LOCALIZATION IN RANDOM GEOMETRIC GRAPHS WITH TOO MANY EDGES

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Abstract. Consider a random geometric graph $G(\chi_n, r_n)$, given by connecting two vertices of a Poisson point process $\chi_n$ of intensity $n$ on the unit torus whenever their distance is smaller than the parameter $r_n$. The model is conditioned on the rare event that the number of edges observed, $|E|$, is greater than $(1 + \delta)\mathbb{E}(|E|)$, for some fixed $\delta > 0$. This article proves that upon conditioning, with high probability there exists a ball of diameter $r_n$ which contains a clique of at least $\sqrt{2\delta \mathbb{E}(|E|)}(1 - \epsilon)$ vertices, for any $\epsilon > 0$. Intuitively, this region contains all the “excess” edges the graph is forced to contain by the conditioning event, up to lower order corrections. As a consequence of this result, we prove a large deviations principle for the upper tail of the edge count of the random geometric graph.

1. Introduction

The Random Geometric Graph is a simple stochastic model, first studied in [7], for generating a graph: given the parameters $n$ and $r$, consider a Poisson point process of intensity $n$ on the unit torus, equipped with some norm $\|\cdot\|$, and declare an edge between any two vertices that are distance $\leq r$ from each other.

Unlike the well-known Erdős–Rényi random graph, the random geometric graph’s definition leads to strong dependence between edges: if three vertices form a “V” shaped graph, they are far more likely to have the third edge of the triangle than if no assumption were made on the other edges, as a consequence of the triangle inequality.

Many properties of this graph model have been studied. A comprehensive survey of laws of large numbers and central limit theorems for many graph-theoretical functions of random geometric graphs, such as subgraph counts, independence number, and chromatic number, may be found in the classic monograph of Mathew Penrose [15] (see also [16, 17]). Besides this, there have been investigations into other probabilistic features, such as threshold functions for cover times and mixing times [2] and a characterization of sharp thresholds for many monotone graph functions [8]. The random geometric

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graph is also highly related to the random-connection continuum percolation model. In that model, the vertex set is given by an (almost surely infinite) Poisson point process of fixed intensity on $\mathbb{R}^d$, and the radius of connection is fixed at 1. The objects of interest in this model are the existence of an infinite connected component, as well as the behavior of the subset $\mathbb{R}^d$ that is at distance at most 1 from one of the vertices of the graph (the so-called “Poisson blob”). Continuum percolation is treated in detail in a book-length monograph by Meester and Roy [11], as well as in the more general work of Grimmett [9].

Most of the work done on random geometric graphs is concerned with either the behavior of a typical graph — the graph we are likely to see for a given $r$ as $n$ goes to infinity — or typical deviations from that behavior, i.e. central limit theorems. In this paper, we are concerned with the behavior of the model conditioned on a rare event. Specifically, we are concerned with the random geometric graph conditioned on having more edges than is expected (a formal description will follow). The large deviation regime of the upper tail of the any subgraph count of the random geometric graph is not well studied. Janson [10] has studied concentration inequalities for $U$-statistics, a general class of statistics which includes the subgraph counts we are interested in. These upper bounds work in very general settings, but do not produce a tight upper bound, even up to constants in the exponent. Large deviation principles have been proven for functionals of random point processes in which the contribution of any particular vertex is uniformly bounded [19], but no such bound is known for functionals with a possibly large influence. Finally, we will see that the conditional graph model is characterized by the appearance of a “giant clique” in the graph. The clique number of the random geometric graph, which falls under the general class scan statistics, has been shown to focus on two values with high probability for certain values of the threshold parameter $r$ (see [14], [12]); however, these works do not explore the large deviation regime.

The fact that a sum of i.i.d. random variables may be very large due to the presence of very few large variables was known relatively early in the history of large deviation theory: a survey by Nagaev [13], summarizing a series of papers written in the Soviet Union in the 1970’s, includes this fact as a corollary. A similar phenomenon occurs in this context. We show that the random geometric graph, conditioned on having too many edges, exhibits localization: a small number of vertices will contribute almost all the extra edges that we require the graph to exhibit, while the “bulk” of the graph of the graph will have lower order changes. Furthermore, we will show that the geometry of the localized region has the shape of a ball in the given norm. The proof uses techniques from large deviations, concentration inequalities, convex analysis, and geometric measure theory. A key component in the proof is a technique for proving localization that has previously appeared in [21] and [4].
2. Main result

Let \( \chi_n \) be a Poisson Point Process of intensity \( n \) on the \( d \)-dimensional unit torus \( \mathbb{T}^d = [0, 1]^d \). Let \( N := |\chi_n| \). Recall that \( N \) is a Poisson random variable with mean \( n \), and conditional on \( N \), \( \chi_n \) is just a set of \( N \) points chosen independently and uniformly at random. Let \( r_n \) be a sequence that decreases to 0 as \( n \to \infty \), and \( \| \cdot \| \) be some norm on \( \mathbb{R}^d \) that induces a translation-invariant metric on \( \mathbb{T}^d \). We define the random geometric graph \( G(\chi_n, r_n) := (V, E) \), where \( V = \chi_n = \{v_1, \ldots, v_N\} \), enumerated arbitrarily, and \( E \) is the set of unordered pairs \( \{i, j\} \) such that \( \|v_i - v_j\| \leq r_n \). Figure 1 shows a particular instance of \( G(\chi_{150}, 0.1) \).

Letting \( 1_{i,j} \) be the indicator that there is an edge between \( v_i \) and \( v_j \), we can calculate the expected value of \( |E| \), the number of edges in the graph:

\[
\mathbb{E}(|E|) = \mathbb{E}\left( \sum_{1 \leq i < j \leq N} 1_{i,j} \right) = \mathbb{E}\left( \binom{N}{2} \mathbb{E}(1_{1,2} \mid N) \right)
\]

\[= \frac{n^2}{2} \mathbb{P}(\|v_1 - v_2\| \leq r_n). \]

By the symmetry of the torus, \( \mathbb{P}(\|v_1 - v_2\| \leq r_n) \) is simply \( \nu r_n^d \), where \( \nu \) is the volume of the unit ball of the given norm, \( r_n \). Thus,

\[
\mathbb{E}(|E|) = \frac{n^2 \nu r_n^d}{2} =: \mu_n.
\]

For the rest of the article, we suppose the existence of a fixed constant \( \delta^* > 0 \) such that, for all sufficiently large \( n \),

\[
(1) \quad n^{(\delta^* - 2)/d} \leq r_n \leq n^{-\delta^* / d}
\]

The lower bound ensures that the expected number of edges is polynomially large in \( n \); the upper bound excludes the possibility of \( r_n \) vanishing as the inverse of a polynomial in \( \log n \), but is not much stronger than the initial requirement of \( r_n \) vanishing in as \( n \) grows. We define a parameter \( p \) as

\[
(2) \quad p := \lim_{n \to \infty} \frac{\log \mu_n}{\log n},
\]

implicitly assuming that the limit exists. This ensures that \( \mu_n = f(n)n^p \) for some slowly varying \( f(n) \). Notice that \( p \) depends only on the tail behavior of \( r_n \), and is completely independent of \( n \). Furthermore,

\[
(3) \quad \delta^* \leq p \leq 2 - \delta^*,
\]

thanks to (1).

The following theorem is the main result of the paper:

**Theorem 2.1.** Let \( G(\chi_n, r_n) \) be a random geometric graph model on the \( d \)-dimensional torus with respect to some norm \( \| \cdot \| \) and a threshold parameter that satisfies \( n^{(\delta^* - 2)/d} \leq r_n \leq n^{-\delta^*/d} \) for some fixed \( \delta^* > 0 \). Let \( \tau_n = \nu (r_n/2)^d \), where \( \nu \) is the volume of the unit ball under the norm \( \| \cdot \| \). (That
is, \( \tau_n \) is the volume of the ball of diameter \( r_n \).) Fix \( \delta > 0 \) and \( \epsilon > 0 \). Let \( F_n \) be the event that the following happen: (a) There is a ball \( A \) of diameter \( r_n \) such that any convex set \( S \subseteq A \) with \( \lambda(S) > (\epsilon/16)\tau_n \) (where \( \lambda \) denotes Lebesgue measure) satisfies

\[
\frac{|\chi_n(S)|}{\sqrt{2\delta \mu_n}} - \frac{\lambda(S)}{\tau_n} < \epsilon,
\]

and (b) for any convex set \( S \subseteq A^c \) with \( \lambda(S) > (\epsilon/16)\tau_n \), such that \( S \) lies in some ball of diameter \( r_n \),

\[
\frac{|\chi_n(S)|}{\sqrt{2\delta \mu_n}} < \epsilon \frac{\lambda(S)}{\tau_n}.
\]

Then, if \( \epsilon \) is sufficiently small (depending only on \( \delta \)) the conditional probability of the event \( F_n \) given that \( |E| \geq (1 + \delta)\mu_n \) tends to 1 as \( n \to \infty \).

Note that the set \( S \) can be taken to be equal to \( A \), which implies our original claim that “almost all extra edges are between points in \( A \)”. The theorem actually says more: it says that the extra points are not only almost all in \( A \), but that they are more or less equi-distributed inside \( A \).

The convexity requirement is probably not optimal, but forces the set \( S \) to be sufficiently “nice” to preclude sets which are either sparse but of large measure (such as generalized Cantor sets) or have boundaries that take up a large amount of space. We also force \( S \) to be large to make sure it cannot pick up the lower order uneveness in the conditional process. Probing the Poisson Point Process with such sets is sufficient to show that graph will
appear uniform, up to sets of measure equal to arbitrarily small multiples of $\tau_n$.

As a consequence of Theorem 2.1, we prove that the upper tail of the edge count of random geometric graphs satisfies a large deviation principle. Recall that $|E|/\mu_n$ satisfies an upper tail large deviation principle with speed $f(n)$ and rate function $I(x)$ if, for any closed set $F \subset (1, \infty)$,

$$\limsup_{n \to \infty} \frac{1}{f(n)} \log \mathbb{P}(|E|/\mu_n \in F) \leq -\inf_{x \in F} I(x),$$

and for any open set $G \subset [1, \infty)$,

$$\liminf_{n \to \infty} \frac{1}{f(n)} \log \mathbb{P}(|E|/\mu_n \in G) \geq -\inf_{x \in G} I(x).$$

**Theorem 2.2.** Let $G(\chi_n, r_n)$ be a random geometric graph model on the $d$-dimensional torus, with the same assumptions as in Theorem 2.1. Define

$$I(x) := \left(\frac{2 - p}{2}\right) \sqrt{2(x - 1)},$$

where $p$ is defined as in (2). Then the random variable $|E|/\mu_n$ satisfies an upper tail large deviation principle with speed $\sqrt{\mu_n \log n}$ and rate function $I(x)$.

Note that this rate function, like all other claims in this paper, is only valid for events in which the number of edges exceeds its mean. The lower tail of the edge count is likely to satisfy Poisson-like statistics, and hence the speed of its rate function is expected to behave like $\mu_n$, not $\sqrt{\mu_n \log n}$.

3. The $s$-Graded Model

Henceforth in the manuscript, we will suppress the subscript $n$ and write $\chi, \mu, \tau$ and $r$ instead of $\chi_n, \mu_n, \tau_n$ and $r_n$.

We now present an approximation of the random geometric model which allows us to replace the Poisson point process with a sequence of independent Poisson random variables. To do, we first discretize space, and then produce a metric on the resulting “cells” that approximates the norm $\| \cdot \|$ on the unit torus. We call this the $s$-graded model.

Fix an integer $s$, and define

$$m := \lfloor s/r \rfloor,$$

so that

$$\frac{s}{r} - 1 \leq m \leq \frac{s}{r}.$$

Let $T = \{1, 2, \ldots, m\}^d$. Pick $I = (i_1, i_2, \ldots, i_d) \in T$, and define

$$A_I = \left[\frac{i_1 - 1}{m}, \frac{i_1}{m}\right] \times \cdots \times \left[\frac{i_d - 1}{m}, \frac{i_d}{m}\right].$$
The $A_I$’s partition the unit torus into $m^d$ cubes (ignoring sets of measure 0), each of volume $1/m^d$, and therefore, $X_I = |\chi(A_I)|$ is a Poisson random variable of mean
\[ D := \frac{n}{m^d}. \]
As a function of $n$, $m^d$ grows as $n^{2-p}$, up to a possible multiplicative factor that is slowly growing in $n$ (recall that $s$ is fixed); this implies that $D$ grows as $n^{p-1}$. Note that $s$ is exactly the minimal number of cells that must be traversed in any one direction to ensure that we exceed the distance $r$. This parameter takes the place of $r$ in the $s$-graded model.

We now define a metric on $T$, induced by the norm on torus:
\[ d(I, J) = \inf_{x \in A_I^o, y \in A_J^o} \left\lceil m \|x - y\| \right\rceil \]
where the circles indicate the interiors of the sets. Note that the distance is always an integer. Moreover, $d(I, J) = z$ if $z$ is the smallest integer such that some point in $A_I^o$ and some point in $A_J^o$ are less than $z$ away, measured in units of $1/m$, the side length of the cubes. We force the points to be in the interior to prevent “trivialities”, such as two adjacent cells being distance 0, since they share a boundary.

We are now ready to define the $s$-graded random geometric graph. Let $G_s(\chi, r) = (V, E_s)$ have the same vertex set as the original graph. For each vertex $v$, let $I_v$ be the index in $T$ such that $v \in A_{I_v}$; there is ambiguity on the boundary of the $A_I$’s, but that set has Lebesgue measure 0, and therefore it has no vertices of $\chi$, almost surely. We say $(v, w) \in E_s$ whenever $d(I_v, I_w) \leq s$. Essentially, the $s$-graded model allows every point to wander inside a cubical “cage” of side-length $1/m$, and connects any two points that might be connected after we allow this mobility. In this framework, it is clear that $E_s$ becomes smaller as $s$ decreases. In fact, for sufficiently large $s$, $E_s$ is identical to $E$; unfortunately, this $s$ will depend on $n$. Due to uniformity issues, we will instead fix $s$ and later show that, even when $s$ is finite but large, the approximation is good.

The major benefit of the $s$-graded model is that its edge count is very simple to express in terms of $X_I$, the number of points in each $A_I$:
\[
|E_s| = \sum_{I \in T} \left( \left( \frac{X_I}{2} \right) + \frac{1}{2} \sum_{J : 0 < d(I, J) \leq s} X_I X_J \right)
= \frac{1}{2} \sum_{I \in T} X_I \left( \sum_{J : d(I, J) \leq s} X_J - 1 \right).
\]
This random variable is defined in terms of i.i.d. random variable, which eases the analysis greatly. Furthermore, the ambient geometry of the torus is encoded completely by the metric $d$.

For any $I \in T$, let $N_I := \{ J : d(I, J) \leq s \}$. Thanks to translation invariance of $d$, the cardinality of this set is independent of the choice of $I$. 

The $A_I$’s partition the unit torus into $m^d$ cubes (ignoring sets of measure 0), each of volume $1/m^d$, and therefore, $X_I = |\chi(A_I)|$ is a Poisson random variable of mean
\[ D := \frac{n}{m^d}. \]
Using this parameter, we can compute the expected number of edges in the $s$-graded random geometric graph easily:

$$
\mu_s := \mathbb{E}(|E_s|)
$$

$$
= \sum_{I \in T} \mathbb{E} \left( \frac{(X_I)^2}{2} \right) + \frac{1}{2} \sum_{J: 0 < d(I,J) \leq s} \mathbb{E}(X_I)\mathbb{E}(X_J)
$$

$$
= \frac{|N_I|m^dD^2}{2} = \frac{|N_I|n^2}{2m^d}.
$$

As before, we are interested in conditioning the $s$-graded model on the event \{$|E_s| > (1 + \tilde{\delta})\mu_s$\}. Again, the appropriate geometric notion is that of the largest set of diameter $s$. We call a set of indices a maximal clique set if it is a subset of $T$ with diameter $s$ that achieves the maximal cardinality of all such sets. Let $\tau_s$ be the cardinality of such a set.

We can now state the equivalent to Theorem 2.1 for the $s$-graded model:

**Theorem 3.1.** Fix an integer $s$ and let $G_s(\chi, r)$ be an $s$-graded random geometric graph, with $n^{(2-\delta^*)/d} \leq r \leq n^{-\delta^*/d}$ for some fixed $\delta^* > 0$. Fix $\tilde{\delta} > 0$ and $\bar{\epsilon} > 0$. Let $F$ be the event that the following happen: (a) There is a maximal clique set $\Psi$ such that for all $I \in \Psi$,

$$
\left| \frac{\tau_s X_I}{(2\tilde{\delta}\mu_s)^{1/2}} - 1 \right| < \bar{\epsilon},
$$

and (b) for all $J \in \Psi^c$,

$$
\frac{\tau_s X_J}{(2\tilde{\delta}\mu_s)^{1/2}} \leq \bar{\epsilon}.
$$

Then if $\bar{\epsilon}$ is sufficiently small (depending only on $\tilde{\delta}$), the conditional probability of $F$ given that $|E_s| \geq (1 + \tilde{\delta})\mu_s$ tends to 1 as $n \to \infty$.

