Heavy Quark Potential at Finite Temperature in AdS/CFT

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Abstract
A calculation of the heavy quark potential at finite temperature at strong coupling based on the AdS/CFT correspondence is presented. The calculation relies on the method of complex string trajectories previously introduced in [1], and on the introduction of a modified renormalization subtraction. The obtained potential is smooth, negative definite for all quark-antiquark separations, and develops an imaginary part for $r > r_c = 0.870/\pi T$. At large separations the real part of the potential does not exhibit the exponential Debye fall off expected from perturbation theory and instead falls off as a power law, proportional to $1/r^4$.

The heavy quark potential at finite temperature is a very important quantity in the study of Quark Gluon Plasma formation in Heavy Ion Collisions carried out at RHIC. Until recently it has been calculated either analytically at small coupling using perturbation theory, or numerically using lattice simulations, hence in Euclidean time. However, the matter formed in RHIC collisions is rapidly evolving in time and, according to latest analyses, it is strongly coupled. Thus, calculational tools for strongly coupled, real-time QCD dynamics are needed. Such tools are now available via the Anti-de-Sitter space/conformal field theory (AdS/CFT) correspondence, albeit for a theory different to QCD, namely $N = 4$ supersymmetric Yang-Mills theory.

The first calculation of a heavy quark potential in vacuum for $N = 4$ SYM theory was carried out by Maldacena in [2]. There, the potential is obtained from the expectation value of a static temporal Wilson loop which, on the gravity side, corresponds to the action of the worldsheet spanned by an open string connecting the quark and the antiquark. Due to conformal invariance of $N = 4$ SYM theory, the potential is of Coulomb type:

$$V_0(r) = -\frac{\sqrt{\lambda}}{2\pi c_0^2 r}$$

with $r$ is the distance between the quark and the anti-quark, $\lambda$ the 't Hooft coupling and $c_0 = \Gamma^2 \left(\frac{1}{4}\right)/(2\pi)^{3/2}$. Soon after [2] calculations of the heavy quark potential for $N = 4$ SYM theory at finite temperature appeared in [3,4]. The potential obtained in these calculations starts out at small $r$ being close to the vacuum potential of Eq. (1), but rises steeper than the vacuum potential, becoming zero at a separation $r^* = 0.754/\pi T$. For larger separations, i.e., for $r > r^*$, the authors of [3,4] argue that the string “melts”, and the dominant configuration corresponds to two straight strings stretching from the quark and the anti-quark down to the black hole horizon. The resulting potential is thus zero for $r > r^*$ and has a kink (a discontinuity in its derivative) at $r = r^*$. Here I summarize the results presented in [5], where some modifications to the calculations in [3,4] were proposed in order to avoid the unphysical features of the resulting finite-$T$ potential, namely...
its trivial infrared behavior plus the presence of discontinuities in its derivative (which leads to
infinite forces).

In SU(N_c) gauge theory in Euclidean time, the heavy quark potential or, more precisely,
the free energy, is calculated through the connected correlator of two Polyakov loops at spatial separation \( \vec{r} \):
\[
\langle L(0) L^t(\vec{r}) \rangle_c = \frac{e^{-\beta V(r)} + (N^2_c - 1)e^{-\beta V_{adj}(r)}}{N^2_c} ;
L(\vec{r}) = \frac{1}{N_c} \text{Tr} \left[ P \exp \left( i g \int_0^{\beta} d\tau A_4(\vec{r}, \tau) \right) \right], 
\] (2)
where \( \beta = 1/T \). Eq. (2) is the definition of singlet \( V_1(r) \) and adjoint \( V_{adj}(r) \) potentials. In the
AdS/CFT set up, the color decomposition of Eq. (2) can be read off from the different open
string configurations attached to the quark and the antiquark. One can have a "hanging" string
connecting the quark and the antiquark or, alternatively, two "straight" strings going from the
(anti)quark to the black hole horizon. \( N_c \)-counting in AdS indicates that the hanging string con-
figuration gives \( V_1(r) \), since there is just one way to connect the quark and the antiquark, while the
two straight strings give \( V_{adj}(r) \), since there are \( N^2_c - 1 \) different ways in which two independent
strings can be connected to the \( N_c \) D3-branes living at the bottom of the AdS space. Moreover,
the two straight strings can only be connected through graviton exchange in the bulk. Therefore
the contribution of the two straight string configuration to the potential is \( 1/N^2_c \) suppressed, in
agreement with the analysis of [6].

