DYSON’S DISORDERED LINEAR CHAIN FROM A RANDOM MATRIX THEORY VIEWPOINT

PETER J. FORRESTER

Abstract. The first work of Dyson relating to random matrix theory, "The dynamics of a disordered linear chain", is reviewed. Contained in this work is an exact solution of a so-called Type I chain in the case of the disorder variables being given by a gamma distribution. The exact solution exhibits a singularity in the density of states about the origin, which has since been shown to be universal for one-dimensional tight binding models with off diagonal disorder. We discuss this context and also point out some universal features of the weak disorder expansion of the exact solution near the band edge. Further, a link between the exact solution, and a tridiagonal formalism of anti-symmetric Gaussian $\beta$-ensembles with $\beta$ proportional to $1/N$, is made.

1. Introduction

In the early 1960’s Dyson, starting with the publication [19] and building on work of Wigner from the 1950’s, developed a theory of random matrices for applications to universal aspects of quantum spectra as determined by global symmetries. For reference, we remark that these early works are conveniently reprinted and reviewed in a book edited by Porter [39]. Whereas Wigner focussed on modelling the Hamiltonian using Hermitian random matrices, Dyson considered ensembles of unitary matrices more fundamental due to there being a unique invariant measure; see Section I and the beginning of Section II of [19], and also the review [15]. In addition to putting in place the mathematical framework, an extensive theory was developed in relation to the statistical properties of the eigenvalues of the new ensembles.

It is no exaggeration to say these contributions of Dyson to random matrix theory and its applications are celebrated achievements. Lesser known is the fact that these series of works were not the first time Dyson had use for random matrices, nor the first time that he had the need to develop theory relating to random matrices in a pioneering fashion. The title to these claims goes instead to Dyson’s 1953 work “The dynamics of a disordered linear chain” [18]. From the viewpoint of foundational knowledge, revisiting [18] provides a valuable lesson in the methods and motivations of random matrices. And with Dyson’s recent passing at age 96 on February 28th 2020, drawing attention to [18] is also a contribution to paying tribute to his seminal contributions to the field generally.

There has been an earlier contextual review of [18], in the book of reprints with introductory text on mathematical physics in one-dimension written in the mid 1960’s [32, Ch. 2] by Lieb and Mattis. In addition to discussing follow up works from the original aims and objectives of [18], this book also contains reprints of the those papers. These follow ups are in relation to the properties of the disordered chain. A mathematical follow up written in 1960, with the aim of putting all limiting procedures in [18] on a rigorous footing, can be found in the work [40] by a student of...
V. Marčenko, cited in the 1973 survey of Pastur [38] on the spectra of random self adjoint operators. This latter reference also contains a discussion of [18].

In Section 2 an account is given of the salient content of [18] from a random matrix theory viewpoint. Some subsequent refinements to aspects of the working are covered in Section 3, and also attention is drawn to universal features of the exact solvable case found by Dyson. These are typically from the literature on localisation and the one-dimensional Anderson model. In Section 4 a link between Dyson’s solvable case, and a tridiagonal formalism of anti-symmetric Gaussian \( \beta \)-ensembles, with \( \beta \) proportional to \( 1/N \) and for \( N \rightarrow \infty \), is made.

2. Overview of Dyson’s paper

2.1. Coupled harmonic oscillators and tridiagonal matrices. Dyson’s Introduction in [18] makes it clear that his motivation was to present a mathematical model of a disordered system. The particular choice made was a system of \( N \) masses \( \{m_i\}_{i=1}^N \), confined to a line and each coupled to their nearest neighbour by (fictitious) springs with corresponding springs constants \( \{K_j\}_{j=1}^N \), and obeying Hooke’s law. With free boundary conditions, the displacements from equilibrium of the positions \( \{u_i\}_{i=1}^N \) of each mass obey the coupled set of Newton’s equations

\[
m_j \ddot{u}_j = K_j(u_{j+1} - u_j) + K_{j-1}(u_{j-1} - u_j).
\]

Here \( K_0 = K_N = 0 \) in keeping with free boundary conditions. The disorder is introduced by choosing the masses, or the spring constants, or possibly a combination of both from a probability distribution function.

With \( \mathbf{a} = [a_{ij}]_{i=1}^N \), introduce the notation \( \text{diag} \mathbf{a} \) for a matrix with entries given by \( \mathbf{a} \) along the diagonal and zero elsewhere. And with \( \mathbf{b} = [b_{ij}]_{i=1}^{N-1} \), introduce too the notation \( \text{diag}^+ \mathbf{b} \) ( \( \text{diag}^- \mathbf{b} \) ) for a matrix with non-zero entries only on the first diagonal above (below) the main diagonal, with those entries given by \( \mathbf{b} = [b_{ij}]_{i=1}^{N-1} \). To make use of this notation, set

\[
\mathbf{u} = [u_i]_{i=1}^N, \quad \alpha_0 = [-K_j/m_j - K_{j-1}/m_j]_{j=1}^N,
\]

\[
\alpha_1 = [K_j/m_j]_{j=1}^{N-1}, \quad \alpha_{-1} = [K_j/m_{j+1}]_{j=1}^{N-1}.
\]

We then have that the system (2.1) is equivalent to the second order matrix differential equation

\[
\ddot{\mathbf{u}} = \mathbf{A} \mathbf{u}, \quad \mathbf{A} = \text{diag} \alpha_0 + \text{diag}^+ \alpha_1 + \text{diag}^- \alpha_{-1}.
\]

Separating variables by writing \( \mathbf{u} = e^{i\omega t} \mathbf{U} \), where \( \mathbf{U} \) is independent of \( t \), shows the allowed values of \( -\omega^2 \) are given by the eigenvalues of the tridiagonal matrix \( \mathbf{A} \).

Instead of considering this eigenvalue problem, Dyson chose to first transform the second order system (2.2) into a first order system by changing variables \( y_j = m_j^{1/2} u_j \) (\( j = 1, \ldots, N \)), then defining \( \{z_j\}_{j=1}^{N-1} \) by

\[
\ddot{z}_j = -\lambda_j^{1/2} y_j + \lambda_{j+1}^{1/2} y_{j+1},
\]

where

\[
\lambda_{2j-1} = K_j/m_j, \quad \lambda_{2j} = K_j/m_{j+1}.
\]

Introducing too

\[
\mathbf{Y} = [y_1 \ z_1 \ y_2 \ z_2 \ \cdots \ y_n]^T
\]

and with \( \mathbf{A} \) the \((2N-1) \times (2N-1)\) anti-symmetric tridiagonal matrix

\[
\mathbf{A} = \text{diag}^+ [\lambda_j^{1/2} \ |_{j=1}^{2N-1}] - \text{diag}^- [\lambda_j^{1/2} \ |_{j=1}^{2N-1}],
\]

Observe too that by choosing in (2.2) \( \lambda_j = -\lambda_{j+1} \), the corresponding matrices \( \mathbf{A} \) and \( \mathbf{Y} \) become symmetric.
the second order matrix differential equation (2.2) is seen to be equivalent to the first order matrix differential equation

\[ \dot{y} = \Lambda y. \]  

Separating variables by writing \( y = e^{i\omega t}Y \), where \( Y \) is independent of \( t \), shows the allowed values of \( \omega \) are given by the \((N - 1)\) positive eigenvalues of the matrix \( iA \), as well as the zero eigenvalue. The latter occurs due to the choice of free boundary conditions.

