Generalized limits for parameter sensitivity via quantum Ziv–Zakai bound

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Abstract
We study the generalized limit for parameter sensitivity in quantum estimation theory considering the effects of repeated and adaptive measurements. Based on the quantum Ziv–Zakai bound, we derive some lower bounds for parameter sensitivity when the Hamiltonian of a system is unbounded and when the adaptive measurements are implemented on the system. We also prove that the parameter sensitivity is bounded by the limit of the minimum detectable parameter. In particular, we examine several known states in quantum phase estimation with non-interacting photons and show that they cannot perform better than the Heisenberg limit in a much simpler way with our result.

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(Some figures may appear in colour only in the online journal)

1. Introduction
Quantum theory enables us to estimate a parameter more precisely than classical theory [1]. For the quantum phase estimation, it is known that the phase sensitivity with non-interacting photons is improved from the usual shot-noise limit (SNL), namely $\Delta \theta \simeq 1/\sqrt{\langle n \rangle}$, to the Heisenberg limit (HL), namely $\Delta \theta \simeq 1/\langle n \rangle$, where $\langle n \rangle$ is the mean number of photons [2]. The underlying reason is the superposition principle that plays an essential role. Although a photon-number state $|n\rangle$ is useless for estimating phase, its superposition with the vacuum state, $(|0\rangle + |n\rangle)/\sqrt{2}$, is optimal for phase sensitivity with $n$ available photons, i.e. $\Delta \theta = 1/n$. More recently, there appear some counter examples that were used to beat the HL for phase sensitivity without limits [3–5]. However, all these proposals were based on either crude statistical arguments or a non-achievable lower bound, such as the quantum Cramer–Rao (CR) bound [6]. More careful calculations reveal that they approach but never beat the HL.
As for the CR bound, it sets only a lower limit for phase sensitivity, and whether it can be achieved can only be checked by details, that is to say, sometimes the CR bound may not be achievable. On the other hand, by splitting the total mean number of photons $N_T = M \langle n \rangle$ into $M$ independent and identical samplings for repeated measurements, the Fisher theorem tells us that the CR bound in this case is approached asymptotically as $M \to \infty$. Hence, whether the HL, i.e. $\Delta \theta \simeq 1/N_T$, is beaten or not becomes more tricky. It is argued in [7, 8] that the optimal sensitivity with maximum likelihood estimation for $N_T$ photons will occur at $M \simeq M_{\text{knee}}$, where $M_{\text{knee}}$ is the turning point after which the CR bound is asymptotically approached. Within this scheme, [7] examined the Shapiro–Shepard–Wong (SSW) state [3] which was proposed to beat the HL and found that the SSW state performs even worse than the HL. Similar arguments can also be used for other states to check if they beat the HL. However, this scheme has its own inconvenience because it needs cumbersome calculations, numerically or analytically.

It is thus very convenient to have a condition that can check if the sensitivity with a given state can achieve the HL more simply, and that is independent of the estimation scheme. It is proposed in [9] that by loosely relating the phase sensitivity to the minimum detectable phase shift, such a condition can be expressed as the fidelity between the two output states—undergoing zero and the minimal detectable phase shift $\theta_m$, respectively—should be significantly different from unity as $\theta_m \simeq 1/N_T$ and $N_T \to \infty$. This condition applies to the single measurement of phase shift as well as the repeated ones, where the states are taken as the direct products of states over all measurements. However, [9] does not present a rigorous relation indicating that the phase sensitivity is bounded by the scaling of the minimum detectable phase shift over $N_T$. In this paper, we obtain such a relation in general cases and prove some lower bounds for the parameter sensitivity in terms of the quantum Ziv–Zakai (ZZ) bound [10, 11] when the Hamiltonian of the system is unbounded and when the adaptive measurements are implemented on the system.

In section 2, we review the quantum ZZ bound and some recent results for the parameter sensitivity. This bound is then applied to the more general cases in section 3 and to obtain our main result. In section 4, we discuss some known examples of quantum phase estimation showing that they cannot perform better than the HL. Finally, a summary is given in section 5.

2. Quantum Ziv–Zakai bound

In parameter estimation theory, the ZZ bound provides a lower bound for the parameter sensitivity, namely the root-mean-square error other than the usual CR bound [10]. The ZZ bound connects the parameter sensitivity to the error probability in a binary decision problem and is often tighter than the CR bound in a highly non-Gaussian regime. Both the ZZ and CR bounds can also follow toward each other asymptotically as the number of the repeated measurements increases to infinity.

