REGULARIZED MODIFIED LOG-SOBOLEV INEQUALITIES, AND
COMPARISON OF MARKOV CHAINS

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Abstract. In this work, we develop a comparison procedure for the Modified log-Sobolev Inequality (MLSI) constants of two reversible Markov chains on a finite state space. Efficient comparison of the MLSI Dirichlet forms is a well known obstacle in the theory of Markov chains. We approach this problem by introducing a regularized MLSI constant which, under some assumptions, has the same order of magnitude as the usual MLSI constant yet is amenable for comparison and thus considerably simpler to estimate in certain cases. As an application of this general comparison procedure, we provide a sharp estimate of the MLSI constant of the switch chain on the the set of simple bipartite regular graphs of size $n$ with a fixed degree $d$. Our estimate implies that the total variation mixing time of the switch chain is of order $O_d(n \log n)$. The result is optimal up to a multiple depending on $d$ and resolves a long-standing open problem. We expect that the MLSI comparison technique implemented in this paper will find further applications.

1. Introduction

Let $\Omega$ be a finite state space, and let $Q$ be a Markov generator of a reversible chain on $\Omega$ with a stationary distribution $\pi$. We say that $(\Omega, \pi, Q)$ satisfies a Modified Logarithmic Sobolev Inequality (MLSI) with a constant $\alpha$ if for any function $f : \Omega \to (0, \infty)$ we have

$$\text{Ent}_\pi(f) := E_\pi[f(\log f - \log E_\pi f)] \leq \alpha \mathcal{E}_\pi(\log f, f),$$

where $\mathcal{E}_\pi(\log f, f) = \frac{1}{2} \sum_{\omega, \omega' \in \Omega} \pi(\omega) Q(\omega, \omega')(f(\omega) - f(\omega')) \log \frac{f(\omega)}{f(\omega')}$ is the corresponding Dirichlet form. We refer to the smallest $\alpha$ in the above inequality as the MLSI constant and denote it by $\alpha_{\text{MLSI}}(Q)$.

Similarly to the log-Sobolev inequality $\text{Ent}_\pi(f^2) \leq \alpha_{\text{LSI}} \mathcal{E}_\pi(f, f)$, the MLSI is known to imply sub-Gaussian concentration via the Herbst argument (see, for example, [18, Chapter 5]). Moreover, it constitutes a powerful tool allowing to capture the mixing time of the underlying Markov chain (see [3]). More precisely, for every $\varepsilon \in (0, 1)$,

$$t_{\text{mix}}(Q, \varepsilon) \leq \alpha_{\text{MLSI}}(Q) \left( \log \log \frac{1}{\pi_{\text{min}}} + \log \frac{1}{2\varepsilon^2} \right),$$

where $t_{\text{mix}}(Q, \varepsilon)$ denotes the total variation $\varepsilon$-mixing time of $Q$ and $\pi_{\text{min}} = \min_{x \in \Omega} \pi(x)$. While sharing similar properties with the log-Sobolev inequality, the MLSI often holds with a much smaller constant than the log-Sobolev inequality, and thus allows to get stronger concentration and mixing estimates.

Estimating the relaxation time or the log-Sobolev constant of a Markov chain by comparing it with another random process is a well developed technique which has been successfully used in a variety of situations (see for instance [5, 6, 21, 7] as well as a recent paper [23] by the authors for details). The main idea is that when the stationary measures of two Markov chains are “close” to each other, the Poincaré and log-Sobolev constants of the chains can be related by comparing the corresponding Dirichlet forms of the two chains. The canonical path (or the flow) method

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Often, the MLSI constant is defined as inverse of the above; however, we prefer to use the given definition.

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[22] aims at providing an efficient relation between the Dirichlet forms. While this comparison procedure has been widely used to obtain bounds on the Poincaré and the log-Sobolev constants, the case of the MLSI constant turns out to be fundamentally different. Indeed, let \((\pi, Q)\) and \((\tilde{\pi}, \tilde{Q})\) be two reversible irreducible Markov generators on a finite state space \(\Omega\). It is known (see [21, Chapter 4]) that there exists a constant \(C\) (depending on \(Q, \tilde{Q}\)) such that for any \(f : \Omega \to \mathbb{R}_+\) one has \(\tilde{E}(f, f) \leq C\mathcal{E}(f, f)\) where \(\tilde{E}\) (resp. \(E\)) denotes the Dirichlet form associated with \((\tilde{\pi}, \tilde{Q})\) (resp. \((\pi, Q)\)). On the other hand, under the same assumptions, there does not in general exist a constant \(C\) such that for all \(f : \Omega \to \mathbb{R}_+\) one has \(E(f, \log f) \leq C\mathcal{E}(f, \log f)\) (see [12, Page 74] for a counter-example).

In this paper, we develop a comparison procedure for the MLSI constants based on a notion of a regularized Modified log-Sobolev Inequality, which is the MLSI restricted to a special class of functions. Given a Markov chain on \(\Omega\), we show that the MLSI and its regularized version hold with constants having the same order of magnitude. We then show that under certain assumptions the Dirichlet forms of two Markov chains evaluated on the regular functions can be efficiently compared.

Below, we provide a rigorous description of our method. Given a reversible Markov generator \(Q\) on a state space \(\Omega\) with a stationary distribution \(\pi\), we equip \(\Omega\) with the graph structure induced by \(Q\), namely, two distinct vertices \(\omega, \omega' \in \Omega\) are connected by an edge if and only if \(Q(\omega, \omega') \neq 0\). Given \(r \geq 1\), define \(\mathcal{R}(Q, r)\) as the collection of all functions \(f : \Omega \to (0, \infty)\) such that

\[
\frac{f(\omega)}{f(\omega')} \leq r^{\text{dist}(\omega, \omega')} \quad \text{for all vertices } \omega, \omega' \text{ of } \Omega,
\]

where \(\text{dist}(\omega, \omega')\) is the usual graph distance between \(\omega\) and \(\omega'\). We call these functions \(r\)-regular. Note that if a function is \(r\)-regular then it is also \(r'\)-regular for any \(r' \geq r\). Moreover, any positive function is \(\infty\)-regular while constant functions are 1-regular.

We say that \((\Omega, \pi, Q)\) satisfies the \(r\)-regularized MLSI with a constant \(\alpha_r\) if for any function \(f \in \mathcal{R}(Q, r)\) we have

\[
\text{Ent}_\pi(f) \leq \alpha_r \mathcal{E}_\pi(\log f, f).
\]

As before, we refer to the best constant in the above inequality as the \(r\)-regularized MLSI constant of \(Q\). Note that with our notations, the \(\infty\)-regularized MLSI is the “usual” Modified log-Sobolev inequality, and in view of the above, whenever \((\Omega, \pi, Q)\) satisfies the “usual” MLSI, it also satisfies the \(r\)-regularized MLSI with the same constant for any \(r \geq 1\). Our first main result shows that there exists \(1 < r < \infty\) for which the reverse is true.

**Theorem 1.1.** Let \(Q\) be a reversible Markov generator with a stationary measure \(\pi\) on a finite state space \(\Omega\). Define

\[
(2) \quad \gamma := \frac{\max_{\omega \in \Omega} \pi(\omega)}{\min_{\omega \in \Omega} \pi(\omega)} \quad \text{and} \quad \Upsilon := \frac{16\gamma^2 \max_{\omega} |\{\omega' \neq \omega : Q(\omega, \omega') \neq 0\}|}{\min_{\omega \neq \omega', Q(\omega, \omega') \neq 0} Q(\omega, \omega')}.\]

If \((\Omega, \pi, Q)\) satisfies the \(\Upsilon\)-regularized MLSI with a constant \(\alpha_\Upsilon\), then \((\Omega, \mu, Q)\) satisfies MLSI with constant \(C\alpha_\Upsilon\), where \(C > 0\) is a universal constant.

While the above is satisfactory for the application we have in mind, it would be interesting to find the “best” value of the parameter \(r\) for which the \(r\)-regularized MLSI implies MLSI. We did not pursue this line of research in the current work.

Let \((\pi, Q)\) and \((\tilde{\pi}, \tilde{Q})\) be two reversible Markov generators on a finite set \(\Omega\). For each \(x, y \in \Omega\) with \(\tilde{Q}(x, y) > 0\), we let \(\mathcal{P}_{x,y}\) be the set of all paths \(x_0 = x, x_1, \ldots, x_k = y\) (of arbitrary lengths \(k \geq 1\)) such that \(Q(x_i, x_{i+1}) > 0\) for all \(i = 0, \ldots, k - 1\). We define \(\Gamma(Q, \tilde{Q}) := \bigcup_{x,y \in \Omega, \tilde{Q}(x,y) > 0} \mathcal{P}_{x,y}\). Recall that a weight function \(W : \Gamma(Q, \tilde{Q}) \to [0, 1]\) is called a \((Q, \tilde{Q})\)-flow
if for every \( x, y \) with \( \tilde{Q}(x, y) > 0 \) we have
\[
\sum_{P \in \mathcal{P}_{x,y}} W(P) = \tilde{\pi}(x)\tilde{Q}(x, y)
\]
(see [6 Section 2C]). The second main result in the paper is the following theorem.

**Theorem 1.2.** Let \((\pi, Q)\) and \((\tilde{\pi}, \tilde{Q})\) be two reversible Markov generators on a finite set \(\Omega\), let \(W\) be a \((Q, \tilde{Q})\)-flow, and suppose that \(\pi(\omega) \leq a\tilde{\pi}(\omega)\) for every \(\omega \in \Omega\) for some parameter \(a > 0\). If \((\Omega, \tilde{\pi}, \tilde{Q})\) satisfies MLSI with constant \(\tilde{\alpha}(\tilde{Q})\), then for any \(r \geq 2\) the \(r\)-regularized modified log-Sobolev constant \(\alpha_r(Q)\) of \((\Omega, \pi, Q)\) satisfies
\[
\alpha_r(Q) \leq Ca A(W, r) \tilde{\alpha}_r(\tilde{Q}),
\]
where
\[
A(W, r) = \max_{(\omega, \omega') \in Q(\omega, \omega') > 0} \frac{1}{\pi(\omega)Q(\omega, \omega')} \sum_{P \in \Gamma(Q, \tilde{Q})} \frac{W(P)}{\pi(\omega)} (1 + (\text{len}(P) - 1)^2 \log r),
\]
and \(C\) is a universal constant.

Note that without imposing the \(r\)-regularization on functions on \(\Omega\) (i.e when considering the setting \(r = \infty\)), the above comparison result in itself does not produce a useful estimate. However, when combined with Theorem 1.1, a comparison statement for the “usual” MLSI constants readily follows. This fills a gap in the literature by providing a result for MLSI similar to classical comparison statements for Poincaré and log-Sobolev inequalities (see for example [21 Chapter 4]).