Essentially, Theorem 3.1 produces a maximal clique set, and whose entries sum up to $(2\tilde{\delta}\mu_s)^{1/2}$, up to lower order corrections. This creates all the ‘extra edges’ in the graph. As in Theorem 2.1, this set will contain almost all the edges not expected to be the in unconditional $s$-graded model, and the rest of graph will look unaltered, at least in the scale of $\sqrt{\mu_s}$.

### 4. Outline of the Proof

Before embarking on a proper proof, we sketch the main ideas required. The first step is to show that the $s$-graded model is, in fact, a good approximation for the random geometric graph. To do so, we first show that we can approximate any convex subset $S$ of a ball of diameter $r$ from both the inside and the outside by a union of $A_I$’s. Next, we use the classical isodiametric inequality to show that the $A_I$’s associated with a maximal clique set approximate a ball of diameter $r$, in the sense of the Hausdorff metric. Armed with these facts, showing that Theorem 3.1 implies Theorem 2.1 is a matter of careful “epsilontics.”
We then turn to directly analyzing the $s$-graded model, conditioned on the event

$$L := \{|E_s| \geq (1 + \tilde{\delta})\mu_s\}.$$ 

For notational convenience, let $q = \sqrt{2\delta\mu_s}$, $w = \tau_s D$. Define

$$\mathcal{J} = \{I \in T : X_I > \max\{Dn^a, n^a\}\},$$

where $a$ is a small positive number. This is the set of “large” indices, in the sense that their associated $X_I$’s exceed their expected value by a fixed polynomial factor in $n$. Furthermore, define

$$Y_I := X_I (\log(X_I/D) - 1) + D,$$

and

$$Q(\mathcal{J}) := \frac{2}{q^2} \left( \sum_{I \in \mathcal{J}} \frac{X_I}{2} + \frac{1}{2} \sum_{J \in N_I \cap \mathcal{J}} X_I X_J \right).$$

The former quantity is an appropriately chosen convex function of the $X_I$’s, while the latter is a scaled version of the number of edges with both endpoints in the $A_I$’s associated with $\mathcal{J}$. Consider the event

$$\left\{ Q(\mathcal{J}) > 1 - \frac{\xi}{\log n} \right\} \cap \left\{ \frac{1}{q} \sum_{I \in \mathcal{J}} Y_I \leq \log(q/w) - 1 + \xi \right\},$$

where $\xi$ is an arbitrary positive constant. By comparing a lower bound on the probability of $L$ to an upper bound on the probability of the complement of the above event, we can show that this event occurs with high probability in the $s$-graded model conditional on $L$.

We now have a set of indices $\mathcal{J}$ which satisfies both a quadratic lower bound and a convex upper bound with high probability. From here on, the analysis is completely deterministic, characterizing configurations that satisfy the simultaneous inequalities above. We wish to look at the largest elements of $\mathcal{J}$. Specifically, we wish to take the smallest set that includes roughly $q$ vertices of the Poisson point process. To make this precise, define

$$V(\mathcal{A}) := \frac{1}{q} \sum_{I \in \mathcal{A}} X_I,$$

for any subset of indices $\mathcal{A}$. Now, order the elements of $\mathcal{J}$ by size, so that $X_1$ is the largest, $X_2$ is the second largest, etc. Letting $\mathcal{I}_k = \{X_1, X_2, \ldots X_k\}$, define

$$\mathcal{I} := \text{The first } \mathcal{I}_k \text{ such that } V(\mathcal{I}_k) > 1 - \frac{2\xi}{\log n}.$$ 

Careful use of minimality and Jensen’s inequality can establish that

$$V(\mathcal{I}) \leq 1 + \phi(\mathcal{I}),$$

where $\phi(\mathcal{I})$ is bounded above by $\xi/3$. More importantly, moving from $\mathcal{J}$ to $\mathcal{I}$ does not force us to discard too many edges; formally,

$$Q(\mathcal{I}) \geq 1 - \psi(\mathcal{I}),$$
with \( \psi(\mathcal{I}) \) bounded above by \( \xi \).

We now define

\[
\mathcal{P} := \left\{ I \in \mathcal{I} : X_I > \frac{\xi^{1/4} q}{\tau s}\right\}.
\]

If \( \mathcal{I} \) was the set large indices, \( \mathcal{P} \) is the set of very large indices — that is, those whose associated \( X_I \)'s are commensurate with \( q \). We can show that this set cannot have diameter that strictly exceeds \( s \) without contradicting either the lower bound \( Q(\mathcal{I}) \) or the upper bound on \( V(\mathcal{I}) \). Furthermore, technical estimates ensure that \( |\mathcal{P}| \geq \tau s \). As a set of diameter at most \( s \) and cardinality at least \( \tau s \), \( \mathcal{P} \) must be a maximal clique set, by definition. Moreover, a quantitative version of Jensen’s inequality allows us to claim that all the \( X_I \)'s associated with this set are roughly equal. Thus, \( \mathcal{P} \) is the maximal clique set described in Theorem 3.1. Finally, we can show that its complement only includes lower order variables. This completes the proof.

In the last section, we prove the large deviation principle. We use the first stipulation of 2.1 and the \( s \)-graded model to compute the upper bound. The lower bound is derived directly from the Poisson Point Process.

5. The \( s \)-Graded Model Approximates the Random Geometric Graph

Before proving Theorem 3.1, we show that it implies Theorem 2.1. To do so, we first define three operations to go between subsets of \( T \), the natural objects in the \( s \)-graded model, and subsets of \([0,1]^d\). For \( \mathcal{B} \subset T \), let

\[
\Omega(\mathcal{B}) := \bigcup_{I \in \mathcal{B}} A_I
\]

the set associated with the index set \( \mathcal{A} \). In the other direction, we cannot be as exact; for \( K \subset [0,1]^d \), let \( \mathcal{O}(K) \) and \( \mathcal{R}(K) \) be the outer and inner hulls of \( K \), defined as the maximal and minimal subsets of \( T \) such that

\[
\Omega(\mathcal{O}(K)) \subset K \subset \Omega(\mathcal{R}(K))
\]

With these operators in place, we show that the \( s \)-graded parameters are good approximations of their respective properties in the usual random geometric graph:

**Lemma 5.1.** Let \( G(\chi, r) = (V, E) \) be the usual random geometric graph, and \( G_s(\chi, r) = (V, E_s) \) be the \( s \)-graded model. As before, \( \mu = \mathbb{E}(|E|) \), \( \mu_s = \mathbb{E}(|E_s|) \) and \( \tau = \nu(r/2)^d \). Let \( \tau_s \) be the cardinality of a maximal clique set. Then \( E \subset E_s \), and there exist constants \( C_{at} \) and \( s_0 \), depending only on the dimension and the chosen norm of the torus, such that if \( s \geq s_0 \) then

\[
\mu \leq \mu_s \leq \mu \left( 1 + \frac{C_{at}}{s} \right)
\]

and

\[
m_d \tau \leq \tau_s \leq m_d \tau \left( 1 + \frac{C_{at}}{s} \right)
\]
Furthermore, both $|N_I|$ and $\tau_s$ are uniformly bounded in $n$.

Proof. Pick an arbitrary $I$ and consider $\mathcal{U}(N_I)$. By definition of $d$ and $s$, this set includes a ball of radius $r$ around any point in $A_I$. Therefore, any pair $(v, w) \in E$ must also be in $E_s$, giving the first stipulation. Since this inclusion holds for any configuration of the underlying Poisson Point process, this also gives $\mu \leq \mu_s$.

Now, let $\rho$ be the diameter of the unit cube under the norm $\| \cdot \|$. Then, for any $x \in A_I$ and $y \in \mathcal{U}(N_I)$,

$$\|x - y\| \leq \frac{rs}{s-r} + \frac{2\rho}{m}$$

Therefore, $\mathcal{U}(N_I)$ is contained in a ball of radius $r + 2\rho/m$ around any point in $A_I$, which implies that

$$|N_I| = m^d \lambda(\mathcal{U}(N_I)) \leq \nu m^d r^d \left(\frac{s}{s-r} + \frac{2\rho}{rm}\right)^d \leq \nu m^d r^d \left(1 + \frac{r + 4d\rho}{s-r}\right)$$

where the final inequality follows because $(1 + x)^d \leq 1 + (2d)x$ for all sufficiently small $x$, and the fact that $rm \geq s-r$ by definition of $m$. Substituting this into the definition of $\mu_s$ produces the desired inequality on $\mu_s$ (assuming, without loss, that $r \leq \rho$).

The lower bound on $\tau_s$ follows similarly, by noticing that a ball of radius $r/2$ can always be included in some set of diameter $s$ with respect to the metric on indices. For the upper bound, let $\mathcal{W}$ be a set of indices such that

$$\lambda(\mathcal{U}(\mathcal{W})) \geq \tau\left(1 + \frac{C_{at}}{s}\right)$$

Applying the isodiametric inequality for finite dimensional normed spaces \cite[p. 93]{3} and choosing $C_{at}$ and $s_0$ sufficiently large gives

$$\text{diam}(\mathcal{U}(\mathcal{W})) \geq r \left(1 + \frac{C_{at}}{s}\right)^{1/d} \geq r + \frac{4\rho}{m}$$

This implies that the diameter of $\mathcal{W}$ is at least $s + 1$ in the $d$ metric, and therefore the set cannot be a maximal clique set. Translating the measure bound to a cardinality bound on the size $\mathcal{W}$ gives the desired upper bound on $|\mathcal{W}|$.

Finally, $m^d \leq s^d/\rho^d$, giving the uniform upper bound on $|N_I|$. Since $\tau_s \leq |N_I|$, the final claim of the lemma follows. \qed

As a simple corollary to the lemma, we see that the hypothesis of Theorem 2.1 implies the hypothesis of Theorem 3.1 as long as $\tilde{\delta} < \delta$ and $s$ is sufficiently large:

**Corollary 5.2.** For any $\tilde{\delta} < \delta$, there exists $s_0$ depending only on $\delta$, the chosen norm, and the dimension, such that whenever $s \geq s_0$, $|E| > (1 + \delta)\mu$ implies $|E_s| > (1 + \delta)\mu_s$. 
Proof. Pick $s$ large enough to ensure $(1 + \tilde{\delta})(1 + C_{\text{cat}}/s) \leq 1 + \delta$. Lemma 5.1 guarantees that $|E_s| \geq |E|$ and

$$|E| \geq (1 + \delta)\mu \geq (1 + \tilde{\delta})(1 + C_{\text{cat}}/s)\mu \geq (1 + \tilde{\delta})\mu_s$$

This proves the corollary. □

Next, we get some quantitative bounds on the geometric approximation of certain subsets of $[0, 1]^d$ by the $A_I$'s. Let $S$ be a convex subset of a ball of diameter $r$. Assume that

$$z(S) := \lambda(S) \geq \frac{1}{16}. \tag{6}$$

Recall that $\tau$ is the volume of a ball of diameter $r$. We rescale Lebesgue measure to bypass factors of $\tau$ throughout this section.

Let $\rho$ be the diameter of the unit cube under the chosen norm. We wish to estimate the measure of the set

$$(\partial S)_{\rho/m} := \left\{ y : \|\partial S - y\| \leq \frac{\rho}{m} \right\}.$$ 

that is, the set of all points which may share an $A_I$ with some point in the boundary of $S$; the distance to the boundary is defined as the infimum of distance from $y$ to $x \in \partial S$. Specifically, we wish to show that it can be made to have arbitrarily small Lebesgue measure, when normalized by $\tau$. If we were to allow $s$ to depend on the particular geometry of $S$, this would follow from continuity of Lebesgue measure. Instead, we wish to find a single value of $s$ such that $(\partial S)_{\rho/m}$ has measure $\epsilon \tau$ for all convex subsets $S$ of a ball of diameter $r$. To do this, we appeal to geometric measure theory.

To get a uniform bound on the “fattening” of the boundary, we first note that, by definition of $m$,

$$(\partial S)_{\rho/m} \subset (\partial S)_{\rho r/(s - r)}.$$ 

Now, we consider a function $f : \mathbb{R}^d \to \mathbb{R}$ that is Lipschitz with respect to the Euclidean distance, and a Borel set $A$. The Euclidean coarea formula [6, pg. 248] states that, with the functions as above,

$$\int_A \|Df(x)\|_2 dx = \int_{-\infty}^{\infty} H^{d-1}(A \cap f^{-1}(y)) dy,$$

where $\| \cdot \|_2$ is the Euclidean norm, and $H^{d-1}$ is the Hausdorff measure on the surface (effectively the surface area) and $Df$ is the gradient of $f$, which exists since $f$ is almost everywhere differentiable. To formally define this, pick a set $A$ and let $U_\delta(A)$ be the set of all coverings $\{U_i\}$ of $A$, where $U_i$ have diameter at most $\delta$. By definition, the Hausdorff measure of $A$ is

$$H^{d-1}(A) = C_{d-1} \lim_{\delta \to 0} \inf_{\{U_i\} \in U_\delta(A)} \sum [\text{diam } (U_i)]^{d-1},$$

where $C_{d-1}$ is some constant depending on dimension, used to normalize the measure appropriately to be compatible with Lebesgue measure. This limit
is well defined \[6, \text{pg. 170}\], but may be 0 or infinity for a general \(A\); in fact, it is infinite for any set with positive \(d\)-dimensional Lebesgue measure.

Now, consider the function \(f(x) = \|\partial A - x\|\), where \(\|B - x\|\) is shorthand for \(\inf_{b \in B} \|b - x\|\) for any set \(B\). First, we recall the classical fact that all norms are equivalent in finite dimensional space, i.e. there exists two positive constants \(c\) and \(C\) such that, for all \(x\) and \(y\),

\[
c \|x - y\| \leq \|x - y\|_2 \leq C \|x - y\|.
\]

To see that \(f\) is Lipschitz with respect to the Euclidean norm, pick two points \(x\) and \(y\), and let \(a \in \partial A\) be the point such that \(f(x) = \|x - a\|\) (this point exists because \(\partial A\) is closed). Then

\[
f(y) - f(x) \leq \|y - a\| - \|x - a\| \leq \|x - y\| \leq C \|x - y\|_2.
\]

Similarly, \(f(x) - f(y) \leq C \|x - y\|_2\). Thus, \(f\) is differentiable almost everywhere. Pick an \(x\) where the function is differentiable, and again let \(a\) be as before. Then, for any \(t \in (0, 1)\),

\[
f(x + t(a - x)) \leq f(x) - t\|a - x\|,
\]

by the properties of norms. Subtracting \(f(x)\) from both sides, dividing by \(t\) and letting \(t \to 0\), we get

\[
\langle a - x, Df \rangle \leq -\|a - x\|,
\]

where \(\langle \cdot, \cdot \rangle\) is the Euclidean inner product (or in this case, the directional derivative). Applying the Cauchy-Schwarz inequality, we conclude that

\[
\|a - x\| \leq \|a - x\|_2 \|Df\|_2,
\]

which implies, by the equivalence of norms, that

\[
\|Df\|_2 \geq \frac{\|a - x\|}{\|a - x\|_2} \geq c.
\]

Substituting this into the coarea formula, we see that

\[
c \lambda(A) \leq \int_{-\infty}^{\infty} H^{d-1}(A \cap f^{-1}(y)) \, dy,
\]

Letting \(A = \{x : f(x) \leq \rho r/(s - r)\}\), we deduce that

\[
\lambda[(\partial S)_{\rho r/(s - r)}] \leq C \int_{0}^{\rho r/(s - r)} H^{d-1}(\{y : \|\partial S - y\| = z\}) \, dz,
\]

where \(C\) is some (possibly different) universal constant. Note that, for sufficiently small \(z\) and \(T\) convex, the set \(\{y : \|\partial T - y\| = z\}\) has two parts: one inside \(T\) and the other outside of it. Luckily, the external one is the boundary of \(T_z\), which is convex as it is the affine sum of \(T\) and a ball of radius \(z\). The internal one is boundary of the set

\[
T^{(z)} := \{x : x \in T, \|\partial T - x\| \geq z\}.
\]

This set is also convex; if it weren’t, we could find \(x, y \in T^{(z)}\) such that \(w = tx + (1 - t)y \not\in T^{(z)}\) for some \(t \in (0, 1)\). Let \(v\) be the minimal length
vector such that \( w + v \in \partial T \). Then, since \( \|v\| < z \) by definition, we can find an \( \epsilon \) sufficiently small such that \( w + (1 + \epsilon)v \notin T \), while \( x + (1 + \epsilon)v \), \( y + (1 + \epsilon)v \) are both in \( T \), contradicting convexity of \( T \). Therefore, controlling the “fattening” of \( \partial S \) reduces to controlling the boundaries of convex sets.

Suppose \( T \) is a convex subset of a Euclidean ball \( B \) centered around some \( x \) in the interior of \( T \). Define \( P_x : \partial B \to \partial T \) by defining \( P_x(y) \) to be the intersection of the ray from \( x \) to \( y \) with \( \partial T \). Since \( T \) is convex, this is well defined; suppose, by way of contradiction, that some ray emanating from \( x \) hits a pair of distinct points \( b_1, b_2 \in \partial T \), in this order. Since \( x \) is an interior point of \( T \), there is some open set \( O \) that includes \( x \) entirely inside \( T \). Now, define the set

\[
\{ t : t = \lambda y + (1 - \lambda)b_2, y \in O, \lambda \in (0, 1) \}.
\]

This set is open, and contains \( b_1 \). By convexity of \( T \), it is completely inside \( T \), and thus \( b_1 \) is not a boundary point, contradicting the assumption. This also implies that the map \( P_x \) is bijective.

