Performance Analysis for Reconfigurable Intelligent Surface Assisted MIMO Systems

Likun Sui®, Zihuai Lin®, Senior Member, IEEE, Pei Xiao®, Senior Member, IEEE, and Branka Vucetic®, Life Fellow, IEEE

Abstract—This paper investigates the maximal achievable rate for a given maximal error probability, and blocklength for the reconfigurable intelligent surface (RIS) assisted multiple-input and multiple-output (MIMO) system. The result consists of a finite blocklength and finite alphabet constraints channel coding achievability and converse bounds based on the Berry-Esseen theorem, the Mellin transform and the closed-form expression of the mutual information and the unconditional variance. The numerical evaluation shows a fast speed of convergence to the maximal achievable rate as the blocklength increases and also proves that the channel variance is a sound measure of the backoff from the maximal achievable rate due to finite blocklength.

Index Terms—RIS, MIMO, finite blocklength, achievable rate, achievability bound, converse bound.

I. INTRODUCTION

RECENTLY, there has been a prodigious increase in the demand for higher data rates in wireless communication networks due to the escalating number of mobile and IoT devices, together with dramatically increased services. To this end, many candidate solutions have been proposed to deal with this demand, such as multiple-input multiple-output (MIMO) and millimeter-wave (mmWave) communications. These technologies offer significant data rate gains but have power and hardware cost limitations. Generally speaking, they can be regarded as a way to achieve higher data rates by altering transmitter and receiver features without influencing the propagation channel.

A possible approach to overcome the issues mentioned above lies in the use of the recently-developed reconfigurable intelligent surface (RIS), which consists of a massive array of scattering elements [1], [2], [3]. The array of elements can be configured by controllers to reflect radio waves towards arbitrary angles so that we can apply phase shifts and modify polarization [4]. Unlike existing relay technologies, the RIS can turn the hostile propagation environment into a favorable one due to its unique properties, which ameliorate the signal quality at the receiver side without consuming additional power.

Most prior works have demonstrated the advantages of the RIS in terms of the bit-error-rate performance and cell coverage. In contrast, this paper takes a more fundamental information-theoretic perspective on the performance of RIS-assisted MIMO communication systems at the finite blocklength regime.

Related Work: In [5], a broad mathematical framework of the RIS-assisted wireless communication system over the Rayleigh fading channel was presented. Then a theoretical upper bound on bit error probability was derived. Moreover, the authors presented the relationship between the received signal-to-noise ratio (SNR) and the number of reflecting elements, indicating that the received SNR grew as the number of reflecting elements increased. Thus reliable transmission over a noisy channel could still be accomplished at low SNRs with the support of RIS elements. The authors of [6] investigated the coverage expansion achieved by the RIS-assisted wireless communication system over quasi-static flat Rayleigh fading channels. Furthermore, compared with both direct link and relay-assisted wireless communication systems, the SNR gain and the delay outage rate of the RIS were investigated. In [7], the authors studied the RIS’s placement optimization in a cellular network to maximize cell coverage. They developed a coverage maximization algorithm (CMA) to obtain the optimal RIS’s orientation distance. The authors of [8], [9], and [10] focused on the RIS-assisted multiple-input single-output (MISO) wireless communication system, for which efficient algorithms, such as Lagrangian dual transform, active and passive beamforming, were studied to address the non-convex maximization problem of the weighted sum-rate that can be achieved by all groups. The authors of [11] statistically characterized the RIS-assisted wireless communication system under the premise that all cascaded fading channels between the transmitter, RIS and receiver follow the Rayleigh distribution. Furthermore, the closed-form expression of theoretical outage probability was derived, and the accuracy of their results was validated. In [12], the authors focused on a downlink multi-user system in which the transmission from a base station with multiple antennas to multiple users is assisted by an RIS. Applying a hybrid beamforming scheme solved the sum-rate maximization problem of the practical case with a
limited number of discrete phase shifts is solved. The authors of [13] proposed a complete circuit-based reflection-refraction model for a new surface, i.e., the intelligent omni-surface (IOS) in terms of the physical structure.

**Contribution:** We use the Berry-Esseen theorem, mutual information and unconditional information variance as the fundamental mathematical basis to obtain the achievability and converse bounds for the maximal achievable rate $R$ given a fixed maximal error probability $\epsilon$ and blocklength $n$ for an RIS MIMO system. To derive the achievability bound, we use the Berry-Esseen theorem and some other inequalities and show the channel output’s exact probability density function (PDF). To derive our converse bound, we combine the maximum of the auxiliary channel’s PDF, which is a product of $m$ copies of the PDF of Gamma distributed variables by the Mellin transform and Meijer G-function, and the maximum of its output space by the Lebesgue measure. Furthermore, we utilize the saddle point approximation and the Taylor series expansion to find the closed forms for both the mutual information and the conditional information variance. In order to complete our achievability and converse bounds, we utilize different modulation schemes in our RIS MIMO system and compare the performance of each modulation scheme mainly in two aspects. One is the required blocklength to achieve a certain level of the maximal achievable rate, and the other is how the unconditional information variance affects the convergence’s speed to the maximal achievable rate.

**Notation:** The modulus, real portion, and imaginary part of a scalar complex number $y$ are denoted by $|y|$, $\Re\{y\}$ and $\Im\{y\}$, respectively. A random vector is denoted by a bold capital letter, and its realization is denoted by a bold lowercase symbol. The identity matrix of dimension $n \times n$ is denoted as $I_n$. The Hermitian transposition of a matrix $Y$ is denoted by the superscript $Y^H$. The trace of the matrix $Y$ is represented by $\text{tr}(Y)$. A complex Gaussian distribution with a mean of $\mu$ and a variance of $\sigma^2$ is denoted as $CN(\mu, \sigma^2)$. The Frobenius norm of a matrix $Y$ is $\|Y\| = \sqrt{\text{tr}(YY^H)}$. The nonnegative real line is denoted by $\mathbb{R}_+$. The nonnegative orthant of the $m$-dimensional real Euclidean spaces is denoted by $\mathbb{R}_+^m$. $\mathbb{E}[\cdot]$ and $\mathbb{P}[\cdot]$ represent the statistical expectation and the probability of an event, respectively.

The remainder of this paper is structured as follows. The system model is described in Section II, and the concept of a channel code is reviewed. The achievability and converse bounds for our system are derived in Section III. The extension to more general cases is presented in Section IV. In Section V, numerical findings are presented. Finally, Section VI draws the conclusion.

## II. System Model

We consider an RIS-assisted wireless communication system with $t$ transmit and $r$ receive antennas shown in Fig. 1. Both the transmitter and receiver have multiple antennas, which are placed as uniform linear arrays (ULAs). The direct link is blocked by an obstacle (i.e., a wall or building) that is situated between the transmit and receive antennas. A rectangular RIS of $N_{\text{ris}}$ elements is utilized to improve the whole system performance, and only reflection-type RIS is considered in this paper. We assume that all the RIS elements are ideal, which means that each of them can independently influence the phase and the reflection angle of the impinging wave.

We let $m = \min\{t, r\}$. The signal vector at the receive antenna array is given by

$$Y = HX + W,$$

where $H \in \mathbb{C}^{t \times N}$ is the channel matrix, $X \in \mathbb{C}^{t \times n}$ is the transmit signal over $n$ channel uses, $Y \in \mathbb{C}^{t \times n}$ is the corresponding received signal, and $W \in \mathbb{C}^{t \times n}$ is the additive noise at the receiver, which is independent of $H$ and has independent and identically distributed (i.i.d.) $CN(0, 1)$ entries.

