Percolation on the product graph of a regular tree and a line does not satisfy the triangle condition at the uniqueness threshold

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Abstract

We consider Bernoulli bond percolation on the product graph of a regular tree and a line. We show that the triangle condition does not hold at the uniqueness threshold.

1 Introduction

Let $G = (V, E)$ be a connected, quasi-transitive and infinite graph, where $V$ is the set of vertices, $E$ is the set of edges. In Bernoulli bond percolation, each edge will be open with probability $p$, and closed with probability $1 - p$ independently, where $p \in [0, 1]$ is a fixed parameter. Let $\Omega = \{0, 1\}^E$ be the set of samples, where $\omega(e) = 1$ means $e$ is open. Each $\omega \in \Omega$ is regarded as a subgraph of $G$ consisting of all open edges. The connected components of $\omega$ are referred to as clusters. Let $p_c = p_c(G)$ be the critical probability for Bernoulli bond percolation on $G$, that is,

$$p_c = \inf \{ p \in [0, 1] \mid \text{there exists an infinite cluster almost surely} \},$$

and let $p_u = p_u(G)$ be the uniqueness threshold for Bernoulli bond percolation on $G$, that is,

$$p_u = \inf \{ p \in [0, 1] \mid \text{there exists a unique infinite cluster almost surely} \}.$$

For $p \in [0, 1]$ and $x, y \in V$, let $\tau_p(x, y)$ be the probability that $x$ and $y$ are connected in $\omega$, that is, $x$ and $y$ belong to the same cluster. Let $\chi_p(v)$ be the expected volume of the cluster containing $v$ which is defined by

$$\chi_p(v) = \sum_{x \in V} \tau_p(v, x).$$

This expected volume is a monotone increasing function of $p$, and diverges at $p_c$. Aizenman and Newman [11] introduced the triangle condition. They analyzed the critical behavior of $\chi_p(v)$ if $G = \mathbb{Z}^d$ and the triangle condition holds at $p_c$. Let $\nabla_p(v)$ be the triangle diagram which is defined by

$$\nabla_p(v) = \sum_{x, y \in V} \tau_p(v, x)\tau_p(x, y)\tau_p(y, v).$$

We say that $G$ satisfies the triangle condition at $p$ if $\nabla_p(v) < \infty$ for every $v$. When $G = \mathbb{Z}^d$, Hara and Slade [3] showed that the triangle condition holds at $p_c$ for all $d \geq 19$. This result was improved by Fitzner and van der Hofstad [2] for $d \geq 11$. It is known $p_c = p_u$ if $G = \mathbb{Z}^d$. Then above result also says that the triangle condition holds at $p_u$. When $G$ is the product graphs, Kozma [6] showed
that the product graph of two $d$-regular trees $T_d \Box T_d$ for $d \geq 3$ holds the triangle condition at $p_c$. In 2017, Hutchcroft [5] showed more general cases, $G$ is the product graph of finitely many regular trees $T_{d_1} \Box T_{d_2} \Box \cdots \Box T_{d_N}$ for $d_i \geq 3$. Hutchcroft [4] also showed that the triangle condition holds at $p_c$ if $G$ is nonunimodular. Furthermore, Hutchcroft showed that $p_c < p_u$ holds if $G$ is nonunimodular. Then we do not have the result as to whether the triangle condition holds at $p_u$.

A nonunimodular class contains $T_d \Box \mathbb{Z}$ for $d \geq 3$. Therefore, we consider percolation on $T_d \Box \mathbb{Z}$ and focus on the triangle condition holds or does not. This graph is a vertex transitive graph. Then we only consider $v = o$ where $o$ is a fixed origin. Our main result is the following theorem.

**Theorem 1.1.** Let $G = T_d \Box \mathbb{Z}$ for $d \geq 3$. Then we have

$$\nabla_p(o) \begin{cases} < \infty & (p < p_u) \\ = \infty & (p = p_u) \end{cases}$$

To lead this result, we use a certain function $\alpha(p)$ which is defined by

$$\alpha(p) = \alpha_d(p) = \lim_{n \to \infty} \tau_p(o, (v_n, 0))^{\frac{1}{n}},$$

where $v_n$ is a vertex on $T_d$ with $n$ distance from the origin. From a homogeneity of $T_d$, $\alpha(p)$ does not depend on a choice of $v_n$. We abbreviate $v_n$ as $n$. We check on the existence of a limit. From FKG inequality, we have

$$\tau_p(o, (n + l, 0)) \geq \tau_p(o, (n, 0))\tau_p(o, (l, 0))$$

for all $n, l \geq 0$. By using Fekete’s subadditive lemma, the existence of the limit is ensured, and we have

$$\alpha(p) = \lim_{n \to \infty} \tau_p(o, (n, 0))^{\frac{1}{n}} = \sup_{n \geq 1} \tau_p(o, (n, 0))^{\frac{1}{n}}.$$

This function was introduced by Schonmann [7], who showed the following inequality.