Thus through either (2.2) or (2.6) Dyson was faced with the problem of quantifying the eigenvalue distribution for a (random) tridiagonal matrix. It was immediately realised that simplifying features could be expected in the limit \( N \to \infty \). Thus define \( M(\mu) \) as the proportion of frequencies \( \{\omega_j\} \) with \( \omega_j^2 \leq \mu \). Dyson hypothesised that for \( N \to \infty \) [18, Eq. (10)]

\[ D(\mu) := \frac{dM}{d\mu} \]  

is well defined, with \( D(\mu) \) corresponding to the density of states for the square eigenvalues. Under this assumption the function

\[ \Omega(x) := \lim_{N \to \infty} \frac{1}{2N-1} \sum_{j=1}^{N-1} \log(1 + x\omega_j^2), \]  

referred to in [18] as the characteristic function of the chain, is also well defined and can expressed in terms of \( D(\mu) \) according to [18, Eq. (11)]

\[ \Omega(x) = \int_0^{\infty} \log(1 + x\mu)D(\mu)\,d\mu. \]  

Moreover, it is noted that this can be inverted (differentiate and apply the Sokhotski–Plemelj — also associated with Stieltjes–Perron — formula) to deduce [18, Eq. (13)]

\[ D(1/x) = -x^2 \lim_{\epsilon \to 0^+} \frac{1}{\pi} \text{Im} \Omega'(-x + i\epsilon). \]  

It is also noted that in the limit \( x \) tends to \( -z \) on the negative real axis from above, \( \text{Im} \log(1+x\mu) = 0 \) \((i\pi)\) for \( z\mu < 1 \) \((z\mu > 1)\) and thus [18, Eq. (12)]

\[ \text{Im} \frac{1}{\pi} \lim_{\epsilon \to 0^+} \Omega(-z + i\epsilon) = \int_{1/z}^{\infty} D(\mu)\,d\mu = 1 - M(1/z). \]  

Note the consistency between (2.11) and (2.10).

### 2.2. A continued fraction formula for \( \Omega(x) \).

Dyson expands the logarithm in (2.8) to deduce

\[ \sum_{j=1}^{N-1} \log(1 + x\omega_j^2) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n \text{Tr} \Lambda^n. \]  

After some intricate combinatorial analysis of \( \text{Tr} \Lambda^n \), it is shown that for large \( N \) (2.12) can be expressed in terms of the continued fraction [18, Eq. (33)]

\[ \xi(a) = x\lambda_a/(1 + x\lambda_{a+1}/(1 + x\lambda_{a+2}/(1 + \cdots) \]  

Substituting in (2.8), this leads to the formula [18, Eq. (34)]

\[ \Omega(x) = \lim_{N \to \infty} \frac{1}{N} \sum_{a=1}^{2N-1} \log(1 + \xi(a)). \]
In §3.1 below, subsequent simplified derivations of (2.14) will be given [2, 12] which make use of algebraic rather than combinatorial properties of $\Lambda$.

From the structure of (2.13) and (2.14), it is observed in [18] that the simplest type of disorder to impose is to choose $\{\lambda_a\}$ from a common probability distribution. The coupled spring and masses system is then referred to as a Type I disordered chain. With the probability density function (PDF) of the continued fraction (2.13) denoted $F(\xi)$, (2.14) then reads [18, Eq. (46)]

$$\Omega(x) = 2 \int_0^\infty F(\xi) \log(1 + \xi) \, d\xi. \quad (2.15)$$

An alternative type of disorder introduced in [18], giving rise to what is termed a Type II disordered chain, is when each mass $m_j$ is an independent identically distributed random variable chosen with PDF $G(m_j)$, and with the spring constants all equal to the same value $K$. Then, from (2.4) [18, Eq. (47)],

$$\lambda_{2j-2} = \lambda_{2j-1} = K/m_j$$

so the random variables $\{\lambda_j\}$ are constrained to be equal in pairs, while from (2.13) $\xi(2j)$ and $\xi(2j-1)$ have different distributions. Defining [18, Eq. (48)]

$$\eta_j = \frac{1}{\xi(2j)},$$

and with $F(\eta)$ denote the corresponding PDF, manipulation of (2.14) shows [18, Eq. (53)]

$$\Omega(x) = \int_0^\infty d\eta F(\eta) \int_0^\infty d\tilde{m} G(\tilde{m}) \log \left(1 + (1/\eta) + x(K/m)\right). \quad (2.16)$$

As a distribution of masses of special interest, suppose [18, Eq. (54)]

$$G(\tilde{m}) = p\delta(\tilde{m} - m) + (1 - p)\delta(\tilde{m} - M) \quad (2.17)$$

so that the chain consists of two masses $m, M$ with concentrations $p$ and $(1 - p)$ respectively. Then (2.16) reads [18, Eq. (56)]

$$\Omega(x) = \int_0^\infty d\eta F(\eta) \left(p \log \left(1 + (1/\eta) + x(K/m)\right) + (1 - p) \log \left(1 + (1/\eta) + x(K/M)\right)\right). \quad (2.18)$$

2.3. A functional equation in the case of a Type I chain and an exact solution. The continued fraction (2.13) obeys the functional equation [18, Eq. (43)]

$$\xi(a) = x\lambda_a/(1 + \xi(a + 1)). \quad (2.19)$$

For a Type I chain, the random variables $\lambda_a$ and $\xi(a + 1)$ are uncorrelated and moreover $\xi(a)$ and $\xi(a + 1)$ have the same distribution, leading to the equality in law between random variables $\xi$, and a combination of $\lambda$ and $\xi$,

$$\xi \overset{\text{d}}{=} x\lambda/(1 + \xi). \quad (2.20)$$

Recalling now that the PDF for $\xi$ has been denoted $F(\xi)$ above (2.15), and with the PDF for the distribution of $\lambda$ to be denoted $G(\lambda)$, we see that (2.20) implies [18, equivalent to Eq. (44)]

$$F(t) = \int_0^\infty d\lambda G(\lambda) \int_0^\infty d\xi F(\xi) \delta \left(t - \frac{x\lambda}{1 + \xi}\right). \quad (2.21)$$

With $\alpha, \kappa > 0$, suppose now [18, equivalent to eq. (57)]

$$G(\lambda) = \frac{\kappa^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\kappa\lambda}. \quad (2.22)$$
Then, by taking the Mellin transform of both sides of (2.21) it is straightforward to verify the fact that the solution of (2.21) is [18, Eq. (59)]

$$F(t) = F_\alpha(t) = \frac{1}{K_\alpha(x)} \frac{t^{\alpha - 1}}{(1 + t)^\alpha} e^{-\kappa t/x},$$

(2.23)

where \( K_\alpha(x) \) is the normalisation given as an integral by

$$K_\alpha(x) = \int_0^\infty \frac{t^{\alpha - 1}}{(1 + t)^\alpha} e^{-\kappa t/x} dt.$$

(2.24)

Substituting in (2.15) shows [18, Eq. (60)]

$$\Omega(x) = 2 L_\alpha(x) K_\alpha(x),$$

(2.25)

where \( K_\alpha(x) \) is given by (2.24) and \( L_\alpha(x) \) is given by

$$L_\alpha(x) = \int_0^\infty \frac{t^{\alpha - 1}}{(1 + t)^\alpha} \log(1 + t) e^{-\kappa t/x} dt.$$

(2.26)

Remark 2.1.