The above result becomes interesting when the MLSI and the log-Sobolev constants of \((\tilde{\pi}, \tilde{Q})\) have different orders of magnitude. Indeed, it is always possible to bound \(\alpha_r(Q)\) by the log-Sobolev constant \(\alpha_{LSI}(Q)\) and then use standard comparison procedures (see, in particular, [21 Theorem 4.2.5]) to bound the latter by the log-Sobolev constant of \((\tilde{\pi}, \tilde{Q})\) multiplied by a function of the flow similar to the one in Theorem 1.2. One particular example is when \(\tilde{Q}\) is the trivial Markov generator on \((\Omega, \pi)\), with \(\tilde{Q}(\omega, \omega') = \pi(\omega') = \tilde{\pi}(\omega')\) for any \(\omega \neq \omega'\). It is known that this chain satisfies a log-Sobolev inequality with \(\alpha_{LSI}(\tilde{Q}) = O\left(\frac{1}{\min_{\omega \in \Omega} \pi(\omega)}\right)\) and the Modified log-Sobolev Inequality with constant 1. It follows (see [6 Theorem 2.3] and [21 Section 4.2]) that any triple \((\Omega, \pi, Q)\) satisfies a log-Sobolev inequality (and thus MLSI) with a constant
\[
C \log \frac{1}{\min_{\omega \in \Omega} \pi(\omega)} \max_{(\omega, \omega') \in Q(\omega, \omega') > 0} \frac{1}{\pi(\omega)Q(\omega, \omega')} \sum_{P \in \mathcal{P}_{x,y}, x \neq y} W(P)\text{len}(P).
\]

The bound provided by combining Theorems 1.1 and 1.2 improves the last estimate in many situations of interest, as it replaces the “global” factor \(\log \frac{1}{\min_{\omega \in \Omega} \pi(\omega)}\) by a “local” parameter \(\log \Upsilon\), at the price of squaring the lengths of the paths in the flow.

To illustrate the power of the comparison procedure introduced in this paper, we will apply this concept to derive a sharp Modified log-Sobolev Inequality for the switch chain on the set of regular bipartite graphs. This chain uses a standard local operation called the simple switching which takes two non-incident edges \((i_1, j_1)\) and \((i_2, j_2)\) of the graph uniformly at random, destroys them, and replaces them by their “crossed” counterparts \((i_1, j_2)\) and \((i_2, j_1)\) whenever possible. Formally, given \(n \in \mathbb{N}\) and \(2 \leq d \leq n/2\), we denote by \(\Omega^B_n(d)\) the set of all simple bipartite \(d\)-regular graphs on the vertex set \([n]^{(\ell)} \sqcup [n]^{(r)}\) (where we use the superscripts “(\ell)” and “(r)” for sets of left and right vertices), and we equip it with the uniform probability measure \(\pi_u\). The
switch chain is defined through its Markov generator $Q_u$ as follows:

$$Q_u(G_1, G_2) := \begin{cases} \frac{|\mathcal{N}(G_1)|}{nd(nd-1)/2}, & \text{if } G_1 = G_2; \\ \left((nd(nd - 1)/2)\right)^{-1}, & \text{if } G_2 \in \mathcal{N}(G_1); \\ 0, & \text{otherwise.} \end{cases}$$

Here, $\mathcal{N}(G_1)$ denotes the set of all graphs in $\Omega_n^B(d)$ which can be obtained from $G_1$ by the simple switching operation. The mixing time of this chain was first investigated in [17], followed by papers [4, 13, 14, 20] which studied the switch chain for several graph models. We refer to [11] for a recent account of this line of research and a comprehensive reference list.

Recently, the authors [23] established a sharp Poincaré inequality for the chain $(\Omega_n^B(d), \pi_u, Q_u)$ for any degree $d \geq 3$ satisfying $d \leq n^c$, for some small universal constant $c$. When $d$ is fixed, they also established a log-Sobolev inequality with a constant $C_d n \log n$ and showed that the dependence of the LSI constant on $n$ is sharp. The strategy employed in [23] is a double comparison procedure with the standard random transposition model and the switch chain on the configuration model. The main challenge in [23] is that in the regime $d \to \infty$, the configuration model and the space $(\Omega_n^B(d), \pi_u, Q_u)$ do not admit standard comparison techniques for Markov chains without incurring a loss of precision. To overcome this issue, a delicate construction of function extensions with induced “controlled” fluctuations was introduced [23]. When $d$ is fixed, the standard comparison techniques can be employed, and the main technical task is to construct canonical path and a flow with a small congestion. This was carried out in [23] and allowed the authors to obtain the sharp log-Sobolev inequality which implied in particular that the total variation mixing time of the switch chain is bounded above by $C_d n \log^2 n$ for some constant depending only on $d$. Previously, the best known bound in this regime was $C_d n^7 \log n$ [8]. The mixing time bound obtained in [23] is off by a factor $\log n$ from the conjectured optimal estimate. That in itself is not surprising since the approach relied on a comparison with the random transposition model, and it is known that the log-Sobolev constant for that model fails to capture the correct total variation mixing time [12]. On the other hand, the sharp MLSI constant for the random transposition model was calculated in [12] and it was shown that it does yield the optimal TV mixing time bound. This is one particular instance where the modified log-Sobolev inequality offers an advantage over the classical log-Sobolev inequality. To summarize, the comparison techniques developed in this paper allowed us to prove the following.

**Theorem 1.3.** For every fixed $2 \leq d \leq n/2$, the triple $(\Omega_n^B(d), \pi_u, Q_u)$ satisfies the Modified log-Sobolev Inequality with a constant $C_d n$, where $C_d > 0$ depends only on $d$.

**Corollary 1.4.** For every fixed $2 \leq d \leq n/2$, the total variation mixing time $t_{mix} := t_{mix}(Q_u, 1)$ of the switch chain $(\Omega_n^B(d), \pi_u, Q_u)$ is bounded above by $C_d n \log n$, for some constant $C_d$ depending only on $d$.

The above bound is sharp (see Proposition 4.2) and was previously conjectured in [4]. With the techniques developed in [23] and the present paper, we believe it is possible to derive sharp bounds on the mixing time of the switch chain for other graph models of interest, including simple undirected $d$-regular graphs.

The paper is organized as follows: Section 2 is devoted for the proof of Theorem 1.1, while Theorem 1.2 is proved in Section 3. Finally, the proof of Theorem 1.3 is carried in Section 4.

2. MLSI and Function Regularization

Before proceeding with the proof of Theorem 1.1 we consider another statement in the same spirit aiming at restricting the class of functions on which the MLSI needs be verified. The next lemma may be of independent interest, and will be used in Section 4 when proving the MLSI for the switch chain.
Lemma 2.1. There are universal constants $c, C > 0$ with the following property. Assume that a reversible Markov chain $(\Omega, Q, \pi)$ satisfies
\[
\text{Ent}_\pi f \leq K \mathcal{E}_\pi(f, \log f)
\]
for every positive function $f$ on $\Omega$ with $f(\omega) \geq c$ for all $\omega \in \Omega$ and $\mathbb{E}_\pi f = 1$. Then $(\Omega, Q, \pi)$ satisfies the MLSI with a constant $C_K$.

Proof. We will assume that the constant $c$ is sufficiently small so that in particular $1 - c + c \log c \geq \frac{1}{2}$ and $c/(1 - c) \leq 1/2$.

Fix any non-constant positive function $f : \Omega \to (0, \infty)$ with $\mathbb{E}_\pi f = 1$, and define an auxiliary function $f'$ as follows:
\[
f'(\omega) := \begin{cases} 
\max(f(\omega), c), & \text{if } f(\omega) \leq 1; \\
\alpha + (1 - \alpha)f(\omega), & \text{if } f(\omega) > 1,
\end{cases}
\]
where the parameter $\alpha \in [0, 1)$ is chosen so that $\mathbb{E}_\pi f' = 1$. Note that in view of our assumptions,
\[
\text{Ent}_\pi f' \leq K \mathcal{E}_\pi(f', \log f').
\]

We first estimate the value of the parameter $\alpha$. Let $T_0, T_{[c, 1]}, T_{> 1}$ be the partition of the space $\Omega$ into subsets of points $\omega$ where $f(\omega) < c$, $f(\omega) \in [c, 1]$ and $f(\omega) > 1$, respectively. Thus,
\[
1 = \mathbb{E}_\pi f' = c\pi(T_0) + \sum_{\omega \in T_{[c, 1]}} f(\omega)\pi(\omega) + \alpha\pi(T_{> 1}) + (1 - \alpha) \sum_{\omega \in T_{> 1}} f(\omega)\pi(\omega),
\]
implying that
\[
\alpha \left( \sum_{\omega \in T_{> 1}} f(\omega)\pi(\omega) - \pi(T_{> 1}) \right) = c\pi(T_0) - \sum_{\omega \in T_0} f(\omega)\pi(\omega).
\]
It remains to observe that
\[
\sum_{\omega \in T_{> 1}} f(\omega)\pi(\omega) - \pi(T_{> 1}) = \pi(T_0) - \sum_{\omega \in T_0} f(\omega)\pi(\omega) + \pi(T_{[c, 1]}) - \sum_{\omega \in T_{[c, 1]}} f(\omega)\pi(\omega) \geq (1 - c)\pi(T_0)
\]
to conclude that $\alpha \leq \frac{c}{1 - c} \leq 1/2$.

The next step of the argument is to compare the entropies of the functions $f$ and $f'$. We will use the representations of the entropies
\[
\text{Ent}_\pi f = \sum_{\omega \in \Omega} (1 - f(\omega) + f(\omega)\log f(\omega))\pi(\omega);
\]
\[
\text{Ent}_\pi f' = \sum_{\omega \in \Omega} (1 - f'(\omega) + f'(\omega)\log f'(\omega))\pi(\omega),
\]
which have the advantage that the convex function $x \to 1 - x + x \log x$, $x \in (0, \infty)$, is non-negative, allowing term-by-term comparison of the expressions on the right side. Clearly, for every $\omega \in T_{[c, 1]}$, the respective terms agree. Further, for any $\omega \in T_0$, in view of the conditions on $c$,
\[
1 - f'(\omega) + f'(\omega)\log f'(\omega) = 1 - c + c \log c \geq \frac{1}{2} \geq \frac{1}{2}((1 - f(\omega) + f(\omega)\log f(\omega)),
\]
where we also used that $x \to 1 - x + x \log x$ is bounded above by 1 on $(0, 1)$. Finally, for $\omega \in T_{> 1}$ we consider two cases. If $f(\omega) \geq 10$ then $f'(\omega) \geq f(\omega)/2 \geq 5$, and we have
\[
1 - f(\omega) + f(\omega)\log f(\omega) \leq f(\omega)\log f(\omega), \quad 1 - f'(\omega) + f'(\omega)\log f'(\omega) \geq \frac{1}{2} f'(\omega)\log f'(\omega).
\]
At the same time, since \( f'(\omega) \geq f(\omega)/2 \) and \( f(\omega) \geq 10 \), we have \( f'(\omega) \log f'(\omega) \geq \frac{1}{2} f(\omega) \log f(\omega) \).
Thus, whenever \( f(\omega) \geq 10 \), we have
\[
1 - f'(\omega) + f'(\omega) \log f'(\omega) \geq \frac{1}{6} (1 - f(\omega) + f(\omega) \log f(\omega)).
\]
In the remaining case \( f(\omega) \in (1, 10) \), we observe that
\[
1 - f(\omega) + f(\omega) \log f(\omega) \leq (f(\omega) - 1)^2, \quad 1 - f'(\omega) + f'(\omega) \log f'(\omega) \geq \frac{1}{6} (f'(\omega) - 1)^2,
\]
and thus we conclude that \( 1 - f'(\omega) + f'(\omega) \log f'(\omega) \geq \frac{(1 - a)^2}{6} (1 - f(\omega) + f(\omega) \log f(\omega)) \).
To summarize, we have shown that
\[
C \text{Ent}_\pi f' \geq \text{Ent}_\pi f
\]
for some constant \( C > 0 \).

