Abstract

We consider self-duality equation of $U(1)$ gauge fields on Euclidean $AdS_4$ space, and find a simple finite action solution. With a suitable ansatz, we are able to embed this solution into the 10d supergravity background of $AdS_4 \times CP^3$. Further, we show that the solution can be uplifted to an exact solution in 11d supergravity background of $AdS_4 \times SE_7$. In the context of ABJM model, we also discuss the coupling to the boundary conserved currents.
1 Introduction

Instantons have played an important role in understanding the nonperturbative effects in quantum field theories and string theory. AdS/CFT correspondence [1], on the other hand, has provided a new perspective on instantons in terms of D-brane solutions in string theory. In particular, Yang-Mills instantons have been identified with D(-1)-brane solutions in type IIB supergravity. Following this identification, it has been possible to trace over the corresponding nonperturbative effects on both sides of the duality, and hence testing the AdS/CFT duality beyond the perturbative level [2, 3].

The AdS/CFT correspondence has further been generalized by Aharony, Bergman, Jafferis, and Maldacena (ABJM) to M-theory (and type IIA theory upon compactification) [4]. Therefore, to investigate the nonperturbative characteristics of this duality, it is important to look for some exact D-brane solutions in the corresponding supergravity backgrounds, and try to identify the dual instantons on the boundary Chern-Simons theory. Recently, we succeeded in constructing such dual instanton configurations in the antимembranes theory ignoring the backreaction on the metric. This construction further led us to propose that the antимembranes boundary theory is related to the ABJM model by swapping the $s$ and $c$ representations of the $SO(8)$ global symmetry [5]. In the present paper, however, we provide the first examples of exact solutions on $AdS_4 \times CP^3$, and $AdS_4 \times SE_7$ for the type IIA and M-theory backgrounds, respectively.

We start with the self-duality equation on $AdS_4$ space, and find a solution which has a finite action. Because of the self-duality, the energy-momentum tensor of this solution vanishes and hence there will be no backreaction on the metric. This implies that we have in fact an exact solution to the equations of motion coming from the Maxwell-Einstein action. In Sec. 3, we provide an ansatz for a system of D0-D2 brane configuration and discuss how the self-dual gauge fields can be embedded into an exact solution of 10d supergravity on $AdS_4 \times CP^3$. As in four dimensions, the energy-momentum tensor of individual branes along $AdS_4$ vanishes, whereas the components along $CP^3$ add up to zero. Hence the background metric will not change in the presence of branes (fluxes). In Sec. 4, we use the consistent truncation of [6] to uplift our 4-dimensional solution to an exact solution on $AdS_4 \times SE_7$. We will see explicitly how this comes about by looking at the reduced four-dimensional equations. In Sec. 5, we discuss the coupling to the boundary operators. We will determine the dual operators by examining the symmetry properties of the supergravity ansatzs under the isometry group of the metric.
2 \( U(1) \) Instantons on AdS\(_4\)

To discuss the self-dual gauge fields on Euclidean AdS\(_4\),\(^1\) we use the Poincaré coordinates for the metric:

\[ ds^2 = \frac{1}{\rho^2} (d\rho^2 + dx_1^2 + dx_2^2 + dx_3^2), \]

which reflects the conformal flatness of the metric. On the other hand, the (anti)self-duality condition is invariant under the conformal transformations of the metric, so on AdS\(_4\) we can write

\[ F_{\mu\nu} = -\frac{\sqrt{g}}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} = -\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \delta^\rho_\xi \delta^\sigma_\eta F_{\xi\eta}, \]

just as on flat \( \mathbb{R}^4 \).

To find a solution, we make the following ansatz for the \( U(1) \) gauge field \( A_\mu \):

\[ A_1 = x_2 h(r, \rho), \quad A_2 = -x_1 h(r, \rho), \quad A_3 = g(r, \rho), \quad A_4 = -x_3 h(r, \rho), \]

where \( r = \sqrt{x_1^2 + x_2^2 + x_3^2} \). Note that this ansatz respects the \( SO(3) \) symmetry along the 3-dimensional space orthogonal to the radial direction. It also resembles an ansatz used in [8] for solving the noncommutative \( U(1) \) instanton equation on \( \mathbb{R}^4 \). For the field strengths we get

\[ F_{12} = -2h - \frac{h'}{r} (x_1^2 + x_2^2), \quad F_{34} = -h - \frac{h'}{r} x_3^2 - \dot{g}, \]

\[ F_{23} = \frac{1}{r} (g' x_2 + h' x_1 x_3), \quad F_{14} = -\frac{h'}{r} x_1 x_3 - x_2 \dot{h}, \]

\[ F_{24} = -\frac{h'}{r} x_2 x_3 + x_1 \dot{h}, \quad F_{31} = \frac{h'}{r} x_2 x_3 - \frac{g'}{r} x_1, \]

where prime and dot indicate the differentiation with respect to \( r \) and \( \rho \), respectively. Now let us impose the self-duality conditions. From \( F_{12} = -F_{34} \) we obtain

\[ -3h - rh' = \dot{g}, \quad (4) \]

or

\[ \frac{1}{r^2} \frac{\partial}{\partial r} (r^3 h) + \frac{\partial g}{\partial \rho} = 0, \quad (5) \]

while \( F_{23} = -F_{14}, \) or \( F_{24} = -F_{31} \) yield

\[ \dot{h} = \frac{g'}{r}, \quad (6) \]

\(^1\)\( U(1) \) instantons on AdS\(_4\) have also been discussed in [7]. Here, however, we take a different approach.
Taking the derivative of the above equation with respect to $\rho$ and using (5), we have

$$\frac{\partial}{r \partial r} \left[ \frac{1}{r^2} \frac{\partial}{\partial r} (r^3 h) \right] + \frac{\partial^2 h}{\partial \rho^2} = 0. \quad (7)$$

Let us further make an assumption that $h$ depends on $r$ and $\rho$ through $z \equiv (\rho + \rho_0)^2 + r^2$, i.e.,

$$h = h(z) = h((\rho + \rho_0)^2 + r^2). \quad (8)$$

Plugging this ansatz into (7), we get

$$z \frac{\partial^2 h}{\partial z^2} + 3 \frac{\partial h}{\partial z} = 0, \quad (9)$$

which has the simple solution of

$$h(r, \rho) = C_1 + \frac{C_2}{[(\rho + \rho_0)^2 + r^2]^2}. \quad (10)$$

Note that with $\rho_0 \neq 0$ we will have a smooth solution. Actually here lies the difference with the $U(1)$ instantons on flat $\mathbb{R}^4$, where there is no way to avoid singularities of the solution. Furthermore, as we will see presently, with $\rho_0$ in (10) the action turns out to be finite. Let us further set $C_1 = 0$, then we can write

$$g(r, \rho) = (\rho + \rho_0) h(r, \rho), \quad (11)$$

which satisfies both (4) and (6). The full moduli of the solution can be seen by writing $A_\mu$ as

$$A_1 = (x_2 - x_0^2) h(r, \rho), \quad A_2 = -(x_1 - x_0^1) h(r, \rho),$$
$$A_3 = (\rho + \rho_0) h(r, \rho), \quad A_4 = -(x_3 - x_0^3) h(r, \rho), \quad (12)$$

now with $r^2 = (x_1 - x_1^0)^2 + (x_2 - x_2^0)^2 + (x_3 - x_3^0)^2$, and hence

$$h(r, \rho) = \frac{C_2}{[(\rho + \rho_0)^2 + (x_1 - x_1^0)^2 + (x_2 - x_2^0)^2 + (x_3 - x_3^0)^2]^2}. \quad (13)$$

So we have (12) as our solution of the self-duality equation on $AdS_4$.

