Asymptotic Shape of Quantum Markov Semigroups for Compact Uniform Trees

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Abstract

We give locally finite Markov trees in \(L^p\)-compact, separable Hilbert, supersymmetric process: \([0, \infty) \times \mathbb{R}^{|\mathcal{A}^\otimes n|}/\mathcal{A}^\otimes m\) on quantum \(U(|\mathcal{A}^\otimes m|)\) semigroups. In full automorphism group \(\text{Aut}(T)\) of modular subgroup, asymptotic-ergodicity is entropy-worthy \(\mathbb{R}\) shape for uniform partition.

Keyword: asymptotic-uniform-trees, quantum-Markov-semigroups, compact-partition-entropy

\(GL(n)\) theory for discrete-, continuous-time, quantum \(U(|\mathcal{A}^\otimes m|)\) Markov trees

Adhering (\(\cdot, \cdot\))_\Phi, \(\mathcal{X} \equiv L^p(\mathcal{X}, \Phi)\)-compact space \((\Omega, \mathcal{F}, \mathcal{P})\) of usual \([0, \infty) \times \mathbb{R}^{|\mathcal{A}^\otimes m|}\) Markov trees, is \(\mathcal{X}\)-acting process, positive semi-definite \(H\) set \(\mathcal{P}\) of density (normalized), subnormalized states:

\[
\{ \mathbf{T}: \text{tr}(\mathbf{T}) = 1 \} \subseteq \mathcal{P}, \text{ resp. } \{ \mathbf{T}: \text{tr}(\mathbf{T}) \leq 1 \} \subseteq \mathcal{P}
\]

in supersymmetric basis \(\mathcal{A}^\otimes m\) representation of the real Lie group, unitary group \(U(|\mathcal{A}^\otimes m|)\) for \(\nu\)-partite Hilbert space \(\mathcal{X}_{X_0 \cdots X_{n-1}} = X_0 \oplus \cdots \oplus X_{n-1}\), and affine functional \(\Gamma \rightarrow \Phi_\Gamma(F), F \in \mathcal{F}\):

\[
\Gamma_X = \text{tr}_{\mathcal{X}}(\Gamma_{X_0 \cdots X_{n-1}}), \forall \nu = 0, \ldots, \nu - 1 \quad \Phi_{\Gamma, y}(F) \neq \Phi_{\Gamma, y}(y); \text{ i.e. } \exists \Gamma \rightarrow \Phi_\Gamma, \Phi_{\Gamma, x} \neq \Phi_{\Gamma, y}, \forall \Gamma_x \neq \Gamma_y
\]

where \(\Gamma_X, \ldots, \Gamma_{X_{n-1}}\) are reduced states of unique mixed state \(\Gamma_{X_0 \cdots X_{n-1}} \in \mathcal{P}_{X_0 \cdots X_{n-1}}\) on compact (closed and bounded) space. For events \(\mathcal{X} \ni x \in \mathcal{X}\), measurable subsets \(\mathcal{X} \subset \mathcal{X}\) all form \(\sigma\)-field \(\mathcal{F}(\mathcal{X})\) of measurable space \((\mathcal{X}, \mathcal{F})\) for all outcomes. In real, Euclidean \(|\mathcal{A}^\otimes m|\)-dimensional space \(\mathbb{R}^{|\mathcal{A}^\otimes m|}\), \(\mathcal{X}\) is a domain with Borel \(\sigma\)-field generated by multi-dimensional open sets (intervals). Main result here is limiting affine bilinear mapping \(\gamma: \mathcal{P} \rightarrow \Phi\); by \(\Phi: \mathbb{R}^{|\mathcal{A}^\otimes m|} \rightarrow \mathbb{R}+, \forall \| T - T^\dagger \|_{\text{L}(\mathcal{P}^{\geq 1}(\gamma))} \leq \varepsilon\):

\[
\Phi_X = \sum_{x \in (\mathcal{A}^\otimes m \cup \{I_{\mathcal{A}^\otimes m}\})} \mathbf{P}_x \text{tr}(x^\dagger x \Gamma_x), \text{ resp. } \Phi_X = \int_{\mathbb{R}^{|\mathcal{A}^\otimes m|}} \det(\sum_{x \in (\mathcal{A}^\otimes m \cup \{I_{\mathcal{A}^\otimes m}\})} x^\dagger x \exp(-iH_{\Gamma}(\theta_x)) \mathbf{P}_t(\theta_x^* \in d\theta_x)
\]

\[
T_X = \sum_{x \in (\mathcal{A}^\otimes m \cup \{I_{\mathcal{A}^\otimes m}\})} \mathbf{P}_x x^\dagger x, \text{ resp. } T_X = \sum_{x \in (\mathcal{A}^\otimes m \cup \{I_{\mathcal{A}^\otimes m}\})} x^\dagger x \exp(-\delta(x)) \mathbf{P}_t(\theta_x^* \in d\theta_x).
\]

Dirac-delta \(\delta(x)\) thus assumes zero Hamiltonian of the generic Lagrangian gauge fixings (quantum gravity). Diagonal state \(\Gamma\) is on eigenfunction as unique probabilities \(\mathbf{P}_x = \mathbf{P}_X(x) = \mathbf{P}(X = x)\) \(x \in \mathcal{X}\), resp. \(\mathbf{P}_t\) on i.i.d. \(\delta(\epsilon_{ij})\) random \(X\) from set \(\mathcal{X}\). For all infinite dimensional analogue: \((n+1)\)-mixture is convex hull of a set of vertices: \(n\)-simplex extreme points \(\{\Phi_{x_0}^{2m} \}_{x_0=0} (\text{Fig. 1})\); a simplex of mixture sequence \(\{\Gamma\} \subset \mathcal{P}\) gets modular algebraic closure \([2, 4\text{–}6]\) of stellations and spherical tilings in weights \(\{\mathbf{P}_x\}\) convex mixture, by the one-to-one map of \(\mathcal{P}\) into convex subset of linear space. The bipartite conditional states \(\Gamma_{Q|X=x}\) \(x \in \mathcal{X}\) operator is feasible as:

\[
\Gamma_{XQ} = \sum_{x \in (\mathcal{A}^\otimes m \cup \{I_{\mathcal{A}^\otimes m}\})} \mathbf{P}_x x^\dagger x \otimes \Gamma_{Q|X=x} = \mathbf{P}_x x^\dagger x \exp(-iH_{\Gamma_{Q|X=x}}(\theta_x)) \mathbf{P}_t(\theta_x^* \in d\theta_x)
\]
Definition 1. Let \((P_x, x \in \mathcal{X})\) be a family of \(\gamma\) probability measures for Markov process \((X_t)\) taking values in \(\mathcal{X}\) set up on limit shape probability space \((\Omega, \mathcal{F})\); then \(P_x\) is the \(L^p(\gamma)\), \(p \geq 1\), distribution law of random variable \(X\) under initial condition \(X_0 = x\). For \(\sigma\)-algebra \(\mathcal{X}\) on state space \(\mathcal{X}\) of \(X\):

(i) \((t, \omega) \mapsto X_t(\omega)\) is \(\mathbb{B} \otimes \mathcal{F}/\mathcal{X}\)-measurable mapping of \([0, \infty) \times \Omega\) into \(\mathcal{X}\) by \([0, \infty)\) Borel \(\sigma\)-algebra \(\mathbb{B}\)

(ii) \(x \mapsto P_x\{F\}\) is \(\mathcal{X}\)-measurable for each \(F \in \mathcal{F}\); in particular, for topological space \((\mathcal{X}, 2^{\mathcal{X}})\).

