An Accelerated Convergence Algorithm for Sparse-View CT Image Reconstruction

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Abstract. In order to reduce radiation dose during CT scanning, sparse sampling is an effective way. Although the TV-based iterative reconstruction algorithm is a breakthrough to solve the problem of sparse-view CT image reconstruction, its applicability is still limited by huge computational burden. It is necessary to study the acceleration method of TV-based iterative algorithm. This paper show that the FISTA acceleration method is not suitable for POCS-TV algorithm, meanwhile, an improved acceleration method, IFISTA, is proposed to accelerate the convergence rate of POCS-TV. Numerical experiments show that the convergence rate of POCS-TV-IFISTA is about 35% faster than POCS-TV.

1. Introduction

At present, computed tomography (CT) has been widely used in clinical diagnosis, however, X-ray radiation dose is a potential risk for diseases such as cancer [1]. In order to reduce radiation dose during CT scanning, sparse sampling is an effective way, but it also destroys the completeness of projection data [2]. For sparse-view CT image reconstruction, conventional analytic algorithms such as FBP suffers serious artifacts because of the incompleteness of projections; iterative algorithms like ART and SART can obtain better reconstruction quality than FBP, but the fuzzy image still has no practical meaning [3].

In recent years, more and more studies have shown that the TV-based iterative reconstruction algorithm is a breakthrough to reconstruct high quality CT image form incomplete projection data [4, 5, and 6]. For TV-based algorithm, sparse-view CT image reconstruction is a constrained optimization problem as follows:

\[
\min _{f} \| \tilde{f} \|_{TV} \quad s. t. \quad M \tilde{f} = \tilde{g}, \quad \tilde{f} \geq 0 \quad (1)
\]

\[
\| f_{s,t} \|_{TV} = \sum _{s,t} | \nabla f_{s,t} | = \sum _{s,t} \sqrt{(f_{s,t} - f_{s-1,t})^2 + (f_{s,t} - f_{s,t-1})^2} \quad (2)
\]

In formula (1), \( \tilde{f} \) represents the discrete image vector, \( \| \tilde{f} \|_{TV} \) represents the TV norm of \( \tilde{f} \) and its definition is given by formula (2), M represents the system matrix and \( \tilde{g} \) represents the measured
projection data. A large number of iterative algorithms have been developed to solve formula (1), POCS-TV algorithm proposed by Sidky et al. [7, 8], is one of the most classical among these. POCS-TV algorithm alternately minimizes TV norm (TV-step) and imposes data consistency or positive constraints (POCS-step), iteratively, to find the minimum TV solution satisfying all constraints. However, its applicability is limited, due to it needs too many iterations to reach convergence. Therefore, it is necessary to study the method of accelerating the convergence of POCS-TV.

Some study show that the convergence can be accelerated effectively by adding appropriate prediction step called FISTA in the iterative updating process of CT image reconstruction [9]. However, in our study, we found FISTA cannot steadily accelerate POCS-TV algorithm, there will be undesirable divergence after few iterations! After studying the divergence process of POCS-TV-FISTA, we propose an improved FISTA acceleration method (IFISTA), and numerical experiments prove our IFISTA method can steadily accelerate POCS-TV algorithm.

2. Method

2.1. FISTA acceleration method and its defects

Traditional image updating method only uses the current iteration result, the key idea of FISTA is to use the linear combination of the current and last iteration results to update the image. When FISTA is used to accelerate POCS-TV algorithm, after TV-step, formula (3a-3c) is simply added. The flow chart of POCS-TV algorithm can refer to appendix A.

\[
t = \frac{1 + \sqrt{1 + 4t_0^2}}{2}, \quad \text{(3a)}
\]

\[
\tilde{f}^{(k+1)} = \tilde{f}^{(k+1)} + \frac{t_0}{t} (\tilde{f}^{(k+1)} - \tilde{f}^{(k)}), \quad \text{(3b)}
\]

\[
t_0 = t \quad \text{(3c)}
\]

In the appendix B, we give a schematic diagram of FISTA acceleration method. Theoretically, the appropriate linear combination of \( \tilde{f}^{(k+1)} \) and \( \tilde{f}^{(k)} \), expressed in \( \tilde{f}^{(k+1)}_{\text{FISTA}} \), is closer to the feasible region, so the iterative process may be accelerated. Unfortunately, as shown in Figure 1, when POCS-TV-FISTA is used for reconstruction, error starts to accumulate slowly after few iterations, eventually leading to unexpected divergence. This phenomenon illustrates that FISTA method cannot steadily accelerate POCS-TV algorithm! For the two algorithms, the parameters we used is shown in table1. And we use relative error (RE) defined in formula (4) to quantitatively evaluate quality and convergence rate of reconstruction image. Obviously, the smaller the RE is, the faster the algorithm converges.