### 2.1 The Boundary Term

To complete our discussion, let us look at the variation of the action in the presence of a boundary. This proves to be useful in our study of the coupling to the boundary operators of the dual CFT. For the variation of the Maxwell action we have

$$\delta S = \frac{1}{2} \int d^4x \, F_{\mu\nu} \delta F^{\mu\nu}$$
$$= - \int d^4x \, \delta A_\nu (\partial_\mu F^{\mu\nu}) + \int d^4x \, \partial_\mu (\delta A_\nu F^{\mu\nu})$$
$$= - \int d^4x \, \delta A_\nu (\partial_\mu F^{\mu\nu}) - (\delta A_\nu F^{\nu i})|_{\rho = 0}, \quad (14)$$
here $i,j,... = 1,2,3$, indicate the boundary tangent indices, and $\rho$ is the radial direction so that the boundary is at $\rho = 0$. So for having the equation of motion in the bulk, on the boundary we have to have either $\delta A_i = 0$ (Dirichlet boundary condition) or $F_{\rho i} = 0$ (Neumann boundary condition). However, neither of these boundary conditions are consistent with self-duality condition (2) in the bulk. For this to happen, we deform the Maxwell action as follows:

$$S = \frac{1}{4} \int d^4x F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} \int d^3x \varepsilon_{ijk} A^i F^{jk}, \quad (15)$$

so that the variation of the action now reads

$$\delta S = -\int d^4x \delta A_{\mu}(\partial_{\mu} F^{\mu\nu}) - \delta A_i F^{\rho i} + \frac{1}{2} \varepsilon_{ijk} F^{jk})|_{\rho=0}. \quad (16)$$

Now, on the boundary we can demand

$$F_{\rho i} = -\frac{1}{2} \varepsilon_{ijk} F^{jk} \quad (17)$$

which is also consistent with the self-duality condition in the bulk. This is a sort of mixed boundary condition as it relates the electric field to the gauge invariant part of the gauge potential.

### 2.2 The Action

With our ansatz (3), we can compute the action of instantons on $AdS_4$:

$$S = \frac{1}{4} \int \sqrt{g} d^4x F_{\mu\nu} F^{\mu\nu} - \frac{1}{8} \int d^4x \varepsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$$

$$= -\frac{1}{2} \int d^4x (F_{12}F_{34} + F_{23}F_{14} + F_{24}F_{31})$$

$$= \frac{1}{2} \int d^4x \left[ 4h^2 + (x_1^2 + x_2^2)(h'^2 + h^2 + \frac{4hh'}{r}) \right]$$

$$= 4\pi \int r^2 dr d\rho \left[ 2h^2 + \frac{r^2}{3} (h'^2 + h^2 + \frac{4hh'}{r}) \right],$$

where in the last line we have performed the integral over $\theta$ and $\phi$. Using solution (10), with $C_1 = 0$, results in

$$S = 8\pi C_2^2 \int \frac{r^2 dr d\rho}{[(\rho + \rho_0)^2 + r^2]^4} = \frac{\pi^2 C_2^2}{2} \int_0^\infty \frac{d\rho}{2(\rho + \rho_0)^5} = \frac{\pi^2 C_2^2}{16\rho_0^3}. \quad (18)$$

If $\rho_0$ is going to be a modulus of the solution, we need to choose $C_2 \sim \rho_0^2$. This makes $S$ independent of $\rho_0$, and gives $A_{\mu}$ the right dimension of one. We have therefore obtained a finite action solution of equations of motion, i.e., a $U(1)$ instanton in $AdS_4$. Moreover, being self-dual, the solution has a vanishing energy-momentum tensor and hence it provides an exact solution to the Maxwell-Einstein equations in four dimensions.
3 Uplift to 10d Type IIA Supergravity Solutions

In this section we will see how the self-dual gauge fields can be embedded into a solution of type IIA supergravity. In fact, our construction of $U(1)$ instantons on $AdS_4$ in the previous section was motivated by our search for an exact type IIA solution on $AdS_4 \times CP^3$ background.

To begin the discussion, let us recall the Euclidean action of type IIA supergravity in the string frame

$$S_{IIA} = \frac{1}{2\kappa^2} \int d^{10}x e^{-2\phi} \sqrt{g} R + \frac{1}{2\kappa^2} \int \left( e^{-2\phi} (4d\phi \wedge *d\phi - \frac{1}{2} H \wedge *H) - \frac{1}{2} F_2 \wedge *F_2 - \frac{1}{2} \tilde{F}_4 \wedge *\tilde{F}_4 + \frac{i}{2} B \wedge F_4 \wedge F_4 \right).$$  \hspace{1cm} (19)$$

For the field equations we have

$$d\tilde{F}_4 = -F_2 \wedge H, \quad d\wedge*\tilde{F}_4 = -\tilde{F}_4 \wedge H, \quad dH = 0, \quad dF_2 = 0,$$
$$d*(e^{-2\phi} H) = -F_2 \wedge *\tilde{F}_4 - \frac{i}{2} \tilde{F}_4 \wedge \tilde{F}_4, \quad d*F_2 = H \wedge *\tilde{F}_4,$$
$$d*d\phi - d\phi \wedge *d\phi - \frac{1}{8} H \wedge *H + \frac{1}{4!} 3! R \epsilon_4 \wedge J^3 = 0,$$
$$\tilde{F}_4 = F_4 - A_1 \wedge H. \hspace{1cm} (20)$$

With the Euclidean signature, we have the following background solution\(^2\)

$$ds^2 = \frac{R^3}{k^4} \left( \frac{1}{4} ds_{AdS_4}^2 + ds_{CP^3}^2 \right),$$
$$e^{2\phi} = \frac{R^3}{k^3}, \quad F_4 = -\frac{3i}{8} R^3 \epsilon_4, \quad F_2 = 2kJ, \hspace{1cm} (21)$$

where $J$ is the Kähler form on $CP^3$.

Having had the background solution (21), we would like to look for a new solution on this background, i.e., a D-instanton. So to proceed, let us make the following ansatz:

$$F_2 = 2kJ + F, \quad F_4 = -\frac{3i}{8} R^3 \epsilon_4 + i\alpha J \wedge F, \quad H = 0,$$  \hspace{1cm} (22)$$

with $\alpha$ a constant parameter, and $F$ a 2-form in $AdS_4$. The extra terms in $F_2$ and $F_4$ can be thought to be sourced by a D0-brane and D2-brane, respectively.

Setting $H = 0$, the field equations (20) now read

$$dF_4 = 0, \quad d*F_4 = 0, \quad dF_2 = 0, \quad d*F_2 = 0,$$
$$0 = d*(e^{-2\phi} H) = -F_2 \wedge *F_4 - \frac{i}{2} F_4 \wedge F_4. \hspace{1cm} (23)$$

As we will discuss in the following, the extra terms in our ansatz (22) will have no backreaction on the metric so that the background dilaton $\phi$ continues to satisfy the

\(^2\)We have explicitly checked for the factor of 2 in $F_2$. 

