Global analysis of muon decay measurements

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We have performed a global analysis of muon decay measurements to establish model-independent limits on the space-time structure of the muon decay matrix element. We find limits on the scalar, vector and tensor coupling of right- and left-handed muons to right- and left-handed electrons. The limits on those terms that involve the decay of right-handed muons to left-handed electrons are more restrictive than in previous global analyses, while the limits on the other non-standard model interactions are comparable. The value of the Michel parameter η found in the global analysis is −0.0036±0.0069, slightly more precise than the value found in a more restrictive analysis of a recent measurement. This has implications for the Fermi coupling constant $G_F$.

I. INTRODUCTION

Muon decay, $\mu \to e\nu\bar{\nu}$, is an excellent laboratory to investigate the weak interaction. The energy and angular distributions of the electrons emitted in polarized muon decay are specified by the muon decay parameters $\rho$, $\eta$, $\xi$, and $\delta$ \cite{2}, conventionally referred to as the Michel parameters. The additional decay parameters $\xi'$ and $\xi''$ determine the longitudinal polarization of the outgoing electrons, and the parameters $\alpha$, $\beta$, $\alpha'$, and $\beta'$ determine the transverse polarization \cite{2}. Radiative muon decay, $\mu \to e\nu\bar{\nu}\gamma$, provides access to another decay parameter, $\eta$ \cite{2}. These 11 decay parameters, which are not all independent, together with the muon lifetime provide a complete description of muon decay if the neutrinos are not observed.

The decay parameters are related to the underlying space-time structure of muon decay by the most general, local, derivative-free, Lorentz-invariant transition matrix element, which can be written in terms of helicity-preserving amplitudes as \cite{4}:

$$M = \frac{4G_F}{\sqrt{2}} \sum_{\gamma=S,V,T;\tau=\mu,\bar{\mu}} g_{\mu}^\gamma \langle \bar{e}_\gamma | \Gamma | \nu_\tau \rangle \langle \bar{\nu}_\tau | \Gamma | \mu_\mu \rangle. \quad (1)$$

In this equation, $G_F$ is the Fermi coupling constant, determined from the muon lifetime, and the $g_{\mu}^\gamma$ specify the scalar, vector and tensor couplings between $\mu$-hand ed muons and $e$-handed electrons. Deviations from this expression due to the non-local nature of the standard model weak interaction are $O(m_\mu^2/m_W^2)$ and, thus, are negligible compared to our current experimental knowledge of the muon decay parameters. Equation (1) is particularly convenient because the standard model, with pure $V - A$ coupling, implies $g_{V}^{\nu \bar{\nu}} = 1$ and all the other coupling constants are zero. In contrast, additional coupling constants are non-zero in many extensions to the standard model. For example, $g_{R}^{\nu \bar{\nu}}$, $g_{L}^{\nu \bar{\nu}}$, and $g_{R}^{\nu \bar{\nu}}$ are non-zero in left-right symmetric models \cite{3}. For recent reviews of muon decay, see \cite{3, 13, 18}.

The coupling constants $g_{\mu}^\gamma$ represent 18 independent real parameters, so it is not possible to determine all of them from the existing muon decay observables. A global analysis of all the existing muon decay measurements can nonetheless set stringent model-independent experimental limits on the non-standard model contributions \cite{4, 3, 10, 11} that can then be compared to model calculations such as those in \cite{12}, which predict that $g_{S,V,T}^{\nu \bar{\nu}}$ and $g_{R}^{\nu \bar{\nu}}$ should be very small based on their contributions to neutrino masses. Recently, new measurements have been performed for the Michel parameters $\rho$ \cite{12} and $\delta$ \cite{12} and for the transverse polarization parameters $\eta$, $\eta''$, $\alpha'$, and $\beta'$ \cite{17}. ($\eta$ and $\eta''$ parametrize the momentum-dependence of the $CP$-allowed transverse polarization in a manner different from, but equivalent to, $\alpha$ and $\beta$. See Eq. (5) below.) Each of these new results is a factor of $\sim 2.5$ more precise than the previous best measurement \cite{16}. This makes a new global analysis timely. In this paper, we present such an analysis to establish updated limits on possible non-standard model contributions to muon decay.

