Scattering Equations and Global Duality of Residues

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Abstract

We examine the polynomial form of the scattering equations by means of computational algebraic geometry. The scattering equations are the backbone of the Cachazo-He-Yuan (CHY) representation of the S-matrix. We explain how the Bezoutian matrix facilitates the calculation of amplitudes in the CHY formalism, without explicitly solving the scattering equations or summing over the individual residues. Since for $n$-particle scattering, the size of the Bezoutian matrix grows only as $(n - 3) \times (n - 3)$, our algorithm is very efficient for analytic and numeric amplitude computations.
I. INTRODUCTION AND REVIEW

Cachazo, He and Yuan (CHY) recently proposed an intriguing representation of massless tree-level scattering amplitudes in any space-time dimensions in terms of a multidimensional complex contour integral of a certain rational function over an auxiliary coordinate space [1–5]. This approach is an alternative to the traditional program based on Feynman diagrams. The contour \( O \) in question is restricted to enclose the simultaneous solutions of a set of algebraic constraints referred to as the scattering equations. Let \( n \) denote the number of external particles. The auxiliary coordinate space consists of \( n \) puncture points \( z_i \in \mathbb{CP}^1 \) on the Riemann sphere, and the scattering equations take the form,

\[
f_a(z, k) = \sum_{b \neq a} \frac{s_{ab}}{z_a - z_b} = 0, \quad s_{ab} \equiv (k_a + k_b)^2, \quad a \in A = \{1, 2, \ldots, n\}.
\]  

(1.1)

For the benefit of the reader, let us recall a few basic properties previously reported elsewhere [1, 2]. It can be shown that eqs. (1.1) are invariant under \( \text{SL}(2, \mathbb{C}) \) transformations, 

\[
z_a \mapsto \zeta_a = \frac{\alpha z_a + \beta}{\gamma z_a + \delta}, \quad a \in A,
\]

(1.2)

by virtue of momentum conservation. This implies that only \( (n - 3) \) of the scattering equations are independent. The \( \text{SL}(2, \mathbb{C}) \) invariance allows us to specify three arbitrary coordinates \( z_r, z_s \) and \( z_t \), say, \( z_1 \to \infty \), \( z_2 \) fixed and \( z_n \to 0 \). An important observation is that the number of solutions to the scattering equations grows factorially as \( (n - 3)! \).

Let us return to the construction of tree amplitudes within the CHY formalism. The rational integrand consists of a universal part \( d\Omega_{\text{CHY}} \) and a purely theory-dependent factor denoted \( \mathcal{I} \). The universal part is constructed from the \( \text{SL}(2, \mathbb{C}) \) invariant integration measure and the rational functions \( f_a(z, k) \). It is responsible for localizing the integrand onto the joint solutions of the scattering equations upon integration. The precise form of \( \mathcal{I} \) has recently been explored for a large variety of quantum field theories including \( \varphi^3 \)-theory, Yang-Mills, Einstein gravity and Dirac-Born-Infeld [3–5]. We write schematically,

\[
\mathcal{A}_{\text{tree}} = \oint_O d\Omega_{\text{CHY}} \mathcal{I}(z, k),
\]

(1.3)

where

\[
d\Omega_{\text{CHY}} \equiv \frac{d^n z}{\text{vol} \left( \text{SL}(2, \mathbb{C}) \right)} \prod_a' \frac{1}{f_a(z, k)} = \prod_{a \in A \setminus \{r, s, t\}} dz_a(z_r z_s z_t z_{tr})(z_{ij} z_{jk} z_{ki}) \prod_{a \in A \setminus \{i, j, k\}} \frac{1}{f_a(z, k)}.
\]

(1.4)
Note that the scattering constraints are imposed in a permutation invariant manner as the measure is independent of both \( \{i, j, k\} \) and \( \{r, s, t\} \). Throughout the paper \( z_{ab} \equiv z_a - z_b \).

The natural question to address is how to actually calculate the amplitudes in this formalism. Actually it is in principle very straightforward to carry out any CHY integral of the form (1.3) without specifying the form of the theory-dependent part of the integrand. In fact, the CHY integral simply reduces to the sum of the \((n-3)!\) nondegenerate multivariate residues evaluated at the simultaneous zeros \( S \) of the denominator factors in eq. (1.4),

\[
A_n^{\text{tree}} = \sum_{z^* \in S} J^{-1}(z, k)(z_{ij}z_{jk}z_{ki})(z_{rs}z_{st}z_{tr})\mathcal{I}(z, k)\big|_{z = z^*}.
\] (1.5)

The Jacobian associated with the individual residues is

\[
J(z, k) = \det_{a \neq i, j, k, b \neq r, s, t} \left( \frac{\partial f_a}{\partial z_b} \right).
\] (1.6)

There exists a closed-form expression for this determinant [2], but it is not particularly illuminating for our purposes. Although the CHY formula (1.4) is extremely compact, it suffers from a practical limitation. Indeed, the instruction to sum over the \((n-3)!\) residues makes this formalism intractable already at relatively low multiplicities. The major problem is that the solutions can be very complicated and are inevitably irrational beyond five external particles [6]. Analytic expressions of the solutions are in general not attainable. Nevertheless, the final result computed from the \((n-3)!\) multivariate residues is always a simple rational function. Our motivation is to employ the Bezoutian matrix method from computational algebraic geometry to directly evaluate the sum of residues (1.5) without solving the scattering equations explicitly. (The Bezoutian matrix method has previously proven valuable in multiloop generalized unitarity cuts [32].)

The scattering equations and the CHY formalism have received extensive attention in the literature recently. Here we attempt to provide a brief overview of the most important developments. A proof of the CHY representation for \( \varphi^3 \)-theory and Yang-Mills based on Britto-Cachazo-Feng-Witten (BCFW) recursion relations [7, 8] and a generalization of the scattering equations to massive particles were presented in ref. [9]. Ref. [10] discussed scattering equations and fermions whereas an equivalent polynomial form of the scattering equations was derived in ref. [11]. Interestingly, new rules and techniques for evaluating amplitudes from CHY representations have been established, see papers by Cachazo and
Gomez [12] and Baadsgaard, Bjerrum-Bohr, Bourjaily and Damgaard [13]. Moreover, the authors of ref. [14] uncovered a link between CHY integrals and individual Feynman diagrams. Additional novel insight was achieved through string theory [15–23]. Despite the fact that the present paper is concentrated on tree-level amplitudes, we should mention that, very recently, Geyer, Mason, Monteiro and Tourkine [24] generalized the scattering equations at one loop [25] and suggested how to extend the result to arbitrary loop order. This analysis was further generalized by Baadsgaard, Bjerrum-Bohr, Bourjaily, Damgaard and Feng [26] for one-loop amplitudes in $\phi^3$-theory.

