Energy equality in the isentropic compressible Navier-Stokes equations allowing vacuum

Yulin Ye∗, Yanqing Wang† and Wei Wei‡

Abstract

It is well-known that a Leray-Hopf weak solution in $L^4(0, T; L^4(\Omega))$ for the incompressible Navier-Stokes system is persistence of energy due to Lions [19]. In this paper, it is shown that Lions’s condition for energy balance is also valid for the weak solutions of the isentropic compressible Navier-Stokes equations allowing vacuum under suitable integrability conditions on the density and its derivative. This allows us to establish various sufficient conditions implying energy equality for the compressible flow as well as the non-homogenous incompressible Navier-Stokes equations. This is an improvement of corresponding results obtained by Yu in [32, Arch. Ration. Mech. Anal., 225 (2017)], and our criterion via the gradient of the velocity partially answers a question posed by Liang in [18, Proc. Roy. Soc. Edinburgh Sect. A (2020)].

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1 Introduction

In this paper, we are concerned with the classical isentropic compressible Navier-Stokes equations in a periodic domain $\Omega = \mathbb{T}^d$ with $d = 2, 3$

\[
\begin{aligned}
\rho_t + \nabla \cdot (\rho v) &= 0, \\
(\rho v)_t + \text{div} (\rho v \otimes v) + \nabla P(\rho) - \mu \Delta v - (\mu + \lambda) \nabla \text{div} v &= 0,
\end{aligned}
\]

(1.1)

where $\rho \geq 0$ stands for the density of the flow, $v$ represent the fluid velocity field and $P(\rho) = \rho^\gamma$ with $\gamma > 1$ is the scalar pressure. The viscosity coefficients $\mu$ and $\lambda$ satisfy $\mu \geq 0$ and $2\mu + d\lambda > 0$. This model usually describes the motion of a compressible viscous barotropic fluid. We complement equations (1.1) with initial data

\[
\rho(0, x) = \rho_0(x), \quad (\rho v)(0, x) = (\rho_0 v_0)(x), \quad x \in \Omega,
\]

(1.2)
where we define \( v_0 = 0 \) on the sets \( \{ x \in \Omega : \rho_0 = 0 \} \).

Recently, originated from Yu’s work \([32]\), the energy conservation of weak solutions in the compressible fluid equations attracts a lot of attention (see e.g. \([1, 5, 6, 12, 18, 22–24, 29, 32]\)). Towards critical regularity for a weak solution keeping energy equality, the main result in \([32]\) is that if a weak solution \((\rho, v)\) in the sense of Definition 2.1 satisfies

\[
0 \leq \rho \leq c < \infty, \quad \nabla \sqrt{\rho} \in L^\infty(0,T;L^2(\mathbb{T}^d)), \quad v \in L^p(0,T;L^q(\mathbb{T}^d)), \quad p \geq 4 \quad \text{and} \quad q \geq 6, \quad \text{and} \quad v_0 \in L^{q_0}, \quad q_0 \geq 3,
\]

the following energy equality is valid

\[
\int_{\mathbb{T}^d} \left( \frac{1}{2} \rho |v|^2 + \frac{\rho^\gamma}{\gamma - 1} \right) dx + \int_0^T \int_{\mathbb{T}^d} \left[ \mu |\nabla v|^2 + (\mu + \lambda) |\text{div} v|^2 \right] dx dt \\
= \int_{\mathbb{T}^d} \left( \frac{1}{2} \rho_0 |v_0|^2 + \frac{\rho_0^\gamma}{\gamma - 1} \right) dx.
\]

The famous Onsager’s conjecture \([25]\) is what is the critical regularity for a weak solution conserving energy in the incompressible Euler equations. In this direction, a celebrated result due to Constantin-E-Titi \([9]\) states that a weak solution of the Euler system conserves the energy if \( v \) is in \( L^3(0,T;B^3_{3,\infty}) \) with \( \alpha > 1/3 \). The negative results of Onsager’s conjecture can be found in \([4, 14]\). For the 3D incompressible Navier-Stokes, a kind of well-known positive energy conservation class is Lions-Shinbrot type criterion in terms of integrability of the velocity \( v \). Particularly, in \([19]\), Lions showed that a weak solution \( v \) satisfies the energy equality in the incompressible Navier-Stokes in any spatial dimension if the velocity \( v \) meets

\[
v \in L^4(0,T;L^4(\mathbb{T}^d)).
\]

Lions’s energy conservation criterion was generalized by Shinbrot in \([20]\) to

\[
v \in L^p(0,T;L^q(\mathbb{T}^d)), \quad \text{with} \quad \frac{1}{p} + \frac{3}{q} = 1 \quad \text{and} \quad 3 < q < 4.
\]

For \( q < 4 \), Lions-Shinbrot type criterion in tri-dimensional space states

\[
v \in L^p(0,T;L^q(\mathbb{T}^3)), \quad \text{with} \quad \frac{1}{p} + \frac{3}{q} = 1 \quad \text{and} \quad 3 < q < 4,
\]

which was studied by Taniuchi \([28]\) and Beirao da Veiga-Yang \([2]\). Recent progress about Lions-Shinbrot type criterion can be found in \([3, 7, 8, 29–31, 33, 34]\).

Now, we turn attention back to compressible fluid. For equations \((1.1)\) posed an open bounded domain \( \Omega \) with Dirichlet boundary condition, Chen-Liang-Wang-Xu \([6]\) showed that \((1.3)\) yields the energy balance law \((1.4)\). For the general compressible models including \((1.1)\) in the absence of vacuum, it is shown that the following conditions

\[
0 < c_1 \leq \rho \leq c_2 < \infty, v \in L^\infty(0,T;L^2(\mathbb{T}^d)), \quad \nabla v \in L^2(0,T;L^2(\mathbb{T}^d)), \quad v \in L^4(0,T;L^4(\mathbb{T}^d)),
\]

ensure that a weak solution meets the energy equality by Nuye-Nguye-Tang in \([22]\). The energy conservation criterion \((1.8)\) is still valid if both \((1.8)_2\) and \((1.8)_3\) are replaced by \((1.6)\).
or (1.7), which was recently proved in [29]. We also refer the readers to [18, 29] for energy equality class via the gradient of the velocity. Roughly speaking, from the aforementioned works, the Lions-Shinbrot type energy conservation class for the isentropic compressible Navier-Stokes equations in the case away from vacuum is the same as the incompressible Naiver-Stokes equations. A natural question is whether this type energy conservation class is valid when the initial data of system (1.1) contain vacuums. The objective of this paper will be devoted to this issue.

We state our first result as follows.

**Theorem 1.1.** The energy equality (1.4) of weak solutions \((\rho, v)\) to the isentropic compressible Navier-Stokes equations (1.1)-(1.2) holds if one of the following four conditions is satisfied

1. \(0 \leq \rho \leq c^2 < \infty\), \(\nabla \sqrt{\rho} \in L^4(0, T; L^4(T^d))\), \(\nabla v \in L^2(0, T; L^2(T^d))\), \(v_0 \in L^2(T^d)\), \(v \in L^4(0, T; L^4(T^d))\); (1.9)

2. \(0 \leq \rho \leq c^2 < \infty\), \(\nabla \sqrt{\rho} \in L^4(0, T; L^4(T^d))\), \(\nabla v \in L^2(0, T; L^2(T^d))\), \(v_0 \in L^2(T^d)\), \(v \in L^p(0, T; L^q(T^d))\) with \(\frac{1}{p} + \frac{3}{q} = 1, 3 < q \leq 4\); (1.10)

3. \(0 \leq \rho \leq c^2 < \infty\), \(\nabla \sqrt{\rho} \in L^4(0, T; L^4(T^d))\), \(\nabla v \in L^2(0, T; L^2(T^d))\), \(v_0 \in L^3(T^d)\), \(v \in L^4(0, T; L^6(T^d))\); (1.11)

4. \(0 \leq \rho \leq c^2 < \infty\), \(\nabla \sqrt{\rho} \in L^4(0, T; L^4(T^d))\), and \(\nabla v \in L^2(0, T; L^2(T^d))\), \(v_0 \in L^2(T^d)\), \(\nabla v \in L^p(0, T; L^r(T^d))\) with \(\frac{1}{p} + \frac{3}{r} = 1 + \frac{3d}{d+3} < r \leq \frac{4d}{d+4}\). (1.12)

**Remark 1.1.** This theorem extends the well-known Lions’s energy equality criterion to the isentropic compressible equations (1.1) with vacuum. Sufficient conditions (1.7) implying energy equality are also generalized from the incompressible fluid to the compressible flow allowing vacuum.

