Classical Radiation Reaction Off-Shell
Corrections to the Covariant Lorentz Force

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Abstract:
It has been shown by Gupta and Padmanabhan that the radiation reaction force of the Abraham-Lorentz-Dirac equation can be obtained by a coordinate transformation from the inertial frame of an accelerating charged particle to that of the laboratory. We show that the problem may be formulated in a flat space of five dimensions, with five corresponding gauge fields in the framework of the classical version of a fully gauge covariant form of the Stueckelberg-Feynman-Schwinger covariant mechanics (the zero mode fields of the 0, 1, 2, 3 components correspond to the Maxwell fields). Without additional constraints, the particles and fields are not confined to their mass shells. We show that in the mass-shell limit, the generalized Lorentz force obtained by means of the retarded Green's functions for the five dimensional field equations provides the classical Abraham-Lorentz-Dirac radiation reaction terms (with renormalized mass and charge). We also obtain general coupled equations for the orbit and the off-shell dynamical mass during the evolution. The theory does not admit radiation if the particle remains identically on-shell. The structure of the equations implies that mass-shell deviation is bounded when the external field is removed.

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Gupta and Padmanabhan\textsuperscript{4} have shown that the motion of a charged particle in an electromagnetic field can be described in the inertial frame of the particle with a time varying non-trivial background metric. Using the general covariant form of the Maxwell equations and transforming back to the inertial frame of the laboratory, they obtained the Abraham-Lorentz-Dirac radiation reaction term as a consequence of this geometrical picture. This result demonstrates that the description of the motion of a charged particle in acceleration must include the radiation terms of the Abraham-Lorentz-Dirac equation\textsuperscript{5}.

Alternatively, one can develop the mechanics in a flat space of higher dimension, an approach that we shall take. We shall work with the manifestly covariant mechanics of Stueckelberg\textsuperscript{6}, which provides a description of dynamical systems under the influence of forces (which may be represented in terms of potentials or gauge fields) in a framework which is Lorentz covariant. This theory admits, on a classical level, deviations from the particle’s mass shell during interaction, as in quantum field theory. A similar approach was used by Mendonça and Oliveira e Silva\textsuperscript{7}, who studied the motion of a relativistically kicked oscillator in the $E, t$ plane using what they called a “super Hamiltonian.” One can, in fact derive the relativistic Lorentz force

$$m \ddot{x}^\mu = F^\mu_{\nu} \dot{x}^\nu$$  \hspace{1cm} (1)

from such a Hamiltonian.

Consider the Hamiltonian\textsuperscript{5,6}(we take $c = 1$ henceforth)

$$K = \frac{(p^\mu - eA^\mu(x))(p_\mu - eA_\mu(x))}{2M}$$  \hspace{1cm} (2),

where $x \equiv x^\mu$. The Hamilton equations (generalized to the four-dimensional symplectic mechanics\textsuperscript{6}) are

$$\frac{dx^\mu}{d\tau} = \frac{\partial K}{\partial p_\mu} = \frac{p^\mu - eA^\mu(x)}{M}$$

$$\frac{dp^\mu}{d\tau} = -\frac{\partial K}{\partial x_\mu} = e \frac{\partial A^\lambda(x)}{\partial x_\mu} \frac{p^\lambda - eA^\lambda(x)}{M},$$  \hspace{1cm} (3)

where $\tau$ is the absolute (universal) invariant time parametrizing the path of the particle in spacetime\textsuperscript{6}. Computing $\frac{dp^\mu}{d\tau}$ from the first of these, one finds Eq. (1). It moreover follows from the first of Eqs.(3) that

$$\frac{dx^\mu}{d\tau} \frac{dx_\mu}{d\tau} = \frac{(p^\mu - eA^\mu(x))(p_\mu - eA_\mu(x))}{M^2};$$  \hspace{1cm} (4)

