MILNOR INVARIANTS, $2n$-MOVES AND $V^n$-MOVES
FOR WELDED STRING LINKS

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Abstract. In a previous paper, the authors proved that Milnor link-homotopy
invariants modulo $n$ classify classical string links up to $2n$-move and
link-homotopy. As analogues to the welded case, in terms of Milnor invariants,
we give here two classifications of welded string links up to $2n$-move and self-
crossing virtualization, and up to $V^n$-move and self-crossing virtualization,
respectively.

1. Introduction

In [19, 20] J. Milnor defined a family of classical link invariants, known as Milnor
$\mu$-invariants. Given an $m$-component classical link $L$, Milnor invariants $\mu_L(I)$ are
indexed by a finite sequence of elements in $\{1, \ldots, m\}$. In [12] N. Habegger and
X.-S. Lin introduced the notion of classical string links and defined Milnor invari-
ants for classical string links. These invariants are called Milnor $\mu$-invariants. It
is remarkable that $\mu$-invariants for non-repeated sequences classify classical string
links up to link-homotopy [12] (whereas $\mu$-invariants are not enough strong to clas-
sify classical links with four or more components up to link-homotopy [17]). Here
the link-homotopy is the equivalence relation on classical (string) links generated
by the self-crossing change and ambient isotopy [19].

A $2n$-move is a local move as illustrated in Figure 1.1. The $2n$-moves were
probably first studied by S. Kinoshita in 1957 [15]. Since then, $2n$-moves have been
well-studied in Knot Theory. In particular, the connections between $2n$-moves and
classical link invariants are increasingly well-understood; see for example [16, 23, 8,
9]. Recently, the authors [22] established the relation between Milnor link-homotopy
invariants and $2n$-moves as follows.

\begin{center}
\begin{tikzpicture}
\draw[thick] (0,0) .. controls (1,1) and (2,-1) .. (0,0);
\draw[thick] (2,0) .. controls (1,1) and (0,-1) .. (2,0);
\draw[thick] (3,0) .. controls (2,1) and (1,-1) .. (3,0);
\end{tikzpicture}
\end{center}

\textbf{Figure 1.1.} $2n$-move

Theorem 1.1 ([22 Theorem 1.1]). Let $n$ be a positive integer. Two classical string
links $\sigma$ and $\sigma'$ are $(2n + lh)$-equivalent if and only if $\mu_\sigma(I) \equiv \mu_{\sigma'}(I) \pmod{n}$ for
any non-repeated sequence $I$.

\begin{itemize}
\item 2010 Mathematics Subject Classification. 57M25, 57M27.
\item Key words and phrases. Welded string links, Milnor invariants, $2n$-moves, $V^n$-moves, self-
crossing virtualization, arrow calculus.
\item The second author was supported by a Grant-in-Aid for JSPS Research Fellow (\#17J08186)
of the Japan Society for the Promotion of Science.
\item The third author was partially supported by a Grant-in-Aid for Scientific Research (C)
(\#17K05264) of the Japan Society for the Promotion of Science and a Waseda University Grant
for Special Research Projects (\#2018S-077).
\end{itemize}
Here, the \((2n+lh)\)-equivalence is the equivalence relation generated by the \(2n\)-move, self-crossing change and ambient isotopy.

The set \(\mathcal{SL}(m)\) of \(m\)-component classical string links has a monoid structure under the \textit{stacking product}. Habegger and Lin proved that the set of link-homotopy classes of \(\mathcal{SL}(m)\) forms a torsion free group of rank \(s_m = \sum_{r=2}^{m} (r-2)!\binom{m}{r}\) \cite{12} Section 3. We remark that the quotient \(\mathcal{SL}(m)/(2n+lh)\) of \(\mathcal{SL}(m)\) under \((2n+lh)\)-equivalence forms a finite group generated by elements of order \(n\), and that the order of the group is \(n^s_m\) \cite{22} Corollary 1.2.

The notion of \textit{welded string links} was introduced by R. Fenn, R. Rimányi and C. Rourke \cite{10}. M. Goussarov, M. Polyak and O. Viro essentially proved that two classical string links are equivalent as welded objects if and only if they are equivalent as classical objects \cite{11} Theorem 1.B]. Therefore, welded string links can be viewed as a natural extension of classical string links. The study of welded string links has recently become an area of active interest; see for example \cite{5, 6, 1, 2, 3, 18}.

In \cite{2} B. Audoux, P. Bellingeri, J.-B. Meilhan and E. Wagner defined Milnor invariants, denoted by \(\mu^w\), for welded string links and proved that the \(\mu^w\)-invariants for non-repeated sequences classify welded string links up to sv-equivalence. Here, the \textit{sv-equivalence} is the equivalence relation generated by the self-crossing virtualization and welded isotopy. A \textit{crossing virtualization} is a local move replacing a classical crossing with virtual one, and a \textit{self-crossing virtualization} is a crossing virtualization involving two strands of a single component. The sv-equivalence is indeed a natural extension of link-homotopy in the sense that two classical string links are sv-equivalent if and only if they are link-homotopic \cite{1} Theorem 4.3.

As analogues of Theorem \cite{11, 11} to the welded case, we show the following two theorems.

\textbf{Theorem 1.2.} Let \(n\) be a positive integer. Two \(m\)-component welded string links \(\sigma\) and \(\sigma'\) are \((2n+sv)\)-equivalent if and only if \(\mu^w_{\sigma}(I) \equiv \mu^w_{\sigma'}(I) \pmod{n}\) for any non-repeated sequence \(I\), and \(\mu^w_{\sigma}(ij) - \mu^w_{\sigma'}(ji) \equiv \mu^w_{\sigma}(ij) - \mu^w_{\sigma'}(ji) \pmod{n}\) for any \(1 \leq i < j \leq m\).

Here, the \((2n+sv)\)-equivalence is the equivalence relation on welded string links generated by the \(2n\)-move, self-crossing virtualization and welded isotopy.

