Quantum and Floer cohomology have the same ring structure

Sergey Piunikhin

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Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02139 USA
e-mail serguei@math.mit.edu

Abstract

The action of the total cohomology $H^*(M)$ of the almost Kahler manifold $M$ on its Floer cohomology, introduced originally by Floer, gives a new ring structure on $H^*(M)$. We prove that the total cohomology space $H^*(M)$, provided with this new ring structure, is isomorphic to the quantum cohomology ring. As a special case, we prove the formula for the Floer cohomology ring of the complex grassmanians conjectured by Vafa and Witten

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1. INTRODUCTION
Floer cohomology $HF^*(M)$ of the free loop-space of the symplectic manifold $M$ is under extensive study by both mathematicians and physicists.

Originally the machinery of Floer cohomology was developed in [F1] in order to prove the classical Arnold’s Conjecture giving a lower bound on the number of fixed points of a symplectomorphism in terms of Morse theory. Later Floer cohomology appeared in “topological sigma-models” of string theory [Wi1],[BS].

Quantum cohomology rings of Kahler (or more generally, almost-Kahler) manifolds were introduced by Witten [Wi2] and Vafa [Va1] using moduli spaces of holomorphic curves [CKM],[Gr],[McD1],[Ru2].

Recently by Ruan and Tian [RT] by adjusting Witten’s degeneration argument, proved that these rings are associative

It is difficult to give a rigorous and self-consistent mathematical definition of quantum cohomology rings because the moduli spaces of holomorphic curves are non-compact and may have singularities, the facts often ignored by physicists.

The proof of associativity in [RT] for quantum cohomology ring uses a definition of multiplication that involves the moduli spaces of solutions of inhomogenous Cauchi-Riemann equations [Ru1].

The quantum cohomology rings have been computed for:

a) complex projective spaces [Wi2]

b) complex Grassmanians (the formula was conjectured by Vafa [Va1], is proved in the present paper, and also independently in [AS] and [ST] )

c) toric varieties [Ba]

d) flag varieties [GK]

e) more general hermitian symmetric spaces [AS]

Quantum cohomology rings of Calabi-Yau 3-fold “computed” in several examples by “mirror symmetry” [COGP] not considered rigorous. To justify these “computations” is a great challenge for algebraic geometers.

The linear map $m_F : H^*(M) \to End(HF^*(M))$ or, equivalently, the action of the classical cohomology of the manifold $M$ on its Floer cohomology, was defined by Floer himself. He computed this action for the case $M = \mathbb{C}P^n$ and noticed the following fact:

For any two cohomology classes $A$ and $B$ in $H^*(\mathbb{C}P^n)$ the product $m_F(A)m_F(B)$ of the linear operators $m_F(A)$ and $m_F(B)$ acting on Floer cohomology $HF^*(\mathbb{C}P^n)$ has the form $m_F(C)$ for some cohomology class $C$ in $H^*(\mathbb{C}P^n)$. This gives us a new ring structure on $H^*(\mathbb{C}P^n)$, which is known to be different from the classical cup-product. We will call this new multiplication law Floer multiplication.

Floer conjectured that the same phenomenon might be true for all semi-positive symplectic (or at least Kahler) manifolds, thus providing a new ring
structure on the total cohomology $H^*(M)$ our symplectic manifold $M$.

Using path-integral arguments, V. Sadov [S] “proved” that when $M$ is a semi-positive symplectic manifold, then:

1) “The operator algebra should close”, i.e., for any two cohomology classes $A$ and $B$ in $H^*(M)$ the product $m_F(A)m_F(B)$ always has the form $m_F(C)$ for some cohomology class $C$ in $H^*(M)$.

and

2) **Floer multiplication** coincides with **quantum multiplication**.

See also [Fu1], [CJS], [GK] and [McD S] for alternative definitions of cup-products in Floer cohomology and their conjectural relation with the quantum multiplication.

The purpose of the present note is to give a rigorous proof of Sadov’s statements. We prove

**The Main Theorem.** *For semi-positive symplectic manifold $M$ the ring structure on $H^*(M)$ inherited from its action $m_F$ on the Floer cohomology coincides with the the quantum multiplication on $H^*(M)$ defined as in [RT]*.

Another problem we will encounter is that some components of the moduli space of holomorphic curves may not have have its dimension equal to the “virtual dimension” which can be computed using Atiyah-Singer index theorem. This problem was artificially avoided in [RT] paper by introducing the moduli spaces of solutions of inhomogenous Cauchy-Riemann equations.

The definition of “contribution to the quantum cup-product” from these “exceptional” components of the moduli space of holomorphic curves is different from the definition of contribution to the quantum cup-product from the other components.

When all these irreducible components are smooth , Witten showed in [Wi4] how to define this contribution.

Witten introduces “the bundle of the fermion zero-modes” over the “exceptional” component of the moduli space of holomorphic curves and takes the Euler class of this bundle. He wedges this Euler class with the wedge-product of “geometric” classes and then “integrates” over the whole component of the moduli space ( taking care with the compactification of this component).

See also [AM] for the sheaf-theoretic computations of contribution from “multiply-covered curves” using this definition.

In the case when these “exceptional” components occur, it was not known whether the quantum cup-product defined as in [Wi4] and [AM] was associative.

Assuming that the compactified moduli spaces admit smooth desingularization such that “the compactification divisor” of that desingularization has
codimension at least two we prove associativity by establishing equivalence of this definition with the definition used in [RT].

The Floer picture gives a geometric way to calculate the Euler class which encounters in Witten’s definition.

2. MODULI SPACES OF J-HOLOMORPHIC SPHERES AND ITS COMPACTIFICATION

Definition. The manifold $M$ is called is called an almost-Kahler manifold if it admits an almost-complex structure $J$ and a symplectic form $\omega$ such that for any two tangent vectors $x$ and $y$ to $M$, $\omega(x; y) = \omega(J(x); J(y))$ and for any non-zero tangent vector $x$ to the following inequality holds:

$$\omega(x; J(x)) > 0 \quad (2.1)$$

Definition. An almost-complex structure $J$ and a closed 2-form $\omega$ on $M$ are called compatible if for any tangent vector $x$ to $M$ $\omega(x; J(x)) \geq 0$ and the equality takes place only if $\omega(x; y) = 0$ for any tangent vector $y$.

In particular, the symplectic form $\omega$ is compatible with $J$ iff (2.1) holds.

Let $M$ be a compact almost-Kahler manifold of dimension $2n$ which we assume (for simplicity) to be simply-connected. Let us fix an almost-complex structure $J_0$ on $M$ and let us consider the space $\tilde{K}$ of all $J_0$-compatible symplectic forms and its image $K$ in the cohomology $H^2(M, \mathbb{R})$.

If it will not lead to a confusion, we will denote the closed $J_0$-compatible two-form and the corresponding cohomology class by the same symbol.

It follows directly from definitions that if $M$ is an almost-Kahler manifold (which is equivalent to the fact that $\tilde{K}$ is non-empty) then:

1) $\tilde{K}$ is an open convex cone in the space of all closed 2-forms on $M$. The set $\tilde{K}$ does not contain any nontrivial linear subspace (otherwise $\omega$ and $-\omega$ would be simultaneously $J_0$-compatible which is impossible).

2) $K$ is an open convex cone in $H^2(M, \mathbb{R})$ which does not contain any nontrivial linear subspace.

The openness of $K$ follows from the fact that small perturbations of any given $J_0$-compatible symplectic form, are themselves $J_0$-compatible and symplectic.

Since a symplectic form compatible with $J_0$ (and in fact any symplectic form) on a compact oriented manifold $M$ cannot be cohomologically trivial then $K$ cannot contain a nontrivial linear subspace in $H^2(M, \mathbb{R})$.

Let us consider symplectic forms $\{\omega_1, ..., \omega_s\}$ such that:
1) they lie inside $\tilde{K}$.
2) their cohomology classes form a basis in $H^2(M, R)$.
3) the elements of this basis are represented by integral cohomology classes
4) $\{\omega_1, ..., \omega_s\}$ generate $H^2(M, Z)$ as an abelian group.

We can always find such a collection of symplectic forms since any open convex cone in $H^2(M, R)$ contains such a collection.

If we fix a homotopy class of $J_0$ in the space of almost-complex structures on $M$ it will make sense to talk about the **first Chern class** $c_1(TM)$ of the tangent bundle of $M$ as an element in the second cohomology group of $M$.

**Definition.** The almost-complex manifold $M$ is called semi-positive if $c_1(TM)$ can be represented by a closed 2-form compatible with the almost-complex structure $J_0$.

The almost-complex manifold $M$ is called **positive** if $c_1(TM)$ can be represented by a symplectic form compatible with the almost-complex structure $J_0$.

The above definitions should be thought as generalizations the notion of (semi)-positive simply-connected Kahler manifold.

It sometimes appears to be useful to perturb slightly the complex structure on $M$ (or on $CP^1 \times M$) and to work with non-integrable almost-complex structures. It is much easier to prove transversality results if we are allowed to work in this larger category.

Let us consider the complex projective line $CP^1$ with its standard comolex structure $i$ and Fubiny-Study Kahler form $\Omega$. Let us take the product $CP^1 \times M$ in the almost-Kahler category. Let $J$ be the space of all smooth almost-complex structures on $CP^1 \times M$ such that the projection on the first factor $CP^1 \times M \rightarrow CP^1$ is holomorphic. Let us equip this space with the $W_{2n}^5$-Sobolev norm topology. This means that all the partial derivatives up to order $5n$ should be square-integrable on $CP^1 \times M$.

Let $J_0$ be a neighborhood of $i \times J_0$ in $J$ consisting of almost-complex structures compatible with symplectic forms $\{1 \otimes \omega_1, ..., 1 \otimes \omega_s, \Omega \otimes 1\}$ and with some differential form representative of $1 \otimes c_1(TM)$. Since the notion of compatibility with a 2-form is an open condition in $J$, such a neighborhood always exists.

Let us consider the vector bundle over the product $CP^1 \times M$ consisting of $i \times J_0$-antilinear maps from $T(CP^1)$ to $TM$. “$i \times J_0$-antilinear” means that for any $g \in \mathcal{G}$ we have $J_0 g = -gi$. Let $\mathcal{G}$ be the space consisting of all $W_{2n-1}^5$-sections of the above-defined vector bundle.

Equivalently, $\mathcal{G}$ can be thought as a space of all $(0, 1)$-forms on $CP^1$ with the coefficients in the tangent bundle to $M$. 

If $g$ is any such $(0, 1)$-form, we can construct an almost-complex structure $J_g$ on $CP^1 \times M$ which is written in coordinates as follows:

$$J_g = \begin{pmatrix} i & -g \\ 0 & J_0 \end{pmatrix}$$

Here we wrote the matrix of $J_g$ acting on $T(CP^1) \oplus TM$.

Thus, we have an embedding $G \subset J$. Let $G_0$ be the intersection of $G$ and $J_0$.

We will assume both $J_0$ and $G_0$ to be contractible.

Presumably, the introduction of almost-complex structures can be avoided. We use them to modify the proofs of some analytic lemmas.

What we really need is to one fixed almost-complex structure $J_0$ on $M$ (which in all examples will be an actual complex structure) and perturbations of the product (almost)-complex structure on $CP^1 \times M$ of the form (2.2).

So, we desided to use more complicated notations to simplify the proofs.

Let $J \in J_0$ be an almost-complex structure on $CP^1 \times M$

**Definition (Gromov).** A $J$-holomorphic sphere in $M$ is any almost-complex submanifold in $CP^1 \times M$ of real dimension two (or “complex dimension one”) which projects isomorphically onto the first factor $CP^1$.

Equivalently, a $J$-holomorphic sphere in $M$ can be defined as a pseudo-holomorphic section of the the (pseudo-holomorphic) bundle $M \times CP^1$ over $CP^1$ where the almost-complex structure $J$ on $M \times CP^1$ is a perturbation of the product almost-complex structure. Topologically this is the trivial bundle over $CP^1$ with the fiber $M$ but (pseudo)-holomorphically it is not trivial.

Any $J$-holomorphic sphere can be thought as a map $\varphi$ from $CP^1$ to $M$ which satisfies a non-linear PDE

$$\bar{\partial}_J \varphi = 0$$

If our almost-complex structure $J_g$ has the form (2.2) than the equation of a $J_g$-holomorphic sphere $\varphi$ can be rewritten as

$$\bar{\partial}_{J_0} \varphi = g$$

Here $\bar{\partial}_{J_0}$ is the usual $\bar{\partial}$-operator on $M$ associated with our original (almost)-complex structure $J_0$.

We assume our manifold $M$ to be **semi-positive** with respect to the almost-complex structure $J_0$ (and thus, the manifold $CP^1 \times M$ will be semi-positive with respect to all almost-complex structures in $J_0$). Semi-positivity implies
that the integrals of the first Chern class \( c_1(TM) \) over all \( J \)-holomorphic curves in \( M \) are non-negative if \( J \in J_0 \).

Let \( C \subset H_2(M, \mathbb{R}) \) be the closure of the convex cone generated by the images of homology classes of \( J \)-holomorphic spheres (for all \( J \in J_0 \)). Then \( C \) will lie in the closure of the convex dual of the cone \( K \subset H^2(M, \mathbb{R}) \).

Following [Ru1], we will call a non-zero cohomology class \( A \in H^2(M, \mathbb{Z}) \) an effective class if \( A \) lies inside the closed cone \( C \).

Let \( q_1, \ldots, q_s \) be the dual to \( w_1, \ldots, w_s \) basis in \( H^2(M, \mathbb{R}) \). We will write the elements of \( H^2(M) = H^2(M, \mathbb{Z}) \) in multiplicative notation. The monomial \( q^d = q_1^{d_1} \cdots q_s^{d_s} \) is by definition the sum \( \sum_{i=1}^s d_1 q_{i} \in H_2(M) \). Here \( d \) is a vector of integers \( (d_1, \ldots, d_s) \) and \( q = (q_1, \ldots, q_s) \) is a multi-index. Then the group ring \( Z[H_2(M)] \) is a commutative ring generated (as an abelian group) by monomials of the form \( q^d = q_1^{d_1} \cdots q_s^{d_s} \).

The group ring \( Z[H_2(M)] \) which is isomorphic to the ring \( Z[q_1^{\pm 1}, \ldots, q_s^{\pm 1}] \) of Laurent polynomials, has an important subring \( Z[C] \). The semi-positivity of \( M \) implies that

\[
Z[C] \subset Z[q_1^{\pm 1}, \ldots, q_s^{\pm 1}]
\]

i.e., that monomials \( q_1^{d_1} \cdots q_s^{d_s} \) may appear in \( Z[C] \) only if all \( (d_1, \ldots, d_s) \) are non-negative.

The ring \( Z[C] \) has a natural augmentation \( I : Z[C] \to Z \) which sends all non-constant monomials in \( \{q_i\} \) to zero. Thus, we can consider its completion \( Z_{<C>} \) with respect to the \( I \)-adic topology. This completion lies naturally in the ring \( Z_{<q_1, \ldots, q_s>} \) of formal power series in \( \{q_i\} \).

For our future purposes let us introduce the following ring

\[
N = Z_{<q_1, \ldots, q_s>} \otimes Z_{[H_2(M)]}
\]

The ring \( N \) is called Novikov ring. The similar ring appeared in Novikov’s study of Morse theory of multivalued functions [No]. Novikov’s refinement of Morse theory is almost exactly the kind of Morse theory we need in our study of of Floer homology (see also [Hs]).

Let us consider the abelian group \( H^*(M, \mathbb{Z}) \otimes Z_{<C>} \). It has an obvious structure of a \( Z_+ \)-graded ring inherited from the usual grading in cohomology, provided that all the elements of the augmentation ideal \( I(Z_{<C>}) \) have degree zero.

The same abelian group \( H^*(M) \otimes Z_{<C>} \) has another \( Z_+ \)-graded ring structure which can be constructed as a \( q \)-deformation of the classical cohomology ring \( H^*(M) \) with non-trivial grading of the “deformation parameters”.
To be more concrete, let us define a \( Z \)-grading on \( H^\ast(M) \otimes Z_{<q_1,\ldots,q_s>} \) as follows: any element \( A \) from \( H^\ast(M) \otimes Z_{<q_1,\ldots,q_s>} \) can be obtained as a (possibly infinite) sum of “bihomogeneous pieces” \( A = \sum_{m,d} A^{m,d} \otimes q^d \) where \( A^{m,d} \in H^m(M,Z) \). Then let us define

\[
\text{deg}[A^{m,d} \otimes q^d] = m + 2 < c_1(TM); q^d > \tag{2.4A}
\]

where the last term means evaluation of the 2-cocycle \( c_1(TM) \) on the 2-cycle \( q^d \).

The formula (2.4A) can be rewritten in more elegant way:

\[
\text{deg}[A^{m,d} \otimes q^d] = m + 2 \sum_{i=1}^s d_i < c_1(TM); q_i > \tag{2.4B}
\]

Using the semi-positivity condition

\[
\text{deg}[q_i] = 2 < c_1(TM); q_i > \geq 0 \tag{2.5}
\]

we see that our \( Z \)-grading is actually a \( Z_+ \)-grading on \( H^\ast(M) \otimes Z_{<q_1,\ldots,q_s>} \) and on its subring \( H^\ast(M) \otimes Z_{<C>} \). This \( Z \)-grading can be extended to the \( Z \)-grading on \( H^\ast(M) \otimes N \) (when we “extend the scalars”).

**Definition.** For each multi-index \( d = (d_1,\ldots,d_s) \) let \( \text{Map}_d \) be the space of all Sobolev maps from \( CP^1 \) to \( M \) of a given homotopy type specified by “the generalized degree” \( d = (d_1,\ldots,d_s) \).

“Sobolev” means that the the map \( \varphi \in \text{Map}_d \) should have the square-integrable partial derivatives up to order \( 5n \). (The first derivative of the map \( \varphi \) from \( CP^1 \) to \( M \) is a one-form on \( CP^1 \) with the values in \( \varphi^\ast(TM) \)).

“Homotopy type specified by the generalized degree” \( d = (d_1,\ldots,d_s) \) means that \( \int_{\varphi(CP^1)} \omega_i = d_i \) for each \( \varphi \in \text{Map}_d \) and for each \( i = 1,\ldots,s \).

The space \( \text{Map}_d \) thus has a natural structure of a (connected) Hilbert manifold (see [McD1] and [McD S] for the proof) which is homotopically equivalent to the space of all smooth (or all continuous) maps from \( CP^1 \) to \( M \) of a given homotopy type. This space is a connected component of the larger space \( \text{Map} = \bigcup_d \text{Map}_d \) of all Sobolev maps from \( CP^1 \) to \( M \) (regardless of homotopy type) which is also a Hilbert manifold.

Let us introduce an infinite-dimensional Hilbert bundle \( \mathcal{H} \) over \( \text{Map} \times J_0 \). The fiber \( \mathcal{H}_{J,\varphi} \) of the bundle \( \mathcal{H} \) over the “point” \( (\varphi,J) \in \text{Map} \times J_0 \) will be the space of all \( (0,1) \)-forms on \( CP^1 \) with the values in the complex \( n \)-dimensional vector bundle \( \varphi^\ast(TM) \). All \( (0,1) \)-forms on \( CP^1 \) means all \( (0,1) \)-forms on \( CP^1 \) lying in the Sobolev space \( W^2_{5n-1} \). The almost-complex structure \( J \) on \( M \) provides the tangent bundle \( TM \) with the structure of the complex \( n \)-dimensional vector bundle.
The bundle \( \mathcal{H} \) is provided with a section \( \bar{\partial} \), given by the formula

\[
(\varphi, J) \rightarrow \bar{\partial}_J(\varphi)
\] (2.6)

The above-defined section \( \bar{\partial} \) is actually a nonlinear \( \bar{\partial} \)-operator.