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field equations. If we choose $F$ to be a self-dual 2-form in $AdS_4$, the field equations of $F_2$ and $F_4$ are obviously satisfied. What about the last equation in (23)? Let us expand the right hand side of this equation:

$$F_2 \wedge \ast F_4 + \frac{i}{2} F_4 \wedge F_4 =$$

$$= (2k J + F) \wedge \ast \left( -\frac{3i}{8} R^3 \epsilon_4 + i \alpha J \wedge F \right) - \frac{i \alpha^2}{2} F \wedge F \wedge J^2$$

$$= (2k J + F) \wedge \left( -\frac{i R^6}{k} J^3 + \frac{i \alpha}{2} J^2 \wedge \ast_4 F \right) - \frac{i \alpha^2}{2} F \wedge F \wedge J^2$$

$$= i \alpha R^3 \ast_4 F \wedge J^3 - \frac{i R^6}{k} F \wedge J^3 + \frac{i \alpha R^3}{2k} F \wedge \ast_4 F \wedge J^2 - \frac{i \alpha^2}{2} F \wedge F \wedge J^2. \tag{24}$$

Hence, for (24) to vanish we must have

$$F = \ast_4 F, \tag{25}$$

and

$$\alpha = \frac{R^3}{k}. \tag{26}$$

The nice thing about ansatz (22) is that the indices of $F$ and $J \wedge F$ do not contract with those of the background fields. So in discussing the energy-momentum tensors, we need only to be concerned with the contributions of the these terms. Further, with $F$ a self-dual 2-form the energy-momentum tensor along $AdS_4$ vanishes. Let $\mu, \nu, \ldots$ and $\alpha, \beta, \ldots$ indicate the tangent indices on $AdS_4$ and $CP^3$, respectively, then we have

$$T_{F_2}^{F_2} = \frac{1}{2} \frac{1}{2!} \left[ 2 F_{\mu \rho} F_{\nu}^{\rho} - \frac{1}{2} g_{\mu \nu} F^{\eta \rho \sigma} F_{\eta \rho \sigma} \right] = 0 \tag{27}$$

as $F$ is self-dual. For $F_4$ we have

$$T_{F_4}^{F_4} = -\frac{1}{2 \cdot 4!} \frac{\alpha^2 k^2}{R^6} \left[ 4 \cdot 3 F_{\mu \rho \alpha \beta} F_{\nu}^{\rho \alpha \beta} - \frac{1}{2} \cdot 6 g_{\mu \nu} F_{\eta \rho \alpha \beta} F^{\eta \rho \alpha \beta} \right]$$

$$= -\frac{3}{2 \cdot 4!} \left[ 4 F_{\mu \rho} F_{\nu}^{\rho} - g_{\mu \nu} F^{\eta \rho \sigma} F_{\eta \rho \sigma} \right] 2 J_{\alpha \beta} J^{\alpha \beta}$$

$$= -\frac{3}{4} \left[ 2 F_{\mu \rho} F_{\nu}^{\rho} - \frac{1}{2} g_{\mu \nu} F^{\eta \rho \sigma} F_{\eta \rho \sigma} \right] = 0. \tag{28}$$

If we use the complex coordinate on $CP^3$, for the energy-momentum tensor of $F_2$ along $CP^3$ we have

$$T_{F_2}^{F_2} = -\frac{1}{8} \frac{R^3}{k} g_{\alpha \beta} F_{\mu \nu} F^{\mu \nu}. \tag{29}$$

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Note that $\epsilon_4 = vol(AdS_4)$, $vol(CP^3) = \frac{1}{6} J \wedge J \equiv \frac{1}{6} J^3$, $\ast_6 J = \frac{1}{2} J \wedge J \equiv \frac{1}{2} J^2$, and $J \wedge J^3 = 0$. We use $\ast_4$ and $\ast_6$ to indicate the Hodge star operation with respect to the metrics of $AdS_4$ and $CP^3$, respectively.
whereas, for $F_4$ it reads

$$
T_{\alpha\beta}^{F_4} = -\frac{1}{2 \cdot 4!} \frac{\alpha^2 k}{R^3} \left[ 4 \cdot 3 F_{\mu\nu} F^{\mu\nu} J_{\bar{\alpha}\bar{\gamma}} J_{\bar{\beta}\bar{\gamma}} - \frac{1}{2} \cdot 6 g_{\alpha\beta} F_{\eta\rho} F^{\eta\rho} J^2 \right]
$$

$$
= + \frac{1}{8} R^3 \frac{3}{k} g_{\alpha\beta} J_{\mu\nu},
$$

(30)

where we used

$$
J_{\alpha\gamma} J_{\bar{\beta}\bar{\gamma}} = g_{\alpha\beta}, \quad J_{\alpha\beta} J^{\alpha\beta} = 3.
$$

(31)

Therefore we conclude that

$$
T_{\alpha\beta}^{F_2} + T_{\alpha\beta}^{F_4} = 0.
$$

(32)

So we have showed that with ansatz (22) the energy-momentum tensor of D0-D2 brane configuration vanishes and there will be no backreaction on the background metric, i.e., the Einstein equations are the same as before. Also note that the dilaton has its original value of $e^{2\phi} = R^3/k^3$. In conclusion, we have obtained an exact solution of the form (22) with $F$ a self-dual gauge field explicitly constructed in Sec. 2.

## 4 Uplift to 11d Supergravity Solutions

The $U(1)$ instanton that we found in Sec. 2, can further be uplifted to an exact solution in 11d supergravity. Fortunately, the consistent truncation of 11d supergravity to four dimensions of [6] makes this uplift much simpler, and in fact it is more general than our discussion in the previous section. To begin with, consider the following ansatz for the metric of 11d supergravity

$$
ds^2 = ds_4^2 + e^{2U} ds_6^{2} + e^{2V} (\eta + A_1) \otimes (\eta + A_1),
$$

(33)

where $ds_4^2$ is an arbitrary metric on a 4d spacetime, and $ds_6^{2}$ is the metric of a 6d Kähler-Einstein space. $U$ and $V$ are scalar fields and $A_1$ is a one-form defined on the four-dimensional space. If we set $U = V = A_1 = 0$, then the the last two factors of the metric constitute the Sasaki-Einstein metric

$$
ds_{SE}^2 = ds_6^{2} + \eta \otimes \eta,
$$

(34)

where $\eta$ is the globally defined one-form dual to the Reeb Killing vector. On a Sasaki-Einstein manifold one can globally define a 2-form $J$ and a holomorphic 3-form so that

$$
d\eta = 2J, \quad d\Omega = 4i\eta \wedge \Omega.
$$

(35)

The 4-form $F_4$ is taken to be

$$
F_4 = f \text{vol}_4 + H_3 \wedge (\eta + A_1) + H_2 \wedge J + H_1 \wedge J \wedge (\eta + A_1)
$$

$$
+ 2h J \wedge J + \sqrt{3} [\chi_1 \wedge \Omega + \chi(\eta + A_1) \wedge \Omega + \text{c.c.}],
$$

(36)
where $f$ and $h$ are real scalars, $H_p$, $p = 1, 2, 3$, are real $p$-forms, $\chi_1$ is a complex one-form, and $\chi$ is a complex scalar on the four-dimensional spacetime. With the ansatzs (33) and (36), the 11-dimensional supergravity field equations all reduce to 4-dimensional equations, so that any solution to the reduced equations can be lifted to an 11-dimensional supergravity solution. These 4-dimensional equations are given in [6].

