Perturbed Coulomb potentials in the Klein-Gordon equation via the shifted-$l$ expansion technique

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Abstract

A shifted-$l$ expansion technique is introduced to calculate the energy eigenvalues for Klein-Gordon (KG) equation with Lorentz vector and/or Lorentz scalar potentials. Although it applies to any spherically symmetric potential, those that include Coulomb-like terms are only considered. Exact eigenvalues for a Lorentz vector or a Lorentz scalar, and an equally mixed Lorentz vector and Lorentz scalar Coulombic potentials are reproduced. Highly accurate and rapidly converging ground-state energies for Lorentz vector Coulomb with a Lorentz vector or a Lorentz scalar linear potential, \( V(r) = -A_1/r + kr \), and \( V(r) = -A_1/r \) and \( S(r) = kr \), respectively, are obtained. Moreover, a simple straightforward closed-form solution for a KG-particle in Coulombic Lorentz vector and Lorentz scalar potentials is presented in appendix A.
I Introduction

The Klein-Gordon (KG) and the Dirac equations with Lorentz scalar (added to the mass term) and/or Lorentz vector (coupled as the 0-component of the four-vector potential) potentials are of interest in many branches of physics. For example, Lorentz scalar or equally mixed Lorentz scalar and Lorentz vector potentials have considerable interest in quark-antiquark mass spectroscopy [1-5]. Lorentz vector potentials have great utility in atomic, nuclear, and plasma physics [6,7]. Therefore many attempts have been made to develop approximation techniques to treat relativistic particles in the KG and the Dirac equations [1-7].

Very recently we have introduced a shifted - \( l \) expansion technique (SLET) to solve for the Schrödinger [8], and the Dirac equations for some model potentials [9]. SLET is a reformulation of the existing shifted - N expansion technique (SLNT) [1,10-12, and references therein]. SLET simply consists of using \( 1/\bar{l} \) as an expansion parameter where \( \bar{l} = l - \beta \), \( \beta \) is a suitable shift, \( l \) is the angular momentum quantum number for spherically symmetric potentials, and \( l = |m| \) for cylindrically symmetric potentials, where \( m \) is the magnetic quantum number. As such, one does not need to construct the \( N \)-dimensional form, required to perform SLNT, of the wave equation of interest. With SLET we simply expand through the quantum number in the centrifugal term of that equation. Unlike other perturbation methods [13-16], SLET puts no constraints on the coupling constants of the potential or on the quantum numbers involved. Above all, it yields very accurate and rapidly converging eigenvalues without the need of wave functions or matrix elements.

In this paper we shall be concerned with the shifted - \( l \) expansion for the KG equation with radially symmetric Lorentz scalar, \( S(r) \), and/or Lorentz vector, \( V(r) \), potentials that include Coulomb-like terms. We shall examine SLET and calculate the energy eigenvalues for the KG equation with the following potential mixtures: (i) \( V(r) = -A_1/r \) and \( S(r) = 0 \), which represents a \( \pi^- \) meson in a Coulomb potential. (ii) \( V(r) = 0 \) and \( S(r) = -A_2/r \), which has no experimental evidence, to the best of our knowledge, thus our calculations are only of academic interest. (iii) \( V(r) = S(r) = -A/r \) which represents not only a KG-particle in an equally mixed Lorentz scalar and Lorentz vector potential but also a Dirac particle in the same potential mixture, where \( l = j + 1/2 \) and the radial KG wave function represents the
radial large - component of the Dirac - spinor \([4,9]\). (iv) \(V(r) = -A_1/r + kr\) and \(S(r) = 0\) representing a \(\pi^-\) meson in a Coulomb potential perturbed by a linear Lorentz vector interaction \(kr\). (v) \(V(r) = -A_1/r\) and \(S(r) = kr\) describing a \(\pi^-\) meson in a Coulomb potential perturbed by a linear Lorentz scalar potential \(kr\).

In Sec II we shall introduce SLET for the KG equation with any spherically symmetric Lorentz scalar and/or Lorentz vector potentials that include Coulomb - like interactions. We shall cast SLET’s analytical expressions in such a way that allows the reader to use them without proceeding into their derivations. In Sec III we shall show that these expressions yield closed - form solutions to the KG equation for the mixtures (i), (ii), and (iii). Ground - state energies for the mixtures (iv) and (v) will be calculated and compared with those of McQuarrie and Vrscay [13] in the same section. We conclude and remark in Sec IV.