this quantity is absolutely conserved, since $K$ does not depend explicitly on $\tau$. It follows, since the square of the proper time $ds^2 = -dx^\mu dx_\mu$, that $ds$ is proportional to $d\tau$, independently of the acceleration of the particle. The numerator of (4) is the mass-squared of the particle; we infer that this result is associated with the restriction of the particle to a sharp mass shell.
Taking into account full $U(1)$ gauge invariance, the Stueckelberg-Schrödinger equation\(^6\) (including a compensation field for the $\tau$-derivative) is
\[
(i \frac{\partial}{\partial \tau} + e_0 a_5) \psi_\tau(x) = \frac{(p^\mu - e_0 a^\mu(x, \tau))(p_\mu - e_0 a_\mu(x, \tau))}{2M} \psi_\tau(x),
\]
where the gauge fields may depend on $\tau$ and $e_0$ is a dimensionless coupling. The corresponding classical Hamiltonian then has the form
\[
K = \frac{(p^\mu - e_0 a^\mu(x, \tau))(p_\mu - e_0 a_\mu(x, \tau))}{2M} - e_0 a_5(x, \tau).
\]
The equations of motion for the field variables are given (for both the classical and quantum theories) by\(^8\)
\[
\lambda \partial_\alpha f_{\beta\alpha}(x, \tau) = e_0 j^\beta(x, \tau),
\]
where $\alpha, \beta = 0, 1, 2, 3, 5$, the last corresponding to the $\tau$ index, and $\lambda$, of dimension $\ell^{-1}$, is a factor on the terms $f_{\alpha\beta}f_{\alpha\beta}$ in the Lagrangian associated with (6) (including degrees of freedom of the fields), required by dimensionality, as we shall see below. The field strengths are
\[
f_{\alpha\beta} = \partial_\alpha a_\beta - \partial_\beta a_\alpha,
\]
and the current satisfies the conservation law\(^8,10\)
\[
\partial_\alpha j^\alpha(x, \tau) = 0.
\]
Writing out (9) explicitly ($j^5 \equiv \rho$, the density of events in spacetime),
\[
\partial_5 \rho + \partial_\mu j^\mu = 0;
\]
integrating over $\tau$ on $(-\infty, \infty)$, and assuming that $j^5(x, \tau)$ vanishes\(^8\) at $|\tau| \to \infty$, one finds that
\[
\partial_\mu J^\mu(x) = 0,
\]
where (for some dimensionless $\eta$)
\[
J^\mu(x) = \eta \int_{-\infty}^{\infty} d\tau j^\mu(x, \tau).
\]
We identify this $J^\mu(x)$ with the Maxwell conserved current. In ref. 9, for example, this expression occurs with
\[
j^\mu(x, \tau) = \dot{x}^\mu(\tau) \delta^4(x - x(\tau)),
\]
and $\tau$ is identified with the proper time of the particle (an identification which can be made for the motion of a free particle).

Integrating the $\mu$-components of Eq. (7) over $\tau$ (assuming $f^{\mu 5}(x, \tau) \to 0$ for $\tau \to \pm \infty$), we obtain the Maxwell equations with the Maxwell charge $e = e_0/\eta$ and the Maxwell fields given by
\[
A^\mu(x) = \lambda \int_{-\infty}^{\infty} a^\mu(x, \tau) d\tau.
\]
The Hamiltonian of Stueckelberg\textsuperscript{6} and Mondonça and Oliveira e Silva\textsuperscript{8} can be recovered in the limit of the zero mode of the fields

$$a^\mu(x, \tau) = \int ds \hat{a}^\mu(x, s)e^{-is\tau}. \quad (14)$$

In the zero mode limit, when the Fourier transform of the fields have support only in the neighborhood $\Delta s$ of $s = 0$, the vector potential takes on the form $a^\mu(x, \tau) \sim \Delta s \hat{a}^\mu(x, 0) = (\Delta s/2\pi \lambda)A^\mu(x)$, and we identify $e = (\Delta s/2\pi \lambda)e_0$. The zero mode therefore emerges when the inverse correlation length of the field satisfies the relation $\eta \Delta s = 2 \pi \lambda$. We remark that in this limit, the fifth equation obtained from (7) decouples; the zero mode of the $\tau$ derivative of $a^\mu(x, \tau)$ vanishes. If the parameter $\lambda$ is independent of the dynamical structure of the fields, then the effective width of $\hat{a}^\mu(x, s)$, when it is well-defined, affects the value of the charge $e$, as well as the relation between the effective Maxwell current and the microscopic current $j^\mu$. This effect, occurring when a Maxwell type theory is a good approximation, can be understood as a classical analog of charge renormalization, where the effective charge is a function of momentum transfer.

Again, writing the Hamilton equations for the Hamiltonian (6), we find the generalized Lorentz force\textsuperscript{10}

$$M\ddot{x}^\mu = e_0 f^\mu_{\nu}\dot{x}^{\nu} + f^\mu_{5} = e_0 (f_{\text{self}}^\mu_{\nu}\dot{x}^{\nu} + f_{\text{self}}^\mu_{5} + f_{\text{ext}}^\mu_{\nu}\dot{x}^{\nu} + f_{\text{ext}}^\mu_{5}). \quad (15)$$

Multiplying this equation by $\dot{x}_{\mu}$, one obtains

$$M\dot{x}_{\mu}\ddot{x}^\mu = e_0 \dot{x}_{\mu} f^\mu_{5} = e_0 (\dot{x}_{\mu} f_{\text{self}}^\mu_{5} + \dot{x}_{\mu} f_{\text{ext}}^\mu_{5}); \quad (16)$$

this equation therefore does not necessarily lead to the trivial relation between $ds$ and $d\tau$ discussed above in connection with Eq. (4). The $f^\mu_{5}$ term has the effect of moving the particle off-shell.

In the following we use the Green’s functions for (7) to calculate the radiation reaction force directly, as, for example, in the derivation of Sokolov and Ternov\textsuperscript{11}. In the limit for which the particle stays on its mass shell during the interaction, we show that this formula reduces to the known Abraham-Lorentz-Dirac formula\textsuperscript{9,12} for the Maxwell self-interaction problem. We furthermore show that the deviation from mass shell is stable. We shall use the retarded Green’s function and treat divergences by renormalization of charge and the mass parameter $M$.

