Fixity of elusive groups and the polycirculant conjecture

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Abstract. Let $G \leq \text{Sym}(\Omega)$ be transitive. Then $G$ is called elusive on $\Omega$ if it has no fixed point free element of prime order. The 2-closure of $G$, denoted by $G^{(2)},$ is the largest subgroup of $\text{Sym}(\Omega)$ whose orbits on $\Omega \times \Omega$ are the same orbits of $G$. $G$ is called 2-closed on $\Omega$ if $G = G^{(2)},$. The polycirculant conjecture states that there is no 2-closed elusive group. In this paper, we study the fixity of elusive groups, where the fixity of $G$ is the maximal number of fixed points of a non-trivial element of $G$. In particular, we prove that there is no 2-closed elusive solvable group of fixity at most 5, a partial answer to the polycirculant conjecture.

Keywords: Fixity, elusive group, polycirculant conjecture.

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1. Introduction

The concept of fixity was introduced by Rosen in 1980 to study of permutation groups of prime power order. Let $G$ be a permutation group on a set $\Omega$. We say that $G$ has fixity $f = f(G)$ if no non-trivial element of $G$ fixes more than $f$ letters, and there is a non-trivial element of $G$ fixing exactly $f$ letter [11].

Let $G$ be a transitive permutation group. Then $f(G) = 0$ if and only if $G$ is regular. Furthermore, $f(G) = 1$ if and only if $G$ is a Frobenius group. For some applications of the fixity of permutation groups, we refer the reader to [8, 9, 11].

A permutation $g$ of a set $\Omega$ is said to be a derangement if it has no fixed-point on $\Omega$, equivalently for each $\alpha \in \Omega$, $\alpha^g \neq \alpha$. A transitive permutation group $G \leq \text{Sym}(\Omega)$ is called elusive on $\Omega$ if it has no fixed point free element of prime order. Let $G$ be a permutation group on a set $\Omega$. The 2-closure of $G,$
denoted by $G^{(2),\Omega}$, is the largest subgroup of $\text{Sym}(\Omega)$ whose orbits on $\Omega \times \Omega$ are the same orbits of $G$. Then $G$ is called 2-closed on $\Omega$ if $G = G^{(2),\Omega}$.

In 1981, Marušič asked whether there exists a vertex-transitive digraph without a non-identity automorphism having all of its orbits of the same length [10, Problem 2.4]. In 1988, independently, the above problem was again proposed by Jordan [6]. In the 15th British combinatorial conference, in 1995, Klin proposed a more general question in the context of 2-closed groups [7]: “Is there a 2-closed transitive permutation group containing no fixed-point-free element of prime order?” After decades, not only has a positive answer to Klin’s question not been found, based on the evidence, mathematicians conjecture that there is no positive answer to this question. This conjecture is known as polycirculant conjecture. Equivalently, the polycirculant conjecture states that no 2-closed transitive permutation group is elusive. The conjecture is still open. We refer the reader to [1] to a survey on the recent results and future directions of the polycirculant conjecture.

In this paper, we study the fixity of elusive groups. Then we prove that every transitive 2-closed solvable permutation group of fixity at most 5 confirms the polycirculant conjecture.

2. Main Results

First we collect some notations we need later. Let $G$ be a finite group and $\Omega$ be a non-empty set. We denote the center of $G$ and the set of all prime divisors the order of $G$ by $Z(G)$ and $\pi(G)$, respectively. Also $\text{Sym}(\Omega)$ denotes the group of all permutations on $\Omega$. Let $G$ acts on $\Omega$ and $\alpha \in \Omega$. We denote the stabilizer of $\alpha$ in $G$ and the orbit of $G$ containing $\alpha$ by $G_\alpha$ and $\alpha^G$, respectively. Also $\text{Fix}_\Omega(G)$ denotes the fixed points of $G$ on $\Omega$, the set of all elements of $\Omega$ which fixes by all elements of $G$. Furthermore, $G^{\Omega}$ denotes the homomorphic image of the action of $G$ on $\Omega$ which is a subgroup of $\text{Sym}(\Omega)$. Recall that $G$ is called a permutation group on $\Omega$ if $G$ is isomorphic to $G^{\Omega}$. For the notations and terminology not defined here, we refer the reader to [2].

