Exact solution of $\mathbb{Z}_n$ Belavin model with open boundary condition

W. -L. Yang$^{a,b}$ and R. Sasaki$^b$

$^a$ Institute of Modern Physics, Northwest University
Xian 710069, P.R. China

$^b$ Yukawa Institute for Theoretical Physics,
Kyoto University, Kyoto 606-8502, Japan

Abstract

$\mathbb{Z}_n$ Belavin model with open boundary condition is studied. The double-row transfer matrices of the model are diagonalized by algebraic Bethe ansatz method in terms of the intertwiner and the face-vertex correspondence relation. The eigenvalues and the corresponding Bethe ansatz equations are obtained.

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1 Introduction

Exactly solvable two-dimensional lattice models have been attracting a great deal of interest from both physicists and mathematician. Bethe ansatz method has been proved to be the most powerful and (probably) unified tool to construct the common eigenvectors of commuting families of operators (so-called transfer matrices) for various models [1, 2, 3, 4]. Two-dimensional exactly solvable lattice models have traditionally been solved by imposing periodic boundary condition. The Yang-Baxter equation [5, 3]

\[ R_{12}(u_1 - u_2)R_{13}(u_1 - u_3)R_{23}(u_2 - u_3) = R_{23}(u_2 - u_3)R_{13}(u_1 - u_3)R_{12}(u_1 - u_2), \]

(1.1)

together with such boundary condition then leads to families of commuting one-row transfer matrices and hence solvability [3]. The work of Sklyanin [6] shows that, by using the reflection equation (RE) introduced by Cherednik [7]

\[ R_{12}(u_1 - u_2)K_1(u_1)R_{21}(u_1 + u_2)K_2(u_2) = K_2(u_2)R_{12}(u_1 + u_2)K_1(u_1)R_{21}(u_1 - u_2), \]

(1.2)

it is also possible to construct families of commuting double-row transfer matrices for the models with open boundary condition [6, 8, 9]. Both the one-row (for periodic boundary condition) and the double-row (for open boundary condition) transfer matrices for rational and trigonometric integrable models can be very successfully diagonalized by algebraic Bethe ansatz method [10, 11, 12].

In contrast to the rational and trigonometric models, the elliptic models of vertex type, such as the eight-vertex model (or \( Z_2 \) Belavin model) and \( Z_n \) Belavin model [13], had been problematic due to the fact that the corresponding pseudo-vacuum state could not be constructed directly in vertex form [3, 14, 15, 16]. Or from the representation theories point of view, the highest weight representations of the underlying algebras—\( Z_n \) Skyanin algebras [17, 18] were not properly defined. Therefore, the algebraic Bethe ansatz method had not been applied to the elliptic integrable models directly.

It is well-known that in Baxter’s original work [19] for the eight-vertex model with periodic boundary condition, he elegantly used the intertwiner to transform the eight-vertex model (vertex type) to \( A_1^{(1)} \) solid-on-solid (SOS) model (face type) through the face-vertex correspondence relation, then he succeeded in constructing the corresponding pseudo-vacuum state and in diagonalizing the transfer matrices by algebraic Bethe ansatz method. This
method was later generalized to diagonalize $\mathbb{Z}_n$ Belavin model with periodic boundary condition \cite{15} by using the generalized intertwiner and the face-vertex correspondence relation given by \cite{20}. In \cite{14}, it was further generalized to diagonalize the eight-vertex model with open boundary condition.

In this paper, we will extend the above construction of Bethe ansatz method to the generic $\mathbb{Z}_n$ Belavin model with open boundary condition. In section 2, we first review $\mathbb{Z}_n$ Belavin model and reflecting open boundary condition; the model under consideration in this paper is constructed from the corresponding double-row transfer matrices. In section 3, we introduce the intertwiner vectors and accompanying face-vertex correspondence relations which will play key roles in transforming the model in “vertex language” to the one in the “face language”. After finding the pseudo-vacuum state, we use the algebraic Bethe ansatz method to diagonalize the transfer matrices of the model in section 4. Section 5 is for conclusions. Some detailed technical calculations are given in appendix A-C.

2 $\mathbb{Z}_n$ Belavin model and integrable boundary condition

2.1 $\mathbb{Z}_n$ Belavin R-matrix

Let us fix $\tau$ such that $Im(\tau) > 0$ and a generic complex number $w$. Introduce the following elliptic functions

$$\theta\left(\begin{array}{c}a \\ b\end{array}\right)(u, \tau) = \sum_{m=-\infty}^{\infty} \exp \left\{\sqrt{-1} \pi \left[(m+a)^2 \tau + 2(m+a)(u+b)\right]\right\}, \quad (2.1)$$

$$\theta^{(j)}(u) = \theta\left[\begin{array}{c}1/2 - \frac{j}{n} \\ 1/2\end{array}\right](u, n\tau), \quad \sigma(u) = \theta\left[\begin{array}{c}1/2 \\ 1/2\end{array}\right](u, \tau). \quad (2.2)$$

Among them the $\sigma$-function\footnote{Our $\sigma$-function is the $\vartheta$-function $\vartheta_1(u)$ \cite{21}. It has the following relation with the Weierstrassian $\sigma$-function if denoted it by $\sigma_w(u)$: $\sigma_w(u) \propto e^{\eta_1 u^2} \sigma(u)$, $\eta_1 = \pi^2 \left(\frac{1}{8} - 4 \sum_{n=1}^{\infty} \frac{n \sigma_{2n}^2}{1 - q^{2n}}\right)$ and $q = e^{\sqrt{-1} \tau}$.} satisfies the following identity:

$$\sigma(u+x)\sigma(u-x)\sigma(v+y)\sigma(v-y) - \sigma(u+y)\sigma(u-y)\sigma(v+x)\sigma(v-x)$$

$$= \sigma(u+v)\sigma(u-v)\sigma(x+y)\sigma(x-y), \quad (2.3)$$

which will be useful in deriving equations in the following.

The $\mathbb{Z}_n$ Belavin R-matrix \cite{13} is given by \cite{20}

$$R^B(u) = \sum_{i,j,k,l} R^B_{ij}(u) E_{ik} \otimes E_{lj}, \quad (2.4)$$
in which $E_{ij}$ is the matrix with elements $(E_{ij})^l_k = \delta_{jk}\delta_{il}$. The coefficient functions are

$$R^{kl}_{ij}(u) = \begin{cases} \frac{h(u)\sigma(w)\delta^{(i-j)}(u+w)}{\sigma(u+w)\delta^{(k-l)}(u)\theta^{(k-j)}(u)} & \text{if } i + j = k + l \mod n, \\ 0 & \text{otherwise.} \end{cases} \quad (2.5)$$

Here we have set

$$h(u) = \frac{\prod_{j=0}^{n-1} \theta^{(j)}(u)}{\prod_{j=1}^{n-1} \theta^{(j)}(0)}. \quad (2.6)$$

The R-matrix satisfies the quantum Yang-Baxter equation (1.1) and the following unitarity and crossing-unitarity relations [22]

Unitarity : \quad $R^B_{12}(u)R^B_{21}(-u) = id$, \quad (2.7)

Crossing-unitarity : \quad $(R^B)^t_{21}(-u-nw)(R^B)^t_{12}(u) = \frac{e^{\sqrt{-1}nw} \sigma(u)\sigma(u+ nw)}{\sigma(u+w)\sigma(u+ nw - w)} id$, \quad (2.8)

where $t_i, i = 1, 2$ denotes transposition in the $i$-th space.

We introduce “row-to-row” monodromy matrix $T(u)$, which is an $n \times n$ matrix with elements being operators acting on $(\mathbb{C}^n)^{\otimes N}$, from the R-matrix by the standard way [4]

$$T_0(u) = R^B_{01}(u + z_1)R^B_{02}(u + z_2) \cdots R^B_{0N}(u + z_N). \quad (2.9)$$

Here $\{z_i|i = 1, \cdots, N\}$ are arbitrary free complex parameters which are usually called inhomogeneous parameters. One can show that $T(u)$ satisfies the so-called “RLL” relation from the Yang-Baxter equation (1.1)

$$R^B_{12}(u - v)T_1(u)T_2(v) = T_2(v)T_1(u)R^B_{12}(u - v). \quad (2.10)$$

### 2.2 Integrable boundary condition

We proceed to study $\mathbb{Z}_n$ Belavin model with open boundary, by following Sklyanin’s work [3]. Let us introduce a pair of K-matrices $K^-(u)$ and $K^+(u)$. The former $K^-(u)$ satisfies RE (1.2), namely,

$$R^B_{12}(u_1 - u_2)K_1^-(u_1)R^B_{21}(u_1 + u_2)K_2^-(u_2) = K_2^-(u_2)R^B_{12}(u_1 + u_2)K_1^-(u_1)R^B_{21}(u_1 - u_2), \quad (2.11)$$
and the latter $K^+(u)$ satisfies its dual equation

$$R_{12}^B(u_2 - u_1)K^+_1(u_1)R_{21}^B(-u_1 - u_2 - nw)K^+_2(u_2) = K^+_2(u_2)R_{12}^B(-u_1 - u_2 - nw)K^+_1(u_1)R_{21}^B(u_2 - u_1).$$

(2.12)

Various integrable boundary conditions are described by different solutions of $K^-(u)$ and $K^+(u)$ [22, 23]. In this paper, we shall consider the solution of RE (2.11) $K^-(u)$ given by [24]

$$K^-(u) = \sum_{i=1}^n \frac{\sigma(\lambda_i^- w + \xi - u)}{\sigma(\lambda_i^- w + \xi + u)} \phi^{(s)}_{\lambda^-,\lambda(-)}(u) \phi^{(t)}_{\lambda^-,\lambda(-)}(-u),$$

(2.13)

and the solution of the dual RE (2.12) given by [25]

$$K^+(u) = \sum_{i=1}^n \left\{ \prod_{k \neq i} \frac{\sigma((\lambda_i^+ - \lambda_k^+)w - w)}{\sigma((\lambda_i^+ - \lambda_k^+)w)} \right\} \frac{\sigma(\lambda_i^+ w + \xi + u + nw)}{\sigma(\lambda_i^+ w + \xi - u - nw)} \times \phi^{(s)}_{\lambda^+,\lambda(+)}(-u) \tilde{\phi}^{(t)}_{\lambda^+,\lambda(+)}(u).$$

(2.14)

They depend on free parameters $\{\lambda_i^-\}_{i=1}^n$ and $\xi$ which specify the left integrable boundary condition (resp. the right integrable boundary condition). It is very convenient to introduce two vectors $\lambda^\pm = \sum_i \lambda_i^\pm \epsilon_i$ associated with the boundary parameters $\{\lambda_i^\pm\}$, where $\{\epsilon_i, i=1,\cdots,n\}$ is the orthonormal basis of the vector space $\mathbb{C}^n$ such that $\langle \epsilon_i, \epsilon_j \rangle = \delta_{ij}$. In equations (2.13, 2.14), $\phi, \tilde{\phi}, \tilde{\phi}$ are intertwiners which will be specified later in section 3. We consider the generic $\{\lambda_i^\pm\}$ such that $\lambda_i^\pm w \neq \lambda_j^\pm w$ (modulo $\mathbb{Z} + \tau \mathbb{Z}$) for $i \neq j$. This condition is necessary for the non-singularity of $K^+(u)$. Let us remark that a further restriction

$$\lambda^+ = \lambda^- = \lambda_0, \quad \text{a generic vector } \lambda_0 \in \mathbb{C}^n,$$

(2.15)

is necessary for the application of the algebraic Bethe ansatz method in section 4. Hereafter we will consider only the above case.