We now use the map \( P_x \) to control the \((d - 1)\) dimensional Hausdorff measure of the boundary of \( T \). To do so, we consider \( \{U_i \} \in \mathcal{U}_d(\partial B) \) - i.e. a particular covering of the boundary of the Euclidean ball \( B \). Without loss of generality, we may assume that all the \( U_i \)'s are subsets of \( \partial B \); otherwise, we simply intersect the \( U_i \)'s with \( \partial B \), possibly decreasing the diameter. Define \( \{U'_i \} \) to be the image of the \( U_i \)'s under \( P_x \). The \( \{U'_i \} \)'s cover \( \partial T \), since \( P_x \) is bijective, and the diameter of \( U'_i \) is no greater than the diameter of \( U_i \).

Taking a minimizing sequence of coverings of \( \partial B \), we conclude that

\[
H^{d-1}(\partial T) \leq H^{d-1}(\partial B).
\]

Note that, formally, it is possible that either of these measures is infinite. However, the Hausdorff measure of the Euclidean ball is well known to be finite [6, pg. 171], meaning \( H^{d-1}(\partial T) \) is finite.

We now return to the original question: \( S \) is a convex set that is a subset of some ball of radius \( r \). Therefore, by equivalence of norms, \( S \) is also a subset of Euclidean ball \( B \) of radius \( C r \) centered at some point in its interior, for some \( C \); in fact, for all sufficiently small \( z \), both \( S_z \) and \( S(z) \) are also subsets of this ball. Therefore, using the above inequality,

\[
H^{d-1}\{y : \|\partial S - y\| = z\} = H^{d-1}(\partial S_z) + H^{d-1}(\partial S(z)) \leq C H^{d-1}(\partial B).
\]

Since \( B \) is a ball of radius \( C r \), there exists some constant that depends only on the norm and the dimension such that

\[
H^{d-1}(\partial B) = C' r^{d-1}
\]

Plugging in these estimates into the Euclidean coarea formula, we get

\[
\lambda((\partial S)_{pr/(s-r)}) \leq C'' \int_0^{pr/(s-r)} r^{d-1} dz \leq \frac{C''' r^d}{s-r},
\]

where \( C'' \) and \( C''' \) are possibly different constants depending only on the dimension and norm. Thus, for \( s \) sufficiently large and depending only on
$d$, the norm, and $\epsilon$, we conclude that
\[ z\left((\partial S)_{\rho/m}\right) \leq \frac{\epsilon^2}{64}. \]
By definition,
\[ S \subset \mathfrak{U}(\mathfrak{R}(S)) \cup (\partial S)_{\rho/m} \]
as any parts of $S$ that might not be covered by the inner hull of $S$ are at
most $\rho/m$ away from the boundary. Thus
\[ z(S) - \frac{\epsilon^2}{64} \leq z(\mathfrak{U}(\mathfrak{R}(S))) \leq z(S) \]
Similarly, the corresponding inequality
\[ z(S) \leq z(\mathfrak{U}(\mathfrak{O}(S))) \leq z(S) + \frac{\epsilon^2}{64} \]
hold for the outer hull of $S$.

As a final preliminary, we prove a lemma about the geometry of maximal
clique sets:

**Lemma 5.3.** Let $\mathfrak{P}$ be a maximal clique set. Then, for any $\epsilon > 0$, $\mathfrak{U}(\mathfrak{P})$
contains a ball of diameter $(1 - \epsilon)r$, for some sufficiently large $s$.

**Proof.** To prove this lemma, we go through the abstract framework of Haus-
dorff convergence of subsets of a metric space. Consider an abstract metric
space $X$ imbued with metric $\iota$, and, for any $S \subset X$, define the $\epsilon$-fattening of
$S$ as before, using the metric $\iota$ to measure distance. For any two $A, B \subset X$,
the Hausdorff distance is defined as
\[ \iota_H(A, B) := \inf\{\epsilon : A \subset B_\epsilon, B \subset A_\epsilon\}. \]
If $X$ is compact in the topology defined by $\iota$, the space of closed subsets of
$X$ makes a compact space with respect to this metric [18, page 294].

Let $A$ be a cube of side $3r$ in the $d$-dimensional torus. Note that $\mathfrak{U}(\mathfrak{R}(A))$
includes a cube of side $(3 - 2/s)r$. If $s > 2$, such a set will also include a
ball of diameter $r$, and therefore at least one maximal clique set. For each
$s > 2$, let $\mathfrak{P}_s$ be some maximal clique set that is a subset of $\mathfrak{R}(A)$.

Let $W_s = \mathfrak{U}(\mathfrak{P}_s)$. The diameter of $W_s$ is bounded above by $r(1 + C_{at}/s)$,
and its measure is bounded below by $\tau$ from Lemma 5.1. Let $\tilde{A}$ and $\tilde{W}_s$
be the same sets, with distances scaled by $1/r$. Then $\tilde{W}_s$ are subsets of a
cube of side-length $\leq 3$. Their diameter cannot exceed $(1 + C_{at}/s)$ and
their Lebesgue measure is at least $\nu/2^d$, where the final bound follows from
the definition of $\tau$. Note that, after rescaling, every $\tilde{W}_s$ is a subset of the
same metric space $\tilde{A}$, which is independent of $n$. Thus, any convergence
in the Hausdorff distance associated with $\tilde{A}$ is automatically uniform in $n$.
Since $\tilde{A}$ is a compact metric space and $\tilde{W}_s$ is a sequence of closed sets, the
compactness result above guarantees that some subsequence $\tilde{W}_{s_k}$ converges
to a limit $\tilde{W}$ in the Hausdorff metric. Passing through to the limit, we see
that $\tilde{W}$ must have measure at least $\nu/2^d$, and its diameter cannot exceed $1$. 
However, it is also a subset of a cube of side-length \( \leq 3 \), and therefore we can embed \( \tilde{W} \) into the finite dimensional normed space \( (\mathbb{R}^d, \| \cdot \|) \) isometrically. Quoting the isodiametric inequality again [3, pg 94],

\[
\lambda(\tilde{W}) \leq \nu \left( \frac{\text{diam}(\tilde{W})}{2} \right)^d.
\]

Thus, \( \tilde{W} \) satisfies equality for the isodiametric inequality. Luckily, the isodiametric inequality also characterizes all sets that achieve equality as balls in the given norm.

Fix \( \epsilon > 0 \). For all sufficiently large \( k \), the Hausdorff convergence of \( \tilde{W}_{sk} \) guarantees that, for some \( B(x) \), a ball of diameter 1 centered at \( x \), we have

\[
B(x) \subset (\tilde{W}_{sk})_{\epsilon/2}.
\]

Now, consider the set \( B(x) \setminus \tilde{W}_{sk} \). The distance between \( x \) and this set must exceed \( 1/2 - \epsilon/2 \); otherwise, \( B(x) \) would not be inside the \( \epsilon/2 \)-fattening of \( \tilde{W}_{sk} \). Thus, a ball of diameter \( 1 - \epsilon \) must be inside \( \tilde{W}_{sk} \). Scaling by \( r \) completes the proof of Lemma 5.3.

Proof of Theorem 2.1, assuming Theorem 3.1. Fix \( \delta > 0 \) and \( \epsilon > 0 \), and assume that \( |E| > (1 + \delta)\mu \). Let \( \tilde{\delta} = \delta(1 - \epsilon/16) \). By 5.2, \( |E_s| > (1 + \tilde{\delta})\mu_s \) occurs for some sufficiently large \( s \). Let \( \tilde{\epsilon} = \epsilon/8 \). Assume that the event \( F \) described in Theorem 3.1 happens. For ease of notation, let \( q = (2\tilde{\delta}\mu_s)^{1/2} \).

Then there is a maximal clique set \( \mathcal{P} \subseteq T \) such that for all \( I \in \mathcal{P} \),

\[
\left| \frac{\tau_s X_I}{q} - 1 \right| < \tilde{\epsilon}.
\]

Now, let \( A \) be some ball of diameter \( (1 - \epsilon/128)r \) contained in \( U(\mathcal{P}) \) - for sufficiently large \( s \), such as set exists by Lemma 5.3. Pick \( S \) to be a convex subset of this ball with \( z(S) \geq \epsilon/16 \). We now wish to show that

\[
(z(S) - 15\epsilon/16)\sqrt{2\tilde{\delta}\mu} \leq |\chi(S)| \leq (z(S) + 15\epsilon/16)\sqrt{2\tilde{\delta}\mu}.
\]

This would be sufficient to prove the first stipulation of Theorem 2.1 if \( A \) were a ball of radius \( r \); because of the error, we must get slightly better bounds and strengthen them later. To get an upper bound, we look at the number of vertices in the outer hull of \( S \); the corresponding lower bound will arrive via a bound on the inner hull.

First, we want to get an upper bound on \( |\mathcal{D}(S)| \). By the upper bound on the Lebesgue measure of \( U(\mathcal{D}(S)) \) from above, we conclude that

\[
|\mathcal{D}(S)| \leq (md) \left( z(S) + \frac{\epsilon^2}{64} \right) \leq z(S) \tau_s \left( 1 + \frac{\epsilon^2}{64z(S)} \right) \leq z(S) \tau_s \left( 1 + \frac{\epsilon}{4} \right),
\]
where the penultimate inequality follows from Lemma 5.1 and the final one from the assumed lower bound on $\zeta(S)$. If we assume that the $X_I$ associated with every index in $\mathcal{O}(S)$ takes on the maximal value allowed by Theorem 3.1 we deduce the inequality

$$|\chi(S)| \leq |\chi(\mathcal{U}(\mathcal{O}(S)))|$$

$$\leq \zeta(S)\tau_s \left(1 + \frac{\epsilon}{4}\right) \left[\left(\frac{a}{\tau_s}\right) (1 + \tilde{\epsilon})\right]$$

$$\leq \zeta(S)q \left(1 + \frac{\epsilon}{2}\right).$$

This is nearly the desired upper bound; we simply need to replace $q$ with $\sqrt{2 \delta \mu}$:

$$q = (2\delta \mu)^{1/2} \leq \sqrt{2 \delta \mu} \left(1 + \frac{C_{at}}{s}\right)^{1/2}.$$

Increasing $s$ sufficiently to ensure that

$$\left(1 + \frac{C_{at}}{s}\right)^{1/2} \leq \left(1 + \frac{\epsilon}{4}\right)$$

and substituting into the earlier inequality, we produce the desired upper bound on $|\chi(S)|$.

The lower bound follows similarly. First, we get a lower bound on $|\mathcal{R}(S)|$:

$$|\mathcal{R}(S)| \geq \left(\zeta(S) - \frac{\epsilon^2}{64}\right) (m^d r) \geq \zeta(S)\tau_s \left(1 + \frac{C_{at}}{s}\right)^{-1} \left(1 - \frac{\epsilon}{4}\right).$$

If every $X_I$ takes on its minimal value, we get the following lower bound for $|\chi(S)|$:

$$|\chi(S)| \geq |\chi(\mathcal{U}[\mathcal{R}(S)])| \geq \zeta(S)\tau_s \left(1 + \frac{C_{at}}{s}\right)^{-1} \left(1 - \frac{\epsilon}{4}\right) \left[\left(\frac{a}{\tau_s}\right) (1 - \tilde{\epsilon})\right].$$

The lower bound on $q$

$$q = (2\delta \mu)^{1/2} \geq \left(\frac{2 \delta \mu}{1 - \epsilon/16}\right)^{1/2}$$

follows from the Lemma 5.1 bound $\mu_s \geq \mu$. Substituting this in and increasing $s$ sufficiently gives the desired lower bound.

As mentioned above, this nearly completes the first stipulation of Theorem 2.1; the only difference is that the ball $A$ is of diameter $(1 - \epsilon/128)r$ instead of $r$. Let $A'$ be an arbitrary ball of diameter $r$ containing $A$, and pick a convex set $S'$ in $A'$. By definition, $z(A'/A) \leq \epsilon/32$. Defining $S = S' \cap A$, all the previous assertions follow. Furthermore, the lower bound

$$|\chi(S')| \geq |\chi(S)| \geq (\zeta(S) - 15\epsilon/6)\sqrt{2 \delta \mu}$$

follows from before. Finally, $z(S) \geq z(S') - \epsilon/32$, as $z(S'/S) \leq z(A'/A)$, which implies the lower bound

$$|\chi(S')| \geq (z(S) - \epsilon)\sqrt{2 \delta \mu}.$$
For the upper bound, we note that

\[ |\chi(S'|S)| \leq |\chi(S)| + |\chi(\mathcal{O}(S'|S))| \]

\[ \leq (z(S) + 15\epsilon/16) \left( \sqrt{2\delta\mu} \right) + |\chi(\mathcal{O}(S'|S))| \]

Again, assuming all elements of \( \mathcal{O}(S'|S) \) intersect the indices with largest density, we can conclude that

\[ |\chi(\mathcal{O}(S'|S))| \leq z(S') \left( 1 + \frac{\epsilon}{4} \right) \left( \frac{q}{\tau_s} \right) (1 + \tilde{\epsilon}) \leq \left( \frac{\epsilon}{16} \right) \left( \sqrt{2\delta\mu} \right) \]

following the same derivations as above. Therefore,

\[ (z(S') - \epsilon) \sqrt{2\delta\mu} \leq |\chi(S')| \leq (z(S') + \epsilon) \sqrt{2\delta\mu} \]

for any convex set \( S' \) of \( A' \) (including \( A' \) itself), proving the first stipulation of 2.1.

For the second part of Theorem 2.1, consider a convex set \( S \subset A^c \), with \( z(S) > \epsilon/16 \) that lies completely inside a ball of radius \( r \). For sufficiently large \( s \), \( \mathcal{R}(S) \) will be completely disjoint from \( \mathcal{P} \); however, \( \mathcal{O}(S) \) may not be. Thanks to the bounds on the measure of the outer and inner hulls,

\[ |\mathcal{O}(S)/\mathcal{R}(S)| \leq \left( \frac{c^2}{32} \right) m^d \tau_s \leq \left( \frac{c^2}{26} \right) \tau_s \]

To get an upper bound on \( |\chi(S)| \), we assume that \( \mathcal{O}(S) \) has the maximal intersection with \( \mathcal{P} \), and that the remaining elements take on the maximal value allowed by Theorem 3.1. Specifically, this gives

\[ |\chi(S)| \leq |\mathcal{M}(\mathcal{O}(S)))| \]

\[ \leq |\mathcal{O}(S) \cap \mathcal{P}| \left( \frac{q}{\tau_s} \right) (1 + \tilde{\epsilon}) + |\mathcal{O}(S)| \left( \frac{\epsilon q}{\tau_s} \right) \]

\[ \leq z(S) \left[ \frac{c^2}{26z(S)} (1 + \tilde{\epsilon}) + K \left( 1 + \frac{\epsilon}{4} \right) 2\tilde{\epsilon} \right] q \]

using the earlier bound on the cardinality of the outer hull and the upper bound on the \( X_I, J \notin \mathcal{P} \) from Theorem 3.1. Thanks to the lower bound on \( z(S) \) and the definition of \( \tilde{\epsilon} \), the bracketed expression can be bounded above by \( 3\epsilon/4 \). Using the bound \( q \leq (1 + \epsilon/4) \sqrt{2\delta\mu} \) for all large \( s \), derived above, and dividing through by \( \sqrt{2\delta\mu} \), gives the desired result. This completes the proof of Theorem 2.1.

6. Proof of Theorem 3.1

The bulk of the remaining length of the paper is dedicated to the proof of Theorem 3.1. The main tool of the proof is matching upper and lower bounds. As an illustration, suppose we could produce a subset \( \mathcal{E} \subset T \) with strictly positive \( X_I \)'s such that

\[ \sum_{I \in \mathcal{E}} X_I \leq (2\delta\mu_s)^{1/2} \]
and
\[ \sum_{I \in \mathcal{E}} \sum_{j \in \mathcal{E}, d(I, J) \leq s} X_I X_J \geq 2\tilde{\delta} \mu_s. \]

If we square the upper bound, we see that these bounds are, indeed, matching. In fact, subtracting the lower bound from the square of the upper bound, we conclude that
\[ \sum_{I \in \mathcal{E}} \sum_{j \in \mathcal{E}, d(I, J) > s} X_I X_J \leq 0. \]

Notice that the sum is over indices whose distance is larger than \( s \). Since all the \( X_I \)'s are strictly positive, no two indices in \( \mathcal{E} \) can be more than \( s \) away from one another! This gives us a diameter restriction of \( \mathcal{E} \).

While this scenario is very simple, its also unrealistic. First, the argument presented above does not account for entropy: it is agnostic as to whether all the required vertices are in one \( A_I \) or are spread out over \( \tau_s \) different \( A_I \)'s. Intuitively, the latter setup is much more likely, as the Poisson Point Process would not be as heavily penalized for spreading the same number of vertices over a larger area. In fact, the correct quantity to bound is the sum of \( X_I (\log[X_I/D] - 1) + D \), the rate function of the Poisson distribution of mean \( n/m^0 \). The convexity of this function would penalize overly concentrated setups. However, even with this modification, the argument above relies on precisely matching lower and upper bounds. As with most probabilistic setups, we can only prove similar bounds with lower order corrections. Therefore, we must produce a stability-style argument to show that, whenever a sum over “far away” indices is small in comparison to \( \mu_s \), the geometry of the set is a maximal clique set.

