EXISTENCE OF ARBITRARILY SMOOTH SOLUTIONS OF THE LLG EQUATION IN 3D WITH NATURAL BOUNDARY CONDITIONS

MICHAEL FEISCHL AND THANH TRAN

Abstract. We prove that the Landau-Lifshitz-Gilbert equation in three space dimensions with homogeneous Neumann boundary conditions admits arbitrarily smooth solutions, given that the initial data is sufficiently close to a constant function.

1. Introduction

The Landau-Lifshitz-Gilbert (LLG) equation is widely considered as a valid model of micromagnetic phenomena occurring in, e.g., magnetic sensors, recording heads, and magneto-resistive storage device [12, 14, 20]. It describes the precessional motion of magnetization in ferromagnets. The main difficulty of the LLG equation is its strongly non-linear character.

Classical results concerning existence and non-uniqueness of solutions can be found in [5, 22]. The existence of weak solutions is proved for 2D and 3D in [2]. It is known that weak solutions are in general not unique but exist globally. Throughout the literature, there are various works on weakly-convergent numerical approximation methods for the LLG (coupled to the Maxwell-equations) equations [2, 4, 6, 7, 9, 15, 16] (the list is not exhausted) even without an artificial projection step [1, 11].

This paper considers the question of existence of arbitrarily smooth strong solutions of this equation. For the case of the 2D torus, the book [20] gives an exhaustive overview on results concerning the existence and regularity of strong solutions. A brief summary of the state of the art for 2D domains with periodic boundary conditions could be phrased as follows: There exist arbitrarily smooth solutions provided that the initial data is sufficiently close to a constant function. Moreover, there exist arbitrarily smooth local-in-time solutions for initial data of finite energy (see, e.g., [13]). For the 3D case, much less is known in terms of strong solvability. For the 3D torus (with periodic boundary conditions) [8] proves $H^2$-regularity local in time for the coupled system of LLG and Maxwell-equations. The work [3] proves global existence of strong solutions for small initial energies on small ellipsoids. The survey article [21] summarizes results in the context of the evolution of harmonic maps (which however does not cover the LLG equation). A recent paper [19] studies the existence, uniqueness and asymptotic behavior of solutions in the whole spatial space $\mathbb{R}^3$.

To the authors best knowledge, this work is the first which proves existence of arbitrarily smooth (non-trivial) solutions on bounded 3D domains. It also gives a first result on existence of arbitrarily smooth strong solutions with natural boundary conditions (in 2D and 3D). It is worth mentioning that the proof is constructive in the sense that a convergent sequence of approximate solutions is designed algorithmically. The limit of this sequence turns out to be a smooth strong solution of the LLG equation.

The main motivation to prove existence of smooth strong solutions for the LLG equation originated in the recent work [11] by the authors. There, we proved a priori error
estimates for a time integrator for the LLG equations (as well as the coupled LLG-Maxwell system) which imply strong convergence of the numerical method in case of smooth strong solutions. Thus, the present work justifies the assumptions in [11].

2. The Landau-Lifshitz-Gilbert Equation

Consider a bounded smooth domain $D \subset \mathbb{R}^3$ with connected boundary $\Gamma$ having the outward normal vector $n$. Note that all the results in this paper also hold true for $D \subset \mathbb{R}^n$, $n \geq 2$. For brevity of presentation, however, we only consider the physically most relevant case $n = 3$. We define $D_T := (0, T) \times D$ and $\Gamma_T := (0, T) \times \Gamma$ for $T > 0$. We start with the LLG equation which reads as

$$m_t - \alpha m \times m_t = -C_e m \times \Delta m \quad \text{in } D_T$$

for some constant $C_e > 0$. Here the parameter $\alpha$ is a positive constant. It follows from eq. (1) that $|m|$ is constant. We follow the usual practice to normalize $|m|$. The following conditions are imposed on the solution of eq. (1):

$$\partial_n m = 0 \quad \text{on } \Gamma_T,$$

$$|m| = 1 \quad \text{in } D_T,$$

$$m(0, \cdot) = m^0 \quad \text{in } D,$$  

where $\partial_n$ denotes the normal derivative.

The initial data $m^0$ satisfies $|m^0| = 1$ in $D$. The condition eq. (2b) together with basic properties of the cross product leads to the following equivalent formulation of eq. (1):

$$\alpha m_t + m \times m_t = C_e \Delta m - C_e (m \cdot \Delta m)m \quad \text{in } D_T.$$  

Before stating the main result of the article, we set some notations. Bold letters (e.g. $v$) will be used for vector functions. However, as there is no confusion, we still use $L^2(D_T)$ to denote the Lebesgue space of vector functions taking values in $\mathbb{R}^3$, i.e., we will write $v \in L^2(D_T)$ instead of $v \in L^2(D_T)^3$. The same rule applies to other function spaces.

The following function spaces will be frequently used. For any non-negative integer $k \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, we define

$$H^{k, 2k}(D_T) := \{v \in L^2(D_T) : \|v\|_{H^{k, 2k}(D_T)} < \infty\}$$

where the norm is defined by

$$\|v\|_{H^{k, 2k}(D_T)} := \sum_{\ell=0}^{k} \|\partial_\ell^T v\|_{L^2(0,T;H^{2\ell-2k}(D))}.$$  

The corresponding seminorm is

$$|v|_{H^{k, 2k}(D_T)} := \left( \sum_{\ell=1}^{2k} \|D^\ell v\|_{L^2(D_T)}^2 \right)^{1/2} + \sum_{\ell=1}^{k} \|\partial_\ell^T v\|_{L^2(0,T;H^{2\ell-2k}(D))},$$

where $D^\ell$ denotes $\ell$th-order partial derivatives with respect to the spatial variables.

Finally, we define

$$H^1_\Delta(D) := \{v \in H^1(D) : \Delta v \in L^2(D) \quad \text{and} \quad \partial_n v = 0 \text{ on } \Gamma\}.$$  

We are now ready to state the main result of the paper.

**Theorem 1.** Assume that the initial data $m^0$ satisfies $|m^0| = 1$ in $D$ and, for some integer $k \geq 3$, 

(i) \( m^0 \in H^{2k}(D) \cap H^1(D) \);
(ii) \( D^j m^0 \in H^1(D) \) for all \( j/2 \leq k - 1 \);
(iii) \( \|m^0\|_{H^{2k}(D)} \) is sufficiently small.

Then the problem eq. (1)–eq. (2) has a smooth strong solution \( m \in H^{k,2k}(D_T) \) which satisfies

\[
\|m\|_{H^{k,2k}(D_T)} \leq C_{\text{smooth}} \|m^0\|_{H^{2k}(D)}, \tag{5}
\]

where \( C_{\text{smooth}} > 0 \) depends only on \( \alpha, C_\epsilon, T, \) and \( k \).

### 3. Auxiliary Results

For the reader’s convenience, we state in the following lemma some well-known results regarding Sobolev embeddings and traces.

**Lemma 2.**

(i) The embeddings \( H^1(D) \hookrightarrow L^6(D) \) as well as \( H^{1,2}(D_T) \hookrightarrow L^2(0,T;L^\infty(D)) \cap L^\infty(0,T;L^2(D)) \) are continuous.

(ii) The embedding \( H^{k+2,2k+4}(D_T) \hookrightarrow W^{k,\infty}(D_T) \) is continuous for all \( k \in \mathbb{N}_0 \).

(iii) If \( w \in H^{k,2k}(D_T) \) for \( k \geq 1 \) then \( \partial_t^i D^j w(0) \in H^1(D) \) for all \( i + j/2 \leq k - 1 \).

**Proof.** We first prove (i). The embedding \( H^1(D) \hookrightarrow L^6(D) \) follows from the standard Sobolev inequality. By definition of \( H^{1,2}(D_T) \), there holds

\[ H^{1,2}(D_T) := H^1(0,T;L^2(D)) \cap L^2(0,T;H^2(D)). \]

The well-known embeddings \( H^1(0,T;L^2(D)) \hookrightarrow L^\infty(0,T;L^2(D)) \) and \( L^2(0,T;H^2(D)) \hookrightarrow L^2(0,T;L^\infty(D)) \) (since \( D \subset \mathbb{R}^3 \)) conclude (i).

