On Malcev algebras nilpotent by Lie center and corresponding analytic Moufang loops.

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Abstract

In this note we describe the structure of finite dimensional Malcev algebras over a field of real numbers \( \mathbb{R} \), which are nilpotent modulo its Lie center. It is proved that the corresponding analytic global Moufang loops are nilpotent modulo their nucleus.

Key words: Malcev algebras, Moufang loops, Global Moufang loops.

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1 Introduction

The theory of analytic loops started with the work of A.I. Malcev in [Ma1]. In this article, the correspondence between the analytic local diassociative loops and the binary Lie algebras was established. A loop is a set \( M \), endowed with a binary operation \( M \times M \to M \), with the neutral element \( e \in M \) and the condition that the equations \( ax = b, ya = b \) for all \( a, b \in M \) have a unique solution. A loop is called diassociative, if every two elements of this loop generate a subgroup. Malcev showed that for an analytic loop, with the Moufang identity: \( (xy)(zx) = x(yz)x \) the corresponding tangent algebra satisfies the following identities:

\[
x^2 = [J(x, y, z), x] - J(x, y, [x, z]) = 0,
\]

where \( J(x, y, z) = [[x, y], z] + [[x, z], x] + [[z, x], y] \) (See [1]). Algebras with these defining identities are currently called Malcev algebras. The more difficult question, if every finite dimensional Malcev \( \mathbb{R} \)-algebra is the tangent algebra of some local analytic Moufang loop was solved positively by E.Kuzmin in 1969 in [Kuzm2].

Let us consider a pair \((a, \mathcal{L})\), where \( a \) is some subvariety of binary Lie algebras (in particular Malcev algebras) and \( \mathcal{L} \) is a subvariety of diassociative loops (in particular, Moufang loops).

A pair \((a, \mathcal{L})\), will be called locally dual, if it satisfies the following conditions:
• every tangent algebra of a local analytic loop from the variety $\mathcal{L}$ belongs to the variety $\mathfrak{a}$,

• every finite-dimensional $\mathbb{R}$-algebra from the variety $\mathfrak{a}$ is a tangent algebra of some local analytic loop from the variety $\mathcal{L}$

The description of all locally dual pairs is a meaningful and difficult problem.

If a locally dual pair $(\mathfrak{a}, \mathcal{L})$ satisfies a stronger condition, namely, if for every global analytic loop $S$ such that its local loop belongs to $\mathcal{L}$, is in $\mathcal{L}$ too, we will call such a pair $(\mathfrak{a}, \mathcal{L})$ globally dual.

Kerdman [Ker] showed that a pair $(\mathfrak{m}, \mathfrak{M})$ is globally dual if $\mathfrak{m}$ is the variety of all Malcev algebras, and $\mathfrak{M}$ is a variety of all Moufang loops. In this note we study the duality of a pair formed by two varieties: the first one, $NL_k$, is the subvariety of the variety $\mathfrak{m}$, which is defined by the identity $J([x_1, x_2, ..., x_k], y, z) = 0$, where in the product $[x_1, x_2, ..., x_k]$ a distribution of parentheses is arbitrary and the second one, $NG_k$, is the subvariety of the variety $\mathfrak{M}$, defined by the identity $([x_1, x_2, ..., x_k], y, z) = 1$. Here $[x_1, x_2, ..., x_k]$ is a commutator of length $k$ with an arbitrary distribution of parentheses and $(a, b, c) = ((ab)c)(a(bc))^{-1}$ is an associator.

For a Malcev algebra $M$ we define $M^1 = M$, $M^n = \sum_{i,j>0} [M^i, M^j]$. Let $F$ be a free Malcev algebra, then $NL_k$ is a variety of Malcev algebras defined by all identities of the type $J(w, x, y) = 0$, where $w \in F^k$, $k \geq 2$.

2 Structure of finite dimensional Malcev algebras nilpotent by Lie center.

Let $M$ be a finite dimensional Malcev algebra over a field $\mathbb{C}$, and let $G$ be a solvable radical of $M$. Then there exists a semisimple subalgebra (Levi factor) $S$, such that $M = S \oplus G$ ([Gri1],[Kuzm1],[Car]).