**Remark 1.2.** Compared with Yu’ class (1.3), the condition (1.11) relaxes the integrability of the density.

**Remark 1.3.** In [18, Remark 1.7, Proc. Roy. Soc. Sect. A (2020)], Liang mentioned a question that one considers the energy equality sufficient condition via the gradient of the velocity when the vacuum state is allowed. From (1.12), we see that this result gives an affirmative answer to this question.

**Remark 1.4.** By means of a global mollification combined with an independent boundary cut-off used in [6], one may generalize Theorem 1.1 from periodic boundary to Dirichlet boundary.

We give some comments on the proof of this theorem. In contrast with the works of [22, 29] without vacuum, it seems that the test function and the Constantin-E-Titi type commutators on mollifying kernel break down in the presence of vacuum. The test function
Corollary 1.2. Let \( \rho \) and \( v \) be the three-dimensional isentropic compressible Navier-Stokes equations (1.1). This is the critical point to get (1.12).

Assume the density \( \rho \) satisfies one of the following conditions

(1) \( 0 \leq \rho \leq c_2 < \infty, \nabla \sqrt{\rho} \in L^4(0, T; L^4(\mathbb{T}^3)) \);

(2) \( \nabla \sqrt{\rho} \in L^\infty(0, T; L^4(\mathbb{T}^3)) \);

(3) \( 0 \leq \rho \leq c < \infty, \nabla \sqrt{\rho} \in L^\infty(0, T; L^2(\mathbb{T}^3)) \) and \( \nabla \sqrt{\rho} \in L^{p_1}(0, T; L^{q_1}(\mathbb{T}^3)) \), with \( \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{2}, q_1 \geq 4 \);

and the velocity meet one of the following

(i) \( \nabla v \in L^2(0, T; L^2(\mathbb{T}^3)) \) \( v \in L^p(0, T; L^q(\mathbb{T}^3)) \) with \( \frac{1}{p} + \frac{3}{q} = 1, 3 < q \leq 4 \), \( v_0 \in L^2(\mathbb{T}^3) \);

(ii) \( \nabla v \in L^2(0, T; L^2(\mathbb{T}^3)) \) \( \nabla v \in L^p(0, T; L^r(\mathbb{T}^3)) \) with \( \frac{1}{p} + \frac{3}{r} = 2, \frac{3}{2} < r \leq \frac{12}{7} \), \( v_0 \in L^2(\mathbb{T}^3) \).

As mentioned above, the key point for the proof of Theorem 1.1 is the following energy equality condition in terms of the velocity and its gradient. Indeed, Theorem 1.1 is an immediate consequence of the following theorem.

Theorem 1.3. For any \( p, q \geq 4 \) and \( dp < 2q + 3d \) with \( d \geq 2 \), the energy equality (1.4) of weak solutions \((\rho, v)\) to the isentropic compressible Navier-Stokes equation (1.1)-(1.2) is valid provided

\[
0 \leq \rho < c < \infty, \nabla \sqrt{\rho} \in L^{\frac{p}{p-3}}(0, T; L^\infty(\mathbb{T}^d)),
\]

\[
v \in L^p(0, T; L^q(\mathbb{T}^d)), \nabla v \in L^{\frac{q}{q-3}}(0, T; L^\infty(\mathbb{T}^d)), \quad v_0 \in L^\frac{q}{2}(\mathbb{T}^d).
\]
Finally, following the path of the above theorem, one can consider the non-homogenous incompressible Navier-Stokes equations below

\[
\begin{align*}
\rho_t + \text{div} (\rho v) &= 0, \\
(\rho v)_t + \text{div} (\rho v \otimes v) - \mu \Delta v + \nabla P &= 0, \\
\text{div} v &= 0,
\end{align*}
\]

and derive a similar result as Theorem 1.3. Since the velocity is divergence-free, the pressure part in the proof becomes trivial. Hence, we just formulate the corresponding result here.

**Theorem 1.4.** Let \((\rho, v)\) be a weak solution to the Navier-Stokes equation (1.14). Assume that the pair \((\rho, v)\) obeys

\[
0 \leq \rho \leq c_2 < \infty, \quad \nabla \sqrt{\rho} \in L^4(0, T; L^4(\mathbb{T}^3)), \quad \nabla v \in L^2(0, T; L^2(\mathbb{T}^3)),
\]

\[
v \in L^p(0, T; L^q(\mathbb{T}^3)) \quad \text{with} \quad \frac{1}{p} + \frac{3}{q} = 1, \quad 3 < q \leq 4, \quad \text{and} \quad v_0 \in L^2(\mathbb{T}^3);
\]

then, the energy equality below holds

\[
\int_{\mathbb{T}^3} \frac{1}{2} \rho |v|^2 \, dx + \int_0^T \int_{\mathbb{T}^3} \mu |\nabla v|^2 \, dxdt = \int_{\mathbb{T}^3} \frac{1}{2} \rho_0 |v_0|^2 \, dx.
\]

**Remark 1.5.** When the density \(\rho\) becomes a constant, Theorem 1.4 reduces to the Lions’s classical result (1.5).

**Remark 1.6.** Previous results on energy balance proved in [17, 22, 29] are all without vacuum. It seems that this theorem will be the first criterion on energy equality of weak solutions to non-homogenous incompressible Navier-Stokes equations in the presence of vacuum.

The remainder of this paper is organized as follows. Section 2 is devoted to the auxiliary lemmas involving mollifier and the key inequality. In Section 3, we first present the proof of Theorem 1.3. Then, based on Theorem 1.3 we complete the proof of Theorem 1.4 and Corollary 1.2.

## 2 Notations and some auxiliary lemmas

First, we introduce some notations used in this paper. For \(p \in [1, \infty]\), the notation \(L^p(0, T; X)\) stands for the set of measurable functions on the interval \((0, T)\) with values in \(X\) and \(\|f(t, \cdot)\|_X\) belonging to \(L^p(0, T)\). The classical Sobolev space \(W^{k,p}(\Omega)\) is equipped with the norm \(\|f\|_{W^{k,p}(\Omega)} = \sum_{|\alpha|=0}^k \|D^\alpha f\|_{L^p(\Omega)}\). For simplicity, we denote by

\[
\int_0^T \int_\Omega f(t, x) \, dxdt = \int_0^T \int f \quad \text{and} \quad \|f\|_{L^p(0,T;X)} = \|f\|_{L^p(X)}.
\]

Let \(\eta\) be non-negative smooth function supported in the space-time ball of radius 1 and its integral equals to 1. We define the rescaled space-time mollifier \(\eta_\varepsilon(t, x) = \frac{1}{\varepsilon^{d+1}} \eta(\frac{t}{\varepsilon}, \frac{x}{\varepsilon})\)

\[
f_\varepsilon(t, x) = \int_0^T \int_\Omega f(s, y) \eta_\varepsilon(t - s, x - y) \, dyds.
\]
Definition 2.1. A pair \((\rho, v)\) is called a weak solution to (1.1) with initial data (1.2) if \((\rho, v)\) satisfies

(i) equation (1.1) holds in \(D'(0, T; \Omega)\) and

\[
P(\rho), \rho |v|^2 \in L^\infty(0, T; L^1(\Omega)), \quad \nabla v \in L^2(0, T; L^2(\Omega)),
\]

(2.1)

(ii) the density \(\rho\) is a renormalized solution of (1.1) in the sense of [10].

(iii) the energy inequality holds

\[
E(t) + \int_0^T \int_\Omega \left( \mu |\nabla v|^2 + (\mu + \lambda) |\text{div } v|^2 \right) dx dt \leq E(0),
\]

(2.2)

where \(E(t) = \int_\Omega \left( \frac{1}{2} \rho |v|^2 + \frac{\rho^2}{2} \right) dx\).

