Constructive Renormalization of the 2-dimensional Grosse-Wulkenhaar Model with Multi-scale Loop Vertex Expansions

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Abstract

In this paper we construct the noncommutative Grosse-Wulkenhaar model on 2-dimensional Moyal plane with the method of loop vertex expansion. We treat renormalization with this new tool, adapt Nelson’s argument and prove Borel summability of the perturbation series. This is the first non-commutative quantum field theory model to be built in a non-perturbative sense.

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1 Introduction

Quantum field theories on noncommutative space time became popular after the discovery that they may arise as effective regimes of string theory either due to the compactification [1] or due to the presence of the Green-Schwarz $B$ field for open strings [2, 3]. The simplest non-commutative space is the Moyal space, which could be considered also as a low energy limit of open string theory. However the quantum field theories on the Moyal space are non-renormalizable due to a phenomenon called UV/IR mixing, namely when we integrate out the high scale fields for non planar graphs, there exist still infrared divergences and the divergent terms cannot be compensated by counter-terms [4].

Several years ago H. Grosse and R. Wulkenhaar made a breakthrough by introducing a harmonic oscillator term in the propagator so that the theory fully obeys a new symmetry called the Langmann-Szabo duality. They have proved in a series of papers [5, 6, 7] (see also [8]) that the noncommutative $\phi^4$ field theory possessing the Langmann-Szabo duality (which we call the $GW_4$ model hereafter) is perturbatively renormalizable to all orders.
After their work many other QFT models on Moyal space \cite{9, 10, 11} or degenerate Moyal space \cite{12, 13} have also been shown to be \textit{perturbatively} renormalizable. More details could be found in \cite{15}.

The $GW_4$ model is not only perturbatively renormalizable but also asymptotically safe due to the vanishing of the $\beta$ function in the ultraviolet regime \cite{17}, \cite{18}, \cite{19}, and the renormalization flow of the coupling constant is bounded, which means that it is even better behaved than its commutative counterpart. In the commutative case we either have the Landau ghost for the $\phi_4^4$ theory and $QED$ in the ultraviolet regime or we have confinement for non Abelian gauge theory in the infrared regime. This makes the non-perturbative construction of these commutative field theories models very difficult, if not impossible; Recently Grosse and Wulkenhaar proved that this model can be exactly solved \cite{20, 21} in the limit $\theta \to \infty$, by using the method of integral models and the fixed point methods. What’s more, in the $\theta \to \infty$ limit the correlation functions are proved to satisfy the Osterwalder-Schrader axioms, except that the reflection positivity still needs to be proved. This makes it the unique candidate until now to be fully constructed in four dimensional Minkowski space.

But this would not be the whole story of GW4, since after taking the limit $\theta \to \infty$ all non-planar graphs vanish; only planar graphs survive. Combinatorially planar graphs are not so different from trees. On the other hand, in the process of taking the limit $\theta \to \infty$ non-planar graphs always exist. It seems that their method is not very suitable for summing over those nonplanar graphs and one should use the techniques from constructive renormalization theory to treat them.

Constructive field theory or constructive renormalization theory builds the exact Green’s functions whose Taylor expansions correspond to perturbative quantum field theory. The traditional techniques for Bosonic constructive theories are the cluster and Mayer expansions which divide the space into cubes and require locality of the interaction. However due to the noncommutativity of the coordinates in Moyal plane, the naive division of space into cubes seems unsuited in this case. What’s more, due to the non-locality of the interaction vertex, it is not clear to which cube an interaction vertex should belong. So it seems very difficult, if not impossible, to construct the Grosse-Wulkenhaar model with the traditional methods of cluster and Mayer expansion.

In this paper we shall construct the Grosse-Wulkenhaar model in the 2 dimensional Moyal plane or $GW_2$, as a warm-up towards building non-perturbatively the full $GW_4$ model. The method we use is the Multi-scale Loop Vertex Expansion (MLVE for short) \cite{40} which was invented precisely for overcoming these difficulties of the traditional methods \cite{22}. This method has been applied to ordinary $\phi_4^4$ model in a companion paper \cite{27} and to the construction of tensor field theory models \cite{44, 48}.

Here we briefly summarize the main idea of the MLVE and our construction. The MLVE is an advanced version of LVE \cite{22, 24} which is a combination of the intermediate field technique and the BKAR forest formula in that the interaction vertex is also sliced by scales and hard core constraints are naturally added for the scaling parameters, before performing the BKAR Taylor expansions. Due to the hardcore constraint the Taylor expansion of the partition function into polynomials of sliced interaction vertex is actually a finite expansion. That the order of the Taylor expansion being controlled by the scaling indices is the main feature of the MLVE.

The $GW_2$ model is more difficult to treat than the $\phi_4^4$ model in that the Wick-ordered
interaction vertex contains not only logarithmically divergent term but also linearly divergent term (see section 2). So that we shall also perform the slice-testing expansions, which is the renormalized process adapted to MLVE to compensate the tadpoles with counter-terms. Another benefit from the slice-testing expansions is that we can generate a set of marked propagators from each of them we can gain a convergent factor, with which we can finally bound the linear divergent term from the Wick-ordered interaction terms and the nonperturbative combinatorial factors. These are the preparation steps for the MLVE and are studied in sections 3 and section 4.

Then we shall perform the BKAR forest formula twice (see section 5), one for decoupling the integration measure and another for getting rid of the hardcore constraints, and we obtain the perturbation series for the connected function, which contain not only the normal propagators, counter terms but also new kind of operators called the resolvents. The resolvents are bounded in the cardioid domain and are getting rid of by recursively using the Cauchy-Schwarz inequality. Finally we gather the all the convergent factor to beat the non-perturbative bound and the combinatorial factors and prove the Borel summability of the perturbation series in the cardioid domain. These are the materials of section 6. In the appendix we have shown an example of the second order slice testing expansions.

2 Moyal space and Grosse-Wulkenhaar Model

2.1 The Moyal space

The \( D \)-dimensional Moyal space \( \mathbb{R}^D_\theta \) for \( D \) even is generated by the non-commutative coordinates \( x^\mu \) that obey the commutation relation \([x^\mu, x^\nu] = i\Theta^{\mu\nu}\), where \( \Theta \) is a \( D \times D \) non-degenerate skew-symmetric matrix such that \( \Theta^{\mu\nu} = -\Theta^{\nu\mu} \). It is the simplest and best studied model of non-commutative space (see \[14, 15\] for more details).

Consider the Moyal algebra of smooth and rapidly decreasing functions \( \mathcal{S}(\mathbb{R}^D) \) defined on \( \mathbb{R}^D_\theta \). \( \mathcal{S}(\mathbb{R}^D) \) is equipped with the non-commutative Groenwald-Moyal product defined by:

\[
(f \star_\Theta g)(x) = \frac{1}{\pi^D|\det \Theta|} \int_{\mathbb{R}^D} d^D y d^D z f(x+y)g(x+z)e^{-2iy\Theta^{-1}z}.
\]

We now set \( D = 2 \) in this paper. Define the creation and annihilation operators in terms of the coordinates of the Moyal plane \( \mathbb{R}^2_\theta \) \[6, 7, 14\]:

\[
a = \frac{1}{\sqrt{2}}(x_1 + ix_2), \quad \bar{a} = \frac{1}{\sqrt{2}}(x_1 - ix_2),
\]

\[
\frac{\partial}{\partial a} = \frac{1}{\sqrt{2}}(\partial_1 - i\partial_2), \quad \frac{\partial}{\partial \bar{a}} = \frac{1}{\sqrt{2}}(\partial_1 + i\partial_2).
\]
such that for any function \( f \in \mathcal{S}_D \) we have
\[
(a \star f)(x) = a(x)f(x) + \frac{\theta}{2} \frac{\partial f}{\partial a}(x), \quad \quad \quad (f \star a)(x) = a(x)f(x) - \frac{\theta}{2} \frac{\partial f}{\partial a}(x),
\]
\[
(\bar{a} \star f)(x) = \bar{a}(x)f(x) - \frac{\theta}{2} \frac{\partial f}{\partial \bar{a}}(x), \quad \quad \quad (f \star \bar{a})(x) = \bar{a}(x)f(x) + \frac{\theta}{2} \frac{\partial f}{\partial \bar{a}}(x). \quad (3)
\]

With the creation and annihilation operators we could build the matrix basis \( f_{mn}(x) \) for the functions \( \phi(x) \in \mathcal{S}(\mathbb{R}^2) \) defined on \( \mathbb{R}^D_\theta \), such that:
\[
\phi(x) = \sum_{m,n \in \mathbb{N}} \phi_{mn} f_{mn}(x),
\]
where \( f_{mn}(x) \) is defined as:
\[
f_{mn}(x) := \frac{1}{\sqrt{n!m!\theta^{m+n}}} \bar{a}^m \ast f_0 \ast a^n \]
\[
= \frac{1}{\sqrt{n!m!\theta^{m+n}}} \sum_{k=0}^{\min(m,n)} (-1)^k \binom{m}{k} \binom{n}{k} k! 2^{m+n-2k} \theta^k \bar{a}^{m-k} a^{n-k} f_0,
\]
and \( f_0(x) = 2e^{-\frac{1}{2}(x_1^2 + x_2^2)} \).

The matrix basis defined above has the following properties:
\[
(f_{mn} \ast f_{kl})(x) = \delta_{nk} f_{ml}(x), \quad \quad \quad \int d^2 x f_{mn}(x) = 2\pi \theta \delta_{mn}. \quad (6)
\]

### 2.2 The 2-dimensional Grosse-Wulkenhaar Model

The 2-dimensional Grosse-Wulkenhaar model (GW₂ for short) is defined by the action:
\[
S = \int d^2 x \left[ \frac{1}{2} \partial_\mu \phi \ast \partial^\mu \phi + \frac{\Omega^2}{2} (\tilde{x}_\mu \phi) \ast (\tilde{x}^\mu \phi) + \frac{\mu^2}{2} \phi \ast \phi + \frac{\lambda}{4} \phi \ast \phi \ast \phi \ast \phi \right].
\]

where \( \phi(x) \) is a real scalar field, \( \tilde{x}_\mu = 2(\Theta^{-1})_{\mu\nu} x^\nu \) and the Euclidean signature has been used. We take the Wick ordering to the interaction term in order to compensate the tadpoles. The explicit form of the Wick ordered interaction in the matrix basis is given by \( (16) \).

The action \( (7) \) has a remarkable symmetric property \( [16] \) such that if we exchange the position and momentum operator:
\[
p_\mu \leftrightarrow \tilde{x}_\mu, \quad \phi(p) \leftrightarrow \pi \sqrt{|\det \Theta|} \phi(x),
\]
where
\[
\hat{\phi}(p) := \int d^2 x e^{-ip\Theta x} \phi(x), \quad p\Theta x := p_\mu \Theta^{\mu\nu} x_\nu,
\]
then the action \( (7) \) is invariant up to a scalar factor:
\[
S[\phi; \mu, \lambda, \Omega] \mapsto \Omega^2 S[\phi; \frac{\mu}{\Omega}, \frac{\lambda}{\Omega}, \frac{1}{\Omega}]. \quad (10)
\]
This symmetry is called the Langmann-Szabo duality \[16\]. It is is essential for curing the UV/IR mixing problem and proving the renormalizability of this model \[7, 8, 29\]. Ω = 1 is a fixed point of this model \[18\], at which the β function is vanishing \[17, 18\] and the flow of the coupling constant is bounded. \[19\]

The scalar fields become $n \times n$ Hermitian matrices $\phi_{mn}$ in the matrix basis \(5\) and the action can be written as:

$$S[\phi] = 2\pi\theta \sum_{m,n,k,l \in \mathbb{N}} \left[ \frac{1}{2} \phi_{mn} \Delta_{mn,kl} \phi_{kl} + \frac{\lambda}{4} : \phi_{mn} \phi_{nk} \phi_{kl} \phi_{lm} : \right], \quad (11)$$

where $\Delta$ is the kinetic matrix with elements $\Delta_{mn,kl}$:

$$\Delta_{mn,kl} = [\mu^2 + 2(1 + \Omega^2)(m+n+1)]\delta_{mk}\delta_{nl} - \frac{2}{\theta}(1 - \Omega^2) \times \left[ \sqrt{(m+1)(n+1)}\delta_{m+1,k}\delta_{n+1,l} + \sqrt{mn}\delta_{m-1,k}\delta_{n-1,l} \right]. \quad (12)$$

The Feynman graphs become Ribbon graphs \[7, 31, 32\] in the matrix basis.

We introduce an ultraviolet cutoff $\Lambda \in \mathbb{N}^+$ for the matrix indices such that $0 \leq m, n \leq \Lambda$, and the Laplacian becomes an $\Lambda^2 \times \Lambda^2$ dimensional matrix. The covariance $C_{sr,kl}$ is defined by the following equation: \[6\] :

$$\sum_{r,s=0}^{\Lambda} \Delta_{mn,rs} C_{sr,kl} = \delta_{ml}\delta_{nk}. \quad (13)$$

When $\Omega = 1$, the elements of the kinetic matrix becomes the diagonal matrix:

$$\Delta_{mn,kl} = 2\pi\theta \left[ \mu^2 + \frac{4}{\theta}(m+n+1) \right]\delta_{ml}\delta_{nk}, \quad (14)$$

The covariance matrix also becomes diagonal:

$$C_{mn,kl} = \frac{1}{2\pi\theta} \mu^2 + \frac{1}{\theta}(m+n+1) \delta_{ml}\delta_{nk} := \frac{1}{2\pi\theta} C_{mn} \delta_{ml}\delta_{nk}, \quad (15)$$

Setting $\theta = 4$, $\mu^2 = 0$ and forget inessential factor $\frac{1}{8\pi}$ for simplicity, then we have $C_{mn} = \frac{1}{m+n+1}$. Since $\Omega = 1$ is a fixed point of this model \[18\], we shall consider only the case $\Omega = 1$ in the rest of this paper.

