Classification of Bicovariant Differential Calculi on the Quantum Groups $SL_q(n+1)$ and $Sp_q(2n)$

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Abstract

For transcendental values of $q$ all bicovariant first order differential calculi on the coordinate Hopf algebras of the quantum groups $SL_q(n+1)$ and $Sp_q(2n)$ are classified. It is shown that the irreducible bicovariant first order calculi are determined by an irreducible corepresentation of the quantum group and a complex number $ζ$ such that $ζ^{n+1} = 1$ for $SL_q(n+1)$ and $ζ^2 = 1$ for $Sp_q(2n)$. Any bicovariant calculus is inner and its quantum Lie algebra is generated by a central element. The main technical ingredient is a result of the Hopf algebra $R(G_q)^*$ for arbitrary simple Lie algebras.

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1 Introduction

The study of non-commutative geometry on quantum groups requires to select a first order differential calculus (that is, a FODC) on the corresponding coordinate Hopf algebra. It is natural to assume that this differential calculus is compatible with left and right translations of the quantum group. Such calculi are called bicovariant (see Section 2.1 for the precise definition) and have been invented by S. L. Woronowicz in his seminal paper [18]. In the meantime there exists a well developed general theory of such calculi (see, for instance, [11], Chapter 14).

Whereas there exists a unique distinguished "classical" differential calculus on any $C^\infty$-manifold, no functorial method is known in order to get a canonical differential calculus on a quantum group. Thus the problem of classifying all possible finite dimensional bicovariant FODC on a Hopf algebra arises. This problem was investigated under various additional assumptions in [16,17,14] and [2]. The present paper gives a complete solution of the classification problem for the coordinate Hopf algebras $O(G_q)$ of the quantum groups $G_q = SL_q(n+1), Sp_q(2n)$ when the parameter $q$ is transcendental. Our main result (Theorem 4.1) says that up to forming direct sums all finite dimensional bicovariant FODC on $O(SL_q(n+1))$ and $O(Sp_q(2n))$ can be obtained by the same general method. In special examples, this method was first used in [4] and [10]. It was extended to arbitrary coquasitriangular Hopf algebras in [11], 14.5-6, where the structure of such calculi is developed in detail. More precisely, any irreducible finite dimensional bicovariant FODC is parametrized by a pair $(v, ζ)$ of an irreducible matrix corepresentation $v = (v_{ij})_{i,j=1,...,k}$ of $G_q = SL_q(n+1), Sp_q(2n)$ and a complex number $ζ$ such that $ζ^{n+1} = 1$ for $SL_q(n+1)$ and $ζ^2 = 1$ for $Sp_q(2n)$. The dimension

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of the corresponding FODC is $k^2$. A remarkable byproduct of the classification theorem is that any finite dimensional bicovariant FODC is inner, that is, there is a biinvariant one form $\theta$ such that $da = \theta a - a \theta$ for all $a \in \mathcal{O}(G_q)$. Another important corollary is that the quantum Lie algebra of any such calculus can be obtained by means of a central element of the Hopf dual $\mathcal{O}(G_q)^*$ (see 2.4).

The proof of our main classification theorem 3.3 relies essentially on two deep algebraic results. The first one is a theorem due to A. Joseph and G. Letzter 3.1 on the locally finite part for the left adjoint action of the corresponding quantized universal enveloping algebra $\mathcal{U}_q(g)$ (see 3.3) and the second is a result of A. Joseph 3.1 on the description of the Hopf dual of $\mathcal{R}(G_q)$ (see 3.3).

This paper is organized as follows. Section 2 contains general notions and facts on bicovariant FODC which are needed later. Proposition 2.2 relates the bicovariant FODC on a Hopf algebra $\mathcal{A}$ to $\text{ad}_R$-invariant right coideals of the Hopf dual $\mathcal{A}^\circ$. Moreover, the general method for the construction of such calculi from 1.1 is briefly recalled. In Section 3 for an arbitrary simple Lie algebra the dual of the Hopf algebra $\mathcal{R}(G_q)$ (see Proposition 3.1) and its finite dimensional $\text{ad}_R$-invariant coideals (see Theorem 3.3) are described when $q$ is transcendental. The latter result is of interest in itself and it provides a classification of all quantum Lie algebras of finite dimensional bicovariant differential calculi on the Hopf algebra $\mathcal{R}(G_q)$. In Section 4 the main results of this paper (Theorem 4.1 and Corollaries 4.2–4.4) on the classification problem for the Hopf algebras $\mathcal{O}(SL_n(n+1))$ and $\mathcal{O}(Sp_q(2n))$ are stated and proved. It is shown therein that all finite dimensional bicovariant calculi can be given by the general methods developed in 2.3 and 2.4. As a byproduct we prove that the coquasitriangular Hopf algebras $\mathcal{O}(SL_q(n+1))$ and $\mathcal{O}(Sp_q(2n))$ are factorizable.

In the formulation given in 4.1 the classification results are no longer valid for the quantum groups $\mathcal{O}(O_q(N))$. For instance, the two four dimensional irreducible bicovariant FODC on the Hopf algebra $\mathcal{O}(O_q(3))$ constructed in 1.7 are not of the form $\Gamma_\xi(v)$ for some corepresentation $v$ of $O_q(3)$. The reason for this failure lies in the fact that the tensor product $\mathcal{O}(O_q(N)) \otimes \mathbb{C}Z_2^n$ is not the full Hopf dual of $\mathbb{C}Z_2^n \cong \mathcal{U}_q(so_N)$, because it does not contain the matrix coefficients of the spinor representations. However, large parts of the proofs given in this paper carry over with some modifications to the orthogonal case. In fact, if the coordinate Hopf algebra $\mathcal{O}(O_q(N))$ is replaced by the larger Hopf algebra generated by the matrix coefficients of the spinor representations, then the results in 4.3 are also valid for this Hopf algebra.

We close this introduction by fixing some assumptions and notations that are used in the sequel. All algebras and Lie algebras are over the complex field. We shall use the convention to sum over repeated indices belonging to different terms. The symbol $\varepsilon$ marks the end of a proof, an example or a remark. Let $\mathcal{A}$ be a Hopf algebra. The comultiplication, the counit and the antipode of $\mathcal{A}$ are denoted by $\Delta$, $\varepsilon$ and $S$, respectively. Throughout we shall use the Sweedler notation in the form $\Delta(a) = a_{(1)} \otimes a_{(2)}$. By a right coideal of $\mathcal{A}$ we mean a linear subspace $\mathcal{C}$ of $\mathcal{A}$ such that $\Delta(\mathcal{C}) \subseteq \mathcal{C} \otimes \mathcal{A}$. As usual, $\mathcal{A}^\circ$ denotes the Hopf dual of $\mathcal{A}$. For $a, b \in \mathcal{A}$, let $\text{ad}_R(a)b = S(a_{(1)})ba_{(2)}$. Then the map $a \mapsto \text{ad}_R(a)$ is a representation of the algebra $\mathcal{A}$ on itself, called the right adjoint action of $\mathcal{A}$. The map $\text{Ad}_R : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ defined by $\text{Ad}_R(a) = a_{(2)} \otimes S(a_{(1)})a_{(3)}$, $a \in \mathcal{A}$, is the right adjoint coaction of $\mathcal{A}$. A matrix $v = (v^i_j)_{i,j=1,...,m}$ of elements $v^i_j \in \mathcal{A}$ is called a matrix corepresentation of $\mathcal{A}$ if $\varepsilon(v^i_j) = \delta_{ij}$ and $\Delta(v^i_j) = v^i_k \otimes v^k_j$ for $i, j = 1, \ldots, m$. For a corepresentation $v$ of $\mathcal{A}$, let $\mathcal{C}(v)$ denote the subcoalgebra of $\mathcal{A}$ spanned by the matrix elements of $v$.

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2 Construction of bicovariant first order differential calculi

2.1 Bicovariant first order differential calculi: basic concepts

We shall use the general framework of bicovariant differential calculi developed by S. L. Woronowicz [13], see also [11], Chapter 14. In this subsection we collect the main notions and facts needed in what follows. Suppose that $A$ is a Hopf algebra.

A first order differential calculus (abbreviated, a FODC) over $A$ is an $A$-bimodule $\Gamma$ equipped with a linear mapping $d : A \to \Gamma$, called the differentiation of the FODC, such that:

(i) $d$ satisfies the Leibniz rule $d(ab) = a \cdot d b + d a \cdot b$ for any $a, b \in A$,

(ii) $\Gamma$ is the linear span of elements $a \cdot d b \cdot c$ with $a, b, c \in A$.

A FODC $\Gamma$ over $A$ is called bicovariant if there exist linear mappings $\Delta_L : \Gamma \to A \otimes \Gamma$ and $\Delta_R : \Gamma \to \Gamma \otimes A$ such that

$$\Delta_L(ab) = 1 \otimes \Delta_L(b) \quad \text{and} \quad \Delta_R(ab) = \Delta_R(a)(d \otimes \text{id}) \Delta_R(b) \quad \text{for all} \ a, b \in A. \quad (1)$$

A bicovariant FODC $\Gamma$ is called inner if there exists a biinvariant one form $\theta \in \Gamma$ (that is, $\Delta_L(\theta) = 1 \otimes \theta$ and $\Delta_R(\theta) = \theta \otimes 1$) such that

$$da = \theta a - a \theta, \quad a \in A. \quad (2)$$

By the dimension of a bicovariant FODC we mean the dimension of the vector space $\text{inv}\Gamma = \{ \omega \in \Gamma | \Delta_L(\omega) = 1 \otimes \omega \}$ of left invariant one forms. All bicovariant FODC occurring in this paper are assumed to be finite dimensional.

Let $\Gamma$ be a bicovariant FODC over $A$. Then the set

$$R_{\Gamma} := \{ a \in \ker \varepsilon \subset A | S(a(1))da(2) = 0 \} \quad (3)$$

is an $\text{Ad}_R$-invariant right ideal of $\ker \varepsilon$. Conversely, for any $\text{Ad}_R$-invariant right ideal $R$ of $\ker \varepsilon$ there exists a bicovariant FODC $\Gamma$ such that $R = R_{\Gamma}$ ([11], Proposition 14.7). The linear subspace

$$X_{\Gamma} := \{ X \in A' | X(1) = 0 \text{ and } X(a) = 0 \text{ for } a \in R_{\Gamma} \} \quad (4)$$

is called the quantum Lie algebra of $\Gamma$.

**Proposition 2.1.** (i) $X_{\Gamma}$ is an $\text{ad}_R$-invariant vector space of the dual Hopf algebra $A^\circ$ satisfying $\Delta(X) - \varepsilon \otimes X \in X_{\Gamma} \otimes A^\circ$ for all $X \in X_{\Gamma}$.

(ii) If $A^\circ$ separates the elements of $A$, then any finite dimensional $\text{ad}_R$-invariant vector space $X \subset A^\circ$ satisfying $\Delta(X) - \varepsilon \otimes X \in X \otimes A^\circ$ and $X(1) = 0$ for all $X \in X$ is the quantum Lie algebra of a unique bicovariant FODC over $A$.

**Proof.** [11], Corollary 14.10.

For the considerations below the following characterization of quantum Lie algebras of bicovariant FODC will be more convenient. Let us define a projection $P_\varepsilon : A^\circ \to A^\circ$ by $P_\varepsilon(f) := f - f(1)\varepsilon$. Obviously, $P_\varepsilon(A^\circ) = \{ f \in A^\circ | f(1) = 0 \}$. For a linear subspace $X$ of $A^\circ$ we set $\tilde{X} := X + \mathbb{C}\varepsilon$. 

Proposition 2.2. (i) If $\mathcal{X}_\Gamma$ is the quantum Lie algebra of a bicovariant FODC over $A$, then $\mathcal{X}_\Gamma$ is an $\text{ad}_R$-invariant right coideal of $A$. 
(ii) Suppose that $A^\circ$ separates the elements of $A$. If $\mathcal{X}$ is a finite dimensional $\text{ad}_R$-invariant right coideal of $A^\circ$ containing $\varepsilon$, then $\mathcal{X} := P_\varepsilon(\mathcal{X})$ is the quantum Lie algebra of a unique bicovariant FODC over $A$.

