Large time behavior of differential equations with drifted periodic coefficients and modeling Carbon storage in soil

Stephane Cordier (1), Le Xuan Truong (2), Nguyen Thanh Long (3), Alain Pham Ngoc Dinh (1)

Abstract

This paper is concerned with the linear ODE in the form $y'(t) = \lambda \rho(t)y(t) + b(t)$, $\lambda < 0$ and which represents a simplified model of storage of the carbon in the soil. In the first part, we show that, for a periodic function $\rho(t)$, a linear drift in the coefficient $b(t)$ involves a linear drift for the solution of this ODE. In the second part, we give sufficient conditions on the coefficients to ensure the existence of a unique periodic solution of this differential equation. Numerical examples are given.

Keywords: Ordinary differential equations, T-periodic function, linear drift, Cauchy sequence, series’ estimates, AMS subject classification: 34E05, 40A05.

1 Introduction

A lot of phenomena of evolution are described using ordinary differential equations ODE or systems in which the coefficients and/or the source terms are periodic. Let us mention some applications in physics (e.g. the harmonic oscillator, the resonance phenomena due to oscillatory source terms), electricity (let us mention the famous RLC circuit with an oscillatory generator), in biology (circadian cycle), in agricultural studies (due to seasonal effects).

The main question which is addressed here is when or under which conditions a slow perturbation of the coefficient in the ODE will induce a similar behavior on the solutions, in large time. More precisely, we are looking for conditions to ensure that a linear drift in the (periodic) coefficients of the ODE will lead to a linear deviation (and thus unbounded) of the solutions or, on the contrary, what kind of perturbation in the coefficients are compatible with a stable (bounded, periodic in large time) solutions.

Although these questions can be applied to large number of applications, our original motivation concerned with the effect of climate change on the seasonal variations in the organic carbon contained in the soil as claimed at the end of the conclusion of Martin et al. [13].
Since the readers may be not familiar with this domain. Let us recall some basis about the issues of the soil organic carbon (SOC) modelling.

The spreading is one of opportunities for organic materials of human origin (sludge of filtering treatment station and derivative products) and agricultural (manure). The organic matter spread contain significant amounts of organic carbon which, after application, a fraction is permanently stored in the soil. The remainder is returned to the atmosphere as CO2. The spreading can be accessed through the storage of organic carbon in the soil, helping to reduce CO2 emissions (a major greenhouse gas effect) compared for example to the incineration of organic matter that returns all carbon into the atmosphere. The optimization of spreading of organic materials is important in reducing emissions of greenhouse gas effect.

The dynamics of carbon in the soil, which determines the amount of organic carbon stored in soil and returned in the form of CO2, depends on soil type, agricultural practices, climate and quantities spread. The soil organic carbon (SOC) plays an important role in several environmental and land management issues. One of the most important issues is the role that SOC plays as part of the terrestrial carbon and might play as a regulation of the atmospheric CO2. Many factors are likely, in a near future, to modify the SOC content, including changes in agricultural practices [3,18,2] and global climate changes [9,5,8,10,11].

Understanding SOC as a function of soil characteristics, agricultural management and climatic conditions is therefore crucial and many models have been developed in this perspective. These models are used in a variety of ways and after for long term studies [6,15,16,17]. The behavior of the SOC system, over a long term and assuming that the environment of the system (inputs of organic carbon, climatic conditions) is stable, is reported to tend towards steady or periodic state.

Some of the SOC models have been formulated mathematically [14,4,1,12,13]. Here we consider the RothC model [7,13] which consists in splitting the soil carbon into four active compartments. Under a continuous form it can be written as

\[
\frac{dC(t)}{dt} = \rho(t)AC(t) + B(t),
\]

where \( C(t) \) is a vector with 4 components, each corresponding to a compartment storage of carbon in the soil: DPM (decomposable plant material), RPM (resistant plant material), BIO (microbial community) and HUM (humus). These components indicate the amount of carbon stored at the moment \( t \).

In (1.1)

i) \( \rho(t) \) is a function indicating the speed of mineralization of soil, which results in CO2 emissions.

ii) \( A \) is a matrix that can represent the percentage of clay in the soil and speeds of mineralization in each compartment of \( C \) in the soil.

iii) \( B(t) \) indicates the amount of carbon brought in the soil (amount spread per unit time).

Let us recall that the initial goal of this study was to understand how long term evolution on climatic data imply variation in large time on the solution of ODE with periodic coefficient and/or source terms with a (linear) drift. The mathematical tools involved in this paper are rather classical and simple but there are, up to our knowledge, very few literature on the subject.

In the reminder of this paper, we shall consider a simplified case with a single (scalar) equation but the extension to a diagonalizable system (as it is the case for the \( A \) matrix in [13], see eq (5)).

The main results in this paper are, first (section 2), to prove that a linear drift in the coefficients leads to a linear drift in the solution over a large time behavior. Second (section 3), we try to find sufficient conditions on the coefficients in order to get the existence of an unique periodic solution. These results are illustrated by numerical tests in section 4.

The extension to linear differential system is straightforward (by diagonalizing the system). The extension to partial differential equation are under study. For example, in the case of the linear heat equation, using Fourier Series, one can expect the same kind of results.
2 Asymptotic behavior of the solution

In this paper, we consider the linear differential equation

\[ y'(t) = \lambda \rho(t) y(t) + b(t), \quad 0 \leq t < +\infty, \tag{2.1} \]

where \( \lambda < 0, \rho(t) \) and \( b(t) \) are given functions satisfying the following conditions

\( (A_1) \) \( \rho(t) \) is an \( T \)-periodic function, with \( T > 0 \) fixed.

