IDENTITIES INVOLVING BESSEL POLYNOMIALS ARISING
FROM LINEAR DIFFERENTIAL EQUATIONS

TAEKYUN KIM AND DAE SAN KIM

ABSTRACT. In this paper, we study linear differential equations arising from
Bessel polynomials and their applications. From these linear differential equations, we give some new and explicit identities for Bessel polynomials.

1. Introduction

As is well known, the Bessel differential equation is given by
\begin{equation}
\frac{d^2 y}{dx^2} + x \frac{dy}{dx} + \left( x^2 - \alpha^2 \right) y = 0, \quad (\text{see } [17]).
\end{equation}
for an arbitrary complex number \( \alpha \).

The Bessel functions of the first kind \( J_\alpha (x) \) are defined by the solution of (1.1).

For \( n \in \mathbb{Z} \), \( J_n (x) \) are sometimes also called cylinder function or cylindrical harmonics.

It is known that
\begin{equation}
J_n (x) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l! (n+l)!} \left( \frac{x}{2} \right)^{2l+n}, \quad (\text{see } [1, 16, 17]).
\end{equation}

The generating function of Bessel functions is given by
\begin{equation}
e^\frac{x}{2} (t-1/t) = \sum_{n=-\infty}^{\infty} J_n (x) t^n,
\end{equation}
and \( J_n (x) \) can be also represented by the contour integral as
\begin{equation}
J_n (x) = \frac{1}{2\pi i} \oint e^{t/-1} t^{-n-1} dt, \quad (\text{see } [17]),
\end{equation}
where the contour encloses the origin and is traversed in a counterclockwise direction.

The Bessel polynomials are defined by the solution of the differential equation
\begin{equation}
\frac{d^2 y}{dx^2} + 2 (x+1) \frac{dy}{dx} - n (n+1) y = 0, \quad (\text{see } [1, 6, 13, 16]).
\end{equation}

Indeed, the solutions of (1.5) are given by
\begin{equation}
y_n (x) = \sum_{k=0}^{n} \frac{(n+k)!}{(n-k)! k!} \left( \frac{x}{2} \right)^k
= \sqrt{\frac{2}{\pi x}} e^{1/2} K_{-n-1/2} \left( \frac{1}{x} \right), \quad (\text{see } [1, 15, 17]),
\end{equation}

2010 Mathematics Subject Classification. 05A19, 33C10, 34A30.
Key words and phrases. Bessel polynomials, linear differential equation.
where
\[
K_{\nu}(z) = \frac{\Gamma\left(\nu + \frac{1}{2}\right)}{\sqrt{\pi}} (2z)^\nu \int_0^\infty \frac{\cos t}{(t^2 + z^2)^{\nu + \frac{1}{2}}} dt.
\]

We note that \( y_n(x) \) are very similar to the modified spherical Bessel function of the second kind. The first few are given as
\[
y_0(x) = 1, \quad y_1(x) = x + 1, \quad y_2(x) = 3x^2 + 3x + 1, \\
y_3(x) = 15x^3 + 15x^2 + 6x + 1, \\
y_4(x) = 105x^4 + 105x^3 + 45x^2 + 10x + 1, \ldots.
\]

Carlitz reverse Bessel polynomials are defined by
\[
p_n(x) = x^n y_{n-1}\left(\frac{1}{x}\right), \quad (n \in \mathbb{N} \cup \{0\}), \quad (\text{see } [4, 15]).
\]

These polynomials are also given by the generating function as
\[
e^{x(1-\sqrt{1-t})} = \sum_{n=0}^{\infty} p_n(x) \frac{t^n}{n!}.
\]

The explicit formulas for them are
\[
p_n(x) = \sum_{k=1}^{n} \frac{(2n-k-1)!}{2^{n-k} (k-1)! (n-k)!} x^k
\]
\[
= (2n-3)!! x \, _1F_1\left(1-n; 2-2n; 2x\right), \quad (\text{see } [1, 15, 16]),
\]

where
\[
n!! = \begin{cases} 
(n(n-2) \cdots 5 \cdot 3 \cdot 1) & \text{if } n > 0 \text{ odd}, \\
(n(n-2) \cdots 6 \cdot 4 \cdot 2) & \text{if } n > 0 \text{ even}, \\
1 & \text{if } n = -1, 0,
\end{cases}
\]

and
\[
_1F_1(a; b; z) = 1 + \frac{a}{b} z + \frac{a(a+1) z^2}{b(b+1) 2!} + \cdots
\]
\[
= \sum_{k=0}^{\infty} \frac{a(a+1) \cdots (a+k-1) z^k}{b(b+1) \cdots (b+k-1) k!}
\]
\[
= \frac{\Gamma(b)}{\Gamma(b-a) \Gamma(a)} \int_0^1 e^{zt} t^{a-1} (1-t)^{b-a-1} dt.
\]

