EXISTENCE AND MULTIPlicity OF SOLUTIONS FOR DOUBLE-PHASE ROBIN PROBLEMS

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Abstract. We consider a double phase Robin problem with a Carathéodory nonlinearity. When the reaction is superlinear but without satisfying the Ambrosetti-Rabinowitz condition, we prove an existence theorem. When the reaction is resonant, we prove a multiplicity theorem. Our approach is Morse theoretic, using the notion of homological local linking.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial \Omega$. In this paper we study the following two phase Robin problem

\begin{equation}
\begin{cases}
- \text{div} \left( a_0(z)|Du|^{p-2}Du \right) - \Delta_q u + \xi(z)|u|^{p-2}u = f(z,u) & \text{in } \Omega \\
\frac{\partial u}{\partial n_\theta} + \beta(z)|u|^{p-2}u = 0 & \text{on } \partial \Omega,
\end{cases}
\end{equation}

where $1 < q < p \leq N$.

In this problem, the weight $a_0 : \Omega \rightarrow \mathbb{R}$ is Lipschitz continuous and $a_0(z) > 0$ for all $z \in \Omega$. The potential function $\xi \in L^\infty(\Omega)$ satisfies $\xi(z) \geq 0$ for a.a. $z \in \Omega$, while the reaction term $f(z,x)$ is Carathéodory (that is, for all $x \in \mathbb{R}$ the mapping $z \mapsto f(z,x)$ is measurable and for a.a. $z \in \Omega$ the function $x \mapsto f(z,x)$ is continuous). Let $F(z,x)$ be the primitive of $f(z,x)$, that is, $F(z,x) = \int_0^x f(z,s)ds$. We assume that for a.a. $z \in \Omega$, $F(z,\cdot)$ is $q$-linear near the origin. On the other hand, near $\pm \infty$, we consider two distinct cases for $f(z,\cdot)$:
(i) for a.a. $z \in \Omega$, $f(z,\cdot)$ is $(p-1)$-superlinear but without satisfying the Ambrosetti-Rabinowitz condition (the AR-condition for short), which is common in the literature when dealing with superlinear problems;
(ii) for a.a. $z \in \Omega$, $f(z,\cdot)$ is $(p-1)$-linear and possibly resonant with respect to the principal eigenvalue of the weighted $p$-Laplacian

$$u \mapsto - \text{div} \left( a_0(z)|Du|^{p-2}Du \right)$$

with Robin boundary condition.

In the boundary condition, $\frac{\partial u}{\partial n_\theta}$ denotes the conormal derivative of $u$ corresponding to the modular function $\theta(z,x) = a_0(z)x^p + x^q$ for all $z \in \Omega$, all $x \geq 0$. We interpret this derivative via the nonlinear Green identity (see Papageorgiou, Rădulescu and Repovš [18, p. 34]) and

$$\frac{\partial u}{\partial n_\theta} = [a_0(z)|Du|^{p-2} + |Du|^{q-2}] \frac{\partial u}{\partial n} \text{ for all } u \in C^1(\overline{\Omega}),$$

with $n(\cdot)$ being the outward unit normal on $\partial \Omega$. The boundary coefficient $\beta$ satisfies $\beta \in C^{0,\alpha}(\partial \Omega)$ with $0 < \alpha < 1$ and $\beta(z) \geq 0$ for all $z \in \partial \Omega$.

The differential operator in problem (1) is a weighted $(p,q)$-Laplace operator and it corresponds to the energy functional

$$u \mapsto \int_\Omega [a_0(z)|Du|^p + |Du|^q]dz.$$
Since we do not assume that the weight function $a_0(z)$ is bounded away from zero, the continuous integrand $\theta_0: \Omega \times \mathbb{R}^N \to \mathbb{R}_+$ of this integral functional exhibits unbalanced growth, namely

$$|y|^q \leq \theta_0(z, y) \leq c_0(1 + |y|^p)$$

for a.a. $z \in \Omega$, all $y \in \mathbb{R}^N$ and some $c_0 > 0$.

Such integral functionals were first investigated by Marcellini [14] and Zhikov [22], in connection with problems in nonlinear elasticity theory. Recently, Baroni, Colombo and Mingione [3] and Colombo and Mingione [6, 7] revived the interest in them and produced important local regularity results for the minimizers of such functionals. A global regularity theory for such problems remains elusive.

In this paper, using tools from Morse theory (in particular, critical groups), we prove an existence theorem (for the superlinear case) and a multiplicity theorem (for the linear resonant case). Existence and multiplicity results for two phase problems were proved recently by Cencelj, Rădulescu and Repovš [19] (multiple solutions for superlinear problems), Papageorgiou, Vetro [20] (parametric Dirichlet problems). The approach in all the aforementioned works is different and the hypotheses on the reaction are more restrictive.

Finally, we mention that $(p, q)$-equations arise in many mathematical models of physical processes. We refer to the very recent works of Bahrouni, Rădulescu and Repovš [1, 2] and the references therein.

2. Mathematical background

The study of two-phase problems requires the use of Musielak-Orlicz spaces. So, let $\theta: \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ be the modular function defined by

$$\theta(z, x) = a_0(z)x^p + x^q$$

for all $z \in \Omega$, $x \geq 0$.

This is a generalized N-function (see Musielak [16]) and it satisfies

$$\theta(z, 2x) \leq 2^p \theta(z, x)$$

for all $z \in \Omega$, $x \geq 0$,

that is, $\theta(z, \cdot)$ satisfies the $(\Delta_2)$-property (see Musielak [16, p. 52]). Using the modular function $\theta(z, x)$, we can define the Musielak-Orlicz space $L^\theta(\Omega)$ as follows:

$$L^\theta(\Omega) = \left\{ u: \Omega \to \mathbb{R}; \text{ $u$ is measurable and } \int_\Omega \theta(z, |u|)dz < \infty \right\}.$$

This space is equipped with the so-called “Luxemburg norm” defined by

$$\|u\|_{\theta} = \inf \left\{ \lambda > 0 : \int_\Omega \theta(z, \frac{|u|}{\lambda})dz \leq 1 \right\}.$$

Using $L^\theta(\Omega)$, we can define the following Sobolev-type space $W^{1,\theta}(\Omega)$, by setting

$$W^{1,\theta}(\Omega) = \{u \in L^\theta(\Omega) : |Du| \in L^\theta(\Omega)\}.$$

We equip $W^{1,\theta}(\Omega)$ with the norm $\| \cdot \|$ defined by

$$\|u\| = \|u\|_{\theta} + \|Du\|_{\theta},$$

where $\|Du\|_{\theta} = \| |Du| \|_{\theta}$. The spaces $L^\theta(\Omega)$ and $W^{1,\theta}(\Omega)$ are separable and uniformly convex (hence reflexive) Banach spaces.

