Asymptotic Behaviour of Discretised Functionals of Long-Range Dependent Functional Data

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The paper studies the asymptotic behaviour of weighted functionals of long-range dependent data over increasing observation windows. Various important statistics, including sample means, high order moments, occupation measures can be given by these functionals. It is shown that in the discrete sampling case additive functionals have the same asymptotic distribution as the corresponding integral functionals for the continuous functional data case. These results are applied to obtain non-central limit theorems for weighted additive functionals of random fields. As the majority of known results concern the discrete sampling case the developed methodology helps in translating these results to functional data without deriving them again. Numerical studies suggest that the theoretical findings are valid for wider classes of long-range dependent data.

Keywords Long-range dependence; Non-central limit theorems; Hermite distributions; Additive functional; Functional data

Mathematics Subject Classification 60G60, 62M30, 62M40, 60F99

1 Introduction

Recent advances in technology allowed collecting big data at high frequency (effectively continuous) rates that led to the ubiquity of functional data (samples of curves or surfaces) (\textsuperscript{[Ramsay and Silverman (2005); Wang et al. (2016)]}). Handling such new complex data is essential in various applications, for example, in earth, environmental, ecological sciences, cosmology and image analysis. However, most of classical statistical models and results were developed for discretely sampled data.

Discretisation and corresponding additive models are often used as powerful dimension reduction tools in the analysis of functional data which are intrinsically infinite dimensional. The discretisation is a common strategy for approximating statistics of

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such data (see, for example, § 6.4.3 in [Ramsay and Silverman (2005)]). Also, in practice, the functional curves or surfaces are often observed only at a finite number of points.

Note, that various statistics of functional data can be expressed by integral functionals of these data or their transformations. For instance, some well-known examples of such statistics include sample moments and sample sojourn measures (Minkowski functionals) (see [Leonenko and Olenko (2014)]). Another popular model in various applications (especially in engineering and signal processing) is stochastic processes that are obtained as outputs of filters, i.e. defined mathematically by a convolution integral operator. Another example is functional linear regression models defined by weighted integral functionals. These models found numerous statistical applications in medicine, linguistics, chemometrics (see [Ramsay and Silverman (2005); Crambes et al. (2009); Zhang (2014)]).

In all above applications it is usually assumed that the discretisation error is negligible with respect to the estimation error. However, there are almost no known results that rigorously prove it. This paper addresses this problem and investigates discretisation errors for weighted functionals of long-range dependent spatial processes. Their rates of decay for the case of increasing observation windows are found. It is shown that both additive and integral functionals converge to the same limit distribution. It is proved that these distributions are non-Gaussian. These results provide a constructive method for determining the number of discretisation nodes for a given accuracy.

Various results in statistical inference of random fields were first obtained by [Yadrenko (1983)]. Recently, considerable attention has been paid to asymptotic behaviour of non-linear statistics of random processes and fields (see [Ivanov and Leonenko (2008); Bai and Taqqu (2013); Leonenko and Olenko (2014); Anh et al. (2019) and the references therein]). Direct probability techniques were used to study these statistics, for example, in regression models. Asymptotic distributions of these statistics were discussed by [Ivanov and Leonenko (1989)] and it was shown that central and non-central limit theorems hold for particular models. However, no results about discretisations were given.

There are many practical situations in which non-Gaussian random processes and fields are appropriate for statistical modeling. We deal with an important class of models defined by non-linear functions of Gaussian random fields. This class is widely used in modeling non-Gaussian data. It can be analysed using Wiener chaos expansions that give good data approximations in many cases (see [De Oliveira et al. (1997); Vio et al. (2001)]).

This research deals with asymptotic behaviour of integral and additive non-linear functionals of random fields with long-range dependence. Long-range dependence is an empirical phenomenon which has been observed in different applied fields including cosmology, economics, geophysics, air pollution, image analysis, earth sciences, just to mention a few examples. For this reason, great effort has been devoted to studying models based on long-range dependent random fields (see [Ivanov and Leonenko (1989); Wackernagel (1998); Doukhan et al. (2002); Frias et al. (2008)]). Weighted functionals of long-range dependent random fields were considered in [Ivanov and Leonenko (1989); Ivanov et al. (2013); Olenko (2013)]. These functionals can have non-Gaussian limits that
are known as Hermite or Hermite-Rosenblatt distributions (Rosenblatt (1961); Taqqu (1975); Dobrushin and Major (1979); Taqqu (1979)). Their asymptotic distributions can be characterised by either multiple Wiener-Itô integrals representations or characteristic functions (see Dobrushin and Major (1979); Taqqu (1979); Leonenko and Taufer (2006)).

In various applications, it is natural to consider statistics of random fields and to study their limit behaviour over increasing spatial windows. In these cases, integrals of non-linear functionals of spatial functional data and additive non-linear functionals for discrete observations on a bounded region were studied in numerous papers (see, for example, Major (1981); Leonenko and Olenko (2014); Anh et al. (2015, 2019)). When we deal with the asymptotic behaviour of discretised functionals of functional data, it is important to know how asymptotics of these integrals are related to additive functionals. To the best of our knowledge only particular cases of this correspondence have been addressed in Leonenko and Taufer (2006) and Alodat and Olenko (2017) for rectangular observation windows. However, in many applications spatial data is not necessarily available over rectangles, but rather over irregularly-shaped regions (Cressie (1993); Lahiri et al. (1999)). Therefore, it is important to obtain theoretical results about asymptotics for general types of observation windows. In this paper we extend results of Leonenko and Taufer (2006) and Alodat and Olenko (2017) under more general conditions. More precisely, we consider weighted functionals of random fields of the form

\[ d_{r}^{-1}\int_{\Delta_n(r)} g(x)H_\kappa(\xi(x))dx, \quad r \to \infty, \]

where \( \xi(x), x \in \mathbb{R}^n \), is a long-range dependent random field, \( g(x) \) and \( H_\kappa(\cdot) \) are non-random functions, \( \Delta_n \subset \mathbb{R}^n \) is an observation window and \( d_{r}^{-1} \) is a normalising factor. We show that these integrals and the corresponding discretised versions have same non-Gaussian limit distributions.

The article is organised as follows. In Section 2 we introduce main notations and outline necessary background from the theory of random fields. In Section 3 we recall some assumptions and auxiliary results from the spectral and correlation theory of random fields. In Section 4.1 we study the case of two-dimensional functionals. Section 4.2 gives a general multidimensional version of the results. Proofs are provided in Section 5. In Section 6 some simulations studies are presented to confirm theoretical results. Conclusions and directions for future research are presented in Section 7.

2 Definitions and Notations

This section provides basic definitions and notations that are used in this article.

In what follows \(|\cdot|\), \(\|\cdot\|\), \(\lfloor\cdot\rfloor\) and \(\lceil\cdot\rceil\) are used for the Lebesgue measure, the Euclidean distance in \(\mathbb{R}^n\), the floor and ceiling functions, respectively. The symbols \(C, \varepsilon\) and \(\delta\) (with subscripts) will be used to denote constants that are not important for our discussion. Note, that the same symbol \(C\) may be used for different constants appearing in the same proof. For a set \(A \subset \mathbb{R}^n, n \geq 1\), we denote by \(A^o\) and \(A^c\) the interior and the exterior of
the set \( A \) respectively. Moreover, it is assumed that all random variables are defined on a fixed probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \).

We consider a measurable mean-square continuous zero-mean homogeneous isotropic real-valued random field \( \xi(x) \), \( x \in \mathbb{R}^n \), with the covariance function

\[
B(r) = B(\|x\|) := \mathbb{E} \left( \xi(0)\xi(x) \right), \quad x \in \mathbb{R}^n.
\]

It is well known that there exists a bounded non-decreasing function \( \Phi(u) \), \( u \geq 0 \), (see Yadrenko (1983); Ivanov and Leonenko (1989)) such that

\[
B(r) = (2/r)^{(n-2)/2} \frac{n}{\Gamma(n/2)} \int_0^\infty J_{(n-2)/2}(ru) u^{(2-n)/2} d\Phi(u),
\]

where \( J_v(\cdot) \) is the Bessel function of the first kind of order \( v > -1/2 \).

The function \( \Phi(\cdot) \) is called the isotropic spectral measure of the random field \( \xi(x) \), \( x \in \mathbb{R}^n \). If there exists a function \( \varphi(u) \), \( u \in [0, \infty) \), such that

\[
u^{n-1} \varphi(u) \in L_1([0, \infty)), \quad \Phi(u) = \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^u z^{n-1} \varphi(z) dz,
\]

then the function \( \varphi(\cdot) \) is called the isotropic spectral density of the field \( \xi(x) \).

The field \( \xi(x) \) with an absolutely continuous spectrum has the following isonormal spectral representation

\[
\xi(x) = \int_{\mathbb{R}^n} e^{i(\lambda \cdot x)} \sqrt{\varphi(\|\lambda\|)} W(d\lambda),
\]

where \( W(\cdot) \) is the complex Gaussian white noise random measure on \( \mathbb{R}^n \) (see Yadrenko (1983); Ivanov and Leonenko (1989)).

The Hermite polynomials \( H_m(x) \), \( m \geq 0 \), are defined by

\[
H_m(x) := (-1)^m \exp \left( \frac{x^2}{2} \right) \frac{d^m}{dx^m} \exp \left( -\frac{x^2}{2} \right).
\]

The first few Hermite polynomials are \( H_0(x) = 1, H_1(x) = x, H_2(x) = x^2 - 1, H_3(x) = x^3 - 3x \).

The Hermite polynomials \( H_m(x) \), \( m \geq 0 \), form a complete orthogonal system in the Hilbert space \( L_2(\mathbb{R}, \phi(\omega)d\omega) := \{ G : \int_{\mathbb{R}} G^2(\omega)\phi(\omega)d\omega < \infty \} \), where \( \phi(\omega) \) is the probability density function of the standard normal distribution. An arbitrary function \( G(\omega) \in L_2(\mathbb{R}, \phi(\omega)d\omega) \) admits the mean-square convergent expansion

\[
G(\omega) = \sum_{j=0}^{\infty} C_j H_j(\omega), \quad C_j := \int_{\mathbb{R}} G(\omega) H_j(\omega) \phi(\omega)d\omega.
\]

By Parseval’s identity it holds

\[
\sum_{j=0}^{\infty} \frac{C_j^2}{j!} = \int_{\mathbb{R}} G^2(\omega)\phi(\omega)d\omega.
\]
Definition 2.1. (Taqqu (1975)) Let $G(\cdot) \in L_2\left(\mathbb{R}, \phi(\omega) d\omega\right)$. Assume that there exists an integer $\kappa \geq 1$, such that $C_j = 0$ for all $0 < j \leq \kappa - 1$, but $C_\kappa \neq 0$. Then $\kappa$ is called the Hermite rank of $G(\cdot)$ and is denoted by $H\text{rank}G(\cdot)$.

Note, that by (2.1.8) in Ivanov and Leonenko (1989) we get $\mathbb{E}(H_m(\xi(x))) = 0$ and

$$
\mathbb{E}\left(H_{m_1}(\xi(x))H_{m_2}(\xi(y))\right) = \delta_{m_1}^{m_2}m_1!B^{m_1}(\|x - y\|), \quad x, y \in \mathbb{R}^n,
$$

where $\delta_{m_1}^{m_2}$ is the Kronecker delta function.

Definition 2.2. (Bingham et al. (1989)) A measurable function $L : (0, \infty) \rightarrow (0, \infty)$ is slowly varying at infinity if for all $t > 0$, $\lim_{r \rightarrow \infty} L(tr)/L(r) = 1$.

3 Assumptions and Auxiliary Results

This section gives some assumptions and results from the spectral and correlation theory of random fields that will be used in the following sections.

Assumption 3.1. Let $\xi(x), x \in \mathbb{R}^n$, be a homogeneous isotropic Gaussian random field with $\mathbb{E}\xi(x) = 0$ and the covariance function $B(x)$, such that $B(0) = 1$ and

$$
B(x) = \mathbb{E}\left(\xi(0)\xi(x)\right) = \|x\|^{-\alpha}L_0\left(\|x\|\right), \quad \alpha > 0,
$$

where $L_0(\|\cdot\|)$ is a function slowly varying at infinity.

If $\alpha \in (0, n)$, then the covariance function $B(x)$ satisfying Assumption 3.1 is not integrable, which corresponds to the long-range dependence case (Anh et al. (2015)).

