CENTRAL VALUES OF ADDITIVE TWISTS OF MODULAR $L$-FUNCTIONS

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Abstract. Additive twists of a modular $L$-function are important invariants associated to a cusp form, since the additive twists encode the Eichler-Shimura isomorphism. In this paper we prove that additive twists of $L$-functions associated to cusp forms $f$ of even weight are asymptotically normally distributed. This generalizes a recent breakthrough of Petridis and Risager concerning the arithmetic distribution of modular symbols. Furthermore we present applications to the moments of $L(f \otimes \chi, 1/2)$ supplementing recent work of Blomer-Fouvry-Kowalski-Michel-Miličević-Sawin.

1. Introduction

The study of central values of $L$-function is a prominent subject in number theory. In this paper we study the arithmetic distribution of the central values of additive twists of the $L$-function associated to a cusp form of even weight. The additive twists of modular $L$-functions carry deep arithmetical information since they encode the Eichler-Shimura isomorphism. We show that the central values are asymptotically normally distributed when given an arithmetical ordering. We also present applications to the first and second moment of multiplicative twists of modular $L$-functions. The second moment result gives an average version (including all moduli) of the results of Blomer-Fouvry-Kowalski-Michel-Miličević-Sawin from [2] and [16].

In another direction the results of this paper can be seen as a higher weight analogue of a recent paper by Petridis and Risager [21], in which they settle a conjecture of Mazur and Rubin [17] concerning modular symbols. This conjecture predicts the arithmetic distribution of the modular symbol map

$$\{\infty, a\} \mapsto \langle a, f \rangle := 2\pi i \int_{\infty}^{a} f(z)dz,$$

where $f \in S_2(\Gamma_0(q))$ is a cusp form of weight $k$ and level $q$ and $\{\infty, a\}$ is the homology class of curves between the cusps $\infty$ and $a$. Petridis and Risager prove that this map is asymptotically normally distributed when ordered by the denominator of the cusp and appropriately normalized [21, Theorem 1.10].

To state our main results, fix a cofinite, discrete subgroup $\Gamma$ of $SL_2(\mathbb{R})$ with a cusp at $\infty$ of width 1. Define the set (following [21])

$$T_{\Gamma} = T := \left\{ r \equiv \frac{a}{c} \mod 1 \mid \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \Gamma_{\infty} \backslash \Gamma / \Gamma_{\infty}, c > 0 \right\}.\quad (1.1)$$

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In [21] it is shown that $r$ uniquely determines an element in the double quotient and we therefore define $c(r)$ as the left-lower entry of any representative. From this we define

$$T(X) := \{ r \in T \mid c(r) \leq X \}. \tag{1.2}$$

Now let $f \in S_k(\Gamma)$ be a cusp form of even weight $k$ with Fourier expansion (at $\infty$) given by

$$f(z) = \sum_{n \geq 1} a_f(n)n^{(k-1)/2}q^n,$$

where $q = e^{2\pi iz}$. Then for $r \in \mathbb{R}$ we define the additive twist of the $L$-function associated to $f$ as

$$L(f \otimes e(r), s) := \sum_{n \geq 1} \frac{a_f(n)e(nr)}{n^s},$$

where $e(x) = e^{2\pi ix}$. This converges absolutely for $\text{Re } s > 1$ and for specific choices of $r$ these Dirichlet series satisfy analytic continuation and a functional equation (see Section 3.3). The main results of this paper are the following two theorems.

**Theorem 1.1.** Let $f \in S_k(\Gamma)$ be a cusp form of even weight $k$ and let $n, m$ be non-negative integers. Then we have

$$\sum_{r \in T(X)} L(f \otimes e(r), 1/2)^n L(f \otimes e(r), 1/2)^m \ll \log(X)^{\min(n,m)} X^2. \tag{1.3}$$

When $n = m$ we furthermore have

$$\sum_{r \in T(X)} |L(f \otimes e(r), 1/2)|^{2n} = P_n(\log X)X^2 + O_{\varepsilon}(X^{\max(4/3, 2 \text{ Re } s_1) + \varepsilon}), \tag{1.4}$$

where $s_1$ is the first small eigenvalue of the Laplacian $\Delta_\Gamma$ and $P_n$ is a polynomial of degree $n$ with leading coefficient

$$\frac{2^{n!}n!}{\pi \text{ vol}(\Gamma)} (C_f)^n,$$

with

$$C_f = \frac{(4\pi)^k ||f||^2}{(k-1)! \text{ vol}(\Gamma)},$$

where $||f||$ denotes the Petersen-norm of $f$ and $\text{ vol}(\Gamma)$ is the hyperbolic volume of $\Gamma \backslash \mathbb{H}$.

**Remark 1.2.** S. Bettin [1] has considered the some-what similar situation of the Estermann function

$$D(s; r) := \sum_{n \geq 1} \frac{d(n)e(na/c)}{n^s},$$

where $d(n)$ is the divisor function and $a/c \in \mathbb{Q}$. He manages to calculate all moments averaging over $a \in (\mathbb{Z} \backslash c\mathbb{Z})^\times$ using an approximate functional equation. He applied his results to studying certain moments of $L(\chi, 1/2)$. This can be seen as the $GL_1$-case (or Eisenstein-case) of the theorem above.

**Remark 1.3.** The constant $C_f$ is a higher weight analogue of the variance slope defined by Mazur and Rubin (see [21] Theorem 1.9).

From Theorem 1.1 we easily conclude using a classical result from probability theory due to Fréchet-Shohat [23, p.17] the following distribution result.
Theorem 1.4. Let \( f \in S_k(\Gamma) \) be a cusp form of even weight \( k \). Then we have
\[
\# \left\{ r \in T(X) \mid x_1 \leq \text{Re} \left( \frac{L(f \otimes e(r), 1/2)}{C_f \log c(r)} \right) \leq x_2, y_1 \leq \text{Im} \left( \frac{L(f \otimes e(r), 1/2)}{C_f \log c(r)} \right) \leq y_2 \right\} \rightarrow \frac{1}{2\pi} \int_{x_1}^{x_2} \int_{y_1}^{y_2} e^{-(x^2+y^2)/2} \, dx \, dy,
\]
as \( X \rightarrow \infty \).

If the cusp \( a \) is represented by the real number \( r \in \mathbb{R} \) (i.e. \( r \) is fixed by the parabolic subgroup \( \Gamma_a \)), then it follows by the integral representation (3.9) that for \( f \in S_2(\Gamma) \)
\[
(a, f) = L(f \otimes e(r), 1/2).
\]

Thus the additive twists generalize modular symbols to higher weight forms and one immediately sees that Theorem 1.4 generalizes [21, Theorem 1.10].

Remark 1.5. The proof of Theorem 1.4 relies on determining the analytic properties of a certain Eisenstein series \( E^{n,m}(z, s) \) generalizing a series introduced by Goldfeld in [9] and [10]. Determining the location of the dominating pole, the corresponding pole order and leading Laurent coefficient of the original Goldfeld Eisenstein series was firstly achieved by Petridis and Risager in [20] using perturbation theory and the analytic properties of the hyperbolic resolvent. This allowed them to prove normal distribution for a certain more geometrically flavored ordering of the modular symbols (ordered by \( c^2 + d^2 \), where \( c, d \) are the lower entries of the matrix \( \gamma \)). In order to prove the conjecture of Mazur and Rubin, they essentially had to derive the analytic properties of the constant Fourier coefficient of \( E^{n,m}(z, s) \). This is somewhat reminiscent of the Shahidi-Langlands method [8, Section 8]. The strategy of proof in this paper is inspired by the overall strategy invented by Petridis and Risager.

Remark 1.6. Our results generalize those of Petridis and Risager [21] in that we derive explicit error-terms in (1.4). In the appendix we have explained the complex analysis involved. Although the method is very well known, the details with explicit error-term are hard to find in the literature, so we included them for convenience.

1.1. Applications to multiplicative twists. The motivation for the conjecture by Mazur and Rubin was to gain information about the non-vanishing of the central value \( L(E \otimes \chi, 1) \) where \( E/\mathbb{Q} \) is an elliptic curve and \( \chi \) is a primitive Dirichlet character. By the conjectures of Birch-Swinnerton-Dyer this is related to studying when
\[
\text{rank } E(K) > \text{rank } E(\mathbb{Q})
\]
for cyclic extensions \( K/\mathbb{Q} \) (see [17] for details). If \( \chi \) is primitive mod \( c \) then we have by the Birch-Stevens formula (see [22, Eq. 2.2])
\[
\tau(\chi)L(E \otimes \chi, 1) = \sum_{a \in (\mathbb{Z}/c\mathbb{Z})^\times} \chi(a)\langle a/c, f_E \rangle,
\]
where \( f_E \) is the primitive form corresponding to \( E \) via modularity and \( \tau(\chi) \) is a Gauss-sum. This led Mazur and Rubin to the study of the distribution of modular symbols, and based on computational experiments they made certain conjectures about the distribution of modular symbols.

Following these lines of thinking, we apply Theorem 1.1 to the study of the family
\[
\{ L(f \otimes \chi, 1/2) \mid \chi \text{ mod } c \}.\]
For this we specialize to the case $\Gamma = \Gamma_0(q)$ where $q$ is square-free and when choosing $n = m = 1$, we get the following.

**Corollary 1.7.** Let $f \in S_k(\Gamma_0(q))$ be a primitive form of even weight $k$ and square-free level $q$. Then we have

$$
\sum_{c \leq X} \frac{1}{\varphi(c)} \sum_{\chi \mod c} c(\chi) |\nu(f, \chi^*, c/c(\chi))|^2 |L(f \otimes \chi^*, 1/2)|^2
$$

$$
= \frac{(4\pi)^k |f|^2}{\pi(k-1)! (\pi/3) \text{vol}(\Gamma_0(q))} (\log X)^2 + \beta_f X^2 + O_{\varepsilon}(X^{4/3+\varepsilon}),
$$

(1.5)

where $\chi$ is induced by the primitive character $\chi^* \mod c(\chi)$ and

$$
\nu(f, \chi, n) = \sum_{n_1 n_2 n_3 = n} \chi(n_1) \mu(n_1) \chi(n_2) \mu(n_2) \lambda_f(n_3) n_3^{1/2}.
$$

And for the 1st moment.

**Corollary 1.8.** Let $f \in S_k(\Gamma_0(q))$ be a primitive form of even weight $k$ and square-free level $q$. Then we have

$$
\sum_{c \leq X} \frac{1}{\varphi(c)} \sum_{\chi \mod c} \tau(\chi) \tau(\chi^*) |\nu(f, \chi^*, c/c(\chi))| L(f \otimes \chi^*, 1/2)
$$

$$
= \frac{3}{\pi} X^2 + O_{\varepsilon}(X^{4/3+\varepsilon}),
$$

(1.6)

with $\chi^* \mod c(\chi)$ and $\nu(f, \chi, n)$ as above.

**Remark 1.9.** Similarly Bettin [1] uses his results on the Estermann function to study certain iterated moments of Dirichlet $L$-functions.

Corollary 1.7 should be compared with the following result proved in [2] and [16].

**Theorem 1.10 (Blomer-Fouvry-Kowalski-Milčević-Sawin).** Let $f \in S_k(\Gamma_0(1))$ be a cusp form of weight $k$ and level 1. Then we have

$$
\frac{1}{\varphi^*(p)} \sum_{\chi \mod p \chi \text{ primitive}} |L(f \otimes \chi, 1/2)|^2 = \frac{6\pi^{k-3} 4^k |f|^2}{(k-1)! \log p} \log p^{144} + \beta_f + O_{\varepsilon}(p^{-1/144+\varepsilon})
$$

as $p \to \infty$ runs through the primes.

The methods in [2] and [16] rely on deep input from algebraic geometry via the theory of trace functions, whereas our method is much more general. Furthermore our methods naturally incorporate all moduli.

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2. Method of proof

In this section we will describe the overall strategy of the proof of Theorem 1.4. By the classical result of Fréchet and Shohat [23, p.17], it is enough to show that the asymptotic moments of the central values

$$\sum_{r \in T(X)} L(f \otimes e(r), 1/2)^n L(f \otimes e(r), 1/2)^m,$$

agree with those of the complex Gaussian as $X \to \infty$.

By a standard complex analysis argument, this can be reduced to understanding the analytic properties of the following Dirichlet series

$$D_{n,m}(f,s) := \sum_{r \in T} \frac{L(f \otimes e(r), 1/2)^n L(f \otimes e(r), 1/2)^m c(r)^2 s}{c(r)^{2s}}.$$

We will derive the analytic properties of $D_{n,m}(f,s)$ by studying the following generalized Goldfeld Eisenstein series

$$E_{n,m}(z,s) := \sum_{\gamma \in \Gamma \setminus \infty} L(f \otimes e(\gamma \infty), 1/2)^n L(f \otimes e(\gamma \infty), 1/2)^m \Im(\gamma z)^s,$$

where we consider $\Gamma$ as a subgroup of $PSL_2(\mathbb{R})$. Observe that $L(f \otimes e(\gamma \infty), 1/2)$ is well-defined in the coset $\Gamma \setminus \infty$ and we will see that $E_{n,m}(z,s)$ converges a priori absolutely for $\Re s \gg 1$ (see (3.11)), but we will show that in fact it converges absolutely for $\Re s > 1$. The series (2.1) and (2.2) are connected since the constant Fourier coefficient of $E_{n,m}(z,s)$ is given by

$$\frac{\pi^{1/2} y^{1-s} \Gamma(s - 1/2)}{\Gamma(s)} D_{n,m}(f,s)$$

(see Lemma 5.3). This will allow us to pass analytic information from $E_{n,m}(z,s)$ to $D_{n,m}(f,s)$ as one does in the Langlands-Shahidi method.

In order to get a hold of the analytic properties of $E_{n,m}(z,s)$ we will employ a technique inspired by the one developed by Petridis and Risager in [20] and [21]. Furthermore we also use ideas from an unpublished paper by Chinta and O’Sullivan [3]. Firstly we express $E_{n,m}(z,s)$ as a linear combination of certain Poincaré series $G_{A,B,l}(z,s)$, which are elements of $L^2(\Gamma \setminus \mathbb{H}, l)$. Then we will use the analytic properties of the weight $k$-resolvent to recursively understand the pole order at $s = 1$ and the leading Laurent coefficient of $G_{A,B,l}(z,s)$.

The overall strategy can be illustrated as follows.

\{Analytic properties of weight $k$-resolvents\} \rightarrow \text{Induction argument}

\{Analytic properties of the Poincaré series $G_{A,B,l}(z,s)$\} \rightarrow \text{A formula for the central value}

\{Analytic properties of $E_{n,m}(z,s)$\} \rightarrow \text{Fourier expansion}

\{Analytic properties of $D_{n,m}(f,s)$\} \rightarrow \text{Contour integration}

\{Asymptotic moments of $L(f \otimes e(r), 1/2)$\} \rightarrow \text{Fréchet-Shohat}

\{Normal distribution of $L(f \otimes e(r), 1/2)/(C_f \log c(r))^{1/2}$\}.
3. Background: Weight $k$ Laplacians and Additive Twists

In this section we will recall some facts about higher weight Laplacians and additive twists of modular $L$-functions. For this, fix a discrete and co-finite subgroup $\Gamma$ of $SL_2(\mathbb{R})$ with a cusp at $\infty$ of width 1.