The Wick ordering of the interaction with ultraviolet cutoff $\Lambda$ reads:

$$\sum_{mnkl} : \phi_{mn} \phi_{nk} \phi_{kl} \phi_{lm} := \sum_{mnkl} \phi_{mn} \phi_{nk} \phi_{kl} \phi_{lm} - \sum_{mp} 4\phi_{mp} \phi_{pm} T^\Lambda_m + 2 \sum_m (T^\Lambda_m)^2. \quad (16)$$

where

$$T^\Lambda_m = \sum_q C_{mq,qm} = \sum_{q=0}^{\Lambda} \frac{1}{q + m + 1} = \log \frac{\Lambda + m}{m + 1} \sim \log \Lambda, \text{ for } 0 \leq m \ll \Lambda, \quad (17)$$

and

$$\Pi^\Lambda = \sum_m \left( \sum_p \frac{1}{m + p + 1} \right) \left( \sum_q \frac{1}{m + q + 1} \right) \sim c_1 \Lambda + c_2 \ln^2 \Lambda + c_3 \ln \Lambda < 2c_1 \Lambda, \quad (18)$$

where $c_1, c_2, c_3$ are positive constants such that $1 < c_1 < 3$. 

3 The intermediate field representation and the Loop vertex expansion

3.1 The intermediate field representation

The partition function for the matrix model reads:

\[ Z(\lambda, \Lambda) = \int d\mu^\Lambda(\phi) e^{-S[\phi]}, \] (19)

where

\[ d\mu^\Lambda(\phi) = \pi^{-\frac{\Lambda^2}{2}} \det C^{-1} e^{-\frac{1}{4} \text{Tr} \phi \Delta \phi} \prod_{m \leq n}^{\Lambda} \text{Re}(\phi_{mn}) \prod_{m < n}^{\Lambda} \text{Im}(\phi_{mn}). \] (20)

is the normalized Gaussian measure of the matrix fields \( \phi \) with covariance \( C = \Delta^{-1} \) (see [15]) and \( S[\phi] \) is the Wick ordered interaction term. Before proceeding some remarks on the Gaussian integrations are in order: Gaussian integrations over \( \Lambda \times \Lambda \) dimensional Hermitian matrices \( [A_{mn}]_{\Lambda \times \Lambda} \) are considered as integrations over the \( \Lambda^2 \times 1 \) dimensional vectors \( \vec{A} \) formed by listing the elements of the matrices \( [A_{mn}] \) as \( \vec{A} = (A_{11}, A_{12}, \ldots, A_{1\Lambda}, A_{21}, \ldots, A_{2\Lambda}, \ldots, A_{\Lambda1}, \ldots, A_{\Lambda\Lambda}) \), one row after another; The transposed vectors \( \vec{A}^t \) are defined similarly by listing all the columns of \( A \), one after another, and in general are not equal to the transpose of \( \vec{A} \) in general. Gaussian integrals over \( \Lambda \times \Lambda \) matrices results in an \( \Lambda^2 \times \Lambda^2 \) dimensional determinant.

The main difficulty for calculating non perturbatively the partition functions and vacuum connected Schwinger functions is that the action is not positive. Observe that for \( \lambda > 0 \),

\[ e^{-\frac{1}{4} \text{Tr} [\phi^4 - 4\phi^2 T^\Lambda + 2 (T^\Lambda)^2]} = e^{-\frac{1}{4} \text{Tr} [(\phi^2 - 2 T^\Lambda)^2] - 2 (T^\Lambda)^2} \leq e^{\frac{1}{2} \Pi^\Lambda}, \] (21)

reproducing the Nelson’s divergent bound as \( \Lambda \to \infty \),

\[ | Z(\lambda, \Lambda) | \leq e^{\lambda O(1)\Lambda}. \] (22)
Now we introduce the $\Lambda \times \Lambda$ dimensional Hermitian matrices $\sigma = [\sigma_{mn}]$ as intermediate fields so as to writing the interaction $\int d\mu(\phi) \exp[-\lambda \text{Tr} \phi^4]$ as $\int d\mu(\phi) d\mu(\sigma) \exp[-i\sqrt{\lambda} \text{Tr} \phi^2 \sigma]$. Integrating out the matrix fields $\phi$, we can write the partition function as:

$$Z(\lambda) = \int d\mu(\sigma) \ e^{\frac{i\sqrt{\lambda}}{2} \text{Tr} \ T^\lambda (I \otimes \sigma + \sigma^t \otimes I) - \frac{i}{2} \text{Tr} \log[1 + i \sqrt{\lambda} (I \otimes \sigma + \sigma^t \otimes I)] + \frac{i}{2} \lambda \Lambda^\lambda},$$

(23)

where

$$d\mu(\sigma) = \pi^{-\Lambda^2/2}e^{-1/2 \text{Tr} \sigma^2} \prod_{m \leq n} d\text{Re}(\sigma_{mn}) \prod_{m < n} d\text{Im}(\sigma_{mn}),$$

(24)

is the normalized Gaussian measure with covariance $<\sigma_{mn}, \sigma_{kl}> = \int d\mu(\sigma) \sigma_{mn} \sigma_{kl} = \delta_{nk} \delta_{ml} \exp\{-\text{Tr} \log[1 + i \sqrt{\lambda} (I \otimes \sigma + \sigma^t \otimes I)]\}$ means the $\Lambda^2 \times \Lambda^2$ dimensional determinant of the the Gaussian integration with $\sigma^t$ the transpose of the matrix $[\sigma_{mn}]$. Define the matrix operator $\hat{\sigma} = I \otimes \sigma + \sigma^t \otimes I$ with elements:

$$\hat{\sigma}_{mn,pq} = [I \otimes \sigma + \sigma^t \otimes I]_{mn,pq} = \sigma_{np} \delta_{qm} + \sigma_{qm} \delta_{np},$$

(25)

which acts on the Hilbert space $\mathcal{H}^\Lambda \otimes \mathcal{H}^\Lambda$ with basis $e_m \otimes e_n$ by the following rule:

$$\sum_k \sigma_{mk} e_k \otimes e_n + e_m \otimes \sum_k \sigma_{kn} e_n.$$

(26)

It is useful to identify the two boarders of the ribbon as the two Hilbert spaces that $\hat{\sigma}$ acts on. So we can define the border on which the operator $I \otimes \sigma$ acts as the inner boarder of the ribbon and the border on which $\sigma^t \otimes I$ acts as the outer boarder.

The interaction term $\text{Tr} \log[1 + i \sqrt{\lambda} C \hat{\sigma}]$ is called the loop vertex [22] defined in the operator sense by the expansions $\text{Tr} \sum \frac{(-1)^{n+1}}{n} (i \sqrt{\lambda} C \hat{\sigma})^n$, where the matrix elements of $C \hat{\sigma}$ read:

$$(C \hat{\sigma})_{mn,kl} = \sum_{pq} C_{mn,pq} (I \otimes \sigma + \sigma^t \otimes I)_{qp,kl} = C_{mn} \sigma_{nk} \delta_{lm} + \sigma_{lm} C_{mn} \delta_{nk}.$$ 

(27)

$\text{Tr} \ T^\lambda ((I \otimes \sigma + \sigma^t \otimes I)) = 2 \sum_m T^\lambda \sigma_{mm}$, where the factor 2 reflects the fact that there are two different ways of decomposing the quartic term $\text{Tr} \phi^4$ into the cubic term $\text{Tr} \phi^2 \sigma$, see Figure 2.

The perturbation theory in terms of $\sigma$ is indexed by the intermediate field Feynman graphs that are ribbon graphs whose vertices are the loops obtained by expansion of [23]. The former $\phi^4$ propagators, which are now attached at the corner of the loop vertices, are called c-propagators. (see Figure 3 for an example of first order expansion) The $\sigma$ propagators corresponding to the former $\phi^4$ vertices of ordinary perturbation expansion. The perturbative order of an intermediate graph is the total number of $\sigma$-propagators. In the case of vacuum graphs, which is the only one considered in this paper, it is also half the number of c-propagators.

Remark that the graphic meaning of (27) is that the intermediate field $\sigma$ can hook to both the inner boarder and the outer boarder of a ribbon.
Figure 2: Two different ways of decomposing a $\phi^4$ vertex. The dotted lines stand for the $\sigma$ propagators.

Before explicitly calculating the partition function we first of all expand the loop vertex term as:

$$\frac{-1}{2} \text{Tr} \log [1 + i \sqrt{\frac{2\lambda}{2}} C \hat{\sigma}] = \frac{-i}{2} \text{Tr} \left( \sqrt{2\lambda} C \hat{\sigma} \right) - \frac{1}{2} \text{Tr} \log_2 [1 + i \sqrt{\frac{2\lambda}{2}} C \hat{\sigma}],$$  \hspace{1cm} (28)

where $\log_n(x)$ for $n \geq 2$ is defined as the $n$-th Taylor remainder of the function $\log(x)$:

$$\log_n(x) = \log(x) - \left[ x - x^2/2 + x^3/3 \cdots + (-1)^{n+1} x^n/n \right].$$

Using (27) we have

$$-i/2 \text{Tr} \left( \sqrt{2\lambda} C \hat{\sigma} \right) = -i \sqrt{2\lambda/2} \sum_m T_m^\Lambda \sigma_{mm}$$

which could partially cancel the counter-term. So we can write the interaction vertex as:

$$V(\sigma) = -\frac{1}{2} \text{Tr} \log_2 \left[ 1 + i \sqrt{\frac{2\lambda}{2}} C \hat{\sigma} \right] + i \sqrt{\frac{2\lambda}{2}} \sum_m T_m^\Lambda \sigma_{mm} + \frac{1}{2} \lambda \Pi^\Lambda.$$

(29)

Remark that since the operators $C$ are positive, we can define naturally the operator $C^{1/2}$ with elements $[C^{1/2}]_{mn,kl} = \left( \frac{1}{m+n+1} \right)^{1/2} \delta_{ml} \delta_{nk}$ and equivalently write (29) as:

$$V(\sigma) = -\frac{1}{2} \text{Tr} \log_2 \left[ 1 + i \sqrt{\frac{2\lambda}{2}} C^{1/2} \hat{\sigma} C^{1/2} \right]$$

$$+ i \sqrt{\frac{2\lambda}{2}} \sum_m T_m^\Lambda \sigma_{mm} + \frac{1}{2} \lambda \Pi^\Lambda.$$

(30)

Now we consider the derivatives of the loop vertex w.r.t the intermediate fields. Using (27) we have

$$\frac{\partial}{\partial \sigma_{\alpha\beta}} \text{Tr} [1 + i \sqrt{\frac{2\lambda}{2}} C \hat{\sigma}] = \frac{\partial}{\partial \sigma_{\alpha\beta}} \sum_{mn,mm} \left[ \sum_N \frac{(-1)^{N+1}}{N} (i \sqrt{\frac{2\lambda}{2}})^N \left( C \hat{\sigma} \right)^N \right]_{mn,mm}$$

$$= i \sqrt{\frac{2\lambda}{2}} \left( \sum_m R_{m\beta,am} C_{ma} + \sum_n R_{an,n\beta} C_{\beta n} \right),$$

(31)
$R$ is called the resolvent matrix with elements

$$R_{mn,pq} = \left[ \frac{1}{1 + i\frac{\sqrt{2}λ}{2} \hat{C}^\dagger} \right]_{mn,pq} := \sum_n (-i\frac{\sqrt{2}λ}{2})^n (\hat{C}^\dagger)^n, \quad (\text{30})$$

and we have

$$\frac{\partial}{\partial \sigma^{ab}} R_{mn,pq} = -i\frac{\sqrt{2}λ}{2} \sum_s \left[ R_{mn,αs} C_sα R_{sβ,pq} + R_{mn,sβ} C_βs R_{αs,pq} \right], \quad (\text{32})$$

which could be symbolically written as

$$\frac{\partial}{\partial \sigma} R = -i\frac{\sqrt{2}λ}{2} RCR. \quad (\text{33})$$

4 The scale decompositions and indices sets

4.1 Sliced propagators and the index sets

Let $I_Λ = \{1, 2, \ldots, Λ\}$ be the set of matrix indices. Define $S := \{0, 1, \ldots, j_{\text{max}}\}$ as the set of coarse scaling indices, where for a fixed integer $M \geq 2$, $j_{\text{max}}$ is defined as the integer part of the number $\tilde{j}_{\text{max}}$ such that $M\tilde{j}_{\text{max}} = Λ$. Then $S$ divides $I_Λ$ into $j_{\text{max}} + 1$ slices:

$$I_Λ = I_0 \cup I_1 \cup I_2 \cup \ldots \cup I_{j_{\text{max}}},$$

each of which is made of the integers $ω(j) \in I_j = [M^j, M^{j+1} - 1]$, called the refined scaling indices. Define also $I_{≤j} = \cup_{k=0}^j I_k$. Let $J \subseteq S$ be subset of $S$, we can define the union of index sets $I_J = \cup_{j \in J} I_j$.

We can slice the propagator as:

$$C_{mn} = \sum_{ω \in I_Λ} C^ω = \sum_{j=0}^{j_{\text{max}}} \sum_{ω(j) \in I_j} C^ω_{mn}, \quad (\text{34})$$

where the propagator of refined scaling index $ω(j)$ is defined by:

$$C^ω_{mn} := C_{mn} \cdot \mathbb{I}_{m \in I_{≤j}, n \in I_{≤j}, m+n=ω(j) \in I_j}, \quad (\text{35})$$

where $\mathbb{I}_x$ is the characteristic function for the event $x$ defined by:

$$\mathbb{I}_x = \begin{cases} 1, & \text{if } x \text{ is true}, \\ 0, & \text{otherwise}. \end{cases} \quad (\text{36})$$

It is easy to find that

$$O(1)M^{-j-1} \leq |C^ω_{mn}| \leq O(1)M^{-j}, \quad (\text{37})$$

where $O(1)$ is a general name for the inessential constants of order 1.
Similarly we can slice the counter term as

\[ T^\omega(j) = \sum_{m \in \mathcal{I}_{j}} C^\omega(j) \big|_{m \in \mathcal{I}_{\leq j}} \]  

where

\[ T^\omega(j) m = \sum_{n \in \mathcal{I}_{j}} C^\omega(j) \big|_{m \in \mathcal{I}_{\leq j}, n \in \mathcal{I}_{j}, m+n = \omega(j) \in \mathcal{I}_{j}} = \frac{1}{\omega(j) + 1}. \]  

(38)

Define \( T^j_m = \sum_{\omega(j) \in \mathcal{I}_j} T^\omega(j) \). Using the fact that \( \omega(j) \in \mathcal{I}_j = [M^j, M^{j+1} - 1] \) and \( |\mathcal{I}_j| = (M - 1)M^j \), we have:

\[ T^j_m = \sum_{\omega(j) = M^j}^{M^{j+1} - 1} \frac{1}{\omega(j) + 1} = O(1), \quad T^\Lambda_m = \sum_{j} T^j_m = O(1) \ln \Lambda, \]  

(39)

and

\[ \Pi^\Lambda = \sum_{j=0}^{j_{\text{max}}} \sum_{m \in \mathcal{I}_{\leq j}} (\sum_{\omega(j)} T^\omega(j))^2 \leq \sum_{j=0}^{j_{\text{max}}} O(1)M^j = O(1)' \Lambda, \]  

(40)

where \( O(1) \) and \( O(1)' \) are the inessential order 1 constants.

