Heat fluctuations in equilibrium

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Abstract. The characteristic function for heat fluctuations in a nonequilibrium system is characterised by a large deviation function whose symmetry gives rise to a fluctuation theorem. In equilibrium the large deviation function vanishes and the heat fluctuations are bounded. Here we consider the characteristic function for heat fluctuations in equilibrium, constituting a sub-leading correction to the large deviation behaviour. Modelling the system by an oscillator coupled to an explicit multi-oscillator heat reservoir we evaluate the characteristic function.

Keywords: transport processes/heat transfer, transport properties, fluctuation phenomena, stochastic thermodynamics

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1. Introduction

There is a current interest in the thermodynamics and statistical mechanics of small fluctuating systems in contact with heat reservoirs and driven by external forces. This interest stems from the recent possibility of the direct manipulation of nano systems and biomolecules in nonequilibrium scenarios [1–8]. Together with the advent of the so-called fluctuation theorems which impose symmetry relations on the nonequilibrium heat and work probability distributions [9–26]. In recent years there has also been increased interest in the above nonequilibrium issues for open quantum systems, see e.g. [27, 28].

As an illustrative example we consider a single oscillator coupled to two heat reservoirs at temperatures $T_1$ and $T_2$ and characterised by the damping constants $\Gamma_1$ and $\Gamma_2$, respectively [30, 31]. The representative Langevin equations for the position $x$, the momentum $p$, force constant $\kappa$, heat flux $dQ/dt$ from reservoir 1, and noises $\xi_1$ and $\xi_2$ are then given by (setting the mass $m = 1$)

\[
\frac{dx}{dt} = p, \quad (1.1)
\]
\[
\frac{dp}{dt} = - (\Gamma_1 + \Gamma_2)p - \kappa x + \xi_1 + \xi_2, \quad (1.2)
\]
\[
\frac{dQ}{dt} = - \Gamma_1 p^2 + p \xi_1, \quad (1.3)
\]
\[
\langle \xi_1 \xi_1 \rangle (t) = 2 \Gamma_1 T_1 \delta (t), \quad (1.4)
\]
\[
\langle \xi_2 \xi_2 \rangle (t) = 2 \Gamma_2 T_2 \delta (t). \quad (1.5)
\]

The characteristic function describing the long time behaviour of the heat fluctuations is given by

\[
\tilde{C}(k, t) = \langle \exp (kQ(t)) \rangle = C(k) \exp (t \mu (k)), \quad (1.6)
\]

where $\mu (k)$ is the large deviation function. The general Gallavotti–Cohen fluctuation theorem applying to nonequilibrium systems implies the fundamental symmetry [18, 31]

\[
\mu (k) = \mu (1/T_1 - 1/T_2 - k). \quad (1.7)
\]

For the above example we have explicitly [30, 31]

\[
\mu (k) = \left[ \Gamma_1 + \Gamma_2 - \sqrt{(\Gamma_1 + \Gamma_2)^2 + 4 \Gamma_1 \Gamma_2 f(k)} \right], \quad (1.8)
\]
\[
f (k) = T_1 T_2 k (1/T_1 - 1/T_2 - k), \quad (1.9)
\]

where the form of $f (k)$ implies the fluctuation theorem. Disconnecting for example reservoir 1 by setting $\Gamma_1 = 0$ the large deviation function $\mu (k)$ vanishes. The resulting system with an oscillator interacting with a single reservoir is an equilibrium system with bounded heat fluctuations and $\tilde{C}(k) = C(k)$.
The purpose of the present paper is to investigate further the case of fluctuations in equilibrium described by the characteristic function $C(k)$. In the case of a thermodynamic variable $x$ the understanding is well known and follows from the Boltzmann–Gibbs scheme [29, 32]. The probability distribution $P(x)$ is determined by the entropy $S(x)$ according to $P(x) \propto \exp(S(x))$ [29, 32]. Expanding $S(x)$ about its maximum value corresponding to equilibrium, i.e., $S(x) = S(x_0) - \text{const.}(x - x_0)^2$, we arrive at the Gaussian distribution $P(x) \propto \exp(-x^2/2\langle x^2 \rangle)$, where $\langle x^2 \rangle$ is the mean square fluctuation.

