A BOURGAIN-BREZIS-MIRONESCU-DÁVILA THEOREM IN GROUPS OF
HEISENBERG TYPE

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Abstract. In this note we prove the following theorem in any group of Heisenberg type $G$:

$$\lim_{s \to 1/2} (1 - 2s)\Psi_{H,s}(E) = \frac{4}{\sqrt{\pi}} \Psi_H(E).$$

Here, $\Psi_H(E)$ represents the horizontal perimeter of a measurable set $E \subset G$, whereas the nonlocal horizontal perimeter $\Psi_{H,s}(E)$ is a heat based Besov seminorm. This result represents a dimensionless sub-Riemannian counterpart of a famous characterisation of Bourgain-Brezis-Mironescu and Dávila.

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1. Introduction

In their celebrated papers [4], [5] (see also [7]) Bourgain, Brezis and Mironescu discovered a new characterisation of the spaces $W^{1,p}$ and BV as suitable limits of the fractional Aronszajn-Gagliardo-Slobedetzky spaces $W^{s,p}$. We also mention the earlier work [39], in which the authors had already settled the case $p = 2$ of their limiting theorem, and the subsequent work [40], in
which Maz’ya & Shaposhnikova extended the results in \[5\]. Keeping in mind the definition of the seminorm in \( W^{s,p} \), see \[1\],

\[
[f]_{p,s} = \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n+ps}} \, dx \, dy \right)^{1/p}, \quad p \geq 1, \quad 0 < s < 1,
\]

and denoting by \( 1_E \) the indicator function of a measurable set \( E \subset \mathbb{R}^n \), we recall that such set is said to have finite nonlocal \( s \)-perimeter if \( |E| < \infty \) and

\[
(1.1) \quad P_s(E) \overset{\text{def}}{=} |1_E|_{2,s}^2 = |1_E|_{1,2s} < \infty.
\]

This notion appeared in the above mentioned works \[4\], \[5\], \[7\], as well as in Maz’ya’s paper \[38\], and in the work of Caffarelli, Roquejoffre and Savin \[9\], in which these authors have first studied the Plateau problem for the relevant nonlocal minimal surfaces. It is well-known that every non-empty open set has infinite \( s \)-perimeter as soon as \( 1/2 \leq s < 1 \). For instance, if we denote by \( B = \{ x \in \mathbb{R}^n \mid |x| < 1 \} \), then it was observed in \[24\] that

\[
P_s(B) = \frac{n\pi^n \Gamma(1 - 2s)}{s \Gamma\left(\frac{n}{s} + 1\right) \Gamma(1 - s) \Gamma\left(\frac{N - 2s}{2}\right)},
\]

where for \( x > 0 \) we have indicated by \( \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt \) the Euler gamma function. It is clear from this formula, as well as from those appeared in \[22\] and \[18\], that \( s \rightarrow P_s(B) \) has a pole in \( s = 1/2 \) (and also in \( s = 0 \)), and that moreover one has the limiting relation

\[
\lim_{s \nearrow 1/2} (1 - 2s)P_s(B) = \frac{2\pi n - 1}{\Gamma\left(\frac{n+1}{2}\right)} P(B),
\]

where we have denoted by \( P(B) = \frac{2\pi n}{\Gamma\left(\frac{n}{2}\right)} \) the standard perimeter of \( B \). This limit relation is in fact a special case of the following result proved by J. Dávila in \[15\], Theor. 1\], and conjectured in \[4\]:

\[
(1.2) \quad \lim_{s \nearrow 1/2} (1 - 2s)P_s(E) = \left( \int_{\mathbb{R}^n-1} | < e_n, \omega > | \, d\sigma(\omega) \right)^2 \, P(E),
\]

where \( e_n = (0, \ldots, 0, 1) \), and \( P(E) \) indicates the perimeter of \( E \) according to De Giorgi, see \[16\]. The limiting behaviour of the fractional perimeter was also studied in \[2\] and \[10\]. All these results underscore an important aspect of the nonlocal minimal surfaces: they asymptotically converge to the classical ones.

To introduce the results in the present paper, we now make the crucial observation that theorem \( (1.2) \) admits a dimension-free formulation using the heat semigroup \( P_t^{\Delta} f(x) = e^{-t\Delta} f(x) = (4\pi t)^{-n/2} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} f(y) \, dy \). For \( s > 0 \) and \( 1 \leq p < \infty \), consider the following caloric Besov seminorm

\[
(1.3) \quad \mathcal{N}_{s,p}^{\Delta}(f) = \left( \int_0^\infty \frac{1}{t^{\frac{n}{2} + 1}} \int_{\mathbb{R}^n} P_t^{\Delta} (|f - f(x)|^p) (x) \, dx \, dt \right)^{\frac{1}{p}}.
\]
Seminorms such as (1.3) were first considered by Taibleson in his works \cite{45,46} for quite different purposes than those in the present note. We leave it as an easy exercise for the reader to recognise that (1.4)

\[ \mathcal{N}_{s,p}(f)^p = \frac{2^p \Gamma\left( \frac{n+sp}{2} \right)}{\pi^{\frac{n}{2}}} \|f\|_{s,p}^p. \]

