Integrability of the spatial restricted three–body problem near collisions

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Abstract

We prove the integration of the spatial circular restricted three-body problem in a neighbourhood of its collision singularities by extending an idea of Tullio Levi-Civita.

1 Introduction

The circular restricted three-body problem is defined by the motion of a body $P$ of infinitesimally small mass in the gravitation field of two massive bodies $P_1$ and $P_2$, the primary and secondary body respectively, which rotate uniformly around their common center of mass. In a rotating frame we consider the Hamiltonian:

$$h(x, y, z, p_x, p_y, p_z) = \frac{p_x^2 + p_y^2 + p_z^2}{2} + p_x y - p_y x - \frac{1 - \mu}{r_1} - \frac{\mu}{r_2}, \quad (1)$$

where $r_1 = \sqrt{(x + \mu)^2 + y^2 + z^2}$ and $r_2 = \sqrt{(x - 1 + \mu)^2 + y^2 + z^2}$ denote the distances of $P$ from $P_1, P_2$; notice that as usual the units of mass, length and time have been chosen so that the masses of $P_1$ and $P_2$ are $1 - \mu$ and $\mu$ ($\mu \leq 1/2$) respectively, their coordinates are $(x_1, 0, 0) = (-\mu, 0, 0)$, $(x_2, 0, 0) = (1 - \mu, 0, 0)$ and their revolution period is $2\pi$.

For $\mu > 0$, no smooth constants of motion independent of the Hamilton function $h$ are known, and this represents the major obstruction to the lack of explicit uniform representations of solutions of the problem. There is a long history around the existence/non-existence of first integrals for the three-body problem as well as for general Hamiltonian systems. Theorems of non-existence of such constant of motions are due to Bruns [3] (whose result concerns algebraic first integrals) and Poincaré [15], revisited in [16, 10]. Actually, whenever we discuss about the theorem of non-existence of Poincaré for the restricted three
body problem, we are speaking precisely of first integrals analytic with respect to the mass parameter \( \mu \), in domains which, when represented using the Delaunay variables \((L, G, l, g)\) (for the planar problem), have the form \( D \times \mathbb{T}^2 \) where \( D \subseteq \mathbb{R}^2 \) is any open subset of the actions \( L, G \) with \( L > 0 \), \([1]\). The theorem of Poincaré leaves the door open for the integration of the system in domains which are not invariant under translations of the angles \((l, g)\). The interest in these kind of integrations depends on the specific domain. For example, when the domain is a neighbourhood of the collision set:

\[
C_j = \{(x, y, z, p_x, p_y, p_z) : (x, y, z) = (x_j, 0, 0)\}, \quad j = 1, 2,
\]
even restricted to constant energy levels \( C_j(E) = C_j \cap h^{-1}(E) \), the integration would allow to solve the (open) problem of close encounters, which we formulate as follows. Let \( \sigma \) be arbitrarily small; for any motion \((x(t), y(t), z(t))\) entering the ball \( B(x_j, 0, 0)(\sigma) \subseteq \mathbb{R}^3 \) (centered at \((x_j, 0, 0)\) of radius \( \sigma \)) at time \( t = t_1 \) and leaving it at time \( t_2 \), express \((x(t_2), y(t_2), z(t_2), p_x(t_2), p_y(t_2), p_z(t_2))\) as an explicit function of \((x(t_1), y(t_1), z(t_1), p_x(t_1), p_y(t_1), p_z(t_1))\).

In a remarkable paper \([13]\) T. Levi-Civita performed the integration of the planar circular restricted three-body problem in a neighbourhood of a collision set \( C_j \) through the introduction of a transformation which nowadays bears the name of Levi-Civita (LC hereafter) regularization; explicitly:

\[
\begin{align*}
    x &= x_j + u_1^2 - u_2^2, \quad (2) \\
    y &= 2u_1u_2, \quad (3) \\
    dt &= r_j \, ds, \quad (4)
\end{align*}
\]

where \((2), (3)\) are equivalent to the complex transformation: \( \xi = \zeta^2, \xi = (x - x_j) + iy, \zeta = u_1 + iu_2 \), while \([1]\) is a parametrization of the physical time \( t \) into the proper time \( s \). In the last part of the paper \([13]\) Levi-Civita proved the existence of an integral of the Hamilton-Jacobi equation of the Hamiltonian representing the planar circular restricted three-body problem regularized with \((2), (3), (4)\), which we call the Levi-Civita Hamiltonian, in a neighbourhood of the collision singularity at \( P_j \). The complete integral is constructed as a series analytic at \((u_1, u_2) = (0, 0)\), whose coefficients can be explicitly computed iteratively up to any arbitrary large order. From this series, he proved the existence of a second first integral for the problem, independent of \( h \), defined in a neighbourhood of the the collision singularity at \( P_j \). Therefore, the problem of planar close encounters can be solved explicitly\(^1\)

The regularization of the spatial restricted three-body problem has been done by Kustaanheimo and Stiefel \([11, 12]\) many decades after Levi-Civita, but the

\(^1\)As a matter of fact, Levi-Civita constructed the solution of the Hamilton–Jacobi equation only for the collision singularity at \( P_1 \). Nevertheless, Levi-Civita’s argument is valid also in a neighbourhood of the singularity at the secondary body \( P_2 \). Formally this extension is achieved by exchanging \( 1 - \mu \) with \( \mu \) within the series. We notice that while the series at \( P_1 \) is analytic also in \( \mu = 0 \), the series at \( P_2 \) is not.
integrability of the regularized Hamiltonian, which we call the Kustaanheimo-Stiefel Hamiltonian, has never been addressed. Here, our purpose is precisely to extend to the fully spatial case the point of view followed by Levi-Civita, thus offering a complete integrability of the spatial problem near collisions.

Regularizations of spatial problems are dramatically more complicated than regularizations of the planar problem, see [14]. As for the Levi-Civita regularization, the Kustaanheimo-Stiefel regularization (KS hereafter) is defined by the introduction of a transformation on the space variables and by a time-reparametrization; but the KS space transformation is more complicated than the LC space transformation, since it is a map from a space of redundant variables $u_1, u_2, u_3, u_4$ to a space of Cartesian variables $q_1, q_2, q_3$. Precisely, following [11, 12], we introduce the projection map:

$$\pi : \mathbb{R}^4 \rightarrow \mathbb{R}^3\quad (u_1, u_2, u_3, u_4) \mapsto \pi(u_1, u_2, u_3, u_4) = (q_1, q_2, q_3),$$

where $(q_1, q_2, q_3, 0) = A(u)u$, and:

$$A(u) = \begin{pmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \\ u_4 & -u_3 & u_2 & -u_1 \end{pmatrix},$$

is a matrix which plays a central role in the KS regularization, it is a linear homogeneous function of $u_1, \ldots, u_4$ and satisfies $A(u)A^T(u) = |u|^2 I$. Matrices with such properties exist only for $n = 1, 2, 4, 8$ (see [9]), and the lack of this result for $n = 3$ is the reason for the definition of the KS regularization in a 4-dimensional space. Then, for any motion in the KS variables we introduce the parametrization of time [14]; notice that we have $r_j = |u|^2$. The space and time transformations (4), (5) have been used to represent the regularized equations of motions of the spatial circular restricted three-body problem in various forms (see [3] for a review of the subject).

To better accomplish the technique of integration introduced in [13] we first perform the phase-space translation

$$X = x - x_j, \quad Y = y, \quad Z = z, \quad P_x = p_x, \quad P_y = p_y - x_j, \quad P_z = p_z,$$

conjugating $h$ to the Hamiltonian (to fix ideas we present all these computations for $j = 2$, so that the reference system defined above will be called planetocentric):

$$H(X, Y, Z, P_x, P_y, P_z) = \frac{P_x^2 + P_y^2 + P_z^2}{2} + P_x Y - P_y X - \frac{\mu}{\sqrt{X^2 + Y^2 + Z^2}} - (1 - \mu) \left( \frac{1}{\sqrt{(X + 1)^2 + Y^2 + Z^2}} - 1 + X \right) - (1 - \mu) - \frac{(1 - \mu)^2}{2},$$

(8)
the constant terms being kept for comparison with the values of the original Hamiltonian \( h \). The KS regularization is obtained from the space transformation (5) with \((q_1, q_2, q_3) = (X, Y, Z)\), and can be expressed in the following Hamiltonian form:

\[
K(u, U; E) = \frac{1}{8} |U - b_{(0,0,1)}(u)|^2 - \frac{1}{2} |u|^2 |(0, 0, 1) \times \pi(u)|^2 - |u|^2 E_\mu - \mu
- (1 - \mu) |u|^2 \left( \frac{1}{|\pi(u) + (1, 0, 0)|} - 1 + \pi(u) \cdot (1, 0, 0) \right),
\]

where \( U = (U_1, U_2, U_3, U_4) \) denote the conjugate momenta to \( u = (u_1, u_2, u_3, u_4) \), the vector potential \( b_\omega(u) \) (in (9) we have \( \omega = (0, 0, 1) \)), is defined by

\[
b_\omega(u) = 2A^T(u)\Lambda_\omega A(u)u, \quad \Lambda_\omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 & 0 \\ \omega_3 & 0 & -\omega_1 & 0 \\ -\omega_2 & \omega_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]

and:

\[
E_\mu = E + (1 - \mu) + \frac{(1 - \mu)^2}{2}.
\]

The Hamiltonian \( K(u, U; E) \) is a regularization of the spatial three-body problem at \( P_2 \). This means that the solutions \((u(s), U(s))\) of the Hamilton equations of \( K(U, u; E) \) with initial conditions satisfying:

(i) \( u(0) \neq 0 \);

(ii) \( l(u(0), U(0)) = 0 \), where

\[
l(u, U) = u_4U_1 - u_3U_2 + u_2U_3 - u_1U_4
\]

is called the bilinear form;

(iii) \( K(u(0), U(0); E) = 0 \),

are conjugate, for \( s \) in a small neighbourhood of \( s = 0 \), via equations (4), (5) to solutions \((X(t), Y(t), Z(t), P_X(t), P_Y(t), P_Z(t))\) of the Hamilton equations of (8).

Our integration of the spatial circular restricted three-body problem is established on the construction of a complete integral \( W(u, \nu; E, \mu) \) of the Hamilton-Jacobi equation of Hamiltonian \( K(u, U; E) \), defined for all the values of the parameters \( \nu = (\nu_1, \ldots, \nu_4) \) in a neighbourhood of the sphere \( |\nu| = 1 \), and analytic in a neighbourhood of \( u = 0 \). Our main result is the following:

**Theorem 1.** For fixed values of \( E_\star \) and of \( \mu_\star > 0 \), there exists a complete integral \( W(u, \nu; E, \mu) \) of the Hamilton-Jacobi equation:

\[
\mathcal{K} \left( u, \frac{\partial W}{\partial u}(u, \nu; E, \mu); E \right) = \mu(|\nu|^2 - 1)
\]

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depending on the four parameters $\nu$ and on $E, \mu$, which is analytic for $E, \mu, \nu$ in the set:

$$\{|\mu - \mu_*| < a, \ |E - E_*| < b, \ |\nu| - 1| < c\}$$

and $u$ in the (complex) ball:

$$\{u \in \mathbb{C}^4 : |\nu| < d\}$$

with suitable constants $a, b, c, d > 0$ (depending only on $E_*, \mu_*$). The coefficients of the Taylor expansions of $W$ with respect to the variables $u$ can be explicitly computed iteratively to any arbitrary order; in particular we have:

$$W = \sqrt{8\mu} \sum_{j=1}^{4} \nu_j u_j + O_3(u). \quad (13)$$

The complete integral $W$ of the Hamilton-Jacobi equation will be used to define a canonical transformation through the system

$$U_\ell = \frac{\partial W}{\partial u_\ell}(u, \nu; E, \mu), \ \ell = 1, \ldots, 4 \quad (14)$$

$$n_\ell = \frac{\partial W}{\partial \nu_\ell}(u, \nu; E, \mu), \ \ell = 1, \ldots, 4. \quad (15)$$

conjugating $\mathcal{K}(u, U; E)$ to the Hamiltonian:

$$\hat{\mathcal{K}}(n, \nu) = \mu(\nu^2 - 1),$$

so that by denoting with $(n, \nu) = (\hat{n}(u, U; E, \mu), \hat{\nu}(u, U; E, \mu))$ the canonical transformation, the solutions $(u(s), U(s))$ of the Hamilton equations of $\mathcal{K}(u, U; E)$ are obtained from:

$$(n(0) + 2\mu \nu(0)s, \nu(0)) = (\hat{n}(u(s), U(s); E), \hat{\nu}(u(s), U(s); E)). \quad (16)$$

Formula (16) provides all the solutions of the spatial circular restricted three-body problem in a neighbourhood of the collision set $C_2$.