The proof of Theorem 3.1 is organized as follows: First, we use concentration inequalities and large deviations estimate to show that, in the \( s \)-graded model conditioned to have too many edges, three inequalities hold with high probability. Next, we show that these inequalities imply the existence of a small set of indices such that almost all the “extra” edges have both endpoints in the union of its associated \( A_I \)'s, and satisfies a convex upper bound. The next two subsections require delicate estimates which produce a subset which satisfies a similar quadratic lower bound, a termwise lower bound, and an additional upper bound on the total number of vertices in the set. Finally, simple convex analysis will show that this set will be a maximal clique set, and that the vertices are roughly equidistributed among the \( A_I \)'s in this set.

6.1. Probabilistic Analysis. For this section, \( C \) will indicate any multiplicative constant that is uniformly bounded in \( n \). It may depend on all other parameters, including \( s \), and may take on different values from line to line.

Let \( L \) be the event \( \{|E_s| > (1 + \tilde{\delta}) \mu_s\} \). As before, let \( q = (2\tilde{\delta} \mu_s)^{1/2} \) and \( w = \tau_s D \). To show that certain events occur with high probability in the
conditional s-graded model, we must first produce a lower bound for $\mathbb{P}(L)$. To do so, we manufacture a configuration that implies $L$. By the definition of $p$, the inequalities in (3), and Lemma 5.1, it is clear that $\log \mu_s \sim p \log n$ as $n \to \infty$. Set $z = \max\{p/4, 3p/4 - 1/2\}$. Pick a maximal clique set $\mathfrak{K}$ and define the event $H$ by

$$H := \left\{ \forall I \in \mathfrak{K}, X_I \geq \left\lfloor \frac{q + n^z}{s} \right\rfloor \right\}$$

Note that, when $p \leq 1$, each $X_I$ in $\mathfrak{K}$ will be at most $(q/\tau^s) + 1$. Also note that by (3), the correction term $n^z$ is of lower order than $q$, since $3p/4 - 1/2 < p/2$.

Assume that $H_{\mathfrak{K}}$ occurs. This implies that the number of edges (in the graph $G_s$) with both endpoints in $\mathcal{U}(\mathfrak{K})$ is at least

$$\left( q + n^z \right) \geq \delta \mu_s + \frac{qn^z}{2}$$

Therefore, if $H_{\mathfrak{K}}$ occurs then $L$ would be true if the total number of edges with at least one endpoint in $\mathcal{U}(\mathfrak{K})$ would exceed $\mu_s - \frac{qn^z}{2}$. Let $|E_s'|$ be the number of edges with neither endpoints in $\mathcal{U}$. Then

$$\mathbb{E}(|E_s'|) \geq \sum_{I : d(I, \mathfrak{K}) > s} \mathbb{E} \left[ \left( \frac{X_I}{2} \right) + \frac{1}{2} \sum_{J : 0 < d(I, J) \leq s} X_I X_J \right].$$

This is a lower bound as we ignore pairs of nonzero elements which are close to $\mathfrak{K}$ but not in $\mathfrak{K}$. The set $\{I : d(I, \mathfrak{K}) \leq s\}$ has diameter $\leq 3s$, and therefore its cardinality must be some constant independent of $n$. Therefore by (5),

$$\mathbb{E}(|E_s'|) \geq \left( m^d - 3^{d+1} \tau_s^s \right) \frac{N_I |n^2}{2m^2 d} = \mu_s - \frac{C \mu_s}{m^d}.$$

We now wish to bound the probability

$$\mathbb{P}(|E_s'| < \mu_s - q n^z / 2) \leq \mathbb{P} \left( |E_s'| < \mathbb{E}(|E_s'|) - q n^z / 2 + \frac{C \mu_s}{m^d} \right),$$

where the inequality follows from the above relation between the mean of $|E_s'|$ and $\mu_s$. By Lemma 5.1, $\mu_s/m^d$ grows as $n^{2(p-1)}$ (on the logarithmic scale). If $p < 1$, this is a vanishing quantity, and therefore

$$qn^z / 2 - \frac{C \mu_s}{m^d} \leq q n^z / 4.$$

If $p \geq 1$, then $qn^z$ grows as $n^{5p/4 - 1/2}$. Since $p < 2 - \delta^*$ by the restrictions on $r$, $2(p - 1) < 5p/4 - 1/2$, and, for sufficiently large $n$, the inequality above still holds.

Using the sum definition of $|E_s|$, a straightforward calculation can show that

$$\text{Var}(|E_s'|) \leq \text{Var}(|E_s|) \leq C m^d \left( D^3 + D^2 \right).$$
Referring back to the growth bounds on $m^d$ and $D$, we see that the variance grows as $n^p$ if $p < 1$ and $n^{2p-1}$ if $p \geq 1$. By Chebyshev’s inequality,
\[ \mathbb{P}(|E_s'| < \mathbb{E}(|E_s'|) - qn^z/4) \leq \frac{16 \text{Var}(|E_s'|)}{(qn^z)^2}. \]

For all value of $p$ of interest, this fraction vanishes as $n^{−p/2}$, up to possible logarithmic factors. Thus, the probability of the event
\[ \mathbb{P}(|E_s'| \geq \mu_s - qn^z/2) > 1 - \epsilon \]

for any $\epsilon$ positive, for sufficiently large $n$.

Since the above event implies $L$ once $H$ occurs, this allows us to conclude that
\[ \mathbb{P}(L) \geq \mathbb{P}(H) \mathbb{P}(L \mid H) \geq (1 - \epsilon) \mathbb{P}(H). \]

Thus, the lower bound on the probability of $L$ will come from a good lower bound on the probability of $H$. By definition,
\[
\begin{align*}
\mathbb{P}(H) &= \left[ \mathbb{P}\left( X_I \geq \left\lceil \frac{q + n^z}{\tau_s} \right\rceil \right) \right]^{\tau_s} \\
&\geq \left[ \mathbb{P}\left( X_I = \left\lceil \frac{q + n^z}{\tau_s} \right\rceil \right) \right]^{\tau_s} \\
&\geq \left( e^{-\tau_s D(q+2n^z)/\tau_s} / [(q + 2n^z)/\tau_s]! \right)^{\tau_s}
\end{align*}
\]

where the final inequality follows by removing the ceiling function and compensating by adding a factor of two to the correction term. Now, by Stirling’s approximation,
\[
\left( [(q + 2n^z)/\tau_s]! \right)^{-1} \geq \exp \left( -\left( \frac{q + 3n^z}{\tau_s} \right) \left[ \log \left( \frac{q + 3n^z}{\tau_s} \right) - 1 \right] \right),
\]

where the changed constant in front of the $n^z$ term is to compensate for the omitted polynomial term in the approximation. Substituting this above gives
\[
\mathbb{P}(H) \geq \exp \left( -\tau_s D - (q + 3n^z) \left[ \log \left( \frac{q + 3n^z}{\tau_s D} \right) - 1 \right] \right).
\]

Since $n^z$ and $\tau_s D$ are both lower order correction, we can find an absolute constant $C$ such that
\[
\mathbb{P}(L) \geq \exp \left( -q(\log(q/w) - 1) - Cn^z \log n \right),
\]

using the definition of $w$.

Let $a = \delta^*/20$ and $M = \max\{Dn^a, n^a\}$ We will say an index $I \in T$ is in the bulk if
\[ X_I \leq M. \]

First, we wish to show that not too many indices are outside the bulk. Let
\[ \alpha = \min\{1 - p/2 - a/2, p/2 - a/2\} \]

\[
\frac{16 \text{Var}(|E_s'|)}{(qn^z)^2}. \]
and define $A$ to be the event 
\[ \{ \exists S \subset T, |S| > n^\alpha \text{ such that } \forall I \in S, X_I > M \}. \]
Note that it is sufficient to show that no such set of cardinality exactly $\lceil n^\alpha \rceil$ exists. We can bound the probability of this event in a straightforward manner. First, we establish the classical large deviation bound for Poisson random variables. To do so, note that 
\[ E[\exp(\lambda X_I)] = e^{-D} \sum_{k=0}^{\infty} \frac{\exp(\lambda k) D^k}{k!} = \exp \left[ D \left( e^\lambda - 1 \right) \right] \]
By Chebychev’s inequality, this implies that, for any positive $\lambda$,
\[ \mathbb{P}[X_I > t] \leq \exp \left[ D \left( e^\lambda - 1 \right) - \lambda t \right] \]
Setting $\lambda = \log(t/D)$ gives
\[ (7) \quad \mathbb{P}[X_I > t] \leq \exp(-t(\log(t/D) - 1) - D), \quad t > D. \]
A similar procedure, using function $E[e^{-\lambda X_I}]$, leads to
\[ (8) \quad \mathbb{P}[X_I < t] \leq \exp(-t(\log(t/D) - 1) - D), \quad t < D. \]
We now return to bounding the probability of $A$. If $p \geq 1$, we get that
\[ (9) \quad \mathbb{P}(A) \leq \left( \frac{m^d}{n^\alpha} \right) \mathbb{P}(X_I > M)^{n^\alpha} \leq m^{dn^\alpha} \exp(-n^\alpha M \log(M/D)). \]
The polynomial rate of growth of $n^{\alpha+a}D$ is $p/2 + a/2$, whereas $n^\alpha$ grows as $1-p/2-a/2$, which is smaller for all $p \geq 1$. Therefore, replacing $M \log(M/D)$ by $n^aD$, which is strictly smaller, implies that
\[ \mathbb{P}(A) \leq \exp(-n^{\alpha+a}D + dn^\alpha \log m) \leq \exp(-Cn^{p/2+a/2}) \]
If $p < 1$, the same calculation gives
\[ (10) \quad \mathbb{P}(A) \leq \left( \frac{m^d}{n^{p/2-a/2}} \right) \exp(-n^{p/2+a/2}) \leq \exp(-Cn^{p/2+a/2}). \]
As a consequence of (6), (9) and (10), we get
\[ \mathbb{P}(A \mid L) = \frac{\mathbb{P}(A \cap L)}{\mathbb{P}(L)} \leq \frac{\mathbb{P}(A)}{\mathbb{P}(L)} \leq \exp(-Cn^{p/2+a/2} + q \log(q/w) - q + Cdn^z \log n). \]
Since $q$ grows like $n^{p/2}$, the above inequality shows that $\mathbb{P}(A \mid L)$ vanishes in as $n \to \infty$.
Next, we want some control over the behavior of indices outside the bulk. Define
\[ Y_I = X_I \left( \log(X_I/D) - 1 \right) + D, \]
with the convention that $0 \log 0 = 0$. This transformation exactly matches the exponential upper bound First note that $Y_I \geq 0$, as the function $f(x) = x(\log(x/D) - 1) + D$ achieves its minimum at $x = D$. This means that, in
general, \(f(x)\) is not invertible for all positive integer values of \(x\). Therefore, we define two inverses. First, let

\[ g_1(x) : [0, D] \rightarrow [0, D] \text{ be a function such that } (f \circ g_1)(x) = x. \]

Note that this function is decreasing, with \(g_1(0) = D\) and \(g_1(D) = 0\). For any \(x > D\), we say that \(g_1(x) = -\infty\). We define \(g_2\), the second inverse, similarly, except its range is defined to be \((D, \infty)\). This inverse is strictly increasing. We use (8) and (7) to deduce

\[ P[Y_I > t] = P[X_I < g_1(t)] + P[X_I > g_2(t)] \leq 2e^{-t}. \]

Note that, if \(t > D\), the first term contributes nothing to the probability.

These estimates allow us to bound the moment generating function of \(Y_I\): let \(A\) be the set of atoms of \(Y_I\). Then, for any \(\lambda < 1\),

\[ E[\exp(\lambda Y_I)] \leq \sum_{i \in A} 2e^{-i+\lambda i}. \]

If \(p < 1\), \(A\) is more sparse than the integers, in the sense that \(|A \cap [0, n]| \leq n\). Thus, we can increase the expectation by taking the sum over the integers, and possibly adding a multiplicative constant. If \(D \rightarrow \infty\), the minimal distance between atoms in \(A\) is \(C/\log D\). Thus, if \(p < 1\),

\[ E[\exp(\lambda Y_I)] \leq C \sum_{i=1}^{\infty} e^{(\lambda-1)i} = \frac{C}{1 - \exp(\lambda - 1)}. \]

Meanwhile, we can only produce the inferior bound

\[ E[\exp(\lambda Y_I)] \leq C \sum_{i=1}^{\infty} e^{C(\lambda-1)i/\log D} \leq \frac{C}{1 - \exp(C(\lambda-1)/\log D)}. \]

Let \(\beta = p/2 - a/4\). Clearly, \(\alpha < \beta < p/2\); furthermore, \(z < \beta\), as \(3p/4 - 1/2 < p/2 - a/4\) whenever \(1 \geq p < 2 - a\), and \(p/4 < p/2 - a/4\) as long as \(p > a\). Since \(\delta^* \leq p < 2 - \delta^*\), both bounds hold. Now, let \(t = q(\log(q/w) - 1) + n^{\beta}\). Note that \(t\) exceeds the exponent of the lower bound on the probability of \(L\), but only thanks to a slightly larger lower order term. We now define an event

\[ B := \left\{ \exists S \subset T, |S| \leq n^{\alpha} \text{ such that } \sum_{i \in S} Y_I > t \right\}. \]

Again, this is equivalent to finding a set of size exactly \([n^{\alpha}]\) that satisfies the sum lower bound, as the \(Y_I\)'s are positive, and finding a smaller set will automatically give a larger set. First, we bound the probability that the sum of \(Y_I\)'s over a particular set of indices \(S\) of cardinality \([n^{\alpha}]\) exceeds \(t\).
To do so, we use a Chernoff-like strategy. Assume $p \geq 1$. Then

$$
\P \left( \sum_{I \in S} Y_I > t \right) \leq \left( \frac{\exp(\lambda t)}{1 - \exp(C(\lambda - 1)/\log D)} \right)^{n^a} e^{-\lambda t}
$$

We now set $\lambda = 1 - n^a/t$. This leaves an exponential in $n^a/(t \log D)$, which is vanishing in $n$. Therefore, we can bound

$$
\frac{1}{1 - \exp(-Cn^a/(t \log D))} \leq \frac{Cn^a/(t \log D) - (Cn^a/(t \log D))^2}{(t \log D)/n^a + 2}
$$

for all sufficiently large $n$. Therefore,

$$
\P \left( \sum_{I \in S} Y_I > t \right) \leq (C(t \log D)/n^a + 2)^{n^a} e^{-t + n^a} \leq \exp \left( -t + Cn^a \log n \right).
$$

Invoking the union bound, we deduce that

$$
\P(B) \leq \frac{m^d}{n^a} \exp \left( -t + Cn^a \log n \right) \leq \exp \left( -q(\log(q/w) - 1) - Cn^\beta \right)
$$

for all sufficiently large $n$. This bound follows because the positive terms grow slower than $n^\beta$, by definition of $\beta$ and $a$. This implies that $B^c$ occurs with high probability, as

$$
P(B | L) \leq \frac{P(B)}{P(L)} \leq \exp(-C\alpha n^\beta + Cn^\gamma \log n)
$$

which vanishes as $n$ grows, thanks to the choice $\beta > z$.

The final probabilistic step is to show that conditional on $L$, the bulk does not contain very many extra edges. Let

$$
\tilde{X}_I := X_I \cdot 1_{X_I \leq M}
$$

and $|\tilde{E}_s|$ be the define analogously with $|E_s|$ by replacing $X_I$ with its truncated version. In other words, $|\tilde{E}_s|$ is the version of $G_s$ obtained after deleting all vertices in boxes that satisfy $X_I > M$. Fix $\gamma = p - 2a$, and consider the event

$$
D = \{ ||\tilde{E}_s| - \mu_s| > n^\gamma \}.
$$

Bounding the probability of this event will require the use of concentration inequalities; however, we must be careful as $D$ requires $\tilde{E}_s$ to concentrate around the mean of $|E_s|$, not its own mean. Luckily, $\mu_s$ is only slightly larger than the expected value of $|E_s|$, as the truncation operation moves probability mass from positive areas to 0.

