ADAPTIVE REGULATION TO INVARIANT SETS

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Abstract: A new framework for adaptive regulation to invariant sets is proposed. Reaching the target dynamics (invariant set) is to be ensured by state feedback while adaptation to parametric uncertainties is provided by additional adaptation algorithm. We show that for a sufficiently large class of nonlinear systems it is possible to adaptively steer the system trajectories to the desired non-equilibrium state without requiring knowledge or existence of a specific strict Lyapunov function.

Keywords: adaptive systems, non-equilibrium dynamics, invariance, algorithms in finite form

1. INTRODUCTION

Whether adaptive or non-adaptive solutions are sought in control theory, the problem is usually stated in terms of stabilization problem of an equilibrium or tracking of a given reference signal.

In recent years, motivated by problems in physics and natural sciences, slightly different demands came to the surface. Instead of forcing a system to an arbitrary equilibrium one should search for the natural motions in the system which satisfy the control goal the most, and then transform these to the desired state by gentle and small control efforts (Kolesnikov, 1994; Fradkov, 2003). One of the successful examples is the result reported in the seminal paper (Ott et al., 1990) with long-standing theoretical impact and exciting practical applications (Tziperman et al., 1997).

The problem with this and similar methods, however, in the context of adaptive control is that for the given feedback one must know the Lyapunov function ensuring asymptotic stability of the target dynamics. This gives rise to another severe limitation of the conventional Lyapunov-based methodology – the problem with asymptotic behavior of adaptive systems. Roughly speaking, the problem is as follows: while the specific Lyapunov function fits very well the non-adaptive controller design (i.e. ensures that solutions converge asymptotically to the desired state), it may not guarantee the desired asymptotic in the adaptive case. The reason for such is that the Lyapunov function itself is not strict. The breakthrough in this problem has been reported in (Panteley et al., 2002; Astolfi and Ortega, 2003). The problem has been resolved for equilibria that can be made asymptotically stable by state feedback. Yet, non-equilibrium and non-asymptotically stable dynamics were not addressed.

The problems of non-equilibrium control are gaining substantial attention in the recent years, especially in the framework of output regulation.
In (Byrnes and Isidori, 2003) a number of sufficient and necessary conditions assuring existence of the solution to this problem are proposed. Although the internal model principle in the output regulation problem (Byrnes et al., 1997) proves strong bindings between adaptive and output regulation problems, historical and methodological differences in these branches of the control theory do not always allow explicit application of the results from one field to another. This provides additional motivation to our current study in the context of adaptation.

The contribution of our present paper is as follows. First, we aim to formulate the problem of adaptive regulation to the desired non-equilibrium dynamics. This dynamics should in principle be invariant under the system flow. It also should poses certain properties like boundedness of the trajectories and/or partial stability (Vorotnikov, 1998). No asymptotic Lyapunov-like stability conditions are to be imposed a-priori in order to escape the burden of detectability. Second, under these assumptions we shall be able to derive adaptation algorithms which are capable of steering the system trajectories to the desired invariant set. In order to do so we employ the recently developed adaptive algorithms in finite form (Tyukin, 2003). These algorithms guarantee improved performance and are capable of handling nonlinear parametrization of the uncertainty (Tyukin et al., 2003a). The main idea of this approach is to introduce the desired invariant set into the system dynamics (virtual adaptation algorithms) and then realize these algorithms by means of the embedding technique proposed in (Tyukin et al., 2003b; Tyukin et al., 2004; Tyukin et al., 2003a).

The paper is organized as follows: in Section 2 we provide necessary notations and formulate the problem. Section 3 contains the main results of the paper given in Theorem 3. The proof of the theorem is provided in the subsequent subsections. Each of the subsection substitutes the separate step of the construction. Subsection 3.1 addresses design of the virtual algorithms, Subsection 3.2 provides auxiliary system which is necessary for the embedding, Subsection 3.3 contains the main arguments of the proof. Section 4 concludes the paper.

Throughout the paper we will use the following notations: symbol $x(t, x_0, t_0)$ stands for the flow which maps $x_0 \in \mathbb{R}^n, t_0, t \in \mathbb{R}_+$ into $x(t)$. Function $\nu : R_+ \to R$ is said to belong to $L_2$ if $L_2(\nu) = \int_0^\infty \nu^2(\tau) d\tau < \infty$. The value $\sqrt{L_2(\nu)}$ stands for the $L_2$ norm of $\nu(t)$. Function $\nu : \mathbb{R}_+ \to \mathbb{R}$ belongs to $L_\infty$ if $L_\infty(\nu) = \sup_{t \geq 0} \| \nu(t) \| < \infty$, where $\| \cdot \|$ is the Euclidean norm. The value of $L_\infty(\nu)$ stands for the $L_\infty$ norm of $\nu(t)$.

