Abstract

Using concepts of noncommutative probability we show that the Löwner’s evolution equation can be viewed as providing a map from paths of measures to paths of probability measures. We show that the fixed point of the Löwner map is the convolution semigroup of the semicircle law in the chordal case, and its multiplicative analogue in the radial case. We further show that the Löwner evolution “spreads out” the distribution and that it gives rise to a Markov process.

1 Introduction

Intersections of Brownian motions have been studied by mathematicians and physicists for a long time [24, 12]. Since in high dimensions \((d > 4)\) Brownian motions do not see each other, and in dimension \(d = 1\) intersection is certain, dimensions two, three and four are of particular interest. In dimension \(d = 2\) physicists, using arguments from conformal field theory, predicted...
the values of the intersection exponents for Brownian motions \cite{9}, \cite{8}. These results appeared out of reach for mathematicians until recent breakthroughs in joint work of Greg Lawler, Oded Schramm and Wendelin Werner. After establishing a series of properties of the Brownian intersection exponents in \cite{14}, and their universality in \cite{15}, the introduction of the “stochastic Löwner evolution process” in \cite{22} finally allowed the computation of all intersection exponents in a series of papers \cite{16}, \cite{17}, \cite{18}, \cite{19}. These developments brought new attention to the Löwner equation, introduced some 80 years ago in \cite{20} as a step to prove part of the Bieberbach conjecture. The original Löwner evolution equation describes slit mappings interpolating between the identity map and a conformal map of the disk into itself, where the slit grows from the boundary to the interior. In the application to the calculation of Brownian intersection exponents the Löwner evolution is more generally considered as a “hull” growing from the boundary into a domain, \cite{16}. The main point of this paper is to consider the “generalized” slit mappings as Cauchy transforms of probability measures and to study the consequences of this viewpoint.

Cauchy transforms of probability measures play a central role in free probability, a branch of noncommutative probability pioneered by Dan Voiculescu. Besides its applications to operator theory and $C^*$-algebras it also provides a framework for the limit of large random $n \times n$-matrices as $n \to \infty$. \cite{25}. In physics these concepts were used for the construction of the master field, \cite{11}. Interestingly, the physics approach to the calculation of Brownian intersection exponents as presented in \cite{8} also relies on partition functions of large random matrices. I show in this paper that various concepts and central objects of noncommutative probability appear naturally in the context of the Löwner equation once one takes the above-mentioned viewpoint.

The paper is structured as follows. We begin with an introduction to those aspects of noncommutative probability that will be needed later on: free independence, distribution of noncommutative random variables, additive and multiplicative free convolution, Cauchy transform, $R$- and $S$-transform, semi-circle law and convolution semigroups. This section is written with the (classical) probabilist in mind who may not have been exposed to noncommutative probability before. Those already familiar with noncommutative probability may want to browse through those pages since notational conventions are set there.

The next section introduces the chordal and radial Löwner equations. The stochastic Löwner evolution is also described here.
The final section shows how noncommutative processes arise from Löwner’s
equation. We first show that the conformal maps given by the solution of
the Löwner equation indeed are Cauchy transforms of probability measures,
on \mathbb{R} for the chordal Löwner equation and on the unit circle \( \mathbb{T} \) in the ra-
dial case. Thus the chordal case corresponds to self-adjoint random variables
and the radial case to unitary random variables. We show in Theorem 3
that the Löwner equation induces a map from continuous paths in the space
of measures to the continuous paths in the space of probability measures.
The fixed point of this map is shown to be the convolution semigroup for the
semicircle law in the chordal case, and its unitary analogue in the radial case.

Theorem 4 shows that the Löwner evolution corresponds to a spreading out
of a distribution. Figures 1 and 2 make this particularly apparent. Finally,
Theorems 5 and 6 show that the (deterministic) Löwner evolution gives rise
to a Markov process.

2 Free probability

A noncommutative probability space is a unital algebra \( \mathcal{A} \) over \( \mathbb{C} \) together
with a linear functional \( \varphi : \mathcal{A} \to \mathbb{C} \) satisfying \( \varphi(1) = 1 \). Elements \( a \in \mathcal{A} \) are
called random variables.

**Definition 1.** A family of subalgebras containing \( 1, \{ \mathcal{A}_i \}_{i \in I} \), of the noncom-
mute probability space \( (\mathcal{A}, \varphi) \) is freely independent if for any
\( n \in \mathbb{Z}^+ \), and random variables \( a_k \in \mathcal{A}_{i(k)}, 1 \leq k \leq n \),

\[
\varphi(a_1 \cdots a_n) = 0,
\]

whenever \( \varphi(a_k) = 0, 1 \leq k \leq n \) and consecutive indices are distinct, \( i(k) \neq \)
\( i(k+1) \), \( 1 \leq k < n \).

A family of random variables \( \{a_i\}_{i \in I}, a_i \in \mathcal{A} \), is said to be freely indepen-
dent (free) if the unital subalgebras they generate are freely independent.

**Example 1.** Let \( \mathcal{B}, \mathcal{C} \subset \mathcal{A} \) be two free subalgebras, and \( b \in \mathcal{B}, c \in \mathcal{C} \). Set
\( \overline{b} = \varphi(b)1 \) and define \( b' \) by \( b = \overline{b} + b' \). Then \( \varphi(b') = 0 \). Similarly, write
\( c = \overline{c} + c' \). Then,

\[
\varphi(bc) = \varphi(\overline{b}\overline{c}) + \varphi(\overline{b}c') + \varphi(b\overline{c}) + \varphi(b'c').
\]
By freeness, \( \varphi(b'c') = 0 \). Also \( \varphi(bc') = \varphi(\varphi(b)1c') = \varphi(b)\varphi(c') = 0 \), and similarly \( \varphi(b'c) = 0 \). Thus \( \varphi(bc) = \varphi(b)\varphi(c) \). However, if \( b_1, b_2 \in B \), and \( c_1, c_2 \in C \), then, by splitting of means as before, one can show that

\[
\varphi(b_1c_1b_2c_2) - \varphi(b_1)\varphi(c_1)\varphi(b_2)\varphi(c_2) = \varphi(b_1b_2)\varphi(c_1)\varphi(c_2) + \varphi(b_1)\varphi(b_2)\varphi(c_1c_2).
\]

Remark 1. Example \( \square \) shows that free independence does not imply the factorization of expectations (as independence does in classical probability). However, the calculation shows that the expectation of the product \( b_1c_1b_2c_2 \) can be reduced to the computation of expectations in the subalgebras \( B \) and \( C \). More generally the following holds: if \( \{A_i\}_{i \in I} \) is a family of free subalgebras of \( A \), then, for \( n \in \mathbb{Z}^+, \ a_i \in A_{k_i}, 1 \leq i \leq n \), \( \varphi(a_1 \cdots a_n) \) can be computed explicitly from the restriction of \( \varphi \) to the algebras \( A_{k_i}, 1 \leq i \leq n \), see \( [3, \text{Proposition 1}] \).

So far, the constructions have been purely algebraic. To do analysis we need to add more structure. The following definitions and examples are taken from \( [24] \).

**Definition 2.** A unital algebra \( A \) is a \( C^\ast \)-algebra if it is a Banach algebra \( (A, \| \cdot \|) \) with an involution \( a \to a^\ast \), which is isomorphic to an algebra of bounded operators on some Hilbert space with the usual operator norm and involution defined by taking the adjoint.

Thus, if \( B(H) \) denotes the algebra of bounded operators of a Hilbert space \( H \), then, for some Hilbert space \( H \), we can identify \( (A, \| \cdot \|, *) \) with an algebra \( I \in A \subseteq B(H) \) which is norm closed and such that \( T \in A \) implies \( T^\ast \in A \).

**Definition 3.** A state \( \varphi: A \to \mathbb{C} \) is a linear functional such that \( \varphi(1) = 1 \) and \( \varphi(a) \geq 0 \) if \( a \geq 0 \).

Here \( a \geq 0 \) means \( a = x^\ast x \) for some \( x \in A \). Equivalently, \( a \geq 0 \) if and only if \( a = a^\ast \) and the spectrum \( \sigma(a) \subseteq [0, \infty) \). Finally, when \( A \subseteq B(H) \), then \( a \geq 0 \) if and only if \( \langle ah, h \rangle \geq 0 \) for all \( h \in H \).

