NAGATA-ASSOUAD DIMENSION VIA LIPSCHITZ EXTENSIONS

N. BRODSKIY, J. DYDAK, J. HIGES, AND A. MITRA

Abstract. In the first part of the paper we show how to relate several dimension theories (asymptotic dimension with Higson property, asymptotic dimension of Gromov, and capacity dimension of Buyalo [7]) to Nagata-Assouad dimension. This is done by applying two functors on the Lipschitz category of metric spaces: microscopic and macroscopic. In the second part we identify (among spaces of finite Nagata-Assouad dimension) spaces of Nagata-Assouad dimension at most \( n \) as those for which the \( n \)-sphere \( S^n \) is a Lipschitz extensor. Large scale and small scale analogs of that result are given.

Contents

1. Introduction 1
2. Microscopic and macroscopic Nagata-Assouad dimensions 3
3. Spheres as Lipschitz extensors 6
4. Lipschitz extensions and Nagata-Assouad dimension 10
5. Coarsely equivalent metrics and Nagata-Assouad dimension 13
6. References 14

1. Introduction

The large-scale geometry of metric spaces has been the subject of intense research in the last 15 years. For a comprehensive account of the area see Gromov’s paper [12]. There are two large-scale concepts related to the topic of our paper: asymptotic dimension and asymptotic dimension of linear type (Dranishnikov and Zarichnyi [11] refer to the latter as asymptotic dimension with Higson property). Asymptotic dimension is an invariant of the coarse...
category of Roe [19]. Asymptotic dimension of linear type is preserved by bi-Lipschitz functions, so its natural place is in the Lipschitz category.

Notice that our notion of a Lipschitz function is a bit different from that used in [13]. Namely, we allow Lipschitz constant to be smaller than 1. Thus, a Lipschitz function \( f: (X, d_X) \rightarrow (Y, d_Y) \) satisfies \( d_Y(f(x), f(y)) \leq \lambda \cdot d_X(x, y) \) for some \( \lambda \geq 0 \) and all \( x, y \in X \). The infimum of all possible \( \lambda \) is denoted by \( \text{Lip}(f) \). \( f \) is called bi-Lipschitz if there are constants \( \mu, \lambda > 0 \) such that \( \mu \cdot d_X(x, y) \leq d_Y(f(x), f(y)) \leq \lambda \cdot d_X(x, y) \) for all \( x, y \in X \).

In [14] a variation of asymptotic dimension is considered. That invariant of bi-Lipschitz functions was introduced and named Nagata dimension by Assouad [1] as it is closely related to a theorem of Nagata characterizing the topological dimension of metrizable spaces (cf. [[16], Thm. 5] or [[17], p. 138]). In our paper we refer to it as Nagata-Assouad dimension. In contrast to the asymptotic dimension, the Nagata dimension of a metric space is in general not preserved under quasi-isometries, but it is still a bi-Lipschitz invariant and, as it turns out, even a quasisymmetry invariant (see [14]). The class of metric spaces with finite Nagata dimension includes all doubling spaces, metric trees, euclidean buildings, and homogeneous or pinched negatively curved Hadamard manifolds as shown in [14]. One of main results of [14] relates to theorems of Assouad [2] and Dranishnikov [10]. Namely, Theorem 1.3 of [14] states: Every metric space with Nagata-Assouad dimension at most \( n \) admits a quasisymmetric embedding into the product of \( n + 1 \) metric trees.

The result from [14] of major importance to us is Theorem 1.4:

**Theorem 1.1.** Suppose that \( X, Y \) are metric spaces, \( \dim_{NA} X \leq n < \infty \), and \( Y \) is complete. If \( Y \) is Lipschitz \( m \)-connected for \( m = 0, 1, \ldots, n - 1 \), then the pair \((X, Y)\) has the Lipschitz extension property.

Recall that \((X, Y)\) has the Lipschitz extension property if there is a constant \( C > 0 \) such that for any Lipschitz map \( f: A \rightarrow Y \), \( A \) any subset of \( X \), there is a Lipschitz extension \( g: X \rightarrow Y \) of \( f \) such that \( \text{Lip}(g) \leq C \cdot \text{Lip}(f) \). We call \( Y \) a Lipschitz extensor of \( X \) in such a case. \( Y \) is Lipschitz \( m \)-connected if there is a constant \( C_m > 0 \) such that any Lipschitz function \( f: S^m \rightarrow Y \) extends over the \((m + 1)\)-ball \( B^{m+1} \) to \( g: B^{m+1} \rightarrow Y \) so that \( \text{Lip}(g) \leq C_m \cdot \text{Lip}(f) \).

One of the main themes of our paper is characterizing Nagata-Assouad dimension via Lipschitz extensions. In Section 4 we characterize spaces of Nagata-Assouad dimension at most \( n \) (among all spaces of finite Nagata-Assouad dimension) as those for which \( n \)-sphere \( S^n \) is a Lipschitz extensor. In the case of dimension zero the assumption of \( \dim_{NA}(X) \) being finite can be dropped.

Recently, S.Buyalo [7] introduced the capacity dimension of a metric space and proved many analogs of results obtained by U. Lang and T. Schlichenmaier [14] for Nagata-Assouad dimension. It is clear that capacity dimension is the small scale version of Nagata-Assouad dimension. In Section 2
we formalize that observation by introducing microscopic and macroscopic functors on the Lipschitz category. That way many results from Section 3 of [7] can be deduced formally from [14]. Also, the main result of [7] (the asymptotic dimension of a visual hyperbolic space \( X \) is bounded by 1 plus the capacity dimension of the visual boundary of \( X \)) is really about Nagata-Assouad dimension of the visual boundary of \( X \) - see 2.12.

2. Microscopic and macroscopic Nagata-Assouad dimensions

**Definition 2.1.** A metric space \( X \) is said to be of Nagata-Assouad dimension at most \( n \) (notation: \( \dim_{NA}(X) \leq n \)) if there is \( C > 0 \) such that for all \( r > 0 \) there is a cover \( \mathcal{U} = \bigcup_{i=1}^{n+1} \mathcal{U}_i \) of \( X \) so that each \( \mathcal{U}_i \) is \( r \)-disjoint and the diameter of elements of \( \mathcal{U} \) is bounded by \( C \cdot r \).

Nagata-Assouad dimension can be characterized in many ways (see [14]):

**Proposition 2.2.** For a metric space \((X,d)\) the following conditions are equivalent:

1. \( \dim_{NA}(X) \leq n \).
2. There is a constant \( C_1 > 0 \) such that for any \( r > 0 \) there is a cover \( \mathcal{U}_r \) of \( X \) of multiplicity at most \( n + 1 \), of mesh at most \( C_1 \cdot r \), and of Lebesgue number at least \( r \).
3. There is a constant \( C_2 > 0 \) such that for any \( r > 0 \) there is a cover \( \mathcal{V}_r \) of mesh at most \( C_2 \cdot r \) such that each \( r \)-ball \( B(x,r) \) intersects at most \( n + 1 \) elements of \( \mathcal{V}_r \).

If one replaces all \( r > 0 \) in 2.1 by \( r \) sufficiently small, then one gets the concept of capacity dimension of Buyalo [7] for which he proved analog of 2.2. If one replaces all \( r > 0 \) in 2.1 by \( r \) sufficiently large, then one gets the concept of asymptotic dimension of linear type or asymptotic dimension with Higson property. In this section we will show how to introduce those dimensions formally from Nagata-Assouad dimension. That way small scale version and large scale version of 2.2 are in fact consequences of 2.2.

