EXISTENCE OF SOLUTIONS FOR A KIRCHHOFF-TYPE-NONLOCAL OPERATORS OF ELLIPTIC TYPE

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Abstract. In this paper by using the minimal principle and Morse theory, we prove the existence of solutions to the following Kirchhoff nonlocal fractional equation:

\[
\begin{align*}
\left\{ \begin{array}{ll}
M \left( \int_{\mathbb{R}^n \times \mathbb{R}^n} |u(x) - u(y)|^2 K(x-y) \, dx \, dy \right) (-\Delta)^s u = f(x,u(x)), & \text{in } \Omega, \\
u = 0, & \text{in } \mathbb{R}^n \setminus \Omega,
\end{array} \right.
\end{align*}
\]

where \((-\Delta)^s\) is the fractional Laplace operator, \(s \in (0,1)\) is a fix, \(\Omega\) an open bounded subset of \(\mathbb{R}^n\), \(n > 2s\), with Lipschitz boundary, \(f : \Omega \times \mathbb{R} \to \mathbb{R}\) Carathéodory function and \(M : \mathbb{R}^+ \to \mathbb{R}^+\) is a function that satisfy some suitable conditions.

1. Introduction. The aim of this paper is to establish the existence of nontrivial solutions for the following Kirchhoff-type-nonlocal problem

\[
\begin{align*}
\left\{ \begin{array}{ll}
-\mathcal{L}_K u(x) = f(x,u(x)), & \text{in } \Omega, \\
u = 0, & \text{in } \mathbb{R}^n \setminus \Omega,
\end{array} \right.
\end{align*}
\]

where \(\Omega\) an open bounded subset of \(\mathbb{R}^n\) with Lipschitz boundary, \(f : \Omega \times \mathbb{R} \to \mathbb{R}\) is a Carathéodory function, \(\mathcal{L}_K : \mathbb{R}^+ \to \mathbb{R}^+\) is a function that satisfy conditions which will be stated later, and

\[
\mathcal{L}_K u(x) := \int_{\mathbb{R}^n} \left( u(x+y) + u(x-y) - 2u(x) \right) K(y) \, dy, \quad x \in \mathbb{R}^n,
\]

where \(K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)\) is a kernel function satisfying properties that

(K1) \(mK \in L^1(\mathbb{R}^n)\), where \(m(x) = \min\{|x|^2, 1\}\); 
(K2) there exists \(\theta > 0\) such that \(K(x) \geq \theta|x|^{-(n+2s)}\) for any \(x \in \mathbb{R}^n \setminus \{0\}\); 
(K3) \(K(x) = K(-x)\) for any \(x \in \mathbb{R}^n \setminus \{0\}\).

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The homogeneous Dirichlet datum in (1) is given in $\mathbb{R}^n \setminus \Omega$ and not simply on the boundary $\partial \Omega$, consistent with the nonlocal character of the kernel operator $L_K$.

A typical model for $K$ is given by the singular kernel

$$K(x) = |x|^{-n+2s}$$

which gives rise to the fractional Laplace operator $-(-\Delta)^s$ where $s \in (0, 1)$ ($n > 2s$) is fixed, which, up to normalization factors, may be defined as

$$-(-\Delta)^s u(x) := \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} dy, \quad x \in \mathbb{R}^n. \quad (3)$$

The problem (1) in the model case $L_K = -(-\Delta)^s$ becomes

$$\begin{cases}
M \left( \int_{\mathbb{R}^n \times \mathbb{R}^n} |u(x) - u(y)|^2 K(x-y) dx dy \right) (-\Delta)^s u = f(x, u(x)), & \text{in } \Omega, \\
u = 0, & \text{in } \mathbb{R}^n \setminus \Omega. \quad (4)
\end{cases}$$

Recently, several studies have been performed for non-local fractional Laplacian equations substituted by superlinear and subcritical or critical nonlinearities, we refer interested readers to [1, 2, 3, 4, 5, 9, 10, 11, 12, 14, 15, 17, 18, 19] and references therein.

Inspired by the above articles and very recent articles [6, 16], in this paper, we would like to investigate the existence of solutions for problem (1). The technical tool is the minimal principle and Morse theory.

The paper is organized as follows. In Section 2, we give preliminary facts and provide some basic properties which are needed later. Sections 3 and 4 are devoted to our results on existence of one solution by the minimal principle and existence of two nontrivial solutions by Morse theory, respectively.

