Some notes on ideology of waves in plasmas

V. N. Soshnikov

Plasma Physics Dept.,
All-Russian Institute of Scientific and Technical Information
of the Russian Academy of Sciences
(VINITI, Usievitcha 20, 125315 Moscow, Russia)

Abstract

Our last three papers [4, 7, 8] provide an occasion to make some brief notes on ideology of waves in plasmas and to rehabilitate Vlasov prescription to calculate relevant logarithmically divergent integrals in the principal value sense. In this approach asymptotical solutions of plasma oscillations obtained with the method of Laplace transform must be in the form of a sum of wave exponents with amplitudes being selected according to self-consistent boundary physical conditions. Landau damping is absent in this case by definition. Boundary electrical field together with conditions of absence of unphysical backward waves (for boundary problem in half-infinite plasmas) and of kinematical waves define single-valued dependence of boundary distribution function on electron velocity \( \vec{v} \) in the case of transversal waves and on the surface break of the normal electrical field in the case of longitudinal oscillations. We have proposed physically more justified modified iteration procedure of collisional damping calculation and demonstrated some results of damping decrements calculations in a low-collision electron-ion plasma. We have pointed out a possibility of natural selection of boundary conditions, which allows to eliminate simultaneously both backward and kinematical waves as for transversal as well as for longitudinal oscillations. Dispersion smearing of both longitudinal and transversal high-frequency waves, for which the smearing decrement \( \delta_x \) is proportional to \( \Delta \omega / (\omega \sqrt{\omega^2 - \omega^2_L} ) \) (where \( \Delta \omega \) is the frequency interval of exciting boundary field), and non-Maxwellian background distribution function might be the main causes of waves amplitude damping in collisionless plasmas imitating Landau damping.

PACS numbers: 52.25 Dg; 52.35 Fp.

Key words: plasma oscillations; plasma waves; Landau damping; Coulomb collisions; collision damping; dispersion equations; Vlasov equations; plasma dielectric permittivity; kinematical waves; plasma echo; Liouville theorem.

After the basic paper of L. Landau in 1946 [1], which stated that even in collisionless Maxwellian plasmas electron waves are damping (“Landau damping”), there appeared an enormous amount of works, including all textbooks, that use it as a conception. In practice this scientific trend can be considered as being completed without any hope to carry therein something essentially new.

Landau had considered the simplest case of plane half-infinite geometry with the beforehand given form of a solution \( \exp(ikx)f(t) \), where for determination of the function \( f(t) \) he had used Laplace transformation, and had postulated to calculate an appearing in Laplace image \( f_p \) logarithmically divergent integral not in the sense of the principal value (as it had been proposed in 1937 by Vlasov [2] for the analogous integral appearing at solving his

1Krasnodarskaya str., 51-2-168, Moscow 109559, Russia.
“Vlasov equations”) but according to Landau rule of by-passing poles of \( dv_x \)-integrands in the complex-valued plane \( v_x \) (\( v_x \) is an x-component of the electron velocity, \( x \) is a direction of the wave traveling).

However it is well known (see, for instance review \[3\] or Eq. (1) in \[4\]) that asymptotical Landau solution in the form of a solitary plane damping wave \( \exp(i k x - i \omega t) \) with \( \omega = \omega_0 - i \delta \) does not satisfy Vlasov equations neither at these nor at any other real (if considered as a limit at tending imaginary parts of \( \omega, k \) to zero, see, for instance, \[4\]), or complex values \( \omega \) and \( k \). In spite of some paradoxes and hardly to be explained consequences of Landau theory (see, for instance, \[5\]) it was supposed that Landau solution corresponds to reality and appears to be true, so that Vlasov equations must be corrected by additional terms, which are determined by Landau rule of by-passing poles (see, for instance, \[3, 6\]).