Given a metric space \((X,d)\) and \( \epsilon > 0 \) we consider the metric \( \max(d,\epsilon) \) on \( X \). Needles to say, the formula should not be read literally, only in the case of \( x \neq y \). Similarly, we consider the metric \( \min(d,\epsilon) \).

A metric space \((X,d)\) is called discrete if \((X,d)\) is \( \delta \)-discrete for some \( \delta > 0 \). That means \( d(x,y) > \delta \) for all \( x \neq y \).

**Lemma 2.3.** Any discrete metric space \((X,d)\) is bi-Lipschitz equivalent to \((X,\max(d,\epsilon))\) for all \( \epsilon > 0 \).

**Proof.** Suppose \((X,d)\) is \( \delta \)-discrete. Notice the identity map \( \text{id} : (X,\max(d,\epsilon)) \to (X,d) \) is \( 1 \)-Lipschitz and its inverse is \( (1 + \frac{\epsilon}{\delta}) \)-Lipschitz as \( d(x,y) \leq \max(d(x,y),\epsilon) \leq (1 + \frac{\epsilon}{\delta}) \cdot d(x,y) \) for all \( x \neq y \in X \).

**Corollary 2.4.** For any metric space \((X,d)\) and \( \epsilon,\delta > 0 \) the space \((X,\max(d,\epsilon))\) is bi-Lipschitz equivalent to \((X,\max(d,\delta))\).
Proof. Assume $\varepsilon > \delta$ and notice $\max(d_\delta, \varepsilon) = d_\varepsilon$, where $d_a = \max(d, a)$. Use 2.3.

Since Nagata-Assouad dimension is an invariant of the Lipschitz category (see [14] for a stronger result for quasisymmetric embeddings), one gets the following.

Corollary 2.5. For any metric space $(X, d)$ the Nagata-Assouad dimension of $(X, \max(d, \varepsilon))$ does not depend on $\varepsilon > 0$.

Given a metric space $(X, d)$ one can disregard its microscopic features by considering the space $(X, \max(d, 1))$. The macroscopic Nagata-Assouad dimension of $(X, d)$ is defined as $\dim_{NA}(X, \max(d, 1))$.

Corollary 2.6. If $(X, d)$ is a discrete metric space, then its macroscopic Nagata-Assouad dimension equals the Nagata-Assouad dimension $\dim_{NA}(X, d)$ of $(X, d)$.

Lemma 2.7. The macroscopic Nagata-Assouad dimension of a metric space $X$ is at most $n \geq 0$ if and only if there is $C > 0$ such that for sufficiently large $r > 0$ there is a cover $U = \bigcup_{i=1}^{n+1} U_i$ of $X$ so that each $U_i$ is $r$-disjoint and the diameter of elements of $U$ is bounded by $C \cdot r$.

Proof. Suppose $X$ has a constant $C > 0$ such that for all $r > M$, where $M > 0$, there is a cover $U_r = \bigcup_{i=1}^{n+1} U_i$ of $X$ so that each $U_i$ is $r$-disjoint and the diameter of elements of $U_r$ is bounded by $C \cdot r$. Let $d_M = \max(d, M)$. Notice the cover $U_r, r \leq M$, consisting of one-point sets is $r$-disjoint in $(X, d_M)$. Since covers $U_r, r > M$, have the same desired properties in $(X, d_M)$ as they do in $(X, d)$, $\dim_{NA}(X, d_M) \leq n$.

If $\dim_{NA}(X, d_1) \leq n$, then for each $r > 1$ there is a a cover $U_r = \bigcup_{i=1}^{n+1} U_i$ of $X$ so that each $U_i$ is $r$-disjoint (in $(X, d_1)$) and the diameter of elements of $U_r$ is bounded by $C \cdot r$ in $(X, d_1)$. Notice that $U_i$ is also $r$-disjoint in $(X, d)$ and the diameter of elements of $U_r$ is bounded by $(C + 1) \cdot r$ in $(X, d)$. ■

In view of definition of the Higson property in [9] one has the following consequence of 2.7.

Corollary 2.8. If $(X, d)$ is a metric space, then the macroscopic Nagata-Assouad dimension of $(X, d)$ is at most $n$ if and only if $asdim(X) \leq n$ with the Higson property.

In the reminder of this section we will dualize the above results from large scale/macroscopic category to small scale/microscopic category.

Lemma 2.9. Any bounded metric space $(X, d)$ is bi-Lipschitz equivalent to $(X, \min(d, \varepsilon))$ for all $\varepsilon > 0$.

Proof. Suppose $(X, d)$ is $\delta$-bounded. Notice the identity map $id : (X, \min(d, \varepsilon)) \to (X, d)$ is $(1 + \frac{\delta}{\varepsilon})$-Lipschitz and its inverse is $1$-Lipschitz as $\min(d(x, y), \varepsilon) \leq d(x, y) \leq (1 + \frac{\delta}{\varepsilon}) \cdot \min(d(x, y), \varepsilon))$ for all $x \neq y \in X$. ■
Corollary 2.10. For any metric space \((X, d)\) and \(\epsilon, \delta > 0\) the space \((X, \min(d, \epsilon))\) is bi-Lipschitz equivalent to \((X, \min(d, \delta))\).

Proof. Assume \(\epsilon < \delta\) and notice \(\min(d, \epsilon) = d_\epsilon\), where \(d_a = \min(d, a)\). Use 2.9. 

Since Nagata-Assouad dimension is an invariant of the Lipschitz category, one gets the following.

Corollary 2.11. For any metric space \((X, d)\) the Nagata-Assouad dimension of \((X, \min(d, \epsilon))\) does not depend on \(\epsilon > 0\).

Given a metric space \((X, d)\) one can disregard its macroscopic features by considering the space \((X, \min(d, 1))\). The microscopic Nagata-Assouad dimension of \((X, d)\) is defined as \(\dim_{\text{NA}}(X, d)\).

Lemma 2.13. The microscopic Nagata-Assouad dimension of a metric space \(X\) is at most \(n\) if and only if there is \(C > 0\) such that for sufficiently small \(r > 0\) there is a cover \(U = \bigcup_{i=1}^{n+1} U_i\) of \(X\) so that each \(U_i\) is \(r\)-disjoint and the diameter of elements of \(U\) is bounded by \(C \cdot r\).

Proof. Suppose \(X\) has a constant \(C > 0\) such that for all \(r < M\), where \(M > 0\), there is a cover \(U_r = \bigcup_{i=1}^{n+1} U_i\) of \(X\) so that each \(U_i\) is \(r\)-disjoint and the diameter of elements of \(U_r\) is bounded by \(C \cdot r\). Notice the cover \(U_r\), \(r = M\), consisting of the whole \(X\) has diameter at most \((C + 1) \cdot r\) in \((X, d_M)\). Since covers \(U_r\), \(r < M\), have the same desired properties in \((X, d_M)\) as they do in \((X, d)\), \(\dim_{\text{NA}}(X, d_M) \leq n\).