For open boundary condition case, instead of the standard “row-to-row" monodromy matrix $T(u)$ [29], one needs to introduce a “double-row" monodromy matrix $\mathbb{T}(u)$

$$\mathbb{T}(u) = T(u)K^-(u)T^{-1}(-u).$$

(2.16)

Using the “RLL” relation (2.10) and RE (2.11), one can prove that $\mathbb{T}(u)$ satisfies the reflection equation

$$R_{12}^B(u_1 - u_2)\mathbb{T}_1(u_1)R_{21}^B(u_1 + u_2)\mathbb{T}_2(u_2) = \mathbb{T}_2(u_2)R_{12}^B(u_1 + u_2)\mathbb{T}_1(u_1)R_{21}^B(u_1 - u_2).$$

(2.17)
The double-row transfer matrices of $\mathbb{Z}_n$ Belavin model with open boundary are given by:

$$t(u; \xi) = tr(K^+(u)T(u)).$$

Here we have emphasized the dependence on the boundary parameter $\xi$ of the transfer matrices through the boundary K-matrix $K^-(u)$ \((2.13)\). With the help of the unitarity \((2.7)\) and crossing-unitarity relations \((2.8)\) of R-matrix, Yang-Baxter relation \((1.1)\), the RE \((2.11)\) and its dual \((2.12)\), we can prove that the transfer matrices with different spectral parameters commute with each other \([6]\): $[t(u; \xi), t(v; \xi)] = 0$. This ensures the integrability of the system. The aim of this paper is to find the common eigenvalues and eigenvectors of the transfer matrices \((2.18)\) with the special K-matrices $K^\pm(u)$ given by \((2.13)\), \((2.14)\) and \((2.15)\).

3 $A_{n-1}^{(1)}$ SOS R-matrix and face-vertex correspondence

The $A_{n-1}$ simple roots are \(\{\alpha_i = \epsilon_i - \epsilon_{i+1} \mid i = 1, \cdots, n-1\}\) and the fundamental weights \(\{\Lambda_i \mid i = 1, \cdots, n-1\}\) satisfying $\langle \Lambda_i, \alpha_j \rangle = \delta_{ij}$ are given by

$$\Lambda_i = \sum_{k=1}^{i} \epsilon_k - \frac{i}{n} \sum_{k=1}^{n} \epsilon_k.$$ 

Set

$$\hat{i} = \epsilon_i - \epsilon, \quad \bar{\epsilon} = \frac{1}{n} \sum_{k=1}^{n} \epsilon_k, \quad i = 1, \cdots, n, \text{ then } \sum_{i=1}^{n} \hat{i} = 0. \quad (3.1)$$

For each dominant weight $\Lambda = \sum_{i=1}^{n-1} a_i \Lambda_i, \ a_i \in \mathbb{Z}^+$ (the set of non-negative integers), there exists an irreducible highest weight finite-dimensional representation $V_{\Lambda}$ of $A_{n-1}$ with the highest vector $|\Lambda\rangle$. For example the fundamental vector representation is $V_{\Lambda_1}$.

Let $\mathfrak{h}$ be the Cartan subalgebra of $A_{n-1}$ and $\mathfrak{h}^*$ be its dual. A finite dimensional diagonalisable $\mathfrak{h}$-module is a complex finite dimensional vector space $W$ with a weight decomposition $W = \oplus_{\mu \in \mathfrak{h}^*} W[\mu]$, so that $\mathfrak{h}$ acts on $W[\mu]$ by $x v = \mu(x) v, (x \in \mathfrak{h}, \ v \in W[\mu])$. For example, the fundamental vector representation $V_{\Lambda_1} = \mathbb{C}^n$, the non-zero weight spaces $W[\hat{i}] = \mathbb{C} \epsilon_i, \ i = 1, \cdots, n$.

\(^2\)It will be shown in section 4 that the spectral parameter $u$ and the boundary parameter $\xi$ will be shifted for the reduced transfer matrices in each step of the nested Bethe ansatz procedure. Therefore, it is convenient to specify the dependence on the boundary parameter $\xi$ in addition to the spectral parameter $u$. 

6
For a generic $\lambda \in \mathbb{C}^n$, define

$$\lambda_i = \langle \lambda, \epsilon_i \rangle, \quad \lambda_{ij} = \lambda_i - \lambda_j = \langle \lambda, \epsilon_i - \epsilon_j \rangle, \quad i, j = 1, \ldots, n. \quad (3.2)$$

Let $R(u, \lambda) \in \text{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$ be the R-matrix of the $A^{(1)}_{n-1}$ SOS model \cite{20} given by

$$R(u, \lambda) = \sum_{i=1}^{n} R_{ii}^{ii}(u, \lambda) E_{ii} \otimes E_{ii} + \sum_{i \neq j} \left\{ R_{ij}^{ij}(u, \lambda) E_{ii} \otimes E_{jj} + R_{ji}^{ji}(u, \lambda) E_{ji} \otimes E_{ij} \right\}. \quad (3.3)$$

The coefficient functions are

$$R_{ii}^{ii}(u, \lambda) = 1, \quad R_{ij}^{ij}(u, \lambda) = \frac{\sigma(u)\sigma(\lambda_{ij}w - w)}{\sigma(u + w)\sigma(\lambda_{ij}w)}, \quad \text{if } i \neq j. \quad (3.4)$$

and $\lambda_{ij}$ is defined in (3.2). The R-matrix satisfies the dynamical (modified) quantum Yang-Baxter equation

$$R_{12}(u_1 - u_2, \lambda - h(3))R_{13}(u_1 - u_3, \lambda)R_{23}(u_2 - u_3, \lambda - h(1)) = R_{23}(u_2 - u_3, \lambda)R_{13}(u_1 - u_3, \lambda - h(2))R_{12}(u_1 - u_2, \lambda), \quad (3.6)$$

with the initial condition

$$R_{ij}^{kl}(0, \lambda) = \delta_i^k \delta_j^l. \quad (3.7)$$

We adopt the notation: $R_{12}(u, \lambda - h(3))$ acts on a tensor $v_1 \otimes v_2 \otimes v_3$ as $R(u, \lambda - \mu) \otimes \text{id}$ if $v_3 \in W[\mu]$. Moreover, the R-matrix satisfies unitarity and a modified crossing-unitarity relation \cite{20 24}.

Let us introduce an intertwiner—an $n$-component column vector $\phi_{\lambda, \lambda - j}(u)$ whose $k$-th element is

$$\phi_{\lambda, \lambda - j}^{(k)}(u) = \theta^{(k)}(u + nw\lambda_j). \quad (3.8)$$

Using the intertwiner, the face-vertex correspondence can be written as \cite{20}

$$R_{12}^{B}(u_1 - u_2)\phi_{\lambda, \lambda - i}(u_1) \otimes \phi_{\lambda - i, \lambda - i - j}(u_2) = \sum_{k,l} R(u_1 - u_2, \lambda)_{ij}^{kl} \phi_{\lambda - i, \lambda - i - k}(u_1) \otimes \phi_{\lambda, \lambda - i}(u_2). \quad (3.9)$$
Then the Yang-Baxter equation of $\mathbb{Z}_n$ Belavin’s R-matrix $R^B(u)$ (1.1) is equivalent to the dynamical Yang-Baxter equation of $A_{n-1}^{(1)}$ SOS R-matrix $R(u, \lambda)$ (3.6). For a generic $\lambda$, we can introduce other types of intertwiners $\bar{\phi}$, $\tilde{\phi}$ satisfying the following conditions

$$
\sum_{k=1}^{n} \bar{\phi}_{\lambda, \lambda-\hat{\mu}}^{(k)}(u) \phi_{\lambda, \lambda-\hat{\phi}}^{(k)}(u) = \delta_{\mu\nu}, \quad (3.10)
$$

$$
\sum_{k=1}^{n} \tilde{\phi}_{\lambda+\hat{\mu}, \lambda}^{(k)}(u) \phi_{\lambda+\hat{\phi}, \lambda}^{(k)}(u) = \delta_{\mu\nu}. \quad (3.11)
$$

One can derive the following relations from the above conditions

$$
\sum_{\mu=1}^{n} \bar{\phi}_{\lambda, \lambda-\hat{\mu}}^{(i)}(u) \phi_{\lambda, \lambda-\hat{\phi}}^{(j)}(u) = \delta_{ij}, \quad (3.12)
$$

$$
\sum_{\mu=1}^{n} \tilde{\phi}_{\lambda+\hat{\mu}, \lambda}^{(i)}(u) \phi_{\lambda+\hat{\phi}, \lambda}^{(j)}(u) = \delta_{ij}. \quad (3.13)
$$

With the help of the properties of $\phi$, $\bar{\phi}$, $\tilde{\phi}$ (3.10)-(3.13), we can derive the following relations from the face-vertex correspondence relation (3.9)

$$
(\bar{\phi}_{\lambda+\hat{k}, \lambda}^{(1)}(u_1) \otimes 1) R^B_{12}(u_1 - u_2) (1 \otimes \phi_{\lambda+\hat{j}, \lambda}^{(2)}(u_2)) = \sum_{i,j} R(u_1 - u_2, \lambda)^{kl}_{ij} \bar{\phi}_{\lambda+i, \lambda+j}^{(k)}(u_1) \otimes \phi_{\lambda+k, \lambda+j}^{(l)}(u_2), \quad (3.14)
$$

$$
(\bar{\phi}_{\lambda+\hat{k}, \lambda}^{(1)}(u_1) \otimes \tilde{\phi}_{\lambda+\hat{i}, \lambda+k}^{(2)}(u_2)) R^B_{12}(u_1 - u_2) = \sum_{i,j} R(u_1 - u_2, \lambda)^{kl}_{ij} \bar{\phi}_{\lambda+i, \lambda+j}^{(i)}(u_1) \otimes \tilde{\phi}_{\lambda+j, \lambda}^{(j)}(u_2), \quad (3.15)
$$

$$
(1 \otimes \bar{\phi}_{\lambda-\hat{i}, \lambda}^{(1)}(u_2)) R^B_{12}(u_1 - u_2) (\phi_{\lambda, \lambda-i}^{(2)}(u_1) \otimes 1) = \sum_{k,j} R(u_1 - u_2, \lambda)^{kl}_{ij} \bar{\phi}_{\lambda-i, \lambda-k}^{(k)}(u_1) \otimes \bar{\phi}_{\lambda, \lambda-i-j}^{(l)}(u_2), \quad (3.16)
$$

$$
(\tilde{\phi}_{\lambda-\hat{i}, \lambda-k}^{(1)}(u_1) \otimes \bar{\phi}_{\lambda, \lambda-i}^{(2)}(u_2)) R^B_{12}(u_1 - u_2) = \sum_{i,j} R(u_1 - u_2, \lambda)^{kl}_{ij} \tilde{\phi}_{\lambda, \lambda-i}^{(i)}(u_1) \otimes \bar{\phi}_{\lambda-i, \lambda-i-j}^{(j)}(u_2). \quad (3.17)
$$

The face-vertex correspondence relations (3.9) and (3.14)-(3.17) will play an important role to translate all formulas in “vertex language” into their “face language” form so that the algebraic Bethe ansatz method can be applied to diagonalize the transfer matrices.
4 Algebraic Bethe ansatz for $\mathbb{Z}_n$ Belavin model with open boundary condition

As mentioned in Introduction, the intertwiners and the face-vertex correspondence relations (3.8)-(3.17) will play a fundamental role in the construction of the eigenstates of $\mathbb{Z}_n$ Belavin model with open boundary condition specified by the K-matrices $K^\pm(u)$ given in (2.13), (2.14) and (2.15). In order to apply the algebraic Bethe ansatz method, we need to transform the fundamental exchange relation (2.17) of vertex type into its face type so that we can construct the corresponding pseudo-vacuum and the creation operators to construct associated Bethe ansatz states.