2. Preliminaries. In this section, we present some preliminaries and lemmas that are useful to the proof to the main results. For the convenience of the reader, we also present here the necessary definitions.

Let $(X, \| \cdot \|_X)$ be a Banach space, $(X^*, \| \cdot \|_{X^*})$ be its topological dual, and $\varphi : X \to \mathbb{R}$ be a functional. First, we recall the definition of the Palais-Smale condition which plays an important role in our paper.

Definition 2.1. We say that $\varphi$ satisfies the Palais-Smale condition if any sequence $(u_n) \in X$ for which $\varphi(u_n)$ is bounded and $\varphi'(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence.

Before proving the main result, some preliminary material on function spaces and norms is needed. In the following we briefly recall the definition of the functional space $X_0$, firstly introduce in [13], and we give some notations. We denote $Q = \mathbb{R}^{2n} \setminus \mathcal{O}$, where $\mathcal{O} = \mathbb{R}^n \setminus \Omega \times \mathbb{R}^n \setminus \Omega$. We denote the set $X$ by

$$X = \left\{ u : \mathbb{R}^n \to \mathbb{R} : u|_{\Omega} \in L^2(\Omega), (u(x) - u(y))\sqrt{K(x-y)} \in L^2(\mathbb{R}^{2n} \setminus \mathcal{O}) \right\},$$

where $u|_{\Omega}$ represents the restriction to $\Omega$ of function $u(x)$. Also, we denote by $X_0$ the following linear subspace of $X$

$$X_0 = \{ g \in X : g = 0 \ \text{a.e. in } \mathbb{R}^n \setminus \Omega \}.$$

In this paper we will prove the existence of nontrivial weak solutions for problem (1) using variational and topological methods. By a weak solutions of (1) we mean...
a solution of the following problem

\[
\begin{align*}
M \left( \int_{\mathbb{R}^n \times \mathbb{R}^n} |u(x) - u(y)|^2 K(x-y) dxdy \right) & \left[ \int_{\mathbb{R}^n \times \mathbb{R}^n} (u(x) - u(y)) \eta(x) \eta(y) dxdy \right] \\
K(x-y) dxdy & = \int_{\Omega} f(x, u(x)) \eta(x) dx, \quad \forall \eta \in X_0,
\end{align*}
\]

(5)

\[ u \in X_0. \]

We know that \( X \) and \( X_0 \) are nonempty, since \( C^2_0(\Omega) \subseteq X_0 \) by Lemma 11 of [13]. Moreover, the linear space \( X \) is endowed with the norm defined as

\[
\|u\| := \|u\|_{L^2(\Omega)} + \left( \int_{Q} |g(x) - u(y)|^2 K(x-y) dxdy \right)^{\frac{1}{2}}. \tag{6}
\]

It is easy seen that \( \| \cdot \| \) is a norm on \( X \) (see, for instance, [11] for a proof). By Lemmas 6 and 7 of [11], in the sequel we can take the function

\[
X_0 \ni v \mapsto ||v||_{X_0} = \left( \int_{Q} |v(x) - v(y)|^2 K(x-y) dxdy \right)^{\frac{1}{2}} \tag{7}
\]
as norm on \( X_0 \). Also \( (X_0, \| \cdot \|_{X_0}) \) is a Hilbert space, with scalar product

\[
\langle u, v \rangle_{X_0} := \int_{Q} (u(x) - u(y))(v(x) - v(y)) K(x-y) dxdy. \tag{8}
\]

Note that in (7) the integral can be extend to all \( \mathbb{R}^n \times \mathbb{R}^n \), since \( v \in X_0 \) and so \( v = 0 \) a.e. in \( \mathbb{R} \setminus \Omega \).

In what follows, we denote by \( \lambda_1 \) the first eigenvalue of the operator \( \mathcal{L}_K \) with homogeneous Dirichlet boundary data, namely the first eigenvalue of the problem

\[
\begin{align*}
\mathcal{L}_K u = \lambda u, & \quad \text{in } \Omega, \\
u = 0, & \quad \text{in } \mathbb{R}^n \setminus \Omega,
\end{align*}
\]

which \( \lambda_1 \) is positive and that can be characterized as follows

\[
\lambda_1 = \min_{u \in X_0 \setminus \{0\}} \frac{\int_{\mathbb{R}^n \times \mathbb{R}^n} |u(x) - u(y)|^2 K(x-y) dxdy}{\int_{\Omega} |u(x)|^2 dx}. \tag{9}
\]

We refer to Proposition 9 and Appendix A of [12], for the existence and the basic properties of this eigenvalue, where a spectral theory for general integro-differential nonlocal operators was developed.