Choosing the generalized Lorentz gauge $\partial_\alpha a^\alpha = 0$, Eq. (7) becomes

$$\lambda \partial_\alpha \partial^\alpha a^\beta(x, \tau) = (\sigma \partial^2_\tau - \partial^2_t + \nabla^2) a^\beta = -e_0 j^\beta(x, \tau), \quad (17)$$

where $\sigma = \pm 1$ corresponds to the possible choices of metric for the symmetry $O(4,1)$ or $O(3,2)$ of the homogeneous field equations.

The Green’s functions for Eq. (17) can be constructed from the inverse Fourier transform

$$G(x, \tau) = \frac{1}{(2\pi)^5} \int d^4k d\kappa e^{i(k^\mu x_\mu + \kappa \kappa \tau)}/k_\mu k^\mu + \sigma \kappa^2. \quad (18)$$
Integrating this expression over all \( \tau \) gives the Green’s function for the standard Maxwell field. Assuming that the radiation reaction acts causally in \( \tau \), we shall restrict our attention here to the \( \tau \)-retarded Green’s function. In his calculation of the radiation corrections to the Lorentz force, Dirac\(^{12}\) used the difference between advanced and retarded Green’s functions in order to cancel the singularities that they contain. One can, alternatively\(^{11}\), use the retarded Green’s function and “renormalize” the mass in order to eliminate the singularity. In our analysis, we follow the latter procedure.

The \( \tau \)-retarded Green’s function is given by multiplying the principal part of the integral (18) by \( \theta(\tau) \). Carrying out the integrations (on a complex contour in \( \kappa \); we consider the case \( \sigma = +1 \) in the following), one finds (this Green’s function differs from that used in ref. 13, constructed on a complex contour in \( k^0 \))

\[
G(x, \tau) = \frac{2\theta(\tau)}{(2\pi)^3} \left\{ \frac{\tan^{-1}(\sqrt{-\frac{x^2}{x^2-\tau^2}})}{(-x^2-\tau^2)^{\frac{3}{2}}} - \frac{x}{x^2+(x^2-\tau^2)^{\frac{3}{2}}} \right\} \begin{cases} x^2 + \tau^2 < 0; \\ 1 - \frac{1}{(x^2-\tau^2)^{\frac{3}{2}}} \ln \frac{\tau+\sqrt{x^2+x^2}}{-\tau+\sqrt{x^2+x^2}} - \frac{x}{x^2+(x^2-\tau^2)^{\frac{3}{2}}} \end{cases} \quad x^2 + \tau^2 > 0. (19)
\]

With the help of this Green’s function, the solutions of Eq. (17) for the self-fields can be written,

\[
a_{self}^\mu(x, \tau) = \frac{e_0}{\lambda} \int d^4x' d\tau' G(x - x', \tau - \tau') \dot{x}^\mu(\tau') \delta^4(x' - x(\tau')) \\
= \frac{e_0}{\lambda} \int d\tau' \dot{x}^\mu(\tau') G(x - x(\tau'), \tau - \tau') \\
a_{self}^5(x, \tau) = \frac{e_0}{\lambda} \int d^4x' d\tau' G(x - x', \tau - \tau') \delta^4(x' - x(\tau')) \\
= \frac{e_0}{\lambda} \int d\tau' G(x - x(\tau'), \tau - \tau')
\]

where we have used (12) (along with \( j^5(x, \tau) = \delta^4(x - x(\tau)) \)). We have written this Green’s function as a scalar, acting in the same way on all five components of the source \( j^\alpha \); to assure that the resulting field is in Lorentz gauge, however, it should be written as a five by five matrix, with the factor \( \delta_{\beta}^\alpha - k^\alpha k_\beta/k^2 \) \((k_5 = \kappa)\) included in the integrand. Since we are computing only the gauge invariant field strengths here, this extra term will not influence any of the results.

From (8) and (15), it then follows that the generalized Lorentz force for the self-action (the force of the fields generated by the world line on a point \( x^\mu(\tau) \) of the trajectory) is

\[
M \ddot{x}^\mu = \frac{e_0^2}{\lambda} \int d\tau' (\dot{x}^\mu(\tau) \dot{x}_\nu(\tau') \partial^\mu - \dot{x}^\mu(\tau) \dot{x}^\mu(\tau') \partial_\nu) G(x - x(\tau')) |_{x=x(\tau)} \\
+ \frac{e_0^2}{\lambda} \int d\tau' (\partial^\mu - \dot{x}^\mu(\tau') \partial_\tau) G(x - x(\tau')) |_{x=x(\tau)} \\
+ e_0 (f_{ext}^\mu \nu \ddot{x}^\nu + f_{ext}^\mu 5)
\]

We define \( u \equiv (x^\mu(\tau) - x^\mu(\tau'))(x^\mu(\tau) - x^\mu(\tau')) \), so that

\[
\partial_\mu = 2(x^\mu(\tau) - x^\mu(\tau')) \frac{\partial}{\partial u}. \quad \text{(22)}
\]
Eq. (21) then becomes

\[ M \ddot{x}^\mu = 2 \frac{\epsilon_0^2}{\lambda} \int d\tau' \left\{ \dot{x}^\nu(\tau) \dot{x}_\nu(\tau')(x^\mu(\tau) - x^\mu(\tau')) - \dot{x}^\nu(\tau) \dot{x}^\mu(\tau') (x_\nu(\tau) - x_\nu(\tau')) \right\} \frac{\partial}{\partial u} G(x - x(\tau'), \tau - \tau') \bigg|_{x=x(\tau)} \\
+ \frac{\epsilon_0^2}{\lambda} \int d\tau' \left\{ 2(x^\mu(\tau) - x^\mu(\tau')) \frac{\partial}{\partial u} \dot{x}^\mu(\tau') \right\} \frac{\partial}{\partial \tau} G(x - x(\tau'), \tau - \tau') \bigg|_{x=x(\tau)} \right\} \\
+ \epsilon_0 \left( f_{\text{ext}}^\mu \dot{x}^\nu + f_{\text{ext}}^5 \right) \]

