Construction of Multi-Bubble Solutions for a System of Elliptic Equations arising in Rank Two Gauge Theory

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Abstract

We study the existence of multi-bubble solutions for the following skew-symmetric Chern–Simons system

\[
\begin{align*}
\Delta u_1 + \frac{1}{\varepsilon^2} e^{u_2}(1 - e^{u_1}) &= 4\pi \sum_{i=1}^{2k} \delta_{p_{1,i}} \quad \text{in} \quad \Omega, \\
\Delta u_2 + \frac{1}{\varepsilon^2} e^{u_1}(1 - e^{u_2}) &= 4\pi \sum_{i=1}^{2k} \delta_{p_{2,i}}
\end{align*}
\]

where \( k \geq 1 \) and \( \Omega \) is a flat torus in \( \mathbb{R}^2 \). It continues the joint work with Zhang\[20\], where we obtained the necessary conditions for the existence of bubbling solutions of Liouville type. Under nearly necessary conditions (see Theorem 1.1), we show that there exist a sequence of solutions \((u_1, \varepsilon, u_2, \varepsilon)\) to (0.1) such that \( u_1, \varepsilon \) and \( u_2, \varepsilon \) blow up simultaneously at \( k \) points in \( \Omega \) as \( \varepsilon \to 0 \).

1 Introduction

Over the past few decades, various Chern–Simons models have been proposed in the study of condensed matter physics and particle physics, including the relativistic Chern-Simons models of high temperature superconductivity \[21\] \[12\], Lozano-Marqués-Moreno-Schaposnik model \[32\] of bosonic sector of \( \mathcal{N} = 2 \) supersymmetric Chern-Simons-Higgs theory, Gudnason model \[14\] \[15\] of \( \mathcal{N} = 2 \) supersymmetric Yang-Mills-Chern-Simons-Higgs theory, Aharony–Bergman–Jafferis–Maldacena model \[1\] and so on. The self-dual solutions of these Chern-Simons models often can be reduced to systems of elliptic partial differential equations with exponential nonlinearities. We refer the readers to the \[12\] \[40\] \[38\] for exhaustive bibliography.

For the Abelian Chern-Simons models, Hong, Kim and Pac\[18\], and Jackiw and Weinberg\[21\] considered a model with one Chern–Simons gauge field and constructed selfdual Abelian Chern–Simons–Higgs vortices which describe anyonic solitons in 2+1 dimensions. Speilman et al.\[34\] observed no parity breaking in an experiment with high temperature superconductivity. In \[16\] and \[39\], those authors indicated the parity broken may not happen in the field theory with even number of Chern-Simons gauge fields. One of the simplest models of this kind is the \( U(1) \times U(1) \) Chern-Simons model of two Higgs fields, where each of them coupled to one of
two Chern–Simons fields. In this paper, we will investigate the relativistic self-dual \(U(1) \times U(1)\) Chern–Simons model proposed by Kim et al.\(^{23}\). We give a brief description on this model below.

Consider the Lagrangian action density of the \([U(1)]^2\) Chern–Simons model with only mutual Chern–Simons interaction is given in the form

\[
\mathcal{L} = -\frac{\varepsilon}{4} \varepsilon^{\mu
u\alpha} \left( A_\mu^{(1)} F^{(2)}_{\mu\nu} + A_\mu^{(2)} F^{(1)}_{\mu\nu} \right) + \sum_{i=1}^{2} D_\mu \phi_i D^\mu \phi_i - V(\phi_1, \phi_2),
\]

where \(\varepsilon > 0\) is a coupling parameter, \((A^{(1)}_\mu)\) and \((A^{(2)}_\mu)\) are two associated Abelian gauge fields with the electromagnetic fields \(F^{(i)}_{\mu\nu} = \partial_\mu A^{(i)}_\nu - \partial_\nu A^{(i)}_\mu\), \(\phi_1\) and \(\phi_2\) are two Higgs scalar fields with the covariant derivatives \(D_\mu \phi_i = \partial_\mu \phi_i - i A^{(i)}_\mu \phi_i (\mu = 0, 1, 2, i = 1, 2)\), and the Higgs potential \(V(\phi_1, \phi_2)\) is taken as

\[
V(\phi_1, \phi_2) = \frac{1}{4\varepsilon^2} \left( |\phi_2|^2 \left[ |\phi_1|^2 - 1 \right]^2 + |\phi_1|^2 \left[ |\phi_2|^2 - 1 \right]^2 \right).
\]

Set \(u_{i, \varepsilon} = \ln |\phi_i|^2\), and denote the zeros of \(\phi_i\) by \(\{p_{i,1}, \ldots, p_{i,N_i}\}\), \(i = 1, 2\) as in \(^{22}\). After a BPS reduction \(^{2,33}\), we find that \((u_{1, \varepsilon}, u_{2, \varepsilon})\) satisfies

\[
\begin{align*}
\Delta u_1 + \frac{1}{\varepsilon^2} e^{u_2} (1 - e^{u_1}) &= 4\pi \sum_{i=1}^{N_1} \delta_{p_{1,i}} \quad \text{in} \quad \Omega, \\
\Delta u_2 + \frac{1}{\varepsilon^2} e^{u_1} (1 - e^{u_2}) &= 4\pi \sum_{i=1}^{N_2} \delta_{p_{2,i}},
\end{align*}
\]

where \(\delta_p\) is the Dirac measure at \(p\). See \(^{23,13,24}\) for the details of the derivation of \((1.3)\).

In physical literature, \(\Omega\) here is usually refereed to \(\mathbb{R}^2\) or a flat tours in \(\mathbb{R}^2\). In this paper, we consider the case of flat tours. We refer the readers to \(^{24,25,19,17,6}\) and reference therein for the recent developments.

Set \(u_1 \equiv u_2\) and \(\{p_{i,1}\}_{i=1}^{N_1} = \{p_{j,1}\}_{j=1}^{N_2}\), then the system \((1.3)\) is reduced to the Abelian Chern–Simons equation with one Higgs particle proposed by Kim–Pac\(^{18}\) and Jackiw–Weinberg\(^{21}\),

\[
\Delta u + \frac{1}{\varepsilon^2} e^u (1 - e^u) = 4\pi \sum_{s=1}^{N} \delta_{p_s},
\]

which has been extensively studied for more than twenty years. We refer the readers to \(^{33,37,3,7,12,36,4,8,10,11}\) and reference therein for more details.

We introduce the Green function \(G(x, p)\) to eliminate the Dirac measure in \((1.4)\). Here, the Green function \(G(x, p)\) is a doubly periodic function on \(\partial\Omega\) and satisfies

\[
-\Delta G(x, p) = \delta_p - \frac{1}{|\Omega|},
\]

and \(|\Omega|\) is the measure of \(\Omega\). Let \(u_0(x) = -4\pi \sum_{j=1}^{N} G(x, p_j)\). With the transformation \(u \to u + u_0\), \((1.4)\) becomes

\[
\Delta u + \frac{1}{\varepsilon^2} e^{u+u_0} (1 - e^{u+u_0}) = \frac{4N\pi}{|\Omega|},
\]

which is the equation of \((1.5)\).
Choe and Kim[9] showed that, there may have a sequence of solutions to (1.6) satisfying the following:

There is a finite set \( \{ x_{1, \varepsilon}, \ldots, x_{l, \varepsilon} \} \), \( x_{j, \varepsilon} \in \Omega, j = 1, \ldots, l \), such that, as \( \varepsilon \to 0 \)

\[
u_{\varepsilon}(x_{j, \varepsilon}) + 2 \ln \frac{1}{\varepsilon} \to \infty, \ j = 1, \ldots, l,
\]
and

\[
u_{\varepsilon} + 2 \ln \frac{1}{\varepsilon} \to -\infty \text{ uniformly on any compact subset of } \Omega \setminus \{ q_1, \ldots, q_l \},
\]
where \( q_j = \lim_{\varepsilon \to 0} x_{j, \varepsilon} \). Furthermore,

\[rac{1}{\varepsilon^2} e^{u_{\varepsilon} + u_0} (1 - e^{u_{\varepsilon} + u_0}) \to \sum_{j=1}^{l} M_j \delta_{q_j}, \ \text{as} \ \varepsilon \to 0, M_j \geq 8\pi.
\]

Solutions of (1.6) satisfying (1.7) and (1.8) are called 

bubbling solutions and \( q_j \) is called the 

blow-up point of the bubbling solution.

We can classify these blow-up points as follows: After suitable rescaling at the blow-up point \( q_j \), the bubbling solution \( u_{\varepsilon} \) to (1.6) converges to either the entire solution of

\[
\Delta u + |x|^{2m} e^u (1 - |x|^{2m} e^u) = 0,
\]

or the entire solution of

\[
\Delta u + |x|^{2m} e^u = 0.
\]

Here, \( m = 0 \) if \( q_j \) is not a vortex point and \( m = \# \{ p_i : p_i = q_j \} \) if \( q_j \) is a vortex point. The blow-up point is called Chern-Simons type if the limiting equation is (1.10) and is called mean field type if the limiting equation is (1.11). The existence and non-existence of bubbling solutions of (1.6) have been studied in a series of work of Lin and Yan[28, 29, 30].

For (1.3), we may use the following system to construct the bubbling solutions to (1.3) which blow up at the same point \( q \).

(i)

Chern-Simons system:

\[
\begin{align*}
\Delta u_1 + |x|^{m_2} e^{u_2} (1 - |x|^{m_1} e^{u_1}) &= 0, \\
\Delta u_2 + |x|^{m_1} e^{u_1} (1 - |x|^{m_2} e^{u_2}) &= 0 \quad \text{in} \quad \mathbb{R}^2.
\end{align*}
\]

(ii)

Liouville system:

\[
\begin{align*}
\Delta u_1 + |x|^{m_2} e^{u_2} &= 0, \\
\Delta u_2 + |x|^{m_1} e^{u_1} &= 0 \quad \text{in} \quad \mathbb{R}^2.
\end{align*}
\]

Here, \( m_1 = \# \{ p_{1,j} : p_{1,j} = q \} \) and \( m_2 = \# \{ p_{2,j} : p_{2,j} = q \} \). For the case (i), the blow-up point is called the Chern-Simons type; for the case (ii), the blow-up point is called the Liouville type.

For the existence of the bubbling solutions of Chern-Simons type to (1.3), there are two known results. The first one is due to Lin and Yan[28], in which they assume that \( N_1 = N_2 \) and one of the vortex points \( \{ p_{1,i} \}_{i=1, \ldots, N_1} \) and \( \{ p_{1,i} \}_{i=1, \ldots, N_1} \) coincide, then they constructed a Chern-Simons type bubbling solution blow up at that vortex point.
Theorem A. Suppose that $N_1 = N_2 = N > 4$ and $p_{1,1} = p_{2,1} = p_1$ and $p_{1,1} \neq p_{1,j}, p_{2,j}$, $j > 1$. Then, there is an $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$, (1.3) has a solution $(u_{1,\varepsilon}, u_{2,\varepsilon})$, satisfying

\[
 u_{1,\varepsilon} - u_{2,\varepsilon} \to 4\pi \sum_{j=2}^{N} G(p_{1,1}, p_{1,j}) - 4\pi \sum_{j=2}^{N} G(p_{2,1}, p_{2,j}) \tag{1.14}
\]

and

\[
 \frac{1}{\varepsilon^2} e^{u_{i,\varepsilon}} (1 - e^{u_{j,\varepsilon}}) \to 4\pi N \delta_{p_1}, \quad i \neq j, \quad i, j \in \{1, 2\} \tag{1.15}
\]

as $\varepsilon \to 0$.

