UNIFORMIZATION, $\partial$-BILIPSCHITZ MAPS, SPHERICALIZATION, AND INVERSION

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Abstract. We define $\partial$-biLipschitz homeomorphisms between uniform metric spaces and show that these maps are always quasimöbius. We also show that a homeomorphism being $\partial$-biLipschitz is equivalent to the map biLipschitz in the quasi hyperbolic metrics on these spaces. The proofs of these claims require us to uniformize the quasi hyperbolic metric. We further show that all admissible uniformizations of a Gromov hyperbolic space are quasimöbius to one another by the identity map, including those uniformizations that are based at a point of the Gromov boundary. Using the main results we then show that the sphericalization and inversion operations are compatible with uniformization of hyperbolic spaces in a natural sense.

1. Introduction

Our work in this paper has two principal goals. The primary goal is to connect together the bounded uniformizations of Gromov hyperbolic spaces defined by Bonk, Heinonen, and Koskela [2] and the unbounded uniformizations of Gromov hyperbolic spaces defined by the author [4]. We will show that these uniformizations are related by a quasimöbius homeomorphism, much like how the unit disk and upper half plane in $\mathbb{R}^2$ are related by a Möbius transformation. The secondary goal is to provide a useful local criterion for a homeomorphism between uniform metric spaces to be quasimöbius. This criterion follow from classical results in quasiconformal mapping theory in the Euclidean case, and has been generalized by Väisälä to the case of uniform domains in Banach spaces [12]. We take inspiration primarily from [12] in formulating this condition. In the process of proving the main results we establish several new results on unbounded uniform metric spaces that may be of independent interest.

We will state our results regarding the secondary goal first. Before proceeding further we will fix some notation for the rest of the paper. In general when we have positive functions $f, g : X \to (0, \infty)$ defined on a set $X$ we will write $f \asymp_C g$ for a constant $C \geq 1$ if the inequality $C^{-1}g \leq f \leq Cg$ holds on $X$. For a metric space $(\Omega, d)$ and $x \in \Omega$ we write $B_d(x, r) = \{ y \in \Omega : d(x, y) < r \}$ for the open ball of radius $r$ centered at $x$. For a subset $E \subset \Omega$ we write

$$\text{dist}(x, E) = \inf_{y \in E} d(x, y),$$

for the distance from a point to that set. When $\Omega$ is incomplete we write $\bar{\Omega}$ for the completion of $\Omega$ and $\partial \Omega = \bar{\Omega} \setminus \Omega$ for the complement of $\Omega$ in $\bar{\Omega}$, which we will refer to as the metric boundary of $\Omega$. We will continue to write $d$ for the canonical extension of the metric $d$ on $\Omega$ to the completion $\bar{\Omega}$. For $x \in \Omega$ we write $d_{\Omega}(x) = \text{dist}(x, \partial \Omega)$ for the distance of $x$ to the metric boundary of $\Omega$. 

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The key condition that we will consider involves a Lipschitz-type property in which we rescale distances in the domain and range by distance to the metric boundary before comparing them. A similar condition has previously been considered by Väisälä [12, Definition 7.2].

**Definition 1.1.** Let \((\Omega, d)\) and \((\Omega', d')\) be incomplete metric spaces. For a constants \(L \geq 0\) and \(0 < \lambda < 1\), a map \(f : (\Omega, d) \to (\Omega', d')\) is \(\partial\)-Lipschitz with data \((L, \lambda)\) if for each \(x \in \Omega\) we have for all \(y, z \in B_d(x, \lambda d_{\Omega}(x))\),

\[
\frac{d'(f(y), f(z))}{d'_{\Omega'}(f(x))} \leq L \frac{d(y, z)}{d_\Omega(x)}.
\]

We say that \(f\) is \(\partial\)-biLipschitz with data \((L, \lambda)\) if \(f\) is a homeomorphism, \(L \geq 1\), and both \(f\) and \(f^{-1}\) are \(\partial\)-Lipschitz with data \((L, \lambda)\).

We note that if \(f\) is \(\partial\)-biLipschitz with data \((L, \lambda)\) then applying (1.1) for \(f\) followed by \(f^{-1}\) yields that for \(x \in \Omega\) we have for \(y\) and \(z\) belonging to the smaller ball \(B_d(x, L^{-1} \lambda d_{\Omega}(x))\),

\[
\frac{d'(f(y), f(z))}{d'_{\Omega'}(f(x))} \leq L \frac{d(y, z)}{d_\Omega(x)}.
\]

We will consider these conditions in the context of uniform metric spaces, which we now define. We remark that, unlike in our previous work [11], in this paper we will be requiring that uniform metric spaces are locally compact. For a metric space \((\Omega, d)\) and a curve \(\gamma : I \to \Omega\) we write \(\ell(\gamma)\) for the length of \(\gamma\); if \(\ell(\gamma) < \infty\) then we say that \(\gamma\) is rectifiable. For an interval \(I \subset \mathbb{R}\) and \(t \in I\) we write \(I_{\leq t} = \{s \in I : s \leq t\}\) and \(I_{\geq t} = \{s \in I : s \leq t\}\).

**Definition 1.2.** Let \((\Omega, d)\) be an incomplete, locally compact metric space. For a constant \(A \geq 1\) and a compact interval \(I \subset \mathbb{R}\), a curve \(\gamma : I \to \Omega\) with endpoints \(x, y \in \Omega\) is \(A\)-uniform if

\[
\ell(\gamma) \leq Ad(x, y),
\]

and if for every \(t \in I\) we have

\[
\min\{\ell(\gamma|_{I_{\leq t}}), \ell(\gamma|_{I_{\geq t}})\} \leq Ad_{\Omega}(\gamma(t)).
\]

The metric space \(\Omega\) is \(A\)-uniform if any two points in \(\Omega\) can be joined by an \(A\)-uniform curve.

To state our first theorem we will also need to define quasimöbius homeomorphisms. Quasimöbius homeomorphisms were first introduced by Väisälä in [11]; we refer the reader there for a detailed treatment of these maps. The cross-ratio of a quadruple of distinct points \(x, y, z, w\) in a metric space \((\Omega, d)\) is defined by

\[
[x, y, z, w]_d = \frac{d(x, z)d(y, w)}{d(x, y)d(z, w)}.
\]

**Definition 1.3.** A homeomorphism \(f : (\Omega, d) \to (\Omega', d')\) between metric spaces is \(\theta\)-quasimöbius for a homeomorphism \(\theta : [0, \infty) \to [0, \infty)\) if for each quadruple of distinct points \(x, y, z, w \in \Omega\) we have

\[
[f(x), f(y), f(z), f(w)]_{d'} \leq \theta([x, y, z, w]_d).
\]
We will commonly refer to \( \theta \) as a control function. When the control function \( \theta \) does not need to be mentioned we simply say that \( f \) is quasimöbius. It is easy to see via the symmetries of the cross-ratio that inequality (1.6) implies the two-sided inequality

\[
\frac{1}{\theta([x, y, z, w]_d)} \leq [f(x), f(y), f(z), f(w)]_d \leq \theta([x, y, z, w]_d).
\]

We can now state our first theorem.

**Theorem 1.4.** Let \( f : (\Omega, d) \to (\Omega', d') \) be a homeomorphism between two \( A \)-uniform metric spaces that is \( \partial \)-biLipschitz with data \( (L, \lambda) \). Then \( f \) is \( \theta \)-quasimöbius with \( \theta \) depending only on \( A \) and \( L \).

Theorem 1.4 is useful because the \( \partial \)-biLipschitz condition is a local condition that is relatively straightforward to verify in many cases. However it is also somewhat difficult to work with in proofs. Our next theorem (which is used in the proof of Theorem 1.4) transforms the \( \partial \)-biLipschitz condition into another condition that is easier to work with.

We will say that a metric space is rectifiably connected if any two points \( x, y \in \Omega \) can be joined by a rectifiable curve. For an incomplete, rectifiably connected metric space \((\Omega, d)\) we define the quasihyperbolic metric on \( \Omega \) by, for \( x, y \in \Omega \),

\[
k(x, y) = \inf_{\gamma} \int_{\gamma} \frac{ds}{d_\Omega(\gamma(s))},
\]

where the infimum is taken over all rectifiable curves joining \( x \) to \( y \). The metric space \((\Omega, k)\) is then called the quasihyperbolization of the metric space \((\Omega, d)\). The quasihyperbolic metric has a rich history of use in problems in analysis and geometry when \((\Omega, d)\) is a domain in Euclidean space equipped with either the Euclidean metric or an internal path metric; we refer to [2, Chapter 1] for surveys of some of these applications as well as an overview of how they connect to the topics considered in this paper.

For a uniform metric space \((\Omega, d)\) the corresponding quasihyperbolization \((\Omega, k)\) is always a Gromov hyperbolic space [2, Theorem 3.6]. Gromov hyperbolic spaces will be formally defined later in this introduction; for now we observe that transferring problems to the setting of Gromov hyperbolic spaces unlocks a number of additional tools that can be used. For the next theorem we recall that a map \( f : (\Omega, d) \to (\Omega', d') \) between metric spaces is \( H \)-Lipschitz for a constant \( H \geq 0 \) if for all \( x, y \in \Omega \) we have \( d'(f(x), f(y)) \leq H d(x, y) \). The map \( f \) is \( H \)-biLipschitz if \( d'(f(x), f(y)) =_H d(x, y) \).

**Theorem 1.5.** Let \( f : (\Omega, d) \to (\Omega', d') \) be a continuous map between two \( A \)-uniform metric spaces that is \( \partial \)-Lipschitz with data \((L, \lambda)\). Then there is a constant \( H = H(A, L) \) such that the induced map \( f : (\Omega, k) \to (\Omega', k') \) between their quasihyperbolizations is \( H \)-Lipschitz.

Conversely, if \( f : (\Omega, d) \to (\Omega', d') \) is such that the induced map \( f : (\Omega, k) \to (\Omega', k') \) is \( H \)-Lipschitz for a given \( H \geq 0 \) then there is \( L = L(A, H) \) and \( \lambda = \lambda(A, H) \) such that \( f \) is \( \partial \)-Lipschitz with data \((L, \lambda)\).

Thus for homeomorphisms between uniform metric spaces the \( \partial \)-biLipschitz property is equivalent to the induced map between the quasihyperbolizations being biLipschitz. The proof of the first direction is a short calculation (which shows that we may in fact take \( H = A^2 L \)), while the proof of the other direction is significantly more involved as it makes use of uniformizations of the quasihyperbolizations of the spaces. Theorems 1.4 and 1.5 were inspired by a collection of similar results established by Väisälä in his study of mappings between uniform domains in Banach spaces [12, Section 11].
We move now to a discussion of our primary theorems, which concern uniformizations of Gromov hyperbolic spaces. We begin by recalling some definitions from our previous work [4]; we refer the reader to there for further details. For a continuous function \( \rho : X \to (0, \infty) \) we write
\[
\ell_\rho(\gamma) = \int_\gamma \rho \, ds,
\]
for the line integral of \( \rho \) along \( \gamma \). Such a positive continuous function \( \rho \) will be called a density on \( X \).

**Definition 1.6.** Let \((X, d)\) be a rectifiably connected metric space and let \( \rho : X \to (0, \infty) \) be a density on \( X \). The conformal deformation of \( X \) with conformal factor \( \rho \) is the metric space \( X_\rho = (X, d_\rho) \) with metric
\[
d_\rho(x, y) = \inf \ell_\rho(\gamma),
\]
with the infimum taken over all curves \( \gamma \) joining \( x \) to \( y \).

If \( X \) is geodesic then we say further that the density \( \rho \) is admissible for \( X \) with constant \( M \geq 1 \) if for any \( x, y \in X \) and any geodesic \( \gamma \) joining \( x \) to \( y \) we have
\[
\ell_\rho(\gamma) \leq Md_\rho(x, y).
\]
(1.9)

The quasihyperbolization \((\Omega, k)\) of an incomplete metric space \((\Omega, d)\) is an example of a conformal deformation with conformal factor \( \rho(x) = d_\Omega(x)^{-1} \).

A metric space \((X, d)\) is proper if its closed balls are compact. A geodesic \( \gamma : I \to X \) is a curve \( \gamma \) which is isometric as a mapping of the interval \( I \) into \( X \), or in other words, it satisfies \( d(\gamma(s), \gamma(t)) = |s - t| \) for all \( s, t \in I \). We say that \( X \) is geodesic if any two points in \( X \) can be joined by a geodesic. A geodesic triangle \( \Delta \) in \( X \) consists of three points \( x, y, z \in X \) together with geodesics joining these points to one another. Writing \( \Delta = \gamma_1 \cup \gamma_2 \cup \gamma_3 \) as a union of its edges, we say that \( \Delta \) is \( \delta \)-thin for a given \( \delta \geq 0 \) if for each point \( p \in \gamma_i \), \( i = 1, 2, 3 \), there is a point \( q \in \gamma_j \) with \( d(p, q) \leq \delta \) and \( i \neq j \). A geodesic metric space \( X \) is Gromov hyperbolic if there is a \( \delta \geq 0 \) such that all geodesic triangles in \( X \) are \( \delta \)-thin; in this case we will also say that \( X \) is \( \delta \)-hyperbolic.

For a proper geodesic \( \delta \)-hyperbolic space \( X \) its Gromov boundary \( \partial X \) is defined to be the set of all geodesic rays \( \gamma : [0, \infty) \to X \) up to the following equivalence relation: \( \gamma \sim \sigma \) if there is a constant \( c \geq 0 \) such that \( d(\gamma(t), \sigma(t)) \leq c \) for all \( t \geq 0 \). In this case we say that \( \gamma \) and \( \sigma \) are at bounded distance from each other. For a geodesic line \( \gamma : \mathbb{R} \to X \) its endpoints in the Gromov boundary \( \partial X \) are defined to be the equivalence classes \( \xi = [\gamma|_{[0, \infty)}] \) and \( \zeta = [\gamma|_{[0, \infty)}] \), where we write \( \tilde{\gamma}(t) = \gamma(-t) \) for \( \gamma \) with its reversed orientation. We then say that \( \gamma \) starts from \( \zeta \) and ends at \( \xi \).

We note that there is a formal conflict between the notation of \( \partial X \) for the Gromov boundary and the notation \( \partial X = X \setminus X \) for the metric boundary of \( X \). However when \( X \) is proper we always have \( X = \bar{X} \) so that the metric boundary is always empty. Thus there should be no ambiguity in the use of this notation.

We define rough starlikeness next.

**Definition 1.7.** Given \( K \geq 0 \) we say that \( X \) is \( K \)-roughly starlike from a point \( z \in X \) if for each \( x \in X \) there is a geodesic ray \( \gamma : [0, \infty) \to X \) with \( \gamma(0) = z \) and \( \text{dist}(x, \gamma) \leq K \).

We extend this notion to points of the Gromov boundary by saying that \( X \) is \( K \)-roughly starlike from a point \( \omega \in \partial X \) if for each \( x \in X \) there is a geodesic line \( \gamma : \mathbb{R} \to X \) starting from \( \omega \) with \( \text{dist}(x, \gamma) \leq K \).
Remark 1.8. The definition of $K$-rough starlikeness we give here corresponds to condition (1) of $K$-rough starlikeness defined in [4] Definition 2.3. For proper geodesic Gromov hyperbolic spaces condition (2) of that reference always holds, so these definitions are equivalent for proper spaces.

The Busemann function $b_\gamma : X \to \mathbb{R}$ associated to a geodesic ray $\gamma : [0, \infty) \to X$ is defined by the limit
\begin{equation}
(1.10) \quad b_\gamma(x) = \lim_{t \to \infty} d(\gamma(t), x) - t.
\end{equation}
The Busemann function $b_\gamma$ is a 1-Lipschitz function on $X$. We will generally omit $\gamma$ from the notation and write $b = b_\gamma$. We write
\begin{equation}
(1.11) \quad \mathcal{B}(X) = \{b_\gamma + s : \gamma \text{ a geodesic ray in } X, s \in \mathbb{R}\},
\end{equation}
for the set of all Busemann functions associated to geodesic rays in $X$ as well as all translates of these functions by additive constants, which we will also refer to as Busemann functions. For a point $\omega \in \partial X$ we say that a Busemann function $b \in \mathcal{B}(X)$ is based at $\omega$ if $b = b_\gamma + s$ for some $s \in \mathbb{R}$ and some geodesic ray $\gamma$ belonging to the equivalence class of $\omega$. Conversely, given such a Busemann function $b = b_\gamma + s$ we refer to the equivalence class $[\gamma] \in \partial X$ as its basepoint and write $\omega_b = [\gamma]$ for this basepoint.

For $z \in X$ we define $b_z(x) = d(x, z)$ to be the distance from $z$. We augment the set of Busemann functions with the set of translates of distance functions on $X$,
\begin{equation}
(1.12) \quad \mathcal{D}(X) = \{b_z + s : z \in X, s \in \mathbb{R}\}.
\end{equation}
For $b \in \mathcal{D}(X)$ with $b = b_z + s$ for some $z \in X$ and $s \in \mathbb{R}$ we then refer to $z$ as the basepoint of $b$ and will sometimes write $\omega_b = z$ in analogy to the case of Busemann functions. We write $\mathcal{B}(X) = \mathcal{D}(X) \cup \mathcal{B}(X)$. Then all functions $b \in \mathcal{B}(X)$ are 1-Lipschitz. For $\varepsilon > 0$ and $b \in \mathcal{B}(X)$ we define a density $\rho_{\varepsilon,b}(x) = e^{-\varepsilon b(x)}$ on $X$. We write $X_{\varepsilon,b} = X_{\rho_{\varepsilon,b}}$ for the conformal deformation of $X$ with conformal factor $\rho_{\varepsilon,b}$. In the case $b = b_z$ for some $z \in X$ we will also use the notation $\rho_{\varepsilon,z}(x) = e^{-\varepsilon d(x,z)}$ and write $X_{\varepsilon,z}$ for the corresponding conformal deformation of $X$.

By [4] Theorem 1.4, given $b \in \mathcal{B}(X)$ such that $X$ is $K$-roughly starlike from the basepoint $\omega_b$ of $b$ and given $\varepsilon > 0$ such that $\rho_{\varepsilon,b}$ is admissible for $X$ with constant $M$ then $X_{\varepsilon,b}$ is $A$-uniform with $A = A(\delta, K, \varepsilon, M)$ depending only on these parameters as well as the hyperbolicity parameter $\delta$. The space $X_{\varepsilon,b}$ is bounded when $b \in \mathcal{D}(X)$ and unbounded when $b \in \mathcal{B}(X)$. It is natural to inquire as to what extent the uniformization $X_{\varepsilon,b}$ depends on the choice of $\varepsilon > 0$ and $b \in \mathcal{B}(X)$. This is the topic of our next theorem.

Theorem 1.9. Let $X$ be a proper geodesic $\delta$-hyperbolic space. Let $\varepsilon, \varepsilon' > 0$ and $b, b' \in \mathcal{B}(X)$ be given such that $X$ is $K$-roughly starlike from the basepoints of both $b$ and $b'$ and both $\rho_{\varepsilon,b}$ and $\rho_{\varepsilon',b'}$ are admissible for $X$ with the same constant $M$. Then the homeomorphism $X_{\varepsilon,b} \to X_{\varepsilon',b'}$ induced by the identity map on $X$ is $\theta$-biLipschitz with data $(L, \lambda)$ depending only on $\delta, K, \varepsilon, \varepsilon'$, and $M$. Furthermore this map is $\theta$-quasimöbius with
\begin{equation}
\theta(t) = C \max\{t, t^{\frac{\varepsilon'}{\varepsilon}}\},
\end{equation}
where $C = C(\delta, K, \varepsilon, \varepsilon', M)$.

Note in particular that we obtain $\theta(t) = Ct$ in the case $\varepsilon = \varepsilon'$. Theorem 1.9 provides sharper control over the form of the control function $\theta$ than would be given by Theorem 1.4. The uniformizations $X_{\varepsilon,b}$ for $b \in \mathcal{D}(X)$ are a slight generalization of the uniformizations considered by Bonk, Heinonen, and Koskela in their work [2]; Theorem 1.9 thus relates
these uniformizations to the uniformizations associated to Busemann functions built by the author in [4].

We next consider inversion and sphericalization in the context of uniformization. We recall these notions as defined in the work of Buckley, Herron, and Xie [3], starting with inversion. For an incomplete metric space \( \Omega \) we fix a point \( p \in \Omega \). We define a function \( i^p \) on \( \Omega \) by

\[
i^p(x, y) = \frac{d(x, y)}{d(x, p)d(y, p)}
\]

This function may not define a metric on \( \Omega \), but it is 4-biLipschitz to a canonically defined metric \( d^p \) on \( \Omega \) by [3] Lemma 3.2. The inversion of \( \Omega \) about the point \( p \) is defined to be the metric space \( \Omega^p = (\Omega, d^p) \). Since \( p \in \partial \Omega \), \( p \) is not an isolated point of \( \Omega \) and therefore \( \Omega^p \) is unbounded by [3] Lemma 3.2 (a)].