We will use the results and the terminology from [Gri2]. Let $g \in M$, the element $g$ is said to be splitting element if $g = t + n$, where $n$ is a nilpotent element and $t$ is a semisimple element, i.e., the right multiplication operator $R_t$ is diagonalizable and the operators $R_t$ and $R_n$ commute. A Malcev algebra $M$ is said to be splitting if all elements of $M$ are splitting. If $M$ is a finite-dimensional splitting Malcev algebra over a field of characteristic 0, then $M = S \oplus T \oplus N$, where $S$ is a semisimple Levi factor, $T$ is an abelian subalgebra of $M$ such that each element of $T$ is semisimple (toroidal subalgebra), and $N$ is the nilpotent radical of $M$. Additionally $[S, T] = 0$,

$$N = \sum_{\alpha \in \Delta} \bigoplus N_\alpha,$$

where $\Delta \subset T^* = \text{Hom}_k(T, k)$ and

$$N_\alpha = \{ x \in N \mid [x, t] = \alpha(t)x, \quad \forall t \in T \}.$$  \hspace{1cm} (1)
Moreover, \([N_\alpha, N_\beta] \subseteq N_{\alpha+\beta}\), if \(\alpha \neq \beta\), and \([N_\alpha, N_\alpha] \subseteq N_{2\alpha} + N_{-\alpha}\).

Since \([T, S] = 0\), one has that \(N_\alpha\) is an \(S\)-module and hence \(N_0 = N_{01} \oplus N_{00}\), \([S, N_0] = [S, N_{01}] = N_0\) and \([S, N_{00}] = 0\). Set

\[
M_{11} = \left( S \oplus \sum_{\alpha \in \Delta \setminus 0} \bigoplus_{\alpha} N_\alpha \right) \oplus N_{01}
\]

and let us denote by \(M_1\) the subalgebra generated by \(T \oplus M_{11}\). Notice that in general, \(M_1 \neq T \oplus M_{11}\), and \([N_{01}, N_{00}] \subseteq N_{01}\), \([M_{11}, N_0] \subseteq M_{11}\). Hence \(M_1\) is an ideal. Every finite-dimensional Malcev algebra \(M\) over a field of characteristic 0 is contained in some splitting Malcev algebra \(\hat{M}\). If such \(\hat{M}\) does not contain intermediate splitting subalgebra, which contains \(M\), then \(\hat{M}\) is called a splitting of \(M\). Each automorphism of the algebra \(M\) extends uniquely to an automorphism of a splitting of \(M\). If \(M\) is a splitting of \(M\), then \(\hat{M}^2 = \hat{M}^2\), and any ideal of the algebra \(\hat{M}\) is an ideal of the algebra \(M\) and vice versa any ideal of \(M\), which is in \(\hat{M}\) is also the ideal of \(M\). This result is analogous to one for Lie algebras due to A. I. Malcev [Mn2].

In what follows in this article the splitting algebra of an algebra \(M\) we will denote by \(\hat{M}\).

Recall that

\[
\text{Lie}(M) = \{x \in M | J(x, M, M) = 0\}
\]

is the Lie center of \(M\).

**Lemma 1.** In this notation \(\text{Lie}(M) \subseteq \text{Lie}(\hat{M})\).

**Proof.** By the construction \(\hat{M} = \bigcup_{i=1}^n M(i)\), where \(M(1) = M, M(n) = \hat{M}\), \(\dim M(i) = \dim M(i-1) + 1\), \(M(i) = M(i-1) + R_{t_i}, i \geq 2\), where \(t_i\) is a semisimple element, i.e. \(R_{t_i} : M(i-1) \to M(i-1)\) is a diagonalizable operator.

Moreover there exists \(x_i \in M(i-1)\), such that \(n_i = x_i - t_i\) is a nil element, i.e \(R_{n_i}\) is a nilpotent operator and \(R_{t_i} \circ R_{n_i} = R_{n_i} \circ R_{t_i}, [n_i, t_i] = 0\).

Now consider \(l \in \text{Lie}(M(i-1)), x_i \in M(i-1)\). It is sufficient to show that \(J(l, x_i, t_i) = 0\). Suppose that

\[
M(i-1) = \sum_{\alpha \in \Delta} \oplus M(i-1)_{\alpha}
\]

is a Cartan decomposition with respect to the operator \(R_{t_i}\) or \(R_{x_i}\).

Without loss of generality one can assert that

\[
l \in \text{Lie}(M(i-1) \cap M(i-1)_{\alpha}, x_i \in M(i-1)_{\beta}.
\]

If \(\alpha \neq \beta\), then \(J(l, x_i, t_i) = 0\) due to the fact that any Malcev algebra is a binary-Lie algebra.
Consider the case $\alpha = \beta \neq 0$.

