On a mean field approximation for Higgs-Yukawa systems

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We discuss the phase structure of a lattice Higgs-Yukawa system in the variational mean field approximation with contributions of fermionic determinant being calculated in a ladder approximation. In particular, we demonstrate that in this approximation the ferrimagnetic phase in the $Z_2$ model with naive fermions can appear as an artifact of a finite lattice and that the phase diagram for this model on infinite lattice changes qualitatively at space-time dimension $D = 4$ compared with those at $D > 4$.

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1. Introduction

Although mean field method for lattice systems including fermions loses considerably its simplicity and requires further approximations it is still useful to get some idea of the phase structure of the systems and to orientate Monte Carlo simulations towards investigating the most interesting points. In this paper we make an improvement in the approximations within the variational mean field approximation for $Z_2$ Higgs-Yukawa systems by summing up a ladder type contributions to fermionic determinant, including those of the next order in inverse space-time dimension $1/D$. This enables us to observe two new points. As the first one we demonstrate that within our approximation the ferrimagnetic phase in the simplest Higgs-Yukawa model with naive fermions can arise as a finite lattice artifact. The second point is that the value $D = 4$ turns out in a sense to be critical, as the domain of paramagnetic phase just at $D = 4$ becomes disconnected, being connected at $D > 4$.

The paper is organized as follows. The system under consideration is defined in Sect. 2. In Sect. 3 we describe the method and approximations. Results are discussed in Sect. 4.

2. The model

The system is defined on a hyper cubic $D$-dimensional ($D$ is even) lattice $\Lambda$ with sites numbered by $n = (n_1, ..., n_D)$, $-N/2 + 1 \leq n_\mu \leq N/2$ ($N$ is even) and with lattice spacing $a = 1$; $\hat{\mu}$ is the unit vector along the lattice link in the positive $\mu$-direction. Dynamical variables of the model are the fermion $2D/2$-component fields $\psi_n, \bar{\psi}_n$, and scalar field $\phi_n \in Z_2$ (i.e. $\phi_n = \pm 1$). We imply antiperiodic boundary conditions for the fermion and periodic for the scalar fields.

The model is defined by functional integral

$$Z[J] = \sum_{\phi_n \in Z_2} \int \prod_{n \in \Lambda} d\psi_n d\bar{\psi}_n e^{-A[\phi, \psi, \bar{\psi}]} + \sum_n J_n \phi_n$$

(2.1)

with the action

$$A[\phi, \psi, \bar{\psi}] = -2\kappa \sum_{n, \mu} \phi_n \phi_{n+\hat{\mu}} + \sum_{m, n} \bar{\psi}_m (\bar{\phi}_{mn} + y \phi_m \delta_{mn}) \psi_n,$$

(2.2)

where

$$\bar{\phi}_{mn} = \sum_\mu \gamma_\mu \frac{1}{2} (\delta_{m+\hat{\mu}, n} - \delta_{m-\hat{\mu}, n}) = N^{-D} \sum_{p, \mu} e^{ip(m - n)} i \gamma_\mu L_\mu(p),$$

(2.3)
\[ \kappa \in (-\infty, \infty) \] is the hopping parameter, \( y \geq 0 \) is the Yukawa coupling; we use the Hermitean \( \gamma \)-matrices: \( [\gamma_\mu, \gamma_\nu]_+ = 2\delta_{\mu\nu} \); \( L_\mu(p) = \sin p_\mu \), \( p_\mu = \frac{2\pi}{N}(k_\mu - 1/2), -N/2 + 1 \leq k_\mu \leq N/2 \), so that \( p_\mu \in (-\pi/2, \pi/2) \). Operator \( \partial \) satisfies

\[ \partial_{mn} = -\partial_{nm}. \] (2.4)

In the limit of \( N \to \infty \) the sum \( N^{-D} \sum_p \) defines the integral \( \int_{-\pi/2}^{\pi/2} d^D p / (2\pi)^D \).

The action (2.2) is invariant under \( \mathbb{Z}_2 \) global chiral transformations

\[ \phi_n \to -\phi_n, \quad \psi_n \to (-P_L + P_R)\psi_n, \quad \overline{\psi}_n \to \overline{\psi}_n(-P_R + P_L), \] (2.5)

where \( P_{L,R} = (1 \pm \gamma_{D+1})/2 \) are chiral projecting operators.

