A domain wall between single-mode and bimodal states and its transition to dynamical behavior in inhomogeneous systems

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We consider domain walls (DW's) between single-mode and bimodal states, that occur in coupled nonlinear diffusion (NLD), real Ginzburg-Landau (RGL), and complex Ginzburg-Landau (CGL) equations with a spatially dependent coupling coefficient. Group-velocity terms are added to the NLD and RGL equations, which breaks the variational structure of these models. In the simplest case of two coupled NLD equations, we reduce the description of stationary configurations to a single second-order ordinary differential equation. We demonstrate analytically that a necessary condition for existence of a stationary DW is that the group-velocity must be below a certain threshold value. Above this threshold, dynamical behavior sets in, which we consider in detail. In the CGL equations, the DW may generate spatio-temporal chaos, depending on the nonlinear dispersion. A spatially dependent coupling coefficient as considered in this paper can be realized at least in two different convection systems: a rotating narrow annulus supporting two traveling-wave wall modes, and a large-aspect-ratio system with poor heat conductivity at the lateral boundaries, where the two phases separated by the DW are rolls and square cells.

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I. INTRODUCTION

Close to a pattern-forming transition, many hydrodynamic systems can be described by coupled one-dimensional (1D) Ginzburg-Landau equations for some envelope functions (order parameters) $A$ and $B$. These equations were first introduced in an explicit form (with purely real coefficients, but also with the group-velocity terms) by Cross\textsuperscript{[1]}. In the presence of a small over-criticality parameter, and when rewritten in terms of two real amplitudes and two real phases, these equations for two complex order parameters are completely tantamount to a single fourth-order complex equation (the “complex Swift-Hohenberg” equation) introduced still earlier in Ref.\textsuperscript{[2]}. As is well known, the coefficient in front of the cross-coupling (CC) term is an essential parameter of these equations. For a small CC coefficient, homogenous bimodal states of the form $|A| = |B|$ occur, while for sufficiently large CC, this bimodal state is unstable, and the system evolves into a single-mode state in which either $A$ or $B$ is zero\textsuperscript{[3]}. In this work, we consider the case that the CC coefficient is a function of the spatial coordinate $x$. In particular, we focus on the case that the CC coefficient is slightly above 1 for $x \to -\infty$ and slightly below 1 for $x \to \infty$. The simplest state that can occur in this case is a spatial juxtaposition of a single and a bimodal state, separated by a so-called domain-wall. We study this domain-wall in the NLD, RGL, and CGL equations. We show that non-variational effects can destroy the stationary domain-wall and study the dynamical states that occur in this case. In particular we focus on the effect of the group-velocity terms in the coupled equations.

This paper is organized as follows. First we introduce a simplified model for the occurrence of domain-walls. For stationary solutions of this model we derive a perturbation equation that allows us to predict the vanishing of stationary domain-walls when the group-velocity is sufficiently large. This is confirmed by numerical simulations of this model, and we describe the ensuing dynamical states. We study the relevance of this model for coupled RGL and CGL equations by performing numerical simulations. In particular for the CGL equations, a broad spectrum of
dynamical behavior is observed, including spatio-temporal chaos. It is very difficult to characterize these states, and we restrict ourself to a limited exploration of the possible dynamical states.

Some of the essential features of the CGL equations that are important in the study of the domain-wall, i.e., the stability of the single and bimodal states as a function of the CC can also be studied in simpler models. We will first consider two coupled nonlinear diffusion (NLD) equations, to which we add group-velocity terms borrowed from the CGL equation that break the variational representation of the NLD equations (however, the group-velocity terms should not be added in the above-mentioned case of the Rayleigh-Bénard convection in the inhomogeneous large-aspect-ratio system [4]). In the simplest approach, we take the amplitudes real-valued:

\[ \begin{align*}
\partial_t A + c\partial_x A &= A + \partial_x^2 A - (A^2 + g(x)B^2)A, \\
\partial_t B - c\partial_x B &= B + \partial_x^2 B - (B^2 + g(x)A^2)B,
\end{align*} \]

These NLD equations are the simplest set of equations that admit the DW’s that we are interested in. These equations have no direct physical interpretation in terms of amplitude equations, but they have the advantage over more realistic models that the effect of the group-velocity terms on the domain-walls can be studied analytically. Since the group-velocity terms have opposite sign for the \( A \) and the \( B \) equation of the coupled CGL equations (2), we retain this property in the NLD equations. Without the cross-coupling, the group-velocity terms generate a counter propagation of the patterns in \( A \) and \( B \). When the equations for \( A \) and \( B \) are coupled, the effect of the group-velocity terms cannot be given in terms of a simple propagation rule.

We will derive, in a certain limit, a single ordinary differential equation for the order parameter \( \chi := \tan^{-1}(A/B) \) that describes stationary configurations. Using this equation, we will be able to investigate the existence of a stationary DW analytically. In section IV we consider the case that the group-velocity terms are absent. The equations are then variational and for the particular case that the CC coefficient, as a function of the spatial coordinate \( x \), is proportional to \( \tanh(kx) \), an exact analytical solution for the DW can be obtained. Next, we extend in section V the equation for \( \chi \) to include the non-variational group-velocity terms, and when we take \( g(x) \) to be a step function, an analytical solution can be constructed. Analyzing the latter solution, we find a transition from a regime where the DW is stationary to a regime where no stationary DW exists. This transition, which we show to have a nice geometrical interpretation, occurs when the group-velocity \( c \) is equal to the critical value \( c_{crit} := \pm 2\sqrt{|g - 1|} \). We will show that this result remains valid for a broad class of inhomogeneous CC coefficients.

Direct simulations of the coupled NLD equations, presented in section III B, confirm the analytical results and show that there is a value of the group-velocity, \( c_i \), where the stationary domain-wall loses its stability. This instability occurs because the position of the domain-wall diverges when \( c \) approaches \( c_{crit} \). The difference between \( c_{crit} \) and \( c_i \) is of the order of a few percent. For a group-velocity just beyond \( c_i \), the DW is seen to perform small-amplitude chaotic oscillations around a mean position, without essentially disturbing the DW’s shape.

The study of the coupled CGL equations is of a more explorative nature. As a first step towards the study of the full coupled CGL equations, we allow the amplitudes \( A \) and \( B \) to be complex valued. These equations are reminiscent of the RGL equations, but it should be noted that in physical applications, the RGL equations have no group-velocity terms; nevertheless we refer to this model as the coupled RGL equations. We explain that the main properties of the stationary domain-walls in this model can be described by the NLD equations. Simulations of coupled RGL equations, presented in section I V, show a transition between the stationary domain-walls and a dynamical state with the increase of the group-velocity that is similar to the NLD-case. The role of wavenumbers of the initial conditions, which constitutes the main feature that is not present in the NLD equations, is discussed briefly.

For the coupled CGL equations, the numerical results presented in section III show that the nonlinear dispersion terms may render the domain-walls unstable, even when the group-velocity terms are absent. Spatio-temporal disordered states often occur. The oscillations of the DW can be understood qualitatively to arise from the large gradients of \( |A| \) and \( |B| \) around the domain-wall.

It may be pertinent to note that both complex and real Ginzburg-Landau equations with the coefficients in front of the linear terms smoothly depending upon the spatial coordinate have been the subject of many studies [1, 2]. It was found that a parameter ramp in the real or complex GL equation can perform wavenumber selection [3, 4]. In the case of the full CGL equation, it can also render the single-mode traveling waves unstable, and can trap subcritical solitary pulses [5]. However, as far as we know, there have been no studies aimed to consider effects of a spatial dependence in the coefficient in front of the nonlinear CC term.

A spatially dependent CC can be realized in Rayleigh-Bénard convection in a rotating annulus of non-constant width [6]. Convection in rotating systems has recently been the focus of several studies [6, 7], and quasi-1D traveling waves were shown to occur in these systems near the vertical side-walls of the annulus. In a rotating annulus, there are two so-called wall modes, localized, respectively, near its inner and outer side walls. Note that there is only one wall-mode per side-wall (instead of two); this is due to a symmetry breaking that is induced by the rotation [6, 7]. The amplitude equations describing slow modulations of these modes in the co-rotating reference frame are two coupled
cubic complex Ginzburg-Landau (CGL) equations [8]. The strength of the CC between the two wall modes sensitively depends on the width of the annulus. When the annulus is not uniform but has a varying width, the corresponding CC can be made to vary across its critical value as a function of the longitudinal spatial coordinate (which is going along the circumference of the rotating annulus). Other coefficients of the amplitude equations describing such a system will also depend on this coordinate, but if we assume that these coefficients, in contrast to the CC coefficient, are not close to a critical value, their spatial dependence may be ignored, provided that it is smooth enough.