In appendix A we present a simple straightforward closed - form solution for the KG equation with Coulomb - like Lorentz scalar and Lorentz vector potentials. It could be interesting to mention that a similar solution was found by McQuarrie and Vrscay [13], who used a confluent hypergeometric function in their calculation. They have misprinted it though (see appendix of Ref [13]). To the best of our knowledge, such explicit solution has not been reported elsewhere.

II SLET for the KG equation with potentials including Coulombic terms.

In this section we shall consider the three - dimensional KG equation with radially symmetric Lorentz vector and Lorentz scalar potentials, \(V(r)\) and \(S(r)\), respectively. If \(\Psi(r)\) denotes the wave function of the KG particle, a separation of variables \(\Psi(r) = r^{-1}R(r)Y(\theta, \phi)\) yields the following radial equation (in units \(\hbar = c = 1\)) [1]:

\[
\left[ -\frac{d^2}{dr^2} + \frac{l(l + 1)}{r^2} + [S(r) + m]^2 - [E - V(r)]^2 \right] R(r) = 0, \quad (1)
\]
where $E$ is the energy, and $l$ is the angular quantum number. For Coulomb-like potentials one may use the substitutions:

$$V_r(r) = V(r)^2 - A_1^2/r^2,$$

and

$$S_r(r) = S(r)^2 - A_2^2/r^2,$$

so that Eq.(1) becomes

$$\left[-\frac{d^2}{dr^2} + \frac{l'(l' + 1)}{r^2} + \gamma(r) + 2EV(r)\right]R(r) = E^2R(r),$$

where

$$\gamma(r) = -V_r(r) + S_r(r) + 2mS(r) + m^2,$$

$$l'(l' + 1) = l(l + 1) - A_c; \quad l' = -1/2 + \sqrt{(l + 1/2)^2 - A_c},$$

$$A_c = A_1^2 - A_2^2.$$

Hereby, it should be noted that for the case of $V(r) = -A_1/r$ and $S(r) = -A_2/r$ Eq.(4) reduces to a form nearly identical to the Schrödinger equation for a Coulomb field. Its solution can thus be inferred from the known solution of the Schrödinger - Coulomb problem. We do this in appendix A.

If we shift $l'$ through the relation $l' = \bar{l} + \beta$ Eq.(4) reads

$$\left[-\frac{d^2}{dr^2} + \frac{\bar{l}^2 + \bar{l}(2\beta + 1) + \beta(\beta + 1)}{r^2} + \gamma(r) + 2EV(r)\right]R(r) = E^2R(r),$$

where $\beta$ is a suitable shift to be determined and is mainly introduced to avoid the trivial case when $l' = 0$.

To start the systematic $1/\bar{l}$ expansion [8,9] we define

$$\gamma(r) = \frac{\bar{l}^2}{Q} \left[\gamma(r_o) + \gamma'(r_o)r_o x/\bar{l}^{1/2} + \gamma''(r_o)r_o^2 x^2/2\bar{l} + \cdots \right],$$

$$V(r) = \frac{\bar{l}^2}{Q} \left[V(r_o) + V'(r_o)r_o x/\bar{l}^{1/2} + V''(r_o)r_o^2 x^2/2\bar{l} + \cdots \right],$$

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where \( x = \frac{l_1}{2}(r - r_o)/r_o \), \( r_o \) is currently an arbitrary point to do Taylor expansions about, with its particular value to be determined below, and \( Q \) is to be set equal to \( l^2 \) at the end of the calculations. Substituting Eqs (9) - (11) into Eq.(8) implies

\[
\begin{align*}
E &= \frac{\bar{r}^2}{Q} \left[ E_o + E_1/\bar{r} + E_2/\bar{r}^2 + E_3/\bar{r}^3 + \cdots \right]. \\
\end{align*}
\]

