NO FINITE INVARIANT DENSITY FOR MISIUREWICZ EXPONENTIAL MAPS

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Abstract. For exponential mappings such that the orbit of the only singular value 0 is bounded, it is shown that no integrable density invariant under the dynamics exists on $\mathbb{C}$.

1. Introduction

We consider one parameter family of exponential functions $f_\Lambda(z) = \Lambda e^z$, $z \in \mathbb{C}$, $\Lambda \in \mathbb{C}^\ast$. These maps have only one finite singular value 0 whose forward trajectory determines the dynamics on $\mathbb{C}$. From now we assume that the orbit of the asymptotic value 0 is bounded, hence the Julia set $J(f_\Lambda) = \mathbb{C}$. Thus $f_\Lambda$ satisfies so called Misiurewicz condition i.e. the post-singular set $P(f) := \bigcup_{n=0}^{\infty} f^n_\Lambda(0)$ is bounded and $P(f) \cap \text{Crit}(f) = \emptyset$. It follows from [5, Th.1] that $P(f)$ is hyperbolic. The problem of existence of probabilistic invariant measure absolutely continuous with respect to the Lebesgue measure (abbr. pacim) for transcendental meromorphic functions satisfying Misiurewicz condition was discussed in [7]. However this result cannot be applied to entire functions. It is still an open problem whether the simplest entire functions like exponential map $z \to 2\pi i \exp(z)$ have pacim. The main result of this paper is the following theorem.

Theorem 1. Let $f(z) = \Lambda \exp(z)$ with $\Lambda \in \mathbb{C} \setminus \{0\}$ chosen so that the Julia set is the entire sphere and the orbit of 0 under $f$ is bounded. Then $f$ admits no probabilistic invariant measure absolutely continuous with respect to the Lebesgue measure.

However these maps have $\sigma$-finite invariant measure absolutely continuous with respect to the Lebesgue measure (see [4]). A result similar to Theorem 1 has been mentioned to us by other authors, [3].

The proof will proceed by contradiction, so we suppose that such a measure exists and call it $\mu$, while reserving $\lambda$ for the Lebesgue measure of the plane. It follows from [4] that the set of points escaping to $\infty$ has zero Lebesgue’s measure for every map in our family. It is not difficult to prove that for these functions the union $P(f) \cup \{\infty\}$ is not a metric attractor in sense of Milnor with respect to the measure $\lambda$ on $\mathbb{C}$. The results of [1] implies that $f_\Lambda$ is ergodic with respect to $\lambda$. Thus

Fact 1. The measure $\mu$ is ergodic.

2. Proof

For a positive integer $n$ write $A_n := \{z : |\Lambda| e^n < |z| \leq |\Lambda| e^{n+1}, \arg z \neq \arg \Lambda\}$. A fundamental rectangle will refer to any set in the form $\{x + 2\pi i y : k < x < k + 1, l < y < l + 1\}$ for integers $k, l$. Thus, any fundamental rectangle is mapped with bounded distortion and onto some $A_n$.

Lemma 1. For all $n \in \mathbb{Z}_+$, $\inf \text{ess} \left( \frac{d\mu}{d\lambda}(z) : z \in A_n \right) > 0$.

Proof. By [5, Th.1], the post-singular set $P(f)$ has area 0, so it cannot be the support of $\mu$. Additionally, the image of every open set covers $A_n$ after finitely many iterations, so it suffices to have the $\frac{d\mu}{d\lambda}$ essentially bounded away from 0 on any open set. Hence, Lemma 1 follows from the following fact. □

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Lemma 2. Suppose that $F$ is a meromorphic function whose Julia set is the entire sphere, and $\nu$ a probability invariant and ergodic measure absolutely continuous with respect to $\lambda$ and such that the $\nu$-measure of the closure of the post-singular set of $F$ is less than 1. Then, there is an open set $U$ such that

$$\inf \frac{d\nu}{d\lambda}(z) : z \in U > 0.$$ 

Proof. Fix $U$ to be a disk in a positive distance from the orbit of 0 and such that $\eta := \nu(U)$ is positive. Denote $\rho(z) := \frac{d\nu}{d\lambda}$. Pick $\epsilon > 0$. In the argument to follow it is important distinguish between parameters that do or do not depend on $\epsilon$.