The idea in the proof of Theorem A is using the ansatz

\[
 u_1 - u_2 \approx 4\pi \sum_{j=2}^{N} G(p_{1,1}, p_{1,j}) - 4\pi \sum_{j=2}^{N} G(p_{2,1}, p_{2,j})
\]

to reduce the system (1.3) to a single equation.

In a joint work with Han and Lin, we proved the existence of Chern-Simons type bubbling solutions when the blow-up point is not vortex point.

Theorem B. Suppose $(N_1 - 1)(N_2 - 1) > 1$, and $q$ satisfies

\[
 D(N_1 u_{2,0} + N_2 u_{1,0})(q) = 0 \tag{1.16}
\]

and

\[
 \deg(D(N_1 u_{2,0} + N_2 u_{1,0}), q) \neq 0. \tag{1.17}
\]

Then the system (1.3) admits Chern-Simons type bubbling solutions $(u_{1,\varepsilon}, u_{2,\varepsilon})$ blowing up at $q$.

Theorem A and B only discussed the one blow-up case. However, the problem on the existence of multi-bubble solutions of (1.3) is still open. The analysis of linearized system to (1.12) and (1.13) is not fully understood when $(m_1, m_2) \neq (0, 0)$. Thus, we only investigate the issue on the existence of the bubbling solutions of Liouville type to (1.3) whose blow up points are regular points. We give more precise description on this type bubbling solutions below.

Let $(u_{1,\varepsilon}, u_{2,\varepsilon})$ be a sequence of solutions to (1.3) blowing up at $\{q_1, \cdots, q_k\}$ and $q_i \notin \{p_{1,1}, p_{2,j}\}_{1 \leq i \leq N_1, 1 \leq j \leq N_2}$. For a small constant $d > 0$, we define the local mass of $(u_{1,\varepsilon}, u_{2,\varepsilon})$ at $q_i$:

\[
 (m_{1,\varepsilon}, m_{2,\varepsilon}) = \left( \frac{1}{\varepsilon^2} \int_{B_d(q_i)} e^{u_{2,\varepsilon}} (1 - e^{u_{1,\varepsilon}}), \frac{1}{\varepsilon^2} \int_{B_d(q_i)} e^{u_{1,\varepsilon}} (1 - e^{u_{2,\varepsilon}}) \right).
\]

We assume that there exist $\{x_{1,j,\varepsilon}\}_{j=1}^{k}$ and $\{x_{2,j,\varepsilon}\}_{j=1}^{k}$ such that

(a1) $u_{i,\varepsilon}(x_{i,j,\varepsilon}) + 2 \ln \frac{1}{\varepsilon} \to +\infty$ as $\varepsilon \to 0$, $i = 1, 2$, $j = 1, \cdots, k$.

(a2) $u_{i,\varepsilon}(x) + 2 \ln \frac{1}{\varepsilon} \to -\infty$ as $\varepsilon \to 0$ uniformly on any compact set of $\Omega \setminus \{q_1, \cdots, q_k\}$, where $q_j = \lim_{\varepsilon \to 0} x_{1,j,\varepsilon} = \lim_{\varepsilon \to 0} x_{2,j,\varepsilon}$, $j = 1, \cdots, k$. 


(a3) \( \beta_{j,\varepsilon} = \max\{u_{1,\varepsilon}(x_{1,j,\varepsilon}), u_{2,\varepsilon}(x_{2,j,\varepsilon})\} \to -\infty \) as \( \varepsilon \to 0 \).

(a4) \(|u_{1,\varepsilon}(x_{1,j,\varepsilon}) - u_{2,\varepsilon}(x_{2,j,\varepsilon})| = O(1)\)

(a5) \(|x_{i,j,\varepsilon} - q_j| < C\varepsilon e^{-\frac{1}{8}\beta_{j,\varepsilon}}\) for some constant \( C > 0 \).

The solutions \((u_{1,\varepsilon}, u_{2,\varepsilon})\) satisfying \((a1)-(a5)\) are called fully bubbling solutions of Liouville type.

Let \( \mu_{j,\varepsilon} = \varepsilon e^{-\frac{1}{8}\beta_{j,\varepsilon}} \) and assume \( \max\{u_{1,\varepsilon}(x), u_{2,\varepsilon}(x)\} \) attains its maximum at \( x_{j,\varepsilon} \) for \( x \) near \( q_j \). Formally,

\[
(\bar{u}_1(y), \bar{u}_2(y)) = (u_{1,\varepsilon}(\mu_{j,\varepsilon}y + x_{j,\varepsilon}) - \beta_{j,\varepsilon}, u_{2,\varepsilon}(\mu_{j,\varepsilon}y + x_{j,\varepsilon}) - \beta_{j,\varepsilon})
\]

converges to the entire solution \((U_{1,j}, U_{2,j})\) of \((1.13)\) with \( m_1 = m_2 = 0 \). Furthermore, the flux \((M_{1,j}, M_{2,j}) = (\int_{\mathbb{R}^2} e^{U_{2,j}}, \int_{\mathbb{R}^2} e^{U_{1,j}})\) satisfies

\[
\frac{1}{M_{1,j}} + \frac{1}{M_{2,j}} = \frac{1}{4\pi}.
\]

Thus, either \( \min\{M_{1,j}, M_{2,j}\} < 8\pi \) or \( M_{1,j} = M_{2,j} = 8\pi \). The analysis of bubbling solutions of these two types has different phenomena. We refer the readers to [31] for the related bubbling analysis for Liouville system.

In a joint work with Zhang [20], we obtained necessary conditions for the fully bubbling solutions of Liouville type to \((1.3)\). Before we state the main result in [20], we introduce some notations. Let \( q = (q_1, \ldots, q_k), q_i \in \mathbb{R}^2, i = 1, \ldots, k, \)

\[
G_1^i(q) = \sum_{i=1}^k u_{0,1}(q_i) + 8\pi \sum_{1 \leq i < j \leq k} G(q_i, q_j) \quad \text{and} \quad G_2(q) = \sum_{i=1}^k u_{0,2}(q_i) + 8\pi \sum_{1 \leq i < j \leq k} G(q_i, q_j).
\]

Denote the function \( f_{i,j,q}(i = 1, 2, j = 1, \ldots, k) \) as follows.

\[
f_{i,j,q} = 8\pi(\gamma(y, q_j) - \gamma(q_j, q_j)) + \sum_{l \neq j} (G(y, q_l) - G(q_j, q_l)) + u_{0,i}(y) - u_{0,i}(q_j),
\]

where \( \gamma(y, q) \) is the regular part of \( G(y, q) \). So, it is clear that \( \frac{\partial G_1^i(x)}{\partial x_i} = \frac{\partial f_{i,j,q}(x)}{\partial x_i} \) and \( \frac{\partial G_2^i(x)}{\partial x_i} = \frac{\partial f_{i,j,q}(x)}{\partial x_i}. \) We define the quantity \( D^{(2)}(q) \) as follows

\[
D^{(2)}(q) = \lim_{\delta \to 0} \left( \sum_{j=1}^k \frac{\rho_j}{\rho_1} \left( \int_{|x - q_j|^4} \frac{e^{f_{1,j,q} - 1}}{|x - q_j|^4} - \int_{\mathbb{R}^2 \setminus \Omega_j} \frac{1}{|x - q_j|^4} \right) + \sum_{j=1}^k \frac{\rho_j}{\rho_1} \left( \int_{|x - q_j|^4} \frac{e^{f_{2,j,q} - 1}}{|x - q_j|^4} - \int_{\mathbb{R}^2 \setminus \Omega_j} \frac{1}{|x - q_j|^4} \right) \right)
\]

(1) \( \Omega_i \cap \Omega_j = \emptyset, i \neq j, i, j \in \{1, \ldots, k\}, \)
(2) $\bigcup_{j=1}^{k} \Omega_j = \overline{\Omega}$

(3) $B_{d_j}(q_j) \subset \Omega_j$, $j = 1, \ldots, k$

$\rho_j = e^{8\pi(\gamma(q_j,q_j)+\sum_{i\neq j} G(q_i,q_i)+u_{0,1}(q_j))}$, and $\rho_j^* = e^{8\pi(\gamma(q_j,q_j)+\sum_{i\neq j} G(q_i,q_i)+u_{0,2}(q_j)}$.

In particular, when $k = 1$,

$$D^{(2)}(q) = \lim_{\delta \to 0} \left( \int_{\Omega \setminus B_{d_j}} \frac{e^{u_{0,1}(x)-u_{0,1}(q)} - 1}{|x-q|^4} + \frac{e^{u_{0,1}(x)-u_{0,1}(q)} - 1}{|x-q|^4} - \int_{\mathbb{R}^2 \setminus \Omega} \frac{2}{|x-q|^4} \right).$$

(1.21)

When $\Omega$ is a rectangle, $k = 1$ and $p_{i,j} = p(i,j = 1,2)$, it was shown [26] that the Green function $G(x,p)$ has three critical points: two of them are saddle points whose corresponding $D^{(2)}(q) > 0$, and the other is a maximum points whose $D^{(2)}(q) < 0$. We refer to [26, 27] for the related discussion on this kind quantity.

The necessary conditions for the existence of fully bubbling solutions of Liouville type are given as follows.

**Theorem C** [20] Suppose $(u_{1,\varepsilon},u_{2,\varepsilon})$ is a sequence of fully bubbling solution of Liouville type to (1.3) with $N_1 = N_2 = 2k$ and blow up set is $\{q_1, \cdots, q_k\} \notin \{p_{1,i},p_{2,i}\}_{i=1,\cdots,2k}$. Then

1. $(u_{0,1} - u_{0,2})(q_i) = (u_{0,1} - u_{0,2})(q_j)$, $1 \leq i,j \leq k$.
2. $q = (q_1, \cdots, q_k)$ is a critical point of $G_1^*$ and $G_2^*$.
3. $D^{(2)}(q) \leq 0$.

Naturally, we are led to the question whether the conditions obtained in Theorem C are sufficient for the existence of such solutions. In this paper, we will construct a sequence of bubbling solutions whose limiting local masses are $(8\pi, 8\pi)$ and the sufficient condition are nearly necessary. Since we assume that $N_1 = N_2 = 2k$, it is natural to use the entire solution $(U_1,U_2)$ of (1.13) with flux $(M_1, M_2) \approx (8\pi, 8\pi)$ to construct the approximation solutions for (1.3). But $(M_1, M_2)$ must satisfy (1.18), it adds a difficulty in construction of good approximation solutions for multi-bubble case. With the assumption (A.1)(see below), we can choose $(U_1,U_2)$ with $(\int_{\mathbb{R}^2} e^{U_2} dx, \int_{\mathbb{R}^2} e^{U_1} dx) = (8\pi, 8\pi)$ to construct the approximation solutions around each blow-up point and glue them together. It is a crucial step when we apply the invertibility of linear operator in the contraction mapping argument.

**Theorem 1.1.** Assume that

(A.1) $(u_{0,1} - u_{0,2})(q_i) = (u_{0,1} - u_{0,2})(q_j)$, $1 \leq i,j \leq k$.