The notation $\Delta_n \subset \mathbb{R}^n$ will be used to denote a Jordan-measurable compact bounded set, such that $|\Delta_n| > 0$, and $\Delta_n$ contains the origin in its interior. Let $\Delta_n(r), r > 0$, be the homothetic image of the set $\Delta_n$, with the centre of homothety at the origin and the coefficient $r > 0$, that is $|\Delta_n(r)| = r^n|\Delta_n|$ and $\Delta_n = \Delta_n(1)$. For $c \in \mathbb{R}$ and $v \in \mathbb{R}^n$, $n \geq 1$, we define $c\Delta_n := \{cx : x \in \Delta_n\}$ and $\Delta_n - v := \{x - v : x \in \Delta_n\}$.

Let $H\text{rank}G(\cdot) = \kappa$. Denote the random variables $K_r$ and $K_{r,\kappa}$ by

$$
K_r := \int_{\Delta_n(r)} G\left(\xi(x)\right) dx \quad \text{and} \quad K_{r,\kappa} := \frac{C_{\kappa}}{\kappa!} \int_{\Delta_n(r)} H_\kappa\left(\xi(x)\right) dx,
$$

where $C_{\kappa}$ is given by (2.2).

Theorem 3.1. (Leonenko and Olenko (2014)) Suppose that $\xi(x), x \in \mathbb{R}^n$, satisfies Assumption 3.1 and $H\text{rank}G(\cdot) = \kappa \geq 1$. If a limit distribution exists for at least one of the random variables

$$
\frac{K_r}{\sqrt{\text{Var}K_r}} \quad \text{and} \quad \frac{K_{r,\kappa}}{\sqrt{\text{Var}K_{r,\kappa}}},
$$

then the limit distribution of the other random variable also exists, and the limit distributions coincide when $r \rightarrow \infty$. 
By Theorem 3.1 it is enough to study $K_{r,\kappa}$ to get asymptotic distributions of $K_r$. Therefore, we restrict our attention only to $K_{r,\kappa}$.

**Assumption 3.2.** The random field $\xi(x)$, $x \in \mathbb{R}^n$, has the isotropic spectral density

$$\varphi(\|\lambda\|) := c_1(n, \alpha) \|\lambda\|^{\alpha-n} L \left( \frac{1}{\|\lambda\|} \right),$$

where $\alpha \in (0, n)$, $c_1(n, \alpha) := \Gamma \left( \frac{(n - \alpha)/2}{2} \right) / 2^{\alpha-n/2} \Gamma \left( \frac{\alpha}{2} \right)$, and $L(\|\cdot\|) \sim L_0(\|\cdot\|)$ is a locally bounded function which is slowly varying at infinity.

One can find more details on relations between Assumptions 3.1 and 3.2 in Anh et al. (2019).

The function $K_{\Delta_n}(x)$ will be used to denote the Fourier transform of the indicator function of the set $\Delta_n$, i.e.

$$K_{\Delta_n}(x) := \int_{\Delta_n} e^{i\langle u, x \rangle} du, \quad x \in \mathbb{R}^n.$$

**Theorem 3.2.** (Leonenko and Olenko (2014)) Let $\xi(x)$, $x \in \mathbb{R}^n$, be a homogeneous isotropic Gaussian random field. If Assumptions 3.1 and 3.2 hold, $\alpha \in (0, n/\kappa)$, then for $r \to \infty$ the random variables

$$X_{r,\kappa}(\Delta_n) := r^{\kappa\alpha/2-n} L^{-\kappa/2}(r) \int_{\Delta_n(r)} H_\kappa(\xi(x)) \, dx$$

converge weakly to

$$X_\kappa(\Delta_n) := c_1^{\kappa/2}(n, \alpha) \int_{\mathbb{R}^n} K_{\Delta_n} (\lambda_1 + \cdots + \lambda_\kappa) \frac{W(d\lambda_1) \cdots W(d\lambda_\kappa)}{\|\lambda_1\|^{(n-\alpha)/2} \cdots \|\lambda_\kappa\|^{(n-\alpha)/2}}.$$

Here $\int_{\mathbb{R}^n}$ denotes the multiple Wiener-Itô integral with respect to a Gaussian white noise measure, where the diagonal hyperplanes $\lambda_i = \pm \lambda_j$, $i, j = 1, \ldots, \kappa$, $i \neq j$, are excluded from the domain of integration.

Below we present a limit theorem and the corresponding assumptions on the weight function in the integral functionals from Ivanov and Leonenko (1989). These results will be generalised in the subsequent sections. The obtained results on asymptotic equivalence of additive and integral functionals of random fields will be used to obtain limit theorems for the case of discrete observations.

**Assumption 3.3.** (Ivanov and Leonenko (1989)) Let $\vartheta(x) = \vartheta(\|x\|)$ be a radial continuous function that is positive for $\|x\| > 0$ and such that for $\alpha \in (0, n/\kappa)$

$$\lim_{r \to \infty} \int_{\Delta_n} \int_{\Delta_n} \frac{\vartheta(r\|x\|) \vartheta(r\|y\|) dx dy}{\vartheta^2(r) \|x-y\|^{\alpha \kappa}} \in (0, \infty).$$

Let $u(\|\lambda\|) := c_1(n, \alpha) L \left( \frac{1}{\|\lambda\|} \right)$, where $L(\cdot)$ is from Assumption 3.2. In Section 2.10 in Ivanov and Leonenko (1989) the case when the function $u(\|\lambda\|)$ is continuous in a
neighborhood of zero, bounded on \((0, \infty)\) and \(u(0) \neq 0\), was studied. It was assumed that there is a function \(\vartheta(||x||)\) such that

\[
\int_{\mathbb{R}^n} \prod_{j=1}^{\kappa} \|\lambda_j\|^{\alpha-n} \left| \int_{\Delta_n} e^{i(\lambda_1+\cdots+\lambda_{\kappa},x)} \vartheta(x) dx \right|^2 \prod_{j=1}^{\kappa} d\lambda_j < \infty
\]

and

\[
\lim_{r \to \infty} \int_{\mathbb{R}^n} \left| \int_{\Delta_n} e^{i(\lambda_1+\cdots+\lambda_{\kappa},x)} \left( \frac{\vartheta(r||x||)}{\vartheta(r)} \prod_{j=1}^{\kappa} \sqrt{\frac{u(||\lambda_j||r^{-1})}{u(0)}} - \vartheta(x) \right) dx \right|^2 \prod_{j=1}^{\kappa} ||\lambda_j||^{\alpha-n} \prod_{j=1}^{\kappa} d\lambda_j = 0.
\]

Under these assumptions the following result was obtained.

**Theorem 3.3.** (Ivanov and Leonenko (1989)) If Assumption 3.3 holds, then for \(r \to \infty\) the random variables

\[
Y_{r,\kappa} := \frac{1}{r^{n-\kappa\alpha/2} \vartheta(r)u^{\kappa/2}(0)} \int_{\Delta_n(r)} \vartheta(||x||) H_{\kappa}(\xi(x)) \, dx
\]

converge weakly to

\[
Y_\kappa := \int_{\mathbb{R}^n} K_{\Delta_n} \left( \lambda_1 + \cdots + \lambda_{\kappa}; \vartheta \right) \prod_{j=1}^{\kappa} \frac{W(d\lambda_j)}{\prod_{j=1}^{\kappa} ||\lambda_j||^{(n-\alpha)/2}},
\]

where \(\alpha \in (0, \min \left( \frac{n}{\kappa}, \frac{n-1}{2} \right) \) and \(K_{\Delta_n}(\lambda; \vartheta) := \int_{\Delta_n} e^{i(\lambda,x)} \vartheta(x) dx\).

Note, that the above result is a specification of the results on convergence to stochastic processes in Ivanov and Leonenko (1989) where functionals over the observation windows \(\Delta_n(rt^{1/n})\), \(t \in [0, 1]\), were studied. For simplicity, this paper deals with a particular case when \(t = 1\). But the results of this paper can be easily extended to the case of general \(\Delta_n(rt^{1/n})\) and convergence to stochastic processes on \(t \in [0, 1]\).

**4 Main Results**

**4.1 Two-dimensional Case**

In this section we consider integrals of two-dimensional random fields over a bounded increasing observation window \(\Delta_2(r) \subset \mathbb{R}^2\), \(r > 0\). We show that the limit distributions of these integrals and their corresponding additive functionals coincide.

Our setup is as follows. We assume that the set \(\Delta_2\) can be represented as

\[
\Delta_2 = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, f_{l,1}(x) \leq y \leq f_{u,1}(x)\},
\]

where \(a = \min_{(x,y) \in \Delta_2} x\), \(b = \max_{(x,y) \in \Delta_2} x\), \(f_{l,1}(x) < f_{u,1}(x)\), \(x \in (a, b)\), and \(f_{q,1}(x)\), \(q \in \{l, u\}\), are smooth functions (i.e. \(f_{q,1}(x) \in C^1\), \(q \in \{l, u\}\), where \(C^1\) is a class of functions with continuous first derivatives) except of the sets \(M_q = \{x^q_1, \ldots, x^q_{k_q}\} \subset [a, b]\), where these functions have finite jumps. Here \(k_q\) is the number of jumps of \(f_{q,1}(\cdot)\), see, for
example, Figure 1a. That is $x_j^q \in M_q$, $j = 1, \ldots, k_q$, are jump points of the functions $f_{q,1}(\cdot)$, $q \in \{l, u\}$, if $\lim_{x \to x_j^q^+} f_{q,1}(x) \neq \lim_{x \to x_j^q^-} f_{q,1}(x)$ but $\lim_{x \to x_j^q^+} f_{q,1}(x)$ and $\lim_{x \to x_j^q^-} f_{q,1}(x)$ both exist. Note that, by the homothety of $\Delta_2(r)$, $r > 0$, the set $\Delta_2(r)$ can be represented as $\Delta_2(r) = \{(x, y) \in \mathbb{R}^2 : ar \leq x \leq br, f_{l,r}(x) \leq y \leq f_{u,r}(x)\}$, where $f_{q,r}(x) = r f_{q,1}(x/r)$, $q \in \{l, u\}$. As $\Delta_2$ contains the origin in its interior, it follows that $ar < 0 < br$, $[ar] \to -\infty$, and $[br] \to \infty$, as $r \to \infty$. Let $\xi(x, y)$, $x, y \in \mathbb{R}$, be a real-valued homogeneous isotropic Gaussian random field satisfying Assumptions 3.1 and 3.1. We investigate the integrals

$$Y_{r,\kappa}^{(c)} := d_r^{-1} \int_{\Delta_2(r)} g(x, y) H_\kappa(\xi(x, y)) dxdy,$$

as $r \to \infty$, where $g(x, y)$ is a non-random function such that $g(x, y) \neq 0$ when $x = y$, and $d_r^{-1}$ is a normalising factor.

We define the corresponding additive functional to $Y_{r,\kappa}^{(c)}$, $r > 0$, by

$$Y_{r,\kappa}^{(d)} := d_r^{-1} \sum_{i=\lfloor ar \rfloor}^{\lfloor br \rfloor} \sum_{j=\lfloor f_{l,r}^{(d)}(i) \rfloor}^{\lfloor f_{u,r}^{(d)}(i) \rfloor} g(i, j) H_\kappa(\xi(i, j)),$$  \hspace{1cm} (4.1)

where $f_{l,r}^{(d)}(i) := \inf_{x \in [i, i+1]} f_{l,r}(x)$ and $f_{u,r}^{(d)}(i) := \sup_{x \in [i, i+1]} f_{u,r}(x)$, see Figure 1b. It is assumed that for $ar \in (i, i+1)$:

$$\inf_{x \in [i, i+1]} f_{l,r}(x) = \inf_{x \in [ar, i+1]} f_{l,r}(x), \quad \sup_{x \in [i, i+1]} f_{u,r}(x) = \sup_{x \in [ar, i+1]} f_{u,r}(x);$$

for $br \in (i, i+1)$:

$$\inf_{x \in [i, i+1]} f_{l,r}(x) = \inf_{x \in [i, br]} f_{l,r}(x), \quad \sup_{x \in [i, i+1]} f_{u,r}(x) = \sup_{x \in [i, br]} f_{u,r}(x).$$

Example 4.1. Figure 2a visualises a realisation of the long-range dependent Cauchy field $\xi(x, y)$ over the set $\Delta_2$ from Figure 1a. This field satisfies Assumptions 3.1 and 3.1. The corresponding normal Q-Q plot of $Y_{r,\kappa}^{(d)}$ in Figure 2b was obtained by simulating $\xi(x, y)$ 1000 times for the large value $r = 200$ and $g(x, y) \equiv 1$. It is close to the asymptotic distribution and Figure 2b shows its departure from the Gaussian distribution.
Figure 2: (a) A realisation of the Cauchy field $\xi(x,y)$ over $\Delta_2$, (b) The normal Q-Q plot of $Y^{(d)}_{200,2}$.