Proof. (i) We apply Proposition 2.1(i). Because $\mathcal{X}_\Gamma$ is $\text{ad}_R$-invariant, $\mathcal{X}_\Gamma$ is $\text{ad}_R$-invariant as well. Since $\Delta(\varepsilon) = \varepsilon \otimes \varepsilon$ and $\Delta(X) \in \varepsilon \otimes X + \mathcal{X}_\Gamma \otimes A^\circ \subset \mathcal{X}_\Gamma \otimes A^\circ$ for $X \in \mathcal{X}_\Gamma$, we conclude that $\mathcal{X}_\Gamma$ is a right coideal of $A^\circ$.

(ii) For $Y \in \mathcal{X}$ and $f \in A$, we have $\text{ad}_R(f)(Y - Y(1)\varepsilon) = \text{ad}_R(f)Y - f(1)Y(1)\varepsilon \in \mathcal{X}$ and $\text{ad}_R(f)(Y - Y(1)\varepsilon)(1) = 0$. Hence $\mathcal{X}$ is $\text{ad}_R$-invariant. Further, for $X \in \mathcal{X}$ we compute $\Delta(X) - \varepsilon \otimes X = \Delta(X) - \varepsilon \otimes \mathcal{X}(1)X(2) = (P_\varepsilon \otimes \text{id})(\Delta(X) - \varepsilon \otimes X) \in \mathcal{X} \otimes A^\circ$. Since obviously $X(1) = 0$ for $X \in \mathcal{X}$, the assertion follows from Proposition 2.1(ii).

Suppose that $\mathcal{N}$ is a sub-bimodule of the $A$-bimodule $\Gamma$ such that $\Delta_L(\mathcal{N}) \subseteq A \otimes \mathcal{N}$ and $\Delta_R(\mathcal{N}) \subseteq \mathcal{N} \otimes A$. Let $\pi : \Gamma \rightarrow \Gamma/\mathcal{N}$ denote the canonical map. Then the quotient $A$-bimodule $\Gamma/\mathcal{N}$ endowed with the linear mapping $\tilde{d} := \pi \circ d$ is also a bicovariant FODC over $A$, called a bicovariant quotient FODC of $\Gamma$. We shall say that the bicovariant FODC $\Gamma$ is irreducible if $\Gamma$ has no nontrivial bicovariant quotient FODC, that is, if there is no $A$-sub-bimodule $\mathcal{N} \neq \{0\}$ of $\Gamma$ such that $\Delta_L(\mathcal{N}) \subseteq A \otimes \mathcal{N}$ and $\Delta_R(\mathcal{N}) \subseteq \mathcal{N} \otimes A$. An $\text{ad}_R$-invariant right coideal $\mathcal{X}$ of a Hopf algebra is called irreducible if it does not contain an $\text{ad}_R$-invariant right coideal $\mathcal{Y}$ such that $\mathcal{Y} \neq \{0\}$, $\mathcal{X}$. One easily verifies the following lemma.

Lemma 2.3. For a bicovariant FODC $\Gamma$ the following statements are equivalent:

(i) The FODC $\Gamma$ is irreducible.

(ii) There is no $\text{ad}_R$-invariant right subcoideal $\mathcal{Y} \subset \mathcal{X}_\Gamma$ of $A^\circ$ containing $\varepsilon$ such that $\mathcal{Y} \neq \mathcal{C} = \mathcal{X}_\Gamma$.

(iii) There is no $\text{Ad}_R$-invariant right ideal $\mathcal{I}$ of $A$ such that $R\mathcal{I} \subset \mathcal{I} \subset \ker \varepsilon$, $\mathcal{I} \neq R\mathcal{I}$ and $\mathcal{I} \neq \ker \varepsilon$.

For $i \in \{1, \ldots, k\}$, let $\Gamma_i$ be a FODC over $A$ with differentiation $d_i$. Define a linear mapping $d = d_1 + \cdots + d_k$ of $A$ into the direct sum $\tilde{\Gamma} := \Gamma_1 \oplus \cdots \oplus \Gamma_k$ of $A$-bimodules $\Gamma_1, \ldots, \Gamma_k$. Clearly, $\Gamma := A \cdot d\mathcal{A} \cdot A$ is a FODC over $A$ with differentiation $d$. We call this FODC $\Gamma$ the sum of the FODC $\Gamma_1, \ldots, \Gamma_k$.

Suppose that the FODC $\Gamma_1, \ldots, \Gamma_k$ are bicovariant. Then it is not difficult to show that $\Gamma$ is also bicovariant and that $X_{\Gamma} = X_{\Gamma_1} + \cdots + X_{\Gamma_k}$ and $R\mathcal{I} = R\mathcal{I}_1 \cap \cdots \cap R\mathcal{I}_k$. We shall say that $\Gamma$ is a direct sum of the FODC $\Gamma_1, \ldots, \Gamma_k$ if $\tilde{\Gamma} = \Gamma$. One easily verifies

Lemma 2.4. $\Gamma$ is the direct sum of $\Gamma_1, \ldots, \Gamma_k$ if and only if $X_{\Gamma}$ is the direct sum of quantum Lie algebras $X_{\Gamma_1}, \ldots, X_{\Gamma_k}$ (that is, if $X_{\Gamma_1} \cap X_{\Gamma_j} = \{0\}$ for $i \neq j$, $i, j = 1, \ldots, k$).

Let $\Gamma_1$ and $\Gamma_2$ be two bicovariant FODC over $A$. Suppose that $\Gamma_i$ is a sub-bimodule of a possibly larger $A$-bimodule $\tilde{\Gamma}_i$ which contains a bi-invariant element $\theta_i \in \Gamma_i$ such that $d_i a = \theta_i a - a \theta_i$, $a \in A$ for $i = 1, 2$. Let $d$ be the linear mapping of $A$ into the tensor product $\tilde{\Gamma} = \tilde{\Gamma}_1 \otimes \tilde{\Gamma}_2$ of the two $A$-bimodules $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ defined by

$$da = \theta a - a \theta = d_1 a \otimes \theta_2 + \theta_1 \otimes A d_2 a, \quad a \in A$$

where $\theta := \theta_1 \otimes \theta_2$. It is straightforward to check that $\Gamma := A \cdot d\mathcal{A} \cdot A$ is again a bicovariant FODC over $A$ with differentiation $d$. This FODC $\Gamma$ is called the tensor product of the FODC $\Gamma_1$ and $\Gamma_2$. 

4
2.2 Construction of bicovariant FODC on coquasitriangular Hopf algebras

In this subsection we briefly repeat the general method for the construction of bicovariant FODC (see [1], Sect. 14.5, for a detailed treatment). Let \( \mathcal{A} \) be a coquasitriangular Hopf algebra (see [2], [1] or [3]). That is, \( \mathcal{A} \) is Hopf algebra equipped with a linear functional \( r \) on \( \mathcal{A} \otimes \mathcal{A} \) which is invertible with respect to the convolution multiplication and satisfies the following conditions

\[
\begin{align*}
r(ab \otimes c) &= r(a \otimes c_{(1)})r(b \otimes c_{(2)}), & r(a \otimes bc) &= r(a_{(1)} \otimes c)r(a_{(2)} \otimes b), \\
r(a_{(1)} \otimes b_{(1)})a_{(2)}b_{(2)} &= r(a_{(2)} \otimes b_{(2)})b_{(1)}a_{(1)}
\end{align*}
\]

for \( a, b, c \in \mathcal{A} \). Such a linear form \( r \) is called a universal \( r \)-form of the Hopf algebra \( \mathcal{A} \).

Suppose that \( r \) is a fixed universal \( r \)-form and \( v = (v^i_j)_{i,j=1,\ldots,m} \) is a matrix corepresentation of \( \mathcal{A} \). Let \( \tilde{r} \) denote the convolution inverse of \( r \). We define linear functionals \( l^i_j, l^i_j \in \mathcal{A}^\circ \) by

\[
l^{+i}_j(\cdot) = r(\cdot \otimes v^i_j), \quad l^{-i}_j(\cdot) = \tilde{r}(v^i_j \otimes \cdot), \quad l^i_j = S(l^{-i}_k)l^{+k}_j. \tag{6}\]

The \( m \times m \)-matrices \( L^+ \) and \( L^{-,c} \) with entries \( (L^+)^{i}_j = l^{+i}_j \) and \( (L^{-,c})^{i}_j = S(l^{-i}_j) \) are then representations of the algebra \( \mathcal{A} \) and the complex vector space

\[
\mathcal{X}^c(v) := \text{Lin} \{ l^i_j | i, j = 1, \ldots, m \} \tag{7}
\]

of the functionals \( l^i_j \) is an \( \text{ad}_R \)-invariant right coideal of \( \mathcal{A}^\circ \). Let \( v^c \) be the contragredient matrix corepresentation of \( \mathcal{A} \), that is, \( v^c \) is the \( m \times m \)-matrix with entries \( (v^c)^{i}_j = S(v^i_j) \). Then the pairs \( \Gamma = (v, L^{-,c}) \) and \( \Gamma^c = (v^c, L^+) \) are bicovariant bimodules over \( \mathcal{A} \) (see [1], Sect. 14.5, for details). Hence the tensor product

\[
\tilde{\Gamma} := \Gamma \otimes_{\mathcal{A}} \Gamma^c = (v \otimes v^c, L^{-,c} \otimes L^+) \tag{8}
\]

of the bicovariant bimodules \( \Gamma \) and \( \Gamma^c \) is again a bicovariant bimodule. Let us fix \( \{ \theta_{ij} \}_{i,j=1,\ldots,m} \) be a basis of the vector space \( \text{inv} \tilde{\Gamma} \) such that \( \Delta_R(\theta_{ij}) = \theta_{ki} \otimes v^k_j(v^c)^{k}_i \). Obviously, \( \theta := \sum_{ij} \theta_{ij} \) is a biinvariant element of \( \tilde{\Gamma} \). Equipped with the differentiation defined by \( [4] \), \( \Gamma(v) := \mathcal{A} \cdot d\mathcal{A} \cdot \mathcal{A} \) becomes a bicovariant FODC \( \Gamma(v) \) with quantum Lie algebra \( \mathcal{X}(v) \) spanned by the linear functionals

\[
X_{ij} = l^i_j - \delta_{ij} \varepsilon, \quad i, j = 1, \ldots, m. \tag{9}
\]

If the functionals \( X_{ij} \), \( i, j = 1, \ldots, m \), are linearly independent, then we clearly have \( \Gamma(v) = \tilde{\Gamma} \).

**Example 2.5.** If \( v = 1 \), then \( \tilde{\Gamma} = (1, \varepsilon) \) and hence \( \mathcal{X}(v) = \{ 0 \} \), so \( \Gamma(v) \neq \tilde{\Gamma} \) and \( \Gamma(v) \) is the trivial bicovariant FODC with \( da = 0 \) for all \( a \in \mathcal{A} \). \( \blacksquare \)

2.3 Construction of bicovariant first order differential calculi over the Hopf algebras \( \mathcal{O}(SL_q(n + 1)) \) and \( \mathcal{O}(Sp_q(2n)) \)

Let us say that a complex number \( \zeta \) is admissible if \( \zeta^{n+1} = 1 \) for \( G_q = SL_q(n + 1) \) and \( \zeta^2 = 1 \) for \( G_q = Sp_q(2n) \). For any admissible number \( \zeta \) let \( \varepsilon_{\zeta} \) denote the multiplicative linear functional on the algebra \( \mathcal{O}(G_q) \) such that \( \varepsilon_{\zeta}(w^i_j) = \zeta \delta^i_j \), \( i, j = 1, \ldots, N \).
First we define some 1-dimensional bicovariant FODC over \( O(G_q) \). For admissible \( \zeta \), let \( \Gamma^\zeta = (1, \varepsilon_\zeta) \) be the one dimensional bicovariant bimodule over \( O(G_q) \). That is, \( \Gamma^\zeta \) has a free left module basis consisting of a single biinvariant element \( \theta_0 \) and we have \( \theta_0 a = a(1) \varepsilon_\zeta (a(2)) \theta_0 \) for \( a \in O(G_q) \).

If \( \zeta \neq 1 \), then \( \Gamma^\zeta \) is an inner bicovariant FODC over \( O(G_q) \) with differentiation \( \mathfrak{F} \). It can be shown (see Remark 2.4 in [14]) that any one dimensional bicovariant FODC over \( O(G_q) \) is of this form.