2. PROBLEM FORMULATION

**Definition 1.** A point $p \in \mathbb{R}^n$ is called an $\omega$-limit point $\omega(x(t, x_0, t_0))$ of $x_0 \in \mathbb{R}^n$ if there exists a sequence \( \{t_i\}, t_i \to \infty \), such that $x(t, x_0, t_0) \to p$. The set of all limit points $\omega(x(t, x_0, t_0))$ is the $\omega$-limit set of $x_0$.

In order to specify explicitly in our notations which particular flow is referred to in the notion of the $\omega$-limit set we use notations $\omega_{\Omega}(x_0)$ (and $x_{\Omega}(t, x_0, t_0)$) to denote the $\omega$-limit set (and flow) of $x_0$ in the following system $\dot{x} = f(x)$, $x_0 \in X \subset \mathbb{R}^n$. Symbol $\Omega_{\Omega}(x)$ denotes the union of all $\omega_{\Omega}(x_0)$, $x_0 \in X$. Throughout the paper we will refer to set $\Omega_{\Omega}(x)$ as the $\Omega_{\Omega}$-limit set (or simply $\Omega$-limit set if the corresponding flow is defined from the context) of the system.

**Definition 2.** Set $S \subset \mathbb{R}^n$ is invariant (forward-invariant) under the flow $\dot{x}_\Omega(t, x_0, t_0)$ if $x_{\Omega}(t, x_0, t_0) \in S$ for any $x_0 \in S$ for all $t > t_0$.

In our current study we consider the following class of systems:

\[
\begin{align*}
\dot{x} &= f(x) + G_u(\phi(x)\theta + u), \\
\dot{\theta} &= S(\theta), \quad \theta(t_0) = \theta_0 \in \Theta \subset \mathbb{R}^d
\end{align*}
\]

(1)

where $f : \mathbb{R}^n \to \mathbb{R}^n$, $\phi : \mathbb{R}^n \to \mathbb{R}^{m \times d}$, are $C^0$-smooth vector-fields, $G_u \in \mathbb{R}^{n \times m}$, $\theta$ is the vector of unknown time-varying parameters, $S : \mathbb{R}^d \to \mathbb{R}^d$, $S \in C^1$ is known, vector of initial conditions $\theta(t_0) = \Theta$, however, is assumed to be unknown. Without loss of generality we assume that $\Omega_S(\Theta) \subseteq \Theta$, and that $\Theta$ is bounded. Our goal is to steer the state to the target domain:

$\Omega^*(x) \subset \mathbb{R}^n$

Let us introduce the following set of assumptions related to the choice of domain $\Omega^*(x)$.

**Assumption 1.** Set $\Omega^*(x) \subset \mathbb{R}^n$ is the bounded and closed set in $\mathbb{R}^n$.

**Assumption 2.** There exists positive-definite matrix $H = H^T \in \mathbb{R}^{d \times d}$, such that function $S : \mathbb{R}^d \to \mathbb{R}^d$ in (1) satisfies the following inequality:

\[
H \frac{\partial S(\theta)}{\partial \theta} + \frac{\partial S(\theta)\theta}{\partial \theta} \leq 0 \quad \forall \theta \in \mathbb{R}^d
\]

**Assumption 3.** For the given $\Omega^*(x)$ and system (1) there exists control function $u_0(x)$ such that $G_u u_0(x) + f(x) = f_0(x)$ and, furthermore, for any $x_0 \in \mathbb{R}^n$ the following holds: $\Omega^*(x) \subset \Omega_0(x)$, where the flow $x(t, x_0, t)$ is defined by

\[
\dot{x} = f_0(x)
\]

(2)
Let us finally introduce two alternative hypotheses. The first hypothesis is formulated in Assumptions 4, 5, and 6. The second is given by Assumption 7.

**Assumption 4.** There exist functions \( \psi(x) : \mathbb{R}^n \to \mathbb{R}, \varphi : \mathbb{R}^n \to \mathbb{R} \), and induced by function \( \psi(x) \) set:

\[
\Omega_\psi = \{ x \in \mathbb{R}^n | x : \varphi(\psi(x)) = 0 \}
\]
such that the following holds \( \Omega^* \subseteq \Omega_k(\Omega_\psi) \), i.e. \( \Omega^*(x) \) is the largest invariant set of (2) in \( \Omega_\psi \).