Now, as mentioned in that paper one can further consistently truncate by setting $U = V = h = \chi = H_3 = H_1 = 0$. Equations (B.9) to (B.11) of [6], when adapted to Euclidean signature, reduce to

$$-f F + 6 \ast_4 H = 0,$$
$$d \ast_4 H = 0,$$
$$\ast_4 H \wedge F + iH \wedge H = 0,$$  \hspace{1cm} (37)

with $F = dA_1$, $H \equiv H_2$, and $dH = 0$. Equations (B.19), (B.20), and (B.22) reduce to

$$R_{\mu\nu} = \frac{1}{3} g_{\mu\nu} f^2 + \frac{1}{2} F_{\mu\rho} F_{\nu}^{\rho} + \frac{3}{2} (H_{\mu\nu} H_{\nu}^{\rho} - \frac{1}{6} g_{\mu\nu} H_{\rho\sigma} H_{\rho\sigma} ) ,$$
$$d \ast_4 F = 0 ,$$
$$F_{\mu\nu} F^{\mu\nu} + H_{\mu\nu} H^{\mu\nu} = 0 .$$  \hspace{1cm} (38)

The last eqs. of (37) and (38) are satisfied if we set

$$H = i \ast_4 F ,$$  \hspace{1cm} (39)

from which $d \ast_4 H = 0$ and $d \ast_4 F = 0$ are followed by the Bianchi identities for $F$ and $H_2$, respectively. The first equation in (37) then implies $f = 6i$. We are then left with the first equation in (38):

$$R_{\mu\nu} = -12 g_{\mu\nu} + \frac{1}{2} F_{\mu\rho} F_{\nu}^{\rho} + \frac{3}{2} (H_{\mu\nu} H_{\nu}^{\rho} - \frac{1}{6} g_{\mu\nu} H_{\rho\sigma} H_{\rho\sigma} ) ,$$
$$= -12 g_{\mu\nu} + 2 ( F_{\mu\rho} F_{\nu}^{\rho} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} ) ,$$  \hspace{1cm} (40)

where we have used

$$H_{\mu\nu} H_{\nu}^{\rho} = F_{\mu\rho} F_{\nu}^{\rho} - \frac{1}{2} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} ,$$  \hspace{1cm} (41)

implied by (39).

As a final step, let us set

$$F = \ast_4 F ,$$  \hspace{1cm} (42)

for which the second term in (40) (i.e., the energy-momentum tensor of $F$) vanishes, and hence equation (40) is solved when the metric is that of Euclidean $AdS_4$. 

8
5 Coupling to the Boundary Operators

The supergravity backgrounds we have considered so far appear in the ABJM model, where a concrete CFT dual has been proposed. In this model a Chern-Simons-matter theory describes the low energy dynamics of \( N \) M2-branes at the tip of the orbifold \( C_4/\mathbb{Z}_k \). This theory is in turn conjectured to be dual to M-theory on \( AdS_4 \times S^7/\mathbb{Z}_k \), with \( k \) the level of the Chern-Simons term in the gauge theory side. For large \( k \) (\( k^5 >> N \)), the dual theory is better described in terms of type IIA string theory on \( AdS_4 \times CP^3 \) [4]. In this set up, we would now like to examine the effect on the boundary of turning on the gauge fields in the bulk.

According to the standard prescription of AdS/CFT, the boundary value of a bulk mode acts as a source for the dual operator on the boundary theory [9]. In particular, the boundary value of a gauge field couples to a dimension 2 conserved current \( J_i \) through

\[
\int d^3 x A_i J^i .
\]

This is the case when we are imposing Dirichlet boundary conditions. For Neumann boundary conditions, on the other hand, the electric field is held fixed on the boundary and thus couples to gauge fields which are the boundary value of the dynamical bulk gauge field. In fact, Neumann boundary conditions are related to Dirichlet boundary conditions through the action of \( SL(2, \mathbb{Z}) \) duality in the bulk. The generator \( S \) of this group acts through

\[
\left( \begin{array}{c} \vec{E} \\ \vec{B} \end{array} \right) \to \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \left( \begin{array}{c} \vec{E} \\ \vec{B} \end{array} \right) = \left( \begin{array}{c} \vec{B} \\ -\vec{E} \end{array} \right) ,
\]

so in terms of the transformed gauge fields, Dirichlet boundary condition of \( \vec{B} = 0 \) corresponds in terms of original gauge fields to Neumann boundary condition of \( \vec{E} = 0 \) [10]. The mixed boundary condition in (17), however, can be realized by the application of \( ST \). Let \( T \) denote the second generator of \( SL(2, \mathbb{Z}) \)

\[
T = \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right)
\]

so that \( ST \) acts through

\[
\left( \begin{array}{c} \vec{E} \\ \vec{B} \end{array} \right) \to \left( \begin{array}{c} \vec{B} \\ -\vec{E} - \vec{B} \end{array} \right) ,
\]

we see that the Dirichlet condition \((\vec{B} = 0)\) of the transformed gauge fields corresponds in terms of the original gauge fields to the mixed boundary condition

\[
\vec{E} + \vec{B} = 0 ,
\]
which is the same boundary condition as (17). The above $ST$ operation of $SL(2, \mathbb{Z})$ on the bulk gauge fields translates on the boundary to the following change of the Dirichlet coupling:

$$\int d^3 x A_i J^i \rightarrow \int d^3 x \left( A_i J^i + \frac{1}{2\pi} \epsilon^{ijk} C_i \partial_j A_k + \frac{1}{4\pi} \epsilon^{ijk} A_i \partial_j A_k \right).$$

(48)

The last term is the effect of $T$ operation, whereas $S$ operation introduces a dual background gauge field $C_i$ and couples it to the conserved current $\epsilon^{ijk} \partial_j A_k$ and then promotes $A_i$ to a dynamical field [10].

So far, we have determined boundary coupling (48) which is induced by the bulk self-dual gauge fields. The coupling, however, depends on the current $J$ which we are going to discuss next. In the ABJM model, a $U(N) \times U(N)$ Chern-Simons-matter theory describes the dynamics on the boundary. Let $A_i$ and $\hat{A}_i$ indicate the corresponding gauge fields of the $U(N)$ factors. The matter content consists of bosonic fields $Y^A$, and fermionic fields $\psi_A$ transforming in the $4$ and $\bar{4}$ representations of the $SU(4)$ global symmetry, respectively. The global symmetry is in fact $SU(4) \times U(1)_b$, which is enhanced to $SO(8)$ for $k = 1, 2$. Further, for the $U(1)$ parts, let $A_i^\pm = \text{tr} (A_i \pm \hat{A}_i)$, the matter field then couples only to $A^-$, whereas $A^+$ appears in the action through

$$\frac{k}{4\pi} \int \epsilon^{ijk} A_i^- F_{jk}^+. \quad (49)$$

Hence, apart from $U(1)_b$ symmetry, which is generated by

$$J_i = i \text{tr} \left( Y_A^D_i Y^A - Y^A D_i Y_A^+ \right) \quad (50)$$

there is a further global symmetry associated to the shift symmetry of $A^+$ which is generated by the current $\tilde{J} = * F^+$. So, altogether, we can identify two global $U(1)$ symmetries in the theory.

Let us then see to which bulk gauge excitations these currents could couple. A look back at ansatzs (33) and (36) shows that they are only invariant under $SU(4) \times U(1)$ subgroup of $SO(8)$ isometry group (the global symmetry when $k = 1, 2$). So, the excitations must couple to $J_i$, which is also invariant under $SU(4) \times U(1)$ but not the full $SO(8)$ symmetry group. Notice that $\tilde{J}_i$ is invariant under $SO(8)$, so it cannot couple to the bulk gauge field excitations we considered in this paper.
References

[1] J. M. Maldacena, The Large N limit of superconformal field theories and supergravity, Adv. Theor. Math. Phys. 2, 231 (1998), [Int. J. Theor. Phys. 38, 1113 (1999)], [arXiv:hep-th/9711200].