A variant of Luzin’s Theorem. For every $\epsilon > 0$, we can find a continuous function with compact support $\rho_\epsilon : \mathbb{C} \to [0, +\infty)$ such that

(1) \[ \int_{\mathbb{C}} (\rho_\epsilon(w) - \rho(w))_+ \, d\lambda(w) < \epsilon \]

where the plus subscript denote the positive part,

(2) \[ \int_{\mathbb{C}} \min(\rho_\epsilon(z), \rho(z)) \, d\lambda(z) \geq 1 - \eta/10. \]

This statement follows from introductory measure theory.

Proof of Lemma 1 continued. Now for any $k$ consider the set $\Omega_k$ of connected components of $F^{-k}(U)$ which intersect the support of $\rho_\epsilon$. If $V \in \Omega_k$, then $F^k$ maps $V$ onto $U$ univalently and with distortion bounded depending solely on $U$. Denote $d_k = \sup\{\text{diam} V : V \in \Omega_k\}$. Since the Julia set is the whole sphere, $\lim_{k\to\infty} d_k = 0$. Let $G_k$ denote the set of inverse branches of $F^k$ defined on $U$. For $z \in U$

$$\rho_{\epsilon,k}(z) = \sum_{g \in G_k} \inf\{\rho_\epsilon(w) : w = g(z), z \in U\} |g'(z)|^2.$$ 

For any $g$, the ratio of the values of each summand at two points $z_1, z_2$ is equal to the ratio of $|g'|^2$ at these points, hence bounded above by some $Q_0 \geq 1$ which depends solely on the distortion of inverse branches and therefore only on $U$. Consequently,

(3) \[ \frac{\rho_{\epsilon,k}(z_1)}{\rho_{\epsilon,k}(z_2)} \leq Q_0 \]

for every $z_1, z_2 \in U$. Consider a similarly constructed

$$\hat{\rho}_\epsilon(z) = \sum_{g \in G_k} \rho_\epsilon(g(z)) |g'(z)|^2.$$ 

By the change of variable formula

$$\int_U (\hat{\rho}_\epsilon(z) \, d\lambda(z) = \int_{F^{-k}(U)} \rho_\epsilon(w) \, d\lambda(w) \geq \int_{F^{-k}(U)} \min(\rho(w), \rho_\epsilon(w)) \, d\lambda(w) =$$

$$= \int_{\mathbb{C}} \min(\rho(w), \rho_\epsilon(w)) \, d\lambda(w) - \int_{F^{-k}(U)^c} \min(\rho(w), \rho_\epsilon(w)) \, d\lambda(w) \geq$$

(4) \[ \geq 1 - \eta/10 - \nu(F^{-k}(U)^c) = 1 - \eta/10 - (1 - \eta) = \frac{9}{10} \eta \]

where we have also used condition (2). Clearly, $\rho_{\epsilon,k} \leq \hat{\rho}_\epsilon$. Let $\delta_\epsilon$ denote the modulus of continuity of $\rho_\epsilon$.

Then

$$\int_U (\hat{\rho}_\epsilon(z) - \rho_{\epsilon,k}(z)) \, d\lambda(z) \leq \delta_\epsilon(d_k) \int_U \sum_{g \in G_k} |g'(z)|^2 \, d\lambda(z).$$
Here $G_1^e$ denoted the set of only those inverse branches which map onto some $V \in \Omega_k$. By bounded distortion, if $g$ maps on $V$, then for any $z \in U$, $|g'(z)|^2 \leq Q_0 \frac{\lambda(V)}{\lambda(U)}$. Hence, we can further estimate

$$\int_U (\hat{\rho}_e(z) - \rho_{e,k}(z)) \, d\lambda(z) \leq \delta_e(d_k) \lambda(U)^{-1} \sum_{V \in \Omega_k} \lambda(V).$$

Since all $V \in \Omega_k$ must touch the compact support of $\rho_e$ and their diameters tend uniformly to 0 with $k$, their joint area remains bounded depending solely on $U, \epsilon$. Since also $d_k$ tend to 0 with $k$, for all $k \geq k(\epsilon)$,

$$\int_U (\hat{\rho}_e(z) - \rho_{e,k}(z)) \, d\lambda(z) \leq \frac{2^5}{5} \eta.\quad (5)$$