Combining (1.4) with (1.1) and (1.2), and keeping in mind that \( \int_{\mathbb{S}^{n-1}} |\mathbf{e}_n, \omega| |d\sigma(\omega) = \frac{2^{n-1}}{\Gamma\left( \frac{n+1}{2} \right)} \), we now see that the theorem of Bourgain-Brezis-Mironescu-Dávila can be reformulated in terms of the heat seminorm (1.3) in the following suggestive dimension-free fashion:

(1.5) \[
\lim_{s \uparrow \frac{1}{2}} (1 - 2s) \mathcal{N}_{2s,1}(1_E)^2 = \lim_{s \uparrow \frac{1}{2}} \left( 1 - 2s \right) \mathcal{N}_{s,2}(1_E)^2 = \frac{4}{\sqrt{\pi}} P(E).
\]

The present work stems from the desire of understanding what happens to (1.5) if we leave the Euclidean setting and move into sub-Riemannian geometry. Does the Bourgain-Brezis-Mironescu-Dávila phenomenon persist? Our main result proves that, remarkably, the answer is yes and with exactly the same universal constant as in (1.5)! - in the framework of Lie groups of Heisenberg type provided that:

(i) the perimeter of De Giorgi \( P(E) \) is replaced by the sub-Riemannian horizontal perimeter in (2.6) below;
(ii) the nonlocal perimeter \( P_s(E) \) in (1.1) is replaced by a notion of nonlocal horizontal perimeter defined via some Besov seminorms based on the heat semigroup in \( G \), see Definition 3.1 below.

Before stating our main theorem we mention that the basic prototype of the geometric ambients in this note is the Heisenberg group \( \mathbb{H}^n \), which is the primary model of a Sasakian CR manifold with zero Tanaka-Webster Ricci tensor, see \cite{17}. Groups of Heisenberg type were introduced by Kaplan in \cite{34} in connection with hypoellipticity questions. They constitute a direct and important generalisation of the Heisenberg group, as they include, in particular, Iwasawa groups, i.e., the nilpotent component \( N \) in the Iwasawa decomposition \( KAN \) of simple groups of rank one, see in this respect the seminal work of Cowling, Dooley, Korányi and Ricci \cite{13}, and also the visionary address of E. Stein \cite{44} at the 1970 International Congress of Mathematicians in Nice.

Given a group of Heisenberg type \( G \), we indicate with \( \mathcal{P}_H(E) \) the horizontal perimeter of a set \( E \subset G \). Such notion was introduced in \cite{11} in much greater generality than groups, and a corresponding theory of isoperimetric inequalities was subsequently developed in \cite{25}. Next, let us denote with \( \mathcal{P}_{H,s}(E) \) the key notion of nonlocal horizontal \( s \)-perimeter of \( E \) as in Definition 3.1 below. The following is the main theorem of this note.

**Theorem 1.1.** Let \( G \) be a group of Heisenberg type, and let \( E \subset G \) be a measurable set having finite horizontal perimeter and such that \( |E| < \infty \). Then,

\[
\lim_{s \uparrow \frac{1}{2}} (1 - 2s) \mathcal{P}_{H,s}(E) = \frac{4}{\sqrt{\pi}} \mathcal{P}_H(E).
\]
One should compare Theorem 1.1 with the dimension-free version of the Bourgain-Brezis-Mironescu and Dávila theorem in (1.5) above. It is worth mentioning explicitly that our result underscores the critical role of the heat based notion of nonlocal perimeter in Definition 3.1. Our proof of Theorem 1.1 combines the interesting Ledoux type result in the work of Bramanti, Miranda and Pallara [6, Theorem 2.14] with two crucial asymptotic estimates for the nonlocal perimeter $\mathcal{P}_{H,s}(E)$ which are proved in Section 3. We mention that even for $\mathbb{R}^n$ our proof provides a new perspective on (1.2) (based also on the result in [41]), with a dimensionless constant in the right-hand side.

Having stated our main result we briefly describe the organisation of the paper. In Section 2 we collect some basic facts which will be needed in the rest of the paper. In Section 3 we introduce the notion of nonlocal horizontal $s$-perimeter, see Definition 3.1. Then, we prove the two key results of the section, Propositions 3.2 and 3.3. These two results allow us to conclude in Section 4 that the limit in the left hand-side of the equation in Theorem 1.1 does exist, and moreover

\begin{equation}
\lim_{s \to 1/2} (1 - 2s)\mathcal{P}_{H,s}(E) = \lim_{t \to 0^+} \sqrt{\frac{4}{t}} \|P_t1_E - 1_E\|_{L^1(G)},
\end{equation}

provided that the limit of the sub-Riemannian Ledoux functional in the right-hand side of (1.6) exists. At this point, we exploit the geometric measure theoretic result in [6, Theorem 2.14], which states that in every Carnot group of step two one has

\begin{equation}
\lim_{t \to 0^+} \sqrt{\frac{4}{t}} \|P_t1_E - 1_E\|_{L^1(G)} = 8 \int_G \varphi_G(\nu_E) d|\partial_H E|,
\end{equation}

where we have indicated by $d|\partial_H E|$ the horizontal perimeter measure, and by $\varphi_G(\nu_E)$ the function defined in (26) of [6]. Finally, we prove that in any group of Heisenberg type the function $\varphi_G(\nu_E)$ is a universal constant which is independent of both the dimension of the horizontal layer and of that of the vertical layer of $G$. Once this is recognised, the proof of Theorem 1.1 follows.