If \(\dim_{\text{NA}}(X, d_1) \leq n\), then for each \(r < 1\) there is a a cover \(U_r = \bigcup_{i=1}^{n+1} U_i\) of \(X\) so that each \(U_i\) is \(r\)-disjoint in \((X, d_1)\) and the diameter of elements of \(U_r\) is bounded by \(C \cdot r\) in \((X, d_1)\). Notice that \(U_i\) is also \(r\)-disjoint in \((X, d)\) and the diameter of elements of \(U_r\) is bounded by \(C \cdot r\) in \((X, d)\). ■

Since the capacity dimension of Buyalo [7] can be characterized by the condition appearing in 2.13, one derives the following.

Corollary 2.14. If \((X, d)\) is a metric space, then the microscopic Nagata-Assouad dimension of \((X, d)\) is at most \(n\) if and only if the capacity dimension of \(X\) is at most \(n\).

In [14] it is shown that if \(X = A \cup B\), then

\[
\dim_{\text{NA}}(X) = \max(\dim_{\text{NA}}(A), \dim_{\text{NA}}(B)).
\]

Corollary 2.15. Let \(D(Y)\) stand for either the capacity dimension of \(Y\) or the asymptotic dimension of linear type of \(Y\). If \(X = A \cup B\), then

\[
D(X) = \max(D(A), D(B)).
\]
Proof. Switch to either max(d, 1) or min(d, 1) metrics. ■

3. SPHERES AS LIPSCHITZ EXTENSORS

A metric space $E$ is a Lipschitz extensor of $X$ if there is a constant $C > 0$ such that any $\lambda$-Lipschitz function $f : A \to E$, $A$ a subset of $X$, extends to a $C \cdot \lambda$-Lipschitz function $\tilde{f} : X \to E$.

The purpose of this section is to find necessary and sufficient conditions for a sphere $S^m$ to be a Lipschitz extensor of $X$. This is done by comparing existence of Lipschitz extensions in a finite range of Lipschitz constants to existence of Lebesgue refinements in a finite range of Lebesgue constants (see 3.1, 3.3, and 3.5).

Given a cover $\mathcal{U} = \{U_s\}_{s \in S}$ of a metric space $(X, d)$ there is a natural family of functions $\{f_s\}_{s \in S}$ associated to $\mathcal{U}$: $f_s(x) := \text{dist}(x, X \setminus U_s)$. To simplify matters by the local Lebesgue number $L_{\mathcal{U}}(x)$ of $\mathcal{U}$ at $x$ we mean

$$\sup\{f_s(x) \mid s \in S\}$$

and by the (global) Lebesgue number $L(\mathcal{U})$ of $\mathcal{U}$ we mean

$$\inf\{L_{\mathcal{U}}(x) \mid x \in X\}.$$ 

We are interested in covers with positive Lebesgue number. For those the local multiplicity $m_{\mathcal{U}}(x)$ can be defined as $1 + |T(x)|$, where $T(x) = \{s \in S \mid f_s(x) > 0\}$ and the global multiplicity $m(\mathcal{U})$ can be defined as

$$\sup\{m_{\mathcal{U}}(x) \mid x \in X\}.$$ 

If the multiplicity $m(\mathcal{U})$ is finite, then $\mathcal{U}$ has a natural partition of unity $\{\phi_s\}_{s \in S}$ associated to it:

$$\phi_s(x) = \frac{f_s(x)}{\sum_{t \in S} f_t(x)}.$$ 

That partition can be considered as a barycentric map $\phi : X \to N(\mathcal{U})$ from $X$ to the nerve of $\mathcal{U}$. Since each $f_s$ is 1-Lipschitz, $\sum_{t \in S} f_t(x)$ is $2m(\mathcal{U})$-Lipschitz and each $\phi_s$ is $\frac{2m(\mathcal{U})}{L(\mathcal{U})}$-Lipschitz (use the fact that $\frac{u}{u+v}$ is $\frac{\text{max}(\text{Lip}(u), \text{Lip}(v))}{\text{min}(u+v)}$-Lipschitz). Therefore $\phi : X \to N(\mathcal{U})$ is $\frac{4m(\mathcal{U})^2}{L(\mathcal{U})}$-Lipschitz. See [3] and [8] for more details and better estimates of Lipschitz constants.

Most estimates in this paper work well for both the $l_1$ and $l_2$ metrics on $R^n$ and simplicial complexes.

In analogy to $\lambda$-Lipschitz functions we introduce the concept of $r$-Lebesgue cover $\mathcal{U}$. That is simply a shortcut to $r \leq L(\mathcal{U})$.

Proposition 3.1. Suppose $X$ is metric, $m \geq 0$, $C > 0$, and $\lambda_2 > \lambda_1 > 0$. If any $\lambda$-Lipschitz function $f : A \to S^m$, $A$ a subset of $X$ and $\lambda_1 < \lambda < \lambda_2$, extends to a $C \cdot \lambda$-Lipschitz function $\tilde{f} : X \to S^m$, then $t = \frac{1}{2m(m+1)^2(m+1)}$ has the property that any finite $r$-Lebesgue cover $\mathcal{U} = \{U_0, \ldots, U_{m+1}\}$ of $X$
admits a refinement \( \mathcal{V} \) so that \( \mathcal{V} \) is \( t \cdot r \)-Lebesgue and the multiplicity of \( \mathcal{V} \) is at most \( m + 1 \) provided \( \frac{4(m+2)^2}{\lambda_2} < r < \frac{4(m+2)^2}{\lambda_1} \).

**Proof.** Assume \( \frac{4(m+2)^2}{\lambda_2} < r < \frac{4(m+2)^2}{\lambda_1} \) and \( \mathcal{U} \) is \( r \)-Lebesgue. Therefore \( \lambda_1 < \frac{4(m+2)^2}{r} < \lambda_2 \). Consider a barycentric map \( \phi : X \rightarrow \mathcal{N}(\mathcal{U}) = \Delta^{m+1} \).

Notice \( \text{Lip}(\phi) \leq \frac{4(m+2)^2}{r} \). There is \( g : X \rightarrow \partial \Delta^{m+1} \) such that \( \text{Lip}(g) \leq \frac{4C(m+2)^2}{r} \) and \( g(x) = \phi(x) \) for all \( x \in X \) so that \( \phi(x) \in \partial \Delta^{m+1} \). Consider \( V_i = \{ x \in X \mid g_i(x) > 0 \} \). Notice \( \mathcal{V} = \{ V_i \}_{i=0}^{m+1} \) is of multiplicity at most \( m+1 \). Also \( x \in V_i \) implies \( x \in U_i \), so \( \mathcal{V} \) refines \( \mathcal{U} \). Given \( x \in X \) there is \( i \) such that \( g_i(x) \geq \frac{1}{m+1} \). If \( d(x, y) < \frac{r}{4C(m+2)^2(m+1)} \), then \( \sum_{i=0}^{m+1} |g_i(x) - g_i(y)| < \frac{1}{m+1} \) and \( g_i(y) > 0 \). Thus, the ball at \( x \) of radius \( \frac{r}{4C(m+2)^2(m+1)} \) is contained in one element of \( \mathcal{V} \). That proves \( \mathcal{V} \) is \( t \cdot r \)-Lebesgue, where \( t = \frac{1}{4C(m+2)^2(m+1)} \).