To the knowledge of the authors, the following statement about Lions’s type commutators involving space-time mollifier with \(p = q, p_i = q_i, (i = 1, 2)\) can be found in [16], which plays an important role in recent works [3, 6, 32] and whose complete rigorous proof was not presented anywhere (see [20] for original version). To make our work more self-contained and more readable, we shall outline its proof.

Lemma 2.1. Let \(1 \leq p, q, p_1, q_1, p_2, q_2 \leq \infty\), with \(\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}\) and \(\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}\). Let \(\partial\) be a partial derivative in space or time, in addition, let \(\partial_t f, \nabla f \in L^{p_1}(0, T; \overline{L}^{q_1}(\Omega))\), \(g \in L^{p_2}(0, T; \overline{L}^{q_2}(\Omega))\). Then, there holds

\[
\|\partial (fg)^\varepsilon - \partial (fg)^\varepsilon\|_{L^p(0, T; L^q(\Omega))} \leq C \left( \|\partial_t f\|_{L^{p_1}(0, T; L^{q_1}(\Omega))} + \|\nabla f\|_{L^{p_1}(0, T; L^{q_1}(\Omega))} \right) \|g\|_{L^{p_2}(0, T; L^{q_2}(\Omega))},
\]

for some constant \(C > 0\) independent of \(\varepsilon, f\) and \(g\). Moreover,

\[
\partial (fg)^\varepsilon - \partial (fg)^\varepsilon \to 0 \quad \text{in } L^p(0, T; L^q(\Omega)),
\]

as \(\varepsilon \to 0\) if \(p_2, q_2 < \infty\).

Proof. First, we let

\[
\partial (fg)^\varepsilon - \partial (fg)^\varepsilon = \partial (fg)^\varepsilon - f \partial g^\varepsilon - \partial f g^\varepsilon = G^\varepsilon - \partial f g^\varepsilon,
\]

(2.3)

where \(G^\varepsilon = \partial (fg)^\varepsilon - f \partial g^\varepsilon\). Due to the properties of the Mollifier and the integration by parts, we have

\[
|G^\varepsilon(t, x)| = \left| \int \int (f(s, y) - f(t, x)) g(s, y) \frac{1}{\varepsilon^{d+1}} \frac{1}{\varepsilon} \partial \eta \left( \frac{t - s}{\varepsilon}, \frac{x - y}{\varepsilon} \right) dy ds \right|
\]

\[
= \left| \int \int [f(s, y) - f(t, x) + f(t, y) - f(t, x)] g(s, y) \frac{1}{\varepsilon^{d+1}} \frac{1}{\varepsilon} \partial \eta \left( \frac{t - s}{\varepsilon}, \frac{x - y}{\varepsilon} \right) dy ds \right|
\]

\[
\leq C \int_{t-\varepsilon}^{t+\varepsilon} \int_{B(x, \varepsilon)} \frac{1}{\varepsilon^{d+1}} \left| \partial_t f(t + \tau(s - t), y) \right| d\tau |g(s, y)| dy ds
\]

\[
+ \int_{t-\varepsilon}^{t+\varepsilon} \int_{B(x, \varepsilon)} \frac{1}{\varepsilon^{d+1}} \left| \nabla f(t, x + \tau(y - x)) \right| d\tau |g(s, y)| dy ds
\]

\[
= I_1 + I_2,
\]

(2.4)
here, \( B(x, \varepsilon) = \{ y \in \Omega; |y - x| < \varepsilon \} \) is an open ball centered at \( x \). To control \( I_1 \) and \( I_2 \), by the Hölder’s inequality, we know that

\[
I_1 \leq C \int_{t-\varepsilon}^{t+\varepsilon} \frac{1}{\varepsilon} \left( \int_{B(x, \varepsilon)} \frac{1}{\varepsilon^d} \int_0^1 |\partial_t f(t + \tau(s-t), y)|^{s_1} d\tau dy \right)^{\frac{1}{s_1}} \left( \int_{B(x, \varepsilon)} \frac{1}{\varepsilon^d} |g(s, y)|^{s_2} dy \right)^{\frac{1}{s_2}} ds
\]

\[
\leq C \int_{t-\varepsilon}^{t+\varepsilon} \frac{1}{\varepsilon} \left( \int_{B(0, \varepsilon)} \frac{1}{\varepsilon^d} \int_0^1 |\partial_t f(t + \tau(s-t), x-z)|^{s_1} d\tau dz \right)^{\frac{1}{s_1}} \left( \int_{B(0, \varepsilon)} \frac{1}{\varepsilon^d} |g(s, x-z)|^{s_2} dz \right)^{\frac{1}{s_2}} ds,
\]

where \( \frac{1}{s_1} + \frac{1}{s_2} = 1, 1 \leq s_1 \leq q_1 \) and \( 1 \leq s_2 \leq q_2 \). Moreover, notice that

\[
\int_{B(0, \varepsilon)} \frac{1}{\varepsilon^d} \int_0^1 |\partial_t f(t + \tau(s-t), x-z)|^{s_1} d\tau dz
\]

\[
= \int_{\mathbb{R}^d} \int_0^1 |\partial_t f(t + \tau(s-t), x-z)|^{s_1} d\tau \frac{1}{\varepsilon^d} B(0, \varepsilon)(z) dz
\]

\[
= \left( \int_0^1 |\partial_t f(t + \tau(s-t))|^{s_1} d\tau J_\varepsilon(x) \right)(x),
\]

and

\[
\int_{B(0, \varepsilon)} \frac{1}{\varepsilon^d} |g(s, x-z)|^{s_2} dz
\]

\[
= \int_{\mathbb{R}^d} |g(s, x-z)|^{s_2} \frac{1}{\varepsilon^d} B(0, \varepsilon)(z) dz
\]

\[
= (|g(s)|^{s_2} * J_\varepsilon)(x),
\]

where \( J_\varepsilon(z) = \frac{1}{\varepsilon^d} B(0, \varepsilon)(z) \geq 0 \) and \( \| J_\varepsilon \|_{L^1(\mathbb{R}^d)} = \text{measure of } B(0,1) \). Then substituting (2.6) and (2.7) into (2.5), we infer

\[
I_1 \leq C \int_{t-\varepsilon}^{t+\varepsilon} \frac{1}{\varepsilon} \left( \int_0^1 |\partial_t f(t + \tau(s-t))|^{s_1} d\tau J_\varepsilon(x) \right)^{\frac{1}{s_1}} \left( |g(s)|^{s_2} * J_\varepsilon(x) \right)^{\frac{1}{s_2}} ds.
\]

Then using the Minkowski inequality and Hölder’s inequality, one can derive that

\[
\| I_1 \|_{L^q(\Omega)}
\]

\[
\leq C \left( \int_{t-\varepsilon}^{t+\varepsilon} \frac{1}{\varepsilon} \left( \int_0^1 |\partial_t f(t + \tau(s-t))|^{s_1} d\tau J_\varepsilon(x) \right)^{\frac{1}{s_1}} \left( |g(s)|^{s_2} * J_\varepsilon(x) \right)^{\frac{1}{s_2}} ds \right)^{\frac{1}{q}}
\]

\[
\leq C \left( \int_{t-\varepsilon}^{t+\varepsilon} \frac{1}{\varepsilon} \left( \int_0^1 |\partial_t f(t + \tau(s-t))|^{s_1} d\tau J_\varepsilon(x) \right)^{\frac{1}{s_1}} \left( |g(s)|^{s_2} * J_\varepsilon(x) \right)^{\frac{1}{s_2}} ds \right)^{\frac{1}{q}}
\]

\[
\leq C \left( \int_{t-\varepsilon}^{t+\varepsilon} \frac{1}{\varepsilon} \left( \int_0^1 |\partial_t f(t + \tau(s-t))|^{s_1} d\tau J_\varepsilon(x) \right)^{\frac{1}{s_1}} \left( |g(s)|^{s_2} * J_\varepsilon(x) \right)^{\frac{1}{s_2}} ds \right)^{\frac{1}{q}}
\]

