Rotational dynamics in the ensemble of three coupled pendulums

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Abstract

We investigate the nonlinear dynamics of a chain of three coupled pendulums. We demonstrate that the system has two value ranges of the coupling strength, in which a synchronous rotation is not in-phase one. The existence of these modes is shown analytically. We prove that they emerge from the existing in-phase periodic rotation through a period-doubling bifurcation.

For the instability domains mentioned above, and at their intersection, emergence of chaotic dynamics, including chaotic chimera states, is also demonstrated.

I. INTRODUCTION

The study of collective dynamics in ensembles and networks of coupled oscillatory elements in various artificially created or natural objects is one of the hottest topics in modern nonlinear dynamics. It is important both for a theoretical understanding of complex processes and for a wide range of applications in different scientific fields: from physics and chemistry, earth sciences, biology and medicine, business and social sciences to a variety of engineering and technical disciplines. Synchronization is a general phenomenon of collective behavior. Synchronization is usually understood as the process of achieving by the coupled objects of a different nature the collective rhythm of functioning. Synchronization of two or three elements is possible [1–4], as well as of ensembles consisting of hundreds and thousands of elements [5, 6]. The system of coupled pendulums is one of the widely used models in multiple fields of science and technology. Despite the simplicity of this model, it adequately describes not only mechanical objects, but also various processes that occur in electrical circuits [7], semiconductor structures [8], molecular biology [9] etc. The model of coupled pendulums is also used for analyzing phase synchronization systems [10]. This model is often considered as the basis for the theoretical studies of coupled Josephson junctions [11, 12] and granular superconductors [13]. The analogy with coupled pendulums plays a crucial role in the analysis of the problem of rotational vibrations of nitrogenous bases of DNA molecules during the formation of so-called open state that is an essential part of molecule’s functioning [14]. Interest in these problems is caused by a large number of physical applications, where the Frenkel-Kontorova model and its modifications are applicable [8].

Cluster and chimera states in ensembles of various dimensions are of particular interest for the synchronization phenomena investigation. These states were observed in small ensembles e.g. in system of four coupled elements [14, 15]. The authors of the article [17] describe regular and chaotic chimera states in the system of three coupled phase oscillators with inertia. Experimentally these states have been detected in mechanical systems [18, 19] and in optoelectronic oscillator networks [20]. It is worse mentioning that the chaotic chimera states have been also detected numerically and semi-analytically in the resent article [21], where the system of four globally coupled Stuart-Landau oscillators is considered.

In this paper we examine singularities found in rotational dynamics of three nonlinearly coupled pendulums. We are interested in in-phase rotations and synchronous out-of-phase ones. In Section III A we describe the model, state the problem and demonstrate the main observed effect: in-phase periodic motion instability. Then, in Section III B we build an asymptotic theory, developed for an infinitely small dissipation, which explains instability of the in-phase limit rotation mode of the pendulums. Here we also find analytical formulas for the boundaries of the in-phase limit rotation mode instability interval regarding the coupling strength. During the nonlinear stage of this instability a periodic out-of-phase rotation emerges, in particular, a chimera state for which the phases of the two pendulums coincide, while the phase of the third pendulum differs from the rest. In Section III C numerical results that confirm our theoretical findings are presented. In addition to this, in Section III A bifurcations that lead to the appearance and disappearance of the out-of-phase limit rotation modes are analyzed. Bistability of the in-phase and out-of-phase limit periodic modes is established for the system under study. In Section III B scenario of chaotic rotational dynamics emergence is described, including chaotic chimera states. A summary of the main results can be found in Conclusion.

II. BISTABILITY OF PERIODIC ROTATIONS

A. Model. Problem statement. Main effect

Let us consider the chain of three coupled identical pendulums described by the following system of ODEs

\[
\begin{align*}
\dot{\varphi}_1 + \lambda \dot{\varphi}_1 + \sin \varphi_1 &= \gamma + K \sin(\varphi_2 - \varphi_1), \\
\dot{\varphi}_2 + \lambda \dot{\varphi}_2 + \sin \varphi_2 &= \gamma + K \left[ \sin(\varphi_1 - \varphi_2) + \sin(\varphi_3 - \varphi_2) \right], \\
\dot{\varphi}_3 + \lambda \dot{\varphi}_3 + \sin \varphi_3 &= \gamma + K \sin(\varphi_2 - \varphi_3).
\end{align*}
\]