[2] M. Bianchi, M. B. Green, S. Kovacs and G. Rossi, Instantons in supersymmetric Yang-Mills and D instantons in IIB superstring theory, JHEP 9808, 013 (1998), [arXiv:hep-th/9807033].

[3] T. Banks and M. B. Green, Nonperturbative effects in AdS$_5 \times S^5$ string theory and d = 4 SUSY Yang-Mills, JHEP 9805, 002 (1998), [arXiv:hep-th/9804170].

[4] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals, JHEP 0810, 091 (2008), [arXiv:0806.1218 [hep-th]].

[5] A. Imaanpur and M. Naghdi, Dual Instantons in Anti-membranes Theory, Phys. Rev. D 83, 085025 (2011), [arXiv:1012.2554 [hep-th]].

[6] J. P. Gauntlett, S. Kim, O. Varela and D. Waldram, Consistent supersymmetric Kaluza-Klein truncations with massive modes, JHEP 0904, 102 (2009), [arXiv:0901.0676 [hep-th]].

[7] S. de Haro and P. Gao, Electric-magnetic duality and deformations of three-dimensional CFT’s, Phys. Rev. D 76, 106008 (2007), [arXiv:hep-th/0701144].

[8] N. Seiberg and E. Witten, String theory and noncommutative geometry, JHEP 9909, 032 (1999), [arXiv:hep-th/9908142].

[9] E. Witten, Anti-de Sitter space and holography, Adv. Theor. Math. Phys. 2, 253 (1998), [arXiv:hep-th/9802150].

[10] E. Witten, $SL(2,Z)$ action on three-dimensional conformal field theories with Abelian symmetry, In Shifman, M. (ed.) et al.: From fields to strings, vol. 2, 1173-1200, arXiv:hep-th/0307041.
$U(1)$ Instantons on $AdS_4$ and the Uplift to Exact Supergravity Solutions

A. Imaanpur

Department of Physics, School of Sciences
Tarbiat Modares University, P.O.Box 14155-4838, Tehran, Iran

Abstract

We consider self-duality equation of $U(1)$ gauge fields on Euclidean $AdS_4$ space, and find a simple finite action solution. With a suitable ansatz, we are able to embed this solution into the 10d supergravity background of $AdS_4 \times CP^3$. Further, we show that the solution can be uplifted to an exact solution in 11d supergravity background of $AdS_4 \times SE_7$.
1 Introduction

Instantons have played an important role in understanding the nonperturbative effects in quantum field theories as well as string theory. AdS/CFT correspondence [1], on the other hand, has provided a new perspective on instantons in terms of D-brane solutions in string theory. In particular, Yang-Mills instantons have been identified with D(-1)-brane solutions in type IIB supergravity. Following this identification, it has been possible to trace over the corresponding nonperturbative effects on both sides of the duality, and hence testing the AdS/CFT duality beyond the perturbative level [2, 3].

The AdS/CFT correspondence has further been generalized by Aharony, Bergman, Jafferis, and Maldacena (ABJM) to M-theory (and type IIA theory upon compactification) [4]. Therefore, to investigate the nonperturbative characteristics of this duality, it is important to look for some exact D-brane solutions in the corresponding supergravity backgrounds, and try to identify the dual instantons on the boundary Chern-Simons theory. The first steps in a nonperturbative study of ABJM model was taken in [5]. Recently, we succeeded in constructing such dual instanton configurations in the ant membranes theory ignoring the backreaction on the metric. This construction further led us to propose that the ant membranes boundary theory is related to the ABJM model by swapping the s and c representations of the $SO(8)$ global symmetry [6]. This line of study was further followed to construct supergravity monopole solutions in [7]. In the present paper, however, we provide the first examples of exact solutions on $AdS_4 \times CP^3$, and $AdS_4 \times SE_7$ for the type IIA and M-theory backgrounds, respectively. We will see that our construction of D0-D2 brane configuration closely parallels that of D-instantons in type IIB supergravity.

We start with the self-duality equation on $AdS_4$ space, and find a solution which has a finite action. Because of the self-duality, the energy-momentum tensor of this solution vanishes and hence there will be no backreaction on the metric. This implies that we have in fact an exact solution to the equations of motion coming from the Maxwell-Einstein action. In Sec. 3, we provide an ansatz for a system of D0-D2 brane configuration and discuss how the self-dual gauge fields can be embedded into an exact solution of 10d supergravity on $AdS_4 \times CP^3$. As in four dimensions, the energy-momentum tensor of individual branes along $AdS_4$ vanishes, whereas the components along $CP^3$ add up to zero. Hence the background metric will not change in the presence of branes (fluxes). In Sec. 4, we use the consistent truncation of [8] to uplift our 4-dimensional solution to an exact solution on $AdS_4 \times SE_7$. We will see explicitly how this comes about by looking at the reduced four-dimensional equations.
2 U(1) Instantons on AdS$_4$

To discuss the self-dual gauge fields on Euclidean AdS$_4$, we use the Poincare coordinates for the metric:

$$ds^2 = \frac{1}{\rho^2}(d\rho^2 + dx_1^2 + dx_2^2 + dx_3^2),$$  \(\text{(1)}\)

which reflects the conformal flatness of the metric. On the other hand, the (anti)self-duality condition is invariant under the conformal transformations of the metric, so on AdS$_4$ we can write

$$F_{\mu\nu} = -\sqrt{g} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} = -\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \delta^\rho_\xi \delta^\sigma_\eta F_{\xi\eta},$$  \(\text{(2)}\)

just as on flat $\mathbb{R}^4$.

To find a solution, we make the following ansatz for the U(1) gauge field $A_{\mu}$:

$$A_1 = x_2 h(r, \rho), \quad A_2 = -x_1 h(r, \rho), \quad A_3 = g(r, \rho), \quad A_4 = -x_3 h(r, \rho),$$  \(\text{(3)}\)

where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$. Note that this ansatz respects the SO(3) symmetry along the 3-dimensional space orthogonal to the radial direction. It also resembles an ansatz used in [11] for solving the noncommutative U(1) instanton equation on $\mathbb{R}^4$. For the field strengths we get

$$F_{12} = -2h - \frac{h'}{r}(x_1^2 + x_2^2), \quad F_{34} = -h - \frac{h'}{r}x_3^2 + \dot{g},$$
$$F_{23} = \frac{1}{r}(g'x_2 + h'x_1x_3), \quad F_{14} = \frac{-h'}{r}x_1x_3 - x_2\dot{h},$$
$$F_{24} = -\frac{h'}{r}x_2x_3 + x_1\dot{h}, \quad F_{31} = \frac{h'}{r}x_2x_3 - \frac{g'}{r}x_1,$$

where prime and dot indicate the differentiation with respect to $r$ and $\rho$, respectively.