The channel matrix $H$ of our RIS-assisted system can be expressed as

$$H = H_2 \Sigma(\theta) H_1,$$

where $H_1 \in \mathbb{C}^{N_{\text{ris}} \times t}$ represents the channel between the transmitter and the RIS, $H_2 \in \mathbb{C}^{t \times N_{\text{ris}}}$ represents the channel between the RIS and the receiver, and $\Sigma(\theta) = \text{diag}(\theta) \in \mathbb{C}^{N_{\text{ris}} \times N_{\text{ris}}}$, where $\theta = [\theta_1, \ldots, \theta_{N_{\text{ris}}}^T \in \mathbb{C}^{N_{\text{ris}} \times 1}$ represents the signal reflecting coefficient from the RIS. In this paper, similar to the related works [14], [15], [16], we assume that the signal reflection from any RIS element is ideal, i.e., without any power loss. In other words, we may write $\theta_i = \exp\{j\phi_i\}$ for $i = 1, \ldots, N_{\text{ris}}$, where $\phi_i$ is the phase shift induced by the $i$-th RIS element, which can be flexibly adjusted in $[0, 2\pi]$.1

Equivalently, we may write $|\theta_i| = 1, i = 1, \ldots, N_{\text{ris}}$.

Let us consider input and output sets $\mathcal{A}$ and $\mathcal{B}$ and a conditional probability measure $P_{Y|X} : \mathcal{A}^{t \times n} \mapsto \mathcal{B}^{t \times n}$. We denote a codebook with $M$ codewords by $(C_1, \ldots, C_M)$. A decoder is defined as a random transformation $P_{Z|Y} : \mathcal{B}^{t \times n} \mapsto \{1, \ldots, M\}$ which satisfies $P_{Z|X}(j|C_j) \geq 1 - \epsilon, j = 1, \ldots, M$ where $\epsilon$ is the maximal error probability. We also consider that each codeword $C_j$ satisfies the equal power constraint $\|C_j\|^2 = nP$, where $P$ is the transmit power. Then, a codebook and a decoder whose maximal error probability is smaller than $\epsilon$ are termed as an $(n, M, \epsilon)$ code and its coding.

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1To characterize the achievable rate limit of RIS-assisted MIMO systems without perfect phase shifting, we assume that the phase shift by each RIS element can be continuously adjusted, i.e., we assume that the phase shift of the RIS elements obeys the continuous uniform distribution over the interval $[0, 2\pi]$.
rate is defined as \( R = \frac{\log M}{n} \). In this paper, the information density also plays an essential role, which is defined as
\[
i(X; Y) \triangleq \log \frac{P_{X|Y}(y|x)}{P_Y(y)},
\]
where \( P_{X|Y}(y|x) \) denotes the conditional distribution on \( B^{r \times 1} \) for all \( x \in A^{t \times 1} \), and \( P_Y(y) \) represents the output distribution.

III. Achievability and Converse Bounds

In this section, we provide the definitions of achievability and converse bounds. The achievability and converse bounds are important to the proof of the channel coding theorem. The achievability bound is a lower bound on the size of a code that can be guaranteed to exist with a given arbitrary blocklength and error probability. The converse bound is an upper bound on the size of any code with a given arbitrary blocklength and error probability. The mutual information is defined as
\[
I(X; Y) \triangleq \mathbb{E}[i(X; Y)].
\]
Additionally, the unconditional information variance is defined as \( U(X; Y) \triangleq \text{Var}[i(X; Y)] \), where \( \text{Var}[] \) denotes the variance of \( (\cdot) \). Moreover, our achievability and converse bounds for the examined RIS MIMO system are presented below.

**Theorem 1:** We consider a communication system with finite input alphabet \( A \), and the continuous output alphabet \( B \). Let \( p(Y|X,H) \) be the corresponding conditional PDF on \( B \) for all \( X \in A^n \), where \( B \) is a channel matrix. The input distribution \( P(X) \triangleq [q_1, \ldots, q_t]^T \), where \( q_t = [q_{i0}, \ldots, q_{in}] \), \( i = 1, \ldots, t \) with \( q_{ij} \) being equiprobable, i.e., \( q_{ij} = \frac{1}{|A|} \). Then we define the mutual information and the unconditional information variance as \( I(X; Y) \) and \( U(X; Y) \), respectively. Thus for the RIS MIMO channel and arbitrary \( 0 < \epsilon < 1 \), we have the achievability and converse bounds
\[
I(X; Y) - \sqrt{\frac{U(X; Y)}{n}Q^{-1}(\epsilon) + \frac{1}{n} + O(n^{-\frac{1}{2}})} \\
\leq R \leq I(X; Y) - \sqrt{\frac{U(X; Y)}{n}Q^{-1}(\epsilon + \frac{\epsilon}{\sqrt{n}})} + \frac{(m + 1) \log n}{2n} + O(n^{-\frac{1}{2}}),
\]
where \( Q \) is the complementary Gaussian cumulative distribution function \( Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt \).

**Proof:** We give the key proof steps. At first, we prove that the second moment of \( i(X; Y) \) is nonzero and the third moment is always less than infinity. According to the DT bound in \([17]\), we have \( \epsilon \geq \mathbb{P}[i(X^n; Y^n) \leq \log \lambda + \lambda \mathbb{E}[\exp\{-i(X^n; Y^n)\}]1_{\{i(X^n; Y^n) > \log \lambda\}}] \). Applying the Berry-Esseen theorem several times, we have \( \mathbb{P}[i(X^n; Y^n) \leq \log \lambda + \lambda \mathbb{E}[\exp\{-i(X^n; Y^n)\}]1_{\{i(X^n; Y^n) > \log \lambda\}}] \). Applying the Meijer G function and the Lebesgue measure, we have the converse bound of the maximal error probability of the auxiliary channel. According to the binary hypothesis testing in \([17]\), we finally obtain the converse part. The detailed proof of Th. 1 can be found in Appendix A and B.

To accomplish the achievability bound by applying Th. 1, we need to obtain the exact expression of both the mutual information and the unconditional variance. At first, for our system model, the input distribution \( P(X) = [q_1, \ldots, q_t]^T \), where \( q_t = [\frac{1}{2}, \ldots, \frac{1}{2}]^T \) and \( q_t = [\frac{1}{4}, \ldots, \frac{1}{4}]^T \), for BPSK and QPSK, respectively. Additionally, the conditional PDF of a MIMO Rayleigh fading channel, \( p(Y|X,H) \), is given by \([18]\) and \([19]\).

\[
p(Y, H|X) = p(H)p(Y|X,H) = \frac{p(H)}{\det(\pi_r)} \exp\left(- (Y - HX)(Y - HX)^H\right),
\]
where \( I_r \) designates the \( r \times r \) identity matrix and \( \det(.) \) denotes the determinant.