Now, we compare the Dirichlet forms with the functions \( f' \) and \( f \). This step is elementary since it is sufficient for us to confirm that \( \mathcal{E}_\pi(f', \log f') \leq \mathcal{E}_\pi(f, \log f) \), while the construction of \( f' \) guarantees that for every \( \omega, \omega' \in \Omega \) with \( f(\omega) \leq f(\omega') \), we have \( f'(\omega') - f'(\omega) \leq f(\omega') - f(\omega) \) and
\[
\frac{f'(\omega)}{f(\omega)} \geq \frac{f'(\omega')}{f(\omega')}. \tag{7}
\]
The result follows.

The main goal of this section is to prove Theorem 1.1. We first define the notion of \( r \)-regularization. Given a reversible Markov generator \( Q \) on \((\Omega, \pi)\), and a positive function \( f: \Omega \to \mathbb{R}_+ \), the \( r \)-regularization of \( f \) is the function \( f_r \) given by
\[
f_r(\omega) := \max_{\omega' \in \Omega} \frac{f(\omega')}{\text{dist}(\omega, \omega')^r} \quad \text{for every } \omega \in \Omega,
\]
where \( \text{dist}(\cdot, \cdot) \) is the usual distance in the graph \((\Omega, \{(w, w') : w \neq w', Q(w, w') \neq 0\})\).

In what follows, given \( r > 1 \), it will be convenient to associate with every positive function \( f \) on \( \Omega \) a mapping \( \mathcal{R}_f = \mathcal{R}_{f,\Omega,r} \) as follows. Let \( f_r \) be the \( r \)-regularization of \( f \). For every \( \omega \in \Omega \) with \( f_r(\omega) > f(\omega) \) there is at least one vertex \( \bar{\omega} \in \Omega \) such that
\[
f_r(\bar{\omega}) = \frac{f(\bar{\omega})}{\text{dist}(\bar{\omega}, \omega)} = \frac{f_r(\omega)}{\text{dist}(\omega, \omega)}.
\]
Then we set \( \mathcal{R}_f(\omega) := \bar{\omega} \). Thus, \( \mathcal{R}_f \) is a mapping on \( \{\omega : f_r(\omega) > f(\omega)\} \). Note that in general \( \mathcal{R}_f \) does not have to be uniquely defined. For convenience, we will fix a single realization of \( \mathcal{R}_{f,\Omega,r} \) for every triple \((f, \Omega, r)\).

Simple properties of \( r \)-regularizations are collected in the following lemma.

**Lemma 2.2.** Let \( Q \) be a reversible Markov generator on a finite probability space \((\Omega, \pi)\), \( f \) be a positive function on \( \Omega \) and let \( f_r \) be the \( r \)-regularization of \( f \). Then
\[
\text{\quad (i) } f_r \text{ is } r \text{-regular;}
\]
\[
\text{\quad (ii) For any } \omega \in \Omega \text{ with } f_r(\omega) > f(\omega) \text{ there exists a geodesic path } P \text{ on the graph } (\Omega, \{(w, w') : w \neq w', Q(w, w') \neq 0\}) \text{ starting at } \omega \text{ such that}
\]
\[
f_r(P[\tau]) = \frac{f(P[\text{len}(P)])}{\text{len}(P) - \tau}
\]
for all \( \tau \in [0, \text{len}(P)] \).

The strategy of proving Theorem 1.1 is straightforward: for any positive function \( f \) on \( \Omega \) we consider its \( \Upsilon \)-regularization \( f_\Upsilon \) and show that the entropies are related as
\[
\text{Ent}_\pi(f_\Upsilon) \geq C \text{Ent}_\pi(f),
\]
where \( \mathcal{E}_\pi(\log f_\Upsilon, f_\Upsilon) \leq C \mathcal{E}_\pi(\log f, f) \) for some universal constants \( C > 0 \). This immediately implies the required result. The necessary auxiliary statements are verified below.

**Lemma 2.3.** Let \( Q \) be a reversible Markov generator on a finite probability space \((\Omega, \pi)\), \( f \) be a positive function on \( \Omega \), and let \( \gamma \) and \( \Upsilon \) be defined according to (2). Further, let \( f_\Upsilon \) be the \( \Upsilon \)-regularization of \( f \). Then
\[
\mathcal{E}_\pi(\log f_\Upsilon, f_\Upsilon) \leq C \mathcal{E}_\pi(\log f, f),
\]
where \( C > 0 \) is a universal constant.
Proof. For brevity, denote
\[ u := \min_{\omega \neq \omega', Q(\omega, \omega') \neq 0} Q(\omega, \omega'), \]
and
\[ V(\tilde{\omega}) := \sum_{\omega' : \omega'' \neq \omega} Q(\tilde{\omega}, \omega') (f(\tilde{\omega}) - f(\omega')) \log \frac{f(\omega)}{f(\omega')}, \quad \tilde{\omega} \in \Omega. \]
Observe that, in view of the definition of \( \Upsilon \), we have for any \( \tilde{\omega} \in \Omega \):
\[
\sum_{\omega : \omega' \neq \omega} \frac{\pi(\omega)}{u} \frac{\pi(\omega)}{u} \leq \sum_{\omega : \omega' \neq \omega} \frac{\gamma \pi(\omega)}{u \Upsilon(\omega, \omega)} \leq \frac{\pi(\omega)}{8 \gamma}.
\]
Fix any pair of adjacent vertices \( \omega, \omega' \) of \( G_{\Omega, Q} := (\Omega, \{(w, w') : w \neq w', Q(w, w') \neq 0\}) \). Without loss of generality, we can assume that \( f(\omega) \geq f(\omega') \). We shall consider three cases:

- \( f(\omega) = f_T(\omega) \geq f_T(\omega') \). Since \( f_T(\omega') \geq f(\omega) \), in this case we have
  \[
  (f_T(\omega) - f_T(\omega')) \log \frac{f_T(\omega)}{f_T(\omega')} \leq (f(\omega) - f(\omega')) \log \frac{f(\omega)}{f(\omega')},
  \]

- \( f(\omega) < f_T(\omega) \) and \( f_T(\omega) \geq f_T(\omega') \). In this case necessarily there is a vertex \( \tilde{\omega} = R_f(\omega) \) with
  \[
  f_T(\omega) = f_T(\tilde{\omega}) = \frac{f_T(\tilde{\omega})}{\Upsilon(\tilde{\omega}, \omega)} ;
  \]
  moreover, there is a vertex \( \tilde{\omega} \) adjacent to \( \tilde{\omega} \) and with \( \text{dist}(\tilde{\omega}, \omega) = \text{dist}(\tilde{\omega}, \omega') - 1 \) such that \( f_T(\tilde{\omega}) = \frac{1}{T} f(\tilde{\omega}) \) (see Lemma [2.2]). Note that \( f_T(\omega') \geq \frac{f(\tilde{\omega})}{\Upsilon(\tilde{\omega}, \omega)} \geq \frac{f(\tilde{\omega})}{\Upsilon(\tilde{\omega}, \omega)} \). Using this, we can write
  \[
  (f_T(\omega) - f_T(\omega')) \log \frac{f_T(\omega)}{f_T(\omega')} \leq \frac{1}{\Upsilon(\omega, \omega')}(f(\tilde{\omega}) - f(\tilde{\omega})) \log \frac{f(\tilde{\omega})}{f(\tilde{\omega})}.
  \]

- \( f(\omega) < f_T(\omega') \). Similarly to the previous case, there is a vertex \( \tilde{\omega} \) with
  \[
  f_T(\omega') = f_T(\tilde{\omega}) = \frac{f_T(\tilde{\omega})}{\Upsilon(\tilde{\omega}, \omega')},
  \]
  and there is a vertex \( \hat{\omega} \) adjacent to \( \tilde{\omega} \) and with \( \text{dist}(\hat{\omega}, \omega') = \text{dist}(\hat{\omega}, \omega') - 1 \) such that \( f_T(\hat{\omega}) = \frac{1}{T} f(\hat{\omega}) \). Hence,
  \[
  (f_T(\omega) - f_T(\omega')) \log \frac{f_T(\omega)}{f_T(\omega')} \leq \frac{1}{\Upsilon(\omega, \omega')}(f(\hat{\omega}) - f(\hat{\omega})) \log \frac{f(\hat{\omega})}{f(\hat{\omega})}.
  \]

Summing the above estimates and using the chain reversibility, we obtain
\[
\sum_{\omega' \neq \omega} \pi(\omega)Q(\omega, \omega')(f_T(\omega) - f_T(\omega')) \log \frac{f_T(\omega)}{f_T(\omega')}
\leq \sum_{\omega' \neq \omega} \pi(\omega)Q(\omega, \omega')(f(\omega) - f(\omega')) \log \frac{f(\omega)}{f(\omega')} + \sum_{\omega' \neq \omega} \sum_{\hat{\omega} : \hat{\omega} \neq \omega} \frac{4\pi(\omega)Q(\omega, \omega')}{\Upsilon(\omega, \omega')} V(\hat{\omega})
\leq 2E_{\pi}(\log f, f) + \sum_{\hat{\omega} \in \Omega} \sum_{\omega : \omega' \neq \omega} \frac{4\pi(\omega)}{\Upsilon(\omega, \omega')} V(\hat{\omega})
\leq 2E_{\pi}(\log f, f) + \frac{1}{\gamma}E_{\pi}(\log f, f),
\]
where the last inequality follows after using [1].

In order to verify a counterpart comparison inequality for the entropies, we need the following simple relaxation of the duality formula of the entropy.
Lemma 2.4. Let \((\Omega, \pi)\) be a finite probability space and \(f\) be a positive function on \(\Omega\). Then
\[
\Ent_\pi(f) \leq 2 \sup \{ \E_\pi[\hat{f}] \mid \hat{f}: \Omega \rightarrow \mathbb{R} \text{ satisfying } \E_\pi[\hat{f}] = 1 \text{ and } \hat{f} \geq \log(1/2) \}.
\]
Proof. Let \(h\) be any function on \(\Omega\) with \(\E_\pi[e^h] = 1\). Define \(\tilde{h}\) via the relation
\[
\exp(\tilde{h}) = \frac{\exp(h) + 1}{2}.
\]
Clearly, \(\E_\pi[e^{\tilde{h}}] = 1\) and \(\tilde{h} \geq \log(1/2)\). At the same time, it is easy to check that
\[
\tilde{h}(\omega) \geq h(\omega)/2
\]
for every \(\omega \in \Omega\), whence
\[
\E_\pi[\tilde{f}\tilde{h}] \geq \frac{1}{2} \E_\pi[f\tilde{h}].
\]
Applying the variational formula for the entropy \[24, Lemma 3.15\], we get the result. \(\square\)

We are now ready to prove a lemma which, together with Lemma 2.4, yields Theorem 1.1.