Remark 4.1. The functionals $Y_{r,\kappa}^{(c)}$ and $Y_{r,\kappa}^{(d)}$ include various important statistics. For example, the case of $g(x,y) \equiv 1$, $k = 1$, and $H_1(t) = t$ corresponds to the sample mean estimator. High order sample moments can be expressed in terms of $Y_{r,\kappa}^{(c)}$ or $Y_{r,\kappa}^{(d)}$ by using the formula

$$t^\kappa = \kappa! \sum_{m=0}^{\lfloor \kappa/2 \rfloor} \frac{1}{2^m m! (\kappa - 2m)!} H_{\kappa-2m}(t).$$

Another important example, a level excess measure, can be found by using the Hermite series expansion (2.2) for the indicator function $\chi(t > C)$:

$$\chi(t > C) = \sum_{m=0}^{\infty} \frac{C_m^{(C)} H_m(t)}{m!}, \quad C_m^{(C)} = \begin{cases} 1 - \Phi(C), & m = 0, \\ \phi(C) H_{m-1}(C), & m \geq 1, \end{cases}$$

where $\Phi(\cdot)$ and $\phi(\cdot)$ are the cdf and pdf for $\mathcal{N}(0,1)$ respectively. The case of $g(x,y) \not\equiv 1$ corresponds to weighted versions of the above statistics.

Let us define rectangles $S_{q,r}(i)$, $q \in \{l,u\}$, $i = \lfloor ar \rfloor, \ldots, \lceil br \rceil$, as

$$S_{q,r}(i) := [i,i+1) \times \left[ \inf_{x \in [i,i+1)} f_{q,r}(x), \sup_{x \in [i,i+1)} f_{q,r}(x) \right].$$

Then, $Y_{r,\kappa}^{(c)}$ can be rewritten as

$$Y_{r,\kappa}^{(c)} = d_r^{-1} \int_{A(r)} g(x,y) H_\kappa(\xi(x,y))dxdy - d_r^{-1} \sum_{i=\lfloor ar \rfloor}^{\lceil br \rceil} \int_{S_{l,r}(i) \cap \Delta_2^l(r)} g(x,y) H_\kappa(\xi(x,y))dxdy$$

$$- d_r^{-1} \sum_{i=\lfloor ar \rfloor}^{\lceil br \rceil} \int_{S_{u,r}(i) \cap \Delta_2^u(r)} g(x,y) H_\kappa(\xi(x,y))dxdy,$$

where $A(r) := \{ (x,y) \in \mathbb{R}^2 : x \in [i,i+1), y \in [f_r^{(l)}(i), f_r^{(u)}(i)] \}$, $i = \lfloor ar \rfloor, \ldots, \lceil br \rceil$. 

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Assumption 4.1. Let \( g(u, v), \ u, v \in \mathbb{R} \), be such that \( r^{4-\alpha} g^2(r, r) L^\kappa(r) \to \infty \), as \( r \to \infty \), and there exists a function \( g^*(u, v) \) such that for some \( \varepsilon > 0 \) uniformly for \( (u, v) \in \Delta_{2\varepsilon} := \Delta_2(1 + \varepsilon) \) it holds

\[
\lim_{r \to \infty} \frac{|g(r u, r v) - g^*(u, v)|}{g(r, r)} = 0,
\]

where \( \alpha \in (0, 2/\kappa) \) and

\[
\int_{\Delta_{2\varepsilon}} \int_{\Delta_{2\varepsilon}} \frac{|g^*(u_1, u_2)g^*(v_1, v_2)|}{d_{\varepsilon}(u_1 - v_1)^2 + d_{\varepsilon}(u_2 - v_2)^2} < \infty.
\]

Notice that, Assumption 4.1 is more general than Assumption 3.3.

Remark 4.2. It follows from Assumption 4.1 that \( g^*(u, v) \) is bounded on \( \Delta_{2\varepsilon} \).

Remark 4.3. Note, that the conditions on the function \( g(\cdot, \cdot) \) in Assumption 4.1 are met by numerous types of functions that are important in solving various statistical problems, in particular, non-linear regression and M estimators. For example, the functions

- \( g(u, v) = u^{\mu_1} v^{\mu_2} \) with \( g^*(u, v) = u^{\mu_1} v^{\mu_2} \),
- \( g(u, v) = uv \log(\mu_1 + u) \log(\mu_2 + v) \) with \( g^*(u, v) = uv \)

(for some appropriate constants \( \mu_1 \) and \( \mu_2 \)) can be chosen. The case of \( g(u, v) \equiv C > 0 \) corresponds to the classical equally-weighted functionals and non-central limit theorems.

Remark 4.4. To avoid degenerated cases, the condition \( r^{4-\alpha} g^2(r, r) L^\kappa(r) \to \infty \), as \( r \to \infty \), is essential to guarantee the boundedness of the variance of \( d_{-1}^{-1} Y_{r,\kappa} \).

Now, we proceed to the main result.

Theorem 4.1. Let \( 0 < \alpha < 2/\kappa \). If Assumptions 3.1, 3.2 and 4.1 hold, then

\[
\lim_{r \to \infty} \mathbb{E} \left[ \int_{\Delta_2(r)} g(x, y) H_\kappa(\xi(x, y)) dy dx - \sum_{i=1}^{br} \sum_{j=1}^{fr} f^{(i)}(j) g(i, j) H_\kappa(\xi(i, j)) \right]^2 = 0. \tag{4.3}
\]

Remark 4.5. Theorem 4.1 is also true if \( f_q(\cdot) \), \( q \in \{l, u\} \), are Lipschitz functions.

4.2 Multidimensional Case

This section gives a multidimensional version of Theorem 4.1 and a generalisation of Theorem 3.3. Then, we apply the obtained results to show that additive functionals have the same asymptotic distribution as the corresponding integral functionals.

We use the following notations that enable us to obtain the result of this section analogously to the two-dimensional case. Let \( x := (x_1, \ldots, x_n) \in \mathbb{R}^n \), \( I_n := (1, \ldots, 1) \in \mathbb{R}^n \), and the set \( \Delta_n \subset \mathbb{R}^n \), \( n \geq 3 \).
First, we consider the case of \( n = 3 \). We assume that the set \( \Delta_3 \) can be represented as

\[
\Delta_3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2) \in \Delta_2, f_{l,1}(x_1, x_2) \leq x_3 \leq f_{u,1}(x_1, x_2)\},
\]

where \( f_{l,1}(x_1, x_2) < f_{u,1}(x_1, x_2) \), \((x_1, x_2) \in \Delta_2\), and \( f_{q,1}(x_1, x_2), q \in \{l, u\} \), are smooth functions except of the sets where these functions have finite jumps, i.e.

\[
M_q' := \left\{ (x_1, x_2) \in \Delta_2 : x_1 = x^q_{j_1} or x_2 = x^q_{j_2}, \ j_1 = 1, \ldots, k_q^{(1)}, \ j_2 = 1, \ldots, k_q^{(2)} \right\},
\]

where \( x^q_{j_1} \) and \( x^q_{j_2} \), are constants. Here \( k_q^{(1)} \) and \( k_q^{(2)} \) are the number of jumps of \( f_{q,1}(\cdot, \cdot) \). Thus, \( M_q' \) consists of a finite number of two-dimensional line segments.

Note that, by the homothety of \( \Delta_3(r), \ r > 0 \), the set \( \Delta_3(r) \) can be represented as \( \Delta_3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2) \in \Delta_2(r), f_{l,r}(x_1, x_2) \leq x_3 \leq f_{u,r}(x_1, x_2)\} \), where \( f_{q,r}(x_1, x_2) = r f_{q,1}(x_1/r, x_2/r), q \in \{l, u\} \).

For a real-valued homogeneous isotropic Gaussian random field \( \xi(x) \), \( x \in \mathbb{R}^3 \), let

\[
Z_{r,c}^{(c)} := d_r^{-1} \int_{\Delta_3(r)} g(x) H_r(\xi(x)) dx,
\]

where \( g(x), x \in \mathbb{R}^3 \), is a non-random scalar function such that \( g(x_1 I_3) \neq 0 \), and \( d_r^{-1} \) is a normalising factor.

We define the corresponding additive functional to \( Z_{r,c}^{(c)} \), \( r > 0 \), by

\[
Z_{r,c}^{(d)} := d_r^{-1} \sum_{(i_1, i_2, i_3) \in Q_3(\Delta_3(r))} g(i_1, i_2, i_3) H_r(\xi(i_1, i_2, i_3)),
\]

where

\[
Q_3(\Delta_3(r)) := \left\{ (i_1, i_2, i_3) \in \mathbb{Z}^3 : i_1 \in \{ [ar], \ldots, [br] \}, \ i_2 \in \{ f_r^{(l)}(i_1), \ldots, f_r^{(u)}(i_1) \}\right\},
\]

\[
f_r^{(l)}(i_1, i_2) := \left\lfloor \inf_{(x_1, x_2) \in P_2(r, i_1, i_2)} f_{l,r}(x_1, x_2) \right\rfloor \text{ and } f_r^{(u)}(i_1, i_2) := \left\lceil \sup_{(x_1, x_2) \in P_2(r, i_1, i_2)} f_{u,r}(x_1, x_2) \right\rceil,
\]

and

\[
P_2(r, i_1, i_2) := \left\{ (x_1, x_2) \in [i_1, i_1 + 1] \times [i_2, i_2 + 1] \cap \Delta_2(r), \ i_1 \in \{ [ar], \ldots, [br] \}, \ i_2 \in [f_r^{(l)}(i_1), f_r^{(u)}(i_1)] \right\}.
\]

**Remark 4.6.** The intersection with \( \Delta_2(r) \) is required to correctly define the infimum and supremum for the cases when some points in \([i_1, i_1 + 1] \times [i_2, i_2 + 1] \) are outside of \( \Delta_2(r) \), which may happen for the boundary region.
For each \( q \in \{ l, u \} \), define three-dimensional parallelepipeds \( S_{q,r}(i_1, i_2) \) as

\[
S_{q,r}(i_1, i_2) := [i_1, i_1 + 1] \times [i_2, i_2 + 1] \times \inf_{(x_1, x_2) \in P_2(r, i_1, i_2)} f_{q,r}(x_1, x_2),
\]

where \( i_1 = \lfloor ar \rfloor, \ldots, \lceil br \rceil \), \( i_2 = f_r^{(l)}(i_1), \ldots, f_r^{(u)}(i_1) \).

By induction one can extend this construction to an arbitrary dimension \( n \) as follows.

The sets \( \Delta_n \) and \( \Delta_n(r) , n \geq 4 \), can be defined as

\[
\Delta_n = \{(x_1, \ldots, x_{n-1}) \in \Delta_{n-1}, f_{l,1}(x_1, \ldots, x_{n-1}) \leq x_n \leq f_{u,1}(x_1, \ldots, x_{n-1})\},
\]

and \( \Delta_n(r) = \{(x_1, \ldots, x_{n-1}) \in \Delta_{n-1}(r), f_{l,r}(x_1, \ldots, x_{n-1}) \leq x_n \leq f_{u,r}(x_1, \ldots, x_{n-1})\}, \)

where \( f_{q,r}(x_1, \ldots, x_{n-1}) = r f_{q,1}(\frac{x_1}{r}, \ldots, \frac{x_{n-1}}{r}) \), such that \( f_{l,1}(x_1, \ldots, x_{n-1}) < f_{u,1}(x_1, \ldots, x_{n-1}) \)

if \( (x_1, \ldots, x_{n-1}) \in \Delta_{n-1} \), and \( f_{q,1}(x_1, \ldots, x_{n-1}) \), \( q \in \{ l, u \} \), are smooth functions except of the sets where these functions have finite jumps, i.e.

\[
M_q^n := \left\{ (x_1, \ldots, x_{n-1}) \in \Delta_{n-1} : x_1 = x_{j_1}^q \text{ or } x_2 = x_{j_2}^q \text{ or } \ldots \text{ or } x_{n-1} = x_{j_{n-1}}^q \right\},
\]

where \( j_m = 1, \ldots, k_q^{(m)} \), \( m = 1, \ldots, n-1 \), and \( x_{j_m}^q \), \( m = 1, \ldots, n-1 \), is the number of jumps of \( f_{q,1}(\cdot, \ldots, \cdot) \) over dimension \( m \). Thus, \( M_q^n \) consists of a finite number of \((n-1)\)-dimensional hyperplanes sections in \( \Delta_{n-1} \).