In closing we mention the recent work [8] in which we have studied the limiting behaviour as $s \to 0^+$ of Besov seminorms such as (3.1) below, but associated to some non-symmetric and non-doubling semigroups whose generators contain a drift. We also mention two upcoming works that are connected to the present one. In the former [28], we develop in the setting of arbitrary Carnot groups some optimal nonlocal isoperimetric inequalities which involve the notion of nonlocal horizontal perimeter in Definition 3.1. In a related perspective, but with a different framework, the reader should also see [27]. In the latter [29], we provide in the setting of the Heisenberg group $\mathbb{H}^n$ a stronger version of the sub-Riemannian Ledoux limiting relation in (1.7). Our result extends in a nontrivial way a preceding result in [41] in the Euclidean setting, see also the original paper by Ledoux [36], where this circle of ideas originated.

2. Preliminaries

In this section we collect some preliminary material that will be used in the rest of the paper. We begin with introducing the main geometric ambients of this note, referring to [34], [13] and [23, Section 2] for an extensive discussion. A group of Heisenberg type $G$ is a stratified nilpotent
Lie group of step \( r = 2 \), such that if \( \mathfrak{g} = V_1 \oplus V_2 \) is its Lie algebra, with inner product \( (\cdot, \cdot) \) and induced norm \( |\cdot| \), then the Kaplan mapping \( J : V_2 \to \text{End}(V_1) \) defined by
\[
< J(\sigma)z, z' > = < [z, z'], \sigma >,
\]
has the property that \( J(\sigma) \) is orthogonal for every \( \sigma \in V_2 \) such that \( |\sigma| = 1 \). This is equivalent to saying that for every \( z, z' \in V_1 \), and every \( \sigma \in V_2 \), one has
\[
(2.1) \quad < J(\sigma)z, J(\sigma)z' > = |\sigma|^2 < z, z' >.
\]
In particular, when \( \mathbb{G} \) is of Heisengerg type, then (2.1) with \( z = z' \) yields
\[
|J(\sigma)z|^2 = < J(\sigma)z, J(\sigma)z > = |\sigma|^2 |z|^2.
\]
By polarization, one has for every \( \sigma, \sigma' \in V_2 \) and any \( z \in V_1 \)
\[
< J(\sigma)z, J(\sigma')z > = |\sigma|^2 < \sigma, \sigma' >.
\]
In particular, for every \( \ell, \ell' = 1, ..., k \) one has
\[
(2.2) \quad < J(\varepsilon_\ell)z, J(\varepsilon_{\ell'})z > = |z|^2 \delta_{\ell\ell'}.
\]
We let \( k = \text{dim}(V_2) \). Because of the complex structure induced by \( J \), it is well-known that \( \text{dim}(V_1) = 2n \), with \( n \in \mathbb{N} \). We fix orthonormal basis \( \{e_1, ..., e_{2n}\} \) and \( \{\varepsilon_1, ..., \varepsilon_k\} \) for \( V_1 \) and \( V_2 \) respectively, and for points \( z \in V_1 \) and \( \sigma \in V_2 \) we will use either one of the representations \( z = \sum_{j=1}^{2n} z_j e_j \), \( \sigma = \sum_{\ell=1}^{k} \sigma_\ell \varepsilon_\ell \), or also \( z = (z_1, ..., z_{2n}) \), \( \sigma = (\sigma_1, ..., \sigma_k) \). Accordingly, whenever convenient we will identify the point \( g = \exp(z + \sigma) \in \mathbb{G} \) with its logarithmic coordinates \( (z, \sigma) \).

By the Baker-Campbell-Hausdorff formula, see p. 12 of [12],
\[
(2.3) \quad \exp(z + \sigma) \exp(z' + \sigma') = \exp \left( z + z' + \sigma + \sigma' + \frac{1}{2} [z, z'] \right),
\]
we obtain the non-Abelian multiplication in \( \mathbb{G} \)
\[
g \circ g' = (z + z', \sigma + \sigma' + \frac{1}{2} \sum_{\ell=1}^{k} J(\varepsilon_\ell)z, z' > \varepsilon_\ell).
\]
If for \( j = 1, ..., 2n \) we define left-invariant vector fields by the Lie rule \( X_j u(g) = \frac{d}{ds} u(g \exp se_j) \bigg|_{s=0} \),
then by (2.3) one obtains in the logarithmic coordinates \( (z, \sigma) \)
\[
(2.4) \quad X_j = \partial_{z_j} + \frac{1}{2} \sum_{\ell=1}^{k} < J(\varepsilon_\ell)z, e_j > \partial_{\sigma_\ell}.
\]
Given a function \( f \in C^1(\mathbb{G}) \) we will indicate by \( \nabla_H f = (X_1 f, ..., X_{2n} f) \), its horizontal gradient, and set \( |\nabla_H f| = (\sum_{j=1}^{2n} (X_j f)^2)^{1/2} \).
2.1. Horizontal perimeter. We next recall the variational notion of horizontal perimeter that in our main result will replace the perimeter of De Giorgi $P(E)$ in (1.2). Such notion was introduced in [11] (in much greater generality than in stratified nilpotent Lie groups) and it has since occupied a central position in the theory of minimal surfaces in sub-Riemannian geometry. Given an open set $\Omega \subset G$, we denote by