\[
\leq C \left( \int_{t-\varepsilon}^{t+\varepsilon} \frac{1}{\varepsilon} \left( \int_0^1 |\partial_t f(t + \tau(s-t))|^{s_1} d\tau J_\varepsilon(x) \right)^{\frac{1}{s_1}} \left( |g(s)|^{s_2} * J_\varepsilon(x) \right)^{\frac{1}{s_2}} ds \right)^{\frac{1}{q}}
\]

\[
\leq C \left( \int_{t-\varepsilon}^{t+\varepsilon} \frac{1}{\varepsilon} \left( \int_0^1 |\partial_t f(t + \tau(s-t))|^{s_1} d\tau J_\varepsilon(x) \right)^{\frac{1}{s_1}} \left( |g(s)|^{s_2} * J_\varepsilon(x) \right)^{\frac{1}{s_2}} ds \right)^{\frac{1}{q}}
\]

\[
\leq C \left( \int_{t-\varepsilon}^{t+\varepsilon} \frac{1}{\varepsilon} \left( \int_0^1 |\partial_t f(t + \tau(s-t))|^{s_1} d\tau J_\varepsilon(x) \right)^{\frac{1}{s_1}} \left( |g(s)|^{s_2} * J_\varepsilon(x) \right)^{\frac{1}{s_2}} ds \right)^{\frac{1}{q}}
\]

\[
\leq C \left( \int_{t-\varepsilon}^{t+\varepsilon} \frac{1}{\varepsilon} \left( \int_0^1 |\partial_t f(t + \tau(s-t))|^{s_1} d\tau J_\varepsilon(x) \right)^{\frac{1}{s_1}} \left( |g(s)|^{s_2} * J_\varepsilon(x) \right)^{\frac{1}{s_2}} ds \right)^{\frac{1}{q}}
\]
\[ \leq C \left( \| \partial_t f(t - \varsigma) \|_{L^2(\Omega)}^{s_1} \star J_{1\varepsilon}(\varsigma) \right)^{\frac{1}{s_1}} \left( \| g(t - \varsigma) \|_{L^2(\Omega)}^{s_2} \star J_{2\varepsilon}(\varsigma) \right)^{\frac{1}{s_2}}, \]

(2.10)

where \( J_{1\varepsilon} = \int_0^1 \frac{1_{(1-x^2, x)}}{\varepsilon^2}(\varsigma) d\tau \geq 0, \ J_{2\varepsilon} = \frac{1_{(1-x^2, x)}}{\varepsilon^2}(\varsigma) \geq 0 \) and \( \| J_{1\varepsilon} \|_{L^1(\mathbb{R})} = \| J_{2\varepsilon} \|_{L^1(\mathbb{R})} = 2. \)

Using the Hölder’s inequality again, \( 1 \leq s_1 \leq p_1, \) and \( 1 \leq s_2 \leq p_2, \) this immediately gives

\[ \| I_1 \|_{L^p(\mathbb{R}^d)} \leq C \left( \| \partial_t f(t - \varsigma) \|_{L^2(\Omega)}^{s_1} \star J_{1\varepsilon}(\varsigma) \right)^{\frac{1}{s_1}} \left( \| g(t - \varsigma) \|_{L^2(\Omega)}^{s_2} \star J_{2\varepsilon}(\varsigma) \right)^{\frac{1}{s_2}} \]

(2.11)

As the same manner of the derivation above, we see that

\[ I_2 \leq C \int_{t-\varepsilon}^{t+\varepsilon} \frac{1}{\varepsilon} \left( \int_{B(x, \varepsilon)} \int_0^1 |\nabla f(t, x + \tau(y - x))|^{s_1} d\tau dy \right)^{\frac{1}{s_1}} \left( \int_{B(x, \varepsilon)} \int_0^1 |g(s, y)|^{s_2} d\tau dy \right)^{\frac{1}{s_2}} \]

(2.12)

where \( J_{\varepsilon}(z) = \frac{1_{B(0, r\varepsilon)}}{(\tau \varepsilon)^d} \geq 0 \) and \( \| J_{\varepsilon}(z) \|_{L^1(\mathbb{R}^d)} = \) measure of ball \( B(0, 1). \)

Hence, we further get

\[ \| I_2 \|_{L^q(\Omega)} \leq C \left( \int_{t-\varepsilon}^{t+\varepsilon} \frac{1}{\varepsilon} \left( \| \nabla f(t, x - z) \|_{L^2(\Omega)}^{s_1} \star J_{\varepsilon}(z) \right)^{\frac{1}{s_1}} \left( \| g(s, x - z) \|_{L^2(\Omega)}^{s_2} \star J_{\varepsilon}(z) \right)^{\frac{1}{s_2}} \right) \]

(2.13)

which implies that

\[ \| I_2 \|_{L^p(\mathbb{R}^d)} \leq C \left( \| \nabla f(t) \|_{L^2(\Omega)} \left( \| g(t - \varsigma) \|_{L^2(\Omega)} \star J_{2\varepsilon}(\varsigma) \right) \right) \]

(2.14)

Then together with (2.3), (2.11) and (2.14), we obtain that

\[ \| \partial (fg)^{\varepsilon} - \partial (fg)^{\varepsilon} \|_{L^p(\mathbb{R}^d)} \leq C \left( \| G^\varepsilon \|_{L^p(\mathbb{R}^d)} + \| \partial f g^{\varepsilon} \|_{L^p(\mathbb{R}^d)} \right) \]

(2.15)

Furthermore, if \( 1 \leq p_2, q_2 < \infty, \) let \( \{ g_n \} \in C_0^\infty(\Omega) \) with \( g_n \to g \) strongly in \( L^{p_2}(L^{q_2}). \) Thus,
by the density arguments and properties of the standard mollification, we arrive at

\[ \| \partial (fg)^\varepsilon - \partial (fg_\varepsilon) \|_{L^p(L^q)} \]
\[ \leq C \| \partial (f (g - g_n)^\varepsilon - \partial (f (g - g_n)^\varepsilon) + (\partial (fg_n)^\varepsilon - \partial (fg_n^\varepsilon)) \|_{L^p(L^q)} \]
\[ \leq C \left( \| \partial f \|_{L^p(L^1)} + \| \nabla f \|_{L^p(L^1)} \right) \| g - g_n \|_{L^{p^2(q_2, \varepsilon)}} + C \| \partial (fg_n)^\varepsilon - \partial (fg_n^\varepsilon) \|_{L^p(L^q)} \]
\[ + C \| \partial (f(g_n - g_n^\varepsilon)) \|_{L^p(L^q)} \to 0, \quad \text{as } \varepsilon \to 0. \] (2.16)

This concludes the proof. \( \square \)

**Lemma 2.2.** Let \( p, q, p_1, q_1, p_2, q_2 \in [1, +\infty) \) with \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} \). Assume \( f \in L^{p_1}(0, T; L^{q_1}(\Omega)) \) and \( g \in L^{p_2}(0, T; L^{q_2}(\Omega)) \). Then for any \( \varepsilon > 0 \), there holds

\[ \| (fg)^\varepsilon - f^\varepsilon g^\varepsilon \|_{L^p(0, T; L^q(\Omega))} \to 0, \quad \text{as } \varepsilon \to 0. \] (2.17)

**Proof.** By the triangle inequality, one have

\[ \| (fg)^\varepsilon - f^\varepsilon g^\varepsilon \|_{L^p(0, T; L^q(\Omega))} \]
\[ \leq C \left( \| (fg)^\varepsilon - (fg) \|_{L^p(0, T; L^q(\Omega))} + \| f^\varepsilon g - f^\varepsilon g \|_{L^p(0, T; L^q(\Omega))} + \| f^\varepsilon - f^\varepsilon \|_{L^p(0, T; L^q(\Omega))} \right) \]
\[ \leq C \left( \left( \| (fg)^\varepsilon - f^\varepsilon g \|_{L^p(0, T; L^q(\Omega))} + \| f - f^\varepsilon \|_{L^p(0, T; L^{q_2}(\Omega))} \right) \| g \|_{L^{p_2}(0, T; L^{q_2}(\Omega))} \]
\[ + \| f^\varepsilon \|_{L^p(0, T; L^{q_2}(\Omega))} \| g - g^\varepsilon \|_{L^{p_2}(0, T; L^{q_2}(\Omega))} \right), \]

then, together with the properties of the standard mollification, we can obtain (2.17). \( \square \)

Finally, we recall the generalized Aubin-Lions Lemma to extend the energy equality up to the initial time.