Remark. Physical observation suggests \((\Omega, \mathcal{F})\) accepts a random variable \(T_\alpha\mid \alpha > 0\), which, under \(P_x\mid x \in \mathcal{X}\), is independent of \(X\) and has the mixing, exponential distribution of parameter \(\alpha\).

Proposition (Feynman path semigroup). For \(f: \mathbb{R}^{\lvert A^{\otimes m}\rvert} \rightarrow \mathbb{R}_+/A^{\otimes m} \cup \left\{ I_{\lvert A^{\otimes m}\rvert} \right\}\), set \(P_t f_{\mid t \geq 0}\):

\[
\int_{\mathbb{R}^{\lvert A^{\otimes m}\rvert} \times \mathbb{R}^{\lvert A^{\otimes m}\rvert}} f(x_1, \ldots, x_{2m}) \frac{\exp\left(-\frac{1}{2} \sum_{i,j=1}^{\lvert A^{\otimes m}\rvert} (x_i - \mu_i)^{-1}(x_j - \mu_j) dx_1 \cdots dx_{\lvert A^{\otimes m}\rvert}\right)}{\sqrt{\det(2\pi t \Sigma)}} \, dx_1 \cdots dx_{\lvert A^{\otimes m}\rvert}
\]

and put \(P_0 f = f\). Then \((P_t)_{t \geq 0}\) is a semigroup; moreover, on Feynman path "amplitude."

Proof. \(P_s \circ P_t = P_{s+t}\); and, clearly, the integrand is of the form \(e^{-(i/\hbar)S[x]}\). In addition, \(P_t f(x)\) exists since \(P_t f(x) = \mathbb{E}[f(x + \sqrt{t}Z)]\), for \(Z \sim \mathcal{N}[0, 1]\), on space of \((\lvert A^{\otimes m}\rvert \times \lvert A^{\otimes m}\rvert)\) symmetric invertible positive semi-definite \(\Sigma = A^T A\) invariant with respect to action of orthogonal group (i.e. \(U \Sigma = \Sigma\)); and, \(\Sigma = PD\Sigma P^{-1}\) for diagonal \(D\) of eigenvectors.

Remark. By (0.1), integral (7) makes sense for \(f \in L^1\), resp. \(f \in L^\infty\), and defines a quantum, linear contraction operator \(L^1 \rightarrow L^1\), resp. \(L^\infty \rightarrow L^\infty\). Consequently (Riesz-Thorin [18, 19]), \(P_t\) can be thought of as quantum, linear contraction operator \(L^p \rightarrow L^p\) restricted to upper half-plane, for all \(1 \leq p \leq \infty\). For a heat semigroup, by \(f: \mathbb{R}^{\lvert A^{\otimes m}\rvert} \rightarrow \mathbb{R}\), we have,

\[
P_t f(\mu) = \int_{\mathbb{R}^{\lvert A^{\otimes m}\rvert}} f(x) \frac{1}{(2\pi t)^{\frac{1}{2}\lvert A^{\otimes m}\rvert}} \exp\left(-\frac{\|x-\mu\|^2}{2t}\right) dx \bigg|_{0}^{P_0 f = f}. \tag{8}\]

Lemma 1 (tree process). For stopping time \(t\), tree-valued Markov process \((X_t)_{t \geq 0}\), there exists \(P_t\) random process, i.e. random function on sample functions defined by \(t \mapsto W(t, \omega)\) over sample path, as uniform limit of interpolated \(n\)-state continuous functions, on independent increment:

\[
W(t_1) - W(0), \ W(t_2) - W(t_1), \ldots, W(t_n) - W(t_{n-1}), \quad \forall \ l \mid A^{\otimes m} = n \in \mathbb{N}. \tag{9}\]

with respect to diagonal-eigenvalues and states, given by \(\|\lambda - \lambda^*\|\) and \(\|T - T^\dagger\|_{L(L^p(\gamma))}\).

Proof. The idea is to construct in \(n\)-box, a time \([0, 1]\), \(n\)-dimensional (i.e. \(n\)-tuple state) process starting at \((0, \ldots, 0)\) and interpolated on time intervals \(1 \leq T \leq t\), \(\forall T \geq 0\), as \(n\)-tuple random element in space \(C[0, 1]\) of continuous functions, from ensemble of \(n\) paths of respective time \([0, 1]\) planar continuous-state process with state-space \(\mathcal{X}\) starting at zero i.e. from \(n\)-path ensemble of time \([0, 1]\) planar continuous-state random walk of state-space \(\mathcal{X}\) starting at zero.
such that $W_t(0) = 0$, $W_t(1) = Z_1$, $\forall i = 0, \ldots, n$; $W(t) = (W_1(t), \ldots, W_n(t))^T$; $Z_t = (Z_t(1), \ldots, Z_t(nn))^T$; $Z_{t_{m \in (1, \ldots, 2^n)}} \in \mathcal{N}[0, 1]$; and $\forall t_{m} \in \mathcal{I}_{\eta} \mathcal{T}_{\eta-1}$, $k_{\eta} \in (1, \ldots, 2^n)$,

$$W(t_{m}) = \frac{1}{n!} \left( W(k_{m-1} + \frac{1}{f_{m-1}}) + W(k_{m-1}) \right) \cdot k_{m} \in (1, \ldots, 2^n) + \frac{1}{f_{m}} Z_{t_{m} \in (1, \ldots, 2^n)}.$$  

Clearly, in interpolation, the $\mathbb{R}^{3 \times (A \otimes n)}$ process is given by independent increments; first summand is exponential in addition to the whole pieces being independent affine-random process with respect to normal distribution; the necessary task of showing a threshold of uniform convergence for this constructed (interpolated) process $W : \left[0, 1\right] \rightarrow \mathbb{R}^{3 \times (A \otimes n)}$ can then follow.

**Theorem 1 (meromorphic extension).** On continuous $P_t^{\otimes n}$ limit $P(t)$, Markov process $(X_t)_{t \geq 0}$,

$$\lim_{n \rightarrow \infty} \frac{t}{|\mathcal{A}|^{-1}} \left\| P_t^{\otimes n} - P(t) \right\|_{L(L^p(\gamma))} \overset{n \rightarrow \infty}{\longrightarrow} 0 \quad \forall p \geq 1 \quad \left\| X \right\|_{L(L^p(\gamma))} = \left( \mathbb{E} \left[ \left| X \right| ^p \right] \right)^{1/p} = \left( \sum_{X < \infty} (X^X X^{-p}/p) \right)^{1/p}$$

$$\lim_{n \rightarrow \infty} \frac{t}{|\mathcal{A}|^{-1}} \left\| P_t^{\otimes n} - P(t) \right\|_{L(L^\infty(\gamma))} \overset{n \rightarrow \infty}{\longrightarrow} 0 \quad \left\{ \frac{1}{n |\mathcal{A}|} \right\} \left[ f(n) P_t(t) + P(\bar{x}_n \leq x) \right] \rightarrow 0$$

In addition, $P_t$ defines meromorphic extension of locally compact Markov trees, on irreducible Markov chain $(S, P)$ of periodic transitions, for finite state space $S = (0, 1, \ldots, n)$.