We next describe the sphericalization construction in [3]. We will consider sphericalization as a special case of inversion, as described at the end of [3] Section 3.B]. We start with a metric space \((\Omega, d)\) with a fixed choice of point \( p \in \Omega \). We define an auxiliary metric space \( \hat{\Omega} = \Omega \cup_{p \to 0} [0, 1] \) by attaching the interval \([0, 1]\) to \( \Omega \) by identifying \( p \) with \( 0 \) and then giving the resulting space the metric \( d \) which restricts to \( d \) on \( \Omega \), restricts to the Euclidean metric on \([0, 1]\), and for \( x \in \Omega \), \( t \in [0, 1] \) is given by \( \hat{d}(x, t) = d(x, p) + t \). We then let \( \hat{d}^p \) be the metric on \( \hat{\Omega}\{1\} \) defined by taking the inversion of \( \Omega \) about the point \( 1 \in [0, 1] \) and write \( \hat{\Omega}^p = (\hat{\Omega}, \hat{d}^p) \) for the metric space resulting from restricting this metric to \( \Omega \). The metric space \( \hat{\Omega}^p \) is called the sphericalization of \( \Omega \) based at \( p \). The space \( \hat{\Omega}^p \) always satisfies \( \operatorname{diam} \hat{\Omega}^p \leq 1 \) by the discussion in [3] Section 3.B], and in particular is always bounded.

Since inversion produces an unbounded space from a bounded space and sphericalization produces a bounded space from an unbounded space, given a \( \delta \)-hyperbolic space \( X \) as in Theorem 1.9 one may ask whether the bounded uniformizations \( X_{\varepsilon,b} \) for \( b \in D(X) \) and the unbounded uniformizations \( X_{\varepsilon,b} \) for \( b \in B(X) \) can be related to one another by the operations of sphericalization and inversion. Using the work of [3] we will show below that this is indeed the case. To state the result properly we require another definition.

**Definition 1.10.** A homeomorphism \( f : (\Omega, d) \to (\Omega', d') \) between metric spaces is \( \theta \)-quasisymmetric for a homeomorphism \( \theta : [0, \infty) \to [0, \infty) \) if for each triple of distinct points \( x, y, z \in \Omega \) we have

\[
\frac{d'(f(x), f(y))}{d'(f(x), f(z))} \leq \theta \left( \frac{d(x, y)}{d(x, z)} \right).
\]

As with quasimöbius maps, we will call \( \theta \) a control function and will simply say that \( f \) is quasisymmetric if the control function does not need to be mentioned. Quasisymmetric maps are always quasimöbius [11] Theorem 3.2] (with the control function for the quasimöbius condition being quantitative in the control function for the quasisymmetry condition), but the reverse need not always be the case since quasisymmetric maps must take bounded sets to bounded sets [8 Proposition 10.8] while the same is not true of quasimöbius homeomorphisms.

In the statement of Theorem 1.11 below we will restrict to \( b \in D(X) \) of the form \( b_z(x) = d(x, z) \) for some \( z \in X \), and we will only consider Busemann functions \( b \in B(X) \) such that \( b(z) = 0 \). These restrictions are harmless normalizations, as any \( b \in D(X) \) or \( b \in B(X) \) will only differ from such a function by an additive constant. In the formulation of Theorem 1.11 we are implicitly using the fact that for \( z \in X \) the metric boundary \( \partial X_{\varepsilon,z} \) can be identified
with the Gromov boundary \(\partial X\) in a canonical fashion \([4]\) Theorem 1.6). In particular for \(b \in \mathcal{B}(X)\) with basepoint \(\omega \in \partial X\) we can consider \(\omega\) as a point of \(\partial X_{\epsilon,z}\).

**Theorem 1.11.** Let \(X\) be a proper geodesic \(\delta\)-hyperbolic space that is \(K\)-roughly starlike from a point \(z \in X\) and a point \(\omega \in \partial X\). Fix a Busemann function \(b\) based at \(\omega\) satisfying \(b(z) = 0\) and suppose that \(\epsilon > 0\) is chosen such that \(\rho_{\epsilon,z}\) and \(\rho_{\epsilon,b}\) are admissible with the same constant \(M\). Let \(X^\omega_{\epsilon,z}\) be the inversion of \(X_{\epsilon,z}\) about the point \(\omega \in \partial X_{\epsilon,z}\) and let \(X^z_{\epsilon,b}\) be the sphericalization of \(X_{\epsilon,b}\) based at \(z\).

Then the metric spaces \(X^\omega_{\epsilon,z}\) and \(X^z_{\epsilon,b}\) are each \(A\)-uniform with \(A' = A'(\delta, K, \epsilon, M)\) and the maps \(X_{\epsilon,b} \to X^\omega_{\epsilon,z}\) and \(X_{\epsilon,z} \to X^z_{\epsilon,b}\) induced by the identity map on \(X\) are each \(\partial\)-biLipschitz with data \((L, \lambda)\) and \(\theta\)-quasisymmetric with \(L, \lambda, \theta\) depending only on \(\delta, K, \epsilon, \epsilon, \) and \(M\).

The uniformity of the metric spaces \(X^\omega_{\epsilon,z}\) and \(X^z_{\epsilon,b}\) is a straightforward consequence of the fact shown in \([4]\) that sphericalization and inversion preserve uniformity of metric spaces. Theorem 1.4 immediately implies from the \(\partial\)-biLipschitz condition that the maps \(X_{\epsilon,b} \to X^\omega_{\epsilon,z}\) and \(X_{\epsilon,z} \to X^z_{\epsilon,b}\) are quasimöbius, but we are able to obtain with a small amount of additional work that they are actually quasisymmetric with quantitative control on the control function.

**Remark 1.12.** In \([2]\) Chapter 1] the term *quasisimilarity* is used for a notion that is both weaker and stronger than the property of being \(\partial\)-biLipschitz that we define in this paper. They define a homeomorphism \(f : (\Omega, d) \to (\Omega', d')\) of incomplete metric spaces to be a *quasisimilarity* with data \((\theta, L, \lambda)\) if \(f\) is \(\theta\)-quasisymmetric and for each \(x \in \Omega\) there is a constant \(c_x > 0\) such that whenever \(y, z \in B(x, \lambda d_\Omega(x))\) we have

\[
(1.14) \quad d'(f(y), f(z)) \asymp_L c_x d(y, z).
\]

The \(\partial\)-biLipschitz property corresponds to requiring that \(c_x = \frac{d'(\rho_{\epsilon,b}(x))}{d_\Omega(x)}\) (see the comparison (1.2)). With this tweak to the definition, Theorem 1.4 shows in many cases that the comparison (1.14) is sufficient on its own to deduce the quasisymmetry property, including in the case of maps between bounded uniform metric spaces considered in \([2]\) (see (2) of Remark 6.4).

Lastly, it is important to know when the hypotheses of Theorems 1.9 and 1.11 are actually satisfied. The key hypotheses are the rough starlikeness condition and the admissibility of the density \(\rho_{\epsilon,b}\). For rough starlikeness it turns out that rough starlikeness from a single point of \(X\) is sufficient to obtain rough starlikeness from all points of \(X \cup \partial X\), provided that \(\partial X\) contains at least two points.

**Proposition 1.13.** Let \(X\) be a proper geodesic \(\delta\)-hyperbolic space such that \(\partial X\) contains at least two points, and suppose that \(X\) is \(K\)-roughly starlike from a point \(x \in X \cup \partial X\). Then there is a constant \(K' \geq 0\) such that \(X\) is \(K'\)-roughly starlike from all points of \(X \cup \partial X\).

Proposition 1.13 shows that we need not worry about the dependence on the basepoint in the rough starlikeness hypothesis. The claim is false when \(\partial X\) consists of a single point; see Example 2.8. However in this case \(X\) is roughly isometric to the Euclidean half-line \([0, \infty)\) by Proposition 2.9 and most claims of interest can be verified by direct argument.

The nature of the dependence of \(K'\) on the other parameters is summarized in Proposition 2.0. We remark that \(K'\) is not always quantitative solely in \(\delta\) and \(K\), as can be seen in Example 2.7.
Regarding the question of admissibility, we will make use of the following important theorem due to Bonk, Heinonen, and Koskela; we note that the version in the reference is somewhat more general. This is known as a Gehring-Hayman-type theorem, after the namesakes’ corresponding result in the context of Euclidean and hyperbolic metrics on hyperbolic domains in the complex plane [6].

**Theorem 1.14.** [2, Theorem 5.1] Let \((X, d)\) be a geodesic \(\delta\)-hyperbolic space. There is \(\varepsilon_0 = \varepsilon_0(\delta) > 0\) depending only on \(\delta\) such that if a continuous function \(\rho : X \to (0, \infty)\) satisfies for all \(x, y \in X\) and some fixed \(0 < \varepsilon \leq \varepsilon_0\),

\[
e^{-\varepsilon d(x, y)} \leq \frac{\rho(x)}{\rho(y)} \leq e^{\varepsilon d(x, y)},
\]

then \(\rho\) is admissible for \(X\) with constant \(M = 20\).

The inequality (1.15) is known as a Harnack type inequality. For a given \(\varepsilon > 0\) and any \(b \in \hat{B}(X)\) the function \(\rho_{\varepsilon, b}\) satisfies (1.15) inequality since \(b\) is 1-Lipschitz. By Theorem 1.14 admissibility is a non-issue once \(\varepsilon\) is sufficiently small. There are cases in which one must consider values of \(\varepsilon\) larger than the \(\varepsilon_0\) given above, however. Such values of \(\varepsilon\) appear naturally when considering uniformizations of hyperbolic fillings of metric spaces [1], [4].

Combining Proposition 1.13 and Theorem 1.14 we see that, for a given proper geodesic \(\delta\)-hyperbolic space \(X\) (with \(\partial X\) having at least two points) that is \(K\)-roughly starlike from some point of \(X \cup \partial X\), there is always a \(K' \geq 0\) and an \(\varepsilon_0 = \varepsilon_0(\delta)\) such that for any \(b \in \hat{B}(X)\) and any \(0 < \varepsilon \leq \varepsilon_0\) we have that \(X\) is \(K'\)-roughly starlike from the basepoint of \(b\) and the density \(\rho_{\varepsilon, b}\) is admissible for \(X\) with constant \(M = 20\). Thus Theorems 1.9 and 1.11 can be applied freely in this range.

Proposition 1.13 raises the interesting question of whether a similar phenomenon also holds for admissibility of the densities \(\rho_{\varepsilon, b}\) for \(\varepsilon > 0, b \in \hat{B}(X)\).

**Question 1.15.** Suppose that \(X\) is a proper geodesic \(\delta\)-hyperbolic space and suppose that \(\varepsilon > 0\) and \(b \in \hat{B}(X)\) are such that \(\rho_{\varepsilon, b}\) is admissible for \(X\) with constant \(M\). Is there a constant \(M'\) such that for all \(b' \in \hat{B}(X)\) the density \(\rho_{\varepsilon, b'}\) is admissible for \(X\) with constant \(M'\)?

It may be natural to also assume that \(X\) is \(K\)-roughly starlike from \(b\) for some \(K \geq 0\) in Question 1.15. Theorem 1.14 shows that this question has a positive answer once \(\varepsilon\) is sufficiently small. This question is in a sense complementary to a recent theorem of Lindquist and Shanmugalingam [10] that concerns the invariance of the uniformity property of \(X_{\varepsilon, z}\) for a given \(\varepsilon > 0, z \in X\) under rough isometries of Gromov hyperbolic spaces. Another interesting question to consider is whether their theorem also holds for uniformizations by Busemann functions instead.

In Section 2 we review some basic facts about Gromov hyperbolic spaces and recall some results from our previous paper [4]. We also prove Proposition 1.13. In Section 3 we study the quasihyperbolic metric on unbounded uniform metric spaces and prove several claims that will be required later in the paper. Section 4 is devoted to uniformizing the quasihyperbolic metric on a uniform metric space. Section 5 considers what happens if we quasihyperbolize a uniformization of a Gromov hyperbolic space. The results in Section 5 are not required in the rest of the paper. Lastly in Section 6 we complete the proofs of the main theorems.
2. Hyperbolic metric spaces

In this section we recall some standard results regarding Gromov hyperbolic spaces, as well as some facts regarding Busemann functions established in our previous paper. We will also prove Proposition 1.2. The first parts of this section are closely modeled on the corresponding section of our previous paper [4, Section 2], but with appropriate simplifications now that we are assuming our Gromov hyperbolic spaces are proper. The material in this section can be found in any standard reference on Gromov hyperbolic spaces, for instance [5], [7].

2.1. Definitions. Let \( X \) be a set and let \( f, g \) be real-valued functions defined on \( X \). For \( c \geq 0 \) we will write \( f \approx_c g \) if

\[
|f(x) - g(x)| \leq c,
\]

for all \( x \in X \). If the exact value of the constant \( c \) is not important or implied by context we will often just write \( f \approx g \). The relation \( f \approx g \) will sometimes be referred to as a rough equality between \( f \) and \( g \). We will generally stick to the convention throughout this paper of using \( c \geq 0 \) for additive constants and \( C \geq 1 \) for multiplicative constants. To indicate on what parameters – such as \( \delta \) – the constants depend on we will write \( c = c(\delta) \), etc.

Let \( f : (X, d) \to (X', d') \) be a map between metric spaces. We say that \( f \) is isometric if \( d'(f(x), f(y)) = d(x, y) \) for \( x, y \in X \). For a constant \( c \geq 0 \) we say that \( f \) is \( c \)-roughly isometric if \( d(f(x), f(y)) \approx_c d(x, y) \) for \( x, y \in X \). As usual we will omit the constants in these terms when we don’t require them. We recall that a curve \( \gamma : I \to X \) is a geodesic if it is an isometric mapping of the interval \( I \subset \mathbb{R} \) into \( X \).

When dealing with Gromov hyperbolic spaces \( X \) in this paper we will in many cases use the generic distance notation \( |xy| := d(x, y) \) for the distance between \( x \) and \( y \) in \( X \) and the generic notation \( xy \) for a geodesic connecting two points \( x, y \in X \), even when this geodesic is not unique. We recall that a geodesic triangle \( \Delta \) in \( X \) is a collection of three points \( x, y, z \in X \) together with geodesics \( xy, xz, \) and \( yz \) joining these points and serving as the edges of this triangle. We will sometimes alternatively write \( xyz = \Delta \) for a geodesic triangle with vertices \( x, y \) and \( z \).

The Gromov boundary \( \partial X \) of a proper geodesic \( \delta \)-hyperbolic space \( X \) is defined to be the collection of all geodesic rays \( \gamma : [0, \infty) \to X \) up to the equivalence relation of two rays being equivalent if they are at a bounded distance from one another. Using the Arzelà-Ascoli theorem it is easy to see in a proper geodesic \( \delta \)-hyperbolic space that for any points \( x, y \in X \cup \partial X \) there is a geodesic \( \gamma \) joining \( x \) to \( y \). We will continue to write \( xy \) for any such choice of geodesic joining \( x \) to \( y \). We will allow our geodesic triangles \( \Delta \) to have vertices on \( \partial X \), in which case we will still write \( \Delta = xyz \) if \( \Delta \) has vertices \( x, y, z \). We remark that geodesic triangles with vertices in \( X \cup \partial X \) are \( 10\delta \)-thin by [4, Lemma 2.2].

We next state a useful fact regarding Busemann functions. Let \( X \) be a proper geodesic \( \delta \)-hyperbolic space and let \( b : X \to \mathbb{R} \) be a Busemann function based at some point \( \omega \in \partial X \). By [4, Lemma 2.5], if \( b' \) is any other Busemann function based at \( \omega \) then there is a constant \( s \in \mathbb{R} \) such that

\[
2.1 \quad b \approx_{72\delta} b' + s,
\]

with \( s = 0 \) if the geodesic rays associated to \( b \) and \( b' \) have the same starting point. Thus all Busemann functions based at \( \omega \) differ from each other by an additive constant, up to an additive error of \( 72\delta \).
2.2. Gromov products. For $x, y, z \in X$ the Gromov product of $x$ and $y$ based at $z$ is defined by

\[(x|y)_z = \frac{1}{2}(|xz| + |yz| - |xy|).\]

We can also take the basepoint of the Gromov product to be any function $b \in \hat{B}(X)$. For $b \in \hat{B}(X)$ the Gromov product based at $b$ is defined by

\[(x|y)_b = \frac{1}{2}(b(x) + b(y) - |xy|).\]

For $b \in D(X)$, $b(x) = d(x, z) + s$ this reduces to the notion of Gromov product in (2.2), as we have $(x|y)_b = (x|y)_z + s$.

The following statements briefly summarize a more extensive discussion of Gromov products in [4, Section 2]. In particular we give details there of how the precise forms of these statements follow from the corresponding statements in the literature. For these statements it is useful to conceive of the Gromov boundary in an alternative way using Gromov products. Fix $z \in X$. A sequence $\{x_n\} \subset X$ converges to infinity if $(x_n|x_n)_z \to \infty$ as $m, n \to \infty$. Two sequences $\{x_n\}$ and $\{y_n\}$ are equivalent if $(x_n|y_n)_z \to \infty$. These notions do not depend on the choice of basepoint $z$, as can easily be checked by the triangle inequality. For a proper geodesic $\delta$-hyperbolic space $X$ the set of equivalence classes of sequences converging to infinity gives an equivalent definition of the Gromov boundary $\partial X$, with the equivalence being given by sending a geodesic ray $\gamma : [0, \infty) \to X$ to the sequence $\{\gamma(n)\}$. For $\xi \in \partial X$ and a sequence $\{x_n\}$ that converges to infinity we will $\{x_n\} \in \xi$ if $\{x_n\}$ belongs to the equivalence class of $\xi$.

These notions may be extended to Busemann functions $b \in B(X)$ based at a given point $\omega \in \partial X$ [5, Chapter 3]. As above a sequence $\{x_n\}$ converges to infinity with respect to $\omega$ if $(x_n|x_n)_b \to \infty$ as $m, n \to \infty$, and two sequences $\{x_n\}$ and $\{y_n\}$ are equivalent with respect to $\omega$ if $(x_n|y_n)_b \to \infty$ as $n \to \infty$. These definitions do not depend on the choice of Busemann function based at $\omega$ by (2.1). The Gromov boundary relative to $\omega$ is defined to be the set $\partial_\omega X$ of all equivalence classes of sequences converging to infinity with respect to $\omega$. By [5, Proposition 3.4.1] we have a canonical identification of $\partial_\omega X$ with the complement $\partial X \setminus \{\omega\}$ of $\omega$ in the Gromov boundary $\partial X$. We will thus use the notation $\partial_\omega X = \partial X \setminus \{\omega\}$ throughout the rest of the paper. We extend this notation to $\omega \in X$ by setting $\partial_\omega X = \partial X$ in this case.

Gromov products based at functions $b \in \hat{B}(X)$ can be extended to points of $\partial X$ by defining the Gromov product of equivalence classes $\xi, \zeta \in \partial X$ based at $b$ to be

\[
(\xi|\zeta)_b = \inf \liminf_{n \to \infty} (x_n|y_n)_b,
\]

with the infimum taken over all sequences $\{x_n\} \in \xi$, $\{y_n\} \in \zeta$; if $b \in B(X)$ has basepoint $\omega$ then we leave this expression undefined when $\xi = \zeta = \omega$. As a consequence of [5, Lemma 2.2.2], [5, Lemma 3.2.4], and the discussion in [4, Section 2.2], for any choices of sequences $\{x_n\} \in \xi$ and $\{y_n\} \in \zeta$ we have

\[
(\xi|\zeta)_b \leq \liminf_{n \to \infty} (x_n|y_n)_b \leq \limsup_{n \to \infty} (x_n|y_n)_b \leq (\xi|\zeta)_b + c(\delta),
\]

with the constant $c(\delta)$ depending only on $\delta$. One may take $c(\delta) = 8\delta$ for $b \in D(X)$ and $c(\delta) = 600\delta$ for $b \in B(X)$. For $x \in X$ and $\xi \in \partial X$ the Gromov product based at $b$ is defined analogously as

\[
(x|\xi)_b = \inf \liminf_{n \to \infty} (x|x_n)_b,
\]
and the analogous inequality (2.4) holds with the same constants.

For a given $z \in X$ and $b \in D(X)$ defined by $b(x) = |xz|$, we will also write $(\xi|\zeta)_z = (\xi|\zeta)_b$ for $\xi, \zeta \in X \cup \partial X$. We remark that, with the extended definition (2.3), a sequence $\{x_n\}$ belongs to the equivalence class of $\xi \in \partial X$ if and only if $(x_n|\xi)_z \to \infty$ for some (hence any) $z \in X$.