In this paper we consider the space \( \mathbb{R}^n \) with either the \( l_1 \)-metric \( d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i| \) or the \( l_2 \)-metric \( d_2(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \). The inequalities

\[
d_1(x, y) \leq n \cdot \max \{|x_i - y_i|\} \leq n \cdot d_2(x, y)
\]

\[
d_2(x, y) \leq \sqrt{n} \cdot \max \{|x_i - y_i|\} \leq \sqrt{n} \cdot d_1(x, y)
\]

show that the identity map \( (\mathbb{R}^n, d_2) \rightarrow (\mathbb{R}^n, d_1) \) is \( n \)-Lipschitz and its inverse is \( \sqrt{n} \)-Lipschitz.

**Lemma 3.2.** Let \( \Delta \) be a closed convex subset of the space \( (\mathbb{R}^n, d_j), j = 1 \text{ or } 2 \). For any metric space \( X \) any \( \lambda \)-Lipschitz map \( f : A \rightarrow \Delta \) of a subspace \( A \subset X \) can be extended to a \( n^2 \cdot \lambda \)-Lipschitz map \( \hat{f} : X \rightarrow \Delta \).

**Proof.** Fix an orthogonal coordinate system in \( (\mathbb{R}^n, d_2) \). Given a \( \lambda \)-Lipschitz map \( f : A \rightarrow (\mathbb{R}^n, d_2) \), every coordinate map \( f_i : A \rightarrow \mathbb{R} \) is \( \lambda \)-Lipschitz and can be extended to a \( \lambda \)-Lipschitz map \( \hat{f_i} : X \rightarrow \mathbb{R} \). These coordinate extensions define the map \( \hat{f} : X \rightarrow \mathbb{R}^n \) which is \( \lambda \sqrt{n} \)-Lipschitz. Clearly, the nearest point retraction \( r_\Delta : (\mathbb{R}^n, d_2) \rightarrow \Delta \) is \( 1 \)-Lipschitz. Therefore the composition \( \hat{f} = r_\Delta \circ \hat{f} \) is \( \sqrt{n} \lambda \)-Lipschitz.

Since the nearest point retraction \( r_\Delta : (\mathbb{R}^n, d_1) \rightarrow \Delta \) may be multivalued, we proceed as follows. The composition \( id \circ f : A \rightarrow (\Delta, d_1) \rightarrow (\Delta, d_2) \) is \( \sqrt{n} \lambda \)-Lipschitz and admits a \( n \lambda \)-Lipschitz extension \( \hat{f}_2 : X \rightarrow (\Delta, d_2) \) by the first part of the proof. Then the composition \( \hat{f} = id \circ \hat{f}_2 : X \rightarrow (\Delta, d_2) \rightarrow (\Delta, d_1) \) is \( n^2 \lambda \)-Lipschitz.

The idea behind the proof of the next proposition is best understood if one thinks of maps from \( X \) to an \( (m+1) \)-simplex \( \Delta^{m+1} \) as a partition of unity. Since we want to create a map to its boundary \( S^m = \partial \Delta^{m+1} \), a geometrical tool is the radial projection \( r \) which we splice in the form of \( (1 - \beta) \cdot r + \beta \cdot \phi \) with a partition of unity \( \phi \) coming from a covering of \( X \) of multiplicity at most \( m + 1 \).
Proposition 3.3. Suppose $X$ is metric, $m \geq 0$, $t > 0$, and $r_2 > r_1 > 0$. There is $s > 0$ such that if any finite $r$-Lebesgue cover $U = \{U_0, \ldots, U_{m+1}\}$ of $X$, where $r_1 < r < r_2$, admits an $t \cdot r$-Lebesgue refinement $V$ satisfying $m(V) \leq m+1$, then any $\lambda$-Lipschitz function $f : A \to S^m$, $A \subset X$, admits a $C \cdot \lambda$-Lipschitz extension $\tilde{f} : X \to S^m$ provided $\frac{1}{12s\lambda^2(m+2)} < \lambda < \frac{1}{12s\lambda r_1(m+2)}$ and $C = 50(m+2)^2 s + \frac{150s^2(m+2)^5}{t}$.

Proof. $s > 0$ is chosen so that given a $\lambda$-Lipschitz $f : A \to \Delta^{m+1}$ one can extend it to an $s \cdot \lambda$-Lipschitz $g : X \to \Delta^{m+1}$ (see 3.2).

Suppose $\frac{1}{12s\lambda^2(m+2)} < \lambda < \frac{1}{12s\lambda r_1(m+2)}$ and $f : A \to \partial\Delta^{m+1}$ is $\lambda$-Lipschitz. Extend it to an $s \cdot \lambda$-Lipschitz $g : X \to \Delta^{m+1}$. Let $\alpha : X \to [0, 1]$ be defined as $\alpha(x) = (m+2) \cdot \min \{g_i(x) \mid 0 \leq i \leq m+1\}$. Notice $Lip(\alpha) \leq (m+2)s \cdot \lambda$. Let $\beta : [0, 1] \to [0, 1]$ be defined by $\beta(z) = 3z - 1$ on $[1/3, 2/3]$, $\beta(z) = 0$ for $z \leq 1/3$ and $\beta(z) = 1$ for $z \geq 2/3$. Notice $Lip(\beta) \leq 3$.

Put $U_i = \{x \in X \mid g_i(x) > \frac{\alpha(x)}{m+2} \text{ or } \alpha(x) > 2/3\}$ and notice $L(U) \geq r = \frac{1}{12s\lambda(m+2)}$ as follows:

Case 1: $x \in X$ and $\alpha(x) > 3/4$. There is some $y \in X$ with $d(x, y) < \frac{1}{12s\lambda(m+2)}$ one has $\alpha(x) - \alpha(y) \leq 1/12$, so $\alpha(y) \leq 2/3$ is not possible. Thus, in that case, the ball $B(x, \frac{1}{12s\lambda(m+2)})$ is contained in all $U_i$.

Case 2: $\alpha(x) \leq 3/4$. There is $i$ so that $g_i(x) \geq \frac{1}{m+2}$. Since $\psi_i = g_i - \frac{\alpha}{m+2}$ is $2s\lambda$-Lipschitz, for any $y \in X$ satisfying $d(x, y) < \frac{1}{8s\lambda(m+2)}$ one has $\psi_i(x) - \psi_i(y) < \frac{1}{4(m+2)}$ and $\psi_i(y) > 0$ as $\psi_i(x) \geq \frac{1}{4(m+2)}$.

Thus $U$ is $r$-Lebesgue and $r_1 < r < r_2$. Shrink each $U_i$ to $V_i$ so that $m(V) \leq m+1$ and $L(V) \geq \frac{1}{12s\lambda(m+2)}$. The barycentric map $\phi : X \to \partial\Delta^{m+1}$ corresponding to $V$ has $Lip(\phi) \leq \frac{4(m+2)^2}{L(V)} \leq \frac{48s\lambda(m+2)^3}{t}$.

Define $h(x) = \sum_{i=0}^{m+1} (g_i(x) - \frac{\alpha(x)}{m+2}) \cdot \frac{1-\beta(\alpha(x))}{1-\alpha(x)} \cdot e_i + \sum_{i=0}^{m+1} \beta(\alpha(x)) \cdot \phi_i(x) \cdot e_i$.

To show $Lip(h) \leq C \cdot \lambda$ we will use the following observations.