However, in the case of the fluctuating heat $Q$ exchanged between a small system and a single heat reservoir at inverse temperature $\beta = 1/T$ (we have set the Boltzmann constant $k_B = 1$), the heat distribution $P(Q)$, surprisingly, does not take a Gaussian form. As discussed in previous papers [5, 33] addressing an overdamped oscillator, the distribution has the form $P(Q) = (\beta/\pi)K_0(\beta|Q|)$, where $K_0$ is a Bessel function [34, 35]. This distribution only depends on the temperature of the reservoir, exhibits exponential Boltzmann tails $\sim \exp(-\beta|Q|)$ and diverges logarithmically at small $Q$ as $P(Q) \sim -\ln(|Q|)$.

In general the characteristic function, defined according to [32]

$$C(k) = \int dQ P(Q) \exp(kQ),$$

is given by the expression

$$C(k) = \frac{Z(\beta + k)Z(\beta - k)}{Z(\beta)^2},$$

where $Z(\beta)$ is the partition function for the system [29, 32]. We note that in the case of a single overdamped oscillator with one degree of freedom $Z(\beta) \propto 1/\sqrt{\beta}$, i.e., $C(k) = \beta/\sqrt{\beta^2 - k^2}$. Likewise, for a damped harmonic oscillator with two degrees of freedom coupled to a reservoir $Z(\beta) \propto 1/\beta$, yielding the characteristic function $C(k) = \beta^2/(\beta^2 - k^2)$. The corresponding heat distribution is $P(Q) = (\beta/2)\exp(-\beta|Q|)$, decaying exponentially and exhibiting a cusp at $Q = 0$.

We note that in the presence of two temperature-biased heat reservoirs driving the system into a nonequilibrium state, a finite fluctuating heat flux $q = dQ/dt$ will be established. As a result the integrated heat $Q(t) = \int_0^t dt' dq/dt'$ will on average grow linearly in time, i.e., $Q(t) \sim qt$. More precisely, the characteristic function associated with the probability distribution $P(Q, t)$, $\langle \exp(kQ(t)) \rangle \sim \exp(t\mu(k))$, where $\mu(k)$ is the large deviation function; in equilibrium we have $\mu = 0$. In that sense the equilibrium heat distribution constitutes the sub-leading correction to the large deviation result.

In the present paper we extend the analysis in [33] and consider the case of an explicitly defined heat reservoir in evaluating the equilibrium heat distribution. For convenience we consider the case of a single oscillator coupled to a single heat bath. The explicit characterisation of the heat bath in terms of a collection of oscillators is well known and a prerequisite for a quantum treatment. It is also well known that the so-called ohmic approximation is equivalent to a standard Langevin/Fokker Planck
description. However, we believe that the present calculation carried out within the multi-oscillator scheme is novel.

The paper is organised in the following manner. In section 2 we define the heat exchanged between the system and the reservoir. In section 3 we discuss a heuristic derivation of the characteristic function, yielding equation (1.11). In section 4 we discuss the explicit characterisation of the heat bath in terms of a collection of oscillators. In section 5 we turn to an evaluation of the heat characteristic function in the case of a multi-oscillator heat bath. Since the methods we employ are well known we defer technical details to appendix A and B sections.

2. Heat

A small system coupled to a heat reservoir constitutes a closed system. Correspondingly, the total energy of system and reservoir is conserved. However, the small system itself exchanges energy with the reservoir and is in this respect and open system.

Let us characterise the small fluctuating system coupled to the heat reservoir by the fluctuating Hamiltonian \( H_0(t) = H_0(\{x_n(t)\}) \), where \( \{x_n\} \) are the degrees of freedom. The time dependence of \( H_0(t) \) is due to the heat reservoir and not to an applied external time dependent force, i.e., we are not applying an external protocol as is common in the context of fluctuation theorems \([9–26]\).

The fluctuating heat flux \( q(t) \) from the reservoir to the small system is thus given by \( q(t) = dH_0/dt \). Note that since we are in equilibrium the mean value \( \langle q(t) \rangle = 0 \). Consequently, the heat \( Q(t) \) transmitted in a time span \( t \) is \( Q(t) = \int_0^t dq(\tau) \), i.e.,

\[
Q(t) = H_0(t) - H_0(0). \tag{2.1}
\]

The transmitted heat \( Q(t) \) is a fluctuating quantity and the issue is to determine its stationary probability distribution \( P(Q) \), or, equivalently, its characteristic function

\[
C(k) = \langle \exp(kQ(t)) \rangle = \int dQP(Q) \exp(kQ). \tag{2.2}
\]