In this paper, we first introduce the Bezoutian matrix and its application to calculating the total sum of residues algebraically. Then we generalize this method to rational integrands, with the help of an elimination and grevlex monomial ordering. We apply this approach to the tree-level scattering equations, to get the amplitudes without finding the explicit solutions or summing over the residues. In particular, we emphasize that in all examples from the scattering equations we have examined, the dual form from the Bezoutian matrix computation has a strikingly simple form. This leads to a shortcut which greatly enhances the amplitude calculation. Finally, we explicitly show some high-multiplicity examples, like 8-point Yang-Mills amplitudes and 10-point $\phi^3$-theory amplitudes, to demonstrate the strength of our method.

**Note:** During the preparation of this manuscript, an interesting preprint by Feng, He, Huang and Rao [27] appeared. Clearly, the motivation for these authors has been the same as ours, hence there is a natural overlap with the present paper. We remark that the key difference between our approach and ref. [27] is that the principal object in our paper—the Bezoutian matrix—is of size $(n-3) \times (n-3)$ only whereas the companion matrix used in ref. [27] grows factorially as $(n-3)! \times (n-3)!$. Hence our method is able to efficiently evaluate high-multiplicity (for example, 10-point) amplitudes. After this project was completed, we were informed of another similar algorithm [28]. The relation between refs. [27] and [28] has now been investigated [29].

### A. Examples of CHY Representations

For later reference it is worthwhile to supply the reader with a few examples of CHY representations of tree-level amplitudes. Pure Yang-Mills and $\phi^3$-theory give rise to CHY
formulas that are quite representative to the formalism and thus serve as appropriate testing grounds for our method. We denote the canonically ordered \( n \)-gluon color-stripped amplitude as \( A_{n}^{\text{tree}} \) and the \( n \)-point \( \varphi^3 \) amplitude as \( A_{n}^{\varphi^3,\text{tree}} \).

The simplest instance of a CHY representation is found in massless \( \varphi^3 \)-theory. The prescription is simply to insert a squared Parke-Taylor factor into the CHY integrand. In connection with the CHY construction, the (canonically ordered) Parke-Taylor factor is understood as the following expression,

\[
PT(1,2,\ldots,n) \equiv \frac{1}{(z_1 - z_2)(z_2 - z_3) \cdots (z_n - z_1)},
\]

whose structure obviously bears resemblance with the Parke-Taylor denominator of the maximally helicity violating (MHV) gluon tree-level amplitude. Amplitudes in \( \varphi^3 \)-theory can thus be written

\[
A_{n}^{\varphi^3,\text{tree}} = \oint_{\mathcal{O}} \frac{d\Omega_{\text{CHY}}}{(z_1 - z_2)^2(z_2 - z_3)^2 \cdots (z_n - z_1)^2}.
\]

The representation of amplitudes in pure Yang-Mills is slightly more complicated and involves in addition to a Parke-Taylor factor also a Pfaffian which encodes the dependence of the external polarizations \( \{\epsilon_i\} \). According to CHY \[2,3\],

\[
A_{n}^{\text{tree}} = \oint_{\mathcal{O}} \frac{\text{Pf}^\prime \Psi(z,k,\epsilon)}{(z_1 - z_2)(z_2 - z_3) \cdots (z_n - z_1)},
\]

where \( \Psi \) is the \( 2n \times 2n \) antisymmetric matrix,

\[
\Psi = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}.
\]

The components of the \( n \times n \) matrices \( A, B \) and \( C \) are as follows,

\[
A_{ab} = \begin{cases} 
\frac{k_a \cdot k_b}{z_a - z_b} & a \neq b, \\
0 & i = j,
\end{cases} \quad B_{ab} = \begin{cases} 
\frac{\epsilon_a \cdot \epsilon_b}{z_a - z_b} & a \neq b, \\
0 & a = b,
\end{cases} \quad C_{ab} = \begin{cases} 
\frac{\epsilon_a \cdot k_b}{z_a - z_b} & a \neq b, \\
-\sum_{c \neq a} \frac{\epsilon_a \cdot k_c}{z_a - z_c} & a = b.
\end{cases}
\]

Finally, the reduced Pfaffian in eq. (1.9) is defined in terms of the matrix \( \Psi_{ij}^{ij} \), obtained from \( \Psi \) by removing rows \( i, j \) and columns \( i, j \) for \( 1 \leq i < j \leq n \), and is given by

\[
\text{Pf}^\prime \Psi \equiv 2 \frac{(-1)^{i+j}}{(z_i - z_j)} \text{Pf} \Psi_{ij}^{ij}.
\]

Note that \( \text{Pf}^\prime \Psi \) is independent of \( i \) and \( j \).
B. The Polynomial Form of the Scattering Equations

The scattering equations (1.1) proposed by Cachazo, He and Yuan are rational; for our purposes, it is advantageous to reformulate them as polynomial equations in order to enable systematic studies by means of computational algebraic geometry. Here we follow Dolan and Goddard [11], who recently proved that the $n$-point scattering equations, after properly fixing the SL($2$, $\mathbb{C}$) redundancy, are equivalent to a system of $(n - 3)$ polynomial equations,

$$h_m = 0, \quad 1 \leq m \leq n - 3,$$

in $(n - 3)$ variables. Each $h_m$ is a homogeneous polynomial of degree $m$ which is linear in each variable taken separately. More precisely,

$$h_m = \frac{1}{m!} \sum_{\substack{a_1, a_2, \ldots, a_m \in A' \atop a_i \neq a_j}} \sigma_{a_1 a_2 \cdots a_m} z_{a_1} z_{a_2} \cdots z_{a_m},$$

where $A' = \{2, \ldots, n - 1\}$ and $\sigma_{a_1 a_2 \cdots a_m} = k_{1 a_1 a_2 \cdots a_m}^2$. The polynomial equations define a zero-dimensional projective algebraic variety in $\mathbb{CP}^{n - 3}$ and the number of solutions is $(n - 3)!$ for generic kinematics by Bézout’s theorem.