Taking into account estimate (4), for $k \geq k(\epsilon)$, $\int_U \rho_{e,k}(z) \, d\lambda(z) \geq \eta/2$. Based on estimate (3), we conclude that for all $k \geq k(\epsilon)$,

$$\rho_{e,k}(z) \geq Q_1 > 0$$

for all $z \in U$ and $Q_1$ which only depends on $U$. Next, we estimate

$$\int_U (\rho_{e,k}(z) - \rho(z))_+ \, d\lambda(z) \leq \int_U (\hat{\rho}_e(z) - \rho(z))_+ \, d\lambda(z) = \int_C (\rho_e(w) - \rho(w))_+ \, d\lambda(w) < \epsilon$$

where we used a change of variables formula and condition (1). For every $\epsilon > 0$ and $k \geq k(\epsilon)$, we conclude from this and estimate (5) that $\rho(z) < \frac{\eta}{Q_1}$ on a set $\lambda$-measure less than $\frac{2^5}{Q_1}$. Since $\epsilon$ can be made arbitrarily small while $Q_1$ is fixed, then $\rho(z) \geq \frac{\eta}{2}$ on a set of full $\lambda$-measure in $U$. \hfill \Box

2.1. Return times. Introduce the following function $g: \mathbb{R} \to \mathbb{R}$: $g(x) = |x|/\sqrt{e^2}$. 

**Lemma 3.** There exists $N_0$ such that for all $n \geq N_0$, there exist sets $W_+, W_- \subset A_n$ which consist of fundamental rectangles each of which is mapped by $f$ onto some $A_m \subset \{ z \in \mathbb{C} : |z| \geq g(|A|e^n) \}$ in the case of $W_+$, $A_m \subset \{ z \in \mathbb{C} : |z| \geq g(2|A|e^n) \}$ for $W_-$ and such that

$$\lambda(W_+) > \frac{1}{4} \lambda(A_n).$$

**Proof.** For an annulus centered at 0 with inner radius $r$, 1/3 of its area belongs to the half-plane $\Re z > r/2$ and another 1/3 to $\Re z < -r/2$. For $A_n$ with $n$ large enough, almost the entire area, certainly more than 1/4 of the area of the whole annulus, of $A_n \cap \{ z : \Re z > |A| \exp n \}$ can be filled with fundamental rectangles. This defines $W_+$. The set $W_-$ is constructed in the same way. \hfill \Box

The following lemma generalizes Lemma 3.

**Lemma 4.** There are constants $N_1$ and $K_0 > 1$ such that for all $n \geq N_1$ and any integer $p \geq 1$, there is a set $W_p \subset A_n$ such that:

- $W_p$ is the union of sets each of which is mapped by $f^{p-1}$ univalently onto a fundamental rectangle,
- for every $z \in W_p$ and $0 < j < p$, $f^j(z) \in A_m$, with $m \geq n$, while $f^p(z) \in A_m$ with $m \geq g^p(|A|e^n)$,
- $\lambda(W_p) \geq K_0^{-p}$.

**Proof.** Choose $N_1$ at least as large as $N_0$ in Lemma 3 and so large that $g(|A|e^{N_1}) \geq |A|e^{N_1}$. Additionally, the orbit of 0 must fit inside $D(0, |A| \exp (N_1 - 1))$. For $p = 1$ the claim follows from Lemma 3. Assuming now the claim for some $p \geq 1$, we can first split the set $W_p$ into $W_p^m$, $A_m \subset \{ z \in \mathbb{C} : |z| \geq g^p(|A|e^n) \}$, defined by $W_p^m = \{ z \in W_p : f^p(z) \in A_m \}$. The set $W_p^m$ splits into the union of topological disks each of which is initially mapped by $f^{p-1}$ univalently onto a fundamental rectangle and then by $f$. Since by our choice of $N_1$ the post-singular set is far away surrounded by $A_{N_1 - 1}$, the first map has distortion bounded independently of $m, p$ and the distortion of $f$ satisfies the explicit bound of $\epsilon$. Let $Q > 1$ bound the ratio of the squares of the derivatives for any branch of $f^{-p}$ from $A_m$ into $W_p^m$ at any two points of $A_m$. Since $m \geq N_1 \geq N_0$ by our choice of $N_1$,
inside $A_m$ we can find $W$ given by Lemma 3 and then define $W_m^{p+1} := W_m \cap f^{-p}(W)$. As a consequence of the bounded distortion of $f^p$ and Lemma 4, \( \frac{\lambda(W_m^{p+1})}{\lambda(W_m^p)} \geq (4Q)^{-1} \). Now set $W_{p+1} = \bigcup_{m \geq g^p(n)} W_m^{p+1}$. Then \( \lambda(W_{p+1}) \geq (4Q)^{-1} \lambda(W_p) \) so with $K_0 = 4Q$ the last claim of Lemma 3 will persist under induction. The remaining claims follow immediately from Lemma 3 and the construction of $W_{p+1}$. \( \square \)