(1)

Here \( \lambda \) is the damping coefficient responsible for all the dissipative processes in the system, \( \gamma \) is a constant external force identical for all pendulums, \( K \) characterizes the nonlinear coupling strength between the elements.
For certain relations between the parameters $\gamma$ and $K$ the system (1) demonstrates non-trivial behavior. First, it should be mentioned that when at any given time $t$ coordinates $\varphi_1(t), \varphi_2(t)$ and $\varphi_3(t)$ coincide, the system will demonstrate in-phase dynamics, i.e. $\varphi_1(t) = \varphi_2(t) = \varphi_3(t) = \phi(t)$. We shall denote such regime as $(3 : 0)$. All pendulums move synchronously and their dynamics is described by a single equation:

$$\ddot{\phi} + \lambda \dot{\phi} + \sin \phi = \gamma.$$  \hspace{1cm} (2)

The dynamics of this system is well studied [23]. Fig. 1 shows the parameter plane $(\lambda, \gamma)$. The line $\gamma = 1.0$ and the bifurcation curve $T$, which is also called the Tricomi curve [6], divide the plane into three domains $G_1, G_2$ and $G_3$. These domains correspond to different structurally stable phase portraits of the system (2). When $\lambda, \gamma$ are from domain $G_1$, the nonlinear pendulum has only two steady states (in cylindrical phase space): a saddle and a stable foci (node). For values of $\lambda, \gamma$ from $G_2$ there exist a stable 2$\pi$-periodic in $\phi$ motion and a stable foci (node), whose attraction domain is delimited by stable separatrices of a saddle. When $\lambda, \gamma$ are from $G_3$ there are no steady states in the system, and a stable rotational mode is established. We are interested in rotational dynamics of pendulums ensemble, i.e. parameters $\lambda, \gamma$ are from $G_2$ or $G_3$.

Figure 1. Bifurcation diagram of the $(\lambda, \gamma)$ parameter plane and structurally stable phase portraits of the system that obeys (2) for parameter values $\lambda = 0.2$, $\gamma = 0.3$ (right inset), $\lambda = 0.9$, $\gamma = 0.4$ (left inset) and $\lambda = 0.6$, $\gamma = 1.2$ (middle inset). Curve $T$ is the Tricomi bifurcation curve. Its crossing from $G_1$ to $G_2$ leads to a destruction of the saddle-node invariant curve and a limit 2$\pi$-periodic in $\phi$ mode origination which is marked in red in the left and middle insets.

It is obvious that the system (1) has an in-phase rotation motion $\phi(t)$. We have found that for certain parameter values the instability of this motion can be observed. Let us demonstrate this for some fixed parameters $\lambda = 0.4, \gamma = 0.97, K = 1.5$ under very close initial conditions $\varphi_1(0) = 5.0, \varphi_2(0) = 5.00001, \varphi_3(0) = 5.00002, \dot{\varphi}_1(0) = 0.0, \dot{\varphi}_2 = 0.0, \dot{\varphi}_3(0) = 0.0$. As can be seen from Fig. 2 for $20 \leq t \leq 45$ the general velocities $\dot{\varphi}_{1,2,3}$ practically coincide. From the second part of Fig. 2 when $2100 \leq t \leq 2125$, one can already see asynchrony in the oscillations of the pendulums, i.e. the instability of their synchronous rotation mode has developed. The difference between $\dot{\varphi}_{1,2,3}$ is quite noticeable. A new type of limit rotations develops when $7000 \leq t \leq 7025$ with $\dot{\varphi}_{1,2,3}$ changing out-of-phase. Thus, when the coupling parameter $K$ reaches some values, the instability of the synchronous periodic motion develops: a new $4\pi$-periodic limit rotations emerge in the ensemble of pendulums. This new motion is characterized by out-of-phase rotations with two times larger period than the synchronous periodic rotation has, so a period-doubling bifurcation takes place here. It is worth that this effect takes place for the system of only two elements and is considered in our previous article [22]. However, as it will be shown below, the system of three coupled pendulums demonstrates more complex and interesting dynamics, which can be interpreted as chaotic chimera states by analogy with articles [17,19].