4.1 Exchange relations of double-row monodromy matrix of face type

The transfer matrices of $\mathbb{Z}_n$ Belavin model with open boundary condition (2.18) can be rewritten in the face type form by using (3.12) and (3.13)

\[
t(u; \xi) = tr(K^+(u)T(u)) = \sum_{\mu, \nu} tr \left( K^+(u)\phi_{\lambda-\mu+\nu, \lambda-\mu}(u)\phi_{\lambda-\mu}(u) \phi_{\lambda, \lambda-\mu}(u)T(u)\phi_{\lambda, \lambda-\mu}(u) \right) = \sum_{\mu, \nu} \tilde{K}(\lambda|u)^\mu_\nu T(\lambda|u; \xi)^\nu_\mu.
\]

We have introduced the dual K-matrix $\tilde{K}(\lambda|u)$ of face type \cite{25} and the double-row monodromy matrix $T(\lambda|u; \xi)$ of face type as follows

\[
\tilde{K}(\lambda|u)^\mu_\nu = \phi_{\lambda-\mu}(u)K^+(u)\phi_{\lambda+\nu, \lambda}(u) \equiv \sum_{i, j} \tilde{\phi}_{\lambda, \lambda-\mu}(u)^{(j)} \phi_{\lambda-\mu}(u)^{(i)} \phi_{\lambda+\nu, \lambda}(u),
\]

\[
T(\lambda|u; \xi)^\nu_\mu = \phi_{\lambda-\mu+\nu, \lambda}(u) \phi_{\lambda-\mu}(u) \phi_{\lambda, \lambda-\mu}(u)T(u)\phi_{\lambda, \lambda-\mu}(u) \phi_{\lambda+\nu, \lambda}(u) \equiv \sum_{i, j} \tilde{\phi}_{\lambda, \lambda-\mu}(u)^{(j)} \phi_{\lambda-\mu+\nu, \lambda}(u) \phi_{\lambda-\mu}(u)^{(i)} \phi_{\lambda+\nu, \lambda}(u). \tag{4.3}
\]

The dependence on the boundary parameter $\xi$ of $T(\lambda|u; \xi)$ is through the double-row monodromy matrix of vertex type $T(u)$ in (4.3) by the definitions (2.16) and (2.13). One can derive the following exchange relations among $T(\lambda|u; \xi)^\nu_\mu$ from the exchange relation (2.17),
the face-vertex correspondence relation (3.9) and the relation (3.13) (for details, see Appendix A)

\[
\sum_{i_1,i_2,j_1,j_2} R(u_1 - u_2, \lambda)^{i_1,i_2} j_1 j_2 T(\lambda + j_1 + i_2 | u_1; \xi) i_1^{i_1} R(u_1 + u_2, \lambda)^{j_1,j_2} T(\lambda + j_3 + i_3 | u_2; \xi)^{j_2} \\
= \sum_{i_1,i_2,j_1,j_2} T(\lambda + j_1 + i_0 | u_2; \xi) i_0^{i_0} R(u_1 + u_2, \lambda)^{i_1,j_1} T(\lambda + j_2 + i_2 | u_1; \xi) i_2^{i_2} R(u_1 - u_2, \lambda)^{j_2,j_2}.
\]

(4.4)

Next, let us introduce a set of standard notions for convenience:

\[
\mathcal{A}(\lambda|u) = T(\lambda|u)_1, \quad \mathcal{B}_i(\lambda|u) = \frac{\sigma(w)}{\sigma(\lambda_1 w)} T(\lambda|u)_1, \quad i = 2, \cdots, n,
\]

(4.5)

\[
\mathcal{D}_i^j(\lambda|u) = \frac{\sigma(\lambda^{j_1} w - \delta_i w)}{\sigma(\lambda_1 w)} \{ T(\lambda|u)_1 - \delta_i^j R(2u, \lambda + \hat{1}) i_1^1 A(\lambda|u) \}, \\
i, j = 2, \cdots, n.
\]

(4.6)

After some tedious calculation, we have found the commutation relations among \(\mathcal{A}(\lambda|u), \mathcal{D}(\lambda|u)\) and \(\mathcal{B}(\lambda|u)\) (for details, see Appendix B). The relevant commutation relations are

\[
\mathcal{A}(\lambda|u) \mathcal{B}_i(\lambda - \hat{1} + \hat{i}|v) = \frac{\sigma(u + v)\sigma(u - v - w)}{\sigma(u + v + w)\sigma(u - v)} \mathcal{B}_i(\lambda - \hat{1} + \hat{i}|v) \mathcal{A}(\lambda - \hat{1} + \hat{i}|u) \\
- \frac{\sigma(w)\sigma(2v)}{\sigma(u - v)\sigma(2v + w)} \frac{\sigma(u - v - \lambda_1 w + w)}{\sigma(\lambda_1 w - w)} \mathcal{B}_i(\lambda - \hat{1} + \hat{i}|u) \mathcal{A}(\lambda - \hat{1} + \hat{i}|v) \\
- \frac{\sigma(w)}{\sigma(u + v + w)} \sum_{\alpha=2}^n \frac{\sigma(u + v + \lambda_\alpha w + 2w)}{\sigma(\lambda_\alpha w + w)} \mathcal{B}_\alpha(\lambda - \hat{1} + \hat{\alpha}|u) \mathcal{D}_i^\alpha(\lambda - \hat{1} + \hat{i}|v),
\]

(4.7)

\[
\mathcal{D}_i^j(\lambda|u) \mathcal{B}_j(\lambda + \hat{j} - \hat{1}|v) = \frac{\sigma(u - v + w)\sigma(u + v + w)}{\sigma(u - v)\sigma(u + v + w)} \left\{ \sum_{\alpha_1,\alpha_2,\beta_1,\beta_2=2}^n R(u + v + w, \lambda - \hat{j})^\alpha_1^\alpha_2 \beta_2 \beta_1 \times \mathcal{B}_\beta(\lambda - \hat{i} + \hat{k} + \hat{\beta} - \hat{1}|u) \mathcal{D}_\beta^\alpha(\lambda - \hat{1} + \hat{j}|u) \right\} \\
- \frac{\sigma(w)\sigma(2u + 2w)}{\sigma(u - v)\sigma(2u + w)} \left\{ \sum_{\alpha,\beta=2}^n \frac{\sigma(u - v + \lambda_\alpha w - w)}{\sigma(\lambda_\alpha w - w)} R(2u + w, \lambda - \hat{i})^\kappa_\alpha \beta \times \mathcal{B}_\beta(\lambda - \hat{i} + \hat{k} + \hat{\beta} - \hat{1}|u) \mathcal{D}_\beta^\alpha(\lambda - \hat{1} + \hat{j}|v) \right\} \\
+ \frac{\sigma(w)\sigma(2v)\sigma(2u + 2w)}{\sigma(u + v + w)\sigma(2v + w)\sigma(2u + w)} \left\{ \sum_{\alpha=2}^n \frac{\sigma(u + v + \lambda_\alpha w)}{\sigma(\lambda_\alpha w - w)} R(2u + w, \lambda - \hat{i})^\kappa_\alpha \times \mathcal{B}_\alpha(\lambda - \hat{i} + \hat{k} + \hat{\alpha} - \hat{1}|u) \mathcal{A}(\lambda - \hat{1} + \hat{j}|v) \right\},
\]

(4.8)

\(^3\)The scalar factors in the definitions of the operators \(\mathcal{B}(\lambda|u)\) and \(\mathcal{D}(\lambda|u)\) are to make the relevant commutation relations as concise as (4.7)-(4.9).
\[ B_i(\lambda + i - \hat{1}|u)B_j(\lambda + i + \hat{j} - 2\hat{1}|v) = \sum_{\alpha, \beta = 2}^n R(u - v, \lambda - 2\hat{1})_{j_i}^{\beta \alpha} \times B_j(\lambda + \hat{\beta} - \hat{1}|v)B_\alpha(\lambda + \hat{\alpha} + \hat{\beta} - 2\hat{1}|u). \] (4.9)

### 4.2 Pseudo-Vacuum state

The algebraic Bethe ansatz, in addition to the relevant commutation relations (4.7)-(4.9), requires a pseudo-vacuum state (also called reference state) which is the common eigenstate of the operators \( A_i, D_i^j \) and annihilated by the operators \( C_i \). In contrast to the rational and trigonometric models, for elliptic models of vertex type such as the eight-vertex model (or \( Z_2 \) Belavin model) and \( Z_n \) Belavin model, the corresponding pseudo-vacuum state cannot be constructed directly in “vertex language” \([3, 14, 15, 16]\). However, the state can be successfully constructed when one translates it into “face language” (or equivalently after some local gauge transformation \([1]\)). By the same way, we will construct the corresponding pseudo-vacuum state for \( Z_n \) Belavin model with open boundary condition specified by the K-matrices \( K^\pm(u) \) given in \([2, 13]\), \([2, 14]\) and \([2, 15]\).

Before introducing the pseudo-vacuum state, let us introduce a generic state in the quantum space by the intertwiner vector \([3, 8]\)

\[ |i_1, \ldots, i_N \rangle_m^m = \phi_{m_0, m_0 - i_N}^N (-z_N) \phi_{m_0 - i_{N-1}, m_0 - i_{N-1} - i_{N-1}}^{N-1} (-z_{N-1}) \cdots \phi_{m_0 - \sum_{k=1}^N i_k}^{1} (-z_1), \] (4.10)

where the vectors \( m_0, m \in \mathbb{C}^n \) and \( m = m_0 - \sum_{k=1}^N i_k \), the vector \( \phi^k = id \otimes id \cdots \otimes \phi \otimes id \cdots \).

Now let us evaluate the action of the monodromy matrix \( T \) \([13]\) on the state \([1, 10]\). Using the definition of the double-row monodromy matrix \( T \) \([2, 16]\) and relations \([3, 12]-[3, 13]\), we can further write \( T \) in the following form

\[ T(m|u; \xi)^j_i = \tilde{\phi}_{m_0 - m_0 - i}(u)T(u)K^-(u)T^{-1}(-u)\phi_{m_0 - i}(u) \]

\[ = \sum_{\mu, \nu} \tilde{\phi}_{m_0 - i}(u)T(u)\phi_{m_0 - \nu + \mu, m_0 - \nu}(u)\tilde{\phi}_{m_0 - \nu + \mu, m_0 - \nu}(u)K^{-}(u) \]

\[ \times \phi_{m_0 - m_0 - \nu}(u)\tilde{\phi}_{m_0 - m_0 - \nu}(u)T^{-1}(-u)\phi_{m_0 - m_0 - \nu}(u) \]

\[ = \sum_{\mu, \nu} T(m - i, m_0 - \nu|u)^{j\nu}_{\mu} K(m_0|u; \xi)^\mu_{\nu} S(m, m_0|u)^\nu_i. \] (4.11)
Here we have introduced

\[ T(m, m_0|u)_j^i = \tilde{\phi}_{m+j, m}(u) T(u) \phi_{m_0, m_0}(u), \]

(4.12)

\[ S(m, m_0|u)_j^i = \phi_{m_0, m_0-\hat{\mu}}(-u) T^{-1}(-u) \phi_{m, m-\hat{\mu}}(-u), \]

(4.13)

\[ \mathcal{K}(m_0|u; \xi_j^i) = \tilde{\phi}_{m_0-i+j, m_0-i}(u) K^{-}(u) \phi_{m_0, m_0-i}(-u). \]

(4.14)

