Quantum entanglement of anharmonic oscillators

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Abstract
We investigate the quantum entanglement dynamics of undriven anharmonic (nonlinear) oscillators with quartic potentials. We first consider the indirect interaction between two such nonlinear oscillators mediated by a third, linear oscillator and show that this leads to a time-varying entanglement of the oscillators, the entanglement being strongly influenced by the nonlinear oscillator dynamics. In the presence of dissipation, the role of nonlinearity is strongly manifested in the steady-state dynamics of the indirectly coupled anharmonic oscillators. We further illustrate the effect of nonlinearities by studying the coupling between an electromagnetic field in a cavity and one movable mirror which is modelled as a nonlinear oscillator. For this case, we present a full analytical treatment, which is valid in a regime where both the nonlinearity and the coupling due to radiation pressure are weak. We show that, without the need of any conditional measurements on the cavity field, the state of the movable mirror is non-classical as a result of the combined effect of the intrinsic nonlinearity and the radiation-pressure coupling. This interaction is also shown to be responsible for squeezing the movable mirror’s position quadrature beyond the minimum uncertainty state even when the mirror is initially prepared in its ground state.

((Some figures in this article are in colour only in the electronic version)

1. Introduction

The last decade has seen a surge of interest in investigating the quantum properties of micro- or nano-mechanical systems [1]. There has been much effort to prepare entangled or other non-classical states of such mechanical systems. With the advent of techniques such as laser cooling [2], these mechanical systems can be cooled sufficiently close to their ground states and hence tailored to mimic the physics of quantum harmonic oscillators to a very good approximation. There are many proposals which aim to investigate the dynamics of such mechanical systems operating deep in the quantum regime. These include coupling them to other quantum systems such as an ultracold atomic Bose–Einstein condensate (BEC) [3, 4], a Cooper pair box [5, 6] and even entangling two distant oscillators [5–8]. Often the main motivation is to test the foundations of quantum mechanics, but these nano-mechanical systems also turn out to be extremely good candidates for applications such as ultrasensitive measurement devices [9].

The physics of anharmonic oscillators has been studied in great detail by several authors both in the classical and quantum domain [10–13]. In particular, Milburn has investigated the quantum and classical dynamics of an anharmonic oscillator in the phase space [11] and has shown that decoherence-induced state reduction results in quantum to classical crossover in a nonlinear oscillator. Also in [13] a quantum master equation has been derived for a doubly clamped driven nonlinear beam.

Many of the existing schemes, which aim at exploring the quantum dynamics of nano-mechanical systems, treat them as harmonic oscillators. However, an anharmonic (nonlinear) oscillator in the quantum regime offers a number of intriguing new possibilities for quantum state preparation and
manipulation. One of many motivations for studying nonlinear oscillators is that by active cooling techniques, such as laser cooling, the thermal fluctuations of linear nano-mechanical systems can only be reduced to the standard quantum limit. If a reduction in noise is sought beyond this limit, then squeezing the quadratures of the nanomechanical oscillator is required. For this, one typically relies on nonlinearities. There already exist many feasible schemes that explore the possibility of squeezing the state of a mechanical oscillator [14–16]. Moreover, coherent nonlinear effects are of great interest as they turn out to be important resources for universal processing of quantum information with continuous variables [17, 18].

In the present work, however, we will concentrate on the influence of an intrinsic nonlinearity on the entangled states of two indirectly coupled oscillators. We address a situation where the anharmonic oscillators are coupled to an additional quantum system. First we investigate the quantum dynamics of two anharmonic oscillators interacting with a linear oscillator. We show that as a result of indirect interactions mediated by the linear oscillator, the two nonlinear oscillators exhibit a time-varying entanglement. Interestingly, we find that the effect of nonlinearity is much more pronounced for certain initial states. When dissipation is included, the effect of nonlinearity strongly governs the steady-state evolution of the indirectly coupled nonlinear oscillators.

As a second illustration of the effect of the nonlinearities, we investigate the unitary evolution of a cavity mode interacting with a movable mirror. We provide a full analytical treatment of a physical model that describes this interaction in a regime where both the nonlinearity and the coupling due to radiation pressure are weak. We show that unitary evolution results in time-dependent entanglement between the oscillator and the cavity mode. Moreover, under the joint action of radiation-pressure coupling and intrinsic nonlinearity, the movable mirror will also exhibit non-classical dynamics [19].

Nonlinear effects are typically small in nano-cantilevers, since the amplitudes of their oscillations are inevitably small compared to their lengths. Moreover, it is difficult to control the nonlinearities externally. In this paper, we propose to use an electromagnetic setup based on a Helmholtz coil configuration, where the nonlinearity stems from the fact that the energy due to the interaction between the magnetic field produced by the coils and permanent magnets at the tips of the cantilevers has a term that depends on the fourth power of the deflection of a tip from its equilibrium position. This allows us to externally tune the strength of the nonlinearity, which may be difficult in other realizations [15, 16].

The paper is organized as follows. In section 2, we introduce the theoretical model describing, in detail, the indirect interaction of two anharmonic oscillators mediated via a linear oscillator. In section 3, we investigate the coherent interaction between an anharmonic oscillator and a quantized cavity mode. In section 4, we briefly sketch a scheme to induce nonlinearities for the mechanical oscillator, and we present our conclusions in section 5.