(1.1) \hspace{1cm} \alpha(p_u) \leq \frac{1}{\sqrt{b}}

where $b = d - 1$. By using this inequality, Schonmann showed that there exists a.s. no unique infinite cluster at $p_u$. We will show that the equality is established, that is,

(1.2) \hspace{1cm} \alpha(p_u) = \frac{1}{\sqrt{b}}.

We introduce an example of the triangle condition, let $G = T_d$, it is easy to check that

$$\nabla_p(v) \begin{cases} < \infty & (p < \frac{1}{\sqrt{b}}) \\ = \infty & (p = \frac{1}{\sqrt{b}}) \end{cases}.$$
2 Proof

We define the level difference function $L(x, y)$ from $T_d \times T_d$ to $\mathbb{Z}$. Let $\xi$ be a fixed end of $T_d$. The parent of a vertex $x \in T_d$ is the unique neighbor of $x$ that is closer to $\xi$ than $x$ is. We call the other vertices of $x$ its children. If $y$ is parent of $x$, then we define $L(x, y) = 1$. If $y$ is child of $x$, then we define $L(x, y) = -1$. In general cases, for any $x, y$, there exists an unique geodesic $\{x_i\}_{i=0}^{n}$ such that $x_0 = x$ and $x_n = y$, then we define

$$L(x, y) = \sum_{i=1}^{n} L(x_{i-1}, x_i).$$

Note that $L(x, z) = L(x, y) + L(y, z)$ and $L(y, x) = -L(x, y)$ for any $x, y, z \in T_d$. This function is extended to $T_d \square \mathbb{Z}$ naturally. Let $\pi$ be a natural projection from $T_d \square \mathbb{Z}$ to $T_d$. Then we extend $L(x, y)$ as $L(x, y) = L(\pi(x), \pi(y))$. Similarly, we have $L(x, z) = L(x, y) + L(y, z)$ and $L(y, x) = -L(x, y)$ for any $x, y, z \in T_d \square \mathbb{Z}$. We define $\Delta(x, y)$ by

$$\Delta(x, y) = b^{L(x, y)}$$

for all $x, y \in T_d \square \mathbb{Z}$ where $b = d - 1$. Note that $\Delta(x, z) = \Delta(x, y)\Delta(y, z)$ and $\Delta(y, x) = \Delta(x, y)^{-1}$ for any $x, y, z$. We define the tilted susceptibility by

$$\chi_{p, 1/2}(o) = \sum_{x \in T_d \square \mathbb{Z}} \tau_p(o, x)\Delta(o, x)^{1/2}.$$  

Our method is based on [4], if you would like to know more detail of the tilted susceptibility, then please refer to [4]. Hutchcroft showed the following inequality.

$$\nabla_p(o) \leq (\chi_{p, 1/2}(o))^{3}.$$  

Therefore we will show that $\chi_{p, 1/2}(o) < \infty$ for $p < p_u$ to prove the first half of Theorem 1.1. Similar to $\alpha(p)$, the function $\beta(p)$ is defined by

$$\beta(p) = \lim_{m \to \infty} \tau_p(o \leftrightarrow (0, m))^{1/m} = \sup_{m \geq 1} \tau_p(o \leftrightarrow (0, m))^{1/m}.$$  

By FKG inequality and the homogeneity of $T_d \square \mathbb{Z}$, we have

$$\tau_p(o, (n, m)) \leq \alpha(p)^n, \quad \tau_p(o, (n, m)) \leq \beta(p)^{|m|}$$

for each $(n, m)$. For $x \in T_d$, we define $I_x(p)$ by

$$I_x(p) = \sum_{m \in \mathbb{Z}} \tau_p(o, (x, m)).$$

Lemma 2.1 ([8]). If $\alpha(p) < 1/\sqrt{b}$, then we have $\beta(p) < 1$.

By this lemma, the function $I_x(p)$ is well-defined for $p < p_u$.

