ASYMPTOTIC BEHAVIORS OF KLOOSTERMAN SUMS

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Abstract. By virtue of Chebyshev polynomials and the Rademacher-Menchov device, we investigate the asymptotic behaviors of Kloosterman sums over short intervals by making use of the Gallagher-Montgomery refinement on D.A. Burgess' estimate for character sums, and an estimate analogous of N.M. Katz's Sato-Tate law is obtained. Subsequently, we can gain some information on the sign changes and extreme values of Kloosterman sums.

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1. Introduction

Kloosterman sums are a special kind of algebraic exponential sums of the form

\[ S(a,b;c) = \sum_{x \mod c}^* e\left(\frac{ax + bx}{c}\right), \]

where \(a, b, c\) are integers and \(c \geq 1\), \(x^c \equiv 1 \pmod{c}\). They play quite an important role in modern analytic number theory and automorphic forms (see [Iw] and [Mi3] for instance). It turns out that the most fascinating case is that \(c\) is an odd prime.

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For any fixed odd prime $p$, the best possible estimate is due to A. Weil [We]

$$|S(a, 1; p)| \leq 2p^{1/2}.$$ 

In view of this upper bound, we can write

$$\frac{S(a, 1; p)}{2p^{1/2}} =: \cos \theta_p(a),$$

where $0 \leq \theta_p(a) \leq \pi$ is called the Kloosterman sum angle. It is meaningful to investigate the distribution of the Kloosterman sum angles because of its profound applications in many problems of arithmetics and automorphic forms, and a comprehensive and professional introduction could be found in the survey article [Mi3].

An important progress on the investigation of the distribution of Kloosterman sum angles is due to N.M. Katz, who proved the celebrated equidistribution theorem, suggesting that the Kloosterman sum angles are equidistributed on $[0, \pi]$ relatively to the Sato-Tate measure, see Proposition 1 below for details. It turns out that this theorem has became the cornerstone in the investigation of numerous problems, and the aim of this paper is to illustrate this phenomenon. More precisely, we would like to investigate the asymptotic behaviors of Kloosterman sums over short intervals, and the outline will be presented in the next section.

**Notation.** As usual, we write $e(z) = e^{2\pi iz}$, $[x]$ denotes the largest integer not exceeding $x$, $\#X$ denotes the cardinality of a set, $\pi(x)$ denotes the number of primes up to $x$, $\binom{n}{m}$ is the binomial coefficient, $\Gamma(z)$ is the Gamma function, $\mu$ is the Möbius function, $\omega(n)$ denotes the number of distinct prime factors of $n$, $P^-(n)$ denotes the smallest prime factor of $n$ and assume $P^-(1) = +\infty$. $\sum^*$ denotes the summation over primitive elements, $\varepsilon$ denotes a small positive number which may be different at each occurrence. The Landau symbol $f = O(g)$ and Vinogradov’s notation $f \ll g$ are both understood as $|f| \leq cg$ for certain unspecified $c > 0$, $f \gg g$ means $g \ll f$, $f \asymp g$ means both $f \ll g$ and $f \gg g$ holds, and $f \sim g$ means $f/g \to 1$ as the limit of the variable proceeds. Moreover, we write $\ll_{\delta}$ if the implied constant in $\ll$ depends on the parameter $\delta$. The letter $c$ in the subsequent sections denotes an absolute positive number and we write $c(\delta_1, \delta_2, \cdots)$ for a constant $c$ depending on the parameters $\delta_1, \delta_2, \cdots$; of course, the constants are not necessary to be of the same value at different occurrences although usually denoted by the same letter.

**2. Backgrounds and Statement of Results**

**2.1. Katz’s equidistribution theorem and vertical Sato-Tate law.** We shall first introduce the Chebyshev polynomials $U_k(x)$ ($k \geq 0$), which are defined recursively by

$$U_0(x) = 1, \quad U_1(x) = 2x, \quad U_{k+1}(x) = 2xU_k(x) - U_{k-1}(x).$$

It is a remarkable progress that N.M. Katz [Ka] deeply investigated the so-called Kloosterman sheaves, in particular of their monodromy and proved basing on P. Deligne’s results [De] that

**Proposition 1.** For any positive integer $k$, we have

$$\left| \sum_{a \mod p}^* U_k(\cos \theta_p(a)) \right| \leq \frac{1}{2}(k + 1)p^{1/2}.$$
Then it follows that
\[
\lim_{p \to +\infty} \frac{1}{p - 1} \# \{1 \leq a \leq p - 1 : \alpha \leq \theta_p(a) \leq \beta\} = \frac{2}{\pi} \int_{\alpha}^{\beta} \sin^2 u \, du
\]
for any \([\alpha, \beta] \subseteq [0, \pi]\), which predicts the Kloosterman sum angles are equidistributed on \([0, \pi]\) relatively to the Sato-Tate measure \(\frac{2}{\pi} \sin^2 u \, du\). This statement is usually called the \emph{vertical Sato-Tate law} (VST).

The primary task of this paper is to investigate the \emph{short-interval} version of Proposition 1. To this end, we define
\[
\mathcal{D}_k(M, N; p, h) = \sum_{M < a \leq M + N} U_k(\cos \theta_p(ha)),
\]
where \(M\) is arbitrary, \(0 < N < p\), \(k\) is a positive integer and \((h, p) = 1\).

By the completing method or Fourier technique, one can show (combining the simplest case of Lemma 5 below) that
\[
\max_{(h, p) = 1} |\mathcal{D}_k(M, N; p, h)| \ll p^{1/2} \log p
\]
with an implied constant depending on \(k\). Definitely, this shows that
\[
\max_{(h, p) = 1} |\mathcal{D}_k(M, N; p, h)| = o(N)
\]
holds for \(N \geq p^{1/2+\varepsilon}\), which leads us to connect in mind with the Pólya-Vinogradov inequality for incomplete character sums that
\[
\sum_{M < a \leq M + N} \chi(a) \ll p^{1/2} \log p
\]
for any non-principal character \(\chi \mod p\). However, this is a trivial estimate unless \(N \geq p^{1/2+\varepsilon}\), in which case the inequality gives
\[
\left| \sum_{M < a \leq M + N} \chi(a) \right| = o(N).
\]

It has been such a barrier that nobody could beat until D.A. Burgess [Bu] showed half a century later that
\[
\sum_{M < a \leq M + N} \chi(a) \ll N^{1-1/r} p^{(r+1)/4r^2} \log p
\]
for any positive integer \(r\), which yields (2) holds for \(N \geq p^{1/4+\varepsilon}\). Furthermore, there are several refinements on Burgess’ argument, and the most remarkable ones are due to J. Friedlander & H. Iwaniec [FI] and recently P.X. Gallagher & H.L. Montgomery [GM].

In this paper, we shall make use of the Gallagher-Montgomery refinement and the device of H. Rademacher [Ra] and D. Menchov [Me] to show that
\[
\max_{(h, p) = 1} |\mathcal{D}_k(M, N; p, h)| = o(N)
\]
provided \(N \geq p^{1/4+\varepsilon}\). More precisely, write
\[
\omega_r(p, N) = N^{1-1/r} p^{(r+1)/4r^2} \log p,
\]
then we have
Theorem 1. Let $k$ be a fixed positive integer. For any positive integer $r$ and prime number $p$, we have
\[ \max_{(h,p)=1} |\mathcal{D}_k(M,N;h,p)| \ll k^2 \omega_r(p,N), \]
where the implied constant depends only on $r$.