2. Indirectly coupled anharmonic oscillators

In this section, we will explore the quantum dynamics of two anharmonic oscillators, which interact with the same linear oscillator. We will keep the theoretical treatment general at this point, but in section 4 we will discuss a potential realization of the required nonlinearities.

2.1. Unitary dynamics

Consider two identical micro- or nano-mechanical oscillators each of mass $m$ and operating in the quantum regime with the fundamental vibrational frequency $\omega_m$. Denoting the position and momentum operators of each oscillator by $\hat{q}_i$ and $\hat{p}_i$, where $i = 1, 2$, the free evolution of the oscillators is governed by the Hamiltonian

$$\hat{H} = \sum_{i=1}^{2} \left[ \frac{\hat{p}_i^2}{2m} + \frac{\omega_m^2 \hat{q}_i^2}{2} \right].$$

(1)

If we can modulate the potential seen by the oscillator such that there is an additional term proportional to $\hat{q}_i^4$, then this will introduce an effective nonlinearity for the mechanical oscillator. The Hamiltonian of two such independent anharmonic oscillators then takes the form

$$\hat{H}_1 = \sum_{i=1}^{2} \left[ \frac{\hat{p}_i^2}{2m} + \frac{\omega_m^2 \hat{q}_i^2}{2} + \beta \hat{q}_i^4 \right].$$

(2)

where $\beta \hat{q}_i^4$ is the nonlinear interaction energy. Expressing the position and momentum operators of each oscillator as

$$\hat{q}_1 = \sqrt{\frac{\hbar}{2 m \omega_m}} (\hat{a}^\dagger + \hat{a}); \quad \hat{p}_1 = i \sqrt{\frac{\hbar m \omega_m}{2}} (\hat{a}^\dagger - \hat{a})$$

$$\hat{q}_2 = \sqrt{\frac{\hbar}{2 m \omega_m}} (\hat{b}^\dagger + \hat{b}); \quad \hat{p}_2 = i \sqrt{\frac{\hbar m \omega_m}{2}} (\hat{b}^\dagger - \hat{b}),$$

where $\hat{a}^\dagger (\hat{a})$ and $\hat{b}^\dagger (\hat{b})$ are the creation (annihilation) operators for the vibron excitations of the two anharmonic oscillators, and neglecting all the counter-rotating terms, (2) takes the form

$$\tilde{\hat{H}} \hbar \approx \omega_m (\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}) + \beta (\hat{n}_a + \hat{n}_b) + \beta (\hat{n}_a^2 + \hat{n}_b^2).$$

(3)

where $\hat{n}_a$ and $\hat{n}_b$ are the number operators of the two anharmonic oscillators and $\beta$ is the nonlinearity strength. It is worth stressing that in general, for a driven nonlinear oscillator, the oscillation frequency depends on the driving amplitude, although a single resonance frequency can still be a valid approximation in the case of a very weak driving force. Moreover, in this work we consider a system of two undriven nonlinear oscillators, for which assigning a single resonance frequency seems to be a reasonable assumption. Nonetheless, depending on the initial excitation amplitude, the nonlinear oscillator might exhibit multistable behaviour. But as long as the initial average number of excitations $\langle \hat{n} \rangle$ of each oscillator is such that $\omega_m + \langle \hat{n} \rangle \beta \approx \omega_m$, the assumption of a single resonant frequency for each oscillator is still a reasonable

5. A nonlinear oscillator may have an amplitude-dependent oscillation frequency and exhibit, for instance quantum chaotic features. See, e.g., [20] and [21].
approximation. Keeping this in mind in the discussion to follow, we will restrict ourselves to low-excitation subspaces of each oscillator.

We are interested in the indirect interaction between the two nonlinear oscillators mediated by a linear oscillator with quantized energy levels equispaced by $\hbar \omega$. The indirect coupling is advantageous because it allows for accurate control of the interaction strength by manipulating the mediating oscillator, and consequently gives a handle on the quantum dynamics of the two nonlinear oscillators. The importance of the indirect interactions can be further appreciated in the dissipative regime. There, if the dissipation rate of the mediating oscillator is much faster than the thermal relaxation rates of the individual oscillators, then steady-state entangled states of the oscillators can be achieved. The linear oscillator is here assumed to be addressable by electromagnetic radiation created by excess charge or by nano-magnets at the tip of the oscillators, which produces an oscillating electromagnetic field [3]. Making the rotating wave and dipole approximations, the unitary evolution then corresponds to the Hamiltonian

$$\hat{H}_1 = \omega_N (\hat{\alpha}_1^\dagger \hat{\alpha}_1 + \hat{\beta}_1^\dagger \hat{\beta}_1) + \omega_N (\hat{\alpha}_1^\dagger \hat{\alpha}_1 + \hat{\beta}_1^\dagger \hat{\beta}_1) + \sqrt{\kappa} (\hat{\alpha}_1^\dagger \hat{\beta}_1 + \text{h.c.}),$$

