Quantum benchmarks are routinely used to validate the experimental demonstration of quantum information protocols. Many relevant protocols, however, involve an infinite set of input states, of which only a finite subset can be used to test the quality of the implementation. This is a problem, because the benchmark for the finitely many states used in the test can be higher than the original benchmark calculated for infinitely many states. This situation arises in the teleportation and storage of coherent states, for which the benchmark of 50% fidelity is commonly used in experiments, although finite sets of coherent states normally lead to higher benchmarks. Here we show that the average fidelity over all coherent states can be indirectly probed with a single setup, requiring only two-mode squeezing, a 50-50 beamsplitter, and homodyne detection. Our setup enables a rigorous experimental validation of quantum teleportation, storage, amplification, attenuation, and purification of noisy coherent states. More generally, we prove that every quantum benchmark can be tested by preparing a single entangled state and measuring a single observable.

In this article, we show that every quantum benchmark can be tested by a single preparation setup of an entangled state and a single measurement setup on the output. More broadly, we develop a unified framework to test one to test many, which can be applied to a variety of quantum protocols, including the teleportation and storage of finite-dimensional quantum systems. Theoretical values of the benchmarks have been determined in a variety of scenarios, including the teleportation and storage of coherent states and squeezed states. Benchmarks for the amplification of coherent states are important for assessing the realization of deterministic amplifiers, and have been studied in Refs. [22, 23]. Many benchmarks are fidelity-based, meaning that they use the fidelity as the figure of merit. Other benchmarks are entanglement-based, meaning that the figure of merit is (a measure of) the ability to preserve entanglement.

In theory, quantum benchmarks provide rigorous criteria of quantumness. In practice, the application of these criteria can be problematic. The benchmarks often rank quantum devices based on their average performance on an infinite set of input states, such as the set of all coherent states. In a real experiment, however, only a finite subset of inputs can be tested. The evaluation of the performance on each input requires many sessions of data collection, often amounting to a full tomography of the state. Now, the problem is that the value of the benchmark for the finite subset of states used in the experiment can be much larger than the theoretical benchmark. For example, the fidelity benchmark for the teleportation of uniformly distributed coherent states is 50%, while the benchmark for just two coherent states is at least 93.3%, the minimum value over all pairs of coherent states. Comparing the experimental fidelity with the theoretical benchmark requires additional assumptions on the device—e.g., assumptions on how it would have worked if it had been tested on other inputs. But making such assumptions is in contradiction to the purpose of quantum benchmarks, i.e., to certify quantum advantages without having to trust the devices. An alternative approach would be to perform a full tomography of the device, but this would require a large number of measurement settings (or even an infinite number in the case of continuous variable systems).

In this article, we show that every quantum benchmark can be tested by a single preparation setup of an entangled state and a single measurement setup on the output. More broadly, we develop a unified framework...
for quantum benchmarks, including fidelity-based and entanglement-based benchmarks as special cases. We observe that the same benchmark can be tested in multiple equivalent ways, among which one can choose the most experimentally friendly one. Using the idea of equivalent tests, we propose a benchmark setup for the demonstration of continuous-variable quantum memories [7–10] and for the demonstration of quantum-enhanced amplification [17–21]. Our proposal allows one to measure the average fidelity over all possible coherent states, using the set of input states is infinite. Now, we show that the average fidelity (2) exists. Among these indirect measurements, some can be dramatically easier. The merit of Eq. (3) is that it reveals a general structure, suggesting new ways to measure the average fidelity which can be viewed as the special case of Eq. (1) where each POVM \( \{ P_{y}^{(x)} \} \) has an outcome \( y \) associated to the projector \( P_{y}^{(x)} = |\phi_{y}^{(x)}\rangle \langle \phi_{y}^{(x)}| \) and the score \( \omega(x, y) \) is either 1 or 0, depending on whether or not \( y \) is equal to \( y_{x} \).

The benchmark for a genuine quantum implementation has the form \( F_{\text{det}} > F_{C} \), where \( F_{C} \) is the classical fidelity threshold, namely the maximum fidelity achievable by measure-and-prepare channels [1]. The direct way to evaluate the score (1)—or the average fidelity (2)—is to test the action of the channel \( C \) on all the input states \( \{ \rho_{x} \} \) and to use the experimental data to compute the average. However, this approach is not viable when the set of input states is infinite. Now, we show that many indirect ways to experimentally measure the average score (1) or the average fidelity (2) exist. Among these indirect measurements, some can be dramatically simpler than the direct approach of Figure 1.

First of all, we note that every test with random input states can be reformulated as a test with a single, mixed, input state \( \sigma_{AR} \). This is because one can regard the preparation of the state \( \rho_{x} \) with probability \( p_{x} \) as the preparation of a single quantum-classical state \( \sigma = \sum_{x} p_{x} \rho_{x} \otimes |x\rangle \langle x|_{R} \), where \( R \) is an auxiliary system keeping track of the index \( x \). Likewise, one can formally write down a single quantum observable \( O = \sum_{x,y} \omega(x, y) P_{y}^{(x)} \otimes |x\rangle \langle x|_{R} \), so that the average score (1) takes the form

\[
S_{\text{det}} = \text{Tr}[O(C \otimes I_{R})(\sigma_{AR})].
\]  

Per se, this reformulation does not make the problem easier. The merit of Eq. (3) is that it reveals a general structure, suggesting new ways to measure the average score. This reformulation also offers a unified approach, which can be adopted not only for fidelity-based benchmarks, but also for other types of quantum benchmarks, such as the entanglement-based benchmarks [26–29].

The single-input setup for testing quantum channels is depicted in Figure 2. Now, the key observation is that many different tests are equivalent, meaning that they assign the same average score to all possible channels. This observation is important because, among the many
Theorem 1. Every test for deterministic devices is equivalent to a canonical test of the following form:

1. Choose a mixed state $\tau_A$, with the property that the operator $I_A \otimes \tau_A$ is invertible on the support of the operator $\Omega^{T_A}$, where $T_A$ denotes the partial transpose on system $A$.

2. Prepare a purification of $\tau_A$, denoted by $|\Psi\rangle_{AR}$.

3. Apply the channel $C$ on system $A$.

4. Measure systems $A'$ and $R$ with the observable

$$O = \left( I_{A'} \otimes \tau_{R}^{-1/2} T_{AR} \right) \Omega^{T_A} \left( I_{A'} \otimes T_{AR} \tau_{R}^{-1/2} \right).$$

where $\tau_{R} = Tr_{A}[|\Psi\rangle_{AR} \langle \Psi|]$ is the marginal of the state $|\Psi\rangle_{AR}$ on system $R$, and $T_{AR}$ is the partial isometry such that $T_{AR}^{*} T_{RA} = \tau_{R}$.

The best way to understand Theorem 1 is to use it in a concrete example. Consider the problem of amplifying coherent states \[17, 22, 23\]. Here, the task is to transform a generic coherent state $|\alpha\rangle \propto \sum_n \alpha^n |n\rangle/\sqrt{n!}$ into the amplified coherent state $|g\alpha\rangle$, where $g \geq 1$ is the gain of the amplifier. For $g = 1$, the problem is to teleport coherent states \[2\] or to store them in a quantum memory \[7, 10\]. Assuming that the inputs are Gaussian-distributed, the average fidelity is

$$F^{(\text{det})} = \int \frac{d^2 \alpha}{\pi} \lambda e^{-\lambda|\alpha|^2} \langle g\alpha|C(|\alpha\rangle\langle \alpha|)|g\alpha\rangle,$$

where $\lambda \geq 0$ is the inverse of the variance. In practice, the average cannot be evaluated directly, because this would require sampling over an infinite set of input states. Moreover, in the actual experiments \[19\] the fidelity is evaluated through a full tomography of the output state, meaning that each value of $\alpha$ requires a large (ideally infinite) number of experimental settings, making the evaluation of the average fidelity prohibitively expensive. Luckily, Theorem 1 offers a way out. Instead of sampling over all coherent states, it is enough to prepare a two-mode squeezed vacuum state

$$|\Psi\rangle_{AR} = \sqrt{1-x} \sum_n x^{n/2} |n\rangle_{A} \otimes |n\rangle_{R},$$

where the squeezing parameter $x$ can be any number in the interval $(0, 1)$. Instead of evaluating the fidelity on each coherent state, it is enough to measure a single observable, given by Eq. 4 with the performance operator

$$\Omega = \int \frac{d^2 \alpha}{\pi} \lambda e^{-\lambda|\alpha|^2} |g\alpha\rangle\langle g\alpha| \otimes |\alpha\rangle\langle \alpha|.$$

Now, we take advantage of the fact that every value of the squeezing parameter $x$ is allowed, and therefore one can choose the most convenient $x$. Specifically, we notice that the observable \[8\] takes a simple form when $x = 1/(1 + \lambda)$. For $g^2 \leq \lambda + 1$, we find (Appendix C)

$$O = S_{\theta}^{\dagger} (I \otimes G_{\theta}) S_{\theta},$$

where $S_{\theta} = \exp[\theta(ab - a^\dagger b^\dagger)]$ is a two-mode squeezer with $\tan \theta = g/\sqrt{\lambda + 1}$, and $G_{\theta}$ is the Gaussian observable $G_{\theta} = \sum_n (\tan \theta)^{2n} |n\rangle\langle n|$. In practice, this means that
The input mode and a reference are prepared in the two-mode squeezed vacuum (TMSV). After the action of the amplifier, the output mode and the reference are sent through a two-mode squeezer $S_\theta$, followed by a 50-50 beamsplitter and two quadrature measurements on the output modes.

The observable $O$ can be measured by sending the two output modes $A'$ and $R$ through a two-mode squeezer and by measuring the observable $G_\theta$ on the second port. In turn, the observable $G_\theta$ can be measured by sending the mode through a 50-50 beamsplitter, measuring the quadratures $X = (a + a^\dagger)/2$ and $P = (b - b^\dagger)/(2i)$ on the two output modes, respectively, and, finally, averaging the outcomes with a Gaussian weight (see Appendix C for the exact expression). The setup for $g^2 > \lambda + 1$ is identical, except that one has to set $\tanh \theta = \sqrt{\lambda + 1}/g$ and the observable $G_\theta$ is measured on the first output port (Appendix C).