3. Heat distribution—heuristic derivation

Here we present a heuristic derivation of the heat distribution or characteristic function; this approach was also discussed in \([33]\) and is based on the definition in equation (2.1). From equations (2.1) and (2.2) we have

\[
C(k) = \langle \exp(k(H_0(t) - H_0(0))) \rangle. \tag{3.1}
\]

Assuming that the system is in equilibrium at time \( t = 0 \) and at time \( t \) and, moreover, assuming that \( t \) is larger than the characteristic decay time of fluctuations, the energy fluctuations at time \( t = 0 \) and time \( t \) can be assumed to be uncorrelated and we infer \( C(k) \sim \langle \exp(kH_0(t)) \rangle \langle \exp(-kH_0(0)) \rangle \). An interesting side issue is the role of finite
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time correlations; this will, however, not be considered here. Finally, averaging with respect to the canonical ensemble, \( \exp(-\beta H_0)/Z(\beta) \), for a heat reservoir maintained at temperature \( T = 1/\beta \), we arrive at equation (1.11), i.e.,

\[
C(k) = \frac{Z(\beta + k)Z(\beta - k)}{Z(\beta)^2}.
\]

(3.2)

In the case of two degrees of freedom, e.g., for a harmonic oscillator coupled to a heat bath, where \( Z(\beta) \propto 1/\beta \) [29, 32], we obtain

\[
C(k) = \frac{\beta^2}{\beta^2 - k^2}.
\]

(3.3)

4. Multi-oscillator heat bath

Here we derive the characteristic function introducing an explicit representation of the heat reservoir in terms of a system of non-interacting oscillators. In a quantum context this is a standard approach [36–41] and we therefore defer technical details to appendix A. The total system consisting of the oscillator coupled to the heat bath is isolated and globally energy conserving. The oscillator itself can exchange energy with the reservoir and is in this regard an open system. The total system is described by the oscillator Hamiltonian \( H_0 \), a Hamiltonian \( H_1 \) for the heat bath together with an interaction term \( V \) to be specified later. We have

\[
H_0(x, p) = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2x^2,
\]

(4.1)

\[
H_1(\{x_k, p_k\}) = \sum_k \left( \frac{p_k^2}{2M} + \frac{1}{2}M\Omega_k^2x_k^2 \right),
\]

(4.2)

where \( m \) is the mass and \( \omega_0 \) the frequency of the oscillator. The bath oscillators have mass \( M \) and frequencies \( \Omega_k \), where \( k \) is the wave number.

The analysis is simplified by introducing the complex amplitude variables \( a \) and \( b_k \), see appendix A, yielding the Hamiltonians

\[
H_0(a) = \omega_0|a|^2,
\]

(4.3)

\[
H_1(\{b_k\}) = \sum_k \Omega_k|b_k|^2.
\]

(4.4)

For the interaction between the oscillator and the heat bath we choose the linear coupling

\[
V = \sum_k \lambda_k(ab_k^* + a^*b_k),
\]

(4.5)

where the coupling strength \( \lambda_k \) is assumed weak. By appropriate choice of the phases of \( a \) and \( b_k \) we can ensure that \( \lambda_k \) is real. We note that the interaction Hamiltonian

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$V$ differs from the shift interaction $H_1 = \sum_k (p_k^2/2M + (1/2)M\Omega_k^2(x_k - x)^2)$ used by Ford [36–38]. In the present case the interaction $V$ corresponds to the rotating wave approximation employed in quantum optics [39, 42, 43].

In the ohmic approximation [36–41] and at long times the amplitude of the oscillator is given by

$$a(t) = \sum_k \frac{\lambda_k e^{-i\omega_0 t}}{\Omega_k - \omega_0 + i\Gamma} b_k,$$

(4.6)

where the renormalised frequency $\tilde{\omega}_0 = \omega_0 + \Delta$, the shift $\Delta$ and the damping $\Gamma$ are given by

$$\Delta = P\sum_k \frac{\lambda_k^2}{\omega_0 - \Omega_k},$$

(4.7)

$$\Gamma = \pi \sum_k \lambda_k^2 \delta(\omega_0 - \Omega_k).$$

(4.8)

Here $b_k = b_k(0)$ is the initial value of the amplitude of the $k$th reservoir mode. In the limit of a large reservoir time smoothing is implemented and we have $\sum_k \lambda_k^2 \cdots = \int d\Omega g(\Omega) \cdots$, where $\lambda_k^2$ is incorporated in the definition of the density of states $g(\Omega)$.