\( (A_2) \) there exist the \( T \)-periodic function, \( \beta(t) \), such that

\[ b(t + T) = b(t) + \beta(t), \quad \forall t \in [0, \infty). \tag{2.2} \]

The general solution of (2.1) has the form

\[ y(t) = e^{\alpha(t)} \left\{ C_1 + \int_0^t b(s)e^{-\alpha(s)}ds \right\}, \tag{2.3} \]

where \( C_1 \) is a constant and

\[ a(t) \overset{df}{=} \lambda \int_0^t \rho(s)ds. \tag{2.4} \]

In this section, we prove that there exists a unique solution \( y_\infty(t) \) of (2.1) satisfying

\[ y_\infty(t + T) = y_\infty(t) + \gamma(t), \forall t \in [0, \infty), \tag{2.5} \]

where \( \gamma(t) \) is an \( T \)-periodic function. Let first remark that

Lemma 2.1. Let \( b : R^+ \rightarrow R \). The following properties are equivalent:

\( (a) \exists \beta(t) \) periodic with period \( T \) such that \( b(t + T) = b(t) + \beta(t), \forall t \geq 0 \)

\( (b) \exists \tilde{b}(t) \) periodic with period \( T \) such that \( b(t) = \tilde{b}(t) + \frac{T}{T} \beta(t), \forall t \geq 0 \)

Proof. Let us first prove (a)\( \Rightarrow \)(b). Choose \( \tilde{b}(t) = b(t) - \frac{T}{T} \beta(t). \) Then \( \tilde{b}(t) \) is periodic with period \( T \) since

\[ \tilde{b}(t + T) = b(t + T) - \frac{T}{T} \beta(t + T) = b(t) + \beta(t) - (\frac{T}{T} + 1) \beta(t) = b(t) - \frac{T}{T} \beta(t) = \tilde{b}(t). \]

Conversely (b)\( \Rightarrow \)(a). Using \( b(t) = \tilde{b}(t) + \frac{T}{T} \beta(t), \) we have \( b(t + T) = \tilde{b}(t + T) + \frac{T + T}{T} \beta(t + T) = \tilde{b}(t) + (\frac{T}{T} + 1) \beta(t) = \tilde{b}(t) + \frac{T}{T} \beta(t) + \beta(t) = b(t) + \beta(t). \) This concludes the proof. \( \square \)

Let us now prove the main result of this section. First, we state the same lemmas

Lemma 2.2. Let \( (A_1) \) hold. Then

\[ a(t + nT) = a(t) + a(nT) = a(t) + na(T), \forall t \geq 0, n \in \mathbb{N}. \tag{2.6} \]

Proof. From (2.4) we deduce that

\[ a(t + nT) = \lambda \int_0^n \rho(s)ds + \lambda \int_T^{nT} \rho(s)ds = a(nT) + \lambda \int_{nT}^{t+nT} \rho(s)ds. \tag{2.7} \]

On the other hand, by the assumption \( (A_1) \), we have

\[ \lambda \int_{nT}^{T+nT} \rho(s)ds = \lambda \int_0^T \rho(s)ds = \lambda \int_0^T \rho(s)ds = a(T), \tag{2.8} \]

and

\[ a(nT) = \lambda \sum_{k=0}^{n-1} \int_{kT}^{(k+1)T} \rho(s)ds = \lambda \sum_{k=0}^{n-1} \int_0^T \rho(s)ds = na(T). \tag{2.9} \]

Combining (2.7)-(2.9) we have (2.6). \( \square \)
Lemma 2.3. Let assumptions (A₁), (A₂) hold. For \( n \in \mathbb{Z}_+ \) and \( t \in [0, \infty) \), we put

\[
y_n(t) = y(t + nT) = e^{a(t + nT)} \left( y(0) + \int_0^{t + nT} b(s) e^{-a(s)} ds \right).
\]  

(2.10)

Then,

\[
y_n(t) = y_\infty(t) + \delta_n(T) + n \gamma(t),
\]

(2.11)

where

\[
y_\infty(t) = e^{a(t)} \left\{ \frac{e^{a(T)}}{1 - e^{a(T)}} \int_0^T b(s) e^{-a(s)} ds - \frac{e^{a(T)}}{(1 - e^{a(T)})^2} \int_0^T \beta(s) e^{-a(s)} ds + \int_0^t b(s) e^{-a(s)} ds \right\},
\]

(2.12)

\[
\delta_n(T) = e^{a(t)} e^{na(T)} \left\{ y(0) + \int_0^T b(s) e^{-a(s)} ds + \frac{e^{a(T)}}{(1 - e^{a(T)})^2} \int_0^T \beta(s) e^{-a(s)} ds \right\},
\]

(2.13)

and

\[
\gamma(t) = e^{a(t)} \left\{ \frac{e^{a(T)}}{1 - e^{a(T)}} \int_0^T \beta(s) e^{-a(s)} ds + \int_0^t \beta(s) e^{-a(s)} ds \right\}.
\]

(2.14)

Proof. By the assumption (A₂), it follows from (2.10) and the lemma 2.2 that

\[
y_n(t) = e^{a(t)} \left\{ y_0(0) + \int_0^T b(s) e^{-a(s)} ds + n \int_0^t \beta(s) e^{-a(s)} ds \right\}.
\]