In fact, Theorem 1 yields a short-interval version of VST, which we can call the local vertical Sato-Tate law (LVST). Hence we shall state that

Theorem 2. Let $M$ be an arbitrarily given number and $(h,p) = 1$. Then for any $[\alpha, \beta] \subseteq [0, \pi]$, we have
\[ \lim_{N \to +\infty} \frac{1}{N} \# \{ M < a \leq M + N : \alpha \leq \theta_p(a) \leq \beta \} = \frac{2}{\pi} \int_{\alpha}^{\beta} \sin^2 u \, du, \]
where the process of the limitation is also restricted to $N \geq p^\theta$ for certain $\theta > 1/4$.

2.2. Moments of Kloosterman sums over short intervals. By virtue of some basic properties of Chebyshev polynomials, P. Xi & Y. Yi [XY] obtained, by employing VST, several asymptotic formulae and upper bounds for the integral power mean values of $S(a,1;p)$ as $a$ runs over a primitive residue class mod $p$. Following the similar argument in [XY], we can apply Theorem 1 to investigate the mean values of Kloosterman sums over short intervals. Furthermore, the method in [XY] is also valid for non-integral moments. To this end, we define
\[ V_\alpha(M,N;p,h) = \sum_{M < a \leq M + N} S(a,h;p)^\alpha \]
and
\[ \widetilde{V}_\alpha(M,N;p,h) = \sum_{M < a \leq M + N} |S(a,h;p)|^\alpha \]
for $\alpha \in \mathbb{R}^+$ and $(h,p) = 1$.

We shall state without any proof that

Theorem 3. For any fixed $\alpha \in \mathbb{R}^+$ and each large prime number $p$ with $(h,p) = 1$, we have
\[ V_\alpha(M,N;p,h) = \frac{1 + (-1)^\alpha}{2} \frac{\Gamma(\alpha + 1)}{\Gamma(\frac{\alpha}{2} + 2)\Gamma(\frac{\alpha}{2} + 1)} Np^{\alpha/2} + O(p^{\alpha/2}\omega_r(p,N)), \]
and
\[ \widetilde{V}_\alpha(M,N;p,h) = \frac{\Gamma(\alpha + 1)}{\Gamma(\frac{\alpha}{2} + 2)\Gamma(\frac{\alpha}{2} + 1)} Np^{\alpha/2} + O(p^{\alpha/2}\omega_r(p,N)). \]
where the implied constants depend only on $r$.

The proof of Theorem 3 will rely on Lemma 2, however we will not give the details. One can see that Theorem 3 can provide asymptotic formulae for $N \geq p^{1/4+\epsilon}$.

2.3. Horizontal Sato-Tate conjecture and signs of Kloosterman sums. In parallel with VST, motivated by the Sato-Tate conjecture for elliptic curves, N.M. Katz proposed the following conjecture:

Conjecture 1. For any $[\alpha, \beta] \subseteq [0, \pi]$ and fixed integer $a$, we have
\[ \lim_{x \to +\infty} \frac{1}{\pi(x)} \# \{ p \leq x : (p,a) = 1, \alpha \leq \theta_p(a) \leq \beta \} = \frac{2}{\pi} \int_{\alpha}^{\beta} \sin^2 u \, du. \]
This is known as the horizontal Sato-Tate conjecture (HST) for Kloosterman sum angles. Of course, it is intimately connected with some non-trivial analytic information on the associated $L$-functions. Although it seems beyond the approach up to date, there are still some facts which encourage people to believe in this conjecture. Maybe the most remarkable ones are the equidistribution theorem for the angles of cubic Gauss sums due to D.R. Heath-Brown & S.J. Patterson [HP] and the equidistribution theorem for angles of Salié sums due to W. Duke, J. Friedlander & H. Iwaniec [DFI] (with respect to the natural Lebesgue measure).

One may realize the intractability of this conjecture since we still know little about the following question.

**Question 1.** Are there infinitely many primes such that $S(1, 1; p) \not\equiv 0$?

A brilliant consideration is due to E. Fouvry & P. Michel [FM], who investigated the mean value of Kloosterman sums over composite moduli with large prime factors by sieve methods and $\ell$-adic cohomology. They showed that there exists certain $\delta > 0$ such that

$$\sum_{c \leq X} \mu^2(2c) \frac{|S(1, 1; c)|}{c^{1/2}} \geq \frac{\delta X}{\log X}$$

and

$$\left| \sum_{c \leq X} \mu^2(2c) \frac{S(1, 1; c)}{c^{1/2}} \right| \leq (\delta - o(1)) \frac{X}{\log X}$$

with $\kappa = 23.9$, from which they deduce that there exist infinitely many $c$ with $\omega(c) \leq 23$ such that $\mu^2(2c)S(1, 1; c) \geq 0$. The subsequent improvements are due to J. Sivak-Fischler [SF1, SF2] who showed that $\kappa = 22.29$ and $\omega(c) \leq 18$ are admissible. The most recent result of this type is due to K. Matomäki [Ma], who showed that there exist infinitely many $c$ with $\omega(c) \leq 15$ such that $\mu^2(c)S(1, 1; c) \geq 0$.

As another approximation to Question 1, we have

**Theorem 4.** Let $M, N$ be positive numbers with $N \geq x^{1/4+\varepsilon}$. Then for any fixed $h$, there exist a positive proportion of $a \in (M, M+N]$ such that for each $a$ there are

$$\gg x/\log x$$

primes $p$ with $x < p \leq 2x, p \nmid h$ and $S(a, 1; p) \not\equiv 0$.

More precisely, for $N \geq x^{1/4+\varepsilon}$ we have

$$\sum_{M < a \leq M+N} \sum_{x < p \leq 2x} \frac{1}{S(a, h; p) \geq 0} \sim \frac{N x}{2 \log x}.$$

In fact, Theorem 4 is an immediate consequence of the following result.

**Theorem 5.** Let $M, N$ be arbitrary positive numbers and $(h, p) = 1$. Then for any $r \geq 1$ we have

$$\sum_{M < a \leq M+N} \sum_{S(a, h; p) \geq 0} 1 = \frac{1}{2} N + O(\omega_r(p, N)p^\varepsilon),$$

where the implied constant depends only on $r$ and $\varepsilon$. 
We make a remark here that one can obtain

$$\sum_{M < a \leq M + N \atop S(a, h; p) \geq 0} 1 \geq \left( \frac{4}{9\pi^2} - o(1) \right) N,$$

for any fixed $h$ with $(h, p) = 1$ and $N \geq p^{1/4+\varepsilon}$ by using Theorem 3, and this follows from the method used in [XY]. In fact, we expect the proportion of $a \mod p$ such that $S(a, 1; p) \geq 0$ is around 50%, unfortunately $\frac{4}{9\pi^2} < \frac{1}{2}$. Here we develop another approach to show that our expectation is acceptable and this will be completed in Section 5.

Moreover, in order to investigate HST on average, P. Michel [Mi2] succeeded in showing the following estimate.