**Proof.** WLOG, set $P_t^{\otimes n} := P(\frac{1}{\sqrt{n}} \bar{x}_n \leq x)$; $\bar{x}_n = x_1 + \cdots + x_n$; $P_t^{(j)} = P(x_j \leq x)$; then, for some $f$,

$$\lim_{n \rightarrow \infty} \frac{1}{|\mathcal{A}|} \left\| P_t^{\otimes n} - P(t) \right\|_{L(L^\infty(\gamma))} = \frac{1}{|\mathcal{A}|} \left( \left\| P_t^{(j)} \right\|_{L(L(\gamma))} \right)^{\otimes n} \sup_n \left\{ \frac{1}{n |\mathcal{A}|} \left[ f(n) P_t(t) + P(\bar{x}_n \leq x) \right] \right\} \rightarrow 0$$

a.s. as $n \rightarrow \infty$; similarly, on sets $\lambda, \lambda^*$, of all eigenvalues of diagonal states $T$, resp. $T^*$.

Let $P = (P_{i,j})_{i,j \geq 0}$ be transition matrix, defined for $P_t$, where $P_{i,j} = P(X_{n+m} = j \mid X_m = i)$ is the probability that the process goes from state $i$ to state $j$ in $n$ transitions, given only temporally homogeneous process having stationary transition probabilities. Moreover, let

$$\theta_{i,j}^{(n)} = P(X_n = i, X_m \neq i, \forall m = 1, \ldots, n-1 \mid X_0 = i), \forall n \geq 1 \quad \theta_{i,j}^{(0)} : = 0, \forall i$$

**Fig. 2:** $\mathbb{R}^3$ tree process from 3-ensemble of a planar continuous process starting at $(0, 0, 0)$. 

Such construction can be glued, for all $t \in \mathbb{R} \geq 0$, in equivalence class (partition) for all $\frac{t}{2} \in \mathbb{N}_0$, by:

$$\mathcal{T}_\eta = \left\{ \frac{t}{(\frac{1}{2})! 2^{(\eta/2)}} : 0 \leq t \leq f_\eta = \left( \frac{\eta}{2} \right)! 2^{(\eta/2)} \right\} \quad \mathcal{T} = \bigcup_{\eta=0}^{\infty} \mathcal{T}_\eta$$

**Proof.** WLOG,
be the probability that, starting at $i$, the first return to state $i$ occurs at $n$th transition. Clearly,

$$\theta_{i,i}^{(1)} = P_{i,i}. \quad (15)$$

Considering all possibilities of the process in each mutually exclusive event $E_m$, $m = 1, \ldots, n$, given by $X_0 = i$, $X_n = i$, and first return to state $i$ occurring at $m$th transition: Given $\theta_{i,i}^{(m)}$, then for the remaining $n - m$ transitions, we only have possibilities of the process for which $X_n = i$. Hence, by Markov property language

$$P\{E_m\} = P\{\text{first return is at } m\text{th transition }|\ X_0 = i\} \cdot P\{X_n = i \mid X_m = i\} = \theta_{i,i}^{(m)} P_{i,i}^{n - m} \quad (16)$$

$$P_{i,i}^n = P\{X_n = i \mid X_0 = i\} = \sum_{m=1}^{n} P\{E_m\} = \sum_{m=1}^{n} \theta_{i,i}^{(m)} P_{i,i}^{n - m} = \sum_{m=0}^{n} \theta_{i,i}^{(m)} P_{i,i}^{n - m} \mid P_{i,i}^0 = 1, \ \theta_{i,i}^{(0)} = 0. \quad (17)$$

Now, define generating function of sequence $\{P_{i,i}^n\}$, resp. $\{\theta_{i,i}^{(n)}\}$, as follows:

$$\Theta_{i,j}(r) = \sum_{n=0}^{\infty} r^n \theta_{i,j}^{(n)}, \quad P_{i,j}(r) = \sum_{n=0}^{\infty} r^n P_{i,j}^n \mid |r| < 1. \quad (18)$$

Then by

$$\sum_{n=0}^{\infty} r^n P^n = (1 - rP)^{-1} = \frac{1}{\det(1 - rP)} \left(\text{Cofactor}\left(1 - rP\right)\right)^T \quad (19)$$

i.e. on matrix inverse (by transpose of cofactor divided by determinant), it follows that $P_{i,j}(r)$ can be extended to a meromorphic function on complex plane for all $i, j \in S$.

In particular, by

$$\left(\sum_{k=0}^{\infty} a_k r^k\right) \left(\sum_{l=0}^{\infty} b_l r^l\right) = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} a_m b_{n-m}\right) = \sum_{n=0}^{\infty} c_n r^n \mid c_n = \sum_{m=0}^{n} a_m b_{n-m}, \quad (20)$$

we have

$$\Theta_{i,i}(r) P_{i,i}(r) = P_{i,i}(r) - 1, \quad \text{i.e. } \quad P_{i,i}(r) = \frac{1}{1 - \Theta_{i,i}(r)} \mid |r| < 1. \quad (21)$$

By $\sum_{n=0}^{\infty} \theta_{i,i}^{(n)} = 1$, then $\lim_{r \to 1} (-\Theta_{i,i}(r)) = 1$. Hence $P_{i,i}(r)$ has a pole at $r = 1$.

As a result,

$$\Theta_{i,i}(r) = 1 - \frac{1}{P_{i,i}(r)} \quad (22)$$

defines a meromorphic extension of $\Theta_{i,i}(r)$ with removable singularity at $r = 1$, i.e. let $\varrho_{i,i} \in [1, \infty)$ denote radius of convergence of power series for $\Theta_{i,i}(r)$. By $\theta_{i,i}^{(n)} \geq 0, \ \forall n \in \mathbb{N}$, then either $\varrho_{i,i} = \infty$ or $\varrho_{i,i}$ is a singular point for the meromorphic extension of $\Theta_{i,i}(r)$. Thus, $\varrho_{i,i} > 1$ for $i \in S$.  

**Remark.** Since every local field is either a finite algebraic extension of the $p$-adic number field for prime $p$ or finite algebraic extension of the $p$-series field; moreover, a locally compact, non-discrete, topological field which is not totally disconnected is necessarily either the real or the complex numbers; let the rings of integers for $p$-adic numbers and $p$-series field (i.e. field of formal series with coefficients from finite field of $p$ elements) be represented by a $p$-ary tree (although the $p$-adic field has characteristic 0 whereas the $p$-series field has characteristic $p$), then $\{P_{i,i}\}_{i \geq 0}$ satisfies family of meromorphic functions of genus $g > 0$ tree having a single pole (an analog of polynomials for higher genera i.e. with weakened planarity condition).

Thus, we have an object for the flexible classification of meromorphic functions (often called topological classification), which relates, Fig. 3, tree isometry to enumerative algebraic geometry and singularity theory, on the one hand, and to braid group action on constellations, on the other hand. Classically, in [1] Alice, Bob, and eavesdropper Eva known $\{P_{i,i}\}$, for uniform $\{P_{i,i}^*\}$, the total variation distance $\frac{1}{2} \sum_{x \in \mathcal{X}} |P_{x} - P_{x}'| \leq \varepsilon$ implies the uncorrelated $P(x \neq x') \leq \varepsilon, \ \forall x, x' \in \mathcal{X}$. 