(1) If $u, v : X \to [0, M]$, then $Lip(u \cdot v) \leq M \cdot (Lip(u) + Lip(v))$.

(2) In addition, if $v : X \to [k, M]$ and $k > 0$, then

$Lip(\frac{u}{v}) \leq M \cdot \frac{Lip(u) + Lip(v)}{k^2}$

(3) $v(x) = 1 - \alpha(x) \geq 1/3$ if $\frac{1-\beta(\alpha(x))}{1-\alpha(x)} > 0$.

Therefore $Lip(\sum_{i=0}^{m+1} \beta(\alpha(x)) \cdot \phi_i(x) \cdot e_i) \leq 3 \cdot Lip(\alpha) \cdot \sum_{i=0}^{m+1} \beta(\alpha(x)) \cdot \phi_i(x) \cdot e_i \leq 3(m+2) \cdot (m+2) \cdot s \cdot \lambda \cdot \frac{48s\lambda(m+2)^3}{t} \leq \frac{150s^2\lambda(m+2)^5}{t}$. Also, $Lip(\frac{1-\beta(\alpha(x))}{1-\alpha(x)}) \leq 9 \cdot 4 \cdot Lip(\alpha) \leq 36(m+2)s\lambda$, so $Lip(\sum_{i=0}^{m+1} (g_i(x) - \frac{\alpha(x)}{m+2}) \cdot \frac{1-\beta(\alpha(x))}{1-\alpha(x)} \leq (m + 2) \cdot (2s) + 36(m+2)s\lambda \leq 50(m+2)^2 s \lambda$ and $C = 50(m+2)^2 s + \frac{150s^2(m+2)^5}{t}$ works.
It remains to show \( h(X) < \partial \Delta^{m+1} \) and \( h|A = f \). \( h|A = f \) follows from the fact \( \alpha(x) = 0 \) if \( x \in A \). It is clear \( h(x) < \partial \Delta^{m+1} \) if either \( \beta(\alpha(x)) = 0 \) or \( \beta(\alpha(x)) = 1 \), so assume \( 0 < \beta(\alpha(x)) < 1 \). In that case \( \phi_i(x) > 0 \) implies \( g_i(x) - \frac{\alpha(x)}{m+2} > 0 \), so the only possibility for \( h(x) \) to miss \( \partial \Delta^{m+1} \) is when \( g_i(x) - \frac{\alpha(x)}{m+2} > 0 \) for all \( i \) which is not possible.

Propositions 3.1 and 3.4 imply the following

**Corollary 3.4.** If \( X \) is a metric and \( m \geq 0 \), then the following conditions are equivalent:

a. \( S^m \) is a Lipschitz extensor of \( X \).

b. There is \( t > 0 \) such that any finite cover \( U = \{U_0, \ldots, U_{m+1}\} \) of \( X \) admits a refinement \( V \) so that \( L(V) \geq tL(U) \) and the multiplicity of \( V \) is at most \( m+1 \).

**Proposition 3.5.** Suppose \( X \) is a metric space, \( n \geq 0 \), \( 1 > t > 0 \), and \( r_2 > r_1 > 0 \). If every \( r \)-Lebesgue cover \( U = \{U_0, \ldots, U_{n+1}\} \) of \( X \), where \( r_1 < r < r_2 \), admits a \( 4t \cdot r \)-Lebesgue refinement \( V \) satisfying \( m\{V\} \leq n+1 \), then any \( s \)-Lebesgue cover \( W = \{W_0, \ldots, W_{n+2}\} \) of \( X \), where \( 4r_1 < s < 4r_2 \), admits a \( t \cdot s \)-Lebesgue refinement \( V \) of multiplicity at most \( n+2 \).

**Proof.** Suppose \( 4r_1 < s < 4r_2 \). First, let us show that any \( s/2 \)-Lebesgue cover \( U = \{U_i\}_{i=0}^{n+1} \) of \( A \subset X \) consisting of \( n+2 \) elements has a refinement \( V \) such that \( L(V) \geq t \cdot s \) and \( m\{V\} \leq n+1 \). Define \( U'_i = U_i \cup (X \setminus A) \) for \( i \leq n+1 \). Notice \( L(U') \geq s/4 \). Indeed, if \( x \in X \), then \( B(x, s/4) \cap A \) is either empty or is contained in \( B(y, s/2) \) for some \( y \in A \). Since \( B(y, s/2) \cap \{U_i\} \) for some \( i \leq n+1 \), \( B(y, s/2) \subset U'_i \) and \( B(x, s/4) \subset U'_i \). There is a cover \( W \) of \( X \) such that \( W \) refines \( U' \), \( L(W) \geq 4t \cdot s/4 \), and \( m\{W\} \leq n+1 \). Putting \( V = W|A \) finishes the task.

Suppose \( W = \{W_0, \ldots, W_{n+2}\} \) is an \( s \)-Lebesgue cover of \( X \). Let \( A \) be the union of balls \( B(x, s/2) \) such that \( B(x, s) \) is not contained in \( W_{n+2} \). Define \( U_i = W_i \cap A \) for \( i \leq n+1 \) and observe \( L(U) \geq s/2 \) for \( U = \{U_i\}_{i=0}^{n+1} \) as a cover of \( A \). Indeed, if \( x \in A \), then there is \( y \in X \) such that \( B(y, s) \) is not contained in \( W_{n+2} \) and \( x \in B(y, s/2) \). Therefore, \( B(y, s) \subset W_i \) for some \( i \leq n+1 \) which means \( B(x, s/2) \cap A \subset B(y, s) \cap A \subset W_i \cap A = U_i \).

Shrink each \( U_i \) to \( V_i \) so that the intersection of all \( V_i \) is empty and \( L(V) \geq t \cdot s \). Define \( W'_i = V_i \) for \( i \leq n+1 \) and \( W'_{n+2} = W_{n+2} \). The cover \( W' \) is of multiplicity at most \( n+2 \). We want to show \( L(W') \geq t \cdot s \). If \( B(x, s) \subset W_{n+2} \), we are done. Otherwise \( B(x, s/2) \subset A \) and there is \( i \leq n+1 \) such that \( B(x, t \cdot s) \subset V_i \) in which case \( B(x, t \cdot s) \subset W'_i \).

**Corollary 3.6.** Suppose \( X \) is a metric space and \( n \geq 0 \). If \( S^n \) is a Lipschitz extensor of \( X \), then so is \( S^{n+1} \).

**Proof.** By 3.4 there is \( t > 0 \) such that any cover \( U \) of \( X \) consisting of \( n+2 \) elements has a refinement \( V \) satisfying \( L(V) \geq t \cdot L(U) \) and \( m\{V\} \leq n+1 \). Use \( \lambda_2 \) very large and \( \lambda_1 \) very small. We may assume \( t < 1 \), so applying 3.5 and 3.4 completes the proof.
**Definition 3.7.** A metric space $E$ is a large scale Lipschitz extensor (respectively, a small scale Lipschitz extensor) of $X$ if there are constants $C, M > 0$ such that any $\lambda$-Lipschitz function $f : A \to E$, $A$ a subset of $X$, extends to a $C \cdot \lambda$-Lipschitz function $\tilde{f} : X \to E$ for all $\lambda < M$ (respectively, for all $\lambda > M$).

Using 3.1, 3.3, and 3.5 one proves easily the following large/small scale analogs of 3.4.