A \( C^\ast \)-probability space \( (A, \varphi) \) is a noncommutative probability space such that \( A \) is a \( C^\ast \)-algebra and \( \varphi \) is a state. By the Gelfand-Naimark-Segal theorem a \( C^\ast \)-probability space \( (A, \varphi) \) can always be realized in the form \( A \subseteq B(H) \), and such that there is a unit vector \( h \in H \) for which \( \varphi(a) = \langle ah, h \rangle \), for all \( a \in A \).
A $W^*$-algebra or von Neumann algebra $I \in \mathcal{A} \subseteq B(H)$ is a $C^*$-algebra of operators which is weakly closed, i.e., if $\{T_i\}_{i \in I} \subset \mathcal{A}$ is a net such that $\langle T_i h_1, h_2 \rangle$ converges to $\langle Th_1, h_2 \rangle$ for all pairs $h_1, h_2 \in H$, then $T \in \mathcal{A}$.

**Definition 4.** $(\mathcal{A}, \varphi)$ is a $W^*$-probability space if the pair is isomorphic to a $W^*$-algebra and some vector state $\langle \cdot, h \rangle$.

**Example 2.** Let $(\Omega, \mathcal{F}, P)$ be a (classical) probability space. Up to sets of measure zero, the data $(\Omega, \mathcal{F}, P)$ is encoded in $(L^\infty((\Omega, \mathcal{F}, P); \mathbb{C}), \varphi)$, where $\varphi(X) \equiv \mathbb{E}^P[X] = \int X \, dP$. Indeed, $P(A) = \mathbb{E}^P[1_A]$ for all $A \in \mathcal{F}$, and for any element $f$ in the equivalence class of $1_A$ in $L^\infty((\Omega, \mathcal{F}, P); \mathbb{C})$ the set $B \equiv \{f = 1\}$ equals $A$ up to a set of measure zero. Furthermore, if $H \equiv L^2((\Omega, \mathcal{F}, P); \mathbb{C})$, then the multiplication operators $M(X)$ defined for $X \in L^\infty((\Omega, \mathcal{F}, P); \mathbb{C})$ by $Y \in H \mapsto XY \in H$, form a von Neumann algebra and $\varphi(X) = \langle X1, 1 \rangle$, where $1 \in H$ is the constant function with value 1. Thus $(L^\infty((\Omega, \mathcal{F}, P); \mathbb{C}), \varphi)$ is a $W^*$-probability space.

**Remark 2.** Suppose that, in the setup of Example 2, $X$ and $Y$ are $\mathbb{R}$-valued independent (classical) random variables in $L^\infty((\Omega, \mathcal{F}, P); \mathbb{C})$ with mean-value 0. Then

$$\varphi(XYXY) = \mathbb{E}^P[X^2Y^2] = \mathbb{E}^P[X^2]\mathbb{E}^P[Y^2].$$

Thus $X, Y$ will only be free if at least one of them equals 0, (a.s.-$P$). Conversely, if $X, Y \in L^\infty((\Omega, \mathcal{F}, P); \mathbb{C})$ have essentially disjoint support, i.e. $P(\{X = 0\} \cup \{Y = 0\}) = 1$, then, for any $n, m \in \mathbb{Z}^+$, $X^nY^m = 0$, (a.s.-$P$), and so $X$ and $Y$ are free.

**Example 3.** Again, let $(\Omega, \mathcal{F}, P)$ be a (classical) probability space, and denote $M_n$ the algebra of complex $n \times n$ matrices. Let further

$$\mathcal{A}_n = \bigcap_{1 \leq p < \infty} L^p((\Omega, \mathcal{F}, P), M_n)$$

be the algebra of (classical) random variables with values in $M_n$ which are $p$-integrable for $1 \leq p < \infty$. $\mathcal{A}_n$ is not a Banach algebra, but it has a natural involution, $X \mapsto X^*$, where $X^*$ is defined by $X^*_{ij}(\omega) = X_{ji}(\omega)$. Furthermore, if $\varphi_n : \mathcal{A}_n \to \mathbb{C}$ is given by

$$\varphi_n(X) = \mathbb{E}^P[\frac{1}{n}\text{Tr}(X)],$$

then $\varphi_n$ is a state.
Suppose that \((\mathcal{A}, \varphi)\) is a \(C^*\)-probability space. Identify \(\mathcal{A}\) with an algebra of operators on a Hilbert space \(H\) such that \(\varphi(\cdot) = \langle \cdot h, h \rangle\), for some unit vector \(h \in H\). If \(a = a^* \in \mathcal{A}\) is a self-adjoint element, then, by the spectral theorem, there is a projection-valued compactly supported measure \(E(\cdot; a)\) so that for every continuous function \(f\)

\[
f(a) = \int f(t) \, dE((-\infty, t]; a).
\]

Denote \(\nu\) the scalar measure given by \(\nu(\cdot) = \langle E(\cdot; a)h, h \rangle\). Then for any polynomial \(p\) we have

\[
\varphi(p(a)) = \langle (\int p(t) \, dE)h, h \rangle = \int p(t) \, d(Eh, h) = \int p(t) \, \nu(dt).
\]

We call \(\nu\) the distribution of \(a\). Since \(\nu\) has compact support it is uniquely determined by the moments \(\varphi(a^n), n \in \mathbb{N}\).

**Example 4.** In the setting of Example 3 let \(X \in L^\infty((\Omega, \mathcal{F}, P); \mathbb{C})\) be a bounded random variable. \(X\) is self-adjoint if and only if

\[
\mathbb{E}^{P}[X 1_A] = \langle (X 1_A) 1, 1 \rangle = \langle 1_A, X \rangle = \mathbb{E}^{P}[X 1_A],
\]

for all \(A \in \mathcal{F}\). Thus \(X = X^*\) if and only if \(X\) is real-valued. Denote \(\nu = X_* P\) the (classical) distribution of the random variable \(X\). Then for any polynomial \(p\),

\[
\varphi(p(X)) = \mathbb{E}^{P}[p(X)] = \int p(t) \, \nu(dt).
\]

Thus in this case the distribution of \(X\) as a (classical) random variable agrees with the distribution of \(X\) as a noncommutative random variable.

**Example 5.** In the setting of Example 3 let \(X\) be a random variable with values in the complex \(n \times n\) matrices such that \(X = X^*\). For \(\omega \in \Omega\) let \(\lambda_1(\omega) \leq \cdots \leq \lambda_n(\omega)\) be the eigenvalues of the hermitian matrix \(X(\omega)\) and define random variables \(\lambda_1, \ldots, \lambda_n\) accordingly. Then \(\lambda_i \in L^P(P; \mathbb{R})\) for every
where \( \nu_i = \lambda_i P \) is the distribution of \( \lambda_i \). If the moment problem for \( \nu \equiv \frac{1}{n} \sum_1^n \nu_i(dt) \) is determinate (see [23]), then we call \( \nu \) the distribution of the noncommutative random variable \( X \in \mathcal{A}_n \).

**Remark 3.** Suppose that the moment problem for \( \nu \) in Example 3 is determinate. For a Borel set \( A \subset \mathbb{R} \) we have

\[
\left( \frac{1}{n} \sum_1^n \nu_i \right)(A) = \frac{1}{n} \sum_1^n P(\lambda_i \in A)
= \mathbb{E}^P\left[ \frac{1}{n} \sum_1^n 1_{(\lambda_i \in A)} \right] = \mathbb{E}^P\left[ (\frac{1}{n} \sum_1^n \delta_{\lambda_i}) (A) \right],
\]

where \( \frac{1}{n} \sum_1^n \delta_{\lambda_i} \) is a random variable with values in the space \( M_1(\mathbb{R}) \) of probability measures on \( \mathbb{R} \). Note that for each \( \omega \in \Omega \)

\[
\frac{1}{n} \sum_1^n \delta_{\lambda_i(\omega)}
\]

is the empirical distribution on the eigenvalues of the hermitian matrix \( X(\omega) \). Thus the distribution of the noncommutative random variable \( X \) is the \( P \)-average of the empirical distribution of its eigenvalues.