**Proposition 2.1.** The zero set \( \bar{\partial}^{-1}(0) \) consists of the pairs \( (\varphi, J) \) where \( \varphi \) is a \( J \)-holomorphic map.

**Definition.** For each multi-index \( d \) let \( \mathcal{M}_{J,d} \subset \text{Map}_d \) be the space of all solutions of (2.3A) of homotopy type specified by \( d \). Let us and call \( \mathcal{M}_{J,d} \) the moduli space of \( J \)-holomorphic maps from \( \mathbb{C}P^1 \) to \( M \) of “the generalized degree” \( d \).

The above defined Hilbert bundle \( \mathcal{H} \) over \( \text{Map} \times J_0 \) can be (trivially) extended to a Hilbert bundle over the product of \( \text{Map} \times J_0 \) and \( G_0 \). It can also be trivially extended to a Hilbert bundle over the product of \( \text{Map} \times J_0 \) and \( G_0 \times G_0 \times [0; 1] \).

We will denote these three Hilbert bundles by the same symbol \( \mathcal{H} \). We will also denote by \( \mathcal{H} \) the restriction of these Hilbert bundles to connected components \( \text{Map}_d \times \text{[auxilliary space]} \) of their bases.

Since \( G_0 \) is an open subset in the vector space \( G \) which has has a basepoint (zero), then it makes sense to speak about extension of smooth sections of \( \mathcal{H} \) from \( \text{Map} \times J_0 \) to the larger spaces \( \text{Map} \times J_0 \times G_0 \) and \( \text{Map} \times J_0 \times G_0 \times G_0 \times [0; 1] \).

We assume that \( \text{Map} \times J_0 \) is embedded as \( \text{Map} \times J_0 \times \{0\} \) into the product with the auxilliary spaces.

**Proposition 2.2 (Gromov).** If restricted to the subspace \( \text{Map} \times G_0 \) in \( \text{Map} \times J_0 \) the zero set \( \bar{\partial}^{-1}(0) \) consists of the pairs \( (\varphi, g) \) where \( \varphi \) is a solution of inhomogenous Cauchy-Riemann equation (2.3B)

**Definition.** A section \( \Phi \) of the Hilbert bundle \( \mathcal{H} \) over some base Hilbert manifold \( B \) is called regular if its derivative \( D\Phi \) at each point in the zero-locus \( \Phi^{-1}(0) \) is a surjective linear map from the tangent space to \( B \) to the tangent space to the fiber of \( \mathcal{H} \).

It is obvious that the section \( \bar{\partial} \) is regular since its derivative in \( J_0 \)-directions is already surjective linear map from \( T_{G_0} \subset T_{J_0} \) to \( TH \).

Thus, \( \bar{\partial}^{-1}(0) \) is a smooth Hilbert manifold and by the infinite-dimensional version of Sard Theorem we have that for “generic” \( g \in \Omega^{0,1}(TM) \) the space of solutions of inhomogenous Cauchy-Riemann equation (2.3B) is smooth finite-dimensional manifold.
By the same reason, for “generic” almost-complex structure $J \in J_0$ the moduli space $M_{J,d}$ of $J$-holomorphic spheres of “degree $d$” is a smooth finite-dimensional manifold.

Dimension of this manifold is given by the index of the Fredholm linear operator $D\bar{\partial}$ which acts from $T(Map_d)$ to $TH$. The operator $D\bar{\partial}$ is defined as a derivative of the section $\bar{\partial}$ in $Map_d$-directions.

“Generic” here means the Baire second category set.

**Proposition 2.3 (Gromov).** For the “generic” choice of $J$ the moduli space $M_{J,d}$ will be a smooth almost-complex manifold of dimension

$$\dim M_{J,d} = \dim M + \sum_{i=1}^{s} d_i \deg [g_i] \quad (2.7)$$

The idea of the proof of (2.7) is as follows. The operator $D\bar{\partial}$ is actually a (twisted) $\bar{\partial}$-operator on $CP^1$. Then Atiyah-Singer index theorem, applied to any of our “$\bar{\partial}$-operators”, gives us the r.h.s. of (2.7).

To prove that the actual dimension of the moduli space $M_{J,d}$ is equal to its “virtual dimension” given by the index calculation in the r.h.s. of (2.7), we need some analytic lemmas. These lemmas were first proved by Freed and Uhlenbeck [FU]. We also recommend the reader a book [DK].

**Lemma 2.4 (Proposition 4.3.11 of [DK]).** If $\Phi$ is the regular section of the Hilbert bundle $H$ over $Map_d \times \text{[auxillary space]}$ then for “generic” value of the parameter $g$ in the auxillary space the zero-set of $\Phi$ restricted to $Map_d \times g$ will be a smooth submanifold of dimension equal to “the virtual dimension”

Here “the virtual dimension” means the index of the derivative of the section $\Phi$ in $Map_d$-directions (these operators are always Fredholm).

**Lemma 2.5 (Proposition 4.3.10 of [DK]).** Any finite-dimensional pseudo-manifold of parameters in the auxillary space can be perturbed to be made transversal to the projection operator.

Here the projection operator (by definition) projects $\Phi^{-1}(0) \subset Map_d \times \text{[auxillary space]}$ to the second factor (the auxillary space).

The particular case of this lemma will be

**Lemma 2.6.** For the pair $g^1$ and $g^2$ of “the regular values” of parameters in the auxillary space any path $\gamma$ joining them can be perturbed to be made transversal to the projection operator.

The Lemma 2.6 implies that the inverse image of this “transversal path” $\gamma$ gives us a smooth cobordism between $\Phi^{-1}(0) \cap Map_d \times \{g^1\}$.
and $\Phi^{-1}(0) \cap Map_d \times \{g^2\}$.

Using Lemmas 2.4 and 2.6 we have that there exists a smooth cobordism $\mathcal{M}^i$ inside $Map_d \times J_0$ between the moduli spaces $\mathcal{M}_{Jg^1,d}$ and $\mathcal{M}_{Jg^2,d}$ constructed using different “regular” almost-complex structures $J_1$ and $J_2$.

The moduli spaces $\mathcal{M}_{J,d}$ of $J$-holomorphic spheres are not compact.

There are only two (closely related) sources of non-compactness of these moduli spaces.

1) The sequence of unparametrized $J$-holomorphic spheres may “split” into two $J$-holomorphic spheres by contracting of some loop on $CP^1$. The resulting “splitting” is (formally) not in our space which means that the above sequence diverges. This “degeneration” may occur only if both spheres which appear after this “splitting process” have non-trivial homotopy type (and cannot be contracted to a point).

2) The sequence of parametrized $J$-holomorphic spheres in $\mathcal{M}_d$ may diverge by “splitting off” a $J_0$-holomorphic sphere of lower (or the same) degree at some point on $CP^1$. This means that the curvature of our sequence of maps “blows up” at some point on $CP^1$. This phenomenon is the famous Uhlenbeck’s “bubbling off” phenomenon [SU].

The bubbling off may be possible even when the classical splitting is impossible. For example, let us consider the simplest case when $M = CP^1$ with the standard complex structure and $d = 1$. Then the sequence of holomorphic degree-one maps from $CP^1$ to itself may diverge by “bubbling off” any any point on $CP^1$. This will compactify the non-compact space $\mathcal{M}_{J,1}$ (which is diffeomorphic to $PSL(2,C)$ in this example) to a compact space $CP^3$.

**COMPACTIFICATION OF MODULI SPACES**

In order to compactify the moduli space $\mathcal{M}_{J,d}$ in the sense of Gromov, we should, roughly speaking, add to it the spaces of $J$-holomorphic maps of the connected sum of several copies of $CP^1$ to $M$ of total degree $d$.

In other words, the space of “non-degenerate” $J$-holomorphic spheres in $M$ is non-compact but it will be compact if we add to it “degenerate $J$-holomorphic spheres”.

Ruan [Ru1] gave an explicit description how to stratify the compactified moduli spaces $\bar{\mathcal{M}}_{J,d}$.

**Definition (Ruan).** Let us call a degeneration pattern the following set of data $DP1 - DP3$:

- **DP1**) The class $d^0 \in C$, the set $\{d^1; \ldots ; d^k\} \subset C \{0\} \subset H^2(M)$ of
effective classes, and the set \( \{a_1; \ldots; a_k\} \) of positive integers, such that the following identity holds: \( d = d^0 + \sum_{i=1}^k a_i d^i \)

DP2) The set \( \{I_1; \ldots; I_t\} \) of subsets in the set \( \{d^0; d^1; \ldots; d^k\} \). We do not allow one of \( \{I_1; \ldots; I_t\} \) to be the proper subset of another.

Using the set of data \( \{d^0; d^1; \ldots; d^k; I_1; \ldots; I_t\} \), we can construct a graph \( T \) with \( k + 1 + t \) vertices \( \{d^0; d^1; \ldots; d^k; I_1; \ldots; I_t\} \) as follows:

If the class \( d^i \) lies in the set \( I_j \) then we join the vertices \( d^i \) and \( I_j \) by an edge.

DP3) The graph \( T \) obtained by above prescription is a tree.

**Definition.** We will call a \( J_0 \)-holomorphic sphere \( C_i \in \mathcal{M}_{J_0,d^i} \) simple if \( C_i \) cannot be obtained as a branched cover of any other \( J_0 \)-holomorphic sphere.

**Definition.** If the \( J_0 \)-holomorphic sphere \( C_i \) is not simple then we will call it multiple-covered.

We will denote \( \mathcal{M}^\ast_{J_0,d^i} \) the space of all simple \( J_0 \)-holomorphic spheres of “degree \( d \)” in \( M \). According to the theorem of McDuff [McD1] if the almost-complex structure \( J_0 \) on \( M \) is “generic” then \( \mathcal{M}^\ast_{J_0,d^i} \) is a smooth manifold of dimension given by the formula (2.7)

Let \( D_d = \{ \{d^0; d^1; \ldots; d^k\}; \{a_1; \ldots; a_k\}; \{I_1; \ldots; I_t\}; T \} \) be some degeneration pattern. Then let us define \( \mathcal{N}_{I,J,D} \) as a topological subspace in \( \mathcal{M}_{J_0,d^0} \times \prod_{i=1}^k [\mathcal{M}^\ast_{J_0,d^i} / PSL(2,C)] \) as follows:

An element \( \varphi \) in \( \mathcal{N}_{I,J,D} \) consists of one parametrized \( J_0 \)-holomorphic sphere \( C_0 \in \mathcal{M}_{J_0,d^0} \) and \( k \) unparametrized \( J_0 \)-holomorphic spheres \( \{C_i \in [\mathcal{M}^\ast_{J_0,d^i} / PSL(2,C)] \} \). We require that for any subset \( I_j = \{d^{i_1}; \ldots; d^{i_n}\} \) from \( \{I_1; \ldots; I_t\} \) the spheres \( \{C_{i_1}; \ldots; C_{i_n}\} \) have a common intersection point. We do not allow this intersection point to lie on any other sphere \( C_i \subset M \) in our collection.

**Comment 1.**

We can think about parametrized spheres in \( M \) as about unparametrized spheres in \( M \times CP^1 \) which have degree one in \( CP^1 \)-directions.

**Comment 2.**

Degeneration of parametrized \( J_0 \)-holomorphic sphere of degree \( d \) in \( M \) can be translated in this language as splitting of unparametrized \( J_0 \)-holomorphic sphere of degree \( d + [CP^1] \) in \( M \times CP^1 \) in connected sum of several unparametrized \( J_0 \)-holomorphic spheres of total degree \( d + [CP^1] \).

---

1 If \( C_{i_1} \cap \ldots \cap C_{i_n} \cap C_i \neq \emptyset \) then our “degenerate \( J \)-holomorphic sphere” would lie in the other stratum governed by another “degeneration pattern”. 
One of these spheres has degree one in $CP^1$-directions (and should lie in $M_{J_0,d^0}$). All the other spheres have degree zero in $CP^1$-directions. Each of these spheres maps to a point under projection $M \times CP^1 \to M$ and thus, it should lie in $M_{J_0,a_i,d}/PSL(2,C)$.

**Comment 3.**

The numbers $\{a_i\}$ respect the fact that some of $J_0$-holomorphic spheres which appear in “the degeneration process” are $\{a_i\}$-fold branched covers of other $J_0$-holomorphic spheres $\{C_i \in M_{J_0,a_i,d}/PSL(2,C)\}$.

The topological space $N_{J_0,D}$ is not a smooth manifold. However, it admits a smooth desingularization $M_{J_0,D}$ constructed as follows [Ru1]:

For each point $z \in CP^1$ let $ev_z$ be the evaluation at the point $z$ map from $Map$ to $M$ defined as follows: $ev_z(\varphi) = \varphi(z)$

We also have a more general evaluation map from $Map \times (CP^1)^m$ to $M^m$:

$$ev(\varphi, z_1, \ldots, z_m) = \{\varphi(z_1), \ldots, \varphi(z_m)\}$$

Here the symbol $\tilde{\times}$ means taking the product and then moding out by the action of $PSL(2,C)$. The group element $g \in PSL(2,C)$ acts on $Map \times (CP^1)^m$ by the formula:

$$g \cdot (\varphi, z_1, \ldots, z_m) = (\varphi \cdot g^{-1}, g \cdot z_1, \ldots, g \cdot z_m) \quad (2.8)$$

To construct the desired desingularization, we also need “the product evaluation map”, which we will also define by $ev$. This “product map”

$$ev : Map \times (CP^1)^{m_0} \times Map \times (CP^1)^{m_1} \times \cdots \times Map \times (CP^1)^{m_k} \to$$

$$\to M^{m_0} \times (CP^1)^{m_0} \quad (2.9)$$

acts as identity from the factor $(CP^1)^{m_0}$ in the l.h.s. of $(2.9)$ to the factor $(CP^1)^{m_0}$ in the r.h.s. of $(2.9)$.

For any degeneration pattern $D_d$ let us consider the evaluation map

$$ev : \bigcup_{g \in G_0} M_{J_0,d^0} \times (CP^1)^{m_0} \times M_{J_0,d^1} \times (CP^1)^{m_1} \times \cdots \times M_{J_0,d^k} \times (CP^1)^{m_k} \to$$

$$\to M^{m_0} \times (CP^1)^{m_0} \times G_0 \quad (2.10)$$

Here $m_i$ is the valency of the vertex $d^i$ of the “degeneration graph” $T$ of our degeneration pattern (how many other components the given component $C_i$ intersects)
Let us observe that the factors of \( M \) in the r.h.s. of (2.10) are in one-to-one correspondence with the edges of the “degeneration graph” \( T \). The set of these edges can be divided in the union of groups in two different ways:

The first way is to consider two edges lying in the same group iff they have the common vertex of the type \( \{d^0; d^1; ..., d^k\} \). This corresponds to the grouping the factors of \( M \) as in the r.h.s. of (2.10).

The second way is to consider two edges lying in the same group iff they have the common vertex of the type \( \{I_1; ..., I_t\} \). Using this way of grouping the edges, we can regroup the factors of \( M \) in \( M^{m_0 + ... + m_k} \times (CP^1)^{m_0} \) and rewrite the r.h.s. of (2.10) as

\[
M^{m_0 + ... + m_k} \times (CP^1)^{m_0} \times G_0 = M^{m_0 + ... + m_t} \times (CP^1)^{m_0} \times G_0 \quad (2.11)
\]

For each index \( j = 1, ..., t \) let us take the diagonal \( \Delta_j = M \subset M^{m_j} \) and take the product \( \Delta = \prod_{j=0}^{t} \Delta_j \subset M^{m_0 + ... + m_k} \) of these diagonals.

Let \( \pi \) be the projection from

\[
\mathcal{M}_{J_0,d^0} \times (CP^1)^{m_0} \times \mathcal{M}_{J_0,d^i}^* \times (CP^1)^{m_1} \times ... \times \mathcal{M}_{J_0,d^k}^* \times (CP^1)^{m_k}
\]

to \( \mathcal{M}_{J_0,d^0} \times \mathcal{M}_{J_0,d^i}^* \times ... \times \mathcal{M}_{J_0,d^k}^* \)

It follows directly from the definition of \( \mathcal{N}_{J_0,D} \) that \( \pi^{-1}(\mathcal{N}_{J_0,D}) \) lies inside \( ev^{-1}[\Delta \times (CP^1)^{m_0} \times \{g\}] \) (both topological spaces lie inside the manifold

\[
\mathcal{M}_{J_0,d^0} \times (CP^1)^{m_0} \times \mathcal{M}_{J_0,d^i}^* \times (CP^1)^{m_1} \times ... \times \mathcal{M}_{J_0,d^k}^* \times (CP^1)^{m_k}.
\]

Moreover, dimension-counting [Ru1] implies that the map \( \pi \) restricted to \( ev^{-1}[\Delta \times (CP^1)^{m_0} \times \{g\}] \) is a branched covering. Let us denote the topological space \( ev^{-1}[\Delta \times (CP^1)^{m_0} \times \{g\}] \) by \( \mathcal{M}_{J_0,D} \).

It follows from the theorem proved in [McDS] that the image of the evaluation map \( ev \) is transversal to the product of diagonals \( \Delta \).

McDuff and Salamon stated this theorem in slightly different terms without working with inhomogenous Cauchy-Riemann equations and without including an additional factor of \((CP^1)^{m_0}\). However, the transversality result stated here can be derived from their result by taking \( M \times CP^1 \) instead of \( M \) in their considerations.

It follows from the lemma 2.4 that for generic value of \( g \in G_0 \) the space \( \mathcal{M}_{J_0,D} \) is a smooth manifold which gives the desired smooth desingularization of \( \mathcal{N}_{J_0,D} \).
Now we can state explicitly the following

**LIST OF STATEMENTS ABOUT THE COMPACTIFICATION**

**Statement 2.7.** For the “generic” choice of \( g \in G_0 \) the moduli space \( \mathcal{M}_{J_g,d} \) can be compactified as a stratified space \( \overline{\mathcal{M}}_{J_g,d} \) such that each stratum is a smooth manifold.

**Statement 2.8.** The strata of \( \overline{\mathcal{M}}_{J_g,d} \) are labelled by degeneration patterns \( \{D_d\} \) and are diffeomorphic to the manifolds \( \{\mathcal{M}_{J_g,D_d}\} \).

The stratum \( \mathcal{M}_{J_g,D_d} \) lies inside the closure of another stratum \( \mathcal{M}_{J_g,D_e} \) if the degeneration pattern \( D_d \) is a **subdivision** of the degeneration pattern \( D_e \).