**Proposition 2.** Let $\alpha, \beta$ be real numbers with $16/17 < \beta \leq 1$ and $1/2\beta < \alpha < 1/32(1 - \beta)$. Then for any complex numbers $\lambda = (\lambda_a)$ satisfying

$$\sum_{a \leq x^\alpha} |\lambda_a| \gg \delta A^{\beta/2 - \delta} \left( \sum_{a \leq A} |\lambda_a|^2 \right)^{1/2}$$

for any $\delta, A > 0$, we have

$$\sum_{a \leq x^\alpha} \lambda_a \sum_{x < p \leq 2x \atop p \nmid a} S(a, 1; p) \frac{1}{2p^{1/2}} \ll \varepsilon, \alpha, \beta x^{1-\varepsilon} \sum_{a \leq x^\alpha} |\lambda_a|.$$

Ideally, one expects to have the following much stronger result

$$\sum_{a \in I} \lambda_a \sum_{x < \rho \leq 2x \atop \rho \nmid a} U_k(\cos \theta_p(a)) \ll \varepsilon, k x^{1-\varepsilon} \sum_{a \in I} |\lambda_a|$$

for each $k \geq 1$ and any complex numbers $\lambda = (\lambda_a)$ with the interval $I$ as short as possible. In fact, Theorem 1 yields

**Corollary 1.** Let $M, N$ be arbitrary positive numbers with $N \geq p^{1/4+\varepsilon}$. Then there must exist a positive constant $C(\varepsilon)$ and a sufficiently small number $\delta = \delta(\varepsilon)$ such that

$$\left| \sum_{M < a \leq M + N} \sum_{x < \rho \leq 2x \atop \rho \nmid a} U_k(\cos \theta_p(a)) \right| \leq C(\varepsilon) k^2 N x^{1-\delta}$$

Definitely, Corollary 1 is much stronger than Proposition 2 in the special case $\lambda \equiv 1$. In fact, by partial summation, we can prove the following weighted version without any obstacle.

**Corollary 2.** With the same notation as in Corollary 1, for any differential function $\rho(x)$, we have

$$\left| \sum_{M < a \leq M + N} \rho(a) \sum_{x < \rho \leq 2x \atop \rho \nmid a} U_k(\cos \theta_p(a)) \right| \leq C(\varepsilon) k^2 \tilde{\rho}(M, N) N x^{1-\delta}$$

with

$$\tilde{\rho}(M, N) = \rho(M + N) + \int_M^{M+N} |\rho'(x)| dx.$$
2.4. Extremely small/large values of Kloosterman sums. Furthermore, Theorem 1 can also be employed to investigate the distribution of extreme values of Kloosterman sums. In fact, we have

**Theorem 6.** Let \( M, N \) be arbitrary positive numbers and \((h, p) = 1\). Then for any \( r \geq 1 \) and \( \delta = \delta(p) \) with \( 0 < \delta \leq 1 \), we have

\[
\sum_{M < a < M + N \atop |S(a, h; p)| \leq 2\delta p^{1/2}} 1 = 2 \left( \arcsin \delta + \delta \sqrt{1 - \delta^2} \right) N + O(1 + \omega_r(p, N)\delta p^\varepsilon),
\]

where the implied constant depends only on \( r \) and \( \varepsilon \). In particular, this provides an asymptotic formula for \( N \geq p^{1/4+2\varepsilon} \).

This idea of the proof is rather similar to that of Theorem 5, and we shall give the details in Section 6.

In the other direction, we can deduce that

**Theorem 7.** Let \( M, N \) be arbitrary positive numbers and \((h, p) = 1\). Then for any \( r \geq 1 \) and \( \delta = \delta(p) \) with \( 0 < \delta \leq 1 \), we have

\[
\sum_{M < a < M + N \atop |S(a, h; p)| \geq 2\delta p^{1/2}} 1 = 2 \left( \arccos \delta - \delta \sqrt{1 - \delta^2} \right) N + O(1 + \omega_r(p, N)\delta p^\varepsilon),
\]

where the implied constant depends only on \( r \) and \( \varepsilon \). In particular, this provides an asymptotic formula for \( N \geq p^{1/4+2\varepsilon} \).

We can see from Theorems 6 and 7 that for any fixed \( h \) with \((h, p) = 1\), there are indeed a positive proportion of \( a \mod p \) such that the value of \( \frac{1}{2p} S(a, h; p) \) can be extremely close to 0 or extremely close to 1, provided that \( a \) runs over an interval not too short.

It should be mentioned that P. Michel [Mi1] investigated the similar problem in the horizontal direction and showed that for each nonzero integer \( a \), there are a positive proportion of prime pairs \((p, q)\) such that

\[
\left| \frac{S(a, 1; pq)}{4\sqrt{pq}} \right| \geq 0.16,
\]

and a positive proportion of prime pairs \((p, q)\) such that

\[
\left| \frac{S(a, 1; pq)}{4\sqrt{pq}} \right| \leq \delta
\]

for any fixed \( \delta > 0 \).

Now summing over \( p \) with \( x < p \leq 2x \) with \( p \nmid h \) in Theorems 6 and 7, we can obtain that

\[
\sum_{M < a < M + N \atop p \nmid h} \sum_{x < p \leq 2x \atop |S(a, h; p)| \leq 2\delta p^{1/2}} 1 \gg \frac{\delta Nx}{\log x},
\]

\[
\sum_{M < a < M + N \atop p \nmid h} \sum_{x < p \leq 2x \atop |S(a, h; p)| \geq 2\delta p^{1/2}} 1 \gg \frac{(1 - \delta)Nx}{\log x}
\]
for $N \geq x^{1/4+\varepsilon}$ and $0 < \delta \leq 1$, thus we have

**Corollary 3.** Let $M, N$ be positive numbers with $N \geq x^{1/4+\varepsilon}$. Then for any fixed $h$ and $0 < \delta \leq 1$, there exist $\gg \delta N$ many $a \in (M, M + N]$ such that for each $a$ there are $\gg x/\log x$ primes $p$ with $x < p \leq 2x, p \nmid h$ and

$$|S(a, h; p)| \leq 2\delta p^{1/2},$$

and also there exist $\gg (1 - \delta) N$ many $b \in (M, M + N]$ such that for each $b$ there are $\gg x/\log x$ primes $p$ with $x < p \leq 2x, p \nmid h$ and

$$|S(b, h; p)| \geq 2\delta p^{1/2}.$$

### 3. Preliminary Lemmas

#### 3.1. Chebyshev polynomials

We shall present some basic properties of Chebyshev polynomials at first.