The first few polynomials are
\[
p_1(x) = x, \\
p_2(x) = x^2 + x, \\
p_3(x) = x^3 + 3x^2 + 3x, \\
p_4(x) = x^4 + 6x^3 + 15x^2 + 15x, \ldots.
\]

Recently, several authors have studied non-linear differential equations related to special polynomials (see [2, 14]).

The reverse Bessel polynomials are used in the design of Bessel electronic filters.
In this paper, we consider linear differential equations arising from Carlitz reverse Bessel polynomials and give some new and explicit identities for Bessel polynomials.

2. IDENTITIES INVOLVING BESSEL POLYNOMIALS ARISING FROM LINEAR DIFFERENTIAL EQUATIONS

Let us put

\[ F = F(t, x) = e^{x(1 - \sqrt{1 - 2t})}. \]

Thus, by (2.1), we get

\[ F^{(1)} = \frac{d}{dt} F(t, x) = x (1 - 2t)^{-\frac{1}{2}} F, \]

\[ F^{(2)} = \frac{dF^{(1)}}{dt} = \left( x (1 - 2t)^{-\frac{3}{2}} + x^2 (1 - 2t)^{-1} \right) F, \]

\[ F^{(3)} = \frac{d}{dt} F^{(2)} = \left( 3x (1 - 2t)^{-\frac{3}{2}} + 3x^2 (1 - 2t)^{-2} + x^3 (1 - 2t)^{-\frac{5}{2}} \right) F, \]

and

\[ F^{(4)} = \frac{dF^{(3)}}{dt} = \left( 15x (1 - 2t)^{-\frac{7}{2}} + 15x^2 (1 - 2t)^{-3} + 6x^3 (1 - 2t)^{-\frac{5}{2}} + x^4 (1 - 2t)^{-2} \right) F. \]

Continuing this process, we set

\[ F^{(N)} = \left( \frac{d}{dt} \right)^{N} F(t, x) = \left( \sum_{i=0}^{2N-1} a_{i-N}(N, x) (1 - 2t)^{-\frac{i}{2}} \right) F, \]

where \( N = 1, 2, 3, \ldots \).

From (2.6), we note that

\[ F^{(N+1)} = \frac{d}{dt} F^{(N)} = \left( \sum_{i=N}^{2N-1} a_{i-N}(N, x) \left( -\frac{i}{2} \right) (1 - 2t)^{-\frac{i}{2}-1} (-2) \right) F \]

\[ + \sum_{i=N}^{2N-1} a_{i-N}(N, x) (1 - 2t)^{-\frac{i}{2}} F^{(1)} \]
\[
\begin{align*}
= & \left( \sum_{i=N}^{2N-1} ia_{i-N} (N, x) (1 - 2t)^{-\frac{i+1}{2}} \right) F \\
+ & \left( \sum_{i=N}^{2N-1} a_{i-N} (N, x) (1 - 2t)^{-\frac{i}{2}} \right) x (1 - 2t)^{-\frac{1}{2}} F \\
= & \left( \sum_{i=N}^{2N-1} ia_{i-N} (N, x) (1 - 2t)^{-\frac{i+1}{2}} \right) F + \left( \sum_{i=N}^{2N-1} xa_{i-N} (N, x) (1 - 2t)^{-\frac{i+1}{2}} \right) F \\
= & \left\{ xa_0 (N, x) (1 - 2t)^{-\frac{N+1}{2}} + (2N - 1) a_{N-1} (N, x) (1 - 2t)^{-\frac{2N+1}{2}} \\
+ & \sum_{i=N+1}^{2N-1} ((i - 1) a_{i-N-1} (N, x) + xa_{i-N} (N, x)) (1 - 2t)^{-\frac{i+1}{2}} \right\} F.
\end{align*}
\]