1. The use of the Mellin transform in relation (2.21) is natural due to multiplicative nature of the noise; see [11, 43] for recent developments.

2. Denote by \( \Gamma[\alpha, p] \) the gamma distribution with PDF proportional to \( x^{\alpha - 1} e^{-px} \) supported on \( x > 0 \). Denote by \( K[\alpha, \beta, p] \) the Kummer type II distribution with PDF proportional to \( x^{\alpha - 1} e^{-px}/(1 + x)^{\alpha + \beta} \) supported on \( x > 0 \). A result attributed to Letac in an unpublished manuscript (see [28, Remark 2.2]) gives that with \( X \overset{d}{=} \Gamma[\alpha, p] \) and \( Y \overset{d}{=} K[\alpha + \beta, -\beta, p] \),

$$\frac{X}{1 + Y} \overset{d}{=} K[\alpha, \beta, p].$$

(2.27)

With \( \beta = 0 \) this reduces to Dyson’s result relating to (2.20).

In view of (2.10), to compute the density of states, it is necessary to analytically continue both (2.24) and (2.26) for negative \( x \). This is done in [18, Appendix III], and the result is substituted in the integrated form of (2.7) [18, equivalent to final equality in eq. (12)]

$$M(x) = 1 - \int_x^\infty D(\mu) d\mu.$$

(2.28)

In the case that \( \alpha = n \in \mathbb{Z}^+ \), the explicit evaluation of (2.28) was presented [18, Eq. (63)].

Attention was drawn to the \( x \to 0^+ \) singularity [18, consequence of eq. (72)]

$$M(x) \sim \frac{c}{(\log x)^2}$$

(2.29)

for some (explicit) \( c \), now referred to as the Dyson spectral singularity.

Attention was also drawn to the \( \alpha \to \infty \) behaviour. In this limit, after setting \( \kappa = \alpha \) in (2.22), the PDF for \( \{\lambda_j\} \) has the asymptotic form [18, Eq. (58)]

$$G(\lambda) \sim \left( \frac{\alpha}{2\pi} \right)^{1/2} e^{-\alpha(\lambda - 1)^2/2}$$

(2.30)

of a Gaussian centred about \( \lambda = 1 \). To leading order each \( \lambda_j \) is equal to 1, and there is no disorder.

It then follows from (2.20) that [18, Eq. (37) with \( \lambda = 1 \)]

$$\xi = \frac{1}{2} \left( (1 + 4\varepsilon)^{1/2} - 1 \right),$$

(2.31)
which substituted in (2.14) gives [18, Eq. (38)]

\[ \Omega(x) = 2 \log \left( \frac{1}{2}((1 + 4x)^{1/2} + 1) \right). \]  

(2.32)

Substituting this in (2.10) implies for the density of states [18, Eq. (41) with \( \lambda = 1 \)]

\[ D(\mu) = \begin{cases} 
\frac{1}{\pi \sqrt{\mu - \mu^2}}, & \mu < 4 \\
0, & \mu > 4\lambda.
\end{cases} \]  

(2.33)

A corollary of (2.33), obtained by substituting in (2.28), is that the integrated density of states for the chain with no disorder is [18, Eq. (74) with \( \lambda = 1 \)]

\[ M_\infty(x) = \begin{cases} 
\frac{1}{\pi} \text{Arcosh}(1 - x/2), & \mu < 4 \\
1, & \mu > 4.
\end{cases} \]  

(2.34)

In relation to corrections to this behaviour due to disorder, denote by \( M_n(x) \) the integrated density of states in the case \( \{ \lambda_j \} \) are distributed with PDF specialised to \( \alpha = \kappa = n \). From his exact result, Dyson showed that for \( n \to \infty \) [18, Eq. (75)]

\[ M_n(x) \sim \begin{cases} 
\frac{1}{\pi} \text{Arcosh}(1 - x/2) + \frac{1}{2\pi n (4/2 - 1)^{1/2}}, & 0 < x < 4 \\
1 - \frac{2}{n} \exp \left( -\gamma - 2n(\sinh \gamma - \gamma) \right), & x > 4 \\
1 - \left( \frac{1}{(1/3)} \right)^{2} \left( \frac{12}{n} \right)^{1/3}, & x = 4,
\end{cases} \]  

(2.35)

where \( \gamma = \text{Arcosh}((x/2) - 1) \).

3. Some subsequent refinements

3.1. Ratios of characteristic polynomials and Dyson’s continued fraction. It was noted by Bellman [2] and Dean [12] that Dyson’s combinatorial derivation of (2.14) could be simplified by adopting an algebraic approach. For this purpose, in the notation of the paragraph including (2.2) introduce the general Hermitian tridiagonal matrix

\[ T_n = \text{diag} [a_i]_{i=1}^{n} + \text{diag}^{+} [b_i]_{i=1}^{n-1} + \text{diag}^{-} [b_i]_{i=1}^{n-1}. \]  

(3.1)

The corresponding (modified) characteristic polynomial is

\[ P_n(y) = \det(I_n - yT_n) = \prod_{i=1}^{n}(1 - y\lambda_i^{(n)}), \]  

(3.2)

where \( \{ \lambda_i^{(n)} \} \) are the eigenvalues of \( T_n \). By expanding \( \det(I_n - yT_n) \) along the final row, \( \{ P_n(y) \} \) is seen to obey the three-term recurrence

\[ P_n(y) = (1 - ya_n)P_{n-1}(y) - y^2|b_{n-1}|^2P_{n-2}(y), \quad P_0(y) := 1. \]  

(3.3)

In terms of \( r_n(y) := P_n(y)/P_{n-1}(y) \), (3.3) reads

\[ r_n(y) = (1 - ya_n) - y^2|b_{n-1}|^2r_{n-1}(\lambda). \]  

(3.4)

With \( a_j = 0 \) \((j = 1, \ldots, n)\) and upon the relabelling \( b_{n-j} \to b_j \), iteration of (3.4) shows

\[ \lim_{n \to \infty} r_{n+1-j}(y) = 1 - y^2|b_j|^2/(1 - y^2|b_{j+1}|^2)/(1 - y^2|b_{j+2}|^2)/(1 - \cdots). \]  

(3.5)
Furthermore, in terms of \( \{ r_n(\lambda) \} \)
\[
P_n(y) = \prod_{j=1}^{n} r_{n+1-j}(y). \tag{3.6}
\]
The significance of this setting is that in the case \( n = 2N - 1 \), with diagonal entries given by \( b_j = i \lambda_j \), we have that \( T_n = A \) and thus
\[
P_n(y) = \prod_{j=1}^{N-1} (1 - y^2 \omega_j^2). \tag{3.7}
\]
Substituting (3.7) in (3.6) with \(-y^2\) replaced by \(x\), taking the logarithm and dividing by \((2N-1)\), then making use of (3.5) we see that (2.14) is reclaimed.