**Assumption 5.** For the given function \( \psi(x) : \mathbb{R}^n \to \mathbb{R}, \psi(x) \in C^1 \) and vector field \( f_0(x) \) defined in (2) there exists function \( \beta(x) : \mathbb{R}^n \to \mathbb{R}_+ \) such that \( \beta(x) \) is separated from zero and satisfies the following equality:

\[
\psi \frac{\partial \psi(x)}{\partial x} f_0(x) \leq -\beta(x) \varphi(\psi) \psi,
\]

\[
\int_0^\infty \varphi(\sigma) d\sigma \geq 0, \quad \lim_{\psi \to \infty} \int_0^\psi \varphi(\sigma) d\sigma = \infty \tag{3}
\]

**Assumption 6.** For the given function \( \psi(x) : \mathbb{R}^n \to \mathbb{R}, \psi(x) \in C^1 \) the following relation holds:

\[
\psi(x(t)) \in L_\infty \Rightarrow x \in L_\infty
\]

Notice that function \( \psi(x) \) in Assumptions 5, 6 should not necessarily be the (positive) definite function. Function \( \varphi(\psi) \) is also not required to be (positive) definite.

**Assumption 7.** Consider system (2) with additive input \( \varepsilon_0(t) : \mathbb{R} \to \mathbb{R}^n, \varepsilon_0(t) \in C^1 \):

\[
x = f_0(x) + \varepsilon_0(t), \quad \varepsilon_0 \in L_2 \tag{4}
\]

System (4) has finite \( L_2 \to L_\infty \) gain, and in addition \( \Omega^* \subseteq \Omega_k \).

The main question of our current study is that whether or not it is possible to design the adaptation algorithm \( \hat{\theta}(t) \) for system (1) such that the feedback of the following form

\[
u(x, \xi) = u(x, \xi, \hat{\theta}) = f_\xi(x, \xi, \nu), \xi \in \mathbb{R}^k
\]

ensures boundedness of the trajectories in the closed loop system and that \( x(t) \to \Omega^* \) as \( t \to \infty \).

### 3. MAIN RESULTS

The main idea of our approach is two-fold. First, we search for the desired dynamics of the closed loop system with feedback \( u(x, \xi, \hat{\theta}) \) and yet unknown \( \hat{\theta}(t) \), \( \xi(t) \) which ensures desired properties of the controlled system. These properties should allow us to show that under specific conditions \( x(t) \to \Omega^* \) as \( t \to \infty \). Derivative of function \( \hat{\theta}(t) \) with respect to \( t \) at this stage can, in principle, depend on unknown parameters \( \theta \). Family of all such desired subsystems is referred to as virtual adaptation algorithms.

The second stage of our method is to render these algorithms into computable and physically realizable form. In particular, these realizations should neither rely on a-priori unknown parameters, nor should they require measurements of the right-hand side of (1) (i.e. derivatives).

In order to achieve this goal we invoke the algorithms in finite form (Tyukin, 2003; Tyukin et al., 2003a) (physically realizable and computable control) and the embedding argument introduced in (Tyukin et al., 2003b; Tyukin et al., 2004). In general, finite form realizations of virtual adaptation algorithms require analytic solution of a partial differential equation known as explicit realization condition. However, with the embedding technique proposed in our earlier publications it is possible to avoid this difficulty and derive adaptation schemes as a known and well-defined function of \( x, t \). The main result of our current study is formulated in Theorems 3 and 4 below.

**Theorem 3.** Let system (8) be given and Assumptions 1–6 hold. Let, in addition, there exists \( C^1 \)-smooth function \( \kappa(x) \) such that the following estimate holds:

\[
\nu(x, \xi, \nu) \leq |\kappa(x)|. \tag{5}
\]

Then there exists auxiliary system

\[
\dot{\xi} = f_\xi(x, \xi, \nu), \quad \nu = f_\nu(x, \xi, \nu), \quad \xi \in \mathbb{R}^n, \quad \nu \in \mathbb{R}^d
\]

control input \( u(x, \hat{\theta}) = u_0(x) - \phi(\xi)\hat{\theta}(t) \), and adaptation algorithm

\[
\dot{\theta} = (H^{-1}\Psi(\xi)x + \theta_1(t)), \quad \dot{\theta}_1 = S(\hat{\theta}) - H^{-1}\Psi(\xi)
\]

such that the following properties hold:

1) \( \hat{\theta}(t), x(t) \in L_\infty \)

2) trajectories \( x(t) \) converge into the domain \( \Omega^* \) as \( t \to \infty \)

3) if \( G_u\phi(\xi) \) is persistently exciting then \( \hat{\theta}(t), \theta(t, \theta_0, t_0) \) asymptotically converges to \( \theta(t, \theta_0, t_0) \).