Let $x$ be the parameter to be estimated, $y$ be the outcome of the measurement and $X(y)$ be an estimator of $x$ constructed from the outcome $y$. The parameter sensitivity of $x$ is defined as

$$\Delta Y = \left\{ \int dx \int dy \ p(y|x) p(x) [X(y) - x]^2 \right\}^{1/2},$$

where $p(y|x)$ is the conditional probability distribution of obtaining a certain outcome $y$ given $x$, and $p(x)$ is the prior probability distribution. As shown in figure 1, the parameter sensitivity characterizes the uncertainty of the posterior probability distribution after the estimation.
The ZZ bound is then given by \cite{10, 11}
\[
\Delta Y \geq \left\{ \int_{0}^{\infty} d\gamma \gamma \int_{-\infty}^{\infty} dx \min[p(x), p(x + \gamma)] \text{Pr}_e(x, x + \gamma) \right\}^{1/2},
\] (2)
where \( \text{Pr}_e(x, x + \gamma) \) denotes the minimum error probability with an equally likely hypothesis in a binary decision problem.

In the quantum parameter estimation problem, suppose the parameter \( x \) is encoded in the quantum state \( \rho_x \). The binary decision problem then becomes one of discriminating the two possible states given by \( \rho_x \) and \( \rho_x + \gamma \) with equal prior information. For such a problem, the minimum error probability over all possible measurements and estimations is obtained by \cite{12}
\[
\text{Pr}_e(x, x + \gamma) = \frac{1}{2} [1 - D(\rho_x, \rho_x + \gamma)]
\geq \frac{1}{2} \left[ 1 - \sqrt{1 - F(\rho_x, \rho_x + \gamma)^2} \right],
\] (3)
where the distance \( D \) and the fidelity \( F \) for any given two states \( \rho \) and \( \sigma \) are defined by \( D(\rho, \sigma) = \sqrt{\text{tr}[(\rho - \sigma)^2]/2} \leq 1 \) and \( F(\rho, \sigma) = \text{tr}\sqrt{\rho \sigma \sqrt{\rho} \sigma} \leq 1 \). For pure states, \( D(\psi, \chi) = \sqrt{1 - F(\psi, \chi)^2} \) and \( F(\psi, \chi) = |\langle \psi | \chi \rangle| \).

Assume now that \( \rho_x \) is generated by the unitary evolution
\[
\rho_x = e^{-iH} \rho e^{iH},
\] (4)
where \( \rho \) is the input state and \( H \) is an effective Hamiltonian, the ground level of which is chosen to be zero, namely \( H \geq 0 \). The HL in such situation means that the parameter sensitivity \( \Delta Y \) scales with the average effective energy \( \langle H \rangle = \text{tr}[H \rho] \). Under this condition, the fidelity satisfies \( F(\rho_x, \rho_{x+\gamma}) = F(\rho_0, \rho_\gamma) \). For simplicity, assume further that the prior probability distribution is a uniform window with mean \( \mu \) and width \( W \) given by
\[
p(x) = \frac{1}{W} \text{rect}\left( \frac{x - \mu}{W} \right),
\] (5)
the standard deviation of which is thus \( \Delta x = W/\sqrt{12} \). Putting equations (2), (3) and (5) together gives
\[
\Delta Y \geq \Delta Y_{\text{LB}} \equiv \left\{ \int_{0}^{W} d\gamma \gamma \left[ 1 - \frac{\gamma}{W} \right]^{1/2} \left[ 1 - \sqrt{1 - \mathcal{F}(\gamma)^2} \right] \right\}^{1/2},
\] (6)
where \( \mathcal{F} \) is a lower bound of the fidelity \( F \).
Figure 2. Comparison between the lower bounds on the parameter sensitivity for the uniform prior distribution. (a) The solid line is $\Delta Y_{LB}$ defined in equations (7) and (8). The dashed line is the Ht, namely $0.1548/\langle H \rangle$. The dotted line is the initial uncertainty $\Delta x = W/\sqrt{12} = x_0/\sqrt{3}$ by guessing $x$ from the prior information. (b) The solid line is $\Delta Y_{LB}$ defined in equations (9) and (10). The dashed line is the limit $0.3418/\Delta H$. The dotted line is the initial uncertainty $\Delta x$. Here $\langle H \rangle = \Delta H = 1$.