### 3.2. Type II chain and the work of Schmidt.

After separating variables as described below (2.2), the equations of motion (2.1) can be rearranged to read
\[
U_{j+1} = \left( 1 + (K_{j-1}/K_j) - \omega^2 m_j/K_j \right) U_j - (K_{j-1}/K_j) U_{j-1}.
\][3.8]

To do this requires \( K_j \neq 0 \) for each \( j = 1, \ldots, N \). This therefore excludes the free boundary conditions as used by Dyson, since then \( K_N = 0 \) (recall the text below (2.1)). A compatible alternative is to use fixed boundary conditions, specified by \( U_0 = U_{N+1} = 0 \). We remark that with (3.8) multiplied through by \( K_j \) \((j = 1, \ldots, N)\) free and fixed boundary conditions are indistinguishable. Note too, following Schmidt [41], that an equivalent way to write (3.8) is as the \( 2 \times 2 \) matrix recurrence
\[
\begin{bmatrix}
U_{j+1} \\
U_j
\end{bmatrix} = T_j
\begin{bmatrix}
U_j \\
U_{j-1}
\end{bmatrix}, \quad T_j = \begin{bmatrix} 1 + (K_{j-1}/K_j) - \omega^2 m_j/K_j & -K_{j-1}/K_j \\ 0 & 1 \end{bmatrix},
\][3.9]

which implies the matrix product formula
\[
\begin{bmatrix} U_{N+1} \\
U_N
\end{bmatrix} = T_N T_{N-1} \cdots T_1 \begin{bmatrix} U_1 \\
U_0
\end{bmatrix}.
\][3.10]

Iterating (3.8) starting with \( U_0 = 0 \) determines \( \{ U_j \}_{j=1,2,...} \) up to an overall scalar factor \( c \) say. We see that \( U_{j+1} \) is a polynomial of degree \( j \) in \( \omega^2 \). Denoting the corresponding zeros by \( \{ \mu^{(j)}_l \}_{l=1} \), this allows us to write
\[
U_{j+1} = c \prod_{l=1}^{j} (-m_l/K_l)(\omega^2 - \mu^{(j)}_l).
\][3.11]

To obtain a more explicit characterisation of \( \{ \mu^{(j)}_l \}_{l=1} \), in (3.8) change variables by writing
\[
V_{j+1} = \frac{1}{\omega^2 j} \prod_{l=1}^{j} \left( -\frac{K_l}{m_l} \right) U_{j+1}, \quad y = 1/\omega^2.
\][3.12]

This gives
\[
V_{j+1} = 1 - y \left( \frac{K_j}{m_j} - \frac{K_{j-1}}{m_j} \right) V_j - y^2 \left( \frac{K_{j-1}^2}{m_{j-1} m_j} \right) V_{j-1}.
\][3.13]

With \( A \) denoting the tridiagonal matrix specified in (2.2), and \( A_n \) denoting its top \( n \times n \) submatrix, comparison with (3.3) shows
\[
V_{n+1} = c \det(I_n + yA_n),
\][3.14]
where \( c \) is an arbitrary scalar. In particular, it follows that \( \{ \mu^{(j)}_l \}_{l=1} \) are equal to the nonzero eigenvalues of \(-A_{j+1} \). Since \( A_N = A \) it follows that for \( U_{N+1} = 0 \) as required by fixed boundary conditions.
conditions, we must have that $-\omega^2$ in (3.8) corresponds to the eigenvalues of $A$. This has been noted below (3.9) for the case of free boundary conditions.

The above theory implies that upon consideration of the ratios $\tilde{r}_n := U_n/U_{n-1}$, formulas equivalent to (3.6) and (3.7) hold. Thus we have

$$U_{N+1} = c \prod_{l=1}^{N} (-m_l/K_l)(\omega^2 - \omega_l^2) = \prod_{j=0}^{N} \tilde{r}_{N+1-j}. \quad (3.15)$$

However $\{\tilde{r}_n\}$ relate to the matrix $A$, whereas $\{r_n\}$ relate to the anti-symmetric matrix $\Lambda$ so they have different distributions. In fact for a Type II chain, characterised by all spring constants being equal, there are simplifications which result by considering $\{\tilde{r}_n\}$.

First, in the setting of a Type II chain, it follows from (3.8) that

$$\tilde{r}_{j+1} = (2 - \omega^2 m_j/K) - 1/\tilde{r}_j. \quad (3.16)$$

For $n$, large let $w(z) = w(z; \omega^2)$ denote the PDF for the distribution of $\tilde{r}_n$. Proceeding as in the derivation of (2.21), and specialising to the case of a diatomic chain as specified in (2.17) for definiteness, we see that $w(z)$ satisfies the functional equation [41, equivalent to Eq. (II,16)]

$$w(z) = p \frac{1}{z^2} w \left( 2 - \frac{M \mu^2 - y^2}{K} - \frac{1}{z} \right) + (1-p) \frac{1}{z^2} w \left( 2 - \frac{M \mu^2}{K} - \frac{1}{z} \right). \quad (3.17)$$

Next, as a variant of Dyson’s characteristic function (2.8) define

$$\tilde{\Omega}(y^2) = \lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} \log(\omega_j^2 - y^2)$$

$$= \lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} \log U_{N+1} \bigg|_{\omega=y} + \left( p \log m + (1-p) \log M - \log K \right). \quad (3.18)$$

Comparison with (2.8) shows

$$\tilde{\Omega}(-1/z) = -\log z + \Omega(z). \quad (3.19)$$

In contrast to (2.8), $\tilde{\Omega}(y^2)$ is not real for positive real values of the argument $y^2$ ($x$ in (2.8)), since for $\omega_j^2 < y^2$

$$\log(\omega_j^2 - y^2) = \log |\omega_j^2 - y^2| + i\pi. \quad (3.20)$$

A useful consequence is that analogous to (2.28), it follows

$$\text{Im} \frac{1}{\pi} \tilde{\Omega}(y^2) = M(y^2). \quad (3.21)$$

From the second equality in (3.18) and the definition of $w(z)$ it also follows that analogous to (2.15)

$$\tilde{\Omega}(y^2) = \int_{-\infty}^{\infty} (\log z) w(z; y^2) \, dz. \quad (3.22)$$

Taking imaginary parts using (3.21) then gives [41, Eq. (II,26)]

$$M(y^2) = \int_{-\infty}^{0} w(z; y^2) \, dz = \text{Pr} \{ w(z; y^2) \leq 0 \}. \quad (3.23)$$

We remark that due to this development of Schmidt to Dyson’s pioneering work, (3.17), along with (2.21) in the case of the Type I chain, is nowadays typically referred to as an example of a Dyson–Schmidt equation for the stationary distribution of the corresponding stochastic sequences.
Also of note is that the final equality in (3.23) is well suited to numerical approximation, whereas formalisms based on (2.11) require analytic continuation.