Second, we prove (ii). Since \( D \subset \mathbb{R}^3 \), it is well-known that the embeddings

\[ H^1(0,T;H^{\ell+2}(D)) \hookrightarrow H^1(0,T;W^{\ell,\infty}(D)) \hookrightarrow L^\infty(0,T;W^{\ell,\infty}(D)) \]

are continuous for any \( \ell \geq 0 \); see e.g. [17]. On the other hand, we can write

\[ W^{k,\infty}(D_T) = \{ v : \partial_t^i v \in L^\infty(0,T;W^{k-i,\infty}(D)), \ i = 0,\ldots,k \}. \]

Hence the embedding

\[ \{ v : \partial_t^i v \in H^1(0,T;H^{k-i+2}(D)), \ i = 0,\ldots,k \} \hookrightarrow W^{k,\infty}(D_T) \]

is continuous. Consequently, the embedding

\[ \bigcap_{i=0}^k H^{i+1}(0,T;H^{k-i+2}(D)) \hookrightarrow W^{k,\infty}(D_T) \]

is continuous. Since \( H^{k+2,2k+4}(D_T) \subset \bigcap_{i=0}^k H^{i+1}(0,T;H^{k-i+2}(D)) \), part (ii) is proved.

Statement (iii) can be derived from [10, Theorem 4, Section 5.9.2, p. 288] as follows:

\[
\|\partial_t^i D^j w(0)\|_{H^1(D)} \lesssim \|\partial_t^i D^j w\|_{L^2(0,T;H^2(D))} + \|\partial_t^{i+1} D^j w\|_{L^2(0,T;L^2(D))} \\
\lesssim \|\partial_t^i w\|_{L^2(0,T;H^{i+2}(D))} + \|\partial_t^{i+1} w\|_{L^2(0,T;H^i(D))} \\
\lesssim \|w\|_{H^{k,2k}(D_T)}
\]

if \( i + j/2 \leq k - 1 \) and \( k \geq 1 \). The lemma is proved. \( \square \)

The following lemma states some useful inequalities involving the norm and seminorm of \( H^{k,2k}(D_T) \).

**Lemma 3.** Let \( v, w, \mathbf{v} \), and \( \mathbf{w} \) be scalar and vector functions in \( H^{k,2k}(D_T) \) for \( k \geq 2 \).
(i) If \(i, j \in \mathbb{N}_0\) satisfy \(0 < m = \lceil i + j/2 \rceil \leq k\) then \(\partial_t^i D^j v \in H^{k-m,2k-2m}(D_T)\) and
\[
\|\partial_t^i D^j v\|_{H^{k-m,2k-2m}(D_T)} \leq C|v|_{H^{k,2k}(D_T)}.
\] (6)

(ii) Furthermore, \(vw, vw, v \times w, v \cdot w\), and \(|v|^2 - |w|^2\) belong to the corresponding space \(H^{k,2k}(D_T)\) and satisfy
\[
\begin{align*}
\|vw\|_{H^{k,2k}(D_T)} &\leq C\|v\|_{H^{k,2k}(D_T)}\|w\|_{H^{k,2k}(D_T)}, \\
\|vw\|_{H^{k,2k}(D_T)} &\leq C\|v\|_{H^{k,2k}(D_T)}\|w\|_{H^{k,2k}(D_T)}, \\
\|v \times w\|_{H^{k,2k}(D_T)} &\leq C\|v\|_{H^{k,2k}(D_T)}\|w\|_{H^{k,2k}(D_T)}, \\
\|v \cdot w\|_{H^{k,2k}(D_T)} &\leq C\|v\|_{H^{k,2k}(D_T)}\|w\|_{H^{k,2k}(D_T)}, \\
\||v|^2 - |w|^2\|_{H^{k,2k}(D_T)} &\leq C(\|v\|_{H^{k,2k}(D_T)} + \|w\|_{H^{k,2k}(D_T)})\|v - w\|_{H^{k,2k}(D_T)}.
\end{align*}
\] (7)

The constant \(C > 0\) depends only on an upper bound of \(\ell\) and on \(D_T\).

**Proof.** To see eq. (6), we use the definition of the \(H^{k,2k}(D_T)\)-norm and write
\[
\|\partial_t^i D^j v\|_{H^{k-m,2k-2m}(D_T)} = \sum_{\ell=0}^{k-m} \|\partial_t^{\ell+i} D^j v\|_{L^2(0,T;H^{2k-2m-2\ell}(D))}
= \sum_{\ell=i}^{k-m+i} \|\partial_t^i D^j v\|_{L^2(0,T;H^{2k-2m-2\ell}\times(D))}.
\]

Since \(m = \lceil i + j/2 \rceil\), we have
\[
k - m + i \leq k \quad \text{and} \quad 2k - 2m + 2i + j \leq 2k.
\] (8)

Hence, if \(i > 0\) then
\[
\begin{align*}
\|\partial_t^i D^j v\|_{H^{k-m,2k-2m}(D_T)} &\leq \sum_{\ell=1}^{k} \|\partial_t^\ell v\|_{L^2(0,T;H^{2k-2m-2\ell+i+j}(D))} \\
&\leq \sum_{\ell=1}^{k} \|\partial_t^\ell v\|_{L^2(0,T;H^{2k-2\ell}(D))} \leq |v|_{H^{k,2k}(D_T)}.
\end{align*}
\]

If \(i = 0\) then \(1 \leq j \leq 2k\) (as \(0 < m \leq k\)) and thus
\[
\begin{align*}
\|\partial_t^i D^j v\|_{H^{k-m,2k-2m}(D_T)}
&= \sum_{\ell=0}^{k-m} \|\partial_t^\ell D^j v\|_{L^2(0,T;H^{2k-2m-2\ell}(D))} \\
&\leq \|D^j v\|_{L^2(0,T;H^{2k-2m}(D))} + \sum_{\ell=1}^{k} \|\partial_t^\ell v\|_{L^2(0,T;H^{2k-2m-2\ell+j}(D))} \\
&= \left( \sum_{j'=0}^{2k-2m} \|D^{j+j'} v\|_{L^2(D_T)}^2 \right)^{1/2} + \sum_{\ell=1}^{k} \|\partial_t^\ell v\|_{L^2(0,T;H^{2k-2m-2\ell+j}(D))} \\
&= \left( \sum_{j'=j}^{2k-2m+j} \|D^{j'} v\|_{L^2(D_T)}^2 \right)^{1/2} + \sum_{\ell=1}^{k} \|\partial_t^\ell v\|_{L^2(0,T;H^{2k-2m-2\ell+j}(D))} \\
&\leq |v|_{H^{k,2k}(D_T)},
\end{align*}
\]
where in the last step we used eq. (8) and the definition of the seminorm.
We next show eq. (7a). The product rule implies
\[ \|vw\|_{H^{k,2k}(D_T)} = \sum_{\ell=0}^k \|\partial_t^\ell (vw)\|_{L^2(0,T;H^{2k-2\ell}(D))} \]
\[ \lesssim \sum_{\ell=0}^k \sum_{i_1+j_2=\ell} \left( \int_0^T \|\partial_t^{i_1} v (\partial_t^{j_2} w)\|^2_{H^{2k-2\ell}(D)} \, dt \right)^{1/2} \]
\[ \lesssim \sum_{\ell=0}^k \sum_{i_1+j_2=\ell} \sum_{n=0}^{2k-2\ell} \sum_{i_1+i_2=n} \left( \int_0^T \int_D |D^{i_1} \partial_t^{i_1} v|^2 |D^{i_2} \partial_t^{j_2} w|^2 \, dx \, dt \right)^{1/2}. \]

Note that
\[ \frac{i_1}{2} + j_1 + \frac{i_2}{2} + j_2 = \frac{n}{2} + \ell \leq k - \ell + \ell = k. \]