Let $\hat{\theta}(z, x)$ be another modular function. We say that “$\hat{\theta}$ is weaker than $\theta$” and write $\hat{\theta} \prec \theta$, if there exist $c_1, c_2 > 0$ and a function $\eta \in L^1(\Omega)$ such that

$$\hat{\theta}(z, x) \leq c_1 \theta(z, c_2 x) + \eta(z)$$

for a.a. $z \in \Omega$ and all $x \geq 0$.

Then we have

$$L^\theta(\Omega) \hookrightarrow L^{\hat{\theta}}(\Omega)$$

and $W^{1,\theta}(\Omega) \hookrightarrow W^{1,\hat{\theta}}(\Omega)$ continuously.

Combining this fact with the classical Sobolev embedding theorem, we obtain the following embeddings; see Propositions 2.15 and 2.18 of Colasuonno and Squassina [5].
Proposition 2.1. We assume that $1 < q < p < \infty$. Then the following properties hold.

(a) If $q \neq N$, then $W^{1,q}(\Omega) \hookrightarrow L^r(\Omega)$ continuously for all $1 \leq r \leq q^*$, where

\[
q^* = \begin{cases} 
\frac{Nq}{N-q} & \text{if } q < N \\
+\infty & \text{if } q \geq N.
\end{cases}
\]

(b) If $q = N$, then $W^{1,N}(\Omega) \hookrightarrow L^r(\Omega)$ continuously for all $1 \leq r < \infty$.

(c) If $q \leq N$, then $W^{1,q}(\Omega) \hookrightarrow L^r(\Omega)$ compactly for all $1 \leq r < q^*$.

(d) If $q > N$, then $W^{1,q}(\Omega) \hookrightarrow L^\infty(\Omega)$ compactly.

(e) $W^{1,q}(\Omega) \hookrightarrow W^{1,q}(\Omega)$ continuously.

We have

$$L^p(\Omega) \hookrightarrow L^q(\Omega) \hookrightarrow L^q(\Omega) \cap L^q(\Omega)$$

with both embeddings being continuous.

We consider the modular function

$$\rho_\theta(u) = \int_\Omega \theta(|\nabla u|)dz = \int_\Omega [\alpha_0(z)|\nabla u|^p + |\nabla u|^q]dz$$

for all $u \in W^{1,q}(\Omega)$.

There is a close relationship between the norm $\| \cdot \|$ of $W^{1,q}(\Omega)$ and the modular functional $\rho_\theta(\cdot)$; see Proposition 2.1 of Liu and Dai [13].

Proposition 2.2. (a) If $u \neq 0$, then $\|\nabla u\|_\theta = \lambda$ if and only if $\rho_\theta(u) \leq 1$.

(b) If $\|\nabla u\|_\theta < 1$ (resp. $= 1$, $> 1$) if and only if $\rho_\theta(u) < 1$ (resp. $= 1$, $> 1$).

(c) If $\|\nabla u\|_\theta < 1$, then $\|\nabla u\|_\theta \leq \rho_\theta(u) \leq \|\nabla u\|_\theta$.

(d) If $\|\nabla u\|_\theta > 1$, then $\|\nabla u\|_\theta \leq \rho_\theta(u) \leq \|\nabla u\|_\theta$.

(e) $\|\nabla u\|_\theta \to 0$ if and only if $\rho_\theta(u) \to 0$.

(f) $\|\nabla u\|_\theta \to +\infty$ if and only if $\rho_\theta(u) \to +\infty$.

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure, we can define in the usual way the “boundary” Lebesgue spaces $L^s(\partial\Omega)$ for $1 \leq s \leq \infty$. It is well-known that there exists a unique continuous linear map $\gamma_0 : W^{1,q}(\Omega) \to L^s(\partial\Omega)$, known as the “trace map”, such that

$$\gamma_0(u) = u|_{\partial\Omega} \text{ for all } u \in W^{1,q}(\Omega) \cap C(\overline{\Omega}).$$

We have

$$\text{im} \gamma_0 = W^{1,q}(\Omega) \left( \frac{1}{q} + \frac{1}{q'} = 1 \right) \text{ and } \ker \gamma_0 = W^{0,q}(\Omega).$$

Moreover, the trace map $\gamma_0(\cdot)$ is compact into $L^s(\partial\Omega)$ for all $1 \leq s < (N-1)q/(N-q)$ if $q < N$, and for all $1 \leq s < \infty$ if $q \geq N$. In what follows, for the sake of notational simplicity, we drop the use of the trace map $\gamma_0(\cdot)$. All restrictions of the Sobolev functions on the boundary $\partial\Omega$ are understood in the sense of traces.

Let $\langle \cdot, \cdot \rangle$ denote the duality brackets for the pair $(W^{1,q}(\Omega), W^{1,q}(\Omega)^*)$ and $\langle \cdot, \cdot \rangle_{1,q}$ denote the duality brackets for the pair $(W^{1,q}(\Omega), W^{1,q}(\Omega)^*)$. We introduce the maps $A_\alpha^q : W^{1,q}(\Omega) \to W^{1,q}(\Omega)^*$ and $A_q : W^{1,q}(\Omega) \to W^{1,q}(\Omega)^*$ defined by

$$\langle A_\alpha^q(u), h \rangle = \int_\Omega \alpha_0(z)|\nabla u|^{p-2}(\nabla u, Dh)_{\mathbb{R}^N}dz \text{ for all } u, h \in W^{1,q}(\Omega),$$

$$\langle A_q(u), h \rangle_{1,q} = \int_\Omega |\nabla u|^{q-2}(\nabla u, Dh)_{\mathbb{R}^N}dz \text{ for all } u, h \in W^{1,q}(\Omega).$$

We have

$$\langle A_q(u), h \rangle_{1,q} = \langle A_q(u), h \rangle \text{ for all } u, h \in W^{1,q}(\Omega).$$

We introduce the following hypotheses on the weight $\alpha_0(\cdot)$ and on the coefficients $\xi(\cdot)$ and $\beta(\cdot)$.

$H_u$: $\alpha_0 : \overline{\Omega} \to \mathbb{R}$ is Lipschitz continuous, $\alpha_0(z) > 0$ for all $z \in \Omega$, $\xi \in L^\infty(\Omega)$, $\xi(z) \geq 0$ for a.a. $z \in \Omega$, $\beta \in C^{0,\alpha}(\partial\Omega)$ with $0 < \alpha < 1$, $\xi \not\equiv 0$ or $\beta \not\equiv 0$ and $q > Np/(N + p - 1)$.