(4)

where $\hat{\alpha}_1^\dagger$ and $\hat{\alpha}_1$ are respectively the creation and annihilation operators for the single quantized mode of the linear oscillator, which couples symmetrically—with coupling strength $\kappa$—to each of the nonlinear oscillators. The Hamiltonian (4) may, for instance, describe the coherent interaction of two anharmonic oscillators with an ultracold BEC [8]. In which case, in the limit of low atomic excitations, the creation and annihilation operators $\hat{\alpha}_1^\dagger$ and $\hat{\alpha}_1$, respectively, will be analogous to the collective atomic raising and lowering operators $\hat{J}_+^\dagger$, $\hat{J}_-$ [8, 22]. As shown in [8], the indirect coupling strength $\kappa$ between the two oscillators can be made to exceed the direct coupling $\kappa_{\text{direct}}$ between them. Moreover, the nonlinearity strength $\beta$ can also be made stronger than the direct coupling strength $\kappa_{\text{direct}}$ such that $\kappa_{\text{direct}} < \beta < \kappa$, for instance, by following the treatment in [8] and treating the two nano-cantilevers as anharmonic oscillators with a zero-point oscillation amplitude of 50 pm and with a ferromagnet with $10^6$ atoms on each cantilever tip. If the BEC cloud is treated as a linear oscillator trapped at a distance $d = 1 \mu m$ above the nano-cantilevers and contains $N = 10^4$ atoms in the trap centre, then $\kappa/\kappa_{\text{direct}} = 8$ and $\beta/\kappa_{\text{direct}} = 5$ (see section 4).

A general solution of (4) may be found, but since the Hamiltonian (4) conserves the total number of excitations, a significant simplification occurs. If the oscillators can be cooled near to their ground states, we can restrict ourselves to subspaces with few excitations. Furthermore, in order to simplify the analytical and numerical treatment, we will truncate the Hilbert space of the middle linear oscillator to its two lowest excitation subspaces. Previously we have found that this assumption results in a rescaling of the Rabi oscillations without altering the qualitative picture [8].

In the one-excitation subspace, the relevant basis states for the unitary dynamics governed by Hamiltonian (4) are $|1\rangle_a|0\rangle_b$, $|0\rangle_a|1\rangle_b$, and $|0\rangle_a|0\rangle_b$. Here $|1\rangle_a$ is a state where one of the anharmonic oscillators is in its first excited state, while the other nonlinear oscillator and the linear oscillator are in their ground states, and analogously for the other combinations. If we assume that the energy splitting of the linear oscillator can be brought in resonance with the oscillation frequency of the anharmonic oscillator, then in the interaction picture Hamiltonian (4) takes the form

$$\hat{H}_{\text{int}} = \hbar \beta (\hat{n}_a^2 + \hat{n}_a + \hat{n}_b^2 + \hat{n}_b) + \hbar \kappa (\hat{\alpha}_1^\dagger \hat{\beta}_1 + \text{h.c.}).$$

(5)

General initial states of the nonlinear–linear coupled oscillator system in the subspace of one excitation can be written as

$$|\Psi_1(t)\rangle = \sum_{j=0}^{1} C_j |1-j,0\rangle |1-j\rangle_a |j\rangle_b |0\rangle_c$$

(6)

In the one-excitation subspace, with the initial condition $|\Psi(0)\rangle = |1\rangle_a|0\rangle_b|0\rangle_c$, the time-evolved wavefunction due to the coherent interactions between the two anharmonic oscillators and the effective two-level system becomes

$$|\Psi_1(t)\rangle = \alpha_1(t)|1\rangle_a|0\rangle_b|0\rangle_c + \alpha_2(t)|0\rangle_a|1\rangle_b|0\rangle_c + \alpha_3(t)|0\rangle_a|0\rangle_b|1\rangle_c,$$

(7)

where

$$\alpha_1(t) = \left( \frac{1}{2} + \frac{e^{-i\beta t/2}}{2} \cos(K_1 t/2) - i\beta \frac{e^{-i\beta t/2}}{2K_1} \sin(K_1 t/2) \right),$$

$$\alpha_2(t) = \left( -\frac{1}{2} + \frac{e^{-i\beta t/2}}{2} \cos(K_1 t/2) - i\beta \frac{e^{-i\beta t/2}}{2K_1} \sin(K_1 t/2) \right),$$

$$\alpha_3(t) = -\frac{2i}{K_1} \frac{\kappa}{\beta} e^{-i\beta t/2} \sin(K_1 t/2/2)|0\rangle_a|0\rangle_b|1\rangle_c,$$

(8)

(9)

(10)

with $K_1 = \sqrt{\beta^2 + 8\kappa^2}$. In the limit $\beta \rightarrow 0$, we obtain

$$|\Psi_1(t)\rangle = \left( \frac{1}{2} + \cos(\sqrt{2} \kappa t) \right)|1\rangle_a|0\rangle_b|0\rangle_c$$

$$+ \left( -1 + \cos(\sqrt{2} \kappa t) \right)|0\rangle_a|1\rangle_b|0\rangle_c$$

$$- \frac{i}{\sqrt{2}} \sin(\sqrt{2} \kappa t)|0\rangle_a|0\rangle_b|1\rangle_c,$$

(11)

which coincides with the wavefunction that describes the dynamics of two linear oscillators interacting with an effective two-level system as discussed in [8]. It should be noted that the effect of the nonlinearity cannot be fully appreciated in a one-excitation subspace. In this case, the effect of the nonlinearity can be mimicked by making the two oscillators non-resonant with the mediating two-level system. Hence, to better understand the effect of the intrinsic nonlinearity on the quantum dynamics of each oscillator, we have to also study the dynamics of the system in the two- and three-excitation subspaces.