2.3. Visual metrics. Let $X$ be a proper geodesic $\delta$-hyperbolic space. Gromov products based at $b \in B(X)$ can be used to define visual metrics on the Gromov boundary $\partial X$. We refer to [5] Chapters 2-3 as well as [4] Section 2.3 for precise details on this topic. We will summarize the results we need here. For $b \in B(X)$ we let $\omega = \omega_b$ denote the basepoint of $b$. We recall that we write $\partial_b X = \partial X$ when $\omega \in X$ and $\partial_b X = \partial X \setminus \{\omega\}$ when $\omega \in \partial X$.

For $b \in B(X)$ and $q > 0$ we define for $\xi, \zeta \in \partial_b X$,

$$\alpha_{b,q}(\xi, \zeta) = e^{-q(\xi|\zeta)_b}.$$  

This may not define a metric on $\partial_b X$, since the triangle inequality may not hold. However there is always $q_0 = q_0(\delta) > 0$ depending only on $\delta$ such that for $0 < q \leq q_0$ the function $\alpha_{b,q}$ is $4$-biLipschitz to a metric $\alpha$ on $\partial_b X$. We refer to $\alpha$ as a visual metric on $\partial_b X$ based at $b$ and refer to $q$ as the parameter of $\alpha$. We give $\partial X$ the topology associated to a visual metric based at $b$ for any $b \in D(X)$. When equipped with a visual metric $\partial_b X$ is a locally compact metric space. It is always a compact metric space when $b \in D(X)$, or when $b \in B(X)$ and $\omega$ is an isolated point of $\partial X$.

For any visual metrics $\alpha$ and $\alpha'$ on $\partial X$ based at $b, b' \in B(X)$ with parameters $q, q' > 0$ respectively, the identity map on $\partial X$ induces a $\theta$-quasimöbius homeomorphism,

$$(\partial X, \alpha) \to (\partial X, \alpha'),$$

with $\theta$ of the form $\theta(t) = C \left( \frac{\delta}{q} \right)^{\frac{q'}{q}}$ [5 Corollary 5.2.9]; this result is stated in the reference for the special case $q = q'$, but the general case can be deduced immediately from the observation that for a metric space $(\Omega, d)$ and any $0 < a \leq 1$ the identity map $(\Omega, d) \to (\Omega, a^n$) is always $\theta$-quasimöbius with $\theta(t) = t^a$.

2.4. Tripod maps. We let $\Upsilon$ be the tripod geodesic metric space composed of three copies $L_1, L_2$, and $L_3$ of the closed half-line $[0, \infty)$ identified at $0$. We denote this identification point by $o$ and will refer to $o$ as the core of the tripod $\Upsilon$. The space $\Upsilon$ is $0$-hyperbolic and its Gromov boundary $\partial \Upsilon$ consists of three points $\zeta_i, i = 1, 2, 3$, corresponding to the half-lines $L_i$ thought of as geodesic rays starting from $o$. We write $b_\Upsilon := b_{L_1}$ for the Busemann function associated to the half-line $L_1$ thought of as a geodesic ray in $\Upsilon$. A quick calculation shows that $b_\Upsilon$ is given by $b_\Upsilon(s) = -s$ for $s \in L_1$ and $b_\Upsilon(s) = s$ for $s \in L_2$ or $s \in L_3$.

Throughout this section we let $X$ be a proper geodesic $\delta$-hyperbolic space and let $\Delta = xyz$ be a geodesic triangle in $X$ with vertices $x, y, z \in X \cup \partial X$. Let $(\tilde{x}, \tilde{y}, \tilde{z})$ be an ordered triple of points with $\tilde{x} \in yz, \tilde{y} \in xy, \tilde{z} \in xy$, such that $|\tilde{x}y| = |\tilde{z}y|, |\tilde{x}z| = |\tilde{y}z|$, and $|\tilde{x}x| = |\tilde{y}y|$ (we allow some of these distances to be infinite, in which case the equalities becomes trivial). The tripod map $T : \Delta \to \Upsilon$ associated to such a tripod is the unique 1-Lipschitz map characterized by the properties that $T(\tilde{x}) = T(\tilde{y}) = T(\tilde{z}) = o$, that $T$ maps $xy$ isometrically into $L_2 \cup L_3$, $xz$ isometrically into $L_1 \cup L_3$, $yz$ isometrically into $L_1 \cup L_2$, and $T(x) \in L_1, T(y) \in L_2, T(z) \in L_3$. When $x \in \partial X$ this final inclusion should instead be understood as $\partial T(x) = \zeta_1$, recalling that $\zeta_1 \in \partial \Upsilon$ is the point defined by $L_1$, and similarly for $y$ and $z$.

A tripod map associated to $\Delta$ is by definition a $400\delta$-roughly isometric map $T : \Delta \to \Upsilon$ that is a tripod map associated to some triple of points in $\Delta$. By [4] Proposition 3.7 such a tripod map exists for every geodesic triangle in $X$ that has vertices in $X \cup \partial X$. Since we
will use the existence of this tripod map very frequently throughout this paper, we often will not refer back to this section when we use it. For a tripod map $T : \Delta \to \Upsilon$ associated to a geodesic triangle $\Delta = xyz$ the equiradial points for $T$ are by definition the three points $\hat{x} \in yz$, $\hat{y} \in zx$, $\hat{z} \in xy$ used to construct $T$, which can be equivalently thought of as the preimages of the core $o, T^{-1}(o) = \{\hat{x}, \hat{y}, \hat{z}\}$.

The following proposition computes Busemann functions $b \in \mathcal{B}(X)$ on geodesic triangles $\Delta$ in $X$ that have the basepoint of $b$ as one of their vertices. In general we will not being giving explicit constants in our claims here since we do not provide explicit constants in our final results.

**Proposition 2.1.** [4, Proposition 3.9] Let $\Delta = \omega xy$ be a geodesic triangle in $X$ with $\omega \in \partial X$ and $x, y \in X \cup \partial_0 X$. Let $b$ be a Busemann function based at $\omega$. Let $T : \Delta \to \Upsilon$ be a tripod map associated to $\Delta$. Then

$$b(p) \doteq c(\delta) b_T(T(p)) + (x|y)_b,$$

and therefore

$$\inf_{p \in x\gamma} b(p).$$

The following direct consequence of Proposition 2.1 will be more useful for computing in practice.

**Proposition 2.2.** Let $\Delta = \omega x_1 x_2$ be a geodesic triangle in $X$ with $\omega \in \partial X$ and $x_1, x_2 \in X \cup \partial_0 X$. Let $b$ be a Busemann function based at $\omega$. Let $T : \Delta \to \Upsilon$ be a tripod map associated to $\Delta$. Then there are parametrizations $\gamma_i : (-\infty, a_i] \to X$ of $\omega x_i$, $a_i \in [0, \infty]$, $i = 1, 2$, and $\sigma : I \to X$ of $x_1 x_2$ with $0 \in I$ such that the following properties hold,

1. The equiradial points of $T$ on $\gamma_1, \gamma_2$, and $\sigma$ are $\gamma_1(0), \gamma_2(0)$, and $\sigma(0)$ respectively. In particular \(\text{diam}\{\gamma_1(0), \gamma_2(0), \sigma(0)\} \leq c(\delta)\).
2. For $t \leq 0$ we have $|\gamma_1(t)| \gamma_2(t)| \leq c(\delta)$.
3. For $t \in I < 0$ we have $|\gamma_1(-t)\sigma(t)| \leq c(\delta)$ and for $t \in I > 0$ we have $|\gamma_2(t)\sigma(t)| \leq c(\delta)$.
4. For $t \in (-\infty, a_1]$ we have $b(\gamma_1(t)) \approx c(\delta) t + (x_1 x_2)_b$.
5. For $t \in I$ we have $b(\sigma(t)) \approx c(\delta) |t| + (x_1 x_2)_b$.

**Proof.** We identify $L_1 \cup L_2$ and $L_1 \cup L_3$ with $\mathbb{R}$ by reversing the orientation of $L_1$ and identifying it with $(-\infty, 0]$. We then take $(-\infty, a_i]$ to be the image under $T$ of $\omega x_i$ for $i = 1, 2$ and take $\gamma_i : (-\infty, a_i] \to X$ to be the inverse of $T$ restricted to the edge $\omega x_i$. Similarly for the edge $x_1 x_2$ we identify $L_2 \cup L_3$ with $\mathbb{R}$ by reversing the orientation of $L_2$ and identifying it with $(-\infty, 0]$ in this case. We set $I$ to be the image of $x_1 x_2$ under $T$ and define $\sigma : I \to X$ to be the inverse of $T$ restricted to $x_1 x_2$. Properties (1)-(5) of these parametrizations can then be verified directly from the conclusions of Proposition 2.1 and the fact that $T$ is $400\delta$-roughly isometric.

We’ll need a variant of Proposition 2.1 when we consider $\omega$ as a point of $X$ instead.

**Lemma 2.3.** Let $\Delta = xyz$ be a geodesic triangle in $X$ with $x \in X$ and let $T : \Delta \to \Upsilon$ be a tripod map associated to $\Delta$. Let $\hat{y} \in xy$ and $\hat{z} \in xz$ be the points such that $T(\hat{y}) = T(\hat{z}) = o$. Then

$$|xy| \doteq c(\delta) (y|z)_x \doteq c(\delta) |xz|,$$

and consequently,

$$\inf_{\gamma \in x\gamma} b(p).$$

$$|y|_x \doteq c(\delta) \text{dist}(x, yz).$$
Proof. Let $T : \Delta \to \Upsilon$ be a tripod map associated to $\Delta$. By (2.4) we can find $y' \in \bar{y}y$, $z' \in \bar{z}z$ such that $(y'|z')_x \approx_{c(\delta)} (y|z)_x$. Then $(T(y')|T(z'))_{T(x)} \approx_{c(\delta)} (y'|z')_x$. Since $T(y') \in L_2$ and $T(z') \in L_3$, a quick calculation shows that

$$
(T(y')|T(z'))_{T(x)} = |T(x)o| \approx_{c(\delta)} |xy| \approx_{c(\delta)} |xz|.
$$

This gives the first assertion. The second follows from the combined observations that $T$ is $c(\delta)$-roughly isometric, that $o \in T(yz) \subset L_2 \cup L_3$, and that $|T(x)o| = \text{dist}(T(x), L_2 \cup L_3)$. □

Inequality (2.9) actually holds with $c(\delta) = 8\delta$, see [2, (3.2)].

Lastly we record the following useful inequality regarding geodesic rays. The lemma below rephrases the conclusions of [4, Lemma 3.6].

**Lemma 2.4.** Let $\gamma, \sigma : [0, \infty) \to X$ be geodesic rays with the same endpoint in $\partial X$. Then for all $t \geq 0$ we have

$$
|\gamma(t)\sigma(t)| \leq 3|\gamma(0)\sigma(0)| + 8\delta.
$$

2.5. **Rough starlikeness.** In this section we will prove Proposition 1.13. For a proper geodesic $\delta$-hyperbolic space $X$ and any $x \in X$ we define

$$
S(x) = \sup_{\omega \in \partial X} \inf_{\xi \in \partial X} (\omega|\xi)_x.
$$

**Lemma 2.5.** We have $S(x) < \infty$ if $\partial X$ contains at least two points.

Proof. Let $\alpha$ be a visual metric on $\partial X$ that is biLipschitz to $\alpha_{r,q}(\omega, \xi) = e^{-q(|\omega|\xi)}$ for some small enough parameter $q > 0$. With this metric $\partial X$ is a compact metric space. We define

$$
S_\alpha(x) = \sup_{\omega \in \partial X} \inf_{\xi \in \partial X} -q^{-1} \log \alpha(\omega, \xi).
$$

Then $S_\alpha(x) \approx_x S(x)$, with the additive constant $c$ depending only on $\delta$ and $q$. Thus $S(x) < \infty$ if and only if $S_\alpha(x) < \infty$. The quantity $S_\alpha(x)$ is finite if and only if

$$
S_\alpha^*(x) = \inf_{\omega \in \partial X} \sup_{\xi \in \partial X} \alpha(\omega, \xi) > 0.
$$

We prove by contradiction that $S_\alpha^*(x) > 0$. If this does not hold then we can find sequences $\{\omega_n\}, \{\xi_n\} \subset \partial X$ such that $\alpha(\omega_n, \xi_n) \to 0$ as $n \to \infty$ and

$$
(2.10) \quad \alpha(\omega_n, \xi_n) \geq \alpha(\omega_n, \zeta) - \frac{1}{n},
$$

for all $\zeta \in \partial X$. Since $\partial X$ is compact we can assume, by passing to subsequences if necessary, that there are points $\omega, \xi \in \partial X$ such that $\omega_n \to \omega$ and $\xi_n \to \xi$. The condition $\alpha(\omega_n, \xi_n) \to 0$ then forces $\omega = \xi$.

Choose $r > 0$ small enough that the ball $B_\alpha(\omega, 2r)$ in the metric space $(\partial X, \alpha)$ has nonempty complement, which is possible since $\partial X$ has more than one point. For $n$ large enough we have that both $\omega_n$ and $\xi_n$ belong to the smaller ball $B_\alpha(\omega, r)$, so that $\alpha(\omega_n, \xi_n) < 2r$. But since $\partial X \setminus B_\alpha(\omega, 2r)$ is nonempty, it follows immediately that inequality (2.10) fails when $n$ is large enough that $\frac{1}{n} < r$ and $\omega_n, \xi_n \in B_\alpha(\omega, r)$ since there is some $\zeta \in \partial X$ such that $\alpha(\omega, \zeta) \geq 2r$. This gives the desired contradiction. □

The next proposition summarizes the relations between rough starlikeness from points of $X$ and rough starlikeness from points of $\partial X$.

**Proposition 2.6.** Let $X$ be a proper geodesic $\delta$-hyperbolic space such that $\partial X$ contains at least two points. Let $K \geq 0$ be given. Then
(1) If $X$ is $K$-roughly starlike from $\omega \in \partial X$ then it is $(K + 10\delta)$-roughly starlike from any point $x \in X \cup \partial X$.

(2) If $X$ is $K$-roughly starlike from $x \in X$ then it is $(K + S(x) + c(\delta))$-roughly starlike from any point $\omega \in \partial X$.

Proof. We first assume that $X$ is $K$-roughly starlike from a point $\omega \in \partial X$ and let $x \in X \cup \partial X$ be given; we may assume that $x \neq \omega$ as otherwise the claim is trivial. Let $p \in X$ and let $\xi \in \partial X$ be such that $\text{dist}(p, \xi \omega) \leq K$ for some choice of geodesic $\xi \omega$ from $\xi$ to $\omega$. We may assume that $\xi \neq x$, as when $x = \xi$ we immediately obtain the desired estimate.

Let $y \in \xi \omega$ satisfy $|py| \leq K$. We form a geodesic triangle $\Delta = x\xi\omega$ that includes the geodesic $\xi \omega$ containing $y$. Since this triangle is $10\delta$-thin by the discussion in Section 2.1, we can find $z \in x\xi \cup x\omega$ such that $|yz| \leq 10\delta$, from which it follows that $|pz| \leq K + 10\delta$. Regardless of whether $z \in x\xi$ or $z \in x\omega$, we obtain the necessary estimate to conclude that $X$ is $K$-roughly starlike from $x$.

Now assume that $X$ is $K$-roughly starlike from a point $x \in X$ and let $\omega \in \partial X$ be given. Since we assumed that $\partial X$ has at least two points, by Lemma 2.3 we then have $S(x) < \infty$. Let $p \in X$ be given. Let $x\xi$ denote a geodesic ray from $x$ to a point $\xi \in \partial X$ such that $\text{dist}(p, x\xi) \leq K$. We will assume for now that $\xi \neq \omega$. We choose a point $z \in \partial X$ such that $$(\omega|\zeta)_x \leq S(x) + \delta,$$
and consider additional geodesics $x\zeta$ and $\omega\zeta$. We consider two geodesic triangles: a triangle $\Delta_1 = x\xi\omega$ containing the geodesic ray $x\xi$ that has $\omega$ as a vertex and a triangle $\Delta_2 = x\zeta\omega$ that shares the edge $x\omega$ with $\Delta_1$. We let $T_1: \Delta_1 \to Y$ be tripod maps associated to each of these triangles for $i = 1, 2$.

We let $u \in x\xi$ be the equiradial point for $T_1$ on this edge of $\Delta_1$. Let $y$ be a point on $x\xi$ such that $|py| \leq K$. If $y \neq u\xi$ then the fact that $T_1$ is $c(\delta)$-roughly isometric implies that we can find $z \in \omega\xi$ such that $|yz| \leq c(\delta)$.

Then $|zp| \leq K + c(\delta)$, so $\text{dist}(p, \omega\xi) \leq K + c(\delta)$. This implies our desired estimate.

On the other hand, if $y \in xu$ then we let $w_1 \in x\omega$ be the equiradial point for $T_1$ on this edge and let $z \in xu_1$ be such that $|xy| = |xz|$. Since $T_1$ is $c(\delta)$-roughly isometric we then have $|yz| \leq c(\delta)$. We now let $w_2 \in x\omega$ and $s \in \omega\zeta$ be the equiradial points of $T_2$ on these edges. Then by Lemma 2.3 we have

$$|xw_2| \leq c(\delta) (\omega|\zeta)_x \leq S(x) + \delta.$$ 

Thus if $z \in xw_2$ then it follows that

$$|ps| \leq |py| + |yz| + |zs|$$
$$\leq K + c(\delta) + |zs|$$
$$\leq K + |zw_2| + c(\delta)$$
$$\leq K + |xw_2| + c(\delta)$$
$$\leq K + S(x) + c(\delta).$$

This gives the necessary estimate for $p$ since $s \in \omega\zeta$. If $z \in w_2\omega$ instead then since $T_2$ is $c(\delta)$-roughly isometric we can find some $t \in \omega\zeta$ such that $|zt| \leq c(\delta)$. Then

$$|pt| \leq |py| + |yz| + |zt| \leq K + c(\delta).$$

This gives the desired estimate in this case as well. Combining all of these claims together, we conclude that $X$ is $(K + S(x) + c(\delta))$-roughly starlike from $\omega$. 
In the case $\xi = \omega$ we can skip the part of this argument concerning the triangle $\Delta_1$ and just apply all of our arguments to the triangle $\Delta_2$ formed by $x\omega$, $x\zeta$, and $\omega\zeta$, where the geodesic $x\omega$ is now chosen such that $\operatorname{dist}(p, x\omega) \leq K$. We obtain from these arguments the same conclusion as above that $\operatorname{dist}(p, \omega\zeta) \leq K' + S(x) + c(\delta)$. □

Proposition 1.13 follows immediately from Proposition 2.6, since for $x \in \partial X$ it is an immediate consequence of (1) and for $x \in X$ we can apply (2) first to get some $K_0 \geq 0$ and $\omega \in \partial X$ such that $X$ is $K_0$-roughly starlike from $\omega$, and then we can use (1) to conclude that $X$ is $K'$-roughly starlike from all points of $X \cup \partial X$ with $K' = K_0 + 10\delta$.

Simple examples show that the dependence on $S(x)$ cannot be removed in (2) of Proposition 2.6 and that this proposition fails if $\partial X$ consists of a single point, even if we ignore the fact that it is impossible for $X$ to be roughly starlike from that single point of $\partial X$. These examples are given below.

Example 2.7. Let $n \in \mathbb{N}$ be given and let $X_n = \mathbb{R} \cup_{0 \sim 0} [0, n]$ be the geodesic metric space obtained by identifying $0 \in \mathbb{R}$ with $0 \in [0, n]$. Let $p$ denote the origin in $\mathbb{R}$ considered as a point of $X_n$ and let $x$ denote the point $n$ in $[0, n]$ considered as a point of $X_n$. Then $X_n$ is 0-hyperbolic and 0-roughly starlike from $x$. It has two points $\xi_i$, $i = 1, 2$ in its Gromov boundary corresponding to the half-lines $[0, \infty)$ and $(-\infty, 0]$ in $\mathbb{R}$, which are the only geodesic rays in $X_n$ starting from $p$. Furthermore the only geodesic line in $X_n$ is the isometrically embedded copy of $\mathbb{R}$ to which we glued the interval $[0, n]$. Thus $X_n$ is $n$-roughly starlike from $p$ and from either point of $\partial X_n$, but is not $K$-roughly starlike for any $K < n$ from these same points. Note in this example that $(\xi_1, \xi_2)_p = n$.

Example 2.8. Let $X$ be the "half-line with teeth", constructed by starting with the half-line $L = [0, \infty)$ and for each $n \in \mathbb{N}$ gluing a copy $I_n$ of the interval $[0, 1]$ to $L$ by identifying $0 \in I_n$ with $n \in L$. Then $X$ is proper, geodesic, 0-hyperbolic, and clearly 1-roughly starlike from the point $x \in X$ corresponding to the origin $0 \in L$. The Gromov boundary $\partial X = \{\omega\}$ can be identified with the equivalence class of the geodesic ray $\gamma : [0, \infty) \to X$ parametrizing $L$ starting from 0. For each $n \in \mathbb{N}$ we let $x_n \in X$ be the point corresponding to 1 in $I_n$. Then the only geodesic ray from $x_n$ to $\omega$ is given by $I_n \cup [n, \infty)$; this geodesic ray is a distance $n$ from $x$. Thus $X$ is $n$-roughly starlike from $x_n$, but is not $K$-roughly starlike from $x_n$ for any $K < n$.