\textbf{Remark 1.3.} In \cite{3}, Proposition 3.8], Audoux, Bellingeri, Meilhan and Wagner proved Theorem \cite{1, 2} for \(n = 1\). Note that \(\mu^w(ij) - \mu^w(ji)\) is written as \(\text{vlk}_{ij} - \text{vlk}_{ji}\) in \cite{3}.

\textbf{Theorem 1.4.} Let \(n\) be a positive integer. Two welded string links \(\sigma\) and \(\sigma'\) are \((V^n + sv)\)-equivalent if and only if \(\mu^w_{\sigma}(I) \equiv \mu^w_{\sigma'}(I) \pmod{n}\) for any non-repeated sequence \(I\).

Here a \(V^n\)-move, defined by the authors in \cite{21}, is an oriented local move as illustrated in Figure 1.2 which is a generalization of the crossing virtualization. (In fact, the \(V^1\)-move is equivalent to the crossing virtualization.) The \((V^n + sv)\)-equivalence is the equivalence relation on welded string links generated by the \(V^n\)-move, self-crossing virtualization and welded isotopy.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{v_n_move.png}
\caption{\(V^n\)-move}
\end{figure}

Let \(w\mathcal{SL}(m)/(2n + sv)\) and \(w\mathcal{SL}(m)/(V^n + sv)\) denote the quotients of the set \(w\mathcal{SL}(m)\) of \(m\)-component welded string links under \((2n + sv)\)-equivalence and \((V^n + sv)\)-equivalence, respectively.
sv)-equivalence, respectively. Since the set of sv-equivalence classes of $w\mathcal{SL}(m)$ forms a group of rank $w_m = \sum_{r=2}^{m} (r-2)! r^{(m)}$ [Remark 4.9], it is not hard to see that both $w\mathcal{SL}(m)/(2n+sv)$ and $w\mathcal{SL}(m)/(V^n + sv)$ also form groups. In contrast to $\mathcal{SL}(m)/(2n + lh)$, the quotient $w\mathcal{SL}(m)/(2n + sv)$ is not a finite group. On the other hand, we have the following.

**Corollary 1.5.** The quotient $w\mathcal{SL}(m)/(V^n + sv)$ forms a finite group generated by elements of order $n$, and the order of the group is $n^{w_m}$.

Theorems 1.2 and 1.4 indicate that $(2n + sv)$-equivalence implies $(V^n + sv)$-equivalence (Proposition 5.1). Moreover, these theorems together with Theorem 1.1 imply that both natural maps $\iota_1 : \mathcal{SL}(m)/(2n + lh) \to w\mathcal{SL}(m)/(2n + sv)$ and $\iota_2 : \mathcal{SL}(m)/(2n + lh) \to w\mathcal{SL}(m)/(V^n + sv)$ are injective (Proposition 5.7). Figure 1.3 gives the summary of relations between $\mathcal{SL}(m)/(2n + lh)$, $w\mathcal{SL}(m)/(2n + sv)$ and $w\mathcal{SL}(m)/(V^n + sv)$.

![Figure 1.3. Relations between $\mathcal{SL}(m)/(2n + lh)$, $w\mathcal{SL}(m)/(2n + sv)$ and $w\mathcal{SL}(m)/(V^n + sv)$](image)

**2. Welded string links and welded Milnor invariants**

In this section, we review the definitions of welded string links and their Milnor invariants.

### 2.1. Welded string links

An $m$-component virtual string link diagram is an $m$-component string link diagram in the plane, whose transverse double points admit not only classical crossings but also virtual crossings illustrated in Figure 2.1. Throughout the paper, virtual string link diagrams are assumed to be ordered and oriented.

![classical crossing virtual crossing](image)

**Figure 2.1.**

A welded string link is an equivalence class of virtual string link diagrams under welded Reidemeister moves, which consist of Reidemeister moves R1–R3, virtual moves V1–V4 and the overcrossings commute move OC illustrated in Figure 2.2. A sequence of welded Reidemeister moves is called a welded isotopy.

Here, we give the definition of the group of a welded string link. The group $G(D)$ of a virtual string link diagram $D$ is defined via the Wirtinger presentation [14, Section 4], i.e. an arc of $D$ yields a generator, and each classical crossing gives a relation of the form $c^{-1}b^{-1}ab$, where $a$ and $c$ correspond to the underpasses and $b$ corresponds to the overpass at the crossing; see Figure 2.3 (Here, an arc of $D$ is a piece of strand such that each boundary is either a strand endpoint or a classical undercrossing, and the interior does not contain classical undercrossings.)
The group $G(D)$ is preserved under welded isotopy [14, 24], and hence we define the group $G(\sigma)$ of a welded string link $\sigma$ to be $G(D)$ of a virtual diagram $D$ of $\sigma$.

### 2.2. Welded Milnor invariants.

In [2], Audoux, Bellingeri, Meilhan and Wagner defined Milnor invariants for ribbon 2-dimensional string links, i.e., properly embedded annuli in the 4-ball bounding immersed 3-balls with only ribbon singularities. They also defined a welded extension of Milnor invariants, which is an invariant of welded string links, via the Tube map (see [25, 24]) sending welded string links to ribbon 2-dimensional string links. These invariants are called welded Milnor invariants. The construction of welded Milnor invariants is topological, since it is defined via the Tube map. However, applying Milnor’s algorithm given in [20], we can (define and) compute welded Milnor invariants by means of virtual diagrams as follows.