**Theorem 4.** Let system (8) be given and Assumptions 1–3, and 7 hold. Then there exist auxiliary system of type (5), control input \( u(x, \theta) = u_0(x) - \phi(\xi)\theta(t) \) and adaptation algorithm (6) with \( \kappa(\xi) \equiv 0 \) such that statements 1)–3) of Theorem 3 hold.
The proof of the theorems is given in the next subsections. In subsection 3.1 we derive virtual adaptation algorithms which satisfy in part the requirement of the theorem. Subsection 3.2 introduces function \( \xi(t) \) satisfying the embedding assumption from (Tyukin et al., 2003b). (Tyukin et al., 2003a). In subsection 3.3 we combine these results together and complete the proofs.

### 3.1 Design of Virtual Adaptive Algorithms

Let us consider the following dynamic state feedback \( u(x, \theta) = u_0(x) - \phi(\xi)\hat{\theta}(t) \). This feedback renders system (1) into the following form

\[
\dot{x} = f_0(x) + G_u(\phi(x) - \phi(\xi))\dot{\theta} + G_u(\phi(x) - \phi(\xi))\theta(7)
\]

Let us denote \( G_u\phi(x) = \alpha(x) \) and consider the following auxiliary system

\[
\dot{x} = f_0(x) + \alpha(\xi)(\theta - \hat{\theta}) + \varepsilon(t), \quad \dot{\theta} = S(\theta)
\]

Let us denote \( \kappa(\xi) : \mathbb{R}^n \rightarrow \mathbb{R}, \kappa \in C^1 \)

**Lemma 5.** (Virtual Adaptation Algorithm). Let system (8) be given and Assumptions 1–3, 6 hold. Furthermore, let \( \kappa(\xi(t))\varepsilon(t) \in L_2 \), and \( \varepsilon \in L_2 \).

Then the following statements hold:

1) \( \hat{\theta}(t) \) is bounded for every \( \theta(t_0) \in \Theta, \hat{\theta}(t_0) \in \mathbb{R}^d \)

2) \( \kappa(\xi)\alpha(\xi)(\hat{\theta}(t) - \theta(t)), \alpha(\xi)(\hat{\theta}(t) - \theta(t)) \) is bounded

3) Let, in addition, \( \|\frac{\partial \phi(x)}{\partial x}\| \leq |\kappa(\xi)|, \ x - \xi \in L_\infty \) then \( \mathbf{x} \in L_{\infty} \)

4) if, independently on the conditions of statement 3), \( \varepsilon(t) \equiv 0 \) and the function \( \alpha(\xi) \) is persistently exciting, i. e. there exist constants \( \delta, T > 0 \) such that

\[
\int_{t}^{t+T} \alpha(\xi(\tau))T \alpha(\xi(\tau)) \geq \delta I_d
\]

then \( \hat{\theta}(t) \) converges to the solution \( \theta(t, \theta_0, t_0) \) exponentially fast.

**Lemma 5 proof.** Let us show that statements
1) and 2) hold. Consider the following positive-definite function:

\[
\mathbf{V}_0(\theta, \hat{\theta}, t) = \|\theta - \hat{\theta}\|^2 + \varepsilon(\theta - \hat{\theta})^2 H(\theta - \hat{\theta}) + \varepsilon,
\]

where \( \varepsilon(t) = \frac{1}{2} \int_{t}^{t+T} (\kappa^2(\xi(\tau)) + 1)\varepsilon T(\tau)\varepsilon(\tau)d\tau \geq 0 \).