3.1. Weight $k$ Laplacian. Let $k$ be an even integer. The space of automorphic functions of weight $k$ are (measurable) functions $g : \mathbb{H} \to \mathbb{C}$ satisfying

$$g(\gamma z) = j_\gamma(z)^k g(z), \quad \text{for all } \gamma \in \Gamma,$$

where $j_\gamma(z) = j(\gamma, z)/|j(\gamma, z)| = (cz + d)/|cz + d|$ and $c, d$ are the bottom row-entries of $\gamma$. Note that

$$j_{\gamma_1\gamma_2}(z) = j_{\gamma_1}(\gamma_2 z) j_{\gamma_2}(z).$$

Given an automorphic function $g$ of weight $k$, we define the Petersson norm by

$$||g||^2 := \int_{\Gamma \backslash \mathbb{H}} |g(z)|^2 d\mu(z),$$

where $d\mu(z) = dx dy/y^2$ is the hyperbolic measure on $\mathbb{H}$. From this we define the Hilbert space of all square integrable weight $k$ automorphic functions;

$$L^2(\Gamma, k) := \{ g \text{ automorphic of weight } k \mid ||g|| := \sqrt{\langle g, g \rangle} < \infty \}.$$

with inner-product given by

$$\langle g, h \rangle := \int_{\Gamma \backslash \mathbb{H}} g(z)\overline{h(z)} d\mu(z).$$

Maass defined certain raising- and lowering operators \cite[Chapter 4]{[5]} which maps between spaces of different weights. They are defined in local coordinates as

$$K_k := (z - \overline{z}) \frac{\partial}{\partial z} + \frac{k}{2},$$

$$L_k := (z - \overline{z}) \frac{\partial}{\partial \overline{z}} + \frac{k}{2}$$

for $z \in \mathbb{H}$ and they define unbounded operators

$$K_k : D(K_k) \subset L^2(\Gamma \backslash \mathbb{H}, k) \to L^2(\Gamma \backslash \mathbb{H}, k+2),$$

$$L_k : D(L_k) \subset L^2(\Gamma \backslash \mathbb{H}, k) \to L^2(\Gamma \backslash \mathbb{H}, k-2)$$

with certain domains $D(K_k)$ and $D(L_k)$ dense in $L^2(\Gamma, k)$. The raising and lowering operators are adjoint to each other in the following sense

$$\langle K_k g_k, g_{k+2} \rangle = \langle g_k, L_{k+2} g_{k+2} \rangle$$

for $g_k \in D(K_k)$ and $g_{k+2} \in D(L_{k+2})$ (see \cite[section 2.1.2]{[18]} for a reference on all these matters).

Remark 3.1. In most modern expositions the raising operator is denoted $R_k$ (see \cite[Chapter 4]{[5]}), but in order to avoid confusion with the resolvent we follow the notation of Fay \cite{[7]}. We remark that our definition of the lowering operator is equal to minus the one of Fay.

From these two operators the weight $k$-Laplacean is defined as

$$\Delta_k := K_{k-2} L_k + \lambda(k/2) = L_{k+2} K_k + \lambda(-k/2),$$

where $\lambda(s) = s(1-s)$. This defines a symmetric operator with a self-adjoint extension on the Hilbert space $L^2(\Gamma, k)$ with a certain domain $D(\Delta_k)$ dense in $L^2(\Gamma, k)$ (see \cite[Chapter 3.1]{[12]})
4] for the weight 0 case and [5 Chapter 4] for general weights. We denote the unique self-adjoint extension also by $\Delta_k$.

One sees by a direct computation that for $f \in S_k(\Gamma)$, we have $y^{k/2}f(z) \in L^2(\Gamma, k)$ and (3.4)

$$L_k y^{k/2} f(z) = 0.$$ 

This shows that $y^{k/2} f(z)$ is an eigenfunction of $\Delta_k$ with eigenvalue $\lambda(k/2)$ by (3.3).

We parametrize the spectrum of $\Delta_k$ with respect to the $s$-parameter where

$$\lambda = \lambda(s) = s(1 - s)$$

with $\text{Re } s \geq 1/2$ (and $\text{Im } s \geq 0$ on the line $\text{Re } s = 1/2$). The continuous part of the spectrum $\text{spec } \Delta_k$ is always given by the line $\text{Re } s = 1/2$ and furthermore for $k = 0$ the point $s = 1$ is always in the spectrum of $\Delta = \Delta_0$, since the constant function is annihilated by $\Delta$.

Now let

$$\mathcal{P} = \mathcal{P}_\Gamma := \{ s_0 = 1, s_1, s_2, \ldots \}$$

be the eigenvalues of $\Delta$ where

$$\ldots \leq \text{Re } s_2 \leq \text{Re } s_1 < \text{Re } s_0 = 1$$

Recall that we might have embedded eigenvalues i.e. eigenvalues on the line $\text{Re } s = 1/2$. The size of $\text{Re } s_1$ will turn out to control the error-terms of our results.

Using the raising- and lowering operators (see [5 Chapter 4]) one can show that

$$\text{spec } \Delta_k \subset \{ 1/2 + it \mid t \geq 0 \} \cup \mathcal{P} \cup \{ 1, \ldots, k/2 \}$$

where the added eigenvalues correspond to holomorphic cusp forms $S_j(\Gamma)$ with $2 \leq j \leq k$ and $j \equiv k (2)$ (see [5 Chapter 4]).

The eigenvalues $0 < \lambda < 1/4$ (corresponding to points in the spectrum with $1/2 < s < 1$) are called small eigenvalues. It is a famous conjecture of Selberg that there are no small eigenvalues when $\Gamma = \Gamma_0(N)$ is a Hecke congruence subgroup. It is a theorem of Kim and Sarnak that for $\Gamma_0(N)$ the smallest eigenvalue $\lambda_1 > 0$ satisfies

$$\lambda_1 \geq 1/4 - \left( \frac{7}{64} \right)^2$$

which means $s_1 \leq 39/64$ [15].

3.2. **Weight $k$-resolvents.** Associated to the weight $k$-Laplacian, we have the resolvent operator

$$R(\cdot, k) : \mathbb{C} \setminus \text{spec } \Delta_k \to \mathcal{B}(L^2(\Gamma, k))$$

characterized by the property

$$(\Delta_k - \lambda(s))R(s, k) = Id.$$ 

The analytic properties of weight $k$ resolvents have been studied intensively by Fay in [7].

It follows from general properties of resolvents and (3.6) that $R(s, k)$ defines a meromorphic
operator in the half-plane $\Re s > 1/2$ with poles at $P \cup \{1, \ldots, k/2\}$. In a neighborhood of one of these poles $p$, we have the following representation

\[(3.7) \quad R(s,k) = \frac{P_{\lambda(p),k}}{\lambda(s) - \lambda(p)} + R_{\text{reg},p}(s,k),\]

where $P_{\lambda(p),k}$ is the projection to the eigenspace of $\Delta_k$ corresponding to the eigenvalue $\lambda(p)$ and $R_{\text{reg},p}(s,k)$ is regular at $s = p$.

Finally we also quote the following useful bound on the norm of the resolvent (see [12, Appendix A]).

**Lemma 3.2.** For $s \in \mathbb{C}\setminus \text{spec}(\Delta_k)$ we have

\[\|R(s,k)\| \leq \frac{1}{\text{dist}(\lambda(s),\text{spec}(\Delta_k))},\]

where $\| \cdot \|$ is the operator norm and dist is the distance function.

### 3.3. Additive twists of $L$-functions.

Let $f \in S_k(\Gamma)$ be a cusp form of even weight $k$. Then we already defined the additive twists of the associated $L$-function as follows

\[L(f \otimes e(r),s) := \sum_{n \geq 1} \frac{a_f(n)e(nr)}{n^s},\]

where $e(z) = e^{2\pi iz}$ and $r \in \mathbb{R}$. We also define $L(f \otimes e(\infty),s) := 0$. For all $r \in \mathbb{R}$, the additively twisted $L$-function $L(f \otimes e(r),s)$ converges absolutely for $\Re s > 1$ by Hecke’s bound (see [12, Theorem 3.2])

\[(3.8) \quad \sum_{n \leq X} |a_f(n)|^2 \ll_f X.\]

However the twists given by $a/c = \gamma_\infty$ where

\[\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\]

are of special significance. For such twists we have an associated completed $L$-function given by

\[\Lambda(f \otimes e(a/c),s) := \Gamma(s + (k - 1)/2) \left( \frac{c}{2\pi} \right)^{s+(k-1)/2} L(f \otimes e(a/c),s).\]

The completed $L$-function satisfies the following functional equation (generalizing [14, Lemma 1.1]).

**Proposition 3.3.** For $\gamma \in \Gamma$, the completed $L$-function $\Lambda(f \otimes e(a/c),s)$ admits analytic continuation to the entire complex plane and satisfies the functional equation

\[\Lambda(f \otimes e(a/c),s) = (-1)^{k/2} \Lambda(f \otimes e(-d/c),1-s),\]

where $a/c = \gamma_\infty$ and $-d/c = \gamma^{-1}\infty$.

**Proof.** We mimic Hecke’s proof of analytic continuation and functional equation of modular $L$-functions. In the range of absolute convergence of $L(f \otimes e(a/c),s)$, we have the following
where the implied constant also depends on \(f\) which shows that
\[
\Gamma(s) = \left(\frac{2\pi}{c}\right)^{s-1/2} \Gamma(s-1/2) = \Lambda(f \otimes e(a/c), s).
\]

Now observe that
\[
\gamma(-d/c - 1/(iy)) = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) (-d/c - 1/(iy)) = a/c + iy/c
\]
and
\[
j(\gamma, -d/c - 1/(iy)) = -(iy)^{-1}.
\]

This yields
\[
\int_0^\infty f(a/c + iy/c)y^{s+(k-1)/2} \frac{dy}{y} = \int_0^1 f(a/c + iy/c)y^{s+(k-1)/2} \frac{dy}{y} + \int_1^\infty f(\gamma(-d/c - 1/(iy)))y^{s+(k-1)/2} \frac{dy}{y}
\]
\[
= \int_0^1 f(a/c + iy/c)y^{s+(k-1)/2} \frac{dy}{y} + \int_1^\infty j(\gamma, -d/c - 1/(iy))^{k} f(-d/c - 1/(iy))y^{s+(k-1)/2} \frac{dy}{y}
\]
\[
= \int_0^1 f(a/c + iy/c)y^{s+(k-1)/2} \frac{dy}{y} + (1/y)^{k/2} \int_1^\infty f(-d/c - 1/(iy))y^{s+(k-1)/2 - k/2} \frac{dy}{y}
\]
\[
= \int_0^1 f(a/c + iy/c)y^{s+(k-1)/2} \frac{dy}{y} + (-1)^{k/2} \int_0^1 f(-d/c + iy/c)y^{1-s+(k-1)/2} \frac{dy}{y}
\]
using modularity of \(f\) in the second equality and a change of variable \(y \to 1/y\) in the last.

Now we get analytic continuation to the entire complex plane by the exponential decay of \(f\) at the cusps \(a/c\) and \(-d/c\). Furthermore (3.10) yields the functional equation immediately. This completes the proof.

As a first application of the functional equation, we will derive a preliminary bound for the central value of the additive twists at \(s = 1/2\) using the Phragmen-Lindelöf principle (see [13, Theorem 5.53]).

By the absolute convergence of \(L(f \otimes e(a/c), s)\) for \(\Re s > 1\), we get that
\[
\Lambda(f \otimes e(a/c), 1 + \varepsilon + it) \ll_{\varepsilon} c^{k/2 + 1/2 + \varepsilon},
\]
where the implied constant also depends on \(f\) (here we also use Stirling’s approximation, which shows that \(\Gamma(k/2 + 1/2 + \varepsilon + it)\) is bounded in \(t\)). By the functional equation, we
derive similarly that
\[ \Lambda(f \otimes e(a/c), -\varepsilon + it) \ll \varepsilon e^{k/2 + 1/2 + \varepsilon}. \]
Finally by the period integral representation (3.9), we get in the strip $-\varepsilon \leq \Re s \leq 1 + \varepsilon$ that
\[ \Lambda(f \otimes e(a/c), s) \ll \varepsilon. \]
Thus the Phragmen-Lindelöf principle applies and we conclude that
\[ L(f \otimes e(a/c), 1/2) \ll \varepsilon e^{k/2 + 1/2 + \varepsilon} = \varepsilon^{1/2 + \varepsilon}. \]
All though this is a crude bound, it shows together with
\[ \# \left\{ (a, c) \mid 0 \leq a < c, 0 < c \leq X, \left( \begin{array}{c} a \\ c \\ \ast \end{array} \right) \in \Gamma \right\} \ll \Gamma X^2 \]
(see [12, (2.37)]) that the main generating series $D_{n,m}(f, s)$ converges absolutely (and locally uniformly) in some half-plane $\Re s \gg n,m$ to an analytic function. We will see later that in fact $L(f \otimes e(a/c), 1/2) \ll \varepsilon$ for all $\varepsilon > 0$.

Remark 3.4. The additively twisted $L$-functions have been studied in the theory of exponential sums in order to get estimates for sums of the type
\[ \sum_{A \leq n \leq B} a_f(n)e(nr) \]
(see [14]), but they also have arithmetic significance in themselves as they appear in the Eichler-Shimura isomorphism. We recall how this isomorphism is constructed following [24, Section 8.2].

Let $f$ be a cusp form of weight $k$ and level $N$ and let $\gamma \in \Gamma_0(N)$. Then we can associate the following $(k - 1)$-dimensional real vector
\[ u_f(\gamma) = \left( \Re \int_{\gamma \infty} f(z)dz, \Re \int_{\gamma \infty} f(z)zdz, \ldots, \Re \int_{\gamma \infty} f(z)z^{k-2}dz \right). \]
In his study of parabolic cohomology (see [6]) M. Eichler studied what he called the period polynomial defined as
\[ r_{f,\gamma}(X) = \frac{1}{(k-1)!} \int_{-1-\infty}^{\infty} f(z)(z - X)^{k-2}dz \]
and we note that the entries of the above vector are just real part of the coefficients of this polynomial (up to a scaling by a binomial-coefficient).

The map $u_f : \Gamma \rightarrow \mathbb{R}^{k-1}$ defines a parabolic co-cycle, i.e. an element of $Z_{p}^1(\Gamma, X)$ in Shimura’s notation where $X = \mathbb{R}^{k-1}$ is a certain $\Gamma$-module. From this we get a map
\[ f \mapsto \text{cohomology class of } u_f, \]
which by [24 Theorem 8.4] induces an $\mathbb{R}$-linear isomorphism from $S_k(\Gamma)$ to the parabolic cohomology group $H^1_p(\Gamma, X)$. This is what is known as the Eichler-Shimura isomorphism and it can also be described in terms of the period polynomials directly (see [19] for details).
For any $0 \leq l \leq k - 2$ we have
\[ \int_{\gamma \infty} f(z)^ldz = i \int_{0}^{\infty} f(a/c + iy)(a/c + iy)^l dy \]
\[ = \sum_{j=0}^{l} \binom{l}{j} (a/c)^{l-j} \frac{j!}{(-2\pi i)^j+1} L(f \otimes e(a/c), j + 1 - (k - 1)/2), \]
which shows that the special values of additive twists encode the Eichler-Shimura isomorphism.