Our method makes the average fidelity (11) experimentally accessible, thus enabling a rigorous experimental test of the quantum advantage. The same method can be used to test the fidelity of attenuation (22) (37–39), cloning (41–43), purification of displaced thermal states (39–41), and phase conjugation (49), as shown in Appendix C and D. A limitation of the present approach is that the verifier should be able to preserve the reference mode from noise. In the case of quantum memories, this means that the verifier should possess a good quantum memory for the reference mode. Basically, the test of Fig. 3 compares the untrusted quantum memory implemented by the experimenter with a trusted quantum memory in the verifier’s lab.

Canonical tests for nondeterministic devices. Now, let us consider the case of devices that return an output with some nonunit probability. Examples of such devices are the noiseless probabilistic amplifier [30] and the noiseless probabilistic attenuator of Refs. (37) (38) (40). In general, a probabilistic device can be described by a quantum operation $C$ (completely positive trace-nonincreasing linear map). To test the device, one can prepare a single input state $\sigma$ and measure an observable $O$ on the output, as in Figure 2. Sometimes, the device will report failure instead of producing an output. The probability that an output is produced is

$$p_{\text{succ}} = \text{Tr} \left[ (C \otimes I_R)(\sigma_{AR}) \right] = \text{Tr}[C(\sigma_A)],$$

(13)

where $\sigma_A = \text{Tr}_R[\sigma_{AR}]$ is the marginal of $\sigma_{AR}$ on system $A$. The average score is then

$$S(\text{prob}) := \frac{\text{Tr}[O(C \otimes I_R)(\sigma_{AR})]}{\text{Tr}[C(\sigma_A)]}$$

(14)

and can be expressed as

$$S(\text{prob}) = \frac{\text{Tr}[C\Omega]}{\text{Tr}[C(I_A \otimes \sigma_A)]},$$

(15)

where $\Omega$ is the performance operator (6) and $C$ is the Jamiołkowski operator. Note that, now, the score depends both on the performance operator $\Omega$ and on the marginal input state $\sigma_A$, which determines the probability of success via Eq. (13).

It is easy to see that two tests are equivalent in terms of score and success probability if and only if they have the same pair of operators $(\Omega, \sigma_A)$. Leveraging on the equivalence, we can construct a canonical realization.

**Theorem 2.** Every test of probabilistic devices is equivalent to a canonical test of the following form:

1. Prepare a purification of the marginal input state $\sigma_A$, denoted by $|\Phi\rangle_{AR}$.
2. Apply the quantum operation $C$ on system $A$.
3. Measure systems $A'$ and $R$ with the observable $O = (I_{A'} \otimes \tilde{\sigma}_R^{-1/2}T_{AR}^T \sigma_R^{-1/2})\Omega T^A (I_{A'} \otimes T_{AR} \sigma_R^{-1/2})$.

(16)

where $\tilde{\sigma}_R$ is the marginal of the state $|\Phi\rangle_{AR}$ on system $R$ and $T_{AR}$ is the partial isometry such that $T_{AR}^\dagger \sigma_A T_{AR} = \tilde{\sigma}_R$.

Theorem 2 offers the first rigorous way of testing the fidelity benchmark for noiseless nondeterministic amplifiers (18) (21). In this case, the marginal state $\sigma_A$ is

$$\sigma_A = \int \frac{d^2\alpha}{\pi} \lambda e^{-\lambda |\alpha|^2} |\alpha\rangle \langle \alpha|.$$  

(17)

Its purification is a two-mode squeezed vacuum, given by Eq. (10) with $x = 1/(1 + \lambda)$. Then, one can obtain the observable $O$ from Eqs. (16) and (11). Again, the observable has a simple experimental realization. In fact, this is the same realization described in the deterministic case. Using this realization, it is now possible to set up a conclusive demonstration of quantum advantage for noiseless amplifiers. The same holds for nondeterministic attenuation (37) (38) (40).
The fully black box test. We analyzed, separately, the tests of deterministic devices and the tests of probabilistic devices. In practice, however, we may not know the success probability of the tested device. This would be a problem, because the benchmark generally depends on the success probability $A$: in general, the smaller the success probability, the higher the benchmark. A solution to the problem would be to use the highest benchmark, calculated in the limit of vanishing success probability. However, this could set an unreasonably high bar for the experiment. Now, we show that the verifier can devise a fully black box test, where the value of the benchmark is independent of the probability of success.

**Theorem 3** (Appendix F). Given a test $T$ for deterministic devices, one can construct a new test $T'$ for probabilistic devices, with the following properties:

1. $T'$ has the same performance operator as the original test $T$. Therefore, $T'$ assigns the same score as $T$ to all deterministic devices.

2. For probabilistic devices, the benchmark for $T'$ is independent of the success probability.

The new test $T'$ is described by a pair of operators $(\Omega, \sigma_A)$, with the following properties: the performance operator $\Omega$ is chosen to be the same as the performance operator of the old test $T$. This choice guarantees that the test $T'$ assigns the same score as $T$ when applied to deterministic devices. The marginal state $\sigma_A$ is chosen to be the state that reduces the probabilistic benchmark to its minimum: this means that $\sigma_A$ minimizes the best score $[15]$ over all measure-and-prepare channels. The test for amplification or attenuation shown earlier in the article is an example of a fully black box test: the same experimental test and the same benchmark value can be used for both deterministic and probabilistic devices. More examples of this situation are shown in Appendix F, which focusses on the scenario where the test $T$ enjoys a symmetry with respect to a group of physical transformations.

**Conclusions.** In this article we showed that a verifier can experimentally evaluate the performance of a quantum device on an infinite set of inputs, by preparing a single entangled input and measuring a single joint observable. As an application, we constructed a test for the realization of quantum memories, amplifiers, and attenuators of coherent states, and purifiers of displaced thermal states. The test can be realized using two-mode squeezers, beamp splitters, and homodyne detection. Using these ingredients, one can experimentally assess the average fidelity over all possible coherent states (or all possible displaced thermal states), thus providing a fully rigorous demonstration of genuine quantum advantage.

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Appendix A: Derivation of Eq. (5)

Here we show that the average score of the channel $C$ can be expressed in terms of the Jamiołkowski operator as $S^{(\text{det})} = \text{Tr}[C \Omega]$.

We remind the reader that the action of the channel $A$ can be expressed in terms of the Jamiołkowski operator as follows:

$$C(\rho) = \sum_{i,j} \rho_{ij} C(|i\rangle\langle j|)$$

$$= \sum_{i,j} C(|i\rangle\langle j|) \text{Tr} [\rho |j\rangle\langle i|]$$

$$= \sum_{i,j} \text{Tr}_A \left[ C(|i\rangle\langle j|) \otimes (\rho |j\rangle\langle i|) \right]$$

$$= \text{Tr}_A \left[ (I_{A'} \otimes \rho) C \right]. \quad (A-1)$$

The next step is to compute the average score. To keep track of the Hilbert spaces, we will add a subscript to each operator, so that, e.g. $\rho_A$ indicates that the operator $\rho$ acts on the Hilbert space of system $A$. We will also write expressions like $(C_{A:A'} \otimes I_R) (O_{A'R} \otimes I_A)$, with the implicit understanding that the Hilbert spaces are suitably reordered in order to perform the matrix multiplication. With this notation, we have

$$S^{(\text{det})} = \text{Tr}[O_{A'R} (C \otimes I_R)(\sigma_{AR})]$$

$$\text{Tr}[O_{A'R} \text{Tr}_A [(I_{A'} \otimes \sigma_{AR})(C_{A:A'} \otimes I_R)]]$$

$$= \text{Tr}[O_{A'R} \text{Tr}_A [(I_{A'} \otimes \sigma_{AR})(C_{A:A'} \otimes I_R)]]$$

$$= \text{Tr}[O_{A'R} \text{Tr}_A [(C_{A:A'} \otimes I_R)(O_{A'R} \otimes I_A)(I_{A'} \otimes \sigma_{AR})]]$$

$$= \text{Tr}[C_{A:A'} \text{Tr}_A [O_{A'R} \otimes I_A] (I_{A'} \otimes \sigma_{AR})]$$

$$= \text{Tr}[C_{A:A'} \Omega_{AA'}] \quad (A-2)$$

with

$$\Omega_{AA'} := \text{Tr}_R [(O_{A'R} \otimes I_A)(I_{A'} \otimes \sigma_{AR})]. \quad (A-3)$$

Appendix B: Proof of Theorem 1

Theorem 1 states that a (deterministic) test with performance operator $\Omega$ can be implemented through the preparation of a single pure state and the measurement of a single joint observable.

To construct the pure state and the observable, we pick a state $\tau_A$ of system $A$, and we diagonalize it as

$$\tau_A = \sum_n \lambda_n |\phi_n\rangle \langle \phi_n| \quad (B-1)$$

We require that the operator $(I_{A'} \otimes \tau_A)$ be invertible on the support of the operator $O_{A'A}$, in such a way that operators like $(I_{A'} \otimes \tau)^{-1} O_{A'A}$ are well defined.

The input state in our canonical test will be a purification of the state $\tau$, with purifying system $R$. The purification, denoted by $|\Psi\rangle_{AR}$, can be written in the Schmidt decomposition as

$$|\Psi\rangle_{AR} = \sum_n \sqrt{\lambda_n} |\phi_n\rangle_A \otimes |\psi_n\rangle_R , \quad (B-2)$$

where the states $\{|\psi_n\rangle\}$ are orthonormal.

The joint observable in the canonical test is defined as

$$O_{A'R} = (I_{A'} \otimes \tau_R^{-1/2} T_{AR} \tau_R^{-1/2}) O_{A'A} (I_{A'} \otimes \tau_A) , \quad (B-3)$$

where

$$\tau_R = \sum_n \lambda_n |\psi_n\rangle \langle \psi_n|_R \quad (B-4)$$

is the marginal of $|\Psi\rangle_{AR}$ on system $R$, and

$$T_{AR} = \sum_n |\phi_n\rangle_A \langle \phi_n|_R \quad (B-5)$$

is the partial isometry that maps each eigenvector of $\tau_R$ into the corresponding eigenvector of $\tau_A$.

