, we have the following equalities
\[ \text{ch}_\sigma(V_{\vec{\lambda}}) = \text{ch}_\sigma(V_{\lambda_1}) \cdots \text{ch}_\sigma(V_{\lambda_k}) = \text{ch}(W_{\lambda_1}) \cdots \text{ch}(W_{\lambda_k}) = \text{ch}(W_{\vec{\lambda}}) = \sum \dim \text{Hom}_g(W_\mu, W_{\vec{\lambda}}) \text{ch}(W_\mu), \]
where the first equality follows from Lemma 5.4 and the second equality follows from Theorem 5.3. In view of Lemma 5.5, the theorem follows. □

Let $R(g_\sigma)$ be the representation ring of $g_\sigma$.

Proposition 5.7. There is a natural ring isomorphism
\[ R(g, \sigma) \cong R(g_\sigma) \]
by sending $[V_\lambda]_\sigma \mapsto [W_\lambda]$ for any $\sigma$-invariant dominant weight $\lambda$.

Proof. For any $\lambda, \mu \in (P^+)^\sigma$, consider the following two decompositions
\[ [V_\lambda]_\sigma \otimes [V_\mu]_\sigma = \sum_{\sigma(\mu) = \mu} \text{tr}(\sigma|\text{Hom}_g(V_\nu, V_\lambda \otimes V_\mu))[V_\nu]_\sigma, \]
\[ [W_\lambda] \otimes [W_\mu] = \sum_{\sigma(\mu) = \mu} \dim \text{Hom}_g(W_\nu, W_\lambda \otimes W_\mu)[W_\nu]. \]
In view of Theorem 5.6 and Lemma 5.5, we have
\[ \text{tr}(\sigma | \text{Hom}_g(V_\nu, V_\lambda \otimes V_\mu)) = \dim \text{Hom}_{g^\sigma}(W_\nu, W_\lambda \otimes W_\mu). \]
Hence the proposition follows. \( \square \)

5.2. A new definition of \( \sigma \)-twisted fusion ring via Borel-Weil-Bott theory.

**Lemma 5.8.** The operation \([\cdot]_\sigma\) satisfies Euler-Poincaré property, i.e. for any complex of finite dimensional \( g \rtimes \langle \sigma \rangle \)-representations
\[ V^* := \cdots \overset{d_{i-1}}{\longrightarrow} V_i \overset{d_i}{\longrightarrow} V_{i+1} \overset{d_{i+1}}{\longrightarrow} \cdots \]
such that only finite many \( H^i(V^*) \) are nonzero, we have
\[ \sum_i (-1)^i[V^i]_\sigma = \sum_i (-1)^i[H^i(V^*)]_\sigma, \]
where \( H^i(V^*) \) is the \( i \)-th cohomology of this complex.

**Proof.** First of all, we have Euler-Poincaré property in the representation ring \( R(g \rtimes \langle \sigma \rangle) \) of \( g \rtimes \langle \sigma \rangle \), i.e.
\[ \sum_i (-1)^i[V^i] = \sum_i (-1)^i[H^i(V^*)]. \]
Secondly we can define a linear map \( R(g \rtimes \langle \sigma \rangle) \to R(g, \sigma) \) given by \([V] \mapsto [V]_\sigma\). It is well-defined and additive, since any finite dimensional representation of \( g \rtimes \langle \sigma \rangle \) is completely reducible. Hence the lemma follows. \( \square \)

Recall the \( \sigma \)-twisted fusion ring \( R_\ell(g, \sigma) \) defined in Section 3.5. We embed \( R_\ell(g, \sigma) \) into \( R(g, \sigma) \) as free abelian groups by simply sending \( \lambda \) to \([V_\lambda]_\sigma\) for any \( \lambda \in P^\sigma_\ell \). From now on we view \( R_\ell(g, \sigma) \) as a free abelian group with basis \( \{[V_\lambda]_\sigma | \lambda \in P^\sigma_\ell \} \). The fusion product \( \lambda \cdot \mu \) in \( R_\ell(g, \sigma) \) will be written as \([V_\lambda]_\sigma \cdot [V_\mu]_\sigma\).

Given any integral representation \( \mathcal{H} \) of \( g \), we denote by \( \mathcal{H}_{g^-} \) the coinvariant space of \( g^- \) on \( \mathcal{H} \). If \( \mathcal{H} \) is a representation of \( g \rtimes \langle \sigma \rangle \), then the space \( \mathcal{H}_{g^-} \) is naturally a representation of \( g \rtimes \langle \sigma \rangle \). For any \( \lambda, \mu \in P^\sigma_\ell \), we define
\[ [V_\lambda]_\sigma \otimes [V_\mu]_\sigma := [(H^*(\text{Gr}_G, L_\ell(V_\lambda \otimes V_\mu))^\vee)_{g^-}]_\sigma \in R_\ell(g, \sigma), \]
where we view \((H^*(\text{Gr}_G, L_\ell(V_\lambda \otimes V_\mu))^\vee)_{g^-}\) as a complex of \( g \rtimes \langle \sigma \rangle \)-representations with zero differentials.

Note that all representations of \( g \) appearing in \( H^*(\text{Gr}_G, L_\ell(V_\lambda \otimes V_\mu))^\vee \) are of level \( \ell \), and only finite many cohomology groups are nonzero. Hence the above definition makes sense.

Recall the representation \( \mathcal{H}_\nu \otimes V_\mu^\vee \) defined in Section 4.3. The following is a vanishing theorem of Lie algebra cohomology due to Teleman [Te].

**Theorem 5.9** (Teleman). For any \( \lambda, \mu, \nu \in P_\ell \) and for any \( i \geq 1 \), \( V_\lambda \) does not occur in \( H_i(\mathcal{H}_\nu, \mathcal{H}_\nu \otimes V_\mu^\vee) \) as a \( G \)-module.