Then
\[
I(X; Y) = \int_0^\infty \int_0^\infty \sum_{X \in A^n} \left( P(X)p(Y, H|X) \right) dYdH
\]
\[
\times \log \left\{ \sum_{X' \in A^n} P(X')p(Y, H|X') \right\} \right) dYdH
\]
\[
= \sum_{i=1}^{\left| A^t \right|} \frac{P(x_i)}{\det(\pi_r)} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left( \frac{p(h)}{\det(\pi_r)} \right) \exp\left\{- \frac{1}{2} \left| y - hx_i \right|^2 \right\} \right) dYdH
\]
\[
= \sum_{i=1}^{\left| A^t \right|} \frac{P(x_i)}{\det(\pi_r)} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left( \frac{p(h)}{\det(\pi_r)} \right) \exp\left\{- \frac{1}{2} \left| y - hx_i \right|^2 \right\} \right) dYdH
\]
\[
= \sum_{i=1}^{\left| A^t \right|} \frac{P(x_i)}{\det(\pi_r)} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left( \frac{p(h)}{\det(\pi_r)} \right) \exp\left\{- \frac{1}{2} \left| y - hx_i \right|^2 \right\} \right) dYdH
\]
\[
= \sum_{i=1}^{\left| A^t \right|} \frac{P(x_i)}{\det(\pi_r)} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left( \frac{p(h)}{\det(\pi_r)} \right) \exp\left\{- \frac{1}{2} \left| y - hx_i \right|^2 \right\} \right) dYdH
\]
\[
= \sum_{i=1}^{\left| A^t \right|} \frac{P(x_i)}{\det(\pi_r)} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left( \frac{p(h)}{\det(\pi_r)} \right) \exp\left\{- \frac{1}{2} \left| y - hx_i \right|^2 \right\} \right) dYdH
\]
\[
= \sum_{i=1}^{\left| A^t \right|} \frac{P(x_i)}{\det(\pi_r)} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left( \frac{p(h)}{\det(\pi_r)} \right) \exp\left\{- \frac{1}{2} \left| y - hx_i \right|^2 \right\} \right) dYdH
\]
\[
= \sum_{i=1}^{\left| A^t \right|} \frac{P(x_i)}{\det(\pi_r)} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left( \frac{p(h)}{\det(\pi_r)} \right) \exp\left\{- \frac{1}{2} \left| y - hx_i \right|^2 \right\} \right) dYdH
\]
\[
= \sum_{i=1}^{\left| A^t \right|} \frac{P(x_i)}{\det(\pi_r)} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left( \frac{p(h)}{\det(\pi_r)} \right) \exp\left\{- \frac{1}{2} \left| y - hx_i \right|^2 \right\} \right) dYdH
\]
\[
= \sum_{i=1}^{\left| A^t \right|} \frac{P(x_i)}{\det(\pi_r)} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left( \frac{p(h)}{\det(\pi_r)} \right) \exp\left\{- \frac{1}{2} \left| y - hx_i \right|^2 \right\} \right) dYdH
\]
and \( i' \)-th points in the constellation of \( x \). In order to reduce the complexity of the mutual information and the unconditional variance, we give the approximation of (7) and (9). At first, we deal with the mutual information \( I(X;Y) \).

\[
I(X;Y) = t \log |A| + \frac{1}{|A|} \sum_{i=1}^{|A|} \int_0^\infty \int_{-\infty}^\infty \frac{1}{\det(\pi_{l,r})} p(h) \exp\left(-\frac{1}{2} \right) dy dh \\
\times \|y - hx_i\|^2 \log \left\{ \frac{\exp\left(-\frac{1}{2} \|y - hx_i\|^2\right)}{\sum_{i'=1}^{|A|} \exp\left(-\frac{1}{2} \|y - hx_{i'}\|^2\right)} \right\} dy dh
\]

(10)

where (11) comes from Taylor series expansion of (10) and

\[
\Pi(y, h) = \sum_{i'=1}^{|A|} \exp \left\{ \frac{\|y - qhx_i - x_{i'}\|^2}{2q} - \frac{(q + 1) \|h(x_i - x_{i'})\|^2}{2} \right\}
\]

(11)

Before utilizing the saddle point approximation [20], we need to guarantee the existence of the saddle point. For the convenience of notation, we use vector \( c_{i,i'} \) to represent \( h(x_i - x_{i'}) \). Since \( q \) is a positive integer, it is easy for us to validate that

\[
\Pi^{-q}(y) = \left[ \sum_{i'}^{|A|} \exp \left\{ \frac{\|y - qhc_{i,i'}\|^2}{2q} - \frac{(q + 1) \|c_{i,i'}\|^2}{2} \right\} \right]^{-q} > 0
\]

\[
\lim_{y \to -\infty} \left\{ \Pi^{-q}(y) \right\} = 0
\]

\[
\lim_{y \to +\infty} \left\{ \Pi^{-q}(y) \right\} = 0
\]

Thus there exists a maximum value of \( \Pi^{-q}(y) \), which satisfies the condition of the saddle point approximation. Then we can assume that \( \Pi^{-q}(y) \) achieves its maximum at \( y = y_0 \), which \( y_0 \) satisfies \( \frac{\partial}{\partial y} \Pi^{-q}(y) \big|_{y=y_0} = 0 \)

\[
\sum_{i'=1}^{|A|} \frac{2(y_0 - qhc_{i,i'})}{2q} \exp\left\{ \frac{\|y_0 - qhc_{i,i'}\|^2}{2q} - \frac{(q + 1) \|c_{i,i'}\|^2}{2} \right\} = 0
\]

(13)

After solving (13), we have \( y_0 = \sum_{i'=1}^{|A|} \rho_{i,i'} c_{i,i'} \), where \( \rho_{i,i'} = \Pi(y_0) / \sum_{i'=1}^{|A|} \Pi(y_0) \) is a positive number from (0, 1) and satisfies that \( \sum_{i'=1}^{|A|} \rho_{i,i'} = 1 \). Therefore, we have for a non-zero number \( q \), the multiple integrals over the complex number vector \( y \) can be approximated by the saddle point approximation

\[
\int_0^\infty \frac{1}{\det(\pi_{l,r})} \left( \Pi(y, h) \right)^{-q} dy dh
\]

\[
\approx \left\{ \sum_{i'=1}^{|A|} \exp \left\{ -\frac{1}{3} \|h(x_i - x_{i'})\|^2 / 4 \right\} \right\}^{-q}
\]

(14)

Combining (11) and (14), we eliminate the multiple integrals over the complex vector \( y \) as

\[
I(X;Y) \approx t \log |A| - \frac{1}{|A|} \sum_{i=1}^{|A|} \int_0^\infty \int_{-\infty}^\infty \frac{1}{\det(\pi_{l,r})} p(h) \exp\left(-\frac{1}{2} \right) dy dh \\
\times \left[ \sum_{i'=1}^{|A|} \exp \left\{ -\frac{1}{3} \|h(x_i - x_{i'})\|^2 / 4 \right\} \right]^{-q} dh
\]

(15)

Then by observing (15), we take advantage of inverse Taylor series expansion, leading to the following result.

**Lemma 1:** Let \( t \) represent the number of transmit antennas, and \( A \) denotes the input alphabet, and \( p(h) \) represents the channel distribution, and \( x_i \) denotes the transmitted vector in the \( i \)-th transmitter antenna. The mutual information can be approximated as

\[
I(X;Y) \approx t \log |A| - \int_0^\infty p(h) \frac{1}{|A|} \sum_{i=1}^{|A|} \int_0^\infty \frac{1}{\det(\pi_{l,r})} p(h) \exp\left(-\frac{1}{2} \right) dy dh \\
\times \log \left[ \sum_{i'=1}^{|A|} \exp \left\{ -\frac{1}{3} \|h(x_i - x_{i'})\|^2 / 4 \right\} \right] dh
\]

(16)

Moreover, we need to obtain the approximation of the unconditional variance \( U(X;Y) \). The first step is similar to the process of the approximation of \( I(X;Y) \), in which we utilize the Taylor series expansion as follows

\[
U(X;Y) = \left[ I(X;Y) - (t \log |A|) \right] + \frac{1}{\det(\pi_{l,r})} |A| \ln 2 \\
\sum_{i=1}^{|A|} \int_0^\infty \int_{-\infty}^\infty p(h) \left( \sum_{i'=1}^{|A|} \int_{-\infty}^\infty p(h) \exp\left(-\frac{1}{2} \right) dy dh \right) \left( \Pi_2(y, h) \right)^{-q}
\]

(17)
Then the PDF of the product of two Rayleigh random variables

\[ p_{|h|}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_{|h|,|g|}(k) N_{\text{ris}}^{-1} \exp(-j k x)dk \]

and the angle of \( h \) follows the uniform distribution in \([0, 2\pi]\).

Therefore, we obtain the expression of the \( h \)'s PDF.