One bound for the fidelity is given by [13] $F(\gamma) = 1 - \gamma \langle H \rangle$ for $0 \leq \gamma \leq x_0 \equiv 1/\langle H \rangle$ and $F = 0$ for $\gamma \geq 1/\langle H \rangle$. It follows from equation (6) that for $0 \leq z_0 \equiv W/(2x_0) \leq 1/2$,

$$\Delta Y_{LB} = x_0 \left\{ \frac{2}{3} z_0^2 - \frac{15 - 14z_0 - 8z_0^2 + 16z_0^3}{48} \sqrt{1 - z_0} \right\}^{1/2} \rightarrow \Delta x \text{ when } z_0 \rightarrow 0,$$

and for $z_0 \geq 1/2$,

$$\Delta Y_{LB} = x_0 \left\{ \frac{5}{12} - \frac{\pi}{8} - \frac{1}{4} - \frac{5\pi/64}{z_0} \right\}^{1/2} \rightarrow 0.1548/\langle H \rangle \text{ when } z_0 \rightarrow \infty.$$

Another bound for $F$ is given by [13] $F(\gamma) = \cos(\gamma \Delta H)$ for $0 \leq \gamma \leq x_0\pi/2 \equiv \pi/(2\Delta H)$ and $F = 0$ for $\gamma \geq \pi/(2\Delta H)$ with $\Delta H = \text{tr}[H^2\rho] - \langle H \rangle^2$. Then equation (6) gives that for $0 \leq z_0 \leq \pi/4$,

$$\Delta Y_{LB} = x_0 \left\{ \frac{2}{3} z_0^2 + \cos(2z_0) - \frac{1}{2z_0} \right\}^{1/2} \rightarrow \Delta x \text{ when } z_0 \rightarrow 0,$$

and for $z_0 \geq \pi/4$,

$$\Delta Y_{LB} = x_0 \left\{ \frac{\pi^2}{16} - 1/2 - \frac{1}{4} - \frac{\pi/4 + \pi^3/96}{z_0} \right\}^{1/2} \rightarrow 0.3418/\Delta H \text{ when } z_0 \rightarrow \infty.$$  

As shown in figure 2, we can see that the parameter sensitivity $\Delta Y$ is lower bounded by the standard deviation $\Delta x$ of the prior distribution, i.e.

$$\Delta Y \geq \Delta x,$$
when $W \ll 1/(\langle H \rangle)$ or $1/\Delta H$ in the high prior information (HPI) regime, and
$$\Delta Y \geq \max \left[ \frac{0.1548}{\langle H \rangle}, \frac{0.3418}{\Delta H} \right] \quad (12)$$
when $W \gg 1/(\langle H \rangle)$ or $1/\Delta H$ in the low prior information (LPI) regime. As shown in figure 2, we can see that only in the LPI regime do we get the HL, and the sub-HL can be obtained in the HPI and intermediate regimes. However, in the HPI regime, the sub-Heisenberg strategy is useless since one can attain the same sensitivity by just taking a random $x$ subject to the prior distribution. On the other hand, we can only provide a small enhancement over the initial uncertainty in the intermediate regime by a factor of order 1. Therefore, it is not much more effective for practical estimations in the HPI and intermediate regimes where the prior is already large enough to allow for the sub-HL. Similar results can be obtained for other prior distributions [11].

For comparison, the quantum CR bound for the parameter sensitivity defined by equation (1) is [12]
$$\Delta Y \geq \frac{1}{\sqrt{4\Delta H^2 + \Pi}}, \quad (13)$$
where the quantity $\Pi$ is the prior Fisher information,
$$\Pi = \int dx \, p(x) \left( \frac{d \ln p(x)}{dx} \right)^2. \quad (14)$$
For a Gaussian prior distribution with variance $\Delta x^2$, the Fisher information is $\Pi = 1/\Delta x^2$. From equation (13), we see that $\Delta Y \geq \Delta x$, which is the same as the quantum ZZ bound given by equation (9) in the HPI regime, and $\Delta Y \geq 1/(2\Delta H)$, which is more tight than the quantum ZZ bound given by equation (10) in the LPI regime.

3. Main result

In the previous section, we reviewed the known limits based on quantum ZZ and CR bounds when the output state $\rho_x$ is generated by a simple unitary $U_x(t) = e^{-ixHt}$, which does not consider possible decoherence and measurements during the evolution interval. In [15], a bound for parameter sensitivity taking into account such an effect was derived via a quantum CR bound only for the bounded Hamiltonian. In this section, we will study the unbounded Hamiltonian and present our main result on the generalized limit for parameter sensitivity via the quantum ZZ bound taking into account the effect of excess decoherence, repeated and adaptive measurements during the interval [14]. Generally, the quantum dynamics of the input state in such a situation is described by completely positive maps [19], including sequential measurements and feedback according to measurement outcomes.