Now we specialize the procedure from the preceding section to the coordinate Hopf algebra \( O(G_q) \), where \( G_q = SL_q(n+1) \) or \( G_q = Sp_q(2n) \). Recall (see [11], 10.1.2) that these Hopf algebras are coquasitriangular with universal \( r \)-form \( r \) such that

\[
r(u_j^i \otimes u_m^n) = z R_{jm}^{in}, \quad i, j, n, m = 1, \ldots, N. \tag{10}
\]

Here \( u_j^i \) are the matrix entries of the fundamental corepresentation of \( O(G_q) \), \( R \) is the corresponding \( R \)-matrix (see [3], (1.5) and (1.9), or [11], (9.13) and (9.30)) and \( z \) is a fixed complex number such that \( z^N = q^{-1} \) for \( SL_q(N) \) and \( z^2 = 1 \) for \( Sp_q(N) \).

Suppose that \( v = (u_j^i)_{i,j=1,\ldots,m} \) is a matrix corepresentation of \( A = O(G_q) \) and \( \zeta \) is an admissible number. Consider the bicovariant bimodule

\[
\tilde{\Gamma} := \Gamma^\zeta \otimes_A \Gamma_v \otimes_A \Gamma^\zeta_\varepsilon = (1 \otimes v \otimes v^c, \varepsilon_\zeta \otimes L^{-c} \otimes L^+).
\]

(11)

The element \( \theta := \sum_i \theta_0 \otimes_A \theta_{ii} \in \tilde{\Gamma} \) is biinvariant and \( \Gamma^\zeta(v) := A \cdot dA \cdot A \) is a bicovariant FODC over \( A = O(G_q) \) with differentiation \( \mathfrak{F} \). If \( \zeta = 1 \), then \( \Gamma^\zeta(v) \) is just the FODC \( \Gamma(v) \) of \( \mathfrak{F} \). If \( \zeta \neq 1 \), then the FODC \( \Gamma^\zeta(v) \) is the tensor product of the one dimensional FODC \( \Gamma^\zeta \) and the FODC \( \Gamma(v) \).

The quantum Lie algebra \( \mathcal{X}^\zeta(v) \) of the bicovariant FODC \( \Gamma^\zeta(v) \) is then the linear span of the functionals

\[
X_{ij} = \varepsilon_\zeta b_j^i - \delta_{ij} \varepsilon, \quad i, j = 1, \ldots, m. \tag{12}
\]

**Proposition 2.6.** Let \( \zeta, \zeta' \) be admissible numbers for \( G_q \) and \( v, w \) corepresentations of \( O(G_q) \). Then the bicovariant FODC \( \Gamma^\zeta(v \otimes w) \) and \( \Gamma^\zeta(v) \otimes \Gamma^\zeta(w) \) (see [4]) are isomorphic.

**Proof.** Because of \( [\mathfrak{F}] \), the sets \( \{ l(ab) \mid a \in \mathcal{C}(v), b \in \mathcal{C}(w) \} \) and \( \{ l(a)l(b) \mid a \in \mathcal{C}(v), b \in \mathcal{C}(w) \} \) span the \( \text{ad}_R \)-invariant right coideals \( \mathcal{X}^\zeta(v \otimes w) \) and \( \mathcal{X}^\zeta(v) \cdot \mathcal{X}^\zeta(w) \), respectively. We compute

\[
l(ab) = l(a(1)) \text{ad}_R(l^+(a(2)))l(b).
\]

Since \( \Delta : \mathcal{C}(v) \rightarrow \mathcal{C}(v) \otimes \mathcal{C}(v) \) and \( \mathcal{X}^\zeta(w) \) is \( \text{ad}_R \)-invariant, we obtain that \( l(ab) \in \mathcal{X}^\zeta(v) \cdot \mathcal{X}^\zeta(w) \).

On the other hand, by formula (10.29) in [11], we have

\[
l(a)l(b) = r(b(1) \otimes S(a(3)))r(b(3) \otimes a(2))l(a(1)b(2)).
\]

Hence it follows that \( l(a)l(b) \in \mathcal{X}^\zeta(v \otimes w) \) for all \( a \in \mathcal{C}(v) \) and \( b \in \mathcal{C}(w) \). This proves that \( \mathcal{X}^\zeta(v \otimes w) = \mathcal{X}^\zeta(v) \cdot \mathcal{X}^\zeta(w) \).

The quantum Lie algebra of the bicovariant FODC \( \Gamma^\zeta(v \otimes w) \) is given by \( [\mathfrak{F}] \). Clearly, we have \( \mathcal{X}^\zeta(v \otimes w) = P_\mathfrak{F}(\varepsilon = \mathcal{X}^\zeta(v \otimes w)) \). Let \( \theta_1 \) and \( \theta_2 \) be nonzero biinvariant elements in \( \Gamma^\zeta(v) \) and \( \Gamma^\zeta(w) \), respectively. If one of the pairs \( (v, \zeta) \) or \( (w, \zeta') \) is equal to \( (1, 1) \), then for this pair we take the one dimensional bicovariant bimodule \( \hat{\Gamma} = (1, \varepsilon) \) given by [3] and a biinvariant element

\[
\theta_0 \]
therein. By the definition of the tensor product of FODC, the biinvariant element \( \theta_1 \otimes_A \theta_2 \) defines the differentiation of \( \Gamma_\zeta(v) \otimes \Gamma_\zeta'(w) \) (see (3)). Obviously, the quantum Lie algebra of \( \Gamma_\zeta(v) \otimes \Gamma_\zeta'(w) \) is \( P_\zeta(\varepsilon_\zeta \mathcal{X}^\zeta(v) \varepsilon_\zeta' \mathcal{X}^\zeta'(w)) \). Since \( \varepsilon_\zeta \) and \( \varepsilon_\zeta' \) are central and \( \mathcal{X}^\zeta(v \otimes w) = \mathcal{X}^\zeta(v) \cdot \mathcal{X}^\zeta'(w) \) as shown in the preceding paragraph, we have \( P_\zeta(\varepsilon_\zeta \mathcal{X}^\zeta(v) \varepsilon_\zeta' \mathcal{X}^\zeta'(w)) = \mathcal{X}^\zeta(v \otimes w) \). That is, the quantum Lie algebras of the bicovariant FODC \( \Gamma_\zeta(v \otimes w) \) and \( \Gamma_\zeta'(v) \otimes \Gamma_\zeta'(w) \) coincide. Hence the FODC are isomorphic.

Set \( (D^{-1})^j_i := r(S^2(v^n_j) \otimes v^n_i) \) for \( i, j = 1, \ldots, m \). Then the element

\[
c_\zeta(v) := \varepsilon_\zeta \text{Tr} LD^{-1} = \sum_{i,j} \varepsilon_\zeta l^i_j(D^{-1})^j_i.
\]

belongs to the centre of the Hopf dual \( A^\circ \) (11), Proposition 10.15(ii)), so each quantum Lie algebra \( \mathcal{X}^\zeta(v) \) contains the central element

\[
P_\zeta(c_\zeta(v)) = c_\zeta(v) - \sum_{i} (D^{-1})^i_i \varepsilon.
\]

2.4 Bicovariant FODC generated by central elements of \( A^\circ \)

In this brief subsection we recall another general method for the construction of bicovariant FODC which was invented in [3], see also [14]. We state the corresponding result as

**Proposition 2.7.** [3, 14] Suppose that \( A \) is a Hopf algebra such that its Hopf dual \( A^\circ \) separates the elements of \( A \). For any central element \( c \in A^\circ \) there exists a bicovariant FODC \( \Gamma(c) \) over \( A \) with quantum Lie algebra

\[
\mathcal{X}[c] = \text{Lin} \{ \chi_a := c(2)(a)c(1) - c(a)\varepsilon \mid a \in A \}.
\]

**Proof.** See [11], Proposition 14.11.

Note that (14) differs from the corresponding formula in [14], because we use another definition of the quantum Lie algebra.

If we write \( \Delta(c) = \sum_i x_i \otimes y_i \) with \( \{y_i\} \) linearly independent, then \( \mathcal{X}[c] \) is obviously equal to the linear span of functionals \( x_i - x_i(1)\varepsilon \). In particular, we see that the vector space \( \mathcal{X}[c] \) and hence the bicovariant FODC \( \Gamma(c) \) are finite dimensional.

3 The right adjoint action of the Hopf dual \( R(G_q)^\circ \)

In this section \( \mathfrak{g} \) denotes a simple Lie algebra with Cartan matrix \( (a_{ij}) \) and rank \( n \) and \( (d_i \delta_{ij}) \) is a diagonal matrix such that \( (d_i a_{ij}) \) is symmetric. Throughout we suppose that \( q \) is a transcendental complex number. (However, some assertions and arguments require only that \( q \) is not root of unity.)
3.1 The extended Drinfeld-Jimbo algebra $U_q(\mathfrak{g})$

Let $U_q(\mathfrak{g})$ denote the unital complex algebra with generators $E_i$, $F_i$, $K_i$, $K_i^{-1}$, $i = 1, \ldots, n$, subject to the relations

\[
K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1, \\
K_i E_j K_i^{-1} = q_i^{\delta_{ij}} E_j, \quad K_i F_j K_i^{-1} = q_i^{-\delta_{ij}} F_j, \\
E_i F_j - F_j E_i = \delta_{ij} \frac{K_{\alpha_i} - K_{\alpha_i}^{-1}}{q_i - q_i^{-1}},
\]

where

\[
[\begin{array}{c}
m \\
k \end{array}]_q = \frac{[m]_q!}{[k]_q! [m-k]_q!}, \quad [m]_q! = [m]_q [m-1]_q \cdots [1]_q, \quad [0]_q! = 1 \quad \text{and} \quad [m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}.
\]

In equation (15) the symbol $K_{\alpha_i}$ denotes the element $K_{\alpha_i} = \prod_{j=1}^n K_{\alpha_j}^{k_j}$ and $K_{\alpha_i}^{-1}$ its inverse. The generators $K_i$ correspond to the fundamental weights $\omega_i$ of the Lie algebra $\mathfrak{g}$. As usual, $U_q(\mathfrak{n}_+)$ and $U_q(\mathfrak{n}_-)$ are the subalgebras of $U_q(\mathfrak{g})$ generated by the elements $E_i$ resp. $F_i$, $i = 1, \ldots, n$.

There is a Hopf algebra structure on $U_q(\mathfrak{g})$ determined by the formulas

\[
\Delta(K_i) = K_i \otimes K_i, \quad \Delta(K_i^{-1}) = K_i^{-1} \otimes K_i^{-1}, \\
\Delta(E_i) = E_i \otimes K_{\alpha_i} + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_{\alpha_i}^{-1} \otimes F_i, \\
\varepsilon(K_i) = \varepsilon(K_i^{-1}) = 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0, \\
S(K_i) = K_i^{-1}, \quad S(K_i^{-1}) = K_i, \quad S(E_i) = -E_i K_{\alpha_i}^{-1}, \quad S(F_i) = -K_{\alpha_i} F_i.
\]

The Hopf subalgebra $U_q(\mathfrak{g})_{DJ}$ of $U_q(\mathfrak{g})$ generated by the elements $E_i$, $F_i$ and $K_{\alpha_i}$, $i = 1, \ldots, n$, is isomorphic to the "usual" Drinfeld-Jimbo algebra. For the considerations in Section 4 it is crucial to work with the extended Drinfeld-Jimbo algebra $U_q(\mathfrak{g})$, because this algebra is also generated by the so-called $L$-functionals $t^\pm_1$ (see \[3\] or \[1\], 8.5.3, for details).

3.2 The dual Hopf algebra $\mathcal{R}(G_q)^\circ$

Let $\mathcal{R}(G_q)$ denote the Hopf subalgebra of $U_q(\mathfrak{g})^\ast$ spanned by the matrix coefficients of all finite dimensional type 1 highest weight representations of $U_q(\mathfrak{g})$. The purpose of this subsection is to repeat a result of Joseph [7] on the description of the Hopf dual $\mathcal{R}(G_q)^\circ$.

Let $T$ be the multiplicative group in $U_q(\mathfrak{g})$ generated by $K_i$, $i = 1, \ldots, n$. First we describe the Hopf dual of the group algebra $\mathbb{C}T$ (see \[3\], Lemma 9.4.8(i)). Let $(\mathbb{C}^\times)^n$ be the product of $n$ copies of the multiplicative group of nonzero complex numbers. For any $\mu = (\mu_1, \ldots, \mu_n) \in (\mathbb{C}^\times)^n$, there is a unique multiplicative linear functional $f_\mu$ on $\mathbb{C}T$ such that $f_\mu(K_i) = \mu_i$, $i = 1, \ldots, n$. Hence there are numbers $\tilde{\mu}_i \neq 0$ such that $f_\mu(K_{\alpha_i}^{-1}) = \tilde{\mu}_i$. Further, for any $i = 1, \ldots, n$ there is a unique linear functional $g_i$ on $\mathbb{C}T$ such that $g_i(K_j) = (a^{-1})_{ij}$ and $g_i(ab) = \varepsilon(a)g_i(b) + g_i(a)\varepsilon(b)$ for all $a, b \in \mathbb{C}T$. 