**Lemma 1.** Suppose $k, k_1, k_2 \geq 0$. Then

1. (Upper bound) We have $|U_k(x)| \leq k + 1$ uniformly in $x \in [-1, 1]$.
2. (Orthogonality) We have

$$\frac{2}{\pi} \int_{-1}^{1} \sqrt{1 - x^2} U_{k_1}(x) U_{k_2}(x) dx = \begin{cases} 1, & k_1 = k_2, \\ 0, & k_1 \neq k_2. \end{cases}$$

**Proof.** The orthogonality is well known and the upper bound can be seen from the identity

$$U_k(\cos \varphi) = \frac{\sin((k + 1)\varphi)}{\sin \varphi} = \sum_{j=0}^{k} (\cos \varphi)^{k-j} \cos(j\varphi)$$

and the trigonometric inequality. \(\square\)

Moreover, we expect that each nice function defined in $[-1, 1]$ can be represented by a linear combination of such Chebyshev polynomials. In particular, we have

**Lemma 2.** For any $\alpha \in \mathbb{R}^+$ and $x \in [-1, 1]$, we have

$$x^\alpha = \frac{1 + (-1)^\alpha}{2^{\alpha+1}} \frac{\Gamma(\alpha + 1)}{\Gamma(\frac{\alpha}{2} + 2)\Gamma(\frac{\alpha}{2} + 1)} + \sum_{\ell \geq 1} a_{\alpha, \ell} U_\ell(x)$$

and

$$|x|^\alpha = \frac{1}{2^\alpha} \frac{\Gamma(\alpha + 1)}{\Gamma(\frac{\alpha}{2} + 2)\Gamma(\frac{\alpha}{2} + 1)} + \sum_{\ell \geq 1} b_{\alpha, \ell} U_2\ell(x)$$

with

$$a_{\alpha, \ell} = (1 + (-1)^{\alpha+\ell}) \frac{\ell + 1}{2^{\alpha+1}} \frac{\Gamma(\alpha + 1)}{\Gamma(\frac{\alpha+\ell}{2} + 2)\Gamma(\frac{\alpha+\ell}{2} + 1)}$$

and

$$b_{\alpha, \ell} = \frac{2\ell + 1}{2^\alpha} \frac{\Gamma(\alpha + 1)}{\Gamma(\frac{\alpha}{2} + \ell + 2)\Gamma(\frac{\alpha}{2} - \ell + 1)}.$$

Moreover, if $\alpha$ is an odd positive integer, then $a_{\alpha, \ell}$ must vanish unless $\ell$ is odd and $\ell \leq \alpha$; if $\alpha$ is an even positive integer, then $a_{\alpha, \ell}$ must vanish unless $\ell$ is even and $\ell \leq \alpha$. 
Proof. The proof is similar to that of Lemma 2 in [XY]. Here we shall also use an identity (see [RG], 3.633)

$$\int_0^{\pi/2} \sin \theta \sin(\beta \theta) (\cos \theta)^\alpha d\theta = \frac{\beta \pi}{2^{\alpha+2}} \frac{\Gamma(\alpha+1)}{\Gamma(\frac{\alpha+\beta+3}{2}) \Gamma\left(\frac{\alpha-\beta+3}{2}\right)}.$$  

The last statement follows from the fact that $\frac{1}{\Gamma(s)}$ has zeros at non-positive integers. □

Furthermore, if $f$ is approximated by a series in terms of Chebyshev polynomials which is conditionally convergent, one should alternatively make use of its finite approximation. Hence we would like to state the following result.

**Lemma 3.** If $f$ has $A + 1$ continuous derivatives on $[a, b] \subseteq [-1, 1]$, then for any $L > 0$ we have

$$f(x) = \sum_{0 \leq \ell \leq L} \widehat{f}(\ell) U_\ell(x) + O(L^{-A})$$

uniformly in $x \in [a, b]$, where

$$\widehat{f}(\ell) = \frac{2}{\pi} \int_a^b \sqrt{1-x^2} f(x) U_\ell(x) dx.$$

**Proof.** This is essentially Theorem 5.14 in [MH]. □

**Lemma 4.** Let $\{U_{k_j}(x)\}_{j=1}^J$ be a finite sequence of Chebyshev polynomials with $k_j \geq 0$, and define $K = \sum_j k_j$. Then there exist a positive constant $c = c(J)$ and $\{\beta_\ell\}_{\ell \geq 0}$ with

$$\beta_\ell \leq \frac{c}{K+1} \prod_{1 \leq j \leq J} (k_j + 1),$$

such that

$$\prod_{1 \leq j \leq J} U_{k_j}(x) = \sum_{0 \leq \ell \leq K} \beta_\ell U_\ell(x).$$

Moreover, $\beta_0$ vanishes if $K$ is odd.

**Proof.** The left side of (5) can be regarded as an ordinary polynomial of degree $K$, then Lemma 2 yields it can be expressed as a linear combination of Chebyshev polynomials of degrees not exceeding $K$.

Now we turn to investigate the size of $\{\beta_\ell\}_{\ell \geq 0}$. In view of the orthogonality of Chebyshev polynomials, we have

$$\beta_\ell = \frac{2}{\pi} \int_{-1}^1 \sqrt{1-x^2} \prod_{1 \leq j \leq J} U_{k_j}(x) U_\ell(x) dx.$$  

Without loss of the generality, we assume $k_1 = \max_j k_j$. Thus we have

$$\beta_\ell \ll \left| \int_0^{\pi/2} \sin^2 \varphi \prod_{1 \leq j \leq l} U_{k_j}(\cos \varphi) U_\ell(\cos \varphi) d\varphi \right|$$

$$= \left| \int_0^{\pi/2} \sin((k_1+1)\varphi) \sin((\ell+1)\varphi) \prod_{2 \leq j \leq J} U_{k_j}(\cos \varphi) d\varphi \right|. $$
where the second inequality follows from the assumption that \( k_1 = \max_j k_j \), and the implied constant depends only on \( J \). This establishes (4). The vanishing of \( \beta_0 \) can be deduced from Lemma 2. \( \square \)

3.2. High-dimensional analog of Katz’s equidistribution theorem. In the investigation of the pseudorandomness of the signs of Kloosterman sums, E. Fouvry, P. Michel, J. Rivat & A. Sárközy [FMRS] proved a crucial estimate for the sum

\[
\mathcal{D}_k(f, s; p, h) = \sum_{a \mod p} \prod_{1 \leq i \leq s} U_{k_i}(\cos \theta_p(f_i(a))) e\left(\frac{ha}{p}\right),
\]

where \( s \) is a positive integer, \( f = (f_1, \ldots, f_s) \in \mathbb{Z}[x]^s \), \( k = (k_1, \ldots, k_s) \in \mathbb{N}^s \), and \( b \) denotes the summation restricted to those \( a \) with

\[
\prod_{1 \leq i \leq s} f_i(a) \not\equiv 0 \pmod{p}.
\]

This can be regarded as a high-dimensional analog of Katz’s equidistribution theorem and we shall quote it here.

**Lemma 5.** For every positive integer \( s \), there exists a constant \( c = c(s) \) such that

- for every \( s \)-tuple of polynomials \( f = (f_1, \ldots, f_s) \) of the form \( f_i(x) = a_i x + b_i \) with \( a_i, b_i \in \mathbb{Z}(1 \leq i \leq s) \), satisfying \( \prod_{1 \leq i \leq s} a_i \neq 0 \) and \( (a_i, b_i) \neq (a_j, b_j)(1 \leq i < j \leq s) \);
- for every prime \( p \), for every \( s + 1 \)-tuple of integers \( (k_1, \ldots, k_s, h) \), such that \( k_i \geq 0(1 \leq i \leq s) \) and \( (k_1, \ldots, k_s, h) \not\equiv (0, \ldots, 0, 0) \pmod{p} \),

we have the inequality

\[
|\mathcal{D}_k(f, s; p, h)| \leq c \prod_{1 \leq i \leq s} (k_i + 1)p^{1/2}.
\]

By virtue of Lemma 5, we can prove the following general result with a weaker constant depending on the orders of the Chebyshev polynomials.