**Definition.**

A degeneration pattern

\[
D^\beta = \{ ((d^0)^\beta; (d^1)^\beta; \ldots; (d^k)^\beta; I_1^\beta; \ldots; I_T^\beta) \}
\]

is called a **subdivision** of a degeneration pattern

\[
D^\alpha = \{ ((d^0)^\alpha; (d^1)^\alpha; \ldots; (d^k)^\alpha; I_1^\alpha; \ldots; I_T^\alpha) \}
\]

if there is a system of maps

\[
\psi_d : \{(d^0)^\beta; (d^1)^\beta; \ldots; (d^k)^\beta\} \to \{(d^0)^\alpha; (d^1)^\alpha; \ldots; (d^k)^\alpha\}
\]

\[
\psi_I : \{I_1^\beta; \ldots; I_T^\beta\} \to \{I_1^\alpha; \ldots; I_T^\alpha\}
\]

and

\[
\psi_T : T^\beta \to T^\alpha
\]

which are consistent in an obvious sense and satisfy an additional property

\[
\sum_{d, i \in \psi_d^{-1}(d_i)} a_{i, \beta} d_{i, \beta} = a_{i, \alpha} d_{i, \alpha}
\]

**Statement 2.9.** The codimension of the stratum \( \mathcal{M}_{J_g,D_d} \) is always greater or equal to \( 2k \) where \( \{d^0; d^1; \ldots; d^k\} \) is the part of the degeneration pattern \( D_d \).

**Statement 2.10.** For any two generic \( g_1 \) and \( g_2 \) in \( G_0 \) there exists a smooth path \( \gamma : [0; 1] \to G_0 \) joining them, such that for any degeneration pattern \( D_d \) the manifold \( \bigcup_{g \in \gamma} \mathcal{M}_{J_g,D_d} \) gives a smooth cobordism between...
This cobordism has dimension at least one smaller than the moduli space $M_{J,g,d}$. The proof of the statements 2.7 - 2.10 should appear in the paper by Ruan and Tian [RT]. It can also be derived from the analysis carried out by McDuff and Salamon [McD S] as they mentioned at the end of their review. Although we also have a proof of these statements, we give a credit for them to [RT] and [McD S].

3. QUANTUM CUP - PRODUCTS

The total cohomology group $H^*(M)$ has a natural bilinear form given by Poincare duality. We will denote this bilinear form by $<;>$ i.e.,

$$\eta^{AB} = <A; B> \text{ where } A \in H^m(M); B \in H^{2n-m}(M).$$

In order to determine the structure constants $(C^D_{AC})_q$ of the quantum cohomology ring it is sufficient to define “quantum tri-linear pairings” $<A; B; C>_q$ and then put

$$(C^D_{AC})_q = \eta^{BD} <A; B; C>_q\quad (3.1)$$

where we use Einstein notation and sum over the repeated index $B$.

Definition A (Witten).

Let $A, B, C \in H^*(M, Z) \otimes Z_{<C>}$ Then

$$<A; B; C>_{W_1} = \sum d \int_{M_{J,d}} ev_0^*(A) \wedge ev_1^*(B) \wedge ev_\infty^*(C)\quad (3.2)$$

Strictly speaking, the r.h.s. does not make sense because the moduli space $M_{J,d}$ is non-compact and the notion of its top-dimensional homology class is not well-defined.

In order to make it well-defined, the integral in the r.h.s. of (3.2) should be considered as an integral over the compactified moduli space.

Since the evaluation maps $ev_0, ev_1$ and $ev_\infty$ do not extend to the compactification divisor, in order to define the integral in the r.h.s. of (3.2), we should make some choices of differential forms on $M$ representing cohomology classes $A, B$ and $C$. 
In addition we need $ev_0^*(A)$, $ev_\infty^*(B)$ and $ev_1^*(C)$ to be differential forms on $\mathcal{M}_{J,d}$ which should extend (at least as continuous differential forms) to the compactification divisor.

Taubes [Ta2] proved that in this case the integral (3.2) always converges (due to the fact that the compactification of $\mathcal{M}_{J,d}$ is known explicitly and can be “blown up” to a manifold with corners).

In order to show that the integral (3.2) over the compactified moduli space is well-defined, one must prove that it is independent of the choice of differential form representatives of cohomology classes $A$, $B$ and $C$ and on the choice of $J$, assuming the latter to be “generic”.

This analytic problem has not been solved (see [Ta2] for the most advanced treatment of it).

It appears that in order to handle analytic problems related to the non-compactness of the moduli spaces $\mathcal{M}_{J,d}$ it is more convenient to work with cycles on $M$ and their intersections instead of forms on $M$ and their wedge product (if we choose our cycles to be “generic”).

The two approaches by Poincare duality $A \rightarrow \hat{A}$ where $A \in H^m(M)$, $\hat{A} \in H_{2n-m}(M)$.

Let $M$ be a smooth compact $2n$-dimensional manifold. A $d$-dimensional pseudo-cycle of $M$ is a smooth map

$$f : V \rightarrow M$$

where $V = V_1 \cup \ldots \cup V_d$ is a disjoint union of oriented $\sigma$-compact manifolds without boundary such that

$$\overline{f_d(V_d) - f_d(V_d)} \subset \bigcup_{j=0}^{d-2} f_j(V_j), \quad \dim V_j = j, \quad V_{d-1} = \emptyset$$

Of course, the manifolds $V_j$ are not required to be compact.

Every $d$-dimensional singular homology class $\alpha$ can be represented by a pseudo-cycle $f : V \rightarrow M$. To see this represent it by a map $f : P \rightarrow M$ defined on a $d$-dimensional finite oriented simplicial complex $P$ without boundary. This condition means that the oriented faces of its top-dimensional simplices cancel each other out in pairs.\(^2\)

\(^2\)A finite dimensional manifold $V$ is called $\sigma$-compact if it is a countable union of compact sets.

\(^3\)To avoid some technicalities with jiggling (i.e. making maps transverse) caused by the fact that $P$ is not a manifold, one could equally well work with elements in the rational homology $H_*(M,Q)$. Because rational homology is isomorphic to rational bordism $\Omega_*(M) \otimes Q$, there is a basis of $H_*(M,Q)$ consisting of elements which are represented by smooth manifolds. Thus we may suppose that $P$ is a smooth manifold, if we wish.
Thus $P$ carries a fundamental homology class $[P]$ of dimension $d$ and $\alpha$ is by definition the class $\alpha = f_*[P]$. Now approximate $f$ by a map which is smooth on each simplex. Finally, consider the union of the $d$ and $(d-1)$-dimensional faces of $P$ as a smooth $d$-dimensional manifold $V$ and approximate $f$ by a map which is smooth across the $(d-1)$-dimensional simplices.

Pseudo-cycles of $M$ form an abelian group with addition given by disjoint union. The neutral element is the empty map defined on the empty manifold $V = \emptyset$. The inverse of $f : V \to M$ is given by reversing the orientation of $V$. A $d$-dimensional pseudo-cycle $f : V \to M$ is called cobordant to the empty set if there exists a $(d+1)$-dimensional pseudo-cycle with boundary $F : W \to M$ such that $W = \bigcup W_j$ such that

$$\delta W_{j+1} = V_j, \quad F_{j+1}|_{V_j} = f_j$$

for $j = 0, \ldots, d$. Two $d$-dimensional pseudo-cycles $f : V \to M$ and $f' : V' \to M$ are called cobordant if $f \cup f' : (-V) \cup V' \to M$ is cobordant to the empty set.

Two pseudo-cycles $e : U \to M$ and $f : V \to M$ are called transverse if $e_i : U_i \to M$ is transverse to $f_j : V_j \to M$ for all $i$ and $j$.

**Lemma 3.1 (McDuff-Salamon).** Let $e : U \to M$ be an $(m-d)$-dimensional singular submanifold and $f : V \to M$ be a $d$-dimensional pseudo-cycle.

If $e$ is transverse to $f$ then the set $\{(u,x) \in U \times V | e(u) = f(x)\}$ is finite. In this case define

$$e \cdot f = \sum_{u \in U, x \in V \atop e(u) = f(x)} \nu(u,x)$$

where $\nu(u,x)$ is the intersection number of $e_{m-d}(U_{m-d})$ and $f_d(V_d)$ at the point $e_{m-d}(u) = f_d(x)$.

The intersection number $e \cdot f$ depends only on the cobordism classes of $e$ and $f$.

Every $(2n-d)$-dimensional pseudo-cycle $e : W \to M$ determines a homomorphism

$$\Phi_e : H_d(M, Z) \to Z$$

as follows. Represent the class $\alpha \in H_d(M, Z)$ by a pseudo-cycle $f : V \to M$. Any two such representations are cobordant and hence, by Lemma 2.5, the intersection number

$$\Phi_e(\alpha) = e \cdot f$$

is independent of the choice of $f$ representing $\alpha$. The next assertion also follows from Lemma 2.5.

**Lemma 3.2 (McDuff-Salamon).** The homomorphism $\Phi_e$ depends only on the cobordism class of $e$.

Using this isomorphism, “$q$-deformed tri-linear pairings” $< A; B; C >_q$ can be defined as follows:
Definition B (Vafa, Ruan).

\[
<A; B; C >^{VR}_q = \sum_d q^d \sum_{[\varphi \in M_{J,d} \cap ev^{-1}_0(\hat{A}) \cap ev^{-1}_\infty(\hat{B}) \cap ev^{-1}_1(\hat{C})]} \pm 1
\tag{3.3}
\]

Here the sum in the r.h.s. of (3.3) is only over those values of \(d\) that \(\dim A + \dim B + \dim C = \dim M_{J,g,d}\) and only over zero-dimensional components of \(M_{J,d} \cap ev^{-1}_0(\hat{A}) \cap ev^{-1}_\infty(\hat{B}) \cap ev^{-1}_1(\hat{C})\)

The sign \(\pm 1\) is taken according to the orientation of intersection \(M_{J,g,d} \cap ev^{-1}_0(\hat{A}) \cap ev^{-1}_\infty(\hat{B}) \cap ev^{-1}_1(\hat{C})\). This intersection index is unambiguously defined since the moduli space \(M_{J,d}\) is provided with its canonical orientation using the determinant line bundle of the \(\bar{\partial}\)-operator [FH].

The above definition requires several comments:

1) We should make some clever choice of cycles representing the homology classes \(\hat{A}, \hat{B}, \hat{C}\) in order the r.h.s. of (3.3) to be defined (i.e., the intersection of the cycles to be transverse)

2) We should prove that the r.h.s. of (3.3) is independent of this choice

3) We should prove that the r.h.s. of (3.3) is independent of the choice of \(J\) and \(g\) as long as \(J\) and \(g\) are “regular”

“Regular” means that \(J\) is a regular value of the projection map \(\pi_{\mathcal{J}_0}\) from \(\bar{\partial}^{-1}(0) \in \text{Map} \times \mathcal{J}_0\) to \(\mathcal{J}_0\)

“The clever choice of cycles” means that these cycles should be realized by “pseudo-manifolds”.

The proof of “independence of the choices” is expected to be given in [RT]. This proof uses cobordism arguments and relies on the Statements 2.4 - 2.7.

The formula (3.3) for the “\(q\)-deformed tri-linear pairings” was first written by Vafa.

But in [Va] only “unperturbed” holomorphic maps were considered. This makes the formula (3.3) incorrect in when the dimension formula (2.7) does not hold for some components of the moduli space \(\mathcal{M}_{J,d}\).

Lemma 3.3 (Taubes). There exist choices of smooth differential form representatives of cohomology classes \(A, B\) and \(C\) such that

\[
<A; B; C >^{VR}_q = < A; B; C >^{Wi}_q.
\]

Taubes takes differential forms with support near \(\hat{A}, \hat{B}, \hat{C}\) respectively. Then the integral in the r.h.s. of (3.2) is well-defined.

Let \(A, B\) be \(Z_{<C>}\)-valued cohomology classes of \(M\) and let \(A \ast B\) be their quantum cup-product. Then we have:

...
Lemma 3.4. \( \deg(A \ast B) = \deg(A) + \deg(B) \)

Thus, we have a new \( Z \)-graded ring structure on \( H^*(M, Z) \otimes Z_{<C>} \). We will call this new ring the quantum cohomology ring of \( M \) and we will denote it \( HQ^*(M) \).

Let us define the homomorphism \( l^*: HQ^*(M) \to H^*(M) \) as tensor multiplication on the ring \( Z \) over the ring \( Z_{<C>} \) which is induced by the augmentation \( I: Z_{<C>} \to Z \).

**Lemma 3.5.** \( l^* \) is a ring homomorphism which preserves the grading.

Before going to the Floer cohomology ring and proving that it is isomorphic to the quantum cohomology ring let me comment once again about the status of the definitions of the latter ring.

**Comment Only Definition B of the quantum cup-product has well-defined mathematical objects in its r.h.s.**

**THE OPERATION OF QUANTUM MULTIPLICATION**

In Floer theory which will be discussed in the next paragraph there is a linear map \( m_F : H^*(M) \to \text{End}(HF^*(M)) \) or, equivalently, the action of the classical cohomology of the manifold \( M \) on its Floer cohomology \( HF^*(M) \). Latter module is canonically isomorphic with the total cohomology group \( H^*(M) \otimes N \) defined as a module over the Novikov ring \( N \).

There is a natural analog of this Floer’s map \( m_F \) in quantum cohomology: namely, an operation \( m_Q(C) \) of quantum multiplication (from the left) on the cohomology class \( C \in H^*(M) \otimes N \)

\[
m_Q(C) : H^*(M) \otimes N \to H^*(M) \otimes N
\]

In order to obtain the action of \( H^*(M) \otimes N \) on the homology of \( M \) instead of cohomology of \( M \) we should apply Poincare duality to (2.13). Let us fix some (homogenous) basis \( \{ A, B, \ldots \} \) in \( H^*(M, Z) \) and the Poincare dual basis \( \hat{A}, \hat{B}, \ldots \) in \( H_*(M, Z) \subset H_*(M, Z) \otimes N \). Then we can write matrix elements \( < B|m_Q(C)|A > \) of the operator \( m_Q(C) \) in this basis.

**Lemma 3.6.**

\[
<B|m_Q(C)|A >= < A; \eta(B); C >_q
\]

Here \( \eta : H_m(M) \to H_{2n-m}(M) \) is a Poincare duality isomorphism in homology of \( M \).
4. REVIEW OF SYMPLECTIC FLOER HOMOLOGY

Let $\mathcal{L}M$ be the free loop-space of our (compact, simply-connected semi-positive) almost-Kahler manifold $M$ and let $\hat{\mathcal{L}}M$ be its universal cover. The points in $\hat{\mathcal{L}}M$ can be described as pairs $(\gamma; z)$ where $\gamma : S^1 \to M$ be a free-loop in $M$ and $z : D^2 \to M$ be a smooth map from 2-disc $D^2$ which coincides with $\gamma$ at the boundary of the disc $\partial D^2 = S^1$. The two maps $z_1$ and $z_2$ of the disc are considered to be equivalent if they are homotopic to each other and the corresponding homotopy leaves their common boundary loop $\gamma$ fixed.

Following Floer [F1-F8] we can define “the symplectic action functional” $S_{\omega} : \hat{\mathcal{L}}M \to \mathbb{R}$ as follows:

$$S_{\omega}(\gamma; z) = \int_{D^2} z^*(\omega)$$

where $\omega$ is the symplectic form on $M$ and $z^*(\omega)$ is its pull-back to the 2-disc $D^2$.

The tangent vectors to the free loop-space at the point $\gamma \in \mathcal{L}M$ can be described as vector fields $\{\xi, \eta, \ldots\}$ on $M$ restricted to the loop $\gamma$. The free loop-space $\mathcal{L}M$ (and its universal cover) has a natural structure of (infinite-dimensional) almost-Kahler manifold described as follows:

Let $g$ and $\omega$ be the Riemannian metric and the symplectic form on $M$. Then we can define the Riemannian metric $\hat{g}$ and the symplectic form $\hat{\omega}$ on the loop-space $\mathcal{L}M$ by the formulas:

$$\hat{g}(\xi, \eta) = \int_{S^1} g(\xi(\gamma(\theta)); \eta(\gamma(\theta)))d\theta$$

$$\hat{\omega}(\xi, \eta) = \int_{S^1} \omega(\xi(\gamma(\theta)); \eta(\gamma(\theta)))d\theta$$

where $\theta$ is the natural length parameter on the circle $S^1$ defined modulo $2\pi$

The Riemannian metric $\hat{g}$ and the symplectic form $\hat{\omega}$ on the loop-space $\mathcal{L}M$ are related through the almost-complex structure operator $J$. Action of this almost-complex structure operator $\hat{J}$ on the tangent vector $\xi$ to the loop $\gamma$ (which is the vector field restricted to the loop $\gamma$) is defined as the action of the almost-complex structure operator $J$ on the base manifold $M$ on our vector field $\xi$.

**Lemma 4.1. (Givental).** The following statements hold:

A) $S_{\omega}$ is a Morse-Bott function on $\hat{\mathcal{L}}M$
B) All the critical submanifolds of the “symplectic action” $S_\omega$ on the universal cover of $\hat{L}M$ are obtained from each other by the action of the group $\pi_1(\hat{L}M) = \pi_2(M) = H_2(M)$ of covering transformations. The image of (any of) these critical submanifolds under the universal covering map $\pi: \hat{L}M \to L\hat{M}$ is the submanifold $M \subset L\hat{M}$ of constant loops.

If we consider $\hat{L}M$ as a symplectic manifold with the symplectic form $\tilde{\omega}$ given by (3.2B) then:

C) The Hamiltonian flow of the functional $S_\omega$ generates the circle action on $\hat{L}M$ and

D) This circle action is just rotation of the loop $\gamma(\theta) \to \gamma(\theta + \theta^0)$

Let us choose (once and for all) one particular critical submanifold $M \subset \hat{L}M$ of the symplectic action $S_\omega$. Then any other critical submanifold of $S_\omega$ has the form $q^dM$ which means that it is obtained from $M$ by the action of the element $q^d$ of the group $H_2(M)$ of covering transformations.

**Lemma 4.2.** The gradient flow of the symplectic action functional $S_\omega$ on the universal cover of the loop-space (which is provided with its canonical Riemannian metric $\tilde{g}$) depends only on the almost-complex structure $J$ and does not depend on the symplectic form $\omega$.

We assume that the metric $g$ and the symplectic form $\omega$ are related in the standard way through the almost-complex structure $J$.

Let $\dot{\gamma}(\theta)$ be unit the tangent vector field to the loop $\gamma \in L\hat{M}$ (this tangent vector coincides with the generator of the circle action rotating the loop). Then we have

**Lemma 4.3.**

$$\nabla_{\dot{\gamma}(\theta)} S_\omega(\gamma(\theta)) = J(\dot{\gamma}(\theta))$$

(4.3)

Let $\{H_\theta\}: M \to R$ be some (smooth) family of functions on $M$ parametrized by $\theta \in S^1$. This family of functions on $M$ is usually called “periodic time-dependent Hamiltonian” where $\theta$ is “time”. The fact that $\theta \in S^1$ reflects the fact that the time-dependence of our Hamiltonian is periodic. Let $S_{\omega,H}: \hat{L}M \to R$ be a functional on $\hat{L}M$ defined as follows:

$$S_{\omega,H}(\gamma; z) = S_\omega(\gamma; z) - \int_{S^1} H_\theta(\gamma(\theta)) d\theta$$

(4.4)

**Theorem 4.4 (Floer).** For “generic” choice of $H$ and $J$ the functional $S_{\omega,H}$ is a Morse functional on $\hat{L}M$ (which is usually called “the symplectic action functional perturbed by a Hamiltonian term”)

“Generic” here means that the statement is true for the Baire second category set in the product of the space of all functions on $M \times S^1$ and the space of all almost-complex structures on $M$.
The gradient flow trajectory of “the perturbed symplectic action functional” on the universal cover of the loop-space can be defined as a solution of the following PDE:

\[
\frac{\partial \gamma_\tau(\theta)}{\partial \tau} = J \frac{\partial \gamma_\tau(\theta)}{\partial \theta} - \text{grad} \ H_\theta
\]  

(4.5)

where \( \tau \) is the parameter on the gradient flow line, varying from minus infinity to plus infinity, and \( \theta \) be the parameter on the loop.