Lemma 2.5. Let \(Q\) be a reversible Markov generator on a finite probability space \((\Omega, \pi)\), \(f\) be a positive function on \(\Omega\), and let \(\Upsilon\) be defined according to (2). Further, let \(f_\Upsilon\) be the \(\Upsilon\)-regularization of \(f\). Then
\[
\Ent_\pi(f_\Upsilon) \geq c \Ent_\pi(f),
\]
where \(c > 0\) is a universal constant.

Proof. In view of Lemma 2.4, we can find a function \(\tilde{h}\) on \(\Omega\) with \(\E_\pi[e^{\tilde{h}}] = 1\) and \(\tilde{h} \geq \log(1/2)\), such that
\[
\E_\pi[\tilde{f}\tilde{h}] \geq \frac{1}{2} \Ent_\pi(f).
\]
Denote the domain of \(\mathcal{R}_{f,\Omega,\Upsilon}\) by \(S\):
\[
S := \{ \omega \in \Omega : f_\Upsilon(\omega) > f(\omega) \}.
\]
We clearly have
\[
\E_\pi[f_\Upsilon\tilde{h}] = \E_\pi[\tilde{f}\tilde{h}] + \sum_{\omega \in S} \pi(\omega)(f_\Upsilon(\omega) - f(\omega))\tilde{h}(\omega)
\geq \E_\pi[\tilde{f}\tilde{h}] - \log(2) \sum_{\omega \in S} \pi(\omega)f_\Upsilon(\omega)
\geq \frac{1}{2} \Ent_\pi(f) - \log(2) \sum_{\omega \in S} \pi(\omega)f_\Upsilon(\omega).
\]
On the other hand, using the definition of \(f_\Upsilon, \mathcal{R}_{f,\Omega,\Upsilon}\), and relation (4), we get
\[
\sum_{\omega \in S} \pi(\omega)f_\Upsilon(\omega) \leq \sum_{\omega \in \text{Im } \mathcal{R}_{f,\Omega,\Upsilon}} \sum_{\omega : \omega \neq \omega'} \pi(\omega)f(\omega') \text{ dist}(\omega,\omega') \leq \frac{1}{8\gamma} \sum_{\omega \in \text{Im } \mathcal{R}_{f,\Omega,\Upsilon}} \pi(\omega)f(\omega).
\]
Pick a subset \(T \subset \{ \omega : f_\Upsilon(\omega) > f(\omega) \}\) of cardinality \(|\text{Im } \mathcal{R}_f|\) such that \(\mathcal{R}_f(T) = \text{Im } \mathcal{R}_f\). Observe that \(f(\omega) \leq \frac{1}{4\gamma} f(\mathcal{R}_f(\omega)) \leq \frac{1}{16\gamma} f(\mathcal{R}_f(\omega))\) for every \(\omega \in T\). Using this, we can write
\[
\frac{1}{8\gamma} \sum_{\omega \in \text{Im } \mathcal{R}_{f,\Omega,\Upsilon}} \pi(\omega)f(\omega) = \frac{1}{4\gamma} \sum_{\omega \in \text{Im } \mathcal{R}_{f,\Omega,\Upsilon}} \pi(\omega)f(\omega) - \frac{1}{8\gamma} \sum_{\omega \in T} \pi(\mathcal{R}_f(\omega)) f(\mathcal{R}_f(\omega))
\leq \frac{1}{4\gamma} \sum_{\omega \in \text{Im } \mathcal{R}_{f,\Omega,\Upsilon}} \pi(\omega)f(\omega) - 2 \sum_{\omega \in T} \pi(\omega)f(\omega)
\leq \frac{1}{2} \E_\pi[\tilde{f}\tilde{h}],
\]
where

\[ h(\omega) = \begin{cases} 
\frac{1}{2\pi}, & \text{if } \omega \in \text{Im} Rf; \\
-4, & \text{if } \omega \in T; \\
0, & \text{otherwise.}
\end{cases} \]

Using that \( \frac{\pi(\text{Im} Rf)}{\pi(T)} \leq \gamma \), it is easy to check that \( \mathbb{E}_\pi[e^h] \leq 1 \). Thus, by the variational formula of the entropy [24, Lemma 3.15], we deduce that \( \mathbb{E}_\pi[fh] \leq \text{Ent}_\pi(f) \). Using this, together with (5) and (6), we finish the proof. \( \square \)

3. A COMPARISON TECHNIQUE FOR MLSI

The goal of this section is to prove Theorem 1.2. A crucial role in comparison techniques for Markov chains is played by the canonical path method. In its most general setting, we are given a collection of paths \( \mathcal{P} \) on \((\Omega, \pi)\) and a collection of non-negative weights \( \mathcal{W} = (w_P)_{P \in \mathcal{P}} \) indexed over the paths, and would like to bound the weighted sum

\[
\sum_{P \in \mathcal{P}} w_P \left( f(P[\text{len}(P)]) - f(P[0]) \right) \log \frac{f(P[\text{len}(P)])}{f(P[0])}
\]

from above in terms of \( \mathcal{E}_\pi(\log f, f) \). Note that, unlike in the case of squares of differences which are dealt with in the context of the Poincaré or log-Sobolev inequalities, the expression

\[
(f(P[\text{len}(P)]) - f(P[0])) \log \frac{f(P[\text{len}(P)])}{f(P[0])}
\]

does not split into corresponding quantities for adjacent points in the path, unless some assumptions on \( f \) are imposed. Indeed even in the situation when \( \text{len}(P) = 2 \), the above quantity can be arbitrarily large compared to

\[
(f(P[1]) - f(P[0])) \log \frac{f(P[1])}{f(P[0])} + (f(P[2]) - f(P[1])) \log \frac{f(P[2])}{f(P[1])}
\]

(for example, taking \( f(P[2]) = 1, f(P[0]) = \varepsilon \) and \( f(P[1]) = (\log 1/\varepsilon)^{-1} \), we clearly get that \( (f(P[2]) - f(P[0])) \log \frac{f(P[2])}{f(P[0])} = \Theta(\log 1/\varepsilon) \) while \( (f(P[1]) - f(P[0])) \log \frac{f(P[1])}{f(P[0])} + (f(P[2]) - f(P[1])) \log \frac{f(P[2])}{f(P[1])} = \Theta(\log \log 1/\varepsilon) \) when \( \varepsilon \to 0 \).

However, when the function \( f \) is \( r \)-regular in the sense introduced in this paper, the following simple estimate holds:

**Lemma 3.1.** Let \( S \) be a finite set, \( (x_i)_{0 \leq i \leq T} \) be a sequence of elements (not necessarily distinct) in \( S \) and let \( f : S \to \mathbb{R}_+ \) be a function such that \( \max \left( \frac{f(x_i)}{f(x_{i-1})}, \frac{f(x_{i+1})}{f(x_i)} \right) \leq r, 1 \leq i \leq T, \) for some \( r \geq 2. \) Then

\[
(f(x_T) - f(x_0)) \log \frac{f(x_T)}{f(x_0)} \leq C \left( 1 + (T - 1)^2 \log(r) \right) \sum_{t=1}^{T} (f(x_t) - f(x_{t-1})) \log \frac{f(x_t)}{f(x_{t-1})},
\]

where \( C > 0 \) is a universal constant.

**Proof.** Note that if \( T = 1 \) then there is nothing to prove so we assume that \( T > 1 \). Without loss of generality, we may also assume that \( f(x_T) > f(x_0) \) and that \( f(x_T) = \max_{t=0,\ldots,T} f(x_t) \). Indeed, if that was not the case, then we would let \( t_0 \) be such that \( f(x_{t_0}) = \max_{t=0,\ldots,T} f(x_t) \), then write

\[
(f(x_T) - f(x_0)) \log \frac{f(x_T)}{f(x_0)} \leq (f(x_{t_0}) - f(x_0)) \log \frac{f(x_{t_0})}{f(x_0)}.
\]

and work with the truncated sequence \((x_i)_{0 \leq i \leq t_0} \).
Define $H$ as the collection of all indices $t \in [T]$ such that
\[
\frac{f(x_t)}{f(x_{t-1})} - 1 \leq \frac{f(x_T) - f(x_0)}{2T f(x_T)},
\]
and denote by $H^c$ the complement of $H$ in $[T]$. Observe that
\[
\sum_{t \in H} (f(x_t) - f(x_{t-1})) \leq f(x_T) \sum_{t \in H} \left( \frac{f(x_t)}{f(x_{t-1})} - 1 \right)
\leq \frac{1}{2} (f(x_T) - f(x_0))
= \frac{1}{2} \left( \sum_{t \in H} (f(x_t) - f(x_{t-1})) + \sum_{t \in H^c} (f(x_t) - f(x_{t-1})) \right),
\]
whence
\[
f(x_T) - f(x_0) \leq 2 \sum_{t \in H^c} (f(x_t) - f(x_{t-1})).
\]
Therefore, denoting $\delta := \frac{f(x_T)}{f(x_0)}$, we can write
\[
(f(x_T) - f(x_0)) \log \frac{f(x_T)}{f(x_0)} \leq 2 \log(\delta) \sum_{t \in H^c} (f(x_t) - f(x_{t-1}))
\leq \frac{2 \log(\delta)}{\log \left( 1 + \frac{(1-\delta^{-1})}{2T} \right)} \sum_{t \in H^c} (f(x_t) - f(x_{t-1})) \log \frac{f(x_t)}{f(x_{t-1})},
\]
where we used that $\frac{f(x_t)}{f(x_{t-1})} \geq 1 + \frac{(1-\delta^{-1})}{2T}$ when $t \in H^c$. Now using that $\log x \geq \frac{1}{2}(x-1)$ when $1 \leq x \leq 2$, we get
\[
(f(x_T) - f(x_0)) \log \frac{f(x_T)}{f(x_0)} \leq 8T \frac{\log(\delta)}{(1-\delta^{-1})} \sum_{t \in H^c} (f(x_t) - f(x_{t-1})) \log \frac{f(x_t)}{f(x_{t-1})}.
\]
It remains to note that the function $s \to \frac{\log(s)}{(1-s^{-1})}$, $s \geq 1$, is increasing in $s$ and use that $\delta \leq r^T$ to finish the proof.

With this lemma in hand, the proof of Theorem \ref{thm:1.2} will easily follow.

