Determination and Reduction of Large Diffeomorphisms

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For diffeomorphism invariant theories we consider the problem of how to determine and reduce diffeomorphisms which are not in the identity component.

1. THE PROBLEM

We shall consider diffeomorphism invariant theories within the Hamiltonian formulation, where space-time is assumed to be a topological product $M \cong \Sigma \times \mathbb{R}$. The constraints of the theory will only generate the identity component $D^0(\Sigma)$ of some subgroup $D(\Sigma)$ of diffeomorphisms of $\Sigma$. This means that after a reduction by $D^0(\Sigma)$, i.e., implementing the constraints – which itself is a highly non-trivial problem –, one still has a residual action of the discrete and generally non-abelian and infinite group $\mathcal{S}(\Sigma) := D(\Sigma)/D^0(\Sigma)$ on the reduced state space $S_{\text{red}}$. In what follows, we shall always restrict to orientable $\Sigma$ and orientation preserving $D(\Sigma)$. But this is not essential. In principle one is now free to regard $\mathcal{S}(\Sigma)$ either as residual part of the gauge group, i.e., as redundancy, or as proper physical symmetry. In the first case physical observables must lie in the commutant of the von Neumann algebra generated by $\mathcal{S}(\Sigma)$, whereas in the second some physical observables must break the symmetry to render it observable. In the first case the reduction procedure is not completed and we must consider the set of $\mathcal{S}(\Sigma)$-orbits in $S_{\text{red}}$ as faithful space of physically distinguishable states. Whether this orbit space can be given a sufficiently well-behaved structure will depend crucially on the details on $\mathcal{S}(\Sigma)$’s action on $S_{\text{red}}$. A priori there is absolutely no reason to expect the structure and action of a general infinite discrete group to be nice. Also, the analogous problem to the one posed by continuous spectra of the generators of continuous Lie groups also occurs even for the simplest infinite discrete groups. This will be seen in the first example. In addition, if a discrete group is not an abelian extension of a finite group it is not of type I, meaning that general representations may be written in different, mutually disjoint direct integral decompositions of irreducibles. Whether this really implies difficulties in the quantum theory, for example as ambiguities in the determination of sectorial structures by the reduction process, is presently not known to us.

2. FIRST EXAMPLE

In this section we wish to illustrate some typical problems connected with the reduction of large diffeomorphisms.

Einstein gravity in 2+1 dimensions can be considered as $\text{ISO}(2,1)$ Chern-Simons gauge theory \[.\] The point of doing this is that the constraints can be solved and the reduced state space be constructed. We specialize to $\Sigma = T^2$, where $T^2$ denotes the two-torus. In the so-called metric sector the classical reduced state- (or phase-) space $P_{\text{red}}$ is a cotangent bundle $T^*(Q_{\text{red}})$, where the reduced configuration space $Q_{\text{red}} = \mathbb{R}^2/\mathbb{Z}_2$ is the punctured plane ($\mathbb{R}^2 = \mathbb{R}^2 - \{0\}$) with antipodal points identified by the $\mathbb{Z}_2$-action of reflections at the origin. Let $\vec{q} = (q_1, q_2)$ be cartesian coordinates on $\mathbb{R}^2$ and $\vec{p} = (p_1, p_2)$ the conjugate momenta. Their interpretation is as follows: Let $\alpha$ and $\beta$ be two closed curves on $T^2$ whose homotopy classes generate $\pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z}$. Then $(q_1, q_2)$ and $(p_2, -p_1)$ are $\text{ISO}(2,1)$ holonomies along $(\alpha, \beta)$, which are boosts in $x$-direction for the $q$’s and spatial translations in $y$-direction for
the p’s. See e.g., [3] for details. If \( D(T^2) \) denotes the (orientation preserving) diffeomorphisms, one has \( S(T^2) = D(T^2)/D^0(T^2) \cong \text{SL}(2, \mathbb{Z}). \) Its action on \( Q_{\text{red}} \) is just the projection of the defining representation on \( \mathbb{R}^2 \), so that on \( Q_{\text{red}} \) only \( \text{SL}(2, \mathbb{Z})/\pm 1 = \text{PSL}(2, \mathbb{Z}) \cong \mathbb{Z}_2 \ast \mathbb{Z}_3 \) (\(* = \text{free product}\)) acts effectively. On \( P_{\text{red}} \) the action is just given by canonically lifting the action on \( Q_{\text{red}} \).