According to the lemma assumptions function \( \kappa(\xi(t))\varepsilon(t) \in L_{\infty} \). This implies that \( \varepsilon(t) \) is bounded for every \( t > t_0 \) and therefore function \( \mathbf{V}_0 \) is well defined. Let us consider derivative \( \dot{\mathbf{V}}_0 \).

\[
\dot{\mathbf{V}}_0 = (\theta - \hat{\theta})^T H(S(\theta) - S(\hat{\theta})) + (\theta - \hat{\theta})^T \mathbf{C}(\theta - \hat{\theta}) + \varepsilon(t)
\]

Function \( S(\cdot) \) is continuous, therefore, applying Hadamard lemma we can write the difference \( S(\theta) - S(\hat{\theta}) \) as follows:

\[
S(\theta) - S(\hat{\theta}) = \int_{0}^{1} \frac{\partial S(z(\lambda))}{\partial z(\lambda)}d\lambda(\theta - \hat{\theta}), \ z(\lambda) = \theta + \lambda(1 - \lambda).
\]

Applying Mean Value Theorem we derive the following:

\[
S(\theta) - S(\hat{\theta}) = \frac{\partial S(z(\lambda'))}{\partial z(\lambda)}(\theta - \hat{\theta}) \quad \text{for some } \lambda' \in [0, 1].
\]

The last equation leads to the following estimation of \( \dot{\mathbf{V}}_0 \):

\[
\dot{\mathbf{V}}_0 = (\theta - \hat{\theta})^T (\theta - \hat{\theta}) + (\theta - \hat{\theta})^T (\theta - \hat{\theta}) + 0.5\varepsilon(t)\]

Inequality (10) ensures that \( \theta - \hat{\theta} \) is bounded. Taking into account that for every \( \theta \in \Theta \) solutions \( \theta(t, \theta_0, t_0) \subseteq \Omega(\Theta) \subseteq \Theta \) where \( \Theta \) is the bounded set, we can conclude that trajectories \( \hat{\theta}(t) \) are bounded, i.e. \( \hat{\theta}(t) \in L_{\infty} \). Thus statement 1) is proven.

Let us prove statement 2) of the lemma. Notice that function \( V(\theta, \hat{\theta}, t) \) is non-increasing and bounded from below. Therefore \( \kappa(\xi)((\theta - \hat{\theta})^T (\theta - \hat{\theta}) + 0.5\varepsilon(t)) \in L_2 \). Hence function \( \kappa(\xi)((\theta - \hat{\theta})^T (\theta - \hat{\theta}) + 0.5\varepsilon(t)) \) belongs to \( L_2 \) as a sum of two functions from \( L_2 \). The fact that \( \kappa^2(\xi) + 1 \) is separated from zero implies that \( (\theta - \hat{\theta})^T (\theta - \hat{\theta}) + 0.5\varepsilon(t) \) is bounded.

Let us show that \( \mathbf{x}(t) \in L_{\infty} \) under conditions formulated in statement 3) of the lemma. Consider derivative

\[
\dot{\psi} = \frac{\partial \psi}{\partial x} f_0(x) + \frac{\partial \psi}{\partial x} (\alpha(\xi)(\theta - \hat{\theta}) + \varepsilon(t))
\]

\[
= \frac{\partial \psi}{\partial x} f_0(x) + \frac{\partial \psi}{\partial x} (\frac{\partial \psi}{\partial x} - \frac{\alpha(\xi)}{\partial x})(\alpha(\xi)(\theta - \hat{\theta}) + \varepsilon(t))
\]

Notice that \( \psi \in C^1 \), \( \mathbf{x} - \xi \in L_{\infty} \) imply that the norm: \( ||\frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial x}\xi|| \) is bounded. Moreover, \( ||\frac{\partial \psi}{\partial x}|| \leq \kappa(\xi) \). Hence we can rewrite (11) as follows:

\[
\dot{\psi} = \frac{\partial \psi}{\partial x} f_0(x) + \mu(t), \mu(t) \in L_2
\]
Function $\beta(x)$ is separated from zero, i.e. $\exists \delta > 0$ : $\beta(x) > 2\delta \forall x \in \mathbb{R}^n$. Let us consider the following positive-definite function:

$$
V_\varphi = \int_0^\varphi \varphi(\sigma)d\sigma + \frac{1}{4\delta} \int_t^\infty \mu^2(\tau)d\tau
$$

(13)

Taking into account Assumption 5 and equality (12) derivative $\dot{V}_\varphi$ can be estimated as follows:

$$
\dot{V}_\varphi \leq -\beta(x)\varphi^2(\psi) + \varphi(\psi)\mu(t) - \frac{1}{4\delta}\mu^2(t)
$$

$$
\leq -2\delta\varphi^2(\psi) + \varphi(\psi)\mu(t) - \frac{1}{4\delta}\mu^2(t)
$$

$$
= -\delta\varphi^2(\psi) - \delta(\varphi(\psi) - \frac{1}{2}\mu(t))^2 \leq 0
$$

(Boundedness of $x$ then follows explicitly from Assumption 6. This proves statement 3).