**Proof of Theorem \ref{thm:1.2}** Fix $r \geq 2$, $f \in \mathcal{R}(Q,r)$, and let $\mathcal{W}$ be a $(Q,\tilde{Q})$-flow. First recall the following characterization of entropy (see \cite[Problem 3.13a]{[24]}),
\[
\text{Ent}_{\pi}(f) = \inf_{t \geq 0} \mathbb{E}_{\pi} \left[ f \log f - f \log t - f + t \right]
= \inf_{t \geq 0} \sum_{\omega \in \Omega} \left( f(\omega) \log f(\omega) - f(\omega) \log t - f(\omega) + t \right) \pi(\omega)
\]
(with the corresponding formula for $\text{Ent}_{\tilde{\pi}}(f)$), and note that $x \log x - x \log y - x + y \geq 0$ for any $x, y > 0$. The term-wise comparison and the assumption $\pi(\omega) \leq a \tilde{\pi}(\omega)$ then yields $\text{Ent}_{\pi}(f) \leq a \text{Ent}_{\tilde{\pi}}(f)$. It remains to compare the two Dirichlet forms $\mathcal{E}_\pi$ and $\tilde{\mathcal{E}}_\pi$ associated with $(\pi,Q)$ and $(\tilde{\pi},\tilde{Q})$ respectively. To this aim, we write
\[
\tilde{\mathcal{E}}_\pi(f, \log f) = \frac{1}{2} \sum_{\omega, \omega' \in \Omega} \tilde{\pi}(\omega) \tilde{Q}(\omega, \omega')(f(\omega) - f(\omega')) \log \frac{f(\omega)}{f(\omega')}
= \frac{1}{2} \sum_{\omega, \omega' \in \Omega} \sum_{P \in P_{\omega, \omega'}} \mathcal{W}(P) (f(P[T]) - f(P[0])) \log \frac{f(P[T])}{f(P[0])},
\]
where we denoted by $T = T(P)$ the length of a path $P$. Applying Lemma 3.1 we get for some universal constant $C$ that

$$\tilde{\mathcal{E}}(f, \log f) \leq C \sum_{\omega, \omega' \in \omega} \sum_{P \in \mathcal{P}_{\omega, \omega'}} W(P)(1 + (T - 1)^2 \log(r)) \sum_{t=1}^{T} (f(x_t) - f(x_{t-1})) \log \frac{f(x_t)}{f(x_{t-1})} \leq 2C A(W, r) \mathcal{E}(f, \log f),$$

where $A(W, r)$ is given by [3]. Putting together the above estimates, we finish the proof. \(\square\)

4. MLSI for the Switch Chain

4.1. Preliminaries. In this section, we establish an optimal Modified log-Sobolev Inequality for the switch chain on regular bipartite graphs. We start by considering a lower bound for the MLSI constant.

**Proposition 4.1** (Lower bound for the MLSI constant). Let $2 \leq d \leq \frac{n}{2}$. The modified log-Sobolev constant of $(\Omega_n^B, \pi_u, Q_u)$ is at least $cnd$, for some universal constant $c > 0$.

**Proof.** Denote by $\alpha$ the optimal MLSI constant, so that

$$\alpha = \sup \frac{\text{Ent}_{\pi_u}(f)}{\mathcal{E}_{\pi_u}(f, \log f)},$$

where the supremum is taken over all functions $f : \Omega_n^B \to \mathbb{R}_+$. To obtain the required lower bound on $\alpha$, we shall use a test function. Define $f : \Omega_n^B \to \mathbb{R}_+$ as

$$f(G) = \begin{cases} 2, & \text{if } (1,1) \text{ is an edge in } G; \\ 1, & \text{otherwise}. \end{cases}$$

Note that given $G \in \Omega_n^B$ with $(1,1)$ not as an edge, there are at most $d^2$ adjacent graphs $G'$ for $G$ having the edge $(1,1)$. Using this, we can write

$$\mathcal{E}_{\pi_u}(f, \log f) = \frac{\log 2}{2} \sum_{G \in \Omega_n^B} \sum_{G' \sim G, f(G') = 2} \pi_u(G)Q_u(G, G') = \frac{\log 2}{2} \sum_{G \in \Omega_n^B} \pi_u(G) \left|\left\{G' \sim G : f(G') = 2\right\}\right| \leq \frac{2\log 2}{n^2}.$$

On the other hand, it follows from $d$-regularity that

$$\left|\left\{G \in \Omega_n^B : f(G) = 2\right\}\right| = \frac{d}{n} |\Omega_n^B|,$$

whence $\mathbb{E}_{\pi_u} f = \frac{d}{n}$, and

$$\text{Ent}_{\pi_u}(f) = \frac{n - d}{n} \log \frac{n}{n + d} + \frac{2d}{n} \log \frac{2n}{n + d} \geq \frac{c'd}{n}$$

for some universal constant $c' > 0$.

Putting these estimates together, we deduce that

$$\alpha \geq \frac{c'nd}{2\log 2},$$

and finish the proof. \(\square\)
We further note that the lower bound Ω(nd) for the MLSI constant can be obtained indirectly, by bounding the mixing time of the switch chain by Ω(nd log n), and applying relation (1). We include this alternative argument, which also shows that our mixing time upper bound is sharp, for completeness.

**Proposition 4.2 (A lower bound for the mixing time).** There are universal constants C, c > 0 with the following property. Let n ≥ C and let 2 ≤ d ≤ √n. Then the total variation mixing time t_{mix}(Q_u, 1/4) of the switch chain (Ω^B_n(d), π_u, Q_u) is bounded below by cnd log n.

**Proof.** We will assume in the proof that n is sufficiently large. In order to derive a lower bound on the mixing time, we will make use of a distinguishing statistic. We start our chain with a graph G_0 which contains all edges of the form (i, i), 1 ≤ i ≤ n. Let T be a positive integer parameter and let G_1, ..., G_T be the steps of the switch chain starting at G_0. For each 0 ≤ t ≤ T, denote by (ξ_i,t)_{1≤i≤n} the Bernoulli variables indicating the “diagonal” edges in the graph G_t, i.e ξ_i,t = 1 whenever the edge (i, i) is present in G_t. Given 0 ≤ t ≤ T − 1 and 1 ≤ i ≤ n, note that conditioned on a realization of G_t with ξ_i,t = 1, we have ξ_i,t+1 = 1 with conditional probability at least 1 − 2^{−nd}. Thus, we have

\[ E[ξ_{i,t+1}] ≥ (1 − 2^{−nd})E[ξ_{i,t}] \]

Iterating this inequality, we get

\[ E[ξ_{i,T}] ≥ (1 − 2^{−nd})^T, \]

for every i = 1, ..., n.

Now given 1 ≤ i ≠ j ≤ n, and conditioned on (ξ_i,t, ξ_j,t), it is not difficult to check that ξ_{i,t+1,ξ_j,t+1} = 1 with conditional probability at most

\[
\begin{cases}
\frac{c}{n^2} & \text{if } ξ_{i,t}ξ_{j,t} = 0 \\
1 − \frac{4}{nd} + \frac{c}{n^2} & \text{if } ξ_{i,t}ξ_{j,t} = 1,
\end{cases}
\]

for some universal constant c. Putting these estimates together, we get after iteration that

\[ E[ξ_{i,T}ξ_{j,T}] ≤ (1 − 4^{−nd})^T + c'd \]

for some universal constant c'.

We deduce from the above that for every 1 ≤ i ≠ j ≤ n we have

\[ \text{Cov}(ξ_{i,T}, ξ_{j,T}) ≤ \frac{c'd}{n}. \]

Denoting D_T = ∑_{i=1}^{n} ξ_{i,T}, we deduce from the above that

\[ E[D_T] ≥ n(1 − 2^{−nd})^T \quad \text{and} \quad \text{Var}(D_T) ≤ Cnd, \]

for some universal constant C.

On the other hand, the d–regularity immediately implies that the expected number of “diagonal” edges in a uniform random graph on Ω^B_n(d) is d while the variance of that number is at most nd. It remains to apply [19, Proposition 7.9] to finish the proof.

Following the approach from [23], we develop a comparison procedure between (Ω^B_n(d), π_u, Q_u) and the switch chain on multigraphs generated according to the configuration model, which in turn can be compared to the random transposition chain on the set of permutations. We denote
by \( \Omega_n^{BC}(d) \) the set of all \( d \)-regular bipartite multigraphs on \([n^{(d)}] \sqcup [n^{(r)}]\) and equip it with the probability measure \( \pi_{BC} \) induced by the configuration model. Recall that for any \( G \in \Omega_n^{BC}(d) \)

\[
\pi_{BC}(G) = \frac{(d!)^{2n}}{(nd)! \prod_{1 \leq i,j \leq n} \text{mult}_G(i,j)!},
\]

where \( \text{mult}_G(i,j) \) denotes the multiplicity of the edge \((i,j)\) in \( G \). When \( n \) is large enough, we have the following estimate (see [16, Theorem 6.2])

\[
\frac{1}{2} e^{-\frac{(d-1)^2}{2}} \leq \pi_{BC}(\Omega_n^{B}(d)) \leq 2e^{-\frac{(d-1)^2}{2}}.
\]

The generator \( Q_c \) of the switch chain on \( \Omega_n^{BC}(d) \) is defined for any \( G_1, G_2 \in \Omega_n^{BC}(d) \) by

\[
Q_c(G_1, G_2) := \begin{cases} 
    \frac{\text{mult}_{G_1(i,j)} \text{mult}_{G_1(i',j')}}{nd(n^d-1)/2}, & \text{if } G_2 \in \mathcal{N}(G_1) \text{ is obtained from } G_1 \\
    -\sum_{G' \in \mathcal{N}(G_1)} Q_c(G_1, G'), & \text{if } G_1 = G_2; \\
    0, & \text{otherwise}.
\end{cases}
\]

In the above notation, \( (i,i',j,j') \) denotes the switching destroying the edges \((i,j)\) and \((i',j')\), and replacing them with \((i,j')\) and \((i',j)\).

It is known that the random transposition chain on the set of permutations of \([nd]\) satisfies the Modified log-Sobolev Inequality with constant \( nd \) [12]. Exploiting the intimate relation between the random transposition chain and the switch chain on the configuration model, it is easy to derive the following Modified log-Sobolev Inequality for the latter (see [23, Proposition 2.2] for details).

**Proposition 4.3.** For any \( 2 \leq d \leq n/2 \), \((\Omega_n^{BC}(d), \pi_{BC}, Q_c)\) satisfies the Modified log-Sobolev Inequality with constant \( cnd \) for some universal constant \( c > 0 \).

### 4.2. Simple paths and \( s \)-neighborhoods

For the remainder of the paper, fix \( 2 \leq d \leq n/2 \). We will use the last proposition to construct an auxiliary Markov chain on \( \Omega_n^{B}(d) \) satisfying the Modified log-Sobolev Inequality with a constant of order \( O(nd) \), and then use this auxiliary chain with the comparison Theorem [1.2]. In order to verify that the auxiliary chain does satisfy the MLSI with a satisfactory constant, we will construct for any given positive function \( f \) on \( \Omega_n^{B}(d) \) an appropriate extension \( \hat{f} \) to the set \( \Omega_n^{BC}(d) \). Following [23], we interpret \( \Omega_n^{B}(d) \) as a boundary for \( \Omega_n^{BC}(d) \setminus \Omega_n^{B}(d) \) and define \( \hat{f} \) as a “relative” of the standard harmonic extension of \( f \). While the harmonic extension is constructed by launching a random walk from the given point in \( \Omega_n^{BC}(d) \setminus \Omega_n^{B}(d) \) and averaging the values of \( f \) where it hits the boundary, the strategy developed in [23] is to construct specific “direct” paths to the boundary \( \Omega_n^{B}(d) \) which would make the result of the averaging tractable. As these paths crucially depend on some properties of the corresponding starting multigraph, let us start by partitioning \( \Omega_n^{BC}(d) \).