$$\mathcal{F}(\Omega) = \{ \zeta = (\zeta_1, \ldots, \zeta_{2n}) \in C^1_{0}(\Omega; \mathbb{R}^{2n}) \mid \|\zeta\|_\infty = \sup_{g \in \Omega} \left( \sum_{j=1}^{2n} \zeta_j(g)^2 \right)^{1/2} \leq 1 \}.$$  

We say that a function $f \in L^1_{loc}(\Omega)$ has bounded horizontal total variation in $\Omega$ if

$$\text{Var}_H(f, \Omega) \overset{\text{def}}{=} \sup_{\zeta \in \mathcal{F}(\Omega)} \int_{\Omega} f \sum_{j=1}^{2n} X_j \zeta_j dg < \infty.$$  

Hereafter, we denote with $dg$ the bi-invariant Haar measure in $G$ obtained by pushing forward with the exponential map the standard Lebesgue measure in the Lie algebra $\mathfrak{g}$. Such Haar measure interacts with the group non-isotropic dilations, $\delta_\lambda(z, \sigma) = (\lambda z, \lambda^2 \sigma)$, according to the formula

$$d\delta_\lambda(g) = \lambda^Q dg,$$

where $Q = 2n + 2k$ is the group homogeneous dimension, see [21]. The Banach space of $L^1(\Omega)$ functions of bounded horizontal total variation in $\Omega$, with its norm given by

$$||f||_{BV_H(\Omega)} = ||f||_{L^1(\Omega)} + \text{Var}_H(f, \Omega),$$

will be denoted by $BV_H(\Omega)$. Given a set $E \subset G$ such that $|E| < \infty$, we say that $E$ has finite horizontal perimeter with respect to $\Omega$ if $1_E \in BV_H(\Omega)$. If this is the case, the horizontal perimeter of $E$ in $\Omega$ is defined as

$$(2.6) \quad P_H(E; \Omega) = \text{Var}_H(1_E, \Omega) = \sup_{\zeta \in \mathcal{F}(\Omega)} \int_{E \cap \Omega} \sum_{j=1}^{2n} X_j \zeta_j dg.$$  

When $\Omega = G$ we will simply write $\mathfrak{P}_H(E)$, instead of $\mathfrak{P}_H(E; G)$. For the main properties of the space $BV_H$ and general sharp isoperimetric inequalities for the horizontal perimeter, we refer the reader to [25].

2.2. Heat semigroup. The horizontal Laplacian generated by the orthonormal basis $\{e_1, \ldots, e_{2n}\}$ of $V_1$ is the second-order differential operator on $G$ defined by

$$\Delta_H f = \sum_{j=1}^{2n} X_j^2 f,$$

where $X_1, \ldots, X_{2n}$ are given by (2.4). Since $G$ is of Heisenberg type, we obtain from (2.4) and (2.2)

$$(2.7) \quad \Delta_H = \Delta_z + \frac{|z|^2}{4} \Delta_\sigma + \sum_{\ell=1}^{k} \partial_{\sigma_\ell} \Theta_\ell,$$
where $\Delta_z$ and $\Delta_\sigma$ represent the standard Laplacians in the variables $z = (z_1, ..., z_{2n})$ and $\sigma = (\sigma_1, ..., \sigma_k)$, and

$$\Theta_\ell = \sum_{i=1}^{2n} < J(\varepsilon_\ell)z, e_i > \partial_{z_i}.$$ 

The operator $\Delta_H$ fails to be elliptic at every point $g \in \mathbb{G}$. However, thanks to the commutation relation $[X_i, X_j] = \sum_{\ell=1}^{k} (J(\varepsilon_\ell)e_i, e_j)\partial_{\sigma_\ell}$ (which follows from (2.4)), and to the fundamental hypoellipticity theorem of Hörmander in [32], one knows that $\Delta_H$ is hypoelliptic, see also [19].