### 2.1 Additive free convolution

Suppose now that \( a \) and \( b \) are freely independent in the \( W^* \)-probability space \( (\mathcal{A}, \varphi) \). By Remark 1, the restriction of \( \varphi \) to the subalgebra generated by \( \{1, a, b\} \) is determined by the restrictions of \( \varphi \) to the subalgebras generated by \( \{1, a\} \) and \( \{1, b\} \). In particular the moments \( \varphi((a + b)^n), \ n \in \mathbb{N} \), are
determined by the moments $\varphi(a^n), \varphi(b^m), n,m \in \mathbb{N}$. Thus the distribution $\mu_{a+b}$ is completely determined by the distributions $\mu_a$ and $\mu_b$ and we may define a free convolution operation $\boxplus$ on the distributions of noncommutative random variables such that $\mu_a \boxplus \mu_b = \mu_{a+b}$ whenever $a$ and $b$ are freely independent in $(\mathcal{A}, \varphi)$. To compute the free convolution D. Voiculescu found a linearizing map, called the $R$-transform.

**Theorem 1.** [27] If $\mu$ is the distribution of a random variable $a$ in a $W^*$-probability space $(\mathcal{A}, \varphi)$ let

$$G_\mu(z) = \sum_{n=0}^{\infty} \varphi(a^n)z^{-(n+1)}$$

and let $K_\mu$ be the formal inverse $G_\mu(K_\mu(z)) = z$ and let $R_\mu(z) = K_\mu(z) - z^{-1}$. Then

$$R_{\mu \boxplus \nu} = R_\mu + R_\nu.$$

Free convolution extends to unbounded random variables, [3]. Denote $\tilde{\mathcal{A}}$ the algebra of (possibly) unbounded operators affiliated to $\mathcal{A}$, and $\tilde{\mathcal{A}}_{sa}$ the subspace of self-adjoint elements of $\tilde{\mathcal{A}}$. Thus $a \in \tilde{\mathcal{A}}_{sa}$ if and only if all its spectral measures are in $\mathcal{A}$. If $a \in \tilde{\mathcal{A}}_{sa}$, then the distribution of $a$ in the state $\varphi$ is the unique probability measure $\mu_a$ on $\mathbb{R}$ such that $\varphi(f(a)) = \int_\mathbb{R} f(x) \mu_a(dx)$ for any bounded Borel measurable function $f$ on $\mathbb{R}$. Conversely, given a probability measure $\mu$ on $\mathbb{R}$, there is a von Neumann algebra $\mathcal{A}$ with normal faithful trace $\varphi$ and a self-adjoint operator $a \in \tilde{\mathcal{A}}_{sa}$ with distribution $\mu = \mu_a$, [3].

For a compactly supported probability measure $\mu$ on $\mathbb{R}$,

$$G_\mu(z) = \int_\mathbb{R} \frac{\mu(dx)}{z - x}$$

(1)

is the Cauchy transform of $\mu$, which is an analytic function in $\mathbb{C} \setminus \text{supp} \mu$. If we use the right hand side of (1) as the definition of $G_\mu$ for an arbitrary probability measure on $\mathbb{R}$, then the conclusions of the above theorem continue to hold. Set $\mathbb{C}^+ = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$, $\mathbb{C}^- = \{z \in \mathbb{C} : \text{Im}(z) < 0\}$ and note that $G_\mu : \mathbb{C}^+ \to \mathbb{C}^-$. For $\alpha, \beta > 0$, let

$$\Theta_{\alpha,\beta} = \{z \in \mathbb{C}^- : \alpha \text{ Im}(z) < \text{Re}(z) < -\alpha \text{ Im}(z), |z| < \beta\}. \quad (2)$$
Then one can show that for every $\alpha > 0$, there exists a $\beta > 0$ such that $G_\mu$ has a right inverse defined on $\Theta_{\alpha,\beta}$, taking values in some domain of the form

$$\Gamma_{\gamma,\delta} = \{ z \in \mathbb{C}^+ : -\gamma \text{ Im}(z) < \text{Re}(z) < \gamma \text{ Im}(z), |z| > \delta \},$$

with $\gamma, \delta > 0$. Call $K_\mu$ this right inverse, and let $R_\mu(z) = K_\mu(z) - \frac{1}{z}$. The function $R_\mu$ linearizes free convolution. That is $R_{\mu \boxplus \nu} = R_\mu + R_\nu$ on some domain $\Theta_{\alpha,\beta}$ where all three functions are defined.

**Example 6.** The rôle of the Gaussian in classical probability is played by the semicircle law in noncommutative probability. The centered semicircle law $\mu$ with variance $\sigma^2$ has density $\frac{1}{2\pi\sigma} \sqrt{4\sigma^2 - x^2}$ when $-2\sigma < x < 2\sigma$ and 0 elsewhere. Its Cauchy transform is

$$G_\mu(z) = \frac{z - \sqrt{z^2 - 4\sigma^2}}{2\sigma^2},$$

where the branch of square root is chosen so that Im$(z) > 0$ implies Im$(G_\mu(z) < 0)$, and

$$K_\mu(z) = \frac{1}{z} + \sigma^2 z, \quad R_\mu(z) = \sigma^2 z. \quad (3)$$

Note that if $\mu_t$ denotes the centered semicircle law with variance $t > 0$, then, by (3),

$$\mu_{1/n} \boxplus \cdots \boxplus \mu_{1/n} = \mu_1, \quad \text{for any } n \in \mathbb{Z}^+. \quad (4)$$

A distribution $\mu$ that satisfies (3) is said to be freely infinitely divisible and gives rise to a free convolution semigroup. That is a family of distributions $\{\mu_t, t \in [0, \infty)\}$, such that $\mu = \mu_1$, $\mu_{t+s} = \mu_t \boxplus \mu_s$, and $t \mapsto \mu_t$ is weak*-continuous. If $\{\mu_t, t \in [0, \infty)\}$ is a free convolution semigroup and $\nu$ an arbitrary probability measure on $\mathbb{R}$, then the Cauchy transform

$$G(z, t) \equiv G_{\mu_\mu \boxplus \nu}(z)$$

satisfies the equation

$$\frac{\partial G}{\partial t} = -R(G) \frac{\partial G}{\partial z} \quad \text{for } z \in \mathbb{C}^+, t \in [0, \infty), \quad (5)$$

with $R = R_{\mu_1}$ and initial condition $G(z, 0) = G_\nu(z)$. 9
Remark 4. If \( \{\mu_t, t \in [0, \infty)\} \) is the convolution semigroup for the semicircle law, i.e. \( R_{\mu_t}(z) = tz \), then (5) reduces to

\[
\frac{\partial G}{\partial t} = -G \frac{\partial G}{\partial z},
\]

which, by analogy with the Gaussian semigroup, is the “heat equation” in free probability. Note that for \( \nu = \delta_0 \), \( G(z, 0) = \frac{1}{z} \) and the image of \( \mathbb{C}^+ \) under \( G_t(.) = G(\cdot, t) \) is a semicircle in \( \mathbb{C}^- \) centered at the origin and of radius \( \frac{1}{t} \). Thus the convolution semigroup for the semicircle law can also be described by a flow of nested semidisks in the lower halfplane, centered at the origin. Figure 1 is a plot of the semicircle densities for \( t \in [0, 1] \).

![Figure 1: Semicircle law densities](image)

2.2 Multiplicative free convolution

If \( a, b \) are freely independent in \((A, \varphi)\), then, by Remark 4, \( \mu_{ab} \) is determined by \( \mu_a \) and \( \mu_b \). Hence we may define a multiplicative free convolution operation
on the distributions of noncommutative random variables, such that

$$\mu_a \boxtimes \mu_b = \mu_{ab}$$

whenever $a$ and $b$ are freely independent in some noncommutative probability space.