We now show that the product defined in \( (30) \) is exactly the fusion product.
Theorem 5.10. Two products on $R_\ell(\mathfrak{g}, \sigma)$ coincide, i.e. for any $\lambda, \mu \in P_\ell^\sigma$ we have

$$[V_\lambda]_\sigma \otimes [V_\mu]_\sigma = [V_\lambda]_\sigma \cdot [V_\mu]_\sigma.$$ 

Proof. Consider the following decomposition

$$V_\lambda \otimes V_\mu = \bigoplus_{\nu} \text{Hom}_g(V_\nu, V_\lambda \otimes V_\mu) \otimes V_\nu.$$ 

In further we write

$$V_\lambda \otimes V_\mu = \bigoplus_{w \in W_{\ell+\hat{h}}, \nu \in P_\ell} \text{Hom}_g(V_{w^*\nu}, V_\lambda \otimes V_\mu) \otimes V_{w^*\nu}.$$ 

We have the following chain of equalities

$$[V_\lambda]_\sigma \otimes [V_\mu]_\sigma = \sum_i (-1)^i [([H^i(\text{Gr}_G, \mathcal{L}_\ell(V_\lambda \otimes V_\mu))^\vee])_{\tilde{g}^-}]_\sigma$$

$$= \sum_i (-1)^i \sum_{w \in (W_{\ell+\hat{h}})_\sigma} [\text{Hom}_g(V_{w^*\nu}, V_\lambda \otimes V_\mu) \otimes (H^i(\text{Gr}_G, \mathcal{L}_\ell(V_{w^*\nu})))_{\tilde{g}^-}]_\sigma$$

$$= \sum_{w \in (W_{\ell+\hat{h}})_\sigma} (-1)^{\ell(w)} [\text{Hom}_g(V_{w^*\nu}, V_\lambda \otimes V_\mu)]_\sigma$$

$$= \sum_{w \in (W_{\ell+\hat{h}})_\sigma} (-1)^{\ell(w)} \text{tr}(\sigma|\text{Hom}_g(V_{w^*\nu}, V_\lambda \otimes V_\mu)) [V_\nu]_\sigma,$$

where the third isomorphism follows from Corollary 4.14. By Lemma 3.7 and Proposition 3.8, we have the following $\sigma$-equivariant isomorphisms:

$$V_{\hat{g}^-,\ell,\lambda,\mu^*}(\mathbb{P}^1, 0, 1, \infty) \simeq V_{\hat{g}^-,\ell,\lambda,\mu^*}(\mathbb{P}^1, 0, 1, \infty)$$

$$\simeq (H_\nu \otimes V_{\lambda}^\vee \otimes V_{\mu}^\vee)_{\tilde{g}[t^{-1}]}$$

$$\simeq \text{Hom}_g(V_\lambda, H_0(\hat{g}^-, H_\nu \otimes V_{\mu}^\vee)),$$

The following formula follows immediately from Theorem 5.9

$$\text{tr}(\sigma|\text{Hom}_g(V_\lambda, H_0(\hat{g}^-, H_\nu \otimes V_{\mu}^\vee))) = \sum_i (-1)^i \text{tr}(\sigma|\text{Hom}_g(V_\lambda, H_i(\hat{g}^-, H_\nu \otimes V_{\mu}^\vee))).$$

By Lemma 5.8 and Theorem 4.16, we have

$$\sum_i (-1)^i \text{tr}(\sigma|\text{Hom}_g(V_\lambda, H_i(\hat{g}^-, H_\nu \otimes V_{\mu}^\vee)))$$

$$= \sum_i (-1)^i \sum_{w \in (W_{\ell+\hat{h}})_\sigma} \text{tr}(\sigma|\text{Hom}_g(V_\lambda, V_{w^*\nu} \otimes V_{\mu}^\vee))$$

$$= \sum_{w \in (W_{\ell+\hat{h}})_\sigma} (-1)^{\ell(w)} \text{tr}(\sigma|\text{Hom}_g(V_\lambda, V_{w^*\nu} \otimes V_{\mu}^\vee)).$$
It follows that

\[
[V_\lambda]_\sigma : [V_\mu]_\sigma = \sum_{\nu \in P^\sigma} \text{tr}(\sigma|V_{0,\ell,\lambda,\mu,\nu^*}(\mathbb{P}^1, 0, 1, \infty))[V_\nu]_\sigma
\]

\[
= \sum_{w \in (W^\ell_\ell)^\sigma} (-1)^{\ell^*(w)} \text{tr}(\sigma|\text{Hom}(V_\lambda, V_{w^*\nu} \otimes V_\mu^*))[V_\nu]_\sigma.
\]

In the end we need to check that

\[
\text{tr}(\sigma|\text{Hom}(V_{w^*\nu}, V_\lambda \otimes V_\mu)) = \text{tr}(\sigma|\text{Hom}(V_\lambda, V_{w^*\nu} \otimes V_\mu^*)).
\]

In view of Lemma 5.5, it reduces to show that the trace of \(\sigma\) on \(V_{w^*\nu,\lambda,\mu}^g\) and \(V_{\lambda^*,w^*\nu,\mu^*}^g\) are equal. It is a consequence of Lemma 3.6. \(\square\)

From the proof of Theorem 5.10, we get the following analogue of Kac-Watson formula.

**Theorem 5.11.** For any \(\lambda, \mu, \nu \in P^\sigma_\ell\), we have

\[
\text{tr}(\sigma|V_{0,\ell,\lambda,\mu,\nu}(\mathbb{P}^1, 0, 1, \infty)) = \sum_{w \in (W^\ell_\ell)^\sigma} (-1)^{\ell^*(w)} \text{tr}(\sigma|V_{\lambda,\mu,w^*\nu}^g).
\]

**Remark 5.12.** The proof of Theorem 5.10 and Theorem 5.11 does not rely on the fact that the trace on conformal blocks is a fusion rule. In fact Theorem 5.11 is used to show that the trace on conformal blocks gives a fusion rule, see Lemma 3.17.