B. Parametric instability of the synchronous rotation mode

Let us investigate the case where a system has small dissipation (i.e. when $\lambda \ll 1$). Let us also assume $\gamma$ characterizing the external force to be close to 1. In this case, one can build an asymptotic theory that would explain the instability of the in-phase limit rotation regime of three coupled pendulums, a phenomenon observed in the system (1) undergoing forward numerical modeling. To develop an analytic approach for a small parameter $\lambda$ we introduce the formal smallness parameter $\varepsilon$, and $\varepsilon \ll \lambda \ll 1$.

Let us construct an asymptotic solution to Eq. (2) using the Lindstedt-Poincaré method [24], the essence of which is to introduce a new dimensionless time $\tau$, where $t = \omega \tau$ and

$$\omega = \sum_{j=0}^{\infty} \varepsilon^j \omega_j,$$  \hspace{1cm} (3)

is an unknown angular frequency of the sought-for solution allowing to avoid secular terms. Taking for simplicity $\phi(0) = 0$, we represent the solution in the form of the following asymptotic expansion:

$$\phi(\tau) = \tau + \sum_{j=0}^{\infty} \varepsilon^j \phi_j(\tau),$$  \hspace{1cm} (4)

where $\phi_j$ are $2\pi$-periodic functions of the variable $\tau$. By substituting Eqs. (3) and (4) into Eq. (2), expanding both
sides in powers of $\varepsilon$, equating the coefficients of the same powers of $\varepsilon$ and determining $\omega_j$ from the condition of absence of secular terms, we obtain the in-phase rotation solution of the system (1) in the next form:

$$\phi(\tau) = \tau + \frac{\lambda^2}{2\gamma^2} \sin(\tau) + O(\varepsilon^4), \quad (5)$$

where

$$\tau = \left[ \frac{\gamma}{\lambda} - \frac{1}{2} \left( \frac{\lambda}{\gamma} \right)^3 + O(\varepsilon^7) \right] t. \quad (6)$$

Let us find the stability conditions for the in-phase rotation mode. First linearize the system (1) around $\phi(t)$, then $\varphi_i(t) = \phi(t) + \delta\varphi_i(t), \ i = 1, 2, 3$. Next we get the corresponding equations for variations $\delta\varphi_{1,2,3}$:

$$\begin{align*}
\delta\ddot{\varphi}_1 + \lambda\delta\varphi_1 + \cos\phi(t)\delta\varphi_1 &= K(\delta\varphi_2 - \delta\varphi_1), \\
\delta\ddot{\varphi}_2 + \lambda\delta\varphi_2 + \cos\phi(t)\delta\varphi_2 &= K(\delta\varphi_1 - 2\delta\varphi_2 + \delta\varphi_3), \\
\delta\ddot{\varphi}_3 + \lambda\delta\varphi_3 + \cos\phi(t)\delta\varphi_3 &= K(\delta\varphi_2 - 2\delta\varphi_3).
\end{align*} \quad (7)$$

To continue with, we introduce two detuning variables $\xi_{ij} = \delta\varphi_i - \delta\varphi_j$. For $\xi_{12}$ and $\xi_{23}$ we obtain a closed system of equations

$$\begin{align*}
\ddot{\xi}_{12} + \lambda\dot{\xi}_{12} + \cos\phi(t)\xi_{12} &= K(-2\xi_{12} + \xi_{23}), \\
\ddot{\xi}_{23} + \lambda\dot{\xi}_{23} + \cos\phi(t)\xi_{23} &= K(\xi_{12} - 2\xi_{23}), \quad (8)
\end{align*}$$

which admits two simple solutions. First of them $\xi_{12} = \xi_{23}$ corresponds to the regime with pairwise different phases of the oscillators $\varphi_1(t) \neq \varphi_2(t) \neq \varphi_3(t)$. We shall denote this regime as $(1 : 1 : 1)$. Introducing for brevity $\xi = \xi_{12}$, we obtain equation

$$\ddot{\xi} + \lambda\dot{\xi} + (K + \cos\phi(t))\xi = 0. \quad (9)$$

This equation belongs to the Mathieu-type equation. Hence, the parametric instability effects can be observed for some values of the parameter $K$ depended on $\lambda$ and $\gamma$ (22). To find the boundaries of the instability domain of the in-phase rotation mode, we determine the coupling parameter $K$ values for which the Eq. (9) admits a solution with $2\pi$ period or, equivalently, with $\omega/2$ frequency.