The proof of Theorem 1 is achieved through several steps: first, a geometric analysis of the KS Hamiltonian is needed to identify the parameters $\nu_1, \ldots, \nu_4$, providing the conserved momenta of Hamiltonian $\hat{\mathcal{K}}(n, \nu)$; second, an analytic part based on the Cauchy-Kowaleski theorem is used to provide analytic solutions to the Hamilton-Jacobi equation. The geometric analysis is the real original heart of the proof and is completely new with respect to the work of Levi-Civita, since the geometric part required by the planar case is rather simpler; the analytic part is instead the argument that we extend from the integration of the planar problem, with an additional care for the global definition of the family of particular solutions found.
An additional interesting question concerns the existence of Cartesian first integrals \( F(x, y, z, p_x, p_y, p_z) \) independent of \( h(x, y, z, p_x, p_y, p_z) \) defined in a set \( \mathcal{B}\setminus \mathcal{C}_j \), where \( \mathcal{B} \) is a neighbourhood of the collision set \( \mathcal{C}_j \). First, we remark that the existence of Cartesian first integrals is not granted a priori from the existence of first integrals of the KS Hamiltonian; for example \(|\nu|^2\) and \( l(n, \nu) \) do not provide, with evidence, Cartesian first integrals. But neither the momenta \( \nu \ell \) provide Cartesian first integrals. The deep reason is that the map \( \pi \) has not a global smooth inversion defined in a neighbourhood of \( q = (x - x_j, y, z) = 0 \) (see [8], where a similar problem is addressed for the global definition of chaos indicators for the spatial three body problem), so it can happen that functions \( F(n, \nu) \) which are first integrals for \( \hat{K} \) do not define global Cartesian smooth functions in any neighbourhood of the collision set \( \mathcal{C}_j \). Precisely, while we are not able define Cartesian representatives of \( \nu \ell, \ell = 1, \ldots, 4 \), which are smooth functions in a neighbourhood of \( \mathcal{C}_j \), the functions:

\[
N_X = \nu_1 n_4 - \nu_4 n_1 \\
N_Y = \frac{1}{2} (\nu_1 n_3 - n_1 \nu_3 + n_2 \nu_4 - n_4 \nu_2) \\
N_Z = \frac{1}{2} (\nu_1 n_2 - n_1 \nu_2 + n_4 \nu_3 - n_3 \nu_4)
\]  

are first integrals and have Cartesian representatives \( N_X, N_Y, N_Z \) globally defined and smooth in a neighbourhood of the collision sets. We consider the set of three first integrals:

\[
\left( H, N^2 := N_X^2 + N_Y^2 + N_Z^2, N_Z \right)
\]

We notice that, since \( N^2, N_Z \) are first integrals, we have:

\[
\{ H, N^2 \} = 0, \quad \{ H, N_Z \} = 0.
\]

The Poisson bracket \( \{ H, N_Z \} = 0 \) is sufficient to grant the complete integrability of the planar circular restricted three-body problem in a neighbourhood of its collision singularities. It remains to understand if even the spatial problem is completely integrable. At this regard, we notice that in the space of the variables \( n, \nu \) we have:

\[
\{ N^2, N_Z \} = l(n, \nu) a(n, \nu), \quad N^2 = N_X^2 + N_Y^2 + N_Z^2,
\]

so that the two integrals are in involution on the level set \( l(n, \nu) = 0 \). The atypical Poisson bracket in [18] seems a rule for the KS regularization. For example, the elementary Poisson brackets of \( q = \hat{q}(u), p = \hat{p}(u, U) \) defined from \( \hat{q}(u) = \pi(u), (\hat{p}_1, \hat{p}_2, \hat{p}_3, 0) = \frac{1}{2 |u|^2} A(u) U \), satisfy:

\[
\{ \hat{q}_i, \hat{q}_j \} = \delta_{ij}, \quad \{ \hat{q}_i, \hat{q}_j \} = 0, \quad \{ \hat{p}_i, \hat{p}_j \} = l(u, U) \phi_{ij}(u, U), \text{ for } i,j = 1, 2, 3.
\]

Through this paper, whenever we will refer to a subset of the Cartesian phase-space which is a neighbourhood of the collision set, we will precisely refer to a set \( \mathcal{B}\setminus \mathcal{C}_j \), where \( \mathcal{B} \) is a neighbourhood of the collision set.
From (18) and (19) we prove the following:

**Theorem 2.** The set of first integrals \((H, \mathcal{N}^2, \mathcal{N}_Z)\) is complete.

The paper is organized as follows. In Section 2 we revisit the definition of the KS transformation with respect to any spatial frame centered at \(P_j\) and arbitrarily rotated; Section 3 is devoted to the identification of suitable parameters for the definition of a complete integral of the Hamilton-Jacobi equation of \(K(u, U : E)\); in Section 4 we prove the existence of particular solutions of the Hamilton-Jacobi equation; in Section 5 we prove the existence of a complete integral, thus proving Theorem 1, and we use it to define a canonical transformation; in Section 6 we discuss the existence of Cartesian first integrals and we prove Theorem 2; in Appendix 1 we revisit the integration of the planar three body problem done by Levi-Civita in [13]; in Appendix 2 we review a basic formulation of the Cauchy-Kowaleski theorem.

## 2 The KS Hamiltonian revisited

In order to solve the problem of close encounters in the spatial case we need to introduce the KS transformation with respect to any spatial frame centered at \(P_j\) and arbitrarily rotated, while in the usual KS transformation the cartesian coordinates are referred to a rotating spatial frame with \(x\) axis containing the primaries \(P_1, P_2\) and the \(z\) axis orthogonal to their orbit plane. In addition, we consider also an arbitrary scaling of the coordinates by a factor \(\lambda > 0\); the scaling will be needed to define the parameters of the solutions of the Hamilton-Jacobi equation.

**The Lagrangian formulation in the Cartesian variables.** We start from the Lagrange function of the spatial circular restricted three-body problem:

\[
L_C(x, y, z, \dot{x}, \dot{y}, \dot{z}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \dot{y}x - \dot{x}y + \frac{1}{2}(x^2 + y^2) + \frac{1 - \mu}{\sqrt{(x + \mu)^2 + y^2 + z^2}} + \frac{\mu}{\sqrt{(x - 1 + \mu)^2 + y^2 + z^2}}
\]

\[
= \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + (\dot{x}, \dot{y}, \dot{z}) \wedge (0, 0, 1) \cdot (x, y, z) + \frac{1}{2} |(0, 0, 1) \wedge (x, y, z)|^2
\]

\[
+ \frac{1 - \mu}{\sqrt{(x + \mu)^2 + y^2 + z^2}} + \frac{\mu}{\sqrt{(x - 1 + \mu)^2 + y^2 + z^2}}
\]  

(20)

and, for any arbitrary matrix \(R \in SO(3)\) and any \(\lambda > 0\), we define the coordinates transformation:

\[
(x - x_j, y, z) = \lambda Rq,
\]  

(21)
where \( q = (q_1, q_2, q_3) \) and (to fix ideas) \( x_j = x_2 = 1 - \mu \), which extends to the transformation on the generalized velocities:

\[
(x, \dot{y}, \dot{z}) = \lambda \mathcal{R} \dot{q}.
\]  

(22)

By transforming the Lagrangian \( L_C \) with (21), (22), and by dropping the constants as well as the terms which are independent on the \( q_i \) and linear in the \( \dot{q}_i \) (which do not contribute to the Lagrange equations) we obtain the Lagrangian:

\[
L(q, \dot{q}) = \frac{1}{2} \lambda^2 |\dot{q}|^2 + \lambda^2 (\dot{q} \wedge \omega) \cdot q + \frac{1}{2} \lambda^2 |\omega \wedge q|^2 + \frac{\mu}{\lambda |q|}
\]

\[
+ (1 - \mu) \left( \frac{1}{|\lambda q + e|} + \lambda q \cdot e \right),
\]

(23)

where \( \omega = \mathcal{R}^T(0, 0, 1) \), \( e = \mathcal{R}^T(1, 0, 0) \).

**The redundant variables \( u_1, \ldots, u_4 \).** Redundant variables are easily introduced in the Lagrangian formalism (see, for example, [1]). As a first step, we compute the function:

\[
\hat{L}(u, \dot{u}) = L\left(\pi(u), \frac{\partial \pi}{\partial u}(u) \dot{u}\right)
\]

using the formulas:

\[
(q_1, q_1, q_3, 0) = A(u)u
\]

\[
(q_1, q_2, q_3, 0) = A(\dot{u})u + A(u)\dot{u} = 2A(u)\dot{u} - 2(0, 0, 0, l(u, \dot{u}))
\]

where \( l(u, \dot{u}) \) is the bilinear form defined in (11). We obtain:

\[
\hat{L}(u, \dot{u}) = 2\lambda^2 |u|^2 |\dot{u}|^2 - 2\lambda^2 l(u, \dot{u})^2 + \lambda^2 b_\omega(u) \cdot \dot{u}
\]

\[
+ \frac{1}{2} \lambda^2 |\omega \wedge \pi(u)|^2 + \frac{\mu}{\lambda |u|} + (1 - \mu) \left( \frac{1}{|\lambda \pi(u) + e|} + \lambda \pi(u) \cdot e \right),
\]

(24)

where \( b_\omega(u) \), is the vector potential already defined in (10).

Let us compare the solutions of the Lagrange equations of \( \hat{L}(u, \dot{u}) \), which we write in the form:

\[
[\hat{L}]_i(u, \dot{u}, \ddot{u}) = 0 , \quad \forall i = 1, \ldots, 4
\]

where:

\[
[\hat{L}]_i(u, \dot{u}, \ddot{u}) = \frac{d}{dt} \frac{\partial L}{\partial \dot{u}_i} - \frac{\partial L}{\partial u_i},
\]

with the solutions of the Lagrange equations of \( L(q, \dot{q}) \), which we write in the form:

\[
[L]_j(q, \dot{q}, \ddot{q}) = 0 , \quad \forall j = 1, 2, 3
\]

where:

\[
[L]_j(q, \dot{q}, \ddot{q}) = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j}.
\]
Proposition 1. If \( u(t) \) is a solution of the Lagrange equations of \( \hat{L}(u, \dot{u}) \) with \( u(0) \neq 0 \), then \( q(t) = \pi(u(t)) \) is a solution of the Lagrange equations of \( L \) as soon as \( u(t) \neq 0 \).

Proof of Proposition 1. For any smooth curve \( u(t) \), we have:

\[
[\hat{\mathcal{L}}](u(t), \dot{u}(t), \ddot{u}(t)) = \left( \frac{\partial \pi}{\partial u} \right)^T [\mathcal{L}](\pi(u(t)), \frac{d}{dt} \pi(u(t)), \frac{d^2}{dt^2} \pi(u(t)))
\]

where \([\hat{\mathcal{L}}] \in \mathbb{R}^4\), \([\mathcal{L}] \in \mathbb{R}^3\) are the vectors of components \([\hat{\mathcal{L}}]_i\), \([\mathcal{L}]_j\) respectively.

Since for \( u \neq 0 \), the Kernel of the matrix \( \frac{\partial \pi^T}{\partial u} \) contains only the vector \((0, 0, 0)\), any solution \( u(t) \) of the Lagrange equations of \( \hat{L} \) (i.e. satisfying \([\hat{\mathcal{L}}] = (0, 0, 0, 0)\)) projects to a solution \( q(t) = \pi(u(t)) \) of the Lagrange equations of \( \hat{L} \) as soon as \( u(t) \neq 0 \).

The Legendre transform defined by \( \hat{L} \) is not invertible, since the quadratic form \( 2 |u|^2 \dot{u} - 2l(u, \dot{u})^2 \) in the generalized velocities \( \dot{u} \) is degenerate; therefore the definition of the Hamiltonian formalism is more tricky than usual. To remove the degeneracy we consider the modified Lagrangian:

\[
\mathcal{L}(u, \dot{u}) = \hat{L}(u, \dot{u}) + 2\lambda^2 l(u, \dot{u})^2 = 2\lambda^2 |u|^2 \dot{u}^2 + \lambda^2 \mathcal{L}(u)
\]

\[
+ \frac{1}{2} \lambda^2 \omega \wedge \pi(u)^2 + \frac{\mu}{\lambda |u|^2} + (1 - \mu) \left( \frac{1}{|\lambda \pi(u) + e|} + \lambda \pi(u) \cdot e \right),
\]

(25)

whose Legendre transform:

\[
U = \frac{\partial \mathcal{L}}{\partial u} = \lambda^2 (4 |u|^2 \dot{u} + b_\omega(u)),
\]

(26)

where \( U = (U_1, U_2, U_3, U_4) \) denote the momenta conjugate to \( u = (u_1, u_2, u_3, u_4) \), is non-degenerate for \( u \neq 0 \).