**Lemma 6.** For every positive integer \( s \), there exists a constant \( c = c(s) \) such that

- for every \( s \)-tuple of polynomials \( f = (f_1, \ldots, f_s) \) of the form \( f_i(x) = a_i x + b_i \) with \( a_i, b_i \in \mathbb{Z}(1 \leq i \leq s) \), satisfying \( \prod_{1 \leq i \leq s} a_i \neq 0 \);
- for every \( s \)-tuple of integers \( k = (k_1, \ldots, k_s) \), such that \( k_i \geq 0(1 \leq i \leq s) \),
- \( p \nmid h \) or else if we split the polynomials \( f_i(x) \)’s into several groups such that the polynomials of the same values are in the same group, any two polynomials from different groups are distinct, and denote by \( \{g_j(x)\} \) the set of the representatives of these groups, then there exists at least one \( j \) with

\[
2 \nmid \sum_{\substack{1 \leq i \leq s \\backslash f_i(x) = g_j(x)\}} k_i,
\]

we have the inequality

\[
|\mathcal{D}_k(f, s; p, h)| \leq c \prod_{1 \leq i \leq s} (k_i + 1)^2p^{1/2}.
\]
Proof. First we can write
\[
\prod_{1 \leq i \leq s} U_{k_i} (\cos \theta_p (f_i (a))) = \prod_{1 \leq j \leq J} \prod_{1 \leq i \leq s \atop f_i = g_j} U_{k_i} (\cos \theta_p (g_j (a)))
\]
for certain \( J \). From Lemma 4 we have
\[
\prod_{1 \leq i \leq s} U_{k_i} (\cos \theta_p (f_i (a))) = \prod_{1 \leq j \leq J} \sum_{0 \leq \ell_j \leq K_j} \beta_{j, \ell_j} U_{\ell_j} (\cos \theta_p (g_j (a))),
\]
where
\[
K_j = \sum_{1 \leq i \leq s \atop f_i = g_j} k_i.
\]

If there exists a \( j_0 \) such that \( K_{j_0} \) is odd, then Lemma 2 yields \( \prod_j \beta_{j, 0} = 0 \). Moreover, these \( g_j \)'s are distinct, then it follows from Lemma 5 that
\[
|\mathcal{D}_k (f, s; p, h)| \leq \sum_{0 \leq \ell_j \leq K_1} \left( \prod_{1 \leq j \leq J} |\beta_{j, \ell_j}| \right) \left| \sum_{a \mod p} \prod_{1 \leq j \leq J} U_{\ell_j} (\cos \theta_p (g_j (a))) e \left( \frac{ha}{p} \right) \right|
\]
\[
\ll J p^{1/2} \prod_{1 \leq j \leq J} \sum_{0 \leq \ell_j \leq K_j} (\ell_j + 1)|\beta_{j, \ell_j}|.
\]

Now applying (4) to each \( \beta_{j, \ell_j} \), we get
\[
\mathcal{D}_k (f, s; p, h) \ll J p^{1/2} \prod_{1 \leq j \leq J} (K_j + 1) \prod_{1 \leq i \leq s \atop f_i = g_j} (k_i + 1)
\]
\[
\leq c(s) \prod_{1 \leq i \leq s} (k_i + 1)^2 p^{1/2}.
\]

If \( 2 \mid K_j \) for each \( j \), then \( \prod_j \beta_{j, 0} \neq 0 \), in which case we assume \( p \nmid h \). Hence we can apply Lemma 5 to derive the lemma. \( \square \)

3.3. **Equidistribution theorem on average — Gallagher-Montgomery’s approach.** Now we shall follow the approach of P.X. Gallagher & H.L. Montgomery \cite{GM} to prepare several results concerning the Kloosterman sum angles.

**Lemma 7.** For any positive integers \( r, k \) and \( (m, p) = 1 \), define
\[
S_k (h, r; m) = \sum_{a=1}^p \left| \sum_{a < n < a + h} U_k (\cos \theta_p (mn)) \right|^{2r}.
\]

Then for \( 1 \leq h \leq p \), we have
\[
S_k (h, r; m) \ll k^{2r} h^r p + k^{4r} h^{2r} p^{1/2}
\]
for \( r \geq 1 \), where the implied constant depends only on \( r \).

**Proof.** First we write
\[
S_k (h, r; m) = \sum_{a=1}^p \left| \sum_{1 \leq n \leq h} U_k (\cos \theta_p (ma + mn)) \right|^{2r}.
\]
Proof. We shall prove the second estimate. Observing that 
\( h \) function in  
\( r \) on 
\( r \) being an integer depending on 
\( j \) exceeding 
\( \lambda \) with 
\(| \cdot | \)

For any positive integers 
Lemma 8.

Then we have 
\( \sum \sum U_k(\cos \theta_p(f_i(a))) \)

with \( f_i(x) = mx + mn_i \).

Now we can divide the vector \( n = (n_1, \cdots, n_2r) \) into two types. If the value set of the coordinates of \( n \) consists of at most \( r \) distinct integers, each of which occurs an even number of times, we can estimate the contribution to \( S_k(h, r; m) \) trivially by 
\(|(k+1)^2r^2h^rp| \).

Applying Lemma 6, we find that the rest \( n \) produces the contribution bounded by 
\( c(r)(k+1)^4r^2hp^1/2 \)

for certain \( c(r) > 0 \). Collecting the two estimates above we can deduce the lemma. \( \square \)

Next, we shall introduce the Rademacher-Menchov device to prove a maximal analogue of Lemma 7.

Lemma 8. For any positive integers \( r, k \) and \( (m, p) = 1 \), define
\( S_k^r(h, r; m) = \sum_{a=1}^{p} \max_{h' < h} \sum_{a < n \leq a + h'} U_k(\cos \theta_p(mn)) \).

Then we have 
\( S_k^r(h, 1; m) < k^2hp(\log 2h)^2 + k^4h^2p^{1/2} \)

and 
\( S_k^r(h, r; m) < k^2r^2h^rp + k^4r^2p^{1/2} \)

for \( r \geq 2 \), where the first implied constant is absolute and the second one depends only on \( r \).

Proof. We shall prove the second estimate. Observing that \( S_k^r(h, r; m) \) is an increasing function in \( h \), thus it suffices to prove the bound as \( h \) is a power of 2, say \( h = 2^\lambda \).