In the case that $\partial X$ consists of a single point there are often simpler direct arguments available when $X$ is roughly starlike from one of its points. In this case $X$ can be thought of as the half-line $[0, \infty)$ up to a bounded error, as is shown below.

Proposition 2.9. Let $X$ be a proper geodesic $\delta$-hyperbolic space such that $X$ is $K$-roughly starlike from a point $x \in X$ and $\partial X = \{\omega\}$ consists of a single point. Then for any geodesic ray $x\omega$ we have $\operatorname{dist}(p, x\omega) \leq K$ for all $p \in X$ with $K' = K + 8\delta$.

Proof. Let $\gamma, \sigma : [0, \infty) \to X$ be two geodesic rays starting at $x$. By Lemma 2.7 we have $|\gamma(t) - \sigma(t)| \leq 8\delta$ for all $t \geq 0$. The proposition follows from this inequality since any point $p \in X$ is within distance $K$ of some geodesic ray starting from $x$ by the $K$-rough starlikeness assumption from $x$. □

2.6. Uniformization estimates. We summarize in this section the estimates we will need from our previous work regarding uniformization of Gromov hyperbolic spaces. We let $X$ be a proper $\delta$-hyperbolic geodesic metric space, let $b \in B(X)$ be such that $X$ is $K$-roughly starlike from the basepoint $\omega$ of $b$, and suppose that $\varepsilon > 0$ is given such that $\rho_{\varepsilon, b}$ is admissible for $X$ with constant $M$. We write $d_{\varepsilon, b}$ for the metric on $X_{\varepsilon, b}$. 
Since $b$ is 1-Lipschitz we have the Harnack type inequality for $x, y \in X$,

$$e^{-\varepsilon |xy|} \leq \frac{\rho_{\varepsilon,b}(x)}{\rho_{\varepsilon,b}(y)} \leq e^{\varepsilon |xy|}.$$  

Integrating this inequality over a curve joining $x$ to $y$ gives the following inequality for $x, y \in X$ \[\text{Lemma 4.4}],

$$M^{-1} \rho_{\varepsilon,b}(x) e^{-\varepsilon(1 - e^{-\varepsilon|xy|})} \leq d_{\varepsilon,b}(x,y) \leq \rho_{\varepsilon,b}(x) e^{-\varepsilon(1 - e^{-\varepsilon|xy|} - 1)}.$$  

This inequality is stated for $b \in B(X)$ in \[\text{Lemma 4.5}], but the proof only uses inequality (2.11) so this inequality also holds for $b \in B(X)$.

**Lemma 2.10.** For $x, y \in X \cup \partial_\omega X$ we have

$$d_{\varepsilon,b}(x,y) \preceq e^{-\varepsilon(x|y)\kappa} \min\{1, |xy|\},$$  

with $C = C(\delta, K, \varepsilon, M)$.

**Proof.** By \[\text{Lemma 4.5} the comparison

$$d_{\varepsilon,b}(x,y) \preceq e^{-\varepsilon(x|y)\kappa} \min\{1, \varepsilon|xy|\},$$

holds for $b \in B(X)$ and any $x, y \in X$ with $C = C(\delta, K, \varepsilon, M)$. One may then easily verify the comparison for $\kappa \geq 0$ and $\varepsilon > 0$,

$$\min\{1, \varepsilon\kappa\} \preceq C(\varepsilon) \min\{1, \kappa\},$$

with $C(\varepsilon) = \max\{\varepsilon, \varepsilon^{-1}\}$. Combining the comparisons (2.14) and (2.15) gives (2.13) for $b \in B(X)$ and $x, y \in X$. Using (2.14) then extends this comparison to $x, y \in X \cup \partial_\omega X$.

To obtain the estimate (2.13) for $b \in D(X)$ we let $\omega \in X$ denote the basepoint of $b$ and construct the ray augmentation $Y = X \cup \omega \sim [0, \infty)$ of $X$ based at $\omega$, which is a proper geodesic $\delta$-hyperbolic space obtained by gluing the half-line $[0, \infty)$ to $X$ by attaching it at the point $\omega$. We refer to \[\text{Definition 4.12} for more precise details. We let $\hat{b} : Y \to \mathbb{R}$ be the Busemann function associated to the half-line $[0, \infty)$ we glued to $X$, thought of as a geodesic ray in $Y$, and let $\hat{\omega}$ denote the basepoint of $\hat{b}$ in $\partial Y$. By the claims \[\text{Lemmas 4.13-4.16} we have that $\hat{b}|_X = b$, that $Y$ is $K$-roughly starlike from $\hat{\omega}$, that $\rho_{\varepsilon,\hat{b}}$ is admissible for $Y$ with the same constant $M$, that the embedding $X_{\varepsilon,b} \to Y_{\varepsilon,\hat{b}}$ is isometric, and that $\partial X = \partial_\varepsilon Y$. The desired inequality (2.13) then follows from the corresponding inequality on $Y_{\varepsilon,\hat{b}}$.  

\[\text{\hspace{1cm}}\]

We write $d_{\varepsilon,b}(x) = d_{X_{\varepsilon,b}}(x)$ for $x \in X$. We then have the following fundamental estimate \[\text{Proposition 4.7].}

**Proposition 2.11.** For $x \in X$ we have

$$d_{\varepsilon,b}(x) \preceq \rho_{\varepsilon,b}(x),$$  

with $C = C(\delta, K, \varepsilon, M)$.

We have absorbed the factor $\varepsilon^{-1}$ on the right in the reference into the constant $C$. This inequality is only stated for $b \in B(X)$ in \[\text{Lemma 4.6}, but the same ray augmentation argument as was given in the proof of Lemma 2.10 shows that this inequality holds for $b \in D(X)$ as well.
We will need to use the following claim in the proof of Theorem 1.11. This claim formalizes the intuition that the basepoint \( \omega \) of \( b \) serves as the ideal point at infinity for the uniformization \( X_{\varepsilon,b} \).

**Lemma 2.12.** Suppose that \( b \in B(X) \), let \( \omega \) be the basepoint of \( b \), and let \( z \in X \) be given. Then for a sequence \( \{x_n\} \subset X \) we have \( d_{\varepsilon,b}(z,x_n) \to \infty \) if and only if \( (x_n|\omega)_z \to \infty \).

**Proof.** By Proposition 1.13 and Theorem 1.14 we can find \( \varepsilon_0 > 0 \) and \( K_0 \geq 0 \) such that \( X \) is \( K_0 \)-roughly starlike from \( z \) and \( \rho_{\varepsilon_0,z} \) is an admissible density for \( X \) with constant \( M = 20 \). As remarked in [2, Remark 4.14(b)], the completion \( \bar{X}_{\varepsilon_0,z} \) of \( X_{\varepsilon_0,z} \) is compact and a neighborhood basis of the point \( \omega \in \partial X_{\varepsilon_0,z} \) in this topology is given by

\[
N_{\omega,\lambda} = \{ x \in X \cup \partial X : (x|\omega)_z \geq \lambda \},
\]

for \( \lambda \geq 0 \).

We first suppose that \( d_{\varepsilon,b}(z,x_n) \to \infty \). When we consider the sequence \( \{x_n\} \) in \( \bar{X}_{\varepsilon_0,z} \) it must have at least one limit point since \( X_{\varepsilon_0,z} \) is compact. If \( \xi \in \bar{X}_{\varepsilon_0,z}\setminus\{\omega\} \) is a limit point of \( \{x_n\} \) then there will be a subsequence \( \{y_n\} \) that is bounded above by \( 2d_{\varepsilon,b}(z,\xi) \), a contradiction. Thus \( \omega \) is the only possible limit point of \( \{x_n\} \) in \( \bar{X}_{\varepsilon_0,z} \), so we must have \( x_n \to \omega \) in \( \bar{X}_{\varepsilon_0,z} \). The description (2.17) of the neighborhood basis at \( \omega \) then implies that \( (x_n|\omega)_z \to \infty \).

Now suppose that \( (x_n|\omega)_z \to \infty \). The description (2.17) of the neighborhood basis at \( \omega \) then implies that \( x_n \to \omega \) in \( \bar{X}_{\varepsilon_0,z} \). Since \( X_{\varepsilon,b} \) is a uniform metric space, its completion \( \bar{X}_{\varepsilon,b} \) is proper by [2, Proposition 2.20]. Thus if the sequence of distances \( \{d_{\varepsilon,b}(z,x_n)\} \) were bounded then we could find a point \( \bar{\xi} \in \bar{X}_{\varepsilon,b} \) and a subsequence \( \{y_n\} \) such that \( y_n \to \bar{\xi} \) in \( \bar{X}_{\varepsilon,b} \).

We first suppose that \( \xi \in X_{\varepsilon,b} \). Then, since the metrics on \( X_{\varepsilon,b} \) and \( X_{\varepsilon_0,z} \) are locally \( \varepsilon \)-Lipschitz, it follows that \( y_n \to \xi \) in \( X_{\varepsilon_0,z} \). But this contradicts the fact that \( y_n \to \omega \) in \( \bar{X}_{\varepsilon_0,z} \).

Now suppose that \( \xi \in \partial X_{\varepsilon,b} \); note that this forces \( \xi \neq \omega \). Then, using the identification of \( \partial X_{\varepsilon,b} \) with \( \partial X \), the comparison (2.13) implies that \( (y_n|\xi)_b \to \infty \). By [3, Proposition 3.4.1] this implies that \( (y_n|\xi)_z \to \infty \). But this contradicts the fact that \( (y_n|\omega)_z \to \infty \) since \( \xi \neq \omega \).

\( \square \)

### 3. Quasihyperbolization

In this section we will study the quasihyperbolizations of unbounded uniform metric spaces. Our focus will be on generalizing the results in [2, Chapter 3] to the unbounded setting. Previously the unbounded setting has been treated using the method of sphericalization [9] to reduce claims to the bounded setting.

We let \((\Omega, d)\) be an \( A \)-uniform metric space (either bounded or unbounded for now) and write \( Y = (\Omega, k) \) for the quasihyperbolization of \( \Omega \), defined using the quasihyperbolic metric (1.8) on \( \Omega \). Then \( Y \) is a proper geodesic \( \delta \)-hyperbolic space by [2, Theorem 3.6] with \( \delta = \delta(A) \) depending only on \( A \). For clarity we will denote distances between points \( x, y \in Y \) by \( k(x,y) \) as opposed to the standard distance notation \( |xy| \) in \( \delta \)-hyperbolic spaces. We will refer to geodesics in \( Y \) as quasihyperbolic geodesics in \( \Omega \). We denote the distance to the metric boundary of \( \Omega \) by \( d(x) := d_{\Omega}(x) \) for \( x \in \Omega \). By [2, Proposition 2.20] the completion \( \bar{\Omega} \) of \( \Omega \) with respect to the metric \( d \) is proper.
By [2] (2.4) and [2] (2.16) the quasihyperbolic metric $k$ satisfies the inequality for $x, y \in \Omega$,

$$
\log \left(1 + \frac{d(x, y)}{\min\{d(x), d(y)\}}\right) \leq k(x, y) \leq 4A^2 \log \left(1 + \frac{d(x, y)}{\min\{d(x), d(y)\}}\right).
$$

We observe the easily verified inequality for any incomplete metric space $(\Omega, d)$ and $x, y \in \Omega$,

$$
|d_\Omega(x) - d_\Omega(y)| \leq d(x, y),
$$

which leads to the following inequality for the quasihyperbolic metric for $x, y \in \Omega$ [2] (2.3),

$$
\left|\log\frac{d(x)}{d(y)}\right| \leq k(x, y).
$$

For a rectifiable curve $\gamma$ we let $\ell_d(\gamma)$ denote its length measured in the metric $d$ and we let $\ell_k(\gamma)$ denote its length measured in the quasihyperbolic metric $k$. We note by [2] (2.15) that if $\gamma : I \to \Omega$ is an $A$-uniform curve then

$$
\ell_k(\gamma) \leq 4A \log \left(1 + \frac{\ell_d(\gamma)}{\min\{d(x), d(y)\}}\right),
$$

By [2] Theorem 2.10, there is a constant $A_* = A_*(A) \geq 1$ such that geodesic segments $\gamma : I \to Y$ defined on compact intervals $I \subset \mathbb{R}$ are $A_*$-uniform curves in $\Omega$. We will use this fact frequently throughout this and subsequent sections, always denoting the corresponding constant by $A_*$ to distinguish it from the other constants since this constant plays a special role. We remark that an explicit bound $A_* \leq e^{1000.4^6}$ is given in [2].

We now discuss the relationship of the metric boundary $\partial \Omega$ of $\Omega$ to the Gromov boundary $\partial Y$ of $Y$. By [2] Proposition 3.12 any quasihyperbolic geodesic $\gamma : I \to \Omega$ can be continuously extended to the closure $\bar{I}$ of $I$ and then reparametrized by arclength with respect to $d$ in order to obtain an $A_*$-uniform curve $\sigma : [0, \ell_d(\gamma)] \to \Omega$ with $\sigma(t) \in \Omega$ for all $t$ except possibly $t = 0$ and $t = \ell_d(\gamma)$. Here we allow $\ell_d(\gamma) = \infty$, and by $A_*$-uniformity of $\sigma$ we mean that inequalities (3.3) and (3.4) hold for $\sigma$ with $A = A_*$, even though the endpoints $x$ and $y$ of $\sigma$ may belong to $\partial \Omega$ instead of $\Omega$. When $\ell_d(\gamma) < \infty$ this uniformity assertion follows specifically from [2] Proposition 3.12(d). For $\ell_d(\gamma) = \infty$ inequality (3.3) is meaningless, but inequality (3.4) translates into

$$
t = \ell_d(\sigma|_{[0,t]}) \leq A_* d(\sigma(t)),
$$

for all $t \geq 0$, since $\ell_d(\sigma|_{[t,\infty]}) = \infty$. This inequality can be obtained by fixing $t \geq 0$ and then noting that for each $s \geq 0$ the curve $\sigma|_{[0,s]}$ is an $A_*$-uniform curve since it is a reparametrization of a quasihyperbolic geodesic; applying inequality (3.4) to the curve $\sigma|_{[0,2t]}$ for the given parameter $t$ then gives inequality (3.5). If we take the quasihyperbolic geodesic $\gamma$ to have the form $\gamma : (-\infty, a] \to Y$, $a \in (-\infty, \infty]$, with the orientation of $\gamma$ being the reverse of the orientation of $\sigma$, we obtain from inequality (3.5) that for all $t \in (-\infty, a]$,

$$
\ell_d(\gamma|_{[0,a]}) \leq A_* d(\gamma(t)).
$$

We fix a point $\omega \in \Omega \cup \{\infty\}$ that is going to have one of two meanings, depending on whether or not $\Omega$ is bounded. When $\Omega$ is bounded we take $\omega \in \Omega$ to be a point such that $d(\omega) = \sup_{x \in \Omega} d(x)$. When $\Omega$ is unbounded we take $\omega = \infty$ to be an ideal point at infinity for $\Omega$. This meaning of $\omega$ will be fixed for the rest of this section. The claims of [2] Proposition 3.12 imply that we have an identification of $\partial_\omega Y$ with $\partial \Omega$ given by sending a quasihyperbolic geodesic ray $\gamma : [0, \infty) \to \Omega$ with $\ell_d(\gamma) < \infty$ to its endpoint in $\partial \Omega$. We note that $\partial_\omega Y = \partial Y$ when $\Omega$ is bounded and $\partial_\omega Y = \partial Y \backslash \{\omega\}$ when $\Omega$ is unbounded. Furthermore, in the case that $\Omega$ is unbounded, [2] Proposition 3.12(a)] shows that all quasihyperbolic geodesic rays
Let $x_1, x_2 \in \bar{\Omega}$ and let $x_1 x_2$ be a quasihyperbolic geodesic between these points. Let $p \in x_1 x_2$ be such that $d(p) = \sup_{x \in x_1 x_2} d(x)$. Then there is a constant $C = C(A) \geq 1$ such that
\begin{equation}
(3.7) 
\quad d(x_1, x_2) \asymp_C d(p).
\end{equation}

Proof. We must have $\ell_d(x_1 x_2) < \infty$ since $x_1$ and $x_2$ are both points of $\bar{\Omega}$. Set $a = \ell_d(x_1 x_2)$ and let $\sigma : [0, a] \to \Omega$ be a $d$-arc-length parametrization of $x_1 x_2$. We let $p \in x_1 x_2$ be chosen such that $d(p) = \sup_{x \in x_1 x_2} d(x)$. By reversing the orientation of $\sigma$ if necessary, we can assume without loss of generality that $p \in \sigma([0, \frac{a}{2}])$. Since $\sigma$ is an $A_\ast$-uniform curve, we then have
\[
\quad d(p) \leq \ell_d(\sigma|_{[0, \frac{a}{2}]}) \leq A_\ast d\left(\sigma\left(\frac{a}{2}\right)\right) \leq A_\ast d(p).
\]
Thus
\[
\quad \ell_d(\sigma) \asymp A_\ast d(\sigma|_{[0, \frac{a}{2}]}) \asymp A_\ast d(p).
\]
Since $\ell_d(\sigma) \asymp A_\ast d(x_1, x_2)$ by the $A_\ast$-uniformity of $\sigma$, the comparison (3.7) follows. \hfill \Box

Our next claim shows that $Y$ is roughly starlike from $\omega$, quantitatively in the uniformity constant $A$ of $\Omega$.

Proposition 3.2. There is $K = K(A) \geq 0$ such that $Y = (\Omega, k)$ is $K$-roughly starlike from $\omega$.

Proof. When $\Omega$ is bounded (so that $\omega \in \Omega$) this claim follows from [2, Theorem 3.6]. Thus we will focus on the case that $\Omega$ is unbounded. To this end we let $x \in \Omega$ be given. Since the completion $(\bar{\Omega}, d)$ of $(\Omega, d)$ is proper, we can find $\xi \in \partial\Omega$ such that $d(\xi, x) = d(x)$. Let $\xi \omega$ be a quasihyperbolic geodesic from $\xi$ to $\omega$, and let $\gamma : [0, \infty) \to \Omega$ be a $d$-arc-length reparametrization of this geodesic with $\gamma(0) = \xi$. Since $d(\gamma(t)) \to \infty$ as $t \to \infty$ by [2, Proposition 3.12(a)] and $d(\gamma(0)) = 0$, by continuity we can find some $s > 0$ such that $d(\gamma(s)) = d(x)$. We set $y = \gamma(s)$. By (3.3) we then have
\[
\quad d(x, y) \leq d(\gamma|_{[0, s]} + d(\xi, x)
\quad = \ell_d(\gamma|_{[0, s]} + d(\xi, x)
\quad \leq A_\ast d(y) + d(x
\quad = (A_\ast + 1)d(x),
\]
Thus by (3.3), using again that $d(y) = d(x)$,
\[
\quad k(x, y) \leq 4A^2 \log\left(1 + \frac{d(x, y)}{d(x)}\right) \leq 4A^2 \log(2 + A_\ast).
\]
Thus $Y$ is $K$-roughly starlike from $\omega$ with $K = 4A^2 \log(2 + A_\ast)$. \hfill \Box

We will assume for the rest of this section that $\Omega$ is an unbounded $A$-uniform metric space, so that $\omega \in \partial Y$ corresponds to the point at infinity for $\bar{\Omega}$. The next lemma gives us more precise control over the function $t \to d(\gamma(t))$ for a quasihyperbolic geodesic $\gamma$ starting from $\omega$. 
Lemma 3.3. There are constants $C = C(A) \geq 1$ and $0 < u = u(A) \leq 1$ such that if $\gamma : (-\infty, a] \to Y$, $a \in (-\infty, \infty)$, is a quasihyperbolic geodesic starting from $\omega$ then for all $t \leq s \leq a$,

$$e^{-(s-t)} \leq \frac{d(\gamma(s))}{d(\gamma(t))} \leq Ce^{-(s-t)}.$$  

Proof. Let $t \leq s \leq a$ be given. Put $h(x) = -\log d(x)$. The inequality (3.8) is equivalent to the inequality

$$s - t \geq h(\gamma(s)) - h(\gamma(t)) \geq u(s - t) - c,$$

with $c = c(A) \geq 0$ depending only on $A$. We will prove inequality (3.8) in the form (3.9). The left side of (3.9) follows immediately from the fact that $h$ is 1-Lipschitz in the quasihyperbolic metric $k$ on $\Omega$ by (3.3).