Proposition 2. If \( u(t) \) is a solution of the Lagrange equations of \( \mathcal{L}(u, \dot{u}) \) with initial conditions \( u(0), \dot{u}(0) \) satisfying \( u(0) \neq 0 \) and \( l(u(0), \dot{u}(0)) = 0 \), then it is also a solution of the Lagrange equations of \( \hat{L}(u, \dot{u}) \) as soon as \( u(t) \neq 0 \).

Before proving the Proposition, we remark that the Lagrangian \( \mathcal{L}(u, \dot{u}) \) is invariant with respect to the one-parameter family of transformations:

\[
u \mapsto S^0_\alpha u
\]

(27)

where \( S^0_\alpha \in SO(4) \) is defined by

\[
S^0_\alpha = \begin{pmatrix}
\cos \alpha & 0 & 0 & -\sin \alpha \\
0 & \cos \alpha & \sin \alpha & 0 \\
0 & -\sin \alpha & \cos \alpha & 0 \\
\sin \alpha & 0 & 0 & \cos \alpha
\end{pmatrix}
\]

(28)
whose orbits define the fibers of the projection $\pi$, i.e. $\pi(S_{\alpha}^0 u) = \pi(u)$ for all $\alpha$. Precisely, for all $u, \dot{u}, \alpha$, we have:

$$L(S_{\alpha}^0 u, S_{\alpha}^0 \dot{u}) = L(u, \dot{u}).$$

As a consequence, by Noether’s theorem, the function:

$$J(u, \dot{u}) = \left( \frac{d}{d\alpha} S_{\alpha}^0 u \right)_{|\alpha=0} \cdot \frac{\partial L}{\partial \dot{u}} = (-u_4, u_3, -u_2, u_1) \cdot (4\lambda^2 |u|^2 \dot{u} + \lambda^2 b_\omega(u))$$

is a first integral for the Lagrange equations of $L$. Moreover, since: $(-u_4, u_3, -u_2, u_1) \cdot b_\omega(u)$ vanishes identically, then:

$$J_0(u, \dot{u}) = |u|^2 l(u, \dot{u})$$

is a first integral for the Lagrange equations of $L$.

**Proof of Proposition 2.** Let us consider a solution $u(t)$ of the Lagrange equations of $L$ with $u(0) \neq 0$ and $l(u(0), \dot{u}(0)) = 0$. Since $J_0(u, \dot{u})$ is constant along the solution, as soon as $u(t) \neq 0$ we have also $l(u(t), \dot{u}(t)) = 0$, as well as $l(u(t), \ddot{u}(t)) = 0$.

We claim that $u(t)$ solves also the Lagrange equations of $\dot{L}$. In fact, we have:

$$[\dot{L}]_i = [L]_i - 2\lambda^2 \left( \frac{d}{dt} \frac{\partial}{\partial \dot{u}_i} l^2(u, \dot{u}) - \frac{\partial}{\partial u_i} l^2(u, \dot{u}) \right)$$

$$= [L]_i - 4\lambda^2 \left( \frac{d}{dt} \left( l(u, \dot{u}) \frac{\partial}{\partial \dot{u}_i} l(u, \dot{u}) \right) - l(u, \dot{u}) \frac{\partial}{\partial u_i} l(u, \dot{u}) \right)$$

$$= [L]_i - 4\lambda^2 \left( l(u, \ddot{u}) \frac{\partial}{\partial \dot{u}_i} l(u, \dot{u}) + l(u, \dot{u}) \frac{d}{dt} \frac{\partial}{\partial \dot{u}_i} l(u, \dot{u}) - l(u, \dot{u}) \frac{\partial}{\partial u_i} l(u, \dot{u}) \right)$$

and when computed along the solution $u(t)$ (so that $[L]_i = 0, l(u, \dot{u}) = 0, l(u, \ddot{u}) = 0$) we have also:

$$[\dot{L}]_i(u(t), \dot{u}(t), \ddot{u}(t)) = 0.$$

$$\Box$$

Finally, we remark that for any initial condition $(q(0), \dot{q}(0))$ with $q(0) \neq 0$ we have the freedom of choosing the initial conditions $(u(0), \dot{u}(0))$ satisfying:

$$\pi(u(0)) = q(0), \quad \frac{\partial \pi}{\partial u}(u(0)) \dot{u}(0) = \dot{q}(0), \quad l(u(0), \dot{u}(0)) = 0.$$

In fact, if $l(u(0), \dot{u}(0)) \neq 0$, since the Kernel of $\frac{\partial \pi}{\partial \dot{u}}(u)$ is generated by $\dot{u} = (u_4, -u_3, u_2, -u_1)$ we have the freedom of adding to $\dot{u}(0)$ a vector $\xi \dot{u}$ and to select $\xi \in \mathbb{R}$ so that:

$$l(u(0), \dot{u}(0) + \xi \dot{u}) = l(u(0), \dot{u}(0)) + \xi |u(0)|^2 = 0.$$
The KS Hamiltonian. The Legendre transform \((26)\), which is invertible for all \(|u| \neq 0\), conjugates the Lagrangian system defined by \(\mathcal{L}\) to the Hamiltonian system with Hamilton function:

\[
K(u, U) = \frac{1}{8\lambda^2 |u|^2} |U - \lambda^2 b_\omega(u)|^2 - \frac{1}{2} \lambda^2 |\omega \wedge \pi(u)|^2
- \frac{\mu}{\lambda |u|^2} - (1 - \mu) \left( \frac{1}{|\lambda \pi(u) + e|} + \lambda \pi(u) \cdot e \right),
\]

where \(U = (U_1, U_2, U_3, U_4)\) are the conjugate momenta to \(u = (u_1, u_2, u_3, u_4)\).

Let us compute the bilinear equality \(l(u, \dot{u}) = 0\) in the Hamiltonian formulation; for all \(u \neq 0\) we have:

\[
l(u, \dot{u}) = \frac{1}{4\lambda^2 |u|^2} l(u, U - \lambda^2 b_\omega(u)) = \frac{1}{4\lambda^2 |u|^2} (l(u, U) - \lambda^2 l(u, b_\omega(u))).
\]

Since \(l(u, b_\omega(u)) = 0\) identically, the bilinear equality \(l(u, \dot{u}) = 0\) is equivalent to the condition \(l(u, U) = 0\).

The Hamiltonian \(K(u, U)\) is still singular at \(u = 0\); to remove the singularity we perform the iso-energetic reduction. For any value \(E\) we introduce the Hamiltonian:

\[
K_{\lambda R}(u, U) = |u|^2 \left( K(u, U) - E - \frac{(1 - \mu)^2}{2} \right)
= \frac{1}{8\lambda^2 |U - \lambda^2 b_\omega(u)|^2} - \frac{1}{2} \lambda^2 |u|^2 |\omega \wedge \pi(u)|^2 - \mu \lambda^{-1} - |u|^2 \left( E + (1 - \mu) + \frac{(1 - \mu)^2}{2} \right)
- (1 - \mu) |u|^2 \left( \frac{1}{|\lambda \pi(u) + e|} - 1 + \lambda \pi(u) \cdot e \right),
\]

which we call the KS Hamiltonian.

The solutions \(u(s), U(s)\) of the Hamilton equations:

\[
\begin{align*}
\dot{u}_j &= \frac{\partial}{\partial U_j} K_{\lambda R} \\
\dot{U}_j &= -\frac{\partial}{\partial u_j} K_{\lambda R} , \quad j = 1, \ldots, 4
\end{align*}
\]

with initial conditions \(u(0) \neq 0\) and \(K_{\lambda R}(u(0), U(0)) = 0\) are conjugate by the time transformation:

\[
t(s) = \int_0^s |u(\sigma)|^2 d\sigma
\]

to solutions of the Hamilton equations of \(K(u, U)\) as soon as \(u(s) \neq 0\). We also notice that \(K_{\lambda R}(u, U)\) is invariant with respect to the one-parameter family of transformations

\[
(u, U) \mapsto (S_0^0 u, S_0^0 U),
\]
i.e. we have:
\[ K_{\lambda R}(S^0_0 u, S^0_0 U) = K_{\lambda R}(u, U). \]
As a consequence, \( l(u, U) \) is a first integral for this Hamiltonian system.

We remark that for \( \lambda = 1, R = I \) the Hamiltonian:
\[
K_I(u, U) = \frac{1}{8} |U - b_{(0,0,1)}(u)|^2 - \frac{1}{2} |u|^2 |(0, 0, 1) \wedge \pi(u)|^2 - \mu - |u|^2 \left( E + (1 - \mu) + \left( \frac{1 - \mu}{2} \right)^2 \right)
- (1 - \mu) |u|^2 \left( \frac{1}{|\pi(u) + (1, 0, 0)|} - 1 + \pi(u) \cdot (1, 0, 0) \right)
\]
provides an Hamiltonian formulation of the traditional KS regularization; see, for example, [7, 4] for alternative derivations.

3 The Hamilton-Jacobi equation for the KS Hamiltonian: the parameters space

Our aim is to define a complete integral of the Hamilton-Jacobi equation:
\[
K_I \left( u, \frac{\partial W}{\partial u} \right) = \kappa, \tag{33}
\]
which is analytic in a neighbourhood of \( u = 0 \), obtained from a family of solutions of (33) depending on suitable four parameters. Therefore, we proceed by defining families of particular solutions \( \tilde{W} \) of the Hamilton–Jacobi equations:
\[
K_{\lambda R} \left( u, \frac{\partial \tilde{W}}{\partial u} \right) = \kappa
\]
where \( R \in SO(3) \) is an arbitrary rotation matrix of the euclidean three-dimensional space and \( \lambda > 0 \), with \( \tilde{W} \) vanishing identically on an hyperplane defined by the choice of \( R \).

Remark. This procedure depends on four free parameters related to \( \lambda > 0 \) and to the matrix \( R \in SO(3) \), which in the end will provide the four parameters needed to define a complete solution of the Hamilton-Jacobi equation. The first idea to extend the argument of Levi-Civita would seem that of using the group \( SO(4) \) to transform the KS Hamiltonian \( K_I \), and then to define families of particular solutions \( \tilde{W} \) of the Hamilton–Jacobi equations:
\[
\tilde{K} \left( u, \frac{\partial \tilde{W}}{\partial u} \right) = \kappa
\]
where \( \tilde{K}(u, U) = \tilde{K}_I(Su, SU) \) with \( S \in SO(4) \), with \( \tilde{W} \) vanishing identically on an hyperplane defined by the choice of \( S \). The problem is that, for arbitrary
matrix $S \in SO(4)$, the bilinear form $l(u, U)$ is not invariant, i.e. $l(Su, SU) \neq l(u, U)$ on some $u, U$. We therefore follow a different strategy.

We have therefore to find a family of transformations on $\mathbb{R}^4$ such that:

- they project on the linear transformations of the three-dimensional euclidean space
  $$(X, Y, Z) \mapsto \lambda \mathcal{R}(X, Y, Z)$$
  with $\lambda > 0$ and $\mathcal{R} \in SO(3)$;

- their canonical extensions to the momenta leave invariant the diagram about the conjugation of Hamiltonians represented in figure 1;

- their canonical extensions to the momenta leave invariant the bilinear form $l(u, U)$ (up to the multiplication with a constant different from zero).