On the other hand, we have

\[
y_0(0) = y(nT) = e^{a(nT)} \left\{ y(0) + \int_0^{nT} b(s) e^{-a(s)} ds \right\}
\]

\[= e^{na(T)} \left\{ y(0) + \sum_{k=0}^{n-1} \int_0^T b(s + kT) e^{-a(s + kT)} ds \right\}
\]

\[= e^{na(T)} \left\{ y(0) + \int_0^T b(s) e^{-a(s)} ds \sum_{k=0}^{n-1} e^{-ka(T)} + \int_0^T \beta(s) e^{-a(s)} ds \sum_{k=0}^{n-1} ke^{-ka(T)} \right\}.
\]

By using the following equalities

\[
\sum_{k=0}^{n-1} e^{-ka(T)} = \frac{1 - e^{-na(T)}}{1 - e^{-a(T)}},
\]

and

\[
\sum_{k=0}^{n-1} ke^{-ka(T)} = \frac{e^{-a(T)}}{(e^{-a(T)} - 1)^2} - \frac{e^{-(n+1)a(T)}}{(e^{-a(T)} - 1)^2} + n \frac{e^{-na(T)}}{e^{-a(T)} - 1},
\]

thus we obtain

\[
y_0(0) = \frac{e^{a(T)}}{1 - e^{a(T)}} \int_0^T b(s) e^{-a(s)} ds - \frac{e^{a(T)}}{(1 - e^{a(T)})^2} \int_0^T \beta(s) e^{-a(s)} ds
\]

\[+ e^{na(T)} \left\{ y(0) + \frac{e^{a(T)}}{1 - e^{a(T)}} \int_0^T b(s) e^{-a(s)} ds + \frac{e^{a(T)}}{(1 - e^{a(T)})^2} \int_0^T \beta(s) e^{-a(s)} ds \right\}
\]

\[+ n \frac{e^{a(T)}}{1 - e^{a(T)}} \int_0^T \beta(s) e^{-a(s)} ds.
\]

Combining previous equalities, we obtain (2.11). The proof of Lemma is complete.
Now, we state the main theorem

**Theorem 2.4.** Let $(A_1), (A_2)$ hold. Then, there exists a unique solution $y_\infty(t)$ of (2.1) such that

$$y_\infty(t + T) = y_\infty(t) + \gamma(t), \forall t \in [0, \infty),$$  \hspace{1cm} (2.15)

where $\gamma(t)$ is an $T$-periodic function, defined by

$$\gamma(t) = e^{a(t)} \left\{ \frac{e^{a(T)}}{1 - e^{a(T)}} \int_0^T \beta(s) e^{-a(s)} ds + \int_0^t \beta(s) e^{-a(s)} ds \right\}. \hspace{1cm} (2.16)$$

**Proof.** For $n \in \mathbb{Z}_+$ and $t \geq 0$, let us define

$$u_n(t) \overset{def}{=} y_n(t) - n\gamma(t).$$  \hspace{1cm} (2.17)

It follows from (2.11)-(2.14) and (2.17) that

$$\lim_{n \to +\infty} u_n(t) = y_\infty(t), \forall t \in [0, +\infty).$$  \hspace{1cm} (2.18)

It is clear that $y_\infty(t)$ is a solution of equation (2.1) satisfies the initial value

$$y_\infty(0) = \frac{e^{a(T)}}{1 - e^{a(T)}} \int_0^T b(s) e^{-a(s)} ds - \frac{e^{a(T)}}{(1 - e^{a(T)})^2} \int_0^T \beta(s) e^{-a(s)} ds \equiv L(T).$$  \hspace{1cm} (2.19)

By (2.18), we have

$$y_\infty(t + T) = \lim_{n \to +\infty} u_n(t + T) = \lim_{n \to +\infty} \{y_n(t + T) - n\gamma(t + T)\} = \lim_{n \to +\infty} \{y_{n+1}(t) - (n + 1)\gamma(t) + n[\gamma(t) - \gamma(t + T)] + \gamma(t)\}.$$  \hspace{1cm} (2.20)

On the other hand, by the periodicity of $\beta(t)$, we get

$$\gamma(t + T) = e^{a(T)} \left\{ \frac{e^{a(T)}}{1 - e^{a(T)}} \int_0^T \beta(s) e^{-a(s)} ds + \int_0^{t+T} \beta(s) e^{-a(s)} ds \right\}$$

$$= e^{a(t)} \left\{ \frac{e^{a(T)}}{1 - e^{a(T)}} \int_0^T \beta(s) e^{-a(s)} ds + \int_T^{t+T} \beta(s) e^{a(T) - a(s)} ds \right\}$$

$$= e^{a(t)} \left\{ \frac{e^{a(T)}}{1 - e^{a(T)}} \int_0^T \beta(s) e^{-a(s)} ds + \int_0^t \beta(s) e^{-a(s)} ds \right\} = \gamma(t).$$  \hspace{1cm} (2.21)

Combining (2.20) and (2.21), we obtain

$$y_\infty(t + T) = \lim_{n \to +\infty} u_{n+1}(t) + \gamma(t) = y_\infty(t) + \gamma(t).$$  \hspace{1cm} (2.22)