Let \( V = \text{span}\{e_1\} \) be the one-dimensional eigenspace associated with \( \lambda_1 \), where \( e_1 > 0 \) in \( \mathbb{R}^n \) and \( \|e_1\|_{L^2(\mathbb{R}^n)} = 1 \). Taking one subspace \( Y \subset X_0 \) completing \( V \) such that \( X_0 = V \oplus Y \), there exists \( \lambda > \lambda_1 \) such that

\[
\int_{\mathbb{R}^n \times \mathbb{R}^n} |u(x) - u(y)|^2 K(x-y) dxdy \geq \lambda \int_{\Omega} |u(x)|^2 dx.
\]

Let \( H^s(\mathbb{R}^n) \) be the usual fractional Sobolev space endowed with the norm (the so-called Gagliardo norm)

\[
\|u\|_{H^s(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)} + \left( \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|y-x|^{n+2s}} dxdy \right)^{\frac{1}{2}}. \tag{9}
\]

Also, we recall the embedding properties of \( X_0 \) into the usual Lebesgue spaces (see Lemma 8 of [11]). The embedding \( j : X_0 \hookrightarrow L^s(\mathbb{R}^n) \) is continuous for any
$v \in [1, 2^*]$ ($2^* = \frac{2n}{n-2}$), while it is compact whenever $v \in [1, 2^*)$. Hence, for any $v \in [1, 2^*)$ there exists a positive constant $\delta_v$, such that
\[\|v\|_{L^q(\mathbb{R}^n)} \leq \delta_v \|v\|_{X_0},\] (10)
for any $v \in X_0$.

3. Existence solution by the minimal principle. The functional $J : X_0 \rightarrow \mathbb{R}$ corresponding to problem (1) is defined by
\[J(u) = \frac{1}{2} \overline{M} \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} |u(x) - u(y)|^2 K(x-y) dx dy - \int_{\Omega} F(x, u(x)) dx\right)\]
\[= \frac{1}{2} \overline{M} \left(\|u\|_{X_0}^2 - \int_{\Omega} F(x, u(x)) dx,\right)\] (11)
where $\overline{M}(s) = \int_0^s M(t) dt$ and $F(x, t) = \int_0^t f(x, s) ds$ and
\[\Phi(u) = \frac{1}{2} \overline{M} \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} |u(x) - u(y)|^2 K(x-y) dx dy\right).\] (12)

In this section, we assume that we assume that $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function with subcritical growth with respect to $t$, that is:

(F1) $|f(x, t)| \leq C(1 + |t|^q-1)$ hold for some positive constant $C$, for all $x \in \Omega$ and $t \geq 0$, where $1 \leq q < 2^*$;

and $M : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous function satisfying the following condition

(H0) $M(t) \geq mt^{\alpha-1}$ for all $t \in \mathbb{R}^+$, where $m > 0$, $\alpha > 1$ and $q < 2\alpha$.

We recall a convergence property for bounded sequences in $X_0$ (see [11], for this we need a Lipschitz boundary):

Lemma 3.1. Let $K : \mathbb{R}^n \setminus \{0\} \rightarrow (0, +\infty)$ satisfy assumptions (K1)-(K3) and let $\{u_n\}$ be a bounded sequence in $X_0$. Then, there exists $u \in L^p(\mathbb{R}^n)$ such that, up to a subsequence, $u_n \rightarrow u$ in $L^p(\mathbb{R}^n)$, as $n \rightarrow \infty$, for any $p \in [1, 2^*)$.

Now, we can state our main result in this section.

Theorem 3.2. Let $K$ satisfy assumptions (K1)-(K3). Under assumptions (H0) and (F1) hold, then the problem (1) has at least one weak solution in $X_0$.