We find that the matrices:

$$S_\nu = \begin{pmatrix} \nu_1 & -\nu_2 & -\nu_3 & -\nu_4 \\ \nu_2 & \nu_1 & -\nu_4 & \nu_3 \\ \nu_3 & \nu_4 & \nu_1 & -\nu_2 \\ -\nu_3 & \nu_2 & \nu_1 & \nu_4 \end{pmatrix},$$

with $\nu = (\nu_1, \nu_2, \nu_3, \nu_4) \in \mathbb{R}^4 \backslash \{0\}$, satisfy:

$$S_\nu S_\nu^T = |\nu|^2 I,$$

and define linear transformations of $\mathbb{R}^4$ which project on linear transformation of the three-dimensional space so that, for any $u \in \mathbb{R}^4$, we have:

$$\pi(S_\nu u) = \mathcal{R}_\nu \pi(u)$$

where:

$$\mathcal{R}_\nu = \begin{pmatrix} \nu_1^2 - \nu_2^2 - \nu_3^2 + \nu_4^2 & -2(\nu_1 \nu_2 + \nu_3 \nu_4) & -2(\nu_1 \nu_3 - \nu_2 \nu_4) \\ 2(\nu_1 \nu_2 - \nu_3 \nu_4) & \nu_1^2 - \nu_2^2 + \nu_3^2 - \nu_4^2 & -2(\nu_2 \nu_3 + \nu_1 \nu_4) \\ 2(\nu_1 \nu_3 + \nu_2 \nu_4) & -2(\nu_2 \nu_3 - \nu_1 \nu_4) & \nu_1^2 + \nu_2^2 - \nu_3^2 - \nu_4^2 \end{pmatrix}$$

is a matrix satisfying:

$$\mathcal{R}_\nu \mathcal{R}_\nu^T = |\nu|^4 I,$$

which depends on the $\nu_j$ as in the Euler-Rodrigues formula.

Moreover, for all $(u, U) \in T^*\mathbb{R}^4$, we have:

$$l(S_\nu u, S_\nu U) = |\nu|^2 l(u, U).$$

We therefore consider the set of matrices:

$$S = \cup_{\nu \in \mathbb{R}^4 \backslash \{0\}} S_\nu.$$
and the map:

\[ \Pi : S \rightarrow SO(3) \]

\[ S_\nu \mapsto \Pi(S_\nu) = \frac{1}{|\nu|^2} \mathcal{R}_\nu. \]

The map \( \Pi \) is surjective. We have the following:

**Proposition 3.** For any matrix \( S_\nu \in S \) we have the identity:

\[ K_I(S_\nu u, S_\nu^T U) = |\nu|^2 K_{|\nu|^2 \Pi(S_\nu)}(u, U). \] (38)

**Proof of Proposition 3.** Let us denote \( u = S_\nu \tilde{u}, U = S_\nu \tilde{U} \); we have the following identities:

1. \( |u| = |\nu| |\tilde{u}|; \)
2. \( |\pi(u) + (1, 0, 0)| = |\pi(S_\nu \tilde{u}) + (1, 0, 0)| = |\nu|^2 \Pi(S_\nu)\pi(\tilde{u}) + (1, 0, 0)| = |\nu|^2 \pi(\tilde{u}) + \Pi(S_\nu)^T(1, 0, 0); \)
3. \( |\pi(u) \cdot (1, 0, 0)| = |\pi(S_\nu \tilde{u}) \cdot (1, 0, 0)| = |\nu|^2 \pi(\tilde{u}) \cdot \Pi(S_\nu)^T(1, 0, 0); \)
4. \( |(0, 0, 1) \wedge \pi(u)| = |(0, 0, 1) \wedge \pi(S_\nu \tilde{u})| = |\nu|^2 |(0, 0, 1) \wedge \Pi(S_\nu)\pi(\tilde{u})| = |\nu|^2 \Pi(S_\nu)^T(0, 0, 1) \wedge (\pi(\tilde{u})|, \)

which are proved from (33) and (34). Finally, we prove:

\[ \left| S_\nu^{-T} \tilde{U} - b_{(0,0,1)}(S_\nu \tilde{u}) \right|^2 = \frac{1}{|\nu|^2} \left| \tilde{U} - |\nu|^4 b_{\Pi(S_\nu)^T(0,0,1)}(\tilde{u}) \right|^2. \] (39)

From direct computation, for any \( u \in \mathbb{R}^4 \), we obtain:

\[ A(S_\nu u) S_\nu = \mathcal{R}_\nu A(u) \]

with:

\[ \mathcal{R}_\nu = \begin{pmatrix}
\nu_1^2 - \nu_2^2 - \nu_3^2 + \nu_4^2 & -2(\nu_1 \nu_2 + \nu_3 \nu_4) & -2(\nu_1 \nu_3 - \nu_2 \nu_4) & 0 \\
2(\nu_1 \nu_2 - \nu_3 \nu_4) & \nu_1^2 - \nu_2^2 + \nu_3^2 - \nu_4^2 & -2(\nu_2 \nu_3 + \nu_1 \nu_4) & 0 \\
2(\nu_1 \nu_3 + \nu_2 \nu_4) & -2(\nu_2 \nu_3 - \nu_1 \nu_4) & \nu_1^2 + \nu_2^2 - \nu_3^2 - \nu_4^2 & 0 \\
0 & 0 & 0 & |\nu|^2 \end{pmatrix}. \]

As a consequence, using (34) and by recalling the definition (10) of the vector potential \( b_\omega \), we have:

\[ \left| S_\nu^{-T} \tilde{U} - b_{(0,0,1)}(S_\nu \tilde{u}) \right|^2 = \frac{1}{|\nu|^2} S_\nu^{-T} \tilde{U} - 2A(S_\nu \tilde{u})^T \Lambda_{(0,0,1)} A(S_\nu \tilde{u}) S_\nu \tilde{u}. \]
Figure 1: For any $S \in S$ and $\mathcal{R} = \Pi(S)$ the diagram is commutative.

\[
L(\dot{q}, \dot{\hat{q}}) \xrightarrow{q = \mathcal{R}\dot{q}} L(\hat{q}, \dot{\hat{q}}) \\
\mathcal{L}_I(u, \dot{u}) \rightarrow \mathcal{L}_R(\dot{\hat{u}}, \hat{u}) = 0 \\
\mathcal{K}_I(u, U) \rightarrow \mathcal{K}_R(\dot{\hat{u}}, \hat{U}) = 0
\]

\[
\mathcal{K}_I(u, U) \rightarrow \mathcal{K}_R(\dot{\hat{u}}, \hat{U}) = 0
\]

\[
\mathcal{K}_I(u, U) \rightarrow \mathcal{K}_R(\dot{\hat{u}}, \hat{U}) = 0
\]

\[
F_1: \text{For any } S \in S \text{ and } \mathcal{R} = \Pi(S) \text{ the diagram is commutative.}
\]

\[
= \frac{1}{|\nu|^2} \left| \tilde{U} - 2S^T \nu A(S\nu) T \Lambda_{(0,0,1)} A(S\nu) S\nu \tilde{U} \right|^2
\]

\[
= \frac{1}{|\nu|^2} \left| \tilde{U} - 2A(\tilde{u}) T \hat{R}_\nu \Lambda_{(0,0,1)} \hat{R}_\nu A(\tilde{u}) \tilde{U} \right|^2
\]

\[
= \frac{1}{|\nu|^2} \left| \tilde{U} - |\nu|^4 b_{\Pi(S\nu)\tau(0,0,1)}(\tilde{u}) \right|^2
\]

where the last equality is a consequence of the fact that, for any $\omega \in \mathbb{R}^3$, the matrix:

\[
\Lambda_\omega = \begin{pmatrix}
0 & -\omega_3 & \omega_2 \\
\omega_3 & 0 & -\omega_1 \\
-\omega_2 & \omega_1 & 0
\end{pmatrix}
\]

represents the linear transformations of $\mathbb{R}^3$:

\[
\Lambda_\omega \vec{x} = \omega \wedge \vec{x}
\]

for all $\vec{x} \in \mathbb{R}^3$; then, for all $\vec{x} \in \mathbb{R}^3$ we have also:

\[
\Pi(S\nu) T \Lambda_{(0,0,1)} \Pi(S\nu) \vec{x} = \Pi(S\nu) T (0,0,1) \wedge (\Pi(S) \vec{x}) = (\Pi(S\nu) T (0,0,1)) \wedge \vec{x},
\]

and therefore

\[
\Pi(S\nu) T \Lambda_\omega \Pi(S\nu) = \Lambda_{\Pi(S\nu)\tau(0,0,1)}.
\]

From all the previous equalities we obtain (38).

4 The Hamilton-Jacobi equation for the KS Hamiltonian: particular solutions

In this Section we prove the existence of particular solutions $\tilde{W}$ of the Hamilton-Jacobi equation:

\[
\mathcal{K}_{|\nu|^2 \Pi(S\nu)} \left( u, \frac{\partial \tilde{W}}{\partial u} \right) = \frac{\kappa}{|\nu|^2},
\]

with the following properties:
- the solutions $\tilde{W}(u; E, \mu, \kappa, \nu_1, \ldots, \nu_4)$ are defined for any value of the parameters $(E, \mu, \kappa, \nu_1, \ldots, \nu_4)$ in a set $D_{\alpha, \ldots, \delta}(E_*, \mu_*)$ defined by fixed values $E_*$ and $\mu > 0$, and by suitably small $\alpha, \beta, \gamma, \delta > 0$:

\[
\begin{align*}
|\mu - \mu_*| &< \alpha \\
|E - E_*| &< \beta \\
|\kappa| &< \gamma \\
\{\nu \in \mathbb{R}^4 : 1 - \delta < |\nu| < 1 + \delta\},
\end{align*}
\]

and for any value of the parameters in this set it is analytic in the same common domain:

\[
u \in \mathbb{C}^4 : |\nu| < \sigma,
\]

with $\sigma > 0$ (depending only on $E_*, \mu_*, \alpha, \ldots, \delta$).

- they satisfy:

\[
\tilde{W}(0, u_2, u_3, u_4; E, \mu, \kappa, \nu_1, \ldots, \nu_4) = 0
\]  (41)

for all $u_2, u_3, u_4$ in a neighbourhood of 0.

- they are analytic also with respect to the parameters.

We remark that the domain above considered is local in the variables $u$ and in the parameters $E, \mu, \kappa$, but is not local in the parameters $\nu$ which are naturally defined in a neighbourhood of $\mathbb{S}^3$. Therefore, since the proof will be obtained from the Cauchy-Kowaleski theorem, which grants the existence of local analytic solutions of PDE, we have to pay some care in proving the global character of the solutions obtained from the Cauchy-Kowaleski theorem with respect to the parameters $\nu_i$.

In order to apply the Cauchy-Kowaleski theorem, we first rewrite the HJ equation (40) as follows:

\[
\begin{align*}
\frac{\partial \tilde{W}}{\partial u_1} & = |\nu|^4 b_{1, \omega}(u) \pm \sqrt{8} |\nu| \left( \mu + \kappa + \frac{1}{2} |\nu|^6 |u|^2 |\omega \wedge \pi(u)|^2 + |u|^2 |\nu|^2 E_\mu \right) \\
& \quad + (1 - \mu) |u|^2 |\nu|^2 \frac{1}{\left(|\nu|^2 \pi(u) + e\right) + 1 + |\nu|^2 \pi(u) \cdot e} \\
& \quad - \frac{1}{8 |\nu|^2} \sum_{j=2}^4 \left( \frac{\partial \tilde{W}}{\partial u_j} - |\nu|^4 b_{j, \omega}(u) \right)^2 \right)^{1/2} 
\]  (42)

where $E_\mu = E + (1 - \mu) + \frac{(1-\mu)^2}{2}$, $\omega = \Pi(S_{\nu})^T(0, 0, 1)$, $e = \Pi(S_{\nu})^T(1, 0, 0)$. We solve the previous equation by selecting the positive sign in front of the square
root (the minus would provide a different solution), and therefore we consider the function:

\[ F(u_1, \ldots, u_4, p_1, p_2, p_3; E, \mu, \kappa, \nu_1, \ldots, \nu_4) = |\nu|^4 b_{1,\omega}(u) \]

\[ + \sqrt{8} |\nu| \left( \mu + \kappa + \frac{1}{2} |\nu|^6 |u|^2 |\omega \cdot \pi(u)|^2 + |u|^2 |\nu|^2 E_\mu \right) \]

\[ + (1 - \mu) |u|^2 |\nu|^2 \left( \frac{1}{|\nu|^2 \pi(u) + e} - 1 + |\nu|^2 \pi(u) \cdot e \right) \]

\[ - \frac{1}{8 |\nu|^2} \sum_{j=2}^{4} \left( p_{j-1} - |\nu|^4 b_j \omega(u) \right)^2 \]

which depends parametrically on \( E, \mu, \kappa, \nu_1, \ldots, \nu_4 \). For any fixed \( E_*, \mu_* \) with \( \mu_* > 0 \) there exist \( \alpha_0, \ldots, \delta_0 \) and \( \sigma_0 \) such that \( F \) is analytic for all \((E, \kappa, \nu_1, \ldots, \nu_4) \in D_{\alpha_0, \ldots, \delta_0} \) in the set \(|u| < \sigma_0\).