Here \((a^{-1})_{ij}\) are the entries of the inverse Cartan matrix. From the normalization \(g_i(K_j) = (a^{-1})_{ij}\) we get \(g_i(K_n) = \delta_i^n\) which implies the simple commutation rules (14) below between \(E_i, F_i\) and \(g_j\).

Now the Hopf dual \(U_0 := (CT)^\circ\) is the (commutative and cocommutative) Hopf algebra generated by the functionals \(f_\mu\) and \(g_i\) with relations \(f_\mu f_\nu = f_{\mu \nu}, g_i g_j = g_j g_i, f_\mu g_i = g_i f_\mu\) and Hopf algebra structure given by

\[
\Delta(f_\mu) = f_\mu \otimes f_\mu, \quad \varepsilon(f_\mu) = 1, \quad S(f_\mu) = f_{\mu^{-1}}, \\
\Delta(g_i) = g_i \otimes 1 + 1 \otimes g_i, \quad \varepsilon(g_i) = 0, \quad S(g_i) = -g_i.
\]

Here \(\mu^{-1}\) denotes the \(n\)-tuple \((\mu_1^{-1}, \ldots, \mu_n^{-1})\) in \((\mathbb{C}^\times)^n\) if \(\mu = (\mu_1, \ldots, \mu_n)\).

As shown in (15), the Hopf algebra \(U_0\) can be embedded into the Hopf algebra \(R(G_q)\). For notational simplicity let us identify \(U_0\) with its image in \(R(G_q)\). We shall not need the explicit form of this embedding. It suffices to know the following commutation relations (see (16), 9.4.8 and 9.4.9) of the functionals \(f_\mu, g_i \in U_0\) with the generators \(E_i, F_i \in U_q(\mathfrak{g})\) in the algebra \(R(G_q)\):

\[
f_\mu E_i = \bar{\mu}_i E_i f_\mu, \quad F_i f_\mu = \bar{\mu}_i f_\mu F_i, \quad E_i g_j = (g_j + \delta_j^i) E_i, \quad g_j F_i = F_i (g_j + \delta_j^i). \quad \tag{16}
\]

**Proposition 3.1.** (16) The multiplication map \(U_q(n_+) \otimes U_0 \otimes U_q(n_-) \to R(G_q)^\circ\) is an isomorphism of vector spaces.

**Proof.** Proposition 9.4.9 in (16).

### 3.3 The locally finite part for the right adjoint action of \(U_q(\mathfrak{g})\)

In this subsection we restate the fundamental result of Joseph and Letzter (12, 13, 14) about the locally finite part of \(U_q(\mathfrak{g})\). If \(B\) is a subalgebra of a Hopf algebra \(A\), we define

\[
F_B(A) := \{a \in A | \dim \text{ad}_R(B)a < \infty\}.
\]

The vector space \(F(A) := F_A(A)\) is called the locally finite part of the Hopf algebra \(A\). We denote by \(P(\mathfrak{g}) = \{\lambda \in \mathfrak{h}^* | (\alpha_i^\vee, \lambda) \in \mathbb{Z}\}\) the weight lattice of \(\mathfrak{g}\) and by \(P_+(\mathfrak{g})\) the subset of dominant integral weights. Let \(\omega_i \in P_+(\mathfrak{g})\) be given by \((\alpha_i^\vee, \omega_i) = \delta_i\). There is a unique group homomorphism \(\tau : P(\mathfrak{g}) \to T\) such that \(\tau(\omega_i) = K_i\). Further, let \(T_+\) be the subsemigroup of \(T\) generated by the elements \(K_i^2 = \tau(2\omega_i), i = 1, \ldots, n\), and \(\tilde{U}_q(\mathfrak{g})\) the subalgebra of \(R(G_q)^\circ\) generated by the elements \(E_i, F_i\) and \(f_\mu\).

**Proposition 3.2.** (i) \(F(\tilde{U}_q(\mathfrak{g})) = \bigoplus_{\lambda \in P_+(\mathfrak{g})} \text{ad}_R(\tilde{U}_q(\mathfrak{g})) \tau(-2\lambda)\). Further, the subspace \(W(\lambda) := \text{ad}_R(\tilde{U}_q(\mathfrak{g})) \tau(-2\lambda)\) has dimension \((\dim V(\lambda))^2\), where \(V(\lambda)\) is the irreducible representation of \(\tilde{U}_q(\mathfrak{g})\) with highest weight \(\lambda\) with respect to the simple roots \(\{-\alpha_1, \ldots, -\alpha_n\}\).

(ii) There is a subset \(M\) of \((\mathbb{C}^\times)^n\) such that \(\tilde{U}_q(\mathfrak{g}) = \bigoplus_{\mu \in M} F(\tilde{U}_q(\mathfrak{g})) T_+ f_\mu\).

**Proof.** Since a subspace \(E\) of \(\tilde{U}_q(\mathfrak{g})\) is \(\text{ad}_L\)-invariant if and only if \(S^{-1}(E)\) is \(\text{ad}_R\)-invariant, (i) is only a restatement of the corresponding result for the left adjoint action \(\text{ad}_L\) obtained in (16). (ii) follows from (16), 7.1.13.
3.4 Finite dimensional $\text{ad}_R$-invariant right coideals of $\mathcal{R}(G_q)^\circ$

Let $C$ denote the set of all multiplicative linear functionals in the centre of $\mathcal{R}(G_q)^\circ$. Recall we assumed in this section that $q$ is a transcendental complex number.

**Lemma 3.3.** Suppose that $u \in \mathcal{F}(\mathcal{U}_q(\mathfrak{g}))$, $\lambda \in P_+(\mathfrak{g})$ and $\mu \in (\mathbb{C}^\times)^n$. If there is a natural number $k$ such that

$$\text{ad}_R (E^k_i)(u\tau(2\lambda)f_\mu) = 0$$

(17)

for all $i$, then it follows that $u = 0$ or $\tau(2\lambda)f_\mu = c\tau(-2\gamma)$ for some $c \in C$ and $\gamma \in P^\circ(\mathfrak{g})$.

**Proof.** Since $u \in \mathcal{F}(\mathcal{U}_q(\mathfrak{g}))$, there exists a natural number $l$ such that $\text{ad}_R (E^l_i)u = 0$ for all $i$. Let us fix the index $i$ and abbreviate

$$b_{rs} := q_i^{-s(l+k+r-s)}\left[l+k+r\atop s\right] \prod_{j=0}^{l+k+r-s-1} (q_i^{2(\alpha_j^\vee, \lambda)} - q_i^{-2j}).$$

From the relations $\text{ad}_R (E^{l+k+r}_i)(u\tau(2\lambda)f_\mu) = 0$ by (17) and $\text{ad}_R (E^l_i)u = 0$ we obtain

$$\sum_{s=0}^{l-1} b_{rs}\text{ad}_R (E^l_i)u \cdot E^{l+k+r-s}_i\tau(2\lambda)f_\mu = 0, \quad r, 0, \ldots, l - 1.$$  

(18)

By induction one proves that the determinant of the matrix $(b_{rs})_{r,s=0,\ldots,l-1}$ is equal to

$$\det (b_{rs}) = q_i^{-l(l-1)(k+l)} \prod_{j=1}^{l-1} (\mu_j q_i^{2(\alpha_j^\vee, \lambda)} - q_i^{2(l-j)} - q_i^{-2(l-j)} - q_i^{-2j - l + j}).$$

(19)

Therefore, if $\det (b_{rs}) \neq 0$, then it follows from (18) that $\text{ad}_R (E^l_i)u \cdot E^{l+k+r-s}_i\tau(2\lambda)f_\mu = 0$, $r, s = 0, \ldots, l - 1$. In the case $s = 0$ we get $u \cdot E^{l+k+r}_i\tau(2\lambda)f_\mu = 0$ and hence $u = 0$. If $\det (b_{rs}) = 0$, then (19) implies that there are integers $n_i$ such that $\mu_j q_i^{2(\alpha_j^\vee, \lambda)} = q_i^{-2n_i}$ for all $i$. Set $\gamma := \sum_i n_i \omega_i$. From the commutation rules in the algebra $\mathcal{U}_q(\mathfrak{g})$ we see that $c := f_\mu \tau(2\lambda)\tau(2\gamma)$ commutes with all generators $E_i, F_i, f_\mu$, so $c$ is in the centre of $\mathcal{U}_q(\mathfrak{g})$. Obviously, $c$ is a character, because $f_\mu, \tau(2\lambda)$ and $\tau(2\gamma)$ are so. Hence we have $c \in C$ and $\tau(2\lambda)f_\mu = c\tau(-2\gamma)$.

An immediate consequence of Lemma 3.3 is

**Corollary 3.4.** Suppose that $\mu \in (\mathbb{C}^\times)^n$. If there is no $c \in C$ such that $\mathcal{F}(\mathcal{U}_q(\mathfrak{g}))T_+f_\mu \subseteq c\mathcal{F}(\mathcal{U}_q(\mathfrak{g}))T_+f_\mu$, then $\text{ad}_R (E_i)$ is injective on the $\text{ad}_R$-invariant subspace $\mathcal{F}(\mathcal{U}_q(\mathfrak{g}))T_+f_\mu$ for all $i$.

Our next aim is to prove Proposition 3.7 below which was suggested by the referee. The authors would like to thank the referee for his advice.

**Lemma 3.5.** $\mathcal{F}(\mathcal{U}_q(\mathfrak{g})) \subseteq \mathcal{F}(\mathcal{U}_q(\mathfrak{g}))(\mathcal{U}_q(\mathfrak{g}))$.

**Proof.** We first note that because $q$ is not a root of unity the $\text{ad}_R$-right module $\mathcal{F}(\mathcal{U}_q(\mathfrak{g}))(\mathcal{U}_q(\mathfrak{g}))$ of the algebra $\mathcal{U}_q(\mathfrak{g})$ decomposes into a direct sum of simple submodules. Let $V$ be such a submodule. Since the algebra $\mathcal{U}_q(\mathfrak{g})$ is the sum of $\text{ad}_R$-invariant subspaces $\mathcal{F}(\mathcal{U}_q(\mathfrak{g}))(\mathcal{U}_q(\mathfrak{g}))T_+f_\mu$ by Proposition 3.3, there is $\mu \in (\mathbb{C}^\times)^n$ such that $V \subseteq \mathcal{F}(\mathcal{U}_q(\mathfrak{g}))(\mathcal{U}_q(\mathfrak{g}))T_+f_\mu$. Any element $v \in V$ is of the form $v = u\tau(2\lambda)f_\mu$, where $u \in \mathcal{F}(\mathcal{U}_q(\mathfrak{g}))$ and $\lambda \in P_+(\mathfrak{g})$, and satisfies the assumptions of Lemma 3.3. If $u = 0$, then
$v = 0$, so $v$ is trivially in $\mathcal{F}(\tilde{U}_q(\mathfrak{g}))$. If $u \neq 0$, then $v = u \tau(2\lambda) f_\mu = u \tau(-2\gamma) \in cU_q(\mathfrak{g})$ by Lemma 3.4. Since $v \in \mathcal{F}(\tilde{U}_q(\mathfrak{g}))$ by assumption, we get $v \in \mathcal{F}(\tilde{U}_q(\mathfrak{g}))$. \hfill 

Let $\mathcal{G}$ be the subalgebra of $\mathcal{R}(G_q)^o$ generated by the elements $g_i$, $i = 1, \ldots, n$. The elements 
\[ \{g^i = g_1^i, \ldots, g_n^i \mid k_1, \ldots, k_n \in \mathbb{N}_0\}, \mathfrak{t} = (k_1, \ldots, k_n), \] 
form a vector space basis of $\mathcal{G}$. By Proposition 3.1, the multiplication map gives a vector space isomorphism of $\tilde{U}_q(\mathfrak{g}) \otimes \mathcal{G}$ and $\mathcal{R}(G_q)^o$. That is, any $v \in \mathcal{R}(G_q)^o$ is a finite sum $v = \sum u_t g^t$ with uniquely determined elements $u_t \in \tilde{U}_q(\mathfrak{g})$. If $v \neq 0$, then the largest number $|\mathfrak{t}| := k_1 + \cdots + k_n$ such that $u_t \neq 0$ is called the degree of $v$ and denoted by $|v|$. We shall write $v \simeq w$ if the degree of $v - w$ is lower than the degree of $v$ and $w$.