For a real-valued homogeneous isotropic Gaussian random field \( \xi(x), x \in \mathbb{R}^n \), let

\[
Z_{r,\xi}^{(c)} := d_r^{-1} \int_{\Delta_n(r)} g(x) H_n(\xi(x)) dx,
\]

where \( g(x), x \in \mathbb{R}^n \), is a non-random function such that \( g(x_1 I_n) \neq 0 \), and \( d_r^{-1} \) is a normalising factor.

We define the corresponding additive functional to \( Z_{r,\xi}^{(c)} \), \( r > 0 \), by

\[
Z_{r,\xi}^{(d)} := d_r^{-1} \sum_{i \in Q_n(\Delta_n(r))} g(i) H_n(\xi(i)) ,
\]

where \( i = (i_1, \ldots, i_n) \in \mathbb{Z}^n \),

\[
Q_n(\Delta_n(r)) := \left\{ i : (i_1, \ldots, i_{n-1}) \in Q_{n-1}(\Delta_{n-1}(r)), i_n \in \left\{ f_r^{(l)}(i_1, \ldots, i_{n-1}), \ldots, f_r^{(u)}(i_1, \ldots, i_{n-1}) \right\} \right\},
\]

\[
f_r^{(l)}(i_1, \ldots, i_{n-1}) := \inf_{(x_1, \ldots, x_{n-1}) \in P_{n-1}(r, i_1, \ldots, i_{n-1})} f_{l,r}(x_1, \ldots, x_{n-1}) ,
\]

\[
f_r^{(u)}(i_1, \ldots, i_{n-1}) := \sup_{(x_1, \ldots, x_{n-1}) \in P_{n-1}(r, i_1, \ldots, i_{n-1})} f_{u,r}(x_1, \ldots, x_{n-1}) ,
\]

and

\[
P_{n-1}(r, i_1, \ldots, i_{n-1}) := \left\{ (x_1, \ldots, x_{n-1}) \in [i_1, i_1 + 1] \times [i_2, i_2 + 1] \times \cdots \times [i_{n-1}, i_{n-1} + 1] \cap \Delta_{n-1}(r), i_1 = \lfloor ar \rfloor, \ldots, \lceil br \rceil, i_j \in [f_r^{(l)}(i_1, \ldots, i_{j-1}), f_r^{(u)}(i_1, \ldots, i_{j-1})], j = 2, \ldots, n-1 \right\} .
\]
Assumption 4.2. Let $g(u), u \in \mathbb{R}^n$, be such that $r^{2n-\alpha}g^2(rI_n)L^\kappa(r) \to \infty$, as $r \to \infty$, and there exists a function $g^*(u)$ such that for some $\varepsilon > 0$ uniformly for $u \in \Delta_n := \Delta_n(1+\varepsilon)$ it holds
\[
\lim_{r \to \infty} \left| \frac{g(ru)}{g(rI_n)} - g^*(u) \right| \to 0
\]
where $\alpha \in (0, n/\kappa)$ and
\[
\int_{\Delta_n} \int_{\Delta_n} \frac{|g^*(u)g^*(v)|}{\|u-v\|^{\alpha\kappa}} dudv < \infty.
\]

Following the steps analogous to the proof in Section 3 and replacing intervals by multidimensional parallelepipeds we obtain a multidimensional version of Theorem 4.1.

Theorem 4.2. Let $\xi(x), x \in \mathbb{R}^n$, and $\Delta_n$ satisfies assumptions (4.4) and (4.5). If Assumptions 3.1, 3.2 and 4.2 hold, $\alpha \in (0, n/\kappa)$, then
\[
\lim_{r \to \infty} \mathbb{E} \left[ \int_{\Delta_n(r)} g(x)H_\kappa(\xi(x))dx - \sum_{i \in Q_n(\Delta_n(r))} g(i)H_\kappa(\xi(i)) \right]^2 \frac{2^{2n-\alpha}L^\kappa(r)g^2(rI_n)}{r^{2n-\alpha}L^\kappa(r)g^2(rI_n)} = 0.
\]

Next, we give a generalisation of Theorem 3.3.

Assumption 4.3. Let $g^*(x), x \in \mathbb{R}^n$, be a function such that for $\alpha \in (0, n/\kappa)$ it holds
\[
\int_{\mathbb{R}^n} \prod_{j=1}^\kappa \|\lambda_j\|^{\alpha-n}\|K_{\Delta_n}(\lambda;g^*)\|^2 \prod_{j=1}^\kappa d\lambda_j < \infty
\]
and
\[
\lim_{r \to \infty} \int_{\mathbb{R}^n} \left| \int_{\Delta_n} e^{i(\lambda_1+\cdots+\lambda_n,x)} \left( \frac{g(rx)}{g(rI_n)} \prod_{j=1}^\kappa \sqrt{L(r/\|\lambda_j\|)/L(r)} - g^*(x) \right) dx \right|^2 \prod_{j=1}^\kappa \|\lambda_j\|^{\alpha-n} \prod_{j=1}^\kappa d\lambda_j = 0.
\]

Theorem 4.3. Let $\Delta_n(r)$ satisfy (4.4) and (4.5). If Assumptions 3.1, 3.2, 4.2 and 4.3 hold, $\alpha \in (0, n/\kappa)$, then for $r \to \infty$
\[
Z^{(c)}_{r,\kappa} := \frac{1}{r^{n-\alpha\kappa/2}g(rI_n)L^{\kappa/2}(r)c_1^{\kappa/2}(n,\alpha)} \int_{\Delta_n(r)} g(x)H_\kappa(\xi(x))dx
\]
converge weakly to the random variable
\[
Z_\kappa := \int_{\mathbb{R}^{n\kappa}} K_{\Delta_n}(\lambda_1+\cdots+\lambda_n;g^* \prod_{j=1}^\kappa W(\lambda_j) \prod_{j=1}^\kappa \|\lambda_j\|^{(n-\alpha)/2}.
\]

Now we apply the result of Theorem 4.2 to Theorem 4.3 to obtain an analogous result in the discrete case.

Theorem 4.4. Let $\Delta_n(r)$ satisfy (4.4) and (4.5). If Assumptions 3.1, 3.2, 4.2 and 4.3 hold, $\alpha \in (0, n/\kappa)$, then for $r \to \infty$
\[
Z^{(d)}_{r,\kappa} := \frac{1}{r^{n-\alpha\kappa/2}L^{\kappa/2}(r)g(rI_n)c_1^{\kappa/2}(n,\alpha)} \sum_{i \in Q_n(\Delta_n(r))} g(i)H_\kappa(\xi(i))
\]
converge weakly to the random variable $Z_\kappa$. 

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Remark 4.7. If \( \kappa = 1 \) the limit \( Z_\kappa \) is Gaussian. For \( \kappa > 1 \) the random variables \( Z_\kappa \) have non-Gaussian distribution. The most studied case is the Rosenblatt distribution that corresponds \( \kappa = 2 \) and a rectangular \( \Delta_n \), see [Taqqu 2013].

5 Proofs of Results from Sections 4.1 and 4.2

In this section we give proofs of results in Sections 4.1 and 4.2.

Proof of Theorem 4.1 Using (4.1) and (4.2), one can estimate the numerator in (4.3) as

\[
E \left[ \int_{A(r)} g(x, y) H_\kappa(x, y) dydx - \sum_{i=|ar|} [br] \sum_{j=f(i)} \int_{S_{i,r}(i) \cap \Delta_2^2(r)} g(i, j) H_\kappa(x, y) dydx \right]^2
\]

\[
\leq 2E \left[ \int_{A(r)} g(x, y) H_\kappa(x, y) dydx - \sum_{i=|ar|} [br] \sum_{j=f(i)} g(i, j) H_\kappa(x, y) dydx \right]^2
\]

\[
+ 2E \left[ \sum_{i=|ar|} [br] \left( \int_{S_{i,r}(i) \cap \Delta_2^2(r)} g(x, y) H_\kappa(x, y) dydx + \int_{S_{i,r}(i) \cap \Delta_2^2(r)} g(x, y) H_\kappa(x, y) dydx \right) \right]^2
\]

\[
\leq 2E \left[ \int_{A(r)} g(x, y) H_\kappa(x, y) dydx - \sum_{i=|ar|} [br] \sum_{j=f(i)} g(i, j) H_\kappa(x, y) dydx \right]^2
\]

\[
+ 4E \left[ \sum_{i=|ar|} [br] \int_{S_{i,r}(i) \cap \Delta_2^2(r)} g(x, y) H_\kappa(x, y) dydx \right]^2
\]

\[
+ 4E \left[ \sum_{i=|ar|} [br] \int_{S_{i,r}(i) \cap \Delta_2^2(r)} g(x, y) H_\kappa(x, y) dydx \right]^2 = 2J_1 + 4J_2 + 4J_3. \quad (5.1)
\]

We will consider each term in (5.1) separately. By (2.3) we get

\[
\frac{J_2}{r^{4-\kappa}g^2(r, r) L^\kappa(r)} = \frac{E \left[ \sum_{i=|ar|} [br] \int_{S_{i,r}(i) \cap \Delta_2^2(r)} g(x, y) H_\kappa(x, y) dydx \right]^2}{r^{4-\kappa}g^2(r, r) L^\kappa(r)}
\]

\[
= \kappa! \sum_{i,j=|ar|} [br] \int_{S_{i,r}(i) \cap \Delta_2^2(r)} \int_{S_{j,r}(j) \cap \Delta_2^2(r)} \frac{g(x, y) g(x', y') B^\kappa(||x-x', y-y'||)}{r^{4-\kappa}g^2(r, r) L^\kappa(r)} dy'dx' dx dy
\]

\[
\leq \kappa! \sum_{i,j=|ar|} [br] \int_{S_{i,r}(i)} \int_{S_{j,r}(j)} \frac{g(x, y) g(x', y') B^\kappa(||x-x', y-y'||)}{r^{4-\kappa}g^2(r, r) L^\kappa(r)} dy'dx' dx dy.
\]
As by Assumption \[3.1\] \(|B(\cdot)| \leq 1\), we have

\[
\frac{J_2}{r^{4-\kappa}g^2(r,r)L^\kappa(r)} \leq \frac{\kappa!}{r^{4-\kappa}L^\kappa(r)} \sum_{i,j=[ar]}^{[br]} \int S_{l,r}(i) \int S_{l,r}(j) \frac{|g(x,y)g(x',y')|dydy'dxdx'}{g^2(r,r)}.
\]

Using the following transformation

\[
u_1 = x, r\nu_2 = y, rv_1 = x', \text{ and } rv_2 = y',
\]

we get

\[
\frac{J_2}{r^{4-\kappa}g^2(r,r)L^\kappa(r)} \leq \frac{\kappa!}{r^{4-\kappa}L^\kappa(r)} \sum_{i,j=[ar]}^{[br]} \int S_{l,r}(i) \int S_{l,r}(j) \frac{|g(r\nu_1,r\nu_2)g(rv_1,rv_2)|du_1du_2dv_1dv_2}{g^2(r,r)}
\]

(5.3)

where

\[
S_{l,r}^*(i) = \left\{(u_1,u_2) : u_1 \in \left[\frac{i}{r}, \frac{i+1}{r}\right], u_2 \in \left[\inf_{u_1 \in \left[\frac{i}{r}, \frac{i+1}{r}\right]} f_{l,1}(u_1), \sup_{u_1 \in \left[\frac{i}{r}, \frac{i+1}{r}\right]} f_{l,1}(u_1)\right]\right\}.
\]

Note, that the rectangles \(\{S_{l,r}^*(i), i \in [[ar],[br]]\}\) have the same width but different lengths. Denote by \(i' \in [[ar],...,[br]]\) a such index that the rectangle \(S_{l,r}^*(i')\) has the largest length. Let \(\epsilon_* := \min_{i,j=1,...,k_l} |x_i - x_j|\) and for each point of discontinuity \(x_j \in M_l\) the \(\epsilon_*/r\) neighbourhood of \(x_j\) be defined by \(N_{\epsilon_*}(x_j,r) := \left(x_j - \frac{\epsilon_*}{r}, x_j + \frac{\epsilon_*}{r}\right)\).