Proof. Let $(u_n)$ be a sequence that converges weakly to $u$ in $X_0$, so by Lemma 3.1, we have
\[\begin{align*}
\begin{cases}
u_n \rightarrow u, \text{ weakly in } X_0, \\
u_n \rightarrow u, \text{ strongly in } L^p(\mathbb{R}^n) \ (1 \leq p < 2^*), \\
u_n \rightarrow u, \text{ a.e. in } \mathbb{R}^n.
\end{cases}
\end{align*}\] (13)
Therefore, by the weak lower semicontinuous of the norm, one can get
\[\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^n \times \mathbb{R}^n} |u_n(x) - u_n(y)|^2 K(x-y) dx dy \geq \int_{\mathbb{R}^n \times \mathbb{R}^n} |u(x) - u(y)|^2 K(x-y) dx dy.\]

Combining this with the continuity and monotonicity of the function $\overline{M}$, we have
\[\liminf_{n \rightarrow \infty} \Phi(u_n) = \liminf_{n \rightarrow \infty} \frac{1}{2} \overline{M} \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} |u_n(x) - u_n(y)|^2 K(x-y) dx dy\right)\]
\[\geq \frac{1}{2} \overline{M} \left(\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^n \times \mathbb{R}^n} |u_n(x) - u_n(y)|^2 K(x-y) dx dy\right)\]
\[\geq \frac{1}{2} \overline{M} \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} |u(x) - u(y)|^2 K(x-y) dx dy\right) = \Phi(u).\] (14)
By (F1), (10) and the H"older inequality, we get
\[ \int_{\Omega} (F(x, u_n(x)) - F(x, u(x)))dx \]
\[ \leq \int_{\Omega} f(x, u + \theta_n(u_n - u))|u_n - u| dx \]
\[ \leq C \int_{\Omega} (1 + |u + \theta_n(u_n - u)|^{\eta-1})|u_n(x) - u(x)| dx \]
\[ \leq C|\Omega|^{\frac{\eta-1}{\eta}}||u_n - u||_{L^\eta} + C||u + \theta_n(u_n - u)||_{L^\eta}^{\eta-1}||u_n - u||_{L^\eta} \]
\[ \leq Cg_0^q|\Omega|^{\frac{\eta-1}{\eta}}||u_n - u||_{X_0} + Cg_0^{q-1}||u + \theta_n(u_n - u)||_{X_0}^{\eta-1}||u_n - u||_{X_0}, \]
which tends to 0 as \( n \to \infty \), where \( 0 \leq \theta_n(x) \leq 1 \), for all \( x \in \Omega \). From (14) and (15), the functional \( J \) is weakly lower semicontinuous in \( X_0 \).

On the other hand, by assumptions (H0) and (F1), one can get
\[ J(u) = \frac{1}{2} \int_{\Omega} \left( \int_{\mathbb{R}^n \times \mathbb{R}^n} |u(x) - u(y)|^2 K(x - y) dx dy \right) - \int_{\Omega} F(x, u(x)) dx \]
\[ \geq \frac{m}{2\alpha}||u||_{X_0}^{2\alpha} - C \int_{\Omega} (1 + |u(x)|^{\eta}) dx \]
\[ \geq \frac{m}{2\alpha}||u||_{X_0}^{2\alpha} - C'||u||_{L^\eta} - C'' \]
\[ \geq \frac{m}{2\alpha}||u||_{X_0}^{2\alpha} - C'g_0^{q}||u||_{X_0}^{q} - C''. \quad (16) \]
Since \( 1 < q < 2\alpha \), it follows from (16), that the functional \( J \) is coercive. Thus, using the minimal principle, we deduce that the functional \( J \) has at least one weak solution and therefore the problem (1) has at least one weak solution. \( \square \)

4. Existence solution by Morse theory. In this section, we assume that \( M : \mathbb{R}^+ \to \mathbb{R}^+ \) is a continuous function with the following conditions:
(M0) there exists a constant \( m_0 > 0 \) such that \( M(t) \geq m_0 \) for all \( t \geq 0 \);
(M1) there exists a constant \( m_1 > 0 \) such that \( M(t) \leq m_1 \) for all \( t \geq 0 \), where \( 2\frac{m_1}{m_0} \leq 2^* = 2\frac{2\alpha}{n-2\alpha} \).

We note that if \( m_0 = m_1 = 1 \) the problem (1) reduces to the usual non-local fractional problem.

Also, we make the following assumptions:
(F2) there exist \( r > 0, \bar{\lambda} \in (\lambda_1, \lambda_\infty) \) such that \( m_1 \lambda_1 < m_0 \bar{\lambda} \), and \( |u| \leq r \) implies \( m_1 \lambda_1 |u|^2 \leq 2F(x, u) \leq m_0 \bar{\lambda} |u|^2 \);
(F3) \( \limsup_{|t| \to \infty} \frac{2F(x, t)}{|t|^2} < m_0 \lambda_1 \);
(F4) \( \lim_{|t| \to \infty} \frac{2F(x, t)}{|t|^2} = m_0 \lambda_1 \);
(F5) \( \lim_{|t| \to \infty} (f(x, t) - 2F(x, t)) = +\infty \).