We will consider only those solutions of (4.5) which are \( L^2 \)-bounded, i.e. satisfy the estimate

\[
\int_R d\tau \int_{S^1} d\theta \left\| \frac{\partial \gamma_\tau(\theta)}{\partial \tau} \right\|^2 < \infty
\]  

(4.6)

The \( L^2 \)-boundedness condition (4.6) implies that

\[
\gamma_\tau(\theta) \to \gamma_- (\theta) \quad \tau \to -\infty
\]  

(4.7A)

and

\[
\gamma_\tau(\theta) \to \gamma_+ (\theta) \quad \tau \to +\infty
\]  

(4.7B)

where \( \gamma_- (\theta) \) and \( \gamma_+ (\theta) \) are some “critical loops” or, in another words, critical points of the perturbed symplectic action functional on the universal cover of the loop-space.

This means that any \( L^2 \)-bounded solution of (4.5) always extends to some continuous map from \( S^1 \times R \) to \( M \) (which is actually a smooth map with finitely many singular points).

Here \( S^1 \times R \) is identified with \( C^* \) by the map

\[
(\theta (\text{mod}2\pi); \tau) \to \text{exp}(\tau + i\theta)
\]  

(4.8)

Let \( \gamma_+, \gamma_- \in \hat{\mathcal{L}}M \) be two such critical points of \( S_{\omega,H} \).

Let us define \( \mathcal{M}(\gamma_-, \gamma_+) \) as the space of all \( L^2 \)-bounded trajectories of the gradient flow of \( S_{\omega,H} \), joining the critical point \( \gamma_- \) and the critical point \( \gamma_+ \).

In more down-to-earth terms, the space \( \mathcal{M}(\gamma_-, \gamma_+) \) can be defined as the space of all solutions of (4.5), \( 2\pi \)-periodic in \( \theta \) with the asymptotics given by (4.7A) and (4.7B)

\( \mathcal{M}(\gamma_-, \gamma_+) \) can be thought as union of all loops lying on the gradient flow trajectories, and thus, as a topological subspace in \( \hat{\mathcal{L}}M \)

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Theorem 4.6 (Floer). For the any “generic” choice of the function $H$ on $S^1 \times M$ and for any pair $\{\gamma_-, \gamma_+\}$ of the critical points of $S_{\omega, H}$ in $\hat{\mathcal{LM}}$ the following statements hold:

A) The space $\mathcal{M}(\gamma_-, \gamma_+)$ is a smooth submanifold in $\hat{\mathcal{LM}}$

B) The dimension of this submanifold is equal to the spectral flow of the family $\{D_\tau = \bar{\partial} - \text{grad}(H_\theta)\} (-\infty < \tau < \infty)$ of $\bar{\partial}$-operators acting from the space $W^2_{\bar{\partial}n}(S^1, \gamma^*_\omega(TM))$ to the space $W^2_{\bar{\partial}n-1}(S^1, \gamma^*_\omega(TM))$

C) For any element $q^d \in \pi_2(M)$ we have

$$\dim(\mathcal{M}(\gamma_-, q^d \gamma_+)) = \dim(\mathcal{M}(\gamma_-, \gamma_+)) + 2 < c_1(TM); q^d >$$  \hspace{1cm} (4.9)

(this formula follows from the computation of the spectral flow)

Since the Hessian of $S_{\omega, H}$ at any of its critical points has infinitely many positive and infinitely many negative eigenvalues, the usual Morse index of the critical point is not well-defined.

But the relative Morse index of the pair $\gamma_-$ and $\gamma_+$ of the critical points is well-defined as $\text{vdim}(\mathcal{M}(\gamma_-, \gamma_+))$

Here by $\text{vdim}(\mathcal{M}(\gamma_-, \gamma_+))$ we denote “the virtual dimension” of the manifold $\mathcal{M}(\gamma_-, \gamma_+)$ which is defined as a spectral flow of the family $\{D_\tau\}(-\infty < \tau < \infty)$ of $\bar{\partial}$-operators

In the case when $J$ and $H$ are “generic” (or “regular” in the sense of the previous section), this virtual dimension $\text{vdim}$ is equal to actual dimension $\dim(\mathcal{M}(\gamma_-, \gamma_+))$ of this manifold.

But for some choices of $H$ which will be of interest to us this might not be true. In these cases $\mathcal{M}(\gamma_-, \gamma_+)$ is no longer smooth. Different components of $\mathcal{M}(\gamma_-, \gamma_+)$ are allowed to have different dimensions and to meet each other nontransversally.

Lemma 4.7. Let $\gamma_1, \gamma_2, \gamma_3$ be three “critical loops” in $\hat{\mathcal{LM}}$ Then

$$\text{vdim}(\mathcal{M}(\gamma_1, \gamma_3)) = \text{vdim}(\mathcal{M}(\gamma_1, \gamma_2)) + \text{vdim}(\mathcal{M}(\gamma_2, \gamma_3))$$  \hspace{1cm} (3.10)

This formula follows from the spectral flow calculations and from the fact that we are working on the simply-connected space $\hat{\mathcal{LM}}$.

It is worth mentioning that the formula (4.10) is not true if we do not go from $\mathcal{LM}$ to its universal cover $\hat{\mathcal{LM}}$. Without going to the universal cover the formula (4.10) is only true modulo $2\Gamma$ where $\Gamma$ is the least common multiple of the numbers $\{< c_1(TM); q_i >\}$
Lemma 4.8.

\[ \mathcal{M}(q^d\gamma_-, q^d\gamma_+) = q^d[\mathcal{M}(\gamma_-, \gamma_+)] \]  \hspace{1cm} (4.11)

Although Morse index of the critical points \( \{\gamma_i\} \) of \( S_{\omega, H} \) is not defined in the usual sense, the formulas (4.9) and (4.10) allow us to define it by hands.

Let us fix some “basic critical point” \( \gamma_0 \in \mathcal{L}\overline{M} \)

For any other critical point \( \gamma \in \mathcal{L}\overline{M} \) we can always find \( q^d \in H_2(M) \) such that either the manifold \( \mathcal{M}(\gamma_0, q^d\gamma) \) or the manifold \( \mathcal{M}(\gamma, q^d\gamma_0) \) is non-empty. Then we can define

\[ \text{deg} \gamma = \text{deg} \gamma_0 + \text{vdim}(\mathcal{M}(\gamma_0, q^d\gamma)) - \text{deg}[q^d] \]  \hspace{1cm} (4.12A)

\[ \text{deg} \gamma = \text{deg} \gamma_0 - \text{vdim}(\mathcal{M}(\gamma, q^d\gamma_0)) + \text{deg}[q^d] \]  \hspace{1cm} (4.12B)

Here \( \text{deg}[q^d] \) is defined by (2.4)

The formulas (4.12A) and (4.12B) for different \( \{d\} \) are consistent with each other.

So, our grading on the set of critical points of \( S_{\omega, H} \) is defined uniquely up to an additive constant \( \text{deg} \gamma_0 \)

The manifolds \( \{\mathcal{M}(\gamma_-, \gamma_+)\} \) of the gradient flow trajectories are non-compact. There are two basic reasons of their non-compactness:

A) The gradient flow trajectory may goes through the intermediate critical point, i.e., it may “split” into the union of two trajectories

B) The sequence of the gradient flow trajectories in \( \mathcal{M}(\gamma_-, \gamma_+) \) may diverge by “bubbling off” a \( J \)-holomorphic sphere of degree \( d \). The formal limit of this diverging sequence will be of a gradient flow trajectory from \( \mathcal{M}(q^d\gamma_-, \gamma_+) \) (which can be thought as a pseudo-holomorphic cylinder in \( M \) in the sense which will be explained in the next section) and a \( J \)-holomorphic sphere of degree \( d \) attached to this cylinder at some point.

In order to have a good intersection theory on manifolds of gradient flow trajectories (which is the main ingredient in the definition of cup-product in Floer cohomology) we should **compactify** them.

The compactification of the manifold \( \mathcal{M}(\gamma_-, \gamma_+) \) includes:

A) The loops lying inside the product

\[ \mathcal{M}(\gamma_-, \gamma_1) \times \mathcal{M}(\gamma_1, \gamma_2) \times \ldots \times \mathcal{M}(\gamma_{k-1}, \gamma_k) \times \mathcal{M}(\gamma_k, \gamma_+) \]
B) Those trajectories in $\mathcal{M}(q^d, \gamma_-, \gamma_+)$ which can be obtained by bubbling off from some sequences of trajectories in $\mathcal{M}(\gamma_-, \gamma_+)$. 

The part A) of the compactification is easy to handle. We just add this part to $\mathcal{M}(\gamma_-, \gamma_+)$ to obtain a smooth manifold with corners.

The above constructed manifold with corners is desingularized by a canonical Morse-theoretic procedure of “gluing trajectories” (see [CJS1], [AuBr] for a precise construction) to obtain smooth manifold with boundary. The boundary of this “desingularized” manifold consists of the gradient flow trajectories going through the intermediate critical points together with “the gluing data” which corresponds to “blowing up” the corners.

The part B) of the compactification is much more complicated object to work with. It was proved by Floer himself using dimension-counting argument (4.9) that if we bubble off the sphere of degree $d$ such that $\langle c_1(TM); q_i \rangle > 0$ then the corresponding part of the compactification has codimension at least two.

For the case when $\langle c_1(TM); q_i \rangle \geq 0$ this was proved by Hofer and Salamon [HS] (assuming that the almost-complex structure $J_0$ on $M$ is “generic”).

Let us consider the free abelian group $\text{CF}_*(M)$ generated by the critical points of the perturbed symplectic action $S_\omega, H$ in $\hat{\mathcal{L}}M$. This abelian group has a structure of $\mathbb{Z}[H_2(M)]$-module since the group $H_2(M)$ of the covering transformations acts on the set of critical points.

Since the action of the group of covering transformations is free, the module $\text{CF}_*(M)$ is a free module, generated by the finite set of the critical points of the multivalued functional $S_\omega, H$ on the loop-space $\mathcal{L}M$ (before going to the universal cover).

Let us take a completion of this abelian group $\text{CF}_*(M)$ by allowing certain infinite linear combinations of the critical points of $S_\omega, H$ to appear in $\text{CF}_*(M)$. More precisely, let us tensor our $\mathbb{Z}[H_2(M)]$-module $\text{CF}_*(M)$ on the Novikov ring $N$ over the ring $\mathbb{Z}[H_2(M)]$. We will denote this extended abelian group by the same symbol $\text{CF}_*(M)$ (which is actually an $N$-module) and call it a Floer chain complex corresponding to “perturbed symplectic action” $S_\omega, H$.

A Floer chain complex $\text{CF}_*(M)$ has a natural $\mathbb{Z}$-grading $\text{deg}$ induced from the above-defined grading of the critical points.

Let $\{x, y, \ldots\}$ be some set of critical points of $S_\omega, H$ on $\hat{\mathcal{L}}M$. We assume that this set maps isomorphically onto the set of all critical points of $S_\omega, H$ on $\mathcal{L}M$. In other words, we choose one point in the fiber of the universal cover over each critical point.
Now it is time to define a boundary operator \( \delta : CF_* (M) \rightarrow CF_* (M) \) which will:

A) commute with \( N \)-action (i.e. \( \delta \) will be \( N \)-module homomorphism);

B) decrease the \( Z \)-grading \( \text{deg} \) by one.

Let us define

\[
\delta x = \sum_y \sum_d <\delta x; q^d y > q^d y
\]  

(4.13)

where the sum in the r.h.s. of (3.13) is taken only over such values of \( y \) and of \( d \) that the critical points \( x \) and \( q^d y \) have relative Morse index one.

Let \(<\delta x; q^d y >\) be the number of connected components of \( M(x, q^d y) \) (all of them are one-dimensional) counted with \( \pm 1 \)-signs depending on orientations of these components relative to their ends \( x \) and \( q^d y \).

**Lemma 4.9.** The boundary operator \( \delta \) is defined over the Novikov ring \( N \).

This means that for any index \( i = 1, ..., s \) there exists an integer \( N_i \) such that only those values of \((d_1, ..., d_s)\) could contribute to the r.h.s. of (4.13) that \( d_i > -N_i \) for all \( i \).

**Proof.**

By definition of the gradient flow, if the manifold \( M(x, q^d y) \) is non-empty, then \( S_{\omega, H}(x) > S_{\omega, H}(q^d y) \) for any \( J_0 \)-compatible symplectic form \( \omega \) (and in particular for our basic forms \{\(\omega_1, ..., \omega_s\}\)). This means that for any positive real number \( t \) and for any trajectory \( \gamma(\tau, \theta) \in M(x, q^d y) \) we have

\[
S_{\omega_1, H}(x) - S_{\omega_1, H}(q^d y) =
\]

\[
= \int_{S^1 \times R} \gamma^* (\omega_i) + \int_{S^1} H_\theta(y(\theta))d\theta - \int_{S^1} H_\theta(x(\theta))d\theta > 0 \quad (4.14)
\]

Since the values of the integrals \( \int_{S^1} H_\theta(y(\theta))d\theta \) and \( \int_{S^1} H_\theta(x(\theta))d\theta \) are independent of the symplectic form, and \( \int_{S^1 \times R} \gamma^* (\omega_i) \) is a homotopy invariant which depends only on the limit values of \( \gamma \) as \( \tau \rightarrow \pm \infty \), then we can conclude that in our case \( \int_{S^1 \times R} \gamma^* (\omega_i) \) depends only on the value of \( d \).

It follows directly from the fact that \{\(\omega_1, ..., \omega_s\}\} form a basis dual to \{\(q_1, ..., q_s\)\} that if the value of \( d_i \) decreases by one then the value of the integral \( \int_{S^1 \times R} \gamma^* (\omega_i) \) also decreases by one.
This observation implies existence of the lower bound $-N_i$ on the value of $d_i$ in order the inequality (4.14) to hold. This is equivalent to the statement of the Lemma 4.9.

**Theorem 4.10 (Floer).** $\delta^2 = 0$.

The proof of this statement is highly non-trivial and relies heavily on the way how we compactify the manifolds $\{M(\gamma-, \gamma+)\}$ of the gradient flow trajectories. This allows one to prove that the contributions to $\delta^2$ “from the boundary” of the appropriate manifold of the gradient flow trajectories will cancel each other.

**Lemma 4.11.** Homology $HF_*(M)$ of the Floer chain complex inherit both the $N$-module structure and the $\mathbb{Z}$-grading $\deg$ from $CF_*(M)$.

**Theorem 4.12 (Floer).** $HF_*(M) = H_*(M) \otimes N$.

The idea of the proof of this theorem is as follows:

First, Floer proved that the graded module $HF_*(M)$ is well-defined and independent of the choice of “hamiltonian perturbation” $H$ involved in its definition.

Floer constructed an explicit chain homotopy between Floer chain complexes $CF_*(M, H_1)$ and $CF_*(M, H_2)$ constructed from two different hamiltonians $H_1$ and $H_2$ (which are functions from $S^1 \times M$ to $R$).

Second, if we consider $\theta$-independent Hamiltonian $H : M \to R$ which is small in $C^2$-norm, then all the critical points of perturbed symplectic action functional $S_{\omega, H}$ on $\hat{LM}$ can be obtained from the critical points of $H$ on $M$ by covering transformations. Here $M$ is embedded in $\hat{LM}$ as a submanifold of constant loops as specified above.

Saying the same thing in another words, only constant loops can be critical points of $S_{\omega, H}$. These “critical loops” can take values in the critical points of $H$ on the manifold $M$ and only in those points.

The gradient flow trajectories joining these critical points can be of two types:

A) Lying inside submanifold $M \subset \hat{LM}$ of constant loops
B) Not lying inside any submanifold of constant loops

The trajectories of type B) cannot be isolated due to non-triviality of $S^1$-action (which rotates the loop) on the space of those trajectories.

Thus, only trajectories of type A) can contribute to the Floer boundary operator $\delta$. But the chain complex generated by these trajectories is exactly the Morse complex of $M$. 


Thus, the homology of the Floer complex will be the same as homology of $M$ (tensored by the appropriate coefficient ring due to the action of the group of covering transformations)

Before starting to explain cup-product structure, let us define Floer cohomology $HF^*(M)$ and Floer cochain complex $CF^*(M)$ for both perturbed and unperturbed symplectic action. To define those objects we should define:

A) Floer cochain complex $CF^*(M) = \text{Hom}_N(CF^*(M), N)$

B) Coboundaty operator $\delta^*$ in the Floer cochain complex

C) Floer cohomology $HF^*(M)$ as homology of the complex $(CF^*(M); \delta^*)$

**Lemma 4.13.** The following statements hold:

A) $HF^*(M) = \text{Hom}_N(HF^*(M), N)$

B) $HF^*(M) = H^*(M) \otimes N = H^*(M, N)$ i.e. Floer cohomology are isomorphic to ordinary cohomology with the appropriate coefficient ring.

The Floer cochain complex of the perturbed symplectic action functional $S_{\omega, H}$ has a canonical basis corresponding to the critical points $\{q^d x, q^d y, \ldots\}$ of $S_{\omega, H}$. This basis is dual to the basis of the critical points $\{q^{-d} x, q^{-d} y, \ldots\}$ in the Floer chain complex $CF_*(M)$.

Proceeding as above, we can develop the Morse-Bott-Witten theory for the Morse-Bott functional $S_\omega$ on the universal cover of the loop-space $\hat{L}M$ in the same way as Floer developed his theory for Morse functional $S_{\omega, H}$ on the same space.

The main ingredient of such a theory is a Floer chain complex corresponding to the “unperturbed symplectic action” $S_\omega$. Algebraically this chain complex is defined as $H^*(M) \otimes N$.

Geometrically, this Floer chain complex is generated (as an abelian group) by the total homology of all the critical submanifolds $\{q^d M\}$ of the symplectic action functional.

Here, as above, we allow certain infinite linear combinations to appear. The appearance of these infinite linear combinations stands for the fact that we are working over the Novikov ring $N$.

This new Floer chain complex (we will again denote it $CF_*(M)$) also has a $N$-module structure and the $Z$-grading $\text{deg}$. The latter is defined as follows:

$$\text{deg}[q^d \hat{A}] = \text{deg}[\hat{A}] - \sum_{i=1}^{s} d_i \text{deg}[q_i] \quad (4.15)$$

where $\hat{A}$ be some homology class of degree $\text{deg}[\hat{A}]$. 

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Here, as usual, $A \rightarrow \hat{A}$ stands for Poincare duality isomorphism between cohomology class $A \in H^{2n-deg[\hat{A}]}(M)$ and homology class $\hat{A} \in H_{deg[\hat{A}]}(M)$.

Let $q^d \hat{A}$ and $q^{d'} \hat{B}$ be two (bihomogenous) elements of the Floer chain complex $CF_*(M)$. Proceeding as above, we can define:

A) The manifold $\mathcal{M}(q^d \hat{A}, q^{d'} \eta(\hat{B}))$ of the gradient trajectories of $S_\omega$ which flow from the cycle $q^d \hat{A}$ in $q^d M$ as $\tau \rightarrow -\infty$ to the cycle $q^{d'} \eta(\hat{B})$ in $q^{d'} M$ as $\tau \rightarrow +\infty$.

We compactify this manifold by the gradient flow trajectories passing through the intermediate critical submanifolds and by trajectories in $\mathcal{M}(q^d \hat{A}, q^{d'} \eta(\hat{B}))$ obtained by bubbling off.