4.2 The slice-testing expansion

The slice testing expansion is the renormalization procedure adapted to the multi-scale loop vertex expansions (MLVE). Each step of the slice testing expansion generates a propagator of refined scaling index \( \omega(j) \), which is called the marked propagator, accompanied with a resolvent. If the two operators form a loop \( \text{Tr} [C^\omega(j) R] \), then by partially expansion of \( R \) we have \( \text{Tr} [C^\omega(j) R] = \text{Tr} C^\omega(j) + \text{Tr} [C^\omega(j) (R - 1)] \), where \( \text{Tr} C^\omega(j) \) is a tadpole of scale \( \omega(j) \) and we shall compensate it with the corresponding counter term. We may also generate tadpole terms like \( \text{Tr} [C(t) R] = \text{Tr} C^\Lambda + \text{Tr} [C^\Lambda (R - 1)] \), where \( \text{Tr} C^\Lambda = \text{Tr} \sum_{j} \sum_{\omega(j)} C^\omega(j) \) is the full tadpole and can be compensated by the counter term \( T^\Lambda \). If the operator \( [C^\omega(j) R] \) doesn’t form a closed loop then we gain the convergent factor \( O(1)M^{-j} \) from the marked propagator. Marked propagators that are not tadpoles are the sources of the convergent factors which can be used to compensate the divergent Nelson’s bound. However, generating too many marked propagators results in large combinatorial factors. So we can’t expand too much and we have to set the stopping rules for the slice-testing expansions: we perform the expansions from the highest coarse scaling index \( j_{\text{max}} \) down to \( j_{\text{min}} \) and at each scale \( j \) we shall generate \( a M^j \) marked propagators, where \( a \in (0, 1) \) is a order one constant independent of the scaling indices \( j \). Then we stop the expansion at scale \( j \) and restart the expansion at scale \( j - 1 \) with the same stopping rule, and continue until we reach \( j_{\text{min}} \).

We introduce inductively interpolation parameters \( \{ t^\omega(j) \} \in [0, 1], \omega(j) \in \mathcal{I}_j, j \in S \), one for each marked propagator \( C^\omega(j) \) and write simply \( t \) for the family \( \{ t^\omega(j) \} \), which means we
write

\[ C(t) = \sum_j \sum_{\omega(j)} t^{\omega(j)} C^{\omega(j)}, \quad T(t) = \sum_j \sum_{\omega(j)} t^{\omega(j)} T^{\omega(j)}, \]

\[ V(\sigma)(t) = V_\Lambda(\sigma, t) = -\frac{1}{2} Tr \log_2 [1 + i \frac{\sqrt{2} \lambda}{2} \sum_{j=0}^{j_{\text{max}}} \sum_{\omega(j)=M_j} C^{\omega(j)} t^{\omega(j)} \tilde{\sigma}] \]

\[ + i \frac{\sqrt{2} \lambda}{4} Tr [\sum_{j=0}^{j_{\text{max}}} \sum_{\omega(j)=M_j} T^{\omega(j)} t^{\omega(j)} \tilde{\sigma}] + \frac{1}{2} \lambda \sum_{j=0}^{j_{\text{max}}} \sum_{\omega(j)=M_j} T^{\omega(j)} t^{\omega(j)} \tilde{\sigma}^2. \]

Let \( \Omega_j \subseteq I_j \) be the set of refined scaling indices of the marked propagators generated in the slice testing expansions. For \( J \subseteq S = [0, j_{\text{max}}] \), we can define \( \Omega_J = \bigcup_{j \in J} \Omega_j \) to be a subset of \( I_J = \bigcup_{j \in J} I_j \). We can write the partition function as:

\[ Z^{j_{\text{max}}} = \sum_{J \subseteq S} \sum_{\Omega_J \subseteq I_J} \int d\nu(\sigma) \left[ \prod_{j \in J} \prod_{\omega(j) \in \Omega_J} \int_0^1 dt^{\omega(j)} \frac{d}{dt^{\omega(j)}} \right] e^{V(\sigma)(t)} \bigg|_{t^{\omega(j)=0, \ for \ j \notin J, \ \omega(j) \notin \Omega_J}}. \]

Each derivation \( \frac{\partial}{\partial t^{\omega(j)}} \) on \( V(\sigma, t) \) results in:

\[ \frac{\partial}{\partial t^{\omega(j)}} V(\sigma, t) = -i \frac{\sqrt{2} \lambda}{4} Tr [C^{\omega(j)}(I \otimes \sigma + \sigma^t \otimes I)(R - 1)] \]

\[ + i \frac{\sqrt{2} \lambda}{4} Tr [T^{\omega(j)}(I \otimes \sigma + \sigma^t \otimes I)] + \lambda \sum_j T^{\omega(j)}(\sum_{j \in J} \sum_{\omega(j)} T^{\omega(j)} t^{\omega(j)}). \]

Due to the Gaussian measure the linear terms \( \sigma^t \otimes I \) and \( I \otimes \sigma \) in the numerator contract on the resolvents and the vertex \( e^{V(\sigma, t)} \) that are functions of \( \sigma \). Tadpoles will be generated and we shall compensate it with the corresponding counter terms. The resulting terms are free of tadpoles and counter-terms and are called the renormalized amplitudes. The ribbon graphs for the slice-testing expansions are also called the resolvent graphs, as they bear resolvents \( R \). Let \( \Omega_J \) be the set of refined scaling indices. An \( \Omega_J \)-resolvent graph is defined as a resolvent graph in which exactly \(|\Omega_J|\) marked propagators bear marks \( \Omega_J \). An \( \Omega_J \) resolvent graph is called minimal if any connected component of the graph bears at least one mark and the total perturbation order of the graph, i.e. the number of \( \sigma \) propagators, is at most \(|\Omega_J|\). The set of minimal \( \Omega_J \) resolvent graphs is noted \( G(\Omega_J) \) and we denote \( G = \bigcup_J G(\Omega_J) \).

We have the following lemma for the renormalized partition function:

**Lemma 4.1.** For an arbitrary minimal \( \Omega_J \) resolvent graph \( G \) the partition function can be expressed as integral over the renormalized amplitude \( A^R_G \) as:

\[ Z^{j_{\text{max}}} = \sum_{J \subseteq S} \sum_{\Omega_J \subseteq I_J} \sum_{G \subseteq G(\Omega_J)} c(G) \int d\nu(\sigma) \prod_{\omega(j) \in \Omega_J} \int_0^1 dt^{\omega(j)} [e^{V(t)} A^R(t, \sigma)] \bigg|_{t^{\omega(j)=0, \ for \ \omega(j) \notin I_J}}, \]

(44)
where

\[ A^R_G(t,\sigma) = (-\lambda)^{|V|} \text{Tr} \prod_{l \in CP(G), I \text{ tadpole}} \left[ (R(\sigma) - 1)C^{\omega(j)(\ell)} \right] \prod_{l \in CP(G), \ell \text{ not tadpole}} \left[ R(\sigma)C^{\omega(j)(\ell)} \right], \tag{45} \]

is the renormalized amplitude, \( CP(G) \) is the set of c-propagators of \( G \), the sum over \( G(\Omega_J) \) runs over a set of minimal resolvent graphs. The index \( \omega(j)(\ell) \) specifies the markings, that is, restricts the c-propagator \( \ell \) to be \( C^{\omega(j)} \) if that propagator bears the mark \( \omega(j) \). All propagators belonging to a single loop of the form \( \text{Tr} CR \) are renormalized, hence accompanied by an \( R-1 \) resolvent factor. The c-propagator which do not bear any mark are equal to \( C \).

It is useful to consider an example of first order slice-testing expansion before proving this lemma.

**Example 4.1** (The first order slice-testing expansion). Now we consider the first order expansion an an example. We have

\[
I_1 = \int dv(\sigma) \frac{d}{dt^{\omega(j)}} \text{e}^{V(t,\sigma)} = \int dv(\sigma) \text{e}^{V} \left\{ -i \frac{\sqrt{2\lambda}}{4} \text{Tr} \left[ C^{\omega(j)}(I \otimes \sigma + \sigma' \otimes I)(R - 1) \right] + \frac{\sqrt{2\lambda}}{4} \text{Tr} \left[ T^{\omega(j)}(I \otimes \sigma + \sigma' \otimes I) \right] + \lambda \text{Tr} T^{\omega(j)}(\sum_j \omega(j)) \right\}, \tag{46}
\]

where we have used the fact that \( dt^{\omega(j)}/ dt^{\omega(j)'} = \delta_{\omega(j)}, \omega(j') \delta_j, j' \).

Each intermediate field in the numerator in the form of \( \sigma \otimes I \) or \( I \otimes \sigma \) will act either on the vertex \( V \) or the resolvent \( R \) by integration by parts. Remark that the action of \( \sigma \otimes I \) acting on the term \( I \otimes \sigma \) of the resolvent \( R \) will generate a non-planar graphs from which we can gain convergent factors. We have:

\[
I_1 = \int dv(\sigma) \left\{ - \frac{\lambda}{2} \text{Tr} \left[ C^{\omega(j)}(R - 1) \right] \circ \text{Tr} \left[ C(t)(R - 1) \right] + \frac{\lambda}{2} \text{Tr} \left[ C^{\omega(j)}(R - 1) \right] T(t) 
+ \frac{\lambda}{2} \text{Tr} \left[ C(t)(R - 1) \right] T^{\omega(j)} \circ \text{Tr} \left[ C(t)R \right] 
+ \frac{\lambda}{2} \text{Tr} \left[ C^{\omega(j)}R \right] \circ \text{Tr} \left[ C(t)R \right] 
+ \text{non-planar terms} \right\} 
= \int dv(\sigma) \left\{ - \lambda \text{Tr} \left[ C^{\omega(j)}(R - 1) \right] \circ \text{Tr} \left[ C(t)(R - 1) \right] 
+ \text{non-planar terms} \right\}, \tag{47}
\]

where we have used the fact that \( \text{Tr} \left[ C(t)R \right] = \text{Tr} \left[ C(t)(R - 1) + C(t) \right], \text{Tr} C(t) = T^\lambda(t) \) and \( \text{Tr} C^{\omega(j)} = T^{\omega(j)} \). Let \( A \) and \( B \) be two \( \Lambda^2 \times \Lambda^2 \) dimensional matrices with double indices, the product \( \text{Tr} A \circ \text{Tr} B \) is defined as

\[
\text{Tr} A \circ \text{Tr} B := \sum_n \text{Tr} A_{nn} \circ \text{Tr} B_{nn} = \sum_{n=1}^{N} \sum_{m=1}^{N} \sum_{k=1}^{N} \sum_{k=1}^{N} [A_{nm,mn}] [B_{nk,kn}]. \tag{48}
\]

This definition can be easily generalized to the product of more matrices. For an illustration of the first order expansion see Figure 3.
Figure 3: Intermediate Feynman graphs for the first order slice-testing expansions. Here $C^{\omega_1}$ represents a marked propagator of scale $\omega$, $C$ is the full propagator $C(t)$; $R - 1$ is the renormalized resolvent and $R$ is the full resolvent. Graph A as well as its variants $B$, $D$ and $E$ are called dumbbell graphs. Graph B is a non-planar graph as the $\sigma$ fields hook to the inner boarders of two renormalized tadpoles but its amplitude is the same as Graph A.

Figure 4: A nonplanar graph generated by contracting two $\sigma$ fields attached to different boarders of the ribbon.

We have used the duality relation such that the tadpoles in Graph A and Graph B in Figure 5 have the same amplitude. This is a consequence of the fact that each $\phi^4$ interaction can be decomposed in two different ways into $\phi^2 \sigma$ vertex (see Figure 3) and we call this symmetry the flipping symmetry.

The propagator $C^{\omega(j)}$ is called the marked propagator which is the source for the convergent factors. Another source of convergent factors is the crossings, for each of which of scale $\omega(j)$ we can also gain a convergent factor $M^{-j}$.

An explanation of the numerical factors in (47) is in order. We have $-\lambda = -\frac{1}{8} \times 4 - \frac{1}{3} \times 2$, etc.
where $\frac{1}{8}$ is the contribution from the graphs $A$, $B$, $C$ and the total combinatorial factor is 4, and $\frac{1}{4}$ is the contribution from the graphs $D$ and $E$ for which the total combinatorial factor is 2. The slice-testing expansion of second order is given in the appendix.

Proof of Lemma 4.1. We shall prove this Lemma inductively. Suppose that Lemma 4.1 holds up to $n$ slices $\Omega_J = \{\omega_1, \cdots, \omega_{n-1}\}$ and the renormalized amplitude is denoted by

$$A_{G}^{n-1,R}(\sigma, t) = (-\lambda)^{n-1} \prod_{\ell \, \text{tadpoles}} \{[\text{Tr} \, (R - 1)C(\ell)] \circ [\text{Tr} \, (R - 1)C^\omega(\ell)]\} \times \text{Tr} \prod_{\ell \, \text{not tadpoles}} [RC(\ell)RC^\omega(\ell)].$$

(49)

Now we consider

$$\int d\nu(\sigma) \int dt^\omega_n \frac{d}{dt^\omega_n} [A_{G}^{n-1,R} e^{V(\sigma, t)}].$$

(50)

Without losing generality we can write $\int dt^\omega_n A_{G}^{n-1,R} = A_{G}^{n-1,R(\text{rest})} \cdot \frac{d}{dt^\omega_n} [\tilde{R}\tilde{C}]$, where $\tilde{R}$ is a general name for the resolvent $R$ or $(R - 1)$ and $\tilde{C}$ is a general name for $C$ or $C^\omega$.

