A ROBINSON-SCHENSTED CORRESPONDENCE FOR PARTIAL PERMUTATIONS

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Abstract. We study the Steinberg variety associated to matrix Schubert varieties, and develop a Robinson-Schensted type correspondence, \( \tau \leftrightarrow (\Lambda, Q, P) \). Here \( \tau \) is a partial permutation of size \( p \times q \), \( \Lambda \) an admissible signed Young diagram of size \( p + q \), and \( P \) (resp. \( Q \)) a standard Young tableau of size \( p \) (resp. \( q \)) whose shape is determined by \( \Lambda \). By embedding the matrix Schubert variety into a Schubert variety, we find a close relationship between the combinatorics of the classical Robinson-Schensted-Knuth correspondence and our bijection. We also show that an involution \( (\Lambda, Q, P) \mapsto (\Lambda^\vee, P, Q) \) corresponds to projective duality on matrix Schubert varieties.

1. Introduction

The classical Robinson-Schensted correspondence, see [Rob38, Ful97], associates to each permutation on \( n \) letters, a triple \( (\lambda, Q, P) \), with \( \lambda \) a partition of \( n \), and \( Q \) and \( P \) Young tableaux of shape \( \lambda \). In [Ste88], Steinberg presented a geometric interpretation of the Robinson-Schensted correspondence, by showing that both sides of the correspondence count the irreducible components of the Steinberg variety. This work has seen many generalizations, see [Tra05, HT12, Ros12, FN] among others.

We continue in this tradition, developing a bijection for partial permutations by studying the Steinberg variety associated to matrix Schubert varieties.

Let \( F^l(V) \) be the flag variety of a vector space \( V \). Given a nilpotent map \( x \in \mathfrak{gl}(V) \), the variety

\[
\{(x, F_i) \in \mathfrak{gl}(V) \times F^l(V) \mid xF_i \subset F_{i-1} \forall i\}
\]

is called the Springer fibre over \( x \). Following [Spa82], the irreducible components of the Springer fibre are indexed by the standard Young tableaux of shape \( J(x) \), the Jordan type of \( x \).

Let \( V_p \) (resp. \( V_q \)) be a \( p \) (resp. \( q \))-dimensional vector space, and

\[
K = GL(V_q) \times GL(V_p), \quad C = F^l(V_q) \times F^l(V_p) \times \text{Hom}(V_q, V_p).
\]

The \( K \)-orbits in \( C \) are called matrix Schubert varieties; they are indexed by partial permutations of size \( p \times q \), see [Ful92]. In a very general setting, see Section 2, the irreducible components of the corresponding Steinberg variety \( Z \) are known to be in bijection with the \( K \)-orbits, i.e., the matrix Schubert varieties.

Let \( \mathcal{O} \) be the nilpotent cone of the \( \widehat{\mathbb{A}}_2 \) quiver. The \( K \)-orbits of \( \mathcal{O} \) are indexed by signed Young diagrams, cf. [Kem82, Joh10], see also Section 4. We construct a \( K \)-equivariant proper map, \( \text{pr} : Z \to \mathcal{O} \), closely related to the moment map for the \( K \)-action on \( C \). The fibre at each point of this map is a product of Springer fibres.
In particular, the irreducible components of this fibre are indexed by certain pairs of standard Young tableaux.

This suggests an alternate characterization of the irreducible components of \( Z \). For every \( Z' \in \operatorname{Irr}(Z) \), there is a signed Young diagram \( \Lambda \) such that \( \operatorname{pr}(Z') = \mathcal{O}(\Lambda) \); let \( (Q, P) \) be the pair of standard Young tableaux indexing the generic fibre of \( \operatorname{pr} : Z' \to \mathcal{O}(\Lambda) \). The triple \( (\Lambda, Q, P) \) identifies the irreducible component \( Z' \).

Our first result, Theorem A, states that the above procedure does yield a bijection, with one caveat: only a subset of the signed Young diagrams appear as \( \Lambda \); we call these the admissible signed Young diagrams. We give geometric and combinatorial characterizations of admissible signed Young diagrams in Section 4.12 and Theorem 4.13.

**Theorem A.** Let \( \tau \) be a partial permutation of size \( p \times q \), and let \((F_\bullet, G_\bullet, x, y)\) be a generic point in the matrix Schubert variety \( C(\tau) \). Let \( \operatorname{pr}(C(\tau)) = \mathcal{O}(\Lambda) \), and set \( Q = \operatorname{Tab}(yx, F_\bullet) \), \( P = \operatorname{Tab}(xy, G_\bullet) \). We have a bijection,

\[
\mathcal{P}(p, q) = \bigsqcup_{\Lambda \in \operatorname{ASYD}(q, p)} \operatorname{SYT}(\Lambda^+) \times \operatorname{SYT}(\Lambda^-),
\]

given by \( \tau \mapsto (\Lambda, Q, P) \).

The flag \( F_\bullet \) (resp. \( G_\bullet \)) lives in the Springer fibre above the nilpotent matrix \( yx \) (resp. \( xy \)), and \( Q \) (resp. \( P \)) is the standard Young tableau relating the \( F_\bullet \) (resp. \( G_\bullet \)) with \( yx \) (resp \( xy \)), see Section 3.10.

To unravel the combinatorics of the bijection in Theorem A we embed the matrix Schubert variety \( C(\tau) \) into a Schubert variety \( Y(\hat{\tau}) \), see Sections 3.9 and 3.11, via (a simple variation of) the map,

\[
x \mapsto \begin{pmatrix} I_q & 0 \\ x & I_p \end{pmatrix},
\]

see Eq. (6.1.1). The key idea is that the conormal variety of \( C(\tau) \) is closely related to the conormal variety of \( Y(\hat{\tau}) \), see Proposition 6.3. This allows us to leverage the geometric interpretation of the RSK correspondence (cf. Ste88, Ros12). Combining this with the geometric interpretations of various tableau algorithms developed by van Leeuwen, cf. vL00, we obtain, in Theorems B and C, combinatorial descriptions of the bijection in Theorem A.

**Theorem B.** Consider \( \tau \in \mathcal{P}(p, q) \). Let \((\hat{Q}, \hat{P})\) be the tableaux pair corresponding to \( \hat{\tau} \) via the Robinson-Schensted-Knuth correspondence, and set

\[
\lambda = \operatorname{sh}(\hat{Q}), \quad Q = \hat{Q}(q), \quad P = \operatorname{Rect}(\hat{P}, q).
\]

Let \( \Lambda \) be the signed Young diagram given by the following rules:

1. \( \operatorname{sh}(\Lambda) = \operatorname{sh}(Q) + \operatorname{sh}(P) \).
2. The \( i \)-th row of \( \Lambda \) starts with \( \square \) if and only if \( \operatorname{sh}(Q)(i) > \lambda(i) \).

The bijection of Theorem A is made explicit by the relation: \( \tau \mapsto (\Lambda, Q, P) \).

See Eq. (6.2.1) for the definition of \( \hat{\tau} \), Section 6.8 for the definition of \( \hat{Q}(q) \), Section 4.14 for a definition of \( \operatorname{Rect}(\hat{P}, q) \), and Section 4.13 for a discussion of the Robinson-Schensted-Knuth correspondence.
Theorem C. Consider $\Lambda \in \text{ASYD}(q, p)$, $Q \in \text{SYD}(\Lambda^+)$, and $P \in \text{SYD}(\Lambda^-)$, where $\Lambda^+$ and $\Lambda^-$ be as in Section 4.5. Let $\lambda$ be as in Lemma 4.6, let $\hat{Q} = (Q; \lambda)$, $\text{ev}(\hat{P}) = \text{ev}((\text{ev}(P); \lambda))$, $\hat{\tau} \overset{\text{RSK}}{\leftarrow} (\hat{Q}, \hat{P})$. Then $\tau$ is the south-west sub-matrix of $\hat{\tau}$ of size $p \times q$.

The operator $\text{ev}$ is tableau evacuation, also known as the Schützenberger involution, see Section 3.15.

Given a signed Young diagram $\Lambda$, let $\Lambda^\vee$ be the diagram obtained by switching the labelling of the boxes. In Section 7, we show that the involution $(\Lambda, Q, P) \mapsto (\Lambda^\vee, P, Q)$ corresponds to projective duality for matrix Schubert fibres.

The questions tackled here are also discussed in [FN], where a Robinson-Schensted correspondence is developed for partial permutations of size $n \times n$. However, both the bijection and the algorithms presented in [FN, Thm. 7.6] are different from the ones here. Another key difference is that we work with Schubert varieties in a double partial flag variety $F_l(a, V) \times F_l(b, V)$, as contrasted with the Grassmannian sub-varieties used in [FN]. This allows us to factor much of the combinatorics through the combinatorics of the classical Robinson-Schensted-Knuth correspondence.

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2. The Steinberg Variety

We work over $\mathbb{C}$. In this section, we recall some standard results on the Steinberg variety of a $G$-variety. For a detailed discussion, the reader may consult [CG97].