**Uniqueness**

Now, let $\tilde{y}(t)$ be the solution of (2.1) corresponding to the initial value $\tilde{y}(0) = A$ and

$$\tilde{y}(t + T) = \tilde{y}(t) + \tilde{\gamma}(t),$$  \hspace{1cm} (2.23)

where $\tilde{\gamma}(t)$ is a $T$-periodic function. Then $y^*(t) = y_\infty(t) - \tilde{y}(t)$ satisfy

$$\begin{cases} y'(t) = \lambda \rho(t) y(t), & 0 < t < +\infty, \\ y(0) = L(T) - A, \end{cases}$$  \hspace{1cm} (2.24)
and
\[ y^*(t + T) = y^*(t) + \gamma^*(t), \quad \gamma^*(t + T) = \gamma^*(t), \quad \forall t \geq 0. \] (2.25)

It follows from (2.24) that
\[ y^*(t) = (L(T) - A) e^{\alpha(t)}, \forall t \geq 0. \] (2.26)

From (2.25) and (2.26) we deduce that
\[ \gamma^*(t) = - (L(T) - A) \left( 1 - e^{\alpha(T)} \right) e^{\alpha(t)}, \forall t \geq 0. \] (2.27)

Combining (2.25), (2.27) we get \( A = L(T) \). By the uniqueness of Cauchy problem, the proof of theorem 2.8 is complete.

Remark If we consider the equation (2.1) where the assumptions \((A_1)\) and \((A_2)\) are replaced by
\( (A'_1) \) \( b(t) \) is an \( T \)-periodic function, with \( T > 0 \) fixed
\( (A'_2) \) there exist a \( T \)-periodic function, \( \alpha(t) \), such that
\[ \rho(t + T) = \rho(t) + \alpha(t), \quad \forall t \in [0, \infty). \] (2.28)

In that case it is clear that there does not exist a solution which has the same property as the function \( \rho(t) \), for instance if we consider the example with \( b(t) = 0 \). Here the solution of (2.1) tends to 0 as \( t \to +\infty \).

3 Sufficient conditions for the existence of periodic solutions

In this section, we consider the following assumptions for the functions \( \rho(t) \) and \( b(t) \) of the ODE (2.1)

\( (B_1) \) \( \rho \in L^1_{loc}(0, \infty), \rho(t) > 0 \) and there exists a constant \( T > 0 \) such that
\[ \rho(t + iT) = \rho(t) + \alpha_i(t), \quad \forall t \in [0, +\infty), \forall n \in \mathbb{N}, \] (3.1)

\( (B_2) \) Assume that the function \( t \mapsto b(t)e^{-\lambda \int_0^t \rho(s)ds} \) belongs to \( L^1_{loc}(0, \infty) \) and
\[ b(t + iT) = b(t) + \beta_i(t), \quad \forall t \in [0, +\infty), \forall n \in \mathbb{N}, \] (3.2)

where the functions sequences \( \{\alpha_i(t)\} \), \( \{\beta_i(t)\} \) satisfy some conditions specified later. Note that, by (3.1), (3.2) we also deduce that \( \alpha_i \in L^1_{loc}(0, \infty) \) and \( \beta_i(\cdot)e^{-\lambda \int_0^\cdot \rho(s)ds} \in L^1_{loc}(0, \infty) \).

Let us define the function sequence \( \{y_n(t)\}_{n=0}^\infty \) by
\[ y_n(t) = y(t + nT), \] (3.3)

where \( y(t) \) is a solution of equation (2.1) corresponding to the initial value \( y(0) = C_1 \) and defined by (2.3). It can be proved that if \( \rho(t) \) and \( b(t) \) are the \( T \)-periodic functions then the function sequence \( \{y_n(t)\}_{n=0}^\infty \) converge to \( y_\infty(t) \) where
\[ y_\infty(t) = e^{\alpha(t)} \left\{ \frac{e^{\alpha(T)}}{1 - e^{\alpha(T)}} \int_0^T b(s)e^{-\alpha(s)}ds + \int_0^t b(s)e^{-\alpha(s)}ds \right\}, \] (3.4)

\( y_\infty(t) \) being a unique \( T \)-periodic solution of equation (2.1). Now, we will extend this result to the case where \( \rho(t) \) and \( b(t) \) satisfy the conditions \((B_1), (B_2), \) respectively.

Let us now state some lemmas.
Lemma 3.1. Let \((B_1)\) hold. Then we have

\[
a(t + nT) = a(nT) + a(t) + \lambda \int_0^t \alpha_n(s) ds, \forall n \in \mathbb{Z}_+, \forall t \geq 0,
\]

and

\[
a(nT) = na(T) + \lambda \sum_{k=0}^{n-1} \int_0^T \alpha_k(s) ds.
\]

Proof. From (3.1) and (2.4), it follows that

\[
a(t + nT) = \lambda \int_0^n \rho(s) ds + \lambda \int_{nT}^{t+nT} \rho(s) ds
\]

\[
= a(nT) + \lambda \int_0^t (s + nT) ds = a(nT) + a(t) + \lambda \int_0^t \alpha_n(s) ds.
\]

Hence, we obtain (3.5). On the other hand, from (3.5) we can deduce that

\[
a((n + 1)T) = a(nT) + a(T) + \lambda \int_0^T \alpha_n(s) ds.
\]

By recursion, we obtain (3.6) from (3.8).