Remark 2.1. The latter condition on the exponent $q$ implies that $W^{1,q}(\Omega) \hookrightarrow L^p(\partial\Omega)$ compactly and $q < p^*$. 
We introduce the $C^1$-functional $\gamma_p : W^{1,p}(\Omega) \to \mathbb{R}$ defined by
$$
\gamma_p(u) = \int_{\Omega} a_0(z)|Du|^p dz + \int_{\Omega} \xi(z)|u|^{p-2}u + \int_{\partial \Omega} \beta(z)|u|^{p-2}u \, d\sigma \quad \text{for all } u \in W^{1,p}(\Omega).
$$

Then hypotheses $H_0$, Lemma 4.11 of Mugnai and Papageorgiou [15], and Proposition 2.4 of Gasinski and Papageorgiou [10], imply that

$$
c_1 \|u\|^p \leq \gamma_p(u) \quad \text{for some } c_1 > 0, \text{ all } u \in W^{1,p}(\Omega).
$$

We denote by $\hat{\lambda}_1(p)$ the first (principal) eigenvalue of the following nonlinear eigenvalue problem

$$
\begin{cases}
-\text{div}(a_0(z)|Du|^{p-2}Du) + \xi(z)|u|^{p-2}u = \hat{\lambda}|u|^{p-2}u & \text{in } \Omega \\
\frac{\partial u}{\partial n_p} + \beta(z)|u|^{p-2}u = 0 & \text{on } \partial \Omega.
\end{cases}
$$

Here, $\frac{\partial u}{\partial n_p} = |Du|^{p-2} \frac{\partial u}{\partial n}$. The eigenvalue $\hat{\lambda}_1(p)$ has the following variational characterization

$$
\hat{\lambda}_1(p) = \inf \left\{ \frac{\gamma_p(u)}{\|u\|^p} : u \in W^{1,p}(\Omega) \setminus \{0\} \right\} \quad \text{(see [17]).}
$$

Then by (2), we see that $\hat{\lambda}_1(p) > 0$. This eigenvalue is simple (that is, if $\hat{u}$, $\hat{v}$ are corresponding eigenfunctions, then $\hat{u} = \eta \hat{v}$ with $\eta \in \mathbb{R} \setminus \{0\}$) and isolated (that is, if $\hat{\sigma}(p)$ denotes the spectrum of (3), then we can find $\varepsilon > 0$ such that $(\hat{\lambda}_1(p), \hat{\lambda}_1(p) + \varepsilon) \cap \hat{\sigma}(p) = \emptyset$). The infimum in (4) is realized on the corresponding one-dimensional eigenspace, the elements of which have fixed sign. We denote by $\hat{u}_1(p)$ the corresponding positive, $L^p$-normalized (that is, $\|\hat{u}_1(p)\|_p = 1$) eigenfunction. We know that $\hat{u}_1(p) \in L^\infty(\Omega)$ (see Colasuonno and Squassina [5, Section 3.2]) and $\hat{u}_1(p)(z) > 0$ for a.a. $z \in \Omega$ (see Papageorgiou, Vetro and Vetro [19, Proposition 4]).

We will also use the spectrum of the following nonlinear eigenvalue problem

$$
-\Delta u = \hat{\lambda}|u|^{p-2}u \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial \Omega.
$$

It is well known that this problem has a sequence of variational eigenvalues $\{\hat{\lambda}_k(q)\}_{k \geq 1}$ such that $\hat{\lambda}_k(q) \to +\infty$ as $k \to \infty$. We have $\hat{\lambda}_1(q) = 0 < \hat{\lambda}_2(q)$ (see Gasinski and Papageorgiou [9, Section 6.2]).

Let $X$ be a Banach space and $\phi \in C^1(X, \mathbb{R})$. We denote by $K_\phi$ the critical set of $\phi$, that is,

$$
K_\phi = \{u \in X : \phi'(u) = 0\}.
$$

Also, if $\eta \in \mathbb{R}$, then we set

$$
\phi^\eta = \{u \in X : \phi(u) \leq \eta\}.
$$

Consider a topological pair $(A, B)$ such that $B \subseteq A \subseteq X$. Then for every $k \in \mathbb{N}_0$, we denote by $H_k(A, B)$ the $k$th-singular homology group for the pair $(A, B)$ with coefficients in a field $\mathbb{F}$ of characteristic zero (for example, $\mathbb{F} = \mathbb{R}$). Then each $H_k(A, B)$ is an $\mathbb{F}$-vector space and we denote by $\dim H_k(A, B)$ its dimension. We also recall that the homeomorphisms induced by maps of pairs and the boundary homomorphism $\partial$ are all $\mathbb{F}$-linear.

Suppose that $u \in K_\phi$ is isolated. Then for every $k \in \mathbb{N}_0$, we define the “$k$-critical group” of $\phi$ at $u$ by

$$
C_k(\phi, u) = H_k(\phi^c \cap U, \phi^c \cap U \setminus \{u\}),
$$

where $U$ is an isolating neighborhood of $u$, that is, $K_\phi \cap U \cap \phi^c = \{u\}$. The excision property of singular homology implies that this definition is independent of the choice of the isolating neighborhood $U$.

We say that $\phi$ satisfies the “$C$-condition” if it has the following property:

“Every sequence $\{u_n\}_{n \geq 1} \subseteq X$ such that $\{\phi(u_n)\}_{n \geq 1} \subseteq \mathbb{R}$ is bounded and $(1 + \|u_n\|)\phi'(u_n) \to 0$ in $X^*$ as $n \to \infty$, has a strongly convergent subsequence”. 
Suppose that \( \phi \in C^1(X, \mathbb{R}) \) satisfies the C-condition and that \( \inf \phi(K_\phi) > -\infty \). Let \( c < \inf \phi(K_\phi) \).

Then the critical groups of \( \phi \) at infinity are defined by

\[ C_k(\phi, \infty) = H_k(\phi, \phi') \text{ for all } k \in \mathbb{N}_0. \]

On account of the second deformation theorem (see Papageorgiou, Rădulescu and Repovš [18, p. 386], Theorem 5.3.12) this definition is independent of the choice of the level \( c < \inf \phi(K_\phi) \).

Our approach is based on the notion of local \((m,n)\)-linking \((m, n \in \mathbb{N})\), see Papageorgiou, Rădulescu and Repovš [18, Definition 6.6.13, p. 534].

**Definition 2.3.** Let \( X \) be a Banach space, \( \phi \in C^1(X, \mathbb{R}) \), and \( 0 \) an isolated critical point of \( \phi \) with \( \phi(0) = 0 \). Let \( m, n \in \mathbb{N} \). We say that \( \phi \) has a “local \((m,n)\)-linking” near the origin if there exist a neighborhood \( U \) of \( 0 \) and nonempty sets \( E_0, E \subseteq U \), and \( D \subseteq X \) such that \( 0 \notin E_0 \subseteq E \), \( E_0 \cap D = \emptyset \) and

\( a) \) \( 0 \) is the only critical point of \( \phi \) in \( \phi^0 \cup U \);
\( b) \) \( \dim \ker i_* - \dim \ker j_* \geq n \), where

\[ i_* : H_{m-1}(E_0) \to H_{m-1}(X \setminus D) \text{ and } j_* : H_{m-1}(E_0) \to H_{m-1}(E) \]

are the homomorphisms induced by the inclusion maps \( i : E_0 \to X \setminus D \) and \( j : E_0 \to E \);
\( c) \) \( \phi|_E \leq 0 < \phi|_{U \cap D \setminus \{0\}} \).