By replacing \( N \) by \( N + 1 \) in (2.6), we get

\[
F(N+1) = \left( \sum_{i=N+1}^{2N+1} a_{i-N-1} (N+1, x) (1 - 2t)^{-\frac{i}{2}} \right) F \\
= \left( \sum_{i=N}^{2N} a_{i-N} (N+1, x) (1 - 2t)^{-\frac{i+1}{2}} \right) F.
\]

By comparing the coefficients on both sides (2.7) and (2.8), we have

\[
\begin{align*}
a_0 (N+1, x) & = xa_0 (N, x), \\
a_N (N+1, x) & = (2N - 1) a_{N-1} (N, x),
\end{align*}
\]

and

\[
a_{i-N} (N+1, x) = (i - 1) a_{i-N-1} (N, x) + xa_{i-N} (N, x),
\]

where \( N + 1 \leq i \leq 2N - 1 \).

From (2.2) and (2.6), we can derive the following equation (2.11):

\[
(2.12) \quad x (1 - 2t)^{-\frac{1}{2}} F = F^{(1)} = a_0 (1, x) (1 - 2t)^{-\frac{1}{2}} F.
\]

Thus, by (2.12), we have

\[
(2.13) \quad a_0 (1, x) = x.
\]

From (2.9), we note that

\[
a_0 (N+1, x) = xa_0 (N, x) = x^2 a_0 (N - 1, x) = \cdots = x^N a_0 (1, x) = x^{N+1},
\]

and, by (2.10), we see

\[
a_N (N+1, x) = (2N - 1) a_{N-1} (N, x)
= (2N - 1) (2N - 3) a_{N-2} (N - 1, x)
\vdots 
= (2N - 1) (2N - 3) \cdots 1a_0 (1, x)
= (2N - 1)!! x.
\]

The matrix \((a_i (j, x))_{0 \leq i \leq N-1, 1 \leq j \leq N}\) is given by
From (2.11), we obtain

\[(2.16)\]
\[a_1 (N + 1, x) = Na_0 (N, x) + xa_1 (N, x) = Na_0 (N, x) + x (N - 1) a_0 (N - 1, x) + x^2 a_1 (N - 1, x)
\]
\[
\vdots
\]
\[= \sum_{i=0}^{N-2} x^i (N - i) a_0 (N - i, x) + x^{N-1} a_1 (2, x)
\]
\[= \sum_{i=0}^{N-2} x^i (N - i) a_0 (N - i, x) + x^{N-1} x
\]
\[= \sum_{i=0}^{N-1} x^i (N - i) a_0 (N - i, x),
\]

\[(2.17)\]
\[a_2 (N + 1, x)
\]
\[= (N + 1) a_1 (N, x) + xa_2 (N, x)
\]
\[= (N + 1) a_1 (N, x) + x a_1 (N - 1, x) + x^2 a_2 (N - 1, x)
\]
\[
\vdots
\]
\[= \sum_{i=0}^{N-3} x^i (N + 1 - i) a_1 (N - i, x) + x^{N-2} a_2 (3, x)
\]
\[= \sum_{i=0}^{N-3} x^i (N + 1 - i) a_1 (N - i, x) + 3x^{N-2} a_2 (2, x)
\]
\[= \sum_{i=0}^{N-2} x^i (N + 1 - i) a_1 (N - i, x),
\]

and

\[(2.18)\]
\[a_3 (N + 1, x)
\]
\[= (N + 2) a_2 (N, x) + xa_3 (N, x)
\]
\[= (N + 2) a_2 (N, x) + x (N + 1) a_2 (N - 1, x) + x^2 a_3 (N - 1, x)
\]
Continuing this process, we get

$$a_j (N + 1, x) = \sum_{i=0}^{N-j} x^i (N - i + j - 1) a_{j-1} (N - i, x),$$

where $j = 1, 2, \ldots, N - 1$.