**Definition 4.4** (A partition of \( \Omega_n^{BC}(d) \), [23]). Let \( \lfloor \log \log n \rfloor \). We write

\[
\Omega_n^{BC}(d) = \bigsqcup_{k=0}^{m} \text{Cat}_{n,d}(k) \sqcup \mathcal{U}_{n,d}(m),
\]

where \( \mathcal{U}_{n,d}(m) := \text{Cat}_{n,d}([0,m]^c) \), \( \text{Cat}_{n,d}([0,m]) := \bigsqcup_{k=0}^{m} \text{Cat}_{n,d}(k) \), and \( \text{Cat}_{n,d}(k) \) is defined as the set of multigraphs \( G \in \Omega_n^{BC}(d) \) which satisfy all of the following:

- \( G \) has exactly \( k \) multiedges of multiplicity 2;
- None of those multiedges are incident to one another;
- \( G \) has no multiedges of multiplicity three or greater.
Note that with this definition, we have $\Omega_B^n(d) = \text{Cat}_{n,d}(0)$. Multigraphs in $\text{Cat}_{n,d}([1,m])$ have a simple structure allowing to build “direct” paths from them to the boundary. The paths are formed by the simple switchings which destroy the multiedges one at a time while not “interfering” with one and another. Thus, for every $G' \in \text{Cat}_{n,d}(k)$, we construct a unique family of paths from $G'$ to $\Omega_B^n(d)$ of length $k$ where each step of the path destroys a multiple edge. Such paths will be called “simple paths”, and are formally defined as follows.

**Definition 4.5** (Simple paths, [23]). Given $1 \leq k \leq m$ and $G' \in \text{Cat}_{n,d}(k)$, denote by $\{ (i_s,j_s) \}_{1 \leq s \leq k}$ the multiedges of $G'$ of multiplicity 2 arranged in increasing order of $(i_s)_{1 \leq s \leq k}$. A simple path $P$ starting at $G'$ is a path of length $k$ where $P[t+1]$ is obtained from $P[t]$ via the simple switching $(i_{t+1}, i'_{t+1}, j_{t+1}, j'_{t+1})$, such that $i_{t+1} \in [n(t)]$, $j_{t+1} \in [n(t)]$ satisfy all of the following conditions:

- For every $0 \leq t < k$, we have $i_{t+1} \not\in \{ i_s \} _{1 \leq s \leq k}$, $j_{t+1} \not\in \{ j_s \} _{1 \leq s \leq k}$, and all $(i'_s)_{1 \leq s \leq k}$ (resp. $(j'_s)_{1 \leq s \leq k}$) are pairwise distinct.
- For every $0 \leq t < k$, $\text{mult}_{G'}(i_{t+1}, j_{t+1}) = \text{mult}_{G'}(i_{t+1}, j'_{t+1}) = 0$.

It can be verified that (with our choice of $m$) simple paths exist for every $G' \in \text{Cat}_{n,d}(k)$, and each simple path is uniquely determined by its starting point and endpoint (see [23] Section 3) for details. Note that the endpoint of a simple path belongs to $\Omega_B^n(d)$.

**Definition 4.6** ($s$–neighborhood, [23]). The set of all endpoints of simple paths starting at $G' \in \Omega_B^n(d) \setminus \Omega_B^n(d)$ will be denoted by $\mathcal{SN}(G')$ and called the $s$–neighborhood of the graph.

Additionally, when $G' \in \Omega_B^n(d)$, we set $\mathcal{SN}(G') := \{ G' \}$. When $G' \in \text{Cat}_{n,d}(k)$, $1 \leq k \leq m$, the $s$–neighborhood of $G'$ satisfies (see [23] Section 3):

$$|\mathcal{SN}(G)| \in \left[ \frac{(nd)^k}{2}, (nd)^k \right].$$

We will also need to control the number of $s$–neighborhoods which contain a given simple graph: for any given $G \in \Omega_B^n(d)$ we have [23] Section 3]

$$|\{ G' \in \text{Cat}_{n,d}(k) : G \in \mathcal{SN}(G') \}| \leq \frac{(nd)^k}{k!} (d-1)^{2k}, \quad 1 \leq k \leq m.$$  

In the sequel, we will use the notation $T = (G_1, G'_1, G_2, G'_2)$ for any 4–tuple of graphs such that $G_1 \sim G'_2$ are in $\Omega_B^n(d)$, $G_1 \in \mathcal{SN}(G'_1)$, and $G_2 \in \mathcal{SN}(G'_2)$.

**4.3. Matchings and connections.** In order to make use of the comparison Theorem 1.2 we will need to construct a special family of paths between elements of $\Omega_B^n(d)$. As our auxiliary Markov chain on $\Omega_B^n(d)$ will be “inherited” from the switch chain on $\Omega_B^n(d)$, that family of paths will be determined by the structure of $\Omega_B^n(d)$. Again, we shall rely on the constructions from [23]. It was observed in [23] that for a large proportion of adjacent multigraphs in $\Omega_B^n(d)$ there is a natural bijective mapping between their respective $s$–neighborhoods.

**Definition 4.7** (Perfect pairs, [23]). Given $1 \leq k \leq m$, a pair of adjacent graphs $(G_1, G_2) \in \text{Cat}_{n,d}(k) \times \text{Cat}_{n,d}(k)$ is referred to as a perfect pair if the switching $(i, i', j, j')$ used to obtain $G_2$ from $G_1$ satisfies the following conditions:

- Vertices $i, i', j, j'$ are not incident to any multiedges.
- Vertices $i, i', j, j'$ are not adjacent to vertices incident to some multiedges.

Denote by $\mathcal{C}_{n,d}(k)$ the set of perfect pairs from $\text{Cat}_{n,d}(k) \times \text{Cat}_{n,d}(k)$ and set $\mathcal{C}_{n,d} := \bigsqcup_{k=1}^{m} \mathcal{C}_{n,d}(k)$.

**Proposition 4.8** (Matchings, [23] Section 5). Let $G'_1 \in \text{Cat}_{n,d}([1,m])$. Then the following assertions hold.
Let $G'_2 \in \text{Cat}_{n,d}([1,m])$ be such that $(G'_1, G'_2) \in \mathcal{C}_{n,d}$. Then there is a bijective mapping $\psi_{G'_1,G'_2} : SN(G'_1) \to SN(G'_2)$ such that $\psi_{G'_1,G'_2}(\psi_{G'_1,G'_2}(G)) = G$ for all $G \in SN(G'_1)$ and $G$ is adjacent to $\psi_{G'_1,G'_2}(G)$ for all $G \in SN(G'_1)$.

Let $G_1 \in SN(G'_1)$ and let $G_2 \in \Omega^B_n(d)$ be adjacent to $G_1$. Then there exists at most one multigraph $G'_2 \in \text{Cat}_{n,d}([1,m])$ such that $(G'_1, G'_2) \in \mathcal{C}_{n,d}$ and $\psi_{G'_1,G'_2}(G_1) = G_2 \in SN(G'_2)$.

The above proposition will play a crucial role in our comparison procedure as it associates a family of paths of minimal length 1 for most pairs of adjacent multigraphs. When $T = (G_1, G'_1, G'_2, G_2')$ is such that $(G'_1, G'_2) \in \mathcal{C}_{n,d}$, $G'_1 \in SN(G'_1)$, and $G_2' = \psi_{G'_1,G'_2}(G_1)$, we let $\mathcal{P}_T := (G_1, \psi_{G'_1,G'_2}(G_1) = G_2')$ to be the path of length one from $G_1$ to $G_2$.

For the pairs of multigraphs which are not perfect, a different construction is required. Let us define

$$I_{n,d}(m) := \{(G'_1, G'_2) : G'_1 \sim G'_2, G'_1, G'_2 \in \text{Cat}_{n,d}([0,m]),
(G'_1, G'_2) \notin \Omega^B_n(d) \times \Omega^B_n(d) \} \setminus \bigcup_{k=1}^m \mathcal{C}_{n,d}(k),$$

and consider

$$\mathcal{H} := \{(G_1, G'_1, G'_2, G_2') : (G'_1, G'_2) \in I_{n,d}(m), G_1 \in SN(G'_1), G_2 \in SN(G'_2)\}.$$

We recall that in a tuple $T = (G_1, G'_1, G'_2, G_2')$, we may possibly have that $G_1 = G'_1$ or $G_2 = G'_2$. One of the technical contributions in [23] consists in constructing for every given tuple $T = (G_1, G'_1, G'_2, G_2') \in \mathcal{H}$ a path $\mathcal{P}_T$ in $\Omega^B_n(d)$ starting at $G_1$ and ending at $G_2$ (called a connection) having a set of special properties useful in the context of functional inequalities on $\Omega^B_n(d)$. The definition of a connection is very technical (see [23 Definition 6.11]) and we prefer not to include it in this paper. Rather, we provide a proposition which establishes existence of certain paths satisfying properties crucial to us (we refer the reader to [23 Sections 6-7] for a comprehensive treatment):

**Proposition 4.9 (Connections, [23]).** Assuming $n$ is sufficiently large, there exists a collection of paths in $\Omega^B_n(d)$, $(\mathcal{P}_T)_{T \in \mathcal{H}}$, indexed by $\mathcal{H}$ and satisfying the following conditions.

i. For every $0 \leq k_1, k_2 \leq m$ and every $T = (G_1, G'_1, G'_2, G_2') \in \mathcal{H}$ with $G'_1 \in \text{Cat}_{n,d}(k_1), G'_2 \in \text{Cat}_{n,d}(k_2)$, the path $\mathcal{P}_T$ starts at $G_1$, ends at $G_2$, and is of length at most $C(k_1 + k_2)$ for some universal constant $C > 0$.

ii. Given any adjacent graphs $H \sim H'$ in $\Omega^B_n(d)$, we have

$$\sum_{T = (G_1, G'_1, G'_2, G_2') \in \mathcal{H}} \frac{\pi_{BC}(G'_1)Q_c(G'_1, G'_2)}{|SN(G'_1)||SN(G'_2)|} \text{len}(\mathcal{P}_T)^2 \leq C \frac{(H)Q(H,H')}{\sqrt{n}},$$

for some universal constant $C > 0$.