We calculate the singlet potential \( V_1(r) \) according to the definition proposed in [7] in the
real-time formalism: \( V_1(r) \) is given by the expectation value of a static (temporal) Wilson loop via
\[
\langle W \rangle = e^{-\tau V_1(r)}
\] (3)
with the temporal extent of the Wilson loop \( \tau \to \infty \). To calculate the Wilson loop in the
AdS/CFT set up, we start with the AdS\(_5\) black hole metric in Minkowski space
\[
ds^2 = \frac{L^2}{z^2} \left[ -\left( 1 - \frac{z^4}{z_h^4} \right) dt^2 + dx^2 + \frac{dz^2}{1 - \frac{z^4}{z_h^4}} \right],
\] (4)
where \( dx^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \), \( z \) is the coordinate describing the 5th dimension and \( L \)
is the curvature of the AdS\(_5\) space. The horizon of the black hole is located at \( z = z_h \) with
\( z_h = 1/\pi T \). We want to extremize the open string worldsheet for a string attached to a static
quark at \( x^1 = r/2, x^2 = x^3 = 0 \) and an anti-quark at \( x^1 = -r/2, x^2 = x^3 = 0 \). Parameterizing the
static string coordinates by \( X^\mu = [X^0 = t, X^1 = x, X^2 = 0, X^3 = 0, X^4 = z(x)] \) we write the Nambu-
Goto (NG) action as
\[
S_{NG}(r, T) = -\frac{\sqrt{A}}{2\pi} \int_{-r/2}^{r/2} dx \left[ \frac{1 + z'^2}{z^4} - \frac{1}{z_h^4} \right],
\] (5)
where \( z' = dz(x)/dx \). Finally, the action in Eq. (5) contains a UV divergence associated to the
infinite mass of the quarks which has to be subtracted out. Usually, the subtraction contains a
finite piece as well [2,3,4], which may be temperature-dependent in the case at hand. Here we
will use the following subtraction: We define the quark–anti-quark potential by
\[
V(r) = -[S_{NG}(r, T) - \text{Re}[S_{NG}(r = \infty, T)]] / \tau.
\] (6)
Our prescription is different from the one used in [3, 4], where the contribution of two straight strings were subtracted. It insures that the real part of the potential \( V(r) \) goes to zero at infinite separations. The definition in Eq. (6) is consistent with the original prescription outlined in [2] to find the heavy quark potential at zero temperature. Importantly, it is also consistent with our choice of color representation, since only singlet configurations appear in the definition of \( V(r) \).

What follows next is just a problem of classical mechanics: One has to find the string trajectories that minimize the NG action Eq. (5). In order to do so one has to derive and solve the Euler-Lagrange equations with the appropriate boundary conditions, \( z(x = \pm r/2) = 0 \). The details of this calculation are explained in [5]. Importantly, the classical solutions can be parametrized in terms of the quark–antiquark separation \( r \), the temperature \( T \) (both “external” parameters), and the maximum of the string along the 5th dimension, \( z_{\text{max}} \). The latter is determined by the condition \( z_{\text{max}}(r, x = 0) = 0 \). We obtain the following expression for the heavy quark potential of the \( \mathcal{N} = 4 \) SYM theory at finite temperature:

\[
V(r) = \frac{\sqrt{7}}{2c_0 \pi} \left[ -1 - \frac{z_{\text{max}}^4}{z_h^4} \right] F \left( \frac{1}{2}, \frac{3}{4}, \frac{1}{4}, \frac{z_{\text{max}}^4}{z_h^4} \right) + \frac{1}{z_h},
\]

where \( F \) is the hypergeometric function. Importantly, the solutions for \( z_{\text{max}} \) become complex for \( r > r_c = 0.870 z_h \), leading to complex-valued string trajectories \( z(x) \) and potential \( V(r) \). This led the authors of [3, 4] to abandon their solution for \( r > r_c \). We suggest however to interpret the complex-valued saddle points as corresponding to quasi-classical configurations in the classically forbidden region of string coordinates, as previously suggested in [1]. This is similar to the method of complex trajectories used in quasi-classical approximations to quantum mechanics.