Given a measure $\mu$ on $T \equiv \{ z \in \mathbb{C} : |z| = 1 \}$ there is a unitary random variable $U$ in some $W^*$-probability space $(A, \varphi)$ such that

$$\varphi(U^k) = \int_T z^k \mu(dz)$$

for each $k \in \mathbb{N}$. For example, we may take $A = L^\infty(T, \mu)$ and let $U$ be multiplication by $z$. Conversely, if $U$ is a unitary random variable in some $W^*$-probability space $(A, \varphi)$, then there is a unique probability measure on $T$ with the same moments as $U$, the expectation of the spectral measure of $U$. Since the product of two unitary operators is again a unitary operator, $\boxtimes$ defines an operation on probability measures on $T$, [25].

The multiplicative free convolution is computed using the $S$-transform.

**Theorem 2.** [26] If $\mu$ is the distribution of a random variable $a$ in a $W^*$-probability space $(A, \varphi)$ and $\varphi(a) \neq 0$, let

$$\psi_\mu(z) = \sum_{n=1}^{\infty} \varphi(a^n) z^n$$

and let $\chi_\mu$ be the formal inverse $\psi_\mu(\chi_\mu(z)) = z$. Let further $S_\mu(z) = \frac{1+z}{z} \chi_\mu(z)$. Then we have

$$S_{\mu \boxtimes \nu} = S_\mu S_\nu.$$

Note that

$$\psi_\mu(z) = \int_0^{2\pi} \frac{ze^{-i\theta}}{1 - ze^{-i\theta}} \mu(d\theta),$$

where we identified the measure $\mu$ on $T$ with the corresponding measure on $[0, 2\pi)$. With the analogous definitions, there is a notion of infinite divisibility and semigroups relative to $\boxtimes$. Let $M_1(T)'$ denote the probability measures
\( \mu \) on \( \mathbb{T} \) such that \( \int_{\mathbb{T}} z \mu(dz) \neq 0 \). If \( \{\mu_t, t \geq 0\} \) is a semigroup with respect to \( \otimes \) in \( M_1(\mathbb{T})' \), then \( \psi(z, t) = \psi_{\mu_t}(z) \) satisfies the equation
\[
\frac{\partial \psi}{\partial t} = -u(\psi)z \frac{\partial \psi}{\partial z},
\]
where \( S_{\mu_t}(z) = \exp(tu(z)) \). The analogue of the Gaussian family in this context is the family \( \{\mu_t, t \geq 0\} \) with
\[
S_{\mu}(z) = \exp(t(z + \frac{1}{2})), \quad t \geq 0.
\]
In this case the evolution equation (7) simplifies to
\[
\frac{\partial \psi}{\partial t} = -(\psi + \frac{1}{2})z \frac{\partial \psi}{\partial z}.
\]

3 The Löwner equation

3.1 Chordal Löwner evolution

Let \( M^0(\mathbb{R}) \) be the set of nonzero Borel measures \( \mu \) on \( \mathbb{R} \) with finite total mass and bounded support. That is, \( 0 < \mu(\mathbb{R}) < \infty \) and \( \mu(\mathbb{R}\setminus[-m,m]) = 0 \) for some \( m \in \mathbb{Z}^+ \). Let \( M^0(\mathbb{R}) = \{\mu \in M^0(\mathbb{R}) : \mu(\mathbb{R}) = 1\} \) be the set of probability measures on \( \mathbb{R} \) with bounded support. We equip \( M^0(\mathbb{R}) \) and \( M_1^0(\mathbb{R}) \) with the topology of weak convergence (weak-* convergence in functional analysts language) and let \( \mathfrak{P}(M^0(\mathbb{R})) \) be the space \( C([0, \infty); M^0(\mathbb{R})) \) of continuous paths \( s : [0, \infty) \to M^0(\mathbb{R}) \). We will usually write \( s(t) = \mu_t \in M^0(\mathbb{R}) \).

Similarly, let \( \mathfrak{P}(M_1^0(\mathbb{R})) = C([0, \infty); M_1^0(\mathbb{R})) \). Finally, let
\[
\mathfrak{P}_b(M^0(\mathbb{R})) = \{s \in \mathfrak{P}(M^0(\mathbb{R})) : \sup_{t \in [0, \infty)} \mu_t(\mathbb{R}) < \infty\}.
\]

If \( s \in \mathfrak{P}_b(M^0(\mathbb{R})) \), then for every \( z \in \mathbb{C}^+ \) the function
\[
t \in [0, \infty) \mapsto \int_{\mathbb{R}} \frac{\mu_t(dx)}{z-x} \in \mathbb{C}^-
\]
is continuous. For each \( z \in \mathbb{C}^+ \) consider the chordal Löwner differential equation
\[
\partial_t g_t(z) = \int_{\mathbb{R}} \frac{\mu_t(dx)}{g_t(z)-x}, \quad g_0(z) = z.
\]
Since
\[ \left| \int_{\mathbb{R}} \frac{\mu_t(dx)}{z-x} \right| \leq \mu_t(\mathbb{R}) \left( (\text{Im}(z))^2 + \inf_{x \in \text{supp} \mu_t} (\text{Re}(z) - x)^2 \right)^{-1/2}, \tag{11} \]
the solution is well defined up to a time \( T_z \in (0, \infty] \), for each \( z \in \mathbb{C}^+ \). Note that, for \( t < T_z \),
\[ \partial_t \text{Im}(g_t(z)) = \int_{\mathbb{R}} \text{Im} \left( \frac{1}{g_t(z)-x} \right) \mu_t(dx) = -\text{Im}(g_t(z)) \int_{\mathbb{R}} \frac{\mu_t(dx)}{|g_t(z)-x|^2}, \]
and so
\[ \text{Im}(g_t(z)) = \exp\left[ -\int_0^t \int_{\mathbb{R}} \frac{\mu_s(dx)}{|g_s(z)-x|^2} ds \right] \text{Im}(z) \geq 0 \]
for each \( z \in \mathbb{C}^+ \). Thus, if \( \lim_{t \to T^-_z} \text{Im}(g_t(z)) > 0 \), then \( \inf_{s \in [0,T_z]} \text{Im}(g_t(z)) > 0 \) and the solution could be extended beyond \( T_z \). Since this is impossible we get now
\[ \lim_{t \to T^-_z} \text{Im}(g_t(z)) = 0 \text{ and also } \partial_t \text{Im}(g_t(z)) < 0, \text{ for } t < T_z. \tag{12} \]

Let \( K_t \) be the closure of \( \{ z \in \mathbb{C}^+ : T_z \leq t \} \). Since \( s \in \mathbb{R}_b(M^0(\mathbb{R})) \), \( (11) \) implies that \( K_t \) is compact for each \( t \in [0,\infty) \). We quote \[13, Proposition 2.2\]

**Proposition 1.** For every \( t > 0 \), \( g_t \) is a conformal transformation of \( \mathbb{C}^+ \setminus K_t \) onto \( \mathbb{C}^+ \) satisfying
\[ g_t(z) = z + \frac{a(t)}{z} + O\left( \frac{1}{|z|^2} \right), \quad z \to \infty, \]
where \( a(t) = \int_0^t \mu_s(\mathbb{R}) \ ds. \)

The chordal Löwner equation is often written in terms of the inverse transformation \( f_t(z) = g_t^{-1}(z) \). Differentiating the equation \( f_t(g_t(z)) = z \) with respect to \( t \) gives the following alternative form of the Löwner differential equation
\[ \partial_t f_t(z) = -f'_t(z) \int_{\mathbb{R}} \frac{\mu_t(dx)}{z-x}, \quad f_0(z) = z. \tag{13} \]
The set $K_t$ is called a hull in [13]. More generally, a hull is a compact set $K \subset \mathbb{C}^+$ such that $K = K \cap \mathbb{C}^+$ and $\mathbb{C}^+ \setminus K$ is simply connected. In [13], $a(t)$ in Proposition 1 is called the “capacity of $K_t$.” Note that this differs from the definition of capacity or conformal radius in [2] and [10]. The chordal Löwner equation can be viewed as describing the evolution of a hull growing to infinity.