To tackle this problem, we can first use the Kraus representation theorem [19], which implies that any quantum dynamics described by completely positive maps can be reproduced by the unitary evolution of an enlarged system with appropriate ancillas, and then use the principle of deferred measurement [15, 16, 19], which allows us to shuffle the measurements during the evolution time of the enlarged system to the end of the evolution time, while the measurement-based feedback is replaced by coherent-controlled unitaries prior to the overall final measurement of the enlarged probe-ancilla system. Since our analysis below holds for all possible measurements and estimations [12, 15] at the end of the evolution time, we only need to consider the generalized Hamiltonian
$$H_x(t) = xH + H_0(t), \quad (15)$$
where the Hamiltonian $H$ contains a coupling to the parameter of the probe systems, and the auxiliary Hamiltonian $H_0$ collects all parameter-independent parts, such as the free Hamiltonians of the probes and the controlled unitaries induced by adaptive measurements, etc. Then, $\rho_s$ is generated by the transformation
\begin{equation}
\rho_s(t) = U_s(t)\rho U^\dagger_s(t),
\end{equation}
where the unitary operator $U_s$ is the solution of the Schrodinger equation
\begin{equation}
dU_s(t)/dt = -iH(t)U_s(t).
\end{equation}
To find a lower bound for the fidelity $F(\rho_s, \rho_{s+\gamma})$ in this case, let $s = 0$ without loss of generality, since the linear dependence of $H_s$ on $x$.

At first, for a pure input state $\rho = |\Psi\rangle$, as shown in figure 3, the fidelity between the output states $|\Psi_0\rangle$ and $|\Psi_{\gamma}\rangle$ is given by
\begin{equation}
F = |\langle\Psi_0|\Psi_{\gamma}\rangle| = \left|\langle\Psi|U_s^\dagger(t)U_{\gamma}(t)|\Psi\rangle\right|.
\end{equation}
In the interaction picture of $H_{\gamma}$ [17], where $H_0$ is taken as free Hamiltonian and $\gamma H$ as interaction, we can express $U_{\gamma}$ of the form $U_{\gamma}(t) = U_0(t)U_{\gamma}(t)$, where $U_{\gamma}$ satisfies the equation
\begin{equation}
dU_{\gamma}(t)/dt = -i\gamma H(t)U_{\gamma}(t)
\end{equation}
with the interaction Hamiltonian $H = U_0^\dagger H U_0$. The solution of equation (19) can be written as
\begin{equation}
U_{\gamma}(t) = 1 - i\gamma \int_0^t ds H(s)U_{\gamma}(s).
\end{equation}
So equation (18) becomes
\begin{flalign}
F &= |\langle\Psi|U_s^\dagger(t)U_{\gamma}(t)|\Psi\rangle| \\
&= \left|1 - i\gamma \int_0^t ds \langle\Psi|H(s)U_{\gamma}(s)|\Psi\rangle\right| \\
&\geq 1 - \gamma \int_0^t ds |\langle\Psi|H(s)U_{\gamma}(s)|\Psi\rangle|.
\end{flalign}
To proceed, we use the Cauchy inequality $|\langle\psi|x\rangle|^2 \leq \langle\psi|\psi\rangle \langle x|x\rangle$ for $|\psi\rangle = H(t)|\Psi\rangle$ and $|x\rangle = U_{\gamma}|\Psi\rangle$, or $|\psi\rangle = \sqrt{\langle\gamma|\Psi\rangle}$ and $|x\rangle = \sqrt{\langle\gamma|U_{\gamma}|\Psi\rangle}$ to obtain
\begin{equation}
|\langle\Psi|H(t)U_{\gamma}|\Psi\rangle| \leq \min\left(\sqrt{\langle H^2\rangle}, \sqrt{\langle H\rangle[H,U_{\gamma}]}\right),
\end{equation}
where the unitary of $U_{\gamma}$ has been used. Let us consider two types of $H$, depending on whether its possible energy spectra are bounded or not. For the first type of $H$, such as in spin systems [18], we have
\begin{equation}
\min\left(\sqrt{\langle H^2\rangle}, \sqrt{\langle H\rangle[H,U_{\gamma}]}\right) \leq \|H\|,
\end{equation}
where the schematic of single parameter estimation. The effects of repeated and adaptive measurements are taken into consideration by introducing ancillas and controlled unitaries, with an input state $\Psi$, the difference between $U_0$ and $U_{\gamma}$ is encoded in the distinguishability of $\Psi_0$ and $\Psi_{\gamma}$.
where \( \mathcal{H} \) is transformed back to \( H \) and \( \|A\| = \lambda - \lambda \) is the semi-norm of \( A \). Here \( \lambda (\lambda) \) is the largest (smallest) eigenvalue of \( A \). For the second type of \( H \), such as in quantum phase estimation with a coherent state, we make further assumption that the measurements themselves do not change the energy distributions of input state with respect to the energy spectra of \( H \), namely passive measurements, such as in the adaptive phase estimation [14]—only the auxiliary controlled phase shifts are introduced, whereas the energy distributions are left untouched. Under this condition, we have

\[
\langle \mathcal{H}^2 \rangle = \langle H^2 \rangle, \quad \langle \mathcal{H} \rangle = \| \mathcal{H} \mathcal{U}_x \| = \langle H \rangle.
\]  

(24)

Substituting equations (23) and (24) into equations (21) and (22) leads to

\[
F \geq 1 - \frac{\gamma}{\lambda_0},
\]

(25)

where \( \lambda_0^{-1} = \|H\| \) or \( \min \{ \sqrt{\langle H^2 \rangle}, \langle H \rangle \} = \langle H \rangle \) for the bounded or unbounded \( H \), respectively. Here the Cauchy inequality \( \sqrt{\langle H^2 \rangle} \geq \langle H \rangle \) has been used and \( t = 1 \) is assumed for convenience from now on.