In [20] Folland proved, in the more general setting of a stratified nilpotent Lie group $\mathbb{G}$, the existence of a fundamental solution $p(g,g',t)$ for the heat equation $\partial_t u - \Delta_H u = 0$ associated with a horizontal Laplacian on $\mathbb{G}$. We need the heat semigroup defined by

$$P_t u(g) = \int_\mathbb{G} p(g,g',t)u(g')dg'.$$

It is well-known that (2.8) defines a stochastically complete, positive and symmetric semigroup in $L^p(\mathbb{G})$, for any $1 \leq p \leq \infty$, which is contractive

$$||P_t u||_{L^p(\mathbb{G})} \leq ||u||_{L^p(\mathbb{G})}, \quad 1 \leq p \leq \infty.$$ 

Although we will not make explicit use of the following Gaussian estimates (which are corollaries of the general results in [31] and [35]) we state them for completeness

$$Ct^{-Q/2} \exp(-\alpha \frac{d(g,g')^2}{t}) \leq p(g,g',t) \leq C^{-1}t^{-Q/2} \exp(-\beta \frac{d(g,g')^2}{t}).$$

In (2.10) we have indicated with $d(g,g')$ the left-invariant intrinsic distance in $\mathbb{G}$ defined by $d(g,g') \overset{def}{=} \{ f(g) - f(g') \mid f \in C^\infty(\mathbb{G}), \ |\nabla_H f| \leq 1 \}$. Such $d(g,g')$ coincides with the Carnot-Carathéodory distance defined in [42]. If we indicate by $B(g,r) = \{ g' \in \mathbb{G} \mid d(g,g') < r \}$, then by scale invariance we obtain for every $g \in \mathbb{G}$ and $r > 0$, $|B_\rho(g,r)| = \omega r^Q$, where $\omega > 0$ is a universal constant and $Q$ is as in (2.5), and this accounts for the term $t^{-Q/2}$ in (2.10).

In the proof of Theorem 1.1 we will need the following remarkable formula (2.11) which originated with the independent works in $\mathbb{H}^n$ of Hulanicki [33] and Gaveau [30], followed by the works of Cygan [14], Randall [43] and R. Beals, Gaveau and Greiner [3]. We warn the reader that different authors choose different group laws and normalisations. Since for the purpose of Theorem 1.1 it is of paramount importance to have the precise normalisations, in the following result the constants have been carefully adjusted to reflect the framework of Section 2 above. Let then $\mathbb{G}$ be a group of Heisenberg type and denote by $p(g,g',t)$ the heat kernel relative to the horizontal Laplacian in (2.7) above. Then, for $g = (z, \sigma), g' = (z', \sigma') \in \mathbb{G}$ and $t > 0$ we have

$$p(g,g',t) = \frac{2^k}{(4\pi t)^{n+k}} \int_{\mathbb{R}^k} \left( \frac{1}{\sinh |\lambda|} \right)^n e^{\frac{1}{2}(\lambda,\sigma' - \sigma) + \frac{1}{2}J(\lambda)(z', z)} e^{-\frac{|z-z'|^2}{4t} - \frac{|\lambda|}{\sinh |\lambda|}} d\lambda.$$ 

We mention that in (2.11) we have identified the vertical layer $V_2 \subset \mathfrak{g}$ with $\mathbb{R}^k$, and we have used that dim$(V_1) = 2n$. When $\mathbb{G} = \mathbb{H}^n$, then $k = 1$, and we recover from (2.11) (with different constants!) the formula of Hulanicki and Gaveau.
3. The nonlocal horizontal s-perimeter and two key asymptotics

With the notion of horizontal perimeter in hands, we now turn to the second key player in our main result: the nonlocal horizontal perimeter. In this section, using a new functional space based on the heat semigroup, we introduce this object in Definition 3.1. Then, we prove two results that will play a key role in the proof of Theorem 1.1. The former provides an interesting one-sided bound for the limiting case $s = 1/2$ similar to the right-hand side of the Bourgain, Brezis and Mironescu’s bound in [4]. The latter instead contains a lower bound for such limit.

Consider the heat semigroup (2.8). For any $0 < s < 1$ and $1 \leq p < \infty$ we define the horizontal Besov space $\mathcal{B}_{s,p}(G) = \mathcal{B}^{s,p}_{1} (G)$ as the collection of all functions $u \in L^{p}(G)$ such that the seminorm

$$ (3.1) \quad \mathcal{N}_{s,p}(u) = \left( \int_{0}^{\infty} \frac{1}{t^{\frac{p}{2}+1}} \int_{G} P_{t} (|u - u(g)|^{p})(g)dgdt \right)^{\frac{1}{p}} < \infty. $$

The norm $\|u\|_{\mathcal{B}_{s,p}(G)} = \|u\|_{L^{p}(G)} + \mathcal{N}_{s,p}(u)$ turns $\mathcal{B}_{s,p}(G)$ into a Banach space. We emphasise that $\mathcal{B}_{s,p}(G)$ is nontrivial since, for instance, it contains $C_{c}^{\infty}(G)$. In a different, but related perspective, similar semigroup based Besov spaces have been introduced and developed in [26], [27] and [8]. In the framework of stratified nilpotent Lie groups the limiting behaviour in $s$ of fractional Aronszajn-Gagliardo-Slobedetzky seminorms different from (3.1) was studied in [37], without the identification of the sharp constants. We are ready to introduce the central notion in this note.