We now expand the integrands in Taylor series around the most singular point \( \tau = \tau' \). In this neighborhood, keeping the lowest order terms in \( \tau'' = \tau - \tau' \), the variable \( u \) reduces to \( u \approx \dot{x}^\mu \dot{x}_\mu \tau''^2 \). We shall also use the following definition;

\[ \varepsilon \equiv 1 + \dot{x}^\mu \dot{x}_\mu, \tag{24} \]

a quantity that vanishes on the mass shell of the particle (as we have pointed out above). In this case the derivatives of (19) take the form

\[ \frac{\partial G}{\partial u} \approx \frac{\theta(\tau'') f_1(\varepsilon)}{(2\pi)^3 \tau''^5} \]

\[ \frac{\partial G}{\partial \tau''} \approx \frac{\theta(\tau'') f_2(\varepsilon)}{(2\pi)^3 \tau''^4} + \frac{\delta(\tau'') f_3(\varepsilon)}{(2\pi)^3 \tau''^3} \tag{25} \]

where we have used the following definitions:

\[ \varepsilon < 0: \]

\[ f_1(\varepsilon) = \frac{3 \tan^{-1}(\sqrt{-\varepsilon})}{(-\varepsilon)^{\frac{1}{2}}} - \frac{3}{\varepsilon^2(1-\varepsilon)} + \frac{2}{\varepsilon \varepsilon(1-\varepsilon)^2} \]

\[ f_2(\varepsilon) = \frac{3 \tan^{-1}(\sqrt{-\varepsilon})}{(-\varepsilon)^{\frac{1}{2}}} - \frac{1}{\varepsilon^2} - \frac{2 - \varepsilon}{\varepsilon^2(1-\varepsilon)} \tag{26a} \]

\[ f_3(\varepsilon) = \frac{\tan^{-1}(\sqrt{-\varepsilon})}{(-\varepsilon)^{\frac{1}{2}}} + \frac{1}{\varepsilon(1-\varepsilon)} \]

\[ \varepsilon > 0: \]

\[ f_1(\varepsilon) = \frac{3}{2} \ln\left| \frac{1 + \sqrt{\varepsilon}}{1 - \sqrt{\varepsilon}} \right| \left( \varepsilon \right)^{\frac{1}{2}} - \frac{3}{\varepsilon^2(1 - \varepsilon)} + \frac{2}{\varepsilon \varepsilon(1 - \varepsilon)^2} \]

\[ f_2(\varepsilon) = \frac{3}{2} \ln\left| \frac{1 + \sqrt{\varepsilon}}{1 - \sqrt{\varepsilon}} \right| \left( \varepsilon \right)^{\frac{1}{2}} - \frac{1}{\varepsilon^2} - \frac{2 - \varepsilon}{\varepsilon^2(1 - \varepsilon)} \tag{26b} \]

\[ f_3(\varepsilon) = -\frac{1}{2} \ln\left| \frac{1 + \sqrt{\varepsilon}}{1 - \sqrt{\varepsilon}} \right| \left( \varepsilon \right)^{\frac{1}{2}} + \frac{1}{\varepsilon(1 - \varepsilon)} \]
For either sign of \( \varepsilon \), when \( \varepsilon \sim 0 \),
\[
\begin{align*}
    f_1(\varepsilon) &\sim \frac{8}{5} + \frac{24}{7} \varepsilon, \\
    f_2(\varepsilon) &\sim -\frac{2}{5} - \frac{4}{7} \varepsilon, \\
    f_3(\varepsilon) &\sim \frac{2}{3} + \frac{4}{5} \varepsilon,
\end{align*}
\]
(26c)

One sees that the derivatives in (25) have no singularity in \( \varepsilon \) at \( \varepsilon = 0 \).

From (8) and (20), we have
\[
f_{self \mu} \varepsilon M(x(\tau), \tau) = e \int d\tau' \{ 2(x^\mu(\tau) - x^\mu(\tau')) \frac{\partial}{\partial u} - \ddot{x}^\mu(\tau') \partial_\tau \} G(x - x(\tau'), \tau - \tau') |_{x=x(\tau)},
\]
(27)
We see (from (25)) that the main contributions to the integrals come from small \( \varepsilon \). We may therefore expand \( x^\mu(\tau) - x^\mu(\tau') \) and \( \dot{x}^\mu(\tau) - \dot{x}^\mu(\tau') \) in (27) in power series in \( \varepsilon \), and write the integrals formally with infinite limits.