2.1. The Conormal Bundle. Let $X$ be a smooth variety, $Y$ a smooth (not necessarily closed) subvariety of $X$, and $TX$ and $TY$ the corresponding tangent bundles. The conormal bundle of $Y$ in $X$ is a vector bundle, $T^*X_Y \to Y$, whose fibre at a point $p \in Y$ is the annihilator of the tangent subspace $T_pY$ in $T^*_pX$, i.e.,

$$(T^*_XY)_p = \{x \in T^*_pX \mid x(v) = 0, \forall v \in T_pY\}.$$

2.2. Group Actions with Finitely Many Orbits. Let $G$ be a reductive group acting on a (possibly singular) algebraic variety $X$ with finitely many orbits. We denote by $X/G$ the set of $G$-orbits in $X$, and write

$$X = \bigsqcup_{\lambda \in X/G} X(\lambda),$$

for the $G$-orbit decomposition of $X$. The orbit closure inclusion relation induces a partial order on $X/G$, namely $\lambda \preceq \nu \iff \overline{X(\lambda)} \subseteq X(\nu)$.

2.3. The Steinberg Variety. The $G$-action on $X$ induces a $G$-action on the cotangent bundle $T^*X$; The symplectic structure on $T^*X$ admits a $G$-equivariant proper map, $\mu_X : T^*X \to \mathfrak{g}$, called the moment map. The zero fibre of the moment map, $\mu_X^{-1}(0)$, is called the Steinberg variety $Z_X$ of $X$. Its irreducible components are
precisely the closures of the conormal bundles of the $G$-orbits $X(\lambda) \subset X$. We write this concisely as

$$\text{Irr}(Z_X) = \left\{ \mathcal{T}^*_X X(\lambda) \mid \lambda \in X/G \right\}.$$ 

3. The Robinson-Schensted-Knuth Correspondence

Let $V$ be a $N$-dimensional vector space, and let $G = GL(V)$. In this section, we recall some results on the nilpotent cone, flag varieties, and Springer fibres associated to $G$. We also present the Robinson-Schensted-Knuth correspondence from the geometric point of view, and some results relating algorithms on Young tableaux with certain geometric constructions.

3.1. Partitions. A partition of $N$ is a weakly decreasing sequence

$$\lambda = (\lambda(1), \ldots, \lambda(k))$$

of positive integers satisfying $\lambda(1) + \cdots + \lambda(k) = N$. We call $|\lambda| = N$ the size of $\lambda$. We denote by $\text{Par}(N)$ the set of all partitions of size $N$. The Young diagram of $\lambda$ is a collection of boxes, arranged in left-justified rows, with the $i^{th}$-row containing precisely $\lambda(i)$ boxes.

![Figure 3.2. The Young Diagram of $\lambda = (4, 3, 1)$.](image)

3.3. The Nilpotent Cone. Let $\mathcal{N}$ be the nilpotent cone in $\mathfrak{gl}(V)$, i.e,

$$\mathcal{N} = \{ x \in \mathfrak{gl}(V) \mid x^N = 0 \},$$

and let $\mathcal{N}(\lambda) \subset \mathcal{N}$ be the subvariety of nilpotent matrices whose Jordan type is $\lambda$. Following [Mac79, V(2.9)], we have,

$$\text{dim} \mathcal{N}(\lambda) = |\lambda| (|\lambda| + 1) - 2 \sum \lambda(i).$$

The $G$-orbit decomposition of $\mathcal{N}$ is precisely,

$$\mathcal{N} = \bigsqcup_{\lambda \in \text{Par}(N)} \mathcal{N}(\lambda).$$

Further, the closure inclusion order on $\mathcal{N}/G = \text{Par}(N)$ is precisely the so-called dominance order $\preceq$. Recall that for $\lambda, \nu \in \text{Par}(N)$, we have

$$\lambda \preceq \nu \iff \lambda(1) + \cdots + \lambda(i) \leq \nu(1) + \cdots + \nu(i), \quad \forall i.$$

3.4. Sum of Partitions. Given $\nu \in \text{Par}(q)$ and $\lambda \in \text{Par}(p)$, we define

$$\nu + \lambda = (\nu(1) + \lambda(1), \nu(2) + \lambda(2), \cdots) \in \text{Par}(q + p).$$

Observe that the operation $+$ of Section 3.3 respects the dominance order, i.e., for any $\lambda, \nu \in \text{Par}(N)$, and any $\mu \in \text{Par}$, we have $\lambda \preceq \nu$ if and only if $\lambda + \mu \preceq \nu + \mu$. 
3.5. **The Young Lattice.** Given partitions \( \lambda \) and \( \nu \), we say \( \lambda \leq \nu \) if the Young diagram of \( \lambda \) is contained in the Young diagram of \( \nu \). The set of all partitions of all positive integers forms a lattice, called the *Young lattice*, under the partial order \( \leq \). The Young lattice is graded by size, i.e.,

\[
\lambda < \nu \implies |\lambda| < |\nu|.
\]

It is clear that distinct partitions of the same size are mutually incomparable in the Young lattice.

3.6. **Column Strips.** Consider partitions \( \lambda, \nu \) with \( \lambda \leq \nu \). The set-theoretic difference \( \nu / \lambda \) is called a *skew-diagram*, see [Ful97]. A *column strip* is a skew-diagram in which every row contains at most 1 box.

![Figure 3.7: A pair of partitions \( \lambda \) (green boxes only) and \( \nu \) (all boxes). The skew-diagram \( \nu / \lambda \) is a Column Strip.](image)

3.8. **Row-Standard Tableaux.** A *composition* of \( N \) is a sequence \( \underline{a} = (a_1, \cdots, a_n) \) of positive integers satisfying \( a_1 + \cdots + a_n = N \).

A *row-standard tableau* \( T \) of content \( \underline{a} \) is a sequence of partitions,

\[
T = (T(1); \cdots; T(n)),
\]

satisfying for each \( i \), the following conditions:

1. \( |T(i)| = a_1 + \cdots + a_i \).
2. \( T(i) < T(i + 1) \) in the Young lattice.
3. The skew-diagram \( T(i)/T(i-1) \) is a column-strip.

Equivalently, \( T \) is a filling of a Young diagram, which is weakly increasing in each column and strictly increasing in each row, with the integer \( i \) occurring \( a_i \) times, see [Ful97]. We denote by \( SYT(\lambda, \underline{a}) \) the set of row-standard Young tableaux of shape \( \lambda \) and content \( \underline{a} \).

While column-standard tableaux abound in the literature, row-standard tableaux are more natural for our purposes, as will become clear in Sections \( \S.10 \) and \( \S.13 \); see also [Ros12]. All the tableaux we encounter in this paper will be row-standard. Accordingly, we drop the adjective row-standard, and simply call them tableaux.

3.9. **Partial Flag Varieties.** Let \( \underline{a} = (a_1, \cdots, a_n) \) be a composition of \( N \); we denote by \( Fl(\underline{a}, V) \) the variety of partial flags in \( V \) of shape \( \underline{a} \), i.e.,

\[
Fl(\underline{a}, V) = \{ F_\bullet = (F_1 \subset \cdots \subset F_n \subset V) \mid \dim F_i/F_{i-1} = a_i \}.
\]

Note that \( G = GL(V) \) acts transitively on \( Fl(\underline{a}, V) \). Following [Spr09], we have,

\[
T^*Fl(\underline{a}, V) = \{ (F_\bullet, x) \in Fl(\underline{a}, V) \times gl(V) \mid xF_i \subset F_{i-1}, \forall 1 \leq i \leq n \}.
\]
The moment map for the $G$-action on $\mathcal{F}l(\mathfrak{a}, V)$ is given by,

\[(3.9.1) \quad \mu_{\mathcal{F}l(\mathfrak{a}, V)} : T^* \mathcal{F}l(\mathfrak{a}, V) \to \mathfrak{gl}(V), \quad (F_\bullet, x) \mapsto x.\]

In particular, the image of the moment map is contained in $\mathcal{N}$.

3.10. **The Springer Fibre.** Given $x \in \mathcal{N}$, let $S_{\underline{a}}(x) = \mu_{\mathcal{F}l(\mathfrak{a}, V)}^{-1}(x)$, i.e.,

\[S_{\underline{a}}(x) = \{ F_\bullet \in \mathcal{F}l(\mathfrak{a}, V) \mid x F_i \subset F_{i-1} \forall 1 \leq i \leq n \}.\]

We call $S_{\underline{a}}(x)$ the *Springer fibre* over $x$.