What remains is to reexpress the measure of the CHY formula (1.3) in terms of the polynomials (1.14). At this point it is convenient to fix the SL($2$, $\mathbb{C}$) redundancy and specialize to $z_1 \to \infty$, $z_2$ fixed and $z_n \to 0$, and also extract a Parke-Taylor factor from $I$. A short calculation shows that [11]

$$A_{\text{tree}} = \int_O d\tilde{\Omega}_{\text{CHY}} \tilde{I}(z, k),$$

where (up to an overall sign)

$$d\tilde{\Omega}_{\text{CHY}} \equiv \frac{z_2}{z_{n - 1}} \prod_{m=1}^{n-3} \frac{1}{h_m(z, k)} \prod_{2 \leq a < b \leq n - 1} \left( z_a - z_b \right) \prod_{a=2}^{n-2} \frac{z_a d z_{a+1}}{(z_a - z_{a+1})^2}.$$

This formula completes our review of the scattering equations and CHY integrals.

II. MULTIVARIATE RESIDUES AND THE BEZOUTIAN MATRIX

Motivated by the preceding discussion, we continue with a quick introduction to the elementary theory of multivariate residues. For more details, refer to the classical text books by Griffiths and Harris [34] and Hartshorne [35].
A. Local Residues

Suppose that \( f = (f_1, \ldots, f_n) : \mathbb{C}^n \to \mathbb{C}^n \) is a holomorphic function with an isolated common zero at \( z = (z_1, \ldots, z_n) = \xi \in \mathbb{C}^n \). Moreover, let \( N : \mathbb{C}^n \to \mathbb{C} \) be a meromorphic function and assume regularity of \( N \) at \( z = \xi \). Let \( \omega \) be the \( n \)-form,

\[
\omega = \frac{N(z)}{f_1(z) \cdots f_n(z)} dz_1 \wedge \cdots \wedge dz_n.
\]  (2.1)

The (local) residue of \( \omega \) at \( \xi \) with respect to the divisors \( \{f_1, \ldots, f_n\} \) is by definition given by the multidimensional contour integral,

\[
\text{Res}_\xi(\omega) = \text{Res}_{\{f_1, \ldots, f_n\}, \xi}(\omega) = \frac{1}{(2\pi i)^n} \oint_{\Gamma_\delta} \frac{N(z)}{f_1(z) \cdots f_n(z)} dz_1 \wedge \cdots \wedge dz_n,
\]  (2.2)

where \( \Gamma_\delta = \{ z \in \mathbb{C}^n : |f_i(z)| = \delta_i \} \) is the real \( n \)-dimensional cycle around \( \xi \), oriented such that \( d \arg f_1 \wedge \cdots \wedge d \arg f_n \geq 0 \). Here \( \delta_i \) is a sufficiently small positive number. Note that the residue is independent of the \( \delta_i \)’s.

There are three types of multivariate residues: factorizable, nondegenerate and degenerate. Factorizable residues are trivial to calculate: if \( f_i(z) = f_i(z_i) \), the residue factorizes into a product of one-dimensional contour integrals,

\[
\text{Res}_{\{f_1, \ldots, f_n\}, \xi}(\omega) = \frac{1}{(2\pi i)^n} \oint_{|f_i(z_i)| = \delta_i} \frac{d z_1}{f_1(z_1)} \cdots \oint_{|f_n(z_n)| = \delta_n} \frac{d z_n}{f_n(z_n)}.
\]  (2.3)

Generally speaking, the \( f_i \)’s are not univariate functions and to decide whether the residue under consideration is degenerate or nondegenerate, we first evaluate the Jacobian determinant,

\[
J(\xi) \equiv \det i,j \left( \frac{\partial f_i}{\partial z_j} \right) \bigg|_{z = \xi}.
\]  (2.4)

If \( J(\xi) \) is nonvanishing, the residue at \( z = \xi \) is termed nondegenerate. Under these circumstances the residue can easily be evaluated by applying Cauchy’s theorem in higher dimensions, with the result

\[
\text{Res}_{\{f_1, \ldots, f_n\}, \xi}(\omega) = \frac{N(\xi)}{J(\xi)}.
\]  (2.5)

Degenerate residues are more challenging and require algebraic geometry methods, for example the transformation law \[34\] or the Bezoutian matrix \[36\]. Techniques for evaluating degenerate multivariate residues are beyond the scope of the present paper. See instead applications in the context of multiloop unitarity in refs. \[30–33\].
B. Global Residues and the Bezoutian Matrix

Often we are not interested in the details of each individual residue, but rather the global structure of the total sum of the residues. For example, to get tree amplitudes from the scattering equations, we only need to know the sum of the \((n - 3)!\) residues. It is frequently very difficult to calculate individual residues, because complicated algebraic extensions appear in the intermediate steps and the final result. Hence, we introduce an algebraic geometry approach, the Bezoutian matrix method, which allows us to arrive at the total sum of residues directly, without finding the singular point locus or calculating individual residues.

For the purposes of the remainder of this paper, it suffices to specialize to situations involving only rational functions. More precisely, we consider separately \(n\) polynomials \(\{f_1, \ldots, f_n\}\) in the ring \(R = \mathbb{C}[z_1, \ldots, z_n]\) and an arbitrary numerator \(N \in R\). We assume that the ideal \(I = \langle f_1, \ldots, f_n \rangle\) is zero-dimensional, i.e. the zero locus \(\mathcal{Z}(I)\) (set of all simultaneous zeros of the \(f_i\)s) consists only of a finite number of points. Since \(I\) is zero-dimensional, the quotient ring \(R/I\) is a finite-dimensional \(\mathbb{C}\)-linear space.

In view of the above considerations, we simply define the global residue as the sum of all the individual or local residues,

\[
\text{Res}(N) \equiv \sum_{\xi_i \in \mathcal{Z}(I)} \text{Res}_{(f_1, \ldots, f_n)\xi_i} \left( \frac{N(z)dz_1 \wedge \cdots \wedge dz_n}{f_1(z) \cdots f_n(z)} \right). \tag{2.6}
\]

Stokes’ theorem ensures that the values of the residues only depend on the equivalence class \([N]\) of \(N\) in \(R/I\). This equivalence splits polynomials into subclasses of polynomials which have the same remainder after polynomial division. Unlike the familiar division algorithm for univariate polynomials, multivariate polynomial division is only well-defined if performed towards a Gröbner basis (in some monomial order) of the ideal \(I\). (For the benefit of the non-expert reader, we remark that a Gröbner basis is a particular set of generators of an ideal. The theory of Gröbner bases is perhaps the most important practical tool in computational algebraic geometry; it allows for e.g. non-linear generalization of Gaussian elimination and multivariate polynomial division.) Then for any \(N \in R\), there is a unique polynomial \(r \in R\) (the unique remainder, for the monomial order being considered), which

\footnote{The ideal \(I\) generated by a set of polynomials \(\{f_1, \ldots, f_n\}\) is a special subset of the ring \(R = \mathbb{C}[z_1, \ldots, z_n]\), defined as \(I = \langle f_1, \ldots, f_n \rangle \equiv \{f \mid f = \sum_{i=1}^{n} a_i f_i, \ a_i \in R\} \).}
yields a representation of the equivalence class \([N] \in R/I\) such that \(N(z) = q(z) + r(z)\) with \(q \in I\). In particular, if \(N\) coincidentally belongs to \(I\), the residue vanishes identically.