### 3. The method and approximations

To analyze the phase structure of the model we use the variational mean field approximation [4] (see also [5]) which becomes applicable to (2.1) after integrating out the fermions

\[ Z[J] = e^{-W[J]} = \sum_{\phi_n \in \mathbb{Z}_2} e^{2\kappa \sum_{n,\mu} \phi_\nu \phi_{n+\bar{\mu}} + \ln \det [\phi + y\phi] + \sum_n J_n \phi_n}. \] (3.1)

Then for free energy of the system \( F = W[0] \) the method yields the inequality

\[ F \leq F_{MF} = \inf_{h_n} \left[ -\sum_n (u(h_n) - h_n u'(h_n)) - 2\kappa \sum_{n,\mu} \phi_\nu \phi_{n+\bar{\mu}} + \ln \det [\phi + y\phi] \right], \] (3.2)

where \( h_n \) is a mean field, and

\[ u(h_n) = \ln \sum_{\phi_n \in \mathbb{Z}_2} e^{h_n \phi_n} = \ln 2 \cosh h_n, \] (3.3)

\[ \langle O[\phi] \rangle_h = e^{-\sum_n u(h_n)} \sum_{\phi_n \in \mathbb{Z}_2} O[\phi] e^{\sum_n h_n \phi_n}. \]

So, we can get some idea of the system, studying \( F_{MF} \), that is much simpler than that for \( F \). From (3.3) it immediately follows that

\[ \langle \phi_n \rangle = u'(h_n) = \tanh h_n, \]

\[ \langle \phi_m \phi_n \rangle_h = u'(h_m)u'(h_n) + \delta_{mn} u''(h_m), \quad \text{etc.} \] (3.4)
and therefore the main problem is a calculation of the expectation value of the fermionic determinant

\[
\langle \ln \det [\partial + y\phi] \rangle_h = \ln \det [\partial] - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} y^n \sum_{i_1, \ldots, i_n} \text{tr} \left( (\partial^{-1}_{i_1 i_2} \partial^{-1}_{i_2 i_3} \cdots \partial^{-1}_{i_n i_1}) \langle \phi_{i_1} \phi_{i_2} \cdots \phi_{i_n} \rangle_h \right)
\]

where \( \text{tr} \) stands for the trace over spinorial indices; in the first term of the second equation the relation \( \phi_i^{2D/2} = 1 \) has been taken into account.

Following the usual way we consider \( F_{MF} \) for two translation invariant ansätze for \( h_n \)

\[
h_n^{FM} = h, \quad h_n^{AF} = \epsilon_n h, \quad \epsilon_n = (-1)\sum_{\mu} n_{\mu}.
\]

which in fact are the order parameters distinguishing the ferromagnetic (FM: \( h_n^{FM} \neq 0, h_n^{AF} = 0 \)), antiferromagnetic (AF: \( h_n^{FM} = 0, h_n^{AF} \neq 0 \)), paramagnetic (PM: both are zero), and ferrimagnetic (FI: both are nonzero) phases in the system. Then the mean field equations are reduced to

\[
\frac{\partial}{\partial h} F_{MF, AF}^{FM, AF} = 0,
\]

where \( F_{MF, AF}^{FM, AF} \) is the functional of the right-hand side of Eq.(3.2) on ansätze (3.6). Further simplification comes from the observation (see, for example [2]), that as the value \( h = 0 \) is always a solution of Eq. (3.7), and, therefore, second order phase transition lines are determined by equations

\[
\frac{\partial^2}{\partial h^2} F_{MF, AF}^{FM, AF} \big|_{h=0} = 0,
\]

to find them it is sufficient to know \( \langle \ln \det [\partial + y\phi] \rangle_h \) to terms of order of \( h^2 \).

If the problem could be solved exactly both of two representations (3.5) of the fermionic determinant would yield the same answer. But correlations of \( \phi \)'s at coinciding arguments (Eq.(3.4)) make the problem unsolvable exactly, as the contributions of order of \( h^2 \) to (3.5) come from terms of any orders of \( u'' \), as well as from those of order of \( u'^2 \). These contributions shown schematically in Fig. 1. Therefore, we are forced to use some approximations, and, particularly, to use two representations of (3.5) separately for “weak” and “strong” coupling regimes of \( y \), though the exact meaning of this can only be clear a posteriori.
Our approximation involves summing up all diagrams of Fig.1 (a) (proper ladder diagrams) and (b) (crossed ladder diagrams), so we may call it as a ladder approximation. Using property (2.4) of the Dirac operator we find that the contributions to \( F_{MF}^{FM,AF} \) from the fermionic determinant, \( \Delta F_{MF}^{FM,AF} \), have the same functional form for both representations (3.5) and in our approximation read as follows