Let $G$ be a permutation group on a set $\Omega$. Recall that fixity of $G$ is the maximal number of fixed points of a non-trivial element of $G$ on $\Omega$. We start with the following lemma:

**Lemma 2.1.** Let $G$ be a finite permutation group on $\Omega$ of fixity $f \geq 2$, $p > f$ a prime and $\alpha \in \Omega$. If $p \in \pi(G_\alpha)$ then $G_\alpha$ contains a Sylow $p$-subgroup of $G$. In particular, if $G$ is transitive and contains a non-trivial normal $p$-subgroup then $p \notin \pi(G_\alpha)$.

**Proof.** Let $p$ be a prime divisor of order of $G_\alpha$. Then there exists a Sylow $p$-subgroup $P$ of $G$ such that $P_\alpha \neq 1$. We claim that $Z(P) \leq G_\alpha$. Suppose towards a contradiction, that our claim is not true. Then there exists $g \in Z(P)$ such that $g \notin G_\alpha$. Hence $\alpha, \alpha^g, \ldots, \alpha^{g^{p-1}}$ are distinct elements of $\text{Fix}_\Omega(P_\alpha)$. 


because if $\alpha^\delta = \alpha^\gamma$, for some $0 \leq i < j \leq p - 1$, then $g^{j-i} \in G_\alpha$ and so $(j - i, p) = 1$ implies that $g \in G_\alpha$, a contradiction. So $P_\alpha \neq 1$ implies that $p \leq f$, a contradiction. This proves that $Z(P) \leq G_\alpha$.

Now we prove that $P \leq G_\alpha$. Suppose, by a contrary, that $P \notin G_\alpha$. Then $x \notin G_\alpha$ for some $x \in P$. Hence, by a similar argument to the above paragraph, $\alpha, \alpha^x, \ldots, \alpha^{x^{p-1}}$ are distinct. Now $Z(P) \leq G_\alpha$ implies that these elements are all in $\text{Fix}_\Omega(Z(P))$. Let $y$ be a non-trivial element of $Z(P)$. Then

$$p \leq |\text{Fix}_\Omega(Z(P))| = \left| \bigcap_{x \in Z(P)} \text{Fix}_\Omega(x) \right| \leq |\text{Fix}_\Omega(y)| \leq f,$$

which is a contradiction.

Finally, suppose that $G$ is transitive and $N$ is a non-trivial normal $p$-subgroup of $G$. Then is contained in every Sylow $p$-subgroup of $G$. Now if $p$ is a divisor of $|G_\alpha|$, then by the above paragraph, $N \leq G_\alpha$. Since $G$ is transitive, this implies that $N$ is contained in any point-stabilizer of $G$ which implies that $N = 1$, because $G$ is a permutation group. This completes the proof.

**Corollary 2.2.** Let $G$ be an elusive group on $\Omega$ with fixity $f$. If $G$ contains a non-trivial normal $p$-subgroup then $p \leq f$.

**Proof.** Suppose, towards a contradiction, that $p > f$. Then by Lemma 2.1, $p \notin \pi(G_\alpha)$, where $\alpha \in \Omega$. On the other hand, by Lemma 2.1, $\pi(G) = \pi(G_\alpha)$ which is a contradiction.

**Corollary 2.3.** Let $G$ be an elusive group on $\Omega$ of fixity $f$. Then $f \geq 3$.

**Proof.** Suppose, towards a contradiction, that $f \leq 2$. Since $f = 0$ if and only if $G$ is regular on $\Omega$, and since regular groups are not elusive, we have $f = 1, 2$. On the other hand, $f = 1$ if and only if $G$ is a Frobenius group with Frobenius complement $G_\alpha$ on $\Omega$, where $\alpha \in \Omega$. Since the Frobenius kernel of any finite Frobenius group is a regular subgroup, we conclude that $f = 2$.

Let $\alpha \in \Omega$. Then, by Lemma 2.1, $\pi(G) = \pi(G_\alpha)$. Hence, by Lemma 2.1 $|\Omega| = 2^k$, for some $k \geq 1$, which contradicts Lemma 2.6.