Given an $m$-component welded string link $\sigma$, consider its virtual diagram $D_1 \cup \cdots \cup D_m$. Put labels $a_{i1}, a_{i2}, \ldots, a_{ir(i)}$ in order on all arcs of the $i$th component $D_i$ while we go along orientation on $D_i$ from the initial arc, where $r(i)$ denotes the number of arcs of $D_i$ ($i = 1, \ldots, m$). We call the arc $a_{i1}$ of $D_i$ the $i$th meridian. The Wirtinger presentation of $G(\sigma)$ has the form

$$\langle a_{ij} \ (1 \leq i \leq m, 1 \leq j \leq r(i)) \mid a_{ij+1}^{-1} u_{ij}^{-1} a_{ij} u_{ij} (1 \leq i \leq m, 1 \leq j \leq r(i) - 1) \rangle,$$

where the $u_{ij}$ are generators or inverses of generators that depend on the signs of the classical crossings. Here we set

$$v_{ij} = u_{i1} u_{i2} \cdots u_{ij}.$$

We call the product $v_{ir(i)-1}$ an $i$th longitude. Furthermore, we obtain the preferred longitude $l_i$ by multiplying $v_{ir(i)-1}$ by $a_{i1}^s$ on the left for some $s \in \mathbb{Z}$.
Let $G(\sigma)_q$ denote the $q$th term of the lower central series of $G(\sigma)$, and let $\alpha_i$ denote the image of $a_{\alpha i}$ in the quotient $G(\sigma)/G(\sigma)_q$. Since $G(\sigma)/G(\sigma)_q$ is generated by $\alpha_1, \ldots, \alpha_m$ (see [7, 8]), the $i$th preferred longitude $l_i$ is expressed modulo $G(\sigma)_q$ as a word in $\alpha_1, \ldots, \alpha_m$ for each $i \in \{1, \ldots, m\}$. We denote this word by $\lambda_i$.

Let $(\alpha_1, \ldots, \alpha_m)$ denote the free group on $(\alpha_1, \ldots, \alpha_m)$, and let $Z(\langle X_1, \ldots, X_m \rangle)$ denote the ring of formal power series in noncommutative variables $X_1, \ldots, X_m$ with integer coefficients. The Magnus expansion is a homomorphism

$$E : \langle \alpha_1, \ldots, \alpha_m \rangle \longrightarrow Z(\langle X_1, \ldots, X_m \rangle)$$

defined by, for $1 \leq i \leq m$,

$$E(\alpha_i) = 1 + X_i, \quad E(\alpha_i^{-1}) = 1 - X_i + X_i^2 - X_i^3 + \cdots.$$ 

**Definition 2.1.** For a sequence $I = j_1j_2\ldots j_k$ of elements in $\{1, \ldots, m\}$, the **welded Milnor invariant** $\mu_w^w(I)$ of $\sigma$ is the coefficient of $X_{j_1} \cdots X_{j_k}$ in the Magnus expansion $E(\lambda_i)$.

**Remark 2.2 ([10, Theorem 5.4]).** The $\mu_w$-invariant is indeed a welded extension of the (classical) Milnor $\mu$-invariant in the sense that if $\sigma$ is a classical string link, then $\mu_w^w(I) = \mu_x(I)$ for any sequence $I$.

To compute $\mu_w^w(I)$, we need to obtain the word $\lambda_i$ in $\alpha_1, \ldots, \alpha_m$ concretely. In [20], Milnor introduced an algorithm to give $\lambda_i$ by using the Wirtinger presentation of $G(\sigma)$ and a sequence of homomorphisms $\eta_q$ (Although this algorithm was actually given for Milnor invariants of links, it can be applied to those of welded string links.)

Let $\overline{A}$ denote the free group on the Wirtinger generators $\{a_{ij}\}$, and let $A$ denote the free subgroup generated by $a_{11}, a_{21}, \ldots, a_{m1}$. A sequence of homomorphisms $\eta_q : \overline{A} \rightarrow A$ is defined inductively by

$$\eta_1(a_{ij}) = a_{ij}, \quad \eta_{q+1}(a_{i1}) = a_{i1}, \quad \eta_{q+1}(a_{ij+1}) = \eta_q(v_{ij}^{-1}a_{i1}v_{ij}).$$

Let $\overline{A}_q$ denote the $q$th term of the lower central series of $\overline{A}$, and let $N$ denote the normal subgroup of $\overline{A}$ generated by the Wirtinger relations $\{a_{i1}^{-1}1_{ij}^{-1}a_{i1}1_{ij}\}$. Milnor proved that

$$\eta_q(a_{ij}) = a_{ij} \pmod{\overline{A}_qN}.$$ 

Hence, by the congruence above, we can identify $\phi \circ \eta_q(l_i)$ with $\lambda_i$, where $\phi : A \rightarrow \langle \alpha_1, \ldots, \alpha_m \rangle$ is a homomorphism defined by $\phi(a_{i1}) = \alpha_i$ ($i = 1, \ldots, m$).

### 3. Welded Milnor Invariants and $V^n$-Moves

In this section, we discuss the invariance of welded Milnor invariants under $V^n$-moves. We start with the following theorem concerning $\mu_w$-invariants for non-repeated sequences.

**Theorem 3.1.** Let $n$ be a positive integer. If two welded string links $\sigma$ and $\sigma'$ are $(V^n$+sv)-equivalent, then $\mu_w^w(I) \equiv \mu_w^w(I) \pmod{n}$ for any non-repeated sequence $I$.

**Proof.** It is obvious for $n = 1$, and hence we consider the case $n \geq 2$. Since $\mu_w$-invariants for non-repeated sequences are sv-equivalence invariants, we show that their residue classes modulo $n$ are preserved under $V^n$-moves.