**Remark 3.5.** It seems possible that the techniques of this paper can be used to determine the distribution of the map \( u_f \). We hope to return to this in future work.

### 4. Poincaré series defined from antiderivatives of cusp forms

In this section, we will construct a certain Poincaré series \( G_{A,B,l}(z, s) \) from a cusp form \( f \in \mathcal{S}_k(\Gamma) \). Then we will study the analytic properties of these Poincaré series, which will be crucial in proving our main results. The study of these Poincaré series might have independent interest.

#### 4.1. Definitions.

Let \( \Gamma \) be a co-finite, discrete subgroup of \( SL_2(\mathbb{R}) \) with a cusp at infinity of width 1. Let \( f \in \mathcal{S}_k(\Gamma) \) be a cusp form of even weight \( k \). Then we define for \( n \geq 1 \)

\[
I_n(z) = I_n(z; f) := \int_\infty^z \int_\infty^{z_{n-1}} \cdots \int_\infty^{z_1} f(z_0) dz_0 dz_1 \cdots dz_{n-1}
\]

and \( I_0(z) = f(z) \). It is clear that we have \( I_{n+1}(z) = I_n(z) \) and by a simple check we see that

\[
I_n(z) = \frac{(-1)^{n-1}}{(n-1)!} \int_\infty^z f(w)(w-z)^{n-1} dw.
\]

The Poincaré series \( G_{A,B,l}(z, s) \), are indexed by multi-sets (i.e. sets where elements have multiplicities) \( A, B \) with elements contained in \( \{0, \ldots, k/2\} \). We call such a multi-set *positive* if all elements are positive or if the multi-set is empty. We let \(|A|\) and \(|B|\) denote the sizes of the multi-sets counted with multiplicity.

For \( l \) an even integer, we define

\[
G_{A,B,l}(z, s) := \sum_{\gamma \in \Gamma \setminus \Gamma'} j_{\gamma}(z)^{-l} \left( \prod_{a \in A} \frac{I_n(\gamma^2; f)}{(-2i)^n} \left( \prod_{b \in B} \frac{I_b(\gamma^2; f)}{(2i)^b} \right) \right) \operatorname{Im}(\gamma z)^{s+\alpha(A, B)},
\]

where

\[
\alpha(A, B) = \left( \sum_{a \in A} k/2 - a \right) + \left( \sum_{b \in B} k/2 - b \right).
\]

We observe that by (3.1) these Poincaré series satisfy

\[
G_{A,B,l}(\gamma z, s) = j_{\gamma}(z)^l G_{A,B,l}(z, s)
\]

for any \( \gamma \in \Gamma \).

The scaling \( \alpha(A, B) \) has the nice property that

\[
G_{A \cup \{0\}, B, l}(z, s) = y^{k/2} f(z) G_{A,B,l-k}(z, s),
\]

\[
G_{A,B, l \cup \{0\}}(z, s) = y^{k/2} f(z) G_{A,B, l+k}(z, s),
\]

which follows from the modularity of \( f \).

Observe that with \( A \) and \( B \) as above, we always have \( \alpha(A, B) \geq 0 \), which will be crucial in many argument. We also have the following symmetry

\[
G_{A,B,l}(z, s) = G_{B,A,-l}(z, s).
\]

This shows that it is enough to consider the case \( l \geq 0 \).
Firstly we will show that (4.2) defines an element of $L^2(\Gamma \setminus \mathbb{H}, l)$ in some half-plane following unpublished work of Chinta and O’Sullivan [3].

Lemma 4.1. For $|A| + |B| > 0$ the series $G_{A,B,l}(z, s)$ converges absolutely (and locally uniformly in $z$ and $s$) in the half-plane

$$\text{Re } s > 1 + |A| + |B|$$

to an element of $L^2(\Gamma, l)$

Proof. By Hecke’s bound on the coefficients of cusp forms $a_f(n) \ll f n^{1/2}$ (see (??)), we have

$$I_n(z) \ll \sum_{m=1}^{\infty} m^{k/2-n} e^{-2\pi my} \ll \frac{(k/2 - n)!}{y^{k/2 + 1 - n}},$$

using that $r! = \Gamma(r + 1) = \int_0^\infty e^{-x} x^r dx$. This gives

$$\sum_{\gamma \in \Gamma \setminus \Gamma} \left| j_\gamma(z)^{-l} \left( \prod_{a \in A} \frac{I_a(\gamma z)}{(-2)^a} \right) \left( \prod_{b \in B} \frac{I_b(\gamma z)}{(2i)^b} \right) \text{Im}(\gamma z)^{\sigma + \alpha(A,B)} \right|$$

$$\ll_{A,B,k} \sum_{\gamma \in \Gamma \setminus \Gamma} \text{Im}(\gamma z)^{\sigma - |A|-|B|}$$

$$= E(z, \sigma - |A| - |B|),$$

where $s = \sigma + it$. Since the standard Eisenstein series converges absolutely for $\text{Re } s > 1$, we get that $G_{A,B,l}(z, s)$ converges absolutely (and locally uniformly in $z$ and $s$) in the desired half-plane. Furthermore (4.6) shows that $I_n(z) \ll e^{-\pi y}$ as $y \to \infty$. This yields

$$G_{A,B,l}(z, s) \ll e^{-\pi y(|A|+|B|)} y^{\sigma + \alpha(A,B)} + \sum_{\gamma \in \Gamma \setminus \Gamma} \text{Im}(\gamma z)^{\sigma - |A|-|B|},$$

Combining this with the fact that the standard Eisenstein series satisfies the bound (see [12, Corollary 3.8])

$$|E(z, w) - y^w| \ll y^{1-\text{Re } w}, \quad \text{Re } w > 1,$$

we conclude that $G_{A,B,l}(z, s) \to 0$ as $y \to \infty$, which implies that

$$G_{A,B,l}(z, s) \in L^2(\Gamma, l).$$

This finishes the proof. \qed

4.2. The recursion formula. In order to understand the pole structure of $G_{A,B,l}(z, s)$ we will use certain recursion formulas involving the resolvent and the raising- and lowering operators. First of all we will record how the raising and lowering operators act on smooth functions.

Lemma 4.2. For smooth functions $h : \mathbb{H} \to \mathbb{C}$, we have

$$K_l[h(\gamma z) \text{Im}(\gamma z)^{s} j_\gamma(z)^{-l}] = \left( 2i \frac{\partial h}{\partial \overline{z}}(\gamma z) \text{Im}(\gamma z)^{s+1} + (s + l/2) h(\gamma z) \text{Im}(\gamma z)^{s} \right) j_\gamma(z)^{-l-2},$$

$$L_l[h(\gamma z) \text{Im}(\gamma z)^{s} j_\gamma(z)^{-l}] = \left( 2i \frac{\partial h}{\partial z}(\gamma z) \text{Im}(\gamma z)^{s+1} - (s - l/2) h(\gamma z) \text{Im}(\gamma z)^{s} \right) j_\gamma(z)^{-l+2}$$

for any $\gamma \in SL_2(\mathbb{R})$. 
Proof. Using the intertwining relation

\[ K_l \left( j_\gamma(z)^{-l} F(\gamma z) \right) = j_\gamma(z)^{-l-2} (K_l F)(\gamma z), \]
valid for any smooth function \( F : \mathbb{H} \rightarrow \mathbb{C} \), we reduce the problem to the identity

\[ K_l h(z)y^s = y \left( i \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{k}{2} \right) h(z)y^s, \]
and similar for the lowering operator. Now the result follows from simple differentiation. \( \square \)

This yields the following useful formula.

**Corollary 4.3.** For smooth functions \( h : \mathbb{H} \rightarrow \mathbb{C} \), we have

\[
(\Delta_l - \lambda(s))[h(\gamma z) \Im(\gamma z)^s j_\gamma(z)^{-l}] = -4 \frac{\partial^2 h}{\partial z \partial \bar{z}}(\gamma z) \Im(\gamma z)^{s+2} j_\gamma(z)^{-l} - 2i(s - l/2) \frac{\partial h}{\partial z}(\gamma z) \Im(\gamma z)^{s+1} j_\gamma(z)^{-l} + 2i(s + l/2) \frac{\partial h}{\partial z}(\gamma z) \Im(\gamma z)^{s+1} j_\gamma(z)^{-l}
\]
for \( s \in \mathbb{C} \) and \( \gamma \in \SL_2(\mathbb{R}) \).

From the above we will deduce the main recursion formula which will allow us to inductively understand the pole structure of \( G_{A,B,l}(z,s) \). To write down the formula we will introduce the following convenient notation for \( a \in A \)

\[ A_a := (A \setminus \{a\}) \cup \{a-1\}. \]

With this notation we have for positive \( A \)

\[
\frac{\partial}{\partial z} \prod_{a \in A} I_a(z) = \sum_{a \in A} \prod_{a' \in A_a} I_{a'}(z),
\]
by the Leibniz rule. Thus by summing over \( \gamma \in \Gamma_\infty \setminus \Gamma \) and using Lemma 4.1, we arrive at

**Lemma 4.4.** Let \( A, B \) be positive multisets and \( G_{A,B,l}(z,s) \) as above. Then we have

\[
K_l G_{A,B,l}(z,s) = (s + \alpha(A,B) + l/2)G_{A,B,l+2}(z,s) - \sum_{a \in A} G_{A_a,B,l+2}(z,s),
\]
(4.7)

\[
L_l G_{A,B,l}(z,s) = -(s + \alpha(A,B) - l/2)G_{A,B,l-2}(z,s) + \sum_{b \in B} G_{A,B_b,l-2}(z,s)
\]
(4.8)

and

\[
G_{A,B,l}(z,s) = R(s + \alpha(A,B),l) \left( - \sum_{a \in A, b \in B} G_{A_a,B_b,l}(z,s) + (s + \alpha(A,B) - l/2) \sum_{a \in A} G_{A_a,B,l}(z,s) + (s + \alpha(A,B) + l/2) \sum_{b \in B} G_{A,B_b,l}(z,s) \right),
\]
(4.9)
valid for \( \Re s > 1 + |A| + |B| \).

This lemma will turn out to be extremely useful.
Remark 4.5. The recursion formula (4.9) is the reason why we have $2i$ and $-2i$ in the denominators in the definition of $G_{A,B,l}(z,s)$ and why we have the shift $\alpha(A, B)$.

If we define the quantity

$$\Sigma(A, B) = \sum_{a \in A} a + \sum_{b \in B} b,$$

then all elements on the right-hand side in the recursion formula (4.9) have strictly less $\Sigma(A, B)$-value. This will allow us to do an inductive argument.

As a first application of Lemma 4.4, we will show meromorphic continuation of $G_{A,B,l}(z,s)$ to $\Re s > 1/2$. Firstly we will handle the case $l = 0$ using (4.9). This case is easiest to handle since the poles of $R(s, 0)$ all satisfy $\Re s \leq 1$. Then we use (4.7) and (4.8) to get the result for general (even) weights $l$.

Proposition 4.6. For $|A| + |B| > 0$ the series $G_{A,B,l}(z,s)$ has meromorphic continuation to $\Re s > 1/2$ satisfying

(i) $G_{A,B,l}(z,s)$ defines an element of $L^2(\Gamma \backslash \mathbb{H}, l)$ at all regular points

(ii) The poles of $G_{A,B,l}(z,s)$ in $1/2 < \Re s \leq 1$ are contained in $\mathcal{P}$

(iii) $G_{A,B,l}(z,s)$ is regular for $\Re s > 1$

Proof. We prove the claims by an induction on $\Sigma(A, B)$. If $\Sigma(A, B) = 0$ then by (4.3) and (4.4), we can write

$$G_{A,B,l}(z,s) = (y^{k/2} f(z))^{\Gamma A_l} (y^{k/2} f(z))^{\Gamma B} E_{l-k(|A| - |B|)}(z,s).$$

Now $E_{l-k(|A| - |B|)}(z,s)$ is meromorphic in $\Re s > 1/2$ with poles contained in $\mathcal{P}$ and is regular for $\Re s > 1$ (see [5, Chapter 4]). Furthermore since $f(z)$ has exponential decay at all cusps, the above defines an element of $L^2(\Gamma \backslash \mathbb{H}, l)$ at all regular points.

Now let $\Sigma(A, B) > 0$. By using (4.3) and (4.4) we may assume that $A, B$ are positive. Firstly we consider the case $l = 0$.

By (4.9), we can write

$$G_{A,B,0}(z,s) = R(s + \alpha(A, B), 0)(\ldots),$$

where by the induction hypothesis all terms inside the parenthesis satisfy the properties of this proposition. Since the resolvent $R(s + \alpha(A, B), 0)$ is regular in the half-plane $\Re s > 1$ and meromorphic in $\Re s > 1/2$ with poles contained in $\mathcal{P}$, the wanted properties follow for $G_{A,B,0}(z,s)$ as well. (Observe that if $\alpha(A, B) \geq 1$ then the resolvent is actually regular for $\Re s > 1/2$).

Now to get the claim for all positive weights $l$, we do an induction on the weight. For $l \geq 0$, the identity (4.7) gives

$$G_{A,B,l+2}(z,s) = K_l G_{A,B,l}(z,s) + \sum_{a \in A} G_{A_a,B,l+2}(z,s).$$

We know by the induction hypothesis that all the Poincaré series on the right-hand side satisfies the three properties of the proposition. So since

$$(s + \alpha(A, B) + l/2)$$

is non-zero for $\Re s > 1/2$, we see get the claim for $G_{A,B,l+2}(z,s)$ as well. A similar argument applies for negative weights using (4.8).
This finishes the induction and thus the proof. \( \square \)

This allows us to extend the range of validity of Lemma 4.4.

**Corollary 4.7.** The equations (4.7), (4.8) and (4.9) are valid in the half-plane \( \text{Re} \, s > 1/2 \) as equalities of meromorphic functions.

4.3. **Bounds on the pole order at \( s = 1 \).** Next we want to determine the pole order at \( s = 1 \) of \( G_{A,B,l}(z, s) \). In this section we will prove certain bounds on the pole order. We will proceed by induction relying on the formulas (4.7), (4.8) and (4.9). We firstly need the following lemma.

**Lemma 4.8.** Let \( A, B \) be positive multi-sets and \( l \) an even integer. Then we have for \( l \geq 0 \)

\[
(4.11) \quad \langle G_{A,B,lk}(z, s), (y^{k/2} f(z))^l \rangle = \frac{\sum_{a \in A} \langle G_{A_{a},B,lk}(z, s), (y^{k/2} f(z))^l \rangle}{s + \alpha(A, B) + lk/2 - 1},
\]

and for \( l \leq 0 \)

\[
(4.12) \quad \langle G_{A,B,lk}(z, s), (y^{k/2} f(z))^l \rangle = \frac{\sum_{b \in B} \langle G_{A,B_{b},lk}(z, s), (y^{k/2} f(z))^l \rangle}{s + \alpha(A, B) - lk/2 - 1}.
\]

**Proof.** Assume \( l \geq 0 \) then by the identity (4.8), we have

\[
\langle G_{A,B,lk}(z, s), (y^{k/2} f(z))^l \rangle = \langle K_{kl} - 2G_{A,B,lk-2}(z, s) + \sum_{a \in A} G_{A_{a},B,lk}(z, s), (y^{k/2} f(z))^l \rangle + \langle L_{lk}(y^{k/2} f(z))^l \rangle = 0,
\]

using (3.4). This yields the desired formula. The case \( l \leq 0 \) is proved similarly using (4.7). This finishes the proof. \( \square \)

From this we conclude the following key result.