Hence, putting \( I := \{(i_1, i_2, j_1, j_2) \in \mathbb{N}_0 : \frac{i_1}{2} + j_1 + \frac{i_2}{2} + j_2 \leq k\} \) we obtain
\[ \|vw\|_{H^{k,2k}(D_T)} \lesssim \sum_{(i_1, i_2, j_1, j_2) \in I} \left( \int_0^T \int_D |D^{i_1} \partial_t^{i_1} v|^2 |D^{i_2} \partial_t^{j_2} w|^2 \, dx \, dt \right)^{1/2} \leq S_1 + S_2 + S_3, \]
where
\[ S_\nu := \sum_{(i_1, i_2, j_1, j_2) \in I_\nu} \left( \int_0^T \int_D |D^{i_1} \partial_t^{i_1} v|^2 |D^{i_2} \partial_t^{j_2} w|^2 \, dx \, dt \right)^{1/2}, \quad \nu = 1, 2, 3, \]
with
\[ I_1 := \{(i_1, i_2, j_1, j_2) \in I : \frac{i_1}{2} + j_1 \geq 1 \text{ and } \frac{i_2}{2} + j_2 \geq 1\}, \]
\[ I_2 := \{(i_1, i_2, j_1, j_2) \in I : \frac{i_1}{2} + j_1 = 0 \text{ or } \frac{i_2}{2} + j_2 = 0\}, \]
\[ I_3 := \{(i_1, i_2, j_1, j_2) \in I : \frac{i_1}{2} + j_1 = 1/2 \text{ or } \frac{i_2}{2} + j_2 = 1/2\}. \]

Each term in \( S_1 \) is estimated by using the Hölder inequality separately in space and time as
\[ S_1 \leq \sum_{(i_1, i_2, j_1, j_2) \in I_1} \left( \int_0^T \|D^{i_1} \partial_t^{i_1} v(t)\|^2_{L^\infty(D)} \|D^{i_2} \partial_t^{j_2} w(t)\|^2_{L^2(D)} \, dt \right)^{1/2} \]
\[ \leq \sum_{(i_1, i_2, j_1, j_2) \in I_1} \|D^{i_1} \partial_t^{i_1} v\|_{L^2(0,T;L^\infty(D))} \|D^{i_2} \partial_t^{j_2} w\|_{L^\infty(0,T;L^2(D))} \]
\[ \leq \sum_{(i_1, i_2, j_1, j_2) \in I_1} \|D^{i_1} \partial_t^{i_1} v\|_{H^{1,2}(D_T)} \|D^{i_2} \partial_t^{j_2} w\|_{H^{1,2}(D_T)}, \]
where in the last step we used lemma 2 (i). Note that in this index set \( I_1 \) there hold \([i_1/2 + j_1] \leq k - 1\) and \([i_2/2 + j_2] \leq k - 1\). Hence, estimate eq. (6) gives
\[ \|D^{i_1} \partial_t^{i_1} v\|_{H^{1,2}(D_T)} \lesssim \|v\|_{H^{k,2k}(D_T)} \quad \text{and} \quad \|D^{i_2} \partial_t^{j_2} w\|_{H^{1,2}(D_T)} \lesssim \|w\|_{H^{k,2k}(D_T)}, \]
implies \( S_1 \lesssim \|v\|_{H^{k,2k}(D_T)} \|w\|_{H^{k,2k}(D_T)}. \)
The sum $S_2$ is estimated with the help of lemma 2 (ii) by

$$S_2 \leq \sum_{\ell=0}^{k} \sum_{i=0}^{2k-2\ell} \left( \int_0^T \int_D |D^i \partial_t^\ell w|^2 \, dx \, dt \right)^{1/2} + \sum_{\ell=0}^{k} \sum_{i=0}^{2k-2\ell} \left( \int_0^T \int_D |D^i \partial_t^\ell v|^2 \, dx \, dt \right)^{1/2} \lesssim \|v\|_{L^\infty(D_T)} \sum_{\ell=0}^{k} \sum_{i=0}^{2k-2\ell} \|\partial_t^\ell w\|_{L^2(0,T;H^i(D))} + \|w\|_{L^\infty(D_T)} \sum_{\ell=0}^{k} \sum_{i=0}^{2k-2\ell} \|\partial_t^\ell v\|_{L^2(0,T;H^i(D))} \lesssim \|v\|_{H^{k,2k}(D_T)} \|w\|_{H^{k,2k}(D_T)}.$$

Finally, for $S_3$, since the problem is symmetric, we just consider the case when $i_1 = 1$ and $j_1 = 0$. Since $H^1(D) \subseteq L^6(D) \subseteq L^4(D)$ (see lemma 2 (i)) we have

$$\left( \int_0^T \int_D |D^1 v|^2 |D^{i_2} \partial_t^{j_2} w|^2 \, dx \, dt \right)^{1/2} \leq \left( \int_0^T \|D^1 v(t)\|_{L^4(D)}^2 \|D^{i_2} \partial_t^{j_2} w(t)\|_{L^4(D)}^2 \, dt \right)^{1/2} \leq \|D^1 v\|_{L^\infty(0,T;L^4(D))} \|D^{i_2} \partial_t^{j_2} w\|_{L^2(0,T;L^4(D))} \lesssim \|D^1 v\|_{H^1(0,T;H^1(D))} \|D^{i_2} \partial_t^{j_2} w\|_{L^2(0,T;H^1(D))} \leq \|v\|_{H^{1,2k}(D_T)} \|w\|_{H^{k,2k}(D_T)} \leq \|v\|_{H^{1,2k}(D_T)} \|w\|_{H^{k,2k}(D_T)},$$

where in the penultimate step we used eq. (6), noting that $i_2/2 + j_2 < k$. This and the analogous result for $i_2 = 1$ and $j_2 = 0$ prove

$$S_3 \lesssim \|v\|_{H^{k,2k}(D_T)} \|w\|_{H^{k,2k}(D_T)}.$$

Altogether, we obtain eq. (7a).

The remaining multiplicative estimates eq. (7b)–eq. (7d) follow from eq. (7a) by the fact that all of them can be expressed as (sums of) products of scalar functions.

Finally, we show eq. (7e) by using the identity $|v|^2 - |w|^2 = (v + w) \cdot (v - w)$ and the already proved estimate eq. (7d). This concludes the proof. \hfill \square

The following lemma is a slight generalization to the vector case of a well-known result on the existence of solutions of the heat equation.

**Lemma 4.** Let $L: \mathbb{R}^3 \to \mathbb{R}^3$ denote a linear operator which satisfies

$$L a \cdot a \geq c_L |a|^2 \quad \text{for all } a \in \mathbb{R}^3,$$

for some $c_L > 0$. For a given $r \in L^2(D_T)$, the vector-valued heat equation

$$L \partial_t w - \Delta w = r \quad \text{in } D_T; \quad \partial_n w = 0 \quad \text{in } \{0\} \times D,$$

$$\partial_t w = 0 \quad \text{on } \Gamma_T$$

has a weak solution which satisfies

$$\|w\|_{H^{1,2}(D_T)} \leq C_{\text{heat}} \|r\|_{L^2(D_T)}.$$

The constant $C_{\text{heat}} > 0$ depends only on $T, L, \text{ and } D.$
The lemma.