**Remark 3.1.** 1. Consider the Type I disordered chain in Dyson's anti-symmetric tridiagonal matrix formulation. The characteristic polynomial \( Q_n(y) = \det(uI_n - \Lambda_n) \), where \( \Lambda_n \) is the top \( n \times n \) block of \( \Lambda \) as specified by (2.5) satisfies the recurrence

\[
Q_{n+1}(y) = yQ_n(y) - \lambda_n Q_{n-1}(y).
\]

Introducing the ratios \( s_n = Q_n(y)/Q_{n-1}(y) \), then writing \( s_n = y(1 + \tilde{s}_n) \) we see that the random variable \( \tilde{s} \) corresponding to the limiting distribution of \( \tilde{s}_n \) must satisfy the equality in law

\[
\tilde{s} \overset{d}{=} -\frac{(\lambda/y)}{1 + s}. \tag{3.24}
\]

This is identical to (2.20) except that the positive parameter \( x \) is now equal to the negative parameter \(-1/y\). However, in relation to Dyson’s exact solution in the case that the distribution of \( \lambda \) is specified by (2.22), having the parameter positive is an essential ingredient. In particular, no analogous exact solution is known in relation to (3.24).

2. Related to the above point is the simplification — for example the existence of an exact solution — which results by the consideration of \( \Omega(z) \) (or equivalently from (3.19) the consideration of \( \tilde{\Omega}(-1/z) \)) for \( z \) positive and real, and then analytically continuing as required by (2.11) and (2.10).

For related uses of this strategy, applied to multichannel models, see [26, 48].

Schmidt’s reformulation of the second order difference system (3.8) in the matrix form (3.9) is significant as perhaps the first applied problem giving rise to a product of random matrices, as seen in (3.10). In the case of the Type II chain, each matrix in (3.10) is independent and identically distributed. Starting from the early 1960’s, products of random matrices with independent and identically distributed elements attracted much attention in the mathematics literature, and many significant theoretical developments have followed [23, 24, 30, 37]. A key quantity in such studies as they relate to (3.10) is

\[
\gamma := \lim_{N \to \infty} \frac{1}{N} \log \sqrt{U_{N+1}^2 + U_N^2} = \lim_{N \to \infty} \frac{1}{N} \log |U_{N+1}|, \tag{3.25}
\]

referred to as the Lyapunov exponent. In this setting it is usual to refer to the limiting PDF of \( U_{N+1}/U_N \) as specifying an invariant measure.

For Type II chains, it follows from the first equality in (3.15) substituted in (3.25) that

\[
\gamma = \int_0^\infty \log |\omega^2 - \mu|D(\mu) \, d\mu + \frac{1}{K}(\log |m|), \tag{3.26}
\]

where \( D(\mu) \) denotes the density of the squared singular eigenvalues; cf. (2.9). Nearly two decades after Dyson’s work, it was understood by Thouless [46] that through (3.26) there is a link between the density of states and the localisation length in a one-dimensional disordered system — the latter being an interpretation of \( 1/\gamma \); see the review [10]. Due to this conceptual advance, (3.26) is nowadays typically referred to as the Thouless formula, although some authors simultaneously cite both Dyson and Thouless; see e.g. [25].
3.3. **Type I chains near zero frequency.** The natural discretisation of the one-dimensional Schrödinger equation

\[
\left(-\frac{d^2}{dx^2} + V(x)\right)\psi(x) = E\psi(x)
\]

on the integer lattice is

\[
-(\psi_{n+1} + \psi_{n-1} - 2\psi_n) + V_n\psi_n = E\psi_n.
\] (3.27)

This is to be compared with the difference equation (3.8) in the case of a type II chain

\[
-(U_{n+1} + U_{n-1} - 2U_n) - \frac{\omega^2}{K}m_n U_n = 0.
\] (3.28)

While the discretisation of the Laplace operator \(-d^2/dx^2\) is evident, there is no direct analogy between \(\{E, V_n\}\) in (3.27) and \(\{\omega, m_n\}\) in (3.28).

On the other hand, consider Dyson’s Type I chain in the anti-symmetric tridiagonal form (2.6). The corresponding equation for the eigenvalues and eigenvectors can be written

\[
i\lambda_{n+1}^{1/2}\phi_{n-1} - i\lambda_n^{1/2}\phi_n = \omega\phi_n,
\] (3.29)

where \(\{\phi_n\}_{n=1}^{2N-1}\) are the components of the eigenvector. This is recognised as an example of the Schrödinger equation for the tight binding Hamiltonian (one-dimensional Anderson model) with random off diagonal elements and constant diagonal the latter being absorbed into the energy \(E\) to give \(\omega\) in (3.29).

A number of works have given consideration to the limiting \(\omega \to 0^+\) form of the density of states as implied by (3.29) for a general distribution of the non-negative random variable \(\lambda_n\), assuming finite second moment; see for example [4, 14, 20, 45]. The conclusion of these works is that the singularity (2.29) exhibited for the special distribution of \(\lambda_n\) (2.22) actually holds in the general case, and thus is a universal feature of both Dyson’s Type I chain, and the one-dimensional Anderson model with off diagonal disorder. The same singularity is also seen in the density of states for the one-dimensional XY model with random coupling constants [42], and the one-dimensional random mass Dirac Hamiltonian [44], both systems being related to the Dyson’s Type I chain.

A closely related general behaviour can be seen from the solution of (3.29) with \(\omega = 0\) [8, 45, 51]. After iteration, and setting \(\phi_1 = 1\) for normalisation, we see that for \(n\) even

\[
\phi_{n+1} = \prod_{l=1}^{n/2} \frac{\lambda_{2l-1}}{\lambda_{2l+1}}
\]

According to the central limit theorem \(\log|\phi_{n+1}|\) will, to leading order for large \(n\), be proportional to \(\sqrt{n}\), with the proportionality constant given in terms of the variance of \(\lambda_n\). In contrast, typically for \(\omega > 0\) the values of \(\log|\phi_{n+1}|\) obtained by iterating (3.29) will be proportional to \(n\). The modification of this conclusion in the case that the variance diverges has been the subject of the recent work [31]; see too the earlier work [3].