Now let us impose the self-duality conditions. From $F_{12} = -F_{34}$ we obtain

$$-3h - r h' = \dot{g},$$  \(\text{(4)}\)

or

$$\frac{1}{r^2 \frac{\partial}{\partial r}(r^3 h)} + \frac{\partial g}{\partial \rho} = 0,$$  \(\text{(5)}\)

while $F_{23} = -F_{14}$, or $F_{24} = -F_{31}$ yield

$$\dot{h} = \frac{g'}{r}.$$

$^{2}$U(1) instantons on AdS$_4$ have also been discussed in [9]. Here, however, we take a different approach. For Yang-Mills instantons on AdS$_4$ see [10].
Taking the derivative of the above equation with respect to $\rho$ and using (5), we have
\[ \frac{\partial}{\partial r} \left[ \frac{1}{r^2} \frac{\partial}{\partial r} (r^3 h) \right] + \frac{\partial^2 h}{\partial \rho^2} = 0. \tag{7} \]

Let us further make an assumption that $h$ depends on $r$ and $\rho$ through $z \equiv (\rho + \rho_0)^2 + r^2$, i.e.,
\[ h = h(z) = h((\rho + \rho_0)^2 + r^2). \tag{8} \]

Plugging this ansatz into (7), we get
\[ z \frac{\partial^2 h}{\partial z^2} + 3 \frac{\partial h}{\partial z} = 0, \tag{9} \]
which has the simple solution of
\[ h(r, \rho) = C_1 + \frac{C_2}{[(\rho + \rho_0)^2 + r^2]^2}. \tag{10} \]

Note that with $\rho_0 > 0$ we will have a smooth solution (note that $\rho \in [0, \infty)$). Actually here lies the difference with the $U(1)$ instantons on flat $\mathbb{R}^4$, where there is no way to avoid singularities of the solution. Furthermore, as we will see presently, with $\rho_0$ in (10) the action turns out to be finite. Let us further set $C_1 = 0$, then we can write
\[ g(r, \rho) = (\rho + \rho_0) h(r, \rho), \tag{11} \]
which satisfies both (4) and (6). The full moduli of the solution can be seen by writing $A_\mu$ as
\[ A_1 = (x_2 - x^0_2) h(r, \rho), \quad A_2 = -(x_1 - x_1^0) h(r, \rho), \]
\[ A_3 = (\rho + \rho_0) h(r, \rho), \quad A_4 = -(x_3 - x^0_3) h(r, \rho), \tag{12} \]
now with $r^2 = (x_1 - x_1^0)^2 + (x_2 - x_2^0)^2 + (x_3 - x_3^0)^2$, and hence
\[ h(r, \rho) = \frac{C_2}{[(\rho + \rho_0)^2 + (x_1 - x_1^0)^2 + (x_2 - x_2^0)^2 + (x_3 - x_3^0)^2]^2}. \tag{13} \]

So we have (12) as our solution of the self-duality equation on $AdS_4$.

### 2.1 The Boundary Term

To complete our discussion, let us look at the variation of the action in the presence of a boundary. This proves to be useful when we are concerned with the coupling to the boundary operators of the dual CFT. For the variation of the Maxwell action we have
\[
\delta S = \frac{1}{2} \int d^4 x \, F_{\mu\nu} \delta F^{\mu\nu} \\
= - \int d^4 x \, \delta A_\nu (\partial_\mu F^{\mu\nu}) + \int d^4 x \, \partial_\mu (\delta A_\nu F^{\mu\nu}) \\
= - \int d^4 x \, \delta A_\nu (\partial_\mu F^{\mu\nu}) - (\delta A_\mu F^{\mu\nu})|_{\rho=0}, \tag{14}
\]
here \( i,j, \ldots = 1,2,3 \), indicate the boundary tangent indices, and \( \rho \) is the radial direction so that the boundary is at \( \rho = 0 \). So for having the equation of motion in the bulk, on the boundary we have to have either \( \delta A_i = 0 \) (Dirichlet boundary condition) or \( F_{\rho i} = 0 \) (Neumann boundary condition). It is clear that we can always impose the Dirichlet boundary condition, however, Neumann boundary condition is not consistent with self-duality condition (2) in the bulk. For this to happen, we deform the Maxwell action as follows:

\[
S = \frac{1}{4} \int d^4x \, F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} \int d^3x \, \epsilon_{ijk} A^i F^{jk},
\]

so that the variation of the action now reads

\[
\delta S = - \int d^4x \, \delta A_\nu (\partial_\mu F^{\mu\nu}) - \delta A_i (F^{\rho i} + \frac{1}{2} \epsilon_{ijk} F^{jk})|_{\rho=0}.
\]

Now, on the boundary we can demand

\[
F^{\rho i} = -\frac{1}{2} \epsilon_{ijk} F^{jk}
\]

which is also consistent with the self-duality condition in the bulk. This is a sort of mixed boundary condition as it relates the electric field to the gauge invariant part of the gauge potential. More general boundary conditions have been discussed in [9].

### 2.2 The Action

With our ansatz (3), we can compute the action of instantons on \( AdS_4 \):

\[
S = \frac{1}{4} \int \sqrt{g} d^4x \, F_{\mu\nu} F^{\mu\nu} = -\frac{1}{8} \int d^4x \, \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}
\]

\[
= \frac{1}{2} \int d^4x \left[ 4h^2 + (x_1^2 + x_2^2) (h'^2 + \frac{4hh'}{r}) \right]
\]

\[
= 4\pi \int r^2 dr d\rho \left[ 2h^2 + \frac{r^2}{3} (h'^2 + \frac{4hh'}{r}) \right],
\]

where in the last line we have performed the integral over \( \theta \) and \( \phi \). Using solution (10), with \( C_1 = 0 \), results in

\[
S = 8\pi C_2^2 \int \frac{r^2 dr d\rho}{(\rho + \rho_0)^2 + r^2} = \frac{\pi^2 C_2^2}{2} \int_0^\infty \frac{d\rho}{2(\rho + \rho_0)^5} = \frac{\pi^2 C_2^2}{16\rho_0^2}.
\]

If \( \rho_0 \) is going to be a modulus of the solution, we need to choose \( C_2 \sim \rho_0^2 \). This makes \( S \) independent of \( \rho_0 \), and gives \( A_\mu \) the right dimension of one. We have therefore obtained a finite action solution of equations of motion, i.e., a \( U(1) \) instanton in \( AdS_4 \). Moreover, being self-dual, the solution has a vanishing energy-momentum tensor and hence it provides an exact solution to the Maxwell-Einstein equations in four dimensions.
3 Uplift to 10d Type IIA Supergravity Solutions

In this section we will see how the self-dual gauge fields can be embedded into a solution of type IIA supergravity. In fact, our construction of $U(1)$ instantons on $AdS_4$ in the previous section was motivated by our search for an exact type IIA solution on $AdS_4 \times CP^3$ background.

To begin the discussion, let us recall the Euclidean action of type IIA supergravity in the string frame

$$S_{IIA} = \frac{1}{2\kappa^2} \int d^{10}x \, e^{-2\phi} \sqrt{g} R + \frac{1}{2\kappa^2} \int \left( e^{-2\phi} (4d\phi \wedge *d\phi - \frac{1}{2} H \wedge *H) - \frac{1}{2} F_2 \wedge *F_2 - \frac{1}{2} \tilde{F}_4 \wedge *\tilde{F}_4 - \frac{i}{2} B \wedge F_4 \wedge F_4 \right).$$  \hspace{1cm} (19)

For the field equations we have

$$d\tilde{F}_4 = -F_2 \wedge H, \quad d*\tilde{F}_4 = -\tilde{F}_4 \wedge H, \quad dH = 0, \quad dF_2 = 0,$$

$$d*(e^{-2\phi} H) = -F_2 \wedge *\tilde{F}_4 + \frac{i}{2} \tilde{F}_4 \wedge \tilde{F}_4, \quad d*F_2 = H \wedge *\tilde{F}_4,$$

$$d*d\phi - d\phi \wedge *d\phi - \frac{1}{8} H \wedge *H + \frac{1}{4} \cdot 3! R \epsilon_4 \wedge J^3 = 0,$$

$$\tilde{F}_4 = F_4 - A_1 \wedge H.$$  \hspace{1cm} (20)

With the Euclidean signature, we have the following skew-whiffed background solution\(^3\)

$$ds^2 = \frac{R^3}{k^4} \left( \frac{1}{4} ds^2_{AdS_4} + ds^2_{CP^3} \right),$$

$$e^{2\phi} = \frac{R^3}{k^3}, \quad F_4 = \frac{3i}{8} R^3 \epsilon_4, \quad F_2 = 2kJ,$$  \hspace{1cm} (21)

where $\epsilon_4$ indicates the volume form of $AdS_4$, and $J$ is the Kähler form on $CP^3$.