The mathematical question we would now like to pose is how to obtain the value of the global residue \([2.6]\), without completing the instructed summation over the local residues. Given a pair of polynomials \(N_1, N_2\) we can introduce a symmetric inner product,

\[
\langle N_1, N_2 \rangle \equiv \text{Res}(N_1 \cdot N_2) .
\] (2.7)

The following theorem serves as the heart of computations of this paper.

**Theorem 1 (Global Duality)** \(\langle \bullet, \bullet \rangle\) is a nondegenerate inner product in \(R/I\).

The proof is omitted here, but can be found in ref. [34]. Now suppose that \(\{e_i\}\) forms a linear basis of \(R/I\). This merely means that any remainder can be written relative to the monomials \(\{e_i\}\). This theorem implies the existence of a **dual basis** \(\{\Delta_i\}\) in \(R/I\), with the property,

\[
\langle e_i, \Delta_j \rangle = \delta_{ij} .
\] (2.8)

The virtue of the dual basis is that it characterizes the structure of global residues in an explicit manner, without reference to the individual residues or their locations. To realize this, expand the remainder \([N]\) over the canonical linear basis,

\[
[N] = \sum_i \lambda_i e_i , \quad \lambda_i \in \mathbb{C} ,
\] (2.9)

and similarly, decompose unity using the dual basis,

\[
1 = \sum_i \mu_i \Delta_i , \quad \mu_i \in \mathbb{C} .
\] (2.10)

Then, by construction, the global residue is given by \([36]\)

\[
\text{Res}(N) = \langle N, 1 \rangle = \sum_{i,j} \lambda_i \mu_j \langle e_i, \Delta_j \rangle = \sum_i \lambda_i \mu_i .
\] (2.11)

In particular, if one term in the dual basis, \(\Delta_s\), is a constant, then the global residue computation is extremely simple,

\[
\text{Res}(N) = \langle N, \Delta_s \rangle / \Delta_s = \lambda_s / \Delta_s .
\] (2.12)
The partition (2.10) and summation (2.11) are not needed for this case. We have explicitly verified that the tree-level scattering equations have this remarkable feature for at least up to \( n = 10 \).

The canonical linear basis and the dual basis can be obtained in practice by means of the Gr"obner basis method and the Bezoutian matrix \([36]\). The procedure is described in the following.

1. (Canonical Linear Basis of Quotient Ring) Calculate the Gr"obner basis of \( I, G \), in Degree Lexicographic (grlex) or Degree Reverse Lexicographic (grevlex) order. Identify the leading terms \( LT(G) \) for all polynomials in \( G \). The canonical linear basis, \( \{e_i\} \), for \( R/I \) consists of all monomials in \( R \) which are lower than \( LT(G) \), with respect to the monomial order.

2. (Dual Basis of Quotient Ring) Define the \( n \times n \) Bezoutian matrix \( B \) for \( I \),

\[
B_{ij}(z, y) \equiv \frac{f_i(y_1, \ldots, y_{j-1}, z_j, \ldots, z_n) - f_i(y_1, \ldots, y_j, z_{j+1}, \ldots, z_n)}{z_j - y_j}.
\]  

(2.13)

Calculate the determinant of \( B \), \( \det B \). Define \( \tilde{G} \) as \( G \) subject to the replacement \( z_i \rightarrow y_i \) for \( i = 1, \ldots, n \). Carry out the multivariate polynomial division of \( \det B \) over \( G \otimes \tilde{G} \) and obtain the remainder,

\[
\sum_i a_i(y)e_i(z).
\]  

(2.14)

The dual basis \( \{\Delta_i\} \) for \( \{e_i\} \), with respect to \( (\cdot, \cdot) \), is \( \Delta_i = a_i(z) \) \([36]\).

At this moment it is important to underline that all operations related to Gr"obner bases require a certain choice of monomial ordering, for example Lexicographic (lex), Degree Lexicographic (grlex), Degree Reverse Lexicographic (grevlex). Moreover, the choice of monomial ordering may drastically influence on the speed of the calculation. Although some of these monomial orderings are rather self-explanatory, we supply here a concise clarification. Consider the shorthand notation for monomials, \( z^\alpha \equiv z_1^{\alpha_1} \cdots z_n^{\alpha_n} \).

- \( z^\alpha \succ_{\text{lex}} z^\beta \) if the left-most nonzero entry of \( \alpha - \beta \) is positive. It follows that there are \( n! \) nonequivalent lexicographic orderings, corresponding to the particular orderings of the variables. This monomial ordering is often slow to use in connection with Gr"obner bases.

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\[ z^\alpha \succ_{\text{grlex}} z^\beta \] if \(|\alpha| > |\beta|\), or if \(|\alpha| = |\beta|\) and \(z^\alpha \succ_{\text{lex}} z^\beta\). In words, the grlex order first compares total degree and then applies the lexicographic order.

\[ z^\alpha \succ_{\text{grevlex}} z^\beta \] if \(|\alpha| > |\beta|\), or if \(|\alpha| = |\beta|\) and the right-most nonzero entry of \(\alpha - \beta\) is negative. We stress that this monomial order is often the most efficient order for constructing Gröbner bases.

We conclude this subsection with a quick and transparent example of how the Bezoutian matrix can be used in practice to calculate global residues, without the need for obtaining the individual residues.

**Example 1 (Global Residues and the Bezoutian Matrix)** Let \( R = \mathbb{C}[z_1, z_2] \) and consider the zero-dimensional ideal \( I = \langle 2 - z_1 - z_2, z_1 + z_2 + 2z_1z_2 \rangle \subset R \). The zero locus is clearly \( Z(I) = \{(1 - \sqrt{2}, 1 + \sqrt{2}), (1 + \sqrt{2}, 1 - \sqrt{2})\} \). The Gröbner basis of \( I \) in grlex order is \( G = \{z_1 + z_2 - 2, 1 + 2z_2 - z_2^2\} \) and therefore the canonical linear basis of \( R/I \) is \( \{e_i\} = \{z_2, 1\} \). The Bezoutian matrix takes the form

\[
B = \begin{pmatrix}
-1 & -1 \\
1 + 2z_2 & 1 + 2y_1
\end{pmatrix},
\]

whence \( \det B = 2(z_2 - y_1) \). Polynomial division yields the dual basis \( \{\Delta_i\} = \{2, 2(z_2 - 2)\} \) and since \( 1 = \frac{1}{2} \Delta_1 \) we have \( \{\mu_i\} = \{\frac{1}{2}, 0\} \). Let us now pick a numerator, say, \( N(z) = z_2^2 \); for this choice the decomposition over the canonical linear basis is \( \{\lambda_i\} = \{2, 1\} \). The global residue thus takes the value

\[
\text{Res}(N) = \sum_{i=1,2} \lambda_i \mu_i = 1
\]

and the result matches the sum of the two individual residues,

\[
\text{Res}(N) = \frac{1}{8}(4 + 3\sqrt{2}) + \frac{1}{8}(4 - 3\sqrt{2}) = 1.
\]

Note that by the Bezoutian matrix computation, we do not need the algebraic extension from \( \sqrt{2} \). This greatly simplifies computation for more complicated examples.