**Definition 3.1.** Given $0 < s < 1/2$, we say that a measurable set $E \subset G$ has finite horizontal s-perimeter if $1_{E} \in \mathcal{B}_{2s,1}(G)$, and we define

$$ \mathcal{P}_{H,s}(E) \overset{\text{def}}{=} \mathcal{N}_{2s,1}(1_{E}) < \infty. $$

We call the number $\mathcal{P}_{H,s}(E) \in [0, \infty)$ the horizontal s-perimeter of $E$ in $G$.

We have the following result.

**Proposition 3.2.** For every measurable set $E \subset G$, such that $|E| < \infty$, one has

$$ (3.2) \quad \limsup_{s \searrow 1/2} (1 - 2s) \mathcal{P}_{H,s}(E) \leq \limsup_{t \to 0^{+}} \sqrt[4]{t} \|P_{t}1_{E} - 1_{E}\|_{L^{1}(G)}. $$

**Proof.** Henceforth in this note, whenever there is no risk of confusion we simply write $\| \cdot \|_{1}$, instead of $\| \cdot \|_{L^{1}(G)}$. Now, if $L \overset{\text{def}}{=} \limsup_{t \to 0^{+}} \sqrt[4]{t} \|P_{t}1_{E} - 1_{E}\|_{1} = \infty$, then (3.2) is trivially valid. Therefore, we might as well assume that $L < \infty$. This implies the existence of $\varepsilon_{0} > 0$ such that

$$ \sup_{\tau \in (0, \varepsilon_{0})} \frac{1}{\sqrt[4]{\tau}} \|P_{\tau}1_{E} - 1_{E}\|_{1} \leq L + 1. $$

For every $0 < s < 1/2$ we thus have

$$ \int_{0}^{\varepsilon_{0}} \frac{1}{\tau^{1+s}} \|P_{\tau}1_{E} - 1_{E}\|_{1} \leq \sup_{\tau \in (0, \varepsilon_{0})} \frac{1}{\sqrt[4]{\tau}} \|P_{\tau}1_{E} - 1_{E}\|_{1} \int_{0}^{\varepsilon_{0}} \frac{d\tau}{\tau^{1+s-1/2}} \leq (L + 1) \frac{\varepsilon_{0}^{1/2-s}}{1/2 - s} < \infty. $$
On the other hand, since by (2.9) $P_\tau$ is contractive on every $L^p(G)$, with $1 \leq p \leq \infty$, we have
\[
\int_{\mathbb{R}^n} \frac{1}{\tau^{1+s}} \|P_\tau (1_E - \epsilon 1_E)\|_1 d\tau \leq 2 |E| \int_{\mathbb{R}^n} \frac{d\tau}{\tau^{1+s}} < \infty.
\]

The latter two inequalities show that
\[
\int_{0}^{\infty} \frac{1}{\tau^{1+s}} \|P_\tau (1_E - \epsilon 1_E)\|_1 d\tau < \infty.
\]

Using the stochastic completeness of the semigroup, we can recognise by Definition 3.1 and (3.1) that
\[
\Psi_{H,s}(E) = \int_{0}^{\infty} \frac{1}{\tau^{1+s}} \int_G P_t (|1_E - \epsilon 1_E(g)|) (g) dg dt
\]
\[
= \int_{0}^{\infty} \frac{1}{\tau^{1+s}} \left( \int_{G \setminus E} P_t (1_E)(g) dg + \int_{E} (1 - P_t (1_E)(g)) dg \right) dt
\]
\[
= \int_{0}^{\infty} \frac{1}{\tau^{1+s}} \left( \int_{G \setminus E} \left| P_t (1_E)(g) - 1_E(g) \right| dg \right. 
\]
\[
+ \int_{E} \left| 1_E(g) - P_t (1_E)(g) \right| dg \right) dt
\]
\[
= \int_{0}^{\infty} \frac{1}{\tau^{1+s}} \|P_\tau (1_E - \epsilon 1_E)\|_1 dt < \infty,
\]
where in the last inequality we have used (3.3). Thus, assuming $L < +\infty$, we reach the conclusion that the set $E$ has finite horizontal $s$-perimeter for every $0 < s < \frac{1}{2}$. With this being said, given any $\epsilon \in (0, \epsilon_0)$, we now obtain from (3.4)
\[
\Psi_{H,s}(E) = \int_{0}^{\epsilon} \frac{1}{\tau^{1+s}} \|P_\tau (1_E - \epsilon 1_E)\|_1 d\tau + \int_{\epsilon}^{\infty} \frac{1}{\tau^{1+s}} \|P_\tau (1_E - \epsilon 1_E)\|_1 d\tau.
\]

As before, one easily recognises
\[
\int_{\epsilon}^{\infty} \frac{1}{\tau^{1+s}} \|P_\tau (1_E - \epsilon 1_E)\|_1 d\tau \leq \frac{2 |E|}{\epsilon} \epsilon^{-s}.
\]