\[
\Delta F_{MF}^{FM} = N^D 2^{D/2-1} \left( \frac{c^2 u^2 G(0)}{1 + c^2 u'' G(0)} + N^{-D} \sum_q \frac{c^2 u'' G(q)}{1 + c^2 u'' G(q)} \right),
\]

\[
\Delta F_{MF}^{AM} = N^D 2^{D/2-1} \left( \frac{c^2 u^2 G(\pi)}{1 + c^2 u'' G(\pi)} + N^{-D} \sum_q \frac{c^2 u'' G(q)}{1 + c^2 u'' G(q)} \right),
\]

(3.9)

where \( q_\mu = (2\pi/N) l_\mu, -N/2 + 1 \leq l_\mu \leq N/2 \) (so that \( q_\mu \in (-\pi, \pi] \)), while coupling \( c \) and form of function \( G \) depend on the representation. So, for weak coupling regime we have \( c = y \) and

\[
G^W(q) = N^{-D} \sum_p \frac{L(p)L(p+q)}{L^2(p)L^2(p+q)} = N^{-D} \sum_p \frac{\sum_\mu \sin(p)\mu \sin(p+q)\mu}{\sum_\mu \sin^2 p_\mu \sum_\nu \sin^2 (p+q)\nu},
\]

and for the strong coupling they are \( c = y^{-1} \) and

\[
G^S(q) = \frac{1}{N^D} \sum_p L(p)L(p+q) = N^{-D} \sum_{p,\mu} \sin p_\mu \sin (p+q)_\mu.
\]

(3.10)

(3.11)

The first terms in (3.9) come from the diagrams of Fig.1(a), while the second from those of Fig.1(b).

Then, from Eq. (3.8) and the above formulae it follows that critical lines in the system in our approximation are determined by the expressions

\[
\kappa_{cr}^{F(W)} = \frac{1}{4D} \left[ 1 - 2^{D/2} y^2 \left( \frac{G^W(0)}{1 + y^2 G^W(0)} - N^{-D} \sum_q \frac{G^W(q)}{(1 + y^2 G^W(q))^2} \right) \right],
\]

\[
\kappa_{cr}^{AF(W)} = -\frac{1}{4D} \left[ 1 - 2^{D/2} y^2 \left( \frac{G^W(\pi)}{1 + y^2 G^W(\pi)} - N^{-D} \sum_q \frac{G^W(q)}{(1 + y^2 G^W(q))^2} \right) \right];
\]

\[
\kappa_{cr}^{F(S)} = \frac{1}{4D} \left[ 1 - 2^{D/2} \left( \frac{G^S(0)}{y^2 + G^S(0)} - N^{-D} \sum_q \frac{G^S(q)}{(y^2 + G^S(q))^2} \right) \right],
\]

\[
\kappa_{cr}^{AF(S)} = -\frac{1}{4D} \left[ 1 - 2^{D/2} \left( \frac{G^S(\pi)}{y^2 + G^S(\pi)} - N^{-D} \sum_q \frac{G^S(q)}{(y^2 + G^S(q))^2} \right) \right].
\]

(3.12)

We now should make some comments.
(i) The contributions to (3.12) which are proportional to $G(0)$ and $G(\pi)$ are generalization of "double chain" contributions of Ref. [2], as the diagrams of Fig.1(a) are the generalization of the double chains to any configurations of the same topology. They coincide only for $G^S$ because of strict locality of the Dirac operator, but not for $G^W$. More important difference comes from the second terms corresponding to the diagrams of Fig.1(b) (the latter correspond to the generalization of the double chains with coinciding ends), which have not been taken into account in previous calculations (see also [3]). From the well known symmetry of the model under the transformations: $(\psi, \overline{\psi})_n \to \exp(i\epsilon_n \pi/4)(\psi, \overline{\psi})_n$, $\phi_n \to \epsilon_n \phi_n$, $\kappa \to -\kappa$, $y \to -iy$, it follows that $G(\pi) = -G(0)$, and also, that the contributions of the new terms are of even power in $\pm 2$ beginning from $\pm 4$.