**Lemma 4.1.** Assume that (M0) holds. Then \( \Phi' \) is of type \((S_+), \) i.e. if \( u_n \rightharpoonup u \) in \( X \) and
\[ \lim_{n \to +\infty} (\Phi'(u_n) - \Phi'(u), u_n - u) = 0, \]
then \( u_n \to u \) in \( X_0 \).
Lemma 4.2. Assume that $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ satisfy assumptions (K1)-(K3). Under assumptions (m0) and (F1), any bounded sequence $\{u_n\}$ in $X_0$ such that $J'(u_n) \to 0$ in $(X_0)^*$ as $n \to \infty$ has a convergent subsequence.

Proof. Since $(u_n)$ is bounded in $X_0$ and $X_0$ is a reflexive Banach space ($X_0$ is a Hilbert space) and so by passing to a subsequence (for simplicity denoted gain by $\{u_n\}$) if necessary, by Lemma 3.1, we may assume that

\[
\begin{cases}
  u_n \rightharpoonup u, \text{ weakly in } X_0, \\
  u_n \to u, \text{ strongly in } L^p(\mathbb{R}^n) (1 \leq p < 2^*), \\
  u_n \to u, \text{ a.e. in } \mathbb{R}^n.
\end{cases}
\]

Therefore

\[
\langle J'(u_n), u_n - u \rangle \to 0, \quad \int_{\Omega} f(u_n(x))(u_n(x) - u(x))dx \to 0,
\]

so, we get

\[
\varepsilon_n \|u_n - u\| \geq \langle J'(u_n), u_n - u \rangle = \langle \Phi'(u_n), u_n - u \rangle - \int_{\Omega} f(u_n(x))(u_n(x) - u(x))dx \to 0.
\]

with $\varepsilon_n \to 0$. Thus $\limsup_{n \to \infty} \langle \Phi'(u_n), u_n - u \rangle \leq 0$. By (17), it is easy to get

\[
\lim_{n \to \infty} \langle \Phi'(u_n), u_n - u \rangle = 0.
\]

Therefore

\[
\limsup_{n \to \infty} \langle \Phi'(u_n) - \Phi'(u), u_n - u \rangle \leq \limsup_{n \to \infty} \langle \Phi'(u_n), u_n - u \rangle - \liminf_{n \to \infty} \langle \Phi'(u), u_n - u \rangle \leq 0.
\]

Since $\Phi'$ is of type $(S)_+$ (see Lemma 4.1), so we obtain $u_n \to u$ as $n \to \infty$ in $X_0$. 

Lemma 4.3. Assume that $K : \mathbb{R}^n \setminus \{0\} \to (0, +\infty)$ satisfy assumptions (K1)-(K3). Under assumption (M0), (F1) and (F3) (or substitute (F4) and (F5) for (F3)), the functional $J$ is coercive in $X_0$.

Proof. We divide the proof into two steps.

Step 1. Let (F3) hold. By (F1) and (F3) one can get for small enough $\varepsilon > 0$, there exists a constant $C_1 > 0$ such that

\[
F(x, t) \leq \frac{m_0}{2} (\lambda_1 - \varepsilon)|t|^2 + C_1, \quad \forall t \in \mathbb{R}, x \in \Omega \text{ a.e.}
\]

Thus, by definition of $\lambda_1$, for $u \in X_0$

\[
J(u) = \frac{1}{2} \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} |u(x) - u(y)|^2K(x - y)dy \right) dx - \int_{\Omega} F(x, u(x))dx
\geq \frac{m_0}{2} \|u\|_{X_0}^2 - m_0 \left(\frac{\lambda_1 - \varepsilon}{\lambda_1}\right) \|u\|_{L^2}^2 - C_1|\Omega|
\geq \frac{m_0}{2} \left(1 - \frac{\lambda_1 - \varepsilon}{\lambda_1}\right) \|u\|_{X_0}^2 - C_1|\Omega| \to +\infty \text{ as } \|u\|_{X_0} \to \infty.
\]
**Step 2.** If (F4) and (F5) hold, the we can write \( F(x, t) = \frac{m_0}{2} \lambda_1 |t|^2 + G(x, t) \) and \( f(x, t) = m_0 \lambda_1 |t| + g(x, t) \), which

\[
\lim_{|t| \to \infty} \frac{2G(x, t)}{|t|^2} = 0, \quad \lim_{|t| \to \infty} (g(x, t)t - 2G(x, t)) = +\infty.
\]