Let $D$ and $D'$ be virtual diagrams of $m$-component welded string links $\sigma$ and $\sigma'$, respectively. Assume that $D$ and $D'$ are related by a single $V^n$-move in a disk $\Delta$; see Figure [6.1]. Since a $V^n$-move involving two strands of a single component is realized by sv-equivalence, we may assume that two strands in the disk $\Delta$ belong to different components. Put labels $a_{ij}$ ($1 \leq i \leq m$, $1 \leq j \leq r(i)$) on all arcs of $D$ as
described in Section 2.2, and put labels \(a'_{ij} \) on all arcs in \(D' \setminus \Delta \) which correspond to the arcs labeled \(a_{ij} \) in \(D \setminus \Delta \). Also put labels \(b'_1, \ldots, b'_n \) on the arcs of \(D' \) in \(\Delta \) as illustrated in Figure 3.1. Let \(\mathbb{F} \) be the free group on \(\{a'_{ij}\} \cup \{b'_1, \ldots, b'_n\} \) and \(A' \) the free subgroup on \(\{a'_{11}, a'_{21}, \ldots, a'_{m1}\} \). Let \(\eta'_q : \mathbb{F} \to A' \) denote the sequence of homomorphisms associated with \(D' \) given in Section 2.2 and define a homomorphism \(\phi' : A' \to \langle \alpha_1, \ldots, \alpha_m \rangle \) by \(\phi'(a'_{ij}) = \alpha_i \) \((i = 1, \ldots, m)\).

\[\begin{array}{c}
D : \quad \Delta \\
\cdots \quad a_{kl-1} \quad \cdots \quad a_{kl} \quad \cdots \\
\cdots \quad a'_{kl-1} \quad \cdots \quad a'_{kl} \quad \cdots \\
\end{array}\]

\[\begin{array}{c}
D' : \quad \Delta \\
\cdots \quad a'_{gh-1} \quad \cdots \quad a'_{gh} \quad \cdots \\
\cdots \quad b'_1 \quad \cdots \quad b'_2 \quad \cdots \\
\cdots \quad a'_{gh} \quad \cdots \quad a'_{gh+1} \quad \cdots \\
\end{array}\]

\[\begin{array}{c}
\cdots \quad V''\text{-move} \\
\end{array}\]

**Figure 3.1.** \(D \) and \(D' \) are related by a single \(V''\)-move.

Here, for \(P, Q \in \mathbb{Z}[(X_1, \ldots, X_m)] \), we use the notation \(P \overset{(n)}{=} Q \) if \(P - Q \) is contained in the ideal generated by \(n\). For the \(i\)th preferred longitudes \(l_i \) and \(l'_i \) associated with \(D \) and \(D' \), respectively, it is enough to show that

\[(3.1) \quad E(\phi \circ \eta_q(l_i) \overset{(n)}{=} E(\phi'(l'_i)) + O(2)\]

for any \(1 \leq i \leq m \), where \(O(2)\) denotes \(0\) or the terms containing \(X_r \) at least two for some \(r (= 1, \ldots, m) \). Without loss of generality we may assume that \(i = 1 \), i.e. we compare \(l_1 = a_{11}^t v_{t1(1)-1} \) and \(l'_1 = a_{11}^{t'} v_{t'(1)-1} \) \((s, t \in \mathbb{Z}) \). Recall that two strands in \(\Delta \) belong to different components. This implies that \(s = t \).

If \(g \neq 1 \) in Figure 3.1 then \(l'_1 \) is obtained from \(l_1 \) by replacing \(u_{1j} \) with \(u'_{1j} \) \((j = 1, \ldots, r(1)-1) \) and \(a_{11} \) with \(a_{11}' \). If \(g = 1 \) in Figure 3.1 then \(l_1 \) and \(l'_1 \) can be written respectively in the forms

\[l_1 = a_{11}^t u_{11} \cdots u_{1h-1} u_{1h} \cdots u_{1r(1)-1}\]

and

\[l'_1 = a_{11}' u_{11} \cdots u_{1h-1} a_{kl}^t u_{1h} \cdots u_{1r(1)-1} .\]

Therefore, in both cases, Congruence 3.1 follows from the claim below. 

**Claim 3.2.** Let \(n \geq 2 \) be an integer and \(\varepsilon \in \{1, -1\} \). For any \(1 \leq i \leq m \) and \(1 \leq j \leq r(i) \), the following (1) and (2) hold:

1. \(E(\phi' \circ \eta'_q(a_{ij}^{\varepsilon n})) \overset{(n)}{=} 1 + O(2)\).
2. \(E(\phi' \circ \eta'_q(a_{ij})) \overset{(n)}{=} E(\phi'(a'_{ij})) + O(2)\).

**Proof.** By the definition of \(\eta'_q \), it follows that \(\phi' \circ \eta'_q(a_{ij}^{\varepsilon n}) = w^{-1} \alpha_i^\varepsilon w \) for some word \(w \) in \(\alpha_1, \ldots, \alpha_m \). Set \(E(w) = 1 + W \) and \(E(w^{-1}) = 1 + \overline{W} \), where \(W \) and \(\overline{W} \) denote
the terms of degree $\geq 1$ such that $(1 + W)(1 + W) = 1$. Then we have

\[ E \left( \phi' \circ \eta_q \left( a_{ij}^{w_1} \right) \right) = E \left( w^{-1}a_{ij}^{w_1} \right) \]

\[ = \left(1 + W\right)(1 + \varepsilon X_1)(1 + W) + O(2) \]

\[ = 1 + \varepsilon X_1 + \varepsilon X_1 W + \varepsilon W X_1 + \varepsilon W X_1 W + O(2) \]

\[ = 1 + \varepsilon P(X_1) + O(2), \]

where $P(X_1) = X_1 + X_1 W + \varepsilon W X_1 + \varepsilon W X_1 W$. Note that each term in $P(X_1)$ contains $X_1$. Therefore, it follows that

\[ E \left( \phi' \circ \eta_q \left( a_{ij}^{w_1} \right) \right) = \left(1 + \varepsilon P(X_1) + O(2)\right)^n = 1 + \varepsilon n P(X_1) + O(2) \equiv 1 + O(2). \]

This completes the proof of Claim 3.2 (1).