Moreover, the solution for \( z_{\text{max}} \) has several branches. In order to select the physical branch we impose two conditions: i) The right branch should map onto Maldacena’s solution at small \( r \). ii) In order to get a sensible the quantum-mechanical time-evolution we require that \( \text{Im}[V(r)] < 0 \), so that the probability of a state, \( \sim e^{\text{Im}[V]} \), does not exceed unity. These criteria suffice to single just one branch for \( z_{\text{max}} \). Inserting it into Eq. (7), we readily obtain the real and imaginary parts of the resulting potential, plotted in Fig. 1 for two non-zero temperatures, along with the zero-T curve for comparison.

![Figure 1: The real and imaginary parts of the heavy quark potential plotted as functions of the separation \( r \) for several different temperatures. We put \( \lambda = 10 \).](image)

Several comments are in order: First, the real part of the potential is a smooth, negative definite function, going to zero at large inter-quark separations. This latest feature is a direct
consequence of our renormalization prescription Eq. (6). If we had used the renormalization
proposed in [3, 4], then Re[V(r)] would have gone to a positive constant as r → ∞. Moreover,
the non-zero temperature curves exhibit a strong screening compared to the zero temperature
case, while at small r we recover the zero temperature potential of [2]. On the other hand, the
heavy quark potential develops an imaginary part for r > r_c. This means the potential becomes
absorptive, as the q̄q singlet state may decay in the medium, with an absorption rate that increases
with the quark–antiquark separation. The existence of an imaginary part in the heavy quark
potential has been previously observed in perturbation theory in [7].

The nature of the screening exhibited by the lines in Fig. 1 can be better understood from the
following asymptotic expansions for the real and imaginary part of the potential:

\[\text{Re}[V(r)] \bigg|_{T \rightarrow 1} = -\frac{\pi^3 c_0^3}{4} \sqrt{\frac{z_0^3}{r^3}} + o \left( \frac{z_0}{r^2} \right), \quad \text{and} \quad \text{Im}[V(r)] \bigg|_{T \rightarrow 1} = -\frac{\sqrt{\lambda}}{\pi} \frac{1}{2 z_0} \left[ \frac{r}{z_0} - \frac{1}{c_0} + o \left( \frac{z_0}{r} \right) \right].\]

Instead of the exponential falloff with r characteristic of Debye screening expected from perturbation
theory and which has been postulated for N = 4 SYM theory at strong coupling in [8],
the real part of the heavy quark potential falls off as a power, Re[V(r)] ∼ 1/T^3 r^4, at large r. If
our hypothesis of using the complex string configurations is confirmed, this would be an interest-
ing new type of screening for the potential. However, the large negative imaginary part of the
potential leads to exponential decay with time of the heavy quark pair, which may blur the poten-
tial observable consequences of such new screening mechanism. Also, combining the large-
and small-r asymptotics we can interpolate the real part of the potential to write an approximate
formula

\[\text{Re}[V(r)] \approx -\frac{\sqrt{\lambda}}{2 \pi c_0^3} \frac{r_0^3}{(r_0 + r)^3}, \quad \text{with} \quad r_0 = z_0 \pi c_0 \left( \frac{\pi c_0^2}{2} \right)^{1/3} \approx \frac{2.702}{\pi T},\]

where the parameter r_0 given can be interpreted as the screening length.

Several concluding remarks: First, Debye screening is the result of a perturbative calculation.
Therefore, it may not be valid for distances of the order of r ∼ 1/(g^2 T), where non-perturbative
effects may be important. On the other hand, one should also bear in mind that the particle con-
tent of N = 4 SYM is quite different to that of QCD. In particular it does not include particles in
the fundamental representation (i.e. quarks), but only in the adjoint (The heavy quarks in our cal-
culation ought to be regarded as external test charges). Since a charge in the adjoint cannot fully
screen a fundamental charge, it is reasonable to expect a milder screening. Finally, the power-law
fall off of the heavy quark potential has also been suggested in perturbative calculations [9].

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