Let \(N_{\epsilon_*}'(x_j',r) = [x_j' - \frac{\epsilon_*}{r}, x_j' + \frac{\epsilon_*}{r}] \setminus \{x_j\}\) and

\[
SN_1(x_j',r) = N_{\epsilon_*}(x_j',r) \times \left(\inf_{u_1 \in N_{\epsilon_*}'(x_j',r)} f_{l,1}(u_1), \sup_{u_1 \in N_{\epsilon_*}'(x_j',r)} f_{l,1}(u_1)\right).
\]

We also define

\[
T_{k_l} := \left\{i \in [[ar],...,[br]] : x_j' \notin \left[\frac{i}{r}, \frac{i+1}{r}\right] \text{ for all } j = 1,...,k_l\right\},
\]

and for \(x_j' \in \left[\frac{i}{r}, \frac{i+1}{r}\right]\)

\[
SN_2(x_j',r) := \left[x_j', \frac{x_j' - \epsilon_*}{r}\right] \times \left[\inf_{u_1 \in \left[\frac{x_j'}{r} - \frac{\epsilon_*}{r}, \frac{x_j'}{r}\right]} f_{l,1}(u_1), \sup_{u_1 \in \left[\frac{x_j'}{r} - \frac{\epsilon_*}{r}, \frac{x_j'}{r}\right]} f_{l,1}(u_1)\right],
\]

\[
SN_3(x_j',r) := \left(\frac{x_j'}{r} + \frac{\epsilon_*}{r}, \frac{x_j' + 1}{r}\right) \times \left[\inf_{u_1 \in \left[\frac{x_j'}{r} + \frac{\epsilon_*}{r}, \frac{x_j' + 1}{r}\right]} f_{l,1}(u_1), \sup_{u_1 \in \left[\frac{x_j'}{r} + \frac{\epsilon_*}{r}, \frac{x_j' + 1}{r}\right]} f_{l,1}(u_1)\right].
\]

Note, that for each \(j = 1,...,k_l\), \(f_{l,1}(\cdot)\) is a bounded function on \(N_{\epsilon_*}(x_j',r)\). As the number of jumps is finite then there is a constant \(C > 0\), such that for all \(j = 1,...,k_l\) it holds

\[
|SN_1(x_j',r)| \leq \frac{C \epsilon_*}{r} \sup_{u_1 \in N_{\epsilon_*}(x_j',r)} f_{l,1}(u_1) - \inf_{u_1 \in N_{\epsilon_*}(x_j',r)} f_{l,1}(u_1) \leq C \frac{\epsilon_*}{r}.
\]
The smoothness of the function $f_{l,1}(\cdot)$ in $[\frac{t}{r}, x_j^l -\frac{\varepsilon r}{2}]$ and $(x_j^l + \frac{\varepsilon r}{2}, \frac{t+1}{r}]$ gives 
\[ |SN_{e}(x_j^l, r)| \leq \frac{C}{r}, \quad e = 2, 3. \]

Now, using the above results for sufficient large $r$, one can estimate (5.3) as 
\[
\begin{align*}
J_2 \frac{J}{r^{4-\kappa}g^2(r, r)L^\kappa(r)} & \leq \frac{\kappa!}{r^{-7\alpha}L^\kappa(r)} \left( \sup_{i \in [ar, [br]]} \sup_{(u_1, u_2) \in S_i^e} \left| \frac{g(ru_1, ru_2)}{g(r, r)} \right| \right)^2 \\
& \times \left( \left( [br] - [ar] + 1 \right) \sup_{u_1 \in \hat{[v, v+1]}_{r\kappa}} f_{l,1}(u_1) - \inf_{u_1 \in \hat{[v, v+1]}_{r\kappa}} f_{l,1}(u_1) \right)^2 \\
& \leq \frac{C}{r^{2-\kappa}L^\kappa(r)} \left( \sup_{i \in [ar, [br]]} \sup_{(u_1, u_2) \in S_i^e} \left| \frac{g(ru_1, ru_2)}{g(r, r)} \right| \right)^2 \\
& \times \left( \left( [br] - [ar] + 1 \right) \sup_{u_1 \in \hat{[v, v+1]}_{r\kappa}} f_{l,1}(u_1) - \inf_{u_1 \in \hat{[v, v+1]}_{r\kappa}} f_{l,1}(u_1) \right)^2 + C.
\end{align*}
\]

Note, that as the function $f_{l,1}(\cdot) \in C^1$ on $T_{k_1}$, then by the mean-value theorem there exists $u_0 \in \hat{[\frac{t}{r}, \frac{t+1}{r}]}$, $i \in T_{k_1}$, such that 
\[
\left| \sup_{u_1 \in \hat{[v, v+1]}_{r\kappa}} f_{l,1}(u_1) - \inf_{u_1 \in \hat{[v, v+1]}_{r\kappa}} f_{l,1}(u_1) \right| \leq \frac{1}{r} \max_{i \in T_{k_1}} \sup_{u_0 \in \hat{[\frac{t}{r}, \frac{t+1}{r}]}_{r\kappa}} \left| f_{l,1}'(u_0) \right| = \frac{C}{r}.
\]

Then, (5.4) can be estimated as 
\[
\begin{align*}
& J_2 \frac{J}{r^{4-\kappa}g^2(r, r)L^\kappa(r)} \leq \frac{\kappa!}{r^{2-\kappa}L^\kappa(r)} \left( \sup_{i \in [ar, [br]]} \sup_{(u_1, u_2) \in S_i^e} \left| \frac{g(ru_1, ru_2)}{g(r, r)} \right| \right)^2 \\
& \times \left( \frac{C}{r} \left( [br] - [ar] + 1 \right) + C \right)^2.
\end{align*}
\]

As for sufficiently large $r$ the rectangles $S_i^e(i) \subset \Delta_{2\varepsilon}$ for all $i = [ar], \ldots, [br]$, then by Assumption \[4.1\] and Remark \[4.2\] for an arbitrary $\varepsilon > 0$ there exists $r_0$ such that for $r > r_0$ 
\[
\max_{i=[ar], \ldots, [br]} \sup_{(u_1, u_2) \in S_i^e} \left| \frac{g(ru_1, ru_2)}{g(r, r)} \right| \leq \sup_{(u_1, u_2) \in \Delta_{2\varepsilon}} \left| g^*(u_1, u_2) \right| + \varepsilon_1 < \infty.
\]
As \( a = \lim_{r \to \infty} \frac{|ar|}{r} \) and \( b = \lim_{r \to \infty} \frac{|br|}{r} \) it holds that \( \frac{|br| - |ar| + 1}{r} \to b - a < \infty \), when \( r \to \infty \).

Finally, as \( 0 < \alpha < 2/\kappa \) the term \( \frac{1}{r^{2-\alpha} L^\kappa(r)} \to 0 \), when \( r \to \infty \). Hence, the upper bound in (5.5) approaches 0 when \( r \to \infty \).

Similarly, one obtains that \( \frac{J_3}{r^{4-2\alpha} g^2(r, r) L^\kappa(r)} \to 0 \), as \( r \to 0 \).

Therefore, it is enough to investigate the behaviour of

\[
J_1 = \mathbb{E} \left[ \int_{A(r)} g(x, y) H_\kappa(\xi(x, y)) dydx - \sum_{i=[ar]}^{[br]} \sum_{j=f(i)}^{f(i+1)} g(i, j) H_\kappa(\xi(i, j)) \right]^2. \tag{5.6}
\]

One can estimate (5.6) as

\[
J_1 = \mathbb{E} \left( \sum_{i=[ar]}^{[br]} \sum_{j=f(i)}^{f(i+1)} \int_{[0,1]^2} (g(x + i, y + j) H_\kappa(\xi(x + i, y + j)) - g(i, j) H_\kappa(\xi(i, j))) dydx \right)^2 =: \sum_{k=1}^{3} D_r^{(k)}. \tag{5.7}
\]

where

\[
D_r^{(1)} = \mathbb{E} \sum_{i,i'=[ar]}^{[br]} \sum_{j,j'=f(i)}^{f(i+1)} \int_{[0,1]^4} g(x + i, y + j)g(x' + i', y' + j') H_\kappa(\xi(x + i, y + j))
\]
\[
\times H_\kappa(\xi(x' + i', y' + j')) dydy' dxdx',
\]

\[
D_r^{(2)} = -2\mathbb{E} \sum_{i,i'=[ar]}^{[br]} \sum_{j,j'=f(i)}^{f(i+1)} \int_{[0,1]^4} g(x + i', y + j')g(i, j)
\]
\[
\times H_\kappa(\xi(x + i', y + j')) H_\kappa(\xi(i, j)) dydx,
\]

and

\[
D_r^{(3)} = \mathbb{E} \sum_{i,i'=[ar]}^{[br]} \sum_{j,j'=f(i)}^{f(i+1)} g(i, j)g(i', j') H_\kappa(\xi(i, j)) H_\kappa(\xi(i', j')).
\]

Now, using (2.3) and Assumption 1 we can rewrite the first term in (5.7) as follows

\[
D_r^{(1)} = \kappa! \sum_{i,i'=[ar]}^{[br]} \sum_{j,j'=f(i)}^{f(i+1)} \int_{[0,1]^4} g(x + i, y + j)g(x' + i', y' + j')
\]
\[
\times B^\kappa(\|(x' + i' - (x + i), y' + j' - (y + j))\|) dydy' dxdx'
\]
\[
= \kappa! \int_{A(r)} \int_{A(r)} g(x, y)g(x', y') L^\kappa(\|(x' - x, y' - y)\|) dydy' dxdx'
\]
\[
\frac{((x' - x)^2 + (y' - y)^2)^{\alpha/2}}. \tag{5.8}
\]
Using change of variables \((5.2)\) and elementary computations, we obtain
\[
D_r^{(1)} = \kappa! r^{4-\alpha} g^2(r, r) \int_{r^{-1}A(r)} \int_{r^{-1}A(r)} \frac{g(ru_1, ru_2)g(rv_1, rv_2)}{g^2(r, r)} \times \frac{L^*(r\|u_1 - v_1, u_2 - v_2\|)du_1dv_1du_2dv_2}{((u_1 - u_2)^2 + (v_2 - u_2)^2)^{\alpha/2}}.
\]

Adding and subtracting \(g^*(u_1, u_2)\) and \(g^*(v_1, v_2)\) inside the integrals, we obtain
\[
D_r^{(1)} = \kappa! r^{4-\alpha} g^2(r, r)(I_1 + 2I_2 + I_3),
\]
where
\[
I_1 = \int_{r^{-1}A(r)} \int_{r^{-1}A(r)} \left\{ \frac{g(ru_1, ru_2)}{g(r, r)} - g^*(u_1, u_2) \right\} \left\{ \frac{g(rv_1, rv_2)}{g(r, r)} - g^*(v_1, v_2) \right\} \times \frac{L^*(r\|u_1 - v_1, u_2 - v_2\|)du_1dv_1du_2dv_2}{((u_1 - u_2)^2 + (v_2 - u_2)^2)^{\alpha/2}},
\]
\[
I_2 = \int_{r^{-1}A(r)} \int_{r^{-1}A(r)} \left\{ \frac{g(ru_1, ru_2)}{g(r, r)} - g^*(u_1, u_2) \right\} g^*(v_1, v_2) \times \frac{L^*(r\|u_1 - v_1, u_2 - v_2\|)du_1dv_1du_2dv_2}{((u_1 - u_2)^2 + (v_2 - u_2)^2)^{\alpha/2}},
\]
and
\[
I_3 = \int_{r^{-1}A(r)} \int_{r^{-1}A(r)} \frac{L^*(r\|u_1 - v_1, u_2 - v_2\|)g^*(u_1, u_2)g^*(v_1, v_2)du_1dv_1du_2dv_2}{((u_1 - u_2)^2 + (v_2 - u_2)^2)^{\alpha/2}}.
\]

Let us analyse each term \(I_i, i = 1, 2, 3\), separately. The term \(I_1\) can be estimated as
\[
I_1 \leq \int_{r^{-1}A(r)} \int_{r^{-1}A(r)} \left\{ \frac{g(ru_1, ru_2)}{g(r, r)} - g^*(u_1, u_2) \right\} \left\{ \frac{g(rv_1, rv_2)}{g(r, r)} - g^*(v_1, v_2) \right\} \times \frac{L^*(r\|u_1 - v_1, u_2 - v_2\|)du_1dv_1du_2dv_2}{((u_1 - u_2)^2 + (v_2 - u_2)^2)^{\alpha/2}}.
\]