(11)

and \( n_r \) is the radial quantum number. Eq.(12) is a Schrödinger-like equation for the one-dimensional anharmonic oscillator and has been discussed in detail by Imbo et al \[11\]. We therefore quote only the resulting eigenvalue of Ref.[11] and write

\[
\begin{align*}
\mu_{n_r} &= r_o^2 \bar{l} \left[ E_o^2 + \frac{2E_o E_1}{\bar{l}} + \frac{(E_1^2 + 2E_o E_2)}{\bar{l}^2} + \frac{2(E_o E_3 + E_1 E_2)}{\bar{l}^3} + \cdots \right], \\
\end{align*}
\]

(13)

where

\[
\begin{align*}
\mu_{n_r} &= \bar{l} \left[ 1 + \frac{2r_o^2 V(r_o)E_o}{Q} + \frac{r_o^2 \gamma(r_o)}{Q} \right] \\
&\quad + \left[ (2\beta + 1) + \frac{2r_o^2 V(r_o)E_1}{Q} + \left( n_r + \frac{1}{2} \right)w \right] \\
&\quad + \frac{1}{\bar{l}} \left[ \beta(\beta + 1) + \frac{2r_o^2 V(r_o)E_2}{Q} + \alpha_1 \right]
\end{align*}
\]

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where $\alpha_1$ and $\alpha_2$ are given in appendix B of this text. If we compare Eq.(14) with (13), we obtain

$$E_o = V(r_o) \pm \sqrt{V(r_o)^2 + Q/r_o^2 + \gamma(r_o)}, \quad (15)$$

$$E_1 = \frac{Q}{2r_o^2(E_o - V(r_o))} [2\beta + 1 + (n_r + 1/2)w], \quad (16)$$

$$E_2 = \frac{Q}{2r_o^2(E_o - V(r_o))} [\beta(\beta + 1) + \alpha_1], \quad (17)$$

$$E_3 = \frac{Q}{2r_o^2(E_o - V(r_o))} \alpha_2, \quad (18)$$

and

$$E_{n_r} = E_o + \frac{1}{2r_o^2(E_o - V(r_o))} \left[ \beta(\beta + 1) + \alpha_1 + \frac{\alpha_2}{l} \right]. \quad (19)$$

$r_o$ is chosen to be the minimum of $E_o$, i. e.;

$$dE_o/dr_o = 0 \quad \text{and} \quad d^2 E_o/dr_o^2 > 0. \quad (20)$$

Hence, $r_o$ is obtained through the relation

$$2Q = 2(l' - \beta)^2 = b(r_o) + \sqrt{b(r_o)^2 - 4c(r_o)}, \quad (21)$$

where

$$b(r_o) = r_o^3 \left[ 2V(r_o)V'(r_o) + \gamma'(r_o) + r_o V'(r_o)^2 \right], \quad (22)$$

$$c(r_o) = \frac{r_o^6}{4} \left[ \gamma'(r_o)^2 + 4V(r_o)V'(r_o)\gamma'(r_o) - 4\gamma(r_o)V'(r_o)^2 \right] \quad (23)$$

The shifting parameter $\beta$ is determined by requiring $E_1 = 0$ [1, 8-12] to obtain

$$\beta = -[1 + (n_r + 1/2)w]/2, \quad (24)$$
where
\[ w = \left[ 12 + \frac{2r_o^4 \gamma'(r_o)}{Q} + \frac{4r_o^4 V''(r_o) E_o}{Q} \right]^{1/2}. \] 
(25)

It is convenient to summarize the above procedure in the following steps: (a) Calculate \( Q \) from Eq.(21) and substitute it in Eq.(15) to find \( E_o \) in terms of \( r_o \). (b) Substitute \( E_o \) and \( Q \) in Eq.(25) to obtain \( w \). (c) Find \( \beta \) from Eq.(24) to calculate \( r_o \) from Eq.(21). (e) Finally, one can obtain \( E_o \) and calculate \( E_n r \) from Eq.(19). However, one is not always able to calculate \( r_o \) in terms of the potential coupling constants since the analytical expressions become algebraically complicated, although straightforward. Therefore, one has to appeal to numerical computations to find \( r_o \) and hence \( E_o \).

III Applications, results, and discussion.

To show the performance of the analytical expressions of SLET it is best to consider some special cases.