II. FIT PROCEDURE

Several different parametrizations for muon decay have been introduced over the years. One common version \cite{2}, which is based on the expression for the muon decay matrix element in charge-retention order, describes the muon decay observables in terms of 10 constants: $a/A$, $a'/A$, $b/A$, $b'/A$, $c/A$, $c'/A$, $\alpha/A$, $\beta/A$, $\alpha'/A$, $\beta'/A$. Each of these 10 constants is a bilinear combination of the coupling constants $g_{\mu}^\gamma$ \cite{8}. The normalization factor, $A = a + 4b + 6c = 16$, is fixed by adjusting $G_F$ to reproduce the muon lifetime. The 9 remaining linearly independent terms may be determined from the experimental values of the muon decay parameters. This procedure was used in the global analysis performed in \cite{4}.

An alternative parametrization \cite{4} utilizes a different...
set of bilinear combinations of the coupling constants:

\[ Q_{RR} = \frac{1}{4} |g_{RR}|^2 + |g_{RL}|^2, \]
\[ Q_{LR} = \frac{1}{4} |g_{LR}|^2 + |g_{RL}|^2 + \frac{3}{4} |g_{LR}|^2, \]
\[ Q_{RL} = \frac{1}{4} |g_{RL}|^2 + |g_{RL}|^2 + 3 |g_{LR}|^2, \]
\[ Q_{LL} = \frac{1}{4} |g_{RL}|^2 + |g_{LL}|^2, \]
\[ B_{LR} = \frac{1}{16} |g_{LR}|^2 + 6 |g_{LR}|^2 + |g_{RL}|^2, \]
\[ B_{RL} = \frac{1}{16} |g_{RL}|^2 + 6 |g_{RL}|^2 + |g_{RL}|^2, \]
\[ I_\alpha = \frac{1}{4} [g_{LR}(g_{RL} + 6 g_{LR})^* (g_{RL})^* (g_{LR} + 6 g_{LR})] = (\alpha + i \beta')/2A, \]
\[ I_\beta = \frac{1}{2} [g_{LL} (g_{RR})^* + (g_{RR})^* (g_{LL})] = -2(\beta + i \beta')/A. \]

These bilinear combinations satisfy the constraints [4]:

\[ 0 \leq Q_{\epsilon \mu} \leq 1, \quad \text{where } \epsilon, \mu = R, L, \]
\[ 0 \leq B_{\epsilon \mu} \leq Q_{\epsilon \mu}, \quad \text{where } \epsilon \mu = RL, LR, \]
\[ |I_\alpha|^2 \leq B_{LR} B_{RL}, \]
\[ |I_\beta|^2 \leq Q_{LL} Q_{RR}. \]

and the normalization condition,

\[ Q_{RR} + Q_{LR} + Q_{RL} + Q_{LL} = 1. \]

The advantages of this parametrization are that it contains the maximum number of positive semi-definite bilinear combinations of the \( g_{\mu} \) and that the \( Q_{\epsilon \mu} \) are directly interpretable as the total probabilities for \( \mu \)-handed muons to decay into \( \epsilon \)-handed electrons [4]. Furthermore, experimentally it is found that \( Q_{LL} \) is close to unity. Thus, if Eq. (1) is used to eliminate \( Q_{LL} \), the 9 remaining variables are all close to zero. The bilinear combinations \( Q_{RR}, Q_{LR}, Q_{RL}, B_{LR}, B_{RL}, I_\alpha, I_\beta \) were adopted in the global analyses presented in [4, 10, 11].

For convenience, in the present analysis we have adopted a hybrid set of independent variables: \( Q_{RR}, Q_{LR}, Q_{RL}, B_{LR}, B_{RL}, I_\alpha, I_\beta, \eta, \eta'' \) were used in the global analyses presented in [4, 10, 11].

\[ \rho = \frac{3}{4} + \frac{1}{4} (Q_{LR} + Q_{RL}) - (B_{LR} + B_{RL}), \]
\[ \xi = 1 - 2 Q_{RR} - \frac{10}{3} Q_{LR} + \frac{4}{3} Q_{RL} + \frac{16}{3} (B_{LR} - B_{RL}), \]
\[ \xi \delta = \frac{3}{4} - \frac{3}{2} Q_{RR} - \frac{7}{4} Q_{LR} + \frac{1}{4} Q_{RL} + (B_{LR} - B_{RL}), \]
\[ \xi' = 1 - 2 Q_{RR} - 2 Q_{RL}, \]
\[ \xi'' = 1 - 10 \frac{3}{4} (Q_{LR} + Q_{RL}) + 16 \frac{3}{3} (B_{LR} + B_{RL}), \]

