Dark periods and revivals of entanglement in a two qubit system

Z. Ficek\textsuperscript{a} and R. Tanaś\textsuperscript{b}

\textsuperscript{a}Department of Physics, School of Physical Sciences,
The University of Queensland, Brisbane, Australia 4072

\textsuperscript{b}Nonlinear Optics Division, Institute of Physics, Adam Mickiewicz University, Poznań, Poland

(Dated: September 21, 2018)

In a recent paper Yu and Eberly [Phys. Rev. Lett. \textbf{93,} 140404 (2004)] have shown that two initially entangled and afterwards not interacting qubits can become completely disentangled in a finite time. We study transient entanglement between two qubits coupled collectively to a multimode vacuum field and find an unusual feature that the irreversible spontaneous decay can lead to a revival of the entanglement that has already been destroyed. The results show that this feature is independent of the coherent dipole-dipole interaction between the atoms but it depends critically on whether or not the collective damping is present. We show that the ability of the system to revival entanglement via spontaneous emission relies on the presence of very different timescales for the evolution of the populations of the collective states and coherence between them.

PACS numbers: 03.67.Mn, 42.50.Fx, 42.50.Lc

The problem of controlling the evolution of entanglement between atoms (or qubits) that interact with the environment has received a great deal of attention in recent years \cite{1, 2, 3, 4}. The environment may be treated as a reservoir and it is well known that the interaction of an excited atom with the reservoir leads to spontaneous emission that is one of the major sources of decoherence. In light of the experimental investigations, the spontaneous emission leads to irreversible loss of information encoded in the internal states of the system and thus is regarded as the main obstacle in practical implementations of entanglement.

This justifies the interest in finding systems where the spontaneous emission is insignificant. However, in many treatments of the entanglement creation and entanglement dynamics, the coupling of atoms to the environment is simply ignored or limited to the interaction of the atoms with a single mode cavity \cite{5, 6}.

It is well known that under certain circumstances a group of atoms can act collectively that the radiation field emitted by an atom of the group may influence the dynamics of the other atoms \cite{7, 8, 9, 10}. It is a strong dynamical influence on one another through the mutual coupling to the vacuum field \cite{11, 12}.

It has also been predicted that two initially entangled and afterwards not interacting atoms can become completely disentangled in a time much shorter than the decoherence time of spontaneous emission. This feature has been studied by Yu and Eberly \cite{13} and Jakóbczyk and Jamróz \cite{14}, who termed it as the "sudden death" of entanglement, and have elucidated many new characteristics of entanglement evolution in systems of two atoms. Their analysis, however, have concentrated exclusively on systems of independent atoms.

In this paper, we consider a situation where the atoms are coupled to the multimode vacuum field and demonstrate the occurrence of multiple dark periods and revivals of entanglement induced by the irreversible spontaneous decay. We fully incorporate collective interaction between the atoms and study in detail the dependence of the revival time on the initial state of the system and on the separation between the atoms. We emphasize that the revival of entanglement in a pure spontaneous emission process contrasts the situation of the coherent exchange of entanglement between atoms and a cavity mode \cite{15, 16}.

We consider two identical two-level atoms (qubits) having lower levels $|g_i\rangle$ and upper levels $|e_i\rangle$ separated by energy $\hbar\omega_0$, where $\omega_0$ is the transition frequency. The atoms are coupled to a multimode radiation field whose modes are initially in the vacuum state $\{\{0\}\}$. The atoms radiate spontaneously and their radiation field exerts a strong dynamical influence on one another through the vacuum field modes. The time evolution of the system is studied using the Lehmberg–Agarwal \cite{17} master equation, which reads as

$$
\frac{\partial \rho}{\partial t} = -i\omega_0 \sum_{i=1}^{2} [S_i^+, \rho] - i \sum_{i\neq j}^{2} \Omega_{ij} \left( S_i^+ S_j^-, \rho \right) - \frac{1}{2} \sum_{i,j=1}^{2} \gamma_{ij} \left( [\rho S_i^+, S_j^-] + [S_i^+, S_j^- \rho] \right),
$$