(A.2) $q$ be a critical point of $G_1^*$ and $G_2^*$

(A.3) $q$ be non-degenerate critical point of $G_1^* + G_2^*$

(A.4) $D^{(2)}(q) < 0$
Then for \( \varepsilon > 0 \) small, (1.3) has a solution \((u_{1,\varepsilon}, u_{2,\varepsilon})\) satisfies
\[
\frac{1}{\varepsilon^2} e^{u_i}(1 - e^{u_j}) \to 8\pi \sum_{k=1}^{N} \delta_{q_l}, \quad i \neq j, i, j = 1, 2, l = 1, \ldots, k, \quad \text{as} \quad \varepsilon \to 0. \quad (1.22)
\]

When \( k = 1 \), Theorem 1.1 is reduced to the following theorem.

**Theorem 1.2.** Assume \( q \) is a critical point of \( u_{0,1} \) and \( u_{0,2} \) and a non-degenerate critical point of \( u_{0,1} + u_{0,2} \), and \( D^{(2)}(q) < 0 \), then there exists a sequence of solution \((u_{1,\varepsilon}, u_{2,\varepsilon})\) to (1.3) blowing up at \( q \).

It will be interesting to consider the Liouville type bubbling solutions with \( N_1 \neq N_2 \). In this case, \( \frac{k}{N_1} + \frac{k}{N_2} = 1 \) and \( k \geq 2 \). For instance, \( k = 2 \) and \( (N_1, N_2) = (3, 6) \). As suggested in the study of Liouville system\[31\], the quantity of \( D^{(2)}(q) \) should be a slight different form and only depend on \( \min \{ \frac{kN_1}{N_2}, \frac{kN_2}{N_1} \} \). In \[31\], they only consider the simple bubble case, but in our case, it must be multi-bubble. Furthermore, the invertibility of the corresponding linear operator could be another difficulty. Since the height of each bubble is compatible, the dimension of the kernel space to the corresponding linear operator is \( 2k + 1 \). The kernel with respect to height is not a local condition which makes the analysis of linear operator difficult. We will come back to this issue in the future.

The rest of our paper is organized as follows. In Sec. 2, we construct approximate solutions of (1.3). In Sec. 3, we use the contraction mapping theorem to prove the existence of multi-bubble solutions. Appendix is devoted to the invertibility of the linear operator \( L_\mu \).

## 2 The Approximation Solution

In this section, we will construct an approximation solution \((U_{1,\mu}, U_{2,\mu})\) (see (2.5) below) for (1.3) with the assumptions in Theorem 1.1 and give the estimates on the approximation solution. This construction follows the method in \[29\]. Without loss of generality, we assume that \(|\Omega| = 1\).

We consider the solution of mean field equation
\[
V_{x_i,\mu_i}(y) = \ln \frac{8\mu_i^2}{(1 + \mu_i^2 |y - x_i|^2)}, \quad x_i \in \mathbb{R}^2, \quad \mu_i > 0, \tag{2.1}
\]
which satisfies
\[
\begin{cases}
\Delta V_{x_i,\mu_i}(y) + e^{V_{x_i,\mu_i}(y)} = 0 \text{ in } \mathbb{R}^2, \\
\int_{\mathbb{R}^2} e^{V_{x_i,\mu_i}(y)} dy = 8\pi.
\end{cases} \tag{2.2}
\]

In the construction of approximation solution, we always assume that \(|x_i - q_i| < \frac{C_1}{\mu_i}\) for some large \(C_1\) and
\[
\mu_i \in \left[ \frac{\beta_1}{\sqrt{\varepsilon}}, \frac{\beta_2}{\sqrt{\varepsilon}} \right] \quad \text{for some constants } \beta_1, \beta_2 > 0.
\]
Define $\omega_{x_i,\mu}$ and $\omega^*_x,\mu$ as follows.

$$\omega^*_{x_i,\mu}(y) = \begin{cases} V_{x_i,\mu}(d_i) + 8\pi G(y, x_i) - \frac{1}{2\pi} \ln \frac{1}{d_i} (1 - \frac{1}{d_i}), & y \in \Omega \setminus B_d(x_i), \\ V_{x_i,\mu}(y) + 8\pi \gamma(y, x_i) (1 - \frac{1}{d_i}), & y \in B_d(x_i), \end{cases} \quad (2.3)$$

where $\frac{1}{d_i} = \frac{1}{1 + (\mu d_i)}$ which makes $\omega^*_{x_i,\mu} \in C^1(\Omega)$, and

$$\omega^*_{x,\mu} = \sum_{i=1}^k \omega^*_{x_i,\mu}. \quad (2.4)$$

Set $\omega_{\mu}$

$$\omega_{\mu}(y) = \omega^*_{x,\mu}(y) - \int_{\Omega} \omega^*_{x,\mu}. \quad (2.5)$$

For $k \geq 2$, we construct the approximation solution $(U_{1,\mu}, U_{2,\mu})$ as follows:

$$(U_{1,\mu}, U_{2,\mu}) = (\omega_{\mu} + c_{1,\mu}, \omega_{\mu} + c_{2,\mu}), \quad (2.6)$$

where

$$c_{1,\mu} = \ln \frac{16k \pi \varepsilon^2}{\int_{\Omega} e^{u_{0,1} + \omega_{\mu}} \left( 1 + \sqrt{1 - 32k \pi \varepsilon^2} \frac{\int_{\Omega} e^{\sum_{i=1}^k (u_{0,1} + \omega_{\mu})}}{\int_{\Omega} e^{u_{0,1} + \omega_{\mu}} \frac{\int_{\Omega} e^{\sum_{i=1}^k (u_{0,1} + \omega_{\mu})}}{\int_{\Omega} e^{u_{0,1} + \omega_{\mu}}}} \right)},$$

and

$$c_{2,\mu} = \ln \frac{16k \pi \varepsilon^2}{\int_{\Omega} e^{u_{0,2} + \omega_{\mu}} \left( 1 + \sqrt{1 - 32k \pi \varepsilon^2} \frac{\int_{\Omega} e^{\sum_{i=1}^k (u_{0,2} + \omega_{\mu})}}{\int_{\Omega} e^{u_{0,2} + \omega_{\mu}} \frac{\int_{\Omega} e^{\sum_{i=1}^k (u_{0,2} + \omega_{\mu})}}{\int_{\Omega} e^{u_{0,2} + \omega_{\mu}}}} \right)},$$

Remark 2.1. When $k = 1$, the approximation solution for (1.3) has a simpler form:

$$\begin{pmatrix} U_{1,\mu} \\ U_{2,\mu} \end{pmatrix} = \begin{pmatrix} \omega^*_{x_1,\mu_1} - u_{0,1}(x_1) \\ \omega^*_{x_1,\mu_1} - u_{0,2}(x_1) \end{pmatrix}. \quad (2.6)$$

In the construction of approximation, the height $\mu_i$ of the bubble $\omega_{x_i,\mu}$ cannot be an independent variable. So, we set

$$\mu_1 = \mu, \quad \mu_i = \frac{\rho_1}{\rho_i}, \quad i = 2, \cdots, k,$$

for some $\mu > 0$,

where

$$\rho_i = e^{8\pi \gamma(x_i, x_i) + 8\pi \sum_{j \neq i} G(x_i, x_j) + u_{0,1}(x_i)}, \quad i = 1, \cdots, k.$$
Denote \( \rho_i^* = e^{8\pi \gamma(x_i,x_i) + 8\pi \sum_{j \neq i} G(x_i,x_j) + u_{0,2}(x_i)} \), \( i = 1, \cdots, k \). By using \((u_{0,1} - u_{0,2}(x_i)) - (u_{0,1} - u_{0,2}(q_i)) = O(\frac{1}{\mu})\), we obtain

\[
\rho_i^* \mu_i^2 = \rho_i^* \mu_i^2 \left( 1 + O\left( \frac{1}{\mu} \right) \right), \quad i = 1, \cdots, k. \tag{2.8}
\]

In the rest of this section, we will some properties of the approximation solution \((U_{1,\mu}, U_{2,\mu})\).

For \( l = 1, 2 \), \( j = 1, \cdots, k \) and \( y \in B_{d_j}(x_j) \), we have

\[
\begin{align*}
& u_{0,l}(y) - u_{0,l}(x_i) + 8\pi \left( (\gamma(y, x_i) - \gamma(x_i, x_i))(1 - \frac{1}{\theta_j}) + \sum_{m \neq j} (G(y, x_m) - G(x_j, x_m))(1 - \frac{1}{\theta_m}) \right) \\
& = \left< D u_{0,l}(x_i) + 8\pi \sum_{m \neq j} DG(x_i, x_m), y - x_i \right> + O(|y - x_i|^2 + \frac{1}{\mu^2}).
\end{align*}
\tag{2.9}
\]

By this and (2.1), we find that

\[
\int_{B_{d_j}(x_i)} e^{\omega_{\mu}^* + u_{0,1}} \\
= \frac{8^{k-1} \rho_i \mu_i^2}{\mu_1^2 \cdots \mu_k^2} \left( 1 + O\left( \frac{1}{\mu^2} \right) \right) \\
\times \int_{B_{d_i}(x_i)} e^{V_{\epsilon, \mu_i + u_{0,i}(x_j) + u_{0,i}(x_j) + \gamma(x,y) + \gamma(x_j, x_j) - \gamma(y, x_j) + \sum_{m \neq j} (G(y, x_m) - G(x_j, x_m))} \\
= \frac{8^{k-1} \rho_i \mu_i^2}{\mu_1^2 \cdots \mu_k^2} \left( 8\pi + \int_{B_{d_i}(x_i)} e^{V_{\epsilon, \mu_i}|y - x_i|^2 + O\left( \frac{1}{\mu^2} \right)} \right) \tag{2.10}
= \frac{8^{k-1} \rho_i \mu_i^2}{\mu_1^2 \cdots \mu_k^2} \left( 8\pi + O\left( \frac{\ln \mu}{\mu^2} \right) \right). \tag{2.11}
\]

Similarly,

\[
\int_{B_{d_i}(x_i)} e^{\omega_{\mu}^* + u_{0,2}} = \frac{8^{k-1} \rho_i^* \mu_i^2}{\mu_1^2 \cdots \mu_k^2} \left( 8\pi + O\left( \frac{\ln \mu}{\mu^2} \right) \right). \tag{2.11}
\]

By (2.7), (2.8), (2.10) and (2.11), we obtain

\[
\int_{\Omega} e^{\omega_{\mu}^* + u_{0,1}} = \frac{8^{k-1} \rho_i \mu_i^2}{\mu_1^2 \cdots \mu_k^2} \left( 8\pi + O\left( \frac{\ln \mu}{\mu^2} \right) \right) = \frac{8^{k-1} \rho_i \mu_i^2}{\mu_1^2 \cdots \mu_k^2} \left( 8\pi + O\left( \frac{\ln \mu}{\mu^2} \right) \right). \tag{2.12}
\]

Similarly,

\[
\int_{\Omega} e^{\omega_{\mu}^* + u_{0,2}} = \frac{8^{k-1} \rho_i^* \mu_i^2}{\mu_1^2 \cdots \mu_k^2} \left( 8\pi + O\left( \frac{1}{\mu} \right) \right), \tag{2.13}
\]

where (2.8) is used. The asymptotic behaviours of \( \omega_{\mu} \) and \( c_{i,\mu}(i=1,2) \) will be discussed in the following proposition.
Proposition 2.2. (1) The function $\omega_\mu$ defined in (2.4) satisfies that, for $\delta > 0$,

$$\begin{cases}
\max_{y \in B_\delta(x_i)} \omega_\mu(y) = 2 \ln \frac{1}{\varepsilon} + O(1), \\
\omega_\mu(y) = O(1), \ y \in \Omega \setminus \bigcup_{i=1}^k B_\delta(x_i).
\end{cases}$$

(2.14)

(2) The constants $c_{1,\mu}$ and $c_{2,\mu}$ satisfy

$$c_{i,\mu} = -3 \ln \frac{1}{\varepsilon} + O(1), \ i = 1, 2.$$  

(2.15)

Proof. The estimate of $\omega_\mu$ is exactly the same as Proposition 2.1 in [29]. So, we omit the proof.