On the other hand, one has
\[
\int_{0}^{\epsilon} \frac{1}{\tau^{1+s}} \|P_\tau (1_E - \epsilon 1_E)\|_1 d\tau \leq \sup_{\tau \in (0, \epsilon)} \frac{1}{\sqrt{\tau}} \|P_\tau (1_E - \epsilon 1_E)\|_1 \epsilon^{1/2-s} \frac{1}{1/2-s}.
\]

We infer that for every $\epsilon \in (0, \epsilon_0)$ we have
\[
\Psi_{H,s}(E) \leq \frac{1}{(1/2 - s)} \sup_{\tau \in (0, \epsilon)} \frac{1}{\sqrt{\tau}} \|P_\tau (1_E - \epsilon 1_E)\|_1 \epsilon^{1/2-s} + \frac{2 |E|}{\epsilon} \epsilon^{-s}.
\]

Multiplying by $(1 - 2s)$ in (3.5) and taking the limit as $s \nearrow 1/2$, we find for any $\epsilon \in (0, \epsilon_0)$,
\[
\limsup_{s \nearrow 1/2} (1 - 2s) \Psi_{H,s}(E) \leq \sup_{\tau \in (0, \epsilon)} \sqrt{\frac{4}{\tau}} \|P_\tau (1_E - \epsilon 1_E)\|_1.
\]

Passing to the limit as $\epsilon \to 0^+$ in (3.6), we reach the desired conclusion (3.2).
Our next result can be seen as dual to Proposition 3.2.

**Proposition 3.3.** For every measurable set \( E \subset G \) one has

\[
\liminf_{s \nearrow 1/2} (1 - 2s) \mathcal{P}_{H,s}(E) \geq \liminf_{t \to 0^+} \sqrt{\frac{4}{t}} \| P_t 1_E - 1_E \|_{L^1(G)}.
\]

**Proof.** Before starting we remark that (3.4) holds regardless the assumption that the right-hand side be finite. Now, for every \( 0 < s < 1/2 \) and any \( \varepsilon > 0 \), the identity (3.4) yields

\[
(1 - 2s) \mathcal{P}_{H,s}(E) \geq (1 - 2s) \int_0^\varepsilon \frac{1}{t^{1+s}} \| P_t 1_E - 1_E \|_1 dt
\]

\[
\geq (1 - 2s) \inf_{0 < t < \varepsilon} \sqrt{\frac{4}{t}} \| P_t 1_E - 1_E \|_1 \int_0^{\varepsilon} t^{1/2-s-1} dt
\]

\[
= \inf_{0 < t < \varepsilon} \sqrt{\frac{4}{t}} \| P_t 1_E - 1_E \|_1 \varepsilon^{1/2-s}.
\]

Taking the \( \liminf \) as \( s \nearrow 1/2 \) in the latter inequality, gives

\[
\liminf_{s \nearrow 1/2} (1 - 2s) \mathcal{P}_{H,s}(E) \geq \inf_{0 < t < \varepsilon} \sqrt{\frac{4}{t}} \| P_t 1_E - 1_E \|_1.
\]

If we now take the limit as \( \varepsilon \to 0^+ \), we reach the desired conclusion. \( \square \)

4. **Proof of Theorem 1.1**

In this section we finally prove Theorem 1.1. Our first key observation is that, if we knew that for any measurable set \( E \subset G \) with finite horizontal perimeter we have

\[
(4.1) \quad \liminf_{t \to 0^+} \sqrt{\frac{4}{t}} \| P_t 1_E - 1_E \|_1 = \limsup_{t \to 0^+} \sqrt{\frac{4}{t}} \| P_t 1_E - 1_E \|_1 = \frac{4}{\sqrt{\pi}} P_H(E),
\]

then the combination of Propositions 3.2, 3.3 and (4.1) would give

\[
\lim_{s \nearrow 1/2} (1 - 2s) \mathcal{P}_{H,s}(E) = \frac{4}{\sqrt{\pi}} P_H(E),
\]

and Theorem 1.1 would be proved. To establish (4.1) we first appeal to (29) in [6, Theorem 2.14]. This implies that in every stratified nilpotent Lie group \( G \) of step \( r = 2 \) the limit in (4.1) does exist for any set of finite horizontal perimeter \( E \subset G \), and moreover

\[
(4.2) \quad \lim_{t \to 0^+} \sqrt{\frac{4}{t}} \| P_t 1_E - 1_E \|_1 = 8 \int_G \varphi_G(\nu_E) d|\partial E|.
\]

In the right-hand side of (4.2) for every horizontal unit vector \( \nu \) the function \( \varphi_G \) is defined by

\[
(4.3) \quad \varphi_G(\nu) = \int_{T_G(\nu)} p(\hat{g}, e, 1) d\hat{g},
\]

where we have denoted by \( e \in G \) the identity and by \( \hat{g} \) the generic point on the vertical space \( T_G(\nu) \) perpendicular to the horizontal vector \( \nu \), see (26) in [6]. Now, it is easy to see that in
Using the well-known formula
\[ p(\nu, e, 1) = \frac{2^k}{(4\pi)^{n+k}} \int_{\mathbb{R}^k} \left( \frac{1}{\sinh |\lambda|} \right)^n e^{i(\lambda, \sigma)} e^{-\frac{1}{4} \frac{|\lambda|^2}{\sinh^2 |\lambda|}} d\lambda. \]