Now, we give explicit expressions for $a_j (N + 1, x)$ ($j = 1, 2, \ldots, N - 1$). From (2.14) and (2.16), we can easily derive the following equation:

$$a_1 (N + 1, x) = \sum_{i_1=0}^{N-1} x^{i_1} (N - i_1) a_0 (N - i_1, x)$$

$$= x^N \sum_{i_1=0}^{N-1} (N - i_1).$$

By (2.17), (2.18) and (2.19), we get

$$a_2 (N + 1, x) = \sum_{i_2=0}^{N-2} x^{i_2} (N - i_2 + 1) a_1 (N - i_2, x)$$

$$= x^{N-1} \sum_{i_2=0}^{N-2} \sum_{i_1=0}^{N-2-i_2} (N - i_2 + 1) (N - i_2 - i_1),$$

and

$$a_3 (N + 1, x) = \sum_{i_3=0}^{N-3} x^{i_3} (N - i_3 + 2) a_2 (N - i_3, x)$$

$$= x^{N-2} \sum_{i_3=0}^{N-3} \sum_{i_2=0}^{N-3-i_3} \sum_{i_1=0}^{N-3-i_2} (N - i_3 + 2) (N - i_3 - i_2)$$

$$\times (N - i_3 - i_2 - i_1 - 2),$$

and

$$a_4 (N + 1, x) = \sum_{i_4=0}^{N-4} x^{i_4} (N - i_4 + 3) a_3 (N - i_4, x)$$

$$= x^{N-3} \sum_{i_4=0}^{N-4} \sum_{i_3=0}^{N-4-i_4} \sum_{i_2=0}^{N-4-i_3-i_4} \sum_{i_1=0}^{N-4-i_3-i_4} (N - i_4 + 3) (N - i_4 - i_3 + 1)$$

$$\times (N - i_4 - i_3 - i_2 - 1) (N - i_4 - i_3 - i_2 - i_1 - 3).$$
Continuing this process, we get
\begin{equation}
\label{eq:2.24}
a_j(N+1,x) = x^{N-j+1} \sum_{i_j=0}^{N-j} \sum_{i_{j-1}=0}^{N-j-i_j} \cdots \sum_{i_1=0}^{N-j-i_1-\cdots-i_2} (N-i_j-\cdots-i_k-(j-(2k-1))).
\end{equation}

Therefore, we obtain the following theorem.

**Theorem 1.** For $N \in \mathbb{N}$, the linear differential equations
\begin{equation}
\label{eq:2.25}
F^{(N)} = \left( \frac{d}{dt} \right)^N F(t,x) = \left( \sum_{i=N}^{2N-1} a_{i-N}(N,x) (1-2t)^{-\frac{i}{2}} \right) F
\end{equation}
has a solution $F = F(t,x) = e^{x(1-\sqrt{1-2t})}$, where
\begin{align*}
a_0(N,x) &= x^N, \\
 a_{N-1}(N,x) &= (2n-3)!!x,
\end{align*}
and
\begin{align*}
a_j(N,x) &= x^{N-j} \sum_{i_j=0}^{N-j} \sum_{i_{j-1}=0}^{N-j-i_j} \cdots \sum_{i_1=0}^{N-j-i_1-\cdots-i_2} (N-i_j-\cdots-i_k-(j-(2k-1)))
\end{align*}
\begin{align*}
&\quad \times \left( \prod_{k=1}^{j} (N-i_j-i_{j-1}-\cdots-i_k-(j-(2k-2))) \right).
\end{align*}

Recall the reverse Bessel polynomials $p_k(x)$ are given by the generating function as
\begin{equation}
\label{eq:2.26}
F = F(t,x) = e^{x(1-\sqrt{1-2t})}
= \sum_{k=0}^{\infty} p_k(x) \frac{t^k}{k!}.
\end{equation}

Thus, by \eqref{eq:2.26}, we get
\begin{equation}
\label{eq:2.27}
F^{(N)} = \left( \frac{d}{dt} \right)^N F(t,x)
= \sum_{k=N}^{\infty} p_k(x) (k)_N \frac{t^{k-N}}{k!}
= \sum_{k=0}^{\infty} p_{k+N}(x) (k+N)_N \frac{t^k}{(k+N)!}
= \sum_{k=0}^{\infty} p_{k+N}(x) \frac{t^k}{k!}.
\end{equation}

On the other hand, by Theorem 1, we get
\begin{equation}
\label{eq:2.28}
F^{(N)} = \left( \sum_{i=N}^{2N-1} a_{i-N}(N,x) (1-2t)^{-\frac{i}{2}} \right) F
= \sum_{i=N}^{2N-1} a_{i-N}(N,x) \left( \sum_{l=0}^{\infty} \left( -\frac{i}{2} \right)_l \frac{(-2t)^l}{l!} \right) \left( \sum_{m=0}^{\infty} p_m(x) \frac{t^m}{m!} \right)
\end{equation}
\[
= \sum_{k=0}^{\infty} \left\{ \sum_{i=N}^{2N-1} a_{i-N}(N,x) \sum_{l=0}^{k} \binom{k}{l} 2^l \left( \frac{i}{2} + l - 1 \right) p_{k-l}(x) \right\} \frac{t^k}{k!}.
\]

Therefore, by (2.26) and (2.27), we obtain the following theorem.