We have:

$$\int d\nu(\sigma) \int dt^\omega_n \frac{d}{dt^\omega_n} [A_{G}^{n-1,R} e^{V(\sigma, t)}] = \int d\nu(\sigma) \int dt^\omega_n e^{V(\sigma, t)} \{ A_{G}^{n-1,R(\text{rest})} [-i\frac{\sqrt{2}\lambda}{2} R\tilde{\sigma}C^\omega R\tilde{C}] + A_{G}^{n-1,R} \left( -i\frac{\sqrt{2}\lambda}{4} \text{Tr} [C^\omega \tilde{\sigma}(R - 1)] + i\frac{\sqrt{2}\lambda}{4} \text{Tr} [T^\omega \tilde{\sigma}] + \lambda \text{Tr} [T^\omega T(t)] \right) \}. \quad (51)$$

$\tilde{\sigma}$ in the numerator will be contracted with all other terms that are functions of $\sigma$. We shall consider all the possibilities and prove that each tadpole generated in this process is canceled exactly with the corresponding counter term.

We shall start with the term in the first line:

\footnote{Remark that graph $A$ represents two different graphs where the tadpole with marked propagator can appear on the left side or on the right side.}
• if $\hat{\sigma}$ acts on the resolvent next to it, by using formula (32) and $\Tr (RC) = \Tr (R^{-1}C + T)$, we generate a tadpole term with amplitude $\left(-i \frac{\sqrt{2\lambda}}{2}\right) \times \left(-i \frac{\sqrt{2\lambda}}{2}\right) T \times 2 = -\lambda T$.

• if $\hat{\sigma}$ acts on any other resolvents or on the $\log_2[\cdots]$ term in the interaction $V$, the resulting terms contain no tadpoles nor counter terms so that they belong to $A^{n,R}_G$.

• if $\hat{\sigma}$ acts on the term $i \frac{\sqrt{2\lambda}}{4} \Tr \hat{\sigma} T$ from $V(\sigma, t)$ then we generate a term as $\left(-i \frac{\sqrt{2\lambda}}{2}\right) \times \left(i \frac{\sqrt{2\lambda}}{2} \lambda T\right) \times 4 = -\lambda T$. This term cancels exactly with the tadpole generated above.

So after contracting the $\sigma$ fields in the first line of (51) the resulting terms contain no tadpole not counter term, hence belong to $A^{n,R}_G$.

Now we consider the first term in the second line.

• If $\hat{\sigma}$ acts on the adjacent resolvent we get

$$
\Tr \left[-i \frac{\sqrt{2\lambda}}{4} C^{\omega_n} R\right] \circ \Tr \left[-i \frac{\sqrt{2\lambda}}{2} C R\right] \times 2
= -\frac{\lambda}{2} \left[ \Tr [(R - 1)C^{\omega_n}] \circ \Tr [(R - 1)C] + \Tr [(R - 1)C] \cdot T^{\omega_n} + \Tr [(R - 1)C^{\omega_n}] \cdot T + T^{\omega_n} T \right].
$$

(52)

• If $\hat{\sigma}$ acts on the counter term in $V(\sigma, t)$ we get $\frac{1}{2} \Tr [C^{\omega_n}(R-1)]T$ which cancels exactly the third term in (52).

• If $\hat{\sigma}$ acts on $A^{n-1,R}_G$ or on the $\log_2(\cdots)$ term in $V(\sigma, t)$ then no tadpole nor counter term will be generated.

Now we consider the last two terms in (51).

• $\hat{\sigma}$ acting on $\log_2(\cdots)$ generates

$$
\left(-i \frac{\sqrt{2\lambda}}{4} T^{\omega_n}\right) \times \left(i \frac{\sqrt{2\lambda}}{4} \Tr [(R - 1)C]\right) \times 4 = \frac{\lambda}{2} \Tr [(R - 1)C] \cdot T^{\omega_n},
$$

which cancels with the second term in (52);

• $\hat{\sigma}$ acting on the counter term in $V(\sigma, t)$ results in $-\frac{1}{2} T^{\omega_n} T$, which, together with the last term in (52), cancel with the last term in (51).

So all tadpoles generated in the slice-testing expansions cancel exactly with the counter-terms and $A^{R,n}$ contains only terms that are finite. So we have proved this Lemma.

The slice testing expansion cannot go forever as this will generate unbounded combinatorial factors. So we set the stopping rules for the slice testing expansion. Let $n_j \geq 1$ be the number of marked propagators of scale $j$. We perform the slice testing expansions at scale $j$ until $n_j = aM^j$, where $a \in (0, 1)$ is a numerical constant, and and we start the expansion at scale $j - 1$.

Remark that there could be many dumbbell graphs generated in the slice testing expansion which bears only one single marked propagator. We shall treat them separately.
4.3 Factorization of interaction

A resolvent graph $G$ corresponding to a non-empty set of marks $\Omega_J$ generated in the slice-testing expansion is in general not connected. Its amplitude $A^R_G$ factorizes over different connected components of $G$, so does the combinatorial factor $c_G$ and index set $\Omega_J$.

Now we list all the elements of $\Omega_J$ in the increasing order and label them by the ordinal natural numbers as: $\Omega_J := \{\omega_1, \omega_2, \cdots, \omega_h \mid \omega_j > \omega_i, \forall j > i\} = \Omega_J, h = |\Omega_J|$. Let $\Omega_a$ the set of refined scaling indices attributed to each connected component graph $G_a$, we have:

$$\Omega = \bigcup_a \Omega_a.$$ REMARK that the sets $\{\Omega_a\}$ are disjoint, namely $\Omega_a \cap \Omega_b = \emptyset$ for $a \neq b$. This is also called the hardcore constraint.

Now we factorize the action $e^{V(t)}$ into pieces attributed to each slice.

In order to perform this factorization we shall attribute to each loop vertex the index of its highest $c$-propagator and to each interpolation parameter $t_{\omega(j)}$ we introduce an additional auxiliary parameter $u_{\omega(j)}$ which multiplies $t_{\omega(j)}$ (we could also say that we substitute $t_{\omega(j)}(u_{\omega(j)}) = u_{\omega(j)}t_{\omega(j)}$) and we rewrite the interaction with cutoff $\omega(j)$ as

$$V(\sigma)_{\leq \omega(j)} = V_{\leq \omega(j)}(t, \sigma)|_{t_{\omega'(j')}=1}$$

$$= -\frac{1}{2}\text{Tr} \log_2[1 + i\frac{\sqrt{2\lambda}}{2} \sum_j \sum_{j'=0}^{\omega(j)} C^{\omega'(j')}t^{\omega'(j')}\hat{\sigma}]|_{t_{\omega'(j')}=1}$$

$$+ \frac{i\sqrt{2\lambda}}{4}\text{Tr} [\sum_j \sum_{j'=0}^{\omega(j)} T^{\omega'(j')}t^{\omega'(j')}\hat{\sigma}]|_{t_{\omega'(j')}=1}$$

$$+ \frac{1}{2}\lambda \text{Tr} (\sum_j \sum_{j'=0}^{\omega(j)} T^{\omega'(j')}t^{\omega'(j')}^2)|_{t_{\omega'(j')}=1}$$

(53)

The specific part of the interaction which should be attributed to the scale $\omega(j)$ is the sum over all loop vertices with at least one $c$-propagator at scale $\omega(j)$ and all others at scales

\(^2\)Recall that an empty set of marks correspond to the term 1 in the expansion of $Z_{j_{\text{max}}}^j(\lambda)$.

\(^3\)If we were to interchange this factorization and the testing expansion, the cancellation of tadpoles with counter-terms would not be exact.
\[ V_{\omega(j)} \equiv V_{\leq \omega(j)} - V_{\leq \omega(j)}^{-1} = V_{\leq \omega(j)}|_{u_{\omega(j)}=1} - V_{\leq \omega(j)}|_{u_{\omega(j)}=0} \]

\[ = \int_0^1 du_{\omega(j)} \frac{d}{du_{\omega(j)}} \left\{ \frac{1}{2} \sum_{j=0}^j \sum_{\omega(j) \in I} \sum_{\omega(j)=M^{j+1}} T_{\omega(j)} \right\}^2 \]

\[ + \text{Tr} \left[ i \frac{\sqrt{2} \lambda}{4} \sum_{j=0}^j \sum_{\omega(j) \in I} \sum_{\omega(j)=M^{j+1}} C_{\omega(j)} \right] \]

\[ = \frac{1}{2} \log_2[1 + i \frac{\sqrt{2} \lambda}{2} \sum_{j=0}^j \sum_{\omega(j) \in I} \sum_{\omega(j)=M^{j+1}} C_{\omega(j)}] \]

\[ = \int_0^1 t_j du_j \left\{ \lambda \sum_{p \in I} \left( \sum_{\omega(j) \in I} \sum_{\omega(j)=M^{j+1}} T_{\omega(j)} \right) T_{\omega(j)} \right\}

\[ + \text{Tr} \left[ i \frac{\sqrt{2} \lambda}{4} T_{\omega(j)} \hat{\sigma} - i \frac{\sqrt{2} \lambda}{4} [R_{\leq \omega(j)}(t) - 1] C_{\omega(j)} \hat{\sigma} \right], \quad (54) \]

and

\[ R_{\leq \omega(j)}(t) = \left( 1 + i \frac{\sqrt{2} \lambda}{2} \sum_{j=0}^j \sum_{\omega(j) \in I} \sum_{\omega(j)=M^{j+1}} C_{\omega(j)} \right)^{-1}. \quad (55) \]

In order to factorize the interaction vertex it is useful to write the index set \( \Omega_J \) as \( \Omega^J \) introduced earlier and for \( \omega_j \in \Omega^J \) such that \( \omega_j = \omega(j) \) we can define the sliced interaction as \( V_{\omega} := V_{\omega(j)} \) and \( R_{\leq \omega(j)} = \left( 1 + i \frac{\sqrt{2} \lambda}{2} \sum_{\omega=1}^{\omega_j} C_{\omega} t^{\omega} \hat{\sigma} \right)^{-1} \). Since \( t^{\omega} = 0 \) for \( \omega_j \notin \Omega^J \), the total interaction can be written as

\[ V(\sigma, t) = V_{\leq \Lambda}(\sigma, t) := \sum_{j=0}^{j_{\max}} \sum_{\omega(j)=M^j} V_{\omega(j)}(\sigma, t) = \sum_{\omega \in I^\Lambda} V_{\omega}. \quad (56) \]

Now that the combinatorial weights of Feynman graph factorize over connected components, \( c_G = \prod_a c_{G_a} \), we can rewrite the result of the slice-testing expansion as:

\[ Z^{j_{\max}}(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \int d\nu(\sigma) \sum_{\Omega_1, \ldots, \Omega_n, \Omega_a \cap \Omega_b = \emptyset} \prod_{a=1}^n \prod_{\omega \in \Omega_a} \int_0^1 dt^\omega \prod_{a=1}^n \mathcal{I}(\Omega_a, t, \sigma) \quad (57) \]

where

\[ \mathcal{I}(\Omega_a, t, \sigma) = \sum_{G_a \in \mathcal{CG}(\Omega_a)} c_{G_a} \left( \prod_{\omega \in \Omega_a} [e^{\nu(t, \sigma)}] A_{G_a}^R(t, \sigma) \right), \quad (58) \]

\( \mathcal{CG}(\Omega) \) being the set of connected graphs in \( \mathcal{G}(\Omega) \). The \( 1/n! \) in \( (57) \) comes from summing over the sequences \( \Omega_1, \ldots, \Omega_n \) in \( (57) \). The sum over \( n \) is in fact not infinite due to the hardcore constraint, which forces all terms to be zero for \( n \geq M^{j_{\max}} + 1 \). The term \( n = 0 \) corresponds to the factor 1 in the sum (the normalization \( \omega \) of the free theory).
We can now encode the hardcore constraints $\Omega_a \cap \Omega_b = \emptyset$ through Grassmann numbers as

$$Z_{j_{\text{max}}}^\lambda = \sum_{n=0}^\infty \frac{1}{n!} \int dt \int d\nu(\sigma) d\mu_{1,\lambda}(\bar{\chi}, \chi)$$

$$\prod_{a=1}^n \left( \sum_{\Omega_a \subset \Omega} \sum_{G \in CG} c_G \left( \prod_{\omega \in \Omega_a} [\bar{\chi}_\omega e^{V_\omega(t, w, \sigma)} \chi_\Omega A_G^{R, M}(t, w, \sigma) \right) \right)$$

(59)

where $d\mu_{1,\lambda}(\bar{\chi}, \chi) = \prod_{\omega=1}^{\omega_{\text{max}}} d\bar{\chi}_\omega d\chi_\omega e^{-\bar{\chi}_\omega \chi_\omega}$ is the standard normalized Grassmann Gaussian measure with covariance $I_S$, which is the $\Lambda$ by $\Lambda$ dimensional identity matrix. Indeed Grassmann Gaussian variables automatically implement the hardcore constraints, and saturate the Grassmann pairs for $\omega \notin \Omega = \bigcup_a \Omega_a$.

Remark that (59) is the developed expansion for a new type of vertex $W$ which is a sum over slice-subsets $\bar{\Omega}_a$:

$$Z_{j_{\text{max}}}^\lambda = \int dt \int d\nu(\sigma) d\mu_{\bar{\Omega}, \lambda}(\bar{\chi}, \chi) e^W,$$

$$W = \sum_{\Omega \subset \Omega, G \in CG} c_G \left( \prod_{\omega \in \Omega} [\bar{\chi}_\omega e^{V_\omega(t, w, \sigma)} \chi_\Omega A_G^{R, M}(t, w, \sigma) \right)$$

(60)

To distinguish $W$, which contains potentially infinitely many vertices of type $V$, we call it an exp-vertex.

However the sums over the blocks $\Omega_a$ in (57) doesn’t factorize due to the integral over the parameters $t$, the functional integral over $\sigma$ and the hardcore constraint $\Omega_a \cap \Omega_b = \emptyset$. In order to factorize the integration measure and remove the hardcore constraints we shall perform the multi-scale Loop vertex expansions.