The equality $J(l, x, t_i) = 0$ is equivalent to $[x, l] \in M(i - 1)_{2\alpha}$.

Indeed, suppose that $[x, l] \notin M(i - 1)_{2\alpha}$. Then $[x, l] = [x, l]_{2\alpha} + [x, l]_{-\alpha}$, where $[x, l]_{2\alpha} \in M(i - 1)_{2\alpha}$ and $0 \neq [x, l]_{-\alpha} \in M(i - 1)_{-\alpha}$. Recall that $x_i - n_i = t_i$.

Now if $\alpha = \beta = 0$ we have $J(l, x, t_i) = 0$, since $[N_0, N_0] \subseteq N_0$ and $[N_0, t_i] = 0$.

Thus one has $\text{Lie}(M) \subseteq ... \subseteq \text{Lie}(M(i)) \subseteq ... \subseteq \text{Lie}(\hat{M})$.

In the theory of Lie algebras there exists the following construction of decomposable extension. Let $L$ be a Lie algebra and let $N$ be a subalgebra of the Lie algebra $\text{Der}L$ (the algebra of all derivations of $L$). Then the direct sum $N \oplus L$ has a structure of Lie algebra with the multiplication:

$$(a, l) \cdot (b, r) = ([a, b], l^a - r^a + [l, r]).$$

Notice that we are not assuming that $N$ or $L$ is abelian.
This construction has a generalization for Malcev algebras.

Suppose that $M$ is a Malcev algebra such that $M = \tilde{N} + L$, where $L \subseteq \text{Lie}(M)$, and $M$ has an ideal $I \subseteq \tilde{N}$ such that $J(\tilde{N}) \subseteq I$, $[I, L] = 0$, then $\tilde{N}/I \cong N$ is a Lie algebra. It means that $N$ acts on $L$ by derivations. In this case the formula \[\text{(5)}\] defines a Malcev algebra structure on $\tilde{M} = \tilde{N} \oplus L$, where $I$ acts trivially on the Lie algebra $L$ by definition. This construction is called the \textit{decomposable extension of Malcev algebras}. Notice that in this construction $L$ is an ideal contained in the Lie center of $M$.

In what follows the decomposable extension of Malcev algebra will be denoted by $M$.

The aim of this section is to show the following

**Theorem 1.** Let $M$ be a finite dimensional Malcev algebra from the variety $NL_k$ over a field of complex numbers $\mathbb{C}$. Then

1. $M$ may be embedded into the splitting Malcev algebra $\tilde{M} = S \oplus T \oplus N \in NL_k$, where $S$ is a semi-simple Lie subalgebra, $T$ is a toroidal subalgebra, $N$ is a nilpotent ideal.

2. $N = N_{00} \oplus [S, N]$, where $N_{00}$ is a Malcev subalgebra of $N$ and the ideal $M_1$ generated by $S \oplus T \oplus \left( \sum_{\alpha \in \Delta \setminus 0} N_{\alpha} \right) \oplus [S, N]$ is contained in $\text{Lie}(\tilde{M})$.

3. There exists a Malcev algebra $\tilde{M} = N_{00} \oplus M_1$ of the variety $NL_k$, which is a decomposable extension of a nilpotent Malcev algebra $N_{00}$ and a Lie subalgebra $M_1$, such that there exists an epimorphism $\pi: \tilde{M} \longrightarrow M$.

In order to prove this Theorem we need to collect some intermediate results which we will present in the following lemmata.

Put $\bigoplus_{n=1}^{\infty} M^n$ by $M^\omega$. Following introduced notation one has:

**Lemma 2.** Let $M$ be a splitting Malcev algebra from the variety $NL_k$, $k \geq 2$. Then ideal $M_1$, constructed above, contains $M^\omega$ and is contained in the Lie center $\text{Lie}(M)$.

\begin{proof}
By definition of the variety $NL_k$ we get that $M^\omega \subseteq \text{Lie}(M)$ the Lie center of $M$. By construction $M_{11} \subseteq M^\omega \subseteq \text{Lie}(M)$, hence $M' \subseteq \text{Lie}(M)$ where $M'$ is a subalgebra of $M$ generated by $M_{11}$. It is clear that $M_1 = T \oplus M'$, $[T, M'] \subseteq M' = S \oplus V$ with $V \subseteq N$. Hence for proving the lemma it is enough to prove that $J(x, y, z) = 0$, where $x$, $y$ and $z$ are elements of $T \cup (\cup_{\alpha \in \Delta} N_{\alpha})$. If $x, y \in T$, then $z \in N_{\alpha}$ and $J(x, y, z) = [[x, z], y] + [[[y, z]], x] = \alpha(x)\alpha(y)z - \alpha(y)\alpha(x)z = 0$.