As for each \(\epsilon\) there exists \(r_0 > 0\) such that \(r^{-1}A(r) \subseteq r^{-1}\Delta_2(r(1 + \epsilon)) = \Delta_{2\epsilon}\) for all \(r > r_0\), then
\[
I_1 \leq \int_{\Delta_{2\epsilon}} \int_{\Delta_{2\epsilon}} \left\{ \frac{g(ru_1, ru_2)}{g(r, r)} - g^*(u_1, u_2) \right\} \left\{ \frac{g(rv_1, rv_2)}{g(r, r)} - g^*(v_1, v_2) \right\} \times \frac{L^*(r\|u_1 - v_1, u_2 - v_2\|)du_1dv_1du_2dv_2}{((u_1 - u_2)^2 + (v_2 - u_2)^2)^{\alpha/2}} \leq \sup_{(u_1, u_2) \in \Delta_{2\epsilon}} \left\{ \frac{g(ru_1, ru_2)}{g(r, r)} - g^*(u_1, u_2) \right\}^2 \int_{\Delta_{2\epsilon}} \int_{\Delta_{2\epsilon}} \frac{L^*(r\|u_1 - v_1, u_2 - v_2\|)du_1dv_1du_2dv_2}{((u_1 - u_2)^2 + (v_2 - u_2)^2)^{\alpha/2}}.
\]

To estimate the above integral, we consider the uniform distribution on \(\Delta_{2\epsilon}\) with probability density function \(|\Delta_{2\epsilon}|^{-1} \chi_{\Delta_{2\epsilon}}(x), x \in \mathbb{R}^2\), where \(\chi_E(\cdot)\) is the indicator function of a set \(E \subseteq \mathbb{R}^2\). Let \(U = (u_1, u_2)\) and \(V = (v_1, v_2)\) be two random points which are independent and uniformly distributed inside the set \(\Delta_{2\epsilon}\). We denote by \(\psi_{\Delta_{2\epsilon}}(\rho), \rho \geq 0, \)
the pdf of the distance $\rho = \|U - V\|$. Note that in this case $\psi_{\Delta z}(\rho) = 0$ if $\rho > \text{diam}(\Delta z)$, and the Jacobian is equal to $|J| = C\rho$, $C > 0$. Hence, for $\alpha_0 < 1$

$$\int_{\Delta z} \int_{\Delta z} \frac{d\nu_1 d\nu_2 d\nu_2}{\alpha_0} = |\Delta z|^2 E \left( \chi \left( \|x - y\| \leq \text{diam}(\Delta z) \right) \|x - y\|^{-2\alpha_0} \right) = C|\Delta z|^2 \int_0^{\text{diam}(\Delta z)} \rho^{1-2\alpha_0} \psi_{\Delta z}(\rho) d\rho$$

$$\leq C|\Delta z| \int_0^{\text{diam}(\Delta z)} \rho^{1-2\alpha_0} d\rho = C|\Delta z| \left( \frac{\text{diam}(\Delta z)^{2-2\alpha_0}}{2 - 2\alpha_0} \right).$$

As $|\Delta z| = (1 + \varepsilon)^2 |\Delta z|$ and $\text{diam}(\Delta z) = (1 + \varepsilon) \text{diam}(\Delta z)$

$$\int_{\Delta z} \int_{\Delta z} \frac{d\nu_1 d\nu_2 d\nu_2}{\alpha_0} \leq C(1 + \varepsilon)^{1-2\alpha_0} |\Delta z| \left( \frac{\text{diam}(\Delta z)^{2-2\alpha_0}}{2 - 2\alpha_0} \right) < \infty. \quad (5.9)$$

It follows from Assumption 3.1 that $L(\cdot)$ is locally bounded and by Theorem 1.5.3 in Bingham et al. (1989) for an arbitrary $\delta > 0$ there exists $r_0$ and $C > 0$ such that for all $r > r_0$

$$\sup_{0 \leq s \leq r \cdot \text{diam}(\Delta z)} L(s) \leq C.$$ 

Therefore, for all $r > r_0$

$$\int_{\Delta z} \int_{\Delta z} \frac{L^\kappa(r \cdot \|u_1 - u_2, v_2 - v_2\|)}{(c_1 - 1)^2 + (c_2 - 1)^2} d\nu_1 d\nu_2 d\nu_2 = \int_{\Delta z} \int_{\Delta z} \frac{L^\kappa(r \cdot \|u_1 - u_2, v_2 - v_2\|)}{(c_1 - 1)^2 + (c_2 - 1)^2} d\nu_1 d\nu_2 d\nu_2$$

$$\times \sup_{(u_1, u_2) \in \Delta z} \left( \frac{L^\kappa(r \cdot \|u_1 - u_2, v_2 - v_2\|)}{(c_1 - 1)^2 + (c_2 - 1)^2} \right)^{\frac{s}{\delta}} \leq \left( \frac{L^\kappa(r \cdot \|u_1 - u_2, v_2 - v_2\|)}{(c_1 - 1)^2 + (c_2 - 1)^2} \right)^{\frac{s}{\delta}} \leq \left( \frac{L^\kappa(r \cdot \|u_1 - u_2, v_2 - v_2\|)}{(c_1 - 1)^2 + (c_2 - 1)^2} \right)^{\frac{s}{\delta}}.$$

As $\lim_{r \to \infty} L(r \cdot \text{diam}(\Delta z))/L(r) = 1$, one obtains for sufficiently large $r$ that

$$\int_{\Delta z} \int_{\Delta z} \frac{L^\kappa(r \cdot \|u_1 - u_2, v_2 - v_2\|)}{(c_1 - 1)^2 + (c_2 - 1)^2} \leq \frac{C L^\kappa(r)}{(c_1 - 1)^2 + (c_2 - 1)^2}.$$

It follows from the condition $\alpha \kappa < 2$ that there exists $\delta > 0$ such that $\frac{\delta + \alpha \kappa}{2} < 1$. Then, applying the upper bound in (5.9) to the right hand side of the inequality and selecting $\delta$ such that $\frac{\delta + \alpha \kappa}{2} < 1$ we obtain for sufficiently large $r_0$ that

$$\int_{\Delta z} \int_{\Delta z} \frac{L^\kappa(r \cdot \|u_1 - u_2, v_2 - v_2\|)}{(c_1 - 1)^2 + (c_2 - 1)^2} \leq C L^\kappa(r), \quad r > r_0.$$ 

(5.10)
By Assumption 4.1, we get

\[ I_1 \leq CL^\kappa(r) o(1), \quad r \to \infty. \]

Similarly, by Remark 4.2 we get

\[
I_2 \leq \int_{r^{-1}A(r)} \int_{r^{-1}A(r)} \frac{|g(r u_1, r u_2)|}{g(r, r)} \left| g^*(u_1, u_2) - g^*(v_1, v_2) \right| \left \| L^\kappa(r)(u_1 - v_1, u_2 - v_2) \right \| du_1 dv_1 du_2 dv_2 \\
\times \left( (v_1 - u_1)^2 + (v_2 - u_2)^2 \right)^{\alpha \kappa/2} \leq \sup_{(u_1, u_2) \in \Delta_2} \left \| g(r u_1, r u_2) - g^*(u_1, u_2) \right \| \left \| L^\kappa(r)(u_1 - v_1, u_2 - v_2) \right \| du_1 dv_1 du_2 dv_2 \\
\leq \int_{\Delta_2} \int_{\Delta_2} \left \| g^*(u_1, u_2) \right \| \left \| L^\kappa(r)(u_1 - v_1, u_2 - v_2) \right \| du_1 dv_1 du_2 dv_2 \\
\leq \int_{\Delta_2} \int_{\Delta_2} \frac{\left \| g^*(u_1, u_2) \right \|}{\left \| L^\kappa(r)(u_1 - v_1, u_2 - v_2) \right \|} \left \| L^\kappa(r)(u_1 - v_1, u_2 - v_2) \right \| du_1 dv_1 du_2 dv_2 \\
\times \frac{1}{\left( (v_1 - u_1)^2 + (v_2 - u_2)^2 \right)^{\alpha \kappa/2}} < \infty. \]

As \( g^*(\cdot, \cdot) \) is a bounded function on \( \Delta_2 \), by the same reasons as for \( I_1 \) we obtain that \( I_2 \leq CL^\kappa(r) o(1), \) when \( r \to \infty. \)

Note, that as for each \( \varepsilon \) there exists \( r_0 > 0 \) such that \( \Delta_2(r) \subseteq A(r) \subseteq \Delta_2(r) \) for all \( r > r_0 \) and therefore \( \Delta_2 \subseteq r^{-1}A(r) \subseteq \Delta_2 \). So, \( r^{-1}A(r) \) converges to \( \Delta_2 \) when \( r \to \infty. \)

Thus, for sufficiently large \( r \)

\[ I_3 \sim \int_{\Delta_2} \int_{\Delta_2} \frac{L^\kappa(r) \left \| (u_1 - v_1, u_2 - v_2) \right \| g^*(u_1, u_2) g^*(v_1, v_2) du_1 dv_1 du_2 dv_2}{\left( (v_1 - u_1)^2 + (v_2 - u_2)^2 \right)^{\alpha \kappa/2}}. \]

Note, that \( \int_{\Delta_2} \int_{\Delta_2} \frac{g^*(u_1, u_2) g^*(v_1, v_2) du_1 dv_1 du_2 dv_2}{\left( (v_1 - u_1)^2 + (v_2 - u_2)^2 \right)^{\alpha \kappa/2}} < \infty. \) Hence, analogously to Proposition 4.1.2 in [Bingham et al. (1989)] we obtain that

\[ I_3 \sim L^\kappa(r) \int_{\Delta_2} \int_{\Delta_2} \frac{g^*(u_1, u_2) g^*(v_1, v_2) du_1 dv_1 du_2 dv_2}{\left( (v_1 - u_1)^2 + (v_2 - u_2)^2 \right)^{\alpha \kappa/2}} \sim l_{\Delta_2} L^\kappa(r), \quad r \to \infty, \]

where

\[ l_{\Delta_2} := \int_{\Delta_2} \int_{\Delta_2} \frac{g^*(u_1, u_2) g^*(v_1, v_2) du_1 dv_1 du_2 dv_2}{\left( (v_1 - u_1)^2 + (v_2 - u_2)^2 \right)^{\alpha \kappa/2}}. \]

By combining the above results for \( (5.8) \), we obtain

\[ D_r^{(1)} \sim \kappa! r^{4-\kappa \alpha} g^2(r, r) L^\kappa(r)(l_{\Delta_2} + o(1)), \quad r \to \infty. \]

Now, let us consider the second term \( D_r^{(2)} \):

\[ D_r^{(2)} = -2\kappa! \sum_{i=[ar]}^{[br]} \sum_{j=i}^{f^{(0)}(i)} \int_{A(r)} g(x, y) g(i, j) L^\kappa(||(i - x, j - y)||) dxdy \\
\times \left( (i - x)^2 + (j - y)^2 \right)^{\alpha \kappa/2} \]

\[ = -2\kappa! \int_{A(r)} \sum_{i=[ar]}^{[br]} \sum_{j=i}^{f^{(0)}(i)} g(x, y) g(i, j) L^\kappa(||(i - x, j - y)||) dxdy \\
\times \left( (i - x)^2 + (j - y)^2 \right)^{\alpha \kappa/2}. \]
Multiplying and dividing by \(g^2(r, r)\) and using transformation (5.2) again, one obtains

\[
D^{(2)}_r = -2\kappa!r^{4-\kappa/2}g^2(r, r) \int_{r^{-1}A(r)} \left[ \sum_{i=\llbracket ar \rrbracket} \sum_{j=f^{(0)}(i)} f^{(u)}_{r}(i) \right] g(ru_1, ru_2) g(\frac{i}{r}, \frac{j}{r}) \frac{((\frac{i}{r} - u_1)^2 + (\frac{j}{r} - u_2)^2)^{\kappa/2}}{r^2 g^2(r, r)} \times \frac{L^\kappa(r || (\frac{i}{r} - u_1, \frac{j}{r} - u_2)||) du_1 du_2}{r^2 g^2(r, r)}.
\]