**TABLE I: Experimental measurements included in the global analysis.**

| Parameter | Value | Reference |
|-----------|-------|-----------|
| \( \rho \) | 0.7518 ± 0.0026 | [16] |
| \( \delta \) | 0.75080 ± 0.00105 | [13] |
| \( \xi \) | 0.7486 ± 0.0038 | [11] |
| \( \eta \) | 0.74964 ± 0.00130 | [14] |
| \( P_{\rho} \xi \) | 1.0027 ± 0.0005 | [18] |
| \( P_{\rho} \delta / \rho \) | 0.99787 ± 0.00082 | [10, 20] |

- \( \xi' \) = 1.00 ± 0.04 |
- \( \xi'' \) = 0.65 ± 0.36 |
- \( \eta \) = 0.02 ± 0.08 |
- \( \alpha / A \) = 0.015 ± 0.052 |
- \( \beta / A \) = 0.002 ± 0.018 |
- \( \eta'' \) = 0.071 ± 0.037 |
- \( \alpha'' / A \) = -0.047 ± 0.052 |
- \( \beta'' / A \) = 0.017 ± 0.018 |
- \( \beta'' / A \) = -0.0005 ± 0.0005 |

- \( \alpha / A \) = -0.894 in [12].
- \( \beta / A \) = -0.894 in [13].
- \( \eta / A \) = -0.946 in [15].
- \( \eta'' / A \) = -0.946 in [15].
- \( \rho(\alpha / A, \beta / A) = -0.894 \) in [15].
- \( \rho(\eta, \eta'') = +0.946 \) in [15].

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We have computed the joint probability density function of the 9 independent variables using Monte Carlo integration techniques, in a manner similar to that described in [9]. For each experimental input, we have assumed that the corresponding probability distribution takes the form of a one- or two-dimensional Gaussian, truncated to the allowed parameter region if necessary. When statistical and systematic uncertainties have been quoted separately, we have added them in quadrature.

### III. INPUT VALUES

Most of the previous muon decay global analyses [4, 10, 11] utilized the same input parameters — \( \rho, \delta, P_{\rho} \xi \delta / \rho, \xi', \xi'', \alpha / A, \beta / A, \alpha'' / A, \beta'' / A \). For each of these quantities, the previous analyses adopted the single most precise experimental measurement that was available at the time.

We have utilized a different philosophy. We include all of the accepted muon decay parameter measurements that are reported in [10], with two exceptions. When [10] determines a decay constant from a single input, we
TABLE II: Results of the global fit. For the positive semi-definite quantities, 90% confidence level upper limits (lower limit for $Q_{LL}$) are given, and for completeness, the mean and r.m.s. values from the fit are also given in parentheses.

| Parameter | Fit Result ($\times 10^3$) |
|-----------|-----------------|
| $Q_{RR}$  | $<1.14$ (0.60 ± 0.38) |
| $Q_{LR}$  | $<1.94$ (1.22 ± 0.53) |
| $B_{LL}$  | $<1.27$ (0.72 ± 0.40) |
| $Q_{RL}$  | $<44$ (26 ± 13) |
| $B_{RL}$  | $<10.9$ (6.4 ± 3.3) |
| $Q_{LL}$  | $>955$ (973 ± 13) |
| $\alpha/A$ | 0.3 ± 2.1 |
| $\beta/A$  | 2.0 ± 3.1 |
| $\alpha'/A$ | -0.1 ± 2.2 |
| $\beta'/A$  | -0.8 ± 3.2 |

IV. RESULTS

Table II shows the results of the global fits. 90% confidence level upper limits (lower limit for $Q_{LL}$) are given for the positive semi-definite quantities. For completeness, mean and r.m.s. values are also specified for these variables, even though the corresponding probability distributions are far from Gaussian. In contrast, the output probability distributions for $\alpha/A$, $\beta/A$, $\alpha'/A$, and $\beta'/A$ are very close to Gaussian.

Only a subset of the input quantities play a role in constraining each of the fit parameters. $Q_{RR}$ is constrained primarily by the measurement of $P_\mu \xi \delta/\rho$. $Q_{LR}$ and $B_{LL}$ are constrained by the measurements of $\rho$, $\delta$, and $P_\mu \xi \delta/\rho$. $Q_{RL}$ is constrained by the measurement of $\xi'$. $B_{RL}$ is constrained by the combination of $\xi'$, $\rho$, and $\delta$. $\alpha/A$ and $\alpha'/A$ are constrained primarily by the requirement $|I_\alpha|^2 \leq B_{LR}$. Finally, the large reduction in the uncertainties for $\alpha/A$ and $\alpha'/A$ obtained when applying the $I_\alpha$ constraint significantly reduces the correlations between $\alpha/A$ and $\beta/A$ and between $\alpha'/A$ and $\beta'/A$. The fit finds $\rho(\alpha/A, \beta/A) = -0.19$ and $\rho(\alpha'/A, \beta'/A) = -0.20$. The correlations dominate the uncertainties in $\beta/A$ and $\beta'/A$ quoted in Table II so the result is that the transverse polarization measurements become far more precise. Overall, the constraints in Eq. 3 play a crucial role in reducing the uncertainties for a number of the fitted parameters, as was noted in \[.\]