Substituting (23) into the mutual information in (16) and the unconditional variance in (21), respectively. After applying the right Riemann sum to eliminate the multiple integrals over the complex vector \( h \), we obtain the closed-form expression of \( I(X;Y) \) and \( U(X;Y) \) as follows

\[
I(X;Y) 
\approx t \log |\mathcal{A}| - \sum_{k=1}^{N_{\text{ris}}} \Delta_{h_k} \frac{2h_k}{N_{\text{ris}}} \exp\left(-\frac{h_k^2}{N_{\text{ris}}}\right) \frac{|\mathcal{A}|}{|\mathcal{A}|} \sum_{i=1}^{\frac{|\mathcal{A}|}{2}} \log \left\{ - \frac{\|h_k(x_i - x_{i'})\|^2}{2(3 - \exp(-\|h_k(x_i - x_{i'})\|^2 / 8))} \right\}
\]

\[
U(X;Y) 
\approx -[I(X;Y) - (t \log |\mathcal{A}|)]^2 
+ \sum_{k=1}^{N_{\text{ris}}} \Delta_{h_k} \frac{2h_k}{N_{\text{ris}}} \exp\left(-\frac{h_k^2}{N_{\text{ris}}}\right) \frac{|\mathcal{A}|}{|\mathcal{A}|} \sum_{i=1}^{\frac{|\mathcal{A}|}{2}} \log \left\{ - \frac{\|h_k(x_i - x_{i'})\|^2}{2(6 - \exp(-\|h_k(x_i - x_{i'})\|^2 / 8))} \right\},
\]

where \( \Delta_{h_k} = h_k - h_{k-1} \). Finally, we substitute (24) and (25) into (4) to obtain our achievability and converse bounds for the Rayleigh fading channel. To compare with our result, we calculate the capacity of the channel whose input is a circularly symmetric complex Gaussian with zero mean and covariance \( \frac{4}{d} I \). The Theorem is shown below.

**Theorem 2 [21]:** Under the power constraint \( P \), we assume the same channel with the same number of transmitting and receiving antennas as our system model. Its capacity, as determined by the complex Gaussian input, is equal to \( \mathbb{E}_g \left[ \log(1 + \frac{P}{\mathcal{P}} g) \right] = \int_0^\infty \log(1 + \frac{P}{\mathcal{P}} g)dg \), where \( g \) denotes the eigenvalues of the matrix \( \mathbf{H}^H \mathbf{H} \), where \( \mathbf{H} \) is from (2), and its PDF is given by

\[
p_g(x) = \sum_{i=0}^{m} \binom{m}{i} \left( \frac{x}{N_{\text{ris}}} \right)^{\text{max}\{r,t\}-m} \left( \frac{x}{N_{\text{ris}}} \right)^m \exp\left(-\frac{x}{N_{\text{ris}}}\right).
\]

Thus we have

\[
C_{\text{Gaussian}} = \sum_{k=1}^{N_{\text{ris}}} \Delta_{g_k} \log(1 + \frac{P}{\mathcal{P}} g_k) \sum_{i=0}^{m} \binom{m}{i} \left( \frac{x}{N_{\text{ris}}} \right)^{\text{max}\{r,t\}-m} \left( \frac{x}{N_{\text{ris}}} \right)^m \exp\left(-\frac{x}{N_{\text{ris}}}\right).
\]
where \( \Delta_{g_k} = g_k - g_{k-1} \).

**IV. Extension to More General Cases**

In this section, we apply our achievability and converse bounds to several possible cases. We assume perfect channel estimation in Sections IV-A and IV-C. When the channel is not known to the receiver, there exists mismatched decoding, i.e., scaled nearest neighbor (SNN) decoding [22]. We briefly introduce the general notion of mismatched decoding. Consider a conditional probability measure \( P_{Y|X} \) with input \( X \in \mathcal{A}^{r \times n} \) and output \( Y \in \mathcal{B}^{r \times n} \). At coding rate \( R \) and blocklength \( n \), a codebook with \( M \) codewords by \( (C_1, \ldots, C_M) \). For mismatched decoding, we let a function \( d : \mathcal{A}^{r \times n} \times \mathcal{B}^{r \times n} \rightarrow \mathbb{R} \) be a decoding metric, which hence induces the following decoding rule: \( \hat{y} = \arg \min_{j \in \mathcal{M}} d(X, Y, j) \), with ties broken arbitrarily. A coding rate \( R \) is achievable if there exists a sequence of codebook such that the maximal error probability asymptotically vanishes as \( n \rightarrow \infty \), and the supremum of the achievable rate is called the mismatched capacity.

Furthermore, to analyze the effect of the CSI estimation in finite blocklength, there is a received pilot overhead provided to the receiver. For the RIS MIMO system, if the receiver antenna is 1, the overall pilot overhead is \( N_{\text{ris}} + 1 \) [23]. When the pilot overhead is inserted in every channel use, then the resulting achievable rate should be discounted by a factor of \( 1 - N_{\text{ris}}/n \).

**A. Perfect Phase Alignment-Rayleigh Fading Channel**

In this subsection, we consider the case when the phase shift can be perfectly aligned with the channel phases due to the perfect channel estimation. Generally, the perfect phase shift is unknown for capacity maximization over the MIMO channel. However, according to [24] and [25], the discrete-Fourier transform (DFT)-based phase-shifting configuration along with an additional steering direction \( \phi \in \mathbb{C}^{N_{\text{ris}} \times 1} \) could be adopted to obtain a proper design for \( \Phi = [\theta_1, \ldots, \theta_{N_{\text{ris}}} \}, \) where \( \Phi = \text{diag}(\phi) \mathbf{F} \) and \( \mathbf{F} \) is the DFT matrix, which guarantees that the whole channel information in all directions can be estimated. Therefore, to compare with the imperfect phase shift, we assume that there exists a perfect phase shift that can be achieved in the MIMO system. In this case, the channel coefficient \( h \) in (22) is modified to

\[
N_{\text{ris}} \sum_{i=1}^{N_{\text{ris}}} |h_{2,i}| \cdot |h_{1,i}|. \tag{28}
\]

Note that the closed-form approximation of the \( h \)'s PDF can be evaluated as

\[
p_h(x) = \frac{x^a}{b^{a+1} \Gamma(a+1)} \exp \left( - \frac{x}{b} \right), \tag{29}
\]

where \( a = \frac{\pi^2}{8} - 1 \) and \( b = \frac{\pi^2}{2} \) with \( z_1 = \frac{N_{\text{ris}} \pi^2}{16} \) and \( z_2 = N_{\text{ris}}(1 - \frac{\pi^2}{16}). \) Substituting (29) into the closed forms of the mutual information in (24) and the unconditional variance in (25), we have

\[
I(X; Y) 
\approx t \log |\mathcal{A}| - \frac{4h_k}{N_{\text{ris}}(K_1^* + K_2^*)} \exp \left( - \frac{2x^2}{N_{\text{ris}}(K_1^* + K_2^*)} \right), \tag{33}
\]

Substituting (33) into the closed forms of the mutual information and the unconditional variance (24) and (25), we have

\[
I(X; Y) 
\approx \frac{4h_k}{N_{\text{ris}}(K_1^* + K_2^*)} \left( \log |\mathcal{A}| \sum_{i=1}^{N_{\text{ris}}} |A_i^t| + \sum_{i=1}^{N_{\text{ris}}} |A_i^t| \right).
\]

Therefore, we substitute (30) and (31) into (4) to obtain our achievability and converse bounds for the perfect phase alignment-Rayleigh fading channel.

**B. Rician Fading Channel**

In this subsection, we consider the Rician fading channel. Specially, we assume the channel coefficient \( |h_1| \) in (22) follows the Rician distribution, which is \( |h_1| \sim \text{Rician}(\alpha_1, \beta_1) \). The PDF of which is given by

\[
p_{|h_1|}(x) = \frac{x^a}{\alpha_1^a} \exp \left( - \frac{x^2 + \beta_1^2}{2\alpha_1^2} \right) I_0 \left( \frac{\beta_1 x}{\alpha_1^2} \right), \tag{32}
\]

where \( I_0(\cdot) \) is the modified Bessel function of the first kind with order zero, and the shape parameter of the Rician fading \( K_1 = \frac{\beta_1^2}{2\alpha_1^2} \) denotes the ratio of the power contributions by a line-of-sight path to the remaining multipaths, and the scale parameter of the Rician fading \( \Omega = 2\alpha_1^2 + \beta_1^2 \) is the total power received in all paths. We assume that the PDF of \( |h_2| \) in (22) is also Rician distribution which is \( |h_2| \sim \text{Rician}(\alpha_2, \beta_2) \).