**Lemma 3.6.** Let $V$ be a simple submodule of $\mathcal{R}(G_q)^o$ with respect to the right adjoint action $ad_R$ of the algebra $U_q(\mathfrak{g})$. Then we have $V \subseteq \tilde{U}_q(\mathfrak{g})$.

**Proof.** Assume the contrary. Then there exists an element $v \in V$ such that $|v| > 0$. Let us write $v$ in the form $v = \sum u_t g^t + \sum u_h g^h$, where $\mathfrak{t}$ and $\mathfrak{h}$ run over all multindices such that $|\mathfrak{t}| = |v|$ and $|\mathfrak{h}| < |v|$, respectively. One easily verifies that

\[ ad_R (f)(u_t g^t) \simeq (ad_R (f) u_t) g^t \] 

for all $\mathfrak{t}$ and $f \in \tilde{U}_q(\mathfrak{g})$. Hence $V_\mathfrak{t} := ad_R (\tilde{U}_q(\mathfrak{g}))(u_t)$ is an $ad_R$-invariant finite dimensional vector space and $u_t \in \mathcal{F}(\tilde{U}_q(\mathfrak{g}))$ for all $\mathfrak{t}$. Since $ad_R (E_i)$ raises the degree with respect to the usual $\mathfrak{z}$-gradation (that is, $\text{deg} E_i = -\text{deg} F_i = 1$ and $\text{deg} f_\mu = \text{deg} g_i = 0$) by one and the spaces $V_\mathfrak{t}$ are finite dimensional, we can choose $v \in V$ such that $ad_R (E_i) u_t = 0$ for all $\mathfrak{t}$ and all $i$. From the latter and (20) we conclude that $|ad_R (E_i) v| < |v|$. Therefore, $W_\mathfrak{t} := ad_R (\tilde{U}_q(\mathfrak{g}))(ad_R (E_i) v)$ is a submodule of the simple module $V$ such that $W_\mathfrak{t} \neq V$. Thus, $W_\mathfrak{t} = \{0\}$ and hence $ad_R (E_i) v = 0$ for all $i$.

Fix a $\mathfrak{t}'$ such that $u_{\mathfrak{t}'} \neq 0$. Since $|v| = |\mathfrak{t}'| > 0$, there exists an index $l$ such that $k'_l > 0$. Set $h' := \mathfrak{t}' - \mathfrak{t}_l$, where $\mathfrak{t}_l := (k_1, \ldots, k_n)$ and $k_l := \delta_l$. Further, we set $u_j := (k'_j + 1 - \delta'_j) u_{\mathfrak{t}'} + \mathfrak{e}_l - \mathfrak{c}_l$, $u_0 := u_{\mathfrak{h}'}$ and $v' = \sum u_j g_j + u_0$. Comparing the coefficients of $g^h$ in the equality

\[ 0 = ad_R (E_i) v \simeq \sum_{\mathfrak{t}} u_{\mathfrak{t}} ad_R (E_i) g^t + \sum_{\mathfrak{h}} ad_R (E_i) u_{\mathfrak{h}} g^h \] 

by using (20) we obtain

\[ (k'_l + 1 - \delta'_l) u_{\mathfrak{t}'} + \mathfrak{e}_l - \mathfrak{c}_l E_i + ad_R (E_i) u_{\mathfrak{h}'} = 0, \] 

which means that $ad_R (E_i) v' = 0$. Since $|\mathfrak{t}' + \mathfrak{c}_l - \mathfrak{e}_l| = |v|$, we have $u_j \in \mathcal{F}(\tilde{U}_q(\mathfrak{g}))$. Moreover, we also have $ad_R (E_i) u_{\mathfrak{t}} = 0$ by construction and $ad_R (E_i) g_j = \delta_j$. Using these facts we get $0 = ad_R (E_i) v' = u_i E_i + ad_R (E_i) u_0$ and so $ad_R (E_i) u_0 \in \mathcal{F}(\tilde{U}_q(\mathfrak{g})) E_i \subseteq \mathcal{F}(\tilde{U}_q(\mathfrak{g})) T_+$. Recall that $\tilde{U}_q(\mathfrak{g})$ is the direct sum of $ad_R$-invariant subspaces $\mathcal{F}(\tilde{U}_q(\mathfrak{g})) T_+ f_\mu$. Therefore, it follows from Corollary 3.3 and the last relation that the nonvanishing components of $u_0 \in \tilde{U}_q(\mathfrak{g})$ are in $\mathcal{F}(\tilde{U}_q(\mathfrak{g})) T_+$. In particular, $u_0 \tau(-2\lambda) \in \mathcal{F}(\tilde{U}_q(\mathfrak{g}))$ for some $\lambda \in P_+ (\mathfrak{g})$. Hence there exists a natural number $k$ such that $ad_R (E^k) (u_0 \tau(-2\lambda)) = 0$ and $ad_R (E^k) \tau(-2\lambda) = 0$. Since $ad_R (E_i) v' = 0$, the latter implies that $ad_R (E^m) (v' \tau(-2\lambda)) = v' ad_R (E^m) \tau(-2\lambda) = 0$ for all $m \geq k$. Now we compute $ad_R (E^m) (v' \tau(-2\lambda))$ by using the equalities $ad_R (E^m) (u_0 \tau(-2\lambda)) = 0$ and $ad_R (E^k) \tau(-2\lambda) = 0$. We then obtain

\[ \sum_{r=0}^{k-1} q_{t} (m-r) \left[ \begin{array}{c} m \r \end{array} \right]_{q_i} u_{t} \prod_{j=1}^{r} (1 - q_{t}^{-2j+2k}) \tau(-2\lambda) E_{i}^{m-r} \prod_{j=1}^{m-r-1} (1 - q_{t}^{-2j}) E_{i}^{m-r} = 0. \]
Next let us consider the equation
\[
\sum_{r=s}^{k-1} q_{i r}^{-r(m+s-r)} \left[ \begin{array}{c} m + s \\ r - s \end{array} \right] q_{j r}^{m+s-r-1} \prod_{j=1}^{r} (1 - q^{-1}) \prod_{j=1}^{r} (1 - q^{-2j+2k}) u_i = 0
\]  
for \( s = 0, 1, \ldots, k - 1 \). For \( s = 0 \) equation (22) follows from (23), because the algebra \( \hat{U}_q(\mathfrak{g}) \) has no zero divisors. For general \( s \), (22) is easily proved by induction. From (22) applied in the case \( s = k - 1 \) we conclude that \( u_i = 0 \), because \( q_i \) is not a root of unity. Since \( u_1 \) is not a root of unity, we arrived at a contradiction.

\[\text{Proposition 3.7. If } q \text{ is transcendental, then we have } F(R(G_q)^{\circ}) = F(U_q(\mathfrak{g})) C.\]

\[\text{Proof. Since } q \text{ is not a root of unity, the right adjoint action } \text{ad}_R \text{ of the algebra } U_q(\mathfrak{g}) \text{ is a direct sum of simple submodules. Thus we have } F(R(G_q)^{\circ}) \subseteq F(U_q(\mathfrak{g})) C \text{ by Lemma 3.6. In order to prove the converse inclusion it suffices to check that } \text{ad}_R(f_\mu) \text{ and } \text{ad}_R(g_i) \text{ leave each space } cW(\lambda) \text{ for } c \in C \text{ and } \lambda \in P_+(\mathfrak{g}) \text{ invariant. Using the commutation rules (14) between } f_\mu, g_i \text{ and the generators of } U_q(\mathfrak{g}) \text{ this is easily done.} \]

\[\text{Lemma 3.8. For any finite dimensional } \text{ad}_R \text{-invariant right coideal } \mathcal{X} \text{ of } \hat{U}_q(\mathfrak{g}) \text{ there exist elements } c \in C \text{ and } \lambda \in P_+(\mathfrak{g}) \text{ such that } c\tau(-2\lambda) \in \mathcal{X}.\]

\[\text{Proof. Let } \{E_i\} \text{ and } \{F_j\} \text{ be vector space bases of } U_q(n_+) \text{ and } U_q(n_-), \text{ respectively, consisting of monomials in the generators } E_1, \ldots, E_n \text{ and } F_1, \ldots, F_n, \text{ respectively. Let } |i| \text{ resp. } |j| \text{ denote the Z-degree of the monomial } E_i \text{ resp. } F_j. \text{ By Proposition 3.1, the set } \{E_iF_jf_\mu\} \text{ is a vector space basis of } U_q(\mathfrak{g}). \text{ Fix a nonzero element } u \text{ of } \mathcal{X}. \text{ Then } u \text{ is a sum } \sum_{i,j,\mu} \alpha_{i,j,\mu} E_iF_jf_\mu, \text{ where } \alpha_{i,j,\mu} \in C. \text{ We choose indices } i_0 \text{ and } j_0 \text{ such that } \alpha_{i_0,j_0,\mu} \neq 0 \text{ for some } \mu_0 \in (C^*)^n \text{ and that } \alpha_{i,j,\mu} = 0 \text{ if } |i| > |i_0| \text{ and } \alpha_{i_0,j,\mu} = 0 \text{ if } |j| > |j_0|. \text{ Since } \mathcal{X} \text{ is a right coideal, } \Delta(u) \in \mathcal{X} \otimes \hat{U}_q(\mathfrak{g}). \text{ Hence the first tensor factor } \Delta(u) \text{ corresponding to the basis element } E_{i_0}F_{j_0}f_\mu, \text{ in the second tensor factor belongs to } \mathcal{X}. \text{ From the choice of the indices } i_0, j_0, \mu_0 \text{ and the formulas for the comultiplication of the generators } E_i, F_i, f_\mu \text{ it follows that this expression is of the form } \alpha_{i_0,j_0,\mu_0} Kf_\mu, \text{ where } K \in T. \text{ Thus, } f_\mu := Kf_\mu, \in \mathcal{X} \subset F(U_q(\mathfrak{g})). \text{ By Lemma 3.3 applied with } u = 1 \text{ and } \lambda = 0, \text{ there are some } c \in C \text{ and } \gamma \in P(\mathfrak{g}) \text{ such that } f_\mu = c\tau(-2\gamma) \text{. Because } f_\mu = c\tau(-2\gamma) \in F(\hat{U}_q(\mathfrak{g})), \text{ we have } \gamma \in P_+(\mathfrak{g}). \]

The main result of the section is the following

\[\text{Theorem 3.9. Suppose that } q \text{ is transcendental. Then any finite sum of the subspaces } cW(\lambda), \text{ where } c \in C \text{ and } \lambda \in P_+(\mathfrak{g}), \text{ is a finite dimensional } \text{ad}_R \text{-invariant right coideal of } R(G_q)^{\circ}. \text{ Conversely, each finite dimensional } \text{ad}_R \text{-invariant right coideal } \mathcal{X} \text{ of } R(G_q)^{\circ} \text{ is a finite direct sum of subspaces } cW(\lambda). \]

\[\text{Proof. In the proof of Proposition 3.7 we have already noted that each space } cW(\lambda) \text{ is an } \text{ad}_R \text{-invariant subspace of } R(G_q)^{\circ}. \text{ Since } \Delta(\text{ad}_R(X)(c\tau(-2\lambda))) = \text{ad}_R(X(1)(c\tau(-2\lambda))) \otimes S(X(1))c\tau(-2\lambda)X(3) \text{ for any } X \in R(G_q)^{\circ}, cW(\lambda) \text{ is also a right coideal of } R(G_q)^{\circ}. \text{ Now let } \mathcal{X} \text{ be a finite dimensional } \text{ad}_R \text{-invariant right coideal of } R(G_q)^{\circ}. \text{ From Propositions 3.2 and 3.3 it follows that there is a subset } T \text{ of the product set } C \times T_+^{-1} \text{ such that } \]
\[ \mathcal{X} \subset \bigoplus_{(c,\lambda) \in T} cW(\lambda) \subset \tilde{U}_q(\mathfrak{g}). \] By Lemma 3.3, there are elements \( c_0 \in C \) and \( \lambda_0 \in P_+(\mathfrak{g}) \) such that \( c_0 \tau(-2 \lambda_0) \in \mathcal{X} \). Consequently, the whole vector space \( c_0W(\lambda_0) \) is contained in \( \mathcal{X} \). By the direct sum decomposition of \( \mathcal{F}(\mathcal{R}(G_q)^0) \) there is a finite dimensional \( \text{ad}_R \)-invariant subspace \( \mathcal{X}' \) of \( \bigoplus_{(c,\lambda) \in T'} cW(\lambda) \) such that \( \mathcal{X} = c_0W(\lambda_0) \oplus \mathcal{X}' \), where \( T' := T \setminus (c_0, \lambda_0) \). As shown in the preceding paragraph each space \( cW(\lambda) \) is a right coideal. Hence \( \bigoplus_{(c,\lambda) \in T'} cW(\lambda) \) is a right coideal which has trivial intersection with \( c_0W(\lambda_0) \). This implies that \( \mathcal{X}' \) is also a right coideal of \( \mathcal{R}(G_q)^0 \).