This estimate and eq. (13) yield
\[ w_p \quad \text{for all} \quad w \in \mathbb{R}^3. \]  
(12)

Thus, we can reformulate eq. (10) into
\[ \partial_t w - L^{-1} \Delta w = L^{-1} r \quad \text{in} \ D_T, \]
\[ w = 0 \quad \text{in} \ \{0\} \times D, \]
\[ \partial_n w = 0 \quad \text{on} \ \Gamma_T. \]  
(13)

We want to use the result [17, Theorem 3.2]. To that end, and in the notation of [17], we define
\[ A := -L^{-1} \Delta \] and
\[ D(A) := \{ v \in H^2(D) : \partial_n v = 0 \text{ on} \ \Gamma \} \subseteq L^2(D). \]

Define the graph norm \( \| \cdot \|_{D(A)}^2 := \| \cdot \|_{L^2(D)}^2 + \| A(\cdot) \|_{L^2(D)}^2 \). Then, there holds for all \( p \in \mathbb{C} \) satisfying \( \text{Re}(p) \geq p_0 > 0 \) and for all \( v \in D(A) \)
\[ \| (A + p) v \|_{L^2(D)} \leq (1 + |p|) \| v \|_{D(A)} \]
as well as
\[ \| (A + p) v \|_{L^2(D)}^2 = \| Av \|_{L^2(D)}^2 - 2 \text{Re}(p \langle Av, v \rangle_D) + |p|^2 \| v \|_{L^2(D)}^2 \]
\[ = \| Av \|_{L^2(D)}^2 + 2 \text{Re}(p \langle L^{-1} \nabla v, \nabla v \rangle_D) + |p|^2 \| v \|_{L^2(D)}^2. \]

It follows from eq. (12) that
\[ 2 \text{Re}(p \langle L^{-1} \nabla v, \nabla v \rangle_D) \geq 2p_0 \frac{c_L}{\| \cdot \|_2^2} \| \nabla v \|_{L^2(D)}^2 \geq 0, \]
so that
\[ \| (A + p) v \|_{L^2(D)}^2 \geq \| Av \|_{L^2(D)}^2 + |p|^2 \| v \|_{L^2(D)}^2 \geq \min \{ 1, p_0^2 \} \| v \|_{D(A)}^2. \]

Standard elliptic regularity theory (see e.g. [18, Theorem 4.18]) shows that \( A + p : D(A) \rightarrow L^2(D) \) is surjective. Hence, \( A + p : D(A) \rightarrow L^2(D) \) is a bijective isomorphism. Moreover, we have for \( v \in L^2(D) \)
\[ \| (A + p)^{-1} v \|_{L^2(D)} \lesssim \frac{1}{1 + |p|^2} \| v \|_{L^2(D)}. \]

for all \( p \in \mathbb{C} \) satisfying \( \text{Re}(p) > p_0 \). Thus, the requirements of [17, Theorem 3.2] are satisfied which yields the existence of \( w \in L^2(0, T; D(A)) \) satisfying eq. (13) and hence also eq. (10).

Standard elliptic regularity theory (see e.g. [18, Theorem 4.18]) gives
\[ \| w \|_{H^2(D)} \lesssim \| \Delta w \|_{L^2(D)} + \| w \|_{H^1(D)} \quad \text{for all} \quad w \in D(A). \]

Since \( \| w \|_{H^1(D)} \lesssim \| \Delta w \|_{L^2(D)} + \| w \|_{L^2(D)} \) for all functions satisfying \( \partial_n w = 0 \), we deduce that \( w \in L^2(0, T; H^2(D)) \). The proof of [17, Theorem 3.2] also reveals
\[ \| w \|_{L^2(0, T; H^2(D))} \lesssim \| w \|_{L^2(0, T; D(A))} \lesssim \| L^{-1} r \|_{L^2(D_T)} \lesssim \| r \|_{L^2(D_T)}. \]

This estimate and eq. (13) yield \( \| \partial_t w \|_{L^2(0, T; L^2(D))} \lesssim \| r \|_{L^2(D_T)}, \) completing the proof of the lemma. \( \square \)

The next lemma is a result on higher regularity for solutions to eq. (10).
Lemma 5. Under the assumption of lemma 4, if \( r \in H^{k-1,2k-2}(D_T) \) for \( k \geq 2 \) satisfies
\[
\partial_i^j D^j r(0) \in H^1_s(D) \quad \text{for all } i + j/2 \leq k - 2,
\]
then the solution \( w \) of the vector-valued heat equation eq. (10) satisfies
\[
\|w\|_{H^{k,2k}(D_T)} \leq C_r \|r\|_{H^{k-1,2k-2}(D_T)}
\]
and
\[
\partial_t^i D^j w(0) \in H^1_s(D) \quad \text{for all } i + j/2 \leq k - 1.
\]

Proof. We first recall that if \( r \in H^{k-1,2k-2}(D_T) \) then \( \partial_t^i D^j r(0) \in H^1(D) \) for \( i + j/2 \leq k - 2 \); see lemma 2. The proof is an induction on \( k \in \mathbb{N} \), where lemma 4 confirms the case \( k = 1 \). Let \( k > 1 \) and assume that eq. (14) and eq. (15) hold for \( k - 1 \). Then, differentiation reveals that \( v := \partial_t w - L^{-1} r(0) \) is the unique solution of
\[
L \partial_t v - \Delta v = \partial_t r + L^{-1} \Delta r(0) \quad \text{in } D_T,
\]
\[ v = 0 \quad \text{in } \{0\} \times D, \]
\[ \partial_n v = 0 \quad \text{on } \Gamma_T. \]

The right-hand side \( \tilde{r} := \partial_t r + L^{-1} \Delta r(0) \) satisfies \( \partial_t^i D^j \tilde{r}(0) \in H^1_s(D) \) for all \( i + j/2 \leq k - 3 \). The induction hypothesis and lemma 3 show that
\[
\|v\|_{H^{k-1,2k-2}(D_T)} \lesssim \|\partial_t r + L^{-1} \Delta r(0)\|_{H^{k-1,2k-2}(D_T)} \lesssim \|r\|_{H^{k-1,2k-2}(D_T)}
\]
as well as
\[
\partial_t^i D^m v(0) \in H^1_s(D) \quad \text{for all } n + m/2 \leq k - 2.
\]
The definition of \( v \) and estimate eq. (16) imply
\[
\sum_{j=1}^k \|\partial_t^i w\|_{L^2(0,T;H^{2k-2j}(D))} \lesssim \|v\|_{H^{k-1,2k-2}(D_T)} \lesssim \|r\|_{H^{k-1,2k-2}(D_T)}.
\]

Assume for the moment that \( w \) and \( r \) are smooth. Then, we have with elliptic regularity (see, e.g., [18, Theorem 4.18]) and \( -\Delta w = r - L \partial_t w \) that all \( 0 \leq t \leq T \) satisfy
\[
\|w(t)\|_{H^{2k}(D)} \lesssim \|r(t)\|_{H^{2k-2}(D)} + \|\partial_t w(t)\|_{H^{2k-2}(D)}.
\]
Integration over \( t \in (0,T) \) reveals for smooth \( w \) and \( r \)
\[
\|w\|_{L^2(0,T;H^{2k}(D))} \lesssim \|r\|_{H^{k-1,2k-2}(D_T)} + \|\partial_t w\|_{L^2(0,T;H^{2k-2}(D))}.
\]
A density argument now proves \( w \in L^2(0,T;H^{2k}(D)) \) with eq. (19) even for non-smooth \( w \). The combination of eq. (18) and eq. (19) shows
\[
\|w\|_{H^{k,2k}(D_T)} \lesssim \|r\|_{H^{k-1,2k-2}(D_T)}.
\]
To see \( \partial_t^i D^j w(0) \in H^1_s(D) \) for all \( i + j/2 \leq k - 1 \), we distinguish three cases: First, for \( i \geq 2 \), since \( \partial_t^i D^j w = \partial_t^{i-1} D^j v \), property eq. (17) gives with \( n = i - 1 \) and \( m = j \leq 2k - 2 - 2i \) that
\[
\partial_t^i D^j w(0) = \partial_t^i D^m v(0) \in H^1_s(D).
\]
Second, for \( i = 1 \), eq. (17) shows with \( n = 0 \) and \( m = j \leq 2k - 4 \) that
\[
D^m v(0) = \partial_t D^j w(0) - L^{-1} D^j r(0) \in H^1_s(D).
\]
Since \( D^j r(0) \in H^1_s(D) \) for all \( j/2 \leq k - 2 \) by definition, we obtain \( \partial_t D^j w(0) \in H^1_s(D) \) for all \( j/2 \leq k - 2 \). Finally, for \( i = 0 \), we have for \( D^j w(0) = 0 \in H^1_s(D) \) for \( j/2 \leq k - 1 \) by definition. Altogether, this proves \( \partial_t^i \Delta^j w(0) \in H^1_s(D) \) for all \( i + j/2 \leq k - 1 \) and thus concludes the proof.