Verifying the right side of (3.9) is more involved. We will first prove this claim in the case $a = \infty$. We set $t_0 = t$ and for each $n \in \mathbb{N}$ we choose $t_n$ inductively such that $t_n > t_{n-1}$ and

$$\ell_d(\gamma|_{[t_{n-1}, t_n]}) = \frac{1}{2} \ell_d(\gamma|_{[t_{n-1}, a]}).$$

Note that $\ell_d(\gamma|_{[t_{n-1}, t_n]}) > 0$ since $t_n > t_{n-1}$ and that $\ell_d(\gamma|_{[t_{n-1}, \infty)}) < \infty$ since the endpoint of $\gamma$ cannot be $\omega$ (because $\omega$ is the starting point). The above equality implies that

$$\ell_d(\gamma|_{[t_{n-1}, \infty)}) = \frac{1}{2} \ell_d(\gamma|_{[t_{n-1}, \infty]}),$$

which gives us the equality for $n \in \mathbb{N}$,

$$\ell_d(\gamma|_{[t_{n-1}, t_n]}) = 2\ell_d(\gamma|_{[t_{n-1}, t_n+1]}).$$

By (3.6) we then have

$$\ell_d(\gamma|_{[t_n, t_{n+1}]}) \leq \ell_d(\gamma|_{[t_n, a]}) \leq A_d(\gamma(t_n)),$$

We thus obtain from (3.4) and the equality (3.11), for $n \geq 0$,

$$t_{n+1} - t_n = k_d(\gamma|_{[t_n, t_{n+1}]}) \leq 4A \log \left(1 + \frac{\ell_d(\gamma|_{[t_n, t_{n+1}]})}{\min\{d(\gamma(t_n)), d(\gamma(t_{n+1}))\}}\right) \leq 4A \log \left(1 + \frac{A_d(\gamma(t_n))}{\min\{\ell_d(\gamma|_{[t_n, t_{n+1}]})\}}\right) = c_1,$$

with $c_1 = c_1(A) > 0$ depending only on $A$.

For a lower bound, we let $\xi \in \partial\Omega$ be the other endpoint of $\gamma$. Then since $\ell_d(\gamma|_{[t_n, \infty)}) = 2\ell_d(\gamma|_{[t_n, t_{n+1}]})$ we have for all $n \geq 0$,

$$d(\gamma(t_n)) \leq d(\gamma(t_n), \xi) \leq 2\ell_d(\gamma|_{[t_n, t_{n+1}]}).$$
Thus (3.11) implies, using $t_{n+1} - t_n = \ell_k(\gamma|_{[t_n, t_{n+1}]})$ as above,

$$t_{n+1} - t_n \geq \log \left( 1 + \frac{d(\gamma(t_n), \gamma(t_{n+1}))}{\min\{d(\gamma(t_n), d(\gamma(t_{n+1}))\}} \right)$$

$$\geq \log \left( 1 + \frac{A_s^{-1} \ell_d(\gamma|_{[t_n, t_{n+1}]})}{\min\{\ell_d(\gamma|_{[t_n, t_{n+1}]}, \ell_d(\gamma|_{[t_{n+1}, \infty])}\}} \right)$$

$$= \log \left( 1 + \frac{A_s^{-1} \ell_d(\gamma|_{[t_n, t_{n+1}]})}{2 \min\{\ell_d(\gamma|_{[t_n, t_{n+1}]}, \ell_d(\gamma|_{[t_{n+1}, t_{n+2}]})\}} \right)$$

$$= c_0,$$

with $c_0 = c_0(A) > 0$ depending only on $A$. We have thus shown that there are positive constants $c_0$ and $c_1$ depending only on $A$ such that for all $n \geq 0$,

$$c_0 \leq t_{n+1} - t_n \leq c_1. \quad (3.11)$$

On the other hand we have, by $A_s$-uniformity of $\gamma$ and the construction of the subdivision \{t_n\} of [t, a], for each $n \geq 0$ and $m \geq 1$,

$$d(\gamma(t_{n+m})) \leq \ell_d(\gamma|_{[t_{n+m}, \infty])})$$

$$= \frac{1}{2m} \ell_d(\gamma|_{[t_n, \infty)})$$

$$\leq \frac{A_s}{2m} \ell_d(\gamma(t_n)). \quad (3.12)$$

We choose $m = m(A)$ to be the minimal positive integer such that $\frac{1}{2m} < 1$. Then $c_2 = -\log \frac{A_s}{2m}$ is positive and we have from above that

$$h(\gamma(t_{n+m})) \geq h(\gamma(t_n)) + c_2. \quad (3.13)$$

Since $m$ depends only on $A$, we deduce from (3.11) the inequality

$$c_0 \leq t_{n+m} - t_n \leq c_1,$$

with the $c_0$ and $c_1$ being (possibly different) positive constants still depending only on $A$.

Since $h$ is 1-Lipschitz in the quasihyperbolic metric $k$, using (3.12) and (3.13) (and replacing $c_0$ with $c'_0 = \min\{c_0, c_2\}$ if necessary) we conclude that we also have

$$c_0 \leq h(\gamma(t_{n+m})) - h(\gamma(t_n)) \leq c_1,$$

for each $n \geq 0$. There is thus a $C = C(A) \geq 1$ such that

$$h(\gamma(t_{n+m})) - h(\gamma(t_n)) \asymp_C t_{n+m} - t_n.$$

By starting with $n = 0$, recalling $t_0 = t$, and summing this inequality, we obtain for each integer $q \geq 0$ that

$$h(\gamma(t_{qm})) - h(\gamma(t)) \asymp_C t_{qm} - t. \quad (3.14)$$

The inequality (3.13) also shows that $t_{qm} \to \infty$ as $q \to \infty$. There will thus be an integer $q \geq 0$ such that $t_{qm} \leq s < t_{(q+1)m}$, recalling that $t_0 = t \leq s$. Applying (3.13) with $n =qm$ then allows us to conclude that

$$0 \leq s - t_{qm} \leq c_1. \quad (3.15)$$
Since $h$ is 1-Lipschitz in the metric $k$, by combining inequality (3.14) with the comparison (3.14) we obtain that
\[
\begin{align*}
    h(\gamma(s)) - h(\gamma(t)) &= (h(\gamma(s)) - h(\gamma(t))) + (h(\gamma(t)) - h(\gamma(t))) \\
    &\geq t_{qm} - s + C^{-1}(t_{qm} - t) \\
    &\geq C^{-1}(s - t) - c,
\end{align*}
\]
with $c = c(A) \geq 0$ depending only on $A$. This gives the desired inequality upon setting $u = C^{-1}$.

We now consider the case $a < \infty$. We only need to establish the right side of inequality (3.8), as the left side has already been deduced from the fact that $h$ is 1-Lipschitz in the quasi-hyperbolic metric. By Proposition 2.2 we can find a quasi-hyperbolic geodesic $\sigma : \mathbb{R} \to Y$ starting from $\omega$ and parametrized such that $k(\gamma(a), \sigma(a)) \leq K$, with $K = K(A)$. Then Lemma 2.4 implies that for all $t \in (-\infty, a]$ we have
\[
|\gamma(t)| = C + 8\delta = c(A).
\]
Thus by (3.3) we have that $d(\gamma(t)) \asymp c d(\sigma(t))$ for all $t \in (-\infty, a]$ with $C = C(A)$. The right side of inequality (3.8) follows from the corresponding side of the inequality for $\sigma$.

Let $b : Y \to \mathbb{R}$ be a Busemann function based at the distinguished point $\omega \in \partial Y$. Our next lemma shows that maximization of the distance from $\partial \Omega$ and minimization of the Busemann function $b$ occurs at the same point $p$ on a quasi-hyperbolic geodesic between two points $x_1$ and $x_2$ of $\Omega$, up to a bounded error determined only by $A$.

**Lemma 3.4.** Let $x_1, x_2 \in \Omega$ and let $x_1 x_2$ be a quasi-hyperbolic geodesic joining them. Let $\Delta = \omega x_1 x_2$ be a geodesic triangle with the prescribed vertices and let $T : \Delta \to \Upsilon$ be an associated tripod map. Let $p \in x_1 x_2$ be such that $d(p) = \sup_{x \in x_1 x_2} d(x)$. Then for each equiradial point $z \in \Delta$ of $T$ we have $k(p, z) \leq c(A)$ and therefore
\[
(3.16) \quad b(p) \overset{\sim}{=} c(x_1|x_2)_h \overset{\sim}{=} \inf_{x \in x_1 x_2} b(x),
\]
with $c = c(A)$.

**Proof.** Let $x_1, x_2 \in \Omega$ be given. We will consider these as points of $Y \cup \partial_T Y$ using the discussion at the beginning of this section. We let $\gamma_i : (-\infty, a_i) \to X$ and $\sigma : I \to X$ be the parametrizations of $\omega x_i$ for $i = 1, 2$ and of $x_1 x_2$ that are given by Proposition 2.2 applied to a geodesic triangle $\Delta = \omega x_1 x_2$ with associated tripod map $T : \Delta \to \Upsilon$. Since $\delta = \delta(A)$ the conclusions of Proposition 2.2 hold with $c(\delta) = c(A)$; we will implicitly be using variations of this observation throughout the rest of the proof.

By (3) of Proposition 2.2 we have for $t \in I_{\leq 0}$ that $k(\sigma(t), \gamma_1(-t)) \leq c(A)$ and for $t \in I_{\geq 0}$ that $k(\sigma(t), \gamma_2(t)) \leq c(A)$. By (3.8) we then have for $t \in I_{\geq 0}$,
\[
d(\sigma(t)) \asymp c(A) d(\gamma_1(-t)),
\]
and for $t \in I_{\leq 0}$,
\[
d(\sigma(t)) \asymp c(A) d(\gamma_2(t)).
\]
By Lemma 3.3 applied to $\gamma_1$ and $\gamma_2$ with $t = 0$, together with the fact that $d(\gamma_i(0)) \asymp c(A) d(\sigma(0))$ for $i = 1, 2$ since $k(\gamma_i(0), \sigma(0)) \leq c(A)$ by (1) of Proposition 2.2, we thus deduce that for $s \in I$,
\[
(3.17) \quad C^{-1} e^{-|s|} d(\sigma(0)) \leq d(\sigma(s)) \leq C e^{-u|s|} d(\sigma(0)),
\]
with $C = C(A) \geq 1$ and $0 < u = u(A) \leq 1$. 

Let \( p \in x_1x_2 \) be given such that \( d(p) = \sup_{x \in x_1x_2} d(x) \). Let \( t \in I \) be such that \( \sigma(t) = p \). Then \( d(\sigma(0)) \leq d(\sigma(t)) \) and therefore inequality (3.17) implies that
\[
 d(\sigma(0)) \leq d(\sigma(t)) \leq C e^{-u}|s| d(\sigma(0)),
\]
which implies that \( |s| \leq c(A) \). Since \( |s| = k(p, \sigma(0)) \), combining this with another application of (1) of Proposition 2.2 gives that \( k(p, z) \leq c(A) \) for each equiradial point \( z \) for the tripod map \( T \). The rough equality (3.16) follows by Proposition 2.1 and the fact that \( b \) is 1-Lipschitz.

Combining Lemmas 3.3 and 3.4 leads to the following key estimate on ratios of distances for points in \( \Omega \).

**Lemma 3.5.** Let \( x_i \in \bar{\Omega}, i = 1, 2, 3 \) be given distinct points. Let \( \Delta_1 = \omega x_1x_2 \) and \( \Delta_2 = \omega x_1x_3 \) be geodesic triangles that share the edge \( \omega x_1 \) and let \( T_i : \Delta \to \bar{Y} \) be associated tripod maps. Let \( \gamma_{1,i} \) be the parametrizations of \( \omega x_1 \) given by applying Proposition 2.2 to \( \Delta_i \), \( i = 1, 2 \), and define \( s \) such that \( \gamma_{1,1}(s) = \gamma_{1,2}(0) \). Then
\[
 \frac{d(x_1, x_2)}{d(x_1, x_3)} \leq \beta(e^s),
\]
where \( \beta : [0, \infty) \to [0, \infty) \) is a homeomorphism given by \( \beta(t) = C \max\{t, t^u\} \) with \( C = C(A) \geq 1 \) depending only on \( A \) and \( 0 < u = u(A) \leq 1 \) being the constant of Lemma 3.3.

**Proof.** Let \( p_1 \in x_1x_2 \) and \( p_2 \in x_1x_3 \) be such that \( d(p_1) = \sup_{x \in x_1x_2} d(x) \) and \( d(p_2) = \sup_{x \in x_1x_3} d(x) \). Then Lemma 3.4 implies that \( k(p_1, \gamma_{1,1}(0)) \leq c(A) \) and \( k(p_2, \gamma_{1,2}(0)) \leq c(A) \), which implies that \( d(p_1) \leq_C d(\gamma_{1,1}(0)) \) and \( d(p_2) \leq_C d(\gamma_{1,2}(0)) \) with \( C = C(A) \). Combining these claims with Lemma 3.4 then gives
\[
 \frac{d(x_1, x_2)}{d(x_1, x_3)} \leq_C \frac{d(\gamma_{1,1}(0))}{d(\gamma_{1,2}(0))}.
\]

If \( s \leq 0 \) then we can apply the right side of inequality (3.8) together with (3.18) to the quasihyperbolic geodesic \( \gamma_{1,1} \) to obtain that
\[
 \frac{d(x_1, x_2)}{d(x_1, x_3)} \leq C \frac{d(\gamma_{1,1}(0))}{d(\gamma_{1,2}(0))} \leq C(e^s)^u,
\]
with \( C = C(A) \geq 1 \) and \( 0 < u = u(A) \leq 1 \). In this case we can take \( \beta_0(t) = Ct^u \). If \( s \geq 0 \) then we apply the left side of inequality (3.8) and invert the results to get that
\[
 \frac{d(x_1, x_2)}{d(x_1, x_3)} \leq C \frac{d(\gamma_{1,1}(0))}{d(\gamma_{1,2}(0))} \leq Ce^s,
\]
with \( C = C(A) \). Setting \( \beta(t) = \beta(t) = C \max\{t, t^u\} \) and combining the cases \( s \leq 0 \) and \( s \geq 0 \) together gives the conclusion of the lemma.

When \( \Omega \) is bounded it is shown in the final assertion of [2] Theorem 3.6] that the metric \( d \) on \( \partial\Omega \) is quasisymmetrically equivalent to any visual metric \( \alpha \) on the Gromov boundary \( \partial Y \) of \( Y \) that is based at some point \( x \in Y \). We will show that the analogous claim holds in the unbounded case when we consider visual metrics based at a Busemann function \( b \) on \( Y \) that is itself based at the distinguished point \( \omega \in \partial Y \). We refer back to Section 2.3 for details regarding visual metrics.

\[\text{UNIFORMIZATION, \(\partial\)-BILIPSCHITZ MAPS, SPHERICALIZATION, AND INVERSION 23}\]
Proposition 3.6. Let $\alpha$ be a visual metric with parameter $q > 0$ on $\partial Y$ based at a Busemann function $b$ with basepoint $\omega$. Then the identification $(\partial Y, \alpha) \rightarrow (\partial \Omega, d)$ is $\theta$-quasisymmetric with $\theta(t) = C \max\{t^{q^{-1}}, t^{q^{-1}}u\}$, where $C = C(A, q) \geq 1$ and $0 < u = u(A) \leq 1$ is the constant of Lemma 3.3.

Proof. To ease notation in this proof we will write $\times$ and $\setminus$ for $\times_C$ and $\setminus_C$, where the implied constant depends only on $A$. We can assume that $\partial \Omega$ has at least three distinct points, as otherwise the claim is vacuously true. We must show that for any three distinct points $\xi_1, \xi_2, \xi_3 \in \partial_Y$ we have

$$d(\xi_1, \xi_2) - d(\xi_2, \xi_3) \leq \eta_0(e^{-t((\xi_1, \xi_2), (\xi_1, \xi_3))}), \tag{3.19}$$

with the control function $\eta$ having the desired form. Since $\alpha$ is a visual metric with parameter $q$ based at $b$, we have for $\xi, \zeta \in \partial_Y$ that

$$\alpha(\xi, \zeta) \asymp e^{-q(\xi, \zeta)_b},$$

with implied constant independent of $q$ and $A$. Thus it suffices to find a control function $\theta$ of the desired form such that for any $\xi_1, \xi_2, \xi_3 \in \partial_Y$ we have

$$d(\xi_1, \xi_2) - d(\xi_2, \xi_3) \leq \eta_0(e^{-t((\xi_1, \xi_2), (\xi_1, \xi_3))}), \tag{3.20}$$

In fact we need only find a control function $\theta_0$ such that for any $\xi_1, \xi_2, \xi_3 \in \partial_Y$ we have

$$d(\xi_1, \xi_2) - d(\xi_2, \xi_3) \leq \eta_0(e^{-t((\xi_1, \xi_2), (\xi_1, \xi_3))}), \tag{3.19}$$

as then we can set $\theta(t) = \theta_0(t^{q^{-1}})$. Thus it suffices to establish the inequality (3.20) with $\theta_0(t) = C \max\{t, t^u\}$ where $C$ depends only on $A$ and $u$ is the constant of Lemma 3.3.

As in the setup of Lemma 3.3 we let $\Delta_1 = \omega \xi_1 \xi_2$ and $\Delta_2 = \omega \xi_1 \xi_3$ be geodesic triangles that share the edge $\omega \xi_1$ and let $T_i : \Delta \rightarrow Y$ be associated tripod maps. Let $\gamma_{1,i}$ be the parametrizations of $\omega \xi_1$ given by applying Proposition 2.2 to $\Delta_i$, $i = 1, 2$, and define $s$ such that $\gamma_{1,1}(s) = \gamma_{1,2}(0)$. Let $\gamma_2$ and $\sigma_1$ be the parametrizations of $\omega \xi_2$, and $\xi_1 \xi_2$ supplied by applying Proposition 2.2 to $\Delta_1$, and let $\gamma_3$ and $\sigma_2$ be the parametrizations of $\omega \xi_3$, and $\xi_1 \xi_3$ given by applying Proposition 2.2 to $\Delta_2$.

By (4) of Proposition 2.2 we have that $b(\gamma_{1,1}(0)) \doteq (\xi_1, \xi_2)_b$ and $b(\gamma_{1,2}(0)) \doteq (\xi_1, \xi_3)_b$ (recall that $\doteq = \delta(A)$ so that $c(\delta) = c(A)$). Since $\gamma_{1,1}(s) = \gamma_{1,2}(0)$, (4) of Proposition 2.2 gives $b(\gamma_{1,2}(0)) \doteq s + (\xi_1, \xi_2)_b$ and therefore

$$e^{-t((\xi_1, \xi_2)_b, (\xi_1, \xi_3)_b)} \ll e^{-s}.$$

By Lemma 3.5 we thus conclude that

$$d(\xi_1, \xi_2) - d(\xi_2, \xi_3) \leq \beta(e^{-t((\xi_1, \xi_2)_b, (\xi_1, \xi_3)_b)}),$$

with $\beta(t) = C \max\{t, t^u\}$ for $t \geq 0$, with $C = C(A) \geq 1$ and $u$ being the constant of Lemma 3.3. This implies that inequality (3.20) holds with $\theta_0(t) = C \beta(t)$ for $t \geq 0$ for an appropriate constant $C = C(A)$, which implies the proposition. \hfill $\Box$

Since any visual metric on $\partial Y$ based at a Busemann function $b$ is quasimöbius to any visual metric on $\partial Y$ based at a point $x \in Y$ (as discussed at the end of Section 2.3), Proposition 3.6 gives a new proof of a result of Herron, Shammugalingam, and Xie [9], Theorem 6.2] that does not require sphericalizing $\Omega$. 

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4. Uniformizing the quasihyperbolic metric

Let \((\Omega, d)\) be an \(A\)-uniform metric space. As in the previous section we let \(Y = (\Omega, k)\) be the quasihyperbolization of \(\Omega\). We write \(d(x) := d_\Omega(x)\) for \(x \in \Omega\) as in the previous section. As before, if \(\Omega\) is bounded then we let \(\omega \in \Omega\) be such that \(d(\omega) = \sup_{x \in \Omega} d(x)\), and if \(\Omega\) is unbounded then we let \(\omega \in \partial Y\) denote the point corresponding to \(\infty_\Omega\) in \(\partial Y\). When \(\Omega\) is bounded we define \(b(x) = k(x, z)\) and when \(\Omega\) is unbounded we let \(b\) be a Busemann function based at \(\omega\). Proposition 3.3 shows that \(Y\) is \(K\)-roughly starlike from \(\omega\) with \(K = K(A)\). We will assume \(\varepsilon > 0\) is given such that the density \(\rho_{\varepsilon,b}\) is admissible for \(\varepsilon\) with constant \(M\). Since \(Y\) is \(\delta\)-hyperbolic with \(\delta = \delta(A)\) and \(b\) is 1-Lipschitz, by Theorem 1.14 there is always an \(\varepsilon_0 = \varepsilon_0(A)\) and such that \(\rho_{\varepsilon,b}\) is admissible for \(\varepsilon\) with constant \(M = 20\) any \(0 < \varepsilon \leq \varepsilon_0\). We write \(Y_\varepsilon = Y_{\varepsilon,b}\) for the conformal deformation of \(Y\) with conformal factor \(\rho_\varepsilon = \rho_{\varepsilon,b}\). Similarly we drop \(b\) from the notation and write \(d_\varepsilon = d_{\varepsilon,b}\) for the metric on \(Y_\varepsilon\), etc. We write \(B_\varepsilon(x, r) = B_{d_\varepsilon}(x, r)\) for the ball of radius \(r\) centered at \(x \in Y_\varepsilon\) in the metric \(d_\varepsilon\). The notation \((x,y)_b\) always indicates the Gromov product in \(Y\) of \(x\) and \(y\) based at \(b\).