**Definition 2.** Let order $\alpha$ divergence of probability density $f$ from density $f_+$ be $h_\alpha(f\|f_+)$:

$$h_\alpha(f\|f_+) = \frac{1}{1-\alpha} \ln \left( \int_{\Omega} \frac{f^\alpha(x)}{f_+^{\alpha-1}(x)} \, dx \right); \quad \text{resp.} \quad \frac{1}{\ln |\mathcal{A}|^{1-\alpha}} \ln \left( \sum_{x \in X: |x|=|\mathcal{A}|} \frac{f^\alpha(x)}{f_+^{\alpha-1}(x)} \right)$$

in the limit $\lim_{\alpha \to \infty}$ for all $\alpha \geq 0$; where $\Omega \equiv \mathbb{R}$ resp. $\Omega \equiv \mathbb{R}^n$, $\forall \alpha \in \mathbb{N}$.

**Lemma 2 (Ergodic classification).** Ergodic property of the right continuous process $(X_t)_{t \geq 0}$ is classified in terms of entropy and divergence, as follows:

$$h_0(f\|f_+) = -\ln f_+(\{x \mid f(x) > 0\}) \quad (-1 \text{ natural log of conditional probability for } f_+)$$

$$h_{1/2}(f\|f_+) = -2\log \int \sqrt{f(x) f_+(x)} \, dx \quad (-2 \text{ natural log of Bhattacharyya coefficient } [3])$$

$$h_1(f\|1) = -\int f(x) \ln f(x) \, dx = h(f) \quad (\text{differential entropy})$$

$$h_1(f\|f_+) = \int f(x) \ln \frac{f(x)}{f_+(x)} \, dx \quad \text{for continuity} \quad (\text{Kullback-Leibler divergence } [12])$$

$$h_2(f\|f_+) = -\ln \mathbb{E}[\frac{f}{f_+}] \quad (\text{natural log of expectation of the ratio } f/f_+)$$

$$h_{\infty}(f\|f_+) = -\ln \sup_x (f(x)/f_+(x)) \quad (-1 \text{ natural log of maximum of the ratio } f/f_+)$$

$$h_\alpha(f\|1) = \frac{1}{1-\alpha} \ln \left( \int_{\Omega} f^\alpha(x) \, dx \right), \quad \text{resp.} \quad \frac{1}{\ln |\mathcal{A}|^{1-\alpha}} \ln \left( \sum_{x \in X: |x|=|\mathcal{A}|} f^\alpha(x) \right) \quad (\text{modified Rényi } [15]).$$

**Proof.** The proof follows by taking respective limit on the definition. □

**Remark.** The discrete equivalence with $\sum$ replaces $\int$ is Rényi (divergence) entropy classification.

**Theorem 2 (asymptotic equipartition property).** Let $X_1, \ldots, X_n$ be a sequence of random variables drawn i.i.d. according to a probability measure $\mu$ density $f(x)$. Then

$$\frac{1}{n} \ln f^{\otimes n}(X_1, \ldots, X_n) \longrightarrow \mathbb{E}[-\ln f(X)] = h_1(f\|1) = h(f)$$

almost surely, where $f^{\otimes n}$ denotes the $n$th convolution.

**Proof.** This follows directly from the strong law of large numbers, for all $f$ with respect to $\mu$. □

**Remark.** Ergodicity, Fig. 4, defined in terms of 1-divergence (differential entropy), exists in limit:

$$h(f) = h_1(f\|1) = \lim_{n \to \infty} \frac{1}{n} h(f^{\otimes n}) \quad h(f^{\otimes n}) = -\ln f^{\otimes n}(X_1, \ldots, X_n).$$
Definition 3. Let $\mu$ be continuous probability measure on $\mathcal{X}$. We say $\rho$ is $\mu$-entire, if

$$\rho(z) = e^{P_n(z)} = \int_{\Omega} \tilde{\rho}(zx) \mu(dx) \quad \Omega \equiv \mathbb{R} \ (\text{resp.} \ \Omega \equiv \mathbb{R}^n)$$

(33)

holds for all $z \in \mathbb{C}$ (resp. $z \in \mathbb{C}^n$), and degree $n \geq 1$ polynomial $P_n$ with complex coefficients.

It is not a difficult task to show that all bounded $\mu$-entire functions on $\mathcal{X}$ are constant. Thus, for $h(f) < \infty$, the vanishing of entropy $h$ is equivalent to trivialness of Poisson boundary $\Pi(\mathcal{X}, f)$ [11].

Remark (Hadamard’s factorization theorem). Let $\rho(z)$ be entire function. Assume for simplicity that $\rho(0) \neq 0$. Then, the genus of an entire function is the smallest integer $h$ such that $\rho(z)$ can be represented in the form

$$\rho(z) = e^{\theta(z)} \prod_n \left(1 - \frac{z}{a_n}\right) \exp \left(\frac{z}{a_n} + \cdots + \frac{1}{2}(z/a_m)^m\right)$$

(34)

where $\theta(z)$ is degree $\leq m$ polynomial. And, if there is no such representation, the genus is infinite. Denoting by $M(r)$ the maximum of $|\rho(z)|$ on $|z| = r$. The order of the function $\rho(z)$ is

$$\lambda = \limsup_{r \to \infty} \frac{\ln \ln M(r)}{\ln r}.$$  

(35)

In addition, according to this definition, $\lambda$ is the smallest number such that

$$M(r) \leq e^{\lambda + \varepsilon}$$

(36)

for any given $\varepsilon > 0$, as soon as $r$ is sufficiently large. Moreover, it is known (theorem) that the genus and the order of an entire function satisfy

$$h \leq \lambda \leq h + 1.$$  

(37)

Classical (diagonal) operator metrizing for compact uniform Markov trees

Lemma 3 (classical bipartite). Suppose $P_X(x)|x \in \mathcal{X}$, compact separable Hilbert $L^p(\gamma)$ bipartite space $\mathcal{X}_K \oplus \mathcal{X}_G$, for $|\xi| = n+1$ mixture, with unique mixed state, on stopping time $t$ process $(X_t)_{t \geq 0}$ where $K = G(X)$, for uniform-density family $G$ of $|A|$-universal hash functions i.e. $\kappa \in K$, $\forall g$, by

$$G : \mathcal{A}^\otimes m \cong \mathcal{X} \longrightarrow \mathcal{A}^\otimes k \mid \mathcal{A}^\otimes \zeta \cong G.$$  

(38)

For $\alpha = 1$, minimum entropy $h_{\min}$; joint conditional density $T_{KG}$, resp. uniform $T_{KG}^*$, where $p = 1$,

$$|A|^{-1} \|T_{KG} - T_{KG}^*\|_{L^p(\gamma)} \leq \frac{h_{+} - k}{2}, \quad \forall h_{\min} \geq h_{+}.$$  

(39)
Proof. Assuming discrete WLOG, for \( \zeta \neq k \) on \( G \) uniform: \( P_G(g) = |A|^{-\zeta} \). For \( |K| = |A|^k \) by \( G \) universal, then the uniform \( P^*_KG(\kappa, g) \) (by independence) and conditional \( P_{KG}(\kappa, g) \):

\[
P^*_KG(\kappa, g) = |A|^{-k-\zeta}, \quad P_{KG}(\kappa, g) = P_G(g) P_{KG}(\kappa|g) = |A|^{-\zeta} \sum_{x :: \kappa = g(x)} P_x
\]

where the RHS sum is \( P(K = \kappa) \), admitting collision principle: for \( K \) small enough, \( K \) and \( G \) joint state density gets close to uniform (independence, no-correlation), in order for no information to be revealed by some eavesdropped hash function \( g \) in an open receiver-sender channel.