A similar problem may be implemented in a related but different physical system, viz., the Rayleigh-Bénard convection in a (non-rotating) large-aspect-ratio cell. In this case, the two modes are two orthogonal sets of parallel rolls. A single-mode state is stable provided that an effective CC coefficient between the orthogonal rolls is larger than a certain minimum value, while in the opposite case a square-lattice pattern (the bimodal state obtained as a superposition of two orthogonal sets of rolls) is stable [14]. Usually, the actual value of the CC coefficient is well above the above-mentioned minimum, so that the square lattice is unstable. However, in special cases, e.g., for the convection between horizontal surfaces with poor heat conductivity, the CC coefficient may fall below the minimum [4]. One may construct a suitable inhomogeneous system, for example, by means of a variable-thickness lid put on top of the convection layer, such that the local CC coefficient, being a function of the spatial coordinates, is passing through the minimum value. Then one may expect a stationary DW separating the rolls and square lattice, which is impossible in the homogeneous system [12,13].

II. THE VARIATIONAL CASE

In this section we will focus on the coupled NLD and RGL equations, which are the simplest sets of equations where a DW between a single-mode and a bimodal state can occur. The dynamics of these systems without the group-velocity terms are relaxational in the sense that a Lyapunov functional $\mathcal{L}$ exists for each of them, allowing to predict final states by minimizing it. We derive a perturbation theory for a weak inhomogeneity in the cross-coupling coefficients of the coupled NLD equations, that allows us to find a closed-form expression for the DW for a special choice of the inhomogeneity. This perturbation theory will be a starting point in the next subsection, where we will focus on the group-velocity terms that make the equations non-variational.

The equations that we consider in this section are

$$\partial_t A = A + \partial_x^2 A - (|A|^2 + g(x)|B|^2)A,$$
$$\partial_t B = B + \partial_x^2 B - (|B|^2 + g(x)|A|^2)B,$$

where $A$ and $B$ are real-valued in the NLD case, and complex in the RGL case. The critical value of $g$ is 1: for $g < 1$ the bimodal state $|A| = |B| = \sqrt{1/(1+g)}$ is stable, whereas for $g > 1$ this bimodal state loses its stability and the single-mode state with $|A| = 1$ and $B = 0$ (or vice versa) becomes stable [8]. The CC coefficient $g(x)$ is assumed to decrease monotonically from slightly above to slightly below its critical value as a function of the spatial coordinate $x$. In what follows below, we will set $g(x) := 1 + \gamma(x)$, where $\gamma(x)$ is a small monotonically decreasing function of the spatial coordinate $x$, such that $\gamma(x)$ is positive at $x < 0$ and negative at $x > 0$. We assume that $\gamma(x)$ saturates at $x \to \pm \infty$, i.e., it assumes certain asymptotic values $\gamma(-\infty) = \gamma_{\text{max}} > 0$, and $\gamma(+\infty) = \gamma_{\text{min}} < 0$. The solution that one expects for this choice of $g(x)$ is a stationary DW located around $x = 0$ which matches the single-mode and bimodal states existing, respectively, for negative and positive $x$.

The coupled RGL equations can be derived from the Lyapunov functional

$$\mathcal{L} = \int dx \left\{ |\partial_x A|^2 + |\partial_x B|^2 - (|A|^2 + |B|^2) + \frac{1}{2}(|A|^4 + |B|^4) + (1 + \gamma(x))|A|^2|B|^2 \right\}$$

by setting $\partial_t A = -\delta \mathcal{L}/\delta A^*$ and $\partial_t B = -\delta \mathcal{L}/\delta B^*$. As is commonly known, the Lyapunov functional may only decrease in time, and, as it is bounded from below, a final state corresponds to a minimum of $\mathcal{L}$. The final state that corresponds to a global minimum of $\mathcal{L}$ has a zero wave number, which suggests to consider also the particular case of real $A$ and $B$. Thus one obtains the coupled NLD equations. For finite systems with periodic boundary conditions, the RGL equations may evolve to stationary states with nonzero wavenumber. This is discussed in section IIIA.

We will now focus on the stationary solutions of the NLD equations, and set the derivatives with respect to time equal to zero. The ensuing equations can be written as four coupled ordinary differential equations for the real-valued $A$, $\partial_x A$, $B$ and $\partial_x B$. When we identify $A$ and $B$ with position coordinates and their derivatives with generalized momentum coordinates, the equations for a stationary domain-wall can be written as the Hamilton equations corresponding to the Hamiltonian.
where \(x\) is the “time” coordinate. Although the Hamiltonian is similar to the Lyapunov functional, the difference in the signs that occur between these expressions should be noted.

In general we cannot solve the Hamilton equations for arbitrary \(\gamma(x)\). To proceed we will develop a perturbation expansion by taking \(\gamma_{\text{min}}\) and \(\gamma_{\text{max}}\) small. We will complement this condition by the assumption that \(A(x)\) and \(B(x)\) are slowly varying functions of \(x\), so that the diffusive terms in Eqs. (3) are small in comparison with the other terms. Below we will determine, in a self-consistent way, that the spatial scale \(L\) over which \(A\) and \(B\) vary is of order \(1/\sqrt{\gamma}\).

When the diffusive terms and the coupling term \(\gamma(x)A^2B^2\) of the Hamiltonian (3) are small, it follows that \((A^2 + B^2)\) is almost constant [4]. This suggests the following representation for \(A(x)\) and \(B(x)\):

\[
A(x) = R(x) \cos \chi(x); \quad B(x) = R(x) \sin \chi(x).
\]  

The single-mode states are those with \(\chi\) being an integer multiple of \(\pi/2\), and bimodal states have \(\chi = \pi/4 + n\pi/2\) (throughout this paper, \(n\) will represent an integer). When we substitute this representation into the Hamiltonian (3), we obtain

\[
H = R^2 - \frac{1}{2} R^4 + R'^2 + R^2\chi'^2 - 4RR'\chi \cos(\chi) \sin(\chi) - \gamma(x)R^4 \cos(\chi)^2 \sin(\chi)^2,
\]

where a prime denotes differentiation with respect to \(x\). Stationary solutions of the coupled NLD equations are found by determining the minima of the Hamiltonian (4). Since we assumed that \(A\) and \(B\) are slowly varying, and that \(R\) is almost constant, \(R' \ll \chi'\) and in the simplest non-trivial approximation we may set \(R\) equal to one [4].

Differentiating the Hamiltonian (4) with respect to \(x\) and neglecting higher order terms, we obtain the following perturbation equation:

\[
\chi''(x) = \frac{1}{4} \gamma(x) \sin(4\chi),
\]

where the prime stand for \(d/dx\). It follows from here that the relation between the small quantity \(\gamma\) and a large scale \(L\) of variation of the function \(\chi(x)\), which was implicitly assumed above, is \(L^{-2} \sim \gamma\) [4].

As an example, we may solve Eq. (5) exactly for the special choice \(\gamma(x) = -\kappa^2 \tan(\kappa x)\). The solution is then

\[
\chi_0(x) = \frac{1}{2} \tan^{-1}(e^{\kappa x}),
\]

which can be verified by substitution; note the scaling of \(x\) with \(\sqrt{\gamma}\). This choice for \(\gamma(x)\) and the corresponding solution are shown in Fig. 1 for \(|\gamma| = 0.2\). We will proceed by investigating whether this type of the DW solutions exist for more general \(\gamma(x)\) and when the group-velocity terms are included.

### III. THE NLD EQUATIONS WITH THE GROUP-VELOCITY

In this section we will investigate what happens to the domain-walls when the group-velocity terms that are similar to those in the CGL equation are included in the NLD equations. These terms destroy the variational structure of the NLD equations, hence their final states need no longer be stationary. We will show that stationary DW’s cannot exist when the group-velocity is above a certain critical value. Numerical simulations of the NLD equations corroborate this prediction, and show that, beyond the threshold, time-periodic or disordered states occur.