Lemma 3.2. Let \((B_1), (B_2)\) hold. Then

\[
y_n(t) = e^{a(t)+\lambda \tilde{\alpha}_n(t)} \left\{ y_n(0) + \tilde{\beta}_n(t) \right\},
\]

where

\[
\tilde{\alpha}_i(t) = \int_0^t \alpha_i(s) ds, \quad \text{and} \quad \tilde{\beta}_i(t) = \int_0^t (b(s) + \beta_i(s)) e^{-a(s) - \lambda \tilde{\alpha}_i(s)} ds, \quad i \in \mathbb{N}.
\]

Proof. From (3.3), (2.3) and (3.5), we deduce that

\[
y_n(t) = e^{a(nT+t)} \left\{ C_1 + \int_0^{nT+t} b(s)e^{-a(s)} ds \right\}
\]

\[
= e^{a(nT)+a(t)+\lambda \int_0^t \alpha_n(s) ds} \left\{ C_1 + \int_0^{nT} b(s)e^{-a(s)} ds + \int_{nT}^{nT+t} b(s)e^{-a(s)} ds \right\}
\]

\[
= e^{a(t)+\lambda \int_0^t \alpha_n(s) ds} \left\{ e^{a(nT)} \left[ C_1 + \int_0^{nT} b(s)e^{-a(s)} ds \right] + e^{a(nT)} \int_{nT}^{nT+t} b(s)e^{-a(s)} ds \right\}
\]

\[
= e^{a(t)+\lambda \int_0^t \alpha_n(s) ds} \left\{ y_n(0) + e^{a(nT)} \int_{nT}^{nT+t} b(s)e^{-a(s)} ds \right\}.
\]

On the other hand, by change variable, it follows from the assumptions \((B_2)\) and (3.5) that

\[
\int_{nT}^{nT+t} b(s)e^{-a(s)} ds = \int_0^t b(\tau + nT)e^{-a(\tau + nT)} d\tau
\]

\[
= \int_0^t (b(\tau) + \beta_n(\tau)) e^{-a(nT) - a(\tau) - \lambda \int_0^\tau \alpha_n(s) ds} d\tau.
\]

Combining (3.11) and (3.12), we obtain (3.9) - (3.10). The proof of lemma 3.6 is complete.
Now, replace \( t = T \) in (3.9), we get
\[
y_{n+1}(0) = y_n(T) = \mu_n \left( y_n(0) + \hat{\beta}_n \right),
\]
where
\[
\mu_n = e^{a(T) + \lambda \hat{\alpha}_n},
\]
and
\[
\hat{\alpha}_i \equiv \hat{\alpha}_i(T), \quad \hat{\beta}_i \equiv \hat{\beta}_i(T), \quad \forall i \in \mathbb{N}.
\]
By the recurrent, from (3.13), we obtain the following corollary

**Corollary 3.3.** Let \((B_1), (B_2)\) hold. Then
\[
y_{n+1}(0) = \sum_{i=1}^{n} \delta_i^{(n)} \hat{\beta}_{n-i+1} + \delta_n^{(n)} y(T),
\]
where
\[
\delta_i^{(n)} = e^{\alpha_i(T) + \lambda (\hat{\alpha}_{n-1} + \ldots + \hat{\alpha}_{n-i+1})}, \forall i = 1, 2, \ldots, n.
\]

Now, to obtain the convergence of sequence \( \{y_n(0)\}_{n=0}^{\infty} \), we make the following assumptions

Assume that \( \{\alpha_i(t)\}_{i=1}^{\infty} \) and \( \{\beta_i(t)\}_{i=1}^{\infty} \) satisfies
\[
0 < \alpha_{i+1}(t) \leq \alpha_i(t), \forall t \geq 0 \quad \text{and} \quad \sum_{i=1}^{\infty} e^{-\alpha_i(T)} \hat{\alpha}_i < +\infty,
\]
\((B_3)\)
\[
0 < \beta_{i+1}(t) \leq \beta_i(t), \forall t \geq 0 \quad \text{and} \quad \sum_{i=1}^{\infty} e^{-\alpha_i(T)} \hat{\beta}_i < +\infty.
\]

**Remark 3.4.** By the assumption \((B_3)\) we deduce that
\[
\lim_{n \to \infty} \hat{\alpha}_n(t) = 0, \forall t \geq 0,
\]
and
\[
\lim_{n \to \infty} \hat{\beta}_n(t) = \int_{0}^{t} b(s) e^{-a(s)} ds, \forall t \geq 0.
\]
Indeed, the proof of (3.20) is straightforward and we omit it. From (3.10) we have
\[
\hat{\beta}_n(t) - \int_{0}^{t} b(s) e^{-a(s)} ds = \int_{0}^{t} \left( b(s) + \beta_n(s) \right) e^{-a(s)} \left( e^{-\lambda \hat{\alpha}_n(s)} - 1 \right) ds + \int_{0}^{t} \beta_n(s) e^{-a(s)} ds.
\]
Using the following inequality
\[
|e^X - 1| \leq |X| e^{|X|}, \quad \forall X \in \mathbb{R},
\]
and
\[
|\lambda| e^{-\lambda \hat{\alpha}_1} \int_{0}^{t} \left( |b(s)| + \beta_1(s) \right) e^{-a(s)} ds \leq \int_{0}^{t} \beta_n(s) e^{-a(s)} ds,
\]
\((3.15)\)
it follows from (3.18), (3.19) and (3.22) that
\[
\left| \hat{\beta}_n(t) - \int_{0}^{t} b(s) e^{-a(s)} ds \right| \leq \left\{ |\lambda| e^{-\lambda \hat{\alpha}_1} \int_{0}^{t} \left( |b(s)| + \beta_1(s) \right) e^{-a(s)} ds \right\}. \hat{\alpha}_n(t) + \int_{0}^{t} \beta_n(s) e^{-a(s)} ds.
\]
\((3.24)\)
Hence, we obtain (3.21) from (3.22) by (3.19) and (3.20).