**Proposition 4.9.** The pole order of \( G_{A,B,l}(z, s) \) at \( s = 1 \) is bounded by

\[
\min(\#\{a \in A \mid a = k/2\}, \#\{b \in B \mid b = k/2\}) + 1.
\]

**Proof.** We proceed by an induction on \( \Sigma(A, B) \). If \( \Sigma(A, B) = 0 \) then the result is clear by the properties of the standard Eisenstein series. In general by applying modularity as in (4.3) and (4.4), we may assume that both \( A \) and \( B \) are positive. By the symmetry (4.5) we may also assume that \( |A| \geq |B| \).

Assume that \( \Sigma(A, B) > 0 \) and that we have proved the claim for all smaller \( \Sigma(A, B) \)-values.

We begin with the case \( l = 0 \). Then (4.9) gives

\[
G_{A,B,0}(z, s) = R(s + \alpha(A, B), 0) (...) ,
\]

where the terms inside the parenthesis satisfy the claim of the proposition by the induction hypothesis. If \( \alpha(A, B) > 0 \) then the claim also follows for \( G_{A,B,0}(z, s) \), since the resolvent above is regular at \( s = 1 \). If \( \alpha(A, B) = 0 \), then we have

\[
A = \{k/2, \ldots, k/2\}, \quad B = \{k/2, \ldots, k/2\}.
\]
for some \( n \geq m \geq 0 \).

Now we claim that \( \langle G_{A,B,0}(z,s), 1 \rangle \) has a pole of order at most \( m + 1 \).

To see this we do an induction on \( m \). If \( m = 0 \) then by Lemma 4.8 we see directly that
\[
\langle G_{A,B,0}(z,s), 1 \rangle = 0.
\]

If \( m > 0 \) then we get by Lemma 4.8
\[
\langle G_{A,B,0}(z,s), 1 \rangle = \frac{m \langle G_{A,A,k/2,0}(z,s), 1 \rangle}{s - 1}
\]
and by the induction hypothesis \( G_{A,A,k/2,0}(z,s) \) has a pole of order at most \( m \), which proves the claim.

Now we observe that if \( G_{A,B,0}(z,s) \) has a pole of order greater than \( m + 1 \), then by (4.9) there has to be an increase in the pole order coming from the pole in the singular expansion of the resolvent (3.7). But we just showed that \( \langle G_{A,B,0}(z,s), 1 \rangle \) has a pole of order at most \( m \), which proves the induction in the case \( l = 0 \).

By using (4.7) and (4.8) as in the proof of Proposition 4.6 we get by induction the pole bound for all even weights \( l \) as well.

This finishes the induction and hence the proof. \( \square \)

4.4. **Finding the leading pole.** For \( n \neq m \) Proposition 4.9 yields the desired bound needed to prove (1.3) in Theorem 1.1. In order to prove (1.4), we need to determine the exact pole order and leading Laurent coefficient of \( G_{A,A,0}(z,s) \) at \( s = 1 \) where
\[
A = \{k/2, \ldots, k/2\}_{n}.
\]
This is because \( G_{A,A,0}(z,s) \) will contribute with the dominating pole to the generalized Goldfeld series \( E^{n,n}(z,s) \).

By Proposition 4.9 the pole order is bounded by \( n + 1 \) and we will see that this bound is sharp.

**Theorem 4.10.** Let
\[
A = \{k/2, \ldots, k/2\}_{n}.
\]
Then \( G_{A,A,0}(z,s) \) has a pole of order \( n + 1 \) at \( s = 1 \) with leading Laurent coefficient
\[
\frac{(n!)^2 ||f||^{2n}}{((k-1)!)^n \text{vol}(\Gamma)^{n+1}}.
\]

**Proof.** We do an induction on \( n \). For \( n = 0 \) the claim follows by the analytic properties of the standard Eisenstein series [12 (6.33)].

Now assume \( n > 0 \). First of all we see by 4.9 that
\[
G_{A,A,0}(z,s) = R(s,0) \left( -n^2 G_{A_k/2,A_k/2,0}(z,s) + nsG_{A_k/2,A,0}(z,s) + nsG_{A,A_k/2,0}(z,s) \right)
\]

By bounds on the pole order from Proposition 4.9 all the terms inside the parentheses above has a pole of order at most \( n \). This shows that if \( G_{A,A,0}(z,s) \) has a pole of order \( n + 1 \) then the pole is contained in the image of the projection onto the constant subspace,
since (as above) the increase in the pole order has to come from the resolvent. Now we will show that indeed
\[ \langle G_{A,B,0}(z, s), 1 \rangle / \langle 1, 1 \rangle, \]
has a pole of order \( n + 1 \) at \( s = 1 \) with the claimed leading Laurent coefficient.

Applying Lemma 4.8 twice, we get by Proposition 4.9
\[ \langle G_{A,A,0}(z, s), 1 \rangle = \frac{nG_{A,k/2,0}(z, s, 1)}{(s - 1)s} \]

By repeated application of Lemma 4.8 we arrive at
\[ G_{A,A,0}(z, s) = nG_{A,A',k,0}(z, s) + (\text{pole of order at most } n - 1 \text{ at } s = 1). \]

By the induction hypothesis, we know that \( G_{A',A',0}(z, s) \) has a pole of order \( n \) at \( s = 1 \) with leading Laurent coefficient given by
\[ \frac{((n - 1)!)^2 ||f||^{2n-2}}{((k-1)!)^{n-1} \text{vol}(\Gamma)^n}. \]

Thus we see that
\[ \langle G_{A,A,0}(z, s), 1 \rangle / \langle 1, 1 \rangle = \frac{n^2 \left( \frac{((n-1)!)^2 ||f||^{2n-2}}{((k-1)!)^{n-1} \text{vol}(\Gamma)^n} \cdot y^k |f(z)|^2 \right)}{(k-1)! (s - 1)^{n-1} \text{vol}(\Gamma)} + (\text{pole of order at most } n \text{ at } s = 1), \]

which yields the wanted. This finishes the proof.

With this theorem established we can improve Proposition 4.9 in the following special case.

**Corollary 4.11.** Let
\[ A = \{ k/2, \ldots, k/2 \} \]

and \( l \neq 0 \). Then the order of the pole of \( G_{A,A,l}(z, s) \) at \( s = 1 \) is at most \( n \).
Proof. By the symmetry (4.5), it is enough to prove it for \( l > 0 \). We prove it by induction on \( l \). For \( l = 2 \) we get by (4.8)

\[
G_{A,A,2}(z, s) = \frac{K_0 G_{A,A}(z, s) + n G_{A_k/2,A}(z, s)}{s}.
\]

From Theorem 4.10 we know that the leading Laurent coefficient of \( G_{A,A,0}(z, s) \) is constant, and thus it is annihilated by \( K_0 \). Furthermore we know by Proposition 4.9 that \( G_{A_k/2,A,2}(z, s) \) has a pole of order at most \( n \) at \( s = 1 \). Thus we conclude that also \( G_{A,A,2}(z, s) \) has a pole of order at most \( n \) at \( s = 1 \).

Now assume that \( l > 2 \), we get again by (4.8)

\[
G_{A,A,l+2}(z, s) = \frac{K_l G_{A,A,l}(z, s) + n G_{A_k/2,A,l+2}(z, s)}{s + l/2}
\]

by the induction assumption and Proposition 4.9 we see that also \( G_{A,A,l+2}(z, s) \) has a pole of order at most \( n \) at \( s = 1 \). This finishes the induction and hence the proof.

4.5. Growth on vertical lines. In this section we will prove bounds on the \( L^2 \)-norm of \( G_{A,B,l}(z, s) \) with \( s \) in a horizontal strip, bounded away from the eigenvalues of \( \Delta \). This we will use to get bounds on vertical lines for the main generating series \( D^{n,m}(f, s) \) defined in (2.1). This we need in order to apply Theorem 3 in the appendix.

We will firstly consider the case \( \Sigma(A, B) = 0 \). We will use the idea of the proof of [20, Lemma 3.1] and following Petridis and Risager, we will assume that \( \Gamma \) has only one cusp for simplicity. This assumption can easily be removed.

Lemma 4.12. Let \( s = \sigma + it \) with \( 1/2 + \varepsilon \leq \sigma \leq 3/2 \) and \( \text{dist}(s, \mathcal{P}) \geq \varepsilon \). Then we have for \( |A| + |B| > 0 \) and \( \Sigma(A, B) = 0 \)

\[
||G_{A,B,l}(z, s)|| \ll \varepsilon 1,
\]

where the implied constant depends on \( |A|, |B|, l \).

Proof. By the assumption \( \Sigma(A, B) = 0 \), we can write

\[
G_{A,B,l}(z, s) = (y^{k/2} f(z)) |A| (y^{k/2} f(z)) |B| E_{l'}(z, s),
\]

with \( l' \) appropriately adjusted.

For \( \text{Re } s > 1/2 \) and \( z \in F \), we write (following Colin de Verdière [4])

\[
E_{l'}(z, s) = h(y) y^s + g(z, s),
\]

where \( g(z, s) \in L^2(F) \) and \( h(y) \in C^\infty(0, \infty) \) is smooth with \( h(y) = 1 \) near the cusp at \( \infty \) (\( F \) is a fundamental domain for \( \Gamma \backslash \mathbb{H} \) with a cusp at infinity).

Since \( E_{l'}(z, s) \) is a formal eigenfunction for the Laplacian, we have

\[
(\Delta_{l'} - \lambda(s)) g(z, s) = (\Delta_{l'} - \lambda(s))(E_{l'}(z, s) - h(y) y^s)
\]

\[
= \lambda(s) h(y) y^s + sh'(y) y^{s+1} + h''(y) y^{s+2} - \lambda(s) h(y) y^s
\]

\[
= sh'(y) y^{s+1} + h''(y) y^{s+2}.
\]

Now we extend \( g(z, s) \) periodically to an element of \( L^2(\Gamma, l') \). Then the above yields

\[
g(z, s) = R(s, l')(sh'(y) y^{s+1} + h''(y) y^{s+2}),
\]
i.e. \( g(z, s) \) equals the resolvent applied to a function with compact support. Now by the bound on the norm of the resolvent from Lemma 3.2, we get
\[
\| g(z, s) \| \leq \frac{\| sh'(y)y^{s+1} + h''(y)y^{s+2} \|}{\text{dist}(\lambda(s), \text{spec}\Delta_t)}.
\]

Since \( \Delta_t \) is self adjoint, all eigenvalues are real. Thus using the assumption \( \text{dist}(s, \mathcal{P}) \geq \varepsilon \), we get
\[
\text{dist}(\lambda(s), \text{spec}\Delta_t) \gg |\text{Im}(\lambda(s))| + \varepsilon = (2\sigma - 1)|t| + \varepsilon.
\]

This gives
\[
\| g(z, s) \| \ll \frac{\| sh'(y)y^{s+1} + h''(y)y^{s+2} \|}{(2\sigma - 1)|t| + \varepsilon} \ll \frac{|s|}{|t| + \varepsilon} \ll_\varepsilon 1.
\]

Now by the above, we have
\[
\| G_{A,B,l}(z, s) \| \leq \frac{\| (y^{k/2}f(z))^{A}(y^{k/2}\overline{f(z)})^{B}h(y)y^s \| + \| (y^{k/2}f(z))^{A}(y^{k/2}\overline{f(z)})^{B}g(z, s) \|}{\text{dist}(\lambda(s + \alpha(A, B)), \text{spec}(\Delta_l))}.
\]

The second term is bounded by what we showed above and by the exponential decay of \( f \) the first term is bounded uniformly in \( s \) as well. Thus we conclude
\[
\| G_{A,B,l}(z, s) \| \ll_\varepsilon 1,
\]
as wanted. This finishes the proof.

With this done, we can do the general case by induction on \( \Sigma(A, B) \) using the recursion formula (4.9) and the bound on the operator norm of the resolvent in Lemma 3.2.

**Proposition 4.13.** Let \( s = \sigma + it \) with \( 1/2 + \varepsilon \leq \sigma \leq 3/2 \) and \( \text{dist}(s, \mathcal{P}) \geq \varepsilon \). Then we have for \( |A| + |B| > 0 \)
\[
\| G_{A,B,l}(z, s) \| \ll_\varepsilon 1,
\]
where the implied constant depends on \( |A|, |B|, l \).

**Proof.** We proceed by induction. Above we have done the base case so let \( \Sigma(A, B) > 0 \). By applying modularity we may assume that \( A \) and \( B \) are positive (since \( y^{k/2}f(z) \) is bounded). Now by (4.9) and Lemma 3.2 we get
\[
\| G_{A,B,l}(z, s) \| \leq \frac{\| \text{RHS of (4.9)} \|}{\text{dist}(\lambda(s + \alpha(A, B)), \text{spec}(\Delta_l))}.
\]

By the induction assumption and the triangle inequality, we see that
\[
\| \text{RHS of (4.9)} \| \ll_\varepsilon |t| + 1,
\]
using
\[
s + \alpha(A, B) \pm l/2 \ll |t| + 1,
\]
where the implied constant depends on \( |A|, |B|, l \).

Now since the spectrum of \( \Delta_l \) is real, we get
\[
\text{dist}(\lambda(s + \alpha(A, B)), \text{spec}(\Delta_l)) \gg_\varepsilon |t(2\sigma - 1)| + \varepsilon \gg_\varepsilon |t| + \varepsilon
\]
using \( \text{dist}(s, \mathcal{P}) \geq \varepsilon \). This gives
\[
\| G_{A,B,l}(z, s) \| \ll_\varepsilon \frac{|t| + 1}{|t| + \varepsilon} \ll_\varepsilon 1,
\]
as wanted.

This finishes the proof. \( \square \)
5. Central values of additive twists

In this section we will prove the promised formula, which links the generalized Goldfeld series $E^{n,m}(z,s)$ defined in (2.7) with the Poincaré series $G_{A,B,l}(z,s)$ studied in the preceding section. This will be done by expressing the central value

$$L(f \otimes e(r), 1/2)$$

as a linear combination of anti-derivatives of $f(z)$. Recall the definition of $I_n(z)$ in (4.1).

5.1. A formula for the central value. The starting point is the period integral representation of $L(f \otimes e(\gamma \infty), s)$ given in (5.10). A slight variation of this yields with $a/c = \gamma \infty$

$$\int_{\gamma \infty}^{i \infty} f(w) \left( \frac{w - a/c}{i} \right)^{(k-2)/2} dw = \int_{\gamma \infty}^{i \infty} f(w)(\text{Im } w)^{(k-2)/2} dw$$

$$= i \int_{0}^{\infty} f(a/c + iy) y^{(k-2)/2} dy$$

$$= e^{-k/2} i L(f \otimes e(a/c), 1/2)$$

$$= (2\pi)^{-k/2} \Gamma(k/2) i L(f \otimes e(a/c), 1/2).$$

Observe that the integrand above is holomorphic (here it is crucial that $k$ is even). Thus it follows by the exponential decay of $f$ that we can shift the contour and arrive at

$$(-2\pi i)^{-k/2} \Gamma(k/2) L(f \otimes e(a/c), 1/2)$$

(5.1)

$$= \int_{\gamma \infty}^{\gamma z} f(w)(w - a/c)^{(k-2)/2} dw + \int_{\gamma z}^{i \infty} f(w)(w - a/c)^{(k-2)/2} dw$$

for any $z \in \mathbb{H}$.