The NLD equations with the group-velocity terms are

\[
\begin{align*}
\partial_t A + c\partial_x A &= A + \partial_x^2 A - (A^2 + g(x)B^2)A, \quad (9a) \\
\partial_t B - c\partial_x B &= B + \partial_x^2 B - (B^2 + g(x)A^2)B, \quad (9b)
\end{align*}
\]

As was demonstrated in Ref. [4], one can use a balance equation for the Hamiltonian (3) to treat effects of the small group-velocity terms on the stationary solutions. Differentiating (4) and making use of the stationary version of Eqs. (3), one finds

\[
\frac{dH}{dx} = c[(\partial_x A)^2 - (\partial_x B)^2].
\]  

(10)
Using the “polar” representation defined by Eqs. (11) yields an effective equation for \( \chi \), which is a generalization of Eq. (9) (the same equation, albeit for constant \( \gamma \), was obtained in Ref. [13]):

\[
\chi''(x) - \frac{1}{4} \gamma(x) \sin(4\chi) + c \chi' \cos(2\chi) = 0.
\]  

This equation is the basis for the perturbative analysis presented below.

It should be noted that this equation is invariant under a scale transformation \( \gamma \rightarrow \delta \gamma, \ x \rightarrow x/\sqrt{\delta} \) and \( c \rightarrow \sqrt{\delta} c \). This freedom can in principle be used to scale out \( c \), but we will not do this; this scale-invariance is reflected, however, in the formula for \( c_{\text{crit}} \) that is obtained below.

\[\text{A. Phase-space analysis}\]

It will be convenient to rewrite Eq. (11) in the form of a two-dimensional non-autonomous dynamical system:

\[
\frac{d\chi}{dx} = \psi, \\
\frac{d\psi}{dx} = -c \psi \cos(2\chi) + \frac{1}{4} \gamma(x) \sin(4\chi).
\]

Fixed points of Eqs. (12) are \( (\chi, \psi) = (n\pi/4, \psi = 0) \), which correspond to single-mode stationary solutions of equations of Eqs. (9) at even \( n \), and to bimodal solutions at odd \( n \). Hetero-clinic orbits going from one fixed point at pseudo-time \( x = -\infty \) to another fixed point at \( x = \infty \) correspond to the DW’s that we are interested in. Since we have chosen \( \gamma(x) \) to be a decreasing function of \( x \), these orbits go from a fixed point with even \( n \) to one with odd \( n \).

The system (12) is invariant with respect to the following symmetry transformations:

(i) \( \chi \rightarrow \chi + n\pi, \)
(ii) \( \chi \rightarrow \chi + \pi/2, \ c \rightarrow -c. \)
(iii) \( x \rightarrow -x, \ \psi \rightarrow -\psi, \ c \rightarrow -c, \ \gamma(x) \rightarrow \gamma(-x). \)

With regard to these symmetries, it is sufficient to consider only the hetero-clinic orbit that goes from the fixed point \( (\chi, \psi) = (0, 0) \), to be referred to as FP0, to the bimodal fixed point \( (\chi, \psi) = (\pi/4, 0) \), which will be called FP1; the other DW’s can be obtained by applying a combination of the transformations (i), (ii) and (iii) to this hetero-clinic orbit.

An essential feature of the dynamical system (12) is that the direction of the phase-flow in the \( (\chi, \psi) \) plane is not fixed, but depends on the current value of \( \gamma(x) \). Therefore, even when the functional form of \( \gamma \) is fixed, the possible orbits of equation (12) through a certain point form, in general, a one-parameter family.

However, when we take \( \gamma(x) \) to be a step function, i.e.,

\[
\gamma(x) = \gamma_{\text{max}} > 0, \ x < 0, \\
\gamma(x) = \gamma_{\text{min}} < 0, \ x > 0,
\]

there are only 2 orbits through a certain point: one for \( \gamma = \gamma_{\text{max}} \ (x < 0) \) and one for \( \gamma = \gamma_{\text{min}} \ (x > 0) \). Notice that the lack of continuity of \( \gamma(x) \) does not contradict the assumption that the stationary solution for \( \chi \) is a smooth function of \( x \); we will see below that solutions corresponding to this discontinuous \( \gamma(x) \) are smooth indeed. In fact, the scaling properties of equation (11), that also hold for the dynamical system (12) yield that when we rescale \( \gamma \rightarrow \delta \gamma \), with \( \delta < 1 \), the function \( \gamma(x) \) becomes effectively steeper due to the rescaling of the spatial-coordinate; when \( \delta \downarrow 0 \), \( \gamma(x) \) becomes, in a sense, infinitely small and infinitely steep.

We are now interested in the behavior of the hetero-clinic orbits as a function of \( c \). A hetero-clinic trajectory corresponding to the DW exists provided that the the outgoing (unstable) manifold of FP0, which we will refer to as \( W_{0}^{\text{out}} \), intersects the ingoing (stable) manifold of FP1, to be referred to as \( W_{1}^{\text{in}} \) (see Figs. 2 and 3). Obviously, \( W_{0}^{\text{out}} \) and \( W_{1}^{\text{in}} \) have to be calculated, respectively, for \( \gamma = \gamma_{\text{max}} \) and \( \gamma = \gamma_{\text{min}} \). The question of existence of a domain-wall has thus been reduced to checking whether \( W_{0}^{\text{out}} \) and \( W_{1}^{\text{in}} \) intersect. The point of intersection then gives the values of \( \chi \) and \( \psi \) at \( x = 0 \).

This is illustrated in the Figs. 2 and 3, where we have plotted, for \( \gamma_{\text{min}} = -0.1 \) and \( \gamma_{\text{max}} = 0.1 \), these two manifolds for a range of values of the group-velocity. The point of intersection has by definition \( x = 0 \), and so from these Figs.
we can read off the values of $\chi$ and $\chi'$ at $x = 0$ as a function of $c$. The point of intersection of $W^{(out)}_0$ and $W^{(in)}_1$ shifts towards FP0 when $c$ is decreased. As a consequence, the location of the domain-wall, i.e., the value of $x$ where $\chi = \pi/8$, shifts to larger and larger values. In fact, when $c$ approaches a certain value which we define as $c_{crit}$, the intersection of $W^{(out)}_0$ and $W^{(in)}_1$ approaches FP0 and the location of the domain-wall, that corresponds to the hetero-clinic orbit, diverges to infinity (see Fig. 4).

Below we will find that $c_{crit} = -2\sqrt{|\gamma_{min}|}$. Note that the square-root dependence follows from the scaling properties; only the numerical factor $-2$ needs to be determined. There are two complementary methods for the determination of the critical velocity. The first method involves analytically solving the flow-line equations of the dynamical system (13) and the second method involves an inspection of the geometry of the flow-lines. After a description of these two methods we discuss the validity of $c_{crit}$ for inhomogeneities that are not of the form (13).

1. Analytic expression for the flow-lines.

The manifolds $W^{(out)}_0$ and $W^{(in)}_1$ can be found exactly when $\gamma$ is piecewise constant, as in Eqs. (13), because we can solve the equation for the trajectories of the dynamical system (12) explicitly when $\gamma$ is a constant. This equation is obtained by dividing Eq. (12b) by Eq. (12a) which yields for constant $\gamma$:

$$ \frac{d\psi}{d\chi} = -c \cos(2\chi) + \frac{\gamma}{4\psi} \sin(4\chi). $$

(14)

To solve equation (14), we first perform a coordinate transformation by introducing $\zeta := \frac{1}{2} \sin(2\chi)$, which yields

$$ \frac{d\psi}{d\zeta} = -c + \gamma \zeta. $$

(15)

Eq. (15) is homogeneous, and by defining $\lambda = \psi/\zeta$ and some rewriting, we find

$$ \frac{1}{\zeta}d\zeta = \frac{1}{-c + \gamma/\lambda - \lambda} d\lambda. $$

(16)

The manifold $W^{(out)}_0$ corresponds to the case that $\lambda$ is a constant, which occurs for $-c + \gamma/\lambda - \lambda = 0$. By transforming this solution back to the $\chi, \psi$ coordinates we obtain for $W^{(out)}_0$:

$$ \psi = \lambda_0/2 \sin(2\chi), $$

(17)

where $\lambda_0$ satisfies $-c + \gamma/\lambda_0 - \lambda_0 = 0$, and $\gamma = \gamma_{max}$.