Adding and subtracting \(g^*(u_1, u_2)\) and \(g^*\left(\frac{i}{r}, \frac{j}{r}\right)\) inside the integral, we get

\[
D^{(2)}_r = -2\kappa!r^{4-\kappa/2}g^2(r, r) \left( \hat{I}_1 + \hat{I}_2 + \hat{I}_2' + \hat{I}_3 \right),
\]

where

\[
\hat{I}_1 = \int_{r^{-1}A(r)} \left[ \sum_{i=\llbracket ar \rrbracket} \sum_{j=f^{(0)}(i)} f^{(u)}_{r}(i) \right] \frac{g(ru_1, ru_2)}{g(r, r)} - g^*(u_1, u_2) \left[ \frac{g(\frac{i}{r}, \frac{j}{r})}{g(r, r)} - g^*\left(\frac{i}{r}, \frac{j}{r}\right) \right] \times \frac{L^\kappa(r || (\frac{i}{r} - u_1, \frac{j}{r} - u_2)||) du_1 du_2}{r^2 \left( (\frac{i}{r} - u_1)^2 + (\frac{j}{r} - u_2)^2 \right)^{\kappa/2}}.
\]

\[
\hat{I}_2 = \int_{r^{-1}A(r)} \left[ \sum_{i=\llbracket ar \rrbracket} \sum_{j=f^{(0)}(i)} f^{(u)}_{r}(i) \right] \frac{g(ru_1, ru_2)}{g(r, r)} - g^*(u_1, u_2) g^*\left(\frac{i}{r}, \frac{j}{r}\right) \times \frac{L^\kappa(r || (\frac{i}{r} - u_1, \frac{j}{r} - u_2)||) du_1 du_2}{r^2 \left( (\frac{i}{r} - u_1)^2 + (\frac{j}{r} - u_2)^2 \right)^{\kappa/2}}.
\]

\[
\hat{I}_2' = \int_{r^{-1}A(r)} \left[ \sum_{i=\llbracket ar \rrbracket} \sum_{j=f^{(0)}(i)} f^{(u)}_{r}(i) \right] g^*(u_1, u_2) \left[ \frac{g(\frac{i}{r}, \frac{j}{r})}{g(r, r)} - g^*\left(\frac{i}{r}, \frac{j}{r}\right) \right] \times \frac{L^\kappa(r || (\frac{i}{r} - u_1, \frac{j}{r} - u_2)||) du_1 du_2}{r^2 \left( (\frac{i}{r} - u_1)^2 + (\frac{j}{r} - u_2)^2 \right)^{\kappa/2}}.
\]

and

\[
\hat{I}_3 = \int_{r^{-1}A(r)} \left[ \sum_{i=\llbracket ar \rrbracket} \sum_{j=f^{(0)}(i)} f^{(u)}_{r}(i) \right] g^*(u_1, u_2) g^*\left(\frac{i}{r}, \frac{j}{r}\right) \frac{L^\kappa(r || (\frac{i}{r} - u_1, \frac{j}{r} - u_2)||) du_1 du_2}{r^2 \left( (\frac{i}{r} - u_1)^2 + (\frac{j}{r} - u_2)^2 \right)^{\kappa/2}}.
\]

Similarly to the upper bounds for \(I_i, \ i = 1, 2, 3\), we can obtain estimates for the terms \(\hat{I}_1, \hat{I}_2, \hat{I}_2'\) and \(\hat{I}_3\). For example, for sufficiently large \(r\),

\[
\hat{I}_1 \leq \int_{r^{-1}A(r)} \left[ \sum_{i=\llbracket ar \rrbracket} \sum_{j=f^{(0)}(i)} f^{(u)}_{r}(i) \right] \frac{g(ru_1, ru_2)}{g(r, r)} - g^*(u_1, u_2) \left| \frac{g(\frac{i}{r}, \frac{j}{r})}{g(r, r)} - g^*\left(\frac{i}{r}, \frac{j}{r}\right) \right|
\]
\[ \times \frac{L^\kappa \left( r \left\| \left( \frac{i}{r} - u_1, \frac{j}{r} - u_2 \right) \right\| \right) du_1 du_2}{r^2 \left( \left( \frac{i}{r} - u_1 \right)^2 + \left( \frac{j}{r} - u_2 \right)^2 \right)^{\alpha/2}} \leq \int_{\Delta_2} \sum_{i=|ar|}^{[br]} \sum_{j=f^{(u)}(i)} g(ru_1, ru_2) \frac{g(r, r)}{g(r, r)} - g^*(u_1, u_2) \]
\[ \times \left| g \left( \frac{i}{r}, \frac{j}{r} \right) - g^* \left( \frac{i}{r}, \frac{j}{r} \right) \right| \leq \frac{L^\kappa \left( r \left\| \left( \frac{i}{r} - u_1, \frac{j}{r} - u_2 \right) \right\| \right) du_1 du_2}{r^2 \left( \left( \frac{i}{r} - u_1 \right)^2 + \left( \frac{j}{r} - u_2 \right)^2 \right)^{\alpha/2}} \]
\[ \leq \sup_{(u_1, u_2) \in \Delta_2} \left| g(ru_1, ru_2) \frac{g(r, r)}{g(r, r)} - g^*(u_1, u_2) \right|^2 \]
\[ \times \max_i \{ |ar|, \ldots, |br| \} \max_j \{ f^{(i)}(i), \ldots, f^{(u)}(i) \} \int_{\Delta_2} \frac{L^\kappa \left( r \left\| \left( \frac{i}{r} - u_1, \frac{j}{r} - u_2 \right) \right\| \right) du_1 du_2}{r^2 \left( \left( \frac{i}{r} - u_1 \right)^2 + \left( \frac{j}{r} - u_2 \right)^2 \right)^{\alpha/2}}. \] (5.12)

Analogously to the upper bound in (5.9) one obtains

\[ \max_{i \in \{ |ar|, \ldots, |br| \}} \int_{\Delta_2} \frac{du_1 du_2}{r^2 \left( \left( \frac{i}{r} - u_1 \right)^2 + \left( \frac{j}{r} - u_2 \right)^2 \right)^{\alpha/2}} \]
\[ \leq \max_{i \in \{ |ar|, \ldots, |br| \}} \int_{\Delta_2} x \left( \| x - \left( \frac{i}{r}, \frac{j}{r} \right) \| \leq \text{diam} (\Delta_2) \right) \| x - \left( \frac{i}{r}, \frac{j}{r} \right) \|^2 dx \]
\[ \leq \max_{i \in \{ |ar|, \ldots, |br| \}} \int_{\Delta_2} \| y \|^{-2\alpha_0} dy \leq \int_{\Delta_2 (2 \text{diam}(\Delta_2))} \| y \|^{-2\alpha_0} dy < \infty. \] (5.13)

Using (5.12), (5.13) and similar steps to the proof of the upper bound in (5.10) for sufficiently large \( r \) we get

\[ \hat{I}_1 \leq C \sup_{(u_1, u_2) \in \Delta_2} \left| g(ru_1, ru_2) \frac{g(r, r)}{g(r, r)} - g^*(u_1, u_2) \right|^2 \frac{\Delta_2(r)}{r^2} L^\kappa(r) \]
\[ \leq C |\Delta_2| \sup_{(u_1, u_2) \in \Delta_2} \left| g(ru_1, ru_2) \frac{g(r, r)}{g(r, r)} - g^*(u_1, u_2) \right|^2 L^\kappa(r). \]

It follows from Assumption 4.1 that \( \hat{I}_1 \leq o(1)L^\kappa(r), \quad r \to \infty. \)

Also, we have

\[ \hat{I}_2 \leq \int_{r^{-1}A(r)} \sum_{i=|ar|}^{[br]} \sum_{j=f^{(i)}(i)} g(ru_1, ru_2) \frac{g(r, r)}{g(r, r)} - g^*(u_1, u_2) \left| g \left( \frac{i}{r}, \frac{j}{r} \right) - g^* \left( \frac{i}{r}, \frac{j}{r} \right) \right| \]
\[ \times \frac{L^\kappa \left( r \left\| \left( \frac{i}{r} - u_1, \frac{j}{r} - u_2 \right) \right\| \right) du_1 du_2}{r^2 \left( \left( \frac{i}{r} - u_1 \right)^2 + \left( \frac{j}{r} - u_2 \right)^2 \right)^{\alpha/2}} \leq \sup_{(u_1, u_2) \in \Delta_2} \left| g(ru_1, ru_2) \frac{g(r, r)}{g(r, r)} - g^*(u_1, u_2) \right| \]

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\[ \times \int_{\Delta_2} \frac{[br]}{\sum_{i=|ar|} f^{(i)}(i)} \sum_{j=\Omega}^{|G|} g^* \left( \frac{r}{x}, \frac{r}{y} \right) \left| L^r \left( r \| (\frac{r}{x} - u_1, \frac{r}{x} - u_2) \right) \right| \, du_1 du_2. \]

As the function \( g^* (\cdot, \cdot) \) is bounded on \( \Delta_2 \), analogously to (5.12) we obtain for sufficiently large \( r \) that
\[ \int_{\Delta_2} \frac{[br]}{\sum_{i=|ar|} f^{(i)}(i)} \sum_{j=\Omega}^{|G|} g^* \left( \frac{r}{x}, \frac{r}{y} \right) \left| L^r \left( r \| (\frac{r}{x} - u_1, \frac{r}{x} - u_2) \right) \right| \, du_1 du_2 \leq CL^r(r). \]

Hence, \( \hat{I}_2 \leq o(1) L^r(r), \ r \to \infty. \) Similarly, \( \hat{I}_2' \leq o(1) L^r(r), \ r \to \infty. \) Also, analogously to the proof for \( I_3 \), one obtains \( \hat{I}_3 \sim l_{\Delta_2} L^r(r), \ r \to \infty. \)

By combining these results for (5.11), we have
\[ D^{(2)}_r \sim -2\kappa L^r \alpha L^r(r) g^2(r, r)(l_{\Delta_2} + o(1)), \quad r \to \infty. \]

Using similar arguments as for the sums in \( D^{(2)}_r \) we obtain
\[ D^{(3)}_r \sim \kappa L^r \alpha L^r(r) g^2(r, r)(l_{\Delta_2} + o(1)), \quad r \to \infty. \]

By dividing the estimates of \( D^{(i)}_r \), \( i = 1, 2, 3 \), by \( r^{4-\alpha} L^r(r) g^2(r, r) \), for sufficiently large \( r \) we get
\[ \frac{D^{(i)}_r}{r^{4-\alpha} L^r(r) g^2(r, r)} \sim \kappa (l_{\Delta_2} + o(1)), \quad i = 1, 3, \quad r \to \infty, \]
and
\[ \frac{D^{(2)}_r}{r^{4-\alpha} L^r(r) g^2(r, r)} \sim -2\kappa (l_{\Delta_2} + o(1)) \quad r \to \infty. \]

Hence, the ratio in (4.3) converges to zero, which completes the proof. \( \square \)

**Proof of Theorem 4.3** Note, that from the isonormal spectral representation (2.1) and the Itô formula
\[ H_\kappa (\xi(x)) = \int_{\mathbb{R}^\infty} e^{i(\lambda_1 + \cdots + \lambda_\nu, x)} \prod_{j=1}^\nu \sqrt{\varphi(\lambda_j)} W(\lambda_j) \]

it follows that
\[ Z^{(c)}_{r, \kappa} = \frac{c_{1\kappa/2}^n(n, \alpha)}{r^{n-\alpha/2} L^{r/2}(r)} \int_{\Delta_2} \int_{\mathbb{R}^\infty} g(x) e^{i(\lambda_1 + \cdots + \lambda_\nu, x)} \prod_{j=1}^\nu \sqrt{\varphi(||\lambda_j||)} W(\lambda_j) dx. \]