So, for any \( M > 0 \), there is \( R_M > 0 \) such that

\[ g(x, t)t - 2G(x, t) \geq M, \quad \forall |t| \geq R_M, \ x \in \Omega \ a.e. \]

Integrating the equality

\[ \frac{d}{dt} \frac{G(x, t)}{|t|^2} = \frac{g(x, t)t - 2G(x, t)}{|t|^3} \]

over the interval \([t, T] \subset [R_M, +\infty)\), we have

\[ \frac{G(x, T)}{|t|^2} - \frac{G(x, t)}{|t|^2} \geq \frac{M}{2} \left( \frac{1}{|t|^2} - \frac{1}{|T|^2} \right), \]

Letting \( T \to +\infty \), we have \( G(x, t) \leq -\frac{M}{2} t \) for \( t \geq R_M \), a.e. \( x \in \Omega \). In a similar way, we have \( G(x, t) \leq -\frac{M}{2} t \) for \( t \leq R_M \), a.e. \( x \in \Omega \). So we can see that

\[ \lim_{|t| \to \infty} G(x, t) = -\infty, \quad \text{a.e. } x \in \Omega. \]  

(18)

We suppose that, to the contrary, there exists a sequence \( \{u_n\} \subset X_0 \) such that \( \|u_n\|_{X_0} \to \infty \) as \( n \to \infty \), but \( J(u_n) \leq C_2 \) for some constant \( C_2 \in \mathbb{R} \). Set \( v_n = \frac{u_n}{\|u_n\|_{X_0}} \) then up to a subsequence, we assume that there is some \( v_0 \in X_0 \) such that

\[
\begin{cases}
  v_n \rightharpoonup v_0, \quad \text{weakly in } X_0, \\
v_n \to v_0, \quad \text{strongly in } L^p(\mathbb{R}^n) \ (1 \leq p < 2^*), \\
v_n \to v_0, \quad \text{a.e. in } \mathbb{R}^n.
\end{cases}
\]

Moreover one can get

\[
\begin{align*}
&\geq \frac{C_2}{\|u_n\|_{X_0}^2} \\
&\geq \frac{J(u_n)}{\|u_n\|_{X_0}^2} - \frac{1}{2\|u_n\|_{X_0}^2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u_n(x) - u_n(y)|^2 K(x - y) dx dy \\
&\quad - \frac{1}{\|u_n\|_{X_0}^2} \int_{\Omega} F(x, u_n(x)) dx \\
&\geq \frac{m_0}{2} \left( \int_{\mathbb{R}^n} |v_n(x) - v_n(y)|^2 K(x - y) dx dy - \lambda_1 \int_{\Omega} |v_n(x)|^2 dx \right) \\
&\quad - \frac{1}{\|u_n\|_{X_0}^2} \int_{\Omega} G(x, u_n(x)) dx \\
&\geq \frac{m_0}{2} \left( \int_{\mathbb{R}^n} |v_n(x) - v_n(y)|^2 K(x - y) dx dy - \lambda_1 \int_{\Omega} |v_n(x)|^2 dx \right) + \frac{M|\Omega|}{2\|u_n\|_{X_0}^2} \\
&\quad - \frac{1}{\|u_n\|_{X_0}^2} \int_{\{|u_n(x)| \leq R_M\}} G(x, u_n(x)) dx \\
&\geq \frac{m_0}{2} \left( \int_{\mathbb{R}^n} |v_n(x) - v_n(y)|^2 K(x - y) dx dy - \lambda_1 \int_{\Omega} |v_n(x)|^2 dx \right) - \frac{C_3}{\|u_n\|_{X_0}^2},
\end{align*}
\]
where $C_3$ is a positive constant, which implies
\[
\limsup_{n \to \infty} \int_{\mathbb{R}^n \times \mathbb{R}^n} |v_n(x) - v_n(y)|^2 K(x-y) \, dx \, dy \leq \lambda_1 \int_{\Omega} |v_0(x)|^2 \, dx. \tag{19}
\]
By the weakly semicontinuous property of $u \mapsto ||u||_{X_0}$ in the weak topology $X_0$ and the definition of $\lambda_1$, we have
\[
\lambda_1 \int_{\Omega} |v_0(x)|^2 \, dx \leq \int_{\mathbb{R}^n \times \mathbb{R}^n} |v_0(x) - v_0(y)|^2 K(x-y) \, dx \, dy \\
\leq \liminf_{n \to \infty} \int_{\mathbb{R}^n \times \mathbb{R}^n} |v_n(x) - v_n(y)|^2 K(x-y) \, dx \, dy \\
\leq \limsup_{n \to \infty} \int_{\mathbb{R}^n \times \mathbb{R}^n} |v_n(x) - v_n(y)|^2 K(x-y) \, dx \, dy. \tag{20}
\]
By (19) and (20), $\int_{\mathbb{R}^n \times \mathbb{R}^n} |v_0(x) - v_0(y)|^2 K(x-y) \, dx \, dy = \lambda_1 \int_{\Omega} |v_0(x)|^2 \, dx$ and $v_n \to v_0$ in $X_0$ with $|v_0|_{X_0} = 1$. Hence $v_0 = \pm e_1$. Take $v_0 = e_1$; then $u_n \to +\infty$ a.e. on $\Omega$, which implies that $G(x, u_n(x)) \to -\infty$ by (18). So we have
\[
C_2 \geq J(u_n) = \frac{1}{2} \int_{\mathbb{R}^n} |u_n(x) - u_n(y)|^2 K(x-y) \, dx \, dy - \int_{\Omega} F(x, u_n(x)) \, dx \\
\geq \frac{m_0}{2} \left( \int_{\mathbb{R}^n} |u_n(x) - u_n(y)|^2 K(x-y) \, dx \, dy - \lambda_1 \int_{\Omega} |u_n(x)|^2 \, dx \right) \\
- \int_{\Omega} G(x, u_n(x)) \, dx \\
\geq - \int_{\Omega} G(x, u_n(x)) \, dx \to +\infty, \quad \text{as } n \to \infty,
\]
which is a contradiction. So we have that $J$ is coercive in $X_0$. \hfill \Box