5.3. Ring homomorphism from \(\sigma\)-twisted representation ring to \(\sigma\)-twisted fusion ring. We first construct a \(\mathbb{Z}\)-linear map

\[
\pi_\sigma : R(\mathfrak{g}, \sigma) \to R_\ell(\mathfrak{g}, \sigma).
\]

For any finite dimensional \(\mathfrak{g} \rtimes \langle \sigma \rangle\)-representation \(V\), we define

\[
\pi_\sigma([V]_\sigma) := [(H^*(\text{Gr}_G, \mathcal{L}(V))^{\nu})_{\hat{\beta}}]\sigma \in R_\ell(\mathfrak{g}, \sigma).
\]

**Lemma 5.13.** For any \(w \in (W^\ell_\ell)^\sigma\) and \(\lambda \in (P^+_\ell)^\sigma\), we have

\[
[(H^*(\text{Fl}_G, \mathcal{L}_\ell(w \ast \lambda))^{\nu})_{\hat{\beta}}]_\sigma = (-1)^{\ell^*(w)}[(H^*(\text{Fl}_G, \mathcal{L}_\ell(\lambda))^{\nu})_{\hat{\beta}}]_\sigma.
\]

**Proof.** We can write \(\lambda = y \ast \lambda_0\) where \(y \in (W^\ell_\ell)^\sigma\) and \(\lambda_0 \in (P_\ell)^\sigma\). Then \(w \ast \lambda = (wy) \ast \lambda_0\). In view of Theorem 4.11 and Theorem 4.10, we have

\[
[(H^*(\text{Fl}_G, \mathcal{L}_\ell(w \ast \lambda))^{\nu})_{\hat{\beta}}]_\sigma
\]

\[
= (-1)^{\ell^*(wy)}[(H^*(\text{Fl}_G, \mathcal{L}_\ell(\lambda_0))^{\nu})_{\hat{\beta}}]_\sigma
\]

\[
= (-1)^{\ell^*(w)}[(H^*(\text{Fl}_G, \mathcal{L}_\ell(\lambda))^{\nu})_{\hat{\beta}}]_\sigma.
\]

Hence the lemma follows. \(\square\)
Proposition 5.14. Given a finite dimensional representation $V$ of $\mathfrak{g} \rtimes \langle \sigma \rangle$. For any $\lambda \in P^\sigma_\ell$ and $w \in (W^\sigma_{\ell+\hat{h}})^\sigma$, we have

$$\left[ (H^* (\mathrm{Gr}_G, \mathcal{L}_\ell (V_{w \lambda} \otimes V))^\vee \right]_{\hat{g}^-} = (-1)^{\ell \sigma (w)} \left[ (H^* (\mathrm{Gr}_G, \mathcal{L}_\ell (V_{\lambda} \otimes V))^\vee \right]_{\hat{g}^-} \sigma.$$ in $R_{\ell} (\mathfrak{g}, \sigma)$.

Proof. In view of Lemma 4.12, it suffices to show that

$$\left[ (H^* (\mathrm{Fl}_G, \mathcal{L}_\ell (C_{w \lambda} \otimes V|_{B \times (\sigma)})^\vee \right]_{\hat{g}^-} = (-1)^{\ell \sigma (w)} \left[ (H^* (\mathrm{Fl}_G, \mathcal{L}_\ell (C_{\lambda} \otimes V|_{B \times (\sigma)})^\vee \right]_{\hat{g}^-} \sigma.$$

Note that there exists a filtration of $B \rtimes \langle \sigma \rangle$-representations

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_k = V$$
on $V$, such that for each $i$,

$$V_i / V_{i-1} \simeq \begin{cases} V(\mu) & \text{if } \sigma(\mu) = \mu \\ \bigoplus_{i=0}^{r-1} V(\sigma^i(\mu)) & \text{otherwise} \end{cases} ,$$

where $V(\mu)$ denotes the $\mu$-weight space of $V$.

By Lemma 5.1, it is easy to check that

$$\left[ (H^* (\mathrm{Fl}_G, \mathcal{L}_\ell (C_{\lambda} \otimes \bigoplus_{i=0}^{r-1} V(\sigma^i(\mu)))^\vee \right]_{\hat{g}^-} \sigma = \begin{cases} \text{tr} (\sigma | V(\mu)) \left[ (H^* (\mathrm{Fl}_G, \mathcal{L}_\ell (\lambda + \mu))^\vee \right]_{\hat{g}^-} \sigma & \text{if } \sigma(\mu) = \mu \\ 0 & \text{otherwise} \end{cases} .$$

Hence we get the following isomorphisms

$$\left[ (H^* (\mathrm{Fl}_G, \mathcal{L}_\ell (C_{\lambda} \otimes \bigoplus_{i=0}^{r-1} V(\sigma^i(\mu)))^\vee \right]_{\hat{g}^-} \sigma = \sum_i \left[ (H^* (\mathrm{Fl}_G, \mathcal{L}_\ell (C_{\lambda} \otimes V_i / V_{i-1}))^\vee \right]_{\hat{g}^-} \sigma$$

$$= \sum_{\mu \in P^\sigma} \text{tr} (\sigma | V(\mu)) \left[ (H^* (\mathrm{Fl}_G, \mathcal{L}_\ell (\lambda + \mu))^\vee \right]_{\hat{g}^-} \sigma .$$

Similarly we have

$$\left[ (H^* (\mathrm{Fl}_G, \mathcal{L}_\ell (C_{w \lambda} \otimes V|_{B \times (\sigma)})^\vee \right]_{\hat{g}^-} \sigma = \sum_{\mu \in P^\sigma} \text{tr} (\sigma | V(\mu)) \left[ (H^* (\mathrm{Fl}_G, \mathcal{L}_\ell (w \lambda + \mu))^\vee \right]_{\hat{g}^-} \sigma .$$

We can write $w$ as $w = \tau_\beta y^{-1}$, where $y \in W^\sigma$ and $\tau_\beta$ is the translation for $\beta \in (\ell + \hat{h})Q^\sigma$. It is easy to check that $w \lambda + \mu = w \lambda (\lambda + y \cdot \mu)$.

Since $V$ is a representation of $\mathfrak{g} \rtimes \langle \sigma \rangle$, we have

$$\text{tr} (\sigma | V(\mu)) = \text{tr} (\sigma | V(y \cdot \mu)),$$ for any $y \in W^\sigma$, where $V(\mu)$ and $V(y \cdot \mu)$ denote the weight spaces of $V$ as representation of $\mathfrak{g}$. 37
Therefore we have the following chain of equalities
\[
[(H^*(Gr_G, L_\ell(C_{w*w} \otimes V|_{B*(\sigma)}))^\vee)_\delta^-]_\sigma = \sum_{\mu \in P^\sigma} tr(\sigma[V(\mu)])[(H^*(Gr_G, L_\ell(w*(\lambda + \mu)))^\vee)_\delta^-]_\sigma = \sum_{\mu \in P^\sigma} tr(\sigma[V(\mu)])(-1)^{f_\sigma(w)}[(H^*(Gr_G, L_\ell(\lambda + \mu)))^\vee)_\delta^-]_\sigma = (-1)^{f_\sigma(w)}[(H^*(Gr_G, L_\ell(C_\lambda \otimes V|_{B*(\sigma)})^\vee)_\delta^-]_\sigma,
\]
where the second isomorphism follows from Lemma 5.13. It finishes the proof.