Using some aspects of perturbation theory, taking results (5) and (6) and searching for a solution of Eq. (9) with $\omega/2$ frequency, we get boundaries $K_{1,2}$ for the first instability domain

$$K_{1,2} = 1 \left[ \frac{\gamma^2}{\lambda^2} + 2\sqrt{1 - \gamma^2} + \frac{1}{2} \frac{\lambda^2}{\gamma^2} \right] + O(\varepsilon^4). \quad (10)$$

Another solution $\xi_{12} = -\xi_{23}$ corresponds to the regime with $\varphi_1(t) = \varphi_3(t) \neq \varphi_2(t)$, then two oscillators form in-phase synchronous cluster and the third one rotates separately with some delay. It is regime $(2 : 1)$. Such behavior of the system is called a chimera (22). Introducing again $\xi = \xi_{12}$, we obtain equation for detuning $\xi$:

$$\ddot{\xi} + \lambda\dot{\xi} + (3K + \cos\phi(t))\xi = 0. \quad (11)$$

Similarly to the previously examined case, we get boundaries $K_{1,2}$ for the second instability domain

$$K_{1,2} = \frac{1}{12} \left[ \frac{\gamma^2}{\lambda^2} + 2\sqrt{1 - \gamma^2} + \frac{1}{2} \frac{\lambda^2}{\gamma^2} \right] + O(\varepsilon^4). \quad (12)$$

Thus, for a chain of three pendulums, there can exist two intervals of coupling strength $K$ values, corresponding to the regimes $(2 : 1)$ and $(1 : 1 : 1)$, for which in-phase periodic rotation becomes parametrically unstable.

### III. OUT-OF-PHASE ROTATIONAL DYNAMICS

Let us consider in detail the development of the instability of the in-phase synchronous regime. As a characteristic of the degree of synchronization, we consider the value $\Xi$, which is the frequency lag of pendulums:

$$\Xi = \frac{1}{3} \sum_{1 \leq i < j \leq 3} \max_{0 < t < T} |\dot{\varphi}_i(t) - \dot{\varphi}_j(t)|, \quad (13)$$

where $T$ is the period of rotational mode. It follows from the definition (13) that $\Xi$ takes non-negative values, and $\Xi = 0$ only in the case of in-phase mode. In the case of a out-phase regime, when there exists such a pair of pendulums that $\varphi_i \neq \varphi_j$, where $i$ and $j$ are the numbers of pendulums, $\Xi > 0$.

#### A. Regular dynamic

From the expressions (10) and (12) we see that in the case of small values of $\lambda$ for $\gamma \approx 1.0$, two regions of instability of the in-phase mode arise. Next we will investigate the case $\gamma = 0.97$. Let us consider the situation when the instability regions are separated from each other. Fig. 3 shows a bifurcation diagram obtained by numerical simulation of the system. The diagram shows the dependence of synchronism characteristics $\Xi$ from magnitude of the coupling strength $K$ at $\gamma = 0.97, \lambda = 0.4$. The horizon-

![Figure 3. Bifurcation diagram of synchronous rotational regimes of the system (1). Here and below: blue markers - stable regimes, red markers - unstable regimes. Lines without markers – $2\pi$-periodic regimes. Round markers – $4\pi$-periodic regimes. Parameters: $\gamma = 0.97, \lambda = 0.4$.](image-url)
tual segments $A_1$, $A_3$, $A_5$ correspond to the synchronous in-phase regime ($\Xi = 0$). There are two regions $A_2$ and $A_4$ of the values of the parameter $K$, when this regime becomes unstable. As shown above, in the course of the asymptotic consideration (the expressions (10) and (12)), it is for the values of the coupling parameter $K$ that the parametric instability of the in-phase periodic motion develops from these intervals.