By definition of $c_{1,\mu}$, we deduce that, for $\mu > 0$ large,

$$e^{c_{1,\mu}} = \frac{16k\pi \varepsilon^2}{\int_{\Omega} e^{\omega_\mu + u_0,1} \left(2 - 16k\pi \varepsilon^2 \frac{\int_{\Omega} e^{\sum_{i=1}^{2} (\omega_\mu + u_{0,1})}}{\int_{\Omega} e^{\omega_\mu + u_{0,2}}} + O((\varepsilon^2 \frac{\int_{\Omega} e^{\sum_{i=1}^{2} (\omega_\mu + u_{0,1})}}{\int_{\Omega} e^{\omega_\mu + u_{0,2}}})^2)\right)}.$$  

(2.16)

On the other hand,

$$\frac{\int_{\Omega} e^{\sum_{i=1}^{2} (\omega_\mu + u_{0,1})}}{\int_{\Omega} e^{\omega_\mu + u_{0,1}} \int_{\Omega} e^{\omega_\mu + u_{0,2}}} = \frac{\int_{\Omega} e^{\sum_{i=1}^{2} (\omega_\mu^* + u_{0,1})}}{\int_{\Omega} e^{\omega_\mu^* + u_{0,1}} \int_{\Omega} e^{\omega_\mu^* + u_{0,2}}} = O(\mu^2).$$  

(2.17)

So, by (2.16) and (2.17),

$$e^{c_{1,\mu}} = \frac{8k\pi \varepsilon^2}{\int_{\Omega} e^{\omega_\mu + u_{0,1}}} \left(1 + 8k\pi \varepsilon^2 \frac{\int_{\Omega} e^{\sum_{i=1}^{2} (\omega_\mu + u_{0,1})}}{\int_{\Omega} e^{\omega_\mu + u_{0,2}}} + O(\varepsilon^4 \mu^4)\right),$$  

(2.18)

and thus

$$c_{1,\mu} = \ln(8k\pi) + 2 \ln \varepsilon - \ln \int_{\Omega} e^{\omega_\mu + u_{0,1}} + 8k\pi \varepsilon^2 \frac{\int_{\Omega} e^{\sum_{i=1}^{2} (\omega_\mu + u_{0,1})}}{\int_{\Omega} e^{\omega_\mu + u_{0,2}}} + O(\varepsilon^4 \mu^4)$$

$$= 2 \ln \varepsilon - 2 \ln \mu + O(1)$$

$$= -3 \ln \frac{1}{\varepsilon} + O(1).$$  

(2.19)

Similarly, $c_{2,\mu} = -3 \ln \frac{1}{\varepsilon} + O(1)$.

In the end of this section we estimate

$$-\Delta U_{1,\mu} + \frac{1}{\varepsilon^2} e^{U_{1,\mu} + u_{0,2}} (e^{U_{1,\mu} + u_{0,1}} - 1) + 8k\pi$$

and

$$-\Delta U_{2,\mu} + \frac{1}{\varepsilon^2} e^{U_{1,\mu} + u_{0,1}} (e^{U_{2,\mu} + u_{0,2}} - 1) + 8k\pi$$

which will be useful in the next section.
Proposition 2.3. Let \((U_{1,\mu}, U_{2,\mu})\) be defined in (2.3). Then

\[
- \Delta U_{1,\mu} + \frac{1}{\varepsilon^2} e^{U_{2,\mu} + u_0,2} (e^{U_{1,\mu} + u_0,1} - 1) + 8k\pi \\
= \sum_{i=1}^{k} \left( 1_{B_{d_i}}(x_i) e^{V_{x_i,\mu_i}} (1 - e^{f_{1,i,x,\mu}}) + \frac{8\pi}{\theta_i} \right) \\
+ O \left( \frac{\ln \mu}{\mu^2} + \sum_{i=1}^{k} \left( \frac{1}{\mu^2} e^{V_{x_i,\mu_i} + u_0,2} + \frac{1}{\mu^4} e^{2V_{x_i,\mu_i} + u_0,1 + u_0,2} \right) \right) 
\]

(2.20)

and

\[
- \Delta U_{2,\mu} + \frac{1}{\varepsilon^2} e^{U_{1,\mu} + u_0,1} (e^{U_{2,\mu} + u_0,2} - 1) + 8k\pi \\
= \sum_{i=1}^{k} \left( 1_{B_{d_i}}(x_i) e^{V_{x_i,\mu_i}} (1 - e^{f_{1,i,x,\mu}}) + \frac{8\pi}{\theta_i} \right) \\
+ O \left( \frac{\ln \mu}{\mu^2} + \sum_{i=1}^{k} \left( \frac{1}{\mu^2} e^{V_{x_i,\mu_i} + u_0,1} + \frac{1}{\mu^4} e^{2V_{x_i,\mu_i} + u_0,1 + u_0,2} \right) \right). 
\]

where

\[
f_{1,i,x,\mu}(y) = 8\pi \left( (\gamma(y, x_i) - \gamma(x_i, x_i))(1 - \frac{1}{\theta_i}) + \sum_{j \neq i} (G(y, x_i) - G(x_i, x_j))(1 - \frac{1}{\theta_j}) \right) \\
+ u_{0,1}(y) - u_{0,1}(x_i) 
\]

(2.22)

\[
f_{2,i,x,\mu}(y) = 8\pi \left( (\gamma(y, x_i) - \gamma(x_i, x_i))(1 - \frac{1}{\theta_i}) + \sum_{j \neq i} (G(y, x_i) - G(x_i, x_j))(1 - \frac{1}{\theta_j}) \right) \\
+ u_{0,2}(y) - u_{0,2}(x_i) 
\]

(2.23)

Proof. We only prove (2.20) since the proof of (2.21) is similar.

By the definition of \(U_{1,\mu}\), we find that

\[-\Delta U_{1,\mu} = -\Delta \omega_{\mu} \\
= \sum_{i=1}^{k} \left( 1_{B_{d_i}}(x_i) e^{V_{x_i,\mu_i}} - 8\pi (1 - \frac{1}{\theta_i}) \right) 
\]

(2.24)

where \(1_A = 1\) in \(A\) and \(1_A = 0\) otherwise.

Using (2.12) and (2.13), we have

\[
\frac{1}{\varepsilon^2} e^{U_{2,\mu} + u_0,2} \\
= 8k\pi e^{\omega_{\mu} + u_0,2} \left( 1 + 8k\pi \varepsilon^2 \frac{\int_\Omega e^{\sum_{i=1}^{k} (\omega_{\mu} + u_{0,i})}}{\int_\Omega e^{\omega_{\mu} + u_0,1}} \right) + O(\varepsilon^4 \mu^4) 
\]

(2.25)
and
\[
\frac{1}{\varepsilon^2} \sum_{i=1}^{2} (U_{i,\mu} + u_{0,i}) = 64k^2 \pi^2 \varepsilon^2 \left( \frac{e^{\omega_{\mu} + u_{0,1}} e^{\omega_{\mu} + u_{0,2}}}{\int_{\Omega} e^{\omega_{\mu} + u_{0,1}} \int_{\Omega} e^{\omega_{\mu} + u_{0,2}}} \right) \left( 1 + 16k^2 \pi^2 \frac{\int_{\Omega} e^{\sum_{i=1}^{2} (\omega_{\mu} + u_{0,i})} \int_{\Omega} e^{\omega_{\mu} + u_{0,2}} + O(\varepsilon^4 \mu^4)}{\int_{\Omega} e^{\omega_{\mu} + u_{0,1}} \int_{\Omega} e^{\omega_{\mu} + u_{0,2}}} \right). \tag{2.26}
\]

Combining (2.24), (2.25), and (2.26), we obtain
\[
- \Delta U_{1,\mu} + \frac{1}{\varepsilon^2} e^{U_{2,\mu} + u_{0,2}} (e^{U_{1,\mu} + u_{0,1}} - 1) + 8k\pi = \sum_{i=1}^{k} \left( 1 + e^{2} e^{\omega_{\mu} + u_{0,1}} e^{\omega_{\mu} + u_{0,2}} \right) \left( \frac{\int_{\Omega} e^{2\omega_{\mu} + u_{0,1} + u_{0,2}}}{\int_{\Omega} e^{\omega_{\mu} + u_{0,1} + u_{0,2}}} \right) - \frac{8k\pi e^{\omega_{\mu} + u_{0,2}}}{\int_{\Omega} e^{\omega_{\mu} + u_{0,2}}} + K_{1,\mu} + R_{1,\mu} \tag{2.27}
\]

where
\[
K_{1,\mu} = \frac{64k^2 \pi^2 \varepsilon^2}{\int_{\Omega} e^{\omega_{\mu} + u_{0,2}}} \int_{\Omega} e^{\omega_{\mu} + u_{0,2}} \left[ \frac{e^{2\omega_{\mu} + u_{0,1} + u_{0,2}}}{\int_{\Omega} e^{2\omega_{\mu} + u_{0,1} + u_{0,2}}} - \frac{e^{\omega_{\mu} + u_{0,2}}}{\int_{\Omega} e^{\omega_{\mu} + u_{0,2}}} \right]
\]

and
\[
R_{1,\mu} = O \left( \frac{\mu^4 e^{4 \omega_{\mu} + u_{0,2}}}{\int_{\Omega} e^{\omega_{\mu} + u_{0,2}}} + \frac{e^{4 \omega_{\mu} + u_{0,1} + u_{0,2}}}{\int_{\Omega} e^{\omega_{\mu} + u_{0,2}}} \left( \frac{\int_{\Omega} e^{2\omega_{\mu} + u_{0,1} + u_{0,2}}}{\int_{\Omega} e^{\omega_{\mu} + u_{0,1} + u_{0,2}}} + \frac{e^{\omega_{\mu} + u_{0,2}}}{\int_{\Omega} e^{\omega_{\mu} + u_{0,2}}} \right) \right).
\]

Note that
\[
\int_{\Omega} e^{\sum_{i=1}^{2} (\omega_{\mu} + u_{0,i})} = O \left( \frac{\mu^2}{\mu^{4(k-1)}} \right)
\]

By this, and (2.12) and (2.13), we obtain
\[
K_{1,\mu} = O \left( \frac{1}{\mu^2} e^{V_{x_i,\mu} + u_{0,2}} + \frac{1}{\mu^4} e^{2V_{x_i,\mu} + u_{0,1} + u_{0,2}} \right)
\]

and
\[
R_{1,\mu} = O \left( \frac{1}{\mu^4} e^{V_{x_i,\mu} + u_{0,2}} + \frac{1}{\mu^6} e^{2V_{x_i,\mu} + u_{0,1} + u_{0,2}} \right)
\]

Next, by (2.13)
\[
\frac{8k\pi e^{\omega_{\mu} + u_{0,2}}}{\int_{\Omega} e^{\omega_{\mu} + u_{0,2}}} = \frac{8k\pi e^{\omega_{\mu} + u_{0,2}}}{\int_{\Omega} e^{\omega_{\mu} + u_{0,2}}} = \sum_{i=1}^{k} 1B_{d_{i}(r_i)} e^{V_{x_i,\mu} + f_{1,i,\mu}} + O(\frac{1}{\mu^2}) \tag{2.28}
\]

By this, we are led to (2.20).