Note that if we change the distribution of entries in (2) from the Rayleigh distribution to the Rician distribution, then the process is similar to the Rayleigh part which is omitted for simplicity. We have the PDF of \( |h| \) as

\[
p_{|h|}(x) = \frac{4x}{N_{\text{ris}}(K_1^* + K_2^*)} \exp \left( - \frac{2x^2}{N_{\text{ris}}(K_1^* + K_2^*)} \right), \tag{33}
\]

Substituting (33) into the closed forms of the mutual information and the unconditional variance (24) and (25), we have

\[
I(X; Y) 
\approx t \log |\mathcal{A}| - \frac{4h_k}{N_{\text{ris}}(K_1^* + K_2^*)} \left( \log |\mathcal{A}| \sum_{i=1}^{N_{\text{ris}}} |A_i^t| + \sum_{i=1}^{N_{\text{ris}}} |A_i^t| \right).
\]
\begin{align}
\times \exp \left\{ - \frac{\|h_k(x_i - x_{i'})\|^2}{2(3 - \exp\{- \|h_k(x_i - x_{i'})\|^2/8\})} \right\} \right) \right) \right)}
\end{align}

Therefore, we substitute (34) and (35) into (4) to obtain our achievability and converse bounds for the Rician fading channel.

**C. Perfect Phase Alignment-Rician Fading Channel**

In this subsection, we consider the perfect phase alignment case with the Rician fading channel. In this case, the channel coefficient \( h \) in (28) is given by

\[ h = \sum_{i=1}^{N_{\text{ris}}} |h_{2,i}| \cdot |h_{1,i}|. \tag{36} \]

where \( |h_1| \sim \text{Rician}(\alpha_1, \beta_1) \) and \( |h_2| \sim \text{Rician}(\alpha_2, \beta_2) \), respectively. Note that the closed-form approximation of PDF of the channel coefficient \( h \) can be evaluated as

\[ p_h(x) = \frac{x^a}{b^a \Gamma(a + 1)} \exp \left\{ - \frac{x}{b} \right\}, \tag{37} \]

where \( a = \frac{z_1^2}{2} - 1 \) and \( b = \frac{2z_2}{z_1} \) with \( z_1 = N_{\text{ris}}(K_1 + K_2 + \frac{\pi}{18}) \) and \( z_2 = N_{\text{ris}}(K_1 + K_2 + 1 - \frac{\pi}{18}) \). Substituting (29) into the closed forms of the mutual information and the unconditional variance (24) and (25), we have the same equations as (30) and (31) with different values of \( z_1 \) and \( z_2 \). Therefore, we substitute the above mutual information and unconditional variance into (4) to obtain our achievability and converse bounds for the perfect phase alignment-Rician fading channel.

**V. NUMERICAL RESULTS**

**A. Evaluation of the Derived Bounds**

In this section, we consider an RIS MIMO system consisting of multiple transmitter antennas, a rectangular RIS of \( N_{\text{ris}} \) elements and multiple receive antennas. We assume all the channels, i.e., the channels between the transmitter and the RIS, the RIS and the receiver, and the transmitter and the receiver, are independent with the maximal error probability \( \epsilon = 10^{-3} \). Assuming that all the channels are Rayleigh fading channels, the number of the transmit and receive antennas are \( t = 2 \) and \( r = 2 \), respectively, and the SNR is \(-10\) dB. Figs. 2(a) and 2(b) show the numerical results of the derived bounds with BPSK and QPSK modulated signals and the capacity for \( N_{\text{ris}} = 16 \) and 32, respectively. From Fig. 2(a), we can see that \( C_{\text{Gaussian}} = 2.2509 \) bit/(channel use), which is calculated based on Th. 2, and the maximal achievable rate for the BPSK modulated signal is 1.3283 bit/(channel use) from (24). The required blocklength \( n \) to achieve above 80% and 90% of its maximal achievable rate starts at \( n = 100 \) and \( n = 400 \), respectively. With the QPSK modulation, the required blocklengths are \( n = 240 \) and \( n = 940 \), respectively. In Fig. 2(b), we only change the RIS element from \( N_{\text{ris}} = 16 \) to 32. The capacity is 3.3145 bit/(channel use), and the required blocklengths decrease dramatically to 40 and 160, respectively. For QPSK modulation, the required blocklengths are \( n = 100 \) and \( n = 400 \), respectively.

The channel variance can be treated as the unconditional information variance \( U(X; Y) \) in (25). It shows how quickly the performance converges to the maximal achievable rate as blocklength \( n \) grows. When \( \epsilon > Q(\frac{(m+1)\log n}{2n\sqrt{\log(1+2\epsilon)}}) \), the converse bound will first decrease and then converge to its achievability part, while \( 0 < \epsilon < Q(\frac{(m+1)\log n}{2n\sqrt{\log(1+2\epsilon)}}) \), the converse bound will be monotonic increasing along with the increase of the blocklength. Additionally, if the target is to transmit at a fraction of the maximal achievable rate...
0 < \eta < 1 with a maximal error probability \epsilon, the relationship between the required blocklength \( n \) and the channel variance is 
\[ n \approx \frac{1}{(X|Y)\xi}\left(\frac{\epsilon}{1-\eta}\right)^2. \]

The performance of the 2 \times 2 MIMO case over different channels, i.e., perfect phase alignment Rayleigh fading channel in Section IV-A, Rician fading channel in Section IV-B and perfect phase alignment Rician fading channel in Section IV-C, in terms of the required blocklength, with the different number of RIS elements and different SNR level are summarized in Table I. Moreover, the gap between the two bounds and the maximal achievable rate of the 2 \times 2 MIMO case over different channels with a specific number of RIS elements \( N_{\text{ris}} = 16 \), a given maximal error probability \( \epsilon = 10^{-3} \), and the blocklength \( n = 256 \) are summarized in Table II. From Fig. 2 and Table I and II, we can conclude that: 1) as \( N_{\text{ris}} \) increases, the overall channel between the transmitter and the receiver becomes better. That means that the gap between the maximal achievable rate for different modulation schemes and the capacity increases and vice versa at the same SNR level; 2) the required blocklength \( n \) falls significantly to achieve a given fraction of the maximal achievable rate as the number of RIS elements increases.

In Figs. 3 and 4, we demonstrate the performance of the 3 \times 3 MIMO case over the four different channels. From these figures, we observe that: 1) the Rician fading channel is much better than the Rayleigh fading channel regardless of whether it is perfect phase alignment or not; 2) when \( N_{\text{ris}} \) increases, the RIS element’s effect on the QPSK modulated signal is more significant than that on the BPSK modulated signal; 3) to achieve the same performance, the required SNR for the channel with perfect phase alignment is approximately 20 dB smaller than the one without perfect phase alignment.

Table I: Required blocklengths to achieve 80% and 90% of the maximal achievable rate for an RIS MIMO system over different channels and transmit antennas \( t = 2 \) and receive antennas \( r = 2 \), \( \epsilon = 10^{-3} \), and the blocklength \( n = 256 \)

|              | \( N_{\text{ris}} = 16 \) | \( N_{\text{ris}} = 32 \) |
|--------------|-----------------------------|-----------------------------|
| BPSK         | QPSK                        | BPSK                        |
| Perfect phase alignment-Rayleigh fading channel\(^1\) | 80 220 20 40 | 360 860 80 140 |
| Rician fading channel\(^2\) | 180 380 80 180 | 720 1560 300 720 |
| Perfect phase alignment-Rician fading channel\(^3\) | 100 220 20 40 | 400 920 10 140 |

\(^1\) The SNR = -20 dB. \(^2\) The SNR = -30 dB. \(^3\) The SNR = -40 dB.