Here our treatment will differ based on $p$. First, assume that $p < 1$. In these scenarios, the cutoff is at $n^a$, meaning that $|\tilde{E}_s|$ has a Lipschitz constant of $Cn^{2a}$ in all its coordinates with respect to the Hamming metric when seen as a function of the $\tilde{X}_I$'s.
We use Talagrand’s convex concentration inequality [20, Theorem 4.1.1]. First, let us define the setting: let \( \Omega = \prod_{i=1}^{N} \Omega_i \), where \( \Omega_i \) are all probability spaces, the measure on \( \Omega \) is the product measure, and \( X \) is a random variable. For a set \( A \subset \Omega \), define the set
\[
U_A(x) := \{ \{ s_i \} \in \{0, 1\}^N : \exists y \in A, s_i = 0 \Rightarrow x_i = y_i \}.
\]
Let \( V_A(x) \) be the convex hull of \( U_A(x) \), and \( d_c(A, x) \) is the \( \ell^2 \) distance of \( V_A(x) \) to the origin. For any set \( A \), we denote \( A_t \) be the \( t \) blowup of \( A \) with respect to this metric, i.e.
\[
A_t := \{ x \in \Omega : d_c(A, x) \leq t \}.
\]
We can now state the inequality:

**Theorem 6.1 (Talagrand’s Inequality).** If \( \Omega, \mathbb{P}[,], A \) and \( A_t \) are as above, then
\[
\mathbb{P}[A] (1 - \mathbb{P}[A_t]) \leq e^{-t^2/4}.
\]

We will not apply this theorem directly; instead, we use a corollary of this theorem frequently used in discrete settings [1, Theorem 7.7.1]. To do so, we consider \( f \), a function from the natural numbers to the natural numbers. We say that \( f \) is a witness function for \( X \) if, whenever \( X(\omega) \geq t \), there exists \( I \subset [n] \) with \( |I| \leq f(t) \), such that every \( \omega' \) that agrees with \( \omega \) in all \( i \in I \) has \( X(\omega') \geq t \). Furthermore, we assume that \( X(\omega) \) is \( K \)-Lipschitz with respect to the Hamming distance — that is, \( |X(\omega) - X(\omega')| \leq K \) whenever \( \omega \) and \( \omega' \) differ in at most one coordinate.

**Theorem 6.2.** Let \( \Omega \) be a product space, and \( X \) a real valued function on \( \Omega \) with Lipschitz constant \( K \) with respect to the Hamming distance. If \( f \) is witness function for \( X \) as above, then, for any \( b \) and \( t \),
\[
\mathbb{P}[X > b + tK \sqrt{f(b)}] \mathbb{P}[X \leq b] \leq \exp(-t^2/4).
\]

We now apply this theorem on \( X = |\tilde{E}_s| \). Since each coordinate is bounded above by \( n^a \), \( X \) is Lipschitz with \( K = Cn^{2a} \). The function \( f(t) = 2t \) is a witness function for \( |\tilde{E}_s| \); to see this, note that \( |\tilde{E}_s| \) is the edge count of the \( s \)-graded geometric random graph, after we remove any \( X_I \) with very high density. As such, we can “witness” the existence of \( t \) edges by finding at most \( 2t \) vertices; the flexibility of the setup allows us to pick these vertices judiciously, avoiding all the isolated ones. Finding \( 2t \) vertices will require at most \( 2t \) distinct coordinates, if each one of them vertices lies in a distinct \( A_I \).

We apply the theorem with \( b = \mu_s + qn^z \) and
\[
t = \frac{n^\gamma - qn^z}{n^{2a} \sqrt{\mu_s + qn^z}},
\]
with $z$ defined as in the beginning of the section, and deduce that
\[ \mathbb{P}(D) \mathbb{P}(|\tilde{E}_s| \leq \mu_s + qn^z) \leq \exp \left( -\frac{C|n\gamma - qn^z|^2}{(\mu_s + qn^z)n^{4a}} \right) \]
\[ \leq \exp \left( -C'n^{2\gamma - p - 8a} \right), \]
where the final inequality follows because $qn^z$ is a lower order correction to $\mu_s$, by definition of $z$.

Note that this inequality is only possible when $f(\mu_s) \leq m^d$; $f(\mu_s)$ represents the number of coordinates necessary to witness the property $|\tilde{E}_s|$, so it must be bounded by the total number of random variables available. Since $p < 1$, this is guaranteed. Now, we must bound the probability that $|\tilde{E}_s|$ is below $\mu_s + qn^z$. First, the mean of $|\tilde{E}_s|$ is strictly smaller than $\mu_s$. Furthermore, its variance is essentially equal to the variance $|E_s|$. To see this, note that
\[ \text{Var}[|\tilde{E}_s|] \leq \mathbb{E}[|\tilde{E}_s| - \mu_s]^2, \]
since the function $t \to \mathbb{E}[(X - t)^2]$ is minimized at the mean of $X$ for any random variable. The variance computation is identical to the previous computation over the event $\{X_I = \tilde{X}_I, \forall I \in T\}$. By the union bound, the probability of the complement of this event is bounded above by $m^d \exp(-Cn^a \log n)$. Since the maximal value of $(|\tilde{E}_s| - \mu_s)^2$ is $m^{2d}n^{4a}$ by definition, we deduce that
\[ \text{Var}[|\tilde{E}_s|] \leq \text{Var}[|E_s|] + m^{3d}n^{4a} \exp(-Cn^a \log n) \leq Cm^dD^2, \]
where the final inequality holds for all sufficiently large values of $n$, and relies on the earlier calculation of variance. Thus, By Chebyshev’s inequality,
\[ \mathbb{P}(|\tilde{E}_s| \geq \mu_s + qn^z) \leq \frac{Cm^dD^2}{(qn^z)^2} \leq C'n^{-p/2}, \]
with the final inequality following from the definition of $z$. Thus, for sufficiently large $n$, the corollary to Talagrand’s inequality gives
\[ \mathbb{P}(D) \leq (1 - \epsilon)^{-1} \exp(-C'n^{2\gamma - p - 8a}) \]
As a function of $n$, the exponent in the Talagrand bound grows as $n^{p-8a}$. As long as $p > 16a$, the product of the upper bound and $\exp(q \log(q/w))$ will still go to zero, as $q$ grows as $n^{p/2}$. Since we assumed that $p > \delta^* = 20a$, we have shown that $D^c$ occurs with high probability in the conditional model, as long as $p < 1$.

The case of $p \geq 1$ requires more care. First, assume $p > 1$. As $p$ grows, the number of coordinates decreases while the Lipschitz constant increases. Unfortunately, the concentration inequality that will be used here, the Azuma-Hoeffding inequality, scales with the sum of the squares of the Lipschitz constants. Thus, we cannot naively apply this inequality to the $\tilde{X}_I$’s, but instead must manufacture a scheme to decrease the Lipschitz constants, at the price of making the probability space larger.
Specifically, recall that $A_I$ is a cube of measure $1/m^d$. Since $p > 1$, this measure is much larger than $1/n$. Let $t$ range over the integers smaller than $\lceil D \rceil$, and define $\{F_{I,t}\}$ to be a partition of $A_I$ into smaller cubes, where each subcube has measure $1/n$. Since $D$ might not be an integer, we allow for the possibility that the last $F_{I,t}$ overlaps some of the other $F_{I,t}$, but demand that the rest are disjoint. Let $W_{I,t} = |\chi(F_{I,t})|$, and condition on $\sum_t W_{I,t} \leq M$.

The conditional probability of the event $\{W_{I,t} > x\}$ is bounded above by the usual Poisson distribution, as the truncation moves some of the mass to 0, and thus (at worse) decreases the probability of the variables being large. Also note that summing the $W_{I,t}$’s over $t$ gives $\tilde{X}_I$, by the geometric construction. Now, define $\mathcal{B}$ to be all the double indices $(I,t) \in T \times [\lceil D \rceil]$ such that $W_{I,t} > na$. To get a bound on $\mathbb{P}(D)$, we wish to show that there aren’t many vertices in $\mathcal{B}$. First, we show that there are few indices in $\mathcal{B}$:

$$\mathbb{P}(D) = \mathbb{P}(D \cap \{|\mathcal{B}| > q/n^{a/2}\}) + \mathbb{P}(D \cap \{|\mathcal{B}| \leq q/n^{a/2}\})$$

$$\leq \mathbb{P}(|\mathcal{B}| > q/n^{a/2}) + \mathbb{P}(D \mid |\mathcal{B}| \leq q/n^{a/2}).$$

The first term is easy to bound by direction calculation, similar to the analysis of the event $A$ above:

$$\mathbb{P}(|\mathcal{B}| > q/n^{a/2}) \leq \left(\frac{n}{q/n^{a/2}}\right) \mathbb{P}[\text{Poisson}(1) \geq n] \leq \exp(-qn^{a/2}/2).$$

Next, let

$$R = \sum_{(I,t) \in \mathcal{B}} W_{I,t}$$

We split the conditional probability as follows:

$$\mathbb{P}(D \mid |\mathcal{B}| \leq q/n^{a/2}) \leq \mathbb{P}(R > qn^{a/4} \mid |\mathcal{B}| \leq q/n^{a/2})$$

$$+ \mathbb{P}(D \mid \{|\mathcal{B}| \leq q/n^{a/2}\} \cap \{R \leq qn^{a/4}\}),$$

by the same reasoning as before. Again, the first probability on the right-hand side is easy to bound; $R$ is a Poisson variable of mean $|\mathcal{B}|$. By definition,

$$\mathbb{P}(R > qn^{a/4} \mid |\mathcal{B}| \leq q/n^{a/2}) \leq \frac{\mathbb{P}(\{R > qn^{a/4}\} \cap \{|\mathcal{B}| \leq q/n^{a/2}\})}{\mathbb{P}(|\mathcal{B}| \leq q/n^{a/2})}$$

The denominator has probability of at least $1/2$, for all sufficiently large $n$, from earlier computation. To compute the probability of the numerator, we pick a set of cardinality $k$ in $T \times [\lceil D \rceil]$, and computer the probability that, assuming this set is $\mathcal{B}$, $R$ exceeds $qn^{a/4}$; we then multiply over the possible ways of picking such a set, and add over all $k$’s smaller than $q/n^{a/2}$, to get
an upper bound for the desired probability. Thus,

\[ \mathbb{P}(\{R > qn^{a/4}\} \cap \{|B| \leq q/n^{a/2}\}) \leq \sum_{k=1}^{q/n^{a/2}} \left( \frac{n}{k} \right) \mathbb{P}([\text{Poisson}(k) > qn^{a/4}]) \]

\[ \leq \sum_{k=1}^{q/n^{a/2}} \exp\left( k \log n - qn^{a/4} \log \left[ \frac{qn^{a/4}}{k} \right] + qn^{a/4} - k \right), \]

where we use (7) to evaluate the tail probability. Letting \( k \) take up its maximal value at each step, we deduce that

\[ \mathbb{P}(\{R > qn^{a/4}\} \cap \{|B| \leq q/n^{a/2}\}) \leq \left( \frac{q}{n^{a/2}} \right) \exp(-Cqn^{a/4} \log n + q \log n) \]

\[ \leq \exp(-C'qn^{a/4} \log n), \]

for some sufficiently small \( C' \).

Now, assume that \( R \leq qn^{a/4} \) and consider the sum

\[ G := \sum_{(I,t) \in B} W_{I,t} \left( \sum_{J : d(I,J) \leq s} \tilde{X}_J \right). \]

Since \( \tilde{X}_J \) is bounded above by \( M \), this sum is bounded above by \( q|N_I|Dn^{5a/4} \) (using the definition of \( M \)). Moreover, note that \( G \) is an upper bound for all terms in \( |\tilde{E}_s| \) that involve a \( W_{I,t} \) whose index is in \( B \). Let \( |\tilde{E}_s| \) be the restriction of \( |\tilde{E}_s| \) to indices not in \( B \) — that is, set \( W_{I,t} = 0 \) whenever \((I,t) \in B \) and compute \( |\tilde{E}_s| \). The definition of \( G \) ensures that

\[ |\tilde{E}_s| - G \leq |\tilde{E}_s| \]

Using the bound on \( G \), we deduce that

\[ \mathbb{P}(D \mid \{|B| \leq q/n^{a/2}\} \cap \{R \leq qn^{a/4}\}) \]

\[ \leq \mathbb{P}(|\tilde{E}_s| - \mu_s > (n^\gamma - q|N_I|Dn^{5a/4}) \mid \{|B| \leq q/n^{a/2}\} \cap \{R \leq qn^{a/4}\}) . \]

For all sufficiently large \( n \),

\[ n^\gamma - q|N_I|Dn^{5a/4} \geq n^\gamma / 2 \]

as the negative term grows as \( n^{3p/2 - 1 + 5a/4} \), whereas \( n^\gamma = n^{p-2a} \) has a larger exponent as long as \( p < 2 - 7a \). Thus, we must only provide an upper bound for

\[ \mathbb{P}(|\tilde{E}_s| - \mu_s > n^\gamma / 2 \mid \{|B| \leq q/n^{a/2}\} \cap \{R \leq qn^{a/4}\}) \]

\[ \leq \frac{\mathbb{P}(|\tilde{E}_s| - \mu_s > n^\gamma / 2) \cap \{R \leq qn^{a/4}\})}{\mathbb{P}(\{|B| \leq q/n^{a/2}\} \cap \{R \leq qn^{a/4}\})} . \]

As we saw before, the probabilities of both \( \{|B| > q/n^{a/2}\} \) and \( \{R > qn^{a/4}\} \) are exponentially small, so the complement of their union has probability at
least $1/2$ for all sufficiently large $n$. We use Azuma-Hoeffding to bound the probability of the numerator: the function $|\tilde{E}_s|$ has a Lipschitz constant of $|N_I| |D| n^{2a}$

with respect to the Hamming metric when seen as a function of the $W_I,t$'s. Since there are at most $|T| |D|$ coordinates, the Azuma-Hoeffding inequality gives that

$$P(|\tilde{E}_s| - \mu_s| > n^\gamma/2) \leq \exp \left( -\frac{C'n^{2\gamma}}{m^{d|D|^4} n^{4a}} \right)$$

for some universal constant $C'$. Note that applying Azuma-Hoeffding to the $\tilde{X}_I$ variables without this analysis would have increase the denominator by a factor of $D$. Combining all these bounds, we conclude that

$$P(D) \leq \exp(-qn^{a/2}/2) + \exp \left( -C'qn^{a/4} \log n \right) + (1 - \epsilon)^{-1} \exp \left( -\frac{C'n^{2\gamma}}{m^{d|D|^4} n^{4a}} \right).$$

We now compare this upper bound to the lower bound on the probability of $L$. The first two terms clearly vanish even when multiplied by $e^{q(\log(q/w)-1)}$. The exponent in the final expression is $n^{1-4a}$, which exceeds $p/2$ as long as $p < 2 - 8a$. This shows that $D^c$ occurs with high probability if $p > 1$.

This leaves the final case, $p = 1$. Here, we can apply Azuma-Hoeffding directly to $|\tilde{E}_s|$. The Lipschitz constants will be $C(Dn^a)^2$, and a naive application of the Azuma-Hoeffding gives

$$P[D] \leq \exp \left( -\frac{Cn^{2\gamma}}{m^{d|D|^4} n^{4a}} \right)$$

However, since $D$ is slowly growing if $p = 1$, this bound is identical to the earlier bound, up to multiplicative factors that grow more slowly than any polynomial, and therefore it is sufficient to show that $D^c$ occurs with high probability.

6.2. Showing Partial Localization. We now turn to proving the geometric consequences of conditioning the s-graded model on $L = \{|E_s| > (1 + \delta)\mu_s\}$. By the union bound, the event $A^c \cap B^c \cap D^c$ occurs with high probability after conditioning on $L$. We assume that these four events occur, adding the caveat that the claims made in this section hold not almost surely, but only with high probability. Also, notice that the definition of “with high probability” holds trivially for all small $n$. Thus, we may freely increase $n$ finitely many times with no cost. Our first task is to show that, up to smaller order corrections, all the excess value of $|E_s|$ that $L$ forces into the model will be between vertices in a set of at most $n^a$ indices, all of which have many vertices. This result is far from optimal, but will be used to derive stronger results later.
For any $W \subset T$, let
\[ Q(W) := \frac{2}{q^2} \sum_{I \in W} \left( \frac{X_I}{2} + \frac{1}{2} \sum_{J \in N_I \cap W} X_I X_J \right) . \]
This is the number of edges with both endpoints in $U(W)$, normalized by $q^2/2$. With this normalization, showing that most extra edges have both endpoints in $U(W)$ is equivalent to showing that $Q(W)$ is bounded below by something close to 1. Similarly, for two disjoint index sets $W$ and $W'$, we let
\[ Q(W, W') := \frac{2}{q^2} \sum_{I \in W} \sum_{J \in N_I \cap W'} X_I X_J . \]
This counts the edges between $W$ and $W'$ with the same normalization as above.