Following existence, and for all central limit of subsequences, by the trace norm

\[
\|T_{KG} - T^*_KG\|_{L^1(\gamma)} = \sum_{\kappa, g} |P_{\kappa, g} - |A|^{-k-\zeta}|
\]

then

\[
|A|^{-1} \|T_{KG} - T^*_KG\|_{L^1(\gamma)} = |A|^{-1} \sum_{\kappa, g} \left| |A|^{-\zeta} \sum_{x :: \kappa = g(x)} P_x - |A|^{-k-\zeta} \right| = |A|^{-\zeta - 1} \sum_{\kappa, g} \left| \sum_{x :: \kappa = g(x)} P_x - |A|^{-k-\zeta} \right| = |A|^{-\zeta - 1} \sum_{\kappa, g} \left| \sum_{x :: \kappa = g(x)} P_x - |A|^{-k-\zeta} \right| \Rightarrow
\]

i.e., by Cauchy-Bunyakovsky-Schwarz inequality

\[
\sum_{x \in X} a_x b_x \leq \left( \sum_{x \in X} a_x^2 \right)^{1/2} \left( \sum_{x \in X} b_x^2 \right)^{1/2} = \sqrt{|X|} \left( \sum_{x \in X} a_x^2 \right)^{1/2} \Rightarrow
\]

\[
|A|^{-\zeta - 1} \cdot |A|^{((k+\zeta)/2)} \left( \sum_{\kappa, g} \left| \sum_{x :: \kappa = g(x)} P_x - |A|^{-k-\zeta} \right|^2 \right)^{1/2} = |A|^{((k/2)-1)} \left( |A|^{-\zeta} \sum_{\kappa, g} \left| \sum_{x :: \kappa = g(x)} P_x - |A|^{-k-\zeta} \right|^2 \right)^{1/2}
\]

\[
= |A|^{((k/2)-1)} \left( |A|^\zeta \sum_{\kappa, g} \left| P_{\kappa, g} - |A|^{-k-\zeta} \right|^2 \right)^{1/2} = |A|^{((k/2)-1)} \left( |A|^\zeta \sum_{\kappa, g} \left( P^2_{\kappa, g} - 2 P_{\kappa, g} \cdot |A|^{-k-\zeta} + |A|^{-2(k+\zeta)} \right) \right)^{1/2}
\]

\[
= |A|^{((k/2)-1)} \left( |A|^\zeta \left( \sum_{\kappa, g} P^2_{\kappa, g} - 2 \cdot |A|^{-k-\zeta} + |A|^{-2(k+\zeta)} \right) \right)^{1/2} = |A|^{((k/2)-1)} \left( |A|^\zeta \sum_{\kappa, g} P^2_{\kappa, g} - |A|^{-k} \right)^{1/2}. \quad (*)
\]

In addition,

\[
\sum_{\kappa, g} P^2_{\kappa, g} = \sum_{\kappa, g} |A|^{-2} \sum_{x, x'} P_x P_{x'} = |A|^{-2} \left( \sum_{\kappa, g} \left( \sum_{x \neq x' : \kappa = g(x) = g(x')} P_x P_{x'} + \sum_{x = x' : \kappa = g(x) = g(x')} P^2_x \right) \right) = |A|^{-k-\zeta} \sum_{x \neq x'} P_x P_{x'} + |A|^{-\zeta} \sum_{x} P^2_x \ll
\]

for \( \kappa \)-partition, \( \kappa \cdot \sum_{x: \kappa = g(x)} \sum_{x: \kappa = g(x')} \leq \sum_{x} \), on \( P[G(x) = G(x')] \leq |A|^{-k} \), \( \{g: g(x) = g(x')\} \leq |A|^{|\zeta-k|} \).

Moreover, the sum \( \sum_{\kappa, g} P^2_{\kappa, g} \), called collision probability, is bounded by \( |A|^{-k-\zeta} \) as follows:

\[
\sum_{\kappa, g} \sum_{x \neq x': \kappa = g(x) = g(x')} P_x P_{x'} = \sum_{g} \sum_{x \neq x': g(x) = g(x')} P_x P_{x'} = \sum_{x \neq x'} \sum_{g(x) = g(x')} P_x P_{x'} = \sum_{x \neq x'} P_x P_{x'} \sum_{g(x) = g(x')} 1 \leq \sum_{x \neq x'} P_x P_{x'} \cdot |A|^{|\zeta-k|} \leq \sum_{x, x'} |A|^{|\zeta-k|} = |A|^{|\zeta-k|}
\]

where the last inequality stems from all i.i.d \( \sum_{i=0}^{m-1} P_{\xi_i} = 1 \mid P_{\xi_0...\xi_{m-1}} = \prod_{i=0}^{m-1} P_{\xi_i} \),

\[
\sum_{\xi_0, ..., \xi_{m-1}} \prod_{i=0}^{m-1} P_{\xi_i} = \sum_{\xi_0 = \cdots = \xi_{m-1}} \prod_{i=0}^{m-1} P_{\xi_i} + \sum_{j=1}^{m-1} (j+1)! \sum_{\xi_0 < \cdots < \xi_{j} : \xi_{j+1} = \cdots = \xi_{m-1}} \prod_{i=0}^{m-1} P_{\xi_i}
\]

which equals 1 if \( \sum_{i=0}^{m-1} P_{\xi_i} = 1 \); moreover, for all random graph \( \text{sgn}(\pi_\xi : \xi = (\xi_0, \ldots, \xi_{m-1})) \):
The proof follows directly from the prior theorem 3.