The proof of Claim 3.2 (2) is done by induction on $q$. The assertion certainly holds for $q = 1$. Recall that

\[ \phi \circ \eta_{q+1} \left( a_{ij+1} \right) = \phi \circ \eta_q \left( a_{ij}^{w_1} v_{ij} \right) \]

and

\[ \phi' \circ \eta_{q+1} \left( a_{ij+1}' \right) = \phi' \circ \eta_q \left( a_{ij}^{w_1} v_{ij}' \right). \]

If $v_{ij}$ does not pass through $\Delta$ or $g \neq i$ in Figure 3.3 then $v_{ij}'$ is obtained from $v_{ij}$ by replacing $a_{ij}$ with $a_{ij}'$. Hence, $E \left( \phi \circ \eta_q \left( v_{ij} \right) \right) \equiv E \left( \phi' \circ \eta_q \left( v_{ij}' \right) \right) + O(2)$ by the induction hypothesis. This implies that

\[ E \left( \phi \circ \eta_{q+1} \left( a_{ij+1} \right) \right) \equiv E \left( \phi \circ \eta_q \left( a_{ij}^{w_1} v_{ij} \right) \right) \]

\[ \equiv E \left( \phi' \circ \eta_q \left( a_{ij}^{w_1} v_{ij}' \right) \right) + O(2) \]

\[ = E \left( \phi' \circ \eta_{q+1} \left( a_{ij+1}' \right) \right) + O(2). \]

If $v_{ij}$ passes through $\Delta$ and $g = i$ in Figure 3.3 then $v_{ij}$ and $v_{ij}'$ can be written respectively in the forms

\[ v_{ij} = u_{ij} \ldots u_k \ldots u_{ij} \]

and

\[ v_{ij}' = u_{ij}' \ldots u_{ij-k} a_{kl} u_{ij-k} \ldots u_{ij}'. \]

By Claim 3.2 (1) and the induction hypothesis, it follows that $E \left( \phi \circ \eta_q \left( v_{ij} \right) \right) \equiv E \left( \phi' \circ \eta_q \left( v_{ij}' \right) \right) + O(2)$. This completes the proof of Claim 3.2 (2). \qed

**Proposition 3.3.** Let $n$ be a positive integer. If two $m$-component welded string links $\sigma$ and $\sigma'$ are $(2n + sv)$-equivalent, then $\mu^w_n(I) \equiv \mu^w_n(I) \pmod{n}$ for any non-repeated sequence $I$, and $\mu^w_n(ij) - \mu^w_n(ij) = \mu^w_n(ij) - \mu^w_n(ij)$ for any $1 \leq i < j \leq m$.

**Proof.** As mentioned in Section 3 (2n+sv)-equivalence implies $(V^n+sv)$-equivalence (Proposition 3.3.1). This together with Theorem 3.3.1 implies that the residue class of $\mu^w_n(I)$ modulo $n$ is preserved under $(2n + sv)$-equivalence.

By a single $2n$-move involving two strands of the $k$th and the $l$th components, both of the changes of $\mu^w_n(ij)$ and $\mu^w_n(ji)$ are $\varepsilon n$ ($\varepsilon \in \{1, -1\}$) if $\{k, l\} = \{i, j\}$ and 0 otherwise. Furthermore, since $\mu^w_n(ij)$ and $\mu^w_n(ji)$ are sv-equivalence invariants, the integer $\mu^w_n(ij) - \mu^w_n(ji)$ is preserved under $(2n + sv)$-equivalence. This completes the proof. \qed

For $\mu^w$-invariants possibly with repeated sequences, we have the following.

**Proposition 3.4.** Let $p$ be a prime number. If two welded string links $\sigma$ and $\sigma'$ are related by $V^p$-moves, then $\mu^w_n(I) \equiv \mu^w_n(I) \pmod{p}$ for any sequence $I$ of length $\leq p$. 

Proof. Let $D$ and $D'$ be virtual diagrams of $m$-component welded string links $\sigma$ and $\sigma'$, respectively. Assume that $D$ and $D'$ are related by a single $V^p$-move. We use the same notation as in the proof of Theorem 3.1. It is enough to show that, for any $1 \leq i \leq m$,

$$E(\phi \circ \eta_q(l_i)) \equiv E(\phi' \circ \eta'_q(l'_i)) + \text{(terms of degree} \geq p).$$

By arguments similar to those in the proof of Theorem 3.1, $l'_i$ is obtained from $l_i$ by replacing $a_{kl}$ with $a'_{kl}$ for all $k, l$ and inserting the $p$th powers of elements in the free group $\mathcal{F}$. The following claim, which was proved in [22], completes the proof. □

Claim 3.5 ([22, Claim 3.6]). (1) For any word $w$ in $\alpha_1, \ldots, \alpha_m$, we have

$$E(w^p) \equiv 1 + \text{(terms of degree} \geq p).$$

(2) For any $1 \leq i \leq m$ and $1 \leq j \leq r(i)$, we have

$$E(\phi \circ \eta_q(a_{ij})) \equiv E(\phi' \circ \eta'_q(a'_{ij})) + \text{(terms of degree} \geq p).$$

4. Arrow calculus

To show Theorems 1.2 and 1.4 we will use arrow calculus, introduced by Meilhan and the third author in [18], which is a welded version of the theory of clasps [13]. In this section, we briefly recall the basic notions of arrow calculus from [18].

4.1. Definitions.

Definition 4.1. Let $D$ be a virtual string link diagram. An immersed connected uni-trivalent tree $T$ in the plane of the diagram is called a $w$-tree for $D$ if it satisfies the following:

1. The trivalent vertices of $T$ are pairwise disjoint and disjoint from $D$.
2. The univalent vertices of $T$ are pairwise disjoint and are contained in $D \setminus \{\text{crossings of } D\}$.
3. All edges are oriented such that each trivalent vertex has two ingoing and one outgoing edge.
4. All singularities of $T$ and those between $D$ and $T$ are virtual crossings.
5. Each edge of $T$ has a number (possibly zero) of decorations •, called twists, which are disjoint from all vertices and crossings.