Now, it remains to prove that the performance operator of our test is exactly $\Omega$.

Let us provisionally denote the performance operator of our test by $\tilde{\Omega}$. The goal is to show the equality $\tilde{\Omega} = \Omega$.

For this purpose, the operator $\tilde{\Omega}$ can be computed using Eq. (A-3), with $\sigma_{AR} = |\Psi\rangle\langle \Psi|_{AR}$ and $O_{A'A}$ defined in Eq. (B-3). Explicitly, we have

$$\tilde{\Omega}_{A'A} = \text{Tr}_R [(O_{A'R} \otimes I_A)(I_{A'} \otimes |\Psi\rangle \langle \Psi|_{AR})]$$

$$= \text{Tr}_{A} \left[ (\tilde{\Omega}_{A'A} \otimes I_A)(I_{A'} \otimes |\Gamma\rangle \langle \Gamma|_{A\tilde{A}}) \right] . \quad (B-6)$$

where $\tilde{A}$ is a second copy of system $A$, and $|\Gamma\rangle_{A\tilde{A}}$ is the (unnormalized) vector defined as

$$|\Gamma\rangle_{A\tilde{A}} := (I_A \otimes T_{AR} \tau_R^{-1/2}) |\Psi\rangle_{AR}$$

Continuing from Eq. (B-6), we obtain

$$|\Gamma\rangle_{A\tilde{A}} := \sum_n |\phi_n\rangle_A \otimes |\phi_n\rangle_{\tilde{A}} \quad (B-7)$$
\[ \Omega'_{A'A} = \sum_{m,n} \text{Tr}_{\tilde{A}} \left[ \left( T^\dagger_{A'\tilde{A}} \otimes I_A \right) \left( I_{A'} \otimes |\phi_m\rangle \langle \phi_n| \otimes |\phi_m\rangle \langle \phi_n|_A \right) \right] \]

\[ = \sum_{m,n} \left( I_{A'} \otimes |\phi_m\rangle \langle \phi_n| \otimes |\phi_m\rangle \langle \phi_n|_A \right) \Omega_{A'\tilde{A}}^{T^\dagger} \left( I_{A'} \otimes |\phi_m\rangle \langle \phi_n| \otimes |\phi_m\rangle \langle \phi_n|_A \right) \]

\[ = \sum_{m,n} \left( I_{A'} \otimes |\phi_m\rangle \langle \phi_n| \rangle \Omega_{A'\tilde{A}}^{T} \left( I_{A'} \otimes |\phi_m\rangle \langle \phi_n| \otimes |\phi_m\rangle \langle \phi_n|_A \right) \right] \]

\[ = \Omega_{A'A} . \]

This concludes the proof of Theorem 1.

Appendix C: Canonical test for (noisy) coherent states

Here we work out the explicit expression of the canonical test for storage, teleportation, amplification, attenuation, cloning, and purification of (noisy) coherent states.

All the above can be subsumed into a single task. In this task, the experimenter is given a displaced thermal state

\[ \rho_{\alpha,\mu} = \int \frac{d^2 \beta}{\pi} \mu e^{-|\beta|^2} |\alpha + \beta\rangle \langle \alpha + \beta| , \]

where \( \mu \) is a known parameter specifying the amount of noise in the input, while \( \alpha \) is an unknown parameter specifying the modulation of the signal. The experimenter’s goal is to transform the displaced thermal state \( \rho_{\alpha,\mu} \) into the pure coherent state \( |g\alpha\rangle \), where \( g \) is the gain of the amplification (or the attenuation parameter, if \( g < 1 \)).

For \( \mu \rightarrow \infty \), the goal is to transform the pure coherent state \( |\alpha\rangle \) into the coherent state \( |g\alpha\rangle \). Depending on whether \( g > 1, g = 1, \) or \( g < 1 \), this task is amplification, teleportation/storage, or attenuation. For finite \( \mu \), the task is (ideally) to transform a displaced thermal state into a pure coherent state, while amplifying, preserving, or attenuating the signal. Finally, since \( N \) copies of the state \( \rho_{\alpha,\mu} \) can be reversibly converted into a single copy of the state \( \rho_{C,\alpha,\mu} \), the above task encompasses various tasks of cloning and purification.

The figure of merit is the average fidelity

\[ F = \int \frac{d^2 \alpha}{\pi} \lambda e^{-\lambda|\alpha|^2} \langle g\alpha| C(\rho_{\alpha,\mu}) |g\alpha\rangle , \]

where \( C \) is the channel used by the experimenter, and \( \rho_{\lambda}(d^2 \alpha) = \lambda e^{-\lambda|\alpha|^2} d^2 \alpha/\pi \) is the probability distribution of the signal. The performance operator of the fidelity test is

\[ \Omega = \int \frac{d^2 \alpha}{\pi} \lambda e^{-\lambda|\alpha|^2} |g\alpha\rangle \langle g\alpha| \otimes \rho_{\alpha,\mu} . \]

This is consistent with Eq. 7 of the main text.

We now use Theorem 1 to construct a new test for the average fidelity \( \langle C-2 \rangle \). First of all, we have to choose a state \( \tau_A \) such that \( I_A \otimes \tau_A \) is invertible on the support of \( \Omega^{T^\dagger} \). Here we choose a generic thermal state, decomposed as

\[ \tau_A = (1-x) \sum_{n=0}^{\infty} x^n |n\rangle \langle n|_{A} \quad x \in (0,1) , \]

where \( \{|n\rangle\} \) is the Fock basis. The canonical test of Theorem 1 uses a purification of \( \tau_A \). Specifically, we choose the two-mode squeezed vacuum

\[ |\Psi_x\rangle_{AR} = \sqrt{1-x} \sum_{n=0}^{\infty} x^n/2 |n\rangle_A \otimes |n\rangle_R , \]

also known as the twin-beam state \([51]\). Note that the purifying system \( R \) is another Bosonic mode, and therefore \( \mathcal{H}_R \simeq \mathcal{H}_A \). The isomorphism is implemented by the unitary operator \( \tau_{AR} = \sum_{n} |n\rangle_A \langle n| \).

In the following we will construct the observable \( O_{A'R} \), using Eq. 8 of the main text. It is convenient to separate two cases, depending on whether the input states are pure or mixed.

**Pure inputs**

For pure input states, the observable \( O_{A'R} \) of Eq. \( B-8 \) reads

\[ O_{A'R} = \left( I_{A'} \otimes \tau_{R}^{-1/2} T^\dagger_{AR} \right) \Omega_{A'\tilde{A}} T_{AR} \left( I_{A'} \otimes \tau_{R}^{-1/2} T_{AR} \right) \]

\[ = \int \frac{d^2 \alpha}{\pi} \lambda e^{-\lambda|\alpha|^2} |g\alpha\rangle \langle g\alpha| \otimes \left( \tau_{R}^{-1/2} |\bar{\alpha}\rangle \langle \bar{\alpha}| \tau_{R}^{-1/2} \right) \]

\[ = \int \frac{d^2 \alpha}{\pi} \frac{\lambda}{1-x} e^{-(\lambda+1-x)|\alpha|^2} \]

\[ \times |g\alpha\rangle \langle g\alpha| \otimes \left| \frac{\bar{\alpha}}{\sqrt{x}} \right\rangle \langle \frac{\bar{\alpha}}{\sqrt{x}} \right| . \]

The observable takes a simple form when

\[ x = \frac{1}{\lambda+1} , \]
in which case we have
\[ O_{A'R} = \int \frac{d^2\gamma}{\pi} \left| \frac{g\gamma}{\sqrt{\lambda + 1}} \right\rangle \langle \frac{g\gamma}{\sqrt{\lambda + 1}} \otimes |\gamma\rangle \langle \gamma| \right. \] (C-8)

A more explicit expression comes from the relation
\[
S_\theta \left( D(\alpha) \otimes D(\beta) \right) S_\theta^\dagger \\
= D \left( \cosh \theta \alpha - \sinh \theta \beta \right) \otimes D \left( \cosh \theta \beta - \sinh \theta \alpha \right) \\
\] (C-9)

where \( S_\theta \) is the two-mode squeezing operation
\[
S_\theta := \exp \left[ \theta (ab - a^\dagger b^\dagger) \right] .
\] (C-10)

For \( g \leq \sqrt{\lambda + 1} \), we choose
\[
\theta = \tanh^{-1} \left( \frac{g}{\sqrt{\lambda + 1}} \right) ,
\] (C-11)

obtaining
\[
O_{A'R} = \int \frac{d^2\gamma}{\pi} S_\theta^\dagger \left[ I \otimes D \left( \frac{\gamma}{\cosh \theta} \right) \right] \\
\times \left( S_\theta |0\rangle \langle 0| \otimes |0\rangle \langle 0| S_\theta^\dagger \right) \left[ I \otimes D \left( \frac{\gamma}{\cosh \theta} \right) \right]^\dagger S_\theta \\
= S_\theta^\dagger \left( I \otimes G_\theta \right) S_\theta ,
\] (C-12)

where \( G_\theta \) is the Gaussian observable
\[
G_\theta = \sum_n (\tanh \theta)^{2n} |n\rangle \langle n| .
\] (C-13)

Similarly, for \( g > \sqrt{\lambda + 1} \), we choose
\[
\theta = \tanh^{-1} \left( \frac{\sqrt{\lambda + 1}}{g} \right) ,
\] (C-14)

obtaining
\[
O_{A'R} = \left( \tanh \theta \right)^2 S_\theta^\dagger \left( G_\theta \otimes I \right) S_\theta ,
\] (C-15)

where \( G_\theta \) is the Gaussian observable defined in Eq. (C-13), now with \( \tanh \theta = \sqrt{\lambda + 1}/g \).