For \( h' < h = 2^\lambda \) we write \( h' = h \sum 2^{j-i} \), where \( j \) are distinct positive integers not exceeding \( \lambda \). It follows that
\[ \sum_{a < n \leq a + h'} U_k(\cos \theta_p(mn)) = \sum_{j} \sum_{a + \nu 2^{-j} < n \leq a + (\nu+1)2^{-j}} U_k(\cos \theta_p(mn)) \]

with
\[ \nu = \nu_j = \sum_{i < j} 2^{j-i} \]

being an integer depending on \( j \) and \( h' \). By Hölder inequality, for certain \( B > 0 \) we have
\[ \left| \sum_{a < n \leq a + h'} U_k(\cos \theta_p(mn)) \right|^{2r} = \left| \sum_{j} B^{-j} B^{j} \sum_{a + \nu 2^{-j} < n \leq a + (\nu+1)2^{-j}} U_k(\cos \theta_p(mn)) \right|^{2r} \leq \left( \sum_{j} B^{-2rj/(2r-1)} \right)^{2r-1} \sum_{j} B^{2rj} \left| \sum_{n} U_k(\cos \theta_p(mn)) \right|^{2r} , \]
where \( j \) and \( \nu \) are still restricted to those values associated with the binary expansion of \( h'/h \). In order to obviate the dependence of \( \nu \) on \( j \) and \( h' \), we allow \( j \) to run over all integers in \([1, \lambda]\) and \( \nu \) to run over all the integers in \([0, 2^j]\). Hence we can deduce that

\[
\left| \sum_{0 < n \leq a + h'} U_k(\cos \theta_p(mn)) \right|_{2r} \leq c(B, r) \sum_{j \leq \lambda} B^{2rj} \sum_{0 \leq \nu < 2^j} \sum_{a + \nu h2^{-j} < n \leq a + (\nu + 1)h 2^{-j}} U_k(\cos \theta_p(mn)) \left| \right|_{2r}
\]

holds for certain constant \( c(B, r) > 0 \). Note that the bound holds uniformly in \( 0 \leq h' < h \), thus

\[
S_k^*(h, r; m) \leq c(B, r) \sum_{j \leq \lambda} 2^j B^{2rj} S_k(h2^{-j}, r; m).
\]

Now applying Lemma 7 and taking \( B = 2^{1/5} \), we get the estimate for \( S_k^*(h, r; m) \) as \( r \geq 2 \). And the proof for the case of \( r = 1 \) is rather similar, so we omit it here. \( \square \)

**Lemma 9.** Let \( \{I_t\}_{t=1}^T \) be nonoverlapping intervals in \([0, p]\) and \( k, r \geq 1 \). Define

\[
W_k(r) = \sum_{t \leq T} \max_{(a, p) = 1} \left| \sum_{m \in I_t} U_k(\cos \theta_p(am)) \right|_{2r}.
\]

If \( h < \#I_t \leq 2h \) for all \( t \leq T \), then we have

\[
W_k(1) \ll k^2 p (\log 2h)^2 + k^4 h p^{1/2}
\]

and

\[
W_k(r) \ll k^{2r} h^{r-1} p + k^{4r} h^{2r-1} p^{1/2}
\]

for \( r \geq 2 \), where the first implied constant is absolute and the second one depends only on \( r \).

**Proof.** Let \( \{\alpha_m\} \) be a finite sequence supported on \((M, M + N]\). It is proved in [GM] (Lemma 4) that for any \( h \leq N \),

\[
(7) \quad \left| \sum_m \alpha_m \right|_{2r}^2 \leq \frac{(2N)^{2r-1}}{h^{2r}} \sum_{M < n \leq M + N} \left( \max_{h' \leq h} \left| \sum_{n \leq m < n + h'} \alpha_m \right|_{2r} \right)^2 + \max_{h' \leq h} \left| \sum_{n-h' < m < n} \alpha_m \right|_{2r}^2.
\]

Furthermore, if \( N \leq 2h \), then

\[
\frac{(2N)^{2r-1}}{h^{2r}} \leq \frac{16^r}{h}.
\]

Now taking \( \alpha_m = U_k(\cos \theta_p(am)) \) in (7) we have

\[
\left| \sum_{m \in I_t} U_k(\cos \theta_p(am)) \right|_{2r}^2 \leq \frac{16^r}{h} \left( \sum_{n \in I_t} \left( \max_{h' \leq h} \left| \sum_{n-h' < m < n} U_k(\cos \theta_p(am)) \right|_{2r} \right)^2 \right) + \max_{h' \leq h} \left| \sum_{n-h' < m < n} U_k(\cos \theta_p(am)) \right|_{2r}^2,
\]

where

\[
\frac{\sum_{m \in I_t} \left| U_k(\cos \theta_p(am)) \right|_{2r}}{h} \leq \frac{16^r}{h} \left( \sum_{n \in I_t} \left( \max_{h' \leq h} \left| \sum_{n-h' < m < n} U_k(\cos \theta_p(am)) \right|_{2r} \right)^2 \right).
\]
it follows that

\[
W_k(r) \leq \frac{16^r}{h} \sum_{t \leq T} \max_{(a,p) = 1} \left| \sum_{n \in J_t} \left( \max_{h' \leq h} \left| \sum_{n \leq m < n+h'} U_k(\cos \theta_p(am)) \right| \right) + \max_{h' \leq h} \left| \sum_{n-h' < m < n} U_k(\cos \theta_p(am)) \right| \right|^{2^r}.
\]

(8)

Now we aim to show that we can move max to the outer of the summation over \(t\).

Let \(\mathfrak{A}\) consist of all the closed subintervals of \([0, p)\) with integral boundaries and all the singleton sets are also included. Clearly, \(\mathfrak{A}\) is a finite set for any fixed \(p\), and we can arrange the elements and name them \(J_\ell, \ell \leq L\). For any fixed \(r \geq 1\), \((a,p) = 1\) and each \(\ell \leq L\), we define

\[
V(J_\ell) = V(J_\ell; a, r) = \sum_{n \in J_\ell} \max_{h' \leq h} \left| \sum_{n \leq m < n+h'} U_k(\cos \theta_p(am)) \right|^{2^r}.
\]

We shall state the following claim:

**Claim.** \(V(J_\ell)\) is an increasing function in \(#J_\ell\), in the sense that

(a) For \(#J_{\ell_1} = #J_{\ell_2}\), we have \(V(J_{\ell_1}) \asymp V(J_{\ell_2})\);

(b) For \(#J_{\ell_1} < #J_{\ell_2}\), we have \(V(J_{\ell_1}) \ll V(J_{\ell_2})\);

where all the implied constants depend at most on \(r\).

**Proof of the claim.** Since \(V(J_\ell)\) vanishes if and only if \(J_\ell\) is empty, we shall always deal with nonempty intervals. It is clear that \(V(J_\ell) > 0\) for each \(\ell \leq L\). Now suppose \(\ell = \ell_0\) minimizes \(V(J_\ell)\). Then we have \(#J_{\ell_0} = 1\), and for any other \(\ell \leq L\), we have \(#J_\ell \geq #J_{\ell_0}\).

Get rid of all the \(J_\ell\) from \(\mathfrak{A}\) with \(V(J_\ell) \asymp V(J_{\ell_0})\) with a constant depending at most on \(r\), and denote by \(\mathfrak{A}_1\) the set consisting of the rest intervals. Repeat the process above, we can find \(J_{\ell_1} \in \mathfrak{A}_1\) which minimizes the rest \(V(J_\ell)\)'s. Then one finds that each \(J_\ell \in \mathfrak{A}_1\) satisfies \(#J_\ell \geq #J_{\ell_1}\). Or else, one can write \(J_{\ell_1} = J_{\ell'} \sqcup J_{\ell''}\), and from inclusion-exclusion principle, at least one of \(V(J_{\ell'})\) and \(V(J_{\ell''})\) is of a larger order of magnitude than \(V(J_{\ell_0})\), say, \(V(J_{\ell'})\). Then one can conclude that \(J_{\ell'} \in \mathfrak{A}_1\) and \(V(J_{\ell'}) < V(J_{\ell_1})\), which contradict the definition of \(J_{\ell_1}\).