**Lemma 2.3** ([27]). Let \( X \hookrightarrow B \hookrightarrow Y \) be three Banach spaces with compact imbedding \( X \hookrightarrow \hookrightarrow Y \). Further, let there exist \( 0 < \theta < 1 \) and \( M > 0 \) such that

\[ \| v \|_B \leq M \| v \|_X^{1-\theta} \| v \|_Y^\theta \quad \text{for all } v \in X \cap Y. \] (2.18)

Denote for \( T > 0 \),

\[ W(0, T) := W^{s_0, r_0}((0, T), X) \cap W^{s_1, r_1}((0, T), Y) \] (2.19)

with

\[ s_0, s_1 \in \mathbb{R}; \quad r_0, r_1 \in [1, \infty], \]
\[ s_\theta := (1 - \theta) s_0 + \theta s_1, \quad \frac{1}{r_\theta} := \frac{1 - \theta}{r_0} + \frac{\theta}{r_1}, \quad s^* := s_\theta - \frac{1}{r_\theta}. \] (2.20)

Assume that \( s_\theta > 0 \) and \( F \) is a bounded set in \( W(0, T) \). Then, we have

If \( s_* \leq 0 \), then \( F \) is relatively compact in \( L^p((0, T), B) \) for all \( 1 \leq p < p^* := \frac{1}{s_*}. \)

If \( s_* > 0 \), then \( F \) is relatively compact in \( C((0, T), B). \)
3 Proof of Theorem 1.1, Theorem 1.3 and Corollary 1.2

In this section, we first present the proof of Theorem 1.3. Then, making the use of interpolation inequality and the Poincaré inequality, we prove the Theorem 1.1 and Corollary 1.2 by the results of Theorem 1.3.

Proof of Theorem 1.3 Let \( \phi(t) \) be a smooth function compactly supported in \((0, +\infty)\). Multiplying (1.1) by \((\phi v^\varepsilon)^\varepsilon\), then integrating over \((0, T) \times \Omega\), we infer that

\[
\int_0^T \int \phi(t) v^\varepsilon \left[ \partial_t (\rho v)^\varepsilon + \text{div} (\rho v \otimes v)^\varepsilon + \nabla P(\rho)^\varepsilon - \mu \Delta v^\varepsilon - (\mu + \lambda) \nabla (\text{div} v)^\varepsilon \right] = 0. \tag{3.1}
\]

To pass the limit of \( \varepsilon \), we reformulate every term of the last equation. A straightforward computation leads to

\[
\int_0^T \int \phi(t) v^\varepsilon \partial_t (\rho v)^\varepsilon = \int_0^T \int \phi(t) v^\varepsilon \left[ \partial_t (\rho v)^\varepsilon - \partial_t (\rho v^\varepsilon) \right] + \int_0^T \int \phi(t) v^\varepsilon \partial_t (\rho v^\varepsilon) \\
= \int_0^T \int \phi(t) v^\varepsilon \left[ \partial_t (\rho v)^\varepsilon - \partial_t (\rho v^\varepsilon) \right] + \int_0^T \int \phi(t) \rho \partial_t |v^\varepsilon|^2 + \frac{1}{2} \rho \|\nabla v^\varepsilon\|^2. \tag{3.2}
\]

It follows from integration by parts and the mass equation (1.1.1) that

\[
\int_0^T \int \phi(t) v^\varepsilon \text{div} (\rho v \otimes v)^\varepsilon \\
= \int_0^T \int \phi(t) v^\varepsilon \text{div} [(\rho v \otimes v)^\varepsilon - (\rho v \otimes v^\varepsilon)] + \int_0^T \int \phi(t) v^\varepsilon \text{div} (\rho v \otimes v^\varepsilon) \\
= -\int_0^T \int \phi(t) \nabla v^\varepsilon [(\rho v \otimes v)^\varepsilon - (\rho v \otimes v^\varepsilon)] + \int_0^T \int \phi(t) \left( \text{div} (\rho v)|v^\varepsilon|^2 + \frac{1}{2} \rho v^2 \|\nabla v^\varepsilon\|^2 \right) \\
= -\int_0^T \int \phi(t) \nabla v^\varepsilon [(\rho v \otimes v)^\varepsilon - (\rho v \otimes v^\varepsilon)] + \frac{1}{2} \int_0^T \int \phi(t) \text{div} (\rho v)|v^\varepsilon|^2 \\
= -\int_0^T \int \phi(t) \nabla v^\varepsilon [(\rho v \otimes v)^\varepsilon - (\rho v \otimes v^\varepsilon)] - \frac{1}{2} \int_0^T \int \phi(t) \partial_t |v^\varepsilon|^2. \tag{3.3}
\]

We rewrite the pressure term as

\[
\int_0^T \int \phi(t) v^\varepsilon \nabla (\rho^\gamma)^\varepsilon = \int_0^T \int \phi(t) [v^\varepsilon \nabla (\rho^\gamma)^\varepsilon - v \nabla (\rho^\gamma)] + \int_0^T \int \phi(t) v \nabla (\rho^\gamma). \tag{3.4}
\]

Then using the integration by parts and mass equation (1.1.1) again, we find

\[
\int_0^T \int \phi(t) v \cdot \nabla (\rho^\gamma) = -\int_0^T \int \phi(t) \rho^{-1} \rho \text{div} v \\
= \int_0^T \int \phi(t) \rho^{-1} (\partial_t \rho + v \cdot \nabla \rho) \\
= \frac{1}{\gamma} \int_0^T \int \phi(t) \partial_t \rho^\gamma + \frac{1}{\gamma} \int_0^T \int \phi(t) v \cdot \nabla \rho^\gamma,
\]

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which in turn means that
\[
\int_0^T \int \phi(t) v \cdot \nabla (\rho^\varepsilon) = \frac{1}{\gamma - 1} \int_0^T \int \phi(t) \partial_t \rho^\varepsilon. \tag{3.5}
\]

Thanks to integration by parts, we arrive at
\[
- \mu \int_0^T \int \phi(t) v^\varepsilon \Delta v^\varepsilon = \mu \int_0^T \int \phi(t) |\nabla v^\varepsilon|^2,
\]
\[
- (\mu + \lambda) \int_0^T \int \phi(t) v^\varepsilon \nabla v^\varepsilon = (\mu + \lambda) \int_0^T \int \phi(t) |\text{div } v^\varepsilon|^2. \tag{3.6}
\]

Plugging (3.9) and (3.10) into (3.8), we get
\[
- \int_0^T \int \phi(t) v^\varepsilon \left[ \partial_t (\rho v)^\varepsilon - \partial_t (\rho v^\varepsilon) \right] + \int_0^T \int \phi(t) v^\varepsilon [\rho v \otimes v^\varepsilon - (\rho v) \otimes v^\varepsilon]
\]
\[
- \int_0^T \int \phi(t) [v^\varepsilon \nabla (\rho^\varepsilon) - v \nabla (\rho^\varepsilon)].
\]

It is enough to prove that the terms on the right hand-side of (3.7) tend to zero as \(\varepsilon \to 0\).