(Here, as above, $\hat{B} \rightarrow \eta(\hat{B})$ stands for Poincare duality in homology of $M$.)

B) Relative Morse index of $q^d \hat{A}$ and $q^{d'} \hat{B}$ as the virtual dimension of $\mathcal{M}(q^d \hat{A}, q^{d'} \eta(\hat{B}))$ defined as $\text{deg}(q^{d'} \eta(\hat{B})) - \text{deg}(q^d \hat{A})$.

C) Z-grading $\text{deg}$ on the Floer chain complex $CF_*(M)$ (defined by the formula (4.15)) such that the relative Morse index of $q^d \hat{A}$ and $q^{d'} \hat{B}$ is equal to the difference of their degrees.

D) Floer boundary operator $\delta : CF_*(M) \rightarrow CF_*(M)$ which commutes with $N$-action and decreases the Z-grading $\text{deg}$ by one.

This Floer boundary operator is defined as

$$\delta \hat{A} = \sum_{\hat{B}} \sum_d <\delta \hat{A}; q^d \hat{B}> q^d \hat{B} \quad (4.16)$$

Here (as usual) $<\delta \hat{A}; q^d \hat{B}>$ counts the number (weighted with $\pm 1$-signs depending on orientation) of isolated gradient flow trajectories inside the manifold $\mathcal{M}_d(\hat{A}, \eta(\hat{B}))$ defined as

$$\mathcal{M}_d(\hat{A}, \eta(\hat{B})) = \mathcal{M}(\hat{A}, q^d \eta(\hat{B})) \quad (4.17)$$

Here the r.h.s. of (4.17) gives the definition to its l.h.s.

**Lemma 4.14 (Givental).** $\delta = 0$.

The proof of this lemma relies on the fact that any Morse-Bott function which is a hamiltonian of an $S^1$-action has this property.

Thus, we have
Lemma 4.15. Floer homology $HF_\ast(M)$ coincide with the Floer chain complex $CF_\ast(M)$ of the unperturbed symplectic action functional.

The Floer cochain complex of the unperturbed symplectic action functional $S_\omega$ also has a canonical basis $\{q^d A, q^d B\}$ where $\{A, B, \ldots\}$ is some (homogenous) basis in the cohomology of $M$. This basis is dual to the basis $\{q^{-d} \hat{A}, q^{-d}(B), \ldots\}$ in the Floer chain complex $CF_\ast(M)$.

Later on we will use these two bases in these two Floer cochain complexes when we will work with Floer cohomology instead of Floer homology.

5. CUP-PRODUCTS IN FLOER COHOMOLOGY

Original Floer’s motivation for introducing the object which is now known as “symplectic Floer cohomology” was to give an interpretation of fixed points of the symplectomorphism of $M$ in terms of Morse theory.

In order to have such an interpretation, one has to develop some Morse theory on the loop space $LM$ instead of the usual Morse theory on $M$. By identifying the fixed points of our symplectomorphism (constructed canonically from “the periodic time-dependent Hamiltonian” $H_\theta : S^1 \times M \rightarrow \mathbb{R}$) with the critical points of Floer’s “perturbed symplectic action functional” on the loop-space, we have such a Morse-theoretic interpretation.

If we assume all the fixed points of our symplectomorphism to be non-degenerate (which is the case only if “the Hamiltonian” $H$ is “generic” in the sense of Lemma 4.4), and use the fact that homology of our Morse-Floer complex $CF_\ast(M)$ are isomorphic to the classical homology of $M$, then the lower bound on the number of the fixed points of our symplectomorphism will be given by usual Morse inequalities. This was one part of the Arnond’s Conjecture which Floer proved.

The other part of the same Arnond’s Conjecture was: what will be if we drop the non-degeneracy assumption on the Jacobian at the fixed points? Classical Morse theory gives us the lower bound on the number of (not necessarily non-degenerate) critical points of the function $H$ on the compact manifold $M$ in terms of the so-called cohomological length of $M$.

Definition. The cohomological length of the topological space $M$ is an integer $k \in \mathbb{Z}_+$ such that:

A) There exist $k - 1$ cohomology classes $\alpha_1, \ldots, \alpha_{k-1}$ on $M$ of positive degrees such that $\alpha_1 \wedge \ldots \wedge \alpha_{k-1} \neq 0$ in $H^\ast(M)$ and
B) There are no $k$ cohomology classes on $M$ with this property.

Thus we see that in order to try to prove this part of the Arnond’s Conjecture in the framework of Floer’s Morse theory, one needs to invent some multiplicative structure in Floer cohomology. A kind of such a multiplicative structure was also constructed by Floer [F1] and successfully applied to this part of Arnond’s Conjecture in another Floer’s paper [F2].

However, Hofer [Ho2] have found a proof of this part of Arnond’s Conjecture without using Floer homology.

Using the fact that Floer cohomology $HF^*(M)$ are canonically isomorphic (as an abelian group) to the ordinary cohomology $H^*(M) \otimes N$ the following five statements are equivalent:

A) we have a multiplication in Floer cohomology

$$HF^*(M) \otimes HF^*(M) \rightarrow HF^*(M)$$

which is $N$-module homomorphism and which preserves the $Z$-grading;

B) we have an action

$$HF^*(M) \rightarrow \text{End}(HF^*(M))$$

of Floer cohomology on itself (by left multiplication) which is $N$-module homomorphism and which preserves the $Z$-grading;

C) we have an action

$$H^*(M) \rightarrow \text{End}(HF^*(M))$$

of classical cohomology of the manifold $M$ on its Floer cohomology which preserves the $Z$-grading;

D) we have an action

$$H_*(M) \rightarrow \text{End}(HF^*(M))$$

of classical homology of the manifold $M$ (related by Poincare duality with the cohomology of $M$) on its Floer cohomology which preserves the $Z$-grading;

E) we have an action

$$\Omega_*(M) \rightarrow \text{End}(CF^*(M))$$

of the space of singular chains in $M$ which can be realized by pseudo-cycles on the Floer cochain complex of $M$. This action commutes with the boundary operator and preserves the $Z$-grading.
Later on we will denote all these four maps (5.1A) – (5.1E) by the same symbol $m_F$ and call them the Floer multiplication.

In order to define the Floer multiplication $m_F$ in the form (5.1E) it is enough to define its matrix elements $< y| m_F(\tilde{C}) | x >$ where $x, y \in CF^*(M)$, $\tilde{C} \in \Omega_*(M)$ and then put

$$m_F(\tilde{C})(x) = \sum_{y, d} < q^d y | m_F(\tilde{C}) | q^d x >$$

(5.2)

where the sum in the r.h.s. of (5.2) is taken over the basis $\{x, y, \ldots\}$ in the $N$-module $CF^*(M)$.

Let $x, y$ be two “basic” critical point of the perturbed symplectic action $S_{\omega, H}$ on $\hat{LM}$, and let $q^d x$ and $q^d y$ be the corresponding elements of the Floer cochain complex $CF^*(M)$. Then let us put

$$< q^d y | m_F(\tilde{C}) | q^d x > = \mathcal{M}(q^d x; q^d y) \cap \tilde{cv}_1^{-1}(\tilde{C})$$

(5.3)

The r.h.s. of (5.3) is defined here as an intersection index.

Here $\tilde{ev} : S^1 \times \hat{LM} \to M$ is the standard “evaluation map” where the circle $S^1$ is assumed to be embedded as a unit circle $|z| = 1$ in the complex plane $C$. The map $\tilde{cv}_1$ means evaluation of the loop at the point $z = 1$.

**Theorem 5.1.** The following two statements hold:

A) The action $m_F$ of $\Omega_*(M)$ on $CF^*(M)$ defined by (5.2) and (5.3) descends to the action $m_F$ of $H_*(M)$ on $HF^*(M)$;

B) The induced action $m_F : H_*(M) \to \text{End}(HF^*(M))$ does not depend on the choice of “the Hamiltonian” $H$ assuming that this Hamiltonian is “generic” in the sense of Lemma 4.4.

For the case when $M$ is a positive almost-Kahler manifold the Theorem 5.1 was proved by Floer himself [F1]. The same proof works with some modifications for Calabi-Yau and the general semi-positive case.

We will reproduce here the main steps of the proof of the Theorem 5.1. In the next section the techniques which is used in this proof will be applied to prove equivalence of Floer’s and quantum multiplication.

The main idea behind this proof is to consider “$\tau$-dependent Hamiltonian perturbation” of the equation (4.5). More precisely, let $H$ be some smooth function on $R \times S^1 \times M$. Here, as above, the real line $R$ is equipped with the parameter $\tau$, varying from minus infinity to plus infinity, and the circle $S^1$ is equipped with the arclength parameter $\theta$.

Let us restrict ourselves to the functions on $R \times S^1 \times M$ which are $\tau$-independent in the region $-\infty < \tau < -1$ and in the region $1 < \tau < +\infty$. 33
This condition means that there exist two functions $H_-$ and $H_+$ on $S^1 \times M$ such that

$$H(\tau; \theta) = H_-(\theta) \quad \text{if} \quad \tau \leq -1 \quad (5.4A)$$

$$H(\tau; \theta) = H_+(\theta) \quad \text{if} \quad \tau \geq 1 \quad (5.4B)$$

Let us denote the space of all such functions $G_{H_-,H_+}$.

Then we can study the space of solutions of the following PDE

$$\frac{\partial \gamma_\tau(\theta)}{\partial \tau} = J \frac{\partial \gamma_\tau(\theta)}{\partial \theta} - \text{grad } H_\theta \quad (5.5)$$

which are $L^2$-bounded in the sense of (4.6).

The same argument as in the section 4 shows that for any $L^2$-bounded solution of (5.5) there exist a critical point $\gamma_+$ of $S_{\omega,H_+}$ and a critical point $\gamma_-$ of $S_{\omega,H_-}$ such that

$$\gamma_\tau(\theta) \to \gamma_-(\theta) \quad \tau \to -\infty \quad (5.6A)$$

and

$$\gamma_\tau(\theta) \to \gamma_+(\theta) \quad \tau \to +\infty \quad (5.6B)$$

In the same way as in the section 4, we consider the space $\mathcal{M}_H(\gamma_-,\gamma_+)$ of $\tau$-dependent gradient flow trajectories.

The virtual dimension of $\mathcal{M}_H(\gamma_-,\gamma_+)$ is again given by the spectral flow of the appropriate family of $\bar{\partial}$-operators and coincides with the actual dimension for “generic” $J$ and $H$.

The moduli spaces $\{\mathcal{M}_H(\gamma_-,\gamma_+)^C\}$ of solutions of (4.5) are compactified by adding gradient flow trajectories obtained by “splitting” and by “bubbling off”. The compactified moduli spaces $\{\bar{\mathcal{M}}_H(\gamma_-,\gamma_+)^C\}$ have the structure of stratified spaces such that each stratum is a smooth manifold with boundary.

**Statement 5.2.** “The compactification divisor” $\bar{\mathcal{M}}_H(\gamma_-,\gamma_+)^C - \mathcal{M}_H(\gamma_-,\gamma_+)$ has codimension at least two.

For the case of positive symplectic manifold this statement was proved in [F1]. In the semi-positive case the proof was given in [HS] when $H$ was $\tau$-independent. The same arguments as in [Hs] give the proof in the general case.

The analogues of (4.9) − (4.11) also hold for the moduli spaces of
τ-dependent gradient flow trajectories. This implies that we can fix the
additive constant ambiguities in the gradings of the critical points of \( \{ S_{\omega, H} \} \)
for all Hamiltonians simultaneously such that

\[
\text{vdim } \mathcal{M}_H(\gamma_-, \gamma_+) = \text{deg}_{CF^*(M,H_-)}(\gamma_+) - \text{deg}_{CF^*(M,H_+)}(\gamma_-)
\]  (5.7)

**Theorem 5.3.** For any two “τ-dependent Hamiltonians” \( H^{(0)} \) and \( H^{(1)} \)
lying in the space \( \mathcal{G}_{H_- H_+} \) the manifolds of trajectories \( \mathcal{M}_{H^{(0)}}(\gamma_-, \gamma_+) \) and \( \mathcal{M}_{H^{(1)}}(\gamma_-, \gamma_+) \) are cobordant to each other as stratified spaces.

More precisely, there exists a path \( \{ H^{(t)} \} \) \( (0 \leq t \leq 1) \) in \( \mathcal{G}_{H_- H_+} \)
joining \( H^{(0)} \) and \( H^{(1)} \) such that

\[
\bigcup_{0 \leq t \leq 1} \mathcal{M}_{H^{(t)}}(\gamma_-, \gamma_+)
\]

gives us the desired cobordism.

Since Floer boundary operator \( \delta \) (in general) acts nontrivially on \( \gamma_- \) and
Floer coboundary operator \( \delta^* \) acts nontrivially on \( \gamma_+ \) then the manifolds
\( \{ \mathcal{M}_{H^{(t)}}(\gamma_-, \gamma_+) \} \) are the manifolds with boundary and the cobordism
\( \bigcup_{0 \leq t \leq 1} \mathcal{M}_{H^{(t)}}(\delta \gamma_-, \gamma_+) \) and \( \bigcup_{0 \leq t \leq 1} \mathcal{M}_{H^{(t)}}(\gamma_-, \delta^* \gamma_+) \)

These extra components appear due to the presence of intermediate critical
points of \( S_{\omega, H_-} \) and \( S_{\omega, H_+} \).

The lemma 2.6 implies that such a path \( \{ H^{(t)} \} \) \( (0 \leq t \leq 1) \) in \( \mathcal{G}_{H_- H_+} \)
joining \( H^{(0)} \) and \( H^{(1)} \) exists. The Statement 5.2. implies that the corresponding
smooth cobordism can be compactified and the compactification divisor will
have codimension at least two.

Let \( \mathcal{J}_M \) be the space of all almost-complex structures on \( M \) compatible
with symplectic forms \( \{ \omega_1, \ldots, \omega_s \} \) and with some differential form representative of 
\( c_1(TM) \). By the theorem of Gromov, \( \mathcal{J}_M \) is an open contractible
set containing \( J_0 \).

We should consider the space \( \text{Map}(\gamma_-, \gamma_+) \) of all \( W^{2, 5n} \)-Sobolev maps
from \( R \times S^1 \) to \( M \) with the assymptotics (5.6) as \( \tau \to \pm \infty \) and the
infinite-dimensional Hilbert bundle \( \mathcal{H} \) over \( \text{Map}(\gamma_-, \gamma_+) \times \mathcal{J}_M \) The fibre
of the bundle \( \mathcal{H} \) over the point \( (\gamma; J) \) in \( \text{Map}(\gamma_-, \gamma_+) \times \mathcal{J}_M \) will be the
space of all \( W^{2, 5n-1} \)-Sobolev \( (0, 1) \)-forms on \( R \times S^1 \) with the coefficients in
\( \gamma^*(TM) \) which tend to zero as \( \tau \to \pm \infty \).

We can consider the pull-back of this Hilbert bundle to
\( \text{Map}(\gamma_-, \gamma_+) \times \mathcal{J}_M \times \times \mathcal{G}_{H_- H_+} \) and construct a (canonical) section \( \Phi \) as follows:

\[
\Phi(\gamma) = \frac{\partial \gamma}{\partial \tau} - J \frac{\partial \gamma}{\partial \theta} - \text{grad } H(\tau, \theta) \ d\bar{z}
\]  (4.8)
Here $dz$ is a canonical $(0,1)$-form on $R \times S^1 = C^\ast$. The identification between $R \times S^1$ and $C^\ast$ is given by the map (4.8).

The arguments of McDuff [McD] show that if the function $H$ does not admit any holomorphic symmetries with respect to parameters on $R \times S^1$ then the section $\Phi$ is regular over $\text{Map}(\gamma_-, \gamma_+) \times J_M \times \{H\}$.

Since the space of the functions $\{H\}$ with this property is open and dense in $G_{H-, H_+}$ this means that the section $\Phi$ is regular over $\text{Map}(\gamma_-, \gamma_+) \times J_M \times G_{H-, H_+}$. This allows us to apply Lemmas 2.4 and 2.6 to prove existence of the above cobordism. The theorem 5.2 is proved.

Now let us remember that the manifolds $\{M(q^d_1 x; q^d_2 y)\}$ of gradient flow trajectories can be thought either as submanifolds in the loop-space or as submanifolds in the space $\text{Map}(q^d_1 x; q^d_2 y)$ of maps from the cylinder $R \times S^1$ to $M$ with the fixed “boundary values” at $\tau \to \pm \infty$. Thus, we have a commutative diagram

$$
\begin{array}{ccc}
\mathcal{M}(q^d_1 x; q^d_2 y) & \xrightarrow{\text{ev}_0} & \mathcal{L}M \\
\downarrow & & \downarrow \\
\text{Map}(R \times S^1; M) & \xrightarrow{\text{ev}_{0,1}} & M
\end{array}
$$

Having this commutative diagram in mind, we can rewrite the definition (5.3) for the matrix element of the Floer multiplication as:

$$
<q^d_2 y | m_F(\hat{C}) | q^d_1 x> = \mathcal{M}(q^d_1 x; q^d_2 y) \cap \text{ev}^{-1}_{0,1}(\hat{C}) \quad (5.9)
$$

Here $\text{ev}_{0,1}$ is the “evaluation at the point $(0;1)$” map from $\text{Map}(R \times S^1; M)$ to $M$.

The formula (5.9) for the matrix element $<q^d_2 y | m_F(\hat{C}) | q^d_1 x>$ of the Floer multiplication admits the following generalization:

Let $H_+, H_-, \gamma_+, \gamma_-$ and $H$ are defined as above. Then let us put

$$
<q^d_2 \gamma_+ | m_F(\hat{C}) | q^d_1 \gamma_-> = \mathcal{M}_H(q^d_1 \gamma_+; q^d_2 \gamma_-) \cap \text{ev}^{-1}_{0,1}(\hat{C}) \quad (5.10)
$$

where the r.h.s., as usual, means the intersection index.

Any cycle $x$ in the Floer Chain complex $CF_i(M; H_-)$ can be written as a sum $\sum_k n_k x_k$ where $x_k$ are (possibly coinciding) critical points of $S_{\omega, H_-}$ and $n_k = \pm 1$. The same is true for the cycle $y = \sum_i m_i y_i$ in $CF_i(M; H_+)$.

We can consider the manifolds

$$
\mathcal{M}_H(q^d_1 x; q^d_2 y) = \bigcup_{k,l} n_k m_l \mathcal{M}_H(q^d_1 x_k; q^d_2 y_l)
$$
Here the factor $n_k m_l = \pm 1$ in front means that the component $\mathcal{M}_H(q^d x_k; q^{d_2} y_l)$ should be taken with the appropriate orientation.

If we glue all the components of $\mathcal{M}_H(q^d x; q^{d_2} y)$ together, we will obtain a smooth $\text{deg}(q^{d_2} y) - \text{deg}(q^d x)$-dimensional pseudo-manifold without boundary (or pseudo-cycle).