Using this notation, for any $W \subset T$,
\[ |E_s| = \frac{q^2}{2} (Q(W) + Q(W, W^c) + Q(W^c)) > (1 + \tilde{\delta})\mu_s \]
where the inequality follows from $L$ occurring. Define $\mathcal{J} \subset T$ as
\[ \mathcal{J} := \{ I : X_I > M \} . \]
From $D^c$, we know that
\[ \left| \frac{q^2}{2} Q(\mathcal{J}^c) - \mu_s \right| \leq n^\gamma \]
Applying the first inequality to $\mathcal{J}$ and assuming $Q(\mathcal{J}^c)$ takes on its maximal value, we deduce that
\[ Q(\mathcal{J}) + Q(\mathcal{J}, \mathcal{J}^c) \geq 1 - \frac{2n^\gamma}{q^2} \]
The next step is to bound $Q(\mathcal{J}, \mathcal{J}^c)$ from above. By $A^c$,
\[ |\mathcal{J}| \leq n^\alpha . \]
This implies that $B^c$ applies to $\mathcal{J}$, and therefore
\[ \sum_{I \in \mathcal{J}} Y_I \leq q(\log(q/w) - 1) + Cn^\beta \]
Besides this convex upper bound, we also want to keep track of the linear upper bound, as this directly counts the number of vertices in the $A_I$'s associated with $\mathcal{J}$. For any $W \subset T$, define
\[ V(W) := \frac{1}{q} \sum_{I \in W} X_I \]
With this normalization, we can see that $V(W)^2 \geq Q(W)$. We also normalize the cardinality of sets by $\tau_s$, that is, for any $W \subset T$,
\[ h(W) := \frac{|W|}{\tau_s} . \]
We now phrase Jensen’s inequality, an essential tool of this analysis, in terms of these quantities:

**Lemma 6.3.** For any \( W \subset T \),
\[
\frac{1}{q} \sum_{I \in W} Y_I \geq V(W) \left[ \log \left( \frac{q}{w} \right) + \log V(W) - \log h(W) - 1 \right]
\]

**Proof.** This is a direct application of Jensen’s inequality. \( \square \)

Applying this to \( I \) and the upper bound \[15\] given by \( B^c \), we deduce that
\[
V(I) \left( \log \left( \frac{q}{wh(I)} \right) - 1 \right) + V(I) \log V(I) \leq \log(q/w) - 1 + C n^{\beta}/q
\]

If \( V(I) \leq 1 \), we have a (very good!) bound on \( V(I) \). Otherwise, the final term on the left-hand side is positive, and the bound still holds if we ignore it. This gives
\[
V(I) \leq \frac{\log(q/w) - 1 + C n^{\beta}/q}{\log(q/(wh(I))) - 1}.
\]

By the upper bound \[14\] on \( |I| \), \( q/(wh(\mathcal{I})) \geq C r_b n^{\alpha/2} \) for all sufficiently large \( n \) and some \( C r_b \). Meanwhile, \( q/w \leq C' r_b n^{1-p/2} / q \). Therefore,
\[
V(I) \leq \frac{(1 - p/2) \log n + \log C' r_b}{a/4 \log n + \log C' r_b} \leq \frac{8(1 - p/2)}{a}
\]
for sufficiently large \( n \). Let \( C_{bb} \) be the rightmost constant above. Notice that this constant depends only on \( \delta^* \) and \( p \). The exact value of this upper bound is not important; its lack of dependence on \( n \) is.

With the bound on the number of vertices in \( \mathcal{I} \), we can bound \( Q(\mathcal{I}, \mathcal{I}^c) \):
\[
Q(\mathcal{I}, \mathcal{I}^c) \leq \frac{2}{q^2} \sum_{I \in \mathcal{I}} X_I \left( \sum_{J \in N_I} \max_{J^c \in \mathcal{I}^c} X_J \right) \leq \frac{2 |N_I| M}{q} \left( \frac{1}{q} \sum_{I \in \mathcal{I}} X_I \right).
\]

Fix \( \xi > 0 \). We can increase \( n \) to ensure that
\[
\frac{2 |N_I| M}{q} \leq \frac{\xi}{2 C_{bb} \log n}
\]
Note the \( \log n \) term in the right-hand side; this is a generous overestimate, since the left-hand side vanishes at least as fast as \( n^{-a/2} \). Combining this with the above bound implies that \( Q(\mathcal{I}, \mathcal{I}^c) \leq \xi/(2 \log n) \). Similarly, we increase \( n \) so that \( n^\alpha/q^2 \leq \xi/(2 \log n) \) and \( n^\beta/q \leq \xi \). Combining all the results of this section, we conclude that if \( L, A, B \) and \( D \) occur, then for any \( \xi > 0 \), and \( n \) sufficiently large (depending on \( \xi \)),
\[
Q(\mathcal{I}) \geq 1 - \frac{\xi}{\log n},
\]
\[
V(\mathcal{I}) \leq C_{bb},
\]
\[
\frac{1}{q} \sum_{I \in \mathcal{I}} Y_I \leq \log(q/w) - 1 + \xi.
\]
6.3. **Improving the Linear Bound.** Our goal is to produce an upper bound on the linear sum that matches the convex lower bound already in place for \( J \). To do so, we find a subset of \( J \) that has a similar number of “internal” edges, but which only involves approximately \( q \) vertices. Using the convex upper bound from above, we construct such a subset in the simplest way possible: taking the largest elements of \( J \), and stopping when the total number of vertices gets close to 1. However, although we hope that the cardinality of this set has a uniform upper bound in \( n \), our argument is too rough to prove such a statement. Therefore, we ensure that the bounds get tighter as the size of the set increases; later these bounds will allow us to enforce an upper bound on the number of indices we are concerned with.

**Lemma 6.4.** Let \( J \) be the set defined in equation \((12)\), and assume that \((16), (17)\) and \((18)\) are true. Then there exists \( \Sigma \subset J \) such that

\[ Q(\Sigma) \geq 1 - \psi(\Sigma) \]

and

\[ 1 - \frac{2\xi}{\log n} \leq V(\Sigma) \leq 1 + \phi(\Sigma) \]

where

\[ \phi(\Sigma) = \min \left\{ \frac{\xi}{3}, \frac{2}{|\Sigma|} - \frac{2\xi}{\log n} \right\} \]

and

\[ \psi(\Sigma) = \min \left\{ \xi, \frac{C_{lh}}{|\Sigma|} \right\} \]

for some \( C_{lh} \) independent of \( n \).

**Proof.** Since \( Q(J) \geq 1 - \xi / \log n \) by \((16)\), we know that

\[ V(J) \geq \sqrt{1 - \xi / \log n} \geq 1 - \xi / \log n \]

for all sufficiently large \( n \). Now, we order the elements of \( J \) by size; for the purposes of this proof, we label the \( X_I \)'s by integers: let \( X_1 \) the largest \( X_I \) among all \( I \in J \), \( X_2 \) be the second largest, and so on. Pick an integer \( k \), and define \( J_k \) to be the first \( k \) terms of \( J \) with respect to this order. Finally, define

\[ \Sigma := \text{The first } J_k \text{ such that } V(J_k) > 1 - 2\xi / \log n . \]

that is, the smallest subset of \( J \) that has at least \( q(1 - 2\xi / \log n) \) vertices of \( \chi \) in its associated \( A_I \)'s. By minimality,

\[ V(\Sigma \setminus \{X_{|\Sigma|}\}) \leq 1 - \frac{2\xi}{\log n} \]

Furthermore, \( X_{|\Sigma|} \) must be the minimal element of \( \Sigma \), and therefore,

\[ \frac{X_{|\Sigma|}}{q} \leq \frac{V(\Sigma)}{|\Sigma|} . \]
Substituting these two inequalities into the identity $V(\mathcal{X}) = V(\mathcal{X}\setminus\{X_{|\mathcal{X}|}\}) + X_{|\mathcal{X}|}/q$ and solving for $V(\mathcal{X})$ gives

$$V(\mathcal{X}) \leq 1 + \frac{2}{|\mathcal{X}|} - \frac{2\xi}{\log n}.$$ 

Note that this inequality may be derived from the previous one only if $|\mathcal{X}|$ is larger than a universal constant. That, however, is not a problem; Corollary 6.5 at the end of the proof shows that $|\mathcal{X}|$ is large enough, simply using the definition of $\mathcal{X}$ and Lemma 6.3.

This is one half of the desired upper bound on $V(\mathcal{X})$. In fact, it is the only one that matters as long as $|\mathcal{X}| > 1/(6\xi)$.

To get the second half of the upper bound, assume that $|\mathcal{X}| < 1/(6\xi)$, and apply Lemma 6.3 to $\mathcal{X}$:

$$V(\mathcal{X}) \left[ \log \left( \frac{q}{w} \right) - \log h(\mathcal{X}) - 1 \right] + V(\mathcal{X}) \log V(\mathcal{X})$$

$$\leq \frac{1}{q} \sum_{I \in \mathcal{X}} Y_I \leq \log(q/w) - 1 + \xi$$

where the last upper bound follows by (18) and the fact that $\mathcal{X} \subset \mathcal{I}$. From the above inequality it follows that either $V(\mathcal{X}) \leq 1$, or

$$V(\mathcal{X}) \leq \frac{\log(q/w) - 1 + \xi}{\log(q/w) - \log h(\mathcal{X}) - 1}.$$ 

Thanks to the assumed upper bound on $|\mathcal{X}|$ and the fact that $|\log(q/w)| \to \infty$ as $n \to \infty$, this implies the second upper bound on $V(\mathcal{X})$.

Next, we need to show that $Q(\mathcal{X})$ is large. In the remaining part of the proof, $\mathcal{X}^c$ will denote the complement of $\mathcal{X}$ in $\mathcal{I}$, and not in all of $T$. It is sufficient to bound

$$Q(\mathcal{X}, \mathcal{X}^c) + Q(\mathcal{X}^c)$$

from above, by partitioning the sum that defines $Q(\mathcal{I})$, as in the previous section. The maximum element in $\mathcal{X}^c$ cannot exceed $(qV(\mathcal{X}))/|\mathcal{X}|$, as it must be smaller than the minimum of the elements in $\mathcal{X}$, by definition. Therefore, for any $I \in \mathcal{I}$,

$$\sum_{J \in N_I \cap \mathcal{X}^c} X_J \leq |N_I| \max_{J \in \mathcal{X}} X_J \leq |N_I| \frac{qV(\mathcal{X})}{|\mathcal{X}|}.$$ 

Expanding the definition of both terms, we see that

$$(20) \quad Q(\mathcal{X}, \mathcal{X}^c) + Q(\mathcal{X}^c) \leq \frac{4}{q^2} \sum_{I \in \mathcal{I}} X_I \sum_{J \in N_I \cap \mathcal{X}^c} X_J \leq \frac{|N_I|}{|\mathcal{X}|}.$$ 

Setting $C_{th} = 2C_{bb}|N_I|$ and assuming $|\mathcal{X}| > C_{th}/\xi$, we see that this

$$Q(\mathcal{X}, \mathcal{X}^c) + Q(\mathcal{X}^c) \leq \psi(\mathcal{X}),$$

noting that $V(\mathcal{X}) \leq 1 + \phi(\mathcal{X}) \leq 2.$
Finally, assume that $|\mathcal{I}| \leq C_{th}/\xi$. We go back to the upper bound on $Y_I$'s in $\mathcal{I}$:

$$\frac{1}{q} \left( \sum_{I \in \mathcal{I}} Y_I + \sum_{J \in \mathcal{I}^c} Y_J \right) \leq \log(q/w) - 1 + \xi.$$  

We apply Lemma 6.3 to the term involving $\mathcal{I}$ in the above inequality to conclude that

$$V(\mathcal{I}) \left[ \log \left( \frac{q}{w} \right) + \log V(\mathcal{I}) - \log h(\mathcal{I}) - 1 \right] + \frac{1}{q} \sum_{J \in \mathcal{I}^c} Y_J$$

$$\leq \log(q/w) - 1 + \xi.$$  

Next, we substitute the lower bound on $V(\mathcal{I})$ (from definition). After some algebraic manipulation, we conclude that

$$\frac{1}{q} \sum_{J \in \mathcal{I}^c} Y_J \leq \xi + \frac{3\xi}{\log n} + \log h(\mathcal{I})$$

$$+ \frac{2\xi}{\log n} \left[ \log \left( \frac{q}{w} \right) - \frac{3\xi}{\log n} - \log h(\mathcal{I}) - 1 \right].$$

(To get this, we bounded $\log(1 - 2\xi/\log n)$ from below by $-3\xi/\log n$, an estimate that holds for all sufficiently large $n$.) Thanks to the assumed upper bound on $|\mathcal{I}|$, the right-hand size can be bounded above by $2 \log(1/\xi)$ for all sufficiently small $\xi$ and sufficiently large $n$, as all terms except $h(\mathcal{I})$ are bounded above by a constant multiple of $\xi$.

With this upper bound, we can now apply Lemma 6.3 to $\mathcal{I}^c$ to conclude that

$$V(\mathcal{I}^c) \left[ \log \left( \frac{q}{wh(\mathcal{I}^c)} \right) - 1 \right] + V(\mathcal{I}^c) \log V(\mathcal{I}^c) \leq 2 \log(1/\xi).$$

Since the absolute minimum of the function $x \log x$ is $-1/e$, we can replace $V(\mathcal{I}^c) \log V(\mathcal{I}^c)$ by this minimum and maintain the lower bound. We also know that $wh(\mathcal{I}^c)$ is bounded above by $Dn^{\alpha}$ (since $\mathcal{I}^c \subset \mathcal{I}$ and the event $A^c$ is assumed to have happened). Recalling the definition of $\alpha$, we now arrive at the inequality

$$V(\mathcal{I}^c) \leq \frac{2 \log(1/\xi) + 1/e}{\log(q/(Dn^{\alpha})) - 1} \leq \frac{3 \log(1/\xi)}{a/2 \log n} \leq \frac{\xi}{C_{bb}}.$$  

The final inequality follows since the numerator is bounded in $n$, while the denominator grows, and therefore the fraction can be made smaller than any constant. From here, a bound on $Q(\mathcal{I}, \mathcal{I}^c) + Q(\mathcal{I}^c)$ is easy: For any $I \in \mathcal{I}$,

$$\frac{1}{q} \sum_{J \in N_I \cap \mathcal{I}^c} X_J \leq V(\mathcal{I}^c).$$

Therefore,

$$Q(\mathcal{I}, \mathcal{I}^c) + Q(\mathcal{I}^c) \leq \frac{4}{q^2} \sum_{I \in \mathcal{I}} X_I \sum_{J \in N_I \cap \mathcal{I}^c} X_J \leq \left( \frac{\xi}{C_{bb}} \right) V(\mathcal{I}) \leq \xi.$$
Completing the proof of the lemma. □

**Corollary 6.5.** There exists $C_{ts}$ independent of $n$ such that

$$|\mathcal{T}| > \tau_s(1 - 2C_{ts}\xi).$$

**Proof.** Since $\mathcal{T} \subset \mathcal{I}$, (18) applies to $\mathcal{T}$ as well. We now apply Lemma 6.3 to this set, and conclude that

$$V(\mathcal{T}) \left[ \log \left( \frac{q}{w} \right) + \log V(\mathcal{T}) - \log h(\mathcal{T}) - 1 \right] \leq \log \left( \frac{q}{w} \right) + \xi - 1$$

From the definition of $\mathcal{T}$ in the proof of Lemma 6.4, we have a lower bound on $V(\mathcal{T})$. Noting that

$$\log \left( 1 - \frac{2\xi}{\log n} \right) \geq \frac{-4\xi}{\log n}$$

for all sufficiently large $n$, we can conclude that

$$- \left( 1 - \frac{2\xi}{\log n} \right) \log(h(\mathcal{T})) \leq \xi + \frac{2\xi \log(q/w)}{\log n} + \frac{C\xi}{\log n}$$

We recall that $q/w \leq C'_{rb}n^{1-p/2}$, and therefore there exists a constant $C_{ts}$ such that

$$- \log(h(\mathcal{T})) \leq C_{ts}\xi.$$  

Inverting the negative logarithm gives

$$h(\mathcal{T}) \geq \exp(-C_{ts}\xi) \geq 1 - C_{ts}\xi$$

Multiplying through by $\tau_s$ gives the desired bound. □

### 6.4. Removing Lower Order Terms

The set $\mathcal{T}$ satisfies a set of nearly matching lower and upper bounds; if $\xi$ were exactly zero, then $\mathcal{T}$ would have to have exactly $q$ vertices spread out over at most $\tau_s$ indices, by the argument that was presented in the beginning of the section. Unfortunately, $\xi$ is strictly positive. Still, the following series of inequalities still holds:

$$1 - \psi(\mathcal{T}) \leq Q(\mathcal{T}) \leq (V(\mathcal{T}))^2 \leq (1 + \phi(\mathcal{T}))^2.$$

For notational ease, let

$$P_I(W) := \frac{1}{q} \sum_{J \in W, d(I,J) > s} X_J$$

for some fixed $I \in \mathcal{T}$ and any $W \subset \mathcal{T}$. Note that this sum is over indices $J$ that are far away from $I$, not the near ones, as in the sum defining $Q$. If we take the above series of inequalities and subtract $Q(\mathcal{T})$ from all of them, we deduce that

$$\frac{1}{q} \sum_{I \in \mathcal{T}} X_I P_I(\mathcal{T}) \leq (1 + \phi(\mathcal{T}))^2 - 1 + \psi(\mathcal{T}) \leq 3\phi(\mathcal{T}) + \psi(\mathcal{T}).$$

Note that a proper algebraic expansion would give $2\phi(\mathcal{T}) + \phi^2(\mathcal{T}) + \psi(\mathcal{T})$. However, for sufficiently small $\xi$, $\phi^2 \leq \phi$. Recalling the definition of $\phi$ and
ψ, we see that this bound is smaller than $4\xi$ when $|\mathcal{I}|$ is small, and improves when $|\mathcal{I}| > C_{th}/\xi$.

Our next step is to trim $\mathcal{I}$ by adding a lower bound to the individual elements. Our goal is to show that this removes few vertices, and slightly improves the upper bound on the $Y_I$’s.