**Remark 2.2.** The notion of “local \((m,n)\)-linking” was introduced by Perera [21] as a generalization of the concept of local linking due to Liu [12]. Here we introduce a slightly more general version of this notion.

3. Superlinear case

In this section we treat the superlinear case, that is, we assume that the reaction \( f(z, \cdot) \) exhibits \((p-1)\)-superlinear growth near \( \pm \infty \).

The hypotheses on \( f(z, x) \) are the following.

\( H_1: \) \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function such that \( f(z, 0) = 0 \) for a.a. \( z \in \Omega \) and

\( i) \) \( |f(z, x)| \leq \hat{a}(z)(1 + |x|^{-1}) \) for a.a. \( z \in \Omega \) and all \( x \in \Omega \), with \( \hat{a} \in L^\infty(\Omega), p < r < q^* \);

\( ii) \) if \( |F(z, x)| = \int_0^x f(z, s)ds \), then \( \lim_{x \to \pm \infty} |F(z, x)| = +\infty \) uniformly for a.a. \( z \in \Omega \);

\( iii) \) if \( \eta(z, x) = f(z, x)x - pF(z, x) \), then there exists \( e \in L^1(\Omega) \) such that

\[ \eta(z, x) < \eta(z, y) + e(z) \text{ for a.a. } z \in \Omega \text{ and all } 0 \leq x \leq y \text{ or } y \leq x \leq 0; \]

\( iv) \) there exist \( \delta > 0, \theta \in L^\infty(\Omega) \) and \( \tilde{\lambda} > 0 \) such that

\[ 0 \leq \theta(z) \text{ for a.a. } z \in \Omega, \theta \neq 0, \lambda \leq \tilde{\lambda} \text{ for a.a. } z \in \Omega \text{ and all } |x| \leq \delta. \]

**Remark 3.1.** Evidently, hypotheses \( H_1(ii), (iii) \) imply that for a.a. \( z \in \Omega \), the function \( f(z, \cdot) \) is superlinear. However, to express this superlinearity, we do not invoke the usual AR-condition. We recall that the AR-condition says that there exist \( \tau > p \) and \( M > 0 \) such that

\[ 0 < \tau F(z, x) \leq f(z, x)x \text{ for a.a. } z \in \Omega \text{ and all } |x| \geq M; \text{ and} \]

\[ 0 < \text{essinf}_z F(z, \cdot, \pm M). \]

Integrating (5) and using (6), we obtain a weaker condition, namely

\[ c_2 |x|^\tau \leq F(z, x) \text{ for a.a. } z \in \Omega, \text{ all } |x| \geq M \text{ and some } c_2 > 0, \]

\[ \Rightarrow c_3 |x|^\tau \leq f(z, x)x \text{ for a.a. } z \in \Omega, \text{ all } |x| \geq M \text{ and with } c_3 = \tau c_2 > 0. \]

Therefore the AR-condition implies that, eventually, \( f(z, \cdot) \) has at least \((\tau - 1)\)-polynomial growth.
In the present work, instead of the AR-condition, we use the quasimonotonicity hypothesis $H_1(iii)$, which is less restrictive and incorporates in our framework also $(p-1)$-superlinear nonlinearities with slower growth near $\pm \infty$ (see the examples below). Hypothesis $H_1(iii)$ is a slight generalization of a condition which can be found in Li and Yang [11]. There are very natural ways to verify the quasimonotonicity condition. So, if there exists $M > 0$ such that for a.a. $z \in \Omega$, either the function
\[ x \mapsto \frac{f(z, x)}{|x|^{q-2}x} \]
is increasing on $x \geq M$ and decreasing on $x \leq -M$
or the mapping
\[ x \mapsto \eta(z, x) \]
is increasing on $x \geq M$ and decreasing on $x \leq -M$,
then hypothesis $H_1(iii)$ holds.

Hypothesis $H_1(iv)$ implies that for a.a. $z \in \Omega$, the primitive $F(z, \cdot)$ is $q$-linear near 0.

**Examples.** The following functions satisfy hypotheses $H_1$. For the sake of simplicity we drop the $z$-dependence:
\[
f_1(x) = \begin{cases} \mu \frac{|x|^{q-2}x}{|x|} & \text{if } |x| \leq 1 \\ \mu |x|^{q-2}x & \text{if } |x| > 1 \end{cases} \quad (0 < \mu \leq \lambda_2(q) \text{ and } p < r < q^*)
\]
\[
f_2(x) = \begin{cases} \mu |x|^{q-2}x & \text{if } |x| \leq 1 \\ \mu |x|^{q-2}x \ln x + \mu |x|^{q-2}x & \text{if } |x| > 1 \end{cases} \quad (0 < \mu \leq \lambda_2(q) \text{ and } 1 < \tau < p).
\]

Note that only $f_1$ satisfies the AR-condition, whereas the function $f_2$ does not satisfy this growth condition.

The energy functional for problem (1) is the $C^1$-functional $\varphi : W^{1,\theta}(\Omega) \to \mathbb{R}$ defined by
\[
\varphi(u) = \frac{1}{p} \gamma_p(u) + \frac{1}{q} \|Du\|_q^q - \int_{\Omega} F(z, u)dz \text{ for all } u \in W^{1,\theta}(\Omega).
\]

Next, we show that $\varphi(\cdot)$ satisfies the C-condition.

**Proposition 3.1.** If hypotheses $H_0$, $H_1$ hold, then the functional $\varphi(\cdot)$ satisfies the C-condition.

**Proof.** We consider a sequence $\{u_n\}_{n \geq 1} \subseteq W^{1,\theta}(\Omega)$ such that
\[
|\varphi(u_n)| \leq c_4 \text{ for some } c_4 > 0 \text{ and all } n \in \mathbb{N},
\]
\[
(1 + \|u_n\|)\varphi'(u_n) \to 0 \text{ in } W^{1,\theta}(\Omega)^* \text{ as } n \to \infty.
\]

From (8) we have
\[
\left| \langle A_p^{a_0}(u_n), h \rangle + \langle A_q(u_n), h \rangle + \int_{\Omega} \xi(z)|u_n|^{p-2}u_nhdz + \int_{\partial\Omega} \beta(z)|u_n|^{p-2}u_nh \right| \\
\leq \varepsilon_n \frac{\|h\|}{1 + \|u_n\|},
\]
for all $h \in W^{1,\theta}(\Omega)$, with $\varepsilon_n \to 0$.