**Theorem 5.4.** For any two $\tau$-dependent Hamiltonians $H^{(0)}$ and $H^{(1)}$ from $\mathcal{G}_{H_{-},H_{+}}$ we have

$$\mathcal{M}_{H^{(0)}}(q^d x; q^{d_2} y) \bigcap ev^{-1}_{(0;1)}(\hat{C}) = \mathcal{M}_{H^{(1)}}(q^d x; q^{d_2} y) \bigcap ev^{-1}_{(0;1)}(\hat{C}) \quad (5.11)$$

The Theorem 5.3 provides us with a cobordism $M^t$ between $\mathcal{M}_{H^{(0)}}(q^d x; q^{d_2} y)$ and $\mathcal{M}_{H^{(1)}}(q^d x; q^{d_2} y)$. The fact that both $x$ and $y$ are cycles in the appropriate Floer complexes means that the cobordism $M^t$ between them does not have other boundary components. (All the “extra boundary components” of cobordisms $\bigcup_{0 \leq t \leq 1} \mathcal{M}_{H^{(t)}}(q^d x_k; q^{d_2} y_l)$ for different $k$ and $l$ will cancel each other after we glue them together).

The theorem 2.1 of [McD S] which claims that the map $ev_{(0;1)}$ from $\mathcal{M}_{H^{(t)}}(q^d x_k; q^{d_2} y_l) \times \mathcal{J}_M \times \mathcal{G}_{H_{-},H_{+}}$ to $M$ is surjective, allows us to apply Lemma 2.5. By applying this lemma to the evaluation map $ev_{(0;1)}$ taken as “projection operator” we have that the cobordism $M^t$ intersects transversally with $ev^{-1}_{(0;1)}(\hat{C})$ and the corresponding intersection gives us smooth one-dimensional submanifold (with boundary).

This submanifold does not intersect the “compactification divisor” $M^t - M^t$ since the later has codimension $\geq 2$ and we have in our hands the freedom of putting everything “in general position”.

Thus, $M^t \bigcap ev^{-1}_{0;1}(\hat{C})$ gives us the desired compact one-dimensional cobordism between $\mathcal{M}_{H^{(0)}}(q^d x; q^{d_2} y) \bigcap ev^{-1}_{0;1}(\hat{C})$ and $\mathcal{M}_{H^{(1)}}(q^d x; q^{d_2} y) \bigcap ev^{-1}_{0;1}(\hat{C})$. The statement of the Theorem 5.4. follows.

The same cobordism and transversality arguments proves the following

**Lemma 5.5.** If $\hat{C}_1$ and $\hat{C}_2$ be two pseudo-manifolds in $M$ homologous to each other (which implies that they are actually cobordant to each other in the class of pseudo-manifolds) and if $(\tau_0, \theta_0)$ and $(\tau_1, \theta_1)$ be any two points on the cylinder $R \times S^3$ then

$$\mathcal{M}_{H^{(0)}}(q^d x; q^{d_2} y) \bigcap ev^{-1}_{\tau_0;\theta_0}(\hat{C}) = \mathcal{M}_{H^{(1)}}(q^d x; q^{d_2} y) \bigcap ev^{-1}_{\tau_1;\theta_1}(\hat{C}) \quad (5.12)$$

Now we are ready to prove the Theorem 5.1. In order to prove it, we should (following Floer):

37
A) Construct a chain homotopy \( h_H : CF^*(M, H_-) \rightarrow CF^*(M, H_+) \) (which depends on the choice of the function \( H \in G_{H_-, H_+} \)).

B) Prove that the chain homotopy \( h_H \) gives a well-defined homomorphism \( h_{H_-, H_+} : HF^*(M, H_-) \rightarrow HF^*(M, H_+) \) on the level of homology, and this homomorphism is independent on the choice of \( H \).

C) Prove that \( h_{H_1, H_3} = h_{H_2, H_3}h_{H_1, H_2} \) for any triple of “generic” Hamiltonians \( H_1, H_2, H_3 \) defined as functions from \( S^1 \times M \) to \( R \).

D) Prove that for any singular homology class \( \hat{C} \) in \( M \)

\[
h_{H_-, H_+}(m^H_-(\hat{C})) = m^H_+(\bar{C})h_{H_-, H_+} \tag{5.13}
\]

Here \( m^H_- \) and \( m^H_+ \) are operators of the action of \( H_*(M) \) on the Floer cohomology \( HF^*(M, H_-) \) and \( HF^*(M, H_+) \) respectively.

Let \( \{x_1, x_2, \ldots \} \) and \( \{y_1, y_2, \ldots \} \) be the bases (over \( \mathbb{Z} \)) of critical points of \( S_\omega, H_- \) and \( S_\omega, H_+ \) respectively.

Then the matrix element \( <q^d y|h_H|q^d x> \) of the desired chain homotopy \( h_H \) is by definition the number of zero-dimensional components of \( \mathcal{M}_H(q^d x; q^d y) \) taken with appropriate orientation. This number is non-zero only if \( \deg(q^d x) = \deg(q^d y) \). by our convention, the difference \( \deg(q^d y) - \deg(q^d x) \) is given by the spectral flow.

The Theorem 5.4 and Lemma 5.5 imply that above defined \( h_H \) is really a chain homotopy such that the statements A) and B) above hold.

The statement D) above is equivalent to the fact that

\[
\mathcal{M}_H(q^d x; q^d y) \cap ev^{-1}_{(2;1)}(\bar{C}) = \mathcal{M}_H(q^d x; q^d y) \cap ev^{-1}_{(-2;1)}(\bar{C}) \tag{5.14}
\]

The l.h.s. of \( (5.14) \) coincide with the l.h.s. of \( (5.13) \) because of \( H(2, \theta) = H_+(\theta) \). The r.h.s. of \( (5.14) \) coincide with the r.h.s. of \( (5.13) \) because of \( H(-2, \theta) = H_-(\theta) \). Thus, we reduced the statement D) to the particular case of the Lemma 5.5.

The statement C) above is a consequence of the procedure of “gluing trajectories” [AuBr]. Namely, let us glue two half-cylinders \( S^1 \times (-\infty; T] \) and \( S^1 \times [-T; +\infty) \) along their boundaries. Since we have a “\( \tau \)-dependent Hamiltonian” \( H_{12} \) on the first half-cylinder and a “\( \tau \)-dependent Hamiltonian” \( H_{23} \) on the second half-cylinder such that

\[
H_{12}(\tau; \theta) = H_1(\theta) \text{ if } \tau \leq -T - 1
\]

\[
H_{12}(\tau; \theta) = H_2(\theta) \text{ if } \tau \geq -T + 1
\]

\[38\]
\[ H_{23}(\tau; \theta) = H_2(\theta) \quad \text{if} \quad \tau \leq T - 1 \]

\[ H_{23}(\tau; \theta) = H_3(\theta) \quad \text{if} \quad \tau \geq T + 1 \]

then we can glue them together to obtain a new \( \tau \)-dependent Hamiltonian \( H_{13}^T \) which is defined as

\[ H_{13}^T(\tau; \theta) = H_{12}(\tau; \theta) \quad \text{if} \quad \tau \leq 0 \]

\[ H_{13}^T(\tau; \theta) = H_{23}(\tau; \theta) \quad \text{if} \quad \tau \geq 0 \]

If \( x_1 \) and \( x_3 \) are any two cycles in \( CF_*(M, H_1) \) and in \( CF_*(M, H_3) \) respectively of relative Morse index zero then the lemma 5.5. implies that

\[ <x_3|h_{H_1,H_3}|x_1> = \chi(M_{H_{13}}^T(x_1;x_3)) \quad (4.15) \]

for any value of the gluing parameter \( T \). Here \( \chi \) means the Euler characteristics of the zero-dimensional manifold.

Now if we tend \( T \) to infinity then any trajectory in \( \mathcal{M}_{H_{13}}^T \) will “split” into connected sum of a gradient flow trajectory of \( H_{12} \) and a gradient flow trajectory of \( H_{23} \). These two trajectories are glued together in some point \( x_2 \in \hat{LM} \) which has to be a critical point of \( S_{\omega,H_2} \) due to the \( L^2 \)-boundedness condition.

This observation implies that C) holds which proves the Theorem 4.1.

Thus, we have a well-defined map

\[ m_F : H^*(M) \otimes HF^*(M) \to HF^*(M) \]

Since the Floer cohomology \( HF^*(M) \) are isomorphic to the classical cohomology \( H^*(M) \otimes N \) then this “Floer multiplication” gives us some bilinear operation

\[ m_F : H^*(M) \otimes H^*(M) \to H^*(M) \otimes N \]

in classical cohomology.

In order to calculate this bilinear operation and prove that it coincides with the quantum cup-product, we should examine more closely how the isomorphism between \( HF^*(M) \) and \( H^*(M) \otimes N \) is constructed. We will do this in the next section.
6. The Proof of the Main Theorem

For each cohomology class \( C \in H^*(M) \) two linear operators

\[
m_Q(C) : H^*(M) \to H^*(M) \otimes N
\]

and

\[
m_F(C) : H^*(M) \to H^*(M) \otimes N
\]

were defined in the previous three sections. The map \( m_Q(C) \) was called quantum multiplication (from the left) on the cohomology class \( C \). The map \( m_F(C) \) was called Floer multiplication (from the left) on the cohomology class \( C \).

The Main Theorem 6.1. Quantum multiplication coincides with the Floer multiplication.

Let us fix be some (homogenous) basis \( \{ A, B, ... \} \) in the total cohomology of \( M \). To prove that the homomorphisms (6.1A) and (6.1B) are in fact equal, it is sufficient to prove that all their \((N\text{-valued})\) matrix elements

\[
< B | m_Q(C) | A > = \sum_d q^d < B | m_Q(\mu) | A >
\]

and

\[
< B | m_F(C) | A > = \sum_d q^d < B | m_F(\mu) | A >
\]

are the same.

Let \( H \) be some smooth function on \( S^1 \times R \times M \) such that

A) \( H \) vanishes in the region \(||\tau|| > 1\)

B) \( H \) is not invariant under any holomorphic automorphism of \( S^1 \times R \)

(\( \)which is identified with \( \mathbb{C}^* \))

Following the logic of the previous section, we can consider the space of \( L^2 \)-bounded trajectories \( \mathcal{M}_H(\hat{A}; q^d\eta(\hat{B})) \) and compactify it as a stratified space. We assume that “the statement 5.2” holds in this case also. The proof of this generalization of the statement 5.2 repeats the proof of the original statement.

Theorem 6.2.

\[
\mathcal{N}_H(\hat{A}; q^d\eta(\hat{B})) = \mathcal{N}_{\text{grad}Hdz,d} \bigcap \text{ev}_0^{-1}(\hat{A}) \bigcap \text{ev}_{\infty}^{-1}(\eta(\hat{B}))
\]

Let \( \gamma = \gamma(\tau, \theta) \) be any \( L^2 \)-bounded solution of (5.5) with \( H = 0 \) in the region \(||\tau|| > 1\). Then \( \gamma \) (considered as a map from the cylinder...
$S^1 \times \mathbb{R}$ to $M$) can be continuously extended from the cylinder $S^1 \times \mathbb{R}$ to the 2-sphere $S^2$ since the limit value of $\gamma$ at $\tau \to \pm \infty$ should be constant loops.

Ellipticity of the gradient flow equation (5.5) with the prescribed boundary conditions at $\tau \to \pm \infty$ implies that this extension is actually smooth. Now the statement of the Theorem 6.2. follows directly from the definitions of the l.h.s and the r.h.s. of (6.3).

The fact that (6.3) is an isomorphism at the level of compactifications (as stratified spaces) can be observed by comparing the explicit description of these compactifications that we have.

The previous theorem means that the matrix element of quantum multiplication can be written as

$$< B| m_Q(C) | A > = \sum_d q^d M_H(\hat{A}; q^d \eta(\hat{B})) \bigcap e v_{0,1}^i(\hat{C}) \quad (6.4)$$

where the number in the r.h.s., as usual, is understood as intersection index.

**Remark.** The r.h.s. of (6.4) can be thought as a of Floer multiplication operation defined for unperturbed symplectic action.

This remark implies that in order to prove the Main Theorem, it is enough to generalize the program implemented in the previous section as follows:

A) Construct the chain homotopies $\{ h_{H,0} : CF^*(M, H_i) \to CF^*(M, 0) \}$ (i = 1; 2) from the Floer complexes $\{ CF^*(M, H_i) \}$ of perturbed symplectic action $S_{\omega, H_i}$ to the Floer complex $CF^*(M, 0)$ of unperturbed symplectic action $S_\omega$ and the chain homotopies $\{ h_{0, H_i} \} : CF^*(M, 0) \to CF^*(M, H_i)$ going in the opposite direction

B) Prove that these chain homotopies gives us well-defined homomorphisms on the level of cohomology which is independent on the choice of $\tau$-dependent hamiltonians by means of which they are constructed.

C) Prove that for any pair of “generic” Hamiltonians $H_1$ and $H_2$ considered as functions on $S^1 \times M$ we have functoriality property $h_{H_1, H_2} = h_{0, H_2} h_{H_1, 0}$ and prove that $h_{H_1, 0} h_{0, H_1}$ gives us identity map in the Floer cohomology group (defined by unperturbed symplectic action).

D) Prove that for any singular homology class $\hat{C}$ in $M$ we have

$$h_{H_1, 0}(m^0_F(\hat{C})) = m^0_F(\hat{C}) h_{H_1, 0} \quad (6.5A)$$

and

$$h_{0, H_1}(m^0_F(\hat{C})) = m^0_F(\hat{C}) h_{0, H_1} \quad (6.5B)$$
The proof of the Statements A - D above goes exactly the same way as the proof of analogous statements in the section five.

Thus, \( h_{0,H} \) gives us an isomorphism between classical and Floer cohomology which maps quantum multiplication \( m_Q(C) \) to the Floer multiplication \( m_F(C) \) for any cohomology class \( C \in H^*(M) \).

This proves our Main Theorem.

7. FLOER COHOMOLOGY OF COMPLEX GRASSMANIANS

As an example of applications of our Main theorem, let us give a rigorous proof of the formula for Floer cohomology ring of the complex Grassmanian \( G(k,N) \) of \( k \)-planes in complex \( N \)-dimensional vector space \( V \). The formula for the quantum cohomology ring \( HQ^*(G(k,N)) \) was assumed long ago by Vafa [Va1]. More detailed analysis of quantum cohomology of Grassmanians was worked out by Intrilligator [I] and recently by Witten [Wi5] in relation with the Verlinde algebra. Witten also mentioned that Floer cohomology ring of the Grassmanian should be given by the same formula.

Now we need to discuss the cohomology of \( G(k,N) \). We begin with the classical cohomology. Over \( G(k,N) \) there is a “tautological” \( k \)-plane bundle \( E \) (whose fiber over \( x \in G(k,N) \) is the \( k \) plane in \( V \) labeled by \( x \)) and a complementary bundle \( F \) (of rank \( N-k \)):

\[
0 \rightarrow E \rightarrow V^* = C^N \rightarrow F \rightarrow 0
\]

Obvious cohomology classes of \( G(k,N) \) come from Chern classes. We set

\[
x_i = c_i(E^*)
\]

where * denotes the dual. (It is conventional to use \( E^* \) rather than \( E \), because \( \det E^* \) is ample.) This is practically where Chern classes come from, as \( G(k,N) \) for \( N \rightarrow \infty \) is the classifying space of the group \( U(k) \). It is known that the \( x_i \) generate \( H^*(G(k,N)) \) with certain relations. The relations come naturally from the existence of the complementary bundle \( F \) in Let \( y_j = c_j(F^*) \), and let \( c_t(\cdot) = 1 + tc_1(\cdot) + t^2c_2(\cdot) + \ldots \). Then \( H^*(G(k,N)) \) is generated by the \( \{x_i,y_j\} \) with relations

\[
c_t(E^*)c_t(F^*) = 1 \quad (7.1)
\]

Since the left hand side of (7.1) is \textit{a priori} a polynomial in \( t \) of degree \( N \) the classical relations are of degree \( 2, 4, \ldots, 2N \). The first \( N-k \) of these relations (uniquely) express the \( \{y_j\} \) in terms of the \( \{x_i\} \). This means that the classical cohomology ring of \( H^*(G(k,N)) \) is generated by the \( k \) generators \( \{x_i\} \) with \( k \) relations of degree \( 2N-2k+2, 2N-2k+4, \ldots, 2N \).
Let us now work out the quantum cohomology ring $HQ^\ast(G(k,N))$ of the Grassmannian. We can consider a subring in $HQ^\ast(G(k,N))$ generated by $\{x_i, y_j\}$

**Conjecture (Vafa).** quantum cohomology ring $HQ^\ast(G(k,N))$ of the Grassmannian is generated by $\{x_i, y_j\}$ with “deformed relations”

$$c_t(E^\ast)c_t(F^\ast) = 1 + q(-1)^{N-k}t^N \quad (7.2)$$

where $q$ is (the unique) Kahler class in $H^2(G(k,N),\mathbb{Z})$.

To prove this Vafa’s conjecture it is sufficient to prove that

A) $\{x_i\}$ generate the whole quantum cohomology ring

B) $\{y_j\}$ are expressed in terms of the $\{x_i\}$ by the same formulas as in the classical cohomology ring

C) The relations on $\{x_i\}$ in our quantum cohomology ring form an ideal

D) This ideal of relations is generated by $k$ relations of degree $2N-2k+2, 2N-2k+4, \ldots, 2N$ coming from expansion of the l.h.s. of (7.2) in powers of $t$ and taking coefficients of degree $2N-2k+2, 2N-2k+4, \ldots, 2N$ without any extra relations

The fact that $\{y_j\}$ are expressed in terms of the $\{x_i\}$ by the same formulas as in the classical cohomology ring and the fact that these $k$ Vafa’s relations indeed take place was proved (rigorously) by Witten [Wi5] by examining the fact that

a) The classical relations of degree $2, 4, \ldots, 2N-2$ cannot deform since $deg[q] = 2N$, and

b) There is a “quantum correction” to the the “top” relation

$$c_k(E^\ast)c_{N-k}(F^\ast) = 0$$

of degree $2N$ in the classical cohomology. This “deformed relation” has the form $c_k(E^\ast)c_{N-k}(F^\ast) = a$ for some number $a$ which can be computed by examining degree-one rational curves in the Grassmannian. The value of this unknown number $a$ was (rigorously) computed by Witten and was shown to be equal to $(-1)^{N-k}$.

The statement C) that the relations on $\{x_i\}$ in our quantum cohomology ring form an ideal will follow from the associativity of the quantum cohomology ring (which was proved rigorously after [Wi5] was finished).

Thus, the only things we need to prove after Witten are

A) $\{x_i\}$ generate the whole quantum cohomology ring, and

D) that there are no extra relations (in degree higher than $2N$) on these generators.

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The statement A) can be proved inductively by the degree \( \text{deg} \). Let us assume that all the elements in \( HQ^*(G(k,N)) \) of degree less than \( m \) can be expressed as polynomials in \( \{x_i\} \). Let us prove that this also holds for all the elements in \( HQ^*(G(k,N)) \) of degree \( m \).

Let \( A \in H^m(G(k,N),Z) \subset HQ^*(G(k,N)) \) be some homogenous element of degree \( m \). Then we know that in the classical cohomology ring we have

\[
A = P_m(x_1, \ldots, x_k)
\]

for some polynomial \( P_m \) of degree \( m \). The fact that \( \text{deg}[q] = 2N \) is positive means that in the quantum cohomology ring we have

\[
A = P_m(x_1, \ldots, x_k) + \sum_d q^d A_d
\]

for some (unknown) cohomology classes \( A_d \in H^{m-2Nd}(G(k,N), Z) \) of degree \( m - Nd \).

But by our induction hypothesis we know that all \( \{A_d\} \) can be expressed as some polynomials in \( \{x_i\} \). This simple observation proves the statement A).