**Lemma 2.4.** Let $G$ be an elusive group of fixity $f \geq 3$ on a finite set $\Omega$. Let $|\Omega| = p_1^{n_1} \cdots p_k^{n_k}$, where $n_i \geq 1$ and $k \geq 2$. Then

(i) for each $1 \leq i \leq k$, $p_i \leq f$,

(ii) if $p \in \{p_1, \ldots, p_k\}$ then every $p$-element in $G$ is either fixed-point free or fixes $mp$ points, where $1 \leq m \leq f/p$.

**Proof.** (i) Let $\alpha \in \Omega$ and $p$ be a prime divisor of $|\Omega| = |G: G_\alpha|$. By Lemma 2.1 if $p > f$ then $G_\alpha$ contains a Sylow $p$-subgroup of $G$. Hence $p$ can not divide $|\Omega|$, a contradiction.

(ii) Let $p \in \{p_1, \ldots, p_k\}$ and $x \in G$ be a $p$-element. Then $x$ fixes at most $f$ points. Since the length of any orbit of $\langle x \rangle$ on $\Omega$ is a power of $p$, by (i), $|\Omega| = |\text{Fix}_\Omega(\langle x \rangle)| + lp$, for some positive integer $l$. Hence $p$ divides $|\text{Fix}_\Omega(\langle x \rangle)|$. 

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On the other hand, \( \text{Fix}_\Omega((x)) = \text{Fix}_\Omega(x) \) which implies that \( x \) is either fixed-point free or fixes \( mp \) points, where \( 1 \leq m \leq f/p \). \( \square \)

**Corollary 2.5.** Let \( G \) be an elusive group on \( \Omega \) with fixity \( f \). If \( |\Omega| \) is odd, then \( f \geq 5 \). In particular, if \( |G| \) is odd then \( f \geq 5 \).

**Proof.** Since \( |\Omega| \) is a divisor of \( |G| \), the second part is a direct consequence of the first part. Suppose, by contrary, that \( f \leq 4 \). Then by Corollary 2.3 \( f \in \{3, 4\} \). Now by Lemma 2.4 \( |\Omega| = 2^m3^n \), where \( m, n \geq 0 \) are integers. On the other hand, by [1, Lemma 2.6], \( m, n \neq 0 \). Let \( x \) be an element of order 3 of \( G \). Then, by Lemma 2.4, \( x \) is fixed-point free, which is a contradiction. \( \square \)

For a finite group \( 1 \neq G \), we denote the smallest prime divisor of \( |G| \) by \( p_G \).

Then we have the following lemma.

**Lemma 2.6.** Let \( G \) be an elusive group on a finite set \( \Omega \) with fixity \( f \). Then

\[
|\Omega| \leq f \min\{|H| - 1 \mid 1 \neq H \trianglelefteq G\}.
\]

**Proof.** Let \( H \) be a non-trivial normal subgroup of \( G \). Let \( \Omega_1, \ldots, \Omega_m \) be all \( H \)-orbits on \( \Omega \). By [2, Theorem 1.6, A], for \( i \neq j \), \( H^{\Omega_i} \) is permutation isomorphic to \( H^{\Omega_j} \). If \( H \) acts regularly on one orbit, then it acts regularly on all of its orbits and so every element of prime order in \( H \) must be fixed-point free which is a contradiction. Hence \( H \) does not act regularly on its orbits. Thus by [11, Lemma 2.6], \( |\Omega| \leq f(|H| - 1)/(p_H - 1) \), which completes the proof. \( \square \)

**Lemma 2.7.** Let \( G \) be an elusive group on a finite set \( \Omega \) with fixity \( f \). If \( G \) has a non-trivial normal abelian subgroup \( N \) and \( p \) is the smallest prime divisor of \( |N| \), then

\[
\begin{align*}
(1) \quad |N| &\leq pf, \\
(2) \quad |\Omega| &\leq f(pf - 1)/(p - 1) \leq f(2f - 1).
\end{align*}
\]