Proof. Let $S$ be a tree-valued process, and let $S'$ be its projection to space $H$. The supersymmetric tree process can be encoded by a quantum operator metrizing for compact uniform Markov trees. Hence, the Von Neumann uncertainty dualities to estimate entropy $h_{\min}$ are given as follows:

$$ h_{\min}(A|B) + h_{\max}(A|C) = 0; \quad h_{\alpha}(A|B) + h_{\beta}(A|C) = 0; \quad (\alpha \beta)^{-1}(\alpha + \beta) = 2; \quad h_{\frac{1}{2}} \leq h_{0}; \quad h_{\frac{1}{2}} \leq h_{0} \quad (41) \)

$$ h_{\min}(X|B) + h_{\max}(Z|C) \geq \log(c^{-1}); \quad h_{2}(X|B) + h_{2}(Z|C) \geq \log(c^{-1}); \quad (42) \)

$$ h_{\max}(Z|C) \leq h_{0}(Z|C); \quad h_{\frac{1}{2}}(Z|C) \leq h_{0}(Z|C) \quad (43) \)

Theorem 3 (classical bipartite). Following the prior lemma 3, for $p = 1$,

$$ \lim_{n_t \uparrow \infty} (n_t |A|)^{-1} ||T_{KG} - T_{KG}^*||_{L^p(\gamma)}^{\otimes n_t} \leq |A|^{-2} \left(\frac{k - h_{+} - \kappa}{2}\right), \quad \forall h_{\min} \geq h_{+}. \quad (44) \)

Proof. The proof follows from the prior lemma in strong law of large numbers. \qed

Corollary 1 (diagonal bipartite). The maximum extractable key length

$$ \max(k) = \left[ h_{+} - \frac{2}{\ln |A|} \ln \left( |A|^{-1} \lim_{n_t \uparrow \infty} (n_t)^{-1} ||T_{KG} - T_{KG}^*||_{L^p(\gamma)}^{\otimes n_t} \right) \right], \quad \forall h_{\min} \geq h_{+}. \)

for all tree-valued process $h_{\min}$ low-enough metric-strong protocol of indistinguishable $\kappa$ in reality. Proof. The proof follows directly from the prior theorem 3.

Quantum operator metrizing for compact uniform Markov trees

The supersymmetric tree process can be encoded by $(\mathcal{P}, \gamma)$ bilinear form $\Phi$ in pairwise keys. Fig. 5 does it exactly: $\Phi$ is measure on pairwise ring-blocks of Hopf fibration on torus $S^2$: in stereographic projection of $S^3$ to $\mathbb{R}^3$ and compression of $\mathbb{R}^3$ to torus $S^2$ boundary, following embedding of knot fiber (space $S^1$, circle) bundle in total space $(S^3, 3$-sphere$)$; for properties of trace distance:

$$ |A|^{-1} ||\Phi(T) - \Phi(T^*)||_{L^1(\gamma)} \leq |A|^{-1} ||T - T^*||_{L^1(\gamma)} \leq \varepsilon \quad (45) \)

$$ |A|^{-1} tr( |\Phi(T) - \Phi(T^*)| ) = |A|^{-1} tr( |\Phi(T) - T^*| ) ; \quad tr(\Phi(T)) = tr(T); \quad tr(\Phi(T)) \neq tr(|T|) \quad (46) \)

i.e. on $[2, 5]$ formalism of real-tree random metric $[7]$ through $\Phi$-channel quantum-ensemble

$$ \{P_x, \tilde{T}_x\}_{x \in \mathcal{X}}; \quad \forall x \equiv x_{\xi_0} \cdots x_{\xi_{m-1}} = |x_{\xi_0} \otimes \cdots \otimes x_{\xi_{m-1}}|; \quad \xi = 0, \ldots, n-1. \quad (46) \)
Theorem 4 (Φ existence). The measure Φ exists.

Proof. Let Φ: A ⊗m → R+ satisfy the natural property for every observable Γ, resp. Γ'. Then there is a unique state density T, resp. T', such that, just as in the continuous state,

\[ \Phi(\Gamma) = \sum_x p_x \text{tr}(T_x \Gamma_x) = \sum_x \text{tr}(T_x \Gamma_x) = \text{tr}(T \Gamma) = E^{gen}[\Gamma] \quad | \quad T_x = |x\rangle\langle x| \]

\[ T = \sum_x T_x = \sum_x p_x T_x \quad | \quad \Gamma_x = T^{-1/2} T_x T^{-1/2} = T^{-1/2} (p_x T_x) T^{-1/2} \geq 0, \quad \sum_x \Gamma_x = I_n. \]

Hence, the conclusion follows by Tab. 2, under (0.2).

\[ \square \]

Derivation. If n = 2, i.e. |ξ| = 3, m = 1, in particular, for A ⊗ (|0⟩, |1⟩), then Φ(Γ) = E^{gen}[Γ] = \frac{5}{9}.

Theorem 5 (optimality). For an observable Γ | {Γ_x}x∈X̃, general E^{gen}[Γ], resp. optimal E^{opt}[Γ],

\[ E^{gen}[\Gamma] \geq (E^{opt}[\Gamma])^2 \quad \text{i.e.} \quad E^{opt}[\Gamma] \leq \sqrt{E^{gen}[\Gamma]}. \]

Proof.

\[ E^{gen}[\Gamma] = \sum_x \text{tr}(T_x \Gamma_x) = \sum_x \text{tr}\left((T^{1/2} \Gamma_x T^{1/2})(T^{-1/2} T_x T^{-1/2})\right) \]

\[ \leq \sum_x \left(\text{tr}(T^{1/2} \Gamma_x T^{1/2})\right)^{1/2} \left(\text{tr}(T^{-1/2} T_x T^{-1/2})\right)^{1/2} \]

so that by the Cauchy-Bunyakovskiy-Schwarz inequality for matrix-space Hilbert-Schmidt product

\[ |\text{tr}(AB)| \leq \sqrt{\text{tr}(AA')} \sqrt{\text{tr}(BB')} \]

therefore

\[ \leq \left(\sum_x \text{tr}(T^{1/2} \Gamma_x T^{1/2})\right)^{1/2} \left(\sum_x \text{tr}(T^{-1/2} T_x T^{-1/2})\right)^{1/2} \leq \sqrt{E^{gen}[\Gamma]} \]

where

\[ \text{tr}(T^{1/2} \Gamma_x T^{1/2}) \leq \text{tr}(T \Gamma_x), \quad \text{tr}(T^{-1/2} T_x T^{-1/2}) = \text{tr}(T^{1/2} \Gamma_x T^{1/2}) \]

\[ \square \]

Theorem 6 (quantum tripartite). In ε-rescaling of R^3 on finite X tree tangle T_1 ⊗_ε T_2, Fig. 6, for [8–10, 16, 17] on the noted K = G(X), uniform-density |A|'-universal hash functions family G:

\[ G \ni g: A^\otimes m \cong X \rightarrow A^\otimes k \quad | A^\otimes \zeta \cong G \]

for h_{min}(X|Q) ≥ h_+, there exists the metric

\[ |A|^{-1} \left\| T_{KQ} - |A|^{-k} I_K \otimes T_{QQ} \right\|_{L(L^1(\gamma))} \leq \left| A \right| \left( - \frac{h_+ - k}{2} \right) = \varepsilon \quad 0 < \varepsilon \leq 1. \tag{47} \]
Fig. 6: $\varepsilon$-rescaling of $\mathbb{R}^3: (z, t)\mapsto (\varepsilon z, t)$ on $\varepsilon$-parameterized tree tangle $T_1 \otimes T_2$: inter-tangle distance $1-\varepsilon$.