Now consider $(x, F_\bullet) \in T^* \mathcal{F}l(\mathfrak{a}, V)$. For each $i$, we have $x F_i \subset F_{i-1}$, and hence a nilpotent map $x | F_i \in \mathfrak{gl}(F_i)$ obtained by restricting $x$ to $F_i$. Let $J(x| F_i)$ denote the Jordan type of $x| F_i$, and let $\text{Tab}(x, F_\bullet)$ denote the tableau.

\[\text{Tab}(x, F_\bullet) = (J(x| F_1); J(x| F_2); \cdots; J(x| F_n)).\]

It is clear that the content of $\text{Tab}(x, F_\bullet)$ is $\underline{a}$. We have a decomposition,

\[S_{\underline{a}}(x) = \bigsqcup_{T \in \text{SYT}(J(x), \underline{a})} S^T_{\underline{a}}(x),\]

where the subvariety $S^T_{\underline{a}}(x) \subset S_{\underline{a}}(x)$ is given by,

\[S^T_{\underline{a}}(x) = \{ F_\bullet \in S_{\underline{a}}(x) \mid \text{Tab}(x, F_\bullet) = T \}.\]

Following [Spa82], we have,

\[\text{Irr} \left( S_{\underline{a}}(x) \right) = \left\{ S^T_{\underline{a}}(x) \mid T \in \text{SYT}(J(x), \underline{a}) \right\}.\]

Recall that a variety is said to be pure-dimensional if all of its components have the same dimension. Following [Spa82], the variety $S_{\underline{a}}(x)$ is pure-dimensional, with the dimension given by,

\[(3.10.1) \quad 2 \dim S^T_{\underline{a}}(x) = 2 \dim \mathcal{F}l(\mathfrak{a}, V) - \dim \mathcal{N}(J(x)).\]

3.11. **Schubert Varieties.** Let $\underline{a} = (a_1, \cdots, a_n)$ and $\underline{b} = (b_1, \cdots, b_m)$ be a pair of compositions of $N$. The group $G$ acts diagonally on $Y = \mathcal{F}l(\mathfrak{a}, V) \times \mathcal{F}l(\mathfrak{b}, V)$ with finitely many orbits, the so-called *Schubert cells*.

The Schubert cells $Y(\sigma) \subset Y$ are indexed by $m \times n$ matrices $\sigma$ satisfying the following conditions:

- Each entry of $\sigma$ is a non-negative integer.
- The sum of the entries in the $i^{th}$ row is $b_i$.
- The sum of the entries in the $i^{th}$ column is $a_i$.

The Schubert cell $Y(\sigma)$ corresponding to $\sigma$ is then precisely,

\[Y(\sigma) = \left\{ (E_\bullet, F_\bullet) \in Y \mid \text{rk} \ (E_i \cap F_j) = \sum_{k=1}^{i} \sum_{l=1}^{j} \sigma_{kl} \right\}.\]

The closure of a Schubert cell is called a *Schubert variety*. We have,

\[\overline{Y(\sigma)} = \left\{ (E_\bullet, F_\bullet) \in Y \mid \text{rk} \ (E_i \cap F_j) \geq \sum_{k=1}^{i} \sum_{l=1}^{j} \sigma_{kl} \right\}.\]
3.12. The Steinberg Variety of the Double Flag Variety. The moment map, 
\[ \mu_Y : T^* F_l(\mathfrak{g}, V) \times T^* F_l(\mathfrak{h}, V) \to \mathfrak{gl}(V), \]
for the \( G \) action on \( Y = F_l(\mathfrak{g}, V) \times F_l(\mathfrak{h}, V) \) is given by 
\[ ((F_\bullet, x), (E_\bullet, y)) \mapsto x + y. \]
Consequently, the corresponding Steinberg variety \( Z_Y \) is precisely, 
\[ Z_Y = \{(F_\bullet, x, E_\bullet, y) \in T^* F_l(\mathfrak{g}, V) \times T^* F_l(\mathfrak{h}, V) \mid y = -x\}, \]
\[ = \{(F_\bullet, E_\bullet, x) \in Y \times N \mid xF_i \subset F_{i-1}, xE_i \subset E_{i-1}, \forall i\}. \]
For \( \sigma \in Y/G \), let \( Z_Y(\sigma) = T^*_Y Y(\sigma) \). Following Section 2.3, we have, 
\[ \text{Irr}(Z_Y) = \left\{ Z_Y(\sigma) \mid \sigma \in Y/G \right\}. \]

3.13. Geometry of the Robinson-Schensted-Knuth correspondence. Following [Knu70, Ste88, Ros12], we have a bijection, 
\[ Y/G \xrightarrow{{\sim}} \bigsqcup_{\lambda \vdash N} SYT(\lambda, \mathfrak{g}) \times SYT(\lambda, \mathfrak{h}), \]
\[ \sigma \mapsto (\text{Tab}(x, E_\bullet), \text{Tab}(x, F_\bullet)), \]
where \( x \in N \) is chosen to be generic for the property \( (E_\bullet, F_\bullet, x) \in T^*_Y Y(\sigma) \). This bijection is a simple variation of the correspondence discussed in [Ful97], see [Ros12] for details.

Suppose \( \sigma \in Y/G \) corresponds to the tableaux pair \((P, Q)\) via the above correspondence. We will denote
\[ (3.13.1) \quad Z^\sigma_Y = \{(E_\bullet, F_\bullet, x) \mid P = \text{Tab}(x, E_\bullet), Q = \text{Tab}(x, F_\bullet)\}. \]
Following [Ste88], \( Z^\sigma_Y \) is dense open subset of \( Z_Y(\sigma) \).

3.14. Rectification. Given a tableau \( T \), let \( F_\bullet \) be a generic point in \( S^T_\bullet(x) \), see Section 3.10 Fix \( 0 \leq i \leq n \), and consider the partial flag, 
\[ F_\bullet / F_i = 0 \subset F_{i+1} / F_i \subset \cdots \subset F_n / F_i. \]
Since we have \( xF_i \subset F_{i-1} \subset F_i \), we obtain an induced nilpotent map, 
\[ x : F_n / F_i \to F_n / F_i, \quad \text{satisfying} \quad x(F_j / F_i) \subset F_{j-1} / F_i, \forall i \leq j \leq n. \]
This yields the tableau, 
\[ \text{Rect}(T, i) := \text{Tab}(x|F_n / F_i, F_\bullet / F_i). \]
Following [KL00], \( \text{Rect}(T, i) \) is precisely the rectification of the skew-tableau \( T/T(i) \) as described in [Ful97].

3.15. Evacuation. Let \( F_\bullet \) and \( x \) be as in Section 3.14. The sequence, 
\[ (J(x|F_n / F_{n-1}); J(x|F_n / F_{n-2}); \cdots; J(x|F_n / F_0)), \]
is a tableau of content \( (\alpha_n, \alpha_{n-1}, \ldots, \alpha_1) \). We denote this tableau by \( \text{ev}(T) \). Following [KL00], \( \text{ev}(T) \) is precisely the evacuation (also known as the Schützenberger involution) of \( T \), see [BB05, A3.8] for more details, including an algorithm to compute \( \text{ev}(T) \).
4. Signed Young Diagrams

In this section, we study some combinatorial, geometric, and representation theoretic aspects of signed Young diagrams. We also define and characterize the admissible signed Young diagrams.

4.1. The Ring $R$. Consider the non-commutative graded ring,

$$R = \mathbb{C}(\alpha, \beta)/\langle \alpha^2, \beta^2 \rangle,$$

$$\text{deg}(\alpha) = \text{deg}(\beta) = 1.$$

Let $M$ be a finite dimensional graded $R$-module, with degrees the principal $\mathbb{Z}/2$-space \{$\mp$\}, and satisfying the following extra condition: $\alpha$ (resp. $\beta$) acts trivially on $M$ ($\mp$) (resp. $M$ ($\mp$)). For $k \geq 1$, we have $k$-dimensional indecomposable graded modules,

$$U^+_k = \frac{R^+}{\langle \cdots xyx, y \rangle}, \quad U^-_k = \frac{R^-}{\langle \cdots yxy, x \rangle}.$$

We have a decomposition of graded modules,

$$(4.1.1) \quad M = \bigoplus M_i,$$

where each $M_i$ is isomorphic to some $U^\pm_k$.\footnote{The modules $M$ discussed here are precisely the nilpotent representations of the $\hat{A}_2$ quiver. For a discussion of the general theory, see \textsc{Kir16}.}

Lemma 4.2. Let $M$ be an indecomposable $R$-module. The element $z = \beta + \alpha \beta \in \mathfrak{gl}(M)$ is nilpotent, and its Jordan type is given by

$$J(z|U^+_k) = (n+1,1,\cdots,1),$$

$$J(z|U^-_k) = (n+1,1,\cdots,1),$$

$$J(z|U^+_n) = (n,1,\cdots,1).$$

Proof. A simple calculation yields $z^k = \beta(\alpha \beta)^{k+1} + (\alpha \beta)^k$. We decompose $M$ as a sum of an indecomposable $\mathbb{C}[z]$ sub-module and a subspace on which $z$ acts as 0.