We first apply the Cauchy-Kovalevskaya theorem to the first-order PDE:

\[ \frac{\partial \tilde{W}}{\partial u_1} = F \left( u_1, \ldots, u_4, \frac{\partial \tilde{W}}{\partial u_2}, \frac{\partial \tilde{W}}{\partial u_3}, \frac{\partial \tilde{W}}{\partial u_4}; E, \mu, \kappa, \nu_1, \ldots, \nu_4 \right) \]

where \( E, \mu, \kappa, \nu_1, \ldots, \nu_4 \) are fixed in some set \( D_{\alpha_1, \ldots, \delta_1} \), with the boundary condition (41):

\[ \tilde{W}(0, u_2, u_3, u_4; E, \mu, \kappa, \nu_1, \ldots, \nu_4) = 0 \]

for \( u_2, u_3, u_4 \) in a neighbourhood of \( u = 0 \). We obtain (see Section 8) the existence of a unique solution \( \tilde{W}(u; E, \mu, \kappa, \nu_1, \ldots, \nu_4) \) of such PDE problem which is analytic in a neighbourhood of \( u = 0 \), and the radius of convergence of the series:

\[ \tilde{W} = \sum_{i_1, \ldots, i_4 \geq 0} c_{i_1, \ldots, i_4}(E, \mu, \kappa, \nu) u_1^{i_1} \ldots u_4^{i_4} \]

is common for all the values of the parameters in the set \( D_{\alpha_1, \ldots, \delta_1} \). The coefficients \( c_{i_1, \ldots, i_4}(E, \mu, \kappa, \nu) \) can be computed iteratively in the order \( i_1 + \ldots + i_4 \), and since they are functions globally defined in \( D_{\alpha_1, \ldots, \delta_1} \), the series (45) is globally defined in the \( D_{\alpha_1, \ldots, \delta_1} \). In particular, we have:

\[ \tilde{W} = \sqrt{8(\mu + \kappa)} |\nu|^2 u_1 + \frac{E_\mu |\nu|^3}{\sqrt{\mu + \kappa}} u_1 \sqrt{2} \left( \frac{u_1^2}{3} + u_2^2 + u_3^2 + u_4^2 \right) + u_1 O_3(u). \]

It remains to establish the regularity of the function \( \tilde{W} \) defined the series (45) with respect to the parameters \( E, \mu, \kappa, \nu \). Therefore we apply a second time the Cauchy-Kovalevskii theorem to the first-order PDE (44) by considering the independent variables \((u_1, u_2, u_3, u_4, E, \mu, \kappa, \nu_1, \ldots, \nu_4) \) in a neighbourhood of
\((u_1, u_2, u_3, u_4, E, \mu, \kappa, \nu_1, \ldots, \nu_4) = (0, 0, 0, 0, E^*, \mu_*, 0, \nu_1^*, \ldots, \nu_4^*)\) with \(\nu^* \in S^3\), with the boundary condition:

\[
\tilde{W}(0, u_2, u_3, u_4; E, \mu, \kappa, \nu_1, \ldots, \nu_4) = 0
\]

for \(u_2, u_3, u_4\) in a neighbourhood of \(u = 0\) and for all \(E, \mu, \kappa, \nu\) in a neighbourhood of \(E^*, \mu_*, 0, \nu^*\). We obtain (see Section 8) the existence of a unique solution \(\tilde{W}_1(u; E, \mu, \kappa, \nu_1, \ldots, \nu_4)\) of such PDE problem which is analytic in a neighbourhood of \((u, E, \mu, \kappa, \nu) = (0, E^*, \mu_*, 0, \nu_*)\), with series expansion:

\[
\tilde{W}_1 = \sum_{i_1, \ldots, i_{11} \geq 0} d_{i_1, \ldots, i_{11}} (E_*, \mu_*, \nu_*) u_1^{i_1} \ldots u_4^{i_4} (E - E_*)^{i_5} (\mu - \mu_*)^{i_6} \kappa^{i_7} (\nu_1 - \nu_1^*)^{i_8} \ldots (\nu_4 - \nu_4^*)^{i_{11}}
\]

converging within a radius \(\rho(E_*, \mu_*, \nu_*)\) depending only on \((E_*, \mu_*, \nu_*)\). But since \(\tilde{W}_1\) is also a solution of the PDE problem where the \(E, \mu, \kappa, \nu\) are given parameters, and \(\tilde{W}_1\) satisfy the same boundary condition (41), from uniqueness we obtain

\[
\tilde{W}_1(u; E, \mu, \kappa, \nu_1, \ldots, \nu_4) = \tilde{W}(u; E, \mu, \kappa, \nu_1, \ldots, \nu_4),
\]

and this proves the analyticity of the global solution \(\tilde{W}\) for any value of the parameters in some \(D_{\alpha, \ldots, \beta}\) and for some \(|u| \leq \sigma\).

5 The Hamilton-Jacobi equation for the KS Hamiltonian: a complete integral

Theorem 1 follows from the following:

**Proposition 4.** For fixed values of \(E_*\) and \(\mu_* > 0\), there exists a complete integral \(W(u, \nu; E, \mu)\) of the Hamilton-Jacobi equation (33) depending on the four parameters \(\nu\) and two additional parameters \(E, \mu\), with

\[
\kappa = \mu(|\nu|^2 - 1).
\]

and analytic for \(E, \mu, \nu\) in the set:

\[
\{|\mu - \mu_*| < a, \ |E - E_*| < b, \ \nu \in \mathbb{R}^4 : ||\nu| - 1| < c\}
\]

and \(u\) in the (complex) ball:

\[
B_\sigma = \{u \in \mathbb{C}^4 : |\nu| < d\}
\]

with suitable \(a, b, c, d > 0\). The coefficients of the Taylor expansions of \(W\) with respect to the variables \(u\) can be explicitly computed iteratively; in particular we have:

\[
W = \sqrt{S\mu} \sum_{j=1}^4 \nu_j u_j + O_3(u).
\]

(47)
Proof of Proposition 4. The complete integral is defined by:

\[ W(u; E, \mu, \nu) = \tilde{W}(|\nu|^2 S^T_\nu u; E, \mu, \kappa_\nu, \nu), \]

with \( \kappa_\nu = \mu(|\nu|^2 - 1) \), where \( \tilde{W}(\tilde{u}; E, \mu, \kappa, \nu) \) denotes the solution of the Hamilton-Jacobi equation (40):

\[ K|\nu|^2 \Pi(S_\nu) \left( \tilde{u}, \frac{\partial \tilde{W}}{\partial \tilde{u}}(\tilde{u}, E, \mu, \kappa_\nu, \nu) \right) = \kappa \frac{|\nu|^2}{|\nu|^2}, \quad (48) \]

as it has been defined in the previous section. In fact, since we have:

\[ \frac{\partial W}{\partial u}(u; E, \mu, \nu) = |\nu|^{-2} S_\nu \frac{\partial \tilde{W}}{\partial \tilde{u}}(|\nu|^{-2} S^T_\nu u, E, \mu, \kappa_\nu, \nu), \]

using Proposition 1, and setting \( u = S_\nu \tilde{u} \), we obtain

\[ K_I \left( u, \frac{\partial W}{\partial u}(u; E, \mu, \nu) \right) = K_I \left( S_\nu \tilde{u}, S^T_\nu \frac{\partial \tilde{W}}{\partial \tilde{u}}(\tilde{u}, E, \mu, \kappa_\nu, \nu) \right) \]

\[ = |\nu|^2 K|\nu|^2 \Pi(S_\nu) \left( \tilde{u}, \frac{\partial \tilde{W}}{\partial \tilde{u}}(\tilde{u}, E, \mu, \kappa_\nu, \nu) \right) = \kappa_\nu = \mu(|\nu|^2 - 1). \]

By replacing in \( (46) \) \( \kappa \) with \( \kappa_\nu \) and \( u \) with \( |\nu|^{-2} S^T_\nu u \) we obtain \( (47) \). Therefore, the determinant:

\[ j_4(u, \nu; E, \mu) = \det \left( \frac{\partial W}{\partial u_i \partial \nu_j} \right) \]

satisfies:

\[ j_4(0, \nu; E, \mu) = 64 \mu^2. \quad (49) \]

Therefore, \( W \) is a complete integral of the Hamilton-Jacobi equation in a neighbourhood of \( u = 0 \). \( \square \)

Let us analyze some consequences of Theorem 1.

For any \( \nu \in S^3 \), which corresponds to \( \kappa = 0 \), the function \( W \) defines the foliation:

\[ \Gamma_\nu = \left\{ (u, U) \in T^*\mathbb{R}^4 : |u| < \sigma, \quad U_j = \frac{\partial W}{\partial u_j}(u; E, \mu, \nu) \right\}, \]

which is locally invariant (the solutions with initial conditions in a leaf \( \Gamma_\nu \) can flow out of it in the future and/or in the past). Since we are interested in motions of the KS Hamiltonian \( K_I \) which project on motions of the three–body problem, and since the leaves \( \Gamma_\nu \) are foliated by the first integral \( l(u, U) \), we consider:

\[ \tilde{\Gamma}_\nu = \left\{ (u, U) \in T^*\mathbb{R}^4 : |u| < \sigma, \quad U_j = \frac{\partial W}{\partial u_j}(u; E, \mu, \nu), \quad l(u, U) = 0 \right\}. \]
**Proposition 5.** For any $\nu \in \mathbb{S}^3$, $\tilde{\Gamma}_\nu$ is a manifold of dimension 3 in a neighbourhood of $(u, U) = (0, \sqrt{8\mu} \nu)$.

**Proof of Proposition 5.** The set $\tilde{\Gamma}_\nu$ is obtained from the solutions $(u, U)$ of the system:

$$F_1(u, U) = 0, \quad F_5(u, U) = 0,$$

where:

$$F_j(u, U) = U_j - \frac{\partial W}{\partial u_j}(u; E, \mu, \nu), \quad j = 1, \ldots, 4$$

$$F_5(u, U) = l(u, U),$$

with $|u| < \sigma$. Since from (47) we have:

$$F_j = U_j - \sqrt{8\mu} \nu_j + O_2(u), \quad j = 1, \ldots, 4,$$

the restriction of the Jacobian matrix of the map $F = (F_1, \ldots, F_5)$ to $\tilde{\Gamma}_\nu$ has the representation:

$$\mathcal{J}(u, U)|_{\tilde{\Gamma}_\nu} = \begin{pmatrix} \nabla_u F_1 & \nabla_u F_2 & \nabla_u F_3 & \nabla_u F_4 & \nabla_u F_5 \end{pmatrix} |_{\tilde{\Gamma}_\nu} + O_1(u) = \begin{pmatrix} 0 & 0 & 0 & 0 & \nu_4 \\ 0 & 0 & 0 & 0 & -\nu_3 \\ 0 & 0 & 0 & 0 & \nu_2 \\ 0 & 0 & 0 & 0 & -\nu_1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} + O_1(u).$$

Since $|\nu| = 1$, the rank of the matrix $\mathcal{J}(u, U)|_{\tilde{\Gamma}_\nu}$ is equal to 5 in a neighbourhood of $(u, U) = (0, \sqrt{8\mu} \nu)$. □

It remains therefore to represent explicitly the motions on the 3-dimensional locally invariant manifolds $\tilde{\Gamma}_\nu$, and this will be done by defining from the function $W$ a suitable canonical transformation. Precisely, we consider the system:

$$U_i = \frac{\partial W}{\partial u_i}(u, \nu; E, \mu), \quad i = 1, \ldots, 4 \quad (50)$$

$$n_i = \frac{\partial W}{\partial \nu_i}(u, \nu; E, \mu), \quad i = 1, \ldots, 4. \quad (51)$$

which is well defined since the function $W$ can be differentiated with respect to the variables $\nu_i$. 20
Inversion of the sub-system (50). We first consider the sub-system formed by equations (50):

\[ U_i = \frac{\partial W}{\partial u_i}(u, \nu; E, \mu), \quad i = 1, 2, 3, 4. \]  

(52)

From (49), (47) and the analyticity of \( W \) with respect to \( u, \nu \), for any \( \nu^* \in S^3 \) we have the local inversion of the sub-system (50) with respect to the variables \( \nu \), in a neighbourhood of \((u, \nu) = (0, \nu^*)\):

\[ \nu = \hat{\nu}(u, U; E, \mu), \]

and the functions \( \hat{\nu} \) are analytic. As a matter of fact, we have the stronger result:

**Lemma 1.** The sub-system (50) has a global analytic inversion:

\[ \nu = \hat{\nu}(u, U; E, \mu), \]

defined for \( u, U \) so that \( u \) is in some complex ball \( B(d_0) \) and for \( U \) in the image of the map:

\[ \nu \mapsto \frac{\partial W}{\partial u}(u, \nu; E, \mu) \]

with \( \nu \in \Omega_{c_0} = \{ \nu : ||\nu| - 1| < c_0 \} \) with some suitable \( c_0, d_0 \).