Proof. Write: 

$$T_{KGQ} = \sum_{\kappa} |\kappa\rangle \langle \kappa| \otimes T_{GQ, x}$$

where 

$$T_{GQ, x} = |A|^{-\zeta} \sum_{x, g} |x\rangle \langle x| \otimes |g\rangle \langle g| \otimes T_{Q, x}$$

for matrices $T_{GQ, x}$ for collision probability, 

$$T_{KGQ} = |A|^{-\zeta} \sum_{x, g} |x\rangle \langle x| \otimes |g\rangle \langle g| \otimes T_{Q, x}$$

where 

$$T_{XQ} = \sum_{x} P_x |x\rangle \langle x| \otimes \tilde{T}_{Q, x} = \sum_{x} |x\rangle \langle x| \otimes \tilde{T}_{Q, x}$$

For additional register (state), 

$$|0\rangle \langle 0| = T_{XQ} \otimes T_{G} = |A|^{-\zeta} \sum_{x, g} |0\rangle \langle 0| \otimes |x\rangle \langle x| \otimes |g\rangle \langle g| \otimes T_{Q, x}$$

Furthermore, 

$$T_{KGQ} = |A|^{-\zeta} \sum_{x, g} |g(x)\rangle \langle g(x)| \otimes |x\rangle \langle x| \otimes |g\rangle \langle g| \otimes T_{Q, x}$$

where 

$$\sum_{|g(x)\rangle \langle g(x)|} = \sum_{x, g} |\kappa\rangle \langle \kappa| \otimes |g\rangle \langle g| \otimes |x\rangle \langle x| \otimes |g\rangle \langle g| \otimes T_{Q, x}$$

$$\sum_{x, g} |g(x)\rangle \langle g(x)| \otimes |x\rangle \langle x| \otimes |g\rangle \langle g| \otimes T_{Q, x}$$

$$T_{KGQ} = |A|^{-\zeta} \sum_{\kappa} |\kappa\rangle \langle \kappa| \otimes \sum_{x, g} |g\rangle \langle g| \otimes \sum_{x, g} |x\rangle \langle x| \otimes |g\rangle \langle g| \otimes T_{Q, x}$$

$$T_{GQ} = tr_{\kappa}(T_{KGQ}) = \sum_{\kappa} T_{GQ, \kappa} = |A|^{-\zeta} \sum_{x} T_{Q, x} = |A|^{-\zeta} \sum_{x} |g\rangle \langle g| \sum_{x} T_{Q, x}$$

$$= |A|^{-\zeta} \sum_{x} |g\rangle \langle g| \otimes T_{Q} = |A|^{-\zeta} I_{G} \otimes T_{Q}$$

$$T_{Q, \kappa} = |A|^{-\zeta} \sum_{x, g} |g\rangle \langle g| \otimes |x\rangle \langle x| \otimes |g\rangle \langle g| \otimes T_{Q, x}$$

That is, for collision probability, on (Eva, et al.) a priory known state $T_{Q}$, then the bound 

$$|A|^{-1} \left\| T_{KGQ} - |A|^{-k-\zeta} I_{KG} \otimes T_{Q} \right\|_{L^1(\gamma)} \leq |A|^{-1} \left\| T_{KG}(k, g) - |A|^{-k-\zeta} T_{Q} \right\|_{L^1(\gamma)} \leq$$

where, for matrices $\tau, A; A=\tau^{-1/4} A \tau^{-1/4}$, $B=\tau^{1/2}$, $tr(\tau) = 1$, $\tau = \tau^{1}$, the Cauchy-Bunyakovsky 

$$|tr(AB)| \leq (tr(AA^{1})^{1/2} (tr(BB^{1}))^{1/2}$$

implies 

$$|A|^{-1} \sum_{\kappa, g} \left( tr(T_{Q, \kappa} - |A|^{-k-\zeta} T_{Q}) \right)^{2} \leq |A|^{(k+\zeta)/2 - 1} \left( \sum_{\kappa, g} tr(T_{Q, \kappa} - |A|^{-k-\zeta} T_{Q}) \right)^{2}$$

$$\leq$$
\[ |A|^{k/2 - 1} \left( |A|^{\zeta} \sum_{\kappa, g} \text{tr} \left( T_{Q,x} T_{Q}^{-1/2} T_{Q,x'} T_{Q}^{-1/2} \right) \right) - |A|^{-k} \right)^{1/2} \]  \quad \text{(**) (59)}

for \( \sum_{\kappa, g} T_{Q,x} = 1 \). That is, (Eva, et al.) collision probability \( P_{\text{col}} \) is given by trace “tr” formula.

Precisely, for the \( K \) and \( G \) joint collision probability \( P_{\text{col}} \), with respect to expectation \( \mathbb{E}^{\text{gen}} \):

\[ P_{\text{col}} = |A|^{-2\zeta} \sum_{\kappa, g} \sum_{x, x'} \text{tr} \left( T_{Q,x} T_{Q}^{-1/2} T_{Q,x'} T_{Q}^{-1/2} \right) = |A|^{-2\zeta} \sum_{\kappa, g} \left( \sum_{\kappa, g} \# + \sum_{\kappa, g} \#\# \right) \]  \quad \text{(60)}

\[ = |A|^{-k+\zeta} \sum_{x \neq x'} \text{tr} \left( T_{Q,x} T_{Q}^{-1/2} T_{Q,x'} T_{Q}^{-1/2} \right) + |A|^{-\zeta} \sum_{x} \text{tr} \left( T_{Q,x} T_{Q}^{-1/2} T_{Q,x} T_{Q}^{-1/2} \right) \]  \quad \text{(61)}

\[ \leq |A|^{-k+\zeta} + |A|^{-\zeta-h_{+}} \quad \left| T_{Q}^{-1/2} T_{Q,x} T_{Q}^{-1/2} = \Gamma_{x}, \quad T_{Q}^{-1/2} T_{Q,x} T_{Q}^{-1/2} = \Gamma_{x} \right. \]  \quad \text{(62)}

that is,

\[ \sum_{x} \text{tr} \left( T_{Q,x} \Gamma_{x} \right) = \mathbb{E}^{\text{gen}}[\Gamma] \leq \mathbb{E}^{\text{opt}}[\Gamma] = |A|^{-h_{\text{min}}(X|Q)} \leq |A|^{-h_{+}}. \quad \square \]  \quad \text{(63)}

By (**) , then

\[ |A|^{-1} ||\cdots||_{L^{2}(\gamma)} \leq |A|^{k/2 - 1 - k/2} \leq |A|^{(k-h_{+})/2 - 1} \cdot |A|^{(k-h_{+})/2}. \quad \square \]  \quad \text{(64)}

**Remark.** As a result, the tree process hardness is proved for bb84 [1] protocol strength.

**Addendum to asymptotic shape approximation with random trees**

Approximating locally finite tree shapes is a growing area of interest; using the trace-metric process methods, given in this article, can provide rather faster convergence techniques for analytic asymptotics, on trees of same topology (that is, “shape”), given by limit on increasing family of real (rooted, finite) trees which appear in the construction of dynamical approximation techniques, on interval \([0, 1]\). Moreover, the process discussed is just a random elements of the space \( C[0, 1] \) of continuous functions on \([0, 1]\), by a finite field set with respect to supersymmetric (quantum) basis structure measures, under suitable metric of stepwise intervals on right joint state asymptotics; all other processes have duality to the given fundamental nature of the interpolation (construction).

A necessary task is to show a **threshold of uniform convergence** for the equation (11). Such endeavor can be useful in tree asymptotics of braid (string) invariants for knot dualities: Fig. 6.