We then have the following proposition that generalizes [2, Proposition 4.28].

**Proposition 4.1.** Let \(\varepsilon > 0\) be such that \(\rho_\varepsilon\) is an admissible density for \(\varepsilon\) with constant \(M\). Then the map \(\Omega \to Y_\varepsilon\) induced by the identity map on \(\Omega\) is \(\partial\)-biLipschitz with data \((L, \lambda)\) and is \(\theta\)-quasisymmetric with \(L, \lambda\), and \(\theta\) depending only on \(A, \varepsilon,\) and \(M\).

Proposition 4.1 gives a direct quantitative relation between \(\Omega\) and a uniformization of its quasihyperbolization \(Y\) that we will make use of in the proofs of the main theorems in this paper. As remarked above, by Theorem 1.14 we will always be able to apply Proposition 4.1 for \(\varepsilon\) sufficiently small with \(M = 20\). Thus there is always a uniformization of \(Y\) to which this proposition can be applied to. We will focus primarily on the case that \(\Omega\) is unbounded.

We will deduce the bounded case from this using a ray augmentation of \(\Omega\), as was done in the proof of Lemma 2.10.

**Remark 4.2.** Throughout the remainder of this paper we will be using [2, Proposition A.7], which for a geodesic metric space \(X\) and a continuous function \(\rho : X \to (0, \infty)\) allows us to compute the lengths \(\ell_\rho(\gamma)\) in the conformal deformation \(X_\rho\) of curves \(\gamma : I \to X\) parametrized by arclength in \(X\) as

\[
\ell_\rho(\gamma) = \int_I \rho \circ \gamma \, ds,
\]

with \(ds\) denoting the standard length element in \(\mathbb{R}\).

We will use the following lemma for verifying that a map is \(\partial\)-biLipschitz.

**Lemma 4.3.** Let \(f : (\Omega, d) \to (\Omega', d')\) be a homeomorphism of incomplete metric spaces. Suppose that there is \(L \geq 1\) and \(0 < \lambda < 1\) such that for any \(x \in \Omega\) and \(y, z \in B_d(x, \lambda d_\Omega(x))\),

\[
\frac{d'(f(y), f(z))}{d_{\Omega'}(f(x))} \geq L \frac{d(y, z)}{d_\Omega(x)},
\]

and that for all \(x \in \Omega\) we have

\[
B_{d'}(f(x), L^{-1} \lambda d_{\Omega'}(f(x))) \subseteq f(B_d(x, \lambda d_\Omega(x))).
\]

Then \(f\) is \(\partial\)-biLipschitz with data \((L, L^{-1}\lambda)\).

**Proof.** Inequality (4.2) clearly implies that \(f\) is \(\partial\)-Lipschitz with data \((L, \lambda)\). Thus we must show that \(f^{-1}\) is \(\partial\)-Lipschitz with data \((L, L^{-1}\lambda)\). Let \(w \in \Omega'\) be given. Applying (4.3) to
\( x = f^{-1}(w) \) and then applying \( f^{-1} \) to each side gives
\[
\begin{align*}
\quad f^{-1}(B_{d'}(w, L^{-1}\lambda d_{\Omega'}(w))) \subseteq B_d(x, \lambda d_{\Omega}(x))
\end{align*}
\]
Thus if \( y, z \in B_{d'}(w, L^{-1}\lambda d_{\Omega'}(w)) \) then \( f^{-1}(y), f^{-1}(z) \in B_d(x, \lambda d_{\Omega}(x)) \). We can thus apply (4.2) to obtain
\[
\begin{align*}
\quad d'(y, z) \leq L d(f^{-1}(y), f^{-1}(z)) \quad \quad d'_{\Omega'}(w) \leq L d_{\Omega}(f^{-1}(w))
\end{align*}
\]
This implies in particular that \( f^{-1} \) is \( \partial \)-Lipschitz with data \( (L, L^{-1}\lambda) \).

We can now prove Proposition 4.1.

**Proof of Proposition 4.1** We will first consider the case that \( \Omega \) is unbounded. Then \( \omega \in \partial Y \) is the ideal point at infinity for \( \Omega \) and \( b \) is a Busemann function based at \( \omega \). We recall that \( Y \) is \( \delta \)-hyperbolic with \( \delta = \delta(A) \) and \( K \)-roughly starlike from \( \omega \) with \( K = K(A) \). We will first verify that the identity map \( \Omega \to Y_\omega \) is \( \partial \)-biLipschitz. We begin by noting that there is \( \lambda = \lambda(A) \in (0, 1) \) such that \( k(y, z) \leq 1 \) for \( y, z \in B_d(x, \lambda d(x)) \). This is because \( d(y, z) \leq 2\lambda d(x) \) and
\[
\quad \min\{d(y), d(z)\} \geq (1 - \lambda)d(x),
\]
and therefore by (3.1),
\[
\quad k(y, z) \leq C(A) \log \left( 1 + \frac{2\lambda}{1 - \lambda} \right).
\]
Thus if \( \lambda \) is sufficiently small, dependent only on \( A \), we will have \( k(y, z) \leq 1 \). We apply inequality (2.13) to obtain from this that for \( y, z \in B_d(x, \lambda d(x)) \),
\[
\quad d_c(y, z) \leq C \rho_c(x)k(y, z) \leq C d_c(x)k(y, z),
\]
with \( C = C(A, \varepsilon, M) \), with the second comparison following from Proposition 2.11 we are using here that \( (y|z)_b \approx b(x) \) since \( k(x, y) \leq 1 \) and \( k(x, z) \leq 1 \) by our choice of \( \lambda \).

In this next part we will use the following inequality,
\[
\quad a \log(1 + t) \geq t \log(1 + a),
\]
valid for \( 0 \leq t \leq a \) (see [2, (4.31)]), as well as the inequality \( \log(1 + t) \leq t \) for \( t \geq 0 \). Applying (3.1) again to \( y, z \in B_d(x, \lambda d(x)) \),
\[
\quad k(y, z) \leq C(A) \log \left( 1 + \frac{d(y, z)}{(1 - \lambda)d(x)} \right) \leq C(A) \frac{d(y, z)}{(1 - \lambda)d(x)},
\]
and by (3.1) together with (3.3),
\[
\quad k(y, z) \geq \log \left( 1 + \frac{d(y, z)}{(1 + \lambda)d(x)} \right) \geq \frac{\log(1 + a) d(y, z)}{a (1 + \lambda)d(x)},
\]
with
\[
\quad a = a(A) := \frac{2\lambda}{1 + \lambda} \geq \frac{d(y, z)}{(1 + \lambda)d(x)}.
\]

We conclude that
\[
\quad k(y, z) \approx C \frac{d(y, z)}{d(x)},
\]
with \( C = C(A) \). Combining this with (4.1) gives

\[
\frac{d_z(y, z)}{d_z(x)} \simeq_L \frac{d(y, z)}{d(x)},
\]

with \( L = L(A, \varepsilon, M) \). Putting \( z = x \) in (4.6) shows that if \( y \in B_d(x, L^{-1} \lambda d(x)) \) then \( y \in B_{\tilde{\Omega}}(x, \lambda d(x)) \). It then follows from Lemma 4.3 that the identity map \( \Omega \to Y_\varepsilon \) is \( \partial \)-bilipschitz with data \((L, L^{-1} \lambda)\) depending only on \( A, \varepsilon, \) and \( M \).

On the other hand, Lemma 3.4 implies that \( k \leq \sup p \) completes the proof of Proposition 4.1 in the case that \( \Omega \) is unbounded.

It remains to show that the identity map \( \Omega \to Y_\varepsilon \) is \( \theta \)-quasisymmetric with \( \theta \) depending only on \( A, \varepsilon, \) and \( M \). Since the metric spaces \( \Omega \) and \( Y_\varepsilon \) are both uniform, by [12, Theorem 6.6] it suffices to show that there is some \( C = C(A, \varepsilon, M) \) such that, for \( x_1, x_2, x_3 \in \Omega \),

\[
d(x_1, x_2) \leq d(x_1, x_3) \implies d(x_1, x_2) \leq C d(x_1, x_3).
\]

Recalling that \( \omega \in \partial Y \) is the basepoint of the Busemann function \( b \), similarly to the proof of Proposition 3.6 we form geodesic triangles \( \Delta_1 = x_1 x_2 \omega \) and \( \Delta_2 = x_1 x_3 \omega \) sharing \( x_1 \omega \) as a common edge and let \( T_1 : \Delta_1 \to \Gamma \) be associated tripod maps. We let \( \gamma_{i,j} \) be the parametrizations of \( x \times x \) given by applying Proposition 2.2 to \( \Delta_i \), \( i = 1, 2 \), and define \( s \) such that \( \gamma_{i,j}(s) = \gamma_{i,2}(0) \). We conclude by Lemma 3.5 that

\[
\frac{d(x_1, x_2)}{d(x_1, x_3)} \leq \beta(s),
\]

with \( \beta(t) = C(A) \max\{t, t^u\} \) for \( t \geq 0, 0 < u = u(A) \leq 1 \). Thus it suffices to find a constant \( c = c(A, \varepsilon, M) \geq 0 \) such that we always have \( s \leq c \). Since this inequality is trivial when \( s \leq 0 \) we can assume that \( s \geq 0 \).

Let \( p_1 \in x_1 x_2 \) and \( p_2 \in x_1 x_3 \) be such that \( d(p_1) = \sup x_2 x_1 d(x_1, x_2) \) and \( d(p_2) = \sup x_3 x_1 d(x_1, x_3) \). Lemma 3.1 implies that we have \( d(p_1) \leq C d(x_1, x_2) \) and \( d(p_2) \leq C d(x_1, x_3) \) with \( C = C(A) \). The inequality \( d(x_1, x_2) \leq d(x_1, x_3) \) then implies that

\[
d(p_1) \leq C(A) d(p_2).
\]

On the other hand, Lemma 3.2 implies that \( k(p_1, \gamma_{1,i}(0)) \leq c(A) \) for \( i = 1, 2 \), which implies that \( d(p_1) \geq C(A) d(\gamma_{1,i}(0)) \) for \( i = 1, 2 \) with \( C = C(A) \) by (3.3). Thus we conclude that

\[
\frac{d(\gamma_{1,2}(0))}{d(\gamma_{1,1}(0))} \geq C^{-1},
\]

with \( C = C(A) \) depending only on \( A \). Since \( s \geq 0 \) we can combine this with the right side inequality of Lemma 3.3 to conclude that, for \( C = C(A) \geq 1 \) and \( u = u(A) > 0 \),

\[
C^{-1} \leq \frac{d(\gamma_{1,2}(0))}{d(\gamma_{1,1}(0))} \leq C e^{-us}
\]

Rearranging this inequality gives \( s \leq c \) with \( c = c(A) \) depending only on \( A \), as desired. This completes the proof of Proposition 4.1 in the case that \( \Omega \) is unbounded.

For the case that \( \Omega \) is bounded the corresponding uniformization \( Y_\varepsilon = Y_{\varepsilon, b} \) is given by \( b \in D(X) \) of the form \( b(x) = k(x, z) \) for a point \( z \in \Omega \) such that \( d(z) = \sup x \in \Omega d(x) \). As in the proof of Lemma 2.10, we let \( \tilde{\Omega} = \Omega \cup_{z=0} \cup_{0, \infty} \) be the ray augmentation of \( \Omega \) based at \( z \in \Omega \). Then \( \tilde{\Omega} \) is clearly also an \( A \)-uniform metric space with \( \partial \tilde{\Omega} = \partial \Omega \). The quasihyperbolization \( \hat{Y} \) of \( \tilde{\Omega} \) is isometric to the ray augmentation \( Y \cup_{z=0} \cup_{0, \infty} \) of the quasihyperbolization \( Y \) of \( \Omega \) based at \( z \) by [4, Lemma 4.14]. We let \( \tilde{b} \) be the Busemann function associated to the ray \([0, \infty)\) we glued onto \( Y \) and let \( \omega \in \partial \hat{Y} \) be the basepoint of \( \tilde{b} \). We let \( \hat{Y}_\varepsilon \) be the conformal deformation of \( \hat{Y} \) with conformal factor \( \rho_{\varepsilon, \tilde{b}} \).
As the discussion in the proof of Lemma 2.10 shows, we have that \( \tilde{Y} \) is \( \delta \)-hyperbolic, that \( \tilde{Y} \) is \( K \)-roughly starlike from \( \omega \), and that \( \rho_{\varepsilon, b} \) is admissible for \( \tilde{Y} \) with constant \( M \). We can then apply the case \( b \in B(X) \) that we established above to conclude that the map \( \tilde{\Omega} \rightarrow \tilde{Y} \) induced by the identity map on \( \tilde{\Omega} \) is \( \partial \)-biLipschitz with data \((L, \lambda)\) and is \( \eta \)-quasisymmetric, where \( L, \lambda, \) and \( \eta \) depend only on \( A, \varepsilon \) and \( M \). Since the natural embeddings \( \Omega \rightarrow \tilde{\Omega} \) and \( Y_{\varepsilon} \rightarrow \tilde{Y}_{\varepsilon} \) are isometric and we have both \( \partial \Omega = \partial \tilde{\Omega} \) and \( \partial Y_{\varepsilon} = \partial \tilde{Y}_{\varepsilon} \), we conclude that the identity map \( \Omega \rightarrow Y_{\varepsilon} \) is \( \partial \)-biLipschitz with data \((L, \lambda)\) and is \( \eta \)-quasisymmetric, where \( L, \lambda, \) and \( \eta \) depend only on \( A, \varepsilon \) and \( M \).

5. Quasihyperbolizing the uniformization

One may also consider the reverse direction from Section 4. We can start with a proper geodesic \( \delta \)-hyperbolic space \( X \) and a function \( b \in \hat{B}(X) \) such that \( X \) is \( K \)-roughly starlike from the basepoint \( \omega_b \) of \( b \), let \( \varepsilon > 0 \) be given such that \( \rho_{\varepsilon, b} \) is an admissible density on \( X \) with constant \( M \), consider the resulting uniformization \( X_{\varepsilon, b} \) of \( X \), and then take the quasihyperbolization \( Y \) of \( X_{\varepsilon, b} \). For the case \( b \in D(X) \) this situation is considered under slightly more restrictive hypotheses in [2, Proposition 4.37], in which case it is shown that the map \( X \rightarrow Y \) induced by the identity map on \( X \) is biLipschitz. We will generalize their result to the case of Busemann functions \( b \in B(X) \) here, with nearly the same proof. This result will not be needed elsewhere in the paper.

**Proposition 5.1.** The identity map \( X \rightarrow Y \) is \( H \)-biLipschitz with \( H = H(\delta, K, \varepsilon, M) \).

**Proof.** We will prove this result in the case \( b \in B(X) \). The case \( b \in D(X) \) can then be deduced from this using a ray augmentation argument as we did at the end of the proof of Proposition 4.11. Since this result will not be playing an important role in this paper, we leave the details of this deduction to the reader.

Thus we will assume that \( b \in B(X) \). We write \( X_{\varepsilon} = X_{\varepsilon, b} \), \( \rho_{\varepsilon} = \rho_{\varepsilon, b} \), \( d_{\varepsilon} = d_{\varepsilon, b} \), etc. We write \( \ell_{\varepsilon}(\gamma) = \ell_{d_{\varepsilon}}(\gamma) \) for the length of a rectifiable curve \( \gamma \) in the metric \( d_{\varepsilon} \). For \( x, y \in X \) we will denote their distance in \( X \) by \(|xy|\) and their distance in \( Y \) by \( k(x, y) \). We start by showing that the identity map \( X \rightarrow Y \) is Lipschitz. Let \( x, y \in X \) and let \( \gamma \) be a geodesic in \( X \) joining them, parametrized by arc length. Let

\[
L_{\varepsilon}(t) = \int_0^t \rho_{\varepsilon}(\gamma(s)) \, ds,
\]

be the length measurement of this curve in \( X_{\varepsilon} \). Then by Proposition 2.11

\[
k(x, y) \leq \int_\gamma \frac{|dz|}{d_{\varepsilon}(z)}
= \int_\gamma \frac{dL_{\varepsilon}(t)}{d_{\varepsilon}(z)}
= \int_0^{|xy|} \frac{\rho_{\varepsilon}(\gamma(t))}{d_{\varepsilon}(\gamma(t))} \, dt
\leq C|xy|,
\]

with \( C = C(\delta, \varepsilon, K, M) \). It follows that the identity map \( X \rightarrow Y \) is \( C \)-Lipschitz.
For the lower bound on $k(x, y)$, we apply the admissibility inequality (1.9) and the inequality (3.1) to the geodesic $\gamma$ and then use Proposition 2.11 as well to obtain

$$k(x, y) \geq \log \left( 1 + \frac{d_x(x, y)}{\min \{d_x(x), d_x(y)\}} \right)$$

$$\geq \log \left( 1 + C^{-1} \frac{\ell_x(\gamma)}{\min \{d_x(x), d_x(y)\}} \right)$$

$$\geq \log \left( 1 + C^{-1} \frac{\ell_x(\gamma)}{\min \{\rho_x(x), \rho_x(y)\}} \right),$$

with $C = C(\delta, K, \varepsilon, M) \geq 1$. We now restrict to the case $\varepsilon |xy| \leq 1$. By the Harnack inequality (2.11) we then have that $\rho_x(x) \approx \rho_x(z)$ for all $z \in \gamma$. This implies that

$$\ell_x(\gamma) \geq e^{-1} \rho_x(x)|xy|,$$

Therefore,

$$k(x, y) \geq \log \left( 1 + C^{-1} |xy| \right),$$

still with $C = C(\delta, K, \varepsilon, M)$. Using the inequality (1.9) with $t = C^{-1} |xy|$ and $a = 1$, noting that since $\varepsilon |xy| \leq 1$ we can increase the constant $C$ by an amount depending only on $\varepsilon$ such that $C^{-1} \varepsilon^{-1} \leq 1$ and therefore $t \leq 1$, we conclude that

$$k(x, y) \geq C^{-1} |xy|,$$

with $C = C(\delta, K, \varepsilon, M)$. This handles the case $\varepsilon |xy| \leq 1$.

We can thus move to the case $\varepsilon |xy| > 1$. We consider a geodesic triangle $\Delta = \omega xy$ with vertices $\omega, x, y$ that has $\gamma$ as an edge. We let $T : \Delta \to \Upsilon$ be an associated tripod map and assume that $\gamma$ is parametrized in accordance with Proposition 2.22. We set $u = \gamma(0)$. A straightforward computation using (5) of Proposition 2.22 shows that

$$\ell_x(\gamma) \geq C^{-1} \rho_x(u) \left( \int_0^{\varepsilon |xy|} e^{-ct} \, dt + \int_0^{\varepsilon |xy|} e^{-ct} \, dt \right)$$

$$= C^{-1} \varepsilon^{-1} \rho_x(u) \left( 2 - e^{-\varepsilon |xy|} - e^{-\varepsilon |xy|} \right)$$

$$\geq C^{-1} \varepsilon^{-1} \rho_x(u) \left( 1 - e^{-\varepsilon |xy|} \right)$$

$$\geq C^{-1} \varepsilon^{-1} \rho_x(u),$$

with $C = C(\delta, K, \varepsilon, M) \geq 1$, using that $\varepsilon |xy| \geq 1$ so that $1 - e^{-\varepsilon |xy|} \geq 1 - e^{-1}$.

By (5) of Proposition 2.22 we have $|xu| \approx c b(x) - b(u)$ and $|yu| \approx c b(y) - b(u)$ with $c = c(\delta)$. By switching the roles of $x$ and $y$ if necessary we can assume that $|xu| \geq |yu|$ and therefore $|xu| \geq \frac{1}{2} |xy|$. Then

$$\rho_x(x) \approx C(\delta) \rho_x(u) e^{-\varepsilon |xu|} \leq \rho_x(u) e^{-\varepsilon |xy|},$$

so that

$$\min \{\rho_x(x), \rho_x(y)\} \leq \rho_x(u) e^{-\frac{\varepsilon}{2} |xy|}.$$ 

Picking up from the inequality

$$k(x, y) \geq \log \left( 1 + C^{-1} \frac{\ell_x(\gamma)}{\min \{\rho_x(x), \rho_x(y)\}} \right),$$

it now follows that

$$k(x, y) \geq \log \left( 1 + C^{-1} e^{\frac{\varepsilon}{2} |xy|} \right),$$
with \( C = C(\delta, K, \varepsilon, M) \geq 1 \). We apply the elementary inequality \([2] (2.12)\), which states for \( a \geq 1, t \geq 0 \),

\[(5.1) \quad a^{-1} \log(1 + at) \leq \log(1 + t),\]

with \( a = C, t = C^{-1} e^{xy} \), to obtain

\[k(x, y) \geq C^{-1} \log(1 + e^{xy}) \geq C^{-1} |xy|,\]

with \( C = C(\delta, K, \varepsilon, M) \). This gives the desired lower bound on \( k(x, y) \).

\[\Box\]

6. The Main Theorems

In this final section we will complete the proofs of the main theorems. The key step will be Proposition 6.3, which may be of independent interest.