The manifold $W^{(in)}_1$ is found by a straightforward integration of equation (16), which yields the unpleasant equation

$$ \ln(\zeta) = \frac{c \text{ atanh} \frac{c + 2\lambda}{\sqrt{4\gamma + c^2}}} {\sqrt{4\gamma + c^2}} - \frac{\ln(-\lambda + c\lambda + \lambda^2)}{2} + K, $$

(18)

where $K$ is an arbitrary constant of integration, and $\gamma = \gamma_{min}$. $K$ has to be chosen so that the trajectory is passing through the point FP1, where $(\zeta, \lambda) = (1, 0)$. Although we do not have explicit expressions for the manifolds in terms of the original variable $\chi$, we can find from Eqs. (17) and (18) under which conditions the manifolds $W^{(out)}_0$ and $W^{(in)}_1$ intersect. The intersection point can be found by substituting $\lambda = \lambda_0$ into Eq. (18).

Simply solving the ensuing equation numerically yields that $W^{(in)}_1$ and $W^{(out)}_0$ intersect only (apart from the intersection at FP0 itself) when

$$ c > c_{crit} := -2\sqrt{-\gamma_{min}}, $$

(19)

where it should be noted that $c$ is assumed to be negative.

This means that if the group-velocity $c$, which is amenable for the non-variational effects in this model, is below the threshold $-2\sqrt{-\gamma_{min}}$, there can not be a stationary DW. It should be noted that we have focused here on the domain-wall that has a single-mode state for negative $x$ and a bimodal state for positive $x$. When we reflect $x$, and study a domain-wall of opposite chirality, i.e., going from a bimodal state at $x \to -\infty$ to a single-mode state at $x \to +\infty$, according to symmetry (iii) the critical velocity changes sign. When $\gamma(x)$ goes from negative values for $x < 0$ to positive values for $x > 0$, the ensuing stationary domain-walls can not exist for sufficiently large positive $c$. We will encounter domain-walls of both chiralities in the numerical simulations presented below.
2. Geometrical interpretation.

A simple geometric interpretation of the threshold is illustrated in the Figs. 2 and 3. As is illustrated in the phase-portraits (see Figures 3a and 3b), \( W_1^{(in)} \) intersects \( W_0^{(out)} \) for all positive \( c \), and therefore we will concentrate now on the case of negative \( c \). The central point is that for negative \( c \) and negative \( \gamma \), FP0 is a spiral when \( c > c_{\text{crit}} \), and a saddle when \( c < c_{\text{crit}} \). Therefore, the motion of \( W_1^{(in)} \), for which \( \gamma = \gamma_{\text{min}} < 0 \), in the neighborhood of FP0 is spiral-like when \( c > c_{\text{crit}} \); this is not visible on the scale of Fig. 2 but can be seen in Fig. 3. As long as \( W_0^{(out)} \) spirals around FP0, it has to intersect \( W_1^{(in)} \), which means that there is a hetero-clinic orbit and therefore a stationary domain-wall. When \( c \) approaches \( c_{\text{crit}} \), this spiraling motion becomes less prominent, and consequently, the intersection of \( W_0^{(out)} \) and \( W_1^{(in)} \) shifts to FP0 (see Fig. 3). As a consequence, the position of the domain-wall shifts to large values of \( x \).

When \( c \) has crossed the critical value \( c_{\text{crit}} \), FP0 is a saddle and it turns out that \( W_0^{(out)} \) and \( W_1^{(in)} \) only intersect in FP0 itself (see Figure 3b); this corresponds to a domain-wall that is shifted to infinity.

3. Validity of \( c_{\text{crit}} \).

There are two different assumptions involved in the calculation of \( c_{\text{crit}} \). First of all, \( \gamma \) needs to be small, in order for the perturbation equation \( \square \) to hold. Secondly, we used for \( \gamma(x) \) a step-function. We will now discuss the validity of \( c_{\text{crit}} \) when \( \gamma(x) \) is of more general form.

The latter assumption is, due to the scaling properties of equation \( \square \), not crucial, as we have indicated above. When we rescale \( \gamma \rightarrow \delta \gamma \), with \( \delta < 1 \), we can compensate for this by rescaling the spatial coordinate; effectively, the function \( \gamma(x) \) becomes steeper then. When \( \delta \downarrow 0 \), \( \gamma(x) \) becomes infinitely small and infinitely steep. Therefore, the results that are obtained when \( \gamma \) is a step-function, are still valid when \( \gamma(x) \) is an arbitrary monotonically decreasing function, provided that \( |\gamma| \) is small.

It is tempting to try to extend the phase-space analysis to the non-perturbative case, i.e., when \( \gamma \) and \( c \) are not small. The stationary version of the coupled NLD equations \( \square \) can then be written as a 4-dimensional dynamical system. The equations for the trajectories cannot be solved in this case, but we can inspect the eigenvalues of the fixed points that correspond to the single-mode and bimodal states. The latter fixed point always has two positive and two negative eigenvalues. The fixed point corresponding to the single-mode state changes from a saddle to a spiral when \( c \) shifts to large values of \( x \).

When \( c \) is decreased beyond \( c_{\text{crit}} \), the fixed point with one positive, one negative and two complex conjugated eigenvalues. Although this does not prove that the stationary DW disappears, this is at least an indication that the previously obtained value \( c_{\text{crit}} \) may be also valid when \( \gamma \) and \( c \) are not small.

B. Numerical simulations

To test the validity of the perturbation equation \( \square \) and to verify our predictions, we have performed numerical simulations of the coupled NLD equations \( \square \) in a periodic system of size \( L = 400 \). Because of the periodicity of the system, there are two domain-walls but we will, as before, focus attention on the domain-wall that connects a single-mode state \( B = 0 \) to the left with a bimodal state to the right. The results reported here were obtained by using a pseudo-spectral code with the time step 0.05 and, typically, 256 modes; runs with a higher number of modes up to 1024 were performed to check the results. The code was such that \( A \) and \( B \) could either be real or complex-valued in order to simplify the comparison between the results for the NLD equations and the RGL and CGL equations later on.

When \( c \downarrow c_{\text{crit}} \), the intersection point of the manifolds approaches FP0, and consequently the position of the DW shifts to \( +\infty \). In this limiting case, a single-mode state intrudes into the domain where \( g < 1 \), where this state is unstable. Therefore, one might expect that the DW is unstable when \( c \) is still slightly above \( c_{\text{crit}} \). We find below that this is the case indeed, and we will denote the value of the group-velocity where the domain-wall turns unstable by \( c_i \).

The simulations carried out below focus on three items. First of all, we will verify that the stationary DW predicted by the phase-space analysis exists and is stable when \( c \) is not too close to \( c_{\text{crit}} \). Subsequently, we determine, for various choices of the inhomogeneity, the value of \( c \) where the domain-wall turns unstable, and compare it with \( c_{\text{crit}} \). Finally, we investigate the dynamical states that occur when \( c \) is decreased beyond \( c_i \).
1. Stationary states

In this section we present the results of the simulations of the NLD equations for $|c| < |c_{\text{crit}}|$ and step-like $\gamma$. Most of the phase-space analysis presented above was based on the assumption that $\gamma$ and $c$ were small and we therefore take $\gamma(x) = 0.1$ at $x < 0$, and $-0.1$ at $x > 0$; in this case, the critical value of the group-velocity is $c_{\text{crit}} = -2\sqrt{-\gamma_{\text{min}}} \approx -0.632$.

Because of the periodicity of the system, there is also a step of $\gamma$ and, hence, another DW at $x = \pm 200$, but we will, as before, focus attention on the DW solution near $x = 0$ that connects a single-mode state ($B = 0$) to the left with a bimodal state to the right.

We have found that for $|c| \gtrsim 0.6$ the system relaxes to one of the stationary DW’s that are shown in Fig. 4. These domain-walls were obtained by numerical simulations with 1024 modes. These numerical results clearly demonstrate that the stationary DW’s are stable when $c$ is not very close to $c_{\text{crit}}$; this fact cannot be derived from the phase-space analysis alone.

Comparing the domain-walls obtained from direct simulations of the NLD equations with those obtained by numerical integration of the ordinary differential equation $d\psi/dx = \psi$ along the analytically obtained flow-lines, we found that the shape of the domain-wall is predicted very well for all values of $c$ that we consider here. The largest deviations occur for $|c| = 0.6$, when the DW’s obtained by the two aforementioned methods have a relative spatial shift $\approx 2$. The quantity $R^2$ (see Eq. 5) is equal to $1 \pm 0.005$ for the single-mode, and $0.95 \pm 0.005$ for the bimodal state (note that from $|A| = |B| = \sqrt{1/\gamma}$ it follows that $R^2 = 0.95$ in the bimodal state).