Applying the same reasoning with \( \mathcal{X} \) replaced by \( \mathcal{X}' \), the assertion follows by induction.

An consequence of the preceding proof is the following

**Corollary 3.10.** If \( q \) is transcendental, then the irreducible \( \text{ad}_R \)-invariant right coideals of \( \mathcal{R}(G_q)^0 \) are precisely the subspaces \( cW(\lambda) \), where \( \lambda \in P_+(\mathfrak{g}) \) and \( c \in C \).

Recall that by definition \( \mathcal{U}_q(\mathfrak{g}) \) separates the elements of \( \mathcal{R}(G_q) \). Hence \( \mathcal{R}(G_q)^0 \) separates the elements of \( \mathcal{R}(G_q) \). Thus, by Proposition 2.2, there is a one-to-one correspondence between finite dimensional \( \text{ad}_R \)-invariant right coideals of \( \mathcal{R}(G_q)^0 \) and finite dimensional bicovariant differential calculi on \( \mathcal{R}(G_q) \). Therefore, the assertion of Theorem 3.9 gives a classification of all bicovariant differential calculi on the Hopf algebra \( \mathcal{R}(G_q) \) by describing all possible quantum Lie algebras of such calculi. In particular, by Corollary 3.10 the subspaces \( cW(\lambda) \) are precisely the quantum Lie algebras of irreducible bicovariant differential calculi on \( \mathcal{R}(G_q) \).

### 4 Classification of bicovariant differential calculi on \( \mathcal{O}(G_q) \)

Throughout this section \( \mathcal{O}(G_q) \) denotes the coordinate Hopf algebra for one of the quantum groups \( G_q = SL_q(n+1) \) or \( G_q = Sp_q(2n) \) as defined in \( [3] \) (see also \( [1] \), Chap. 9) and \( \mathfrak{g} \) is the corresponding Lie algebra \( \mathfrak{sl}_{n+1} \) or \( \mathfrak{sp}_{2n} \). Further, we set \( N = n+1 \) if \( G_q = SL_q(n+1) \) and \( N = 2n \) if \( G_q = Sp_q(2n) \).

#### 4.1 The main results

Let us say that two pairs \((\zeta, v)\) and \((\tilde{\zeta}, \tilde{v})\) of admissible \( \zeta \) and \( \tilde{\zeta} \) and of matrix corepresentations \( v \) and \( \tilde{v} \) of \( \mathcal{O}(G_q) \) are equivalent if \( \zeta = \tilde{\zeta} \) and \( v \) and \( \tilde{v} \) are equivalent. The main result of this paper is

**Theorem 4.1.** Let \( \mathcal{O}(G_q) \) be the coordinate Hopf algebra for one of the quantum groups \( SL_q(n+1) \) or \( Sp_q(2n) \). Suppose that \( q \) is a transcendental complex number. Then each finite dimensional bicovariant FODC \( \Gamma \) over \( \mathcal{O}(G_q) \) is isomorphic to a direct sum of bicovariant FODC \( \Gamma_{\zeta_1}(v_1), \ldots, \Gamma_{\zeta_k}(v_k) \) (see \( [3] \), where \( (v_i, \zeta_i) \) are mutually inequivalent pairs of admissible numbers \( \zeta_i \) for \( G_q \) and finite dimensional irreducible matrix corepresentations \( v_i \) of \( \mathcal{O}(G_q) \) such that \( (v_i, \zeta_i) \neq (1, 1) \). The number \( k \) and the pairs \((v_i, \zeta_i)\) are uniquely determined up to permutation of indices and we have

\[
\dim \Gamma = \sum_{i=1}^k (\dim v_k)^2.
\]

The quantum Lie algebra \( \mathcal{X}_\Gamma \) of \( \Gamma \) is of the form \( \{c\zeta_1(v_1) + \cdots + c_{\zeta_k}(v_k), \text{ where } c_{\zeta_i}(v_i) \text{ is the central element of } \mathcal{O}(G_q)^0 \text{ defined by } [3] \).\]

Retaining the assumptions of Theorem 4.1 we state three corollaries.

**Corollary 4.2.** Any finite dimensional bicovariant FODC of \( \mathcal{O}(G_q) \) is inner.
Corollary 4.3. For any irreducible finite dimensional bicovariant FODC $\Gamma$ there exists an irreducible matrix corepresentation $v$ of $O(G_q)$ and an admissible number $\zeta$ such that $\Gamma$ is isomorphic to $\Gamma_\zeta(v)$. In particular, we then have $\dim \Gamma = (\dim v)^2$.

Corollary 4.4. Let $v$ be a corepresentation of $O(G_q)$ and $\zeta$ an admissible number. Suppose that $v$ is the direct sum $\bigoplus_{i=0}^k n_i v_i$ of multiples of pairwise inequivalent irreducible corepresentations $v_0, \ldots, v_k$, where $v_0$ denotes the trivial corepresentation and $n_1 \geq 1, \ldots, n_k \geq 1$. If $\zeta \neq 1$ and $n_0 \geq 1$, then the FODC $\Gamma_\zeta(v)$ is the direct sum of the $k+1$ irreducible FODC $\Gamma_{1,i}(v_0), \Gamma_\zeta(v_1), \ldots, \Gamma_\zeta(v_k)$. Otherwise, $\Gamma_\zeta(v)$ is the direct sum of the $k$ irreducible FODC $\Gamma_{1,i}(v_1), \ldots, \Gamma_\zeta(v_k)$.

Remark. The assertions of Proposition 3.6 and Corollary 4.4 are also valid for the quantum groups $G_q = O_q(N)$.

4.2 Proof of the main results

It is well known ([11], Section 9.4) that there exists a unique dual pairing $\langle \cdot, \cdot \rangle$ of the Hopf algebras $U_q(g)$ and $O(G_q)$ such that $\langle t^+_{ij}, u^i_k \rangle = R^{ki}_{lj}$ and $\langle t^{-}_{ij}, u^i_k \rangle = R^{-1}_{lj}^k$ for $i, j, k, l = 1, \ldots, N$, where $t^+_{ij}$ are the functionals (2) in case $w^j = v^j$ and $R$ is the $R$-matrix for the vector representation of $U_q(g)$. Since $q$ is transcendental, the pairing $\langle \cdot, \cdot \rangle$ is non-degenerate ([11], Corollary 11.23). By the definition of this pairing this implies that the Hopf algebras $O(G_q)$ and $R(G_q)$ are isomorphic.

From the formulas for the comultiplication in $U_q(g)$ and the relations (16) it follows easily that the set $C$ of central characters in $R(G_q)^? \cong O(G_q)^? \cong \mathbb{C}^N$ consists of all functionals $f_{\mu}, \mu \in \mathbb{C}^N$, for which $\mu_i = 1$ for all $i$. These are precisely the functionals $\varepsilon_\zeta$ with admissible $\zeta$.

In the proof given below we shall use some facts from the corepresentation theory of the Hopf algebra $O(G_q)$ (see, for instance, ([11]). One of the main results needed in the sequel is that there is a one-to-one correspondence $v \rightarrow \varphi_v$ between irreducible corepresentations $v$ of $O(G_q)$ and irreducible representations $\varphi_v$ of $U_q(g)_{DJ}$ with dominant integral weights $\lambda(v) \in P_+(g)$ ([11], Theorem 11.22 and Proposition 7.20). Using this correspondence the decomposition of tensor products of irreducible corepresentations into direct sums of irreducible components are given by the same formulas as in the classical case. They are described by the graphical method in terms of the Young frames ([11]).

Lemma 4.5. Let $v$ be an irreducible corepresentation of $O(G_q)$ with matrix elements $v^i_j$, $i, j = 1, \ldots, r$, with respect to some basis of the underlying vector space. Suppose that there exists an element $a(v) \in C(v)$ such that $l(a(v)) = \tau(-2\lambda(v))$. Then for each admissible number $\zeta$ we have:

(i) $X^\zeta(v) = W(\lambda(v))$ and the set $\{l(v^i_j) \mid i, j = 1, \ldots, r\}$ is a basis of the vector space $X^\zeta(v)$.

(ii) $X^\zeta(v) \oplus \mathbb{C} \varepsilon = \varepsilon \lambda W(\lambda(v)) + \mathbb{C} \varepsilon$ and if $(v, \zeta) \neq (1, 1)$, then the set $\{\varepsilon \lambda l(v^i_j), \varepsilon \mid i, j = 1, \ldots, r\}$ is a basis of the vector space $X^\zeta(v) \oplus \mathbb{C} \varepsilon$.

Proof. First we prove (i). By Corollary 3.10, $W(\lambda(v))$ is an irreducible ad$_R$-invariant right coideal of $R(G_q)^? \cong O(G_q)^?$. Since $l(a(v)) = \tau(-2\lambda(v)) \in X^\zeta(v) \cap W(\lambda(v))$ by assumption and $X^\zeta(v) \cap W(\lambda(v))$ is an ad$_R$-invariant right coideal contained in $W(\lambda(v))$, it follows that $W(\lambda(v)) \subseteq X^\zeta(v)$. Since $\dim W(\lambda(v)) = r^2$ by Proposition 3.2 and $\dim X^\zeta(v) \leq r^2$, the preceding implies that $X^\zeta(v) = W(\lambda(v))$ and that the generating set $\{l(v^i_j)\}$ of $X^\zeta(v)$ is a vector space basis of $X^\zeta(v)$.

Now we turn to (ii). By ([12]), we have $X^\zeta(v) \oplus \mathbb{C} \varepsilon = \varepsilon \lambda X^\zeta(v) + \mathbb{C} \varepsilon$. Since $X^\zeta(v) = W(\lambda(v))$ by (i), the first assertion follows. If $(v, \zeta) \neq (1, 1)$, then the sum $\varepsilon \lambda W(\lambda(v)) + \mathbb{C} \varepsilon$ is direct by Theorem 3.8.
Hence \( \dim(\mathcal{X}(v) + \mathbb{C} \varepsilon) = \dim(\varepsilon W(\lambda(v)) + \mathbb{C} \varepsilon) = r^2 + 1 \). Therefore, the generating set \( \{ l(v^i_j), \varepsilon \} \) of \( \mathcal{X}(v) + \mathbb{C} \varepsilon \) consisting of \( r^2 + 1 \) functionals is a vector space basis of \( \mathcal{X}(v) + \mathbb{C} \varepsilon \).

In the following example we shall show that for the irreducible corepresentation \( v_k \) with Young frame \([1^k, 0^{n-k}]\), \( k = 1, \ldots, n \), there is an element \( a(v_k) \in \mathcal{C}(v_k) \) satisfying \( l(a(v_k)) = \tau(-2\lambda(v_k)) \).

**Example 4.6.** For \( G_q = SL_q(N) \) and \( G_q = Sp_q(N) \), respectively, let \( V_q \) denote the quantum vector space \( \mathcal{C}_q \) and the quantum symplectic space \( Sp_q^N \), respectively, and \( \Lambda(V_q) \) their exterior algebras (see [2] or [1], Definitions 9.4 and 9.12). Let \( y_1, \ldots, y_N \) be the generators of \( \Lambda(V_q) \) and \( \Lambda(V_q)_k \) the subspace of \( \Lambda(V_q) \) of homogeneous elements \( y \in \Lambda(V_q) \) of degree \( k \). Recall that the right coaction \( \varphi_R \) of \( \mathcal{O}(G_q) \) on \( \Lambda(V_q) \) is the algebra homomorphism \( \varphi_R : \Lambda(V_q) \to \Lambda(V_q) \otimes \mathcal{O}(G_q) \) determined by \( \varphi_R(y_i) = y_i \otimes u_i^1 \). Let \( \varphi_{R,k} \) denote the restriction of \( \varphi_R \) to the invariant subspace \( \Lambda(V_q)_k \). Since the set \( \{ y_I \mid I = (i_1, \ldots, i_k), i_1 < i_2 < \cdots < i_k \} \) forms a basis in \( \Lambda(V_q)_k \), there exist elements \( D_I^J \in \mathcal{O}(G_q) \), such that \( \varphi_{R,k}(y_I) = y_J \otimes D_I^J \). Then we have

\[
\Delta(D_I^J) = D_M^I \otimes D_M^J, 
\]

where the summation is over all sets \( M = (m_1, \ldots, m_k) \) of integers such that \( 1 \leq m_1 < m_2 < \cdots < m_k \leq N \). In the case \( G_q = SL_q(N) \), this is proved in [1], 9.2.2, but all considerations remain valid for \( G_q = Sp_q(N) \) as well.