$$U^+_k = \langle 1, \alpha, \alpha \beta, \cdots, \alpha(\alpha \beta)^{n-1} \rangle$$

$$= \langle z^i \alpha \mid 0 \leq i \leq n-1 \rangle \oplus \langle (\alpha \beta)^i \mid 0 \leq i \leq n-1 \rangle.$$

$$U^-_k = \langle 1, \beta, \beta \alpha, \cdots, \beta(\beta \alpha)^{n-1} \rangle$$

$$= \langle z^i \beta \mid 0 \leq i \leq n-1 \rangle \oplus \langle (\alpha \beta)^i \mid 1 \leq i \leq n-1 \rangle.$$

$$U^+_n = \langle 1, \alpha, \alpha \beta, \cdots, (\alpha \beta)^n \rangle$$

$$= \langle z^i \alpha \mid 0 \leq i \leq n \rangle \oplus \langle (\alpha \beta)^i \mid 0 \leq i \leq n-1 \rangle.$$

$$U^-_n = \langle 1, \beta, \beta \alpha, \cdots, (\beta \alpha)^n \rangle$$

$$= \langle z^i \beta \mid 0 \leq i \leq n \rangle \oplus \langle (\alpha \beta)^i \mid 1 \leq i \leq n-1 \rangle.$$

Counting the dimensions of the summands, we obtain the desired result. \qed
4.3. Signed Young Diagrams. A $\pm$-filling of a Young diagram is an assignment of $\square$ or $\blacksquare$ to each box of the Young diagram such that adjacent boxes in the same row have opposite signs. The signature of a $\pm$-filling is the integer pair $(q, p)$, where $q$ is the number of $\square$ boxes, and $p$ is the number of $\blacksquare$ boxes.

We define an equivalence relation $\sim$ on $\pm$-fillings by setting two $\pm$-fillings to be equivalent if and only if one can be obtained from the other by some permutation of equi-sized rows. The equivalence class of a $\pm$-filling under $\sim$ is called a signed Young diagram.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure4.4}
\caption{Two equivalent $\pm$-fillings with signature $(8, 7)$.}
\end{figure}

We denote by $\text{SYD}(q, p)$ the set of signed Young diagrams of signature $(q, p)$. Given a signed Young diagram $\Lambda$ of signature $(q, p)$, we will denote by $\text{sh}(\Lambda)$ the shape of $\Lambda$, i.e., the Young diagram in $\Par(q+p)$ underlying $\Lambda$.

Consider an $R$-module $M$ as in Section 4.1. We associate a signed Young diagram $\Lambda$ to $M$ as follows:
- If $M \cong U^+_k$ (resp. $M \cong U^-_k$), we set $\Lambda$ to be a one-row signed Young diagram with $k$ boxes, and set the rightmost box of the row to be $\square$ (resp. $\blacksquare$).
- For general $M$, we construct $\Lambda$ as a union of the one-row diagrams corresponding to its indecomposable components, see Eq. (4.1.1).

It is clear that signed Young diagrams classify the isomorphism classes of the graded $R$-modules described in Section 4.1.

Let $M(\square)$ (resp. $M(\blacksquare)$) be the $\square$ (resp. $\blacksquare$) graded component of $M$, and set $q = \dim M(\square)$ (resp. $p = \dim M(\blacksquare)$). The signature of $\Lambda$ is precisely $(q, p)$.

4.5. The Partitions $\Lambda^{\pm}$. The subspace $M(\square)$ (resp. $M(\blacksquare)$) is stable under the action of $\beta \alpha$ (resp. $\alpha \beta$). We define the partitions,
\[ \Lambda^+ = J(\beta \alpha | M(\square)) \in \Par(q), \quad \Lambda^- = J(\alpha \beta | M(\blacksquare)) \in \Par(p). \]

Given the decomposition, $M = \bigoplus M_i$, see Eq. (4.1.1), we obtain a decomposition of $M(\square)$ and $M(\blacksquare)$ into indecomposables,
\[ M(\square) = \bigoplus (M_i \cap M(\square)), \quad M(\blacksquare) = \bigoplus (M_i \cap M(\blacksquare)). \]

It follows that $\Lambda^+$ (resp. $\Lambda^-$) can be obtained from a $\pm$-filling in the equivalence class $\Lambda$ by removing all $\square$ (resp. $\blacksquare$) boxes, (and possibly rearranging the rows if necessary). This yields the Young diagram of $\Lambda^+$ (resp. $\Lambda^-$).

**Lemma 4.6.** Let $\Lambda \in \text{SYD}(q, p)$ be the signed Young diagram associated to an $R$-module $M$ as in Section 4.3. Consider $z \in \text{gl}(M)$, given by $z = \beta + \alpha \beta$. We can obtain the Jordan type of $z$ as follows.

Remove all $\square$ which do not appear in the first column of $\Lambda$, then left-align (and possibly rearrange) the rows to obtain the partition $\lambda'$. Let $\lambda \in \Par(q+p)$ be the partition obtained by attaching to $\lambda'$ a new row of size one for every $\square$ removed.
4.7. The Variety $O$. Fix positive integers $p$ and $q$. Let $V_p$ (resp. $V_q$) be a $p$ (resp. $q$)-dimensional vector space, and let $H = \text{Hom}(V_q, V_p)$. We study the action of the group $K = GL(V_q) \times GL(V_p)$ on $H$ given by the formula $(g,h) \cdot x = hxg^{-1}$.

Let $H^\vee = \text{Hom}(V_p, V_q)$. The non-degenerate bilinear pairing,

$$\langle \ , \ \rangle : H \times H^\vee \to \mathbb{C},$$

yields the identification $T^*H = H \times H^\vee$; with this identification, the induced $K$-action on $T^*H$ is given by

$$(g,h) \cdot (x,y) = (hxg^{-1}, gyh^{-1}).$$

We are interested in studying the $K$-action on the nilpotent cone,

$$(4.7.1) \quad O = \{(x,y) \in H \times H^\vee \mid xy \text{ is nilpotent} \}.$$

To each $(x,y) \in O$, we associate a $\Lambda \in \text{SYD}(q,p)$. Consider the graded $R$-module structure on $V = V_q \oplus V_p$, given by $\alpha \mapsto x$, $\beta \mapsto y$, and

$$\deg(V_q) = \emptyset, \quad \deg(V_p) = \boxtimes$$

We set $\Lambda$ to be the signed Young diagram associated to this module, see Section 4.3

Let $O(\Lambda)$ be the set of all $(x,y) \in O$ for which the associated signed Young diagram is $\Lambda$. Following [Kem82, Joh10], the $(O(\Lambda))$ are precisely the $K$-orbits in $O$, i.e., $O/K = \text{SYD}(q,p)$. Further, we have $(x,y) \in O(\Lambda)$ if and only if

$$J(yx|V_q) = \Lambda^+, \quad J(xy|V_p) = \Lambda^-,$$

$$\dim \ker(xyx)^i|V_q) = \# \{ \emptyset \in \Lambda \text{ contained in the first } 2i+1 \text{ columns} \},$$

$$\dim \ker(yxy)^i|V_p) = \# \{ \emptyset \in \Lambda \text{ contained in the first } 2i+1 \text{ columns} \}.$$

We will denote the closure inclusion order on $O/K$ by $\preceq$, i.e.,

$$(4.7.2) \quad \Lambda \preceq \Upsilon \iff \overline{O(\Lambda)} \subset \overline{O(\Upsilon)}.$$

Following [Kem82], the closure $\overline{O(\Lambda)} \subset O$ is given by the following conditions:

$$J(yx|V_q) \preceq \Lambda^+, \quad J(xy|V_p) \preceq \Lambda^-,$$

$$\dim \ker(xyx)^i|V_q) \geq \# \{ \emptyset \in \Lambda \text{ contained in the first } 2i+1 \text{ columns} \},$$

$$\dim \ker(yxy)^i|V_p) \geq \# \{ \emptyset \in \Lambda \text{ contained in the first } 2i+1 \text{ columns} \}.$$

In particular, the map $\Lambda \mapsto (\Lambda^+, \Lambda^-)$ is a poset homomorphism.

**Lemma 4.8 ([Vog91]).** We have $\dim O(\Lambda) = \nicefrac{1}{2} \dim \mathcal{N}(\text{sh}(\Lambda))$.

**Lemma 4.9.** The map $\text{sh} : \text{SYD}(q,p) \to \text{Par}(q+p)$ is a poset homomorphism, i.e, $\Lambda \preceq \Upsilon \iff \text{sh}(\Lambda) \preceq \text{sh}(\Upsilon)$.