\section{The Reduction and The Existence}

In this section, we will use the contraction mapping theorem to show that there exist \( x = (x_1, \ldots, x_k) \) and \( \mu = (\mu_1, \ldots, \mu_k) \) such that (1.3) has a solution of this form
\[
\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \left( \begin{array}{c} U_{1,\mu} \\ U_{2,\mu} \end{array} \right) + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}
\]

$(\omega_{1,\mu}, \omega_{2,\mu})$ where $(\omega_{1,\mu}, \omega_{2,\mu})$ is a perturbation term. For this purpose, we will use the linear operator $L_{\mu}$ (see (3.1) below). However, the linear operator $L_{\mu}$ has non-trivial kernel, hence, we only can use the contraction mapping theorem to solve (1.3) up to its kernel. Then we use the non-degeneracy of $G_{1} + G_{2}$ at the blow-up point $q$ and $D^{(2)}(q) < 0$ to find suitable $x$ and $\mu$ such that $L_{\mu}$ has a real solution whose blow-up set is $\{q_{1}, \cdots, q_{k}\}$. Here, we only present the multi-bubble case ($k \geq 2$). For the case $k = 1$, it can be easily deduced from the multi-bubble case by using the approximation solution (2.6).

In view of $U_{1,\mu}, U_{2,\mu}$, we consider the following simplified operator

$$L_{\mu}\begin{pmatrix} v_{1} \\ v_{2} \end{pmatrix} = \begin{pmatrix} 0 & \sum_{i=1}^{k} B_{d_{i}(x_{i})} e^{V_{x_{i},\mu_{i}}} \\ \sum_{i=1}^{k} B_{d_{i}(x_{i})} e^{V_{x_{i},\mu_{i}}} & 0 \end{pmatrix}\begin{pmatrix} v_{1} \\ v_{2} \end{pmatrix}$$

(3.1)

Then, $(\omega_{1,\mu}, \omega_{2,\mu})$ satisfies

$$L_{\mu}\begin{pmatrix} \omega_{1,\mu} \\ \omega_{2,\mu} \end{pmatrix} = \begin{pmatrix} g_{1,\mu} \\ g_{2,\mu} \end{pmatrix},$$

(3.2)

where

$$g_{1,\mu}(x, t_{1}, t_{2}) = \sum_{i=1}^{k} B_{d_{i}(x_{i})} e^{V_{x_{i},\mu_{i}}} t_{2} - \frac{1}{\varepsilon} \left( e^{U_{2,\mu}+u_{0,2}+t_{2}} - e^{U_{1,\mu}+u_{0,1}+t_{1}} \right) - \Delta U_{1,\mu} + 8k\pi,$$

and

$$g_{2,\mu}(x, t_{1}, t_{2}) = \sum_{i=1}^{k} B_{d_{i}(x_{i})} e^{V_{x_{i},\mu_{i}}} t_{1} - \frac{1}{\varepsilon} \left( e^{U_{2,\mu}+u_{0,2}+t_{2}} - e^{U_{2,\mu}+u_{0,2}+t_{2}} \right) - \Delta U_{2,\mu} + 8k\pi.$$

To apply the contraction argument, we first introduce two function spaces: $X_{\alpha,\mu,2}$ and $Y_{\alpha,\mu,2}$.

Fix a small fixed constant $\alpha > 0$, we define

$$\rho(x) = (1 + |x|)^{1+\frac{\alpha}{2}}, \quad \hat{\rho}(x) = \frac{1}{(1 + |x|)(\log(2 + |x|))^{1+\frac{\alpha}{2}}}.$$

Denote $\Omega' = \bigcup_{j=1}^{k} B_{d_{j}(x_{j})}$. We say $(\xi_{1}, \xi_{2})$ is in $X_{\alpha,\mu,2}$ if

$$\left\| \begin{pmatrix} \xi_{1} \\ \xi_{2} \end{pmatrix} \right\|_{X_{\alpha,\mu,2}}^{2} = \sum_{j=1}^{k} \sum_{i=1}^{2} \left( \|\Delta \xi_{i,j}\rho\|_{L^{2}(B_{2\xi_{i,j}\rho})}^{2} + \|\xi_{i,j}\bar{\rho}\|_{L^{2}(B_{2\xi_{i,j}\rho})}^{2} \right)$$

(3.3)

$$+ \sum_{i=1}^{2} \left( \|\Delta \xi_{i}\|_{L^{2}(\Omega')}^{2} + \|\xi_{i}\|_{L^{2}(\Omega')}^{2} \right) < +\infty,$$

(3.4)
where \( \tilde{\xi}_{i,j}(y) = \xi(q_j + \frac{1}{\mu_j} y) \) and \( B_d = B_d(0) \); \( \xi \) is in \( Y_{\alpha,\mu} \) if
\[
\left\| \left( \begin{array}{c} \xi_1 \\ \xi_2 \end{array} \right) \right\|_{Y_{\alpha,\mu}} = \sum_{j=1}^{N} \sum_{i=1}^{2} \left( \frac{1}{\mu_j} \right)^{2} \left( \tilde{\xi}_{i,j}(\rho) \right)^{2} \leq \| \xi \|_{L^2((\Omega,\gamma))}^2 < +\infty. \tag{3.5}
\]

We also denote \( \|\xi\|_{X_{\alpha,\mu}} = \left\| \left( \begin{array}{c} \xi \\ 0 \end{array} \right) \right\|_{X_{\alpha,\mu,2}} \) and \( \|\xi\|_{Y_{\alpha,\mu}} = \left\| \left( \begin{array}{c} \xi \\ 0 \end{array} \right) \right\|_{Y_{\alpha,\mu,2}} \). So, we say \( \xi \in X_{\alpha,\mu} \) if \( \|\xi\|_{X_{\alpha,\mu}} < +\infty \) and \( \xi \in Y_{\alpha,\mu} \) if \( \|\xi\|_{Y_{\alpha,\mu}} < +\infty \).

Consider the cut-off function \( \chi_j \in C^\infty(\mathbb{R}^2) \) satisfying
\[
\chi_j(x) = \begin{cases} 1 & \text{for } |x| \leq d_j \\ 0 & \text{for } |x| \geq 2d_j \end{cases}
\]
and \( 0 \leq \chi_j \leq 1 \). Next, we define the approximated kernel for \( L_\mu \) as follows:
\[
\begin{pmatrix} Y_{1,0} \\ Y_{2,0} \end{pmatrix} = \begin{pmatrix} \frac{1}{\mu_1} + \sum_{i=1}^{k} \frac{2 \nu_i (y-x_i)}{\mu_i \mu_i (1+\mu_i |y-x_i|^2)} \\ \frac{1}{\mu_1} + \sum_{i=1}^{k} \frac{2 \nu_i (y-x_i)}{\mu_i \mu_i (1+\mu_i |y-x_i|^2)} \end{pmatrix}
\]
\[
\begin{pmatrix} Y_{1,i,j} \\ Y_{2,i,j} \end{pmatrix} = \chi_j(|x-p_{j,\epsilon}|) \begin{pmatrix} \mu_i^2 (y_i-x_{j,i})^2 \\ \mu_i^2 (y_i-x_{j,i})^2 \end{pmatrix} \begin{pmatrix} \nu_i (y_i-x_{j,i})^2 \\ \nu_i (y_i-x_{j,i})^2 \end{pmatrix}, \quad i=1,2, \quad j=1,\ldots,k. \tag{3.7}
\]

After calculation, it is not difficult to see that
\[
L_\mu \begin{pmatrix} Y_{1,0} \\ Y_{2,0} \end{pmatrix} = O\left( \frac{1}{\mu^3} \right), \quad L_\mu \begin{pmatrix} Y_{1,i,j} \\ Y_{2,i,j} \end{pmatrix} = O(1), \quad i=1,2, \quad j=1,\ldots,k.
\]

Let
\[
\begin{pmatrix} Z_{1,0} \\ Z_{2,0} \end{pmatrix} = \begin{pmatrix} \Delta Y_{1,0} \\ \Delta Y_{2,0} \end{pmatrix} \tag{3.8}
\]

and
\[
\begin{pmatrix} Z_{1,i,j} \\ Z_{2,i,j} \end{pmatrix} = \begin{pmatrix} \Delta Y_{1,i,j} \\ \Delta Y_{2,i,j} \end{pmatrix}, \quad i=1,2, \quad j=1,\ldots,k. \tag{3.9}
\]

Note that \( \begin{pmatrix} Y_{1,0} \\ Y_{2,0} \end{pmatrix}, \begin{pmatrix} Y_{1,i,j} \\ Y_{2,i,j} \end{pmatrix}, \begin{pmatrix} Z_{1,0} \\ Z_{2,0} \end{pmatrix} \) and \( \begin{pmatrix} Z_{1,i,j} \\ Z_{2,i,j} \end{pmatrix} \) are doubly periodic. Let
\[
E_{\mu,2} = \left\{ \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \in X_{\alpha,\mu,2} : \int_{\Omega} \left\langle \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}, \begin{pmatrix} Z_{1,0} \\ Z_{2,0} \end{pmatrix} \right\rangle = \int_{\Omega} \left\langle \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}, \begin{pmatrix} Z_{1,i,j} \\ Z_{2,i,j} \end{pmatrix} \right\rangle = 0, \quad i=1,2, \quad j=1,\ldots,k \right\}
\]
and
\[
F_{\mu,2} = \left\{ \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \in Y_{\alpha,\mu,2} : \int_{\Omega} \left\langle \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}, \begin{pmatrix} Y_{1,0} \\ Y_{2,0} \end{pmatrix} \right\rangle = \int_{\Omega} \left\langle \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}, \begin{pmatrix} Y_{1,i,j} \\ Y_{2,i,j} \end{pmatrix} \right\rangle = 0, \quad i=1,2, \quad j=1,\ldots,k \right\}.
\]

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Lemma 3.1. There is a constant $C > 0$, independent of $x$ and $\mu$, such that

$$
\left\| Q_\mu \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right\|_{Y_{\alpha,\mu,2}} \leq C \left\| \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right\|_{Y_{\alpha,\mu,2}}.
$$

(3.12)

By using the contraction mapping argument, we obtain the following theorem.

Theorem 3.2. Assume (A.1)-(A.4) in Theorem 1.1 hold. There exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$, $|x - q| < \frac{C_1}{\mu}$, and $\mu \in \left( \frac{\beta_1}{\sqrt{\varepsilon}}, \frac{\beta_2}{\sqrt{\varepsilon}} \right)$ for some constants $C_1, \beta_1, \beta_2 > 0$, then there exists $\left( \frac{\omega_{1,\mu}}{\omega_{2,\mu}} \right) \in E_{\mu,2}$ satisfying

$$
Q_\mu \left( L_\mu \left( \frac{\omega_{1,\mu}}{\omega_{2,\mu}} \right) - \left( \frac{g_{1,\mu}}{g_{2,\mu}} \right) \right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
$$

(3.13)

Furthermore, $\left( \frac{\omega_{1,\mu}}{\omega_{2,\mu}} \right)$ is a $C^1$ map of $(x, \mu_1, \cdots, \mu_k)$ to $X_{\alpha,\mu,2}$ and

$$
\left\| Q_\mu \left( \frac{\omega_{1,\mu}}{\omega_{2,\mu}} \right) \right\|_{\text{L}^\infty(\Omega)} + \left\| Q_\mu \left( \frac{\omega_{1,\mu}}{\omega_{2,\mu}} \right) \right\|_{X_{\alpha,\mu,2}} \leq C \frac{\ln \mu}{\mu^2 - \frac{\mu}{2}},
$$

(3.14)

where the constant $C$ is the same constant as in $X_{\alpha,\mu,2}$.