Let $	ilde{K}_t = \{z \in \mathbb{C} : z \in K_t \text{ or } \bar{z} \in K_t\}$. By the Schwarz reflection principle, we can extend $g_t$ to a map $\tilde{g}_t : \mathbb{C} \setminus \tilde{K}_t \to \mathbb{C}$. By the Riemann mapping theorem there is a conformal map $\phi : \mathbb{C} \setminus \tilde{K}_t \to \{\zeta \in \mathbb{C} : |\zeta| > \rho\}$ of the form

$$\phi(z) = z + c_0 + c_1 z^{-1} + c_2 z^{-2} + \cdots,$$

(14)

see [10, Section 10.2]. $\rho$ is known as the conformal radius, which is shown to agree with the capacity and the transfinite diameter in [2, Chapter 2].

**Lemma 1.** The image of $\tilde{g}_t$ is the complex plane minus a closed interval of the real axis of length $d = 4\rho$ and centered at $-c_0$, that is

$$\tilde{g}_t(\mathbb{C} \setminus \tilde{K}_t) = \mathbb{C} \setminus [-c_0 - 2\rho, c_0 + 2\rho].$$

**Proof.** By the principle of boundary correspondence it is clear that the image is $\mathbb{C}$ minus a collection of closed intervals on the real axis. Because $K = K \cap \mathbb{C}^+$ the omitted set is also connected, i.e. a single closed interval $I$ on the real axis. Since $|g_t'(z)|$ is near one for $z$ large, this interval is finite ($K$ is compact). Denote $d$ the length and $e$ the center of $I$. Let $\psi = g_t \circ \phi^{-1}$ be the conformal map from the complement of the disk of radius $\rho$ to the complement of $I$. $\psi$ has the form

$$\psi(z) = \alpha(uz + \rho^2/(uz)) + \beta$$

where $\alpha > 0$, $|u| = 1$ and $\beta$ real. In fact with the above notation, $\beta = e$ and $4\alpha\rho = d$. Using now the series expansions for $\phi$, $g_t$ and comparing coefficients in the identity $\psi \circ \phi = g_t$ gives $u = 1$, $d = 4\rho$ and $e = -c_0$. \qed

**Remark 5.** Comparing the next coefficient leads to the identity

$$a(t) = \rho^2 + c_1$$

Once it is known that the omitted set is a closed interval on the real axis, the fact that $d = 4\rho$ follows directly because a conformal map such as $g_t$ with
an expansion of the type $z + a_0 + a_1z^{-1} + a_2z^{-2} + \cdots$ preserves the capacity, i.e. the capacity of the omitted set in the domain equals the capacity of the omitted set in the image, and the capacity of a line segment of length $d$ is $d/4$, see [1].

Given a (classical) real-valued continuous random process $\{U_t, t \geq 0\}$ on a (classical) probability space $(\Omega, \mathcal{F}, P)$ such that $U_0 = 0$, set $\mu_t = 2\delta_{U_t}$. Then $t \in [0, \infty) \mapsto \mu_t \in \mathcal{M}^0(\mathbb{R})$ is continuous in the topology of weak convergence. The Löwner equation leads now to a collection of random maps $g_t(\omega, \cdot)$, $\omega \in \Omega$, satisfying

$$\partial_t g_t(\omega, z) = \frac{2}{g_t(\omega, z) - U_t(\omega)}, \quad g_0(\omega, z) = z.$$  

Assume that $U_t$ has independent, identically distributed increments, and is symmetric about the origin and for $0 \leq s \leq t$ define $h_{s,t} \equiv g_t \circ g_s^{-1}$ and $\tilde{h}_{s,t}(z) = h_{s,t}(z + U_s) - U_s$. Then, for every $s < t$,

- $\tilde{h}_{s,t}$ is independent of $\{g_r, 0 \leq r \leq s\}$,
- $\tilde{h}_{s,t}$ has the same distribution as $g_{t-s}$,
- the distribution of $g_t$ is invariant under the map $x + iy \mapsto -x + iy$,  

(15) see [13]. It is well known that the only continuous process $U$ with the above properties is driftless Brownian motion. That is, $U_t = B_{\kappa t}$, $t \geq 0$, where $B$ is a standard Brownian motion and $\kappa \in (0, \infty)$ is a free parameter. The process $\{g_t, t \geq 0\}$ resulting from the choice $U_t = B_{\kappa t}$ has been introduced by Oded Schramm in [22]. It is called the stochastic Löwner evolution with parameter $\kappa$, $(SLE_\kappa)$. Because of the properties (15), SLE_\kappa can be thought of as a Brownian motion on the set of conformal maps $g_t$. It has been used to calculate the intersection exponents of two-dimensional Brownian motion and is believed to provide scaling limits for certain random walks [10], [17], [18], [19], [22], [21].

**Remark 6.** If $h : D \to \mathbb{C}^+$ is a conformal homeomorphism from some simply connected domain $D$, and if $\{h_t, t \geq 0\}$ is the solution of (14) with $h_0(z) = h(z)$, then $\{h_t, t \geq 0\}$ is called the $SLE_\kappa$ in $D$ starting at $h$. Note that if $\{g_t, t \geq 0\}$ is the solution of (14) with $g_0(z) = z$, then $h_t = g_t \circ h$, and if $K_t$ is the hull associated with $g_t$, then the hull associated with $h_t$ is $h^{-1}(K_t)$.

Since $h^{-1}(\infty) \in \partial D$, the chordal Löwner equation more generally describes a hull growing from the boundary of a domain to a boundary point.
3.2 Radial Löwner evolution

The radial Löwner equation describes the evolution of a hull from the boundary of a domain to an interior point. We choose the unit disk $\mathbb{D}$ and use the origin as the interior point. Here, a hull is a compact set $K \subset \overline{\mathbb{D}} \setminus \{0\}$ such that $K = \overline{K} \cap \mathbb{D}$ and $\mathbb{D} \setminus K$ is simply connected. Let $t \mapsto \mu_t$ be a piecewise continuous function from $[0, \infty)$ to the set of positive Borel measures on $\mathbb{T}$ such that $\mu_t(\mathbb{T})$ is uniformly bounded. For each $z \in \mathbb{D}$ consider the initial value problem

$$\frac{\partial}{\partial t} g_t(z) = g_t(z) \int_0^{2\pi} \frac{e^{i\theta} + g_t(z)}{e^{i\theta} - g_t(z)} \mu_t(d\theta), \quad g_0(z) = z. \quad (16)$$

The solution exists up to a time $T_z \in (0, \infty]$, and if $T_z < \infty$, then $\lim_{t \to T_z^-} |g_t(z)| = 1$. Let $K_t$ be the closure of $\{z \in \mathbb{D} : T_z \leq t\}$. Then, see [13], one can show

**Proposition 2.** For all $t \in [0, \infty)$, $g_t$ is the unique conformal transformation from $\mathbb{D} \setminus K_t$ to $\mathbb{D}$ such that $g_t(0) = 0$ and $g_t'(0) > 0$. In fact,

$$\ln g_t'(0) = \int_0^t \mu_s(\mathbb{T}) \, ds.$$

If $f_t \equiv g_t^{-1}$, then by differentiating the relation $f_t(g_t(z)) = z$ with respect to $t$ we find

$$\frac{\partial}{\partial t} f_t(z) = -zf_t'(z) \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \mu_t(d\theta), \quad f_0(z) = z. \quad (17)$$

4 Noncommutative processes from Löwner’s evolution equation

We will now use the chordal Löwner equation to construct processes of self-adjoint random variables, and the radial Löwner equation to construct processes of unitary random variables.