The univalent vertices of $T$ with outgoing edges are called tails, and the unique univalent vertex of $T$ with an ingoing edge is called the head. Tails and the head are also called endpoints when we do not need to distinguish between them. The terminal edge of $T$ is the edge which is incident to the head. We say that $T$ is a $w$-tree of degree $k$ or $w_k$-tree if $T$ has $k$ tails. In particular, a $w_1$-tree is called a $w$-arrow.

Given a uni-trivalent tree, picking a univalent vertex as the head uniquely determines an orientation on all edges respecting the above rule. Hence, we may only indicate the orientation on $w$-trees at the terminal edge.

For a union of $w$-trees, vertices are assumed to be pairwise disjoint, and crossings among edges are assumed to be virtual. Hereafter, diagrams are drawn with bold lines, while $w$-trees are drawn with thin lines. Furthermore, we do not draw small circles around virtual crossings between $w$-trees and between $w$-trees and diagrams, while we keep small circles between diagrams.
4.2. Surgery along w-trees. The w-trees are equipped with surgery operations on virtual diagrams. This subsection gives the definition of surgery along w-trees.

We first consider the case of w-arrows. Let $A$ be a union of w-arrows for a virtual string link diagram $D$. Surgery along $A$ on $D$ yields a new virtual string link diagram, denoted by $D_A$, as follows. Assume that there exists a disk in the plane which intersects $D \cup A$ as illustrated in Figure 4.1. Then the figure indicates the result of surgery along a w-arrow of $A$ on $D$. We emphasize that the surgery operation depends on the orientation of the strand of $D$ containing the tail of the w-arrow.

![Figure 4.1. Surgery along a w-arrow of $A$ on $D$](image)

If a w-arrow of $A$ intersects a (possibly the same) w-arrow (resp. $D$), then the result of surgery is essentially the same as above but each intersection introduces virtual crossings illustrated in the left-hand side (resp. center) of Figure 4.2. Furthermore, if a w-arrow of $A$ has some twists, then each twist is converted to a half-twist whose crossing is virtual; see the right-hand side of Figure 4.2.

![Figure 4.2.](image)

An arrow presentation for a virtual string link diagram $D$ is a pair $(V, A)$ of a virtual string link diagram $V$ without classical crossings and a union $A$ of w-arrows for $V$ such that $V_A$ is welded isotopic to $D$. Any virtual string link diagram has an arrow presentation because any classical crossing can be replaced by a virtual one with a w-arrow; see Figure 4.3. Two arrow presentations $(V, A)$ and $(V', A')$ are equivalent if $V_A$ and $V'_A$ are welded isotopic. In [18, Section 4.3], Meilhan and the third author gave a list of local moves on arrow presentations, which are called arrow moves. They proved that two arrow presentations are equivalent if and only if they are related by a sequence of arrow moves [18, Theorem 4.5].

![Figure 4.3. Any classical crossing can be replaced by a virtual one with a w-arrow.](image)

Now we define surgery along w-trees. We start with some preliminary definitions. A subtree of a w-tree is a connected union of edges and vertices of the w-tree. Let $S$ be a subtree of a w-tree $T$ for a virtual string link diagram $D$ (possibly $T$ itself). For each endpoint $e$ of $S$, consider a point $e'$ on $D$ which is adjacent to $e$ such that we meet $e$ and $e'$ consecutively in this order when going along orientation on $D$. Joining these new points by a copy of $S$, we can form a new subtree $S'$ such that
$S$ and $S'$ run parallel and cross only at virtual crossings. Then $S$ and $S'$ are called two parallel subtrees.

The expansion move $(E)$ for a $w_k$-tree, having two variations, produces four $w$-trees of degree $\leq k - 1$ illustrated in Figure 4.4. In the figure, the dotted lines on the left-hand side of $(E)$ represent two subtrees, which form the $w_k$-tree together with represented part. The dotted parts on the right-hand side represent parallel copies of both subtrees.

![Figure 4.4. Expansion move](image)

Applying $(E)$ recursively, we can turn any $w$-tree into a union of $w$-arrows. We call the union of $w$-arrows the expansion of the $w$-tree. The surgery along a $w$-tree is surgery along its expansion. As before, $D_T$ denotes the result of surgery on $D$ along a union $T$ of $w$-trees.

As a natural generalization of arrow presentations, a $w$-tree presentation for a virtual string link diagram $D$ is defined as a pair $(V, T)$ of a virtual string link diagram $V$ without classical crossings and a union $T$ of $w$-trees for $V$ such that $V_T$ is welded isotopic to $D$. Two $w$-tree presentations $(V, T)$ and $(V', T')$ are equivalent if $V_T$ and $V'_T$ are welded isotopic. Then arrow moves are extended to a set of local moves on $w$-tree presentations, which are called $w$-tree moves. It is proved that two $w$-tree presentations are equivalent if and only if they are related by a sequence of $w$-tree moves [18, Theorem 5.21].

In Section 5, we will use three kinds of $w$-tree moves, inverse, tails exchange and heads exchange moves illustrated in Figure 4.5. The inverse move yields or deletes two parallel $w$-trees which only differ by a twist on the terminal edge, the tails exchange move makes an exchange of two consecutive tails of $w$-trees, and the heads exchange move makes an exchange of two consecutive heads of $w$-trees at the expense of an additional $w$-tree illustrated in the lower right of Figure 4.5.

![Figure 4.5.](image)
4.3. **w-tree moves up to sv-equivalence.** To study virtual diagrams up to welded isotopy, we can work on w-tree presentations together with w-tree moves. Considering virtual diagrams up to sv-equivalence, we can use some additional moves on w-tree presentations. In this subsection, we recall these moves from [18, Section 9.1]. (Note that sv-equivalence is called homotopy in [18].)