Eqs. (C-12) and (C-15) imply that we can measure the observable \( O_{A'R} \) by

1. performing the two-mode squeezing operation \( S_\theta \) on the modes \( A' \) and \( R \)

2. discarding one of the two output modes (the first mode, if \( g \leq \sqrt{\lambda + 1} \), or the second mode, if \( g > \sqrt{\lambda + 1} \)), and measuring the Gaussian observable \( G_\theta \) on the other.

In turn, the measurement of the Gaussian observable \( G_\theta \) can be implemented in different ways. When \( \tanh \theta \) is small, the observable \( G \) can be accurately approximated using a photon counter that distinguishes Fock states with low photon number. In general, the Gaussian observable \( G \) can be measured with a heterodyne setup, corresponding to the POVM
\[
P(d^2\gamma) = |\gamma\rangle \langle \gamma| \frac{d^2\gamma}{\pi} .
\] (C-16)

Upon obtaining the outcome \( \gamma \), one can average the outcomes with the Gaussian weight
\[
w(\gamma) = (\tanh \theta)^2 e^{-|\gamma|^2/(\sinh \theta)^2} .
\] (C-17)

Finally, the heterodyne measurement can be implemented with two homodyne detectors, using the following procedure

1. mix the target mode with the vacuum in a 50-50 beamsplitter

2. measure the quadrature \( X = (a + a^\dagger)/2 \) on the first output mode and the quadrature \( P = (b - b^\dagger)/(2i) \) on the second output mode

3. if the outcomes of the two quadrature measurements are \( x \) and \( p \), declare the outcome \( \gamma = \sqrt{2}(x + ip) \).

By construction, the expectation value of the above measurement is equal to the average fidelity of Eq. (C-2).

Mixed inputs

When the input states are mixed, the observable \( O_{A'R} \) is
\[
O_{A'R} = \left( I_{A'} \otimes \tau_{AR}^{-1/2} T_{AR}^\dagger \right) \Omega_{A'A}^\dagger \left( I_{A'} \otimes T_{AR} \tau_{AR}^{-1/2} \right) \\
= \int \frac{d^2\alpha}{\pi} \frac{d^2\beta}{\pi} \lambda e^{-\lambda|\alpha|^2} |\alpha\rangle \langle \alpha| \otimes \left( \tau_{R}^{-1/2} \rho_{\pi,\mu} \tau_{R}^{-1/2} \right) \\
= \int \frac{d^2\alpha}{\pi} \frac{d^2\beta}{\pi} \lambda e^{-\lambda|\alpha|^2} \mu e^{-\mu|\beta|^2} \frac{e^{-\lambda|\alpha+\beta|^2} - e^{-\lambda|\alpha|^2}}{1 - x} \\
\times |\alpha\rangle \langle \alpha| \otimes \frac{\alpha + \beta}{\sqrt{x}} |\alpha + \beta| \\
= \int \frac{d^2\alpha}{\pi} \frac{d^2\gamma}{\pi} \lambda e^{-\lambda|\alpha|^2} \mu e^{-\mu|\beta|^2} \frac{e^{-\lambda|\alpha+\beta|^2} - e^{-\lambda|\alpha|^2}}{1 - x} \\
\times |\alpha\rangle \langle \alpha| \otimes \frac{|\alpha|^2}{\sqrt{x}} |\alpha| \\
= \int \frac{d^2\alpha}{\pi} \frac{d^2\gamma}{\pi} \lambda e^{-\lambda|\alpha|^2} \mu e^{-\mu|\beta|^2} \frac{e^{-\lambda|\alpha+\beta|^2} - e^{-\lambda|\alpha|^2}}{1 - x} \\
\times |\alpha\rangle \langle \alpha| \otimes |\gamma\rangle \langle \gamma| \\
= \int \frac{d^2\alpha}{\pi} \frac{d^2\gamma}{\pi} \lambda e^{-\lambda(|\alpha|^2 + \mu|\beta|^2)} \mu e^{-\mu|\beta|^2} \frac{e^{-\lambda|x|^2} - e^{-\lambda|\alpha|^2}}{1 - x} \\
\times |\alpha\rangle \langle \alpha| \otimes |\gamma\rangle \langle \gamma| ,
\] (C-18)
with
\[ k = \frac{\mu \sqrt{x}}{\lambda + \mu} \quad \text{(C-19)} \]
\[ l = \mu x + x - 1 - \frac{\mu^2 x}{\lambda + \mu}. \quad \text{(C-20)} \]

We observe that the expression can be simplified if we set \( l = 0 \), corresponding to the choice
\[ x = \frac{\lambda + \mu}{\lambda + \mu + \lambda \mu}. \quad \text{(C-21)} \]

Defining \( \delta = g (\alpha - k \gamma) \), we obtain
\[ O_{A'R} = \int \frac{d^2 \gamma}{\pi} \int \frac{d^2 \delta}{\pi} (\lambda + \mu) e^{-\frac{\lambda + \mu}{\nu} |\delta|^2} \]
\[ \times |\delta + gk\gamma\rangle \langle \delta + gk\gamma| \otimes |\gamma\rangle \langle \gamma| \]
\[ = \int \frac{d^2 \gamma}{\pi} N(|gk\gamma\rangle \langle gk\gamma|) \otimes |\gamma\rangle \langle \gamma|, \quad \text{(C-22)} \]
where \( N_\nu \) is the Gaussian-additive-noise channel defined by
\[ N_\nu(\rho) = \int \frac{d^2 \delta}{\pi} \nu e^{-\nu |\delta|^2} D(\delta) \rho D(\delta)^\dagger \quad \text{(C-23)} \]
and
\[ \nu = \frac{\lambda + \mu}{g^2} \quad \text{(C-24)} \]

More concisely, the observable \( O_{A'R} \) can be expressed as
\[ O_{A'R} = (N_\nu \otimes Z_R)(Z_{A'R}), \quad \text{(C-25)} \]
with
\[ Z_{A'R} = \int \frac{d^2 \gamma}{\pi} |gk\gamma\rangle \langle gk\gamma| \otimes |\gamma\rangle \langle \gamma|. \quad \text{(C-26)} \]

Note that we have the relation
\[ \text{Tr}[O_{A'R} \rho] = \text{Tr}[(N_\nu \otimes Z_R)(Z_{A'R}) \rho] \]
\[ = \text{Tr}[Z_{A'R} (N_\nu \otimes Z_R) (\rho)], \quad \text{(C-27)} \]
valid for every \( \rho \). Operationally, this means that the measurement of the observable \( O_{A'R} \) can be realized by first applying the Gaussian channel \( N_\nu \) and then measuring the observable \( Z_{A'R} \). Also, note that the observable \( Z_{A'R} \) has the same form of the observable \( O_{A'R} \) in Eq. (C-8), with the only difference that \( 1/\sqrt{\lambda + 1} \) is now replaced by \( k \). Hence, we know that it can be measured by performing a two-mode squeezing operation on modes \( A' \) and \( R \), discarding one of the modes, and measuring the single-mode Gaussian observable \( G_\theta \) [Eq. (C-13)] on the other mode. Putting everything together, and using the homodyne realization of the observable \( G_\theta \), we obtain the Gaussian setup shown in Fig. [1]

**FIG. 4. Canonical test for the amplification/purification of noisy coherent states.** The two input modes \( A \) and \( B \) are prepared in a two-mode squeezed vacuum state (TMSV), with suitably chosen squeezing parameter. Then, system \( A \) is input into the black box \( C \). Once the black box has acted, the output mode \( A' \) is sent through the noisy channel \( N_\nu \), the output of which is sent through a two-mode squeezer. Then, one mode is discarded and the other is sent through a 50-50 beamsplitter, after which the two quadratures \( X \) and \( P \) are measured.

**Appendix D: Test for the complex conjugation of (noisy) coherent states**

In this section we design a test for the complex conjugation of coherent states, and for various combinations of this task with the tasks of storage, teleportation, amplification, attenuation, cloning, and purification. As in the previous section, all the tasks in question can be subsumed into a single task, where the experimenter has to transform a displaced thermal state \( \rho_{\alpha,\mu} \) into the pure coherent state \( |g\alpha\rangle \).

The figure of merit is the average fidelity
\[ F = \int \frac{d^2 \alpha}{\pi} \lambda e^{-\lambda |\alpha|^2} \langle g\alpha| \mathcal{C}(\rho_{\alpha,\mu}) |g\alpha\rangle, \quad \text{(D-1)} \]

where \( \mathcal{C} \) is the channel used by the experimenter, and the performance operator of the fidelity test is
\[ \Omega = \int \frac{d^2 \alpha}{\pi} \lambda e^{-\lambda |\alpha|^2} \rho_{\alpha,\mu} \otimes |g\alpha\rangle \langle g\alpha| \quad \text{(D-2)} \]

For the input state, we choose the same two-mode squeezed state of the previous section. Let us construct now the observable \( O_{A'R} \), using Eq. (8) of the main text.
Here, we have

\[ O_{A'R} = (I_A' \otimes \tau_R^{-1/2} T_{AR}^T) \Omega_{A'A}^{T_{AR}} (I_A' \otimes T_{AR} \tau_R^{-1/2}) \]

\[ = \int \frac{d^2 \alpha}{\pi} \lambda e^{-\lambda |\alpha|^2} |g\alpha\rangle \langle g\alpha| \otimes (\tau_R^{-1/2} \rho_{\alpha,\mu} \tau_R^{-1/2}) \]

\[ = \int \frac{d^2 \alpha}{\pi} \int \frac{d^2 \beta}{\pi} \lambda e^{-\lambda |\alpha|^2} \mu e^{-\mu |\beta|^2} \]

\[ \times |g\alpha\rangle \langle g\alpha| \otimes (\tau_R^{-1/2} |\alpha + \beta\rangle \langle \alpha + \beta| \tau_R^{-1/2}) \]

\[ = \int \frac{d^2 \alpha}{\pi} \int \frac{d^2 \beta}{\pi} \lambda e^{-\lambda |\alpha|^2} \mu e^{-\mu |\beta|^2} \]

\[ \times |g\alpha\rangle \langle g\alpha| \otimes \left( \frac{\alpha + \beta}{\sqrt{x}} \right) \left( \frac{\alpha + \beta}{\sqrt{x}} \right) \]

\[ = \int \frac{d^2 \alpha}{\pi} \int \frac{d^2 \beta}{\pi} \lambda e^{-\lambda |\alpha|^2} \mu e^{-\mu |\beta|^2} \]

\[ \times |g\alpha\rangle \langle g\alpha| \otimes |\gamma\rangle \langle \gamma| \]

where \( U_{gk} \) is a suitable beamsplitter operator. Using this relation, we obtain

\[ U_{gk} Z_{A'R} U_{gk}^\dagger \]

\[ = \int \frac{d^2 \gamma}{\pi} \left( \sqrt{g^2 k^2 + 1} \gamma \right) \left( \sqrt{g^2 k^2 + 1} \gamma \right) \otimes |0\rangle \langle 0| \]

\[ = \frac{1}{g^2 k^2 + 1} \left( I \otimes |0\rangle \langle 0| \right). \]  