**C. Our Proposal for CHY Integrals**

The Bezoutian matrix method provides us with a highly efficient technique for computing global residues of differential forms of the kind eq. (2.1), where the numerator and denominator factors are polynomials, without the need for calculating the local residues individually.
The only obstacle to immediately apply the theory of global residues in connection with
the polynomial scattering equations (1.14) and the CHY representation (1.15) is the presen-
tce of extra denominator factors, for example the Parke-Taylor factors. Phrased slightly
differently, the numerator \( N \) in our problem is not a polynomial, but a rational function.

The trick is to replace the extra denominator by its polynomial inverse in \( R/I \) in the
numerator. Let \( N = h/g \) where \( h, g \in R \). For a finite residue, \( g \) should not vanish on \( Z(I) \)
so \( \{ f_1, \ldots, f_n, g \} \) have no common zero. By Hilbert’s Nullstellensatz, there exist polynomials
\( a_1, \ldots, a_n, \tilde{g} \in R \) such that,

\[
a_1 f_1 + \cdots + a_n f_n + \tilde{g} g = 1.
\]

Suppose that all residues of \( I \) are nondegenerate. Then by (2.18),

\[
\text{Res}_\xi (N) = \text{Res}_\xi (h\tilde{g}) + \sum_{i=1}^{n} \text{Res}_\xi (a_if_i h/g) = \text{Res}_\xi (h\tilde{g}) ,
\]

because \( a_i f_i h/g \) is in the ideal generated by the \( f_i \)'s in the ring of germs of holomorphic
functions around \( \xi \) \[34\]. So for residue computations, we are free to replace \( g^{-1} \) by the
polynornial \( \tilde{g} \). If \( g \) factorizes, the polynomial inverse of \( g \) equals the product of the inverses. This
elementary observation typically greatly simplifies the problem, especially for the scattering
equations.

Consequently, the rational numerator is converted to a purely polynomial form. The
above discussion is implemented in Mathematica using Macaulay2 \[37\] via the Mathemati-
cMa2 package \[42\].

**Algorithm 1 (Polynomial Inverse)** Let \( I = \langle f_1, \ldots, f_n \rangle \subset \mathbb{C}[z] \) be a zero-dimensional
ideal and suppose \( \langle g \rangle + I = R \). Calculate the generator of the Gröbner basis of the
ideal \( \langle f_1, \ldots, f_n, g \rangle \) in some monomial order and record the converting matrix, so that \( 1 = a_1 f_1 + \cdots + a_n f_n + \tilde{g} g \). The polynomial inverse of \( g \), with respect to \( I \), is \( \tilde{g} \).

However, this algorithm requires the converting matrix for the Gröbner basis compu-
tation. In some cases, it is time and memory consuming. Therefore we may use a more
efficient algorithm.

\[2\] The converting matrix of the Gröbner basis is not directly provided by Mathematica.

\[3\] That is to say, the polynomials \( \{ f_1, \ldots, f_n, g \} \) have no simultaneous zero.
Algorithm 2 (Polynomial Inverse, Enhanced) Let \( I = \langle f_1, \ldots, f_n \rangle \subset \mathbb{C}[z] \) be a zero-dimensional ideal and suppose \( \langle g \rangle + I = R \). Introduce the auxiliary variable \( w \) and the ideal \( J = \langle f_1, \ldots, f_n, wg - 1 \rangle \) in \( \mathbb{C}[w, z_1, \ldots, z_n] \). Define a monomial order \( T \) that (1) compares the degrees of \( w \) first and (2) applies grevlex order for \( \{ z_1, \ldots, z_n \} \) as \( z_1 \succ \cdots \succ z_n \). Then calculate the Gröbner basis \( G(J) \) of \( J \) in the order of \( T \). Inside \( G(J) \), there must be a polynomial linear in \( w \),
\[
    w - \tilde{g}(z_1, \ldots, z_n) \in G(J).
\]
Then \( \tilde{g}(z_1, \ldots, z_n) \) is the inverse of \( g \), with respect to \( I \).

The correctness of the output is guaranteed by the definition of \( T \). This algorithm does not need the converting matrix. Fast routines for Gröbner basis computations, like Faugère’s F4 and F5 algorithms [38, 39], can be applied. In practice, we use the FGb package [40].

Example 2 (Polynomial Inverses and Global Residues) For brevity we will merely revisit the problem in Example 1 but now with a rational function \( N(z) = 1/z_1^2 \) in the numerator. The corresponding polynomial inverse in \( R/I \) is quickly calculated using Algorithm 1. The Gröbner basis computation gives
\[
    1 = \frac{1 + 2z_1 + 2z_2 + 4z_1z_2(2 - z_1 - z_2)}{2} + \frac{-3 + 2z_2}{2}(z_1 + z_2 + 2z_1z_2) + (1 + 2z_2)(z_1^2). \quad (2.21)
\]
It is immediately clear that the inverse of \( 1/z_1^2 \) with respect to \( I \) is \( 1 + 2z_2 \). Following the steps of Example 1 determines the global residue to be 1.

Alternatively we can apply Algorithm 2. Introduce the auxiliary variable \( w \), and define \( J = I + \langle wz_1^2 - 1 \rangle \). Considering the ordering \( T \), the Gröbner basis is
\[
    G(J) = \{-2 + z_1 + z_2, 1 - 2z_2 + z_2^2, w - 1 - 2z_2\}. \quad (2.22)
\]
Hence the inverse of \( 1/z_1^2 \) with respect to \( I \) is \( 1 + 2z_2 \).