A self-arrow is a w-arrow whose tail and head are attached to a single component of a virtual diagram. More generally, a repeated w-tree is a w-tree having two endpoints attached to a single component of a virtual diagram. Clearly, adding or deleting a self-arrow on w-tree presentations corresponds to a self-crossing virtualization on virtual diagrams, i.e. surgery along a self-arrow does not change the sv-equivalence class of a virtual diagram. This holds also for repeated w-trees.

**Lemma 4.2** ([18, Lemma 9.2]). Surgery along a repeated w-tree does not change the sv-equivalence class of a virtual diagram.

Exchanging a head and a tail of w-trees of arbitrary degree can be achieved at the expense of an additional w-tree as follows.

**Lemma 4.3** ([18, Lemma 9.3]). Let \( T_1 \) be a \( w_k \)-tree for a virtual diagram \( D \), and let \( T_2 \) be a \( w_l \)-tree for \( D \). Let \( T_1^S \cup T_2^S \) be obtained from \( T_1 \cup T_2 \) by exchanging a tail of \( T_1 \) and the head of \( T_2 \). Then \( D_{T_1^S \cup T_2^S} \) is sv-equivalent to \( D_{T_1 \cup T_2 \cup Y} \), where \( Y \) denotes the \( w_{k+1} \)-tree \( T \) for \( D \) illustrated in Figure 4.6.

**Figure 4.6.** Head-tail exchange move. Here, the notation \( \sim_{sv} \) denotes that the virtual diagrams obtained by surgery along w-trees are sv-equivalent.

The modification of Figure 4.6 is called a head-tail exchange move. Heads, tails and head-tail exchange moves are also referred to as ends exchange moves.

5. **Proofs**

In this section, we give the proofs of Theorems 1.2 and 1.4.

Now we consider three local moves A, B and C on w-tree presentations illustrated in Figures 5.1 and 5.2. Surgery along an A-move is equivalent to a \( 2n \)-move whose strands are oriented parallel. On the other hand, surgery along a B-move is equivalent to a \( V^n \)-move. Furthermore, it is not hard to see that a B-move is equivalent to a C-move; see [21].

Using w-tree presentations, we show the following.

**Proposition 5.1.** Let \( n \) be a positive integer. If two welded string links are \((2n + sv)\)-equivalent, then they are \((V^n + sv)\)-equivalent.

**Proof.** A \( 2n \)-move involving two strands of a single component is realized by link-homotopy\(^1\). Furthermore, as seen in the proof of [22, Theorem 3.1], a \( 2n \)-move whose two strands are oriented antiparallel is realized by link-homotopy and a

\(^1\)The equivalence relation on welded string links generated by the self-crossing change and welded isotopy is also referred to as link-homotopy.
2n-move whose strands are oriented parallel. Since link-homotopy implies sv-equivalence, we may now assume that the orientations of the strands of a 2n-move are always parallel.

Up to (V^n + sv)-equivalence, by Lemmas 4.2 and 4.3 we can use w-tree moves, B-, C-moves and ends exchange moves, and delete repeated w-trees on w-tree presentations. Hence, it is enough to show that an A-move is realized by a sequence of these operations, since a 2n-move is realized by surgery along an A-move. Figure 5.3 indicates the proof. In the sequence of Figure 5.3 (a)–(c), we obtain (b) from (a) by B- and C-moves, and (c) from (b) by head-tail exchange moves and deleting repeated w-trees. □

For each integer i ∈ {1, . . . , m}, let S_k(i) denote the set of all sequences j_1 . . . j_k of k distinct integers in {1, . . . , m} \ {i} such that j_r < j_k for all r (= 1, . . . , k − 1). For I ∈ S_k(i), let T_{I_i} be the w_k-tree for the trivial m-component string link diagram 1_m illustrated in Figure 5.4 and let T_{I_i} be obtained from T_{I_i} by inserting a twist in the terminal edge. Set W_{I_i} = (1_m)_{T_{I_i}} and W_{I_i} = (1_m)_{T_{I_i}}. We remark that W_{I_i} * W_{I_i}^{-1} is welded isotopic to 1_m by applying an inverse move to T_{I_i} ∪ T_{I_i}, where the notation “*” denotes the stacking product. In [18], a complete list of representatives for welded string links up to sv-equivalence was given in terms of w-trees and welded Milnor invariants as follows.
Theorem 5.2 ([8] Theorem 9.4). Let \( \sigma \) be an \( m \)-component welded string link. Then \( \sigma \) is sv-equivalent to \( \sigma_1 \cdots \sigma_{m-1} \), where for each \( k \),

\[
\sigma_k = \prod_{i=1}^{m} \prod_{l \in S_k(i)} (W_{li})^{x_l}, \text{ with } x_l = \begin{cases} \mu_{\sigma_k}^w(j_l) & (k = 1, \, I = j), \\ \mu_{\sigma_k}^w(i_l) - \mu_{\sigma_1 \cdots \sigma_{k-1}}^w(i_l) & (k \geq 2). \end{cases}
\]

The following plays an important role to show Theorems 1.2 and 1.4.

Lemma 5.3. Let \( n \) be a positive integer and \( \varepsilon \in \{1, -1\} \). Then, for any \( I \in S_k(i) \) and \( k \geq 2 \), \( (W_i)^{n} \) is \((2n + sv)\)-equivalent to \( 1_m \).

Proof. Since \( W_i^{n-1} \ast W_i^n \) is welded isotopic to \( 1_m \), it suffices to show the case \( \varepsilon = 1 \), i.e., \( (W_i)^n \) is \((2n + sv)\)-equivalent to \( 1_m \) for a sequence \( I = j_1 \cdots j_k \in S_k(i) \) \((k \geq 2)\). Up to \((2n + sv)\)-equivalence, we can use w-tree moves, A-moves and ends exchange moves, and delete repeated w-trees on w-tree presentations. We relate \( 1_m \) to a w-tree presentation for \( (W_i)^n \) by a sequence of these operations.