**Proof of lemma 1.** We first proof that for fixed \( E, \mu \), for all \( u \) suitably close to \( u = 0 \), and for suitably small \( c_1 \), the map:

\[ \Psi_u : \Omega_{c_1} \to \mathbb{R}^4 \]

\[ \nu \mapsto \frac{\partial W}{\partial u}(u, \nu; E, \mu) \]  

(53)

is injective. From (47), we have the representation:

\[ \Psi_u(\nu) = \sqrt{8\mu} \nu + \psi_u(u, \nu; E, \mu) \]

with \( \psi_u(u, \nu; E, \mu) = O_2(u) \).

For arbitrary \( k \geq 3 \), we extend the map \( \Psi_u \) to a map

\[ \Psi^k_u : B(1 + c_1) \to \mathbb{R}^4 \]

\[ \nu \mapsto \Psi^k_u(\nu) = \sqrt{8\mu} \nu + \phi^k(|\nu|)\psi_u(u, \nu; E, \mu) \]  

(54)

where \( B(1 + c_1) \) is the real ball centered at \( \nu = 0 \) of radius \( 1 + c_1 \) and

\[ \phi^k : [0, 1 + c_1) \to \mathbb{R}^4 \]

is a \( C^k \)-smooth function such that \( \phi^k(x) = 1 \) if \( x \in [1 - c_1/2, 1 + c_1) \), \( \phi^k(x) = 0 \)

if \( x \in [0, 1 - c_1) \), and in the interval \((1 - c_1, 1 - c_1/2) \) increases smoothly and monotonically from 0 to 1. For any fixed \( k \), by restricting eventually the domain of \( u \), we have that the map \( \Psi^k_u \) is convex in the set \( B(1 + c_1) \). Then, from a result
on the global inversion of convex maps (see Theorem 4.2, page 137, of [2]), the map $\Psi^k_u$ is injective. But this implies that the also the map:

$$\Psi_u : \Omega_{\frac{3}{4}} \rightarrow \mathbb{R}^4$$

$$\nu \mapsto \frac{\partial W}{\partial \nu}(u, \nu; E, \mu)$$

(55)
is injective (in fact, if $\Psi_u(\nu') = \Psi_u(\nu'')$ with $\nu', \nu'' \in \Omega_{\frac{3}{4}}$, then we have also $\Psi^k_u(\nu') = \Psi^k_u(\nu'')$ and therefore $\nu' = \nu''$) and therefore has the inverse:

$$\Psi_u^{-1} : \Psi_u(\Omega_{\frac{3}{4}}) \rightarrow \Omega_{\frac{3}{4}}.$$  

From the local inversion theorem the inverse map is analytic. □

The canonical transformation. The inversion of the system of equations (50) provides the functions:

$$\nu_i = \hat{\nu}_i(u, U; E, \mu)$$

$$n_i = \hat{n}_i(u, U; E, \mu) \quad i = 1, 2, 3, 4$$

(56)

which define a canonical transformation:

$$(n, \nu) = \chi_4(u, U)$$

conjugating $\mathcal{K}_T$ to the Hamiltonian:

$$\hat{\mathcal{K}}(n, \nu) = \mu(\nu^2 - 1).$$

Therefore, the momenta $\nu_i$ are constants of motion and the solutions $(u(s), U(s))$ of the Hamilton equations of $\mathcal{K}_T$ are obtained from the inversion of:

$$(n(0) + 2 \mu \nu(0)s, \nu(0)) = \chi_4(u(s), U(s)).$$

The bilinear relation. From the identity:

$$W(u, \nu; E, \mu) = W(S^0_u u, S^0_\alpha \nu; E, \mu), \quad \forall \alpha \in \mathbb{R}$$

by differentiating both sides with respect to $\alpha$ and computing in $\alpha = 0$ we obtain:

$$l \left( \frac{\partial W}{\partial u}(u, \nu, E, \mu), u \right) + l \left( \frac{\partial W}{\partial \nu}(u, \nu, E, \mu), \nu \right) = 0$$

and therefore we have $l(u, U) = 0$ if and only if $l(\hat{n}, \hat{\nu}) = 0$. Consistently, $l(n, \nu)$ is a first integral of the Hamilton equations of $\hat{\mathcal{K}}(n, \nu).$
6 The first integrals in the space of the Cartesian variables

In the previous section we have constructed four first integrals \( \hat{\nu}_i(u,U;E,\mu) \) of the KS Hamiltonian which are analytic in a neighbourhood of the collision set, represented in the space of coordinates \( u,U \) by:

\[
C = \{ (u,U) \in T^*\mathbb{R}^4 : u = 0, \|U\| = \sqrt{8\mu} \}.
\]

It is therefore interesting to know if, from the \( \hat{\nu}_i \), it is possible to construct first integrals \( N_i(X,Y,Z,P_X,P_Y,P_Z) \) defined in the Cartesian phase–space of the variables \( (X,Y,Z,P_X,P_Y,P_Z) \) introduced in Section 2, eq. (7).

Following [8], we first show that from each \( \hat{\nu}_i \) we construct a family of local first integrals defined only in a neighbourhood of any point \( (X,Y,Z,P_X,P_Y,P_Z) \), with \( (X,Y,Z) \) in a neighbourhood of \( (0,0,0) \); from this family, we construct 2 first integrals (independent on the energy \( E \)) which are globally defined in a complete neighbourhood of \( (X,Y,Z) = (0,0,0) \).

A phase-spaces projection. We introduce a projection from the space:

\[
T^*\mathbb{R}^4_0 = \{ (u,U) \in T^*\mathbb{R}^4 : |u| \neq 0, \ l(u,U) = 0 \}
\]

to the Cartesian phase space of the variables \( (X,Y,Z,P_X,P_Y,P_Z) \) introduced in Section 2, eq. (7). We denote:

\[
(X,Y,Z,P_X,P_Y,P_Z) = \tilde{\pi}(u,U)
\]

where \( (X,Y,Z) = \pi(u) \) and:

\[
(P_X,P_Y,P_Z,0) = \frac{1}{2|u|^2} A(u) U
\]

(57)

Local inversions of the phase-space projection. We consider a local inversion of \( (X,Y,Z) = \pi(u) \):

\[
\pi^{-1} : \mathcal{W} \rightarrow \mathbb{R}^4
\]

\[
(X,Y,Z) \mapsto u = \pi^{-1}(X,Y,Z)
\]

with \( \mathcal{W} \subset \mathbb{R}^3 \setminus 0 \) open set, and define:

\[
(u,U) = \chi(X,Y,Z,P_X,P_Y,P_Z)
\]

where \( u = \pi^{-1}(X,Y,Z) \) and, from (57):

\[
U = 2A(u)^T(P_X,P_Y,P_z,0).
\]
We introduce the matrix:

\[
\Omega = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix},
\]  

(58)

so that \( l(u, U) = u \cdot \Omega U \). We notice that we have:

\[
l(u, U) = 2u \cdot \Omega A(u)^T (P_X, P_Y, P_Z, 0) = 2(A(u)\Omega^T u) \cdot (P_X, P_Y, P_Z, 0) = 0,
\]

since \( A(u)\Omega^T u \) is a four dimensional vector with only the fourth component different from zero. Therefore, for any choice of \( \pi^{-1} \), the phase-space local inversion \( \chi \) is well defined in \( T^*\mathbb{R}^4_0 \).

**An atlas of local inversions.** Following [8] (where a similar result is proved between the Cartesian state-space with coordinates \( x, y, z, \dot{x}, \dot{y}, \dot{z} \) and the state-space of the KS variables \( u, u' \)) we define an atlas of two local inversions of the map \( \pi \) defined in \( \mathbb{R}^3 \setminus \{(0, 0, 0)\) :

**Lemma 2.** Consider the maps

\[
\pi^\pm_1 : D^\pm = \mathbb{R}^3 \setminus \{(X, 0, 0) : X \geq 0\} \longrightarrow \mathbb{R}^4
\]

\[
\pi^\pm_1 : D^\pm = \mathbb{R}^3 \setminus \{(X, 0, 0) : X \leq 0\} \longrightarrow \mathbb{R}^4
\]

defined by

\[
\pi^\pm_1 (X, Y, Z) = \left(\frac{Y}{\sqrt{2(r - X)}}, \frac{\sqrt{r - X}}{\sqrt{2}}, \frac{Z}{\sqrt{2(r - X)}}, 0\right),
\]

where \( r = \sqrt{X^2 + Y^2 + Z^2} \), as well as their phase-space extensions:

\[
\chi^\pm_1 : (\mathbb{R}^3 \setminus \{(0)\} \times \mathbb{R}^3 \longrightarrow T^*\mathbb{R}^4_0
\]

\[
(X, Y, Z, P_X, P_Y, P_Z) \longmapsto (u, U) = \chi^\pm_1 (X, Y, Z, P_X, P_Y, P_Z)
\]

defined by:

\[
\chi^{-1}_\pm (X, Y, Z, P_X, P_Y, P_Z) = (\pi^{-1}_\pm (X, Y, Z), 2A(\pi^{-1}_\pm (X, Y, Z))^T (P_X, P_Y, P_Z, 0))
\]

Then, for every \( (X, Y, Z, P_X, P_Y, P_Z) \) in the domain of \( \chi^{-1}_\pm \) we have:

\[
(X, Y, Z, P_X, P_Y, P_Z) = \chi \circ \chi^{-1}_\pm (X, Y, Z, P_X, P_Y, P_Z),
\]
for every \((X,Y,Z,P_X,P_Y,P_Z)\) in the domain of \(\chi^{-1}_+\) we have:

\[
(X,Y,Z,P_X,P_Y,P_Z) = \chi \circ \chi^{-1}_+(X,Y,Z,P_X,P_Y,P_Z),
\]

and, for every \((X,Y,Z,P_X,P_Y,P_Z)\) in the intersection of the domains of \(\chi^{-1}_-\) and \(\chi^{-1}_+\) exists \(\alpha \in \mathbb{R}\) (depending only on \((X,Y,Z)\)) such that, by denoting

\[
(u_\pm,U_\pm) = \chi^{-1}_-(X,Y,Z,P_X,P_Y,P_Z),
\]

we have:

\[
 u_+ = S_\alpha^0 u_-, \quad U_+ = S_\alpha^0 U_- . \tag{59}
\]

\textbf{Proof of Lemma 2.} We prove that indeed we have \(U_+ = S_\alpha^0 U_-\). Since \(u_+ = S_\alpha^0 u_-\), we have:

\[
 U_+ = 2A(u_+)^T(P_X,P_Y,P_Z) = 2A(S_\alpha^0 u_-)^T(P_X,P_Y,P_Z,0)
 = S_\alpha^0 A(u_-)^T(P_X,P_Y,P_Z,0) = S_\alpha^0 U_-. \]

\[\square\]

\textbf{Cartesian representatives of the \(\hat{\nu}_i, \hat{n}_i\).} Let us fix \(E,\mu\), and consider the set \(\mathcal{D}_E \subseteq (D\setminus\{0\}) \times \mathbb{R}^3\) where \(D\) is a suitable small neighbourhood of \((0,0,0)\) and for any \((X,Y,Z,P_X,P_Y,P_Z) \in \mathcal{D}_E\) we have \(H(X,Y,Z,P_X,P_Y,P_Z) = E\). In the sets:

\[
\mathcal{D}^\pm_E = \mathcal{D}_E \cap (D_\pm \times \mathbb{R}^3)
\]

we define:

\[
 \hat{\nu}_{i,\pm}(X,Y,Z,P_X,P_Y,P_Z) = \nu_i(\chi^{-1}_\pm(X,Y,Z,P_X,P_Y,P_Z); E,\mu)
\]

\[
 \hat{n}_{i,\pm}(X,Y,Z,P_X,P_Y,P_Z) = \tilde{n}_i(\chi^{-1}_\pm(X,Y,Z,P_X,P_Y,P_Z); E,\mu).
\]