For the former, let $s \in \mathfrak{S}_b(\mathcal{M}^0(\mathbb{R}))$. For $\mu_t \equiv s(t)$, $t \in [0, \infty)$, let $g_t$ and $f_t$ be the solutions of the chordal Löwner equations (10), (13), respectively, and denote $K_t$ the associated hull. Define the functions $K_t$ and $G_t$ by

$$K_t(z) = g_t\left(\frac{1}{z}\right), \quad G_t(z) = \frac{1}{f_t(z)}. \quad (18)$$
By Remark 6, \( K_t \) is the solution of the chordal Löwner equation (10) in \( \mathbb{C}^- \) started at the function \( z \mapsto K_0(z) = \frac{1}{z} \) and \( G_t \) is its inverse. For \( G_t \) we have the following

**Lemma 2.** For each \( t \in [0, \infty) \) there exists a unique probability measure \( \nu_t \) on \( \mathbb{R} \) such that

\[
G_t(z) = \int_{\mathbb{R}} \frac{\nu_t(dx)}{z-x}.
\]

**Proof.** By Proposition 1, \( G_t \) is a conformal map from \( \mathbb{C}^+ \) to \( \mathbb{C}^- \) satisfying

\[
G_t(z) = \frac{1}{z - \frac{a(t)}{z} + O\left(\frac{1}{|z|^2}\right)}, \quad \text{as } z \to \infty.
\]

Thus

\[
\lim_{z \to \infty} zG_t(z) = 1,
\]

and it follows from [1, Satz 3, Teil 59, Kapitel VI] that there is a unique, finite, positive Borel measure \( \nu_t \) on \( \mathbb{R} \) such that

\[
G_t(z) = \int_{\mathbb{R}} \frac{\nu_t(dx)}{z-x}.
\]

But then

\[
1 = \lim_{y \to \infty} iyG_t(iy) = \lim_{y \to \infty} \int_{\mathbb{R}} \frac{iy}{iy-x} \nu_t(dx) = \int_{\mathbb{R}} \nu_t(dx),
\]

since \( |iy/(iy-x)| \leq 1 \).

Note that the fact that \( \nu_t \) has total mass one is a consequence of the so-called hydrodynamic normalization at infinity:

\[
\lim_{z \to \infty} g_t(z) - z = 0.
\]

In the radial case, let \( t \mapsto \mu_t \) be a piecewise continuous function from \([0, \infty)\) to the set of positive Borel measures on \( \mathbb{T} \) such that \( \mu_t(\mathbb{T}) \) is uniformly bounded. Let \( g_t \) and \( f_t \) be the solutions to the radial Löwner equations (16),
respectively, and denote $K_t$ the associated hull. Define the functions $\chi_t$ and $\psi_t$ by

$$
\chi_t(z) = g_t(z), \quad \psi_t(z) = \frac{f_t(z)}{1 - f_t(z)}.
$$

(20)

Then $\chi_t$ is the solution of the radial Löwner equation (10) in $\{z \in \mathbb{C} : \text{Re}(z) > -\frac{1}{2}\}$ starting at $z \mapsto \chi_0(z) = \frac{1}{1 - z}$ and $\psi_t$ is its inverse. For $\psi_t$ we have

**Lemma 3.** For each $t \in [0, \infty)$ there exists a unique probability measure $\nu_t$ on $\mathbb{T}$ such that

$$
\psi_t(z) = \int_0^{2\pi} \frac{ze^{-i\theta}}{1 - ze^{-i\theta}} \nu_t(d\theta).
$$

**Proof.** Since $z \mapsto (1 + z)(1 - z)$ maps $\mathbb{D}$ onto $\{z \in \mathbb{C} : \text{Re}(z) > 0\}$ and $f_t : \mathbb{D} \to \mathbb{D}$, it follows that $(1 + f_t)/(1 - f_t)$ is an analytic function with positive real part. By the Herglotz representation, there is a positive Borel measure $\nu_t$ such that

$$
\frac{1 + f_t(z)}{1 - f_t(z)} = i\beta_t + \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \nu_t(d\theta),
$$

where $\beta_t$ is a real constant. Since $f(0) = 0$, we have

$$
1 = i\beta_t + \nu_t(\mathbb{T}).
$$

Thus $\beta_t = 0$ and $\nu_t$ is a probability measure. Finally, since

$$
\frac{z}{1 - z} = \frac{1}{2} \left( \frac{1 + z}{1 - z} - 1 \right),
$$

we get

$$
\psi_t(z) = \frac{1}{2} \left( \frac{1 + f_t(z)}{1 - f_t(z)} - 1 \right) = \frac{1}{2} \int_0^{2\pi} \left( \frac{e^{i\theta} + z}{e^{i\theta} - z} - 1 \right) \nu_t(d\theta)
$$

$$
= \int_0^{2\pi} \frac{z}{e^{i\theta} - z} \nu_t(d\theta) = \int_0^{2\pi} \frac{ze^{-i\theta}}{1 - ze^{-i\theta}} \nu_t(d\theta).
$$
Theorem 3. The chordal Löwner equation induces a map

\[ \mathcal{CL} : \mathcal{P}_b(M^0(\mathbb{R})) \to \mathcal{P}(M_1^0(\mathbb{R})), \]

such that \((\mathcal{CL}(s))(0) = \delta_0\) for each \(s \in \mathcal{P}_b(M^0(\mathbb{R}))\). Its unique fixed point \(s\) is the convolution semigroup for the semicircle law. That is \(\mathcal{CL}(s) = s\) if and only if \(\mu_t \equiv s(t)\) has \(R\)-transform \(R(z) = tz\) for each \(t \in [0, \infty)\). The radial Löwner equation induces a map

\[ \mathcal{RL} : \mathcal{P}_b(M(\mathbb{T})) \to \mathcal{P}(M_1(\mathbb{T})), \]

such that \((\mathcal{RL}(s))(0) = \delta_1\) for each \(s \in \mathcal{P}_b(M(\mathbb{T}))\). Furthermore, if \(s \in \mathcal{P}_b(M(\mathbb{T}))\), then \(\mathcal{RL}(s) = s\) if and only if \(\mu_t \equiv s(t)\) has \(S\)-transform \(\exp(2t(z + \frac{1}{2}))\) for each \(t \in [0, \infty)\).

Proof. For the first statement, using Lemma 2, we need to show that \(\nu_t \equiv (\mathcal{CL}(s))(t)\) has compact support for each \(s \in \mathcal{P}_b(M^0(\mathbb{R}))\), and that \(t \in [0, \infty) \mapsto \nu_t \in M_0^0(\mathbb{R})\) is continuous. By Stieltjes’ inversion formula

\[ \nu_t((a, b)) + \nu_t([a, b]) = -\frac{2}{\pi} \lim_{\epsilon \to 0^+} \int_a^b \text{Im}(G_t(x + i\epsilon)) \, dx, \]

and so the support of \(\nu_t\) is the finite closed interval \(I = [-c_0 - 2\rho, c_0 + 2\rho]\), see Lemma 4.

Since \(\nu_t\) has compact support, it follows that its Cauchy transform \(G_t\) is given by

\[ G_t(z) = \frac{1}{z} + \frac{m_1(t)}{z^2} + \frac{m_2(t)}{z^3} + \cdots, \]

where \(m_k(t) = \int_{\mathbb{R}} x^k \nu_t(dx), k \in \mathbb{Z}^+\), is the \(k\)-th moment of \(\nu_t\). Since \(G_t\) solves the chordal Löwner equation, the coefficients \(m_k(t)\) are absolutely continuous in \(t\). Suppose now that \(\{t_n\}_{1}^{\infty} \subset [0, \infty)\) is a sequence with \(\lim_{n \to \infty} t_n = t \in [0, \infty)\). Then \(\lim_{n \to \infty} m_k(t_n) = m_k(t)\) for each \(k \in \mathbb{Z}^+\). Since the moment problem for \(\nu_t\) is determinate it follows that \(\nu_{t_n} \to \nu_t\), see [3, Theorem 30.2].

If \(\mathcal{CL}(s) = s\), then \(G_t\), which solves \(\mathcal{L}(s)\), also solves \(\mathcal{L}(s)\) with \(G_0(z) = \frac{1}{z}\) for \(z \in \mathbb{C}^+\). Hence \(R_t(z) = tz\) and \(\{\nu_t, t \in [0, \infty)\}\) is the convolution semigroup for the semicircle law. The converse is clear.

Concerning the radial Löwner equation, using Lemma 3, we need to show that \(t \in [0, \infty) \mapsto \nu_t \in M_1(\mathbb{T})\) is continuous, and the argument we used in
the chordal case works here as well. Finally, concerning the fixed point, if \(RL(s) = s\), then
\[
\frac{\partial}{\partial t} \psi_t(z) = -z\psi_t'(z) \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \nu_t(d\theta) = -z\psi_t'(z) \left(2(\psi_t(z) + \frac{1}{2})\right).
\]
But this is equation (9) run at twice the speed. Thus \(S_{\mu_t}(z) = \exp(2t(z + \frac{1}{2}))\) for \(t \geq 0\) and \(z \in \mathbb{D}\).