5 The Multiscale Loop Vertex Expansion

5.1 The Two-Level Jungle Formula

We perform now the two-level jungle expansion [40], starting from (59). Here we just summarize the main steps, referring to [40] for details.

First of all we introduces Bosonic replicas for all the exp-vertices such that each vertex $W_a$ has its own Bosonic field $\sigma^a$ and parameters $\{t_a\}$ which are denoted by $\bar{t}_a$. Let $W = \{1, \cdots, n\}$ the set of labels for the exp-vertices, we have

$$Z_{j_{\text{max}}}^\lambda = \sum_{n=0}^\infty \frac{1}{n!} \int dt_W \int d\nu_W(\sigma, \bar{\chi}, \chi) \prod_{a=1}^n W_a(\bar{t}_a, \sigma_a, \bar{\chi}, \chi) ,$$

(62)

The replicated measure is completely degenerate:

$$d\nu_W(t) = d\nu_{1, W}(\{\bar{t}_a\}) ,$$

$$d\nu_W = d\nu_{1, W}(\{\sigma_a\}) d\mu_{\bar{\Omega}, \lambda}(\bar{\chi}_\omega, \chi_\omega)$$

(63)

$$W_a(\sigma_a, \bar{\chi}, \chi) = \sum_{\Omega \subset \Omega_a} \sum_{G \in CG(\Omega)} c_G \left( \prod_{\omega \in \Omega} [\bar{\chi}_\omega e^{V_\omega(t, \sigma_a)} \chi_{\Omega} A_G^{R, M}(t, \sigma_a) \right) .$$

(64)
where $1_{\mathcal{W}}$ is the $n$ by $n$ matrix with coefficients 1 everywhere. Remark that the sum over $n$ is in fact finite due to the hardcore constraint.

The obstacle to factorize the functional integral $Z$ over vertices lies in the Bosonic degenerate blocks $1_{\mathcal{W}}$ and in the Fermionic fields which couple the vertices $W_a$, and the integration over the parameters $\{t\}$. In order to remove these two obstacles we need to apply two successive forest formulas \cite{33, 34}, one Bosonic, the other Fermionic.

To factorize the Bosonic measure $dv$ over the blocks $1_{\mathcal{W}}$ we introduce coupling parameters $x_{ab} = x_{ba}, x_{aa} = 1$ between the Bosonic vertex replicas. We introduce also the interpolating parameters $x_{ab}$ for the parameters $\{t_a\}$ as follows: consider an integral as

$$F = \int dt_1 \cdots dt_n \prod_{a=1}^n f_a(t_1, \ldots, t_n), \quad (65)$$

Introducing for each pair of labels $(a, b)$, $1 \leq a < b \leq n$, a pair interpolation parameters $x_{ab}$ to simultaneously multiply the $t_b$ dependence of $f_a$ and the $t_a$ dependence of $f_b$, and we have:

$$F = \int dt_1 \cdots dt_n \prod_{a=1}^n f_a(x_{1,a}t_1, \ldots, x_{a-1,a}t_{a-1}, t_a, x_{a,a+1}t_{a+1}, \ldots, x_{a,n}t_n) |_{x_{a,b}=1} \forall a, b. \quad (66)$$

Here the functions $f_a$ are identified as $W_a$ and $t_a$ the list of $|\Omega_a|$ parameters $t^\omega$ for $\omega \in \Omega_a$. Representing Gaussian integrals as derivative operators we have

$$Z_{\lambda}^{\max} = \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \exp \left( \frac{\lambda}{2} \sum_{a=1}^n x_{ab} \frac{\partial}{\partial \sigma_a} \frac{\partial}{\partial \sigma_b} + \sum_{\omega \in \Omega} \frac{\partial}{\partial \chi_b} \frac{\partial}{\partial \bar{\chi}} \right) \prod_{a=1}^n \left[ \int dt_a \right] W_a(\sigma_a, \bar{\chi}, \chi; \{ x t^\omega \}) \right]_{\sigma \chi \bar{\chi} = 0}. \quad (67)$$

The next step applies the standard Taylor forest formula of \cite{33, 34} to the parameters $\{x_{ab}\}$. We denote by $F_B$ a Bosonic forest with $n$ vertices labeled $\{1, \ldots n\}$. For $\ell_B$ a generic edge of the forest we denote by $a(\ell_B), b(\ell_B)$ the end vertices of $\ell_B$. The result of the Taylor forest formula is:

$$Z_{\lambda}^{\max} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathcal{F}_B}^{\ell_B} \left[ \prod_{\ell_B \in \mathcal{F}_B} \left( \frac{\partial}{\partial \sigma_a(\ell_B)} \frac{\partial}{\partial \sigma_b(\ell_B)} \right) \left( \prod_{\ell=(a,b) \in \mathcal{F}} \frac{\partial}{\partial x_{a,b}} \right) \prod_{a=1}^n W_a(\sigma_a, \bar{\chi}, \chi; x_{ab} w_{\ell_B}) \right]_{\sigma \chi \bar{\chi} = 0} \quad (68)$$

where $X_{ab}(w_{\ell_B})$ is the infimum over the parameters $w_{\ell_B}$ in the unique path in the forest $F_B$ connecting $a$ to $b$. This infimum is set to 1 if $a = b$ and to zero if $a$ and $b$ are not connected by the forest \cite{33, 34}.

The forest $F_B$ partitions the set of vertices into blocks $\mathcal{B}$ (which we also call it the effect vertex and denote it by $\{a\}/\mathcal{F}_B$) corresponding to its connected components. The integration measures $dv_{\mathcal{W}}(t)$ and $dv_{\mathcal{W}}(\sigma)$ factorize over the blocks. The edges of each block $\mathcal{B}$ form a spanning tree. Remark that any vertex $a$ belongs to a unique Bosonic block. The corresponding graph is shown in Figure 6 in which each block represents a tree.
The next step introduces replica Fermionic fields $\chi_B^\omega$ for these blocks of $\mathcal{F}_B$ and replica coupling parameters $y_{BB'} = y_B^B$. The last step applies (once again) the forest formula, this time for the $y$'s, leading to a set of Fermionic edges $\mathcal{L}_F$ forming a forest in $\{n\}/\mathcal{F}_B$. Denoting $\mathcal{L}_F$ a generic Fermionic edge connecting blocks and $\mathcal{B}(\mathcal{L}_F), \mathcal{B}'(\mathcal{L}_F)$ the end blocks of $\mathcal{L}_F$, we obtain a two-level-jungle formula \[34\]. It writes

$$Z^\text{max}_j(\lambda) = \sum_{n=0}^\infty \frac{1}{n!} \sum_\mathcal{J} \int dw_\mathcal{J} \int dv_\mathcal{J} \partial_\mathcal{J} \left[ \prod_{a \in \mathcal{B}} \prod_{b \in \mathcal{B}'(\mathcal{L}_F)} W_a(\sigma_a, \chi_B^\omega, \bar{\chi}_B^\omega) \right], \quad (69)$$

where

- A two-level jungle $\mathcal{J}$ is defined as an ordered pair $\mathcal{J} = (\mathcal{F}_B, \mathcal{F}_F)$ of two (each possibly empty) disjoint forests on $\mathcal{W}$, such that $\mathcal{F}_B$ is a forest, $\mathcal{F}_F$ is a forest and the union of two forests $\mathcal{J} = \mathcal{F}_B \cup \mathcal{F}_F$ is still a forest on $\mathcal{W}$. The forests $\mathcal{F}_B$ and $\mathcal{F}_F$ are called the Bosonic and Fermionic components of $\mathcal{J}$, respectively. The sum runs over all two-level jungles $\mathcal{J}$.

- $\int dw_\mathcal{J}$ means integration from 0 to 1 over parameters $w_\ell$, one for each edge $\ell \in \mathcal{J}$, namely $\int dw_\mathcal{J} = \prod_{\ell \in \mathcal{J}} \int_0^1 dw_\ell$. There is no integration for the empty forest since by convention an empty product is 1. A generic integration point $w_\Omega$ is therefore made of $|\mathcal{J}|$ parameters $w_\ell \in [0,1]$, one for each $\ell \in \mathcal{J}$.

- Any derivative $\frac{\partial}{\partial x_a}$ in $\partial^\mathcal{F}$ can either act either on $e^{V_\sigma}$ or on an resolvent or $A_{\sigma_a}^B(t, \sigma)$.

$$\partial_\mathcal{J} = \prod_{\ell_B \in \mathcal{F}_B} \left( \frac{\partial}{\partial \sigma_a} \frac{\partial}{\partial \sigma_b} \right) \prod_{\ell_F \in \mathcal{F}_F} \sum_{\omega_{\ell_F} = 0}^{\omega_{\text{max}}} \left( \frac{\partial}{\partial \chi_B(d)} \frac{\partial}{\partial \chi_B(e)} + \frac{\partial}{\partial \bar{\chi}_B(e)} \frac{\partial}{\partial \bar{\chi}_B(d)} \right), \quad (70)$$

where $\mathcal{B}(d)$ denotes the Bosonic block to which the vertex $d$ belongs.
The measure $d\nu_J$ has covariance $X(w_{\ell_B})$ on Bosonic variables and $Y(w_{\ell_F}) \otimes I_S$ on Fermionic variables, hence
\[
\int d\nu_J F = \left[ e^{\frac{1}{2} \sum_{a,b=1}^n X_{ab}(w_{\ell_B}) \frac{\partial}{\partial \sigma_a} \frac{\partial}{\partial \sigma_b} + \sum_{B,B'} Y_{BB'}(w_{\ell_F}) \sum_{\omega \in \Omega} \frac{\partial}{\partial \bar{\chi}^B} \frac{\partial}{\partial \chi^{B'}} F} \right]_{\sigma=\bar{\chi}=\chi=0}.
\]
(71)

- $X_{ab}(w_{\ell_B})$ is the infimum of the $w_{\ell_B}$ parameters for all the Bosonic edges $\ell_B$ in the unique path $P_{a \rightarrow b}$ in $\mathcal{F}_B$. This infimum is set to zero if such a path does not exist and to 1 if $a = b$.

- $Y_{BB'}(w_{\ell_F})$ is the infimum of the parameters $w_{\ell_F}$ for all the Fermionic edges $\ell_F$ in any of the paths $P_{a \rightarrow b} \cup P_{a \rightarrow b'}$ from some vertex $a \in \mathcal{B}$ to some vertex $b \in \mathcal{B}'$. This infimum is set to 0 if there are no such paths, and to 1 if such paths exist but do not contain any Fermionic edges.

Remark that the Gaussian measure $d\nu_J$ is well-defined, since the symmetric $n$ by $n$ matrix $X_{ab}(w_{\ell_B})$ is positive for any value of $w_{\ell_B}$.

Since the slice assignments, the fields, the measure and the integrand are now factorized over the connected components of $\mathcal{J}$, the logarithm of $Z$ is easily computed as exactly the same sum but restricted to two-levels spanning trees, denoted by $\mathcal{J}^T$, on $W = [1, \cdots, n]$. We have:
\[
\log Z_{j_{\text{max}}}^{\text{max}}(\lambda) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathcal{J}^T} \int dw_{\mathcal{J}^T} \int d\nu_{\mathcal{J}^T} \partial_{\mathcal{J}^T} \left[ \prod_B \prod_{a \in \mathcal{B}} W_a(\sigma_a, \chi^B, \bar{\chi}^B) \right].
\]
(72)

The main result is the convergence of this representation uniformly in $j_{\text{max}}$ for $\lambda$ in a certain domain, allowing to perform in this domain the ultraviolet limit of the theory. More precisely

**Theorem 5.1.** Fix $\rho > 0$ small enough. The series (72) is absolutely convergent, uniformly in $j_{\text{max}}$, for $\lambda$ in the small open cardioid domain $\text{Card}_\rho$ defined by $|\lambda| < \rho \cos^2[(\text{Arg } \lambda)/2]$ (see Figure 7). Its ultraviolet limit $\log Z(\lambda) = \lim_{j_{\text{max}} \to \infty} \log Z_{j_{\text{max}}}^{\text{max}}(\lambda)$ is therefore well-defined and analytic in that cardioid domain; furthermore it is the Borel sum of its perturbation series in powers of $\lambda$. 

\[21\]
5.2 Combinatorial Properties of MLVE

In this part we shall study the graphic representations and combinatorial properties of the MLVE graphs. A typical such graph is a forest made of two level trees $J^T$ whose vertices are the Bosonic blocks (with Bosonic edges inside) and whose edges are the Fermionic lines. A two level tree made of one single vertex (without Fermionic or Bosonic edges) is allowed. Figure 5.2 shows an example of a MLVE forest made of two trees. Now we consider the combinatorial properties of the two level trees $J^T$ and we have the following Proposition:

**Proposition 5.1.** The number of two level trees $J^T$ over $n \geq 1$ vertices is bounded by $2^{2n}n^{n-2}$.

**Proof** This proposition is well known and the number is exactly $2^{n-1}n^{n-2}$. We shall just sketch the proof by listing some combinatorial properties of the MLVE Graphs. The interested reader could consult [40] for more details.

From the discussion of the previous sections we know that the MLVE create in the first step the Bosonic forest made of Bosonic blocks $B$, over the set of vertices $\{1, 2, \cdots, n\}$, and then connect them with Fermionic edges to form the two level forests which factorize into two level trees $J^T$.

- Let $P = \{B|B \subseteq \{1, \cdots, n\} \text{ into different disjoint subsets } B\}$. Let $|P|$ be the number of blocks $B$ in the partition. We shall choose one Bosonic tree $T$ in each block $B$. So the total number of $T$ is $\prod_{B \in P} |B|^{(|B|-2)}$.

- We can choose a set of Fermionic edges $L_F$ connecting these Bosonic blocks to form a two level tree. Let the coordination number at each block $B \in P$ be $D_B$, $B \in P$. The total number of labeled trees with $|P|$ vertices and attribution of coordination number $D_B$ for each block $B$ is $\sum_{D_B, \sum_B D_B=2|P|-2} \frac{(|P|-2)!}{\prod_{B \in P} (D_B-1)!}$. 

Figure 8: A Fermionic forest, which is made of two trees.
• Since there exist $|B|$ vertices in each block $B$, each Fermionic edge has $|B|$ choices to hook to a block. So the total number of choices is $\prod_{B \in \mathcal{P}} |B|^{D_B}$.