Let us prove that estimate $\theta(t)$ converges to $\hat{\theta}$ exponentially fast under assumption of persistent excitation and assuming that $\varepsilon \equiv 0$. Consider the following subsystem

$$
\dot{\theta} = S(\theta) - S(\hat{\theta}) - H^{-1}(\kappa^2(\xi) + 1)x
$$

$$
\alpha(\xi)^T \alpha(\xi) \hat{\theta} = \left( \int_0^1 \frac{\partial S(z(\lambda))}{\partial z(\lambda)} d\lambda - \int_0^1 \frac{\partial S(z(\lambda))}{\partial z(\lambda)} d\lambda \right)
$$

$$
H^{-1}(\kappa^2(\xi) + 1) \alpha(\xi)^T \alpha(\xi) \hat{\theta}
$$

where $\hat{\theta} = \theta - \hat{\theta}$. According to equations (8) system (14) describe dynamics of $\theta(t) - \hat{\theta}(t)$. Solution of (14) can be derived in the following form $\theta(t) = e^{\int_0^t \frac{\partial S(z(\tau))}{\partial z(\tau)} d\tau} e^{-H^{-1}(\kappa^2(\xi(\tau)) + 1)x \alpha(\xi(\tau)) d\tau} \times (t_0)$, where $\theta'(\tau) = (\theta(\tau) - \hat{\theta}(\tau) (1 - \lambda \nu))$ for some $\lambda \in [0, 1]$. It follows from Assumption 2 that the induced matrix norm of $e^{\int_0^t \frac{\partial S(z(\tau))}{\partial z(\tau)} d\tau}$ is bounded, i.e. there exists some positive $D_0 > 0$ such that $\|e^{\int_0^t \frac{\partial S(z(\tau))}{\partial z(\tau)} d\tau}\| \leq D_0$ for all $t \geq 0$. On the other hand, for every $t > T$ there exists integer $n > 0$ such that $t = nT + r$, $r \in \mathbb{R}^+$, $T$, and the following estimation holds:

$$
\|e^{-H^{-1} \int_0^t \frac{\partial S(z(\tau))}{\partial z(\tau)} d\tau}\| \leq D_0 \|e^{-H^{-1} \frac{Ld + 1}{2}d\tau}\| \leq \|e^{-H^{-1} \frac{Ld + 1}{2}d\tau}\|.
$$

Hence we can bound the norm $\|\theta(t)\|$ as follows:

$$
\|\theta(t)\| \leq D_0 \|e^{-H^{-1} \frac{Ld + 1}{2}d\tau}\| \|\theta(t_0)\|
$$

The lemma is proven.

### 3.2 Embedding (design of the extension)

In this section we show that for the class of systems given by (1) with locally Lipschitz $\phi_i(x)$:

$$
\phi(x) : \mathbb{R}^n \rightarrow \mathbb{R}^{d \times m},
$$

$$
\phi(x) = \begin{pmatrix}
\phi_{1,1}(x), \ldots, \phi_{1,d}(x) \\
\vdots, \ldots, \vdots \\
\phi_{m,1}(x), \ldots, \phi_{m,d}(x)
\end{pmatrix},
$$

$$
\phi_i(x) = (\phi_{i,1}(x), \ldots, \phi_{i,d}(x))
$$

one can design $C^1$-smooth function $\xi(t)$ such that $$(\alpha(x) - \alpha(x))\theta(t), \kappa(x)(\alpha(x) - \alpha(x))\theta(t) \in L_2,$$

**Lemma 6.** Let system (1) be given and functions $\phi_i(x)$ defined as in (15) be locally Lipschitz:

$$
\|\phi_i(x) - \phi_i(x)\| \leq \lambda_i(x, \xi) \|x - \xi\|
$$

where $\lambda(x, \xi) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_+$. $\lambda(x, \xi)$ is locally bounded w.r.t. $x, \xi$. Let, furthermore, Assumption 2 hold. Then there exist system

$$
\dot{\xi} = f(x) + G_u u + \lambda(x, \xi)(x - \xi) + G_u \phi(x) \nu
$$

$$
\hat{\nu} = S(\nu) + H^{-1}(G_u \phi(x)^T (x - \xi)^T)
$$

$$
\lambda(x, \xi) = 1 + \sum_{i=1}^m \lambda_i^2(x, \xi)(1 + \kappa_i^2(x))
$$

(15)

such that the following hold:

1. $\|\alpha(x) - \alpha(x)\| \in L_2, \|\kappa(x)(\alpha(x) - \alpha(x))\| \in L_2$ for every bounded $\theta$.
2. $x \in L_\infty \Rightarrow \xi \in L_\infty, \lim_{t \rightarrow \infty} x(t) - \xi(t) = 0$

**Proof of Lemma 6.** To prove the lemma it is enough to consider the following positive definite function $V_\xi$:

$$
V_\xi = 0.5 \|x - \xi\|^2 + 0.5 \| \theta - \nu \|^2
$$

(14)

Its derivative can be written as follows:

$$
\dot{V}_\xi \leq -\lambda(x, \xi) \|x - \xi\|^2 + \lambda(x, \xi)^T G_u \phi(x)(\theta - \nu) + \theta - \nu)^T G_u \phi(x)^T (x - \xi) \leq -\lambda(x, \xi) \|x - \xi\|^2
$$

The last inequality implies that $\lambda_i(x, \xi) \|x - \xi\|, \kappa(x) \lambda_i(x, \xi) \|x - \xi\| \in L_2$. Hence

$$
\|\phi_i(x) - \phi_i(x)\| \leq \lambda_i(x, \xi) \|x - \xi\| \Rightarrow \|\phi_i(x) - \phi_i(x)\|, \kappa(x) \|\phi_i(x) - \phi_i(x)\| \in L_2
$$

Therefore, boundedness of $\theta(t)$ and finiteness of the induced norm $G_u$ there ensure that $\|G_u(\phi_i(x) - \phi_i(x))\theta(t)\|, \|\kappa(x) G_u(\phi_i(x) - \phi_i(x))\theta(t)\| \in L_2$.

In order to complete the proof we notice that function $V_\xi$ is nonincreasing and radially unbounded. This guarantees that $\xi$ is bounded as long as $x$ remains bounded. The fact that $\lambda(x, \xi) > 1$ implies that $x - \xi \in L_2$. Under assumptions of the lemma, the right-hand side of the system is locally bounded. This leads to uniform continuity of $\|x - \xi\|^2$, which guarantees that $\lim_{t \rightarrow \infty} (x - \xi) = 0$. The lemma is proven.

### 3.3 Embedding (proof of Theorems 3, 4)

In this section we provide technical proof of the main results of our paper.

**Proof of Theorem 3.** According to Lemma 6 there exist system (15):

$$
\dot{\xi} = f(x) + G_u u + \lambda(x, \xi)(x - \xi) + G_u \phi(x) \nu
$$

$$
\hat{\nu} = S(\nu) + H^{-1}(G_u \phi(x)^T (x - \xi)^T)
$$

(16)
such that $\|G_u(\phi(x) - \phi(\xi))\|, \|\kappa(\xi)G_u(\phi(x) - \phi(\xi))\| \in L_2$ for every bounded $\theta(t)$ and trajectory $x(t)$ generated by

$$\dot{x} = f(x) + G_u(\phi(x)\theta + u); \quad \dot{\theta} = S(\theta) \tag{17}$$

Using the notation introduced in the previous subsections: $\alpha(\xi) = G_u\phi(\xi)$, taking into account that $u(x, \theta) = u_0(x) - \phi(\xi)\theta(t)$, and denoting $\varepsilon(t) = (\alpha(x) - \alpha(\xi))\theta(t)$ we rewrite (17) as follows:

$$\dot{x} = f_0(x) + \alpha(\xi)(\theta - \dot{\theta}(t)) + \varepsilon(t)$$

Taking into account equation (18) and expression (6) specifying the function $\theta(t)$ we can derive the time-derivative $\dot{\theta}$:

$$\dot{\theta} = S(\hat{\theta}) + H^{-1}(\kappa^2(\xi) + 1)\alpha^T(\xi)(\alpha(\xi)(\theta - \dot{\theta}) + \varepsilon(t)) \tag{19}$$

Then applying Lemma 5 we can conclude that both $x(t)$ and $\theta$ are bounded, i.e. $x(t), \theta(t) \in L_\infty$. On the other hand, according to Lemma 6, boundedness of $x$ implies boundedness of $\xi(t)$. Hence statement 1) of the theorem is proven.