**Lemma 3.5.** Let \((B_1) - (B_3)\) hold. Then, the sequence \( \{y_n(0)\}_{n=0}^{\infty} \) is convergence and we have
\[
\lim_{n \to +\infty} y_{n+1}(0) = \frac{e^{a(T)} - 1}{e^{a(T)}} \int_{0}^{T} b(s) e^{-a(s)} ds \equiv L(T).
\]
\((3.25)\)
Using the following inequality
\[ |\beta \| \leq \exp \left( \sum_{\alpha=1}^{n} \lambda \int_{0}^{T} \rho(s) ds + \alpha_{1} + \alpha_{2} + \ldots + \alpha_{n} \right) \rightarrow 0, \] (3.26)
when \( n \to \infty \). Hence,
\[ \lim_{n \to \infty} \delta^{(n)} g(T) = 0. \] (3.27)

Now, put
\[ Z_{n} = \sum_{i=1}^{n} \delta^{(n)} \beta_{n-i+1}. \] (3.28)

We shall prove that \( \{ Z_{n} \} \) is a Cauchy sequence. Let \( m, n \in \mathbb{Z}_{+}, m \geq n, \) then
\[ Z_{m} - Z_{n} = \sum_{i=n+1}^{m} \delta^{(m)} \beta_{m-i+1} + \sum_{i=1}^{n} \left( \delta^{(m)} \beta_{m-i+1} - \delta^{(n)} \beta_{n-i+1} \right) \] (3.29)

It is clear that
\[ \delta^{(k)} = e^{i a(T)} e^{\lambda (\alpha_{k} + \alpha_{k-1} + \ldots + \alpha_{k-i+1})} \leq \rho_{1}, \quad \forall k \in \mathbb{N}, \quad i = 1, 2, \ldots, k, \] (3.30)
with \( \rho_{1} = e^{\alpha(T)} \in (0, 1) \) and by (3.18)-(3.19), we have
\[ |\beta_{i}| \leq \int_{0}^{T} (|b(s)| + \beta_{1}(s)) e^{-\alpha(s)-\lambda \alpha_{1}(s)} ds \equiv K(T), \quad \forall i \in \mathbb{N}. \] (3.31)

**Estimate J_{1}**

It follows from (3.30) and (3.31), that
\[ |J_{1}| \leq \sum_{i=n+1}^{m} \delta^{(m)} \beta_{m-i+1} \leq K(T) \sum_{i=n+1}^{m} \rho_{1}^{i} \rightarrow 0, \] (3.32)
when \( m, n \to \infty \), by \( \sum_{i=1}^{\infty} \rho_{1}^{i} < +\infty \).

**Estimate J_{2}**

By (3.30)-(3.31), we can estimate \( J_{2} \) as follows
\[ |J_{2}| \leq \sum_{i=1}^{n} \delta_{i}^{(m)} \left| \beta_{m-i+1} - \beta_{n-i+1} \right| + \sum_{i=1}^{n} \left| \delta_{i}^{(m)} \beta_{m-i+1} - \delta_{i}^{(n)} \beta_{n-i+1} \right| \] (3.33)
\[ \leq \sum_{i=1}^{n} \rho_{1}^{i} \left| \beta_{m-i+1} - \beta_{n-i+1} \right| + K(T) \sum_{i=1}^{n} \left| \delta_{i}^{(m)} - \delta_{i}^{(n)} \right|. \]

On the other hand, it follows from (3.10) that
\[ \beta_{m-i+1} - \beta_{n-i+1} = \int_{0}^{T} \left\{ b(s) + \beta_{m-i+1}(s) \right\} e^{-\alpha(s)} \left( e^{-\lambda \alpha_{m-i+1}(s)} - e^{-\lambda \alpha_{n-i+1}(s)} \right) ds \] (3.34)
\[ + \int_{0}^{T} \left\{ \beta_{m-i+1}(s) - \beta_{n-i+1}(s) \right\} e^{-\alpha(s)} e^{-\lambda \alpha_{n-i+1}(s)} ds. \]