For $\Delta$ and $\Gamma$ we then obtain

$$\Delta = P \int d\Omega \frac{\lambda_k^2}{\omega_0 - \Omega},$$

(4.9)

$$\Gamma = \pi \int d\Omega g(\Omega) \delta(\omega_0 - \Omega) = \pi g(\omega_0).$$

(4.10)

It is instructive to show that the oscillator locks onto the reservoir temperature at long times. Inserting the solution in equation (4.6) in equation (4.3) and using equipartition of the $k$-th mode, i.e., $\Omega_k\langle|b_k|^2\rangle = T$, we obtain for the mean value of $H_0$

$$\langle H_0 \rangle = \omega_0 \sum_k \frac{T}{\Omega_k (\Omega_k - \tilde{\omega}_0)^2 + \Gamma^2} \sim \omega_0 T \int d\Omega \frac{g(\Omega)}{(\Omega - \tilde{\omega}_0)^2 + \Gamma^2} \sim T,$$

(4.11)

demonstrating equipartition for the oscillator due to coupling to the heat reservoir.

### 5. Heat distribution

Regarding the characteristic function for the heat we obtain, inserting $Q(t) = H_0(t) - H_0(0)$ from equation (2.1) and averaging over the initial reservoir states $b_k$ according to $H_1(0) = \sum_k \Omega_k |b_k(0)|^2$, the functional integral

$$C(k) = \int \prod \langle db_k |^2 \ e^{-\beta H_1(0)} \ e^{k(H_0(t) - H_0(0))} / \int \prod \langle db_k |^2 \ e^{-\beta H_1(0)},$$

(5.1)
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Figure 1. We depict the heat distribution function $P(Q)$ as function of $Q$ and the characteristic function $C(ik)$ as function of $k$. We have set $\beta = 1$.

where we have used $\prod_k dp_k dx_k \propto \prod_k db_k^* db_k$. Inserting the solution $a(t)$ in equation (4.6) in $H_0\{a\}$ in equation (4.3) the functional integral (5.1) has a Gaussian form and can be performed using standard techniques [44]. By mean of the identity $\int \prod_k |db_k|^2 \exp\left(-\sum_{k\neq l} b_k^* A_{k l} b_l\right) \propto (\det A)^{-1}$ the evaluation of (5.1) is reduced to an eigenvalue problem. Deferring details to appendix B we obtain the characteristic function in equation (3.3), i.e., $C(k) = \beta^2 / (\beta^2 - k^2)$.

6. Discussion

In this paper we have discussed heat fluctuations in equilibrium for an oscillator driven by a single heat reservoir. A simple heuristic argument yields the characteristic function $C(k) = \beta^2 / (\beta^2 - k^2)$ only depending on the inverse temperature $\beta$. However, the main purpose of the present work is to demonstrate that this result is also obtained by an explicit representation of the heat reservoir as a collection of independent oscillators; a representation of a heat bath often used in a quantum mechanical context. We believe this approach is novel in the context of heat fluctuations. We have discussed the problem in terms of complex amplitude variables, the classical counterpart of creation and annihilation operators for quantum oscillators. The linear coupling to the reservoir is implemented within the rotating wave approximation, used in quantum optics. We, moreover, notice that the ohmic approximation, yielding the usual Langevin description with damping and white noise, here is equivalent to the quasi particle approximation, known from quantum many body theory. In figure 1 we have depicted the characteristic function and the associated heat distribution.

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Appendix A. Multi-oscillator heat bath

The oscillator and the heat bath are described by the Hamiltonian $H_0$ in equation (4.1) and $H_1$ in equation (4.2), where the coordinates and momenta satisfy the Poisson brackets [45]

$$\{x, p\} = 1, \quad \{x_k, p_p\} = \delta_{pk}. \quad (7.1)$$

Introducing the complex amplitude variables [46]

$$a = \sqrt{\frac{m\omega_0}{2}} x + \frac{1}{\sqrt{2m\omega_0}} p, \quad (7.3)$$

$$b_k = \sqrt{\frac{M\Omega_k}{2}} x_k + \frac{1}{\sqrt{2M\Omega_k}} p_k, \quad (7.4)$$