![Figure 1](image1.png)

**Figure 1.** Comparison of reconstruction images. (a) reconstructed by POCS-TV-FISTA, (b) reconstructed by POCS-TV, (c) comparison of relative error, (d) contribution in POCS-TV-FISTA.
Table 1. The parameters used in POCS-TV and POCS-TV-FISTA algorithms.

| Algorithm      | λ   | α | numTV | t₀ |
|----------------|-----|---|-------|----|
| POCS-TV-FISTA  | 1.0 | 0.15 | 20    | 1.0 |
| POCS-TV        | 1.0 | 0.15 | 20    |     |

\[
RE = \frac{\|\hat{f} - f^\star\|_2}{\|f^\star\|_2} \times 100\% \quad (4)
\]

2.2. IFISTA acceleration method

We speculate that the reason why POCS-TV-FISTA tends to diverge is that the contribution of FISTA-step is too large, which destroys the harmony of “inward projection” and “outward motion”. According to Figure 4(c) and 4(d), it can be seen that if the contribution of FISTA-step is greater than TV-step, corresponding iteration will be the turning point of divergence. Therefore, A good idea to improve FISTA acceleration method is: 1) In each iteration, the contribution of FISTA-step is constrained to be less than TV-step; 2) based on 1), increase the contribution of FISTA-step appropriately to accelerate iterative convergence. The improved FISTA method is called IFISTA. We give the pseudo-code of POCS-TV-IFISTA algorithm in appendix A, now, we explain the details.

We introduce a new parameter \( \eta \) to restrict the multiplier factor \( (t_0 - 1)/t \) in formula (3b). In general, the new multiplier factor \( \eta (t_0 - 1)/t \) not only keeps the feature that IFISTA-step’s contribution increases with the number of iterations, but also ensures that it is less than TV-step. Specifically, to implement the above idea 1), at the divergence turning point of \( ifista(k) \geq tv(k) \), we reduce \( \eta \) in time through the new parameter \( \eta_{red} \) to avoid iterative divergence (pseudo code line 15). To implement the above idea 2), it’s important to note that because POCS-TV algorithm guarantees convergence, as long as the gain of IFISTA-step’s contribution (represented by \( \nu_{ifista} \)) is less than POCS-TV process (represented by \( \nu_{pocs, tv} \)) between two adjacent iterations, POCS-TV-IFISTA algorithm as a whole still tends to converge. So, \( \eta \) can be appropriately increased to accelerate convergence rate (pseudo code line 19-20), and vice versa (pseudo code line 21-22). After a series of experiments, we find that when the multiplier used to change \( \eta \) is related to the relative variation of POCS-TV process between two adjacent iterations (represented by \( r_{pocs, tv} \)), the algorithm has better robustness.

As for the parameter selection of IFISTA acceleration method: we set the initial value of \( \eta \) as 1.0, and we also suggest that \( \eta_{red} \) and \( \beta \) be taken as smaller positive integers. Because only a small \( \eta_{red} \) can reduce IFISTA-step’s contribution to an acceptable level in time at the divergence turning point, and a small \( \beta \) can ensure that the change of \( \eta \) is more stable. In this paper, after a lot of experiments, we set \( \eta_{red} = 0.1, \beta = 0.09 \).

3. Numerical experiment

In this section, we study the performance of ART, POCS-TV, and POCS-TV-IFISTA by numerical experiments. Without losing generality, we choose a fan beam configuration to obtain the projection data. In order to simulate the sparse sampling situation, the Shepp-Logan head model is used to collect 20 projections uniformly from 0 to 180 degree. Both the noise-free projection and the projection with 0.2% Gaussian noise are reconstructed respectively, and the results are shown in Figure 2. The iteration number for all algorithms in this experiment is 200, which makes sure each algorithm reaches convergence. For POCS-TV-IFISTA, except for the newly introduced parameters that we have explained clearly, the settings of other parameters are the same as POCS-TV-FISTA displayed in Table 1. For POCS-TV, its parameters setting also has been displayed in Table 1. For ART, it only has one parameter \( \lambda = 1.0 \).

In Figure 2(a) and 2(d), due to insufficiency of projection data, the quality of image reconstructed by ART is very low, including very serious noise and artifacts. In Figure 2(b) and 2(e), because of the use
of sparse prior information, the image reconstructed by POCS-TV is of good quality, and even the three tissues at the bottom of the image are clearly visible. In Figure 3(c) and 3(f), superficially, it can be determined that the reconstruction quality of POCS-TV-IFISTA is not inferior to POCS-TV. In order to further compare the reconstruction quality, we show the relative error curves of reconstructed images with different reconstruction algorithm at different iteration numbers in Figure 3. As shown in Figure 3, it is clear that the relative error of POCS-TV-IFISTA is less than POCS-TV at the same iteration numbers, which illustrates that IFISTA method can indeed accelerate the convergence of POCS-TV in the case of sparse sampling.