**Proposition 5.15.** If \( \lambda \in (P^+)^\sigma \) and \( \lambda + \rho \) is in an affine wall of \( W_{\ell+h} \), then
\[
[(H^*(Gr_G, L_\ell(V_\lambda \otimes V))^\vee)_\delta^-]_\sigma = 0.
\]

**Proof.** By the part (3) of Proposition 2.8 \( \lambda + \rho \) is in an affine wall of \( W_{\ell+h}^\sigma \). We can assume \( \lambda + \rho \) is in the following affine wall of \( W_{\ell+h}^\sigma \) in \( P^\sigma \otimes \mathbb{R} \)
\[
H_{\alpha_{\sigma}, a} = \{ \lambda + \rho \in P^\sigma \otimes \mathbb{R} \mid \langle \lambda + \rho, \alpha_{\sigma} \rangle = a \},
\]
where \( \alpha_{\sigma} \) is a root of \( g_\sigma \) and \( \bar{\alpha}_{\sigma} \) is the coroot of \( \alpha_{\sigma} \), moreover \( a \in \frac{1}{2}\mathbb{Z} \) and \( a\alpha_{\sigma} \in (\ell + \tilde{h})Q^\sigma \). Then
\[
(\tau_{a\alpha_{\sigma}} \cdot s_{\alpha_{\sigma}}) * (\lambda) = (s_{\alpha_{\sigma}} \cdot \tau_{a\alpha_{\sigma}}) * (\lambda) = \lambda,
\]
where \( s_{\alpha_{\sigma}} \) is the reflection with respect to \( \alpha_{\sigma} \) in \( W_{\ell+h}^\sigma \) and \( \tau_{a\alpha_{\sigma}} \) is the translation by \( a\alpha_{\sigma} \). Moreover
\[
(-1)^{f_\sigma(\tau_{a\alpha_{\sigma}} \cdot s_{\alpha_{\sigma}})} = (-1)^{f_\sigma(\tau_{a\alpha_{\sigma}})} (-1)^{f_\sigma(s_{\alpha_{\sigma}})} = -1,
\]
since by Lemma 2.9 \( \ell_\sigma(\tau_{a\alpha_{\sigma}}) \) is an even integer.

By Proposition 5.14 we have
\[
[(H^*(Gr_G, L_\ell(V_\lambda \otimes V))^\vee)_\delta^-]_\sigma = -[(H^*(Gr_G, L_\ell(V_\lambda \otimes V))^\vee)_\delta^-]_\sigma.
\]
Hence \( [(H^*(Gr_G, L_\ell(V_\lambda \otimes V))^\vee)_\delta^-]_\sigma = 0 \).

**Theorem 5.16.** The linear map \( \pi_\sigma : R(\mathfrak{g}, \sigma) \to R_\ell(\mathfrak{g}, \sigma) \) is a ring homomorphism.

**Proof.** By Theorem 5.10 we can use the product \( \otimes_\ell \) for \( R_\ell(\mathfrak{g}, \sigma) \). We need to check that for any \( \lambda, \mu \in (P^+)^\sigma \),
\[
\pi_\sigma([V_\lambda \otimes V_\mu]_\sigma) = \pi_\sigma([V_\lambda]_\sigma) \otimes_\ell \pi_\sigma([V_\mu]_\sigma).
\]
If \( \lambda + \rho \) or \( \mu + \rho \) is in an affine Wall, then by Proposition 5.15 both sides of (32) are zero. Hence (32) holds.
If \( \lambda + \rho \) and \( \mu + \rho \) are not in any affine Wall, let \( \lambda_0 \in \mathbb{P}_\ell^\sigma \) such that \( w_\lambda \ast \lambda_0 = \lambda \) and let \( \mu_0 \in \mathbb{P}_\ell^\sigma \) such that \( w_\mu \ast \mu_0 = \mu \) where \( w_\lambda, w_\mu \in (W_{\ell+h})^\sigma \), then
\[
\pi_\sigma([V_\lambda \otimes V_\mu]_\sigma) = [(H^*(\text{Gr}_G, \mathcal{L}_\ell(V_\lambda \otimes V_\mu)))^\gamma_\beta]_\sigma \\
= (-1)^{\ell_\sigma(w_\lambda)}[(H^*(\text{Gr}_G, \mathcal{L}_\ell(V_{\lambda_0} \otimes V_{\mu_0}))^\gamma_\beta]_\sigma \\
= (-1)^{\ell_\sigma(w_\lambda) + \ell_\sigma(w_\mu)}[(H^*(\text{Gr}_G, \mathcal{L}_\ell(V_{\lambda_0} \otimes V_{\mu_0}))^\gamma_\beta]_\sigma \\
= (-1)^{\ell_\sigma(w_\lambda) + \ell_\sigma(w_\mu)}[V_{\lambda_0}]_\sigma \otimes_{\ell} [V_{\mu_0}]_\sigma \\
= \pi_\sigma([V_\lambda]_\sigma) \otimes_{\ell} \pi_\sigma([V_\mu]_\sigma),
\]
where the second, the third and the fifth equalities follows from Proposition 5.14 and the forth equality is the definition (30). It finishes the proof of the theorem. \( \square \)

We can explicitly describe the map \( \pi_\sigma \).