**Proof.** Consider $\Lambda \preceq \Upsilon$, so that

$$O(\Lambda) \subset \overline{O(\Upsilon)} \subset \overline{\mathcal{N}(\text{sh}(\Upsilon))}.$$ 

Observe that $\overline{\mathcal{N}(\text{sh}(\Upsilon))}$ is $G$-stable; hence we have,

$$\mathcal{N}(\text{sh}(\Lambda)) = G \cdot O(\Lambda) \subset \overline{\mathcal{N}(\text{sh}(\Upsilon))},$$

which is equivalent to $\text{sh}(\Lambda) \preceq \text{sh}(\Upsilon)$.  \hfill $\square$
4.10. The Moment Map \( \mu_H \). The \( K \)-action on \( H \) admits a moment map,
\[
\mu_H : H \times H^\vee \to \mathfrak{gl}(V_q) \times \mathfrak{gl}(V_p),
\mu_H(x, y) = (yx, xy).
\]
The \( K \)-orbit decomposition of the nilpotent cone \( M \) of \( K \) is given by
\[
M = \bigsqcup_{\nu \vdash q, \lambda \vdash p} M(\nu, \lambda), \quad M(\nu, \lambda) = N(\nu) \times N(\lambda).
\]
It is clear from Section 4.5 that for any \( \Lambda \in \text{SYD}(q, p) \), we have,
\[
\mu_H(O(\Lambda)) = M(\Lambda^+, \Lambda^-).
\]
In particular, we have \( \mu_H(O) \subset M \).

**Lemma 4.11.** For any \( \Lambda \in \text{SYD}(q, p) \), we have,
\[
\text{sh}(\Lambda) \preceq \Lambda^+ + \Lambda^-.
\]
Equality holds if and only if for each odd integer \( i \), all the rows of size \( i \) in \( \Lambda \) have the same signature.

**Proof.** Fix some \( \pm \)-filling corresponding to \( \Lambda \), and let \( a_i^+ \) (resp. \( a_i^- \)) be the number of \( \Box \) boxes (resp. \( \square \) boxes) in the \( i \)th row of this filling. We have,
\[
\text{sh}(\Lambda) = (a_1^+, a_1^-, a_2^+, a_2^-, \cdots)
\]
Now, observe that the sequence \((a_1^+, a_2^+, \cdots)\) is some permutation of the weakly decreasing sequence \((\Lambda_1^+, \Lambda_2^+, \cdots)\), and hence, we have,
\[
\Lambda^+(1) + \cdots + \Lambda^+(i) \geq a^+(1) + \cdots + a^+(i), \quad \forall i.
\]
This yields the claimed inequality, \( \text{sh}(\Lambda) \preceq \Lambda^+ + \Lambda^- \), with equality holding if and only if equality holds for all \( i \) in Eq. (4.11.1). This happens if and only if the sequences \((a_1^+, \ldots)\) and \((a_1^-, \ldots)\) are weakly decreasing, which happens if and only if any two consecutive rows of the same odd length in the \( \pm \)-filling have the same signature. \( \square \)

4.12. Admissible Signed Young Diagrams. We call \( \Lambda \in \text{SYD}(q, p) \) admissible if \( \text{sh}(\Lambda) = \Lambda^+ + \Lambda^- \), and we denote by \( \text{ASYD}(q, p) \) the set of all admissible signed Young diagrams of signature \((q, p)\).

**Theorem 4.13.** The following conditions are equivalent for \( \Lambda \in \text{SYD}(q, p) \):

1. \( \Lambda \) is admissible.
2. \( 2 \dim O(\Lambda) = 2|\Lambda^+| |\Lambda^-| + \dim M(\Lambda^+, \Lambda^-) \).
3. \( \Lambda \) is maximal in \( \{ \Upsilon \in \text{SYD}(q, p) \mid (\Upsilon^+, \Upsilon^-) = (\Lambda^+, \Lambda^-) \} \).
4. \( O(\Lambda) \) is an irreducible component of \( \mu_H^{-1}(M(\Lambda^+, \Lambda^-)) \).

**Proof.** (1) \( \implies \) (2): Suppose (1) holds. We see from Lemma 4.11 that
\[
2 \dim O(\Lambda) = \dim \mathcal{N}(\text{sh}(\Lambda)) = \dim \mathcal{N}(\Lambda^+ + \Lambda^-).$

Using Eq. (3.3.1), we compute,
\[
2 \dim O(\Lambda) = (q + p)(q + p + 1) - 2 \sum i(\Lambda^+(i) + \Lambda^-(i)) \\
= p(p + 1) + p(p + 1) + 2qp - 2 \sum i\Lambda^+(i) - 2 \sum i\Lambda^-(i) \\
= 2pq + \dim N(\Lambda^+) + \dim N(\Lambda^-) \\
= 2|\Lambda^+|\Lambda^-| + \dim M(\Lambda^+, \Lambda^-).
\]

(2) \implies (3): Suppose (2) holds, and (3) does not hold, i.e., there exists \( \Upsilon \succ \Lambda \) satisfying \((\Upsilon^+, \Upsilon^-) = (\Lambda^+, \Lambda^-)\). Using Lemmas \ref{lem:A} and \ref{lem:B}, we deduce a contradiction:
\[
2 \dim O(\Lambda) < 2 \dim O(\Upsilon) = \dim N(\sh(\Upsilon)) \leq 2 \dim N(\Upsilon^+ + \Upsilon^-) \\
= 2pq + \dim M(\Upsilon^+, \Upsilon^-) = 2pq + \dim M(\Lambda^+, \Lambda^-).
\]

(3) \implies (1): Suppose (1) does not hold. It follows from Lemma \ref{lem:B} that some \(-\)-filling corresponding to \( \Lambda \) contains two consecutive rows of the same odd length with opposite signs. We construct \( \Upsilon \), by moving a box in \( \Lambda \), (and rearranging some rows if necessary), see below:

\[
\begin{array}{ccccccc}
\hline
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
- & & - & & - & & - \\
\hline
\end{array} \quad \rightarrow \quad
\begin{array}{ccccccc}
\hline
& & & & & & \\
& & & & & & \\
& & & & & & \\
- & & - & & - & & - \\
\hline
\end{array}
\]

It is clear that \((\Upsilon^+, \Upsilon^-) = (\Lambda^+, \Lambda^-)\). Comparing with Eq. (4.7.2), we deduce \( \Lambda \prec \Gamma \). We deduce that if (1) does not hold, then neither does (3).

(3) \iff (4): This is a direct consequence of the fact that \( \preceq \) is the closure inclusion order. \(\square\)

**Corollary 4.14.** Consider \( \Lambda \in \text{ASD}(q, p) \). For any \( \pm \)-filling in the equivalence class \( \Lambda \), the following is true: The number of \( \square \) (resp. \( \blacksquare \)) in the \( i \)th-row is \( \Lambda^+(i) \) (resp. \( \Lambda^-(i) \)).

5. Matrix Schubert Varieties

Let \( V_q, V_p, H = \text{Hom}(V_q, V_p), K = GL(V_q) \times GL(V_p) \), and \( \mathcal{O} \) be as in Section 4.7. Set \( X = X_q \times X_p \), where
\[
X_q = \{ F_\bullet = (F_1 \subset \cdots \subset F_q \subset V_q) \mid \dim F_i = i \}, \\
X_p = \{ F_\bullet = (F_1 \subset \cdots \subset F_p \subset V_p) \mid \dim F_i = i \},
\]
are the variety of full flags in \( V_q, V_p \) respectively.

5.1. Partial Permutations. A partial permutation \( \tau \) is a matrix with entries in the set \{0, 1\} in which every row and every column contains at most one non-zero entry. We denote by \( \mathcal{PP}(p, q) \) the set of partial permutations of size \( p \times q \).
5.2. Matrix Schubert Varieties. Consider the variety,

\[ C = X \times H = X_q \times X_p \times \text{Hom}(V_q, V_p), \]

along with the \( K \)-action given by,

\[ (g, h) \cdot (E_*, F_*, x) = (gE_*, hF_*, hxg^{-1}). \]

For \( \tau \) a partial permutation of size \( p \times q \), let

\[ C(\tau) = \left\{ (E_*, F_*, x) \in C \mid \dim(xE_i + F_j) = j + \sum_{l \leq i} \sum_{k > j} \tau_{kl} \right\}. \]

Following [Ful92], the \( C(\tau) \) are precisely the \( K \)-orbits in \( C \), so that we have,

\[ C/K = \mathcal{P}(p, q). \]

The closure \( \overline{C(\tau)} \) is called a matrix Schubert variety; we have,

\[ \overline{C(\tau)} = \left\{ (E_*, F_*, x) \in C \mid \dim(xE_i + F_j) \leq j + \sum_{l \leq i} \sum_{k > j} \tau_{kl} \right\}. \]

5.3. The Steinberg Variety for Matrix Schubert Varieties. Following Sections 3.9 and 4.10 and Eq. (3.9.1), the \( K \)-action on \( C \) admits a moment map,

\[ \mu_C : T^*X_q \times T^*X_p \times T^*H \to \mathfrak{gl}(V_q) \times \mathfrak{gl}(V_p), \]
\[ \mu_C((E_*, \alpha), (F_*, \beta), (x, y)) = (\alpha + yx, \beta + xy). \]