3.4. **Weak disorder limit.** We know from (2.30) that for \(\alpha \to \infty\) the special PDF (2.22) for the couplings \(\{\lambda_n\}\) is to leading order a Gaussian centred at \(\lambda = 1\) with variance \(1/\alpha\). This circumstance, which perturbs about the chain with no disorder, is referred to as weak disorder. Systematic weak disorder expansion methods have been devised (see e.g. [6] and references therein), typically specialised to the setting of the discrete Schrödinger equation (3.27) and so not directly applicable to disordered chains. Nonetheless, comparison of the results which follow from Dyson’s exactly solvable Type I chain for large \(\alpha\) with results from the weak disorder expansion relating to
We consider first the Lyapunov exponent (3.26). A result of Thouless [47] gives that in the weak disorder limit of (3.27), with the variance of \(\{V_n\}\) equal to \(1/\alpha\), the leading large \(\alpha\) form for \(|E| < 2\) is

\[
\gamma \sim \frac{1}{8\alpha (1 - (E/2)^2)}. \tag{3.30}
\]

For Type I chains, we see from the definition (3.25) and (3.29) that in terms of \(D(\mu)\)

\[
\gamma = \frac{1}{4} \int_0^\infty \log |\omega^2 - \mu| D(\mu) \, d\mu - \frac{1}{2} \langle \log |\lambda| \rangle. \tag{3.31}
\]

Making use of (2.33) it follows that with no disorder

\[
\gamma = \frac{1}{4\pi} \int_{-\infty}^0 \frac{1}{4\mu - \mu^2} \log |\omega^2 - \mu| \, d\mu = 0, \tag{3.32}
\]

where the final equality is valid for \(\omega^2 < 4\); see e.g. [21, §1.4.2]. This fact could also be deduced directly from the definition (3.25) since without disorder the components of the eigenvectors do not exponentially increase or decrease but rather oscillate, in keeping with the underlying coupled spring model having all masses equal.

Let \((\gamma)_1\) denote the term proportional to \(1/\alpha\) in the large \(\alpha\) expansion of \(\gamma\), and similarly the meaning of \((\Omega(x))_1\). We see from (3.31) and (2.9) that

\[
(\gamma)_1 = \frac{1}{4\pi} \lim_{\epsilon \to 0^+} \text{Re} \Omega(-1/\omega^2 + i\epsilon) \tag{3.33}
\]

Making use of (2.25) and workings in [18, Appendix IV] gives, for \(\alpha \in \mathbb{Z}^+\)

\[
\lim_{\epsilon \to 0^+} \text{Re} \Omega(-1/\omega^2 + i\epsilon) = 2L_\alpha(-\omega^2)/K_\alpha(-\omega^2), \tag{3.34}
\]

where, with

\[
f(\xi, \omega) = \log \xi - \log(1 + \xi) + \xi \omega^2, \quad g^{(0)}_\alpha(\xi) = \frac{1}{\xi}, \quad g^{(1)}_\alpha(\xi) = \frac{1}{\xi} \log(1 + \xi), \tag{3.35}
\]

we have [18, Eqns. (A.17) and (A.19)]

\[
L_\alpha(-\omega^2) = \int_{-\infty}^{-\omega} g^{(1)}_\alpha(\xi) e^{\alpha f(\xi \omega)} \, d\xi \tag{3.36}
\]

\[
K_\alpha(-\omega^2) = \int_{\omega}^{\infty} g^{(0)}_\alpha(\xi) e^{\alpha f(\xi \omega)} \, d\xi \tag{3.37}
\]

In both (3.36) and (3.37) the contours of integration are to run along the upper half plane side of the negative real axis.

Moreover, it is noted in [18, Appendix IV] that for \(0 < \omega^2 < 4\) there is a single saddle point in the upper half plane [18, Eq. (A.21)]

\[
\eta = \frac{1}{2} \left( -1 + i(4/\omega^2) - 1 \right)^{1/2}. \tag{3.38}
\]

It is noted too that by deforming the contours in (3.36), (3.37) to pass through this point at angle \(3\pi/4\), the large \(\alpha\) asymptotic expansion follows by expanding the integrand about this point. Doing
this, setting \( f(\xi, \omega) = f(\xi) \) for notational convenience, and evaluating the corresponding Gaussian integrals gives
\[
\lim_{\epsilon \to 0^+} \text{Re} \Omega(-1/\omega^2 + i\epsilon) = -\frac{1}{2} \frac{1}{\eta(1 + \eta)} \frac{f'''(\eta)}{|f'''(\eta)|^2} + \text{Re} \frac{i}{2} \frac{1}{(1 + \eta)^2} \frac{f''(\eta)}{|f''(\eta)|}
\]
\[
= 1 + \frac{1}{2((4/\omega)^2 - 1)}.
\]
Now substituting in (3.33) gives
\[
(\gamma)_1 = \frac{1}{8((4/\omega)^2 - 1)}.
\] (3.39)
Comparing with (3.30) shows the functional forms agree for \( E = \omega \to 2^- \).

For the discrete Schrödinger equation (3.27) it is known [7, 13, 22, 27, 29] that in the limit \( E \to 2 \) the leading weak disorder expansion of the Lyapunov exponent obeys the scaling law in terms of Airy functions
\[
\gamma \sim \left(\frac{1}{2\alpha}\right)^{1/3} F \left((2\alpha)^{3/3}(|E| - 2)\right),
\] (3.40)
where
\[
F(x) = \frac{\text{Ai}(x) \text{Ai}'(x) + \text{Bi}(x) \text{Bi}'(x)}{(\text{Ai}(x))^2 + (\text{Bi}(x))^2} = \text{Re} e^{-2\pi i/3} \frac{\text{Ai}'(e^{-2\pi i/3}x)}{\text{Ai}(e^{-2\pi i/3}x)}.
\] (3.41)
Precisely this scaling function was obtained by Smith [42, Eq. (4.14)] in the case of Dyson’s exactly solvable Type I chain with \( \alpha \to \infty \). It was found by extending the asymptotic analysis of Dyson to uniformly account for there being two coalescing saddle points as \( \omega \to 2 \).

Remark 3.2. It is noted in [7, §4.1] that
\[
\text{Im} e^{-2\pi i/3} \frac{\text{Ai}'(e^{-2\pi i/3}x)}{\text{Ai}(e^{-2\pi i/3}x)} = \frac{1}{\pi((\text{Ai}(x))^2 + (\text{Bi}(x))^2)}.
\] (3.42)
The working in [42] then implies that (3.42) plays the role of the scaling function \( F \) in (3.40) for the weak disorder expansion of the density of states near \( \omega = 2 \) for Dyson’s type I chain. Also of interest is the fact that for large \( x \) the leading asymptotics of (3.42) is \( x^{1/2} e^{-4x^{3/2}/3} \), where the particular functional form of the exponential is known in the theory of disordered systems as a Lifshitz tail; see [9, §7.1].

4. INHOMOGENEOUS TYPE I CHAINS RELATED TO GAUSSIAN ANTI-SYMMETRIC MATRICES

Let \( X \) be a real \( N \times N \) standard Gaussian matrix. The corresponding symmetric matrix \( \frac{1}{2}(X + X^T) \) is said to be a member of the Gaussian orthogonal ensemble; see e.g. [21, §1.1]. This ensemble relates to the broader theme of disordered chains and tight binding Hamiltonians through the property that it permits a similarity transformation (using Householder reflection matrices) to a tridiagonal matrix with entries on and above the diagonal again independent [50]. On the diagonal they are unchanged, each being given by a standard Gaussian. On the sub-diagonal directly above the diagonal they are distributed by \( (\hat{\chi}_{N-1}, \hat{\chi}_{N-2}, \ldots, \hat{\chi}_1) \), where \( \hat{\chi}_k \) denotes the square root of the gamma distribution \( \Gamma[k/2, 1] \). Moreover, with the latter generalised to \( (\hat{\chi}_{(N-1)\beta}, \hat{\chi}_{(N-2)\beta}, \ldots, \hat{\chi}_\beta) \), where \( \beta > 0 \) is a parameter, it was shown in [16] that the eigenvalue PDF can be explicitly computed, and is proportional to
\[
\prod_{l=1}^{N} e^{-x_l^2/2} \prod_{1 \leq j < k \leq N} |x_k - x_j|^\beta.
\]
This is the definition of the Gaussian $\beta$-ensemble.