In (9) we choose $h = u_n \in W^{1,\theta}(\Omega)$ and obtain for all $n \in \mathbb{N}$
\[
-\int_{\Omega} a_0(z)|Du_n|^p dz - \|Du_n\|_q^q - \int_{\Omega} \xi(z)|u_n|^{p}dz - \int_{\partial\Omega} \beta(z)|u_n|^{p}d\sigma + \int_{\Omega} f(z, u_n)u_n dz \leq \varepsilon_n.
\]

Also, by (7) we have for all $n \in \mathbb{N}$,
\[
\int_{\Omega} a_0(z)|Du_n|^p dz + \frac{p}{q} \|Du_n\|_q^q + \frac{p}{q} \int_{\Omega} \xi(z)|u_n|^{p}dz + \int_{\Omega} \beta(z)|u_n|^{p}d\sigma - \int_{\Omega} pF(z, u_n)dz \leq pc_4.
\]

We add relations (10) and (11). Since $q < p$, we obtain
\[
\int_{\Omega} \eta(z, u_n)dz \leq c_5 \text{ for some } c_5 > 0 \text{ and all } n \in \mathbb{N}.
\]

**Claim.** The sequence $\{u_n\}_{n \geq 1} \subseteq W^{1,\theta}(\Omega)$ is bounded.
We argue by contradiction. Suppose that the claim is not true. We may assume that
\[ \|u_n\| \to \infty \text{ as } n \to \infty. \]
We set \( y_n = u_n/\|u_n\| \) for all \( n \in \mathbb{N} \). Then \( \|y_n\| = 1 \) and so we may assume that
\[ y_n \xrightarrow{w} y \text{ in } W^{1,\theta}(\Omega) \text{ and } y_n \to y \text{ in } L^r(\Omega) \text{ and in } L^p(\partial\Omega), \]
see hypotheses \( H_0 \), Proposition 2.1 and Remark 2.1.

We first assume that \( y \neq 0 \). Let
\[ \Omega_+ = \{ z \in \Omega : y(z) > 0 \} \text{ and } \Omega_- = \{ z \in \Omega : y(z) < 0 \}. \]
Then at least one of these measurable sets has positive Lebesgue measure on \( \mathbb{R}^N \). We have
\[ u_n(z) \to +\infty \text{ for a.a. } z \in \Omega_+ \text{ and } u_n(z) \to -\infty \text{ for a.a. } z \in \Omega_- \]

Let \( \hat{\Omega} = \Omega_+ \cup \Omega_- \) and let \( |\cdot|_N \) denote the Lebesgue measure on \( \mathbb{R}^N \). We see that \( |\hat{\Omega}|_N > 0 \) and on account of hypothesis \( H_1(ii) \), we have
\[ \frac{F(z, u_n(z))}{\|u_n\|^p} = \frac{F(z, y_n(z))}{\|y_n\|^p} |y_n(z)|^p \to +\infty \text{ for a.a. } z \in \hat{\Omega}, \]
(15)
\[ \Rightarrow \int_{\hat{\Omega}} \frac{F(z, u_n(z))}{\|u_n\|^p} dz \to +\infty \text{ by Fatou's lemma.} \]

Hypotheses \( H_1(i), (ii) \) imply that
\[ F(z, x) \geq -c_6 \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R} \text{ and some } c_6 > 0. \]

Thus we obtain
\[ \int_{\Omega} \frac{F(z, u_n)}{\|u_n\|^p} dz = \int_{\Omega} \frac{F(z, u_n)}{\|u_n\|^p} dz + \int_{\hat{\Omega}} \frac{F(z, u_n)}{\|u_n\|^p} dz \]
\[ \geq \int_{\hat{\Omega}} \frac{F(z, u_n)}{\|u_n\|^p} dz - c_6|\hat{\Omega}|_N \text{ (see (16))}, \]
\[ \Rightarrow \lim_{n \to \infty} \int_{\Omega} \frac{F(z, u_n)}{\|u_n\|^p} dz = +\infty \text{ (see (15) and (13)).} \]

By (7), we have
\[ \int_{\Omega} \frac{pF(z, u_n)}{\|u_n\|^p} dz \leq c_7 \frac{1}{\|u_n\|^p} \left[ \gamma_p(y_n) + \frac{p}{q} \|Dy_n\|^q_q + \frac{c_4}{\|u_n\|^p} \right] \leq c_7, \]
for some \( c_7 > 0 \) and all \( n \in \mathbb{N} \) (see (13) and recall that \( \|y_n\| = 1 \).

We compare relations (15) and (18) and arrive at a contradiction.

Next, we assume that \( y = 0 \). Let \( \mu > 0 \) and set \( v_n = (p\mu)^{1/p}y_n \) for all \( n \in \mathbb{N} \). Evidently, we have
\[ v_n \to 0 \text{ in } L^r(\Omega) \text{ (see (14))}, \]
(19)
\[ \Rightarrow \int_{\Omega} F(z, v_n) dz \to 0 \text{ as } n \to \infty. \]

Consider the functional \( \psi : W^{1,\theta}(\Omega) \to \mathbb{R} \) defined by
\[ \psi(u) = \frac{1}{p} \gamma_p(u) - \int_{\Omega} F(z, u) dz \text{ for all } u \in W^{1,\theta}(\Omega). \]

Clearly, \( \psi \in C^1(W^{1,\theta}(\Omega), \mathbb{R}) \) and
\[ \psi \leq \varphi. \]
We can find \( t_n \in [0, 1] \) such that
\[ \psi(t_n u_n) = \min \{ \psi(t u_n) : 0 \leq t \leq 1 \} \text{ for all } n \in \mathbb{N}. \]

Because of (13), we can find \( n_0 \in \mathbb{N} \) such that
\[ 0 < \frac{(p\mu)^{1/p}}{\|u_n\|} \leq 1 \text{ for all } n \geq n_0. \]
Therefore
\[ \psi(t_n u_n) \geq \psi(v_n) \quad (\text{see } (21), (22)) \]
\[ \geq \mu \gamma_p(y_n) - \int_{\Omega} F(z, v_n) dz \]
\[ \geq \mu c_1 - \int_{\Omega} F(z, v_n) dz \quad (\text{see } (2) \text{ and recall that } \|y_n\| = 1) \]
\[ \geq \frac{\mu}{2} c_1 \text{ for all } n \geq n_1 \geq n_0 \quad (\text{see } (19)). \]

Since \( \mu > 0 \) is arbitrary, it follows that
\[ \psi(t_n u_n) \to +\infty \text{ as } n \to \infty. \]

Note that
\[ \psi(0) = 0 \text{ and } \psi(u_n) \leq c_4 \text{ for all } n \in \mathbb{N} \quad (\text{see } (7), (20)). \]

By (23) and (24) we can infer that
\[ t_n \in (0, 1) \text{ for all } n \geq n_2. \]

From (21) and (25), we can see that for all \( n \geq n_2 \) we have
\[ 0 = t_n \frac{d}{dt} \psi(t_n \eta_n)|_{t=t_n} \]
\[ = \langle \psi'(t_n \eta_n), t_n \eta_n \rangle \quad (\text{by the chain rule}) \]
\[ = \gamma_p(t_n \eta_n) - \int_{\Omega} f(z, t_n \eta_n)(t_n \eta_n) dz. \]

It follows that
\[ 0 \leq t_n u_n^+ \leq u_n^+ \text{ and } -u_n^- \leq -t_n u_n^- \leq 0 \text{ for all } n \in \mathbb{N} \]
(recall that \( u_n^+ = \max\{u_n, 0\} \) and \( u_n^- = \max\{-u_n, 0\} \)).