**Proof.** We define $(\mathcal{P}_T)_{T \in \mathcal{H}}$ as the set of connections [23 Definition 6.11]. The first assertion of the proposition can be deduced from [23, Remark 6.13]. For the second assertion, denote

$$\gamma := \sum_{T = (G_1, G'_1, G'_2, G_2') \in \mathcal{H}} \frac{\pi_{BC}(G'_1)Q_c(G'_1, G'_2)}{|SN(G'_1)||SN(G'_2)|} \text{len}(\mathcal{P}_T)^2.$$
In view of the first part of the proposition, the definitions of \((\pi_{BC}, Q_e)\) and \((\pi_u, Q_u)\), and using (9), we get
\[
\gamma \leq C_1 m^2 \sum_{0 \leq k_1, k_2 \leq m} \sum_{T = (G_1, G_1', G_2, G_2') \in \mathcal{H}} \frac{\pi_{BC}(G_1')Q_e(G_1', G_2')}{(nd)^{k_1+k_2}}
\]
\[
\leq C_2 m^2 \pi_{BC}(\Omega_n^B(d)) \pi_u(H)Q_u(H, H') \sum_{0 \leq k_1, k_2 \leq m} \sum_{T = (G_1, G_1', G_2, G_2') \in \mathcal{H}} \frac{1}{2^k_1 (nd)^{k_1+k_2}},
\]
where \(C_1, C_2 > 0\) are universal constants. By combining [29, Proposition 6.24], [29, Proposition 6.25] (while bounding the parameter \(r\) there by \(2m\)) and [29, Proposition 6.26], we get
\[
\gamma \leq \left(\frac{C_d m}{n}\right)^C \pi_{BC}(\Omega_n^B(d)) \pi_u(H)Q_u(H, H'),
\]
where the constant \(C_d\) depends only on \(d\) and \(C' > 0\) is a universal constant. By the choice of \(m\), we get the result provided \(n\) is large enough. \(\square\)

4.4. The function extension and the auxiliary chain. Now, we define the aforementioned auxiliary chain to compare it with the switch chain on the configuration model. Now, given \(G_1' \sim G_2'\) in \(\text{Cat}_{n,d}([0, m])\) and \(G_1 \in \mathcal{SN}(G_1'), G_2 \in \mathcal{SN}(G_2'),\) set
\[
\beta_{G_1',G_2'}(G_1, G_2) := \begin{cases} \frac{1}{|\mathcal{SN}(G_1')|} & \text{if } G_1', G_2' \in \text{Cat}_{n,d}([1, m]) \text{ and } (G_1', G_2') \in C_{n,d}, \\ 1 & \text{otherwise.} \end{cases}
\]
Note that for any pair \(G_1' \sim G_2'\) in \(\text{Cat}_{n,d}([0, m]),\)
\[
\sum_{G_1 \in \mathcal{SN}(G_1'), G_2 \in \mathcal{SN}(G_2')} \beta_{G_1',G_2'}(G_1, G_2) = 1,
\]
and that for any 4-tuple \((G_1, G_1', G_2, G_2')\), \(\beta_{G_1',G_2'}(G_1, G_2) = \beta_{G_2',G_1}(G_2, G_1').\) We define a Markov generator \(\tilde{Q}_u\) on \((\Omega_n^B(d), \pi_u)\) by setting for every \(G_1 \neq G_2 \in \Omega_n^B(d),\)
\[
\tilde{Q}_u(G_1, G_2) := \frac{1}{4\pi_u(G_1)} \sum_{G_1', G_2' \in \text{Cat}_{n,d}([0, m]): \ G_1' \sim G_2'} \pi_{BC}(G_1')Q_e(G_1', G_2') \beta_{G_1',G_2'}(G_1, G_2),
\]
and taking \(\tilde{Q}_u(G_1, G_1) := - \sum_{G_2 \neq G_1} \tilde{Q}_u(G_1, G_2).\) Note that \(\tilde{Q}_u(G_1, G_2) = \tilde{Q}_u(G_2, G_1)\) for all \(G_1, G_2.\) Further, for every \(G_1 \in \Omega_n^B(d),\) we get in view of (9), (11), and (8)
\[
\sum_{G_2: G_2 \neq G_1} \tilde{Q}_u(G_1, G_2) \leq \frac{1}{4\pi_u(G_1)} \sum_{G_1', G_2' \in \text{Cat}_{n,d}([0, m]): \ G_1' \sim G_2'} \pi_{BC}(G_1')Q_e(G_1', G_2') \sum_{G_2 \in \mathcal{SN}(G_2')} \beta_{G_1',G_2'}(G_1, G_2)
\]
\[
\leq \frac{1}{4\pi_u(G_1)} \sum_{G_1' \in \text{Cat}_{n,d}([0, m]): \ G_1' \in \mathcal{SN}(G_1')} \pi_{BC}(G_1') \frac{|\mathcal{SN}(G_1')|}{|\mathcal{SN}(G_1')|} \leq \frac{m}{k!} \sum_{k=0}^{m} \frac{(d-1)^{2k_1} e^{-\frac{(d-1)^2}{2}}}{2^k} \leq 1.
\]
Thus, the generator $\tilde{Q}_u$ is well defined and is reversible with respect to $\pi_u$. Next, we prove that the above auxiliary chain satisfies the Modified log-Sobolev Inequality with constant of order $O_d(n)$. For the rest of the subsection, we denote by $\Phi : \mathbb{R}^2_+ \to \mathbb{R}$ the function defined by $\Phi(x, y) = (x - y) \log \frac{x}{y}$ (note that the function is convex in two variables). We first need the following lemma.

**Lemma 4.10.** Let $(\Omega, Q, \pi)$ be a reversible Markov chain, and let $f$ be a positive function on $\Omega$, with $\mathbb{E}_\pi f = 1$ and $f(\omega) \geq \delta$ for all $\omega \in \Omega$ and some parameter $\delta \in (0, 1/2]$. Then

$$\sum_{\omega \in \Omega} \pi(\omega) \left((f(\omega) - 1) \log f(\omega)\right) \leq C' \left| \log \delta \right| \text{Ent}_{\pi} f$$

for a universal constant $C' > 0$.

**Proof.** We write

$$\text{Ent}_{\pi} f = \sum_{\omega \in \Omega} \left(1 - f(\omega) + f(\omega) \log f(\omega)\right) \pi(\omega),$$

where $1 - f(\omega) + f(\omega) \log f(\omega) \geq 0$ for all $\omega$, and compare the terms $1 - f(\omega) + f(\omega) \log f(\omega)$ and $(f(\omega) - 1) \log f(\omega)$. We consider several cases.

- $f(\omega) \in [1/2, 10]$. We have $1 - f(\omega) + f(\omega) \log f(\omega) \geq \frac{1}{6}(f(\omega) - 1)^2$ while at the same time $(f(\omega) - 1) \log f(\omega) \leq 2(f(\omega) - 1)^2$. Thus, in this regime we have

$$1 - f(\omega) + f(\omega) \log f(\omega) \geq \frac{1}{12} (f(\omega) - 1) \log f(\omega).$$

- $f(\omega) > 10$. Then

$$1 - f(\omega) + f(\omega) \log f(\omega) \geq \frac{1}{2}f(\omega) \log f(\omega),$$

$$(f(\omega) - 1) \log f(\omega) \leq f(\omega) \log f(\omega),$$

implying that

$$1 - f(\omega) + f(\omega) \log f(\omega) \geq \frac{1}{2} (f(\omega) - 1) \log f(\omega).$$

- $f(\omega) < 1/2$. In this range we have $(f(\omega) - 1) \log f(\omega) \leq -\log f(\omega)$ whereas $1 - f(\omega) + f(\omega) \log f(\omega) \geq \frac{1}{6}$. In view of the assumptions on $f$, this implies

$$1 - f(\omega) + f(\omega) \log f(\omega) \geq \frac{1}{8|\log \delta|} (f(\omega) - 1) \log f(\omega).$$

Combining the above estimates, we get the result. \qed

**Proposition 4.11** (The MLSI for the auxiliary chain). For every fixed $d \geq 2$ there are $n_d, C_d > 0$ depending only on $d$ such that, assuming $n \geq n_d$, $(\Omega_n^B(d), \pi_u, \tilde{Q}_u)$ satisfies the Modified log-Sobolev Inequality with constant $C_d n$.

**Proof.** We will deduce the result by an appropriate comparison with the switch chain on the configuration model. We shall verify that for any positive function $f : \Omega_n^B(d) \to \mathbb{R}_+$,

$$\text{Ent}_{\pi_u} f \leq c_d n \mathcal{E}_{\pi_u}(f, \log f),$$

for some appropriate constant $c_d$. Note that in view of Lemma 2.1, we can assume without loss of generality that $\mathcal{E}_{\pi_u}(f) = 1$ and $f(G) \geq e$ for all $G \in \Omega_n^B(d)$, for some universal constant $e > 0$. Using the characterization of entropy in [7], we get that for any extension $\tilde{f}$ of $f$ to $\Omega_n^{BC}(d)$,

$$\text{Ent}_{\pi_u}(f) \leq \max_{G \in \Omega_n^B(d)} \frac{\pi_u(G)}{\pi_{BC}(G)} \text{Ent}_{\pi_{BC}}(\tilde{f}) \leq C_d \text{Ent}_{\pi_{BC}}(\tilde{f}),$$
In what follows, we estimate each of the terms above. Using Proposition 4.3, we deduce that
\[
\text{Ent}_{\pi_n}(f) \leq C_d n \mathcal{E}_{\pi_n}(\tilde{f}, \log \tilde{f}),
\]
for any extension \( \tilde{f} \). Here \( C_d > 0 \) is a constant depending only on \( d \). Note that whenever \( f' \sim f \), \( G_1' \sim G_2' \), \( G_1 \sim G_2 \), and \( G_1' \sim G_2' \), we have
\[
\tilde{f}(G') = \sum_{G_1 \in \mathcal{SN}(G')} \beta_{G_1', G_2'}(G_1, G_2) f(G_1) = \sum_{G_1 \in \mathcal{SN}(G')} \beta_{G_1', G_2'}(G_1, G_2) f(G_2).
\]
Using reversibility and the symmetry of \( \Phi \), we can write
\[
\mathcal{E}_{\pi_n}(f, \log f) = \frac{1}{2} \sum_{G_1', G_2' \in \text{Cat}_{n,d}([0, m])} \pi_{BC}(G_1', G_2') \Phi(\tilde{f}(G_1'), \tilde{f}(G_2'))
\]
where by \( W \) we denote the set
\[
W := \{ (G_1', G_2') : G_1' \sim G_2', G_1' \in \text{Cat}_{n,d}([0, m]), G_2' \notin \text{Cat}_{n,d}([0, \infty)) \}.
\]
In what follows, we estimate each of the terms above.