Now we suppose that \( k \leq n \). Let us take a closer look at the elements \( D_I^J \). If \( i_k \leq n \), then

\[
D_I^J = \sum_{\sigma \in \mathcal{P}_k} (-q)^{\ell(\sigma)} u_{j_1}^{\sigma(i_1)} \cdots u_{j_k}^{\sigma(i_k)},
\]

where \( \mathcal{P}_k \) is the group of permutations of the set \( \{ i_1, \ldots, i_k \} \) and \( \ell(\sigma) \) is the length of the permutation \( \sigma \). For \( G_q = SL_q(n + 1) \) the latter is proved in [1], Proposition 9.7. For \( G_q = Sp_q(2n) \) the proof is similar (it suffices to use the first two equations in [1], Proposition 9.17(i)). Let \( \mathcal{O}_>(G_q) \) denote the non-unital subalgebra of \( \mathcal{O}(G_q) \) generated by the matrix elements \( u_{j_1}^j, i > n, j \leq n \). We easily see that if \( i_k > n \) and \( j_k \leq n \), then \( D_I^J \in \mathcal{O}_>(G_q) \).

Now let \( v_k \) be the irreducible corepresentation with Young frame \([1^k, 0^{n-k}]\), \( 1 \leq k \leq n \). Set

\[
D_{q,k} := D_I^J = \sum_{\sigma \in \mathcal{P}_k} (-q)^{\ell(\sigma)} u_1^{\sigma(1)} \cdots u_k^{\sigma(k)},
\]

where \( I = J = (1, \ldots, k) \). Obviously, \( D_{q,k} \in \mathcal{C}(u^{\otimes k}) \). The Drinfeld-Jimbo algebra \( \mathcal{U}_q(\mathfrak{g})_{DJ} \) acts on \( \mathcal{O}(G_q) \) by \( \varphi(\mathfrak{f})a = a_1(\mathfrak{f}, a_2(\mathfrak{f})) \), \( a \in \mathcal{O}(G_q) \), \( \mathfrak{f} \in \mathcal{U}_q(\mathfrak{g})_{DJ} \). From the formula \( \Delta(F_i) = F_i \otimes 1 + K_{a_i}^{-1} \otimes F_i \) for the comultiplication of \( \mathcal{U}_q(\mathfrak{g})_{DJ} \) we obtain

\[
\Delta^{(k-1)}(F_i) = \sum_{j=1}^k K_{a_i}^{-1} \otimes F_i \otimes 1 \otimes k-j \quad k \geq 2.
\]

Recall that \( \Delta(D_{q,k}) \) is given by [2]. Since \( \langle F_i, u_s^r \rangle = \delta_i^s \delta_{i+1}^r \) for \( s \leq n \) and \( \langle K_{a_i}, u_s^r \rangle = 0 \) for \( r \neq s \), this implies that \( \varphi(F_i)D_I^J = 0 \) for all \( M = (m_1, \ldots, m_k) \) and \( i = 1, \ldots, n \). Hence we have \( \varphi(F_i)D_{q,k} = 0 \) for \( i = 1, \ldots, n \). Using the explicit formula \( \langle K_{a_i}, u_s^r \rangle = q_i^{\delta_i^r + \delta_{i+1}^r} \) for \( r \leq n \) (see [1], Section 9.4) one easily verifies that \( \varphi^{(K_{a_i})^{-1}D_{q,k}} = q_i^{\delta_i^r + \delta_{i+1}^r}D_{q,k} \). That is, \( D_{q,k} \) is a highest weight vector with highest weight \( \omega_k \) with respect to the ordered sequence of simple roots \( -a_1, \ldots, -a_n \) for the representation \( \varphi \) of \( \mathcal{U}_q(\mathfrak{g})_{DJ} \). Therefore, by the Peter-Weyl decomposition of the coordinate Hopf algebra \( \mathcal{O}(G_q) \) ([1], Theorem 11.22), \( D_{q,k} \) belongs to the coalgebra \( \mathcal{C}(v_k) \).
Suppose that \( I = J = \{1, \ldots, k\} \). From (23) we get \( l(D_j^I) = \sum_M S(l^- (D_M^I)) l^+ (D_j^M) \). If \( M \) is a multi-index such that \( m_k > n \), then we have \( D_j^M \in \mathcal{O}(G_q) \) and hence \( l^+ (D_j^M) = 0 \). Therefore, it suffices to sum over multi-indices \( M = (m_1, \ldots, m_k) \) with \( m_k \leq n \). Further, from the explicit formulas (see for instance [1], 8.5.4) we know that \( K_i = l^{-1} \cdots l_{-i} \), \( i = 1, \ldots, n \). Now we compute

\[
l(D_j^I) = \sum_{m_1 < m_2 < \cdots < m_k \leq n} (q)^{\ell(\sigma) + \ell(\sigma')} S(l^- (u_{m_1}^{\sigma(1)} \cdots u_{m_k}^{\sigma(k)}) l^+ (u_1^{\sigma'(m_1)} \cdots u_k^{\sigma'(m_k)})
\]

\[
= \sum_{m_1 < m_2 < \cdots < m_k \leq n} (q)^{\ell(\sigma) + \ell(\sigma')} S(l^- (\sigma(k) \cdots \sigma(m_k)) l^+ (\sigma'(m_k)) \cdots l^+ (\sigma'(m_1))
\]

\[
= S(l^{-k} \cdots l_{-1}^+ l^+ \cdots l^+_{k} = (l^+_{k} \cdots l^+_{1})^2 = \tau(-2\omega_k) = \tau(-2\lambda(v_k)).
\]

Moreover, the last reasoning shows also that

\[
S(l^- (a(v)(1))) \otimes l^+ (a(v)(2)) = l^+_{1} \cdots l^+_{k} \otimes l^+_{1} \cdots l^+_{k}.
\] (27)

The crucial step in the proof of Theorem 4.4 is the following lemma.

**Lemma 4.7.** For each irreducible corepresentation \( v \) of \( \mathcal{O}(G_q) \) there exists an element \( a(v) \in \mathcal{C}(v) \) such that \( l(a(v)) = \tau(-2\lambda(v)) \).

**Proof.** For \( v = 1 \) we obtain \( l(1) = \tau(0) = \varepsilon \). If \( v = v_k \) is the irreducible corepresentation corresponding to the Young frame \([1^k, 0^{n-k}]\), \( k = 1, \ldots, n \), then we have shown in Example 4.4 that \( a(v_k) = D_{q,k} \) is an element of \( \mathcal{C}(v) \) such that \( l(a(v_k)) = \tau(-2\lambda(v_k)) = \tau(-2\omega_k) \).

The general case will be treated by an induction procedure with respect to an ordering of the Young frames. In order to do so we first introduce some notation.

Let us consider the Young frame of an irreducible corepresentation \( v \) of \( \mathcal{O}(G_q) \). Let \( m_j(v) \) denote the number of columns of length \( j \). Then \( m(v) := \sum_j m_j(v) \) is the number of columns of the Young frame of \( v \) and \( \lambda(v) := \sum_j m_j(v)\omega_j \) is the highest weight corresponding to \( v \) with respect to the simple roots \( -\alpha_1, \ldots, -\alpha_n \). Let \( k(v) \) be the number of squares in the last (i.e. \( m(v) \)-th) column. Let \( \succ \) denote the lexicographic ordering of pairs \((m,k)\), where \( m \in \mathbb{N}, \ k \in \{1, \ldots, n\} \). That is, \((m,k) \succ (m',k')\) if either \( m > m' \) or \( m = m' \), \( k > k' \). As shown in the first part of this proof, the assertion is true for all irreducible corepresentations \( v \) such that \( m(v) \leq 1 \). Now let \( v \) be an arbitrary irreducible corepresentation of \( \mathcal{O}(G_q) \) such that \((m(v),k(v)) \succ (1,n) \). Suppose that the assertion holds for all irreducible corepresentations \( w \) such that \((m(v),k(v)) \succ (m,w), (w,k(w)) \). We shall prove the assertion for \( v \), that is, we have to show that \( \tau(-2\lambda(v)) \in A^c(v) \).

Let \( v_1 \) be the irreducible corepresentation with Young frame consisting of the last column of \( v \) (i.e. with Young frame \([1^k(v), 0^{n-k(v)}]\)) and let \( w \) be the irreducible corepresentation which is obtained if the last column in the Young frame of \( v \) is cancelled. Then we have \( \lambda(v) = \lambda(w) + \lambda(v_1) \) and \( \lambda(v_1) = \omega_{k(v)} \). Since \((m(v),k(v)) \succ (m,w), (w,k(w)) \), by assumption there is an element \( a(w) \in \mathcal{C}(w) \) such that \( l(a(w)) = \tau(-2\lambda(w)) \). For \( a, b \in \mathcal{O}(G_q) \), we easily compute that \( l(ab) = S(l^-(a(1))) l(b) l^+ (a(2)) \). Setting \( a = a(v_1) = D_{q,k_1(v)} \) and \( b = a(w) \) and using (27) we get

\[
l(a(v_1) a(w)) = l^+_{1} \cdots l^+_{k(v)} l^+_{1} (\tau(-2\lambda(w)) l^+_{1} \cdots l^+_{k(v)})_{k(v)}
\]

\[
= \tau(-2\lambda(w)) (l^+_{1} \cdots l^+_{k(v)})^2 = \tau(-2\lambda(w) - 2\omega_{k(v)}) = \tau(-2\lambda(v)).
\] (28)
Since the Hopf algebra $\mathcal{O}(G_q)$ is cosemisimple ([8], or [11], Theorem 11.22), the corepresentation $v_1 \otimes w$ of $\mathcal{O}(G_q)$ decomposes into a direct sum $\bigoplus_j w_j$ of irreducible corepresentations $w_j$. From the decomposition rules of the tensor product in terms of Young frames (see, for instance, [1], §8, c) we see that precisely one summand, say $w_0$, has the same Young frame as $v$ and that $(m(v), k(v)) > (m(w_j), k(w_j))$ for all other summands. Thus, $w_0$ is equivalent to $v$, so that $\mathcal{C}(v) = \mathcal{C}(w_0)$. By the induction hypothesis combined with Lemma 4.5(i) we conclude that $X^c(w_j) = W(\lambda(w_j)), j \neq 0$. The element $a(v_1)\alpha(v)$ is in $\mathcal{C}(v_1 \otimes w)$ by construction and hence $\tau(-2\lambda(v)) = l(a(v_1)\alpha(v)) \in X^c(v_1 \otimes w)$ by (28). Since $W(\lambda(v))$ is the irreducible $\text{ad}_R$-invariant right coideal generated by $\tau(-2\lambda(v))$ and $X^c(v_1 \otimes w)$ is also an $\text{ad}_R$-invariant right coideal, the latter implies that $W(\lambda(v)) \subset X^c(v_1 \otimes w)$. From the preceding and the facts that $v_1 \otimes w = \bigoplus_j w_j$ and $\mathcal{C}(v) = \mathcal{C}(w_0)$ we obtain

$$W(\lambda(v)) \subset X^c(v) + \sum_{j \neq 0} W(\lambda(w_j)). \tag{29}$$

On the other hand, $X^c(v)$ is an $\text{ad}_R$-invariant right coideal of $\mathcal{U}_q(g)$ and hence a sum $\sum_i W(\lambda_i)$ of irreducible $\text{ad}_R$-invariant right coideals $W(\lambda_i), \lambda_i \in P_+(g)$. Therefore, since the sum $\sum_{\lambda \in P_+(g)} W(\lambda)$ is a direct sum by Proposition 3.2, equation (29) implies that $W(\lambda(v)) \subset X^c(v)$. Hence we have $\tau(-2\lambda(v)) \in X^c(v)$, that is, $\tau(-2\lambda(v)) = l(a(v))$ for some element $a(v) \in \mathcal{C}(v). \quad \blacksquare$

**Corollary 4.8.** Let $v = (v^i_{ij})_{i,j=1,\ldots,m}$ be an irreducible corepresentation of $\mathcal{O}(G_q)$ and let $\zeta \in \mathbb{C}$ be admissible for $G_q$ such that $(v, \zeta) \neq (1,1)$. Then the linear functionals $X_{ij} = \varepsilon_\zeta(l(v^i_j)) - \delta_{ij}\varepsilon, i, j = 1, \ldots, m$, are linearly independent. Moreover, the sets of linear functionals $\{\varepsilon_\zeta(l(v^i_j)), \varepsilon \mid i, j = 1, \ldots, m\}$ and $\{l(v^i_j) \mid i, j = 1, \ldots, m\}$ are also linearly independent.