Table II: The gap between the achievability and converse bounds and the maximal achievable rate for an RIS MIMO system over different channels and the number of RIS elements \( N_{\text{ris}} = 16 \), transmit antennas \( t = 2 \) and receive antennas \( r = 2 \), \( \epsilon = 10^{-3} \), and the blocklength \( n = 256 \)

| Maximal achievable rate | Gap to the achievability bound | Gap to the converse bound |
|-------------------------|-------------------------------|----------------------------|
| BPSK                    | QPSK                          | BPSK                        |
| Rayleigh fading channel\(^1\) | 1.3283 1.9254 | 0.3042 0.5204 |
| Perfect phase alignment-Rayleigh fading channel\(^2\) | 1.3435 1.9843 | 0.3028 0.5246 |
| Rician fading channel\(^3\) | 1.0945 1.4743 | 0.2774 0.4598 |
| Perfect phase alignment-Rician fading channel\(^4\) | 1.2865 1.8978 | 0.2942 0.5122 |

\(^1\) The SNR = -10 dB. \(^2\) The SNR = -20 dB. \(^3\) The SNR = -30 dB. \(^4\) The SNR = -40 dB.
Fig. 3. The comparison of achievability and converse bounds between Rayleigh fading channel, Rician fading channel whose two shape parameters are $K_1 = 10$ and $K_2 = 5$ respectively and transmit antennas $t = 3$ and receive antennas $r = 3$ for $\epsilon = 10^{-3}$ and SNR = $-20$ dB with $N_{\text{ris}} = 16$ and $N_{\text{ris}} = 32$, respectively.

is shown below.

$$
\xi Q \left( \frac{I(X;Y) + \frac{m+1}{2} \log n - R(1 - \xi)}{\sqrt{U(X;Y)}} \right) \leq \epsilon' 
$$

$$
\leq \xi Q \left( \frac{I(X;Y) - R(1 - \xi)}{\sqrt{U(X;Y)}} \right).
$$

By utilizing the polar code with a successive cancellation list (SCL) decoder and the extended BCH code with an ordered statistic decoder (OSD), we validate our derived results. In Fig. 5(a), we set the number of RIS elements $N_{\text{ris}} = 16$, the number of the transmitter and receiver antennas to 2, the modulation scheme to BPSK, the coding rate $R = 0.5$, and the blocklength $n = 128$. All the simulations are averaged over $10^6$ Monte Carlo realizations. We choose two coding methods: one is the $(128, 2^{64}, \epsilon')$-polar code with SCL decoder (the list size is $L = 32$), and the other is the $(128, 2^{64}, \epsilon')$-EBCH code with OSD decoder (the order is chosen to 4). We observe that the simulation result of the EBCH code is slightly better than the one of the polar code at the blocklength $n = 128$. Furthermore, as long as the simulation results are below the upper bound, they would validate our derived results. In Fig. 5(b), we change the number of the transmitter and receiver antennas to 3, and the rest parameters remain the same as in Fig. 5(a). At first, we compare the performance of two codes in different MIMO systems, i.e., $2 \times 2$ MIMO and $3 \times 3$ MIMO systems. At the same EbNo level of $-14$ dB, the average error probability drops from 0.3248 to 0.0607 for the EBCH code and from 0.4441 to 0.0754 for the polar code. At the same average error probability level of $10^{-2}$, the gaps between the simulation result and the lower bound are 2 dB and 2.5 dB for the EBCH code and polar code, respectively. We observe that the gaps increase to 2.5 dB and 3 dB for the EBCH code and the polar code. The performance of bounds over the $3 \times 3$ MIMO system is better than that over the $2 \times 2$ MIMO system.
Fig. 5. The lower and upper bounds for \((128, 2^{64}, \epsilon')\) codes for the RIS MIMO system over a Rayleigh fading channel, the number of RIS elements \(N_{\text{ris}} = 16\), and the different modulation scheme with the number of antennas \(t = r = 2\) and \(t = r = 3\), respectively.

B. Rate vs SNR

In Fig. 6, we illustrate the capacity, the maximal achievable rate of QPSK and BPSK modulated signals over Rayleigh fading channel with the different number of RIS elements \(N_{\text{ris}} = 4\) and 32 and different transmit antennas \(t = 2\) and 3, respectively. Fig. 6(a) shows the tightness of the mutual information in Lemma 1. Moreover, it shows that the capacity achieved by circularly symmetric complex Gaussian inputs increases without any boundary as the SNR increases, and the gaps between the Gaussian input and the different modulated signals increase as the SNR increases. In Fig. 6(b), we change the number of the transmit antennas to \(t = 2\) and choose \(N_{\text{ris}} = 4\) and 32. As SNR increases, there exists a limit of the achievable rate for each modulation scheme and the number of transmitter and receiver antennas, i.e., for \(2 \times 1\) MIMO with BPSK modulation, the limit \(t \log |A| = 2\) and for \(3 \times 1\) MIMO with QPSK modulation, the limit \(t \log |A| = 6\). Moreover, the effect of the number of RIS elements on the achievable rate is that the speed approaching the limit increases as the number of RIS elements increases.

VI. CONCLUSION

In this paper, we have established achievability and converse bounds on the maximal achievable rate \(R\) at a given block-length \(n\) and a maximal error probability \(\epsilon\) for an RIS MIMO system. The analytical results demonstrated that the number of transmit and receive antennas and the channel variance \(U(X; Y)\) would affect the convergence speed to the maximal achievable rate as the blocklength \(n\) increases. For the future work, we will investigate our derived results into the new surfaces, such as the intelligent omni-surface [13].

APPENDIX A

THE ACHIEVABILITY PART OF TH. 1

We need to introduce an important tool for proving Th. 1, that is the Berry-Esseen theorem.

Theorem 3 (Berry-Esseen Theorem [26]): Let \(X_k, k = 1, \ldots, n\) be independent with

\[
\mu_k = \mathbb{E}[X_k], \quad \sigma_k^2 = \text{Var}[X_k], \quad t_k = \mathbb{E}[|X_k - \mu_k|^3],
\]

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\[ \sigma^2 = \sum_{k=1}^{n} \sigma_k^2 \quad \text{and} \quad T = \sum_{k=1}^{n} t_k. \]

Then for any \(-\infty < \tau < \infty\)
\[ \left| \mathbb{P} \left[ \sum_{k=1}^{n} (X_k - \mu_k) \geq \tau \sigma \right] - Q(\tau) \right| \leq \frac{6T}{\sigma^2} . \] (38)

For the proof of Th. 1, we first need to prove that the second moment of \( i(X;Y) \) in (3) is nonzero and its third moment is always less than infinite.

\[ U(X;Y) = \mathbb{E}[|i(X;Y) - I(X;Y)|^2] \]
\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{X' \in A^t} \left( P(X)p(Y,H|X)(1-p(Y,H|X)) \right) \cdot \log^2 \left\{ \frac{p(Y,H|X)}{\sum_{X' \in A^t} P(X')p(Y,H|X')} \right\} dY dH \]
\[ > 0, \] (39)

where (39) follows from \( 2(1/2)!/|A| = |A| - 1 \) and \( P(X) = 1/|A|^t \) and (40) follows from \( 1 - p(Y,H|X) > 0 \).