Repeating the process above for sufficiently many times, one can find that a larger \(V(J_\ell)\) comes from a longer \(J_{\ell_1}\), and a longer \(J_\ell\) produces a larger \(V(J_\ell)\) up to a constant factor in \(r\). This establishes (b).

Moreover, for \(\ell_1 \neq \ell_2\) with \(#J_{\ell_1} = #J_{\ell_2}\), we can assume \(V(J_{\ell_1}) \ll V(J_{\ell_2})\) without the loss of generality. On the other hand, we can construct a new interval \(J_{\ell_3} \in \mathfrak{A}\) by replacing the right boundary of \(J_{\ell_1}\), say, \(w\), by \(w+1\). It follows from (b) and the fact \(#J_{\ell_3} = 1 + #J_{\ell_2}\) that \(V(J_{\ell_3}) \gg V(J_{\ell_2})\). Note that \(V(J_{\ell_1}) \asymp V(J_{\ell_3})\), thus we have \(V(J_{\ell_1}) \gg V(J_{\ell_2})\), which yields \(V(J_{\ell_1}) \asymp V(J_{\ell_2})\).

This completes the proof of the claim.

Now apply the claim to \(\{I_t\}_{t=1}^T\). Note that \(h < #I_t \leq 2h\) for each \(t \leq T\), we have \(V(I_{t_1}) \asymp V(I_{t_2})\) for any \(t_1, t_2 \leq T\). In particular, for any fixed \(a\) with \((a,p) = 1\) we have

\[
\max_{t \leq T} V(I_t) := \max_{t \leq T} V(I_t; a, r) \ll \min_{t \leq T} V(I_t; a, r) =: \min_{t \leq T} V(I_t).
\]
Hence we can deduce from (8) that

\[
W_k(r) \leq \frac{T \cdot 16^r}{h} \max_{(a, p) = 1} \max_{t \leq T} \sum_{n \in \mathcal{I}_t} \left( \max_{h' \leq h} \sum_{n \leq m < n + h'} U_k(\cos \theta_p(am)) \right)^{2r} \\
+ \max_{h' \leq h} \sum_{n - h' < m < n} U_k(\cos \theta_p(am)) \right)^{2r} \\
\leq \frac{c(r)}{h} \max_{(a, p) = 1} \sum_{t \leq T} \sum_{n \in \mathcal{I}_t} \left( \max_{h' \leq h} \sum_{n \leq m < n + h'} U_k(\cos \theta_p(am)) \right)^{2r} \\
+ \max_{h' \leq h} \sum_{n - h' < m < n} U_k(\cos \theta_p(am)) \right)^{2r},
\]

thus

\[
W_k(r) \leq \frac{c(r)}{h} \max_{(a, p) = 1} S_k^* (h, r; a),
\]

from which and Lemma 8 we can deduce the lemma. \(\square\)

### 4. Proof of Theorem 1 — Local VST

The case of \(r = 1\) can be deduced from the completing method or Fourier technique together with Lemma 5, so we only deal with the case \(r \geq 2\). Suppose \(q\) is a prime different from \(p\). Then

\[
\mathcal{D}_k(M, N; p, h) = \sum_{b \mod q \atop M < a \leq M + N} \sum_{a \equiv b \mod q} U_k(\cos \theta_p(ha)) \\
= \sum_{b \mod q \atop M < a \leq M + N} \sum_{a \equiv -bp \mod q} U_k(\cos \theta_p(ha)).
\]

Now we write

\[
\mathcal{I}(b, q) = \left\lfloor \frac{b}{q} + \frac{M}{pq}, \frac{b}{q} + \frac{M + N}{pq} \right\rfloor.
\]

Thus we have

\[
\mathcal{D}_k(M, N; p, h) = \sum_{b \mod q \atop m / pq \in \mathcal{I}(b, q)} \sum_{m \equiv bp \mod q} U_k(\cos \theta_p(hmq)).
\]

Denote by \(\mathcal{P}\) the set of primes \(q \in (Q, 2Q]\), where \(Q\) is a parameter to be chosen later such that

\[
2QN \leq p.
\]
Now we sum (9) over each \( q \in \mathcal{P} \), getting

\[
\left| \mathcal{D}_k(M, N; p, h) \right| \leq \frac{1}{\# \mathcal{P}} \sum_{q \in \mathcal{P}} \sum_{b \mod q} \left| \sum_{m/p \in \mathcal{I}(b, q)} U_k(\cos \theta_p(hm)) \right| \\
\leq \frac{1}{\# \mathcal{P}} \sum_{q \in \mathcal{P}} \sum_{b=1}^{q-1} U_k(\cos \theta_p(hm)) + \frac{1}{\# \mathcal{P}} \sum_{q \in \mathcal{P}} \sum_{\frac{M}{q} < m \leq \frac{M+N}{q}} U_k(\cos \theta_p(hm))
\]

Estimating the summation over \( m \) in the second term trivially and from Hölder inequality, we get

\[
\mathcal{D}_k(M, N; p, h) \ll \frac{1}{\# \mathcal{P}} \sum_{q \in \mathcal{P}} \sum_{b=1}^{q-1} \max_{(h, p) = 1} \left| \sum_{m/p \in \mathcal{I}(b, q)} U_k(\cos \theta_p(hm)) \right| + \frac{(k + 1)N}{Q}
\]

\[
\ll \frac{1}{\# \mathcal{P}} \left( \sum_{q \in \mathcal{P}} q \right)^{1 - 1/2r} \left( \sum_{q \in \mathcal{P}} \sum_{b=1}^{q-1} \max_{(h, p) = 1} \left| \sum_{m/p \in \mathcal{I}(b, q)} U_k(\cos \theta_p(hm)) \right| \right)^{2r} \frac{1}{2r}
\]

\[
+ \frac{(k + 1)N}{Q}.
\]

Note that

\[
\# \mathcal{P} \approx \frac{Q}{\log Q}, \quad \sum_{q \in \mathcal{P}} q \approx \frac{Q^2}{\log Q},
\]

and the intervals \( \mathcal{I}(b, q) \) are disjoint. In fact, for \((b, q) \neq (b', q')\), we have

\[
\left| \frac{b}{q} - \frac{b'}{q'} \right| = \frac{|b'q - b'q|}{qq'} \geq \frac{1}{qq'} \geq \frac{1}{2qQ} \geq \frac{N}{pq}
\]

by (10). Now applying Lemma 9 with \( T = \sum_{q \in \mathcal{P}} (q - 1) \) and \( h = [N/2Q] \), we have

\[
\left| \mathcal{D}_k(M, N; p, h) \right| \leq c(r)k\frac{1/2}{r} (NQ)^{1/2 - 1/2r} p^{1/2r} (\log Q)^{1/2r} + c(r)k^2 N^{1-1/2r} Q^{-1/2r} p^{1/4r} (\log Q)^{1/2r} + (k + 1)NQ^{-1}
\]

for \( r \geq 2 \).