Following Buyalo and Schroeder \([5, \text{Chapter 4}]\), we define the cross-difference of four points \( x, y, z, w \) in a metric space \( X \) by

\[(6.1) \quad (x, y, z, w) = \frac{1}{2} (|xz| + |yw| - |xy| - |zw|).\]

The significance of the expression \((6.1)\) lies in the following lemma, which is a simple calculation that we leave to the reader.

**Lemma 6.1.** Let \( X \) be a proper geodesic \( \delta \)-hyperbolic space. For \( b \in \hat{B}(X) \) we have for any \( x, y, z, w \in X \),

\[(6.2) \quad - (x|z)_b - (y|w)_b + (x|y)_b + (z|w)_b = \langle x, y, z, w \rangle.\]

We will also require a lemma regarding the behavior of the distance function \( d_{\varepsilon, b} \) on balls of the form \( B_{d_{\varepsilon, b}}(x, \lambda d_{\varepsilon, b}(x)) \) for \( \lambda > 0 \) sufficiently small.

**Lemma 6.2.** Let \( X \) be a proper geodesic \( \delta \)-hyperbolic space and let \( b \in \hat{B}(X) \). We suppose that \( X \) is \( K \)-roughly starlike from the basepoint \( \omega_b \) of \( b \) and that we are given \( \varepsilon > 0 \) such that \( \rho_{\varepsilon, b} \) is an admissible density for \( X \) with constant \( M \). Then there exists \( 0 < \lambda < 1 \) with \( \lambda = \lambda(\delta, K, \varepsilon, M) \) such that for any \( x \in X \) and any \( y, z \in B_{d_{\varepsilon, b}}(x, \lambda d_{\varepsilon, b}(x)) \) we have that

\[(6.3) \quad d_{\varepsilon, b}(y, z) \leq C \frac{d_{\varepsilon, b}(x)}{|yz|},\]

with \( C = C(\delta, K, \varepsilon, M) \).

**Proof.** To simplify notation we will write \( B_{\varepsilon}(x, r) = B_{d_{\varepsilon, b}}(x, r) \) for the ball of radius \( r \) centered at \( x \) in \( X_{\varepsilon, b} \). We will write \( X_{\varepsilon} = X_{\varepsilon, b}, d_{\varepsilon} = d_{\varepsilon, b} \), etc. Let \( 0 < \lambda < 1 \) be given. We first observe by inequality \((3.2)\) that for \( y \in B(x, \lambda d_{\varepsilon}(x)) \) we have

\[d_{\varepsilon}(y) \leq d_{\varepsilon}(x) + d_{\varepsilon}(x, y) \leq (1 + \lambda) d_{\varepsilon}(x),\]

and

\[d_{\varepsilon}(y) \geq d_{\varepsilon}(x) - d_{\varepsilon}(x, y) \geq (1 - \lambda) d_{\varepsilon}(x).\]

By making the restriction \( \lambda \leq \frac{1}{2} \) we can then conclude that \( d_{\varepsilon}(y) \geq 2 d_{\varepsilon}(x) \) for \( y \in B_{\varepsilon}(x, \lambda d_{\varepsilon}(x)) \). We will impose this restriction in what follows.

Thus if \( y, z \in B_{\varepsilon}(x, \lambda d_{\varepsilon}(x)) \) then combining inequality \((2.12)\) with Proposition \((2.11)\) implies that

\[1 - e^{-\varepsilon |yz|} \leq C \frac{d_{\varepsilon}(y, z)}{d_{\varepsilon}(y)} \leq C \frac{d_{\varepsilon}(y, z)}{d_{\varepsilon}(x)} \leq C \lambda,\]

with \( C = C(\delta, K, \varepsilon, M) \), which can be rearranged to

\[(6.4) \quad e^{-\varepsilon |yz|} \geq 1 - C \lambda.\]
Thus by further decreasing \( \lambda \), depending only on \( \delta, K, \varepsilon, \) and \( M \), we can assume that \( y, z \in B_\varepsilon(x, \lambda d(x)) \) implies that \( e^{-\varepsilon|yz|} \geq e^{-\varepsilon} \), which implies that \( |yz| \leq 1 \). This gives the first claim.

For the second claim we let \( \lambda \) be as determined in the first claim and let \( y, z \in B_\varepsilon(x, \lambda d(x)) \). Since \( |yz| \leq 1 \), by the comparison \( \ref{2.3} \) we have
\[
d_\varepsilon(y, z) \asymp e^{-\varepsilon(y|z|)} |yz|,
\]
with \( C = C(\delta, K, \varepsilon, M) \). Since \( b \) is 1-Lipschitz we have
\[
|b(y) - (y|z|)u| \leq |yz| \leq 1,
\]
and therefore \( \rho_\varepsilon(y) \asymp e^{-\varepsilon(y|z|)} \). Thus
\[
d_\varepsilon(y, z) \asymp C \rho_\varepsilon(y)|yz| \asymp C d_\varepsilon(y)|yz| \asymp 2 d_\varepsilon(x)|yz|,
\]
with \( C = C(\delta, K, \varepsilon, M) \), where we have used Proposition \( \ref{2.11} \) once more. The comparison \( \ref{6.3} \) follows.

The following proposition is inspired by \cite{2} Proposition 4.15. However our methods are somewhat different.

**Proposition 6.3.** Let \( f : X \to X' \) be an \( H \)-biLipschitz map between proper geodesic \( \delta \)-hyperbolic spaces and let \( b \in \text{mathcal}(B(X'), b' \in \text{B}(X') \) be given such that \( X \) and \( X' \) are \( K \)-roughly starlike from their basepoints \( \omega_b \) and \( \omega_{b'} \) respectively. We suppose that we are given \( \varepsilon, \varepsilon' > 0 \) such that \( \rho_{\varepsilon,b} \) and \( \rho_{\varepsilon',b'} \) are admissible densities for \( X \) and \( X' \) respectively with the same constant \( M \).

Then the induced map \( f : X_{\varepsilon,b} \to X_{\varepsilon',b'} \) is \( \partial \)-biLipschitz with data \( (L, \lambda) \) and is \( \theta \)-quasimöbius with \( \theta(t) = C \max\{t, t^{C_1}, t^{C_2}\} \). The constants \( L, \lambda, \) and \( C \) depend only on \( \delta, K, \varepsilon, \varepsilon', M, \) and \( H \). The constants \( C_0 \) and \( C_0 \) have the form \( C_1 = \frac{\varepsilon}{\varepsilon'} C_0^{-1} \) and \( C_2 = \frac{\varepsilon}{\varepsilon'} C_0 \), where \( \varepsilon_0 = C_0(\delta, H) \) depends only on \( \delta \) and \( H \).

**Proof.** To simplify notation in the proof we will write \( x' = f(x) \) for the image of a point under \( f : X \to X' \). However we will not apply this convention to the relation of \( b' \) to \( b, \) i.e., even if \( b \in X, b' \in X' \) we still do not assume that \( b' = f(b) \). We write \( d_{\varepsilon',b'} = d_{\varepsilon',b} \), etc.

for the uniformized distance on \( X' \). We write \( B_\varepsilon = B_{\varepsilon,b} \) for balls in \( X_{\varepsilon,b} \) and \( B_{\varepsilon',b'} = B_{\varepsilon',b'} \) for balls in \( X_{\varepsilon',b'} \). To avoid repeatedly writing out long lists of parameter dependencies, we will use the expression “the given data” to indicate the parameters \( \delta, K, \varepsilon, \varepsilon', M, \) and \( H \).

We will first establish that the induced map \( f : X_{\varepsilon,b} \to X_{\varepsilon',b'} \) is \( \partial \)-biLipschitz with data \( (L, \lambda) \) depending only on the given data. Since \( f^{-1} : X' \to X \) is also \( H \)-biLipschitz, it suffices by symmetry to show that \( f \) is \( \partial \)-Lipschitz with data \( (L, \lambda) \) depending only on the given data. For use later we will prove this claim under the weaker hypothesis that \( f : X \to X' \) is a continuous map that is \( H \)-Lipschitz for some \( H \geq 0 \).

We let \( \lambda \) be sufficiently small that the conclusions of Lemma \( \ref{6.2} \) hold on both \( B_\varepsilon(x, \lambda d_{\varepsilon,b}(x)) \) and \( B'_{\varepsilon}(x', \lambda d_{\varepsilon',b'}(x')) \) for \( x \in X \); by that lemma we can choose \( \lambda \) to depend only on the given data. We claim that there is a \( 0 < \lambda \leq \lambda \) also depending only on the given data such that if \( y \in B_\varepsilon(x, \lambda d_{\varepsilon,b}(x)) \) then \( y' \in B'_{\varepsilon}(x', \lambda d_{\varepsilon',b'}(x')) \). The computation of Lemma \( \ref{6.2} \) (specifically inequality \( \ref{6.3} \)) shows that for any \( \kappa > 0 \) we can, by sufficiently decreasing \( \lambda \), find \( \lambda \) depending only on \( \kappa \) and the given data such that if \( y \in B_\varepsilon(x, \lambda d_{\varepsilon,b}(x)) \) then \( |xy| \leq \kappa \). Plugging this into inequality \( \ref{2.12} \) applied on \( X_{\varepsilon',b'} \) together with the fact that \( f \) is \( H \)-Lipschitz, we conclude that
\[
d_{\varepsilon',b'}(x', y') \leq C(e^{H\varepsilon'} - 1)d_{\varepsilon',b'}(x'),
\]
with $C = C(\delta, K, \varepsilon', M)$. We then choose $\kappa$ close enough to 0 that $C(e^{H\varepsilon'\kappa} - 1) < \lambda'$ in the above inequality and then choose $\lambda \leq \lambda'$ based on this value of $\kappa$.

Now let $y, z \in B_x(\lambda d_{\varepsilon, b}(x))$ be given. Then by Lemma 6.2 we have
\[ d_{\varepsilon, b}(y, z) \geq C d_{\varepsilon, b}(x)|yz|, \]
with $C = C(\delta, K, \varepsilon, M)$. Meanwhile, the work of the previous paragraph shows that $y', z' \in B_{\varepsilon'}(x', \lambda d_{\varepsilon', b}(x'))$ as well, which implies by a second application of Lemma 6.2 that
\[ d_{\varepsilon', b}(y', z') \geq C d_{\varepsilon', b}(x)|y'z'|, \]
with $C = C(\delta, K, \varepsilon, M)$. Since $|y'z'| \leq H|yz|$, the desired inequality (6.7) immediately follows with a constant $L$ depending only on the given data. This implies that $f$ is $\theta$-Lipschitz with data $(L, \lambda)$ as desired.

We now return to assuming that $f : X \to X'$ is an $H$-biLipschitz homeomorphism. We will show that $f$ is $\theta$-quasimöbius with the control function $\theta$ having the indicated form. By the comparison (2.13) we have for $x, y \in X$
\[ d_{\varepsilon, b}(x, y) \geq C \varepsilon^{-1} e^{-\varepsilon(x|y)b} \min\{1, |xy|\}, \]
with $C = C(\delta, K, \varepsilon, M)$. Applying this to the cross-ratio of four distinct points $x, y, z, w \in X_{\varepsilon, b}$ and using Lemma 6.4 we obtain
\begin{equation}
[x, y, z, w]_{d_{\varepsilon, b}} \geq C \varepsilon^{c(x, y, z, w)} \frac{\min\{1, |xz|\} \min\{1, |yw|\}}{\min\{1, |xy|\} \min\{1, |zw|\}},
\end{equation}
with $C = C(\delta, K, \varepsilon, M)$. Applying this same calculation to $X_{\varepsilon', b'}$, recalling that $x' = f(x)$ denotes images of points under $f$, we obtain that
\begin{equation}
[x', y', z', w']_{d_{\varepsilon', b'}} \geq C \varepsilon^{c(x', y', z', w')} \frac{\min\{1, |x'z'|\} \min\{1, |y'w'|\}}{\min\{1, |x'y'|\} \min\{1, |z'w'|\}},
\end{equation}
with $C = C(\delta, K, \varepsilon', M)$.

Since $f$ is an $H$-biLipschitz map between geodesic $\delta$-hyperbolic spaces, by [5] Theorem 4.4.1 we have that $f$ roughly quasi-preserves the cross-difference in the following sense: there are constants $C_0 \geq 1$ and $c_0 \geq 0$ depending only on $H$ and $\delta$ such that the cross-difference of the points $x, y, z, w$ compared to the image points $x', y', z', w'$ satisfies
\begin{equation}
C_0^{-1} \langle x, y, z, w \rangle - c_0 \leq \langle x', y', z', w' \rangle \leq C_0 \langle x, y, z, w \rangle + c_0,
\end{equation}
whenever $\langle x, y, z, w \rangle \geq 0$. Thus in the case $\langle x, y, z, w \rangle \geq 0$ there is $C_1 = \frac{\varepsilon'}{\varepsilon} C_0^{-1}$ and $C_2 = \frac{\varepsilon}{\varepsilon'} C_0$ such that
\begin{equation}
C_1^{-1} (e^{c(x, y, z, w)}) \leq e^{c(x', y', z', w')} \leq C_2 (e^{c(x, y, z, w)}),
\end{equation}
with $C = C(\delta, K, \varepsilon, \varepsilon', M, H)$. We note that $C_1$ and $C_2$ only depend on $\varepsilon$ and $\varepsilon'$ through the ratio $\frac{\varepsilon}{\varepsilon'}$. Note also that $C_1 = C_2 = \frac{\varepsilon'}{\varepsilon}$ if $X = X'$ and $f : X \to X$ is the identity map, since in this case we can take $C_0 = 1$ and $c_0 = 0$.

We set
\[ G(x, y, z, w) = \frac{\min\{1, |xz|\} \min\{1, |yw|\}}{\min\{1, |xy|\} \min\{1, |zw|\}}. \]
Let us first assume in addition to $\langle x, y, z, w \rangle \geq 0$ that we also have $G(x, y, z, w) \geq 1$. Then we have
\[ G(x, y, z, w)^{C_2} \leq G(x, y, z, w) \leq G(x, y, z, w)^{C_1}, \]
which implies by inequality (6.8) and the comparison (6.5) that
\[
\tag{6.9}
C^{-1}[x, y, z, w]_{d_{e,b}}^{C_2} \leq e^\varepsilon(x', y', z', w') \frac{\min\{1, |xz|\} \min\{1, |yw|\}}{\min\{1, |xy|\} \min\{1, |zw|\}} \leq C[x, y, z, w]_{d_{e,b}}^{C_1},
\]
with \( C = C(\delta, K, \varepsilon, \varepsilon', M, H) \). Since \( f \) is \( H \)-biLipschitz we have
\[
\tag{6.10}
\frac{\min\{1, |x'| \} \min\{1, |y'|\}}{\min\{1, |x'y'|\} \min\{1, |z'|w'|\}} \approx_{H^4} \frac{\min\{1, |xz|\} \min\{1, |yw|\}}{\min\{1, |xy|\} \min\{1, |zw|\}},
\]
which when combined with inequalities (6.9) and (6.6) gives us
\[
\tag{6.11}
C^{-1}[x, y, z, w]_{d_{e,b}}^{C_2} \leq [x', y', z', w']_{d_{e,b'}} \leq C[x, y, z, w]_{d_{e,b}}^{C_1},
\]
with \( C' \) depending only on the given data.

Let’s now assume instead that \( \langle x, y, z, w \rangle \geq 0 \) and \( G(x, y, z, w) < 1 \). Then it must be the case that either \( |xz| < 1 \) or \( |yw| < 1 \). Since we have the equality \( \langle x, y, z, w \rangle = \langle y, x, w, z \rangle \) when we interchange the roles of the pairs \((x, z)\) and \((y, w)\), we can assume without loss of generality that \( |xz| \leq 1 \). Then by the triangle inequality we have \( |zw| = 1 |xw| \) and therefore
\[
\langle x, y, z, w \rangle = \frac{1}{2}( |yw| - |xy| - |xw| ) \leq 0.
\]
But since we’ve also assumed that \( \langle x, y, z, w \rangle \geq 0 \), it then follows that \( 0 \leq \langle x, y, z, w \rangle \leq 1 \). Applying inequality (6.7) then gives that \( |[x', y', z', w']| \leq c, \) with \( c = c(\delta, H) \) depending only on \( \delta \) and \( H \). Thus in this case, by combining the comparison (6.10) with the comparisons (6.9) and (6.6) we conclude simply that
\[
\tag{6.12}
[x', y', z', w']_{d_{e,b'}} \approx_{C'} [x, y, z, w]_{d_{e,b}},
\]
with \( C' \) depending only on the given data.

Now suppose that \( \langle x, y, z, w \rangle \leq 0 \). Then
\[
\langle w, y, z, x \rangle = -\langle x, y, z, w \rangle \geq 0,
\]
and \( G(w, y, z, x) = G(x, y, z, w) \). We thus conclude that inequality (6.11) holds if \( G(x, y, z, w) \geq 1 \) and inequality (6.12) holds if \( G(x, y, z, w) < 1 \), with \( x \) and \( w \) swapping places in these inequalities. But since \( [w, y, z, x]_{d_{e,b}} = [x, y, z, w]_{d_{e,b}}^{-1} \) and \( [w', y', z', x']_{d_{e,b'}} = [x', y', z', w']_{d_{e,b'}}^{-1} \), it follows from this that for \( \langle x, y, z, w \rangle \leq 0 \) and \( G(x, y, z, w) \geq 1 \) we have
\[
\tag{6.13}
C^{-1}[x, y, z, w]_{d_{e,b}}^{C_1} \leq [x', y', z', w']_{d_{e,b'}} \leq C[x, y, z, w]_{d_{e,b}}^{C_2},
\]
and if \( \langle x, y, z, w \rangle \leq 0 \) and \( G(x, y, z, w) < 1 \) then
\[
\tag{6.14}
[x', y', z', w']_{d_{e,b'}} \approx_{C'} [x, y, z, w]_{d_{e,b}},
\]
with \( C, C', C_1, \) and \( C_2 \) the same as in (6.11) and (6.12). Combining (6.11), (6.12), (6.13), and (6.14), we conclude that \( f \) is \( \theta \)-quasimöbius with \( \theta(t) = C \max\{t, t^{C_1}, t^{C_2}\} \), \( C = C(\delta, K, \varepsilon, \varepsilon', M, H) \).

\[\tag*{\Box}\]

**Remark 6.4.** There are some special cases in which the conclusions of Proposition 6.3 can be improved.

1. If \( X = X' \) and \( f : X \to X \) then we can take \( C_0 = 1 \), as is noted in the proof. This implies that the control function \( \theta \) has the form \( \theta(t) = C \max\{t, t^{C_1}, t^{C_2}\} \). This proves Theorem 1.9.
Proposition 3.2. Let \( \not\) of incomplete metric spaces such that \( \not\) will take

\[ \infty \]

Proposition 6.5. Let \( f_1 : (\Omega, d) \to (\Omega', d') \) and \( f_2 : (\Omega', d') \to (\Omega'', d'') \) be homeomorphisms of incomplete metric spaces such that \( f_i \) is \( \partial\)-Lipschitz with data \( (L_i, \lambda_i) \), \( i = 1, 2 \). Then \( f_2 \circ f_1 \) is \( \partial\)-Lipschitz with data \( (L, \lambda) \) where \( L = L_2 L_1 \) and \( \lambda = \min\{\lambda_1, L_1^{-1} \lambda_2\} \).

Similarly if \( f_i \) is \( \partial\)-biLipschitz with data \( (\lambda_i, \lambda_i) \), \( i = 1, 2 \), then \( f_2 \circ f_1 \) is \( \partial\)-biLipschitz with data \( (L, \lambda) \) where \( L = L_2 L_1 \) and \( \lambda = \min\{L_2^{-1} \lambda_1, L_1^{-1} \lambda_2\} \).