III. EXAMPLES

We present several explicit examples of how to employ the Bezoutian matrix method to evaluate scattering amplitudes in the CHY formalism. Without loss of the main features we will primarily be interested in massless \( \varphi^3 \)-theory and pure Yang-Mills. In the beginning we consider analytic calculations for lower multiplicity kinematics. As the complexity of
the intermediate and final analytic expressions increases significantly with the number of particles, it is instructive to analyze higher-point examples using numerical data for the kinematic invariants.

A. Four-Point Amplitudes

There is only one independent scattering equation for four external particles and thus a single univariate residue to compute. The pole in $z_3$ is trivial to locate,

$$h_1 = \sigma_2 z_2 + \sigma_3 z_3 = 0 \quad \Rightarrow \quad \frac{z_3}{z_2} = -\frac{\sigma_2}{\sigma_3} = -\frac{s_{12}}{s_{13}}. \quad (3.1)$$

We can now effortlessly extract the four-scalar amplitude from the CHY representation (1.15), with the familiar result

$$A_{4}^{\varphi, \text{tree}} = -\oint_{\mathcal{O}} \frac{dz_3}{\sigma_2 z_2 + \sigma_3 z_3 (z_2 - z_3) z_3} = \frac{1}{s_{12}} + \frac{1}{s_{14}}, \quad (3.2)$$

Even though this example is very simple, let us for the sake of completeness work it out using the Bezoutian matrix approach. We readily arrive at $\{e_i\} = \{1\}$, $\det B = \sigma_3$ and $\{\Delta_i\} = \{\sigma_3\}$. Denote $g_1 = z_2 - z_3$ and $g_2 = z_3$. Then by Algorithm 1,

$$\tilde{g}_1 = \frac{\sigma_3}{(\sigma_2 + \sigma_3) z_2}, \quad \tilde{g}_2 = -\frac{\sigma_3}{\sigma_2 z_2}, \quad (3.3)$$

and therefore, as expected,

$$A_{4}^{\varphi, \text{tree}} = \oint_{\mathcal{O}} \frac{dz_3}{\sigma_2 z_2 + \sigma_3 z_3 (z_2 + \sigma_2 + \sigma_3)} = \frac{\sigma_3}{\sigma_2 (\sigma_2 + \sigma_3)}, \quad (3.4)$$

where in the last step we invoked eq. (2.12).

B. Five-Point Amplitudes

The five-particle case gives rise to the first nontrivial instance of the scattering equations and provides a prime example of the intermediate steps of the proposed method. We examine the two independent polynomial scattering equations,

$$h_1 = \sigma_2 z_2 + \sigma_3 z_3 + \sigma_4 z_4 = 0,$$

$$h_2 = \sigma_{23} z_2 z_3 + \sigma_{24} z_2 z_4 + \sigma_{34} z_3 z_4 = 0, \quad (3.5)$$
whose two solutions we quote for later reference (setting \( z_2 = 1 \) for simplicity),

\[
S_1 : \begin{cases} 
  z_3 &= -\frac{\sigma_2 \sigma_{24} + \sigma_3 \sigma_{23} - \sigma_4 \sigma_{23} + \sqrt{\Delta}}{2 \sigma_3 \sigma_{24}} , \\
  z_4 &= -\frac{\sigma_2 \sigma_{34} - \sigma_3 \sigma_{24} + \sigma_4 \sigma_{23} - \sqrt{\Delta}}{2 \sigma_3 \sigma_{24}} , 
\end{cases} \tag{3.6}
\]

\[
S_2 : \begin{cases} 
  z_3 &= -\frac{\sigma_2 \sigma_{34} + \sigma_3 \sigma_{24} - \sigma_4 \sigma_{23}}{2 \sigma_3 \sigma_{24}} , \\
  z_4 &= -\frac{\sigma_2 \sigma_{34} - \sigma_3 \sigma_{24} + \sigma_4 \sigma_{23} + \sqrt{\Delta}}{2 \sigma_3 \sigma_{24}} . 
\end{cases} \tag{3.7}
\]

The discriminant is given by

\[
\Delta = (\sigma_2 \sigma_{34} + \sigma_3 \sigma_{24} - \sigma_4 \sigma_{23})^2 - 4 \sigma_2 \sigma_3 \sigma_{24} \sigma_{34} . \tag{3.8}
\]

According to the CHY prescription (1.15), the five-point amplitude in \( \varphi^3 \)-theory is computed by the two-dimensional contour integral,

\[
\mathcal{A}_{5}^{\varphi^3,\text{tree}} = \oint_{\mathcal{O}} dz_3 dz_4 \frac{z_3(1-z_4)}{h_1 h_2 (1-z_3)(z_3-z_4)z_4} , \tag{3.9}
\]

where the contour \( \mathcal{O} \) encloses the two simultaneous zeros (3.6)-(3.7) of \( h_1 \) and \( h_2 \), but no other singularities. The denominator of the integrand contains three additional linear factors \( g_1 = 1-z_3, g_2 = z_3-z_4 \) and \( g_3 = z_4 \). Let \( I = \langle h_1, h_2 \rangle \subset R = \mathbb{C}[z_3, z_4] \). Using Algorithm 1 it takes Mathematica only a split second to find the following expressions for the polynomial inverses,

\[
\tilde{g}_1 = -\frac{\sigma_3 \sigma_{24} + \sigma_3 \sigma_{34} - \sigma_4 (\sigma_{23} + \sigma_{34} z_4)}{(\sigma_2 + \sigma_3)(\sigma_{24} + \sigma_{34}) - \sigma_4 \sigma_{23}} , \tag{3.10}
\]

\[
\tilde{g}_2 = -\frac{\sigma_3 \sigma_{34} - (\sigma_3 + \sigma_4)(\sigma_3 \sigma_{24} - \sigma_4 (\sigma_{23} + \sigma_{34} z_4))}{\sigma_2(\sigma_2 \sigma_{34} - (\sigma_3 + \sigma_4)(\sigma_{23} + \sigma_{24}))} , \tag{3.11}
\]

\[
\tilde{g}_3 = \frac{\sigma_3 (\sigma_{24} + \sigma_{34} z_3) - \sigma_4 \sigma_{23}}{\sigma_2 \sigma_{23}} . \tag{3.12}
\]

The amplitude can then be rewritten in terms of a polynomial numerator function \( N \),

\[
\mathcal{A}_{5}^{\varphi^3,\text{tree}} = \oint_{\mathcal{O}} dz_3 dz_4 \frac{N}{h_1 h_2} , \quad N(z_3, z_4) = z_3(1-z_4) \prod_{i=1}^{3} \tilde{g}_i . \tag{3.13}
\]