(i) \( V(r) = -A_1/r \) and \( S(r) = 0 \)

A pionic atom in a Coulomb potential obeys the KG equation with \( V(r) = -A_1/r \) and \( S(r) = 0 \). To calculate its bound-state energies, which are simply the bound state energies of a \( \pi^- \) meson in a Coulomb potential, we follow the SLET procedure and find
\[ E_o = \frac{-A_1^2 \pm (A_1^2 + Q)}{A_1 r_o}. \] 
(26)

Here we have to choose the positive sign since states with negative energies correspond to anti-particles. Furthermore, the negative sign yields a contradiction to Eq.(21). Hence \( w = 2 \),
\[ Q = (l - \beta)^2 = \left[ n_r + 1/2 + \sqrt{(l + 1/2)^2 - A_1^2} \right]^2. \] 
(27)
\[ r_o = \sqrt{\frac{Q^2 + QA_1^2}{m^2 A_1^2}}, \] 
(28)
and

\[ E_o = m[1 + \frac{A_2^2}{\tilde{n}^2}]^{-1/2}, \]  

(29)

where \( \tilde{n} = \sqrt{Q} \). Eq. (29) represents the well known closed - form solution of the KG equation for a \( \pi^- \) meson in a Coulomb potential [17]. It should be noted that higher - order terms of the energy eigenvalues vanish identically, i.e. \( E_2 = 0 \) and \( E_3 = 0 \). Hence \( E_{n_r} = E_o \).

(ii) \( V(r) = 0 \) and \( S(r) = -A_2/r \)

Since there is no experimental evidence, to the best of our knowledge, for such a long - range interaction, our calculations are only of academic interest. Following the above procedure we find

\[ r_o = \frac{(l' - \beta)^2}{mA_2}, \]  

(30)

and

\[ E_o = \pm m[1 - \frac{A_2^2}{\tilde{n}^2}]^{1/2}, \]  

(31)

where \( \tilde{n} = n_r + 1/2 + \sqrt{(l + 1/2)^2 + A_2^2} \). Again the higher - order terms of the energy eigenvalues vanish identically. Thus \( E_{n_r} = E_o \).

Obviously there exist two branches of solutions in the bound region and they exhibit identical behaviour, which reflects the fact that the Lorentz scalar interaction does not distinguish between particle and antiparticle. The particle and antiparticle states, positive and negative energies, respectively, approach each other with increasing coupling constant, without touching. Therefore, spontaneous pair creation never occurs, no matter how strong the potential chosen.

(iii) \( V(r) = S(r) = -A/r \)

This type of potential mixture, \( V(r) = S(r) \), has considerable interest in quarkonium spectroscopy [4,9,18]. For the particular case \( V(r) = S(r) = -A/r \), SLET yields

\[ E_o = -A/r_o \pm m, \]  

(32)

and
\[(E_o^2 - m^2)r_o^2 = A^2 \mp (Q + A^2).\] (33)

Eq.(33) can be satisfied if and only if the negative sign is chosen, otherwise it contradicts Eq.(21). The only valid sign in Eq.(32) is thus the positive one, and hence

\[E_o = -A/r_o + m.\] (34)

Which in turn implies that

\[r_o = \frac{n^2 + A^2}{2mA}; \quad n = n_r + l + 1,\] (35)

and

\[E_{n_r} = E_o = m[1 - \frac{2A^2}{n^2 + A^2}].\] (36)

Where higher - order terms of the energy eigenvalues vanish identically, and \(n\) is the principle quantum number. For \(A \to \infty\), \(E_{n_r}\) approaches the value \(-m\) asymptotically, but the state never diverges into the negative continuum.

To show that Eq.(36) yields the energy eigenvalues for Dirac particle in the same potential mixture, we replace \(l\) by \(j + 1/2\) to obtain

\[E_{n_r} = m[1 - \frac{2A^2}{(n_r + |\kappa| + 1)^2 + A^2}].\] (37)

where \(|\kappa| = j + 1/2\) [19].

(iv) \(V(r) = -A_1/r + kr\) and \(S(r) = 0\)

This potential represents a \(\pi^-\) meson in a Coulomb potential perturbed by a linear Lorentz vector potential \(kr\). In this case

\[\gamma(r) = -k^2r^2 + 2A_1k + m^2.\] (38)

Eq.(38) when substituted in (21), (15), (25), (24), and again in (21), respectively, yields a very involved algebraic equation for \(r_o\). We solve this equation numerically with a maximum error of order \(\sim 10^{-15}\) to calculate for \(r_o\). Once \(r_o\) is calculated, \(Q, E_o, w, \beta\), and hence \(E_{n_r}\) can be obtained.