Having had the background solution (21), we would like to look for a new solution on this background, i.e., a D-instanton. So to proceed, let us make the following ansatz:

$$F_2 = 2kJ + F, \quad F_4 = \frac{3i}{8} R^3 \epsilon_4 - i\alpha J \wedge *_4 F, \quad H = 0,$$  \hspace{1cm} (22)

with $\alpha$ a constant parameter, and $F$ a 2-form in $AdS_4$. The extra terms in $F_2$ and $F_4$ can be thought to be sourced by a D0-brane and D2-brane, respectively.

Setting $H = 0$, the field equations (20) now read

$$dF_4 = 0, \quad d*F_4 = 0, \quad dF_2 = 0, \quad d*F_2 = 0,$$

$$0 = d*(e^{-2\phi} H) = -F_2 \wedge *F_4 + \frac{i}{2} F_4 \wedge F_4.$$  \hspace{1cm} (23)

As we will discuss in the following, the extra terms in our ansatz (22) will have no backreaction on the metric so that the background dilaton $\phi$ continues to satisfy the

\(^3\)We have explicitly checked for the factor of 2 in $F_2$. 


field equations. If we choose $F$ to be a self-dual 2-form in $AdS_4$, the field equations of $F_2$ and $F_4$ are obviously satisfied. What about the last equation in (23)? Let us expand the right hand side of this equation:

$$-F_2 \wedge *F_4 + \frac{i}{2} F_4 \wedge F_4 =$$

$$= -(2k J + F) \wedge * \left( \frac{3i}{8} R^3 \epsilon_4 - i \alpha J \wedge J^2 \right) + \frac{i \alpha^2}{2} F \wedge F \wedge J^2$$

$$= -(2k J + F) \wedge \left( \frac{iR^6}{k} J^3 - \frac{i \alpha R^3}{2k} J^2 \wedge F \right) + \frac{i \alpha^2}{2} F \wedge F \wedge J^2$$

$$= i \alpha R^3 F \wedge J^3 - \frac{iR^6}{k} F \wedge J^3 + \frac{i \alpha R^3}{2k} F \wedge F \wedge J^2 - \frac{i \alpha^2}{2} F \wedge F \wedge J^2.$$  \hspace{1cm} (24)

Hence, for (24) to vanish we must have

$$\alpha = \frac{R^3}{k}. \hspace{1cm} (25)$$

It is interesting to note that the indices of $F$ and $J \wedge F$ do not contract with those of the background fields. So in discussing the energy-momentum tensors, we need only to be concerned with the contributions of these terms. Further, with $F$ a self-dual 2-form the energy-momentum tensor along $AdS_4$ vanishes. Let $\mu, \nu, \ldots$ and $\alpha, \beta, \ldots$ indicate the tangent indices on $AdS_4$ and $CP^3$, respectively, then we have

$$T^F_{\mu \nu} = \frac{1}{2 \cdot 2!} \left[ 2F_{\mu \rho} F^\rho_{\nu} - \frac{1}{2} g_{\mu \nu} F^{\eta \rho \sigma} F_{\eta \rho \sigma} \right] = 0 \hspace{1cm} (26)$$

as $F$ is self-dual. For $F_4$ we have

$$T^{F_4}_{\mu \nu} = -\frac{1}{2 \cdot 4!} \frac{\alpha^2 k^2}{R^6} \left[ 4 \cdot 3 F_{\mu \rho \alpha \beta} F^\rho_{\nu \sigma \alpha \beta} - \frac{1}{2} \cdot 6 g_{\mu \nu} F_{\eta \rho \alpha \beta} F^{\eta \rho \alpha \beta} \right]$$

$$= -\frac{3}{2 \cdot 4!} \left[ 4 F_{\mu \nu} - g_{\mu \nu} F_{\eta \rho} F^{\eta \rho} \right] 2 J_{\alpha \bar{\beta}} J^{\alpha \bar{\beta}} = 0. \hspace{1cm} (27)$$

If we use the complex coordinate on $CP^3$, for the energy-momentum tensor of $F_2$ along $CP^3$ we have

$$T^{F_2}_{\alpha \bar{\beta}} = -\frac{1}{8} \frac{R^3}{k} g_{\alpha \bar{\beta}} F_{\mu \nu} F^{\mu \nu} \hspace{1cm} (28)$$

whereas, for $F_4$ it reads

$$T^{F_4}_{\alpha \bar{\beta}} = -\frac{1}{8} \frac{R^3}{k} g_{\alpha \bar{\beta}} F_{\mu \nu} F^{\mu \nu}. \hspace{1cm} (29)$$

\footnote{Note that $\epsilon_4 = vol(AdS_4)$, $vol(CP^3) = \frac{1}{3!} J \wedge J \wedge J = \frac{1}{3} J^3$, $*_6 J = \frac{1}{2} J \wedge J = \frac{1}{2} J^2$, and $J \wedge J = 0$. Also we have $J_{\alpha \gamma} J_{\bar{\beta}}^{\bar{\gamma}} = g_{\alpha \bar{\beta}}$, $J_{\alpha \bar{\beta}} J_{\gamma \bar{\delta}} = 3$. We use $*_4$ and $*_6$ to indicate the Hodge star operation with respect to the metrics of $AdS_4$ and $CP^3$, respectively.}
Therefore we conclude that

\[ T^{F_2}_{\alpha \beta} + T^{F_4}_{\alpha \beta} = 0. \]  

(30)

So we have showed that with ansatz (22) the energy-momentum tensor of D0-D2 brane configuration vanishes and there will be no backreaction on the background metric, i.e., the Einstein equations are the same as before. This is analogous to the D-instanton construction in type IIB supergravity. For type IIB D-instantons, the energy-momentum tensors of dilaton and axion cancel against each other so that there is no backreaction on the metric. In the type IIA case, we observe a similar cancellation between the energy momentum tensors of D0 and D2 branes as implied by (30).

In conclusion, we have obtained an exact solution of the form

\[ ds^2 = R^3_k (1/4 ds^2_{AdS4} + ds^2_{CP^3}), \]

\[ e^{2\phi} = R^3_k, \quad F_2 = 2k J + F, \quad F_4 = 3i R^3 \epsilon_4 - i R^3 J \wedge \ast_4 F, \]  

(31)

with all other form fields set to zero, and \( F \) the self-dual gauge field strength constructed in Sec. 2. Note that the dilaton has its original value of \( e^{2\phi} = R^3 / k^3 \).

4 Uplift to 11d Supergravity Solutions

The \( U(1) \) instanton that we found in Sec. 2, can further be uplifted to an exact solution in 11d supergravity. Fortunately, the consistent truncation of 11d supergravity to four dimensions of [8] makes this uplift much simpler, and in fact it is more general than our discussion in the previous section as it involves more general metrics.