We can evaluate the action of the operator \( T(m, m_0|u) \) on the state \(|i_1, \ldots, i_N\rangle_m \) from the definition (2.3) and the face-vertex correspondence relation (3.9)

\[
T(m, m_0|u)_j^i|i_1, \ldots, i_N\rangle_m = \tilde{\phi}_{m+j, m}(u) R_{01}^B(u + z_1) \phi_0^{1} m_0 - \sum_{k=2}^{N} \sum_{i_k, m_0 - \sum_{k=1}^{N} :} (-z_1) \\
\times \cdots R_{0N}^B(u + z_N) \phi_N^{N} m_0 - \tilde{\mu}, m_0 (u) \\
= \sum_{\beta_1, i_{N}'} \tilde{\phi}_{m+j, m}(u) R_{01}^B(u + z_1) \phi_0^{1} m_0 - \sum_{k=2}^{N} \sum_{i_k, m_0 - \sum_{k=1}^{N} :} (-z_1) \\
\times \cdots R_{0N-1}^B(u + z_{N-1}) \phi_0^{N} m_0 - \tilde{\mu}, m_0 (u) \\
\times R(u + z_N, m_0 - \tilde{i}_N)^{\alpha_N} \phi_N^{N} m_0 + \tilde{\mu}, m_0 - \tilde{i}_N (-z_N) \\
\vdots \\
= R(u + z_1, m)^{\beta_1} i_{N} R(u + z_2, m + \tilde{i}_1)^{\beta_{N-1} i_2} \cdots \\
\times R(u + z_N, m_0 - \tilde{i}_N)^{\beta_1 i_{N}'} \mid i_1', \ldots, i_{N}'}^{m_0 + \tilde{j}}.
\]

(4.15)

We adopt the convention that the repeated indices imply summation over 1, 2, \ldots n in the last equation. The property of the R-matrix

\[ R(u, \lambda)^{kl}_{ij} = R(u, \lambda \pm (\tilde{i} + \tilde{j}))^{kl}_{ij} = R(u, \lambda \pm (\tilde{k} + \tilde{l}))^{kl}_{ij}, \]

(4.16)

is helpful to derive the above equation. Noting the unitarity of \( R^B \) (2.7), \( T^{-1}(-u) \) can be written

\[ T_{0}^{-1}(-u) = R_{N0}^B(u - z_N) \cdots R_{10}^B(u - z_1). \]

(4.17)

Then we can evaluate the action of the operator \( S(m, m_0|u) \) on the state \(|i_1, \ldots, i_N\rangle_m \) from the face-vertex correspondence relation (3.9) as we have done for \( T(m, m_0|u) \):

\[
S(m, m_0|u)^{\mu}_{\alpha} i_1, \ldots, i_N\rangle_m = R(u - z_N, m_0)^{\tilde{i}_N}_{\alpha_N} R(u - z_{N-1}, m_0 - \tilde{i}_N)^{\tilde{i}_{N-1}}_{\alpha_{N-1}} \cdots \\
\times R(u - z_1, m_0 - \sum_{k=2}^{N} \tilde{i}_k)^{\alpha_1}_{i_1} \mid i_1, \ldots, i_{N}^{m_0 - \tilde{\mu}}.
\]

(4.18)
We also adopt the convention that the repeated indices imply summation over 1, 2, \cdots n in the above equation. Similarly, we obtain the action of $\mathcal{T}$ on the state $|i_1, \cdots, i_N\rangle^m_{m_0}$ from the decomposition relation (4.11) and equations (4.14), (4.18)

$$\mathcal{T}(m|u; \xi)^j_1|i_1, \cdots, i_N\rangle^m_{m_0} \equiv \mathcal{T}(m, m_0|u; \xi)^j_1|i_1, \cdots, i_N\rangle^m_{m_0}$$

$$= T(m - \hat{i}, m_0 - \hat{\nu}|u)^j_1 K(m_0|u; \xi)^j_1 S(m, m_0|u)^j_1|i_1, \cdots, i_N\rangle^m_{m_0}$$

$$= R(u + z_1, m - \hat{i})_\beta_{\beta - i}^{\nu K}(m_0|u; \xi)^j_1 R(u + z_2, m - \hat{i} + \hat{\nu})_\beta_{\beta - i}^{\nu K} \cdots$$

$$\times R(u + z_N, m - \hat{i} + \sum_{k=1}^{N-1} \hat{\nu}_k |\nu^\alpha_{\alpha - 1} R(u - z_N, m + \sum_{k=1}^{N-1} \hat{\nu}_k) |\nu^\alpha_{\alpha - 1} \cdots R(u - z_1, m + \hat{\nu})_\nu^\alpha_{\alpha - 1}$$

$$\times \cdots \times R(u - z_{N-1}, m + \sum_{k=1}^{N-1} \hat{\nu}_k |\nu^\alpha_{\alpha - 1} \cdots R(u - z_1, m + \hat{\nu})_\nu^\alpha_{\alpha - 1}$$

$$\times R(u - z_{N-1}, m + \sum_{k=1}^{N-1} \hat{\nu}_k) |\nu^\alpha_{\alpha - 1} \cdots R(u - z_1, m + \hat{\nu})_\nu^\alpha_{\alpha - 1}$$

(4.19)

Here we also adopt the convention that the repeated indices imply summation over 1, 2, \cdots n in the above equation.

If one chooses $\lambda = \lambda_0$ in equation (4.12) and $m_0 = \lambda_0$ in equation (4.14), where $\lambda_0$ is related to the boundary parameters by (2.13), the corresponding face type boundary $K$-matrices $K(\lambda_0|u; \xi)$ and $\tilde{K}(\lambda_0|u)$ simultaneously become diagonal ones from equations (2.13), (2.14) and the restriction (2.15)

$$K(\lambda_0|u; \xi)^j_1 = \delta^j_1 k(\lambda_0|u; \xi), \quad \tilde{K}(\lambda_0|u)^j_1 = \delta^j_1 \tilde{k}(\lambda_0|u).$$

(4.20)

The functions $k(\lambda_0|u; \xi), \tilde{k}(\lambda_0|u)$, are given by

$$k(\lambda_0|u; \xi)^j_1 = \frac{\sigma((\lambda_0)_i w + \xi - u)}{\sigma((\lambda_0)_i w + \xi + u)},$$

(4.21)

$$\tilde{k}(\lambda_0|u)^j_1 = \left\{ \prod_{k \neq i} \frac{\sigma((\lambda_0)_i k w - w)}{\sigma((\lambda_0)_i k w)} \right\} \frac{\sigma((\lambda_0)_i w + \xi + u + \frac{n w}{2})}{\sigma((\lambda_0)_i w + \xi - u - \frac{n w}{2})}.$$

(4.22)

The fact that both $K(\lambda_0|u; \xi)$ and $\tilde{K}(\lambda_0|u)$ have diagonal form will enable us to construct the pseudo-vacuum state of the model and apply the algebraic Bethe ansatz method to diagonalize the \textit{double-row transfer matrices} (2.18) later.

Now, let us construct the pseudo-vacuum state $|\Omega\rangle$:

$$|\Omega\rangle \equiv |vac\rangle_{\lambda_0 - N^1} = |1, \cdots, 1\rangle_{\lambda_0 - N^1},$$

(4.23)
where $\lambda_0$ is related to the boundary parameters of the boundary K-matrices $\mathcal{K}^\pm$ in (2.15).

Then we find that the actions of the operators $T(\lambda_0 - N\hat{1}, \lambda_0 | u)$ given by (4.12) and $S(\lambda_0 - N\hat{1}, \lambda_0 | u)$ given by (4.13) on the pseudo-vacuum state (4.23) become

$$T(\lambda_0 - N\hat{1}, \lambda_0 | u)_i^1|\text{vac}\rangle_{\lambda_0}^{\lambda_0 - N\hat{1}} = |\text{vac}\rangle_{\lambda_0 + 1}^{\lambda_0 - N\hat{1} + i}, \quad (4.24)$$

$$T(\lambda_0 - N\hat{1}, \lambda_0 | u)_i^j|\text{vac}\rangle_{\lambda_0}^{\lambda_0 - N\hat{1}} = 0, \quad i = 2, \ldots, n, \quad (4.25)$$

$$T(\lambda_0 - N\hat{1}, \lambda_0 | u)_i^j|\text{vac}\rangle_{\lambda_0}^{\lambda_0 - N\hat{1}} = \delta_j^i \prod_{k=1}^{N} R(u + z_k, \lambda_0 + \hat{j} - (N - k)\hat{1})_{j1}^i|\text{vac}\rangle_{\lambda_0 + j}^{\lambda_0 - N\hat{1} + j}, \quad i, j = 2, \ldots, n, \quad (4.26)$$

$$S(\lambda_0 - N\hat{1}, \lambda_0 | u)_i^1|\text{vac}\rangle_{\lambda_0}^{\lambda_0 - N\hat{1}} = |\text{vac}\rangle_{\lambda_0 - 1}^{\lambda_0 - N\hat{1} - 1}, \quad (4.27)$$

$$S(\lambda_0 - N\hat{1}, \lambda_0 | u)_i^j|\text{vac}\rangle_{\lambda_0}^{\lambda_0 - N\hat{1}} = 0, \quad i = 2, \ldots, n, \quad (4.28)$$

$$S(\lambda_0 - N\hat{1}, \lambda_0 | u)_i^j|\text{vac}\rangle_{\lambda_0}^{\lambda_0 - N\hat{1}} = \delta_j^i \prod_{k=1}^{N} R(u - z_k, \lambda_0 - (N - k)\hat{1})_{j1}^i|\text{vac}\rangle_{\lambda_0 - j}^{\lambda_0 - N\hat{1} - j}, \quad i, j = 2, \ldots, n, \quad (4.29)$$

from equations (4.15) and (4.18). Noting that the diagonal form of $\mathcal{K}(\lambda_0 | u; \xi)$ (4.20) and the above equations, we can derive

$$\mathcal{T}(\lambda_0 - N\hat{1}, \lambda_0 | u; \xi)_i^1|\text{vac}\rangle_{\lambda_0}^{\lambda_0 - N\hat{1}} = k(\lambda_0 | u; \xi)_1^1|\text{vac}\rangle_{\lambda_0}^{\lambda_0 - N\hat{1}}, \quad (4.30)$$

$$\mathcal{T}(\lambda_0 - N\hat{1}, \lambda_0 | u; \xi)_i^j|\text{vac}\rangle_{\lambda_0}^{\lambda_0 - N\hat{1}} = 0, \quad i = 2, \ldots, n. \quad (4.31)$$

Moreover, after a tedious calculation, we have (for details, see Appendix C)

$$\mathcal{T}(\lambda_0 - N\hat{1}, \lambda_0 | u; \xi)_i^j|\text{vac}\rangle_{\lambda_0}^{\lambda_0 - N\hat{1}} = \delta_j^i \left\{ k(\lambda_0 | u; \xi)_1(R(2u, \lambda_0 - (N - 1)\hat{1})_i^1_j) 
- R(2u, \lambda_0 + \hat{1})_{j1}^1_i \prod_{k=1}^{N} R(u - z_k, \lambda_0 - (N - k)\hat{1})_{1j}^1_i R(u + z_k, \lambda_0 - (N - k)\hat{1})_{j1}^1_i 
+ k(\lambda_0 | u; \xi)_j \prod_{k=1}^{N} R(u - z_k, \lambda_0 - (N - k)\hat{1})_{j1}^1_i R(u + z_k, \lambda_0 - (N - k)\hat{1})_{j1}^1_i \right\} \times |\text{vac}\rangle_{\lambda_0}^{\lambda_0 - N\hat{1}}, \quad i, j = 2, \ldots, n. \quad (4.32)$$