**Corollary 3.8.** If $X$ is metric and $m \geq 0$, then the following conditions are equivalent:

a. $S^n$ is a large scale Lipschitz extensor (respectively, a small scale Lipschitz extensor) of $X$.

b. There are constants $t, M > 0$ such that any finite $r$-Lebesque cover $\mathcal{U} = \{U_0, \ldots, U_{m+1}\}$ of $X$, where $r > M$ (respectively, $r < M$), admits a $t \cdot r$-Lebesque refinement $\mathcal{V}$ of multiplicity at most $m + 1$.

As in 3.6 one proves its large/small scale analogs:

**Corollary 3.9.** Suppose $X$ is a metric space and $n \geq 0$. If $S^n$ is a large scale Lipschitz extensor (respectively, a small scale Lipschitz extensor) of $X$, then so is $S^{n+1}$.

4. **Lipschitz extensions and Nagata-Assouad dimension**

**Theorem 4.1.** Suppose $X$ is a metric space of finite Nagata-Assouad dimension. If $n \geq 0$, then the following conditions are equivalent:

a. $S^n$ is a Lipschitz extensor of $X$.

b. $\dim_{NA}(X) \leq n$.

**Proof.** The direction (b) $\implies$ (a) follows from 3.4 as follows. Given a cover $\mathcal{U}$ of $X$ of Lebesque number $L(\mathcal{U}) > 0$ pick $\mathcal{V}$ of mesh($\mathcal{V}$) < $L(\mathcal{U})$ so that $L(\mathcal{V}) > L(\mathcal{U})/C$ and $m(\mathcal{V}) \leq m + 1$. Notice that $\mathcal{V}$ refines $\mathcal{U}$ and $L(\mathcal{V}) > L(\mathcal{U})/C$.

(a) $\implies$ (b). Without loss of generality (in view of 3.6), we may assume $\dim_{NA}(X) \leq n + 1$. $k > 0$ is chosen so that given a $\lambda$-Lipschitz $f : A \to \partial \Delta^{n+1}$ one can extend it to an $k \cdot \lambda$-Lipschitz $g : X \to \partial \Delta^{n+1}$. Let $c > 1$ be a constant such that for any $r > 0$ there is a cover $\mathcal{U}$ of $X$ of mesh at most $c \cdot r$, Lebesque number at least $r$, that can be expressed as a union $\bigcup_{i=1}^{n+2} \mathcal{U}_i$ so that each $\mathcal{U}_i$ is $r$-disjoint. For such a cover pick a barycentric map $f : X \to \mathcal{N}(\mathcal{U})$ to the nerve of $\mathcal{U}$ such that $Lip(f) \leq \frac{4(n+2)^2}{r}$. Given an $(n+1)$-simplex $\Delta$ in $\mathcal{N}(\mathcal{U})$ we look at $f | f^{-1}(\partial \Delta)$ and extend it over $f^{-1}(\Delta)$ to obtain $g_\Delta : f^{-1}(\Delta) \to \partial \Delta$ of Lipschitz number at most $k \cdot \frac{4(n+2)^2}{r}$. Paste all $g_\Delta$ together to $g : X \to \mathcal{N}(\mathcal{U})$. Our goal it to estimate the mesh and Lebesque number of $g^{-1}(st(v))$, $v$ a vertex of $\mathcal{N}(\mathcal{U})$.

The mesh of $\{g^{-1}(st(v))\}$ is at most twice that of $\{f^{-1}(st(v))\}$. Indeed, if $g(x) \in st(w)$ and $g(x) \neq f(x)$, then $f(x)$ must belong to the interior of a
simplex $\Delta$ containing $w$. Thus $g(x)$ belongs to the star of $st(w)$ in $\{st(v)\}$ and $x$ belongs to the star of $U_w$ in $\mathcal{U}$ ($U_w$ is the element of $\mathcal{U}$ corresponding to $w$). That star is of size at most $4cr$.

Suppose $C$ is a subset of $X$ of diameter less than $\frac{r}{4kn^3}$. Pick all elements $U_0, \ldots , U_m$ of $\mathcal{U}$ intersecting $C$ and let $v_i$ be corresponding vertices of $N(\mathcal{U})$. Notice $m \leq n+1$ and $g(C)$ is contained in the simplex $[v_0, \ldots , v_m]$ of $N(\mathcal{U})$. Pick $x_0 \in C$ and, without loss of generality, assume the barycentric coordinate $\phi_{v_0}(x_0)$ of $g(x_0)$ corresponding to $v_0$ is at least $\frac{1}{n+2}$: Suppose $g(x)$ does not belong to $st(v_0)$ for some $x \in C$. Thus $\phi_{v_0}(x) = 0$ and

$$\frac{1}{n+2} \leq |\phi_{v_0}(x) - \phi_{v_0}(x_0)| \leq (k \frac{(n+3)^2}{r}) \cdot d(x,x_0) \leq (k \cdot \frac{4(n+3)^2 r}{r}) \cdot \frac{r}{4kn^3} = \frac{1}{n+3},$$

a contradiction.

By setting $d = \frac{r}{4kn^3}$ the argument above shows the existence of a cover $\mathcal{V}$ of $X$ of multiplicity at most $n+1$, of Lebesque number at least $d$, and of mesh at most $d \cdot 16c \cdot k \cdot (n+3)^3$. That means $\dim_{NA}(X) \leq n$. □

Adjusting the proof of 4.1 one can deduce the following.

**Theorem 4.2.** Suppose $X$ is a metric space of finite capacity dimension (respectively, of finite asymptotic dimension with Higson property). If $n \geq 0$, then the following conditions are equivalent:

a. $S^n$ is a small scale (respectively, large scale) Lipschitz extensor of $X$.

b. The capacity dimension (respectively, the asymptotic dimension with Higson property) of $X$ is at most $n$.

**Problem 4.3.** Suppose $X$ is a metric space such that $S^n$ is a Lipschitz extensor of $X$. Is $\dim_{NA}(X)$ at most $n$?

In [6] the authors proved the following.

**Theorem 4.4.** For a metric space $X$ the following conditions are equivalent:

a. $\dim_{NA}(X) \leq 0$.

b. Every metric space $Y$ is a Lipschitz extensor of $X$.

c. The 0-sphere $S^0$ is a Lipschitz extensor of $X$.

Thus, for $n = 0$, the answer to 4.3 is positive.

In the remainder of this section we extend 4.4 to large and small scales.

**Corollary 4.5.** For a metric space $X$ the following conditions are equivalent:

a. The capacity dimension of $X$ is at most 0.

b. Each bounded metric space $Y$ is a small scale Lipschitz extensor of $X$.

c. The 0-sphere $S^0$ is a small scale Lipschitz extensor of $X$.