If $x \in T$, $y \in N_{\alpha}, z \in N_{\beta}$ and $\alpha \neq 0$ or $\beta \neq 0$, then $y \in M_{11} \subseteq \text{Lie}(M)$ or $z \in \text{Lie}(M)$, hence $J(x, y, z) = 0$. At last, in the case $\alpha = \beta = 0$ we get $J(x, y, z) = 0$ since $[N_0, N_0] \subseteq N_0$. \hfill \qed
\end{proof}
Lemma 3. Let $M$ be a Malcev algebra from the variety $NL_k$.
Then $[J(M), M^\omega] = 0$.

Proof. By the result of Filippov (see [Fi], page 236) one has $[J(M), \text{Lie}(M)] = 0$. On the other hand, $M^\omega \subseteq \text{Lie}(M)$ since $M \in NL_k$. 

Since $M_1 \subseteq M^\omega$ by Lemma 3 we have $[J(M), M_1] = 0$. Then the subalgebra $N_{00}$ acts on the ideal $M_1$ by derivations, hence it is possible to construct, as above, a Malcev algebra $\hat{M} = N_{00} \oplus M_1$ with a product given by $[\cdot, \cdot]$. It is easy to see that the morphism $\varphi : M \rightarrow M$, $\varphi(n, m) = n + m$ is an epimorphism of Malcev algebras.

Lemma 4. The Malcev algebra $\hat{M} = N_{00} \oplus M_1$ with the product given by $[\cdot, \cdot]$ is a Malcev algebra of the variety $NL_k$.

Proof. Since $M_1 \subseteq \text{Lie}(\hat{M})$, one has $J(\hat{M}) = J(N_{00})$. By construction, $[M_1, J(N_{00})] = 0$. In this case, $\hat{M}$ is a Malcev algebra of the variety $NL_k$ if and only if $N_{00}$ is a Malcev algebra of the variety $NL_k$; which is exactly our case.

Remark. In general, if we have a Malcev algebra $P = P_0 + P_1$, where $P_0$ is a nilpotent subalgebra, $P_1 \subseteq \text{Lie}(P)$ is an ideal contained in the Lie center of $P$ and $P/P_1$ is a Malcev algebra of the variety $NL_k$, then $P$ is not necessarily a Malcev algebra of the the variety $NL_k$. It is possible that $P \in NL_{k+1} \setminus NL_k$.

Example. Set $P_1 = \mathbb{R}\{t, a, b, c \mid [a, t] = a, [b, t] = -b, [a, b] = c, [c, t] = [a, c] = [b, c] = 0\}$ and let it be a splitting Lie algebra. Choose any nilpotent Malcev algebra $P_0$ which is not a Malcev algebra from the variety $NL_k$, but $P_0/I$ is. Here $Z = \mathbb{R}z$ is some central ideal of $P_0$. It is easy to construct an algebra with those properties. Let us consider $\hat{P} = P_0 \oplus P_1$ and $P = \hat{P}/I$, where $I = \mathbb{R}(c-z)$ is a central ideal. It is clear that $P$ is not a Malcev algebra from the variety $NL_k$. But $P = \pi(P_0) + \pi(P_1)$, where $\pi : \hat{P} \rightarrow P$ is a canonical homomorphism. Notice that $\pi(P_0) \cong P_0, \pi(P_1) \cong P_1 \subseteq N(P)$ and $P/P_1 = P_0/Z$ is a Malcev algebra from the variety $NL_k$.