□
The next technical result will be used to prove that the solution of some nonlinear parabolic problem satisfies condition eq. (2b) for all $t > 0$ if it satisfies that condition at $t = 0$.

**Lemma 6.** Let $u \in H^1(0,T;L^2(D))$ such that $u(t) \in W^{2,\infty}(D)$ for all $0 \leq t \leq T$ with $u|_{(0) \times D} = 1$ be a strong solution of

$$
\beta \partial_t u - u \Delta u = 0 \quad \text{in} \ D_T,
$$

$$
\partial_n u = 0 \quad \text{on} \ \Gamma_T
$$

for some constant $\beta > 0$. Then, there holds $u = 1$ in $D_T$.

**Proof.** Define $e := u - 1$. There holds

$$
\beta \partial_t e - e \Delta e - \Delta e = 0 \quad \text{and} \quad \partial_n e = 0 \text{ on } \Gamma_T.
$$

Multiplication by $e$ and integration by parts over $D$ shows

$$
\frac{\beta}{2} \partial_t \|e(t)\|^2_{L^2(D)} + \|\nabla e\|^2_{L^2(D)} \leq \|\Delta e(t)\|_{L^\infty(D)} \|e(t)\|^2_{L^2(D)} \lesssim \|e(t)\|^2_{L^2(D)},
$$

by use of the regularity assumptions for the last inequality. Thus, we have

$$
\partial_t \|e(t)\|^2_{L^2(D)} \lesssim \|e(t)\|^2_{L^2(D)} \quad \text{for all } 0 \leq t \leq T.
$$

Gronwall’s inequality proves $\|e(t)\|_{L^2(D)} \lesssim \|e(0)\|_{L^2(D)} = 0$, which concludes the proof. \(\square\)

We next define a residual operator which will be used to generate a sequence $\{m_k\}$ converging to a solution $m$ of eq. (1)–eq. (2).

**Definition 7.** Let $x_0$ be an arbitrary point in $D$ and $m^0$ be the initial data given in eq. (2c). For any $v \in H^{k,2k}(D_T)$ for some $k > 0$, we define the residual

$$
\mathcal{R}(v) := \alpha v_t + v \times v_t - C_e|v|^2 \Delta v - C_e|\nabla v|^2 v.
$$

We also define a linear operator $L: \mathbb{R}^3 \to \mathbb{R}^3$ by

$$
L a := L_{m^0(x_0)} a := \alpha a + m^0(x_0) \times a, \quad a \in \mathbb{R}^3.
$$

It is easy to see that $L$ satisfies eq. (9) with $c_L = \alpha$ and that

$$
\mathcal{R}(v) = \alpha v_t + v \times v_t - C_e \Delta v + C_e (1 - |v|^2) \Delta v - C_e |\nabla v|^2 v \\
= \alpha v_t + m^0(x_0) \times v_t + (v - m^0(x_0)) \times v_t - C_e \Delta v \\
+ C_e (1 - |v|^2) \Delta v - C_e |\nabla v|^2 v \\
= L v_t + (v - m^0(x_0)) \times v_t - C_e \Delta v + C_e (1 - |v|^2) \Delta v - C_e |\nabla v|^2 v,
$$

where $L$ is applied pointwise in time and space.

The following lemma gives some mapping properties of the operator $\mathcal{R}$. (We recall the definition of $H^1_+(D)$ in eq. (4).)

**Lemma 8.**

(i) The residual operator $\mathcal{R}$ defined in definition 7 is continuous from $H^{k,2k}(D_T)$ into $H^{k-1,2k-2}(D_T)$ for $k \geq 3$. More precisely, there holds

$$
\|\mathcal{R}(v) - \mathcal{R}(w)\|_{H^{k-1,2k-2}(D_T)} \leq C_R (1 + \|v\|^2_{H^{k,2k}(D_T)} + \|w\|^2_{H^{k,2k}(D_T)}) \|v - w\|_{H^{k,2k}(D_T)}.
$$
(ii) For $k \geq 3$, if $w \in H^{k,2k}(D_T)$ satisfies
\[
\partial_t^i D^j w(0) \in H^1_+(D) \quad \text{for all } i + j/2 \leq k - 1,
\]
then
\[
\partial_t^i D^j R(w)(0) \in H^1_+(D) \quad \text{for all } i + j/2 \leq k - 2.
\]

Proof. Statement (i) is proved by using lemma 3 (which is applicable because $k \geq 3$) as follows:
\[
\begin{align*}
\|R(v) - R(w)\|_{H^{k-1,2k-2}(D_T)} & \leq \|\partial_t (v - w)\|_{H^{k-1,2k-2}(D_T)} + \|v \times \partial_t (v - w)\|_{H^{k-1,2k-2}(D_T)} + \|v \times v\|_{H^{k-1,2k-2}(D_T)} + \|v \Delta (v - w)\|_{H^{k-1,2k-2}(D_T)} + \|v \cdot \nabla (v - w)\|_{H^{k-1,2k-2}(D_T)} \\leq (1 + \|v\|_{H^{k,2k}(D_T)} + \|w\|_{H^{k,2k}(D_T)})^2 \|v - w\|_{H^{k,2k}(D_T)}.
\end{align*}
\]

To prove (ii) we note that since $R(w) \in H^{k-1,2k-2}(D_T)$ for $k \geq 3$, lemma 2 gives $\partial_t^i D^j R(w)(0) \in H^1(0)$ for all $i + j/2 \leq k - 2$. It remains to show that $\Delta \partial_t^i D^j R(w)(0) \in L^2(D)$ and that the normal derivative of $\partial_t^i D^j R(w)(0)$ is zero. It is easy to see from the definition eq. (20) of $R$ and the product rule that for $i + j/2 \leq k - 2$, the derivative $\partial_t^i D^j R(w)$ is a sum of terms of the form
\[
(\partial_t^{n_1} D^{m_1} v_1) \otimes (\partial_t^{n_2} D^{m_2} v_2) \otimes (\partial_t^{n_3} D^{m_3} v_3) \tag{23}
\]

with $n_1 + n_2 + n_3 + (m_1 + m_2 + m_3)/2 \leq k - 1$ and $v_s \in \{w, 1\}$, $s = 1, 2, 3$, where $\otimes_1$ and $\otimes_2$ denote either the scalar, dot, or crossproduct. Thus at least 2 elements in the set $\{(n_1, m_2), (n_2, m_2), (n_3, m_3)\}$ satisfy $n_i + m_i/2 \leq (k - 1)/2$. Without loss of generality we assume $i = 2, 3$. Lemma 3 gives
\[
\partial_t^{n_i} D^{m_i} w \in H^{k-[(k-1)/2],2(k-[(k-1)/2])}(D_T) \subseteq H^{2,4}(D_T)
\]
(because $k \geq 3$). Lemma 2 (iii) and (i) imply
\[
D^2(\partial_t^{n_i} D^{m_i} w(0)) = \partial_t^{n_i} D^{m_i+2} w(0) \in H^1(0) \subseteq L^6(D) \subseteq L^4(D),
\]
and thus
\[
\partial_t^{n_i} D^{m_i} w(0), \partial_t^{n_i} D^{m_i+1} w(0) \in H^2(D) \subseteq L^\infty(D), \quad i = 1, 2.
\]
The product rule shows (with the definition $\Delta^{1/2} := D^1$) that $\Delta \partial_t^i D^j R(w)(0)$ is a sum of terms of the form
\[
(\partial_t^{n_1} \Delta^{r_1} D^{m_1} v_1(0)) \otimes (\partial_t^{n_2} \Delta^{r_2} D^{m_2} v_2(0)) \otimes (\partial_t^{n_3} \Delta^{r_3} D^{m_3} v_3(0)) \tag{24}
\]
with \( r_s \in \{0, 1/2, 1\}, \ s = 1, 2, 3, \text{ satisfying } r_1 + r_2 + r_3 = 1.\) This and the considerations above together with the assumption \( \partial_t^2 D^{m} v_1(0) \in H^1_{\ast}(D) \) show