The action of \( \text{SL}(2, \mathbb{Z}) \) on \( \mathbb{R}^2 \) is wild indeed. For example, the stabilizer subgroups of points \((x, y)\) with irrational \( y/x \) are trivial whereas they are \( \cong \mathbb{Z} \) for rational \( y/x \). Since points with rational slopes lie dense, the isomorphism classes of stabilizer subgroups are nowhere locally constant and hence the quotient is nowhere even locally a manifold. However, the lifted action on \( T^*(\mathbb{R}^2) \) is free and properly discontinuous on the open and dense set \( \{(\vec{q}, \vec{p}) \mid \vec{q} \cdot \vec{p} \neq 0\} \).

Due to the fact that our classical phase space is a cotangent bundle, the quantization of this model may naturally proceed with the realization of the Hilbert space as \( L^2 \)-functions on \( Q_{\text{red}} \):

\[
\mathcal{H} = L^2_+(\mathbb{R}^2; dq_1 dq_2),
\]

where + indicates that the functions must be invariant under \( \vec{q} \rightarrow -\vec{q} \). The choice of the Lebesgue measure is not arbitrary: \( ISO(2, 1) \) is the cotangent-bundle-group of \( SO(2, 1) \) and the space of flat \( ISO(2, 1) \) connections is the cotangent bundle over the space of flat \( SO(2, 1) \) connections. The latter has itself a natural symplectic structure whose associated Liouville measure one may generally use in \( 2+1 \) dimensions to define the inner product (see p. 272 and 276 of [3]). Applied to our case this results in \( dq_1 dq_2 \).

Its \( \text{SL}(2, \mathbb{Z}) \)-invariance implies that we have a unitary representation \( S(T^2) \times \mathcal{H} \rightarrow \mathcal{H} \) given by \( T : ([g], \psi) \rightarrow T_g \psi := \psi \circ g^{-1} \), where \( g \in \text{SL}(2, \mathbb{Z}) \) is any preimage of \([g] \in \text{PSL}(2, \mathbb{Z})\) under the natural projection. It is possible to explicitly decompose this representation into a direct integral of unitary irreducibles [4]:

\[
\mathcal{T} = \int_{-\infty}^{\infty} ds \mathcal{T}_s,
\]

\[
\mathcal{H} = \int_{-\infty}^{\infty} ds \mathcal{H}_s,
\]

where \( \mathcal{H}_s \cong L^2(S^1, d\phi) \). The irreducibles \( \mathcal{T}_s \) are restrictions to \( \text{SL}(2, \mathbb{Z}) \) of the irreducibles \( C^\phi_q \) of \( \text{SL}(2, \mathbb{R}) \) from the principal series with Casimir invariant \( q = (s^2 + 1)/4 \). The interesting features of this decomposition are 1.) the absence of the trivial representation, 2.) the absence of finite dimensional irreducibles, 3.) the purely continuous Casimir spectrum. There are no irreducible subspaces in \( \mathcal{H} \) but closed invariant subspaces are given by \( \mathcal{H}_\Delta = \int_{\Delta} ds \mathcal{H}_s \) for any measurable set \( \Delta \subset \mathbb{R} \).