**Corollary 5.17.** The map \( \pi_\sigma : R(\mathfrak{g}, \sigma) \to R_{\ell}(\mathfrak{g}, \sigma) \) can be described as follows, for any \( \lambda \in (P^+)^\sigma \) we have
\[
\pi_\sigma([V_\lambda]_\sigma) = \begin{cases} 
0 & \text{if } \lambda + \rho \text{ belongs to an affine Wall of } W_{\ell+h} \text{ in } P_R \\
(-1)^{\ell_\sigma(w)}[V_{w^{-1} \lambda}]_\sigma & \text{for some } w \in (W_{\ell+h})^\sigma.
\end{cases}
\]

**Proof.** The corollary is an immediate consequence of Corollary 4.14, Proposition 5.14 and Proposition 5.15. \( \square \)

### 5.4. Characters of the \( \sigma \)-twisted fusion ring

Section 5.4 and Section 5.5 basically follow the arguments in [36], Section 9. However our the arguments of Lemma 5.21 and Proposition 5.23 are substantially different, since in our setting there is no natural identification between \( P_\sigma/(\ell + h)\iota(Q^\sigma) \) and \( T_{\sigma,\ell} \).

Recall that \( P_\sigma \) (resp. \( Q_\sigma \)) is the weight lattice (resp. root lattice) of \( \mathfrak{g}, \sigma \), and the bijection map \( \iota : P_\sigma \cong P^\sigma \) defined in Section 2.4.

Let \( \mathbb{Z}[P_\sigma] \) be the group ring of \( P_\sigma \); we denote by \( (e^\lambda)_{\lambda \in P_\sigma} \) its basis so that the multiplication in \( \mathbb{Z}[P_\sigma] \) obeys the rule \( e^\lambda e^\mu = e^{\lambda+\mu} \). The action of \( W_\sigma \) and \( W_{\ell+h}^\sigma \cong W_\sigma \ltimes (\ell + h)\iota(Q^\sigma) \) on \( P_\sigma \) extends to \( \mathbb{Z}[P_\sigma] \). We denote by \( \mathbb{Z}[P_\sigma]_{W_\sigma} \) (resp. \( \mathbb{Z}[P_\sigma]_{W_{\ell+h}^\sigma} \)) the quotient of \( \mathbb{Z}[P_\sigma] \) by the sublattice spanned by \( e^\lambda - (-1)^{\ell_\sigma(w)}e^{w \ast \lambda} \) for any \( w \in W_\sigma \) (resp. \( w \in W_{\ell+h}^\sigma \)). Let \( p : \mathbb{Z}[P_\sigma]_{W_\sigma} \to \mathbb{Z}[P_\sigma]_{W_{\ell+h}^\sigma} \) be the projection map.

**Lemma 5.18.** The kernel \( \ker(p) \) is spanned by \( e^{\lambda+\alpha} - e^\lambda \) for \( \lambda \in P_\sigma \) and \( \alpha \in (\ell + h)\iota(Q^\sigma) \), and \( 2e^\lambda \) for \( \lambda \in P_\sigma \) and \( \lambda + \rho_\sigma \) is in an affine wall.

**Proof.** We can write any element \( w \in W_{\ell+h}^\sigma \) as \( w = y \tau_\beta \) where \( \beta \in (\ell + h)Q^\sigma \) and \( y \in W^\sigma \). Then the lemma follows from Lemma 2.9. \( \square \)

By Proposition 5.7 and Theorem 5.10 we get a ring homomorphism \( \hat{\pi}_\sigma : R(\mathfrak{g}, \sigma) \overset{\sim}{\to} R(\mathfrak{g}, \sigma) \overset{R_{\ell}(\mathfrak{g}, \sigma) \to \mathbb{Z}[P_\sigma](\mathfrak{g}, \sigma) \to \mathbb{Z}[P_\sigma]_{W_{\ell+h}^\sigma} \to \text{class of } e^\lambda) \). Similarly let \( \phi_{\sigma,\ell} \) be the map \( R_{\ell}(\mathfrak{g}, \sigma) \to \mathbb{Z}[P_\sigma]_{W_{\ell+h}^\sigma} \) sending \( [V_\lambda]_\sigma \) to the class of \( e^\lambda \).
$e^\lambda$ for any $\lambda \in P_\ell^\sigma$. By the same arguments as in [Be, Section 8], $\phi_\sigma$ and $\phi_{\sigma, t}$ are bijections. As a consequence of Corollary 5.17, the following diagram commutes

$$
\begin{align*}
R(\mathfrak{g}_\sigma) \xrightarrow{\tilde{\pi}_\sigma} R_\ell(\mathfrak{g}, \sigma) \\
\phi_\sigma \downarrow \downarrow \phi_{\sigma, t} \\
Z[P_\sigma]_{W_{\sigma}} \xrightarrow{p} Z[P_\sigma]_{W_{\ell + h}}
\end{align*}
$$

(33)

For any $\lambda \in P_\sigma$, put $J(e^{\lambda + \rho}) = \sum_{w \in W_\sigma} (-1)^{\ell_\sigma(w)} e^{w(\lambda + \rho_\sigma)}$, where $\rho_\sigma$ is the sum of all fundamental weights of $\mathfrak{g}_\sigma$. Recall that $\iota(\rho) = \rho_\sigma$ via the bijection $\iota : P_\sigma \simeq P_\sigma$. By Weyl character formula, for any $\lambda \in P_\sigma^+$ and $t \in T_\sigma$,\[ \text{tr}(t|W_\lambda) = \frac{J(e^{\lambda + \rho_\sigma})(t)}{J(e^{\rho_\sigma})(t)}. \]

Let $T_{\sigma, t}$ be the finite subgroup of $T_\sigma$ given by

$$
T_{\sigma, t} := \{ t \in T_\sigma \mid e^\alpha(t) = 1, \alpha \in (\ell + \hat{h})\iota(Q^\sigma) \}.
$$

**Proposition 5.19.** For any $t \in T_{\sigma, t}$, the character $\text{tr}(t|\cdot)$ factors through $\tilde{\pi}_\sigma : R(\mathfrak{g}_\sigma) \to R_\ell(\mathfrak{g}, \sigma)$.