**Proposition 1.** There exist constants $N_2$ and $K_0, K_1 > 1$ such that for each $n \geq N_2$ and $p \geq 1$, $A_n$ contains a subset $V_p$, such that $V_p$ are pairwise disjoint for different $p$ and for every $z \in V_p$, \( |f^i(z)| \geq |\Lambda|e^n \) for $i = 0, \ldots, p$ while \(|f^{p+1}(z)| \leq g(-g^p(|\Lambda|e^n)) \). Additionally, for each $p$, \( \lambda(V_p) \geq K_1^{-1}K_0^{-p}\lambda(A_n) \).

**Proof of the Proposition.** We choose $N_2$ at least equal to $N_1$ from Lemma 4 such that $g(|\Lambda|e^n) \geq |\Lambda|e^n$ if $n \geq N_2$ and so big that the orbit 0 fits inside $D(0, |\Lambda|e^{N_2-1})$ and at least 1. By the last choice, the pairwise disjointness of sets $V_p$ will follow automatically from the conditions on orbits from $V_p$. Consider first the set $W_p$ obtained from Lemma 1. It consists of sets $U_j$ which are univalent preimages of fundamental rectangles, each of which is mapped with bounded distortion onto $A_m \subset \{ z \in \mathbb{C} : |z| \geq g^p(|\Lambda|e^n) \}$. Thus, a portion of $U_j$ of area at least $K_1^{-1}1(\lambda(U_j)$ with $K_1$ a constant, is occupied by the preimage by $f^p$ of the set $W_\lambda$ from Lemma 3. It is immediate that every $z$ from this preimage satisfies the demands of Proposition 1. $V_p$ is the union of such preimages for all $U_j$ and hence its measure is bounded below as claimed in the Proposition.

**Proof of Theorem**

**Lemma 5.** For all $x \geq N_3$ for some $N_3$ and every $\gamma > 0$, $\lim_{p \to \infty} g^p(x)\gamma^{-p} = +\infty$.

**Proof.** Evidently, $g(x)/x$ tends to $\infty$, so pick $N_3$ so that for all $x \geq N_3$, $g(x) \geq 2x$. Then $g^p(x) \geq 2^p x$ for all $p \geq 1$, in particular $g^p(x) - g^{p-1}(x) \to +\infty$. But $\frac{g^{p+1}(x)}{g^p(x)} = \exp(g^p(x) - g^{p-1}(x))$. \( \square \)

Consider a slit annulus $A_n$ for $n$ at least equal to the constant $N_2$ of Proposition 1 and $|\Lambda|e^n \geq N_3$ of Lemma 5. Let $\tau(z)$ for $z \in A_n$ be the first return time to $A_n$. Note that $\mu$-almost every point returns since open sets return and $\mu$ is ergodic. Clearly $\tau$ is $\mu$-integrable, but then also $\lambda$-integrable in view of Lemma 1. Similarly, $\lambda$-almost every point returns. If $z \in D(0, r)$ then it takes at least $k \geq K_2 \log r^{-1}$ for $f^k(z)$ to get in the distance at least 1 away from the orbit of 0. $K_2$ is a positive constant which depends on the maximum modulus of the derivative of $f$ on some compact set. It follows that on each set $V_p$ from Proposition 1 the return time is at least $K_2(\log |\Lambda| + g^p(|\Lambda|e^n))$. Since the measure of $V_p$ is only exponentially small with $p$, by Lemma 5 the return time is not $\lambda$-integrable which gives us the final contradiction.

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