This expression will allow us to prove the formula for the central value.

Lemma 5.1. For any $z \in \mathbb{H}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, we have

$$L(f \otimes e(\gamma \infty), 1/2) = \left( -1 \right)^{k/2} \sum_{0 \leq j \leq (k-2)/2} \frac{(k-2)!}{j!} c^{-j} j(\gamma, z)^{-j} I_{k/2-j}(\gamma z)$$

$$+ \sum_{0 \leq j \leq (k-2)/2} \left( -1 \right)^{j} \frac{(k-2)!}{j!} c^{-j} j(\gamma, z)^{j} I_{k/2+j}(z) \left( -2\pi i \right)^{k/2} \Gamma(k/2).$$

Proof. We treat the two integrals in (5.1) separately. Using the fact that

$$a/c = \gamma \infty = \gamma z + \frac{c^{-1}}{j(\gamma, z)}$$

we get

$$\int_{\gamma z}^{i \infty} f(w)(w - a/c)^{(k-2)/2} dw = \int_{\gamma z}^{i \infty} f(w)(w - \gamma z + \gamma z - a/c)^{(k-2)/2} dw$$

$$= \int_{\gamma z}^{i \infty} f(w) \left( w - \gamma z - \frac{c^{-1}}{j(\gamma, z)} \right)^{(k-2)/2} dw$$

$$= \left( -1 \right)^{k/2} \sum_{0 \leq j \leq (k-2)/2} \frac{(k-2)!}{j!} c^{-j} j(\gamma, z)^{-j} I_{k/2-j}(\gamma z)$$
using the integral representation (4.1) of $I_j(z)$. To threat the other integral we use the identity

$$w - a/c = \gamma \gamma^{-1}w - a/c = -\frac{e^{-1}}{j(\gamma, \gamma^{-1}w)}$$

which yields

$$\int_{\gamma^z} f(w)(w - a/c)^{(k-2)/2} dw = \int_{\gamma^z} f(\gamma \gamma^{-1}w) \left(-\frac{e^{-1}}{j(\gamma, \gamma^{-1}w)}\right)^{(k-2)/2} dw$$

$$= \int_{\gamma^z} f(w) \left(-\frac{e^{-1}}{j(\gamma, w)}\right)^{(k-2)/2} j(\gamma, w)^{-2} dw$$

after the change of variable $w \rightarrow \gamma^{-1}w$. Now by using modularity of $f$ and the following identity

$$\frac{j(\gamma, w')}{c} = w' - z + \frac{j(\gamma, z)}{c}$$

the above equals

$$(-1)^{(k-2)/2} \int_{\infty}^{z} f(w') j(\gamma, w')^{-k-2} \left(-\frac{e^{-1}}{j(\gamma, w')}\right)^{(k-2)/2} dw'$$

$$= (-1)^{(k-2)/2} \int_{\infty}^{z} f(w') \left(w' - z + \frac{j(\gamma, z)}{c}\right)^{(k-2)/2} dw'$$

$$= \sum_{0 \leq j \leq (k-2)/2} (-1)^j \frac{(k-2)/2)!}{j!} c^{-j} j(\gamma, z)^j I_{k/2-j}(z).$$

This finishes the proof. \hfill \square

We would now like to take the formula in Lemma 5.1 and sum over $\gamma \in \Gamma_\infty \backslash \Gamma$. Then use the identity

$$(5.3) \quad c = \frac{j(\gamma, z) - \overline{j(\gamma, z)}}{2iy}$$

and the binomial formula to express $E^{n,m}(z, s)$ as a sum of the Poincaré series $G_{A,B,I}(z, s)$. The only slight complication is that we have negative powers of $c$. In order to avoid this we multiply by a power $c^N$ on both sides of (5.2) for some even $N \geq (k-2)/2$. With this in mind, we define the following $N$-shifted Goldfeld Eisenstein series

$$E^{n,m}(z, s; N) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} c^N L(f \otimes e(\gamma \infty), 1/2)^n L(f \otimes e(\gamma \infty), 1/2)^m \Im(\gamma z)^s.$$
are contained in the set \( \{ p + N/2 \mid p \in \mathcal{P} \} \).

(ii) The pole order of \( E^{n,m}(z, s; N) \) at \( s = N/2 + 1 \) is bounded by \( \min(n, m) + 1 \).

(iii) \( E^{n,n}(z, s; N) \) has a pole at \( s = N/2 + 1 \) of order \( n + 1 \) with leading Laurent coefficient

\[
(4\pi)^{n_k} y^{-N/2} \frac{(N/2)^N}{2} \frac{(n!)^2 \| f \|^2_{2n}}{(k - 1)! n_{\text{vol}}(\Gamma)^{n+1}}.
\]

Proof. By (5.2) we can write

\[
c^N L(f \otimes e(\gamma \infty), 1/2)^n L(f \otimes e(\gamma \infty), 1/2)^m
\]
as a linear combination of terms of the type

\[
h(z) j(\gamma, z)^t \bar{j}(\gamma, z)^s \cdot I_{k/2-j_1}(\gamma z) \cdots I_{k/2-j_{n'}}(\gamma z) I_{k/2-j_{n'+1}}(\gamma z) \cdots I_{k/2-j_{n'+m'}}(\gamma z)
\]
for some smooth function \( h : \mathbb{H} \to \mathbb{C} \) (this will be a product of \( I_j(z) \) for \( 1 \leq j \leq k/2 \)), integers \( t, t' \), non-negative integers \( n' \leq n, m' \leq m \) and finally the important condition \( N' \geq 0 \). By inspecting (5.2), we see that \( t, t' \) and \( N' \) satisfy

\[
t + t' + N' = N - 2 \sum_{v=1}^{n'+m'} j_v.
\]

Now we use (5.3) and expand using the binomial formula to get terms of the type

\[
h(z) j(\gamma, z)^t \bar{j}(\gamma, z)^s \cdot I_{k/2-j_1}(\gamma z) \cdots I_{k/2-j_{n'}}(\gamma z) I_{k/2-j_{n'+1}}(\gamma z) \cdots I_{k/2-j_{n'+m'}}(\gamma z)
\]
where now

\[
t + t' = N - 2 \sum_{v=1}^{n'+m'} j_v.
\]

Now we multiply by \( \text{Im}(\gamma z)^s \) and use the identity

\[
j(\gamma, z)^t \bar{j}(\gamma, z)^s \text{Im}(\gamma z)^s = y^{l(t+t')/2} j_v(z)^{l-t'} \text{Im}(\gamma z)^{s-(t+t')/2}.
\]

Thus summing over \( \gamma \in \Gamma \), we can express \( E^{n,m}(z, s; N) \) as a linear combination of terms of the type

\[
h(z) G_{A,B,l}(z, s - N/2)
\]
where again \( h : \mathbb{H} \to \mathbb{C} \) is some smooth function (now a product of powers of \( y \) and \( I_j(z)'s \), \( |A| \leq n, |B| \leq m \) and \( l \) is even (which follows from (5.4)).

Notice that (5.4) fits beautifully with the the factor \( \alpha(A, B) \) in the definition of \( G_{A,B,l}(z, s) \), which implies that we get the argument \( s - N/2 \) for all terms.

Now it follows directly from Proposition 4.6 that \( E^{n,m}(z, s; N) \) has meromorphic continuation to \( \text{Re } s > N/2 + 1/2 \) satisfying Proposition 5.2 (i). Furthermore by Proposition 4.9 it follows that all terms \( G_{A,B,l}(z, s - N/2) \) and thus also \( E^{n,m}(z, s; N) \) has a pole of order at most

\[
\min(n, m) + 1
\]
at \( s = 1 \).
Now finally let us consider the diagonal case \( n = m \). We see by Corollary 4.11 and Proposition 4.9 that all terms \( 5.3 \) have a pole of order at most \( n \), except the one with
\[
A = B = \{ k/2, \ldots, k/2 \}
\]
and \( l = 0 \). Now let us calculate the coefficient of \( G_{A,A,0}(z, s - N/2) \).
We have
\[
(2\pi)^{nk}e^{N|I_{k/2}(\gamma z)|2n} = \frac{(2\pi)^{nk}}{(2\pi)^N}|I_{k/2}(\gamma z)|^{2n} \sum_{v=1}^{N} (-1)^v \binom{N}{v} j(\gamma, z)^v j(\gamma, z)^{N-v}.
\]
We multiply by \( \text{Im}(\gamma z)^a \) and sum over \( \gamma \in \Gamma_\infty \setminus \Gamma \). By the pole bound from Corollary 4.11 we see that only the term with \( v = N/2 \) can contribute with a pole of order \( n + 1 \) at \( s = N/2 + 1 \). Thus we can write
\[
E^{n,m}(z, s; N) = (4\pi)^{nk} y^{-N/2} \frac{N/2}{2N} G_{A,A,0}(z, s - N/2) + \text{terms with a pole of order at most } n \text{ at } s = N/2 + 1,
\]
where
\[
A = \{ k/2, \ldots, k/2 \}.
\]
(The factor \( 2^{nk} \) comes from the \( 2i \) and \( -2i \) in the denominator of the definition of \( G_{A,A,0}(z, s) \)). Thus the result follows from Theorem 4.10. This finishes the proof. \( \square \)

5.2. Analytic properties of \( D^{n,m}(f, s) \). Using the above we can now extract analytical information about the main generating series \( D^{n,m}(f, s) \) defined in \( (2.1) \). Firstly we will prove that \( D^{n,m}(f, s) \) is essentially the constant Fourier coefficient of \( E^{n,m}(z, s; N) \).

**Lemma 5.3.** Let \( N \geq 0 \) be an even integer. Then the constant Fourier coefficient of \( E^{n,m}(z, s; N) \) is given by
\[
\frac{\pi^{1/2} y^{1-s} \Gamma(s-1/2)}{\Gamma(s)} D^{n,m}(f, s - N/2).
\]

**Proof.** By the double coset decomposition (see [11 Prop. 2.7]), we have
\[
\Gamma_\infty \setminus \Gamma / \Gamma_\infty \leftrightarrow \left\{ (c, d) \mid 0 \leq d < c, \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma \right\} \cup \{(0, 1)\}
\]
Now since \( L(f \otimes \epsilon(\gamma \infty)), 1/2 \) is well-defined in the above double coset and \( L(f \otimes \epsilon(\infty)), 1/2 = 0 \)
per definition, we can write
\[
E^{n,m}(z, s; N) = \sum_{c > 0} \sum_{0 \leq d < c} e^{N L(f \otimes \epsilon(\gamma_{c,d} \infty)), 1/2} y^s \sum_{l \in \mathbb{Z}} \frac{e^{x(z + l)} + d^{2s}}{|c(z + l) + d|^{2s}},
\]
where \( \gamma_{c,d} \) is any representative of \( (c, d) \) in \( \Gamma_\infty \setminus \Gamma / \Gamma_\infty \). By Poisson summation we can express the inner sum as
\[
\sum_{l \in \mathbb{Z}} \frac{y^s}{|c(z + l) + d|^{2s}} = y^s \sum_{l \in \mathbb{Z}} \int_{-\infty}^{\infty} |c(z + t) + d|^{-2s} e(-lt) dt.
\]
Now after the change of variable $t \mapsto t - x - d/c$, we get
\begin{equation}
\sum_{l \in \mathbb{Z}} \frac{y^s}{|c(z + l) + d|^{2s}} = y^s \sum_{l \in \mathbb{Z}} e(xl + \frac{d}{c}l)c^{-2s} \int_{-\infty}^{\infty} (t^2 + y^2)^{-s} e(-lt) dt.
\end{equation}
From this we see that the constant Fourier coefficient is given by
\begin{equation}
y^s \int_{-\infty}^{\infty} (t^2 + y^2)^{-s} dt \sum_{r \in \mathbb{Z}} L(f \otimes c(r), 1/2)^n L(f \otimes c(r), 1/2)^m 
\end{equation}
and a calculation shows
\begin{equation}
\int_{-\infty}^{\infty} (t^2 + y^2)^{-s} dt = \frac{\pi^{1/2} y^{1-2s} \Gamma(s - 1/2)}{\Gamma(s)}.
\end{equation}
This finishes the proof.

With this lemma at our disposal, we can easily derive the analytic properties of $D^{n,m}(f,s)$ from the results already established.

**Theorem 5.4.** The Dirichlet series $D^{n,m}(f,s)$ has meromorphic continuation to $\Re s > 1/2$ satisfying the following.

(i) $D^{n,m}(f,s)$ is regular for $\Re s > 1$ and the poles in the strip

\[ \frac{1}{2} < \Re s \leq 1 \]

are contained in $\mathcal{P}$.

(ii) The pole order of $D^{n,m}(f,s)$ at $s = 1$ is bounded by $\min(n,m) + 1$.

(iii) $D^{n,n}(f,s)$ has a pole of order $n + 1$ at $s = 1$ with leading Laurent coefficient

\[ \frac{(n!)^2 (4\pi)^n |f|^2 n}{\pi ((k - 1)!)^n \text{vol}(\Gamma)^{n+1}}. \]

(iv) For $s = \sigma + it$ with $1/2 + \varepsilon \leq \sigma \leq 2$ and $\text{dist}(\lambda(s), \mathcal{P}) \geq \varepsilon$, we have the following bound

\[ D^{n,m}(f,s) \ll \varepsilon \ (1 + |t|)^{1/2}, \]

where the implied constant depends also on $n,m$.

**Proof.** Let $N \geq (n + m)(k - 2)/2$ be even. By Lemma \[ \ref{lem:main}, \] we have

\[ D^{n,m}(f,s - N/2) = \frac{\Gamma(s)}{\pi^{1/2} y^{1-s} \Gamma(s - 1/2)} \int_0^1 E^{n,m}(z, s; N) dx. \]

Thus we get meromorphic continuation of $D^{n,m}(f,s)$ and the claim about the position of the possible poles directly from Proposition \[ \ref{prop:poles}, \] together with the bound on the order of the pole at $s = 1$.

Now to treat the case $n = m$, we recall that the leading Laurent coefficient of $E^{n,n}(z, s; N)$ at $s = N/2 + 1$ is constant. Thus we see directly from Proposition \[ \ref{prop:poles}, \] that $D^{n,n}(f,s)$ has a pole of order $n + 1$ at $s = 1$ with leading Laurent coefficient

\[ \frac{\Gamma(N/2 + 1)}{\pi^{1/2} y^{-(N/2+1)} \Gamma(N/2 + 1/2)} (4\pi)^n y^{-N/2} \left( \frac{N}{N/2} \right)^{N/2} \frac{n!^n |f|^2 n}{((k - 1)!)^n \text{vol}(\Gamma)^{n+1}}. \]
Now using that for even \( N \), we have
\[
\Gamma\left(\frac{N}{2} + 1\right) = \frac{\pi^{1/2} (N - 1) \cdots 3 \cdot 1}{2^{N/2}}, \quad \Gamma\left(\frac{N}{2} + 1\right) = \frac{(N - 1) \cdots 3 \cdot 1}{2^{N/2}(N/2)!}
\]
the claim about the leading Laurent coefficient follows.