Nevertheless one can formulate a problem of finding asymptotical solutions of Vlasov equations and also equations of propagating transversal electromagnetic waves (with the same difficulty of logarithmically divergent integrals) without these additions, which appear to be artificial. So, the paradoxes of Landau solution are thought to be generated by Landau representation of logarithmically divergent integrals and, in the main, by the single-exponent form (suggested by Landau as well as by Vlasov, however with no paradoxes in the last case ) of an asymptotical solution.

Such asymptotical solutions have been written in our last papers \[4, 7, 8\] proceeding from the following propositions:

1. The logarithmically divergent integrals appearing at Laplace transformation have to be calculated in Vlasov prescription of the principal value sense.

2. For evaluation of principal values of integrals we use a series of successful approximations of the type

\[
\int_{-\infty}^{\infty} e^{-\frac{m v_x^2}{2 k T_e} \cdot \| \mathbf{v} \|^2} \, d^3 \mathbf{v} \equiv -8 \int_{0}^{\infty} e^{\frac{m v_x^2}{2 k T_e} \cdot v_x^2} \, d^2 \mathbf{v} \approx \left( \frac{2 \pi k T_e}{m} \right)^{3/2} \frac{p_2 \bar{v}_x^2}{p_1^2 - v_x^2 p_2^2}.
\]

After this the obtained asymptotical solutions with the collisional damping take the form of a sum of exponents of the type \( \exp(\pm i k x \pm i \omega t + x \delta) \) and \( \exp(\pm i k x \mp i \omega t - x \delta) \), \( \delta > 0 \), corresponding to double poles \( p_1, p_2 \).

3. The given boundary condition \( E(0, t) \) (for instance, in purely boundary problem) defines also, in a single-valued way, the boundary (and initial) conditions for the perturbed distribution function \( f_1(\vec{v}, 0, t), f_1(\vec{v}, x, 0) \) according to linear integral equations following from the condition of absence of kinematical waves \[8\]. It guaranties the physically justified proportionality of the boundary and initial distribution function to the boundary electric field.

4. Selecting free boundary parameter \( F_{p_1} \), being a Laplace transform of \( \partial E(x, t)/\partial x|_{x=0} \) one can exclude (cancel) unphysical (for the case of half-infinite plasmas) backward waves contained in the solution in the form of residua exponential sum. These waves are divergent at \( x \to \infty \) in the case of a low-collision plasma.

5. In the case of longitudinal waves at the given boundary field \( E(0, t) \) the boundary self-consistent function \( f_1(\vec{v}, 0, t) \) is determined analogously from the condition of absence
of kinematical waves. But for elimination of the unphysical backward waves in this case one must inevitably assume a break in the normal constituent of electrical field \( E(0, t) \) at plasma surface (see \[S\] for a way to calculation its value).

6. In general case the solution can not be represented as a single exponent even if the boundary condition \( E(0, t) \) has such a form. At the boundary field frequency \( \omega \) the asymptotic solution must contain at least two simultaneously existing forward wave exponents \( \exp(\pm i\omega t \mp ikx) \), corresponding to the frequencies \( +\omega \) and \( -\omega \).

7. Amplitudes of all modes with different \( k_n(\omega) \) corresponding to the given exciting field \( E(0, t) \) are strongly correlated. Cancellation of kinematical and then backward waves must be achieved for all modes in the general asymptotical solution.

8. Solutions of the wave equations must be real-valued whenever the boundary and initial conditions are such ones.

Only after proceeding these procedures one can calculate strong relations between amplitudes of the different forward waves including, in the general case, both electron and also ion and hybrid ion-electron branches.

This allows to avoid all paradoxes at calculation of \( E(x, t) \) and paradoxical tangling of the distribution function in \( v_x \) with generating some strange electrical field.

When all these conditions being fulfilled one can easily construct, using the method of two-dimensional Laplace transform usual solutions for the poles of image \( EP_1P_2 \), correspondingly, the long before known dispersion relations \( k(\omega) \) \[4, 7, 8\]. Thus, for the case of collisionless plasma mystic Landau damping and kinematical waves are really absent by definition. The solution appears as a sum of exponents with amplitudes selected in accordance with boundary and physical conditions.