\[
(\partial_t^{r_1} \Delta^{r_1} D^{m_1} v_1(0), \partial_t^{r_2} \Delta^{r_2} D^{m_2} v_2(0), \partial_t^{r_3} \Delta^{r_3} D^{m_3} v_3(0))
\]

\[
\in \begin{cases}
    L^2(D) \times L^\infty(D) \times L^\infty(D) & \text{for } r_1 \in \{1/2, 1\}, \ r_2, r_3 \in \{0, 1/2\}, \\
    H^1(D) \times L^4(D) \times L^\infty(D) & \text{for } r_1 = 0, \ r_2 = 1, \ r_3 = 0, \\
    H^1(D) \times L^\infty(D) \times L^4(D) & \text{for } r_1 = 0, \ r_2 = 0, \ r_3 = 1, \\
    H^1(D) \times L^\infty(D) \times L^\infty(D) & \text{for } r_1 = 0, \ r_2 = r_3 = 1/2,
\end{cases}
\]

\[
\in \begin{cases}
    L^2(D) \times L^\infty(D) \times L^\infty(D) & \text{for } r_1 \in \{1/2, 1\}, \ r_2, r_3 \in \{0, 1/2\}, \\
    L^4(D) \times L^4(D) \times L^\infty(D) & \text{for } r_1 = 0, \ r_2 = 1, \ r_3 = 0, \\
    L^4(D) \times L^\infty(D) \times L^4(D) & \text{for } r_1 = 0, \ r_2 = 0, \ r_3 = 1, \\
    L^2(D) \times L^\infty(D) \times L^\infty(D) & \text{for } r_1 = 0, \ r_2 = r_3 = 1/2.
\end{cases}
\]

Hence the product eq. (24) is in \( L^2(D). \) This implies that \( \Delta \partial_t^2 D^j \mathcal{R}(w)(0) \in L^2(D). \) Moreover, the normal derivatives of each factor of eq. (23) are zero by definition, and thus the product rule implies that also \( \partial_t^2 D^j \mathcal{R}(w)(0) = 0, \) completing the proof of the lemma.

The following lemma gives sufficient conditions for a given function \( m \) satisfying \( \mathcal{R}(m) = 0 \) to be a solution to eq. (1)–eq. (2).

**Lemma 9.** If \( m \in H^{k,2k}(D_T) \) for \( k \geq 3 \) satisfies

\[
\begin{align*}
\mathcal{R}(m) &= 0 \text{ on } D_T, \\
\partial_n m &= 0 \text{ on } \Gamma_T, \\
|m| &= 1 \text{ on } \{0\} \times D, \\
m(0, \cdot) &= m^0 \text{ in } D,
\end{align*}
\]

(25)

then \( m \) is a strong solution to eq. (1)–eq. (2).

**Proof.** It suffices to show that \( m \) satisfies eq. (2b) and eq. (3). The first property is shown by invoking lemma 6. To this end, let \( u := |m|^2, \) lemma 2 (ii) shows \( m \in W^{1,\infty}(D_T) \) and hence \( u \in W^{1,\infty}(D_T). \) lemma 2 (iii) proves \( m(t) \in H^5(D) \subseteq W^{2,\infty}(D), \) which implies \( u(t) \in W^{2,\infty}(D) \) for all \( 0 \leq t \leq T. \) Moreover, by using

\[
\Delta|m|^2 = 2\Delta m \cdot m + 2|\nabla m|^2,
\]

(26)

together with eq. (20) and eq. (25) we obtain

\[
\frac{\alpha}{2} \partial_t u - \frac{C_e}{2} u \Delta u = \frac{\alpha}{2} \partial_t |m|^2 - \frac{C_e}{2} |m|^2 \Delta|m|^2
\]

\[
= \alpha m_t \cdot m - C_e |m|^2 \Delta m \cdot m - C_e |\nabla m|^2 |m|^2
\]

\[
= \mathcal{R}(m) \cdot m = 0 \quad \text{in } D_T.
\]

Assumption eq. (25) also implies \( \partial_n u = \partial_n |m|^2 = 2\partial_n m \cdot m = 0 \) on \( \Gamma_T. \) Hence, lemma 6 yields \( u = 1 \) in \( D_T, \) i.e. eq. (2b) holds, which in turn together with \( \mathcal{R}(m) = 0 \) implies

\[
\alpha m_t + m \times m_t = C_e \Delta m + C_e |\nabla m|^2 m.
\]

It follows from eq. (26) that \( |\nabla m|^2 = -\Delta m \cdot m \) so that \( m \) satisfies eq. (3), completing the proof of the lemma.

Finally, since \( \mathcal{R} \) is not linear, we need the following lemma to estimate \( \mathcal{R}(v - w). \)
Lemma 10. Let $v, w \in H^{k,2k}(D_T)$ for $k \geq 3$. Then, there holds
\[
\|\mathcal{R}(v - w)\|_{H^{k-1,2k-2}(D_T)} \lesssim \|\mathcal{R}(v) - (L \partial_t - C_e \Delta)w\|_{H^{k-1,2k-2}(D_T)} \\
+ \|v - m^0(x_0)\|_{H^{k-1,2k-2}(D_T)} \|w\|_{H^{k,2k}(D_T)} \\
+ \|(1 - |v|^2)\Delta w\|_{H^{k-1,2k-2}(D_T)}^2 \\
+ \|w\|_{H^{k,2k}(D_T)} \|v\|_{H^{k,2k}(D_T)}(1 + \|v\|_{H^{k,2k}(D_T)}) \\
+ \|w\|_{H^{k,2k}(D_T)}^2 (1 + \|v\|_{H^{k,2k}(D_T)}) + \|w\|_{H^{k,2k}(D_T)}^3.
\] (27)
The hidden constant depends only on $C_e$ and on the constants from lemma 3.