Proof. By the Theorem 1.1, the equation (3.13) can be rewritten as

$$
\left( \frac{\omega_{1,\mu}}{\omega_{2,\mu}} \right) = B_\mu \left( \frac{\omega_{1,\mu}}{\omega_{2,\mu}} \right) := (Q_\mu L_\mu)^{-1} Q_\mu \left( \frac{g_{1,\mu}}{g_{2,\mu}} \right).
$$

(3.15)

Define

$$
S_\mu = \left\{ \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} : \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \in E_\mu, \left\| \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \right\|_{\text{L}^\infty(\Omega)} + \left\| \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \right\|_{X_{\alpha,\mu,2}} \leq \frac{1}{\mu} \right\}.
$$

Firstly, we show that $B_\mu$ maps $S_\mu$ to $S_\mu$. By Lemma 3.1 and (4.2), we have

$$
\left\| B_\mu \left( \frac{\omega_{1,\mu}}{\omega_{2,\mu}} \right) \right\|_{\text{L}^\infty(\Omega)} + \left\| \frac{\omega_{1,\mu}}{\omega_{2,\mu}} \right\|_{X_{\alpha,\mu,2}} \leq C \ln \mu \left\| Q_\mu \left( \frac{g_{1,\mu}}{g_{2,\mu}} \right) \right\|_{Y_{\alpha,\mu,2}} \leq C \ln \mu \left\| \frac{g_{1,\mu}}{g_{2,\mu}} \right\|_{Y_{\alpha,\mu,2}}.
$$

(3.16)
Thus we need to estimate $\| (g_{1,\mu} g_{2,\mu}) \|_{Y_{\alpha,\mu,2}}$.

Note that

\[
e^{U_{2,\mu} + u_{0,2} + t_2} (1 - e^{U_{1,\mu} + u_{0,1} + t_1})
= e^{U_{2,\mu} + u_{0,2}} (1 - e^{U_{1,\mu} + u_{0,1}}) + e^{U_{2,\mu} + u_{0,2}} (1 - e^{U_{1,\mu} + u_{0,1}}) t_2
- e^{\sum_{i=1}^2 U_{i,\mu} + u_{0,i}} t_1
+ O \left( (e^{U_{1,\mu} + u_{0,1}} + e^{U_{2,\mu} + u_{0,2}} + e^{U_{2,\mu} + u_{0,1}})(t_1^2 + t_2^2) \right)
\]

Thus, by (3.17), we obtain

\[
g_{1,\mu} = h_{\mu} \omega_{2,\mu} - \frac{1}{\varepsilon^2} e^{U_{2,\mu} + u_{0,2} + \omega_{2,\mu}} (1 - e^{U_{1,\mu} + u_{0,1} + \omega_{1,\mu}}) + 8k \pi - \Delta U_{1,\mu}
\]

\[
= (h_{\mu} - \frac{1}{\varepsilon^2} e^{U_{2,\mu} + u_{0,1} + \omega_{2,\mu}}) \omega_{2,\mu} + O \left( \frac{1}{\varepsilon^2} e^{U_{1,\mu} + u_{0,1}} (|\omega_{1,\mu}| + |\omega_{2,\mu}|) \right)
+ O \left( (\frac{1}{\varepsilon^2} e^{U_{1,\mu} + u_{0,1}} + \frac{1}{\varepsilon^2} e^{U_{2,\mu} + u_{0,2}}) (|\omega_{1,\mu}|^2 + |\omega_{2,\mu}|^2) \right)
+ O \left( \sum_{i=1}^k e^{V_{x_i} \mu} (D(u_{0,2}(x_i) + 8\pi \sum_{j \neq i} G(x_j, x_i))|y - x_i| + |y - x_i|^2) \right)
+ O \left( \ln \frac{\mu}{\varepsilon^2} + \sum_{i=1}^k \left( \frac{1}{\mu^4} e^{V_{x_i} \mu} + \frac{1}{\mu^4} e^{2V_{x_i} \mu} \right) \right).
\]

By this, we have

\[
\frac{1}{\mu^4} \| g_{1,\mu} (\mu^{-y} + x_i, \omega_1, \mu^{-y} + x_i), \omega_2, \mu^{-y} + x_i) \rho(y) \|_{L^2(B_{2d_i})}^2
\leq C \left( \frac{\ln^2 \mu}{\mu^4} + \varepsilon^4 \mu^4 \right) \left( \frac{\| \omega_1 \|}{\| \omega_2 \|} \right)_{L^\infty(\Omega)} + \frac{C |DG|^2(\mathbf{x})^2}{\mu^2}
\leq \frac{C}{\mu^{1-\alpha}}.
\]

On the other hand, by the definition of $(U_{1,\mu}, U_{2,\mu})$

\[
\| g_{1,\mu} \|_{L^2(\Omega \setminus \Gamma)} \leq \frac{C}{\mu^2}.
\]

Similarly, we have

\[
\frac{1}{\mu^4} \| g_{2,\mu} (\mu^{-y} + x_i, \omega_1, \mu^{-y} + x_i), \omega_2, \mu^{-y} + x_i) \rho(y) \|_{L^2(B_{2d_i})}^2
\leq \frac{C}{\mu^{1-\alpha}}
\]

and

\[
\| g_{2,\mu} \|_{L^2(\Omega \setminus \Gamma)} \leq \frac{C}{\mu^2}.
\]
By (3.19), (3.20), (3.21) and (3.22),
\[
\left\| \begin{bmatrix} g_{1,\mu} \\ g_{2,\mu} \end{bmatrix} \right\|_{Y,\alpha,\mu,2} \leq \frac{C}{\mu^{2-\frac{\alpha}{2}}}.
\] (3.23)

Next, we show that \( B_\mu \) is a contraction map. For any \((\omega_{1,\mu}, \omega_{2,\mu})\) and \((\tilde{\omega}_{1,\mu}, \tilde{\omega}_{2,\mu})\) \(\in S_\mu\), as the calculation above, we have
\[
\left\| B_\mu \left( \begin{bmatrix} \omega_{1,\mu} \\ \omega_{2,\mu} \end{bmatrix} \right) - B_\mu \left( \begin{bmatrix} \tilde{\omega}_{1,\mu} \\ \tilde{\omega}_{2,\mu} \end{bmatrix} \right) \right\|_{L^\infty(\Omega)} + \left\| B_\mu \left( \begin{bmatrix} \omega_{1,\mu} \\ \omega_{2,\mu} \end{bmatrix} \right) - B_\mu \left( \begin{bmatrix} \tilde{\omega}_{1,\mu} \\ \tilde{\omega}_{2,\mu} \end{bmatrix} \right) \right\|_{X,\alpha,\mu,2} 
\leq C \ln \mu \left\| \begin{bmatrix} g_{1,\mu}(x, \omega_{1,\mu}, \omega_{2,\mu}) \\ g_{2,\mu}(x, \omega_{1,\mu}, \omega_{2,\mu}) \end{bmatrix} - \begin{bmatrix} g_{1,\mu}(x, \tilde{\omega}_{1,\mu}, \tilde{\omega}_{2,\mu}) \\ g_{2,\mu}(x, \tilde{\omega}_{1,\mu}, \tilde{\omega}_{2,\mu}) \end{bmatrix} \right\|_{X,\alpha,\mu,2}.
\] (3.24)

Combining (3.23) and (3.24), we have proved that \( B_\mu \) is a contraction map. Furthermore, by the contraction mapping theorem, we know that there exists \((\omega_{1,\mu}, \omega_{2,\mu})\) \(\in S_\mu\) satisfying
\[
\left\| \begin{bmatrix} \omega_{1,\mu} \\ \omega_{2,\mu} \end{bmatrix} \right\|_{L^\infty(\Omega)} + \left\| \begin{bmatrix} \omega_{1,\mu} \\ \omega_{2,\mu} \end{bmatrix} \right\|_{X,\alpha,\mu,2} \leq \left\| \begin{bmatrix} g_{1,\mu} \\ g_{2,\mu} \end{bmatrix} \right\|_{Y,\alpha,\mu,2} \leq \frac{C \ln \mu}{\mu^{2-\frac{\alpha}{2}}},
\] (3.25)

By the above theorem, we obtain that for \( |x - q| < \frac{1}{\mu} \) and \( \mu \in \left[ \frac{\beta_0}{\sqrt{\epsilon}}, \frac{\beta_2}{\sqrt{\epsilon}} \right] \), there exists a doubly periodic function \((\omega_{1,\mu}, \omega_{2,\mu})\), \(c_0\) and \(c_{i,j}\), \(i = 1, 2, j = 1, \ldots, k\), such that
\[
\Delta \left( \begin{bmatrix} U_{1,\mu} + \omega_{1,\mu} \\ U_{2,\mu} + \omega_{2,\mu} \end{bmatrix} \right) + \sum_{i=1}^{k} \sum_{j=1}^{k} \left( \begin{bmatrix} Z_{1,i,j} \\ Z_{2,i,j} \end{bmatrix} \right) = c_0 \left( \begin{bmatrix} Z_{1,0} \\ Z_{2,0} \end{bmatrix} \right) + \sum_{i=1}^{k} \sum_{j=1}^{k} \left( \begin{bmatrix} Z_{1,i,j} \\ Z_{2,i,j} \end{bmatrix} \right).
\] (3.26)

To obtain a true solution for (1.3), we need to choose suitable \(x\) and \(\mu\) such that
\[
c_0 = c_{i,j} = 0, \quad i = 1, 2, \quad j = 1, \ldots, k.
\] (3.27)

We begin with the following simple observation.

**Lemma 3.3.** If
\[
\int_{\Omega} \begin{bmatrix} (\Delta(U_{1,\mu} + \omega_{1,\mu})) + \frac{1}{\epsilon^2} U_{2,\mu} + \omega_{2,\mu} + u_{i,0} (1 - e^{U_{1,\mu} + \omega_{1,\mu} + u_{i,0}} - 8k\pi) \\ (\Delta(U_{2,\mu} + \omega_{2,\mu})) + \frac{1}{\epsilon^2} U_{1,\mu} + \omega_{1,\mu} + u_{0,i} (1 - e^{U_{2,\mu} + \omega_{2,\mu} + u_{0,i}} - 8k\pi) \end{bmatrix}, \begin{bmatrix} Y_{1,0} \\ Y_{2,0} \end{bmatrix} = 0
\] (3.28)
and for \(i = 1, 2, \quad j = 1, \ldots, k\),
\[
\int_{\Omega} \begin{bmatrix} (\Delta(U_{1,\mu} + \omega_{1,\mu})) + \frac{1}{\epsilon^2} U_{2,\mu} + \omega_{2,\mu} + u_{i,0} (1 - e^{U_{1,\mu} + \omega_{1,\mu} + u_{i,0}} - 8k\pi) \\ (\Delta(U_{2,\mu} + \omega_{2,\mu})) + \frac{1}{\epsilon^2} U_{1,\mu} + \omega_{1,\mu} + u_{0,i} (1 - e^{U_{2,\mu} + \omega_{2,\mu} + u_{0,i}} - 8k\pi) \end{bmatrix}, \begin{bmatrix} Y_{1,i,j} \\ Y_{2,i,j} \end{bmatrix} = 0,
\] (3.29)
then \(c_0 = c_{i,j} = 0, \quad i = 1, 2, \quad j = 1, \ldots, k\).
We calculate the left hand side of (3.28) and (3.29) in the following two lemmas.