Next, for a mixed input state \( \rho = \sum_k p_k |\Psi^{(k)}_1\rangle \langle \Psi^{(k)}_1| \) with \( \sum_k p_k = 1 \), we find that with

\[
F_k = |\langle \Psi^{(k)}_1| H_{\gamma} |\Psi^{(k)}_1 \rangle|,
\]

\[
F(\rho_0, \rho_\gamma) \geq \sum_k p_k F_k = \sum_k p_k F_k,
\]

(26)

where the convex property of fidelity [19] was used. Putting equation (25) into equation (26), we obtain

\[
F(\rho_0, \rho_\gamma) \geq 1 - \frac{\gamma}{\lambda_0},
\]

(27)

Here \( \lambda_0^{-1} = \|H\| \) or \( \langle H \rangle \), where the Cauchy inequality \( \sum_k p_k \sqrt{\langle H^2 \rangle_k} = \sum_k \sqrt{p_k} \sqrt{\langle H^2 \rangle_k} \leq \sqrt{\langle H^2 \rangle} \) has been used and \( \langle H^2 \rangle_k = \langle |\Psi^{(k)}_1\rangle \langle H^2 \rangle |\Psi^{(k)}_1 \rangle \). Equation (27) obviously reduces to equation (25) for the pure state. Because the derivation of equation (27) does not depend on the assumption of \( x = 0 \), we thus have

\[
F(\rho_0, \rho_{x+\gamma}) \geq \mathcal{F}(\gamma)
\]

(28)

for an arbitrary \( x \), where \( \mathcal{F}(\gamma) = 1 - \gamma/\lambda_0 \) for \( 0 \leq \gamma \leq \lambda_0 \) and \( \mathcal{F}(\gamma) = 0 \) for \( \gamma \geq \lambda_0 \).

Substituting equation (28) into equation (6), we obtain identical expressions for \( \Delta Y_{\text{LB}} \) with equations (7) and (8). Therefore, we obtain

\[
\Delta Y \geq \Delta x
\]

(29)

in the HPI regime, and

\[
\Delta Y \geq \frac{0.1548}{\|H\|} \quad \text{or} \quad \frac{0.1548}{\langle H \rangle}
\]

(30)

in the LPI regime for the bounded or unbounded Hamiltonian \( H \), respectively. In the intermediate regime, we can draw similar conclusions as in section 2.

We note that equation (30) resembles the results of [15, 20]. In [15], a much tighter generalized limit based on the quantum CR bound was obtained for bounded \( H \), i.e.

\[
\Delta Y \geq \frac{1}{\|H\|},
\]

(31)

which does not cover the cases of quantum phase estimation with coherent and squeezed states, etc. On the other hand, [20] claims that

\[
\Delta Y \geq \max \left[ \frac{1}{M(n)}, \frac{1}{\sqrt{M(n^2)}} \right]
\]

(32)
for the phase sensitivity with a linear two-mode interferometer and \( M \) repeated independent measurements. However, it conflicts with the result of [23], where the phase sensitivity with the parity detection is found to be

\[
\Delta Y = \frac{1}{\sqrt{\langle n \rangle (\langle n \rangle + 2)}} \leq \frac{1}{\langle n \rangle}
\]

for \( M = 1 \) and also fails to explain the remarkable fact in [14] where the phase sensitivity with \( M \) separated photons and a proper adaptive protocol can even achieve the HL, i.e.

\[
\Delta Y \approx \frac{4.9009}{M} \quad \text{for} \quad M \gg 1.
\]

For this example, we note that \( \langle n \rangle = \sqrt{\langle n^2 \rangle} = 1 \) and equation (32) gives \( \Delta Y \geq \frac{1}{\sqrt{\langle n \rangle}} \), namely the SNL. For such an experiment with \( M \) separated photons, the Hamiltonian generating phase shift can be expressed as \( H = \sum_{k=1}^{M} \hat{n}_k \) with \( \hat{n}_k = a_k^\dagger a_k \) being the photon-number operator and \( a_k \) being the annihilation operator. The relevant input state can be obtained by tracing the photon state in the reference arm over the total state \( |\psi_1\rangle = \left[ (|0\rangle + |1\rangle)/\sqrt{2} \right]^\otimes M \) after the first beam splitter in the interferometer. This leads to \( \rho = (|0\rangle\langle 0| + |1\rangle\langle 1|)/2 \). We thus have \( \langle H \rangle = M/2 \) and equation (30) implies \( \Delta Y \geq 0.3096/M \), namely the HL. The reason why adaptive measurements with separated photons could achieve the HL can be ascribed to the correlations between photons induced by adaptive measurements, i.e. controlled unitaries.