By hypothesis \( H_1(iii) \), we have
\[ \eta(z, t_n u_n^+) \leq \eta(z, u_n^+) + e(z) \text{ for a.a. } z \in \Omega \text{ and all } n \in \mathbb{N}, \]
\[ \eta(z, -t_n u_n^-) \leq \eta(z, -u_n^-) + e(z) \text{ for a.a. } z \in \Omega \text{ and all } n \in \mathbb{N}. \]

From these two inequalities and since \( u_n = u_n^+ - u_n^- \), we obtain
\[ \eta(z, t_n u_n) \leq \eta(z, u_n) + e(z) \text{ for a.a. } z \in \Omega \text{ and all } n \in \mathbb{N}, \]
\[ \Rightarrow f(z, t_n u_n)(t_n u_n) \leq \eta(z, u_n) + e(z) + pF(z, t_n u_n) \text{ for a.a. } z \in \Omega \text{ and all } n \in \mathbb{N}. \]

We return to (26) and apply (27). Then
\[ \gamma_p(t_n u_n) \geq p \int_{\Omega} F(z, t_n u_n) dz \leq \int_{\Omega} \eta(z, u_n) dz + \|e\|_1 \text{ for all } n \in \mathbb{N}, \]
\[ \Rightarrow p \psi(t_n u_n) \leq c_8 \text{ for some } c_8 > 0 \text{ and all } n \in \mathbb{N} \quad (\text{see } (12)). \]

We compare (23) and (28) and arrive at a contradiction.

This proves the claim.

On account of this claim, we may assume that
\[ u_n \rightharpoonup u \text{ in } W^{1,\theta}(\Omega) \text{ and } u_n \to u \text{ in } L^r(\Omega) \text{ and in } L^p(\partial\Omega) \]
(see hypotheses \( H_0 \)).

From (29) we have
\[ Du_n \rightharpoonup Du \text{ in } L^p_{\text{loc}}(\Omega, \mathbb{R}^N) \text{ and } Du_n(z) \to Du(z) \text{ a.a. } z \in \Omega. \]

In (9) we choose \( h = u_n - u \in W^{1,\theta}(\Omega) \), pass to the limit as \( n \to \infty \) and use (30) and the monotonicity of \( \Delta_p(\cdot)^{\alpha_0} \). We obtain
\[ \limsup_{n \to \infty} (A^\alpha_p(u_n), u_n - u) \leq 0, \]
\[ \Rightarrow \limsup_{n \to \infty} \|Du_n\|_{L^p_{\text{loc}}(\Omega, \mathbb{R}^N)} \leq \|Du\|_{L^p_{\text{loc}}(\Omega, \mathbb{R}^N)}. \]
On the other hand, from (30) we have
\[
\liminf_{n \to \infty} \|Du_n\|_{L^p_0(\Omega, \mathbb{R}^N)} \geq \|Du\|_{L^p_0(\Omega, \mathbb{R}^N)}.
\]
Therefore we conclude that
\[
\|Du_n\|_{L^p_0(\Omega, \mathbb{R}^N)} \to \|Du\|_{L^p_0(\Omega, \mathbb{R}^N)}.
\]

The space \(L^p_0(\Omega, \mathbb{R}^N)\) is uniformly convex, hence it has the Kadec-Klee property (see Papageorgiou, Rădulescu and Repovš [18, Remark 2.7.30, p. 127]). So, it follows from (30) and (31) that
\[
Du_n \to Du \text{ in } L^p_0(\Omega, \mathbb{R}^N),
\]
\[
\Rightarrow Du_n \to Du \text{ in } L^q(\Omega, \mathbb{R}^N) \text{ since } L^p_0(\Omega, \mathbb{R}^N) \hookrightarrow L^q(\Omega, \mathbb{R}^N) \text{ continuously},
\]
\[
\Rightarrow \rho_\theta(|Du_n - Du|) \to 0 \text{ (see Proposition 2.2)},
\]
\[
\Rightarrow \|u_n - u\| \to 0 \text{ (see (29) and Proposition 2.2)},
\]
\[
\Rightarrow \varphi \text{ satisfies the C-condition}.
\]
The proof is now complete. \(\square\)

**Proposition 3.2.** If hypotheses \(H_0, H_1\) hold, then the functional \(\varphi(\cdot)\) has a local \((1,1)\)-linking at 0.

**Proof.** Since the critical points of \(\varphi\) are solutions of problem (1), we may assume that \(K_\varphi\) is finite or otherwise we already have infinitely many nontrivial solutions of (1) and so we are done.

Choose \(\rho \in (0, 1)\) so small that \(K_\varphi \cap \bar{B}_\rho = \{0\}\) (here, \(B_\rho = \{u \in W^{1,q}(\Omega) : \|u\| < \rho\}\)). Let \(V = \mathbb{R}\) and let \(\delta > 0\) as postulated by hypothesis \(H_1(iv)\). Recall that on a finite dimensional normed space all norms are equivalent. So, by taking \(\rho \in (0, 1)\) even smaller as necessary, we have
\[
(32) \quad \|u\| \leq \rho \Rightarrow \|u\| \leq \delta \text{ for all } u \in V = \mathbb{R}.
\]
Then for \(u \in V \cap \bar{B}_\rho\), we have
\[
\varphi(u) \leq \frac{1}{p} \gamma_p(u) - \frac{|u|^q}{q} \int_\Omega \theta(z)dz \text{ (see (32) and Hypothesis } H_1(iv))
\]
\[
= \frac{1}{p} \|u\|^p \left( \int_\Omega \xi(z)dz + \int_{\partial\Omega} \beta(z)d\sigma \right) - \frac{|u|^q}{q} \int_\Omega \theta(z)dz
\]
\[
\leq c_0 \|u\|^p - c_{10} \|u\|^q \text{ for some } c_0, c_{10} > 0 \text{ (see hypotheses } H_0 \text{ and } H_1(iv)).
\]
Since \(q < p\), choosing \(\rho \in (0, 1)\) small, we conclude that
\[
(33) \quad \varphi|_{V \cap \bar{B}_\rho} \leq 0.
\]
Let
\[
D = \{u \in W^{1,q}(\Omega) : \|Du\|_q^q \geq \hat{\lambda}_2(q)\|u\|_q^q\}.
\]
For all \(u \in D\) we have
\[
\varphi(u) = \frac{1}{p} \gamma_p(u) + \frac{1}{q} \int_{\{|u| \leq \delta\}} F(z,u)dz - \int_{\{|u| > \delta\}} F(z,u)dz
\]
\[
\geq \frac{1}{p} \gamma_p(u) + \frac{1}{q} \left( \|Du\|_q^q - \int_{\{|u| \leq \hat{\lambda}_2(q)\} \|u\|_q^q} F(z,u)dz \right)
\]
\[
(\text{see hypotheses } H_1(iv))
\]
\[
\geq \frac{1}{p} \gamma_p(u) + \frac{1}{q} \int_{\Omega} (\hat{\lambda}_2(q) - \hat{\lambda})|u|^qdz - c_{11} \|u\|^r
\]
\[
\text{for some } c_{11} > 0 \text{ (since } u \in D \text{ and see hypotheses } H_1(iv)).
\]
Since \(p < r\), for small \(\rho \in (0, 1)\) we have
\[
(34) \quad \varphi|_{D \cap \bar{B}_\rho \setminus \{0\}} > 0.
\]
Let \(U = \bar{B}_\rho, E_0 = V \cap \partial B_\rho, E = V \cap \bar{B}_\rho\) and \(D\) as above. We have \(0 \notin E_0, E_0 \subseteq E \subseteq U = \bar{B}_\rho\) and \(E_0 \cap D = \emptyset\) (see Definition 2.3).