**Proof.** Let $j_t : Z[P_\sigma]_{W_\sigma} \to \mathbb{C}$ be the additive map such that for any $\lambda \in P_\sigma$

$$
\tilde{j}_t(e^\lambda) = \frac{J(e^{\lambda + \rho_\sigma})(t)}{J(e^{\rho_\sigma})(t)}.
$$

By Weyl character formula, the following diagram commutes:

$$
\begin{align*}
R(\mathfrak{g}_\sigma) \xrightarrow{\phi_\sigma} Z[P_\sigma]_{W_\sigma} \\
\text{tr}(t|\cdot) \downarrow \\
\mathbb{C} \xrightarrow{j_t}
\end{align*}
$$

By the commutativity of the diagram (33) and Lemma 5.18 to show $\text{tr}(t|\cdot)$ factors through $\tilde{\pi}_\sigma$, we need to check that $j_t$ takes zero on $e^{\lambda + \sigma} - e^\lambda$ for any $\lambda \in P_\sigma$ and $\sigma \in (\ell + \hat{h})\iota(Q^\sigma)$, and $2e^\lambda$ for any $\lambda \in P_\sigma$ such that $\lambda + \rho$ is in an affine wall. Since $t$ satisfies that $e^\alpha(t) = 0$ for any $\alpha \in (\ell + \hat{h})\iota(Q^\sigma)$, it is clear that $j_t$ takes zero on the first type elements. For the second type elements $2e^\lambda$ where $\lambda + \rho_\sigma$ is in an affine Wall, by the formula (31) we have

$$
2e^\lambda = e^\lambda + e^{(\tau_{\alpha^\sigma} \cdot s_\alpha) \star \lambda} = e^\lambda + e^{s_\alpha \star \lambda} - e^{s_\alpha \cdot \lambda} + e^{(\tau_{\alpha^\sigma} \cdot s_\alpha) \star \lambda},
$$

for some root $\alpha$ of $\mathfrak{g}_\sigma$ which satisfies $\langle \lambda + \rho_\sigma, \tilde{\alpha} \rangle = a$ and $a\alpha \in (\ell + \hat{h})\iota(Q^\sigma)$. Note that

$$
e^{s_\alpha \star \lambda} - e^{(\tau_{\alpha^\sigma} \cdot s_\alpha) \star \lambda} = e^{s_\alpha \cdot \lambda} - e^{s_\alpha (\lambda + \alpha)}
$$

is a first type element. Hence $j_t(2e^\lambda) = 0$. It finishes the proof. \[\square\]
An element \( t \in T_\sigma \) is called regular if the stabilizer of \( W_\sigma \) at \( t \) is trivial. We denote by \( T^{\text{reg}}_{\sigma,\ell} \) the set of regular elements in \( T_{\sigma,\ell} \). Let \( \hat{\rho}_\sigma \) denotes the summation of all fundamental coweights of \( g_\sigma \). Consider the short exact sequence

\[
0 \to 2\pi i \tilde{Q}_\sigma \to t_\sigma \to T_\sigma \to 1,
\]

where \( \tilde{Q}_\sigma \) denote the dual root lattice of \( g_\sigma \) and \( t_\sigma \) is the Cartan subalgebra of \( g_\sigma \). Let \( \tilde{L}_\sigma \) be the dual lattice of \( \iota(Q^\sigma) \) in \( t_\sigma \). We have the following natural isomorphism

\[
T_{\sigma,\ell} \simeq (\frac{1}{\ell + \hat{h}} \tilde{L}_\sigma)/\tilde{Q}_\sigma \simeq \tilde{L}_\sigma/(\ell + \hat{h})\tilde{Q}_\sigma.
\]

(34)

For any \( \mu \in \tilde{L}_\sigma \), we denote by \( t_\mu \) the associated element of \( \mu + \hat{\rho}_\sigma \) in \( T_{\sigma,\ell} \).

We put

\[
\tilde{P}_{\sigma,\ell} := \{ \mu \in \tilde{P}_\sigma^+ \mid \langle \mu, \theta_\sigma \rangle_\sigma \leq \ell \},
\]

where \( \theta_\sigma \) denotes the highest root of \( g_\sigma \) and \( \tilde{P}_\sigma^+ \) denotes the set of dominant coweights of \( g_\sigma \).

**Lemma 5.20.** Assume that \( g \neq A_{2n} \). Then there exists a bijection \( \tilde{P}_{\sigma,\ell} \simeq T^{\text{reg}}_{\sigma,\ell}/W_\sigma \) with the map given by \( \mu \mapsto t_\mu \).

**Proof.** When \( g \neq A_{2n} \), by Lemma [2.3] \( \iota(Q^\sigma) = Q_\sigma \). Hence \( \tilde{L}_\sigma = \tilde{P}_\sigma \). We observe that \( \langle \rho_\sigma, \theta_\sigma \rangle = \hat{h} - 1 \) where \( \hat{h} \) is the dual Coxeter number of \( g_\sigma \). It can be read from [Hu2, Table 2,p.66]). It follows that

\[
\tilde{P}_{\sigma,\ell} = \{ \mu \in \tilde{P}_\sigma^+ \mid \langle \mu + \hat{\rho}_\sigma, \theta_\sigma \rangle_\sigma < \ell + \hat{h} \},
\]

i.e. \( \tilde{P}_{\sigma,\ell} \) consists of all points of \( P_\sigma \) sitting in the interior of the fundamental alcove with respect to the action of the affine Weyl group \( W_\sigma \rtimes (\ell + \hat{h})\tilde{Q}_\sigma \). From the isomorphism (34), we can see that any \( W_\sigma \)-orbit in \( T^{\text{reg}}_{\sigma,\ell} \) has a unique representative in \( \tilde{P}_{\sigma,\ell} \). Hence the lemma follows.

**Lemma 5.21.** The cardinality of \( T^{\text{reg}}_{\sigma,\ell}/W_\sigma \) is the equal to the cardinality of \( P^\sigma_\ell \).

**Proof.** When \( g \) is of type \( A_{2n} \), by Lemma [2.3] \( \iota(Q^\sigma) = \frac{1}{2}Q_\sigma,\ell \) where \( Q_\sigma,\ell \) is the lattice spanned by long roots of \( G_\sigma \). The proof of this lemma is exactly the same as the proof of [Be, Lemma 9.3]. We omit the detail.

Now we assume \( g \neq A_{2n} \). Put

\[
P_{\sigma,\ell} := \{ \lambda \in P_\sigma^+ \mid \langle \lambda, \tilde{\theta}_\sigma \rangle_\sigma \leq \ell \},
\]

where \( \tilde{\theta}_\sigma \) denotes the highest coroot of \( g_\sigma \). In view of (7) and Lemma [2.1] the map \( \iota \) induces a natural bijection \( \iota : P^\sigma_\ell \simeq P_{\sigma,\ell} \).