Finally, we prove that the parameter sensitivity is bounded by the scaling limit of the minimum detectable parameter defined in [9] versus \( \langle H \rangle \) for the state \( \rho_x = e^{-iHx/\hbar} \rho e^{iHx/\hbar} \). The minimum detectable parameter \( \gamma_m \) is corresponding to the situation when the two states \( \rho_0 \) and \( \rho_{\gamma_m} \) can be distinguished efficiently. Since the error probability of discriminating the two states is \( \Pr_e = (1 - D(\rho_0, \rho_{\gamma_m})) / 2 \), it can be seen that \( \rho_0 \) and \( \rho_{\gamma_m} \) are able to be distinguished efficiently when \( \Pr_e \approx 0 \) and the distance

\[
D(\rho_0, \rho_{\gamma_m}) \approx 1
\]

for the inequality \( D \leq \sqrt{1 - F^2} \). Combining equations (27) and (36) leads to

\[
\gamma_m \geq x_0 = \frac{1}{\langle H \rangle}
\]

up to some unimportant factor of order 1.

Suppose the solution of equation (36) is \( \gamma_m \approx \langle H \rangle^{-\alpha} \). The lower bound provided by equation (37) puts a constraint on the exponent \( \alpha \leq 1 \). That is to say, for any infinitesimal \( \epsilon > 0 \),

\[
\lim_{\langle H \rangle \to \infty} F(\rho_0, \rho_{\gamma_m})|_{\gamma_m = \langle H \rangle^{-\alpha}} = 1.
\]

In the regime of \( \langle H \rangle \gg 1 \), we can use equations (6) and (38) to obtain

\[
\Delta Y_{LB} \approx \sqrt{\frac{1}{2} \int_0^{\min[W, \langle H \rangle^{-\alpha}]} d\gamma \gamma \left( 1 - \frac{\gamma}{W} \right)^{1/2}}
\]

\[
= \Delta x = \frac{W}{\sqrt{12}}
\]

for \( W \ll \langle H \rangle^{-\alpha-\epsilon} \) and

\[
\Delta Y_{LB} \approx \langle H \rangle^{-\alpha-\epsilon}
\]
for $W \gg \langle H \rangle^{-\alpha - \epsilon}$ in the practical estimations. Thus, from the continuity of equation (40) with respect to any positive infinitesimal $\epsilon$, we can conclude that

$$\Delta Y \geq O((H)^{-\alpha}) \simeq \gamma_m.$$  \hspace{1cm} (41)

It indicates that the parameter sensitivity $\Delta Y$ is bounded by the scaling of the minimum detectable parameter $\gamma_m$ over $\langle H \rangle$ as $\langle H \rangle \to \infty$. Moreover, we see that equation (41) could be tighter than equation (12) since $\alpha \leq 1$.

4. Applications

Some known states for quantum phase estimation that were proposed to beat the HL have been examined in [10, 11] and it was found that they cannot perform better than the HL when the prior information is appropriately considered. In the following, we use equation (41) to re-examine these states and some other states in [4, 5, 23] in a much simpler way. That is, we first find the minimum detectable phase shift $\theta_m$, and then from equation (41) we can tell that the phase sensitivity $\Delta \theta$ is lower bounded by the scaling of $\theta_m$ over the average photon number in the LPI regime.

We first consider the single mode cases with $U_0 = e^{-i \theta \hat{a}}$. For the coherent state [21], $|\alpha\rangle = e^{i(\alpha - \hat{a})/2} |0\rangle$ with $\alpha$ being a real number, we can see that

$$F = |\langle \alpha | U_0 | \alpha \rangle| = | \exp (\alpha^2 (e^{-i \theta} - 1)) | \simeq e^{-|\alpha|\theta/2}.$$ \hspace{1cm} (42)

The minimum detectable phase shift $\theta_m$ corresponds to the condition that $F$ must be significantly different from unity; thus $\theta_m \simeq 1/\sqrt{\langle n \rangle}$ as $\langle n \rangle = \alpha^2 \gg 1$, namely the SNL. If we consider the superposition of coherent and vacuum (SCV) states, $|\langle 0 \rangle + |\alpha \rangle \rangle / \sqrt{2}$ with $\langle n \rangle = \alpha^2 / 2$, we find that

$$F = \left| 1 + e^{\alpha^2 (e^{-i \theta} - 1)} \right|/2 \simeq (1 + \cos \alpha^2 \theta)/2$$ \hspace{1cm} (43)

and $\theta_m \simeq 1/\langle n \rangle$, namely the HL. Because for the coherent state the phase factor $e^{-i \alpha^2 \theta}$ in $\langle \alpha | U_0 | \alpha \rangle$ does not contribute to $F$, while for the SCV state this term is preserved in $F$, they give different limits for $\theta_m$. This provides one method to construct states with higher sensitivity.