As we remarked before, geometrically, the Löwner equation corresponds to a hull growing from the boundary. In terms of the maps \(\mathcal{C}L\) and \(\mathcal{R}L\), it corresponds to “spreading out” a distribution. In (classical) probability theory we are used to distributions “spreading out” via the heat equation. In that case even a delta mass in \(\mathbb{R}^n\) is “spread” instantaneously into a distribution supported on all of \(\mathbb{R}^n\). By contrast, the “spreading” from the Löwner equation occurs at “finite speed.” In fact, according to the proof of Theorem 6, the support of the distributions cannot grow faster than linearly. Before making this “spreading out” more precise in the chordal case, we give another example.

**Example 7.** Take \(s \equiv 2\delta_0 \in \mathcal{P}_b(M^0(\mathbb{R}))\), a point mass of size 2 fixed at \(x = 0\). Then the chordal Löwner equation becomes
\[
\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z)}, \quad g_0(z) = z,
\]
which has the solution \(g_t(z) = \sqrt{z^2 + 4t}\). Then
\[
G_t(z) = \frac{1}{\sqrt{z^2 - 4t}}, \quad z \in \mathbb{C}^+
\]
and so
\[
\nu_t(dx) = \frac{1}{\pi \sqrt{4t - x^2}} dx, \quad x \in (-2\sqrt{t}, 2\sqrt{t})
\]
is the arcsine law with variance \(2t\), supported in \([-2\sqrt{t}, 2\sqrt{t}]\). Note that
\[
G_t(\mathbb{C}^+) = \mathbb{C}^-=\{iy : y \in [-\frac{1}{2\sqrt{t}}, -\infty)\}.
\]
Figure 2 is a plot of the Arcsine densities for \(t \in (0, 1]\).
Theorem 4. For \( s \in \mathcal{P}_b(M^0(\mathbb{R})) \) set \( \mu_t = s(t) \) and \( \nu_t = (\mathcal{C}\mathcal{L}(s))(t) \). Then \( \nu_t \) has mean-value 0 and variance \( \int_0^t \mu_s(\mathbb{R}) \, ds \) for \( t \in [0, \infty) \), and its forth and sixth moment are strictly increasing in \( t \). If, in addition, \( \mu_t \equiv s(t) \) is symmetric for each \( t \in [0, \infty) \), then all even moments of \( \nu_t \) are strictly increasing in \( t \). Finally, if \( s(t) = \delta_{U_t} \), where \( U : [0, \infty) \to \mathbb{R} \) is continuous, then \( \text{supp} \, \nu_t \subset \text{supp} \, \nu_t_0 \) whenever \( 0 \leq t < t_0 \).

Proof. Denote \( m_k(t) \) the \( k \)-th moment of \( \mu_t \), where \( t \in [0, \infty) \) and \( k \in \mathbb{N} \). Since \( \mu_t \) has compact support

\[
\int_{\mathbb{R}} \frac{\mu_t(dx)}{z-x} = \sum_{k=0}^{\infty} m_k(t) z^{-(k+1)}.
\]

Write \( G_t(z) = z^{-1} + a_1(t) z^{-2} + a_2(t) z^{-3} + \cdots \) and note that \( a_k(t) = \int_{\mathbb{R}} x^k \nu_t(dx) \).

At least for \( |z| \) large enough we then have \( \frac{\partial}{\partial t} G_t(z) = \sum_{k=1}^{\infty} a_k(t) z^{-(k+1)} \). Expanding in power series, differentiating term by term, and comparing coefficients, the chordal Löwner equation for \( G_t \), (13), leads to a sequence of
equations for the coefficients $a_k(t)$:

$$a_0(t) \equiv 1, \quad a_1(t) \equiv 0,$$

and

$$\dot{a}_{n+2}(t) = \sum_{k=0}^{n} a_k(t)m_{n-k}(t), \text{ for } n \in \mathbb{N}. \quad (21)$$

From this we obtain readily the expression for the variance and also that the forth and sixth moment are strictly increasing. If $\mu_t$ is symmetric for $t \in [0, \infty)$, then all its odd moments vanish and its even moments are non-negative. The 0-th moment, that is $\mu_t(\mathbb{R})$, is strictly positive by assumption and an induction argument now shows that in the symmetric case all even moments of $\nu_t$ are strictly increasing.

Finally, let $t_0 \in (0, \infty)$. Since $\mathcal{K}_{t_0}$ is a hull, that is, a compact subset of $\mathbb{C}^+$ such that $\mathcal{K}_{t_0} = \overline{\mathcal{K}_{t_0}} \cap \mathbb{C}^+$ and $\mathbb{C}^+ \backslash \mathcal{K}_{t_0}$ is simply connected, it follows that $\mathcal{K}_{t_0} \cap \{z \in \mathbb{C} : \text{Im}(z) = 0\}$ is a finite closed interval $[a, b]$ on the real axis. Let

$$A = g_{t_0}(\{z \in \mathbb{C} : \text{Im}(z) = 0, \text{Re}(z) \notin [a, b]\}),$$

which is a well defined subset of the real axis since, by the Schwarz reflection principle, $g_{t_0}$ extends analytically across $\{z \in \mathbb{C} : \text{Im}(z) = 0, \text{Re}(z) \notin [a, b]\}$. Then $\text{Im}(G_{t_0}(z)) = 0$ for each $z \in A$ and in fact $A \mathbb{C} = \text{supp } \nu_{t_0}$. If $\mu_t = \delta_{U_t}$ for $t \in [0, \infty)$, then the chordal Löwner equation (10) is given by

$$\frac{\partial}{\partial t} g_t(z) = \frac{1}{g_t(z) - U_t}, \quad g_0(z) = z,$$

and, by the definition of $\mathcal{K}_{t_0}$,

$$a \leq \min_{t \in [0, t_0]} U_t \leq \max_{t \in [0, t_0]} U_t \leq b.$$

Let now $x > b$. Then $\text{Im}(g_t(x)) = 0$ for $t \in [0, t_0]$ and

$$\frac{\partial}{\partial t} g_t(x) = \frac{1}{g_t(x) - U_t} > 0,$$

first for $t = 0$, and then for all $t \in [0, t_0]$. Thus $g_t(x) < g_{t_0}(x)$ for $t \in [0, t_0)$. Similarly, if $x < a$, then

$$\frac{\partial}{\partial t} g_t(x) = \frac{1}{g_t(x) - U_t} < 0,$$

22
for all $t \in [0, t_0]$ and so $g_t(x) > g_{t_0}(x)$ for $t \in [0, t_0)$. Taken together, we get

$$B \equiv g_t\{z \in \mathbb{C} : \text{Im}(z) = 0, \text{Re}(z) \notin [a, b]\} \supseteq A$$

for $t \in [0, t_0)$ and so, since $\text{supp } \nu_t \subseteq B \mathbb{C}$,

$$\text{supp } \nu_t \subset \text{supp } \nu_{t_0} \quad \text{for } t \in [0, t_0).$$

We can realize $\{\nu_t\}_{t \in [0, \infty)}$ as a noncommutative process by taking $X_t \in L^\infty(([0, 1), \mathcal{B}_{[0, 1]}, \lambda_{[0, 1])}; \mathbb{C})$ to be a (classical) random variable with distribution $\nu_t$, for $t \in [0, \infty)$ (see Example 2). The associated processes possess a Markovian property, see [4], [25].

**Theorem 5 (Chordal Markovianity).** With the definitions from above, for each $0 \leq s \leq t$, there exists a conformal map $H_{s,t} : \mathbb{C}^+ \rightarrow \mathbb{C}^+$ such that $G_s \circ H_{s,t} = G_t$. The family of maps $\{H_{s,t}\}$ uniquely defines a Markov transition kernel $\{k_{s,t}(x; du)\}$ on $\mathbb{R}$ by

$$\int_{\mathbb{R}} k_{s,t}(x; du) \frac{z - u}{H_{s,t}(z) - x} = 1, \quad z \in \mathbb{C}^+, x \in \mathbb{R}. \quad (22)$$

Then $\nu_t(du) = k_{0,t}(0; du)$.