Note that whenever \( G_1', G_2' \in \text{Cat}_{n,d}([0, m]) \) and \( G_1' \sim G_2' \), we have
\[
\tilde{f}(G_1') = \sum_{G_1 \in \mathcal{SN}(G_1')} \beta_{G_1', G_2'}(G_1, G_2) f(G_1) \quad \text{and} \quad \tilde{f}(G_2') = \sum_{G_1 \in \mathcal{SN}(G_2')} \beta_{G_1', G_2'}(G_1, G_2) f(G_2).
\]
Using the last inequality and the definition of \( \tilde{Q}_u \), we deduce
\[
\sum_{G_1', G_2' \in \text{Cat}_{n,d}([0, m])} \pi_{BC}(G_1', G_2') \Phi(\tilde{f}(G_1'), \tilde{f}(G_2')) \leq \sum_{G_1 \in \mathcal{SN}(G_1')} \sum_{G_2 \in \mathcal{SN}(G_2')} \pi_{BC}(G_1') \beta_{G_1', G_2'}(G_1, G_2) \Phi(f(G_1), f(G_2)) \leq 2 \sum_{G_1, G_2 \in \Omega^B_n(d)} \pi_u(G_1, G_2) \Phi(f(G_1), f(G_2)) = 4 \mathcal{E}_{\tilde{Q}_u, \pi_u}(f, \log f).
\]
Further, consider the sum
\[
\sum_{G_1' \in \text{Cat}_{n,d}(\{0, m\}), G_2' \in \text{Cat}_{n,d}(m+1,m+2)} \pi_{BC}(G_1') Q_c(G_1', G_2') \Phi(\tilde{f}(G_1'), \tilde{f}(G_2')).
\]

Note that any given $G_1' \in \text{Cat}_{n,d}(\{0, m\})$ has at most $nd(d-1)^2$ adjacent multigraphs $G_2' \in \text{Cat}_{n,d}(m+1,m+2)$. Using formulas (9) and (10), convexity of $\Phi$, and the definition of $Q_c$, we get
\[
\sum_{G_1' \in \text{Cat}_{n,d}(\{0, m\}), G_2' \in \text{Cat}_{n,d}(m+1,m+2)} \pi_{BC}(G_1') Q_c(G_1', G_2') \Phi(\tilde{f}(G_1'), \tilde{f}(G_2')) \\
\leq \frac{4d}{n} \sum_{G_1' \in \text{Cat}_{n,d}(\{0, m\})} \pi_{BC}(G_1') \Phi(\tilde{f}(G_1'), 1) \\
\leq \frac{4d}{n} \sum_{G_1' \in \text{Cat}_{n,d}(\{0, m\})} \frac{\pi_{BC}(G_1')}{|SN(G_1')|} \sum_{G_1 \in SN(G_1')} \Phi(f(G_1), 1) \\
+ \frac{4d}{n} \sum_{G_1' \in \text{Cat}_{n,d}(\{0, m\})} \frac{\pi_{BC}(G_1')}{|SN(G_1')|} \sum_{G_1 \in SN(G_1')} \Phi(f(G_1), 1) \\
\leq \frac{16d(d-1)^2 m}{n} \left(\sum_{G_1 \in \Omega^{\mu}(d)} \Phi(f(G_1), 1) \right) \pi_u(G_1) \\
\leq \frac{\tilde{C}_d}{n \log n} \text{Ent}_{\pi_u} \tilde{f},
\]

where in the last inequality we applied Lemma 4.10 our definition of $m$ and that $n$ is large enough. Here also $\tilde{C}_d$ is a constant depending only on $d$.

Consider now the sum
\[
\sum_{(G_1', G_2') \in W} \pi_{BC}(G_1') Q_c(G_1', G_2') \Phi(\tilde{f}(G_1'), \tilde{f}(G_2')).
\]

Note that we always have $Q_c(G_1', G_2') \leq 4n^{-2}$ provided $n$ is large enough. Further, $(G_1', G_2') \in W$ only if the graph $G_2'$ is obtained from $G_1'$ by either adding an edge of multiplicity three or introducing a multiedge incident to one of the existing multiedges in the graph. This implies that for every $G_1' \in \text{Cat}_{n,d}(\{0, m\})$, the number of graphs $G_2'$ such that $(G_1', G_2') \in W$, is at most $c'_d m$ for some constant $c'_d$ depending only on $d$. Thus, using formulas (9) and (10), convexity of $\Phi$, and the definition of $\pi_{BC}$, we can write
\[
\sum_{(G_1', G_2') \in W} \pi_{BC}(G_1') Q_c(G_1', G_2') \Phi(\tilde{f}(G_1'), \tilde{f}(G_2')) \\
\leq \frac{4c'_d m}{n^2} \sum_{G_1' \in \text{Cat}_{n,d}(\{0, m\})} \pi_{BC}(G_1') \Phi(\tilde{f}(G_1'), 1) \\
\leq \frac{8c'_d m}{n^2} \sum_{k=0}^{m} \frac{(d-1)^{2k}}{2^k k!} \sum_{G_1 \in \Omega^{\mu}(d)} \pi_{BC}(G_1) \Phi(f(G_1), 1) \\
\leq \frac{16c'_d m}{n^2} \sum_{G_1 \in \Omega^{\mu}(d)} \Phi(f(G_1), 1) \pi_u(G_1) \\
\leq \frac{C''_d m}{n^2} \text{Ent}_{\pi_u} \tilde{f},
\]
where the last inequality follows from Lemma 4.10 with some constant $C_d''$ depending only on $d$.

Combining the above estimates, we obtain

$$
\mathcal{E}_{\pi_{BC}}(\hat{f}, \log \hat{f}) \leq 4 \mathcal{E}_{\tilde{Q}, \pi_u}(f, \log f) + \frac{\tilde{c}_d}{n \log n} \text{Ent}_{\pi_u} f,
$$

whence

$$
\text{Ent}_{\pi_u} f \leq C_d'' n \left( \mathcal{E}_{\tilde{Q}, \pi_u}(f, \log f) + \frac{\tilde{c}_d}{n \log n} \text{Ent}_{\pi_u} f \right),
$$

for some constants $\tilde{c}_d$ and $C_d''$ depending only on $d$. The result follows.

\[ \square \]

4.5. **Proof of Theorem 1.3.** The strategy of the proof is to apply the comparison Theorem 1.2 with the auxiliary chain $(\pi_u, \tilde{Q}_u)$ defined in (11). To this aim, we will use the family of paths introduced in Subsection 4.3 in order to define a $(Q_u, \tilde{Q}_u)$-flow. In what follows, for every 4-tuple $T = (G_1, G_1', G_2, G_2')$ such that $G_1, G_2 \in \text{Cat}_{n,d}([0, m])$, $G_1' \sim G_2'$, and $G_1 \in \mathcal{SN}(G_1'), G_2 \in \mathcal{SN}(G_2')$, we write $P_T$ for

- the [trivial] path of length one from $G_1$ to $G_2$ in the case when $(G_1', G_2') \in \mathcal{C}_{n,d}$ and $G_2 = \psi_{G_1', G_2'}(G_1)$;
- the empty path, when $(G_1', G_2') \in \mathcal{C}_{n,d}$ and $G_2 \neq \psi_{G_1', G_2'}(G_1)$;
- the connection $P_T$ from the statement of Proposition 4.9 when $T \in \mathcal{H}$.

**Definition 4.12.** Consider the two Markov generators $Q_u$ and $\tilde{Q}_u$ on $(\Omega_n^B(d), \pi_u)$ and define $W : \Gamma(Q_u, \tilde{Q}_u) \to [0, 1]$ as follows. Given $G_1, G_2 \in \Omega_n^B(d)$ with $\tilde{Q}_u(G_1, G_2) > 0$ and a valid $Q_u$-path $P$ between $G_1$ and $G_2$, we set

$$
W(P) := \frac{1}{4} \sum_{T = (G_1, G_1', G_2, G_2') : P = P_T} \pi_{BC}(G_1') Q_c(G_1', G_2) \beta_{G_1', G_2}(G_1, G_2).
$$

In view of the definition of $\tilde{Q}_u$ in (11), for every $G_1, G_2 \in \Omega_n^B(d)$ with $\tilde{Q}_u(G_1, G_2) > 0$,

$$
\sum_{P \text{ valid } Q_u\text{-path between } G_1 \text{ and } G_2} W(P) = \frac{1}{4} \sum_{T = (G_1, G_1', G_2, G_2')} \pi_{BC}(G_1') Q_c(G_1', G_2') \beta_{G_1', G_2'}(G_1, G_2)
$$

$$
= \pi_u(G_1) \tilde{Q}_u(G_1, G_2),
$$

so that the weight function is indeed a $(Q_u, \tilde{Q}_u)$-flow. In order to make use of Theorem 1.2, we need to calculate a version of the flow congestion. The following lemma helps in this respect.

**Lemma 4.13.** Consider the two Markov generators $Q_u$ and $\tilde{Q}_u$ on $(\Omega_n^B(d), \pi_u)$, and let $W$ be the $(Q_u, \tilde{Q}_u)$-flow defined above. Let $H, H' \in \Omega_n^B(d)$ be such that $Q_u(H, H') > 0$. Then for any $t \geq 1$,

$$
\sum_{P \in \Gamma(Q_u, \tilde{Q}_u) : (H, H') \in P} W(P) (1 + (\text{len}(P) - 1)^2 t) \leq C (1 + \frac{t}{\sqrt{n}}) \pi_u(H) Q_u(H, H'),
$$

for some universal constant $C > 0$.

**Proof.** Denote

$$
\delta := \sum_{P \in \Gamma(Q_u, \tilde{Q}_u) : (H, H') \in P} W(P) (1 + (\text{len}(P) - 1)^2 t).
$$
In view of the definition of $W$, we have $\delta = \frac{1}{2} \sum_{(H,H') \in \mathcal{P}_T} \delta_T$, where for $T = (G_1, G_1', G_2, G_2')$ we defined

$$\delta_T := \pi_{BC}(G_1')Q_e(G_1', G_2')\beta_{G_1', G_2'}(G_1, G_2)(1 + (\text{len}(\mathcal{P}_T) - 1)^2 t).$$

Now note that whenever $(G_1', G_2') \in \mathcal{C}_{n,d}$ and $(H, H') \in \mathcal{P}_T$, necessarily $G_1 = H$, $G_2 = H'$, and $\text{len}(\mathcal{P}_T) = 1$. Therefore, we have

$$\sum_{T = (H, G_1', H', G'_2)} \delta_T = \sum_{(G_1', G_2') \in \mathcal{C}_{n,d}} \frac{\pi_{BC}(G_1')Q_e(G_1', G_2')}{|\mathcal{SN}(G_1')|} \sum_{H \in \mathcal{SN}(G_1'), H' \in \mathcal{SN}(G'_2)} \frac{1}{|\mathcal{SN}(G_1')|},$$

where we made use of (8). Now applying the second point of Proposition 4.8, we deduce that

$$\sum_{T = (H, G_1', H', G'_2)} \delta_T \leq 2e^{-\frac{(d-1)^2}{2}} \pi_u(H)Q_u(H, H') \sum_{G_1' \in \text{Cat}_{n,d}(1,m); H \in \mathcal{SN}(G_1')} \frac{1}{|\mathcal{SN}(G_1')|},$$

Making use of (9) and (10), we get that

$$\sum_{T = (H, G_1', H', G'_2)} \delta_T \leq 4\pi_u(H)Q_u(H, H').$$

On the other hand, using Proposition 4.9, we have

$$\sum_{T \in \mathcal{H}; (H, H') \in \mathcal{P}_T} \delta_T \leq t \sum_{T = (G_1, G_1', G_2, G'_2) \in \mathcal{H}} \frac{\pi_{BC}(G_1')Q_e(G_1', G_2')\beta_{G_1', G_2'}(G_1, G_2)}{|\mathcal{SN}(G_1')||\mathcal{SN}(G_2')|} \text{len}(\mathcal{P}_T)^2 \leq C_d t \pi_u(H)Q_u(H, H'),$$

for some universal constant $C$. Combining the above estimates, we finish the proof. □

Proof of Theorem 1.3 Without loss of generality, $n \geq n_d$ where $n_d$ is taken from Proposition 4.11. Combining Theorem 1.2, Proposition 4.11 and Lemma 4.13, we deduce that for any $r \geq \rho$, $(\Omega^R(d, \pi_u, Q_u), \pi_u, Q_u)$ satisfies an $r$-regularized Modified log-Sobolev inequality with a constant $C_d\left(1 + \frac{\log r \sqrt{n}}{n} \right)$. Here $C_d > 0$ depends only on $d$. It remains to apply Theorem 1.1 to finish the proof. □

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