Then, we need to show the third moment is less than infinite.

\[ T(X;Y) \]
\[ \leq \mathbb{E}[|P(Y,H|X)|^3] + \mathbb{E}[\left| \sum_{X' \in A^t} P(X)p(Y,H|X') \right|^3] \]
\[ + 2I(X;Y)^3 \] (41)
\[ \leq |B|(3e^{-1} \log e)^3 + 2I(X;Y)^3, \] (42)

where (41) follows from Holder’s inequality and (42) follows from \( \max_{0<x<1}\{x \log^2 x\} = 0 \) at \( x = 1 \) and \( \max_{0<x<1}\{x \log^3 x\} = (3e^{-1} \log e)^3 \) at \( x = e^{-3} \).

We denote \( i_X^n;Y^n \) = \( \sum_{i=1}^{n} i(X;Y) \), and let its second
moment \( \sum_{U} U(X;Y) \) be nonzero and its third moment \( \sum_{n} \mathbb{E}[i(X;Y) - I(X;Y)]^3 < \infty \). Thus, Th.3 is still applicable to \( i_X^n;Y^n \).

According to the DT’ bound in [17],
\[ \epsilon \leq \mathbb{E}[\exp \{ -[i_X^n;Y^n] - \log \frac{M-1}{\sigma} \}], \]
where \( [\cdot] \) denotes \( \max\{0,\cdot\} \).

In the sequel, we prove that there exist some \( \lambda \) values, so that
\[ \epsilon \geq \mathbb{E}[\exp \{0\} 1_{\{i_X^n;Y^n > \log \lambda \leq 0\}}] \]
\[ + \mathbb{E}[\exp \{-i_X^n;Y^n + \log \lambda\} 1_{\{i_X^n;Y^n > \log \lambda > 0\}}]. \] (43)

\[ = \mathbb{P}[i_X^n;Y^n \leq \log \lambda] \]
\[ + \lambda \mathbb{E}[\exp \{-i_X^n;Y^n\} 1_{\{i_X^n;Y^n > \log \lambda\}}]. \] (44)

The first step is to obtain the maximum of the first part of the right-hand side of (44). After applying Th. 3, we have
\[ \mathbb{P}[i_X^n;Y^n \leq nI(X;Y) - \tau \sqrt{nU(X;Y)}] \]
\[ \leq \frac{6T(X;Y)}{\sqrt{nU(X;Y)}^2} + Q(\tau). \] (45)

We assume
\[ \log \lambda = nI(X;Y) - \tau \sqrt{nU(X;Y)}, \] (46)

and
\[ \mathbb{P}[i_X^n;Y^n \leq \log \lambda] \leq \frac{6T(X;Y)}{\sqrt{nU(X;Y)}^2} + Q(\tau). \] (47)

The maximum of the second part of the right-hand side of (44) is given below. For \( 0 < \tau < \infty \) and any \( \Delta > 0 \),
\[ \mathbb{P}[i_X^n;Y^n > \log \lambda + i\Delta] \leq \log \lambda + (i + 1)\Delta \]
\[ \leq \frac{12T(X;Y)}{\sqrt{nU(X;Y)}^2} + Q(\tau + \log \frac{1+\Delta}{\sqrt{nU(X;Y)}}), \] (48)

where (49) is obtained by applying Th. 3 twice. Then,
\[ \mathbb{E}[\exp \{-i_X^n;Y^n\} 1_{\{i_X^n;Y^n > \log \lambda\}}] \]
\[ = \exp \{ -i_X^n;Y^n\} \exp \{ -i_X^n;Y^n\} \] (50)

where (51) is a result of the Riemann integral and (52) follows from the fact that for any \( \sigma \), \( Q(\frac{\lambda}{\sigma}) - Q(\frac{\lambda + \Delta}{\sigma}) \leq \frac{\Delta}{\sqrt{2}\pi\sigma} \). Thus, we have

\[ \lambda \mathbb{E}[\exp \{-i_X^n;Y^n\} 1_{\{i_X^n;Y^n > \log \lambda\}}] \]
\[ \leq \frac{\Delta}{\sqrt{2}\pi\sqrt{nU(X;Y)}} + \frac{12T(X;Y)}{\sqrt{nU(X;Y)}^2} \] (51)

while (53) follows for any \( \exp \{x\} > 1 \), \( \sum_{i=0}^{\infty} \exp \{-ix\} = \frac{1}{\exp(x)} - 1 \). Substituting (47) and (53) into (44), we have
\[ \mathbb{P}[i_X^n;Y^n \leq \log \lambda] + \lambda \mathbb{E}[\exp \{-i_X^n;Y^n\} \] (54)
\[ \times 1_{\{i_X^n;Y^n > \log \lambda\}}] \leq Q(\tau) + \frac{1}{\sqrt{nU(X;Y)}^2} \] (55)

Based on (44), we can assume that the right hand side of (54) equals to \( \epsilon \), then we obtain the value of \( \tau \)
\[ \tau = Q^{-1}\left( \epsilon - \frac{1}{\sqrt{nU(X;Y)}^2} \right) \] (56)
For large $n$, the second item inside the Q function of (55) vanishes. Therefore, we can obtain $\tau = Q^{-1}(\epsilon) + O(\frac{1}{\sqrt{n}})$. Then, we have $\log \lambda = nI(X;Y) - Q^{-1}(\epsilon)\sqrt{nU(X;Y)} + O(\frac{1}{\sqrt{n}})$.

Thus, 

$$\begin{align*}
R \geq I(X;Y) - Q^{-1}(\epsilon)\sqrt{\frac{U(X;Y)}{n}} + 1/n + O(n^{-\frac{1}{2}}). 
\end{align*}$$

\section*{Appendix B}

\textbf{The Converse Part of Th. 1}

We assume the transmitter is not aware of the realizations of the channel matrix $H$. We denote the average power constraint $p(X) \triangleq \frac{1}{n}XX^H$. Based on \cite{27, 28, 29}, to evaluate the converse bound of an auxiliary channel, we need to obtain the lower bound of $\epsilon'$, where $\epsilon'$ is the maximal error probability over the corresponding auxiliary channel. We thus denote the auxiliary channel $Q$ as:

$$Q_{Y|X,H} \triangleq \prod_{j=1}^{n} Q_{Y_j|X,H},$$

where

$$Q_{Y_j|X,H} = CN(0, I_r + Hp(X)H^H),$$

We denote $B \triangleq I_r + Hp(X)H^H$ and let its eigenvector $\omega = [\omega_1, \ldots, \omega_m] = \lambda_{\text{max}}(B)$. Note that $P = p(X)$ is the only factor that affects the output of the channel $Y|X,H$. Let the space $S \triangleq p(Y) = \frac{1}{n}YY^H$ and its entry is defined as the square of the norm of $Y$ and is then normalized by the blocklength $n$, which is shown below

$$S_j = \frac{\omega_j}{n} \sum_{i=1}^{n} |Z_{j,i}|^2, \quad j = 1, \ldots, m,$$

where $Z_{j,i} \sim CN(0,1)$. $S$ can be seen as the statistical expression of the receiver’s detection of $X$ from $(Y, H)$. Thus the auxiliary channel $Q_{Y|X,H}$ can be seen as $Q_{S|B}$. From (59), we note that the $S_j$ follows the Gamma distribution, and its corresponding PDF is given by

$$q_{S_j|B_j}(s_j|\omega_j) = \frac{n^n}{(\omega_j)^n \Gamma(n)} s_j^{n-1} \exp \left\{ -\frac{ns_j}{\omega_j} \right\}.$$

Moreover, as $Q_{S|B}$ is a product of $m$ copies of the PDF of $S_j$, We can obtain the PDF of $Q_{S|B}$ by the theorem shown below [30].