In Tables 1 and 2 we list our results for the ground - state energies along with those of McQuarrie and Vrscay [13], who have used hypervirial and
Hellmann - Feynman theorems to construct Rayleigh - Schrödinger (RS) perturbation expressions to an arbitrary order. Our results are given in such a way that the contributions of the second - and third - order corrections, \( E_2/\bar{l}^2 \) and \( E_3/\bar{l}^3 \), respectively, to the energy eigenvalues are made clear. The results are in excellent agreement with those of Ref.[13].

(v) \( V(r) = -A_1/r \) and \( S(r) = kr \)

A \( \pi^- \) meson in a Coulomb potential perturbed by a linear scalar interaction is described by \( V(r) = -A_1/r \) and \( S(r) = kr \) potential mixture in the KG equation. In this case

\[
\gamma(r) = k^2r^2 + 2mkr + m^2. \tag{39}
\]

Following the same steps of (iv) we numerically solve for \( r_o \), to a maximum error of order \( \sim 10^{-15} \), \( Q, E_o, w, \beta, \) and \( E_n \). Our results for the ground - state energies are presented in Tables 3 and 4 in such a way that the convergence of SLET is made clear. We compare them with those of Ref.[13]. They are in excellent agreement.

In view of the above results the following observations deserve to be recorded.

The closed - form solutions, Eqs.(29), (31), and (36), being obtained by the leading term of the energy series, Eq.(11), where higher - order terms vanished identically, reveals how rapidly converging are the results of SLET.

The numerical results of SLET, in Tables 1 - 4, imply that the contributions of the second - and third - order corrections to the energy eigenvalues are almost negligible. The convergence of SLET is thus out of question. However, the RS coefficients \( E^p \) for the eigenvalue

\[
E = \sum_{p=0}^\infty E^{(p)} k^p, \tag{40}
\]

used in Ref.[13], as well as their continued - fraction (CF) counterparts \( c_p \) were computed numerically to large - order, \( p \sim 100 \) and \( p \sim 50 \) for the Lorentz vector linear and the Lorentz scalar linear perturbations, respectively. Numerical ratio tests showed that the perturbation series are divergent [13]. As well, the \( c_p \) had suffered from occasional eruptions reversing the roles of the upper and lower bounds of the energy eigenvalues. Moreover, the gap between the two bounds increases with increasing coupling constant \( k \), same is the uncertainty of the energy eigenvalues.
IV Conclusions and remarks

In this paper we have introduced SLET to solve for the eigenvalues of KG equation with Lorentz vector and Lorentz scalar potentials including Coulombic terms. Although it applies to any spherically symmetric potential, those that include Coulomb-like terms were only considered. We have reproduced closed-form solutions for a Lorentz vector or a Lorentz scalar, and for an equally mixed Lorentz vector and Lorentz scalar Coulombic potentials [20]. Compared to those of Ref.[13] our results are highly accurate and rapidly convergent.

The conceptual soundness of our SLET is obvious. It is highly accurate and efficient with respect to computers time. It does not need the wave functions or matrix elements, but when necessary wave functions can be calculated. It puts no constraints on the coupling constants of the potential or on the quantum numbers. It simply consists of using $1/l$ as an expansion parameter rather than the coupling constant of the potential. It is to be understood as being an expansion through not only the angular momentum quantum number but also through any existing quantum number in the centrifugal-like term of any Schrödinger-like equation, Eq.(4).

A general observation concerning the method used by McQuarrie and Vrscay [13] is in order. Unlike our approach their method involves expansions through the coupling constant $k$, Eq.(40). Thus, whereas their computations for the ground-state energies are beyond doubt, the same need not be true for the case of strong coupling constant $k > 1$, in Eq.(40), for example.