2. Instability of the domain-wall

Now that we have checked that the predications that follow from the perturbation theory are correct when $c$ is not too close to $c_{\text{crit}}$, we will investigate what happens when $c \downarrow c_{\text{crit}}$. The numerical simulations that we will present below reveal that the strength and the steepness of the inhomogeneity, the value of $c$ and the initial conditions all may influence the dynamics. To sample the parameter space without going into too much detail, we have eventually restricted ourselves to inhomogeneities of the form

$$g(x) = 1 + \Delta g \tanh(s(x - 0.25L)) \tanh(s(x + 0.25L)),$$

where $s$ stands for the steepness of the inhomogeneity and $L$ is the size of the system. The middle part of the system is where the bimodal state exists. We have shifted the inhomogeneities away from $x = 0$ because this makes the pictures for the time evolution of $A$ and $B$ that are presented below more clear.

We take values of the inhomogeneity strength $\Delta g$ of 0.05, 0.1, 0.2 and 0.3 in order to obtain information on the dependence of the dynamics on the strength of the inhomogeneity. For the steepness $s$, two different values were selected: 0.1 for a smooth inhomogeneity, and 10 for a steep inhomogeneity; for the numerical simulation presented below, this last value is practically equivalent to a step-like inhomogeneity.

As can be seen in Fig. 4, the location of the domain-wall shifts to larger values of $x$ when $c$ approaches $c_{\text{crit}}$, in agreement with the phase-space analysis. In this case, a large patch of the unstable single-mode state intrudes in the $g < 1$ domain. Eventually, this renders the domain-wall state unstable, and we define $c_i$ as the group-velocity for which this instability occurs (see Fig. 3).

We found that the primary instability indeed occurs when $c$ has approached $c_{\text{crit}}$ within a few percent. The large patch of unstable single-mode state turns then convectively unstable, but the total domain-wall state is absolutely stable. This means that noise, which is mainly due to the discretization of space and time in the numerics, is amplified in the region where we have the unstable single-mode state. Due to the group-velocity, these fluctuations are advected towards the stable bimodal state (mode $A$) or the stable single-mode state (mode $B$), where they are dissipated. To monitor where this instability occurs, we have followed the time evolution of the norm $\int dx A$ as a function of the group-velocity. When the instability of the domain-wall state occurs, the norm starts to oscillate with well-defined frequency. We checked for hysteric effects but could find none, and we conclude that the domain-wall turns unstable via a forward Hopf-bifurcation. However, the fact that in this case the system is still absolutely stable, yields that hidden line plots of the domain-wall hardly show a clear dynamic state. When $c$ is close the $c_i$, one can observe the instability best in the norm $\int dx A$.

We have measured $c_i$ for $s = 10$ and various values of $\Delta g$, and the results are listed below.

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The error-bar on the measurements of \( c_i \) is of order 0.01. The most important conclusion that we can draw from this measurements is that \( c_i \) is very close to \( c_{\text{crit}} \), and that this is independent of the value of \( \gamma \).

### 3. Dynamical states

In this section we investigate the fate of the domain-wall when \( c \) is decreased beyond \( c_i \). It is important to emphasize that, unlike many other transitions to dynamical behavior, there is, according to our phase-space analysis, no stationary albeit unstable state when \( |c| > |c_{\text{crit}}| \). Thus, the single-mode and bimodal states are stable, each in its own domain, but there is no stationary domain-wall to connect these two asymptotic states.

We study again inhomogeneities of the form (20), and take the group-velocity \( 1.1, 1.2, 1.5 \) and 2 times the critical value \( v_{\text{crit}} \). As initial conditions, we always took

\[
A(x) = 0.85 + 0.15 \tanh((x - 0.25L)/10) \tanh((x + 0.15L)/10),
\]

\[
A(x) = 0.35 - 0.35 \tanh((x - 0.25L)/10) \tanh((x + 0.15L)/10),
\]

which is close to the stationary state for \( c = 0.9c_{\text{crit}} \).

Most of the dynamical states fall into two distinct types of behavior. For the weak inhomogeneities, the DW moves irregularly back and forth around an average position. For stronger inhomogeneities, we often found that traveling kinks connecting states with the opposite signs of the amplitude \( B \) were generated in the bimodal regime.

We will describe in detail the dynamics that are found for the inhomogeneity with the steepness \( s = 10 \), which is the value of \( s \) that we will focus on; the other value of \( s \) will merely serve to explore the generality of the behavior found for \( s = 10 \).

At \( \Delta g = 0.05 \), we observe a disorderedly moving DW, and this is also the case for \( \Delta g = 0.1 \). We will focus now on \( c = 1.1c_{\text{crit}} \) and \( \Delta g = 0.1 \). The dynamics of the amplitudes \( A \) and \( B \) and the quantity \( N := \int_0^{\infty} dx A < \int_0^{\infty} dx B \) are shown in Fig. 6. As can be clearly seen in this figure, an essential dynamical degree of freedom is the position of the DW that moves irregularly around a certain mean position. This mean position shifts to the right when \( c \) decreases. By extending the simulations to longer time intervals we have checked that the motion remains disordered.

To inspect whether the dynamics are chaotic, we have studied effects of small perturbations in the initial conditions. We have performed two runs, one (the unperturbed run) starting from initial conditions obtained as the final state of previous simulations in the disordered regime; for the perturbed run, the initial profile of \( A(x) \) was unaltered, but the value of \( B \) at \( x = 0 \) was diminished by \( 10^{-3} \). The data pertaining to the perturbed run will be distinguished by a prime. We have plotted \( |A - A'| \) in Fig. 8, and \( N \) and \( N' := \int_0^{\infty} A' dx / < \int_0^{\infty} dx A > \) in Fig. 8. Clearly, the dynamics are sensitive to the small perturbation. The perturbation of the initial data is seen to grow and spread out, but only an area around the DW is affected (Fig 8b). When a perturbation far from the DW is initiated, the same effect is eventually observed, but after a longer transient time, which is presumably the time the perturbation needs to reach the region around the DW. For the motion to be chaotic, we need to find an exponential divergence between \( A \) and \( A' \) for some time-interval. A close inspection of the dynamics for \( A \) and \( A' \) reveals that this is not the case; \( A \) and \( A' \) diverge but not exponentially. Therefore, we suspect the disordered motion of the domain-wall to be due to the convectively amplified discretization noise. This scenario is somewhat similar to the dynamics described in 7.

It seems that the dynamics, at least at the lowest order, can be described by a disordered drift of the DW, which leaves its shape intact. To check this, we have collapsed 100 snapshots from the time evolution of \( A \), plotting \( A \) vs. \( dA/dx \). All the curves fall, with a good approximation, on top of each other, which indicates that, to the lowest order, the dynamics can be adequately modeled by the only degree of freedom, viz., the position of the DW.

When the strength of the inhomogeneity is increased, a different type of behavior is observed. As an example, we will consider what happens at \( \Delta g = 0.2 \), although qualitatively the same behavior occurs at larger \( \Delta g \). The motion that occurs at \( c = 1.1c_{\text{crit}} \) is shown in Fig. 6. A close inspection of the data reveals that, to the left of the DW, where \( B \) is very small, zeroes of the function \( B(x) \) are generated periodically. Each of them is then convected and amplified, leading to the generation of a traveling kink. We have found that the frequency of the kink generation goes approximately linearly with \( c \), and that it is nonzero at \( c = c_{\text{crit}} \). The velocity of the kinks is slightly smaller than the group-velocity. This is not surprising: associated with the kink in \( B \) is a small "bump" in \( A \) (see Fig 6). Without

| \( \Delta g = |\gamma_{\text{min}}| \) | \( c_i \) | \( c_{\text{crit}} \) | \( c_1/c_{\text{crit}} \) |
|-----------------|------|------|-----------------|
| 0.05            | -0.43| -0.447| 0.96            |
| 0.1             | -0.62| -0.632| 0.98            |
| 0.2             | -0.87| -0.894| 0.97            |
| 0.3             | -1.06| -1.095| 0.97            |
this bump, the kink would travel with precisely the group-velocity of $B$, but since the group-velocity of $A$ is opposite to that of $B$, the small bump somewhat lowers the velocity of the kink.