we have the Poisson brackets

$$\{a, a^*\} = -i, \quad (7.5)$$

$$\{b_k, b_k^*\} = -i\delta_{kp}, \quad (7.6)$$

and the Hamiltonians $H_0$, $H_1$, and $V$ in equations (4.3)–(4.5). Noting that the total Hamiltonian $H = H_0 + H_1 + V$ is time independent we obtain from the general equation of motion $dA/dt = \{A, H\}$:

$$i\frac{da}{dt} = \omega_0 a + \sum_k \lambda_k b_k, \quad (7.7)$$

$$i\frac{db_k}{dt} = \Omega_k b_k + \lambda_k a. \quad (7.8)$$

Introducing the Laplace transform [34, 47]

$$\tilde{a}(\omega) = \int_0^\infty dt \exp(i\omega t)a(t), \quad (7.9)$$

$$a(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \exp(-i\omega t)\tilde{a}(\omega), \quad (7.10)$$

with $\omega$ just above the real axis, i.e., $\omega \to \omega + i\epsilon$, the equations of motion take the form

$$(\omega - \omega_0)\tilde{a}(\omega) = ia + \sum_k \lambda_k \tilde{b}_k(\omega), \quad (7.11)$$

$$(\omega - \Omega_k)\tilde{b}_k(\omega) = ib_k + \lambda_k \tilde{a}(\omega), \quad (7.12)$$
where \( a = a(0) \) and \( b_k = b_k(0) \) denote the initial values. Solving for \( \tilde{a}(\omega) \) and \( \tilde{b}_k(\omega) \) we find

\[
(\omega - \omega_0 - \Sigma(\omega))\tilde{a}(\omega) = ia + \sum_k \frac{ib_k \lambda_k}{\omega - \Omega_k},
\]

\[
\sum_p ((\omega - \Omega_p)\delta_{kp} - \Sigma_{kp}(\omega))\tilde{b}_p(\omega) = ib_k + \frac{ia \lambda_k}{\omega - \omega_0},
\]

where we have introduced the self energies

\[
\Sigma(\omega) = \sum_k \frac{\lambda_k^2}{\omega - \Omega_k},
\]

\[
\Sigma_{kp}(\omega) = \frac{\lambda_k \lambda_p}{\omega - \omega_0}.
\]

Using the Plemelj formula \( 1/(\omega + i\epsilon) = P(1/\omega) - i\pi\delta(\omega) \) we obtain for \( \Sigma(\omega) \)

\[
\Sigma(\omega) = \Delta(\omega) - i\Gamma(\omega),
\]

\[
\Delta(\omega) = P\sum_k \frac{\lambda_k^2}{\omega - \Omega_k},
\]

\[
\Gamma(\omega) = \pi \sum_k \lambda_k^2 \delta(\omega - \Omega_k).
\]

In the weak coupling limit we can make the quasi particle approximation well known in many body theory [48] and replace \( \omega \) by \( \omega_0 \), i.e.,

\[
\Delta(\omega) \sim \Delta(\omega_0) = \Delta,
\]

\[
\Gamma(\omega) \sim \Gamma(\omega_0) = \Gamma.
\]

In the present context the weak coupling quasi particle approximation corresponds to the ohmic approximation [36–38]. Absorbing the shift \( \Delta \) in a renormalisation of \( \omega_0 \), i.e.,

defining \( \tilde{\omega}_0 = \omega_0 + \Delta \), we obtain for the oscillator amplitude

\[
\tilde{a}(\omega) = \frac{ia}{\omega - \tilde{\omega}_0 + i\Gamma} + \sum_k \frac{i\lambda_k}{(\omega - \tilde{\omega}_0 + i\Gamma)(\omega - \Omega_k)} b_k,
\]

and in time

\[
a(t) = a e^{-i\tilde{\omega}_0 t - i\Gamma t} + \sum_k \frac{\lambda_k}{\Omega_k - \tilde{\omega}_0 + i\Gamma} (e^{-i\Omega_k t} - e^{-i\tilde{\omega}_0 t - i\Gamma t}) b_k.
\]