Proof. It can be easily derived from eq. (22) that
\[
\mathcal{R}(v - w) - \mathcal{R}(v) = -(L \partial_t - C_e \Delta)w + (v - m^0(x_0)) \times w_t + w \times (v_t - w_t) \\
- C_e (1 - |v - w|^2) \Delta w - C_e (|w|^2 - 2(v \cdot w)) \Delta v \\
+ C_e (|\nabla v - \nabla w|^2 v - C_e (|\nabla w|^2 - 2 \nabla v \cdot \nabla w) v,
\]
so that
\[
\mathcal{R}(v - w) = \mathcal{R}(v) - (L \partial_t - C_e \Delta)w + T_1 + \cdots + T_6.
\]
Hence
\[
\|\mathcal{R}(v - w)\|_{H^{k-1,2k-2}(D_T)} \\
\lesssim \|\mathcal{R}(v) - (L \partial_t - C_e \Delta)w\|_{H^{k-1,2k-2}(D_T)} + \sum_{i=1}^6 \|T_i\|_{H^{k-1,2k-2}(D_T)}.
\]
Denoting $T_i = \|T_i\|_{H^{k-1,2k-2}(D_T)}$, lemma 3 yields
\[
T_1 \lesssim \|v - m^0(x_0)\|_{H^{k-1,2k-2}(D_T)} \|w\|_{H^{k,2k}(D_T)} \\
\leq \|v - m^0(x_0)\|_{H^{k-1,2k-2}(D_T)} \|w\|_{H^{k,2k}(D_T)},
\]
\[
T_2 \lesssim \|w\|_{H^{k-1,2k-2}(D_T)} \|v\|_{H^{k,2k}(D_T)} + \|w\|_{H^{k,2k}(D_T)} \\
\leq \|v\|_{H^{k,2k}(D_T)} \|w\|_{H^{k,2k}(D_T)} + \|w\|_{H^{k,2k}(D_T)}^2,
\]
\[
T_3 \lesssim \|(1 - |v|^2)\Delta w\|_{H^{k-1,2k-2}(D_T)} \|v\|_{H^{k,2k}(D_T)} + \|w\|_{H^{k,2k}(D_T)} \|w\|_{H^{k,2k}(D_T)} \\
+ \|w\|_{H^{k-1,2k-2}(D_T)} \|w\|_{H^{k,2k}(D_T)}^2 + \|w\|_{H^{k,2k}(D_T)}^3 \\
\leq \|(1 - |v|^2)\Delta w\|_{H^{k-1,2k-2}(D_T)} + \|v\|_{H^{k,2k}(D_T)} \|w\|_{H^{k,2k}(D_T)} + \|w\|_{H^{k,2k}(D_T)}^3,
\]
\[
T_4 \lesssim \|w\|_{H^{k-1,2k-2}(D_T)} \|v\|_{H^{k,2k}(D_T)} + \|w\|_{H^{k,2k}(D_T)} \|w\|_{H^{k-1,2k-2}(D_T)} \|v\|_{H^{k,2k}(D_T)} \\
\leq \|v\|_{H^{k,2k}(D_T)} \|w\|_{H^{k,2k}(D_T)} + \|v\|_{H^{k,2k}(D_T)} \|w\|_{H^{k,2k}(D_T)} \|w\|_{H^{k,2k}(D_T)} \\
+ \|v\|_{H^{k,2k}(D_T)} \|w\|_{H^{k,2k}(D_T)} \|w\|_{H^{k,2k}(D_T)} \|w\|_{H^{k,2k}(D_T)} \\
\leq \|v\|_{H^{k,2k}(D_T)} \|w\|_{H^{k,2k}(D_T)} + \|v\|_{H^{k,2k}(D_T)} \|w\|_{H^{k,2k}(D_T)} \|w\|_{H^{k,2k}(D_T)} + \|w\|_{H^{k,2k}(D_T)}^3,
\]
\[
T_5 \lesssim \|w\|_{H^{k,2k}(D_T)} \|w\|_{H^{k-1,2k-2}(D_T)} + \|v\|_{H^{k,2k}(D_T)} \|w\|_{H^{k,2k}(D_T)} \|w\|_{H^{k,2k}(D_T)} \|w\|_{H^{k,2k}(D_T)} \\
+ \|w\|_{H^{k,2k}(D_T)} \|w\|_{H^{k,2k}(D_T)} \|w\|_{H^{k,2k}(D_T)} \|w\|_{H^{k-1,2k-2}(D_T)} \\
\leq \|v\|_{H^{k,2k}(D_T)} \|w\|_{H^{k,2k}(D_T)} + \|v\|_{H^{k,2k}(D_T)} \|w\|_{H^{k,2k}(D_T)} \|w\|_{H^{k,2k}(D_T)} + \|w\|_{H^{k,2k}(D_T)}^3,
\]
\[
T_6 \lesssim \|w\|_{H^{k,2k}(D_T)} \|v\|_{H^{k-1,2k-2}(D_T)} + \|v\|_{H^{k,2k}(D_T)} \|w\|_{H^{k,2k}(D_T)} \|w\|_{H^{k,2k}(D_T)} \|w\|_{H^{k,2k}(D_T)} \\
\leq \|v\|_{H^{k,2k}(D_T)} \|w\|_{H^{k,2k}(D_T)} \|w\|_{H^{k,2k}(D_T)} + \|v\|_{H^{k,2k}(D_T)} \|w\|_{H^{k,2k}(D_T)} \|w\|_{H^{k,2k}(D_T)} \|w\|_{H^{k,2k}(D_T)}.
Collecting all the terms we obtain the desired estimates, completing the proof. □

4. Proof of the Main Result

This is a constructive proof. Starting with the initial guess \( m_0(t, x) := m^0(x) \) for all \((t, x) \in D_T,\) we define a sequence \((m_\ell)_{\ell \in \mathbb{N}_0} \) as follows. Having defined \( m_\ell, \ell = 0, 1, 2, \ldots, \) the construction involves the following tasks:

- Define \( r_\ell := \mathcal{R}(m_\ell), \)
- Solve
  \[
  L\partial_t R_\ell - C_\varepsilon \Delta R_\ell = r_\ell \quad \text{in } D_T, \\
  \partial_n R_\ell = 0 \quad \text{on } \Gamma_T, \\
  R_\ell = 0 \quad \text{on } \{0\} \times D.
  \]
- Define \( m_{\ell + 1} := m_\ell - R_\ell. \)

First we note that the above iteration is well-defined. Indeed, the assumptions on the initial data \( m^0 \) imply that the initial guess \( m_0 \) belongs to \( H^{k, 2k}(D_T) \) and satisfies \( \partial_i D^j m_0(0) \in H^1(D) \) for all \( i + j/2 \leq k - 1. \) Lemmas 5 and 8 then imply that \( R_0 \) also has the same smoothness properties, and so does \( m_1. \) By repeating the same argument, all functions \( m_\ell \) have the same smoothness properties as \( m_0, \) and the sequence \( \{m_\ell\} \) is well constructed. Next we note that, \( m_\ell|_{\{0\} \times D} = m_0|_{\{0\} \times D} = m^0 \) and \( \partial_n m_\ell = \partial_n m_0 = 0 \) for all \( \ell \in \mathbb{N}. \) Note also that due to lemma 5

\[
\|R_\ell\|_{H^{k, 2k}(D_T)} \leq C_\varepsilon \|r_\ell\|_{H^{k-1, 2k-2}(D_T)}, \quad \ell = 0, 1, 2, \ldots \tag{28}
\]

We will show that the sequence \((m_\ell)_{\ell \in \mathbb{N}_0}\) converges to a function \( m. \) lemma 8 then yields the convergence of \( \mathcal{R}(m_\ell) \) to \( \mathcal{R}(m) \) as \( \ell \to \infty. \) lemma 9 will then be used to conclude that \( m \) is a strong solution of eq. (1)–eq. (2).

To show that \( \{m_\ell\} \) is a Cauchy sequence we note that for \( 0 \leq \ell' \leq \ell \)

\[
\|m_\ell - m_{\ell'}\|_{H^{k, 2k}(D_T)} \leq \sum_{j=\ell'-1}^{\ell-2} \|R_{j+1}\|_{H^{k, 2k}(D_T)}. \tag{29}
\]

Denoting

\[
R_j := \|R_j\|_{H^{k, 2k}(D_T)}, \quad m_j := \|m_j\|_{H^{k, 2k}(D_T)}, \quad m_{j, 0} := \|m_j\|_{H^{k, 2k}(D_T)},
\]

in order to estimate each term in the sum on the right hand side of eq. (29) we use eq. (28) and invoke lemma 10 with \( v = m_j \) and \( w = R_j, \) noting that

\[
\mathcal{R}(m_j) = r_j = (L\partial_t - C_\varepsilon \Delta)R_j,
\]

to obtain

\[
R_{j+1} \lesssim \|r_{j+1}\|_{H^{k-1, 2k-2}(D_T)} = \|\mathcal{R}(m_{j+1})\|_{H^{k-1, 2k-2}(D_T)} = \|\mathcal{R}(m_j - R_j)\|_{H^{k-1, 2k-2}(D_T)} \leq R_j \|m_j - m^0(0)\|_{H^{k, 2k}(D_T)} + \|(1 - |m_j|^2)\|_{H^{k, 2k}(D_T)} \Delta R_j + \|R_j\|_{H^{k, 2k}(D_T)} + R_j m_{j, 0}(1 + m_j) + R_j^2(1 + m_j) + R_j^3.
\]

For the first term on the right hand side of eq. (30) we note that \( m_j(0, x_0) - m^0(x_0) = 0, \) and hence lemma 2 (i) yields (since \( k \geq 3 \))

\[
|m_j(t, x) - m^0(x)| \leq (\text{diam}(D)^2 + T^2)^{1/2} \|\partial_t^n, \nabla\| m_j \|_{L^\infty(D_T)} \lesssim \|\partial_t, \nabla\| m_j \|_{H^{k-1, 2k-2}(D_T)} \lesssim \|m_j\|_{H^{k, 2k}(D_T)}.
\]
This implies
\[ \| m_j - m^0(x_0) \|_{H^{k-1,2k-2}(D_T)} \leq \| m_j - m^0(x_0) \|_{L^2(D_T)} + |m_j|_{H^{k-1,2k-2}(D_T)} \lesssim m_{j,0}. \] (31)