In view of Hölder’s inequality and Lemma (2.1), we know that
\[
\int_s^t \phi(t) t \left( \frac{|v^\varepsilon|^2}{2} + \frac{1}{\gamma - 1} \rho^\varepsilon \right) + \int_0^T \int (\mu |\nabla v^\varepsilon|^2 + (\mu + \lambda)|\text{div } v^\varepsilon|)^2
\]
\[
= - \int_0^T \int \phi(t) v^\varepsilon \left[ \partial_t (\rho v)^\varepsilon - \partial_t (\rho v^\varepsilon) \right] + \int_0^T \int \phi(t) v^\varepsilon [\rho v \otimes v^\varepsilon - (\rho v) \otimes v^\varepsilon]
\]
\[
- \int_0^T \int \phi(t) [v^\varepsilon \nabla (\rho^\varepsilon) - v \nabla (\rho^\varepsilon)]. \tag{3.7}
\]

To bound \(\rho_t\) and \(\nabla \rho\), we employ mass equation to obtain
\[
\rho_t = -2 \sqrt{\rho v} \cdot \nabla \sqrt{\rho} - \rho \text{div } v, \text{ and } \nabla \rho = 2 \sqrt{\rho} \nabla \sqrt{\rho}.
\]

As a consequence, the triangle inequality and Hölder’s inequality guarantee that
\[
\|\rho_t\|_{L^p(T, L^2)} \leq C \left( \| - 2 \sqrt{\rho v} \cdot \nabla \sqrt{\rho} \| + \|\rho \text{div } v\| \right)_{L^p(T, L^2)}
\]
\[
\leq C \left( \|v\|_{L^p(T)} \|\nabla \sqrt{\rho}\|_{L^p(T)} + \|\nabla v\|_{L^p(T)} \right), \tag{3.9}
\]

and
\[
\|\nabla \rho\|_{L^p(T, L^2)} \leq C \|\sqrt{\rho} \nabla \sqrt{\rho}\|_{L^p(T)} \leq C \|\nabla \sqrt{\rho}\|_{L^p(T)} \leq C \|\nabla \sqrt{\rho}\|_{L^p(T)} \leq C \|\nabla \sqrt{\rho}\|_{L^p(T)}. \tag{3.10}
\]

Plugging (3.9) and (3.10) into (3.8), we get
\[
\int_0^T \int \phi(t) v^\varepsilon \left[ \partial_t (\rho v)^\varepsilon - \partial_t (\rho v^\varepsilon) \right]
\]
\[
\leq C \|v\|_{L^p(T)}^2 \left( \|v\|_{L^p(T)}^2 + \|\nabla \sqrt{\rho}\|_{L^p(T)} + \|\nabla v\|_{L^p(T)} \right) \leq C. \tag{3.11}
\]

From Lemma (2.1), we end up with, as \(\varepsilon \to 0\),
\[
\int_0^T \int \phi(t) v^\varepsilon \left[ \partial_t (\rho v)^\varepsilon - \partial_t (\rho v^\varepsilon) \right] \to 0.
\]
Using the integration by parts, we observe that

\[
\left| \int_0^T \phi(t) \nabla v^\varepsilon \left[ (\rho v \otimes v)^\varepsilon - (\rho v) \otimes v^\varepsilon \right] \right|
\leq C \| \nabla v^\varepsilon \|_{L^\infty_t(L^{\frac{q}{p}})} \left( \| (\rho v \otimes v)^\varepsilon - (\rho v) \otimes v^\varepsilon \|_{L^\infty_t(L^{\frac{q}{p}})} \right)
\leq C \| \nabla v \|_{L^\infty_t(L^{\frac{q}{p}})} \left( \| (\rho v \otimes v)^\varepsilon - (\rho v \otimes v) \|_{L^\infty_t(L^{\frac{q}{p}})} + \| v \otimes v - v \otimes v^\varepsilon \|_{L^\infty_t(L^{\frac{q}{p}})} \right)
\leq C \| \nabla v \|_{L^\infty_t(L^{\frac{q}{p}})} \left( \| (\rho v \otimes v)^\varepsilon - (\rho v \otimes v) \|_{L^\infty_t(L^{\frac{q}{p}})} + \| v \|_{L^p_t(L^q)} \| v - v^\varepsilon \|_{L^p_t(L^q)} \right).
\]

(3.12)

Hence, by the standard properties of the mollification, we have

\[
\int_0^T \phi(t) \nabla v^\varepsilon \left[ (\rho v \otimes v)^\varepsilon - (\rho v) \otimes v^\varepsilon \right] \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0.
\]

According to the upper bound of the density, Hölder’s inequality on bound domain, we observe that

\[
\| \nabla (\rho^\gamma) \|_{L^\infty_t(L^{\frac{q}{p}})} \leq C \| \nabla \sqrt{\rho} \|_{L^\infty_t(L^{\frac{q}{p}})} \leq C \| \nabla \sqrt{\rho} \|_{L^\infty_t(L^{\frac{q}{p}})}
\]

which in turn implies that

\[
\int_0^T \phi(t) [v^\varepsilon (\nabla (\rho^\gamma))^\varepsilon - v \nabla (\rho^\gamma)] \rightarrow 0,
\]

(3.13)

where we have used lemma [2.2]

Then together with (3.11) - (3.13), passing to the limits as \( \varepsilon \rightarrow 0 \), we know that

\[
- \int_0^T \phi(t) \left( \frac{1}{2} \rho |v|^2 + \frac{\rho^\gamma}{\gamma - 1} \right) + \int_0^T \phi(t) \left( \mu |\nabla v|^2 + (\mu + \lambda) |\text{div} v|^2 \right) = 0.
\]

(3.14)

The next objective is to get the energy equality up to the initial time \( t = 0 \) by the similar method in [6] and [32], for the convenience of the reader and the integrity of the paper, we give the details. First, we derive from the mass equation (1.1) that

\[
\partial_t (\rho^\gamma) = -\gamma \rho^\gamma \text{div} v - 2 \gamma \rho^\gamma \frac{1}{2} v \cdot \nabla \sqrt{\rho},
\]

(3.15)

and

\[
\partial_t (\sqrt{\rho}) = \frac{\sqrt{\rho}}{2} \text{div} v + v \cdot \nabla \sqrt{\rho},
\]

(3.16)

which together with (3.13) gives

\[
\partial_t (\rho^\gamma, \sqrt{\rho}) \in L^\infty_t (0, T; L^{\frac{q}{p}}(\Omega)), \quad \nabla (\rho^\gamma, \sqrt{\rho}) \in L^\infty_t (0, T; L^{\frac{q}{p}}(\Omega)).
\]

Hence, using the Aubin-Lions Lemma [2.3], we can obtain

\[
(\rho^\gamma, \sqrt{\rho}) \in C([0, T]; L^{\frac{q}{p}}(\Omega)), \quad \text{for} \ dp < 2q + 3d, \ p \geq 4 \text{ and } q \geq 4.
\]

(3.17)

Furthermore, by the momentum equality (1.1) 2, we know that

\[
\rho v \in L^\infty(0, T; L^2(\Omega)) \cap H^1(0, T; H^{-1}(\Omega)),
\]
then, in light of the Aubin-Lions Lemma \[2.3\] we conclude that
\[
\rho v \in C([0,T]; L^2_{\text{weak}}(\Omega)).
\]

Meanwhile, using the natural energy (2.2), (3.17) and (3.18), we have

\[
0 \leq \lim_{t \to 0} \int |\sqrt{\rho}v - \sqrt{\rho_0}v_0|^2 \, dx
= 2\lim_{t \to 0} \left( \int \left( \frac{1}{2} \rho |v|^2 + \frac{1}{\gamma - 1} \rho^\gamma \right) \, dx - \int \left( \frac{1}{2} \rho_0 |v|^2 + \frac{1}{\gamma - 1} \rho_0^\gamma \right) \, dx \right)
+ 2\lim_{t \to 0} \left( \int \sqrt{\rho_0}v_0 (\sqrt{\rho_0}v_0 - \sqrt{\rho}v) \, dx + \frac{1}{\gamma - 1} \int (\rho_0^\gamma - \rho^\gamma) \, dx \right)
\leq 2\lim_{t \to 0} \int \sqrt{\rho_0}v_0 (\sqrt{\rho_0}v_0 - \sqrt{\rho}v) \, dx
= 2\lim_{t \to 0} \int v_0 (\rho_0 v_0 - \rho v) \, dx + 2\lim_{t \to 0} \int v_0 \sqrt{\rho}v_0 (\sqrt{\rho} - \sqrt{\rho_0}) \, dx = 0,
\]

from which it follows
\[
\sqrt{\rho}v(t) \to \sqrt{\rho}v(0) \quad \text{strongly in } L^2(\Omega) \text{ as } t \to 0^+.
\]

Similarly, one has the right temporal continuity of \(\sqrt{\rho}v\) in \(L^2(\Omega)\), hence, for any \(t_0 \geq 0\), we infer that
\[
\sqrt{\rho}v(t) \to \sqrt{\rho}v(t_0) \quad \text{strongly in } L^2(\Omega) \text{ as } t \to t_0^+.
\]