The Gröbner basis of \( I \) in the grlex order \( z_3 \succ z_4 \) is

\[
G = \{ \sigma_2 + \sigma_3 z_3 + \sigma_4 z_4, \sigma_3 \sigma_{24} z_4 - (\sigma_{23} + \sigma_{34} z_4)(\sigma_2 + \sigma_4 z_4) \} \tag{3.14}
\]

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whence the quotient ring basis \( \{ p_i \} = \{ z_4, 1 \} \) of \( R/I \) can be read off immediately. Now, from the Bezoutian matrix,
\[
B(z, y) = \begin{pmatrix}
\sigma_3 & \sigma_4 \\
\sigma_23 + \sigma_34z_4 & \sigma_24 + \sigma_34y_3
\end{pmatrix},
\]
we calculate the determinant,
\[
\det B = \sigma_3(\sigma_24 + \sigma_34y_3) - \sigma_4(\sigma_23 + \sigma_34z_4),
\]
and thus by polynomial division over \( G \otimes \tilde{G} \), we derive the dual basis,
\[
\{ \Delta_i \} = \{-\sigma_4\sigma_34, -\sigma_2\sigma_34 + \sigma_3\sigma_24 - \sigma_4(\sigma_23 + \sigma_34z_4)\}.
\]

The physical singularities of the five-point amplitude are the five independent Mandelstam invariants \( s_{12}, s_{23}, s_{34}, s_{45} \) and \( s_{51} \). Rewriting the \( \sigma \)-variables using simple kinematic identities immediately leads to the well known Feynman diagram result,
\[
A^{\varphi^3}_{5, \text{tree}} = \frac{1}{s_{12}s_{34}} + \frac{1}{s_{12}s_{45}} + \frac{1}{s_{23}s_{51}} + \frac{1}{s_{23}s_{45}} + \frac{1}{s_{34}s_{51}}.
\]

Direct evaluation of the individual residues of course yields the same answer. However, the symbolic manipulations and cancellations are quite involved due to the presence of square roots in eqs. (3.6) and (3.7). Indeed, for comparison, we have the intermediate result
\[
A^{\varphi^3}_{5, \text{tree}} = \int_0^1 \frac{dz_3dz_4}{h_1h_2} \frac{z_3(1-z_4)}{(1-z_3)(z_3-z_4)z_4} = \text{Res}_1 + \text{Res}_2.
\]
where the residues are the rather complicated expressions,

\[
\text{Res}_1 = - \frac{2\sigma_3\sigma_4\sigma_3\sigma_4(\sigma_2\sigma_3 + \sigma_3\sigma_4 - \sigma_4\sigma_3 + \sqrt{\Delta})}{\sqrt{\Delta}[(\sigma_3 + \sigma_4)(\sigma_3\sigma_4 - \sigma_4\sigma_3 + \sqrt{\Delta}) - \sigma_2(\sigma_3 - \sigma_4)\sigma_3\sigma_4]} \\
\times \frac{\sigma_3\sigma_4 - \sigma_4\sigma_3 - (\sigma_2 + 2\sigma_4)\sigma_3\sigma_4 + \sqrt{\Delta}}{(\sigma_2\sigma_3 - \sigma_3\sigma_4 + \sigma_4\sigma_3 + \sqrt{\Delta})(\sigma_3\sigma_4 - \sigma_4\sigma_3 + (\sigma_2 + 2\sigma_3)\sigma_3\sigma_4 + \sqrt{\Delta})} .
\]

\[
\text{Res}_2 = + \frac{2\sigma_3\sigma_4\sigma_3\sigma_4(\sigma_2\sigma_3 + \sigma_3\sigma_4 - \sigma_4\sigma_3 - \sqrt{\Delta})}{\sqrt{\Delta}[(\sigma_3 + \sigma_4)(\sigma_3\sigma_4 - \sigma_4\sigma_3 - \sqrt{\Delta}) - \sigma_2(\sigma_3 - \sigma_4)\sigma_3\sigma_4]} \\
\times \frac{\sigma_3\sigma_4 - \sigma_4\sigma_3 - (\sigma_2 + 2\sigma_4)\sigma_3\sigma_4 - \sqrt{\Delta}}{(\sigma_2\sigma_3 - \sigma_3\sigma_4 + \sigma_4\sigma_3 + \sqrt{\Delta})(\sigma_3\sigma_4 - \sigma_4\sigma_3 + (\sigma_2 + 2\sigma_3)\sigma_3\sigma_4 - \sqrt{\Delta})} .
\]

C. Eight-Point Amplitudes

We will now illustrate and validate the Bezoutian matrix method in a more difficult situation, namely for an eight-gluon amplitude in pure Yang-Mills theory. For the sake of simplicity we will restrict to four dimensions and numerically study the MHV configuration where two of the gluons \(i\) and \(j\) have negative helicity and the rest have positive helicity. The result is thus straightforward to compare with the known answer due to the Parke-Taylor formula,

\[
A_{n,ij}^{\text{tree}} = \frac{\langle i \ j \rangle^4}{\langle 1 \ 2 \ \langle 2 \ 3 \cdots \ n \ 1 \rangle} .
\]

High-multiplicity kinematics is conveniently generated through momentum twistors. For this example we will proceed with the following values for the momenta,

\[
k_1^\mu = (5/2, 5/2, -i/2, 1/2) , \quad k_2^\mu = (-3/4, 3/4, -3i/4, -3/4) , \quad k_3^\mu = (-1/6, 0, 0, 1/6) , \quad k_4^\mu = (-19/84, 11/28, 9i/28, 1/84) , \quad k_5^\mu = (10/21, -55/42, 95i/42, 40/21) , \quad k_6^\mu = (23/12, -1/12, -i/12, 23/12) , \quad k_7^\mu = (-19/28, 1/28, -43i/28, -47/28) , \quad k_8^\mu = (-43/14, -16/7, 2i/7, -29/14) ,
\]

for which the amplitude (3.23) becomes,

\[
A_8^{\text{tree}}(1^-, 2^-, 3^+, \ldots, 8^+) = \frac{4}{441} .
\]

At eight points, there are five independent scattering equations \(h_m = 0, 1 \leq m \leq 5\), in the variables \(z_3, \ldots, z_7\), with \((8 - 3)! = 120\) simultaneous solutions. Here we have gauge
fixed $z_1 \to \infty$, $z_2 \to 1$ and $z_8 \to 0$ as usual. Needless to say, it is almost impossible to compute the desired amplitude in practice by evaluating the sum over the individual residues. The scattering equations are a bit lengthy for problems with many particles, but otherwise elementary to write down explicitly.