The fit results may be used to determine limits on the coupling constants $g_{\mu}$. These are given in Table III. Limits are also given for certain linear combinations of scalar and tensor interactions. Muon decay measurements alone are unable to separate contributions of $g_{LL}^{S}$ from the standard model term, $g_{LL}^{V}$. Other experimental results, such as the branching ratio for $\pi \rightarrow e\nu\nu$, argue that the scalar contribution must be very small, but this requires assumptions that go beyond the model-independent experimental limits that we are exploring here. In \[ it was noted that the rate for inverse muon decay, $\nu_{\mu} e^{-} \rightarrow \mu^{-} \nu_{e}$, may be used to distinguish between $g_{LL}^{S}$ and $g_{LL}^{V}$. In Table III the limits quoted for $g_{LL}^{S}$ and $g_{LL}^{V}$ come from \[, which utilized the inverse muon decay measurements in \[.\]

Table III shows that the present limits on $|g_{LL}^{S}|$, $|g_{LL}^{V}|$, and $|g_{LR}|$ are significantly better than those in \[, which arises from the inclusion of the new measurements of $\rho$.
TABLE III: 90% confidence limits on the muon decay coupling constants in Eq. (1) are compared to the previous accepted values from Ref. [8]. Limits are also given for certain scalar-tensor interference combinations.

| Coupling Constant | | | Present work |
|-------------------|------------------|---------------|
| $|g_{RR}'|^{2}$    | <0.066           | <0.067        |
| $|g_{RR}'|^{2}$    | <0.033           | <0.034        |
| $|g_{LR}'|^{2}$    | <0.125           | <0.088        |
| $|g_{LR}'|^{2}$    | <0.060           | <0.036        |
| $|g_{LL}'|^{2}$    | <0.036           | <0.025        |
| $|g_{LL}'|^{2}$    | <0.424           | <0.417        |
| $|g_{RR}'|^{2}$    | <0.110           | <0.104        |
| $|g_{RR}'|^{2}$    | <0.122           | <0.104        |
| $|g_{LL}'|^{2}$    | <0.550           | <0.550        |
| $|g_{LL}'|^{2}$    | >0.960           | >0.960        |

Thus, the ±0.013 uncertainty in the accepted value of $\eta$ [12] leads to an additional uncertainty in $G_F$ due to $\Delta G_F/G_F = 1.3 \times 10^{-4}$. The recent measurement of the transverse polarization of the electrons emitted in muon decay [12] included two analyses of the results. The general (model-independent) analysis led to the values of $\eta''$, $\alpha'/A$, and $\beta'/A$ quoted in Table III. A second, restricted analysis assumed that muon decay can be described with only two coupling constants, $g_{LL}'$ and $g_{RR}'$. Equations (2) and (5) imply that this assumption requires $\alpha/A = \alpha'/A = 0$ and $\eta'' = -\eta$. The restricted analysis concluded $\eta = -0.0021 \pm 0.0070 \pm 0.0010$ and $\beta'/A = -0.0013 \pm 0.0035 \pm 0.0006$. As noted above, the allowed ranges of $\alpha/A$ and $\alpha'/A$ are severely constrained by other muon decay data in the present global analysis. Thus, we find a model-independent result, $\eta = -0.0036 \pm 0.0069$, with slightly better precision than that of the model-dependent restricted analysis in [12]. This reduces the contribution of $\eta$ to the uncertainty in $G_F$ to $\Delta G_F/G_F = 6.7 \times 10^{-5}$. Note that all of the new measurements [13] play important roles in reducing the uncertainty in $\eta$.

V. CONCLUSION

We have performed a new global analysis of all the existing data on muon decay that do not include observation of the outgoing neutrinos, including recent measurements of the Michel parameters $\rho$ and $\delta$ and the muon decay transverse polarization parameters $\eta$, $\eta''$, $\alpha'/A$, and $\beta'/A$. The global analysis finds that the upper limits on the coupling constants $|g_{LL}'|$, $|g_{LR}'|$, and $|g_{RR}'|$ are more restrictive than the previous accepted values. It also finds that $\eta = -0.0036 \pm 0.0069$, which is nearly a factor of two more precise than the previous accepted value.

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