Proof. Let \( \lambda = \min\{\lambda_1, L_1^{-1} \lambda_2\} \). Then for \( y, z \in B_d(x, \lambda d_{\Omega}(x)) \) we have

\[
\frac{d''(f_1(y), f_1(z))}{d'_{\Omega'}(f_1(x))} \leq L_1 \frac{d(y, z)}{d_{\Omega}(x)} < \lambda_2.
\]

Thus we can apply the inequality (1.1) for \( f_2 \) to \( f_1 \) and \( f_1 \) to get for \( y, z \in B_d(x, \lambda d_{\Omega}(x)) \),

\[
\frac{d''(f_2(f_1(y)), f_2(f_1(z)))}{d''_{\Omega''}(f_2(f_1(x)))} \leq L_2 \frac{d'(f_1(y), f_1(z))}{d'_{\Omega'}(f_1(x))} \leq L_1 L_2 \frac{d(y, z)}{d_{\Omega}(x)}.
\]

Thus inequality (1.1) holds for \( f_2 \circ f_1 \) on \( B_d(x, \lambda d_{\Omega}(x)) \) with constant \( L = L_2 L_1 \). The claim regarding \( \partial\)-biLipschitz maps follows from applying this argument to the composition \( f_1^{-1} \circ f_2^{-1} = (f_2 \circ f_1)^{-1} \) and noting that in this case we have \( L_i^{-1} \leq 1 \), \( i = 1, 2 \).

Proof of Theorem 1.5. We let \( Y = (\Omega, k) \) and \( Y' = (\Omega', k') \) denote the quasi hyperbolizations of \( \Omega \) and \( \Omega' \) respectively. We will use the notation \( d(x) = d_{\Omega}(x) \) for \( x \in \Omega \) and \( d'(x) = d_{\Omega'}(x) \) for \( x \in \Omega' \). We first assume that \( f : \Omega \to \Omega' \) is \( \partial\)-Lipschitz with data \( (L, \lambda) \). Since \( Y \) is geodesic it suffices to prove that the induced map \( f : Y \to Y' \) is locally \( H\)-Lipschitz, i.e., that for any \( x \in Y \) there is an open neighborhood \( U_x \) of \( x \) on which \( f \) is \( H\)-Lipschitz. We will take \( U_x = B_d(x, \lambda d(x)) \). Then by inequalities (1.1) and (3.1) as well as inequality (5.4),

\[
k'(f(x), f(y)) \leq 4A^2 \log \left( \frac{1 + \frac{d'(f(x), f(y))}{\min\{d'(x), d'(y)\}}}{1 + \frac{d(x, y)}{\min\{d(x), d(y)\}}} \right)
\]

\[
\leq 4A^2 \log \left( 1 + L \frac{d(x, y)}{\min\{d(x), d(y)\}} \right)
\]

\[
\leq 4A^2 L \log \left( 1 + \frac{d(x, y)}{\min\{d(x), d(y)\}} \right)
\]

\[
\leq 4A^2 L k(x, y).
\]

We conclude that \( f \) is \( H\)-Lipschitz with \( H = 4A^2 L \).

We now assume that the induced map \( f : Y \to Y' \) is \( H\)-Lipschitz for some \( H \geq 0 \). If \( \Omega \) is bounded then we let \( \omega \in \Omega \) be such that \( d(\omega) = \sup_{x \in \Omega} d(x) \) and set \( b(x) = k(x, \omega) \), while if \( \Omega \) is unbounded then we let \( b \) be a Busemann function on \( Y \) based at the point \( \omega \in \partial Y \) corresponding to the ideal point \( \infty \) for \( \Omega \). We define \( b' \in \hat{B}(Y') \) similarly. Then \( Y \) and \( Y' \) are each \( K\)-roughly starlike from the basepoints of \( b \) and \( b' \) respectively with \( K = K(A) \) by Proposition 5.2.
By Theorem 1.4, we can find $\varepsilon = \varepsilon(A)$ such that the densities $\rho_{\varepsilon,b}$ and $\rho_{\varepsilon,b}$ are admissible on $Y$ and $Y'$ respectively with constant $M = 20$. Let $Y_{\varepsilon,b}$ and $Y'_{\varepsilon,b'}$ be the respective uniformizations of $Y$ and $Y'$. The first part of the proof of Proposition 6.3 implies that the induced map $f : Y_{\varepsilon,b} \to Y'_{\varepsilon,b'}$ is $\partial$-Lipschitz with data $(L, \lambda)$, where $L$ and $\lambda$ depend only on $A$ and $H$ (since $\delta = \delta(A)$, $K = K(A)$, $\varepsilon = \varepsilon(A)$, and $M = 20$). By Proposition 4.1, the identity maps $\Omega \to Y_{\varepsilon,b}$ and $\Omega' \to Y'_{\varepsilon,b'}$ are each $\partial$-biLipschitz with data $(L_0, \lambda_0)$ depending only on $A$ since $\varepsilon = \varepsilon(A)$ and $M = 20$. Since inverses of $\partial$-biLipschitz maps are $\partial$-biLipschitz and compositions of $\partial$-Lipschitz maps are $\partial$-Lipschitz (by Proposition 6.3), we conclude that $f : \Omega \to \Omega'$ is $\partial$-Lipschitz with data $(L, \lambda)$ depending only on $A$ and $H$. □

We can also prove Theorem 1.4. We use the same notation as in the proof of Theorem 1.5.

Proof of Theorem 1.4. By Theorem 1.5 there is an $H = H(A, L)$ such that the induced map $f : Y \to Y'$ is $H$-biLipschitz. By Proposition 6.3, the induced map $f : Y_{\varepsilon,b} \to Y'_{\varepsilon,b'}$ is $\theta_0$-quasimöbius with $\theta_0$ depending only on $A$ and $H$ (and therefore only on $A$ and $L$). By Proposition 4.1, the identity maps $\Omega \to Y_{\varepsilon,b}$ and $\Omega' \to Y'_{\varepsilon,b'}$ are both $\theta_0$-quasimöbius with $\theta_1$ depending only on $A$. Since quasimöbius maps are quasimöbius with quantitative control over the control function [11, Theorem 3.2] and both compositions and inverses of quasimöbius maps are quasimöbius, it follows that $f : \Omega \to \Omega'$ is $\theta$-quasimöbius with $\theta$ depending only on $A$ and $L$. □

Our final task is to prove Theorem 1.11. For all of the claims below we fix a proper geodesic $\delta$-hyperbolic space $X$ that is $K$-roughly starlike from some $z \in X$ and $\omega \in \partial X$. We fix a Busemann function $b$ based at $\omega$ that is chosen such that $b(z) = 0$ (this can always be done by adding a constant). We let $\varepsilon > 0$ be given such that the densities $\rho_{\varepsilon,z}$ and $\rho_{\varepsilon,b}$ are admissible for $X$ with constant $M$. By [11, Theorem 1.4] the metric spaces $X_{\varepsilon,z}$ and $X_{\varepsilon,b}$ are each $A$-uniform with $A = A(\delta, K, \varepsilon, M)$. We will consider the cases of inversion and sphericalization separately.

We start with the case of inversion. We recall that $X_{\varepsilon,z}$ denotes the inversion of $X_{\varepsilon,z}$ based at the point $\omega \in \partial X_{\varepsilon,z}$ defined prior to Theorem 1.11. We will need the following lemma.

Lemma 6.6. There is a constant $C = C(\delta, K, \varepsilon, M) \geq 1$ such that $\text{diam } X_{\varepsilon,z} \leq C \text{diam } \partial X_{\varepsilon,z}$.

Furthermore we have

$$\text{diam } X_{\varepsilon,z} \leq C \text{diam } \partial X_{\varepsilon,z}.$$ 

Proof. By a simple calculation [2, (4.3)] we have $\text{diam } X_{\varepsilon,z} \leq 2\varepsilon^{-1}$. On the other hand, since $X$ is $K$-roughly starlike from $\omega$ we can find a geodesic line $\gamma : \mathbb{R} \to X$ starting from $\omega$ that is parametrized such that $|z\gamma(0)| \leq K$. Let $\xi \in \partial X$ be the other endpoint of $\gamma$. Then, since $X_{\varepsilon,z}$ is $A$-uniform with $A = A(\delta, K, \varepsilon, M)$, by Lemma 3.1 and Proposition 2.11 we have

$$d_{\varepsilon,z}(\xi, \omega) \leq C \text{ sup } d_{\varepsilon,z}(\gamma(t)) \leq C \text{ sup } \rho_{\varepsilon,z}(\gamma(t)).$$

with $C = C(\delta, K, \varepsilon, M)$. By the Harnack inequality [11, Proposition 2.11] we conclude that

$$d_{\varepsilon,z}(\xi, \omega) \geq C^{-1}\rho_{\varepsilon,z}(\gamma(0)) \approx_{\varepsilon,K} \rho_{\varepsilon,z}(z) = 1.$$ 

Thus $d_{\varepsilon,z}(\xi, \omega) \geq C^{-1}$, where $C = C(\delta, K, \varepsilon, M)$. By continuity it follows that

$$d_{\varepsilon,z}(\gamma(t), \gamma(-t)) \geq \frac{1}{2}C^{-1}.$$
for all sufficiently large $t$. The lower bound $\text{diam } X_{\varepsilon, z} \geq \frac{1}{2} C^{-1}$ follows. This gives the first claim of the lemma. The second claim of the lemma follows since 

$$C^{-1} \leq d_{\varepsilon, z}(\xi, \omega) \leq \text{diam } \partial X_{\varepsilon, z}.$$ 

\[ \square \]

**Proposition 6.7.** The metric space $X_{\varepsilon, z}$ is $A'$-uniform with $A' = A'(\delta, K, \varepsilon, M)$. The identity map $X_{\varepsilon, b} \rightarrow X_{\varepsilon, z}$ is $\partial$-biLipschitz with data $(L, \lambda)$ and $\theta$-quasisymmetric with $L$, $\lambda$, and $\theta$ depending only on $\delta, K, \varepsilon, M$.

**Proof.** By [3, Theorem 5.1(a)] the metric space $\hat{X}_{\varepsilon, b}$ is $A'$-uniform with $A'$ depending only on the uniformity parameter $A_{\varepsilon, z}$. The identity map $X_{\varepsilon, z} \rightarrow X_{\varepsilon, b}$ is $\partial$-biLipschitz with data depending only on $\delta, K, \varepsilon, M$ by Proposition [6.3] hence is $H$-biLipschitz in the quasihyperbolic metrics on these spaces with $H = H(\delta, K, \varepsilon, M)$ by Theorem [4.7] the identity map $X_{\varepsilon, z} \rightarrow X_{\varepsilon, b}$ is $H'$-biLipschitz in the quasihyperbolic metrics with $H'$ depending only on the uniformity parameter $A = A(\delta, K, \varepsilon, M)$ of $X_{\varepsilon, z}$ and the ratio

$$\frac{\text{diam } X_{\varepsilon, z}}{\text{diam } \partial X_{\varepsilon, z}}.$$ 

By Lemma [6.6] this ratio is bounded above in terms of $\delta, K, \varepsilon, M$, so it follows that $H' = H'(\delta, K, \varepsilon, M)$.

We thus conclude that there is a constant $H'' = H''(\delta, K, \varepsilon, M)$ such that the identity map $X_{\varepsilon, b} \rightarrow X_{\varepsilon, z}$ is $H''$-biLipschitz in the quasihyperbolic metrics on these uniform spaces. By Theorems [1, 3], and [1.5] it then follows that this identity map is $\partial$-biLipschitz with data $(\lambda, L)$ and $\theta$-quasisymmetric with $\lambda$, $L$, and $\theta$ depending only on $\delta, K, \varepsilon, M$. To complete the proof we will show that the identity map $X_{\varepsilon, z} \rightarrow X_{\varepsilon, b}$ is actually $\theta$-quasisymmetric with the same control function $\theta$. Since both $X_{\varepsilon, z}$ and $X_{\varepsilon, b}$ are unbounded, it suffices by [11, Theorem 3.10] to show that if $\{x_n\}$ is any sequence in $X$ then $d_{\varepsilon, b}(x, x_n) \rightarrow \infty$ if and only if $d_{\varepsilon, x}(x, x_n) \rightarrow \infty$.

We recall from the definition (1.13) of the inversion of $X_{\varepsilon, z}$ based at $\omega$ that we have for any $x, y \in X_{\varepsilon, z}$,

$$d_{\varepsilon, z}(x, y) \approx d_{\varepsilon, z}(x, \omega) \frac{d_{\varepsilon, z}(x, \omega)}{d_{\varepsilon, z}(y, \omega)}.$$ 

From this comparison we see that $d_{\varepsilon, z}(x, x_n) \rightarrow \infty$ if and only if $d_{\varepsilon, z}(x_n, \omega) \rightarrow 0$. The comparison [2.13] shows that this happens if and only if $(x_n | \omega)_z \rightarrow \infty$. By Lemma [2.12] this occurs if and only if $d_{\varepsilon, z}(x, x_n) \rightarrow \infty$. This completes the proof. \[ \square \]

In the next proposition we handle the sphericalization side of Theorem [1.11]. Theorem [1.11] then follows by combining Propositions [6.7] and [6.8].

**Proposition 6.8.** The metric space $\tilde{X}_{\varepsilon, b}$ is $A'$-uniform with $A' = A'(\delta, K, \varepsilon, M)$. The identity map $X_{\varepsilon, z} \rightarrow \tilde{X}_{\varepsilon, b}$ is $\partial$-biLipschitz with data $(\lambda, L)$ and $\theta$-quasisymmetric with $L$, $\lambda$, and $\theta$ depending only on $\delta, K, \varepsilon, M$.

**Proof.** By [3, Theorem 5.5(a)] the metric space $\tilde{X}_{\varepsilon, b}$ is $A'$-uniform with $A'$ depending only on $A$ and an upper bound on $d_{\varepsilon, b}(z)$. Since by Proposition [2.11]

$$d_{\varepsilon, b}(z) \approx e^{-\varepsilon b(z)} = 1,$$

with $C = C(\delta, K, \varepsilon, M)$, we conclude that $\tilde{X}_{\varepsilon, b}$ is $A'$-uniform with $A' = A'(\delta, K, \varepsilon, M)$. The identity map $X_{\varepsilon, b} \rightarrow X_{\varepsilon, z}$ is $\partial$-biLipschitz with data depending only on $\delta, K, \varepsilon, M$ by
Proposition 6.3, hence is $H$-biLipschitz in the quasihyperbolic metrics on these spaces with $H = H(\delta, K, \varepsilon, M)$ by Theorem 1.3.

Viewing sphericalization as a special case of inversion as we did prior to the statement of Theorem 1.14 [3] we conclude that the identity map $X_{\varepsilon, b} \to \hat{X}_{\varepsilon, b}^z$ is $H'$-biLipschitz in the quasihyperbolic metrics with $H'$ depending only on the uniformity parameter $A = A(\delta, K, \varepsilon, M)$ of $X_{\varepsilon, b}$; we emphasize for the purposes of applying the theorem in the reference that the augmented metric space $X_{\varepsilon, b} \cup_{z=0} [0, 1]$ to which we are applying it is unbounded.

We thus conclude as in the case of Proposition 6.7 that the identity map $X_{\varepsilon, z} \to \hat{X}_{\varepsilon, b}^z$ is $H''$-biLipschitz in the hyperbolic metrics for $H'' = H''(\delta, K, \varepsilon, M)$. Therefore by combining Theorems 1.4 and 1.5 we conclude that the identity map $X_{\varepsilon, z} \to \hat{X}_{\varepsilon, b}^z$ is $\theta$-biLipschitz with data $(L, \lambda)$ and $\theta$-quasimöbius with $L$, $\lambda$, and $\theta$ depending only on $\delta$, $K$, $\varepsilon$, and $M$.

Lastly we must upgrade the identity map $X_{\varepsilon, z} \to \hat{X}_{\varepsilon, b}^z$ from being $\theta$-quasimöbius to being $\theta'$-quasisymmetric with $\theta'$ depending only on $\delta$, $K$, $\varepsilon$, and $M$. We will employ a criterion of Väisälä [11] Theorem 3.12] which shows that it suffices to find a constant $\kappa = \kappa(\delta, K, \varepsilon, M) > 0$ and a triple of points $x_i \in X_{\varepsilon, z}$, $i = 1, 2, 3$, such that we have

$$d_{\varepsilon, z}(x_i, x_j) \geq \kappa \text{ diam } X_{\varepsilon, z},$$

for $i \neq j$, and

$$\hat{d}_{\varepsilon, b}^z(x_i, x_j) \geq \kappa \text{ diam } \hat{X}_{\varepsilon, b}^z,$$

for $i \neq j$. Since diam $X_{\varepsilon, z} \asymp C 1$ with $C = C(\delta, K, \varepsilon, M)$ by Lemma 6.6 and diam $\hat{X}_{\varepsilon, b}^z \asymp 1$ by [3] Section 3.B], it suffices to produce a triple of points $x_i \in X$ and $\kappa > 0$ such that for $i \neq j$,

$$\min\{d_{\varepsilon, z}(x_i, x_j), \hat{d}_{\varepsilon, b}^z(x_i, x_j)\} \geq \kappa$$

As in the proof of Lemma 6.6 we let $\gamma : \mathbb{R} \to X$ be a geodesic line starting from $\omega$ such that $\text{dist}(\gamma(0), z) \leq K$. We will set $x_1 = \gamma(0)$, $x_2 = \gamma(t)$, and $x_3 = \gamma(-t)$ for a sufficiently large $t \geq 0$.

We take $t \geq 1$ so that $|x_i x_j| \geq 1$ for $i \neq j$. The triangle inequality shows that we have

$$(x|y)_z \equiv_K (x|y)_{\gamma(0)}$$

for any $x, y \in X$. The comparison (2.13) then gives

$$d_{\varepsilon, z}(\gamma(0), \gamma(t)) \asymp_C e^{-\varepsilon(\gamma(0)\gamma(t))_{\gamma(0)}} = e^{-\varepsilon},$$

with $C = C(\delta, K, \varepsilon, M)$. Similar calculations give that $d_{\varepsilon, z}(\gamma(0), \gamma(-t)) \asymp_C e^{-\varepsilon}$ and $d_{\varepsilon, z}(\gamma(t), \gamma(-t)) \asymp_C e^{-\varepsilon}$ with the same constant $C$. This takes care of the estimates corresponding to $d_{\varepsilon, z}$ in (6.15).

The distance $\hat{d}_{\varepsilon, b}^z$ in the sphericalization has the form for $x, y \in X$,

$$\hat{d}_{\varepsilon, b}^z(x, y) \asymp_4 \frac{d_{\varepsilon, b}(x, y)}{(1 + d_{\varepsilon, b}(x, z))(1 + d_{\varepsilon, b}(y, z))}.$$

Since $b$ is 1-Lipschitz we have $(x|z)_b \equiv_K (x|\gamma(0))_b$ for any $x \in X$. Since $\gamma$ is a geodesic starting from $\omega$ we have by [4] Lemma 2.6] that for $t \in \mathbb{R}$,

$$b(\gamma(t)) \equiv_{c(\delta)} t + b(\gamma(0)) \equiv_K t + b(z) = t.$$

The comparison (2.13) then yields $d_{\varepsilon, b}(\gamma(0), z) \asymp_C 1$, $d_{\varepsilon, b}(\gamma(t), z) \asymp_C e^{-\varepsilon t}$, and the comparison $d_{\varepsilon, b}(\gamma(-t), z) \asymp_C e^{\varepsilon t}$. By similar calculations we obtain that $d_{\varepsilon, b}(\gamma(0), \gamma(t)) \asymp_C 1$, $d_{\varepsilon, b}(\gamma(0), \gamma(-t)) \asymp_C e^{\varepsilon t}$, and $d_{\varepsilon, b}(\gamma(t), \gamma(-t)) \asymp_C e^{-\varepsilon t}$. Plugging these comparisons into
\((6.16)\) gives the following three inequalities, all with constants \(C = C(\delta, K, \varepsilon, M)\) (recall that \(t \geq 1\)),
\[
\hat{d}_{\varepsilon,b}^z(\gamma(0), \gamma(t)) \geq \frac{C^{-1}}{(1 + C)(1 + Ce^{-\varepsilon t})} \geq \frac{C^{-1}}{(1 + C)(1 + C)}.
\]
and
\[
\hat{d}_{\varepsilon,b}^z(\gamma(0), \gamma(-t)) \geq \frac{C^{-1}e^{\varepsilon t}}{(1 + C)(1 + Ce^{\varepsilon t})} \geq \frac{C^{-1}}{(1 + C)(1 + C)}.
\]
and finally
\[
\hat{d}_{\varepsilon,b}^z(\gamma(t), \gamma(-t)) \geq \frac{C^{-1}e^{\varepsilon t}}{(1 + Ce^{-\varepsilon t})(1 + Ce^{\varepsilon t})} \geq \frac{C^{-1}}{(1 + C)(1 + C)}.
\]
These three inequalities complete the proof of inequality \((6.15)\). We conclude that the identity map \(X_{\varepsilon,z} \to \hat{X}_{\varepsilon,b}^x\) is \(\theta\)-quasisymmetric with \(\theta\) depending only on \(\theta\) and \(\kappa\), and therefore only on \(\delta\), \(K\), \(\varepsilon\), and \(M\). \(\Box\)

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