If we use the coherent-squeezed (CS) state [21] as an input, $|\alpha, r \rangle = e^{i(\alpha - \hat{a})} e^{i \hat{a}^2 / 2} |0\rangle$ with $\langle n \rangle = \alpha^2 + \sinh^2 r$ and $\Delta n = \sqrt{\alpha^2 e^{2r} + 2 \cosh^2 r \sinh^2 r}$, where the displacement $\alpha \gg 1$ and the squeezing parameter $r \gg 1$, we have

$$F \simeq \exp \left( -\alpha^2 e^{-\theta} / \left(1 + \beta^2 \theta^2 \right) \right) \left(1 + \beta^2 \theta^2 \right)^{1/4},$$ \hspace{1cm} (44)

where $\beta = 1/(1 - \tanh r) \simeq e^{2r}/2$. For the asymptotically coherent state, $\alpha^2 \gg \sinh^2 r$,

$$F \simeq \exp (-e^{2r} \alpha^2 \theta^2 / 2)$$ \hspace{1cm} (45)

and $\theta_m \simeq e^{-r}/\sqrt{\langle n \rangle}$. On the other hand, at the optimal point $\alpha^2 \simeq \sinh^2 r$,

$$F \simeq \exp \left( -\frac{\langle n \rangle^2 \theta^2 / 2}{1 + \langle n \rangle^2 \theta^2} \right) \left(1 + \langle n \rangle^2 \theta^2 \right)^{1/4}$$ \hspace{1cm} (46)

and $\theta_m \simeq 1/\langle n \rangle$. As with its two-mode analogue, one can use the coherent and squeezed-vacuum state to reach the HL in the Mach–Zehnder interferometer (MZI) [22].

As the first proposed state to beat the HL, the SSW state is [3, 7, 8]

$$| \Psi \rangle = \frac{1}{\xi(2)} \sum_{n=0}^{\Lambda} \frac{1}{n + 1} | n \rangle,$$ \hspace{1cm} (47)
where $\zeta(x)$ is the Riemann zeta function and $\Lambda \gg 1$. Its mean number is $\langle n \rangle = \ln \Lambda/\zeta(2)$ and variance is $\Delta n = \sqrt{\Lambda/\zeta(2)}$. Here we only keep terms up to the leading order of $\Lambda$. The fidelity is thus

$$ F \simeq |\text{Li}_2(e^{-i\theta})|/\zeta(2) \simeq 1 - 3\theta/\pi $$

(48)

around $\theta = 0$, where $\text{Li}_n(x)$ is the $n$th polynomial logarithm. Hence, the SSW state cannot be used to detect a small phase shift [7, 9] even if $\langle n \rangle \to \infty$, because $F \to 1$ as $\theta \to 0$, referring to figure 4. However, for $M$ identical repeated measurements, we find that

$$ F \simeq (1 - 3\theta/\pi)^M \to e^{-3\theta/\pi} < 1 $$

(49)

as $\theta_m \simeq 1/\Lambda$ and $M \simeq \Lambda$. This implies that the minimum detectable phase shift with the total photon number is $\theta_m \simeq (\text{logarithmic corrections})/N_T$ associated with $N_T = M\langle n \rangle = \Lambda \ln \Lambda/\zeta(2)$.

Now we examine the small peak model in [4, 8] for $M$ repeated measurements, $|\Psi\rangle = |\psi\rangle^\otimes M$ and $|\psi\rangle = (|0\rangle + \nu|\alpha\rangle)/\sqrt{1 + \nu^2}$, assuming $\nu \ll 1$ and $\alpha \gg 1$. The quantum CR bound gives $\Delta \theta^2 \geq v^2/(M\langle n \rangle)$, where $\langle n \rangle = \nu^2\alpha^2$. In [4], the following parameters were chosen at will, namely $M \simeq N_T \simeq 1/\nu$ and $\langle n \rangle \simeq 1$, then the quantum CR bound leads to $\Delta \theta^2 \geq 1/N_T^2$, and then it was claimed in [4] that the HL is beaten. However, as noted above, the quantum CR bound is only a lower bound and sometimes only achievable for properly chosen parameters. If we keep $\langle n \rangle \simeq 1$ fixed, then the fidelity is

$$ F = \left|(1 + \nu^2 e^{-i\theta(1-e^{-i\theta})})/(1 + \nu^2)\right|^M $$

$$ \simeq (1 - \nu^2(1 - \cos\alpha^2\theta))^M. $$

(50)

In order to make equation (50) differ from unity significantly, we have to choose $\theta_m \simeq 1/\nu^2$ and $M \simeq 1/\nu^2$, which implies that $M \simeq N_T \simeq \alpha^2$ and $1/\nu \simeq \sqrt{N_T}$, and therefore the minimum detectable phase shift should be $\theta_m \simeq 1/\sqrt{N_T}$.