**Lemma 3.4.** For \(i = 1, 2, j = 1, \cdots, k\), there exist \(A_{i,j} > 0\), such that

\[
\int_\Omega \left( \begin{pmatrix} \Delta (U_{1,\mu} + \omega_{1,\mu}) \\ \Delta (U_{2,\mu} + \omega_{1,\mu}) \end{pmatrix} + \frac{1}{\pi} \int e^{U_{1,\mu} + \omega_{1,\mu} + u_0,2}(1 - e^{U_{1,\mu} + \omega_{1,\mu} + u_0,1} - 8k\pi) \right) \left( \begin{pmatrix} Y_{1,i,j} \\ Y_{2,i,j} \end{pmatrix} \right) = A_{i,j} \left( \frac{\partial G_1^*}{\partial x_{i,j}} + \frac{\partial G_2^*}{\partial x_{i,j}} \right) + O\left( \frac{\ln \mu}{\mu^2} \right) \tag{3.30}
\]

**Proof.** We mainly use (2.20), (2.21) and (3.25) to obtain the estimate (3.30). We calculate the case \(j = 1\) only. The others are similar.

Step 1. We first calculate this term

\[
\int_\Omega \left( \begin{pmatrix} \Delta U_{1,\mu} - 8k\pi \\ \Delta U_{2,\mu} - 8k\pi \end{pmatrix} \left( \begin{pmatrix} Y_{1,i,1} \\ Y_{2,i,1} \end{pmatrix} \right) \right) \tag{3.31}
\]

By the definition of \((U_{1,\mu}, U_{2,\mu})\) and symmetry, we find that

\[
\int_\Omega \left( \begin{pmatrix} \Delta U_{1,\mu} - 8k\pi \\ \Delta U_{2,\mu} - 8k\pi \end{pmatrix} \left( \begin{pmatrix} Y_{1,i,1} \\ Y_{2,i,1} \end{pmatrix} \right) \right) = \int \left[ B_{d_1(x_1)} e^{V_{1,\mu_1} Y_{1,i,1}} + 1_{B_{d_1(x_1)}} e^{V_{1,\mu_1} Y_{2,i,1}} + \sum_{i=1}^k \frac{8\pi}{\theta_i} \int_\Omega Y_{i,i,1} + Y_{2,i,1} = 0 \right] \tag{3.32}
\]

Next,

\[
\frac{8k\pi}{\int_\Omega e^{\omega_{\mu_1} + u_0,1} Y_{1,i,1}} + \frac{8k\pi}{\int_\Omega e^{\omega_{\mu_2} + u_0,2} Y_{2,i,1}}
\]

\[
= \frac{\mu_2^2 \cdots \mu_k^2}{\int_\Omega e^{\omega_{\mu_1} + u_0,1} Y_{1,i,1}} + \frac{\mu_2^2 \cdots \mu_k^2}{\int_\Omega e^{\omega_{\mu_2} + u_0,2} Y_{2,i,1}} + O\left( \frac{1}{\mu^2} \right) \tag{3.33}
\]

where \((A_2)\) is used. We thus denote \(A_{i,j} = \int_{\mathbb{R}^2} \frac{8}{(1 + |y|^2)^2} |y|^2 dy\).

Step 2. Next, we estimate the remainder term. By mean value theorem, there exist \(t_i\) between
0 and \( \omega_{i,\mu} (i = 1, 2) \), such that

\[
\int_{\Omega} \left\langle \left( \Delta \omega_{1,\mu} + \frac{1}{\varepsilon^2} e^{U_{2,\mu} + \omega_{1,\mu} + u_{0,2}} - \frac{1}{\varepsilon^2} e^{U_{2,\mu} + u_{0,2}} \right) \right| (Y_{1,i,1}) \rangle = \int_{\Omega} \left\langle \left( \Delta \omega_{2,\mu} + \frac{1}{\varepsilon^2} e^{U_{1,\mu} + \omega_{1,\mu} + u_{0,1}} - \frac{1}{\varepsilon^2} e^{U_{1,\mu} + u_{0,1}} \right) \right| (Y_{2,i,1}) \rangle
\]

\[= \int_{\Omega} \left\langle \left( \Delta \omega_{1,\mu} + \frac{1}{\varepsilon^2} e^{U_{2,\mu} + \omega_{2,\mu} + 2\omega_{1,\mu}} \right) \right| (Y_{1,i,1}) \rangle = \int_{\Omega} \left\langle \left( \Delta \omega_{2,\mu} + \frac{1}{\varepsilon^2} e^{U_{1,\mu} + \omega_{1,\mu} + 2\omega_{1,\mu}} \right) \right| (Y_{2,i,1}) \rangle + O \left( \frac{\mu}{\varepsilon^2} \right)
\]

(3.34)

Since \((\omega_{1,\mu}, \omega_{2,\mu})\) and \((Y_{1,i,1}, Y_{2,i,1})\) are doubly periodic,

\[
\int_{\Omega} \left\langle \left( \Delta \omega_{1,\mu} + \sum_{m=1}^{k} B_{dm}(x_m) e^{V_{x_m,\mu} \omega_{2,\mu}} \right) \right| (Y_{1,i,1}) \rangle + O \left( \frac{\mu}{\varepsilon^2 \omega_{2,\mu}} \right)
\]

(3.35)

\[
= O \left( \frac{\mu}{\varepsilon^2 \omega_{2,\mu}} \right)
\]

Similarly, we have

\[
\int_{\Omega} \left\langle \left( \frac{1}{\varepsilon^2} \sum_{j=1}^{2} e^{U_{j,\mu} + \omega_{j,\mu} + u_{0,j}} \right) \right| (Y_{1,i,1}) \rangle = O \left( \varepsilon^2 \mu^2 + \varepsilon^2 \mu^3 \right)
\]

(3.36)

By (3.25), (3.34), (3.35) and (3.36), we conclude that

\[
\int_{\Omega} \left\langle \left( \Delta (U_{1,\mu} + \omega_{1,\mu}) + \frac{1}{\varepsilon^2} e^{U_{2,\mu} + \omega_{1,\mu} + u_{0,2}} \right) \right| (Y_{1,i,1}) \rangle = O \left( \frac{\ln \mu}{\mu^{3/2}} \right)
\]

(3.37)

**Lemma 3.5.** There exists \( B > 0 \), such that

\[
\int_{\Omega} \left\langle \left( \Delta (U_{1,\mu} + \omega_{1,\mu}) + \frac{1}{\varepsilon^2} e^{U_{2,\mu} + \omega_{1,\mu} + u_{0,2}} (1 - e^{U_{1,\mu} + \omega_{1,\mu} + u_{0,2}} - \frac{1}{\varepsilon^2} e^{U_{2,\mu} + u_{0,2}} \right) \right| (Y_{1,0}) \rangle \right| (Y_{2,0}) \rangle
\]

\[= \frac{8}{\mu^3 \rho_1} \sum_{i=1}^{k} \rho_i \left( \int_{\Omega \setminus \bar{B}_\delta(x_i)} \frac{e^{f_{1,i,x}} - 1}{|y - x_i|^4} - \int_{\mathbb{R}^2 \setminus \Omega_i} \frac{1}{|y - x_i|^4} \right)
\]

(3.38)

\[+ \frac{8}{\mu^3 \rho_1} \sum_{i=1}^{k} \rho_i \left( \int_{\Omega \setminus \bar{B}_\delta(x_i)} \frac{e^{f_{2,i,x}} - 1}{|y - x_i|^4} - \int_{\mathbb{R}^2 \setminus \Omega_i} \frac{1}{|y - x_i|^4} \right)
\]

\[+ B \varepsilon^2 + O \left( \frac{\delta^2}{\rho_1} \right) + O \left( \frac{\ln \mu}{\mu^{3/2}} \right)
\]

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Proof. As in the proof of Lemma (3.4), we firstly calculate

\[
\int \left( \frac{\Delta (U_{1, \mu} + \omega_{1, \mu})}{\Delta (U_{2, \mu} + \omega_{1, \mu})} + \frac{1}{2} e^{U_{2, \mu} + \omega_{2, \mu} + u_{0, 2}} (1 - e^{U_{1, \mu} + \omega_{1, \mu} + u_{0, 1}} - 8k\pi \right), \left( Y_{1, 0}, Y_{2, 0} \right) \right]
\]

\[
= \sum_{i=1}^{k} \int_{B_{d_{i}}(x_{i})} e^{y_{i} (e^{j_{2, \mu}} - 1)} Y_{1, 0} + e^{y_{i} (e^{j_{1, \mu}} - 1)} Y_{2, 0}
\]

\[
+ \int \frac{8\pi}{\theta_{i}} (Y_{1, 0} + Y_{2, 0}) + \int_{\Omega^{\prime}} e^{\omega_{\mu}^{+} + u_{0, 2}} Y_{1, 0} + \int_{\Omega^{\prime}} e^{\omega_{\mu}^{+} + u_{0, 1}} Y_{2, 0}
\]

For \( i = 1, \cdots, n, \) we write

\[
e^{y_{j, \mu} (y)} = 1 = e^{y_{j, \mu} (y)} - 1 + e^{y_{j, \mu} (y)} - e^{y_{j, \mu} (y)}
\]

Then

\[
\int_{B_{d_{i}}(x_{i})} e^{y_{i} (e^{j_{2, \mu}} - 1)} Y_{1, 0} + e^{y_{i} (e^{j_{1, \mu}} - 1)} Y_{2, 0}
\]

A straightforward calculation shows that

\[
\sum_{i=1}^{k} \int \frac{8\pi}{\theta_{i}} (Y_{1, 0} + Y_{2, 0}) = -\frac{16\pi}{\mu} \sum_{i=1}^{k} 1 + O\left( \frac{\ln \mu}{\mu^{5}} \right).
\]

and

\[
\frac{\int e^{\omega_{\mu}^{+} + u_{0, 2}} Y_{1, 0}}{\int e^{\omega_{\mu}^{+} + u_{0, 1}}} + \frac{\int e^{\omega_{\mu}^{+} + u_{0, 1}} Y_{2, 0}}{\int e^{\omega_{\mu}^{+} + u_{0, 1}}}
\]

\[
= -\frac{8}{\rho_{1} \mu^{4}} \left( 1 + O\left( \frac{1}{\mu} \right) \right) \int_{\Omega^{\prime}} e^{u_{0, 2} + 8\pi \sum_{i=1}^{k} G(y_{i})} \left( -1 + O\left( \frac{1}{\mu^{3}} \right) \right)
\]

\[
- \frac{8}{\rho_{1} \mu^{4}} \left( 1 + O\left( \frac{1}{\mu^{3}} \right) \right) \int_{\Omega^{\prime}} e^{u_{0, 1} + 8\pi \sum_{i=1}^{k} G(y_{i})} \left( -1 + O\left( \frac{1}{\mu^{3}} \right) \right)
\]

\[
= -\frac{8}{\rho_{1} \mu^{4}} \int_{\Omega^{\prime}} e^{u_{0, 2} + 8\pi \sum_{i=1}^{k} G(y_{i})} - \frac{8}{\rho_{1} \mu^{3}} \int_{\Omega^{\prime}} e^{u_{0, 1} + 8\pi \sum_{i=1}^{k} G(y_{i})} + O\left( \frac{1}{\mu^{3}} \right)
\]