For \( t = 0 \) we have \( a(t) = a \); at long times for \( t \gg 1/\Gamma \) we obtain equation (4.6), i.e.,

\[
a(t) = \sum_k \frac{\lambda_k}{\Omega_k - \tilde{\omega}_0 + i\Gamma} b_k.
\]
Regarding the reservoir modes we obtain solving equation (7.14) for $\tilde{b}(\omega)$

$$
\tilde{b}_p(\omega) = \frac{i b_p}{\omega - \Omega_p} + \frac{\lambda_p}{(\omega - \Omega_p)(\omega - \tilde{\omega}_0 + i\Gamma)} \left( \sum_k \lambda_k i b_k \omega - \Omega_k + i\Gamma \right).
$$

(7.25)

In time we have

$$
b_p(t) = b_p e^{-i\Omega_p t} + a\lambda_p \frac{e^{-i\Omega_p t} - e^{-i\tilde{\omega}_0 t - i\Gamma t}}{\Omega_p - \tilde{\omega}_0 + i\Gamma} + \sum_k \frac{\lambda_p \lambda_k b_k}{\Omega_p - \Omega_k} \left( \frac{e^{-i\Omega_p t} - e^{-i\tilde{\omega}_0 t - i\Gamma t}}{\Omega_p - \tilde{\omega}_0 + i\Gamma} - \frac{e^{-i\Omega_k t} - e^{-i\tilde{\omega}_0 t - i\Gamma t}}{\Omega_k - \tilde{\omega}_0 + i\Gamma} \right).
$$

(7.26)

For $t = 0$ we have $b_p(t) = b_p$; at long times for $t \gg 1/\Gamma$ we obtain

$$
b_p(t) = b_p e^{-i\Omega_p t} + a\lambda_p \frac{e^{-i\Omega_p t}}{\Omega_p - \tilde{\omega}_0 + i\Gamma} + \sum_k \frac{\lambda_p \lambda_k b_k}{\Omega_p - \Omega_k} \left( \frac{e^{-i\Omega_p t}}{\Omega_p - \tilde{\omega}_0 + i\Gamma} - \frac{e^{-i\Omega_k t}}{\Omega_k - \tilde{\omega}_0 + i\Gamma} \right).
$$

(7.27)

In the limit of a large heat reservoir the discrete spectrum of reservoir modes labelled by the wavenumber $k$ becomes a continuum. Replacing $\sum_k \lambda_k^2 \cdots$ by the integral $\int d\Omega g(\Omega) \cdots$, where the density of states $g(\Omega)$ incorporates the coupling $\lambda_k$, we obtain

$$
\Sigma(\omega) = \int d\Omega \frac{g(\Omega)}{\omega - \tilde{\omega}},
$$

(7.28)

$$
\Delta(\omega) = P \int d\Omega \frac{g(\Omega)}{\omega - \tilde{\omega}},
$$

(7.29)

$$
\Gamma(\omega) = \pi \int d\Omega g(\Omega) \delta(\omega - \Omega) = \pi g(\omega).
$$

(7.30)

The continuum limit thus automatically implies irreversibility and the separation of time scales. These two assumptions are encoded in the standard classical Langevin/Fokker Planck approach. The long time expressions for the fields $a(t)$ and $b_p(t)$ in equation (7.24) and (7.27) are easy to interpret. Regarding $a(t)$ we notice that the dependence on the initial value $a$ drops out and $a(t)$ is entirely driven by the heat reservoir thus depending on the initial heat reservoir characterised by $b_k$; the heat reservoir gives rise to damping and at the same time locks $a(t)$ onto $b_k(0)$. Regarding the reservoir amplitudes $b_k(t)$ there is a first order “back action” on the $k$th mode from the oscillator and an induced second order interaction between the modes.

**Appendix B. Multi-oscillator heat bath derivation of $P(Q)$**

In order to evaluate the distribution $P(Q)$ we must keep track of the energy flow or heat $Q$ between the reservoir and the oscillator. In the weak coupling limit we can
ignore the energy stored in the interaction term $V$ given by (4.5) and identify the heat $Q$ with the increase of the oscillator energy in the time span $t$. Consequently, $Q$ is given by Equation (2.1), i.e. $Q(t) = H_0(t) - H_0(0)$, where $H_0(t) = \omega_0 |a(t)|^2$. Inserting the solution $a(t)$ in equation (7.24) we arrive at

\begin{equation}
Q(t) = \sum_{kp} b_k^* b_p B_{kp}(t),
\end{equation}

\begin{equation}
B_{kp}(t) = A_k^*(t)A_p(t) - A_k^*(0)A_p(0),
\end{equation}

\begin{equation}
A_k(t) = \sqrt{\omega_0 \over \Omega_k - \tilde{\omega}_0 + i \Gamma}.
\end{equation}