As previously explained in Section I A, Yang-Mills amplitudes in the CHY representation involve a Pfaffian. Recall that for the CHY formula with polynomial denominators, we pulled out a Parke-Taylor factor from the CHY integrand in eq. (1.15). Therefore, the gauge fixed Yang-Mills integrand is given by the limit,

$$
\tilde{I}(z, k) = \prod_{a=2}^{n-1} (z_a - z_{a+1}) \lim_{z_1 \to \infty} \left( z_1^2 \text{Pf}^\prime \Psi \right).
$$

(3.26)

The numerator of $\tilde{I}$ is a huge polynomial whose explicit form is not particularly important for exposing the essential steps of the calculation. On the other hand, the denominator of $\tilde{I}$ is the simple polynomial function (up to an overall constant),

$$
D_{YM} = z_3 z_4 (z_3 - z_5) (1 - z_4) z_5 (1 - z_6) (z_3 - z_6)
$$

$$
(z_4 - z_6) (1 - z_6) z_6 (z_3 - z_7) (z_4 - z_7) (z_5 - z_7) (1 - z_7).
$$

(3.27)

Moreover, there is a contribution to the denominator coming from $d\tilde{\Omega}_{\text{CHY}}$ (1.16),

$$
D_{DG} = (z_2 - z_3) (z_3 - z_4) (z_4 - z_5) (z_5 - z_6) (z_6 - z_7) z_7.
$$

(3.28)

We invert the polynomial factors of these denominators via Algorithm [2]. The Gröbner basis computations are rapidly performed using the FGb library [40] in Maple. For the dual basis $\{\Delta_i\}$, determined by applying the Bezoutian matrix method, we explicitly observe that $\Delta_1$ is a constant, so we can use eq. (2.12). The corresponding term in the canonical linear basis $e_1 = z_7^{10}$. Consequently,

$$
\mathcal{A}^\text{free}_{8}(1^-, 2^-, 3^+, \ldots, 8^+) = \left[ \frac{N_{DG} N_{YM}}{D_{DG} D_{YM}} \right]_{z_7^{10}} / \Delta_1,
$$

(3.29)

where the subscript indicates that we only take the coefficient for $z_7^{10}$. The square brackets refer to the canonical form with respect to $I$, in grevlex order. The end result of this calculation agrees with the value in eq. (3.25).
D. Ten-point Amplitudes

To fully demonstrate the power of our method, we calculate as a final example the 10-point $\varphi^3$ amplitude in four dimensions by applying the Bezoutian matrix and the enhanced inversion algorithm (Algorithm 2). Using momentum twistors, we consider the numeric phase point,

$$k_1^\mu = (6/5, 6/5, -3i/5, -3i/5), \quad k_2^\mu = (7/4, 3/4, 9i/4, -11/4),$$

$$k_3^\mu = (-5/4, -5/4, 15i/4, 15/4), \quad k_4^\mu = (-1, 1/8, i/8, -1),$$

$$k_5^\mu = (1/10, -5/8, 47i/40, -1), \quad k_6^\mu = (17/5, -7/2, -13i/10, 1),$$

$$k_7^\mu = (-2, 3, 3i, 2), \quad k_8^\mu = (-1, 5, -7i, -5),$$

$$k_9^\mu = (-11/4, -6, 6i, 11/4), \quad k_{10}^\mu = (31/20, 13/10, i/10, 17/20).$$

(3.30)

There are 7 scattering equations in the variables $z_1, \ldots, z_7$. The quotient ring $R/I$ has the dimension $(10 - 3)! = 5040$. In this case, Gröbner basis computation is heavy. So we use the fast Gröbner basis computation package [40]. Furthermore, to reduce memory usage, we apply the finite field technique:

1. Calculate the global residue with the coefficients in the finite field $\mathbb{Z}/p$, where $p$ is a prime number. Repeat this process several times for different prime numbers. For this particular amplitude, we find that it is enough to use 12 prime numbers, each of which is around the order of $10^4$.

2. Use the modular method [41] to lift the global residue evaluated in finite fields, to a rational number, i.e. the physical value.

We calculate the dual basis, $\{\Delta_i\}$, in grevlex order, for $I$ via the Bezoutian matrix method. For this example we also find that one of the dual basis terms $\Delta_1$ is constant. The corresponding term in canonical basis is $e_1 = z_5^{21}$. By eq. (2.12),

$$A_{10}^{\varphi^3, \text{tree}} = \left[ \frac{N}{D} \right] z_5^{21} / \Delta_1,$$

(3.31)

where

$$N = z_3(1 - z_4)z_4(1 - z_5)(z_3 - z_5)z_5(1 - z_6)(z_3 - z_6)(z_4 - z_6)z_6(1 - z_7)(z_3 - z_7)(z_4 - z_7)(z_5 - z_7)z_7(1 - z_8)(z_3 - z_8)(z_4 - z_8)(z_5 - z_8)(z_6 - z_8)z_8(1 - z_9)(z_3 - z_9)(z_4 - z_9)(z_5 - z_9)(z_6 - z_9)(z_7 - z_9)$$

(3.32)
and
\[ D = (1 - z_3)(z_3 - z_4)(z_4 - z_5)(z_5 - z_6)(z_6 - z_7)(z_7 - z_8)(z_8 - z_9)z_9. \] (3.33)

Again, we invert the factors in \( D \) one by one to get a purely polynomial form of the remaining integrand. The final result for this phase point is
\[ A_{10}^{\varphi^3, \text{tree}} = -\frac{24890770333766902407787}{24536182021587817097932800}. \] (3.34)

IV. CONCLUSION

In summary we have employed computational algebraic geometry to study the polynomial form of the scattering equations and to calculate various amplitudes from CHY representations of pure Yang-Mills and \( \varphi^3 \)-theory. The main result is a completely general technique to directly carry out the sum over the \((n-3)!\) multivariate residues evaluated at the simultaneous solutions of the scattering equations without solving them explicitly. Our approach is essentially based on global duality of residues and the Bezoutian matrix. The validity of the method has been verified through several examples with \( n \leq 10 \) particles. Another salient aspect is that rationality of all final results is automatically manifest.

This paper suggests several interesting directions for future research on scattering equations and the CHY formalism. First of all, it is worthwhile to compare more thoroughly with other recent papers [12, 13, 26, 27] which address the same problem. The very clean and symmetric form of polynomial scattering equations (1.14) is an immediate invitation to further systematic studies using algebraic geometry. We believe that a much deeper understanding of the scattering equations and the CHY formalism may be gained from a recursive construction. For instance, is it possible to perform the required polynomial inversions by induction? Moreover, it is intriguing to investigate the physical meaning of Bezoutian matrices from scattering equations. We expect that the procedure presented here may be generalized to loop level in the near future.

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