Keeping the definition of operators $\mathcal{A}$ (4.5) and $\mathcal{D}_j^\prime$ (4.6) in mind, and using the relations (4.30)-(4.32), we find that the pseudo-vacuum state given by (4.23) satisfies the following equations as required

$$\mathcal{A}(\lambda_0 - N\hat{1} | u)|\Omega\rangle = k(\lambda_0 | u; \xi)_1|\Omega\rangle, \quad (4.33)$$
\[ \mathcal{D}_j^i(\lambda_0 - N\hat{1}|u|\Omega) = \delta_j^i\beta^{(1)}(u)k(\lambda_0 |u + \frac{w}{2}; \xi - \frac{w}{2})_j \]
\[ \times \left\{ \prod_{k=1}^{N} \frac{\sigma(u + z_k)\sigma(u - z_k)}{\sigma(u + z_k + w)\sigma(u - z_k + w)} \right\} |\Omega\rangle, \quad i, j = 2, \ldots, n, \]  
\[ (4.34) \]
\[ \mathcal{C}_i(\lambda_0 - N\hat{1}|u|\Omega) = 0, \quad i = 2, \ldots, n, \]  
\[ (4.35) \]
\[ \mathcal{B}_i(\lambda_0 - N\hat{1}|u|\Omega) \neq 0, \quad i = 2, \ldots, n. \]  
\[ (4.36) \]

The function \( \beta^{(1)}(u) \) is
\[ \beta^{(1)}(u) = \frac{\sigma(2u)\sigma((\lambda_0)_1w + u + w + \xi)}{\sigma(2u + w)\sigma((\lambda_0)_1w + u + \xi)}. \]  
\[ (4.37) \]

In deriving the equation (4.34), we have used the following equation
\[ k(\lambda_0 |u; \xi)_j - k(\lambda_0 |u; \xi)_1R(2u, \lambda_0 + \hat{1})_{jj} = \beta^{(1)}(u)\frac{\sigma((\lambda_0)_1w)\sigma((\lambda_0)_jw + \xi - w - u)}{\sigma((\lambda_0)_1w + w)\sigma((\lambda_0)_jw + \xi + u)}, \]  
\[ (4.38) \]
which is a consequence of the identity of (2.3), the definitions (3.5) and (4.21).

Therefore, we have constructed the pseudo-vacuum state \(|\Omega\rangle\) which is the common eigenstate of the operators \( \mathcal{A}, \mathcal{D}_i^i, \quad i = 2, \ldots, n, \) and annihilated by the operators \( \mathcal{C}_i, \quad i = 2, \ldots, n. \) The operators \( \mathcal{B}_i, \quad i = 2, \ldots, n, \) will play the role of creation operators to generate the Bethe ansatz states.

### 4.3 Nested Bethe ansatz

After deriving the relevant commutation relations (4.7)-(4.9) and constructing the pseudo-vacuum state (4.23), we now apply the algebraic Bethe ansatz method (in this case, it is usually called nested Bethe ansatz) to solve the eigenvalue problem for the transfer matrices (2.18) of the \( Z_n \) Belavin model with open boundary condition specified by the K-matrices \( K^\pm(u) \) given in (2.13), (2.14) and (2.15). We assume that \( N = n \times l \) with \( l \) being a positive integer so that the algebraic Bethe ansatz method can be applied as in elliptic integrable models [3, 15, 14, 16].

Let us introduce a set of integers:
\[ N_i = (n - i) \times l, \quad i = 0, 1, \ldots, n - 1, \]  
\[ (4.39) \]
and \( \frac{n(n-1)}{2}l \) complex parameters \( \{v_k^{(i)} | k = 1, 2, \ldots, N_{i+1}\}, \quad i = 0, 1, \ldots, n - 2 \) for convenience. Like usual nested Bethe ansatz method [10, 14, 12, 16], the parameters \( \{v_k^{(i)}\} \) will be used to specify the eigenvectors of the corresponding reduced transfer matrices (see
below). They will be constrained later by the Bethe ansatz equations (4.57) and (4.64). For convenience, we adopt the following convention:

\[ v_k = v_k^{(0)}, \ k = 1, 2, \cdots, N_1. \]  

(4.40)

We will seek the common eigenvectors of the transfer matrices in the form

\[
|v_1, \cdots, v_{N_1} \rangle = \sum_{i_1, \cdots, i_{N_1} = 2}^n F^{i_1, i_2, \cdots, i_{N_1}} B_{i_1}(\lambda_0 + \hat{i}_1 - \hat{1}|v_1) B_{i_2}(\lambda_0 + \hat{i}_1 + \hat{i}_2 - 2\hat{1}|v_2) \\
\times \cdots B_{i_{N_1}}(\lambda_0 + \sum_{k=1}^{N_1} \hat{i}_k - N_1\hat{1}|v_{N_1})|\Omega\rangle,
\]  

(4.41)

in which \( F^{i_1, i_2, \cdots, i_{N_1}} \) are coefficients to be determined later by (4.56). The indices in the above equation should satisfy the following condition: the number of \( i_k = j \), denoted by \#(j), being

\[
\#(j) = l, \quad j = 2, \cdots, n.
\]  

(4.42)

With the above restriction, one can derive

\[
\lambda_0 + \sum_{k=1}^{N_1} \hat{i}_k - N_1\hat{1} = \lambda_0 + l \sum_{k=2}^n \hat{k} - (n - 1)l\hat{1} \\
= \lambda_0 + l \sum_{k=1}^n \hat{k} - nl\hat{1} = \lambda_0 - N\hat{1},
\]  

(4.43)

using the identity (3.1).

Noting the diagonal form of the K-matrix \( \tilde{K}(\lambda_0|u) \) (4.20) and its expression (4.22), and the decomposition form (4.1) of the transfer matrices, we can further rewrite the transfer matrices in the following desired form

\[
t(u; \xi) = \sum_{\nu=1}^n \tilde{k}(\lambda_0|u)_\nu T(\lambda_0|u; \xi)_{\nu}^u \\
= \tilde{k}(\lambda_0|u)_1 A(\lambda_0|u) + \sum_{i=2}^n \tilde{k}(\lambda_0|u)_i T(\lambda_0|u; \xi)_i \\
= \tilde{k}(\lambda_0|u)_1 A(\lambda_0|u) + \sum_{i=2}^n \tilde{k}(\lambda_0|u)_i R(2u, \lambda_0 + \hat{1})_{1i}^{1i} A(\lambda_0|u) \\
+ \sum_{i=2}^n \tilde{k}(\lambda_0|u)_i (T(\lambda_0|u; \xi)_i - R(2u, \lambda_0 + \hat{1})_{1i}^{1i} A(\lambda_0|u)).
\]
\[
= \sum_{i=1}^{n} \tilde{k}(\lambda_0|u) R(2u, \lambda_0 + 1)^{i_1}_{i} A(\lambda_0|u) \\
+ \sum_{i=2}^{n} \tilde{k}(\lambda_0|u + \frac{w}{2}) \cdot \frac{\sigma((\lambda_0)_{i} w - w)}{\sigma((\lambda_0)_{i} w)} (\mathcal{T}(\lambda_0|u; \xi)_{i} - R(2u, \lambda_0 + 1)^{i_1}_{i} A(\lambda_0|u)) \\
= \alpha^{(1)}(u) A(\lambda_0|u) + \sum_{i=2}^{n} \tilde{k}(\lambda_0|u + \frac{w}{2})_{i} D(\lambda_0|u)_{i}.
\]

(4.44)

We have used the definition (4.40) and introduced function \(\alpha^{(1)}(u)\) given by

\[
\alpha^{(1)}(u) = \sum_{i=1}^{n} \tilde{k}(\lambda_0|u) R(2u, \lambda_0 + 1)^{i_1}_{i},
\]

and reduced K-matrix \(\tilde{\mathcal{K}}^{(1)}(\lambda|u)\) as follows

\[
\tilde{\mathcal{K}}^{(1)}(\lambda_0|u)_{i}^{j} = \delta_{i}^{j} \tilde{k}(\lambda_0|u)_{i}, \quad i, j = 2, \ldots, n,
\]

(4.46)

\[
\tilde{k}(\lambda_0|u)_{i} = \left\{ \prod_{k \neq i, k=2}^{n} \frac{\sigma((\lambda_0)_{i} k w - w)}{\sigma((\lambda_0)_{i} k w)} \right\} \frac{\sigma((\lambda_0)_{i} w + \tilde{\xi} + u + \frac{(n-1)w}{2})}{\sigma((\lambda_0)_{i} w + \tilde{\xi} - u - \frac{(n-1)w}{2})}. \tag{4.47}
\]

Now, let us evaluate the action of \(\mathcal{D}^{(1)}(\lambda_0|u)\) on the state \(|v_1, \ldots, v_{N_1}\rangle\) given in (4.31). Many terms will appear when we move \(\mathcal{D}^{(1)}(\lambda_0|u)\) from the left to the right of \(\mathcal{B}(v_k)'s\). They can be classified into two types: wanted terms and unwanted terms. The wanted terms in \(\mathcal{D}^{(1)}(\lambda_0|u)|v_1, \ldots, v_{N_1}\rangle\) can be obtained by retaining the first term in the commutation relation (4.8). The unwanted terms arising from the second and third terms of (4.8), have some \(\mathcal{B}(v_k)\) replaced by \(\mathcal{B}(u)\). One unwanted term where \(\mathcal{B}(v_1)\) is replaced by \(\mathcal{B}(u)\) can be obtained by using firstly the second and third terms of (4.8), then repeatedly using the first term of (4.8) and (4.9). Thanks to the commutation relation (4.9), one can easily obtain the other unwanted terms where the other \(\mathcal{B}(v_k)\) is replaced by \(\mathcal{B}(u)\). Keeping the equation (4.43) and the properties of pseudo-vacuum state (4.33)-(4.34) in mind, we find the action of \(\mathcal{D}^{(1)}(\lambda_0|u)\) on the state \(|v_1, \ldots, v_{N_1}\rangle\)

\[
\mathcal{D}^{(1)}(\lambda_0|u)|v_1, \ldots, v_{N_1}\rangle = \prod_{k=1}^{N_1} \frac{\sigma(u - v_k + w)\sigma(u + v_k + 2w)}{\sigma(u - v_k)\sigma(u + v_k + w)} \prod_{k=1}^{N} \frac{\sigma(u + z_k)\sigma(u - z_k)}{\sigma(u + z_k + w)\sigma(u - z_k + w)} \\
\times \beta^{(1)}(u) T^{(1)}(\lambda_0|u + \frac{w}{2}; \tilde{\xi} - \frac{w}{2})_{i_1, \ldots, i_{N_1}} B^0_{v_1}(\lambda_0 + \xi - 1|v_1) \\
\times B^0_{v_2}(\lambda_0 + \xi - 1|v_2) \cdots B^0_{v_{N_1}}(\lambda_0 - N|v_{N_1})|\Omega \rangle F^{i_1, \ldots, i_{N_1}} \\
- \frac{\sigma(w)\sigma(2u + 2w)\sigma(u - v_1 + (\lambda_0)_{1\alpha} w - w)}{\sigma(u - v_1)\sigma(2u + w)\sigma((\lambda_0)_{1\alpha} w - w)} \prod_{k=2}^{N_1} \frac{\sigma(v_1 - v_k + w)\sigma(v_1 + v_k + 2w)}{\sigma(v_1 - v_k)\sigma(v_1 + v_k + w)}
\]