• We shall organize the blocks by cardinals. Let $B_q$ be the block with $q$ vertices and $m_q$ be the number of such blocks, then we have $\sum_q |B| = \sum_q m_q |B_q| = \sum_q q m_q = n$. The number of such partitions is $\frac{n!}{\prod_{q \geq 1} (m_q!)^{q m_q}}$.

Combine all these facts and summing over all possible partitions $\mathcal{P}$ we can easily prove that

$$\left| \sum_{\mathcal{P}} \sum_{D_B, \sum_{B \in \mathcal{P}} D_B = 2|\mathcal{P}|-2} \frac{(|\mathcal{P}| - 2)!}{(D_B - 1)!} \left( \prod_{B \in \mathcal{P}} |B|^{D_B} \right) \left( \prod_{B \in \mathcal{P}} |B|^{|B|-2} \right) \right| \leq 2^{2n} n^{n-2}. \quad (73)$$

The graphic structure of the Bosonic blocks is more complex. Each Bosonic bloc is a tree over $n$ vertices made of marked and unmarked propagators, resolvents with decoration of counter-terms. Figure 3 shows a general Bosonic tree graph of MLVE. The black dots represent the counter terms; The leaves made of one circle are the renormalized resolvents $\text{Tr}(R - 1)C$. Dash lines represent the $\sigma$ propagators. The segments sandwiched by the dash lines in circles that are not leaves are the resolvent propagators $RC$. Marked and unmarked propagators are not distinguished in this Figure. A MLVE graph is in general non-planar due to the crossing lines of the $\sigma$ propagators generated in the slice-testing expansions. This graphic representation of the multi-scale LVE is also called the direct representations. The amplitudes of a LVE graph in the direct representation and its corresponding dual representation are equal.

For each MLVE graph in the direct representation there is a unique corresponding dual representation, which is represented as a single big circle divided by dotted lines representing the $\sigma$ propagator. Each region enclosed by the $\sigma$ propagators in the dual graph represents a loop vertex in the direct representation, see Figure 9. The cyclic ordering for the counter-terms is the same as in the direct representation.

Figure 9: The dual representation of a Bosonic block. Here all the tadpoles and counter-terms are hooked to one side of the ribbon graph, for simplicity.
6 Proof of Theorem 5.1

In this section we shall prove that the perturbation series for log $Z_{\text{max}}$ is Borel summable in the cardioid domain. The key steps to prove this theorem is to get the optimal bounds for the Fermionic integral and the Bosonic integrals. While the former is essentially algebraic and the procedure for getting the bound is very similar to [27], the latter is new due to the linear divergence of the vacuum tadpole term from the Wick ordering.

6.1 Grassmann Integrals

The Grassmann Gaussian part of the functional integral (72) is treated as in [40], resulting in a similar computation. Let us for the moment fix the forest $F$, hence also the two ends $a, b$ of each Fermionic link $\ell F$, and let us also fix the scale $\omega_{\ell F}$ of each Fermionic link $\ell F$. All these data will be summed later. Defining the natural $n$ by $n$ extension of the matrix $Y_B$ by $Y_{ab} = Y_B(a)Y_B'(a)_{b}$, we can evaluate

$$
\int \left[ \prod_B \prod_\omega (d\overline{\chi}_B d\chi_B) \right] e^{-\sum_{\omega=0}^{\omega_{\text{max}}} \overline{\chi}_B Y_B'(w_{\ell F}) \overline{\chi}_B} \prod_{\ell F \in F_{\omega}(a,b)} \left( \overline{\chi}_{\omega_{\ell F}} \lambda_{\omega_{\ell F}} + \chi_{\omega_{\ell F}} \lambda_{\omega_{\ell F}} \right)
$$

$$
= \left( \prod_B \prod_{a,b \in B} 1(\Omega_a \cap \Omega_b = \emptyset) \right) \left( Y_{\hat{a}_1 \ldots \hat{b}_k} + Y_{\hat{b}_1 \ldots \hat{a}_k} + \ldots + Y_{\hat{b}_1 \ldots \hat{b}_k} \right),
$$

where

$$
Y_{\hat{a}_1 \ldots \hat{b}_k} = \int \left( \prod_i d\overline{\chi}_i d\chi_i \right) e^{-\sum_{i,j} \overline{\chi}_i Y_{ij} \chi_j} \prod_{i=1}^k \overline{\psi}_{\hat{a}_i} \psi_{\hat{b}_i},
$$

is the minor of $Y$ with the lines $a_1 \ldots a_k$ and columns $b_1 \ldots b_k$ deleted, $k = |F|$ is the total number of vertices of the Fermionic forest and the sum runs over the $2^k$ ways to exchange the ends $a_i$ and $b_i$ of each $\ell F$, and the $Y$ factors are (up to a sign) the minors of $Y$ with the lines $b_1 \ldots b_k$ and the columns $a_1 \ldots a_k$ deleted. The most important factor in (74) is $\prod_B \prod_{a,b \in B} 1(\Omega_a \cap \Omega_b = \emptyset)$ which means the hard core constraint inside each block, ensures the disjointness of the slices in each block.

Positivity of the $Y$ covariance means that the $Y$ minors are all bounded by 1 [49, 40], namely for any $a_1, \ldots a_k$ and $b_1, \ldots b_k$,

$$
\left| Y_{\hat{a}_1 \ldots \hat{b}_k} \right| \leq 1.
$$

So the Grassmann integral is bounded by $O(1) \cdot 2^k$.

6.2 Bosonic Integrals

The main problem is now the evaluation of the Bosonic integral in (72). Since it factorizes over the Bosonic blocks, it is sufficient to bound separately this integral in each fixed block $B$. Consider such a block $B$. Let the graphs associated with that block be $G(\Omega(B)) = \cup_{a \in B} G(\Omega_a)$
with associated index sets $\Omega(B) = \bigcup_{a \in B} \Omega_a$. Now the Bosonic forest $F_B$ restricts to a Bosonic tree $T_B$, and the Bosonic Gaussian measure $d\nu(\sigma)$ restricts to $d\nu_B$ defined by:

$$\int d\nu_B F_B = \left[ e^{\frac{1}{2} \sum_{a,b \in B} X_{ab}(w(B))} \frac{\partial}{\partial \sigma_a} \frac{\partial}{\partial \sigma_b} F_B \right]_{\sigma=0}. \quad (77)$$

The Bosonic integrand is obtained by evaluating the action of the derivatives on the $exp$-vertices:

$$F_B = \prod_{a \in B} \left[ \prod_{e \in E^{a}_B} \left( \frac{\partial}{\partial \sigma_e} \right) W_a \right] \prod_{a \in B} \left[ \prod_{e \in E^{a}_B} \left( \frac{\partial}{\partial \sigma_e} \right) \prod_{\omega \in \Omega_a} \int_0^1 dt \omega V(t,\sigma_a) \right] A_{G_a}(t, \sigma_a), \quad (78)$$

where $\Omega_a$ is the set of refined scaling indices of vertex $a$, $E^{a}_B$ runs over the set of all edges in $T_B$ which end at vertex $a$, hence $|E^{a}_B| = d_a(T_B)$, the degree or coordination of the tree $T_B$ at vertex $a$. The derivatives $\left( \prod_{e \in E^{a}_B} \frac{\partial}{\partial \sigma_e} \right)$ act either on the amplitudes $A_{G_a}$ or derive new loop vertices from the exponential $\prod_{\omega \in \Omega_a} e^{V(\sigma_a)}$. When the block $B$ is reduced to a single vertex $a$, there is no derivative to compute and the integrand reduces simply to the one of the slice-testing expansion. This case is easy.

When $B$ has more than one vertex, since $T_B$ is a tree, each vertex $a \in B$ is touched by at least one derivative. We can evaluate the derivatives in (78) through the Faà di Bruno formula:

$$\left( \prod_{e \in E} \frac{\partial}{\partial \sigma_e} \right) f(g(\sigma)) = \sum_{\pi} f^{|\pi|}(g(\sigma)) \prod_{B \in \pi} [(\prod_{e \in B} \frac{\partial}{\partial \sigma_e}) g(\sigma)], \quad (79)$$

where $\pi$ runs over the partitions of the set $\Omega(B)$ and $B$ runs through the blocks of the partition $\pi$.

The resulting terms after performing the the forest formula are as follows:

- the exponential $\prod_{a \in B} \prod_{\omega \in \Omega_a} e^{V(\sigma_a)}$ cannot disappear since the exponential function is its own derivative,

- the derivatives which act on the graph integrand $A_{G_a}(t, \sigma_a)$ must act on the $R$ resolvent factors and create therefore new propagators sandwiched by resolvents, through $\frac{\partial}{\partial \sigma} R = i \frac{\sqrt{2}}{2} R C R$,

- the derivatives which act on the exponential create new loop vertices of the type $i \frac{\sqrt{2}}{2} C^\omega(R-1)$ or the counter terms $T^\omega$.

Again tadpoles $\text{Tr}\tilde{C}R = \text{Tr}\tilde{C} + \text{Tr}(R-1)C$ will be generated in the forest expansion, where $\tilde{C}$ stands for a marked propagator $C^{\Omega_1}$ or the full propagator $C(t)$. We shall compensate the tadpoles by the corresponding counter terms, in the same way as in the slice testing expansions and the renormalized amplitude is free of tadpoles or counter-terms. The resulting expression is almost the same as (45), namely it can be written as product of renormalized
resolvents, $RC$ or $(R-1)C$ depends on if the resolvent forms a tadpole, with the marked propagators $C^\omega$. Hence we can write the Bosonic integrand after action of derivatives as

$$\int dw_B \int d\nu_B F_B = \sum_{\Omega_B} \prod_{\ell \in B} \int_0^1 dw_\ell \int d\nu_B e^{V(B,\Omega_B)} \sum_{G_B \in \mathcal{G}(B)} A_{G_B}(\sigma, \Omega_B).$$

(80)

where the graphs in $\mathcal{G}(B)$ are trees that are resolvent graphs over all vertices of $B$.

Since each derivation $\partial / \partial \sigma$ of the Forest formula brings down exactly one marked propagator and the coupling constant of order $\lambda$, the tree of $|B|-1$ derivatives connecting $B$ brings down exactly the coupling constant of order $\lambda^{|B|-1}$ and $|B|-1$ $\sigma$-propagators.

Then we perform a Cauchy-Schwarz inequality with respect to the positive measure $d\nu_B$ to separate the graphs from the remaining interaction:

$$\sum_{\Omega_B} \sum_{G_B \in \mathcal{G}(B)} \left| \int dw_B \int d\nu_B e^{V(B,\Omega)}(\sigma)A_{G_B}(\sigma) \right| \leq \sum_{\Omega_B} \sum_{G_B \in \mathcal{G}(B)} \left( \int dw_B \int d\nu_B |A_{G_B}(\sigma)|^2 \right)^{1/2} \left( \int dw_B \int d\nu_B \prod_a e^{2[V(B,\Omega_a)]} \right)^{1/2}.$$  

(81)

6.3 Non-Perturbative Bounds I: Getting Rid of Resolvents

In this subsection we explain how to bound, for a fixed graph $G_B$ the factor

$$I = \left( \int d\nu_B(\sigma)|A_{G_B}(\sigma)|^2 \right)^{1/2}$$

(82)

from the Cauchy-Schwarz inequality (81). This is still a non-perturbative problem, since the resolvents in the $A_{G_B}(\sigma)$, if expanded in power series of $\sigma$ and integrated out with respect to $d\nu_B$, would lead to infinite divergent series of Feynman graphs. Hence we shall use the norm bound (90) to get rid of these resolvents.

We write

$$I^2 = \int d\nu_B A_{G_B}(\sigma)A_{\hat{G}_B}(\sigma)$$

(83)

In the last section we find that any graph $G_B$ is obtained from a minimal graphs $G$ (not connected) by using the BKAR formula which effectively adds $(|B|-1)$ tree lines to $G$. Now we fix $G_B$ and $G$. Let the set of refined scaling index of $G$ be $\Omega(G)$ with cardinal $|\Omega(G)| = p$. Due to the hardcore constraint the number of disconnected components in $G$ is bounded by the cardinality of $\Omega(G)$, namely $|B| \leq p$.

The amplitudes for these graphs, which we call $G_B$ are given by (45) hence still include typically many renormalized tadpoles in the form of $\text{Tr } (R-1)C$. In order to get the correct bounds for them we write each $R-1$ as $R-1 = -i\sqrt{2\lambda}C\hat{\sigma}R$. Each $\sigma$ field in the numerator contracts either with another $\sigma$ field in the numerator or the $\sigma$ field hidden in a resolvent $R$. In either case the number of $\sigma$ propagators is increased by one and the number of $\sigma$ fields is decreased by two or one. Let the set of renormalized tadpoles in $B$ be $\tau$ and the set of newly generated $\sigma$ propagators be $\gamma$, then we have $|\tau| \leq |B| \leq p$ and $|\gamma| \leq |\tau|$. Let us call $n$ the order of perturbation theory for $G_B$ (i.e. the order of $\lambda$ which is equal to the number of
the \( \sigma \) propagators). Then we have \( n = p + (|\mathcal{B}| - 1) + |\gamma| \leq 3p \). The order of perturbation for \( G_B \cup \bar{G}_B \) is 2\( n \).

The amplitude for the resulting graphs can be written as:

\[
A_{G_B}^R = \left[ (-\lambda)^n \right] \text{Tr} \left[ \prod_{\ell \in T(\mathcal{B})} R(\sigma)C R(\sigma)C_\omega \prod_{\ell \in \gamma} R(\sigma)C R(\sigma)C_{\omega(\ell)} \right],
\]

which contain typically many resolvents \( R \). The next step is to get rid of all the resolvents \( R(\sigma) \) by using the fact that their norm is bounded by \( \cos^{-1}(\phi/2) \) in the cardioid domain. This is not trivial and we shall rely on the technique of recursive Cauchy-Schwartz (CS) inequalities of \([24, 25, 27]\) and we shall go to the dual representation of the LVE graphs. The key definition to define the CS inequality is that of a balanced cut.