Substituting (27) into (16), we obtain (note that \( x^\mu \) and its derivatives are evaluated at the point \( \tau \), and are not subject to the \( \varepsilon \) integration), after integrating by parts using \( \delta(\varepsilon) \sim \frac{\partial}{\partial \varepsilon} \theta(\varepsilon') \),
\[
M \ddot{x}_\nu \ddot{x}^\nu = \frac{2e^2}{\lambda(2\pi)^3} \int_{-\infty}^{\infty} d\tau'' \left\{ \left( \frac{f_1 - f_2 - 3f_3}{\tau''^3} \ddot{x}_\nu \ddot{x}^\nu - \frac{1}{2} \frac{f_1 - f_2 - 2f_3}{\tau''} \dot{x}_\nu \ddot{x}^\nu \right) \frac{1}{2} \frac{f_1 - f_2 - 2f_3}{\tau''} \ddot{x}_\nu \ddot{x}^\nu \right\} \theta(\varepsilon') + e_0 \dot{x}_\mu f_{ext \mu} \varepsilon.
\]
(28)
The integrals are divergent at the lower bound \( \tau'' = 0 \) imposed by the \( \theta \)-function; we therefore take these integrals to a cut-off \( \mu > 0 \). Eq.(28) then becomes
\[
\frac{M}{2} \ddot{x} = \frac{2e^2}{\lambda(2\pi)^3} \left\{ \left( \frac{f_1 - f_2 - 3f_3}{3\mu^3} \ddot{x}_\nu \ddot{x}^\nu - \frac{1}{2} \frac{f_1 - f_2 - 2f_3}{4\mu^2} \ddot{x}_\nu \ddot{x}^\nu \right) \frac{1}{2} \frac{f_1 - f_2 - 2f_3}{\mu} \ddot{x}_\nu \ddot{x}^\nu + e_0 \dot{x}_\mu f_{ext \mu} \varepsilon \right\}.
\]
(29)
Following a similar procedure, we obtain from (23)
\[
\begin{align*}
    M \ddot{x}_\mu &= \frac{2e^2}{\lambda(2\pi)^3} \left\{ \left( \frac{-f_1}{4\mu^2} (1-\varepsilon) \ddot{x}_\mu + \frac{1}{2} \ddot{x}_\mu \right) + \frac{f_1}{3\mu} (\ddot{x}_\nu \ddot{x}^\nu \ddot{x}_\mu + (1-\varepsilon) \ddot{x}_\mu) \\
    &+ \left( \frac{f_1 - f_2 - 3f_3}{3\mu^3} \ddot{x}_\mu - \frac{1}{2} \frac{f_1 - f_2 - 2f_3}{2\mu^2} \ddot{x}_\mu \ddot{x}^\nu + \frac{1}{2} \frac{f_1 - f_2 - 2f_3}{\mu} \ddot{x}_\mu \ddot{x}^\nu \right) \right\} + e_0 \left\{ f_{ext \mu \nu} \ddot{x}^\nu + f_{ext \mu} \varepsilon \right\}.
\end{align*}
\]
(30)
Using (29) to substitute for the coefficient of the $\frac{1}{\mu}$ term in (30), we obtain (for $\varepsilon \neq 1$)

$$
M(\varepsilon)\dddot{x}^\mu = -\frac{1}{2} \frac{M(\varepsilon)}{1 - \varepsilon} \dddot{x}^{\mu} + \frac{2e_0^2}{\lambda(2\pi)^3} F(\varepsilon) \left\{ \dddot{x}^{\mu} + \frac{1}{1 - \varepsilon} \dddot{x}_\nu \dddot{x}^\nu \dddot{x}^\mu \right\}
$$

$$
+ \frac{e_0 \dddot{x}^{\mu} \dddot{x}_\nu f_{ext}^\nu}{1 - \varepsilon} + e_0 f_{ext}^{\mu \nu} \dddot{x}^\nu + e_0 f_{ext}^5 \dddot{x}^\mu,
$$

(31)

where

$$
F(\varepsilon) = \frac{f_1}{3} (1 - \varepsilon) + \left( \frac{1}{6} f_1 - \frac{1}{2} f_2 - \frac{1}{2} f_3 \right). 
$$

(32)

Here, the coefficients of $\dddot{x}^{\mu}$ have been grouped into a renormalized (off-shell) mass term, defined (as in the procedure of Sokolov and Ternov) as

$$
M(\varepsilon) = M + \frac{e^2}{2\mu} \left[ \frac{f_1 (1 - \varepsilon)}{2} + \frac{1}{2} f_1 - f_2 - 2 f_3 \right]
$$

(33)

where, as we shall see below,

$$
e^2 = \frac{2e_0^2}{\lambda(2\pi)^3 \mu},
$$

(34)

can be identified with the Maxwell charge by studying the on-shell limit.