**Lemma 6.6.** Let $\mathcal{I}$ be as above, and define $\mathcal{P}$ such that

$$\mathcal{P} := \left\{ I \in \mathcal{I} : X_I > \frac{\xi^{1/4}q}{\tau_s} \right\}$$

Then, for sufficiently small $\xi > 0$,

$$1 - \xi^{1/5} \leq V(\mathcal{P}) \leq 1 + \phi(\mathcal{I})$$

and

$$\frac{1}{q} \sum_{I \in \mathcal{P}} Y_I \leq V(\mathcal{P})(\log(q/w) - 1) + 2\xi^{1/10}.$$  

*Proof.* We split $\mathcal{I}$ into three sets: $\mathcal{P}$ is defined as above, while the rest of the indices are split as follows:

$$\mathcal{L}_1 := \left\{ I \in \mathcal{I} : X_I \leq \frac{\xi^{1/4}q}{\tau_s \log|\mathcal{I}|} \right\}$$

and

$$\mathcal{L}_2 := \left\{ I \in \mathcal{I} : \frac{\xi^{1/4}q}{\tau_s \log|\mathcal{I}|} \leq X_I \leq \frac{\xi^{1/4}q}{\tau_s} \right\}.$$  

Since the three sets partition $\mathcal{I}$, we can prove the lemma as with sufficiently good upper bounds on $V(\mathcal{L}_i)$ and lower bounds on the sum of the $Y_I$’s in both sets.

To bound $V(\mathcal{L}_1)$, we first need to bound $P_I(\mathcal{L}_1)$ from below. The worst case scenario is that the distance restriction removes the $|N_I|$ largest elements of $\mathcal{L}_1$. Therefore,

$$P_I(\mathcal{L}_1) \geq V(\mathcal{L}_1) - \frac{1}{q} |N_I| \max_{J \in \mathcal{L}_1} X_J \geq V(\mathcal{L}_1) - \frac{|N_I| \xi^{1/4}}{\tau_s \log|\mathcal{I}|}.$$  

Since $P_I(W) \leq P_I(W')$ whenever $W \subset W'$, we see that equation (21) implies that

$$\frac{1}{q} \sum_{I \in \mathcal{I}} X_I P_I(\mathcal{L}_1) \leq 3\phi(\mathcal{I}) + \psi(\mathcal{I}).$$  

Replacing $P_I(\mathcal{L}_1)$ with its minimum and recalling that $|N_I|/\tau_s$ is uniformly upper bounded (by Lemma 5.1), we see that

$$\left( V(\mathcal{L}_1) - \frac{C\xi^{1/4}}{\log|\mathcal{I}|} \right) V(\mathcal{I}) \leq 3\phi(\mathcal{I}) + \psi(\mathcal{I}).$$  

Using the (very suboptimal) lower bound of $1/2$ for $V(\mathcal{I})$, we conclude that

$$V(\mathcal{L}_1) \leq 6\phi(\mathcal{I}) + 2\psi(\mathcal{I}) + \frac{C_{th}\xi^{1/4}}{\log|\mathcal{I}|}. $$
for some $C_{lo}$ independent of $n$. Repeating this analysis with $\mathcal{L}_2$ yields the inequality
\[
V(\mathcal{L}_2) \leq 6\phi(\mathcal{T}) + 2\psi(\mathcal{T}) + \frac{C_{lo}\xi^{1/4}}{|\mathcal{T}|}.
\]
Since both $\phi$ and $\psi$ are bounded above by $\xi$, we get
\[
(23) \quad \max\{V(\mathcal{L}_1), V(\mathcal{L}_2)\} \leq \xi^{1/5}/3.
\]
Now,
\[
V(\mathcal{P}) = V(\mathcal{T}) - V(\mathcal{L}_1) - V(\mathcal{L}_2) \geq 1 - \frac{2\xi}{\log n} - \frac{2\xi^{1/5}}{3} \geq 1 - \xi^{1/5}
\]
This establishes the lower bound on $V(\mathcal{P})$. The upper bound follows trivially from $\mathcal{P} \subset \mathcal{T}$.

Finally, we need to improve the upper bound on the $Y_I$’s associated with $\mathcal{P}$. Not surprisingly, we will use Jensen’s inequality to get lower bounds on the $Y_I$ associated with $\mathcal{L}_1$ and $\mathcal{L}_2$. By inclusion, we know that
\[
\frac{1}{q} \sum_{I \in \mathcal{P}} Y_I \leq \left(\log\left(\frac{q}{w}\right) - 1 + \xi\right) - \sum_{I \in \mathcal{L}_1 \cup \mathcal{L}_2} Y_I
\]
Suppose that
\[
(24) \quad \frac{1}{q} \sum_{I \in \mathcal{L}_i} Y_I \geq V(\mathcal{L}_i) \left(\log\left(\frac{q}{w}\right) - 1\right) - \xi^{1/10}/2
\]
for $i = 1, 2$. The inequality
\[
1 - V(\mathcal{L}_1) - V(\mathcal{L}_2) \leq V(\mathcal{P}) + (2\xi)/\log n
\]
follows from partitioning $\mathcal{T}$ into its three constituent sets, and the lower bound on $V(\mathcal{T})$ from Lemma 6.4. Substituting the two inequalities into the earlier statement gives
\[
\frac{1}{q} \sum_{I \in \mathcal{P}} Y_I \leq V(\mathcal{P}) \left(\log\left(\frac{q}{w}\right) - 1\right) + \xi + \frac{2\xi}{\log n} \left(\log\left(\frac{q}{w}\right) - 1\right) + \xi^{1/10}
\]
\[
\leq V(\mathcal{P}) \left(\log\left(\frac{q}{w}\right) - 1\right) + 2\xi^{1/10}.
\]
using the fact that $\log(q/w)$ is bounded above by a constant multiple $\log n$. Since this is the required inequality in the statement of the lemma, we will be done if we can prove the inequality (24).

First, we prove that $\mathcal{L}_2$ cannot be too large. Assume $\mathcal{L}_2 > |N_I|$ (otherwise, we will use $|N_I|$ to bound its size — a far better bound than the one we will derive below). Since $\mathcal{L}_2$ has a termwise lower bound, we can get a new lower bound on $X_I P_I(\mathcal{L}_2)$ when $I \in \mathcal{L}_2$:
\[
\frac{1}{q} X_I P_I(\mathcal{L}_2) \geq \frac{1}{q^2} \left(|\mathcal{L}_2| - |N_I|\right) \left(\min_{J \in \mathcal{L}_2} X_J\right)^2 \geq \frac{C_{lo}\xi^{1/2} (|\mathcal{L}_2| - |N_I|)}{r_s^2 (\log |\mathcal{T}|)^2}.
\]
Summing over $I$ and applying (21), we get
\[ |\mathcal{L}_2| - |N_I| \leq \frac{\tau_s^2 (\log|\mathcal{X}|)^2 (3\phi(\mathcal{X}) + \psi(\mathcal{X}))}{C_{th} \xi^{1/2}}. \]

By definition, both $\phi$ and $\psi$ are bounded above by a constant multiple of $2/|\mathcal{X}|$. Thus, by Lemma 5.1, the above inequality shows that $|\mathcal{L}_2|$ is bounded above by a constant multiple of $\xi^{-1/2}$. Therefore, we may apply Lemma 6.3 to $\mathcal{L}_2$ to conclude that
\[ \frac{1}{q} \sum_{I \in \mathcal{L}_2} Y_I \geq V(\mathcal{L}_2) (\log(q/w) - 1) + V(\mathcal{L}_2) \log \left( \frac{\xi^{1/2} V(\mathcal{L}_2)}{C_{th}} \right) \]
for some constant $C_{th}$. Applying (23) and choosing $\xi$ sufficiently small, this gives (24) for $i = 2$.

If $|\mathcal{X}| \leq C_{th}/\xi$, then inclusion forces $h(\mathcal{L}_1)$ to also be small, and therefore we can repeat this exact same argument to get the desired inequality for $\mathcal{L}_1$. This leaves the case in which $|\mathcal{X}| > C_{th}/\xi$. By the inequality (22), the upper bounds on $\phi$ and $\xi$, and this assumed lower bound on $|\mathcal{X}|$, we arrive at the inequality
\[ \frac{\xi^{1/5}}{\log |\mathcal{X}|}, \]
assuming, as always, that $\xi$ is sufficiently small.

Since the map $x \mapsto x \log x$ is decreasing in a small right-hand neighborhood of 0, therefore by (23),
\[ V(\mathcal{L}_1) \log V(\mathcal{L}_1) \geq (\xi^{1/5}/3) \log(\xi^{1/5}/3), \]
provided that $\xi$ is small enough. On the other hand by (25),
\[ V(\mathcal{L}_1) \log h(\mathcal{L}_1) \leq \frac{\xi^{1/5} (\log |\mathcal{L}_1| + \log \tau_s)}{\log |\mathcal{X}|} \leq C \xi^{1/5}, \]
since $\mathcal{L}_1 \subseteq \mathcal{X}$, $\tau_s$ is bounded, and $\log |\mathcal{X}|$ is bounded away from zero by the assumption that $|\mathcal{X}| \geq C_{th}/\xi$. Combining (26), (27), and Lemma 6.3 for the set $\mathcal{L}_1$, the proof of (24) for $i = 1$ is complete. This completes the proof of the lemma. \qed

6.5. Convex Analysis. Somewhat surprisingly, the very technical provisions of Lemma 6.6 have the following geometric consequence:

**Corollary 6.7.** The set $\mathcal{P}$ is a maximal clique set.

**Proof.** Assume that there exists a pair of indices $I, J \in \mathcal{P}$ such that $d(I, J) > s$. By the definition of $\mathcal{P}$,
\[ \frac{X_I X_J}{q^2} \geq \frac{\xi^{1/2}}{\tau_s^2}. \]
By \(^{(21)}\), we know this cannot exceed \(3\phi(T) + \psi(T)\), which is bounded above by \(4\xi\). This forces

\[
\frac{\xi^{1/2}}{\tau_s^2} \leq 4\xi.
\]

However, the above inequality can easily be rendered false by choosing \(\xi\) so small that \(\xi \leq (2\tau_s)^{-4}\). This can be done, because by Lemma \(5.1\), \(\tau_s\) is uniformly bounded by a constant. With such a choice of \(\xi\), the diameter of \(\mathfrak{P}\) is bounded above by \(s\) (for sufficiently large \(n\), assuming that \(A^c \cap B^c \cap D^c\) holds).

Combining Lemma \(6.3\) and \(5.6\) gives

\[
V(\mathfrak{P}) (\log(q/w) - 1) + V(\mathfrak{P}) (\log V(\mathfrak{P}) - \log h(\mathfrak{P})) \leq V(\mathfrak{P}) (\log(q/w) - 1) + 2\xi^{1/10},
\]

and therefore

\[
h(\mathfrak{P}) \geq V(\mathfrak{P}) \exp\left(-\frac{2\xi^{1/10}}{V(\mathfrak{P})}\right) \geq V(\mathfrak{P}) \left(1 - \frac{2\xi^{1/10}}{V(\mathfrak{P})}\right)
\]

using the standard estimate \(e^{-x} \geq 1 - x\). Combining this with the lower bound on \(V(\mathfrak{P})\) from Lemma \(6.6\) forces

\[
|\mathfrak{P}| = \tau_s h(\mathfrak{P}) \geq \tau_s (1 - 3\xi^{1/10})
\]

Again, since \(\xi\) is under our control, we set \(\xi < (2\tau_s)^{-10}\). This implies that

\[
|\mathfrak{P}| \geq \tau_s - \frac{3}{2^{10}}.
\]

Cardinality must be an integer, and therefore \(|\mathfrak{P}| \geq \tau_s\). Therefore, we have shown that \(\mathfrak{P}\) is a set of diameter at most \(s\), with at least \(\tau_s\) elements. But, by definition, \(\tau_s\) is the largest possible cardinality of any set of diameter \(s\). Thus, \(\mathfrak{P}\) must have diameter exactly \(s\) — otherwise, consider the union of \(\mathfrak{P}\) with one of the indices in \(T\) at distance 1 from \(\mathfrak{P}\). By the triangle inequality, the diameter of this set is at most \(s\), but its cardinality is \(\tau_s + 1\), contradicting maximality. Thus, \(\mathfrak{P}\) is a maximal clique set.

We are nearly done with the proof: all that remains is to show that the elements of \(\mathfrak{P}\) are close to \(q/\tau_s\), and to improve the upper bound for maximal clique sets far away from \(\mathfrak{P}\). First, to show “equipartition”, we expand \(Y_I\) about the value \(q/\tau_s\): let \(f(x) = x (\log(x/D) - 1) + D\), so that

\[
Y_I = f\left(\frac{q}{\tau_s}\right) + f'\left(\frac{q}{\tau_s}\right) \left(X_I - \frac{q}{\tau_s}\right) + \frac{f''(L(X_I))}{2} \left(X_I - \frac{q}{\tau_s}\right)^2
\]

where \(L(X_I)\) is some number between \(X_I\) and \(q/\tau_s\). Differentiating \(f(x)\) explicitly and simplifying algebraically, we see that

\[
Y_I = D - \frac{q}{\tau_s} + X_I \log(q/w) + \frac{1}{2L(X_I)} \left(X_I - \frac{q}{\tau_s}\right)^2.
\]
Next, we sum over $P$ and use the upper bound from Lemma 6.6, which gives

$$
\frac{1}{q} \sum_{I \in P} \left[ D - \frac{q}{\tau_s} + X_I \log(q/w) + \frac{1}{2L(X_I)} \left( X_I - \frac{q}{\tau_s} \right)^2 \right] \leq V(P) \left( \log(q/w) - 1 \right) + 2\xi^{1/10}.
$$

Recalling that $P$ has exactly $\tau_s$ elements, a bit of algebraic manipulation yields

\begin{equation}
\frac{1}{q} \sum_{I \in P} \frac{1}{2L(X_I)} \left( X_I - \frac{q}{\tau_s} \right)^2 \leq 1 - \frac{w}{q} - V(P) + 2\xi^{1/10} \leq 3\xi^{1/10}
\end{equation}

where the final inequality follows from the lower bound on $V(P)$ from Lemma 6.6 and the fact that $w/q$ vanishes in $n$, and therefore can be dominated by any positive power of $\xi$ for sufficiently large $n$.

Now, define

$$
\mathcal{M}_1 := \left\{ I \in \mathcal{P} : X_I \geq (1 + \xi^{1/40})q/\tau_s \right\}
$$

and

$$
\mathcal{M}_2 := \left\{ I \in \mathcal{P} : X_I \leq (1 - \xi^{1/40})q/\tau_s \right\}.
$$

On $\mathcal{M}_1$, the function $1/L(X_I)$ is bounded below by $1/X_I$. Furthermore, if we restrict its domain to $x > (1 + \xi^{1/40})q/\tau_s$, the function

$$
\frac{1}{2x} \left( x - \frac{q}{\tau_s} \right)^2
$$

is minimized at its left endpoint. Therefore,

$$
\frac{1}{q} \sum_{I \in \mathcal{M}_1} \left[ \frac{1}{2L(X_I)} \left( X_I - \frac{q}{\tau_s} \right)^2 \right] \geq \frac{\left| \mathcal{M}_1 \right| \xi^{1/20}}{2\tau_s (1 + \xi^{1/20})}
$$

by assuming every element takes on the minimal value. Using the upper bound (28), we conclude that

$$
|\mathcal{M}_1| \leq 3\xi^{1/20} \cdot 2\tau_s (1 + \xi^{1/20}).
$$

The analysis is equally simple in the case of $\mathcal{M}_2$; we bound $1/L(X_I)$ from below by $\tau_s/q$, and deduce that

$$
|\mathcal{M}_2| \leq 3\xi^{1/20} \cdot 2\tau_s
$$

Setting $\xi < (8\tau_s)^{-20}$ ensure both sets are empty. If we further enforce $\xi^{1/40} \leq \tilde{\epsilon}$, we have proven that all elements of $\mathcal{P}$ are within an $1 \pm \tilde{\epsilon}$ error of $q/\tau_s$, as required by Theorem 3.1.

To finish the proof, we need to show that all elements outside $\mathcal{P}$ are small. The elements outside $\mathcal{J}$ are small by the definition of $\mathcal{J}$. The elements of concern are those in $\mathcal{J} \setminus \mathcal{I}$. If $\mathcal{K} \setminus \mathcal{P}$ is nonempty, we are done, as the element in this set is at most $\xi^{1/4}/q/\tau_s$, and, since the construction of $\mathcal{K}$ took the largest elements in $\mathcal{J}$, the maximal element in $\mathcal{J} \setminus \mathcal{I}$ is bounded above by the value of the smallest element in $\mathcal{I}$.
Next, assume that $\mathcal{T} = \mathcal{P}$. Then $h(\mathcal{T}) = 1$, and, using Lemma 6.3 and the lower bound in Lemma 6.6, we deduce that
\[
\frac{1}{q} \sum_{I \in \mathcal{T}} Y_I \geq V(\mathcal{P}) (\log(q/w) + \log(V(\mathcal{P}))) - 1 \\
\geq (1 - \xi^{1/5}) \left( \log(q/w) - 1 - 2\xi^{1/5} \right).
\]
To get control of the elements in $\mathcal{I} \setminus \mathcal{T}$, we go back to the inequalities on $\mathcal{I}$. Subtracting out the $Y_I$'s in $\mathcal{T}$ from (18) and substituting the lower bound that was just derived, we see that
\[
\frac{1}{q} \sum_{I \in \mathcal{I} \setminus \mathcal{T}} Y_I \leq \left[ \log(q/w) - 1 + \xi \right] - \left(1 - \xi^{1/5}\right) \left[ \log(q/w) - 1 - 2\xi^{1/5} \right] \\
\leq 2\xi^{1/5} \log(q/w).
\]
Since all the $Y_I$'s are positive, this implies that no individual $Y_I$ can exceed the upper bound. Suppose $X_I > \xi^{1/10}q/\tau_s$ for some $I \in \mathcal{I} \setminus \mathcal{T}$. Then, the $Y_I$ associated with it satisfies
\[
\frac{Y_I}{q} = \left( \frac{X_I}{q} \right) (\log(X_I/D) - 1) + \frac{D}{q} \\
\geq \frac{\xi^{1/10}}{\tau_s} \left( \log(q/w) + \log(\xi^{1/10}) - 1 \right) \\
\geq \frac{\xi^{1/10} \log(q/w)}{2\tau_s},
\]
where we bound $(\log \xi^{1/10} - 1)$ below by $-\log(q/w)/2$; this bound holds for all sufficiently large $n$. If $\xi < (1/\tau_s)^20$, then, for all sufficiently large $s$, this contradicts the upper bound above. By possibly shrinking $\xi$ until $\xi^{1/10} \leq \tilde{\epsilon}$, we satisfy the requirement on a generic index outside of $\mathcal{P}$, and completes the proof of Theorem 3.1.