Taking a smoother inhomogeneity ($s = 0.1$) suppresses the generation of kinks, and if they are still generated, this process is mixed with the irregular motion of the DW, as is shown in Fig. 9. The generation of kinks can even become irregular.

In conclusion, when $c$ is beyond $c_{\text{crit}}$, the dynamical state may be represented by the disordered DW, regularly generated and traveling kinks, or a mixture between the two.

IV. THE RGL EQUATIONS WITH THE GROUP-VELOCITY

In this section we relax the condition that the order parameters $A$ and $B$ are real, and study the behavior of a DW in a system of two coupled RGL equations with the group-velocity terms:

$$\partial_t A + c \partial_x A = A + \partial^2_x A - (|A|^2 + (1 + \gamma(x))|B|^2)A,$$

$$\partial_t B - c \partial_x B = B + \partial^2_x B - (|B|^2 + (1 + \gamma(x))|A|^2)B.$$  \hspace{1cm} (23a)

The main question here is to see whether the results obtained for the NLD equations are relevant for the RGL equations.

The complex order parameter poses the question of wavenumber selection, and the possibility of nonzero wavenumbers constitutes the main difference with the NLD equations. We have found that, for the stationary states, the wavenumbers often, but not always, relax to zero when the initial conditions have nonzero wavenumbers. When the wavenumbers of $A$ and $B$ are constant and equal, the NLD equations can be used to describe the stationary states; when the wavenumbers are more general, modified NLD equations model the stationary domain-walls, as we will discuss below. In the dynamical regime, small wavenumbers may be generated starting from initial conditions with zero wavenumbers. Note that the usual Eckhaus band of stable wavenumbers for a single RGL equation \[18\] will be affected by the coupling between the two RGL equations. In particular, since for $\gamma \to 0$ both the single-modes and the bimodal states with zero wavenumber are only marginally stable, one may expect that the size of the band of stable wavenumbers is directly related to $\gamma$; and that for $\gamma \to 0$ the band of stable wavenumbers closes. We have not pursued this question further.

The wavenumbers of $A$ and $B$, which we refer to as $q_A$ and $q_B$, also have an influence on the stability borders of the single and bimodal states. When $q_A = q_B$, the crossover between single and bimodal states occurs at $g = 1$, just as when $q_A = q_B = 0$. This can be shown by substituting plane-wave solutions with equal wavenumbers in the RGL equations and performing a rescaling of $x$, $t$ and the amplitudes, similar to the rescaling that is used to scale out the growth-rate $\varepsilon$. Apart from this scale transformation, the equations for zero wavenumber and equal wavenumber are equivalent. When $q_A \neq q_B$, the stability borders for both the homogeneous single-mode and bimodal states shift away from $g = 1$, and in particular, there exists a tiny parameter regime $g' < g < g''$, for which both the single-mode and bimodal states are linearly unstable. The differences between $g'$ and $g''$ and 1 are of order $q^2$, and we will not go into this in detail.

A. Stationary states

In this section we focus on the case $|c| < |c_{\text{crit}}|$ and explore numerically the mechanisms by which the wavenumber relaxes. The values of $c_t$ that we found are very close to those found in the simulations of the NLD equations (we will come back to this below).

The simulations of Eqs. (23) were performed for a periodic system of the size 400. The inhomogeneity was of the form given in Eq. (22) with $s = 10$. The initial conditions were chosen to be DW-like, similar to Eq. (21), but the wavenumbers $q_A$ and $q_B$ of the complex variables $A(x)$ and $B(x)$ in the initial states were allowed to be nonzero but constant, to study the wavenumber relaxation.

In the variational case ($c = 0$) the system (23a) has a Lyapunov functional $\mathcal{L}$ with a minimum at $q_A = q_B = 0$. However, such states cannot always be reached, since the periodic boundary conditions lead to conservation of the total phase difference across the system, as long as the amplitude remains nonzero. So, the wavenumber of the, e.g., mode $A(x)$ can be relaxed through the so-called phase-slips occurring at the points where $A = 0$ vanishes \[14\]. When $A$ is not close to zero, generation of such a phase-slip may lead to a significant increase of $\mathcal{L}$, which is forbidden. It is then possible for the system to end up in a local instead of a global minimum of $\mathcal{L}$. But when $A$ is small in some region, phase-slips easily occur, and the wavenumber relaxes to zero.

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When $c \neq 0$, the RGL equations are no longer variational, but for the stationary domain-walls the tendency to evolve to states with zero wavenumber is still present. We found that, the wavenumber of the $B$ mode always relaxes to zero, while for the $A$ mode, the wavenumber occasionally remains nonzero.

The $B$ mode is close to zero in the single-mode region. Phase slips can easily occur in this regime, and we have observed that $q_B$ always relaxes to zero, both for $v_y = 0$ and $v_y \neq 0$. At $c = 0$, the local wavenumber diffuses, and, at $c \neq 0$, the wavenumber is advected to the single-mode region, where it relaxes. When the initial $q_B$ is large, the phase slips of $B$ occasionally occur in the double-mode region, but only as a transient.

For the $A$ mode, two different mechanisms for relaxation of $q_A$ are observed. In the simplest case, the phase slips occur close to the DW, until $q_A$ falls to a level $\sim 0.1$. It may also occur that the phase slips lead to a single-mode state consisting of alternating patches with $A = 0$ and $B = 0$. In that case, both $q_A$ and $q_B$ relax to zero. The kink that is generated between these two different single-mode states is not a global, but rather a local minimum of $L$ corresponding to a metastable state.

The stationary domain-walls of the coupled RGL equations are very similar to those of the NLD equations. When $q_A = q_B$, we can show this by substituting plane-waves of the form $A = |A| \exp(iq x)$ and $B = |B| \exp(iq x)$ into the RGL equations, which yields that the RGL equations are similar to the NLD equations, up to the coefficient $(1 - q^2)$ in front of the linear term. By the aforementioned scaling we can scale this factors out, and this yields the NLD equations for $|A|$ and $|B|$. Therefore, when $q_A = q_B$, the stationary domain-walls produced by the RGL equations can be described by the NLD equations. The critical value of the group-velocity where the position of the domain-wall diverges is then also precisely given by the value of $c_{crit}$ obtained for the NLD equations. The effect of nonzero wavenumber on $c_i$ can in principle be more subtle, but in practice we could not find a difference between the values of $c_i$ for the NLD equations and the RGL equations with $q_A = q_B$.

When $q_A \neq q_B$, we cannot scale out the wavenumbers. Substituting plane-waves of the form $A = |A| \exp(iq_{Ax} x)$ and $B = |B| \exp(iq_{Bx} x)$ into the RGL equations, yields two NLD-like equations were the first terms of the right-hand side $A$ and $B$ are replaced by $(1 - q_A^2)A$ and $(1 - q_B^2)B$. Only one of these pre-factors can be scaled out, so the ordinary NLD equations are not correct. However, since the difference of $q_A$ and $q_B$ that occurs in stationary domain-walls is of order 0.1 at most, and the effect on the RGL equations is of order $q^2$, the NLD equations are still valuable to give a lowest order description of the ensuing domain-walls. In principle, one could carry out the geometrical analysis for the equations with the $(1 - q_A^2)A$ and $(1 - q_B^2)B$ terms; we will not give all the details here, but it can be shown that $c_{crit}$ is perturbed by terms of order $q^2$. This is consistent with the numerically observed increase of the value of $c_i$ with $q_A$ (when $q_B = 0$); this value increases at most a few percent when $q_A = 0.1$.

**B. Dynamical states**

To investigate the dynamical solutions of the RGL equations, we used the same inhomogeneities and values of $c$ as for the NLD equations. The same initial conditions were used, so we restrict ourselves to the zero initial wavenumbers. For a steepness $s = 10$, we have found for almost all $c$ beyond $c_{crit}$ and $\Delta g$ a disordered fluctuating DW state. The mean position of the DW shifts to the right with decreasing $c$; this effect becomes weaker when $\Delta g$ becomes larger. A slightly different state was observed for $\Delta g = 0.1$ and $c = 2c_{crit}$, when the oscillations of the DW showed a strong periodic component but are nevertheless disordered; the reason for this is not clear. We found that when $\Delta g$ is increased, the fluctuations of the position of the domain-wall in general decrease.