We now obtain, from (31),

$$
M(\varepsilon)\dddot{x}^\mu = -\frac{1}{2} \frac{M(\varepsilon)}{1 - \varepsilon} \dddot{x}^{\mu} + F(\varepsilon) e^2 \left\{ \dddot{x}^{\mu} + \frac{1}{1 - \varepsilon} \dddot{x}_\nu \dddot{x}^\nu \dddot{x}^\mu \right\}
$$

$$
+ e_0 f_{ext}^{\mu \nu} \dddot{x}^\nu + e_0 \left( \frac{\dddot{x}^{\mu} \dddot{x}_\nu}{1 - \varepsilon} + \delta^{\mu\nu}_\nu \right) f_{ext}^\nu.
$$

(35)

We remark that when one multiplies this equation by $\dddot{x}_\mu$, it becomes an identity (all of the terms except for $e_0 f_{ext}^{\mu \nu} \dddot{x}^\nu$ may be grouped to be proportional to $\left( \frac{\dddot{x}^{\mu} \dddot{x}_\nu}{1 - \varepsilon} + \delta^{\mu\nu}_\nu \right)$); one must use Eq. (29) to compute the off-shell mass shift $\varepsilon$ corresponding to the longitudinal degree of freedom in the direction of the four velocity of the particle. Eq. (35) determines the motion orthogonal to the four velocity. Equations (29) and (35) are the fundamental dynamical equations governing the off-shell orbit.

We now show that the standard relativistic Lorentz force, with radiation corrections, can be obtained from these equations when $\mu \dddot{x} << \varepsilon << 1$ and $\dddot{x}^\mu f_{ext}^5$ are small. In this case, Eq. (29) becomes

$$
(M - \frac{1}{15\mu}) \frac{\dddot{x}}{2} \cong e^2 \left\{ -\frac{8\varepsilon}{15\mu^2} + \frac{2}{15} \dddot{x}_\nu \dddot{x}^\nu \right\}
$$

(36)

The left hand side can be neglected if

$$
\left[ M/(\frac{e^2}{\mu}) \right](\mu \dddot{x}) << \varepsilon.
$$

(37)
We shall see below that we must have $0.68 e^2/\mu < M$ for stability of $\varepsilon$, but if $e^2/\mu$ is not too small, the inequality (37) is consistent with our assumed inequalities, and it then follows that

$$4\varepsilon/\mu^2 \approx \ddot{x}_\nu \ddot{x}^\nu.$$  

(38)

If, furthermore, $\dddot{\varepsilon}$ is small, then

$$\dot{x}_\mu \dddot{x}^\mu = \dddot{\varepsilon} - \dddot{x}_\nu \dddot{x}^\nu \approx -\dddot{x}_\mu \dddot{x}^\mu,$$  

(39)

the known expression associated with radiation. Since $\varepsilon/\mu^2$ may be appreciable even if $\varepsilon$ is small, the inequalities we have assumed can admit a significant contribution of this type. Under these conditions equation (34) becomes,

$$M_{ren} \ddot{x}^\mu = \frac{2}{3} e^2 \{ \dddot{x}^\mu - \dddot{x}_\nu \dddot{x}^\nu \} + e_0 f_{\text{ext}} \mu \ddot{x}^\nu,$$  

(40)

where $M_{ren} = M(\varepsilon)|_{\varepsilon=0} = M + e^2/3\mu$.

This result is of the form of the standard relativistic Lorentz force with radiation reaction.$^{9,11,12,14}$

We now study the stability of the variations of the off-shell parameter $\varepsilon$ when the external field is removed. First, we construct an equation of motion for $\varepsilon$. We define the functions

$$F_1(\varepsilon) = \frac{1}{3\mu^2}(\varepsilon - 1)(f_1 - f_2 - 3f_3)$$

$$F_2(\varepsilon) = \frac{1}{4\mu}(\frac{1}{2}f_1 - f_2 - 2f_3)$$

$$F_3(\varepsilon) = \frac{1}{6}f_1 - \frac{1}{2}f_2 - \frac{1}{2}f_3$$

(41)

so equation (29), in the absence of external fields, becomes:

$$\frac{M}{2} \ddot{\varepsilon} = e^2 \{ F_1(\varepsilon) + F_2(\varepsilon)\dot{\varepsilon} + F_3(\varepsilon)\dot{x}_\mu \ddot{x}^\mu \}.$$  

(42)

Solving for the explicit $x$ derivatives in (42) and differentiating with respect to $\tau$, one obtains

$$\dot{x}_\mu \dddot{x}^\mu + \dddot{x}_\mu \dddot{x}^\mu = \frac{1}{F_3} \{ F_2' \dddot{x}^2 + \dddot{\varepsilon} \left( \frac{M}{2e^2} + F_2 \right) - F_1' \dddot{\varepsilon} \}$$

$$- \frac{F_3'}{F_3^2} \left( F_2 + \frac{M}{2e^2} \ddot{\varepsilon} - F_1 \right) \dddot{\varepsilon} \equiv H.$$  

(43)

Together with

$$\dot{x}_\mu \dddot{x}^\mu + 3\dddot{x}_\mu \dddot{x}^\mu = \frac{1}{2} \dddot{\varepsilon}$$

one finds, from (43),

$$\dddot{x}_\mu \dddot{x}^\mu = \frac{1}{4} \dddot{\varepsilon} - \frac{1}{2} H(\varepsilon, \dot{\varepsilon}, \ddot{\varepsilon})$$  

(44)