(1)

where $S_i^+$ ($S_i^-$) are the dipole raising (lowering) operators and $S_i^z$ is the energy operator of the $i$th atom, $\gamma_{ii} \equiv \gamma$ are the spontaneous decay rates of the atoms caused by their direct coupling to the vacuum field. The parameters $\gamma_{ij}$ and $\Omega_{ij}$ ($i \neq j$) depend on the distance between the atoms and describe the collective damping and the dipole-dipole interaction defined, respectively, by

$$
\gamma_{ij} = \frac{3}{2} \left( \frac{\sin kr_{ij}}{kr_{ij}} + \frac{\cos kr_{ij}}{(kr_{ij})^2} - \frac{\sin kr_{ij}}{(kr_{ij})^3} \right),
$$

(2)

and

$$
\Omega_{ij} = \frac{3}{4} \left( -\frac{\cos kr_{ij}}{kr_{ij}} + \frac{\sin kr_{ij}}{(kr_{ij})^2} + \frac{\cos kr_{ij}}{(kr_{ij})^3} \right),
$$

(3)

The occurrence of multiple dark periods and revivals of entanglement has been studied by Yu and Eberly \cite{13} and Jakóbczyk \cite{14}.
where \( k = \omega_0/c \), and \( r_{ij} = | \vec{r}_j - \vec{r}_i | \) is the distance between the atoms. Here, we assume, with no loss of generality, that the atomic dipole moments are parallel to each other and are polarized in the direction perpendicular to the interatomic axis. The effect of the collective parameters on the time evolution of the entanglement in the system is the main concern of this paper.

It will prove convenient to work in the basis of four collective states, so-called Dicke states, defined as

\[
|e\rangle = |e_1\rangle \otimes |e_2\rangle,
|g\rangle = |g_1\rangle \otimes |g_2\rangle,
|s\rangle = (|g_1\rangle \otimes |e_2\rangle + |e_1\rangle \otimes |g_2\rangle) / \sqrt{2},
|a\rangle = (|g_1\rangle \otimes |e_2\rangle - |e_1\rangle \otimes |g_2\rangle) / \sqrt{2}.
\]

(4)

In this basis, the two-atom system behaves as a single four-level system with the ground state \(|g\rangle\), two intermediate states \(|s\rangle\) and \(|a\rangle\), and the upper state \(|e\rangle\). As a result, the problem of entanglement evolution in the two qubit system can be determined in terms of populations and coherences between the collective levels.

In order to determine the amount of entanglement between the atoms and the entanglement dynamics, we use concurrence that is the widely accepted measure of entanglement. The concurrence introduced by Wootters is defined as

\[
C(t) = \max \left( 0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4} \right),
\]

(5)

where \( \{ \lambda_i \} \) are the eigenvalues of the matrix

\[
R = \tilde{\rho} \tilde{p}, \quad \text{with} \quad \tilde{\rho} = \sigma_y \otimes \sigma_y \tilde{p} \sigma_y \otimes \sigma_y,
\]

(6)

and \( \sigma_y \) is the Pauli matrix. The range of concurrence is from 0 to 1. For unentangled atoms \( C(t) = 0 \) whereas \( C(t) = 1 \) for the maximally entangled atoms.

The density matrix, which is needed to calculate \( C(t) \) is readily evaluated from the master equation \( \dot{\rho} \). Following Yu and Eberly, we choose the atoms to be at the initial time \( (t=0) \) prepared in an entangled state of the form

\[
| \Psi_0 \rangle = \sqrt{p} |e\rangle + \sqrt{1-p} |g\rangle,
\]

(7)

where \( 0 \leq p \leq 1 \). The state \( | \Psi_0 \rangle \) is a linear superposition of only those states of the system in which both or neither of the atoms is excited. As discussed in Refs. [12, 13], in the absence of the coupling between the qubits, the initial entangled state of the form \( | \Psi_0 \rangle \) disentangles in a finite time. They termed this feature as the sudden death of entanglement.