We deduce that the Steinberg variety, \( Z_C = \mu_C^{-1}(0) \), has the following description:

\[ Z_C = \{(E_*, F_*, x, y) \in X \times O \mid yx E_i \subset E_{i-1}, \ xy F_i \subset F_{i-1}, \ \forall i\} \]
\[ = \{(E_*, F_*, x, y) \in X \times O \mid (E_*, yx) \in T^*X_q, \ (F_*, xy) \in T^*X_p \}. \]

For \( \tau \in \mathcal{P}(p, q) \), let \( Z_C(\tau) = T^*_C(\tau) \), i.e., the conormal bundle of \( C(\tau) \). Following Section 2.3 we have,

\[ \text{Irr}(Z_C) = \left\{ \overline{Z_C(\tau)} \mid \tau \in C/K \right\}. \]

5.4. The Map \( \text{pr} \). The \( K \)-equivariant map,

\[ \text{pr} : Z_C \to O, \quad (E_*, F_*, x, y) \mapsto (x, y), \]

is proper; in particular, it sends the \( K \)-stable irreducible component \( \overline{Z_C(\tau)} \) to a closed irreducible \( K \)-stable subvariety of \( O \). Since \( O/K \) is finite, this subvariety is necessarily a \( K \)-orbit closure. We obtain a map, \( \text{pr} : C/K \to O/K \), characterized by

\[ \text{pr}(\overline{Z_C(\tau)}) = \overline{\text{O}(\text{pr}(\tau))}. \]

Recall the identifications \( C/K = \mathcal{P}(p, q) \) and \( O/K = SYD(q, p) \). With an abuse of notation, we also denote the induced map, \( \text{pr} : \mathcal{P}(p, q) \to SYD(q, p) \).
5.5. The Fibre Bundle $Z_C(\Lambda) \to \mathcal{O}(\Lambda)$. Recall from Sections 3.12 and 4.10 the moment maps,
\[ \mu_X : T^*X \to M \quad \text{and} \quad \mu_H : T^*H \to M. \]
Given $\Lambda \in SY D(q, p)$, let $Z_C(\Lambda) = \text{pr}^{-1}(\mathcal{O}(\Lambda))$, and consider the induced map,
\[ \text{pr} : Z_C(\Lambda) \to \mathcal{O}(\Lambda). \]
We see from Section 5.3 that the fibre of $\text{pr}$ over the point $(x, y) \in \mathcal{O}(\Lambda)$ is precisely
\[ \mu_X^{-1}(yx) \times \mu_X^{-1}(xy) = \mu_X^{-1}(yx, xy) = \mu_H(\mu_H(x, y)). \]
It follows that
\[ \text{Irr}(Z_C(\Lambda)) \cong SY T(J(yx)) \times SY T(J(xy)) = SY T(\Lambda^+) \times SY T(\Lambda^-). \]
Further, the pure-dimensionality of Springer fibres, see Section 3.10, implies that $Z_C(\Lambda)$ is a pure-dimensional fibre bundle over $\mathcal{O}(\Lambda)$.

**Theorem 5.6.** The image of the map $\text{pr} : \mathcal{P}P(p, q) \to SY D(q, p)$ is precisely the set of admissible signed Young diagrams.

**Proof.** Observe that $\Lambda \in \text{pr}(\mathcal{P}P(p, q))$ if and only if $Z_C(\Lambda)$ contains at least one irreducible component of $Z_C$. Since $Z_C$ is pure-dimensional, this is equivalent to
\[ \dim Z_C(\Lambda) = \dim C = \dim X + \dim H. \]
Following Sections 4.10 and 5.5, and Eq. (3.10.1), we compute,
\[ \dim Z_C(\Lambda) = \dim \mathcal{O}(\Lambda) + \dim \mu_X^{-1}(yx) + \dim \mu_X^{-1}(xy) \]
\[ = \dim \mathcal{O}(\Lambda) + (\dim X_q - 1/2 \dim \mathcal{N}(\Lambda^+)) + (\dim X_p - 1/2 \dim \mathcal{N}(\Lambda^-)) \]
\[ = \dim \mathcal{O}(\Lambda) + \dim X - 1/2 \dim \mathcal{N}(\Lambda^+) - 1/2 \dim \mathcal{N}(\Lambda^-). \]
It follows that $\Lambda \in \text{pr}(\mathcal{P}P(p, q))$ if and only if
\[ \dim \mathcal{O}(\Lambda) - 1/2 \dim \mathcal{N}(\Lambda^+) - 1/2 \dim \mathcal{N}(\Lambda^-) = \dim H, \]
\[ \iff 2 \dim \mathcal{O}(\Lambda) = \dim \mathcal{N}(\Lambda^+) + \dim \mathcal{N}(\Lambda^-) + 2pq, \]
which is one of the characterizations of admissibility, see Theorem 4.13. \qed

We now have all the pieces needed to prove the main result of this section.

**Theorem A.** Consider $\tau \in \mathcal{P}P(p, q)$, and let $(E_+, F_+, x, y)$ be a generic point in $C(\tau)$. Let $\Lambda = \text{pr}(\tau)$, $Q = \text{Tab}(yx, E_+)$, and $P = \text{Tab}(xy, F_+)$. We have a bijection,
\[ \mathcal{P}P(p, q) = \bigsqcup_{\Lambda \in \text{ASYD}(q, p)} SY T(\Lambda^+) \times SY T(\Lambda^-), \]
given by $\tau \mapsto (\Lambda, Q, P)$.

**Proof.** The bijection $\text{Irr}(Z_C) = \mathcal{P}P(p, q)$ is a consequence of Sections 2.3 and 5.2. Now consider $\tau \in \mathcal{P}P(p, q)$, and let $\Lambda = \text{pr}(\tau)$. It follows from Theorem 5.6 that $\Lambda \in \text{ASYD}(q, p)$. Combined with Section 5.5, this yields
\[ \text{Irr}(Z_C) = \bigsqcup_{\Lambda \in \text{ASYD}(q, p)} SY T(\Lambda^+) \times SY T(\Lambda^-). \]
Finally, we see from Section 3.10 that the indexing of \( \operatorname{Irr}(Z_C) \) is given precisely by the conditions \( \text{Tab}(xy, E_\bullet) = Q \) and \( \text{Tab}(xy, F_\bullet) = P \).

\[ \square \]

6. Combinatorial Description of the Bijection

Let \( V_q \) and \( V_p \) be as in Section 4 and set \( V = V_q \oplus V_p \). We identify \( V_q \) and \( V_p \) with the subspaces \( V_q \oplus 0 \) and \( 0 \oplus V_p \) of \( V \) respectively. We fix \( a = (1, \cdots, 1, p) \) and \( b = (q, 1, \cdots, 1) \), compositions of \( N = p + q \), and denote

\[ Y = \mathcal{F}(a, V) \times \mathcal{F}(b, V). \]

We identify \( K = \text{GL}(V_q) \times \text{GL}(V_p) \) as a subgroup of \( G = \text{GL}(V) \) via the natural diagonal embedding. As in Section 5, we set

\[ X_q = \mathcal{F}(V_q), \quad X_p = \mathcal{F}(V_p), \quad X = X_q \times X_p, \]

\[ H = \text{Hom}(V_q, V_p), \quad H^\vee = \text{Hom}(V_p, V_q), \quad C = X \times H. \]

In this section, we prove Theorems B and C. Our main tool is Proposition 6.3, which relates the Steinberg variety \( Z_Y \) of \( Y \) with the Steinberg variety \( Z_C \) of \( C \).

6.1. The Embedding \( C \hookrightarrow Y \). Given \( x \in \text{Hom}(V_q, V_p) \), let \( \tilde{x} \in \mathfrak{gl}(V) \) be the invertible linear map given by

\[ \tilde{x}(v, w) = (v, xv + w). \]

In matrix form, we have,

\[ \tilde{x} = \begin{pmatrix} I_q & 0 \\ x & I_p \end{pmatrix}, \quad \tilde{x}^{-1} = \begin{pmatrix} I_q & 0 \\ -x & I_p \end{pmatrix}, \]

where \( I_q \) and \( I_p \) are identity matrices. Next, for \( (E_\bullet, F_\bullet) \in X_q \times X_p \), let

\[ \tilde{E}_\bullet = (\tilde{E}_1, \cdots, \tilde{E}_{q+1}), \quad \tilde{F}_\bullet = (\tilde{F}_0, \cdots, \tilde{F}_p), \]

be the partial flags given by

\[ \tilde{E}_i = \begin{cases} E_i & \text{for } 1 \leq i \leq q, \\ V & \text{for } i = q + 1, \end{cases} \quad \tilde{F}_i = \begin{cases} V_q & \text{for } i = 0, \\ V_q \oplus F_i & \text{for } 1 \leq i \leq p. \end{cases} \]

Observe that the embedding,

\[ (6.1.1) \quad \phi : C \hookrightarrow Y, \quad (E_\bullet, F_\bullet, x) \mapsto (\tilde{x}\tilde{E}_\bullet, \tilde{F}_\bullet), \]

is \( K \)-equivariant. This is most easily seen via matrix calculation,

\[ \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} \begin{pmatrix} I_q & 0 \\ x & I_p \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}^{-1} = \begin{pmatrix} I_q & 0 \\ hxy^{-1} & I_p \end{pmatrix}. \]
6.2. **Induced Map on Orbits.** Let \( C/K \hookrightarrow Y/K \) be the map on \( K \)-orbits induced from \( \phi \), and let \( Y/K \to Y/G \) be the natural projection. We have a composite map,

\[
\tau : C/K \hookrightarrow Y/K \to Y/G,
\]

Here \( \tau \in \mathcal{P}(p,q) \), and \( \tilde{\tau} \) is a \((p + 1) \times (q + 1)\) matrix satisfying the conditions described in Section 3.11.