In summary, we constructed a procedure that allows us to experimentally measure the average fidelity \( \mathcal{F}_{A'} \) through the following steps

1. Prepare the two-mode squeezed state \( |\Psi_x\rangle_{AR} \) with parameter \( x = (\lambda + \mu)/(\lambda + \mu + \lambda \mu) \).
2. Apply the channel \( C \) on the input mode \( A \).
3. Apply the noisy channel \( \mathcal{N}_\nu \) \( \{ \text{Eq. (C-23)} \} \) to the output mode \( A' \).
4. Let modes \( A' \) and \( R \) go through a beamsplitter described by the unitary operator \( U_{gk} \), with \( k \) as in Eq. (D-4).
5. Discard mode \( A' \) and send \( R \) to photodetector.
6. If no photon is detected, assign score 1/\((g^2 k^2 + 1)\).

If one or more photons are detected, assign score 0. By construction, the expected frequency of the no detection events, divided by \( g^2 k^2 + 1 \), is equal to the average fidelity of Eq. (C-2). The procedure is illustrated in Figure 5.

Note that the third step (application of the channel \( \mathcal{N}_\nu \)) is trivial for pure input states. This is because the case of pure input states corresponds to the limit \( \mu \to \infty \), in which case Eq. (C-23) yields \( \nu \to \infty \) and \( \mathcal{N}_\nu \to I_A' \). The resulting setup is illustrated in Fig. 5.
Appendix E: Proof of Theorem 3

To prove Theorem 3, we have to show that, given a test $T$ for deterministic devices, one can construct a new test $T'$ for probabilistic devices, with the following properties:

1. $T'$ has the same performance operator as the original test $T$.
2. The benchmark for $T'$ is independent of the success probability of the tested device.

In the proof, we restrict our attention to tests represented by a performance operator $\Omega$ with positive partial transpose (PPT), cf. [52, 53]. This can be done without loss of generality, as long as the performance operator $\Omega$ corresponds to a test where the experimenter prepares an input state $\sigma$ and measures a bounded observable $O$. In this case, one can always replace the original observable by the positive observable $O' = O + \|O\|_\infty I$, so that the resulting operator $\Omega'$ has PPT, as one can verify from the definition (A-3) using simple algebra.

For PPT performance operators, we have two general expressions for the deterministic and probabilistic benchmark:

**Lemma 1.** For a test with PPT performance operator $\Omega$, the maximum of $\text{Tr}[C \Omega]$ over all $C$’s that are Jamiołkowski operators of measure-and-prepare channels is

$$S_{\text{M&P}}^{(\text{det})} = \inf_{\sigma_A} \Lambda^\otimes \left( I_{A'} \otimes \tau_A^{-1/2} \right) \Omega \left( I_{A'} \otimes \tau_A^{-1/2} \right),$$

(E-1)

where the infimum is taken over all states $\tau_A$ such that $I_{A'} \otimes \tau_A$ is invertible on the support of $\Omega$, and $\Lambda^\otimes$ is the product numerical range [54, 55], defined as

$$\Lambda^\otimes(O) = \sup_{|\alpha\rangle, \langle\alpha|} \langle\alpha| O |\alpha\rangle |\alpha\rangle, \quad \text{supremum being over all unit vectors } |\alpha\rangle \text{ and } |\alpha\rangle.$$  

(E-2)

The two lemmas are proven in the following subsections. Here we show how they can be used to prove Theorem 3.

**Proof of Theorem 3.** Let us see how to construct the test $T'$. Let $\tau_{\text{min}}$ be the state that minimizes the right hand side of Eq. (E-1). Then, the pair $(\Omega, \tau_{\text{min}})$ defines an equivalence class of tests for probabilistic devices, with the equivalence relation defined in the main text (i.e. two tests are equivalent if they give the same score and the same probability of success for all quantum operations). Now, consider the canonical test in this class, as defined by Theorem 2 in the main text. This is the desired test $T'$: by construction, $T'$ has performance operator $\Omega$, which is the performance operator of $T$. In addition, Eqs. (E-1) and (E-3) imply that, for $T'$, the benchmark for arbitrary probabilistic devices (with arbitrarily small probability of success) coincides the benchmark for deterministic devices. Hence, the benchmark for $T'$ is independent of the probability of success.

In summary, the test $T'$ sets a single threshold, independent of the success probability of the tested device. In the main text, we called a test with this property a fully black box test. Fully black box tests allow us to detect quantum advantages without knowing what is the probability that the tested device produces an output.

The proof of Theorem 3 gives us a complete characterization of the fully black box tests:

**Corollary 1.** Let $T'$ be a test for probabilistic devices, with PPT performance operator $\Omega$ and marginal input state $\sigma_A$. The test $T'$ is fully black box if and only if $\sigma_A$ is equal to $\tau_{\text{min}}$, where $\tau_{\text{min}}$ is the minimizer of the function

$$f(\tau) = \Lambda^\otimes \left( I_{A'} \otimes \tau_A^{-1/2} \right) \Omega \left( I_{A'} \otimes \tau_A^{-1/2} \right).$$

(E-4)

In the following sections, we give the proof of Lemmas 1 and 2.
Proof of Lemma 1

We have to compute the maximum of $\text{Tr}[C \Omega]$ over all Jamiołkowski operators $C$ corresponding to measure-and-prepare channels. To this purpose, we use the technique developed in the proof of Theorem 1 in [23]. Since $C$ is a channel, the operator $C$ must satisfy the condition $\text{Tr}_A[C] = I_A[56]$. Since in addition the channel $C$ is measure-and-prepare, the operator $C$ must be positive and separable [56]. To handle the separability condition, we use the following property:

**Proposition 1.** [57] A positive operator $C$ on $\mathcal{H}_{A'} \otimes \mathcal{H}_A$ is separable if and only if for every $n \in \mathbb{N}$ there exists a positive operator $C_n$ on $\mathcal{H}_{A'}^n \otimes \mathcal{H}_A$ s.t.

1. $C_n$ is an extension of $C$, namely $\text{Tr}_{n-1}[C_n] = C$, where $\text{Tr}_{n-1}$ denotes partial trace over first $n - 1$ copies of $\mathcal{H}_{A'}$;

2. $C_n$ is symmetric on $\mathcal{H}_{A'}^n$, namely $(\Pi_n \otimes I_A)C_n = C_n$, where $\Pi_n$ is the permutation-twirling operator $\Pi_n = \frac{1}{n!} \sum_{\pi \in S_n} U_{\pi} \rho U_{\pi}^T$ where $U_{\pi}$ is a unitary operator that permutes the $n$ modes of $\mathcal{H}_{A'}$ according to $\pi$.

Using the above result, the supremum over all Jamiołkowski operators of measure-and-prepare channels can be written as

$$S_{\text{M&P}}^{(\text{det})} = \sup_{C} \text{Tr}[C \Omega]$$

$$= \inf_{n \in \mathbb{N}} \sup_{C_n} \text{Tr} \left[ \text{Tr}_{n-1}[C_n] \Omega \right]$$

$$= \inf_{n \in \mathbb{N}} \sup_{C_n : C_n \geq 0} \text{Tr} \left[ C_n \left( I_{A'}^{\otimes (n-1)} \otimes I_A \right) \right]$$

$$= \inf_{n \in \mathbb{N}} \sup_{C_n : C_n \geq 0} \text{Tr} \left[ C_n \left( I_{A'}^{\otimes (n-1)} \otimes I_A \right) \right]$$

where $C_n'$ is a generic (not necessarily permutationally invariant) operator.

Now, the optimization over $C_n'$ is a semidefinite program. Its optimal value is equal to the optimal value of the dual program

$$\inf \text{Tr}[\Lambda_n]$$

s.t. $(I_{A'}^{\otimes n} \otimes \Lambda_n) \geq (\Pi_n \otimes I_A)(I_{n-1} \otimes \Omega)$.

Note that, since $\Lambda_n$ is an arbitrary positive operator acting on system $A$, and since $\Omega$ has PPT, the dual program can be equivalently written as

$$\inf \text{Tr}[\Lambda_n]$$

s.t. $(I_{A'}^{\otimes n} \otimes \Lambda_n) \geq (\Pi_n \otimes I_A)(I_{n-1} \otimes \Omega)$.

with the advantage that now the operator $\Omega$ is positive. Hence, we can express the quantum benchmark as

$$S_{\text{M&P}}^{(\text{det})} = \inf_{n \in \mathbb{N}} \text{Tr}[\Lambda_n].$$

We also observe that, since $\Omega$ is positive, the condition

$$(I_{A'}^{\otimes n} \otimes \Lambda_n) \geq (\Pi_n \otimes I_A)(I_{n-1} \otimes \Omega)$$

guarantees that the operator on the left hand side is invertible on the support of the operator on the right hand side. Note also that the operator $\Lambda_n$ must be nonnegative and therefore it can be written as

$$\Lambda_n = \lambda_n \tau_n,$$

where $\tau_n$ is a density operator on the Hilbert space $\mathcal{H}_A$ and $\lambda_n$ is a non-negative constant.