Notice also that according to Lemma 6 the following holds: $x(t) - \xi(t) \to 0$ as $t \to \infty$. This fact together with uniform asymptotic stability of unperturbed system (19) (i.e. when $\varepsilon(t) \equiv 0$) imply that $\theta(t, \theta_0, t_0) \to \theta(t, \theta_0, t_0)$ as $t \to \infty$. This proves statement 3) of the theorem.

Let us prove that $x(t) \to \Omega^*$ as $t \to \infty$. In order to do this let us rewrite the closed-loop system in the following form:

$$\dot{x} = f_0(x) + \alpha(\xi)(\theta - \dot{\theta}(t)) + \varepsilon(t)$$

Taking into account equation (18) and expression (6) specifying the function $\theta(t)$ we can derive the time-derivative $\dot{\theta}$:

$$\dot{\theta} = S(\hat{\theta}) + H^{-1}(\kappa^2(\xi) + 1)\alpha^T(\xi)(\alpha(\xi)(\theta - \dot{\theta}) + \varepsilon(t))$$

Therefore, applying LaSalle invariance principle (LaSalle, 1976) we can conclude that $(x(t), \theta(t))$ converge (as $t \to \infty$) to the largest invariant set in $\Omega_\nu \times \Omega_\theta$, where $\Omega_\nu = \{x \in \mathbb{R}^n | x: \varphi(x) = 0\}$, and $\Omega_\theta = \{\theta \in \mathbb{R}^d | \theta: \alpha(\xi)(\theta - \dot{\theta}) + \varepsilon(t) = 0\}$. For the trajectory $x(t)$ this set is defined as the largest invariant set of system

$$\dot{x} = f_0(x) + \varepsilon(t), \quad (21)$$

under restriction that $x(t) \in \Omega_\nu$. According to Assumption 4 the largest invariant set of (21) in $\Omega_\nu$ is $\Omega^*$. Q.E.D.

Proof of Theorem 4. Consider system (16). It follows from Lemma 6 and Assumption 2 that $G_u(\phi(x) - \phi(\xi))\theta \in L_2$. Then boundedness of $\theta(t)$ follows explicitly from the proof of Theorem 3 (let $\kappa(\xi) \equiv 0$ in (10)). Furthermore, Lemma 5 ensures that $G_u\phi(\xi)(\theta - \dot{\theta}) \in L_2$. Hence denoting $\epsilon_0(t) = G_u\phi(\xi)(\theta - \dot{\theta}) + G_u(\phi(x) - \phi(\xi))\theta$ we obtain that trajectories $x(t)$ in system (1) satisfy the following equation:

$$\dot{x} = f_0(x) + \varepsilon(t), \quad (22)$$

where $\varepsilon(t) \in L_2$. System (22), however, has finite $L_2 \to L_\infty$ gain and therefore $x(t)$ is bounded. Therefore, statement 1) of the theorem is proven.

Statement 3) follows explicitly from Lemma 5. Let us show that $x(t) \to \Omega^*$ as $t \to \infty$. In order to do so let us consider system (20) excluding the equation for $\epsilon_0$. We have already shown that solutions of system (20) are bounded. Define $V = \|\theta - \dot{\theta}\|^2_\mathcal{H} + \frac{1}{2}\epsilon_1(t) + \epsilon_2(t)$. Its time-derivative satisfies the following inequality: $\dot{V} \leq -\|\alpha(\xi)(\theta - \dot{\theta}) + \varepsilon(t)\|^2$ and therefore, applying LaSalle invariance principle (LaSalle, 1976) we obtain that $x(t) \to \Omega^*$ as $t \to \infty$. The theorem is proven.

4. CONCLUSION

In this paper we have proposed a new framework for adaptive regulation to invariant sets. The main advantage of our approach is that we do not require knowledge of the strict Lyapunov functions for design of the adaptation schemes. Our method also handles non-equilibrium desired regimes of the system. In addition it does not assume asymptotic Lyapunov stability of the target dynamics.

The number of the additional equations required for implementation of our method is $(n + 2d)\nu$ which compares favorably with $(nd + d + n)$ in (Panteley et al., 2002). Though the conditions we require differ from that of (Panteley et al., 2002), we believe that our results naturally complement the existing ones without too much of additional restrictions.
In the present study we considered linear parameterizations of the uncertainties. On the other hand, the machinery we use in the proofs allows to extend the results to nonlinear parameterized systems (Tyukin et al., 2003c; Tyukin et al., 2003a). This together with robustness analysis are currently the topics of our future studies.

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