By Lemmas 4.2 and 4.3, it follows that:

**Lemma 4.4.** Under assumptions (M0), (F1) and (F3) (or substitute (F3) for both (F4) and (F5)), the functional $J$ satisfies the Palais-Smale condition.

Now, we recall the definition of local linking to proceed with our proof.

**Definition 4.5.** We say that a functional $\varphi$ has a local linking to the decomposition of the space $X = V \oplus Y$ near the origin 0 iff there is a small ball $B_\rho$ with the center at 0 and small radius $\rho > 0$ such that
\[
\varphi(v_1) > \varphi(0), \quad \text{for } v_1 \in B_\rho \cap Y \setminus \{0\}; \\
\varphi(v_2) \leq \varphi(0), \quad \text{for } v_2 \in B_\rho \cap V.
\]

**Lemma 4.6.** Let $K$ satisfy assumptions (K1)-(K3), then under assumptions (M0), (M1), (F1) and (F2), the functional $J$ has a local linking at the origin with respect to $X_0 = V \oplus Y$, where $V$ and $Y$ are functional subspaces of $X_0$ in the Introduction Section.

**Proof.** First, we take $u \in V$; since $V$ is finite dimensional, we can see that $||u||_{X_0} \leq \rho$ implies $|u| \leq r$, for all $x \in \Omega$ and $r > 0$ small enough. Thus by (F2), for $||u||_{X_0} \leq \rho$, we have...
we get
\[
J(u) = \frac{1}{2} \mathcal{M} \left( \int_{\mathbb{R}^n} |u(x) - u(y)|^2 K(x - y) dx dy \right) - \int_{\Omega} F(x, u(x)) \, dx
\]
\[
\leq \frac{m_1}{2} \lambda_1 \int_{\Omega} |u(x)|^2 \, dx - \int_{\Omega} F(x, u(x)) \, dx
\]
\[
= \int_{|u| \leq r} \left( \frac{m_1}{2} \lambda_1 |u(x)|^2 - F(x, u(x)) \right) \, dx \leq 0.
\]