**Proof.** a) $\implies$ b). Pick $\epsilon > 0$. Let $d = \min(d_X, \epsilon)$ and let $Y$ be $S$-bounded. Define the relation between $M$ and $\epsilon$ by $M = \frac{\epsilon}{4}$ and notice that for $\lambda \geq M = \frac{\epsilon}{4}$ a function $f : (A, d_X|A) \to (Y, d_Y)$ is $\lambda$-Lipschitz if and only if the function $f : (A, d|A) \to Y$ is $\lambda$-Lipschitz.
Let $C > 1$ be a constant such that any $\lambda$-Lipschitz $f : (A, d|A) \rightarrow (Y, d_Y)$ extends to $C \cdot \lambda$-Lipschitz $\tilde{f} : (X, d) \rightarrow (Y, d_Y)$. Given $\lambda$-Lipschitz function $f : (A, d_X|A) \rightarrow (Y, d_Y)$, $\lambda \geq M = \frac{2}{\epsilon}$, the function $f : (A, d|A) \rightarrow Y$ is also $\lambda$-Lipschitz. Extend it to a $C \cdot \lambda$-Lipschitz $\tilde{f} : (X, d) \rightarrow Y$ and notice that $\tilde{f} : (X, d_X) \rightarrow Y$ is also $C \cdot \lambda$-Lipschitz.

b) $\Rightarrow$ c) is obvious.

c) $\Rightarrow$ a). Let $M > 1$ be a number such that any $f : A \rightarrow S^0$ satisfying $\text{Lip}(f) \geq M$ has an extension $g : X \rightarrow S^0$ so that $\text{Lip}(g) \leq C \cdot \text{Lip}(f)$, where $C > 1$. Suppose $r < \frac{1}{(C+1)M}$. Consider the equivalence classes determined by $x \sim y$ if and only if $x$ can be connected to $y$ by a chain of points separated by at most $r$. If any of them has diameter bigger that $Cr$, then there are points $x$ and $y$ in that particular class such that $Cr \leq d(x, y) \leq (C+1)r$. Pick injection $f : \{x, y\} \rightarrow S^0$. Its Lipschitz constant is at least $\frac{1}{Cr} > M$ and at most $\frac{1}{C^r}$. By extending it to $g : B \rightarrow S^0$ of Lipschitz constant at most $\frac{1}{r}$ we arrive at a contradiction that points of the chain joining $x$ and $y$ are mapped to the same point by $g$. By 2.13 the capacity dimension of $X$ is at most 0.

**Corollary 4.6.** For a metric space $X$ the following conditions are equivalent:

a. The asymptotic dimension of $X$ is at most 0 with the Higson property.

b. Each discrete metric space $Y$ is a large scale Lipschitz extensor of $X$.

c. The 0-sphere $S^0$ is a large scale Lipschitz extensor of $X$.

**Proof.** a) $\Rightarrow$ b). Pick $\epsilon > 0$. Let $d = \max(d_X, \epsilon)$ and let $Y$ be $\delta$-discrete. Define the relation between $M$ and $\epsilon$ by $M = \frac{\delta}{\epsilon}$ and notice that for $\lambda \leq M = \frac{\delta}{\epsilon}$, a function $f : (A, d_X|A) \rightarrow (Y, d_Y)$ is $\lambda$-Lipschitz if and only if the function $f : (A, d|A) \rightarrow Y$ is $\lambda$-Lipschitz.

Let $C > 1$ be a constant such that any $\lambda$-Lipschitz $f : (A, d|A) \rightarrow (Y, d_Y)$ extends to $C \cdot \lambda$-Lipschitz $\tilde{f} : (X, d) \rightarrow (Y, d_Y)$. Given $\lambda$-Lipschitz function $f : (A, d_X|A) \rightarrow (Y, d_Y)$, $\lambda \leq M = \frac{\delta}{\epsilon}$, the function $f : (A, d|A) \rightarrow Y$ is also $\lambda$-Lipschitz. Extend it to a $C \cdot \lambda$-Lipschitz $\tilde{f} : (X, d) \rightarrow Y$ and notice that $\tilde{f} : (X, d_X) \rightarrow Y$ is also $C \cdot \lambda$-Lipschitz.

b) $\Rightarrow$ c) is obvious.

c) $\Rightarrow$ a). Let $M > 0$ be a number such that any $f : A \rightarrow S^0$ satisfying $\text{Lip}(f) \leq M$ has an extension $g : X \rightarrow S^0$ so that $\text{Lip}(g) \leq C \cdot \text{Lip}(f)$, where $C > 1$. Suppose $r > \frac{1}{CM}$. Consider the equivalence classes determined by $x \sim y$ if and only if $x$ can be connected to $y$ by a chain of points separated by less than $r$. If any of them has diameter bigger that $Cr$, then there are points $x$ and $y$ in that particular class such that $Cr > d(x, y)$. Pick injection $f : \{x, y\} \rightarrow S^0$. Its Lipschitz constant is less than $\frac{1}{C^r} < M$. By extending it to $g : B \rightarrow S^0$ of Lipschitz constant less than $\frac{1}{r}$ we arrive at a contradiction
that points of the chain joining $x$ and $y$ are mapped to the same point by $g$. By 2.8 the asymptotic dimension of $X$ with Higson property is at most 0.

A way to probe solving 4.3 would be to investigate, for a given $n > 0$, the class of metric spaces $X$ such that $S^n$ is a Lipschitz extensor of $X$. One faces immediately the question of extending 2.15:

**Problem 4.7.** Suppose $X = A \cup B$ is a metric space such that $S^n$ is a Lipschitz extensor of $A$ and $B$. Is $S^n$ is a Lipschitz extensor of $X$?

5. **Coarsely equivalent metrics and Nagata-Assouad dimension**

In this section we characterize asymptotic dimension of Gromov in terms of Nagata-Assouad dimension.

**Theorem 5.1.** For an unbounded metric space $(X, d)$ the following conditions are equivalent:

a. $\text{asdim}(X) \leq n$.

b. There is a hyperbolic metric $(X, d_h)$ coarsely equivalent to $(X, d)$ such that $\dim_{NA}(X, d_h) \leq n$ and the Gromov boundary $\partial_\infty X$ of $X$ consists of one point.

c. There is a metric $(X, d_1)$ coarsely equivalent to $(X, d)$ such that Nagata-Assouad dimension $\dim_{NA}(X, d_1)$ of $(X, d_1)$ is at most $n$.

**Proof.** a) $\Rightarrow$ b). Pick a sequence of covers $U_i$ of $X$, $i \geq 1$, of multiplicity at most $n + 1$ such that $\text{mesh}(U_i) \to \infty$, $L(U_i) \to \infty$, and $2\text{mesh}(U_i) < L(U_{i+1})$ for all $i$. If $x \neq y$, define $d_h(x, y)$ as the smallest $i$ so that there is $U \in U_i$ containing both $x$ and $y$. Clearly, $d_h$ is coarsely equivalent to $d$: if $d_h(x, y) \leq i$, then $d(x, y) \leq \text{mesh}(U_i)$. Also, $d(x, y) \leq L(U_i)$ implies $d_h(x, y) \leq i$.

Notice that for any triangle in $(X, d_h)$ with sides $a \geq b \geq c$ one has $a \leq b + 1$. The reason for this is that $x, y \in U \in U_i$ and $y, z \in V \in U_i$ implies existence of $W \in U_{i+1}$ containing all three points $x, y, z$ as $U \cup V$ is of diameter less than the Lebesgue number of $U_{i+1}$. Thus, in any triangle of $(X, d_h)$ the difference of any two sides that are not minimal is either $-1$, $0$, or $1$.

Fix $x_0 \in X$ and consider the Gromov product

$$(x|y) = \frac{d_h(x, x_0) + d_h(y, x_0) - d_h(x, y)}{2}.$$

To show $(X, d_h)$ is Gromov hyperbolic it suffices to prove

$$(x|z) \geq \min((x|y), (y|z)) - 4$$

for all $x, y, z \in X$. Equivalently, the smallest product in a triangle is at least the medium one minus 4.