Before we go any further, it should be noted that (3.14) remains valid for function \(\phi\) belonging to \(W^{1,\infty}\) rather than \(C^1\), then for any \(t_0 > 0\), we redefine the test function \(\phi\) as \(\phi_\tau\) for some positive \(\tau\) and \(\alpha\) such that \(\tau + \alpha < t_0\), that is
\[
\phi_\tau(t) = \begin{cases} 
0, & 0 \leq t \leq \tau, \\
\frac{t - \tau}{\alpha}, & \tau \leq t \leq \tau + \alpha, \\
1, & \tau + \alpha \leq t \leq t_0, \\
\frac{t_0 - t}{\alpha}, & t_0 \leq t \leq t_0 + \alpha, \\
0, & t_0 + \alpha \leq t.
\end{cases}
\]

Then substituting this test function into (3.14), we arrive at
\[
- \int_{\tau}^{\tau + \alpha} \int \frac{1}{\alpha} \left( \frac{1}{2} \rho v^2 + \frac{1}{\gamma - 1} \rho^\gamma \right) \, dx + \frac{1}{\alpha} \int_{t_0}^{t_0 + \alpha} \int \left( \frac{1}{2} \rho v^2 + \frac{1}{\gamma - 1} \rho^\gamma \right) \, dx
+ \int_{\tau}^{t_0 + \alpha} \int \phi_\tau (\mu |\nabla v|^2 + (\mu + \lambda) |\text{div } v|^2) \, dx = 0.
\]

Taking \(\alpha \to 0\) and using the fact that \(\int_0^t (\mu |\nabla v|^2 + (\mu + \lambda) |\text{div } v|^2)\) is continuous with respect to \(t\) and the Lebesgue point Theorem, we deduce that
\[
- \int \left( \frac{1}{2} \rho v^2 + \frac{1}{\gamma - 1} \rho^\gamma \right) (\tau) \, dx + \int \left( \frac{1}{2} \rho v^2 + \frac{1}{\gamma - 1} \rho^\gamma \right) (t_0) \, dx
+ \int_{\tau}^{t_0} \int (\mu |\nabla v|^2 + (\mu + \lambda) |\text{div } v|^2) \, dx \, ds = 0.
\]
Finally, letting $\tau \to 0$, using the continuity of $\int_0^t \int (\mu |\nabla v|^2 + (\mu + \lambda) |\text{div} v|^2)$, (3.17) and (3.20), we can obtain
\[
\int \left( \frac{1}{2} \rho v^2 + \frac{1}{\gamma - 1} \rho^\gamma \right) (t_0) dx + \int_0^{t_0} \int (\mu |\nabla v|^2 + (\mu + \lambda) |\text{div} v|^2) dx ds = \int \left( \frac{1}{2} \rho_0 v_0^2 + \frac{1}{\gamma - 1} \rho_0^\gamma \right) dx.
\] (3.25)

Then we complete the proof of Theorem 1.3.

Next, with the help of Theorem 1.3, we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. (1) Choose $p = q = 4$ in (1.13), we obtain Lions type energy conservation condition.
\[
0 \leq \rho < c < \infty, \nabla \sqrt{\rho} \in L^4(L^4),
\]
\[
v \in L^4(L^4), \nabla v \in L^2(L^2) \text{ and } v_0 \in L^2(\mathbb{T}^d).
\] (3.26)

(2) With the help of interpolation, we show (1.10) can be reduced to (1.9). Indeed, taking advantage of the Gagliardo-Nirenberg inequality on bounded domains, Hölder’s inequality and Young’s inequality, we know that
\[
\|v\|_{L^4(0,T;L^4(\mathbb{T}^d))} \leq C \|v\|_{L^2(0,T;L^6(\mathbb{T}^d))} \|v\|_{L^p(0,T;L^q(\mathbb{T}^d))}
\]
\[
\leq C \left( \|\nabla v\|_{L^2(0,T;L^2(\mathbb{T}^d))} + \|v\|_{L^p(0,T;L^q(\mathbb{T}^d))} \right)^{\frac{3(q-4)}{2(6-q)}} \|v\|_{L^p(0,T;L^q(\mathbb{T}^d))}^{\frac{q}{2}}
\]
\[
\leq C \left( \|\nabla v\|_{L^2(0,T;L^2(\mathbb{T}^d))} \|v\|_{L^p(0,T;L^q(\mathbb{T}^d))} + \|v\|_{L^p(0,T;L^q(\mathbb{T}^d))} \right).
\] (3.27)

From the the result just proved, we obtain energy equality via (1.9). We finish the proof of this case.

(3) Taking $p = 4, q = 6$ in (1.13), we immediately get the desired result.

(4) We move to the proof of the last case. Firstly, we temporarily assume that the following fact is valid,
\[
\int_\Omega v dx \in L^p(0,T).
\] (3.28)

It follows from the triangle inequality and Poincaré-Sobolev inequality that, for $q < d$,
\[
\|v\|_{L^p(\Omega;L^q)} \leq \left\| v - \frac{1}{|\Omega|} \int_\Omega v dy \right\|_{L^q(\Omega;L^q)} + \left\| \frac{1}{|\Omega|} \int_\Omega v dy \right\|_{L^p(\Omega;L^q)}
\]
\[
\leq C \|\nabla v\|_{L^p(\Omega)} + C \left\| \int_\Omega v dy \right\|_{L^p(0,T)}.
\]

From the result case (2) in just proved, it is enough to prove the quality (3.28) we have assumed. The Hölder inequality and classical Poincaré inequality ensure that
\[
\left\| \int_\Omega \rho \left( v - \frac{1}{|\Omega|} \int_\Omega v dy \right) dx \right\| \leq \|\rho\|_{L^{\frac{q}{q-1}}} \left\| v - \frac{1}{|\Omega|} \int_\Omega v dy \right\|_{L^q}
\]
\[
\leq C \|\rho\|_{L^{\frac{q}{q-1}}} \|\nabla v\|_{L^q}.
\] (3.29)
Using the Hölder’s inequality once again and the upper bound of the density, we find

\[ \int \rho v dx \leq C \| \sqrt{\rho v} \|_{L^2}. \]  

(3.30)

In view of the triangle inequality, (3.29) and (3.30), we infer that

\[
\left| \int_{\Omega} v dy \right| = \left| \Omega \right| \left( \int_{\Omega} v dy \right) - v |_{\Omega} dx \leq C \| \nabla v \|_{L^q} + C \| \sqrt{\rho v} \|_{L^2}.
\]

which means

\[
\left\| \int_{\Omega} v dy \right\|_{L^p(0,T)} \leq C \| \nabla v \|_{L^p(L^q)} + C \| \sqrt{\rho v} \|_{L^\infty(L^2)}.
\]

Thus, it follows from the triangle inequality and Poincaré-Sobolev inequality that

\[
\| v \|_{L^p(L^q)} \leq \left\| v - \frac{1}{|\Omega|} \int_{\Omega} v dy \right\|_{L^p(L^q)} + \frac{1}{|\Omega|} \int_{\Omega} v dy \left\| \nabla v \right\|_{L^p(L^q)}
\]

\[
\leq C \| \nabla v \|_{L^p(L^q)} + C \left( \left\| v \right\|_{L^p(0,T)} \right),
\]

which means \( v \in L^p(D^{1,q}) \) implies \( v \in L^p(W^{1,q}) \). which implies

\[ v \in L^p(0, T; W^{1,q}(\mathbb{T}^d)), \quad q < d. \]

In summary, we have shown

\[ \nabla v \in L^p(0, T; L^r(\mathbb{T}^d)) \text{ with } \frac{1}{p} + \frac{3}{r} = 1 + \frac{3}{d}, \quad \frac{3d}{d+3} < r \leq \frac{4d}{d+4}, \]

turns out

\[ v \in L^p(0, T; L^q(\mathbb{T}^d)) \text{ with } \frac{1}{p} + \frac{3}{q} = 1, \quad 3 < q \leq 4. \]

At this stage, we complete the proof of Theorem 1.1. \( \square \)

Remark 3.1. To prove (1.12), one may directly employ the following Poincaré inequality which can be proved by slightly modifying the proof of [11, Lemma 3.2, page 47]:

Let \( v \in W^{1,q}(\Omega) \) with \( q \geq 2 \), and let \( \rho \) be a non-negative function such that

\[ 0 < M \leq \int_{\Omega} \rho dx, \quad \int_{\Omega} \rho^{\gamma} dx \leq E_0 < \infty, \]  

(3.31)

where \( \Omega \subset \mathbb{R}^d \) is a bounded domain and \( \gamma > 1 \). Then there exists a constant \( c \) depending solely on \( M \) and \( E_0 \) such that

\[ \| v \|^q_{L^q(\Omega)} \leq c(E_0, M) \left( \| \nabla v \|^q_{L^q(\Omega)} + \left( \int_{\Omega} \rho |v| dx \right)^q \right). \]  

(3.32)

Next, with the help of Theorem 1.1, we prove Corollary 1.2.
Proof of Corollary 1.2. According to Theorem 1.1, it suffices to show that all the density hypotheses yield

\[ 0 \leq \rho \leq c_2 < \infty, \nabla \sqrt{\rho} \in L^4(0,T; L^4(\mathbb{T}^3)). \]

On the one hand, for the case (2), it is enough to show \( 0 \leq \rho \leq c_2 < \infty \). The Sobolev embedding theorem ensures that \( \nabla \sqrt{\rho} \in L^\infty(0,T; L^4(\mathbb{T}^3)) \) and \( \rho \in L^\infty(0,T; L^\gamma(\mathbb{T}^3)) \) with \( \gamma > \frac{3}{2} \) (due to the theory of the global existence of weak solutions to (1.1)-(1.2) obtained by Feireisl- Novotný-Petzeltová in [13]) leads to

\[ 0 \leq \rho \leq c_2 < \infty. \]

For the case (3), it follows from the well-know interpolation inequality and Hölder’s inequality that, for \( \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{2} \), with \( q_1 \geq 4 \),

\[
\| \nabla \sqrt{\rho} \|_{L^4(0,T; L^4(\mathbb{T}^3))} \leq C \| \nabla \sqrt{\rho} \|_{L^\infty(0,T; L^2(\mathbb{T}^3))} \| \nabla \sqrt{\rho} \|_{L^{q_1}(0,T; L^{q_1}(\mathbb{T}^3))} < \infty.
\]

This completes the proof of this corollary. \( \square \)

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References

[1] I. Akramov, T. Debiec, J. W. D. Skipper and E. Wiedemann, Energy conservation for the compressible Euler and Navier-Stokes equations with vacuum. Anal. PDE. 13 (2020), 789–811

[2] H. Beirao da Veiga and J. Yang, On the Shinbrot's criteria for energy equality to Newtonian fluids: a simplified proof, and an extension of the range of application. Nonlinear Anal. 196 (2020), 111809, 4 pp.

[3] L. C. Berselli and E. Chiodaroli, On the energy equality for the 3D Navier-Stokes equations. Nonlinear Anal. 192 (2020), 111704, 24 pp.

[4] T. Buckmaster, C. De Lellis, L Jr Székelyhidi and V. Vicol, Onsager’s conjecture for admissible weak solutions. Commun. Pure Appl. Math. 72 (2019), 229–274.

[5] R. M. Chen and C. Yu, Onsager’s energy conservation for inhomogeneous Euler equations, J. Math. Pures Appl. 131 (2019), 1–16.

[6] M. Chen, Z. Liang, D. Wang and R. Xu, Energy equality in compressible fluids with physical boundaries. SIAM J. Math. Anal. 52 (2020), 1363–1385.
[7] A. Cheskidov and P. Constantin, S. Friedlander and R. Shvydkoy, Energy conservation and Onsager’s conjecture for the Euler equations. Nonlinearity, 21 (2008), 1233–52.

[8] A. Cheskidov and X. Luo, Energy equality for the Navier-Stokes equations in weak-in-time Onsager spaces. Nonlinearity, 33 (2020), 1388–1403.

[9] P. Constantin, E. Weinan and E.S. Titi, Onsager’s conjecture on the energy conservation for solutions of Euler’s equation. Commun. Math. Phys. 165 (1994), 207–209.

[10] R. J. DiPerna and P. L. Lions, Ordinary differential equations, transport theory and Sobolev spaces, Invent. Math., 98 (1989), 511–547.

[11] E. Feireisl, Dynamics of Viscous Compressible Fluids, Oxford University Press, 2004.

[12] E. Feireisl, P. Gwiazda, A. Świerczewska-Gwiazda and E. Wiedemann, Regularity and energy conservation for the compressible euler equations. Arch Ration Mech Anal. 223 (2017), 1375–1395.

[13] E. Feireisl, A. Novotný, H. Petzeltová, On the existence of globally defined weak solutions to the Navier-Stokes equations, J. Math. Fluid Mech. 3 (2001), 358–392.

[14] P. Isett, A proof of Onsager’s conjecture. Ann. of Math. 188 (2018), 871–963.

[15] Q. Jiu, Y. Wang and Y. Ye, Refined blow-up criteria for the full compressible Navier-Stokes equations involving temperature. J. Evol. Equ. 21 (2021), 1895–1916.

[16] I. Lacroix-Violet and A. Vasseur, Global weak solutions to the compressible quantum Navier-Stokes equation and its semi-classical limit, J. Math. Pures Appl. 114 (2018), 191–210.

[17] T. M. Leslie and R. Shvydkoy, The energy balance relation for weak solutions of the density-dependent Navier-Stokes equations J. Differential Equations. 261 (2016), 3719–3733.

[18] Z. Liang, Regularity criterion on the energy conservation for the compressible Navier-Stokes equations. Proc. Roy. Soc. Edinburgh Sect. A, (2020), 1–18.

[19] J. L. Lions, Sur la régularité et l’unicité des solutions turbulentes des équations de Navier Stokes. Rend. Semin. Mat. Univ. Padova, 30 (1960), 16–23.

[20] P. L. Lions, Mathematical Topics in Fluid Mechanics, vol. 1. Incompressible Models, Oxford University Press, New York, 1998.

[21] P. L. Lions, Mathematical Topics in Fluid Mechanics, vol. 2. Compressible Models, Oxford University Press, New York, 1998.

[22] Q. Nguyen, P. Nguyen and B. Tang, Energy equalities for compressible Navier-Stokes equations. Nonlinearity 32 (2019), 4206–4231.

[23] Q. Nguyen, P. Nguyen and B. Tang, Onsager’s conjecture on the energy conservation for solutions of Euler equations in bounded domains. J. Nonlinear Sci. 29 (2019), 207–213.
[24] Q. Nguyen, P. Nguyen and B. Tang, Energy conservation for inhomogeneous incompressible and compressible Euler equations. J. Differential Equations, 269 (2020), 7171–7210.

[25] L. Onsager, Statistical hydrodynamics, Nuovo Cim. (Suppl.) 6 (1949), 279–287.

[26] M. Shinbrot, The energy equation for the Navier-Stokes system. SIAM J. Math. Anal. 5 (1974), 948–954.

[27] J. Simon, Compact sets in the space $L^p(0,T;B)$, Ann. Mat. Pura Appl., 146 (1987), 65–96.

[28] Y. Taniuchi, On generalized energy equality of the Navier-Stokes equations. Manuscripta Math. 94 (1997), 365–384.

[29] Y. Wang and Y. Ye, Energy conservation via a combination of velocity and its gradient in the Navier-Stokes system. arXiv: 2106.01233.

[30] W. Wei, Y. Wang and Y. Ye, Gagliardo-Nirenberg inequalities in Lorentz type spaces and energy equality for the Navier-Stokes system. arXiv: 2106.11212.

[31] C. Yu. A new proof to the energy conservation for the Navier-Stokes equations. arXiv: 1604.05697.

[32] C. Yu. Energy conservation for the weak solutions of the compressible Navier-Stokes equations. Arch. Ration. Mech. Anal. 225 (2017), 1073–1087.

[33] C. Yu. The energy equality for the Navier-Stokes equations in bounded domains. arXiv: 1802.07661.

[34] Z. Zhang, Remarks on the energy equality for the non-Newtonian fluids. J. Math. Anal. Appl. 480 (2019), 123443, 9 pp.