Proof of the Theorem.
Consider $\hat{M} = S \oplus T \oplus N$ as splitting algebra of $M = S \oplus G$. We will show that $\hat{M} \in NL_k$ if $M \in NL_k$. Due to the construction of $\hat{M}$ ([Gri2]) any ideal $I \triangleleft \hat{M}$ is also the ideal of $\hat{M}$ and therefore $M^k = \hat{M}^k, k \geq 2$. Now since $\text{Lie}(M) \subseteq \text{Lie}(\hat{M})$ and $M^k \subseteq \text{Lie}(M)$ one has $M^k \subseteq \text{Lie}(M) \subseteq \text{Lie}(M^k)$. This means $M^k \in NL_k$. As it was noticed above $N$ is a semisimple $S \oplus T$-module, therefore $N_{00} = \text{Ann}_N(S \oplus T)$ is the nilpotent subalgebra of $M$. Moreover

$$M_{11} \subseteq \hat{M}^\omega \subseteq \text{Lie}(\hat{M})$$

Since $[T, N_{00}] = 0$ one has $N_{00} \subseteq N_0$. In other hand in general case $N_{00} \cap M_1 \neq 0$. Recall that $M_1$ is the ideal generated by $T \oplus M_{11}$. Finally one gets $M =
$N_{00} \oplus M_1$.  

3 Malcev algebras and global Moufang loops.

A variety $\mathbf{M}$ of Malcev algebras will be called (locally) smooth, if there exists a variety of Moufang loops $\mathbf{L}$ such that the pair $(\mathbf{M}, \mathbf{L})$ is (locally) globally dual. Analogously, a variety of Moufang loops $\mathbf{L}$ is (locally) smooth if there exists a variety of Malcev algebras $\mathbf{M}$, such that the pair $(\mathbf{M}, \mathbf{L})$ is (locally) globally dual. A dual pair $(\mathbf{M}, \mathbf{L})$ will be called global if for any local analytic loop $G$ of the variety $\mathbf{L}$, there exists a global analytic loop $\tilde{G}$ from the variety $\mathbf{L}$ which is locally isomorphic to $G$. It is clear, that not all varieties of Moufang loops are smooth. For example, the variety $\mathbf{B}_n$ of Moufang loops of exponent $n$ is not smooth, since every analytic Moufang loop of a positive dimension is not periodic. Nevertheless we have the following Conjecture:

**Conjecture 1.** Every dual pair $(\mathbf{M}, \mathbf{L})$ of Malcev algebras and their corresponding Moufang loops is global.

Notice that if the pair $(\mathbf{M}, \mathbf{L})$ is locally dual and the variety $\mathbf{M}$ contains only Lie algebras, then all finite dimensional Lie algebras from $\mathbf{M}$ are solvable. Indeed, if $M \in \mathbf{M}$ is not solvable finite dimensional then $M = S \oplus G$, where $S$ is semisimple Lie subalgebra. Hence $S$ contains some simple 3–dimensional Lie subalgebra $L$. But the corresponding Lie group $G(L)$ contains free subgroup. Hence $\mathbf{L}$ is variety of all groups and $\mathbf{M}$ is the variety of all Lie algebras.

We will prove the Conjecture 1 for the pairs $(\text{NL}_k, G_k)$, where $\text{NL}_k$ is the variety defined in the last section and $G_k$ is a variety of Moufang loops defined by all identities of the type $(w, x, y) = 1$, where $w \in F^k$, $k \geq 2$ and $F$ is an infinite free generated Moufang loop such that $F^1 = F$, and $F^k$ is the normal subloop generated by $\Pi \prod_{i=1}^{k-1} [F^i, F^{k-i}]$.

**Proposition 1.** The pair $(\text{NL}_k, G_k)$ is dual for any $k \geq 2$.

**Proof.** Let $M$ be a Malcev $\mathbb{R}$-algebra of dimension $n$ of the variety $\text{NL}_k$. Then $M \cong \mathbb{R}^n$. There exists a small ball $M_\epsilon = \{ x \in M \mid |x| \leq \epsilon \}$, which is a local Moufang loop with the product given by the Campbell-Hausdorff formula

\[ x \cdot y := \text{CH}(x, y) = x + y + \frac{1}{2}[x, y] + \cdots. \]  

(6)

Notice that the element 0 of $M$ is the unit of this local analytic loop. From [9] we have that for every subalgebra $P$ of $M$ the corresponding local subgroup is given by $P_\epsilon = P \cap M_\epsilon$. The subgroup $P_\epsilon$ is normal if and only if $P$ is an ideal of $M$.  