**Proof.** The second assertion follows from Lemma 4.7 and Lemma 4.5(ii) and (i). The first one is a consequence of the linear independence of functionals $\varepsilon_\zeta(l(v^i_j)), \varepsilon$. \quad \blacksquare

**Lemma 4.9.** Let $v = (v^i_{ij})_{i,j=1,\ldots,m}$ be an irreducible corepresentation of $\mathcal{O}(G_q)$ and let $\zeta \in \mathbb{C}$ be admissible for $G_q$ such that $(v, \zeta) \neq (1,1)$. Then the central functionals $c_\zeta(v) - c_\zeta(v)(1)\varepsilon \in X_\zeta^c(v)$ defined by (13) are nonzero.

**Proof.** By (13) the element $c_\zeta(v) - c_\zeta(v)(1)\varepsilon$ is a linear combination of functionals $\varepsilon_\zeta(l(v^i_j)), i, j = 1, \ldots, m$, and $\varepsilon$. By Corollary 4.8 these functionals are linearly independent. Therefore, since the matrix $D^{-1}$ is invertible, $c_\zeta(v) - c_\zeta(v)(1)\varepsilon \neq 0$. \quad \blacksquare

Now we are able to give the proofs of our main results.

**Proof of Theorem 4.1.** Let $\Gamma$ be a finite dimensional bicharacteristic FODC over $\mathcal{O}(G_q)$ and $X_\Gamma$ be its quantum Lie algebra. By Proposition 2.2(i), $X_\Gamma + \mathbb{C}e$ is an $\text{ad}_R$-invariant right coideal of the Hopf dual $\mathcal{O}(G_q)^* = \mathcal{R}(G_q)^*$. Therefore, by Theorem 4.3, $X_\Gamma + \mathbb{C}e$ is a direct sum $\bigoplus_{i=0}^k \varepsilon_c W(\lambda_i)$, where $\lambda_i \in P_+(g)$ and $\zeta_i$ is an admissible number for $G_q$. Without loss of generality we can assume that $\mathbb{C}e = \varepsilon_{\zeta_0} W(\lambda_0)$, that is, $\zeta_0 = 1$ and $\lambda_0 = 0$. Let $v_i$ denote the irreducible corepresentation of $\mathcal{O}(G_q)$ which corresponds to the irreducible representation of $\mathcal{U}_q(g)$ with highest weight $\lambda_i$ (with respect to the ordered sequence $\{-\alpha_1, \ldots, -\alpha_n\}$ of simple roots). Since $\bigoplus_{i=0}^k \varepsilon_c W(\lambda_i)$ is a direct sum, it follows that the pairs $(\zeta_i, v_i), i = 0, \ldots, k$, are mutually inequivalent. From Lemma 4.5(ii) and Lemma 4.7 we conclude that $X_\zeta(v_i) + \mathbb{C}e = \varepsilon_c W(\lambda_i) + \mathbb{C}e$ for $i = 1, \ldots, k$. Therefore, since $\mathbb{C}e + \bigoplus_{i=1}^k X_\zeta(v_i) = \bigoplus_{i=0}^k \varepsilon_c W(\lambda_i) = \mathbb{C}e + X_\Gamma$, it follows that $\sum_{i=1}^k X_\zeta(v_i)$ is a direct sum and equal to $X_\Gamma$. Since $X_\zeta(v_i)$ is the quantum Lie algebra of the FODC $\Gamma_\zeta(v_i)$, this implies that the
sum of bicovariant FODC $\Gamma_\zeta(v_i)$, $i = 1, \ldots, k$, is a direct sum and that the quantum Lie algebras of the FODC $\bigoplus_{i=1}^k \Gamma_\zeta(v_i)$ and $\Gamma$ coincide. Hence $\Gamma$ is isomorphic to $\bigoplus_{i=1}^k \Gamma_\zeta(v_i)$.

Next we prove that $X_\Gamma = \mathcal{X}[c]$ with $c = c_\zeta(v_1) + \cdots + c_\zeta(v_k)$. Since $c_i := c_\zeta(v_i) - c_\zeta(v_i)(1)\epsilon \in X_\zeta(v_i)$ as noted in 2.3, we have $c - c(1)\epsilon \in X_\Gamma$ and so $c \in X_\Gamma \oplus \mathbb{C}\epsilon$. Let us write $\Delta(c) = \sum x_j \otimes y_j$ with $\{y_j\}$ linearly independent. Since $X_\Gamma \oplus \mathbb{C}\epsilon$ is a right coideal by Proposition 2.7(i), $x_j \in X_\Gamma \oplus \mathbb{C}\epsilon$ and hence $x_j - x_j(1)\epsilon \in X_\Gamma$. As noted in 2.4, $\mathcal{X}[c]$ is the linear span of functionals $x_j - x_j(1)\epsilon$, so we conclude that $\mathcal{X}[c] \subset X_\Gamma$ and hence

$$
\mathcal{X}[c] \oplus \mathbb{C}\epsilon \subset X_\Gamma \oplus \mathbb{C}\epsilon = \bigoplus_{i=1}^k X_\zeta(v_i) \oplus \mathbb{C}\epsilon = \bigoplus_{i=0}^k \mathbb{C}\epsilon W(\lambda_i).
$$

(30)

On the other hand, from Proposition 2.7 and Proposition 2.2(i) we obtain that $\mathcal{X}[c] \oplus \mathbb{C}\epsilon$ is an $ad_R$-invariant right coideal of $\mathcal{O}(G_q)^\circ$ and so a direct sum of irreducible right coideals $\mathbb{C}\epsilon W(\lambda_i)$. Equation (31) implies that $\mathcal{X}[c] \oplus \mathbb{C}\epsilon = \bigoplus_{I \in \mathcal{I}} \mathbb{C}\epsilon W(\lambda_i) \oplus \mathbb{C}\epsilon$ for some subset $I$ of $\{1, \ldots, k\}$. Setting $a = 1$ in (14) we get $c - c(1)\epsilon \in \mathcal{X}[c]$ and hence $c = c_1 + \cdots + c_k + c(1)\epsilon \in \mathcal{X}[c] \oplus \mathbb{C}\epsilon = \bigoplus_{I \in \mathcal{I}} X_\zeta(v_i) \oplus \mathbb{C}\epsilon$. Since $c_i \neq 0$ by Lemma 4.3 and $c_i \in X_\zeta(v_i)$ for $i = 1, \ldots, k$, the latter is only possible if $I = \{1, \ldots, k\}$. This implies at once that $\mathcal{X}[c] = X_\Gamma$ and completes the proof of Theorem 4.1.

**Proof of Corollary 4.2.** Since the linear functionals $[I]$ for any FODC $\Gamma_\zeta(v_i)$, $i = 1, \ldots, k$, are linearly independent by Corollary 1.3, we have $\Gamma_\zeta(v_i) = \Gamma$ (in the notation of 2.3, see (11)). Hence the corresponding biinvariant element $\theta_i$ belongs to $\Gamma_\zeta(v_i)$. By the definition of the direct sum FODC, $\theta := \sum_{i=1}^k \theta_i$ is a biinvariant element of $\bigoplus_{i=1}^k \Gamma_\zeta(v_i)$ which defines the differentiation of the FODC $\bigoplus_{i=1}^k \Gamma_\zeta(v_i)$ to $\Gamma$ by (2). Thus $\Gamma$ is inner.

**Proof of Corollary 4.3.** Since the bicovariant FODC $\Gamma$ is irreducible, by Lemma 2.3 there is no $ad_R$-invariant right coideal $\mathcal{Y}$ of $\mathcal{O}(G_q)^\circ$ such that $\mathbb{C}\epsilon \subset \mathcal{Y} \subset X_\Gamma \oplus \mathbb{C}\epsilon$ and $\mathcal{Y} \neq \mathbb{C}\epsilon, X_\Gamma \oplus \mathbb{C}\epsilon$. Therefore we have $k = 1$ in the proof of Theorem 4.1 and so $\Gamma$ is isomorphic to $\Gamma_\zeta(v_1)$.

**Proof of Corollary 4.4.** The assumption $v = \bigoplus_{i=0}^k n_i v_i$ yields that $X_\zeta(v) = \sum_{i=0}^k X_\zeta(v_i)$ for $\zeta \neq 1$ and $n_0 > 0$ and $X_\zeta(v) = \sum_{i=0}^k X_\zeta(v_i)$ otherwise. Since the corepresentation $v_0, \ldots, v_k$ are pairwise inequivalent, $\lambda(v_i) \neq \lambda(v_j)$ for $i \neq j$. Hence $\sum_{i=0}^k W(\lambda_i)$ is a direct sum. This implies that $\mathcal{X}_\zeta(v) = \bigoplus_{i=0}^k X_\zeta(v_i)$ ($\mathcal{X}_\zeta(v) = \bigoplus_{i=1}^k X_\zeta(v_i)$, resp.) is a direct sum. By Lemma 2.4, $\Gamma_\zeta(v)$ is a direct sum of $\Gamma_\zeta(v_0), \ldots, \Gamma_\zeta(v_k)$ ($\Gamma_\zeta(v_1), \ldots, \Gamma_\zeta(v_k)$, resp.).

**5 Factorizability of the Hopf algebras $\mathcal{O}(G_q)$**

Following [13], we say that a pair $(\mathcal{A}, r)$ of a coquasitriangular Hopf algebra $\mathcal{A}$ and a universal $r$-form $r$ is factorizable if the bilinear form $\mathbf{q}$ on $\mathcal{A} \otimes \mathcal{A}$ defined by

$$
\mathbf{q}(a \otimes b) = r(b_{(1)} \otimes a_{(1)})r(a_{(2)} \otimes b_{(2)}), \quad a, b \in \mathcal{A},
$$

is non-degenerate. The corresponding dual notion for quasitriangular Hopf algebras was introduced in [12].

The functionals $l^j_i$ (see [1]) are of the form $l^j_i(\cdot) = \mathbf{q}(\cdot \otimes v^j_i)$, see [1]. For arbitrary $a \in \mathcal{A}$, we define a linear functional $l(a)$ on $\mathcal{A}$ by $l(a)(\cdot) := \mathbf{q}(\cdot \otimes a)$. 

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Theorem 5.1. Suppose that $q$ is transcendental. Let $r$ be the canonical universal $r$-form on $\mathcal{O}(G_q)$ determined by (14). Then the pair $(\mathcal{O}(G_q), r)$, $G_q = SL_q(n + 1), Sp_q(2n)$, is factorizable.

Proof. Suppose that $l(a) = 0$ for some $a \in \mathcal{O}(G_q)$. Let $v(\lambda)$ denote the corepresentation of $\mathcal{O}(G_q)$ corresponding to $\lambda \in P_+(g)$. Because $\mathcal{O}(G_q)$ is cosemisimple (see [3] or [11]), $a$ is a finite sum of elements $a_\lambda \in C(v(\lambda))$. Recall that $l(a_\lambda) \in \mathcal{X}^c(v(\lambda))$ by definition. Since $\mathcal{X}^c(v(\lambda)) = W(\lambda(v))$ by Lemmas 4.5 and 1.7 and the sum of spaces $W(\lambda)$ is direct, the assumption $l(a) = \sum l(a_\lambda) = 0$ implies that $l(a_\lambda) = 0$ for all $\lambda$. Since the functionals $l(v(\lambda))$ form a basis of $\mathcal{X}^c(v(\lambda))$ by Lemmas 4.5 and 1.7, $a_\lambda = 0$ for all $\lambda$. Thus we have shown that $l(a)(b) = q(a \otimes b) = 0$ for all $b \in \mathcal{O}(G_q)$ implies that $a = 0$. Since $q(S(b) \otimes S(a)) = q(a \otimes b)$ and $S$ is invertible, $q$ is non-degenerate.

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