Using the following inequality
\[ |e^{-X} - e^{-Y}| \leq |X - Y|, \forall X, Y \geq 0, \] (3.35)
we deduce from (3.18), (3.19) and (3.34) that
\[
|\hat{\beta}_{m-i+1} - \hat{\beta}_{n-i+1}| \leq |\lambda| \int_0^T (|b(s)| + \beta_1(s)) e^{-\alpha(s)} (\hat{\alpha}_{n-i+1} - \hat{\alpha}_{m-i+1}) ds 
+ \int_0^T \{\beta_{n-i+1}(s) - \beta_{m-i+1}(s)\} e^{-\alpha(s)} e^{-\lambda \hat{\alpha}_1(s)} ds 
\leq |\lambda| \left\{ \int_0^T (|b(s)| + \beta_1(s)) e^{-\alpha(s)} ds \right\} \hat{\alpha}_{n-i+1} + e^{-\lambda \hat{\alpha}_1} \int_0^T \beta_{n-i+1}(s) e^{-\alpha(s)} ds.
\]
Hence,
\[
\sum_{i=1}^n \rho_i^1 |\hat{\beta}_{m-i+1} - \hat{\beta}_{n-i+1}| \leq |\lambda| \left\{ \int_0^T (|b(s)| + \beta_1(s)) e^{-\alpha(s)} ds \right\} \sum_{i=1}^n \rho_i^1 \hat{\alpha}_{n-i+1} + e^{-\lambda \hat{\alpha}_1} \sum_{i=1}^n \rho_i^1 \int_0^T \beta_{n-i+1}(s) e^{-\alpha(s)} ds.
\]
On the other hand, we have
\[
\sum_{i=1}^n \rho_i^1 \hat{\alpha}_{n-i+1} = \rho_1^{n+1} \sum_{i=1}^n \rho_1^i \hat{\alpha}_i \leq \rho_1^{n+1} \sum_{i=1}^\infty \rho_1^i \hat{\alpha}_i \rightarrow 0,
\]
and
\[
\sum_{i=1}^n \rho_i^1 \int_0^T \beta_{n-i+1}(s) e^{-\alpha(s)} ds = \rho_1^{n+1} \sum_{i=1}^n \rho_1^i \int_0^T \beta_i(s) e^{-\alpha(s)} ds \leq \rho_1^{n+1} \sum_{i=1}^\infty \rho_1^i \int_0^T \beta_i(s) e^{-\alpha(s)} ds \rightarrow 0.
\]
By the assumption (B₃), it follows from (3.37) and (3.39) that
\[
\sum_{i=1}^n \rho_i^1 |\hat{\beta}_{m-i+1} - \hat{\beta}_{n-i+1}| \rightarrow 0, \quad \text{when} \quad m, n \rightarrow \infty.
\]
Now, we shall estimate the term \(\sum_{i=1}^n |\hat{\delta}_i^{(m)} - \hat{\delta}_i^{(n)}|\). It follows from (3.17) and the inequality (3.35) that
\[
|\hat{\delta}_i^{(m)} - \hat{\delta}_i^{(n)}| \leq |\lambda| \rho_i^1 \left( \sum_{j=m-i+1}^n \hat{\alpha}_j - \sum_{j=m-i+1}^m \hat{\alpha}_j \right) \leq |\lambda| \rho_i^1 \sum_{j=m-i+1}^n \hat{\alpha}_j.
\]
Therefore, we deduce from (3.41) that
\[
\sum_{i=1}^n |\hat{\delta}_i^{(m)} - \hat{\delta}_i^{(n)}| \leq |\lambda| \rho_i^1 \sum_{j=m-i+1}^n \hat{\alpha}_j = |\lambda| \rho_i^1 \sum_{j=m-i+1}^{n-j+1} \hat{\alpha}_j = |\lambda| \rho_i^1 \sum_{j=m-i+1}^{n-j+1} \hat{\alpha}_j \rightarrow 0, \quad \text{when} \quad m, n \rightarrow \infty.
\]
Combining (3.29), (3.32), (3.33), (3.40) and (3.42), we deduce that \(\{Z_n\}\) is Cauchy sequence. Hence, it is clear that there exists the \(\lim_{n \rightarrow \infty} y_{n+1}(0)\), by (3.16) and (3.20). Passing to the limit in (3.13) by (3.20), we have
\[
\lim_{n \rightarrow +\infty} y_{n+1}(0) = \frac{e^{\alpha(T)}}{1 - e^{\alpha(T)}} \int_0^T b(s)e^{-\alpha(s)} ds \equiv L(T).
\]
The lemma 3.5 is completely proved. \(\square\)
Theorem 3.6. Let \((B_1)-(B_3)\) hold. Then there exists a unique \(T\)-periodic solution of equation (2.1), \(y_\infty(t)\), defined by
\[
y_\infty(t) = e^{a(t)} \left[ L(T) + \int_0^t b(s)e^{-a(s)}ds \right].
\] \hspace{1cm} (3.44)

Proof. First, by passing to the limit in \((3.9)\), it follows from \((3.20)\), \((3.21)\) and \((3.25)\) that
\[
\lim_{n \to \infty} y_n(t) = y_\infty(t), \quad \forall \ t \geq 0.
\]
It is clear that \(y_\infty(t)\) is a solution of equation (2.1) and satisfies the initial value \(y(0) = L(T)\). Moreover, since
\[
y_\infty(t + T) = \lim_{n \to \infty} y_n(t + T) = \lim_{n \to \infty} y(t + T + nT) = \lim_{n \to \infty} y_{n+1}(t) = y_\infty(t), \quad \forall \ t \in [0, +\infty),
\]
we deduce that \(y_\infty(t)\) is a \(T\)-periodic function. We can also show as in section 2 that there is uniqueness of such a solution \(y_\infty(t)\).