Let \(Y\) be the topological complement of \(V\). We have that
\[
W^{1,q}(\Omega) = V \oplus Y \text{ (see [18, pp. 73, 74])}.
\]
So, every \( u \in W^{1,\theta}(\Omega) \) can be written in a unique way as
\[
u = v + y \text{ with } v \in V, y \in Y.
\]

We consider the deformation \( h : [0, 1] \times (W^{1,\theta}(\Omega) \setminus D) \to W^{1,\theta}(\Omega) \setminus D \) defined by
\[
h(t, u) = (1 - t)u + t\bar{\rho} \frac{v}{\|v\|} \text{ for all } t \in [0, 1], u \in W^{1,\theta}(\Omega) \setminus D.
\]

We have
\[
h(0, u) = u \text{ and } h(1, u) = \rho \frac{v}{\|v\|} \in V \cap \partial B_\rho = E_0.
\]

It follows that \( E_0 \) is a deformation retract of \( W^{1,\theta}(\Omega) \setminus D \) (see Papageorgiou, Rădulescu and Repovš [18, Remark 6.1.26, p. 468]). Hence
\[
i_* : H_0(E_0) \to H_0(W^{1,\theta}(\Omega) \setminus \{0\})
\]
is an isomorphism (see Eilenberg and Steenrod [8, Theorem 11.5, p.30] and Papageorgiou, Rădulescu and Repovš [18, Remark 6.1.6, p. 460]).

The set \( E = V \cap B_\rho \) is contractible (it is an interval). Hence \( H_0(E, E_0) = 0 \) (see Eilenberg and Steenrod [8, Theorem 11.5, p. 30]). Therefore, if \( j_* : H_0(E_0) \to H_0(E) \), then \( \dim im j_* = 1 \) (see Papageorgiou, Rădulescu and Repovš [8, Remark 6.1.26, p. 468]). So, finally we have
\[
\dim im i_* - \dim im j_* = 2 - 1 = 1,
\]
\( \Rightarrow \varphi(\cdot) \) has a local \((1,1)\)-linking at 0, see Definition 2.3.

The proof is now complete. \( \Box \)

From Proposition 3.2 and Theorem 6.6.17 of Papageorgiou, Rădulescu and Repovš [18, p. 538], we have
\[
(35) \quad \dim C_1(\varphi, 0) \geq 1.
\]

Moreover, Proposition 3.9 of Papageorgiou, Rădulescu and Repovš [18] leads to the following result.

**Proposition 3.3.** If hypotheses \( H_0, H_1 \) hold, then \( C_k(\varphi, \infty) = 0 \) for all \( k \in \mathbb{N}_0 \).

We are now ready for the existence theorem concerning the superlinear case.

**Theorem 3.4.** If hypotheses \( H_0, H_1 \) hold, then problem (1) has a nontrivial solution \( u_0 \in W^{1,\theta}(\Omega) \cap L^\infty(\Omega) \).

**Proof.** On account of (35) and Proposition 3.3, we can apply Proposition 6.2.42 of Papageorgiou, Rădulescu and Repovš [18, p. 499]. So, we can find \( u_0 \in W^{1,\theta}(\Omega) \) such that
\[
u_0 \in K_\varphi \setminus \{0\},
\]
\( \Rightarrow \nu_0 \in W^{1,\theta}(\Omega) \cap L^\infty(\Omega) \) is a solution to problem (1), see [18, Section 3.2].

The proof is now complete. \( \Box \)

### 4. Resonant case

In this section we are concerned with the resonant case (\( p \)-linear case). Our hypotheses allow resonance at \( \pm \infty \) with respect to the principal eigenvalue \( \hat{\lambda}_1(p) > 0 \).

The new conditions on the reaction \( f(z, x) \) are the following.

**H2.** \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function such that \( f(z, 0) = 0 \) for a.a. \( z \in \Omega \) and

(i) \( |f(z, x)| \leq \hat{a}(z)(1 + |x|^{p-1}) \) for a.a. \( z \in \Omega \), all \( x \in \mathbb{R} \), with \( \hat{a} \in L^\infty(\Omega), p < r < q^* \);

(ii) if \( F(z, x) = \int_0^x f(z, s)ds \), then \( \lim_{x \to \pm \infty} pF(z, x)/|x|^p \leq \hat{\lambda}_1(p) \) uniformly for a.a. \( z \in \Omega \);

(iii) we have
\[
f(z, x)x - pF(z, x) \to +\infty \text{ uniformly for a.a. } z \in \Omega, \text{ as } x \to \pm \infty;
\]

(iv) there exist \( \delta > 0, \theta \in L^\infty(\Omega) \) and \( \hat{\lambda} > 0 \) such that
\[
0 \leq \theta(z) \text{ for a.a. } z \in \Omega, \theta \not\equiv 0, \hat{\lambda} \leq \hat{\lambda}_2(g),
\]
\( \theta(z)|x|^q \leq qF(z, x) \leq \hat{\lambda}|x|^q \) for a.a. \( z \in \Omega \) and all \( |x| \leq \delta \).
Remark 4.1. Hypothesis $H_2(ii)$ implies that at $\pm \infty$, we can have resonance with respect to the principal eigenvalue of the operator $u \mapsto -\text{div}(a_0(z)|Du|^{p-2}Du) - \Delta u$ with Robin boundary condition.

Proposition 4.1. If hypotheses $H_0$, $H_2$ hold, then the energy functional $\varphi(\cdot)$ is coercive.