Now for the growth on vertical lines we need to somehow bring the bounds on the \( L^2 \)-norms of \( G_{A,B,l}(z, s) \) in Proposition 4.13 into play. First step is to integrate \( D^{n,m}(f, s - N/2) \) with respect to \( y \) over some finite segment, say \([1, 2]\), which gives
\[
D^{n,m}(f, s - N/2) = \frac{\Gamma(s)}{\pi^{1/2} \Gamma(s - 1/2)} \int_1^2 \int_0^1 y^{1-s} E^{n,m}(z, s; N) dxdy.
\]
By the proof of Proposition 5.2 we can write the above as a linear combination of terms of the type
\[
\frac{\Gamma(s)}{\Gamma(s - 1/2)} \int_1^2 \int_0^1 y^{1-s} h(z) G_{A,B,l}(z, s - N/2) dxdy.
\]
Clearly \( h(z) \) is bounded in the region \([0, 1] \times [1, 2]\) since it is a smooth function and hence Cauchy-Schwarz implies
\[
\int_1^2 \int_0^1 y^{1-s} h(z) G_{A,B,l}(z, s - N/2) dxdy \ll \sqrt{\left( \int_1^2 \int_0^1 y^{4-2\sigma} |h(z)|^2 dxdy \right) \cdot \left( \int_1^2 \int_0^1 |G_{A,B,l}(z, s - N/2)|^2 \frac{dxdy}{y^2} \right)}.
\]
Thus for \( s = \sigma + it \) with
\[
1/2 + N/2 + \varepsilon \leq \sigma < 3/2 + N/2
\]
and \( s - N/2 \) bounded away from \( P \), we get by Proposition 4.13 that
\[
D^{n,m}(f, s - N/2) \ll \varepsilon \frac{|\Gamma(s)|}{|\Gamma(s - 1/2)|},
\]
where the implied constant depends also on \( n, m \).

Now Stirling’s formula implies for \( s \) in the given range
\[
\frac{\Gamma(s)}{\Gamma(s - 1/2)} \ll_N (1 + |t|)^{1/2},
\]
and thus
\[
D^{n,m}(f, s) \ll \varepsilon (1 + |t|)^{1/2}
\]
for \( s = \sigma + it \) with \( 1/2 + \varepsilon \leq \sigma < 1 \) and \( s \) bounded away from \( P \). This finishes the proof.

This result implies the Lindelöf Hypothesis in the \( c \)-aspect for additive twists.

**Corollary 5.5.** We have
\[
L(f \otimes e(r), 1/2) \ll \varepsilon c(r) \varepsilon^c
\]
where the implied constant depends on \( f \).
Proof. By combining Landau’s Lemma \cite[Lemma 5.56]{13} and Theorem 5.4 we see that

\[ D_{n,n}(f,s) = \sum_{r \in T(X)} \frac{|L(f \otimes e(r), 1/2)|^{2n}}{c(r)^{2s}} \]

converges absolutely for \( \text{Re } s > 1 \) for all \( n \). In particular this implies that

\[ \frac{|L(f \otimes e(r), 1/2)|^{2n}}{c(r)^{2+\varepsilon}} \]

is bounded as \( c(r) \to \infty \). This means that

\[ L(f \otimes e(r), 1/2) \ll_{c,n} c(r)^{1/n+\varepsilon}. \]

This finishes the proof. \qed

5.3. Normal distribution. In this section we will show that the central values

\[ L(f \otimes e(r), 1/2) \]

with a suitable normalization are normally distributed when ordered by the size of \( c(r) \). This is done by determining all asymptotic moments and then appealing to a classical result of Fréchet and Shohat \cite[Theorem B on p. 17]{23}.

To evaluate the asymptotic moments, we firstly apply Theorem 8.2 to \( D_{n,m}(f,s) \). This allows us to prove Theorem 1.1

Proof of Theorem 1.1. The bound (1.3) follows directly from Theorem 5.4 using a standard contour integration argument (even though the coefficients of \( D_{n,m}(f,s) \) are not positive numbers, we do not need to be careful, since we do not care about error-term).

For \( n = m \) we apply Theorem 8.2 to \( D_{n,m}(f,s) \) with

\[ S_{\text{poles}} = \{2s_0 = 2, 2s_1, \ldots\}, \quad a = 1 \quad \text{and} \quad A = 1/2. \]

We know from Theorem 5.4 that \( s \mapsto D_{n,n}(f,s/2) \) has a pole at \( s = 2 \) of order \( n + 1 \) with leading Laurent coefficient

\[ b_{n+1} = \frac{2^{n+1}(n!)^2(4\pi)^nk\|f\|^{2n}}{\pi((k-1)!)^n \text{vol}(\Gamma)^{n+1}}. \]

(The factor \( 2^{n+1} \) stems from \( (s/2 - 1)^{-n-1} = 2^{n+1}(s - 2)^{-n-1} \)).

Thus it follows directly from Theorem 8.2 that, we get the wanted asymptotic formula (1.4) with error-term

\[ O(X^{\max((1+2 \cdot 1/2)/(1+1/2),2 \Re s_1)+\varepsilon}) = O(X^{\max(4/3,2 \Re s_1)+\varepsilon}). \]

This finishes the proof. \qed

Remark 5.6. If we instead considered smooth moments, we would just get the error-term \( O_{\varepsilon}(X^{2 \Re s_1}) \) using Theorem 8.1 in the appendix.

Remark 5.7. One can check that in weight 2 case, this agrees with Petridis and Risager \cite[Corollary 7.7]{21}.

From the above we can deduce the asymptotic moments of \( L(f \otimes e(a/c), 1/2) \) by partial summation.
Corollary 5.8. For \( n \neq m \) we have

\[
\sum_{r \in T(X)} \left( \frac{L(f \otimes e(r), 1/2)}{(C_f \log c(r))^{1/2}} \right)^n \left( \frac{L(f \otimes e(r), 1/2)}{(C_f \log c(r))^{1/2}} \right)^m \rightarrow 0,
\]
as \( X \rightarrow \infty \).

For \( n = m \), we have

\[
\sum_{r \in T(X)} \left| \frac{L(f \otimes e(r), 1/2)}{(C_f \log c(r))^{1/2}} \right|^{2n} \rightarrow 2^n n!,
\]
as \( X \rightarrow \infty \).

**Proof.** The corollary follows immediately from Theorem 1.1 and the asymptotic formula [21, Lemma 3.5];

\[
\#T(X) \sim \frac{X^2}{\pi \text{vol}(\Gamma)},
\]
using partial summation. \(\square\)

Recall that for a standard 2-dimensional normal distribution \( (Y, Z) \) the moments are given by

\[
E(Y^n Z^m) = \begin{cases} (n-1)!!(m-1)!! & \text{if } n \text{ and } m \text{ are even} \\ 0 & \text{otherwise} \end{cases},
\]

where \((n-1)!! = (n-1)(n-3)\cdots 1\). By taking linear combinations of the moments in Corollary 5.8 it follows that the asymptotic moments of

\[
\left( \begin{array}{c} \text{Re} \frac{L(f \otimes e(r), 1/2)}{(C_f \log c(r))^{1/2}} \\ \text{Im} \frac{L(f \otimes e(r), 1/2)}{(C_f \log c(r))^{1/2}} \end{array} \right), \quad r \in T(X)
\]
as \( X \rightarrow \infty \) are the same as those of the 2-dimensional standard normal. This fact and the above corollary allow us to prove the main theorem.

**Proof of Theorem 1.4.** We would like to use the theorem of Fréchet and Shohat from probability theory [23, p. 17] mentioned before. To make it fit into the probability theoretical framework of the Fréchet-Shohat theorem, we consider for each \( X > 0 \) the 2-dimensional random variable

\[
\left( \begin{array}{c} Y_X(r) \\ Z_X(r) \end{array} \right) = \left( \begin{array}{c} \text{Re} \frac{L(f \otimes e(r), 1/2)}{(C_f \log c(r))^{1/2}} \\ \text{Im} \frac{L(f \otimes e(r), 1/2)}{(C_f \log c(r))^{1/2}} \end{array} \right), \quad r \in T(X)
\]

where the outcome space \( T(X) \) is equipped with the discrete \( \sigma \)-algebra and the uniform measure. Note that [23, p. 17] is only directly applicable for 1-dimensional distribution functions, but we can get around this as follows:

It follows from Corollary 5.8 that for \( (a, b) \in \mathbb{R}^2 \setminus (0, 0) \) the moments of the random variables \( aY_X + bZ_X \) converges to the moments of a normal distribution with mean 0 and variance \( a^2 + b^2 \) as \( X \rightarrow \infty \). Thus it follows by Fréchet-Shohat that \( aY_X + bZ_X \) converges in distribution to the normal distribution with mean 0 and variance \( a^2 + b^2 \) (the normal distribution is uniquely determined by its moments).

Now by the Cramér-Wold Theorem (see [23, p. 18]) it follows that \( \left( \begin{array}{c} Y_X \\ Z_X \end{array} \right) \) converges in
distribution to the 2-dimensional standard normal distribution as $X \to \infty$.

This finishes the proof. \qed

6. Applications to the 1st and 2nd moment of $L(f \otimes \chi, 1/2)$

We now apply our results to the 1st and 2nd moment of $L(f \otimes \chi, 1/2)$.

In this section we show an average version of the result of [2] and [16] incorporating also composite moduli. The connection between multiplicative and additive twists is for primitive characters given by the Birch-Stevens formula [22, Eq. 2.2], but some cleverness has to be applied in order to deal with non-primitive characters.

Our results apply to a primitive cusp form $f \in S_k(\Gamma_0(q))$ of even weight $k$ and level $q$ with Fourier expansion

$$f(z) = \sum_{n \geq 1} \lambda_f(n) n^{(k-1)/2} q^n.$$

It what follows it is essential that $f$ is a primitive form, i.e. that it is an eigenform for all Hecke operators.

6.1. Second moment of multiplicative twists.

The first step is to establish a connection between additive twists and multiplicative ones. The formula below should be seen as a generalization of the Birch-Stevens formula [22, Eq. 2.2].

**Proposition 6.1.** Let $f \in S_k(\Gamma_0(q))$ be a primitive form and $\chi$ a character mod $c$. Then we have

$$\tau(\chi) \nu(f, \chi^*, c/c(\chi)) L(f \otimes \chi^*, 1/2) = \sum_{a \in (\mathbb{Z}/c\mathbb{Z})^\times} \tau(a) L(f \otimes e(a/c), 1/2),$$

where $\chi$ is induced by the primitive character $\chi^* \mod c(\chi)$ and

$$\nu(f, \chi, n) = \sum_{n_1 n_2 n_3 = n} \chi(n_1) \mu(n_1) \chi(n_2) \mu(n_2) \lambda_f(n_3) n_3^{1/2}.$$

**Proof.** We have for $\Re s > 1$

$$\sum_{a \in (\mathbb{Z}/c\mathbb{Z})^\times} \tau(a) L(f \otimes e(a/c), s) = \sum_{n \geq 1} \lambda_f(n) n^s \left( \sum_{a \in (\mathbb{Z}/c\mathbb{Z})^\times} \tau(a) e(an/c) \right).$$

The inner sum is a Gauss sum and by [24, Lemma 3], we get

$$\sum_{a \in (\mathbb{Z}/c\mathbb{Z})^\times} \tau(a) e(an/c) = \tau(\chi^*) \sum_{d \mid (n, c/c(\chi))} \frac{e}{c(\chi) d} \mu(\frac{e}{c(\chi) d}) \chi^*(\frac{n}{d}).$$

Plugging this into (6.2), interchanging the sums and putting $n = dl$, we arrive at

$$\sum_{a \in (\mathbb{Z}/c\mathbb{Z})^\times} \chi(a) L(f \otimes e(a/c), s) = \tau(\chi^*) \sum_{d \mid c/c(\chi)} \chi^*(\frac{e}{c(\chi) d}) \mu(\frac{e}{c(\chi) d}) d \sum_{l > 0} \frac{\lambda_f(dl)}{(dl)^s} \chi^*(l).$$

Now we use that $f$ is a Hecke eigenform, which implies that

$$\lambda_f(ld) = \sum_{h \mid (l, d)} \mu(h) \lambda_f(\frac{l}{h}) \lambda_f(\frac{d}{h}).$$
With \(m = l/h\) and \(\delta = d/h\), we get
\[
\tau(\overline{\chi}) \sum_{\delta h | c/c(\chi)} \chi^*(c_{c(\chi)} \delta | h) \mu(c_{c(\chi)} \delta | h) \lambda_f(\delta) \lambda_f(m) \chi^*(m) \mu(h) \sum_{m > 0} \chi^*(m) \lambda_f(m) / m^s
\]
\[
= \tau(\overline{\chi}) L(f \otimes \chi, 1/2) \sum_{\delta h | c/c(\chi)} \chi^*(c_{c(\chi)} \delta | h) \mu(c_{c(\chi)} \delta | h) \lambda_f(\delta) \lambda_f(m) \chi^*(m) \mu(h) \sum_{m > 0} \chi^*(m) \lambda_f(m) / m^s.
\]
Since the sum above is finite, this equality is also true at \(s = 1/2\) by analytic continuation. This finishes the proof. \(\square\)

From this we conclude easily.

**Lemma 6.2.** Let \(f \in S_k(\Gamma_0(q))\) be a primitive form and \(c > 0\) a positive integer. Then we have
\[
\sum_{a \in (\mathbb{Z}/c^*)^*} |L(f \otimes e(a/c), 1/2)|^2 = \frac{1}{\varphi(c)} \sum_{\chi \mod c} c(\chi) |\nu(f, \chi^*, c/c(\chi))|^2 |L(f \otimes \chi^*, 1/2)|^2,
\]
with \(\chi^* \mod c(\chi)\) and \(\nu(f, \chi, n)\) as in Proposition 6.4.

**Proof.** This follows by orthogonality of characters and the fact that \(|\tau(\overline{\chi})|^2 = c(\chi)\). \(\square\)

**Remark 6.3.** If we instead try to just sum over primitive characters on the right hand side, we do not get additive twists on the left due to lack of orthogonality.

From this we get immediately the following corollary of Theorem 5.4.

**Corollary 6.4.** Let \(f \in S_k(\Gamma_0(q))\) be a primitive form of even weight \(k\) and level \(q\). Then we have
\[
\sum_{c \leq X, q | c} \frac{1}{\varphi(c)} \sum_{\chi \mod c} c(\chi) |\nu(f, \chi^*, c/c(\chi))|^2 |L(f \otimes \chi^*, 1/2)|^2
\]
\[
= \frac{(4\pi)^k \|f\|^2}{\pi (k - 1)! \log(\Gamma_0(q))^2} (\log X) X^2 + \beta_{f,1} X^2 + O_{\varepsilon}(X^{4/3 + \varepsilon})
\]
with \(\chi^* \mod c(\chi)\) and \(\nu(f, \chi, n)\) as in Proposition 6.4, where \(\beta_{f,1}\) is an explicit constant.