We define a balanced \( X - Y \) cut for \( G_B \) as a partition of the graph \( G_B \) into two pieces which we call the top and bottom chains, \( H_t \) and \( H_b \), each containing the same number of resolvent propagators (up to one unit if the initial number of resolvent propagators is odd). \( H_t \) and \( H_b \) are each made of a chain of \( \lfloor n/2 \rfloor \) or \( \lceil n/2 \rceil - 1 \) resolvent propagators, where the function \( \lfloor y \rfloor \) takes the integer part of \( y \in \mathbb{R} \), plus the two half resolvent propagators \( X \) and \( Y \) at the ends of the chain, plus an arbitrary number of cleaned propagators (it needs not be the same number in \( H_t \) and \( H_b \)). To these two chains are hooked the same number \( q \) of half \( \sigma \)-propagators which cross the cut, plus certain number of \( \sigma \)-propagators that do not cross the cut, which we call the inner \( \sigma \)-propagators. Remark that \( \sigma \) propagators have no reason to occur at symmetric positions along the top and bottom chains (see Figure 10).

Now let \( (G, R) \) be an intermediate vacuum graph with resolvent set \( R \). Balanced cuts for a \( (G, R) \) with \( R \neq \emptyset \) can be obtained in many different ways. A nice way to define such cuts is to first go to the direct representation and select a spanning tree of \( \sigma \)-propagators of \( G \). Turning around the tree provides a well defined cyclic ordering for the resolvent propagators and pure propagators of the graph (jumping over the \( \sigma \)-propagators not in the tree). Then
we go back to the dual representation. We shall choose the balanced cuts by first contracting all marked propagators along the cycle, and selecting an antipodal pair \((X, Y)\) among the resolvent propagators left in that cycle. We then cut the cycle across that pair (see Figure 10).

To any such balanced cut is associated a Cauchy-Schwartz (CS) inequality. It bounds the resolvent amplitude \(A_{G_0}^\sigma\) (see (84)) by the geometric mean of the amplitudes of the two graphs \(G_t = H_t \cup \bar{H}_t\) and \(G_b = H_b \cup \bar{H}_b\). These two graphs are obtained by gluing \(H_t\) and \(H_b\) with their mirror image along the cut. Remark that in this gluing the \(\sigma\) propagators crossing the cut are fully disentangled: in \(G_t\) and \(G_b\) they no longer cross each other, see Figure 11. Remark also that the right hand side of the CS inequality, hence the bound obtained for \(A_G\), is a priori different for different balanced cuts.

Now the two propagators \(X\) and \(Y\) crossed by the balanced cut at the end of the top and bottom chain, which had values \(RC\) in the amplitude \(A_G\), can be replaced by two ordinary propagators \(C\) in the amplitudes of \(G_t\) and \(G_b\), loosing simply a factor \(\cos^{-2}(\phi/2)\) for the two norms of \(R\). Hence they are cleaned. Here we have used the fact that in the cardioid domain we have (see (90)) \(\|R\| \leq \cos^{-1}(\phi/2)\).

**Lemma 6.1.** For any balanced \(X - Y\) cut

\[
|A_{G,R}(\sigma)| \leq \frac{1}{\cos^2(\phi/2)} \sqrt{A_{G_t,R-\{X\cup Y\}}(\sigma)} \sqrt{A_{G_b,R-\{X\cup Y\}}(\sigma)} \quad (85)
\]

uniformly in \(\sigma\). Hence we have cleaned the two resolvent propagators \(X\) and \(Y\) crossed by the cut.

**Proof** Each CS inequality is simply obtained by writing

\[
<H_t, O H_b > \leq \|O\| \sqrt{<H_t H_t>} \sqrt{<H_b H_b>} \quad (86)
\]

in the tensor product of \(2 + q\) Hilbert spaces corresponding to the two end c-propagators and the \(q\) crossing \(\sigma\)-propagators. We symmetrize first the operators \(RC\) of the two cut

---

4Or almost antipodal if the number of resolvents is not even; this can happen only at the first CS step.
propagators, writing them as $C^{1/2}BC^{1/2}$ with $B = C^{-1/2}RC^{1/2}$. The operator $O = B \otimes \Pi \otimes B$ in \cite{86} is the tensor product of the two end operators $B$ and of a permutation operator $\Pi$ for the remaining $H^q$ tensor product of the $q$ crossing $\sigma$-propagators. Therefore $\|O\| \leq \|B\|\|\Pi\|$. Any permutation operators has eigenvalues which are roots of unity, hence has norm bounded by 1, and $\|B\| = \|R\| \leq \cos^{-1}(\phi/2)$.

Starting with a full resolvent graph, the inductive Cauchy-Schwarz inequalities of \cite{25} consist in iterating lemma 6.1 until no resolvents are left anywhere. In this way we can therefore reach a bound made of a geometric mean of $2^m$ ordinary perturbation amplitudes for an initial resolvent graph of of order $m$. To understand the result of the induction, let us observe that

- Only at the first step the number of resolvent propagators can be odd. In that case we choose an almost antipodal pair (antipodal up to half a unit): but at all later stages the mirror gluing creates an even number of resolvents and we can choose truly antipodal pairs.

- The result of $m$ complete inductive layers of CS steps applied to a starting graph $G$ of order $m$ is a family $F^C_m(G)$ of $2^m$ graphs, which depends on the inductive choices, noted by $C$, of all the balanced cuts of the induction. Each layer of graphs and can be pictured to stand at the leaves of a rooted binary tree, with the initial graph $G$ standing at the root.

- Although the graphs in the family $F^C_q(G)$ may have very different orders, they all have the same number of resolvents (up to one at most, if the initial number of resolvents was odd).

- No matter which inductive choice $C$ is made, every $c$-propagator $\ell$ of the initial graph $G$ gets finally copied into exactly $2^m$ $c$-propagators in the union of all graphs of $F^C_m(G)$. Notice that all these copies have the same set of refined scaling indices than the initial propagator. But they are not at all evenly distributed among the members of the family,

This is summarized in the following lemma.

**Lemma 6.2.** For any choice $C$ of $m$ recursive cuts

$$|A_G(\sigma)| \leq \left[ \prod_{G' \in F^C_m(G)} |A_{G'}| \right]^{2^{-m}}$$

uniformly in $\sigma$. The amplitudes $A_{G'}$ are computed with coupling constants $\rho$ instead of $|\lambda|$.

**Proof** Straightforward induction using lemma 6.1. We bound all the factors $\cos^{-2}(\phi/2)$ generated by the CS inequalities by changing the factor $|\lambda^{V(G')}|$ into $|\rho^{V(G')}|$. Indeed for each pair of resolvents destroyed by a CS inequality there is an independent coupling constant factor $|\lambda|$, and in the cardioid we have $|\lambda| \cos^{-2}(\phi/2) \leq \rho$.

Remark that no additional combinatorial factor is generated in the course of removing the resolvents since the mirror graphs are generated symmetrically and the cuts are chosen by following the cyclic ordering. The amplitudes $A_{G'}$ have no resolvent factors any more, hence are ordinary perturbative amplitudes no longer depending on $\sigma$. 

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6.4 Non-Perturbative Bound II: The Remaining Interaction

In this subsection we bound \( \int d\nu_{\mathcal{B}} \prod_{a} e^{2\text{Im}(\mathcal{B},\Omega_{a})} \) in (81). We shall first bound the term \( |e^{\text{Tr} \, V(\hat{\sigma})}| \). We have the following lemma:

**Lemma 6.3.** For \( g \) in the cardioid domain \( \text{Card}_{\rho} \) defined in Theorem 5.1 and \( \omega \in I_{i} \) we have

\[
|\exp(\text{Tr} \, V(\hat{\sigma}))| \leq \exp \left( \rho |\lambda|^{1/2} \sin(\phi/2) M^{-j} \text{Tr} \hat{\sigma} + \rho \text{Tr} \left( C_{\leq \omega} \hat{\sigma} C_{\omega} \hat{\sigma} \right) \right).
\]

(88)

**Proof** Using (54) we write, putting Arg \( \lambda = \phi \), we have:

\[
|\exp(\text{Tr} \, V(\hat{\sigma}))| = \exp \Re \left( \int_{0}^{1} t^{\omega} d\nu_{\omega} \left[ \lambda \sum_{p \in I_{j}} \sum_{\omega_{1} = M^{j} + 1} T_{p}^{\omega_{1}}(t) T_{p}^{\omega} \right.ight.
\]

\[ + i \frac{\sqrt{2\lambda}}{4} \text{Tr} \hat{\sigma} - i \frac{\sqrt{2\lambda}}{4} \text{Tr} \left[ R_{\leq \omega}(t) - 1 \right] C_{\omega} \hat{\sigma} \right] \]

\[ \left. \leq \exp \left( O(1) \left( \rho + |\lambda|^{1/2} \sin(\phi/2) M^{-j} \text{Tr} \hat{\sigma} + |\lambda| \text{Tr} \left( C_{\omega} \hat{\sigma} R_{\leq \omega} C_{\omega} \hat{\sigma} \right) \right) \right) \right] \]

\[ \leq \exp \left( O(1) \left( \rho + |\lambda|^{1/2} \sin(\phi/2) M^{-j} \text{Tr} \hat{\sigma} + \rho \text{Tr} \left( C_{\leq \omega} \hat{\sigma} C_{\omega} \hat{\sigma} \left( C_{\omega} \hat{\sigma} R_{\leq \omega} C_{\omega} \hat{\sigma} \right) \right) \right) \right) \]

(89)

From the first to second line we used that \( \left( R_{\leq \omega}(t) - 1 \right) C_{\omega} \hat{\sigma} = -\frac{\sqrt{2\lambda}}{4} i R_{\leq \omega} C_{\omega} \hat{\sigma} C_{\omega} \hat{\sigma} \) and the bounds \( |T_{p}^{\omega}| \leq O(1) M^{-j} \), \( \sum_{p \in I_{j}} 1 \leq O(1) M^{j} \). We also used the fact that for a positive Hermitian operator \( A \) and a bounded operator \( B \) we have \( |\text{Tr} AB| \leq \|B\| \text{Tr} A \). Indeed if \( B \) is diagonalizable with eigenvalues \( \mu_{i} \), computing the trace in a diagonalizing basis we have \( |\sum_{i} A_{ii} \mu_{i}| \leq \max_{i} |\mu_{i}| \sum_{i} A_{ii} \); if \( B \) is not diagonalizable we can use a limit argument.

We can now remark that for any Hermitian operator \( L \) we have, if \( |\text{Arg} \lambda| = |\phi| < \pi \),

\[
\| (1 - i\sqrt{\lambda} L)^{-1} \| \leq \frac{1}{\cos(\phi/2)}.
\]

We can therefore apply these arguments to \( A = C_{\leq \omega}^{1/2} \hat{\sigma} C_{\omega} \hat{\sigma} C_{\leq \omega}^{1/2} \) (which is Hermitian positive) and \( B = C_{\leq \omega}^{-1/2} R_{\leq \omega} C_{\omega}^{-1/2} \hat{\sigma} C_{\omega} \hat{\sigma} C_{\leq \omega}^{-1/2} \).

Indeed

\[
\| B \| = \| R_{\leq \omega} \| = \| (1 - i\sqrt{\lambda} C_{\leq \omega}^{1/2} \hat{\sigma} C_{\omega} \hat{\sigma} C_{\leq \omega}^{1/2})^{-1} \| \leq \frac{1}{\cos(\phi/2)}.
\]

(90)

We conclude since in the cardioid \( \frac{|\lambda|}{\cos^{2}(\phi/2)} \leq \rho \). As a corollary we have

**Corollary 6.1.** For \( g \) in the cardioid domain \( \text{Card}_{\rho} \) defined in Theorem 5.1 we have

\[
|\exp(\text{Tr} \sum_{\omega \in \Omega_{j}} V_{\omega}(\hat{\sigma}))| \leq \exp \left( O(1) \rho M^{j} + |\lambda|^{1/2} \sin(\phi/2) \text{Tr} \hat{\sigma} + \rho \text{Tr} \left( \sum_{\omega \in \Omega_{j}} C_{\leq \omega} \hat{\sigma} C_{\omega} \hat{\sigma} \right) \right) \right).
\]

(91)

Now we can bound the first term in the Cauchy-Schwarz inequality (81).

\[^{5}\text{We usually simply say positive for "non-negative", i. e. each eigenvalue is strictly positive or zero.}\]
Theorem 6.1 (Bosonic Integration). For $\rho$ small enough and for any value of the $w$ interpolating parameters

$$\left( \int d\nu_{B} e^{2 \sum_{\omega \in \Omega(B)} |V_{\omega}(\sigma_{a})|} \right)^{1/2} \leq e^{O(1)\rho \sum_{j \in J(B)} M^{j}}. \quad (92)$$

Proof In order to get the correct bound we shall work on the indices set $\Omega(j(B))$ instead of $\Omega(B)$. Let $J(B)$ be the set of coarse scaling indices for the block $B$. Now we identify each scale index $\omega \in \Omega_{a} \subset \Omega(B)$ with the corresponding index $\omega(j) \in \Omega_{j}$ and we have:

$$\sum_{a \in B} \sum_{\omega \in \Omega_{a}} \sum_{j \in J(B)} \omega(j) \in \Omega_{j} \quad 1 = \sum_{a \in B} \sum_{\omega \in \Omega_{a}} \sum_{j \in J(B)} \omega(j) \in \Omega_{j}, \quad (93)$$

so that

$$\int d\nu_{B} e^{2 \sum_{\omega \in \Omega_{a}} |V_{\omega}(\sigma_{a})|} \leq \int d\nu_{B} e^{\sum_{a \in B} \sum_{\omega(j) \in \Omega_{j}} |V_{\omega(j)}(\sigma_{a})|}. \quad (94)$$

Then it is easy to find that the first term in (91) gives precisely a bound $O(1)\rho \sum_{j \in J(B)} M^{j}$. So it remains to check that

$$\left( \int d\nu_{B} e^{\sum_{j} O(1)(|\lambda|^{1/2} \sin(\phi/2) \text{Tr} \sigma + \rho \text{Tr}(C_{\leq j} \sigma C_{j} \sigma))} \right)^{1/2} \leq e^{O(1)\rho \sum_{j \in J(B)} M^{j}}. \quad (95)$$