This situation is well known from continuous groups with continuous spectra of their generators. Consider for example the translations in \( y \)-direction acting on \( L^2(\mathbb{R}^2, dz dy) \). But in this example the trivial representation occurs in the direct integral decomposition. So if we wanted to interpret the \( y \)-translations as gauge redundancy we could identify the reduced quantum state space with the integrand \( \cong L^2(\mathbb{R}, dx) \) carrying the trivial representation. This is just what the procedure of “group averaging” leads to [3]. In our example such a simple identification does not seem possible. Note that if we just kept \( \mathcal{H} \) as state space and implemented the unobservability of the transformations in \( \mathcal{S} \) by restricting the algebra of observables \( \mathcal{O} \) to the commutant \( \{S(T^2)\}' \) of \( S(T^2) \) in \( B(\mathcal{H}) \) (bounded Operators), then \( \mathcal{H} \) would not contain a single pure state for \( \mathcal{O} \). The proof is simple: Let \( \psi \in \mathcal{H}_\Delta \) be normalized. We can always find disjoint measurable sets \( \Delta_{1,2} \) of non-zero measure such that \( \Delta = \Delta_1 \cup \Delta_2 \), hence \( \mathcal{H} = \mathcal{H}_{\Delta_1} \oplus \mathcal{H}_{\Delta_2} \), and associated decomposition \( \psi = \lambda_1 \psi_1 + \lambda_2 \psi_2 \) with normalized \( \psi_1,2 \). Since each \( \mathcal{H}_{\Delta_i} \), reduces \( \mathcal{O} \), the density matrices \( \rho = |\psi\rangle \langle \psi| \) and \( \rho_{1,2} = |\psi_{1,2}\rangle \langle \psi_{1,2}| \) obey \( \rho = |\lambda_1|^2 \rho_1 + |\lambda_2|^2 \rho_2 \) as linear functionals on \( \mathcal{O} \). Hence \( \rho \) is a non-trivial convex sum and therefore a mixed state on \( \mathcal{O} \).

We ask: 1.) regarding \( S(T^2) \) as a proper physically symmetry group implies observables outside \( \{S(T^2)\}' \). How could these be justified physically? 2.) Regarding \( S(T^2) \) as part of the gauge group necessitates further reduction. How?

3. THE GENERAL 3+1 CASE

Except in cosmology we are usually not interested in closed 3-manifolds representing space. Appropriate for the description of isolated gravi-
tating configurations are 3-manifolds $\Sigma$ with one regular end, i.e., there exists a compact set $K \subset \Sigma$ so that $\Sigma - K$ is homeomorphic to $\mathbb{R} \times S^2$. This is precisely the condition that the one-point-compactification $\Sigma = \Sigma \cup \infty$ ($\infty$ is the added point) is again a manifold. The mapping class group we are interested in is then conveniently characterized by using the fiducial manifold $\bar{\Sigma}$ (e.g. [8]). Let $D_F(\bar{\Sigma})$ be the diffeomorphisms of $\Sigma$ that fix the frames at $\infty$ and $D^0_F(\bar{\Sigma})$ its identity component. We define

$$S(\Sigma) := D_F(\bar{\Sigma})/D^0_F(\bar{\Sigma}).$$

One studies $S(\Sigma)$ by considering the group homomorphism

$$h_F : S(\Sigma) \rightarrow \text{Aut}(\pi_1(\Sigma, \infty))$$

$$h_F([\phi])([\gamma]) := [\phi \circ \gamma],$$

where $\gamma$ is a loop based at $\infty$, $[\gamma]$ its homotopy class, $\phi \in D_F(\bar{\Sigma})$, and $[\phi]$ its class in $S(\Sigma)$. The strategy is to obtain $S(\Sigma)$ from 1.) $\text{Ker}(h_F) = \text{kernel of } h_F$, 2.) $\text{Im}(h_F) = \text{image of } h_F$, 3.) a prescription to extend $\text{Im}(h_F)$ by $\text{Ker}(h_F)$. Given the connected sum decomposition of $\Sigma$, it is indeed possible to explicitly present $S(\Sigma)$ for a large class of 3-manifolds. The generating diffeomorphisms fall into three classes: 1.) internal-, 2.) exchange-, and 3.) slide diffeomorphisms. To explain this we recall that any compact orientable 3-manifold is uniquely built as finite connected sum of so-called prime manifolds (see [8] for details and references): $\bar{\Sigma} = P_1 \uplus \cdots \uplus P_n$. Then $\pi_1(\bar{\Sigma}) \cong \pi_1(P_1) \ast \cdots \ast \pi_1(P_n)$. In this way $\Sigma$ is represented by a 3-disk $D$ to which primes $P_i \neq S^1 \times S^2$ are glued by removing an open 3-disk from $P_i$ and $B$ and identifying the resulting 2-sphere boundaries so as to match the given orientations. If $P_i = S^1 \times S^2$ we remove two open 3-disks from $B$ and identify the boundaries left with the boundary 2-spheres of $[0, 1] \times S^2$. We thus view $\bar{\Sigma}$ as a configuration of $n$ elementary objects connected to the base $B$ by 2-spheres, like particles with internal structure “moving” in $B$. Internal diffeomorphisms are those which (up to isometry) have support within the primes. Exchanges are those non-internal ones that leave $B$ and the interiors of the $P_i$’s setwise invariant (i.e. permutations of diffeomorphic primes). Finally, slide generators account for the fact that primes can penetrate and “move” through each other. They mix exterior and interior points. Roughly speaking, each connecting sphere of a prime can be slid a full turn within a closed tube whose axis-loop generates an element of the fundamental group of another prime. Homotopic loops define isotopic slides. Slides form an invariant subgroup $G^S \subset S(\Sigma)$ (see [8] and its references).