**Fig. 7:** Tree measure construction (left) for orientable-manifold, framed-knot spin-structure duality (right).
Appendix

(0.1) Closure of the $|\mathcal{A}^\otimes m|$ normalization

The 1 is trivial. For $(2 \times 2)$, i.e. $|\mathcal{A}^\otimes m| = 2$, write $t \Sigma$ and $t^{-1} \Sigma^{-1}$ (by cofactor or symmetric group):

$$t \Sigma = t \begin{pmatrix} \Sigma_{11} & \rho_{12}\sqrt{\Sigma_{11}\Sigma_{22}} \\ \rho_{12}\sqrt{\Sigma_{11}\Sigma_{22}} & \Sigma_{22} \end{pmatrix}, \quad t^{-1} \Sigma^{-1} = \frac{1}{t \Sigma_{11}\Sigma_{22} (1 - \rho_{12}^2)} \begin{pmatrix} \Sigma_{22} & -\rho_{12}\sqrt{\Sigma_{11}\Sigma_{22}} \\ -\rho_{12}\sqrt{\Sigma_{11}\Sigma_{22}} & \Sigma_{11} \end{pmatrix}$$

then we require the closure

$$\int_R \int_R \frac{1}{2\pi t \sqrt{\Sigma_{11}\Sigma_{22} (1 - \rho_{12}^2)}} \exp \left\{ -\frac{1}{2t} \left( \frac{(x_1 - \mu_1)^2}{(1 - \rho_{12}^2) \Sigma_{11}} - 2\frac{(x_1 - \mu_1)(x_2 - \mu_2) \rho_{12}}{(1 - \rho_{12}^2) \sqrt{\Sigma_{11}\Sigma_{22}}} + \frac{(x_2 - \mu_2)^2}{(1 - \rho_{12}^2) \Sigma_{22}} \right) \right\} dx_1 dx_2$$

$$= \int_R \int_R \frac{1}{2\pi t \sqrt{\Sigma_{11}\Sigma_{22} (1 - \rho_{12}^2)}} \exp \left\{ -\frac{1}{2t} \left( \frac{1}{\sqrt{1 - \rho_{12}^2}} \left( \frac{x_1 - \mu_1}{\sqrt{\Sigma_{11}}} - \frac{(x_2 - \mu_2) \rho_{12}}{\sqrt{\Sigma_{22}}} \right)^2 + \frac{(x_2 - \mu_2)^2}{\sqrt{\Sigma_{22}}} \right) \right\} dx_1 dx_2$$

which is the closure of the origin-centered symmetry

$$\mathbb{S}^{n-1} = \left\{ y_1, \ldots, y_n \in \mathbb{R} \mid 0 < r = \sqrt{\sum_{i=1}^{n} y_i^2} \right\}$$

for $n = 2$.

The closure of $2^m \mid m > 1$ follows by induction from the $(3 \times 3)$:

$$t \Sigma = t \begin{pmatrix} \Sigma_{11} & \rho_{12}\sqrt{\Sigma_{11}\Sigma_{22}} & \rho_{13}\sqrt{\Sigma_{11}\Sigma_{33}} \\ \rho_{12}\sqrt{\Sigma_{11}\Sigma_{22}} & \Sigma_{22} & \rho_{23}\sqrt{\Sigma_{22}\Sigma_{33}} \\ \rho_{13}\sqrt{\Sigma_{11}\Sigma_{33}} & \rho_{23}\sqrt{\Sigma_{22}\Sigma_{33}} & \Sigma_{33} \end{pmatrix}$$

$$t^{-1} \Sigma^{-1} = \frac{1}{t \Sigma_{11}\Sigma_{22}\Sigma_{33} (1 - \rho_{12}^2 - \rho_{13}^2 - \rho_{23}^2 + 2\rho_{12}\rho_{13}\rho_{23})} \times$$

$$\times \begin{pmatrix} (1 - \rho_{23}^2) \Sigma_{22}\Sigma_{33} & (\rho_{13}\rho_{23} - \rho_{12}) \Sigma_{33}\sqrt{\Sigma_{11}\Sigma_{22}} & (\rho_{12}\rho_{23} - \rho_{13}) \Sigma_{22}\sqrt{\Sigma_{11}\Sigma_{33}} \\ (\rho_{13}\rho_{23} - \rho_{12}) \Sigma_{33}\sqrt{\Sigma_{11}\Sigma_{22}} & (1 - \rho_{13}^2) \Sigma_{11}\Sigma_{33} & (\rho_{12}\rho_{13} - \rho_{23}) \Sigma_{11}\sqrt{\Sigma_{22}\Sigma_{33}} \\ (\rho_{12}\rho_{23} - \rho_{13}) \Sigma_{22}\sqrt{\Sigma_{11}\Sigma_{33}} & (\rho_{12}\rho_{13} - \rho_{23}) \Sigma_{11}\sqrt{\Sigma_{22}\Sigma_{33}} & (1 - \rho_{12}^2) \Sigma_{11}\Sigma_{22} \end{pmatrix}.$$ 

In general, by symmetric group $\mathcal{S}_n$,

$$\det(\Sigma^{n \times n}) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{t(\sigma)} \prod_{k=1}^{n} \Sigma_{k,\sigma(k)} \mid \Sigma_{i,j} = \Sigma_{j,i}; \quad t(\sigma) := (\sigma(1), \ldots, \sigma(n)) \rightarrow (1, \ldots, n)$$

for number of transpositions $t(\sigma)$ in $(\sigma(1), \ldots, \sigma(n)) \rightarrow (1, \ldots, n)$, for $\mathcal{S}_n$ automorphism $\sigma$.

| $n$ | $\Sigma_{11}\Sigma_{22}(1 - \rho_{12}^2)$ |
|-----|-------------------------------------|
| $n = 2$ | $\Sigma_{11}\Sigma_{22}\Sigma_{33}(1 - \rho_{12}^2 - \rho_{13}^2 - \rho_{23}^2 + 2\rho_{12}\rho_{13}\rho_{23})$ |
| $n = 4$ | $\Sigma_{11}\Sigma_{22}\Sigma_{33}\Sigma_{44}(1 - \rho_{12}^2 - \rho_{13}^2 - \rho_{14}^2 - \rho_{23}^2 - \rho_{24}^2 - \rho_{34}^2 + \rho_{12}\rho_{14}\rho_{24}^2 + \rho_{13}\rho_{24}^2 + \rho_{23}\rho_{24}^2 - 2\rho_{12}\rho_{13}\rho_{24}^2 - 2\rho_{12}\rho_{14}\rho_{23}^2 - 2\rho_{12}\rho_{14}\rho_{23}^2 - 2\rho_{12}\rho_{13}\rho_{23}^2$ |

Tab. 1: Table of $\det(\Sigma)$
\[ \Phi(\Gamma) = \frac{1}{n+1} \left( \sum_{\xi=0}^{n-1} |x_0 \otimes \cdots \otimes x_{\xi-1} \rangle \langle x_0 \otimes \cdots \otimes x_{\xi-1} | + \frac{1}{n+1} \right) = \frac{1}{n+1} \left( \frac{n}{n+1} + \frac{1}{n+1} \right) = \frac{n^2+1}{(n+1)^2} = \mathbb{E}_\text{gen}[\Gamma] \]

Tab. 2: An existence of \( \Phi \) for \(|\mathcal{A}^{\otimes m}|+1 \) quantum states, i.e. \((|\mathcal{A}^{\otimes m}|+1)\)-mixture, including mixed state.

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**Acknowledgment**

Research and authors were supported by the Lynn Bit Foundation in State of California.