Finally, we would like to remark that SLET is also applicable to more complicated potentials. For example, the screened Coulomb potentials which have great utility in atomic, nuclear, and plasma physics. The equally mixed Lorentz scalar and Lorentz vector logarithmic potential which has significant interest in quarkonium spectroscopy [4].
Appendix A
The KG equation with Coulomb - like Lorentz scalar and
Lorentz vector potentials

In this section we present a simple solution for a KG particle in Coulomb
-like Lorentz scalar and Lorentz vector potentials, i. e. \( V(r) = -A_1/r \) and
\( S(r) = -A_2/r \). For this particular problem the KG equation reduces to

\[
\left[ -\frac{d^2}{dr^2} + \frac{l'(l' + 1)}{r^2} - \frac{2(mA_2 + EA_1)}{r} \right] R(r) = [E^2 - m^2] R(r). \tag{41}
\]

It is obvious that this equation is in a form nearly identical to the Schrödinger
equation for a Coulomb potential. Its solution can thus be inferred from the
known Schrödinger - Coulomb solution. Therefore, one may obtain its solu-
tion through the relation

\[
E^2 - m^2 = \frac{-(2mA_2 + 2EA_1)^2}{(2\tilde{n})^2}. \tag{42}
\]

This equation is quadratic in \( E \) and thus admits a solution of the form

\[
E_{n_r} = m \left[ \frac{-A_1 A_2 \pm \sqrt{A_1^2 A_2^2 + (\tilde{n}^2 + A_1^2)(\tilde{n}^2 - A_2^2)}}{\tilde{n}^2 + A_1^2} \right], \tag{43}
\]

where \( \tilde{n} = n_r + l' + 1 \). Hereby, it should be pointed out that this result reduces
to those obtained in Sec.III.1,2, and 3. Although McQuarrie and Vrscay [13]
have used a confluent hypergeometric function to obtain this result, they
have misprinted it (see the appendix of Ref.[13]).
Appendix B

\(\alpha_1\) and \(\alpha_2\) in Eq. (14)

The definitions of \(\alpha_1\) and \(\alpha_2\) appeared in Eq. (14) are:

\[\begin{align*}
\alpha_1 &= [(1 + 2n_r)e_2 + 3(1 + 2n_r + 2n_r^2)e_4] \\
&\quad - w^{-1}[e_1^2 + 6(1 + 2n_r)e_1e_3 + (11 + 30n_r + 30n_r^2)e_3^2], \quad (44) \\
\alpha_2 &= (1 + 2n_r)d_2 + 3(1 + 2n_r + 2n_r^2)d_4 \\
&\quad + 5(3 + 8n_r + 6n_r^2 + 4n_r^3)d_6 \\
&\quad - w^{-1}[(1 + 2n_r)e_2^2 + 12(1 + 2n_r + 2n_r^2)e_2e_4 + 2e_1d_1 \\
&\quad + 2(21 + 59n_r + 51n_r^2 + 34n_r^3)e_4^2 + 6(1 + 2n_r)e_1d_3 \\
&\quad + 30(1 + 2n_r + 2n_r^2)e_1d_5 + 6(1 + 2n_r)e_3d_1 \\
&\quad + 2(11 + 30n_r + 30n_r^2)e_3d_3 + 10(13 + 40n_r + 42n_r^2 + 28n_r^3)e_3d_5] \\
&\quad + w^{-2}[4e_1e_2^2 + 36(1 + 2n_r)e_1e_2e_3 + 8(11 + 30n_r + 30n_r^2)e_2e_3^2 \\
&\quad + 24(1 + n_r)e_1^2e_4 + 8(31 + 78n_r + 78n_r^2)e_1e_3e_4 \\
&\quad + 12(57 + 189n_r + 225n_r^2 + 150n_r^3)e_3^2e_4] \\
&\quad - w^{-3}[8e_1^3e_3 + 108(1 + 2n_r)e_1e_3^2 + 48(11 + 30n_r + 30n_r^2)e_1e_3^3 \\
&\quad + 30(31 + 109n_r + 141n_r^2 + 94n_r^3)e_3^4], \quad (45)
\end{align*}\]
where

\[ e_j = \frac{\varepsilon_j}{w^{j/2}} \quad \text{and} \quad d_i = \frac{\delta_i}{w^{i/2}}, \]  