**Proof.** By the approximation towards Selberg’s conjecture by Kim and Sarnak [12 p. 167], we know that \(\text{Re } s_1 \leq 39/64 < 2/3\). Thus by Theorem 1.1 with \(n = m = 1\), we get the result directly from Lemma 5.2 \(\square\)

**Remark 6.5.** We will see below how to remove the restriction \(q \mid c\).

6.2. 1st moment. To relate the moments of the additive twists to the 1st moment of \(L(f \otimes \chi, 1/2)\), we will prove a formula similar to Lemma 6.2. This will lead to the study of the analytic properties of \(G_{l}^{(n)}, \theta, l(z, s)\) for \(n \leq k/2\). By Proposition 4.9 we know that the pole order is at most 1, but actually we will prove below that for \(l < k\) these Poincaré series are regular at \(s = 1\).

The connection between the 1st moment of \(L(f \otimes \chi, 1/2)\) and additive twists is again through an application of Proposition 6.1.
Proposition 6.6. Let \( f \in S_k(\Gamma_0(q)) \) be a primitive form and \( c > 0 \) a positive integer. Then we have

\[
\frac{1}{\varphi(c)} \sum_{\chi \mod c} \tau(\chi) \tau(\chi^*) \nu(f, \chi^*, c/c(\chi)) L(f \otimes \chi^*, 1/2)
\]

(6.5)

\[
= \sum_{a \in (\mathbb{Z}/c\mathbb{Z})^\times} L(f \otimes e(a/c), 1/2)e(-a/c),
\]

with \( \chi^* \mod c(\chi) \) and \( \nu(f, \chi, n) \) as above.

Remark 6.7. Observe that for \( \chi \) primitive, we have

\[
\overline{\tau(\chi)} \tau(\chi^*) \nu(f, \chi^*, c/c(\chi)) = \tau(\chi) \tau(\chi) = c.
\]

Proof. We apply Proposition 6.1, unfold \( \tau(\chi) \), interchange the sums and use orthogonality of characters

\[
\frac{1}{\varphi(c)} \sum_{\chi \mod c} \tau(\chi) \tau(\chi^*) \nu(f, \chi^*, c/c(\chi)) L(f \otimes \chi, 1/2)
\]

\[
= \frac{1}{\varphi(c)} \sum_{\chi \mod c} \sum_{a \in (\mathbb{Z}/c\mathbb{Z})^\times} \tau(\chi) L(f \otimes e(a/c), 1/2)
\]

\[
= \sum_{a, b \in (\mathbb{Z}/c\mathbb{Z})^\times} L(f \otimes e(a/c), 1/2)e(-b/c) \frac{1}{\varphi(c)} \sum_{\chi \mod c} \tau(\chi) \chi(b)
\]

(6.6)

\[
\sum_{a \in (\mathbb{Z}/c\mathbb{Z})^\times} L(f \otimes e(a/c), 1/2)e(-a/c).
\]

This finishes the proof. \( \square \)

This allows us to study the 1st moment of \( L(f \otimes \chi, 1/2) \) since the generating series for (6.6) is essentially given by the 1st Fourier coefficient of \( E^{1,0}(z, s) \) and more generally of \( E^{1,0}(z, s; N) \).

To see this we examine the proof of Proposition 5.3; picking off the terms with \( l = 1 \) in (6.6), we see that the 1st Fourier coefficient of \( E^{1,0}(z, s; N) \) is exactly

\[
y^s \int_{-\infty}^{\infty} (t^2 + y^2)^{-s} e(-t) dt \sum_{\gamma \in \Gamma_\infty \backslash \Gamma/\Gamma_\infty} \frac{L(f \otimes e(\gamma \infty), 1/2)e(-\gamma^{-1} \infty)}{c^{2s-N}},
\]

(6.7)

where \( c \) is the lower left entry of \( \gamma \).

Now by applying the functional equation from Theorem 3.3, we see that the Dirichlet series above is equal to

\[
\sum_{\gamma \in \Gamma_\infty \backslash \Gamma/\Gamma_\infty} \frac{L(f \otimes e(\gamma \infty), 1/2)e(-\gamma \infty)}{c^{2s-N}} \sum_{r \in \Gamma} \frac{L(f \otimes e(r), 1/2)e(-r)}{c(r)^{2s-N}}
\]

as claimed.

Now we apply Lemma 5.1 with \( (k - 2)/2 \leq N < k \) even, use \( c = (j(\gamma, z) - j(\gamma, z))/2i\mu \), multiply by \( (\text{Im } \gamma z)^n \) and sum over \( \gamma \in \Gamma_\infty \backslash \Gamma \).

This allows us to express \( E^{1,0}(z, s; N) \) as a sum of terms of the type

\[
h_1(z) E_l(z, s - N/2)
\]

and

\[
h_2(z) G_{(n), 1}(z, s)
\]
with \( n \leq k/2 \) and \( |l| \leq N \), where \( h_1, h_2 : \mathbb{H} \to \mathbb{C} \) are smooth functions. Now we will show that the last type is actually regular at \( s = 1 \).

**Lemma 6.8.** Let \( l < k \) be an even integer and \( n \leq k/2 \). Then \( G_{\{n\}, \emptyset, l}(z, s) \) is regular at \( s = 1 \).

**Proof.** We have by (4.9) that

\[
G_{\{n\}, \emptyset, 0}(z, s) = R(s + k/2 - n, 0)\cdots R(s + k/2 - 1, 0)(s + k/2 - 1)y^{k/2}f(z)E_{-k}(z, s).
\]

For \( n < k/2 \) all terms above are regular at \( s = 1 \). For \( n = k/2 \) we get a pole exactly if \( \langle y^{k/2}f(z)E_{-k}(z, s), 1 \rangle \neq 0 \).

By the product rule of the lowering operator and the fact that \( L_k y^{k/2}f(z) = 0 \), we get

\[
y^{k/2}f(z)E_{-k}(z, s) = y^{k/2}f(z)L_{-k+2}\frac{-E_{-k+2}(z, s)}{s} = L_0 \left( \frac{-y^{k/2}f(z)E_{-k+2}(z, s)}{s} \right).
\]

Thus by the adjoint properties of the lowering and raising operators, we see that

\[
\langle y^{k/2}f(z)E_{-k}(z, s), 1 \rangle = \left\langle L_0 \left( \frac{-y^{k/2}f(z)E_{-k+2}(z, s)}{s} \right), 1 \right\rangle
= \left\langle \frac{-y^{k/2}f(z)E_{-k+2}(z, s)}{s}, K_0 1 \right\rangle
= 0.
\]

This finishes the proof when \( l = 0 \). For negative weights it follows directly from (4.8) since \( s + k/2 - n - l/2 \) is never zero at \( s = 1 \) for \( l \leq 0 \).

For \( l > 0 \) we proceed by induction on \( l + n \). The base case is done above. So assume \( n + l > 1 \). By (4.7) we have

\[
G_{\{n\}, \emptyset, l+2}(z, s) = \frac{K_0 G_{\{n\}, \emptyset, l}(z, s) + G_{\{n-1\}, \emptyset, l+2}(z, s)}{(s + k/2 - n + l/2)}.
\]

For \( n \neq 1 \) the result follows by induction since \( s + k/2 - n + l/2 \) is non-zero at \( s = 1 \).

If \( n = 1 \) then, we have

\[
G_{\{0\}, \emptyset, l+2}(z, s) = y^{k/2}f(z)E_{l+2-k}(z, s)
\]

and since \( l + 2 < k \) we know that \( E_{l+2-k}(z, s) \) is regular at \( s = 1 \). This finishes the induction and thus the proof. \( \square \)

Thus we conclude
$$E^{1,0}(z, s; N) = (2\pi i)^{k/2} \sum_{0 \leq j \leq N/2} \frac{(-1)^j}{j!} (2i)^j y^j N/2 I_{k/2-j}(z) \sum_{0 \leq l \leq N-j} (N-j)^{l} E_{2l-N}(z, s) + \text{(regular at } s = 1)$$

$$=(2\pi i)^{k/2} \sum_{0 \leq j \leq N/2} \frac{(-1)^j}{j!} (2i)^j y^j N/2 I_{k/2-j}(z) \left(\frac{N-j}{N/2}\right)^{N/2} (\text{regular at } s = 1).$$

Now we use the following $q$-expansion

$$I_{k/2-j}(z) = \sum_{n \geq 1} \frac{a_f(n)}{(2\pi in)^{k/2-j}} q^n,$$

to arrive at the following different expression for the 1st Fourier coefficient of $E^{1,0}(z, s; N)$

$$(6.8) \quad \frac{1}{(s-1)\text{vol}(\Gamma)} \sum_{0 \leq j \leq N/2} \frac{(-1)^j}{j!} (2i)^j y^j N/2 \left(\frac{N-j}{N/2}\right)^{N/2} (\text{regular at } s = 1).$$

Now we compare (6.7) and (6.8). By a complex integration argument one can calculate that for $s = N/2 + 1$

$$\int_{-\infty}^{\infty} (t^2 + y^2)^{-s} e(-t) dt$$

is exactly (as a function of $y$) equal to $\pi \cdot \text{vol}(\Gamma)$ times the residue in (6.8). This yields the following result.

**Proposition 6.9. The Dirichlet series**

$$D^{1,0}(f, s; 1) := \sum_{r \in \mathcal{T}} \frac{L(f \otimes e(r), 1/2) e(-r)}{e(r)^{2s-N}}$$

**has meromorphic continuation to $\text{Re } s > 1/2$ satisfying the following.**

(i) $D^{1,0}(f, s; 1)$ is regular for $\text{Re } s > 1$ and the poles contained in the segment

$$1/2 < \text{Re } s \leq 1$$

are contained in $\mathcal{P}$. 

(ii) $D^{1,0}(f, s; 1)$ has a simple pole at $s = 1$ with residue

$$\frac{1}{\pi \text{vol}(\Gamma)}.$$

We can prove bounds on vertical lines for $D^{1,0}(f, s; 1)$ exactly as we did in the proof of Theorem 5.4.

Now we restrict to the case $\Gamma = \Gamma_0(q)$. Combining Proposition 6.9, Theorem 8.2, and the formula in Lemma 6.6, we conclude
6.11 Remark

in \[21\]) is to consider a general pair of cusps analogous to those in the preceding sections. This can be done easily using Lemma 3.5 in \[21\] and Corollary 5.5.

We will only sketch the proofs, since they are all completely analogous to the case \((8.8)\) is satisfied. This can be done easily using Lemma 3.5 in \[21\] and Corollary 5.5.

7. Further generalizations

In this section we will describe a few generalizations, which can easily be handled by the methods we have described. We will only sketch the proofs, since they are all completely analogous to those in the preceding sections.

7.1. General pairs of cusps. One obvious generalization (which was also carried out in \[21\]) is to consider a general pair of cusps \(a\) and \(b\) of \(\Gamma\) (instead of \(\infty\) and \(\infty\)).

In this case, we apply Lemma 5.1 with \(\sigma^{-1}\gamma\) and sum over \(\gamma \in \Gamma_a \backslash \Gamma\). This shows that

\[
E_{a,m}^n(z,s) = \sum_{\gamma \in \Gamma_a \backslash \Gamma} L(f \otimes e(\sigma^{-1}_a \gamma \infty), 1/2)^n L(f \otimes e(\sigma^{-1}_a b \gamma \infty), 1/2)^m (\text{Im} \sigma^{-1}_a \gamma z)^s
\]

can be expressed as a sum of Poincaré series of the following type

\[
G_{a,A,B,l}(z,s) = \sum_{\gamma \in \Gamma_a \Gamma} j_{\sigma^{-1}_a \gamma}(z)^{-l} \left( \prod_{\alpha \in A} I_a(\sigma^{-1}_a \gamma z; f) \right) \left( \prod_{b \in B} I_b(\sigma^{-1}_a \gamma z; f) \right) \left( \text{Im} \sigma^{-1}_a \gamma z \right)^{s + \alpha(A,B)}.
\]

Following \[21\] we define

\[
T_{ab} := \{ r \equiv a \mod c \mid \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_\infty \backslash \sigma^{-1}_a \Gamma / \Gamma_\infty, c > 0 \}
\]

and

\[
T_{ab}(X) := \{ r \in T_{ab} \mid c(r) \leq X \}.
\]

Then the constant Fourier coefficient at \(b\) of \(E_{a,m}^n(z,s)\) is exactly given by

\[
\frac{\pi^{1/2} \Gamma(s - 1/2)}{\Gamma(s)} D_{ab}^{n,m}(f,s)
\]

where

\[
D_{ab}^{n,m}(f,s) := \sum_{r \in T_{ab}} L(f \otimes e(r), 1/2)^n L(f \otimes e(r), 1/2)^m / c(r)^{2s}.
\]

By an argument completely analogous to the case \(a = b = \infty\), we get

Theorem 7.1. We have

\[
\sum_{r \in T_{ab}(X)} L(f \otimes e(r), 1/2)^n L(f \otimes e(r), 1/2)^m \ll X^2 (\log X)^{\min(n,m)}
\]
and
\[ \sum_{r \in T_{ab}(X)} |L(f \otimes e(r), 1/2)|^{2n} = P_n(|\log X|X^2 + O_\varepsilon(X^{\max(4/3, 2 \Re s_1)} + \varepsilon)), \]

where \( P_n \) is an explicit polynomial of degree \( n \) with leading coefficient
\[ \frac{2^n n!}{\pi \text{vol}(\Gamma)} (C_f)^n, \]

where
\[ C_f = \frac{(4\pi)^k ||f||^2}{(k - 1)! \text{vol}(\Gamma)}. \]

From this we conclude

**Theorem 7.2.** We have
\[ \# \left\{ r \in T_{ab}(X) \mid x_1 \leq \Re \left( \frac{L(f \otimes e(r), 1/2)}{(C_f \log c(r))^{1/2}} \right) \leq x_2, y_1 \leq \Im \left( \frac{L(f \otimes e(r), 1/2)}{(C_f \log c(r))^{1/2}} \right) \leq y_2 \right\} \]
\[ \rightarrow \frac{1}{2\pi} \int_{x_1}^{x_2} \int_{y_1}^{y_2} e^{-(x^2 + y^2)/2} dx dy, \]
as \( X \to \infty. \)

In particular in the case of \( \Gamma = \Gamma_0(q) \) and \( q \) squarefree a set of inequivalent cusps is given by \( \{ a_d = 1/d \mid d | q \} \) with scaling matrices
\[ \sigma_{1/d} = \begin{pmatrix} \sqrt{v} & 0 \\ d\sqrt{v} & 1/\sqrt{v} \end{pmatrix} \]
where \( q = dv \) (see [21, Section 8]). This implies that
\[ T_{+\infty} = \{ r = \frac{a}{c} \mod 1 \mid (a, c) = 1, (c, q) = d \} \]
and \( 1/d \). Thus we conclude that for \( f \in S_k(\Gamma_0(q)) \) we have
\[ \sum_{c < X, (c, q) = d} \sum_{a \in (\mathbb{Z}/c\mathbb{Z})^{\times}} |L(f \otimes e(a/c), 1/2)|^2 \]
\[ = \sum_{r \in T_{+\infty}(\sqrt{X})} |L(f \otimes (a/c), 1/2)|^2 \]
\[ = \frac{2}{\pi (k - 1)! \text{vol}(\Gamma_0(q))^{1/2}} (\log X)X^2 + \beta f, dX^2 + O_\varepsilon(X^{4/3 + \varepsilon}). \]