Next, we consider the two-mode cases with the field operators $a$ and $b$, such as the MZI. For two-mode squeezed vacuum (TMSV) [23],

$$ |\Psi\rangle = \sqrt{1 - t} \sum_{n} \sqrt{n!}|n\rangle_{a}|n\rangle_{b} $$

(51)

with $t = \langle n \rangle/(\langle n \rangle + 2)$, the action of the MZI is described by the unitary transformation $U_0 = \exp(\theta(a^\dagger b - b^\dagger a)/2)$. The fidelity is then given by

$$ F = (1 - t) \sum_{n} n^n P_n(\cos \theta) $$

$$ = 1/\sqrt{1 + \langle n \rangle(\langle n \rangle + 2) \sin^2(\theta/2)} $$

(52)
in terms of Legendre polynomials \( P_n \), and thus \( \theta_m \simeq 1/\sqrt{\langle n \rangle} \). Similarly, the entangled coherent state was proposed to reach the HL in [24], which can be expressed as \( |\Psi\rangle = (|\alpha\rangle_a |0\rangle_b + |0\rangle_a |\alpha\rangle_b) / \sqrt{2} \) with \( \langle n \rangle = \alpha^2 \) right after the first beam splitter. The corresponding fidelity is the same as that of the SCV state, \( \theta_m \simeq 1/\langle n \rangle \).

In [5], the following two states after the first beam splitter in the MZI are introduced to beat the HL, i.e. the noon-like state:

\[
|\Psi\rangle = \sum_{n=1}^{\infty} (|n\rangle_a |0\rangle_b + |0\rangle_a |n\rangle_b) / \sqrt{2} \zeta(3/n^3) \tag{53}
\]

and the dual-Fock-like state:

\[
|\Psi\rangle = \sum_{n=1}^{\infty} |n\rangle_a |n\rangle_b / \sqrt{2} \zeta(3/n^3). \tag{54}
\]

It was claimed that these two states can be used to realize unlimited phase sensitivity because they noted \( \langle n^2 \rangle \to \infty \). However, equation (37) tells us that \( \theta_m \geq 1/\langle n \rangle \), which is of order 1 since \( \langle n \rangle = \zeta(2)/\zeta(3) \). Therefore, they cannot even reach the HL. If we calculate their corresponding fidelities,

\[
F = |1 + \text{Li}_3(e^{-i\theta})|/\zeta(3) \tag{55}
\]

for the noon-like state and

\[
F = |\text{Li}_3(e^{-i\theta})|/\zeta(3) \tag{56}
\]

for the dual-Fock-like state, as shown in figure 4, they only differ from unity significantly at \( \theta \approx 1 \). The two states in [5] thus cannot beat the HL.

Finally, we consider a mixed input state in the MZI [23], namely

\[
\rho = (1 - p)|0, 0\rangle\langle 0, 0| + p|n, n\rangle\langle n, n| \tag{57}
\]

with \(|n, n\rangle = |n\rangle_a |n\rangle_b\), which has \( \langle n \rangle = 2pn \). The distance measure is

\[
D(\rho, U_{\theta} \rho U_{\theta}^+) = p\sqrt{1 - P_n^2(\cos \theta)} \simeq p\sqrt{1 - J_0^2(n\theta)}, \tag{58}
\]

where we have used the asymptotical expression of \( P_n(\cos \theta) \) for large \( n \) and \( J_0(x) \) is the Bessel function. In order to use this state for efficient phase estimation, it is required that \( p \approx 1 \) and then \( \theta_m \approx 1/\langle n \rangle \). Otherwise, repeated measurements are preferred. So we have verified that neither one of the above examples can perform better than the HL.

5. Summary

In conclusion, we investigated the generalized limits for the parameter sensitivity via the quantum Ziv–Zakai bound, which provides a lower bound in terms of the error probability in a quantum binary decision problem. Such a lower bound takes into account possible correlations induced by adaptive measurements. We also proved that the parameter sensitivity is bounded by the scaling of the minimum detectable parameter over the expectation value of the Hamiltonian. Finally, we examined several known states in quantum phase estimation with non-interacting photons and verified that neither one of them can perform better than the HL.

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