With the result of the unitary evolution for all three excitation subspaces in hand, we can now attempt to characterize the entanglement between the nano-cantilevers, and by doing so try to understand the influence of the inherent nonlinearities. The time-dependent state of the two
anharmonic oscillators is a mixed state found by tracing over the degrees of freedom of the linear oscillator. To quantify the entanglement in a bipartite system in an overall mixed state, we use the Peres criterion [23]. We compute the negativity for the reduced density matrix, defined as

$$N = \frac{1}{2} \left( \sum_{i=1}^{n} |\lambda_i| - 1 \right),$$

(12)

where $\sum_{i=1}^{n} |\lambda_i|$ is the sum of the absolute values of all the eigenvalues of the partially transposed reduced density matrix [24] of size $n$. A non-zero value of negativity $N$ ensures inseparability of a bipartite system.

As a result of the coherent exchange of excitation(s) between the two anharmonic oscillators mediated indirectly via the linear oscillator, the two nonlinear oscillators become entangled. As shown in figure 1, the system of two nonlinear oscillators exhibits time-varying entanglement, and at certain instants the entanglement is found to be maximal or nearly maximal in both excitation subspaces. For the sake of comparison, we also plot the negativity for two indirectly coupled linear oscillators. As can be seen from figure 1, for an initial state given by $|1\rangle_a|0\rangle_b|0\rangle_c$, the nonlinear oscillators exhibit stronger entanglement as compared to their linear counterparts. On the other hand, we find that for a symmetric initial state $|0\rangle_a|0\rangle_b|1\rangle_c$, a stronger nonlinearity strength $\beta$ leads to a more weakly entangled state of the two oscillators.

The time evolution of the entanglement for the indirectly coupled nonlinear oscillators in subspaces of higher excitations is shown in figures 2 and 3. As can be seen, in subspaces of two and three excitations, the effect of nonlinearity is clearly imprinted on the entangled state of the two oscillators. All these results indicate that the effect of nonlinearity is much more pronounced for certain initial states. The dynamics is becoming more complex in subspaces with higher excitations.

As mentioned before, the particular form of nonlinearity that we are interested in is clearly manifested in subspaces with higher excitations. As an illustration of this case, we plot the logarithmic negativity for an initial mixed state of two indirectly coupled nonlinear oscillators in figure 4. As expected, the degree of entanglement is reduced considerably as compared to the case of initially pure states. Furthermore, crucially depending on the initial state, a non-zero value of $\beta$ may or may not enhance quantum entanglement between the oscillators.

2.2. Dissipation of the oscillator

Every physical system is susceptible to dissipation. A more realistic approach will therefore take decoherence induced by an environment into account. Dissipation can either occur through the thermalization of the two anharmonic oscillators or through decay in the effective two-level system. As a first approximation we assume that the two nonlinear oscillators and the linear oscillator are coupled to independent zero-temperature heat baths with the coupling rates $\gamma_{a,b}$ and $\gamma_c$. 

**Figure 1.** Degree of entanglement, as measured by the negativity for $\beta/\kappa = 0$ (solid) and $\beta/\kappa = 0.5$ (dashed). The initial states are (a) $C_{1,0,0}(0) = 1$ and (b) $C_{0,0,1}(0) = 1$.

**Figure 2.** Degree of entanglement, as measured by the negativity for $\beta/\kappa = 0$ (solid) and $\beta/\kappa = 0.5$ (dashed). The initial states are (a) $C_{2,0,0}(0) = 1$ and (b) $C_{1,1,0}(0) = 1$. 
respectively, the effect of dissipation on their evolution—under the Born–Markov approximation—is well described by a Lindblad-type master equation of the form

\[ \frac{\partial}{\partial t} \rho = -\frac{i}{\hbar} [\hat{H}_{\text{int}}, \rho] + \mathcal{L}_a \rho + \mathcal{L}_b \rho + \mathcal{L}_c \rho. \] (13)

Here \( \rho \) is the density matrix of the system, and

\[ \mathcal{L}_a \rho \equiv \frac{\gamma_a}{2} (2\hat{a} \rho \hat{a}^\dagger - \hat{a}^\dagger \hat{a} \rho - \rho \hat{a}^\dagger \hat{a}), \] (14)

\[ \mathcal{L}_b \rho \equiv \frac{\gamma_b}{2} (2\hat{b} \rho \hat{b}^\dagger - \hat{b}^\dagger \hat{b} \rho - \rho \hat{b}^\dagger \hat{b}), \] (15)

\[ \mathcal{L}_c \rho \equiv \frac{\gamma_c}{2} (2\hat{c} \rho \hat{c}^\dagger - \hat{c}^\dagger \hat{c} \rho - \rho \hat{c}^\dagger \hat{c}). \] (16)

are the Lindblad operators representing the coupling of the two anharmonic oscillators and the linear oscillator to their independent zero-temperature heat baths.

A typical numerical solution of (13) in the one-excitation subspace is shown in figure 5. As can be seen from the figure,
the effect of the nonlinearity is clearly imprinted on the entangled state of the two oscillators even when they undergo dissipation. As for the case of unitary evolution, the effect of an inherent nonlinearity of the oscillator is much more pronounced for certain initial states.

The intrinsic nonlinearity of the two oscillators has another dramatic effect in the sense that it determines the dissipative dynamics of the coupled oscillators in higher excitation subspaces. To see this, we solve (13) in the two-excitation subspace. This time we only allow the mediating linear oscillator to undergo dissipation on the time scale of interest. A related problem has been studied in [8], where under similar conditions long-lived entangled states of two linear oscillators were achieved.