Proof. We have
\[
\frac{d}{dx} \left( \frac{F(z,x)}{|x|^p} \right) = \frac{f(z,x)|x|^p - p|x|^{p-2}xF(z,x)}{|x|^{2p}} = \frac{|x|^{p-2}x[f(z,x)x - pF(z,x)]}{|x|^{2p}}.
\]

On account of hypothesis $H_2(iii)$, given any $\gamma > 0$, we can find $M_1 = M_1(\gamma) > 0$ such that
\[
f(z,x)x - pF(z,x) \geq \gamma \text{ for a.a. } z \in \Omega \text{ and all } |x| \geq M_1.
\]

Hence we obtain
\[
\frac{d}{dx} \left( \frac{F(z,x)}{|x|^p} \right) \begin{cases} 
\geq \frac{\gamma}{x^{p+1}} & \text{if } x \geq M_1 \\
\leq -\frac{\gamma}{|x|^{p+1}} & \text{if } x \leq -M_1.
\end{cases}
\]

Integrating, we obtain
\[
\frac{F(z,x)}{|x|^p} - \frac{F(z,x)}{|u|^p} \geq -\frac{\gamma}{p} \left( \frac{1}{|x|^p} - \frac{1}{|u|^p} \right) \text{ for a.a. } z \in \Omega \text{ and all } |x| \geq |u| \geq M_1.
\]

On account of hypothesis $H_2(ii)$, given $\varepsilon > 0$, we can find $M_2 = M_2(\varepsilon) > 0$ such that
\[
F(z,x) \leq \frac{1}{p} (\hat{\lambda}_1(p) + \varepsilon)|x|^p \text{ for a.a. } z \in \Omega \text{ and all } |x| \geq M_2.
\]

Using this inequality in (36) and letting $|x| \to \infty$ we obtain
\[
\frac{1}{p} (\hat{\lambda}_1(p) + \varepsilon) - \frac{F(z,u)}{|u|^p} \geq \frac{\gamma}{p} \frac{1}{|u|^p} \text{ for a.a. } z \in \Omega \text{ and all } |u| \geq M = \max\{M_1, M_2\},
\]

Arguing by contradiction, suppose that $\varphi(\cdot)$ is not coercive. Then we can find $\{u_n\}_{n \geq 1} \subseteq W^{1,\theta}(\Omega)$ such that
\[
\|u_n\| \to \infty \text{ and } \varphi(u_n) \leq M_0 \text{ for some } M_0 > 0 \text{ and all } n \in \mathbb{N}.
\]

Let $y_n = u_n/\|u_n\|$ for all $n \in \mathbb{N}$. Then $\|y_n\| = 1$, hence we may assume that
\[
y_n \rightharpoonup y \text{ in } W^{1,\theta}(\Omega) \text{ and } y_n \to y \text{ in } L^p(\Omega) \text{ and in } L^p(\partial \Omega).
\]

From (38) we have
\[
\frac{1}{p} \gamma_p(y_n) + \frac{1}{q} \frac{1}{\|u_n\|^{p-q}} \int_{\Omega} |Dy_n|^q dz \leq \int_{\Omega} \frac{F(z,u_n)}{\|u_n\|^p} dz \leq \frac{M_0}{\|u_n\|^p},
\]
\[
\Rightarrow \gamma_p(y_n) + \frac{1}{q} \frac{1}{\|u_n\|^{p-q}} \int_{\Omega} |Dy_n|^q dz \leq \tau_n + (\hat{\lambda}_1(p) + \varepsilon) \|y_n\|^p \text{ with } \tau_n \to 0, \text{ see (37),}
\]
\[
\Rightarrow \gamma_p(y_n) \leq \left( \hat{\lambda}_1(p) + \varepsilon \right) \|y_n\|^p \text{ (see (39)),}
\]
\[
\Rightarrow \gamma_p(y_n) \leq \lambda_1(p) \|y_n\|^p \text{ (since } \varepsilon > 0 \text{ is arbitrary),}
\]
\[
y = \mu \hat{u}_1(p) \text{ for some } \mu \in \mathbb{R} \text{ (see (4)).}
\]

If $\mu = 0$, then $y = 0$ and so $\gamma_p(y_n) \to 0$. Hence, as in the proof of Proposition 3.1, we have $y_n \to 0$ in $W^{1,\theta}(\Omega)$, contradicting the fact that $\|y_n\| = 1$ for all $n \in \mathbb{N}$.

So, $\mu \neq 0$ and since $\hat{u}_1(p)(z) > 0$ for a.a. $z \in \Omega$, we have $|u_n(z)| \to +\infty$ for a.a. $z \in \Omega$. By (38) and (4) we have
\[
\int_{\Omega} \left[ \frac{1}{p} \hat{\lambda}_1(p)|u_n|^p - F(z,u_n) \right] dz \leq M_0 \text{ for all } n \in \mathbb{N}.
\]
However, from (37) and since $\gamma > 0$ is arbitrary, we can infer that
\[
\frac{1}{p} \lambda_1(p) |u_n|^p - F(z, u_n) \to +\infty \text{ for a.a. } z \in \Omega, \text{ as } n \to \infty,
\]
(41)
\[
\Rightarrow \int_\Omega \left[ \frac{1}{p} \lambda_1(p) |u_n|^p - F(z, u_n) \right] dz \to +\infty \text{ by Fatou's lemma}.
\]

Comparing (40) and (41) we arrive at a contradiction. Therefore we can conclude that $\varphi(\cdot)$ is coercive.

Using Proposition 4.1 and Proposition 5.1.15 of Papageorgiou, Rădulescu and Repovš [18, p. 369], we obtain the following result.

**Corollary 4.2.** If hypotheses $H_0$, $H_2$ hold, then the energy functional $\varphi(\cdot)$ is bounded below and satisfies the C-condition.

Now we are ready for the multiplicity theorem in the resonant case.

**Theorem 4.3.** If hypotheses $H_0$, $H_2$ hold, then problem (1) has at least two nontrivial solutions $u_0, \hat{u} \in W^{1,\theta}(\Omega) \cap L^\infty(\Omega)$.

**Proof.** By Proposition 3.2 we know that $\varphi(\cdot)$ has a local (1,1)-linking at the origin. Note that for that result mattered only the behavior of $f(z, \cdot)$ near zero and this is common in hypotheses $H_1$ and $H_2$. Also, we know that $\varphi(\cdot)$ is sequentially weakly lower semicontinuous. This fact in conjunction with Proposition 4.1, permit the use of the Weierstrass-Tonelli theorem. So, we can find $u_0 \in W^{1,\theta}(\Omega)$ such that
\[
\varphi(u_0) = \min \{ \varphi(u) : u \in W^{1,\theta}(\Omega) \}.
\]
(42)

On account of hypothesis $H_2(iv)$ and since $q < p$, we have
\[
\varphi(u_0) < 0 = \varphi(0),
\]
\[
\Rightarrow u_0 \neq 0 \text{ and } u_0 \in K_\varphi,
\]
\[
\Rightarrow u_0 \in K_\varphi \cap L^\infty(\Omega) \text{ is a nontrivial solution of (1)}.
\]

Moreover, by Corollary 6.7.10 of Papageorgiou, Rădulescu and Repovš [18, p. 552], we can find $\hat{u} \in K_\varphi$, $\hat{u} \not\in \{0, u_0\}$. Then $\hat{u} \in W^{1,\theta}(\Omega) \cap L^\infty(\Omega)$ is the second nontrivial solution of problem (1). \qed

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