We describe this map in some detail. Fix a basis \( e_1, \ldots, e_q \) of \( V_q \), and a basis \( e_{q+1}, \ldots, e_{q+p} \) of \( V_p \). Given \( \tau \in \mathcal{P}(p,q) \), we interpret \( \tau \) as an element of \( \text{Hom}(V_q, V_p) \) with respect to the basis \( e_1, \ldots, e_q \) and \( e_{q+1}, \ldots, e_{q+p} \). Then, we have \((E_\bullet, F_\bullet, \tau) \in C_\tau \), where \( E_\bullet \) and \( F_\bullet \) are the flags given by

\[
E_i = \langle e_1, \ldots, e_i \rangle, \quad F_j = \langle e_{q+1}, \ldots, e_{q+j} \rangle.
\]

We can now directly compute the matrix \( \tilde{\tau} \). It is obtained from \( \tau \) by adjoining a row to the top and a column to the right,

\[
\tilde{\tau} = \begin{pmatrix}
\tau_{11} & \cdots & \tau_{1q} & m_1 \\
\vdots & \ddots & \vdots & \vdots \\
\tau_{pt} & \cdots & \tau_{pq} & m_p
\end{pmatrix}
\tag{6.2.1}
\]

The entries in the top row and rightmost column of \( \tilde{\tau} \) are uniquely determined by the conditions described in Section 3.11. In particular, we have

\[
l_j = 1 - \sum_i \tau_{ij}, \quad m_i = 1 - \sum_j \tau_{ij}, \quad r = \text{rk} (\tau) = \sum \tau_{ij}.
\]

Observe that \( l_j \) (resp. \( m_i \)) is 0 if the \( j^{th} \)-column (resp. \( i^{th} \)-row) of \( \tau \) contains a non-zero entry, and 1 otherwise.

As a corollary, we deduce that

\[
\text{Irr}(Z_Y) = \{ Z_Y(\tilde{\tau}) \mid \tau \in \mathcal{P}(p,q) \}.
\]

Indeed, the above calculation shows that the map \( \tau \mapsto \tilde{\tau} \) is an isomorphism between \( \mathcal{P}(p,q) \) and \( Y/G \). The latter indexes \( \text{Irr}(Z_Y) \), see Section 2.3.

**Proposition 6.3.** Consider \((E_\bullet, F_\bullet, x) \in C \), and \( z \in \text{gl}(V) \) given by

\[
z = \begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix} \in \begin{pmatrix}
\text{gl}(V_q) & \text{Hom}(V_q, V_p) \\
\text{Hom}(V_q, V_p) & \text{gl}(V_p)
\end{pmatrix} = \text{gl}(V).
\]

Then \((\bar{x} E_\bullet, F_\bullet, z) \in Z_Y \), if and only if

\[\alpha = \gamma = \delta - x \beta = 0, \quad \text{and} \quad (E_\bullet, F_\bullet, x, \beta) \in Z_C.\]

In particular, the fibre of \( Z_Y(\tilde{\tau}) \to Y \) at the point \((\bar{x} E_\bullet, F_\bullet) \) is isomorphic to the fibre of \( Z_C(\tau) \to C \) at the point \((E_\bullet, F_\bullet, x) \).

**Proof.** Following Section 5.12 we have \((\bar{x} E_\bullet, F_\bullet, z) \in Z_Y \) if and only if

\[
(6.3.1) \quad z F_i \subset F_{i-1}, \quad \text{for } 0 \leq i \leq p,
\]

\[
(6.3.2) \quad z \bar{x} E_i \subset \bar{x} E_{i-1} \iff \bar{x}^{-1} z \bar{x} E_i \subset \bar{x} E_{i-1}, \quad \text{for } 1 \leq i \leq q + 1.
\]
Equation (6.3.1) is equivalent to
\[ \alpha = \gamma = 0, \quad \text{and} \quad \delta F_i \subset F_{i-1}, \ \forall 1 \leq i \leq p. \]

For Eq. (6.3.2), we compute,
\[ \bar{x}^{-1} z \bar{x} = \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} \begin{pmatrix} 0 & \beta \\ 0 & \delta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} \beta x & \beta \\ -x \beta x + \delta x & -x \beta + \delta \end{pmatrix}. \]

It follows that Eq. (6.3.2) is equivalent to
\[ \delta = x \beta, \quad \text{and} \quad \beta x E_i \subset E_i, \ \forall 1 \leq i \leq q. \]

We see that Eqs. (6.3.1) and (6.3.2) hold if and only if Eqs. (6.3.3) and (6.3.4) hold.

Theorem B. Consider \( \tau \in \mathcal{P}(p, q) \), set \( \Lambda = \text{pr}(\tau) \), and let
\[ (Q, P) \in \text{SYT}(\Lambda^+) \times \text{SYT}(\Lambda^-) \]
be the tableaux pair corresponding to \( \tau \) via the bijection in Theorem A. Let \( \tilde{\tau} \) be given by Eq. (6.2.1), and let \( (\tilde{Q}, \tilde{P}) \) be the tableaux pair corresponding to \( \tilde{\tau} \) via the Robinson-Schensted-Knuth correspondence. Let \( \lambda = \text{sh}(\tilde{Q}) = \text{sh}(\tilde{P}) \). Then
\[ Q = \tilde{Q}(q), \quad P = \text{Rect}(\tilde{P}, q), \quad \text{sh}(\Lambda) = \text{sh}(Q) + \text{sh}(P), \]
and a \( \pm \)-filling in the equivalence class \( \Lambda \) is given by the following rule: the first box of the \( i^{th} \) row is \[ if \( \text{sh}(P)(i) < \lambda(i); \) otherwise it is \[. \]

Proof. Consider \((E_\bullet, F_\bullet, x, y) \in Z_C(\tau)\) for some \( \tau \in \mathcal{P}(p, q) \), and set
\[ z = \begin{pmatrix} 0 & y \\ 0 & xy \end{pmatrix} \in \begin{pmatrix} \text{gl}(V_q) & \text{Hom}(V_p, V_q) \\ \text{Hom}(V_q, V_p) & \text{gl}(V_p) \end{pmatrix} = \text{gl}(V). \]

Let \( \tilde{\tau} \) be given by Eq. (6.2.1), and let \( (\tilde{Q}, \tilde{P}) \) be the tableaux pair corresponding to the matrix \( \tilde{\tau} \) via the Robinson-Schensted-Knuth correspondence. Recall that \( \tilde{Q} \in \text{SYT}(\lambda, \underline{\underline{\alpha}}), \) and \( \tilde{P} \in \text{SYT}(\lambda, \underline{\underline{\beta}}) \), where \( \lambda \) is the Jordan type of \( z \).

Let \( Z' \) be the fibre of \( Z_Y(\tilde{\tau}) \) over the point \((\bar{x} \tilde{E}_\bullet, \bar{E}_\bullet)\). Recall the dense open subset \( Z'_Y(\tilde{\tau}) \subset Z_Y(\tilde{\tau}) \) from Eq. (3.13.1). Since \( Y(\tilde{\tau}) \) is \( G \)-homogeneous, the intersection \( Z'_Y(\tilde{\tau}) \cap Z' \) is a dense open subset of \( Z' \). Following Proposition 6.3, \( Z' \) is isomorphic to the fibre of \( Z_C(\tau) \rightarrow C \) at the point \((E_\bullet, F_\bullet, x) \); hence for \( \text{generic} \ (E_\bullet, F_\bullet, x, y) \in Z_C(\tau) \), we have \( z \in Z'_Y(\tilde{\tau}) \), i.e.,
\[ \text{Tab}(z, \bar{E}_\bullet) = \tilde{P}, \quad \text{Tab}(\bar{x}^{-1} z \bar{x}, \bar{E}_\bullet) = \text{Tab}(z, \bar{E}_\bullet) = \tilde{Q}. \]

Combined with Section 3.14 this yields \( \text{Tab}(x, F_\bullet) = \text{Rect}(\tilde{P}, q) \), see also Eq. (6.1.1). A further calculation,
\[ \bar{x}^{-1} z \bar{x} = \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} \begin{pmatrix} 0 & y \\ 0 & xy \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} yx & y \\ 0 & 0 \end{pmatrix}, \]
yields the claim \( \text{sh}(x, E_\bullet) = \text{Q}(q) \), see Section 3.10.