If $\beta = 0$, then $\hat{a} - \hat{b}$ is a constant of motion of Hamiltonian (5). Exploiting this fact, one can obtain steady states of the two linear oscillators which are entangled [8]. On the other hand, for nonlinear oscillators this does not hold true. Here, depending on the initial state, one may or may not see that a steady-state entangled state of the two nonlinear oscillators is developed. To prove this statement, a numerical solution of (13) is shown in figure 6. As can be seen, for an initial asymmetric state the steady state is an entangled state of the two nonlinear oscillators, while for an initial symmetric state the steady state is separable. These observations can thus also be used as an indirect signature of the state of the nonlinear oscillator.

As is clear from this section, the degree of inseparability of the two indirectly coupled oscillators has a non-trivial dependence on both the nonlinearity parameter $\beta$ and the initial state. Depending on the initial state, a non-zero value of nonlinearity strength $\beta$ can enhance or suppress the degree of entanglement of indirectly coupled oscillators. This interesting behaviour holds true both when the system undergoes unitary evolution and in the dissipative regime.

3. Interaction with a quantized cavity mode

In the previous section, we saw how an intrinsic nonlinearity can strongly affect the entanglement between the oscillators. Here we will discuss a second physical scenario where nonlinearities play a key role in the quantum dynamics. The first setup we have in mind is an anharmonic oscillator coupled to a mode of a Fabry–Pérot cavity with one fixed and one movable mirror.

In [25, 26], the problem of a cavity with a movable mirror has been discussed in great detail. In these studies, the movable mirror was treated as a simple harmonic oscillator. This work was purely analytical and showed that the coherent interaction of a movable mirror with the cavity mode generates various non-classical states of the cavity mode and the mirror.

Here we are interested in probing the quantum features of an anharmonic oscillator. We will model the movable mirror as a nonlinear oscillator with a nonlinearity proportional to $x^4$, where $x$ is the displacement of the mirror from its equilibrium position. In what follows we study the coherent interaction between a single quantized cavity mode and a nonlinear mirror coupled by the radiation pressure [27]. We derive a closed analytical expression for the time-evolved state of the cavity field and the movable anharmonic mirror, which is valid in the limit of weak nonlinearity and low radiation-pressure coupling.

If we assume that leakage of photons through the cavity can be neglected, then the main source of decoherence is the coupling of the mirror to its surroundings, which to some extent can also be avoided [28]. In what follows we therefore neglect dissipation and consider only unitary evolution of the system of the coupled cavity- and nonlinear-mirror system.

Consider a single quantized cavity mode with the creation and annihilation operators $\hat{k}$ and $\hat{k}^\dagger$, respectively, and the resonance frequency $\omega_k = 2\pi c/L$, where $L$ is the length of the cavity. We assume that the movable mirror has been cooled near to its ground state and thus operates in its quantum regime. Under the action of cavity photon-induced radiation pressure, the movable mirror will oscillate about its equilibrium position.

If we assume that the mirror moves a distance $x$ along the cavity axis such that the displacement is much smaller than the wavelength of the cavity mode in one cavity round-trip time, then the scattering of photons to other cavity modes can be safely neglected [27]. The length of the cavity then becomes $L + x$ so that the resonance frequency of the cavity is of the form $\omega_k^\prime = 2\pi c/(L + x)$. The Hamiltonian of the cavity can then be rewritten as

$$\hat{H}_{\text{cav}} = \hbar \omega_k^\prime \hat{k}^\dagger \hat{k} = 2\pi \hbar \frac{c}{L + x} \hat{k}^\dagger \hat{k},$$  \hspace{1cm} (17)
which, in a quantum description of the mirror motion, becomes
\begin{equation}
\hat{H}_{\text{av}} = \hbar \omega_0 \hat{k}^2 \hat{k} - \hbar g_k \hat{k}^2 \hat{a}(\hat{a}^\dagger + \hat{a}),
\end{equation}
where it is assumed that \( x/L \ll 1 \), and \( g_k = (\omega_0/L)\sqrt{\hbar/2\mu_0 \omega_m} \) is the radiation-pressure coupling constant between the nonlinear mirror and the cavity field. Thus, the unitary dynamics of the above physical system is governed by the Hamiltonian
\begin{equation}
\hat{H}_{\text{trans}} = e^{i\hat{H}_2 \hat{t}} e^{-i\hat{H}_2},
\end{equation}
where the unitary operator \( \hat{S} \) is given by
\begin{equation}
\hat{S} = -\frac{g_k}{\omega_0 + \beta} \hat{k} \hat{k}^\dagger (\hat{a}^\dagger - \hat{a}).
\end{equation}

Consequently, the operators \( \hat{a} \) and \( \hat{k} \) transform as
\begin{equation}
\hat{a} \rightarrow \hat{a} + \frac{g_k}{\omega_0 + \beta} \hat{k} \hat{k}^\dagger,
\end{equation}
\begin{equation}
\hat{k} \rightarrow \hat{k} \exp \left[ \frac{g_k}{\omega_0 + \beta} (\hat{a}^\dagger - \hat{a}) \right].
\end{equation}