Fix a small positive constant \( \delta \) with \( \delta \ll d_{i}, \) \( i = 1, \cdots, k. \) Note that \( \Delta e^{y_{j, \mu} x} = e^{y_{j, \mu} x} |Df_{j, \mu}|, \) \( i = 1, \cdots, k, \) \( j = 1, 2. \) By the symmetry and Taylor expansion on \( e^{y_{j, \mu} x} - 1, \) we obtain

\[
\int_{B_{d_{i}}(x_{i})} e^{y_{i} (e^{j_{2, \mu}} - 1)} Y_{1, 0} + \int_{B_{d_{i}}(x_{i})} e^{y_{i} (e^{j_{1, \mu}} - 1)} Y_{2, 0}
\]

\[
= O\left( \frac{1}{\mu} \int_{B_{d_{i}}(x_{i})} e^{y_{i} (e^{j_{2, \mu}} x_{i})^{2} |y - x_{i}|^{2} + |Df_{j, \mu}(x_{i})|^{2} |y - x_{i}|^{2} + |y - x_{i}|^{2}} \right)
\]

\[
= O\left( \frac{\delta^{2} + \ln \mu}{\mu^{5}} \right)
\]
By this, we find that
\[\begin{align*}
\int_{B_d(x_i)} e^{V_{z,\nu_1}}(e^{f_{2,\nu}} - 1)Y_{1,0} + \int_{B_d(x_i)} e^{V_{z,\nu_1}}(e^{f_{1,\nu}} - 1)Y_{2,0} \\
= \int_{B_d(x_i) \setminus B_3(x_i)} e^{V_{z,\nu_1}}(e^{f_{2,\nu}} - 1)Y_{1,0} + \int_{B_d(x_i) \setminus B_3(x_i)} e^{V_{z,\nu_1}}(e^{f_{1,\nu}} - 1)Y_{2,0} + O\left(\frac{\delta^2}{\mu^3} + \frac{\ln\mu}{\mu^5}\right)
\end{align*}\]

For \( i = 1, \cdots, k \), we have
\[e^{u_{0,1}(y) + 8\pi \sum_{i=1}^k G(y, x_i)} = \frac{\rho_i}{|y - x_i|^4} e^{f_{1,\nu}(y)}\] (3.46)
and
\[e^{u_{0,2}(y) + 8\pi \sum_{i=1}^k G(y, x_i)} = \frac{\rho_i}{|y - x_i|^4} e^{f_{2,\nu}(y)}\] (3.47)

Recall that \( \frac{1}{\theta_i} = \frac{1}{1 + \langle \mu, d_i \rangle^2} \). So, we have
\[\frac{1}{\theta_i} = \int_{\mathbb{R}^2 \setminus B_d(x_i)} \frac{1}{|y - x_i|^4} dy + O\left(\frac{1}{\mu^2}\right)\] (3.48)

Combining (3.42) (3.43) (3.44) (3.45) (3.46) (3.47) and (3.48), we are led to
\[\int_{\Omega} \left( \left( \Delta U_{1,\mu} + \frac{\delta}{\varepsilon^2} e^{U_{2,\mu} + u_{0,2}} \right) \left( Y_{1,0} \right) + \left( \Delta U_{2,\mu} + \frac{\delta}{\varepsilon^2} e^{U_{1,\mu} + u_{0,1}} \right) \left( Y_{2,0} \right) \right) \right) \]
\[= -\frac{8}{\mu^3} \frac{1}{\rho_i^*} \left( \sum_{i=1}^k \rho_i \left( \int_{\Omega_i \setminus B_d(x_i)} \frac{e^{f_{2,\nu}} - 1}{|y - x_i|^4} - \int_{\mathbb{R}^2 \setminus \Omega_i} \frac{1}{|y - x_i|^4} \right) \right) \]
\[\left( \sum_{i=1}^k \rho_i \left( \int_{\Omega_i \setminus B_d(x_i)} \frac{e^{f_{1,\nu}} - 1}{|y - x_i|^4} - \int_{\mathbb{R}^2 \setminus \Omega_i} \frac{1}{|y - x_i|^4} \right) \right) + O\left(\frac{\delta^2}{\mu^3} + \frac{1}{\mu^2}\right)\] (3.49)

By the argument in Lemma 3.3
\[\int_{\Omega} \left( -\frac{\delta}{\varepsilon^2} e^{U_{1,\mu} + u_{0,1} + U_{2,\mu} + u_{0,2}} \right) \left( Y_{1,0} \right) = B\varepsilon^2 \mu + O\left(\frac{1}{\mu^5}\right),\] (3.50)
for some \( B > 0 \), and
\[\int_{\Omega} \left( \frac{\Delta \omega_{1,\mu} + \frac{1}{\varepsilon^2} R_{1,\mu}}{\Delta \omega_{2,\mu} + \frac{1}{\varepsilon^2} R_{2,\mu}} \right) \left( Y_{1,0} \right) = O\left(\frac{\ln\mu}{\mu^4 - \frac{\mu}{2}}\right),\] (3.51)
where
From this and (A.2), we find that (3.52) and (3.53) have a solution and equivalent to
and
\[ R_1,µ = e^{U_2,µ + u_{0,2} + ω_{2,µ}} (1 - e^{U_1,µ + u_{0,1} + ω_{1,µ}}) - e^{U_2,µ + u_{0,2}} (1 - e^{U_1,µ + u_{0,1}}) \]
and
\[ R_2,µ = e^{U_1,µ + u_{0,1} + ω_{1,µ}} (1 - e^{U_2,µ + u_{0,2} + ω_{2,µ}}) - e^{U_1,µ + u_{0,1}} (1 - e^{U_2,µ + u_{0,2}}). \]

**Proof of Theorem 1.1** From (3.30) and (3.38), we observe that (3.28) and (3.29) are equivalent to
\[ DG^*_i(x) = O \left( \frac{\ln μ}{µ^2} \right), \quad i = 1, 2, \] (3.52)
and
\[ \sum_{i=1}^{k} \frac{ρ_i}{ρ_1} (\int_{Ω \setminus B_δ(x_i)} \frac{e^{f_{1,i}} - 1}{|y - x_i|^4} - \int_{R^2 \setminus Ω} \frac{1}{|y - x_i|^4}) + \sum_{i=1}^{k} \frac{ρ_i^*}{ρ_1^*} (\int_{Ω \setminus B_δ(x_i)} \frac{e^{f_{2,i}} - 1}{|y - x_i|^4} - \int_{R^2 \setminus Ω} \frac{1}{|y - x_i|^4}) + Bε² μ \] (3.53)
\[ = \frac{1}{μ^2} O ((|DG^*_1(x)|^2 + |DG^*_2(x)|^2) \ln μ + δ²) + O \left( \frac{1}{µ^5} \right). \]

Since we assume \( D^{(2)}(q) < 0 \), for small \( δ > 0 \), there exists \( x \) close to \( q \) such that
\[ \sum_{i=1}^{k} \left( \frac{ρ_i}{ρ_1} (\int_{Ω \setminus B_δ(x_i)} \frac{e^{f_{1,i}} - 1}{|y - x_i|^4} - \int_{R^2 \setminus Ω} \frac{1}{|y - x_i|^4}) + \frac{ρ_i^*}{ρ_1^*} (\int_{Ω \setminus B_δ(x_i)} \frac{e^{f_{2,i}} - 1}{|y - x_i|^4} - \int_{R^2 \setminus Ω} \frac{1}{|y - x_i|^4}) \right) + O(δ²) < 0. \] (3.54)

From this and (A.2), we find that (3.32) and (3.33) have a solution \( x_ε \) and \( (μ_1,ε, \cdots, μ_k,ε) \) satisfies
\[ |DG^*_1(x)| + |DG^*_2(x)| \leq C \frac{\ln μ}{µ^2}, \quad μ_i,ε ∈ \left( \frac{β_0}{\sqrt{ε}}, \frac{β_1}{\sqrt{ε}} \right), \quad i = 1, \cdots, k. \] (3.55)

### 4 Appendix

In this section, we will discuss the invertibility of the linear operator \( L_µ \). From our construction of the approximation solutions, we can split the associated linear operator \( L_µ \) into two parts \( L_1 \) and \( L_2 \) (see (4.3) below). Apply Theorem A.2 in [29] and Theorem B.1 in [28] to \( L_1 \) and \( L_2 \) respectively, we obtain the invertibility of \( L_µ \).

**Theorem 4.1.** (1) The operator \( Q_µ L_µ \) is an isomorphism from \( E_µ,2 \) to \( F_µ,2 \).
(2) If \( \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \in E_{\mu,2} \) and \( \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in F_{\mu,2} \) satisfies
\[
L_\mu \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}.
\]

Then there exists a constant \( C > 0 \), independent of \( \mathbf{x} \) and \( \mu \), such that
\[
\left\| \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \right\|_{L^\infty(\Omega)} + \left\| \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \right\|_{X_{\alpha,\mu,2}} \leq C \ln \mu \left\| \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right\|_{Y_{\alpha,\mu,2}}.
\]

Proof. Since \( Z_{1,0} = Z_{2,0} \), \( Z_{1,i,j} = Z_{2,i,j} \), \( Y_{1,0} = Y_{2,0} \), and \( Y_{1,i,j} = Y_{2,i,j} \), \( i = 1, 2, j = 1, \cdots, k \). We denote
\[
E_\mu = \left\{ \omega \in X_{\alpha,\mu} : \int_\Omega \omega Z_{1,0} = \int_\Omega \omega Z_{1,i,j} = 0, i = 1, 2, j = 1, \cdots, k \right\}
\]
and
\[
F_\mu = \left\{ \omega \in Y_{\alpha,\mu} : \int_\Omega \omega Y_{1,0} = \int_\Omega \omega Y_{1,i,j} = 0, i = 1, 2, j = 1, \cdots, k \right\}.
\]
We use the same notation for the project operator \( Q_\mu : Y_\mu \to F_\mu \).

We can rewrite (4.1) as
\[
\mathcal{L}_1(\omega_1 + \omega_2) := \Delta (\omega_1 + \omega_2) + \sum_{i=1}^k 1_{B_i(x_i)} e^{V_{i,x_i}} (\omega_1 + \omega_2) = h_1 + h_2
\]
\[
\mathcal{L}_2(\omega_1 - \omega_2) := \Delta (\omega_1 - \omega_2) - \sum_{i=1}^k 1_{B_i(x_i)} e^{V_{i,x_i}} (\omega_1 - \omega_2) = h_1 - h_2
\]

By Theorem A.2 in [29], \( Q_\mu \mathcal{L}_1 \) is an isomorphism from \( E_\mu \) to \( F_\mu \); by Theorem B.1 in [28], \( \mathcal{L}_2 \) is an isomorphism from \( X_{\alpha,\mu} \) to \( Y_{\alpha,\mu} \). Furthermore, by Theorem A.2 in [29] and Theorem B.1 in [28] again, we obtain
\[
\| \omega_1 + \omega_2 \|_{L^\infty(\Omega)} + \| \omega_1 + \omega_2 \|_{X_{\alpha,\mu}} \leq C \ln \mu \| h_1 + h_2 \|_{Y_{\alpha,\mu}} \tag{4.4}
\]
and
\[
\| \omega_1 - \omega_2 \|_{L^\infty(\Omega)} + \| \omega_1 - \omega_2 \|_{X_{\alpha,\mu}} \leq C \ln \mu \| h_1 - h_2 \|_{Y_{\alpha,\mu}} \tag{4.5}
\]
which imply (4.2).

\[\square\]

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