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where \( o.u.t \) stands for the other unwanted terms. We adopt here and in the following part of this subsection the convention that the repeated indices imply summation over 2, \( \cdots \), \( n \) (but we do not take the summation for the index \( i \) in the above equation). We have introduced reduced monodromy matrix \( \mathcal{T}^{(1)}(\lambda|u; \xi) \):

\[
\mathcal{T}^{(1)}_i(\lambda|u + \frac{w}{2}; \xi - \frac{w}{2}) = R(u + \frac{w}{2} + z^{(1)}_i, \lambda - i \rangle_{\beta_{N_i-1}}^{j} \rangle_{\alpha_{N_i-1}}^{i}
\times R(u + \frac{w}{2} + z^{(1)}_2, \lambda - i \rangle_{\beta_{N_i}}^{j} \rangle_{\alpha_{N_i-1}}^{i} \cdots R(u + \frac{w}{2} + z^{(1)}_1, \lambda - i \rangle_{\beta_{N_i}}^{j} \rangle_{\alpha_{N_i-1}}^{i} + \sum_{k=1}^{N_i-1} \hat{z}_k(\lambda)_{\alpha_{N_i-1}}^{i} \rangle_{\beta_{N_i-1}}^{j} \rangle_{\alpha_{N_i-1}}^{i}
\times R(u + \frac{w}{2} - z^{(1)}_{N_i-1}, \lambda + \sum_{k=1}^{N_i-1} \hat{z}_k(\lambda)_{\alpha_{N_i-1}}^{i} \rangle_{\beta_{N_i-1}}^{j} \rangle_{\alpha_{N_i-1}}^{i} \cdots R(u + \frac{w}{2} - z^{(1)}_1, \lambda + \hat{z}_1(\lambda)_{\alpha_{N_i-1}}^{i}
\]

(4.49)

where

\[
z^{(1)}_k = v_k + \frac{w}{2}, \quad k = 1, \cdots, N_i.
\]

(4.50)

The following property of \( R \)-matrix has been used to derive the equation \[4.48\]

\[
R(u, \lambda + \hat{1})^{ij}_{kl} = R(u, \lambda)^{ij}_{kl}, \quad i, j, k, l > 1.
\]

Similarly, we find the action of \( A(\lambda_0|u) \) on the state \( |v_1, \cdots, v_{N_i} \rangle \)

\[
A(\lambda_0|u)|v_1, \cdots, v_{N_i} \rangle = k(\lambda_0|u; \xi) \prod_{k=1}^{N_i} \frac{\sigma(u + v_k)\sigma(u - v_k - w)}{\sigma(u + v_k + w)\sigma(u - v_k)} |v_1, \cdots, v_{N_i} \rangle
\]
Using the results when $A(\lambda_0|u)$ and $D^j_i(\lambda_0|u)$ act on the state $|v_1,\ldots,v_{N_1}\rangle$ (4.51) and (4.48), and the equation (4.44), we can find the action of the transfer matrices on the state $|v_1,\ldots,v_{N_1}\rangle$

$$t(u;\xi)|v_1,\ldots,v_{N_1}\rangle = (\alpha^{(1)}(u)A(\lambda_0|u) + \sum_{i=2}^n \hat{k}^{(i)}(\lambda_0|u + \frac{w}{2})D(\lambda_0|u)^i_1)|v_1,\ldots,v_{N_1}\rangle$$

$$= \alpha^{(1)}(u)k(\lambda_0|u;\xi_1)\prod_{k=1}^{N_1} \frac{\sigma(u + v_k)\sigma(u - v_k - w)}{\sigma(u + v_k + w)\sigma(u - v_k)}|v_1,\ldots,v_{N_1}\rangle$$

$$+ \beta^{(1)}(u)\prod_{k=1}^{N_1} \frac{\sigma(u - v_k + w)\sigma(u + v_k + 2w)}{\sigma(u - v_k)\sigma(u + v_k + w)}\prod_{k=1}^{N_1} \frac{\sigma(u + z_k)\sigma(u - z_k)}{\sigma(u + z_k + w)\sigma(u - z_k + w)}$$

$$\times t^{(1)}(u + \frac{w}{2};\xi - \frac{w}{2})^{\hat{v}_1''\cdots\hat{v}_{N_1}''}_i|\Omega\rangle F^{i_1\cdots i_{N_1}}$$

$$+ u.t., \quad (4.52)$$

where $u.t.$ stands for the unwanted terms. Reduced transfer matrices $t^{(1)}(u;\xi)$ are given in terms of the reduced monodromy matrix (4.49) and the reduced K-matrix (4.46) and (4.47)

$$t^{(1)}(u;\xi) = \sum_{i=2}^n \hat{k}^{(i)}(\lambda_0|u);T^{(1)i}_j(\lambda_0|u;\xi). \quad (4.53)$$

The equation (4.52) tells that the state $|v_1,\ldots,v_{N_1}\rangle$ is not an eigenvector of the transfer matrices $t(u;\xi)$ unless $F's$ are the eigenvectors of the reduced transfer matrices $t^{(1)}(u;\xi)$ as (4.56). The condition that the unwanted terms should cancel each other, will give a
restriction on the $N_1$ parameters $\{v_k\}$, the so-called Bethe ansatz equation. Hence we arrive at the final results:

$$t(u; \xi | v_1, \ldots, v_{N_1}) = \Lambda(u; \xi, \{v_k\}| v_1, \ldots, v_{N_1}).$$  \hfill (4.54)

The eigenvalue reads

$$\Lambda(u; \xi, \{v_k\}) = \alpha^{(1)}(u) \frac{\sigma((\lambda_0)_1 w + \xi - u)}{\sigma((\lambda_0)_1 w + \xi + u)} \prod_{k=1}^{N_1} \frac{\sigma(u + v_k)\sigma(u - v_k - w)}{\sigma(u + v_k + w)\sigma(u - v_k)}
+ \frac{\sigma(2u)\sigma((\lambda_0)_1 w + u + w + \xi)}{\sigma(2u + w)\sigma((\lambda_0)_1 w + u + \xi)} \prod_{k=1}^{N_1} \frac{\sigma(u - v_k + w)\sigma(u + v_k + 2w)}{\sigma(u - v_k)\sigma(u + v_k + w)} \times \prod_{k=1}^{N_0} \frac{\sigma(u + z_k)\sigma(u - z_k)}{\sigma(u + z_k + w)\sigma(u - z_k + w)} \Lambda^{(1)}(u + \frac{w}{2}; \xi - \frac{w}{2}, \{v^{(1)}\}),$$  \hfill (4.55)

in which the functions $\alpha^{(1)}(u)$ is given in (4.45) and $\Lambda^{(1)}(u; \xi, \{v^{(1)}_k\})$ is the eigenvalue of the reduced transfer matrices defined in (4.53)

$$t^{(1)}(u; \xi)i^{i_1 \cdots \iota_{N_1}}_1 \Lambda^{(1)}(u; \xi, \{v^{(1)}_k\}) F^{i_1 \cdots \iota_{N_1}}_1 = \Lambda^{(1)}(u; \xi, \{v^{(1)}_k\}) F^{i_1 \cdots \iota_{N_1}}_1. \hfill (4.56)$$

The parameters $\{v_k|k = 1, \ldots, N_1\}$ satisfy so-called Bethe ansatz equation

$$\alpha^{(1)}(v_s) \frac{\sigma((\lambda_0)_1 w + \xi - v_s)\sigma(2v_s + w)}{\sigma((\lambda_0)_1 w + \xi + v_s + w)\sigma(2v_s + 2w)} \prod_{k \neq s, k=1}^{N} \frac{\sigma(v_s + v_k)\sigma(v_s - v_k - w)}{\sigma(v_s + v_k + 2w)\sigma(v_s - v_k + w)} \times \prod_{k=1}^{N} \frac{\sigma(v_s + z_k)\sigma(v_s - z_k)}{\sigma(v_s + z_k + w)\sigma(v_s - z_k + w)} \Lambda^{(1)}(v_s + \frac{w}{2}; \xi - \frac{w}{2}, \{v^{(1)}_k\}).$$  \hfill (4.57)

The parameters $\{v^{(1)}_k|k = 1, \ldots, N_2\}$ will be specified later by the further Bethe ansatz equation (4.64).

The diagonalization of the transfer matrices $t(u; \xi)$ of $\mathbb{Z}_n$ Belavin model with open boundary condition specified by the K-matrices $K^{\pm}(u)$ given in (2.13), (2.14) and (2.15) is now reduced to the diagonalization of the reduced transfer matrices $t^{(1)}(u; \xi)$ in (4.56). The explicit expression of $t^{(1)}(u; \xi)$ given in (4.53) and (4.49) implies that $t^{(1)}(u; \xi)$ can be considered as the transfer matrices of a $\mathbb{Z}_{n-1}$ Belavin model with open boundary condition: the corresponding quantum space (resp. inhomogeneous parameters $\{z_k\}$ in (2.16) ) is replaced by $\mathbb{C}^{n-1}$ resp. $\{z^{(1)}_k\}$); the boundary parameter $\xi$ for $K(\lambda_0|u; \xi)$ given in (4.21) is replaced by $\xi^{(1)} = \xi - \frac{w}{2}$ (see (4.49)); at the same time the corresponding $\bar{K}(\lambda_0|u)$ is replaced by the
Finally, we obtain all the eigenvalues of the reduced transfer matrices \( t^{(1)}(u; \xi) \) as we have done for the diagonalization of \( t(u; \xi) \). Repeating the above procedure further \( n - 2 \) times, one can reduce to the last reduced transfer matrices \( t^{(n-1)}(u; \xi) \) which are trivial to get the eigenvalues. This is so-called nested Bethe ansatz. At the same time we need to introduce the \( \frac{n(n-1)}{2} l \) parameters \( \{v_k^{(i)}| k = 1, \cdots, N_i+1 \}, i = 0, 1, \cdots, n - 2 \) to specify the eigenvectors of the corresponding reduced transfer matrices \( t^{(i)}(u; \xi) \) (including the original one \( t(u; \xi) = t^{(0)}(u; \xi) \)), and the further reduced K-matrices \( \tilde{K}^{(m)}(\lambda_0|u) \) (including the original one \( \tilde{K}(\lambda_0|u) = \tilde{K}^{(0)}(\lambda_0|u) \)) as we have done in (4.46) and (4.47). The reduced K-matrices are given as follows

\[
\hat{K}^{(m)}(\lambda_0|u)^2 = \delta_{ij}^2 \hat{k}^{(1)}(\lambda_0|u)_i, \quad i, j = m + 1, \cdots, n, \quad m = 0, \cdots, n - 1, \quad (4.58)
\]

\[
\hat{k}^{(m)}(\lambda_0|u)_i = \left\{ \prod_{k \neq i, k = m+1}^{n} \frac{\sigma((\lambda_0)_{ik} w - w)}{\sigma((\lambda_0)_{ik} w)} \right\} \frac{\sigma((\lambda_0)_{i} w + \xi + u + \frac{(n-m)w}{2})}{\sigma((\lambda_0)_{i} w + \xi - u - \frac{(n-m)w}{2})}. \tag{4.59}
\]

Like (4.45), we introduce a set of functions \( \{\alpha^{(m)}(u)|m = 1, \cdots, n-1 \} \) related to the reduced K-matrices \( \hat{K}^{(m)}(\lambda_0|u) \)

\[
\alpha^{(m)}(u) = \sum_{i=m}^{n} R(2u, \lambda_0 + \tilde{m})^{m} \hat{k}^{(m-1)}(\lambda_0|u)_i, \quad m = 1, \cdots, n. \tag{4.60}
\]