Let us consider processes occurring in a chain when $K$ takes values from the $A_2$ and when $K$ escapes from it. As the parameter $K$ increases, the in-phase periodic motion undergoes period-doubling bifurcation ($K \approx 0.4505$), while from the stable in-phase $2\pi$-periodic in $\varphi = (\varphi_1, \varphi_2, \varphi_3)^T$ of motion, a stable $4\pi$-periodic motion in $\varphi$ corresponding to the synchronous regime of dynamics is generated when $\varphi_1(t) = \varphi_3(t) \neq \varphi_2(t) (2 : 1)$, and $2\pi$-periodic motion loses stability. The branch $B_1$ corresponds to this regime on the bifurcation diagram. In addition to stable periodic motions, there is also an unstable out-of-phase $4\pi$-periodic motion in $\varphi$ (branch $B_2$), which is generated from an unstable $2\pi$-periodic motion as a result of a subcritical period doubling bifurcation ($K \approx 0.5435$) with increasing $K$. Further, for $K \approx 0.6145$, the stable (branch $B_1$) and the unstable (branch $B_2$) out-of-phase periodic motions merge and disappear as a result of the saddle-node bifurcation.

Similarly, for the instability zone $A_4$. As the parameter $K$ decreases as a result of period doubling bifurcation ($K \approx 1.6305$), the in-phase periodic motion loses stability and $4\pi$-periodic in $\phi$ motion occurs ($B_4$ branch), corresponding to a completely out-of-phase regime $\varphi_1(t) \neq \varphi_2(t) \neq \varphi_3(t) (1 : 1 : 1)$. For $K \approx 1.327$, the stable $4\pi$-periodic motion (branch $B_4$) merges with the $4\pi$-periodic unstable motion (branch $B_3$) as a result of the saddle-node bifurcation. An unstable $4\pi$-periodic motion (the $B_3$ branch) arises as $K$ decreases from an unstable $2\pi$-periodic in-phase motion ($A_4$ domain) at the subcritical period doubling bifurcation ($K \approx 1.3505$).

Thus, when the in-phase mode is unstable, out-of-phase $4\pi$-periodic ($2 : 1$) and ($1 : 1 : 1$) regimes are realized in the system.

**B. Chaotic dynamics**

As the dissipation parameter increases, the regions of instability of the in-phase regime approach each other. At the same time, chaotic dynamics is possible in the chain of three pendulums. In the paragraph (III), we considered the case of coexistence of two regions of instability of the in-phase regime with the parameter $\gamma = 0.97$, $\lambda = 0.4$ was described, and with the loss of stability of the in-phase regime, only $4\pi$-periodic out-of-phase regimes were appeared. Let us now investigate the case $\gamma = 0.97$, $\lambda = 0.7$. Fig. (a) shows the bifurcation diagram of $2\pi$- and $4\pi$-periodic regimes. Here segments $A_1$, $A_2$, $A_3$, $A_4$, $A_5$ and branches $B_1$, $B_2$, $B_3$, $B_4$ correspond to regimes similar to cases $\gamma = 0.97$, $\lambda = 0.4$. Let us consider the stable $4\pi$-periodic ($1:1:1$) regime (branch $B_4$). As the $K$ decreases, the $4\pi$-periodic motion loses stability through the pitch-fork bifurcation ($K \approx 0.5933$), and from it two stable $4\pi$-periodic motions arise, which in Fig. (b) the branch $B_8$ corresponds to, and the unstable $4\pi$-periodic motion (branch $B_8$). Further, with decreasing values of the coupling parameter $K$, a sequence of period doubling bifurcations occurs (Fig. (d)), which results in a transition to chaotic dynamics. Fig. (c) shows several first bifurcations in this sequence at $K \approx 0.5785$, $K \approx 0.5729$, when $8\pi$- and $16\pi$-periodic motions respectively are generated (branches $C_2$ and $D_2$). Branches $C_3$ and $D_3$ correspond to $8\pi$, and $16\pi$-periodic rotations that lost stability after period doubling bifurcations. Fig. (e) shows the largest Lyapunov exponent of the system, depending on the coupling strength $K$. It is to see that at $0.478 < K < 0.572$.