(ii) These terms can become dominating when $y^2$ is close to the values $1/G^W(0)$ or $G^S(0)$ which are singular points of the expressions under the sum, even though in weak coupling regime they are of $O(D^{-1})$ compared with the first ones. Thereby these terms determine domains of the "weak" and "strong" coupling regimes also for $\kappa_{cr}^F$. They are domains of analyticity of functions $\kappa_{cr}^W(y)$ and $\kappa_{cr}^S(y)$, that is $y^2 < 1/G^W(0)$ and $y^2 > G^S(0)$, respectively, coinciding for $\kappa_{cr}^F$ and $\kappa_{cr}^{AF}$.

(iii) We have no strict arguments why other diagrams which we did not take into account could be neglected compared with the ladder ones. In particular, in strong coupling regime they can give contributions to $\kappa_{cr}$ of the same order in $1/D$ as the latter. But because those diagrams come into play in higher orders in $y^\pm 1$, at least from the order of $y^\pm 6$, the assumption that their contributions are suppressed and less singular looks plausible.

Finally, it worth noting that the formulae (3.12) are applicable to any lattice fermion actions, including non-local ones, whose Dirac operators satisfy property (2.4) [4].

4. Results and discussion

Let us now compare the phase diagrams determined by the expressions (3.12) for $D = 4$ for finite $N$ and for the limiting case of $N \to \infty$. The new terms are always negative and therefore increase the contributions of the first terms for $\kappa_{cr}^F$ and decrease them for $\kappa_{cr}^{AF}$. The question is of how much.

For $N = 4$ we have $G^W(0) = 0.5$, $G^S(0) = 2$, so that the domain of inapplicability of our formulae shrinks to the point $y = 2^{1/2}$, and the phase diagram is shown in Fig. 2. The curves $\kappa_{cr}^F(y)$ and $\kappa_{cr}^{AF}(y)$ intersect each other forming narrow domain with the
ferrimagnetic phase around $y = 2^{1/2}$, which is spreaded from $-\infty$ to $\infty$ in $\kappa$. It is natural to assume, that contributions of other diagrams (Fig.1(c)) smooth the negative contribution of those of Fig. 1(b), so that the PM-AF phase transition line in Fig. 2(b) becomes continuous. Then, as a result we would have a familiar picture, typical for SU(2) models (see, for example, [3]), with FI phase lying below this line.

In the limit of $N \to \infty$ we have $G^W(0) \simeq 0.62$, $G^S(0) = 2$, but the picture is changed qualitatively. The phase diagram is shown in Fig. 3. The curves do not touch each other even at $\kappa \to -\infty$ and FI phase does not appear.

To clear up why this happens let us consider behaviour of functions of $y^2$ determined by the sums in (3.12) near the points $1/G^W(0)$ and $G^S(0)$. Let us define positive $\delta = y^2 - G^S(0)$ or $1/G^W(0) - y^2$. Then a simple analysis shows that at a finite $N$ and a small $\delta$'s these functions is of order of $O(D^k N^{-D} \delta^{-2})$, so the intersections of the curves $\kappa_{cr}^F(y)$ and $\kappa_{cr}^{AF}(y)$ always occur at the points $\delta = O(D^l N^{-D})$, $-\kappa = O(D^m 2^{D/2} N^D)$, where $k$, $l$, $m$ are some (negative or non-negative) powers. But at $N \to \infty$, when the sums go over to integrals, this functions become of order of $\ln \delta$ for $D = 4$, and even of $O(\delta^0)$ at $D > 4$. This means that at $D > 4$ we can continue lines $\kappa_{cr}^{F(W)}(y)$ and $\kappa_{cr}^{F(S)}(y)$ until they intersect each other, so that the phase diagram in this case looks like in Fig. 4, that reproduces the result of ref.[2].

Thus, this example demonstrates importance of summing up contributions to fermionic determinant including those of the next order in $1/D$ for $D = 4$ systems. Another point is that even though we cannot definitely conclude whether the FI phase in this example is an artifact only of a finite lattice or also of the mean field approximation, this gives one one more caution in what concerns finite lattice effects.

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Fig. 1. Diagrams contributed to $\Delta F$ to order $h^2$. Solid lines denote $\hat{\phi}$ or $\hat{\phi}^{-1}$, each solid circle stands for $u'$, dashed line for $u''$. 
Fig. 2. (a) Phase diagram of the model at $D = 4$, $N = 4$. Intersections of FM-PM phase transition line (solid) with PM-AF phase transition line (gray) form FI phase in the narrow region around the point $y = 2^{1/2}$ shown in (b) in more detail.
Fig. 3. Phase diagram of the model at $D = 4$, $N \to \infty$.

Fig. 4. Qualitative picture of the phase diagram of the model at $D > 4$, $N \to \infty$. 