Finally, we obtain all the eigenvalues of the reduced transfer matrices \( t^{(i)}(u; \xi) \) with the eigenvalue \( \Lambda^{(i)}(u; \xi, \{v_k^{(i)}\}) \) in a recurrence form

\[
\Lambda^{(i)}(u; \xi, \{v_k^{(i)}\}) = \alpha^{(i+1)}(u) \frac{\sigma((\lambda_0)_{i+1} w + \xi + u - w)}{\sigma((\lambda_0)_{i+1} w + \xi + u)} \prod_{k=1}^{N_i+1} \frac{\sigma(u + v_k^{(i)} + w)\sigma(u - v_k^{(i)} - w)}{\sigma(u + v_k^{(i)} + w)\sigma(u - v_k^{(i)} - w)} \\
+ \frac{\sigma(2u)\sigma((\lambda_0)_{i+1} w + u + \xi + u)}{\sigma(2u + w)\sigma((\lambda_0)_{i+1} w + \xi + u)} \prod_{k=1}^{N_i+1} \frac{\sigma(u - v_k^{(i)} + w)\sigma(u + v_k^{(i)} + 2w)}{\sigma(u - v_k^{(i)} + w)\sigma(u + v_k^{(i)} + 2w)} \\
\times \prod_{k=1}^{N_i} \frac{\sigma(u + z_k^{(i)} + w)\sigma(u - z_k^{(i)} + w)}{\sigma(u + z_k^{(i)} + w)\sigma(u - z_k^{(i)} + w)} \Lambda^{(i+1)}(u + \frac{w}{2}, \xi + \frac{w}{2}, \{v^{(i+1)}\}),
\]

\[
\Lambda^{(n-1)}(u; \xi^{(n-1)}) = \frac{\sigma((\lambda_0)_n w + \xi + u + \frac{w}{2})\sigma((\lambda_0)_n w + \xi^{(n-1)} - u)}{\sigma((\lambda_0)_n w + \xi - u - \frac{w}{2})\sigma((\lambda_0)_n w + \xi^{(n-1)} + u)}. \tag{4.61}
\]

The reduced boundary parameters \( \{\xi^{(i)}\} \) and inhomogeneous parameters \( \{z_k^{(i)}\} \) are given by

\[
\xi^{(i+1)} = \xi^{(i)} - \frac{w}{2}, \quad z_k^{(i+1)} = v_k^{(i)} + \frac{w}{2}, \quad i = 0, \cdots, n - 2. \tag{4.62}
\]
We adopt the convention: \( \xi = \xi^{(0)}, \ z_{k}^{(0)} = z_{k} \). The \( \{v_{k}^{(i)}\} \) satisfy the following Bethe ansatz equations

\[
\alpha^{(i+1)}(v_{s}^{(i)}) \frac{\sigma((\lambda_{0})_{i+1}w + \xi^{(i)} - v_{s}^{(i)})\sigma(2v_{s}^{(i)} + w)}{\sigma((\lambda_{0})_{i+1}w + \xi^{(i)} + v_{s}^{(i)} + w)\sigma(2v_{s}^{(i)} + 2w)}
\times \prod_{k \neq s, k=1}^{N_{k+1}} \frac{\sigma(v_{s}^{(i)} + v_{k}^{(i)})\sigma(v_{s}^{(i)} - v_{k}^{(i)} - w)}{\sigma(v_{s}^{(i)} + v_{k}^{(i)} + 2w)\sigma(v_{s}^{(i)} - v_{k}^{(i)} + w)}
\]

\[
= \prod_{k=1}^{N_{i}} \frac{\sigma(v_{s}^{(i)} + z_{k}^{(i)})\sigma(v_{s}^{(i)} - z_{k}^{(i)})}{\sigma(v_{s}^{(i)} + z_{k}^{(i)} + w)\sigma(v_{s}^{(i)} - z_{k}^{(i)} + w)} \Lambda^{(i+1)}(v_{s}^{(i)} + \frac{w}{2}; \xi^{(i)} - \frac{w}{2}; \{v_{k}^{(i+1)}\}),
\]

(4.64)

5 Conclusions

We have studied \( Z_{n} \) Belavin model with integrable open boundary condition which are described by the boundary K-matrix \( K^{-}(u) \) given in (2.13) and its dual \( K^{+}(u) \) given in (2.14) with restriction (2.15). The total number of the independent free boundary parameters among \( \xi, \bar{\xi} \) and \( (\lambda_{0})_{i}, i = 1, \ldots, n \), is actually \( n + 1 \). Although the K-matrices are non-diagonal in the vertex form, they simultaneously become diagonal ones in the face type after the face-vertex transformation which are given in (4.20)-(4.22) (c.f. (2.13) and (2.14)). This fact enables us to successfully construct the corresponding pseudo-vacuum state \( |\Omega\rangle \) (4.23) and apply the algebraic Bethe ansatz method to diagonalize the corresponding double-row transfer matrices after using the intertwiner vectors and face-vertex correspondence relation. The eigenvalues of the transfer matrices and associated Bethe ansatz equations are given by (4.55), (4.61)-(4.62), and (4.57), (4.64). For the special case of \( n = 2 \) (or the eight-vertex model case), our result recovers that of [14].

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\footnote{One can fix one of \( \{(\lambda_{0})_{i}\} \), for an example, \( (\lambda_{0})_{n} = 0 \) by shifting \( \xi \rightarrow \xi - (\lambda_{0})_{n}w \) in (1.21).}
Appendix A: The exchange relation of $\mathcal{T}$

The starting point for deriving the exchange relations (A.4) among $\mathcal{T}(\lambda|u; \xi)_{\mu}^\nu$ is the exchange relation (2.17). Multiplying both sides of equation (2.17) from the right by $\phi_{\lambda+i_3, \lambda}(-u_1) \otimes \phi_{\lambda+i_3+j_3, \lambda+i_3}(-u_2)$, we can derive the following equations with the help of the face-vertex correspondence relation (3.3), the definition of $\mathcal{T}(\lambda|u; \xi)_{\mu}^\nu$ (1.3), and relation (3.13)

$$L.H.S. = R^B_{i_2}(u_1 - u_2)T_1(u_1)R^B_{i_2}(u_1 + u_2)(\phi_{\lambda+i_3, \lambda}(-u_1) \otimes T(u_2)\phi_{\lambda+i_3+j_3, \lambda+i_3}(-u_2))$$

$$= R^B_{i_2}(u_1 - u_2)T_1(u_1)R^B_{i_2}(u_1 + u_2)(\phi_{\lambda+i_3, \lambda}(-u_1) \otimes 1)$$

$$\times (1 \otimes \{ \sum_{j_2} \phi_{\lambda+i_3+j_2, \lambda+i_3}(u_2)\phi_{\lambda+i_3+j_2, \lambda+i_3}(u_2)T(u_2)\phi_{\lambda+i_3+j_3, \lambda+i_3}(-u_2) \})$$

$$= \sum_{j_2} R^B_{i_2}(u_1 - u_2)T_1(u_1)R^B_{i_2}(u_1 + u_2)(\phi_{\lambda+i_3, \lambda}(-u_1) \otimes \phi_{\lambda+i_3+j_2, \lambda+i_3}(u_2))$$

$$\times \mathcal{T}(\lambda + \hat{i}_3 + \hat{j}_3|u_2; \xi)_{j_3}^{j_2}$$

$$= \sum_{i_2, j_1, j_2} \sum_{t_1, t_2, t_3} R^B_{i_2}(u_1 - u_2)T_1(u_1)R(u_1 + u_2, \lambda)^{j_1}_1 \mathcal{T}(\lambda + \hat{i}_3 + \hat{j}_3|u_2; \xi)_{j_3}^{j_2}$$

$$\times R(u_1 + u_2, \lambda)^{j_1}_1 \mathcal{T}(\lambda + \hat{i}_3 + \hat{j}_3|u_2; \xi)_{j_3}^{j_2}.$$  \hfill (A.1)

Similarly, we have

$$R.H.S. = \sum_{i_0, j_0} (\phi_{\lambda+i_0, \lambda}(-u_1) \otimes \phi_{\lambda+i_0+j_0, \lambda+i_0}(u_2)) \sum_{i_1, i_2, j_1, j_2} R(u_1 - u_2, \lambda)^{i_0}_1 \mathcal{T}(\lambda + \hat{i}_0 + \hat{j}_1|u_1; \xi)_{j_1}^{i_1} R(u_1 + u_2, \lambda)^{i_2}_1 \mathcal{T}(\lambda + \hat{i}_1 + \hat{j}_2|u_2; \xi)_{j_2}^{i_2}$$

$$\times T(\lambda + \hat{i}_0 + \hat{j}_1|u_1; \xi)_{j_1}^{i_1} R(u_1 - u_2, \lambda)^{j_1}_i \mathcal{T}(\lambda + \hat{i}_1 + \hat{j}_2|u_2; \xi)_{j_2}^{i_2}.$$  \hfill (A.2)

Noting the fact that intertwiners are linearly independent of each other with a generic $\lambda$, we obtain the exchange relation (A.4) by comparing (A.1) with (A.2).

Appendix B: The relevant commutation relations

Let us adopt the following notation for convenience

$$A(\lambda|u) = \mathcal{T}(\lambda|u; \xi)_{1}^{1}, \quad B_i(\lambda|u) = \mathcal{T}(\lambda|u; \xi)_{1}^{i}, \quad i = 2, \cdots, n,$$  \hfill (B.1)

$$D_j^i(\lambda|u) = \mathcal{T}(\lambda|u; \xi)_{j}^{i}, \quad i, j = 2, \cdots, n.$$  \hfill (B.2)
The starting point for deriving the commutation relations among $A(u)$, $D_i^j(u)$ and $B_i(u)$ ($i, j = 2, \ldots, n$) is the exchange relation (4.4).

For $i_0 = j_0 = j_3 = 1$, $i_3 = i \neq 1$, we obtain

\[ A(\lambda + 2\hat{1}|u)B_i(\lambda + \hat{1} + \hat{i}|u) = \frac{\sigma(u + v)\sigma(u - v + w)}{\sigma(u + v + w)\sigma(u - v)}B_i(\lambda + \hat{1} + \hat{i}|u)A(\lambda + \hat{1} + \hat{i}|u) - \frac{\sigma(w)\sigma(u + v)\sigma(\lambda_1 w + u - v)}{\sigma(u - v)\sigma(u + v + w)\sigma(\lambda_1 w)}B_i(\lambda + \hat{1} + \hat{i}|u)A(\lambda + \hat{1} + \hat{i}|u) - \sum_{j=2}^{n} \frac{\sigma(w)\sigma(u + v + \lambda_j w)}{\sigma(u + v + w)\sigma(\lambda_j w)}B_j(\lambda + \hat{1} + \hat{j}|u)D_i^j(\lambda + \hat{1} + \hat{i}|u). \quad (B.3) \]

One can derive the following equation

\[
\frac{R(u + v, \lambda)_{ij}^1 R(u - v, \lambda)_{i1}^1}{R(u + v, \lambda)_{i1}^1 R(u - v, \lambda)_{i1}^1} + \frac{R(u + v, \lambda)_{ij}^1 R(2u, \lambda + \hat{1})_{i1}^1}{R(u + v, \lambda)_{i1}^1 R(2u, \lambda + \hat{1})_{i1}^1} = \frac{\sigma(w)\sigma(2u)\sigma(\lambda_1 w + u - v + w)}{\sigma(u - v)\sigma(2u + w)\sigma(\lambda_1 w + w)}. \quad (B.4)
\]

from the identity (2.3), the definitions (3.4) and (3.5). The commutation relation (4.4) is a simple consequence of the equations (B.3) and (B.4), the definitions (1.5) and (1.6).