To begin with, consider the following ansatz for the metric of 11d supergravity

\[ ds^2 = ds^2_4 + e^{2U} ds^2_{KE_6} + e^{2V} (\eta + A_1) \otimes (\eta + A_1), \]  

(32)

where \( ds^2_4 \) is an arbitrary metric on a 4d spacetime, and \( ds^2_{KE_6} \) is the metric of a 6d Kähler-Einstein space. \( U \) and \( V \) are scalar fields and \( A_1 \) is a one-form defined on the four-dimensional space. If we set \( U = V = A_1 = 0 \), then the last two factors of the metric constitute the Sasaki-Einstein metric

\[ ds^2_{SE_7} = ds^2_{KE_6} + \eta \otimes \eta, \]  

(33)

where \( \eta \) is the globally defined one-form dual to the Reeb Killing vector. On a Sasaki-Einstein manifold one can globally define a 2-form \( J \) and a holomorphic 3-form so that

\[ d\eta = 2J, \quad d\Omega = 4i \eta \wedge \Omega. \]  

(34)
The 4-form $F_4$ is taken to be

$$F_4 = f \text{vol}_4 + H_3 \wedge (\eta + A_1) + H_2 \wedge J + H_1 \wedge (\eta + A_1) + 2h J \wedge J + \sqrt{3} \left[ \chi_1 \wedge \Omega + \chi (\eta + A_1) \wedge \Omega + \text{c.c.} \right],$$

where $f$ and $h$ are real scalars, $H_p$, $p = 1, 2, 3$, are real $p$-forms, $\chi_1$ is a complex one-form, and $\chi$ is a complex scalar on the four-dimensional spacetime. With the ansatzes \eqref{32} and \eqref{35}, the 11-dimensional supergravity field equations all reduce to 4-dimensional equations, so that any solution to the reduced equations can be lifted to an 11-dimensional supergravity solution. These 4-dimensional equations are given in \cite{8}.

Now, one can further consistently truncate by setting $U = V = h = \chi = H_3 = H_1 = 0$. Equations (B.9) to (B.11) of \cite{8}, when adapted to Euclidean signature, reduce to

$$-fF + 6 *_4 H = 0,$$

$$d *_4 H = 0,$$

$$*_4 H \wedge F + iH \wedge H = 0,$$

with $F = dA_1$, $H \equiv H_2$, and $dH = 0$. Equations (B.19), (B.20), and (B.22) reduce to

$$R_{\mu\nu} = \frac{1}{3} g_{\mu\nu} f^2 + \frac{1}{2} F_{\mu\rho} F_{\nu}^{\rho} + \frac{3}{2} (H_{\mu\rho} H_{\nu}^{\rho} - \frac{1}{6} g_{\mu\nu} H_{\rho\sigma} H^{\rho\sigma}),$$

$$d *_4 F = 0,$$

$$F_{\mu\nu} F^{\mu\nu} + H_{\mu\nu} H^{\mu\nu} = 0.$$  \hspace{1cm} \text{(37)}

The last eqs. of \eqref{36} and \eqref{37} are satisfied if we set

$$H = i *_4 F,$$

from which $d *_4 H = 0$ and $d *_4 F = 0$ are followed by the Bianchi identities for $F$ and $H_2$, respectively. The first equation in \eqref{36} then implies $f = 6i$. We are then left with the first equation in \eqref{37}:

$$R_{\mu\nu} = -12g_{\mu\nu} + \frac{1}{2} F_{\mu\rho} F_{\nu}^{\rho} + \frac{3}{2} (H_{\mu\rho} H_{\nu}^{\rho} - \frac{1}{6} g_{\mu\nu} H_{\rho\sigma} H^{\rho\sigma})$$

$$= -12g_{\mu\nu} + 2(F_{\mu\rho} F_{\nu}^{\rho} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}),$$

where we have used

$$H_{\mu\rho} H_{\nu}^{\rho} = F_{\mu\rho} F_{\nu}^{\rho} - \frac{1}{2} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma},$$

implied by \eqref{38}.

As a final step, let us set

$$F = *_4 F,$$
for which the second term in (39) (i.e., the energy-momentum tensor of $F$) vanishes, and hence equation (39) is solved when the metric is that of Euclidean $AdS_4$. So, in summary, we arrive at the following exact 11d supergravity solution:

$$ds^2 = \frac{1}{4} ds^2_{AdS_4} + ds^2_{KE_6} + (\eta + A_1) \otimes (\eta + A_1),$$

(42)

together with the 4-form $F_4$

$$F_4 = \frac{3i}{8} \epsilon_4 + IF \wedge J,$$

(43)

with $F = dA_1$, the self-dual gauge field strength explicitly constructed in Sec. 2.

5 Conclusions

In this paper we constructed the first examples of exact D-instanton solutions in type IIA supergravity as well as M-theory. The construction crucially depended on the existence of $U(1)$ instantons on $AdS_4$. This motivated us to reexamine such configurations in Sec. 2. Our ansatzs included form fields for which the energy-momentum tensors were vanishing along the $AdS_4$ factor, whereas they added up to zero in the orthogonal direction. As a result, the background metric was left unchanged. As in the case of D-instantons in type IIB theory, the solutions provide the necessary tools for a nonperturbative study of the type IIA supergravity and M-theory. Further, it is interesting to study them in the context of AdS/CFT correspondence. This requires a construction of dual instanton configurations on the boundary theory so that one can investigate the nonperturbative effects on both sides of the duality.

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References

[1] J. M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. 2, 231 (1998), [Int. J. Theor. Phys. 38, 1113 (1999)], [arXiv:hep-th/9711200].

[2] M. Bianchi, M. B. Green, S. Kovacs and G. Rossi, *Instantons in supersymmetric Yang-Mills and D instantons in IIB superstring theory*, JHEP 9808, 013 (1998), [arXiv:hep-th/9807033].

[3] T. Banks and M. B. Green, *Nonperturbative effects in AdS$_5 \times S^5$ string theory and d = 4 SUSY Yang-Mills*, JHEP 9805, 002 (1998), [arXiv:hep-th/9804170].

[4] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, *N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals*, JHEP 0810, 091 (2008), [arXiv:0806.1218 [hep-th]].

[5] K. Hosomichi, K. M. Lee, S. Lee, S. Lee, J. Park and P. Yi, *A Nonperturbative Test of M2-Brane Theory*, JHEP 0811, 058 (2008), [arXiv:0809.1771 [hep-th]].

[6] A. Imaanpur and M. Naghdi, *Dual Instantons in Antimembranes Theory*, Phys. Rev. D 83, 085025 (2011), [arXiv:1012.2554 [hep-th]].

[7] M. Naghdi, *A Monopole Instanton-Like Effect in the ABJM Model*, Int. J. Mod. Phys. A 26, 3259 (2011), [arXiv:1106.0907 [hep-th]].

[8] J. P. Gauntlett, S. Kim, O. Varela and D. Waldram, *Consistent supersymmetric Kaluza–Klein truncations with massive modes*, JHEP 0904, 102 (2009), [arXiv:0901.0676 [hep-th]].

[9] S. de Haro and P. Gao, *Electric magnetic duality and deformations of three-dimensional CFT’s*, Phys. Rev. D 76, 106008 (2007), [arXiv:hep-th/0701144].

[10] O. Sarioglu and B. Tekin, *Self dual solutions of Yang-Mills theory on Euclidean AdS*, Phys. Rev. D 79, 104024 (2009), [arXiv:0903.3803 [hep-th]].

[11] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, JHEP 9909, 032 (1999), [arXiv:hep-th/9908142].