In what follows, we examine the time evolution of the entanglement of two atoms coupled to the multimode vacuum field. If the atoms are initially prepared in the state \( | \Psi_0 \rangle \), it is not difficult to verify that the initial one-photon coherences are zero, i.e. \( \rho_{ee}(0) = \rho_{ee}(0) = \rho_{gg}(0) = \rho_{ag}(0) = \rho_{aa}(0) = 0 \). Moreover, the coherences remain zero for all time, that they cannot be produced by spontaneous decay. This implies that for all times, the density matrix of the system represented in the collective basis \( \{ \Psi_i \} \), is given in the block diagonal form

\[
\rho(t) = \begin{pmatrix}
\rho_{ee}(t) & \rho_{eg}(t) & 0 & 0 \\
\rho_{eg}(t) & \rho_{gg}(t) & 0 & 0 \\
0 & 0 & \rho_{ss}(t) & 0 \\
0 & 0 & 0 & \rho_{aa}(t)
\end{pmatrix},
\]

(8)

with the density matrix elements evolving as

\[
\rho_{ee}(t) = p e^{-2\gamma t},
\rho_{eg}(t) = \sqrt{p(1-p)} e^{-\gamma t},
\rho_{ss}(t) = p e^{-\gamma t} \frac{\gamma + \gamma_{12}}{\gamma - \gamma_{12}} (e^{-\gamma_{12} t} - e^{-\gamma t}),
\]

(9)

\[
\rho_{aa}(t) = p e^{-\gamma t} \frac{\gamma + \gamma_{12}}{\gamma - \gamma_{12}} (e^{\gamma_{12} t} - e^{-\gamma t}),
\]

subject to conservation of probability \( \rho_{gg}(t) = 1 - \rho_{ss}(t) - \rho_{aa}(t) - \rho_{ee}(t) \). Note that the evolution of the density matrix elements is independent of the dipole-dipole interaction between the atoms, but it is profoundly affected by the collective damping \( \gamma_{12} \). This collective behavior leads to two distinct timescales of the evolution of the populations of the symmetric and antisymmetric states, the former much shorter and the later much longer than that for independent atoms.

Given the density matrix, Eq. (8), we can now calculate the concurrence \( C(t) \) to which we shall later refer as concurrence in the full sense, and examine the transient dynamics of the entanglement. First, we find that the square roots of the eigenvalues of the matrix \( R \) are

\[
\sqrt{\lambda_{1,2}}(t) = | \rho_{ge}(t) | \pm ( \rho_{ss}(t) + \rho_{aa}(t) )
\]

\[
\sqrt{\lambda_{3,4}}(t) = ( \rho_{ss}(t) - \rho_{aa}(t) ) \pm \sqrt{\rho_{gg}(t) \rho_{ee}(t)}
\]

(10)

from which it is easily verified that for a particular value of the matrix elements there are two possibilities for the largest eigenvalue, either \( \sqrt{\lambda_1} \) or \( \sqrt{\lambda_3} \). The concurrence is thus given by

\[
C(t) = \max \{ 0, C_1(t), C_2(t) \},
\]

(11)

with

\[
C_1(t) = 2 | \rho_{ge}(t) | - ( \rho_{ss}(t) + \rho_{aa}(t) )
\]

\[
C_2(t) = | \rho_{ss}(t) - \rho_{aa}(t) | - 2 \sqrt{\rho_{gg}(t) \rho_{ee}(t)}
\]

(12)

From this it is clear that the concurrence \( C(t) \) can always be regarded as being made up of the sum of nonnegative contributions of the weights \( C_1(t) \) and \( C_2(t) \) associated with two different classes of entangled states that can be generated in a two qubit system. From the form of the entanglement weights it is obvious that \( C_1(t) \) provides a measure of an entanglement produced by linear superpositions involving the ground \(|g\rangle\) and the upper \(|e\rangle\) states of the system, whereas \( C_2(t) \) provides a measure of an entanglement produced by a distribution of the population between the symmetric and antisymmetric states.
Inspection of Eq. (12) shows that for $C_1(t)$ to be positive it is necessary that the two-photon coherence $\rho_{eg}$ is different from zero, whereas the necessary condition for $C_2(t)$ to be positive is that the symmetric and antisymmetric states are not equally populated.

We consider first the effect of the collective damping on the sudden death of an initial entanglement determined by the state $\ket{7}$. The entanglement weights $C_1(t)$ and $C_2(t)$, which are needed to construct $C(t)$ are readily calculated from Eqs. (7) and (12). We see that the system initially prepared in the state $\ket{7}$ can be entangled according to the criterion $C_1$, and the degree to which the system is initially entangled is $C_1(0) = 2\sqrt{p}(1-p)$.