Since these functions are constructed using the local inversions \(\chi^{-1}_\pm\), they satisfy the identity:

\[
l(\tilde{n}_{\pm}(X,Y,Z,P_X,P_Y,P_Z),\tilde{\nu}_{\pm}(X,Y,Z,P_X,P_Y,P_Z)) = 0,
\]

and since they are constructed from the solutions of the Hamilton-Jacobi equation on the zero energetic level of the KS Hamiltonian, they also satisfy:

\[
|\tilde{\nu}_{i,\pm}(X,Y,Z,P_X,P_Y,P_Z)| = 1.
\]

Let us denote by:

\[
(u_\pm,U_\pm) = \chi^{-1}_\pm(X,Y,Z,P_X,P_Y,P_Z)
\]

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the pre-images, by $\alpha$ the angle such that:

$$ u_+ = S^0_\alpha u_-, \quad U_+ = S^0_\alpha U_-, $$

and:

$$ \nu_+ = \hat{\nu}(u_+, U_+; E, \mu), \quad \nu_- = \hat{\nu}(u_-, U_-; E, \mu) $$

$$ n_+ = \hat{n}(u_+, U_+; E, \mu), \quad n_- = \hat{n}(u_-, U_-; E, \mu). $$

We prove:

$$ \nu_+ = S^0_\alpha \nu_-, \quad n_+ = S^0_\alpha n_- \quad (60) $$

Since:

$$ U_+ = \frac{\partial W}{\partial u}(u_+, \nu_+; E, \mu) $$

$$ U_- = \frac{\partial W}{\partial u}(u_-, \nu_-; E, \mu) $$

we have:

$$ \frac{\partial W}{\partial u}(u_-, \nu_-; E, \mu) = (S^0_\alpha)^T \frac{\partial W}{\partial u}(S^0_\alpha u_-, \nu_+; E, \mu). \quad (61) $$

We use the previous equation to establish the relation between $\nu_+$ and $\nu_-$. Let us consider the complete integral $W(u, \nu; E, \mu)$ of the Hamilton-Jacobi equation:

$$ \mathcal{K}_I \left( u, \frac{\partial W}{\partial u}(u, \nu; E, \mu) \right) = \mu(|\nu|^2 - 1) $$

defined in Section 5. In particular, for any $\nu$ in a suitable small neighbourhood of the sphere $|\nu| = 1$, the function $W(u, \nu; E, \mu)$ is analytic in a neighbourhood of $u = 0$ and, if also $u \cdot \nu = 0$, we have:

$$ W(u, \nu; E, \mu) = 0. $$

For any $\alpha \in \mathbb{R}$, let us define the function:

$$ \tilde{W}_\alpha(u, \nu; E, \mu) = W(S^0_\alpha u, S^0_\alpha \nu; E, \mu). $$

We prove:

$$ W(u, \nu; E, \mu) = \tilde{W}_\alpha(u, \nu; E, \mu). $$

In fact, since $S^0_\alpha$ acts as a symmetry for the Hamiltonian $\mathcal{K}_I$,

$$ \mathcal{K}_I (S^0_\alpha u, S^0_\alpha U) = \mathcal{K}_I (u, U), $$

we have:

$$ \mathcal{K}_I \left( u, \frac{\partial \tilde{W}_\alpha}{\partial u}(u, \nu; E, \mu) \right) = \mathcal{K}_I \left( u, (S^0_\alpha)^T \frac{\partial W}{\partial u}(S^0_\alpha u, S^0_\alpha \nu; E, \mu) \right). $$
\[
K_2 \left( S_0^\alpha u, \frac{\partial W}{\partial u} (S_0^\alpha u, S_0^\alpha \nu; E, \mu) \right) = \mu (|S_0^\alpha \nu|^2 - 1) = \mu (|\nu|^2 - 1)
\]
and therefore \( \hat{W}_\alpha (u, \nu; E, \mu) \) is a solution of the Hamilton-Jacobi equation. Also, \( \hat{W}_\alpha (u, \nu; E, \mu) = 0 \) on the hyperplane \( u \cdot \nu = 0 \). Therefore, \( W, \hat{W}_\alpha \) are both solutions of the same Hamilton-Jacobi equation; they are both analytic in a common neighbourhood of \( u = 0 \); they both vanish on the same hyperplane. Therefore, they concide in their common domain:
\[
W(u, \nu; E, \mu) = W(S_0^\alpha u, S_0^\alpha \nu; E, \mu),
\]
and in particular we have the identity:
\[
\frac{\partial W}{\partial u} (u, \nu; E, \mu) = (S_0^\alpha)^T \frac{\partial W}{\partial u} (S_0^\alpha u, S_0^\alpha \nu; E, \mu).
\]
Therefore, from eq. (61), we have:
\[
\frac{\partial W}{\partial u} (S_0^\alpha u_-, S_0^\alpha \nu_-; E, \mu) = \frac{\partial W}{\partial u} (S_0^\alpha u_-, \nu_+; E, \mu)
\]
and from Lemma 1: \( \nu_+ = S_0^\alpha \nu_- \). Finally, we have:
\[
n_- = \frac{\partial W}{\partial \nu} (u_-, \nu_-; E, \mu) = (S_0^\alpha)^T \frac{\partial W}{\partial \nu} (S_0^\alpha u_-, S_0^\alpha \nu_-; E, \mu)
\]
\[
= (S_0^\alpha)^T \frac{\partial W}{\partial \nu} (u_+, \nu_+; E, \mu) = (S_0^\alpha)^T n_+.
\]

**From local to global first integrals.** The functions \( \tilde{\nu}_\pm, \tilde{n}_\pm \) constructed above indeed depend on the chart \( D_\pm \), and therefore are not globally defined in \( D_E \). We here aim to construct, from the functions \( \nu(u, U, E), n(u, U, E) \), first integrals in the cartesian coordinates which are globally defined in \( D_E \). First of all, we consider the dynamics in the \( \nu, n \) variables:
\[
\nu_i(s) = \nu_i(0) \quad , \quad n_i(s) = n_i(0) + 2\mu \nu_i(0)s
\]
and we notice that the functions:
\[
N_X = \nu_1 n_4 - \nu_4 n_1
\]
\[
N_Y = \frac{1}{2}(\nu_1 n_3 - n_1 \nu_3 + n_2 \nu_4 - n_4 \nu_2)
\]
\[
N_Z = \frac{1}{2}(\nu_1 n_2 - n_1 \nu_2 + n_4 \nu_3 - n_3 \nu_4)
\]
are first integrals. Since they are all invariant by composition with the map \( (n, \nu) \mapsto (S_0^\alpha n, S_0^\alpha \nu) \) for any \( \alpha \), their local representatives:
\[
N_X^\pm(X, Y, Z, P_X, P_Y, P_Z) = (\tilde{\nu}_4^\pm \tilde{n}_1^\pm - \tilde{\nu}_1^\pm \tilde{n}_4^\pm)(X, Y, Z, P_X, P_Y, P_Z)
\]
\[ N_X^\pm (X, Y, Z, P_X, P_Y, P_Z) = \frac{1}{2} (\dot{\varphi}_1^\pm \hat{n}_3^\pm - \dot{\varphi}_2^\pm \hat{n}_1^\pm + \dot{\varphi}_4^\pm \hat{n}_2^\pm - \dot{\varphi}_3^\pm \hat{n}_4^\pm) (X, Y, Z, P_X, P_Y, P_Z) \]

\[ N_Y^\pm (X, Y, Z, P_X, P_Y, P_Z) = \frac{1}{2} (\dot{\varphi}_1^\pm \hat{n}_2^\pm - \dot{\varphi}_2^\pm \hat{n}_1^\pm + \dot{\varphi}_4^\pm \hat{n}_3^\pm - \dot{\varphi}_3^\pm \hat{n}_4^\pm) (X, Y, Z, P_X, P_Y, P_Z) \]

satisfy, for all \((X, Y, Z, P_X, P_Y, P_Z) \in D_E^+ \cap D_E^-:\)

\[ N_X^- (X, Y, Z, P_X, P_Y, P_Z) = N_X^+ (X, Y, Z, P_X, P_Y, P_Z) \]

\[ N_Y^- (X, Y, Z, P_X, P_Y, P_Z) = N_Y^+ (X, Y, Z, P_X, P_Y, P_Z) \]

\[ N_Z^- (X, Y, Z, P_X, P_Y, P_Z) = N_Z^+ (X, Y, Z, P_X, P_Y, P_Z) \]

and therefore are the local representatives of a functions \(N_X, N_Y, N_Z\) globally defined in \(D_E\). Now we allow \(E\) change in a small neighbourhood \(E\) of a given \(E_*\), and we consider the set of three first integrals:

\[ \left( H , \; N^2 := N_X^2 + N_Y^2 + N_Z^2 , \; N_Z \right) \]

defined in \(\cup_{E \in E} D_E\). We have the following:

**Theorem.** The set of first integrals \((H, N^2, N_Z)\) is complete.

**Proof.** Let us prove that \((H, N^2, N_Z)\) are independent in a set \(\cup_{E \in E} D_E\). We first prove that \(N^2, N_Z\) are independent on \(E\), by showing that they are not constant on the energy levels \(H(X, Y, Z, P_X, P_Y, P_Z) = E\).

For any arbitrary small \(\varepsilon\), in the set

\[ \{(X, Y, Z, P_X, P_Y, P_Z) \in \cup_{E \in E} D_E, \quad 0 < \| (X, Y, Z) \| < \varepsilon \} \tag{62} \]

we have:

\[ N_X = P_Y Z - P_Z Y + D_X \]

\[ N_Y = P_X Z - P_Z X + D_Y \]

\[ N_Z = P_Y Y - P_Y X + D_Z \]

\[ N^2 = (P_Y Z - P_Z Y)^2 + (P_X Z - P_Z X)^2 + (P_X Y - P_Y X)^2 + D^2 \tag{63} \]

where the functions \(D_X, D_Y, D_Z\) have sup-norm boundend by order \(\varepsilon^3\) and \(D^2\) bounded by order \(\varepsilon^4\) in the set \(\text{[62]}\). Therefore, if we fix the value of \(E\) and one between \(N_Z, N^2\), the third integral is not constant in the level set of the first two.

Let us now compute the Poisson brackets. Since \(N_Z, N^2\) are first integrals for the Hamilton equations of \(H\), we have:

\[ \{ H, N_Z \} = 0 , \quad \{ H, N^2 \} = 0 \quad . \]

It remains to compute the Poisson bracket \(\{ N^2, N_Z \}\). Let us denote by \(\dot{q}(u), \dot{p}(u, U)\) the functions defined by:

\[ \dot{q}(u) = \pi(u) , \quad \dot{p}(u, U) = \frac{1}{2 |u|^2} A(u) U . \]

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We notice the remarkable property of the Poisson brackets:

\[ \{ \hat{q}_i, \hat{p}_j \} = \delta_{ij}, \quad \{ \hat{q}_i, \hat{q}_j \} = 0, \quad \{ \hat{p}_i, \hat{p}_j \} = l(u, U)\phi_{ij}(u, U) \quad i, j = 1, 2, 3. \]  

(64)

and from:

\[ \{ N^2, N_Z \} = l(n, \nu) a(n, \nu), \]

(65)

we prove \{N^2, N_Z\} = 0. In fact, since \{N^2, N_Z\} they are invariant by composition with the map \((n, \nu) \mapsto (S_0^n n, S_0^n \nu)\) for any \(\alpha\), we have:

\[ \hat{N}_Z(u, U; E) = N_z(\chi_4(u, U; E)) = N_Z(\hat{q}(u, U), \hat{p}(u, U)) \]

\[ \hat{N}^2(u, U; E) = N^2(\chi_4(u, U; E)) = N^2(\hat{q}(u, U), \hat{p}(u, U)). \]

By denoting with \(E_k\) the standard symplectic matrix of \(\mathbb{R}^{2k}\) and \(q = (X, Y, Z), p = (P_X, P_Y, P_Z)\), we have:

\[ \{ \hat{N}^2(u, U; E), \hat{N}_Z(u, U; E) \} = \left( \frac{\partial N^2}{\partial u}, \frac{\partial N^2}{\partial U} \right) \cdot \left( E_4 \left( \frac{\partial N_Z}{\partial u}, \frac{\partial N_Z}{\partial U} \right) \right) \]

\[ = \left( \frac{\partial N^2}{\partial q}, \frac{\partial N^2}{\partial p} \right) \cdot \left( J(u, U)^T E_4 J(u, U) \left( \frac{\partial N_Z}{\partial q}, \frac{\partial N_Z}{\partial p} \right) \right) \]

where \(J\) is the \(8 \times 6\) Jacobian matrix of \((\hat{q}(u), \hat{p}(u, U)).\) From the Poisson brackets (64) we notice that the \(6 \times 6\) matrix \(J(u, U)^T E_4 J(u, U)\) is not identically equal to \(E_3,\) but when it is computed on \((u, U)\) satisfying \(l(u, U) = 0\) we have: \(J(u, U)^T E_4 J(u, U) = E_3.\) But, from (65), for \(l(u, U) = 0\) we also have \{\hat{N}^2(u, U; E), \hat{N}_Z(u, U; E)\} = 0. Finally, since have identified the preimages of \(\hat{q}, \hat{p}\) satisfying \(l(u, U) = 0,\) we have: \{\hat{N}^2, \hat{N}_Z\} = 0.