We also have developed the more physically justified modification of iteration process in the presence of collision terms of kinetic equation (low-collision electron-ion plasmas). By this method we have obtained damping decrements as for electron longitudinal waves as well as for two branches (low- and high-frequency ones) of transversal electromagnetic waves with the unusual\(^2\) decrement for the low-frequency branch. The collisional dissipative absorption in the high-frequency branch at \( \omega > \omega_L \) is proportional to \( 1/\sqrt{1 - \omega^2/\omega^2_L} \) and grows to infinity at \( \omega \rightarrow \omega_L \) \[8\].

In this connection we think that Van Kampen waves (see \[3\]) are called to compensate the erroneous consequences of Landau theory.

It is highly believed that Landau damping is detected and verified experimentally. In this respect we should note the relatively small number of such verifications and the necessary extreme delicateness of similar experiments. The theoretically absent Landau damping in fact might be imitated by a series of secondary effects. These ones can be:

(i) the difference of the background distribution from Maxwellian one due to electron collisions with and recombinations on the walls of discharge tube;

(ii) the presence of longitudinal magnetic field and cyclotron motion and transversal diffusion to walls;

(iii) effects of the method of plasma oscillations exciting;

(iv) effects of reflecting from walls;

\(^2\)nonlinear in concentrations of charged particles.
effects of the base electric field supporting discharge;

growing to infinity of the Coulomb collision damping at \( \omega \to \omega_L + 0 \);

non-harmonic composed waveform and signal dispersion and its smearing in \( \omega \);

experimental requirements to electron distribution function to be Maxwellian and electrons to be collisionless are intrinsically contradictory;

geometrical (diffraction) effects (declination from the proposed one-dimensional problem with plane waves since the length of the discharge tube is much more than its diameter).

The decrement of the amplitude damping at dispersion smearing of a wave can be calculated according to expression

\[
\delta_x(\omega) = \frac{\Delta \omega}{2\pi} k(\omega) \frac{\partial^2 \omega / \partial k^2}{(\partial \omega / \partial k)^2} = \frac{\omega_L^2 \Delta \omega}{2\pi \beta \omega \sqrt{\omega^2 - \omega_L^2}},
\]

where \( \beta = c \) or \( \beta = \sqrt{v_x^2} \) for transversal or longitudinal waves, correspondingly; \( \Delta \omega \) is spectral width of the boundary exciting field \( E(0,t) \). This smearing might be the main cause of the wave amplitude damping in collisionless plasma both for longitudinal and transversal waves that imitates Landau damping.

The return to Vlasov prescription of calculating relevant integrals in the principal value sense and the proper determination and using of mutually dependent boundary (or initial) conditions in the self-consistent manner allow to solve and remove all paradoxes of “Landau damping”.

Thus, the right natural and simple but non-traditional ideology of waves in plasmas reduces to combine the method of Laplace transformation with the proper account for boundary/initial conditions, and Vlasov prescription of the principal value sense of relevant integrals.

Mathematical correctness of the represented solution follows from the way of its construction. However one can analogously construct also other mathematically irreproachable solutions, for instance using “Landau functions”, that is with integrals in the principal value sense plus results of passing around the running with \( p_1 \) and \( p_2 \) poles \( v_x = -p_1/p_2 \) in the lower or upper half-plane (in either case of fixed \( p_0^1 \) with two roots for the further calculated poles \( p_0^2 \)).

It appears that the only criterion to pick out a physical solution from the set of mathematically correct solutions to be only physical reasons, namely the definition of logarithmically divergent integrals and related together boundary conditions in a such manner to avoid the paradoxes of divergent at \( x \to \infty \) wave solutions, appearance of kinematical and backward waves, and waves not related to the boundary electrical field. The stability of the physical solution might be also considered as an additional requirement.

Namely, these and only these criterions justify constructing the solution in terms of functions with integrals in the principal value sense as an optimal and apparently the unique variant.