If the atoms radiate independently, $\gamma_{12} = 0$, and then we find from Eq. (9) that $\rho_{ss}(t) = \rho_{aa}(t)$. It is clear by inspection of Eq. (12) that in this case $C_2(t)$ is always negative, so we immediately conclude that no entanglement is possible according to the criterion $C_2$, and the atoms can be entangled only according to the criterion $C_1$.

The initial entanglement decreases in time because of the spontaneous emission and disappears at the time

$$t_d = \frac{1}{\gamma} \ln \left( \frac{p + \sqrt{p}(1-p)}{2p - 1} \right),$$

(13)

from which we see that the time it takes for the system to disentangle is a sensitive function of the initial atomic conditions. We note from Eq. (13) that the sudden death of the entanglement of independent atoms is possible only for $p > 1/2$. Since $\rho_{ee}(0) = p$, we must conclude that the entanglement sudden death is ruled out for the initially not inverted system.

The most interesting consequence of the collective damping is the possibility of the entanglement revival. We now use Eqs. (9) and (12) to discuss the ability of the system to revive entanglement in the simple process of spontaneous emission. Figure 2 shows the deviation of the time evolution of the concurrence for two interacting atoms from that of independent atoms. In both cases, the initial entanglement falls as the transient evolution damps by the spontaneous emission. For independent atoms we observe the collapse of the entanglement without any revivals. However, for interacting atoms, the system collapses over a short time and remains disentangled until a time $t_r \approx 1.7/\gamma$ at which, somewhat counterintuitively, the entanglement revives. This revival then decays to zero, but after some period of time a new revival begins. Thus, we see two time intervals (dark periods) at which the entanglement vanishes and two time intervals at which the entanglement revives. To estimate the death and revival times, we use Eqs. (12) and (9), and find that for $\gamma_{12} \approx \gamma$, the entanglement weight $C_1(t)$ vanishes at times satisfying the relation

$$\gamma t \exp(-\gamma t) = \sqrt{\frac{1-p}{p}},$$

(14)

which for $p > 0.88$ has two nondegenerate solutions, $t_d$ and $t_r > t_d$. The time $t_d$ gives the collapse time of the entanglement beyond which the entanglement disappears. The death zone of the entanglement continues until the time $t_r$ at which the entanglement revives. Thus, for the parameters of Fig. 2 the entanglement collapses at $t_d = 0.6/\gamma$ and revives at the time $t_r = 1.7/\gamma$.

The origin of the dark periods and the revivals of the entanglement can be understood in terms of the populations of the collective states and the rates with which the populations and the two-photon coherence decay. One can note from Eq. (9) that for short times $\rho_{aa}(t) \approx 0$, but $\rho_{ss}(t)$ is large. Thus, the entanglement behavior can be analyzed almost entirely in terms of the population of the symmetric state and the coherence $\rho_{eg}(t)$.

Figure 3 shows the time evolution of $C(t)$, the population $\rho_{ss}(t)$, and the coherence $\rho_{eg}(t)$. As can be seen from the

![FIG. 1: The death time of the entanglement prepared according to the criterion $C_1$ and plotted as a function of $p$ for different separations between the atoms: $\gamma_{12} = \lambda$ (solid line), $\gamma_{12} = \lambda/3$ (dashed line), $\gamma_{12} = \lambda/6$ (dashed-dotted line), $\gamma_{12} = \lambda/20$ (dotted line).](image)

![FIG. 2: Transient evolution of the concurrence $C(t)$ for the initial state $|\Psi_0\rangle$ with $p = 0.9$. The solid line represents $C(t)$ for the collective system with the interatomic separation $r_{12} = \lambda/20$. The dashed line shows $C(t)$ for independent atoms, $\gamma_{12} = 0$.](image)
graphs, the entanglement vanishes at the time at which the population of the symmetric state is maximal and remains zero until the time \( t_r \), at which \( \rho_{ss}(t) \) becomes smaller than \( \rho_{eg}(t) \). We may conclude that the first dark period arises due to the significant accumulation of the population in the symmetric state. The impurity of the state of the two-atom system is rapidly growing and entanglement disappears.