To prove the last remaining statement D) let us note that the rank (over the ring \( Z_{<q>} \)) of the quantum cohomology of the Grassmanian \( HQ^*(G(k,N)) \) should be equal to the rank (over \( Z \)) of the classical cohomology \( H^*(G(k,N)) \).

If there were some extra relations among the generators \( \{x_i\} \) this would mean that the rank (over the ring \( Z_{<q>} \)) of the free polynomial ring in \( \{x_i\} \) moded out by the ideal generated by the coefficients of the l.h.s. of (6.2) would be strictly greater than the rank of \( H^*(G(k,N)) \).

But we know that any two \( Z \)-graded rings generated by \( k \) homogenous generators \( \{x_1, \ldots, x_k\} \) of degrees \( \{2, 4, \ldots, 2k\} \) with \( k \) homogenous relations of degrees \( 2N - 2k + 2, 2N - 2k + 4, \ldots, 2N \) should have the same rank.

This proves the statement D) and the Vafa’s conjecture.

The arguments presented here together with the results of Ruan and Tian [RT] who proved “the handle-gluing formula” of Witten [Wi1] give a complete proof to a more refined formula of Intrilligator [I] for the certain intersection numbers (known as Gromov-Witten invariants) on the moduli space of holomorphic maps of higher genus curves to the Grassmanian. The proof of this formula was previously known only for the special case \( (G(2,N)) \) of the Grassmanians of 2-planes and is due to Bertram,Daskaloupulos and Wentworth [BDW],[Be]. Our arguments prove this formula in the full generality.

8. QUANTUM COHOMOLOGY REVISITED
The significant drawback of the Definition B of quantum cup-product (which is the only completely justified definition available at the moment) is that the Definition B invokes the moduli spaces of $J_g$-holomorphic spheres instead of the moduli spaces of $J_0$-holomorphic spheres when $J_0$ is an actual complex structure (which is much more interesting object from an algebro-geometric point of view).

It is hard to prove that some particular choice of $g$ is “regular” and to make any calculations using this definition.

In order to give a definition of quantum cup-product which uses only the moduli spaces $\{M_{J_0,d}\}$ of holomorphic curves we need to introduce some more notations.

Let us suppose that:

**Statement 8.1.** For any “generalized degree” $d$ the moduli space $\mathcal{M}_d = \mathcal{M}_{J_0,d}$ will be a finite union of smooth strata (of possibly different dimensions) such that each stratum is a smooth almost-complex manifold.

**Statement 8.2.** Each manifold $\mathcal{M}_d$ can be compactified (by adding “degenerate $J$-holomorphic curves”) as a stratified space $\bar{\mathcal{M}}_d$ such that each stratum is a smooth almost-complex manifold.

**Statement 8.3.** “The compactification divisor” $\bar{\mathcal{M}}_d - \mathcal{M}_d$ and “the singular strata” have codimension at least two (or “complex codimension one”) in each irreducible component of the compactified moduli space $\bar{\mathcal{M}}_d$.

**Theorem 8.4 (Gromov [Gr2]).** The statements (8.1) – (8.3) always hold if $M$ is Kahler manifold with its actual complex structure.

If the almost-complex structure $J$ on $M$ is non-integrable, we will state (8.1) – (8.3) as assumptions.

**Note.** If $M$ is algebraic the Theorem 8.4 follows from the fact that in the case $\mathcal{M}_{J,d}$ can be defined in algebraic terms as a Hilbert scheme.

Let us note that the formal tangent space to the moduli space $\mathcal{M}_{J,d}$ at the point $\varphi \in \mathcal{M}_{J,d}$ is equal to the kernel of the linearised $\bar{\partial}$-operator, acting from the space of $W_2^{2\alpha}$-sections of (holomorphic) vector bundle $\varphi^*(TM)$ over $CP^1$ to the space of $W_0^{2\alpha-1}$-one-forms with the coefficients in this vector bundle.

Equivalently, the formal tangent space to $\mathcal{M}_{J,d}$ at the point $\varphi$ is isomorphic to $H^0[\varphi^*(TM)]$.

The cokernel of the same $\bar{\partial}$-operator is isomorphic to $H^1[\varphi^*(TM)]$.

In general, the formal tangent space $H^0[\varphi^*(TM)]$ to the moduli space of
$J$-holomorphic spheres is not necessarily equal to the actual tangent space to this moduli space at the point $\varphi$.

There may be “an obstruction” to integration of the formal tangent vector to the local deformation of the moduli space in the direction of this formal tangent vector. This obstruction is a non-linear map from $H^0[\varphi^*(TM)]$ to $H^1[\varphi^*(TM)]$ with vanishing first derivative.

The existence of a non-trivial obstruction corresponds to the singularity of our moduli space $M_{J,d}$ at the point $\varphi$.

According to the theorem 8.4 (or according to the assumption 8.3 if $J$ is non-integrable) the space of $\varphi \in M_d$ with obstructed deformations has codimension at least two. This means that on the complement to the lower-dimensional singular startum there is no obstruction and the moduli space $M_d$ is a smooth manifold of dimension $2\text{dim}(H^0[\varphi^*(TM)])$.

On the complement to this singular startum the cokernel of $\bar{\partial}$-operator has constant dimension $\text{dim}H^1[\varphi^*(TM)]$.

By applying Riemann-Roch theorem to the vector bundle $\varphi^*(TM)$ over $\mathbb{C}P^1$ we see that

$$2\text{dim}(H^0[\varphi^*(TM)]) - 2\text{dim}(H^1[\varphi^*(TM)]) = \text{dim}M + \sum_{i=1}^{s} d_i \text{deg}[q_i]$$

which is equal to the r.h.s of (2.7)

So, the r.h.s. of (8.1) reproduces us “the virtual dimension” $\text{vdim}[M_d]$ of the moduli space $M_d$. This “virtual dimension” $\text{vdim}$ is equal to the actual dimension of this moduli space if and only if the first cohomology $H^1[\varphi^*(TM)]$ is zero-dimensional (or, equivalently, if the $\bar{\partial}$-operator is surjective).

Unfortunately, this situation almost never takes place if we do not consider $g$-perturbed $\bar{\partial}$-operator. Usually the different irreducible components of the moduli space $M_d$ have different dimensions. Unlike the case of moduli space of $J_g$-holomorphic spheres, these components may intersect each other. This why we call them “irreducible components” instead of “connected components”.

This difference between the actual and the virtual dimension of any irreducible component $M_c$ of the moduli space $M_d$ is always non-negative and is given by the number $2b_c = 2\text{dim}(H^1[\varphi^*(TM)])$ computed at the complement to the singular locus of $M_c$. Physicists would call the number $b_c$ “the number of the fermion zero-modes”.

The dimension $\text{dim}(H^1[\varphi^*(TM)])$ usually is not constant as $\varphi$ varies over $M_c$. The “jumping divisor” coincides with the singular locus $\text{sing}_c$ of $M_c$. 

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If the number of the fermion zero-modes $b_c$ is positive, then let us introduce $b_c$-dimensional complex vector bundle $F_c$ over $\mathcal{M}_c - \text{sing}_c$ which assigns to each holomorphic map $\varphi \in \mathcal{M}_c$ the vector space $H^1(\varphi^*(TM))$. This vector space varies holomorphically as $\varphi$ varies. Physicists would call this bundle “the bundle of the fermion zero-modes”.

Let $\chi(F_c)$ be the Euler class of the bundle $F_c$ (formally) considered as a cohomology class in $H^b_c(\mathcal{M}_c)$. The precise meaning of this Euler class will be specified later in this section.

In the case when $M$ is a projective algebraic manifold, we can think about $\overline{\mathcal{M}}_{J,d}$ as a Hilbert scheme and about $\overline{\mathcal{M}}_c$ as its irreducible component. Then we have a coherent algebraic sheaf $\mathcal{F} = H^1(\varphi^*(TM))$ over $\overline{\mathcal{M}}_{J,d}$. This sheaf is not locally free. Its restriction on any irreducible component $\mathcal{M}_c$ (compactified algebraically) of our Hilbert scheme is also not locally free.

But if we restrict our sheaf $\mathcal{F}$ on $\mathcal{M}_c - \text{sing}_c$ we will obtain a locally free sheaf of rank $b_c$. This sheaf over $\mathcal{M}_c - \text{sing}_c$ will be the sheaf of sections of the bundle $\mathcal{F}_c$ of fermion zero-modes (considered as an algebraic vector bundle).

Thus, in algebraic situation we can think about $\chi(F_c)$ as an algebraic Euler class of the sheaf on $\overline{\mathcal{M}}_c$ which is a well-defined mathematical object.

Now let us give two more definitions of the quantum tri-linear pairings $\langle A; B; C \rangle_q$ using only the moduli spaces $\{\mathcal{M}_d\}$

**Definition C (Witten).**

Let $A, B, C \in H^*(M, Z) \otimes Z_{<C>}$ Then

$$\langle A; B; C \rangle_q = \sum_d q^d \sum_c \int_{\mathcal{M}_c} \text{ev}_0^*(A) \wedge \text{ev}_\infty^*(B) \wedge \text{ev}_1^*(C) \wedge \chi(F_c) \quad (8.2)$$

Here the second sum is over the irreducible components $\{\mathcal{M}_c\}$ of the moduli space $\mathcal{M}_d$.

Strictly speaking, the r.h.s. of (8.2) does not make sense by the same reason as the r.h.s. of (3.2) does not make sense (we should integrate over non-compact moduli spaces and extend $\chi(F_c)$ to the singular part of $\mathcal{M}_c$).

It is again possible, following [SeSi] and [Wi4], to choose differential form representative of $\chi(F_c)$ computed through the curvature and try to extend it over the compactification divisor and over the singularities.

But in order to show that the integral (8.2) over the compactified moduli space is well-defined, we should prove its independence of the choice of differential form representatives of cohomology classes $A, B$ and $C$ and $\chi(F_c)$.
Since the structure of the compactification (and singularities) of $M_d$ is enormously complicated and not well-understood, the proof of “independence of the choices” is not yet available.

To get rid of this problem, we will give another definition in the spirit of Vafa, using the Poincare dual language of intersection of cycles.

But even in this approach, we need to choose some model $\hat{M}_c$ for desingularization of $\bar{M}_c$. Our strategy will be to give a formula using some choice of desingularization and then prove that the answer is actually independent of this choice. More precisely, we need the following:

**Theorem 8.5.** The following five statements hold:

1) There exists a (non-unique) desingularization $\hat{M}_c$ of $\bar{M}_c$ which coincides with $M_c$ on the complement to the small tubular neighborhood $B'_c$ of the singular locus. This desingularization is constructed analytically through Kuranishi obstruction theory.

2) The vector bundle $F_c$ extends canonically to any of these desingularizations as a smooth $b_c$-dimensional vector bundle.

3) Different irreducible components $\{\hat{M}_c\}$ of desingularized moduli space $\hat{M}_d$ do not intersect each other.

4) Any two “desingularizations constructed a la Kuranishi” are cobordant to each other.

5) The bundle of the fermion zero-modes $F_c$ can be extended as the vector bundle over the cobordism between two different desingularizations of $\hat{M}_c$.

**Proof.**

Following analogous constructions by Taubes [Ta1], [Ta2] for the moduli space of anti-self-dual connections on a four manifold, we can argue as follows:

For any $\varphi \in \text{Map}_d$ we have two linear operators: $\bar{\partial}$-operator acting from $T_\varphi(\text{Map}_d)$ to $\mathcal{H}_\varphi$ and the adjoint $\bar{\partial}^*$ operator acting from $\mathcal{H}_\varphi$ to $T_\varphi(\text{Map}_d)$ (since we choose our elliptic differential operators to act in Hilbert spaces).

The non-zero spectrum of the “Laplace operators”

$\bar{\partial}^* \bar{\partial} : T_\varphi(\text{Map}_d) \to T_\varphi(\text{Map}_d)$ and $\bar{\partial} \bar{\partial}^* : \mathcal{H}_\varphi \to \mathcal{H}_\varphi$ is the same.

The kernel of $\bar{\partial}^* \bar{\partial}$ is isomorphic to the kernel of $\bar{\partial}$ and the kernel of $\bar{\partial} \bar{\partial}^*$ is isomorphic to the cokernel of $\bar{\partial}$.

For any positive number $\lambda$ let $\mathcal{M}_d^\lambda$ be the topological subspace in $\text{Map}_d$ consisting of those $\varphi \in \text{Map}_d$ for which the operator $\bar{\partial}^* \bar{\partial}$ has eigenvalues less than $\lambda$. In particular, $\mathcal{M}_d \subset \mathcal{M}_d^\lambda$.

Since the spectrum of any of “Laplace operators” (parametrized by the points in $\text{Map}_d$) is discrete and changes smoothly as $\varphi$ varies then for any
particular choice of $\varphi \in \text{Map}_d$ such that $\lambda$ is not in the spectrum the following statements hold:

1) The intersection of $\mathcal{M}_d^\lambda$ with the small ball $B^\epsilon(\varphi)$ of radius $\epsilon$ with the center $\varphi$ is a smooth finite-dimensional submanifold in $\text{Map}_d$ and

2) $\mathcal{M}_d^\lambda \cap B^\epsilon(\varphi)$ is a smooth submanifold of $\mathcal{M}_d^{\lambda_2} \cap B^\epsilon(\varphi)$ if $\lambda_1 < \lambda_2$ and both $\lambda_1$ and $\lambda_2$ are not in the spectrum

3) If restricted to the smooth part of $\mathcal{M}_d^\lambda$, the infinite-dimensional Hilbert bundle $\mathcal{H}$ splits into direct orthogonal sum $\mathcal{H}_{<\lambda} \oplus \mathcal{H}_{>\lambda}$.

Here $\mathcal{H}_{<\lambda}$ is a finite-dimensional subbundle in $\mathcal{H}$ spanned by the eigenvectors of the Laplacian $\bar{\partial}\bar{\partial}^*$ with the eigenvalues less than $\lambda$.

$\mathcal{H}_{>\lambda}$ is an infinite-dimensional subbundle in $\mathcal{H}$ spanned by the eigenvectors of $\bar{\partial}\bar{\partial}^*$ with the eigenvalues greater than $\lambda$.

Moreover, Kuranishi-type techniques [Ku],[Ta1] gives us that:

4) If $\varphi$ lies in $\mathcal{M}_d$ then there exists a preferred section $\Psi_\lambda$ of the bundle $\mathcal{H}_{<\lambda}$ over $\mathcal{M}_d \cap B^\epsilon(\varphi)$ such that

$$\mathcal{M}_d \cap B^\epsilon(\varphi) = \Psi_\lambda^{-1}(0) \cap \mathcal{M}_d^\lambda \cap B^\epsilon(\varphi)$$

(8.3)

This preferred section $\Psi_\lambda$ is called “Kuranishi map” [Ku],[Ta1]

5) If the point $\varphi \in \mathcal{M}_c$ does not lie in the singular locus $\text{sing}_c$ then the derivative of the Kuranishi map $\Psi_\lambda$ has the constant rank. The corank of this derivative is equal to the number of fermion zero-modes $b_c$.

If the point $\varphi \in \mathcal{M}_c$ lies in the singular locus than the corank of the derivative of $\Psi_\lambda$ jumps near the singular locus $\text{sing}_c$ and is equal to $b_c$ outside the singular locus.

6) If $\lambda_1 < \lambda_2$ then the corresponding “Kuranishi maps” are related as

$$\Psi_{\lambda_2} = \Psi_{\lambda_1} \oplus \Pi_{\lambda_1}^{\lambda_2} \partial$$

(8.4)

where $\Pi_{\lambda_1}^{\lambda_2}$ is the orthogonal projector on the subbundle in $\mathcal{H}$ generated by eigenvectors of Laplacian $\partial\partial^*$ with eigenvalues between $\lambda_1$ and $\lambda_2$.

Since the Gromov compactification of the moduli space $\mathcal{M}_d$ consists of degenerate curves and we know from the work of Seeley and Singer [Se Si] that $\partial$-operator can be continously extended to such degenerate curves, then the above described Kuranishi machinery is also applicable to the case when $\varphi$ lies in the compactification of $\mathcal{M}_c$.

We can cover the neighborhood of the compactification divisor by a system of charts $\{\mathcal{M}_d^\lambda \cap B^\epsilon(\varphi)\}$. We also have a Kuranishi map on each chart such
that the piece of $\mathcal{M}_c$ covered by this chart coincides with the zero locus of the corresponding Kuranishi map.

Now we can construct the desired desingularization.

For any point $\varphi \in \text{sing}_c$ we can perturb slightly the Kuranishi map $\Psi_\lambda$ on $\mathcal{M}_c^\lambda \cap B^r(\varphi)$ without changing it outside the ball $B^{r/2}(\varphi)$. We make the perturbation such that the new map will have constant rank $\text{dim}(\mathcal{H}_{<\lambda}) - b_c$. Let us denote this “perturbed Kuranishi map” $\Psi'_\lambda$.

Moreover, we can do these perturbations on all charts $\{\mathcal{M}_c^\lambda \cap B^r(\varphi)\}$ covering the singular locus $\text{sing}_c$ simultaneously. We can make these perturbations on different charts consistent with each other in the sense of (8.4).

Thus, we will have a new smooth compact manifold $\hat{\mathcal{M}}_c$ which we will call a desingularization of $\mathcal{M}_c$.

The extension of the vector bundle $\mathcal{F}_c$ on $\hat{\mathcal{M}}_c$ is defined as follows:

let us define the restriction of $\mathcal{F}_c$ on the chart $(\Psi'_\lambda)^{-1}(0) \cap \mathcal{M}_c^\lambda \cap B^r(\varphi)$ as cokernel of derivative of the perturbed Kuranishi map $\Psi'_\lambda$.

This definition is consistent with the definition of the bundle $\mathcal{F}_c$ over the region $\mathcal{M}_c \cap \mathcal{M}_c$ where the Kuranishi map is not perturbed.

Moreover, the bundle $\mathcal{F}_c$ on $\hat{\mathcal{M}}_c$ is by its construction a factor-bundle of the infinite-dimensional Hilbert bundle $\mathcal{H}$ restricted to $\hat{\mathcal{M}}_c$. Since $\mathcal{H}$ is a Hilbert bundle then we can also think of $\mathcal{F}_c$ as a subbundle of $\mathcal{H}|_{\hat{\mathcal{M}}_c}$.

Thus, we have already proved parts one and two of the Theorem 8.5. Part three follows from the transversality arguments. What remains to be proved is parts four and five.

Let $\hat{\mathcal{M}}_c$ and $\hat{\mathcal{M}}'_c$ be two different desingularizations corresponding to two different perturbations $\Psi'_\lambda$ and $\Psi'_\lambda'$ of the Kuranishi map.

Although we denote the (perturbed) Kuranishi map by one symbol $\Psi'_\lambda$, we understand that actually we have a system of perturbed Kuranishi maps on different charts which match on their intersection.

Then by a version of Sard Lemma we can construct a function $\Psi_\lambda(t)$ on $[\text{the union of charts}] \times [0; 1]$ which interpolates between $\Psi'_\lambda$ at $t = 0$ and $\Psi'_\lambda'$ at $t = 1$. Here $t$ is the parameter on $[0; 1]$.

We can choose the interpolating function $\Psi_\lambda(t)$ to coincide with the unperturbed Kuranishi map outside the tubular neighborhood $B^{r/2}(\text{sing}_c)$ of the singular locus and to be independent of $t$ in that region. We can also choose $\Psi_\lambda(t)$ to have constant corank $b_c$.

This proves the existence of the required cobordism and the part four of the
theorem.