Violation of the uniqueness and existence theorem is related with singularity of the Laplace image of \( f_1^{(e)} \) due to logarithmic divergence of the integral in \( dv_x \) at points \( v_x = -p_1/p_2 \), where \( p_1 \) and \( p_2 \) are the running Laplace transform parameters.
Attempts to establish functional constraints on the maximal value of the divergent integral naturally prove to be artificial and can be justified only by special physical considerations.

As a conclusion, our principally new approach in the plasma wave problem is the prescription of self-consistent calculation of the lacking boundary and initial conditions that gives a possibility to calculate the correct relative amplitudes of all relevant oscillatory modes and to eliminate the known paradoxes. Here acts a simple and robust principle: in the self-consistent problem there must be no perturbations which are not induced by (and as a consequence nonlinearly related with) the perturbing boundary electrical field.

Transition to the real-valued boundary condition $E(0, t) = E_0 \cos(\omega t)$ in the asymptotical two-wave solution $a \cdot \exp(i \omega t - ikx) + a^* \cdot \exp(-i \omega t + ikx)$ is equivalent to representing the solution as (cf. [2])

\[
E(x, t) = E'_0 \cos[\omega t - kx + \varphi(k)] ;
\]

\[
f_1(v, v_x, x, t) = F(v, v_x) \cos[\omega t - kx + \psi(k)] ,
\]

where $E'_0$, $\varphi(k)$, $F(v, v_x)$, $\psi(k)$ are real-valued amplitudes and phases which have to be determined. The substitution of $E(x, t)$ and $f_1(v, v_x, x, t)$ into Vlasov equations at $\varphi(k) - \psi(k) = \pm \pi/2$ leads to the traditional equations

\[
F(v, v_x) = \pm \left| e \right| \left( \frac{E'_0}{\omega - kv_x} \right) \frac{\partial f_0(v)}{\partial v_x} ,
\]

with real roots $\pm k$ of Eq. (6) at $\omega > \omega_L$. If Eqs. (5) and (6) are satisfied, then $E(x, t)$ and $f_1(v, v_x, x, t)$ are asymptotical solutions of Vlasov equations.

The case of the totally imaginary $k$ ($\text{Re}(k) = 0$, $\omega < \omega_L$) in collisionless plasmas corresponds to the total wave reflection without energy dissipation. According to the aforementioned theory the solution reduces to a sum of terms, which equally exponentially damp at $x \to \infty$. As can be verified by direct substitution into Vlasov equations, elementary general asymptotical solution has the following form

\[
E(x, t) = E'_0 e^{-|k|x} \left[ \cos(\omega t + \varphi) + \cos(\omega t + \psi) \right] ;
\]

\[
f_1(v, v_x, x, t) = e^{-|k|x} \left[ F(v, v_x) \cos(\omega t + \varphi) + F(v, -v_x) \cos(\omega t + \psi) \right] ,
\]

\[
F(v, v_x) = \mp \left| e \right| \left( \frac{\partial f_0(v)}{\partial v_x} -\frac{E'_0}{\omega^2 + |k|^2 v_x^2} \right) ,
\]

with $\varphi - \psi = \pm \pi/2$ and dispersion equation being defined by Eq. (3).

In dependence on the sense of improper integral in Eq. (6) one obtains different solutions $k(\omega)$, with functions (3) and (4) being indeed the asymptotical solutions of Vlasov equations. From the real-valued character of the solution it follows directly the necessity to exclude in Eq. (3) additive imaginary constituents related to passing around the pole $v_x = \omega/k$, that is, to exclude the Landau prescription how to calculate this integral. From the physical symmetry of frequency and velocity of propagation of the forward and backward waves with respect to interchange $k \to -k$ it follows also that one has to calculate the integral in Eq. (3) in the sense of principal value. This becomes more evident when one uses our

\footnote{Since real physical values are limited by different physical processes which are not accounted in the original equations in the given formulation of the problem.}
approximate expression for the principal value of this integral. For calculation of the phase \( \varphi(k) \), amplitudes and initial and boundary values \( f_1(v, v_x, x, 0), f_1(v, v_x, 0, t) \) in order to take into account several modes \( k_1(\omega), k_2(\omega), \ldots \) one ought to use the general procedure of Laplace transform method with exclusion of kinematical and backward waves.