Applying Lemma 6.3 we get

$$\int d\nu_{B} \prod_{a \in B} e^{2 \sum_{\omega \in \Omega_{a}} |V_{\omega}(\sigma_{a})|} = \int d\nu_{B} \prod_{a \in B} \prod_{j \in J_{B}} e^{2 \sum_{\omega(j) \in \Omega_{j}} |V_{\omega(j)}(\sigma_{a})|} \leq \int d\nu_{B} e^{\frac{1}{2} \langle \sigma, Q \sigma \rangle + \langle \sigma, P \rangle} \quad (96)$$

where $Q$ is diagonal in replica space and $Q$ and $P$ are defined by the equations

$$\langle \sigma, Q \sigma \rangle = \sum_{a \in B} \sum_{j \in J_{B}} \sum_{\omega(j) \in \Omega_{j}} < \sigma_{a}, Q_{\omega(j)} \sigma_{a} >, \quad < \sigma_{a}, Q_{\omega(j)} \sigma_{a} > \equiv O(1)\rho \text{ Tr}(C_{\leq \omega(j)} \sigma_{a} C_{\omega(j)} \sigma_{a}),$$

$$\langle \sigma, P \rangle = \sum_{a \in B} \text{Tr} T^{a} \sigma_{a} = \sum_{a \in B} \sum_{j \in J_{B}} \sum_{\omega(j) \in \Omega_{j}} O(1) \frac{1}{\omega(j)} \text{Tr} \sigma_{a} = O(1) \sum_{a \in B} \sum_{j \in J_{a}} \text{Tr} \sigma_{a}, \quad (97)$$

where each $Q_{\omega(j)}$ is positive. Using the bounds (37) it is easy to check that the kernel of $Q_{j}$ is bounded by

$$Q_{mn,mn}^{\omega(j)} \leq O(1)\rho M^{-j} e^{-M^{-j}(m+n)}. \quad (98)$$

The following lemma follows easily:

Lemma 6.4. Uniformly in $j_{\max}$

$$\text{Tr} Q^{\omega(j)} \leq O(1)\rho, \quad (99)$$

$$\|Q^{\omega(j)}\| \leq O(1)\rho M^{-j}. \quad (100)$$
The covariance $X$ of the Gaussian measure $d\nu_B$ is a symmetric matrix on the big space $V$, which is the tensor product of the identity in space times the matrix $X_{ab}(w_{\ell_B})$ in the replica space. Defining $A \equiv XQ$ and performing the Gaussian integration over $\sigma$ we have

$$\int d\nu_B e^{\frac{1}{2}\langle \sigma, Q\sigma \rangle + \langle \sigma, P \rangle} = e^{\frac{1}{2}P, X(1-A)^{-1}P} \left[ \det(1-A) \right]^{-1/2} \quad (101)$$

**Lemma 6.5.** The following bounds hold uniformly in $j_{\max}$

$$\text{Tr} A \leq O(1)\rho \sum_{j \in J(B)} M^j, \quad (102)$$

$$\|A\| \leq O(1)\rho. \quad (103)$$

**Proof** Since $Q = \sum_{j \in J_B} \sum_{\omega(j) \in \Omega_j} X^{\omega(j)}$ is diagonal in replica space we find that

$$\text{Tr} A = \sum_{a \in B} \sum_{\omega \in \Omega_a} \text{Tr} XQ^\omega = \sum_{j \in J_B} \sum_{\omega(j) \in \Omega_j} \text{Tr} XQ^{\omega(j)} = \sum_{j \in J_B} \sum_{\omega(j) \in \Omega_j} \text{Tr} Q^{\omega(j)} \leq O(1)\rho \sum_{j \in J(B)} M^j. \quad (104)$$

where in the last inequality we used (99) and the fact that $|\Omega_j| \leq O(1)M^j$. Furthermore by the triangular inequality in (97) and using (100)

$$\|A\| \leq \sum_{a \in B} \sum_{\omega(j) \in \Omega_j} \|Q^{\omega(j)}\| = \sum_{a \in B} \sum_{j \in J_a} O(1)\rho M^{-j} \leq O(1)\rho. \quad (105)$$

We can now complete the proof of Theorem 6.1. Since $\text{Tr} A^n \leq \text{Tr} A \|A\|^{n-1}$, by (103) for $\rho$ small enough the series $\sum_{n=1}^{\infty} \text{Tr} A^n$ converges and is bounded by $2\text{Tr} A$. So we have:

$$\left[ \det(1-A) \right]^{-1/2} = e^{\frac{1}{2} \sum_{n=1}^{\infty} (\text{Tr} A^n) / n} \leq e^{\text{Tr} A} \leq e^{O(1)\rho \sum_{j \in J(B)} M^j}. \quad (106)$$

Moreover

$$e^{\frac{1}{2}P, X(1-A)^{-1}P} \leq e^{\frac{1}{2}\|1-A\|^{-1}\langle P, XP \rangle} \leq e^{O(1)\rho \sum_{a \in B} \sum_{j \in J_a, j' \in J_a} X_{ab}(w_{\ell_B})} \leq e^{O(1)\rho \sum_{a \in B} \sum_{j \in J_B} |J_a| |J_B|} \leq e^{O(1)\rho \sum_{j \in J(B)} 1^2} \quad (107)$$

where we used that $X_{ab}(w_{\ell_B}) \leq 1$ for any $\{w_{\ell_B}\}$. It is easily to find that

$$e^{O(1)\rho \sum_{j \in J(B)} 1^2} \leq e^{O(1)\rho \sum_{j \in J(B)} M^1}. \quad (108)$$

So we proved Theorem 6.1. 

**Corollary 6.2.** As a corollary we can easily find that for $\rho$ small enough and for any refined scaling index $\omega(j)$ we have:

$$\left( \int d\nu_B e^{2V_{\omega(j)}(\sigma)} \right)^{1/2} \leq e^{O(1)\rho}. \quad (109)$$
6.5 Perturbative Bounds, Combinatorics and Final Bound

Now we gather all the convergent factors and combinatorial factors for an arbitrary fixed connected graph $G$ of order $n$. We write the summing over refined scaling indices $\omega \in \Omega_B$ in (81) as (93) and gather the factors scale by scale, from $j_{\text{max}} \in J(G)$ to $j_{\text{min}} \in J(G)$, to get the final bound.

- Let the number of marked propagators of $G$ at scale $j$ be $n^j_G$ and the total number of marked propagator contained in $G$ be $n_G = \sum_{j \in J(G)} n^j_G$. Since at each scale $j$ we have generated $n^j = aM^j$ marked propagators in total from the slice-testing expansions, we have $n_G \leq n_j$. 

- The marked propagators at scale $j$ are either associated with a renormalized tadpole resolvent, which contain an intermediate field in the numerator hence contracts with other terms that contain $\sigma$ fields, or associated with a crossing or a resolvent that is not a tadpole. Let the number of the renormalized propagators at scale $j$ be $n_1$ and the number of non-tadpole propagators be $n_2$, the combinatorial factors for generating the marked propagators and contracting the $n_1 \sigma$ fields is $n^j_G! = n^j_1! n^j_2! \times n^j_1! n^j_2! = n^j_G!$. 

- The nonperturbative bound for each scale $j$ is $e^{\rho O(1)|n^j_G|}$ (see Formula (91)). Each counter-term at scale $j$ generated in the MLVE is bounded by $O(1)$, hence we have the bound $O(1)n^j_G$ in total.

- Since from each marked propagator of scale $j$ we gain the convergent factor $M^{-j}$, we gain in total $\prod_j M^{-jn^j_G}$. This convergent factor is good enough to compensate both the combinatorial factor and the nonperturbative bound:

\[
e^{\rho O(1)|n^j_G|} (n^j_G!) M^{-jn^j_G} \sim e^{\rho O(1)|n^j_G|} e^{n^j_G \ln n^j_G} M^{-jn^j_G} \leq e^{-O(1)n^j_G},
\]

as long as $n^j_G \leq e^{-\rho O(1)M^j}$. The resulting convergent factor is also good enough to bound the counter-terms if $M$ is large enough; so that summing over $j \in J(G)$ is bounded. This condition can be certainly verified since at each scale we generate no more than $aM^j$ marked propagators and we can choose $a = e^{-\rho O(1)}$.

Example 6.1. As an example we could consider a connected graph which contains exactly $aM^j$ marked propagators at order $j$. Then the nonperturbative bounds and the combinatorial factors reads

\[
\prod_j e^{\rho O(1)M^j} (aM^j)! \sim \prod_j e^{\rho O(1)M^j} a^{aM^j} M^{ajM^j},
\]

while the convergent factor gained from the marked propagator reads $\prod_j M^{-ajM^j}$. It is easy to find that this factor can bound the divergent factor in Formula (111) as long as the following equation for $a$ is fulfilled:

\[
a \ln a \leq -\rho O(1).
\]
• The order of the two level Forest formula is bounded by \( n \), the number of two level trees is bounded by \( 2^{2n}n^{n-2} \sim 2^{2n}n! \), which is partially canceled by \( \frac{1}{n!} \) from the expansion of \( \log Z \).

So we find that the amplitude of the MLVE graph at any order \( n \) is bounded by polynomials \( (O(1)\rho)^n \). Finally we have the theorem:

**Theorem 6.2.** The perturbation series is Borel summable [35, 40] in the Cardioid domain, uniformly in \( j_{max} \).

We have proved that the polynomial bounds for perturbation series. Now we consider the remainder term. The easier way to obtain the combinatorial factors for the remainder term is to go back to the direct representation and contract all the fields \( \phi \) by Wick’s rule, hence we can easily obtain the bound \( n!(O(1)\rho)^n \) for the remainder term and hence prove the Borel summability of the perturbation series.

### 7 Appendix: Second Order Expansions

In the appendix we shall consider the second order slice-testing expansion. In the previous section we have calculated

\[
I_1 = \int d\sigma e^{V(\sigma,t)} \left\{ -\lambda \text{Tr} \left[ C^{(j)}_\omega (R-1) \right] \circ \text{Tr} \left[ C(t)(R-1) \right] \right.
\]

\[
+ \text{non-planar terms} \right\}.
\]

(113)

We shall forget the contributions from the non-planar graphs as their amplitudes are not divergent. The second order expansion reads:

\[
I_2 = \frac{d}{dt^{(j_2)}} I_1 = \int d\sigma e^{V(\sigma,t)} \left\{ -\lambda \text{Tr} \frac{d}{dt^{(j_2)}} \left[ C^{(j_1)}_\omega (R-1) \right] \circ \text{Tr} \left[ C(t)(R-1) \right] \right.
\]

\[
- \lambda \text{Tr} \left[ C^{(j_1)}(R-1) \right] \circ \text{Tr} \frac{d}{dt^{(j_2)}} \left[ C(t)(R-1) \right] \right.
\]

\[
- \lambda \text{Tr} \left[ C^{(j_1)}(R-1) \right] \circ \text{Tr} \left[ C(t)(R-1) \right] \frac{d}{dt^{(j_2)}} V(t, \sigma) \right\}.
\]

(114)

Using the same method as the the first order slice-testing expansion, including using the flipping symmetry (see Figure 5) for the tadpoles and after some lengthy but straightforward calculation we have:
Figure 12: The graphs for the second order slice-testing expansions. We have only shown the case such that all the \( \sigma \) propagators are hooked to the outer border of the ribbon graphs.

Figure 13: The graphs for the second order slice-testing expansions. We have only shown the case such that all the \( \sigma \) propagators are hooked to the outer border of the ribbon graphs.

\[
I_2 = \int d\nu(\sigma) \ e^{V(\sigma,t)} \left\{ 2\lambda^2 \left[ \sum_{mn} \text{Tr}[(R - 1)C^{\omega(j_1)}]_{mn} \circ \text{Tr}[RC(t)RC^{\omega(j_2)}]_{mm,nn} \circ \text{Tr}[(R - 1)C(t)]_{nn} \\
+ \sum_{mn} \text{Tr}[(R - 1)C^{\omega(j_2)}]_{mm} \circ \text{Tr}[RC(t)RC^{\omega(j_1)}]_{mm,nn} \circ \text{Tr}[(R - 1)C(t)]_{nn} \\
+ \sum_{mn} \text{Tr}[(R - 1)C(t)]_{mm} \circ \text{Tr}[RC^{\omega(j_1)}RC^{\omega(j_2)}]_{mm,nn} \circ \text{Tr}[(R - 1)C(t)]_{nn} \\
+ \sum_{mn} \text{Tr}[(R - 1)C^{\omega(j_1)}]_{mm} \circ \text{Tr}[RC(t)RC^{\omega(j_2)}]_{mm,nn} \circ \text{Tr}[(R - 1)C^{\omega(j_2)}]_{nn} \\
+ \lambda^2 \sum_{mn,pq} \left[ [RC^{\omega(j_1)}RC^{\omega(j_2)}]_{mn,pq} [RC(t)RC(t)]_{pq,nn} + [RC^{\omega(j_1)}RC(t)]_{mn,pq} [RC^{\omega(j_2)}RC(t)]_{qp,nn} \right] \\
+ \lambda^2 \text{Tr}[(R - 1)C^{\omega(j_1)}] \circ \text{Tr}[(R - 1)C(t)] \times \text{Tr}[(R - 1)C^{\omega(j_2)}] \circ \text{Tr}[(R - 1)C(t)] \\
- \lambda \text{Tr}[(R - 1)C^{\omega(j_1)}] \circ \text{Tr}[(R - 1)C^{\omega(j_2)}] \right\},
\]

(115)
where $[\text{Tr}A]_{nn}$ for an $\Lambda^2 \times \Lambda^2$ dimensional matrix $A$ with double indices means that we don’t sum over the index $n$ for the external face; similarly we can define $\text{Tr}A_{mn,nn} := \sum_k \sum_q A_{mk,km;nm,qn}$. The product $\circ$ is defined in Formula $\circ$; $\times$ means the terms are not connected (see Graph $G_2(\omega_1, \omega_2)$ in Figure 12). The terms in the last line of (115) correspond to Graph $G_1(\omega_1, \omega_2)$ in Figure 12 while the terms in the previous line correspond to Graph $G_2(\omega_1, \omega_2)$ in Figure 12. The first four terms correspond to the first four graphs in Figure 13 while the two following two terms correspond to the last two terms in Figure 13. Here for simplicity we have only shown the graphs such that all the $\sigma$ propagators are hooked to the outer boarder of the planar graphs. The other half such that all the $\sigma$ fields are hooked to the inner boarder of the ribbon graph has been omitted.

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