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Multiplying Eq.(35) by \( \ddot{x}_\mu \) (with no external fields) and using (42) and (44), we obtain
\[
\ddot{\varepsilon} - A(\varepsilon)\dot{\varepsilon} + B(\varepsilon)\dot{\varepsilon}^2 + C(\varepsilon)\dot{\varepsilon} - D(\varepsilon) = 0,
\] (45)
where
\[
A(\varepsilon) = \frac{2}{F_3} \left( \frac{M}{2e^2} + F_2 \right) + \frac{2M(\varepsilon)}{e^2 F(\varepsilon)},
\]
\[
B = \frac{2F'_3}{F_3} \left( \frac{M}{2e^2} + F_2 \right) - \frac{2F'_2}{F_3} + \frac{2}{1 - \varepsilon} \frac{1}{F_3} \left( \frac{M}{2e^2} + F_2 \right) - \frac{M(\varepsilon)}{e^2 F(\varepsilon)} \frac{1}{1 - \varepsilon},
\]
\[
C = \frac{4M(\varepsilon)}{e^2 F(\varepsilon)} \frac{1}{F_3} \left( \frac{M}{2e^2} + F_2 \right) - \frac{2}{F_3} F_1 F'_3
\]
\[
- \frac{2F_1}{(1 - \varepsilon)F_3} + \frac{2}{F_3} F_1',
\]
\[
D = \frac{4M(\varepsilon)}{e^2 F(\varepsilon)} \frac{F_1}{F_3}.
\] (46)

We first study the possibility of having a solution of the form \( \varepsilon \equiv \varepsilon_0 \), a constant. In this case \( \ddot{\varepsilon} = 0 \) implies,
\[
\dot{x}^\mu \ddot{x}_\mu = -\ddot{x}_\mu \ddot{x}^\mu.
\]
Since all the derivatives of \( \varepsilon \) are zero we also find from (44),
\[
\ddot{x}_\mu \ddot{x}^\mu = 0
\]
. Multiplying eq.(35) by \( \ddot{x}_\mu \) and substituting these last two results we get
\[
\ddot{x}_\mu \ddot{x}^\mu = \dddot{x}_\mu \dddot{x}^\mu = 0
\]
From (42) we find then that
\[
\dddot{x}_\mu \dddot{x}^\mu = -\frac{F_1}{F_3} = 0.
\]
From (41) and (26b), one sees that this equation can be satisfied only if \( \varepsilon = 0 \) (\( F_1 = 0 \)) or \( \varepsilon = 1 \) (\( F_3 = \infty \)).

Since \( \ddot{\varepsilon} = 0 \) we find that \( \dddot{t} = \frac{1}{\dot{t}} |\dot{\varepsilon}|^2 |\dot{x}|^2 \cos^2 \theta \). Together with \( \dddot{x}_\mu \dddot{x}^\mu = \dddot{t}^2 - |\dddot{x}|^2 = 0 \) this implies
\[
\dddot{t}^2 = |\dot{x}|^2 \dddot{t}^2 \cos^2 \theta
\]
The solution \( \dddot{t} = |\dot{x}| = 0 \) implies \( \dddot{x}_\mu = \text{const} \). The other solution \( \dddot{t} = |\dot{x}| \cos^2 \theta \) implies that \( |\dot{x}|^2 (1 - \cos^2 \theta) = \varepsilon - 1 \); since the left hand side is positive, \( \varepsilon \) cannot be zero, and the only possibility for a constant solution is then \( \varepsilon = 1 \), motion on the light cone. We shall show below that the trajectory cannot reach this boundary.

The mass shell condition \( \varepsilon = 0 \), in the theoretical framework we have given here, implies that the particle motion must be with constant velocity, and that no radiation \( \dddot{x}_\mu \dddot{x}^\mu = 0 \) is possible, i.e., in order to radiate, the particle must be off-shell. This result
is also true in the presence of an external field. In particular, it follows from Eq. (29) that
for $\varepsilon \equiv 0$,
\[ -\frac{2}{15\mu} \hat{x}^{\mu} \hat{x}_{\mu} \equiv \frac{2}{15\mu} \bar{x}^{\mu} \bar{x}_{\mu} \]
\[ = e_0 \hat{x}_{\mu} \hat{f}_{\text{ext}}^{\mu} \, 5. \]

From Eq.(15), however, it follows (in case $\dot{\varepsilon} = 0$) that $\hat{x}_{\mu} \hat{f}_{\text{ext}}^{\mu} \, 5 = -\hat{x}_{\mu} \hat{f}_{\text{self}}^{\mu} \, 5$, so that the nonvanishing value of $\bar{x}^{\mu} \bar{x}_{\mu}$ corresponds only to a self-acting field $f_{\text{self}}^{\mu} \, 5$ (driven by $\hat{f}_{\text{ext}}^{\mu} \, 5$), and not to radiation.

We now show that, in general, $\varepsilon$ is bounded when the external fields are turned off. For the case $\varepsilon < 0$ the function $F_3$ is zero at $\varepsilon = -0.735$. In this case eq.(42) becomes
\[ \dot{\varepsilon}(-0.735) = \frac{F_1(-0.735)}{2e^2} + F_2(-0.735). \]
If $\dot{\varepsilon} > 0$ at this value of $\varepsilon$, then $\varepsilon$ cannot cross this boundary. Since $F_1(-0.735) = \frac{624}{\mu^2}$, $F_2(-0.375) = -\frac{259}{\mu}$, this condition implies that
\[ \mu > 0.68 e^2 \frac{1}{M}. \]
Setting $M, e$ equal to the electron mass (the lowest mass charged particle) and charge one finds that $\mu > 10^{-23} \text{sec}$, a cut-off of reasonable size for a classical theory.