Now, the so-called Fuchs-Rabinovich presentation for $\text{Aut}(G_i \ast \cdots \ast G_n)$ allows to explicitly present $\text{Im}(h_F)$, once we have presentations for each $S(P_i)$ (see [8] and its references). Given this, we obtain a presentation for $S(\Sigma)/\text{Ker}(h_F)$. The problem is now to determine $\text{Ker}(h_F)$ and the way it extends $\text{Im}(h_F)$. If all $P_i$ satisfy that homotopic diffeomorphisms are also isotopic (no prime violating this seems to be known) and no $P_i$ is a homotopy sphere then it is known that $\text{Ker}(h_F)$ consists of rotations parallel to connecting spheres $[9]$. Then $S(\Sigma) = \text{Ker}(h_F) \ltimes \text{Im}(h_F)$ where only the permutations in $\text{Im}(h_F)$ act non-trivially (in the obvious way) on $\text{Ker}(h_F)$. In this way presentations for connected sums of an arbitrary number of $\mathbb{R}P^3$’s or an arbitrary number of $S^1 \times S^2$’s were obtained [8] in terms of three and four generators respectively.

The obvious semi-direct product of the internal symmetry group $G^I = S(P_1) \times \cdots \times S(P_n)$ and the permutations form the so-called “particle subgroup” $G^P \subset S(\Sigma)$ [8]. $G^P$ and $G^S$ together exhaust $S(\Sigma)$ but they may intersect non-trivially. It has been shown that iff $P_i \neq S^1 \times S^2 \forall i$, $G^P \cap G^S = \{e\}$ and $S(\Sigma) \cong G^S \ltimes G^P$, and that $G^S$ is perfect if more than two primes are $S^1 \times S^2$.

4. SECOND EXAMPLE

We consider the connected sum of two real projective spaces $\Sigma = \mathbb{R}P^3 \uplus \mathbb{R}P^3$ to illustrate the determination of $S(\Sigma)$ according to the general scheme outlined above.

One way to understand the manifold $\bar{\Sigma}$ is to look at the fundamental domain $F = \{ \bar{x} \in \mathbb{R}^3 \mid 1 \leq ||\bar{x}|| \leq 3 \}$. We label points in $F$ by $(r, \bar{n})$ where $r = ||\bar{x}||$ and $\bar{n} = \bar{x}/r$. Let $S_{\bar{n}r}$ denote the sphere $r = r'$ and $\sigma(\bar{n}') \subset F$ the radial segment
\(\vec{n} = \vec{n}'\). To obtain \(\bar{\Sigma}\) we identify antipodal points on \(S_1\) and on \(S_2\). The sets \(C(\pm \vec{n}) := \sigma(\vec{n}) \cup \sigma(-\vec{n})\) define a \(\mathbb{R}P^2\) worth of circles which establishes \(\bar{\Sigma}\) as circle bundle over \(\mathbb{R}P^2\) (which is not principal). \(S_2\) may be taken as the sphere along which the connected sum of the two \(\mathbb{R}P^3\)'s is taken. Obviously it cuts each fiber twice. We also place \(\infty\) on \(S_2\), say at \(\vec{n} = 2\vec{e}_z\).