Note, that solution at \( v_x \to \omega/k \) tends to \( \pm \infty \), violating the initial perturbative condition \( |f_1| \ll f_0 \). However, one can assume from the physical point of view that there occurs some saturation of \( |f_1| \) growth at periodically sign-alternating \( f_1 \) with limitation \( |f_1(v_x = \omega/k \pm \varepsilon)| \leq f_0(v_x = \omega/k) \) at \( \varepsilon \to 0 \), and presumably \( f_1(v_x = \omega/k) = 0 \) with linear dependence \( f_1 \) on \( v_x \) in this region of unapplicability of the kinetic equation and possible discontinuity of \( \partial f_1/\partial v_x \) near \( f_1 \simeq \pm f_0 \), so, at least, there are no physical reasons for appearance of any imaginary part in the principal value of the integral and in the function \( f_1 \). All these appear to be additional arguments in favor of calculation of the divergent integral in the principal value sense. The case of complex \( k \) in forward waves (\( \text{Re} k > 0 \), collision damping) was considered in our previous papers using the general Laplace transform method. In this case for real-valued boundary (and initial) conditions and \( \omega \) the solution is defined with a pair combination of exponents of the type \( a \cdot \exp(i\omega t - ikx) + a^* \cdot \exp(-i\omega t + ik^*x) \).

Owing to the symmetry \( k \to -k \) in isotropic medium, the wave number \( k \) must enter into dispersion equation only through \( k^2 \). In collisionless plasmas asymmetry of \( |f_1| \) relative to the neighbourhood of the point \( v_x = \omega/k \pm \varepsilon \) leads to the possibility of appearing complex \( k \), which define both damping and growing waves. Therefore this asymmetry leads to the paradoxical availability of waves (both in forward and backward directions) with exponential damping as well as with exponential growing due to the fact that asymptotic solution in this case contains both exponents \( \pm i(\omega t + kx) \) and \( \pm i(\omega t - kx) \).

The fact of appearance of the indefinitely divergent integrals is the direct consequence of incomplete information in original differential equations. There must be additional exclusively physical (non-mathematical) considerations at calculation of these integrals, since all different possible ways lead to different equally right (by their Laplace transform construction) mathematical solutions with correspondingly different dispersion relations \( k(\omega) \) and different wave amplitudes. But these solutions can be as physical as well as unphysical ones and have to be selected.

To resume, the functions \( \Phi \) and \( \Psi \), as well as \( \Omega \) and \( \Gamma \), are evidently the very simple illustration of the existence of asymptotical non-damping or non-dissipative real-valued solutions for plasma wave equations.

In the resonance region, where the condition \( f_0(v, v_x, x, t) + f_1(v, v_x, x, t) \geq 0 \) is violated, the original, linearized or precise, equations are inapplicable. The divergence of \( f_1 \) at \( v_x = \omega/k \) can not be removed when one uses expansion of the precise kinetic equation with the quadratic term in multiple overtones \( F_n \cos[n(\omega t - kx) + \varphi_n] \) and \( E_n \cos[n(\omega t - kx) + \psi_n] \). Thus, the original equations, linearized or not, are not sufficient to obtain single-valued solution of the problem without introducing into the kinetic equation (which details Liouville theorem) some additional physical specifications.