In view of Lemma [5.20] we are reduced to show that \( \tilde{P}_{\sigma,\ell} \) and \( P_{\sigma,\ell} \) have the same cardinality. If \( g_\sigma \) is not of type \( B_n \) or \( C_n \), it is obviously true, because in this case weight lattice and coweight lattice, root lattice and coroot lattice can be identified. Otherwise if \( g_\sigma \) is of type \( B_n \) or \( C_n \), by comparing the highest roots of \( B_n \) and \( C_n \) (see [Hu2, Table 2,p.66]), we conclude that \( \tilde{P}_{\sigma,\ell} \) and \( P_{\sigma,\ell} \) indeed have the same cardinality.

□
The following proposition completely describes all characters of \( R_{\ell}(g, \sigma) \).

**Proposition 5.22.** \{\( \text{tr}(t|\cdot) \mid t \in T_{\sigma, \ell}^{\text{reg}}/W_{\sigma} \)\} gives a full set of characters of \( R_{\ell}(g, \sigma) \).

**Proof.** It is an immediate consequence of Proposition 5.19 and Lemma 5.21. \( \square \)

### 5.5. Proof of Theorem 1.2

Denote by \( \check{T}_{\sigma, \ell} \) the finite abelian subgroup
\[
\check{T}_{\sigma, \ell} := P_{\sigma}/(\ell + \check{h})Q_{\sigma}.
\]
For any \( \lambda \in P_{\sigma} \), we denote by \( \check{t}_{\lambda} \) the associated element of \( \lambda + \rho_{\sigma} \) in \( \check{T}_{\sigma, \ell} \).

In the following lemma we determine \( \chi(\omega_{\sigma}) \) for each \( \chi = \text{tr}(t|\cdot) \), where \( \omega_{\sigma} \) is the Casimir element defined in (21).

**Proposition 5.23.** For any \( t \in T_{\sigma, \ell}^{\text{reg}} \), we have
\[
\sum_{\lambda \in P_{\ell}^{\sigma}} |\text{tr}(t|W_{\lambda})|^2 = \frac{|\check{T}_{\sigma, \ell}|}{\Delta_{\sigma}(t)}.
\]
where \( \Delta_{\sigma} \) is defined in (1) in the introduction.

**Proof.** When \( g = A_{2n} \), the proof of this lemma is exactly the same as the proof of [Be, Lemma 9.7]. We omit the detail.

Now we assume \( g \neq A_{2n} \). In this case,
\[
T_{\sigma, \ell} = P_{\sigma}/(\ell + \check{h})Q_{\sigma}, \quad \text{and} \quad T_{\sigma, \ell} \simeq \check{P}_{\sigma}/(\ell + \check{h})\check{Q}_{\sigma}.
\]
For any \( \lambda \in P_{\sigma} \) and \( \check{\mu} \in \check{L}_{\sigma} = \check{P}_{\sigma} \),
\[
J(e^{\lambda+\rho_{\sigma}})(t_{\check{\mu}}) = \sum_{w \in W_{\sigma}} (-1)^{\ell_{\sigma}(w)} e^{2\pi i \langle (\lambda+\rho_{\sigma}, w(\check{\mu}+\check{\rho}_{\sigma}))_{\sigma} / \check{h} \rangle} = J(e^{\check{\mu}+\check{\rho}_{\sigma}})(\check{t}_{\lambda}),
\]
where we put
\[
J(e^{\check{\mu}+\check{\rho}_{\sigma}}) = \sum_{w \in W_{\sigma}} (-1)^{\ell_{\sigma}(w)} e^{w(\check{\mu}+\check{\rho}_{\sigma})}.
\]
By Weyl character formula, we have
\[
\sum_{\lambda \in P_{\ell}^{\sigma}} |\text{tr}(t_{\check{\mu}}|W_{\lambda})|^2 = \frac{1}{\Delta_{\sigma}(t_{\check{\mu}})} \sum_{\lambda \in P_{\ell}^{\sigma}} |J(e^{\check{\mu}+\check{\rho}_{\sigma}})(\check{t}_{\lambda})|^2.
\]
We now introduce an inner product \( (\cdot, \cdot) \) on the space \( L^{2}(\check{T}_{\sigma, \ell}) \) of functions on the finite abelian group \( \check{T}_{\sigma, \ell} \),
\[
(\phi, \psi) := \frac{1}{|\check{T}_{\ell}|} \sum_{\check{t} \in \check{T}_{\sigma, \ell}} \overline{\phi(\check{t})} \psi(\check{t}),
\]
for any functions \( \phi, \psi \) on \( \check{T}_{\sigma, \ell} \).

The function \( J(e^{\check{\mu}+\check{\rho}_{\sigma}}) \) on \( \check{T}_{\sigma, \ell} \) is \( W_{\sigma} \)-antisymmetric, i.e.
\[
J(e^{w(\check{\mu}+\check{\rho}_{\sigma})}) = (-1)^{\ell_{\sigma}(w)} J(e^{\check{\mu}+\check{\rho}_{\sigma}}).
\]
It shows that if $t$ is not regular, then for any $\dot{t} \in \dot{T}_{\sigma,\ell}$, $J(e^{\dot{\mu} + \dot{\rho}_\sigma})(\dot{t}) = 0$. It follows that

$$
\sum_{\lambda \in P^\sigma_\ell} |J(e^{\dot{\mu} + \dot{\rho}_\sigma})(\dot{t}_\lambda)|^2 = \frac{|T_{\sigma,\ell}|}{|W_\sigma||J(e^{\dot{\mu} + \dot{\rho}_\sigma})|},
$$

where $|J(e^{\dot{\mu} + \dot{\rho}_\sigma})| = \sqrt{(J(e^{\dot{\mu} + \dot{\rho}_\sigma}), J(e^{\dot{\mu} + \dot{\rho}_\sigma}))}$.