The reason for the occurrence of the first revival, seen in Fig. 2 is that the two-photon coherence \( \rho_{eg}(t) \) decays more slowly than the population of the symmetric state. Once \( \rho_{ss}(t) \) falls below \( 2|\rho_{eg}(t)| \), entanglement emerges again. Thus, the coherence can become dominant again and entanglement regenerated over some period of time during the decay process. This is the same coherence that produced the initial entanglement. Therefore, we may call the first revival as an "echo" of the initial entanglement that has been unmasked by destroying the population of the symmetric state. It is interesting to note that the entanglement revival appears only for large values of \( p \), and is most pronounced for \( p > 0.88 \). This is not surprising because for \( p > 1/2 \) the system is initially inverted that increases the probability of spontaneous emission.

We have seen that the short time behavior of the entanglement is determined by the population of the symmetric state of the system. A different situation occurs at long times. As it is seen from Fig. 2, the entanglement revives again at longer times and decays asymptotically to zero as \( t \to \infty \). The second revival has completely different origin than the first one. At long times both \( \rho_{ss}(t) \) and \( \rho_{eg}(t) \) are almost zero. However, the population \( \rho_{aa}(t) \) is sufficiently large as it accumulates on the time scale \( t = 1/(\gamma - \gamma_{12}) \) which is very long when \( \gamma_{12} \approx \gamma \). A careful examination of Eq. 4 shows that \( C_1(t) < 0 \) at long times, so that the long time entanglement is determined solely by the weight \( C_2 \), which is negative for short times, and it becomes positive after a finite time \( t_{r2} \) (second revival time) given approximately by the formula

\[
t_{r2} \approx \frac{1}{\gamma_{12}} \ln \left( \frac{1}{\sqrt{p}} \frac{4\gamma}{\gamma - \gamma_{12}} \right),
\]

It follows from the above analysis and Fig. 2 that the entanglement prepared according to the criterion \( C_1 \) is rather short-lived affair compared with a long-lived entanglement prepared along the criterion \( C_2 \). Asymptotically, the concurrence is equal to the population \( \rho_{aa}(t) \).

In summary, we have examined the transient evolution of the entanglement in a two atom system coupled to the multimode vacuum field. We have predicted the occurrence of dark periods and revivals of entanglement induced by the irreversible process of spontaneous emission. The results show that the revivals are independent of the dipole-dipole interaction between the qubits but crucially depend on the collective damping. We have shown that this unusual behavior of the entanglement results from a significant modification of the spontaneous emission rates of the symmetric and antisymmetric states. This work was supported in part by the Australian Research Council and the Polish Ministry of Education and Science grant 1 P03B 064 28.

[1] M. B. Plenio, S. F. Huelga, A. Beige, and P. L. Knight, Phys. Rev. A 59, 2468 (1999).
[2] Z. Ficek and R. Tanaś, Phys. Rep. 372, 369 (2002).
[3] V. S. Malinovsky and I. R. Sola, Phys. Rev. Lett. 93, 190502 (2004).
[4] A. Serafini, S. Mancini, and S. Bose, Phys. Rev. Lett. 96, 010503 (2006).
[5] S.-B. Zheng and G.-C. Guo, Phys. Rev. Lett. 85, 2392 (2000).
[6] S. Osnaghi et al., Phys. Rev. Lett. 87, 037902 (2001).
[7] R. H. Dicke, Phys. Rev. 93, 99 (1954).
[8] R. H. Lehmberg, Phys. Rev. A 2, 883 (1970).
[9] G. S. Agarwal, Quantum Statistical Theories of Spontaneous Emission and their Relation to other Approaches, edited by G. Höhler, Springer Tracts in Modern Physics, Vol. 70, (Springer-Verlag, Berlin, 1974).
[10] Z. Ficek and S. Swain, Quantum Interference and Coherence: Theory and Experiments (Springer, New York, 2005).
[11] G. K. Brennen et al., Phys. Rev. A 61, 062309 (2000).
[12] Z. Ficek and R. Tanaś, J. Mod. Opt. 50, 2765 (2003) and references therein.
[13] T. Yu and J. H. Eberly, Phys. Rev. Lett. 93, 140404 (2004).
[14] L. Jakóbczyk and A. Jamróz, Phys. Lett A 333, 35 (2004); A. Jamróz, quant-ph/0602128 (2006).
[15] W. K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).