**Proof.** For any $0 \leq s \leq t$ define the map $H_{s,t}$ by $H_{s,t} = g_s \circ g_t^{-1}$. Then $H_{s,t}$ is the unique conformal transformation of $\mathbb{C}^+$ onto $\mathbb{C}^+ \setminus K_{s,t}$, where $K_{s,t} = g_s(\mathbb{C}^+ \setminus K_t) \cap \mathbb{C}^+$, such that

$$H_{s,t}(z) = z - \frac{a(t) - a(s)}{z} + O\left(\frac{1}{|z|^2}\right), \quad z \to \infty.$$ 

Thus, for any real $x$,

$$\frac{1}{H_{s,t}(\cdot) - x} : \mathbb{C}^+ \to \mathbb{C}^-$$

and $\lim_{z \to \infty} z/(H_{s,t}(z) - x) = 1$. It follows from the proof of Lemma 2 that there exists a family of probability measures $\{k_{s,t}(x; du)\}$ such that

$$\int_{\mathbb{R}} k_{s,t}(x; du) \frac{z - u}{H_{s,t}(z) - x} = 1.$$ 

23
Note that since $K_t$ is compact each $k_{s,t}$ is compactly supported and the moment problem is determinate. For $s = t$,

$$\int_{\mathbb{R}} \frac{k_{t,t}(x; du)}{z-u} = \frac{1}{z-x}$$

and so $k_{t,t}(x; du) = \delta_x$. Next, for fixed $0 \leq s \leq t$, the moment generating function of $k_{s,t}(x; du)$ varies continuously in $x$. It follows that

$$x \in \mathbb{R} \mapsto k_{s,t}(x; B) \in [0, 1]$$

is continuous in $x$ for any Borel set $B \subseteq \mathbb{R}$. Finally, if $0 \leq s < r < t$, then

$$\frac{1}{H_{s,t}(z) - x} = \frac{1}{H_{s,t}(H_{r,t}(z)) - x} = \int_{\mathbb{R}} \frac{k_{s,r}(x; du)}{H_{r,t}(z) - u} = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{k_{r,t}(u; dv)}{z-v} k_{s,r}(x; du) \int_{\mathbb{R}} k_{r,t}(x; du) k_{r,t}(u; dv). \quad (23)$$

Since $1/(H_{s,t}(\cdot) - x)$ is the Cauchy transform of a unique probability measure it follows that

$$k_{s,t}(x; du) = \int_{\mathbb{R}} k_{s,r}(x; du) k_{r,t}(u; dv)$$

and hence that $\{k_{s,t}(x; du)\}$ is a Markov transition kernel. \hfill \Box

We have the analogous statement in the radial case.

**Theorem 6 (Radial Markovianity).** For each $0 \leq s \leq t$ there exists a conformal map $H_{s,t}: D \rightarrow D$ such that $G_s \circ H_{s,t} = G_t$. The family of maps $\{H_{s,t}\}$ uniquely defines a Markov transition kernel $\{k_{s,t}(\xi; d\zeta)\}$ on $\mathbb{T}$ by

$$\int_{\mathbb{T}} \frac{z}{1-z\zeta} k_{s,t}(\xi; d\zeta) = \frac{H_{s,t}(z) \xi}{1 - H_{s,t}(z) \xi}.$$

Furthermore, $\nu_t(d\zeta) = k_{0,t}(1; d\zeta)$.

**Proof.** For any $\xi \in \mathbb{T}$, $z \in D$, it follows from $H_{s,t} \equiv g_s \circ g_t^{-1}$ that $H_{s,t}(z) \xi \in D$ and $H_{s,t}(0) \xi = 0$. By the proof of Lemma 3 and equation 21 there exists a unique family of probability measures $\{k_{s,t}(\xi; d\zeta)\}$ on $\mathbb{T}$ such that

$$\int_{\mathbb{T}} \frac{z}{1-z\zeta} k_{s,t}(\xi; d\zeta) = \frac{H_{s,t}(z) \xi}{1 - H_{s,t}(z) \xi}.$$

Now the Markov property is shown as in the proof for the chordal case. \hfill \Box

24
Remark 7. The stochastic Löwner equation thus gives a “Brownian motion” whose paths are noncommutative Markov processes. Note that noncommutative Markov processes contain (classical) Markov processes as a special case ([4]).

References

[1] N. I. Achieser, I. M. Glasmann, Theorie der linearen Operatoren im Hilbert-Raum, Akademie-Verlag, Berlin, (1954).

[2] L. V. Ahlfors, Conformal Invariants, McGraw-Hill, New York, (1973).

[3] H. Bercovici, D. Voiculescu, Free convolution of measures with unbounded support, Indiana Univ. Math. J. 42 (1993), 733–773.

[4] P. Biane, Processes with free increments, Math. Z. 227 (1998), 143–174.

[5] P. Biane, Free probability for probabilists, arXiv:math.PR/9809193 v1.

[6] P. Billingsley, Probability and Measure, 2nd ed., Wiley, New York, (1986).

[7] J. B. Conway, Functions of One Complex Variable, 2nd ed., Springer-Verlag, New York, (1978).

[8] B. Duplantier, Random walks and quantum gravity in two dimensions, Phys. Rev. Lett. 82 (1998), 5489–5492.

[9] B. Duplantier, K.-H. Kwon, Conformal invariance and intersection of random walks, Phys. Rev. Lett. 61 (1988), 2514–2517.

[10] P. L. Duren, Univalent functions, Springer, New York, Berlin (1983).

[11] R. Gopakumar, D. J. Gross, Mastering the master field, arXiv:hep-th/9411021 v1 2 Nov 94.

[12] G. F. Lawler, The probability of intersection of independent random walks in four dimensions, Commun. Math. Phys. 86, no. 4 (1982), 539–554.
[13] G. F. Lawler, *An introduction to the stochastic Loewner evolution*, preprint, (2001).

[14] G. F. Lawler, W. Werner, *Intersection exponents for planar Brownian motion*, Ann. Probab. 27, no. 4 (1999), 1601–1642.

[15] G. F. Lawler, W. Werner *Universality for conformally invariant intersection exponents*, J. Eur. Math. Soc. 2 (2000), 291–328.

[16] G.F. Lawler, O. Schramm, W. Werner, *Values of Brownian intersection exponents I: Half-plane exponents*, Acta Math. 187 (2001), 237–273.

[17] G.F. Lawler, O. Schramm, W. Werner, *Values of Brownian intersection exponents II: Plane exponents*, Acta Math. 187 (2001), 275–308.

[18] G.F. Lawler, O. Schramm, W. Werner, *Values of Brownian intersection exponents III: Two-sided exponents*, Ann. Inst. H. Poincaré Probab. Statist. 38, no. 1 (2002), 109–123.

[19] G.F. Lawler, O. Schramm, W. Werner, *Analyticity of intersection exponents for planar Brownian motion*, arXiv:math.PR/0005295 v1 31 May 2000.

[20] K. Löwner, *Untersuchungen über schlichte konforme Abbildungen des Einheitskreises*, I. Math. Ann. 89 (1923), 103–121.

[21] S. Rohde, O. Schramm, *Basic properties of SLE*, arXiv:math.PR/0106036 v1 5 Jun 2001.

[22] O. Schramm, *Scaling limits of loop-erased random walks and uniform spanning trees*, Israel J. Math. 118 (2000), 221–288.

[23] B. Simon, *The classical moment problem as a self-adjoint finite difference operator*, arXiv:math-ph/9906008 v1.

[24] K. Symanzyk, *Euclidean quantum field theory*, in “Local Quantum Theory,” ed. R. Jost, Academic Press, London, New York (1969).

[25] D. Voiculescu, *Lectures on free probability theory*, in “Ecole d’Eté de Probabilités de Saint-Flour XXYIII,” ed. P. Bernard, Lecture Notes in Mathematics, no. 1738, Springer, Heidelberg (2000).
[26] D. Voiculescu, *Multiplication of certain non-commuting random variables*, J. Operator Theory **18** (1987), 223–235.

[27] D. Voiculescu, *Addition of certain non-commuting random variables*, J. Funct. Anal. **66** (1986), 323–346.