**Theorem 2.** For \( k \in \mathbb{N} \cup \{0\} \), and \( N \in \mathbb{N} \), we have

\[
p_{k+N}(x) = \sum_{i=N}^{2N-1} a_{i-N}(N,x) \sum_{l=0}^{k} \binom{k}{l} 2^l \left( \frac{i}{2} + l - 1 \right) p_{k-l}(x),
\]

where \( (x)_n = x(x-1)(x-2)\cdots(x-n+1) \), \( (n \geq 1) \), and \( (x)_0 = 1 \).

**References**

1. W. A. Al-Salam and L. Carlitz, *Bernoulli numbers and Bessel polynomials*, Duke Math. J. 26 (1959), 437–445. MR 0105516 (21 #4256)
2. M. J. Atia and S. Chneguir, *The exceptional Bessel polynomials*, Integral Transforms Spec. Funct. 25 (2014), no. 6, 470–480. MR 3172058
3. G. Bevilacqua, V. Biancalana, Y. Dancheva, T. Mansour, and L. Moi, *A new class of sum rules for products of Bessel functions*, J. Math. Phys. 52 (2011), no. 3, 033508, 9. MR 2814858 (2012c:33018)
4. R. P. Boas, *Book Review: Bessel polynomials*, Bull. Amer. Math. Soc. (N.S.) 1 (1979), no. 5, 799–800. MR 1567180
5. L. Carlitz, *A note on the Bessel polynomials*, Duke Math. J. 24 (1957), 151–162. MR 0085360 (19,27d)
6. P. Duan and J. Du, *Riemann-Hilbert characterization for main Bessel polynomials with varying large negative parameters*, Acta Math. Sci. Ser. B Engl. Ed. 34 (2014), no. 2, 557–567. MR 3174101
7. L.-C. Jang and B. M. Kim, *On identities between sums of Euler numbers and Genocchi numbers of higher-order*, J. Comput. Anal. Appl. 20 (2016), 1240–1247.
8. D. Kang, J. Jeong, S.-J. Lee, and S.-H. Rim, *A note on the Bernoulli polynomials arising from a non-linear differential equation*, Proc. Jangjeon Math. Soc. 16 (2013), no. 1, 37–43. MR 3059283
9. D. S. Kim and T. Kim, *A note on non-linear Changhee differential equations*, Russ. J. Math. Phys., (to appear).
10. T. Kim, *Identities involving Laguerre polynomials derived from umbral calculus*, Russ. J. Math. Phys. 21 (2014), no. 1, 36–45. MR 3182545
11. T. Kim, *Identities involving Frobenius-Euler polynomials arising from non-linear differential equations*, J. Number Theory 132 (2012), no. 12, 2854–2865. MR 2965196
12. T. Kim, D. S. Kim, T. Mansour, S.-H. Rim, and M. Schork, *Umbral calculus and Sheffer sequences of polynomials*, J. Math. Phys. 54 (2013), no. 8, 083504, 15. MR 3135486
13. T. Kim and T. Mansour, *Umbral calculus associated with Frobenius-type Eulerian polynomials*, Russ. J. Math. Phys. 21 (2014), no. 4, 484–493. MR 3284958
14. J.-W. Park, *On the q-analogue of λ-Daehee polynomials*, J. Comput. Anal. Appl. 19 (2015), no. 6, 966–974. MR 3309750
15. S. Roman, *The umbral calculus*, Pure and Applied Mathematics, vol. 111, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1984. MR 741185 (87c:05015)
16. H. M. Srivastava, S.-D. Lin, S.-J. Liu, and H.-C. Lu, *Integral representations for the Lagrange polynomials, Shively’s pseudo-Laguerre polynomials, and the generalized Bessel polynomials*, Russ. J. Math. Phys. **19** (2012), no. 1, 121–130. MR 2892608

17. D. G. Zill and W. S. Wright, *Advanced Engineering Mathematics*, Jones & Bartlett Publishers, 2009.

Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea
E-mail address: tkkim@kw.ac.kr

Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea
E-mail address: dskim@sogang.ac.kr