![Bifurcation Diagram](image)

Figure 4. Bifurcation diagram of synchronous rotational regimes of the system. (a) $2\pi$- and $4\pi$-periodic regimes. (b) $4\pi$-periodic regimes. (c) $4\pi$-, $8\pi$- and $16\pi$-periodic regimes. Triangular markers – $8\pi$-periodic regimes. Diamond-shaped markers – $16\pi$-periodic regimes. (d) Local maxima of $\dot{\varphi}_2$. (e) The largest Lyapunov exponent $\Lambda_1$. Parameters: $\gamma = 0.97$, $\lambda = 0.7$. 
a chaotic regime is observed in the system. With a further
decrease of the parameter $K$, after the escape from the
region of chaotic dynamics, a sequence of period dou-
bling bifurcations is observed in the reverse order. Sev-
eral bifurcations in this sequence shows on the Fig. 4(c).
$16\pi$, $8\pi$- and $4\pi$-periodic regimes become stable as a re-
result of doubling bifurcations at $K \approx 0.479$, $K \approx 0.4742$, $K \approx 0.456$. The $8\pi$- and $4\pi$-periodic regimesmerge in Fig. 4(b) into $B_0$.

As the values of the dissipation parameter $\lambda$ increase,
the regions of instability of the in-phase periodic motion
$A_2$ and $A_4$ approach each other. At a critical value of the parameter $\lambda \approx 0.75$, the regions of instability touch each other, after that they begin to overlap (Fig. 5(a)). The $A_2$ region disappears, bistability of out-of-phase regimes is observed: regimes $(2 : 1)$ and $(1 : 1 : 1)$ coexist. When the regions of instability of the in-phase regime approach the first instability region $A_2$ through the cascade of dou-
bling bifurcations (Fig. 5(b)), a chaotic regime arises.

We note that when chaotic dynamics appears in the first
instability region, the regime $(2 : 1)$ is first observed, and the dynamics of the variables $\phi_{1,2,3}(t)$ is irregular (see Fig. 6(a)). This regime can be interpreted as a chaotic chimera [17]. If the coupling strength parameter $K$ takes the value from the region $A_4$, the chaotic regime $(1 : 1 : 1)$ is realized (see Fig. 6(b)). With a further
cade of period-doubling bifurcations (see Figs. 6(a) and 8). Further, at $K \approx 0.019$, the chaotic $(2 : 1)$ chimera be-
comes unstable, and the regime is realized in the system
when the long time intervals for which the phases of the
pendulums $\phi_1(t) \approx \phi_3(t)$, alternate with short intervals, where $\phi_1(t)$ and $\phi_3(t)$ do not coincide (see Figs. 7(b), 8), i.e. there is an intermittency of chaotic oscillations $(2 : 1)$ and $(1 : 1 : 1)$. With further increase of $K$, only chaotic $(1 : 1 : 1)$ regime is realized (see figure 7(c), 8).

Figure 5. (a) Bifurcation diagram of the synchronous ro-
tational regimes of the system. (b) Local maxima of $\dot{\phi}_2$. (c) The largest Lyapunov exponent $\Lambda_1$. Parameters: $\gamma = 0.97$, $\lambda = 0.76$.

Figure 6. Time dynamics of instantaneous frequencies $\phi_i$ $(i = 1, 2, 3)$ of the three pendulums in the system. (a) Chaotic chimera $(2 : 1)$ regime at $K = 0.1524$. (b) Chaotic $(1 : 1 : 1)$ regime at $K = 0.4$. Parameters: $\gamma = 0.97$, $\lambda = 0.76$.

Figure 7. (a) Bifurcation diagram of the synchronous ro-
tational regimes of the system. (b) Local maxima of $\dot{\phi}_2$. (c) The largest Lyapunov exponent $\Lambda_1$. Parameters: $\gamma = 0.97$, $\lambda = 0.86$.

IV. CONCLUSION

We have studied the dynamics of an ensemble of three
identical coupled pendulums. A relatively simple model
demonstrates a great variety of regular and chaotic in-
phase and out-of-phase regimes. Parametric instability
of the synchronous rotation regime is found and theo-
retically proved. It is shown that in the system with
the growth of the coupling strength, the generation of out-
of-phase rotational periodic motions occurs. Note that
there are two such instability regions. Bistability [20]
of in-phase and out-of-phase rotational periodic motions can also be observed. With increasing the dissipation parameter, regions of the instability approach each other and chaos throw the cascade period-doubling bifurcations arises here. Moreover, other chimera state regime can appear in this region. With a further increase of the dissipation parameter, the regions of instability overlap and, as a result, the regions with chaotic dynamics also overlap. Further, only the chaos corresponding to the globally out-of-phase oscillations remains.

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