\[ \square \]

7 Appendix 1: a revisitation of the integrability of the 
LC Hamiltonian in a neighbourhood of the collision 
singularities

Let us consider the Hamiltonian of the planar circular restricted three-body problem in the planetocentric reference frame (see [S] for comparison):

\[ H_2(X, Y, P_x, P_y) = \frac{P_x^2 + P_y^2}{2} + P_x Y - P_y X - \frac{\mu}{\sqrt{X^2 + Y^2}} \]

\[ - (1 - \mu) \left( \frac{1}{\sqrt{(X + 1)^2 + Y^2}} - 1 + X \right) - (1 - \mu) - \frac{(1 - \mu)^2}{2}. \]  

(66)

Following Levi-Civita we first define the canonical transformation:

\[ (X, Y, P_x, P_y) = \mathcal{Y}(u_1, u_2, U_1, U_2) \]
where:

\[ X = u_1^2 - u_2^2, \quad Y = 2u_1u_2 \]

represents the equations (2), (3) in the planetocentric reference frame, and:

\[ P_x = \frac{U_1u_1 - U_2u_2}{2|u|^2}, \quad P_y = \frac{U_1u_2 + U_2u_1}{2|u|^2} \]

the canonical extension to the momenta \( U_1, U_2 \). The transformation \( Y \) conjugates \( H_2 \) to the Hamiltonian:

\[
K_2(u_1, u_2, U_1, U_2) = \frac{1}{8|u|^2} \left( U_1 + 2|u|^2 u_2 \right)^2 + \frac{1}{8|u|^2} \left( U_2 - 2|u|^2 u_1 \right)^2 - \frac{1}{2} |u|^4 - \frac{\mu}{|u|^2} - (1 - \mu) \left( \frac{1}{\sqrt{1 + 2(u_1^2 - u_2^2) + |u|^4}} - 1 + u_1^2 - u_2^2 \right) - (1 - \mu) \left( \frac{1 - \mu)^2}{2} \right). \tag{67}
\]

To remove the singularity at \( u = 0 \) we perform the iso-energetic reduction: for any value \( E \) of the Hamiltonian, we introduce the LC Hamiltonian:

\[
K_2(u_1, u_2, U_1, U_2; E) = |u|^2 \left( K_2(u, U; E) - E \right) = \frac{1}{8} \left( U_1 + 2|u|^2 u_2 \right)^2 + \frac{1}{8} \left( U_2 - 2|u|^2 u_1 \right)^2
- \frac{1}{2} |u|^4 - \mu - |u|^2 \left( E + (1 - \mu) + \frac{(1 - \mu)^2}{2} \right)
- (1 - \mu) \left( \frac{1}{\sqrt{1 + 2(u_1^2 - u_2^2) + |u|^4}} - 1 + u_1^2 - u_2^2 \right). \tag{68}
\]

The LC Hamiltonian is regular at \( u = (0, 0) \), and the solutions \((u(s), U(s))\) of the Hamilton equations of \( K_2(u, U) \):

\[
\frac{du_i}{ds} = \frac{\partial K_2}{\partial U_i}, \quad \frac{dU_i}{ds} = -\frac{\partial K_2}{\partial u_i},
\]

with initial conditions satisfying \( u(0) \neq 0 \) and \( K_2(u, U) = 0 \), are conjugate, in a neighbourhood of \( s = 0 \), to solutions \((X(t), Y(t), P_x(t), P_y(t))\) of the Hamilton equations of (66) after the replacement of the proper time \( s \) with the time \( t \) through the formula:

\[
t(s) = \int_0^s |u(\tau)|^2 \, d\tau. \tag{69}
\]

A complete integral of the Hamilton-Jacobi equation:

\[
K_2 \left( u, \frac{\partial W}{\partial u}; E \right) = \kappa, \tag{70}
\]

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is a family of solutions of (70) depending on two parameters, satisfying the usual non-transversality property. We identify the two parameters in $\kappa$ and, following T. Levi-Civita, an angle $\alpha$ related to the rotations of the plane $u_1,u_2$. We have the following proposition: there exists a family of solutions $W(u_1,u_2,\alpha,\kappa)$ of the Hamilton-Jacobi equation (70), defined for $\alpha \in \mathbb{S}^1$, $\kappa$ in a neighbourhood of $\kappa = 0$, and analytic for $u_1,u_2$ in a neighbourhood of $(0,0)$ of radius $\sigma > 0$, with $\sigma$ depending from $E$ and $\mu$. Moreover:

(i) the Taylor series of $W$:

$$W = \sum_{n_1,n_2} W_{n_1,n_2}(\alpha,\mu,\kappa,E)u_1^{n_1}u_2^{n_2}$$

has coefficients periodic in $\alpha$, which can be computed iteratively to any order $n_1 + n_2$. In particular, we have:

$$W = \sqrt{8(\mu + \kappa)}(u_1 \cos \alpha + u_2 \sin \alpha) + O_2(u_1,u_2). \quad (71)$$

(ii) By denoting:

$$j_2(u_1,u_2,\alpha,\kappa) = \det \left( \begin{array}{cc} \frac{\partial W}{\partial u_1} & \frac{\partial W}{\partial \kappa} \\ \frac{\partial W}{\partial u_2} & \frac{\partial W}{\partial \alpha} \end{array} \right),$$

from (71), we obtain $j_2(0,0,\alpha,\kappa) = 4$. Therefore we have $j_2(0,0,\alpha,\kappa) \neq 0$ in a neighbourhood of $(u_1,u_2) = (0,0)$, uniformly in $\alpha$ and for $\kappa$ in a small neighbourhood of $\kappa = 0$.

As a consequence, the system:

$$U_1 = \frac{\partial W}{\partial u_1}(u_1,u_2,\alpha,\kappa)$$
$$U_2 = \frac{\partial W}{\partial u_2}(u_1,u_2,\alpha,\kappa)$$
$$\beta = -\frac{\partial W}{\partial \alpha}(u_1,u_2,\alpha,\kappa)$$
$$K = s + \frac{\partial W}{\partial \kappa}(u_1,u_2,\alpha,\kappa) \quad (72)$$

defines by inversion a $s$-dependent canonical transformation

$$(\alpha, \kappa, \beta, K) = \chi_2(s, \alpha, \kappa, \beta, K),$$

conjugating the Hamiltonian $\mathcal{K}_2$ to the zero-value Hamiltonian $\hat{\mathcal{K}}_2(s, \alpha, \kappa, \beta, K) = 0$. In particular, by selecting the value $\kappa = 0$, equations (72) provide the solution to the problem of planar close encounters.

3In [13] only the case $\kappa = 0$ was considered, which actually is the only value which grants the conjugation between the solutions of the regularized and non-regularized equations.
The proof of the existence of the complete integral $W$ has been done in [13] as follows. Consider the canonical transformation

$$u = R_\alpha \tilde{u}, \quad U = R_\alpha \tilde{U}$$

where

$$R_\alpha = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix},$$

(73)

conjugating $K_2(u, U; E)$ to the Hamiltonian:

$$\tilde{K}_2(\tilde{u}, \tilde{U}; \alpha, E) = \frac{1}{8} \left( \tilde{U}_1 + 2 |\tilde{u}|^2 \tilde{u}_2 \right)^2 + \frac{1}{8} \left( \tilde{U}_2 - 2 |\tilde{u}|^2 \tilde{u}_1 \right)^2 - \frac{1}{2} |\tilde{u}|^2 - \mu - |u|^2 \left( E + (1 - \mu) + \frac{(1 - \mu)^2}{2} \right)$$

$$- (1 - \mu) |\tilde{u}|^2 \left( \frac{1}{\sqrt{1 + 2(\tilde{u}_1^2 - \tilde{u}_2^2) \cos 2\alpha + |\tilde{u}|^4}} - 1 + (\tilde{u}_1^2 - \tilde{u}_2^2) \cos 2\alpha \right),$$

(74)

and look for a particular solution $\tilde{W}(\tilde{u}, \alpha, \kappa)$ of the Hamilton–Jacobi equation:

$$\tilde{K}_2(\tilde{u}, \frac{\partial \tilde{W}}{\partial u}; \alpha, E) = \kappa$$

satisfying:

$$\tilde{W}(0, \tilde{u}_2, \alpha, \kappa) = 0$$

(75)

for all $u_2$ in a neighbourhood of 0. The existence of a solution to this problem which is analytic in a neighbourhood of $\tilde{u} = 0$, (with a common analyticity radius to all $\alpha$) is quoted in [13] as a consequence on a general result about the regularity of the solutions of first order PDE, which we identify in the Cauchy–Kowaleski theorem (see [6], and the Appendix). The complete integral $W(u; \alpha, \kappa)$ is then defined by:

$$W = \tilde{W}(R_\alpha^T u; \alpha, \kappa).$$

As it is usual in the Cauchy–Kowaleski theorem, the coefficients of the series expansion of $W$ in $u_1, u_2$ can be computed iteratively up to any arbitrary order.

8 Appendix 2: the Cauchy-Kowaleski theorem

We consider the first order PDE:

$$\frac{\partial W}{\partial q_1} = F \left( q_1, q_2, \ldots, q_n, \frac{\partial W}{\partial q_2}, \ldots, \frac{\partial W}{\partial q_n} \right)$$

(76)

where $F(q_1, \ldots, q_n, p_1, \ldots, p_{n-1})$ is analytic in a neighbourhood of $q = (q_1, \ldots, q_n) = 0$, $(p_1, \ldots, p_{n-1}) = 0$. We call the plane $q_1 = 0$ the initial plane in the space of
the variables $q$; then, we consider the Cauchy’s problem of finding a solution $W(q)$ of the PDE (76) satisfying the given initial condition:

$$W(0, q_2, \ldots, q_n) = \phi(q_2, \ldots, q_n)$$ (77)

in a suitable neighbourhood of $(q_2, \ldots, q_n) = (0, \ldots, 0)$, where $\phi$ is a given function analytic in a neighbourhood of $(q_2, \ldots, q_n) = (0, \ldots, 0)$. The Cauchy-Kowaleski theorem states (see for example [6]) that the Cauchy problem has a unique solution analytic in a suitable small neighbourhood of $q = 0$. We will continue our discussion in the case which is useful for our purposes, defined by the special choice of the initial condition:

$$\phi(q_2, \ldots, q_n) = 0.$$ 

The proof is obtained by constructing first a formal series expansion:

$$W = \sum_{i_1, \ldots, i_n \geq 0} c_{i_1, \ldots, i_n} u_1^{i_1} \cdots u_n^{i_n}$$ (78)

for the solution as follows. From: $W(0, q_2, \ldots, q_n) = \phi(q_2, \ldots, q_n) = 0$ we immediately obtain:

$$\frac{\partial^{i_2}}{\partial q_{j_2}^{i_2}} \cdots \frac{\partial^{i_n}}{\partial q_{j_n}^{i_n}} W(0, q_2, \ldots, q_n) = 0 \quad j_2, \ldots, j_n = 2, \ldots, n$$

for all $i_2, \ldots, i_n \geq 1$; correspondingly, we have $c_{0, i_2, \ldots, i_n} = 0$. The coefficients with $i_1 \neq 0$ are computed iteratively on the order $i_1 + \ldots + i_n$ by differentiating (76) and by computing the result at $q = 0$.

Then, the proof of the absolute convergence of the expansion (78) in a neighbourhood of $q = 0$ is obtained by using the method of majorants. To apply the method, one first observes that the terms $c_{i_1, \ldots, i_n}$ computed as indicated above can be represented as polynomials of the terms of the Taylor expansions of $F$ and $\phi$ at $q = 0$, and the coefficients of these polynomials are non negative numbers. By exploiting this property one constructs a PDE problem whose solution can be given explicitly (and so its analyticity can be directly checked) and is a majorant of $W$. For the purposes of our paper, it is crucial to remark that all the differential equations whose solution have the same majorant converge in a common domain of $q = 0$.

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