The above construction actually proves the part five as well since the cokernel of derivative of $\Psi_{\lambda}(t)$ is the desired bundle extension.

At this point it is time to state the following

Conjecture. If $M$ is a projective algebraic manifold then any of algebraic resolutions of singularities of $\bar{M}_d$ considered as a Hilbert Scheme can be obtained by the above described Kuranishi-type construction.

We are not going to rely on this conjecture anywhere in the present paper.

The proof of the above conjecture will after certain work give us equivalence between the definition (8.2) of the quantum cup-product interpreted algebraically (or sheaf-theoretically) and the Gromov-type Ruan’s definition (3.3) which has symplectic origin.

We are going to work out these matters in a separate publication.

Let us choose some desingularization $\hat{M}_c$ of $\bar{M}_c$ and some smooth section $f_c$ of the vector bundle $F_c$ over this desingularization

We choose this section to be “regular” in the sense that its zero-set is a smooth $\text{dim}(\hat{M}_c) - 2b_c$-dimensional submanifold in $\hat{M}_c$

Assumption 8.6. We assume that “regularity at infinity” holds, i.e., $\hat{M}_c \cap f_c^{-1}(0) - \text{Map}_d$ has codimension two in $\hat{M}_c \cap f_c^{-1}(0)$

Definition D. \( <A; B; C >_q = \)

\[
= \sum_d q^d \sum_c \sum_{\varphi \in \hat{M}_c \cap \text{ev}_0^{-1}(\hat{A}) \cap \text{ev}_2^{-1}(\hat{B}) \cap \text{ev}_1^{-1}(\hat{C}) \cap f_c^{-1}(0)} \pm 1 \quad (8.5)
\]

One can prove by the standard cobordism methods that the Definition D is independent of the choices of the pseudo-manifold representatives of $\hat{A}, \hat{B}, \hat{C}$ and the choices of the sections $\{f_c\}$.

The part 4) of the theorem 2.14 implies that the r.h.s. of (8.5) is also independent on the choice of desingularization $\hat{M}_c$

What requires a careful proof is

Theorem 8.7. For any choice of the desingularization $\hat{M}_c$ and the sections $\{f_c\}$ such that assumption 2.15 holds we have \( <A; B; C >_q = <A; B; C >_{qR} \)

To prove the Theorem 8.7 it is enough to prove that:
A) there exists a smooth cobordism $\mathcal{M}^t$ inside $\text{Map}_d$ between the manifolds $\mathcal{M}_{J,g,d}$ and $\bigcup_c \mathcal{M}_c \cap f_c^{-1}(0) \cap \text{Map}_d$

B) The compactification of $\mathcal{M}_{J,g,d}$ and of $\bigcup_c \mathcal{M}_c \cap f_c^{-1}(0) \cap \text{Map}_d$ can be extended to the compactification $\overline{\mathcal{M}}^t$ of the cobordism $\mathcal{M}^t$ (considered as a stratified space) such that “the compactification divisor” $\overline{\mathcal{M}}^t - \mathcal{M}^t$ has codimension at least two.

C) The finite-codimensional cycles $\text{ev}^{-1}_0(\hat{A})$; $\text{ev}^{-1}_\infty(\hat{B})$ and $\text{ev}^{-1}_1(\hat{C})$ in $\text{Map}_d$ intersect the cobordism $\mathcal{M}^t$ transversally

D) $\mathcal{M}^t \cap \text{ev}^{-1}_0(\hat{A}) \cap \text{ev}^{-1}_\infty(\hat{B}) \cap \text{ev}^{-1}_1(\hat{C})$ is a compact one-dimensional cobordism between $\mathcal{M}_{J,g,d} \cap \text{ev}^{-1}_0(\hat{A}) \cap \text{ev}^{-1}_\infty(\hat{B}) \cap \text{ev}^{-1}_1(\hat{C})$ and $\bigcup_c (\mathcal{M}_c \cap f_c^{-1}(0)) \cap \text{ev}^{-1}_0(\hat{A}) \cap \text{ev}^{-1}_\infty(\hat{B}) \cap \text{ev}^{-1}_1(\hat{C})$

This cobordism lies inside $\mathcal{M}^t$ and does not touch the compactification divisor $\overline{\mathcal{M}}^t - \mathcal{M}^t$

If we prove A) - D), we will be done since the statement D) already implies the Theorem 8.5.

Let us consider the Hilbert manifold $\text{Map}_d \times G_0 \times G_0 \times [0; 1]$

The general element of this space has the form $(\varphi; g_1; g_2; s)$

This space contains two submanifolds $\text{Map}_d \times G_0 \times \{0\} \times \{0\}$ and $\text{Map}_d \times \{0\} \times G_0 \times \{1\}$. Both of them are (canonically) diffeomorphic to $\text{Map}_d \times G_0$

We have a Hilbert bundle $\mathcal{H}$ over $\text{Map}_d \times G_0 \times G_0 \times [0; 1]$ and a canonical section $\partial = \partial_{J_0} - g$ of the bundle $\mathcal{H}$ restricted to $\text{Map}_d \times G_0 \times \{0\} \times \{0\}$.

The zero set of this section $\partial$ restricted to $\text{Map}_d \times \{g_1\} \times \{0\} \times \{0\}$ will be the moduli space $\mathcal{M}_{J_0,1,d}$

The section $\partial$ is “regular” which means that the image of the derivative of $\partial$ on its zero set is surjective linear operator.

Our strategy of constructing a cobordism will be to extend the section $\partial$ to some “regular” (in the sense defined above) section $\Phi$ over $\text{Map}_d \times G_0 \times G_0 \times [0; 1]$ such that:

A) The zero set of $\Phi$ restricted to $\text{Map}_d \times \{0\} \times \{g_2\} \times \{1\}$ will be a union of smooth submanifolds inside some desingularization of $\mathcal{M}_d$ (in the sense defined above).

B) For each irreducible component $\mathcal{M}_c$ of $\mathcal{M}_d$ there exists a section $f_c$ of the “bundle of the fermion zero-modes” $\mathcal{F}_c$ over $\mathcal{M}_c$ such that
The sections \( \{ f_c \} \) will depend on the choice of \( g_2 \) and vary smoothly as \( g_2 \in G_0 \) varies.

We already have a canonical section \( \bar{\partial} = \bar{\partial}_{J_0} - g \) of the bundle \( H \) considered as a bundle over \( Map_d \times G_0 \).

We can take a pull-back \( \Phi_1 \) of this section to \( Map_d \times G_0 \times G_0 \times [0; 1] \) with respect to the projection \( Map_d \times G_0 \times G_0 \times [0; 1] \to Map_d \times G_0 \) on the product of the first two factors.

By construction of the section \( \Phi_1 \) (which is the ordinary \( \bar{\partial} \)-operator if restricted to \( Map_d \times \{ 0 \} \times G_0 \times \{ 0 \} \)) we have

\[
\Phi^{-1}(0) \cap \{ Map_d \times \{ 0 \} \times G_0 \times \{ 0 \} \} = \mathcal{M}_d \times \{ 0 \} \times G_0 \times \{ 0 \}
\]

For any irreducible component \( \hat{\mathcal{M}}_c \) of desingularized moduli space \( \hat{\mathcal{M}}_d \) we have a finite-dimensional bundle \( F_c \) (of fermion zero-modes) over it. This finite-dimensional bundle is constructed as a quotient of an infinite-dimensional Hilbert bundle \( H \) restricted to \( \hat{\mathcal{M}}_c \). Using the inner product in the fibers of \( H|_{\hat{\mathcal{M}}_c} \) we can think of \( F_c \) as a subbundle of \( H \).

Let us consider a tubular neighborhood of the finite-dimensional manifold \( \mathcal{M}_c \) inside the infinite-dimensional manifold \( Map_d \). Let us denote this tubular neighborhood \( \mathcal{M}_c^\delta \). Inside this tubular neighborhood let us consider some smaller tubular neighborhood of the desingularized component \( \hat{\mathcal{M}}_c \). Let us denote this smaller tubular neighborhood \( \hat{\mathcal{M}}_c^\delta \).

We assume that the tubular neighborhoods \( \{ \hat{\mathcal{M}}_c^\delta \} \) of different irreducible components \( \hat{\mathcal{M}}_c \) do not intersect each other. (We can make this assumption by the part three of the theorem 8.5).

By definition of “perturbed Kuranishi maps” the zero locus of which defines “the finite part of our desingularization” \( \mathcal{M}_c \cap Map_d \) we can match together different “perturbed Kuranishi maps” \( \{ \Psi_\lambda \} \) defined as sections of finite-dimensional bundles \( H_{<\lambda} \) on the corresponding charts. The result will be a section \( \Psi \) of the infinite-dimensional Hilbert bundle \( H \) defined over \( \mathcal{M}_c^\delta \).

The finite part of the desingularization \( \hat{\mathcal{M}}_c \cap Map_d \) is (by its definition) a zero-locus of this “perturbed section” \( \Psi \) defined over \( \mathcal{M}_c^\delta \).

Since the normal bundle to \( \hat{\mathcal{M}}_c \cap Map_d \) in \( Map_d \) (as any other infinite-dimensional Hilbert bundle) is trivial, the tubular neighborhood \( \hat{\mathcal{M}}_c^\delta \) is dif-
feomorphic to a product of $\hat{\mathcal{M}}_c \cap \text{Map}_d$ and an infinite-dimensional ball. Let $\pi : \hat{\mathcal{M}}_c^\delta \to \hat{\mathcal{M}}_c$ be the corresponding projection.

**Lemma 8.8.** We can extend the subbundle $F_c \subset H$ from $\hat{\mathcal{M}}_c$ to $\hat{\mathcal{M}}_c^\delta$ such that:

A) $F_c$ will be a smooth $b_c$-dimensional subbundle in $\hat{\mathcal{M}}_c^\delta$

B) For any point $\varphi \in \hat{\mathcal{M}}_c^\delta$, the fiber $F_c|_\varphi$ will be a $b_c$-dimensional subspace in $H|_\varphi$ which is orthogonal to the value of the section $\Psi$ at the point $\varphi$.

Since any two infinite-dimensional Hilbert bundles over any topological space are isomorphic to each other (and trivial), let us trivialize the bundle $H$ over $\hat{\mathcal{M}}_c$. The bundle $H|_{\hat{\mathcal{M}}_c^\delta}$ will thus be isomorphic to the bundle $\pi^*(H|_{\hat{\mathcal{M}}_c})$.

Let us fix some isomorphism between these two Hilbert bundles and let $F'_c$ be the image of $\pi^*(F_c|_{\hat{\mathcal{M}}_c})$ under this isomorphism.

So, we have constructed some $b_c$-dimensional subbundle $F'_c$ over $\hat{\mathcal{M}}_c^\delta$.

Then over $\hat{\mathcal{M}}_c^\delta$ we have a codimension-one-subbundle $H' \subset H$ whose fiber at the point $\varphi$ is defined as orthogonal complement to the one-dimensional subspace generated by the value of the section $\Psi$ at the point $\varphi$. We can also define an operator $\pi'$ of orthogonal projection onto $H' \subset H$.

Let us put $\tilde{F}_c = \pi'(F'_c)$ to be the subbundle in $H$ defined over $\hat{\mathcal{M}}_c^\delta - \hat{\mathcal{M}}_c$.

We claim that the subbundle $\tilde{F}_c$ can be smoothly extended to $\hat{\mathcal{M}}_c$ if we put it equal to $F_c$ there.

Now let us check that this bundle has the properties required by the Lemma 8.8.

The property B of the Lemma 8.8 is obvious by construction.

Since $F_c$ is orthogonal to the image of the derivative of the section $\Psi$, the neighboring to $\hat{\mathcal{M}}_c$ fibers of the bundle $F_c$ are “closed to be orthogonal” to the image of $\Psi$. This implies that $\tilde{F}_c$ smoothly extends to $\hat{\mathcal{M}}_c$ (which is the zero-set of $\Psi$ in $\mathcal{M}_c^\delta$)

The property A) and the Lemma 8.8 follows.

Now it is time to define “a perturbed section” $\Phi$ over $\text{Map}_d \times \{0\} \times G_0 \times \{1\}$

For each irreducible component $\mathcal{M}_c$ of $\mathcal{M}_d$ we have a $b_c$-dimensional subbundle $\tilde{F}_c$ in $H|_{\hat{\mathcal{M}}_c^\delta}$

Let us take some “cut-off function” $h$ on the positive real axis which is identically one on $[0; \delta/2]$ and identically zero on $[\delta; +\infty)$
Then for any \( g_2 \in G_0 \) we can:

1) Take a section \( g_2 \) of \( H|_{\tilde{\mathcal{M}}^\delta_c} \)

2) Take a projection \( f_c(g_2) \) of this section to the subbundle \( \tilde{\mathcal{F}}_c \subset H \) to have a smooth section of \( \tilde{\mathcal{F}}_c \)

3) Multiply the above section of \( \tilde{\mathcal{F}}_c \) by a cut-off function \( h(||\varphi - M_c||) \) to have another smooth section of a subbundle \( \tilde{\mathcal{F}}_c \) in \( H \) which can be extended (by zero) from \( \tilde{\mathcal{M}}^\delta_c \) to \( Map_\delta \) as a some smooth section of \( H \). Let us denote this section \( (g_2)_c \)

4) Take the sum \( \tilde{g}_2 = \sum_c (g_2)_c \) over different components of \( M_d \)

If we put \( \Phi = \Psi - \tilde{g}_2 \) as a section of \( H \) over \( Map_\delta \times \{0\} \times \{g_2\} \times \{1\} \) and consider it as a section of \( H \) over \( Map_\delta \times \{0\} \times G_0 \times \{1\} \) (as \( g_2 \in G_0 \) varies) then this section will be “regular” by construction.

Here, as usual, “regular” means that the image of derivative of the section \( \Phi \) is surjective over its zero-set.

This section \( \Phi \) can be extended from \( Map_\delta \times \{0\} \times G_0 \times \{1\} \) to \( Map_\delta \times G_0 \times G_0 \times \mathbb{R} \) by the formula

\[
\Phi = (1 - s)\bar{\partial}J_0 + s\Psi - g_1 - \tilde{g}_2 
\]  \hspace{1cm} (8.6)

**Lemma 8.9.** The section \( \Phi \) is regular

**Lemma 8.10.** The zero-set of \( \Phi \) restricted to \( Map_\delta \times \{0\} \times G_0 \times \{1\} \) lies inside

\( \tilde{\mathcal{M}}_d \times \{0\} \times G_0 \)

**Lemma 8.11.** \( \Phi^{-1}(0) \cap \mathcal{M}_c \times \{0\} \times \{g_2\} \times \{1\} \) is the zero-set of the section \( f_c(g_2) \) of the bundle \( \mathcal{F}_c \)

**Lemma 8.12.** For a generic choice of \( g_2 \in G_0 \) all the sections \( \{f_c(g_2)\} \) of the bundles \( \{\mathcal{F}_c\} \) are “regular” and their zero-sets are smooth manifolds of dimension equal to \( \text{vdim}[\mathcal{M}_d] \).

**Assumption 8.13.** If \( g_1 + g_2 \) lies in \( G_0 \) then the zero-submanifolds \( \Phi^{-1}(0) \cap Map_\delta \times \{g_1\} \times \{g_2\} \times \{1\} \) can be compactified as stratified spaces such that their “compacification divisors” have codimension at least two.

This “assumption” can be proved using estimates on norms of our “perturbed \( \bar{\partial} \)-operators”

Using Lemmas 2.4 and 2.6 we have that there exists a smooth cobordism \( \mathcal{M}^t \) inside \( Map_\delta \times G_0 \times G_0 \times \mathbb{R} \) between the manifolds

\( \mathcal{M}_{J,g_1,d} \times \{g_1\} \times \{0\} \times \{0\} \) and \( \{\mathcal{M}_d \cap f^{-1}(0)\} \times \{0\} \times \{g_2\} \times \{1\} \)
Assuming that 8.13 holds we have that this cobordism can be compactified and “the compactification divisor” $\mathcal{M}^t - M^t$ has codimension at least two in the total space of the cobordism.

Thus, we have established the statements A and B after the theorem 8.7. Now let us observe that the statement D (and the Theorem 8.7 itself) will follow from the statement C. But the statement C follows from the Lemma 2.5.

The Theorem 8.7 is proved.

The above proof of the Theorem 8.7 is very similar to the proof of Ruan [Ru2] of the following fact (we state it using our notation):

Let us take a non-regular almost-complex structure $J$ on $M$ satisfying Assumptions 8.1 - 8.3. Let us assume that we can be perturbed $J$ inside the space of almost-complex structures on $M$ (without going to $CP^1 \times M$) such that “generic” perturbation is regular, then the r.h.s. of (8.5) and the r.h.s. of (3.3) are equal as formal power series in $\{q_i\}$.

Here we take the expression (8.5) for a a non-regular almost-complex structure $J$. The expression (3.3) is taken for a regular generic perturbation of $J$ inside the space of almost-complex structures on $M$ (if such a perturbation exists).

The advantage of our approach is that it handles the problem what to do if there are no regular perturbations of the almost-complex structure $J$. Such situation almost always happens when $J$-holomorphic curve $\varphi$ is a multiple branched cover of some other $J$-holomorphic curve of lower degree (see [McD1] for the best exposition of this analytic problem).

9. DISCUSSION

The Main Theorem 6.1, proved in the present paper can be thought as mathematical implementation of the program of Vafa [Va] of understanding quantum cohomology through geometry of the loop space. The notion of “BRST-quantization on the loop space” considered by string theorists (see [Wi1] for the best treatment), can be put in the mathematically rigorous framework of symplectic Floer homology.

If we are studying geometry of Kahler manifold $M$ from the point of view of the string propagating on it, we can extract more algebrogeometrical information on $M$ than is contained in its quantum cohomology ring $HQ^*(M)$.

The String Theory on $M$ also provides us with
A) Deformation of the classical cohomology ring $H^*(M)$ with respect to all (and not just two-dimensional) cohomology classes

B) Some explicitly constructed cohomology classes of the moduli spaces of punctured curves known as Gromov-Witten classes

We recommend the reader a very interesting recent paper by Kontsevich and Manin [KM] with new developments in “quantum cohomology” (in this broader sense) and applications of these new invariants to classical problems in algebraic geometry

During the preparation of the present paper we also received two very interesting papers by Fukaya [Fu1],[Fu2] with new developments in Floer homology. What Fukaya did is that he constructed analogues of the classical Massey products in Floer homology of Lagrangian intersections. In order to construct these “quantized Massey products”, Fukaya used the loop space generalization of a finite-dimensional Morse-theoretic construction, which was not known before.

These new results together with the work of Cohen-Jones-Segal [CJS2] and Betz-Cohen [BK] give a hope to understand what is “quantum homotopy type” and “Floer homotopy type” of a semi-positive almost-Kahler manifold.

If our Kahler “manifold” $M$ is the moduli space of flat connections on a two-dimensional surface (which is actually a stratified space and not a manifold), symplectic Floer homology of this “manifold” is conjectured [A] to be isomorphic to instanton Floer homology of a circle bundle over this surface (see [DS],[Y] for the proof of this conjecture and [Ta2] for further developments).

The multiplicative structure in symplectic Floer homology corresponds under this isomorphism to relative Donaldson invariants of some 4-dimensional manifolds with boundary. Thinking about these relative Donaldson invariants as some matrix elements of quantum multiplication on the moduli space of flat connections we can interpret gluing formulas [BrD] and recursion relations [KrM] for Donaldson invariants as recursion relations coming from associativity of quantum multiplication.

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