Under this fact, we can rewrite the average score as

$$S_{\text{M&P}}^{(\text{det})} = \inf_{n \in \mathbb{N}} \text{Tr}[\Lambda_n]$$

$$= \inf_{n \in \mathbb{N}} \inf_{\tau_n > 0, \text{Tr}[\tau_n] = 1} \text{inf} \left\{ \lambda_n : (I_{A'}^{\otimes n} \otimes \tau_n) \geq (\Pi_n \otimes I_A)(I_{n-1} \otimes \Omega) \right\}$$

$$= \inf_{n \in \mathbb{N}} \inf_{\tau_n > 0, \text{Tr}[\tau_n] = 1} \left\{ \lambda_n : (I_{A'}^{\otimes n} \otimes \Pi_n \otimes I_A)(I_{n-1} \otimes \Omega) \right\}.$$

Note that the infimum over $\lambda_n$ in Eq. (E-13) is the operator norm of the operator $(\Pi_n \otimes I_A)(I_{n-1} \otimes \Omega)$. Hence, we can rewrite the average score as

$$S_{\text{M&P}}^{(\text{det})} = \inf_{n \in \mathbb{N}} \text{Tr}[\Lambda_n]$$

$$= \inf_{n \in \mathbb{N}} \inf_{\tau_n > 0, \text{Tr}[\tau_n] = 1} \left\{ \lambda_n : (I_{A'}^{\otimes n} \otimes \Pi_n \otimes \tau_n) \right\}.$$

Note also that $\tau_n$ is a generic density operator on the Hilbert space $\mathcal{H}_A$, and therefore the dependence on $n$ can be removed: one has

$$S_{\text{M&P}}^{(\text{det})} = \inf_{\tau > 0, \text{Tr}[\tau] = 1} \left\{ \lambda : (I_{A'}^{\otimes n} \otimes \Pi_n \otimes \tau) \right\}.$$
where \( \tau \) is a generic density operator on the Hilbert space \( \mathcal{H}_A \). Continuing the chain of equalities, we get

\[
S_{M\&P}^{(\text{det})} = \inf_{n \in \mathbb{N}} \inf_{\tau > 0} \frac{\| (\Pi_n \otimes I_A) (I_{n-1} \otimes \Omega_{I_{\tau}}^T) \|_{\infty}}{\text{Tr}[\tau]} = \inf_{n \in \mathbb{N}} \inf_{\tau > 0} \frac{\| (\Pi_n \otimes I_A) (I_{n-1} \otimes \Omega_{I_{\tau}}^T) \|_{\infty}}{\text{Tr}[\tau]} = \inf_{n \in \mathbb{N}} \inf_{\tau > 0} \text{Tr}[\rho_n (\Pi_n \otimes I_A) (I_{n-1} \otimes \Omega_{I_{\tau}}^T)]
\]

This concludes the proof of Lemma 1.

Continuing the chain of equalities, we get

\[
\tau > 0 \quad \inf_{n \in \mathbb{N}} \sup_{\rho_n \geq 0} \text{Tr}[\rho_n (\Pi_n \otimes I_A) (I_{n-1} \otimes \Omega_{I_{\tau}}^T)] = 1
\]

This concludes the proof of Lemma 1.

**Proof of Lemma 2**

For any probabilistic measure-and-prepare device \( C \) with Jamiołkowski operator \( C \), we define

\[
\Gamma = I_{A'} \otimes \sigma_A \\
\rho = \frac{\Gamma^{1/2} C \Gamma^{1/2}}{\text{Tr}[\Gamma^{1/2} C \Gamma^{1/2}]} \\
\Omega_{\sigma_A} = \Gamma^{-1/2} \Omega \Gamma^{-1/2}.
\]

Note that \( \rho \) is a quantum state. Then, we write the score of \( C \) as

\[
S_{M\&P}^{(\text{det})} = \frac{\text{Tr}[C \Omega]}{\text{Tr}[C T]} = \text{Tr}[\Gamma^{-1/2} \rho \Gamma^{-1/2} \Omega] = \text{Tr}[\rho \Omega_{\sigma_A}] \leq \sup_{\rho \text{ separable}} \text{Tr}[\rho \Omega_{\sigma_A}] = \Lambda^{\otimes} (\Omega_{\sigma_A})
\]

The last inequality holds because for classical \( C \), \( C \) is separable, and thus \( \rho \) is separable, too.

Eq. \((E-18)\) gives an upper bound on the benchmark \( S_{M\&P}^{(\text{det})} \). It remains to prove that, in fact, the upper bound holds with the equality sign. To this purpose, pick the unit vectors \(|\psi\rangle \in \mathcal{H}_A \) and \(|\phi\rangle \in \mathcal{H}_A \) such that

\[
\sup_{\rho \text{ separable}} \Lambda^{\otimes} (\Omega_{\sigma_A}) = \langle \psi | \langle \phi | \Omega_{\sigma_A} | \psi \rangle | \phi \rangle.
\]

Let

\[
Q = \frac{\sigma_A^{-1/2} |\phi\rangle \langle \phi | \sigma_A^{-1/2}}{\text{Tr}[\sigma_A^{-1/2} |\phi\rangle \langle \phi | \sigma_A^{-1/2}]},
\]

and we pick the Jamiołkowski operator as

\[
C = |\psi\rangle \langle \psi | \otimes Q
\]

where \( C \) correspond to a probabilistic measure-and-prepare device that performs the projective measurement POVM \( \{Q, I - Q\} \), and, if the outcome corresponds to the projector \( Q \), outputs the state \(|\psi\rangle \langle \psi | \).

By construction, the probabilistic measure-and-prepare device with Jamiołkowski operator \( C \) achieves score

\[
\text{Tr}[C \Omega] = \frac{\text{Tr}[\langle \psi | \otimes \sigma_A^{-1/2} |\phi\rangle \langle \phi | \sigma_A^{-1/2}] \Omega]}{\text{Tr}[\langle \psi | \otimes \sigma_A^{-1/2} |\phi\rangle \langle \phi | \sigma_A^{-1/2} (I_{A'} \otimes \sigma_A)]} = \text{Tr}[\langle \psi | \otimes |\phi\rangle \langle \phi | \Omega_{\sigma_A}] = \Lambda^{\otimes} (\Omega_{\sigma_A})
\]

This concludes the proof of Lemma 2.

**Appendix F: Tests with symmetry**

In this section we consider tests that exhibit a symmetry with respect to a group of physical transformations. We also provide further examples of fully black box tests, where the value of the benchmark is independent of the probability of success of the tested setup.
Definitions and examples

In the following we will assume that a certain group of physical transformations $G$ acts on the systems $A$ and $A'$. For example, $A$ and $A'$ could be qubits and the group $G$ could be the group of rotations of the Bloch sphere, or the group of rotations around the $z$ axis. More generally, $A$ and $A'$ could be two systems of different dimension. We denote by $(U_g)_{g \in G}$ and $(U'_g)_{g \in G}$ the two unitary (projective) representations of $G$ acting on the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_{A'}$, respectively.

Using the above notation, we define what it means for a test to have symmetry:

Definition 1. (Covariant tests for deterministic devices) Let $T$ be a test for deterministic devices and let $\Omega \in \mathcal{H}_{A'} \otimes \mathcal{H}_A$ be the corresponding performance operator. We say that the test $T$ is covariant with respect to the action of $G$ iff the performance operator $\Omega$ satisfies the condition

$$[\Omega, U_g \otimes U'_g] = 0 \quad \forall g \in G. \quad (F-1)$$

When the group $G$ is compact, an example of covariant test is the fidelity test for the transformation $\rho_g \rightarrow |\psi_g\rangle \langle \psi_g|$, where $g$ is chosen at random according to the normalized Haar measure $dg$ and the states are defined as

$$\rho_g := U_g \rho U_g^\dagger, \quad \text{and} \quad |\psi_g\rangle := U'_g |\psi\rangle, \quad (F-2)$$

$\rho$ being a fixed density matrix and $|\psi\rangle$ being a fixed unit vector. In this case, the performance operator is

$$\Omega = \int d g \ |\psi_g\rangle \langle \psi_g| \otimes \rho_g \quad (F-3)$$

and satisfies Eq. (F-1) due to the invariance of the Haar measure.

Two examples of tests with symmetry are presented in the following:

1. Fidelity test for the teleportation of pure states. Consider the task of transmitting a generic pure state $|\psi\rangle$ of a $d$-dimensional system. The fidelity test for the teleportation of pure states [3, 40] has performance operator

$$\Omega = \int d \psi \ |\psi\rangle \langle \psi| \otimes |\psi\rangle \langle \psi| = \frac{P_+}{\text{Tr}[P_+]}, \quad (F-4)$$

where $P_+$ is the projector on the symmetric subspace, and $d \psi$ is the normalized invariant measure on the pure states. In this case, the performance operator satisfies the condition $[\Omega, U \otimes U] = 0$ for arbitrary unitary gates.

2. CHSH test. It is important to stress that a test can be covariant even if its is not testing a transformation of the form $\rho_g \rightarrow |\psi_g\rangle \langle \psi_g|$. As an example, consider the following entanglement-based test, designed to test the preservation of a Bell inequality:

(a) prepare an input qubit and a reference qubit in the entangled state $|\Phi^+\rangle = (|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)/\sqrt{2}$

(b) send the input qubit through the tested device

(c) test the CHSH inequality on the output qubit and the reference.

The CHSH test corresponds to the two-qubit observable

$$O = Z \otimes \frac{Z + X}{\sqrt{2}} + Z \otimes \frac{Z - X}{\sqrt{2}} + X \otimes \frac{Z + X}{\sqrt{2}} - X \otimes \frac{Z - X}{\sqrt{2}}$$

$$= \sqrt{2} \left( \frac{Z + X}{\sqrt{2}} \otimes \frac{Z + X}{\sqrt{2}} + \frac{Z - X}{\sqrt{2}} \otimes \frac{Z - X}{\sqrt{2}} \right). \quad (F-5)$$

Using Eq. (A-3), one obtains the performance operator

$$\Omega = \frac{O}{2}, \quad (F-6)$$

where the operator $O$ on the right hand side is interpreted as acting on qubits $A'$ and $A$. It is easy to see that the performance operator satisfies the commutation relations

$$[\Omega, X \otimes X] = 0, \quad [\Omega, Y \otimes Y] = 0, \quad [\Omega, Z \otimes Z] = 0,$$

meaning that the test is covariant with respect to the action of the Pauli group.