As discussed in [5], this class of random tridiagonal matrices is of interest from the viewpoint of stochastic Schrödinger operators in one-dimension with a random potential decaying as $|x|^{-\alpha}$. It is known that the exponent $\alpha$ equalling $1/2$ separates localised and extended states, and it turns out that the random tridiagonal matrices giving rise to the Gaussian $\beta$-ensemble corresponds to this critical case.

An anti-symmetric Hermitian matrix can be formed out of a real Gaussian matrix $X$ by forming $i$ times $\frac{1}{2}(X - X^T)$. It is known [17] that reduction of the latter to an anti-symmetric tridiagonal form using Householder transformations gives for the entries directly above the diagonal the same distribution as in the symmetric case, $(\tilde{\chi}_{N-1}, \tilde{\chi}_{N-2}, \ldots, \tilde{\chi}_1)$. Furthermore, as a generalisation, if an anti-symmetric tridiagonal matrix is constructed with entries directly above the diagonal distributed by

$$(\tilde{\chi}_{(N-1)\beta/2}, \tilde{\chi}_{N-2}/2, \ldots, \tilde{\chi}_0), \quad (4.1)$$

it was shown in [17] that the eigenvalue PDF can be explicitly determined. The precise functional form depends on the parity of $N$. Replacing $N$ by $2N + 1$ so the size of the matrix is odd, there is one zero eigenvalue, with the remaining eigenvalues coming in pairs $\{\pm ix_j\}_{j=1}^N$, $x_j > 0$, and their squares $x_j^2 =: y_j$ distributed according to the PDF proportional to

$$\prod_{l=1}^N y_l^{3\beta/4-1} e^{-y_l} \prod_{1 \leq j < k \leq N} |y_k - y_j|^\beta. \quad (4.2)$$

We remark that up to scaling, with $\beta = \alpha$ the distribution $\tilde{\chi}_\beta$ in the final entry of (4.1) is precisely that which underlies the exactly solvable case of a Type I chain identified by Dyson — recall (2.22).

The PDF (4.2) is an example of the Laguerre $\beta$-ensemble; see e.g. [21, §3.10]. After scaling the squared singular values $y_j \to \beta y_j/2$ the density of states corresponding to (4.2) has for large $N$ a Marchenko-Pastur functional form. The latter is independent of $\beta$, and is given by (see e.g. [21, §3.4.1])

$$D(\mu) = \frac{2}{\pi \mu^{1/2}} \frac{1}{(1 - \mu)^{1/2}}, \quad (4.3)$$

where $\mu = y/4N$, supported on $0 < Y < 1$; cf. (2.32).

Another limiting procedure is possible. Following [1] (see also [34, 35, 49]) set $\beta = c/N$, then take $N \to \infty$. Note from (4.1) that all elements in the top $k \times k$ ($k$ fixed) sub-block have the same distribution $\tilde{\chi}_{c/2}$, which with $\alpha = c/2$ (and up to a scaling) agrees with the distribution specifying Dyson’s exactly solvable Type I chain. In this regime we know from [1, Eq. (3.49) with $\lambda/2 = \mu$] that

$$D(\mu) = \frac{1}{\Gamma(c) \Gamma(c + 1)} \frac{1}{|W_{c+1/2,0}(-\mu)|^2}, \quad (4.4)$$

where $W_{\kappa,\mu}(z)$ denotes the Whittaker function in usual notation. From the known $z \to 0$ asymptotics of the latter [36, Eq. 13.14.99] it follows that for $\mu \to 0^+$

$$D(\mu) \sim \frac{e}{\mu (\log \mu)^2}. \quad (4.5)$$

In comparison, it follows by differentiating (2.29) that for Dyson’s exactly solvable Type I chain, and more generally to all chains in this class where the distribution has finite second moment
(recall the discussion in §3.3), that for $\mu \to 0^+$

$$D(\mu) \sim \frac{\tilde{c}}{\mu (\log \mu)^3}, \quad (4.6)$$

for some proportionality $\tilde{c}$. Thus up to a factor of $1/\log \mu$, for $\mu \to 0^+$ the density $D(\mu)$ has a functional form characteristic of a Dyson Type I chain with all random variables drawn from the same distribution.

**Remark 4.1.** With the Whittaker function in (4.4) expressed in terms of the Kummer function, and up to a factor of $\mu/c$, the RHS is known in the context of Dyson’s exactly solvable Type I chain, being obtained in a calculation of a particular spectral density (in distinction to the density of states) [9, second last displayed equation in §7.2 with $q = 1, p = c$].

**Acknowledgements.** This research is part of the program of study supported by the Australian Research Council Centre of Excellence ACEMS, and the Discovery Project grant DP210102887. I thank M. Sodin for providing the references [38], [40], K. Borovkov for the English translation of the bibliographical details of the latter, and Ch. Texier for helpful remarks on the first draft.

**References**

[1] R. Allez, J.-P. Bouchaud, S. N. Majumdar, P. Vivo, *Invariant $\beta$-Wishart ensembles, crossover densities and asymptotic corrections to the Marchenko-Pastur law*, J. Phys. 46, 015001 (2013).

[2] R. Bellman, *Dynamics of a disordered linear chain*, Phys. Rev. 101 (1956), 19.

[3] T. Bienaimé and C. Texier, *Localization for one-dimensional random potentials with large fluctuations*, J. Phys. A 41, (2008) 475001.

[4] A. Bovier, *Perturbation expansion for a one-dimensional Anderson model with off-diagonal disorder*, J. Stat. Phys. 56 (1989), 645–668.

[5] J. Breuer, P.J. Forrester, and U. Smilansky, *Random Schrödinger operators from random matrix theory*, J. Phys. A 40 (2007), F1–F8.

[6] A. Crisanti, G. Paladin, and A. Vulpiani, *Products of random matrices*, Springer series in solid-state sciences, vol. 104, Springer-Verlag, Berlin Heidelberg, 1993.

[7] A. Comtet, J.M. Luck, C. Texier, and Y. Tourigny, *The Lyapunov exponent of products of random $2 \times 2$ matrices close to the identity*, J. Stat. Phys. 150 (2013), 13–65.

[8] A. Comtet and C. Texier, *One-dimensional disordered supersymmetric quantum mechanics: a brief survey*, in “Supersymmetry and Integrable Models”, edited by H. Aratyn, T. D. Imbo, W.-Y. Keung and U. Sukhatme, Lecture Notes in Physics, Vol. 502, pp. 313–328, Springer (1998).