7. Proof of the Upper Tail Large Deviation Principle

As usual, we write $\mu$, $\tau$, and $r$ instead of $\mu_n$, $\tau_n$ and $r_n$. We now prove Theorem 2.2 which claims that the function
\[
I(x) := \left( \frac{2 - p}{2} \right) \sqrt{2(x - 1)}
\]
is the rate function for the random variable $|E|/\mu$ with speed $\sqrt{\mu} \log n$. Recall that we restrict our attention to subsets of the interval $(1, \infty)$, as our result only holds for events in which $|E|$ exceeds its expectation. For notational convenience, for any $t > 1$, we define the event
\[
R_t := \{|E| \geq t\mu\}.
\]
We claim that it is sufficient to prove that

\[
\lim_{n \to \infty} \frac{\log \mathbb{P}[R_t]}{\sqrt{\mu \log n}} = -I(t).
\]

This follows easily: pick \( F \) to be a closed subset of \((1, \infty)\), and let its leftmost endpoint be \( a_F \). Since \( I(x) \) is increasing, its infimum over \( F \) is achieved at \( a_F \). Furthermore, \( F \subset [a_F, \infty) \), and therefore,

\[
\mathbb{P}\left[ \frac{|E|}{\mu} \in F \right] \leq \mathbb{P}\left[ \frac{|E|}{\mu} \in [a_F, \infty) \right] = \mathbb{P}[R_{a_F}].
\]

Taking the logarithm, dividing by \( \sqrt{\mu \log n} \), and applying (29) gives the upper bound for \( F \). Similarly, take any \( b \in G \). Since \( G \) is open, there exists an \( \epsilon \) such that \([b, b + \epsilon) \subset G\). Therefore,

\[
\mathbb{P}\left[ \frac{|E|}{\mu} \in G \right] \geq \mathbb{P}\left[ \frac{|E|}{\mu} \in [b, b + \epsilon) \right] = \mathbb{P}[R_b] - \mathbb{P}[R_{b+\epsilon}].
\]

Applying (29) twice, we deduce that, for any \( \delta > 0 \), there is an \( n \) sufficiently large to ensure that

\[
\mathbb{P}\left[ \frac{|E|}{\mu} \in G \right] \geq \exp \left( -I(b)\sqrt{\mu \log n}(1 + \delta) \right) - \exp \left( -I(b + \epsilon)\sqrt{\mu \log n}(1 - \delta) \right).
\]

It is easy to see from this how we can get the desired lower bound on the probability of \( \frac{|E|}{\mu} \in G \).

We now prove the desired limit (29) on the probability of \( R_t \). We first prove that the \( \limsup \) is bounded above as desired. Fix \( \epsilon > 0 \). For an arbitrary pair of events \( A \) and \( B \), assume that, conditional on \( A \), the event \( B \) occurs with probability at least \( 1 - \epsilon \). This implies that \( \mathbb{P}[A] \leq \left( \frac{1}{1 - \epsilon} \right) \mathbb{P}[B] \).

By Theorem 2.1, there exists a sufficient large \( n \) such that conditioning on \( R_{t} \) implies that the random geometric graph has a clique of size at least \( \sqrt{2(t - 1)\mu}(1 - \epsilon) \) with probability at least \( 1 - \epsilon \). This means that, for any \( s \), there exists a maximal clique set \( \mathcal{P} \subset T \) with at least as many vertices as in the clique of the random geometric graph. Since every maximal clique set has \( \tau_s \) indices, there can be at most \( m^{dr_s} \) distinct maximal clique sets. This is an egregious overcount, but we have no need for finer control. Thus, by the union bound, the probability that there exists a maximal clique set with \( \sqrt{2\delta\mu} \) vertices is bounded above by \( m^{dr_s} \) times the probability that a single one has the same property. The number of vertices in a maximal clique set is distributed as a Poisson random variable of mean \( \tau_s D = w \). Therefore, the chain of implication allows us to conclude that

\[
\mathbb{P}[R_t] \leq \left( \frac{m^{dr_s}}{1 - \epsilon} \right) \mathbb{P} \left[ \text{Poisson}(w) > \sqrt{2(t - 1)\mu}(1 - \epsilon) \right].
\]
Let \( v := \sqrt{2(t - 1)\mu}(1 - \epsilon) \). Applying (7) to the right-hand side above gives

\[
\mathbb{P}[R_t] \leq \left( \frac{m^{d_s}}{1 - \epsilon} \right) \exp \left( -v \left[ \log \left( \frac{v}{w} \right) - 1 \right] + w \right) 
\leq \exp \left( -(1 - 2\epsilon)\sqrt{2(t - 1)\mu} \log \left( \frac{\sqrt{n}}{w} \right) \right),
\]

where we the second inequality follows for all sufficiently large \( n \) by noting that all the missing terms vanish in comparison to \( \sqrt{\mu} \log n \), and can therefore be absorbed at the cost of changing \( \epsilon \) to \( 2\epsilon \). By the definitions of \( \mu \), \( p \) and \( w \),

\[
\frac{\sqrt{\mu}}{w} = n^{(2-p)/2}h(n),
\]

for some slowly growing \( h(n) \). Therefore,

\[
\frac{1}{\sqrt{\mu} \log n} \log \mathbb{P}[R_{t,n}] \leq -(1 - 2\epsilon) \left( \frac{2 - p}{2} \right) \sqrt{2(t - 1)} + \frac{\log h(n)}{\log n}.
\]

Since \( \epsilon \) is arbitrary, we conclude that the lim sup is bounded above as desired.

To calculate the lim inf, we find a configuration that implies \( R_t \). The strategy follows very similarly to the derivation of the lower bound on \( L \); however, the \( s \)-graded model estimates \( |E| \) from above, and therefore we must rederive the same type of estimates directly from the Poisson point process. Let \( B \) be a ball of diameter \( r \), and let \( H' \) be the event that there are \( \sqrt{2(t - 1)\mu + n^2} \) vertices in \( B \), where \( z \) is defined as in the beginning of Section 6.1. Since the number of vertices in \( B \) is a Poisson random variable of mean \( n\tau \), we can explicitly compute that

\[
\mathbb{P}[H'] = \mathbb{P}[\text{Poisson}(n\tau) > \sqrt{2(t - 1)\mu + n^2}] 
\geq \mathbb{P}[\text{Poisson}(n\tau) = \lceil \sqrt{2(t - 1)\mu + n^2} \rceil].
\]

As discussed in Section 6.1, this shows that

\[
\mathbb{P}[H'] \geq \exp \left( -(\sqrt{2(t - 1)\mu + 3n^2}) \log \left( \frac{\sqrt{2(t - 1)\mu + 3n^2}}{n\tau} \right) \right).
\]

Absorbing all terms that grow in \( n \) more slowly than \( \sqrt{\mu} \log n \), we can deduce that, for sufficiently large \( n \),

\[
(30) \quad \mathbb{P}[H'] \geq \exp(-(1 + \epsilon)I(t)\sqrt{\mu} \log n),
\]

where we bound the rate of growth of \( \sqrt{\mu}/(n\tau) \) using the definitions of \( p \) and \( \tau \).

The event \( R_t \) will follow if the number of edges with at most one endpoint in \( B \) exceeds \( \mu - 2\sqrt{2(t - 1)\mu n^2} \). Let \( |E'| \) be the number of edges with no endpoints in \( B \). Letting \( 1_{i,j} \) be the indicator of an edge between vertices \( i \)
and $j$, we can see that
\[
\mathbb{E}(|E|') = \mathbb{E} \left[ \binom{N}{2} \mathbb{E}(1_{1,2 \cdot 1_{v_1,v_2 \notin B}} \mid N) \right] 
= \frac{n^2}{2} \mathbb{P}(\{\|v_1 - v_2\| \leq r\} \cap \{v_1, v_2 \notin B\}),
\]
where $N$ is the total number of points in the torus, as before, and the probability measure in the second equality is given by the uniform process. For notational convenience, let $1_{B_{i,j}}$ be the indicator of the event $\{\|v_1 - v_2\| \leq r\} \cap \{v_1, v_2 \notin B\}$, and $\mu_B$ be its expectation under the measure of the uniform process (by symmetry, this is independent of the indices $i$ and $j$). If $v_1$ is at a distance greater than $r$ from $B$, the second condition holds trivially. For a fixed $B$, the probability that $v_1$ is within distance $r$ of $B$ is a constant multiple of $r^d$. Thus, $\mu_B \geq (1 - Cr^d)\nu r^d$, for some $C$ that depends only on the norm and the dimension. Thus, the expected value of $|E|'$ is bounded below by $\mu(1 - Cr^d)$. Therefore, by definition of $z$, we have the inequality
\[
(31) \quad \mu - 2\sqrt{2(t - 1)\mu n^2} \leq \mathbb{E}[|E|'] - \frac{\sqrt{2(t - 1)\mu n^2}}{4}
\]
We now need a variance estimate for $|E|'$:
\[
\text{Var}(|E|') = \mathbb{E}(\text{Var}(|E|' \mid N)) + \text{Var}(\mathbb{E}(|E|' \mid N)).
\]
We have already calculated the expectation of $|E|'$ given $N$ above; since $\mu_B$ does not depend on $N$, we deduce that
\[
\text{Var}(\mathbb{E}(|E|' \mid N)) = (\mu_B)^2 \text{Var} \left[ \binom{N}{2} \right].
\]
A standard calculation will show that the variance of $\binom{N}{2}$ is $n^3 + n^2/2$. Meanwhile,
\[
\mu_B \leq \mathbb{P}(\|v_1 - v_2\| \leq r) = \nu r^d.
\]
Combining these facts gives
\[
\text{Var}(\mathbb{E}(|E|' \mid N)) \leq C r^{2d} n^3,
\]
for some universal constant $C$.
Next, we estimate the expression $\mathbb{E}(\text{Var}(|E|' \mid N))$. We can write this variance as
\[
\text{Var}(|E|' \mid N) = \mathbb{E} \left[ \left( \sum_{1 \leq i < j \leq N} (1_{i,j} - \mu_B) \right)^2 \mid N \right].
\]
We now decompose this sum into three sums by distributing the square: one sum over pairs of the form $(i,j),(k,l)$ with four distinct indices, one with pairs of the form $(i,j),(i,k)$ where one index repeats, and the final over
perfect squares of terms involving \((i, j)\). The expectation of the first one is zero, as the event that \((i, j)\) form an edge with both endpoints outside of \(B\) is completely independent of the same event occurring over distinct vertices \((k, l)\). For a fixed choice of \((i, j)\) and \((i, k)\), we can bound
\[
\mathbb{E}[1_{B}^{B} \cdot 1_{B}^{B}] \leq \mathbb{P}[(\|v_{i} - v_{j}\| \leq r, \|v_{i} - v_{k}\| \leq r)] = (\nu r^{d})^{2},
\]
where the first inequality follows by removing the requirement that the vertices lie outside of \(B\), and thus increasing the probability. There are \(N(N - 1)(N - 2)\) ways to choose a pair of indices that overlap in exactly one entry. Thus,
\[
\sum \mathbb{E}((1_{B}^{B} - \mu_{B})(1_{B}^{B} - \mu_{B})) = \sum (\mathbb{E}(1_{B}^{1_{B}} \cdot 1_{B}^{1_{B}}) - (\mu_{B})^{2}) \leq C'r^{2d}N^{3},
\]
for some universal constant \(C'\). Again, this overestimates the value of this sum dramatically, but is sufficient for our purposes. Finally, the contribution of terms of the form \((1_{B}^{B} - \mu_{B})^{2}\) to the sum is exactly \((\frac{N}{2})((\mu_{B} - \mu_{B})^{2})\), which is bounded above by \(C''r^{d}N^{2}\). Combining these results, taking expectations over \(N\), and then adding the contribution of the variance of the expectation from before, we conclude that
\[
\text{(32) } \text{Var}(|E'|) \leq C'''r^{d}n^{2} + r^{2d}n^{3},
\]
for yet another universal constant \(C'''\). Now, \(r^{d}n^{2}\) grows as \(n^{p}\), while \(r^{2d}n^{3}\) grows as \(n^{2p-1}\) — both up to slowly growing functions. Thus, the variance of \(|E'|\) is \(n^{p}f(n)\) if \(p \leq 1\), and \(n^{2p-1}g(n)\) when \(p > 1\), with \(f(n)\) and \(g(n)\) being slowly growing functions.

By Chebychev’s inequality and (31),
\[
\mathbb{P}[|E'| < \mu - 2\sqrt{2(t-1)\mu n^{2}}] \leq \mathbb{P}
\left[
|E'| < \mathbb{E}[|E'|] - \frac{\sqrt{2(t-1)\mu n^{2}}}{4}
\right]
\leq \frac{8\text{Var}(|E'|)}{(t-1)\mu n^{2}}.
\]

Regardless of the value of \(p\), this quantity grows as \(n^{-p/2}\) up to logarithmic factors (just as it did in Section 6.1), and therefore, with probability \(1 - \epsilon\), \(|E'|\) exceeds \(\mu - 2\sqrt{2(t-1)\mu n^{2}}\) for all sufficiently large \(n\). Following the chain of implication, we see that
\[
\mathbb{P}[R_{t}] \geq \mathbb{P}[H'](1 - \epsilon).
\]

Substituting the earlier bound (30) on the probability of \(H'\) completes the proof.

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References

[1] Alon, N. and Spencer, J. H. (2008) The Probabilistic Method. John Wiley and Sons, Hoboken, New Jersey.
[2] Avin, C., and Ercal, G. (2007). On the cover time and mixing time of random geometric graphs. Theoretical Computer Science, 380(1), 2–22.
[3] Burago, Y. D. and Zalgaller, V. A. (1998). Geometric Inequalities. Leningrad.
[4] Chatterjee, S. (2010). A note about the uniform distribution on the intersection of a simplex and a sphere. Preprint. Available at http://arxiv.org/abs/1011.4043.
[5] Dembo, A. and Zeitouni, O. (2010). Large Deviations Techniques and Applications. (Vol 38). Springer, New York.
[6] Federer, H. (1996). Geometric Measure Theory. Springer, Berlin.
[7] Gilbert, E. N. (1961). Random Plane Networks. Journal of the Society of Industrial and Applied Mathematics, 9(4),533–543.
[8] Goel, A., Rai, S., and Krishnamachari B. (2005). Monotone properties of random geometric graphs have sharp thresholds. Annals of Applied Probability, 15(4), 2535–2552.
[9] Grimmett, G. (1999). Percolation. Springer, Berlin.
[10] Janson, S. (2004). Large deviations for sums of partly dependent random variables. Random Structures and Algorithms, 24(3), 234–248.
[11] Meester, R. and Roy, R. (1996). Continuum Percolation. (Vol. 119) Cambridge University Press, Cambridge.
[12] Muller, T. (2006). Two-point concentration in random geometric graphs. Combinatorica, 28(5), 529–545.
[13] Nagaev, S. V. (1979). Large deviations of sums of independent random variables. The Annals of Probability, 7(5),745–789.
[14] Penrose, M. D. (2002). Focusing of the scan statistic and geometric clique number. Advances in Applied Probability, 739–753.
[15] Penrose, M. (2003). Random Geometric Graphs. (Vol. 5) Oxford University Press, Oxford.
[16] Penrose, M. D. and Yukich, J.E. (2003). Weak laws of large numbers in geometric probability. The Annals of Applied Probability, 13(1), 277–303.
[17] Penrose, M. D. and Yukich, J. E. (2005). Approximation in geometric probability. Steins Method and Applications, Lecture Note Series, Institute for Mathematical Sciences, National University of Singapore, 37–58.
[18] Petersen, P. (2006). Riemannian Geometry. (Vol. 171). Springer, New York.
[19] Schreiber, T. and Yukich, J. E. (2005). Large deviations for functionals of spatial point processes with applications to random packing and spatial graphs. Stochastic Processes and Their Applications, 115(8), 1332–1356.
[20] Talagrand, M. (1996). Concentration of measures and isoperimetric inequalities in product spaces. Publications Mathematiques de l’I.H.E.S., 81(1),73–205.
[21] Talagrand, M. (2003). Spin Glasses: A Challenge for Mathematicians. Springer, Berlin.

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