Using the spherical symmetry of \( p(g, e, 1) \) in the horizontal variable \( z \), it is now clear that if \( \nu \) and \( \eta \) are two horizontal vectors, there exists an orthogonal transformation in the horizontal layer \( V_1 \equiv \mathbb{R}^{2n} \) that takes \( \nu \) into \( \eta \). By the invariance of Lebesgue measure in \( T_G(\nu) \) with respect to orthogonal transformations in the horizontal layer, this shows that \( \varphi_G(\nu) = \varphi_G(\eta) \), and therefore \( \varphi_G(\nu) \equiv \overline{\varphi_G} \). Using this observation in (4.2) we conclude that, in any group of Heisenberg type \( G \), we have
\[ \lim_{t \to 0^+} \sqrt{\frac{4}{t}} |P_t 1_E - 1|_1 = 8 \overline{\varphi_G} \int_G d|\partial E| = 8 \overline{\varphi_G} P_H(E). \]

In light of (4.5) it is at this point clear that, in order to finish the proof of Theorem 1.1 we must show that
\[ \varphi_G(\nu) = \overline{\varphi_G} = \frac{1}{\sqrt{4\pi}}. \]

We remark that (4.6) is saying that the function \( \varphi_G(\nu) \) in (4.3) is universal, in the sense that, besides being independent of the horizontal unit vector \( \nu \), it is also independent of the group \( G \).

It is clear from the above discussion that in order to evaluate \( \varphi_G \) it suffices to compute \( \varphi_G(e_{2n}) \). In such case \( T_G(e_{2n}) \) can be identified with \( \mathbb{R}^{2n-1} \times \mathbb{R}^k \), and we can write the points of such space as \( \check{g} = (\check{z}, \check{\sigma}) \), where \( \check{z} \in \mathbb{R}^{2n-1} \). From (4.3) and (4.4) we thus have
\[ \varphi_G = \frac{2^k}{(4\pi)^{n+k}} \int_{\mathbb{R}^{2n-1}} \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} \left( \frac{|\lambda|}{\sinh |\lambda|} \right)^n e^{i(\lambda, \sigma)} e^{-\frac{1}{4} \frac{|\lambda|^2}{\sinh^2 |\lambda|}} d\lambda d\check{z} d\check{\sigma} \]
\[ = \frac{2^k}{(4\pi)^{n+k}} \int_{\mathbb{R}^{2n-1}} \int_{\mathbb{R}^k} \left( \frac{|\lambda|}{\sinh |\lambda|} \right)^n e^{i(\lambda, \sigma)} \int_{\mathbb{R}^k} e^{-\frac{1}{4} \frac{|\lambda|^2}{\sinh^2 |\lambda|}} d\lambda d\check{\sigma}. \]

Using the well-known formula \( \int_{\mathbb{R}^N} e^{-|\zeta|^2} d\zeta = \pi^{N/2} \), an easy calculation now shows that
\[ \int_{\mathbb{R}^{2n-1}} e^{-\frac{1}{4} \frac{|\lambda|^2}{\sinh^2 |\lambda|}} d\lambda = \left( \frac{4 \tanh |\lambda|}{|\lambda|} \right)^{n-1/2} \pi^{n-1/2}. \]

Substitution in the above formula gives
\[ \varphi_G = \frac{1}{2^{k+1 + \frac{n+1}{2}}} \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} \left( \frac{1}{\cosh |\lambda|} \right)^n \left( \frac{1}{\tanh |\lambda|} \right)^{1/2} e^{i(\lambda, \sigma)} d\lambda d\sigma \]
\[ = \frac{1}{\sqrt{4\pi}} \int_{\mathbb{R}^k} \hat{f}(\sigma) d\sigma, \]
where we have let
\[ f(\lambda) = \left( \frac{1}{\cosh |\lambda|} \right)^n \left( \frac{|\lambda|}{\tanh |\lambda|} \right)^{1/2}, \]
and we are using the following definition of Fourier transform
\[ \hat{f}(\sigma) = \int_{\mathbb{R}^k} e^{-2\pi i \langle \sigma, \lambda \rangle} d\lambda. \]

We now notice that \( f \in C^\infty(\mathbb{R}^k) \) and \( f(0) = 1 \). Furthermore, because of the factor \((\cosh |\lambda|)^{-n}\), it is not difficult to verify that all derivatives of \( f \) decay exponentially at infinity, and therefore \( f \) belongs to the Schwartz class \( \mathcal{S}(\mathbb{R}^k) \). We can then apply the inversion theorem for the Fourier transform and conclude that
\[ 1 = f(0) = \int_{\mathbb{R}^k} \hat{f}(\sigma) d\sigma, \]
which completes the proof of (4.6).

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