The definition of covariant tests, formulated in the deterministic case, can be extended to the probabilistic case:

Definition 2. (Covariant tests for probabilistic devices) Let $T$ be a test for probabilistic devices, let $\Omega$ be the performance operator, and let $\sigma_A$ be the marginal input state on system $A$. We say that the test $T$ is covariant with respect to the action of $G$ iff the performance operator $\Omega$ satisfies the condition Eq. (F-7) and the marginal state $\sigma_A$ satisfies the condition

$$[\sigma_A, U_g] = 0. \quad (F-7)$$

For a compact group $G$, a fidelity test for the transformation $\rho_g \rightarrow |\psi_g\rangle \langle \psi_g|$ (with $g$ chosen according to the Haar measure) is covariant. This is because the marginal input state is

$$\sigma_A = \int d g \rho_g, \quad (F-8)$$
which obviously satisfies the relation $[\sigma_A, U_g] = 0$. In the teleportation example, the marginal input state is
\begin{equation}
\sigma_A = \frac{I}{d},
\end{equation}
where $I$ is the identity matrix on $\mathcal{H}_A$ and $d$ is the dimension of $\mathcal{H}_A$. Clearly, the state $\sigma_A$ commutes with every unitary operator.

Again, it is important to stress that there are other examples of covariant tests other than the tests for transformations of the form $\rho_0 \rightarrow \langle \psi | \psi \rangle \rho_0$. For example, the CHSH test, viewed as a test on probabilistic devices, is also covariant (with respect to the Pauli group): indeed, the marginal input state is
\begin{equation}
\sigma_A = \text{Tr}_R(\langle \Phi^+ | \Phi^+ \rangle) = \frac{I}{2},
\end{equation}
and clearly commutes with the Pauli matrices $X, Y,$ and $Z$.

**Examples of fully black box tests**

Here we show a class of examples where it is easy to construct the fully black box test. In all these examples, the original test is covariant and the representation acting on the input system is irreducible. We recall that the representation $(U_g)_{g \in G}$ is called irreducible if no subspace is invariant under its action, except for the trivial subspace $\{0\}$ and the whole Hilbert space $\mathcal{H}_A$. The Schur’s lemma then guarantees that, for every operator $X$, the condition
\begin{equation}
[X, U_g] = 0 \quad \forall g \in G
\end{equation}
implies that $X$ has the form
\begin{equation}
X = c I_A,
\end{equation}
where $c \in \mathbb{C}$ is a suitable constant.

**Lemma 3.** Let $T$ be a test for deterministic devices with PPT performance operator $\Omega$. Suppose that the input system $A$ has dimension $d < \infty$. If $T$ is covariant under the action of $G$ and if and the group representation acting on $A$ is irreducible, then the deterministic benchmark is
\begin{equation}
S^{(\text{det})}_{\text{M&P}} = d \Lambda^\otimes (\Omega).
\end{equation}
Every test $T'$ with performance operator $\Omega$ and marginal input state $\sigma_A = I_A/d$ is a fully black box test.

**Proof.** To derive Eq. (F-13) we use the dual program
\begin{equation}
S^{(\text{det})}_{\text{M&P}} = \inf_{n \in \mathbb{N}} \inf_{\Lambda_n} (\langle I^\otimes_n \otimes \Lambda_n \rangle \geq \langle I_n \otimes \mathcal{I}_A \rangle (I_{n-1} \otimes \Omega^\otimes A)) \text{Tr}[\Lambda_n],
\end{equation}
coming from Eq. (E-8). Observe that the right hand side of the inequality on $\Lambda_n$ commutes with the group representation $(U_g^\otimes n \otimes U_g)_{g \in G}$. By twirling both sides of the inequality with respect to this representation, we obtain a new operator $\Lambda'_n$, which commutes with each $U_g$ and has the same trace of $\Lambda_n$. Hence, the benchmark can be rewritten as
\begin{equation}
S^{(\text{det})}_{\text{M&P}} = \inf_{n \in \mathbb{N}} \Lambda'_n : (I^\otimes_n \otimes \Lambda'_n) \geq \langle I_n \otimes \mathcal{I}_A \rangle (I_{n-1} \otimes \Omega^\otimes A) \text{Tr}[\Lambda'_n].
\end{equation}
Now, the Schur’s lemma implies that $\Lambda'_n$ is proportional to the identity matrix. We write it as $\Lambda'_n = \lambda_n I/A$. The rest follows by substituting $\tau$ with $I/d$ in the proof of Lemma 3. Following the steps of the proof we obtain
\begin{equation}
S^{(\text{det})}_{\text{M&P}} = \inf_{\tau > 0, \text{Tr}[\tau] = 1} \Lambda^\otimes (\Omega^\otimes A),
\end{equation}
with $\Omega^\otimes A = (I_A \otimes \tau^{-1/2}) \Omega^\otimes A (I_A \otimes \tau^{-1/2})$. Substituting $\tau = I/d$ we then obtain the desired expression
\begin{equation}
S^{(\text{det})}_{\text{M&P}} = d \Lambda^\otimes (\Omega^\otimes A) = d \Lambda^\otimes (\Omega).
\end{equation}
Now, note that, by construction, the infimum over $\tau$ is achieved by the maximally mixed state $\tau_{\text{min}} = I/d$. Hence, the characterization of Corollary 3 implies that the pair $(\Omega, I/d)$ defines a fully black box test. \qed

It is useful to illustrate the theorem in a few examples.

1. **Fidelity test for the teleportation of pure states.** Consider the fidelity test for the teleportation of arbitrary pure states in dimension $d$. As we have seen in the previous Section, this test has operators
\begin{equation}
\Omega = \frac{P_+}{\text{Tr}[P_+]} \quad \text{and} \quad \sigma_A = \frac{I}{d}
\end{equation}
and is covariant under the action of the group $U(d)$ of all unitary operators in dimension $d$. Note that the representation of the group on the input system is irreducible. Using Lemma 3 we obtain the benchmark
\begin{equation}
S^{(\text{det})}_{\text{M&P}} = d \Lambda^\otimes (\Omega) = d \sup_{||\alpha|| = 1, ||\beta|| = 1} \langle \alpha | \langle \beta | \Omega | \alpha \rangle | \beta \rangle = \frac{d}{\text{Tr}[P_+]} = \frac{2}{d+1}.
\end{equation}
This value coincides with the known fidelity benchmark for pure states [3, 46]. Since the state $\sigma_A$ is maximally mixed, we know that the fidelity test is fully black box, meaning that the benchmark $2/(d+1)$ holds independently of the probability of success of the tested device.
2. **The CHSH test.** Another interesting example of fully black box test is the entanglement-based CHSH test, also described in the previous Section. The CHSH test has operators

\[
\Omega = \frac{1}{\sqrt{2}} \left( \frac{Z + X}{\sqrt{2}} \otimes \frac{Z + X}{\sqrt{2}} + \frac{Z - X}{\sqrt{2}} \otimes \frac{Z - X}{\sqrt{2}} \right)
\]

\[
\sigma_A = \frac{I}{2}.
\]  

(F-18)

In this case, the test is covariant under the action of the Pauli group, which is irreducible on the input space. The performance operator is not PPT, but can be transformed into a PPT operator by adding a constant term proportional to \(I \otimes I\). Since this transformation only offsets the products numerical range by a constant, we still can use Lemma 3 to compute the benchmark, obtaining

\[
S_{\text{M&P}}^{(\text{det})} = 2 \Lambda \otimes (\Omega)
\]

\[
= 2 \sup_{\|\alpha\|=1,\|\beta\|=1} \text{Tr} \left[ \left( |\alpha\rangle \langle \alpha| \otimes |\beta\rangle \langle \beta| \right) \Omega \right],
\]

which can be evaluated explicitly using the Bloch representation

\[
|\alpha\rangle \langle \alpha| = \frac{I + m_x X + m_y Y + m_z Z}{2}
\]

\[
|\beta\rangle \langle \beta| = \frac{I + n_x X + n_y Y + n_z Z}{2},
\]  

(F-19)

\(m = (m_x, m_y, m_z)\) and \(n = (n_x, n_y, n_z)\) are unit vectors in \(\mathbb{R}^3\), and \(X, Y, Z\) are defined as follows:

\[
\begin{align*}
X &= \frac{Z + X}{\sqrt{2}}, & Y &= \frac{Z - X}{\sqrt{2}}, & Z &= Y.
\end{align*}
\]

(F-20)

Using this notation, the performance operator can be rewritten as

\[
\Omega = \frac{\tilde{X} \otimes \tilde{X} + \tilde{Z} \otimes \tilde{Z}}{\sqrt{2}}
\]

(F-21)

and one has

\[
S_{\text{M&P}}^{(\text{det})} = \sqrt{2} \Lambda \otimes (\tilde{X} \otimes \tilde{X} + \tilde{Z} \otimes \tilde{Z})
\]

\[
= \sqrt{2} \sup_{m,n} \{m_x n_x + m_z n_z\}
\]

\[
= \sqrt{2}.
\]  

(F-22)

Note that the benchmark is strictly smaller than the Bell inequality value, which is equal to 2. The reason is that here we are restricting the optimization over the set of two-qubit separable states, while the measurements are fixed. Lemma 3 guarantees that preparing a maximally entangled input state, letting the unknown device act, and measuring the CHSH observable on the output is a fully black box test: any experimental value above \(\sqrt{2}\) guarantees that the tested device has performance above the performance of every measure-and-prepare device, even allowing measure-and-prepare devices that postselect on some subset of favourable outcomes.