The claim \( \text{sh}(\Lambda) = \text{sh}(\Lambda^+) + \text{sh}(\Lambda^-) = \text{sh}(Q) + \text{sh}(P) \) follows from the admissibility of \( \Lambda \), see Theorems 4.13 and 5.6.

Let \( V = V_1 \oplus V_2 \oplus \cdots \oplus V_n \) be decomposition of \( V \) as in Eq. (4.1.1). Without loss of generality, we may assume that the indexing \( V_i \) satisfies the following rules:
We construct a ±-filling in the equivalence class \( \Lambda \) by setting the \( i^{th} \)-row to be the signed Young diagram corresponding to the indecomposable subspace \( V_i \).

Since \( \Lambda \) is admissible, the indexing ensures that for all \( i \), we have,

\[
\Lambda^+(i) = \dim(V_i \cap V_{q}), \quad \Lambda^-(i) = \dim(V_i \cap V_{p}),
\]

see Lemma 4.11 and Section 4.12. For \( 1 \leq i \leq k \), let \( \alpha_i = J(z|V_i)(1) \), the first entry of the partition \( J(z|V_i) \). The indexing ensures that the sequence \( \alpha_i \) is weakly decreasing. Following Lemma 4.6 we see that \( \lambda = (\alpha_1, \ldots, \alpha_n, 1, 1, \ldots, 1) \).

Now, for each \( i \), precisely one of the following holds:

\[
\begin{align*}
\Lambda^+(i) &= \Lambda^-(i) = \alpha_i - 1 = l, & \text{if } & V_i \cong U^+_{2l}, \\
\Lambda^+(i) &= \Lambda^-(i) + 1 = \alpha_i = l + 1, & \text{if } & V_i \cong U^+_{2l+1}, \\
\Lambda^+(i) &= \Lambda^-(i) = \alpha_i = l, & \text{if } & V_i \cong U^-_{2l}, \\
\Lambda^+(i) + 1 &= \Lambda^-(i) = \alpha_i = l + 1, & \text{if } & V_i \cong U^-_{2l+1}.
\end{align*}
\]

In the first two cases, the first box of the signed Young diagram corresponding to \( V_i \) is [ ]. In the other cases, it is [ ]. We see that this ±-filling is precisely as described in the theorem.

**Theorem C.** Consider \( \Lambda \in ASYD(q, p), Q \in SYD(\Lambda^+), \) and \( P \in SYD(\Lambda^-) \). Let \( \lambda \) be as in Lemma 4.6 and let

\[
\hat{Q} = (Q; \lambda), \quad \hat{P} = ev((ev(P); \lambda)), \quad \hat{\tau} \xrightarrow{RSK} (\hat{Q}, \hat{P}).
\]

Then \( \tau \) is the south-west sub-matrix of \( \hat{\tau} \) of size \( p \times q \).

**Proof.** Let \((E_\bullet, F_\bullet, x, y)\) be a generic point in \( Z_C(\tau) \), and set \( z = \begin{pmatrix} 0 & y \\ 0 & xy \end{pmatrix} \), so that

\[
\text{Tab}(z, \tilde{F}_\bullet) = \hat{P}, \quad \text{Tab}(\bar{x}^{-1}z\bar{x}, \tilde{E}_\bullet) = \hat{Q},
\]

see Eq. (6.3.5). Following Lemma 4.6 we obtain

\[
(6.3.7) \quad \hat{Q}(q + 1) = \text{sh}(\hat{Q}) \lambda, \quad \hat{P}(p + 1) = \text{sh}(\hat{P}) = \lambda.
\]

Recall from Theorem A that \( Q = \text{Tab}(yx, E_\bullet) \) and \( P = \text{Tab}(xy, F_\bullet) \). Following Eq. (6.3.6) we have \( z|E_\bullet = yx \), and hence

\[
\hat{Q}(i) = Q(i), \quad \forall 1 \leq i \leq q,
\]

Combined with Eq. (6.3.7), this yields \( \hat{Q} = (Q; \lambda) \).

Next, we see from Section 3.15 that

\[
ev(\hat{P}) = (J(z|\tilde{F}_r/\tilde{F}_{r-1}); J(z|\tilde{F}_r/\tilde{F}_{r-2}); \cdots; J(z|\tilde{F}_r/\tilde{F}_1); J(z|\tilde{F}_r/\tilde{F}_0))
\]

Following Eqs. (6.1.11) and (6.3.5), we have

\[
J(z|\tilde{F}_r/\tilde{F}_1) = J(xy|F_r/F_1), \quad \forall 1 \leq i \leq p,
\]

and hence,

\[
ev(\hat{P})(i) = ev(P)(i), \quad \forall 1 \leq i \leq p.
\]
Combined with Eq. (6.3.7), this yields, \( \text{ev}(\hat{P}) = (\text{ev}(P); \lambda) \). Since evacuation is an involution, we obtain the desired formula, \( \hat{P} = \text{ev}((\text{ev}(P); \lambda)) \).

Finally, the relationship between \( \hat{\tau} \) and \( \tau \) follows from Eq. (6.2.1). □

7. Projective Duality for Matrix Schubert Varieties

Consider the natural duality \( SYD(q, p) \rightarrow SYD(p, q) \), denoted \( \Lambda \mapsto \Lambda^\vee \), which simply switches the \( \Box \) and \( \blacksquare \) boxes. In this section, we study the duality \( (\Lambda, Q, P) \mapsto (\Lambda^\vee, P, Q) \), and show that this recovers the projective duality on (projectivized) matrix Schubert fibres.

Let \( K = GL(V_q) \times GL(V_p) \), \( H = \text{Hom}(V_q, V_p) \), \( X = \mathcal{F}l(V_q) \times \mathcal{F}l(V_p) \), and \( C = X \times H \) be as in Section 5.

7.1. Matrix Schubert Fibres. Fix \( (E_\bullet, F_\bullet) \in X \), and let

\[
B_K = \text{Stab}_K(E_\bullet, F_\bullet).
\]

For \( \tau \in PP/(q, p) \), let \( H(\tau) \) be the fibre of \( C(\tau) \rightarrow X \) at the point \((E_\bullet, F_\bullet)\). The \( H(\tau) \) are precisely the \( B_K \) orbits in \( H \), i.e., \( B/K = PP/(p, q) \). We call \( H(\tau) \) a matrix Schubert fibre in \( H \).

7.2. Duality for partial permutations. Recall from Section 4.4 the identification \( T^*H = H \times H^\vee \). We identify \( T^*H \) as the cotangent bundle of \( H^\vee \) also, via the projection map \( \pi : H \times H^\vee \rightarrow H^\vee \). Let \( T^*_H H(\tau) \) be the conormal bundle of \( H(\tau) \) in \( H \), and set

\[
\left( (H(\tau))^\vee \right) = \pi(T^*_H H(\tau)).
\]

Now, since \( T^*_H H(\tau) \) is \( B_K \)-stable and irreducible, the same is true of \( \left( (H(\tau))^\vee \right) \).

Consequently, \( \left( (H(\tau))^\vee \right) \) is the closure of some matrix Schubert fibre in \( H^\vee \), i.e.,

\[
\left( (H(\tau))^\vee \right) = H^\vee(\tau^\vee),
\]

for some \( \tau^\vee \in PP/(p, q) \). Following [GKZ08, 1.3.C], \( T^*_H H(\tau) \) is also the conormal variety of \( H^\vee(\tau^\vee) \) in \( H^\vee \), and \( PP H^\vee(\tau^\vee) \) is in projective duality with \( PP H(\tau) \).

**Proposition 7.3.** Let \( \Lambda^\vee \) be obtained from \( \Lambda \) by switching the \( \Box \) and \( \blacksquare \) boxes. If \( \tau \in PP/(p, q) \) corresponds to the triple \( (\Lambda, Q, P) \) via Theorem 4, then \( \tau^\vee \) corresponds to \( (\Lambda^\vee, P, Q) \).

**Proof.** Let \( C^\vee = H^\vee \times X \). Consider a generic point \((x, y, F_\bullet, G_\bullet)\) in \( Z_C(\tau) \). Following Sections 7.1 and 7.2 we have \( Z_C(\tau) = Z_{C^\vee}(\tau^\vee) \), and hence \((x, y, F_\bullet, G_\bullet)\) is generic in \( Z_{C^\vee}(\tau^\vee) \). It follows that the triple is determined again by Theorem 13 with the roles of \( V_q \) and \( V_p \) switched, i.e., with \( \text{deg}(V_q) = \Box \) and \( \text{deg}(V_p) = \blacksquare \). This yields the claimed correspondence \( \tau^\vee \leftrightarrow (\Lambda^\vee, P, Q) \). □
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