\textbf{Theorem 4:} Given $N$ independent Gamma-distributed random variables $x_i$ and that their shape parameter $k$ and scale parameter $\theta$ are all the same, we have the PDF of $x_i$ as

$$f_i(x_i) = \frac{1}{\Gamma(k)\theta^k} x_i^{k-1} e^{-\frac{x_i}{\theta}}.$$

We denote $z$ as the product of $N$ independent gamma variables $x_i$. Therefore, the PDF of $z = x_1x_2 \ldots x_N$ is a normalized Meijer $G$-function as $g(z) = KG_{0,N}^{0,0}(k -1 | \frac{z}{\theta})$, where $K$ is a normalizing factor $K = (\frac{1}{\theta})^n \prod_{i=1}^{n} \frac{1}{\Gamma(k)}$, and

$$G_{m,n}^{p,q}(\begin{array}{c} j_1, j_2, \ldots, j_p \\ k_1, k_2, \ldots, k_q \end{array} | z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} z^{-s} \prod_{j=p}^{m} \Gamma(s + k_j) \cdot \prod_{j=1}^{n} \Gamma(1 - j - s) \cdot \prod_{j=m+1}^{n} \Gamma(1 - k - s - s) \cdot ds,$$

where $c$ is a vertical contour in the complex plane chosen to separate the poles of $\Gamma(s + k_j)$ from those of $\Gamma(1 - j - s)$.

We set two parameters, the shape parameter $k = n$ and the scale parameter $\theta_j = \frac{n}{\omega_j}$. The number of copies in our case is $N = m$. Then we can apply Th. 4 to calculate the PDF of $Q_{S|B}$ as $q_{S_j|B_j}(s_j|\omega_j) = KG_{0,m}^{0,0}(n - 1 | s_j \frac{n}{\omega_j})$, where $K = (\frac{n}{\omega_j})^m \prod_{i=1}^{m} \frac{1}{\Gamma(1)}$, and

$$G_{0,m}^{m,0}(n - 1 | s_j \frac{n}{\omega_j}) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (s_j \frac{n}{1 + \omega_j})^m \exp \left\{ -\frac{ns_j}{\omega_j} \right\} \prod_{j=1}^{m} \Gamma(z + n - 1)dz.$$

Consider an arbitrary code for the auxiliary channel $Q$. The decoding sets corresponding to the $M$ codewords is denoted by $D_i$, $i = 1, \ldots, M$. $\epsilon'$ is the maximal error probability over the auxiliary channel $Q$. Then we have

$$1 - \epsilon' \leq \frac{1}{M} E_H \left[ \sum_{i=0}^{M} \int_{D_i} q_{S|B}(s)ds \right] \leq E_H \left[ \int_{D_0} q_{S|B}(s)ds \right] \leq E_H \left[ \max\{q_{S|B}(s)\} \times \textbf{Leb}(D_0) \right].$$

Next we need to provide the maximum of the output space of an arbitrary decoding set, $\textbf{Leb}(D_0)$. Due to the power allocation vector $p(X)$, the space $P$ can be bounded by a certain ball in $\mathbb{R}^m$. Based on the definition of $S$, its space is a slightly larger ball than the space $P$. Thus we can obtain the maximum of the Lebesgue measure \cite{31} of $D_0$,

$$\textbf{Leb}(D_0) \leq \textbf{Leb}(S) \leq \frac{K}{M},$$

where $\textbf{Leb}$ is the Lebesgue measure and $K$ is a constant.

Then the decoding set of any codeword has a Lebesgue measure space which is always smaller than $\frac{K}{M}$. Therefore, we have

$$1 - \epsilon' \leq \frac{1}{M} E_H \left[ \max\{q_{S|B}(s)\} \times \frac{K}{M} \right] = \frac{1}{M} \left( \frac{(n - 1)^n \exp\left\{ -(n - 1) \right\} }{\Gamma(n)} \right)^m \times \int_0^\infty \prod_{i=1}^{m} (\omega_j)p(g)dg \leq \frac{n^{m/2}}{M}.$$
According to the binary hypothesis testing in [17], we have
\[
\Lambda(\epsilon) \geq \frac{1}{\chi} \left( \epsilon - \mathbb{P}[\hat{Y} \geq Y^n] \leq \log \lambda \right) \geq \frac{1}{\chi} \left( \epsilon - \frac{6T(X; Y)}{\sqrt{nU(X; Y)^2}} - Q(\tau) \right),
\]
where \( \Lambda(\epsilon) \) denotes the maximal probability of error under \( P_{Y|X; H} \) if the probability of error under \( Q_{Y|X; H} \) is \( \epsilon \) and (65) follows from (47). Then,
\[
\log \Lambda(\epsilon) \geq -nI(X; Y) + \tau \sqrt{nU(X; Y)} + \log \left( \epsilon - \frac{6T(X; Y)}{\sqrt{nU(X; Y)^2}} - Q(\tau) \right),
\]
where (66) follows from (46). We assume \( \tau = Q^{-1}(\epsilon(1 + \frac{1}{\sqrt{n}})) - \frac{Q(\frac{6T(X; Y)}{\sqrt{nU(X; Y)^2}})}{2} \). Thus,
\[
\log \Lambda(\epsilon) \geq -nI(X; Y) + \sqrt{nU(X; Y)}Q^{-1}(\epsilon(1 + \frac{1}{\sqrt{n}})) - \frac{6T(X; Y)}{\sqrt{nU(X; Y)^2}} - \frac{1}{2} \log n.
\]
Due to the fact that \( \log \Lambda(\epsilon) \leq 1 - \epsilon' \), we have
\[
-nI(X; Y) + \sqrt{nU(X; Y)}Q^{-1}(\epsilon(1 + \frac{\epsilon}{\sqrt{n}})) - \frac{1}{2} \log n + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right) \leq 1 - \epsilon'.
\]
Thus substituting (68) into (64), we have
\[
R \leq \frac{I(X; Y) - \frac{U(X; Y)}{n}Q^{-1}(\epsilon(1 + \frac{\epsilon}{\sqrt{n}}))}{\frac{(m + 1) \log n}{2n} + \mathcal{O}(n^{-2})}.
\]

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Likun Sui received the M.Eng. degree in electrical engineering from The University of Sydney, Sydney, NSW, Australia, in 2017, where he is currently pursuing the Ph.D. degree with the Centre for IoT and Telecommunications. His research interests include information theory, RIS-aided communications, AmBC communications, and channel coding.

Zihuai Lin (Senior Member, IEEE) received the Ph.D. degree in electrical engineering from the Chalmers University of Technology, Sweden, in 2006. He was with Ericsson Research, Stockholm, Sweden. He was a Research Associate Professor with Aalborg University, Denmark. He is currently an Associated Professor with the School of Electrical and Information Engineering, The University of Sydney, Australia. His research interests include source/channel/network coding, coded modulation, MIMO, radio resource management, cooperative communications, small-cell networks, 5G/6G, the IoT, ECG and EEG signal analysis, and Radar imaging. He is an Associate Editor of IEEE ACCESS and IEEE SENSORS JOURNAL.

Pei Xiao (Senior Member, IEEE) received the Ph.D. degree from the Chalmers University of Technology, Gothenburg, Sweden, in 2004. He was with Newcastle University and with Queen’s University Belfast. He held a position with Nokia Networks, Finland. He is currently a Professor of wireless communications with the Institute for Communication Systems (ICS), University of Surrey. He is also a Technical Manager of 5GIC/6GIC, leading the research team in the new physical layer work area, and coordinating/supervising research activities across all the work areas. He has published extensively in the fields of communication theory, RF and antenna design, signal processing for wireless communications, and is an inventor on over 15 recent 5GIC patents addressing bottleneck problems in 5G systems.

Branka Vucetic (Life Fellow, IEEE) is currently an Australian Laureate Fellow, a Professor of telecommunications, and the Director of the Centre for IoT and Telecommunications, The University of Sydney. Her current research interest include wireless networks and Industry 5.0. In the area of wireless networks, she works on communication system design for 6G and wireless AI. In the area of Industry 5.0, her research is focused on the design of cyber-physical-human systems and wireless networks for applications in healthcare, energy grids, and advanced manufacturing. She is a Fellow of IEEE, the Australian Academy of Technological Sciences and Engineering, and the Australian Academy of Science.