Taking \( Q = Np^{-1/2r} \) to optimize the size of each term, we obtain that

\[
\mathcal{D}_k(M, N; p, h) \ll k^2 N^{1-1/r} p^{(r+1)/4r^2} \log p
\]

for \( r \geq 2 \) with an implied constant depending only on \( r \).
5. Proof of Theorem 5 — Balance of the Signs of Kloosterman Sums

Now we introduce the characteristic function $\phi(x)$ of $[0, 1]$, and suppose it has continuous derivatives of order $\leq A + 1$ on $[0, 1]$. Then for any fixed $h$ with $(h, p) = 1$, we have

$$\sum_{M < a \leq M + N \atop S(a, h; p) \geq 0} 1 = \sum_{M < a \leq M + N} \phi(\pm \cos \theta_p(ah)).$$

From Lemma 3, we find that

$$\sum_{M < a \leq M + N \atop S(a, h; p) \geq 0} 1 = \hat{\phi}(0) N + \sum_{1 \leq \ell \leq L} \hat{\phi}(\ell) \sum_{M < a \leq M + N} U_\ell(\cos \theta_p(ah)) + O(NL^{-A}) + O(1)$$

for certain $L > 0$ to be chosen later, where

$$\hat{\phi}(\ell) = \frac{2}{\pi} \int_{-1}^{1} \sqrt{1 - x^2} \phi(\pm x) U_\ell(x) dx.$$

Thus $\hat{\phi}(0) = 1/2$ and for $\ell \geq 1$, $\hat{\phi}(\ell) \ll \ell^{-1}$. It follows from Theorem 1 that

$$\sum_{M < a \leq M + N \atop S(a, h; p) \geq 0} 1 = \frac{1}{2} N + O(\omega_r(p, N) L^2) + O(NL^{-A}).$$

Now taking $L = p^{\varepsilon/2}$ and

$$A = \left[ \frac{2}{\varepsilon} \cdot \frac{\log N}{\log p} \right],$$

we can derive Theorem 5.

6. Proof of Theorem 6 — Extremely Small Values of Kloosterman Sums

Now we introduce the characteristic function $\eta(x)$ of $[-\delta, \delta]$ and suppose it has continuous derivatives of order $\leq B + 1$ on $[-\delta, \delta]$. Then for any fixed $h$ with $(h, p) = 1$, we have

$$\sum_{M < a \leq M + N \atop |S(a, h; p)| \leq 2\delta p^{1/2}} 1 = \sum_{M < a \leq M + N} \eta(\cos \theta_p(ah)).$$

From Lemma 3, we find that

$$\sum_{M < a \leq M + N \atop |S(a, h; p)| \leq 2\delta p^{1/2}} 1 = \hat{\eta}(0) N + \sum_{1 \leq \ell \leq L} \hat{\eta}(\ell) \sum_{M < a \leq M + N} U_\ell(\cos \theta_p(ah)) + O(NL^{-B}) + O(1)$$

for certain $L > 0$ to be chosen later. Moreover, we have

$$\hat{\eta}(0) = \frac{2}{\pi} \int_{-\delta}^{\delta} \sqrt{1 - x^2} dx = \frac{2}{\pi} \left( \arcsin \delta + \delta \sqrt{1 - \delta^2} \right),$$

and trivially

$$\hat{\eta}(\ell) = \frac{2}{\pi} \int_{-\delta}^{\delta} \sqrt{1 - x^2} U_\ell(x) dx \ll \ell \delta, \quad \ell \geq 1.$$
Thus it follows from Theorem 1 that
\[
\sum_{M < a \leq M + N} 1 = \frac{2}{\pi} \left( \arcsin \delta + \delta \sqrt{1 - \delta^2} \right) N + O(\omega_r(p, N)\delta^4) + O(NL^{-B}) + O(1).
\]

Now taking \( L = p^{\varepsilon/4} \) and
\[
B = \left[ \frac{4}{\varepsilon} \cdot \frac{\log N}{\log p} \right],
\]
we can derive Theorem 6.

7. Final Remarks

The constant in the upper bound (6) depends essentially on Lemma 4. Hence it is important to obtain a sharp estimate for \( \beta_\ell \) in Lemma 4, or alternatively an estimate for the mean value
\[
\sum_{0 \leq \ell \leq K} |\beta_\ell|.
\]
For example, if we assume \( k_1 \geq k_2 \geq \cdots \geq k_J \), the argument in the proof of Lemma 4 is fine if \( k_2 \leq \ell \), or else we should apply the identity \( U_m(\cos \varphi) = \sin((m + 1)\varphi)/\sin \varphi \) to \( m = k_1, k_2 \) instead of \( m = k_1, \ell \). However, it seems that this delicate argument can not yield a better bound in (6). Anyway, the upper bound with the weaker constant is sufficient for applications in this paper and we don’t seek the best possible constant here.

On the other hand, one can find that the bounds for \( S_k(h, r) \) in Lemma 7 also serve the following general sum
\[
\sum_{a=1}^{p} \left| \sum_{a<n\leq a+h} \psi(n)U_k(\cos \theta_p(mn)) \right|^{2r}
\]
with \( \psi \) being an additive character mod \( p \). Especially, we may take \( \psi(n) = e(mn/p) \), thus one can follow the arguments in this paper to derive the estimate for
\[
D_k(M, N; h, m, p) := \sum_{M < a \leq M + N} U_k(\cos \theta_p(ha))e\left( \frac{ma}{p} \right)
\]
for any integers \( h, m \) with \( (h, p) = 1 \).

To be precise, we have

**Theorem 8.** Let \( k \) be a fixed positive integer. For any positive integer \( r \) and prime number \( p \), we have
\[
\max_{(h, p)=1} |D_k(M, N; h, m, p)| \ll k^2 \omega_r(p, N),
\]
where the implied constant depends only on \( r \).

Based on Theorem 8, we can obtain a nontrivial estimate for the incomplete Kloosterman sum
\[
\sum_{M < a \leq M + N} e\left( \frac{an + bm}{p} \right).
\]
Since this is of special interests, we shall illustrate such an idea in another occasion [Xi].

Moreover, it should be mentioned that H. Niederreiter [Ni] gave a quantitative version of VST with an explicit error term such as \( O(p^{-1/4}) \). It can be seen that the method in
proving Theorem 7 together with Lemma 3 yields LVST with an explicit error term such as \( O(N^{-1}\omega_r(p,N)p^\varepsilon) \), which generalizes and improves the result of H. Niederreiter.

We would like to note at last that the argument in this paper is also valid to bound the following general sum

\[
\sum_{M < a \leq M+N} \prod_{1 \leq i \leq s} U_{k_i}(\cos \theta_p(f_i(a)))
\]

for any fixed \( r \geq 1 \), where \( s \in \mathbb{N}^+ \), \( f_i(x) \in \mathbb{Z}[x] \), \( k_i \in \mathbb{N} \) for \( 1 \leq i \leq s \), the symbol \( \mathcal{b} \) denotes the summation restricted to those \( a \) with

\[
\prod_{1 \leq i \leq s} f_i(a) \not\equiv 0 \pmod{p}.
\]

We note here that such an upper bound for (11) together with Lemma 3 can imply better estimates in [FMRS].

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