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From (6) we get
\[
\{x, y\} = x^{-1} \cdot y^{-1} \cdot x \cdot y = [x, y] + \sum_{s=3}^{\infty} a_s(x, y),
\] (7)
where \(a_s(x, y) \in M^s\) if \(x, y \in M\). Hence every commutator \(w\) in the local Moufang loop \((M, \cdot)\) of length \(k \geq 3\) has the form \(w = \sum_{i=k}^{\infty} w_i\), with \(w_i \in M^i\). Since \(M \in NL_k\), then \(M^s \subseteq Lie(M)\) for \(s \geq k\). Hence the corresponding commutator subloop \(M^k\) of local Moufang loop \(M_e\) is contained in \(Lie(M)\).

E. Kuzmin proved [?], that in a local Moufang loop \((M, \cdot)\) the associator can be expressed as:
\[
(x, y, z) = \frac{1}{6} J(x, y, z) + \sum_{i=7}^{\infty} a_i(x, y, z),
\] (8)
where \(a_i(x, y, z)\) is an element of degree \(i\) of the ideal \(J(M) \subset M\).

By (8) we get that \(Lie(M) \cap M_e \subseteq Nuc(M_e)\), where \(Nuc(M_e) = \{ x \in M_e | [x, a, b] = 0, \forall a, b \in M_e \}\). Hence \(M^k_e \subseteq Lie(M) \cap M_e \subseteq Nuc(M^k_e)\). It means that \((M_e, \cdot) \subseteq G_k\).

Now suppose that \((M_e, \cdot) \subseteq G_k\). Following the previous notation, we have:

**Lemma 5.** \(M^k \cap M_e = (M_e, \cdot)^k\), where \((M_e, \cdot)^k\) is a commutator subloop of the local loop \((M_e, \cdot)\) of degree \(k\).

**Proof.** From the construction of the local loop \((M_e, \cdot)\), for every ideal \(I\) of the Malcev algebra \(M\) there is a corresponding normal subloop \(I_e = I \cap M_e\) of \((M_e, \cdot)\). It is clear that for nilpotent of class \(k\) Malcev algebra \(M/M^k\) the corresponding local Moufang loop is \(((M/M^k)_e, \cdot)\), which is nilpotent of class \(k\). Hence \((M_e, \cdot)^k \subseteq M^k\).

Suppose that \((M_e, \cdot)\) is a nilpotent local loop of class \(k\). By induction we prove that the Malcev algebra \(M\) is nilpotent of the same class \(k\). It is clear for \(k = 1\). If the Malcev algebra \(M\) is not nilpotent of degree \(k\) then for some \(x_1, ..., x_k \in M\) we have \(w = [x_1, ..., x_k] \neq 0\) for some distribution of parentheses. By (7) we get in \((M_e, \cdot)\): \(u_i = \{tx_1, tx_2, ..., tx_k\} = t^i w + \sum_{s > k} t^j w_i\), where \(w_i\) is an element of \(M^i\) and \(t \in R\). Since \((M, \cdot)\) is nilpotent of degree \(k\), \(u = 0\) in \((M, \cdot)\). Then \(w = 0\) in \(M\) which yields to a contradiction. \(\square\)

With all considerations above Lemma 5 is proved.

Now we can finish the proof of Proposition 1. Let \(w = [x_1, ..., x_k] \in M^k\), we have to prove that \(w \in Lie(M)\). For some \(t \in R\) we have by Lemma 5 that \(t^k w = [tx_1, ..., tx_k] \in M^k \cap M_e \subseteq (M_e, \cdot)^k \subseteq Nuc(M_e)\). Here we used that \((M_e, \cdot) \subseteq G_k\). Hence \((t^k w, x, y) = 0\) for all \(x, y \in M_e\). By (8) we get that \(J(t^k w, x, y) = 0\). It means that \(w \in Lie(M)\). Proposition 1 is proved. \(\square\)
Now we are ready to prove the main result of this paper.

**Theorem 2.** The dual pair \((NL_k, G_k)\) is global.

**Proof.** Let \(G_0 \in G_k\) be a local analytic loop and let \(L(G_0) = M \in NL_k\) be its corresponding Malcev algebra. Let \(\varphi : M \mapsto \hat{M}\) be an embedding of \(M\) in a splitting Malcev algebra \(\hat{M} = S \oplus T \oplus N\), (see notation of Theorem 1). By definition \(\hat{M}\) is minimal with this property. Hence \([M, \hat{M}] = [M, M]\). Since \(M \in NL_k\) then \(\hat{M} \in NL_k\) by Theorem 1. By \([\text{Ker}]\) there exists the corresponding to \(M\) global analytic simply connected Moufang loop \(\hat{G}\).