Neglecting the counter-rotating terms, the transformed Hamiltonian in (20) becomes
\begin{equation}
\frac{\hat{H}_{\text{trans}}}{\hbar} = \omega_0 \hat{k}^2 \hat{k} + (\omega_0 + \beta) \hat{a}^\dagger \hat{a} - \frac{g_k^2 \omega_m}{(\omega_0 + \beta)^2} (\hat{k} \hat{k})^2 + \beta (\hat{a}^\dagger \hat{a})^2 + \frac{4g_k^2 \beta}{(\omega_0 + \beta)^2} (\hat{a}^\dagger \hat{a})(\hat{k} \hat{k}^\dagger)^2 + 2\beta \left( \frac{g_k}{\omega_0 + \beta} \right)^3 (\hat{k} \hat{k}^\dagger)^3 (\hat{a}^\dagger + \hat{a})^2 + \frac{g_k^4 \beta}{(\omega_0 + \beta)^4} (\hat{k} \hat{k}^\dagger)^4.
\end{equation}

To further simplify the analysis, we assume that both the nonlinearity and the radiation-pressure coupling are weak, so that quadratic and higher orders terms in \( g_k/\omega_0 + \beta \) can be neglected. This can be justified since a cavity of length \( L \sim 10^3 \) m and a movable mirror with oscillation frequency \( \omega_m \sim 10^6 \) Hz and zero-point oscillation amplitude 50 pm give \( g_k/\omega_m \sim 0.01 \). Thus, (24) reduces to
\begin{equation}
\frac{\hat{H}_{\text{trans}}}{\hbar} = \omega_0 \hat{k}^2 \hat{k} + (\omega_0 + \beta) \hat{a}^\dagger \hat{a} - \frac{g_k^2 \omega_m}{(\omega_0 + \beta)^2} (\hat{k} \hat{k})^2 + \beta (\hat{a}^\dagger \hat{a})^2.\end{equation}

It should be noted that \( \hat{n}_k = \hat{k} \hat{k} \) and \( \hat{n}_a = \hat{a}^\dagger \hat{a} \) are constants of motion since \( [\hat{H}_{\text{trans}}, \hat{n}_k] = [\hat{H}_{\text{trans}}, \hat{n}_a] = 0 \). The transformed unitary time-evolution operator corresponding to \( \hat{H}_{\text{trans}} \) takes the form
\begin{equation}
\hat{U}_{\text{trans}}(t) = \exp \left[ -i\omega_0 t \hat{k} \hat{k} + i \frac{g_k^2 \omega_m}{(\omega_0 + \beta)^2} t (\hat{k} \hat{k})^2 \right] \times \exp \left[ -i(\omega_0 + \beta) t \hat{a}^\dagger \hat{a} - i\beta t (\hat{a}^\dagger \hat{a})^2 \right].
\end{equation}

The corresponding untransformed time evolution operator is \( \hat{U}(t) = e^{-i\hat{H}_{\text{trans}}(t) \hat{t}} \). See the appendix for technical details regarding the exact form of \( \hat{U}(t) \).

Under the assumption of weak nonlinearity and low radiation-pressure coupling, \( \hat{U}(t) \) describes the undamped motion of an anharmonic oscillator interacting with a cavity mode. If we assume that both the cavity mode and the oscillator are prepared in coherent states with amplitudes \( \alpha \) and \( \eta \), respectively, then the state of the combined system evolves as
\begin{equation}
|\Psi(t)\rangle = \hat{U}(t) |\Psi(0)\rangle
\end{equation}
\begin{equation}
= \sum_{n=0}^{\infty} e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} e^{i(\frac{\alpha^2}{2} - \sin(\alpha + \eta) t)} x_{\text{trans}}(n, \eta) |\eta(t)\rangle_{\text{a}},
\end{equation}
where the state of the mirror is a mixture of Fock states given by
\begin{equation}
|\eta(t)\rangle_{\text{a}} = \sum_{m=0}^{\infty} [\eta(t)]^m e^{-|\eta(t)|^2/2} e^{-i\beta m} \frac{1}{\sqrt{m!}} |m\rangle_{\text{a}}.
\end{equation}

In the limit \( \beta \rightarrow 0 \), we retrieve the result obtained in [25, 26] where the mirror state reduces to a mixture of coherent states. It is worth noting that even in the weakly nonlinear regime, the effect of the nonlinearity is clearly imprinted on the state of the movable mirror which is now in a mixture of Fock states. Also evident from (28) is the inseparable state of the anharmonic oscillator and the cavity mode. As can be seen from (27), the anharmonic oscillator exhibits periodic entanglement with the cavity mode, except at certain instants where the state of the oscillator is completely separable from the cavity mode. This happens when \( (\omega_0 + \beta)t = 2q\pi \) for \( q \in \mathbb{N} \). The reduced density matrix of the state of the mirror is thus given by
\begin{equation}
\hat{\rho}_{\text{mirror}}(t) = \text{Tr}_k(|\Psi(t)\rangle \langle \Psi(t)|)
\end{equation}
\begin{equation}
= \sum_{m,q,n=0}^{\infty} e^{-|\alpha|^2/2} e^{-|\eta(t)|^2/2} e^{-i\beta m} \frac{1}{\sqrt{m!}} |m\rangle_{\text{a}} \langle q|.
\end{equation}

It is now a straightforward exercise to compute the Wigner function \( W(\lambda, \lambda^*) \) [29] of the mirror, which we plot in figure 7.