For the characteristic function in equation (3.1) we then obtain

\begin{equation}
C(k) = \int \prod_k db_k db_k^* e^{-\beta H_1(0)} e^{i Q(t)} / \int \prod_k db_k db_k^* e^{-\beta H_1(0)},
\end{equation}

where we average over the bath amplitudes $b_k$ at time $t = 0$. Using the identity

\begin{equation}
\int \prod_k db_k db_k^* e^{-\sum_m A_m b_m b_m^*} \propto (\det A)^{-1},
\end{equation}

and inserting $H_1(0)$ from equation (4.4) we have

\begin{equation}
C(k) = \frac{\det(\beta \Omega_k \delta_{kp})}{\det(\beta \Omega_k \delta_{kp} - k B_{kp})}.
\end{equation}

In order to evaluate the determinant we consider the eigenvalue problem

\begin{equation}
\sum_l (\beta \Omega_k \delta_{kl} - k B_{kl}) \Phi_l = \mu \Phi_k.
\end{equation}

Inserting $B_{kl}$ from equations (7.32) and (7.33) we have

\begin{equation}
(\beta \Omega_k - \mu) \Phi_k = k (A_k^*(t) K - A_k^*(0) L),
\end{equation}

\begin{equation}
K = \sum_l A_l(t) \Phi_l,
\end{equation}

\begin{equation}
L = \sum_l A_l(0) \Phi_l.
\end{equation}

Solving equation (7.38) for $\Phi_l$ and inserting in equations (7.39) and (7.40) we obtain the linear system

\begin{equation}
K = K \sum_l \frac{k |A_l(t)|^2}{\beta \Omega_l - \mu} - L \sum_l \frac{k A_l(t) A_l^*(0)}{\beta \Omega_l - \mu},
\end{equation}

\begin{equation}
L = K \sum_l \frac{k A_l(0) A_l^*(t)}{\beta \Omega_l - \mu} - L \sum_l \frac{k |A_l(0)|^2}{\beta \Omega_l - \mu}.
\end{equation}
implying the determinantal condition

\[
\left( 1 - \sum_k k |A_k(t)|^2 \right) \left( 1 + \sum_k k |A_k(0)|^2 \right) + \sum_k k A_k(t) A_k^*(0) + \sum_p k A_p(0) A_p^*(t) = 0.
\]  

(7.43)

Further reduction inserting \( A_p \) yields the condition

\[
k^2 \sum_{kp} \left[ \frac{\omega_0^2 \lambda_k^2 \lambda_p^2 (1 - e^{i(\Omega_k - \Omega_p)t})}{(\beta \Omega_k - \mu)(\beta \Omega_p - \mu)((\Omega_k - \tilde{\omega}_0)^2 + \Gamma^2)((\Omega_p - \tilde{\omega}_0)^2 + \Gamma^2)} \right] = 1,
\]  

(7.44)

determining the eigenvalues \( \mu \). In the continuum limit inserting the density of states \( g(\Omega) \) we have

\[
k^2 \int d\Omega d\Omega' \frac{g(\Omega)g(\Omega')\omega_0^2 (1 - e^{i(\Omega - \Omega')t})}{(\beta \Omega - \mu)(\beta \Omega' - \mu)((\Omega - \tilde{\omega}_0)^2 + \Gamma^2)((\Omega' - \tilde{\omega}_0)^2 + \Gamma^2)} = 1.
\]  

(7.45)

Integrating over \( \Omega \) and \( \Omega' \) to leading order in \( \lambda_k \) setting \( \tilde{\omega}_0 \sim \omega_0 \) and introducing \( \Gamma = \pi g(\omega_0) \) we obtain

\[
k^2 g(\omega_0) \omega_0^2 (\pi/\Gamma)^2 \frac{1}{(\beta \omega_0 - \mu)^2} = 1,
\]  

(7.46)

or the eigenvalues

\[
\mu_{\pm} \propto \beta \pm k,
\]  

(7.47)

yielding

\[
C(k) = \frac{\beta^2}{\beta^2 - k^2},
\]  

(7.48)

in agreement with \( C(k) \) in equation (3.3).