The phase of $A$ remains zero in the dynamical state. The phase of $B$ departs from zero, and $B$ slowly develops wavenumbers $\sim 0.01$. At longer times, these wavenumbers often remain constant in space and time. A clear exception is the aforementioned state with $\Delta g = 0.1$ and $c = 2c_{crit}$, where periodic modulations of the wavenumber persist.

For smoother inhomogeneities ($s=0.1$) and small $\Delta g$, the wave number generation is suppressed, and a traveling kink state is observed. This state is similar to the states produced by the NLD equations, in the sense that the phase difference across such a kink is exactly $\pi$. However, the generation of kinks is now no longer periodic. When $\Delta g > 0.1$, these kinks are not produced, and the local wavenumber is generated instead, just as in the case of the steeper inhomogeneity.

In conclusion, both the stationary states and the critical value of $c$ for the RGL equations are almost the same as for the NLD equations, while the dynamical states are somewhat different. The main mechanisms that play a role for the domain-walls of the RGL equations can be described by the NLD equations.

**V. THE CGL EQUATIONS**

In this section we will study the coupled CGL equations:
\[
\begin{align*}
\partial_t A + c \partial_x A &= A + (1 + ic_1) \partial_x^2 A - \left[ (1 - ic_3) |A|^2 + (1 - ic_2) g(x) |B|^2 \right] A , \\
\partial_t B - c \partial_x B &= B + (1 + ic_1) \partial_x^2 B - \left[ (1 - ic_3) |B|^2 + (1 - ic_2) g(x) |A|^2 \right] B .
\end{align*}
\]

These are the generic amplitude equations for left- and right traveling waves, and the group-velocity terms appear here naturally. The coefficients \(c_1, c_2\) and \(c_3\) can be obtained from a systematic expansion of the underlying equations of motion and play an essential role for the dynamics of the CGL equations. The nonlinear dispersion coefficients \(c_2\) and \(c_3\) are, in general, not equal, and we will find below that their difference will be a crucial parameter for the DW’s. The behavior of a single CGLE is already incredibly rich \([22]\), and the situation for the coupled CGLE’s is of course not simpler. As a function of the coefficients \(c_1\), typical states in homogeneous coupled CGLE’s include single and bimodal phase-winding solutions, periodic solutions and spatio-temporal chaotic solutions \([3,20]\). For fixed values of the coefficients, different states can coexist. In the following we will restrict ourselves to describing some of the interesting states that occur in the numerical simulations of the coupled CGLE’s when there is a CC that passes through its critical value.

The numerical simulations were carried out similarly to the RGL model, in a periodic system of the size 400. The inhomogeneity was of the form given in Eq. \([21]\), and we focus on the case \(s = 0.1\), although we have performed some runs with \(s = 10\). It turns out that only the details of the dynamics are different for this larger value of \(s\). As initial conditions we use, as before, the stationary state that is obtained when \(c = c_1 = c_2 = c_3 = 0\). This means in particular that the critical value for \(g\) is 1; it has been shown by Sakaguchi \([20]\), that for periodic or disordered states the transition between single and bimodal behavior can occur for values different from 1.

Even with these restrictions, the parameter-space of the CGLE’s is too large to warrant a complete overview of the dynamical behavior. The nonlinear dispersion is essential for the fate of the domain-wall, and to restrict the search in parameter-space, we restrict ourselves to three different cases: (A) \(c_2 = c_3\), (B) \(c_2 = -c_3\), (C) \(c_2 = 0\). The role of the group-velocity and \(c_1\) is discussed briefly for each case.

We have found many interesting dynamical states; in many cases, the homogeneous single and bimodal states appear to be invaded by more complicated states that grow from the initial DW. By carefully adjusting the various parameters, it seems possible to smoothly proceed from regular periodic to completely disordered states. Below we will only sketch a few of the dynamical regimes that we have found. A more systematic exploration is left for a further work. It is relevant to note that, when the CC constant is far from its critical value 1, we did not find new behavior.

A. The case \(c_2 = c_3\)

When the nonlinear dispersion coefficients \(c_2\) and \(c_3\) are equal, the nonlinear term of Eq. \([23]\) (similar for \([24b]\)) can be written as \((1 - ic_2)(|A|^2 + g(x)|B|^2)A\). Just as in the RGL case, \((|A|^2 + g(x)|B|^2)\) differs only slightly from the value 1 for a DW state, so the nonlinear dispersion acts similar to the linear dispersion. The main effect of the dispersion is then to shift the wavenumber of the \(B\)-mode, but apart from this, the behavior of the CGL’s is qualitatively similar to that of the coupled RGL’s.

When \(g_B \neq 0, |B|\) is smaller then \(|A|\) in the bimodal regime, and as a result, the critical group-velocity is seen to increase (see the discussion on the effect of nonzero wavenumber on \(v_1\) for the RGL equations). For example, when \(c_2 = c_3 = 0\) and \(c_1 = \pm 1\), the selected wavenumber of \(B\) is close to 0.1, and \(|B|\) is a few percent smaller than \(|A|\). The critical value of the group-velocity is then increased by a few percent. When both \(c_1\) and \(c_2 = c_3\) are different from zero, the wavenumber of \(B\) can be large enough to shift the critical group-velocity substantially. For \(c_1 = -1\) and \(c_2 = c_3 = 0.5\), the wavenumber of \(B\) is close to 0.2, and the amplitude of \(B\) is approximately 20 percent smaller than that of \(A\). The critical group-velocity is then found to be between \(c = -0.84\) and \(c = -0.85\). Although this is quite different from the case where the wavenumbers are zero, the essence of the transition to the dynamical DW’s is still given by the analysis for the NLD equations. When the difference between \(c_2\) and \(c_3\) is small, we find qualitatively the same behavior as for \(c_2 = c_3\).

B. The case \(c_2 \neq c_3\)

When \(c_2 \neq c_3\), the nonlinear dispersion is no longer spatially independent, and this leads to oscillatory, spatially periodic or chaotic behavior. When we take both the linear dispersion and the group-velocity equal to zero, we already find various types of behavior, as shown in Fig. \([10]\). The oscillatory behavior shown in Fig. \([11b]\), and in more detail in Fig. \([11]\) arises from a feedback mechanism between local wavenumbers and amplitudes. When \(c_2 \neq c_3\), gradients of \(|A|\) and \(|B|\) generate local wavenumber (see Fig. \([11b]\)), and this local wavenumber suppresses, via the diffusive term, the amplitudes. As a result of this feedback mechanism, the DW becomes oscillatory.
The strength of the aforementioned feedback mechanism grows with the nonlinear dispersion, and above a certain threshold, we find that the oscillations do not stay confined around the DW, but spread out into the single-mode region. For instance, for \( c_3 = -c_2 = 0.2 \) we found that a periodic state is generated (Fig. 10b), that is similar to the periodic state described recently by Sakaguchi [20]. The DW itself becomes disordered. It should be noted that for these values of the coefficients, there are asymptotic single and bimodal phase-winding solutions that are linearly stable.

Finally, the nonlinear dispersion can become so large that also the bimodal state becomes periodic. The disorder seen in Fig. 10c may be either a transient behavior or an established state; our simulations were not conclusive, and we leave this for further work. Here the coefficients of the CGL equations are such, that homogenous phase-winding solutions are unstable, and that even in homogeneous systems, i.e., for \( \gamma \) fixed at \( \pm 0.1 \), periodic states arise. The domain-wall here is disordered, and so this state shows the competition between linearly stable periodic states and a disordered domain-wall.

The linear dispersion has a damping effect on the dynamics as for \( c_1 \neq 0 \) similar behavior is observed for slightly higher values of the nonlinear dispersion. A nonzero group-velocity has a more complicated effect on the dynamics, as it breaks the reflection symmetry. The oscillations that occur for small nonlinear dispersion are damped, because the group-velocity terms advect the local wavenumbers away from the DW, and therefore they suppress the aforementioned feedback effect. For instance, for \( c_1 = -1 \) and \( c = 0.5 \), we have observed stationary states up to \( c_3 = -c_2 = 0.1 \). On the other hand, the symmetry breaking can have a destabilizing effect on the dynamics: the periodic state shown in Fig. 10b becomes disordered when \( c \) is nonzero.