As can be seen from this figure, the negativity of the Wigner function \( W(\lambda, \lambda^*) \) clearly identifies the non-classical state of the mirror. It should be contrasted with the case of a linear oscillator interacting with a cavity mode. There the state of the mirror is a mixture of coherent states and always characterized by a positive Wigner function. The evolution of the mirror into a non-classical state is an effect of the combination of an intrinsic nonlinearity of the mirror and the radiation-pressure coupling with the cavity mode. This feature should be compared with results obtained in [26] where it has been shown that only a conditional measurement on the cavity mode can project the linear mirror into a non-classical state. One would expect the amplitude and the phase quadratures of the movable mirror to be influenced by the intrinsic nonlinearity in the mirror. In order to quantify this, we define two Hermitian
operators $\hat{Q}$ and $\hat{P}$, which correspond to the amplitude and phase quadratures of the movable mirror and are given by

$$\hat{Q}(t) = \langle \hat{a}^\dagger(t) + \hat{a}(t) \rangle,$$

$$\hat{P}(t) = i\langle \hat{a}^\dagger(t) - \hat{a}(t) \rangle.$$ \hspace{1cm} (31) (32)

The coherent interaction of the cavity with the anharmonic oscillator should be reflected in the variance of the quadratures $\hat{P}$ and $\hat{Q}$ defined by

$$\Delta P^2(t) = \langle \hat{P}^2(t) \rangle - \langle \hat{P}(t) \rangle^2,$$

$$\Delta \hat{Q}^2(t) = \langle \hat{Q}^2(t) \rangle - \langle \hat{Q}(t) \rangle^2.$$ \hspace{1cm} (33) (34)

We analytically solve for $\hat{P}^2(t)$ and $\hat{Q}^2(t)$ and plot the variance of the quadratures $\hat{P}$ and $\hat{Q}$ in figure 8. As can be seen there, the coherent interaction between a quantized cavity mode and an anharmonic oscillator induces a time-dependent squeezing in one of the mirror quadrature beyond the minimum uncertainty limit.

It is worth pointing out that the squeezing in the variance of the mirror quadratures beyond the minimum uncertainty limit is the result of a combined effect of the intrinsic nonlinearity and the radiation-pressure coupling. This can be understood from the fact that if the mirror is initially prepared in its vacuum state, then it is known that a nonlinearity of the form (3) alone cannot induce squeezing in the mirror quadratures \cite{15}. As a result of joint coherent interaction with the cavity mode and intrinsic nonlinearity, an initial vacuum state of the nonlinear mirror evolves into a mixture of Fock states which exhibit time-dependent squeezing beyond the minimum uncertainty limit.

4. Origin of the nonlinearities

The harmonic oscillator is often the result of an approximation of a more complicated potential landscape. Nonlinear force terms are often naturally present in many physical systems, but they are of higher order, hence small. Here we will outline a physical scheme for inducing nonlinearity of a mechanical oscillator in the form of a nano-cantilever\cite{6}, where the nonlinear quartic term appears as a lowest order approximation. In order to obtain a nonlinear oscillator, we propose to use a hybrid system which relies on the electromagnetic coupling between nano-magnets located at the tip of the cantilever and external magnetic fields. Consider a setup consisting of two identical circular magnetic coils of radii $R$ placed a distance $R/2$ apart, with their common axis along the $x$ direction. This Helmholtz coil configuration is known to produce a very uniform magnetic field near the centre. A nano-cantilever cooled near to its vibrational ground state and fabricated with a strong ferromagnet of magnetic moment $\vec{\mu}$ attached to its tip is placed at the centre of the Helmholtz coil setup.

\begin{itemize}
  \item We note that our analysis could equally well be applied to a doubly clamped resonator or a membrane.
\end{itemize}
The magnetic field experienced by the ferromagnet at the tip of the nano-cantilever, placed at the centre of the Helmholtz coils, is given by
\[
B(x) = \frac{\mu_0 I_{\text{trans}}}{2R} \left\{ \left[ 1 + \frac{(\frac{x}{R} - x)^2}{R^2} \right]^{-\frac{1}{2}} + \left[ 1 + \frac{(\frac{x}{R} + x)^2}{R^2} \right]^{-\frac{1}{2}} \right\},
\] (35)
where \(x\) is the displacement of the tip of the oscillator from the centre and \(I\) is the current in the pair of coils. Simplifying (35) for \(x/R \ll 1\), we obtain
\[
B(x) = 8\mu_0 I_{\text{trans}} \frac{\mu I}{5R\sqrt{5}} \left[ 1 - \frac{144}{125} \left( \frac{x}{R} \right)^4 \right].
\] (36)
Thus, the interaction energy of the ferromagnet is given by
\[
H_{\text{int}} = -\mu \cdot B(x) \approx 0.8\mu_0 I_{\text{trans}} \left( \frac{x}{R} \right)^4.
\] (37)
Representing the quantized motion of the oscillator in terms of the creation and annihilation operators \(\hat{a}^\dagger\) and \(\hat{a}\), respectively, the interaction Hamiltonian takes the form
\[
H_{\text{int}} = \hbar\beta(\hat{a}^\dagger + \hat{a})^4,
\] (38)
where \(\beta = 1.28\mu_0\mu_B N_{\text{mag}} a_0^4/\mu_0 R^5\), \(N_{\text{mag}}\) is the number of atoms in the ferromagnet and \(a_0\) is the zero-point amplitude of the nano-mechanical cantilever.