Now we sum up (7.1) for \( d \mid q \) and use
\[ \frac{1}{\text{vol}(\Gamma_0(q))} \sum_{d \mid q} \frac{q}{d} = \frac{1}{\text{vol}(\Gamma_0(1))} = \frac{3}{\pi}. \]
This proves Corollary 1.7 and similarly we can prove Corollary 6.6

**Remark 7.3.** It is an interesting problem to determine an explicit expression for the constant \( \beta_f \) in (1.5). Our methods do yield some kind of expression involving integrals of resolvents, but a much nicer expression was derived in the weight 2 case by Petridis and Risager, see [21, Theorem 6.3]. We hope to return to this in future works.
7.2. The distribution of a basis of cusp forms. Instead of considering a single cusp form \( f \), we can consider an orthogonal basis
\[
f_1, \ldots, f_d
\]
of \( S_k(\Gamma) \). We will restrict to the case \( a = b = \infty \).
In this case for any two sequences \( \underline{g} = (g_1, \ldots, g_n), \quad \underline{h} = (h_1, \ldots, h_m) \)
with \( g_j, h_j \in \{ f_1, \ldots, f_d \} \), we define the corresponding generalized Goldfeld Eisenstein series
\[
E^{\underline{g}, \underline{h}}(z, s) = \sum_{\gamma \in \Gamma \setminus \Gamma} \left( \prod_{j=1}^n L(g_j \otimes e(\gamma \infty), 1/2) \right) \left( \prod_{j=1}^m L(h_j \otimes e(\gamma \infty), 1/2) \right) (\text{Im } \gamma z)^s.
\]
The constant Fourier coefficient of \( E^{\underline{g}, \underline{h}}(z, s) \) is given by
\[
\frac{\pi^{1/2} y^{1-s} \Gamma(s - 1/2)}{\Gamma(s)} D^{\underline{g}, \underline{h}}(s),
\]
where
\[
D^{\underline{g}, \underline{h}}(s) := \sum_{r \in T} \left( \prod_{j=1}^n L(g_j \otimes e(r), 1/2) \right) \left( \prod_{j=1}^m L(h_j \otimes e(r), 1/2) \right) e(r)^{2s}.
\]
We can express the Goldfeld Eisenstein series as a linear combinations of certain Poincaré series, which generalizes \( G_{A, B, L}(z, s) \). These are indexed by tuples
\[
\underline{u} = (u_1, \ldots, u_{n'}) \quad \underline{v} = (v_1, \ldots, v_{m'})
\]
with \( n' \leq n, m' \leq m \) and \( 0 \leq u_j, v_j \leq k/2 \) and defined as
\[
G_{\underline{u}, \underline{v}}(z, s) := \sum_{\gamma \in \Gamma \setminus \Gamma} j_\gamma(z)^{-1} \left( \prod_{j=1}^{n'} \frac{I_{u_j}(\gamma z; g_j)}{(-2i)^{u_j}} \right) \left( \prod_{j=1}^{m'} \frac{I_{v_j}(\gamma z; h_j)}{(2i)^{v_j}} \right) \text{Im}(\gamma z)^{s + \alpha(\underline{u}, \underline{v})},
\]
where
\[
\alpha(\underline{u}, \underline{v}) = \left( \sum_j k/2 - u_j \right) + \left( \sum_j k/2 - v_j \right).
\]
Then the analogue of Proposition 4.9 holds for the above Poincaré series as well. Using this, it can be shown by the methods from the preceding sections that \( D^{\underline{g}, \underline{h}}(s) \) has a pole of order at most \( \min(n, m) + 1 \) at \( s = 1 \). Furthermore when \( n = m \), we have
\[
D^{\underline{g}, \underline{h}}(s) = \left( \sum_{\sigma, \sigma' \in S_n} \prod_{j=1}^n \langle g_{\sigma(j)}, h_{\sigma'(j)} \rangle \right) \frac{(4\pi)^{nk}}{(s - 1)^{n+1} \pi((k - 1)!)^n \text{vol}(\Gamma)^{n+1}} + (\text{pole order at most } n \text{ at } s = 1),
\]
where \( S_n \) denotes the group of permutation on \( n \) letters. Observe that this generalizes our previous results since \( |S_n| = n! \). In particular \( D^{\underline{g}, \underline{h}}(s) \) has a pole of order \( n + 1 \) exactly if
$g$ and $h$ are permutations of each other.

Now consider the random variable

$$
\begin{pmatrix}
Y_{1,X} \\
Z_{1,X} \\
\vdots \\
Y_{n,X} \\
Z_{n,X}
\end{pmatrix}
$$

with outcome space $T(X)$ defined as

$$Y_{j,X}(r) = \Re L(f_j \otimes e(r), 1/2)/\sqrt{C_f \log c(r)},$$

$$Z_{j,X}(r) = \Im L(f_j \otimes e(r), 1/2)/\sqrt{C_f \log c(r)}$$

for $r \in T(X)$. Then by the above we can evaluate all asymptotic moments and show (using Fréchet-Shohat) that as $X \to \infty$, this random variable converges in distribution to the $2n$-dimensional standard normal.

**Theorem 7.4.** Let $f_1, \ldots, f_d$ be an orthogonal basis for $S_k(\Gamma)$. Then we have

$$\# \left\{ r \in T(X) \mid x_{11} \leq \Re \left( \frac{L(f_1 \otimes e(r), 1/2)}{(C_{f_1} \log c(r))^{1/2}} \right) \leq x_{12}, y_{11} \leq \Im \left( \frac{L(f_1 \otimes e(r), 1/2)}{(C_{f_1} \log c(r))^{1/2}} \right) \leq y_{12}, \right. \left. \ldots, x_{d1} \leq \Re \left( \frac{L(f_d \otimes e(r), 1/2)}{(C_{f_d} \log c(r))^{1/2}} \right) \leq x_{d2}, y_{d1} \leq \Im \left( \frac{L(f_d \otimes e(r), 1/2)}{(C_{f_d} \log c(r))^{1/2}} \right) \leq y_{d2} \right\} / #T(X)$$

$$\to \frac{1}{(2\pi)^d} \int_{x_{11}}^{x_{12}} \cdots \int_{x_{d1}}^{x_{d2}} \int_{y_{11}}^{y_{12}} \cdots \int_{y_{d1}}^{y_{d2}} e^{-|x|^2 + |y|^2} / \sqrt{\pi^d} \, dx \, dy,$$

as $X \to \infty$, where $x, y \in \mathbb{R}^d$.

8. **Appendix A: Contour integration with explicit error-terms**

This appendix follows closely an unpublished note of M. Risager. We are grateful to Risager for allowing us to include it here.

8.1. **Smooth cut-offs.** For a compactly supported smooth function $\psi$ on $[0, \infty)$, we define the Mellin transform as

$$\hat{\psi}(s) = \int_0^\infty \psi(y) y^{s-1} \, dy$$

which converges absolutely for $\Re s > 0$. We have the following inversion formula

$$\psi(y) = \frac{1}{2\pi i} \int_{(c)} \hat{\psi}(s) y^{-s} \, ds$$

valid for $c > 0$. By the compact support of $\psi$ and repeated partial integration, we get the bound

$$\hat{\psi}(s) \ll_N |s|^{-N}$$

for any $N \geq 1$.

Now let $(c_n)_{n \geq 1}$ be a sequence of positive real numbers and $(a_n)_{n \geq 1}$ a sequence of complex numbers such that the Dirichlet series

$$D(s) = \sum_{n \geq 1} \frac{a_n}{c_n^s}$$

...
converges absolutely in the half-plane $\Re s > \sigma_0 > 0$. Assume further that $D(s)$ admits meromorphic continuation to $\Re s > a - \varepsilon > 0$ for some $\varepsilon > 0$ and that there is a finite number of poles in the half-plane $\Re s > a - \varepsilon$. Denote these poles by 

$$S_{\text{poles}} = \{s_0, \ldots, s_m\}$$

with $\Re s_0 = \sigma_0 \geq \cdots \geq \Re s_m$ and let the singular expansion at $s = s_m$ be given by

$$D(s) = \sum_{j=0}^{m} \frac{b_{m,j}}{(s-s_m)^j} + r_m(s),$$

where $r_m(s)$ is regular at $s = s_m$. Assume further that we have the following bound on the growth

$$(8.3) \quad D(s) \ll (1 + |t|)^A$$

valid for $a \leq \Re s \leq \sigma_0 + \varepsilon$. Under these conditions we have

**Theorem 8.1.** For a compactly supported smooth function $\psi$ on $[0, \infty)$, we have for $X > 0$

$$(8.4) \sum_{n \geq 1} a_n \psi\left(\frac{n}{X}\right) = \sum_{m=0}^{M} P_m(\log X) X^{s_n} + O\left(X^a \int_{-\infty}^{\infty} |\psi(a + it)| (1 + |t|)^A dt\right),$$

where $P_m$ are explicit polynomials of degree $d_m - 1$ given by

$$P_m(x) = \sum_{k=1}^{d_m-1} \frac{1}{k!} \left(\sum_{l=0}^{d_m-k} \frac{\psi^{(l)}(s_m)}{l!} b_{m,k+l+1} x^k\right).$$

**Proof.** Follows directly from Mellin inversion by moving the contour to $\Re s = a$. \[\square\]

In many respects the above smooth cut-off sum, may be the more natural result, but it is desirable to also obtain asymptotic formulas for the sharp cut-off

$$\sum_{c_n \leq X} a_n.$$  

For this we let $\psi$ be an approximation of the indicator function of $[0, 1]$. Below we present an explicit such construction and work out the exact error-terms.

### 8.2. Sharp cut-offs

We will now restrict to the case where $a_n \geq 0$ for all $n$. First step is to construct a smooth approximation to the indicator function $1_{[0,X]}$. The first step is a smooth approximation of the Dirac measure at $t = 0$. So let $\varphi : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be a smooth function supported in $[-1, 1]$ with $\int_{-1}^1 \varphi(t) dt = 1$.

Then for $\delta < 1/2$ we define

$$\varphi_{\delta}(t) = \delta^{-1} \varphi(t/\delta),$$

which is supported in $[-\delta, \delta]$ and satisfies $\int_{-1}^1 \varphi_{\delta}(t) dt = 1$. This will serve as an approximation of the Dirac measure at $t = 0$.

From this we define the functions $\psi_{\delta, \pm} : \mathbb{R}_+ \to \mathbb{R}$ as the following convolutions

$$\psi_{\delta, \pm}(y) = \int_{0}^{\infty} 1_{[0,1+\delta]}(yt) \varphi_{\delta}(t-1) dt.$$  

Observe that the support of $\psi_{\delta, +}$ is contained in $[0, (1 + \delta)/(1 - \delta)]$ and for $y < 1$ we have $\psi_{\delta, +}(y) = 1$ and similarly $\psi_{\delta, -}(y) = 0$ for $y > 1$ and $\psi_{\delta, -}(y) = 1$ for $y \in [0, (1 - \delta)/(1 + \delta)]$. The functions $\psi_{\delta, \pm}(y)$ will serve as respectively an upper and lower bound for the indicator function $1_{[0,X]}$. In the end we will use the parameter $\delta$ to
Now let us bound the error-term

\[ \int_{-\infty}^{\infty} |\hat{\psi}_{\delta,\pm}(a + it)| (1 + |t|)^A dt. \]  

(8.6)

For this we need estimates for \( \hat{\psi}_{\delta,\pm}(a + it) \). By using the definition of \( \psi_{\delta,\pm} \) we get

\[ \psi_{\delta,\pm}^{(n)}(y) \ll_{n} (y\delta)^{-n}. \]

For \( n \geq 1 \) the support of \( \psi_{\delta,\pm}^{(n)}(y) \) is contained in

\[ \left( \left[ \frac{1 - \delta}{1 + \delta} \right], \left[ \frac{1 + \delta}{1 - \delta} \right] \right). \]

Thus we get by repeated partial integration

\[ \hat{\psi}_{\delta,\pm}(a + it) \ll \int_{0}^{\infty} \frac{|\hat{\psi}_{\delta,\pm}^{(n)}(y)|}{|a + it|^n} dy \ll \int_{\left( \frac{1 + \delta}{1 - \delta} \right)}^{\left( \frac{1 - \delta}{1 + \delta} \right)} \frac{1}{(\delta y)^n} \frac{y^{a+n}}{|a + it|^n} dy \ll \frac{\sqrt{\delta}}{(\delta(a + it))^n} \]

for any \( n \geq 1 \). By interpolation this is true for all \( r \in \mathbb{R}_{\geq 1} \). Now we need to choose a small \( n \) in order to make \( \delta^{1-n} \) small but large enough so that the integral in (8.6) converges, i.e. \( r = A + 1 + \varepsilon \). This yields

\[ \int_{-\infty}^{\infty} |\hat{\psi}_{\delta,\pm}(a + it)| (1 + |t|)^A dt \ll e^{\varepsilon \delta^{-A-\varepsilon}}. \]

By a straight forward computation, we see that

\[ \psi_{\delta,\pm}^{(n)}(s_j) = (-1)^n n! s_{m-1}^{n-1} + O(\delta^{n+1}). \]

Now since

\[ \sum_{n} a_n \psi_{\delta,\pm}(c_n/X) \leq \sum_{c_n \leq X} a_n \leq \sum_{n} a_n \psi_{\delta,\pm}(c_n/X), \]

we get by Theorem 8.1

\[ \sum_{c_n \leq X} a_n = \sum_{m=0}^{M} P_m(\log X) X^{s_m} + O_{\varepsilon}(\delta X^{\sigma_m + \varepsilon} + \delta^{-A-\varepsilon} X^{a}) \]

where

\[ P_m(x) = \sum_{k=1}^{d_m-1} \frac{1}{k!} \left( \sum_{l=0}^{d_m-1-k} (-1)^l s_m^{l-1} b_{m,k+l+1} \right) x^k. \]

To balance the error-term we put

\[ \delta = X^{-(\sigma_m - a)/(1+A)} \]

which yields an error-term \( \ll_{\varepsilon} X^{(a + \sigma_m A)/(1+A) + \varepsilon} \). Absorbing \( P_m \) for \( m \neq 0 \) into the error-term, we arrive at
Theorem 8.2. Let $D(s)$ be a Dirichlet series as above with positive coefficients $a_n$. Then we have
\begin{equation}
\sum_{c_n \leq X} a_n = P(\log X)X^{s_0} + O\left(X^{\max\left(\sigma_0, A/(1+A), \Re s_1\right)} + \varepsilon\right),
\end{equation}
where $P = P_0$ is a polynomial of degree $d_0 - 1$ with leading coefficient $b_{d_0}/s_0(d_0 - 1)!$.

Remark 8.3. If the coefficients $a_n$ are not positive, then one needs the extra assumption
\begin{equation}
\sum_{X < c_n \leq (1+\delta)X} |a_n| \ll \delta X^{\sigma_0} + X^{\Re s_1},
\end{equation}
but then the conclusion of Theorem 8.2 still holds. This assumption is needed to control the error
\[
\sum_{n \leq X} a_n - \sum_{n \geq 1} a_n \psi_{\delta_+}(n/X).
\]
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