**Proof.** It is obvious that the fixity is at most \( f \). On the other hand, by the proof of Lemma 2.6, \( N \) has no regular orbit on \( \Omega \). Hence, by [11, Lemma 2.7], \( |N| \leq pf \). The second part follows from Lemma 2.6. \( \square \)

**Corollary 2.8.** Let \( G \) be a 2-closed elusive solvable group of fixity \( f \). Then \( f \geq 6 \).

**Proof.** Let \( G \) be a 2-closed elusive group on \( \Omega \). Suppose that \( N \) is a minimal normal subgroup of \( G \). Then \( N \cong \mathbb{Z}_p^k \) for some \( k \geq 1 \), where \( p \) is a prime.

Suppose, towards a contradiction, that \( f \leq 5 \). Then, by Lemma 2.4 \( |\Omega| \leq 45 \) which contradicts [5, Proposition 6.1]. \( \square \)

**Corollary 2.9.** Let \( G \) be an elusive group of fixity \( f \) on \( \Omega \) and \( N \neq 1 \) be an abelian normal subgroup of \( G \) of order \( p_1^{n_1}p_2^{n_2} \cdots p_k^{n_k} \), where \( p_1 < p_2 < \ldots < p_k \) are primes, \( n_i \geq 1 \) and \( k \geq 1 \). Then

\[
(1) \quad \text{If } k = 1 \text{ then } n_1 \neq 1. \quad \text{Also } (n_1, \ldots, n_k) \neq (1, \ldots, 1).
\]
(2) \( p_1^{a_1} \ldots p_k^{a_k} \leq f \). In particular, \( p_1^{k-1} \leq f \).

(3) If \( f = 3 \), then \( p_1 = 2 \) or \( 3 \) and \( N \cong \mathbb{Z}_2^2 \) or \( \mathbb{Z}_3^2 \), respectively.

(4) If \( f = 4 \) then \( p_1 = 2 \) or \( 3 \). In the first case \( N \) is isomorphic to one of the groups \( \mathbb{Z}_2^2 \), \( \mathbb{Z}_2 \times \mathbb{Z}_2^2 \) or \( \mathbb{Z}_2^2 \times \mathbb{Z}_p \), where \( p \) is an odd prime. In the later case, \( N \cong \mathbb{Z}_2^3 \).

Proof. (1) If \( |N| = p_1 \) or \( p_1 p_2 \ldots p_k \) then there exists a non-trivial normal cyclic subgroup of \( G \) which contradicts \([1, \text{Lemma 2.20}]\).

(2) It is an immediate consequence of Lemma \( \text{2.7} \).

(3) Since \( G \) is elusive, \( N \) is not cyclic by \([1, \text{Lemma 2.20}]\). Now (3) is a consequence of (1) and (2).

(4) Let \( f = 4 \). Then (2) implies that \( p_1 = 2 \) or \( p_1 = 3 \). If \( k = 1 \) then by Lemma \( \text{2.7} \) \( N \cong \mathbb{Z}_2^3 \). If \( k \geq 2 \) the the result follow from (1) and (2). \( \square \)

Corollary 2.10. Let \( G \) be a transitive 2-closed permutation group on a set \( \Omega \) of fixity 4 and \( N \neq 1 \) be a normal \( p \)-subgroup of \( G \). Then \( G \) admits fixed-point free element.

Proof. Suppose, towards a contradiction, that \( G \) is elusive. Then, by Corollary \( \text{2.9} \) we have \( N \cong \mathbb{Z}_2^3 \) or \( N \cong \mathbb{Z}_2^2 \). Let \( \alpha^N \) be an orbit of \( N \) on \( \Omega \). Then \( |\alpha^N| \in \{1, p, p^2\} \), where \( p \in \{2, 3\} \). If \( |\alpha^N| = 1 \), then \( N = N_\alpha \leq G_\alpha \) which implies that \( N = 1 \), a contradiction. If \( |\alpha^N| = p^2 \) then \( N_\alpha = 1 \) which contradicts \([1, \text{Lemma 2.7}]\). Hence \( |\alpha^N| = p \) which contradicts \([1, \text{Theorem 2.11}]\). \( \square \)

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