If all distances $d_h(x, x_0)$, $d_h(y, x_0)$, and $d_h(z, x_0)$ are within 1.5 from a number $t$, then, as $d_h(z, x)$ cannot be larger than both $d_h(y, x) + 1$ and
we may assume \( d_h(z, x) \leq d_h(y, x) + 1 \). Now \( (x|z) \geq (t-1.5 + t-1.5 - (d_h(y, x)+1))/2 = ((t+1.5) + (t+1.5) - d_h(y, x))/2 - 3.5 \geq (x|y) - 4. \\

Arrange points \( x, y, \) and \( z \) as \( u, v, \) and \( w \) so that \( s = d_h(u, x_0) \leq m = d_h(v, x_0) \leq l = d_h(w, x_0) \). We may assume \( s \leq l - 4 \) (otherwise \( s, m, \) and \( l \) lie within 1.5 from \((s + l)/2\)).

Case 1: \( m \leq l - 2 \). Now \( d_h(w, v) \) must be within 1 from \( l \), so \( 2(w|v) = m + l - d_h(u, v) \) is contained between \( m - 1 \) and \( m + 1 \). Similarly, \( s - 1 \leq 2(u|v) \leq s + 1 \). Since \( d_h(u, v) \leq m + 1, 2(u|v) \geq m + s -(m+1) = s - 1 \). The only possibility for the smallest of Gromov products for the triple \( uvw \) to be less than the medium one minus 4 is if \( s \leq m - 2 \). In that case \( d_h(u, v) \geq m - 1 \), so \( s + 1 = m + s -(m - 1) \geq 2(u|v) = m + s -(m+1) = s - 1 \) and the smallest of Gromov products for the triple \( uvw \) is larger than the medium one minus 4.

Case 2: \( m \geq l - 1 \). Now \( d_h(u, v) \) must be within 1 from \( m \), so \( 2(u|v) = m + s - d_h(u, v) \) is contained between \( s - 1 \) and \( s + 1 \). Similarly, \( s - 1 \leq 2(u|v) \leq s + 1 \). Since \( d_h(u, v) \leq l + 1, 2(u|v) \geq m + l -(l+1) = m - 1 \geq s + 2 \), and the smallest of Gromov products for the triple \( uvw \) is larger than the medium one minus 4.

To prove \( \dim_{FA}(X, d_h) \leq n \) we plan to define covers \( V_r \) of \((X, d_h)\) such that \( \text{mesh}(V_r) \leq r, L(V_r) \geq r/4, \) and \( m(V_r) \leq n + 1 \). If \( r \leq 4 \), we define \( V_r \) as all singletons of \( X \), otherwise we put \( V_r = U_i \) with \( i \) being the integral part of \( r \). Indeed, for \( r > 4 \), \( \text{mesh}(U_i) \leq i \leq r \) and \( L(U_i) \geq i - 1 \geq r - 2 \geq r/4 \).

Let us show the Gromov boundary \( \partial_{\infty} X \) of \( X \) consists of one point. Given two sequences of points \( \{x_i\} \) and \( \{y_i\} \) such that \( (x_m|x_k) \to \infty \) and \( (y_m|y_k) \to \infty \) we need to prove \( (x_m, y_m) \to \infty \). However, \( d_h(x_m, y_m) \) is smaller than \( \max(d_h(x_m, x_0), d_h(y_m, x_0)) + 2 \), so \( 2(x_m|y_m) \geq \min(d_h(x_m, x_0), d_h(y_m, x_0)) - 2 \to \infty \).

Since b) \( \implies \) c) and c) \( \implies \) a) are obvious, we are done. ■

References

[1] P. Assouad, *Sur la distance de Nagata*, C. R. Acad. Sci. Paris Ser. I Math. **294** (1982), no. 1, 31–34.

[2] P. Assouad, *Plongements lipschitziens dans \( \mathbb{R}^n \)*, Bull. Soc. Math. France **111** (1983), 429–448.

[3] G. Bell and A. Dranishnikov, *On asymptotic dimension of groups acting on trees* Geom. Dedicata **103** (2004), 89–101.

[4] Y. Benyamini and J. Lindenstrauss, *Geometric nonlinear functional analysis*. Vol. 1. American Mathematical Society Colloquium Publications, 48. American Mathematical Society, Providence, RI, 2000. xii+488 pp.

[5] N. Brodskiy, J. Dydak, *Coarse dimensions and partitions of unity*, preprint arXiv:math.GT/0506547.

[6] N. Brodskiy, J. Dydak, J. Higes, A. Mitra, *Dimension zero at all scales*, in preparation.

[7] S. Buyalo, *Asymptotic dimension of a hyperbolic space and capacity dimension of its boundary at infinity*, Algebra i analis (St. Petersburg Math. J.), v.17 (2005), 70–95 (in Russian).

[8] S. Buyalo and V. Schroeder, *Hyperbolic dimension of metric spaces*, arXiv:math.GT/040525
[9] A. Dranishnikov, *Asymptotic topology*, Russian Math. Surveys 55 (2000), no.6, 1085–1129.

[10] A. Dranishnikov, *On hypersphericity of manifolds with finite asymptotic dimension*, Trans. Amer. Math. Soc. 355 (2003), 155–167.

[11] A. Dranishnikov and M. Zarichnyi, *Universal spaces for asymptotic dimension*, Topology and its Appl. 140 (2004), no.2-3, 203–225.

[12] M. Gromov, *Asymptotic invariants for infinite groups*, in Geometric Group Theory, vol. 2, 1–295, G. Niblo and M. Roller, eds., Cambridge University Press, 1993.

[13] J. Heinonen, *Lectures on analysis on metric spaces*, Universitext, Springer, 2001.

[14] U. Lang, T. Schlichenmaier, *Nagata dimension, quasisymmetric embeddings, and Lipschitz extensions*, arXiv:math. MG/0410048 (2004).

[15] U. Lang, C. Plaut, *Bilipschitz embeddings of metric spaces into space forms*, Geom. Dedicata 87 (2001), 285–307.

[16] J. Nagata, *Note on dimension theory for metric spaces*, Fund. Math. 45 (1958) 143–181.

[17] J. Nagata, *Modern Dimension Theory*, North-Holland 1965.

[18] P. Ostrand, *A conjecture of J. Nagata on dimension and metrization*, Bull. Amer. Math. Soc. 71 (1965), 623–625.

[19] J. Roe, *Lectures on coarse geometry*, University Lecture Series, 31. American Mathematical Society, Providence, RI, 2003.

[20] E.J. McShane, *Extension of range of functions*, Bull. Amer. Math. Soc. 40 (1934), 837–842.

University of Tennessee, Knoxville, TN 37996, USA  
*E-mail address*: brodskiy@math.utk.edu

University of Tennessee, Knoxville, TN 37996, USA  
*E-mail address*: dydak@math.utk.edu

Departamento de Geometría y Topología, Facultad de CC. Matemáticas, Universidad Complutense de Madrid, Madrid, 28040 Spain  
*E-mail address*: josemhiges@yahoo.es

University of Tennessee, Knoxville, TN 37996, USA  
*E-mail address*: ajmitra@math.utk.edu