(46)

with \( j = 1, 2, 3, 4, i = 1, 2, 3, 4, 5, 6, \) and

\begin{align*}
\varepsilon_1 &= -2(2\beta + 1), \quad \varepsilon_2 = 3(2\beta + 1), \\
\varepsilon_3 &= -4 + \frac{r_o^5}{6Q} \left[ \gamma''(r_o) + 2V''(r_o)E_o \right], \\
\varepsilon_4 &= 5 + \frac{r_o^6}{24Q} \left[ \gamma'''(r_o) + 2V'''(r_o)E_o \right], \\
\delta_1 &= -2\beta(\beta + 1) + \frac{2r_o^3V'(r_o)E_2}{Q}, \\
\delta_2 &= 3\beta(\beta + 1) + \frac{r_o^4V''(r_o)E_2}{Q}, \\
\delta_3 &= -4(2\beta + 1), \quad \delta_4 = 5(2\beta + 1), \\
\delta_5 &= -6 + \frac{r_o^7}{120Q} \left[ \gamma''''(r_o) + 2V''''(r_o)E_o \right], \\
\delta_6 &= 7 + \frac{r_o^8}{720Q} \left[ \gamma''''(r_o) + 2V''''(r_o)E_o \right].
\end{align*}

(47)-(54)

The terms including \( E_1 \) have been dropped from the expressions above since \( E_1 = 0. \)
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Table 1: Ground-state energies of a $\pi^-$ meson in $V(r) = -A_1/r + kr$ and $S(r) = 0$ (in $\hbar = c = m = 1$ units). The lower bounds to the energies $E$ of Ref.[13] are obtained by replacing the last $j$ digits of the upper bounds with the $j$ digits in parentheses.

| $A_1$ | $k$  | Ref.[13]  | $E_o$    | $E_o + E_2/l^2$ | Eq.(11)    |
|-------|------|-----------|----------|-----------------|------------|
| 0.2   | 0.0  | 0.97890631293 | 0.97890631293 | 0.97890631293 | 0.97890631293 |
|       | 0.01 | 1.027622(19)    | 1.029590     | 1.027995     | 1.027641    |
|       | 0.05 | 1.152(48)       | 1.1541       | 1.1514       | 1.1504      |
|       | 0.1  | 1.277(48)       | 1.2681       | 1.2648       | 1.2634      |
|       | 0.2  | 1.50(37)        | 1.4478       | 1.4436       | 1.4416      |
|       | 0.3  | 1.73(45)        | 1.5959       | 1.5907       | 1.5882      |
| 0.3   | 0.0  | 0.9486832981    | 0.9486832981 | 0.9486832981 | 0.9486832981 |
|       | 0.01 | 0.9843795380(78)| 0.9861391713 | 0.9844578890 | 0.9843836237 |
|       | 0.05 | 1.08612(08)     | 1.09160      | 1.08710      | 1.08611     |
|       | 0.1  | 1.18398(08)     | 1.19111      | 1.18517      | 1.18356     |
|       | 0.2  | 1.345(34)       | 1.3500       | 1.3421       | 1.3397      |
|       | 0.3  | 1.487(52)       | 1.4820       | 1.4723       | 1.4693      |
Table 2: Ground - state energies of a $\pi^-$ meson in $V(r) = -A_1/r + kr$ and $S(r) = 0$ (in $\hbar = c = m = 1$ units). The lower bounds to the energies $E$ of Ref.[13] are obtained by replacing the last $j$ digits of the upper bounds with the $j$ digits in parentheses.

| $A_1$ | $k$   | Ref.[13]   | $E_o$     | $E_o + E_2/l^2$ | Eq.(11) |
|------|------|------------|----------|-----------------|--------|
| 0.4  | 0.0  | 0.894427191| 0.894427191| 0.894427191     | 0.894427191 |
|      | 0.01 | 0.9190495619| 0.9199389105| 0.9190185899   | 0.9190592557 |
|      | 0.05 | 0.99735023(19) | 1.00292492 | 0.99763976     | 0.99732540 |
|      | 0.1  | 1.0759666(05)| 1.08503730 | 1.07689328    | 1.07585618 |
|      | 0.2  | 1.20488(59)   | 1.21825   | 1.20662       | 1.20453  |
|      | 0.3  | 1.3138(20)    | 1.32987   | 1.31551       | 1.31261  |
| 0.5  | 0.0  | 0.70710678119| 0.70710678119| 0.70710678119 | 0.70710678119 |
|      | 0.01 | 0.7174441845 | 0.7175816778| 0.7174395344  | 0.7174446067 |
|      | 0.05 | 0.7548104279 | 0.7569796562| 0.7546116940  | 0.7548645285 |
|      | 0.1  | 0.795714744277(07) | 0.8013654436 | 0.7951332112  | 0.7959080738 |
|      | 0.2  | 0.86613531(67) | 0.87841386 | 0.86508153    | 0.86648455 |
|      | 0.3  | 0.9269(68)    | 0.9450    | 0.92586       | 0.92734  |
Table 3: Ground - state energies of a $\pi^-$ meson in $V(r) = -A_1/r$ and $S(r) = kr$ (in $\hbar = c = m = 1$ units). The lower bounds to the energies $E$ of Ref.[13] are obtained by replacing the last $j$ digits of the upper bounds with the $j$ digits in parentheses.