Lemma 2.2 ([8]). For any $p$ such that $\alpha(p) < 1/\sqrt{b}$, we have

$$\lim_{|x| \to \infty} I_x(p)^{1/|x|} = \alpha(p).$$
By this lemma, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that
\[ I_x(p) \leq (\alpha(p) + \epsilon)|x| \]
for any $x \in T_d$ such that $|x| \geq N$. For any $p < p_u$, we choose $\epsilon$ such that $\alpha(p) + \epsilon < 1/\sqrt{b}$. Then we have
\[
\chi_{p,1/2}(o) = \sum_{x \in T_d} I_x(p) \Delta(o,x)^{1/2} \\
\leq \sum_{|x|<N} I_x(p) \Delta(o,x)^{1/2} + \sum_{|x|\geq N} (\alpha(p) + \epsilon)|x| \Delta(o,x)^{1/2}.
\]
For $r > 0$ and $z \in \mathbb{R}_{>0}$, we define the function $J(r,z)$ by
\[ J(r,z) = \sum_{x \in T_d} r^{|x|} z^{L(o,x)}. \]

**Lemma 2.3** ([5]). For any $r < 1/\sqrt{b}$ and $z \in (br, 1/r)$, we have $J(r,z) < \infty$.

**Remark 2.4.** In [5], using level function based on the origin $o$, it equal to $-L(o,x)$. Then $z$ appeared in [5] means $z^{-1}$ in this paper.

By this lemma, let $r = \alpha(p) + \epsilon$ and $z = \sqrt{b}$. Then we have
\[
\sum_{|x|\geq N} (\alpha(p) + \epsilon)|x| \Delta(o,x)^{1/2} \leq J(r,z) < \infty.
\]
Therefore, we have $\nabla_p(o) < \infty$ for all $p < p_u$. If $p > p_u$, then there exists a constant $C(p) > 0$ such that $\tau_p(x,y) \geq C(p)$ for all $x,y$. Hence, we have $\chi_{p,1/2}(o) = \infty$ for $p > p_u$. Hutchcroft [4] showed that the set $\{p \in [0,1] : \chi_{p,1/2}(o) < \infty\}$ is open in $[0,1]$. Then we have $\chi_{p_u,1/2}(o) = \infty$. That means $\alpha(p)$ must equal to $1/\sqrt{b}$. Then we have the equation [12]. By using this result, we will show that $\nabla_{p_u}(o) = \infty$. Similar to $I_x(p)$, we define the function $\mathcal{I}_x(p)$ by
\[ \mathcal{I}_x(p) = \sum_{m \in \mathbb{Z}} \tau_p(o,x)^2. \]
From a homogeneity of $T_d \square \mathbb{Z}$, we have $\mathcal{I}_x(p) = \mathcal{I}_y(p)$ for any $x, y$ such that $|x| = |y|$. For $|x| = n$, we denote $\mathcal{I}_x(p)$ as $\mathcal{I}_n(p)$. By using BK inequality, we have
\[ \mathcal{I}_{n+1}(p) \leq \sum_{m \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \tau_p(o,(n,k))^2 \tau_p(o,(l,m-k))^2 = \mathcal{I}_n(p) \mathcal{I}_l(p) \]
for all $n, l \geq 1$. Then by using Fekete’s subadditive lemma, we have
\[
\inf_{n \geq 1} \mathcal{I}_n(p)^{1/n} = \lim_{n \to \infty} \mathcal{I}_n(p)^{1/n}.
\]
Since $\mathcal{I}_n(p) \geq \tau_p(o,(n,0))^2$, we have
\[ \mathcal{I}_n(p)^{1/n} \geq \tau_p(o,(n,0))^2. \]
By taking the limit, we have
\[ \lim_{n \to \infty} \mathcal{I}_n(p)^{1/n} \geq \alpha(p)^2. \]
By above equation and inequality, we obtain
\begin{equation}
\mathbb{I}_n(p) \geq \alpha(p)^{2n}
\end{equation}
for all $n \geq 0$. From FKG inequality, we have
\begin{equation}
\nabla_p(o) \geq \sum_{x \neq y} \tau_p(o, x) \tau_p(x, o) \tau_p(y, o) \tau_p(y, o) = \left( \sum_x \tau_p(o, x)^2 \right)^2.
\end{equation}
If $\nabla_{p_u}(o) < \infty$, then also $\sum \tau_{p_u}(o, x)^2 < \infty$. Hence $\mathbb{I}_x(p_u)$ is well-defined. On the other hand, by inequality (2.2) and equation (1.2) we have
\begin{equation}
\nabla_{p_u}(o) = \sum_{x \in T_d} \mathbb{I}_x(p_u) \geq \left( \sum_{x \in T_d} \alpha(p_u)^{2|\mathfrak{F}|} \right)^2 \geq \left( \sum_{n \geq 1} b^n \cdot \frac{1}{b^n} \right)^2 = \infty
\end{equation}
Therefore, we have a contradiction. That means $\nabla_{p_u}(o) = \infty$. It ends the proof of (1.1).

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