We have \(\pi_1(\bar{\Sigma}) = \mathbb{Z}_2 \ast \mathbb{Z}_2 = \{a, b \mid a^2 = b^2\}\), where \(e\) denotes the identity. \(a\) and \(b\) may be generated by meridians on \(S_1\) and \(S_3\) respectively. One then sees from a picture of \(F\) that a circle fibre generates \(ab\), which itself generates a subgroup \(\cong \mathbb{Z} \subset \pi_1(\bar{\Sigma})\). The conjugacy class of \(ab\) in \(\pi_1(\bar{\Sigma})\) consists only of \(ab\) and \(ba = (ab)^{-1}\). The map \(ab \to ba\) generates \(\text{Aut}(\mathbb{Z}) \cong \mathbb{Z}_2\) and may be interpreted as the action of the fundamental group of the base \(\mathbb{R}P^2\) on the fundamental group of the fibre \(S^1\). In fact, \(\pi_1(\bar{\Sigma})\) is the semi-direct product of these two groups: \(\mathbb{Z}_2 \ast \mathbb{Z}_2 \cong \mathbb{Z} \ltimes \mathbb{Z}_2\). To give an explicit isomorphism let \((n, p) \in \mathbb{Z} \ltimes \mathbb{Z}_2\) with \(p \in \{1, -1\}\) (multiplicative notation). Then \((n', p')(n, p) = (n' + p' n, p' p)\) and an isomorphism \(\phi: \mathbb{Z} \ltimes \mathbb{Z}_2 \to \mathbb{Z}_2 \ast \mathbb{Z}_2\) may be defined by \(\phi(n, 1) = (ab)^n\) and \(\phi(n, -1) = (ab)^n a\). One easily verifies the homomorphism property. In- and surjectivity are obvious.

It is known that \(S(\mathbb{R}P^3) = \{e\}\) so that \(G^P \cong \mathbb{Z}_2\) is generated by exchanging the two primes. In \(F\) the exchange can be defined by a reflection at \(S_2\) followed by a reflection at the \(yz\)-plane. This still rotates tangent vectors at \(\infty\) by \(\pi\) in the \(y\) direction, but a slight modification by rotating back a small 3-disk about \(\infty\) renders this diffeomorphism an element of \(D_P(\bar{\Sigma})\). It defines a generator \(\omega\) of \(S(\bar{\Sigma})\) satisfying \(\omega^2 = e\). There are two slides, \(\mu_{12}\) and \(\mu_{21}\), corresponding to sliding the second through the first prime along some generator of its fundamental group and vice versa. It suffices to define one of them: In \(\bar{\Sigma}\) consider the closed solid tori \(T_{1,2} = F \cap \{\vec{x} \in \mathbb{R}^3 \mid y^2 + z^2 \leq R_{1,2}, 1 < R_1 < R_2 < 2\}\) and a diffeomorphism with support in the closure of \(T_2 - T_1\) that slides \(T_1\) against \(T_2\) a full turn parallel to their common axis. This defines a slide of the “inner” \((r < 2)\) through the “outer” \((r > 2)\) prime. Since the prime’s fundamental group is \(\mathbb{Z}_2\), we have \(\mu_{12} = e = \mu_{21}\). The Fuks-Rabinovich presentation implies that there is no other relation between the slides (this would change if we considered more than two primes \([8]\)). Hence \(G^S \cong \mathbb{Z}_2 \ast \mathbb{Z}_2\). Finally \(S(\bar{\Sigma}) = G^S \ltimes G^P\) where \(G^P\)'s action on \(G^S\) is \(\omega \mu_{12} \omega^{-1} = \mu_{21}\). We can use this last relation to eliminate \(\mu_{21}\) from the presentation and just retain \(\omega\) and \(\mu = \mu_{12}\) with no other relation except their idempotency. Hence

\[
S(\bar{\Sigma}) \cong \mathbb{Z}_2 \ast \mathbb{Z}_2 \cong \mathbb{Z} \ltimes \mathbb{Z}_2
\]

As shown above, the generators of \(\mathbb{Z}\) and \(\mathbb{Z}_2\) in the semi-direct product may be identified with \(\omega \mu\) and \(\mu\) respectively. Compared to the group \(\mathbb{Z}_2 \ast \mathbb{Z}_3\) considered in the first example, \(\mathbb{Z}_2 \ast \mathbb{Z}_2\) has a much simpler representation theory being a semi-direct product. (It is clearly of type I by the criterion mentioned above). There are the obvious four one-dimensional irreducible representations and the more interesting one-parameter \((0 < t < \pi)\) family of two-dimensional ones given by \(\omega \mapsto \tau_3\) and \(\mu \mapsto \tau_1 \sin t + \tau_3 \cos t\), where \(\tau_i\) are the Pauli Matrices \([10]\).

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