On the other hands, we take \( u \in Y \); from (F1), (F2) and the definition of \( \bar{\lambda} \), one can get
\[
J(u) = \frac{1}{2} \mathcal{M} \left( \int_{\mathbb{R}^n} |u(x) - u(y)|^2 K(x - y) dx dy \right) - \int_{\Omega} F(x, u(x)) \, dx
\]
\[
\geq \frac{m_0}{2} \left( \int_{\mathbb{R}^n} |u(x) - u(y)|^2 K(x - y) dx dy - \bar{\lambda} \int_{\Omega} |u(x)|^2 \, dx \right)
\]
\[
- \int_{\{|u| \leq r\}} \left( F(x, u(x)) - \frac{\bar{\lambda}m_0}{2} |u(x)|^2 \right) \, dx
\]
\[
- \int_{\{|u| > r\}} \left( F(x, u(x)) - \frac{\bar{\lambda}m_0}{2} |u(x)|^2 \right) \, dx
\]
\[
\geq \frac{m_0}{2} \left( 1 - \frac{\bar{\lambda}}{\lambda} \right) \|u\|^2_{X_0} - C_3 \int_{\{|u| > r\}} |u(x)|^s \, dx
\]
\[
\geq \frac{m_0}{2} \left( 1 - \frac{\bar{\lambda}}{\lambda} \right) \|u\|^2_{X_0} - C_3 \|u\|^s_{X_0}, \quad (2 < s < 2^*).
\]
So we can derive that when \( u \in X \) and \( 0 < \|u\|_{X_0} \leq \rho \) and \( \rho > 0 \) small, \( J(u) > 0 \), which completes the proof. \( \square \)

**Remark 1.** From the proof of Lemma 4.3, we can get a stronger result: there exists a \( \rho_0 > 0 \) such that for any \( 0 < \rho < \rho_0 \), \( B_{\rho} \) satisfies all the conditions required by the definition of local linking. From this point of view, we can conclude that \( 0 \in X_0 \) is the unique critical point of our \( J \) in a ball that is small enough.

For an isolated critical point \( u \) of a \( C^1 \) functional \( f : E \rightarrow \mathbb{R} \), we define a the critical group of \( f \) at \( u \) as follows:

**Lemma 4.7.** ([7, Theorem 2.1]). Let \( E \) be a Banach space and \( f : E \rightarrow \mathbb{R} \) a \( C^1 \) functional satisfying the Palais-Smale condition. Suppose that \( E \) has a decomposition \( E = W \oplus Z \), where \( W \) is a finite dimensional subspace, say \( \dim W = m < \infty \). Suppose that there exists a small ball \( B_{\rho} \) with its center at the origin \( 0 \) and small radius \( \rho > 0 \) such that
\[
f(z) > f(0), \quad \text{for } z \in B_{\rho} \cap Z \setminus \{0\};
f(w) \leq f(0), \quad \text{for } w \in B_{\rho} \cap W.
\]
If \( 0 \in E \) is the unique critical point of \( f \) in \( B_{\rho} \), then
\[
C_m(f, 0) = H_m(f_c \cap B_{\rho}, f_c \cap B_{\rho} \setminus \{0\}) \neq 0.
\]

Here, since \( \dim W = 1 < \infty \), by Lemma 4.6, Remark 1 and Lemma 4.7, we have the following lemma:
Lemma 4.8. Let $K$ satisfy assumptions (K1)-(K3), then under assumptions (M0), (M1), (F1) and (F2), 0 is a critical point of $J$ and $C_1(J, 0) \neq 0$.

Now, we can state our main result.

**Theorem 4.9.** Let $K$ satisfy assumptions (K1)-(K3). Assume (M0), (M1), (F1), (F2) and (F3) hold, then the problem (1) has at least two nontrivial weak solutions in $X_0$.

**Theorem 4.10.** Let $K$ satisfy assumptions (K1)-(K3). Assume (m0), (m1), (F1), (F2), (F4) and (F5) hold, then the problem (1) has at least two nontrivial weak solutions in $X_0$.

For the proof of Theorems 4.9 and 4.10, we present the following theorem from [8].

**Lemma 4.11.** Let $X$ be a real Banach space and let $J \in C^1(X, \mathbb{R})$ satisfy the Palais-Smale condition and be bounded from below. If $J$ has a critical point that is homologically nontrivial and is not the minimizer of $J$, then $J$ has at least three critical points.

**Proof of Theorems 4.9 and 4.10.** By Lemmas 4.3 and 4.4, $J$ is coercive and satisfies the Palais-Smale condition. Thus $J$ is bounded below. By Lemma 4.8, $0 \in X_0$ is a homologically nontrivial critical point of $J$ but not a minimizer. Then the conclusion follows from Lemma 4.11.

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