By construction of \(\hat{G}\) in \([\text{Ker}]\) we get that \(\hat{G} = P \cdot Q\), where \(P\) is simply connected semisimple Lie group with corresponding Lie algebra \(S\) and \(Q = Q_0 \cdot Q_1\) is simply connected solvable Moufang loop with corresponding Malcev algebra \(T \oplus N\), and \(Q_0 \cong \mathbb{R}^1 \cong T\) is abelian vectorial group Lie, corresponding to Lie subalgebra \(T\), \(Q_1\) is simply connected nilpotent normal subloop corresponding to the nilpotent ideal \(N\). By Theorem 1 we have that \(\hat{M} = S \oplus T \oplus N = N_{00} + M_1\), where \(N_{00} \subseteq N\) is a nilpotent subalgebra and \(M_1\) is an ideal of \(M\) that is contained in \(\text{Lie}(\hat{M})\). Since exponential map from \(N\) to \(Q_1\) is a bijection then \(\exp(N_{00}) = Q_2 \subseteq Q_1\) is a nilpotent simply connected subloop of \(\hat{G}\). Since \(\hat{M} = N_{00} + M_1\) then \(\hat{G} = Q_2 \times G_1\), where \(G_1\) is simply connected normal group Lie corresponding to the ideal \(M_1 \subseteq \text{Lie}(\hat{M})\).

Since \(\hat{M}\) is a splitting by Theorem 1 there exists a Malcev algebra \(\hat{M} = N_{00} \oplus M_1 \in NL_k\) and an epimorphism \(\pi : M \mapsto \hat{M}\).

Let \(\hat{G}\) be simply connected analytic Moufang loop corresponding to Malcev algebra \(\hat{M}\). Then \(\hat{G} = Q_2 \times G_1\), where \(Q_2\) and \(G_1\) are subloops of \(\hat{G}\). Notice that multiplication in \(\hat{G}\) may be given the following analogue of (3).

\[(r_1, g_1) \cdot (r_2, g_2) = (r_1 r_2, g_1^{g_2} g_2)\]  \hspace{1cm} (9)

Where \(g_1^{g_2} = r_2^{-1} g_1 r_2\) is natural action of Moufang loop \(Q_2\) on the Lie group \(G_1\) by automorphisms, since \(A(Q_2) = J(N_{00})\) acts trivially on \(G_1\). Here \(A(Q_2)\) is an associate of \(Q_2\) and we used that \([J(\hat{M}), M_1] \subseteq [J(\hat{M}), \text{Lie}(\hat{M})] = 0\).

It is clear that \(G_1\) is contained in the nucleus of \(\hat{G}\) and \(\hat{G} \in G_k\) if and only if \(Q_2 \in G_k\).

**Lemma 6.** Let \(N\) be a nilpotent finite dimensional Malcev algebra and \(R\) be the corresponding simply connected analytic Moufang loop.

If the corresponding local analytic Moufang loop \(R_e\) satisfies some identity \(f(x_1, \ldots, x_n) = 1\), then the global analytic loop \(R\) satisfies the same identity.

In particular, \(N \in NL_k\) if and only if \(R \in G_k\).

**Proof.** It is possible to assume that \(R = N \cong \mathbb{R}^m\) with multiplication \(\cdot\). Let \(f\) be an identity of the local analytic loop \(R_e\). Then \(f(x_1, \ldots, x_n) = \sum_j f_j\), where \(f_j = f_j(x_1, \ldots, x_n)\) is a Lie word in \(x_1, \ldots, x_n\). Let \(v_1, \ldots, v_m\) be a basis of \(N\). Then \(f = \sum_{j=1}^m g_j v_j\), where \(g_j = g_j(x_1, \ldots, x_n)\) is a polynomial function in \(x_1, \ldots, x_n\). Since \(f\) is a local identity then \(g_j = 0\) if \(|x_s| < \varepsilon, s = 1, \ldots, n\) and \(\varepsilon\) is small enough. But any polynomial function, which is equal
to zero in some neighborhood of $\tilde{0}$ is equal to zero for all values of the variables. Hence $f = 1$ is an identity of the loop $R$.

Returning to the proof of the Theorem 2 we have that $\tilde{G} \in G_k$ due to Lemma 6. Hence $\hat{G} \in G_k$, since $\hat{G}$ is a homomorphic image of $\tilde{G}$. With this the proof of the Theorem 2 is done.

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