4 Numerical results

In this section the following Cauchy problem (2.1) with the following choice
\[
\lambda = -1, \quad y_0 = 1, \quad \rho(t) = \sin^2 t, \quad b(t) = t
\]
Clearly, \(\rho(t)\) and \(b(t)\) satisfy \(\rho(t + \pi) = \rho(t)\) and \(b(t + \pi) = b(t) + \pi\) i.e. the assumptions of section 2 with \(T = \pi\) and \(\beta(t) = \pi\). Therefore the calculus of the different elements defined in section 2 give
\[
a(t) = -\int_0^t \rho(s)ds = -\frac{t}{2} + \frac{\sin 2t}{4}, \quad a(T) = -\frac{\pi}{2}, \quad a_0 = \frac{e^{a(T)}}{1 - e^{a(T)}} \int_0^T \beta(s)e^{-a(s)}ds \approx 6.752
\]
\[
L(T) = \frac{e^{a(T)}}{1 - e^{a(T)}} \int_0^T b(s)e^{-a(s)}ds - \frac{e^{a(T)}}{(1 - e^{a(T)})^2} \int_0^T \beta(s)e^{-a(s)}ds \approx 4.0912
\]

In fig.1 we have drawn the graph of the function \(t \to \gamma(t)\), \(\gamma(t) = \left[ a_0 + \int_0^t \beta(s)e^{-a(s)}ds \right]e^{a(t)}\).

![Fig1. The periodic function \(\gamma(t)\)](image-url)
The fig. 2 indicates the graphs $t \to y_\infty(t)$ and $t \to y_\infty(t + T)$ where $y_\infty(t) = \left[ t(T) + \int_0^t b(s)e^{-a(s)}ds \right]e^{a(t)}$

\[ y(t) = y(0) + \int_0^t b(s)e^{-a(s)}ds \]

So for $n = 1$ the two graphs coincide exactly for $t \geq 5$!
Finally in fig. 4 we have put the graphs of the functions $y_n(t)$ and $y_n(t + \pi)$ with $n = 5$ and here we also note the drift property for the solution of (2.1) taking initial value $y_0 = 1$ at $t = 0$.

References

[1] W.T. Baisden and R. Amundson, An analytical approach to ecosystem biogeochemistry modeling, Ecological Applications, 13, (2003), 649-653.

[2] P.H. Bellamy, P.J. Loveland, R.I. Bradley, R.M. Lark and G.J.D. Kirk, Carbon losses from soils across England and Wales 1978, Nature 437, 2003, 245-248.

[3] R.A. Betts, Offset of the potential carbon sink from boreal forestation by decreases in surface albedo, Nature, 408, (2000), 187-190.

[4] B.M. Bolker, S.W. Pakala and W.J. Parton, Linear analysis of soil decomposition: insights from the century model, Ecological Applications, 8, 1998, 425-439.

[5] M.K. Cao and F.I. Woodward, Dynamic responses of terrestrial ecosystem carbon cycling to global climate, Nature, 393 (1998), 249-252.

[6] K. Coleman and D.S. Jenkinson, ROTHC-26.3, a model for the turnover of carbon in soil. Model description and users guide, Lawes Agricultural Trust, Harpenden (1995).

[7] K. Coleman, D.S. Jenkinson, G.J. Crocker, P.R. Grace, J. Kliir, M. Korschens, P.R. Poulton and D.D. Richter, Simulating trends in soil organic carbon in long-term experiments using ROTHC-26.3, Geoderma, 81 (1997), 29-44.

[8] P.M. Cox, R.A. Betts, C.D. Jones, S.A. Spall and I.J. Totterdell Acceleration of global warming due to carbon-cycle feedbacks in a coupled climate model, Nature, 408, (2000), 184-187.

[9] D.S. Jenkinson, D.E. Adams and A. Wild, Model estimates of CO2 emissions from soil in response to global warming, Nature, 351, (1991), 304-306.
[10] C. Jones, C. McConnell, K. Coleman, P. Cox, P. Falloon, D. Jenkinson and D. Powlson Global climate change and soil carbon stocks: predictions from two contrasting models for the turnover of organic carbon in soil, Global Change Biology, 11, (2005), 154-166.

[11] W. Knorr, I.C. Prentice, J.I. House and E.A. Holland Long-term sensitivity of soil carbon turnover to global warming, Nature, 433, (2005), 298-301.

[12] S. Manzoni, A. Porporato, P. D'Odorico, F. Laio and I. Rodriguez-Iturbe, Soil nutrient cycles as a nonlinear dynamical system, Nonlinear Processes in Geophysics, 11, (2004), 589-598.

[13] M.P. Martin, S. Cordier, J. Balesdent, D. Arrouays, Periodic solutions for soil carbon dynamic equilibriums with time varying forcing variables, HAL, (2007).

[14] A. Parshotam, The Rothamsted soil-carbon turnover model-Discrete to continuous form, Ecological Modelling, 86, (1996), 283-289.

[15] Y. Shirato, Testing the suitability of the DNDC model for simulating long-term soil organic carbon dynamics in Japanese paddy soils, Soil Science and Plant Nutrition, 51, (2005), 183-192.

[16] Y. Shirato and M. Yokozawa Applying the Rothamsted Carbon Model for long-term experiments on Japanese paddy soils and modifying it by simple mining of the decomposition rate, Soil Science and Plant Nutrition, 51, (2005), 405-415.

[17] Y. Shirato, K. Paisancharoen, P. Sangtong, C. Nakviro, M. Yokozawa and N. Matsumoto, Testing the Rothamsted Carbon Model against data from long-term experiments on upland soils in Thailand, European Journal of Soil Science, 56, (2005), 179-188.

[18] L.M. Vleeshouwers and A. Verhagen Carbon emission and sequestration by agricultural land use: a model study for Europe, Global Change Biology, 8, (2002), 519-530.