Using the transformation \( x' = x/r \) we get
\[ Z^{(c)}_{r, \kappa} = \frac{r^n c_{1\kappa/2}(n, \alpha)}{r^{n-\alpha/2} L^{r/2}(r)} \int_{\Delta_2} \int_{\mathbb{R}^\infty} g(x) e^{i(\lambda_1 + \cdots + \lambda_\nu, x')} \prod_{j=1}^\nu \sqrt{\varphi(||\lambda_j||)} W(\lambda_j) dx'. \quad (5.14) \]
Note that, for any fixed real number \( r > 0 \) the function \( \frac{g(rx)}{g(r I_n)} \) is bounded on \( \Delta_n \). Also, for sufficiently large \( r \), it follows by Assumption 4.2 that the function \( g^* (\cdot) \) is bounded on \( \Delta_n \) and therefore
\[
\int_{\Delta_n} \left| \frac{g(rx)}{g(r I_n)} \right| \, dx = \int_{\Delta_n} \left| \frac{g(rx)}{g(r I_n)} - g^*(x) + g^*(x) \right| \, dx \\
\leq |\Delta_n| \sup_{x \in \Delta_n} \left| \frac{g(rx)}{g(r I_n)} - g^*(x) \right| + \int_{\Delta_n} |g^*(x)| \, dx = C < \infty, \quad r \to \infty.
\]
By Assumption 3.2 it follows that \( \prod_{j=1}^\kappa \sqrt{\varphi(\|\lambda^j\|)} \in L_2(\mathbb{R}^n) \). So, one can apply the stochastic Fubini’s theorem to interchange the order of integration in (5.14) (see Theorem 5.13.1 in Peccati and Taqqu (2011)), which results in
\[
Z_{r,\kappa}^{(c)} = \frac{1}{r^{-\alpha n/2} L^{\kappa/2}(r) c_{1/2}^{\kappa/2}} \int_{\mathbb{R}^n} \left( \int_{\Delta_n} g(r x) e^{i(\lambda_1 + \cdots + \lambda_\kappa) x} \, dx \right) \prod_{j=1}^\kappa \sqrt{\varphi(\|\lambda^j\|/\sqrt{r})} W(d\lambda^j).
\]
Using the transformation \( \lambda^{(j)} = r \lambda_j, \ j = 1, \ldots, \kappa \), and the self-similarity of the Gaussian white noise we get
\[
Z_{r,\kappa}^{(c)} = \frac{r^{\alpha n/2 - \kappa n/2}}{L^{\kappa/2}(r) c_{1/2}^{\kappa/2} \alpha} \int_{\mathbb{R}^n} \left( \int_{\Delta_n} g(r x) e^{i(\lambda_1 + \cdots + \lambda_\kappa) x} \, dx \right) \prod_{j=1}^\kappa \sqrt{L(r/\|\lambda^j\|)/L(r)} W(d\lambda^j).
\]
By Assumption 3.2 it follows that
\[
Z_{r,\kappa}^{(c)} = \frac{1}{L^{\kappa/2}(r)} \int_{\mathbb{R}^n} \left( \int_{\Delta_n} g(r x) e^{i(\lambda_1 + \cdots + \lambda_\kappa) x} \, dx \right) \prod_{j=1}^\kappa \sqrt{L(r/\|\lambda^j\|)/L(r)} W(d\lambda^j).
\]
By the isometry property of multiple stochastic integrals
\[
R_r := \mathbb{E} \left| Z_{r,\kappa}^{(c)} - Z_\kappa \right|^2 = \int_{\mathbb{R}^n} \int_{\Delta_n} e^{i(\lambda_1 + \cdots + \lambda_\kappa) x} \left( \frac{g(rx)}{g(r I_n)} \right)^\kappa \prod_{j=1}^\kappa \sqrt{L(r/\|\lambda_j\|)/L(r)} - g^*(x) \, dx \right| \, dx \prod_{j=1}^\kappa \|\lambda_j\|^\alpha \prod_{j=1}^\kappa d\lambda_j.
\]
It follows from Assumption 4.3 that \( R_r \to 0 \) as \( r \to \infty \), which completes the proof.

**Proof of Theorem 4.4** Note, that to obtain the result of the theorem it is sufficient to prove that
\[
\hat{R}_r := \mathbb{E} (Z_{r,\kappa}^{(d)} - Z_\kappa)^2 = 0, \quad r \to \infty.
\]
One can estimate \( \hat{R}_r \) as
\[
\hat{R}_r = \mathbb{E} (Z_{r,\kappa}^{(d)} - Z_{r,\kappa}^{(c)} + Z_{r,\kappa}^{(c)} - Z_\kappa)^2 \leq 2 \mathbb{E} (Z_{r,\kappa}^{(c)} - Z_\kappa)^2 + 2 \mathbb{E} (Z_{r,\kappa}^{(d)} - Z_{r,\kappa}^{(c)})^2.
\]
By Theorem 4.3 the term \( \mathbb{E} (Z_{r,\kappa}^{(c)} - Z_\kappa)^2 \to 0 \) as \( r \to \infty \). Also, by Theorem 4.2 the term \( \mathbb{E} (Z_{r,\kappa}^{(d)} - Z_{r,\kappa}^{(c)})^2 \to 0 \) as \( r \to \infty \). Hence, \( \hat{R}_r \to 0 \) as \( r \to \infty \), which completes the proof. □
6 Numerical Studies

This section presents numeric examples confirming that the obtained theoretical results are valid even for wider classes of cyclic long-range dependent fields with spectral singularities outside the origin. It is demonstrated that the mean square distance between additive and the corresponding integral functionals approaches zero when \( r \to \infty \).

We also present simulation studies of convergence rates. All simulations were performed by using parallel computing on the NCI’s high-performance computer Raijin and the R package ‘RandomFields’ (Schlather et al. (2015)). A reproducible version of the code in this paper is available in the folder "Research materials" from the website https://sites.google.com/site/olenkoandriy/.

For numerical examples in this section we used the cyclic long-range dependent Bessel random field. Its realisations \( \xi(x, y) \) on the squares \( \Delta_2(r) := [-r, r]^2, r > 0 \), were simulated. The covariance function of this field has the form

\[
B(\sqrt{x^2 + y^2}) = 2^v \Gamma(v + 1) \frac{J_v(\sqrt{x^2 + y^2})}{(x^2 + y^2)^{v/2}}, \quad x, y \in \mathbb{R}, \ 0 \leq v < \frac{1}{2}.
\]

Note, that for the range \( v \in [0, \frac{1}{2}) \) the covariance function \( B(r) \) oscillates with the amplitude \( r^{-v-0.5}, r \to \infty \), and is not integrable which means that the long-range dependent case is considered.

As computer simulations are possible only on a discrete grid the integrals functionals in Theorem 4.1 were approximated by the Riemann sums. Each unit interval was uniformly split by equidistant points with the step length \( h \). Then the corresponding approximation of \( Y_{r, \kappa}^{(c)} \) is

\[
\tilde{Y}_{r, \kappa}^{(c)} := h^2 \sum_{i=0}^{2r} \sum_{j=0}^{2r} \frac{g(-r + hi, -r + hj)H_{\kappa}(\xi(-r + hi, -r + hj))}{r^{2-\alpha\kappa/2} L_{\kappa/2}(r) g(r, r)}.
\] (6.1)

The weight function \( g(x, y) = 1 + (x + y)^2, \ x, \ y \in \mathbb{R} \), was used. The function \( g(\cdot, \cdot) \) satisfies Assumption 4.1 and \( g^*(x, y) = \frac{1}{4}(x + y)^2, \ x, \ y \in \mathbb{R} \). The value \( v = 0 \) was used to simulate Bessel random fields. In this case the Bessel covariance function oscillating and has the asymptotic hyperbolic decay rate \( (\sqrt{x^2 + y^2})^{-0.5} \). Hence, the long-range dependence parameter can be chosen \( \alpha = 0.5 \). For simulations we used \( \kappa = 2 \) and \( h = 0.1 \). Hence, \( H_2(x) = x^2 - 1 \) and (6.1) becomes

\[
\tilde{Y}_{r, 2}^{(c)} = (0.1)^2 \sum_{i=0}^{20r} \sum_{j=0}^{20r} \frac{1 + (-2r + 0.1(i + j))^2}{r^{3/2}(1 + 4r^2)} \left( \xi^2(-r + 0.1i, -r + 0.1j) - 1 \right).
\] (6.2)

The corresponding additive functional is

\[
Y_{r, 2}^{(d)} = \sum_{i=-[r]}^{[r]} \sum_{j=-[r]}^{[r]} \frac{(1 + (i + j)^2)(\xi^2(i, j) - 1)}{r^{3/2}(1 + 4r^2)}.
\] (6.3)

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Each random variable (6.2) and (6.3) was simulated 100 times for \( r = 10, 20, \ldots, 80 \). Then, for each \( r \) we calculated the sample mean square distance \( \hat{L}_2(r) \) between \( \tilde{Y}_{r,2}^{(c)} \) and \( Y_{r,2}^{(d)} \).

Figure 3 shows box plots of \( \hat{L}_2(r) \) as a function of \( r \) by repeating the simulation steps 50 times, while Table 1 shows the sample averages of \( \hat{L}_2(r) \) for each \( r \). Figure 3 and Table 1 confirm that the mean squared distances quickly approaches zero when \( r \) increases.

![Box plots of \( \hat{L}_2(r) \)](image)

**Figure 3:** Box plots of the mean square distances between \( \tilde{Y}_{r,2}^{(c)} \) and \( Y_{r,2}^{(d)} \).

| \( r \) | 10  | 20  | 30  | 40  | 50  | 60  | 70  | 80  |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|
| Average of \( \hat{L}_2(r) \) | 0.755 | 0.322 | 0.184 | 0.123 | 0.095 | 0.073 | 0.056 | 0.046 |

As the limit random variable \( Z_2 \) in Theorem 4.3 we considered the random variable \( \tilde{Y}_{R,2}^{(c)} \), where \( R \) is sufficiently large. For simulations we used \( \kappa = 2 \) and \( h = 0.2 \). The random variables \( \tilde{Y}_{R,2}^{(c)} \) and \( Y_{r,2}^{(d)} \) were simulated 100 times for \( r = 40, 80, \ldots, 200 \), and \( R = 200 \). Using the simulated values, the sample mean square distance \( \hat{L}_2(200, r) \) between \( \tilde{Y}_{200,2}^{(c)} \) and \( Y_{r,2}^{(d)} \) was calculated for each \( r \). Figure 4 shows box plots of \( \hat{L}_2(200, r) \) as function of \( r \) and the corresponding box plots of the logarithms of \( \hat{L}_2(200, r) \) by repeating the above simulation steps 50 times. The sample averages of \( \hat{L}_2(200, r) \) for each \( r \) is listed in Table 2.

Figure 4a and Table 2 confirm that \( \hat{L}_2(200, r) \) approaches zero when \( r \) increases.

From Figure 4b one can see that the means form a declining slope, which suggests that the exact rate of convergence might be a power or even exponential function of \( r \). By fitting the linear regression model to \( \log(\hat{L}_2(200, r)) \) values we obtained the following log-transformed models:

\[
\log(\hat{L}_2(200, r)) \approx -3.43 - 2.019 \log(r) \quad \text{and} \quad \log(\hat{L}_2(200, r)) \approx -2.89 - 0.830r
\]

for the power and exponential cases respectively. The fitted models are shown in Figure 4b as the solid blue (power) and dashed green (exponential) lines.

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Figure 4: (a) Box plots of $\tilde{L}_2(200, r)$, (b) Box plots of $\log(\tilde{L}_2(200, r))$.

Table 2: Sample averages of values in the box plots from Figure 4a.

| $r$   | 40      | 80      | 120     | 160     | 200     |
|-------|---------|---------|---------|---------|---------|
| Average of $\tilde{L}_2(200, r)$ | 0.0846  | 0.0264  | 0.0122  | 0.0056  | 0.0043  |

7 Conclusion and Directions for Future Research

This paper discussed the asymptotic behaviour of additive and integral functionals of long-range dependent random fields over increasing observation windows. It is shown that both additive and integral functionals converge to the same non-Gaussian distribution. The results were obtained under rather general assumptions on the weight functions and random fields.

The main results in Sections 4.1 and 4.2 were obtained for random fields with a spectral singularity at the zero frequency. The simulations studies in Section 6 suggest to study the case of cyclic long-range dependent random fields that have a singularity at a non-zero frequency.

Simulation results in Section 6 suggest that the rate of convergence might be a power or exponential function of $r$. It would be interesting to obtain the exact rate of convergent for additive functionals using the approaches developed for integral functionals by Anh et al. (2019).

Furthermore, the results in this paper were obtained for functionals over increasing observation windows. It would be interesting to derive similar results for high frequency asymptotics where the observation window is the same but the sampling rate increases.

Also, it would be important to obtain similar results for the case of functionals of vector data, see Olenko and Omari (2019).
Acknowledgements

This research was partially supported under the Australian Research Council’s Discovery Project DP160101366. This research includes extensive simulation studies using the computational cluster Raijin of the National Computational Infrastructure (NCI), which is supported by the Australian Government and La Trobe University.

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