[9] A. Comtet and Y. Tourigny, *Impurity models and products of random matrices*, in “Stochastic Processes Random Matrices: Lecture Notes of the Les Houches Summer School 2015, edited by G. Schehr, A. Altland, Y.V. Fyodorov, N. O’Connell and L.F. Cugliandolo Vol. 104, 474–547 (2017).

[10] A. Comtet, C. Texier, and Y. Tourigny, *Lyapunov exponents, one-dimensional Anderson localisation and products of random matrices*, J. Phys. A 46 (2013), 254003.

[11] A. Comtet, C. Texier, and Y. Tourigny, *Representation theory and products of random matrices in SL(2,R)*, arXiv:1911.00117.
[12] P. Dean, The spectral distribution of a Jacobi matrix, Proceedings of Cambridge Philos. Soc. 52 (1956) 752–755.
[13] E. Derrida and E. Gardner, Lyapounov exponent of the one dimensional Anderson model: weak disorder expansions, J. Phyiques. 45 (1984), 1283–1295.
[14] D. Dhar, Central peak in the density of states of a disordered linear chain, Physica A 102, (1980) 370–378.
[15] P. Diaconis and P.J. Forrester, Hurwitz and the origin of random matrix theory in mathematics, Random Matrix Th. Appl. 6 (2017), 1730001.
[16] I. Dumitriu and A. Edelman, Matrix models for beta ensembles, J. Math. Phys. 43 (2002), 5830–5847.
[17] I. Dumitriu and P.J. Forrester, Tridiagonal realization of the antisymmetric Gaussian $\beta$-ensemble, J.Math. Phys. 51 (2010), 093302.
[18] F.J. Dyson, The dynamics of a disordered linear chain, Phys. Rev. 92 (1953), 1331–1888.
[19] F.J. Dyson, Statistical theory of energy levels of complex systems I, J. Math. Phys. 3 (1962), 140–156.
[20] T.P. Eggarter and R. Riedinger, Singular behavior of tight chains with off-diagonal disorder, Phys. Rev. B 18 (1978), 569–575.
[21] P.J. Forrester, Log-gases and random matrices, Princeton University Press, Princeton, NJ, 2010.
[22] H. L. Frisch and S. P. Lloyd, Electron levels in a one-dimensional random lattice Phys. Rev. 120 (1960), 1175–1189.
[23] H. Furstenberg and H. Kesten, Products of random matrices. Ann.Math. Stat. 31 (1960), 457–469.
[24] H. Furstenberg, Noncommuting random products, Trans. Am. Math. Soc. 108 (1963), 377–428.
[25] E.J. Gardner, C. Itzykson and B. Derrida, The Laplacian on a random one-dimensional lattice, J. Phys. A 17, 1093–1109.
[26] A. Grabsch and C. Texier, Capacitance and charge relaxation resistance of chaotic cavities — Joint distribution of two linear statistics in the Laguerre ensemble of random matrices, Europhys. Lett. 109 (2015), 50004.
[27] B.I. Halperin, Green’s functions for a particle in a one-dimensional random potential, Phys. Rev. 139, A104, 104–117.
[28] M. Hamza and P. Vallois, On Kummer’s distribution of type two and a generalized beta distribution, Statistics & Prob. Lett., 118, (2016) 60–69.
[29] F.M. Izrailev, S. Rufio, S., and L. Tessier, Classical representation of the one-dimensional Anderson model, J. Phys. A 315 315 (1998), 5263–5270.
[30] J.F.C. Kingman, Subadditive ergodic theory, Ann. Prob. 1 (1973), 883–909.
[31] A. Krishna and R.N. Bhatt, Beyond universal behavior in the one-dimensional chain with random nearest neighbor hopping, Phys. Rev. B 101, (2020) 224203.
[32] E. Lieb and D.C. Mattis, Mathematical physics in one dimension, Academic Press, NewYork, 1966.
[33] J.M. Luck, Scaling laws for weakly disordered 1D flat bands, J.Phys.A 52, 205301.
[34] G. Mazzuca, On the mean density of states of some matrices related to the beta ensembles and an application to the Toda lattice, arXiv:2008.04604.
[35] P. Mergny and M. Potters, *The HCIZ at high temperature: interpolating between classical and free convolutions*, arXiv:2101.01810

[36] NIST Digital Library of Mathematical Functions.

[37] V.I. Oseledec, *A multiplicative ergodic theorem. Lyapunov characteristic numbers for dynamical systems*, Trans. Moscow Math. Soc. 19 (1968), 197–231.

[38] L. A. Pastur, *Spectra of random self adjoint operators*, Russ. Math. Surv. 28, (1973) 1–67.

[39] C.E. Porter, *Statistical theories of spectra: fluctuations*, Academic Press, New York, 1965.

[40] F.S. Rofe-Beketov, *On the limiting distribution of the eigenfrequencies of a disordered chain* [in Russian], Communications of the Dept. of Mathematics of the Faculty of Physics and Mathematics of the Kharkov A.M. Gorky State University and the Kharkov Math. Soc. Vol. XXVI, ser. 4 (1960), 143–153.

[41] H. Schmidt, *Disordered one-dimensional crystals*, Phys. Rev. 105 (1957), 425–441.

[42] E.R. Smith, *One-dimensional X-Y model with random coupling constants. I. Thermodynamics* J. Phys. C. 3 (1970), 1419–1432.

[43] C. Texier, *Fluctuations of the product of random matrices and generalized Lyapunov exponent*, J. Stat. Phys., 181 990?1051, 2020.

[44] C. Texier and C. Hagendorf, *Effect of boundaries on the spectrum of a one-dimensional random mass Dirac Hamiltonian*, J. Phys. A 43, (2010) 025002.

[45] G. Theodorou, M. H. Cohen, *Extended states in a one-dimensional system with off-diagonal disorder*, Phys. Rev. B 13 (1976), 4597–4601.

[46] D.J. Thouless, *A relation between the density of states and range of localisation for one dimensional random systems*, J. Phys. C 5 (1972), 77–81.

[47] D.J. Thouless *Percolation and localization in ‘Ill-Condensed Matter’, ed. R. Balian, R. Maynard and G. Toulouse, Amsterdam: North-Holland, 1979.

[48] M. Titov, P.W. Brouwer, A. Furusaki, and C. Mudry, *Fokker-Planck equations and density of states in disordered quantum wires*, Phys. Rev. B, 63 (2001), 235318.

[49] H.D. Trinh and K.D. Trinh, *Beta Laguerre ensembles in global regime*, arXiv:1907.12267.

[50] H.F. Trotter, *Eigenvalue distributions of large Hermitian matrices: Wigner’s semi-circle law and a theorem of Kac, Murdock and Szegö*, Adv. Math. 54 (1984), 67–82.

[51] T.A.L. Ziman, *Localization and spectral singularities in random chains*, Phys. Rev. Lett. 49, (1982) 337—340.

School of Mathematical and Statistics, ARC Centre of Excellence for Mathematical and Statistical Frontiers, The University of Melbourne, Victoria 3010, Australia

Email address: pjforr@unimelb.edu.au