3. **Fidelity test for the teleportation of pure states on the equator of the Bloch sphere.**

Consider the set of qubit pure states

\[
|\phi_k\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle + e^{2\pi i k/N} |1\rangle \right),
\]

(F-23)

with \(k \in \{0, 1, \ldots, N - 1\}\) and \(N > 2\) (we exclude the trivial case \(N = 2\), in which the states are orthogonal and the teleportation task can be achieved perfectly by measuring the input state). The above states are generated by the action of the cyclic group \(C_N\) on the state \(|\phi_0\rangle = (|0\rangle + |1\rangle)/\sqrt{2}\). The group action represented by the unitary matrices

\[
U_k := e^{2\pi i k/N} Z.
\]

(F-24)

Consider the fidelity test for the teleportation of the states \(|\phi_0\rangle\). In such test, the verifier prepares a state \(|\phi_k\rangle\) chosen with uniform probability \(p_k = 1/N\), lets the tested device act, and finally measures the fidelity with the state \(|\phi_k\rangle\).

According to the general formula for fidelity tests, the performance operator

\[
\Omega = \frac{1}{N} \sum_k |\phi_k\rangle \langle \phi_k| \otimes |\phi_k\rangle \langle \phi_k|
\]

\[
= \frac{1}{4} \left( |0\rangle \langle 0| \otimes |0\rangle \langle 0| + |1\rangle \langle 1| \otimes |1\rangle \langle 1| + 2|\Psi^+\rangle \langle \Psi^+| \right),
\]

with \(|\Psi^+\rangle = (|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle)/\sqrt{2}\). On the other hand, the marginal input state reads

\[
\sigma_A = \frac{1}{N} \sum_k |\phi_k\rangle \langle \phi_k|
\]

\[
= \frac{I}{2}.
\]  

(F-25)

Here there is an interesting point to make. The test is based on a teleportation task, corresponding to a transformation of the form \(\rho_g \rightarrow |\psi_g\rangle \langle \psi_g|\), where \(g\) is an element of the cyclic group. As a consequence (see the previous Section), the test is covariant under the action of the cyclic group. But in fact, the symmetries of the test are even larger: Indeed, it is easy to check that one has

\[
[\Omega, X \otimes X] = [\Omega, Y \otimes Y] = [\Omega, Z \otimes Z] = 0,
\]

(F-26)
and

\[ [\sigma_A, X] = [\sigma_A, Y] = [\sigma_A, Z] = 0, \]

meaning that the test is covariant under the action of the Pauli group.

It is interesting to observe that for odd \( N \), the set of states (F-23) is not invariant under the action of the Pauli group. In other words, the symmetries of the test are larger than the symmetries of the original set of states that the test is designed to probe. This example illustrates the usefulness of our unified approach, in which the high-level structure of the benchmark (the operators \( \Omega \) and \( \sigma \)) reveals symmetries that are not visible at the level of the original task that was meant to perform.

It is also important to stress that the tests \( \Omega \) and \( \sigma_A \) are independent of \( N \). This means that tests with different numbers of input states are equivalent in terms of score and probability of success. In practice, this means that one can test the fidelity over all the pure states on the equator, by actually testing only the three states defined by Eq. (F-23) with \( N = 3 \). Alternatively, one can devise an equivalent test consisting in the preparation of the two-qubit maximally entangled state |\( \Phi^+ \rangle \rangle, followed by the measurement of the two-qubit observable

\[
O = 2 \Omega^T_A
\]

\[
= \frac{1}{2} \left( |1\rangle \langle 1| \otimes |0\rangle \langle 0| + |0\rangle \langle 0| \otimes |1\rangle \langle 1| + 2|\Phi^+\rangle \langle \Phi^+| \right)
\]

\[
= \frac{1}{2} \left( |\Psi^+\rangle \langle \Psi^+| + |\Psi^-\rangle \langle \Psi^-| + 2|\Phi^+\rangle \langle \Phi^+| \right),
\]

with |\( \Psi^- \rangle = (|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle) / \sqrt{2}. The expectation value of the observable \( O \) can be measured through a Bell measurement, or, indirectly, using the relation \( O = (I \otimes I + O_1 - O_2)/2 \), where \( O_1 \) and \( O_2 \) are the observables

\[
O_1 = |\Phi^+\rangle \langle \Phi^+| \quad \text{and} \quad O_2 = |\Phi^-\rangle \langle \Phi^+|,
\]

with |\( \Phi^- \rangle = (|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle) / \sqrt{2}. Using the above relation, the average fidelity over all pure states on the equator can be evaluated as

\[
F = \frac{1 + \langle O_1 \rangle - \langle O_2 \rangle}{2},
\]

where \( \langle O_1 \rangle \) and \( \langle O_2 \rangle \) are the expectation values of \( O_1 \) and \( O_2 \), respectively. Also in this case, we can see the advantage of a more high-level formulation of the problem, which allowed us to find a setup that measures the average fidelity over all pure states on the equator by actually performing Bell measurements.

Now, we have seen that the fidelity test for the states (F-23) is covariant under the action of the Pauli group, which is irreducible on the input system. Since the performance operator is PPT, Lemma 3 yields the benchmark expression

\[
S_{M&P}^{(\text{det})} = 2 \Lambda^{\otimes 2}(\Omega)
\]

\[
= 2 \sup_{\|\alpha\|=1,\|\beta\|=1} \text{Tr} \left[ \left( |\alpha\rangle \langle \alpha| \otimes |\beta\rangle \langle \beta| \right) \Omega \right],
\]

where the supremum can be evaluated using the Bloch representation

\[
|\alpha\rangle \langle \alpha| = \frac{I + m_x X + m_y Y + m_z Z}{2}
\]

\[
|\beta\rangle \langle \beta| = \frac{I + n_x X + n_y Y + n_z Z}{2}.
\]

Explicitly, we obtain

\[
\Lambda^{\otimes 2}(\Omega) = \sup_{m,n} \left\{ \frac{4 + 2(m_x n_x + m_y n_y)}{16} \right\}
\]

\[
= \frac{3}{8},
\]

so that the benchmark value is

\[
S_{M&P}^{(\text{det})} = \frac{3}{4}.
\]

Thanks to the symmetry of the problem, we know that the fidelity test is fully black box. Hence, every experimental fidelity above the classical threshold 3/4 indicates a quantum advantage over arbitrary measure-and-prepare strategies, even including strategies that postselect on a subset of outcomes with arbitrary small probability.

The optimal measure-and-prepare strategy

We have seen that tests that are covariant with respect to an irreducible representation are fully black box. In this case, it is also possible to find an explicit expression for the measure-and-prepare channel that achieves the benchmark.

**Lemma 4.** Let \( \mathcal{T} \) be a test for deterministic devices with PPT performance operator \( \Omega \). Suppose that the input system \( A \) has dimension \( d < \infty \). If \( \mathcal{T} \) is covariant under the action of \( G \) and if and the group representation acting on \( A \) is irreducible, then the optimal measure-and-prepare channel is

\[
C(\rho) = \int dg \text{Tr}[P_g \rho] |\psi_g\rangle \langle \psi_g|
\]
where \( \{ P_g \}_{g \in G} \) is the POVM defined by

\[
P_g = d U_g |\phi \rangle \langle \phi | U_g^\dagger,
\]

for some unit vector \( |\phi \rangle \), and the output states \( \{|\psi_g \rangle \}_{g \in G} \) have the form

\[
|\psi_g \rangle = U_g' |\psi \rangle,
\]

for another fixed unit vector \( |\psi \rangle \).

**Proof.** Consider a generic measure-and-prepare strategy with measurement of the POVM \( \{ P_i \} \) and preparation of the states \( \{ \rho_i \} \). The corresponding channel, denoted by \( C \), is

\[
C(\rho) = \sum_i \text{Tr}[P_i \rho] \rho_i,
\]

and its Jamiołkowski operator is

\[
C = \sum_i \rho_i \otimes P_i.
\]

Hence, the score is

\[
S^{(\text{det})}(C) := \sum_i \text{Tr}[(\rho_i \otimes P_i) \Omega]
\]

Without loss of generality, we assume that each operator \( P_i \) is rank-one, because one can always split a non-rank-one operator into the sum of rank-one operators, without affecting the total score. Likewise, we assume that each state \( \rho_i \) is pure, because one can always include the randomization of pure states by adding dummy outcomes to the measurement, and use these outcomes to randomize over pure states.

Now, since the test is covariant, we have the equality

\[
\Omega = \int d g (U_g' \otimes U_g) \Omega (U_g' \otimes U_g)^\dagger
\]

Inserting this relation into Eq. (F-38), we obtain

\[
S^{(\text{det})}(C) = \sum_i \text{Tr}[(\rho_i \otimes P_i) \int d g (U_g' \otimes U_g) \Omega (U_g' \otimes U_g)^\dagger]
\]

\[
= \sum_i \int d g \text{Tr}[(U_g' \rho_i U_g' \otimes U_g' \rho_i U_g) \Omega]
\]

\[
= \sum_i p_i \left( \int d g \text{Tr}[(\rho_g^{(i)} \otimes P_g^{(i)}) \Omega] \right),
\]

having defined

\[
p_i := \frac{\text{Tr}[P_i]}{d} \rho_g^{(i)} := U_g \rho_i U_g^\dagger
\]

\[
P_g^{(i)} := \frac{d}{\text{Tr}[P_i]} U_g P_i U_g^\dagger.
\]

Now, it is easy to check that

1. the numbers \( \{ p_i \} \) form a probability distribution

2. for every given \( i \), the operators \( \{ P_g^{(i)} \} \) form a POVM, normalized as

\[
\int d g P_g^{(i)} = I_A.
\]

For given \( i \), the POVM \( \{ P_g^{(i)} \} \) and the states \( \{ \rho_g^{(i)} \} \) define a measure-and-prepare channel \( C_i \), with

\[
C_i(\rho) = \int d g \text{Tr} [P_g^{(i)} \rho] \rho_g^{(i)}.
\]

With this notation, Eq. (F-40) can be rewritten as

\[
S^{(\text{det})}(C) = \sum_i p_i S^{(\text{det})}(C_i)
\]

\[
\leq \max_i S^{(\text{det})}(C_i).
\]

Hence, the maximum score must be achieved by a measure-and-prepare channel \( C_i \) of the form (F-43). Since by construction the POVM operators \( P_g^{(i)} \) are rank-one, and since the states \( \rho_g^{(i)} \) are pure, this concludes the proof. \( \square \)