For the second term on the right hand side of eq. (30), we first observe that since
\[ m_j = m_0 - \sum_{i=0}^{j-1} R_i \quad \text{and} \quad |m_0(t, \cdot)| = |m^0| = 1, \] (32)
there holds
\[ |m_j|^2 = |m_0|^2 - 2m_0 \cdot \sum_{i=0}^{j-1} R_i + |\sum_{i=0}^{j-1} R_i|^2 \]
so that
\[ 1 - |m_j|^2 = 2m_0 \cdot \sum_{i=0}^{j-1} R_i - |\sum_{i=0}^{j-1} R_i|^2. \]

Thus, with the help of lemma 3, we obtain
\[ \|(1 - |m_j|^2) \Delta R_j\|_{H^{k-1,2k-2}(D_T)} \lesssim R_j \left( m_0 \sum_{i=0}^{j-1} R_i + |\sum_{i=0}^{j-1} R_i|^2 \right). \] (33)

Altogether, eq. (30)–eq. (33) imply
\[ R_{j+1} \leq \tilde{C} R_j \left( (m_{j,0} + R_j)(1 + m_j) + R_j^2 + m_0 \sum_{i=0}^{j-1} R_i + |\sum_{i=0}^{j-1} R_i|^2 \right) \]
\[ =: \tilde{C} Q_j R_j, \] (34)
for some constant \( \tilde{C} > 0 \), where \( Q_j \) is the sum of all the terms in the brackets. We will show that for all \( q \in (0,1) \) there exists \( \varepsilon > 0 \) such that \( |m^0|_{H^{2k}(D)} \leq \varepsilon \) implies
\[ \tilde{C} Q_j \leq q \quad \text{for all} \ j \in \mathbb{N}_0. \] (35)

Given \( q \in (0,1) \) (and with the constants \( C_R \) from lemma 8 (i), and \( C_r \) from eq. (28)), we define \( C_{r,R} := C_r C_R (3 + 2|D| + |D|^{1/2}) \) and choose \( 0 < \varepsilon < 1 \) sufficiently small such that
\[ \varepsilon \left( 1 + C_{r,R} + \frac{C_{r,R}}{1 - q} \right) \left( 1 + \varepsilon + |D|^{1/2} + \frac{C_{r,R}}{1 - q} \right) + \left( C_{r,R} \varepsilon \right)^2 
\[ + \left( \varepsilon + |D|^{1/2} \right) \frac{C_{r,R} \varepsilon}{1 - q} + \frac{C_{r,R} \varepsilon}{1 - q} \right)^2 \leq \tilde{C}^{-1} q. \] (36)

This allows us to prove eq. (35) by induction. By assumption, \( |m^0|_{H^{2k}(D)} \) is sufficiently small such that
\[ m_{0,0} = |m^0|_{H^{2k}(D)} \leq \varepsilon \]
and
\[ \| m_0 - c \|_{H^{k,2k}(D)} = \| m^0 - c \|_{H^{2k}(D)} \leq C_{pc} |m^0|_{H^{2k}(D)} \leq \varepsilon \]
with \( c := |D|^{-1} \int_D m^0 \in \mathbb{R}^3 \) where the Poincaré constant \( C_{pc} > 0 \) depends only on \( D \). By definition, we have \( |c| = 1 \) and hence
\[ m_0 \leq \| m^0 - c \|_{H^{k,2k}(D)} + \| c \|_{H^{k,2k}(D)} \leq \varepsilon + |D|^{1/2}. \]
Moreover, since $\mathcal{R}(c) = 0$ we have, noting eq. (28),
\[
R_0 \leq C_r \|\mathcal{R}(m_0)\|_{H^{k-1,2k-2(D_T)}} = C_r \|\mathcal{R}(m_0) - \mathcal{R}(c)\|_{H^{k-1,2k-2(D_T)}} \\
\leq C_r C_R (1 + m_0^2 + \|c\|_{H^{k,2k(D_T)}}^2) \|m_0 - c\|_{H^{k,2k(D_T)}} \\
\leq C_r C_R (3 + 2|D| + |D|^{1/2}) \epsilon = C_r C_R \epsilon.
\]
Hence
\[
Q_0 = (m_{0,0} + R_0)(1 + m_0) + R_0^2 \leq \epsilon(1 + C_r C_R)(1 + |D|^{1/2} + \epsilon) + (C_r C_R \epsilon)^2.
\]
Our choice of $\epsilon$ guarantees $\tilde{C}Q_0 \leq q$. To conclude the induction, assume that $\tilde{C}Q_i \leq q$ for all $i = 0, \ldots, j-1$. Then the induction assumption and eq. (34) give
\[
R_j \leq qR_{j-1} \leq \cdots \leq q^j R_0 \leq q^j C_r C_R \epsilon,
\]
which implies
\[
\sum_{i=0}^{j-1} R_i \leq \sum_{i=0}^{j-1} q^i R_0 \leq \frac{C_r C_R \epsilon}{1-q}.
\]
Hence eq. (32) proves
\[
m_{j,0} \leq m_{0,0} + \sum_{i=0}^{j-1} R_i \leq \epsilon \left( 1 + \frac{C_r C_R}{1-q} \right)
\]
as well as
\[
m_j \leq m_0 + \sum_{i=0}^{j-1} R_i \leq \epsilon + |D|^{1/2} + \frac{C_r C_R \epsilon}{1-q}.
\]
It then follows from the definition of $Q_j$ and $\epsilon > 0$ that eq. (35) holds for all $j$. This concludes the induction and proves eq. (35) for all $j \in \mathbb{N}_0$.

We now prove that $\{m_\ell\}$ is a Cauchy sequence. It follows from eq. (29), eq. (37) that
\[
\|m_\ell - m_{\ell'}\|_{H^{k,2k(D_T)}} \leq \sum_{j=\ell-1}^{\ell-2} \sum_{i=0}^{j-1} q^{j+1} R_0 \leq \frac{C_r C_R \epsilon}{1-q} q^{\ell'} \to 0 \quad \text{as } \ell' \to \infty.
\]
Therefore, $\{m_\ell\}$ converges to some $m \in H^{k,2k(D_T)}$ which satisfies, by passing to the limit in the first inequality in eq. (38),
\[
\|m\|_{H^{k,2k(D_T)}} \leq \|m_0\|_{H^{k,2k(D_T)}} + \sum_{j=0}^{\infty} \|R_j\|_{H^{k,2k(D_T)}} \lesssim \|m_0\|_{H^{k,2k(D_T)}} + \frac{R_0}{1-q}.
\]
It remains to prove that $\mathcal{R}(m) = 0$, which can easily be seen from the continuity of $\mathcal{R}$ (see lemma 8) and the definition of $R_\ell$:
\[
\|\mathcal{R}(m)\|_{H^{k-1,2k-2(D_T)}} = \lim_{\ell \to \infty} \|\mathcal{R}(m_\ell)\|_{H^{k-1,2k-2(D_T)}} \\
= \lim_{\ell \to \infty} \|L \partial_t R_\ell - C_\epsilon \Delta R_\ell\|_{H^{k-1,2k-2(D_T)}} \lesssim \lim_{\ell \to \infty} \|R_\ell\|_{H^{k,2k(D_T)}} \\
\lesssim \lim_{\ell \to \infty} q^\ell = 0.
\]
As argued at the beginning of this proof, this shows that $m|_{D_T}$ is a strong solution of eq. (1).

Finally, to show eq. (5) we note that eq. (28), the continuity of $\mathcal{R}$, and the fact that $\mathcal{R}(0) = 0$ yield
\[
R_0 \lesssim \|r_0\|_{H^{k-1,2k-2(D_T)}} = \|\mathcal{R}(m_0) - \mathcal{R}(0)\|_{H^{k-1,2k-2(D_T)}} \lesssim \|m_0\|_{H^{k,2k(D_T)}}.
\]
Hence eq. (5) follows from eq. (39), completing the proof of the theorem.

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