![Figure 2. Sparse-view CT images reconstructed by different algorithms. ((a)-(c)) Reconstructed from noise-free data: (a) reconstructed by ART, (b) reconstructed by POCS-TV, (c) reconstructed by POCS-TV-IFISTA; ((d)-(f)) Reconstructed from noisy data: (a) reconstructed by ART, (b) reconstructed by POCS-TV, (c) reconstructed by POCS-TV-IFISTA.](image)

![Figure 3. Relative Error curves of different algorithms. (a) reconstructed from noise-free projection data, (b) reconstructed from noisy projection data.](image)
4. Conclusion
As shown in Figure 3, for sparse-view CT image reconstruction, POCS-TV-IFISTA only needs about 130 iterations to achieve the same reconstruction quality as 200 iterations of POCS-TV, and its convergence rate is about 35% faster than POCS-TV. There is no doubt that IFISTA acceleration method can be used for reference in reducing the computational burden and promoting the popularization of iterative reconstruction algorithm.

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Appendix A

The pseudo-code of POCS-TV-IFISTA algorithm:

1: \( \lambda := 1.0; \alpha := 0.15; num_{TV} := 20; t_0 := 1.0; \eta := 1.0; \eta_{\text{red}} := 0.1; \beta := 0.09; \tilde{f} := 0 \)

4: Repeat main loop; iteration index \( k := 1, 2, ..., K \)

5: \( \tilde{f}_{\text{pre}} := \tilde{f} \)

6: for \( i = 1, m \) do: \( \tilde{f} = \tilde{f} + \lambda \cdot M_i \cdot \frac{g_i - M_i \tilde{f}}{\|M_i\|^2} \) \( \% \) ART algorithm

7: for \( i = 1, N \) do: if \( f_i < 0 \) then \( f_i = 0 \) \( \% \) positivity constraint

8: \( \tilde{f}_{\text{pocs}} := \tilde{f}; \text{pocs}(k) := \|\tilde{f} - \tilde{f}_{\text{pre}}\|_2 \)

9: for \( i = 1, \text{num}_{TV} \) do: \( \% \) TV-steepest descent

10: \( \tilde{d} := \nabla_{TV} \|\tilde{f}\|_TV; \tilde{d} := \tilde{d} / \|\tilde{d}\|_2; \tilde{f} := \tilde{f} - \alpha \cdot \text{pocs}(k) \cdot \tilde{d} \)

12: \( \tilde{f}_{TV} := \tilde{f}; TV(k) := \|\tilde{f} - \tilde{f}_{\text{pre}}\|_2; \text{pocs}_{TV}(k) := \|\tilde{f} - \tilde{f}_{\text{pre}}\|_2 \)

13: \( t := \left( 1 + \sqrt{1 + 4 t_0^2} \right) / 2; c := (t_0 - 1) / t \) \( \% \) IFISTA step

14: \( \tilde{f} := \tilde{f} + \eta \cdot c \cdot (\tilde{f} - \tilde{f}_{\text{pre}}); \text{ifista}(k) := \|\tilde{f} - \tilde{f}_{TV}\|_2 \)

15: if \( \text{ifista}(k) \geq TV(k) \) then \( \eta = \eta_{\text{red}} \cdot \eta \)

16: if \( \text{ifista}(k) < TV(k) \) \&\& \( k > 1 \) then:

17: \( v_{\text{pocs}_{TV}} = \frac{\text{pocs}_{TV}(k)}{\text{pocs}_{TV}(k-1)}; v_{\text{ifista}} = \frac{\text{ifista}(k)}{\text{ifista}(k-1)} \)

18: \( r_{\text{pocs}_{TV}} = \left| \frac{\text{pocs}_{TV}(k-1) - \text{pocs}_{TV}(k)}{\text{pocs}_{TV}(k-1)} \right| \)

19: if \( v_{\text{pocs}_{TV}} > v_{\text{ifista}} \):

20: \( \eta := \eta \cdot (1 + \beta \cdot r_{\text{pocs}_{TV}}) \)

21: else:

22: \( \eta := \eta / (1 + \beta \cdot r_{\text{pocs}_{TV}}) \)

23: \( t_0 := t; k := k + 1 \)

24: Until \{ stopping criteria \}; Return \( \tilde{f} \)

Appendix B

Figure 4. Schematic diagram of FISTA acceleration method