Figure 5.4 (a)–(c) describes the intermediates in the sequence between \( 1_m \) and a w-tree presentation for \( (W_i)^n \). In the sequence, we obtain (a) from \( 1_m \) by an inverse move, and (b) from (a) by an A-move. We obtain (c) from (b) by ends exchange moves on the \( j_{k-1} \)-th component of \( 1_m \) and deleting repeated w-trees. Finally, we obtain a w-tree presentation for \( (W_i)^n \) from (c) by an inverse move and an A-move.

Proposition 5.4. Let \( \sigma \) be an \( m \)-component welded string link, and let \( x_l \) be as in Theorem 5.2. Then \( \sigma \) is \((2n + sv)\)-equivalent to \( \tau_1 \cdots \tau_{m-1} \), where

\[
\tau_1 = \prod_{1 \leq i < j \leq m} (W_{ji}^{y_j} \ast W_{ij}^{z_i})
\]

for some \( y_j, z_i \in \mathbb{Z} \) with \( 0 \leq y_j < n \) and \( y_j \equiv x_j \pmod{n} \), and where for each \( k \geq 2 \),

\[
\tau_k = \prod_{i=1}^{m} \prod_{l \in S_k(i)} (W_{li})^{y_l}
\]

with \( 0 \leq y_l < n \) and \( y_l \equiv x_l \pmod{n} \).

Proof. It follows from Theorem 5.2 that \( \sigma \) is sv-equivalent to \( \sigma_1 \cdots \sigma_{m-1} \), where for each \( k \) \((k \geq 1)\),

\[
\sigma_k = \prod_{i=1}^{m} \prod_{l \in S_k(i)} (W_{li})^{x_l}.
\]

For each \( k \geq 2 \), by Lemma 5.3 and the fact that \( W_i^{x} \ast W_i^{-x} \) is welded isotopic to \( 1_m \) \((x \in \{1, -1\})\), \( \sigma_k \) is \((2n + sv)\)-equivalent to \( \tau_k \).
Now consider the case $k = 1$. Performing ends exchange moves and deleting repeated w-trees, a w-tree presentation for $\sigma_1$ is deformed into one for $\sigma_1'$, where

$$\sigma_1' = \prod_{1 \leq i < j \leq m} (W_{ji}^{x_j} \cdot W_{ij}^{x_i}).$$

By Lemmas 4.2 and 4.3, $\sigma_1$ is sv-equivalent to $\sigma_1'$. Furthermore, a w-tree presentation for $\sigma_1'$ is deformed into one for $\tau_1$ by using A-moves, ends exchange moves and inverse moves. Therefore, $\sigma_1'$ is $(2n + sv)$-equivalent to $\tau_1$. This completes the proof.

**Proposition 5.5.** Let $\sigma$ be an $m$-component welded string link, and let $x_I$ be as in Theorem 5.2. Then $\sigma$ is $(V^n + sv)$-equivalent to $\tau_1 \cdots \tau_{m-1}$, where for each $k$,

$$\tau_k = \prod_{i=1}^{m} \prod_{I \in S_k(i)} (W_{II})^{y_I}$$

with $0 \leq y_I < n$ and $y_I \equiv x_I \pmod{n}$.

**Proof.** For any $j \in S_i(i)$, we see that $(W_{ji})^{\pm n}$ and $1_m$ are related by a single $V^n$-move. This together with Proposition 5.1 and Lemma 5.3 implies that $(W_{II})^{\pm n}$ is $(V^n + sv)$-equivalent to $1_m$ for any $I \in S_k(i)$ and $k \geq 1$. Therefore, the proof can be done by arguments similar to those in the proof of Proposition 5.4.

**Proof of Theorem 1.2** This follows from Propositions 3.3 and 5.4.

**Proof of Theorem 1.3** This follows from Theorem 5.1 and Proposition 5.5.
Remark 5.6. By Theorem 5.2 (resp. Theorem 5.4), we can conclude that Proposition 5.5 (resp. Proposition 5.6) gives a complete list of representatives for welded string links up to $(2n + sv)$-equivalence (resp. $(V^n + sv)$-equivalence).

Proof of Corollary 5.5 This follows from Remark 5.6 and Lemma 5.3.

In the rest of this paper, we discuss the relations between $(2n + lh)$-, $(2n + sv)$- and $(V^n + sv)$-equivalence.

It is not hard to see that $(2n + lh)$-equivalence implies $(2n + sv)$-equivalence. Furthermore, $(2n + sv)$-equivalence implies $(V^n + sv)$-equivalence by Proposition 5.1.

The converse implications hold for classical string links as follows.

Proposition 5.7. Let $n$ be a positive integer, and let $\sigma$ and $\sigma'$ be classical string links. The following assertions (1), (2) and (3) are equivalent:

1. $\sigma$ and $\sigma'$ are $(2n + lh)$-equivalent.
2. $\sigma$ and $\sigma'$ are $(2n + sv)$-equivalent.
3. $\sigma$ and $\sigma'$ are $(V^n + sv)$-equivalent.

Proof. It is enough to show the implications $(2) \Rightarrow (1)$ and $(3) \Rightarrow (1)$.

The proof of the implication $(2) \Rightarrow (1)$ is given as follows. If $\sigma$ and $\sigma'$ are $(2n + sv)$-equivalent, then $\mu^w_\sigma(I) \equiv \mu^w_\sigma(I) \pmod{n}$ for any non-repeated sequence $I$ by Theorem 5.2. It follows from Remark 5.2 that $\mu^w_\sigma(I) = \mu^w_\sigma(I)$ and $\mu^w_\sigma(I) = \mu^w_\sigma(I)$. Hence, Theorem 5.3 completes the proof.

Using Theorem 5.4 instead of Theorem 5.2, the implication $(3) \Rightarrow (1)$ is similarly shown.

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