When we move away from the line \( c_2 = -c_3 \), we find that spatio-temporal chaos occurs quite easily. We have focused on the case \( c_1 = c_2 = 0 \). For \( c_3 = 0.2 \), a periodic state similar to the one depicted in Figure 10b occurs. When \( c_3 \) is increased, both the periodic and the bimodal states become gradually disordered (Figs. 12a and 12b).

When \( c = 0 \), the chaotic state consists of more or less stationary, irregularly growing and decaying pulses, but when \( c \neq 0 \), this quasi-stationary character is destroyed. We have checked that in the disordered regime, two slightly different initial conditions diverge throughout the whole domain, which shows that these states are an example of spatio-temporal chaos. Note that this occurs for values of \( c_1, c_2 \) and \( c_3 \) for which the homogeneous single and bimodal state are linearly stable. Apparently the DW acts as a “seed” for the disorder, that then spreads out and completely destroys the plain-wave state.

We have concentrated here on the case \( c_2 = -c_3 \), although it should be stressed that when \( c_2 \) is not exactly \( -c_3 \), we observed similar behavior. The choice \( c_2 = -c_3 \) merely serves to limit ourselves in exploring the parameter-space.

VI. CONCLUSIONS

In this paper we have considered domain walls between single-mode and bimodal states for three types of coupled equations with a spatially dependent coupling coefficient. In the simplest case of two coupled NLD equations with the group-velocity terms, we were able to reduce the description of stationary configurations to a single non-autonomous second-order ordinary differential equation, that was used to determine analytically a necessary condition for the existence of a stationary DW in terms of the group-velocity. We have found that our prediction for the destabilization of such a stationary DW is in good agreement with numerical simulations, and we have found chaotically oscillating DW’s in the case when the group-velocity is beyond the corresponding threshold. For two coupled RGL equations we have found a similar scenario. Finally, for the coupled CGL equations, we have found that, in most cases, the DW’s are unstable, even when the group-velocity is zero, and spatio-temporal disordered states often occur in this model.

In the future, it would be interesting to investigate the competition between the various states of the coupled CGL equations. In particular, not much is known about the periodic states that seem to play an important role here. Possible research subjects include the development of analytical solutions, the development of counting arguments and the competition between periodic and phase-winding solutions. The effect of an inhomogeneity as studied in this chapter on the various states may be a valuable tool in probing the states that occur for constant \( g \). The effect of nonzero group-velocity on the domain-walls in the CGL equations is poorly understood; it would be interesting to see whether the divergence and subsequent instability of the domain-wall, as observed for the NLD equations and CGL equations, still has some relevance for the CGL equations when \( c_2 \) and \( c_3 \) are sufficiently different.

VII. ACKNOWLEDGMENT

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FIG. 1. The exact DW solution [8] to the reduced NLD equations without the group-velocity terms, when the inhomogeneity is chosen as \( \gamma(x) = -\kappa^2 \tanh(\kappa x) \), with \( \kappa = 0.2 \).

FIG. 2. Trajectories of the two-dimensional dynamical system [12] for the variables \( \chi \) and \( \psi \) in the case when \( \gamma(x) \) is the step function [13] with \( \gamma_{\text{min}} = -0.1 \) and \( \gamma_{\text{max}} = 0.1 \). The fat dots represent the fixed points FP0 and FP1, while the bold and dashed curves represent, respectively, their unstable and stable manifolds \( W_0^{\text{out}} \) and \( W_1^{\text{in}} \).

FIG. 3. The behavior of \( W_0^{\text{out}} \) (bold curve) and \( W_1^{\text{in}} \) (dashed curve) around FP0 (black dot) for a range of values for the group-velocity. In this case we have taken \( \gamma_{\text{min}} = -0.1 \), which yields a critical velocity \( c_{\text{crit}} \approx 0.632 \).

FIG. 4. The function \( \chi(x) \) corresponding to the domain-wall solutions for various choices of the group-velocity \( c \). The domain walls were obtained by direct simulations of the coupled NLD equations [8]. The dashed curve corresponds to the DW for \( c = 0.6 \), while the fat curve corresponds to \( c = -0.6 \). The DW’s in between correspond to \( c_{\text{crit}}, \) the position of the domain-wall diverges (bold curve). For \( c = c_{\text{crit}} \), the domain-wall turns unstable via the occurrence of a Hopf bifurcation. When \( c \) is decreased even further, various dynamical states occur (dotted curve). For \( |c| < |c_{\text{crit}}| \), there is, according to the phase-space analysis, a stationary, but unstable domain-wall (dashed curve); for \( c > c_{\text{crit}} \), there is no stationary domain-wall.

FIG. 5. Schematic representation of the bifurcation structure as a function of the group-velocity for weak inhomogeneities. The vertical axis symbolizes the position of the domain-wall. When \( c \) is decreased (this corresponds to moving to the right on the vertical axis) towards values close to \( c_{\text{crit}} \), the position of the domain-wall diverges (bold curve). For \( c = c_{\text{crit}} \), the domain-wall turns unstable via the occurrence of a Hopf bifurcation. When \( c \) is decreased even further, various dynamical states occur (dotted curve). For \( |c| < |c_{\text{crit}}| \), there is, according to the phase-space analysis, a stationary, but unstable domain-wall (dashed curve); for \( c > c_{\text{crit}} \), there is no stationary domain-wall.

FIG. 6. The upper panel shows the chaotic fluctuations of \( \int_{-\infty}^{+\infty} dx A/ < \int_{-\infty}^{+\infty} dx A > \), where the average is over the entire time of the simulation. The plots of \( A \) and \( B \) for this chaotic state are shown in the lower panels. Subsequent snapshots have a time difference of 80 and are shifted in the upward direction by a distance of 0.05. We used an inhomogeneity of the form [20] with \( s = 10 \) and \( \Delta q = 0.1 \), and set the group-velocity \( c \) to 1.1\( c_{\text{crit}} \). The left DW is seen to fluctuate chaotically whereas the right DW is completely stationary.

FIG. 7. The sensitivity of the DW to a perturbation of the initial conditions is demonstrated using evolution of two close initial conditions (see the text). The upper panel shows the divergence of the time evolution of \( \int_{-\infty}^{+\infty} dx A \) (the thin curve) and \( \int_{-\infty}^{+\infty} dx A' \) (the fat curve) divided by \( < \int_{-\infty}^{+\infty} dx A > \). The lower panel shows plots of the difference between \( A \) and \( A' \); consecutive snapshots have a time difference of 40 and are shifted in the upward direction by a distance of 0.001.

FIG. 8. The upper panel shows the periodic fluctuations of \( \int_{-\infty}^{+\infty} dx A/ < \int_{-\infty}^{+\infty} dx A > \). The plots of \( A \) and \( B \) for this periodic traveling kink state are shown in the lower panels. Consecutive snapshots have a time difference of 20 and are shifted in the upward direction by a distance of 0.1. We used an inhomogeneity of the form [20] with \( s = 10 \) and \( \Delta q = 0.2 \), and took the group-velocity \( c = 1.2c_{\text{crit}} \).

FIG. 9. Mixture of chaotically oscillating DW and traveling kinks in NLD model.

FIG. 10. Three examples for the dynamics for \( c = c_1 = 0 \) and \( c_2 = -c_3 \), all for the time interval 2500. Separate sets have a time difference 50: (a) \( c_3 = 0.02 \), an oscillatory state; (b) \( c_3 = 0.2 \), the domain wall becomes chaotic and nucleates a periodic state, that becomes stationary at longer times; (c) \( c_3 = 1 \), also the case when the bimodal state becomes unstable; it is not clear whether a stationary periodic state sets in finally or not.

FIG. 11. The oscillatory domain wall: (a) the value of \( \int dx |A|/ < \int dx |A| > \) as a function of time; (b) hidden-line plot of \( |A| \) around \( x = -100 \) for the first three oscillations (\( t \) from 0 to 1250); the hidden-line plots of the local wavenumbers of \( A \) (c) and \( B \) (d).
FIG. 12. Two examples of a spatiotemporal chaos, for $c = c_1 = c_2 = 0$: (a) $c_3 = 0.5$, the periodic single-mode state has become chaotic, but the single-mode state seems rather passive, only disturbed by ingoing perturbations generating by the fluctuating domain wall; (b) $c_3 = 1$, both the single and bimodal state have become spatio-temporally chaotic.
\[ g(x) = 1 - \kappa^2 \tanh(\kappa x) \]

\[ \chi(x) = \frac{\text{atan}(e^{\kappa x})}{2} \]