If $t$ is regular, in view of Lemma 5.20 we can assume $t = t_{\dot{\mu}}$ where $\dot{\mu} \in \dot{P}_{\sigma,\ell}$. Now we show that the restriction of $e^{w(\dot{\mu} + \dot{\rho}_\sigma)}$ on $\dot{T}_{\sigma,\ell}$ are all distinct. For any two distinct elements $w, w' \in W_\sigma$, if $e^{w(\dot{\mu} + \dot{\rho}_\sigma)}$ and $e^{w'(\dot{\mu} + \dot{\rho}_\sigma)}$ are the same on $\dot{T}_{\sigma,\ell}$, it means that the pairing $\langle w(\dot{\mu} + \dot{\rho}_\sigma) - w'(\dot{\mu} + \dot{\rho}_\sigma), \lambda \rangle_\sigma \in (\ell + \dot{h})\mathbb{Z}$ for any $\lambda \in P_\sigma$. Equivalently $w(\dot{\mu} + \dot{\rho}_\sigma) - w'(\dot{\mu} + \dot{\rho}_\sigma) \in (\ell + \dot{h})Q_\sigma$. It is impossible as $\dot{\mu} + \dot{\rho}_\sigma$ is in the fundamental alcove of the affine Weyl group $W_\sigma \ltimes (\ell + \dot{h})Q_\sigma$.

By orthogonality relation for the characters of $\dot{T}_{\sigma,\ell}$, we have $|J(e^{\dot{\mu} + \dot{\rho}_\sigma})| = |W_\sigma|$. Hence we have

$$
\sum_{\lambda \in P^\sigma_\ell} |\text{tr}(t_{\dot{\mu}}|W_\lambda)|^2 = \frac{|T_{\sigma,\ell}|}{\Delta_\sigma(t_{\dot{\mu}})}.
$$

From the non-degeneracy of the pairing $\dot{T}_{\sigma,\ell} \times \dot{T}_{\sigma,\ell} \to \mathbb{C}^\times$ given by $(\dot{t}_\lambda, t_{\dot{\mu}}) \mapsto e^{2\pi i \langle \dot{t}_\lambda, t_{\dot{\mu}} \rangle / (\ell + \dot{h})}$, we have $|T_{\sigma,\ell}| = |\dot{T}_{\sigma,\ell}|$. Hence it concludes the proof of the Proposition.

Finally Theorem 1.2 follows from Proposition 3.20, Proposition 5.23 and Proposition 5.22.

5.6. A corollary of Theorem 1.2 Let $\sigma$ be a nontrivial diagram automorphism on $g = sl_{2n+1}$. Then the orbit Lie algebra $g_\sigma$ is isomorphic to $sp_{2n}$.

**Theorem 5.24.** With the same setting as in Theorem 1.2 If $\ell$ is an odd positive integer, then we have the following formula

$$
\text{tr}(\sigma|V_{sl_{2n+1},\ell,\ell}(C, \tilde{p})) = \dim V_{sp_{2n},\ell,\ell}(C, \tilde{p}).
$$

**Proof.** By assumption for any $\lambda$, $\langle i(\lambda), \tilde{\theta}_\sigma \rangle \leq \ell$. In view of (7) and Lemma 2.1 we have $\langle i(\lambda), \tilde{\theta}_{\sigma,\ell} \rangle \leq \ell/2$, where $\tilde{\theta}_{\sigma,\ell}$ is the coroot of the highest root $\theta_{\sigma}$ of $g_\sigma$. Since $\ell$ is odd and $\langle i(\lambda), \tilde{\theta}_{\sigma} \rangle$ is an integer, it follows that $\langle i(\lambda), \tilde{\theta}_{\sigma,\ell} \rangle \leq \ell/2$.

Note that $P_\sigma = 1/2Q_{\sigma,\ell}$ where $Q_{\sigma,\ell}$ is the lattice spanned by long roots of $g_\sigma$. Moreover $\dot{h} = 2n + 1$ and $\dot{h}_\sigma = n + 1$ where $\dot{h}_\sigma$ is the dual Coxeter number of $g_\sigma$. Combining the Verlinde formula (2) and Theorem 1.2, the corollary follows. 

**References**

[Be] A. Beauville, Conformal blocks, fusion rules and the Verlinde formula. Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993), 75-96, Israel Math. Conf. Proc., 9, Bar-Ilan Univ., Ramat Gan, 1996.

[Bel] P. Belkale, Quantum generalization of the Horn conjecture. J.Amer.Math.Soc.21 (2008), 365-408.
[Lus] G. Lusztig, Classification of unipotent representations of simple p-adic groups. II. Represent. Theory 6 (2002), 243-289.

[Loo] E. Looijenga, From WZW models to modular functors. Handbook of moduli. Vol. II, 427-466, Adv. Lect. Math. 25, Int. Press, Somerville, MA, 2013.

[Sw] D. Swinarski, conformalblocks: a macaulay2 package for computing the dimension of conformal blocks. 2010. Version 1.1, [http://www.math.uiuc.edu/Macaulay2/](http://www.math.uiuc.edu/Macaulay2/).

[MV] I. Mirković and K. Vilonen, Geometric Langlands duality and representations of algebraic groups over commutative rings. Ann. of Math. (2) 166 (2007), no. 1, 95-143.

[N1] S. Naito, Twinning character formula of Borel-Weil-Bott type. Journal of Mathematical Sciences-The University of Tokyo, 9 (2002), 637-658.

[N2] S. Naito, Twinning characters and Kostant’s homology formula. Tohoku Math.J. 55 (2003), 157-173.

[So] C. Sorger, La formule de Verlinde. Séminaire N.Bourbaki, 1994-1995, exp. n° 794, p.87-114.

[Te] C. Teleman, Lie algebra cohomology and the Fusion Rules. Commun.Math.Phys.173, 265-311 (1995).

[Ts] Y. Tsuchimoto, On the coordinate-free description of the conformal blocks. J. Math. Kyoto Univ. 33 (1993), no. 1, 29-49.

[TUY] A. Tsuchiya, K. Ueno and Y. Yamada, Conformal field theory on universal family of stable curves with gauge symmetries, Advanced Studies in Pure Mathematics 19, 459-566, 1989.

[Ue] K. Ueno, Conformal Field Theory with Gauge Symmetry. Fields Institute Monographs, AMS 2008.

[V] E. Verlinde, Fusion rules and modular transformations in 2d conformal field theory, Nucl.Phys.B.300[FS22], 360-376, 1988.

Department of Mathematics, University of North Carolina at Chapel Hill, Chapel Hill, NC 27599-3250, U.S.A.

E-mail address: jiuzu@email.unc.edu

45