In the theory of “Landau damping” velocity \( v_x \) can take, according to Cauchy theorem, some arbitrary complex values at arbitrary deformation of the integration contour bypassing real-valued pole \( v_x = \omega/k \) (with \( \omega > \omega_L \), \( \text{Im}(k) = 0 \)) in the complex plane, that leads simultaneously to the appearance of exponentially damping and growing wave solutions with non-zero \( \pm \text{Im}(k) \). According to \( \Omega \) the latter is related with the small non-zero

\[ \int |f_1|d\varepsilon = |\Delta n_e| \text{ can be treated as the real-valued concentration of perturbed electrons (} |\Delta n_e| \leq n_e). \]

\[ ^4 \text{In this case } \int |f_1|d\varepsilon = |\Delta n_e| \text{ can be treated as the real-valued concentration of perturbed electrons (} |\Delta n_e| \leq n_e). \]
values $\text{Im}(k)$ resulting from the next derived quadratic dispersion equation. It evidences the intrinsic discrepancy and non-self-consistency of the Landau damping theory, since complex $k$, according to [1], is a result of solving the dispersion equation, contrary to the before assumed real-valued $k$ with the only one pole $\omega/k$ on the real axis $v_x$, thus the pole bypassing in the complex plane $v_x$ ought e.g. to be made now not along the half-circle (as it is presented in [1]), but along the total circle [9], or must not be made at all, with not clear grounds after all, for taking any value of $k(\omega)$.

Conclusion

The uniqueness theorem for solutions of the plasma wave equations, both for longitudinal and transversal waves at definite boundary and initial conditions, is violated due to the presence of logarithmically divergent integral in dispersion equations. To avoid the appearing of exponentially divergent solution terms, e.g. for considered plane waves in isotropic Maxwellian plasmas in the plasma half-infinite slab, one ought to treat this divergent integral in the principal value sense. The physical requirements for the solution select the only solution out of all mathematically possible ones. We have presented elementary general real-valued solutions of Vlasov equations for collisionless plasmas at real-valued boundary and initial conditions in the form of a trigonometric function sum both for non-damping plasma waves ($\text{Im}(k) = 0, \omega > \omega_L$) and for the total reflection case ($\text{Re}(k) = 0, \omega < \omega_L$) with dissipationless damping (evanescence) proportional to $\exp(-|k(\omega)|x)$.

Finiteness of the wave solution in the neighbourhood of the point $v_x = \omega/k \pm \varepsilon$ results by no means from energy dissipation in the sense of contradictory Landau damping but just from the purely kinematical effect of the electron density limitations with providing proper corrections to the kinetic equation.

Existence of the general real-valued non-damping or dissipationless finite solutions of Vlasov equations is the direct proof of incorrectness of Landau damping theory.

Acknowledgements The author is thankful to Dr. A. P. Bakulev for criticism and assistance in preparing the paper in LATEX style.

References

[1] Landau L. D., J. Phys. (USSR), 10 (1946) 25; JETP (USSR), 16 (1946) 574 (in Russian); Uspekhi Fiz. Nauk, 93 (1967) 527 (reprint, in Russian).

[2] Vlasov A. A., JETP (USSR), 8 (1938) 291 (in Russian); Uspekhi Fiz. Nauk, 93 (1967) 444 (reprint, in Russian).

[3] Pavlenko V. N., Sitenko A. G., ”Echo-phenomena in Plasma and Plasma-like Media”, Nauka, Moscow (1988) (in Russian).

[4] Soshnikov V. N., ”Damping of plasma-electron oscillations and waves in low-collision electron-ion plasmas”, physics/0105040 (http://xxx.lanl.gov/ e-print).

[5] Clemmow P. C., Dougherty J. P., “Electrodynamics of Particles and Plasmas”, 2-nd ed., Addison-Wesley, NY (1990); (Rus. transl. Moscow, Mir, 1996).
[6] Alexandrov A. F., Bogdankevich L. S., Rukhadze A. A., “Foundations of Electrodynamics of Plasma”, 2nd ed., Vysshaya Shkola, Moscow (1988) (in Russian).

[7] Soshnikov V. N., ”Damping of transversal plasma-electron oscillations and waves in low-collision electron-ion plasmas”, physics/0111014 (http://xxx.lanl.gov/e-print).

[8] Soshnikov V. N., ”Damping of electromagnetic waves in low-collision electron-ion plasmas”, physics/0205035 (http://xxx.lanl.gov/e-print).

[9] Alexeff I., Rader M., Int. J. Electronics, 68 (1990) 385.