We now show that $\varepsilon$ is bounded from above by unity. The full classical Hamiltonian, obtained by adding the contribution of the fields to the expression on the right hand side of (6), is a conserved quantity. In the absence of external fields, all the field quantities are related to the source particle through the Green’s functions. In the absence of external fields, as the particle motion approaches the light cone, there are infinite contributions arising from the fields evaluated on the particle trajectory. In this case, it follows from (4) that $(p^{\mu} - e_0 a^{\mu}(x,\tau))(p^{\nu} - e_0 a_{\nu}(x,\tau)) = 0$. The $a_5$ self-field term is less singular than the $f_{\mu\nu} f^{\mu\nu}$ and $f_{\mu5} f^{\mu5}$ terms, which involve derivatives of the Green’s functions, as in (25), squared. As seen from (26b), the most singular contribution arises from $f_i^2$. Since the total Hamiltonian $K$ is conserved, the coefficient of this singularity must vanish. The coefficients involve just $\varepsilon$ (and its square) and $\dot{\varepsilon}$; one finds a simple nonlinear differential equation for which only $\varepsilon = 0$ can be a solution. It follows that the conservation law restricts the evolution of $\varepsilon$ to values less than unity, i.e. the particle trajectory cannot pass through the light cone.

This bound manifests itself in the structure of the differential equation (45) for $\varepsilon$. In the limit that $\varepsilon \to 1$, the coefficients $A, B, C, D$ are all finite; however the behavior of the linearized solution depends on the derivates of these coefficients, and, in this limit, $B'$ is singular, driving the solution away from the light cone.

Numerical studies are under way to follow the motion of this highly nonlinear system both in the presence and absence of external fields.

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References
1. G.M. Zaslavskii, M.Yu. Zakharov, R.Z. Sagdeev, D.A. Usikov, and A.A. Chernikov, Zh. Eksp. Teor. Fiz 91, 500 (1986) [Sov. Phys. JEPT 64, 294 (1986)].
2. D.W. Longcope and R.N. Sudan, Phys. Rev. Lett. 59, 1500 (1987).
3. H. Karimabadi and V. Angelopoulos, Phys. Rev. Lett. 62, 2342 (1989).
4. A. Gupta and T. Padmanabhan, Phys. Rev. D57,7241 (1998). An approach using non-locality has been proposed by B. Mashoon,Proc. VII Brazilian School of Cosmology and Gravitation, Editions Frontières (1944); Phys. Lett. A 145, 147 (1990); Phys. Rev.A47, 4498 (1993). We thank J. Beckenstein for bringing the latter works to our attention.
5. Y. Ashkenazy and L.P. Horwitz, chao-dyn/9905013, submitted.
6. E.C.G. Stueckelberg, Helv. Phys. Acta 14, 322 (1941); 14, 588 (1941); J.R.P. Feynman, Rev. Mod. Phys. 20, 367 (1948); R.P. Feynman, Phys. Rev. 80, 440(1950); J.S. Schwinger, Phys. Rev. 82, 664 (1951);L.P. Horwitz and C. Piron, Helv. Phys. Acta 46, 316 (1973).
7. J.T. Mendonça and L. Oliveira e Silva, Phys. Rev E 55, 1217 (1997).
8. D. Saad, L.P. Horwitz and R.I. Arshansky, Found. of Phys. 19, 1125 (1989); M.C. Land, N. Shnerb and L.P. Horwitz, Jour. Math. Phys. 36, 3263 (1995); N. Shnerb and L.P. Horwitz, Phys. Rev A48, 4068 (1993). We use a different convention for the parameters here.
9. See, for example, J.D. Jackson, Classical Electrodynamics, 2nd edition, John Wiley and Sons, New York(1975); F. Rohrlich, Classical Charged Particles, Addison-Wesley, Reading, (1965); S. Weinberg, Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity, Wiley, N.Y. (1972).
10. M.C. Land and L.P. Horwitz, Found. Phys. Lett. 4, 61 (1991); M.C. Land, N. Shnerb and L.P. Horwitz, Jour. Math. Phys. 36, 3263 (1995).
11. For example, A.A. Sokolov and I.M. Ternov, Radiation from Relativistic Electrons, Amer. Inst. of Phys. Translation Series, New York (1986).
12. P.A.M. Dirac, Proc. Roy. Soc. London Ser. A, 167, 148(1938).
13. M.C. Land and L.P. Horwitz, Found. Phys. 21, 299 (1991).
14. L.D. Landau and E.M. Lifshitz, The Classical Theory of Fields 4th ed., (Pergamon Pr., Oxford, 1975).
15. A.O. Barut and Nuri Unal, Phys. Rev A40, 5404 (1989) found non-vanishing contributions of the type $\dot{x}_\nu \ddot{x}^\nu$ to the Lorentz-Dirac equation in the presence of spin.