Using the physical setup described above, a nonlinearity of strength \(\beta\) can be induced in the nano-mechanical oscillator provided the zero-point motion of the cantilever can be made large (see [30] for a review of the present state-of-the-art manufacturing of nano-mechanical oscillators). For a set of parameters where \(R \sim 80\) nm, \(I \sim 1\) mA, \(N_{\text{mag}} \sim 10^6\) and \(a_0 \sim 50\) pm, one obtains a nonlinearity strength of the order of \(\beta \sim 250\) Hz, where we have neglected any finite-size effects stemming from the nano-magnet and coils.

5. Conclusions

We have studied the dynamics of anharmonic oscillators with a quartic nonlinearity in two different physical settings. We have described, in detail, the quantum evolution of two such anharmonic oscillators interacting indirectly via an effective two-level system. The two-level system could also, for example, be represented by some chosen collective excitations of a BEC. We have shown that indirect coherent interactions cause the two anharmonic oscillators to exhibit time-dependent entanglement. Inherent nonlinearities in the nano-mechanical systems are found to strongly influence the entangled state of the two oscillators. Interestingly, the effect of nonlinearity is much more pronounced for certain initial states. The signature of nonlinearity is clearly imprinted on the entangled state of the two anharmonic oscillators even when these oscillators are subject to decoherence. Nonlinearity also plays a crucial role in determining the steady-state evolution of the indirectly coupled harmonic oscillators.

The coupling strength between two oscillators can be characterized by the connectivity [12]. Connectivity as defined in [12] is the ratio of the coupling strength between the oscillators and the frequency difference between them, and diverges in the limit of identical oscillators. A high value of the connectivity corresponds to coherent exchange of excitations between the oscillators, which is desirable in order to operate in the strong coupling regime where coherent interactions supersede all the losses in the system. In the particular physical model studied here, however, we have found that a larger value of the coupling strength does not always guarantee a strongly entangled state of the two oscillators. The strength of these quantum correlations also depends on the nonlinearity parameter and the initial state distribution. In addition, a very large coupling strength also makes the rotating wave approximation questionable.

As a second illustration of the effect of nonlinearity, we have studied the coherent interaction between a single quantized cavity mode and a weakly nonlinear oscillator in the form of a movable mirror. In this case, we have been able to find an analytical solution for the unitary evolution of the state of such an anharmonic oscillator interacting with a single quantized cavity mode. In particular, we have shown that non-classical states of the mirror arise as a result of the combination of the radiation-pressure coupling and the intrinsic nonlinearity in the mirror. A non-classical state of the mirror can be generated both for initial ground and coherent states. Unlike in previously studied cases, non-classical states of the mirror can be generated without the need of conditional measurements [25, 26]. In addition, we have shown how squeezing appears in the variance of the quadratures beyond the minimum uncertainty state. It should be stressed that for an initial ground state of a single nonlinear mirror, no squeezing will be generated. Squeezing only occurs due to the interaction between the nonlinear mirror and the cavity field.

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Appendix. The evolution operator

The unitary operator \(\hat{U}\) in (21) which is used to transform the Hamiltonian \(\hat{H}_2\) in (25) gives the corresponding transformed time evolution operator
\[
\hat{U}_{\text{trans}}(t) = \exp \left[ -i\omega_m t \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} + i \frac{\omega_m}{2\xi} t (\hat{\mathbf{k}}^\dagger \hat{\mathbf{k}})^2 \right]
\times \exp \left[ -i \xi t \hat{\mathbf{a}}^\dagger \hat{\mathbf{a}} - i \beta t (\hat{\mathbf{a}}^\dagger \hat{\mathbf{a}})^2 \right],
\] (A.1)
where \(\xi = \omega_m + \beta\). The untransformed operator \(\hat{U}(t)\) then becomes
\[
\hat{U}(t) = e^{-\frac{i}{\hbar} \hat{U}_{\text{trans}}(t)} e^{\hat{\mathbf{S}}}
\]
\[
= \exp \left[ -i\omega_m t \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} + i \frac{\omega_m}{2\xi} t (\hat{\mathbf{k}}^\dagger \hat{\mathbf{k}})^2 \right]
\times \exp(\hat{\mathbf{S}}) \exp( -i \xi t \hat{\mathbf{a}}^\dagger \hat{\mathbf{a}} - i \beta t (\hat{\mathbf{a}}^\dagger \hat{\mathbf{a}})^2) \exp(\hat{\mathbf{S}}).\] (A.2)
Using the Baker–Campbell–Hausdorff expansion [29] together with making the rotating wave approximation, and also neglecting quadratic and higher order terms in $g_{c}/\zeta$, (A.2) simplifies to

$$\hat{U}(t) = \exp \left\{ -i \left[ \omega_k t \hat{k}^{\dagger} \hat{k} - \frac{g_k^2}{\zeta^2} [\omega_m t - \sin(\zeta t)] (\hat{k}^{\dagger} \hat{k})^2 ight. ight. \\
+ \left. \left. \beta t (\hat{a}^{\dagger} \hat{a})^2 \right] \exp \left[ \frac{g_k}{\zeta} \hat{k}^{\dagger} \hat{k} (\hat{a}^{\dagger} - \hat{a}) ight. \right. \\
- \left. \left. \frac{g_k}{\zeta} \hat{k}^{\dagger} \hat{k} (\hat{a}^{\dagger} e^{-i\zeta t} - \hat{a} e^{i\zeta t}) \right] \exp[-i\zeta t \hat{a}^{\dagger} \hat{a}]. \right\}$$

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