For $i_0 = k \neq 1$, $j_0 = 1$, $i_3 = i \neq 1$ and $j_3 = j \neq 1$, we obtain

\[
D_i^k(\lambda + \hat{1} + \hat{i}|u)B_j(\lambda + \hat{j} + \hat{i}|v) = \sum_{\alpha_1, \alpha_2, \beta_1, \beta_2 = 2}^{n} \frac{R(u + v, \lambda)_{ij}^{k\beta_2} R(u - v, \lambda)_{ij}^{\beta_1\alpha_1}}{R(u - v, \lambda)_{ij}^{k\alpha_1} R(2u, \lambda + \hat{1})_{i1}^{\beta_1}} \times B_{\alpha_2}(\lambda + \hat{k} + \hat{\beta}_2 |u)D_{\alpha_1}^{\beta_2}(\lambda + \hat{i} + \hat{j}|u) + \sum_{\alpha_2 = 2}^{n} \frac{R(u + v, \lambda)_{ij}^{k1} R(u - v, \lambda)_{ij}^{\alpha_2\alpha_1}}{R(u - v, \lambda)_{ij}^{k1} R(2u, \lambda + \hat{1})_{i1}^{\alpha_1}} A(\lambda + \hat{k} + \hat{1}|u)B_{\alpha}(\lambda + \hat{i} + \hat{j}|u) - \sum_{\alpha_1, \beta_2 = 2}^{n} \frac{R(u + v, \lambda)_{ij}^{k1} R(u + v, \lambda)_{ij}^{\alpha_1\beta_2}}{R(u - v, \lambda)_{ij}^{k1} R(2u, \lambda + \hat{1})_{i1}^{\beta_2}} A(\lambda + \hat{k} + \hat{1}|u)B_{\alpha}(\lambda + \hat{i} + \hat{j}|u) - \frac{R(u + v, \lambda)_{ij}^{k1} R(u + v, \lambda)_{ij}^{1\beta_2}}{R(u - v, \lambda)_{ij}^{k1} R(2u, \lambda + \hat{1})_{i1}^{1}} A(\lambda + \hat{k} + \hat{1}|u)B_{\alpha}(\lambda + \hat{i} + \hat{j}|u). \quad (B.5)
\]

In order to separate the contribution of $A$ and $D_i^j$ in the above relations, we need to introduce (cf. (6)):

\[
\bar{D}_i^k(\lambda|u) = D_i^k(\lambda|u) - \delta_i^k R(2u, \lambda + \hat{1})_{k1}^{1}A(\lambda|u), \quad i, k = 2, \ldots, n. \quad (B.6)
\]

We can derive the commutation relations among $\bar{D}_i^k$ and $B_j$ from (B.5)

\[
\bar{D}_i^k(\lambda|u)B_j(\lambda + \hat{j} - \hat{1}|v) = \sum_{\alpha_1, \alpha_2, \beta_1, \beta_2 = 2}^{n} \frac{R(u + v, \lambda - \hat{1} - \hat{i})_{\alpha_2\beta_2} R(u - v, \lambda - \hat{1} - \hat{i})_{\alpha_1\beta_1}^1}{R(u - v, \lambda - \hat{1} - \hat{i})_{\alpha_2\beta_2} R(2u, \lambda + \hat{1} - \hat{i})_{\alpha_1\beta_1}^{11}} A(\lambda + \hat{k} + \hat{1}|u)B_{\alpha}(\lambda + \hat{i} + \hat{j}|u). \quad (B.7)
\]
which leads to the commutation relation (4.9).

\[
\sum_{\alpha,\beta=2}^{n} \frac{R(u - v, \lambda - \hat{1} - \hat{i}^{\hat{1}}_{\hat{1}1}R(u + v, \lambda - \hat{1} - \hat{i}^{\hat{1}}_{\hat{1}1})B_{\alpha}(\lambda)\beta}{R(u - v, \lambda - \hat{1} - \hat{i}^{\hat{1}}_{\hat{1}1}R(u + v, \lambda - \hat{1} - \hat{i}^{\hat{1}}_{\hat{1}1})}B_{\alpha}(\lambda + \hat{1} + \hat{j})|v) \bar{D}_{\hat{1}1}^{\hat{1}}(\lambda - \hat{1} + \hat{j})|v) \\
- \sum_{\alpha,\beta=2}^{n} \frac{R(u - v, \lambda - \hat{1} - \hat{i}^{\hat{1}}_{\hat{1}1}R(u + v, \lambda - \hat{1} - \hat{i}^{\hat{1}}_{\hat{1}1})B_{\alpha}(\lambda + \hat{1} + \hat{j})|v) \bar{D}_{\hat{1}1}^{\hat{1}}(\lambda - \hat{1} + \hat{j})|v) \\
+ \sum_{\alpha,\beta=2}^{n} \frac{R(u + v, \lambda - \hat{1} - \hat{i}^{\hat{1}}_{\hat{1}1}R(u - v, \lambda - \hat{1} - \hat{i}^{\hat{1}}_{\hat{1}1})B_{\alpha}(\lambda + \hat{1} + \hat{j})|v) \bar{D}_{\hat{1}1}^{\hat{1}}(\lambda - \hat{1} + \hat{j})|v) \\
- \sum_{\alpha=2}^{n} \frac{R(u - v, \lambda - \hat{1} - \hat{i}^{\hat{1}}_{\hat{1}1}R(u + v, \lambda - \hat{1} - \hat{i}^{\hat{1}}_{\hat{1}1})B_{\alpha}(\lambda + \hat{1} + \hat{j})|v) \bar{D}_{\hat{1}1}^{\hat{1}}(\lambda - \hat{1} + \hat{j})|v) \\
+ \sum_{\alpha=2}^{n} \frac{R(u + v, \lambda - \hat{1} - \hat{i}^{\hat{1}}_{\hat{1}1}R(u - v, \lambda - \hat{1} - \hat{i}^{\hat{1}}_{\hat{1}1})B_{\alpha}(\lambda + \hat{1} + \hat{j})|v) \bar{D}_{\hat{1}1}^{\hat{1}}(\lambda - \hat{1} + \hat{j})|v) \\
\sigma(u - v + w)\sigma(u + v + w)\sigma(2u + \lambda_{1k}w)\sigma(\lambda_{1k}w)\sigma(w) \\
\sigma(u - v)\sigma(u + v)\sigma(2u + w)\sigma(\lambda_{1k}w - w)\sigma(\lambda_{1k}w + w) \\
\times \delta^k A(\lambda - \hat{1} + \hat{k})|u)B_j(\lambda + \hat{j} - \hat{1})|v). \\
(B.7)
\]

We have used the following equation

\[
\frac{R(u - v, \lambda - \hat{1} - \hat{i}^{\hat{1}}_{\hat{1}1}R(u + v, \lambda - \hat{1} - \hat{i}^{\hat{1}}_{\hat{1}1})B_{\alpha}(\lambda)\beta}{R(u - v, \lambda - \hat{1} - \hat{i}^{\hat{1}}_{\hat{1}1}R(u + v, \lambda - \hat{1} - \hat{i}^{\hat{1}}_{\hat{1}1})}B_{\alpha}(\lambda + \hat{1} + \hat{j})|v) \bar{D}_{\hat{1}1}^{\hat{1}}(\lambda - \hat{1} + \hat{j})|v) \\
= \frac{\sigma(u - v + w)\sigma(u + v + w)\sigma(2u + \lambda_{1k}w)\sigma(\lambda_{1k}w)\sigma(w) \\
\sigma(u - v)\sigma(u + v)\sigma(2u + w)\sigma(\lambda_{1k}w - w)\sigma(\lambda_{1k}w + w),}{(B.8)}
\]

to derive the last term on the right side of the above equation. The equation \(B.8\) is a consequence of the identity \((B.3)\). Again using the identity \((B.3)\), after some long tedious calculation, we finally obtain the commutation relation \((B.8)\) from the above exchange relation by noting the definitions \((B.5)\) and \((B.6)\).

For \(i_0 = j_0 = 1, i_3 = i \neq 1\) and \(j_3 = j \neq 1\), we obtain

\[
B_i(\lambda + \hat{i} + \hat{1})|u)B_j(\lambda + \hat{i} + \hat{j})|v) = \sum_{\beta,\alpha=2}^{n} \frac{R(u - v, \lambda)\beta\alpha R(u + v, \lambda)\alpha\beta}{R(u - v, \lambda)\beta\alpha R(u + v, \lambda)\alpha\beta} \\
\times B_\beta(\lambda + \hat{i} + \hat{j})|v)B_\alpha(\lambda + \hat{i} + \hat{j})|u), \\
(B.9)
\]

which leads to the commutation relation \((B.9)\).
Appendix C: The action of $T^i_j$ on the pseudo-vacuum state

Using the same method as applied for the calculation of the actions of $T^1_1$ (4.30) and $T^i_1$ (4.31) on the pseudo-vacuum state (4.28) in subsection 4.2, we have

\[
\begin{align*}
T(\lambda_0 - N\hat{1}, \lambda_0|u; \xi)_j^i|\text{vac}\rangle^{\lambda_0 - N\hat{1}} &= k(\lambda_0|u; \xi)_1 T(\lambda_0 - N\hat{1} - \hat{i}, \lambda_0 - \hat{i}|u)_1^i \\
&\times S(\lambda_0 - N\hat{1}, \lambda_0|u)_j^i|\text{vac}\rangle^{\lambda_0 - N\hat{1}} \\
&+ \delta^i_j k(\lambda_0|u; \xi)_j^i \prod_{k=1}^{N} R(u - z_k, \lambda_0 - (N - k)\hat{1})^i_j R(u + z_k, \lambda_0 - (N - k)\hat{1})^i_j \\
&\times |\text{vac}\rangle^{\lambda_0 - N\hat{1}}, \ i, j = 2, \ldots, n. \tag{C.1}
\end{align*}
\]

The first term on the right hand of the above equation cannot be calculated directly by the same method. However, we can calculate the first term by the following way.

We can derive the following exchange relations from “RLL” relation (2.10)

\[
T_1(u)R^{B}_{12}(2u)T_2^{-1}(-u) = T_2^{-1}(-u)R^{B}_{12}(2u)T_1(u). \tag{C.2}
\]

Multiplying both sides of the above equation from the left by $\tilde{\phi}_{\lambda_0 - N\hat{1} - \hat{i}, \lambda_0 - \hat{i}}(u) \otimes \tilde{\phi}_{\lambda_0 + \hat{i}, \lambda_0}(u)$ and from the right by $\phi_{\lambda_0 + \hat{i}, \lambda_0}(u) \otimes \phi_{\lambda_0 - N\hat{1} - \hat{i}}(-u)$, we obtain the following exchange relation from the face-vertex correspondence relations (3.9) and (3.14) - (3.17)

\[
\sum_{\alpha=1}^{n} R(2u, \lambda_0 + \hat{i})^1_\alpha T(\lambda_0 - N\hat{1} - \hat{i}, \lambda_0 - \hat{i}|u)_\alpha^i S(\lambda_0 - N\hat{1} + \hat{i}, \lambda_0 + \hat{i}|u)_\alpha^i \\
= \sum_{\alpha, \beta=1}^{n} R(2u, \lambda_0 - N\hat{1} + \hat{i})^\beta_\alpha S(\lambda_0 - N\hat{1} + \hat{i}, \lambda_0 + \hat{i}|u)_\beta^i T(\lambda_0 - N\hat{1}, \lambda_0|u)_\alpha^i. \tag{C.3}
\]

Acting both sides on the pseudo-vacuum state $|\text{vac}\rangle^{\lambda_0 - N\hat{1}}$, and using the equations (4.24) - (4.29), we obtain

\[
\begin{align*}
T(\lambda_0 - N\hat{1} - \hat{i}, \lambda_0 - \hat{i}|u)_j^i S(\lambda_0 - N\hat{1}, \lambda_0|u)_i^j|\text{vac}\rangle^{\lambda_0 - N\hat{1}} &= \delta^i_j \left\{ R(2u, \lambda_0 - (N - 1)\hat{1})_i^i \\
&- R(2u, \lambda_0 + \hat{i})_i^i \prod_{k=1}^{N} R(u - z_k, \lambda_0 - (N - k)\hat{1})_j^j R(u + z_k, \lambda_0 - (N - k)\hat{1})_j^j \right\} \\
&\times |\text{vac}\rangle^{\lambda_0 - N\hat{1}}. \tag{C.4}
\end{align*}
\]

The equation (4.32) is a simple consequence of the equations (C.1) and (C.4).
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