| $A_1$ | $k$   | Ref.[13]     | $E_o$   | $E_o + E_2/l^2$ | Eq.(11)   |
|-------|-------|--------------|---------|-----------------|-----------|
| 0.2   | 0.0   | 0.9789063129 | 0.9789063129 | 0.9789063129   | 0.9789063129 |
|       | 0.01  | 1.0266839(09)| 1.02871502  | 1.02708579     | 1.02671248 |
|       | 0.05  | 1.145795(48) | 1.1479170   | 1.1449853      | 1.1439393  |
|       | 0.1   | 1.263(34)    | 1.2543     | 1.2506         | 1.2491     |
|       | 0.2   | 1.47(34)     | 1.418      | 1.413          | 1.411      |
|       | 0.3   | 1.68(41)     | 1.551      | 1.545          | 1.542      |
| 0.3   | 0.0   | 0.94868329805| 0.94868329805 | 0.94868329805 | 0.94868329805 |
|       | 0.01  | 0.9834119450(49)| 0.9852632146 | 0.9835164748 | 0.9834118201 |
|       | 0.05  | 1.079797(62) | 1.085772   | 1.081002       | 1.079875   |
|       | 0.1   | 1.170(69)    | 1.17826    | 1.17182        | 1.16999    |
|       | 0.2   | 1.316(05)    | 1.3227     | 1.31403        | 1.31133    |
|       | 0.3   | 1.443(09)    | 1.4404     | 1.42998        | 1.42666    |
Table 4: Ground-state energies of a $\pi^-$ meson in $V(r) = -A_1/r$ and $S(r) = kr$ (in $\hbar = c = m = 1$ units). The lower bounds to the energies $E$ of Ref.[13] are obtained by replacing the last $j$ digits of the upper bounds with the $j$ digits in parentheses.

| $A_1$ | $k$     | Ref.[13] | $E_o$          | $E_o + E_2/l^2$ | Eq.(11) |
|-------|---------|----------|----------------|-----------------|---------|
| 0.4   | 0.0     | 0.894427191 | 0.894427191    | 0.894427191     | 0.894427191 |
|       | 0.01    | 0.9180779227 | 0.9190603882   | 0.9180537998    | 0.9180878824 |
|       | 0.05    | 0.9915050(49) | 0.997801768    | 0.99204395      | 0.99150796  |
|       | 0.1     | 1.063490(84)  | 1.07395616     | 1.06503089      | 1.06352424  |
|       | 0.2     | 1.1791(88)    | 1.19492        | 1.18212         | 1.17926     |
|       | 0.3     | 1.2755(37)    | 1.29449        | 1.27883         | 1.27503     |
| 0.5   | 0.0     | 0.7071067812  | 0.7071067812   | 0.7071067812    | 0.7071067812 |
|       | 0.01    | 0.7168151723  | 0.7169998899   | 0.7168006675    | 0.7168157252 |
|       | 0.05    | 0.7514351538  | 0.7544233036   | 0.7512342784    | 0.7514976377 |
|       | 0.1     | 0.78877520361(59) | 0.7967081101 | 0.7883649991   | 0.7889588588 |
|       | 0.2     | 0.852303690(62) | 0.869749249    | 0.852125792     | 0.852493296  |
|       | 0.3     | 0.9068223(16)  | 0.9322673      | 0.9074421       | 0.906889    |