References

[1] Trepagnier E, Jarzynski C, Ritort F, Crooks G, Bustamante C and Liphardt J 2004 Proc. Natl Acad. Sci. U S A. 101 15038
[2] Collin D, Ritort F, Jarzynski C, Smith S B, Jr I T and Bustamante C 2005 Nature 437 231
[3] Tietz C, Schuler S, Speck T, Seifert U and Wrachtrup J 2006 Phys. Rev. Lett. 97 050602
[4] Blickle V, Speck T, Helden L, Seifert U and Bechinger C 2006 Phys. Rev. Lett. 96 070603
[5] Imparato A, Peliti L, Pesce G, Rusciano G and Sasso A 2007 Phys. Rev. E 76 050101R
[6] Douarche F, Joubaud S, Garnier N B, Petrosyan A and Ciliberto S 2006 Phys. Rev. Lett. 97 140603
[7] Garnier N and Ciliberto S 2007 Phys. Rev. E 71 060101(R)
[8] Imparato A, Jop P, Petrosyan A and Ciliberto S 2008 J. Stat. Mech P10017
[9] Jarzynski C 1997 Phys. Rev. Lett. 78 2690
[10] Kurchan J 1998 J. Phys. A: Math. Gen. 31 3719
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[11] Gallavotti G 1996 Phys. Rev. Lett. 77 4334
[12] Crooks G E 1999 Phys. Rev. E 60 2721
[13] Crooks G E 2000 Phys. Rev. E 61 2361
[14] Seifert U 2005 Phys. Rev. Lett. 95 040602
[15] Seifert U 2005 Europhys. Lett. 70 36
[16] Evans D J, Cohen E G D and Morriss G P 1993 Phys. Rev. Lett. 71 2401
[17] Evans D J and Searles D J 1994 Phys. Rev. E 60 2721
[18] Seifert U 2005 Phys. Rev. E 61 2361
[19] Lebowitz J L and Spohn H 1999 J. Stat. Phys. 95 333
[20] Gaspard P 2004 J. Stat. Phys. 117 599
[21] Imparato A and Peliti L 2006 Phys. Rev. E 74 026106
[22] van Zon R and Cohen E G D 2004 Phys. Rev. Lett. 92 130601
[23] van Zon R and Cohen E G D 2003 Phys. Rev. E 67 046102
[24] van Zon R and Cohen E G D 2004 Phys. Rev. E 69 056121
[25] Speck T and Seifert U 2005 Euro. Phys. J. B 43 521
[26] Salazar D, Macedo A and Vasconcelos G 2019 Phys. Rev. E 99 022133
[27] Denzler T and Lutz E 2018 Phys. Rev. E 98 052106
[28] Landau L and Lifshitz E 1980 Statistical Physics Part 1 (Oxford: Pergamon)
[29] Derrida B and Brunet E 2005 Einstein Aujourd’Hui (Les Ulis: EDP Sciences)
[30] Fogedby H C and Imparato A 2011 J. Stat. Mech. P05015
[31] Reichl L E 1998 A Modern Course in Statistical Physics (New York: Wiley)
[32] Fogedby H C and Imparato A 2009 J. Phys. A: Math. Theor. 42 475004
[33] Lebedev N N 1972 Special Functions and Their Applications (New York: Dover)
[34] Gradshteyn I S and Ryzhik I M 1965 Table of Integrals, Series, and Products (New York: Academic)
[35] Ford G W, Kac M and Mazur P 1965 J. Math. Phys. 6 504
[36] Ford G and Kac M 1987 J. Stat. Phys. 46 803
[37] Ford G W, Lewis J T and O’Connell R F 1988 Phys. Rev. A 37 4419
[38] Glauber R and Man’ko V I 1984 JETP 60 450
[39] Caldeira A O and Leggett A J 1983 Ann. Phys., NY 149 374
[40] Caldeira A O 1983 Physica A 121 587
[41] Gardiner C W 1997 Handbook of Stochastic Methods (New York: Springer)
[42] Fogedby H C 1993 Phys. Rev. A 47 4364
[43] Zinn-Justin J 1989 Quantum Field Theory and Critical Phenomena (Oxford: Oxford University Press)
[44] Landau L and Lifshitz E 1959 Mechanics (Oxford: Pergamon)
[45] Breuer H P and Petruccione F 2002 The Theory of Open Quantum Systems (Oxford: Oxford University Press)
[46] Mahan G D 1990 Many Particle Physics (New York: Plenum)